GLOBAL SOLUTIONS TO CHEMOTAXIS-NAVIER-STOKES EQUATIONS IN CRITICAL BESOV SPACES

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ABSTRACT. In this article, we consider the Cauchy problem to chemotaxis model coupled to the incompressible Navier-Stokes equations. Using the Fourier frequency localization and the Bony paraproduct decomposition, we establish the global-in-time existence of the solution when the gravitational potential \( \phi \) and the small initial data \((u_0, n_0, c_0)\) in critical Besov spaces under certain conditions. Moreover, we prove that there exist two positive constants \( \sigma_0 \) and \( C_0 \) such that if the gravitational potential \( \phi \in \dot{B}_{p,1}^{3/p}(\mathbb{R}^3) \) and the initial data \((u_0, n_0, c_0) := (u_0^h, u_0^3, n_0, c_0)\) satisfies

\[
\left( \|u_0^h\|_{\dot{B}_{p,1}^{-1+3/p}(\mathbb{R}^3)} + \|(n_0, c_0)\|_{\dot{B}_{q,1}^{-2+3/q}(\mathbb{R}^3) \times \dot{B}_{q,1}^{3/q}(\mathbb{R}^3)} \right)^2 \leq \sigma_0
\]

for some \( p, q \) with \( 1 < p, q < 6, \frac{1}{p} + \frac{1}{q} > \frac{3}{2} \) and \( \frac{1}{\min\{p,q\}} - \frac{1}{\max\{p,q\}} \leq \frac{1}{2} \), then the global existence results can be extended to the global solutions without any small conditions imposed on the third component of the initial velocity field \( u_0^3 \) in critical Besov spaces with the aid of continuity argument. Our initial data class is larger than that of some known results. Our results are completely new even for three-dimensional chemotaxis-Navier-Stokes system.

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3427
1. Introduction and main results. Consider the following Chemotaxis-Navier-Stokes equations proposed in [47]:

\[
\begin{cases}
  u_t + \kappa u \cdot \nabla u - \mu \Delta u + \nabla P = -n \nabla \phi, \quad \nabla \cdot u = 0, & \text{in } (0, T) \times \Omega, \\
  n_t + u \cdot \nabla n - \nu \Delta n = -\nabla \cdot (\chi(c)n \nabla c), & \text{in } (0, T) \times \Omega, \\
  c_t + u \cdot \nabla c - \delta \Delta c = -f(c)n, & \text{in } (0, T) \times \Omega,
\end{cases}
\]

where $T \in (0, \infty], c(t, x) : (0, T) \times \Omega \to \mathbb{R}^+$, $n(t, x) : (0, T) \times \Omega \to \mathbb{R}^+$, $u(t, x) : (0, T) \times \Omega \to \mathbb{R}^d$ and $P(t, x) : (0, T) \times \Omega \to \mathbb{R}$ denote the oxygen concentration, cell concentration, fluid velocity, and scalar pressure, respectively. $\Omega \subset \mathbb{R}^d$ or $\mathbb{R}^2$ is a spatial domain where the cells or bacteria and fluid move and interact. Constants $\nu, \delta$ and $\mu$ are the corresponding diffusion coefficients for the cells, substrate and fluid. The nonnegative function $f(c)$ denotes the oxygen consumption rate and the nonnegative function $\chi(c)$ denotes chemotactic sensitivity. The parameter $\kappa \in \mathbb{R}$ measures the strength of nonlinear fluid convection. The time-independent function $\phi = \phi(x)$ denotes the potential function produced by different physical mechanisms, e.g., the gravitational force or centrifugal force.

The system (1) describes a biological process, in which the swimming bacteria move towards higher concentration of oxygen according to mechanism of chemotaxis and meanwhile the movement of fluid is under the influence of gravitational force generated by bacteria themselves. Both the oxygen concentration and bacteria density are transported by the fluid and diffuse through the fluid.

The model has been extensively studied by many authors and the main issue of investigation is the existence of (1). Duan, Lorz and Markowich [12] constructed global existence of weak solutions to the Cauchy problem in spatial dimension two. For the same Cauchy problem in $\mathbb{R}^2$, Liu and Lorz [31] removed the smallness assumption and obtained global existence of weak solutions with large data when $\kappa = 1$. In bounded convex domains $\Omega \subset \mathbb{R}^2$, Winkler [53] proved that the initial-boundary value problem of (1) possesses unique global classical solutions.

Minsuk, Lkhagvasuren, Choe [35] and Lkhagvasuren, Choe [7] established the existence of solutions to (1) in critical Besov spaces. Lorz [33] studied local-in-time weak solutions in a bounded domain in $\mathbb{R}^d, d = 2, 3$ with no-flux boundary condition and in $\mathbb{R}^2$ with inhomogeneous Dirichlet conditions for oxygen. Chae, Kang and Lee [1] established the local regular existence and presented some blow-up criterions of solutions when $(u_0, n_0, c_0) \in H^m(\mathbb{R}^d) \times H^{m-1}(\mathbb{R}^d) \times H^m(\mathbb{R}^d)$ with $m \geq 3$ to the Cauchy problem of (1) in $\mathbb{R}^d$ for $d = 2, 3$, in particular, for two dimensional Chemotaxis-Navier-Stokes equations, regular solutions constructed locally in time are, in reality, extended globally under some assumptions. Moreover, Lorz [34] obtained global existence of a system of the elliptic-parabolic Keller-Segel equations coupled with Stokes equations ($\kappa = 0$) with small initial data in $\mathbb{R}^d$ ($d = 2, 3$). Chae, Kang and Lee [2] established the local existence of regular solutions for both cases that equations of oxygen concentration is of parabolic or hyperbolic type and also proved global existence under the some smallness conditions about initial data. Zhang [61] obtained existence and uniqueness of smooth solutions in inhomogeneous Besov spaces for (1) in $\mathbb{R}^d$ ($d = 2, 3$). Zhang and Zheng [64] showed that there exist global weak solutions to the Cauchy problem of (1) in $\mathbb{R}^2$ with a large class of initial data. If $u = 0$ in the system (1), Tao [42] showed that there exist unique, global and bounded solutions if $\chi$ is sufficiently small. For stability and asymptotic behaviors on the system (1), we refer the reader to [2, 3, 12, 30, 50, 63].
For the three-dimensional case, Duan, Lorz and Markowich [12] proved the global-in-time existence of smooth solutions of (1) when the initial data is close to the constant equilibrium states in $H^3(\mathbb{R}^3)$. Global weak solutions of the system (1) were constructed when $\kappa = 0$ [53] or when $\kappa = 1$ [54].

For the existence results of the Chemotaxis-Navier-Stokes system with nonlinear diffusion for the cell density (a porous medium type $\Delta u ^m$, instead of $\Delta u$), see [8, 13, 15, 33, 43, 44, 62], and the results of which have recently been (partially) extended, including information on large time behavior and in presence of general sensitivities, by Winkler [51].

A classical model about Chemotaxis equations introduced by Patlak [40] and Keller- Segel [22, 23], is given as

\[
\begin{cases}
    n_t - \Delta n + \nabla \cdot (n\chi \nabla c) = 0, & \text{in } (0, T) \times \mathbb{R}^3,
    \\
    \alpha c_t - \Delta c + \tau c - n = 0, & \text{in } (0, T) \times \mathbb{R}^3,
\end{cases}
\]

where $\chi$ is the sensitivity and $\tau^{-1/2}$ represents the activation length. The system (2) has been extensively studied by many authors for the past decades; see [18, 19, 36, 37, 49]. For instance, Winkler [52] studied the finite-time blow-up for the Neumann initial-boundary value problem for the fully parabolic Keller-Segel system in bounded domains.

The numerical experiments in [34] indicate an effect of the fluid interaction on possible occurrence of blow-up in two-dimensional cases. Also, three papers by Kiselev and his collaborators [9, 24, 25] indicate the possibility of blow-up prevention, as well as some subtle further qualitative effects, that fluid interaction may have on cross-diffusive dynamics. Variant of (2) coupled to (Navier-)Stokes equations and including logistic sources has been studied by Espejo and his collaborators [14], Tao and Winkler [44, 45]. It is clear from (1) that the coupling of chemotaxis and fluid is realized through both the transport of cells and chemical substrates $u \nabla n$, $u \cdot \nabla c$ and the external force $-n \nabla \phi$ exerted on the fluid by cells. In particular, if the chemotaxis effects are ignored ($n = c = 0$), for $\Omega = \mathbb{R}^3$ and $\kappa = 1$, then the induced system (1) becomes the problems related to the classical Navier-Stokes equations:

\[
\begin{cases}
    u_t + u \cdot \nabla u - \mu \Delta u + \nabla P = 0, & \text{in } (0, T) \times \mathbb{R}^3, \\
    \nabla \cdot u = 0, & \text{in } (0, T) \times \mathbb{R}^3, \\
    u |_{t=0} = u_0, & \text{in } \mathbb{R}^3,
\end{cases}
\]

which has been widely studied during the past eighty more years. Leray [26] proved the global existence of weak solutions of the system (3), but the uniqueness and regularity of weak solutions are remaining big open problems. It has been proved that the Cauchy problem to (3) is globally well-posed for the small initial data in a series of function spaces including particularly the following critical spaces:

$\dot{H}^\frac{1}{2}(\mathbb{R}^3) \hookrightarrow L^3(\mathbb{R}^3) \hookrightarrow \dot{B}_{p,\infty}^{-\frac{3}{p} + \frac{3}{2}}(\mathbb{R}^3)$ ($3 < p < \infty$) $\hookrightarrow \text{BMO}^{-1}(\mathbb{R}^3)$;

see Fujita and Kato [16], Kato [21], Kozono and Yamazaki [29], Koch and Tataru [27].

Using the algebraic structure of (3), the papers [4, 5, 57, 59, 32, 20, 38, 48, 39, 65, 58] proved the global well-posedness of (3) (or other related equations) with a large vertical component provided that the horizontal component is small enough.
Define the set \( \tilde{B}_{\alpha} \) of tempered distributions. Here \( \tilde{B}_{\alpha} \) is a critical space for the system (4) if the norm of \( (u_0, n_0, c_0) \) is \( \tilde{B}_{\alpha} \) for the initial data \( (u_0, n_0, c_0) \).

In this article, we shall consider the simplified Chemotaxis-Navier-Stokes equations, namely, choosing \( \chi \) valued in \( S_\infty(\mathbb{R}^3) \) be a space of smooth compactly supported functions on the domain \( \Omega \). There

We say that \( (u, n, c) \) is a critical space associated with (4) if the norm of \( (u_0, n_0, c_0) \) in \( \Lambda \times B \times C \) is invariant for all \( \lambda > 0 \). We observe that

and \( \tilde{B}_{p,1}^\beta(\mathbb{R}^3) \) are critical spaces associated with (4) for the initial data \( (u_0, n_0, c_0) \) and the gravitational potential \( \phi \).

Before going further, we recall the functional spaces we are going to use. Let

and \( \tilde{B}_{p,1}^\beta(\mathbb{R}^3) \) be the Schwartz class of rapidly decreasing functions, \( S'(\mathbb{R}^3) \) be the space of tempered distributions. Here \( \mathcal{F} \) and \( \mathcal{F}^{-1} \) denote Fourier and inverse Fourier transforms of \( L^1(\mathbb{R}^3) \) functions, respectively, which are defined by

More generally, the Fourier transform of any \( f \in S'(\mathbb{R}^3) \), given by \( (\mathcal{F}f, g) = (f, \mathcal{F}g) \) for any \( g \in S(\mathbb{R}^3) \). Let \( \mathcal{C} \) be the annulus \( \{\xi \in \mathbb{R}^3 : \frac{3}{4} \leq |\xi| \leq \frac{5}{4} \} \) and \( \mathcal{D}(\Omega) \) be a space of smooth compactly supported functions on the domain \( \Omega \). There

Define the set \( \tilde{C} = B(0, \frac{3}{2}) + \mathcal{C} \). We have

\[ |j - j'| \geq 5 \Rightarrow 2^j \tilde{C} \cap 2^j \mathcal{C} = \emptyset. \]
Furthermore, we have
\[
\forall \xi \in \mathbb{R}^3, \quad \frac{1}{2} \leq \chi^2(\xi) + \sum_{j \geq 0} \varphi^2(2^{-j} \xi) \leq 1; \quad \forall \xi \in \mathbb{R}^3 \setminus \{0\}, \quad \frac{1}{2} \leq \sum_{j \in \mathbb{Z}} \varphi^2(2^{-j} \xi) \leq 1.
\]
From now on, we write \( h = \mathcal{F}^{-1} \varphi \) and \( \tilde{h} = \mathcal{F}^{-1} \chi \). The homogeneous dyadic blocks \( \Delta_j \) and \( S_j \) are defined by
\[
\Delta_j u := \varphi(2^{-j}D)u := 2^{3j} \int_{\mathbb{R}^3} h(2^jy)u(x-y)dy \quad \text{if } j \in \mathbb{Z};
\]
\[
S_j u := \chi(2^{-j}D)u := 2^{3j} \int_{\mathbb{R}^3} \tilde{h}(2^jy)u(x-y)dy \quad \text{if } j \in \mathbb{Z}.
\]
Here \( D := (D_1, D_2, D_3) \) and \( D_j := i^{-1} \partial_{x_j} (i^2 = -1) \). The set \( \{ \Delta_j, S_j \}_{j \in \mathbb{Z}} \) is called the Littlewood-Paley decomposition. Formally, \( \Delta_j = S_j - S_{j-1} \) is a frequency projection to the annulus \( \{ \xi \in \mathbb{R}^3 : 2^{j-1} \leq |\xi| \leq 2^j \} \), and \( S_j = \sum_{j' \leq j-1} \Delta_{j'} \) is a frequency projection to the ball \( \{ \xi \in \mathbb{R}^3 : |\xi| \leq 2^j \} \). Let \( S'_s(\mathbb{R}^3) \) be the space of tempered distributions \( f \) such that \( \lim_{j \to -\infty} S_j f = 0 \) in \( S'(\mathbb{R}^3) \). Recall that for any \( s \in \mathbb{R} \) and \((p, r) \in [1, \infty] \times [1, \infty] \), the homogeneous Besov space \( \dot{B}^s_{p,r}(\mathbb{R}^3) \) is defined by
\[
\dot{B}^s_{p,r}(\mathbb{R}^3) := \left\{ f \in S'_s((0, T), S'_0(\mathbb{R}^3)) : \| f \|_{L^p(0, T; \dot{B}^s_{p,r}(\mathbb{R}^3))} < \infty \right\},
\]
where
\[
\| f \|_{\dot{B}^s_{p,r}(\mathbb{R}^3)} := \left\{ \sum_{k \in \mathbb{Z}} \left[ 2^{ks} \| \Delta_k f \|_{L^p(\mathbb{R}^3)} \right]^p \right\}^{\frac{1}{p}}, \quad \text{if } 1 \leq p \leq \infty, 1 \leq r < \infty, s \in \mathbb{R},
\]
\[
\sup_{k \in \mathbb{Z}} \left[ 2^{ks} \| \Delta_k f \|_{L^p(\mathbb{R}^3)} \right], \quad \text{if } 1 \leq p \leq \infty, r = \infty, s \in \mathbb{R}.
\]
It is well-known that if either \( s < 3/p \) or \( s = 3/p \) and \( r = 1 \), then \( \dot{B}^s_{p,r}(\mathbb{R}^3) \) is a Banach space.

