Shock waves in one-dimensional Heisenberg ferromagnets.

V. V. Konotop† *, M. Salerno‡ † and S. Takeno§

†Department of Physics, University of Madeira,
Praça do Município, 9000 Funchal, Portugal and Center of Science and Technology of Madeira (CITMA), Rua da Alfândega, 75-5’, Funchal, P-9000 Portugal

‡Department of Physical Sciences ”E.R. Caianiello”, University of Salerno,
I-84100, Salerno, Italy

§Faculty of Information Science, Osaka Institute of Technology,
1-79-1 Hirakata, Osaka 573-01, Japan

We use the $SU(2)$ coherent state path integral formulation with the stationary phase approximation to investigate, both analytically and numerically, the existence of shock waves in the one-dimensional Heisenberg ferromagnets with anisotropic exchange interaction. As a result we show the existence of shock waves of two types, ”bright” and ”dark”, which can be interpreted as moving magnetic domains.

Typeset Using \textit{REVTEX}

* Also at Center of Mathematical Sciences, University of Madeira, Praça do Município, P-9000 Funchal, Portugal.

† Also at Istituto Nazionale di Fisica della Materia (INFM), Unità di Salerno, I-84100 Salerno, Italy.
The Heisenberg model is certainly one of the most important models of condensed matter physics on which a large amount of work has been done [1]. In spite of this, the study of its properties, both at the quantum and at the classical level, is still a non exhausted subject of ever-continuing interest. In the one dimensional case the model is exactly solvable and it is known that the excitation spectrum consists of quantum solitons which in the semiclassical limit can be seen as bound states of many magnons. On the other hand, it has recently been shown that nonlinear lattices, besides solitons, may support other kinds of excitation which behave at the initial stages of their evolution like shock waves in liquids or gases. Such waves have been reported both in integrable [2–4] and nonintegrable [5–8] lattices.

The aim of the present paper is to show the existence of shock waves in the one-dimensional Heisenberg ferromagnet. To this end we use the $SU(2)$ coherent state [9] path integral formulation with the stationary phase approximation [10], to derive classical equation of motion out of the original quantum spin model. Within the framework of this approximation the system is shown to be described by a discrete nonlinear Schrödinger (DNLS) like equation which has a hamiltonian structure and a non standard Poisson bracket. Moreover, the classical system preserves the conservation of the analogue of the quantum total spin (for isotropic case) operator as well as of its z-projection. We find that for suitable condition the excitations of this system naturally display shock waves with sharp rectangular profiles moving on uniform backgrounds. Such waves can exist both above background (bright shock) and below background (dark shock) and on the contrary of other excitations, which may decay into soliton trains or background radiation, they are very stable. We remark that similar solutions were found also in a deformable DNLS system [8,11] and in a chain of two level atoms describing the propagation of Frenkel excitons [12]. We give a numerical and an analytical description of these phenomena both in terms of dispersion relations and in terms of a small amplitude multiscale expansion.

The quantum Heisenberg Hamiltonian is written as
\[
H = - \sum_{(m,n)} J(n, m) \left[ (\hat{S}_n^x \hat{S}_m^x + \hat{S}_n^y \hat{S}_m^y) + \lambda \hat{S}_n^z \hat{S}_m^z \right]
\]  

(1)

where \( \hat{S}_n = (\hat{S}_n^x, \hat{S}_n^y, \hat{S}_n^z) \) are spin operators of spin magnitude \( S \), \( J(n, m) \) is the exchange interaction constant and \( \lambda \) is the anisotropy of the exchange XY-like (\( \lambda < 1 \)) and Ising-like (\( \lambda > 1 \)) interactions. In what follows we consider the case of nearest neighbor interaction \( J(n, m) = J \cdot (\delta_{n,m+1} + \delta_{n,m-1}) \) with \( J > 0 \) and we denote, as usual, with \( \hat{S}_n^\pm = \hat{S}_n^x \pm i \hat{S}_n^y \) the raising and lowering operators. To derive classical equation of motion it is suitable to use \( SU(2) \) coherent states

\[
|\mu_n\rangle = \frac{\exp(\mu_n \hat{S}_n^+)}{\sqrt{1 + |\mu_n|^2}} |\downarrow\rangle_n
\]

(2)

in terms of which we write the state of the system (1) in the form

\[
|\Lambda\rangle = \prod_n |\mu_n\rangle.
\]

(3)

Here \( \mu_n \) is a complex variable and \( |\downarrow\rangle_n \) denotes the ground state of a single site. The time evolution operator of the Heisenberg hamiltonian between the initial state \( |\Lambda\rangle_i \) at time \( t_i \) and a final state \( |\Lambda\rangle_f \) at time \( t_f \) can be written in terms of path integral as

\[
i\langle\Lambda| \exp(-i\frac{H t}{\hbar})|\Lambda\rangle_f = \int \partial(L) \exp(i\frac{\hbar}{S})
\]

