Lower Bound on the Chromatic Number by Spectra of Weighted Adjacency Matrices

Paweł Wocjan,* Dominik Janzing, Thomas Beth

Institut für Algorithmen und Kognitive Systeme, Universität Karlsruhe,
Am Fasanengarten 5, D-76128 Karlsruhe, Germany

Abstract

A lower bound on the chromatic number of a graph is derived by majorization of spectra of weighted adjacency matrices. These matrices are given by Hadamard products of the adjacency matrix and arbitrary Hermitian matrices.

1 Introduction

Let $G = (V, E)$ be a graph on $n$ vertices $V = \{0, \ldots, n-1\}$. An admissible vertex coloring of $G$ is an assignment of colors to its vertices such that no two adjacent vertices receive the same color. The minimum number of colors needed for such a coloring is called the chromatic number of $G$, and is denoted $\chi$. Determining the chromatic number of a graph is known to be NP-hard. Besides its theoretical significance as a canonical NP-hard problem, graph coloring arises naturally in a variety of applications such as e.g. register allocation and scheduling. Therefore it is important to have bounds on the chromatic number.

Bounds are often computed with the help of the adjacency matrix. Let $A = (a_{ij})$ denote the adjacency matrix of $G$, i.e.

$$a_{kl} = \begin{cases} 1 & \text{if vertices } k \text{ and } l \text{ are adjacent} \\ 0 & \text{otherwise} \end{cases}$$

and $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ denote the eigenvalues of $A$. The upper bound $\lambda_1 + 1 \geq \chi$ has been shown in [8]. The lower bound

$$\chi \geq \frac{\lambda_1}{|\lambda_n|} + 1$$

(1)

has been shown in [4]. The following Theorem proved in [1] gives a generalization of the bound (1).

*e-mail: \{wocjan, janzing, eiss_office\}@ira.uka.de
Theorem 1  Let $A$ denote the adjacency matrix for a connected graph $G$ on $n$ vertices, and let $D = \text{diag}(d_1, \ldots, d_n)$ be a diagonal matrix such that $A + D$ is positive semidefinite. Then each $d_i$ is positive, and the largest eigenvalue of the matrix $D^{-\frac{1}{2}}AD^{-\frac{1}{2}} + I$ is a lower bound for $\chi$.

Note that the bound (1) can be obtained from Theorem 1 by taking $D = |\lambda_n|I$.

2  Lower bounds for $\chi$

We derive a method to obtain lower bounds on the chromatic number. This method contains the above results on the lower bound as special cases.

First we define the problem to reverse the sign of a matrix and the cost of doing this. We consider weighted adjacency matrices, i.e. special matrices constructed from the adjacency matrix of a graph $G$, and show that $\chi - 1$ is an upper bound on the cost. Then a lower bound is derived on the cost in terms of the spectra of the weighted adjacency matrices. This yields a lower bound on the chromatic number.

Definition 1 (Sign reversal)  Let $M \in \mathbb{C}^{n \times n}$ be any traceless matrix. A sign reversal map $\mathcal{I} := (r_1, U_1; r_2, U_2; \ldots; r_N, U_N)$ for $M$ is characterized by $N$ positive real numbers $r_j$ and unitary matrices $U_j \in \mathbb{C}^{n \times n}$ such that

$$
\sum_{j=1}^{N} r_j U_j^{-1} MU_j = -M.  
$$

We call $c(\mathcal{I}) := \sum_{j=1}^{N} r_j$ the cost of the sign reversal map $\mathcal{I}$.

Using basic results of representation theory we can show that this is always possible. Let $G$ be a group acting irreducibly on $\mathbb{C}^{n}$ via the unitary representation $g \mapsto U_g$. Let $M \in \mathbb{C}^{n \times n}$ matrix. By Schur’s Lemma \[6\] we have

$$
\frac{1}{|G|} \sum_{g \in G} U_g^{-1} MU_g = \frac{\text{tr}(M)}{n} 1,  
$$

where $\text{tr}(M)$ denotes the trace of $M$ and $1$ is the identity matrix of size $N$.

In the following the matrix $M$ is always traceless. Then by conjugating $M$ with all elements of $G$ but the identity we obtain

$$
\sum_{G \setminus \{1\}} U_g MU_g = -M
$$
since $M$ is traceless.