Let us now recall the definition of the Chemin-Lerner space \( \mathcal{L}^\rho(0, T; \dot{B}^s_{p,r}(\mathbb{R}^3)) \) with \( 0 < T \leq \infty, s \in \mathbb{R} \) and \( 1 \leq p, r, \rho \leq \infty \) (with the usual convention if \( r = \infty \) or \( \rho = \infty \)). The Chemin-Lerner space is defined by
\[
\mathcal{L}^\rho(0, T; \dot{B}^s_{p,r}(\mathbb{R}^3)) := \left\{ f \in S'((0, T), S'_0(\mathbb{R}^3)) : \| f \|_{\mathcal{L}^\rho(0, T; \dot{B}^s_{p,r}(\mathbb{R}^3))} < \infty \right\},
\]
where
\[
\| f \|_{\mathcal{L}^\rho(0, T; \dot{B}^s_{p,r}(\mathbb{R}^3))} := \left\| f \right\|_{\mathcal{L}^\rho(0, T; \dot{B}^s_{p,r}(\mathbb{R}^3))} := \left\{ \sum_{k \in \mathbb{Z}} \left[ 2^{ks} \| \Delta_k f \|_{L^\rho_L(\mathbb{R}^3)} \right]^\rho \right\}^{\frac{1}{\rho}},
\]
with the usual modification made when \( r = \infty \) or \( \rho = \infty \); we equip the space
\[
L^\rho(0, T; \dot{B}^s_{p,r}(\mathbb{R}^3)) := \left\{ f \in S'((0, T), S'_0(\mathbb{R}^3)) : \| f \|_{L^\rho(0, T; \dot{B}^s_{p,r}(\mathbb{R}^3))} < \infty \right\},
\]
with the following norm
\[
\| f \|_{L^\rho(0, T; \dot{B}^s_{p,r}(\mathbb{R}^3))} := \left\{ \int_0^T \left[ \sum_{k \in \mathbb{Z}} \left[ 2^{ks} \| \Delta_k f \|_{L^\rho(\mathbb{R}^3)} \right]^\rho \right] dt \right\}^{\frac{1}{\rho}},
\]
where
\[
\| f \|_{L^\rho_L(\mathbb{R}^3)} := \left[ \int_0^T \| f \|_{L^\rho(\mathbb{R}^3)}^\rho \ dt \right]^{\frac{1}{\rho}}.
\]
and the usual modifications are needed when \( r = \infty \) or \( \rho = \infty \). From Minkowski’s inequality, it is easy to deduce that

\[
\|f\|_{L^p(0,T;\dot{B}_{p,r}^\theta(\mathbb{R}^3))} \leq \|f\|_{L^p(0,T;\dot{B}_{p,r}^\theta(\mathbb{R}^3))} \quad \text{when } \rho \leq r,
\]

\[
\|f\|_{L^p(0,T;\dot{B}_{p,r}^\theta(\mathbb{R}^3))} \leq \|f\|_{L^p(0,T;\dot{B}_{p,r}^\theta(\mathbb{R}^3))} \quad \text{when } r \leq \rho.
\]

(5)

If \( s_1 \) and \( s_2 \) are real numbers such that \( s_1 < s_2 \) and \( \theta \in (0,1) \), then we have, for any \( (p, r, \rho, \rho_1, \rho_2) \in [1, \infty]^5 \) and \( 1/\rho = \theta/\rho_1 + (1 - \theta)/\rho_2 \),

\[
\begin{align*}
\|f\|_{L^p(0,T;\dot{B}_{p,r}^{\theta,1+(1-\theta)s_2}(\mathbb{R}^3))} &\leq C \|f\|_{L^p(0,T;\dot{B}_{p,r}^{\theta,s_2}(\mathbb{R}^3))}^{1-\theta} \|f\|_{L^p(0,T;\dot{B}_{p,r}^{\theta,s_1}(\mathbb{R}^3))}^{\theta}, \\
\|f\|_{L^p(0,T;\dot{B}_{p,r}^{\theta,1+(1-\theta)s_2}(\mathbb{R}^3))} &\leq C \|f\|_{L^p(0,T;\dot{B}_{p,r}^{\theta,s_2}(\mathbb{R}^3))}^{1-\theta} \|f\|_{L^p(0,T;\dot{B}_{p,r}^{\theta,s_1}(\mathbb{R}^3))}^{\theta}, \\
\|f\|_{L^p(0,T;\dot{B}_{p,r}^{\theta,1+(1-\theta)s_2}(\mathbb{R}^3))} &\leq C \|f\|_{L^p(0,T;\dot{B}_{p,r}^{\theta,s_2}(\mathbb{R}^3))}^{1-\theta} \|f\|_{L^p(0,T;\dot{B}_{p,r}^{\theta,s_1}(\mathbb{R}^3))}^{\theta},
\end{align*}
\]

(6)

where \( C \) is a positive constant independent of \( f \).

For \( \dot{B}_{p,1}^{3/p}(\mathbb{R}^3) \) is embedded in \( L^\infty(\mathbb{R}^3) \), we deduce that whenever \( 1 \leq p \leq \infty \), the product of two functions in \( \dot{B}_{p,1}^{3/p}(\mathbb{R}^3) \) is also in \( \dot{B}_{p,1}^{3/p}(\mathbb{R}^3) \) and such that for some constant \( C > 0 \),

\[
\|uv\|_{\dot{B}_{p,1}^{3/p}(\mathbb{R}^3)} \leq C \|u\|_{\dot{B}_{p,1}^{3/p}(\mathbb{R}^3)} \|v\|_{\dot{B}_{p,1}^{3/p}(\mathbb{R}^3)}.
\]

(7)

The homogeneous paraproduct of \( v \) and \( u \) is defined by

\[
T_u v := \sum_{j \in \mathbb{Z}} S_{j-1} u \Delta_j v.
\]

The homogeneous remainder of \( v \) and \( u \) is defined by

\[
R(u, v) := \sum_{|k-j| \leq 1} \Delta_k u \Delta_j v := \sum_{k \in \mathbb{Z}} \Delta_k u \Delta_k v,
\]

where

\[
\Delta_k := \Delta_{k-1} + \Delta_k + \Delta_{k+1}.
\]

For two tempered distributions \( f \) and \( g \),

\[
\supp \mathcal{F} (\Delta_k f \hat{\Delta}_k g) \subset \{ \xi \in \mathbb{R}^3 : |\xi| \leq 8 \times 2^k \},
\]

\[
\supp \mathcal{F} (\Delta_j f) \subset \{ \xi \in \mathbb{R}^3 : 4 \times 3^j \leq |\xi| \leq 8 \times 2^j \}.
\]

Then, up to finitely many terms, we obtain

\[
\Delta_j R(f, g) = \sum_{k \geq j-2} \Delta_j (\Delta_k f \hat{\Delta}_k g).
\]

(8)

We have the following Bony decomposition:

\[
uw := T_u v + R(u, v) + T_u u.
\]

(9)

Let

\[
E_0 := \dot{B}_{p,1}^{-1+3/p}(\mathbb{R}^3) \times \dot{B}_{q,1}^{-2+3/q}(\mathbb{R}^3) \times \dot{B}_{q,1}^{3/q}(\mathbb{R}^3).
\]

We introduce a vector space \( \Theta_T := \mathbb{X}_T \times \mathbb{Y}_T \times \mathbb{Z}_T \) and with the usual product norm

\[
\|(u, n, c)\|_{\Theta_T} := \|u\|_{\mathbb{X}_T} + \|n\|_{\mathbb{Y}_T} + \|c\|_{\mathbb{Z}_T}.
\]
where
\[
\mathcal{X}_T := \left\{ u : u \in L^1 \left(0, T; \dot{B}^{-1+3/p}_{p,1}(\mathbb{R}^3) \right) \cap L^\infty \left(0, T; \dot{B}^{-1+3/p}_{p,1}(\mathbb{R}^3) \right) \right\},
\]
\[
\mathcal{Y}_T := \left\{ n : n \in L^1 \left(0, T; \dot{B}^{-2+3/q}_{q,1}(\mathbb{R}^3) \right) \cap L^\infty \left(0, T; \dot{B}^{-2+3/q}_{q,1}(\mathbb{R}^3) \right) \right\},
\]
\[
\mathcal{Z}_T := \left\{ c : c \in L^1 \left(0, T; \dot{B}^{3/q}_{q,1}(\mathbb{R}^3) \right) \cap L^\infty \left(0, T; \dot{B}^{3/q}_{q,1}(\mathbb{R}^3) \right) \right\},
\]
and
\[
\|u\|_{\mathcal{X}_T} := \|u\|_{L^\infty(0,T;\dot{B}^{-1+3/p}_{p,1}(\mathbb{R}^3))} + \|u\|_{L^1(0,T;\dot{B}^{-1+3/p}_{p,1}(\mathbb{R}^3))};
\]
\[
\|n\|_{\mathcal{Y}_T} := \|n\|_{L^\infty(0,T;\dot{B}^{-2+3/q}_{q,1}(\mathbb{R}^3))} + \|n\|_{L^1(0,T;\dot{B}^{-2+3/q}_{q,1}(\mathbb{R}^3))};
\]
\[
\|c\|_{\mathcal{Z}_T} := \|c\|_{L^\infty(0,T;\dot{B}^{3/q}_{q,1}(\mathbb{R}^3))} + \|c\|_{L^1(0,T;\dot{B}^{3/q}_{q,1}(\mathbb{R}^3))}.
\]

Meanwhile, for any \(T \in (0, \infty)\), let
\[
\Theta^C_T = C \left(\left[0, T\right]; \dot{B}^{-1+3/p}_{p,1}(\mathbb{R}^3)\right) \times C \left(\left[0, T\right]; \dot{B}^{-2+3/q}_{q,1}(\mathbb{R}^3)\right) \times C \left(\left[0, T\right]; \dot{B}^{3/q}_{q,1}(\mathbb{R}^3)\right).
\]

For the notational simplification, when \(T = \infty\), we denote \(\mathcal{X}_\infty := \mathcal{X}, \mathcal{Y}_\infty := \mathcal{Y}, \mathcal{Z}_\infty := \mathcal{Z}, \Theta_\infty := \Theta\) and \(\Theta^C_\infty := \Theta^C\).

Now, we state the first results of this paper.

**Theorem 1.1.** Let \(1 \leq p, q < \infty, \frac{1}{p} + \frac{1}{q} > \frac{3}{2}, 1 \leq q < 6\) and \(\frac{1}{\min(p,q)} - \frac{1}{\max(p,q)} \leq \frac{1}{3}\). If \(\nabla \cdot u_0 = 0, (u_0, n_0, c_0) \in E_0, \phi \in \dot{B}^{3/p}_{p,1}(\mathbb{R}^3)\) and
\[
\|(u_0, n_0, c_0)\|_{E_0} \leq \epsilon_0,
\]
for some sufficiently small \(\epsilon_0\), then the system (4) admits a unique global solution such that
\[
(u, n, c) \in \Theta \cap \Theta^C.
\]

**Remark.** Recently, when \(1 \leq p < 3\), Chae and Lkhagvasuren [7] showed that if \(\phi \in \dot{B}^{3/p}_{p,1}(\mathbb{R}^3)\) and
\[
(u_0, n_0, c_0) \in \dot{B}^{-1+3/p}_{p,1}(\mathbb{R}^3) \times \dot{B}^{-2+3/p}_{p,1}(\mathbb{R}^3) \times \dot{B}^{3/p}_{p,1}(\mathbb{R}^3)
\]
with small norm, then system (4) exists a global-in-time solution. From Theorem 1.1, when \(p = q\), we can obtain if \(\phi \in \dot{B}^{3/p}_{p,1}(\mathbb{R}^3)\) and
\[
(u_0, n_0, c_0) \in \dot{B}^{-1+3/p}_{p,1}(\mathbb{R}^3) \times \dot{B}^{-2+3/p}_{p,1}(\mathbb{R}^3) \times \dot{B}^{3/p}_{p,1}(\mathbb{R}^3)
\]
with small norm for \(1 \leq p < 6\), then system (4) exists a global-in-time solution, which in particular implies the global well-posedness result in [7]. Therefore, our initial data class is larger than that of [7] and Theorem 1.1 may be regarded as a new global existence theorem on the system (4).

We can relax the smallness condition in Theorem 1.1 so that system (4) has a unique global solution. The main object of this paper is to prove the following theorem.

**Theorem 1.2.** Let \(1 < p, q < 6, \frac{1}{p} + \frac{1}{q} > \frac{2}{3} \) and \(\frac{1}{\min(p,q)} - \frac{1}{\max(p,q)} \leq \frac{1}{3}\). Let \(\phi \in \dot{B}^{3/p}_{p,1}(\mathbb{R}^3)\) and \((u_0, n_0, c_0) \in E_0\) with \(\nabla \cdot u_0 = 0\) and \(u_0 := (u^{1}_0, u^{2}_0, u^{3}_0) := (u^{h}_0, u^{3}_0)\).
There exist two positive constants $\sigma_0$ and $C_0$ such that if the initial data $(u_0, n_0, c_0)$ satisfies
\[
\left( \|u_0\|_{\dot{B}^{2+3/p}_{p,1}(\mathbb{R}^3)} + \|(n_0, c_0)\|_{\dot{B}^{2+3/q}_{q,1}(\mathbb{R}^3) \times \dot{B}^{2/q}_{q,1}(\mathbb{R}^3)} \right) \times \exp \left\{ C_0 \left( \|u_0\|_{\dot{B}^{-1+3/p}_{p,1}(\mathbb{R}^3)} + 1 \right)^2 \right\} \leq \sigma_0,
\]
then the system (4) admits a unique global-in-time solution such that $(u, n, c) \in \Theta \cap \Theta^C$. Moreover, there exist $C > 0$ and $c_2 > 0$, such that
\[
\|u^h\|_{L^\infty(\mathbb{R}^+; \dot{B}^{-1+3/p}_{p,1}(\mathbb{R}^3))} + \|u^h\|_{L^1(\mathbb{R}^+; \dot{B}^{1+3/p}_{p,1}(\mathbb{R}^3))}
+ \|(n, c)\|_{L^\infty(\mathbb{R}^+; \dot{B}^{-2+3/q}_{q,1}(\mathbb{R}^3) \times L^\infty(\mathbb{R}^+; \dot{B}^{3/q}_{q,1}(\mathbb{R}^3))}
+ \|(n, c)\|_{L^1(\mathbb{R}^+; \dot{B}^{2+3/q}_{q,1}(\mathbb{R}^3) \times L^1(\mathbb{R}^+; \dot{B}^{3/q}_{q,1}(\mathbb{R}^3))} \leq C\ell,
\]
\[
\|u^3\|_{L^\infty(\mathbb{R}^+; \dot{B}^{-1+3/p}_{p,1}(\mathbb{R}^3))} + \bar{c} \|u_3\|_{L^1(\mathbb{R}^+; \dot{B}^{1+3/p}_{p,1}(\mathbb{R}^3))} \leq 2 \|u_0\|_{\dot{B}^{-1+3/p}_{p,1}(\mathbb{R}^3)} + c_2.
\]

**Remarks.**

(i) Our results do not impose any smallest conditions on the third component $u^3_0$ of the initial velocity field. We emphasize that our proof uses in a fundamental way the algebraical structure of (4), that is $\nabla \cdot u = 0$, which will be one of the key ingredients in the proof of Theorem 1.2 in section 4 below. We also remark that the first equation of (4) on the vertical component of the velocity is a linear equation with coefficients depending on the horizontal components of the velocity, $n$ and $\phi$. Therefore, the equation on the vertical component does not demand any smallness condition. Instead of (2), the considered problem (4) here is more related to the corresponding fluid-free variant in which, as in (4) but unlike in (2), the signal is consumed rather than produced by the cells. For that problem, in the three-dimensional case up to now also smallness conditions on the data seem to be required in order to achieve global classical solvability, which can be found in a 2012 paper by Tao [46]. According to the theories of heat equation, the present analysis of us here can not perhaps yield even less restrictive conditions for global existence the system (4) without fluid coupling and we still need the smallness condition for initial data. However, the question whether the solutions of the system (4) with large initial data exist globally or may blow up appears to remain an open and challenging topic in the three-dimensional case.

(ii) Suppose furthermore that $u_0, n_0, c_0, \phi$ are homogeneous of degree $-1, -2, 0$ and $0$, i.e., they satisfy the relations $u_0(x) = \lambda u_0(\lambda x), n_0(x) = \lambda^2 n_0(\lambda x), c_0(x) = c_0(\lambda x)$ and $\phi(x) = \phi(\lambda x)$ for all $x \in \mathbb{R}^3$ and $\lambda > 0$. Then the global solution obtained through Theorem 1.2 is self-similar, that is, $(u, n, c)$ of system (4) given by Theorem 1.2 satisfies $u(t, x) = \lambda u(\lambda^2 t, \lambda x), n(t, x) = \lambda^2 n(\lambda^2 t, \lambda x)$ and $c(t, x) = c(\lambda^2 t, \lambda x)$.

**Notations.** Throughout the paper, $C$ stands for a harmless constant, and we sometimes use the notation $a \lesssim b$ as an equivalent of $a \leq Cb$. Let $A$ and $B$ be two operators, we denote $[A; B] = AB - BA$. For Banach space $X$ and interval $I$, we denote by $C(I; X)$ in the set of continuous functions on $I$ with value in $X$. The symbol $(d_j)_{j \in \mathbb{Z}}$ is a generic element of $\ell^1(\mathbb{Z})$ so that $d_j \geq 0$ and $\sum_{j \in \mathbb{Z}} d_j = 1$.

