Edge-Cuts and Rooted Spanning Trees

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Abstract

We give a closed form formula to determine the size of a $k$-respecting cut. Further, we show that for any $k$, the size of the $k$-respecting cut can be found only using the size of 2-respecting cuts.

Keywords: 

1. Introduction

An edge-cut of a graph is said to $k$ respect a given spanning tree, if the cut shares $k$ edges with the tree. The technique of finding cuts that $k$-respect a given set of spanning trees is used in designing algorithms to find the size of cuts. The pioneering use appears in two breakthrough results by Karger (ACM STOC 1996 and JACM 2000) and Thorup (ACM STOC 2001 and Combinatorica 2007). The former \cite{Kar00} gives the first linear time algorithm to find the size of a min-cut, whereas the later \cite{Tho07} gives the first fully dynamic algorithm for min-cuts.

A common technique among these is to find the size of a minimum 2-respecting cut in a given set of spanning trees. Over the years this technique of finding a 2-respecting cut have found applications in designing algorithms to find min-cuts in several different settings and computational models: centralized, parallel, distributed, streaming, and dynamic.

In the centralized setting post the breakthrough result by Karger, recent results by Kwarabayashi and Thorup \cite{KT15}, improved further by Henzinger et al. \cite{HRW20} give a deterministic linear time. Several simpler algorithms

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have been designed that find the size of a min cut using new algorithms to find the size of a 2-respecting cut given a set of trees $[\text{BLS20}, \text{GMW20}, \text{Sar21}]$. Further, a recent breakthrough result that finds all pairs max flow in $O(n^2)$ uses 4-respecting cuts $[\text{AKL+21}]$.

In the distributed setting the first sub linear algorithm $[\text{DHNS19}]$ uses the concept of 2-respecting cut. Here the algorithms to find the min-cut has two parts: algorithm to reduce the size of the graph through a contraction mechanism and given a set of trees provide efficient algorithm to find the size of a 2-respecting cut. The result in $[\text{DHNS19}]$ is further improved by providing better algorithm for one of the two parts, even leading to an optimal algorithm to find the size of min-cut $[\text{DEMN21}, \text{GZ22}, \text{GNT20}, \text{MN20}]$. Though, the underlying similarity of finding the size of a minimum 2-respecting cut is common among all. There are several open problems that still remain here, for example the size of a small cut $[\text{PT11}, \text{Par19}]$, for a constant $k$.

Most of the aforementioned results rely on a closed form expression to find size of 2-respecting cuts, and the fact that only a small number of trees are required to be constructed in order to find a tree that 2-respects the minimum cut. The guarantees regarding the small number of trees come from Nash Williams, which states that the number of disjoint trees in a $k$-connected graph is at most $k/2$ $[\text{CMW+94}, \text{Die12}]$. In this paper, we extend the closed form expression to find size of any $k$-respecting cut. Furthermore, we show that the size of any $k$-respecting cut can just be found using the size of 2-respecting cuts. Our results rely on the cut-space concept from graph-theory. First, we give a closed form expression that finds the size of any $k$-respecting cut (Theorem 1.1). Secondly, we show that the size of any $k$-respecting cut can be find if we know the size of 1-respecting cuts, and 2-respecting cuts (Theorem 1.2).

Let $G = (V, E)$ be the given tree. Given a rooted spanning tree $T$, let $E(T)$ be the edges in the tree, and let $e_T(v)$, be the tree edge between $v$ and its parent for all $v \in V$, except the root. For any $A \subset V$, let $\delta(A)$ be the edges in the cut $(A, V \setminus A)$. For any rooted spanning tree $T$, let $v \downarrow T$ denote the set of vertices that are decedents of $v$ in $T$, including $v$ itself.
Theorem 1.1. Let $G = (V, E)$ be a given graph, let $T$ be a rooted spanning tree of $G$ and $A \subset V$. Suppose $E(T) \cap \delta(A) = \{ e_T(v_1), \ldots, e_T(v_k) \}$, for some vertices $S = \{ v_1, \ldots, v_k \}$. Then

$$|\delta(A)| = \sum_{l=1}^{k} (-1)^{l-1} 2^{l-1} \sum_{S' \subseteq [k]} \left| \bigcap_{i \in S'} \delta(v_i^{1T}) \right|$$

Further using some combinatorial arguments we show that the size of any $k$-respecting cut can just be found using pair-wise 2-respecting cuts.

