Quantum Groupoids and their Hopf Cyclic Cohomology

Mohammad Hassanzadeh * and Bahram Rangipour *

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Abstract

We define $\times$-Hopf coalgebras as a quantization of groupoids. We develop Hopf cyclic theory with coefficients in stable-anti-Yetter-Drinfeld modules for $\times$-Hopf coalgebras. We use $\times$-Hopf coalgebras to study coextensions of coalgebras. Finally, we define equivariant $\times$-Hopf coalgebra coextensions and apply them as functors between categories of stable anti-Yetter-Drinfeld modules over $\times$-Hopf coalgebras involved in the coextension.

Introduction

It is known that $\times$-Hopf algebras quantize groupoids [BO]. However, this generalization is not comprehensive enough [BS]. One considers the free module generated by the nerve of a groupoid as a cyclic module and observes that the Hopf cyclic complex of the groupoid algebra viewed as a $\times$-Hopf algebra does not coincide with the nerve generated cyclic complex.

To remedy this lack of consistency, one may consider the groupoid coalgebra of a groupoid instead of groupoid algebra. This way one obtains an object that we call in this paper $\times$-Hopf coalgebra. Such $\times$-Hopf coalgebras are naturally generalizations of Hopf algebras and week Hopf algebras. The simplest example of such an object is the enveloping coalgebra of a coalgebra.

After defining $\times$-Hopf coalgebras axiomatically, we define Hopf cyclic theory for $\times$-Hopf coalgebras with coefficients in stable-anti-Yetter-Drinfeld modules. We observe that the cyclic complex of groupoid coalgebra perfectly coincides with the nerve generated cyclic complex.

*Department of Mathematics and Statistics, University of New Brunswick, Fredericton, NB, Canada
The use of Hopf algebras in studying (co)algebra (co)extensions is not new, for example see [BH]. Similarly \(\times\)-Hopf coalgebras are used to define appropriate Galois extension of algebras over algebras [BS]. We use \(\times\)-Hopf algebras to define Galois coextensions of coalgebras over coalgebras. By the work of Jara-Stefan [J-S] and Böhm-Stefan [BS] one knows that Hopf Galois extensions are an interesting source of stable-anti-Yetter-Drinfeld modules. This construction was generalized by our previous work [HR]: any equivariant Hopf Galois extension defines a functor between the category of stable-anti-Yetter-Drinfeld modules of the \(\times\)-Hopf algebras involved in the extension. To state a similar result we introduce equivariant Hopf coalgebras Galois coextensions and use them as a functor between the category of stable anti-Yetter-Drinfeld modules of the \(\times\)-Hopf coalgebras associated with the coextension. Finally, as a special case we cover Hopf algebra coextensions separately.

**Notations:** In this paper all objects are vector spaces over \(C\), the field of complex numbers. Coalgebras are denoted by \(C\) and \(D\), and Hopf algebras are denoted by \(H\). We use \(K\) and \(B\) as left and right \(\times\)-Hopf coalgebra, respectively. We denote left coactions over of coalgebras by \(\nabla(a) = a_{\langle -1 \rangle} \otimes a_{\langle 0 \rangle}\), left coactions of right \(\times\)-Hopf coalgebras by \(\nabla(a) = a_{\langle -1 \rangle} \otimes a_{\langle 0 \rangle}\), and left coaction of left \(\times\)-Hopf coalgebra by \(\nabla(a) = a_{\langle -1 \rangle} \otimes a_{\langle 0 \rangle}\). We use similar notations for right coactions of the above objects. If \(X\) and \(Y\) are right and left \(C\)-comodules, we define their cotensor product \(X \square_C Y\) as follows,

\[
X \square_C Y := \left\{ x \otimes y; \quad x_{\langle -0 \rangle} \otimes x_{\langle -1 \rangle} \otimes y = x \otimes y_{\langle -1 \rangle} \otimes y_{\langle 0 \rangle} \right\}
\]

**Contents**

1 Motivation: Groupoid homology revisited \hspace{1cm} 3

2 \(\times\)-Hopf coalgebras \hspace{1cm} 6
   2.1 Bicoalgebroids and \(\times\)-Hopf coalgebras \hspace{1cm} 6
   2.2 Examples \hspace{1cm} 12
   2.3 SAYD modules over \(\times\)-Hopf coalgebras \hspace{1cm} 15
   2.4 Symmetries via bicoalgebroids \hspace{1cm} 21

3 Hopf cyclic cohomology of \(\times\)-Hopf coalgebras \hspace{1cm} 22
   3.1 Cyclic cohomology of comodule coalgebras \hspace{1cm} 22
   3.2 Cyclic cohomology of comodule rings \hspace{1cm} 25
1 Motivation: Groupoid homology revisited

In this section we briefly recall groupoids and their homology and state the main motivation of $\times$-Hopf coalgebras as a quantization of groupoids. It is known that the Hopf cyclic homology, dual theory, of Hopf algebras generalizes group homology [KR02]. To any Hopf algebra equipped with an extra structure such as modular pair in involution, and stable anti Yetter-Drinfeld module in general, one associates a cyclic module [CM98] (see also [KR02, HKRS1, HKRS2]). When the Hopf algebra is the group algebra one observes that the corresponding cyclic module coincides with the cyclic module associated to the group in question [KR02].

We would like to find appropriate Hopf algebraic structure and define its cyclic theory to generalizes groupoids and their cyclic theory. One knows that groupoid algebra of a groupoid is $\times$-Hopf algebras. However its cyclic complex do not coincide with the cyclic complex of the groupoid generated by the nerve [BS].

We denote a groupoid by $\mathcal{G} = (\mathcal{G}^1, \mathcal{G}^0)$ where $\mathcal{G}^1$ denotes the set of morphisms and $\mathcal{G}^0$ is the set of objects. We denote the source and target maps by $t, s : \mathcal{G}^1 \to \mathcal{G}^0$, i.e., for $g \in \text{Mor}(x, y)$, $s(g) = x$, and $t(g) = y$.

A groupoid is called cyclic if for any $A \in \mathcal{G}^0$ there is a morphism $\theta_A \in \text{Mor}(A, A)$ such that $f \theta_A = \theta_B f$ for any $f \in \text{Mor}(A, B)$. Any groupoid is cyclic via $\theta_A = \text{Id}_A$. We denote a groupoid equipped with a cyclic structure $\theta$ by $(\mathcal{G}, \theta)$. Let us recall from [Bur] that for a groupoid $\mathcal{G} = (\mathcal{G}^1, \mathcal{G}^0)$ we denote by $\mathcal{G}_n$, for $n \geq 2$, the nerve of $\mathcal{G}$, as the set of all $(g_1, g_2, \ldots, g_n)$ such that $s(g_i) = t(g_{i+1})$, and $\mathcal{G}_i = \mathcal{G}^i$, for $i = 0, 1$. Then one easily observes that for any cyclic groupoid $(\mathcal{G}, \theta)$, $\mathcal{G}_\bullet$ becomes a cyclic set with the following morphisms.

$$\partial_i : \mathcal{G}_\bullet \to \mathcal{G}_{\bullet-1}, \quad 0 \leq i \leq \bullet \quad (1.1)$$

$$\sigma_j : \mathcal{G}_\bullet \to \mathcal{G}_{\bullet+1}, \quad 0 \leq j \leq \bullet \quad (1.2)$$

$$\tau : \mathcal{G}_\bullet \to \mathcal{G}_\bullet \quad (1.3)$$
which are defined by
\begin{align*}
\partial_0 [g_1, \ldots, g_n] &= [g_2, \ldots, g_n], \\
\partial_i [g_1, \ldots, g_n] &= [g_1, \ldots, g_i g_{i+1}, \ldots, g_n], \\
\partial_n [g_1, \ldots, g_n] &= [g_1, \ldots, g_n-1], \\
\sigma_j [g_1, \ldots, g_n] &= [g_1, \ldots, g_{i-1}, \text{Id}_{s(g_i)}, g_i, \ldots, g_n], \\
\tau [g_1, \ldots, g_n] &= [\theta_{s(g_n)} g_n^{-1} \cdots g_1^{-1}, g_1, \ldots, g_{n-1}].
\end{align*}

The special cases \( \partial_0, \partial_1 : \mathcal{G}_1 \to \mathcal{G}_0 \) are defined by \( \partial_0 = s \), and \( \partial_1 = t \). Finally \( \sigma_0 : \mathcal{G}_0 \to \mathcal{G}_1 \) is defined by \( \sigma_0(A) = \text{Id}_A \). We denote this cyclic module by \( C_{\bullet}(\mathcal{G}, \theta) \) and its cyclic homology by \( HC_{\bullet}(\mathcal{G}, \theta) \).

Now let \( \mathcal{B} = \mathbb{C} \mathcal{G}^1 \) be the coalgebra generated by \( \mathcal{G}^1 \), i.e. as a module it is the free module generated by \( \mathcal{G} \) with \( \Delta(x) = x \otimes x \), \( \varepsilon(x) = 1 \). Similarly we set \( \mathcal{C} := \mathbb{C} \mathcal{G}^0 \) as the coalgebra generated by \( \mathcal{G}^0 \).

The source and target maps induce the following map of coalgebras
\[ \alpha, \beta : \mathcal{B} \to \mathcal{C}, \quad \alpha(g) := s(g), \quad \beta(g) := t(g). \]

The maps \( \alpha \) and \( \beta \) induce a \( \mathcal{C} \)-bicomodule structure on \( \mathcal{C} \) as follows,
\begin{align*}
\nabla_L : \mathcal{B} &\to \mathcal{C} \otimes \mathcal{B}, \quad \nabla_L(g) = \beta(g) \otimes g \\
\nabla_R : \mathcal{B} &\to \mathcal{B} \otimes \mathcal{C}, \quad \nabla_R(g) = g \otimes \alpha(g).
\end{align*}

One then identifies \( \mathcal{B} \square_c \mathcal{C} \mathcal{B} \) with \( \mathcal{G}_2(\mathcal{G}) \) in the usual way. Here the cotensor space is defined by
\[ \mathcal{B} \square_c \mathcal{C} \mathcal{B} := \left\{ g \otimes h \in \mathcal{B} \otimes \mathcal{B} \mid g^{(1)} \otimes \alpha(g^{(2)}) \otimes h = g \otimes \beta(h^{(1)}) \otimes h^{(2)} \right\} \]

One then uses the composition rule of \( \mathcal{G} \) to define a multiplication
\[ \mu : \mathcal{B} \square_c \mathcal{C} \mathcal{B} \to \mathcal{B}, \quad \mu(g \otimes h) = gh. \]

There is also a map \( \eta : \mathcal{C} \to \mathcal{B} \) defined by \( \eta(A) = \text{Id}_A \). The inverse of \( \mathcal{G} \) induces the following map
\[ \nu^{-1} : \mathcal{B} \square_{\text{cop}} \mathcal{B} \to \mathcal{B} \square_c \mathcal{B}, \quad \nu^{-1}(g \otimes h) = g \otimes g^{-1} h; \]

which defines an inverse for the map
\[ \nu : \mathcal{B} \square_c \mathcal{B} \to \mathcal{B} \square_{\text{cop}} \mathcal{B}, \quad \nu(g \otimes h) = g \otimes gh. \]
In the next step we define our coefficients as $C$ to be acted and coacted by $B$. First we define the coaction of $B$ on $C$ by

\[ \nabla : C \to B \otimes C, \quad \nabla(A) = \theta_A \otimes A. \] (1.16)

We also let $B$ act on $C$ via

\[ C \otimes B \to C, \quad c \cdot g = \varepsilon(c)s(g) \] (1.17)

Via this information we offer a new cyclic module

\[ C_n(B, \theta_C) := \underbrace{B \Box C \Box \cdots \Box C}_\text{\# times} \Box_B C. \] (1.18)

Here $C^n(B, \theta_C)$ can be identified with the vector space generated by all elements $[g_0, \ldots, g_n, c] \in G^1 \times \cdots \times G^1 \times G^0$ such that

\[ s(g_i) = t(g_{i+1}), s(g_n) = t(g_0) \quad \text{and} \quad g_0 \cdots g_n = \theta_c. \] (1.19)

One easily observe that $C_n(B, \theta_C)$ is a cyclic module via the following morphisms.

\[ \partial_i([g_0, \ldots, g_n, c]) = [g_0, \ldots, g_ig_{i+1}, \ldots, g_n, c], \] (1.20)

\[ \partial_n([g_0, \ldots, g_n, c]) = [g_n, g_0, g_1, \ldots, g_{n-1}, c \cdot g_n^{-1}], \] (1.21)

\[ \sigma_j([g_0, \ldots, g_n, c]) = [g_0, \ldots, g_1, \text{Id}_{t(g_i)}, g_{i+1}, \ldots, g_n, c], \] (1.22)

\[ \tau([g_0, \ldots, g_n, c]) = [g_n, g_0, \ldots, g_{n-1}, c \cdot g_n^{-1}]. \] (1.23)

As the final step one sees that $C_n(B, \theta_C)$ and $C_n(G, \theta)$ are isomorphic as cyclic modules via

\[ I : C_n(B, \theta_C) \to C_n(G, \theta), \quad I([g_0, \ldots, g_n, c]) = [g_1, g_2, \ldots, g_n]. \] (1.24)

with inverse given by $I^{-1} : C_n(G, \theta) \to C_n(B, \theta_C),$

\[ I^{-1}([g_1, \ldots, g_n]) = [\theta_{s(g_n)}(g_1 \cdots g_n)^{-1}, g_1, \ldots, g_n, s(g_n)]. \] (1.25)

As we see above, we are faced with a new structure which is not $\times$-Hopf algebra. Our aim in the sequel sections is to study such objects and define a cyclic theory for them (see Example 2.12). We understand also that the desired cyclic theory is capable to handle coefficients and the above cyclic structure $\theta$ defines such a coefficients (see Remark 2.27).
2 \textit{\times-}Hopf coalgebras

In this section we introduce \textit{\times-}Hopf coalgebras as a formal dual of \textit{\times-}Hopf algebras. We carefully show that they define symmetries for (co)algebras with several objects. We apply them to define Hopf cyclic cohomology of (co)algebras with several objects under symmetry of \textit{\times-}Hopf coalgebras.

2.1 Bicoalgebroids and \textit{\times-}Hopf coalgebras

It is well known that the category of \textit{\times-}Hopf algebras contains Hopf algebras as a full subcategory. However, \textit{\times-}Hopf algebras are not a self-dual alike Hopf algebras. In fact this lack of self-duality is passed on to the \textit{\times-}Hopf algebras from bialgebroids.

There are many generalizations of Hopf algebras in the literature and the most natural one is \textit{\times-}Hopf algebras \cite{Sch}. In this paper we are interested in the formal dual notion of \textit{\times-}Hopf algebras. For the convenience of the reader for comparing them with the main object of this paper we present a very short description of \textit{\times-}Hopf algebras. Let $R$ and $K$ be algebras with two algebra maps $s : R \rightarrow K$ and $t : R^{op} \rightarrow K$, such that their ranges commute with one another. We equip $K$ and $K \otimes R K$ with $R$-bimodule structures using the source and target maps. We assume that there are $R$-bimodule maps $\Delta : K \rightarrow K \otimes R K$ and $\varepsilon : K \rightarrow R$ via which $K$ is an $R$-coring. The data $(K, s, t, \Delta, \varepsilon)$ is called a left $R$-bialgebroid if for $k_1, k_2 \in K$ and $r \in R$ the following identities hold

i) $k^{(1)}(1) t(r) \otimes_R k^{(2)} = k^{(1)} \otimes_R k^{(2)} s(r)$,

ii) $\Delta(1_K) = 1_K \otimes_R 1_K$, and $\Delta(k_1 k_2) = k^{(1)}_1 k^{(1)}_2 \otimes_R k^{(2)}_1 k^{(2)}_2$.

iii) $\varepsilon(1_K) = 1_R$ and $\varepsilon(k_1 k_2) = \varepsilon(k_1 s(\varepsilon(k_2)))$.

A left $R$-bialgebroid $(K, s, t, \Delta, \varepsilon)$ is said to be a left $\textit{\times-R}$-Hopf algebra if the Galois map

$$\nu : K \otimes_{R^{op}} K \rightarrow K \otimes_R K, \quad k \otimes_{R^{op}} k' \mapsto k^{(1)} \otimes_R k^{(2)} k', \quad (2.1)$$

is bijective. The role of antipode in $\textit{\times-R}$-Hopf algebras is played by the following map.

$$K \rightarrow K \otimes_{R^{op}} K, \quad k \mapsto \nu^{-1}(k \otimes_R 1_K). \quad (2.2)$$

One similarly has the definition of right bialgebroids and right $\textit{\times-}$Hopf algebras.
Now we define our main object of this paper. We start with recalling the notion of bicoalgebroid from \[BM\]. Roughly speaking, bicoalgebroids are dual notion of bialgebroids as they are defined by reversing the arrows in the definition of bialgebroids. Let \(K\) and \(C\) be two coalgebras with coalgebra maps \(\alpha : K \to C\) and \(\beta : K \to C_{\text{cop}}\), such that their images cocommute, i.e.

\[
\alpha(h^{(1)}) \otimes \beta(h^{(2)}) = \alpha(h^{(2)}) \otimes \beta(h^{(1)}). \tag{2.3}
\]

These maps endow \(K\) with a \(C\)-bicomodule structure, via left and right \(C\)-coactions

\[
\nabla_L(h) = \alpha(h^{(1)}) \otimes h^{(2)}, \quad \nabla_R(h) = h^{(2)} \otimes \beta(h^{(1)}). \tag{2.4}
\]

The next we assume two \(C\)-bicomodule maps \(\mu_K : K \Box_C K \to K\) and \(\eta_K : C \to K\) called the multiplication and the unit, respectively, which satisfy the following axioms:

\begin{enumerate}
  \item \(\sum_i \mu(g_i \otimes h_i^{(1)}) \otimes \alpha(h_i^{(2)}) = \mu(g_i^{(1)} \otimes h_i) \otimes \beta(g_i^{(2)}),\) \(\tag{2.5}\)
  \item \(\Delta \circ \mu \left( \sum_i g_i \otimes h_i \right) = \sum_i \mu(g_i^{(1)} \otimes h_i^{(1)}) \otimes \mu(g_i^{(2)} \otimes h_i^{(2)}),\) \(\tag{2.6}\)
  \item \(\varepsilon(g) \varepsilon(h) = \varepsilon \circ \mu(g \otimes h),\) \(\tag{2.7}\)
  \item \(\mu \circ (\eta \Box \text{Id}_K) \circ \nabla_L = \text{Id}_K = \mu \circ (\text{Id}_K \Box \eta) \circ \nabla_L,\) \(\tag{2.8}\)
  \item \(\Delta(\eta(c)) = \eta(c^{(1)}) \otimes \eta(\alpha(\eta(c^{(2)}))) = \eta(c^{(1)}) \otimes \eta(\beta(\eta(c^{(2)}))),\) \(\tag{2.9}\)
  \item \(\varepsilon(\eta(c)) = \varepsilon(c).\) \(\tag{2.10}\)
\end{enumerate}

We call \((K, \Delta_K, \varepsilon_K, \mu, \eta, \alpha, \beta, C)\) a left bicoalgebroid \([BM]\) (also see [Ball]). For the convenience of the reader, we use the notation \(\mu(h \otimes k) = hk\) for all \(h, k \in K\).