(4)

where \( S = \int_{t_i}^{t_f} L dt \) with \( L \) given by

\[
L = \frac{i\hbar}{2} \sum_n \frac{1}{1 + |\mu_n|^2} \left( \tilde{\mu}_n \frac{d\mu_n}{dt} - \mu_n \frac{d\tilde{\mu}_n}{dt} \right) - \langle\Lambda| H |\Lambda\rangle.
\]

(5)

Using the stationary phase approximation \( \delta S = 0 \) in (4) one readily obtains the following equation of motion

\[
\frac{d\mu_n}{dt} = -J \left( \frac{\mu_{n+1}^2 - \mu_n^2 \tilde{\mu}_{n+1}}{1 + |\mu_{n+1}|^2} + \frac{\mu_{n-1}^2 - \mu_n^2 \tilde{\mu}_{n-1}}{1 + |\mu_{n-1}|^2} \right)
\]

\[
+ \lambda J \left( \frac{\mu_n (1 - |\mu_{n+1}|^2)}{1 + |\mu_{n+1}|^2} + \frac{\mu_n (1 - |\mu_{n-1}|^2)}{1 + |\mu_{n-1}|^2} \right).
\]

(6)

It is of interest to note that (6) and its complex conjugated follow from Hamilton’s equations
\[
\frac{d\mu_n}{dt} = \{\mu_n, H_c\}, \quad \frac{d\bar{\mu}_n}{dt} = \{\bar{\mu}_n, H_c\}
\]  
(7)

with the noncanonical Poisson bracket

\[
\{f, g\} = -\frac{i}{\hbar} \sum_n (1 + |\mu_n|^2)^2 \left[ \frac{\partial f}{\partial \mu_n} \frac{\partial g}{\partial \bar{\mu}_n} - \frac{\partial f}{\partial \bar{\mu}_n} \frac{\partial g}{\partial \mu_n} \right]
\]  
(8)

and with the classical hamiltonian \(H_c\) given by \(\langle \Lambda | H | \Lambda \rangle\) i.e.

\[
H_c = -J \sum_n \left( \frac{\bar{\mu}_{n+1} \mu_n + \bar{\mu}_n \mu_{n+1}}{(1 + |\mu_n|^2)(1 + |\mu_{n+1}|^2)} + \frac{\lambda}{2} \frac{(1 - |\mu_n|^2)(1 - |\mu_{n+1}|^2)}{(1 + |\mu_n|^2)(1 + |\mu_{n+1}|^2)} \right)
\]  
(9)

One readily checks that the conservation of the z-component of the quantum total spin is reflected in the classical system \([\text{Eq. 6}]\) in the conservation of the quantity

\[
s_z \equiv \langle \Lambda | \sum_n \hat{S}_n^z | \Lambda \rangle = \frac{1}{2} \sum_n \frac{(1 - |\mu_n|^2)}{(1 + |\mu_n|^2)}.
\]  
(10)

In the isotropic case \((\lambda = 1)\) besides \([\text{Eq. 10}]\) there is another conserved quantity which is just the analogue of the total spin of the quantum system

\[
s^2 \equiv \langle \Lambda | \hat{S}^2 | \Lambda \rangle = \sum_{n,m} \frac{\bar{\mu}_n \mu_m}{(1 + |\mu_n|^2)(1 + |\mu_m|^2)}
\]  
(11)

[in an infinite chain with \(|\mu_n| \to \rho\) as \(n \to \pm \infty\), expressions \([\text{Eq. 10}]\) and \([\text{Eq. 11}]\) should be properly normalized to be finite quantities]. The classical system \([\text{Eq. 6}]\) keeps, therefore, some important symmetry of the original quantum model [note also that the dynamical variables \((\mu_n, \bar{\mu}_n)\) can be related by an inverse stereographic projection to the motion of vectors on a sphere (classical spins)]. In order to investigate shock solutions of \([\text{Eq. 6}]\) it is suitable to consider solution propagating against a nonzero background of the form \(\mu_n^{(0)} = \rho \exp(-i\omega t + ik n)\) where \(\rho < 1\) and

\[
\omega = J(\lambda - 2 \cos(k)) \frac{1 - \rho^2}{1 + \rho^2}.
\]  
(12)