Theorem 1.2. Let $G = (V, E)$ be a given graph, let $T$ be a rooted spanning tree of $G$ and $A \subset V$. Suppose $E(T) \cap \delta(A) = \{ e_T(v_1), \ldots, e_T(v_k) \}$, for some vertices $S = \{ v_1, \ldots, v_k \}$. Then $|\delta(A)|$ can be determined if the following is known

- $|\delta(x^{1T})|$ $\forall x \in S$,
- $|\delta(x^{1T}) \cap \delta(y^{1T})|$ $\forall x, y \in S$ and
- the path from root of $T$ to $x$ for all $x \in S$.

2. Preliminaries

For any rooted spanning tree $T$, we denote $r^T$ as its root. For any $v \in V$, let $v^{1T}$ be the vertex set that are decedents of $v$ in the tree $T$ including itself. Similarly, $v^{1T}$ is the set of vertices which are on the path $r^T$ to $v$. For all vertices $v \in V \setminus r^T$, we use $\pi_T(v)$ to denote the parent of $v$ in $T$ and $e_T(v)$ to denote the tree edge between $v$ and $\pi_T(v)$. Let $\ell_T(v)$ be the distance of vertex $v$ from the root $r^T$ in the tree $T$. Let $\text{children}_T(v)$ denote the children of the vertex $v$ in the tree $T$. If $v$ is a leaf node, then define $\text{children}_T(v)$ to be $\emptyset$. For any two vertices $v$ and $u$, we say that $v$ and $u$ are independent w.r.t the tree $T$, denoted using $v \perp_T u$ iff $v^{1T} \cap u^{1T} = \emptyset$. If they are not then we say that they are not independent, and denote it using $v \not\perp_T u$. 

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We will use $\oplus$ to denote the symmetric difference operator. More precisely, for any sets $A_1, A_2, \ldots, A_k$, we have $a \in \bigoplus_{i=1}^{k} A_{k'}$ iff $|\{k' \in [k] : a \in A_{k'}\}|$ is odd. Throughout this paper, when we use $\{\}$, we mean it to be set, and not multi-set. That is each entry in $\{\}$ occurs exactly once. Whenever we use an index $k$, it means a whole number less than the number of vertices in the graph.

We stress the readers to familiarize with $\oplus$ operator. We make the following simple preposition to do the same. These qualify how the symmetric difference operator appears when two sets are considered.

**Proposition 2.1.** Let $A_1$ and $A_2$ be any two sets. Suppose $A_1 \cap A_2 = \emptyset$, then $A_1 \oplus A_2 = A_1 \cup A_2$. Further, suppose, $A_2 \subseteq A_1$, then $A_1 \oplus A_2 = A_1 \setminus A_2$.

We use a well-known result that states that cut-spaces are a vector space with respect to the $\oplus$ operator. This can be found in standard graph theory books, for ex. Bondy and Murty \[BM76\].

**Lemma 2.2** (Also noted as Proposition 2.1 in \[PT11\]). Let $T$ be a given spanning tree and $v_1, v_2, \ldots, v_k \in V$. Then $\delta(v_1^{iT} \oplus v_2^{iT} \oplus \cdots \oplus v_k^{iT}) = \delta(v_1^{iT}) \oplus \delta(v_2^{iT}) \oplus \cdots \oplus \delta(v_k^{iT})$

We also mention a set-theoretic result for the cardinality of xor operation of $k$ sets.

**Proposition 2.3.** Suppose $A_1, \ldots, A_k$ are some sets. Then

$$\left| \bigoplus_{i=1}^{k} A_i \right| = \sum_{l=1}^{k} (-1)^{l-1}2^{l-1} \sum_{|S|=l} \left| \bigcap_{i \in S} A_i \right|.$$ 

### 3. Cut Characterization Lemma

In this section, we prove the characterization given in Theorem 1.1. We know that $\delta(A) = \delta(V \setminus A)$. We show that for any $A \subset V$, if $\delta(A) \cap E(T) = \{e_T(v_1), \ldots, e_T(v_k)\}$ for some vertex set $S = \{v_1, \ldots, v_k\}$, where $v_i \in V \setminus r_T$, then either $A = v_1^{iT} \oplus \cdots \oplus v_k^{iT}$ or $V \setminus A = v_1^{iT} \oplus \cdots \oplus v_k^{iT}$. This together with Lemma 2.2 and Proposition 2.3 leads to Theorem 1.1.
Proposition 3.1. Let $T$ be a rooted spanning tree. For any vertex set $A \subset V$, either $A$ or $V \setminus A$ is equal to $\bigoplus_{v \in S} v^iT$ for some $S \subseteq V \setminus r_T$.

