Behaviour of the Stokes operators under domain perturbation

Sylvie Monniaux

Abstract

Depending of the geometry of the domain, one can define –at least– three different Stokes operators with Dirichlet boundary conditions. We describe how the resolvents of these Stokes operators converge with respect to a converging sequence of domains.

1 Introduction

Let \( \Omega \) denote an open connected subset of \( \mathbb{R}^d \). We do not impose any regularity of the boundary \( \partial \Omega \) of the domain \( \Omega \) and possibly \( \Omega \) is unbounded. To avoid too many cases, we will however assume that the \( d \)-dimensional Hausdorff measure of \( \partial \Omega \) is zero.

We denote by \( D = C_\infty^\infty(\Omega, \mathbb{R}^d) \) the space of smooth vector fields with compact support in \( \Omega \). Let \( D' \) denote its dual, the space of (vector valued) distributions on \( \Omega \).

Acknowledgements

The author acknowledges the partial support by the ANR project INFAMIE ANR-15-CE40-001. The understanding of this subject has benefited from discussions with A.F.M. ter Elst. The author would also like to thank the anonymous referee whose remarks greatly improved this manuscript.

2 Setting

2.1 The Leray orthogonal decomposition of \( L^2 \)

We start with a very important and profound result due to de Rham [8, Chapter IV §22, Theorem 17’]; see also [11, Chapter I §1.4, Proposition 1.1].

**Theorem 2.1** (de Rham). Let \( T \in D' \) be a distribution. Then the following two properties are equivalent.

(i) \( D' \langle T, \varphi \rangle_D = T(\varphi) = 0 \) for all \( \varphi \in D \) with \( \text{div} \varphi = 0 \).

(ii) There exists a scalar distribution \( S \in C_\infty^\infty(\Omega)' \) such that \( T = \nabla S \) in \( D' \).

De Rham’s theorem has the following corollary.

**Corollary 2.2.** Let \( T \in H^{-1}(\Omega, \mathbb{R}^d) \). Then the following are equivalent.

*2010 Mathematics Subject Classification* Primary: 35J15, Secondary: 35Q30

†Keywords and phrases: domain perturbation, Stokes operator, Dirichlet boundary conditions

‡Aix Marseille Université, CNRS, Centrale Marseille, I2M UMR 7373, 13453, Marseille, France - sylvie.monniaux@univ-amu.fr
(i) $H^{-1}(\Omega, \mathbb{R}^d) \langle T, \varphi \rangle_{H^{1}_{0}(\Omega, \mathbb{R}^d)} = 0$ for all $\varphi \in \mathcal{D}$ with $\text{div} \varphi = 0$.

(ii) There exists a scalar distribution $\pi \in L^2_{\text{loc}}(\Omega)$ such that $T = \nabla \pi$ in $\mathcal{D}'$.

Proof. We only have to show (i) $\Rightarrow$ (ii). By Theorem 2.1 there exists $S \in \mathcal{C}_c^\infty(\Omega)'$ such that $T = \nabla S$. Then $\nabla S \in H^{-1}(\Omega, \mathbb{R}^d)$. Consequently, $S \in L^2_{\text{loc}}(\Omega)$ by [11, Proposition 1.2] (for a direct proof, see also [9, Lemma 2.2.1]).

Denote by $H = L^2(\Omega, \mathbb{R}^d)$ the square integrable vector fields on $\Omega$. We endow the vector-valued space $H$ with the scalar product

$$
\langle u, v \rangle_H := \int_{\Omega} u \cdot v = \sum_{j=1}^d \int_{\Omega} u_j v_j, \quad u, v \in H.
$$

Then $H$ is a Hilbert space. We define the subspace $\mathcal{G}$ of $H$ consisting of gradients by

$$
\mathcal{G} := \{ \nabla \pi; \pi \in L^2_{\text{loc}}(\Omega), \nabla \pi \in H \}. \quad (2.1)
$$

As a consequence of Corollary 2.2, $\mathcal{G}$ is a closed subspace of $H$. We denote by $H$ the orthogonal subspace of $\mathcal{G}$ in $H$, that is

$$
H = \mathcal{G} \quad (2.2)
$$

Obviously, $H$ is a Hilbert space and one has the orthogonal decomposition

$$
H = \mathcal{H} \oplus \mathcal{G}. \quad (2.3)
$$

The orthogonal projection from $H$ to $\mathcal{H}$ denoted by $P$ is called the Leray projection. It is the adjoint of the canonical embedding $J : \mathcal{H} \hookrightarrow H$; it verifies $PJu = u$ for all $u \in \mathcal{H}$. Next, define the subspace

$$
\mathcal{D} := \{ u \in \mathcal{D}; \text{div} u = 0 \text{ in } \Omega \}. \quad (2.4)
$$

Then $\mathcal{D} \subset \mathcal{H}$ and by De Rham’s theorem, $\mathcal{D} = \mathcal{G}$, so that $\mathcal{D}$ is dense in $\mathcal{H}$ with respect to the $L^2$-norm of $\mathcal{H}$.

