SVZ Sum Rules in Quantum Mechanics

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Abstract. In this work we obtained ground state energy level of well known harmonic oscillator in quantum mechanics by using SVZ sum rules. We calculated energy eigenvalues with and without an external electric field.

1. Sum Rules

Sum rules were an important tool in the early days of quantum mechanics. A sum rule is simply (not many times) a formula for transitions between energy levels, in which the sum of the transition strengths is expressed in a formula. Sum rules have been used in many areas of physics such as atomic, molecular, solid state, nuclear and mostly in particle physics.

Quantum mechanical sum rules (Thomas-Reiche-Kuhn (TRK) sum rule [1-4] and Bethe sum rule [4-6]) were widely used in many branches of physics as stated above. TRK sum rule for energy eigenstates,

\[ \sum_k (E_k - E_n) |\langle n | x | k \rangle|^2 = \frac{\hbar^2}{2m} \]

was used to study the physics of electric-dipole interactions with atoms [4]. Another sum rule was formulated by Bethe as

\[ \sum_k (E_k - E_n) |\langle n | e^{i q x} | k \rangle|^2 = \frac{\hbar^2 q^2}{2m} \]

This was used to study for energy loss mechanisms of charged particles in matter. Sum rules are derived mostly from Heisenberg's view of quantum mechanics and renowned uncertainty principle [4]. Although these sum rules are derived in the early days of quantum mechanics, there can be found modern applications in [7-12].

Shifman-Vainshtein-Zakharov (SVZ) sum rule (mostly known as QCD (Quantumchromodynamics) Sum Rule) was derived in [13] and is being used widely in hadron physics. It is a semi-phenomenological way of determining the characteristics of hadrons. It is a perturbative method. The ground state energy and wave function can be found with an accuracy of 10%-20%. Since it was formulated, it has been applied to numerous problems in high energy physics especially properties of hadrons such as masses, decay and coupling constants, form factors, magnetic moments etc. In this study we applied SVZ method to linear harmonic oscillator in an electric field. We obtained energy levels for this system. A general discussion about SVZ sum rules for bound states can be found in [14]. We compared our results with that study.
2. SVZ sum rules in quantum mechanics

The following consideration is so close to [14]. We used SVZ sum rules to determine ground state energy of harmonic oscillator in quantum mechanics. For this we write the Euclidean Green function

\[ G(x_2, t_2; x_1, t_1) = \sum_{k=0}^{\infty} \psi_k(x_2) \psi_k^*(x_1) e^{-iE_k(t_2-t_1)} \]

is the Green function for time-dependent Schrödinger equation and \( \tau \) is the time parameter. The sum rule for Euclidean Green functions is

\[ M(\tau) = |\psi_0(0)|^2 e^{-E_0 \tau} + M_c(\tau) \]

where \( M_c(\tau) \) shows continuum contribution to \( M(\tau) \) (contribution of states lying above the ground state). It is given by

\[ M_c(\tau) = \sum_{k=1}^{\infty} |\psi_k(0)|^2 e^{-E_k \tau}. \]

L.h.s and r.h.s of (2.3) can be approximately equal if there exist a perturbative expansion, \( M_{\text{pert}}(\tau) \) which is the first few terms of the Born series for \( M(\tau) \) (small Euclidean time, \( \tau < \tau_\text{p} \)) and if \( M(\tau) \) is dominated by ground state contribution (large Euclidean times, \( \tau > \tau_\text{c} \)). Given \( \tau \) as \( \tau_\text{c} < \tau < \tau_\text{p} \) the ground state energy is given by

\[ E_0 = -\frac{d}{d\tau} \ln \left( M_{\text{pert}}(\tau) - M_c(\tau) \right). \]

2.1. Linear harmonic oscillator in an external electric field

In this part we study a generic example of quantum mechanics: energy levels of linear harmonic oscillator. The Hamiltonian in an electric field for 1D is given by

\[ H = -\frac{1}{2m} \frac{d^2}{dx^2} + \frac{1}{2} m \omega^2 x^2 - eEx. \]

