On the non-existence of extended perfect codes and some perfect colorings∗

Evgeny Bespalov †

Abstract

In this paper we obtain the necessary condition for the existence of perfect k-colorings (equitable k-partitions) in Hamming graphs $H(n, q)$, where $q = 2, 3, 4$ and Doob graphs $D(m, n)$. As an application, we prove the non-existence of extended perfect codes in $H(n, q)$, where $q = 3, 4$, $n > q + 2$, and in $D(m, n)$, where $2m + n > 6$.

1. Introduction

A $k$-coloring of a graph $G = (V, E)$ is a surjective function from the vertex set $V$ into a color set of cardinality $k$, usually denoted by $\{0, 1, \ldots, k - 1\}$. This coloring is called perfect if for any $i, j$ the number of vertices of color $j$ in the neighbourhood of vertex $x$ of color $i$ depends only on $i$ and $j$, but not on the choice of $x$. An equivalent concept is an equitable $k$-partition, which is a partition of the vertex set $V$ into cells $V_0, \ldots, V_{k-1}$, where these cells are the preimages of the colors of some perfect $k$-coloring. Also the perfect colorings are the particular cases of the perfect structures, see e.g. [1]. In this paper, we consider perfect colorings in Hamming graphs $H(n, q)$ (mainly focusing on the case $q = 2, 3, 4$) and Doob graphs $D(m, n)$. Remind that the Hamming graph $H(n, q)$ is the direct product of $n$ copies of the complete graph $K_q$ on $q$ vertices, and the Doob graph $D(m, n)$, where $m > 0$, is the direct product of $m$ copies of the Shrikhande graph and $n$ copies of $K_4$. These graphs are distance-regular; moreover, the Doob graph $D(m, n)$ has the same intersection array as $H(2m + n, 4)$. Many combinatorial objects can be defined as perfect colorings with corresponding parameters, for example, MDS codes with distance 2; latin squares and latin hypercubes; unbalanced boolean functions attending the correlation-immunity.

∗This work was funded by the Russian Science Foundation (Grant 18-11-00136).
†Sobolev Institute of Mathematics, Novosibirsk, Russia. E-mail: bespalovpes@mail.ru
bound \[^2\]; orthogonal arrays attaining the Bierbrauer–Friedman bound \[^3, 4\]; boolean-value functions on Hamming graphs and orthogonal arrays that attach some other bounds \[^5, 6, 7\]; some binary codes attending the linear-programming bound that are cells of equitable partitions into 4, 5, or 6 cells \[^8, 9\].

One important class of objects that corresponding to perfect colorings is the 1-perfect codes. It is generally known \[^10\] Ch. 6, Th. 37\] that if \( q = p^m \) is a prime power, then there is a 1-perfect code in \( H(n, q) \) if and only if \( n = (q^l - 1)/(q - 1) \) for some positive integer \( l \). In the case when \( q \) is not prime power, there is very little known about the existence of 1-perfect codes. It is known that there are no 1-perfect codes in \( H(7, 6) \) \[^11\], Theorem 6\] (since there are no pair of orthogonal latin squares of order 6). Heden and Roos obtained the necessary condition \[^12\] on the non-existence of some 1-perfect codes, which in particular implies the non-existence of 1-perfect codes in \( H(19, 6) \). Also we mention result of Lenstra \[^13\], which generalized Lloyd’s condition (see \[^14, 10\]) for a non-prime power \( q \). This result implies that if there is a 1-perfect code in \( H(n, q) \), then \( n = kq + 1 \). Krotov \[^15\] completely solved the problem of the existence of 1-perfect codes in Doob graphs. Namely, he proved that there is a 1-perfect code in \( D(m, n) \) if and only if \( 2m + n = (4^l - 1)/3 \) for some positive integer \( l \). Note that the existence of a 1-perfect code in \( D(m, n) \) not always implies the existence of linear or additive 1-perfect codes in this graph (the set of admissible parameters of unrestricted 1-perfect codes in Doob graphs is essentially wider than that of linear \[^16\] or additive \[^17\] 1-perfect codes).

Another important class of codes corresponding to perfect colorings is the completely regular codes. A code \( C \) is completely regular if the distance coloring with respect to \( C \) (a vertex \( v \) has the color that is equal to the distance from \( v \) to \( C \)) is perfect. These codes originally were defined by Delsarte \[^18\], but here we use the different equivalent definition from \[^19\]. For more information about completely regular codes and problem of its existence, we refer to the survey \[^20\], papers \[^21, 22\] (for codes with covering radius \( \rho = 1 \)), and the small-value tables of parameters \[^23\].

In the current paper, we stay on the class of completely regular codes that correspond to extended 1-perfect codes. An extended 1-perfect code is a code with code distance 4 obtained by appending an additional symbol (this operation we call an extension) to the codewords of some 1-perfect code (the rigorous definition will be given in the next section). It is known that there is an extended 1-perfect code in \( H(2^m, 2) \) and in \( H(2^m + 2, 2^m) \) for any positive integer \( m \) (see \[^10, 20\]). It was mentioned in \[^24\] Section 4\] that a result from \[^25\] implies the non-existence of extended 1-perfect codes in \( H(n, q) \) obtained from the Hamming codes, except the case when \( (n, q) = (2^m, 2) \) or \( (n, q) = (2^m + 2, 2^m) \). An extended 1-perfect code in \( H(q + 2, q) \) (or in \( D(m, n) \), where \( 2m + n = 6 \)) is also an MDS code with distance 4. There is
a characterization of all extended 1-perfect codes in $H(6, 4)$ [26] and in Doob graphs $D(m, n)$ [27], where $2m + n = 6$, $m > 0$. Ball showed [28] that if $q$ is odd prime, then there are no linear extended 1 prefect codes in $H(q + 2, q)$. The non-existence of extended 1-perfect codes in $H(7, 5)$ and $H(9, 7)$ was proved in [29]. In [30] it was shown that any extended 1-perfect code in $H(10, 8)$ is equivalent to a linear code. The non-existence of extended 1-perfect codes in $H(14, 3)$ follows from the bound established in [31]. For completeness, note that formally codes consisting from one vertex in $H(2, q)$ are also extended 1-perfect codes. Such codes are called trivial.

