SURFACE BUNDLES OVER SURFACES WITH A FIXED SIGNATURE

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Abstract. The signature of a surface bundle over a surface is known to be divisible by 4. It is also known that the signature vanishes if the fiber genus \( \leq 2 \) or the base genus \( \leq 1 \). In this article, we construct new smooth 4-manifolds with signature 4 which are surface bundles over surfaces with small fiber and base genera. From these we derive improved upper bounds for the minimal genus of surfaces representing the second homology classes of a mapping class group.

1. Introduction

By a surface bundle over a surface we mean an oriented fiber bundle whose fibers and base are both compact, oriented 2-manifolds. When we study the topology of fiber bundles, the fundamental question is how the topological invariants of the total space, the fiber space, and the base space are related. Even though it is an elementary fact that the Euler characteristic is multiplicative for fiber bundles, for the signature, the same does not hold in general. As the first counterexamples, Atiyah\[1\] and, independently, Kodaira\[21\] provided surface bundles over surfaces with nonvanishing signature. In these classical examples, the fiber genus \( f \) or the base genus \( b \) was fairly big. For example, in Atiyah’s example, \( f = 6 \) and \( b = 129 \).

After that, there have been many efforts to find out the smallest possible genera of surface bundle over surface for which the signature is nonzero.\[8, 3, 2, 31, 6\]

In the early constructions of surface bundles, the signature of the total space was computed by using the signature formula for ramified coverings created by Hirzebruch\[15\]. However, not all of the bundles can be constructed by using the branched covering method. Instead, in general, the monodromy information of a surface bundle allows us to compute its signature, with the help of Meyer’s signature cocycle\[26\] which is a 2-cocycle of the symplectic group \( \text{Sp}(2g, \mathbb{R}) \). Using the signature cocycle and Birman-Hilden’s relations of mapping class group, Meyer proved that if the fiber genus \( f \leq 2 \) or the base genus \( b \leq 1 \), then the signature vanishes. Hence, for a nonzero signature, we only need to consider the case when \( f \geq 3 \) and \( b \geq 2 \). He also proved that for every \( f \geq 3 \) and every \( 4n \in 4\mathbb{Z} \), there exists a \( \Sigma_f \) bundle over \( \Sigma_b \) with signature \( 4n \) for some \( b \geq 0 \). Based on his result, Endo\[8\] studied the following refined question which is very similar to Problem 2.18A in Kirby’s problem list \[17\].

Problem 1.1. For each \( f \geq 3 \) and each \( n \in \mathbb{Z} \), let \( b(f, n) \) be the minimal base genus \( b \) over which a surface bundle with fiber genus \( f \) and signature \( 4n \) exists. Determine the value \( b(f, n) \).

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In [8], Endo showed that \( b(f, n) \leq 111|n| \) for any \( f \geq 3 \). In [31], Stipsicz showed that \( b(f, 2f + 2) \leq 4f + 20 \). In [6], Endo, Kotschick, Korkmaz, Ozbagci, and Stipsicz proved that \( b(f, n) \leq 8|n| + 1 \) for any \( f \geq 3 \) and any \( n \neq 0 \). In this paper, we improve this upper bound for \( b(f, n) \).

**Theorem 1.2.** (a) For every \( f \geq 3 \) and \( n \neq 0 \), \( b(f, n) \leq 7|n| + 1 \). In particular, there exists a smooth 4-manifold with signature 4 which is a \( \Sigma_3 \)-bundle over \( \Sigma_8 \).

(b) For every \( f \geq 5 \) and \( n \neq 0 \), \( b(f, n) \leq 6|n| + 1 \). In particular, there exists a smooth 4-manifold with signature 4 which is a \( \Sigma_5 \)-bundle over \( \Sigma_7 \).

(c) For every \( f \geq 6 \) and \( n \neq 0 \), \( b(f, n) \leq 5|n| + 1 \). In particular, there exists a smooth 4-manifold with signature 4 which is a \( \Sigma_6 \)-bundle over \( \Sigma_6 \).

Our constructions of surface bundles rely on various computations in mapping class groups, which we will introduce in Section 3. From a geometric point of view, these computations correspond to monodromy factorizations of Lefschetz fibrations. From Lefschetz fibrations, by taking neighborhoods of singular fibers out and gluing them along isomorphic boundaries via fiber-preserving diffeomorphisms, we can construct surface bundles over surfaces. This method was introduced in [6] to construct a \( \Sigma_3 \) bundle over \( \Sigma_9 \) with signature 4. A key ingredient in this paper is that a clever use of different embeddings of relations in mapping class groups gives rise to more economical, in the sense of small genera, surface bundles with a fixed signature 4.

**Remark 1.3.** [23] We may think of \( b(f, n) \) as the minimal genus of the surfaces representing the \( n \) times generator of \( H_2(\text{Mod}(\Sigma_f) : \mathbb{Z})/\text{Tor} \) for fixed \( f \geq 3 \) and \( n \).

On the other hand, the lower bound for \( b(f, n) \) was also investigated. Kotschick [23] proved \( b(f, n) \geq \frac{2|n|}{f-1} + 1 \), and Hamenstadt [13] proved \( b(f, n) \geq \frac{3|n|}{f-2} + 1 \). Combining the latter with our result, we have \( 3 \leq b(3, 1) \leq 8 \), \( 2 \leq b(5, 1) \leq 7 \), and \( 2 \leq b(6, 1) \leq 6 \).

It is not hard to see that \( \frac{b(f, n)}{n} \) converges. Now we define \( G_f := \lim_{n \to \infty} \frac{b(f, n)}{n} \) and improve a priori the upper bound for \( G_f \) that appeared in [6].

**Theorem 1.4.** For every odd \( f \geq 3 \), \( G_f \leq \frac{14}{f-1} \).

**Remark 1.5.** As far as I know, this is the best known upper bound for \( f = 3 \) or every odd \( f \) of the form \( 3k + 1, 3k + 2 \). In fact, for some other \( f \)’s, better upper bounds are known: for even \( f \geq 4 \), \( G_f \leq \frac{6}{f-2} \) [2], and for \( f = 3k \geq 6 \), \( G_f \leq \frac{9}{f-2} \) [3].

2. Preliminaries

**2.1. Signature.** Let \( M \) be a compact oriented topological manifold of dimension 4\( m \). Since \( M \) is oriented, it admits the fundamental class \( [M] \in H_{4m}(M, \partial M) \).

**Definition.** The symmetric bilinear form \( Q_M : H^{2m}(M, \partial M) \times H^{2m}(M, \partial M) \to \mathbb{Z} \) defined by \( Q_M(a, b) := \langle a \cup b, [M] \rangle \) is called the intersection form of \( M \).

**Remark 2.1.** In the smooth case, we can understand \( Q_M \) above as the algebraic intersection number of smoothly embedded oriented submanifolds in \( M \) representing the Poincaré duals of \( a \) and \( b \).