(here and in the following we fix \(\hbar = 1\)). The stability of the background can be studied with help of the substitution \(\mu_n = (1 + \psi_n) \rho \exp(-i\omega t + ik n)\), where \(|\psi_n| \ll |\mu_n|\), in \([\text{Eq. 6}]\). By linearization we obtain the dispersion relation \(\Omega(K)\) associated with the linear equation for \(\psi_n \left[ \times \exp(-i\Omega t + iKn) \right]\)
\[ \Omega = 2 \frac{1 - \rho^2}{1 + \rho^2} J \sin(k) \sin(K) \pm 2 \sqrt{2} \frac{J \sin \left( \frac{K}{2} \right)}{1 + \rho^2} \times \left\{ \cos^2(k)(1 + \rho^2)^2 - \cos^2(k) \cos(K)(1 - \rho^2)^2 - 4\lambda \rho^2 \cos(k) \cos(K) \right\}^{1/2}. \] (13)

Thus the background is stable (i.e. \( \Omega \) is real at all \( K \)), if \( \cos k > \lambda \) and \( 0 > \cos k > -\frac{2\rho^2}{1 + \rho^2} \lambda \).

Naturally, in what follows the analysis will be restricted to this region of the parameters.

In order to get the equation governing the initial stages of the evolution of a shock wave we use the small amplitude expansion \( \mu_n = (\rho + a_n) \exp[i(-\omega t + kn - \phi_n)] \), where the two real quantities \( a_n \) and \( \phi_n \) are considered depending on slow variables \( X = \gamma n, T = \gamma t, \) and \( \tau = \gamma^3 t, \) (with \( \gamma \ll 1 \)) and are represented in the form \( a_n = \gamma^2 a_n^{(0)} + \gamma^4 a_n^{(1)} + \ldots, \)

\( \phi_n = \gamma \phi_n^{(0)} + \gamma^2 \phi_n^{(1)} + \ldots. \) Collecting all the terms of the same order in \( \gamma \) we arrive at a series of equations. In the zero order we recover the dispersion relation (12). In the second and third orders we get the following equations

\[ \frac{\partial \phi^{(0)}}{\partial T} = \frac{8\rho J}{(1 + \rho^2)^2} (\cos k - \lambda) a^{(0)} - 2J \sin(k) \frac{1 - \rho^2}{1 + \rho^2} \frac{\partial \phi^{(0)}}{\partial X}, \] (14)

\[ \frac{\partial a^{(0)}}{\partial T} = \rho J \cos(k) \frac{\partial^2 \phi^{(0)}}{\partial X^2} - 2\sin(k) \frac{1 - \rho^2}{1 + \rho^2} \frac{\partial a^{(0)}}{\partial X}. \] (15)

It is suitable to introduce new variables \((\xi_\pm, T)\) instead of \((X, T)\), where \( \xi_\pm = X - c_\pm T \) and the velocities \( c_\pm \) are given by

\[ c_\pm = \frac{2J}{(1 + \rho^2)} \left[ (1 - \rho^2) \sin k \pm \rho \sqrt{2 \cos k(\cos k - \lambda)} \right]. \] (16)

Comparing this result with (13) one sees that \( c_\pm = d\Omega_\pm/dK \) at \( K = 0 \), i.e. \( c_\pm \) are group velocities of two branches of the spectrum in the center of the BZ. Then it follows from (14), (13) that solutions \( a^{(0)} = a^{(0)}(\xi_\pm) = a_\pm \) and \( \phi^{(0)} = \phi^{(0)}(\xi_\pm) \) are related by

\[ a_\pm = \mp(1 + \rho^2) \frac{\sqrt{\cos k}}{2 \sqrt{2(\cos k - \lambda)}} \frac{\partial \phi^{(0)}}{\partial \xi_\pm}. \] (17)

For brevity we drop out the explicit form of the equations appearing in the forth and fifth orders of \( \gamma \) and present simply the condition of their compatibility. This condition has the form of a Korteweg - de Vries (KdV) equation.
\[
\frac{\partial a_{\pm}}{\partial \tau} + \alpha(k)a_{\pm} \frac{\partial a_{\pm}}{\partial \xi_{\pm}} + \beta(k)\frac{\partial^3 a_{\pm}}{\partial \xi_{\pm}^3} = 0
\]  
(18)

with

\[
\alpha(k) = \frac{4J}{(1 + \rho^2)^2} \left[ -5\rho \sin k + 2\lambda \tan k 
\right.
\pm(1 - 3\rho^2) \sqrt{2 \cos k (\cos k - \lambda)} \pm \frac{\lambda \sqrt{2 \cos^3 k}}{2\sqrt{\cos k - \lambda}} (3 - \rho^2) \bigg]\]  
(19)

and

\[
\beta(k) = \frac{J}{4\rho(1 + \rho^2)\sqrt{\cos k - \lambda}} \left\{ \frac{8}{3}\rho \sin k \sqrt{\cos k - \lambda}(1 - \rho^2) 
\pm\sqrt{2 \cos k} \left[ \frac{2\rho^2 (\cos k - \lambda) - 4\rho^2 \lambda - \cos k(1 - \rho^2)^2} \right] \bigg\} .
\]  
(20)

