Overlap for 2D chiral U(1) models

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The overlap formulation is applied to an anomaly free combination of chiral fermions coupled to U(1) gauge fields on a 2D torus. Evidence is presented that gauge averaging the overlap phases in these models produces correct continuum results.

1. Introduction

Two dimensional chiral gauge theories provide a convenient testing ground for non-perturbative regularizations since they are much simpler to simulate numerically than four dimensional ones. As in four dimensions, there are anomalies in two dimensions that have to be cancelled and anomaly free theories can have global charges that are broken by topologically non-trivial gauge fields. In this talk, we will discuss a specific anomaly free abelian chiral gauge theory in two dimensions. The non-perturbative regularization we will employ is the overlap formulation on the lattice [1]. The abelian chiral gauge theory we consider contains four left handed Weyl fermions and one right handed Weyl fermion living on an $l \times l$ torus. All four left handed fermions have a U(1) charge equal to 1 and the right handed fermion has a U(1) charge equal to 2.

The model is expected to be gauge invariant in the continuum since we have chosen the charges to cancel the perturbative anomaly. On the torus the model is sensitive to the boundary conditions. Even in the absence of an electric field, non-perturbative violations of gauge invariance under certain singular gauge transformations can occur if the boundary conditions are not chosen with care [2]. A choice free of problems is that the right handed fermions obey anti-periodic boundary conditions while the left handed fermions obey:

\[
\psi_{L1}(x+l\hat{\mu}) = i\psi_{L1}(x), \psi_{L3}(x+l\hat{\mu}) = -i\mu_\psi_{L3}(x), \\
\psi_{L2}(x+l\hat{\mu}) = i\mu_\psi_{L2}(x), \psi_{L4}(x+l\hat{\mu}) = -i\psi_{L4}(x).
\]

where $s_1 = 1$ and $s_2 = -1$.

Although the theory is chiral, with the above choice, the fermion determinant becomes real for any vector potential. While the fermion determinant is real in the continuum limit, the lattice overlap is a complex number reflecting the absence of exact gauge invariance at finite lattice spacing. When the gauge fields are integrated over, gauge invariance is restored by group averaging. We shall show that gauge integration along a fixed typical orbit reproduces the continuum result for that orbit.

2. Formulation on the lattice

We embed an $L \times L$ lattice in the continuum $l \times l$ torus. To the plaquette with corners at $n$, $n + \hat{\mu}$, $n + \hat{\nu}$ and $n + \hat{\mu} + \hat{\nu}$ we associate an angle $\phi(n)$. $\phi(n)$ is a discretization of the continuum $\phi(x)$ related to the electric field by $E(x) = \partial^2 \phi(x)$. We will restrict ourselves to the zero topological sector and therefore $\sum_n \phi(n) = 0$. Given $E(x)$ we can solve for $\phi(x)$ where $\phi(x)$ is a periodic function on the torus with no zero modes. The parallel transporters are

\[
U_\mu(n) = g(n + \hat{\mu})g^*(n)e^{i\mu_\nu[\phi(n) - \phi(n - \hat{\nu})]}e^{i\mu_\nu h_\mu}.
\]

The $h_\mu$'s are the zero modes of the gauge potential restricted to $[-1/2, 1/2]$. Gauge orbits will be labeled by $E(n) = (\partial_1^2 + \partial_2^2)\phi(n)$, $h_1$ and $h_2$. 

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$g(n)$ is a $U(1)$ valued group variable on the site $n$ and labels points on the orbit. $g(n) = 1$ corresponds to the gauge field in the Landau gauge. The Wilson gauge action is

$$S_g^w = \frac{1}{e^2} \sum_n \text{Re} \{ 1 - \cos((\partial_1^* \partial_1 + \partial_2^* \partial_2) \phi(n))) \}$$

The fermionic path integral is defined on the lattice using the overlap formalism and we refer the reader to [2] for details.

