High-Dimensional Semiparametric Selection Models: Estimation Theory with an Application to the Retail Gasoline Market

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Abstract

This paper proposes a multi-stage projection-based Lasso procedure for the semiparametric sample selection model in high-dimensional settings under a weak nonparametric restriction on the form of the selection correction. In particular, the number of regressors in the main equation, \( p \), and the number of regressors in the selection equation, \( d \), can grow with and exceed the sample size \( n \). The analysis considers the exact sparsity case where the number of non-zero components in the vectors of coefficients is bounded above by some integer which is allowed to grow with \( n \) but slowly compared to \( n \), and also considers the approximate sparsity case, where the vectors of coefficients can be approximated by exactly sparse vectors. The main theoretical results of this paper are finite-sample bounds from which sufficient scaling conditions on the sample size for estimation consistency and variable-selection consistency (i.e., the multi-stage high-dimensional estimation procedure correctly selects the non-zero coefficients in the main equation with high probability) are established. A technical issue related to a set of high-level assumptions on the regressors for estimation consistency and selection consistency arises in the multi-stage estimation procedure from allowing the number of regressors in the main equation to exceed \( n \) and this paper provides analysis to verify these conditions. These verifications also provide a finite-sample guarantee of the population identification condition required by the semiparametric sample selection models. Statistical efficiency of the proposed estimators is studied via lower bounds on minimax risks and the result shows that, for a family of models with exactly sparse structure on the coefficient vector in the main equation, one of the proposed estimators attains the smallest estimation error up to the \((n, d, p)\)-scaling among a class of procedures in worst-case scenarios. Inference procedures for the coefficients of the main equation, one based on a pivotal Dantzig selector to construct non-asymptotic confidence sets and one based on a post-selection strategy (when perfect or near-perfect selection of the high-dimensional coefficients is achieved), are discussed. Other theoretical contributions of this paper include establishing the non-asymptotic counterpart of the familiar asymptotic “oracle” type of results from previous literature: the estimator of the coefficients in the main equation behaves as if the unknown nonparametric component were known, provided the nonparametric component is sufficiently smooth. Small-sample performance of the high-dimensional multi-stage estimation procedure is evaluated by Monte-Carlo simulations and illustrated with an empirical application to the retail gasoline market in the Greater Saint Louis area. Proofs are included in the online supplementary material (https://sites.google.com/site/yingzhu1215/home/JobMar_Proofs.pdf).

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1 Introduction

The past decade has witnessed research activities in high-dimensional statistics that considers inference for models in which the dimension of the parameters of interests is comparable to or even larger than the sample size. The rapid advance of data collection technology is a major driving force of the development of high-dimensional statistics: it allows for not only more observations but also more explanatory variables to be collected. Recently, high-dimensional estimation techniques have been studied in several popular econometric models and some first applications of these techniques are now available in economics. However, a very important class of models, sample selection models, have not been considered in high-dimensional settings even though they are central in many economic and marketing applications. For example, on the demand side, consumers often face choosing a service or brand followed by the amount of utilization or the number of quantities to purchase conditional on the chosen service or brand. On the supply side, firms first decide on the product positioning and then a pricing scheme based on the chosen product type. Selection models are also seen in the auction literature. In estimating the underlying selection models to study these empirical problems, only a low-dimensional set of explanatory variables has been considered even though the actual information available to these empirical problems can be far richer than what has been used by the researchers. The lack of estimation methods that deal with these “data-rich” selection problems may have limited the use of high-dimensional techniques in many economics and marketing problems. This paper aims to provide estimation tools together with their theoretical guarantees for this important but little studied topic.

Observational studies are rarely based on pure random samples. When a sample, intentionally or unintentionally, is based in part on values taken by a dependent variable (e.g., Gronau, 1973; Heckman, 1974), parameter estimates without corrective measures may be inconsistent. Such samples can be broadly defined as selected samples. Selection may be due to self-selection, with the outcome of interest determined in part by individual choice of whether or not to participate in the activity of interest. It can also result from endogenous stratification, with those who participate in the activity of interest deliberately oversampled - an extreme case being sampling only participants.

In the classical low-dimensional selection models, parameter estimates obtained from OLS may be inconsistent unless corrective measures are taken. For the parametric case where the error terms are jointly normally distributed and homoskedastic, the most well-known estimator is Heckman’s two-step procedure (1974, 1976). For semiparametric estimation of the parameters of selection models when the joint distribution of the error terms is of unknown form, many estimators have adopted the two-step estimation strategy similar to Heckman’s, under an additional “single-index” restriction on the form of the selection equation. Several estimators of the parameters of the selection equation have been proposed in the literature on semiparametric estimation; while some of these methods sidestep estimation of the unknown distribution function of the errors (e.g., Manski, 1975 and 1985; Han, 1987), others use nonparametric regression methods to estimate this distribution function along with the parameters of the underlying regression function (e.g., Cosslett, 1981; Ichimura, 1987; Klein and Spady, 1987). Similarly, the methods for estimation of the parameters in the second stage also involve nonparametric regression methods, which are applied either to estimation of the selection correction function directly (Lee, 1982; Cosslett, 1991; Gallant and Nychka, 1987; Ichimura and Lee, 1991; Newey, 1991) or to estimation of other regression functions which depend upon the estimated single index (Powell, 1989; Ahn and Powell, 1993).

The object of interest of this paper is a class of high-dimensional selection models under a weak nonparametric restriction on the form of the selection correction. Statistical estimation and variable selection in the high-dimensional setting concerns models in which the dimension of the parameters of interests is comparable or even larger than the sample size. In the past decade, an increase of research activities in
this field has been facilitated by the advances in data collection technology. In the literature on high-dimensional sparse linear regression models, a great deal of attention has been given to the $l_1$-penalized least squares. In particular, the Lasso and the Dantzig selector are the most studied techniques (see, e.g., Tibshirani, 1996; Candès and Tao, 2007; Bickel, Ritov, and Tsybakov, 2009; Belloni, Chernozhukov, and Wang, 2011; Belloni and Chernozhukov, 2011b; Loh and Wainwright, 2012; Negahban, Ravikumar, Wainwright, and Yu, 2012). Variable selection when the dimension of the problem is larger than the sample size has also been studied in the likelihood method setting with penalty functions other than the $l_1$-norm (see, e.g., Fan and Li, 2001; Fan and Lv, 2011). Lecture notes by Koltchinskii (2011), as well as recent books by Bühlmann and van de Geer (2011) and Wainwright (2015) have given a more comprehensive introduction to high-dimensional statistics.

Recently, these $l_1$-penalized techniques have been applied in a number of econometric papers. Caner (2009) studies a Lasso-type GMM estimator. Rosenbaum and Tsybakov (2010) study the high-dimensional errors-in-variables problem where the non-random regressors are observed with additive error and they present an application to hedge fund portfolio replication. Belloni and Chernozhukov (2011a) study the $l_1$-penalized quantile regression and illustrate its use on an international economic growth application. Fan, Lv, and Li (2011) review the literature on sparse high-dimensional econometric models including the vector autoregressive model for measuring the effects of monetary policy, panel data model for forecasting home price, and volatility matrix estimation in finance. Their discussion is not restricted to $l_1$-based regularization methods. Manresa (2014) considers settings where outcomes depend on an agent’s own characteristics and on the characteristics of other agents in the data and applies a Lasso type estimator to study individuals generating spillovers and their strength using panel data on outcomes and characteristics. Bonaldi, Hortacsu, and Kastl (2014) propose a new measure of systemic risk based on estimating spillovers between funding costs of individual banks with a Lasso type procedure applied to the panel of each individual bank to recover the financial network. Lecture notes by Belloni and Chernozhukov (2011b) discuss the $l_1$-based penalization methods with various econometric problems including earning regressions and instrumental selection in Angrist and Krueger data (1991). Belloni, Chen, Chernozhukov, and Hansen (2012) estimate the optimal instruments using the Lasso and in an empirical example dealing with the effect of judicial eminent domain decisions on economic outcomes, they find the Lasso-based instrumental variable estimator outperforms an intuitive benchmark. Belloni, Chernozhukov, and Hansen (2014) propose robust methods for inference on the effect of a treatment variable on a scalar outcome in the presence of many controls with an application to abortion and crime. In many economic applications, the number of endogenous regressors is also large relative to the sample size. The case of many endogenous regressors and many instrumental variables has been studied by Gautier and Tsybakov (2011), Zhu (2013), and Fan and Liao (2014).

While previous literature has extended the estimation theories and applications of several popular econometric models from the classical low-dimensional settings to the high-dimensional settings, selection models have not been considered in high-dimensional settings even though many economic applications actually fit into this setup. On the demand side, selection models are used in the context where a consumer faces choosing a service (such as electricity, cell phone service, etc.) or brand followed by the amount of utilization or the number of quantities to purchase conditional on the chosen service or brand (e.g., Krishnamurth and Raj, 1988; Chintagunta, 1993; Fox, Kim, and Yang, 2013). On the firms’ side, selection models are useful for situations where a firm first decides on its product positioning and then a pricing scheme based on the chosen product type. For example, a grocery store sometimes needs to choose which products to put on sales or promotions and then the amount of discount on these chosen products; a gas station first chooses to be either a two-product station offering both self-service and full-service
gasoline or a single-product station offering only full-service or self-service gasoline, and then decides on a pricing scheme conditional on the choice of the station type (Iyer and Seetharaman, 2003). Selection models are also seen in the auction literature (e.g., Roberts and Sweeting, 2011, 2012); in particular, by estimating a Heckman selection model with the exclusion restriction that potential competition affects a bidder’s decision to enter an auction, but has no direct effect on the values of the bids, Roberts and Sweeting (2011) presents reduced form evidence that the auction data are best explained by a selection model.

In estimating the underlying selection models to study these empirical problems, analysis has been restricted to only a low-dimensional set of explanatory variables in both the selection equation and the main equation. However, the actual information available to these empirical problems can be far richer than what has been used by the researchers. More importantly, economic theory is not always explicit about the variables that belong to the true model (e.g., Sala-i-Martin, 1997 concerning development economics). In the selection models used for consumer demand estimation, the number of explanatory variables formed by the characteristics (and the transformations of these characteristics) of a service or brand can be very large. In the grocery store example, when choosing whether to put a product on sale and the amount of discount, the store often considers not only the own characteristics of this product but also characteristics of other products. All these characteristics can potentially exceed the number of products chosen to be on sale (namely, the sample size of the observations in the main-equation), which makes it a high-dimensional selection problem. Similarly, in the bidder example, when deciding whether to enter an auction, a bidder considers potential competition from other bidders; when deciding on the values of its bid upon the entry decision, the bidder may still consider competition from the set of other “enters”. Consequently, the number of explanatory variables entering the selection equation and the main equation may grow with the number of bidders.

In the gasoline example mentioned above, besides the large number of station characteristics and demographic characteristics which amount to approximately 400 regressors with only 700 gas stations in the data studied by Iyer and Seetharaman (2008), more explanatory variables can be obtained by utilizing the geographic information and spatial data. In particular, geographic information may be used to analyze the interaction between different gas stations and identify the competitive market structure, as will be shown in Section 6 of this paper. Despite that the explanatory variables in the examples above are high-dimensional, it is plausible that only a small set of these variables (relative to the sample size) matter to the underlying response variables but which variables constitute the relevant regressors are unknown to the researchers.

The following sets up the models of interests and highlights the major contributions made by this paper. In particular, we consider estimation and selection of regression coefficients in the class of selection models captured by the following system: for all \( i = 1, ..., n \),

\[
\begin{align*}
y_{1i} & = \mathbb{I}\{w_i^T\theta^* + \epsilon_{1i} > 0\}, \\
y_{2i} & = y_{1i} (x_i^T\beta^* + \epsilon_{2i}) , \\
\mathbb{E}(\epsilon_{2i}|w_i, x_i, y_{1i} = 1) & = g(w_i^T\theta^*),
\end{align*}
\]

where \( x_i \) is a \( p \)-dimensional vector of explanatory variables and the dimension \( p \) of \( \beta^* \) is large relative to the sample size \( n \) (namely, \( p \asymp n \) or even \( p \gg n \)). Furthermore, \( g(\cdot) \) is an unknown function and \( w_i \) is a \( d \)-dimensional vector of explanatory variables with an unknown coefficient vector \( \theta^* \). Note here the dimension \( d \) of \( \theta^* \) can also be large relative the sample size \( n \) (namely, \( d \asymp n \) or even \( d \gg n \)). The third equation in (1) is known as the “single-index” restriction used in Powell (1989), Newey (1991), and Ahn
and Powell (1993). Newey (1991) and Powell (1994) discuss sufficient conditions for this restriction. In particular, it is implied by independence of the errors \((\epsilon_{1i}, \epsilon_{2i})\) and the regressors \((w_{1i}, x_{i})\). Note that the second equation of model \((\mathbb{1})\) implies
\[
y_{2i} = x_i^T \beta^* + g(w_i^T \theta^*) + \eta_i \quad \text{whenever } y_{1i} = 1,
\]
where by construction \(E[\eta_i | w_i, x_i, y_{1i} = 1] = 0\). Throughout the rest of this paper, when it is clear from the context that only the selected sample is of our interests, the notation \(y_{1i} = 1\) will be suppressed. In addition, the values of \(n\) will vary according to whether we are working with the whole sample (the observations in the selection equation) or the selected sample (the observations in the main equation).

Motivated by the Frisch-Waugh Theorem, applying a projection idea used in Robinson’s semilinear models (1988) yields the following equivalent model
\[
v_{i0} = v_i \beta^* + \eta_i,
\]
where
\[
\begin{align*}
v_i &= (x_{i1} - E(x_{i1} | w_i^T \theta^*), ..., x_{ip} - E(x_{ip} | w_i^T \theta^*)) , \\
v_{i0} &= y_{2i} - E(y_{2i} | w_i^T \theta^*) .
\end{align*}
\]
For convenience, the first equation in \((\mathbb{1})\) is referred to as the selection equation and the second equation in \((\mathbb{1})\) as the main equation.

High dimensionality arises in selection model \((\mathbb{1})\) when the dimension \(p\) of \(\beta^*\) is large relative to the sample size \(n\) (namely, \(p \asymp n\) or \(p \gg n\)) in the main equation. In addition, a weak nonparametric restriction is imposed on the form of the selection correction. Specifically, the selection effect is assumed to depend on the linear combination of some observable selection variables. The selection model under this nonparametric restriction on the form of the selection correction when \(p \geq n\) has apparently not been studied in the literature. As in classical low-dimensional selection models where parameter estimates obtained from OLS may be inconsistent, direct implementation of the Lasso or the Dantzig selector fails as sparsity of coefficients in the main equation in \((\mathbb{1})\) may not correspond to sparsity of linear projection coefficients and “bias” from parameter estimates by the direct Lasso procedure without corrective measures is found to be only exacerbated in the high-dimensional setting. This evidence is given by the Monte-Carlo simulation results in Section 5. The selection equation in \((\mathbb{1})\) is a linear latent variable model and the selection bias \(g(\cdot)\) is assumed to be an unknown function of the single index \(w_i^T \theta^*\). This setup allows us to consider special cases where the dimension \(d\) of \(\theta^*\) is also large relative to the sample size \(n\) (namely, \(d \asymp n\) or \(d \gg n\)) in the selection equation described by some of the most popular binary response models. It is worth noting that the general results provided by this paper also hold for the more general structure where \(E(\epsilon_{2i} | w_i, y_{1i} = 1) = g(h(w_i^T, \theta^*))\) and \(h(w_i^T, \theta^*)\) is a scalar unobservable index, under appropriate identification assumptions.

The proposed estimation procedure for the high-dimensional linear coefficients in the main equation in this paper is the penalized version of a projection-type strategy. In the first-stage, given consistent estimates \(\hat{\theta}\) of \(\theta^*\) in the selection equation obtained using one of several methods recently proposed in the high-dimensional statistics literature, estimates \(w_i^T \hat{\theta}\) of the “single index” variables \(w_i^T \theta^*\) are formed. In the second-stage, nonparametric regression is performed to obtain estimate \(\hat{E}(x_{ij} | w_i^T \hat{\theta})\) of \(E(x_{ij} | w_i^T \theta^*\) for \(j = 1, ..., p\) and \(\hat{E}(y_{2i} | w_i^T \hat{\theta})\) of \(E(y_{2i} | w_i^T \theta^*)\); then the estimated residuals \(\hat{v}_i = \sum_{j=1}^{p} \hat{E}(x_{ij} | w_i^T \hat{\theta}) \hat{E}(y_{2i} | w_i^T \hat{\theta}) \) for \(i = 1, ..., n\).
\[ (x_{i1} - \hat{E}(x_{i1}|w_i^T\hat{\theta}), \ldots, x_{ip} - \hat{E}(x_{ip}|w_i^T\hat{\theta})) \] of \( v_i \) and \( \hat{v}_{i0} = y_{2i} - \hat{E}(y_{2i}|w_i^T\hat{\theta}) \) of \( v_{i0} \) are formed. This step is motivated by the estimator of Robinson (1988) for semilinear models. The second-stage estimation in this paper involves \( p + 1 \) nonparametric regressions where \( p \times n \) or \( p \gg n \), and in contrast to the classical low-dimensional settings (e.g., Robinson 1988), a more careful control for the noise from the \( p + 1 \) nonparametric regressions is required. In particular, the prediction errors of the nonparametric procedures are shown in this paper to satisfy

\[
P \left\{ \sqrt{n} \sum_{i=1}^{n} \left[ \hat{E}(z_{ij}|w_i^T\hat{\theta}) - \hat{E}(z_{ij}|w_i^T\hat{\theta}) \right]^2 \geq t \right\} \leq c \exp(-nt^2)
\]

where \( z_{ij} = x_{ij} \) for \( j = 1, \ldots, p \) and \( z_{i0} = y_{2i} \), and as a consequence,

\[
P \left\{ \max_{j=1,\ldots,p} \sqrt{n} \sum_{i=1}^{n} \left[ \hat{E}(z_{ij}|w_i^T\hat{\theta}) - \hat{E}(z_{ij}|w_i^T\hat{\theta}) \right]^2 \geq t \right\} \leq c \exp(-nt^2 + \log p) = O \left( \frac{1}{p} \right)
\]

where the last equality holds provided \( n \) is sufficiently large. The tail bounds above can be ensured by considering the family of nonparametric least squares estimators or regularized nonparametric least squares estimators defined in van de Geer (2000). This family of estimators include linear regression as the simplest case, sparse linear regressions, convex regression, Lipschitz and Isotonic regression, kernel ridge regression based on reproducing kernel Hilbert spaces, estimators based on series expansion, sieves and spline methods. A procedure based on Lipschitz regression for the second-stage nonparametric estimation is illustrated in this paper for a leading case example.

In the third-stage, regressing \( \hat{v}_{i0} \) on \( \hat{v}_i \) with \( l_1 \)-regularization to estimate the main-equation coefficients \( \beta^\ast \). In particular, for the third-stage estimation, this paper considers a non-pivotal Lasso procedure whose regularization parameter depends on the unknown variance of \( \eta_i \), and a pivotal Dantzig selector whose regularization parameter does not involve the unknown variance of \( \eta_i \). This pivotal Dantzig selector was originally proposed by Gautier and Tsybakov (2011) in the context of instrumental variables regression. A by-product of the pivotal procedure is a set of non-asymptotic confidence intervals (which also do not involve the unknown variance of \( \eta_i \)). Upon the availability of estimates of the high-dimensional linear coefficients, two different estimation strategies for the selection bias function are proposed: one is a closed form estimator and the other is a nonparametric least squares estimator. Despite that the nonparametric least squares estimator of \( g(w^T\theta^\ast) \) is computationally more involved relative to the closed-form estimator, its rate of convergence turns out to be faster. In particular, when \( \beta^\ast \) is approximately sparse with \( q_2 = 1 \), the closed-form estimator cannot achieve \( \text{MSE} \)-consistency even if \( n \to \infty \) while the nonparametric least squares estimator is consistent in \( \text{MSE} \) when \( q_2 = 1 \).

While existing semiparametric estimation techniques for the selection models limit the number of regressors entering the selection equation and the main equation, the multi-stage estimation procedure with \( l_1 \)-regularization in the first- and third-stage are more flexible and particularly powerful for applications in which the vector of parameters of interests is high-dimensional but sparse and there is lack of information about the relevant explanatory variables. Moreover, this above-mentioned high-dimensional multi-stage estimation procedure is intuitive and can be easily implemented using existing software packages. In particular, it decomposes the joint search of the optimal values for the high-dimensional linear coefficients and the nonparametric selection bias component into several sequential searches with each search defined...
over a much smaller parameter space. In particular, the second-stage estimation incurs a computational cost linear in \( p \) as it involves solving \( p + 1 \) independent subproblems and each subproblem can be in general solved with a polynomial-time algorithm. The computational efficiency of the first-stage and third-stage estimations is guaranteed by existing algorithms developed for solving the Lasso or the Dantzig program. Upon the availability of estimates of the high-dimensional linear coefficients, the estimator for the selection bias function is simply a closed form estimator or a nonparametric least squares estimator. In additional to the computational efficiency, as we will see in Section 4.4 that, under some conditions and when \( \beta^* \) is exactly sparse, the proposed procedures for estimating \( \beta^* \) and \( g(\cdot) \) are overall statistically efficient up to the \( (n, d, p) \)-factors, relative to any procedure constructed based on model (2) for estimating model (1), regardless of its computational cost.

The main theoretical results of this paper are finite-sample bounds from which sufficient scaling conditions on the sample size for estimation consistency in \( l_2 \)-norm and variable-selection consistency (i.e., the multi-stage high-dimensional estimator correctly selects the non-zero coefficients in the main equation with high probability) are established. These results imply that the estimate from performing the Lasso-type procedures in the third-stage estimation is \( l_2 \)-consistent as long as \( \beta^* \) is \( l_{q_2} \)-sparse with \( q_2 \in [0, 1] \) but inconsistent when \( q_2 > 1 \). A technical issue related to a set of high-level assumptions on the regressors for estimation consistency and selection consistency arises in the multi-stage estimation procedure from allowing the number of regressors in the main equation to exceed \( n \) and this paper provides analysis to verify these conditions. These verifications also provide a finite-sample guarantee of the population identification condition required by the semiparametric selection models. It is worth mentioning that the multi-stage estimator and the general results in this paper can be applied to other high-dimensional sparse semiparametric models. Section 4.5 discusses estimation of a certain type of high-dimensional semilinear models with the proposed multi-stage strategy when the number of parametric and (additive) nonparametric components are large relative to the sample size (the details are left to one of the PhD thesis chapters by Zhu, 2015). Statistical efficiency of the proposed estimators is studied via lower bounds on minimax risks and the result shows that, for a class of models with exactly sparse \( \beta^* \), the overall convergence rate of the estimator of the high-dimensional linear coefficients in the main equation and the nonparametric least squares estimator of the selection bias function matches the theoretical lower bound up to the \( (n, d, p) \)-factors, and exceeds it at most by a factor of \( k_2^{3/2} \). This statistical efficiency result, however, does not apply to the case where \( \beta^* \) is approximately sparse.

Other theoretical contributions of this paper include establishing the non-asymptotic "oracle" type of results from previous literature: the estimator of the coefficients in the main equation behaves as if the unknown nonparametric component were known, provided the nonparametric component is sufficiently smooth. This new "oracle" result holds for a unified framework of nonparametric least squares estimators and regularized nonparametric least squares estimators considered in the second-stage estimation. In general, for a semiparametric model with two additive components one parametric and the other nonparametric, when the prediction error or the \( \sqrt{\text{MSE}} \) (the square root of the mean squared error) of the nonparametric estimation per se is \( O_p(t_n) \), this paper shows that the error arising from not knowing the functional form contributes \( O_p(t_n^2) \) in the \( l_2 \)-error of the estimator of \( \beta^* \). The driver behind this "oracle" result lies in the projection strategy. An application of this general result to classical low-dimensional semilinear models would imply that the nonparametric component needs to be estimated at a rate no slower than \( O\left(\left(\frac{1}{n}\right)^{\frac{3}{2}}\right) \) in order for the estimator of the parametric component to achieve the rate of \( O\left(\sqrt{\frac{1}{n}}\right) \). In contrast to the semilinear models, the low-dimensional selection models require the rate of the nonparametric component to be at least \( O\left(\left(\frac{1}{n}\right)^{\frac{1}{5}}\right) \) because the nonparametric
component in the selection model involves the unknown parameters $\theta^*$ that also need to be estimated.

The high-dimensional multi-stage procedure is illustrated with an application to the retail gasoline market in the Greater Saint Louis area. Gasoline stations choose to be one of the two types: a two-product station offering both self-service and full-service gasoline or a single-product station offering only full-service or self-service gasoline. A two-product station, by charging different prices for full- and self-service gasoline, induces consumers with different valuations to self-select the product that is consistent with their preferences. In other words, a two-product station engages in price-discrimination. A single-product station, on the other hand, is unable to price discriminate. Similar to Iyer and Seetharaman (2003), this paper models a retailer’s incentive to price discriminate by choosing either single-product or multi-product as a function of market and station characteristics and then models the retailer’s pricing decision, conditional on the choice of the product type. However, Iyer and Seetharaman (2003) did not account for interactions between the gas stations in their empirical analysis. This paper uses geographic information and spatial data to introduce, in the main equation related to the retailers’ pricing decisions, a set of variables that are high-dimensional to control for interactions between the gas stations and employ a proposed estimator to identify the competitive market structure. In contrast to other heuristic ways

Section 2 presents identification assumptions required for model (1) in high-dimensional settings. The estimation procedures are introduced in Section 3. Theoretical results are established in Section 4. Small-sample performance of the proposed multi-step high-dimensional estimator is evaluated with Monte-Carlo simulations in Section 5 and applied to the retail gasoline market in Section 6. Section 7 concludes this paper. Proofs of the main results are collected in Appendix I, with the remaining proofs of technical lemmas contained in Appendix II. The appendices are included in the online supplementary material (https://sites.google.com/site/yingzhu1215/home/JobMar_Proofs.pdf).

