CAFFARELLI-KOHN-NIRENBERG IDENTITIES, INEQUALITIES AND THEIR STABILITIES

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Abstract. We set up a one-parameter family of inequalities that contains both the Hardy inequalities (when the parameter is 1) and the Caffarelli-Kohn-Nirenberg inequalities (when the parameter is optimal). Moreover, we study these results with the exact remainders to provide direct understandings to the sharp constants, as well as the existence and non-existence of the optimizers of the Hardy inequalities and Caffarelli-Kohn-Nirenberg inequalities. As an application of our identities, we establish some sharp versions with optimal constants and theirs attainability of the stability of the Heisenberg Uncertainty Principle and several stability results of the Caffarelli-Kohn-Nirenberg inequalities.

1. Introduction

Our starting point is the classical Hardy inequality that plays important roles in many areas of analysis, mathematical physics and partial differential equations: for $N \geq 3$, we have that

$$\int_{\mathbb{R}^N} |\nabla u|^2 \, dx \geq \left( \frac{N-2}{2} \right)^2 \int_{\mathbb{R}^N} \frac{|u|^2}{|x|^2} \, dx, \quad u \in C_0^\infty (\mathbb{R}^N),$$

(1.1)

with the sharp constant $\left( \frac{N-2}{2} \right)^2$.

A very interesting fact about the Hardy inequalities that has attracted a lot of attention is that though the constant $\left( \frac{N-2}{2} \right)^2$ in (1.1) is optimal, the equality of (1.1) cannot occur for nontrivial functions such that both sides of (1.1) are finite. For instance, to explain for the aforementioned fact, many researchers have tried to study the improvements of the Hardy type inequalities. In particular, in the pioneering work [9], in order to study the stability of certain singular solutions of nonlinear elliptic equations, Brezis and Vázquez proved the following improved version of the Hardy inequality on bounded domains:

Theorem (Brezis-Vázquez [9]). For any bounded domain $\Omega \subset \mathbb{R}^N$, $N \geq 2$, and every $u \in H_0^1 (\Omega)$,

$$\int_{\Omega} |\nabla u|^2 \, dx - \left( \frac{N-2}{2} \right)^2 \int_{\Omega} \frac{|u|^2}{|x|^2} \, dx \geq z_0^2 \omega_N^\frac{2}{N} |\Omega|^{-\frac{2}{N}} \int_{\Omega} \frac{|u|^2}{|x|^2} \, dx$$

(1.2)

where $z_0 = 2.4048...$ is the first zero of the Bessel function $J_0 (z)$ and $\omega_N$ is the volume of the unit ball in $\mathbb{R}^N$. The constant $z_0^2 \omega_N^\frac{2}{N} |\Omega|^{-\frac{2}{N}}$ is optimal when $\Omega$ is a ball but is not achieved in the Sobolev space $H_0^1 (\Omega)$.

The fact that $z_0^2 \omega_N^\frac{2}{N} |\Omega|^{-\frac{2}{N}}$ is optimal when $\Omega$ is a ball, but still is not achievable by nontrivial functions in (1.2) led Brezis and Vázquez to ask whether $z_0^2 \omega_N^\frac{2}{N} |\Omega|^{-\frac{2}{N}}$ is just the
first term of an infinite series of remainder terms. This question has drawn the attention and has been addressed by a lot of researchers. The interested reader is referred to the monographs [2, 39, 42, 43, 50, 54], for instance, that are standard references on the subject. In particular, in [37], Frank and Seiringer provided a general method in terms of nonlinear ground state representations to derive the sharp local and nonlocal Hardy inequalities. We also note that the improved Hardy type inequalities have also been investigated in the form of identities in, for instance, [28, 36, 45, 46].

In this paper, we will present another look at the Hardy type inequalities. More precisely, we will set up a one-parameter family of inequalities in which the Hardy inequalities correspond to the case when the parameter is 1, while if we optimize the parameter, we obtain the Caffarelli-Kohn-Nirenberg inequalities.

In other words, Hardy inequalities can be regarded as the non-optimal (scale non-invariant) Caffarelli-Kohn-Nirenberg inequalities. It is worthy to note that the Caffarelli-Kohn-Nirenberg inequalities have been established by Caffarelli, Kohn and Nirenberg in their celebrated work [10] to generalize many well-known and important inequalities in analysis such as Gagliardo-Nirenberg inequalities, Hardy-Sobolev inequalities, Nash’s inequalities, Sobolev inequalities, etc. Due to their important roles in many areas of mathematics, the Caffarelli-Kohn-Nirenberg type inequalities and their applications have seen a surge of research activity in recent years. We do not attempt a survey of the extensive literature, but refer the reader to [16, 22, 23, 26, 35, 44, 47, 48, 56], to name just a few.

An important subfamily of the CKN inequality is the following $L^2$-Caffarelli-Kohn-Nirenberg inequality:

$$
\int_{\mathbb{R}^N} \frac{|\nabla u|^2}{|x|^{2b}} \, dx \int_{\mathbb{R}^N} \frac{|u|^2}{|x|^{2a}} \, dx \geq C^2(N, a, b) \left( \int_{\mathbb{R}^N} \frac{|u|^2}{|x|^{a+b+1}} \, dx \right)^2, \quad u \in C_0^\infty(\mathbb{R}^N \setminus \{0\}). \tag{1.3}
$$

In particular, this subclass contains the Heisenberg Uncertainty Principle ($a = -1, b = 0$), the Hydrogen Uncertainty Principle ($a = b = 0$) and the Hardy inequality ($a = 1, b = 0$), that play important roles in quantum mechanics. We mention here that the optimal constant $C(N, a, b) > 0$ of the $L^2$-Caffarelli-Kohn-Nirenberg inequality (1.3) was first studied by Costa in [19] for a particular range of parameters using expanding-the-square method, and then by Catrina and Costa in [13] for the full range of parameters using spherical harmonics decomposition and a Kelvin type transform. Very recently, the authors provided in [14] a very simple and direct proof to derive the best constant $C(N, a, b)$ for the whole range of parameters and to characterize all optimizers.

As mentioned earlier, we will prove in this article that the Hardy inequalities and the Caffarelli-Kohn-Nirenberg inequalities belong to the same family of inequalities in which the Caffarelli-Kohn-Nirenberg inequalities appear to be the optimal ones. Hence, our results can be used to explain for the attainability/unattainability of the sharp constants and the existence of optimizers/virtual optimizers of the Hardy inequalities. Actually, we set up these theorems with the exact remainders. Hence, our results can be applied to identify and study the existence and non-existence of the optimizers of the Hardy inequalities and the Caffarelli-Kohn-Nirenberg inequalities. We will also show in this paper that our results can also be used to derive sharp stability estimates of the Heisenberg Uncertainty Principle as well as some stability versions of the Caffarelli-Kohn-Nirenberg inequalities.
In 1985, Brezis and Lieb asked in [8] whether the difference of the two terms in the Sobolev inequalities controls the distance to the family of extremal functions. This question has initiated the studies of quantitative stability results for classical inequalities in mathematics that have been investigated extensively and intensively in the literature. In [5], Bianchi and Egnell provided an affirmative answer to the question of Brezis and Lieb for functions in $W^{1,2}(\mathbb{R}^N)$ by making use of the fact that this function space is a Hilbert space: there is a constant $c_{BE} > 0$ such that
\[
\int_{\mathbb{R}^N} |\nabla u|^2 \, dx - S_N \left( \int_{\mathbb{R}^N} |u|^\frac{2N}{N-2} \, dx \right)^{\frac{N-2}{N}} \geq c_{BE} \inf_{u^*} \int_{\mathbb{R}^N} |\nabla u - \nabla u^*|^2 \, dx.
\]
Here $S_N = \frac{1}{4} N (N - 2) |\mathbb{S}^N|^{\frac{2}{N}}$ is the optimal Sobolev constant and $u^*(x) = \alpha \left( \beta + |x - x_0|^2 \right)^{-\frac{N-2}{2}}$, $\alpha \in \mathbb{C}$, $\beta > 0$, $x_0 \in \mathbb{R}^N$, are the Aubin-Talenti functions. It is worth pointing out that the stability constant $c_{BE}$ has not been investigated until very recently. Indeed, in the paper [21], Dolbeault, Esteban, Figalli, Frank and Loss established some lower and upper bounds for the stability constant $c_{BE}$. In particular, in [41], König has proved that $c_{BE}$ is strictly smaller the spectral gap constant $\frac{1}{N-4}$, which is the best constant of the local stability of the Sobolev inequality [17].

The strategy in [5] and its generalizations were also used by the fourth author and Wei in [49] to study the stability of the second order Sobolev inequality, by Bartsch, Weth and Willem in [3] to investigate the stability of the higher order Sobolev inequality, by Chen, Frank and Weth in [17] to establish the stability of Sobolev inequality for fractional orders, etc. The case on the Sobolev space $W^{1,p}(\mathbb{R}^N)$, $p \neq 2$, is much more complicated and has just been established recently by, for instance, Cianchi, Fusco, Maggi and Pratelli in [18], Figalli and Neumayer in [34], Neumayer in [52] using new approaches.

In [51], McCurdy and Venkatraman studied the stability of the (scale invariant) Heisenberg Uncertainty Principle. More precisely, they applied the concentration-compactness arguments to show that there exist universal constants $C_1 > 0$ and $C_2(N) > 0$ such that for all $u \in \{ u \in W^{1,2}(\mathbb{R}^N) : \| xu \|_2 < \infty \}$:
\[
\delta_2(u) \geq C_1 \left( \int_{\mathbb{R}^N} |u|^2 \, dx \right) d_1(u, E)^2 + C_2(N) d_1(u, E)^4.
\]
Here
\[
\delta_2(u) := \left( \int_{\mathbb{R}^N} |\nabla u|^2 \, dx \right) \left( \int_{\mathbb{R}^N} |x|^2 |u|^2 \, dx \right) - \frac{N^2}{4} \left( \int_{\mathbb{R}^N} |u|^2 \, dx \right)^2
\]
is the Heisenberg deficit, $E = \{ ce^{-\alpha|x|^2} : c \in \mathbb{R}, \alpha > 0 \}$ is the set of the Gaussian functions, and $d_1(u, E) := \inf \left\{ \left( \int_{\mathbb{R}^N} \left| u - ce^{-\alpha|x|^2} \right|^2 \, dx \right)^\frac{1}{2} : c \in \mathbb{R}, \alpha > 0 \right\}$ is the $L^2$ distance to the set of the optimizers. A short and constructive proof of this result has been given by Fathi in [29] using the direct estimates via classical Gaussian functional inequalities. More exactly, Fathi set up the following stability version that provides explicit constants.
of the result in [51]: for all $u \in \{ u \in W^{1,2}(\mathbb{R}^N) : \| xu \|_2 < \infty \}$, there holds

$$\delta_2(u) \geq \frac{1}{4} \left( \int_{\mathbb{R}^N} |u|^{2} \, dx \right)^{\frac{1}{2}} \left( \int_{\mathbb{R}^N} |\nabla u|^{2} \, dx \right)^{\frac{1}{2}} d_1(u, E)^2 + \frac{1}{16} d_1(u, E)^4.$$ 

However, this inequality is not optimal. In this paper, we will show that by working on the scale non-invariant Heisenberg Uncertainty Principle and then shifting to its optimal version using the one-parameter family of inequalities, we are able to obtain a sharp version with optimal stability constants of the above estimate (Theorem 1.3) as a consequence of our main results. Moreover, we also show that our optimal stability version can be achieved by nontrivial functions.

It is also worth mentioning that the stability of the Gagliardo-Nirenberg inequalities has also been studied in [6, 7, 11, 24, 53, 55], to name just a few. We also refer the readers to [12, 15, 25, 27, 30, 31, 32, 33, 38, 40, 57], and references therein, for the stability results of many other functional and geometric inequalities.

The second purpose of our paper is to study the stability of the Caffarelli-Kohn-Nirenberg inequalities. We will use the aforementioned family of inequalities and the following approach: We will first establish a weighted version of the Poincaré inequality for the log-concave probability measure. Then, by combining this new Poincaré inequality with the exact remainders of the non-optimal (scale non-invariant) Caffarelli-Kohn-Nirenberg inequalities, we obtain some versions of the stability of the non-optimal (scale non-invariant) Caffarelli-Kohn-Nirenberg inequalities. Then, by switching to the scale invariant ones, we deduce, among others, the following version of the stability of the optimal Caffarelli-Kohn-Nirenberg inequalities:

**Theorem 1.1.** Let $0 \leq b < \frac{N-2}{2}$, $a \leq \frac{Nb}{N-2}$ and $a + b + 1 = \frac{2bN}{N-2}$. There exists a universal constant $C(N,a,b) > 0$ such that

$$\left( \int_{\mathbb{R}^N} \frac{|u|^2}{|x|^{2a}} \, dx \right)^{\frac{1}{2}} \left( \int_{\mathbb{R}^N} \frac{|
abla u|^2}{|x|^{2b}} \, dx \right)^{\frac{1}{2}} - \left( \frac{N - a - b - 1}{2} \right) \left( \int_{\mathbb{R}^N} \frac{|u|^2}{|x|^{a+b+1}} \, dx \right) \geq C(N,a,b) \inf \left\{ \int_{\mathbb{R}^N} |u - \alpha \exp(-\frac{\beta}{b+1-a}|x|^{b+1-a})|^{\frac{2}{a+b+1}} \, dx \right\}.$$ 

Here the infimum is taken over the set of all $\alpha \in \mathbb{R}$ and $\beta > 0$ such that $\int_{\mathbb{R}^N} \frac{|u|^2}{|x|^{a+b+1}} \, dx = \int_{\mathbb{R}^N} \frac{|\alpha \exp(-\frac{\beta}{b+1-a}|x|^{b+1-a})|^2}{|x|^{a+b+1}} \, dx$.