The energy eigenvalues are given as

\[ E_k = \left( k + \frac{1}{2} - \epsilon \right) \omega \] where \( \epsilon \equiv \frac{(eE)^2}{2m\omega^2} \), \( k = 0, 1, 2, \ldots \)

and eigenfunctions are given as

\[ \psi_k(x) = \phi_k(x - x_0) ; x_0 = \frac{eE}{m\omega^2} \]

where \( \phi_k(x) \) are eigenfunctions of 1D linear harmonic oscillator without electric field. The Euclidean Green function for this example is

\[ M(\tau) = e^{\epsilon \omega \tau} \sum_{k=0}^{\infty} \phi_k(-x_0) \phi_k^*(-x_0) e^{-(k+\frac{1}{2})\omega \tau}. \]

Now let’s focus on the potential \( V(x) = \frac{1}{2} m \omega^2 x^2 - eEx \) and its Born series

\[ M_{\text{pert}}(\tau) = M_0(\tau) \left( 1 - \frac{(\omega \tau)^2}{12} + \epsilon \frac{(\omega \tau)^3}{12} + \frac{(\omega \tau)^4}{160} + \ldots \right) \]

Here \( M_0(\tau) = \sqrt{\frac{m}{2\pi \tau}} \) is the Euclidean Green function of a free particle. Finally we can write down the lowest field-induced correction to \( M(\tau) \) as

\[ M_{\text{pert}}(\tau) = M_0(\tau) e^{(\omega \tau)^3 \frac{12}{2}}. \]

So we have the first part for r.h.s. of (2.5). To have the second part i.e. \( M_c(\tau) \), we use free motion approximation such as
The ground state energy level shift can be obtained using (2.11) and (2.12) in (2.5). For very small fields we use equation (2.5). For normal fields we use

\[ \Delta E_0 = -\frac{d}{d\tau} \ln \left( 1 + \frac{\Delta M_{\text{pert}}(t) - \Delta M_c(t)}{\left| \psi_0(0) \right|^2 e^{-E_0\tau}} \right) \]

where \( \Delta M_{\text{pert}} = M'_{\text{pert}} - M_{\text{pert}} \) and \( \Delta M_c = M'_c - M_c \). Primed (unprimed) quantities refer to the case with (without) electric field.

3. Numerical results

For very small fields we used equation (2.5). For normal fields it is better to use (2.14). Table 1 shows with the results using (2.5) and Table 2 shows with the results using (2.14).

**Table 1.** Ground state energy level shift (in units of \( \omega \)) using (2.5)

| \( \varepsilon \) | \( (eE)^2 \) | \( E_c \) | \( E_0 \) (this work) | \( E_0 \) (ref. 14) | \( \Delta E_0 \) (this work) | \( \Delta E_0 \) (ref. 14) | \( \Delta E_0 \) (exact) |
|---|---|---|---|---|---|---|---|
| 0.01 | 0.90 | 0.397 | 0.387 | -0.01 | -0.018 | -0.01 |
| 0.02 | 0.72 | 0.375 | 0.368 | -0.022 | -0.037 | -0.02 |
| 0.03 | 0.63 | 0.364 | 0.349 | -0.033 | -0.056 | -0.03 |

**Table 2.** Ground state energy level shift using (2.14)

| \( \varepsilon \) | \( (eE)^2 \) | \( E_c \) | \( \Delta E_0 \) (this work) | \( \Delta E_0 \) (ref. 14) | \( \Delta E_0 \) (exact) |
|---|---|---|---|---|---|
| 0.01 | 1.43 | -0.011 | -0.011 | -0.01 |
| 0.02 | 1.36 | -0.021 | -0.023 | -0.02 |
| 0.03 | 1.29 | -0.035 | -0.036 | -0.03 |
| 0.04 | 1.22 | -0.047 | ? | -0.04 |
| 0.05 | 1.15 | -0.058 | -0.061 | -0.05 |

We can see from Table 1. for a zero external field the ground state energy is \( E_0 = 0.397 \) (in units of \( \omega \)) where the exact value is 0.5. In a non-zero electric field, the ground state energy level shift i.e. \( \Delta E_0 = E_0(\varepsilon) - E_0(0) \) was calculated where \( E_0(\varepsilon) \) is the ground state energy with a non-zero field. The energy level shift values of our work well agrees with the exact values.

From Table 2 we can see that by using refined formula, energy level shift values of our work is consistent well with exact values also with the values of [14]. The question mark in the fourth column means that authors of [14] didn’t have \( \Delta E_0 \) value for \( \varepsilon = 0.04 \) in their paper.

To sum up, we obtained ground state energy of harmonic oscillator with and without electric field by using SVZ sum rules. The key point is that we didn’t use directly Schrödinger equation to find energy eigenvalues.

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