2 Identification assumptions

Notation. The $l_q$ norm of a vector $v \in p \times 1$ is denoted by $|v|_q$, $1 \leq q \leq \infty$ where $|v|_q := \left(\sum_{i=1}^{p} |v_i|^q\right)^{1/q}$ when $1 \leq q < \infty$ and $|v|_q := \max_{i=1,...,p} |v_i|$ when $q = \infty$. For a matrix $A \in \mathbb{R}^{p \times p}$, write $|A|_\infty := \max_{i,j} |a_{ij}|$ to be the elementwise $l_\infty$-norm of $A$. The $l_2$—operator norm, or spectral norm of the matrix $A$ corresponds to its maximum singular value; i.e., it is defined as $\|A\|_2 := \sup_{v \in S} |Av|_2$, where $S = \{v \in \mathbb{R}^p \mid |v|_2 = 1\}$. The $l_\infty$ matrix norm (maximum absolute row sum) of $A$ is denoted by $\|A\|_\infty := \max_i \sum_j |a_{ij}|$ (note the difference between $|A|_\infty$ and $\|A\|_\infty$). For a square matrix $A$, denote its minimum eigenvalue and maximum eigenvalue by $\lambda_{\min}(A)$ and $\lambda_{\max}(A)$, respectively. The $L_2(P)$—error of a vector $\Delta(x)$, denoted by $|\Delta|_{L_2(P)}$, is given by $\|\mathbb{E}_X(\Delta(x))^2\|^\frac{1}{2}$. Define $P_n := \frac{1}{n} \sum_{i=1}^{n} \delta_{x_i}$, that places a weight $\frac{1}{n}$ on each observation $x_i$ for $i = 1, ..., n$, and the associated $L_2(P_n)$—norm of the vector $\Delta := \{\Delta(x_i)\}_{i=1}^{n}$, denoted by $|\Delta|_{L_2(P_n)}$, is given by $\left[\frac{1}{n} \sum_{i=1}^{n} (\Delta(x_i))^2\right]^\frac{1}{2}$. For a vector $v \in \mathbb{R}^p$, let $J(v) = \{j \in \{1, ..., p\} \mid v_j \neq 0\}$ be its support, i.e., the set of indices corresponding to its non-zero components $v_j$. The cardinality of a set $J \subseteq \{1, ..., p\}$ is denoted by $|J|$. For functions $f(n)$ and $g(n)$, write $f(n) \gtrsim g(n)$ to mean that $f(n) \geq cg(n)$ for a universal constant $c \in (0, \infty)$ and similarly, $f(n) \lesssim g(n)$ to mean that $f(n) \leq c'g(n)$ for a universal constant $c' \in (0, \infty)$, and $f(n) \asymp g(n)$ when $f(n) \gtrsim g(n)$ and $f(n) \lesssim g(n)$ hold simultaneously. Also denote $\max\{a, b\}$ by $a \vee b$ and $\min\{a, b\}$ by $a \wedge b$. 

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The following assumptions are imposed on model (1).

**Assumption 2.1 (Sampling):** The data \( \{y_{1i}, y_{2i}, w_i, x_i\} \) are i.i.d. with finite second moments.

**Remark.** The identicalness of \( \{y_{1i}, y_{2i}, w_i, x_i\} \) in Assumption 2.1 can be relaxed with the condition that \( \{\epsilon_{1i}, \eta_i\} \) are identically distributed but \( \{w_i, x_i\} \) are not.

**Assumption 2.2 (Sparsity):** The coefficient vector \( \beta^* \in \mathbb{R}^p \) belongs to the \( l_{q_2} \)−"balls" \( B^p_{q_2}(R_{q_2}) \) for a "radius" of \( R_{q_2} \) and some \( q_2 \in [0, 1] \), where the \( l_q \)−"balls" of "radius" \( R \) for \( q \in [0, 1] \) are defined by

\[
B^p_q(R) := \left\{ \beta \in \mathbb{R}^p \mid |\beta|_q = \left( \sum_{j=1}^{p} |\beta_j|^q \right)^{1/q} \leq R \right\}
\]

for \( q \in (0, 1] \)

\[
B^p_0(R) := \left\{ \beta \in \mathbb{R}^p \mid |\beta|_0 = \sum_{j=1}^{p} \mathbb{1} [\beta_j \neq 0] \leq R \right\}
\]

for \( q = 0 \).

**Remark.** Assumption 2.2 requires the coefficient vector to be "sparse". As one might expect, if the high-dimensional model lacks any additional structure, then there is no hope of obtaining consistent estimators of \( \beta^* \) when the ratio \( \frac{p}{n} \) stays bounded away from 0. For this reason, when working in settings in which \( p > n \), it is necessary to impose some type of sparsity assumptions on the unknown coefficient vector \( \beta^* \in \mathbb{R}^p \). Assumption 2.2 formalizes the sparsity condition by considering the \( l_q \)−"balls" \( B^p_q(R_q) \) of "radius" \( R_q \) where \( q \in [0, 1] \). The exact sparsity on \( \beta^* \) corresponds to the case of \( q = q_2 = 0 \) with \( R_{q_2} = k_2 \) (in this paper, the subscript "2" is generally reserved for the main-equation related parameters and the subscript "1" for the selection-equation related parameters), which says that \( \beta^* \) has at most \( k_2 \) non-zero components, where the sparsity parameter \( k_2 \) is also allowed to increase to infinity with \( n \) but slowly compared to \( n \). In the more general setting \( q_2 \in (0, 1] \), membership in \( B^p_{q_2}(R_{q_2}) \) has various interpretations and one of them involves how quickly the ordered coefficients decay. When \( q_2 \in [0, 1) \), the set \( B^p_{q_2}(R_{q_2}) \) is non-convex and the \( l_1 \)−ball is the closest convex approximation of these non-convex sets. In terms of algorithm design, the idea of approximating non-convex problems with their closest convex member (so called "convex relaxation") provides a tremendous computational advantage. This is one of the reasons for favoring the \( l_1 \)−penalization techniques such as the Lasso (in solving high-dimensional regression problems with sparsity described by the \( l_q \)−constraint where \( q \in [0, 1] \) ) over estimators based on the \( l_q \)−penalty with \( q \in [0, 1] \) which are computationally more difficult (see the Bridge estimator in Huang, Horowitz, and Ma, 2008 as an example of these nonconvex penalization procedures) and estimators based on \( l_q \)−penalty with \( q > 1 \) (such as the ridge-penalty) which are not the closest convex approximations. On the other hand, if the coefficient vector belongs to an Euclidean ball (the \( l_2 \)−ball), then it would make more sense to apply a ridge penalty. The focus of this paper is on high-dimensional sparse \( \beta^* \) that belongs to \( B^p_{q_2}(R_{q_2}) \) for \( q_2 \in [0, 1] \).

**Assumption 2.3 (Restricted Identifiability):** For a subset \( S \subseteq \{1, 2, ..., p\} \) and all non-zero \( \Delta \in \mathbb{C}(S; q_2, \varphi) \cap S) \) where

\[
\mathbb{C}(S; q_2, \varphi) := \{ \Delta \in \mathbb{R}^p : |\Delta_S|_1 \leq \varphi |\Delta_S|_1 + (\varphi + 1)|\beta^*_S|_1 \}
\]

for some constant \( \varphi \geq 1 \),
(with $\Delta_S$ denoting the vector in $\mathbb{R}^p$ that has the same coordinates as $\Delta$ on $S$ and zero coordinates on the complement $S^c$ of $S$) and

$$S_\delta := \{ \Delta \in \mathbb{R}^p : |\Delta|_2 \geq \delta \},$$

the matrix $E[y_1v_i^Tv_i]$ satisfies

$$\frac{\Delta^T E[y_1v_i^Tv_i] \Delta}{|\Delta|_2^2} \geq \kappa_L > 0,$$

where

$$v_i = (x_{i1} - E(x_{i1}|w_i^T \theta^*), ..., x_{ip} - E(x_{ip}|w_i^T \theta^*)).$$

**Remark.** Assumption 2.3 is the high-dimensional counterpart of the familiar identification assumption in the low-dimensional selection model literature (e.g., Powell 1989; Newey, 1991; Ahn and Powell, 1993), which assumes the matrix $E[y_1v_i^Tv_i]$ is positive definite uniformly over all $\Delta \in \mathbb{R}^p \setminus \{0\}$. When $v_i$ is a zero-mean Gaussian matrix with covariance $E[y_1v_i^Tv_i] = \sigma^2 I_{p \times p}$, the smallest eigenvalue of $E[y_1v_i^Tv_i]$ is $\sigma^2$, so the traditional identification condition in the low-dimensional case naturally carries to the high-dimensional case. However, for more general structures on $E[y_1v_i^Tv_i]$, while this traditional identification condition is plausible for small $p$, it may become harder to be satisfied when $p$ is large. Assumption 2.3 relaxes the uniform positive definiteness but only requires it to hold over a restricted set $C(S; q_2, \varphi) \cap S_\delta$ so that the special case of $x_i \subset w_i$ is allowed even in the high-dimensional settings (the choices of $\delta$ and $S$ will be made clear in Section 4 when the theoretical results are presented.). If $x_i \subset w_i$, Assumption 2.3 says that for any non-zero vector $\lambda \in C(S; q_2, \varphi) \cap S_\delta$, there is no measurable function $f(w_i^T \theta^*)$ such that $x_i^T \lambda = f(w_i^T \theta^*)$ when $y_{1i} = 1$. Consequently, there is at least one component $w_{ij}$ with $\theta_j^*$ in the support set of $\theta^*$ (namely, the set of non-zero components in $\theta^*$) such that $w_{ij}$ is excluded from $x_i$. This necessary condition is the high-dimensional extension of the familiar “exclusion restriction” condition in the low-dimensional selection model literature.

When $\beta^*$ is exactly sparse (namely, $q_2 = 0$), we can take $\delta = 0$ and choose $S = J(\beta^*)$ (where $J(\beta^*)$ denotes the support of $\beta^*$), which reduces the set $C(S; q_2, \varphi) \cap S_\delta$ to the following cone:

$$C(J(\beta^*); 0, \varphi) := \{ \Delta \in \mathbb{R}^p : |\Delta|_{J(\beta^*)} \leq \varphi |\Delta|_{J(\beta^*)} \}. $$

The sample analog of Assumption 2.3 over the cone $C(J(\beta^*); 0, \varphi)$ is the so-called restricted eigenvalue condition on the Gram matrix $\frac{v_i^Tv_i}{n}$ studied in Bickel, et. al. (2009), Meinshausen and Yu (2009), Raskutti, et al. (2010), Bühlmann and van de Geer (2011), Loh and Wainwright (2012), Negahban, et. al. (2012), etc. Note that in the low-dimensional setting where $p < n$, as long as $\text{rank}(v) = p$, we are guaranteed that the Gram matrix $\frac{v_i^Tv_i}{n}$ is positive definite. In the high-dimensional setting with $p > n$, the matrix $\frac{v_i^Tv_i}{n}$ is a $p \times p$ matrix with rank at most $n$, so it is impossible to have the uniform positive definiteness. It is well-known that the restricted eigenvalue assumption, defined more precisely below, is a sufficient condition for the $l_2$-consistency of the Lasso estimator for the sparse linear regression models in high-dimensional settings. To motivate the restricted set $C(J(\beta^*); 0, \varphi)$, note that the vectors $\Delta$ in this cone have a substantial part of their “mass” concentrated on a set of the cardinality of $J(\beta^*)$. The vectors $\Delta$ of interests often concern the error $\hat{\beta} - \beta^*$ where $\hat{\beta}$ is some estimate of $\beta^*$. When the high-dimensional sparse linear regression models are estimated by the $l_1$-penalized techniques, an appropriate choice of the regularization parameter would generally ensure the error $\hat{\beta} - \beta^*$ to be in this restricted set.

The following discussion provides a review of the restricted eigenvalue condition in literature for consistent estimation of both exactly sparse and approximately sparse regression models in high-dimensional
settings with the Lasso or Dantzig selector. Consider the high-dimensional sparse linear models

$$y_i = x_i^T \beta^* + \epsilon_i = \sum_{j=1}^{p} x_{ij} \beta_j^* + \epsilon_i, \quad i = 1, ..., n,$$

(4)

where $\mathbb{E}(x_i \epsilon_i) = 0$ for $i = 1, ..., n$. Assume $p$, the number of regressors, in the above equation grows with and exceeds the sample size $n$. Again, the focus here is the class of sparse models with $\beta^* \in B_q^{\delta}(R_q)$ for $q \in [0, 1]$. The Lasso procedure is a combination of the residual sum of squares and a $l_1$-regularization defined by the following program

$$\hat{\beta}_{Las} \in \arg \min_{\beta \in \mathbb{R}^p} \left\{ \frac{1}{2n} |y - X \beta|^2_2 + \lambda_n |\beta|_1 \right\},$$

(5)

where $\lambda_n > 0$ is some regularization or tuning parameter. Denote the minimizer to the above program by $\hat{\beta}_{Las}$. A necessary and sufficient condition of $\hat{\beta}_{Las}$ is that 0 belongs to the subdifferential of the convex function $\beta \mapsto \frac{1}{2n} |y - X \beta|^2_2 + \lambda_n |\beta|_1$. This implies that the Lasso solution $\hat{\beta}_{Las}$ satisfies the constraint

$$\left| \frac{1}{2n} X^T (y - X \hat{\beta}_{Las}) \right|_\infty \leq \lambda_n.$$

The Dantzig selector of the linear regression function is defined as a vector having the smallest $l_1$-norm among all $\beta$ satisfying the above constraint, i.e.,

$$\hat{\beta}_{Dan} \in \arg \min \left\{ |\beta|_1 : \left| \frac{1}{2n} X^T (y - X \beta) \right|_\infty \leq \lambda_n \right\}.$$

Under the exact sparsity assumption, Bickel et al., 2009 shows that the Lasso and the Dantzig selector exhibit similar behavior.

In the high-dimensional setting, a sufficient condition for the $l_2$-consistency of the Lasso estimator $\hat{\beta}_{Las}$ is the restricted eigenvalue (RE) condition related to the positive definiteness of the Gram matrix $\frac{X^T X}{n}$ over a restricted set (see, e.g., Bickel, et. al., 2009; Meinshausen and Yu, 2009; Raskutti, et al., 2010; Bühlmann and van de Geer, 2011; Loh and Wainwright 2012; Negahban, et. al., 2012; etc.). Consider the following definition of the RE condition given by Negahban, et. al. (2012) and Wainwright (2015).

**Definition 2.1 (RE condition).** For $q \in [0, 1]$, the matrix $X \in \mathbb{R}^{n \times p}$ satisfies the RE condition over a subset $S \subseteq \{1, 2, ..., p\}$ with parameters $(q, \delta, \kappa, \varphi)$ if

$$\frac{1}{n} |X \Delta|^2_2 \geq \kappa > 0 \quad \text{for all nonzero } \Delta \in \mathbb{C}(S; q, \varphi) \cap S_\delta,$$

(6)

where $\mathbb{C}(S; q, \varphi) \cap S_\delta$ is defined in Assumption 2.3.

As discussed previously, when the unknown vector $\beta^* \in \mathbb{R}^p$ is exactly sparse, the set $\mathbb{C}(S; q, 0) \cap S_\delta$ is reduced to the cone $\mathbb{C}(J(\beta^*); 0, 3)$. When $\beta^*$ is approximately sparse (namely, $q \in (0, 1]$), in sharp contrast to the case of exact sparsity, the set $\mathbb{C}(S; q, 3)$ is no longer a cone but rather contains a ball centered at the origin. As a consequence, it is never possible to ensure that $\frac{1}{n} |X \Delta|^2_2$ is bounded from below for all vectors $\Delta$ in the set $\mathbb{C}(S; q, 3)$ (see Negahban, et. al., 2012 for a geometric illustration of this issue). For this reason, in order to obtain a general applicable theory, it is crucial to further restrict the
set $\mathbb{C}(S; q, 3)$ for $q \in (0, 1]$ by introducing the set
\[
S_\delta := \{ \Delta \in \mathbb{R}^p : |\Delta|_2 \geq \delta \},
\]
where $\delta > 0$ is some parameter depending on the choice of the regularization parameter $\lambda_n$ in the Lasso program \[5\]. Provided the parameter $\delta$ and the set $S$ are suitably chosen, the intersection $\mathbb{C}(S; q, 3) \cap S_\delta$ excludes many “flat” directions (with eigenvalues of 0) in the space for the case of $q \in (0, 1]$. To the best of my knowledge, the necessity of this additional set $S_\delta$, essential for the approximately sparse case of $q \in (0, 1]$, is first recognized explicitly in Negahban, et. al. (2012).

Raskutti et. al. (2010) shows that the RE condition \[5\] is satisfied by the design matrix $X \in \mathbb{R}^{n \times p}$ formed by independently sampling each row $X_i \sim N(0, \Sigma)$. Rudelson and Zhou (2011) as well as Loh and Wainwright (2012) extend the verification of the RE condition from the case of Gaussian designs to the case of sub-Gaussian designs. The sub-Gaussian assumption says that the explanatory variables need to be drawn from distributions with well-behaved tails like Gaussian. In contrast to the Gaussian assumption, sub-Gaussian variables constitute a more general family of distributions. In this paper, we make use of the following definition for a sub-Gaussian matrix.

**Definition 2.2 (Sub-Gaussian variables and matrices).** A random variable $X$ with mean $\mu = \mathbb{E}[X]$ is sub-Gaussian if there is a positive number $\sigma$ such that
\[
\mathbb{E}[\exp(t(X - \mu))] \leq \exp(\sigma^2 t^2/2) \quad \text{for all } t \in \mathbb{R},
\]
and a random matrix $A \in \mathbb{R}^{n \times p}$ is sub-Gaussian with parameters $(\Sigma_A, \sigma_A^2)$ if (a) each row $A_i \in \mathbb{R}^p$ is sampled independently from a zero-mean distribution with covariance $\Sigma_A$; (b) for any unit vector $u \in \mathbb{R}^p$, the random variable $u^T A_i^T$ is sub-Gaussian with parameter at most $\sigma_A^2$.

When applying the proposed multi-stage procedure in this paper to estimate the high-dimensional selection models, there is no guarantee that the random matrix $\hat{\beta}_n^T \hat{A}$ (where $\hat{v}_i$ are the estimates of $v_i = x_i - \mathbb{E}(x_i | w_i^T \theta^*)$ for $i = 1, ..., n$) would automatically satisfy these previously established conditions for estimation consistency. For a broad class of sub-Gaussian matrices formed by the true residuals $v_i = x_i - \mathbb{E}(x_i | w_i^T \theta^*)$ for $i = 1, ..., n$ whenever $y_{1i} = 1$, this paper provides results that imply the RE condition \[5\] holds for $\hat{v}^T \hat{v}$ with high probability provided Assumption 2.3 is satisfied. Verifications of the RE condition provide a finite-sample guarantee of Assumption 2.3 when the unknown residuals $\hat{v}$ are replaced with their estimate $\hat{v}$ and the expectation is replaced with a sample average.

While the RE assumption is a natural sufficient condition for analyzing $l_2$-consistency of the Lasso estimator $\hat{\beta}_{Las}$, $l_2$-consistency of the Dantzig selector $\hat{\beta}_{Dan}$ can be related to a different sufficient condition, the sensitivity characteristics, on the term $|X^T X v|_\infty$. These sensitivity characteristics were originally introduced in Ye and Zhang (2010) as the cone invertibility factors and used in Gautier and Tsybakov (2011) for high-dimensional instrumental variable regressions. Gautier and Tsybakov (2011) shows that the sensitivity characteristics can be larger than the usual RE condition of Bickel, et. al. (2009) and therefore the Dantzig-type estimators may lead to better results in certain cases. The analysis of a pivotal Dantzig selector in this paper for estimating the high-dimensional linear coefficients relies on the following definition based on Gautier and Tsybakov (2011):

\[1\]Recently, another weaker version of the RE condition tailored to the square-root Lasso is developed in Belloni, Chernozhukov, and Wang (2014).
**Definition 2.3** \((l_2\text{-sensitivity})\). The matrix \(X \in \mathbb{R}^{n \times p}\) satisfies the \(l_2\text{-sensitivity}\) condition over a subset \(S \subseteq \{1, 2, \ldots, p\}\) with parameters \((q, \delta, \kappa', \varphi)\) if

\[
\frac{1}{n} |X^T X \Delta|^2_{\infty} \geq \kappa' > 0 \quad \text{for all nonzero } \Delta \in \mathbb{C}(S; q, \varphi) \cap S_\delta
\]

where \(\mathbb{C}(S; q, \varphi) \cap S_\delta\) is defined in Assumption 2.3.

When \(y_i\) in (4) is a latent variable with only an observable sign, other models such as the high-dimensional binary response models may be considered. In analyzing these models, the RE condition can be replaced with a similar notion, the restricted strong convexity (RSC) condition, originally formalized by Negahban, et. al. (2012) in the context of the regularized \(M\)-estimation with a general, convex and differentiable loss function. The following definition from Negahban, et. al. (2012) is adopted in this paper to analyze the theoretical properties of an estimator for the high-dimens ional logit and probit model:

**Definition 2.4** \((RSC\ text{condition})\). A convex and differentiable loss function \(L(\theta; z^n_1)\) satisfies the RSC condition over a subset \(S \subseteq \{1, 2, \ldots, p\}\) with parameters \((q, \delta, \kappa'', \varphi)\) where \(\kappa'' > 0\) if

\[
L(\theta^* + \Delta; z^n_1) - L(\theta^*; z^n_1) - \langle \nabla L(\theta^*; z^n_1), \Delta \rangle \geq \kappa'' |\Delta|^2_2 \quad \text{for all nonzero } \Delta \in \mathbb{C}(S; q, \varphi) \cap S_\delta.
\]

where \(\nabla L(\theta^*; z^n_1)\) denotes the derivative of \(L(\theta; z^n_1)\) evaluated at \(\theta = \theta^*\), and \(\mathbb{C}(S; q, \varphi) \cap S_\delta\) is defined in Assumption 2.3.

### 3 Estimation procedures

This section presents a 3-stage estimation procedure for the high-dimensional linear coefficients in the main equation and two estimators of the selection bias function. In terms of applicability, the proposed estimators enjoy many computational advantages and can be easily implemented using existing software packages.

#### 3.1 The multi-stage estimator of the high-dimensional linear coefficients

To facilitate the presentation of the multi-stage estimator, we reverse the order of the three stages when discussing the estimation procedure; in particular, we will introduce the third-stage estimator and then followed by the second-stage and the first-stage estimators. Note that the second- and third-stage estimations concern only the selected sample (observations with \(y_{1i} = 1\)) and the first-stage estimation concerns the entire sample. For the third-stage estimation, this paper considers a non-pivotal Lasso procedure whose regularization parameter depends on the unknown variance of \(\eta_i\), and a pivotal Dantzig selector (Gautier and Tsybakov, 2011) whose regularization parameter does not involve the unknown variance of \(\eta_i\).

**Non-pivotal third-stage estimation**

Revisiting equation (3) in Section 1 suggests that if an estimate of

\[
(E \left( x_{i1} | w_i^T \theta^* \right), \ldots, E \left( x_{ip} | w_i^T \theta^* \right))
\]
is available to us, then we can form estimates
\[
\hat{v}_i = (x_{i1} - \hat{E}(x_{i1}|w_i^T \theta), \ldots, x_{ip} - \hat{E}(x_{ip}|w_i^T \theta)),
\]
\[
\hat{v}_{i0} = y_{2i} - \hat{E}(y_{2i}|w_i^T \theta).
\]
of the nonparametric residuals
\[
v_i = (x_{i1} - E(x_{i1}|w_i^T \theta^*), \ldots, x_{ip} - E(x_{ip}|w_i^T \theta^*)),
\]
\[
v_{i0} = y_{2i} - E(y_{2i}|w_i^T \theta^*).
\]

Then, an estimator of the high-dimensional linear coefficients in the main equation (the **third-stage** estimator) can be obtained by performing the following Lasso program:
\[
\hat{\beta}_{HSEL} \in \arg\min_{\beta \in \mathbb{R}^p} : \frac{1}{2n} |\hat{v}_0 - \hat{v}\beta|^2 + \lambda_{n,3} |\beta|_1,
\]
where \(\lambda_{n,3} > 0\) is some regularization parameter whose choice is to be discussed in Section 4. In general, the choice of \(\lambda_{n,3}\) depends on \(E(v_{ij}^2)\) and \(E(\eta_i^2)\). To make \(\lambda_{n,3}\) and the estimate \(\hat{\beta}_{HSEL}\) independent of the effect from \(E(v_{ij}^2)\), we can impose weights on the penalty term as follows
\[
\min_{\beta \in \mathbb{R}^p} : \frac{1}{2n} |\hat{v}_0 - \hat{v}\beta|^2 + \lambda_{n,3} \sum_{j=1}^p \hat{\sigma}_{v_j} |\beta_j|,
\]
where \(\hat{\sigma}_{v_j} := \sqrt{\frac{1}{n} \sum_{i=1}^n \hat{v}_{ij}^2}\). To make \(\lambda_{n,3}\) not depend on the unknown variance of \(\eta_i\), we can consider the pivotal version of the Dantzig selector as in Gautier and Tsybakov (2011).