We note that

$$\delta_{1,a,b}(u) := \left( \int_{\mathbb{R}^N} \frac{|u|^2}{|x|^{2a}} \, dx \right)^{\frac{1}{2}} \left( \int_{\mathbb{R}^N} \frac{|
abla u|^2}{|x|^{2b}} \, dx \right)^{\frac{1}{2}} - \left( \frac{N - a - b - 1}{2} \right) \left( \int_{\mathbb{R}^N} \frac{|u|^2}{|x|^{a+b+1}} \, dx \right)$$

is the Caffarelli-Kohn-Nirenberg deficit. Also, we will show in Section 3 that $d_{1,a,b}(u, E, a, b) := \inf_{v \in E, a, b} \left( \int_{\mathbb{R}^N} \frac{|v-u|^2}{|x|^{a+b+1}} \, dx \right)^{\frac{1}{2}}$ is the distance from $u$ to $E, a, b$. Here

$$E, a, b \equiv \left\{ v(x) = \alpha \exp(-\frac{\beta}{b+1-a}|x|^{b+1-a}); \alpha \in \mathbb{R}, \beta > 0 \right\}.$$
is the set of optimizers for the scale invariant Caffarelli-Kohn-Nirenberg inequalities (3.4). Therefore, Theorem 1.1 implies that

$$\delta_{1,a,b}(u) \geq C(N, a, b) d_{1,a,b}(u, E_{a,b})^2.$$  

In the special case $a = -1$, $b = 0$ (that is, the Heisenberg Uncertainty Principle), we actually obtain the explicit constants that do not depend on the dimension:

**Theorem 1.2.** For all $u \in \{ u \in W^{1,2}(\mathbb{R}^N) : \|xu\|_2 < \infty \}$, we have

$$\delta_1(u) \geq \inf_{c \in \mathbb{R}, \alpha > 0} \left\{ \|u - ce^{-|x|\alpha}\|^2 \right\}$$

and

$$\delta_1(u) \geq \frac{1}{2} \inf_{c \in \mathbb{R}, \alpha > 0} \left\{ \|u - ce^{-|x|\alpha}\|^2 : \|u\|_2 = \|ce^{-|x|\alpha}\|^2 \right\}.$$  

These inequalities are sharp and the equalities can be attained by nontrivial functions.

Here we use the following Heisenberg deficit

$$\delta_1(u) := \left( \int_{\mathbb{R}^N} |\nabla u|^2 \, dx \right)^{\frac{1}{2}} \left( \int_{\mathbb{R}^N} |x|^2 |u|^2 \, dx \right)^{\frac{1}{2}} - \frac{N}{2} \int_{\mathbb{R}^N} |u|^2 \, dx.$$  

We note that the stability results in [29, 51] use the Heisenberg deficit

$$\delta_2(u) = \left( \int_{\mathbb{R}^N} |\nabla u|^2 \, dx \right) \left( \int_{\mathbb{R}^N} |x|^2 |u|^2 \, dx \right) - \frac{N^2}{4} \left( \int_{\mathbb{R}^N} |u|^2 \, dx \right)^2.$$  

Actually, our stability version with the Heisenberg deficit $\delta_1(u)$ implies the stability results with the Heisenberg deficit $\delta_2(u)$ in [29, 51]. Moreover, we are able to establish the optimal stability constants and their attainabilities. Indeed, from

$$\delta_1(u) \geq d_1(u, E)^2,$$

we deduce

$$\left( \int_{\mathbb{R}^N} |\nabla u|^2 \, dx \right)^{\frac{1}{2}} \left( \int_{\mathbb{R}^N} |x|^2 |u|^2 \, dx \right)^{\frac{1}{2}} \geq \frac{N}{2} \int_{\mathbb{R}^N} |u|^2 \, dx + d_1(u, E)^2.$$  

Therefore

$$\left( \int_{\mathbb{R}^N} |\nabla u|^2 \, dx \right) \left( \int_{\mathbb{R}^N} |x|^2 |u|^2 \, dx \right) \geq \frac{N^2}{4} \left( \int_{\mathbb{R}^N} |u|^2 \, dx \right)^2 + N \left( \int_{\mathbb{R}^N} |u|^2 \, dx \right) d_1(u, E)^2 + d_1(u, E)^4.$$  

That is

$$\delta_2(u) \geq N \left( \int_{\mathbb{R}^N} |u|^2 \, dx \right) d_1(u, E)^2 + d_1(u, E)^4.$$  

Similarly, let $d_2(u, E) := \inf_{c \in \mathbb{R}, \alpha > 0} \left\{ \|u - ce^{-|x|\alpha}\|^2 : \|u\|_2 = \|ce^{-|x|\alpha}\|^2 \right\} \geq d_1(u, E)$. Then since

$$\delta_1(u) \geq \frac{1}{2} d_2(u, E)^2,$$
That is weighted version of the L^2-Nirenberg inequalities by proving the
and we deduce

\[
\left( \int_{\mathbb{R}^N} |\nabla u|^2 \, dx \right)^{\frac{1}{2}} \left( \int_{\mathbb{R}^N} |x|^2 |u|^2 \, dx \right)^{\frac{1}{2}} \geq \frac{N}{\sqrt{2}} \int_{\mathbb{R}^N} |u|^2 \, dx + \frac{1}{2} d_2 (u, E)^2.
\]

Therefore

\[
\left( \int_{\mathbb{R}^N} |\nabla u|^2 \, dx \right) \left( \int_{\mathbb{R}^N} |x|^2 |u|^2 \, dx \right) \geq \frac{N^2}{4} \left( \int_{\mathbb{R}^N} |u|^2 \, dx \right)^2 + \frac{N^2}{2} \left( \int_{\mathbb{R}^N} |u|^2 \, dx \right) d_2 (u, E)^2 + \frac{1}{4} d_2 (u, E)^4.
\]

That is

\[
\delta_2 (u) \geq \frac{N}{2} \left( \int_{\mathbb{R}^N} |u|^2 \, dx \right) d_2 (u, E)^2 + \frac{1}{4} d_2 (u, E)^4.
\]

Therefore, we have the following sharp results:

**Theorem 1.3.** For all \( u \in \{ u \in W^{1,2} (\mathbb{R}^N) : \| x u \|_2 < \infty \} \), we have

\[
\delta_2 (u) \geq N \left( \int_{\mathbb{R}^N} |u|^2 \, dx \right) d_1 (u, E)^2 + d_1 (u, E)^4
\]

and

\[
\delta_2 (u) \geq \frac{N}{2} \left( \int_{\mathbb{R}^N} |u|^2 \, dx \right) d_2 (u, E)^2 + \frac{1}{4} d_2 (u, E)^4.
\]

These inequalities are sharp and the equalities can be attained by nontrivial functions.

It was also showed in [51] that for any two nonnegative constants \( C_1 \) and \( C_2 \) such that

\( C_1^2 + C_2^2 > 0 \), there exists \( u \in \{ u \in W^{1,2} (\mathbb{R}^N) : \| x u \|_2 < \infty \} \), \( \| u \|_2 = 1 \) and \( u^* \in E \) such that

\[
\delta_2 (u) \leq C_1 \| \nabla (u - u^*) \|_2^2 + C_2 \| x (u - u^*) \|_2^2.
\]

That is, there is no quantitative stability version of the scale invariant Heisenberg Uncertainty Principle if we use the norm \( \| \nabla (u - u^*) \|_2 \) or \( \| x (u - u^*) \|_2 \) as distance functions. In this paper, we will also show that this is not the case for the scale non-invariant Heisenberg Uncertainty Principle:

**Theorem 1.4.** For all \( u \in \{ u \in W^{1,2} (\mathbb{R}^N) : \| x u \|_2 < \infty \} \), then

\[
\int_{\mathbb{R}^N} |\nabla u|^2 \, dx + \int_{\mathbb{R}^N} |x|^2 |u|^2 \, dx - N \int_{\mathbb{R}^N} |u|^2 \, dx
\]

\[
\geq \frac{2}{N + 3} \inf_{c \in \mathbb{R}} \left( \int_{\mathbb{R}^N} \left| \nabla \left( u - ce^{-\frac{1}{2}|x|^2} \right) \right|^2 \, dx + \int_{\mathbb{R}^N} |x|^2 \left| u - ce^{-\frac{1}{2}|x|^2} \right|^2 \, dx + \int_{\mathbb{R}^N} |u - ce^{-\frac{1}{2}|x|^2}|^2 \, dx \right).
\]

Moreover, the inequality is sharp and the equality can be attained by nontrivial functions.

We end this introduction with the following remark. Recently, Anh Do and the second, third and fourth authors have established in [20] the stability of the \( L^p \)-Caffarelli-Kohn-Nirenberg inequalities by proving the \( L^p \)-Caffarelli-Kohn-Nirenberg identities and the weighted version of the \( L^p \)-Poincaré inequality for log-concave measures.

Our paper is organized as follows: In Section 2, we will set up a general identity and use it to establish the \( L^2 \)-Hardy and \( L^2 \)-Caffarelli-Kohn-Nirenberg identities. Several
examples will also be provided in Section 2. In Section 3, we will use some of these identities to study the sharp stability results for the Heisenberg Uncertainty Principle as well as several quantitative results about the stability of the Caffarelli-Kohn-Nirenberg inequalities.

2. \( L^2 \)-Hardy and \( L^2 \)-Caffarelli-Kohn-Nirenberg identities on \( \mathbb{R}^N \)

In this section we will set up general versions of the \( L^2 \)-Hardy and \( L^2 \)-Caffarelli-Kohn-Nirenberg identities on \( \mathbb{R}^N \). Denote \( R_u (x) := \frac{x}{|x|} \cdot \nabla u (x) \). This is the radial derivative. That is, in the polar coordinate \( x = r \sigma \), \( R_u = \partial_r u \). Let \( H \in C^1 (0, R) \), \( 0 < R \leq \infty \). By direct computation, we have

\[
\text{div} \left( H \left( \frac{|x|}{|x|} \right) \frac{x}{|x|} \right) = \sum_{i=1}^{N} \frac{\partial}{\partial x_i} \left( H \left( \frac{|x|}{|x|} \right) \frac{x_i}{|x|} \right)
\]

\[
= N \frac{H (|x|)}{|x|} + \frac{|x| H' (|x|) - H (|x|)}{|x|}
\]

\[
= H' (|x|) + (N-1) \frac{H (|x|)}{|x|}.
\]

Hence for \( u \in C^\infty_0 (B_R \setminus \{0\}) \), we have by the Divergence Theorem that

\[
- \int_{B_R} \left[ H' (|x|) + (N-1) \frac{H (|x|)}{|x|} \right] |u|^2 \, dx
\]

\[
= - \int_{B_R} \text{div} \left( H \left( \frac{|x|}{|x|} \right) \frac{x}{|x|} \right) |u|^2 \, dx
\]

\[
= \int_{B_R} H (|x|) \frac{x}{|x|} \cdot \nabla |u|^2 \, dx
\]

\[
= \int_{B_R} 2H (|x|) u (x) \left( \frac{x}{|x|} \cdot \nabla u (x) \right) \, dx
\]

\[
= \int_{B_R} 2H (|x|) u (x) R_u (x) \, dx. \tag{2.1}
\]

We will now apply the above identity for

\[
H (r) = A (r) B (r).
\]

Note that in this case,

\[
H' (r) + (N-1) \frac{H (r)}{r} = A' (r) B (r) + A (r) B' (r) + (N-1) \frac{A (r) B (r)}{r}
\]

\[
= \frac{1}{\alpha^2} B^2 (r) + \left[ C (r) + B^2 (r) - \frac{1}{\alpha^2} B^2 (r) \right]
\]

for some \( \alpha \neq 0 \). Here

\[
C (r) = (A (r) B (r))' + (N-1) \frac{A (r) B (r)}{r} - B^2 (r).
\]
Lemma 2.1. Hence, we get the following families of identities:

\[- \int_{B_R} \frac{1}{\alpha^2} B^2 (|x|) |u(x)|^2 \, dx - \int_{B_R} \left[ C(|x|) + B^2(|x|) - \frac{1}{\alpha^2} B^2(|x|) \right] |u(x)|^2 \, dx \]

\[= \int_{B_R} 2 \left( \frac{1}{\alpha} B(|x|) u(x) \right) \left( \alpha A(|x|) \mathcal{R} u(x) \right) \, dx. \tag{2.2} \]

Equivalently, by using the identity \(-2ab = a^2 + b^2 - (a + b)^2\):

\[\alpha^2 \int_{B_R} A^2(|x|) |\nabla u(x)|^2 \, dx \]

\[= \int_{B_R} \left[ C(|x|) + B^2(|x|) - \frac{1}{\alpha^2} B^2(|x|) \right] |u(x)|^2 \, dx + \int_{B_R} \left| \alpha A(|x|) \nabla u(x) + \frac{1}{\alpha} B(|x|) u(x) \frac{x}{|x|} \right|^2 \, dx \]

and

\[\alpha^2 \int_{B_R} A^2(|x|) |\mathcal{R} u(x)|^2 \, dx \]

\[= \int_{B_R} \left[ C(|x|) + B^2(|x|) - \frac{1}{\alpha^2} B^2(|x|) \right] |u(x)|^2 \, dx + \int_{B_R} \left| \alpha A(|x|) \mathcal{R} u(x) + \frac{1}{\alpha} B(|x|) u(x) \right|^2 \, dx. \]

Hence, we get the following families of identities:

**Lemma 2.1.** Let \(0 < R \leq \infty, A \) and \(B \) be \(C^1\)-functions on \((0, R)\) and let

\[C(r) = (A(r) B(r))' + (N - 1) \frac{A(r) B(r)}{r} - B^2(r). \]

Then for all \(\alpha \in \mathbb{R} \setminus \{0\} \) and \(u \in C_0^\infty (B_R \setminus \{0\})\), we have

\[\alpha^2 \int_{B_R} A^2(|x|) |\nabla u(x)|^2 \, dx + \frac{1}{\alpha^2} \int_{B_R} B^2(|x|) |u(x)|^2 \, dx \]

\[= \int_{B_R} \left[ C(|x|) + B^2(|x|) \right] |u|^2 \, dx + \int_{B_R} \left| \alpha A(|x|) \nabla u + \frac{1}{\alpha} B(|x|) u \frac{x}{|x|} \right|^2 \, dx \tag{2.3} \]

and

\[\alpha^2 \int_{B_R} A^2(|x|) |\mathcal{R} u(x)|^2 \, dx + \frac{1}{\alpha^2} \int_{B_R} B^2(|x|) |u(x)|^2 \, dx \]

\[= \int_{B_R} \left[ C(|x|) + B^2(|x|) \right] |u(x)|^2 \, dx + \int_{B_R} \left| \alpha A(|x|) \mathcal{R} u(x) + \frac{1}{\alpha} B(|x|) u(x) \frac{x}{|x|} \right|^2 \, dx. \tag{2.4} \]

By choosing \(\alpha = 1\), we obtain the \(L^2\)-Hardy identities while by optimizing \(\alpha\), we get the \(L^2\)-Caffarelli-Kohn-Nirenberg identities.
2.1. **L²-Hardy identities.** By choose $\alpha = 1$ in the Lemma [2.1], we have the following L²-Hardy identities

**Theorem 2.1.** Let $0 < R \leq \infty$, $A$ and $B$ be $C^1$-functions on $(0, R)$ and let

$$C (r) = (A (r) B (r))' + (N - 1) \frac{A (r) B (r)}{r} - B^2 (r).$$

Then for all $u \in C_0^\infty (B_R \setminus \{0\})$, we have

$$\int_{B_R} A^2 (|x|) |\nabla u|^2 \, dx = \int_{B_R} C (|x|) |u|^2 \, dx + \int_{B_R} |A (|x|) \nabla u + B (|x|) u \frac{x}{|x|}|^2 \, dx \quad (2.5)$$

and

$$\int_{B_R} A^2 (|x|) \nabla u^2 \, dx = \int_{B_R} C (|x|) |u|^2 \, dx + \int_{B_R} \left| A (|x|) \nabla u + B (|x|) u \frac{x}{|x|}\right|^2 \, dx. \quad (2.6)$$

Here are some examples that follow from Theorem 2.1 immediately by checking the appropriate pairs of $A$, $B$ and $C$.

**Corollary 2.1.** Let $A = 1$, $B = \frac{N - 2}{2} r$. Then $C = \left( \frac{N - 2}{2} r \right)^2$. Then by the Hardy identities (2.5) and (2.6), for all $u \in C_0^\infty (\mathbb{R}^N \setminus \{0\})$:

$$\int_{\mathbb{R}^N} |\nabla u|^2 \, dx = \left( \frac{N - 2}{2} \right)^2 \int_{\mathbb{R}^N} \frac{|u|^2}{|x|^2} \, dx + \int_{\mathbb{R}^N} |\nabla u |^2 \, dx$$

and

$$\int_{\mathbb{R}^N} \nabla u^2 \, dx = \left( \frac{N - 2}{2} \right)^2 \int_{\mathbb{R}^N} \frac{|u|^2}{|x|^2} \, dx + \int_{\mathbb{R}^N} \left| \nabla u + \frac{B (|x|) u x}{|x|} \right|^2 \, dx.$$
Corollary 2.3. $A = 1$, $B = r$, and $C = N - r^2$. By the Hardy identities (2.5) and (2.6), for all $u \in C_0^\infty (\mathbb{R}^N)$:
\[
\int_{\mathbb{R}^N} |\nabla u|^2 \, dx = \int_{\mathbb{R}^N} (N - |x|^2) |u|^2 \, dx + \int_{\mathbb{R}^N} |\nabla u + x|^2 \, dx.
\]

These identities imply the scale noninvariant Uncertainty Principle:
\[
\int_{\mathbb{R}^N} |\nabla u|^2 \, dx + \int_{\mathbb{R}^N} |x|^2 |u|^2 \, dx \geq N \int_{\mathbb{R}^N} |u|^2 \, dx.
\]

Corollary 2.4. If $a + b \neq N - 1$, choose $A = \text{sign} (N - a - b - 1) r^{-b}$, $B = r^{-a}$. Then $C = [(N - 1 - a - b) r^{-a - b - 1} - r^{-2a}]$. By the Hardy identities (2.5) and (2.6), for all $u \in C_0^\infty (\mathbb{R}^N)$:
\[
\int_{B_R} |\nabla u|^2 \, dx + \int_{B_R} |u|^2 \, dx - |N - 1 - a - b| \int_{B_R} \frac{|u|^2}{|x|^{a+b+1}} \, dx
\]
\[
= \int_{B_R} \left| \text{sign} (N - a - b - 1) \frac{\nabla u}{|x|^b} + \frac{u}{|x|^a} \right|^2 \, dx
\]
and
\[
\int_{B_R} |\nabla u|^2 \, dx + \int_{B_R} |u|^2 \, dx - |N - 1 - a - b| \int_{B_R} \frac{|u|^2}{|x|^{a+b+1}} \, dx
\]
\[
= \int_{B_R} \left| \text{sign} (N - a - b - 1) \frac{\nabla u}{|x|^b} + \frac{u}{|x|^a} \frac{x}{|x|} \right|^2 \, dx.
\]

Corollary 2.5 (Hardy inequalities with Bessel pairs). Let $0 < R \leq \infty$, $V \geq 0$ and $W$ be $C^1$-functions on $(0, R)$. Assume that $(r^{N-1} V, r^{N-1} W)$ is a Bessel pair on $(0, R)$, that is there exists a positive function $\varphi$ such that
\[
(r^{N-1} V \varphi')' + r^{N-1} W \varphi = 0 \text{ on } (0, R).
\]

In this case, choose $A = \sqrt{V}$, $B = -\frac{\varphi'}{\varphi} \sqrt{V}$. Then $C = W$. Hence by the Hardy identities (2.5) and (2.6), we deduce for all $u \in C_0^\infty (B_R \setminus \{0\})$:
\[
\int_{B_R} V(|x|) |\nabla u|^2 \, dx = \int_{B_R} W(|x|) |u|^2 \, dx + \int_{B_R} \sqrt{V(|x|)} |\nabla u - \frac{\varphi'(|x|)}{\varphi(|x|)} \sqrt{V(|x|)} u|^2 \, dx
\]
\[
= \int_{B_R} W(|x|) |u|^2 \, dx + \int_{B_R} V(|x|) \varphi^2(|x|) \left| \nabla \left( \frac{u(x)}{\varphi(|x|)} \right) \right|^2 \, dx.
\]
and
\[
\begin{align*}
\int_{B_R} V(|x|)|\nabla u|^2 \, dx &= \int_{B_R} W(|x|)|u|^2 \, dx + \int_{B_R} \sqrt{V(|x|)} \nabla u - \frac{\varphi'(|x|)}{\varphi(|x|)} \sqrt{V(|x|)} u \frac{x}{|x|}^2 \, dx \\
&= \int_{B_R} W(|x|)|u|^2 \, dx + \int_{B_R} V(|x|) \varphi^2(|x|) \left| \nabla \left( \frac{u(x)}{\varphi(|x|)} \right) \right|^2 \, dx.
\end{align*}
\]

We note that Hardy type inequalities and identities have been investigated by many researchers. See, \cite{28, 39, 36}, for instance. We also refer the reader to the papers \cite{45, 46} in which the Hardy type inequalities and identities have been studied for more general distance functions.

### 2.2. $L^2$-Caffarelli-Kohn-Nirenberg identities

By optimizing $\alpha$ in the Lemma \ref{lemma:1}, that is by choosing
\[
\alpha \left( \int_{B_R} A^2(|x|)|\mathcal{R}u|^2 \, dx \right)^{\frac{1}{2}} = \frac{1}{\alpha} \left( \int_{B_R} B^2(|x|)|u|^2 \, dx \right)^{\frac{1}{2}}
\]
in \eqref{1.10} and
\[
\alpha \left( \int_{B_R} A^2(|x|)|\nabla u|^2 \, dx \right)^{\frac{1}{2}} = \frac{1}{\alpha} \left( \int_{B_R} B^2(|x|)|u|^2 \, dx \right)^{\frac{1}{2}}
\]
in \eqref{1.11}, we have the following Caffarelli-Kohn-Nirenberg identities

**Theorem 2.2.** Let $0 < R \leq \infty$, $A$ and $B$ be $C^1$-functions on $(0, R)$ and let
\[
C(r) = (A(r)B(r))' + (N - 1) \frac{A(r)B(r)}{r} - B^2(r).
\]
Then for all $u \in C_0^\infty(B_R \setminus \{0\})$, we have
\[
\begin{align*}
\left( \int_{B_R} A^2(|x|)|\mathcal{R}u|^2 \, dx \right)^{\frac{1}{2}} &\left( \int_{B_R} B^2(|x|)|u|^2 \, dx \right)^{\frac{1}{2}} \\
&= \frac{1}{2} \int_{B_R} \left[ C(|x|) + B^2(|x|) \right] |u|^2 \, dx \\
&\quad + \frac{1}{2} \int_{B_R} \|Bu\|_{\frac{3}{2}} \frac{1}{\|A\mathcal{R}u\|_{\frac{3}{2}}} A(|x|) \mathcal{R}u + \|A\mathcal{R}u\|_{\frac{3}{2}} B(|x|) u \right|^2 \, dx
\end{align*}
\]
and
\[
\begin{align*}
\left( \int_{B_R} A^2(|x|)|\nabla u|^2 \, dx \right)^{\frac{1}{2}} &\left( \int_{B_R} B^2(|x|)|u|^2 \, dx \right)^{\frac{1}{2}} \\
&= \frac{1}{2} \int_{B_R} \left[ C(|x|) + B^2(|x|) \right] |u|^2 \, dx
\end{align*}
\]
\[ + \frac{1}{2} \int_{\mathbb{R}^N} \left( \int_{\mathbb{R}^N} |\mathcal{R}u|^2 \, dx \right)^\frac{1}{2} \left( \int_{\mathbb{R}^N} |x|^2 |u|^2 \, dx \right)^\frac{1}{2} - \frac{N}{2} \int_{\mathbb{R}^N} |u|^2 \, dx \]

and

\[ = \frac{1}{2} \int_{\mathbb{R}^N} \left( \int_{\mathbb{R}^N} |\nabla u|^2 \, dx \right)^\frac{1}{2} \left( \int_{\mathbb{R}^N} |x|^2 |u|^2 \, dx \right)^\frac{1}{2} - \frac{N}{2} \int_{\mathbb{R}^N} |u|^2 \, dx \]

\[ = \frac{1}{2} \int_{\mathbb{R}^N} \left( \int_{\mathbb{R}^N} |\nabla u|^2 \, dx \right)^\frac{1}{2} \left( \int_{\mathbb{R}^N} |x|^2 |u|^2 \, dx \right)^\frac{1}{2} - \frac{N}{2} \int_{\mathbb{R}^N} |u|^2 \, dx \]

\[ = \frac{\lambda^2}{2} \int_{\mathbb{R}^N} \left( \left( u e^{-2|2x|^2} \right) \right)^2 \, dx. \]

Here \( \lambda = \left( \int_{\mathbb{R}^N} |x|^2 |u|^2 \, dx \right)^\frac{1}{4} \left( \int_{\mathbb{R}^N} |\nabla u|^2 \, dx \right)^\frac{1}{4}. \)

**Corollary 2.6.** Choose \( A = 1, \) \( B = r. \) Then \( H = r, \) \( H'(r) + (N - 1) \frac{H(r)}{r} = N \) and \( C = N - r^2. \) From the Caffarelli-Kohn-Nirenberg identities, we have the scale invariant Heisenberg Uncertainty Principles:

\[ \left( \int_{\mathbb{R}^N} |\mathcal{R}u|^2 \, dx \right)^\frac{1}{2} \left( \int_{\mathbb{R}^N} |x|^2 |u|^2 \, dx \right)^\frac{1}{2} - \frac{N}{2} \int_{\mathbb{R}^N} |u|^2 \, dx \]

\[ = \frac{1}{2} \int_{\mathbb{R}^N} \left( \int_{\mathbb{R}^N} |\nabla u|^2 \, dx \right)^\frac{1}{2} \left( \int_{\mathbb{R}^N} |x|^2 |u|^2 \, dx \right)^\frac{1}{2} - \frac{N}{2} \int_{\mathbb{R}^N} |u|^2 \, dx \]

\[ = \frac{\lambda^2}{2} \int_{\mathbb{R}^N} \left( \left( u e^{-2|2x|^2} \right) \right)^2 \, dx. \]