Let us briefly explain why the conditions i), . . . , vi) make sense. The relation \((2.5)\) is dual of the Takeuchi product. This relation makes sense because for \(g \otimes h \in K \Box_C K\), we have

\[
g^{(2)} \otimes \beta(g^{(1)}) \otimes h = g \otimes \alpha(h^{(1)}) \otimes h^{(2)}. \tag{2.11}\]

Applying \(\Delta \otimes \text{Id} \otimes \text{Id}\) on the both sides of \((2.11)\) we get

\[
g^{(2)} \otimes g^{(3)} \otimes \beta(g^{(1)}) \otimes h = g^{(1)} \otimes g^{(2)} \otimes \alpha(h^{(1)}) \otimes h^{(2)}, \tag{2.12}\]

\[
g^{(2)} \otimes g^{(3)} \otimes \beta(g^{(1)}) \otimes h = g^{(1)} \otimes g^{(2)} \otimes \alpha(h^{(1)}) \otimes h^{(2)}, \tag{2.12}\]
which implies $g(1) \otimes h \otimes g(2) \in \mathcal{K} \square C(\mathcal{K} \otimes \mathcal{K})$. This shows that the right hand side of (2.5) is well-defined. Here we consider $\mathcal{K} \otimes \mathcal{K}$ as a left $C$-comodule by $h \otimes h' \mapsto \nabla_C(h) \otimes h' = \alpha(h(1)) \otimes h(2) \otimes h'$. Similarly by applying $\text{Id} \otimes \text{Id} \otimes \Delta$ on the both sides of (2.11) we obtain
\[
g(2) \otimes \beta(g(1)) \otimes h(1) \otimes h(2) = g \otimes \alpha(h(1)) \otimes h(2) \otimes h(3). \tag{2.13}
\]
This implies
\[
g \otimes h(1) \otimes h(2) \in \mathcal{K} \square C(\mathcal{K} \otimes \mathcal{K}), \tag{2.14}
\]
which shows that the left hand side of (2.5) is well-defined. Also (2.12) and (2.13) imply $g(1) \otimes h(1) \otimes g(2) \otimes h(2) \in \mathcal{K} \square C(\mathcal{K} \otimes \mathcal{K} \otimes \mathcal{K})$, which shows (2.6) is well-defined. Here the left coaction of $C$ on $\mathcal{K} \otimes \mathcal{K} \otimes \mathcal{K}$ is defined by $\alpha$ on the very left argument.

**Lemma 2.1.** Let $\mathcal{K} = (\mathcal{K}, C)$ be a left bicoalgebroid. Then the following properties hold for all $h, k \in \mathcal{K}$.
\[
i) \alpha(\eta(h)) = h.
\]
\[
ii) \beta(\eta(h)) = h.
\]
\[
iii) \alpha(hk) = \alpha(h) \varepsilon(k).
\]
\[
iv) \beta(hk) = \varepsilon(h) \beta(k).
\]

**Proof.** The relation i) and ii) are obtained by applying $\varepsilon \otimes \varepsilon$ on the both sides of (2.9) and then using (2.10). The relations iii) and iv) are proved in [Bal page 8].

One notes that the relations i) and ii) in the above lemma are dual to the relations $\varepsilon(t(r)) = \varepsilon(s(r)) = r$ and relations iii) and iv) are dual to the relations $\Delta(s(r)) = s(r) \otimes 1$ and $\Delta(t(r)) = 1 \otimes t(r)$ for left $\times R$-bialgebroids.

**Definition 2.2.** A left bicoalgebroid $\mathcal{K}$ over the coalgebra $C$ is said to be a left $\times C$-Hopf coalgebra if the following Galois map
\[
\nu : \mathcal{K} \square C \mathcal{K} \longrightarrow \mathcal{K} \square \mathcal{C}_{\text{cop}} \mathcal{K}, \quad k \otimes k' \mapsto kk''(1) \otimes k''(2), \tag{2.15}
\]
is bijective. Here in the codomain of $\nu$, the $\mathcal{C}_{\text{cop}}$-comodule structures are given by right and left coactions defined by $\beta$, i.e.
\[
k \mapsto \beta(k(1)) \otimes k(2), \quad k \mapsto k(1) \otimes \beta(k(2)). \tag{2.16}
\]
While in the domain of \( \nu \), the \( \mathcal{C} \)-comodule structures are given by the original right and left coactions defined by \( \alpha \) and \( \beta \), i.e.

\[
k \mapsto \alpha(k^{(1)}) \otimes k^{(2)}, \quad k \mapsto k^{(2)} \otimes \beta(k^{(1)}).
\] (2.17)

To show that \( \nu \) is well-defined, we use (2.14) and prove that \( kk^{(1)} \otimes k^{(2)} \in K \square_{\mathcal{C}_{\text{cop}}} K \). Indeed,

\[
(kk^{(1)})^{(1)} \otimes \beta((kk^{(1)})^{(2)}) \otimes k^{(2)} = k^{(1)}k^{(1)} \otimes \beta(k^{(2)} \otimes k^{(2)}) \otimes k^{(3)} = k^{(1)} \varepsilon(k^{(2)}) \otimes \beta(k^{(2)}) \otimes k^{(3)} = kk^{(1)} \otimes \beta(k^{(2)}(1)) \otimes k^{(2)(2)}.
\]

Here Lemma (2.1)(iv) is used in the second equality.

We use the following summation notation for the image of \( \nu^{-1} \),

\[
\nu^{-1}(k \otimes k') = \nu^{-}(k, k') \otimes \nu^{+}(k, k').
\] (2.18)

We set \( \nu^{-1} = \nu^{-} \otimes \nu^{+} \), where there is no confusion.

**Lemma 2.3.** Let \( \mathcal{C} \) be a coalgebra and \( \mathcal{K} \) a left \( \times_{\mathcal{C}} \)-Hopf coalgebra. Then the following properties hold for \( \nu \) and \( \nu^{-1} \).

i) \( \nu^{-} \nu^{+(1)} \otimes \nu^{+(2)} = \text{Id}_{\mathcal{K} \square_{\mathcal{C}_{\text{cop}}} \mathcal{K}} \).

ii) \( \nu^{-}(kk^{(1)}, k^{(2)}) \otimes \nu^{+}(kk^{(1)}, k^{(2)}) = k \otimes k' \).

iii) \( \varepsilon(\nu^{-}(k \otimes k'))\varepsilon(\nu^{+}(k \otimes k')) = \varepsilon(k)\varepsilon(k') \).

iv) \( \varepsilon(\nu^{-}(k \otimes k'))\nu^{+}(k \otimes k') = \varepsilon(k)k' \).

v) \( \nu^{-}(k \otimes k')\nu^{+}(k \otimes k') = \varepsilon(k')k \).

**Proof.** The relation i) is equivalent to \( \nu \nu^{-1} = \text{Id} \). The relation ii) is equivalent to \( \nu^{-1} \nu = \text{Id} \). The relation iii) is proved by applying \( \varepsilon \otimes \varepsilon \) to the both sides of i) and then using (2.7). The relation iv) is obtained by applying \( \varepsilon \otimes \text{Id} \) on both hand sides of i). The relation v) is shown by applying \( \text{Id} \otimes \varepsilon \) on both hand sides of i). 

One notes that for a left \( \times_{\mathcal{C}} \)-Hopf coalgebra \( \mathcal{K} \), the maps \( \nu \) and hence \( \nu^{-1} \) are both right \( \mathcal{K} \)-comodule maps where the right \( \mathcal{K} \)-comodule structures of \( \mathcal{K} \square_{\mathcal{C}} \mathcal{K} \) and \( \mathcal{K} \square_{\mathcal{C}_{\text{cop}}} \mathcal{K} \) are given by \( k \otimes k' \mapsto k \otimes k^{(1)} \otimes k^{(2)} \). The right \( \mathcal{K} \)-comodule map property of \( \nu^{-1} \) is equivalent to

\[
\nu_{-}(k \otimes k') \otimes \nu_{+}(k \otimes k')^{(1)} \otimes \nu_{+}(k \otimes k')^{(2)} = \nu_{-}(k \otimes k^{(1)} \otimes k^{(2)}).$


Definition 2.4. A right bicoalgebroid \((\mathcal{B}, \Delta, \varepsilon, \mu, \eta, \alpha, \beta, C)\) consists of coalgebras \(\mathcal{B}\) and \(C\) with coalgebra maps \(\alpha : \mathcal{B} \to C\) and \(\beta : \mathcal{B} \to C_{\text{cop}}\), such that their images cocommute, i.e. \(\alpha(b^{(1)}) \otimes \beta(b^{(2)}) = \alpha(b^{(2)}) \otimes \beta(b^{(1)})\). These maps furnish \(\mathcal{B}\) with a \(C\)-bicomodule structure, via left and right \(C\)-coactions

\[
\nabla_L(b) = \beta(b^{(2)}) \otimes b^{(1)}, \quad \nabla_R(b) = b^{(1)} \otimes \alpha(b^{(2)})
\]

\((2.19)\)

\(C\)-bicomodule maps \(\mu_{\mathcal{B}} : \mathcal{B} \square_C \mathcal{B} \to \mathcal{B}\) and \(\eta_{\mathcal{B}} : C \to \mathcal{B}\) making \(\mathcal{B}\) an algebra in \(C M^C\), satisfying the following properties:

i) \[\sum_i \mu(b_i \otimes b_i^{(2)}) \otimes \beta(b_i^{(1)}) = \mu(b \otimes b') \otimes \alpha(b^{(1)})\] \((2.20)\)

ii) \[\Delta \circ \mu(\sum_i b_i \otimes b_i') = \sum_i \mu(b_i \otimes b_i^{(1)}) \otimes \mu(b_i \otimes b_i^{(2)})\] \((2.21)\)

iii) \(\varepsilon(b)\varepsilon(b') = \varepsilon \circ \mu(b \otimes b')\) \((2.22)\)

iv) \(\mu \circ (\text{Id} \square \eta) \circ^R \nabla_C = \mathcal{B} = \mu \circ (\eta \square \text{Id}) \circ^L \nabla_C\) \((2.23)\)

v) \(\Delta(\eta(c)) = \eta(\alpha(\eta(c)^{(1)})) \otimes \eta(c)^{(2)} = \eta(\beta(\eta(c)^{(1)})) \otimes \eta(c)^{(2)}\) \((2.24)\)

vi) \(\varepsilon(\eta(c)) = \varepsilon(c)\) \((2.25)\)

For the convenience of the reader, we use the notation \(\mu(b \otimes b') = bb'\) for all \(b, b' \in \mathcal{B}\). One notes that the relation \((2.20)\) makes sense because if \(b \otimes b' \in \mathcal{B} \square_C \mathcal{B}\), then

\[b \otimes \beta(b^{(2)}) \otimes b^{(1)} = b^{(1)} \otimes \alpha(b^{(2)}) \otimes b'.\] \((2.26)\)

By applying \(\text{Id} \otimes \text{Id} \otimes \varepsilon\) on the both sides of \((2.20)\) we get

\[b \otimes \beta(b^{(3)}) \otimes b^{(1)} \otimes b^{(2)} = b^{(1)} \otimes \alpha(b^{(2)}) \otimes b^{(1)} \otimes b^{(2)},\] \((2.27)\)

which is equivalent to \(b \otimes b^{(2)} \otimes b^{(1)} \in \mathcal{B} \square_C \mathcal{B} \otimes \mathcal{B}\). So the left hand side of \((2.20)\) is well-defined. To show that the right hand side of \((2.20)\) is well-defined, one applies \(\Delta \otimes \text{Id} \otimes \text{Id}\) on the both sides of \((2.26)\) to obtain

\[b^{(1)} \otimes b^{(2)} \otimes \beta(b^{(3)}) \otimes b^{(1)} = b^{(1)} \otimes b^{(2)} \otimes \alpha(b^{(3)}) \otimes b'.\]

One also notes that the unit map \(\eta : C \to \mathcal{B}\) is a right \(C\)-comodule map, i.e.

\[\eta(c)^{(1)} \otimes c^{(2)} = \eta(c)^{(1)} \otimes \alpha(\eta(c)^{(2)}),\] \((2.28)\)

where \(C\) is a right \(C\)-comodule by \(\Delta_C\) and the right \(C\)-comodule structure of \(\mathcal{B}\) is given in \((2.19)\).
Lemma 2.5. Let $C$ be a coalgebra and $B$ be a right $\times_C$-bicoalgebroid. The following properties hold for all $b, b' \in B$.

i) $\alpha(\eta(b)) = b$.

ii) $\beta(\eta(b)) = b$.

iii) $\alpha(bb') = \varepsilon(b)\alpha(b')$.

iv) $\beta(bb') = \beta(b)\varepsilon(b')$.

Definition 2.6. Let $C$ be a coalgebra. A right $\times_C$-bicoalgebroid $B$ is said to be a right $\times_C$-Hopf coalgebra provided that the map

$$\nu : B \Box_C B \rightarrow B \Box_{C_{\text{cop}}} B, \quad b \otimes b' \mapsto b^{(1)} \otimes b^{(2)} b', \quad (2.29)$$

is bijective. In the codomain of the map in (2.29), $C_{\text{cop}}$-comodule structures are given by right and left coaction by $\beta$, i.e.

$$b \mapsto \beta(b^{(1)}) \otimes b^{(2)}, \quad b \mapsto b^{(1)} \otimes \beta(b^{(2)}). \quad (2.30)$$

In the domain of the map in (2.29), $C$-comodule structures are given by the original right and left coactions by $\alpha$ and $\beta$, i.e.

$$b \mapsto \beta(b^{(2)}) \otimes b^{(1)}, \quad b \mapsto b^{(1)} \otimes \alpha(b^{(2)}). \quad (2.31)$$

The map $\nu$ introduced in (2.29) is well-defined by (2.27) and the fact that one has

$$b^{(1)} \otimes b^{(2)} b' \in B \Box_{C_{\text{cop}}} B.$$

In fact,

$$b^{(1)} \otimes \beta(b^{(2)}) \otimes b^{(3)} b'^{(1)} b'^{(2)} = b^{(1)} \otimes \beta(b^{(2)}) b^{(1)(1)} \otimes b^{(3)} b'^{(2)} = b^{(1)} \otimes \beta(b^{(2)}) \otimes b^{(3)} b'.$$

We use Lemma (2.3) (iv) in the second equality. We denote the inverse map $\nu^{-1}$ by the following summation notation,

$$\nu^{-1}(b \Box_{C_{\text{cop}}} b') = \nu^{-}(b, b') \otimes \nu^{+}(b, b') \quad (2.32)$$

If there is no confusion we use $\nu^{-1} := \nu^{-} \otimes \nu^{+}$. Similar to the Lemma (2.3), one can prove the following lemma.
Lemma 2.7. Let \( C \) be a coalgebra and \( B \) a right \( \times_C \)-Hopf coalgebra. Then the following properties hold.

1. \( \nu^{-1} \otimes \nu^{-2} \nu^+ = \text{Id}_{B \square C_{\text{cop}}} \).
2. \( \nu^{-}(b^{(1)}, b^{(2)} b') \otimes \nu^{+}(b^{(1)}, b^{(2)} b') = b \otimes b' \).
3. \( \varepsilon(\nu(b \otimes b'))\varepsilon(\nu^+(b \otimes b')) = \varepsilon(b)\varepsilon(b') \).
4. \( \varepsilon(\nu^+(b \otimes b'))\nu^-(b \otimes b') = \varepsilon(b')b \).
5. \( \nu^{-}(b \otimes b')\nu^+(b \otimes b') = \varepsilon(b)b' \).

One notes that for any right \( \times \)-Hopf coalgebra \( B \), the map \( \nu \) and therefore \( \nu^{-1} \) are both left \( B \)-comodule maps where the left \( B \)-comodule structures of \( B \square C \) and \( B \square C_{\text{cop}} \) are given by

\[
\nu^{-}(b \otimes b') \mapsto - \rightarrow b^{(1)} \otimes b^{(2)} \otimes b'.
\]

The left \( B \)-comodule structure of \( \nu^{-1} \) is equivalent to

\[
\nu^{-}(b \otimes b') \nu^{+}(b \otimes b') = \nu^{-}(b \otimes b') \nu^{+}(b \otimes b'). \tag{2.33}
\]

2.2 Examples

Example 2.8. The \( \times \)-Hopf coalgebras extend the notion of Hopf algebras. Recall that a left Hopf algebra is a bialgebra \( B \) endowed with a left antipode \( S \), i.e. \( S : B \rightarrow B \) where \( S(b^{(1)})b^{(2)} = b \), for all \( b \in B \) [GNT]. Similarly a right Hopf algebra is a bialgebra endowed with a right antipode. Obviously any Hopf algebra is both a left and a right Hopf algebra. If \( H \) is a bialgebra, it is a left \( \times_C \)-Hopf coalgebra, if and only if

\[
\nu(h \otimes h') = hh'^{(1)} \otimes h'^{(2)}, \quad \nu^{-1}(h \otimes h') = hS(h'^{(1)}) \otimes h'^{(2)}. \tag{2.34}
\]

\( \nu^{-1} \nu = \text{id} \) implies that \( H \) is a right Hopf algebra and \( \nu \nu^{-1} = \text{id} \) is equivalent to the fact that \( H \) is a left Hopf algebra. Therefore \( H \) should be a Hopf algebra. Also \( H \) is a right \( \times_C \)-Hopf coalgebra if and only if

\[
\nu(b \otimes b') = b'^{(1)} \otimes b'^{(2)} b', \quad \nu^{-1}(b \otimes b') = b'^{(1)} \otimes S(b'^{(2)}) b'. \tag{2.35}
\]

Example 2.9. The simplest example of a \( \times \)-coalgebra which is not a Hopf algebra comes as follows. The co-enveloping coalgebra \( C^e = C \otimes C_{\text{cop}} \) is a left \( \times_C \)-Hopf coalgebra where the source and target maps are given by

\[
\alpha : C \otimes C_{\text{cop}} \rightarrow C, \quad c \otimes c' \mapsto c \varepsilon(c'); \quad \beta : C \otimes C_{\text{cop}} \rightarrow C_{\text{cop}}, \quad c \otimes c' \mapsto \varepsilon(c)c'.
\]
multiplication by 
\[ C \otimes C_{\text{cop}} \square C \otimes C_{\text{cop}} \rightarrow C \otimes C_{\text{cop}}, \quad c \otimes c' \square d \otimes d' \mapsto \varepsilon(c')\varepsilon(d)c \otimes d', \]
unit map by 
\[ \eta : C \rightarrow C \otimes C_{\text{cop}}, \quad c \mapsto c^{(1)} \otimes c^{(2)}, \]
and 
\[ \nu(c \otimes c' \square C d \otimes d') = \varepsilon(c)c \otimes d^{(2)} \square C_{\text{cop}} d \otimes d^{(1)}, \]
\[ \nu^{-1}(c \otimes c' \square C_{\text{cop}} d \otimes d') = \varepsilon(c)c \otimes d^{(1)} \square C d^{(2)} \otimes d'. \]

**Example 2.10.** The co-enveloping coalgebra \( C^e = C \otimes C_{\text{cop}} \) is a right \( \times_C \)-Hopf coalgebra where the source and target maps are given by 
\[ \alpha : C \otimes C_{\text{cop}} \rightarrow C, \quad c \otimes c' \mapsto c\varepsilon(c'), \]
\[ \beta : C \otimes C_{\text{cop}} \rightarrow C_{\text{cop}}, \quad c \otimes c' \mapsto \varepsilon(c)c', \]
multiplication, unite, and inverse maps by 
\[ C \otimes C_{\text{cop}} \square C \otimes C_{\text{cop}} \rightarrow C \otimes C_{\text{cop}}, \quad c \otimes c' \square d \otimes d' \mapsto \varepsilon(c)\varepsilon(d')d \otimes c', \]
\[ \eta : C \rightarrow C \otimes C_{\text{cop}}, \quad c \mapsto c^{(2)} \otimes c^{(1)}, \]
\[ \nu(c \otimes c' \square C d \otimes d') = \varepsilon(c_4) c_1^{(2)} \square C_{\text{cop}} c_2^{(1)} \otimes c_3, \]
\[ \nu^{-1}(c \otimes c' \square C_{\text{cop}} d \otimes d') = \varepsilon(c_4) c_1^{(1)} \otimes c_2 \square C_{\text{cop}} c_3 \otimes c_1^{(2)}. \]

The following example introduce a subcategory of \( \times \)-Hopf coalgebras. They are dual notion of Hopf algebroids defined in [BO], hence we call them Hopf coalgebroids.