The canonical embedding $J_0 : \mathcal{D} \hookrightarrow \mathcal{D}$ is the restriction of $J$ to $\mathcal{D}$. Its adjoint $J_0^* = P_1 : \mathcal{D}' \rightarrow \mathcal{D}'$ is therefore an extension of the Leray projection $P$. A reformulation of de Rham’s theorem (Thm 2.1) is

$$
\ker P_1 = \{ T \in \mathcal{D}'; P_1 T = 0 \} = \{ \nabla S; S \in \mathcal{C}_c^\infty(\Omega)' \}. \quad (2.5)
$$

2.2 Another orthogonal decomposition of $L^2$

Since we made the assumption that the $d$-dimensional Hausdorff measure of $\partial \Omega$ is zero, we can identify $H = L^2(\Omega, \mathbb{R}^d)$ with $\{ U_{\mid \Omega}; U \in L^2(\mathbb{R}^d, \mathbb{R}^d), U = 0 \text{ a.e. in } \Omega \}$ and define the space $\mathcal{E}$ to be the closure in $L^2(\Omega, \mathbb{R}^d)$ of

$$
\mathcal{W} := \{ U_{\mid \Omega}; U \in H^1(\mathbb{R}^d, \mathbb{R}^d), U = 0 \text{ a.e. in } \Omega \text{ and } \text{div} U = 0 \text{ in } \mathbb{R}^d \}. \quad (2.5)
$$

The space $\mathcal{E}$ is closed in $H$ by definition and contains $\mathcal{D}$, and then $\mathcal{H}$. The following decomposition of $\mathcal{H}$ holds

$$
\mathcal{H} = \mathcal{E} \oplus \mathcal{F}, \quad (2.6)
$$

2
where \( \mathcal{F} = \mathcal{E}^\perp \). Since \( \mathcal{H} \subset \mathcal{E} \), it is obvious that \( \mathcal{F} \subset \mathcal{I} \). It is also obvious that \( \{ \nabla q|_\Omega; q \in \dot{H}^1(\mathbb{R}^d) \} \subset \mathcal{F} \); let \( u = U|_\Omega \in \mathcal{W} \) and \( q \in \dot{H}^1(\mathbb{R}^d) \); then

\[
(u, \nabla q)_\mathcal{H} = (U, \nabla q)_{L^2(\mathbb{R}^d, \mathbb{R}^d)} = 0.
\]

For further use, we will denote by \( L : \mathcal{E} \hookrightarrow \mathcal{H} \) the canonical embedding; its adjoint \( L^* = Q : \mathcal{H} \hookrightarrow \mathcal{E} \) is the orthogonal projection from \( \mathcal{H} \) to \( \mathcal{E} \). The operators \( L \) and \( Q \) verify \( QL = u = u \) for all \( u \in \mathcal{E} \), as do \( J \) and \( \mathcal{P} \) in the above setting.

**Remark 2.3.** When \( \Omega \subset \mathbb{R}^d \) is bounded and smooth enough, say with Lipschitz boundary, the spaces \( \mathcal{H} \) and \( \mathcal{E} \) coincide: they are equal to

\[
L^2_{\sigma}(\Omega) := \{ u \in L^2(\Omega, \mathbb{R}^d); \text{div} u = 0 \text{ in } \Omega \text{ and } \nu \cdot u = 0 \text{ on } \partial \Omega \},
\]

where \( \text{div} u \) is to be taken in the sense of distributions and \( \nu(x) \) denotes the exterior normal unit vector at \( x \in \partial \Omega \), defined for almost every \( x \) in the case of a Lipschitz boundary \( \partial \Omega \). Here, \( \nu \cdot u \in \dot{H}^{-1/2}(\partial \Omega) \) is defined via the integration by parts formula

\[
\dot{H}^{-1/2}(\nu \cdot u, \varphi)_{\dot{H}^{1/2}} = \int_\Omega u \cdot \nabla \phi + \int_\Omega \text{div} u \cdot \phi
\]

for all \( \varphi \in \dot{H}^{1/2}(\partial \Omega) \) and \( \Phi \in \dot{H}^1(\Omega) \) satisfying \( \text{Tr}_{|_\partial \Omega} \Phi = \varphi \).