In this paper, we obtain a necessary condition for the existence of perfect colorings in Hamming graphs $H(n, q)$, where $q = 2, 3, 4$, and Doob graphs. We apply it to extended 1-perfect codes and prove that there are no such codes in $H(n, q)$, $q = 3, 4$, $n > q + 2$, and in $D(m, n)$, $2m + n > 6$. This completes the characterization of such codes in these graphs (see Theorem [3]). In addition, we prove that extended 1-perfect codes can exist in $H(n, q)$ only if $n$ is even, which particularly implies the non-existence of some MDS codes with distance 4. We hope that this method can be applied for a proof of the non-existence of some other perfect $k$-colorings (but for perfect 2-colorings it does not add something new to results from [22]).

The paper is organized as follows. In Section 2 we give main definitions and simple observations. In Section 3 we obtain a necessary condition (Theorem [1]) for the existence of perfect colorings in Doob graphs and Hamming graphs $H(n, q)$, where $q = 2, 3, 4$. In Section 4 we prove that any extended 1-perfect code in $H(n, q)$ is a completely regular code with intersection array $(n(q-1), (n-1)(q-1); 1, n)$, and vice versa; similar results are shown for Doob graphs. This allows us to apply Theorem [1] to prove the non-existence of some extended 1-perfect codes in Section 5. Finally, we describe all parameters for which there is an extended 1-perfect code in $D(m, n)$ and $H(n, 3)$ in Theorem [3].

2. Preliminaries

Given a graph $G$, we denote by $\nu G$ its vertex set. A surjective function $f : \nu G \to \{0, 1, \ldots, k-1\}$ on the vertex set of $G$ is called a $k$-coloring of a graph $G$ in the colors $0, 1, \ldots, k-1$. If for all $i, j$ every vertex $x$ of color $i$ has exactly $s_{i,j}$ neighbours of color $j$, where $s_{i,j}$ does not depend on the choice of $x$, then the coloring $f$ is called a perfect $k$-coloring with quotient matrix $S = (s_{i,j})$.

Let $G$ be a connected graph. A code $C$ in $G$ is an arbitrary nonempty subset of $\nu G$. The distance $d(x, y)$ between two vertices $x$ and $y$ is the length of the shortest path between $x$ and $y$. The code distance $d$ of a code $C$ is the minimum distance between two different vertices of $C$. The distance $d(A, B)$ between two sets of vertices
A and B equals \( \min\{d(x, y) : x \in A, y \in B\} \). The \textit{covering radius} of a code \( C \) is \( \rho = \max_{v \in V_G} d\{v\}, C\} \). Let \( C \) be a code in a graph \( G \). The \textit{distance coloring} with respect to \( C \) is the coloring \( f \) defined in the following way: \( f(x) \) is equal to the distance between \( \{x\} \) and \( C \). If \( f \) is a perfect coloring with quotient matrix \( S \), then \( C \) is called a \textit{completely regular code} with quotient matrix \( S \). In this case, the matrix \( S \) is tridiagonal. A connected regular graph \( G \) is called distance-regular if for any vertex \( x \) of \( G \) the set \( \{x\} \) is a completely regular code with quotient matrix \( S \) that does not depend on the choice of \( x \). The sequence \((b_0, \ldots, b_{\rho-1}; c_1, \ldots, c_\rho) = (s_{0,1}, \ldots, s_{\rho-1, \rho}, s_{1,0}, \ldots, s_{\rho, \rho-1}) \) is called the intersection array.

The \textit{Shrikhande graph} \( Sh \) is a Cayley graph with the vertex set \( \mathbb{Z}_4^2 \), where two vertices \( x \) and \( y \) are adjacent if and only if their difference \( (x - y) \) belongs to the connecting set \( \{01, 10, 03, 30, 11, 33\} \). The complete graph \( K_q \) on \( q \) vertices can be represented as a Cayley graph, where the vertex set is \( \mathbb{Z}_q \), and two vertices \( x \) and \( y \) are adjacent if and only if their difference \( (x - y) \) belongs to the connecting set \( \{1, 2, \ldots, q - 1\} \). The Hamming graph \( H(n, q) \) is the direct product \( K_q^n \) of \( n \) copies of \( K_q \). The vertex set of \( H(n, q) \) can be represented as \( \mathbb{Z}_q^n = \{(x_1, \ldots, x_n) : x_i \in \mathbb{Z}_q\} \). Denote by \( D(m, n) = Sh^m \times K_4^n \) the direct product of \( m \) copies of the Shrikhande graph \( Sh \) and \( n \) copies of the complete graph \( K_4 \). If \( m > 0 \), then this graph is called \textit{Doob graph}. The vertex set of \( D(m, n) \) can be represented as \((\mathbb{Z}_4^2)^m \times \mathbb{Z}_4^n = \{(x_1, \ldots, x_m; y_1, \ldots, y_n) : x_i \in \mathbb{Z}_4^2, y_j \in \mathbb{Z}_4\}\). The Hamming graph \( H(n, q) \) is distance-regular with intersection array \((n(q-1), (n-1)(q-1), \ldots, q-1; 1, 2, \ldots, n) \). The Doob graph \( D(m, n) \) is distance-regular with the same intersection array as \( H(2m + n, 4) \).