We consider matrices constructed from the adjacency matrix of a graph $G$ and derive a lower bound on the cost of sign reversal in terms of the chromatic number of $G$. To do this we need the definition of the Hadamard product. If $A = (a_{ij}) \in \mathbb{C}^{n \times n}$ and $B = (b_{ij}) \in \mathbb{C}^{n \times n}$ are given, then the Hadamard product of $A$ and $B$ is the matrix $A \ast B = (a_{ij}b_{ij}) \in \mathbb{C}^{n \times n}$ \[5\].
Lemma 1 Let $A = (a_{kl})$ be the adjacency matrix of a graph $G = (V, E)$ with $n$ vertices and chromatic number $\chi$. For an arbitrary matrix $W = (w_{kl}) \in \mathbb{C}^{n \times n}$ the sign of the Hadamard product $M := W \ast A$ can be reversed with cost $\chi - 1$.

Proof: Choose a partition (disjoint union) of the vertices $V = V_0 \cup V_1 \cup \ldots \cup V_{\chi - 1}$ corresponding to a (minimal) coloring. The set $V_c$ contains all vertices of color $c$ ($c \in \{0, \ldots, \chi - 1 \}$). We assume that the vertices are ordered according to their colors, i.e. first come the vertices of color 0, then of color 1, etc.

Let $\omega \in \mathbb{C}$ be a primitive $\chi$-th root of unity, i.e. $\omega^\chi = 1$. Define a diagonal matrix $D := \text{diag}(d_0, d_1, \ldots, d_{n - 1})$ where $d_i := \omega^c$ if $i \in V_c$. We show that

$$
\bar{M} := \sum_{j=0}^{\chi - 1} D^{-j} M D^j
$$

is the zero matrix. Let $M = (m_{kl})_{k,l=0,\ldots,n-1}$ and $\bar{M} = (\bar{m}_{kl})_{k,l=0,\ldots,n-1}$. The entries of $\bar{M}$ are given by

$$
\bar{m}_{kl} = \sum_{j=0}^{\chi - 1} \omega^{-kj} m_{kl} \omega^{lj}.
$$

Let $k, l \in V_c$ with $k \neq l$. The vertices $k$ and $l$ have the the same color and consequently they cannot be adjacent, i.e. $a_{kl} = 0$. Therefore we have $\bar{m}_{kl}$ since $m_{kl} = w_{kl} a_{kl} = 0$. Note that $\bar{m}_{kk} = 0$ since $a_{kk} = 0$ (the diagonal entries of the adjacency matrix are all zero).

Now let $k \in V_c$ and $l \in V_{\bar{c}}$ with $c \neq \bar{c}$. We have

$$
\bar{m}_{kl} = \sum_{j=0}^{\chi - 1} \omega^{-kj} m_{kl} \omega^{lj} = 0
$$

since the vectors $(\omega^c, \omega, \ldots, \omega^{(\chi - 1)})$ and $(\omega^{\bar{c}}, \omega^{\bar{c}-1}, \ldots, \omega^{(\chi - 1)})$ are orthogonal. Note that they are rows of the matrix of the discrete Fourier transform of size $\chi$. \qed

To derive a lower bound on the cost of a sign reversal map we introduce the concept that is known as spectral majorization of matrices [2]. Let $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$ be two $n$-dimensional real vectors. We introduce the notation $\downarrow$ to denote the components of a vector rearranged into non-increasing order, so $x^\downarrow = (x_1^\downarrow, \ldots, x_n^\downarrow)$, where $x_1^\downarrow \geq x_2^\downarrow \geq \ldots \geq x_n^\downarrow$. We say that $x$ is majorized by $y$ and write $x < y$, if

$$
\sum_{i=1}^{m} x_i^\downarrow \leq \sum_{i=1}^{m} y_i^\downarrow,
$$

for $m = 1, \ldots, n - 1$ and $\sum_{i=1}^{n} x_i^\downarrow = \sum_{i=1}^{n} y_i^\downarrow$.

Let $\text{Spec}(X)$ denote the spectrum of the Hermitian matrix $X$, i.e. the vector of eigenvalues. Recall that the eigenvalues of a Hermitian matrix are real. Ky Fan’s maximum principle [2] gives rise to a useful constraint on the eigenvalues of a sum of Hermitian matrices $A_j$ for $j = 1, \ldots, N$:

$$
\text{Spec} \left( \sum_{j=1}^{N} A_j \right) \prec \sum_{j=1}^{N} \text{Spec}(A_j).
$$

(4)
The following lemma gives a lower bound on the cost of sign reversal in terms of the spectrum of $M$.