The fact that \( \mathcal{H} = L^2_{\sigma}(\Omega) \) in the case of a bounded domain with Lipschitz boundary was proved in [11, Thm 1.4]. If \( \Omega \subset \mathbb{R}^d \) has a continuous boundary as in [2, Prop. 2.2] (see also [10], \( \mathcal{W} = \{ u \in H^1_0(\Omega, \mathbb{R}^d); \text{div} u = 0 \} \). According to [11, Thm 1.6], this latter space is the closure of \( \mathcal{D} \) in \( H^1(\Omega, \mathbb{R}^d) \) if the boundary of \( \Omega \) is Lipschitz, so that \( \mathcal{E} = L^2_{\sigma}(\Omega) = \mathcal{H} \).
3.1 Weak- and pseudo-Dirichlet Laplacians

We now briefly describe how to define the Laplacian with homogeneous Dirichlet boundary conditions in a weak sense: depending on how the boundary conditions are modelled, different operators appear. Recall that since we do not impose any regularity on the boundary of our domain Ω, it does not make sense to talk about traces. We start by defining the bilinear form $a : \mathcal{W} \times \mathcal{W} \to \mathbb{R}$ by

$$a(u, v) := \langle \nabla u, \nabla v \rangle_{\mathcal{H}} = \sum_{j=1}^{d} \langle \partial_j u, \partial_j v \rangle_{\mathcal{H}}.$$  \hspace{1cm} (3.1)

The forms $a$ and $a|_{\mathcal{V} \times \mathcal{V}}$ are associated with analytic semigroups of contractions on $\mathcal{H}$ (see, e.g., [5, §VI.2]). Let $-\Delta_D^n$ be the operator associated with the form $a|_{\mathcal{V} \times \mathcal{V}}$ and let $-\Delta_D^{\overline{n}}$ be the operator associated with the form $a$. Following [2], we call $\Delta_D^n$ the (weak-)Dirichlet Laplacian and $\Delta_D^{\overline{n}}$ the pseudo-Dirichlet Laplacian. They are self-adjoint (unbounded) operators in $\mathcal{H}$. The interest of considering the weak-Dirichlet and the pseudo-Dirichlet Laplacians lies in particular in domain perturbation problems.

Let $\Omega, \Omega_1, \Omega_2, \ldots$ be bounded open subsets of $\mathbb{R}^d$. We say that $\Omega_n \uparrow \Omega$ as $n \to \infty$ if $\Omega_n \subset \Omega_{n+1}$ for all $n \in \mathbb{N}$ and for each compact subset $K \subset \Omega$ there exists an $n \in \mathbb{N}$ with $K \subset \Omega_n$. We say that $\Omega_n \downarrow \Omega$ as $n \to \infty$ if $\Omega_n \supset \Omega_{n+1}$ for all $n \in \mathbb{N}$ and $\lim_{n \to \infty} (\Omega_n \cap B) \setminus \overline{\Omega} = 0$ for every ball $B$, where $| \cdot |$ denotes the Lebesgue measure in $\mathbb{R}^d$.

If $f \in \mathcal{H}$, then we denote by $\tilde{f} \in L^2(\mathbb{R}^d, \mathbb{R}^d)$ the extension by 0 of $f$ to $\mathbb{R}^d$.

The following results have been established in [2, §3]. See also [1, §6] and [3, §6 and §7].

**Proposition 3.1.** Let $\Omega, \Omega_1, \Omega_2, \ldots$ be bounded open subsets of $\mathbb{R}^d$.

a. Suppose that $\Omega_n \uparrow \Omega$ as $n \to \infty$. Then

$$\lim_{n \to \infty} \left( (I + (-\Delta_D^n))^{-1} (f|_{\Omega_n}) \right)_{\mathcal{H}} = (I + (-\Delta_D^n))^{-1} f \quad \text{and}$$

$$\lim_{n \to \infty} \left( (I + (-\Delta_D^{\overline{n}}))^{-1} (f|_{\Omega_n}) \right)_{\mathcal{H}} = (I + (-\Delta_D^{\overline{n}}))^{-1} f$$

in $\mathcal{H}$ for all $f \in \mathcal{H}$.

b. Suppose that $\Omega_n \downarrow \Omega$ as $n \to \infty$. Then

$$\lim_{n \to \infty} \left( (I + (-\Delta_D^n))^{-1} (\tilde{f}|_{\Omega_n}) \right)_{\mathcal{H}} = (I + (-\Delta_D^n))^{-1} f \quad \text{and}$$

$$\lim_{n \to \infty} \left( (I + (-\Delta_D^{\overline{n}}))^{-1} (\tilde{f}|_{\Omega_n}) \right)_{\mathcal{H}} = (I + (-\Delta_D^{\overline{n}}))^{-1} f$$

in $\mathcal{H}$ for all $f \in \mathcal{H}$.