If \( a \) or \( b \) is a torsion element, then \( Q_M \) vanishes, and hence \( Q_M \) descends to the cohomology modulo torsion.
Definition. The signature of $M$, denoted by $\sigma(M)$, is defined to be the signature of the symmetric bilinear form $Q_M$ on $H^{2n}(M, \partial M)/\text{Tor}$. If the dimension of $M$ is not divisible by 4, $\sigma(M)$ is defined to be zero.

2.2. Mapping class group. Let $\Sigma_g$ be an oriented surface of genus $g$ with $r$ boundary components and let $\Sigma_{g}'$ be a closed oriented surface of genus $g$. The mapping class group $\text{Mod}(\Sigma_{g}')$ of $\Sigma_{g}'$ is defined to be the group of isotopy classes of orientation preserving self-homeomorphisms which are identity on each boundary component. Based on the theorem of Dehn, we have a surjective homomorphism $\pi: F(S) \rightarrow \text{Mod}(\Sigma_g)$, where $F(S)$ is the free group generated by the generating set $S$ consisting of all the Dehn twists over all isotopy classes of simple closed curves on $\Sigma_g$. Set $R := \text{Ker}(\pi)$ and call each word $w$ in the generators $S$ of $\text{Mod}(\Sigma_g)$ a relation of $\text{Mod}(\Sigma_g)$ if $w \in R$. Now, let us review some famous relations of mapping class groups.

Let $a$ and $b$ be two simple closed curves on $\Sigma_g$. If $a$ and $b$ are disjoint, then the supports of the Dehn twists $t_a$ and $t_b$ can be chosen to be disjoint. Hence, there exist commutativity relations $t_at_b^{-1}t_b^{-1}$ for any disjoint simple closed curves $a$ and $b$. If $a$ intersects $b$ transversely at one point, then there exists a braid relation $t_at_b^{-1}t_a^{-1}t_b^{-1}$. It can be derived from more general fact that $ft_a^{-1}t_f^{-1}$ in $\text{Mod}(\Sigma_g)$ for any simple closed curve $a$ on $\Sigma_g$ and any orientation preserving homeomorphism $f$ of $\Sigma_g$. For braid relations, we will take the latter general form $ft_a^{-1}t_f^{-1}$.

Consider the planar surface $\Sigma_0^3$ with boundary components $a, b, c,$ and $d$. On the left hand side of Figure 1, the boundary curves $a, b, c,$ and $d$ are in black and the interior curves $x, y, z$ are in different colors. One can easily check that $t_at_b^{-1}t_c^{-1}t_b^{-1}t_a^{-1}t_b^{-1}t_c^-1t_a$ holds in $\text{Mod}(\Sigma_0^3)$ by applying the Alexander method, and we call the lantern relations for all embedded subsurfaces $\Sigma_0^4 \hookrightarrow \Sigma_g$. For the $k$-chain relations and any other details for mapping class groups, refer to [11]. One can also deduce the star relations $t_{\delta_3}^{-1}t_{\delta_2}^{-1}t_{\delta_1}^{-1}(t_{\alpha_1}t_{\alpha_2}t_{\alpha_3}t_{\beta})^3$ supported on any embedded subsurfaces $\Sigma_1^3 \hookrightarrow \Sigma_g$. See Figure 4 as an example.

We say that two simple closed curves $a$ and $b$ on $\Sigma_g$ are topologically equivalent if there exists a homeomorphism of $\Sigma_g$ sending $a$ to $b$. Similarly, the two collections $\{a_1, \ldots, a_n\}$ and $\{b_1, \ldots, b_n\}$ of simple closed curves on $\Sigma_g$ are called topologically equivalent if there exists a homeomorphism of $\Sigma_g$ sending $a_i$ to $b_i$ simultaneously for all $1 \leq i \leq n$. To simplify the notation in the rest of this paper, we will use the notation $w_1^{w_2}$ for the conjugation $w_2^{-1}w_1w_2$.

2.3. Lefschetz fibrations and surface bundles.

Definition. Let $X$ be a compact oriented 4-manifold, and $B$ a compact oriented 2-manifold. A smooth surjective map $f : X \rightarrow B$ is called a Lefschetz fibration if for each critical point $p \in X$ there are local complex coordinates $(z_1, z_2)$ on $X$ around $p$ and $z$ on $B$ around $f(p)$ compatible with the orientations and such that $f(z_1, z_2) = z_1^2 + z_2^2$.

It follows that $f$ has only finitely many critical points, and we may assume that $f$ is injective on the critical set $C = \{p_1, \ldots, p_k\}$. A fiber of $f$ containing a critical point is called a singular fiber, and the genus of $f$ is defined to be the genus of the regular fiber. Notice that if $\nu(f(C))$ denotes an open tubular neighborhood of the set of critical values $f(C)$, then the restriction of $f$ to $f^{-1}(B - \nu(f(C)))$ is a smooth surface bundle over $B - \nu(f(C))$. 
For a smooth surface bundle $f : E \to B$ with a fixed identification $\phi$ of the fiber over the base point $p$ of $B$ with a standard genus $g$ surface $\Sigma_g$, the monodromy representation of $f$ is defined to be an antimonomorphism $\chi : \pi_1(B, p) \to \text{Mod}(\Sigma_g)$ defined as follows. For each loop $l : [0, 1] \to B$, $l^*(E) \to [0, 1]$ is trivial and hence there exists a parametrization $\Phi : [0, 1] \times \Sigma_g \to f^{-1}([0, 1])$ with $\Phi|_{0 \times \Sigma_g} = \phi^{-1}$. Now define $\chi(l) := [\Phi|_{0 \times \Sigma_g} \circ \Phi|_{1 \times \Sigma_g}]$. For the genus $g$ Lefschetz fibration $f : X \to B$ with a fixed identification of the fiber with $\Sigma_g$, we define the monodromy representation of $f$ to be the monodromy representation of the surface bundle $f : X \to B - f(C)$.

