Weighted core-EP inverse of a rectangular matrix

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In this paper, we revise the weighted core-EP inverse introduced by Ferreyra, Levis and Thome. Several computational representations of the weighted core-EP inverse are obtained in terms of singular-value decomposition, full-rank decomposition and QR decomposition. These representations are expressed in terms of various matrix powers as well as matrix product involving the core-EP inverse, Moore-Penrose inverse and usual matrix inverse. Finally, those representations involving only Moore-Penrose inverse are compared and analyzed via computational complexity and numerical examples.

Keywords: weighted core-EP inverse, core-EP inverse, pseudo core inverse, outer inverse, complexity

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1 Introduction

Let \( \mathbb{C}^{m \times n} \) be the set of all \( m \times n \) complex matrices and let \( \mathbb{C}^{m \times n}_r \) be the set of all \( m \times n \) complex matrices of rank \( r \). For each complex matrix \( A \in \mathbb{C}^{m \times n} \), \( A^* \), \( \mathcal{R}_s(A) \), \( \mathcal{R}(A) \) and \( \mathcal{N}(A) \) denote the conjugate transpose, row space, range (column space) and null space of \( A \), respectively. The index of \( A \in \mathbb{C}^{n \times n} \), denoted by \( \text{ind}(A) \), is the smallest non-negative integer \( k \) for which \( \text{rank}(A^k) = \text{rank}(A^{k+1}) \). The Moore-Penrose inverse (also known as the pseudoinverse) of \( A \in \mathbb{C}^{m \times n} \), Drazin inverse of \( A \in \mathbb{C}^{n \times n} \) are denoted as usual by \( A^\dagger \), \( A^D \) respectively.

The Drazin inverse was extended to a rectangular matrix by Cline and Greville [1]. Let \( A \in \mathbb{C}^{m \times n} \), \( W \in \mathbb{C}^{n \times m} \) and \( k = \max\{	ext{ind}(AW), \text{ind}(WA)\} \). The \( W \)-weighted Drazin inverse of \( A \), denoted by \( A^{D,W} \), is the unique solution to

\[
(AW)^k = (AW)^{k+1}XW, \quad X = XWAWX \text{ and } AWX = XWA.
\]

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Many authors have been focusing on the weighted Drazin inverse and have achieved much in the aspect of representations (see for example, [2–4]).

Baksalary and Trenkler [5] introduced the notion of core inverse for a square matrix of index one. Then, Manjunatha Prasad and Mohana [6] proposed the core-EP inverse for a square matrix of arbitrary index, as an extension of the core inverse. Later, Gao and Chen [7] gave a characterization for the core-EP inverse in terms of three equations. The core-EP inverse of $A \in \mathbb{C}^{n \times n}$, denoted by $A^{\odot}$, is the unique solution to

$$XA^{k+1} = A^k, \quad AX^2 = X \quad \text{and} \quad (AX)^* = AX, \quad (1.1)$$

where $k = \text{ind}(A)$. The core-EP inverse is an outer inverse (resp. $\{2\}$-inverse), i.e., $A^{\odot} AA^{\odot} = A^{\odot}$. The core inverse and core-EP inverse have applications in partial order theory (see for example, [8–10]).

Recently, an extension of the core-EP inverse from a square matrix to a rectangular matrix was made by Ferreyra et al. [11]. Let $A \in \mathbb{C}^{m \times n}$, $W \in \mathbb{C}^{n \times m}$ and $k = \max\{\text{ind}(AW), \text{ind}(WA)\}$. The $W$-weighted core-EP inverse of $A$, denoted by $A^{\odot,W}$, is the unique solution to the system

$$WAWX = (WA)^k[(WA)^k]^\dagger \quad \text{and} \quad \mathcal{R}(X) \subseteq \mathcal{R}((AW)^k). \quad (1.2)$$

Meanwhile, the authors proved that the $W$-weighted core-EP inverse of $A$ can be written as a product of matrix powers involving two Moore-Penrose inverses.

$$A^{\odot,W} = [W(AW)^{l+1}[(AW)^{l}]^\dagger]^\dagger (l \geq k). \quad (1.3)$$

Then, Mosić [12] studied the weighted core-EP inverse of an operator between two Hilbert spaces as a generalization of the weighted core-EP inverse of a rectangular matrix.

In this paper, our main goal is to further study the weighted core-EP inverse for a rectangular matrix and compile its new, computable representations. The paper is carried out as follows. In Section 2, first of all, the weighted core-EP inverse is characterized in terms of three equations. This could be very useful in testing the accuracy of a given numerical method (to compute the weighted core-EP inverse) via residual norms. Then, we derive the canonical form for the $W$-weighted core-EP inverse of $A$ by using the singular value decompositions of $A$ and $W$. Later, representations of the weighted core-EP inverse are obtained via full-rank decomposition, general algebraic structure (GAS) and QR decomposition in conjunction with the fact that the weighted core-EP inverse is a particular outer inverse. These representations are expressed eventually through various matrix powers as well as matrix product involving the core-EP inverse, Moore-Penrose inverse and usual matrix inverse. In Section 3, some properties of the weighted core-EP inverse are exhibited naturally as outcomes of given representations. As mentioned earlier, the weighted core-EP inverse is a particular outer inverse. It is known that the inverse along an element [13] and $(B,C)$-inverse [14] are outer inverses as well. Thus, in Section 4, we wish to reveal the relations among the weighted core-EP inverse, weighted Drazin inverse, the inverse along an element, and $(B,C)$-inverse. In Section 5, the computational complexities of proposed representations involving pseudoinverse are estimated. In the final section, corresponding numerical examples are implemented by using Matlab R2017b.
2 Representations of the weighted core-EP inverse

In this section, we compile some new expressions of the weighted core-EP inverse for a rectangular complex matrix. First, the weighted core-EP inverse is characterized in terms of three equations. This plays a key role in examining the accuracy of a numerical method.

Lemma 2.1. [2, Theorem 2.3] Let $A \in \mathbb{C}^{n \times n}$ and let $l$ be a non-negative integer such that $l \geq k = \text{ind}(A)$. Then $A^{\odot} = A^D A^l(A^l)^\dagger$. In this case, $AA^{\odot} = A^l(A^l)^\dagger$.