**Remark 3.2.** Strictly speaking, Proposition 3.1 has been proved in [2] for scalar valued functions $f \in L^2(\Omega, \mathbb{R})$, and only the first part of (2) (2 Proposition 3.2) and the second part of (2) (2 Proposition 3.5) can be found in that reference. Using [2, Proposition 2.3] establishing monotonicity properties of the resolvents of the weak-Dirichlet Laplacian and the pseudo-Dirichlet Laplacian with respect to the inclusion of domains, the other two limits are immediate.
3.2 The weak-Dirichlet Stokes operator

Since the spaces \( V \) and \( X \) are dense subspaces of the Hilbert space \( H \), one can define two Dirichlet types of Stokes operators in \( H \). Recall the form \( a : W \times W \to \mathbb{R} \) from (3.1)

\[
a(u, v) := \langle \nabla u, \nabla v \rangle_H = \sum_{j=1}^{d} \langle \partial_j u, \partial_j v \rangle_H.
\]

Then \( a|_{X \times X} \) is a positive symmetric densely defined closed form in \( H \). Let \( \mathcal{B} \) be the operator associated with \( a|_{X \times X} \). Then \( \mathcal{B} \) is self-adjoint and \( \mathcal{B} \) is the Stokes operator considered in [7].

Since \( V \subset X \) we can also define \( \mathcal{B}_0 \) to be the self-adjoint operator in \( H \) associated with the form \( a|_{V \times V} \). We call \( \mathcal{B}_0 \) the weak-Dirichlet Stokes operator. This Stokes operator is the one which was considered by H. Sohr in [9, Chapter 3, §2.1].

The operators \( \mathcal{B} \) and \( \mathcal{B}_0 \) are both negative generators of analytic semigroups in \( H \).

Each of the cases above models differently spaces of divergence free vector fields with zero boundary conditions. As already mentioned before, they coincide in the case of bounded Lipschitz domains and consequently then also the two operators \( \mathcal{B} \) and \( \mathcal{B}_0 \) coincide.

The relation between the weak-Dirichlet Laplacian and the weak-Dirichlet Stokes operator is described in the following commutative diagram:

![Diagram]

where \( J_0 \) is the restriction of \( J \) to \( \mathcal{Y} \) and \( \mathcal{P}_1 \), its adjoint operator, is the extension of the Leray projection \( \mathcal{P} \) to \( \mathcal{Y}' \). What this says in particular is that \( \mathcal{B}_0 = \mathcal{P}_1(-\Delta_H^\Omega)J_0 \).

3.3 The pseudo-Dirichlet Stokes operator

If we now restrict the form \( a \) to \( W \times W \) we obtain a positive symmetric densely defined closed form in \( \mathcal{E} \). We then define \( \mathcal{A} \) to be the self-adjoint operator in \( \mathcal{E} \) associated with \( a|_{W \times W} \). We call \( \mathcal{A} \) the pseudo-Dirichlet Stokes operator. It is the negative generator of an analytic semigroup in \( \mathcal{E} \).

As said before, in the case of a bounded domain \( \Omega \) with Lipschitz boundary, the spaces \( \mathcal{X} \), \( \mathcal{Y} \) and \( \mathcal{W} \) coincide as well as the spaces \( \mathcal{H} \) and \( \mathcal{E} \), then so do the operators \( \mathcal{B} \), \( \mathcal{B}_0 \) and \( \mathcal{A} \).

The relation between the pseudo-Dirichlet Laplacian and the pseudo-Dirichlet Stokes operator is described in the following commutative diagram:

![Diagram]
where $L_0$ is the restriction of $L$ to $\mathcal{W}$ and $Q_1$, its adjoint operator, is the extension of the projection $Q$ from $[2,2]$ to $\mathcal{W}$. What this says in particular is that $A = Q_1(-\Delta_{D}^R)L_0$.