A Lefschetz singular fiber can be described by its monodromy. By looking at the local model of the Lefschetz critical point, one can see that the singular fiber is obtained from the regular fiber by collapsing a simple closed curve, called the vanishing cycle. One can also observe that the monodromy along the loop going around one

$\text{Mod} : \pi_1(B, p) \to \text{Mod}(\Sigma_g)$

of the fiber

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Suppose that \( f : f^{-1}(D_1) \rightarrow D_1 \) and \( g : g^{-1}(D_2) \rightarrow D_2 \) are isomorphic where \( D_1 \subset B_1 \) is a disk containing some critical values \( q_1^{(1)}, \ldots, q_k^{(1)} \) and \( D_2 \subset B_2 \) is a disk containing \( q_1^{(2)}, \ldots, q_k^{(2)} \). Then, the manifolds \( X \setminus f^{-1}(D_1) \) and \( Y \setminus g^{-1}(D_2) \) have a diffeomorphic boundary, and after reversing the orientation of one of them, this diffeomorphism can be chosen to be fiber-preserving and orientation-reversing. Fix such a diffeomorphism \( \phi \) and then glue \( Y \setminus g^{-1}(D_2) \) with the reversed orientation, to \( X \setminus f^{-1}(D_1) \) using this diffeomorphism \( \phi \). Note that the resulting manifold, denoted by \( X - Y \), inherits a natural orientation and admits a smooth fibration \( f \cup g : X \setminus f^{-1}(D_1) \cup Y \setminus g^{-1}(D_2) \rightarrow B_1 \# B_2 \). This is a Lefschetz fibration with \( m - k \) singular fibers. In particular, for \( k = m \), we get a surface bundle over a surface. In general, after repeatedly subtracting Lefschetz fibrations, we get \( X = Y_1 - Y_2 - \cdots - Y_n \), a surface bundle over a surface, under the following assumptions. Let \( f : X \rightarrow B_0 \) be a Lefschetz fibration with \( m \) critical values \( \{q_1^{(0)}, \ldots, q_1^{(n)}\}, \ldots, \{q_1^{(0)}, \ldots, q_n^{(0)}\} \) and \( g : Y_1 \rightarrow B_1 \), \( \cdots, g_n : Y_n \rightarrow B_n \) be Lefschetz fibrations with critical values \( \{q_1^{(1)}, \ldots, q_1^{(l)}\}, \ldots, \{q_1^{(1)}, \ldots, q_n^{(1)}\} \), respectively. We assume that \( k_1 + \cdots + k_n = m \) and that \( f : f^{-1}(D_{0,i}) \rightarrow D_{0,i} \) is isomorphic to \( g_i : g_i^{-1}(D_i) \rightarrow D_i \) for each \( 1 \leq i \leq n \), where each \( D_{0,i} \subset B_0 \) is a disk containing \( q_1^{(0)}, \ldots, q_{k_i}^{(0)} \) and \( D_i \subset B_i \) is a disk containing \( q_1^{(i)}, \ldots, q_{k_i}^{(i)} \).

In order to use the subtraction method explained above, we need to construct the building blocks \( X \) and \( Y_i \)'s. First, we describe various gluing pieces \( Y_i \).

**Proposition 3.1.** \( \square \) Let \( f \geq 3 \) and let \( a \) be a simple closed curve on \( \Sigma_f \). In the mapping class group \( \text{Mod}(\Sigma_f) \),
(a) \( t_a^2 \) can be written as a product of two commutators,
(b) if \( a \) is nonseparating, then \( t_a^4 \) can be written as a product of three commutators.

**Remark 3.2.** This proposition gives us two genus \( f \geq 3 \) Lefschetz fibrations \( Y_1 \rightarrow \Sigma_2 \) and \( Y_2 \rightarrow \Sigma_3 \) whose monodromy factorizations are given by \([f_1, g_1][f_2, g_2]t_a^2 = 1\) and \([f_3, g_3][f_4, g_4][f_5, g_5]t_a^4 = 1\) for some mapping classes \( f_i, g_i \in \text{Mod}(\Sigma_f) \) for \( 1 \leq i \leq 5 \). Generally, for every \( n \), we can obtain a Lefschetz fibration which has \( n \) singular fibers and the monodromy \( t_a^n \) using a daisy relation.

The following two propositions allow us to glue building blocks along more complicated monodromies in the sense that they are products of Dehn twists along distinct simple closed curves.

**Proposition 3.3.** Let \( f \geq 5 \) and let \( b, c \) be disjoint simple closed curves on \( \Sigma_f \) such that \( \Sigma_f - b - c \) is connected. In \( \text{Mod}(\Sigma_f) \), \( t_b^2 t_c^2 \) can be written as a product of three commutators.

**Proof.** We may assume \( b \) and \( c \) are embedded, as shown in Figure 1, because any pair of simple closed curves whose complement in \( \Sigma_f \) is connected is topologically equivalent. On the planar surface \( \Sigma_f \) in Figure 1, the following four lantern relations hold. \( L_1 := t_a t_b t_c t_d t_y t_z t_x \), \( L_2 := t_d t_a t_d t_a t_d t_x t_y t_z \), \( L_3 := t_a t_b t_c t_d t_y t_z t_x \), \( L_4 := t_a t_c t_b t_d t_y t_x t_z \). Here, \( D_1 \) is an interior curve surrounding two boundary curves except \( d_1 \), and all other curves denoted by capital letters are defined similarly. After embedding \( \Sigma_f \) into \( \Sigma_f \) with \( f \geq 5 \), as shown in Figure 1, we have \( 1 = L_1 \cdot L_2 \cdot L_3 \cdot L_4 = t_b t_c t_d t_y t_z t_x \).
Figure 1. Supports of four lantern relations and an embedding of $\Sigma^7_0$ into a genus 5 surface

Figure 2. Supports of two lantern relations embedded in a genus 6 surface

$\begin{align*}
t_b^{-1} t_c^{-1} t_{c_1}^{-1} \text{ in } \text{Mod}(\Sigma_f). & \quad \text{Since both } \Sigma_f - D_2 - d_2 \text{ and } \Sigma_f - D_1 - d_1 \text{ are connected, } \{d_2, D_2\} \text{ and } \{D_1, d_1\} \text{ are topologically equivalent and then } t_{D_2} t_{d_2}^{-1} t_{D_1} t_{d_1}^{-1} = [t_{D_2} t_{d_2}^{-1}, \phi_1] \text{ for some } \phi_1 \in \text{Mod}(\Sigma_f). \\
t_{A_3}^{-1} t_{A_2}^{-1} & \quad \text{Similarly, } t_{A_3}^{-1} t_{A_2}^{-1} = [t_{A_3}^{-1}, \phi_2] \text{ and } t_{C_2}^{-1} t_{C_1}^{-1} = [t_{C_2}^{-1}, \phi_3] \text{ for some } \phi_2, \phi_3 \in \text{Mod}(\Sigma_f). \\
t_b^2 t_c^2 & \quad \text{Therefore, } t_b^2 t_c^2 = [t_{D_2} t_{d_2}^{-1}, \phi_1] (t_{A_3}^{-1})^{-1} [t_{A_3}^{-1}, \phi_2] t_{c_2}^{-1} [t_{C_2}^{-1}, \phi_3]. \quad \square
\end{align*}$