Theorem 2.2. Let $A \in \mathbb{C}^{m \times n}$, $W \in \mathbb{C}^{n \times m}$ and $k = \max \{\text{ind}(AW), \text{ind}(WA)\}$. Then there exists a unique $X \in \mathbb{C}^{m \times n}$ such that

\[ XW(AW)^{k+1} = (AW)^k, \quad AWXWX = X \quad \text{and} \quad (WAWX)^* = WAWX. \] (2.1)

The unique $X$ which satisfies the above equations is $X = A[(WA)^{\odot}]^2$.

Proof. First of all, we can check that $X = A[(WA)^{\odot}]^2$ satisfies the equations

\[ XW(AW)^{k+1} = (AW)^k, \quad AWXWX = X \quad \text{and} \quad (WAWX)^* = WAWX. \]

In fact, in view of Lemma 2.1,

\[ A[(WA)^{\odot}]^2 W(AW)^{k+1} = A(WA)^{\odot} [(WA)^{\odot} (WA)^{k+1}] W = A(WA)^{\odot} (WA)^k W \]
\[ = A(WA)^D (WA)^k [(WA)^{\dagger}]^{\dagger} (WA)^k W \]
\[ = A(WA)^D (WA)^k W, \quad \text{which implies that} \]
\[ A[(WA)^{\odot}]^2 W(AW)^{k+1} = (AW)^D (AW)^{k+1} = (AW)^k, \quad \text{due to} \quad A(WA)^D = (AW)^D A; \]
\[ AW[(WA)^{\odot}]^2 WA[(WA)^{\odot}]^2 = A(WA)^{\odot} WA[(WA)^{\odot}]^2 = A[(WA)^{\odot}]^2; \]
\[ (WAWA[(WA)^{\odot}]^2)^* = WA(WA)^{\odot}. \]

Next, we would give a proof of the uniqueness of $X$. If

\[ XW(AW)^{k+1} = (AW)^k, \quad AWXWX = X \quad \text{and} \quad (WAWX)^* = WAWX \]

and

\[ YW(AW)^{k+1} = (AW)^k, \quad AWYXY = Y \quad \text{and} \quad (WAWX)^* = WAWX, \]

then

\[ X = AWXWX = (AW)^2 (XW)^2 X = (AW)^k (XW)^k X \]
\[ = YW(AW)^{k+1} (XW)^k X = Y(WA)^{k+1} (WX)^{k+1} = Y(WA)^{k+2} (WX)^{k+2} \]
\[ = Y[(WA)^{k+2} (WX)^{k+2} (WA)^{k+2}] (WX)^{k+2} \]
\[ = Y[(WA)^{k+2} (WX)^{k+2}] [(WA)^{k+2} (WX)^{k+2}]^* \]
\[ = Y[(WX)^{k+2}] [(WA)^{k+2} (WX)^{k+2}]^* = Y[(WX)^{k+2}] [(WA)^{k+2}]^* \]
\[ = Y(WAWY)^* = YWAWY = Y(WA)^{k+1} (YW)^{k+1} = YW(AW)^{k+1} (YW)^k Y \]
\[ = (AW)^k (YW)^k Y = AWYXY = Y. \]

This completes the proof. \(\square\)
Theorem 2.3. Let $A, X \in \mathbb{C}^{n \times n}, W \in \mathbb{C}^{n \times m}$ and $k = \max \{\text{ind}(AW), \text{ind}(WA)\}$. Then the following are equivalent:

1. $A^{\odot}W = X$;
2. $XW(AW)^{k+1} = (AW)^k$, $AWXWX = X$ and $(WAW)^* = WA$.

Proof. It suffices to show that $X = A[(WA)^{\odot}]^2$ satisfies condition (1.2). Indeed,

$$WAWA[(WA)^{\odot}]^2 = WA(WA)^{\odot} = WA(WA)^D(WA)^k[(WA)^k]^* = (WA)^k[(WA)^k]^*,$$

$$A[(WA)^{\odot}]^2 = AW[(WA)^{\odot}]^3 = A(WA)^k[(WA)^{\odot}]^{k+2} = (AW)^kA[(WA)^{\odot}]^k+2,$$

i.e.,

$$\mathcal{R}(A[(WA)^{\odot}]^2) \subseteq \mathcal{R}((AW)^k).$$

This completes the proof. 

Corollary 2.4. Let $A \in \mathbb{C}^{m \times n}, W \in \mathbb{C}^{n \times m}$. Then $A^{\odot}W = A$ if and only if $A = A[WAW]$. 

We now give the canonical form for the $W$-weighted core-EP inverse of $A$ by using the singular value decompositions of $A$ and $W$. Let $A \in \mathbb{C}^{m \times n}, W \in \mathbb{C}^{n \times m}$ be of the following singular value decompositions:

$$A = U \left( \begin{array}{cc} \Sigma_1 & 0 \\ 0 & 0 \end{array} \right) V^* \text{ and } W = S \left( \begin{array}{cc} \Sigma_2 & 0 \\ 0 & 0 \end{array} \right) T^*, \quad (2.2)$$

where $U$, $V$, $S$, $T$ are unitary matrices, $\Sigma_1 = \text{diag}(\sigma_1, \cdots, \sigma_r)$, $\sigma_1 \geq \cdots \geq \sigma_r > 0$ and $\Sigma_2 = \text{diag}(\tau_1, \cdots, \tau_s)$, $\tau_1 \geq \cdots \geq \tau_s > 0$.

Theorem 2.5. Let $A \in \mathbb{C}^{m \times n}, W \in \mathbb{C}^{n \times m}$ be of the singular value decompositions as in (2.2). Then

$$A^{\odot}W = U \left[ \begin{array}{cc} \Sigma_1 & 0 \\ 0 & 0 \end{array} \right] H_1[(\Sigma_2 R_1 \Sigma_1 H_1)^{\odot}]^2 \left( \begin{array}{c} 0 \\ 0 \end{array} \right) S^*, \quad (2.3)$$

where $T^*U = \begin{bmatrix} R_1 & R_2 \\ R_3 & R_4 \end{bmatrix}$, $R_1 \in \mathbb{C}^{s \times r}$, $V^*S = \begin{bmatrix} H_1 & H_2 \\ H_3 & H_4 \end{bmatrix}$, $H_1 \in \mathbb{C}^{r \times s}$.