**Example 2.11.** Let \( C \) and \( D \) be two coalgebras. A Hopf coalgebroid over the base coalgebras \( C \) and \( D \) is a triple \((K, B, S)\). Here \((K, \Delta_K, \alpha_K, \beta_K, \eta_K, \mu_K)\) is a left \( \times_C \)-Hopf coalgebra and \((B, \Delta_B, \alpha_B, \beta_B, \eta_B, \mu_B)\) is a right \( \times_D \)-Hopf coalgebra such that as a coalgebra \( K = B = H \) and there exists a \( C \)-linear map \( S : H \rightarrow H \), called antipode. These structures are subject to the following axioms.

i) \[ \beta_B \circ \eta_K \circ \alpha_K = \beta_B, \quad \alpha_B \circ \eta_K \circ \beta_K = \alpha_B, \]
\[ \beta_K \circ \eta_B \circ \alpha_B = \beta_K, \quad \alpha_K \circ \eta_B \circ \beta_B = \alpha_K. \]

ii) \[ \mu_B \circ (\mu_K \square D \text{Id}_H) = \mu_K \circ \text{Id}_H \square C \mu_B, \]
\[ \mu_K \circ (\mu_B \square C \text{Id}_H) = \mu_B \circ \text{Id}_H \square D \mu_K. \]
iii) \( \beta_K(S(h)^{(1)}) \otimes S(h)^{(2)} \otimes \beta_B(S(h)^{(3)}) = \alpha_K(h^{(3)}) \otimes S(h)^{(2)} \otimes \alpha_B(h^{(1)}). \)

iv) \( \mu_K \circ (S \square C \text{Id}_H) \circ \Delta_K = \eta_B \circ \alpha_B, \quad \mu_B \circ (\text{Id}_H \square D S) \circ \Delta_B = \eta_K \circ \alpha_K. \)

**Example 2.12.** Let \( G = (G^1, G^0) \) be a groupoid and \( B := C^G_1 \) and \( C := C^G_0 \) be the groupoid coalgebra as defined in Section [I] with trivial coalgebra structure. Then via the source and target maps \( s, t : B \to C \) defined by \( \alpha(g) := s(g) \), and \( \beta(g) := t(g) \), where \( s \) and \( t \) are source and target maps of \( G \), it is easy to verify that \( B \) is a right \( \times_C \)-Hopf coalgebra if the antipode is defined by

\[ \nu^{-1}(g \otimes h) = g \otimes g^{-1}h. \]

**Example 2.13.** Any weak Hopf algebras defines a \( \times \)-Hopf coalgebra as follows. First let us recall that a weak bialgebra \( B \) is an unital algebra and counital coalgebra with the following compatibility conditions between the algebra and coalgebra structures,

\[
(\Delta(1_B) \otimes 1_B)(1_B \otimes \Delta(1_B)) = (\Delta \otimes \text{Id}_B) \circ \Delta(1_B) =
(1_B \otimes \Delta(1_B))(\Delta(1_B) \otimes 1_B),
\]

\[
\varepsilon(b^{(1)}')\varepsilon(1^{(2)}b') = \varepsilon(bb') = \varepsilon(b^{(1)}') \varepsilon(1^{(1)}b').
\]

The first condition generalizes the the axioms of unitality of coproduct \( \Delta \) and the second one generalizes the algebra map property of \( \varepsilon \) of a bialgebras. In fact its coproduct is not an unital map, i.e. \( \Delta(1_B) = 1^{(1)} \otimes 1^{(2)} \neq 1_B \otimes 1_B. \)

A weak Hopf algebra \([BO]\) is weak bialgebra \( B \) equipped with an antipode map \( S : B \to B \), satisfying the following conditions

\[
b^{(1)}S(b^{(2)}) = \varepsilon(1^{(1)}b)1^{(2)}B, \quad S(b^{(1)})b^{(2)} = 1^{(1)}B \varepsilon(b1^{(2)}), \quad S(b^{(1)})b^{(2)}S(b^{(3)}) = S(b).\]

Every weak Hopf algebra \( B \) is a \( \times_C \)-Hopf coalgebra. Here \( C = B/\ker \xi \) where \( \xi : B \to B \) is given by \( \xi(b) = \varepsilon(1^{(1)}b)1^{(2)} \). The source and target maps are given by

\[
\alpha(b) = \pi(b), \quad \beta(b) = \pi(S^{-1}(b)),
\]

where \( \pi : B \to C \) is the canonical projection map. The multiplication \( \mu \) is given by the original multiplication of the algebra structure of \( B \). The unit map of the \( \times_C \)-Hopf coalgebra is given by \( \eta = \xi \). Also \( \nu(b \otimes b') = bb'^{(1)} \otimes b'^{(2)} \) and \( \nu^{-1}(b \otimes b') = bS(b'^{(1)}) \otimes b'^{(2)}. \)
2.3 SAYD modules over $\times$-Hopf coalgebras

In this subsection, we define modules, comodules, and stable anti Yetter-Drienfeld (SAYD) modules over $\times$-Hopf coalgebras. Then we present examples of stable anti Yetter-Drienfeld modules over the enveloping $\times_C$-Hopf coalgebra $C \otimes C_{\text{cop}}$. We also show that for a cyclic groupoid $(G, \theta)$ the coefficients $\theta C$ defined in Section 1 defines a SAYD module over the groupoid $\times$-Hopf coalgebra $C G^1$. At the end we define the symmetries produced by $\times$-Hopf coalgebras on the algebras and coalgebras.

**Definition 2.14.** A right module $M$ over a left $\times_C$-Hopf coalgebra $K$ is a right $\mathbb{C}$-comodule where the action $\triangleright : M \square C K \to M$ is a right $\mathbb{C}$-comodule map. Here the right $\mathbb{C}$-comodule structure of $M \square C K$ is given by $m \otimes k \mapsto - \otimes m \otimes \beta(k(1))$. Therefore the right $\mathbb{C}$-comodule property of the action is equivalent to 

$$ (m \triangleright k)_{<0>} \otimes (m \triangleright k)_{<1>} = m \triangleright k^{(2)} \otimes \beta(k^{(1)}), \quad m \in M, k \in K. \quad (2.36) $$

**Definition 2.15.** A left module $M$ over a left $\times_C$-Hopf coalgebras $\mathcal{K}$ is a left $\mathbb{C}$-comodule where the action $\triangleleft : \mathcal{K} \square C M \to M$ is a left $\mathbb{C}$-comodule map where the left $\mathbb{C}$-comodule structure of $\mathcal{K} \square C M$ is given by $k \otimes m \mapsto \alpha(k(1)) \otimes k^{(2)} \otimes m$.

A left $\mathcal{K}$-module $M$ can be equipped with a $\mathbb{C}$-bicomodule structure by introducing the following right $\mathbb{C}$-coaction

$$ m \mapsto \eta(m_{<\beta>}^{(1)} \triangleright m_{<\alpha>} \otimes \alpha(m_{<\beta>}^{(2)}). \quad (2.37) $$

One checks that via these comodule structures, $M$ becomes a $\mathbb{C}$-bicomodule and the left $\mathcal{K}$-action is a $\mathbb{C}$-bicomodule map, i.e.

$$ (k \triangleright m)_{<\beta>} \otimes (k \triangleright m)_{<\alpha>} = \alpha(k^{(1)}) \otimes k^{(2)} \triangleright m \quad (2.38) $$

$$ (k \triangleright m)_{<\alpha>} \otimes (k \triangleright m)_{<\beta>} = k^{(1)} \triangleright m \otimes \alpha(k^{(2)}), \quad (2.39) $$

and for all $m \in M$ and $k \in \mathcal{K}$,

$$ k \triangleright m_{<\alpha>} \otimes m_{<\beta>} = k^{(1)} \triangleright m \otimes \beta(k^{(2)}). \quad (2.40) $$

**Definition 2.16.** Let $\mathcal{K}$ be a left $\times_C$-Hopf coalgebra. A right $\mathcal{K}$-comodule $M$ is a right $\mathcal{K}$-comodule, $\rho : M \to M \otimes \mathcal{K}$, where $\mathcal{K}$ is considered as a coalgebra. A right $\mathcal{K}$-comodule $M$ can be naturally equipped with a $\mathbb{C}$-bicomodule where the right $\mathbb{C}$-coaction is given by

$$ m_{<\alpha>} \otimes m_{<\beta>} := m_{<0>} \otimes \alpha(m_{<1>}), \quad (2.41) $$

15
and the left $C$-coaction by

$$m_{<\tau>} \otimes m_{<\sigma>} := \beta(m_{<1>}) \otimes m_{<0>}. \tag{2.42}$$

One observes that the map $\rho$ actually lands in $M \square_{C} K$, and the induced map $\rho : M \rightarrow M \square_{C} K$, is coassociative, counital, and $C$-bicomodule map.

**Definition 2.17.** Let $K$ be a left $\times_{C}$-Hopf coalgebra. A left $K$-comodule $M$ is a left $K$-comodule, $\rho : M \rightarrow K \otimes M$, where $K$ considered as a coalgebra. A left $K$-comodule $M$ can be naturally equipped with a $C$-bicomodule by

$$\rho_{C}^{R}(m) := m_{<0>} \otimes \beta(m_{<1>}), \quad \rho_{C}^{L}(m) := \alpha(m_{<1>}) \otimes m_{<0>}. \tag{2.43}$$

The map $\rho$ lands in $K \square_{C} M$, and the induced map $\rho : M \rightarrow K \square_{C} M$ which is a coassociative, counital, and $C$-bicomodule map.

Now we are ready to define stable anti Yetter-Drinfeld modules for $\times$-Hopf coalgebras.

**Definition 2.18.** Let $K$ be a left $\times_{C}$-Hopf coalgebra, $M$ a left $K$-module and a right $K$-comodule. We call $M$ a left-right anti Yetter-Drinfeld module over $K$ if

i) The right coaction of $C$ induced on $M$ via (2.41) coincides with the canonical coaction defined in (2.37).

ii) For all $k \otimes m \in K \square_{C} M$ we have

$$(k \triangleright m)_{<0>} \otimes (k \triangleright m)_{<1>} = \nu^{+}(m_{<1>}, k^{(1)}) \triangleright m_{<0>} \otimes k^{(2)} \nu^{-}(m_{<1>}, k^{(1)}). \tag{2.44}$$

We call $M$ stable if $m_{<1>} m_{<0>} = m$.

This definition generalizes the definition of stable anti Yetter-Drinfeld modules over Hopf algebras [HKRS1, Definition 4.1]. One similarly defines modules and comodules over right $\times$-Hopf coalgebras.

**Definition 2.19.** Let $B$ be a right $\times_{C}$-Hopf coalgebra, $M$ a right $B$-module and a left $B$-comodule. We call $M$ a right-left anti Yetter-Drinfeld module over $B$ if

i) The left coaction of $C$ on $M$ is the same as the following canonical coaction

$$m \rightarrow \alpha(\eta(m_{<\tau>})^{(1)}) \otimes m_{<\sigma>} \triangleleft \eta(m_{<\tau>})^{(2)}. \tag{2.45}$$

ii) For all $m \otimes b \in M \square_{C} B$ we have

$$(m \triangleright b)_{<1>} \otimes (m \triangleright b)_{<0>} = \nu^{+}(b^{(2)}, m_{<1>}) b^{(1)} \otimes m_{<0>} \triangleleft \nu^{-}(b^{(2)}, m_{<1>}). \tag{2.46}$$

We call $M$ stable if $m_{<0>} \triangleleft m_{<1>} = m$. 

16
**Definition 2.20.** A right group-like of a left $\times_C$-Hopf coalgebra $\mathcal{K}$ is a linear map $\delta : C \rightarrow \mathcal{K}$ satisfying the following conditions

\begin{align*}
\delta(c)^{(1)} \otimes \alpha(\delta(c)^{(2)}) &= \delta(c)^{(1)} \otimes c^{(2)}, \\
\delta(\alpha(\delta(c)^{(1)})) \otimes \delta(c)^{(2)} &= \delta(c)^{(1)} \otimes \delta(c)^{(2)}, \\
\varepsilon(\delta(c)) &= \varepsilon(c). 
\end{align*}

**Example 2.21.** The unit map $\eta$ is a right group-like of a right $\times_C$-Hopf coalgebra. The above three conditions for $\eta$ are equivalent to (2.28), (2.24) and (2.25), respectively. One also notes that for a Hopf algebra the above definition reduces to the original definition of group-like elements.

**Definition 2.22.** A character of a $\times_C$-Hopf coalgebra $\mathcal{K}$, left or right, is a ring morphism $\sigma : \mathcal{K} \rightarrow C$, where $\sigma(\eta_{\mathcal{K}}(c)) = c$. Here $C$ is considered to be a ring on itself, i.e. the product $C \Box C \rightarrow C$ is given by $x \otimes y \mapsto \varepsilon(x)y$.

As an example, the source and target maps $\alpha$ and $\beta$ are characters of the $\times_C$-Hopf coalgebra $\mathcal{K} = C \otimes C_{\text{cop}}$.

Let $\mathcal{K}$ be a left $\times_C$-Hopf coalgebra, $\delta : C \rightarrow \mathcal{K}$ a right group-like, and $\sigma : \mathcal{K} \rightarrow C$ a character of $\mathcal{K}$. The following define an action and a coaction

\begin{equation}
\begin{aligned}
&c \mapsto \alpha(\delta(c)^{(1)}) \otimes \delta(c)^{(2)}, & k \cdot c &= \alpha(k^{(1)})\varepsilon(c)\varepsilon(\sigma(k^{(2)})).
\end{aligned}
\end{equation}

**Lemma 2.23.** The action defined in (2.50) is associative.

**Proof.** For any $k_1, k_2 \in \mathcal{K}$ and $c \in C$ we have,

\begin{align*}
k_1 k_2 \cdot c &= \alpha(k_1 k_2)^{(1)} \varepsilon(c)\varepsilon(\sigma(k_1 k_2)^{(2)}) \\
&= \alpha(k_1^{(1)} k_2^{(1)}) \varepsilon(c)\varepsilon(\sigma(k_1^{(2)} k_2^{(2)})) \\
&= \alpha(k_1^{(1)})\varepsilon(k_2^{(1)})\varepsilon(c)\varepsilon(\sigma(k_1^{(2)}))\varepsilon(\sigma(k_2^{(2)})) \\
&= \alpha(k_1^{(1)})\varepsilon(\sigma(k_2^{(1)}))\varepsilon(c)\varepsilon(\sigma(k_1^{(2)}))\varepsilon(\sigma(k_2^{(2)})) \\
&= k_1 \cdot [\alpha(k_2^{(1)})\varepsilon(c)\varepsilon(\sigma(k_2^{(2)}))] = k_1 \cdot (k_2 \cdot c).
\end{align*}

We use (2.6) in the second equality. We apply Lemma 2.1(iii), the multiplicity of the ring map $\sigma$, and the multiplication rule in $C$ in the third equality. Finally we use the counitality of $\alpha$ in the fourth equality. \hfill $\square$

**Lemma 2.24.** The coaction defined in (2.50) is coassociative and counital.
Proof. It is straightforwardly seen that,

\[ c_{<0>} \otimes c_{<1>}(1) \otimes c_{<1>}(2) = \alpha(\delta(c)(1)) \otimes \delta(c)(2) \otimes \delta(c)(3) \]

\[ = \alpha(\delta(\alpha(\delta(c)(1))))(1) \otimes \delta(\alpha(\delta(c)(1)))(2) \otimes \delta(c)(2) = c_{<0>} \otimes c_{<0> \otimes c_{<1>}}. \]

Here we use (2.48) in the second equality. The counitality of the coaction comes as follows.

\[ \varepsilon(\delta(c)(2)) \alpha(\delta(c)(1)) = \alpha(\delta(c)) = c. \]

For the last equality, we apply \( \varepsilon \otimes \text{Id} \) on both hand sides of (2.47) and then we use (2.49). \( \square \)

**Proposition 2.25.** The action and coaction defined in (2.50) amount to an AYD module over \( K \) if and only if

\[ \varepsilon(\sigma(\nu^+(\delta(c) \otimes k))) \varepsilon(\nu^-(\delta(c) \otimes k)) \alpha(\nu^+(\delta(c) \otimes k)) \]

\[ = \varepsilon(c) \varepsilon(\sigma(\delta(\nu^+(\delta(c))))), \]

and

\[ \alpha(\eta(c)(2)) \varepsilon(\sigma(\eta(c)(1))) = \alpha(\delta(c)). \]

(2.51)

It is stable if and only if

\[ \alpha(\delta(c)(1)) \varepsilon(\sigma(\delta(c)(2))) = c. \]

(2.52)

Proof. The proof is easy if one notices that the right \( K \)-comodule structure of \( C \) is given by \( c \mapsto c(1) \otimes \delta(c)(2) \). \( \square \)