**Corollary 2.7.** If \( a + b \neq N - 1, \) choose \( A = \text{sign} \,(N - a - b - 1) \, r^{-b}, \) \( B = r^{-a}. \) Then \( C = \left[ |N - 1 - a - b| \, r^{-a+b-1} - r^{-2a} \right]. \) From the Caffarelli-Kohn-Nirenberg identities, we have

\[ \left( \int_{\mathbb{R}^N} \frac{1}{|x|^{2b}} |\mathcal{R}u|^2 \, dx \right)^\frac{1}{2} \left( \int_{\mathbb{R}^N} \frac{1}{|x|^{2a}} |u|^2 \, dx \right)^\frac{1}{2} \]

\[ = \left| \frac{N - 1 - a - b}{2} \right| \int_{\mathbb{R}^N} \frac{1}{|x|^{a+b+1}} |u|^2 \, dx \]

\[ + \frac{1}{2} \int_{\mathbb{R}^N} \left( \text{sign} \,(N - a - b - 1) \frac{1}{|x|^{a+b}} \right)^\frac{1}{2} \frac{1}{|x|^{a}} \mathcal{R}u + \frac{\mathcal{R}u}{|x|^{a}} \left( \frac{1}{|x|^{a}} \right)^\frac{1}{2} \frac{1}{|x|^{b}} |u|^2 \, dx. \]
and
\[
\left( \int_{\mathbb{R}^N} \frac{1}{|x|^{2b}} |\nabla u|^2 \right)^{\frac{1}{2}} \left( \int_{\mathbb{R}^N} \frac{1}{|x|^{2a}} |u|^2 \right)^{\frac{1}{2}} \\
= \left| N - 1 - a - b \right| \int_{\mathbb{R}^N} \frac{1}{|x|^{a+b+1}} |u|^2 \, dx \\
+ \frac{1}{2} \int_{\mathbb{R}^N} \text{sign} \, (N - a - b - 1) \left[ \left\| \frac{u}{|x|} \right\|_2^{\frac{1}{2}} \frac{1}{|x|} \nabla u + \left\| \frac{\nabla u}{|x|^{n-1}} \right\|_2^{\frac{1}{2}} \frac{1}{|x|^{a}} u \right] dx.
\]

These identities imply
\[
\left( \int_{\mathbb{R}^N} \frac{1}{|x|^{2b}} |\nabla u|^2 \, dx \right)^{\frac{1}{2}} \left( \int_{\mathbb{R}^N} \frac{1}{|x|^{2a}} |u|^2 \, dx \right)^{\frac{1}{2}} \\
\geq \left( \int_{\mathbb{R}^N} \frac{1}{|x|^{2b}} |\mathcal{R} u|^2 \, dx \right)^{\frac{1}{2}} \left( \int_{\mathbb{R}^N} \frac{1}{|x|^{2a}} |u|^2 \, dx \right)^{\frac{1}{2}} \\
\geq \frac{1}{2} \left( \int_{\mathbb{R}^N} \frac{1}{|x|^{a+b+1}} |u|^2 \, dx \right)^{\frac{1}{2}}.
\]

**Corollary 2.8** (Caffarelli-Kohn-Nirenberg inequalities with Bessel pairs). Let \(0 < R \leq \infty, V\) and \(W\) be positive \(C^1\)-functions on \((0, R)\). If \((r^{N-1}V, r^{N-1}W)\) is a Bessel pair on \((0, R)\), that is there exists \(\varphi > 0\) on \((0, R)\) such that
\[
(r^{N-1}V \varphi')' + r^{N-1}W \varphi = 0.
\]

Then as above, we choose \(A = \sqrt{V}, B = -\frac{\varphi'}{\varphi} \sqrt{V}, C = W\). Then, we have from the Caffarelli-Kohn-Nirenberg identities that
\[
\left( \int_{B_R} V(|x|) |\mathcal{R} u|^2 \, dx \right)^{\frac{1}{2}} \left( \int_{B_R} \left( \frac{\varphi'}{\varphi} \left( \frac{|x|}{|x|} \right) \right)^2 V(|x|) |u|^2 \, dx \right)^{\frac{1}{2}} \\
= \frac{1}{2} \int_{B_R} \left[ W(|x|) + \left( \frac{\varphi'}{\varphi} \left( \frac{|x|}{|x|} \right) \right)^2 V(|x|) \right] |u|^2 \, dx \\
+ \frac{1}{2} \int_{B_R} \left[ \frac{\varphi'}{\varphi} \sqrt{V} u \right]^{\frac{1}{2}} \sqrt{V(|x|)} \mathcal{R} u - \left[ \frac{\varphi'}{\varphi} \sqrt{V} u \right]^{\frac{1}{2}} \frac{\varphi'}{\varphi} \left( \frac{|x|}{|x|} \right) \sqrt{V(|x|)} u \right] \, dx
\]
and
\[
\left( \int_{B_R} V(|x|) |\nabla u|^2 \, dx \right)^{\frac{1}{2}} \left( \int_{B_R} \left( \frac{\varphi'}{\varphi} \left( \frac{|x|}{|x|} \right) \right)^2 V(|x|) |u|^2 \, dx \right)^{\frac{1}{2}} \\
= \frac{1}{2} \int_{B_R} \left[ W(|x|) + \left( \frac{\varphi'}{\varphi} \left( \frac{|x|}{|x|} \right) \right)^2 V(|x|) \right] |u|^2 \, dx
\]
Corollary

Our goal is to apply the Poincaré inequality to the right hand side of (3.1).

Nirenberg inequalities. Of the Heisenberg Uncertainty Principle, which is a special case of the Caffarelli-Kohn-Nirenberg inequalities. To illustrate our approach, we will start with the stability previous section and the Poincaré inequality to investigate the stability of the Caffarelli-Kohn-Nirenberg inequalities. We will need the following classical Poincaré inequality for Gaussian measures (see [14, CRISTIAN CAZACU, JOSHUA FLYNN, NGUYEN LAM, AND GUOZHEN LU

In this section, we will use the Caffarelli-Kohn-Nirenberg identities derived in the previous section and the Poincaré inequality to investigate the stability of the Caffarelli-Kohn-Nirenberg inequalities. To illustrate our approach, we will start with the stability of the Heisenberg Uncertainty Principle, which is a special case of the Caffarelli-Kohn-Nirenberg inequalities.

3. The stability of the Caffarelli-Kohn-Nirenberg inequalities

In this section, we will use the Caffarelli-Kohn-Nirenberg identities derived in the previous section and the Poincaré inequality to investigate the stability of the Caffarelli-Kohn-Nirenberg inequalities. To illustrate our approach, we will start with the stability of the Heisenberg Uncertainty Principle, which is a special case of the Caffarelli-Kohn-Nirenberg inequalities.

3.1. The stability of the Heisenberg Uncertainty Principle. Recall that from the Corollary [2,1], we have that for \( u \in X := W^{1,2}(\mathbb{R}^N) \cap \left\{ u : \int_{\mathbb{R}^N} |x|^2 |u|^2 \, dx < \infty \right\} \),

\[
\int_{\mathbb{R}^N} |\nabla u|^2 \, dx + \int_{\mathbb{R}^N} |x|^2 |u|^2 \, dx - N \int_{\mathbb{R}^N} |u|^2 \, dx = \int_{\mathbb{R}^N} \left| \nabla \left( u e^{\frac{1}{2} |x|^2} \right) \right|^2 e^{-|x|^2} \, dx. \tag{3.1}
\]

Our goal is to apply the Poincaré inequality to the right hand side of (3.1). Therefore, we will need the following classical Poincaré inequality for Gaussian measure (see [1, 4, for instance]):

**Lemma 3.1.** For all smooth function \( v \):

\[
\int_{\mathbb{R}^N} |\nabla v|^2 (2\pi)^{-\frac{N}{2}} e^{-\frac{1}{2}|x|^2} \, dx \geq \int_{\mathbb{R}^N} \left| v - (2\pi)^{-\frac{N}{2}} \int_{\mathbb{R}^N} ve^{-\frac{1}{2}|x|^2} \, dx \right|^2 (2\pi)^{-\frac{N}{2}} e^{-\frac{1}{2}|x|^2} \, dx.
\]

Obviously, the above classical Poincaré inequality for Gaussian measure implies the following Poincaré inequality for Gaussian type measure

**Lemma 3.2.** Let \( \lambda > 0 \). For all smooth function \( v \):

\[
\int_{\mathbb{R}^N} |\nabla v|^2 e^{-\frac{|x|^2}{\lambda^2}} \, dx \geq \frac{2}{\lambda^2} \inf_{c} \int_{\mathbb{R}^N} |v - c|^2 e^{-\frac{|x|^2}{\lambda^2}} \, dx. \tag{3.2}
\]
Hence, by combining (3.1) and (3.2), we get
\[
\int_{\mathbb{R}^N} |\nabla u|^2 \, dx + \int_{\mathbb{R}^N} |x|^2 |u|^2 \, dx - N \int_{\mathbb{R}^N} |u|^2 \, dx
= \int_{\mathbb{R}^N} \left| \nabla \left( u e^{\frac{1}{2} |x|^2} \right) \right|^2 e^{-|x|^2} \, dx
\geq 2 \inf_{c} \int_{\mathbb{R}^N} \left| u e^{\frac{1}{2} |x|^2} - c \right|^2 e^{-|x|^2} \, dx
= 2 \inf_{c} \int_{\mathbb{R}^N} \left| u - ce^{-\frac{1}{2} |x|^2} \right|^2 \, dx.
\]

Thus, we obtain the following stability version of the scale non-invariant Heisenberg Uncertainty Principle

**Theorem 3.1.** For \( u \in X \), then
\[
\int_{\mathbb{R}^N} |\nabla u|^2 \, dx + \int_{\mathbb{R}^N} |x|^2 |u|^2 \, dx - N \int_{\mathbb{R}^N} |u|^2 \, dx
\geq 2 \inf_{c} \int_{\mathbb{R}^N} \left| u - ce^{-\frac{1}{2} |x|^2} \right|^2 \, dx.
\]

The constant 2 is sharp and can be achieved by nontrivial functions.

**Proof.** Let \( u = x_1 e^{-\frac{1}{2} |x|^2} \). Then by direct computations, we get
\[
\int_{\mathbb{R}^N} |\nabla u|^2 \, dx + \int_{\mathbb{R}^N} |x|^2 |u|^2 \, dx - N \int_{\mathbb{R}^N} |u|^2 \, dx = \pi^\frac{N}{2}
\]
and
\[
\inf_{c} \int_{\mathbb{R}^N} \left| u - ce^{-\frac{1}{2} |x|^2} \right|^2 \, dx
= \inf_{c} \left( \pi^\frac{N}{2} + c^2 \pi^\frac{N}{2} \right)
= \pi^\frac{N}{2}
\]
Therefore,
\[
\int_{\mathbb{R}^N} |\nabla u|^2 \, dx + \int_{\mathbb{R}^N} |x|^2 |u|^2 \, dx - N \int_{\mathbb{R}^N} |u|^2 \, dx = 2 \inf_{c} \int_{\mathbb{R}^N} \left| u - ce^{-\frac{1}{2} |x|^2} \right|^2 \, dx.
\]

Now, let \( E = \{ \alpha e^{-\frac{1}{2} |x|^2}, \alpha \in \mathbb{R}, \beta > 0 \} \) be the manifold of optimizers of the scale invariant Heisenberg Uncertainty Principle
\[
\left( \int_{\mathbb{R}^N} |\nabla u|^2 \, dx \right) \left( \int_{\mathbb{R}^N} |x|^2 |u|^2 \, dx \right) \geq \frac{N^2}{4} \left( \int_{\mathbb{R}^N} |u|^2 \, dx \right)^2.
\]
It was showed in [51] that $E$ forms a closed cone in $L^2(\mathbb{R}^N)$. In particular, for all $u \in X$, there exists a $u^* \in E$ such that
\[
\inf_{v \in E} \|u - v\|_2 = \|u - u^*\|_2.
\]

We also recall the Heisenberg deficit
\[
\delta_1(u) = \left( \int_{\mathbb{R}^N} |\nabla u|^2 \, dx \right)^{\frac{1}{2}} \left( \int_{\mathbb{R}^N} |x|^2 |u|^2 \, dx \right)^{\frac{1}{2}} - \frac{N}{2} \int_{\mathbb{R}^N} |u|^2 \, dx.
\]

Then, by using the Corollary 2.6, we get the quantitative stability of the scale invariant Heisenberg Uncertainty Principle, namely, Theorem 1.2:

**Theorem 3.2.** For all $u \in X$ :
\[
\delta_1(u) \geq \inf_{u^* \in E} \|u - u^*\|_2^2.
\]

Moreover, the inequality is sharp and the equality can be attained by nontrivial functions.