Proposition 3.4. Let $f \geq 6$ and let $\beta, \gamma$ be simple closed curves on $\Sigma_f$ embedded, as shown in Figure 2. In $\text{Mod}(\Sigma_f)$, $t_\beta t_\gamma$ can be written as a product of three commutators.
Proof. Choose two lantern relations with their supports on $\Sigma_f$, as shown in Figure 2: $L_1 := t_\gamma^{-1}t_\delta_1^{-1}t_\delta_3^{-1}t_\delta_5^{-1}t_\alpha t_z$ and $L_2 := t_\alpha t_z t_y t_\gamma^{-1}t_\delta_1^{-1}t_\delta_2^{-1}$. For interior curves, see Figure 3. It follows that $1 = L_1 L_2 = t_\gamma^{-1}t_\delta_1^{-1}t_\delta_3^{-1}t_\delta_2^{-1}t_\alpha t_z t_\gamma^{-1}t_\delta_1^{-1}t_\delta_2^{-1}t_\gamma t_\delta_1^{-1}t_\delta_2^{-1}t_\beta$. In Figure 2 and Figure 3, we can see that $\delta_1$ and $x'$ are separating curves on $\Sigma_f$ and that both $\Sigma_f - z - \delta_1$ and $\Sigma_f - \delta_2' - z'$ are homeomorphic to $\Sigma_f - z - \delta_3$. Hence, we have $t_z t_\delta_1^{-1}t_\alpha t_\gamma^{-1} = [t_z t_\delta_1^{-1}, \phi_2]$ for some $\phi_2$. Similarly, we have $t_\delta_2^{-1}t_\gamma t_\delta_3^{-1}t_x = [t_\delta_2^{-1}t_y, \phi_1]$ and $t_z t_\delta_1^{-1}t_\gamma t_\delta_1^{-1} = [t_z t_\delta_1^{-1}, \phi_3]$ for some $\phi_1$ and $\phi_3$. Therefore, $t_\delta_1 t_\gamma = [t_\delta_1^{-1}t_y, \phi_1]^\alpha [t_z t_\delta_1^{-1}, \phi_2]^\beta [t_z t_\delta_1^{-1}, \phi_3]^\gamma$.

In Proposition 11 of [6], they constructed a genus $f \geq 3$ Lefschetz fibration over a torus with 10 singular fibers using a two-holed torus relation which is also called a 3-chain relation. In the following three Propositions, we generalize this construction of a Lefschetz fibration.

**Proposition 3.5.** Let $f \geq 3$ and let $\{\alpha_1, \alpha_2\}$ be any pair of nonseparating simple closed curves on $\Sigma_f$ such that $\Sigma_f - \alpha_1 - \alpha_2$ is connected. Then there exists a genus $f$ Lefschetz fibration $X$ over $\Sigma_3$ which has six singular fibers, four of which have monodromy $t_{\alpha_1}$ and two of which have monodromy $t_{\alpha_2}$.

Proof. We use the star relation $E := t_\delta_1^{-1}t_\delta_3^{-1}t_\delta_5^{-1}(t_{\alpha_1} t_{\alpha_2} t_{\alpha_3} t_{\alpha_4})^3$ supported on $\Sigma_f$ (Figure 4). Also, consider the following lantern relations whose supports are given in Figure 5: $L_1 := t_{\alpha_1}^{-1}t_{\alpha_2}^{-1}t_{\alpha_3}^{-1}t_{\alpha_4} t_{\alpha_5} t_{\alpha_6}$, $L_2 := t_{\alpha_1}^{-1}t_{\alpha_2}^{-1}t_{\alpha_3}^{-1}t_{\alpha_4} t_{\alpha_5} t_{\alpha_6}$. Let $W_0 := t_{\beta}(t_{\alpha_1} t_{\alpha_2} t_{\alpha_3} t_{\alpha_4})$, $W_1 := t_{\beta} t_{\alpha_1} t_{\alpha_2} t_{\alpha_3} t_{\alpha_4}$, and $W_2 := t_{\beta}$. Then, by using commutativity relations and braid relations,

$$1 = E \cdot (W_0^{-1} L_1 W_0) \cdot (W_1^{-1} L_1 W_1) \cdot (W_2^{-1} L_1 W_2)$$

$$= t_\delta_1^{-1}t_\delta_3^{-1}t_\delta_5^{-1}t_{\alpha_1} t_{\alpha_2} t_{\alpha_3} t_{\alpha_4} t_{\alpha_5} t_{\alpha_6} t_{\alpha_1} t_{\alpha_2} t_{\alpha_3} t_{\alpha_4} t_{\alpha_5} t_{\alpha_6} t_{\alpha_1} t_{\alpha_2} t_{\alpha_3} t_{\alpha_4} t_{\alpha_5} t_{\alpha_6} t_{\alpha_1} t_{\alpha_2} t_{\alpha_3} t_{\alpha_4} t_{\alpha_5} t_{\alpha_6} t_{\alpha_1} t_{\alpha_2} t_{\alpha_3} t_{\alpha_4} t_{\alpha_5} t_{\alpha_6} t_{\alpha_1} t_{\alpha_2} t_{\alpha_3} t_{\alpha_4} t_{\alpha_5} t_{\alpha_6} t_{\alpha_1} t_{\alpha_2} t_{\alpha_3} t_{\alpha_4} t_{\alpha_5} t_{\alpha_6}$$

$$= t_{\alpha_1} t_{\alpha_2} t_{\alpha_3} t_{\alpha_4} t_{\alpha_5} t_{\alpha_6} t_{\alpha_1} t_{\alpha_2} t_{\alpha_3} t_{\alpha_4} t_{\alpha_5} t_{\alpha_6} t_{\alpha_1} t_{\alpha_2} t_{\alpha_3} t_{\alpha_4} t_{\alpha_5} t_{\alpha_6} t_{\alpha_1} t_{\alpha_2} t_{\alpha_3} t_{\alpha_4} t_{\alpha_5} t_{\alpha_6} t_{\alpha_1} t_{\alpha_2} t_{\alpha_3} t_{\alpha_4} t_{\alpha_5} t_{\alpha_6}$$
For the last equality, we need to verify that there exists a self-homeomorphism \( \phi_1 \) of \( \Sigma_f \) sending \( \delta_1, t_{\alpha_1}^{-1}(\sigma_1), \) and \( \delta_2 \) to \( \beta, \delta_1, \) and \( \gamma_1, \) respectively. First, it is easy to check that \( \sigma_1 = t_\beta^{-1} t_{\alpha_2}^{-1} t_{\alpha_3}^{-1} (\beta). \) Hence, the self-homeomorphism \( t_{\alpha_2} t_{\alpha_3}^{-1} t_{\alpha_4} \) sends \( \delta_1, t_{\alpha_1}^{-1}(\sigma_1), \) and \( \delta_2 \) to \( \delta_1, \beta, \) and \( \delta_2, \) respectively. Also, there exists a homeomorphism sending \( \delta_1, \beta, \) and \( \delta_2 \) to \( \beta, \delta_1, \) and \( \gamma_1, \) respectively, because both \( \Sigma_f - \delta_1 - \beta - \delta_2 \) and \( \Sigma_f - \beta - \delta_1 - \gamma_1 \) are homeomorphic to \( \Sigma_f^{0, -3}. \) The composition of these two homeomorphisms is the required \( \phi_1. \) The existence of \( \phi_2 \) and \( \phi_3 \) can be proven in a similar way because \( \sigma_2 = t_\beta^{-1} t_{\alpha_1}^{-1} t_{\alpha_2}^{-1} (\beta). \) Finally, we get the required Lefschetz fibration over \( \Sigma_3 \) with fiber \( \Sigma_f \) whose monodromy factorization is given by \( [t_\delta^{-1} t_{\alpha_1}^{-1}(\sigma_1) t_\delta^{-1}, \phi_1][t_\delta^{-1} t_{\alpha_1}^{-1}(\sigma_1) t_\delta^{-1}, \phi_2][t_\delta^{-1} t_{\alpha_2}^{-1}(\sigma_2) t_\delta^{-1}, \phi_3][t_\delta^{-1} t_{\alpha_3}^{-1}(\sigma_3) t_\delta^{-1}, \phi_4][t_\delta^{-1} t_{\alpha_4}^{-1}(\sigma_4) t_\delta^{-1}, \phi_5][t_\delta^{-1} t_{\alpha_5}^{-1}(\sigma_5) t_\delta^{-1}, \phi_6]=1. \)