For a vertex \( v = (x_1, \ldots, x_{n-1}) \) of \( H(n-1, q) \) and \( a \in \mathbb{Z}_q \), denote by \( v_i^a = (x_1, \ldots, x_{i-1}, a, x_i, \ldots, x_{n-1}) \) the vertex of \( H(n, q) \). Analogously, for a vertex \( v = (x_1, \ldots, x_m; y_1, \ldots, y_{n-1}) \) of \( D(m, n - 1) \) and \( a \in \mathbb{Z}_4 \), denote by \( v_i^a = (x_1, \ldots, x_m; y_1, \ldots, y_{i-1}, a, y_i, \ldots, y_{n-1}) \) the vertex of \( D(m, n) \). The projection (also known as puncturing) \( C_i \) of a code \( C \) in \( H(n, q) \) is the code in \( H(n-1, q) \) defined as follows:

\[
C_i = \{v \in \mathcal{V} H(n-1, q) : v_i^a \in C \text{ for some } a \in \mathbb{Z}_q\}.
\]

Similarly, the projection \( C_{ij} \) of a code \( C \) in the Doob graph \( D(m, n) \), \( n > 0 \), is the code in \( D(m, n - 1) \) defined as follows:

\[
C_{ij} = \{v \in \mathcal{V} D(m, n - 1) : v_i^a \in C \text{ for some } a \in \mathbb{Z}_4\}.
\]

By \( B_e(x) = \{y : d(x, y) \leq e\} \) denote the radius \( e \)-ball with center \( x \). A code \( C \) in a graph \( G \) is called \textit{\( e \)-perfect} if \( |C \cap B_e(x)| = 1 \) for any \( x \in \mathcal{V} G \). In equivalent
definition, an e-perfect code is a code with code distance \( d = 2e + 1 \), whose cardinality achieves the sphere-packing bound. It is known that if \( q = p^m \) is a prime power, then a 1-perfect code in \( H(n, q) \) exists if and only if \( n = (q^l - 1)/(q - 1) \) for some positive integer \( l \) \[^{10}\]. The cardinality of this code is equal to \( q^{n-l} \). It is also known that a 1-perfect code in \( D(m, n) \) exists if and only if \( 2m + n = (4^l - 1)/3 \) for some positive integer \( l \) \[^{15}\]. The cardinality of this code is equal to \( 4^{2m+n-l-1} \).

A code \( C \) in \( H(n, q) \) is called an extended 1-perfect code if its code distance is equal to 4 and the projection \( C_i \) in some position \( i \) is a 1-perfect code. If a code \( C \) has distance \( d > 1 \), then its projection has distance at least \( d - 1 \) and the same cardinality. Therefore, if \( C \) is an extended 1-perfect code, then the projection \( C_i \) is a 1-perfect code for any \( i = 1, \ldots, n \). So, if \( q = p^m \) is a prime power, then an extended 1-perfect code in \( H(n, q) \) can exist only for \( n = (q^l + q - 2)/(q - 1) \), \( l \in \mathbb{N} \). The cardinality of such code equals \( q^n-l-1 \). Similarly, a code \( C \) in \( D(m, n) \) is called an extended 1-perfect code if its code distance equals 4 and the projection \( C_i \), for some position \( i \) is a 1-perfect code. So an extended 1-perfect code in \( D(m, n) \) can exist only if \( 2m + n = (4^l + 2)/3 \), \( l \in \mathbb{N} \). The cardinality of such code equals \( 4^{2m+n-l-1} \).

If \( n = 0 \), then a code \( C \) in \( D(m, 0) \) is called an extended 1-perfect code if it has the same parameters as an extended 1-perfect code in Doob graph of the same diameter, i.e. \( 2m = (4^l + 2)/3 \), the code distance is equal to 4, \( |C| = 4^{2m-l-1} \).

### 3. A necessary condition for the existence of perfect colorings

Given a graph \( G \), let us consider the set of complex-valued functions \( f : \nu G \to \mathbb{C} \) on the vertex set. These functions form a vector space \( U(G) \) with the inner product \( (f, g) = \sum_{x \in \nu G} f(x) \overline{g(x)} \). A function \( f : \nu G \to \mathbb{C} \) is called an eigenfunction of \( G \) if \( Mf = \lambda f \), \( f \neq 0 \), where \( M \) is the adjacency matrix of \( G \), for some \( \lambda \), which is called an eigenvalue of \( G \). Denote by \( U_\lambda = \{ f : Mf = \lambda f \} \) the eigensubspace corresponding to \( \lambda \).

Let \( G \) be a Hamming graph \( H(n, q) \) or a Doob graph \( D(m, n) \). Then it is convenient to use the characters to form a basis of each eigensubspace. Let \( \xi \) be the \( q \)-th root of unity, namely \( \xi = e^{\frac{2\pi i}{q}} \). If \( G \) is \( H(n, q) \), then for an arbitrary \( z \in \mathbb{Z}_q^n \) define the function \( \varphi_z(t) = \frac{\xi^{\langle z, t \rangle}}{q^n} \), where \( \langle v, u \rangle = v_1u_1 + \ldots + v_nu_n \mod q \). If \( G \) is \( D(m, n) \), then for an arbitrary \( z \in (\mathbb{Z}_4^n)^m \times \mathbb{Z}_4^n \) define the function \( \varphi_z(t) = \frac{\xi^{\langle z, t \rangle}}{4^{2m+n}/2} \), where \( \langle x, v \rangle = (x_1v_1 + y_1u_1) + \ldots + (x_mv_m + y_mu_m) + r_1s_1 + \ldots + r_ns_n \mod 4 \); \( x = ([x_1, y_1], \ldots, [x_m, y_m]; r_1, \ldots, r_n) \) and \( v = ([v_1, u_1], \ldots, [v_m, u_m]; s_1, \ldots, s_n) \) are vertices in \( D(m, n) \) (we denote by \([a, b]\) an element of \( \mathbb{Z}_4^n \)). It is known that the functions
\(\varphi_z\), where \(z \in \nu G\), are eigenfunctions of \(G\) and these functions form an orthonormal basis of the vector space \(U(G)\).

**Lemma 1.** Let \(f\) be a perfect \(k\)-coloring of a graph \(G\) with quotient matrix \(S\). Let \(f_j = \chi_{f^{-1}(j)}\) be the characteristic function of the set of vertices of color \(j\). Then for any \(t \in \mathbb{N}\)

\[
(M^t f_j, f_j) = s_{j,j}^t \cdot |f^{-1}(j)|,
\]

where \(M\) is the adjacency matrix of \(G\), and \(s_{j,j}^t\) is the \((j,j)\)-th element of the matrix \(S^t\).