4 Domain perturbation

Similar results as those stated in Proposition 3.1 hold for the different Dirichlet Stokes operators described above. Roughly speaking, resolvents of the different Stokes operators converge to the resolvent of the weak-Dirichlet Stokes operator in $\mathcal{H}$ in the case of an increasing sequence of open sets and to the resolvent of the pseudo-Dirichlet Stokes operator in $\mathcal{E}$ in the case of a decreasing sequence of open sets.

4.1 Increasing sequence of domains

**Theorem 4.1.** Let $\Omega, \Omega_1, \Omega_2, \ldots$ be bounded open subsets of $\mathbb{R}^d$. Suppose that $\Omega_n \uparrow \Omega$ as $n \to \infty$. For all $n \in \mathbb{N}$ denote by $\mathbb{P}_n$ the Leray projection from $L^2(\Omega_n, \mathbb{R}^d)$ onto $\mathcal{H}_n$ (the corresponding space of divergence-free vector fields as in (2.2)), $I_n$ the identity operator on $\mathcal{H}_n$, $\mathcal{V}_n$ the corresponding form domain and $\mathcal{B}_0^{(n)}$ the corresponding weak-Dirichlet Stokes operator. Then

$$
\lim_{n \to \infty} \left( (I_n + \mathcal{B}_0^{(n)})^{-1} \mathbb{P}_n(f|_{\Omega_n}) \right)_{|\Omega} = (I + \mathcal{B}_0)^{-1} f
$$

in $L^2(\Omega, \mathbb{R}^d)$ for all $f \in \mathcal{H}$.

**Proof.** For all $n \in \mathbb{N}$ define $u_n = (I_n + \mathcal{B}_0^{(n)})^{-1} \mathbb{P}_n(f|_{\Omega_n})$. Then $u_n \in \mathcal{V}_n$ and $\tilde{u}_{n|\Omega_1} \in \mathcal{V}$ and

$$
\int_{\Omega_n} \nabla u_n \cdot \nabla v + \int_{\Omega_n} u_n \cdot v = \int_{\Omega_n} (\mathbb{P}_n(f|_{\Omega_n})) \cdot v = \int_{\Omega_n} f \cdot v \tag{4.1}
$$

for all $v \in \mathcal{V}_n$. Choosing $v = u_n$ gives

$$
\int_{\Omega_n} |\nabla u_n|^2 + \int_{\Omega_n} |u_n|^2 = \int_{\Omega_n} \mathbb{P}_n(f|_{\Omega_n}) \cdot u_n \leq \|f\|_2 \left( \int_{\Omega_n} |u_n|^2 \right)^{1/2}.
$$

This implies that $(\tilde{u}_{n|\Omega_1})_{n \in \mathbb{N}}$ is a bounded sequence in $\mathcal{V}$. Passing to a subsequence if necessary, there exists a $u \in \mathcal{V}$ such that $\lim_{n \to \infty} \tilde{u}_{n|\Omega_1} = u$ weakly in $\mathcal{V}$. Let $v \in \mathcal{D}$. There exists an $N \in \mathbb{N}$ such that $\tilde{v}_{|\Omega_1} \in \mathcal{D}$ for all $n \geq N$. By definition of $u_n$ we then have for all $n \geq N$ that

$$
\textbf{a}(\tilde{u}_{n|\Omega_1}, v) + \langle \tilde{u}_{n|\Omega_1}, v \rangle_{L^2(\Omega, \mathbb{R}^d)} = \langle f, v \rangle_{L^2(\Omega, \mathbb{R}^d)}.
$$

Taking the limit as $n \to \infty$ we obtain that

$$
\textbf{a}(u, v) + \langle u, v \rangle_{L^2(\Omega, \mathbb{R}^d)} = \langle f, v \rangle_{L^2(\Omega, \mathbb{R}^d)}, \tag{4.2}
$$

This is true for all $v \in \mathcal{D}$. Then by continuity and density, (4.2) is valid for all $v \in \mathcal{V}$. This shows that $u \in \mathcal{D}(\mathcal{B}_0)$ and $u = (I + \mathcal{B}_0)^{-1} f$.