**Proposition 3.6.** Let \( f \geq 4 \) and let \( \{\alpha_2, \alpha_3\} \) be any pair of nonseparating simple closed curves on \( \Sigma_f \) such that \( \Sigma_f - \alpha_2 - \alpha_3 \) is connected. Then there is a genus \( f \) Lefschetz fibration \( Z \) over \( \Sigma_4 \) which has four singular fibers, two of which have monodromy \( t_{\alpha_2} \) and another two of which have monodromy \( t_{\alpha_3}. \)
We use the 4-holed torus relation \(^{20}\) and lantern relations. Let \(E_2 := t_{\delta_4}^{-1} t_{\delta_3}^{-1} t_{\delta_2}^{-1} t_{\delta_1}^{-1} t_{\alpha_1} t_{\alpha_3} t_{\beta} t_{\alpha_4} t_{\beta} t_{\alpha_4} t_{\beta} t_{\alpha_3} t_{\beta} t_{\alpha_2} t_{\alpha_2} t_{\beta}\). We embed the support of this relation into \(\Sigma f\), as shown in Figure 6. Let \(L_5 := t_{\delta_4}^{-1} t_{\delta_3}^{-1} t_{\delta_2}^{-1} t_{\delta_1}^{-1} t_{\alpha_1} t_{\alpha_3} t_{\beta}\) and \(L_6 := t_{\delta_4}^{-1} t_{\delta_3}^{-1} t_{\delta_2}^{-1} t_{\delta_1}^{-1} t_{\alpha_1} t_{\alpha_3} t_{\gamma_3}\). For the supports of lanterns, see Figure 7. Let \(w_1 := t_{\beta} t_{\alpha_2} t_{\alpha_3} t_{\beta} t_{\alpha_4} t_{\beta} t_{\alpha_2} t_{\alpha_4} t_{\beta}\), \(w_2 := t_{\beta} t_{\alpha_1} t_{\alpha_3} t_{\beta} t_{\alpha_2} t_{\alpha_2} t_{\beta}\), and \(w_3 := t_{\beta} t_{\alpha_2} t_{\alpha_4} t_{\beta}\). Then, from commutativity relations and braid relations,

\[
1 = E_2 \cdot L_5^{w_1} \cdot L_6^{w_2} \cdot L_5^{w_3} \cdot L_6^{w_4}
\]

Figure 6. Support of a four-holed torus relation embedded in a genus 4 surface

Figure 7. Supports of two lantern relations

Proof. We use the 4-holed torus relation \(^{20}\) and lantern relations. Let \(E_2 := t_{\delta_4}^{-1} t_{\delta_3}^{-1} t_{\delta_2}^{-1} t_{\delta_1}^{-1} t_{\alpha_1} t_{\alpha_3} t_{\beta} t_{\alpha_4} t_{\beta} t_{\alpha_4} t_{\beta} t_{\alpha_3} t_{\beta} t_{\alpha_2} t_{\alpha_2} t_{\beta}\). We embed the support of this relation into \(\Sigma f\), as shown in Figure 6. Let \(L_5 := t_{\delta_4}^{-1} t_{\delta_3}^{-1} t_{\delta_2}^{-1} t_{\delta_1}^{-1} t_{\alpha_1} t_{\alpha_3} t_{\beta}\) and \(L_6 := t_{\delta_4}^{-1} t_{\delta_3}^{-1} t_{\delta_2}^{-1} t_{\delta_1}^{-1} t_{\alpha_1} t_{\alpha_3} t_{\gamma_3}\). For the supports of lanterns, see Figure 7. Let \(w_1 := t_{\beta} t_{\alpha_2} t_{\alpha_3} t_{\beta} t_{\alpha_4} t_{\beta} t_{\alpha_2} t_{\alpha_4} t_{\beta}\), \(w_2 := t_{\beta} t_{\alpha_1} t_{\alpha_3} t_{\beta} t_{\alpha_2} t_{\alpha_2} t_{\beta}\), and \(w_3 := t_{\beta} t_{\alpha_2} t_{\alpha_4} t_{\beta}\). Then, from commutativity relations and braid relations,
For the fifth equality, we need to find certain $\phi_1, \phi_2, \phi_3$ and $\phi_4$. For $\phi_1$, it is sufficient to verify that $\{\delta_2, t_{\alpha_2}^{-1}(\sigma_2), \delta_3\}$ is topologically equivalent to $\{\beta, \delta_1, \gamma_2\}$. This is because $\{\delta_2, t_{\alpha_2}^{-1}(\sigma_2), \delta_3\}$ is topologically equivalent to $\{\delta_2, \beta, \delta_3\}$, and then $\{\delta_2, \beta, \delta_3\}$ to $\{\beta, \delta_1, \gamma_2\}$. The arguments for $\phi_2, \phi_3$, and $\phi_4$ are similar. For these, we can check that $\{\delta_3, t_{\alpha_3}^{-1}(\sigma_3), \delta_4\}$ is topologically equivalent to $\{\beta, \delta_1, \gamma_3\}, \{\delta_2, t_{\alpha_2}^{-1}(\sigma_2), \delta_3\}$ is topologically equivalent to $\{\beta, \delta_2, \gamma_2\}$, and $\{\delta_3, t_{\alpha_3}^{-1}(\sigma_3), \delta_4\}$ is topologically equivalent to $\{\beta, \delta_3, \gamma_3\}$.

Proposition 3.7. Let $f \geq 6$ and let $\beta, \gamma$ be simple closed curves on $\Sigma_f$ embedded, as shown in Figure 2. Then there is a genus $f$ Lefschetz fibration $W$ over $\Sigma_3$ which has two singular fibers, one of which has monodromy $t_\beta$ and another has monodromy $t_\gamma$.