**Proof.** Let \(F = (f_0, \ldots, f_{k-1})\) be the \(|\nu G| \times k\) matrix, where the \(i\)-th column \(f_i = \chi_{f^{-1}(i)}\) is the characteristic function of the set of vertices of color \(i\). It is known that \(MF = FS\) (see for example [32, Section 5.2]), and consequently \(M^t F = FS^t\) for any \(t\). Hence, \((M^t f_j)(x) = (M^t F)_{x,j} = (FS^t)_{x,j} = s^t_{f(x),j}\) for any vertex \(x \in \nu G\). Since \(f_j(x) = 0\) if \(f(x) \neq j\), we have \((M^t f_j, f_j) = s_{j,j}^t \cdot |f^{-1}(j)|\). □

**Lemma 2.** [32, Section 5.2]. Let \(f\) be a perfect coloring of a graph \(G\) with quotient matrix \(S\). If \(\lambda\) is an eigenvalue of \(S\), then \(\lambda\) is an eigenvalue of \(G\).

**Theorem 1.** Let \(G\) be the Hamming graph \(H(n, q)\), where \(q \in \{2, 3, 4\}\), or the Doob graph \(D(m, n)\). Let \(f\) be a perfect \(k\)-coloring of \(G\) with quotient matrix \(S\) that has eigenvalues \(\lambda_0 > \lambda_1 > \ldots > \lambda_l\). Let \(i\) be a color of \(f\), and let \(s_{i,i}^t\) be the \((i,i)\)-th element of \(S^t\), \(t = 1, \ldots, l - 1\).

Then the linear system of equations

\[
\begin{pmatrix}
1 & 1 & \ldots & 1 \\
\lambda_1 & \lambda_2 & \ldots & \lambda_l \\
\lambda_1^2 & \lambda_2^2 & \ldots & \lambda_l^2 \\
\vdots & \vdots & \ddots & \vdots \\
\lambda_1^{l-1} & \lambda_2^{l-1} & \ldots & \lambda_l^{l-1}
\end{pmatrix}
\begin{pmatrix}
x_1 \\
x_2 \\
x_2 \\
\vdots \\
x_l
\end{pmatrix}
= |f^{-1}(i)|
\begin{pmatrix}
1 \\
1 \\
s_{i,i}^1 \\
s_{i,i}^2 \\
s_{i,i}^{l-1}
\end{pmatrix}

- \frac{|f^{-1}(i)|^2}{|\nu G|}
\begin{pmatrix}
1 \\
\lambda_0 \\
\lambda_1 \\
\vdots \\
\lambda_l
\end{pmatrix}
\]

has a unique solution \((a_1, \ldots, a_l)\). Moreover, \(a_j \cdot |\nu G|\) is a non-negative integer for \(j = 1, \ldots, l\).

**Proof.** The matrix of the system is a transposition of Vandermonde matrix, so the determinant is not equal to 0. Hence the system has a unique solution. Let \(f_i = \chi_{f^{-1}(i)}\) be the characteristic function of color \(i\). By Lemma 2 eigenvalues \(\lambda_0, \ldots, \lambda_l\) are eigenvalues of \(G\). It is known that \(f_i\) belongs to the direct sum of the eigensubspaces corresponding to the eigenvalues of \(S\), i.e., \(f_i \in U_{\lambda_0} \oplus \ldots \oplus U_{\lambda_l}\) (see for example [1].
where $\alpha_z, z \in vG$, are complex coefficients. Therefore,

$$M^t f_i = \sum_{z: \varphi_z \in U_{\lambda_1}} \lambda_1 \alpha_z \varphi_z + \ldots + \sum_{z: \varphi_z \in U_{\lambda_i}} \lambda_i \alpha_z \varphi_z + \lambda_j \alpha_0 \varphi_0.$$ 

This representation implies the following relation for $t = 0, 1, \ldots$

$$(M^t f_i, f_i) = \lambda_1 \sum_{z: \varphi_z \in U_{\lambda_1}} |\alpha_z|^2 + \ldots + \lambda_i \sum_{z: \varphi_z \in U_{\lambda_i}} |\alpha_z|^2 + \lambda_j |\alpha_0|^2.$$ 

Since the basis is orthonormal, we have $\alpha_0 = (f_i, \varphi_0) = |f^{-1}(i)| / |\lambda_0|$, and hence $|\alpha_0|^2 = |f^{-1}(i)|^2 / |\lambda_0|^2$. Since $(M^t f_i, f_i) = |f^{-1}(i)| \cdot s_{t,i}$ by Lemma 1, it is straightforward that $(a_1, \ldots, a_t)$, where $a_j = \sum_{z: \varphi_z \in U_{\lambda_j}} |\alpha_z|^2$, is the solution of the system. On the other hand, as the basis $\{\varphi_z : z \in vG\}$ is orthonormal, we have $\alpha_z = (f_i, \varphi_z)$ for any $z \in vG$. Let us consider subcases.

If $q = 2$, then for any $z \in \mathbb{Z}_2^n$ the function $\varphi_z$ has two distinct values: $\pm 1 / \sqrt{2}$. In this case, $\alpha_z = (f_i, \varphi_z) = x / \sqrt{2}$ and $|\alpha_z|^2 = x^2 / 2$ for some integer $r$. Hence $a_j 2^n = \sum_{z: \varphi_z \in U_{\lambda_j}} 2^n |\alpha_z|^2$ is a non-negative integer.

If $q = 3$, then $\varphi_z$ has three distinct values: $\pm 1 / \sqrt{3}$ and $\pm 2 / \sqrt{3}$. In this case, $\alpha_z = \frac{a + b \sqrt{3} \sqrt{-1}}{2 \sqrt{3} \sqrt{2}}$, were $a$ and $b$ are integers and, moreover, they have the same parity. So $|\alpha_z|^2 = \frac{a^2 + 3b^2}{4} = \frac{r^2}{3}$ for some integer $r$. So $a_j \frac{3^2}{4} = \sum_{z: \varphi_z \in U_{\lambda_j}} 3^n |\alpha_z|^2$ is a non-negative integer.