3. Overlap along gauge orbits

The overlap formula will not be gauge invariant on the lattice but gauge invariance is restored when the lattice spacing goes to zero while the gauge field is fixed. Thus, the gauge breaking is expected to be small. In our previous work [1] we suggested dealing with the extra gauge breaking terms by simply averaging over each gauge orbit. If the breaking is not too large, and if anomaly free chiral gauge theories in the continuum exist also beyond perturbation theory, the most plausible outcome is that the averaging along the orbit simply adds some irrelevant local gauge invariant terms to the rest of the action. For example, a gauge breaking term in an action for a pure gauge theory that has the form of a mass term for the gauge bosons, when averaged over the gauge orbits, induces only effects irrelevant in the infrared, as long as its coefficient is not too large [3].

In this section we shall repeatedly start from some configuration that has a typical gauge invariant content and average over its gauge orbit by computing the overlap for many gauge transformations of the original configuration. The overlap enjoys the nice property that all the gauge breaking is restricted to its phase [1]. The result of averaging this phase will be some complex number, $Z$. We will first focus on the resulting phase $\Phi$ where $Z = |Z| e^{i\Phi}$. We will look at the distribution of the overlap phases along a particular orbit. We will show that this distribution is quite well peaked around zero phase indicating that the continuum determinant is real. The width of the distribution will be a function of the gauge invariant content of the orbit.

Figure 1. $eL = (0.1, 0.25, 0.5, 1.0)\pi$, $h_1 = h_2 = 0$

The simplest background one could imagine is one in which there is no electric field and all Polyakov loops are trivial. The lattice overlap on this “trivial orbit” ($\phi = 0$ and $h_\mu = 0$) for a single chiral fermion was proven to be real in [1]. We will generate a typical electric field configuration by choosing some $eL$ in the Wilson action. We pick $h_\mu$ independently in the range $[-1/2, 1/2)$. In figure 1, we plot the distribution of the phases on orbits where $h_\mu = 0$ and $h_\mu$ = $0$. The x-axis is the phase in units of $\pi$. All the four distributions in the plot are well peaked and centered around zero. As $eL$ gets larger the distributions gets broader. This is because we are going away from the trivial orbit. The plot was obtained on a $6 \times 6$ lattice by randomly generating a total of 1000 points along the orbit. In figure 2, we have set $\phi = 0$ and studied the distribution for two different values of $h_\mu$. The distribution is again peaked around zero.

Figures 1 and 2 show that one can perform an integration along the orbit. Integration will induce an additional term to the effective action. This term will be proportional to the width of the distributions and we can study it as a function of $\phi$. The width characterizes the gauge invariant content of the orbit. Note that the continuum determinant depends on $\phi$ via a factor $\exp -\frac{2}{\pi} \int d^2x \phi \partial^2 \phi [1]$. In two dimensions, a
Thirring term can be induced upon regularization since it is renormalizable. This was found to be the case in a vector theory \[\text{4}\]. Such an effect is expected to be present in the chiral model also since the modulus of the overlap is the same as that of a vector theory. The integration over the orbit also induces a Thirring type interaction. The two Thirring terms modify the coefficient in front of the \[\int d^2x \phi \partial^2 \phi\] in the exponent of the overlap:

\[
\frac{2}{\pi} \rightarrow \frac{2}{\pi + g_1} + g_2 \equiv \frac{2c}{\pi}
\]

\(g_1\) is the effect present in the modulus of the overlap and \(g_2\) is the effect of integration over the orbit. The combined effect is denoted by one coefficient \(c\).

We extract the coefficient \(c\) by plotting the induced term on the lattice (“lattice”), obtained by taking the logarithm of the overlap integrated over the orbit, as a function of \[\frac{2}{\pi} \int d^2x \phi \partial^2 \phi\] (“continuum”) at fixed \(eL = \pi\). The imaginary part of the lattice term, as discussed previously, is consistent with zero. In Figure 3 we present a scatter-plot of the lattice term vs. the continuum one including several typical \(\phi\) configurations and several lattice spacings. Note that the points align quite well along a line with a slope of about unity. The evident correlation between the values on the two axes indicates that indeed one can parameterize the induced action by \(c\) and we find that \(c \approx 1\) in agreement with the continuum. In particular the result implies that the “photon mass” comes out correctly in this model.

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