Proof. Let the root $r_T \notin A$. For any $v \in A$, we know that for any $c \in \text{children}_T(v)$, $c^iT \subset v^iT$. Using Proposition 2.1, $\{v\} = v^iT \bigoplus_{c \in \text{children}_T(v)} c^iT$. Thus, $A = \bigcup_{v \in A} (v^iT \bigoplus_{c \in \text{children}_T(v)} c^iT) = \bigoplus_{v \in A} \left(v^iT \bigoplus_{c \in \text{children}_T(v)} c^iT\right)$.

Here the last equality is true because for any two sets $S_1, S_2$ if $S_1 \cap S_2 = \emptyset$, then $S_1 \cup S_2 = S_1 \oplus S_2$ (see Proposition 2.1). For avoiding multiplicity of occurrences and enforcing $S$ to be a set, we can remove any vertex $v$, if $v^iT$ occurs even number of times, and keep it only once when it occurs odd number of times. When $r_T \in A$, then $V \setminus A$ can be represented similarly.

Proposition 3.2. For any $u, v \in V \setminus r_T$, if $u \neq v$ and $u \in v^iT$, then $\pi_T(u) \in v^iT$.

Proposition 3.3. For any, rooted spanning tree $T$ and $v \in V \setminus r_T$, $\delta(v^iT) \cap E(T) = e_T(v)$

Proof. By definition, $v \in v^iT$ and $\pi_T(v) \notin v^iT$, hence $e_T(v) \in v^iT$. Now for this proof, we need to show that for $u \neq v$, $e_T(u) \notin v^iT$. Suppose not. Then both $u, \pi_T(u)$ are not in $v^iT$ simultaneously. But this cannot be true (see Proposition 3.2).

Now we give a proposition that finds the tree edges in $\delta(v_1^iT \oplus v_2^iT \oplus \ldots \oplus v_k^iT)$, for any $v_1, v_2, \ldots, v_k \in V$.

Proposition 3.4. Let $T$ be a rooted spanning tree. Let $S = \{v_1, \ldots, v_k\} \subset V \setminus r_T$. Then $E(T) \cap \delta(v_1^iT \oplus v_2^iT \oplus \ldots \oplus v_k^iT) = \{e_T(v_1), \ldots, e_T(v_k)\}$.

Proof. From Lemma 2.2, $\delta(v_1^iT \oplus v_2^iT \oplus \ldots \oplus v_k^iT) = \delta(v_1^iT) \oplus \delta(v_2^iT) \oplus \ldots \oplus \delta(v_k^iT)$. From Proposition 3.3, $\delta(v^iT) \cap E(T) = e_T(v)$. Hence,
\{e_T(v_1), \ldots, e_T(v_k)\} are the only tree edge that survives in $\delta(v_1^{i_T}) \oplus \delta(v_2^{i_T}) \oplus \ldots \oplus \delta(v_k^{i_T})$ because each one of these occurs only once, and no other edge in $E(T)$ is in the set $\delta(v_i^{i_T})$.

**Proposition 3.5.** For any $A \subset V$, if $\delta(A) \cap E(T) = \{e_T(v_1), e_T(v_2), \ldots, e_T(v_k)\}$, then either $A$, or $V \setminus A$ is equal to $v_1^{\downarrow} \oplus v_2^{\downarrow} \oplus \ldots \oplus v_k^{\downarrow}$.

**Proof.** We know that either $A$, or $V \setminus A$ can be written as $\bigoplus_{v \in S} v^{i_T}$ (see Proposition 3.1). Also, $\delta(A) = \delta(V \setminus A)$. We claim that $S = \{v_1, v_2, \ldots, v_k\}$, because if it is not, then $\delta(A) \cap E(T) \neq \{e_T(v_1), e_T(v_2), \ldots, e_T(v_k)\}$ (Proposition 3.4).