2. **Preliminaries.** The proofs of Theorem 1.1 in section 3 and likewise, Theorem 1.2 in section 4, require a lot of elementary inequalities which are summarized in the following.
Lemma 2.1. [17] Let \( C \) be an annulus and \( B \) be a ball in \( \mathbb{R}^3 \). There exists a positive constant \( C \) such that for any nonnegative integer \( k \), any couple \((p,q) \in [1, \infty]^2\) with \( q \geq p \geq 1 \), and any function \( u \) of \( L^p(\mathbb{R}^3) \),

(i) if \( \text{supp} \, \hat{u} \subseteq \lambda B \), then

\[
\|D^k u\|_{L^q(\mathbb{R}^3)} := \sup_{|\alpha| \leq k} \|\partial^\alpha u\|_{L^q(\mathbb{R}^3)} \leq C^{k+1} \lambda^{k+3(\frac{2}{p} - \frac{1}{q})} \|u\|_{L^p(\mathbb{R}^3)} ;
\]

(ii) if \( \text{supp} \, \hat{u} \subseteq \lambda C \), then

\[
C^{-k-1} \lambda^k \|u\|_{L^p(\mathbb{R}^3)} \leq \|D^k u\|_{L^p(\mathbb{R}^3)} \leq C^{k+1} \lambda^k \|u\|_{L^p(\mathbb{R}^3)} .
\]

Lemma 2.2. [17] Let \( C \) be an annulus in \( \mathbb{R}^3 \). There exist two positive constants \( \tilde{c} \) and \( \tilde{C} \) such that for any \( p \in [1, \infty] \) and any couple \((t, \lambda) \) of positive numbers, if \( \text{supp} \tilde{f} \subseteq \lambda C \), then

\[
\|e^{t\Delta} f\|_{L^p(\mathbb{R}^3)} \leq \tilde{C} e^{-c \lambda^2 t} \|f\|_{L^p(\mathbb{R}^3)} ,
\]

where \( e^{t\Delta} \) is the heat operator with kernel

\[
G(x, t) = (4t)^{-3/2} \exp \left(-|x|^2 / 4t\right)
\]

for all \( x \in \mathbb{R}^3 \) and \( t \in (0, \infty) \).

Lemma 2.3. [39] Let \( 1 \leq \rho_1 \leq \rho_2 \leq \infty \), and \( s_1 \leq \frac{3}{p_1}, s_2 \leq \frac{3}{p_2} \) with \( s_1 + s_2 > 3 \max(0, \frac{1}{q} + \frac{1}{p_2} - 1) \). If \( a \in B_{\rho_1,1}^{s_1}(\mathbb{R}^3) \) and \( b \in B_{\rho_2,1}^{s_2}(\mathbb{R}^3) \), then \( ab \in B_{\rho_2,1}^{s_1+s_2 - \frac{3}{q}}(\mathbb{R}^3) \) and

\[
\|ab\|_{B_{\rho_2,1}^{s_1+s_2 - \frac{3}{q}}(\mathbb{R}^3)} \leq C \|a\|_{B_{\rho_1,1}^{s_1}(\mathbb{R}^3)} \|b\|_{B_{\rho_2,1}^{s_2}(\mathbb{R}^3)} ,
\]

where \( C \) is a positive constant independent of \( a \) and \( b \).

Lemma 2.4. [60] Let \( 1 \leq \tilde{p} \leq \tilde{q} \leq \infty \), \( s \leq 3/\tilde{q} \) with \( s + 3/\tilde{q} > 3 \max(0, \frac{1}{q} + \frac{1}{p_2} - 1) \). If \( \tilde{a} \in B_{\tilde{q},1}^{3/\tilde{q}}(\mathbb{R}^3) \) and \( \tilde{b} \in B_{\tilde{p},1}^{s} (\mathbb{R}^3) \), then \( \tilde{a}\tilde{b} \in B_{\tilde{p},1}^{s} (\mathbb{R}^3) \) and there holds

\[
\|\tilde{a}\tilde{b}\|_{B_{\tilde{p},1}^{s} (\mathbb{R}^3)} \leq \|\tilde{a}\|_{B_{\tilde{q},1}^{3/\tilde{q}}(\mathbb{R}^3)} \|\tilde{b}\|_{B_{\tilde{p},1}^{s} (\mathbb{R}^3)} ,
\]

where \( C \) is a positive constant independent of \( \tilde{a} \) and \( \tilde{b} \).

Lemma 2.5. [10] Let \( T \in (0, \infty) \), \( s \in \mathbb{R} \) and \( 1 \leq \rho, p, r \leq \infty \). Assume that \( u_0 \in B_p^{s,1}(\mathbb{R}^3) \), \( f \in L^p(0,T; B_{p,r}^{s+2/\rho}(\mathbb{R}^3)) \) and \( u \) solves

\[
\left\{ \begin{array}{ll}
u_t - \mu \Delta u = f & \text{in } \mathbb{R}^3 \times (0, T), \\
u|_{t=0} = u_0 & \text{in } \mathbb{R}^3. 
\end{array} \right.
\]

Denote \( \rho_2 := (1 + 1/\rho_1 - 1/\rho)^{-1} \). Then there exist two positive constants \( \tilde{c} \) and \( \tilde{C} \) such that for all \( \rho_1 \in [\rho_1, +\infty] \), we have

\[
\|u\|_{L^{p_1}(0,T; B_{\rho_1,1}^{s+2/\rho_1}(\mathbb{R}^3))} \leq \tilde{C} \left( \sum_{\eta \in \mathbb{Z}} 2^{q_\eta} \|\Delta_\eta u_0\|_{L^p(\mathbb{R}^3)} \left( \frac{1 - e^{-c \mu_1 T^2 \rho_1}}{c \mu_1} \right)^{\frac{1}{\eta}} \right) + \sum_{\eta \in \mathbb{Z}} 2^{q_\eta (s - 2 + \frac{2}{\rho})} \|\Delta_\eta f\|_{L^p(\mathbb{R}^3)} \left( \frac{1 - e^{-c \mu_1 T^2 \rho_2}}{c \mu_2} \right)^{\frac{1}{\eta}}.
\]
In particular, there holds
\[
\mu \frac{1}{p} \|u\|_{L^p(0,T;\dot{B}^{s+2+2/p}_{p,1}(\mathbb{R}^3))} \leq C \left( \|u_0\|_{\dot{B}^s_{p,1}(\mathbb{R}^3)} + \mu^{\frac{1}{p}-1} \|f\|_{L^p(0,T;\dot{B}^{s+2+2/p}_{p,1}(\mathbb{R}^3))} \right)
\]
and \( u \in C([0,T];\dot{B}^s_{p,1}(\mathbb{R}^3)) \). Moreover, if \( f = 0 \) and \( 1 \leq p_1 < \infty \), then we have
\[
\lim_{T \to 0} \|u\|_{L^p([0,T];\dot{B}^{s+2+2/p}_{p,1}(\mathbb{R}^3))} = 0.
\]

**Lemma 2.6.** Let \( 1 \leq p < \infty \). Assume that \( u \in \mathcal{X}_T \) (see (10)) and \( \nabla \cdot u = 0 \). Arguing by interpolation (6) and the algebra property (7) of Besov spaces yield
\[
\|u \cdot \nabla u\|_{L^q(B^{-1+3/p}_{p,1}(\mathbb{R}^3))} \leq C \|u \otimes u\|_{L^q(B^s_{p,1}(\mathbb{R}^3))} \\
\leq C \|u\|^2_{L^q(B^s_{p,1}(\mathbb{R}^3))} \\
\leq C \|u\|_{L^q(B^{-1+3/p}_{p,1}(\mathbb{R}^3))} \|u\|_{L^q(B^{-1+3/p}_{p,1}(\mathbb{R}^3))},
\]
where \( C \) is a positive constant independent of \( u \).

**Lemma 2.7.** Let \( 1 \leq q < p < \infty \), \( \frac{1}{q} + \frac{1}{p} > \frac{1}{3} \) (or \( 1 \leq p \leq q < \infty \), \( \frac{1}{q} + \frac{1}{p} > \frac{1}{3} \)). Assume that \( n \in \mathcal{Y}_T \) (see (10)) and \( \phi \in \dot{B}^{5/3}_{p,1}(\mathbb{R}^3) \), then we have
\[
\|n\nabla \phi\|_{L^q(B^{-1+3/p}_{p,1}(\mathbb{R}^3))} \leq C \|n\|_{L^q(B^{s+3/3}_{q,1}(\mathbb{R}^3))} \|\nabla \phi\|_{L^q(B^{-1+3/p}_{p,1}(\mathbb{R}^3))} \\
\leq C \|n\|_{L^q(B^{s+3/3}_{q,1}(\mathbb{R}^3))} \|\phi\|_{\dot{B}^{5/3}_{p,1}(\mathbb{R}^3)},
\]
where \( C \) is a positive constant independent of \( n \) and \( \phi \).

**Proof.** When \( 1 \leq q < p < \infty \), \( \frac{1}{q} + \frac{1}{p} > \frac{1}{3} \), applying Lemma 2.3, by choosing \( p_1 = q, p_2 = p, s_1 = 3/q \) and \( s_2 = -1+3/p \), we can obtain
\[
\|n\nabla \phi\|_{L^q(B^{-1+3/p}_{p,1}(\mathbb{R}^3))} \leq C \|n\|_{L^q(B^{s+3/3}_{q,1}(\mathbb{R}^3))} \|\nabla \phi\|_{L^q(B^{-1+3/p}_{p,1}(\mathbb{R}^3))}.
\]
On the other hand, when \( 1 \leq p \leq q < \infty \), \( \frac{1}{q} + \frac{1}{p} > \frac{1}{3} \) and \( \frac{1}{p} - \frac{3}{q} \leq \frac{1}{3} \), applying Lemma 2.4, by choosing \( \tilde{p} = p, \tilde{q} = q \) and \( s = -1+3/p \), we can also get
\[
\|n\nabla \phi\|_{L^q(B^{-1+3/p}_{p,1}(\mathbb{R}^3))} \leq C \|n\|_{L^q(B^{s+3/3}_{q,1}(\mathbb{R}^3))} \|\nabla \phi\|_{L^q(B^{-1+3/p}_{p,1}(\mathbb{R}^3))}.
\]
Combining above estimates and (5) give (16) immediately. The proof of Lemma 2.7 is thus completed.

**Lemma 2.8.** [7] Let \( 1 \leq q < \infty \). Assume that \( n \in \mathcal{Y}_T \) and \( c \in \mathcal{Z}_T \) (see (10)). Then we have
\[
\|cn\|_{L^q(B^{s+3/3}_{q,1}(\mathbb{R}^3))} \leq C \|n\|_{L^q(B^{s+3/3}_{q,1}(\mathbb{R}^3))} \|c\|_{L^p(B^{s+3/3}_{q,1}(\mathbb{R}^3))} \\
+ C \|n\|_{L^q(B^{s+3/3}_{q,1}(\mathbb{R}^3))} \|c\|_{L^p(B^{s+3/3}_{q,1}(\mathbb{R}^3))},
\]
where \( C \) is a positive constant independent of \( n \) and \( c \).

**Lemma 2.9.** Let \( 1/r_1 + 1/r_2 = 1, r_1 \geq \max\{2, \frac{2q}{3pq+3q-3p}\} \). Let \( \frac{1}{p} + \frac{1}{q} > \frac{1}{3} \), \( 1 \leq q \leq p < \infty \), \( \frac{1}{q} - \frac{1}{p} \leq \frac{1}{3} \) (or \( \frac{1}{q} + \frac{1}{p} > \frac{1}{3} \), \( 1 \leq p < q < \infty \)). Assume that \( u \in \mathcal{X}_T \) and \( n \in \mathcal{Y}_T \). Then we have
\[
\|u \cdot \nabla u\|_{L^q(B^{-1+3/p}_{p,1}(\mathbb{R}^3))} \leq C \|u\|_{L^q(B^{-1+3/p}_{p,1}(\mathbb{R}^3))} \|n\|_{L^q(B^{-1+3/p}_{p,1}(\mathbb{R}^3))} \\
+ C \|u\|_{L^q(B^{-1+3/p}_{p,1}(\mathbb{R}^3))} \|n\|_{L^q(B^{-1+3/p}_{p,1}(\mathbb{R}^3))},
\]
where $C$ is a positive constant independent of $n$ and $u$. In particular, when $r_1 = \infty$, we have
\[
\| u \cdot \nabla u \|_{L^q_t(B_{p,1}^{-1+3/q}(R^3))} \leq C \| u \|_{L^1_t(B_{p,1}^{1+3/p}(R^3))} \| n \|_{L^q_t(B_{p,1}^{-1+3/q}(R^3))} + C \| u \|_{L^\infty_t(B_{p,1}^{-1+3/p}(R^3))} \| n \|_{L^q_t(B_{p,1}^{1+3/q}(R^3))}.
\]

Proof. We first get by applying the Bony paraproduct decomposition (9) that
\[
\Delta_j (u \nabla u) = \Delta_j (T_u \nabla n + R(u, \nabla n) + T_{\nabla n} u).
\]
Choosing $1 \leq r_1, r_2 \leq \infty$, such that $1/r_1 + 1/r_1 = 1$, note that $r_1 \geq 2$, applying (6) and Lemma 2.1 give rise to
\[
\| \Delta_j (T_u \nabla n) \|_{L^q_t(L^p)} \leq C \sum_{|j-j'| \leq 4} \sum_{k \leq j'-2} 2^{j'} \| \Delta_k u \|_{L^p_t(L^p)} \| \Delta_j' n \|_{L^q_t(L^q)}
\]
\[
\leq C \sum_{|j-j'| \leq 4} 2^{j'(3-\frac{3}{q} + \frac{3}{p})} d_{j'} \sum_{k \leq j'-2} 2^{(1-\frac{3}{q} + \frac{3}{p})k_j - \frac{k_j}{2} + \frac{k_j}{2}}
\times \| n \|_{L^q_t(B_{p,1}^{-1+3/q}(R^3))} \| \Delta_k u \|_{L^p_t(L^p)}
\leq C 2^{j'(2-3/q)} d_{j'} \| n \|_{L^q_t(B_{p,1}^{-1+3/q+2/r_2}(R^3))} \| u \|_{L^q_t(B_{p,1}^{1+3/p+2/r_1}(R^3))}.
\]
If $1 \leq q \leq p < \infty$, then there exists $1 < \lambda \leq \infty$ such that $\frac{1}{\lambda} = \frac{1}{p} + \frac{1}{q}$, note that
\[
\frac{1}{\lambda} = 3 \pm \frac{3}{q} (\text{as } r_1 \geq \frac{2p}{3p+3q-3p}),
\]
applying (6) and Lemma 2.1, we get that
\[
\| \Delta_j (T_{\nabla u} u) \|_{L^q_t(L^p)}
\leq C \sum_{|j-j'| \leq 4} \sum_{k \leq j'-2} 2^{k(3-\frac{3}{q} - \frac{k_j}{2})} \| \Delta_k n \|_{L^p_t(L^p)} \| \Delta_j' u \|_{L^q_t(L^q)}
\leq C \sum_{|j-j'| \leq 4} 2^{j'(1-3/p-2/r_2)} d_{j'} \| n \|_{L^q_t(B_{p,1}^{1+3/p+2/r_2}(R^3))} \| u \|_{L^q_t(B_{p,1}^{-1+3/q+2/r_1}(R^3))}.
\]
If $1 \leq p < q < \infty$, thanks to $r_1 \geq 2$, Lemma 2.1 and (6), then we arrive at
\[
\| \Delta_j (T_{\nabla u} u) \|_{L^q_t(L^p)}
\leq C 2^{j'(\frac{3}{q} - \frac{k_j}{2})} \sum_{|j-j'| \leq 4} \| \Delta_j' u \|_{L^q_t(L^p)} \| S_{j'-1} \nabla n \|_{L^q_t(L^q)}
\leq C 2^{j'(\frac{3}{q} - \frac{k_j}{2})} \sum_{|j-j'| \leq 4} 2^{j'(1-3/p-2/r_2)} d_{j'} \| n \|_{L^q_t(B_{p,1}^{1+3/p+2/r_2}(R^3))} \| u \|_{L^q_t(B_{p,1}^{-1+3/q+2/r_1}(R^3))}.
\]
To estimate the remaining term $R(u, \nabla n)$, we consider two cases: $\frac{1}{p} + \frac{1}{q} > 1$ and $\frac{1}{p} + \frac{1}{q} \leq 1$. 
Case 1. \( \frac{1}{p} + \frac{1}{q} > 1 \). We can find \( 1 < q' \leq \infty \) such that \( \frac{1}{p} + \frac{1}{q} = 1 \), applying Lemma 2.1 and (6), for fixed constant \( N_0 = 2 \), we infer
\[
\| \Delta_j R(u, \nabla n) \|_{L^1_t(L^p)} \\
\leq C 2^j (1 - 1/q) \sum_{j' \geq j > N_0} \| \tilde{\Delta}_j \nabla n \|_{L^{q'}_t(L^q)} \| \Delta_{j'} u \|_{L^2_t(L^2)} \\
\leq C 2^j (1 - 1/q) \sum_{j' \geq j > N_0} 2^{j - j'} u \| \omega_{q}^2 (B_{q,1}^{-1+3/p+2/r_2}(R^3)) \|_{L^{q'}_t(L^q)} \| \Delta_{j'} u \|_{L^2_t(L^2)} \\
\leq C 2^j (1 - 1/q) \sum_{j' \geq j > N_0} 2^{j - j'} u \| \omega_{q}^2 (B_{q,1}^{-1+3/p+2/r_2}(R^3)) \|_{L^{q'}_t(L^q)} \| \Delta_{j'} u \|_{L^2_t(L^2)} \\
(21)
\]
\[
\| \Delta_j n \|_{L^1_t(L^p)} \\
\leq C 2^j (1 - 1/q) d_j \| n \| \omega_{q}^2 (B_{q,1}^{-1+3/p+2/r_2}(R^3)) \|_{L^1_t(L^p)} \| \Delta_{j'} n \|_{L^1_t(L^p)} \\
\leq C 2^j (1 - 1/q) d_j \| n \| \omega_{q}^2 (B_{q,1}^{-1+3/p+2/r_2}(R^3)) \|_{L^1_t(L^p)} \| \Delta_{j'} n \|_{L^1_t(L^p)} \\
\]