**Pivotal third-stage estimation**

Set \(v_{j*} := \max_{i \in \{1, \ldots, n\}} (\max \{|2x_{ij}|, |\hat{v}_{ij}|\})\) for \(j = 1, \ldots, p\) and denote \(D\) the diagonal \(p \times p\) matrix with diagonal entries \(v_{j*}^{-1}\). Consider the following optimization problem:
\[
\min_{(\beta, \sigma) \in A} : (|D^{-1}\beta|_1 + C\sigma)
\]
where
\[
A = \left\{ (\beta, \sigma) : \beta \in \mathbb{R}^p, \sigma > 0, \frac{1}{n} |D\hat{v}^T(\hat{v}_0 - \hat{v}\beta)|_\infty \leq \sigma \xi, \frac{1}{n} |\hat{v}_0 - \hat{v}\beta|^2 \leq \sigma^2 \right\}
\]
for some tuning parameter \(\xi > 0\) (to be specified in Section 4). The computational aspect of this pivotal estimator is detailed in Gautier and Tsybakov (2011).

**Remark.** The third-stage estimation needs not to be restricted to the Lasso or the Dantzig selector. Other methods with different loss functions (such as the square-root Lasso in Belloni, et. al 2011, 2014) or with different penalty functions (such as the SCAD in Fan and Li, 2001, or the MCP in Zhang, 2010) can be used. This paper focuses on the analysis of the non-pivotal Lasso and the pivotal Dantzig selector laid out above for the third-stage estimation.

**Second-stage estimation**

To simplify the notations in the following, write \(E(x_{ij}|w_i^T \theta) := m_j(w_i^T \theta), \hat{E}(x_{ij}|w_i^T \theta) := \hat{m}_j(w_i^T \theta),\)
\( \mathbb{E}(y_{2i}|w_i^T \theta) := m_0(w_i^T \theta) \), and \( \hat{\mathbb{E}}(y_{2i}|w_i^T \theta) := \hat{m}_0(w_i^T \theta) \). To estimate \( m_j(w_i^T \theta^*) \) for each \( j = 0, \ldots, p \), we first need some estimate \( \hat{\theta} \) of \( \theta^* \) in the selection equation. Supposing such an estimate is available, to obtain a (second-stage) estimator of \( m_j(w_i^T \theta^*) \), we consider the following least squares estimator

\[
\hat{m}_j \in \arg \min_{\hat{m}_j \in \mathcal{F}_j} \left\{ \frac{1}{n} \sum_{i=1}^{n} (z_{ij} - \hat{m}_j(w_i \hat{\theta}))^2 \right\},
\]

or the regularized least-squares estimator

\[
\hat{m}_j \in \arg \min_{\hat{m}_j \in \mathcal{F}_j} \left\{ \frac{1}{n} \sum_{i=1}^{n} (z_{ij} - \hat{m}_j(w_i \hat{\theta}))^2 + \lambda_{nj,2} |\hat{m}_j|^2_{\mathcal{F}_j} \right\},
\]

where \(| \cdot |_{\mathcal{F}_j} \) is a norm associated with the function class \( \mathcal{F}_j \) and \( \lambda_{nj,2} \geq 0 \) is a regularization parameter and \( z_{i0} = y_{2i} \) and \( z_{ij} = x_{ij} \) for each \( j = 1, \ldots, p \). The choice of \( \lambda_{nj,2} \) is specified in Section 4. A nonparametric regression problem based on (11) or (12) is a standard setup in many modern statistics books (e.g., van der Vaart and Wellner, 1996; van de Geer, 2000; Wainwright, 2015, etc).

In words, the solutions to program (11) are least-squares estimators based on imposing explicit constraints on the function class \( \mathcal{F}_j \). The function \( \hat{m}_j \) is chosen such that the vector

\[
( \hat{m}_j(w_1 \hat{\theta}), \ldots, \hat{m}_j(w_l \hat{\theta}), \ldots, \hat{m}_j(w_n \hat{\theta}) )
\]

is closest in \( l_2 \)-norm to the observation \((z_{1j}, \ldots, z_{ij}, \ldots, z_{nj})\) for \( j = 0, \ldots, p \) in terms of the “selected” sample. Examples of (11) include the linear regression as the simplest case, sparse linear regressions, convex regression where \( \mathcal{F}_j \) is the class of convex functions (e.g., Guntuboyina and Sen, 2013), Lipschitz and Isotonic regression where \( \mathcal{F}_j \) is the class of monotone Lipschitz functions (e.g., Kakade, Kalai, Kanade, and Shamir, 2011), etc. In general, this optimization problem defining the non-parametric least squares estimator \( \hat{m}_j \) is infinite-dimensional in nature, since \( \hat{m}_j \) ranges over the function class \( \mathcal{F}_j \). If the function class is “too large”, the solution may not exist, in which case \( \mathcal{F}_j \) is chosen to be a compact subset of some larger function class by introducing a ball radius in some norm. From the computational point of view, it is sometimes more convenient to implement estimators based on explicit penalization or regularization terms as in (12). Examples of (12) include kernel ridge regression where \(| \cdot |_{\mathcal{F}_j} \) is the norm associated with a reproducing kernel Hilbert space (see e.g., Gu, 2002; Berlinet and Thomas-Agnan, 2004; Wainwright, 2015), estimators based on series expansion (e.g., Cencov, 1962; Andrews, 1991; Newey, 1994, 1997), as well as sieves (e.g., van de Geer, 2000; Chen, 2008) and spline methods (e.g., Wahba, 1980, 1990). A procedure based on Lipschitz regression for the second-stage nonparametric estimation is illustrated in Section 4 for a leading case.

It is worth mentioning that although the theoretical guarantees of the multi-stage procedure provided by this paper requires the second-stage estimation to fit into either (11) or (12), other nonparametric methods including kernel density estimators, local polynomials, etc., could also be a valid second-stage estimator for the multi-stage procedure in the context of high-dimensional semiparametric selection models and verifying those methods both theoretically and empirically is an open question for future research.

**First-stage estimation**

Note that in the second-stage estimation of \( m_j(w_i^T \theta^*) \) for each \( j = 0, \ldots, p \), the coefficient vector \( \theta^* \) is unknown and needs to be replaced by some “consistent” first-stage estimate \( \hat{\theta} \). Parametric and semiparametric estimation of \( \theta^* \) in the classical low-dimensional settings when the dimension of \( \theta^* \) is small
relative to the sample size \( n \) is well-studied (see, e.g., Powell 1994; Pagan and Ullah, 1999). In the high-dimensional settings where the dimension of \( \theta^* \) grows with and exceeds \( n \), estimation of \( \theta^* \) in recent development of high-dimensional statistics has been focused on the case where \( \theta^* \) is either exactly sparse or approximately sparse, and a distributional assumption is imposed on the error term in the linear latent utility models in \( \mathbb{R}^d \). Theoretical guarantees have been established for the high-dimensional binary logit models in the context of Generalized Linear Models (GLM) and M-estimation (e.g., van de Geer, 2008; Bühlmann and van de Geer, 2011; Negahban, et. al, 2012; Loh and Wainwright, 2013). While the main theoretical results of this paper concern estimators of the high-dimensional linear coefficients \( \theta^* \) in the main equation and estimators of the selection bias function \( g(\cdot) \), we illustrate here and also in later sections the high-dimensional parametric estimation procedure for the binary logit and probit models as they are considered the work-horse of many empirical literatures and probit models are widely applied to study selection problems.

As for the high-dimensional sparse linear models, it is natural to consider the estimator based on the \( l_1 \)-regularized maximum likelihood for the binary logit and probit models, namely,

\[
\hat{\theta} \in \arg \min_{\theta \in \mathbb{R}^d} \left\{ -\frac{1}{n} \sum_{i=1}^{n} y_{1i} \phi_1(w_i^T \theta) + \frac{1}{n} \sum_{i=1}^{n} \phi_2(w_i^T \theta) + \lambda_{n,1} |\theta|_1 \right\} \tag{13}
\]

where \( n \) is the sample size of all observations. One can easily verify that when \( \phi_1(w_i^T \theta) = w_i^T \theta \) and \( \phi_2(w_i^T \theta) = \log(1 + \exp(w_i^T \theta)) \), the loss function in the above program corresponds to a binary logit model; when \( \phi_1(w_i^T \theta) = \log \frac{\phi(w_i^T \theta)}{1 - \Phi(w_i^T \theta)} \) and \( \phi_2(w_i^T \theta) = - \log \left[ 1 - \Phi(w_i^T \theta) \right] \) (\( \Phi(\cdot) \) is the standard normal c.d.f.), the loss function corresponds to a binary probit model. The loss function in (13) is written in terms of the negative of the likelihood and hence the optimization program is a convex minimization problem. This paper extends the analysis of the theoretical properties of these estimators from the high-dimensional binary logit models to the high-dimensional probit models, and focuses on the semiparametric estimation of \( \beta^* \) instead of \( \theta^* \). Developing semiparametric estimation techniques for the high-dimensional sparse discrete choice models based upon weak restrictions on the error distribution is left for future research.

**Remark.** Upon solving (13), strategies such as the thresholded-Lasso or the post-Lasso may be used before the second-stage estimation, which might boost the performance of the multi-stage estimator in certain situations.

### 3.2 Estimators of the selection bias function

Given the estimates \( \hat{\theta} \) and \( \hat{\beta} \) of \( \theta^* \) and \( \beta^* \), there are two ways to estimate the selection bias function \( g(w_i^T \theta^*) \). Recalling (2) from Section 1,

\[
y_{2i} = x_i^T \beta^* + g(w_i^T \theta^*) + \eta_i.
\]

where by construction \( \mathbb{E}[\eta_i | x_i, y_{1i} = 1] = 0 \). Taking the conditional expectation of the above leads to

\[
\mathbb{E}(y_{2i} | w_i^T \theta^*) = \mathbb{E}(x_i | w_i^T \theta^*) \beta^* + g(w_i^T \theta^*),
\]

and as a result,

\[
g(w_i^T \theta^*) = \mathbb{E}(y_{2i} | w_i^T \theta^*) - \mathbb{E}(x_i | w_i^T \theta^*) \beta^*, \tag{14}
\]
Recall from programs (11) and (12), modification to allow $F$ for notational simplicity, in the main theoretical results presented below, we assume the regime of inter-

4 Main theoretical results

For notational simplicity, in the main theoretical results presented below, we assume the regime of interest is $p \geq n$ and $d \geq n$ (i.e., the number of regressors grows with and exceed the sample size $n$). The modification to allow $p < n$ or $d < n$ is trivial. Also, as a general rule for this paper, all the $b$ constants denote positive constants that are independent of $n$, $p$, $d$, $R_{q_1}$ and $R_{q_2}$ but possibly depending on model specific parameters; all the $c$ constants denote universal positive constants that are independent of $n$, $p$, $d$, $R_{q_1}$ and $R_{q_2}$ as well as model specific parameters. The specific values of these constants may change from place to place.

Recall from programs (11) and (12), $\tilde{m}_j(\cdot) \in \mathcal{F}_j$. Suppose $m_j(\cdot) \in \mathcal{F}^*_j$, which may be different from $\mathcal{F}_j$. Define the shifted version of the function class $\mathcal{F}_j$

$$\bar{\mathcal{F}}_j := \left\{ f = f' - f'' : f', f'' \in \mathcal{F}_j \right\}.$$ 

The following assumptions are imposed to obtain the theoretical results in this section.

Assumption 4.1: For any $j = 0, \ldots, p$, $\bar{\mathcal{F}}_j$ is a star-shaped function class; i.e., for any $f \in \bar{\mathcal{F}}_j$, the entire line $\{ \alpha f, \alpha \in [0, 1] \}$ is also contained within $\bar{\mathcal{F}}_j$.

Remark. The star-shaped condition is often seen in literature of nonparametric statistics (see e.g., van der Vaart and Wellner, 1996; Wainwright, 2015; and other textbooks on mathematical statistics). It is relatively mild; for instance, it is satisfied whenever the set $\bar{\mathcal{F}}_j$ is convex and contains the function $f = 0$. It is also satisfied by various non-convex sets of functions, such as in the case of sparse linear regression.
Assumption 4.2: The random vector $v_j$ for $j = 0, ..., p$ is sub-Gaussian with parameter at most $\sigma_{v_j}$. The matrix $v \in \mathbb{R}^{n \times p}$ is sub-Gaussian with parameters $(\Sigma_v, \sigma^2_v)$ where the $j$th column of $v$ is $v_j$ and $\sigma_v := \max_{j=0,...,p} \sigma_{v_j}$.

Assumption 4.3: The random vector $\eta$ is sub-Gaussian with parameter at most $\sigma_\eta$.

Remark. In the literature of nonparametric estimation, common measures of function complexities associated with sub-Gaussian variables can be controlled with standard maximal inequalities as in van der Vaart and Wellner (1996) and van de Geer (2000), etc. There are some special cases of Assumptions 4.2 are 4.3 where other concentration results (e.g., Maurey, 1991; Ledoux, 1996; Bobkov, 1999; Bobkov and Ledoux, 2000) may provide sharper constants in the tail probability when we relax the identicalness of $\{w_i, x_i\}$ in Assumption 2.1. These special cases include: $v_j$ for $j = 0, ..., p$ and $\eta$ are (i) sub-Gaussian with strongly log-concave distribution (defined below) for some $\gamma_{v_j} > 0$ and $\gamma_\eta > 0$, respectively; or, (ii) a bounded vector such that for every $i = 1, ..., n$, $v_{ij}$ and $\eta$ are supported on the interval $(a'_{v_j}, a''_{v_j})$ with $B_{v_j} := a''_{v_j} - a'_{v_j}$, and on $(a'_{\eta}, a''_{\eta})$ with $B_\eta := a''_{\eta} - a'_{\eta}$; or, (iii) a mixture of (i) and (ii) in terms of its probability measure.

Definition 4.1 (Strongly log-concave distributions). A distribution $\mathbb{P}$ with density $p$ (with respect to the Lebesgue measure) is a strongly log-concave distribution if the function $\log p$ is strongly concave. Equivalently stated, the density can be written in the form $p(x) = \exp(-\psi(x))$, where the function $\psi : \mathbb{R}^n \to \mathbb{R}$ is strongly convex, meaning that there is some $\gamma > 0$ such that

$$\lambda \psi(x) + (1 - \lambda) \psi(y) - \psi(\lambda x + (1 - \lambda) y) \geq \frac{\gamma}{2} \lambda (1 - \lambda) |x - y|_2^2$$

for all $\lambda \in [0, 1]$, and $x, y \in \mathbb{R}^n$.

Remark. It is easy to verify that the distribution of a standard Gaussian vector in $n$ dimensions is strongly log-concave with parameter $\gamma = 1$. More generally, any Gaussian distribution with covariance matrix $\Sigma \succ 0$ is strongly log-concave with parameter $\gamma = \lambda_{\min}(\Sigma^{-1})$. In addition, there are a variety of non-Gaussian distributions that are also strongly log-concave.

4.1 Properties of the non-pivotal Lasso estimator of the high-dimensional linear coefficients

4.1.1 General upper bounds and $l_2$-consistency

The following theorem (Theorem 4.1) provides a general upper bound on the error $|\hat{\beta}_{HSEL} - \beta^*|_2$ when the second-stage estimation concerns a program as in (11). This result is an “oracle-inequality” type which does not assume the unknown function $m(\cdot)$ belongs to the function class over which the nonparametric estimator from (11) is defined. In such settings, the performance of the estimator involves both the estimation error and an approximation error, arising from the fact that $m_j \notin \mathcal{F}_j$.

To state Theorem 4.1, we need to introduce a set of definitions. First, we define a quantity that measures the complexity of the function class $\mathcal{F}_j$ (a notion often used in nonparametric literature; e.g., van der Vaart and Wellner, 1996; van de Geer, 2000; Barlett and Mendelson, 2002; Koltchinski, 2006; A random vector with bounded elements is sub-Gaussian.

2
Wainwright, 2015, etc.). For any radius \( r_j > 0 \), define the conditional local complexity

\[
G_n(r_j; F_j) := \mathbb{E}_{v_j} \left[ \sup_{f \in \Omega(r_j; F_j)} \left| \frac{1}{n} \sum_{i=1}^{n} v_{ij} f(w_i^T \theta^*) \right| |w_i^T \theta^*| \right],
\]

where variables \( \{v_{ij}\}_{i=1}^{n} \) for \( j = 0, ..., p \) are i.i.d. variates that satisfy Assumption 4.2, and

\[
\Omega(r_j; F_j) = \{ f : f \in \bar{F}_j | \|f\|_n \leq r_j \},
\]

where \( |f\|_n := \sqrt{\frac{1}{n} \sum_{i=1}^{n} |f(w_i^T \theta^*)|^2} \). For any star-shaped shifted function class \( \bar{F}_j \), the function \( t \mapsto G_n(t; F_j) \) is non-decreasing on the interval \( (0, \infty) \). Second, let \( T_j^* := \sup_{f \in F_j} \frac{1}{n} \sum_{i=1}^{n} \left[ f(w_i^T \hat{\theta}) - f(w_i^T \theta^*) \right]^2 \), \( T_j := \sup_{f \in F_j} \frac{1}{n} \sum_{i=1}^{n} \left[ f(w_i^T \hat{\theta}) - f(w_i^T \theta^*) \right]^2 \), \( T_j^* := T_j^* \vee T_j \), and

\[
T_1 = \max_{j \in \{0, ..., p\}} \left( T_j^* \vee \sqrt{T_j} \right),
\]

\[
T_2 = \max_{j \in \{0, ..., p\}} i_{nj}^2
\]

\[
T_3 = \max_{j \in \{0, ..., p\}} \inf_{m_j \in F_j} \left( \frac{1}{n} \sum_{i=1}^{n} \left[ \hat{m}_j(w_i^T \hat{\theta}) - m_j(w_i^T \theta^*) \right]^2 + \frac{1}{n} \sum_{i=1}^{n} \left[ \hat{m}_j(w_i^T \theta^*) - m_j(w_i^T \theta^*) \right]^2 \right)
\]

\[
T_4 = \sigma_0 \sigma_n \sqrt{\frac{\log p}{n}}.
\]

Third, recall in Section 2 the set we introduced:

\[
C(S; q_2, 3) := \{ \Delta \in \mathbb{R}^p : |\Delta S^c|_1 \leq 3|\Delta S|_1 + 4|\beta^*_{S^c}|_1 \},
\]

and the spherical set

\[
S_\delta := \{ \Delta \in \mathbb{R}^p : |\Delta|_2 \geq \delta \},
\]

and the intersection of these two sets \( C(S; q_2, 3) \cap S_\delta \). When \( \beta^* \) is approximately sparse (namely, \( q_2 \in (0, 1) \)), we choose \( S \in C(S; q_2, 3) \) to be the following thresholded subset

\[
S_{\varpi} := \{ j \in \{1, 2, ..., p\} : |\beta^*_j| > \varpi \}
\]

with the threshold parameter \( \varpi = \frac{\lambda_{n, 3}}{\kappa_L} \) (recall \( \lambda_{n, 3} \) is the third-stage regularization parameter whose choice is specified in the theorems and the parameter \( \kappa_L \) is defined in Assumption 2.3, Section 2). When \( \beta^* \) is exactly sparse (namely, \( q_2 = 0 \)), we set \( \delta = \varpi = 0 \) and choose \( S = J(\beta^*) \), which reduces the set \( C(S; q_2, 3) \cap S_\delta \) to the following cone:

\[
C(J(\beta^*); 0, 3) := \{ \Delta \in \mathbb{R}^p : |\Delta J(\beta^*)_v|_1 \leq 3|\Delta J(\beta^*)|_1 \}.
\]

**Theorem 4.1:** Let the critical radius \( r_{nj} > 0 \) be the smallest positive quantity satisfying the critical inequality

\[
G_n(r_{nj}; F_j) \leq \frac{r_{nj}^2}{\sigma_{v_j}}.
\]

Suppose the second-stage estimator solves program (\ref{eq:2}) and Assumptions 2.1, 2.2, 4.1-4.3 hold. Addition-
ally, let Assumption 2.3 hold over the restricted set $\mathcal{C}(J(\beta^*); 0, 3)$ for the exact sparsity case ($q_2 = 0$ with $R_{q_2} = k_2$), and over $\mathcal{C}(S_{\Delta}^2; q_2, 3) \cap S_3$ where $\delta \asymp R_{q_2}^{3/4} (\lambda_{n,3})^{1-\frac{2q}{3}}$ and $\bar{r} = \frac{\lambda_{n,3}}{\kappa L}$ for the approximate sparsity case ($q_2 \in (0, 1]$), respectively. For any $t_{nj} \geq r_{nj}$, if the third-stage regularization parameter $\lambda_{n,3}$ satisfies

$$
\lambda_{n,3} \geq b(\sigma_v, \sigma_\eta) |\beta^*|_1 (T_1 + T_2 + T_3) + T_3 := \bar{M},
$$

(17)

where $b(\sigma_v, \sigma_\eta)$ is a known function that only depend on the parameters $\sigma_v$ and $\sigma_\eta$ (and independent of $n, d, p, R_{q_2}$), and the condition

$$
R_{q_2}^{-q_2} \left( \frac{\log p}{n} + T_1 + T_2 + T_3 \right) = O(\kappa L),
$$

(18)

holds, then,

$$
|\hat{\beta}_{HSEL} - \beta^*|_2 \leq \frac{c'' R_{q_2}^{3/4}}{\kappa L} \left[ \bar{M} \vee \lambda_{n,3} \right]^{1-\frac{2q}{3}}
$$

(19)

with probability at least $1 - c_1 \exp(-c_2 \log p) - c_3 \sum_{j=0}^p \exp(-nC_j^* c_{nj}^2)$ for some $C_j^*$ independent of $n, d, p, R_{q_2}$.

The following theorem (Theorem 4.2) provides a general upper bound on the error $|\hat{\beta}_{HSEL} - \beta^*|_2$ when the second-stage estimation concerns a regularized program as in (12). As in Theorem 4.1, this result is an “oracle-inequality" type which does not assume the unknown function $m(\cdot)$ belongs to the function class over which the nonparametric estimator from (12) is defined. For Theorem 4.2, let the local complexity measure $\mathcal{G}_n (r_j; \mathcal{F}_j)$ be defined over the set

$$
\Omega(r_j; \mathcal{F}_j) = \left\{ f : f \in \mathcal{F}_j \mid |f_{\theta^*}|_n \leq r_j, |f|_{\mathcal{F}_j} \leq 1 \right\}
$$

where $|f_{\theta^*}|_n := \sqrt{\frac{1}{n} \sum_{i=1}^n [f(w_i^T \theta^*)]}$ and $j = 0, \ldots, p$. Also define the following quantities:

$$
\begin{align*}
T_1 &= \max_{j \in \{0, \ldots, p\}} \left( T_j' \vee \sqrt{T_j'} \right) \\
T_2 &= \max_{j \in \{0, \ldots, p\}} \frac{R_{nj}^{2q}}{r_{nj}} \\
T_3 &= \max_{j \in \{0, \ldots, p\}} \inf_{\hat{m}_j \in \mathcal{F}_j, \hat{m}_j \mid_{\mathcal{F}_j} \leq R_j} \left( \frac{1}{n} \sum_{i=1}^n \left[ \hat{m}_j(w_i^T \hat{\theta}) - m_j(w_i^T \hat{\theta}) \right]^2 + \frac{1}{n} \sum_{i=1}^n \left[ \hat{m}_j(w_i^T \theta^*) - m_j(w_i^T \theta^*) \right]^2 \right) \\
T_4 &= \sigma_v \sigma_\eta \frac{\sqrt{\log p}}{n},
\end{align*}
$$

where $T_j'$ is defined prior to the presentation of Theorem 4.1.

**Theorem 4.2**: Let the critical radius $r_{nj} > 0$ be the smallest positive quantity satisfying the critical inequality

$$
\mathcal{G}_n (r_{nj}; \mathcal{F}_j) \leq \frac{R_{nj}^{2q}}{\sigma_{v_j}},
$$

where $R_j > 0$ is a user-defined radius. Suppose the second-stage estimator solves the regularized program
and Assumptions 2.1, 2.2, 4.1-4.3 hold. Additionally, let Assumption 2.3 hold over the restricted set \( C(J(\beta^*); 0, 3) \) for the exact sparsity case \((q_2 = 0 \text{ with } R_{q_2} = k_2)\), and over \( C(S_{\tau}; q_2, 3) \cap \delta \) where \( \delta \asymp R_{q_2}^{\frac{1}{2}}(\lambda_{n,3})^{1-\frac{q_2}{2}} \) and \( \tau = \frac{\lambda_{n,3}}{\kappa_L} \) for the approximate sparsity case \((q_2 \in (0, 1])\), respectively. For any \( t_{n,j} \geq r_{n,j} \), if the second-stage regularization parameter \( \lambda_{n,j,2} = 2t_{n,j}^2 + \varsigma \) for any small positive constant \( \varsigma > 0 \) and the third-stage regularization parameter \( \lambda_{n,3} \) satisfies \( (17) \), and condition \( (18) \) holds, then, the upper bound \( (19) \) holds (where the terms \( T_k, k = 1, ..., 4 \) correspond to the ones defined for Theorem 4.2) with probability at least

\[
1 - c_1 \exp \left( -c_2 \log p \right) - c_3 \sum_{j=0}^p \exp \left( -nC_j^* R_{q_2}^2 t_{n,j} \right)
\]

for some \( C_j^* \) independent of \( n, d, p, R_{q_2} \).