We specialize the Proposition 2.25 to the left \( \times_C \)-Hopf coalgebra \( K = C \otimes C_{\text{cop}} \) whose structure is given in Example 2.9. We first investigate all right group-like morphisms of \( C \otimes C_{\text{cop}} \). We claim that a map \( \delta : C \rightarrow C \otimes C_{\text{cop}} \) is a right group-like if and only if \( \delta(c) = c(2) \otimes \theta(c(1)) \) for some counital anti coalgebra map \( \theta : C \rightarrow C \). For this, let \( \delta \) be a right group-like and \( \delta(c) = \sum_i a_i \otimes b_i \). We have

\[ \delta(c) = \sum_i a_i \otimes b_i = \sum_i \varepsilon(b_i)(1) a_i(2) \otimes \varepsilon(a_i)(1) b_i(2) = \]

\[ \sum_i \alpha(a_i(2) \otimes b_i(1)) \otimes \beta(a_i(1) \otimes b_i(2)) = \alpha(\delta(c)(2)) \otimes \beta(\delta(c)(1)) = c(2) \otimes \beta(\delta(c)(1)), \]

Proof. The proof is easy if one notices that the right \( K \)-comodule structure of \( C \) is given by \( c \mapsto c(1) \otimes \delta(c)(2) \). \( \square \)
where we use (2.47) in the last equality. Let us set \( \theta = \beta \circ \delta \). The map \( \theta \) is obviously an anti coalgebra map. It is counital because,

\[
\varepsilon(\theta(c)) = \varepsilon(\beta(\delta(c))) = \varepsilon(\beta(\sum_i a_i \circ b_i)) = \sum_i \varepsilon(a_i) \varepsilon(b_i) = \sum_i \varepsilon(a_i) \varepsilon(b_i) = \varepsilon(c).
\]

Conversely, let \( \theta : C \rightarrow C \) be an anti coalgebra map. One defines a group-like morphism \( \delta : C \rightarrow C \otimes C_{\text{cop}} \) by

\[
c \mapsto c^{(2)} \otimes \theta(c^{(1)}).
\]

(2.54)

Indeed, the following computations show that \( \delta \) satisfies (2.47), (2.48), and (2.49). First we use the definition of \( \alpha \) and counitality of \( \theta \) to see

\[
\delta(c^{(1)}) \otimes \alpha(c^{(2)}) = c^{(3)} \otimes \theta(c^{(2)}) \otimes \alpha(c^{(4)} \otimes \theta(c^{(1)}))
\]

\[
= c^{(3)} \otimes \theta(c^{(2)}) \otimes c^{(4)} \varepsilon(c^{(1)}) = c^{(2)} \otimes \theta(c^{(1)}) \otimes c^{(3)} = \delta(c^{(1)}) \otimes c^{(2)}.
\]

To see (2.48), we have

\[
\delta(\alpha(\delta(c^{(1)}))) \otimes \delta(c^{(2)}) = \delta(\alpha(c^{(3)} \otimes \theta(c^{(2)}))) \otimes c^{(4)} \otimes \theta(c^{(1)})
\]

\[
= \delta(c^{(3)} \varepsilon(\theta(c^{(2)}))) \otimes c^{(4)} \otimes \theta(c^{(1)}) = \delta(c^{(3)} \varepsilon(c^{(2)})) \otimes c^{(4)} \otimes \theta(c^{(1)})
\]

\[
= \delta(c^{(3)} \varepsilon(c^{(2)})) \otimes c^{(4)} \otimes \theta(c^{(1)}) = c^{(3)} \otimes \theta(c^{(2)}) \otimes c^{(4)} \otimes \theta(c^{(1)}) = \delta(c^{(1)}) \otimes \delta(c^{(2)}),
\]

where the counitality of \( \theta \) is used in the third equality. Also

\[
\varepsilon(\delta(c)) = \varepsilon(c^{(2)} \otimes \theta(c^{(1)})) = \varepsilon(c^{(2)}) \varepsilon(\theta(c^{(1)})) = \varepsilon(c^{(2)}) \varepsilon(c^{(1)}) = \varepsilon(c),
\]

shows that \( \delta \) satisfies (2.49), where the counitality of \( \theta \) is used in the third equality. Therefore \( \delta \) is a right group-like morphism.

Now we find all characters \( \sigma : C \otimes C_{\text{cop}} \rightarrow C \). Since \( \sigma \) is a ring morphism, for \( c, d \in C \) and \( c', d' \in C_{\text{cop}} \) we have

\[
\varepsilon(\sigma(c \otimes c')) \sigma(d \otimes d') = \sigma(\mu(c \otimes c' \square c \otimes c')) = \varepsilon(c') \varepsilon(d) \sigma(c \otimes d').
\]

(2.55)

For \( \mathcal{K} = C \otimes C_{\text{cop}} \), the condition \( \sigma(\eta_\mathcal{K}(c)) = c \) is equivalent to

\[
\sigma(c^{(1)} \otimes c^{(2)}) = c, \quad c \in C.
\]

(2.56)

Let us set \( c \otimes c' = a^{(1)} \otimes a^{(2)} \) in (2.55). Using (2.56), we obtain

\[
\varepsilon(a) \sigma(d \otimes d') = \varepsilon(d) \sigma(a \otimes d').
\]

(2.57)
If in the preceding equation we put \( d \otimes d' = b^{(1)} \otimes b^{(2)} \), we have
\[
\sigma(a \otimes b) = \varepsilon(a)b, \quad a, b \in C.
\] (2.58)

Conversely, one easily sees that any \( \mathbb{C} \)-linear morphism \( \sigma : C \otimes C_{\text{cop}} \to C \) satisfying \( (2.58) \) is a character.

**Proposition 2.26.** Let \( K \) be the left \( \times C \)-Hopf coalgebra \( C \otimes C_{\text{cop}}, \delta \) a right group-like, and \( \sigma \) a character of \( K \). Then the following action and coaction
\[
c \mapsto c^{(2)} \otimes c^{(3)} \otimes \theta (c^{(1)}), \quad (c_1 \otimes c_2) \triangleright c_3 = c_1 \varepsilon(c_2) \varepsilon(c_3),
\]
define a left-right \( K \)-SAYD module on \( C \).

**Proof.** By the above characterization of right group-like morphisms of \( K \) the action and coaction defined in \( (2.50) \) reduce to
\[
c \mapsto c^{(2)} \otimes c^{(3)} \otimes \theta (c^{(1)}), \quad (c_1 \otimes c_2) \triangleright c_3 = c_1 \varepsilon(c_2) \varepsilon(c_3).
\]
Let us see the above claim in details,
\[
c \mapsto \alpha(c^{(3)} \otimes \theta (c^{(1)})) \otimes c^{(4)} \otimes \theta (c^{(2)})
= c^{(3)} \varepsilon(c^{(1)}) \otimes c^{(4)} \otimes \theta (c^{(2)}) = c^{(2)} \otimes c^{(3)} \otimes \theta (c^{(1)}).
\]
Using \( (2.58) \), we see
\[
(c_1 \otimes c_2) \triangleright c_3 = \alpha(c^{(1)}_1 \otimes c^{(2)}_2) \varepsilon(c_3) \varepsilon(\sigma(c^{(2)}_1 \otimes c^{(1)}_2))
= c^{(1)}_1 \varepsilon(c^{(2)}_2) \varepsilon(c_3) \varepsilon(\sigma(c^{(2)}_1 \otimes c^{(1)}_2)) = c^{(1)}_1 \varepsilon(c_3) \varepsilon(\varepsilon(c^{(2)}_1) c_2) = c^{(1)}_1 \varepsilon(c_3) \varepsilon(c_2).
\]

**Remark 2.27.** Let \( G = (G^1, G^0) \) be a groupoid endowed with a cyclic structure \( \theta \). We have seen in Example \( 2.7 \) that \( B := \mathbb{C}G^1 \) is a right \( \times C \)-Hopf coalgebra, where \( C := \mathbb{C}G^0 \). We observe that the right module left comodule \( \theta C \), defined in \( (1.17) \) and \( (1.10) \) respectively, offers a SAYD module structure on \( C \). Indeed, the stability condition is obvious since for all \( c \in C \) we have
\[
c \cdot \theta_c = \varepsilon(c) s(\theta_c) = c.
\]
Since \( c \otimes g \in C \square_{C \mathcal{B}} \) using \( (1.9) \) and \( (1.11) \) we obtain \( s(g) = c \). The following computation shows the AYD condition holds.
\[
(c \cdot g)_{< -1>} \otimes (c \cdot g)_{<-1>} = \varepsilon(c) s(g)_{<-1>} \otimes s(g)_{<-1>} = \varepsilon(c) \theta_s(g) \otimes c
= g^{-1} \theta_c g \otimes \varepsilon(c) s(g) = \nu^+(g, \theta_c) g \otimes c \cdot \nu^-(g, \theta_c).
\]
We use \( s(g) = c \) and \( \theta_c g = g \theta_s(g) \) in the third equality and \( (1.13) \) in the last equality.
2.4 Symmetries via bicoalgebroids

In this subsection, based on our needs in the sequel sections, we define the following three symmetries for $\times_C$-Hopf coalgebras.

**Definition 2.28.** Let $\mathcal{K}$ be a left $\times_C$-Hopf coalgebra. A left $\mathcal{K}$-comodule coalgebra $T$ is a coalgebra and a left $\mathcal{K}$-comodule such that for all $t \in T$, the following identities hold,

$$t_{\langle -1 \rangle} \varepsilon_T(t_{\langle 0 \rangle}) = \eta(\alpha(t_{\langle -1 \rangle})) \varepsilon_T(t_{\langle 0 \rangle}),$$

$$t_{\langle -1 \rangle} \otimes t_{\langle 0 \rangle}^{(1)} \otimes t_{\langle 0 \rangle}^{(2)} = t_{\langle -1 \rangle} t_{\langle 0 \rangle}^{(1)} \otimes t_{\langle 0 \rangle}^{(1)} \otimes t_{\langle 0 \rangle}^{(2)},$$

$$\beta(t_{\langle -1 \rangle}^{(1)} \otimes t_{\langle 0 \rangle}^{(1)} \otimes t_{\langle 0 \rangle}^{(2)}) = \alpha(t_{\langle -1 \rangle}^{(2)} \otimes t_{\langle 0 \rangle}^{(1)} \otimes t_{\langle 0 \rangle}^{(2)}).$$

One notes that the equations (2.59) and (2.60) are equivalent to the $\mathcal{K}$-colinearity of the counit and comultiplication of $T$. The equation (2.61) is stating that the comultiplication of $T$ is $C$-cobalanced.

Recall that a $C$-ring $A$ is a $C$-bicomodule endowed with two $C$-bicolinear operators, the multiplication $m : A \Box C A \to A$ and unit map $\eta_A : C \to A$ where the multiplication is associative and unit map is unital, i.e.

$$m(a_{_{\langle \pi \rangle}} \otimes \eta_A(a_{_{\langle T \rangle}})) = a = m(\eta_A(a_{_{\langle \pi \rangle}}) \otimes a_{_{\langle \pi \rangle}}).$$

**Definition 2.29.** Let $\mathcal{B}$ be a right $\times_C$-Hopf coalgebra. A right $\mathcal{B}$-comodule ring $A$ is a $C$-ring and a right $\mathcal{B}$-comodule satisfying the following conditions:

$$\eta_A(c)_{[0]} \otimes \eta_A(c)_{[1]} = \eta_A(\alpha(\eta_B(c)^{(1)})) \otimes \eta_B(c)^{(2)},$$

$$m(a_1 \otimes a_2)_{[0]} \otimes m(a_1 \otimes a_2)_{[1]} = m(a_1)_{[0]} \otimes a_1[1] a_2[1].$$

As an example, $\mathcal{B}$ is a right $\mathcal{B}$-comodule ring where $m = \mu_B$ and the right comodule structure is given by $\Delta_B$. In this case the relations (2.63) and (2.64) are equivalent to (2.21) and (2.24) in the definition of right bicoalgebroids. The following definition is needed later for definition of a $\times_C$-Hopf coalgebras Galois coextension.

**Definition 2.30.** Let $\mathcal{B}$ be a right $\times_C$-Hopf coalgebra and the coalgebra $T$ be a right $\mathcal{B}$-module and $C$-bicocomodule. We say $T$ is a $\mathcal{B}$-module coalgebra if

1) $t^{(1)}_{_{\langle \pi \rangle}} \otimes t^{(1)}_{_{\langle T \rangle}} \otimes t^{(2)}_{_{\langle \pi \rangle}} = t^{(1)}_{_{\langle \pi \rangle}} \otimes t^{(2)}_{_{\langle T \rangle}} \otimes t^{(2)}_{_{\langle \pi \rangle}},$

2) $\varepsilon_T(t < b) = \varepsilon_T(t) \varepsilon_B(b),$
The condition (i) means that the comultiplication of $T$ is $C$-cobalanced. Any right $B$-Hopf coalgebra is a right $B$-module coalgebra where the action of $B$ on itself is defined by the multiplication $\mu_B$. The conditions (i), (ii) and (iii) are equivalent to $C$-cobalanced property of $B$ which comes from $C$-bicomodule structure of $B$, (2.22) and (2.21) equivalently.

3 Hopf cyclic cohomology of $\times$-Hopf coalgebras

In this section we introduce the cyclic cohomology of comodule coalgebras and comodule rings with coefficients in SAYD modules under the symmetry of $\times$-Hopf coalgebras. Also, we illustrate these two theories for Hopf algebras.

3.1 Cyclic cohomology of comodule coalgebras

Let $K$ be a left $\times_S$-Hopf coalgebra, $T$ a left $K$-comodule coalgebra and $M$ a left-right SAYD module over $K$. We set

$$K^C_n(T, M) = M \Box_K T \Box_S (n+1).$$

We define the following cofaces, codegeneracies and cocyclic map.

$$d_i(m \otimes \tilde{t}) = m \otimes t_0 \otimes \cdots \otimes \Delta(t_i) \otimes \cdots \otimes t_n, \quad 0 \leq i \leq n - 1,$$

$$d_n(m \otimes \tilde{t}) =$$

$$t_0^{(2)} \otimes t_{1<1} \cdots t_{n<1} \triangleright m \otimes t_0^{(2)} \otimes t_{1<0} \cdots t_{n<0} \otimes t_0^{(1)}, \quad (3.1)$$

$$s_i(m \otimes \tilde{t}) = m \otimes t_0 \otimes \cdots \otimes \varepsilon(t_i) \otimes \cdots \otimes t_n, \quad 0 \leq i \leq n,$$

$$t_n(m \otimes \tilde{t}) = t_{1<1} \cdots t_{n<1} \triangleright m \otimes t_{1<0} \otimes \cdots \otimes t_{n<0} \otimes t_0.$$

where $\tilde{t} = t_0 \otimes \cdots \otimes t_n$. The left $K$-comodule structure of $T \Box_S (n+1)$ is given by

$$t_0 \otimes \cdots \otimes t_n \mapsto t_{0<1} \cdots t_{n<1} \otimes t_{0<0} \otimes t_{1<0} \otimes \cdots \otimes t_{n<0} \quad (3.2)$$

Proposition 3.1. The morphisms defined in (3.1) make $K^C_n(T, M)$ a cocyclic module.

Proof. We leave to the reader to check that the operators satisfy the commutativity relations for a cocyclic module [C-Book]. However, we check that
the operators are well-defined; this is obvious for the degeneracies and all faces except possibly the very last one. Based on the relations in the cocyclic category, it suffices to check that the cyclic operator is well-defined. Indeed, For \( n = 0 \), the map \( \tau \) is identity map and therefore is well-defined. The following computation shows that the cyclic map is well-defined in general.

\[
(t_{1<1>} \cdots t_{n<1>} \triangleright m)_{<0>} \otimes (t_{1<1>} \cdots t_{n<1>} \triangleright m)_{<1>} \otimes t_{1<0>} \otimes \cdots \\
\cdots \otimes t_{n<0>} \otimes t_0
\]

\[
= \nu^+[m_{<1>}, t_{1<1>}^{(1)} \cdots t_{n<1>}^{(1)}] \triangleright m_{<0>} \otimes t_{1<1>}^{(2)} \cdots t_{n-1<1>}^{(2)} t_{n<1>}^{(2)} \\
\nu^-[m_{<1>}, t_{1<1>}^{(1)} \cdots t_{n-1<1>}^{(1)} t_{n<1>}^{(1)}] \otimes t_{1<0>} \otimes \cdots \otimes t_{n<0>} \otimes t_0
\]

\[
= \nu^+[t_{0<1>} \cdots t_{n<1>}, t_{1<0>}^{(1)} \cdots t_{n<0>}^{(1)}] \triangleright m \otimes \\
(t_{1<0>}^{(2)} \cdots t_{n-1<1>}^{(2)} t_{n<0>}^{(2)} \otimes [t_{0<1>} \cdots t_{n<1>}, t_{1<0>}^{(1)} \cdots t_{n<1>}]_{<1>}, t_{1<1>}^{(2)} \cdots t_{1<1>})_{<1>}
\]

\[
\cdots t_{n<1>}^{(2)}(1) \otimes t_{1<0>} \otimes \cdots \otimes t_{n<0>} \otimes t_0
\]

\[
= \nu^+[t_{0<1>} \cdots t_{n<1>}, t_{1<0>}^{(1)} \cdots t_{n<0>}^{(1)}, t_{1<1>}^{(2)} \cdots t_{n<1>}^{(2)}] \triangleright m \otimes \\
(t_{1<1>}^{(3)} \cdots t_{n<1>}^{(3)} \otimes [t_{0<1>} \cdots t_{n<1>}, t_{1<1>}^{(1)} \cdots t_{n<1>}^{(1)}, t_{1<1>}^{(2)} \cdots t_{n<1>}^{(2)}]
\]

\[
\otimes t_{1<0>} \otimes \cdots \otimes t_{n<0>} \otimes t_0
\]

\[
= \nu^+[t_{0<1>} \otimes \{t_{1<1>} \cdots t_{n<1>}\}^{(1)} \otimes \{t_{1<1>} \cdots t_{n<1>}\}^{(2)}] \triangleright m \otimes \\
\{t_{1<1>} \cdots t_{n<1>}\}^{(3)} \otimes \{t_{1<1>} \cdots t_{n<1>}\}^{(1)} \otimes \{t_{1<1>} \cdots t_{n<1>}\}^{(2)}
\]

\[
\otimes t_{1<0>} \otimes \cdots \otimes t_{n<0>} \otimes t_0
\]

\[
= t_{1<1>}^{(1)} \cdots t_{n<1>}^{(1)} \triangleright m \otimes t_{1<0>}^{(2)} \cdots t_{n<0>}^{(2)} t_{0<1>} \otimes t_{1<0>} \otimes \cdots \\
\cdots \otimes t_{n<0>} \otimes t_0
\]

\[
= t_{1<1>}^{(1)} \cdots t_{n<1>}^{(1)} \triangleright m \otimes t_{1<0>}^{(2)} \cdots t_{n<0>}^{(2)} t_{0<1>} \otimes t_{1<0>} \otimes \cdots \\
\cdots \otimes t_{n<0>} \otimes t_0
\]

We use AYD condition and comultiplicative property \((2.6)\) in the first equality. For the second equality, we use \( m \otimes t \in M \hat{\square} T^{\square S(n+1)} \) or equivalently

23
the following equation,
\[
m_{<0>} \otimes m_{<1>} \otimes t_0 \otimes \cdots \otimes t_n = m \otimes t_{0,<1>} \otimes t_{1,<1>} \otimes \cdots \otimes t_{n,<1>},
\]
We use the comodule property for the elements \(t_1, \cdots, t_n\) in third equality, comultiplicative property (2.6) in fifth equality, Lemma (2.3)(ii) in the sixth equality and the comodule property for the elements \(t_1, \cdots, t_n\) in the last equality.