Combining (18)-(22) and taking summation for \( j \in \mathbb{Z} \), we arrive at (17). We thus conclude the proof of Lemma 2.9. \( \square \)

**Lemma 2.10.** Assume that \( n \in \mathcal{Y}_T \) and \( c \in \mathcal{Z}_T \). Let \( 1 \leq q < 6, r_1 \geq 2 \) and \( 1/r_1 + 1/r_2 = 1 \). Then we have
\[
\| \nabla \cdot (n \nabla c) \|_{L^q_t(B_{q,1}^{-2+3/q/(3q)}(R^3))} \leq C \| n \|_{L^q_t(B_{q,1}^{-2+3/q/(3q)}(R^3))} \| c \|_{L^q_t(B_{q,1}^{-2+3/q/(3q)}(R^3))} \\
+ C \| n \|_{L^q_t(B_{q,1}^{-2+3/q/(3q)}(R^3))} \| c \|_{L^q_t(B_{q,1}^{-2+3/q/(3q)}(R^3))}, \quad (23)
\]
where \( C \) is a positive constant independent of \( n \) and \( c \). In particular, when \( r_1 = \infty \), it holds true that
\[
\| \nabla \cdot (n \nabla c) \|_{L^q_t(B_{q,1}^{-2+3/q/(3q)}(R^3))} \leq C \| n \|_{L^q_t(B_{q,1}^{-2+3/q/(3q)}(R^3))} \| c \|_{L^q_t(B_{q,1}^{-2+3/q/(3q)}(R^3))} \\
+ C \| n \|_{L^q_t(B_{q,1}^{-2+3/q/(3q)}(R^3))} \| c \|_{L^q_t(B_{q,1}^{-2+3/q/(3q)}(R^3))},
\]

**Proof.** Thanks to the Bony paraproduct decomposition (9), we have
\[
\Delta_j (n \nabla c) = \Delta_j (T_n \nabla c + R(n, \nabla c) + T_v c). \]

Taking advantage of Lemma 2.1 and (6), note that \( r_1 \geq 2 \), we thus have
\[
\| \Delta_j (T_n \nabla c) \|_{L^1_t(L^p)} \leq C \sum_{|j-j'| \leq 4} \sum_{k \leq j'-2} 2^{j-k} \| \Delta_k n \|_{L^{q'}_t(L^q)} \| \Delta_{j'} c \|_{L^1_t(L^p)} \\
\leq C \sum_{|j-j'| \leq 4} \sum_{k \leq j'-2} 2^{j-k} (1-2/r_2-3/q) d_{j'} \sum_{k \leq j'-2} 2^{2j-2j} \| \Delta_k n \|_{L^{q'}_t(L^q)} \| \Delta_{j'} c \|_{L^1_t(L^p)} \\
\leq C 2^j (1-3/q) d_j \| n \|_{L^q_t(B_{q,1}^{-2+3/q/(3q)}(R^3))} \| c \|_{L^q_t(B_{q,1}^{-2+3/q/(3q)}(R^3))} \]

Taking advantage of Lemma 2.1 and (6), note that \( r_1 \geq 2 \), we thus have
\[
\| \Delta_j (T_n \nabla c) \|_{L^1_t(L^p)} \leq C \sum_{|j-j'| \leq 4} \sum_{k \leq j'-2} 2^{j-k} \| \Delta_k n \|_{L^{q'}_t(L^q)} \| \Delta_{j'} c \|_{L^1_t(L^p)} \\
\leq C \sum_{|j-j'| \leq 4} \sum_{k \leq j'-2} 2^{j-k} (1-2/r_2-3/q) d_{j'} \sum_{k \leq j'-2} 2^{2j-2j} \| \Delta_k n \|_{L^{q'}_t(L^q)} \| \Delta_{j'} c \|_{L^1_t(L^p)} \\
\leq C 2^j (1-3/q) d_j \| n \|_{L^q_t(B_{q,1}^{-2+3/q/(3q)}(R^3))} \| c \|_{L^q_t(B_{q,1}^{-2+3/q/(3q)}(R^3))} \]

Combining (18)-(22) and taking summation for \( j \in \mathbb{Z} \), we arrive at (17). We thus conclude the proof of Lemma 2.9. \( \square \)
It follows from Lemma 2.1, $r_1 \geq 2$ and (6) that
\[
\|\Delta_j(T_v c)\|_{L^1_T(L^2)} \leq C \sum_{|j-j'| \leq 4 \atop k \leq j'-2} 2^{k+\frac{3}{q}} \|\Delta_k c\|_{L^2_T(L^2)} \|\Delta_j' n\|_{L^2_T(L^4)}
\leq C \sum_{|j-j'| \leq 4 \atop k \leq j'-2} 2^{j'-(2-3/q-2/2)} d_{j'} \sum_{k \leq j'-2} 2^{k(1-2/r_1)}
\times \|n\|_{L^2_T(\dot{B}^{2-2+3/q+2/2}_q(R^3))} 2^{k\left(\frac{3}{q}+\frac{2}{r_1}\right)} \|\Delta_k c\|_{L^2_T(L^4)}
\leq C 2^{j-(3/q)} d_j \|n\|_{L^2_T(\dot{B}^{2-2+3/q+2/2}_q(R^3))} \|c\|_{L^2_T(\dot{B}^{2/r_1+3/q}_q(R^3))}.
\]

To estimate the remaining term $R(n, \nabla c)$, we consider the following two cases.

Case 1. $2 \leq q < 6$. Let $\frac{1}{q} + \frac{1}{p} = 1$, applying Lemma 2.1 and (6), for some fixed constant $N_0$, we see
\[
\|\Delta_j R(n, \nabla c)\|_{L^1_T(L^2)} \leq C 2^{3j/q} \sum_{j' \geq j-N_0} 2^{j'} \|\Delta_j c\|_{L^2_T(L^4)} \|\Delta_j' n\|_{L^2_T(L^4)}
\leq C 2^{3j/q} \sum_{j' \geq j-N_0} 2^{j'(1-6/q)} d_{j'} \|n\|_{L^2_T(\dot{B}^{2-2+3/q+2/2}_q(R^3))}
\times \|c\|_{L^2_T(\dot{B}^{2/r_1+3/q}_q(R^3))}
\leq C 2^{j-(3/q)} d_j \|n\|_{L^2_T(\dot{B}^{2-2+3/q+2/2}_q(R^3))} \|c\|_{L^2_T(\dot{B}^{2/r_1+3/q}_q(R^3))}.
\]

Case 2: $1 \leq q < 2$. There exists $2 < \lambda \leq \infty$ such that $1/q + 1/\lambda = 1$, from Lemma 2.1 and (6), we see that
\[
\|\Delta_j R(n, \nabla c)\|_{L^1_T(L^2)} \leq C 2^{3j(1-1/q)} \sum_{j' \geq j-N_0} 2^{j'2^{3j'(1/q-1/\lambda)}} \|\Delta_j c\|_{L^2_T(L^4)} \|\Delta_j' n\|_{L^2_T(L^4)}
\leq C 2^{3j(1-1/q)} \sum_{j' \geq j-N_0} 2^{j'(1-3)} d_{j'} \|n\|_{L^2_T(\dot{B}^{2-2+3/q+2/2}_q(R^3))}
\times \|c\|_{L^2_T(\dot{B}^{2/r_1+3/q}_q(R^3))}
\leq C 2^{j-(3/q)} d_j \|n\|_{L^2_T(\dot{B}^{2-2+3/q+2/2}_q(R^3))} \|c\|_{L^2_T(\dot{B}^{2/r_1+3/q}_q(R^3))}.
\]
Combining the above estimates (24)-(27) shows the assertion (23), which completes the proof of Lemma 2.10.

Lemma 2.11. Let $1/r_1 + 1/r_2 = 1$, $r_1 \geq \max\{2, \frac{2pq}{pq+3q-3p}\}$. Let $1 \leq q < p < \infty$, $\frac{1}{q} + \frac{1}{p} \leq \frac{1}{3}$ (or $1 \leq p < q < \infty$). Assume that $u \in X_T$ and $c \in Z_T$. Then we have
\[
\|u \cdot \nabla c\|_{L^1_T(\dot{B}^{2/q}_q(R^3))} \leq C \|u\|_{L^r_T(\dot{B}^{-1+3/p+2/r_1}_q(R^3))} \|c\|_{L^q_T(\dot{B}^{2/r_1+3/q}_q(R^3))}
+ C \|u\|_{L^q_T(\dot{B}^{-1+3/p+2/r_2}_q(R^3))} \|c\|_{L^q_T(\dot{B}^{2/r_1+3/q}_q(R^3))},
\]
where $C$ is a positive constant independent of $c$ and $u$. In particular, when $r_1 = \infty$, we have
\[
\|u \cdot \nabla c\|_{L^1_T(\dot{B}^{2/q}_q(R^3))} \leq C \|u\|_{L^q_T(\dot{B}^{-1+3/p}_q(R^3))} \|c\|_{L^q_T(\dot{B}^{2/q}_q(R^3))}
+ C \|u\|_{L^q_T(\dot{B}^{1+3/p}_q(R^3))} \|c\|_{L^q_T(\dot{B}^{q}_q(R^3))}.
\]
Proof. The Bony paraproduct decomposition (9) ensures that
\[ \Delta_j(\nabla v c) = \Delta_j(T_\nabla v c + R(\nabla v c) + T_\nabla v c)). \]

Using Lemma 2.1, \( r_1 \geq 2 \) and (6), we discover that
\[
\|\Delta_j(T_\nabla v c)\|_{L^q_j(L^q)} \leq C \sum_{|j-j'| \leq 4} 2^{2j'} \sum_{k \leq j' - 2} 2^{2k} \|\Delta_j' c\|_{L^q_j(L^q)} \|\Delta_k c\|_{L^q_j(L^q)}
\leq C \sum_{|j-j'| \leq 4} 2^{2j'} 2^{j'(2/p - 3/q)} d_{j'} \|c\|_{L^q_j(B^{-1+3/p+2/r_2}_q(R^3))} \|\Delta_k c\|_{L^q_j(L^q)} \times \sum_{k \leq j' - 2} 2^{k(1 - 2/r_1)} 2^{k(1 + 3/p + 2/r_1)} \|\Delta_k c\|_{L^q_j(L^q)} \leq C 2^{2j'(-3/q)} d_{j'} \|c\|_{L^q_j(B^{-1+3/p+2/r_2}_q(R^3))} \|\Delta_k c\|_{L^q_j(B^{-3/q+2/r_1}_q(R^3))}.
\]

If \( 1 \leq q \leq p < \infty \) and \( 1/q - 1/p \leq 1/3 \), then there exists \( 1 < \lambda \leq \infty \) such that \( 1/q = \frac{1}{p} + \frac{1}{\lambda} \). Note that \( 1 + \frac{3}{p} - \frac{3}{q} \geq \frac{2}{r_1} \) (as \( r_1 \geq \frac{2pq}{pq + 3q - 3p} \)), these, in combination with Lemma 2.1 and (6), show that
\[
\|\Delta_j(T_\nabla v c)\|_{L^q_j(L^q)} \leq C \sum_{|j-j'| \leq 4} \|\Delta_j' u\|_{L^q_j(L^q)} \sum_{k \leq j' - 2} 2^{(1 + 3/p - 3/q)} 2^{k(3/2 + 2/r_1)} \|\Delta_k c\|_{L^q_j(L^q)} \leq C \sum_{|j-j'| \leq 4} d_{j'} 2^{j'(-3/q)} \|u\|_{L^q_j(B^{-1+3/p+2/r_2}_q(R^3))} \|c\|_{L^q_j(B^{-3/q+2/r_1}_q(R^3))}.
\]

If \( 1 \leq p < q < \infty \), note that \( r_1 \geq 2 \), then by Lemma 2.1 and (6), we obtain
\[
\|\Delta_j(T_\nabla v c)\|_{L^q_j(L^q)} \leq C 2^{2j'(-3/q)} \sum_{|j-j'| \leq 4} 2^{j'(-3/p + 2/r_2)} d_{j'} \|u\|_{L^q_j(B^{-1+3/p+2/r_2}_q(R^3))} \times \sum_{k \leq j' - 2} 2^{k(1 - 2/r_1)} 2^{k(3/q + 2/r_1)} \|\Delta_k c\|_{L^q_j(L^q)} \leq C 2^{2j'(-3/q)} d_{j'} \|c\|_{L^q_j(B^{-3/q+2/r_1}_q(R^3))} \|u\|_{L^q_j(B^{-1+3/p+2/r_2}_q(R^3))}.
\]

To estimate the remaining term \( R(\nabla v c) \), we consider the following two cases. In the case \( \frac{1}{p} + \frac{1}{q} > 1 \), we can find \( 1 < q' \leq \infty \) such that \( \frac{1}{q'} + \frac{1}{q'} = 1 \), by Lemma 2.1 and (6), for fixed constant \( N_0 = 2 \), we calculate that
\[
\|\Delta_j R(\nabla v c)\|_{L^q_j(L^q)} \leq C 2^{3j'(-1/q)} \sum_{j' \geq j-N_0} 2^{j'} \|\Delta_j' c\|_{L^q_j(L^q)} 2^{3k(\frac{1}{q'} + \frac{1}{q'})} \|\Delta_j c\|_{L^q_j(L^q)} \leq C 2^{3j'(-1/q)} \sum_{j' \geq j-N_0} d_{j'} 2^{-3j'} \|u\|_{L^q_j(B^{-1+3/p+2/r_2}_q(R^3))} \|c\|_{L^q_j(B^{-3/q+2/r_1}_q(R^3))}.
\]
Since $\frac{1}{p} + \frac{1}{q} \leq 1$, note that $\nabla \cdot u = 0$, with the help of Lemma 2.1 and (6) we obtain

$$\|\Delta_j R(u, \nabla c)\|_{L^2_t(L^p)} \leq C 2^{(3/p+1)j} \sum_{j' \geq j-N_0} 2^{j'(-1-3/p-3/q)2^{j'(-1+3/p+2j/2r)}} \times \|\Delta_j u\|_{L^2_t(L^p)} 2^{j(3/q+2/r)} \\|\Delta_j c\|_{L^2_t(L^q)}$$

$$\leq C 2^{-3j/q} \sum_{j} \|u\|_{\tilde{L}^r_\lambda(B_{p,1}^{-1+3/p+2r/2}(\mathbb{R}^3))} \\|c\|_{\tilde{L}^r_\lambda(B_{p,1}^{3/q+2r/2}(\mathbb{R}^3))}.$$  \hspace{1cm} (33)

Combining (29)-(33), the desired estimate (28), therefore follows and, hence the proof of Lemma 1.11 is finished.

\[\square\]

3. Proof of Theorem 1.2: Global existence with large vertical velocity component. The goal of this section is to present the proof of the global existence of system (4) with large vertical velocity component. Let $u = (u^1, u^2, u^3) = (u^h, u^s)$ and $\text{div} u^h = \partial_t u^1 + \partial_2 u^2$. Denote

$$f_1(t) := \|u^3(t)\|_{\tilde{B}^{1+3/p}(\mathbb{R}^3)}, \quad f_2(t) := \|u^2(t)\|_{\tilde{B}^{1+3/p}(\mathbb{R}^3)}, \hspace{1cm} (34)$$

$$\mathbb{P}_\lambda := \mathbb{P} \exp \left\{ -\lambda \int_0^t f_1(\tau) d\tau - \lambda_2 \int_0^t f_2(\tau) d\tau \right\},$$

where $\lambda_1 > 0$ and $\lambda_2 > 0$, and the similar notations for $u^h, n_\lambda, c_\lambda$. For $\lambda > 0$, let

$$n_\lambda := n \exp \left\{ -\lambda \int_0^t f_1(\tau) d\tau \right\}, \quad c_\lambda := c \exp \left\{ -\lambda \int_0^t f_1(\tau) d\tau \right\}. \hspace{1cm} (35)$$

We also need to introduce the following weighted Chemin-Lerner type norm from [39]. Let $f(t) \in L^1_{loc}(0, +\infty)$ and $f(t) \geq 0$. We define

$$\|u\|_{L^p_{t,x}(\tilde{B}^r_{p,r}(\mathbb{R}^3))} := \left\{ \sum_{k \in \mathbb{Z}} 2^{kr} \left( \int_0^t \|u\|_{L^p_{t,x}(\tilde{B}^r_{p,r}(\mathbb{R}^3))} d\tau \right)^\frac{p}{r} \right\}^{\frac{1}{p}} \hspace{1cm} (36)$$

for $1 \leq p \leq \infty$, $1 \leq \rho, r < \infty, s \in \mathbb{R}$, and with the standard modification for $\rho = \infty$ or $r = \infty$.