**Proof.** Let $u \in X \setminus \{0\}$. By Corollary 2.6, we obtain from the Poincaré inequality for Gaussian type measure (3.2) with $\lambda = \left( \frac{\int_{\mathbb{R}^N} |x|^2 |u|^2 \, dx}{\int_{\mathbb{R}^N} |\nabla u|^2 \, dx} \right)^{\frac{1}{2}}$ that
\[
\left( \int_{\mathbb{R}^N} |\nabla u|^2 \, dx \right)^{\frac{1}{2}} \left( \int_{\mathbb{R}^N} |x|^2 |u|^2 \, dx \right)^{\frac{1}{2}} - \frac{N}{2} \int_{\mathbb{R}^N} |u|^2 \, dx
\]
\[
= \frac{\lambda^2}{2} \int_{\mathbb{R}^N} |\nabla \left( u e^{-\frac{1}{2}|x|^2}\right)|^2 e^{-\frac{|x|^2}{\lambda^2}} \, dx
\]
\[
\geq \inf_{c} \int_{\mathbb{R}^N} \left| u - c e^{-\frac{1}{2\pi |x|^2}} \right|^2 \, dx
\]
\[
\geq \inf_{u^* \in E} \|u - u^*\|_2^2.
\]

Now, let $u = x_1 e^{-\frac{1}{2}|x|^2}$. Then by direct computations:
\[
\nabla u = \langle 1 - x_1^2, -x_1 x_2, \ldots, -x_1 x_N \rangle e^{-\frac{1}{2}|x|^2},
\]
and
\[
\int_{\mathbb{R}^N} |\nabla u|^2 \, dx = \int_{\mathbb{R}^N} |x|^2 |u|^2 \, dx = \frac{N + 2}{4} \pi^{\frac{N}{2}}.
\]

Therefore,
\[
\left( \int_{\mathbb{R}^N} |\nabla u|^2 \, dx \right)^{\frac{1}{2}} \left( \int_{\mathbb{R}^N} |x|^2 |u|^2 \, dx \right)^{\frac{1}{2}} - \frac{N}{2} \int_{\mathbb{R}^N} |u|^2 \, dx
\]
\[
= \frac{1}{2} \pi^{\frac{N}{2}}.
\]

Also,
\[
\int_{\mathbb{R}^N} \left| u - c e^{-\frac{1}{2\pi |x|^2}} \right|^2 \, dx = \frac{1}{2} \pi^{\frac{N}{2}} + |c|^2 \lambda^{\frac{N}{2}}.
\]
That is
\[ \inf_{u^* \in E} \| u - u^* \|_2^2 = \frac{1}{2} \pi^{\frac{N}{2}}. \]

Therefore,
\[
\left( \int_{\mathbb{R}^N} |\nabla u|^2 \, dx \right)^{\frac{1}{2}} \left( \int_{\mathbb{R}^N} |x|^2 |u|^2 \, dx \right)^{\frac{1}{2}} - \frac{N}{2} \int_{\mathbb{R}^N} |u|^2 \, dx = \inf_{u^* \in E} \| u - u^* \|_2^2. \]

\[ \square \]

A more careful analysis leads to the following version of the quantitative stability of the scale invariant Heisenberg Uncertainty Principle, namely, Theorem 1.2:

**Theorem 3.3.** For all \( u \in X \):
\[
\delta_1 (u) \geq \frac{1}{2} \inf_{u^* \in E} \left\{ \int_{\mathbb{R}^N} |u - u^*|^2 \, dx : \int_{\mathbb{R}^N} |u|^2 \, dx = \int_{\mathbb{R}^N} |u^*|^2 \, dx \right\}.
\]

Moreover, the inequality is sharp and the equality can be attained by nontrivial functions.

**Proof.** WLOG, we can assume that \( \int_{\mathbb{R}^N} |u|^2 \, dx = 1 \). Now, if
\[
\delta_1 (u) = \left( \int_{\mathbb{R}^N} |\nabla u|^2 \, dx \right)^{\frac{1}{2}} \left( \int_{\mathbb{R}^N} |x|^2 |u|^2 \, dx \right)^{\frac{1}{2}} - \frac{N}{2} \int_{\mathbb{R}^N} |u|^2 \, dx < 1,
\]
then by Theorem 3.2, we have
\[
\left( \int_{\mathbb{R}^N} |\nabla u|^2 \, dx \right)^{\frac{1}{2}} \left( \int_{\mathbb{R}^N} |x|^2 |u|^2 \, dx \right)^{\frac{1}{2}} - \frac{N}{2} \int_{\mathbb{R}^N} |u|^2 \, dx \geq \inf_{z \in E} \int_{\mathbb{R}^N} |u - z|^2 \, dx.
\]

Moreover, it was showed in [51] that \( E \) is a closed cone (in the \( L^2 \)-norm). Therefore, we can find \( v \in E \) such that \( \inf_{z \in E} \int_{\mathbb{R}^N} |u - z|^2 \, dx = \int_{\mathbb{R}^N} |u - v|^2 \, dx \). Thus,
\[
\int_{\mathbb{R}^N} |u - v|^2 \, dx \leq \delta_1 (u) < 1.
\]

Therefore, since \( \int_{\mathbb{R}^N} |u|^2 \, dx = 1 \), we deduce that \( \int_{\mathbb{R}^N} |v|^2 \, dx \neq 0 \). Let \( \lambda = \left( \int_{\mathbb{R}^N} |v|^2 \, dx \right)^{-\frac{1}{2}} > 0 \). Then \( w = \lambda v \in E \) and \( \int_{\mathbb{R}^N} |w|^2 \, dx = 1 \). Also,
\[
\int_{\mathbb{R}^N} |u - w|^2 \, dx = \int_{\mathbb{R}^N} |u - \lambda v|^2 \, dx = 2 - 2\lambda \int_{\mathbb{R}^N} uv \, dx.
\]
Note that since \( \int_{\mathbb{R}^N} |u - v|^2 \, dx \leq \delta_1 (u) < 1 \), we obtain
\[
1 - 2 \int_{\mathbb{R}^N} uv \, dx + \frac{1}{\lambda^2} \leq \delta_1 (u) < 1
\]
and
\[
0 < \frac{1}{2\lambda^2} \leq \int_{\mathbb{R}^N} uv \, dx.
\]
Now, to show that
\[
\delta_1 (u) \geq \frac{1}{2} \inf_{w^* \in E} \left\{ \int_{\mathbb{R}^N} |u - u^*|^2 dx : \int_{\mathbb{R}^N} |u|^2 dx = \int_{\mathbb{R}^N} |u^*|^2 dx \right\},
\]
it’s enough to prove that
\[
\delta_1 (u) \geq 1 - 2 \int_{\mathbb{R}^N} uv \, dx + \frac{1}{\lambda^2}
\]
\[
\geq 1 - \lambda \int_{\mathbb{R}^N} uv \, dx
\]
\[
= \frac{1}{2} \int_{\mathbb{R}^N} |u - w|^2 \, dx.
\]
That is
\[
(2 - \lambda) \int_{\mathbb{R}^N} uv \, dx \leq \frac{1}{\lambda^2}.
\]
Indeed, note that by the Hölder inequality:
\[
\lambda \int_{\mathbb{R}^N} uv \, dx = \int_{\mathbb{R}^N} uv \, dx \leq 1,
\]
we have
\[
\frac{1}{2\lambda^2} \leq \int_{\mathbb{R}^N} uv \, dx \leq \frac{1}{\lambda}.
\]
If \( \lambda > 2 \), then \( (2 - \lambda) \int_{\mathbb{R}^N} uv \, dx < 0 < \frac{1}{\lambda^2} \). That is \( 1 - \lambda \int_{\mathbb{R}^N} uv \, dx \leq 1 - 2 \int_{\mathbb{R}^N} uv \, dx + \frac{1}{\lambda^2} \leq \delta_1 (u) \).
If \( 0 < \lambda \leq 2 \), then
\[
(2 - \lambda) \int_{\mathbb{R}^N} uv \, dx \leq (2 - \lambda) \frac{1}{\lambda}
\]
\[
= \frac{2}{\lambda} - 1
\]
\[
\leq \frac{1}{\lambda^2}.
\]
If
\[
\delta_1 (u) = \left( \int_{\mathbb{R}^N} |\nabla u|^2 \, dx \right)^{1/2} \left( \int_{\mathbb{R}^N} |x|^2 |u|^2 \, dx \right)^{1/2} - \frac{N}{2} \int_{\mathbb{R}^N} |u|^2 \, dx \geq 1,
\]
then since \( \int_{\mathbb{R}^N} |u - u^*|^2 \, dx + \int_{\mathbb{R}^N} |u + u^*|^2 \, dx = 2 \left( \int_{\mathbb{R}^N} |u|^2 \, dx + \int_{\mathbb{R}^N} |u^*|^2 \, dx \right) \), we get

\[
\inf_{u^* \in E} \left\{ \int_{\mathbb{R}^N} |u - u^*|^2 \, dx : \int_{\mathbb{R}^N} |u|^2 \, dx = \int_{\mathbb{R}^N} |u^*|^2 \, dx = 1 \right\}
\leq \frac{1}{2} \inf_{u^* \in E} \left\{ \int_{\mathbb{R}^N} |u - u^*|^2 \, dx + \int_{\mathbb{R}^N} |u + u^*|^2 \, dx : \int_{\mathbb{R}^N} |u|^2 \, dx = \int_{\mathbb{R}^N} |u^*|^2 \, dx = 1 \right\} = 2.
\]

Therefore

\[
\delta_1 (u) \geq \frac{1}{2} \inf_{u^* \in E} \left\{ \int_{\mathbb{R}^N} |u - u^*|^2 \, dx : \int_{\mathbb{R}^N} |u|^2 \, dx = \int_{\mathbb{R}^N} |u^*|^2 \, dx \right\}.
\]

Now, let \( u = x_1 e^{-\frac{1}{2} |x|^2} \). Then as in the proof of Theorem 3.2, we get

\[
\int_{\mathbb{R}^N} |\nabla u|^2 \, dx = \int_{\mathbb{R}^N} |x|^2 |u|^2 \, dx = \frac{N + 2}{4} \pi^\frac{N}{2}
\]

and

\[
\left( \int_{\mathbb{R}^N} |\nabla u|^2 \, dx \right)^{\frac{1}{2}} \left( \int_{\mathbb{R}^N} |x|^2 |u|^2 \, dx \right)^{\frac{1}{2}} - \frac{N}{2} \int_{\mathbb{R}^N} |u|^2 \, dx = \frac{1}{2} \pi^\frac{N}{2}.
\]

Also,

\[
\int_{\mathbb{R}^N} \left| u - ce^{-\frac{1}{2} |x|^2} \right|^2 \, dx = \frac{1}{2} \pi^\frac{N}{2} + |c|^2 \lambda^N \pi^\frac{N}{2}.
\]

Note that

\[
\int_{\mathbb{R}^N} \left| ce^{-\frac{1}{2} |x|^2} \right|^2 \, dx = |c|^2 \lambda^N \pi^\frac{N}{2}.
\]

Hence

\[
\inf_{u^* \in E} \left\{ \int_{\mathbb{R}^N} |u - u^*|^2 \, dx : \int_{\mathbb{R}^N} |u|^2 \, dx = \int_{\mathbb{R}^N} |u^*|^2 \, dx \right\}
= \inf_{c, \lambda} \left\{ \frac{1}{2} \pi^\frac{N}{2} + |c|^2 \lambda^N \pi^\frac{N}{2} : \frac{1}{2} \pi^\frac{N}{2} = |c|^2 \lambda^N \pi^\frac{N}{2} \right\}
= \frac{\pi^\frac{N}{2}}{2}.
\]

Therefore,

\[
\left( \int_{\mathbb{R}^N} |\nabla u|^2 \, dx \right)^{\frac{1}{2}} \left( \int_{\mathbb{R}^N} |x|^2 |u|^2 \, dx \right)^{\frac{1}{2}} - \frac{N}{2} \int_{\mathbb{R}^N} |u|^2 \, dx
= \frac{1}{2} \pi^\frac{N}{2}.
\]
\[= \frac{1}{2} \inf_{u^* \in E} \left\{ \int_{\mathbb{R}^N} |u - u^*|^2 \, dx : \int_{\mathbb{R}^N} |u|^2 \, dx = \int_{\mathbb{R}^N} |u^*|^2 \, dx \right\}.\]

Next, we note that it was showed by McCurdy and Venkatraman in [51] that there is no quantitative stability version of the scale invariant Heisenberg Uncertainty Principle if we use the norm \(\|\nabla (u - u^*)\|_2\) or \(\|x (u - u^*)\|_2\) on the right hand side. In particular, there is no \(C > 0\) such that

\[
\left( \int_{\mathbb{R}^N} |\nabla u|^2 \, dx \right)^{1/2} \left( \int_{\mathbb{R}^N} |x| \, |u|^2 \, dx \right)^{1/2} - \frac{N}{2} \int_{\mathbb{R}^N} |u|^2 \, dx
\geq C \inf_{u^* \in E} \int_{\mathbb{R}^N} |\nabla (u - u^*)|^2 \, dx + \int_{\mathbb{R}^N} |x|^2 |u - u^*|^2 \, dx + \int_{\mathbb{R}^N} |u - u^*|^2 \, dx.
\]

We now show that this is not the case for the scale non-invariant Heisenberg Uncertainty Principle, namely, Theorem 1.4:

**Theorem 3.4.** For all \(u \in X\), then

\[
\int_{\mathbb{R}^N} |\nabla u|^2 \, dx + \int_{\mathbb{R}^N} |x|^2 |u|^2 \, dx - N \int_{\mathbb{R}^N} |u|^2 \, dx
= \int_{\mathbb{R}^N} \left| \nabla \left( u e^{-\frac{1}{2} |x|^2} \right) \right|^2 e^{-|x|^2} \, dx
\geq \frac{2}{N + 3} \inf_{c \in \mathbb{R}} \left( \int_{\mathbb{R}^N} \left| \nabla \left( u - ce^{-\frac{1}{2} |x|^2} \right) \right|^2 \, dx + \int_{\mathbb{R}^N} |x|^2 |u - ce^{-\frac{1}{2} |x|^2}|^2 \, dx + \int_{\mathbb{R}^N} |u - ce^{-\frac{1}{2} |x|^2}|^2 \, dx \right).
\]

The inequality is sharp and the equality can be attained by nontrivial functions.