The cyclic cohomology of the preceding cocyclic module will be denoted by \(KHC^*(T, M)\).

It should be noted that even for the case \(K := H\) is merely a Hopf algebra and \(T = H\) coacting on itself via adjoint coaction, the above cocyclic module did not appear in the literature. Let us simplify the above cocyclic module in case of right \(\times\)-Hopf coalgebras for Hopf algebras.

Recall that for a Hopf algebra \(H\) a right comodule coalgebra \(C\) is an right \(H\)-comodule and a coalgebra such that the following compatibility conditions holds.

\[
c_{<0>}^{(1)} \otimes c_{<0>}^{(2)} \otimes c_{<1>} = c_{<0>}^{(1)} \otimes c_{<0>}^{(2)} \otimes c_{<1>}^{(1)} \otimes c_{<1>}^{(2)}, \quad \varepsilon(c_{<0>})a_{<1>} = \varepsilon(a)1_H.
\]

As an example, \(H\) is a right \(H\)-comodule coalgebra by coadjoint coaction \(h \mapsto h^{(2)} \otimes S(h^{(1)})h^{(3)}\).

**Proposition 3.2.** Let \(H\) be a Hopf algebra, \(C\) a right \(H\)-comodule coalgebra and \(M\) a right-left SAYD module on \(H\). Let
\[
HC^n(C, M) = C^{\otimes(n+1)} \Box_H M.
\]

The following cofaces, codegeneracies, and cocyclic map defines a cocyclic module for \(HC^n(C, M)\).

\[
\delta_i(c_0 \otimes \cdots \otimes c_n \otimes m) = c_0 \otimes \cdots \otimes \Delta(c_i) \otimes \cdots \otimes c_n \otimes m,
\]
\[
\delta_n(c_0 \otimes \cdots \otimes c_n \otimes m) = c_0^{(2)} \otimes c_1 \otimes \cdots \otimes c_n \otimes c_0^{(1)} \otimes m \triangleleft c_0^{(1)}<1>,
\]
\[
\sigma_i(c_0 \otimes \cdots \otimes c_n \otimes m) = c_0 \otimes \cdots \otimes \varepsilon(c_{i+1}) \otimes \cdots \otimes c_n \otimes m,
\]
\[
\tau_n(c_0 \otimes \cdots \otimes c_n \otimes m) = c_1 \otimes \cdots \otimes c_n \otimes c_0 <0> \otimes m \triangleleft c_0 <1>.
\]

Cyclic cohomology of the preceding cyclic module is denoted by \(HC^*(C, M)\).
3.2 Cyclic cohomology of comodule rings

In this subsection we define cyclic cohomology of comodule rings with coefficients in a SAYD module and symmetry endowed by a $\times$-Hopf coalgebra. Let $B$ be a right $\times_C$-Hopf coalgebra, $A$ a right $B$-comodule ring, and $M$ a right-left SAYD module over $B$. We set

$$\tilde{C}^{B,n}(A, M) := A \Box_C (n+1) \Box_B M,$$

(3.4)
and define the following operators on $\tilde{C}^{B,n}(A, M)$.

$$\delta_i(\tilde{a} \otimes m) = a_0 \otimes \cdots \otimes \eta(a_{i+1}) \otimes a_{i+1,0} \otimes \cdots \otimes a_n \otimes m, \quad 0 \leq i \leq n,$$

$$\sigma_i(\tilde{a} \otimes m) = a_0 \otimes \cdots \otimes m(a_i \otimes a_{i+1}) \otimes \cdots \otimes a_n \otimes m, \quad 0 \leq i \leq n,$$

$$\tau_n(\tilde{a} \otimes m) = a_1 \otimes \cdots \otimes a_n \otimes a_0 \otimes m \triangleleft a_0, \quad 0 \leq i \leq n,$$

(3.5)

where $\tilde{a} = a_0 \otimes \cdots \otimes a_n$. The right $B$-comodule structure of $A \Box_C (n+1)$ is given by

$$\tilde{a} \mapsto a_0\otimes \cdots \otimes a_n \langle a_{0,1}, \ldots, a_{n,1} \rangle.$$

**Proposition 3.3.** Let $B$ be a right $\times_C$-Hopf coalgebra, $M$ a right-left SAYD module for $B$ and $A$ a right $B$-comodule $C$-ring. Then the operators presented in (3.5) define a cocyclic module structure on $\tilde{C}^{B,n}(A, M)$.

**Proof.** It is straightforward to check that the operators satisfy the commutativity relations for a cocyclic module [C-Book]. However, we check that the operators are well-defined; using comodule ring conditions (2.63) and (2.64) one proves that all cofaces except possibly the last one are well-defined. This is readily seen for the codegeneracies. Based on the relations in the cocyclic category, it suffices to check that the cyclic operator is well-defined. If $n = 0$, then $\tau = \text{Id}$ and it is obviously well-defined. The following computation shows that $\tau(\tilde{a} \otimes_B m) \in \tilde{C}^{B,n}(A, M)$. 

25
Proof. We use the AYD condition \((2.46)\) in the first equality, the fact that \(\tilde{a} \otimes m \in \tilde{C}^{\mathcal{B}, n}(A, M)\) in the second equality, Lemma \((2.7)(ii)\) in the fourth equality. 

We denote the cyclic cohomology of this cocyclic module by \(\tilde{HC}^{\mathcal{B}, n}_{\mathcal{B}^*}(A, M)\). It is clear that \(\mathcal{B}\) is a right \(\mathcal{B}\)-comodule \(C\)-ring by considering the comultiplication \(\Delta_{\mathcal{B}}\) as the coaction and \(\mu = \text{id}\). One defines the following map

| Equation | Description |
|----------|-------------|
| (3.6) \(\varphi_n : \tilde{C}^{\mathcal{B}, n}(\mathcal{B}, M) \rightarrow \mathcal{B} \boxtimes C^n \boxtimes c_{\text{cop}} M\) | \(\varphi_n(b_0 \otimes \cdots \otimes b_n \otimes m) = b_0 \otimes \cdots \otimes b_{n-1} \otimes \varepsilon_{\mathcal{B}}(b_n) \otimes m\). |

The right \(C_{\text{cop}}\)-comodule structure of \(\mathcal{B} \boxtimes C^n\) is given by

\[ b_1 \otimes \cdots \otimes b_n \mapsto b_1^{(1)} \otimes b_2 \otimes \cdots \otimes b_n \otimes \beta(b_1^{(2)}), \]  

and the left \(C_{\text{cop}}\)-comodule structure of \(M\) by \(m \mapsto \beta(m_{-1}) \otimes m_0\).

**Proposition 3.4.** The map \(\varphi_n\) defined in \((3.6)\) is a well-defined isomorphism of vector spaces.

**Proof.** Since \(b_0 \otimes \cdots \otimes b_n \otimes m \in \tilde{C}^{\mathcal{B}, n}(\mathcal{B}, M)\), we have

\[ b_0^{(1)} \otimes \cdots \otimes b_n^{(1)} \otimes b_0^{(2)} \cdots b_n^{(2)} \otimes m = b_0 \otimes \cdots \otimes b_n \otimes m_{[-1]} \otimes m_0. \]
We prove that the map $\varphi$ is well-defined. Indeed,

$$
\begin{align*}
&b_0 \otimes \cdots \otimes b_{n-1} \varepsilon(b_n) \otimes \beta(m_{-1}) \otimes m_0 \\
&= b_0^{(1)} \otimes \cdots \otimes b_{n-1}^{(1)} \varepsilon(b_n^{(1)}) \otimes \beta(b_0^{(2)} \cdots b_n^{(2)}) \otimes m \\
&= b_0^{(1)} \otimes \cdots \otimes b_{n-1}^{(1)} \varepsilon(b_n^{(1)}) \otimes \beta(b_0^{(2)} \varepsilon(b_1^{(2)})) \cdots \varepsilon(b_n^{(2)}) \otimes m \\
&= b_0^{(1)} \otimes b_1 \otimes \cdots \otimes b_{n-1} \varepsilon(b_n) \otimes \beta(b_0^{(2)}) \otimes m.
\end{align*}
$$

The equation (3.9) is used in the first equality, and (2.5)(iv) and (2.22) are used in the second equality. The coalgebra properties of $B$ is used in the last one. We claim that the following map defines the two sided inverse map of $\varphi$.

$$
\varphi_n^{-1}(b_1 \otimes \cdots \otimes b_n \otimes m) = b_1^{(1)} \otimes \cdots \otimes b_n^{(1)} \otimes \\
\varepsilon[\nu^{-}(b_1^{(2)} \cdots b_n^{(2)}, m_{<1>})] \nu^{+}(b_1^{(2)} \cdots b_n^{(2)}, m_{<1>}) \otimes m_{<0>}. 
$$

(3.10)

Using Lemma (2.7)(iii), the relation (2.22) and the counital property of the coaction, one easily shows $\varphi \varphi^{-1} = Id$. Since $b_0 \otimes \cdots \otimes b_n \otimes m \in B \square c^{(n+1)} \square gM$, we have

$$
b_0^{(1)} \otimes \cdots \otimes b_n^{(1)} \otimes b_0^{(2)} \cdots b_n^{(2)} \otimes m = b_0 \otimes \cdots \otimes b_n \otimes m_{<1>} \otimes m_{<0>}. 
$$

(3.12)

We check $\varphi^{-1} \varphi = Id$,

$$
\begin{align*}
\varphi^{-1}(\varphi_n(b_0 \otimes \cdots \otimes b_n \otimes m)) &= \varphi^{-1}(b_0 \otimes \cdots \otimes b_{n-1} \varepsilon(b_n) \otimes m) \\
&= b_0^{(1)} \otimes \cdots \otimes b_{n-2}^{(1)} \otimes b_n^{(1)} \varepsilon(b_n) \otimes \\
\varepsilon[\nu^{-}(b_0^{(2)} \cdots b_n^{(2)}, m_{<1>})] \nu^{+}(b_0^{(2)} \cdots b_n^{(2)}, m_{<1>}) \otimes m_{<0>} \\
&= b_0^{(1)} \otimes \cdots \otimes b_{n-2}^{(1)} \otimes b_n^{(1)} \varepsilon(b_n) \otimes \\
\varepsilon[\nu^{-}(b_0^{(2)} \cdots b_n^{(2)}, b_0^{(3)} \cdots b_{n-1}^{(3)} b_n^{(2)})] \nu^{+}(b_0^{(2)} \cdots b_n^{(2)}, b_0^{(3)} \cdots b_{n-1}^{(3)} b_n^{(2)}) \otimes m \\
&= b_0^{(1)} \otimes \cdots \otimes b_{n-2}^{(1)} \otimes b_{n-1}^{(1)} \varepsilon(b_0^{(2)}) \cdots \varepsilon(b_{n-1}^{(2)}) b_n \otimes m \\
&= b_0 \otimes \cdots \otimes b_n \otimes m.
\end{align*}
$$

We use (3.12) in the third equality and $\varepsilon(b_0^{(1)}) b_n^{(2)} = b_n$, (2.22) and Lemma (2.7)(ii) in the fourth equality. This automatically proves that $\varphi^{-1}$ is well-defined.

$\square$
Therefore we obtain the following operators by transferring the cocyclic structure of \( \tilde{C}^{B,n}(B, M) \) on \( B \square C^n \square_{C_{\text{cop}}} M \).

\[
\begin{align*}
\delta_i(b \otimes m) &= b_1 \otimes \cdots \otimes \eta(b_i^{-m}) \otimes b_{i+1} \otimes \cdots \otimes b_n \otimes m, \\
\delta_n(b \otimes m) &= b_1^{(1)} \otimes \cdots \otimes b_n^{(1)} \otimes \\
\varepsilon[\nu^-(b_1^{(2)} \cdots b_{n-1}^{(2)} m_{n-1}^{<1>})] \nu^+(b_1^{(2)} \cdots b_{n-1}^{(2)} m_{n-1}^{<1>}) \otimes m_{n-1}^{<0>}, \\
\sigma_i(b \otimes m) &= b_1 \otimes \cdots \otimes b_i h_{i+1} \otimes \cdots \otimes b_n \otimes m \\
\sigma_n(b \otimes m) &= b_1 \otimes \cdots \otimes b_{n-1} \varepsilon(b_n) \\
\tau_n(b \otimes m) &= b_2^{(1)} \otimes \cdots \otimes b_n^{(1)} \otimes \\
\varepsilon[\nu^-(b_1^{(2)} \cdots b_{n-1}^{(2)} m_{n-1}^{<1>})] \nu^+(b_1^{(2)} \cdots b_{n-1}^{(2)} m_{n-1}^{<1>}) \otimes m_{n-1}^{<0>} \bowtie b_1^{(1)}
\end{align*}
\]

(3.13)

where \( \tilde{b} = b_1 \otimes \cdots \otimes b_n \).

One can specialize this cocyclic module for Hopf algebras to obtain the following.

**Proposition 3.5.** Let \( H \) be a Hopf algebra and \( M \) a right-left SAYD module over \( H \). Set \( C^n(H, M) = H^\otimes_n \otimes M \). The following cofaces, codegeneracies and cyclic maps define a cocyclic module for \( C^n(H, M) \).

\[
\begin{align*}
\delta_i(h \otimes m) &= h_1 \otimes \cdots \otimes h_i \otimes 1_H \otimes \cdots \otimes h_n \otimes m, \\
\delta_n(h \otimes m) &= h_1^{(1)} \otimes \cdots \otimes h_{n-1}^{(1)} \otimes S(h_1^{(2)} \cdots h_{n-1}^{(2)}) m_{n-1}^{<1>} \otimes m_{n-1}^{<0>}. \\
\sigma_i(h \otimes m) &= h_1 \otimes \cdots \otimes h_i h_{i+1} \otimes \cdots \otimes h_n \otimes m \\
\sigma_n(h \otimes m) &= h_1 \otimes \cdots \otimes h_{n-1} \varepsilon(h_n) \\
\tau_n(h \otimes m) &= h_2^{(1)} \otimes \cdots \otimes h_{n-1}^{(1)} \otimes S(h_1^{(2)} \cdots h_{n-1}^{(2)}) m_{n-1}^{<1>} \otimes m_{n-1}^{<0>} \bowtie h_1^{(1)}.
\end{align*}
\]

(3.14)

Here \( \tilde{h} = h_1 \otimes \cdots \otimes h_n \).

The proceeding cocyclic module is in fact dual cyclic module of the one introduced in [KR02]. Let us briefly recall it here. One similarly can define a cyclic module for \( C_n(A, M) = M \square K A \square_{C_{\text{cop}}} C_{n+1} \) where \( M \) is a left-right SAYD module over a left \( \times C \)-Hopf coalgebra \( K \) and \( A \) is a left \( K \)-comodule \( C \)-ring. In the special case when \( K = H \) is a Hopf algebra, the morphisms
of the cyclic module reduce to the following morphisms.

\[ \begin{align*}
    d_0(m \otimes \tilde{h}) &= \varepsilon(h_1)m \otimes h_2 \otimes \cdots \otimes h_n, \\
    d_i(m \otimes \tilde{h}) &= m \otimes h_1 \otimes \cdots \otimes h_i h_{i+1} \otimes \cdots \otimes h_n, \\
    d_n(m \otimes \tilde{h}) &= h_n \triangleright m \otimes h_1 \otimes \cdots \otimes h_{n-1} \\
    \sigma_i(m \otimes \tilde{h}) &= m \otimes h_1 \otimes \cdots \otimes h_i \otimes 1_H \otimes \cdots \otimes h_n, \\
    \tau_n(m \otimes \tilde{h}) &= h_n^{(2)} \triangleright m_{<0>} \otimes m_{<1>} S(h_1^{(1)} \cdots h_n^{(1)}) \otimes h_1^{(2)} \otimes \cdots \otimes h_{n-1}^{(2)},
\end{align*} \]

where \( \tilde{h} = h_1 \otimes \cdots \otimes h_n \).

\section{Equivariant Hopf Galois coextensions}

In this section we define equivariant Hopf Galois coextensions of \( \times \)-Hopf coalgebras. The needed datum for equivariant Hopf Galois coextension is a quadruple \((K, B, T, S)\) satisfying certain properties which is stated in Definition \ref{def:equivariant-Hopf-Galois-coextension}. We show that any equivariant Hopf Galois coextension defines a functor from the category of SAYD modules over \( K \) to the category of SAYD modules over \( B \) such that their Hopf cyclic complexes with corresponding coefficients are isomorphic.

\subsection{Basics of equivariant Hopf Galois coextensions}

Let \( B \) be a right \( \times_C \)-Hopf coalgebra and \( T \) a right \( B \)-module coalgebra as defined in \ref{def:Hopf-module-coalgebra}. We set

\[ I = \{ t \triangleright b - \varepsilon(b)t, \quad b \in B, t \in T \}. \]

One observes that \( I \) is a coideal of \( T \) due to the fact that \( S := T_B = T/I \) is isomorphic to the coalgebra \( T \otimes_B C \). Recall that given coalgebras \( C \) and \( D \), a surjective coalgebra map \( \pi : C \to D \) is called a coalgebra coextension. Thus we have a coalgebra coextension \( \pi : T \to S \).

\begin{definition}
Let \( C \) be a coalgebra, \( B \) a right \( \times_C \)-Hopf coalgebra, \( T \) a right \( B \)-module coalgebra, \( S = T_B \), \( K \) a left \( \times_S \)-Hopf coalgebra, and \( T \) a left \( K \)-comodule coalgebra. Then \( T \) is called a \( K \)-equivariant \( B \)-Galois coextension of \( S \), if the canonical map

\[ \text{can} : T \square_C B \to T \square_S T, \quad t \otimes b \mapsto t^{(1)} \otimes t^{(2)} \triangleleft b, \quad (4.1) \]

is bijective and the right action of \( B \) on \( T \) is \( K \)-equivariant, i.e.

\[ (t \triangleleft b)_{<1>} \otimes (t \triangleleft b)_{<0>} = t_{<1>} \otimes t_{<0>} \triangleleft b, \quad t \in T, b \in B. \quad (4.2) \]
In Definition 4.1 the $S$-bimodule structure of $T$ is given by

$$t \mapsto \pi(t^{(1)}) \otimes t^{(2)}, \quad t \mapsto t^{(1)} \otimes \pi(t^{(2)}),$$  \hspace{1cm} (4.3)

where the coalgebra map $\pi : T \to S$ is the natural quotient map. We note that for any equivariant Hopf Galois coextension we have

$$\pi(t \triangleleft b) = \varepsilon(b)\pi(t), \quad t \in T, b \in B.$$  \hspace{1cm} (4.4)

We denote a $K$-equivariant $B$-Galois coextension in 4.1 by $K_T(S)_B$. One notes that by $K$-equivariant property of the right action of $B$ on $T$, the map $\mathrm{can}$ is a left $K$-comodule map where the left $K$-comodule structures of $T \Box_C B$ and $T \Box_S T$ are given by

$$t \otimes t' \mapsto t_{_{<1>}} \otimes t_{_{<0>}} \otimes t'_{_{<0>}}, \quad t \otimes b \mapsto t_{_{<1>}} \otimes t_{_{<0>}} \otimes b.$$  \hspace{1cm} (4.5)

Also the map $\mathrm{can}$ is a right $B$-module map where the right $B$-module structures of $T \Box_C B$ and $T \Box_S T$ are given by

$$(t \otimes b) \triangleleft b' := t \otimes bb', \quad (t \otimes t') \triangleleft b := t \otimes t' \triangleleft b'.$$  \hspace{1cm} (4.6)

We define the left $C$-comodule structures on $T \Box_S T$ and $T \Box_C B$ by

$$t \otimes t' \mapsto t_{_{<1>}} \otimes t_{_{<0>}} \otimes t'_{_{<0>}}, \quad \beta(b^{(1)}) \otimes t \otimes b^{(2)}.$$  \hspace{1cm} 

Similarly we define the right $C$-comodule structures on $T \Box_S T$ and $T \Box_C B$ by

$$t \otimes t' \mapsto t \otimes t'_{_{<0>}} \otimes t'_{_{<1>}}, \quad t \otimes b \mapsto t \otimes b^{(1)} \otimes \alpha(b^{(2)}).$$

One notes that $\mathrm{can}$ is a right and left $C$-comodule map via the preceding $C$-comodule structures.