Lemma 3.1. [39] Let $1 < p < 6$ and $f_1, f_2$ being given by (34). We obtain

$$\|\Delta_j (u^3 u^h)\|_{L^1_t(L^p)} \lesssim d_j 2^{-3j/p} \|u^h\|_{\tilde{L}^r_\lambda(B^{1+3/p}(\mathbb{R}^3))} \|u^3\|_{\tilde{L}^r_\lambda(B^{-1+3/p}(\mathbb{R}^3))} \hspace{1cm} (37)$$

$$\|\Delta_j (u^3 u^h)\|_{L^1_t(L^p)} \lesssim d_j 2^{-3j/p} \|u^h\|_{\tilde{L}^r_\lambda(B^{1+3/p}(\mathbb{R}^3))} \|u^3\|_{\tilde{L}^r_\lambda(B^{-1+3/p}(\mathbb{R}^3))} \hspace{1cm} (38)$$

$$\|\Delta_j (u^3 \text{div} u^h)\|_{L^1_t(L^p)} \lesssim d_j 2^{j(1-3/p)} \|u^h\|_{\tilde{L}^r_\lambda(B^{1+3/p}(\mathbb{R}^3))} \|u^3\|_{\tilde{L}^r_\lambda(B^{-1+3/p}(\mathbb{R}^3))} \hspace{1cm} (39)$$

and

$$\|\Delta_j (u^3 \text{div} u^h)\|_{L^1_t(L^p)} \lesssim d_j 2^{j(1-3/p)} \|u^h\|_{\tilde{L}^r_\lambda(B^{1+3/p}(\mathbb{R}^3))} \|u^3\|_{\tilde{L}^r_\lambda(B^{-1+3/p}(\mathbb{R}^3))} \hspace{1cm} (40)$$
The estimate of the pressure $P$.

Taking $\text{div}$ to the first equation of the system (4) yields that

\[
- \Delta P = \text{div}_h \text{div}_h (u^h \otimes u^h) + 2\partial_3 \text{div}_h (u^3 u^h) + \partial_3^2 (u^3)^2 + \text{div} (n \nabla \phi).
\]

By virtue of $\nabla \cdot u = 0$ and of the notations given by (34), it is clear that

\[
\nabla P = \nabla (-\Delta)^{-1} \left[ \text{div}_h \text{div}_h (u^h \otimes u^h) + 2\partial_3 \text{div}_h (u^3 u^h) \right] - 2\partial_3 (u^3 \text{div}_h u^h) + \text{div} (n \nabla \phi) \quad \text{(42)}
\]

and

\[
\nabla P_{\chi} = \nabla (-\Delta)^{-1} \left[ \text{div}_h \text{div}_h \left( u^h \otimes u^h_{\chi} \right) + 2\partial_3 \text{div}_h \left( u^3 u^h_{\chi} \right) \right] - 2\partial_3 \left( u^3 \text{div}_h u^h_{\chi} \right) + \text{div} (n_{\chi} \nabla \phi) \quad \text{(43)}
\]

Proposition 1. Let $1 < p, q < 6$, $\frac{1}{p} + \frac{1}{q} > \frac{2}{3}$ and $\frac{1}{\max(p,q)} - \frac{1}{\max(p,q)} \leq \frac{1}{7}$. If $(u, n, c) \in \Theta_T$ with $\nabla \cdot u = 0$, then (42) and (43) have a unique solution $(\nabla P, \nabla P_{\chi}) \in L_t^1 (B_{p,1}^{-1+3/p}(\mathbb{R}^3))^2$ such that for $t \in (0, \infty)$, we obtain

\[
\| \nabla P \|_{L_t^1(B_{p,1}^{-1+3/p}(\mathbb{R}^3))} \lesssim \| u^h \|_{L_t^\infty(B_{p,1}^{-1+3/p}(\mathbb{R}^3))} \| u^3 \|_{L_t^1(B_{p,1}^{-1+3/p}(\mathbb{R}^3))} + \| u^h \|_{L_t^{r_p}(B_{p,1}^{-1+3/p}(\mathbb{R}^3))} \| u^h_{\chi} \|_{L_t^1(B_{p,1}^{-1+3/p}(\mathbb{R}^3))} + \| u^3 \|_{L_t^{r_p}(B_{p,1}^{-1+3/p}(\mathbb{R}^3))} \| \phi \|_{B_{p,1}^3(\mathbb{R}^3)},
\]

and

\[
\| \nabla P_{\chi} \|_{L_t^1(B_{p,1}^{-1+3/p}(\mathbb{R}^3))} \lesssim \| u^h \|_{L_t^\infty(B_{p,1}^{-1+3/p}(\mathbb{R}^3))} \| u^3 \|_{L_t^1(B_{p,1}^{-1+3/p}(\mathbb{R}^3))} + \| u^h \|_{L_t^{r_p}(B_{p,1}^{-1+3/p}(\mathbb{R}^3))} \| u^3 \|_{L_t^1(B_{p,1}^{-1+3/p}(\mathbb{R}^3))} + \| u^h \|_{L_t^{r_p}(B_{p,1}^{-1+3/p}(\mathbb{R}^3))} \| u^h_{\chi} \|_{L_t^1(B_{p,1}^{-1+3/p}(\mathbb{R}^3))} + \| n_{\chi} \|_{L_t^{r_p}(B_{p,1}^{-1+3/p}(\mathbb{R}^3))} \| \phi \|_{B_{p,1}^3(\mathbb{R}^3)}.
\]

Proof. Applying $\Delta_j$ to (43), from $L_t^1(L^p)$ estimates, using (15) from Lemma 2.6, (16) from Lemma 2.7, both (37) and (39) from Lemma 3.1 yield that

\[
\| \Delta_j (\nabla P_{\chi}) \|_{L_t^1(L^p)} \lesssim 2^{j-3} d_j \| u^h \|_{L_t^{r_p}(B_{p,1}^{-1+3/p}(\mathbb{R}^3))} \| u^h_{\chi} \|_{L_t^1(B_{p,1}^{-1+3/p}(\mathbb{R}^3))} + 2^j d_j 2^{-3j/p} \times \left( \| u^h \|_{L_t^1(B_{p,1}^{-1+3/p}(\mathbb{R}^3))} \| u^h \|_{L_t^1(B_{p,1}^{-1+3/p}(\mathbb{R}^3))} \| u^h_{\chi} \|_{L_t^1(B_{p,1}^{-1+3/p}(\mathbb{R}^3))} + \| u^3 \|_{L_t^1(B_{p,1}^{-1+3/p}(\mathbb{R}^3))} \right)
\]
from which, the desired estimate (45) follows readily. Finally, these complete the estimate of $j$ (40) from Lemma 3.1 yield that

\[ \|u_h\|_{L^p_{x,t}(B^{-1+3/p}_{p,1}(\mathbb{R}^3))} \leq 2^{j(1-3/p)} \left( \|u_h\|_{L^p_{x,t}(\mathbb{R}^3)} + \|u_h\|_{L^p_{x,t}(B^{-1+3/p}_{p,1}(\mathbb{R}^3))} \right) \]

Finally, these complete the estimate of $j$ (40) from Lemma 3.1 yield that

\[ \|u_h\|_{L^p_{x,t}(B^{-1+3/p}_{p,1}(\mathbb{R}^3))} \leq 2^{j(1-3/p)} \left( \|u_h\|_{L^p_{x,t}(\mathbb{R}^3)} + \|u_h\|_{L^p_{x,t}(B^{-1+3/p}_{p,1}(\mathbb{R}^3))} \right) \]

\[ \|u_h\|_{L^p_{x,t}(B^{-1+3/p}_{p,1}(\mathbb{R}^3))} + \|u_h\|_{L^p_{x,t}(B^{-1+3/p}_{p,1}(\mathbb{R}^3))} \]

from which, multiplying by $2^{j(-1+3/p)}$ and summing up over $j$ yield the desired result (44).

Keeping in mind the proof of (44), applying the operator $\Delta_j$ to (42), taking the $L^1_x(L^p_t)$ norm, using (15) from Lemma 2.6, (16) from Lemma 2.7, both (38) and (40) from Lemma 3.1 yield that

\[ \|\Delta_j(\nabla P)\|_{L^1_x(L^p_t)} \leq 2^{j-3/p} \left( \|u_h\|_{L^p_{x,t}(B^{-1+3/p}_{p,1}(\mathbb{R}^3))} + \|u_h\|_{L^p_{x,t}(B^{-1+3/p}_{p,1}(\mathbb{R}^3))} \right) \]

\[ + \|u_h\|_{L^p_{x,t}(B^{-1+3/p}_{p,1}(\mathbb{R}^3))} + \|u_h\|_{L^p_{x,t}(B^{-1+3/p}_{p,1}(\mathbb{R}^3))} \]

\[ + \|u_h\|_{L^p_{x,t}(B^{-1+3/p}_{p,1}(\mathbb{R}^3))} + \|u_h\|_{L^p_{x,t}(B^{-1+3/p}_{p,1}(\mathbb{R}^3))} \]

from which, the desired estimate (45) follows readily. Finally, these complete the proof of Proposition 1.

- **The estimate of $u^h$ and $u^3$.**

The first equation of (4) implies that

\[ \partial_t u^h + (\lambda_1 f_1(t) + \lambda_2 f_2(t)) u^h - \Delta u^h = -u \cdot \nabla u^h - \nabla h \cdot P - n \cdot \nabla \phi \]

and

\[ \partial_t u^3 - \Delta u^3 = -u \cdot \nabla u^3 - \partial_3 P - n \partial_3 \phi. \]
Proposition 2. Let $1 < p, q < 6$, $\frac{1}{p} + \frac{1}{q} > \frac{2}{3}$ and $\frac{1}{\min \{p, q\}} - \frac{1}{\max \{p, q\}} \leq \frac{1}{3}$. If $(u, n, c) \in \Theta_T$ with $\nabla \cdot u = 0$, then (48) and (49) satisfy

$$
\|u^3\|_{L^\infty(B_{p,1}^{1+3/p}(\mathbb{R}^3))} + C \left( \|u^h\|_{L^\infty(B_{p,1}^{-1+3/p}(\mathbb{R}^3))} \right)
$$

and

$$
\|u^3\|_{L^\infty(B_{p,1}^{-1+3/p}(\mathbb{R}^3))} + C \left( \|u^h\|_{L^\infty(B_{p,1}^{1+3/p}(\mathbb{R}^3))} \right).
$$

Proof. We first prove (50). When $p \geq 2$, applying $\Delta_j$ to (48) and taking $L^2$ inner product with $|\Delta_j u_h^p|^{p-2} \Delta_j u_h^p$, we conclude that

$$
\frac{1}{p} \frac{d}{dt} \|\Delta_j u_h^p\|_{L^p} + (\lambda_1 f_1(t) + \lambda_2 f_2(t)) \|\Delta_j u_h^p\|_{L^p} - \int_{\mathbb{R}^3} \Delta \Delta_j u_h^p \cdot \Delta_j u_h^p \| \Delta_j u_h^p \|^{p-2} \Delta_j u_h^p dx
$$

$$
= - \int_{\mathbb{R}^3} \Delta_j \left( u \cdot \nabla u_h^p + \nabla h \mathcal{P}_h + n_h \nabla h \phi \right) \Delta_j u_h^p \| \Delta_j u_h^p \|^{p-2} \Delta_j u_h^p dx.
$$

(52)

When $1 < p < 2$, applying $\Delta_j$ to (4.15), and let $T_\varepsilon(x) := \sqrt{x^2 + \varepsilon^2}$, taking $L^2$ inner product of the resulting equation with $(T_\varepsilon(\Delta_j u_h^p))^{p-1} T_\varepsilon'(\Delta_j u_h^p)$, then let $\varepsilon \to 0$, an integration by parts yields (52), for the more details, we can refer to [11].

Thanks to [11, 41], there exists a positive constant $\bar{c}$ fulfilling

$$
- \int_{\mathbb{R}^3} \Delta \Delta_j u_h^p \cdot \Delta_j u_h^p \| \Delta_j u_h^p \|^{p-2} \Delta_j u_h^p dx \geq \bar{c} 2^{2j} \|\Delta_j u_h^p\|_{L^p}.
$$

(53)

Similarly the proceeding as that in [11], from (52) and (53), we thus have

$$
\frac{d}{dt} \|\Delta_j u_h^p\|_{L^p} + (\lambda_1 f_1(t) + \lambda_2 f_2(t)) \|\Delta_j u_h^p\|_{L^p} + \bar{c} 2^{2j} \|\Delta_j u_h^p\|_{L^p}
$$

$$
\leq \|\Delta_j \left( u \cdot \nabla u_h^p \right))\|_{L^p} + \|\Delta_j \nabla h \mathcal{P}_h\|_{L^p} + \|\Delta_j (n_h \nabla h \phi)\|_{L^p}.
$$

After time integrating over $[0, t]$, applying (15) from Lemma 2.6, (16) from Lemma 2.7, (37) from Lemma 3.1 and (44) from Proposition 1, we arrive at
Using the Cauchy-Schwartz inequality \( ab \leq \frac{\varepsilon}{2} a^2 + \frac{1}{2\varepsilon} b^2 \) for \( \varepsilon = \frac{\bar{\varepsilon}}{2\varepsilon_0} \), multiplying (54) by \( 2^j(1-\frac{3}{p}) \) and summing up over \( j \) yield

\[
\| u_j^h \|_{L^2((B_{p, \frac{1+3}{p}}(R^3)))} + \lambda_1 \| u_j^h \|_{L^1(\Omega_j^1 + \Omega^2_j)} + \lambda_2 \| u_j^h \|_{L^1(\Omega_j^1 + \Omega^2_j)} + \| u_j^h \|_{L^1(\Omega_j^1 + \Omega^2_j)} \leq C_0 \left( \| u_j^h \|_{L^2((B_{p, \frac{1+3}{p}}(R^3)))} + \| u_j^h \|_{L^1(\Omega_j^1 + \Omega^2_j)} \right)
\]

from which, we obtain the desired estimate (50).

Finally, as the same proof of (54), applying \( \Delta_j \) to (49), taking \( L^2 \) inner product with \( |\Delta_j u^3|^p - 2\Delta_j u^3 \) (in the case when \( p \in (1, 2) \), we need to make some modifications as that [11]), summing up (16) from Lemma 2.7, both (38) and (40) from
Lemma 3.1, (45) from Proposition 1, we have

\[ \|\Delta_j u^3\|_{L^r_\ast(L^p)} + r^{2j} \|\Delta_j u^3\|_{L^1_\ast(L^p)} \]
\[ \lesssim \|\Delta_j u_0\|_{L^r} + 2^j \|\Delta_j (u \cdot \nabla u^3)\|_{L^1_\ast(L^p)} + \|\Delta_j (u^3 \div u^3)\|_{L^1_\ast(L^p)} \]
\[ + \|\Delta_j \partial \lambda\|_{L^1_\ast(L^p)} + \|\Delta_j (n \partial \lambda)\|_{L^1_\ast(L^p)} \]
\[ \lesssim \|\Delta_j u_0\|_{L^r} + d_j 2^{j(1-3/p)} \left( \|u^h\|_{L^\infty_\ast(B^{-1+3/p}_{p,1}(\mathbb{R}^3))} \|u^3\|_{E^{1}_{1}(B^{-1+3/p}_{p,1})} \right) \]
\[ + \|u^h\|_{E^{1}_{1}(B^{1+3/p}_{p,1}(\mathbb{R}^3))} \|u^3\|_{E^{1}_{1}(B^{1+3/p}_{p,1}(\mathbb{R}^3))} \]
\[ + d_j 2^{j(1-3/p)} \left( \|u^h\|_{L^\infty_\ast(B^{-1+3/p}_{p,1}(\mathbb{R}^3))} \|u^3\|_{E^{1}_{1}(B^{1+3/p}_{p,1}(\mathbb{R}^3))} \right) \]
\[ + \|u^h\|_{L^\infty_\ast(B^{1+3/p}_{p,1}(\mathbb{R}^3))} \|u^3\|_{E^{1}_{1}(B^{1+3/p}_{p,1}(\mathbb{R}^3))} \]
\[ + \|\n\|_{E^{1}_{1}(B^{\frac{1}{q}-\frac{3}{q}}_{q,1}(\mathbb{R}^3))} \|\phi\|_{B^{2/q}_{p,1}(\mathbb{R}^3)} \]  
\[ + d_j 2^{j(1-3/p)} \|\n\|_{E^{1}_{1}(B^{\frac{1}{q}-\frac{3}{q}}_{q,1}(\mathbb{R}^3))} \|\phi\|_{B^{2/q}_{p,1}(\mathbb{R}^3)} , \]

(55)

multiplying (55) by \(2^{j(1-3/p)}\) and summing up over \(j\) yield (51). Therefore, we complete the proof of Proposition 2.

\[ \Box \]

- The estimate of \(n\).