It remains to show that $\lim_{n \to \infty} \tilde{u}_{n|\Omega_1} = u$ strongly in $L^2(\Omega, \mathbb{R}^d)$. Since $\lim_{n \to \infty} \tilde{u}_{n|\Omega_1} = u$ weakly in $L^2(\Omega, \mathbb{R}^d)$,

$$
\liminf_{n \to \infty} \|\tilde{u}_{n|\Omega_1}\|_{L^2(\Omega, \mathbb{R}^d)} \geq \|u\|_{L^2(\Omega, \mathbb{R}^d)}.
$$
Comparing \( \limsup_{n \to \infty} \|u_n\|_2 \) and \( \liminf_{n \to \infty} \|u_n\|_2 \), it suffices to show that

\[
\limsup_{n \to \infty} \|\tilde{u}_n\|_{L^2(\Omega, \mathbb{R}^d)} \leq \|u\|_{L^2(\Omega, \mathbb{R}^d)}.
\]

Let \( n \in \mathbb{N} \). Choose \( v = u_n \) in (4.1). Then

\[
\int_{\Omega} |\tilde{u}_n|_\Omega^2 = \int_{\Omega} f \cdot \tilde{u}_n - \int_{\Omega} |\nabla \tilde{u}_n|_\Omega^2.
\]

Since \( \lim_{n \to \infty} \tilde{u}_n = u \) weakly in \( \mathcal{V} \), and hence in \( H^1_0(\Omega, \mathbb{R}^d) \), one deduces that

\[
\lim_{n \to \infty} \int_{\Omega} f \cdot \tilde{u}_n = \int_{\Omega} f \cdot u \quad \text{and} \quad \lim_{n \to \infty} \partial_k \tilde{u}_n = \partial_k u \text{ weakly in } L^2(\Omega, \mathbb{R}^d) \text{ for all } k \in \{1, \ldots, d\}.
\]

This implies \( \|\partial_k u\|_{L^2(\Omega, \mathbb{R}^d)} \leq \liminf_{n \to \infty} \|\partial_k \tilde{u}_n\|_{L^2(\Omega, \mathbb{R}^d)} \). Consequently,

\[
\limsup_{n \to \infty} \|\tilde{u}_n\|_{L^2(\Omega, \mathbb{R}^d)}^2 = \lim_{n \to \infty} \left( \int_{\Omega} f \cdot \tilde{u}_n \right) - \liminf_{n \to \infty} \|\nabla \tilde{u}_n\|_{L^2(\Omega, \mathbb{R}^d)}^2 \\
\leq (f, u)_{L^2(\Omega, \mathbb{R}^d)} - \|\nabla u\|_{L^2(\Omega, \mathbb{R}^d)}^2 = \|u\|_{L^2(\Omega, \mathbb{R}^d)}^2,
\]

where the last equality follows from (4.2). Then \( \lim_{n \to \infty} \tilde{u}_n = u \) strongly in \( L^2(\Omega, \mathbb{R}^d) \). One concludes by the fact that every sequence for which every subsequence possesses a convergent subsequence to a unique limit is convergent. \( \square \)

### 4.2 Decreasing sequence of domains

If \( f \in L^2(\Omega, \mathbb{R}^d) \), then we denote by \( \tilde{f} \in L^2(\mathbb{R}^d, \mathbb{R}^d) \) the extension by 0 of \( f \) to \( \mathbb{R}^d \).

**Theorem 4.2.** Let \( \Omega, \Omega_1, \Omega_2, \ldots \) be bounded open subsets of \( \mathbb{R}^d \). Suppose that the \( d \)-dimensional Hausdorff measure of \( \partial \Omega \) and \( \partial \Omega_\infty \) for all \( n \in \mathbb{N} \) is zero. Suppose that \( \Omega_\infty \downarrow \Omega \) as \( n \to \infty \). For all \( n \in \mathbb{N} \) denote by \( \mathcal{Q}_n \) the projection from \( L^2(\Omega_n, \mathbb{R}^d) \) onto \( \mathcal{E}_n \) (the corresponding space of divergence-free vector fields as in (2.6)), \( \mathcal{I}_n \) the identity operator on \( \mathcal{E}_n \), \( \mathcal{W}_n \) the corresponding form domain and \( \mathcal{A}_n \) the corresponding pseudo-Dirichlet Stokes operator. Then

\[
\lim_{n \to \infty} \left( (\mathcal{I}_n + \mathcal{A}_n)^{-1} \tilde{f}_{|\Omega_n} \right)_{|\Omega} = (\mathcal{I} + \mathcal{A})^{-1} f
\]

in \( L^2(\Omega, \mathbb{R}^d) \) for all \( f \in \mathcal{E} \).