**Proof of Theorem 1.1.** According to the given condition, and Proposition 3.5 $\delta(A) = \delta(v_1^{\downarrow} \oplus v_2^{\downarrow} \oplus \ldots \oplus v_k^{\downarrow})$. Using Proposition 2.3 concludes the proof.

4. **Proof of Theorem 1.2**

We prove Theorem 1.2 in this section. For any vertex sets $A_1, A_2, A_3, \ldots A_i \subseteq V$ we define $\gamma(A_1, A_2, A_3, \ldots, A_i) \triangleq |\delta(A_1) \cap \delta(A_2) \cap \ldots \cap \delta(A_i)|$. For example $\gamma(A) = |\delta(A)|$, $\gamma(A_1, A_2) = |\delta(A_1) \cap \delta(A_2)|$, $\gamma(A_1, A_2, A_3) = |\delta(A_1) \cap \delta(A_2) \cap \delta(A_3)|$. Note that $\gamma(\cdot)$ is an overloaded function.

The issue with Theorem 1.1 is the involvement of $k$-wise gamma values, that is $\gamma(A_1, \ldots, A_k)$ for $k$ different vertex sets. We give the following lemma which states that pair-wise gamma values are enough to compute these $k$-wise gamma values, when our vertex sets are derived from a spanning tree $T$, and are of the form $x^{i_T}$ for any $x \in V$. This combined with Theorem 1.1 proves Theorem 1.2.

**Lemma 4.1.** Let $T$ be any spanning tree. Let $S = \{x_1, x_2, \ldots, x_k\}$ be a subset of $V \setminus \{r_T\}$, where $k \geq 2$. Then $\gamma(x_1^{i_T}, x_2^{i_T}, \ldots, x_k^{i_T})$ is either equal to 0 or $\gamma(x_i^{i_T}, x_j^{i_T})$, for some $x_i, x_j \in S$.

We describe the proof in this section. We give four exhaustive cases based on the ancestor-descendent relationships of the vertices in $S$ with respect to the
spanning tree $T$. They are shown in Figure 1. We show that the $k$ wise gamma value is zero in two of these cases: CASE-1 and CASE-3. Also, in the remaining two cases: CASE-2 and CASE-4, the $k$ wise gamma value can be written in terms of pair-wise gamma value and $k - 1$ wise gamma value respectively, for any $k \geq 3$. Using recursion, this implies that $k$ wise gamma value can be found from pair-wise gamma value if we know the ancestor-decedent relationship of the vertices.

The cases in Figure 1 are based on the ancestor-decedent relationship of
all $x, y \in S$. Recall that $x \perp_T y$, if $x^T \cap y^T = \emptyset$. We show the following simple observation. This shall enable the reader to get familiarized with the $\perp_T$ operator.

**Proposition 4.2.** Let $T$ be a rooted spanning tree. Let $x, y$ be any nodes. If $x \not\perp_T y$, then either $x^T \subset y^T$ or $y^T \subset x^T$. Further, if $x \not\perp_T y$, and $\ell_T(x) < \ell_T(y)$, then $y^T \subset x^T$.

**Proof.** If $x \not\perp_T y$, then $x^T \cap y^T \neq \emptyset$. The set $x^T$ contains $x$ and its decedents in the rooted spanning tree $T$. If the intersection of $x^T$ and $y^T$ is not empty, it means that $x$ and $y$ have an ancestor-decedent relationship in the tree $T$. Thus $x^T \subset y^T$ or $y^T \subset x^T$. When $\ell_T(x) < \ell_T(y)$, then $y$ is a descendant of $x$, hence $y^T \subset x^T$. \hfill \Box