Proof. Observe that $WA = S \left[ \begin{array}{cc} \Sigma_2 & 0 \\ 0 & 0 \end{array} \right] T^*U \left[ \begin{array}{cc} \Sigma_1 & 0 \\ 0 & 0 \end{array} \right] V^* = S \left[ \begin{array}{cc} \Sigma_2 R_1 \Sigma_1 & 0 \\ 0 & 0 \end{array} \right] V^*$. Now suppose that $\text{ind}(WA) = k$ and $X = S \begin{bmatrix} X_1 & X_2 \\ X_3 & X_4 \end{bmatrix} S^*$ ($X_1 \in \mathbb{C}^{s \times s}$) is the core-EP inverse of $WA$, then $X$ would satisfy condition (1.1). Thus, by computation,

$$(\Sigma_2 R_1 \Sigma_1 H_1 X_1)^* = \Sigma_2 R_1 \Sigma_1 H_1 X_1, \quad \Sigma_2 R_1 \Sigma_1 H_1 X_2 = 0,$$

$$\Sigma_2 R_1 \Sigma_1 H_1 X_3^2 = X_1, \quad X_3 = X_4 = 0,$$

$$X_1 \Sigma_2 R_1 \Sigma_1 (H_1 \Sigma_2 R_1 \Sigma_1)^k = \Sigma_2 R_1 \Sigma_1 (H_1 \Sigma_2 R_1 \Sigma_1)^{k-1},$$

which implies that

$$X_1 (\Sigma_2 R_1 \Sigma_1 H_1)^{k+1} = (\Sigma_2 R_1 \Sigma_1 H_1)^k.$$
These equalities above show that $X_1 = (\Sigma_2 R_1 \Sigma_1 H_1)^\diamond$. As the core-EP inverse is an outer inverse, i.e., $X W A X = X$, then $X_2 = X_1 \Sigma_2 R_1 \Sigma_1 H_1 X_2 = 0$. Hence,
\[
(WA)^\diamond = S \begin{bmatrix} (\Sigma_2 R_1 \Sigma_1 H_1)^\diamond & 0 \\ 0 & 0 \end{bmatrix} S^*.
\]

In light of Theorems 2.2 and 2.3, $A^{\diamond \cdot W} = A[(WA)^\diamond]^2 = U \begin{bmatrix} \Sigma_1 H_1 [(\Sigma_2 R_1 \Sigma_1 H_1)^\diamond]^2 & 0 \\ 0 & 0 \end{bmatrix} S^*$. This completes the proof.

Additional representations of the weighted core-EP inverse can be obtained through the full-rank decomposition. First, let us recall a concerned notion. In 1974, Ben-Israel and Greville \[15\] introduced the notion of generalized inverse with prescribed range and null space. Let $A \in \mathbb{C}^{m \times n}$, $T$ be a subspace of $\mathbb{C}^n$ of dimension $s \leq r$ and let $S$ be a subspace of $\mathbb{C}^m$ of dimension $m - s$. If $A$ has a $\{2\}$-inverse $X$ such that $\mathcal{R}(X) = T$ and $\mathcal{N}(X) = S$, then $X$ is unique and denoted by $A^{(2)}_{T,S}$. Further, Sheng and Chen \[16\] gave a full-rank representation of the generalized inverse $A^{(2)}_{T,S}$, which is based on an arbitrary full-rank decomposition of $G$, where $G$ is a matrix such that $\mathcal{R}(G) = T$ and $\mathcal{N}(G) = S$.

**Lemma 2.6.** \[16\], Theorem 3.1] Let $A \in \mathbb{C}^{m \times n}$, $T$ be a subspace of $\mathbb{C}^n$ of dimension $s \leq r$ and let $S$ be a subspace of $\mathbb{C}^m$ of dimension $m - s$. Suppose that $G \in \mathbb{C}^{n \times m}$ satisfies $\mathcal{R}(G) = T$, $\mathcal{N}(G) = S$. Let $G$ be an arbitrary full-rank decomposition, namely $G = UV$. If $A$ has a $\{2\}$-inverse $A^{(2)}_{T,S}$, then
1. $VAU$ is invertible;
2. $A^{(2)}_{T,S} = U(VAU)^{-1}V$.

The following result shows that the weighted core-EP inverse is a generalized inverse with prescribed range and null space.

**Theorem 2.7.** Let $A \in \mathbb{C}^{m \times n}$, $W \in \mathbb{C}^{n \times m}$ with $\text{ind}(WA) = k$. The $W$-weighted core-EP inverse of $A$ is a $\{2\}$-inverse of $WAW$ with the range $\mathcal{R}(A(WA)^k[(WA)^k]^\dagger)$ and the null space $\mathcal{N}(A(WA)^k[(WA)^k]^\dagger)$ i.e.,
\[
A^{\diamond \cdot W} = (WAW)^{(2)}_{\mathcal{R}(A(WA)^k[(WA)^k]^\dagger),\mathcal{N}(A(WA)^k[(WA)^k]^\dagger)}.
\]