We denote the inverse of the Galois map (4.1) by the following index notation

$$\mathrm{can}^{-1}(t \otimes t') = \mathrm{can}_{_{>-}}(t \otimes t') \otimes \mathrm{can}_{_{<+}}(t \otimes t').$$  \hspace{1cm} (4.7)

If there is no confusion we write $\mathrm{can}^{-1} = \mathrm{can}_{_{>-}} \otimes \mathrm{can}_{_{<+}}$. We record the properties of the maps $\mathrm{can}$ and $\mathrm{can}^{-1}$ by the following lemma.

**Lemma 4.2.** Let $K_T(S)_B$ be a $K$-equivariant $B$-Galois coextension. Then the following properties hold.

i) $\mathrm{can}_{_{>-}}^{(1)} \otimes \mathrm{can}_{_{>-}}^{(2)} \triangleleft \mathrm{can}_{_{<+}} = \text{Id}_{T \Box_S T}$,
ii) \(\text{can}_{<\to} (t^{(1)} \otimes t^{(2)} \triangleleft b) \otimes \text{can}_{<\to} (t^{(1)} \otimes t^{(2)} \triangleleft b) = t \otimes b,\)

iii) \(\text{can}_{<\to} (t \otimes t') \triangleleft \text{can}_{<\to} (t \otimes t') = \varepsilon(t)t'.\)

iv) \(\varepsilon(\text{can}_{<\to} (t \otimes t')) = \text{can}_{<\to} (t \otimes t') = \varepsilon(t').\)

v) \(\text{can}_{<\to} (t \otimes t') \triangleleft \text{can}_{<\to} (t \otimes t') \otimes \text{can}_{<\to} (t \otimes t') = \varepsilon(t)\text{can}_{<\to} (t \otimes t') \otimes \text{can}_{<\to} (t \otimes t').\)

vi) \(\text{can}_{<\to} (t \otimes t') \otimes \text{can}_{<\to} (t \otimes t') = \beta \left[\left(\text{can}_{<\to} (t \otimes t')\right)^{(1)}\right] \otimes \text{can}_{<\to} (t \otimes t') \otimes [\text{can}_{<\to} (t \otimes t')]^{(1)} \).

vii) \(\beta \left(\text{can}_{<\to} (t \otimes t')\right) \otimes \text{can}_{<\to} (t \otimes t') = \varepsilon(t')\text{can}_{<\to} (t \otimes t') \otimes t_{\overline{T}} \otimes t_{\overline{\sigma}}.\)

ix) \(\text{can}_{<\to} (t \otimes t') \otimes \alpha \left(\text{can}_{<\to} (t \otimes t')\right) = \varepsilon(t)^{t_{\overline{\sigma}}} \otimes t_{\overline{T}}’.\)

x) \(\left[\text{can}_{<\to} (t \otimes t')\right]^{(1)} \otimes \left[\left(\text{can}_{<\to} (t \otimes t')\right)^{(2)} \triangleleft \nu^{-} \left(b \otimes \text{can}_{<\to} (t \otimes t')\right)\right] \otimes \nu^{+} \left(b \otimes \text{can}_{<\to} (t \otimes t')\right) = \left(t^{(1)} \otimes \text{can}_{<\to} (t^{(2)} \triangleleft b \otimes t')\right) \otimes \text{can}_{<\to} (t^{(2)} \triangleleft b \otimes t').\)

xii) \(\text{can}_{<\to} (t \otimes t' \triangleleft b) \otimes \text{can}_{<\to} (t \otimes t' \triangleleft b) = \text{can}_{<\to} (t \otimes t') \otimes \text{can}_{<\to} (t \otimes t') b.\)

Proof. The relation i) is equivalent to \(\text{can} \circ \text{can}^{-1} = \text{Id}.\) We see that ii) coincides with \(\text{can}^{-1} \circ \text{can} = \text{Id}.\) One proves iii) by applying \(\varepsilon \otimes \text{Id}\) on both hand sides of i). The relation iv) is derived by applying \(\text{Id} \otimes \varepsilon\) on both hand sides of i) and and then using the right \(B\)-module coalgebra property of \(T.\) It is easy to see that v) is equivalent to the left \(K\)-comodule property of \(\text{can}^{-1}.\) The relations vi) and vii) state the left and right \(C\)-comodule property of \(\text{can}^{-1}.\) One proves viii) by applying \(\text{Id} \otimes \text{Id} \otimes \varepsilon\) on both hand sides of vi). The relation ix) can be obtained by applying \(\text{Id} \otimes \varepsilon \otimes \text{Id}\) on both hand sides of vii). The equality

\[
(Id_T \square g_{\text{can}^{-1}}) \circ (\text{can} \square s \text{Id}_T) = (\text{can} \square C \text{Id}_B) \circ (Id_T \square C \nu^{-1}) \circ (\text{can}^{-1})_{13},
\]

proves x). Here

\[
\text{can}^{-1}_{13}: T \square C \mathcal{B} \square s T \rightarrow T \square C \mathcal{B} \square \text{C}_{\text{cop}} \mathcal{B},
\]

\[
t \otimes b \otimes t' \mapsto \text{can}_{<\to} (t \otimes t') \otimes b \otimes \text{can}_{<\to} (t \otimes t').
\]
One notes that $can_{13}^{-1}$ the inverse map for
\[ can_{13} : t \otimes b \otimes b' \mapsto t^{(1)} \otimes b \otimes t^{(2)} \triangleleft b'. \]

To prove (4.8) we use the bijectivity of all involved maps in the following equation,
\[ (Id_T \square Scan) \circ (can \square C Id_B) = (can \square S Id_T) \circ can_{13} \circ (Id_T \square C \nu). \quad (4.9) \]
We obtain xi) by applying $\varepsilon \otimes \Id \otimes \Id$ on both hand sides of x). Finally xiii) is equivalent to the right $B$-module property of the map $\nu^{-1}$.

For any coalgebra coextension $\pi : T \rightarrow S$, we set
\[ T^S := \left\{ t \in T \mid t_{<\sigma>} \otimes t_{<\tau>} = t_{<\pi>} \otimes t_{<-\tau>} \right\}, \quad T^S := \frac{T}{T^S}. \]
Precisely,
\[ T^S := \left\{ t \in T ; \; t^{(1)} \otimes \pi(t^{(2)}) = t^{(2)} \otimes \pi(t^{(1)}) \right\}. \quad (4.10) \]

**Lemma 4.3.** Let $^K T(S)_B$ be a $K$-equivariant $B$-Galois coextension with the corresponding action $\triangleleft : T \otimes B \rightarrow T$ action. Then, $\triangleleft$ induces a $B$-action, $\triangleleft : T^S \otimes B \rightarrow T^S$.

**Proof.** It is enough to show that if $t \otimes b \in T^S \otimes B$ then $t \triangleleft b \in T^S$. Indeed,
\[
(t \triangleleft b)^{(1)} \otimes \pi((t \triangleleft b)^{(2)}) = t^{(1)} \triangleleft b^{(1)} \otimes \pi(t^{(2)} \triangleleft b^{(2)})
\]
\[
= t^{(1)} \triangleleft b \otimes \pi(t^{(2)}) = t^{(2)} \triangleleft b \otimes \pi(t^{(1)}) = t^{(2)} \otimes \varepsilon(b^{(1)}) \pi(t^{(1)})
\]
\[
= (t \triangleleft b)^{(2)} \otimes \pi((t \triangleleft b)^{(1)}).
\]
We use the $B$-module coalgebra property of $T$ in the first equality and the definition (4.10) in the third and last equalities.

One defines a $S$-bicomodule structure on $T \square S$ as follows.
\[ t \otimes t' \mapsto t \otimes t'^{(1)} \otimes \pi(t'^{(2)}), \quad t \otimes t' \mapsto \pi(t^{(1)}) \otimes t^{(2)} \otimes t'. \quad (4.11) \]

**Lemma 4.4.** Let $^K T(S)_B$ be a $K$-equivariant $B$-Galois coextension with canonical bijective map $\text{can}$. Then the map $\overline{\text{can}}$ induces a bijection
\[ \overline{\text{can}} : T^S \square C B \rightarrow (T \square S)^S, \quad t \otimes b \mapsto \text{can}(t \otimes b). \]

32
Proof. It is enough to show if \( t \otimes b \in T^S \otimes \mathcal{B} \) then \( \text{can}(t \otimes b) \in (T \Box_S T)^S \).

\[
\text{can}(t \otimes b)_{\Sigma} \otimes \text{can}(t \otimes b)_{\Pi} = (t^{(1)} \otimes t^{(2)} \triangleleft b)_{\Sigma} \otimes (t^{(1)} \otimes t^{(2)} \triangleleft b)_{\Pi} = t^{(1)} \otimes (t^{(2)} \triangleleft b)^{(1)} \otimes \pi((t^{(2)} \triangleleft b)^{(2)}) = t^{(1)} \otimes t^{(2)} \triangleleft b^{(1)} \otimes \pi(t^{(3)} \triangleleft b^{(2)}) = t^{(1)} \otimes t^{(2)} \triangleleft b \triangleleft \pi(t^{(3)}) = t^{(2)} \otimes t^{(3)} \triangleleft b \otimes \pi(t^{(1)}) = (t^{(1)} \otimes t^{(2)} \triangleleft b)_{\Sigma} \otimes (t^{(1)} \otimes t^{(2)} \triangleleft b)_{\Pi} = \text{can}(t \otimes b)_{\Sigma} \otimes \text{can}(t \otimes b)_{\Pi}.
\]

We use the \( \mathcal{B} \)-module coalgebra property of \( T \) in the third equality and the fact that \( t \in T^S \), i.e. (4.10) in the sixth equality. Therefore, \( \text{can} \) is well-defined and bijective. \( \square \)

One notes that although \( T \Box_S T \) is not a coalgebra, the subspace \((T \Box_S T)^S\) is a coalgebra by the following coproduct.

\[
\Delta(t \otimes t') = t^{(1)} \Box_S t'^{(2)} \otimes t^{(2)} \Box_S t'^{(1)}, \quad \varepsilon(t \otimes t') = \varepsilon(t) \varepsilon(t'). \tag{4.12}
\]

With respect to (4.11), the coproduct is well-defined. Now we define the following map.

\[
\kappa := (\varepsilon \otimes \text{Id}_B)\text{can}^{-1} : (T \Box_S T)^S \to \mathcal{B}. \tag{4.13}
\]

Lemma 4.5. The map \( \kappa \) is an anti coalgebra map.

Proof. To show that \( \kappa \) is an anti coalgebra map we need to show that \( \Delta \circ \kappa = tw \circ (\kappa \otimes \kappa) \circ \Delta \) and \( \varepsilon \circ \kappa = \varepsilon \). Since \( \text{can} \) is bijective it’s equivalent to show \( \Delta \circ \kappa \circ \text{can} = tw \circ (\kappa \otimes \kappa) \circ \Delta \circ \text{can} \). The following computation proves this for all \( t \otimes b \in T^S \otimes \mathcal{B} \).

\[
tw \circ (\kappa \otimes \kappa) \circ \Delta \circ \text{can}(t \otimes b) = tw \circ (\kappa \otimes \kappa) \circ \Delta(t^{(1)} \otimes t^{(2)} \triangleleft b) = tw \circ (\kappa \otimes \kappa) \left[ t^{(1)} \otimes (t^{(2)} \triangleleft b)^{(2)} \otimes t^{(2)} \otimes (t^{(2)} \triangleleft b)^{(1)} \right] = tw \circ (\kappa \otimes \kappa) \left[ t^{(1)} \otimes t^{(4)} \triangleleft b^{(2)} \otimes t^{(2)} \otimes (t^{(3)} \triangleleft b^{(1)}) \right] = \kappa(t^{(2)} \otimes t^{(3)} \triangleleft b^{(1)}) \otimes \kappa(t^{(1)} \otimes t^{(4)} \triangleleft b^{(2)}) = \varepsilon(t^{(2)})b^{(1)} \otimes \kappa(t^{(1)} \otimes t^{(3)} \triangleleft b^{(2)}) = b^{(1)} \otimes \kappa(t^{(1)} \otimes \varepsilon(t^{(2)})t^{(3)} \triangleleft b^{(2)}) = b^{(1)} \otimes \kappa(t^{(1)} \otimes t^{(2)} \triangleleft b^{(2)}) = \varepsilon(t)b^{(1)} \otimes b^{(2)} = \Delta \circ (\varepsilon \otimes \text{Id}_B)(t \otimes b) = \Delta \circ (\varepsilon \otimes \text{Id}_B) \circ \text{can}^{-1} \text{can}(t \otimes b) = \Delta \circ \text{can}(t \otimes b).
\]
We use Lemma 5.1(ii) in the fifth and the last equalities. Moreover
\[ \varepsilon \circ \kappa(t \otimes t') = \varepsilon \circ (\varepsilon \otimes \text{Id}_B) \circ \text{can}^{-1}(t \otimes t') = \varepsilon \circ (\varepsilon \otimes \text{Id}_B) \circ \text{can}^{-1}(t \otimes t') \]
\[ = \varepsilon \circ (\varepsilon \otimes \text{Id}_B)(\text{can}_-(t \otimes t') \otimes \text{can}_+(t \otimes t')) = \varepsilon(\text{can}_-(t \otimes t') \varepsilon(\text{can}_+(t \otimes t')) \]
\[ = \varepsilon(t) \varepsilon(t') = \varepsilon(t \otimes t'). \]

We use Lemma 5.1(iv) in the fifth equality.

The following lemma introduces some properties of the map \( \kappa \).

**Lemma 4.6.** The map \( \kappa \) satisfies the following properties.

i) \( \kappa(t \otimes t')^{(1)} \otimes \kappa(t \otimes t')^{(2)} = \kappa(t^{(2)} \otimes t'^{(1)}) \otimes \kappa(t^{(1)} \otimes t'^{(2)}). \)

ii) \( t^{(1)} \triangleleft \kappa(t^{(2)} \otimes t') = \varepsilon(t)t'. \)

iii) \( t^{(1)}_{<1>} {t^{(2)}_{<1>}} \otimes \kappa(t^{(2)}_{<0>} \otimes t'^{(2)}_{<0>}) = \varepsilon(t^{(1)}_{<0>})t^{(1)}_{<1>} \otimes \kappa(t^{(2)}_{<0>} \otimes t'). \)

**Proof.** The relation i) is equivalent to the anti coalgebra map property of \( \kappa \).

To prove ii) we apply \( \varepsilon \otimes \text{Id} \otimes \text{Id} \) on Lemma 4.2(v). We obtain
\[ \text{can}^{-1}(t \otimes t') \otimes \text{can}^{+}(t \otimes t') = \varepsilon(\text{can}^{-}(t^{(2)} \otimes t')) \varepsilon(\text{can}^{+}(t^{(2)} \otimes t')) \]
\[ = \varepsilon(t) \varepsilon(t') \varepsilon(t \otimes t'). \]

By applying the right action of \( B \) on \( T \) on the previous equation we have
\[ t^{(1)} \triangleleft \kappa(t^{(2)} \otimes t') = \varepsilon(\text{can}^{-}(t^{(2)} \otimes t'))t^{(1)} \triangleleft \text{can}^{+}(t^{(2)} \otimes t') \]
\[ = \text{can}^{-}(t \otimes t') \triangleleft \text{can}^{+}(t \otimes t') = \varepsilon(t)t'. \]

We use the Lemma 4.2(iii) on the last equality. For the relation iii), we apply (4.14) on Lemma 4.2v.

\[ \square \]

### 4.2 Equivariant Hopf Galois coextension as a functor

Let \( \mathcal{K}T(S)_B \) be a \( \mathcal{K} \)-equivariant \( B \)-Galois coextension, and let \( M \) be a left-right SAYD module over \( \mathcal{K} \). We let \( B \) coacts on \( \tilde{M} := M \square \mathcal{K}T \) from left by
\[ m \otimes t \mapsto \kappa\left((t^{(2)}_{<0>})^{(2)} \otimes t^{(1)}_{<1>} \right) \otimes \left(t^{(2)}_{<1>} \triangleright m \otimes (t^{(2)}_{<0>})^{(1)} \right), \]
and let \( B \) acts on \( \tilde{M} \) from right by
\[ (m \otimes t) \triangleright b = m \otimes (t \triangleright b). \]
One notes that by (4.14) the coaction (4.15) reduces to

\[ m \otimes t \mapsto \text{can}_+ (t^{(2)}_{<0>} \otimes t^{(1)}_{<0>}) \otimes t^{(2)}_{<-1>} \triangleright m \otimes \text{can}_- (t^{(2)}_{<0>} \otimes t^{(1)}_{<0>}). \]  

(4.17)

**Theorem 4.7.** Let \( C \) be a coalgebra, \( B \) a right \( \times_C \)-Hopf coalgebra, \( T \) a right \( B \)-module coalgebra, \( S = T_B \), \( K \) a left \( \times_S \)-Hopf coalgebra, \( T \) a left \( K \)-comodule coalgebra and \( M \) be a left-right \( \text{SAYD} \) module over \( K \). If \( K \)-\( T(S)_B \) is a \( K \)-equivariant \( B \)-Galois coextension, then \( \bar{M} := M \square K T \) is a right-left \( \text{SAYD} \) module over \( B \) by the coaction and action defined in (4.15) and (4.16).