**Lemma 2** A lower bound on the negation cost of a Hermitian matrix $M$ is given by the smallest positive real number $\tau$ such that

$$\text{Spec}(-M) \prec \tau \text{Spec}(M).$$

**(Proof)** Let $\mathcal{I} := (r_1, U_1; r_2, U_2; \ldots r_N, U_N)$ be an arbitrary sign reversal map for $M$, i.e.

$$\sum_{j=1}^{N} r_j U_j^{-1} M U_j = -M. \quad (6)$$

Applying the inequality (5) to eq. (6) yields

$$\text{Spec}(-M) = \text{Spec}\left(\sum_{j=1}^{N} r_j U_j^{-1} M U_j\right) \prec \sum_{j=1}^{N} r_j \text{Spec}(U_j^{-1} M U_j) = c(\mathcal{I}) \text{Spec}(M).$$

Therefore for all sign reversal maps we have $c(\mathcal{I}) \geq \tau$. \hfill \Box

Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the eigenvalues of $M$ with $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$. Then the lower bound $\tau$ is given by

$$\tau := \max_{m=1,\ldots,n-1} \left\{ \sum_{i=1}^{m} \frac{\lambda_i}{\sum_{j=1}^{m+1} \lambda_{n+1-i}} \right\} \quad (7)$$

since for the eigenvalues $\mu_i$ of $-M$ we have $\mu_i = \lambda_{n+1-i} (i = 1 \ldots, n)$.

By using the results on the lower and upper bounds on the cost of sign reversal we obtain:

**Theorem 2 (Lower bound on the chromatic number)** Let $A$ be the adjacency matrix of a graph $G$ with $n$ vertices and $\chi$ its chromatic number. For any Hermitian matrix $W \in \mathbb{C}^{n \times n}$ let $\tau_W$ denote the minimal positive real number such that

$$\lambda(-W * A) \prec \tau_W \lambda(W * A). \quad (8)$$

Let $\tau$ be the maximal $\tau_W$, where the maximum is taken over all Hermitian matrices $W \in \mathbb{C}^{n \times n}$. Then we have

$$\chi \geq \tau + 1. \quad (9)$$

**(Proof)** Lemma 1 shows that the sign of all matrices of the form $W * A$ can be reversed with cost $\chi - 1$. Lemma 2 gives $\tau_W$ in eq. (8) as a lower bound on the sign reversal cost of $W * A$. Consequently, we have $\chi \geq \tau_W + 1$ for all Hermitian matrices. We choose the maximal $\tau_W$ to obtain the best bound on $\chi$. \hfill \Box

Note that we obtain as a special case the well-known lower bound $\chi \geq \frac{\lambda_1}{|\lambda_n|} + 1$. Set $W$ to be the matrix all of whose entries are equal to 1. Then we have $A = W * A$. Instead of taking the maximum over $m = 1, \ldots, n - 1$ consider only $m := 1$ in eq. (5). This corresponds to the maximal and minimal eigenvalues of $A$. 

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We now show that Theorem 1 can also be understood as a special case of Theorem 2. Let \( D := \text{diag}(d_0, \ldots, d_{n-1}) \) be the diagonal matrix defined as in Theorem 1. Take \( W \) to be the matrix with entries \( w_{kl} := \sqrt{d_k} \sqrt{d_l} \). Then we have \( W \star A = D^{-\frac{1}{2}} A D^{-\frac{1}{2}} \). This shows that this modified adjacency matrix can also be expressed with the Hadamard product.

As a conclusion Theorem 2 permits to consider a larger class of modified adjacency matrices and to take into account all eigenvalues (Theorem 1 considers only the maximal and minimal eigenvalues). The advantage of the method presented in \([1]\) is that the matrix \( D \) is the solution of a semidefinite programming problem that can be computed by an algorithm given there. It remains to be shown whether the larger class of modified adjacency matrices permits to obtain better lower bounds and whether there are efficient algorithms to compute them. Nevertheless, the derivation in our approach is easy and gives a generalization of Theorem 2.

We would like to point out that the sign reversal map described here is an abstraction of a problem in quantum physics. A physical system evolving due to the Schrödinger equation

\[
\frac{d}{dt} |\psi(t)\rangle = -iH |\psi(t)\rangle
\]

(where \( H \) is the so-called “Hamilton operator”) can be made evolve backwards in time by interspersing the natural time evolution with external control operations \([2, 3]\). The unitary maps \( U_j \) in Definition 1 are abstractions of control operations. The vertices of the graph represent particles and the edges indicate whether there is an interaction between the two particles or not. The physical meaning of the inversion cost is the time overhead for simulating the inverse evolution, i.e., the factor by which the reverse dynamics is slower than the original one. The cost is therefore the complexity of reversing the dynamical evolution and the chromatic number is an upper bound on this time complexity.

Acknowledgments

This work has been supported by grands of the DFG Schwerpunktprogramm Komplexität und Energie.

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