**Proof.** First, we check that $A[(WA)^\diamond]^2$ is a $\{2\}$-inverse of $WAW$. Indeed,
\[
A[(WA)^\diamond]^2 WAW A[(WA)^\diamond]^2 = A[(WA)^\diamond]^2 WAW A[(WA)^\diamond] = A[(WA)^\diamond]^2.
\]
Then, we show that $\mathcal{R}(A(WA)^k[(WA)^k]^\dagger) = \mathcal{R}(A[(WA)^\diamond]^2)$ and $\mathcal{N}(A(WA)^k[(WA)^k]^\dagger) = \mathcal{N}(A[(WA)^\diamond]^2)$. Indeed,
\[
A(WA)^k[(WA)^k]^\dagger = A[(WA)^\diamond]^2 (WA)^{k+2} [(WA)^k]^\dagger,
\]
i.e., $\mathcal{R}(A(WA)^k[(WA)^k]^\dagger) \subseteq \mathcal{R}(A[(WA)^\diamond]^2)$;
\[
A[(WA)^\diamond]^2 = A(WA)^k[(WA)^\diamond]^{k+2} = A(WA)^k[(WA)^k]^\dagger (WA)^k[(WA)^\diamond]^{k+2},
\]
i.e., $\mathcal{R}(A[(WA)^\diamond]^2) \subseteq \mathcal{R}(A(WA)^k[(WA)^k]^\dagger)$.
If \( X \in \mathcal{N}(A(WA)^k[(WA)^k]^\dagger) \), i.e., \( A(WA)^k[(WA)^k]^\dagger X = 0 \), then
\[
A[(WA)^\odot]^2 X = A(WA)^\odot(WA)^D(WA)^k[(WA)^k]^\dagger X
= A(WA)^\odot[(WA)^D]^2 WA(WA)^k[(WA)^k]^\dagger X = 0,
\]
namely, \( \mathcal{N}(A(WA)^k[(WA)^k]^\dagger) \subseteq \mathcal{N}(A[(WA)^\odot]^2) \);
if \( X \in \mathcal{N}(A[(WA)^\odot]^2) \), i.e., \( A[(WA)^\odot]^2 X = 0 \), then
\[
A(WA)^k[(WA)^k]^\dagger X = AW(AWA)^\odot X = AWAW([WA)^\odot]^2 X = 0,
\]
namely, \( \mathcal{N}(A[(WA)^\odot]^2) \subseteq \mathcal{N}(A(WA)^k[(WA)^k]^\dagger) \).
This completes the proof. 

From Theorem 2.7 it is known that the weighted core-EP inverse is a particular outer inverse, then by applying Lemma 2.6 we derive new representations of the weighted core-EP inverse involving the usual matrix inverse.

**Corollary 2.8.** Let \( A \in \mathbb{C}^{m \times n} \), \( W \in \mathbb{C}^{n \times m} \) with \( \text{ind}(WA) = k \). If \( A(WA)^k[(WA)^k]^\dagger = UV \) is a full-rank decomposition of \( A(WA)^k[(WA)^k]^\dagger \). Then the \( W \)-weighted core-EP inverse of \( A \) possesses the following representation:
\[
A^{\odot,W} = U(VWAWU)^{-1}V.
\] (2.5)

Recall that the general algebraic structures (GAS) of \( A \) and \( W \) are defined as follows (see 3):
\[
A = P \begin{bmatrix} A_{11} & 0 \\ 0 & A_{22} \end{bmatrix} Q^{-1}, \quad W = Q \begin{bmatrix} W_{11} & 0 \\ 0 & W_{22} \end{bmatrix} P^{-1},
\] (2.6)
where \( P, Q, A_{11}, W_{11} \) are non-singular matrices and \( A_{22}, W_{22}, A_{22}W_{22}, W_{22}A_{22} \) are nilpotent matrices.

**Corollary 2.9.** Let \( A \in \mathbb{C}^{m \times n} \), \( W \in \mathbb{C}^{n \times m} \) with \( \text{ind}(WA) = k \) and let \( P = [P_1 \quad P_2] \), \( Q = [L_1 \quad L_2] \), where \( P_1, P_2, L_1, L_2, Q_1, Q_2 \) are appropriate blocks arising from (2.6). Then the \( W \)-weighted core-EP inverse of \( A \) possesses the following representation:
\[
A^{\odot,W} = P_1(L_1^*WAWP_1)^{-1}L_1^*.
\] (2.7)

**Proof.** Suppose that \( Q^{-1} = \begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix} \). From the GAS representations (2.6), it follows that
\[
(WA)^k = Q \begin{bmatrix} (W_{11}A_{11})^k & 0 \\ 0 & 0 \end{bmatrix} Q^{-1} = L_1(W_{11}A_{11})^k Q_1,
\]
\[
[(WA)^k]^\dagger = Q_1^*(Q_1Q_2^{-1})^{-1}[L_1^*L_1]^\dagger L_1^* L_1^{-1} Q_2,
\]
\[
A(WA)^k = P_1A_{11}(W_{11}A_{11})^k Q_1 \quad \text{and} \quad A(WA)^k[(WA)^k]^\dagger = P_1A_{11}(L_1^*L_1)^{-1} L_2^*.
\]
Therefore, it is possible to use the full-rank decomposition \( A(WA)^k[(WA)^k]^\dagger = UV \), where
\[
U = P_1A_{11} \quad \text{and} \quad V = (L_1^*L_1)^{-1}L_1^*.
\]
Then by Corollary 2.8 we obtain
\[
A^\odot_{-2} = P_1A_{11}[(L_1^*L_1)^{-1}L_1^*WAWP_1A_{11}]^{-1}(L_1^*L_1)^{-1}L_1^* = P_1(L_1^*WAWP_1)^{-1}L_1^*. \]
This completes the proof. \( \square \)

The following representation of the weighted core-EP inverse is based on the QR decomposition defined as in [2, 17, 18].