Proof. There is a 9-holed torus relation $E_7 := t_{\delta_1}^{-1} t_{\delta_2}^{-1} \cdots t_{\delta_8}^{-1} t_{\gamma_0}^{-1} t_{\delta_8} t_{\sigma_8} t_{\alpha_1} t_{\beta_5} t_{\alpha_2} t_{\sigma_2} t_{\delta_3} t_{\sigma_3} t_{\alpha_3}$ (see its support in orange in Figure 8 and see Figure 9 for its interior curves), where we use the identifications $(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7, \alpha_8, \alpha_9) \to (\delta_5, \delta_6, \alpha_7, \alpha_8, \alpha_9, \alpha_1, \alpha_2, \alpha_3, \alpha_4)$ to go from Figure 9 in [20] to Figure 9 in this article. Here, each $\beta_i = t_{\alpha_i}(\beta)$ as in [20]. If we combine this relation $E_7$ and one more lantern relation $L_8 := t_{\delta_9}^{-1} t_{\delta_{10}}^{-1} t_{\sigma_9} t_{\alpha_9} t_{\beta_5}^{-1} t_{\alpha_10}$ (see its support in blue in Figure 8), then we get the following 10-holed torus relation $E_8 := t_{\delta_1}^{-1} t_{\delta_2}^{-1} \cdots t_{\delta_{10}}^{-1} t_{\beta_5} t_{\alpha_10} t_{\delta_6} t_{\sigma_6} t_{\delta_7} t_{\sigma_7} t_{\alpha_11} t_{\delta_8} t_{\sigma_8} t_{\delta_9} t_{\sigma_9} t_{\alpha_12}$. Let $\beta_5 = (t_{\sigma_1} t_{\sigma_2} t_{\sigma_3} t_{\sigma_4} t_{\sigma_5})^{-1}(\beta_5)$ and $\beta_2 = (t_{\sigma_1} t_{\sigma_2} t_{\sigma_3} t_{\sigma_4} t_{\sigma_5})^{-1}(\beta_2)$. Then, by using commutativity relations and braid relations,

$$1 = t_{\delta_1}^{-1} t_{\delta_2}^{-1} \cdots t_{\delta_{10}}^{-1} t_{\beta_5} t_{\alpha_10} t_{\delta_6} t_{\sigma_6} t_{\delta_7} t_{\sigma_7} t_{\alpha_11} t_{\delta_8} t_{\sigma_8} t_{\delta_9} t_{\sigma_9} t_{\alpha_12} t_{\delta_1}^{-1} t_{\delta_2}^{-1} t_{\delta_3}^{-1} t_{\delta_4}^{-1} t_{\delta_5}^{-1} t_{\delta_6}^{-1} t_{\delta_7}^{-1} t_{\delta_8}^{-1} t_{\delta_9}^{-1} t_{\alpha_10}$$

$$= t_{\delta_1}^{-1} t_{\delta_2}^{-1} \cdots t_{\delta_{10}}^{-1} t_{\beta_5} t_{\alpha_10} t_{\delta_6} t_{\sigma_6} t_{\delta_7} t_{\sigma_7} t_{\alpha_11} t_{\delta_8} t_{\sigma_8} t_{\delta_9} t_{\sigma_9} t_{\alpha_12} t_{\delta_1}^{-1} t_{\delta_2}^{-1} t_{\delta_3}^{-1} t_{\delta_4}^{-1} t_{\delta_5}^{-1} t_{\delta_6}^{-1} t_{\delta_7}^{-1} t_{\delta_8}^{-1} t_{\delta_9}^{-1} t_{\alpha_10}$$

$$= t_{\delta_1}^{-1} t_{\delta_2}^{-1} \cdots t_{\delta_{10}}^{-1} t_{\delta_1}^{-1} t_{\delta_2}^{-1} t_{\delta_3}^{-1} t_{\delta_4}^{-1} t_{\delta_5}^{-1} t_{\delta_6}^{-1} t_{\delta_7}^{-1} t_{\delta_8}^{-1} t_{\delta_9}^{-1} t_{\alpha_10}$$

$$= \{t_{\delta_1}^{-1} t_{\delta_2}^{-1} t_{\delta_3}^{-1} t_{\delta_4}^{-1} t_{\delta_5}^{-1} t_{\delta_6}^{-1} t_{\delta_7}^{-1} t_{\delta_8}^{-1} t_{\delta_9}^{-1} t_{\alpha_10}\}$$

$$= \{t_{\delta_1}^{-1} t_{\delta_2}^{-1} t_{\delta_3}^{-1} t_{\delta_4}^{-1} t_{\delta_5}^{-1} t_{\delta_6}^{-1} t_{\delta_7}^{-1} t_{\delta_8}^{-1} t_{\delta_9}^{-1} t_{\alpha_10}\}$$

$$= \{t_{\delta_1}^{-1} t_{\delta_2}^{-1} t_{\delta_3}^{-1} t_{\delta_4}^{-1} t_{\delta_5}^{-1} t_{\delta_6}^{-1} t_{\delta_7}^{-1} t_{\delta_8}^{-1} t_{\delta_9}^{-1} t_{\alpha_10}\}$$

For the last equality, we need to verify that $\{\delta_1, t_{\beta_5}(\sigma_3), \delta_3, t_{\beta_5}(\sigma_4), \delta_10, t_{\beta_5}(\sigma_5)\}$ is topologically equivalent to $\{t_{\beta_5}(\sigma_8), \delta_9, t_{\beta_5}(\sigma_7), \delta_7, t_{\beta_5}(\sigma_6), \delta_2\}$. This follows from the fact that both $\Sigma_f - \delta_1 - \delta_3 - \delta_10 - \sigma_3 - \sigma_4 - \sigma_5$ and $\Sigma_f - \delta_2 - \delta_6 - \delta_9 - \sigma_6 - \sigma_7 - \sigma_8$ are connected. For $\phi_2$ and $\phi_3$, it is easy to check that $\Sigma_f - \delta_5 - \sigma_3 - \delta_8 - \delta_10$ is topologically equivalent to $\{\delta_3, t_{\alpha_3}^{-1}(\sigma_3), \delta_4\}$ and that $\{\beta_2, \alpha_8\}$ is topologically equivalent to $\{\beta, \alpha_8\}$ and $\{\alpha_8, t_{\alpha_8}^{-1}(\beta)\}$ is topologically equivalent to $\{\alpha_8, \beta\}$. Finally, observe that $\{\beta_5, t_{\alpha_5}^{-1}(\sigma_9)\}$ is topologically equivalent to $\{\beta, t_{\alpha_5}^{-1}(\sigma_9)\}$ and $t_{\alpha_5}^{-1}(\sigma_9) = \gamma$. □
Figure 8. Supports for a 9-holed torus relation and a lantern relation and their embeddings into a genus 6 surface

Figure 9. Interior curves for a 10-holed torus relation

4. Signature computation

In order to compute the signature of the total space of surface bundles, we first review the definition of Meyer’s signature cocycle.