**Proof.** First note that \( \tilde{f}_{|\Omega_n} \in \mathcal{E}_n \) for all \( f \in \mathcal{E} \) and \( \tilde{u}_{|\Omega_n} \in \mathcal{W}_n \) for all \( u \in \mathcal{W} \) and all \( n \in \mathbb{N} \). Fix \( f \in \mathcal{E} \). For all \( n \in \mathbb{N} \) define \( u_n := (\mathcal{I}_n + \mathcal{A}_n)^{-1} (\tilde{f}_{|\Omega_n}) \). Then \( u_n \in \mathcal{W}_n \) and

\[
\int_{\mathbb{R}^d} \nabla \tilde{u}_n \cdot \nabla \tilde{v} + \int_{\mathbb{R}^d} \tilde{u}_n \cdot \tilde{v} = \int_{\Omega_n} \tilde{f}_{|\Omega_n} \cdot \tilde{v} = \int_{\mathbb{R}^d} \tilde{f} \cdot \tilde{v}, \tag{4.3}
\]

for all \( v \in \mathcal{W}_n \). Choosing \( v = u_n \) gives

\[
\|\nabla \tilde{u}_n\|_{L^2(\mathbb{R}^d, \mathbb{R}^d)}^2 + \|\tilde{u}_n\|_{L^2(\mathbb{R}^d, \mathbb{R}^d)}^2 \leq \|\tilde{f}\|_{L^2(\mathbb{R}^d, \mathbb{R}^d)} \|\tilde{u}_n\|_{L^2(\mathbb{R}^d, \mathbb{R}^d)}.
\]

Hence \( (\tilde{u}_n)_{n \in \mathbb{N}} \) is a bounded sequence in \( H^1(\mathbb{R}^d, \mathbb{R}^d) \). Passing to a subsequence if necessary, there exists a \( U \in H^1(\mathbb{R}^d, \mathbb{R}^d) \) such that \( \lim_{n \to \infty} \tilde{u}_n = U \) weakly in \( H^1(\mathbb{R}^d, \mathbb{R}^d) \).
We next show that $U = 0$ a.e. on $\Omega$. Let $\Phi \in C_c^\infty(\mathbb{R}^d, \mathbb{R})$ and suppose that $\text{supp } \Phi \subset \overline{\Omega}$. If $n \in \mathbb{N}$, then

$$\left| \int_{\mathbb{R}^d} \tilde{u}_n \cdot \Phi \right| \leq \| \tilde{u}_n \|_{L^2(\mathbb{R}^d, \mathbb{R}^d)} \| \Phi \|_{L^2(\Omega_n, \mathbb{R}^d)} \leq \| \tilde{f} \|_{L^2(\mathbb{R}^d, \mathbb{R}^d)} \| \Phi \|_{L^\infty(\mathbb{R}^d, \mathbb{R}^d)} |\Omega_n \setminus \overline{\Omega}|^{1/2}.$$ 

Since $\lim_{n \to \infty} |\Omega_n \setminus \overline{\Omega}| = 0$ it follows that

$$\int_{\mathbb{R}^d} U \cdot \Phi = \lim_{n \to \infty} \int_{\mathbb{R}^d} \tilde{u}_n \cdot \Phi = 0.$$ 

So $U = 0$ a.e. on $\overline{\Omega}$. Set $u = U|_{\Omega}$. Then $u \in \mathcal{W}$.

To prove that $u \in \mathcal{W}$, it remains to prove that $\text{div } U = 0$ in $\mathbb{R}^d$. This is straightforward since for all $\nabla p \in L^2(\mathbb{R}^d, \mathbb{R}^d)$ and for all $n \in \mathbb{N}$,

$$\int_{\mathbb{R}^d} U \cdot \nabla p \leftarrow_{\infty+n} \int_{\mathbb{R}^d} \tilde{u}_n \cdot \nabla p = 0.$$ 

Now, taking the limit as $n$ goes to $\infty$ in (4.3) for $v \in \mathcal{W}$, we obtain that

$$\int_{\mathbb{R}^d} \nabla U \cdot \nabla \tilde{v} + \int_{\mathbb{R}^d} U \cdot \tilde{v} = a(u, v) + \langle u, v \rangle = \langle f, v \rangle = \int_{\mathbb{R}^d} \tilde{f} \cdot \tilde{v}. \quad (4.4)$$