**Proof.** By the use of (2.19) and (2.45), the following computation shows that the left coaction defined in (4.15) is well-defined.

\[
\begin{align*}
\kappa \left( (t^{(2)}_{<0>})^{(2)} \otimes t^{(1)}_{<0>} \right) \otimes \alpha \left\{ \kappa \left( \left( (t^{(2)}_{<0>})^{(1)} \right)_{<0>} \right) \otimes \left( (t^{(2)}_{<0>})^{(1)} \right)_{<0>} \right\} \\
\otimes (t^{(2)}_{<-1>} \otimes t^{(2)}_{<-1>} \otimes m \otimes \left( (t^{(2)}_{<0>})^{(1)} \right)_{<0>}) \\
= \kappa \left( (t^{(2)}_{<0>})^{(3)} \otimes t^{(1)}_{<0>} \right) \otimes \alpha \left\{ \kappa \left( \left( (t^{(2)}_{<0>})^{(2)} \right)_{<0>} \right) \otimes \left( (t^{(2)}_{<0>})^{(1)} \right)_{<0>} \right\} \\
\otimes (t^{(2)}_{<-1>} \otimes m \otimes \left( (t^{(2)}_{<0>})^{(2)} \right)_{<0>}) \\
= \kappa \left( (t^{(4)}_{<0>}) \otimes t^{(1)}_{<0>} \right) \otimes \alpha \left\{ \kappa \left( \left( (t^{(3)}_{<0>})^{(2)} \right) \otimes \left( (t^{(2)}_{<0>})^{(2)} \right) \right) \right\} \\
\otimes (t^{(3)}_{<0>} \otimes m \otimes \left( (t^{(3)}_{<0>})^{(1)} \right)_{<0>}) \\
= \kappa \left( (t^{(3)}_{<0>})^{(3)} \otimes t^{(1)}_{<0>} \right) \otimes \alpha \left\{ \kappa \left( \left( (t^{(3)}_{<0>})^{(2)} \right) \otimes \left( (t^{(2)}_{<0>})^{(2)} \right) \right) \right\} \\
\otimes (t^{(3)}_{<-1>} \otimes m \otimes (t^{(3)}_{<0>})^{(1)}) \right) \otimes \alpha \left\{ \kappa \left( \left( (t^{(2)}_{<0>})^{(2)} \right) \otimes \left( (t^{(1)}_{<0>})^{(2)} \right) \right) \right\} \\
\otimes (t^{(2)}_{<-1>} \otimes m \otimes \left( (t^{(2)}_{<0>})^{(1)} \right)_{<0>}) \\
= \kappa \left( (t^{(2)}_{<0>})^{(2)} \otimes t^{(1)}_{<0>} \right) \otimes \alpha \left\{ \kappa \left( \left( (t^{(2)}_{<0>})^{(2)} \right) \otimes \left( (t^{(1)}_{<0>})^{(2)} \right) \right) \right\} \\
\otimes (t^{(2)}_{<-1>} \otimes m \otimes \left( (t^{(2)}_{<0>})^{(1)} \right)_{<0>}) \\
= \left[ \kappa \left( (t^{(2)}_{<0>})^{(2)} \otimes t^{(1)}_{<0>} \right) \right] \otimes \alpha \left\{ \kappa \left( \left( (t^{(2)}_{<0>})^{(2)} \right) \otimes \left( (t^{(1)}_{<0>})^{(2)} \right) \right) \right\} \\
\otimes (t^{(2)}_{<-1>} \otimes m \otimes \left( (t^{(2)}_{<0>})^{(1)} \right)_{<0>}) \\
= \left[ \kappa \left( (t^{(2)}_{<0>})^{(2)} \otimes t^{(1)}_{<0>} \right) \right] \otimes \alpha \left\{ \kappa \left( \left( (t^{(2)}_{<0>})^{(2)} \right) \otimes \left( (t^{(1)}_{<0>})^{(2)} \right) \right) \right\} \\
\otimes (t^{(2)}_{<-1>} \otimes m \otimes \left( (t^{(2)}_{<0>})^{(1)} \right)_{<0>}) \\
= \left[ \kappa \left( (t^{(2)}_{<0>})^{(2)} \otimes t^{(1)}_{<0>} \right) \right] \otimes \alpha \left\{ \kappa \left( \left( (t^{(2)}_{<0>})^{(2)} \right) \otimes \left( (t^{(1)}_{<0>})^{(2)} \right) \right) \right\} \\
\otimes (t^{(2)}_{<-1>} \otimes m \otimes \left( (t^{(2)}_{<0>})^{(1)} \right)_{<0>})
\end{align*}\]

We use the left \( K \)-comodule coalgebra property of \( T \) in the second equality, left \( K \)-comodule coalgebra property of \( T \) and Lemma (4.6iii) in the fourth equality and the anti coalgebra map property of \( \kappa \) in the last equality.

35
Now we prove that the coaction defined in (4.15) is coassociative.

\[
\begin{align*}
&\left[\kappa\left(t^{(2)}_{<0>} \otimes t^{(1)}\right)\right]^{(1)} \otimes \left[\kappa\left(t^{(2)}_{<0>} \otimes t^{(1)}\right)\right]^{(2)} \\
&\otimes t^{(2)}_{<-1>} \triangleright m \otimes \left(t^{(2)}_{<0>}\right)^{(1)} \\
&= \kappa\left(t^{(2)}_{<0>} \otimes t^{(1)}\right) \otimes \kappa\left(t^{(2)}_{<0>} \otimes t^{(1)}\right) \\
&\otimes t^{(2)}_{<-1>} \triangleright m \otimes \left(t^{(2)}_{<0>}\right)^{(1)} \\
&= \left[\kappa\left(t^{(3)}_{<0>} \otimes t^{(1)}\right)\right] \otimes \left[\kappa\left(t^{(3)}_{<0>} \otimes t^{(2)}\right)\right] \\
&\otimes t^{(3)}_{<-1>} \triangleright m \otimes \left(t^{(3)}_{<0>}\right)^{(1)} \\
&= \kappa\left(t^{(3)}_{<0>} \otimes t^{(1)}\right) \otimes \kappa\left(t^{(3)}_{<0>} \otimes t^{(2)}\right) \\
&\otimes t^{(3)}_{<-1>} \triangleright m \otimes t^{(3)}_{<0>} \\
&= \kappa\left(t^{(5)}_{<0>} \otimes t^{(1)}\right) \otimes \kappa\left(t^{(4)}_{<0}} \otimes t^{(2)}\right) \\
&\otimes t^{(3)}_{<-1>} \triangleright m \otimes t^{(3)}_{<0>} \\
&= \kappa\left(t^{(5)}_{<0>} \otimes t^{(1)}\right) \otimes \kappa\left(t^{(4)}_{<0}} \otimes t^{(2)}\right) \\
&\otimes t^{(3)}_{<-1>} \triangleright m \otimes t^{(3)}_{<0>} \\
&= \kappa\left(t^{(4)}_{<0}} \otimes t^{(1)}\right) \otimes \kappa\left(t^{(3)}_{<0}} \otimes t^{(2)}\right) \\
&\otimes t^{(3)}_{<-1>} \triangleright m \otimes t^{(3)}_{<0>} \\
&= \kappa\left(t^{(3)}_{<0}} \otimes t^{(1)}\right) \otimes \kappa\left(t^{(3)}_{<0}} \otimes t^{(2)}\right) \\
&\otimes t^{(3)}_{<-1>} \triangleright m \otimes t^{(3)}_{<0>} \\
&= \kappa\left(t^{(2)}_{<0}} \otimes t^{(1)}\right) \otimes \kappa\left(t^{(2)}_{<0}} \otimes t^{(2)}\right) \\
&\otimes t^{(2)}_{<-1>} \triangleright m \otimes \left(t^{(2)}_{<0}}\right)^{(1)} \\
&= \kappa\left(t^{(2)}_{<0}} \otimes t^{(1)}\right) \otimes \kappa\left(t^{(2)}_{<0}} \otimes t^{(2)}\right) \\
&\otimes t^{(2)}_{<-1>} \triangleright m \otimes \left(t^{(2)}_{<0}}\right)^{(1)}.
\end{align*}
\]
We use the anti coalgebra map property of $\kappa$ in the first equality, the left $K$-comodule coalgebra property of $T$ in the third equality, Lemma 4.6[iii] and the left $K$-comodule coalgebra property of $T$ in the fifth equality, and finally the left $K$-comodule coalgebra property of $T$ in the seventh equality.

The following shows that the coaction defined in (4.15) is counital.

\[
\varepsilon \left[ \kappa \left( (t_{<0>}^{(2)}) \otimes t^{(1)} \right) \right] t_{<1>}^{(2)} \triangleright m \otimes (t_{<0>}^{(2)})^{(1)} \\
= \varepsilon \left[ \text{can}_- \left( (t_{<0>}^{(2)}) \otimes t^{(1)} \right) \right] \varepsilon \left[ \text{can}_+ \left( (t_{<0>}^{(2)}) \otimes t^{(1)} \right) \right] t_{<1>}^{(2)} \triangleright m \\
\otimes (t_{<0>}^{(2)})^{(1)} \\
\varepsilon(t_{<0>}^{(2)})^{(2)} \varepsilon(t_{<1>}^{(1)}) t_{<1>}^{(2)} \triangleright m \otimes (t_{<0>}^{(2)})^{(1)} = \varepsilon(t_{<0>}^{(2)}) t_{<1>}^{(2)} \triangleright m \otimes t_{<0>}^{(2)} \\
= t_{<1>}^{(2)} \triangleright m \otimes t_{<0>}^{(2)} = m_{<1>} \triangleright m_{<0>} \otimes t = m \otimes t.
\]

We use Lemma 4.6(iv) in the second equality, the fact that $m \otimes t \in M \square K T$ in the penultimate, and eventually the stability condition in the last equality.

The following computation proves the stability condition.

\[
\left[ t_{<1>}^{(2)} \triangleright m \otimes (t_{<0>}^{(2)})^{(1)} \right] \triangleleft \kappa \left( (t_{<0>}^{(2)}) \otimes t^{(1)} \right) \\
= t_{<1>}^{(2)} \triangleright m \otimes \left[ (t_{<0>}^{(2)})^{(1)} \triangleleft \kappa \left( (t_{<0>}^{(2)}) \otimes t^{(1)} \right) \right] \\
= t_{<1>}^{(2)} \triangleright m \otimes \varepsilon(t_{<0>}^{(2)}) t^{(1)} = \varepsilon(t_{<0>}^{(2)}) \eta(\alpha(t_{<1>}^{(2)})) = m \otimes t.
\]

We use Lemma 4.6(ii) in the second equality and left $K$-comodule coalgebra
property of $T$ in the third equality. Here we prove the AYD condition.

\[
[(m \otimes t) < b] <_{-1} > \otimes [(m \otimes t) < b] <_{0} > \\
= \nu^+ \left( (b^{(2)}, (m \otimes t) <_{-1} >) b^{(1)} \otimes (m \otimes t) <_{0} > < b^{(2), (m \otimes t) <_{-1} >} \right) \\
= \nu^+ \left( (b^{(2)}, \kappa \left( (t^{(2)} <_{0} >) \otimes t^{(1)} \right)) b^{(1)} \right) \\
\otimes t^{(2)} <_{-1} > \triangleright m \otimes \left( (t^{(2)} <_{0} >) \triangleleft \nu^+ \left( (b^{(2)}, \kappa \left( (t^{(2)} <_{0} >) \otimes t^{(1)} \right)) \right) \right) \\
= \varepsilon \left\{ \text{can}_- \left[ t^{(2)} <_{0} > \otimes t^{(1)} \right] \right\} \nu^+ \left( (b^{(2)}, \text{can}_+ \left[ t^{(2)} <_{0} > \otimes t^{(1)} \right]) b^{(1)} \right) \\
\otimes t^{(2)} <_{-1} > \triangleright m \otimes \left[ (t^{(2)} <_{0} >) \triangleleft \nu^+ \left( (b^{(2)}, \text{can}_+ \left[ t^{(2)} <_{0} > \otimes t^{(1)} \right]) \right) \right] \\
= \varepsilon \left\{ \text{can}_- \left[ t^{(2)} <_{0} > \otimes t^{(1)} \right] \right\} \varepsilon \left\{ \nu^+ \left( (b^{(3)}, \text{can}_+ \left[ t^{(2)} <_{0} > \otimes t^{(1)} \right]) \right) \right\} \\
\nu^+ \left( (b^{(3)}, \text{can}_+ \left[ t^{(2)} <_{0} > \otimes t^{(1)} \right]) b^{(1)} \otimes t^{(2)} <_{-1} > \triangleright m \otimes \left[ (t^{(2)} <_{0} >) \triangleleft \nu^+ \left( (b^{(2)}) \right) \right] \\
= \varepsilon \left\{ \text{can}_- \left[ t^{(2)} <_{0} > \otimes t^{(1)} \right] \right\} \varepsilon \left\{ \nu^+ \left( (b^{(3)}, \text{can}_+ \left[ t^{(2)} <_{0} > \otimes t^{(1)} \right]) \right) \right\} \\
\nu^+ \left( (b^{(3)}, \text{can}_+ \left[ t^{(2)} <_{0} > \otimes t^{(1)} \right]) b^{(1)} \otimes t^{(2)} <_{-1} > \triangleright m \otimes \left[ (t^{(2)} <_{0} >) \triangleleft \nu^+ \left( (b^{(2)}) \right) \right] \\
= \varepsilon \left\{ \text{can}_- \left[ t^{(2)} <_{0} > \otimes t^{(1)} \right] \right\} \text{can}_+ \left[ (t^{(2)} <_{0} > \triangleleft b^{(3)}) \otimes t^{(1)} \right] b^{(1)} \\
\otimes t^{(2)} <_{-1} > \triangleright m \otimes \left[ (t^{(2)} <_{0} > \triangleleft b^{(3)}) \otimes t^{(1)} \right] b^{(1)} \\
= \varepsilon \left\{ \text{can}_- \left[ t^{(2)} <_{0} > \otimes t^{(1)} \right] \right\} \text{can}_+ \left[ (t^{(2)} <_{0} > \triangleleft b^{(3)}) \otimes t^{(1)} \right] b^{(1)} \\
\text{can}_+ \left[ (t^{(2)} <_{0} > \triangleleft b^{(3)}) \otimes t^{(1)} \right] b^{(1)} \otimes t^{(2)} <_{-1} > \triangleright m \otimes \left[ (t^{(2)} <_{0} > \triangleleft b^{(3)}) \otimes t^{(1)} \right] b^{(1)} \\
= \varepsilon \left\{ \text{can}_- \left[ t^{(2)} <_{0} > \otimes t^{(1)} \right] \right\} \text{can}_+ \left[ (t^{(2)} <_{0} > \triangleleft b^{(3)}) \otimes t^{(1)} \right] b^{(1)} \\
\text{can}_+ \left[ (t^{(2)} <_{0} > \triangleleft b^{(3)}) \otimes t^{(1)} \right] b^{(1)} \otimes t^{(2)} <_{-1} > \triangleright m \\
= \kappa \left[ (t^{(2)} <_{0} > \triangleleft b^{(2)}) \otimes (t^{(2)} <_{0} > \triangleleft b^{(1)}) \otimes t^{(2)} <_{-1} > \triangleright m \\
\otimes (t^{(2)} <_{0} > \triangleleft b^{(2)}) \right] \\
= \kappa \left[ (t \triangleleft b^{(2)}) \otimes (t \triangleleft b^{(1)}) \otimes (t \triangleleft b^{(2)}) \right] \otimes (t \triangleleft b^{(2)}) <_{-1} > \triangleright m \otimes \left[ (t \triangleleft b^{(2)}) <_{0} > \right] \\
= \left[ m \otimes t \triangleleft b \right] <_{-1} > \otimes \left[ m \otimes t \triangleleft b \right] <_{0} > .
\]
We use the AYD condition in the first equality, the coaction defined in (4.15) in the second equality, definition of $\kappa$ in the third equality, the equation obtained by applying $\text{Id} \otimes \varepsilon \otimes \text{Id}$ on the left $\mathcal{B}$ module map property in the fourth equality, left $\mathcal{B}$-module coalgebra property of $T$ in the fifth equality, the Lemma (4.2 xi) in the sixth equality, Lemma (4.2 xi) in the seventh equality, left $\mathcal{B}$-module coalgebra property of $T$ in the eighth equality and $\mathcal{K}$-equivaraint property of (4.2) in the ninth equality.

The following computation shows that the right action $\tilde{\mathcal{M}} \square \mathcal{C}\mathcal{B} \rightarrow \tilde{\mathcal{M}}$, defined in (4.16) is well-defined.

$$m_{<0>} \otimes m_{<1>} \otimes t \triangleleft b = m \otimes t_{<-1>} \otimes t_{<0>} \triangleleft b = m \otimes (t \triangleleft b)_{<-1>} \otimes (t \triangleleft b)_{<0>}.$$  

We use $m \otimes t \otimes b \in M \square_K T \square \mathcal{C}\mathcal{B}$ in the first equality and the $\mathcal{K}$-equivariant property of the action of $\mathcal{B}$ on $T$ on the second equality.

The associativity of the action defined in (4.16) is obvious by the associativity of the right action of $\mathcal{B}$ on $T$.

Let us denote the category of SAYD modules over a left $\times$-Hopf coalgebra $\mathcal{K}$ by $\mathcal{K}_{\text{SAYD}}$. Its object are all left-right SAYD modules over $\mathcal{K}$ and its morphisms are all $\mathcal{K}$-linear-colinear maps. Similarly one denotes by $\mathcal{B}_{\text{SAYD}}$ the category of right-left SAYD modules over a right $\times$-Hopf coalgebra $\mathcal{B}$.

We see that Theorem (4.7) amounts to an object map of a functor $\phi$ from $\mathcal{K}_{\text{SAYD}}$ to $\mathcal{B}_{\text{SAYD}}$. In fact, if $M, N \in _\mathcal{K} \text{SAYD}\mathcal{K}$ then $\phi(M) = \tilde{M}$, $\phi(N) = \tilde{N}$ and if $\phi : M \rightarrow N$ then we denote $\phi \square \text{Id}_T$.

**Proposition 4.8.** The assignment $\phi : _\mathcal{K} \text{SAYD}\mathcal{K} \rightarrow _\mathcal{B} \text{SAYD}\mathcal{B}$ defines a covariant functor.

**Proof.** Using Theorem (4.7) the map $\phi$ is an object map. It is easy to check that $\phi$ is a module and comodule map. 