**Definition.** For any given $A, B \in Sp(2g, \mathbb{R})$, consider the subspace

$$V_{A,B} := \{ (x, y) \in \mathbb{R}^{2g} \times \mathbb{R}^{2g} | (A^{-1} - I_{2g})x + (B - I_{2g})y = 0 \}$$

of the real vector space $\mathbb{R}^{2g} \times \mathbb{R}^{2g}$ and the bilinear form $\langle, \rangle_{A,B} : (\mathbb{R}^{2g} \times \mathbb{R}^{2g}) \times (\mathbb{R}^{2g} \times \mathbb{R}^{2g}) \to \mathbb{R}$ defined by $\langle (x_1, y_1), (x_2, y_2) \rangle_{A,B} := (x_1 + y_1) \cdot J(I_{2g} - B)y_2$, where $\cdot$ is the inner product of $\mathbb{R}^{2g}$ and $J$ is the matrix representing the multiplication by $\sqrt{-1}$ on $\mathbb{R}^{2g} = \mathbb{C}^g$. Since the restriction of $\langle, \rangle_{A,B}$ on $V_{A,B}$ is symmetric, we can define $\tau_g(A, B) := \text{sign}(\langle, \rangle_{A,B}, V_{A,B})$.

We denote by $\psi : \text{Mod}(\Sigma_g) \to Sp(2g : \mathbb{R})$ the symplectic representation of the mapping class group.
Theorem 4.1. Let $E_{A,B} \to P$ be an oriented $\Sigma_g$ bundle over a pair of pants $P$ whose monodromy representation $\chi$ composed with the symplectic representation $\psi$ is given by $\psi \circ \chi : \pi_1(P,*) \to Sp(2g,\mathbb{R})$ sending one generator to $A$ and the other to $B$. Then $\sigma(E_{A,B}) = -\tau_g(A,B)$. We can easily check that $\tau_g$ is a 2-cocycle on the symplectic group $Sp(2g,\mathbb{R})$ using Novikov’s additivity. We call this $\tau_g$ Meyer’s signature cocycle. The pants decomposition of any base surface gives the following signature formula.

**Theorem 4.2.** Let $f : E \to \Sigma_h^g$ be an oriented surface bundle with fiber $\Sigma_g$ and monodromy representation $\chi : \pi_1(\Sigma_h^g) \to \text{Mod}(\Sigma_g)$. Fix a standard presentation of $\pi_1(\Sigma_h^g)$ as follows:

$$\pi_1(\Sigma_h^g) = \langle a_1, b_1, \cdots, a_h, b_h, c_1, \cdots, c_r \rangle \prod_{i=1}^{h} (a_i, b_i) \prod_{j=1}^{r} c_j = 1$$

and let $\tau_g$ be Meyer’s signature cocycle. Then the signature of $E$ is given by the formula

$$\sigma(E) = \sum_{i=1}^{h} \tau_g(\kappa, \beta_i) - \sum_{i=2}^{h} \tau_g(\kappa_i \cdots \kappa_{i-1}, \kappa_i) - \sum_{j=1}^{r-1} \tau_g(\kappa_1 \cdots \kappa_{j} \gamma_j \cdots \gamma_{j-1}, \gamma_j)$$

where $\alpha_i = \psi \circ \chi(a_i), \beta_i = \psi \circ \chi(b_i), \gamma_i = \psi \circ \chi(c_i)$ and $\kappa_i = [\alpha_i, \beta_i]$.

By applying this formula, we can compute the signatures of surface bundles obtained by taking out some neighborhoods of singular fibers from the Lefschetz fibrations constructed in Section 3. We used Mathemtica for computing each term in the above formula.

Meyer also provided another interpretation of the above signature formula. For this, we start with the following diagram.

\[
\begin{array}{ccc}
1 & \to & \tilde{R} \\
\downarrow & & \downarrow \tilde{\pi} \\
R & \to & \tilde{F} \\
\downarrow & & \downarrow \pi \\
1 & \to & \text{Mod}(\Sigma_g) \\
\end{array}
\]

Here, $\pi_1(\Sigma_h^g) = \langle a_1, \cdots, a_h, b_1, \cdots, b_h \rangle \prod_{i=1}^{h} [a_i, b_i] = 1$, $\tilde{F} = \langle \tilde{a}_1, \cdots, \tilde{a}_h, \tilde{b}_1, \cdots, \tilde{b}_h \rangle$, $\tilde{R}$ is the normal closure of $\tilde{r} = \prod_{i=1}^{h} [\tilde{a}_i, \tilde{b}_i]$, and $\tilde{\pi} : \tilde{a}_i \to \tilde{a}_i, \tilde{b}_i \to b_i$. The second row corresponds to the finite presentation of $\text{Mod}(\Sigma_g)$ due to Wajnryb. $F = F(S)$, where $S = \{ y_1, y_2, u_1, \cdots, u_g, z_1, \cdots, z_{g-1} \}$ and $R$ is the normal closure of $A_{i,j}^k, B_{i,j}^k, C_1^1, D_1^1, E_1^1$ (cf. [S] §3). If we have a monodromy representation $\chi : \pi_1(\Sigma_h) \to \text{Mod}(\Sigma_g)$, then there exists a homomorphism $\tilde{\chi} : \tilde{F} \to F$ such that $\chi \circ \tilde{\pi} = \pi \circ \tilde{\chi}$ since $\pi$ is surjective and $\tilde{F}$ is free. Hence we have $\tilde{\chi}(\tilde{r}) \in R \cap [F, F]$. Now define the 1-cochain $c : F \to \mathbb{Z}$ cobounding the 2-cocycle $-\pi^*\psi^*(\tau_g)$ as follows.

$$c(x) := \sum_{j=1}^{m} \tau_g(\psi(\pi(x_j))), \psi(\pi(x_j)))$$

\[
(x = \prod_{i=1}^{m} x_j, \quad x_j \in S \cup S^{-1}, \quad \bar{x}_j = \prod_{i=1}^{j} x_i)
\]

Since $\pi^*\psi^*(\tau_g) |_{R \times R} = 0$, the restriction $c |_{R} : R \to \mathbb{Z}$ is a homomorphism. The values of $c$ for the relations of Wajnryb’s presentation were calculated in [S].
Theorem 4.3. [26]

Let $p : E \to \Sigma_h$ be a $\Sigma_g$-bundle over $\Sigma_h$ and $\chi : \pi_1(\Sigma_h) \to \text{Mod}(\Sigma_g)$ be its monodromy homomorphism. Then the signature of the total space $E$ is given as follows:

$$\sigma(E) = -c \cdot \text{tr}(\tilde{\chi}(\tilde{\tau})) \quad (= \langle \psi^*\tau_g, \tilde{\chi}(\tilde{\tau})[R, F] \rangle)$$

where $\langle, \rangle$ is a pairing on the second cohomology and homology of $\text{Mod}(\Sigma_g)$.

Now, we are ready to prove our main theorem.

