CHARACTERIZATION OF FINITE DIMENSIONAL
SUBSPACES OF COMPLEX FUNCTIONS THAT ARE
INVARIANT UNDER LINEAR DIFFERENTIAL
OPERATORS

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Abstract. The method to solve inhomogeneous linear differential
equations that is usually taught at school relies on the fact that
the right hand side function is the product of a polynomial and an
exponential and that the linear spaces of those functions are invariant
under differential operators (finite or ordinary).

This short note uses Jordan’s canonical decomposition to prove
that the linear spaces spanned by products of polynomial and expo-
nentials are the only linear complex spaces that are invariant under
differential operators, therefore non-homogeneous linear finite diffe-
rence or ordinary differential equations can only be generically solved
when the right hand side belongs to those spaces.

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1. Introduction

We characterize the finite dimensional subspaces of the space of com-
plex sequences which are invariant under every linear finite differences
operator as direct sums of spaces of arithmetic-geometric sequences. We also characterize finite dimensional subspaces of complex functions

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which are invariant under every linear differential operator as spaces of polynomial-exponential spaces. This explains why inhomogeneous linear differential equations can only be generally solved when the right hand sides are sums of exponentials times polynomials.

The following result is equivalent to Jordan’s decomposition of a matrix. Its proof is included for the sake of completeness.

**Theorem 1.** Let \( p(x) = (x - \lambda_1)^{l_1} \cdots (x - \lambda_s)^{l_s} \) be the minimal polynomial of an endomorphism \( \varphi \) of a finite dimensional \( \mathbb{C} \)-vector space \( V \) (i.e., \( p \) is the monic polynomial of minimal degree satisfying \( p(\varphi) = 0 \), \( \lambda_i \in \mathbb{C} \)). Then

\[
V = \bigoplus_{i=1}^{s} \ker(\varphi - \lambda_i I)^{l_i}.
\]

**Proof.** Define polynomials \( p_i(x) = p(x)/(x - \lambda_i)^{l_i} \), whose greatest common divisor is 1. Therefore, by Bezout’s identity (see [3]), there exist polynomials \( r_i \) such that \( 1 = \sum_{i=1}^{s} r_i p_i \). This implies that the identity mapping in \( V \), \( I_V \) can be written as \( I_V = \sum_{i=1}^{s} r_i(\varphi)p_i(\varphi) \), so we deduce that any \( v \in V \) can be written as

\[
v = \sum_{i=1}^{s} r_i(\varphi)p_i(\varphi)(v).
\]

Since

\[
(\varphi - \lambda_i I)^{l_i}r_i(\varphi)p_i(\varphi)(v) = r_i(\varphi)p(\varphi)(v) = 0,
\]

it turns out that \( r_i(\varphi)p_i(\varphi)(v) \in \ker(\varphi - \lambda_i I)^{l_i} \). Since \( v \) is arbitrary, then (1) implies that

\[
(\varphi - \lambda_i I)^{l_i} = I_V.
\]

Assume now that \( \sum_i v_i = 0, v_i \in \ker(\varphi - \lambda_i I)^{l_i} \). For any \( i = 1, \ldots, s \), Bezout’s identity gives polynomials \( s_i, t_i \), such that

\[
s_i(x)(x - \lambda_i)^{l_i} + t_i(x)p_i(x) = 1,
\]

which gives

\[
s_i(\varphi)(\varphi - \lambda_i I)^{l_i} + t_i(\varphi)p_i(\varphi) = I_V.
\]
Since $v_i = - \sum_{j \neq i} v_j$, then, for any $i = 1, \ldots, s$ we get that $p_i(\varphi)v_j = 0$, for $j \neq i$, which implies:

$$v_i = s_i(\varphi)(\varphi - \lambda_i I)^{t_i}(v_i) - \sum_{j \neq i} t_i(\varphi)p_i(\varphi)(v_j) = 0,$$

which yields that the sum (2) is direct. \[\square\]

2. INVARIANT SUBSPACES OF COMPLEX SEQUENCES UNDER LINEAR
FINITE DIFFERENCES OPERATORS

We consider the shift operator $S$ on complex sequences $y = (y_n) = (y_n)_{n \in \mathbb{N}}$ given by $(Sy)_n = y_{n+1}$, where $n$ will denote the independent variable (index) unless otherwise stated. Finite difference operators on complex sequences are polynomials in $S$. We denote by $\Pi_m$ the set of (complex) polynomials of degree at most $m$ and the subspace of complex sequences

$$G_{\lambda,m} = \{ (\lambda^n p(n)) : p \in \Pi_{m-1} \}.$$

Lemma 2. Given $\lambda \in \mathbb{C}$ and $m > 0$, $\ker(S - \lambda I)^m = G_{\lambda,m}$.

Proof. By induction on $k$ it can be easily established that there exist $\alpha_j^{k,r} \in \mathbb{C}$ such that

$$(S - \lambda I)^k(n^r \lambda^n) = \sum_{j=0}^{r-k} \alpha_j^{k,r}(n^j \lambda^n), \quad \alpha_j^{k,r} \neq 0,$$

for any $r \geq k$ and $(S - \lambda I)^k(n^r \lambda^n) = 0$ for $k > r$. This immediately gives that $G_{\lambda,m} \subseteq \ker(S - \lambda I)^m$, for all $m$. We prove the other inclusion by induction on $m$, the case $m = 1$ being trivial. So, assume that $\ker(S - \lambda I)^m = G_{\lambda,m}$ and aim to prove $\ker(S - \lambda I)^{m+1} = G_{\lambda,m+1}$. For this, consider $y \in \ker(S - \lambda I)^{m+1}$, so that $(S - \lambda I)^m y \in \ker(S - \lambda I) = G_{\lambda,1} = \{ (\alpha \lambda^n) : \alpha \in \mathbb{C} \}$ and, therefore,