Let \(n_{\lambda}\) be given by (35). From (4), we have

\[ \partial_t n_{\lambda} + \lambda f_1(t) n_{\lambda} - \Delta n_{\lambda} = -u \cdot \nabla n_{\lambda} - \nabla \cdot (n_{\lambda} \nabla c) . \]

(56)

Proposition 3. Let \(1 < p, q < 6, \frac{1}{p} + \frac{1}{q} > \frac{2}{3}\) and \(\frac{1}{p} - \frac{1}{\min\{p,q\}} - \frac{1}{\max\{p,q\}} \leq \frac{1}{3}\). If \((u, n, c) \in \Theta_{T}\) with \(\nabla \cdot u = 0\), then (56) leads to

\[ \|n_{\lambda}\|_{L^\infty_\ast(B^{\frac{1}{q}-\frac{3}{q}}_{q,1}(\mathbb{R}^3))} + \lambda \|n_{\lambda}\|_{E^{1}_{1,L^\ast(B^{\frac{1}{q}-\frac{3}{q}}_{q,1}(\mathbb{R}^3))}} + \tilde{c} \|n_{\lambda}\|_{E^{1}_{1}(B^{\frac{1}{q}-\frac{3}{q}}_{q,1}(\mathbb{R}^3))} \]
\[ \leq \|n_0\|_{L^\infty_\ast(B^{\frac{1}{q}-\frac{3}{q}}_{q,1}(\mathbb{R}^3))} + C \left( \|n_{\lambda}\|_{E^{1}_{1,L^\ast(B^{\frac{1}{q}-\frac{3}{q}}_{q,1}(\mathbb{R}^3))}} \right) \]
\[ \times \|u^h\|_{E^{1}_{1}(B^{1+3/p}_{p,1}(\mathbb{R}^3))} + \|n_{\lambda}\|_{E^{1}_{1,L^\ast(B^{\frac{1}{q}-\frac{3}{q}}_{q,1}(\mathbb{R}^3))}} \]
\[ + \|n_{\lambda}\|_{E^{1}_{1,L^\ast(B^{\frac{1}{q}-\frac{3}{q}}_{q,1}(\mathbb{R}^3))}} \|c\|_{E^{1}_{1}(B^{\frac{1}{q}-\frac{3}{q}}_{q,1}(\mathbb{R}^3))} \]
\[ + \|n_{\lambda}\|_{E^{1}_{1}(B^{\frac{1}{q}-\frac{3}{q}}_{q,1}(\mathbb{R}^3))} \|c\|_{E^{1}_{1}(B^{\frac{1}{q}-\frac{3}{q}}_{q,1}(\mathbb{R}^3))} \]

(57)

Proof. Applying the operator \(\Delta_j\) to (56), taking \(L^2\) inner product with \(\Delta_j n_{\lambda}\) \(q^{-2}\) \(\Delta_j n_{\lambda}\) (in the case when \(q \in (1, 2)\), we need to make some modifications as that [11]), we obtain

\[ \frac{1}{q} \frac{d}{dt} \|\Delta_j n_{\lambda}\|_{L^q}^2 + \lambda f_1(t) \|\Delta_j n_{\lambda}\|_{L^q}^2 - \int_{\mathbb{R}^3} \Delta \Delta_j n_{\lambda} \cdot |\Delta_j n_{\lambda}|^{q-2} \Delta_j n_{\lambda} \, dx \]
\[ = - \int_{\mathbb{R}^3} \Delta_j (u \cdot \nabla n_{\lambda} + \nabla (n_{\lambda} \nabla c)) \cdot |\Delta_j n_{\lambda}|^{q-2} \Delta_j n_{\lambda} \, dx . \]

(58)
Therefore, substituting (59), (60) and (61) into (58), using an argument for the $L^q$ energy estimate \cite{11}, we arrive at

\[
- \int_{\mathbb{R}^3} \Delta \Delta_j n_\lambda \cdot |\Delta_j n_\lambda|^q - 2 \Delta_j n_\lambda dx \geq \tilde{c} 2^j \|\Delta_j n_\lambda\|_{L^q}^q. \tag{59}
\]

It is clear that, formally, we have the following homogeneous Bony paraproduct decomposition

\[
u \nabla n_\lambda = T_u \nabla n_\lambda + R(u, \nabla n_\lambda) + T_{\nabla^2 n_\lambda} u. \tag{60}
\]

Besides, because $\text{div} u = 0$, by using the standard commutator argument, we obtain

\[
\int_{\mathbb{R}^3} \Delta_j (T_u \nabla n_\lambda) |\Delta_j n_\lambda|^q - 2 \Delta_j n_\lambda dx
= \sum_{|j-j'| \leq 5} \int_{\mathbb{R}^3} |\Delta_j: S_{j'-1} u| \Delta_j' \nabla n_\lambda |\Delta_j n_\lambda|^q - 2 \Delta_j n_\lambda dx
+ \sum_{|j-j'| \leq 5} \int_{\mathbb{R}^3} (S_{j'-1} u - S_{j-1} u) \Delta_j \Delta_j' \nabla n_\lambda |\Delta_j n_\lambda|^q - 2 \Delta_j n_\lambda dx
- \frac{1}{q} \int_{\mathbb{R}^3} S_{j-1} (\text{div} u)|\Delta_j n_\lambda|^q dx.
\tag{61}
\]

Therefore, substituting (59), (60) and (61) into (58), using an argument for the $L^q$ energy estimate \cite{11}, we reach at

\[
\|\Delta_j n_\lambda\|_{L^\infty_t(L^q)} + \lambda \int_0^t f_1(\tau) \|\Delta_j n_\lambda\|_{L^q} d\tau + \tilde{c} 2^j \int_0^t \|\Delta_j n_\lambda\|_{L^q} d\tau
\leq \|\Delta_j n_0\|_{L^q} + C \sum_{|j-j'| \leq 5} \|\Delta_j: S_{j'-1} u| \Delta_j' \nabla n_\lambda\|_{L^1_t(L^q)}
+ C \sum_{|j-j'| \leq 5} \|(S_{j'-1} u - S_{j-1} u) \Delta_j \Delta_j' \nabla n_\lambda\|_{L^1_t(L^q)}
+ C \|\Delta_j (T\nabla n_\lambda u)\|_{L^1_t(L^q)} + C \|\Delta_j (R(u, \nabla n_\lambda))\|_{L^1_t(L^q)}
+ C \|\Delta_j \nabla \cdot (n_\lambda \nabla c)\|_{L^1_t(L^q)}.
\tag{62}
\]

Applying the classical estimate on commutator \cite{11} and Lemma 2.2 yield

\[
\sum_{|j-j'| \leq 5} \|\Delta_j: S_{j'-1} u| \Delta_j' \nabla n_\lambda\|_{L^1_t(L^q)} \lesssim \sum_{|j-j'| \leq 5} d_j 2^{j(2-3/q)} \|u^h\|_{\Sigma^1_1(B^{3+3p}_{p,1}(\mathbb{R}^3))} \|n_\lambda\|_{\Sigma^\infty_1(B^{2+3q}_{q,1}(\mathbb{R}^3))}
+ \sum_{|j-j'| \leq 5} \int_0^t \sum_{k \leq j'-2} 2^{k(1+3/p)} \|u^3(\tau)\|_{L^{3+3p}_{p,1}(\mathbb{R}^3)} \|\Delta_j n_\lambda(\tau)\|_{L^q} d\tau \tag{63}
\lesssim d_j 2^{j(2-3/q)} \|u^h\|_{\Sigma^1_1(B^{3+3p}_{p,1}(\mathbb{R}^3))} \|n_\lambda\|_{\Sigma^\infty_1(B^{2+3q}_{q,1}(\mathbb{R}^3))}
+ d_j 2^{j(2-3/q)} \|n_\lambda\|_{\Sigma^1_{1,1}(B^{2+3q}_{q,1}(\mathbb{R}^3))}.
\]
Now, using the same type of computations as that in (63), we get

\[
\sum_{|j-j'| \leq 5} \| (S_{j'-1} u - S_{j-1} u) \Delta_j \Delta_j' \nabla n_\lambda \|_{L^1_t(L^p)} \\
\leq \sum_{|j-j'| \leq 5} \| S_{j'-1} \nabla h n_\lambda \|_{L^\infty_t(L^{\infty})} \| \Delta_j' \nabla h \|_{L^1_t(L^{p})} + \sum_{|j-j'| \leq 5} \left( \int_0^t \sum_{k \leq j'-2} 2k(1+3/p) \| \Delta_k u^h \|_{L^1_t(L^{p})} \| \Delta_j' n_\lambda \|_{L^\infty_t(L^{p})} \right) \\
+ \sum_{|j-j'| \leq 5} \int_0^t \sum_{k \leq j'-2} 2k(1+3/p) \| \Delta_k u^3(\tau) \|_{L^p} \| \Delta_j' n_\lambda(\tau) \|_{L^\infty} d\tau \\
\leq d_j 2^{j(2-3/q)} \| u^h \|_{L^1_t(B^{1+3/p}_{p,1}(R^3))} \| n_\lambda \|_{C^r_t(B^{-2+3/q}_{q,1}(R^3))} + d_j 2^{j(2-3/q)} \| n_\lambda \|_{L^{1+1}_{t,j_1}(B^{-2+3/q}_{q,1}(R^3))}.
\]

When \( 1 < q \leq p < 6 \), there exists \( 1 < r \leq \infty \) such that \( 1/q = 1/p + 1/r \), applying Lemma 2.2 gives that

\[
\| \Delta_j (T_{n_\lambda} u) \|_{L^1_t(L^p)} \\
\leq \sum_{|j-j'| \leq 5} \| S_{j'-1} \nabla h n_\lambda \|_{L^\infty_t(L^{r})} \| \Delta_j' \nabla h \|_{L^1_t(L^{p})} + \sum_{|j-j'| \leq 5} \left( \int_0^t \sum_{k \leq j'-2} 2k(1+3/p) \| \Delta_k u^3(\tau) \|_{L^p} \| \Delta_j' n_\lambda(\tau) \|_{L^\infty} d\tau \right) \\
\leq d_j 2^{j(2-3/q)} \| u^h \|_{L^1_t(B^{1+3/p}_{p,1}(R^3))} \| n_\lambda \|_{C^r_t(B^{-2+3/q}_{q,1}(R^3))} + d_j 2^{j(2-3/q)} \| n_\lambda \|_{L^{1+1}_{t,j_1}(B^{-2+3/q}_{q,1}(R^3))}.
\]

On the other hand, when \( 1 < p < q < 6 \), applying Lemma 2.2, we obtain

\[
\| \Delta_j (T_{n_\lambda} u) \|_{L^1_t(L^p)} \\
\leq 2^{j(1/p-1/q)} \sum_{|j-j'| \leq 5} d_j 2^{j(2-3/q)} 2^{j(1+3/q)} \| n_\lambda \|_{C^r_t(B^{-2+3/q}_{q,1}(R^3))} \| \Delta_j' u^h \|_{L^1_t(L^p)} + 2^{j(1/p-1/q)} \sum_{|j-j'| \leq 5} d_j 2^{-j'(1+3/p)} \sum_{k \leq j'-2} 2^{3k} 2^{k(-2+3/q)} \\
\times \left( \int_0^t \| u^2 \|_{B^{1+3/p}_{p,1}(R^3)} \| \Delta_k n_\lambda \|_{L^\infty} d\tau \right) \\
\leq d_j 2^{j(2-3/q)} \| u^h \|_{L^1_t(B^{1+3/p}_{p,1}(R^3))} \| n_\lambda \|_{C^r_t(B^{-2+3/q}_{q,1}(R^3))} + d_j 2^{j(2-3/q)} \| n_\lambda \|_{L^{1+1}_{t,j_1}(B^{-2+3/q}_{q,1}(R^3))}.
\]

(66)
For \( \|\Delta_j R(u, \nabla n_\lambda)\|_{L^q_t(L^p)} \), we consider the following two cases. Since \( \text{div}_h u^h \neq 0 \) and \( 2/3 < 1/p + 1/q \leq 1 \), we thus get

\[
\|\Delta_j R(u, \nabla n_\lambda)\|_{L^q_t(L^p)} \\
\leq \|\Delta_j R(u^h, \nabla h n_\lambda)\|_{L^q_t(L^p)} + \|\Delta_j R(u^3, \partial_3 n_\lambda)\|_{L^q_t(L^p)} \\
\lesssim 2^{\delta j/p} \sum_{j' \geq j - N_0} \|\Delta_j' u^h\|_{L^q_t(L^p)} \|\Delta_j \nabla h n_\lambda\|_{L^q_t(L^p)} + 2^{\delta j/p} \sum_{j' \geq j - N_0} \int_0^t \|\Delta_j' u^3\|_{L^p} \|\Delta_j \partial_3 n_\lambda\|_{L^q} \, dt \\
\lesssim 2^{\delta j/p} \sum_{j' \geq j - N_0} 2^{j'(2-3/p-3/q)} d_j' \|u^h\|_{L^q_t(\dot B^{1+3/p}_p(\mathbb{R}^3))} \|n_\lambda\|_{L^q_t(\dot B^{-2+3/q}_q(\mathbb{R}^3))} \\
+ 2^{\delta j/p} \sum_{j' \geq j - N_0} 2^{-3j'/p} \int_0^t \|u^3\|_{L^q_t(\dot B^{1+3/p}_p(\mathbb{R}^3))} \|\Delta_j n_\lambda\|_{L^q} \, dt \\
\leq d_j 2^{j(2-3/q)} \|u^h\|_{L^q_t(\dot B^{1+3/p}_p(\mathbb{R}^3))} \|n_\lambda\|_{L^q_t(\dot B^{-2+3/q}_q(\mathbb{R}^3))} \\
+ d_j 2^{j(2-3/q)} \|n_\lambda\|_{L^q_t(\dot B^{1+3/p}_p(\mathbb{R}^3))}.
\]

(67)

In the case \( \frac{1}{p} + \frac{1}{q} > 1 \), we can thus choose \( 1 < q' \leq \infty \) fulfilling \( \frac{1}{q} + \frac{1}{q'} = 1 \), applying Lemma 2.2, for some fixed constant \( N_0 \), we have

\[
\|\Delta_j R(u, \nabla n)\|_{L^q_t(L^p)} \\
\lesssim 2^{\delta j(1-1/q)} \sum_{j' \geq j - N_0} d_j' 2^{-j'} \|u^h\|_{L^q_t(\dot B^{1+3/p}_p(\mathbb{R}^3))} \|n_\lambda\|_{L^q_t(\dot B^{-2+3/q}_q(\mathbb{R}^3))} \\
+ 2^{\delta j(1-1/q)} \sum_{j' \geq j - N_0} d_j' 2^{-j'} \|n_\lambda\|_{L^q_t(\dot B^{1+3/p}_p(\mathbb{R}^3))} \\
\leq d_j 2^{j(2-3/q)} \|u^h\|_{L^q_t(\dot B^{1+3/p}_p(\mathbb{R}^3))} \|n_\lambda\|_{L^q_t(\dot B^{-2+3/q}_q(\mathbb{R}^3))} \\
+ d_j 2^{j(2-3/q)} \|n_\lambda\|_{L^q_t(\dot B^{1+3/p}_p(\mathbb{R}^3))}.
\]

(68)

Plugging (63)-(68) into (62) and from Lemma 2.10, we conclude

\[
\|\Delta_j n_\lambda\|_{L^q_t(L^p)} + \lambda \int_0^t f_1(\tau) \|\Delta_j n_\lambda\|_{L^q} \, d\tau + c_2 2^{j} \int_0^t \|\Delta_j n_\lambda\|_{L^q} \, d\tau \\
\leq \|\Delta_j n_0\|_{L^q} \\
+ d_j 2^{j(2-3/q)} \|u^h\|_{L^q_t(\dot B^{1+3/p}_p(\mathbb{R}^3))} \|n_\lambda\|_{L^q_t(\dot B^{-2+3/q}_q(\mathbb{R}^3))} + \|n_\lambda\|_{L^q_t(\dot B^{1+3/p}_p(\mathbb{R}^3))} \\
+ \|n_\lambda\|_{L^q_t(\dot B^{-2+3/q}_q(\mathbb{R}^3))} \|c\|_{L^q_t(\dot B^{2+3/q}_q(\mathbb{R}^3))} \\
+ \|n_\lambda\|_{L^q_t(\dot B^{2+3/q}_q(\mathbb{R}^3))} \|c\|_{L^q_t(\dot B^{-2+3/q}_q(\mathbb{R}^3))},
\]

thus multiplying both sides by \( 2^{j(-2+3/q)} \) and then taking the \( \ell^1(\mathbb{Z}) \)-norm, we obtain Proposition 3.

\[\Box\]