**Comments:**

(a) For the probability guarantees in Theorems 4.1 and 4.2, the constant

\[
C_j^* = c \frac{\gamma_{v_j} \wedge (B_{v_j}^2 \lor B_{\eta}^2)^{-1}}{\sigma_{v_j}^2 \lor \sigma_{\eta}^2}
\]

when \( v_j \) for \( j = 1, ..., p \) and \( \eta \) are (i) sub-Gaussian with strongly log-concave distribution for some \( \gamma_{v_j} > 0 \) and \( \gamma_{\eta} > 0 \), respectively; or, (ii) a bounded vector such that for every \( i = 1, ..., n, v_{ij} \) and \( \eta \) are supported on the interval \( (a_{v_j}', a_{v_j}'') \) with \( B_{v_j} := a_{v_j}' - a_{v_j}'' \), and on \( (a_{\eta}', a_{\eta}'') \) with \( B_{\eta} := a_{\eta}' - a_{\eta}'' \); or, (iii) a mixture of (i) and (ii) in terms of its probability measure.

(b) Condition \( (16) \) in Theorems 4.1 and 4.2 ensures that with high probability, \( \hat{\beta}_{R,2} \) satisfies the RE condition \( (10) \) over \( C(J(\beta^*); 0, 3) \) for the exact sparsity case \((q_2 = 0 \text{ with } R_{q_2} = k_2)\), and over \( C(S_{\tau}; q_2, 3) \cap \delta \) where \( \delta \asymp R_{q_2}^\frac{1}{2}(\lambda_{n,3})^{1-\frac{q_2}{2}} \) and \( \tau = \frac{\lambda_{n,3}}{\kappa_L} \) for the approximate sparsity case \((q_2 \in (0, 1])\), respectively. An implication of this scaling condition is that it provides a finite-sample guarantee of the population identification condition (Assumption 2.3) subject to the underlying restricted sets. This result is formalized in the following corollary.

**Corollary 4.3:** Under the assumptions in Theorem 4.1 (respectively, the assumptions in Theorem 4.2), we have, with the same probability guarantees in Theorem 4.1 (respectively, in Theorem 4.2),

\[
\frac{1}{n} \sum_{i=1}^n y_{1i} \left( x_i - \hat{E} \left[ x_i \mid w_i^T \hat{\theta}, y_{1i} = 1 \right] \right) \left( x_i - \hat{E} \left[ x_i \mid w_i^T \hat{\theta}, y_{1i} = 1 \right] \right)^T
\]

is nonsingular on the restricted sets subject to those in Theorem 4.1 (respectively, Theorem 4.2).

**Remarks on Theorems 4.1 and 4.2**

The main proofs for Theorem 4.1, Theorem 4.2, and Corollary 4.3 are provided in Sections A.1-A.4. These theorems imply that if \( \lambda_{n,3} \asymp M \) and

\[
\frac{c^* R_{q_2}^{\frac{1}{2}}}{\kappa_L} \left[ b(\sigma_v, \sigma_{\eta}) \mid \beta^* \right]_1 (T_1 + T_2 + T_3) + T_4 \right]^{1-\frac{q_2}{2}} \to 0,
\]

as \( n \to \infty \), then the two-stage estimator \( \hat{\beta}_{HSEL} \) is \( l_2 \)-consistent for \( \beta^* \). From Theorems 4.1 and 4.2,
it can be seen that the general upper bounds on $|\hat{\beta}_{HSEL} - \beta^*|_2$ depend on four sources of errors, $T_k$, $k = 1, \ldots, 4$. The terms $T_1$, $T_2$, $T_3$, and $T_4$ are related to the statistical error of the first-stage estimation, the statistical error of the second-stage nonparametric regression, the approximation error arising from the fact that $m_j \notin \mathcal{F}_j$, and the statistical error of the third-stage estimation, respectively. Inspecting the error term $T_1$ suggests that, given appropriate identification assumptions, the upper bounds on $|\hat{\beta}_{HSEL} - \beta^*|_2$ in Theorems 4.1 and 4.2 also hold for the more general structure where $\mathbb{E}(\epsilon_{2i}|w_i, y_{1i} = 1) = g(h(w_i^T, \theta^*))$ and $h(w_i^T, \theta^*)$ is a scalar unobservable index.

The extra factor $|\beta^*|_1$ (in the case of exact sparsity,$|\beta^*|_1 \times k_2$) in front of $T_1$, $T_2$, and $T_3$ in the upper bounds on $|\hat{\beta}_{HSEL} - \beta^*|_2$ as well as in the choice of $\lambda_{n,3}$ is unimprovable and arising from the fact that the estimator is a sequential multi-stage procedure based on plugging the first-stage estimator $\hat{\theta}$ in the place of $\theta^*$. When $q_2 = 1$, the extra factor $|\beta^*|_1$ in front of $T_1$, $T_2$, and $T_3$ in the upper bounds on $|\hat{\beta}_{HSEL} - \beta^*|_2$ as well as in the choice of $\lambda_{n,3}$ is crucial in order for the argument in our analysis to go through. To see this, suppose $\sqrt{\log p/n}$ is small relative to $T_1$, $T_2$, and $T_3$, in which case, condition \[1\] can be reduced to

$$R_{q_2}(|\beta^*|_1)^{1-q_2} \max_k \left\{ T_k^{1-q_2} : k = 1, 2, 3 \right\} = O \left( \kappa_L^{1-q_2} \right).$$

When $q_2 = 1$, $R_{q_2} = |\beta^*|_1$ and $\sqrt{R_{q_2}(|\beta^*|_1)^{-q_2}} = 1$, so the above condition holds. On the other hand, when $q_2 \in [0, 1)$, condition \[1\] is easier to be satisfied.

When $T_1$, $T_2$, and $T_3$ are small relative to $T_4$ and $\beta^*$ is exactly sparse with at most $k_2$ non-zero coefficients, if we set $\kappa_L = \lambda_{\min}(\Sigma_v)$, the upper bounds in Theorems 4.1 and 4.2 reduce to $|\hat{\beta}_{HSEL} - \beta^*|_2 \lesssim \frac{\sigma_v\sigma_{\eta}}{\lambda_{\min}(\Sigma_v)} \sqrt{\frac{k_2 \log p}{n}}$. Note that the scaling $\sqrt{\frac{k_2 \log p}{n}}$ is the optimal rate of the Lasso for the usual high-dimensional linear regression model $[4]$ with exact sparsity, and the factor $\frac{\sigma_v\sigma_{\eta}}{\lambda_{\min}(\Sigma_v)}$ has a natural interpretation of an inverse signal-to-noise ratio when $v_i$ is a zero-mean Gaussian matrix with covariance $\Sigma_v = \sigma_v^2 I_p \times p$: one has $\lambda_{\min}(\Sigma_v) = \sigma_v^2$, so $\frac{\sigma_v\sigma_{\eta}}{\lambda_{\min}(\Sigma_v)} = \frac{\sigma_{\eta}}{\sigma_v}$, which measures the inverse signal-to-noise ratio of the regressors.

For the case of approximately sparse $\beta^*$ with $q_1$, $q_2 \in (0, 1]$, the rate

$$\frac{c'' R_{q_2}^{1-q_2} \kappa_L^{1-q_2}}{b(\sigma_v, \sigma_{\eta})} \left( T_1 + T_2 + T_3 + T_4 \right)^{1-q_2}$$

(20)

can be interpreted with the following heuristic. Suppose we choose to the top $s_2$ coefficients of $\beta^*$ in absolute values to estimate, then the fast decay imposed by the $l_{q_2}$—balls condition on $\beta^*$ would mean that the remaining $p - s_2$ coefficients would have relatively little impact. With this intuition, the rate for $q_2 > 0$ can be viewed as the rate that would be achieved by choosing

$$s_2 = \frac{c'' R_{q_2}^{1-q_2} \kappa_L^{1-q_2}}{b(\sigma_v, \sigma_{\eta})} \left( T_1 + T_2 + T_3 + T_4 \right)^{-q_2}$$

and then proceeding as if the problem were an instance of an exactly sparse problem $q_2 = 0$ with $k_2 = s_2$. For such a problem, we would expect to obtain the rate

$$\frac{c'' \sqrt{s_2}}{\kappa_L} \left[ b(\sigma_v, \sigma_{\eta}) \right] \left( T_1 + T_2 + T_3 + T_4 \right).$$

\[4\] Other plug-in type Lasso estimators for the exactly sparse case such as the ones in Rosenbaum and Tsybakov (2011) and the high-dimensional two-stage least-squares estimator in Zhu (2013), also involve the extra factor $|\beta^*|_1$. 

22
which is exactly equal to (20).

Notice that the choice of \( t_{nj} \) incurs a trade-off between \( T_2 \) and the term \( O \left( \sum_{j=0}^{p} \exp \left( -n t_{nj}^2 \right) \right) \) in the probability guarantees in Theorems 4.1 and 4.2. This is a general phenomenon for these tail bounds. For the problems considered in this paper, \( t_{nj} \) may be chosen in the way that \( T_2 \) is dominated by \( T_1, T_3, \) and \( T_4 \) while the probability guarantees are maximized to allow for the least restrictive requirement on the sample size for \( l_2 \)-consistency. Section 4.1.2 provides a specific example in terms of the choice of \( t_{nj} \). When we set \( t_{nj} = r_{nj} \), note that the statistical error related to the second-stage nonparametric regression, \( T_2 \), is on the order of \( O \left( \max_j r_{nj}^2 \right) \) instead of the optimal rate \( O \left( \max_j r_{nj} \right) \) that one would expect from a nonparametric regression as (11) or (12). As long as \( \max_j r_{nj} < 1 \), we have: \( \max_j r_{nj}^2 \leq \max_j r_{nj} \), and provided \( \max_j r_{nj}^2 \) is small relative to \( T_1, T_3, \) and \( T_4 \), the convergence rate of the estimator of the high-dimensional linear coefficients in the main equation behaves as if the unknown nonparametric selection bias were known.

This result establishes the non-asymptotic counterpart of the familiar asymptotic “oracle” type of results from previous literature. One of the drivers behind this oracle result lies on carefully controlling for the term \( \frac{1}{n} \sum_{i=1}^{n} \hat{v}_{ij} \left[ \tilde{m}_{ij} (w_i^T \hat{\theta}) - \tilde{m}_{ij} (w_i^T \hat{\theta}) \right] \) utilizing the fact that \( \hat{v}_{ij} \) estimates the true residual \( v_{ij} \) which is obtained by projecting \( x_{ij} \) onto \( w_i^T \theta^* \), namely, \( v_{ij} = x_{ij} - E \left( x_{ij} | w_i^T \theta^* \right) \) or \( \hat{v}_{ij} = y_{2i} - E \left( y_{2i} | w_i^T \theta^* \right) \).

When this projection procedure is applied to classical low-dimensional semilinear models with fixed \( p \) and \( d \) (in which case, there is no first-stage related error \( T_1 \)), our general upper bounds would imply that the nonparametric component needs to be estimated at a rate no slower than \( O \left( \sqrt{\frac{1}{n}} \right) \) in order for the estimator of the parametric component to achieve the rate of \( O \left( \frac{1}{n} \right) \). In contrast to the semilinear models, low-dimensional selection models require the rate of the nonparametric component to be at least \( O \left( \sqrt{\frac{1}{n}} \right) \) because the nonparametric component in the selection model involves an unknown single index that also needs to be estimated.

### 4.1.2 Upper bounds and \( l_2 \)-consistency for a leading case example

An important consequence of Theorems 4.1 and 4.2 is when \( m_j(\cdot) \in \mathcal{F}_j \) for every \( j = 0, \ldots, p \) and \( \mathcal{F}_j \) in

\[
T_j' := \sup_{f \in \mathcal{F}_j} \frac{1}{n} \sum_{i=1}^{n} \left[ f(w_i^T \hat{\theta}) - f(w_i^T \theta^*) \right]^2
\]

can be restricted to the class of Lipschitz functions, as a result, \( T_3 = 0 \) and \( T_j' = \frac{1}{n} \sum_{i=1}^{n} L^2 \left[ w_i^T \hat{\theta} - w_i^T \theta^* \right] \) is \( \mathcal{L}^2 B' \). Results regarding this leading case are provided in the following corollaries (Corollaries 4.4 and 4.5). Before stating these results, a procedure based on Lipschitz regression for the second-stage estimation is presented and its theoretical guarantees are provided in Corollaries 4.4 and 4.5.

We say that a function \( f : \mathbb{R} \to \mathbb{R} \) is \( L \)-Lipschitz if

\[
| f(t) - f(t') | \leq L | t - t' |
\]

for all \( t, t' \in \mathbb{R} \). When \( \mathcal{F}_j \) satisfies the Lipschitz assumption, we restrict \( \mathcal{F}_j \) in (11) to be the class of
Lipschitz functions and consider \( \hat{m}_j \) in this class only, namely,

\[
\hat{m}_j \in \arg \min_{\tilde{m}_j : \mathbb{R} \to \mathbb{R}} \left\{ \frac{1}{n} \sum_{i=1}^{n} (z_{ij} - \tilde{m}_j(\hat{w}_i \hat{\theta}))^2 \right\} \quad \text{for } j = 0, \ldots, p.
\]

\( \tilde{m}_j \) is \( L \)-Lipschitz

It can be easily verified that \( \hat{F}_j \), the shifted class of Lipschitz functions is also Lipschitz and satisfies Assumption 4.1; i.e., it is star-shaped. By exploiting the structure of Lipschitz functions, the program above can be converted to an equivalent finite-dimensional problem by applying the constraint (21) to each of the sampled points \( \hat{w}_i \hat{\theta} \) so that there must exist a real-valued vector \((\hat{z}_{1j}, \ldots, \hat{z}_{ij}, \ldots, \hat{z}_{nj})\) which satisfies the constraints in the following convex program

\[
(\hat{z}_{1j}, \ldots, \hat{z}_{ij}, \ldots, \hat{z}_{nj}) \in \arg \min_{(\hat{z}_{1j}, \ldots, \hat{z}_{ij}, \ldots, \hat{z}_{nj})} \left\{ \frac{1}{n} \sum_{i=1}^{n} (z_{ij} - \hat{z}_{ij})^2 \right\}
\quad \text{s.t. } \hat{z}_{ij} - z_{ij}^* \leq L \left( w_i - w_i^* \right)^T \hat{\theta} \quad \text{for all } i, i' = 1, \ldots, n.
\]

(22)

Given an optimal solution \((\hat{z}_{1j}, \ldots, \hat{z}_{ij}, \ldots, \hat{z}_{nj})\), a Lipschitz function \( \hat{m}_j \) can be constructed by interpolating linearly between \( \hat{z}_{ij} \)'s and the resulting function \( \hat{m}_j \) is an estimate of \( m_j \) (namely, the second-stage estimator). Moreover, one can easily see that \( \hat{m}_j(w_i^T \hat{\theta}) = \hat{z}_{ij} \). Note that the optimization problem above is a convex program with a quadratic cost function and a total of \( \binom{n}{2} \) linear constraints and \( n \) variables (\( n \) here denotes the sample size of the observations for the main equation). There are many computationally efficient algorithms for solving programs like this (e.g., the interior point method). When \( m_j(\cdot) \) is a monotonic Lipschitz function, we can impose additional monotonicity constraints together with the Lipschitz constraints in the above convex program. Kakade, et. al (2011) provides an algorithm with provable guarantees for this type of minimization problems.

In the case where the Lipschitz constant \( L \) is unknown, cross-validation methods can be used to determine \( L \). For example, we can first solve the optimization problem (22) on a subsample of observations by imposing an additional constraint \( 0 \leq L \leq L^{(0)} \) for a chosen constant \( L^{(0)} \) and obtain \((\hat{z}_{1j}, \ldots, \hat{z}_{ij}, \ldots, \hat{z}_{nj}, L) := \varpi^0 \). We then test for the prediction quality of this optimal solution \( \varpi^0 \) by comparing its predicted values (from interpolating linearly between \( \hat{z}_{ij} \)’s) for the remaining subsample with the actual observed values. If the optimal solution \( \varpi^0 \) returns \( L \approx L^{(0)} \), we can iterate the process by imposing \( 0 \leq L \leq L^{(1)} = 2L^{(0)} \) in (22) and comparing the new optimal solution \( \varpi^1 \) with the previous one \( \varpi^0 \) and also testing for the prediction quality of \( \varpi^1 \).

**Assumption 4.4:** The matrix \( w \) consists of bounded elements\(^4\).

The following proposition (Proposition 4.1) regarding the critical radius \( r_{nj} \) in Theorems 4.1 and 4.2 is based on results from van der Vaart and Wellner (1996), van de Geer (2000), and Wainwright (2015).

**Proposition 4.1:** Let Assumptions 2.1 and 4.4 hold and \( m_j(\cdot) \in \mathcal{F}_j \) for \( j = 0, \ldots, p \). Suppose \( \mathcal{F}_j \) belongs to the class of \( L \)-Lipschitz functions and the Lipschitz regression procedure (22) is applied. Then, for every \( j = 0, \ldots, p \), \( T_3 = 0 \) and \( T'_j = \frac{1}{n} \sum_{i=1}^{n} L^2 \left( w_i^T \hat{\theta} - w_i^T \theta^* \right)^2 := L^2 B' \), and the critical radius

\(^4\)A random matrix with bounded elements is sub-Gaussian.
The following corollaries (Corollaries 4.4 and 4.5) provide results regarding the leading case where for every \( j = 0, \ldots, p \), \( T_j' = \frac{1}{n} \sum_{i=1}^{n} L^2 \left| w_i^T \hat{\theta} - w_i^T \theta^* \right|^2 := L^2 B', T_3 = 0 \), and the critical radius \( r_{n,j} = O \left( \left( \frac{|\beta^*|_1}{n} \right)^{\frac{1}{3}} \right) \), in Theorems 4.1 and 4.2. These conditions are ensured by Proposition 4.1. The two corollaries differ by the upper bounds on the quantity \( B' \). Justifications of these upper bounds on \( B' \) are given by Propositions 4.2 and 4.3. Let \( \Upsilon_{w, \theta^*} \) be a known function depending only on \( w \) and \( \theta^* \). The quantity \( \Upsilon_{w, \theta^*} \) changes according to the assumptions on \( w \), which is to be made clear by Propositions 4.2 and 4.3. To facilitate the discussion and a later comparison with the minimax lower bounds in Section 4.4, the results in Corollaries 4.4 and 4.5 are presented for the case of exact sparsity on \( \beta^* \) and \( \theta^* \) (\( q_1 = q_2 = 0 \)). The case of general sparsity on \( \theta^* \) and \( \beta^* \) (\( q_1, q_2 \in [0, 1] \)) is presented in Corollary 4.6 (which contains Corollary 4.4 as a special case).

**Corollary 4.4** \((q_1 = q_2 = 0)\): Suppose \( \theta^* \) is exactly sparse with at most \( k_1 \) non-zero coefficients. Suppose for every \( j = 0, \ldots, p \), \( T_j' = \frac{1}{n} \sum_{i=1}^{n} L^2 \left| w_i^T \hat{\theta} - w_i^T \theta^* \right|^2 := L^2 B', T_3 = 0 \), and the critical radius \( r_{n,j} = O \left( \left( \frac{k_1}{n} \right)^{\frac{1}{3}} \right) \), and

\[
B' = \frac{1}{n} \sum_{i=1}^{n} \left| w_i^T \hat{\theta} - w_i^T \theta^* \right|^2 \leq c \Upsilon_{w, \theta^*} \frac{k_1 \log d}{n}
\]

with probability at least \( 1 - O \left( \frac{1}{n} \right) \). Assume \( t_{n,j}^2 \) in \( T_2 \) is chosen such that \( |(\beta^*)_1| T_2 \) is at most

\[
O \left( \sqrt{\frac{\log p}{n}} \sqrt{\left( \frac{|\beta^*|_1}{n} \right) \left( \frac{k_1 \log d}{n} \right)} \right)
\]

and \( n t_{n,j}^2 \gtrsim \log p \). Suppose Assumptions 2.1, 4.2-4.4 hold. Additionally, let \( \beta^* \) satisfy the exact sparsity in Assumption 2.2 \((q_2 = 0 \text{ with } R_{q_2} = k_2) \) and Assumption 2.3 hold over the restricted set \( \mathcal{C}(J(\beta^*); 0, 3) \). Assume

\[
\kappa_2 \frac{k_2 \log p}{n} + k_2 \sqrt{\frac{k_1 \log d}{n}} = O(\kappa_1),
\]

for some strictly positive constants \((\kappa_1, \kappa_2)\) depending only on \( \kappa_L, \sigma_v, \Upsilon_{w, \theta^*} \), and \( L \). If the third-stage regularization parameter \( \lambda_{n,3} \) satisfies

\[
\lambda_{n,3} \geq c \left( \sigma_v \sigma_\eta \sqrt{\frac{\log p}{n}} \right) \cap \left( L b(\sigma_v, \sigma_\eta) |(\beta^*)_1| \sqrt{\Upsilon_{w, \theta^*}} \sqrt{\frac{k_1 \log d}{n}} \right) := \mathcal{M}
\]

then, with probability at least \( 1 - O \left( \frac{1}{p^1/3} \right) \), we have

\[
|\hat{\beta}_{HSEL} - \beta^*|_2 \leq \frac{c_1 \sqrt{\kappa_2}}{\kappa_L} \left[ \mathcal{M} \cap \lambda_{n,3} \right]
\]

where \( b(\sigma_v, \sigma_\eta) \) is some known function depending only on \( \sigma_v \) and \( \sigma_\eta \) (and independent of \( n, d, p, k_1, \text{ and } k_2 \)).

The following assumptions and proposition provide an example in which the upper bound on \( B' \) in Corollary 4.4 is achieved. In particular, it requires the eigenvalues of \( \Sigma_w \) to be well-behaved over some restricted
set.

**Assumption 4.5:** In program (13), we have: either (a) \( \phi_1(w_i^T \theta) = w_i^T \theta \) and \( \phi_2(w_i^T \theta) = \log(1 + \exp(w_i^T \theta)) \); namely, the loss function corresponds to a binary logit model. Or, (b) \( \phi_1(w_i^T \theta) = \log \frac{\Phi(w_i^T \theta)}{1 - \Phi(w_i^T \theta)} \) and \( \phi_2(w_i^T \theta) = -\log [1 - \Phi(w_i^T \theta)] \); namely, the loss function corresponds to a binary probit model.

**Assumption 4.6:** The random matrix \( w \) is sub-Gaussian with parameters \( (\Sigma_w, \sigma_w^2) \). For all \( \Delta \in \mathbb{C}(J(\theta^*); 0, 3) \setminus \{0\} \), the matrix \( \Sigma_w \) satisfies

\[
0 < \kappa_L^w \Delta \Sigma_w \Delta \leq \kappa_U^w \Delta \leq \infty
\]

**Proposition 4.2:** Suppose the number of regressors \( d = d_n \) can grow with and exceed the sample size \( n \) and the number of non-zero coefficients in \( \theta^* \) is at most \( k_1(= k_{1n}) \) and \( k_1 \) can increase to infinity with \( n \) but slowly compared to \( n \). Let Assumptions 2.1, 4.5-4.6 hold. If \( \hat{\theta} \) solves program (13) with \( \lambda_{n,1} \geq c \sigma_w \sqrt{1/n} \log d/n \) and \( n \gtrsim k_1 \log d \), then, with probability at least \( 1 - O \left( \frac{1}{d} \right) \),

\[
\frac{1}{n} \sum_{i=1}^{n} \left[ w_i^T (\hat{\theta} - \theta^*) \right]^2 \leq c \frac{\kappa_U^w}{(\kappa_L^w)^2} k_1 \left( \lambda_{n,1} \right)^2 \vee \left( \sigma_w^2 \alpha_u \frac{\log d}{n} \right),
\]

where \( \alpha_u > 0 \) is a scalar such that \( \phi_2''(u) \leq \alpha_u \) for all \( u \in \mathbb{R} \).