**Corollary 2.10.** Let \( A \in \mathbb{C}^{m \times n} \), \( W \in \mathbb{C}^{n \times m} \) with \( \text{ind}(WA) = k \), \( \text{rank}(WA) = r \), \( \text{rank}[A(WA)^k][(WA)^k]^\dagger] = s \), \( s \leq r \). Suppose that the QR decomposition of \( A(WA)^k[(WA)^k]^\dagger \) is of the form
\[
A(WA)^k[(WA)^k]^\dagger P = QR,
\]
where \( P \) is an \( n \times n \) permutation matrix, \( Q \in \mathbb{C}^{m \times m} \), \( Q^*Q = I_m \) and \( R \in \mathbb{C}^{m \times n} \) is an upper trapezoidal matrix. Assume that \( P \) is chosen so that \( Q \) and \( R \) can be partitioned as
\[
Q = [Q_1 \quad Q_2], \quad R = \begin{bmatrix} R_{11} & R_{12} \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} R_1 \\ 0 \end{bmatrix},
\]
where \( Q_1 \) consists of the first \( s \) columns of \( Q \) and \( R_{11} \in \mathbb{C}^{s \times s} \) is non-singular. If \( WA \) has a \( \{2\} \)-inverse \( (WA)^{(2)}_{R(A(WA)^k[(WA)^k]^\dagger)}A(WA)^k[(WA)^k]^\dagger \) = \( A^\odot_{-2} \), then
1. \( R_{11}P^*WAWQ_1 \) is an invertible matrix;
2. \( A^\odot_{-2} = Q_1(R_{11}P^*WAWQ_1)^{-1}R_{11}P^* \);
3. \( A^\odot_{-2} = (WA)^{(2)}_{R(Q_1)}A(WA)^k[(WA)^k]^\dagger \);
4. \( A^\odot_{-2} = Q_1(Q_1^*A(WA)^k[(WA)^k]^\dagger WAWQ_1)^{-1}Q_1^*A(WA)^k[(WA)^k]^\dagger \).

Various generalized inverses of complex matrices can be finally expressed in terms of the matrix product as well as matrix powers involving only Moore-Penrose inverse, so can the weighted core-EP inverse. It is crucial since in that case the operation could be implemented easily by Matlab. The main disadvantage of the representation (1.3) arises from the necessity to calculate Moore-Penrose inverses of two different matrices. The following result derives a representation of \( A^\odot_{-2} \), which involves only one Moore-Penrose inverse.

**Theorem 2.11.** Let \( A \in \mathbb{C}^{m \times n} \), \( W \in \mathbb{C}^{n \times m} \) and let \( l \) be a non-negative integer such that \( l \geq k = \max\{\text{ind}(AW), \text{ind}(WA)\} \). Then \( A^\odot_{-2} \) can be written as follows:
\[
A^\odot_{-2} = (AW)^l[(WA)^l+1]^\dagger; \quad (2.8)
\]
\[
A^\odot_{-2} = A(WA)^l[(WA)^l+2]^\dagger. \quad (2.9)
\]
Proof. From Theorems 2.2 and 2.3 it follows that $A^\oplus W = A[(WA)^\oplus]^2$, together with
\[(WA)^\oplus = (WA)^D(WA)^l[(WA)^l]^\dagger = (WA)^D(WA)^{l+2}[(WA)^{l+2}]^\dagger\]
by Lemma 2.1, then we derive that
\[A^\oplus W = A[(WA)^\oplus]^2 = A[(WA)^D(WA)^{l+2}[(WA)^{l+2}]^\dagger]^2 = A[(WA)^D]^2(WA)^{l+2}(WA)^{l+2} = A(WA)^l[(WA)^{l+2}]^\dagger.\]
One can verify (2.9) by checking three equations in Theorem 2.3. Here we omit the details. □

An expression of the core-EP inverse can be derived as a particular case $W = I$ of Theorem 2.11.

Corollary 2.12. Let $A \in \mathbb{C}^{n \times n}$ and let $l$ be a positive integer such that $l \geq k = \text{ind}(A)$. Then $A^\oplus = A^l(A^{l+1})^\dagger$.

3 Properties of the weighted core-EP inverse

In this section, we study the properties of the weighted core-EP inverse.

Proposition 3.1. Let $A \in \mathbb{C}^{m \times n}$, $W \in \mathbb{C}^{n \times m}$ with $\text{ind}(WA) = k$. Then we have the following facts:
(1) $\mathcal{R}(A^\oplus W) = \mathcal{R}((AW)^k)$;
(2) $\mathcal{N}(A^\oplus W) = \mathcal{N}((WA)^k)^\ast$.

Proof. (1) In view of Theorems 2.2 and 2.3 $A^\oplus W = A[(WA)^\oplus]^2 = A(WA)^k[(WA)^\oplus]^k+2 = (AW)^kA[(WA)^\oplus]^k+2$, i.e., $\mathcal{R}(A^\oplus W) \subseteq \mathcal{R}((AW)^k)$, together with
\[(AW)^k = A[(WA)^\oplus]^2(WA)^{k+2}W(AW)^D = A^\oplus(WA)^k+1W,
\]
i.e., $\mathcal{R}((AW)^k) \subseteq \mathcal{R}(A^\oplus W)$. Thus, $\mathcal{R}(A^\oplus W) = \mathcal{R}((AW)^k)$.

(2) Suppose $Y \in \mathcal{N}(A^\oplus W)$, i.e., $A[(WA)^\oplus]^2Y = 0$, then $[(WA)^k]^\ast(WA)^2[(WA)^\oplus]^2Y = 0$. Thus, $[(WA)^k]^\ast Y = 0$, i.e., $\mathcal{N}(A^\oplus W) \subseteq \mathcal{N}((WA)^k)^\ast$. Conversely, suppose $Z \in \mathcal{N}((WA)^k)^\ast$, i.e., $[(WA)^k]^\ast Z = 0$, then $A[(WA)^\oplus]^2[(WA)^\oplus]^k[(WA)^k]^\ast Z = 0$. Therefore, $A^\oplus WZ = A[(WA)^\oplus]^2Z = 0$, i.e., $\mathcal{N}((WA)^k)^\ast = \mathcal{N}(A^\oplus W)$. Hence $\mathcal{N}((WA)^k)^\ast = \mathcal{N}(A^\oplus W)$. □

Proposition 3.2. Let $A \in \mathbb{C}^{m \times n}$, $W \in \mathbb{C}^{n \times m}$ with $\text{ind}(WA) = k$. Then we have the following facts:
(1) $\mathcal{R}(A^\oplus W) \oplus \mathcal{N}(A^\oplus W)W = \mathbb{C}^m$;
(2) $\mathcal{R}(WA^\oplus W) \oplus \mathcal{N}(WA^\oplus W) = \mathbb{C}^n$. 
\textit{Proof.} (1) Observe that \( A^{\odot W}W = A[(WA)^{\odot}]^2W \). For any \( X \in \mathbb{C}^m \), \( X = A(WA)^{\odot W}X + [I - A(WA)^{\odot W}]X \), where

\[
A(WA)^{\odot W}X = A(WA)^{\odot W}(WA)^{\odot W}X = A(WA)^{\odot}(WA)^{\odot}[(WA)^{\odot}]^kWX \\
= A[(WA)^{\odot}]^2(WA)^{\odot+}[((WA)^{\odot}]^kWX \\
\in \mathcal{R}(A^{\odot W}W),
\]

implies that \([I - A(WA)^{\odot W}]X \in \mathcal{N}(A^{\odot W}W)\).