- **The estimate of \( c \).**

Thanks to the second equation of (4), we have

\[
\partial_1 c_\lambda + \lambda f_1(t)c_\lambda - \Delta c_\lambda = -u \cdot \nabla c_\lambda - cn_\lambda.
\]
Proposition 4. Let $1 < p, q < 6, \frac{1}{p} + \frac{1}{q} > \frac{2}{3}$ and $\frac{1}{\min\{p,q\}} - \frac{1}{\max\{p,q\}} \leq \frac{1}{3}$. If $(u, n, c) \in \Theta_T$ with $\nabla \cdot u = 0$, then (69) satisfies

$$\|\Delta_j c_\lambda\|_{L^p_t(L^q_x)} + \lambda \int_0^t \|\Delta_j c_\lambda\|_{L^q_x} d\tau + c_2 \int_0^t \|\Delta_j c_\lambda\|_{L^q_x} d\tau$$

$$\leq \|\Delta_j c_\lambda\|_{L^p_t(L^q_x)} + C \sum_{|j-j'| \leq 5} \|\Delta_j; S_{j'}-1 u\| \Delta_j; \nabla c_\lambda\|_{L^1_t(L^q_x)} + C \sum_{|j-j'| \leq 5} \|\Delta_j; (T \nabla c_\lambda)\|_{L^1_t(L^q_x)} + C \|\Delta_j (c_\lambda n)\|_{L^1_t(L^q_x)}.$$  

(70)

Proof. Applying the operator $\Delta_j$ to (69), taking $L^2$ inner product with $|\Delta_j c_\lambda|^{q-2}$ $\Delta_j c_\lambda$ (in the case when $q \in (1, 2)$, we need to make some modifications as that [11]), a similar proof of (62) leads to

$$\|\Delta_j c_\lambda\|_{L^p_t(L^q_x)} + \lambda \int_0^t f_1(\tau) \|\Delta_j c_\lambda\|_{L^q_x} d\tau + c_2 \int_0^t \|\Delta_j c_\lambda\|_{L^q_x} d\tau$$

$$\leq \|\Delta_j c_\lambda\|_{L^p_t(L^q_x)} + C \sum_{|j-j'| \leq 5} \|\Delta_j; S_{j'}-1 u\| \Delta_j; \nabla c_\lambda\|_{L^1_t(L^q_x)} + C \|\Delta_j; (T \nabla c_\lambda)\|_{L^1_t(L^q_x)} + C \|\Delta_j (c_\lambda n)\|_{L^1_t(L^q_x)}.$$  

(71)

As the same estimate (63), applying the classical estimate on commutator [10] and Lemma 15, we get

$$\sum_{|j-j'| \leq 5} \|\Delta_j; S_{j'}-1 u\| \Delta_j; \nabla c_\lambda\|_{L^1_t(L^q_x)}$$

$$\leq \sum_{|j-j'| \leq 5} d_j 2^j(3/q) \|u_h\|_{L^1_t(L^q_x)} \{\lambda \|c_\lambda\|_{L^p_t(L^q_x)} \} + C \|\Delta_j; \nabla c_\lambda\|_{L^1_t(L^q_x)} + C \|\Delta_j; (T \nabla c_\lambda)\|_{L^1_t(L^q_x)} + C \|\Delta_j; (c_\lambda n)\|_{L^1_t(L^q_x)}.$$  

(72)

Along the same line to the proof of (64), the same process also ensures

$$\sum_{|j-j'| \leq 5} \|\Delta_j; S_{j'}-1 u\| \Delta_j; \nabla c_\lambda\|_{L^1_t(L^q_x)}$$

$$\leq \sum_{|j-j'| \leq 5} d_j 2^j(3/q) \|u_h\|_{L^1_t(L^q_x)} \|c_\lambda\|_{L^p_t(L^q_x)} + C \|\Delta_j; \nabla c_\lambda\|_{L^1_t(L^q_x)} + C \|\Delta_j; (T \nabla c_\lambda)\|_{L^1_t(L^q_x)} + C \|\Delta_j; (c_\lambda n)\|_{L^1_t(L^q_x)}.$$  

(73)
On the other hand, since $1 < q \leq p < 6, 1/q - 1/p < 1/3$, there exists $1 < r \leq \infty$ such that $1/q = 1/p + 1/r$, applying Lemma 2.2 gives

$$
\|\Delta_j (T\nabla c_\lambda u)\|_{L^1_t(L^r)} \\
\leq \sum_{|j-j'| \leq 5} \sum_{k \leq j'-2} d_k 2^{k(1+3/p-3/q)} \|c_\lambda\|_{L^p_t(B^{3/q}(\mathbb{R}^3))} \|\Delta_j u^h\|_{L^1_t(L^r)} \\
+ 2^{-j(1+3/p)} \sum_{|j-j'| \leq 5} \sum_{k \leq j'-2} 2^{k(1+3/p-3/q)} d_k \|c_\lambda\|_{L^1_t(L^1(B^{3/q}(\mathbb{R}^3)))} \\
\leq d_j 2^{j(-3/q)} \|u^h\|_{L^1_t(B^{1+3/q}(\mathbb{R}^3))} \|c_\lambda\|_{L^p_t(B^{3/q}(\mathbb{R}^3))} + d_j 2^{j(-3/q)} \|c_\lambda\|_{L^1_t(L^1(B^{3/q}(\mathbb{R}^3)))}.
$$

(74)

If $1 < p < q < 6$, then we have

$$
\|\Delta_j (T\nabla c_\lambda u)\|_{L^1_t(L^r)} \\
\leq 2^{3j/(1-p-1/q)} \sum_{|j-j'| \leq 5} d_j 2^{j(-3/p)} 2^{j(1+3/p)} \|c_\lambda\|_{L^p_t(B^{3/q}(\mathbb{R}^3))} \|\Delta_j u^h\|_{L^1_t(L^p)} \\
+ 2^{3j/(1-p-1/q)} \sum_{|j-j'| \leq 5} d_j 2^{j(1+3/p)} \sum_{k \leq j'-2} 2^{k(1+3/q)} \int_0^t \|u^3\|_{L^1_t(B^{1+3/q}(\mathbb{R}^3))} \|\Delta_k c_\lambda\|_{L^q} d\tau
\leq d_j 2^{j(-3/q)} \|u^h\|_{L^1_t(B^{1+3/q}(\mathbb{R}^3))} \|c_\lambda\|_{L^p_t(B^{3/q}(\mathbb{R}^3))} + d_j 2^{j(-3/q)} \|c_\lambda\|_{L^1_t(L^1(B^{3/q}(\mathbb{R}^3)))}.
$$

(75)

To estimate the remaining term $R(u, \nabla c_\lambda)$, we consider the following two cases. In the case that $2/3 < 1/p + 1/q \leq 1$, note that $d_\lambda v^h \neq 0$, we get

$$
\|\Delta_j R(u, \nabla c_\lambda)\|_{L^1_t(L^r)} \\
\leq 2^{3j/p} \sum_{j' \geq j-N_0} \|\Delta_j u^h\|_{L^1_t(L^r)} \|\Delta_j \nabla c_\lambda\|_{L^r_t(L^1)} \\
+ 2^{3j/p} \sum_{j' \geq j-N_0} \int_0^t \|\Delta_j u^h\|_{L^1_t(L^r)} \|\Delta_j \partial_3 c_\lambda\|_{L^1_t(L^q)} d\tau
\leq 2^{3j/p} \sum_{j' \geq j-N_0} 2^{j(-3/p-3/q)} \|u^h\|_{L^1_t(B^{1+3/q}(\mathbb{R}^3))} \|c_\lambda\|_{L^p_t(B^{3/q}(\mathbb{R}^3))} \\
+ 2^{3j/p} \sum_{j' \geq j-N_0} d_j 2^{j(-3/p-3/q)} \|c_\lambda\|_{L^1_t(L^1(B^{3/q}(\mathbb{R}^3)))} \\
\leq d_j 2^{j(-3/q)} \|u^h\|_{L^1_t(B^{1+3/q}(\mathbb{R}^3))} \|c_\lambda\|_{L^p_t(B^{3/q}(\mathbb{R}^3))} + d_j 2^{j(-3/q)} \|c_\lambda\|_{L^1_t(L^1(B^{3/q}(\mathbb{R}^3)))}.
$$

(76)

In the case $1/p + 1/q > 1$, we can find $1 < q' \leq \infty$ such that $1/q + 1/q' = 1$, applying Lemma 2.2, for some fixed constant $N_0$, we get

$$
\|\Delta_j R(u, \nabla c_\lambda)\|_{L^1_t(L^r)} \\
\leq 2^{3j/(1-q')} \sum_{j' \geq j-N_0} \|\Delta_j \nabla c_\lambda\|_{L^q_t(L^1)} \|\Delta_j u^h\|_{L^1_t(L^q)} \\
+ 2^{3j/(1-q')} \sum_{j' \geq j-N_0} \int_0^t \|\Delta_j \partial_3 c_\lambda\|_{L^q} \|\Delta_j u^3\|_{L^{q'}_t(L^q)} d\tau
$$
Multiplying both sides of above inequality with 2-norm, we obtain the desired estimates (70). This proves the Proposition 4.

By inserting (72)-(77) into (71) and applying Lemma 2.10, we conclude

\[
\|\Delta_j c\lambda\|_{L^\infty(L^q)} + \lambda \int_0^t f_1(\tau) \|\Delta_j c\lambda\|_{L^q} d\tau + \bar{c} 2^{2j} \int_0^t \|\Delta_j c\lambda\|_{L^q} d\tau
\leq \|\Delta_j c_0\|_{L^q} + d_j 2^{2(-3/4)} \left( \|u^h\|_{L^\infty(B_{2+3/q}^1(\mathbb{R}^3))} \|c\lambda\|_{L^\infty(B_{2+3/q}^3(\mathbb{R}^3))} + \|n\lambda\|_{L^\infty(B_{2+3/q}^3(\mathbb{R}^3))} \|c\|_{L^\infty(B_{2+3/q}^3(\mathbb{R}^3))} \right).
\]

By inserting (72)-(77) into (71) and applying Lemma 2.10, we conclude

\[
\|\Delta_j c\lambda\|_{L^\infty(L^q)} + \lambda \int_0^t f_1(\tau) \|\Delta_j c\lambda\|_{L^q} d\tau + \bar{c} 2^{2j} \int_0^t \|\Delta_j c\lambda\|_{L^q} d\tau
\leq \|\Delta_j c_0\|_{L^q} + d_j 2^{2(-3/4)} \left( \|u^h\|_{L^\infty(B_{2+3/q}^1(\mathbb{R}^3))} \|c\lambda\|_{L^\infty(B_{2+3/q}^3(\mathbb{R}^3))} + \|n\lambda\|_{L^\infty(B_{2+3/q}^3(\mathbb{R}^3))} \|c\|_{L^\infty(B_{2+3/q}^3(\mathbb{R}^3))} \right).
\]

Multiplying both sides of above inequality with $2^{3j/4}$ and then taking the $\ell^1(\mathbb{Z})$-norm, we obtain the desired estimates (70). This proves the Proposition 4.

Now we show Theorem 1.2.

Proof. Under the conditions from Theorem 1.2, proceeding in the same way as [7, 35] with minor modifications, there exists a positive time $T^*$, such that the system (4) has a unique local solution $(u, n, c) \in \Theta_{T^*} \cap \Theta_T^c$, where $T^*$ is a maximal time of existence, for the more details, we can refer to Appendix A. Hence, to prove Theorem 1.2, we only need to prove that $T^* = \infty$ with $(u, n, c) \in \Theta \cap \Theta^c$ provided that it holds (12).

Let $c_2$ be a small enough positive constant, which will be determined later on, let

\[
\mathcal{T} := \left\{ t \in [0, T^*) : \|u^h\|_{L^\infty(B_{2+3/q}^1(\mathbb{R}^3))} + \|u^h\|_{L^\infty(\mathbb{R}^3(\mathbb{R}^3))} \right\}
\]

Applying Propositions 3 and 4, by taking $\lambda = \lambda_1$ in (57) and (70), we obtain

\[
\|(n_\lambda, c_\lambda)\|_{L^\infty(B_{2+3/q}^1(\mathbb{R}^3))} \times L^\infty(B_{2+3/q}^3(\mathbb{R}^3)) \\
+ \lambda_1 \|(n_{\lambda_1}, c_{\lambda_1})\|_{L^\infty(\mathbb{R}^3(\mathbb{R}^3))} \times L^\infty(\mathbb{R}^3(\mathbb{R}^3)) + \bar{c} \|(n_{\lambda_1}, c_{\lambda_1})\|_{L^\infty(\mathbb{R}^3(\mathbb{R}^3))} \times L^\infty(\mathbb{R}^3(\mathbb{R}^3)) \leq c_2
\]

Applying Propositions 3 and 4, by taking $\lambda = \lambda_1$ in (57) and (70), we obtain

\[
\|(n_\lambda, c_\lambda)\|_{L^\infty(B_{2+3/q}^1(\mathbb{R}^3))} \times L^\infty(B_{2+3/q}^3(\mathbb{R}^3)) \\
+ \lambda_1 \|(n_{\lambda_1}, c_{\lambda_1})\|_{L^\infty(\mathbb{R}^3(\mathbb{R}^3))} \times L^\infty(\mathbb{R}^3(\mathbb{R}^3)) + \bar{c} \|(n_{\lambda_1}, c_{\lambda_1})\|_{L^\infty(\mathbb{R}^3(\mathbb{R}^3))} \times L^\infty(\mathbb{R}^3(\mathbb{R}^3)) \leq c_2
\]
Combining (80) and (81), taking that which, together with (79) and the fact that 
\[ \| \lambda \|_{t/2}^1 \geq \phi \| B_{q,r}^{3/8}(\mathbb{R}^3) \]\[ \times \left( \| u^h \|_{L^\infty(B_{p,r}^{1+3/p}(\mathbb{R}^3))} + \| (n, c) \|_{L^\infty_1(B_{q,r}^{3/8}(\mathbb{R}^3)) \times L^\infty_1(B_{q,r}^{2+3/q}(\mathbb{R}^3))} \right) \]
\[ + \| (n_{\lambda_1}, c) \|_{L^\infty_1(B_{q,r}^{3/8}(\mathbb{R}^3)) \times L^\infty_1(B_{q,r}^{2+3/q}(\mathbb{R}^3))} \| (n, c) \|_{L^\infty_1(B_{q,r}^{2+3/q}(\mathbb{R}^3)) \times L^\infty_1(B_{q,r}^{3/q}(\mathbb{R}^3))} \right). \]

From (79), we have
\[ C_1 \| (n, c) \|_{L^\infty_1(B_{p,r}^{1+3/p}(\mathbb{R}^3)) \times L^\infty_1(B_{q,r}^{3/8}(\mathbb{R}^3))} \leq C_2, \]
\[ C_1 \left( \| u^h \|_{L^\infty_1(B_{p,r}^{1+3/p}(\mathbb{R}^3))} + \| (n, c) \|_{L^\infty_1(B_{q,r}^{3/8}(\mathbb{R}^3)) \times L^\infty_1(B_{q,r}^{2+3/q}(\mathbb{R}^3))} \right) \leq C_1 c_2. \]