We describe the four exhaustive cases from Figure 1. In CASE-1, $x \perp_T y \quad \forall x, y \in S$. In Proposition 4.4, we prove $\gamma(x_1^T, x_2^T, \ldots, x_k^T) = 0$ for this case. CASE-2,3,4 are negation of CASE-1. In these cases, $\exists x, y$ such that $x \not\perp_T y$. Further, these cases are divided into two groups based on the statement: $\exists x' \in S$ such that $\forall y' \in S$, $y' \in x'^T$ ($y'$ is a descendant of $x'$). If this statement is false then it is CASE-4, if it is true, then it is one of CASE-2 or CASE-3. For CASE-4, in Proposition 4.7 we show that k-wise gamma value can be found through $k-1$ wise gamma value. Lastly, CASE-2,3 distinguish between each other based on the statement: $\exists x'' \in S$ such that all $y'' \in S$ are on the tree path from $r_T$ to $x''$ ($y'' \in x''^T$). If this is true, then this is CASE-2, and when it is false it is CASE-3. In Proposition 4.5 we show that for CASE-2, k-wise gamma value can be found through pair-wise gamma value. For CASE-3, in Proposition 4.6 we show that the k-wise gamma value is 0. Based on the aforementioned discussion, we state the following lemma.

**Lemma 4.3.** Let $T$ be any spanning tree. Let $S = \{x_1, x_2, \ldots, x_k\}$ be a subset of $V \setminus \{r_T\}$, where $k \geq 2$. If we know $\gamma(x_i^T, x_j^T)$ for all $x_i, x_j \in S$, and the ancestor decedent relationship between $x_i$ and $x_j$ for all $x_i, x_j \in S$, then $\gamma(x_1^T, x_2^T, \ldots, x_k^T)$ can be found.
Proposition 4.4. Let $T$ be a rooted spanning tree and $S = \{x_1, x_2, \ldots, x_k\} \subset V \setminus \{r_T\}$ such that $|S| = k \geq 3$. If no child-ancestor pair exists in $S$ i.e. $\forall x_i, x_j \in S, x_i \perp_T x_j$, then $\gamma(x_1^T, x_2^T, \ldots, x_k^T) = 0$.

Proof. Here, $\forall x_i, x_j \in S$, we have $x_i \perp_T x_j$. This implies $x_i^T \cap x_j^T = \emptyset$. Since, $|S| \geq 3$, no edge can have endpoints in $x_i^T$ for all $i$ simultaneously (an edge has only two endpoints). Thus $\gamma(x_1^T, x_2^T, \ldots, x_k^T) = 0$.

Proposition 4.5. Let $T$ be a rooted spanning tree and $S = \{x_1, x_2, \ldots, x_k\} \subset V \setminus \{r_T\}$ such that $|S| = k \geq 3$. If

1. $\exists x, y \in S, x \not\perp_T y$ (exists two nodes that are not independent),
2. $\exists x' \in S, \forall y' \in S, y' \in x'^T$ (there exists an $x'$ in $S$ such that all $y'$ in $S$ are decedents of $x$),
3. $\exists x'' \in S, \forall y'' \in S, y'' \in x''^T$ (there exists an $x''$ in $S$ such that all $y''$ in $S$ are ancestors of $x$).

Then $\gamma(x_1^T, x_2^T, \ldots, x_k^T) = \gamma(p^T, q^T)$, where $p = \operatorname{argmax}_{x \in S} \ell_T(x)$ and $q = \operatorname{argmin}_{x \in S} \ell_T(x)$.

Proof. Here, the third condition, subsumes the other two conditions. This is because, if there exists a node $x''$ such that all other nodes are ancestors of it (they are on the path from $x''$ to root $r_T$). Then firstly, all of these are independent. Secondly, there also exists a node such that all the other nodes are decedents of such a node.
Figure 3: Figure shows the orientation of $x_1, x_2, \ldots, x_k$ in terms of their ancestor-decedent relationships for CASE-2.

WLOG, let $\ell_T(x_1) > \ell_T(x_2) > \ldots > \ell_T(x_k)$. Since all vertices in $S$ are on the tree path from $r_T$ to $x_1$, thus we have, $x_1^{i_T} \subset x_2^{i_T} \ldots \subset x_k^{i_T}$.