$$(S - \lambda I)^m y = \alpha(\lambda^n),$$

for some $\alpha \in \mathbb{C}$. On the other hand, by (3)

$$(S - \lambda I)^m (n^m \lambda^n) = \alpha_0^{m,m}(\lambda^n),$$

1. Since $v_i = - \sum_{j \neq i} v_j$, then, for any $i = 1, \ldots, s$ we get that $p_i(\varphi)v_j = 0$, for $j \neq i$, which implies:

$$(S - \lambda I)^k(n^r \lambda^n) = \sum_{j=0}^{r-k} \alpha_j^{k,r}(n^j \lambda^n), \quad \alpha_j^{k,r} \neq 0.$$
which, together with (4), gives:

\[(S - \lambda I)^m (y - \frac{\alpha}{a_0} (n^m \lambda^n)) = 0.\]

The induction hypothesis thus yields \(y - \frac{\alpha}{a_0} (n^m \lambda^n) \in G_{\lambda,m}\), that is \(y \in G_{\lambda,m+1}\) and the proof is complete.

\[\square\]

**Theorem 3.** Let \(V\) be a finite dimensional subspace of complex sequences. Then \(V\) is invariant under every linear finite difference operator if and only if there exists \(\lambda_i, l_i, i = 1, \ldots, s\) such that

\[V = \oplus_{i=1}^s G_{\lambda_i, l_i}.\]

**Proof.** By Lemma 2, \(G_{\lambda,m}\) is invariant under \(S\) for any \(\lambda, m\), thus any subspace of the form \(\oplus_{i=1}^s G_{\lambda_i, l_i}\) is also \(S\)-invariant. Since the linear finite difference operators are polynomials in \(S\), then those subspaces are invariant under those difference operators.

On the other hand, if \(V\) is a finite dimensional subspace which is invariant under every linear finite difference operator, in particular it is invariant under \(S\). Therefore, by Proposition 1 there exist \(\lambda_i, l_i, i = 1, \ldots, s\) such that

\[V = \oplus_{i=1}^s \ker(S|_{V} - \lambda_i I_V)^{l_i} = \oplus_{i=1}^s (\ker(S - \lambda_i I)^{l_i} \cap V), \quad \ker(S - \lambda_i I)^{l_i} \cap V \neq \emptyset,

and we can assume that \(l_i\) is the smallest integer satisfying this equation. Since Lemma 2 implies

\[V = \oplus_{i=1}^s (V \cap G_{\lambda_i, l_i}).\]

the proof will be complete if we establish \(G_{\lambda_i, l_i} \subseteq V\). Since

\[V \cap G_{\lambda_i, l_i-1} \subseteq V \cap G_{\lambda_i, l_i} \neq \emptyset,

we can choose

\[v = \sum_{s=0}^{l_i-1} \beta_s (n^s \lambda^n) \in V \cap G_{\lambda_i, l_i} \setminus V \cap G_{\lambda_i, l_i-1},\]
i.e. $\beta_{i-1} \neq 0$. Equation (3) and the $S$-invariance of $V$ yield for any $k \geq 0$:

$$(S - \lambda I)^k \sum_{s=0}^{l_i-1} \beta_s (n^s \lambda^n) = \sum_{s=0}^{l_i-1} \sum_{j=0}^{s-k} \alpha_{j,s}^k \gamma_{j,k} (n^j \lambda^n) \in V,$$

with $\gamma_{j,k} = \sum_{s=j+k}^{l_i-1} \beta_s \alpha_{j,s}^k$. Since, for any $0 \leq k \leq l_i-1$, we get $\gamma_{l_i-1-k,k} = \beta_{l_i-1-k} \alpha_{l_i-1-k}^k \neq 0$ by (3) and (6), we deduce that $G_{\lambda_i,l_i} \subseteq V$, as claimed. \qed

3. INVARIANT SUBSPACES OF COMPLEX FUNCTIONS UNDER LINEAR DIFFERENTIAL OPERATORS

We consider now the differential operator $D$ on functions $y: K \to \mathbb{C}$ ($K = \mathbb{R}$ or $\mathbb{C}$) given by $Dy = y'$. Linear differential operators are polynomial evaluations of $D$. We denote by

$$H_{\lambda,m} = \{ e^{\lambda t} p(t) : p \in \Pi_{m-1} \}.$$ 

Lemma 4. Given $\lambda \in \mathbb{C}$ and $m > 0$, $\ker(D - \lambda I)^m = H_{\lambda,m}$.

Proof. The proof is similar to that of Lemma 2 and relies on the fact, easily established by induction on $k$, that there exist $\alpha_{j,r}^k \in \mathbb{C}$ such that

$$(D - \lambda I)^k (t^r e^{\lambda t}) = \sum_{j=0}^{r-k} \alpha_{j,r}^k (t^j e^{\lambda t}), \quad \alpha_{r-k}^k \neq 0,$$

for any $r \geq k$ and $(D - \lambda I)^k (t^r e^{\lambda t}) = 0$ for $k > r$. \qed

The proof of the following theorem relies on Lemma 4 and is similar to that of Theorem 3. It explains why inhomogeneous linear differential equations can only be generally solved when the right hand sides are sums of exponentials times polynomials (see [4][1][2]).

Theorem 5. Let $V$ be a finite dimensional subspace of the space of functions $y: K \to \mathbb{C}$. Then $V$ is invariant under every linear differential operator if and only if there exists $\lambda_i, l_i, i = 1, \ldots, s$ such that $V = \bigoplus_{i=1}^s H_{\lambda_i,l_i}$. 
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