**Proof.** Indeed, let \(u = ve^{-\frac{1}{2} |x|^2}\), then

\[
\int_{\mathbb{R}^N} \left| \nabla \left( u - ce^{-\frac{1}{2} |x|^2} \right) \right|^2 \, dx
= \int_{\mathbb{R}^N} \left| \nabla \left[ (v - c) e^{-\frac{1}{2} |x|^2} \right] \right|^2 \, dx
= \int_{\mathbb{R}^N} |\nabla v|^2 e^{-|x|^2} \, dx - 2 \int_{\mathbb{R}^N} (v - c) x \cdot \nabla (v - c) e^{-|x|^2} \, dx + \int_{\mathbb{R}^N} |x|^2 |v - c|^2 e^{-|x|^2} \, dx
= \int_{\mathbb{R}^N} |\nabla v|^2 e^{-|x|^2} \, dx + N \int_{\mathbb{R}^N} |v - c|^2 e^{-|x|^2} \, dx - \int_{\mathbb{R}^N} |x|^2 |v - c|^2 e^{-|x|^2} \, dx.
\]

Therefore

\[
\int_{\mathbb{R}^N} \left| \nabla \left( u - ce^{-\frac{1}{2} |x|^2} \right) \right|^2 \, dx + \int_{\mathbb{R}^N} |x|^2 |u - ce^{-\frac{1}{2} |x|^2}|^2 \, dx + \int_{\mathbb{R}^N} |u - ce^{-\frac{1}{2} |x|^2}|^2 \, dx
= \int_{\mathbb{R}^N} \left| \nabla \left[ (v - c) e^{-\frac{1}{2} |x|^2} \right] \right|^2 \, dx + \int_{\mathbb{R}^N} |x|^2 |v - c|^2 e^{-|x|^2} \, dx + \int_{\mathbb{R}^N} |v - c|^2 e^{-|x|^2} \, dx
\]
\[
\begin{align*}
&= \int_\mathbb{R}^N |\nabla v|^2 e^{-|x|^2} \, dx + (N + 1) \int_\mathbb{R}^N |v - c|^2 e^{-|x|^2} \, dx.
\end{align*}
\]

Noting that by the Poincaré inequality, we get
\[
\int_\mathbb{R}^N |\nabla v|^2 e^{-|x|^2} \, dx \geq 2 \inf_c \int_\mathbb{R}^N |v - c|^2 e^{-|x|^2} \, dx.
\]

Hence
\[
\begin{align*}
\frac{2}{N + 3} \inf_{c \in \mathbb{R}} & \int_\mathbb{R}^N \left| \nabla \left( u - ce^{-\frac{1}{2}|x|^2} \right) \right|^2 \, dx + \int_\mathbb{R}^N |x|^2 \left| u - ce^{-\frac{1}{2}|x|^2} \right|^2 \, dx + \int_\mathbb{R}^N \left| u - ce^{-\frac{1}{2}|x|^2} \right|^2 \, dx \\
& \leq \frac{2}{N + 3} \int_\mathbb{R}^N |\nabla v|^2 e^{-|x|^2} \, dx + \frac{2}{N + 3} (N + 1) \inf_{c \in \mathbb{R}} \int_\mathbb{R}^N |v - c|^2 e^{-|x|^2} \, dx \\
& \leq \int_\mathbb{R}^N |\nabla v|^2 e^{-|x|^2} \, dx \\
& = \int_\mathbb{R}^N \left| \nabla \left( u e^{\frac{1}{2}|x|^2} \right) \right|^2 e^{-|x|^2} \, dx \\
& = \int_\mathbb{R}^N |\nabla u|^2 \, dx + \int_\mathbb{R}^N |x|^2 |u|^2 \, dx - N \int_\mathbb{R}^N |u|^2 \, dx.
\end{align*}
\]

Let \( u = x_1 e^{-\frac{1}{2}|x|^2} \) and \( v = x_1 \). Then
\[
\begin{align*}
\int_\mathbb{R}^N |\nabla u|^2 \, dx + \int_\mathbb{R}^N |x|^2 |u|^2 \, dx - N \int_\mathbb{R}^N |u|^2 \, dx \\
& = \pi \frac{N}{2}
\end{align*}
\]

and
\[
\begin{align*}
\int_\mathbb{R}^N \left| \nabla \left( u - ce^{-\frac{1}{2}|x|^2} \right) \right|^2 \, dx + \int_\mathbb{R}^N |x|^2 \left| u - ce^{-\frac{1}{2}|x|^2} \right|^2 \, dx + \int_\mathbb{R}^N \left| u - ce^{-\frac{1}{2}|x|^2} \right|^2 \, dx \\
& = \left| 1 + (N + 1) \left( \frac{1}{2} + |c|^2 \right) \right| \pi \frac{N}{2}.
\end{align*}
\]

Therefore
\[
\begin{align*}
\inf_{c \in \mathbb{R}} \int_\mathbb{R}^N \left| \nabla \left( u - ce^{-\frac{1}{2}|x|^2} \right) \right|^2 \, dx + \int_\mathbb{R}^N |x|^2 \left| u - ce^{-\frac{1}{2}|x|^2} \right|^2 \, dx + \int_\mathbb{R}^N \left| u - ce^{-\frac{1}{2}|x|^2} \right|^2 \, dx \\
& = \frac{N + 3}{2} \pi \frac{N}{2} \\
& = \frac{N + 3}{2} \left( \int_\mathbb{R}^N |\nabla u|^2 \, dx + \int_\mathbb{R}^N |x|^2 |u|^2 \, dx - N \int_\mathbb{R}^N |u|^2 \, dx \right).
\]
\[\square\]
3.2. The stability of the Caffarelli-Kohn-Nirenberg inequalities. Let \( X_{a,b} \) be the completion of \( C^\infty_0(\mathbb{R}^N) \) under the norm \( \left( \int_{\mathbb{R}^N} \frac{|\nabla u|^2}{|x|^{2a}} \, dx + \int_{\mathbb{R}^N} \frac{|u|^2}{|x|^{2b}} \, dx \right)^{\frac{1}{2}} \). We now recall that by Corollary [2.4] we have the following scale non-invariant Caffarelli-Kohn-Nirenberg inequalities with remainders:

**Lemma 3.3.** Let \( b + 1 - a > 0 \) and \( b < \frac{N-2}{2} \). Then for \( u \in X_{a,b} \):

\[
\left( \int_{\mathbb{R}^N} \frac{|\nabla u|^2}{|x|^{2b}} \, dx \right) \left( \int_{\mathbb{R}^N} \frac{|u|^2}{|x|^{2b}} \, dx \right) \geq \frac{|N-a-b-1|^2}{2} \left( \int_{\mathbb{R}^N} \frac{|u|^2}{|x|^{a+b+1}} \, dx \right)^2.
\]

(3.3)

The equality happens if and only if \( u(x) = \alpha \exp(-\beta |x|^{b+1-a}) \) with \( \alpha \in \mathbb{R}, \beta > 0 \).

Therefore we now can define the Caffarelli-Kohn-Nirenberg deficit

\[
\delta_{1,a,b}(u) := \left( \int_{\mathbb{R}^N} \frac{|u|^2}{|x|^{2b}} \, dx \right)^{\frac{1}{2}} \left( \int_{\mathbb{R}^N} \frac{|\nabla u|^2}{|x|^{2b}} \, dx \right)^{\frac{1}{2}} - \frac{|N-a-b-1|^2}{2} \left( \int_{\mathbb{R}^N} \frac{|u|^2}{|x|^{a+b+1}} \, dx \right)
\]

We use the following Poincaré inequality in [1, Chapter 4]:

**Lemma 3.5.** Let \( \mu \) be a log-concave probability measure on \( \mathbb{R}^N \). Then \( \mu \) satisfies a Poincaré inequality:

\[
\int_{\mathbb{R}^N} |\nabla v|^2 \, d\mu \gtrsim \int_{\mathbb{R}^N} v - \int_{\mathbb{R}^N} v \, d\mu \left| d\mu \right|^2.
\]

We note that when \( a \leq 0 \), then \( \mu(x) = \frac{e^{-\frac{2}{1-a}|x|^{1-a}}}{\int_{\mathbb{R}^N} e^{-\frac{2}{1-a}|x|^{1-a}} \, dx} \) is a log-concave probability measure on \( \mathbb{R}^N \). Therefore, there exists \( C(N,a) > 0 \) such that the following Poincaré inequality holds

\[
\int_{\mathbb{R}^N} |\nabla v|^2 e^{-\frac{2}{1-a}|x|^{1-a}} \, dx \geq C(N,a) \inf_{c \in \mathbb{R}} \int_{\mathbb{R}^N} |v-c|^2 e^{-\frac{2}{1-a}|x|^{1-a}} \, dx.
\]

Therefore, we obtain the following result:

**Theorem 3.5.** Let \( a \leq 0 \). Then we have that for \( u \in X_{a,0} \):

\[
\int_{\mathbb{R}^N} |\nabla u|^2 \, dx + \int_{\mathbb{R}^N} \frac{|u|^2}{|x|^{2a}} \, dx - (N-a-1) \int_{\mathbb{R}^N} \frac{|u|^2}{|x|^{a+1}} \, dx
\]
Proof. For $u \in X_{a,0}$:

$$
\int_{\mathbb{R}^N} |\nabla u|^2 \, dx + \int_{\mathbb{R}^N} \frac{|u|^2}{|x|^2} \, dx - (N - a - 1) \int_{\mathbb{R}^N} |u|^2 \, dx
= \int_{\mathbb{R}^N} \left| \nabla \left( ue^{\frac{1}{2|\alpha|} |x|^{1-a}} \right) \right|^2 e^{-\frac{2}{|\alpha|} |x|^{1-a}} \, dx
\geq C(N, a) \inf_{c \in \mathbb{R}} \int_{\mathbb{R}^N} \left| u - ce^{-\frac{1}{2|\alpha|} |x|^{1-a}} \right|^2 \, dx.
$$

In order to study the stability results for the Caffarelli-Kohn-Nirenberg inequalities, we will now establish a weighted Poincaré inequality for the log-concave probability measure. More precisely, we will prove the following weighted Poincaré inequality:

**Lemma 3.6.** For some $\delta > 0$, $N - 2 > \mu \geq 0$ and $\alpha \geq \frac{N-2-\mu}{N-2}$:

$$
\int_{\mathbb{R}^N} \left| \nabla v(y) \right|^2 \frac{e^{-\delta|y|^\alpha}}{|y|^\mu} \, dy \geq C(N, \alpha, \delta, \mu) \inf_{c \in \mathbb{R}} \int_{\mathbb{R}^N} \left| v(y) - c \right|^2 e^{-\delta|y|^\alpha} \, dy.
$$

**Proof.** Let $\nabla u(x) = \left( \frac{1}{x} \right)^{\frac{1}{2}} v \left( |x|^\lambda - 1 \right)$), Note that for the change of variable $x \rightarrow |x|^\lambda - 1$, x, the Jacobian is $\lambda|\lambda - 1|$. Now, for $\lambda \geq 1$, we can show that

$$
|\nabla \nabla u(x)| \leq \lambda^{\frac{1}{2}} |x|^{\lambda - 1} \left| \nabla v \left( |x|^{\lambda - 1} x \right) \right|.
$$

See [44]. Therefore (set $y = |x|^{\lambda - 1} x$)

$$
\int_{\mathbb{R}^N} \left| \nabla v(y) \right|^2 \frac{e^{-\delta|y|^\alpha}}{|y|^\mu} \, dy = \int_{\mathbb{R}^N} \frac{\left| \nabla v \left( |x|^{\lambda - 1} x \right) \right|^2}{|x|^{\lambda \mu}} e^{-\delta|\lambda|^\alpha} \, dx
\geq \int_{\mathbb{R}^N} \frac{\left| \nabla \nabla u(x) \right|^2}{\lambda|x|^{2(\lambda - 1) + \lambda \mu - N(\lambda - 1)}} e^{-\delta|\lambda|^\alpha} \, dx
= \int_{\mathbb{R}^N} \frac{\left| \nabla \nabla u(x) \right|^2}{|x|^{\lambda(2 + \mu - N) + N - 2}} e^{-\delta|\lambda|^\alpha} \, dx.
$$

So if we choose $\lambda = \frac{N-2}{N-2-\mu} \geq 1$, then by noting that the measure $e^{-\delta|\lambda|^\alpha}$ is log-concave, we obtain

$$
\int_{\mathbb{R}^N} \left| \nabla v(y) \right|^2 \frac{e^{-\delta|y|^\alpha}}{|y|^\mu} \, dy \geq \int_{\mathbb{R}^N} \left| \nabla \nabla u(x) \right|^2 e^{-\delta|\lambda|^\alpha} \, dx.
$$
\[ \geq C(N, \alpha, \delta, \mu) \inf_{c} \int_{\mathbb{R}^{N}} |v(x) - c|^{2} e^{-\delta|x|^\alpha} \, dx \]
\[ = C(N, \alpha, \delta, \mu) \inf_{c} \int_{\mathbb{R}^{N}} \left| \frac{v(|x|^{\lambda-1}) - c}{|x|^{N(\lambda-1)}} \right| e^{-\delta|x|^\alpha} |x|^{N(\lambda-1)} \, dx \]
\[ = C(N, \alpha, \delta, \mu) \inf_{c} \int_{\mathbb{R}^{N}} \left| \frac{v(y) - c}{|y|^\frac{N_{\mu}}{\lambda-1}} \right| e^{-\delta|y|^\alpha} \, dy. \]

\[ \square \]

Let \( E_{a,b} = \{ v(x) = \alpha \exp\left( -\frac{\beta}{b+a-1} |x|^{b+1-a} \right) \} \) with \( \alpha \in \mathbb{R}, \ \beta > 0 \) be the set of optimizers for the scale invariant Caffarelli-Kohn-Nirenberg inequalities (3.4) and \( d_{1,a,b}(u, E_{a,b}) := \inf_{v \in E_{a,b}} \left( \int_{\mathbb{R}^{N}} \frac{|u-v|^{2}}{|x|^{a+b+1}} \, dx \right)^{\frac{1}{2}}. \)