Similarly, $V \setminus x_1^{i_T} \supset V \setminus x_2^{i_T} \supset \ldots \supset V \setminus x_k^{i_T}$. By definition, $\delta(x^{i_T}) = \{(u, v) | u \in x^{i_T}, v \in V \setminus x^{i_T}\}$. Hence,

$$\delta(x_1^{i_T}) \cap \cdots \cap \delta(x_k^{i_T})$$

$$= \{(u, v) | u \in x_1^{i_T}, v \in \bigcap_i (V \setminus x_i^{i_T})\} \cap \cdots \cap \{(u, v) | u \in x_k^{i_T}, v \in \bigcap_i (V \setminus x_k^{i_T})\}$$

$$= \{(u', v') | u' \in x_1^{i_T}, v' \in V \setminus x_1^{i_T}\} \cap \cdots \cap \{(u'', v'') | u'' \in x_k^{i_T}, v'' \in V \setminus x_k^{i_T}\}$$

$$= \delta(x_1^{i_T}) \cap \delta(x_k^{i_T})$$

Hence, $\gamma(x_1^{i_T}, x_2^{i_T}, \ldots, x_k^{i_T}) = |\delta(x_1^{i_T}) \cap \cdots \cap \delta(x_k^{i_T})| = |\delta(x_1^{i_T}) \cap \delta(x_k^{i_T})| = \gamma(x_1^{i_T}, x_k^{i_T}) = \gamma(p^{i_T}, q^{i_T})$, where $p = \arg\max_{x \in S} \ell_T(x)$ and $q = \arg\min_{x \in S} \ell_T(x)$.

Proposition 4.6. Let $T$ be a rooted spanning tree and $S = \{x_1, x_2, \ldots, x_k\} \subset V \setminus \{r_T\}$ such that $|S| = k \geq 3$. If

1. $\exists x, y \in S, x \not\perp_T y$ (exists two vertices that are not independent),
2. \( \exists x' \in S, \forall y' \in S, y' \in x'^{\downarrow T} \) (there exists an \( x' \) in \( S \) such that all \( y' \) in \( S \) are decedents of \( x \)),

3. \( \forall x'' \in S, \exists y'' \in S, y'' \notin x''^{\uparrow T} \) (for all \( x'' \) in \( S \) there exists a \( y'' \) in \( S \) such that \( y'' \) is not ancestor of \( x'' \)).

Then \( \gamma(x_1^{\downarrow T}, x_2^{\downarrow T}, \ldots, x_k^{\downarrow T}) = 0 \).

![Figure 4](image-url)

Figure 4: Figure shows the orientation of \( x_1, x_2, \ldots, x_k \) in terms of their ancestor-decedent relationships for \textit{CASE-3}

**Proof.** Here, condition (2) subsumes condition (1). This is because there exists a node such that all other nodes are its decedents. Let \( x_k \) be such a node. Hence \( x_k \notin_T x_i \) for all \( x_i \in S \), and \( x_i \neq x_k \). Let \( x_1 = \text{argmax}_{x \in S} \) be the node that is furthest from the root \( r_T \) in the tree \( T \). If many such nodes exist, then choose an arbitrary node. Choosing a furthest node implies that there does not exist \( x \in S \), such that \( x \in x_1^{\downarrow T} \). By condition (3), we know that there is a node \( x_2 \notin x_1^{\downarrow T} \). By condition (2), we have \( x_2 \in x_k^{\downarrow T} \). Thus the set relationships of \( x_1^{\downarrow T}, x_2^{\downarrow T}, \) and \( x_k^{\downarrow T} \) is as follows: \( x_1^{\downarrow T} \subset x_k^{\downarrow T}, x_2^{\downarrow T} \subset x_k^{\downarrow T} \) and \( x_1^{\downarrow T} \cap x_2^{\downarrow T} = \emptyset \).

In the remaining part we show \( \delta(x_1^{\downarrow T}) \cap \delta(x_2^{\downarrow T}) \cap \delta(x_k^{\downarrow T}) = \emptyset \). Firstly, \( \delta(x_1^{\downarrow T}) \cap \delta(x_2^{\downarrow T}) \) contains those edges which have one end point in \( x_1^{\downarrow T} \) and other in \( x_2^{\downarrow T} \). Also, \( \delta(x_k^{\downarrow T}) \) contains those edges which have one end point in \( x_k^{\downarrow T} \) and the other point outside of \( x_k^{\downarrow T} \). Because of the aforementioned set relationship, no edge in \( \delta(x_k^{\downarrow T}) \) is contained in \( \delta(x_1^{\downarrow T}) \cap \delta(x_2^{\downarrow T}) \). Hence...
\(\delta(x_1^{\downarrow T}) \cap \delta(x_2^{\downarrow T}) \cap \delta(x_k^{\downarrow T}) = \emptyset.\) Thus \(\delta(x_1^{\downarrow T}) \cap \delta(x_2^{\downarrow T}) \cap \cdots \cap \delta(x_k^{\downarrow T}) = \emptyset\) which implies that \(\gamma(x_1^{\downarrow T}, x_2^{\downarrow T}, \ldots, x_k^{\downarrow T}) = 0.\)