Proof of Theorem 4.3 (a) We apply the subtraction operation to the Lefschetz fibrations $X \to \Sigma_3$, $Y_1 \to \Sigma_2$, and $Y_2 \to \Sigma_3$ constructed in Propositions 3.5 and Proposition 3.1. Let $N_1 \subset X$ be the neighborhood of four singular fibers with coinciding vanishing cycles and $N_2 \subset X$ be the neighborhood of two singular fibers with coinciding vanishing cycles. Then the complement $X \setminus N_1 \setminus N_2$ is the $\Sigma_f$ bundle over $\Sigma_3^2$, and its signature can be computed by applying Theorem 4.2 to this bundle. More precisely to its monodromy representation $\chi : \pi_1(\Sigma_3^2) \to \text{Mod}(\Sigma_f)$ given by

$$\chi(a_1) = (t_{\delta}^{-1}, t_{\delta}^{-1})^{\alpha}, \chi(a_2) = (t_{\delta}^{-1}, t_{\delta}^{-1})^{\alpha}, \chi(a_3) = (t_{\delta}^{-1}, t_{\delta}^{-1})^{\alpha}, \chi(a_4) = (t_{\delta}^{-1}, t_{\delta}^{-1})^{\alpha},$$

where $\alpha$ is the $\Sigma_f$ bundle over $\Sigma_3^2$, and its signature can be computed by applying Theorem 4.2 to this bundle. Alternatively, we can first compute the signature of Lefschetz fibrations $\sigma(Y_1) = -2$ and $\sigma(Y_2) = -4$ (cf. Proposition 15 and Proposition 16 in [3]). In order to compute the signature of taken out parts, apply Theorem 4.1 several times and use the fact that $\sigma(N_{a}(\text{a nonseparating singular fiber})) = 0$ (cf. [28]). From these, we have $\sigma(Y_1 \setminus M_1) = (2) - (1) = -1$ and $\sigma(Y_2 \setminus M_2) = (2) - (1) = -1$. Therefore, $X - Y_1 - Y_2$ is the $\Sigma_f$ bundle over $\Sigma_3$, and $\sigma(X \setminus Y_1 \setminus Y_2) = \sigma(X \setminus N_1 \setminus N_2) + \sigma(Y_1 \setminus M_1) + \sigma(Y_2 \setminus M_2) + 2 + 1 + 4 = 4$ by Novikov additivity. Moreover, if we pullback this bundle (or, with opposite orientation) to unramified coverings of $\Sigma_n$ of degree $|n|$, then we get $b(f \geq 3, n) \leq 7|n| + 1$.

(b) Apply the subtraction operation to the Lefschetz fibrations $Z \to \Sigma_4$ and $Y_3 \to \Sigma_4$, constructed in Proposition 3.6 and Proposition 3.3, respectively. Then, $Z - Y_3$ is the required $\Sigma_f \geq 5$ bundle over $\Sigma_7$. Let $N$ be the neighborhood of all singular fibers in $Z$ and let $M$ be the neighborhood of all singular fibers in $Y_3$. By applying Theorem 4.2 to two surface bundles $Z \setminus N$ and $Y_3 \setminus M$, we get $\sigma(Z - Y_3) = \sigma(Z \setminus N) + \sigma(Y_3 \setminus M) = 2$. Let me give you another proof for verifying $\sigma(Z - Y_3) = 4$ using Theorem 4.3. From Proposition 3.6 and Proposition 3.3, we have $\tilde{\chi}(\tilde{\tau}) \equiv (L_2 \cdot L_{5}^{w_1} \cdot L_{6}^{w_1} \cdot L_{5}^{w_3} \cdot L_{6}^{w_3})^{(L_1 \cdot L_2 \cdot L_3 \cdot L_4)} \cdot \text{modcomp} \cdot \text{bradrel}$, where $g$ is a self-homeomorphism of $\Sigma_f \geq 5$ such that $g(\alpha_2) = b$ and $g(\alpha_3) = c$. Moreover, from [29], $E \equiv L_{10} \cdot (L_{9} \cdot ((C_1^{1})^{-1})^{z_0} \cdot (C_1^{-1})^{z_1}) \cdot \text{modcomp} \cdot \text{bradrel}$. Observe that for each $L_i$, four boundary curves are nonseparating and $\Sigma_f \setminus \text{supp}(L_i)$ is connected. Since the same holds for the relation $(D_1)^{-1}$, there exists a self-homeomorphism $f_i$ of $\Sigma_f$ sending the $\text{supp}((D_i)^{-1})$ to the $\text{supp}(L_i)$ for each $i$. Therefore, $\tilde{\chi}(\tilde{\tau}) \equiv ((D_1)^{-1})^{f_i} \cdot (D_1)^{-1})^{f_i_{z_0z_1}} \cdot ((C_1^{1})^{-1})^{z_0z_1} \cdot ((D_i)^{-1})^{f_i_{z_0z_1}}$. 


4 and the degree of the covering $\Sigma \overset{f}{\rightarrow} b$ degree. After applying this to the genus 3 surface bundle over $\Sigma$ of the base, the resulting surface bundle admits a fiberwise cover.

Every odd genus surface is a covering of genus three surface.

Proof of Theorem 1.4. Every odd genus surface is a covering of genus three surface. By Morita [27], after replacing a given surface bundle by a pullback to some covering of the base, the resulting surface bundle admits a fiberwise covering of any given degree. After applying this to the genus 3 surface bundle over $\Sigma_6$ with signature 4 and the degree of the covering $\Sigma_f \rightarrow \Sigma_3$, we obtain $b_f\left(\frac{f-1}{n}\right) \leq n(b_3(1)-1) + 1$.

Hence, $G_f := \lim_{n \rightarrow \infty} \frac{b_f(n)}{n} \leq \lim_{n \rightarrow \infty} \frac{2n(b_3(1)-1)+2}{(f-1)n} \leq \lim_{n \rightarrow \infty} \frac{14n+2}{15n} = \frac{14}{15}$. 

Remark 4.4. In [14, 22, 30], it was proven that $H_2(\text{Mod}(\Sigma_g) : \mathbb{Z}) \cong \mathbb{Z}$ for every $g \geq 4$ and $H_2(\text{Mod}(\Sigma_3) : \mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z}_2$ for $g = 3$. Meyer [26] proved that each generator of $H_2(\text{Mod}(\Sigma_g))/\text{Tor}$ gives us signature 4 relying on the Theorem 4.3. In order to prove this, Meyer used Birman-Hilden’s presentation of $\text{Mod}(\Sigma_3)$, and Endo [8] reproved this using a simpler presentation due to Wajnryb [32]. By taking $\tilde{\chi}(\tilde{f})$ as different representatives for a generator of $H_2(\text{Mod}(\Sigma_g))/\text{Tor}$, we can construct various surface bundles with a fixed signature 4 as we have seen in the proof of Theorem 1.2. Therefore, the problem to determine $b(f, n)$ is to find the most effective representative $\tilde{\chi}(\tilde{f})$, in the sense of commutator length, for $n$ times generator of $H_2(\text{Mod}(\Sigma_3))/\text{Tor}$.

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