Let us recall that

$$\mathcal{K}C^n(T, M) = M \square_K T \square s^{(n+1)}, \quad \tilde{C}^{\mathcal{B}, n}(\mathcal{B}, \tilde{M}) = \mathcal{B} \square_C^n \square_C \text{cop} \tilde{M}$$

are the cocyclic modules defined in (3.11) and (3.13), respectively. We define the following map,

$$\omega_n : \mathcal{B} \square_C^n \square_C \text{cop} \tilde{M} \rightarrow M \square_K T \square s^{(n+1)}, \quad (4.18)$$

given by

$$\omega_n(b_1 \otimes \cdots \otimes b_n \otimes m \otimes t) = m \otimes t^{(1)} \otimes t^{(2)} \triangleleft b_1^{(1)} \otimes t^{(3)} \triangleleft [b_1^{(2)} b_2^{(1)}] \otimes \cdots \otimes t^{(n)} \triangleleft [b_1^{(n-1)} \cdots b_{n-1}^{(1)}] \otimes t^{(n+1)} \triangleleft \tilde{b}_1 \otimes \cdots \tilde{b}_{n-1} \otimes \tilde{b}_n].$$

39
with an inverse map
\[
\omega_n^{-1}(m \otimes t_0 \otimes \cdots \otimes t_n) \\
= \kappa(t_0^{(2)} \otimes t_1^{(1)}) \otimes \kappa(t_1^{(2)} \otimes t_2^{(1)}) \otimes \cdots \otimes \kappa(t_{n-1}^{(2)} \otimes t_n) \otimes m \otimes t_0^{(1)}.
\]

**Theorem 4.9.** Let \(K T(S)\) be a \(K\)-equivariant \(B\)-Galois coextension, and \(M\) be a left-right SAYD module over \(K\). The map \(\omega\) defines an isomorphism of cocyclic modules between \(KC^n(T, M)\) and \(\tilde{C}B,n(B, \tilde{M})\), which are defined in (3.1) and (3.13) respectively. Here \(\tilde{M} = M \square \kappa T\) is the right-left SAYD module over \(B\) introduced in Theorem 4.7.

**Proof.** One notes that \(\omega = \text{can}^\otimes_n \circ tw_{B^\otimes_n \otimes \tilde{M}}\), where
\[
\text{can}^\otimes_n = (\text{Id}_M \otimes \text{Id}_T \otimes \cdots \otimes \text{Id}_T \otimes \text{can}) \circ \cdots \circ (\text{Id}_M \otimes \text{Id}_T \otimes \text{can} \otimes \text{Id}_S \otimes \cdots \otimes \text{Id}_S) \circ (\text{Id}_M \otimes \text{can} \otimes \text{Id}_S \otimes \cdots \otimes \text{Id}_S).
\]

One uses (4.2)(ii) to easily prove \(\omega^{-1} \circ \omega = \text{Id}\). Also by anti coalgebra map property of \(\kappa\) and (4.14) one can show \(\omega \circ \omega^{-1} = \text{Id}\). By a very routine and long computation reader will check that \(\omega\) is well-defined and it commutes with faces and degeneracies. Here we only show that \(\omega\) commutes with the cyclic maps.
\[ t_n \omega_n (b_1 \otimes \cdots \otimes b_n \otimes m \otimes t) \]

\[
= t_n \{ m \otimes t^{(1)} \otimes t^{(2)} \otimes [b_1^{(1)} \otimes t^{(3)} \otimes [b_1^{(2)} b_2^{(1)}] \otimes \cdots \\
\quad \cdots \otimes t^{(n)} \otimes [b_1^{(n-1)} \cdots b_{n-1}^{(1)}] \otimes t^{(n+1)} \otimes [b_1^{(n)} \cdots b_1^{-(2)} b_n] \} \\
= \left[ t^{(2)} \otimes [b_1^{(1)}] <_{-1} \cdots \left[ t^{(n)} \otimes [b_1^{(n-1)} \cdots b_{n-1}^{(1)}] \right] <_{-1} \right. \\
\quad \left. \left[ t^{(n+1)} \otimes [b_1^{(n)} \cdots b_{n-1}^{(2)} b_n] \right] <_{-1} \triangleright m \right. \\
\quad \otimes \left[ t^{(2)} \otimes [b_1^{(1)}] <_{0} \otimes \cdots \otimes \left[ t^{(n)} \otimes [b_1^{(n-1)} \cdots b_{n-1}^{(1)}] \right] <_{0} \right. \\
\quad \left. \otimes \left[ t^{(n+1)} \otimes [b_1^{(n)} \cdots b_{n-1}^{(2)} b_n] \right] <_{0} \otimes t^{(1)} \right) \\
= t^{(2)} <_{-1} \cdots t^{(n)} <_{-1} t^{(n+1)} \triangleright m \otimes \left[ t^{(2)} <_{0} \otimes [b_1^{(1)}] \otimes \cdots \\
\quad \cdots \otimes \left[ t^{(n)} <_{0} \otimes [b_1^{(n-1)} \cdots b_{n-1}^{(1)}] \right] \otimes t^{(1)} \right) \\
= t^{(2)} <_{-1} \triangleright m \otimes \left[ \left( t^{(2)} <_{0} \right)^{(1)} \otimes [b_1^{(1)}] \otimes \left( t^{(2)} <_{0} \right)^{(2)} \otimes [b_1^{(2)} b_2^{(1)}] \right. \\
\quad \left. \cdots \otimes \left( \left( t^{(2)} <_{0} \right)^{(n)} \otimes [b_1^{(n)} b_2^{(n-1)} \cdots b_{n-1}^{(2)} b_n] \right) \right. \\
\quad \left. \otimes \left( t^{(2)} <_{0} \right)^{(n+1)} \otimes \kappa \left( \left( t^{(2)} <_{0} \right)^{(n+2)} \otimes t^{(1)} \right) \right)
\]

\[
= t^{(2)} <_{-1} \triangleright m \otimes \left[ \left( t^{(2)} <_{0} \right)^{(1)} \otimes [b_1^{(1)}] \otimes \left( t^{(2)} <_{0} \right)^{(2)} \otimes [b_1^{(2)} b_2^{(1)}] \right. \\
\quad \left. \cdots \otimes \left( \left( t^{(2)} <_{0} \right)^{(n+1)} \otimes [b_1^{(n)} b_2^{(n-1)} \cdots b_{n-1}^{(2)} b_n] \right) \right. \\
\quad \left. \otimes \left( t^{(2)} <_{0} \right)^{(n+2)} \otimes \kappa \left( \left( t^{(2)} <_{0} \right)^{(n+2)} \otimes t^{(1)} \right) \right)
\]

\[
= t^{(2)} <_{-1} \triangleright m \otimes \left[ \left( t^{(2)} <_{0} \right)^{(1)} \otimes [b_1^{(1)}] \otimes \left( t^{(2)} <_{0} \right)^{(2)} \otimes [b_1^{(2)} b_2^{(1)}] \right. \\
\quad \left. \cdots \otimes \left( \left( t^{(2)} <_{0} \right)^{(n+1)} \otimes [b_1^{(n)} b_2^{(n-1)} \cdots b_{n-1}^{(2)} b_n] \right) \right. \\
\quad \left. \otimes \left( t^{(2)} <_{0} \right)^{(n+2)} \otimes \kappa \left( \left( t^{(2)} <_{0} \right)^{(n+2)} \otimes t^{(1)} \right) \right)
\]

\[
= t^{(2)} <_{-1} \triangleright m \otimes \left[ \left( t^{(2)} <_{0} \right)^{(1)} \otimes [b_1^{(1)}] \otimes \left( t^{(2)} <_{0} \right)^{(2)} \otimes [b_1^{(2)} b_2^{(1)}] \right. \\
\quad \left. \cdots \otimes \left( \left( t^{(2)} <_{0} \right)^{(n+1)} \otimes [b_1^{(n)} b_2^{(n-1)} \cdots b_{n-1}^{(2)} b_n] \right) \right. \\
\quad \left. \otimes \left( t^{(2)} <_{0} \right)^{(n+2)} \otimes \kappa \left( \left( t^{(2)} <_{0} \right)^{(n+2)} \otimes t^{(1)} \right) \right)
\]
We use the $\mathcal{K}$-equivariant property \([\ref{12}]\) in the third equality, the left $\mathcal{K}$-comodule coalgebra property of $T$ in the fourth equality, Lemma \([\ref{42}]\)(ii) in the fifth equality, the coalgebra property of $B$ in the sixth equality, Lemma \([\ref{27}]\)(v) in the seventh equality, the relation obtained by applying $\Id \otimes \varepsilon \otimes \Id$.
on the left $B$-comodule property \[2.33\] in the eighth equality, the relation \[2.21\] in the ninth equality, the associativity of the comultiplication $B$ in the tenth equality and the right $B$-comodule coalgebra property of $T$ the eleventh equality.

\[\square\]

5 \hspace{1ex} A special case

Extension of algebras in its controlled situation governed by Hopf algebras. Similarly coextension of coalgebras can be modeled by the use of Hopf algebras [Schn, BH]. The ingredients are two coalgebras $C$, $D$ and a Hopf algebra $H$ acting on $C$ as a module coalgebra such that $D$ coincides with the space of coinvariants of this action and the canonical map is an isomorphism. Such a triple is called Hopf Galois coextension. In this section we specialize our main result in the previous section to show that the result of Jara-Stefan in [J-S], on associating a canonical SAYD module to any Hopf-Galois extension, holds for Hopf Galois coextensions.

5.1 Hopf Galois coextension and SAYD modules

The Definition 4.1 of a Equivariant Hopf Galois coextensions for $\times$-Hopf coalgebras generalizes the Hopf Galois coextensions. Indeed, let in Definition 4.1 we set $T = C$ be a coalgebra, $S = D$ be a coalgebra, $K = D^e$, and $B = H$ be a Hopf algebra, then we obtain the following definition of Hopf Galois coextension. Let $H$ be a Hopf algebra, $C$ a right $H$-module coalgebra with the action $\triangleleft : C \otimes H \to C$. One defines the following coideal

\[ I := \{c \triangleleft h - \varepsilon(h)c \mid c \in C, h \in H\}, \tag{5.1} \]

and introduces the coalgebra $D := C/I$. The coextension $C \to D$ is called a (right) Hopf-Galois coextension [Schn] if the canonical map

\[ \beta := (C \otimes \triangleleft) \circ (\Delta \otimes H) : C \otimes H \to C \square_D C, \]

\[ \beta(c, h) = c^{(1)} \otimes c^{(2)} \triangleleft h, \tag{5.2} \]

is a bijection. Such a coextension is denoted by $C(D)^H$. One notes that in this case the map (5.2) is a special case of the map (4.1). If $C(D)^H$ is a Hopf Galois coextension then for all $c \in C$ and $h \in H$ we have

\[ \pi(c \triangleleft h) = \varepsilon(h) \pi(c). \tag{5.3} \]
One notes that $\beta$ is a left $C$-comodule and right $H$-module map. Similar to (4.10) we set

$$C^D := \{ c \in C : c_{<1>} \otimes c_{<2>} = c_{<2>} \otimes c_{<1>} \}, \quad C_D := C^D,$$

or precisely,

$$C^D := \{ c \in C ; \quad c^{(1)} \otimes \pi(c^{(2)}) = c^{(2)} \otimes \pi(c^{(1)}) \}.$$  \hspace{1cm} (5.4)

Let us denote the inverse of the Galois map $\beta$ by the following summation notation.

$$\beta^{-1}(c \otimes c') := \beta_-(c, c') \otimes \beta_+(c, c').$$ \hspace{1cm} (5.5)

If there is no confusion we can simply write $\beta = \beta_- \otimes \beta_+$. We have properties similar to [i], [ii], [iii], [iv] and [xii] of the map can in the lemma 4.2 for the map $\beta$.

**Lemma 5.1.** Let $C$ and $D$ be coalgebras, $H$ be a Hopf algebra, and $C(D)^H$ be a Hopf Galois coextension with canonical bijection $\beta$. Then we have

\begin{itemize}
  \item[i)] $[\beta_-(c_1 \otimes c_2)]^{(1)} \otimes [\beta_-(c_1 \otimes c_2)]^{(2)} \otimes \beta_+(c_1 \otimes c_2) = c_1^{(1)} \otimes \beta_-(c_1^{(2)} \otimes c_2) \otimes \beta_+(c_1^{(2)} \otimes c_2)$.
  \item[ii)] $\beta_-(c_1 \otimes c_2)^{(1)} \otimes \beta_-(c_1 \otimes c_2)^{(2)} \otimes \beta_-(c_1 \otimes c_2)^{(3)} \triangleleft \beta_+(c_1 \otimes c_2)^{(1)} \otimes \beta_-(c_1 \otimes c_2)^{(2)} \otimes \beta_+(c_1 \otimes c_2)^{(3)} = c_1^{(1)} \otimes c_1^{(2)} \otimes c_2^{(1)} \otimes c_2^{(2)}$.
\end{itemize}

**Proof.** The relation i) is equivalent to the left $C$-colinear property of the map $\beta^{-1}$ where the left $C$ comodule structures of $C \otimes H$ and $C \Box_D C$ are given by $c \otimes h \mapsto c^{(1)} \otimes c^{(2)} \otimes h$ and $c_1 \otimes c_2 \mapsto c_1^{(1)} \otimes c_1^{(2)} \otimes c_2$, respectively. The relation ii) holds by applying $\Delta_C \otimes \Delta_C$ on both hand sides of Lemma 4.2 (i) and using $H$-module coalgebra property of $C$. \hfill $\square$

Similar to the case of $\times$-Hopf coalgebras, we define

$$\kappa := (\varepsilon \otimes \text{Id}_H) \circ \tilde{\beta}^{-1} : (C \square_D C)^D \rightarrow H,$$

where

$$\tilde{\beta} : C^D \otimes H \rightarrow (C \square_D C)^D.$$

The map $\kappa$ is an anti coalgebra map and it satisfies the following properties.

**Lemma 5.2.** Let $H$ be a Hopf algebra, $C$ be a $H$-module coalgebra, and $C(D)^H$ be a Hopf Galois coextension by the canonical bijection map $\beta$. Then $\kappa := (\varepsilon \circ H) \circ \tilde{\beta}^{-1} : (C \square_D C)^D \rightarrow H$ has the following properties.
\( i \) \( \kappa(c^{(1)} \triangleleft h \otimes c^{(2)} \triangleleft g) = \varepsilon(c)S(h)g, \quad \forall c \in C, \; g, h \in H, \)

\( ii \) \( \kappa(c \triangleleft h \otimes c') = S(h)\kappa(c \otimes c'), \quad c \otimes c' \in (C \bowtie D \bar{C})^D, \; h \in H. \)

\( iii \) \( c^{(1)} \triangleleft \kappa(c^{(2)} \otimes c') = \varepsilon(c)c'. \)

Proof. Lemma 4.2 xii)(xi) generalizes the relation \( i \). Also the relation \( ii \) is a special case of Lemma 4.2(xi). The relation \( iii \) is a special case of Lemma 5.2 (ii).

Let us introduce the relative cyclic cohomology of coalgebras coextensions. If in the Proposition 2.26 we consider \( \theta = \text{Id} \) then the following action and coaction make \( D \) as a left-right SAYD module on \( D^e = D \otimes D^{\text{cop}} \).

\[ d \mapsto d^{(2)} \otimes d^{(3)} \otimes d^{(1)}, \quad (d_1 \otimes d_2) \triangleright d_3 = \varepsilon(d_1)\varepsilon(d_2)d_3. \]

One specializes Theorem 4.7 for \( T = C, \; S = D, \; K = D^e \) and \( B = H \) to obtain the following. One notes that \( C \) and \( D \) have a left and a right \( D^e = D \otimes D^{\text{cop}} \)-comodule structures by

\[ c \mapsto \pi(c^{(1)}) \otimes \pi(c^{(3)}) \otimes c^{(2)}, \quad d \mapsto d^{(1)} \otimes d^{(3)} \otimes d^{(2)}, \quad (5.6) \]

respectively.

Proposition 5.3. Let \( C(D)^H \) be a Hopf-Galois coextension. Then \( \widetilde{M} := C^D = D \bowtie D^e \bar{C} \) is a right-left SAYD module over \( H \) by the following action and coaction.

\[ (d \otimes c) \triangleleft h = d \otimes c \triangleleft h, \quad d \otimes c \mapsto \kappa(c^{(3)} \otimes c^{(1)}) \otimes d \otimes c^{(2)}. \quad (5.7) \]

If we specialize the cocyclic module defined in the Proposition 3.1, we obtain the cocyclic module \( C^n(C, C^D) = D \bowtie D^e C^{\bowtie D(n+1)} \) with the following cofaces, codegeneracies and cocyclic map.

\[ \delta_i(d \otimes c_0 \otimes \cdots \otimes c_n) = d \otimes c_0 \otimes \cdots \otimes \Delta(c_i) \otimes \cdots \otimes c_n, \quad 0 \leq i \leq n \]
\[ \delta_{n+1}(d \otimes c_0 \otimes \cdots \otimes c_n) = \varepsilon(d)\pi(c_0^{(2)}) \otimes c_0^{(3)} \otimes c_1 \otimes \cdots \otimes c_n \otimes c_0^{(1)} \]
\[ \sigma_i(d \otimes c_0 \otimes \cdots \otimes c_n) = d \otimes c_0 \otimes \cdots \otimes \varepsilon(c_{i+1}) \otimes \cdots \otimes c_n. \]
\[ \tau_n(d \otimes c_0 \otimes \cdots \otimes c_n) = \varepsilon(d)\pi(c_1^{(1)}) \otimes c_1^{(2)} \otimes c_2 \otimes \cdots \otimes c_n \otimes c_0. \quad (5.8) \]

One notes that \( C^{\bowtie D(n+1)} \) is a left \( D^e \)-comodule by

\[ c_0 \otimes \cdots \otimes c_n \mapsto \pi(c_0^{(1)}) \otimes \pi(c_0^{(2)}) \otimes c_0^{(1)} \otimes c_1 \otimes \cdots \otimes c_n \otimes c_0. \quad (5.9) \]

Now we specialize Theorem 4.9 for \( T = C, \; S = D, \; K = D^e \) and \( B = H \) to obtain the following.
Proposition 5.4. Let $C(D)^H$ be a Hopf-Galois coextension. The following map

$$\omega_n : C^n(H, \tilde{M}) \rightarrow C^n(C, C^D).$$

is given by

$$\omega_n(h_0 \otimes h_1 \cdots \otimes h_n \otimes d \otimes c) = 
\varepsilon(h_n) d \otimes c^{(1)} \otimes c^{(2)} \prec h_0^{(1)} \otimes c^{(3)} \prec h_0^{(2)} h_1^{(1)} \otimes \cdots \otimes c^{(n+1)} \prec h_0^{(n)} h_1^{(n-1)} \cdots h_{n-2}^{(2)} h_{n-1},$$

defines an isomorphism of cocyclic modules between $C^n(C, C^D)$ and $C^n(H, \tilde{M})$ defined in (5.8) and (3.14), respectively,

$$HC^*(C, C^D) \cong HC^*(H, \tilde{M}).$$

Here $\tilde{M} = D \square_{D^c} C$ is the right-left SAYD module over $H$ introduced in Proposition 5.3.

The previous proposition is a dual of the [J-S, Theorem 3.7] for Hopf Galois extensions. In fact the cocyclic modules (5.8) and (3.14) are dual to the cocyclic module $Z_*(A/B, A)$ defined in [J-S, Theorem 1.5] and $Z_*(H, A_B)$ defined in [J-S, page 153]. One notes that in our case

$$C^D \cong D \square_{D^c} C,$$

where the isomorphism is given by map $\xi$ where

$$\xi(d \otimes c) = \varepsilon_D(d)c, \quad \text{and} \quad \xi^{-1}(c) = \pi(c^{(1)}) \otimes c^{(2)}.$$

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