**Proposition 4.7.** Let \(T\) be a rooted spanning tree and \(S = \{x_1, x_2, \ldots, x_k\} \subset V \setminus \{r_T\}\) such that \(|S| = k \geq 3.\) If

1. \(\exists x, y \in S, x \not\perp_T y\) (exists two vertices that are not independent),
2. \(\forall x' \in S, \exists y' \in S, y' \notin x'^{\downarrow T}\)

Then \(\gamma(x_1^{\downarrow T}, x_2^{\downarrow T}, \ldots, x_k^{\downarrow T}) = \gamma(y_1^{\downarrow T}, \ldots, y_{k-1}^{\downarrow T})\) where \(\{y_1, y_2, \ldots, y_{k-1}\} = S \setminus \{a\}\) for some \(a \in S.\)

![Figure 5: Figure shows the orientation of \(a, v, u\) in terms of their ancestor-decedent relationships for CASE-4](image)

**Proof.** Let \(a, v\) be two nodes such that \(a \not\perp_T v\) (condition (1)), where \(\ell_T(a) < \ell_T(v).\) Here we choose an \(a, v\) pair such that no other node \(a'\) exists in \(S\) such that \(\ell_T(a') < \ell_T(a)\) and \(a' \not\perp_T a.\) If it exists then our chosen pair is \(a', a.\)

Based on condition (3), we know that \(\exists u\) such that \(u \notin a^{\downarrow T}.\) According to the choice of \(u, v,\) and \(a,\) we show that \(v^{\downarrow T} \subset a^{\downarrow T}, a \perp_T u,\) and \(v \perp_T u.\) Firstly, \(v^{\downarrow T} \subset a^{\downarrow T}\) is true because \(a \not\perp_T v,\) and \(\ell_T(a) < \ell_T(v)\) (see Proposition 4.2). From the choice, \(u \notin a^{\downarrow T}.\) Then either \(a \perp_T u,\) or \(u\) is the ancestor of \(a.\) The latter cannot be true because we have chosen \(a\) such that no \(a' \in S\) exists such that \(\ell_T(a') < \ell_T(a)\) and \(a' \not\perp_T a.\) Thus \(u \perp_T a.\) This also implies that \(u \perp_T v,\) because \(v^{\downarrow T} \subset a^{\downarrow T}.\)

Lastly, in this proof we show that \(\delta(a^{\downarrow T}) \cap \delta(u^{\downarrow T}) \cap \delta(v^{\downarrow T}) = \delta(u^{\downarrow T}) \cap \delta(v^{\downarrow T}).\) This implies that \(\gamma(x_1^{\downarrow T}, x_2^{\downarrow T}, \ldots, x_k^{\downarrow T}) = \gamma(y_1^{\downarrow T}, y_2^{\downarrow T}, \ldots, y_{k-1}^{\downarrow T}).\)
where \( \{y_1, y_2, \ldots, y_{k-1}\} = Q \setminus \{a\} \). That is, we can eliminate \( a^{iT} \) to compute \( \gamma(x_1^{iT}, x_2^{iT}, \ldots, x_k^{iT}) \).

\[
\delta(a^{iT}) \cap \delta(v^{iT}) \cap \delta(u^{iT}) = \big\{(x, y) \mid x \in a^{iT}, y \in V \setminus a^{iT}\big\} \cap \big\{(x, y) \mid x \in v^{iT}, y \in V \setminus v^{iT}\big\} \cap \big\{(x, y) \mid x \in u^{iT}, y \in V \setminus u^{iT}\big\}
\]

The last inequality is true because \( u \perp_T v \) and \( \{(x, y) \mid x \in v^{iT}, y \in u^{iT}\} \) contain all the edges that have an endpoint in \( u^{iT} \) and \( v^{iT} \). \( \square \)

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