From (18),(20) it follows that if

\[
4\sqrt{2}\rho(1 - \rho^2) \sin k \sqrt{\frac{\cos k - \lambda}{\cos k}} = \pm 3[(1 - \rho^2)^2 \cos k + 4\lambda \rho^2] \mp 2\rho^2 (\cos k - \lambda)
\]  
(21)

is satisfied, the coefficient \(\beta(k)\) becomes zero and the KdV equation reduces to the well-known equation

\[
\frac{\partial a_{\pm}}{\partial \tau} + \alpha(k)a_{\pm} \frac{\partial a_{\pm}}{\partial \xi_{\pm}} = 0
\]  
(22)

which support shock solutions. This implies that for parameter values satisfying (21) shock wave should develop in the spin chain.

There are two facts concerning (22) to be mentioned here. Note that from expression (22) it follows that at \(k = 0\) there exists only one background \(0 < \rho < 1\) at which \(\beta(k)\) equals to zero while this is not true for \(k \neq 0\). Moreover, equation (22) is not satisfied for all \(\lambda\) values but there exists a maximal value \(\lambda_{\text{max}} = 1/7\) above which (22) does not have physical meaning.

To check the above predictions, we have numerically integrated (6) on a long chain (we neglect boundary conditions), taking as initial condition a bell shaped bright or dark pulse of the type
\[ \mu_n = \rho e^{ikn} \left( 1 \pm \frac{A}{\cosh[(n - n_0)^2]} \right) \]  

(note that with this initial condition rectangular shock profiles should develop). In Fig. 1 we have reported the profile which develop out from an initial bright pulse of amplitude \(|A| = 3.6\), after an evolution time of \(T = 420\). The background is moving in-phase \((k = 0)\) with \(\rho = 0.8\) and for parameter values given by \(J = 0.8\), and \(\lambda\) derived from (21). From this figure we see the appearance of a leading rectangular shock profile followed by solitons and background radiation. The shock wave connects the uniform background field with a local plateau with two sharp transitions at the edges. If we define the local magnetization as \(M_n = (1 - |\mu_n|^2) / (1 + |\mu_n|^2)\) we have that the local magnetization in the rectangular shock waves of Fig.1 does not change in time and is different from the surroundings. This suggest the interpretation of such solutions as propagating magnetic domains.

A similar result can be obtained starting from an initial dark profile as shown in Fig.2. In this case the shock plateau develops below the background and therefore it can be referred to as a dark shock. Notice that in the above context dark and bright pulses correspond respectively to domains with higher and lower magnetization compared with the magnetization of the background.

Following the time evolution of the shock profiles in Fig.s 1,2 we find that the rectangular waves separate from the other components (solitons and radiation) and stay stable over long time. We have numerically checked that (21) is a necessary condition for creation of shock waves. In Fig.3 we have reported the evolution profile for the same parameter values of Fig.1 except for \(\lambda = 0.185\) not satisfying relation (21). We see that the rectangular shock is destroyed and oscillations develop on the wavefront. The same is observed for other choices of parameters for which Eq. (21) is not satisfied. We also checked that shocks remain stable upon collision with other excitations (the same was found in ref. [8]) this suggesting in them the presence of a strong soliton component. This leads to the interpretation of shocks as bound states of many solitons (in the quantum system they should correspond to long ”string” excitations i.e. to bound states with a large number of
quasiparticles). To check this interpretation, however, further investigations are required.

In conclusion, we have shown both analytically and numerically the existence of shock waves in 1D Heisenberg ferromagnets. Physically, this is a novel example of manifestation of classical fluid dynamics in quantum magnetic systems. Previously, classical behaviors in magnetic systems were reported on multi-magnon instabilities and chaos in pure and doped YIG and some antiferromagnets [13]. In order to observe such shock waves as those obtained here in magnetic systems, generation of macroscopic number of magnons is generally required. One of promising candidates to realize this may be highly pumped YIG.

The work of VVK has been supported by FEDER and by the Program PRAXIS XXI, grant No. PRAXIS/2/2.1/FIS/176/94. MS wishes to acknowledge financial support from INFN (Istituto Nazionale di Fisica della Materia) and from INTAS grant 93-1324.
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FIGURES

FIG. 1. Evolution of a bright shock against nonzero background with $k = 0$ $\rho = 0.8$, and for parameter values $J = 0.8$ and $\lambda$ determined from (21).

FIG. 2. Same as in Fig.1 but for a dark initial condition and for parameter values $\rho = 0.8$ $J = 1$ and $\lambda$ determined from (21).

FIG. 3. Evolution profile for the same parameter values as in Fig.1 but with $\lambda = 0.185$ not satisfying Eq.(21).