**Remark.** From Proposition 4.2, we can set \( \Upsilon_{w,\theta^*} := \frac{\kappa_U^w \sigma_w^2 \alpha_u}{(\kappa_L^w)^2} \) in Corollary 4.4. The boundedness on \( \phi_2''(u) \) holds automatically for the binary logit model and binary probit model. For the logit model, we have \( \phi_2''(u_i) = \frac{\exp(u_i)}{1 + \exp(u_i)} \left( 1 - \frac{\exp(u_i)}{1 + \exp(u_i)} \right) \). For the probit model, note that \( \phi_2''(u_i) = 1 - \text{Var} \left( \epsilon_{i1} \mid \epsilon_{i1} \leq u_i \right) \) when \( y_{i1} = 1 \) and \( 1 - \text{Var} \left( \epsilon_{i1} \mid \epsilon_{i1} \geq -u_i \right) \) when \( y_{i1} = 0 \) and the unconditional variance is normalized to 1. Since truncation always reduces variances (Greene, 2003), \( \phi_2''(u) \) is bounded from above. If \( \lambda_{n,1} \asymp \sigma_w \sqrt{1/n} \log d/n \), then

\[
\frac{1}{n} \sum_{i=1}^{n} \left[ w_i^T (\hat{\theta} - \theta^*) \right]^2 \leq c \frac{\kappa_U^w}{(\kappa_L^w)^2} \sigma_w^2 \alpha_u \frac{k_1 \log d}{n}.
\]

**Corollary 4.5.** Suppose \( \theta^* \) is exactly sparse with at most \( k_1 \) non-zero coefficients. Suppose for every \( j = 0, ..., p \), \( T_j' = \frac{1}{n} \sum_{i=1}^{n} L^2 \left[ w_i^T \hat{\theta} - w_i^T \theta^* \right]^2 := L^2 B', \mathcal{T}_3 = 0 \), the critical radius \( r_{nj} = O \left( \left( \frac{k_{1n}}{n} \right)^{\frac{3}{2}} \right) \), and

\[
B' = \frac{1}{n} \sum_{i=1}^{n} \left[ w_i^T \hat{\theta} - w_i^T \theta^* \right]^2 \leq c \Upsilon_{w,\theta^*} |\theta^*|_1 \sqrt{\frac{\log d}{n}}
\]

with probability at least \( 1 - O \left( \frac{1}{d} \right) \). Assume \( t_{nj}^2 \) in \( \mathcal{T}_2 \) is chosen such that \( |\beta^*|_1 \mathcal{T}_2 \) is at most

\[
O \left( \sqrt{\frac{\log p}{n}} \vee \left( \left| \beta^* \right|_1 \left( \frac{k_{1n}^2 \log d}{n} \right)^{\frac{3}{4}} \right) \right)
\]

and \( n t_{nj}^2 \gtrsim \log p \). Suppose Assumptions 2.1, 4.2-4.4 hold. Additionally, let \( \beta^* \) satisfy the exact sparsity in Assumption 2.2 \( (q_2 = 0 \text{ with } R_{q2} = k_2) \) and Assumption 2.3 hold over the restricted set \( \mathbb{C}(J(\beta^*); 0, 3) \).
Assume

\[ \frac{k_2}{n} \log \frac{\kappa_2}{n} + k_2 \left( \frac{k_2 \log d}{n} \right)^{\frac{1}{4}} = O(\kappa_1), \]

for some strictly positive constants \((\kappa_1, \kappa_2)\) depending only on \(\kappa_L, \sigma_v, \Upsilon_{w, \theta^*}\), and \(L\), if the third-stage regularization parameter \(\lambda_{n,3}\) satisfies

\[ \lambda_{n,3} \geq c' \left( \sigma_v \sigma_\eta \sqrt{\frac{\log p}{n}} \right) \lor \left( Lb(\sigma_v, \sigma_\eta) |\beta^*|_1 \sqrt{\Upsilon_{w, \theta^*}} \left( \frac{|\theta^*|^2 \log d}{n} \right)^{\frac{1}{4}} \right) := \tilde{M} \]

then, with probability at least \(1 - O \left( \frac{1}{n} \right) \), we have

\[ |\hat{\beta}_{HSEL} - \beta^*|_2 \leq \frac{c_2 \sqrt{k_2}}{\kappa_L} [\tilde{M} \lor \lambda_{n,3}] \]

where \(b(\sigma_v, \sigma_\eta)\) is some known function depending only on \(\sigma_v\) and \(\sigma_\eta\) (and independent of \(n, d, p, k_1, \) and \(k_2\)).

The following proposition provides an example in which the upper bound on \(B'\) in Corollary 4.5 is achieved. Let \(\rho_i, \theta := -y_i \phi_1(w_i^T \theta) + \phi_2(w_i^T \theta)\) and \(\rho''_{i, \theta}\) be the second derivative of \(\rho_{i, \theta}\), evaluated at \(\theta = \overline{\theta}\), where \(\overline{\theta}\) is some intermediate value between \(\theta^*\) and \(\theta\), the solution to program \([12]\). Assumption 4.4 implies that there is some \(\alpha_i > 0\) such that \(\rho''_{i, \theta} \geq \alpha_i\) for all \(i = 1, ..., n\).

**Proposition 4.3:** Let Assumptions 2.1, 4.4 and 4.5 hold. Suppose the number of regressors \(d(= d_n)\) can grow with and exceed the sample size \(n\) and the number of non-zero components in \(\theta^*\) is at most \(k_1(= k_{1n})\) and \(k_1\) can increase to infinity with \(n\) but slowly compared to \(n\). If \(\hat{\theta}\) solves program \([12]\) with the regularization parameter \(\lambda_{n,1} \geq c \sqrt{\frac{\log d}{n}}\), then,

\[ \frac{1}{n} \sum_{i=1}^{n} \left[ w_i^T (\hat{\theta} - \theta^*) \right]^2 \leq c_1 \Upsilon_{w, \theta^*} |\theta^*|_1 \left( \sqrt{\frac{\log d}{n}} \lor \lambda_{n,1} \right) \]

with probability at least \(1 - O \left( \frac{1}{n} \right) \), where \(\Upsilon_{w, \theta^*} := \alpha_i^{-1}\).

**Remarks on Corollary 4.4-4.5**

The proofs for Corollaries 4.4-4.5 and Propositions 4.1-4.3 are provided in Sections A.5-A.7.

Corollaries 4.4 and 4.5 imply that if \(\lambda_{n,3} \approx \tilde{M}\) and the upper bounds on \(|\hat{\beta}_{HSEL} - \beta^*|_2\) tend to 0 as \(n \to \infty\), then the two-stage estimator \(\hat{\beta}_{HSEL}\) is \(l_2\)-consistent for \(\beta^*\). The difference between Corollary 4.4 and Corollary 4.5 lies in that the statistical error from the first-stage estimation is smaller in Corollary 4.4 relative to Corollary 4.5 and as a result, the estimator \(\hat{\beta}_{HSEL}\) has a faster rate of convergence in Corollary 4.4. The smaller first-stage statistical error in Corollary 4.4 is at the expense of imposing conditions on the eigenvalues of \(\Sigma_w\), as shown in Proposition 4.2. Consistency of \(\hat{\beta}_{HSEL}\) *per se* does not require restrictions on the eigenvalues of \(\Sigma_w\), which could be useful in certain applications. Proposition 4.3 provides an example where a slower rate of convergence is obtained by the first-stage estimator upon relaxing the assumptions on the eigenvalues of \(\Sigma_w\).

\(^5\) The choice of \(\lambda_{n,1}\) is detailed in Theorems 2.1 or 2.2 in Van de Geer (2008).
By Proposition 4.1, \( \max_j r^2_{nj} = O \left( \left( \frac{k}{n} \right)^{\frac{2}{3}} \right) \). Let us examine various choices of \( t^2_{nj} \geq r^2_{nj} \) in Corollary 4.4 (the analysis for Corollary 4.5 is similar). Setting \( t^2_{nj} \leq \sqrt{\log p/\|\beta^*\|_1} \) and \( \sqrt{n}k_1 \log d \lesssim \log p \) makes the second-stage error \( |\beta^*|_1 \mathcal{T}_2 \) on the same order of \( \sqrt{\log p/\|\beta^*\|_1} \). Under this choice of \( t^2_{nj} \), we require \( \sqrt{n} \log \frac{p}{\|\beta^*\|_1} \lesssim \log p \) in order for the upper bound on \( |\beta^*|_1 \mathcal{T}_2 \) to hold with probability at least \( 1 - O \left( \frac{1}{p^d} \right) \). Setting \( t^2_{nj} \leq \left( \frac{\log p \sqrt{\log \frac{p}{\|\beta^*\|_1} \log d}}{n} \right)^{\frac{2}{\nu}} \lesssim r^2_{nj} \) requires \( n^{\frac{2}{\nu}} \left( \log p \sqrt{\log \frac{p}{\|\beta^*\|_1} \log d} \right)^{\frac{2}{\nu}} \lesssim \log p \) for the upper bound on \( |\beta^*|_1 \mathcal{T}_2 \) to hold with probability at least \( 1 - O \left( \frac{1}{p^d} \right) \). If instead, we set \( t^2_{nj} = r^2_{nj} \), then the probability guarantee of \( 1 - O \left( \frac{1}{p^d} \right) \) would require \( k_1^2 \approx \log p \). Given the exact sparsity of \( \beta^* \) (so \( |\beta^*|_1 \approx k_1^2 \)), if \( k_1^2 \) is sufficiently small relative to \( n \log p \), the first choice of \( t^2_{nj} \) would provide the least restrictive requirement on the sample size. A later result that concerns with the selection consistency of \( \hat{\beta}_{HSEL} \) assumes this choice for \( t^2_{nj} \) and the scaling condition \( \sqrt{n} \log \frac{p}{\|\beta^*\|_1} \lesssim \log p \) on the sample size. When \( p \) and \( d \) are fixed and small relative to \( n \), the analysis above generalizes existing asymptotic “oracle” results in semiparametric estimation of low-dimensional selection models from specific estimators (such as a series estimator) to a unified framework of nonparametric least squares estimators and regularized nonparametric least squares estimators.

More generally, when \( \mathcal{F}_j \) belongs to a Hölder class of order \( \nu > 0 \), we have \( \max_j r^2_{nj} = O \left( \left( \frac{k_1}{n} \right)^{\frac{2\nu}{2\nu + 1}} \right) \). When \( \nu \geq 1 \), \( \mathcal{T}_1 \approx \sqrt{k_1 \log \frac{p}{n}} \) and as long as we choose \( t^2_{nj} \leq \sqrt{\log p/\|\beta^*\|_1} \) and \( \sqrt{n}k_1 \log d \lesssim \log p \), the second-stage error \( |\beta^*|_1 \mathcal{T}_2 \) and consequently the upper bound on \( |\hat{\beta}_{HSEL} - \beta^*|_2 \) would be on the same order of \( \sqrt{\log p/\|\beta^*\|_1} \). On the other hand, when \( \nu \in (0, 1) \), we have \( 2\nu \sqrt{\log \frac{p}{\|\beta^*\|_1} \log d} > \frac{\nu}{2} \) and \( \mathcal{T}_1 \approx \left( \sqrt{\nu} \right)^{\nu} \). Provided \( B' \approx O \left( \frac{1}{n} \right) \) (which is indeed the case for Corollaries 4.4 and 4.5) and the choice of \( t^2_{nj} \leq \mathcal{T}_1 \sqrt{\nu} \), then \( |\hat{\beta}_{HSEL} - \beta^*|_2 \) is bounded above by \( \mathcal{T}_1 \sqrt{\nu} \). Consequently, the minimum requirement for the “oracle” result to hold in the low-dimensional semiparametric selection models with fixed \( p \) and \( d \) is to have \( \nu = 1 \). For the high-dimensional selection models considered in Corollary 4.4, the minimum requirement is to have \( O \left( \sqrt{n} \log d \right)^{\frac{2}{\nu}} = O \left( \sqrt{\log p} \right) \).

In sharp contrast to the low-dimensional semilinear model, the fact that the nonparametric component in the selection model involves an unknown single index that also needed to be estimated increases the requirement on the rate of nonparametric estimation per se.

Note that the regularization parameter \( \lambda_{n,3} \) and the upper bounds on \( |\hat{\beta}_{HSEL} - \beta^*|_2 \) depend on \( \sigma_v \) and \( \sigma_{\eta} \), which is intuitive. It is possible to “remove” the dependence on \( \sigma_v \) from the choice of \( \lambda_{n,3} \) by imposing weights \( \hat{\sigma}_{v_j} := \sqrt{\frac{1}{n} \sum_{i=1}^{n} v^2_{ij}} \), \( j = 1, \ldots, p \) on the penalty term as in \( \mathcal{F}_j \). An application of Lemmas A.11 and A.12 yields \( \max_{j=1, \ldots, p} \hat{\sigma}_{v_j} \leq 2\sigma_v \) with probability at least \( 1 - O \left( \frac{1}{p^d} \right) \). The first-stage estimator \( \hat{\theta} \) in Corollaries 4.4 and 4.5 may be replaced with a post-Lasso estimator where a usual low-dimensional estimation procedure is performed on the regressors selected by \( \hat{\theta} \) (in a spirit similar to Belloni and Chernozhukov, 2011b, for example); and upon perfect selection or near-perfect selection\(^6\) of

\(^6\) Ravikumar, Wainwright, and Lafferty (2010) studies selection of a \( l_1 \)-regularized logistic regression in the high-dimensional setting.
Corollary 4.6 \((q_1, q_2 \in [0, 1])\): Suppose for every \(j = 0, ..., p\), \(T_j = \frac{1}{n} \sum_{i=1}^{n} L^2 \left[ w_i^T \hat{\theta} - w_i^T \theta^* \right]^2 \) := \(L^2 B', \mathcal{T}_3 = 0\), and the critical radius \(r_{nj} = O \left( \left( \frac{\theta^*}{n} \right)^{\frac{1}{2}} \right) \). Also, assume \(\theta^* \in \mathcal{B}^d_q(R_{q_1})\) for \(q_1 \in [0, 1]\) with “radius” \(R_{q_1}\), and

\[B' = \frac{1}{n} \sum_{i=1}^{n} \left[ w_i^T \hat{\theta} - w_i^T \theta^* \right]^2 \leq c \gamma_{w, \theta^*} R_{q_1} \left( \frac{\log d}{n} \right)^{1 - \frac{q_1}{2}}\]

with probability at least \(1 - O \left( \frac{1}{\delta} \right)\). Assume \(t_{nj}^2 \) in \(\mathcal{T}_2\) is chosen such that \(|\beta^*|_1 \mathcal{T}_2\) is at most \(O(\bar{M})\), where

\[\bar{M} := \max \left\{ \sqrt{\frac{\log p}{n}}, |\beta^*|_1 \left( \frac{\theta^*}{n} \right)^{\frac{1}{2}}, |\beta^*|_1 R_{q_1}^\frac{1}{2} \left( \frac{\log d}{n} \right)^{1 - \frac{q_1}{2}} \right\}\]

and \(nt_{nj}^2 \gtrsim \log p\). Moreover, condition \(\text{[15]}\) and Assumptions 2.1, 2.2, 4.2-4.4 hold. Additionally, let Assumption 2.3 hold over the restricted set \(\mathbb{C}(J(\beta^*); 0, 3)\) for the exact sparsity case \((q_2 = 0 \text{ with } R_{q_2} \propto k_2)\), and over \(\mathbb{C}(S_{\bar{S}}; q_2, 3) \cap S_\delta\) where \(\delta \propto R_{q_2}^\frac{1}{2} (\lambda_{n,3})^{1 - \frac{q_2}{2}}\) and \(\tau = \frac{\lambda_{n,3}}{K_L}\) for the approximate sparsity case \((q_2 \in (0, 1])\), respectively. If the third-stage regularization parameter \(\lambda_{n,3} \geq b_0 \bar{M}\), then, with probability at least \(1 - O \left( \frac{1}{p^{\kappa_2}} \right)\), we have

\[|\hat{\beta}_{HSEL} - \beta^*|_2 \leq \frac{b_1 \sqrt{R_{q_2}}}{K_L} \left( \bar{M} \vee \lambda_{n,3} \right)^{1 - \frac{q_2}{2}}\]

where \(b_0\) and \(b_1\) are some known functions depending only on \(\sigma_v, \sigma, \gamma_{w, \theta^*}\), and \(L\) (and independent of \(n, d, p, R_{q_1}\), and \(R_{q_2}\)).

Comment on Corollary 4.6. Corollary 4.6 contains Corollary 4.4 as a special case with \(q_2 = 0, R_{q_2} = k_2\) and \(q_1 = 0\). When \(q_1 = 0\) so that \(R_{q_1} = k_1\) and \(|\theta^*|_1 \propto k_1\), the second term in \(\bar{M}\) is of the order of \(O \left( \left( \frac{\theta^*}{n} \right)^{\frac{1}{2}} \right)\) and therefore dominated by \(\sqrt{\frac{\log p}{n}} \vee |\beta^*|_1 \sqrt{\frac{k_1 \log d}{n}}\), as we have seen previously. For more general sparsity of \(\theta^* (q_1 \in (0, 1])\), the second term in \(\bar{M}\) may still be small relative to the first and third terms and therefore the “oracle” result is likely to hold for a range of scaling conditions on \(n, p, d, R_{q_1}\), and \(|\theta^*|_1\).

4.1.3 Variable-selection consistency of a leading case example with exact sparsity

The following theorem (Theorem 4.7) addresses the question: given \(\hat{\beta}_{HSEL}\), when does \(\hat{\beta}_{HSEL}\) correctly select the non-zero coefficients in the main equation with high probability? This property is referred to as variable-selection consistency, which is relevant to case of exactly sparse \(\beta^*\) (and therefore this section assumes \(\beta^*\) is exactly sparse with at most \(k_2\) non-zero coefficients). We say \(\hat{\beta}_{HSEL}\) achieves perfect selection if \(\mathbb{P}[J(\hat{\beta}_{HSEL}) = J(\beta^*)] \to 1\) and near-perfect selection if \(\mathbb{P}[J(\hat{\beta}_{HSEL}) \supset J(\beta^*)] \to 1\) and the
number of wrong components selected is on the order of $o_p(k_2)$. Upon perfect selection or near-perfect selection of the regressors, we can then apply low-dimensional techniques to estimate and conduct inference on the important coefficients.

In order for the number of wrong components selected by the standard Lasso to be on the order of $o_p(k_2)$ in the context of standard linear regression models, it is known that the so-called “neighborhood stability condition” (Meinshausen and Bühlmann, 2006) on the design matrix, re-formulated in a nicer form as the “irrepresentable condition” by Zhao and Yu, 2006, or the “mutual incoherence condition” by Wainwright (2009), is sufficient and necessary. Furthermore, it can be shown that the “irrepresentable condition” implies the RE condition (see, e.g., Bühlmann and van de Geer, 2011).

**Assumption 4.7:**

\[
\mathbb{E} \left[ v_{1,J(\beta^*)}^T v_{1,J(\beta^*)} \right] \mathbb{E} \left[ (v_{1,J(\beta^*)} v_{1,J(\beta^*)})^{-1} \right] \leq 1 - \phi \text{ for some constant } \phi \in (0, 1].
\]

Assumption 4.7, the so-called “mutual incoherence condition” originally formalized by Wainwright (2009), captures the intuition that the large number of irrelevant covariates cannot exert an overly strong effect on the subset of relevant covariates. In the most desirable case, the columns indexed by $j \in J(\beta^*)^c$ would all be orthogonal to the columns indexed by $j \in J(\beta^*)$ and then we would have $\phi = 1$. In the high-dimensional setting, this perfect orthogonality is hard to achieve, but one can still hope for a type of “near orthogonality” to hold.

Assumptions 2.1 and 4.2 ensure that the left-hand-side of the inequality in Assumption 4.7 always falls in $[0, 1)$. To see this, note that under Assumptions 2.1 and 4.2, each column $v_j$, $j = 1, ..., p$ is consisted of i.i.d. sub-Gaussian variables. Without loss of generality, we can assume $\mathbb{E}(v_{ij}) = 0$ for all $j = 1, ..., p$. Consequently, the normalization $\max_{j=1,...,p} \frac{|v_j|}{\sqrt{n}} \leq \kappa_c$ where $0 < \kappa_c < \infty$ follows from a standard bound for the norms of zero-mean sub-Gaussian vectors and a union bound

\[
P \left[ \max_{j=1,...,p} \frac{|v_j|}{\sqrt{n}} \leq \kappa_c \right] \geq 1 - 2 \exp(-cn + \log p) \geq 1 - 2 \exp(-c'n),
\]

where the last inequality follows from $n > \log p$. For example, if $v_j$ has a Gaussian design, then we have

\[
\max_{j=1,...,p} \frac{|v_j|}{\sqrt{n}} \leq \max_{j=1,...,p} \frac{\log p}{n},
\]

where $\max_{j=1,...,p} \Sigma_{jj}$ corresponds to the maximal variance of any element of $v$ (see Raskutti, et. al, 2011).

**Theorem 4.7:** Under the assumptions in Corollary 4.4 and Assumption 4.7, if $n \gtrsim (k_3^2 \log p) \vee (k_2^2 k_1 \log d)$, $\sqrt{n \log p} \vee \sqrt{n k_1 \log d} \gtrsim \log p$, $\sqrt{k_1 \log d} = o(1)$, and $\lambda_{n,3}$ satisfies

\[
\lambda_{n,3} \geq c \left( \frac{2}{\phi} \right) \left( \sigma_v \sigma_\eta \sqrt{\log p} \right) \vee \left( L \beta^* \|_1 \sqrt{T_{w,\theta} b(\sigma_v, \sigma_\eta)} \sqrt{k_1 \log d} \right),
\]

then, we have: (a) the support $J(\hat{\beta}_{HSEL}) \subseteq J(\beta^*)$; (b) if $\min_{j \in J(\beta^*)} |\beta^*_j| > \tilde{B}$, where

\[
\tilde{B} := \frac{c \sqrt{k_2}}{\lambda_{\min} \left( \mathbb{E} \left[ v_{1,J(\beta^*)}^T v_{1,J(\beta^*)} \right] \right)} \left( \sigma_v \sigma_\eta \sqrt{\log p} \right) \vee \left( L \beta^* \|_1 \sqrt{T_{w,\theta} b(\sigma_v, \sigma_\eta)} \sqrt{k_1 \log d} \right)
\]

then $J(\hat{\beta}_{HSEL}) \supseteq J(\beta^*)$ and hence $\hat{\beta}_{HSEL}$ is variable-selection consistent, i.e., $J(\hat{\beta}_{HSEL}) = J(\beta^*)$, with
probability at least $1 - O \left( \frac{1}{p \log d} \right)$.

**Remark.** The proof for Theorem 4.7 is provided in Section A.8. Part (a) of Theorem 4.7 guarantees that the Lasso does not falsely include elements that are not in the support of $\beta^*$. This result hinges on Assumption 4.7, namely, the mutual incoherence condition. Part (b) implies that as long as the minimum value of $|\beta^*_j|$ over $j \in J(\beta^*)$ is not too small, then the two-stage Lasso does not falsely exclude elements that are in the support of $\beta^*$ with high probability. Combining the claims from (a) and (b), the multi-stage estimator is variable-selection consistent with high probability.

**Inference with perfect or near perfect selection**

When the mutual incoherence condition and the assumption that the true parameters $\beta^*_j$ over $j \in J(\beta^*)$ is well separated from 0 are plausible for the empirical problem of interest, conditioning on the perfect selection or near-perfect selection result from Theorem 4.7, we can then apply low-dimensional techniques to conduct inference on the important coefficients. In the following discussion, we consider the simple case where $k_1$ and $k_2$ are fixed. Then, for example, one can apply the estimator

$$\tilde{\beta} := \left( \hat{v}_j^T \hat{v}_j \right)^{-1} \left( \hat{v}_j^T \hat{v}_0 \right)$$

where $J := J(\hat{\beta}_{HSEL})$. In the multi-stage procedure proposed by this paper, if the second-stage nonparametric estimation uses the series estimator from Newey (1991), then the post-selection estimator $\tilde{\beta}$ can be shown to be algebraically equivalent to the two-stage estimator of Newey (1991) for the semiparametric selection models when the linear coefficients in the main equation is low-dimensional. In deriving the $\sqrt{n}$-consistency and the asymptotic normality of the two-stage estimator, Newey requires $\sqrt{n}$-consistency on the first-stage estimator of the coefficients in the selection equation. This suggests that in order for the results from Newey (1991) to be applied on the estimator $\tilde{\beta}$, perfect selection or near-perfect selection of $\hat{\theta}$ defined in (15) may be required. We may consider a variant of $\hat{\beta}_{HSEL}$. This variant differs from $\hat{\beta}_{HSEL}$ in that, before the second-stage estimation, a post-Lasso procedure is performed on the regressors selected by the first-stage estimator $\hat{\theta}$ to obtain $\tilde{\theta}$, which is then used to form the single index. Rather than imposing perfect selection or near-perfect selection of $\hat{\theta}$, another option is to use the procedure proposed by Ahn and Powell (1993), which does not require $\sqrt{n}$-consistency on the first-stage estimator and may allow imperfect selection of $\hat{\theta}$. For all these post-selection estimators discussed here, the asymptotic covariance matrix is rather complicated as it involves the derivative of the unobservable selection function. Ahn and Powell (1993) proposes a plug-in estimator for the asymptotic covariance matrix. Alternatively, bootstrap variance estimation can be used to obtain the standard errors of these post-selection estimators.

It is worth noting that while selection-consistency is a desirable property of the Lasso that allows us to conduct post-selection inference, it requires assumptions such as the mutual incoherence condition or the irrepresentable condition which might not hold in economic problems where the design matrices exhibit strong (empirical) correlations. When selection consistency is not achieved by the Lasso procedure, other inference procedures may be useful. While it is possible to construct confidence intervals for individual coefficients (e.g., Belloni, Chernozhukov, and Hansen, 2014) and linear combinations of several of them in certain high-dimensional models using a low dimensional projection approach (e.g., El Karoui, 2013; Zhang and Zhang, 2013; Javanmard and Montanari, 2014), general inference theory with high-dimensional data is still underexplored owing to the complexity of the sampling distributions of existing estimators (see e.g., Efron, 2010). Rather than relying on distributional theory to conduct inference, The following section (Section 4.2) provides an alternative way of constructing confidence sets based on the pivotal Dantzig
selector from Section 3. Although developing inference and asymptotic theory for low-dimensional parameters in the high-dimensional selection models is not the focus of this paper, it makes an interesting topic for future research.