Combining (80) and (81), taking \( \lambda_1 \geq 2C_1 \) and \( c_2 \leq \tilde{c} \) := min\{\( \frac{\varepsilon}{\lambda_1}, \frac{1}{\lambda_1} \}\), it follows that
\[ \| (n_{\lambda_1}, c) \|_{L^\infty_1(B_{q,r}^{2+3/q}(\mathbb{R}^3)) \times L^\infty_1(B_{q,r}^{3/8}(\mathbb{R}^3))} + \bar{c} \| (n_{\lambda_1}, c) \|_{L^\infty_1(B_{q,r}^{3/8}(\mathbb{R}^3)) \times L^\infty_1(B_{q,r}^{2+3/q}(\mathbb{R}^3))} \]
\[ \leq \| (n_{\lambda_1}, c) \|_{L^\infty_1(B_{q,r}^{2+3/q}(\mathbb{R}^3)) \times L^\infty_1(B_{q,r}^{3/8}(\mathbb{R}^3))} \]
\[ + \lambda_1 \| (n_{\lambda_1}, c) \|_{L^\infty_1(B_{q,r}^{2+3/q}(\mathbb{R}^3)) \times L^\infty_1(B_{q,r}^{3/8}(\mathbb{R}^3))} \]
\[ + \bar{c} \| (n_{\lambda_1}, c) \|_{L^\infty_1(B_{q,r}^{3/8}(\mathbb{R}^3)) \times L^\infty_1(B_{q,r}^{2+3/q}(\mathbb{R}^3))} \leq 2 \| (n_0, c_0) \|_{B_{q,r}^{2+3/q}(\mathbb{R}^3) \times B_{q,r}^{3/q}(\mathbb{R}^3)}, \]
\[ (82) \]

which, together with (79) and the fact that \( \| n_{\lambda} \|_{L^\infty_1(B_{q,r}^{3/8}(\mathbb{R}^3))} \leq \| n_{\lambda_1} \|_{L^\infty_1(B_{q,r}^{3/8}(\mathbb{R}^3))} \) (see (34) and (35)), shows that
\[ C \| n_{\lambda} \|_{L^\infty_1(B_{q,r}^{3/8}(\mathbb{R}^3))} \leq \| n_{\lambda_1} \|_{L^\infty_1(B_{q,r}^{3/8}(\mathbb{R}^3))} \| \phi \|_{B_{q,r}^{3/p}(\mathbb{R}^3)} \]
\[ \leq C \| \phi \|_{B_{q,r}^{3/p}(\mathbb{R}^3)} \| n_0 \|_{B_{q,r}^{2+3/q}(\mathbb{R}^3)} + C \| \phi \|_{B_{q,r}^{3/p}(\mathbb{R}^3)} \]
\[ \times \left( \| n_{\lambda_1} \|_{L^\infty_1(B_{q,r}^{2+3/q}(\mathbb{R}^3))} \| u^h \|_{L^\infty_1(B_{p,r}^{1+3/p}(\mathbb{R}^3))} \]
\[ + \| n_{\lambda_1} \|_{L^\infty_1(B_{q,r}^{2+3/q}(\mathbb{R}^3))} + \| n_{\lambda_1} \|_{L^\infty_1(B_{q,r}^{2+3/q}(\mathbb{R}^3))} \| c \|_{L^\infty_1(B_{q,r}^{2+3/q})} \]
\[ + \| n_{\lambda_1} \|_{L^\infty_1(B_{q,r}^{2+3/q}(\mathbb{R}^3))} \| c \|_{L^\infty_1(B_{q,r}^{2+3/q}(\mathbb{R}^3))} \right) \]
\[ \leq C \| \phi \|_{B_{q,r}^{3/p}(\mathbb{R}^3)} \left( 1 + 2c_2 + 2 \frac{1}{\lambda_1} + 2 \frac{1}{\bar{c}} \right) \| (n_0, c_0) \|_{B_{q,r}^{2+3/q}(\mathbb{R}^3) \times B_{q,r}^{3/q}(\mathbb{R}^3)} \]
\[ \leq C_2 \| (n_0, c_0) \|_{B_{q,r}^{2+3/q}(\mathbb{R}^3) \times B_{q,r}^{3/q}(\mathbb{R}^3)}, \quad C \| u^h \|_{L^\infty_1(B_{p,r}^{1+3/p}(\mathbb{R}^3))} \leq C_2. \]

Taking \( \lambda_1 \geq 2C, \lambda_2 \geq 2CD \) and \( c_2 \leq \frac{\varepsilon}{4C} \), thanks to (83) and (50) from Proposition 2, we have
\[ \| u^h \|_{L^\infty_1(B_{p,r}^{1+3/p}(\mathbb{R}^3))} + \bar{c} \| u^h \|_{L^\infty_1(B_{p,r}^{1+3/p}(\mathbb{R}^3))} \]
\[ \leq 2 \| u^h \|_{L^\infty_1(B_{p,r}^{1+3/p}(\mathbb{R}^3))} + \bar{c} \| u^h \|_{L^\infty_1(B_{p,r}^{1+3/p}(\mathbb{R}^3))} \]
\[ \leq 2 \| u^h \|_{L^\infty_1(B_{p,r}^{1+3/p}(\mathbb{R}^3))} + \lambda_1 \| u^h \|_{L^\infty_1(B_{p,r}^{1+3/p}(\mathbb{R}^3))} \]
\[ + \lambda_2 \| u^h \|_{L^\infty_1(B_{p,r}^{1+3/p}(\mathbb{R}^3))} + \bar{c} \| u^h \|_{L^\infty_1(B_{p,r}^{1+3/p}(\mathbb{R}^3))} \]
\[ \leq 4C \| u^h \|_{L^\infty_1(B_{p,r}^{1+3/p}(\mathbb{R}^3))} \]
\[ \leq 4C_2 \| u^h \|_{L^\infty_1(B_{p,r}^{1+3/p}(\mathbb{R}^3))}. \]
we have
\[ C \| \phi \|_{L^2_t(B_{q,1}^{s/2}(\mathbb{R}^3))} \leq C_{\lambda_1, \lambda_2} \int_0^t (\phi_1(\tau) + \tilde{\phi}_2(\tau)) \, d\tau \]
\[ \leq \tilde{c} \| u_h \|_{L^2_t(B_{q,1}^{s/2}(\mathbb{R}^3))} + \tilde{c} \| u_h \|_{L^2_t(L^1(B_{p,1}^{s-1/3/q}(\mathbb{R}^3)) \times L^\infty_{x,t}(B_{q,1}^{s/3/q}(\mathbb{R}^3)))} 
+ \| (n, c) \|_{L^\infty_{x,t}(B_{q,1}^{s-2/3/q}(\mathbb{R}^3))) \times L^\infty_{x,t}(B_{q,1}^{s/3/q}(\mathbb{R}^3)))} \exp \left\{ - \int_0^t (\lambda_1 f_1(\tau) + \lambda_2 f_2(\tau)) \, d\tau \right\} \]
\[ \leq \tilde{c} \| u_h \|_{L^2_t(B_{q,1}^{s-1/3/q}(\mathbb{R}^3)))} + \| (n, c) \|_{L^\infty_{x,t}(B_{q,1}^{s-2/3/q}(\mathbb{R}^3))) \times L^\infty_{x,t}(B_{q,1}^{s/3/q}(\mathbb{R}^3)))} \exp \left\{ - \int_0^t (\lambda_1 f_1(\tau) + \lambda_2 f_2(\tau)) \, d\tau \right\} 
+ \| u_h \|_{L^2_t(B_{q,1}^{s-1/3/q}(\mathbb{R}^3)))} \exp \left\{ - \int_0^t (\lambda_1 f_1(\tau) + \lambda_2 f_2(\tau)) \, d\tau \right\} .
\]
We thus complete the proof of Theorem 1.2.

and from (87), we also obtain

\[ \| u_t^h \|_{L^\infty(\mathbb{R}^+; \dot{B}^{-1+3/3}_{p,1}(\mathbb{R}^3))} + \| (n, c) \|_{L^\infty(\mathbb{R}^+; \dot{B}^{-2+3/3}_{q,1}(\mathbb{R}^3)) \times L^\infty(\mathbb{R}^+; \dot{B}^{3/3}_{q,1}(\mathbb{R}^3))} \leq C\ell \]

(91)

and from (87), we also obtain

\[ \| u_0^3 \|_{L^\infty(\mathbb{R}^+; \dot{B}^{-1+3/3}_{p,1}(\mathbb{R}^3))} \leq 2 \| u_0^3 \|_{\dot{B}^{-1+3/3}_{p,1}(\mathbb{R}^3)} + 2C_4c_2. \]

(92)

We thus complete the proof of Theorem 1.2.

\[ \square \]

4. Appendix A. In this appendix, we will give the proof of the assertion (local existence) of first part of Theorem 1.2.

Lemma 4.1. [10] Let \((\chi, \| \cdot \|)\) be a Banach space, \(B : \chi \times \chi \to \chi\) a bilinear operator with norm \(K\) and \(L : \chi \to \chi\) a continuous operator with norm \(M < 1\). Let \(y \in \chi\) satisfy \(4K\| y \|_\chi < (1 - M)^2\). Then the equation \(u = y + L(u) + B(u, u)\) has a unique solution in the ball \(\mathcal{B}(0, \frac{1}{4M})\).

Proposition 5. Let \(u_L = e^{t\Delta}u_0, n_L = e^{t\Delta}n_0\) and \(c_L = e^{t\Delta}c_0\). \((u, n, c)\) is a mild solution of system (4) on \([0, T] \times \mathbb{R}^3\) with initial data \((u_0, n_0, c_0)\) if and only if
\[(u, n, c) = (u_L + \bar{u}, n_L + \bar{n}, c_L + \bar{c}) \text{ with}\]
\[
\begin{align*}
\bar{u} &= -\int_0^t e^{(t-s)\Delta} P(u_L \cdot \nabla u_L + n_L \nabla \phi)(\cdot, s) \, ds \\
- \int_0^t e^{(t-s)\Delta} P(u_L \cdot \nabla \bar{u} + \bar{u} \cdot \nabla u_L + \bar{n} \nabla \phi)(\cdot, s) \, ds \\
- \int_0^t e^{(t-s)\Delta} P(\bar{u} \cdot \nabla \bar{u})(\cdot, s) \, ds = y_1 + L_1((\bar{u}, \bar{n}, \bar{c}), (\bar{u}, \bar{n}, \bar{c})), \\
\bar{n} &= -\int_0^t e^{(t-s)\Delta}(u_L \cdot \nabla n_L + \nabla \cdot (n_L \cdot \nabla c_L))(\cdot, s) \, ds \\
- \int_0^t e^{(t-s)\Delta}(u_L \cdot \nabla \bar{n} + \bar{u} \cdot \nabla n_L + \nabla \cdot (n_L \cdot \nabla \bar{c}) + \nabla \cdot (\bar{n} \cdot \nabla c_L))(\cdot, s) \, ds \\
+ \nabla \cdot (\bar{n} \cdot \nabla c_L)(\cdot, s) - \int_0^t e^{(t-s)\Delta}(\bar{u} \cdot \nabla \bar{n} + \nabla \cdot (\bar{n} \cdot \nabla \bar{c}))(\cdot, s) \, ds = y_2 + L_2((\bar{u}, \bar{n}, \bar{c}), (\bar{u}, \bar{n}, \bar{c})), \\
\bar{c} &= -\int_0^t e^{(t-s)\Delta}(u_L \cdot \nabla c_L + c_L n_L)(\cdot, s) \, ds \\
- \int_0^t e^{(t-s)\Delta}(u_L \cdot \nabla \bar{c} + \bar{u} \cdot \nabla c_L + c_L \bar{n} + \bar{c} n_L)(\cdot, s) \, ds \\
- \int_0^t e^{(t-s)\Delta}(\bar{u} \cdot \nabla \bar{c} + \bar{c} \bar{n})(\cdot, s) \, ds = y_3 + L_3((\bar{u}, \bar{n}, \bar{c}), (\bar{u}, \bar{n}, \bar{c})), \\
(\bar{u}, \bar{n}, \bar{c}) \big|_{t=0} &= (0, 0, 0).
\end{align*}
\]

\[\text{(93)}\]

**Proof.** The proof is easy, so we skip it here.

In the following, we only prove there exists a positive time \(T\), so that \(\text{(93)}\) has a unique solution \((\bar{u}, \bar{n}, \bar{c}) \in \Theta_T\). In order to prove the local existence, applying Lemma 2.3, we note that
\[
\|u_L\|_{L^\infty([0,T];B^{-1+3/p}_{p,1})} \leq C\|u_0\|_{B^{-1+3/p}_{p,1}}, \quad \|n_L\|_{L^\infty([0,T];B^{-2+3/q}_{q,1})} \leq C\|n_0\|_{B^{-2+3/q}_{q,1}}, \\
\|c_L\|_{L^\infty([0,T];B^{3/q}_{q,1})} \leq C\|c_0\|_{B^{3/q}_{q,1}},
\]
and
\[
\lim_{T \to 0} \|u_L\|_{L^1([0,T];B^{3/q}_{q,1})} = 0, \quad \lim_{T \to 0} \|n_L\|_{L^1([0,T];B^{1+3/p}_{p,1})} = 0, \quad \lim_{T \to 0} \|c_L\|_{L^1([0,T];B^{2+3/q}_{q,1})} = 0,
\]
thus, from above we can define
\[
\begin{align*}
\overline{T}_1 &= \sup \left\{T_1 > 0 : \|u_L\|_{L^1([0,T_1];B^{1+3/p}_{p,1})} \leq \eta \right\}, \\
\overline{T}_2 &= \sup \left\{T_1 > 0 : \|n_L\|_{L^1([0,T_1];B^{3/q}_{q,1})} \leq \eta \right\}, \\
\overline{T}_3 &= \sup \left\{T_1 > 0 : \|c_L\|_{L^1([0,T_1];B^{2+3/q}_{q,1})} \leq \eta \right\}.
\end{align*}
\]

We can choose \(\overline{T} = \min\{\overline{T}_1, \overline{T}_2, \overline{T}_3\}\) and we can take \(T \leq \overline{T}\) and \(\eta\) sufficiently small. Let
\[
F_T = L^1([0,T];B^{-1+3/p}_{p,1}) \times L^1([0,T];B^{-2+3/q}_{q,1}) \times L^1([0,T];B^{3/q}_{q,1}).
\]
Therefore thanks to Lemmas 2.3-2.11, there exists a constant $K > 0$ such

$$\|B\|_{\Theta_T} = \|(B_1, B_2, B_3)((\bar{u}, \bar{n}, \bar{c}), (\bar{u}, \bar{n}, \bar{c}))\|_{\Theta_T}$$

\[\lesssim \|(\bar{u} \cdot \nabla \bar{n}, \bar{u} \cdot \nabla \bar{n} + \nabla (\bar{n} \cdot \nabla \bar{c}), \bar{u} \cdot \nabla \bar{c} + \bar{c} n)\|_{F_T}
\]

\[\leq K \|(\bar{u}, \bar{n}, \bar{c})\|_{\Theta_T}^2.
\]

On the other hand, applying (6) and Lemmas 2.3-2.11, there exists a positive constant $C$ such that

$$\|L\|_{\Theta_T} \leq C \|(\bar{u}, \bar{n}, \bar{c})\|_{\Theta_T}(\eta^{1/r_1} + \eta^{1/r_2})(\|n_0\|_{B_{q,1}^{2}}^{1-\frac{1}{r_2}} + \|u_0\|_{B_{q,1}^{2}}^{1-\frac{1}{r_2}} + \|u_0\|_{B_{p,1}^{2}}^{1-\frac{1}{r_2}} + \|n_0\|_{B_{q,1}^{2}}^{1-\frac{1}{r_2}} + \|\bar{c}_0\|_{B_{q,1}^{2}}^{1-\frac{1}{r_2}} + \|\bar{c}_0\|_{B_{q,1}^{2}}^{1-\frac{1}{r_2}} + \|(\bar{u}, \bar{n}, \bar{c})\|_{\Theta_T}(T^{1+3/p} + T^{1+3/p})
$$

where we can choose $M_1 < 1$, if $\eta$ small enough.

In what follows, thus by taking advantage of Lemma 2.5, using the same type of computations as in (98), we discover that

$$\|y\|_{\Theta_T} = \|(y_1, y_2, y_3)\|_{\Theta_T}$$

\[\leq \|(u_L \cdot \nabla u_L + n_L \nabla \phi, u_L \cdot \nabla n_L + \nabla (n_L \cdot \nabla c_L), u_L \cdot \nabla c_L + c_L n_L)\|_{F_T}
\]

\[\leq C\eta \|(u_0, n_0, c_0)\|_{E_0}.
\]

We can take $\eta$ and $T$ small enough such that $C\eta \|(u_0, n_0, c_0)\|_{E_0} \leq \frac{(1-M)^2}{4K}$ for $i = 1$ or $i = 2$, applying Lemma 4.1, we thus get that there exists a positive time $T > 0$ such that the system (93) has a unique solution $(\bar{u}, \bar{n}, \bar{c})$ on $[0, T]$. Therefore, applying Proposition 5 yields that the local existence of solution to the system (4). Note that as we have obtain $(u, n, c) \in \Theta_T$ is a solution of (4), then we can proceed in the same way as in the proof of Lemma 2.6-Lemma 2.11, to obtain that

$$u \cdot \nabla u, \ n \nabla \phi \in \mathcal{L}^1(0, T; \mathcal{B}_{p,1}^{2+3/p}), \ u \cdot \nabla n, \ \nabla \cdot (n \nabla c) \in \mathcal{L}^1(0, T; \mathcal{B}_{q,1}^{2+3/q}),$$

$$u \cdot \nabla c, cn \in \mathcal{L}^1(0, T; \mathcal{B}_{q,1}^{3/q}).$$

Therefore, it follows from Lemma 2.5 that $(u, n, c) \in \mathcal{E}^T_{\mathcal{C}}$.

In the following, we will show the solution can be extended from $I = [0, T]$ to $I_1 = [T, T_1]$ for some $T_1 > T$, here we consider

$$u(t) = e^{(t-T)\Delta} u(T) - \int_T^t e^{(t-s)\Delta} P(u \cdot \nabla u + n \nabla \phi)(\cdot, s)ds,$$

$$n(t) = e^{(t-T)\Delta} n(T) - \int_T^t e^{(t-s)\Delta} (u \cdot \nabla n + \nabla \cdot (n \nabla c))(\cdot, s)ds,$$

$$c(t) = e^{(t-T)\Delta} c(T) - \int_T^t e^{(t-s)\Delta} (u \cdot \nabla c + cn)(\cdot, s)ds,$$
According to the same line with the proof of the local existence (about $T$) above, it follows the Banach’s fixed point Lemma 4.1 that we have the local solution on $[T, T_1]$. We can also extend this solution step by step and finally find a maximal $T_{\text{max}} > 0$.

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