If $G = D(m, n)$, then $\varphi_z$ has four distinct values: $\pm 1 / \sqrt{q(2m+n)/2}, \pm \sqrt{-1} / \sqrt{q(2m+n)/2}$. So $\alpha_z = (f_i, \varphi_z) = \frac{a + b \sqrt{1} \sqrt{-1}}{4 \sqrt{(2m+n)/2}}$ for some integers $a$ and $b$. Hence $|\alpha_z|^2 = \frac{r}{4^{2m+n}}$ for some integer $r$. Hence $a_j 4^{2m+n} = \sum_{z: \varphi_z \in U_{\lambda_j}} 4^{2m+n} |\alpha_z|^2$ is a non-negative integer. \(\blacksquare\)

4. Extended perfect codes are completely regular

Theorem 2.

1. A code $C$ in $H(n, q)$ is extended 1-perfect if and only if $C$ is completely regular
with quotient matrix

\[
\begin{pmatrix}
0 & n(q - 1) & 0 \\
1 & q - 2 & (n - 1)(q - 1) \\
0 & n & n(q - 2)
\end{pmatrix}.
\]

2. A code \( C \) in \( D(m, n) \) is extended 1-perfect if and only if \( C \) is completely regular with quotient matrix

\[
\begin{pmatrix}
0 & 6m + 3n & 0 \\
1 & 2 & 6m + 3n - 3 \\
0 & 2m + n & 4m + 2n
\end{pmatrix}.
\]

Proof. In most parts, the proof for \( D(m, n) \) is similar to the proof for \( H(2m + n, 4) \). So we mainly focus on Hamming graphs, and we consider Doob graphs only in the cases for which the proof is different. Let \( C \) be an extended 1-perfect code in \( H(n, q) \) (\( D(m, n) \)). Let \( f \) be the distance coloring of \( H(n, q) \) with respect to \( C \), i.e. \( f(x) = \min_{y \in C} \{d(x, y)\}, x \in vH(n, q) \). Since the projection of \( C \) in any position is a 1-perfect code that has the covering radius 1, the covering radius of \( C \) equals 2 hence the set of colors is \( \{0, 1, 2\} \).

Define the following \( s_j^i : f^{-1}(i) \to \mathbb{Z} \), where \( s_j^i(x) \) is the number of vertices of color \( j \) in the neighbourhood of \( x \), if \( f(x) = i \), and otherwise is not defined. So, \( f \) is a perfect coloring if and only if \( s_j^i \) is constant for all \( i, j \in \{0, 1, 2\} \). Obviously, \( s_0^0 \equiv 0 \) (as the code distance is 4), and \( s_0^0 \equiv 0, s_2^0 \equiv 0 \) (by the definition).

Let \( y \) be an arbitrary vertex of color \( 1 \). Let us count the values \( s_0^1(y) \) and \( s_1^1(y) \). On the one hand, \( s_0^1(y) \geq 1 \) by the definition. On the other hand, \( s_0^1(y) \leq 1 \) (otherwise we have a contradiction with the code distance). Hence \( s_0^1 \equiv 1 \). Therefore, for any vertex \( x \) of color \( 1 \) we can denote by \( o(x) \) the unique neighbour of \( x \) that has color 0. Any vertex \( y' \) of color \( 1 \) that adjacent to \( y \) belongs to the neighbourhood of \( o(y) \) (indeed, if \( o(y) \neq o(y') \), then \( d(o(y), o(y')) \leq 3 \) that contradicts the code distance). Therefore, all neighbours of \( y \) that have color 1 belong to the neighbourhood of \( o(y) \). The number of common neighbours of two arbitrary adjacent vertices in a distance-regular graph is uniquely determined by the intersection array. For \( H(n, q) \), it is equal to \( q - 2 \) and for \( D(m, n) \), to 2. Hence \( s_0^1 \equiv 1 \) and \( s_1^1 \equiv q - 2 \) (\( s_2^i \equiv 2 \) for a Doob graph). For each vertex \( x \in vH(n, q) \), we have \( s_0^i(x) + s_1^i(x) + s_2^i(x) = n(q - 1) \), where \( i \) is the color of \( x \). Therefore, \( s_0^i \equiv n(q - 1) \) and \( s_2^i \equiv (n - 1)(q - 1) \).

It remains to prove that \( s_2^i \equiv n \) (\( s_2^i \equiv 2m + n \) for \( D(m, n) \)). An edge \( \{v, u\} \) is called an \((i, j)\)-edge if \( v \) has color \( i \) and \( u \) has color \( j \), or vice versa. Denote \( \alpha = \sum_{x \in f^{-1}(2)} s_2^i(x), \)
i.e. the number of $(1,2)$-edges.

Let us calculate the values $|f^{-1}(0)|$, $|f^{-1}(1)|$ and $|f^{-1}(2)|$. The first value is equal to the cardinality of a 1-perfect code in $H(n-1, q)$, i.e. $\frac{q^{n-1}}{(n-1)(q-1)+1}$. From the counting of the number of $(0,1)$-edges, we have $|f^{-1}(1)| = |f^{-1}(0)| n(q-1) = \frac{(n-1)(q-1)+1}{n(q-1)} q^{n-1}$. Counting the number of $(1,2)$-edges, we find $\alpha = (n-1)(q-1)|f^{-1}(1)|$. On the other hand,

$$|f^{-1}(2)| = q^n - |f^{-1}(0)| - |f^{-1}(1)| = \frac{q^{n-1} q((n-1)(q-1)+1) - n(q-1) - 1}{(n-1)(q-1)+1} = \frac{q^{n-1} (n-1)(q-1)^2}{(n-1)(q-1)+1}$$

Hence the average value of $s_1^2$ equals $n$, i.e. $\frac{\alpha}{|f^{-1}(2)|} = n$ (or $2m+n$ for $D(m,n)$).

Let $v$ be a vertex of color 2 in $H(n, q)$. The induced subgraph on the set of its neighbours has $n$ connected components, and every component is a $(q-1)$-clique. Hence $s_1^2(v) \leq n$ (otherwise there are two vertices $u$ and $w$ of color 1 in the same component, but all their common neighbours except $v$ also belongs to this component and one of them is $o(u)$, which has color 0). Since the average value of $s_1^2$ equals $n$, we have $s_1^2 \equiv n$.