Therefore, \( \mathcal{R}(A^{\odot W}W) + \mathcal{N}(A^{\odot W}W) = \mathbb{C}^m \). Further, suppose

\[
Y \in \mathcal{R}(A[(WA)^{\odot}]^2W) \cap \mathcal{N}(A[(WA)^{\odot}]^2W),
\]

that is to say, \( Y = A[(WA)^{\odot}]^2WZ \) for some \( Z \in \mathbb{C}^m \) and \( A[(WA)^{\odot}]^2WY = 0 \). Thus, \( A[(WA)^{\odot}]^2WA[(WA)^{\odot}]^2WZ = 0 \), i.e., \( A[(WA)^{\odot}]^2WZ = 0 \). Pre-multiply this equality by \( WAW \), then \( (WA)^{\odot W}WZ = 0 \), which deduces that \( Y = 0 \). Hence \( \mathcal{R}(A^{\odot W}W) \oplus \mathcal{N}(A^{\odot W}W) = \mathbb{C}^m \).

(2) Note that \( WA^{\odot W} = (WA)^{\odot} \). From \( \mathcal{R}((WA)^{\odot}) = \mathcal{R}(WA(WA)^{\odot}) \) and \( \mathcal{N}((WA)^{\odot}) = \mathcal{N}(WA(WA)^{\odot}) \) as well as \( [WA(WA)^{\odot}]^2 = WA(WA)^{\odot} = [WA(WA)^{\odot}]^* \), it follows clearly that \( \mathcal{R}(WA^{\odot W}) \oplus \mathcal{N}(WA^{\odot W}) = \mathbb{C}^n \). \( \square \)

\textbf{Proposition 3.3.} \( \text{Let } A \in \mathbb{C}^{m \times n}, \text{ } W \in \mathbb{C}^{n \times m}\text{ with } \text{ind}(WA) = k. \text{ Then we have the following facts:} \)

\begin{enumerate}
\item \( WA^{\odot W} \text{ is an orthogonal projector onto } \mathcal{R}((WA)^{k}) \);
\item \( WA^{\odot W} \text{ is an oblique projector onto } \mathcal{R}((WA)^{k}[(WA)^{k}]^{\dagger}WA) \) along \( \mathcal{N}((WA)^{k}[(WA)^{k}]^{\dagger}WA) \).
\end{enumerate}

\textit{Proof.} (1) Since \( A^{\odot W}W = A[(WA)^{\odot}]^2 \) by applying Theorems 2.2 and 2.3, then

\[
WA^{\odot W} = WA(WA)^{\odot} = (WA)^{k}(WA)^{k}^{\dagger}.
\]

Therefore, \( WA^{\odot W} \) is a orthogonal projector onto \( \mathcal{R}((WA)^{k}) \).

(2) Observe that \( WA^{\odot W}WA = (WA)^{\odot W}A \). Since \( (WA)^{\odot} \) is an outer inverse of \( (WA) \), then

\[
[(WA)^{\odot W}W]^2 = (WA)^{\odot W}WA,
\]

together with \( \mathcal{R}((WA)^{\odot W}A) = \mathcal{R}((WA)^{k}[(WA)^{k}]^{\dagger}WA) \) and \( \mathcal{N}((WA)^{\odot W}A) = \mathcal{N}((WA)^{k}[(WA)^{k}]^{\dagger}WA) \), which implies that \( WA^{\odot W}WA \) is a projector onto \( \mathcal{R}((WA)^{k}[(WA)^{k}]^{\dagger}WA) \) along \( \mathcal{N}((WA)^{k}[(WA)^{k}]^{\dagger}WA) \). \( \square \)

\section{4 Relations among the weighted core-EP inverse and other generalized inverses}

In this section, we wish to reveal the relations among the weighted core-EP inverse, weighted Drazin inverse, the inverse along an element, and \((B,C)\)-inverse.
Theorem 4.1. Let $A \in \mathbb{C}^{m \times n}$, $W \in \mathbb{C}^{n \times n}$ with ind$(WA) = k$. Then

1. $A^{\odot,W} = A^{D,W}P_{(WA)^k}$;
2. $A^{D,W} = A^{\odot,W}P_{\mathcal{R}((WA)^k),\mathcal{N}((WA)^k)}$.

Proof. (1) It is known that $A^{\odot,W} = A[(WA)^\odot]^2$, $(WA)^\odot = (WA)^D(WA)^k[(WA)^k]^\dagger$ and $A^{D,W} = A[(WA)^D]^2$. Thus, $A^{\odot,W} = A[(WA)^D]^2(WA)^k[(WA)^k]^\dagger = A^{D,W} = A^{D,W}P_{(WA)^k}$.

(2) Observe that $A^{D,W} = A[(WA)^D]^2 = A[(WA)^D(WA)^k[(WA)^k]^\dagger]^2(WA)^k[(WA)^D]^k = A[(WA)^\odot]^2WA(WA)^D = A^{\odot,W}WA(WA)^D = A^{\odot,W}P_{\mathcal{R}((WA)^k),\mathcal{N}((WA)^k)}$. □

In what follows, we investigate the relations between the weighted core-EP inverse and the inverse along an element, $(B,C)$-inverse respectively. Let us recall two known notions.