4.2 The pivotal Dantzig selector of the high-dimensional linear coefficients and confidence sets

The pivotal Dantzig selector was originally proposed by Gautier and Tsybakov (2011) in the context of high-dimensional IV regression. For the particular case of this paper where the instruments are the fitted regressors $\hat{v}$ themselves, this pivotal estimator is an extension of the Dantzig selector to accommodate for the fact that the variance of the noise $\eta$ is unknown. It can be related to the square-root Lasso of Belloni, Chernozhukov, and Wang (2010) and Belloni, Chernozhukov, and Wang (2014). The non-asymptotic bounds derived in this section only apply to the case of exactly sparse $\beta^*$. However, all these results can be extended to the case of approximately sparse $\beta^*$ by applying analysis similar to those from previous sections. The confidence sets are the by-products of the non-asymptotic bounds on the pivotal estimator. Construction of confidence sets is based on the following theorem (Theorem 4.8), which uses a bound for moderate deviations of self-normalized sums of random variables established by Jing, Shao and Wang (2003). This tool was first applied by Belloni, Chen, and Chernozhukov (2010) and later by Gautier and Tsybakov (2011) as well as Belloni, Chernozhukov, and Wang (2014). The following assumption is needed for this deviation bound to be applied in obtaining Theorem 4.8.

Assumption 4.8: For all $i = 1, ..., n, j = 1, ..., p$ and some constant $\delta' > 0$, $\mathbb{E} \left[ |v_{ij}\eta_i|^{2+\delta'} \right] < \infty$ and neither of $v_{ij}\eta_i$ is almost surely equal to 0.

Define

$$b_{n,\delta'} := \min_{ j=1, \ldots, p} \sqrt{ \sum_{i=1}^{n} \mathbb{E} \left[ v_{ij}^2\eta_i^2 \right] \left( \mathbb{E} \left[ |v_{ij}\eta_i|^{2+\delta'} \right] \right)^{1/(2+\delta')}}.$$  \hspace{1cm} \text{(23)}

Given, for $j = 1, ..., p$, the variables $v_{ij}\eta_i$ are i.i.d., we have

$$b_{n,\delta'} := n^{\frac{\delta'}{4+2\delta'}} \min_{j=1,\ldots,p} \sqrt{ \mathbb{E} \left[ v_{ij}^2\eta_i^2 \right] \left( \mathbb{E} \left[ |v_{ij}\eta_i|^{2+\delta'} \right] \right)^{1/(2+\delta')}}.$$  \hspace{1cm} \text{(24)}

For $\alpha \geq 1$, set

$$\alpha = 2L \left( 1 - \Phi \left( a\sqrt{2 \log p} \right) \right) + 2a_0 \frac{(1 + \alpha \sqrt{2 \log p})^{1+\delta'}}{p^{\frac{1}{2} - \delta'}}$$

where $a_0 > 0$ is the absolute constant from the formula (2.11) in Jing, Shao and Wang (2003), and $\Phi(\cdot)$ is the standard normal c.d.f.

Notation. For Theorem 4.8, define the quantities $\check{Q}(\beta) := \frac{1}{n} |\hat{v}_0 - \hat{v}\beta|_2^2$, and the $l_2$-sensitivity

$$\kappa_{J(\beta^*)}^\ast = \inf_{\Delta \in C(J(\beta^*),\emptyset, \varphi)} \frac{1}{2} |\hat{v}^T \hat{v}\Delta|_\infty |\Delta|_2.$$
for some \( \varphi > 1 \). Recall from Section 4.1 the notation

\[
B' := \frac{1}{n} \sum_{i=1}^{n} \left[ w_i^T \hat{\theta} - w_i^T \theta^* \right]^2 \leq c \Upsilon_{w, \theta^*} \frac{k_1 \log d}{n}
\]

where \( \Upsilon_{w, \theta^*} \) is a known function depending only on \( w \) and \( \theta^* \), and from Section 3 the notations

\[
v_{j*} := \max_{i \in \{1, \ldots, n\}} \{ |2x_{ij}| \lor |\hat{v}_{ij}| \}
\]

for \( j = 1, \ldots, p \), and \( D \) the diagonal \( p \times p \) matrix with diagonal entries \( v_{j*}^{-1} \), \( j = 1, \ldots, p \).

**Remark.** Under Assumptions 4.2 and 4.3, the condition \( \mathbb{E} \left[ |v_{ij} \eta_i|^{2+\delta} \right] < \infty \) is implied by the fact that \( v_{ij} \) (for all \( j = 1, \ldots, p \)) and \( \eta \) are sub-Gaussian. To see this, note that the random variable \( v_{ij} \eta_i \) is sub-Exponential (using the fact that the product of two sub-Gaussian variables is sub-Exponential) and one of the characterizations of sub-Exponential variables says a zero-mean random variable \( X \) is sub-Exponential if and only if the quantity \( \sup_{k \geq 2} \left[ \frac{\mathbb{E}(X^k)}{k^2} \right]^{1/k} \) is finite (see, e.g., Wainwright, 2015).

**Theorem 4.8:** Suppose the assumptions in Corollary 4.4 and Assumption 4.8 hold. For \( a \geq 1 \), choose \( \alpha \) as in (24) and set the tuning parameter

\[
\xi \geq a \max \left\{ c_0 \sqrt{\log p \over n}, \left( \hat{Q}(\beta^*) \right)^{-\frac{1}{2}} \left| \beta^* \right|_1 \min_{j=1, \ldots, p} v_{j*} \right\}
\]

(25)

where \( c_0 > 1 \) and \( b(\sigma_v) \) is some known function depending only on \( \sigma_v \). If \( p \leq \exp \left( \frac{b^2}{2\alpha^2} \right) \), then with probability at least \( 1 - \alpha - O \left( \frac{1}{p^{\alpha}} \right) \), for any solution \((\hat{\beta}, \hat{\sigma})\) of program (10), we have

\[
\left| D^{-1}(\hat{\beta} - \beta^*) \right|_2 \leq \frac{1}{K_{J(\beta^*)}^2} \left[ \frac{Lb(\sigma_v) \sqrt{B'}}{\min_{j=1, \ldots, p} v_{j*}} \left| \beta^* \right|_1 + 2\xi \hat{\sigma} \left[ 1 - \frac{\xi^2}{K_{J(\beta^*)}} \right]^{-1} \right]^2 
\]

\[
\cdot \left[ 1 - \frac{1}{K_{J(\beta^*)}^3} \left( \frac{Lb(\sigma_v) \sqrt{k_2 B'}}{\min_{j=1, \ldots, p} v_{j*}^2} \right) \left[ 1 - \frac{\xi^2}{K_{J(\beta^*)}} \right]^{-1} \right]^{-1}. \tag{26}
\]

and, for all \( j = 1, \ldots, p \),

\[
\left| \hat{\beta}_j - \beta^*_j \right| \leq \frac{1}{v_{j*} K_{J(\beta^*)}} \left[ \frac{Lb(\sigma_v) \sqrt{B'}}{\min_{j=1, \ldots, p} v_{j*}} \left| \beta^* \right|_1 + 2\xi \hat{\sigma} \left[ 1 - \frac{\xi^2}{K_{J(\beta^*)}} \right]^{-1} \right]^2 
\]

\[
\cdot \left[ 1 - \frac{1}{K_{J(\beta^*)}^2} \left( \frac{Lb(\sigma_v) \sqrt{k_2 B'}}{\min_{j=1, \ldots, p} v_{j*}^2} \right) \left[ 1 - \frac{\xi^2}{K_{J(\beta^*)}} \right]^{-1} \right]^{-1}. \tag{27}
\]

Furthermore,

\[
C \hat{\sigma} \leq |\Delta_{J(\beta^*)}|_1 + C \sqrt{\hat{Q}(\beta^*)}
\]
therefore the upper bounds (26)-(27) are:

\[ L_b \eta \]

case, the scaling of the tuning parameter needs to match the scaling of the maximum of the first-stage

\[ \eta \]

has the same scaling as the choice of \( \eta \). In special cases, we may be able to circumvent the fact that

\[ \tilde{\eta} \]

appears in the bounds (26)-(27), we will replace it with \( L_b \). In the term

\[ \log \frac{p}{n} \]

for the non-pivotal Lasso estimator in Corollary 4.4 (in either case, the scaling of the tuning parameter needs to match the scaling of the maximum of the first-stage related error and the third-stage related error) except that the choice of \( \xi \) does not involve the unknown variance of \( \eta \) (and hence pivotal). In addition, notice that the upper bounds (26)- (27) are also pivotal to the unknown variance of \( \eta \). The only terms that can involve unknown parameters in the choice of \( \xi \) and therefore the upper bounds (26)- (27) are:

\[ L_b(\sigma_v)\sqrt{B'} \]

\[ \left( \hat{Q}(\beta^*) \right)^{-\frac{1}{2}} |\beta^*|_1. \]

The term \( L_b(\sigma_v)\sqrt{B'} \) is relatively easy to deal with: \( b(\sigma_v) \) can be replaced with

\[ b(\hat{\sigma}_v) := b \left( \max_j \sqrt{\frac{1}{n} \sum_{i=1}^n \hat{v}_{ij}^2} \right) \]

and an application of Lemma A.11 yields \( \hat{\sigma}_v \leq 2\sigma_v \) with probability at least \( 1 - O \left( \frac{1}{n} \right) \); construction of confidence intervals (that do not contain any unknown parameters) for \( B' \) has been considered in the context of several Generalized Linear models (see, e.g., Van de Geer, 2008) and we will assume in this discussion that these confidence sets \( \hat{B}' \) for \( B' \) are available. Consequently, whenever the term \( L_b(\sigma_v)\sqrt{B'} \) shows up in the bounds (26)-(27), we will replace it with \( L_b(\hat{\sigma}_v)\sqrt{B'} \). In the case where the constant \( L \)

is unknown, Section 4.1.2 discusses methods to determine this constant.

The term \( \left( \hat{Q}(\beta^*) \right)^{-\frac{1}{2}} |\beta^*|_1 \) is the harder one here as \( \beta^* \) is unknown and in fact the parameters we want to estimate. One possibility is to consider the following heuristic:

1. In Step \( k = 0 \) (initialization), solve program (10) with \( \xi^k = c_0 \sqrt{\log \frac{p}{n}} \) to obtain \( \hat{\beta}^k \) for some \( c_0 > 1 \); update \( \xi^k \) with

\[ \xi^{k+1} = a \max \left\{ c_0 \sqrt{\log \frac{p}{n}}, \left( \hat{Q}(\hat{\beta}^k) \right)^{-\frac{1}{2}} |\hat{\beta}^k|_1 \min_{j=1,\ldots,p} v_{jv^*} \right\}. \quad (29) \]

2. In Step \( k + 1 \), solve program (10) with \( \xi^{k+1} \) to obtain \( \hat{\beta}^{k+1} \) and update \( \xi^{k+1} \) with \( \xi^{k+2} \) as in (29).

Establishing theoretical guarantees for the heuristic provided above is pursued in a separate ongoing project. In special cases, we may be able to circumvent the fact that \( \beta^* \) is unknown. For example, when \( p \)
is large relative to $d$ so that $|\beta^*|_1 \sqrt{\frac{k_1 \log d}{n}} \ll \sqrt{\frac{\log p}{n}}$, then the result in Theorem 4.8 is essentially reduced to the case where the pivotal Dantzig selector is applied to the standard high-dimensional linear models with exact sparsity. In a related scenario where a post-Lasso procedure is performed on the regressors selected by the first-stage estimator $\hat{\theta}$ defined in [13], upon perfect selection or near-perfect selection of $\hat{\theta}$, the factor $\sqrt{\frac{k_1 \log d}{n}}$ is reduced to $\sqrt{\frac{k_1}{n}}$ which may be smaller relative to $\sqrt{\frac{\log p}{n}}$.

4.3 Properties of the estimators of the selection bias function

Given the availability of estimates $\hat{\theta}$ and $\hat{\beta}$ of the high-dimensional linear coefficients from either the non-pivotal procedure or the pivotal procedure, two different estimation strategies for the nonparametric selection bias are considered: one is the closed form estimator [15] and the other is the plug-in non-pivotal procedure or the pivotal procedure, two different estimation strategies for the nonparametric to the case where the pivotal Dantzig selector is applied to the standard high-dimensional linear models Section 4.4, we break down the presentations of the results into the case of exact sparsity on $\beta^*$ and $\theta^*$ ($q_1 = q_2 = 0$) in Theorems 4.9 and 4.10, and the case of general sparsity on $\beta^*$ and $\theta^*$ ($q_1, q_2 \in [0, 1]$) in Theorems 4.11 and 4.12 (which contain Theorems 4.9 and 4.10 as special cases, respectively).

**Theorem 4.9** ($q_1 = q_2 = 0$): Let the assumptions in Corollary 4.4 hold. Suppose $g(\cdot)$ belongs to the class $\mathcal{F}$ of Lipschitz functions. For the estimator $\hat{g}(\cdot)$ of $g(\cdot)$ obtained by [15],

$$
\left( \mathbb{E} \left[ \hat{g}(w_i^T \hat{\theta}) - g(w_i^T \theta^*) \right]^2 \right)^{\frac{1}{2}} \leq cb \max \left\{ k_2 \sqrt{\frac{\log p}{n}}, \frac{1}{|\beta^*|_1}, k_2 |\beta^*|_1 \sqrt{\frac{k_1 \log d}{n}} \right\}
$$

where $b$ is some constant depending only on the model-specific structure (and independent of $n$, $d$, $p$, $k_1$, and $k_2$).

**Theorem 4.10** ($q_1, q_2 \in [0, 1]$): Let the assumptions in Corollary 4.6 hold. Suppose $g(\cdot)$ belongs to the class $\mathcal{F}$ of Lipschitz functions. For the estimator $\hat{g}(\cdot)$ of $g(\cdot)$ obtained by [15],

$$
\left( \mathbb{E} \left[ \hat{g}(w_i^T \hat{\theta}) - g(w_i^T \theta^*) \right]^2 \right)^{\frac{1}{2}} \leq cb \max \left\{ R_{q_2} \tilde{M}^{1-q_2}, |\beta^*|_1 \sqrt{\frac{k_1 \log d}{n}} \right\}
$$

where $\tilde{M}$ is defined in Corollary 4.6 and $b$ is some constant depending only on the model-specific structure (and independent of $n$, $d$, $p$, $R_{q_1}$, and $R_{q_2}$).

**Theorem 4.11** ($q_1 = q_2 = 0$): Let the assumptions in Corollary 4.4 hold. Suppose $g(\cdot)$ belongs to the class $\mathcal{F}$ of Lipschitz functions and the random matrix $x$ is sub-Gaussian with parameters $(\Sigma_x, \sigma^2_x)$. For all $\Delta \in \mathbb{C}(J(\beta^*); 0, 3)\setminus\{0\}$, the matrix $\Sigma_x$ satisfies $\frac{\Delta^T \Sigma_x \Delta}{|\Delta|^2} \leq \sigma^2_U < \infty$. For the estimator $\hat{g}(\cdot)$ of $g(\cdot)$ obtained by [16],

$$
\left( \mathbb{E} \left[ \hat{g}(w_i^T \hat{\theta}) - g(w_i^T \theta^*) \right]^2 \right)^{\frac{1}{2}} \leq c \beta^* \max \left\{ \sqrt{\frac{k_2 \log p}{n}}, |\beta^*|_1 \sqrt{\frac{k_1 k_2 \log d}{n}}, \left( \frac{k_1}{n} \right)^{\frac{1}{3}} \right\}
$$

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where \( b' \) is some constant depending only on the model-specific structure (and independent of \( n, d, p, k_1, \) and \( k_2 \)).

**Theorem 4.12** \((q_1, q_2 \in [0, 1])\): Let the assumptions in Corollary 4.6. Suppose \( g(\cdot) \) belongs to the class \( F \) of Lipschitz functions and the random matrix \( x \) is sub-Gaussian with parameters \((\Sigma_x, \sigma_x^2)\). For all non-zero \( \Delta \in C(S_{\Delta}; q_2, 3) \cap S_\delta \) where \( C(S_{\Delta}; q_2, 3) \cap S_\delta \) is defined in Corollary 4.6, the matrix \( \Sigma_x \) satisfies \( \frac{\Delta^T \Sigma_x \Delta}{|\Delta|^2} \leq k_U^x < \infty \). For the estimator \( \hat{g}(\cdot) \) of \( g(\cdot) \) obtained by (16),

\[
\left( \mathbb{E} \left[ \hat{g}(w_i^T \theta) - g(w_i^T \theta^*) \right]^2 \right)^{\frac{1}{2}} \leq c' b' \max \left\{ \sqrt{R_{q_2} \tilde{M}^{1 - q_2}}, \left( \frac{|\theta^*|_1}{n} \right)^{\frac{1}{2}} \right\},
\]

where \( \tilde{M} \) is defined in Corollary 4.6 and \( b' \) is some constant depending only on the model-specific structure (and independent of \( n, d, p, R_{q_1}, \) and \( R_{q_2} \)).

**Remark.** The proofs for Theorems 4.9-4.12 are provided in Sections A.10 and A.11. First let us look at the case of exactly sparse \( \beta^* \) and \( \theta^* \) \((q_1 = q_2 = 0)\). From Theorem 4.11, notice that the terms \( \sqrt{\frac{k_2 \log p}{n}} \) and \( |\beta^*|_1 \sqrt{\frac{k_1 k_2 \log d}{n}} \) are expected from the statistical error of \( \hat{\theta} \) that we plug into the nonparametric regression (16); and the term \( \left( \frac{k_1}{n} \right)^{\frac{1}{4}} \) is expected from the fact that \( g(\cdot) \) belongs to the class \( F \) of Lipschitz functions.

On the other hand, the term \( \left( \frac{k_1}{n} \right)^{\frac{1}{4}} \) is suppressed by \( \left( \frac{k_1 \log d}{n} \right)^{\frac{1}{4}} \) in Theorem 4.9 for the closed-form estimator (15). When \( \beta^* \) is approximately sparse with \( q_2 = 1 \), Theorem 4.10 implies that the \( \sqrt{\text{MSE}} \) of the closed-form estimator (15) is bounded above by \( R_{q_2} \tilde{M}^{1 - q_2} = |\beta^*|_1 \). This upper bound is unimprovable and as a result, it is not possible for (15) to achieve \( \text{MSE}\)-consistency even if \( n \to \infty \) when \( q_2 = 1 \). In contrast to (15), the nonparametric least squares estimator (16) is consistent in \( \text{MSE} \) as \( n \to \infty \) when \( q_2 = 1 \). The key behind the sharp rate achieved by the plug-in nonparametric least squares estimator (16) in Theorems 4.11 and 4.12 lies on the random variables

\[
\frac{1}{n} \sum_{i=1}^{n} \eta_i \left[ \hat{g}(w_i^T \theta^*) - g(w_i^T \theta^*) \right],
\]

and

\[
U_n := \sup_{\delta \in \mathcal{S}(r_1, r_2)} \frac{1}{n} \left| \eta^T w \delta \right|,
\]

where

\[
\mathcal{S}(r_1, r_2) := \left\{ \delta \in \mathbb{R}^d \mid |\delta|_1 \leq r_1, \ |\delta|_2 \leq r_2 \right\}.
\]

The analysis for controlling the first term uses a “local function complexity” argument similar to what is done in the proofs for Theorems 4.1 and 4.2. To upper bound the second term \( U_n \), we can apply a discretization argument over the set \( \mathcal{S}(r_1, r_2) \) together with results on metric entropy and the fact \( \mathbb{E} [\eta_i | w_i] = 0 \). “Small” values of \( r_1 \) and \( r_2 \) are guaranteed by the upper bounds on \( |\hat{\theta} - \theta^*|_2 \) from Lemma A.7. As a result we only need to work with a “small” \( \mathcal{S}(r_1, r_2) \). The sharp rates provided by these types of analysis seem to be driven by the projection nature of the underlying nonparametric least-squares

\[\text{Note that when } \epsilon_{1i} \text{ and } \epsilon_{2i} \text{ in (14) are bivariate normal, the selection bias characterized by the Inverse Mills Ratio is a 1-Lipschitz function (see, e.g., Ruud, 2000). Furthermore, if } m_i(\cdot) \in F_j \text{ and } F_j \text{ is the class of } L \text{-Lipschitz functions for every } j = 1, ..., p, \text{ then } g(\cdot) \text{ is a Lipschitz function.}\]
estimator. As we will see in the following section, the overall convergence rate of the estimator \( \hat{\beta} \) (obtained by either the non-pivotal procedure or the pivotal procedure) and the plug-in nonparametric least squares estimator (16) is minimax optimal in terms of the \((n, d, p)\)-scaling for the case of exactly sparse \( \beta^* \). However, we will also see that this minimax optimality result does not apply to the case of approximately sparse \( \beta^* \) because of the first-stage related estimation error.

### 4.4 Statistical efficiency via lower bounds on minimax risks

This section studies efficiency of the proposed estimators by deriving lower bounds on minimax rate for the case of \( l_2 \)-loss. Complementary to the understanding of computationally efficient procedures are the information-theoretic limitations of statistical estimation, applicable to any procedure regardless of its computational cost. There is a rich literature on such information-theoretic limits, which can have two types of consequences. First, they can reveal gaps between the performance of an optimal procedure in theory and known computationally efficient methods. Second, they can demonstrate regimes in which practical procedures achieve these information-theoretic limits. While one way of interpreting minimax lower bounds is to view the choice of unknown parameters in an adversarial manner, and to compare the estimators based on their worst-case performance, many techniques for deriving minimax lower bounds can be motivated by the Bayesian approach which views the unknown parameters as random variables (e.g., Guntuboyina 2011).

The minimax lower bounds in this section are derived for model (2), implied by the original selection model (1). As a consequence, these lower bounds provide information-theoretic limits for any procedure constructed based on model (2) for estimating model (1), regardless of its computational cost. For \( q_1, q_2 \in [0, 1] \), define \( \mathcal{H} = \mathcal{B}^0_{q_2}(R_{q_2}) \times \mathcal{F} \circ \mathcal{B}^d_{q_1}(R_{q_1}) \), where the \( l_q \)-“ball” is defined in Section 2 and \( \mathcal{F} \) is the class of functions such that \( g \in \mathcal{F} : \mathbb{R} \to \mathbb{R} \). When \( \beta^* \in \mathcal{B}^0_{q_2}(k_2) \) and \( \theta^* \in \mathcal{B}^0_{q_1}(k_1) \), model (2) corresponds to the case of exact sparsity on \( \beta^* \) and \( \theta^* \). When \( \beta^* \in \mathcal{B}^0_{q_2}(R_{q_2}) \) and \( \theta^* \in \mathcal{B}^d_{q_1}(R_{q_1}) \) for \( q \in (0, 1] \), model (2) corresponds to the case of approximate sparsity based upon imposing a certain decay rate on the ordered entries of \( \beta^* \) and \( \theta^* \). Theorem 4.13 (Theorem 4.14) presents a minimix lower bound for the case of exact sparsity \( q_1 = q_2 = 0 \) (respectively, the case of approximate sparsity \( q_1, q_2 \in (0, 1] \)).

**Assumption 4.9:** There exists a constant \( \kappa_\infty > 0 \) and a function \( f_i(R_{q_2}, q_2, n, p) \) such that

\[
\frac{1}{\sqrt{n}} |x|_2 \geq \kappa_\infty |\beta|_2 - f_i(R_{q_2}, q_2, n, p) \quad \text{for all } \beta \in \mathcal{B}^0_{q_2}(R_{q_2}).
\]

**Assumption 4.10:** There is no measurable function \( f(w_1^T \theta) \) such that \( x_i^T \lambda = f(w_1^T \theta) \) when \( y_{1i} = 1 \) for \( \lambda \in \mathcal{B}^d_{q_2}(R_{q_2}) \setminus \{0\} \).

**Remark.** Assumptions 4.9 and 4.10 ensure the identifiability of model (2), without which, lower bounds for high-dimensional linear models usually involve a maximum of two quantities: a term involving the diameter of the null-space restricted to the \( l_q \)-ball, measuring the degree of non-identifiability of the model, and a term arising from the metric entropy structure for \( l_q \)-balls (see Raskutti, et. al, 2011). Assumption 4.9 together with Assumption 4.10 incurs an upper bound on the \( \mathcal{B}^0_{q_2}(R) \)–kernel diameter in \( l_2 \)–norm (this result is formalized in Lemma A.10 and proved in Section A.12), and consequently the identifiability of model (2).