**Lemma 3.7.** Let \( 0 \leq b < \frac{N-2}{2} \) and \( a \leq \frac{Nh}{N-2} \). Then \( E_{a,b} \) is closed under the norm \( \left( \int_{\mathbb{R}^{N}} \frac{|v|^{2}}{|x|^{a+b+1}} \, dx \right)^{\frac{1}{2}} \) and \( d_{1,a,b}(u, E_{a,b}) := \inf_{v \in E_{a,b}} \left( \int_{\mathbb{R}^{N}} \frac{|u-v|^{2}}{|x|^{a+b+1}} \, dx \right)^{\frac{1}{2}} \) is the distance from \( u \) to \( E_{a,b} \). Moreover, for each \( u \in X_{a,b} \), there exists \( u^{*} \in E_{a,b} \) such that \( d_{1,a,b}(u, E_{a,b}) = \left( \int_{\mathbb{R}^{N}} \frac{|u-u^{*}|^{2}}{|x|^{a+b+1}} \, dx \right)^{\frac{1}{2}}. \)

**Proof.** Let \( u \) be such that \( \int_{\mathbb{R}^{N}} \frac{|u|^{2}}{|x|^{a+b+1}} \, dx = 1 \) and \( \inf_{v \in E} \int_{\mathbb{R}^{N}} \frac{|u-v|^{2}}{|x|^{a+b+1}} \, dx = 0 \). That is, there exists a sequence \( v_{j}(x) = \alpha_{j} \exp\left( -\frac{\beta_{j}}{b+a-1} |x|^{b+1-a} \right) \) such that \( \int_{\mathbb{R}^{N}} \frac{|v_{j}-u|^{2}}{|x|^{a+b+1}} \, dx \rightarrow 0 \). Therefore, by taking subsequences, \( v_{j} \rightarrow u \) pointwise a.e. in \( \mathbb{R}^{N} \). Also, since \( \int_{\mathbb{R}^{N}} \frac{|u|^{2}}{|x|^{a+b+1}} \, dx = 1 \) and \( \int_{\mathbb{R}^{N}} \frac{|v_{j}-u|^{2}}{|x|^{a+b+1}} \, dx \rightarrow 0 \), we get that when \( j \) is large enough, \( \int_{\mathbb{R}^{N}} \frac{|v_{j}|^{2}}{|x|^{a+b+1}} \, dx \in \left( \frac{1}{2}, \frac{3}{2} \right) \). Then by using the fact that
\[ \int_{0}^{\infty} r^{x-1} e^{-r \frac{\beta}{y}} \, dr = \left( \frac{y}{\beta} \right)^{\frac{x}{\beta}} \frac{1}{y} \Gamma \left( \frac{x}{\beta} \right) \]
we get
\[ \int_{\mathbb{R}^{N}} \frac{|v_{j}|^{2}}{|x|^{a+b+1}} \, dx = |\alpha_{j}|^{2} |S^{N-1}| \left( \frac{b+1-a}{2\beta_{j}} \right)^{\frac{N-a-b-1}{b+a-1}} \Gamma \left( \frac{N-a-b-1}{b+a-1} \right) \frac{b+1-a}{b+1-a} \]
and
\[ \int_{\mathbb{R}^{N}} \frac{|\nabla v_{j}|^{2}}{|x|^{2a}} \, dx = |\alpha_{j}|^{2} |\beta_{j}|^{2} |S^{N-1}| \left( \frac{b+1-a}{2\beta_{j}} \right)^{\frac{N-2a}{b+1-a}} \Gamma \left( \frac{N-2a}{b+1-a} \right) \frac{b+1-a}{b+1-a} \]
Therefore, \( \frac{|\alpha_j|^2}{|\beta_j|^{2 + \frac{a}{b+1-a}}} = O(1) \). Hence if \( \liminf_{j \to \infty} \beta_j = 0 \), then

\[
\int_{\mathbb{R}^N} \frac{|
abla v_j|^2}{|x|^{2b}} \, dx = O \left( \frac{|\alpha_j|^2}{|\beta_j|^{2 + \frac{a}{b+1-a}}} \right)
\]

\[
= O \left( \frac{|\alpha_j|^2}{|\beta_j|^{2 + \frac{a}{b+1-a}}} \right) |\beta_j| 
\]

\[
\rightarrow 0.
\]

Now, fix \( 0 < r < R < \infty \), then the sequence \( v_j \) is bounded in \( W^{1,2}(B_R \setminus B_r) \). Hence, since \( v_j \to u \) pointwise a.e. in \( \mathbb{R}^N \), we have that \( v_j \to u \) weakly in \( W^{1,2}(B_R \setminus B_r) \). Therefore,

\[
\int_{B_R \setminus B_r} \frac{|
abla u|^2}{|x|^{2b}} \, dx \leq \liminf_{j \to \infty} \int_{B_R \setminus B_r} \frac{|
abla v_j|^2}{|x|^{2b}} \, dx \leq \liminf_{j \to \infty} \int_{\mathbb{R}^N} \frac{|
abla v_j|^2}{|x|^{2b}} \, dx = 0.
\]

Now, sending \( r \to 0 \) and \( R \to \infty \) gives us that

\[
\int_{\mathbb{R}^N} \frac{|
abla u|^2}{|x|^{2b}} \, dx = 0
\]

which is impossible.

If \( \limsup_{j \to \infty} \beta_j = \infty \), then similarly

\[
\int_{\mathbb{R}^N} \frac{|v_j|^2}{|x|^{2a}} \, dx \to 0
\]

and so by Fatou’s lemma

\[
\int_{\mathbb{R}^N} \frac{|u|^2}{|x|^{2a}} \, dx \leq \liminf_{j \to \infty} \int_{\mathbb{R}^N} \frac{|v_j|^2}{|x|^{2a}} \, dx = 0
\]

which is impossible. Therefore, by taking subsequences, \( \lim_{j \to \infty} \beta_j = \beta \in (0, \infty) \). This implies that \( \lim_{j \to \infty} \alpha_j = \alpha \in (0, \infty) \). Therefore, \( v_j(x) = \alpha_j \exp(-\frac{\beta_j}{b+1-a}|x|^{b+1-a}) \to \alpha \exp(-\frac{\beta}{b+1-a}|x|^{b+1-a}) \) pointwise a.e. in \( \mathbb{R}^N \). Hence, \( u(x) = \alpha \exp(-\frac{\beta}{b+1-a}|x|^{b+1-a}) \) a.e. in \( \mathbb{R}^N \). In other words, \( E_{a,b} \) is closed under the norm \( \left( \int_{\mathbb{R}^N} \frac{|u|^2}{|x|^{2a}} \, dx \right)^{\frac{1}{2}} \). Therefore, \( d_{1,a,b}(u, E_{a,b}) \) is the distance from \( u \) to \( E_{a,b} \). That is, \( d_{1,a,b}(u, E_{a,b}) > 0 \) when \( u \notin E_{a,b} \). Moreover, for each \( u \in X_{a,b} \), there exists \( u^* \in E_{a,b} \) such that \( d_{1,a,b}(u, E_{a,b}) = \left( \int_{\mathbb{R}^N} \frac{|u-u^*|^2}{|x|^{a+b+1}} \, dx \right)^{\frac{1}{2}} \). \( \square \)

Now, we are ready to set up some results about the stability for the Caffarelli-Kohn-Nirenberg inequalities, namely, Theorem \( 1.1 \).

**Theorem 3.6.** Let \( 0 \leq b < \frac{N-2}{2} \) and \( a \leq \frac{N b}{N-2} \). There exists a universal constant \( C_1(N,a,b) > 0 \) such that for all \( u \in X_{a,b} \):

\[
\int_{\mathbb{R}^N} \frac{|
abla u|^2}{|x|^{2b}} \, dx + \int_{\mathbb{R}^N} \frac{|u|^2}{|x|^{2a}} \, dx - (N - a - b - 1) \int_{\mathbb{R}^N} \frac{|u|^2}{|x|^{a+b+1}} \, dx
\]
\[ \geq C_1 (N, a, b) \inf_{c} \int_{\mathbb{R}^{N}} \frac{|u - ce^{-\frac{1}{b+1-a} |x|^{b+1-a}}|^2}{|x|^\frac{2bN}{N-2}} \, dx. \]

Furthermore, if \( a + b + 1 = \frac{2bN}{N-2} \), then there exists a universal constant \( C_2 (N, a, b) > 0 \) such that for all \( u \in X_{a,b} \):

\[ \delta_{1,a,b} (u) \geq C_2 (N, a, b) \inf_{v \in E_{a,b}} \left( \int_{\mathbb{R}^{N}} \frac{|u - v|^2}{|x|^{a+b+1}} \, dx \right). \]

**Proof.** By applying Lemma 3.6 with \( \mu = 2b, \alpha = b + 1 - a \) and \( \delta = \frac{2}{b+1-a} \), we have

\[
\int_{\mathbb{R}^{N}} \frac{|\nabla u|^2}{|x|^{2b}} \, dx + \int_{\mathbb{R}^{N}} \frac{|u|^2}{|x|^{2a}} \, dx - (N - a - b - 1) \int_{\mathbb{R}^{N}} \frac{|u|^2}{|x|^{a+b+1}} \, dx
\]

\[
= \int_{\mathbb{R}^{N}} \frac{1}{|x|^{2b}} \nabla \left( ue^{\frac{1}{b+1-a} |x|^{b+1-a}} \right)^2 e^{-\frac{2}{b+1-a} |x|^{b+1-a}} \, dx
\]

\[ \geq C_1 (N, a, b) \inf_{c} \int_{\mathbb{R}^{N}} \frac{\left| u - ce^{-\frac{1}{b+1-a} |x|^{b+1-a}} \right|^2}{|x|^\frac{2bN}{N-2}} \, dx. \]

Now, if \( a + b + 1 = \frac{2bN}{N-2} \), then by applying the above result for \( u_{\lambda} (x) = u (\lambda x) \) and noting that

\[
\int_{\mathbb{R}^{N}} \frac{|\nabla u_{\lambda}|^2}{|x|^{2b}} \, dx = \chi^{2+2b-N} \int_{\mathbb{R}^{N}} \frac{|\nabla u|^2}{|x|^{2b}} \, dx
\]

\[ \int_{\mathbb{R}^{N}} \frac{|u_{\lambda}|^2}{|x|^{2a}} \, dx = \chi^{2a-N} \int_{\mathbb{R}^{N}} \frac{|u|^2}{|x|^{2a}} \, dx
\]

\[ \int_{\mathbb{R}^{N}} \frac{|u_{\lambda}|^2}{|x|^{a+b+1}} \, dx = \chi^{a+b+1-N} \int_{\mathbb{R}^{N}} \frac{|u|^2}{|x|^{a+b+1}} \, dx
\]

\[ \int_{\mathbb{R}^{N}} \frac{|u_{\lambda} - ce^{-\frac{1}{b+1-a} |x|^{b+1-a}}|^2}{|x|^\frac{2bN}{N-2}} \, dx = \chi^{a+b+1-N} \int_{\mathbb{R}^{N}} \frac{|u - ce^{-\frac{1}{b+1-a} |x|^{b+1-a}}|^2}{|x|^\frac{2bN}{N-2}} \, dx,
\]

we get

\[
\chi^{b+1-a} \int_{\mathbb{R}^{N}} \frac{|\nabla u|^2}{|x|^{2b}} \, dx + \frac{1}{\chi^{b+1-a}} \int_{\mathbb{R}^{N}} \frac{|u|^2}{|x|^{2a}} \, dx - (N - a - b - 1) \int_{\mathbb{R}^{N}} \frac{|u|^2}{|x|^{a+b+1}} \, dx
\]

\[ \geq C_1 (N, a, b) \inf_{c} \int_{\mathbb{R}^{N}} \frac{|u - ce^{-\frac{1}{b+1-a} |x|^{b+1-a}}|^2}{|x|^\frac{2bN}{N-2}} \, dx. \]
By choosing
\[ \lambda = \left( \frac{\int_{\mathbb{R}^N} \frac{|u|^2}{|x|^{2a}} \, dx}{(\int_{\mathbb{R}^N} \frac{|
abla u|^2}{|x|^{2b}} \, dx)^{\frac{1}{2}} \int_{\mathbb{R}^N} |u|^2 \, dx} \right)^{\frac{1}{2a+b+1-a}}, \]
we obtain
\[ \left( \int_{\mathbb{R}^N} \frac{|u|^2}{|x|^{2a}} \, dx \right)^{\frac{1}{2}} \left( \int_{\mathbb{R}^N} \frac{|
abla u|^2}{|x|^{2b}} \, dx \right)^{\frac{1}{2}} \geq \frac{N-a-b-1}{2} \left( \int_{\mathbb{R}^N} \frac{|u|^2}{|x|^{a+b+1}} \, dx \right) \]
\[ \geq C_2 \, (N, a, b) \inf_{v \in E_{a,b}} \left( \int_{\mathbb{R}^N} \frac{|u - v|^2}{|x|^{a+b+1}} \, dx \right). \]

Obviously, if we use the Caffarelli-Kohn-Nirenberg deficit
\[ \delta_{2,a,b} (u) := \left( \int_{\mathbb{R}^N} \frac{|u|^2}{|x|^{2a}} \, dx \right)^{\frac{1}{2}} \left( \int_{\mathbb{R}^N} \frac{|
abla u|^2}{|x|^{2b}} \, dx \right)^{\frac{1}{2}} \geq \frac{N-a-b-1}{2} \left( \int_{\mathbb{R}^N} \frac{|u|^2}{|x|^{a+b+1}} \, dx \right)^2, \]
then we get