Let $v = (x_1, \ldots, x_m; y_1, \ldots, y_n)$ be a vertex of color 2 in $D(m,n)$. Denote by $h_{j,v}$ the induced subgraph on the set $\{(x_1, \ldots, x_m; y_1, \ldots, y_{j-1}, b, y_{j+1}, \ldots, y_n) : b \in \mathbb{Z}_4\}$. This graph is the complete graph $K_4$. Denote by $d_{i,v}$ the induced subgraph on the vertex set $\{(x_1, \ldots, x_{i-1}, a, x_{i+1}, \ldots, x_m; y_1, \ldots, y_n) : a \in \mathbb{Z}_2^2\}$. This graph is the Shrikhande graph. Denote by $\alpha_{i,v}$ the number of $(1,2)$-edges in $d_{i,v}$ divided by the number of vertices of color 2 in $d_{i,v}$. Let us prove that for any $i \in \{1, \ldots, m\}$ and $v \in vD(m,n)$ it follows that $\alpha_{i,v} \leq 2$; moreover, if $\alpha_{i,v} = 2$, then any vertex of color 2 in $d_{i,v}$ has exactly two neighbours of color 1 in $d_{i,v}$.

Let $i \in \{1, \ldots, m\}$ and $v \in vD(m,n)$. Consider two cases. If $d_{i,v}$ contains a vertex $u$ of color 0, then $\alpha_{i,v} = 2$. Indeed, all neighbours of $u$ have color 1 and other 9 vertices have color 2 (if some vertex $w$ is at distance 2 from some vertex of color 0, then $f(w) = 2$; otherwise we have a contradiction with the code distance). So any vertex of color 2 has two neighbours of color 1 (because the Shrikhande graph is strongly regular with parameters $(16,6,2,2)$). In the second case, there are no vertices of color 0 in $d_{i,v}$. Then the vertices of color 1 form an independent set (indeed, if some vertices $u$ and $w$ are adjacent, then $o(u)$ is their common neighbour, but these vertices have only two common neighbours, which also belong to $d_{i,v}$). So
\( \alpha_{i,v} = \frac{6x}{16-x} \), where \( x \) is the number of vertices of color 1. A maximum independent set in the Shrikhande graph has cardinality 4; moreover, the characteristic function of a maximum independent set is a perfect coloring, where any vertex that does not belong to this set is adjacent to 2 vertices from this set (see [27, Section 2]). Hence \( \alpha_{i,v} \leq 2 \); moreover, if \( \alpha_{i,v} = 2 \ (x = 4) \), then any vertex of color 2 has exactly two neighbours of color 1 in \( d_{i,v} \). As before, for any \( j \in \{1, \ldots, n\} \) and \( v \in \mathcal{V}(m, n) \) any vertex of color 2 has 0 or 1 neighbours of color 1 in the graph \( h_{j,v} \). Since any \((1,2)\)-edge in \( D(m, n) \) belongs to exactly one subgraph among the subgraphs \( d_{i,v} \) and \( h_{j,v} \), where \( v \in \mathcal{V}(m, n), \ i = 1, \ldots, m, \ j = 1, \ldots, n \), we have \( \alpha_{i,v} \leq 2 \). Moreover, if \( \alpha_{i,v} = 2 \) \((x = 4)\), then any vertex of color 2 has exactly two neighbours of color 1 in the graph \( h_{j,v} \). As before, for any \( j \in \{1, \ldots, n\} \) and \( v \in \mathcal{V}(m, n) \) any vertex of color 2 has 0 or 1 neighbours of color 1 in the graph \( h_{j,v} \). Since any \((1,2)\)-edge in \( D(m, n) \) belongs to exactly one subgraph among the subgraphs \( d_{i,v} \) and \( h_{j,v} \), where \( v \in \mathcal{V}(m, n), \ i = 1, \ldots, m, \ j = 1, \ldots, n \), we have \( \alpha_{i,v} \leq 2 \). Moreover, if \( \alpha_{i,v} = 2 \) \((x = 4)\), then any vertex of color 2 has exactly two neighbours of color 1 in the graph \( h_{j,v} \). As before, for any \( j \in \{1, \ldots, n\} \) and \( v \in \mathcal{V}(m, n) \) any vertex of color 2 has 0 or 1 neighbours of color 1 in the graph \( h_{j,v} \).

5. The non-existence of some extended perfect codes

Now we can apply Theorem \( \Box \) to prove the non-existence of ternary and quaternary extended 1-perfect codes.

Proposition 1.
1. Let $C$ be an extended 1-perfect code in $H(n, 3)$, where $n = \frac{3^l+1}{2}$, $l \in \mathbb{N}$. Then $l \leq 2$.

2. Let $C$ be an extended 1-perfect code in $D(m, n)$ (including the case $D(0, n) = H(n, 4)$), where $2m + n = \frac{4^l+2}{3}$, $l \in \mathbb{N}$. Then $l \leq 3$.

Proof. 1) Let $C$ be an extended 1-perfect code in $H(n, 3)$, where $n = \frac{3^l+1}{2}$ for some positive integer $l$. The cardinality of $C$ is equal to $3^{n-l-1}$. By Theorem 2 and Lemma 3 the distance coloring with respect to $C$ is a perfect coloring with quotient matrix, which has eigenvalues: $\lambda_1 = 1$, $\lambda_2 = -n$ and $\lambda_0 = 2n$. Let us consider the system of equations from Theorem 1

\[
\begin{cases}
a_1 + a_2 = 3^{n-l-1} - 3^{n-2l-2} \\
a_1 - na_2 = -2n \cdot 3^{n-2l-2}.
\end{cases}
\]

From this system we have

\[
a_2 \cdot 3^n = \frac{3^{2n-2l-2}(3^{l+1} + 3^l)}{3^{l+3}} = \frac{3^{2n-3}2^3}{3^l-1 + 1}.
\]

By Theorem 1 the number $a_2 \cdot 3^n$ is integer. Since the denominator $3^{l-1} + 1$ and $3^{2n-l-3}$ are relatively prime, it follows that $3^{l-1} + 1$ is a divisor of 8. This implies $l = 1$ or $l = 2$.