---

*In fact, a recent paper by Chatterjee (2014) shows that the least squares estimators are always admissible up to a universal constant in many modern statistics problems.*
Theorem 4.13 \((q_1 = q_2 = 0)\): Let \(\mathcal{F}\) be the class of \(L\)-Lipschitz functions and Assumptions 4.9-4.10 hold with \(f_l(R_{q_2}, q_2, n, p) = 0\) and \(\kappa_x > 0\). Define the parameter space \(\Theta\) as

\[
\{ \theta \in B_0^p(k_1) : \text{for any } \lambda \in B_0^p(k_2) \setminus \{0\}, \exists \text{ a measurable } g(\cdot) \in \mathcal{F} \text{ s.t. } x_i^T \lambda = g(w_i^T \theta) \text{ when } y_{1i} = 1 \}.
\]

Moreover, \(|w_i\theta| \leq \kappa_u\) for all \(\theta \in \Theta\) and \(|\beta_i| \leq \kappa_u^\prime\) for all \(\beta \in B_0^p(k_2)\). If the vector \(\eta \sim N(0, \sigma_\eta I_{n \times n})\), then, for some constant \(b\) depending only on the model-specific structure (and independent of \(n, d, p, k_1,\) and \(k_2\)),

\[
\min_{\beta, f, \theta} \max_{\theta \in \Theta} \begin{bmatrix} f(\cdot) \in \mathcal{F} \\ \beta \in B_0^p(k_2) \end{bmatrix} \geq b \max \left\{ \sqrt{\frac{k_1 \log d}{n}}, \left(\frac{k_1}{n}\right)^\frac{1}{4}, \sqrt{\frac{k_2 \log p}{n}} \right\}.
\]

Theorem 4.14 \((q_1, q_2 \in (0, 1])\): Let \(\mathcal{F}\) be the class of \(L\)-Lipschitz functions and Assumptions 4.9-4.10 hold with \(f_l(R_{q_2}, q_2, n, p) = o \left(\frac{1}{R_{q_2}^q \left(\log p \right)} \right)^{\frac{3}{4} - \frac{q_2}{2}}\) and \(\kappa_x > 0\). Moreover, \(\frac{1}{\sqrt{n}} \max_{j=1,...,d} |x_j| \leq \kappa_w < \infty\) and \(\frac{1}{\sqrt{n}} \max_{j=1,...,d} |x_j| \leq \kappa_x < \infty\). If the vector \(\eta \sim N(0, \sigma_\eta I_{n \times n})\), then, for some constant \(b^*\) depending only on the model-specific structure (and independent of \(n, d, p, R_{q_1},\) and \(R_{q_2}\))

\[
\min_{\beta, f, \theta} \max_{\theta \in \Theta} \begin{bmatrix} f(\cdot) \in \mathcal{F} \\ \beta \in B_{q_2}^p(R_{q_2}) \end{bmatrix} \geq b^* \max \left\{ \frac{1}{R_{q_1}^q \left(\log d \right)^{\frac{2-q_1}{2}}}, \left(\frac{R^*}{n}\right)^\frac{1}{4}, \frac{R_{q_2}^q \left(\log p \right)}{n} \right\},
\]

where the parameter space \(\Theta\) is defined in Theorem 4.13 with \(B_0^p(k_1)\) replaced by \(B_{q_1}^p(R_{q_1})\) and \(B_0^p(k_2)\) replaced by \(B_{q_2}^p(R_{q_2})\), and \(R^*\) is the “radius" \(R_{q_1}\) when \(q_1 = 1\).

Remark. The proofs for Theorem 4.13 and Theorem 4.14 are provided in Sections A.12 and A.13, respectively. These proofs are information-theoretic in nature and based on Fano’s inequality (see, e.g., Guntuboyina, 2011; Wainwright, 2015) and results on the metric entropy of the \(l_q\)-balls. By Lemma A.10, the conditions on \(f_l(R_{q_2}, q_2, n, p)\) in Theorems 4.13 and 4.14 together with Assumption 4.10 ensure that the kernel diameter for the nullspace of \(B_0^p(R_{q_2})\) is dominated by the term related to the metric entropy of \(B_0^p(R_{q_2})\). In Theorem 4.13, we require Assumption 4.9 to hold with \(f_l(R_{q_2}, q_2, n, p) = 0\) and \(\kappa_x > 0\), which is closely related to the restricted eigenvalue condition on the matrix \(\frac{x^T x}{n}\) over the set \(B_0^p(k_2)\). When \(x\) is a sub-Gaussian matrix with parameters \((\Sigma_x, \sigma^2_x)\) and for all \(\Delta \in B_0^p(k_2) \setminus \{0\}\), the matrix \(\Sigma_x\) satisfies \(\frac{\Delta^T \Sigma_x \Delta}{|\Delta|^2} \geq \kappa_x > 0\), then Lemma B.2 guarantees Assumption 4.9 to hold for \(f_l(R_{q_2}, q_2, n, p) = 0\) with high probability. Additionally, if \(w\) is a sub-Gaussian matrix with parameters \((\Sigma_w, \sigma^2_w)\), \(\frac{\Delta^T \Sigma_w \Delta}{|\Delta|^2} \leq \tilde{\kappa}^\prime w < \infty\)
for all $\theta \in \Theta$, and $\frac{\Delta T \Sigma \Delta}{\Delta T^2} \leq \tilde{\kappa}^2 < \infty$ for all $\beta \in B_0^p(k_2)$, Lemma B.2 also guarantees that $\frac{|w_\theta^2|}{n|\theta|^2} \leq C \tilde{\kappa}^w$ for all $\theta \in \Theta$ and $\frac{|x\beta_2|}{n|\beta|^2} \leq C \tilde{\kappa}^x$ for all $\beta \in B_0^p(k_2)$ hold with high probability (in Theorem 4.13). Similarly, the conditions $\frac{1}{\sqrt{n}} \max_{j=1,\ldots,d} |w_j|_2 \leq \kappa_w < \infty$ and $\frac{1}{\sqrt{n}} \max_{j=1,\ldots,p} |x_j|_2 \leq \kappa_x < \infty$ (in Theorem 4.14) are also implied by Lemma B.2 with high probability given $w_j$ ($j = 1, \ldots, d$) and $x_j$ ($j = 1, \ldots, p$) are sub-Gaussian.

Compare the scaling of the lower bound in Theorem 4.13 with the upper bounds in Corollary 4.4 and Theorem 4.11. In particular, from the previous upper bounds, we have

$$\left( E \left| \hat{\beta}_{HSEL} - \beta^* \right|^2 \right)^{1/2} + \left( E \left[ \hat{g}(w_i^T \hat{\theta}) - g(w_i^T \beta^*) \right]^2 \right)^{1/2} \lesssim \max \left\{ |\beta^*|_1 \sqrt{\frac{k_2 k_1 \log d}{n}}, \left( \frac{k_1}{n} \right)^{1/4}, \sqrt{k_2 \log p} \right\}$$

(The upper bound on $E \left| \hat{\beta}_{HSEL} - \beta^* \right|^2$ is obtained by converting $|\hat{\beta}_{HSEL} - \beta^*|^2$ with a standard integration over the tail probability in the exponential form). Notice that the scaling in the upper bound above matches the lower bound in Theorem 4.13 in terms of $(n, d, p)$--factors. The only difference between these bounds is that the upper bound exceeds the lower bound by a factor of $|\beta^*|_1 \sqrt{k_2} \times k_2^{3/2}$ in the term related to the complexity of the set $\Theta$, which is likely due to the fact that the estimator $\hat{\beta}_{HSEL}$ is a sequential multi-stage procedure based on plugging in the first-stage estimator $\hat{\theta}$ in the place of the unknown coefficient $\theta^*$ in the selection equation. In a different but somewhat related context which concerns with the high-dimensional sparse linear regression models with many endogenous regressors and instruments (see Zhu 2014), it is found that the upper bound on the $\sqrt{MSE}$ of the $l_1$--regularized two-stage estimator exceeds the minimax lower bound in Zhu (2014) by a factor of $k_2$ (where $k_2$ is the sparsity parameter for the second-stage model).

On the other hand, this minimax optimality result does not apply to the case of approximately sparse $\beta^*$ when we compare the scaling of the lower bound in Theorem 4.14 with the upper bounds in Corollary 4.6 and Theorem 4.12 for the case $q_2 \in (0, 1]$. In particular, from the previous upper bounds, we have

$$\left( E \left| \hat{\beta}_{HSEL} - \beta^* \right|^2 \right)^{1/2} + \left( E \left[ \hat{g}(w_i^T \hat{\theta}) - g(w_i^T \theta^*) \right]^2 \right)^{1/2} \lesssim \sqrt{R_{q_2}} \left[ \max \left\{ \sqrt{\frac{\log p}{n}}, |\beta^*|_1 \left( \frac{|\beta^*|_1}{n} \right)^{1/4}, |\beta^*|_1 R_{q_1} \left( \frac{\log d}{n} \right)^{1-\frac{q_1}{2}} \right\} \right]^{1-\frac{q_2}{2}} \cup \left\{ \left( \frac{|\beta^*|_1}{n} \right)^{1/2} \right\}.$$

As in the case of exactly sparse $\beta^*$, the terms $\sqrt{R_{q_2}} \left( \frac{\log p}{n} \right)^{1-\frac{q_1}{2}}$ and $\left( \frac{|\beta^*|_1}{n} \right)^{1/2}$ in the above upper bound match the scalings of the term related to the complexity of the set $B_0^p(R_{q_2})$ and the term related to the complexity of the set $\mathcal{F}$, respectively. In sharp contrast to the case of exactly sparse $\beta^*$ where our sequential multi-stage procedure based on plugging in the first-stage estimator $\hat{\theta}$ only exceeds the minimax optimal result by a factor of $k_2^{3/2}$ in the term related to the complexity of the set $\Theta$, the term $\sqrt{R_{q_2}} \left[ |\beta^*|_1 R_{q_1} \left( \frac{\log d}{n} \right)^{1-\frac{q_1}{2}} \right]^{1-\frac{q_2}{2}}$ in the upper bound above is now worsened by an exponent of $1 - \frac{q_2}{2}$ and a factor of $\sqrt{R_{q_2} (|\beta^*|_1)^{1-\frac{q_2}{2}}}$ when compared to the term related to $\Theta$, $R_{q_2} \left( \frac{\log p}{n} \right)^{1-\frac{q_1}{2}}$, in the lower
bound of Theorem 4.14. When $q_2 \in [0, 1]$, note that

$$
\left[ R_{\tilde{q}_1} \left( \sqrt{\frac{\log d}{n}} \right)^{1-\frac{q_2}{2}} \right]^{1-\frac{q_2}{2}} \geq R_{\tilde{q}_1} \left( \sqrt{\frac{\log d}{n}} \right)^{1-\frac{q_2}{2}}
$$

with “=” holds only if $q_2 = 0$ (the case of exactly sparse $\beta^*$).

The lower bound in either Theorem 4.13 or Theorem 4.14 is a “point” result. Even if the main equation in the original selection model (1) has a normal error, the normality of $\eta$ is plausible in model (2) only if $g(w_i^T \theta^*) = 0$, i.e., when there is no selection activity. Nevertheless, these “point” results provided by Theorems 4.13 and 4.14 are still useful because whether $g(w_i^T \theta^*)$ equals 0 or not would be unknown in general and the error from having to estimate $g(\cdot)$ still appears in the lower bounds. Moreover, even if the “point” result does not hold “globally”, given that the lower bounds are derived for the minimax risks of the high-dimensional linear coefficients together with the nonparametric selection bias function, at least the second and third terms in the lower bounds of Theorems 4.13 and 4.14 should be unimprovable in any “global” result. It is possible to impose distributional assumptions other than normality on $\eta$ but the derivation of the lower bounds in the proofs may involve more difficult computations related to the Kullback-Leibler divergence or the more general $f-$divergence where $f$ is a convex function with $f(1) = 0$ (see Guntuboyina 2011 for a unified treatment of existing techniques for obtaining lower bounds). For this reason, existing literature on minimax lower bounds almost exclusively focuses on the case of normal errors and lower bounds with less restrictive distributional assumptions other than normality (e.g., sub-Gaussianity) on a random vector are in general impossible to obtain. Recent work of efficiency bounds (e.g., Hansen B., 2014) that proposes a shrinking neighborhood analysis may provide a promising direction for extending these “point” results to the case where $g(w_i^T \theta^*)$ is in a shrinking neighborhood of 0.

4.5 Estimation of high-dimensional semilinear models with a two-stage projection strategy

In this section, we discuss how the theory developed in this paper can be applied to the semilinear models in high-dimensional settings. The multi-stage estimator proposed in this paper is also useful for estimating the linear coefficients of the following semilinear model:

$$
y_i = x_i^T \beta^* + g(w_i) + \eta_i
$$

where $x_i$ is a $p-$dimensional vector of regressors (and $p$ can grow with and exceed the sample size $n$). Furthermore, $g(\cdot) : \mathbb{R}^d \rightarrow \mathbb{R}$ is an unknown function and $w_i$ is a $d-$dimensional vector of regressors. Our multi-stage projection strategy is now reduced to a two-stage procedure. Based on the analysis in this paper, it is straightforward to see that Theorems 4.1 and 4.2 remain valid except that there is no first-stage related error $T_1$ in the upper bounds on the estimator of $\beta^*$. As we have mentioned before in Section 4.1, when $p$ and $d$ are fixed and small relative to $n$, as long as $F_j (j = 0, ..., p)$ in Theorems 4.1 and 4.2 are sufficiently smooth so that $\mathbb{E}(x_{ij} \mid w_i)$ and $\mathbb{E}(y_i \mid w_i)$ can be estimated at a rate no slower than $O \left( n^{-\frac{1}{4}} \right)$, the “oracle” property will be achieved.

The case where the dimensions $p$ and $d$ are both large relative to $n$ (namely, $p \geq n$ and $d \geq n$) generalizes the semilinear model considered in Belloni, et. al (2014) in which $d \geq n$ and $p$ remains finite. When $p \geq n$, it is unclear whether the procedure proposed by Belloni, et. al (2014) can be easily extended
because the effect from imperfect selection in the second step of Belloni, et. al (2014) may not be negligible anymore when the number of components in $x_i$ is also large relative to $n$. Instead, the projection strategy proposed in this paper can be used to estimate $\beta^*$ in the semilinear model (30) when $p \geq n$ and $d \geq n$. One way to reduce the curse of dimensionality in the joint multivariate nonparametric component $E(z_{ij} | w_i)$ (recall $z_j = x_j$ for $j = 1, ..., p$ and $z_0 = y$) is to consider the class of additive models of the form (Hastie and Tibshirani, 1999):

$$E(z_{ij} | w_i) := f_j(w_i) = \sum_{l=1}^{d} f_{jl}(w_{il})$$

where $f_{jl}(\cdot) \in F_{jl}$ for $l = 1, ..., d$ and $j = 0, ..., p$.

Let us consider the simplest case of $f_{jl}(w_{il}) = w_{il} \theta^*_l$ where $\theta^*_l$ is a scalar and $\theta^* \in B^d_{q_1}(R_{q_1})$ and $\beta^* \in B^d_{q_2}(R_{q_2})$ for $q_1, q_2 \in [0, 1]$. Theorem 4.1 implies that the $l_2$-error of the two-stage estimator is bounded above by

$$O \left( \left[ \frac{1}{R_{q_2}^d} \left( \frac{\log p}{n} \right)^{1-\frac{1}{q_2}} \right] \vee \left[ \frac{1}{R_{q_2}^d} \left( |\beta^*|_1 R_{q_1} \left( \frac{\log d}{n} \right)^{1-\frac{1}{q_1}} \right) \right] \right).$$

For a more general structure on $f_j$, suppose $J(f_j) := \{l : f_{jl} \neq 0\}$ and $k_{lj} = |J(f_j)|$, the cardinality of $J(f_j)$, can increase to infinity with $n$ but slowly compared to $n$ (i.e., $f_j$ is exactly sparse) and

$$E(z_{ij} | w_{il}) = \sum_{k=1}^{\infty} \vartheta_{jlk} \phi_{jlk}(w_{il})$$

where $B_{jl} = (\phi_{jlk})_{k=1}^{\infty}$ is an orthonormal basis for $F_{jl}$. For a truncation parameter $M$, also define

$$E^M(z_{ij} | w_{il}) = \sum_{k=1}^{M} \vartheta_{jlk} \phi_{jlk}(w_{il}).$$

Let $\Psi_{jl}$ denote the $n \times M$ matrix with $\Psi_{jl}(i, k) = \phi_{jlk}(w_{il})$. For the first-stage estimation in our two-stage procedure, consider the following minimization problem:

$$\min_{\vartheta_{jl} \in \mathbb{R}^M} : \frac{1}{2n} \left| \sum_{l=1}^{d} \Psi_{jl} \vartheta_{jl} \right|^2 + \lambda_n \sum_{l=1}^{d} \sqrt{\frac{1}{n} \vartheta_{jl}^T \Psi_{jl} \vartheta_{jl}}$$

(31)

for some regularization parameter $\lambda_n > 0$. Program (31) is the sample version (with truncation) of the following:

$$\min_{f_{jl} \in F_{jl}} : \frac{1}{2} \mathbb{E} \left( z_{ij} - \sum_{l=1}^{d} f_{jl}(w_{il}) \right)^2 + \lambda_n \sum_{l=1}^{d} \mathbb{E} \left( f_{jl}^2(w_{il}) \right).$$

The optimization program (31) is considered in Ravikumar, Lafferty, and Wasserman (2009) and can be viewed as a functional version of the grouped Lasso (Yuan and Lin, 2006). It can be solved with a coordinate descent algorithm proposed by Ravikumar, et. al (2009). Theoretical properties of the two-stage estimator with the first-stage estimation based on (31) are being analyzed in a chapter of the PhD thesis by Zhu (2015).

to the regression of $y_i$ on $w_i$; second, apply the Lasso to the regression of $x_{ij}$ on $w_i$ for every $j = 1, ..., p$, respectively; and third, regress $y_i$ on $x_i$ and the components of $w_i$ selected by the first and second step.
5 Monte-Carlo simulation

In this section, simulations are conducted to gain preliminary understanding of the small-sample performance of the non-pivotal multi-stage estimator $\hat{\beta}_{HSEL}$; ongoing work involves implementation of the pivotal procedure described in Section 4.2. We consider model (1) where $w \in \mathbb{R}^{n \times d}$ is a matrix consisted of independent uniform zero-mean random variables on $[-2, 2]$ with variance $\sigma_w \approx 1.33$ and $x$ takes on the first $p$ columns of $w$. The i.i.d. errors $\epsilon_{1i} \sim \mathcal{N}(0, 1)$ for $i = 1, \ldots, n$ where $n$ denotes the number of observations generated for the selection equation. We consider two scenarios where $n = 88$ and $n = 200$. Given the setup here, on average 44 (when $n = 88$) and 100 (when $n = 200$) observations, respectively, will be used for estimating the main equation. Conditional on the observations is with $y_{1i} = 1$, the i.i.d. errors $(\epsilon_{1i}, \epsilon_{2i})$ have the following joint normal distribution

$$(\epsilon_{1i}, \epsilon_{2i}) \sim \mathcal{N}\left(\left(\begin{array}{c} 0 \\ 0 \end{array}\right), \left(\begin{array}{cc} 1 & \rho \sigma_2 \\ \rho \sigma_2 & \sigma_2 \end{array}\right)\right),$$

where $\rho \in \{0, 0.9\}$ and $\sigma_2 \in \{0.3, 1, 2\}$. We set $d = 90$, $p = 45$, $k_1 = 4$, and $k_2 = 2$. When $n = 88$, this setup of dimensionality represents a selection model where the number of regressors in the selection equation and the main equation, respectively, exceeds the number of observations used to estimate the corresponding equation, while the number of relevant regressors (ones with nonzero coefficients) is small relative to the sample size. We set $\theta_j^* = 0.5$ for $j = 1, 2, 3, 46$ and the rest of components in $\theta^*$ take on values of 0; set $\beta_1^* = \beta_{45}^* = 1$ and the rest of components in $\beta^*$ take on values of 0. This set up ensures that there is at least one component $w_{ij}$ with $\theta_j^*$ in the support set of $\theta^*$ such that $w_{ij}$ is excluded from $x_i$.

We consider four sets of experiments. The first experiment (Experiment 1) concerns the multi-stage estimator $\hat{\beta}_{HSEL}$. As a benchmark for Experiment 1, Experiment 2 applies a one-step Lasso procedure (without correcting selection bias) to the same main equation. Experiments 3 and 4 are benchmarks concerning classical low-dimensional settings. Experiment 3 applies the Heckman’s 2-step procedure to model (1) where the selection equation and the main equation are in the low-dimensional setting and the supports of the true parameters in both equations are known a priori; Experiment 4 applies the OLS to the same low-dimensional model as Experiment 3. We simulate 100 sets of data following the process described above. For each set $t = 1, \ldots, 100$, we compute the estimates $\hat{\beta}^t$ of the main-equation parameters $\beta^*$, $l_2$-errors of these estimates, $|\hat{\beta}^t - \beta^*|_2$, and selection percentages of $\beta^t$ (computed by the number of the elements in $\hat{\beta}^t$ sharing the same sign as their corresponding elements in $\beta^*$, divided by the total number of elements in $\beta^*$). Results reported in this section include:

(a) the mean of the relevant estimates $\frac{1}{100} \sum_{t=1}^{100} \hat{\beta}^t_1$;
(b) the mean of the relevant estimates $\frac{1}{100} \sum_{t=1}^{100} \hat{\beta}^t_{45}$;
(c) the mean of the averaged irrelevant estimates $\frac{1}{45} \sum_{j \neq 1, 45} \frac{1}{100} \sum_{t=1}^{100} \hat{\beta}^t_j$;
(d) the mean of the $l_2$-errors of the estimates $\hat{\beta}^t$ computed as $\frac{1}{100} \sum_{t=1}^{100} |\hat{\beta}^t - \beta^*|_2$;
(e) the mean of the selection percentages (computed in a similar fashion as the mean of the $l_2$-errors of the estimates);
(f) the mean of the squared $l_2$-errors (i.e., the sample mean squared error, SMSE, computed as $\frac{1}{100} \sum_{t=1}^{100} |\hat{\beta}^t - \beta^*|^2$);
(g) the sample squared bias $\sum_{j=1}^{45} (\tilde{\beta}_j - \beta^*_j)^2$ (where $\tilde{\beta}_j = \frac{1}{100} \sum_{t=1}^{100} \hat{\beta}^t_j$ for $j = 1, \ldots, 45$).

The results in this section regarding Experiment 1 are based on the choices of the regularization pa-
parameter $\lambda_{n,1} = 0.5\sqrt{\frac{\log d}{n}}$ for the first-stage estimation problem (13) and the regularization parameter $\lambda_{n,3} = 0.2k_2\sqrt{\frac{k_1\log d}{n_s}}$ for the third-stage estimation problem (8), where $n_s$ denotes the number of observations with $y_{1i} = 1$. The scalings of $\lambda_{n,1}$ and $\lambda_{n,3}$ are chosen according to Proposition 4.2 and Corollary 4.4, respectively. The choice of $0.2k_2\sqrt{\frac{k_1\log d}{n_s}}$ is also used in Experiment 2 for comparing the performance of the proposed procedure and the Lasso without corrective measures. Note that for $\sigma^2 = 1$ and $\sigma_x = \sigma_w \approx 1.33$, $0.2k_2\sqrt{\frac{k_1\log d}{n_s}}$ is slightly greater than $2\sigma_2 \cdot \sigma_x \sqrt{\frac{\log p}{n_s}}$, the smallest value required for the Lasso estimation of the standard sparse high-dimensional linear models (e.g., Bickel, et. al, 2009). The second-stage estimation in Experiment 1 is based on solving (22) with $L = 1$. Ongoing work involves implementing the cross validation procedure described in Section 4.1.2 to determine $L$ when $\epsilon_{2i}$ has a non-normal distribution.

From Table 5.1, we see that the direct Lasso estimator without correcting selection bias outperforms the multi-stage estimator $\hat{\beta}_{HSEL}$ when $\rho = 0$, and vice versa when $\rho = 0.9$. For the design considered here, in the presence of substantial selection activity ($\rho = 0.9$), the mean of the $l_2$—errors (row d) and the sample squared bias of the estimates (row g) by the direct Lasso procedure without corrective measures are exacerbated in the high-dimensional setting and this exacerbation mainly comes from the poorer estimates of the relevant regressors as the mean of the averaged irrelevant estimates varies little from the case $\rho = 0$ to the case $\rho = 0.9$. Other simulation results (not included here due to space limit) show that when $\sigma_2$ is increased (decreased) from 1 to 2 (respectively, from 1 to 0.3), $\hat{\beta}_{HSEL}$ performs worse (respectively, better) relative to the case $\sigma_2 = 1$, and similar patterns are observed when $w_{ij}$s are drawn from independent uniform zero-mean random variables on $[-1, 1]$ (respectively, on $[-4, 4]$). Also, as $n$ increases from 88 to 200, $\hat{\beta}_{HSEL}$ performs substantially better. These findings are intuitive and expected. It is worth noting that for the design considered here, in terms of the mean of the selection percentages (row e), the direct Lasso procedure without corrective measures is comparable to $\hat{\beta}_{HSEL}$ even in the case $\rho = 0.9$. Ongoing work is exploring situations where variable selection by $\hat{\beta}_{HSEL}$ substantially outperforms variable selection by the direct Lasso procedure.

|                | Exp 1 | Exp 2 | Exp 3 | Exp 4 | Exp 1 | Exp 2 | Exp 3 | Exp 4 |
|----------------|-------|-------|-------|-------|-------|-------|-------|-------|
| $\rho = 0$     | 0.703 | 0.730 | 1.005 | 1.010 | 0.627 | 0.605 | 1.007 | 1.002 |
| $\rho = 0.9$   | 0.742 | 0.736 | 0.996 | 0.994 | 0.762 | 0.757 | 1.006 | 1.020 |
| $\sigma_2 = 0.9$ | 0.001 | 0.001 | 0.969 | 0.965 | 0.946 | 0.961 | 0.981 | 0.979 |
| $\sigma_2 = 2$ | 0.227 | 0.209 | 0.032 | 0.033 | 0.249 | 0.262 | 0.025 | 0.034 |
| $\sigma_2 = 0.3$ | 0.155 | 0.143 | $4 \times 10^{-5}$ | $9 \times 10^{-5}$ | 0.197 | 0.217 | $9 \times 10^{-5}$ | $4 \times 10^{-4}$ |

6 An empirical application to the retail gasoline market

Having established the theoretical properties of the 3-step estimators, we now apply one of these estimators to an empirical example of price-discrimination in the retail gasoline market. When consumers have
different valuations for a good, a firm can increase profits by developing a pricing scheme that distinguish consumers with different valuations. In most cases, a firm knows the distribution of consumer valuations in the market but not the exact valuation of any specific consumer prior to the sale. In these cases, a firm can offer a menu of different prices, appropriately bundled with other aspects of the product (such as product quality), and force consumers to choose bundles consistent with their preferences.