Therefore, $u \in D(A)$. It remains to prove that $u_n|_{\Omega} \xrightarrow{n \to \infty} u$ strongly in $L^2(\Omega, \mathbb{R}^d)$. The proof is similar to the proof of Theorem 4.1 comparing $\liminf_{n \to \infty} \| u_n \|_2$ and $\limsup_{n \to \infty} \| u_n \|_2$. By weak convergence of $(u_n|_{\Omega})_{n \in \mathbb{N}}$ to $u$ in $L^2(\Omega, \mathbb{R}^d)$, the inequality $\liminf_{n \to \infty} \| u_n \|_2 \geq \| u \|_2$ holds. The proof of $\limsup_{n \to \infty} \| u_n \|_2 \leq \| u \|_2$, uses (4.3) with $v = u_n$ and the fact that $(\tilde{u}_n)_{n \in \mathbb{N}}$ converges weakly to $U$ in $L^2(\mathbb{R}^d, \mathbb{R}^d)$, so that

$$\limsup_{n \to \infty} \| u_n \|_2^2 \leq \limsup_{n \to \infty} \| u_n \|_2^2 = \lim_{n \to \infty} \int_{\mathbb{R}^d} \tilde{u}_n \cdot \tilde{f} - \liminf_{n \to \infty} \| \nabla \tilde{u}_n \|_2^2 \leq \int_{\mathbb{R}^d} U \cdot \tilde{f} - \| \nabla U \|_2^2 = \langle u, f \rangle - \| \nabla u \|_2^2 = \| u \|_2^2$$

by (4.4) with $v = u$. \hfill \Box

### 4.3 Comments

The reader may want to compare Theorem 4.2 and Theorem 4.1 with Proposition 3.1 and ask whether one can approximate the pseudo-Dirichlet Stokes operator in $\Omega$ with weak-Dirichlet Stokes operators in $\Omega_n$ where $\Omega_n \downarrow \Omega$ and the weak-Dirichlet Stokes operator in $\Omega$ with pseudo-Dirichlet Stokes operators in $\Omega_n$ where $\Omega_n \uparrow \Omega$. This is obviously true if the approximation domains $\Omega_n$ are smooth and bounded since in this case, the weak-Dirichlet Stokes operator and the pseudo-Dirichlet Stokes operator coincide. In the case of increasing or decreasing sequences of arbitrary domains, the strategy followed by [2] (comparison of resolvents with respect to the inclusion of domains as in Remark 3.2) doesn’t work: the Stokes problem is purely vector-valued and the spaces involved are not Banach lattices.
References

[1] Wolfgang Arendt, *Approximation of degenerate semigroups*, Taiwanese J. Math. 5 (2001), no. 2, 279–295.

[2] Wolfgang Arendt and Daniel Daners, *Varying domains: stability of the Dirichlet and the Poisson problem*, Discrete Contin. Dyn. Syst. 21 (2008), no. 1, 21–39.

[3] Daniel Daners, *Dirichlet problems on varying domains*, J. Differential Equations 188 (2003), no. 2, 591–624.

[4] John G. Heywood, *On uniqueness questions in the theory of viscous flow*, Acta Math. 136 (1976), no. 1-2, 61–102.

[5] Tosio Kato, *Perturbation theory for linear operators*, Classics in Mathematics, Springer-Verlag, Berlin, 1995, Reprint of the 1980 edition.

[6] Olga A. Ladyženskaja and Vsevolod A. Solonnikov, *Some problems of vector analysis, and generalized formulations of boundary value problems for the Navier-Stokes equation*, Zap. Naučn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI) 59 (1976), 81–116, 256, Boundary value problems of mathematical physics and related questions in the theory of functions, 9.

[7] Sylvie Monniaux, *Navier-Stokes equations in arbitrary domains: the Fujita-Kato scheme*, Math. Res. Lett. 13 (2006), no. 2-3, 455–461.

[8] Georges de Rham, *Differentiable manifolds*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 266, Springer-Verlag, Berlin, 1984, Forms, currents, harmonic forms, Translated from the French by F. R. Smith, With an introduction by S. S. Chern.

[9] Hermann Sohr, *The Navier-Stokes equations*, Birkhäuser Advanced Texts: Basler Lehrbücher. [Birkhäuser Advanced Texts: Basel Textbooks], Birkhäuser Verlag, Basel, 2001, An elementary functional analytic approach.

[10] Friedrich Stummel, *Perturbation theory for Sobolev spaces*, Proc. Roy. Soc. Edinburgh Sect. A 73 (1975), 5–49.

[11] Roger Temam, *Navier-Stokes equations*, revised ed., Studies in Mathematics and its Applications, vol. 2, North-Holland Publishing Co., Amsterdam-New York, 1979, Theory and numerical analysis, With an appendix by F. Thomasset.