**Theorem 3.7.** Let \( 0 \leq b < \frac{N-2}{2}, \; a \leq \frac{N b}{N-2} \) and \( a + b + 1 = \frac{2bN}{N-2} \). There exists a universal constant \( C_3 (N, a, b) > 0 \) such that for all \( u \in X_{a,b} \):
\[
\delta_{2,a,b} (u) \geq C_3 \, (N, a, b) \left( \int_{\mathbb{R}^N} \frac{|u|^2}{|x|^{a+b+1}} \, dx \right) d_{1,a,b} (u, E_{a,b})^2 \]
\[ + C_3 (N, a, b) d_{1,a,b} (u, E_{a,b})^4. \]

**Proof.** Since
\[ \delta_{1,a,b} (u) \geq C_2 \, (N, a, b) d_{2,a,b}^2 (u, E_{a,b}), \]
we get
\[ \left( \int_{\mathbb{R}^N} \frac{|u|^2}{|x|^{2a}} \, dx \right)^{\frac{1}{2}} \left( \int_{\mathbb{R}^N} \frac{|
abla u|^2}{|x|^{2b}} \, dx \right)^{\frac{1}{2}} \geq \frac{N-a-b-1}{2} \left( \int_{\mathbb{R}^N} \frac{|u|^2}{|x|^{a+b+1}} \, dx \right)^2 \]
\[ \geq 2 \left( \frac{N-a-b-1}{2} \right) \left( \int_{\mathbb{R}^N} \frac{|u|^2}{|x|^{a+b+1}} \, dx \right) d_{1,a,b} (u, E_{a,b})^2 \]
\[ + C_2 \, (N, a, b) d_{1,a,b} (u, E_{a,b})^4. \]
A more careful analysis now leads to the following quantitative version of the stability of the Caffarelli-Kohn-Nirenberg inequalities:

**Theorem 3.8.** Let $0 \leq b < \frac{N-2}{2}$, $a \leq \frac{Nb}{N-2}$ and $a+b+1 = \frac{2bN}{N-2}$. There exists a universal constant $C_4 (N, a, b) > 0$ such that for all $u \in X_{a, b}$:

$$\delta_{1,a,b} (u) \geq C_4 (N, a, b) \inf_{u^* \in E_{a, b}} \left\{ \int_{\mathbb{R}^N} \frac{|u - u^*|^2}{|x|^{a+b+1}} \, dx : \int_{\mathbb{R}^N} \frac{|u|^2}{|x|^{a+b+1}} \, dx = \int_{\mathbb{R}^N} \frac{|u^*|^2}{|x|^{a+b+1}} \, dx \right\}$$

and

$$\delta_{2,a,b} (u) \geq C_4 (N, a, b) \left( \int_{\mathbb{R}^N} \frac{|u|^2}{|x|^{a+b+1}} \, dx \right)^2 d_{2,a,b} (u, E_{a, b})$$

$$+ C_4 (N, a, b) d_{2,a,b} (u, E_{a, b})^4.$$

Here $d_{2,a,b} (u, E_{a, b}) := \inf_{v \in E_{a, b}} \left\{ \left( \int_{\mathbb{R}^N} \frac{|u-v|^2}{|x|^{a+b+1}} \, dx \right)^{\frac{1}{2}} : \int_{\mathbb{R}^N} \frac{|u|^2}{|x|^{a+b+1}} \, dx = \int_{\mathbb{R}^N} \frac{|u^*|^2}{|x|^{a+b+1}} \, dx \right\}.$$

**Proof.** WLOG, assume that $\int_{\mathbb{R}^N} \frac{|u|^2}{|x|^{a+b+1}} \, dx = 1$.

Now, recall that

$$\left( \int_{\mathbb{R}^N} \frac{|u|^2}{|x|^{2a}} \, dx \right)^{\frac{1}{2}} \left( \int_{\mathbb{R}^N} \frac{|
abla u|^2}{|x|^{2b}} \, dx \right)^{\frac{1}{2}} - \frac{|N - a - b - 1|}{2} \left( \int_{\mathbb{R}^N} \frac{|u|^2}{|x|^{a+b+1}} \, dx \right)$$

$$\geq C_2 (N, a, b) \inf_{u^* \in E_{a, b}} \int_{\mathbb{R}^N} \frac{|u - u^*|^2}{|x|^{a+b+1}} \, dx.$$

Now, if

$$\delta_{1,a,b} (u) = \left( \int_{\mathbb{R}^N} \frac{|u|^2}{|x|^{2a}} \, dx \right)^{\frac{1}{2}} \left( \int_{\mathbb{R}^N} \frac{|
abla u|^2}{|x|^{2b}} \, dx \right)^{\frac{1}{2}} - \frac{|N - a - b - 1|}{2} \left( \int_{\mathbb{R}^N} \frac{|u|^2}{|x|^{a+b+1}} \, dx \right) < \frac{C_2 (N, a, b)}{2},$$

then there exists $v \in E$, $v \neq 0$ such that

$$\int_{\mathbb{R}^N} \frac{|u - v|^2}{|x|^{a+b+1}} \, dx \leq \frac{2}{C_2 (N, a, b)} \delta_{1,a,b} (u) < 1.$$

Let $\lambda = \left( \int_{\mathbb{R}^N} \frac{|v|^2}{|x|^{a+b+1}} \, dx \right)^{\frac{1}{2}}$. Then $w = \lambda v \in E_{a,b}$ and $\int_{\mathbb{R}^N} \frac{|w|^2}{|x|^{a+b+1}} \, dx = 1$. Also,

$$\int_{\mathbb{R}^N} \frac{|u - w|^2}{|x|^{a+b+1}} \, dx = \int_{\mathbb{R}^N} \frac{|u - \lambda v|^2}{|x|^{a+b+1}} \, dx$$

$$= \int_{\mathbb{R}^N} \frac{|u - v + (1 - \lambda) v|^2}{|x|^{a+b+1}} \, dx$$

$$\leq 2 \int_{\mathbb{R}^N} \frac{|u - v|^2}{|x|^{a+b+1}} \, dx + 2 (1 - \lambda)^2 \int_{\mathbb{R}^N} \frac{|v|^2}{|x|^{a+b+1}} \, dx.$$
\[
\delta_{1,a,b}(u) \geq \frac{C_2(N, a, b)}{8} \inf_{u^* \in E_{a,b}} \left\{ \int_{\mathbb{R}^N} \frac{|u - u^*|^2}{|x|^{a+b+1}} \, dx : \int_{\mathbb{R}^N} \frac{|u|^2}{|x|^{a+b+1}} \, dx = \int_{\mathbb{R}^N} \frac{|u^*|^2}{|x|^{a+b+1}} \, dx \right\}. 
\]

If
\[
\delta_{1,a,b}(u) = \left( \int_{\mathbb{R}^N} \frac{|u|^2}{|x|^{2a}} \, dx \right)^{\frac{1}{2}} \left( \int_{\mathbb{R}^N} \frac{\nabla u|^2}{|x|^{2b}} \, dx \right)^{\frac{1}{2}} - \frac{N - a - b - 1}{2} \left( \int_{\mathbb{R}^N} \frac{|u|^2}{|x|^{a+b+1}} \, dx \right) \geq \frac{C_2(N, a, b)}{2}
\]
then since
\[
\inf_{u^* \in E_{a,b}} \left\{ \int_{\mathbb{R}^N} \frac{|u - u^*|^2}{|x|^{a+b+1}} \, dx : \int_{\mathbb{R}^N} \frac{|u|^2}{|x|^{a+b+1}} \, dx = \int_{\mathbb{R}^N} \frac{|u^*|^2}{|x|^{a+b+1}} \, dx \right\} \leq 4,
\]
we get
\[
\delta_{1,a,b}(u) \geq \frac{C_2(N, a, b)}{8} \inf_{u^* \in E_{a,b}} \left\{ \int_{\mathbb{R}^N} \frac{|u - u^*|^2}{|x|^{a+b+1}} \, dx : \int_{\mathbb{R}^N} \frac{|u|^2}{|x|^{a+b+1}} \, dx = \int_{\mathbb{R}^N} \frac{|u^*|^2}{|x|^{a+b+1}} \, dx \right\}.
\]

Our last result is a stronger stability version for the scale non-invariant Caffarelli-Kohn-Nirenberg inequalities:

**Theorem 3.9.** Let \(0 \leq b < \frac{N-2}{2}, a \leq \frac{Nb}{N-2}\) and \(a + b + 1 = \frac{2bN}{N-2}\). There exists a universal constant \(C_5(N, a, b) > 0\) such that for all \(u \in X_{a,b}\):

\[
\int_{\mathbb{R}^N} \frac{\nabla u|^2}{|x|^{2b}} \, dx + \int_{\mathbb{R}^N} \frac{|u|^2}{|x|^{2a}} \, dx - (N - a - b - 1) \int_{\mathbb{R}^N} \frac{|u|^2}{|x|^{a+b+1}} \, dx \\
\geq C_5(N, a, b) \inf_{c} \left[ \int_{\mathbb{R}^N} \frac{|u - ce^{x_1+1-x_2}|^{b+1-a}}{|x|^{a+b+1}} \, dx + \int_{\mathbb{R}^N} \frac{\nabla (u - ce^{x_1+1-x_2})|^2}{|x|^{2b}} \, dx \right]
\]

\[
+ \int_{\mathbb{R}^N} \frac{|u - ce^{x_1+1-x_2}|^2}{|x|^{2a}} \, dx
\]
Proof. Indeed, let \( u = ve^{\frac{1}{b+1-a}|x|^{b+1-a}} \), then

\[
\int_{\mathbb{R}^N} \left| \nabla \left( u - ce^{\frac{1}{b+1-a}|x|^{b+1-a}} \right) \right|^2 dx
\]

\[
= \int_{\mathbb{R}^N} \left| \nabla \left( v - c \right) e^{\frac{1}{b+1-a}|x|^{b+1-a}} \right|^2 dx
\]

\[
= \int_{\mathbb{R}^N} \left| \nabla v \right|^2 e^{-\frac{2}{b+1-a}|x|^{b+1-a}} dx - 2\int_{\mathbb{R}^N} \left( v - c \right) x \cdot \nabla \left( v - c \right) e^{-\frac{2}{b+1-a}|x|^{b+1-a}} dx
\]

\[
+ \int_{\mathbb{R}^N} \left| v - c \right|^2 e^{-\frac{2}{b+1-a}|x|^{b+1-a}} dx
\]

\[
= \int_{\mathbb{R}^N} \left| \nabla v \right|^2 e^{-\frac{2}{b+1-a}|x|^{b+1-a}} dx + (N - a - b - 1) \int_{\mathbb{R}^N} \left| v - c \right|^2 e^{-\frac{2}{b+1-a}|x|^{b+1-a}} dx
\]

Hence

\[
\int_{\mathbb{R}^N} \left| u - ce^{\frac{1}{b+1-a}|x|^{b+1-a}} \right|^2 dx + \int_{\mathbb{R}^N} \left| \nabla \left( u - ce^{\frac{1}{b+1-a}|x|^{b+1-a}} \right) \right|^2 dx + \int_{\mathbb{R}^N} \left| u - ce^{\frac{1}{b+1-a}|x|^{b+1-a}} \right|^2 dx
\]

\[
= \int_{\mathbb{R}^N} \left| \nabla \left( v - c \right) e^{\frac{1}{b+1-a}|x|^{b+1-a}} \right|^2 dx + \int_{\mathbb{R}^N} \left| \nabla v \right|^2 e^{-\frac{2}{b+1-a}|x|^{b+1-a}} dx + \int_{\mathbb{R}^N} \left| v - c \right|^2 e^{-\frac{2}{b+1-a}|x|^{b+1-a}} dx
\]

\[
\leq \int_{\mathbb{R}^N} \left| \nabla v \right|^2 e^{-\frac{2}{b+1-a}|x|^{b+1-a}} dx + (N - a - b) \int_{\mathbb{R}^N} \left| v - c \right|^2 e^{-\frac{2}{b+1-a}|x|^{b+1-a}} dx
\]

Therefore, by the weighted Poincaré inequality Lemma 3.6, we obtain

\[
\inf_{c \in \mathbb{R}} \int_{\mathbb{R}^N} \left| u - ce^{\frac{1}{b+1-a}|x|^{b+1-a}} \right|^2 dx + \int_{\mathbb{R}^N} \left| \nabla \left( u - ce^{\frac{1}{b+1-a}|x|^{b+1-a}} \right) \right|^2 dx + \int_{\mathbb{R}^N} \left| u - ce^{\frac{1}{b+1-a}|x|^{b+1-a}} \right|^2 dx
\]

\[
\leq \int_{\mathbb{R}^N} \left| \nabla v \right|^2 e^{-\frac{2}{b+1-a}|x|^{b+1-a}} dx + (N - a - b) \inf_{c \in \mathbb{R}} \int_{\mathbb{R}^N} \left| v - c \right| e^{-\frac{2}{b+1-a}|x|^{b+1-a}} dx
\]

\[
\leq C(N, a) \int_{\mathbb{R}^N} \frac{1}{|x|^{2b}} \left| \nabla \left( u e^{\frac{1}{b+1-a}|x|^{b+1-a}} \right) \right|^2 e^{-\frac{2}{b+1-a}|x|^{b+1-a}} dx
\]

\[
= C(N, a) \int_{\mathbb{R}^N} \frac{1}{|x|^{2b}} \left| \nabla \left( u e^{\frac{1}{b+1-a}|x|^{b+1-a}} \right) \right|^2 e^{-\frac{2}{b+1-a}|x|^{b+1-a}} dx
\]
\[ C (N, a, b) \left[ \int_{\mathbb{R}^N} \frac{|\nabla u|^2}{|x|^{2b}} \, dx + \int_{\mathbb{R}^N} \frac{|u|^2}{|x|^{2a}} \, dx - (N - a - b - 1) \int_{\mathbb{R}^N} \frac{|u|^2}{|x|^{a+b+1}} \, dx \right]. \]

\[ \square \]

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