2) Let $C$ be an extended 1-perfect code in $D(m, n)$, where $2m + n = \frac{4^l+2}{3}$ for some positive integer $l$.

In this case, we have the following system of equations

\[
\begin{cases}
a_1 + a_2 = 4^{2m+n-l-1} - 4^{2m+n-2l-2} \\
2a_1 - (2m + n)a_2 = -3(2m + n)4^{2m+n-2l-2}.
\end{cases}
\]

From this system we have

\[
a_2 \cdot 4^{2m+n} = \frac{4^{4m+2n-2l-2}(2 \cdot 4^{l+1} - 2 + 4^l + 2)}{4^l+8} = \frac{4^{4m+2n-l-3}3^3}{4^{l+1} + 2}.
\]

By Theorem 1 the number $a_2 \cdot 4^{2m+n}$ is integer. If $l = 1$, then $4^2 \cdot a_2 = 9$. Let $l > 1$. Since the greatest common divisor of the denominator $4^{l-1} + 2$ and $4^{4m+2n-l-3}$ equals 2, it follows that $2 \cdot 4^{l-2} + 1$ divides 27. This implies $l \in \{2, 3\}$. So $l \leq 3$. ▲

The two following propositions solve the remaining cases in $H(n, 3)$ and $D(m, n)$, and codes of odd length in $H(n, q)$ for all $q$. The proofs of these propositions are particular cases of the method described in [33].

**Proposition 2.** Let $C$ be an extended 1-perfect code in $H(n, q)$. Then $n$ is even.
Proof. Let $C$ be an extended 1-perfect code in $H(n,q)$ and $f$ be the distance coloring with respect to $C$. Consider an arbitrary vertex $a$ of color 2. Denote by $W_j^i$ the set of vertices of color $i$ that are at the distance $j$ from $a$ and denote $W_j = W_j^0 \cup W_j^1 \cup W_j^2$. On the one hand, any vertex $x \in W_1^1$ is adjacent to exactly 1 vertex from $W_2^0$. On the other hand, any vertex $y \in W_2^0$ has 2 neighbors in $W_1$ and they have color 1. Hence $|W_1^1| = 2|W_2^0|$, and so $|W_1^1|$ is even. But from Theorem 2 we have $|W_1^1| = n$. ▲

Recall that a code $C$ in $H(n,q)$ is called an MDS code with distance $d$ if its cardinality achieves the Singleton bound, i.e. $|C| = q^{n-d+1}$. In the case $n = q + 2$, the definitions of an extended 1-perfect code and an MDS code with distance 4 are equivalent.

Corollary 1. If $q$ is odd, then there are no MDS codes with distance 4 in $H(q+2,q)$.

Corollary 2. Let $q = p^m$ be an odd prime power, and let $C$ be an extended 1-perfect code in $H(n,q)$. Then $n = \frac{q^l + q - 2}{q - 1}$ for some odd $l$.

Proposition 3. There are no extended 1-perfect codes in $D(m,n)$, where $2m+n = 22$.

Proof. Let $C$ be an extended 1-perfect code in $D(m,n)$, where $2m+n = 22$, and let $f$ be the distance coloring with respect to $C$. Consider an arbitrary vertex $a$ of color 2. Denote by $W_j^i$ the set of vertices of color $i$ that are at the distance $j$ from $a$ and denote $W_j = W_j^0 \cup W_j^1 \cup W_j^2$. By Theorem 2 we have $|W_1^0| = 0$, $|W_1^1| = 22$, and $|W_1^2| = 44$. As in proof of Proposition 2 we have $2|W_2^0| = |W_1^1|$, so $|W_2^0| = 11$. Let us count the number $w$ of edges $(x,y)$ such that $x \in W_1$ and $y \in W_2$. This number is equal to $(22 \cdot 2 + 44 \cdot 22 - 2t - r)$, where $t$ is the number of $(1,1)$-edges and $r$ is the number of $(1,2)$-edges in the induced subgraph on the set of vertices $W_1$. It follows from the intersection array that this subgraph is 2-regular, and hence $2t + r = 2|W_1^1| = 44$. So $w = 22 \cdot 2 + 44 \cdot 22 - 44 = 968$. On the other hand, $w = 2|W_2^1|$, so $|W_2^1| = 484$. Let us count the number of $(0,1)$-edges that are incident to some vertex from $W_1$. This number is equal to $|W_2^0| = 484$; on the other hand, it is equal to $6|W_2^0| + 3|W_3^0| = 66 + 3|W_3^0|$. We find that $3|W_3^0| = 418$. Since $|W_3^0|$ is integer, we have a contradiction. ▲

Remind that formally the singleton from any vertex in $H(2,3)$, $D(0,2)$ or $D(1,0)$ is an extended 1-perfect code, called trivial. Also all extended 1-perfect codes in $D(m,n)$, where $2m+n = 6$, are characterized in [26, 27]. From Propositions 2 and 3 we have the following statement.

Theorem 3.

1. An extended 1-perfect code in $H(n,3)$ exists if and only if $n = 2$. 

12
2. An extended 1-perfect code in $D(m,n)$ (including the case $D(0,n) = H(n,4)$) exists if and only if $(m,n) = (0,2)$, or $(m,n) = (1,0)$, or $(m,n) = (0,6)$, or $(m,n) = (2,2)$.

3. For any $q$, there are no extended 1-perfect codes in $H(n,q)$ if $n$ is odd.

Acknowledgements

The author is grateful to Denis Krotov, Vladimir Potapov, and Ev Sotnikova for helpful remarks and introducing him to some background.

References

[1] A. A. Taranenko. Algebraic properties of perfect structures. E-print 1906.10430v2, arXiv.org, 2020. Available at https://arxiv.org/abs/1906.10430v2.

[2] D. G. Fon-Der-Flaass. A bound on correlation immunity. Sib. Elektron. Mat. Izv., 4:133–135, 2007. Online: http://mi.mathnet.ru/eng/semr149.

[3] J. Bierbrauer. Bounds on orthogonal arrays and resilient functions. J. Comb. Des., 3(3):179–183, 1995. DOI: 10.1002/jcd.3180030304.