Definition 4.2. Let $A \in \mathbb{C}^{n \times m}$ and $D,X \in \mathbb{C}^{m \times n}$. Then $X$ is the inverse of $A$ along $D$ if

$$XAD = D = DAX \text{ and } \mathcal{R}_s(X) \subseteq \mathcal{R}_s(D), \mathcal{R}(X) \subseteq \mathcal{R}(D).$$

Definition 4.3. Let $A \in \mathbb{C}^{n \times m}, B \in \mathbb{C}^{m \times n}, C \in \mathbb{C}^{n \times n}, X \in \mathbb{C}^{m \times n}$. Then $X$ is the $(B,C)$-inverse of $A$ if

$$X \in BC^{m \times n}X \cap XC^{n \times n}C \text{ and } XAB = B, CAX = C.$$
\( \mathcal{R}(A^\circ W) \subseteq \mathcal{R}(A(WA)^k[(WA)^k]^*) \), as well as,

\[
A^\circ W = A[(WA)^\circ]^2 = A[(WA)^D]^2(WA)^k[(WA)^k]^+ \\
= A[(WA)^D]^2((WA)^k)^*[([WA]^k]^*) \\
= A[(WA)^D]^2((WA)^k)[(WA)^k]^*[([WA]^k]^*) \\
= A[(WA)^D]^2((WA)^k)^*[([WA]^k]^*) \
\]

Since \([(WA)^k]^1(WA)^k = [(WA)^k+1]^1(WA)^k\) (see the dual form of Lemma 2.1), then

\[
A^\circ W = A[(WA)^D]^2[(WA)^k]^*[([WA]^k+1]^1(WA)^k]^* \\
= A[(WA)^D]^2((WA)^k)^*[([WA]^k+1]^1WA(WA)^k[(WA)^k]^*) \\
i.e., \mathcal{R}_s(A^\circ W) \subseteq \mathcal{R}_s(A(WA)^k[(WA)^k]^*).
\]

Hence \(A^\circ W\) is the inverse of \(WA\) along \((WA)^k[(WA)^k]^*\), in view of Definition 4.2. \(\square\)

**Theorem 4.5.** Let \(A \in \mathbb{C}^{m \times n}, \; W \in \mathbb{C}^{n \times m}\) with \(k = \max\{\text{ind}(AW), \; \text{ind}(WA)\}\). Then the \(W\)-weighted core-EP inverse of \(A\) (i.e., \(A^\circ W\)) is the \((AW)^k, [(WA)^k]^*\)-inverse of \(WA\).

**Proof.** Clearly, we can verify that

\[
A^\circ W = A[(WA)^\circ]^2 = A(WA)^k[(WA)^\circ]^k+2 = (AW)^kA[(WA)^\circ]^k+2 \\
= (AW)^kA[(WA)^\circ]^{k+1}WA^\circ W \in (AW)^kC^{m \times n}A^\circ W, \\
\]

\[
A^\circ W = A[(WA)^\circ]^2 = A[(WA)^\circ]^{k+1}WA(WA)^\circ \\
= A[(WA)^\circ]^{k+2}WA(WA)^\circ \\
= A[(WA)^\circ]^{k+2}WA(WA)^\circ \\
= A[(WA)^\circ]^{k+2}WA(WA)^\circ \subseteq (\mathcal{C}_{AW}) \subseteq \mathcal{C}_{AW}(WAW)^\circ, \text{ as well as,} \\
A^\circ W(WAW)^k = A[(WA)^\circ]^{k+2}(WA)^{k+1}W = A(WA)^\circ(WA)^kW \\
= A(WA)^\circ(WA)^kW = (AW)^D(AW)^{k+1} = (AW)^k, \\
[(WA)^k]^*WAWA^\circ W = [(WA)^k]^*WA(WA)^\circ = [(WA)^k]^*[WAWA(WA)^\circ] \\
= [WAWA(WA)^\circ]^*[WAWA(WA)^\circ] = [(WA)^k]^*.
\]

The above equalities show that \(A^\circ W\) is the \((AW)^k, [(WA)^k]^*\)-inverse of \(WAW\), in light of Definition 4.3. \(\square\)

5 Computational complexities of representations

Following from [2, 17], the complexity of computation of the pseudoinverse of a singular \(m \times n\) (resp. \(n \times n\)) matrix is denoted by \(\text{pinv}(m,n)\) (resp. \(\text{pinv}(n)\)); the complexity of
multiplying an $m \times n$ matrix by an $n \times k$ matrix is denoted by $M(m,n,k)$, abbreviated to $m \cdot n \cdot k$; the notation $M(n)$ is used instead of $M(n,n,n)$ and is abbreviated to $n^3$. Let $A \in \mathbb{C}^{m \times n}$, $W \in \mathbb{C}^{n \times m}$ with $k = \max\{\text{ind}(WA), \text{ind}(AW)\}$ and let $l$ be a non-negative integer such that $l \geq k$. In general, an $o(\log l)$ algorithm for matrix exponentiation $A^l$ (see [19]) would give an algorithm for computing $(AW)^l$ in $O(m^3 \log l)$ time, so that $O((AW)^l) = O(m^3 \log l)$ (see [2]). Similarly, $O((WA)^l) = O(n^3 \log l)$.

**Table 1: Computational complexity of (2.8)**

| Expression | Additional complexity |
|------------|-----------------------|
| $AW$       | $m \cdot n \cdot m$  |
| $\Lambda_1 = (AW)^l$ | $m^3 \log l$ |
| $\Lambda_2 = (AW)^{l+1} = \Lambda_1(AW)$ | $m^3$ |
| $\Lambda_3 = W(AW)^{l+1} = W\Lambda_2$ | $n \cdot m \cdot m$ |
| $\Lambda_4 = \Lambda^\dagger_3$ | $\text{pinv}(n,m)$ |
| $X = (AW)^l[W(AW)^{l+1}]^\dagger = \Lambda_1\Lambda_4$ | $m \cdot m \cdot n$ |