[4] J. Friedman. On the bit extraction problem. In Foundations of Computer Science, IEEE Annual Symposium on, pages 314–319, Los Alamitos, CA, USA, 1992. IEEE Computer Society. DOI: 10.1109/SFCS.1992.267760.

[5] V. N. Potapov. On perfect 2-colorings of the $q$-ary $n$-cube. Discrete Math., 312(6):1269–1272, 2012. DOI: 10.1016/j.disc.2011.12.004.

[6] V. N. Potapov. On perfect colorings of boolean $n$-cube and correlation immune functions with small density. Sib. Elektron. Mat. Izv., 7:372–382, 2010. In Russian, English abstract.

[7] D. S. Krotov. On the OA(1536,13,2,7) and related orthogonal arrays. Discrete Mathematics, 343(2):111659, 2020. DOI: 10.1016/j.disc.2019.111659.

[8] D. S. Krotov. On the binary codes with parameters of doubly-shortened 1-perfect codes. Des. Codes Cryptography, 57(2):181–194, 2010. DOI: 10.1007/s10623-009-9360-5.
[9] D. S. Krotov. On the binary codes with parameters of triply-shortened 1-perfect codes. Des. Codes Cryptography, 64(3):275–283, 2012. DOI: 10.1007/s10623-011-9574-1.

[10] F. J. MacWilliams and N. J. A. Sloane. The Theory of Error-Correcting Codes. Amsterdam, Netherlands: North Holland, 1977.

[11] S. W. Golomb and E. C. Posner. Rook domains, latin squares, and error-distributing codes. IEEE Trans. Inf. Theory, 10(3):196–208, 1964. DOI: 10.1109/TIT.1964.1053680.

[12] O. Heden and C. Roos. The non-existence of some perfect codes over non-prime power alphabets. Discrete Math., 311(14):1344–1348, 2011. DOI: 10.1016/j.disc.2011.03.024.

[13] H. W. Lenstra, Jr. Two theorems on perfect codes. Discrete Math., 3(1-3):125–132, 1972. DOI: 10.1016/0012-365X(72)90028-3.

[14] S. P. Lloyd. Binary block coding. Bell Syst. Tech. J., 36(2):517–535, 1957. DOI: 10.1002/j.1538-7305.1957.tb02410.x.

[15] D. S. Krotov. The existence of perfect codes in Doob graphs. IEEE Transactions on Information Theory, 66(3):1423–1427, 2020. DOI: 10.1109/TIT.2019.2946612.

[16] D. S. Krotov. Perfect codes in Doob graphs. Des. Codes Cryptography, 80(1):91–102, 2016. DOI: 10.1007/s10623-015-0066-6.

[17] M. Shi, D. Huang, and D. Krotov. Additive perfect codes in Doob graphs. Des. Codes Cryptography, 87(8):1857–1869, 2019. DOI: 10.1007/s10623-018-0586-y.

[18] P. Delsarte. An Algebraic Approach to Association Schemes of Coding Theory, volume 10 of Philips Res. Rep., Supplement. N.V. Philips’ Gloeilampenfabrieken, Eindhoven, Netherlands, 1973.

[19] A. Neumaier. Completely regular codes. Discrete Mathematics, 106-107:353–360, 1992. DOI: 10.1016/0012-365X(92)90565-W.

[20] J. Borges, J. Rifà, and V. A. Zinoviev. On completely regular codes. Problems of Information Transmission, 55(1):1–45, 2019. DOI: 10.1134/S0134347519010017.

[21] D. G. Fon-Der-Flaass. Perfect 2-colorings of a hypercube. Sib. Math. J., 48(4):740–745, 2007. DOI: 10.1007/s11202-007-0075-4 translated from Sib. Mat. Zh., 48(4) (2007), 923–930.
[22] E. Bespalov, D. Krotov, A. Matiushev, A. Taranenko, and K. Vorob’ev. Perfect 2-colorings of Hamming graphs. E-print 1911.13151v2, arXiv.org, 2020. Available at https://arxiv.org/abs/1911.13151v2

[23] J. Koolen, D. Krotov, and W. Martin. Completely regular codes: tables. https://sites.google.com/site/completelyregularcodes/.

[24] R. Ahlswede, H. K. Aydinian, and L. H. Khachatrian. On perfect codes and related concepts. Des. Codes Cryptography, 22(3):221–237, 2001. DOI: 10.1023/A:1008394205999.

[25] R. Hill. Caps and codes. Discrete Math., 22:111–137, 1978. DOI: 10.1016/0012-365X(78)90120-6

[26] T. L. Alderson. (6, 3)-MDS codes over an alphabet of size 4. Des. Codes Cryptography, 38(1):11–40, 2006. DOI: 10.1007/s10623-004-5659-4

[27] E. Bespalov and D. Krotov. MDS codes in Doob graphs. Problems of Information Transmission, 53:136–154, 2017. DOI: 10.1134/S003294601702003X.

[28] S. Ball. On sets of vectors of a finite vector space in which every subset of basis size is a basis. J. Eur. Math. Soc., 14(3):733–748, 2012. DOI: 10.4171/JEMS/316.

[29] J. I. Kokkala, D. S. Krotov, and P. R. J. Östergård. On the classification of MDS codes. IEEE Trans. Inf. Theory, 61(12):6485–6492, December 2015. DOI: 10.1109/TIT.2015.2488659.

[30] J. I. Kokkala and P. R. J. Östergård. Further results on the classification of MDS codes. Adv. Math. Commun., 10(3):489–498, August 2016. DOI: 10.3934/amc.2016020.

[31] D. Gijswijt, A. Schrijver, and H. Tanaka. New upper bounds for nonbinary codes based on the Terwilliger algebra and semidefinite programming. Journal of Combinatorial Theory, Series A, 113(8):1719–1731, 2006. DOI: 10.1016/j.jcta.2006.03.010.

[32] C. D. Godsil. Algebraic Combinatorics. Chapman and Hall, New York, 1993.

[33] D. S. Krotov. On weight distributions of perfect colorings and completely regular codes. Des. Codes Cryptography, 61(3):315–329, 2011. DOI: 10.1007/s10623-010-9479-4.