**Table 2: Computational complexity of (2.9)**

| Expression | Additional complexity |
|------------|-----------------------|
| $WA$       | $n \cdot m \cdot n$  |
| $(WA)^2$   | $n^3$ |
| $\Lambda_1 = (WA)^l$ | $n^3 \log (l-1)$ |
| $\Lambda_2 = (WA)^{l+2} = \Lambda_1(WA)^2$ | $n^3$ |
| $\Lambda_3 = \Lambda^\dagger_2$ | $\text{pinv}(n)$ |
| $X = A(WA)^l[(WA)^{l+2}]^\dagger = AA_1\Lambda_3$ | $2m \cdot n \cdot n$ |

The computational complexity of (2.8) can be estimated from the analysis of Table 1:

$$O(2.8) = 3m^2n + m^3 + m^3 \log l + \text{pinv}(n,m).$$

Likewise, the estimation for the computational complexity of (2.9) comes from Table 2:

$$O(2.9) = 3mn^2 + 2n^3 + n^3 \log (l-1) + \text{pinv}(n).$$

Obviously from $O(2.8)$ and $O(2.9)$, it is more appropriate to use representations involving $AW$ while $m < n$, and use representations involving $WA$ while $m \geq n$. In the following, we consider the case: $(0 < m < n)$.

The computational complexity of (1.3) is estimated from the analysis of Table 3:

$$O(1.3) = 3m^2n + m^3 + m^3 \log l + \text{pinv}(m) + \text{pinv}(n,m).$$

In view of [2] and [20], the complexity $\text{pinv}(m) \geq M(m) = m^3 > 0$. From $\text{pinv}(m) > 0$, it follows that $O(1.3) > O(2.8)$. Hence from this perspective, representation (2.8) is better than representation (1.3).
Table 3: Computational complexity of (1.3)

| Expression | Additional complexity |
|------------|------------------------|
| $AW$       | $m \cdot n \cdot m$    |
| $\Lambda_1 = (AW)^l$ | $m^3 \log l$          |
| $\Lambda_2 = (AW)^{l+1} = \Lambda_1(AW)$ | $m^3$ |
| $\Lambda_3 = \Lambda_1^{\dagger}$ | $\text{pinv}(m)$ |
| $\Lambda_4 = W\Lambda_2\Lambda_3$ | $2 \cdot n \cdot m \cdot m$ |
| $X = [W(AW)^{l+1}[(AW)^l]^\dagger = \Lambda_4^{\dagger}$ | $\text{pinv}(n,m)$ |

Table 4: Comparison of representations (1.3) and (2.8). Entries of $A$, $W$ are uniformly distributed random numbers from 0 to 1

| Equation | Size $m,n$ | $l \geq k$   | CPU Time | $r_1$  | $r_2$  | $r_3$  |
|----------|------------|--------------|----------|--------|--------|--------|
| (1.3)    | 100, 200   | $l = k = 4$  | 0.0300   | 8.7292e+10 | 1.6417e-25 | 3.4789e-16 |
| (2.8)    | 0.0200     | 4.7142e+10   | 8.9982e-26 | 3.7588e-16 |
| (1.3)    | 100, 200   | $l = k + 5$  | 0.0300   | 5.7009e+10 | 1.0516e-25 | 2.1932e-16 |
| (2.8)    | 0.0200     | 1.7365e+10   | 3.2428e-26 | 6.6545e-16 |
| (1.3)    | 100, 200   | $l = k + 15$ | 0.0400   | 4.4537e+10 | 8.1622e-26 | 2.1898e-16 |
| (2.8)    | 0.0200     | 2.5859e+10   | 4.6722e-26 | 2.3790e-16 |
| (1.3)    | 100, 200   | $l = k + 15$ | 0.0300   | 6.1824e+10 | 1.1411e-25 | 2.1261e-16 |
| (2.8)    | 0.0300     | 1.8199e+10   | 3.5081e-26 | 2.6287e-16 |
| (1.3)    | 100, 200   | $l = k + 15$ | 0.0300   | 1.2955e+12 | 1.4910e-29 | 4.6126e-16 |
| (2.8)    | 0.2600     | 5.5178e+11   | 1.9805e-30 | 5.0956e-16 |
| (1.3)    | 0.2600     | 1.2955e+12   | 1.4910e-29 | 4.6126e-16 |
| (2.8)    | 0.2400     | 5.5178e+11   | 1.9805e-30 | 5.0956e-16 |
| (1.3)    | 0.3200     | 1.2142e+12   | 1.3975e-29 | 4.8350e-16 |
| (2.8)    | 0.2400     | 5.5196e+11   | 3.4573e-30 | 5.7581e-16 |
| (1.3)    | 0.4700     | 7.9785e+11   | 9.1222e-30 | 8.3847e-16 |
| (2.8)    | 0.2700     | 5.5190e+11   | 1.1077e-30 | 5.8500e-16 |
6 Numerical examples

Our aim in this section is to test the time efficiency as well as the accuracy of given representations involving only pseudoinverse, namely, Equalities (1.3) and (2.8). For which, randomly generated singular matrices of different sizes are employed. Time efficiency is evaluated by the CPU time and the accuracy is measured by the residual norms. All the numerical tasks have been performed by using Matlab R2017b.

Let \( A \in \mathbb{C}^{m \times n} \) and \( W \in \mathbb{C}^{n \times m} \) with \( \text{ind}(AW) = k \). We assume that \( m < n \). Approximation derived from a numerical method for computing \( A^{\circ, W} \) will be denoted by \( X \), and the residual norms in all numerical experiments are denoted by

\[
    r_1 = ||XW(AW)^{k+1} - (AW)^k||_2, \quad r_2 = ||AWXWX - X||_2 \quad \text{and} \quad r_3 = ||(WAWX)^* - WAWX||_2.
\]

From Table 4, the following overall conclusions can be emphasized:

1. The representation (2.8) gives a better result in the aspect of the computational speed.
2. Representation (2.8) is better in accuracy with respect to the residual norms \( r_1 \) and \( r_2 \).
3. Contrary to the previous conclusion, the representation (1.3) is a better expression in accuracy with respect to norm \( r_3 \).
4. Both (1.3) and (2.8) produce bad results with respect to the norm \( r_1 \). This reason is the numerical instability caused by various matrix powers.

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