When differences in costs incurred to produce various bundles in the menu are small compared to the differences in prices, this menu-based offering is a price discrimination mechanism. Retail gasoline markets present a good context to study price discrimination since different gasoline stations in a market typically face similar costs of procuring gasoline. Therefore, any price differences across gasoline stations are likely due to reasons unrelated to the cost of procuring gasoline. Gasoline retailers can choose to be either a two-product station offering both self-service and full-service gasoline or a single-product station offering only full-service or self-service gasoline. A two-product station, by charging different prices for full- and self-service gasoline, induces consumers with different valuations to choose the products consistent with their preferences, namely, a two-product station engages in price-discrimination. A single-product station, on the other hand, is unable to price discriminate.

Shepard (1991) estimates pricing decisions of gasoline stations without endogenizing their decisions to price discriminate, i.e., their choice to be single versus multi-product. Iyer and Seetharaman (2003) explicitly examines a firm’s incentive to price-discriminate. In doing so, they highlight the importance of accounting for self selectivity considerations in empirical analysis of price discrimination based on market data. Specifically, Iyer and Seetharaman employ a binary probit framework to model a gasoline station’s decision to be single-product or multi-product as a function of market and station characteristics, and then model the prices chosen by the gasoline station for its product(s) by estimating linear regressions with Heckman’s self-selectivity correction conditional on the station’s decision to offer a single- or multi-product. They show that incorrect inferences about the incentive to price discriminate and about the differences in the prices charged between single-product and multi-product stations would result if the endogeneity in the choice of the station-type were ignored in the estimation. Their empirical analysis also shows that a larger income spread in the market implies a greater likelihood of the gasoline station being multi-product. However, Iyer and Seetharaman (2003) did not account for interactions between the gas stations in their empirical analysis. Studies show that pricing decisions of retail gasoline stations may depend on the degree of competitive intensity in the “market” (e.g., Slade, 1992). In the empirical literature on competitive gasoline markets, there have been various ways of defining a “market” (see, e.g., Slade, 1986; Pinkse, Slade and Brett, 2002; Iyer and Seetharaman, 2008). For example, Iyer and Seetharaman (2008) defines mutually exclusive census tracts as local markets, and treat each market as the unit of observation in their empirical analysis. In previous research, markets have been defined based on stations that fall within a circle of half a mile or one mile radius.

One common feature of the previous definitions of competitive markets is that they are subjective heuristics. It would be ideal if one can control for the interactions between different stations without requiring a priori knowledge of the structure of the competitive market. Recent work including Manresa (2014) and Bonaldi, Hortacsu, Kastl (2014) develop econometric models to recover the underlying networks in different applications. Both papers hinge on the availability of panel data for each observation in the cross section. In particular, Manresa considers settings where outcomes depend on an agent’s own characteristics and on the characteristics of other agents in the data. She applies a Lasso type estimator to identify individuals generating spillovers and their strength using panel data on outcomes and characteristics. Bonaldi, et. al proposes a new measure of systemic risk based on estimating spillovers between funding costs of individual banks with a Lasso type procedure, which is applied to the panel of each
individual bank to recover the financial network. However, for the empirical application considered in this paper, panel data of each gas station is not available and as a consequence, the econometric model by either Manresa or Bonaldi is not suitable. Instead, we use geographic information and spatial data to create a set of measures that are high-dimensional to control for the interactions between the gas stations and employ one of our proposed estimators to identify the competitive market structure. The following subsection describes the data followed by the empirical model.

Data and the empirical model

This paper uses the data set from Iyer and Seetharaman (2003). It was collected during July 1998 from a cross-section of 249 gasoline stations in the Greater Saint Louis metropolitan area. Among the 249 stations, 65 are multi-product stations and 172 are single-product self-service stations. In addition, there were 12 single-product full-service stations. In the United States, the low incidence of full-service single product stations is typical and in certain regions full service is required by law. As in Iyer and Seetharaman (2003), we exclude them from the empirical analysis. The survey data include the prices of three grades - 87, 89 and 93 octane levels - of gasoline, along with station-specific characteristics, i.e., number of gasoline pumps, special advertising for cigarettes and soda, presence of convenience store, pay-at-pump facility, car wash, service station, and the number of stations with prices that are visible to a given station. This data set also contains demographic information including income, population density, age distribution, home value, and education levels. This information comes from 1990 U.S. census data, which contain demographic information at the level of each census tract.

The data also records addresses of each station, from which “Bing Maps REST Services” is used to obtain geographic information including longitude and latitude, travel distance in driving mode between any pair of stations, etc. This information can be used to create variables for partially controlling for the interactions between stations. In particular, given any station, we can count the number of stations and/or stations under one of the three national brands (namely, Amoco, Shell and Mobil), that fall within 1km, 1km and 2km, and so on, from this station. Each of the numbers is then divided by the area (in km$^2$) of the corresponding layer. The number of stations with prices that are visible to a given station is another useful measure of interaction between stations and this information is available in the data. Using the number of competitors as a measure of interaction between firms has been seen in previous literature (e.g., Bresnahan and Reiss 1991; Iyer and Seetharaman, 2008). The novelty introduced by this paper lies in the data-driven nature of the approach: rather than assuming a priori knowledge of the structure of the competitive network, it relies on the data to determine the geographic pattern of interaction between stations. If panel data on prices and time-varying instrumental variables for prices are available, we can include prices of other stations in the main equation. Some variants of the 3-step estimators in this paper combined with the high-dimensional IV estimator in Gautier and Tsybakov (2011) or the high-dimensional 2SLS estimator in Zhu (2013) may be considered as an alternative to identify the sets of competitive markets engaged in pricing. However, in the retail gasoline market, it may be difficult to obtain valid time-varying instrumental variables for prices.

As in Iyer and Seetharaman (2003), we use a binary probit model for the selection of service types where $y_{1i} = 1$ in (1) indicates that station $i$ offers multi-service and $y_{1i} = 0$ indicates that station $i$ offers single-self-service. The same set of explanatory variables included in the binary probit model of Iyer and Seetharaman is used here: average income ($AVG$), income spread ($SPREAD$), brand ($BRAND$), pay-at-pump facility ($PAP$), presence of convenience store ($CONV$), car wash ($WASH$), and service station ($SERV$). This paper differs from Iyer and Seetharaman mainly in terms of the specifications of the linear pricing model (the main equation): First, while Iyer and Seetharaman assume the selection bias takes
on the functional form of the Inverse Mills Ratio, we assume the selection bias function to obey the more general nonparametric single index restriction in (1); second, we add a set of measures that are high-dimensional to partially control for the competition effects from other stations. In particular, the following explanatory variables are included in the pricing model: \( AVG, BRAND \), special advertising for cigarettes and soda (\( ADSCC \)), the number of stations with visible prices (\( VISP \)), the total number of stations and the number of stations under one of the three national brands within 1km (\( TOT\_1 \) and \( BRND\_1 \)), 1km and 2km (\( TOT\_2 \) and \( BRND\_2 \)), \cdots, 34km and 35km (\( TOT\_35 \) and \( BRND\_35 \)) from a given station.

In summary, for the pricing equation, we have \( n = 172, p = 74 \) for stations that serve single-self-service grade-87 gasoline, \( n = 168, p = 74 \) for stations that serve single-self-service grade-93 gasoline, \( n = 65, p = 74 \) for stations that serve multi-service grade-87 gasoline, and \( n = 65, p = 74 \) for stations that serve multi-service grade-93 gasoline. While Iyer and Seetharaman include only average income and brand in their pricing model, they suggest that special advertising for cigarettes and soda might be correlated with the retail gasoline prices and hence we include this information in our pricing model. The last group of variables are measures added to partially control for the competition effects from other stations. Iyer and Seetharaman found the indicators of the presence of pay-at-pump facilities and service stations statistically significant and therefore, the exclusion restriction required by the selection model considered in this paper is likely to be satisfied given the setup. Moreover, they justify the exclusion restriction by arguing that a station’s decision pertaining to the configuration of its station characteristics - pay-at-pump, convenience store, car-wash, and service station - involves costly investments that the station owner has made along with the station-type decision while setting up the retail facility. In contrast, the pricing decisions may vary on a daily basis.

The following summarizes the estimation procedure and empirical findings. We briefly discuss the results pertaining to the effects of the service-type decisions and focus mainly on the empirical findings from the pricing regression because the main difference between Iyer and Seetharaman and the empirical analysis in this paper lies in the latter.

### Estimation and empirical findings

A standard maximum likelihood procedure for estimating low-dimensional binary probit models is used to obtain estimates of the selection equation of service-type decisions. The estimation is performed for grade-87 stations and grade-93 stations, respectively, and the results are reported in Table 6.1.

|       | Intercept  | \( AVG \)  | \( SPREAD \) | \( BRAND \)  | \( PAP \)   | \( CONV \)  | \( WASH \)  | \( SERV \)  |
|-------|------------|------------|--------------|-------------|------------|------------|------------|------------|
| Grade-87 | -1.525*** | 0.008*     | -2.591**     | 0.962**     | -0.851**   | -0.341     | -0.012     | 2.341***   |
|         | (0.453)    | (0.005)    | (1.227)      | (0.411)     | (0.414)    | (0.321)    | (0.346)    | (0.292)    |
| Grade-93 | -1.522*** | 0.008*     | -2.583**     | 0.958**     | -0.848**   | -0.338     | -0.004     | 2.335***   |
|         | (0.452)    | (0.005)    | (1.229)      | (0.410)     | (0.413)    | (0.321)    | (0.347)    | (0.292)    |

Because individual-level income is not available for this data set, it is not possible to compute the sample standard deviation in income for each tract. Instead, we use two measures to approximate income spread: one is the absolute difference between the percentages of median-level income group and the low-level income group for each tract; the other is the absolute difference between the percentages of median-level income group and the high-level income group for each tract. A smaller value in the first (second) absolute difference indicates a more evenly distributed population in the low-income (respectively, high-income) group and the median income group. It turns out that the second measure is not statistically significant.
and hence we drop this measure from the probit model. As a consequence, the negative sign of the estimate for \( \text{SPREAD} \) suggests that more heterogeneous income levels below the 50\(^{th}\) percentile in the market implies a greater likelihood of the station being multi-product.

For the linear pricing regression model conditional on the service type, the non-pivotal estimator \( \hat{\beta}_{HSEL} \) based on \( \hat{\eta} \) is used to select the variables with non-zero coefficients and then \( \hat{\beta} := (\hat{\nu}_j^T \hat{\nu}_j)^{-1} (\hat{\nu}_j^T \hat{\nu}_0) \) is computed with \( \hat{J} := J(\hat{\beta}_{HSEL}) \) (this is the Post-Lasso procedure discussed in Section 4.1.3). For the second-stage estimation, program \( (22) \) is solved where the Lipschitz constant \( \Lambda \) is determined by the cross-validation procedure described in Section 4.1.2 and the choice of \( L = 1 \) turns out to be robust. For the third-stage estimation, given the setup of our empirical model, the first-stage related estimation error is likely to be dominated by the third-stage related error in the choice of \( \lambda_{n,3} \) from Corollary 4.4 and hence we choose \( \lambda_{n,3} \) based on the third-stage related error. Program \( (8) \) is first solved with the choice of \( \lambda_{n,3}(t) = 2.001 \cdot \hat{\sigma}_v \hat{\sigma}_v^{-1} \sqrt{\log_p \frac{n}{n_s}} \) for \( t = 0 \) (initialization), where \( \hat{\sigma}_v := \max_{j = 1, \ldots, p} \sqrt{\frac{1}{n} \sum_{i=1}^n \hat{\nu}_i^2} \), \( \hat{\sigma}_v^{-1} = 1 \), and \( n_s \) denotes the number of observations used for the pricing regression. Let \( \hat{\beta}_{HSEL}^t \) denote the resulting estimate based on \( \lambda_{n,3}(t) \) and \( \hat{\sigma}_v^{t+1} \) denote the updated sample standard deviation of the fitted residuals \( \hat{\eta}^{-t}_{i} := \hat{\nu}_0 - \hat{\beta}_{HSEL}^t \) for \( i = 1, \ldots, n_s \). Program \( (8) \) is then solved with the updated \( \lambda_{n,3}(t + 1) = 2.001 \cdot \hat{\sigma}_v \hat{\sigma}_v^{t+1} \sqrt{\log_p \frac{n}{n_s}} \). Repeat this process until a pre-specified tolerance level on \( |\hat{\sigma}_v^{t+1} - \hat{\sigma}_v^t| \) is reached. The result shows that the choice of \( \hat{\sigma}_v \approx 0.04 \) is robust. After experimenting with a range of values around the final choice of \( \lambda_{n,3} \) determined according to the described procedure, the set of variables selected by \( \hat{\beta}_{HSEL} \) for the range of \( \lambda_{n,3} \) and the post Lasso estimates \( \hat{\beta} \) based on these selected variables are reported in Table 6.2 for the following groups:

SSL: single-self-service grade-87 gasoline;
SSH: single-self-service grade-93 gasoline;
MSL: multi-self-service grade-87 gasoline;
MSH: multi-self-service grade-93 gasoline.

The variables with blanks in Table 6.2 correspond to those that are not selected by \( \hat{\beta}_{HSEL} \) in a particular group. The numerical values within parentheses are bootstrapped standard errors for \( \hat{\beta} \). Estimates with three asterisks, two asterisks, and a single asterisk are statistically significant at level \( \alpha = 0.01 \), \( \alpha = 0.05 \), and \( \alpha = 0.1 \), respectively. Note that our second-stage estimation and third-stage estimation use the demeaned explanatory variables and demeaned prices, so the intercept term is excluded from the pricing regression model.

|        | AVG | BRAND | TOT_2  | TOT_4  |
|--------|-----|-------|--------|--------|
| SSL    | 0.030*** | (0.005) | -0.052*** | (0.016) |
| SSH    | 0.049*** | (0.007) | -0.030**  | (0.015) |
| MSL    | 2 x 10^{-4}*** | (1 x 10^{-4}) | -0.070*** | (0.029) |
| MSH    |     |       |        | -0.096*** | (0.032) |
Regarding the results of the pricing regression in Table 6.2, the estimate of BRAND has a positive sign in SSL and SSH. Moreover, AVG (average income) has a positive effect on the pricing decisions in MSL. In Iyer and Seetharaman (2003) which estimated the low-dimensional linear regression counterpart by pooling observations of the single-self-service and multi-self-service each with Heckman’s selectivity correction (for grade-87 and grade-93, respectively), BRAND and AVG are the only two variables included in their pricing model and found to be statistically significant. Based on our empirical results which remove selection bias and partially control for potential interactions between the stations simultaneously, we see that TOT_4 is selected by $\hat{\beta}_{HSEL}$ in all groups and statistically significant at level 0.01 in SSL, MSL, and MSH and at level 0.1 in SSH. TOT_2 is selected by $\hat{\beta}_{HSEL}$ in SSH and statistically significant at level 0.05. The negative sign of the estimate for TOT_4 in all groups (TOT_2 in SSH) suggests that the total number of stations within 3km-4km (respectively, 1km-2km) of a given station has a negative effect on its price. On the other hand, the variable VISP (the number of stations with visible prices) is not selected by $\hat{\beta}_{HSEL}$, which is less intuitive. However, it is possible that in the presence of several clusters of gas stations, there is less competition from adjacent clusters relative to ones that are somewhat further apart. For example, Iyer and Seetharaman (2008) analyzed a similar but richer data set on prices and station characteristics gathered across stations in the Saint Louis metropolitan area and found that closely located retailers who face sufficient heterogeneity in preferences across consumers in a local market may differentiate on product design and pricing strategies (also see, e.g., Png and Reitman, 1994); in contrast, retailers that are farther apart from each other may adopt similar product design and pricing strategies if the market is relatively homogeneous (also see, e.g., Slade 1992). Another explanation for the finding where more competition comes from somewhat intermediate retailers instead of closest ones is that consumers of retail gasoline may travel from their suburban homes located in a neighborhood of one cluster to their work places or central shopping areas located in another cluster that may be somewhat further away; the further located clusters of stations may be linked by routes that are more convenient for commuting (these routes may be more direct or less congested, etc.). This explanation may suggest that retailers consider commuting behavior of their customers when setting the retail price. Investigating this factor requires more substantial empirical analysis and a data set that includes more detailed information on the demographics and business environment, which will be pursued in future research. Nevertheless, the main finding on TOT_2 and TOT_4 suggests that in modeling the pricing decisions of retail gas stations, not only it is useful to account for the self selectivity of service-type but also to take into considerations of potential interactions between stations; in particular, competition effects from retailers that are not in the same local market (e.g., the same census tract or neighborhood within a circle of half a mile or one mile radius, etc.) should not be overlooked.

7 Conclusion

This paper provides estimation tools together with their theoretical guarantees for the semiparametric sample selection model in high-dimensional settings under a weak nonparametric restriction on the form of the selection correction. In particular, the number of regressors in the main equation, $p$, and the number of regressors in the selection equation, $d$, can grow with and exceed the sample size $n$. The main theoretical results of this paper are finite-sample bounds from which sufficient scaling conditions on the sample size for estimation consistency and variable-selection consistency (i.e., the multi-stage high-dimensional estimation procedure correctly selects the non-zero coefficients in the main equation with high probability) are established. Statistical efficiency of the proposed estimators is studied via lower bounds on minimax risks. Inference procedures for the coefficients of the main equation, one based on a pivotal
Dantzig selector to construct non-asymptotic confidence sets and one based on a post-selection strategy (when perfect or near-perfect selection of the high-dimensional coefficients is achieved), are discussed.

Small-sample performance of one of the proposed procedures is evaluated by Monte-Carlo simulations and illustrated with an empirical application to the retail gasoline market in the Greater Saint Louis area. The preliminary simulation results show that the “bias” from not performing the selection correction is exacerbated in high-dimensional settings. For the empirical application, this paper models a firm’s choice of either a single-product or multi-product service as a function of market and station characteristics and then models the station’s pricing decision, conditional on the choice of the station type. Using geographic information and spatial data, a set of variables that are high-dimensional is introduced to control for interactions between the gas stations. The empirical finding suggests that competition effects from retailers that are not in the same local market should not be overlooked.

**Appendix I: Main proofs**

**Appendix II: Technical lemmas and the proofs**

Appendix I and Appendix II can be found in Section A and Section B of the online supplementary material: Proofs to “High-Dimensional Semiparametric Selection Models: Estimation Theory with an Application to the Retail Gasoline Market".

[https://sites.google.com/site/yingzhu1215/home/JobMar_Proofs.pdf](https://sites.google.com/site/yingzhu1215/home/JobMar_Proofs.pdf)
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Supplementary Material: Proofs to “High-Dimensional
Semiparametric Selection Models: Estimation Theory with an
Application to the Retail Gasoline Market”

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A Appendix I: Main proofs

A.1 Lemmas A.1-A.2

Lemma A.1: Let \( \hat{\Gamma} = \frac{1}{n} \hat{v}^T \hat{v} \) and

\[
\begin{align*}
\hat{e} &= \left[ \hat{E} \left( x | \hat{w} \hat{\theta} \right) - \hat{E} \left( x | \hat{w} \hat{\theta} \right) - \hat{E} \left( x | \hat{w} \theta^* \right) \right] \beta^* \\
&\quad - \left[ \hat{E} \left( y_2 | \hat{w} \hat{\theta} \right) - \hat{E} \left( y_2 | \hat{w} \hat{\theta} \right) - \hat{E} \left( y_2 | \hat{w} \theta^* \right) \right] + \eta.
\end{align*}
\]

Suppose the assumptions in Theorems 4.1 or 4.2 hold. If \( \lambda_{n,3} \) in program (8) satisfies

\[
\lambda_{n,3} \geq 2 \left| \frac{1}{n} \hat{v}^T e \right|_\infty > 0,
\]

and

\[
R_{q_2} \frac{\log p}{n} + T_1 + T_2 + T_3 = O(\kappa_L),
\]

then there is a constant \( c > 0 \) such that under Assumption 2.2 (\( q_2 \in [0, 1] \) and when \( q_2 = 0, R_{q_2} := k_2 \)),

\[
\left| \hat{\beta}_{HSEL} - \beta^* \right|_2 \leq c \frac{2 \hat{v}^T e}{n} \left( \lambda_{n,3} \vee \left| \frac{1}{n} \hat{v}^T e \right|_\infty \right)^{1 - \frac{q_2}{2}}.
\]

Proof. First, write

\[
\begin{align*}
v_0 &= \hat{v}_0 + \hat{E} \left( y_2 | \hat{w} \hat{\theta} \right) - \hat{E} \left( y_2 | \hat{w} \hat{\theta} \right) + \hat{E} \left( y_2 | \hat{w} \hat{\theta} \right) - \hat{E} \left( y_2 | \hat{w} \theta^* \right) \\
&= v \beta^* + \eta = \left[ \hat{v} + \hat{E} \left( x | \hat{w} \hat{\theta} \right) - \hat{E} \left( x | \hat{w} \hat{\theta} \right) + \hat{E} \left( x | \hat{w} \hat{\theta} \right) - \hat{E} \left( x | \hat{w} \theta^* \right) \right] \beta^* + \eta,
\end{align*}
\]
thus we have

\[
\dot{v}_0 = \dot{\beta}^* + \left[ \tilde{E}(x|w\theta) - \mathbb{E}(x|w\theta) + \mathbb{E}(x|w\theta) - \mathbb{E}(x|w\theta^*) \right] \beta^* \\
- \left[ \tilde{E}(y_2|w\theta) - \mathbb{E}(y_2|w\theta) + \mathbb{E}(y_2|w\theta) - \mathbb{E}(y_2|w\theta^*) \right] + \eta
\]

where

\[
e = \left[ \tilde{E}(x|w\theta) - \mathbb{E}(x|w\theta) + \mathbb{E}(x|w\theta) - \mathbb{E}(x|w\theta^*) \right] \beta^* \\
- \left[ \tilde{E}(y_2|w\theta) - \mathbb{E}(y_2|w\theta) + \mathbb{E}(y_2|w\theta) - \mathbb{E}(y_2|w\theta^*) \right] + \eta.
\]

Define the thresholded subset

\[
S_{\tau} := \{ j \in \{1, 2, ..., p \} : |\beta^*_j| > \tau \}
\]

where \( \tau = \frac{\lambda_{n,3}}{\kappa_L} \) is the threshold parameter. Define \( \tilde{\Delta} = \hat{\beta}_{HSEL} - \beta^* \) and the Lagrangian \( L(\beta; \lambda_{n,3}) = \frac{1}{\kappa n} |\tilde{\Delta}|^2 + \lambda_{n,3} |\beta^*|_1 \). Since \( \hat{\beta}_{HSEL} \) is optimal, we have

\[
L(\hat{\beta}_{HSEL}; \lambda_{n,3}) \leq L(\beta^*; \lambda_{n,3}) = \frac{1}{2n} |e|^2 + \lambda_{n,3} |\beta^*|_1.
\]

Some algebraic manipulation of the basic inequality above yields

\[
0 \leq \frac{1}{2n} |\dot{\Delta}|^2 \leq \frac{1}{n} e^T \dot{\Delta} + \lambda_{n,3} \left\{ |\beta^*_\tau|_1 + |\beta^*_S|_1 - |(\beta^*_\tau + \Delta_{S_{\tau}} + \beta^*_S + \Delta_{S_{\tau}})|_1 \right\}
\]

\[
\leq |\Delta|_1 \left[ \frac{1}{n} e^T e \right]_{\infty} + \lambda_{n,3} \left\{ |\Delta_{S_{\tau}}|_1 - |\Delta_{S_{\tau}}|_1 + 2|\beta^*_S|_1 \right\}
\]

\[
\leq \frac{\lambda_{n,3}}{2} \left\{ 3|\Delta_{S_{\tau}}|_1 - |\Delta_{S_{\tau}}|_1 + 4|\beta^*_S|_1 \right\}, \tag{A.1}
\]

where the last inequality holds as long as \( \lambda_{n,3} \geq 2\frac{1}{n} e^T e |e|_{\infty} > 0 \). Consequently,

\[
|\Delta|_1 \leq 4|\Delta_{S_{\tau}}|_1 + 4|\beta^*_S|_1 \leq 4 \sqrt{|S_{\tau}|} |\Delta|_2 + 4|\beta^*_S|_1.
\]

We now upper bound the cardinality of \( S_{\tau} \) in terms of the threshold \( \tau \) and \( l_q \)-ball radius of \( R_{q_2} \). Note that we have

\[
R_{q_2} \geq \sum_{j=1}^p |\beta^*_j|^{q_2} \geq \sum_{j \in S_{\tau}} |\beta^*_j|^{q_2} \geq \tau^{q_2} |S_{\tau}|
\]

and therefore \( |S_{\tau}| \leq \tau^{-q_2} R_{q_2} \). To upper bound the approximation error \( |\beta^*_S|_1 \), we use the fact that \( \beta^* \in B^p_{q_2}(R_{q_2}) \) and have

\[
|\beta^*_S|_1 = \sum_{j \in S_{\tau}} |\beta^*_j| = \sum_{j \in S_{\tau}} |\beta^*_j|^{q_2} |\beta^*_j|^{1-q_2} \leq R_{q_2} \tau^{1-q_2}.
\]

Putting the pieces together yields

\[
|\Delta|_1 \leq 4 \sqrt{\tau^{-q_2} R_{q_2}} |\Delta|_2 + 4 R_{q_2} \tau^{1-q_2}. \tag{A.2}
\]
Let us first prove the case of $q_2 \in (0, 1]$. Note that we also have
\[
\frac{1}{2n} |\hat{\Delta}|^2_2 \leq |\hat{\Delta}|_1 \frac{1}{n} |\hat{\beta}^T e|_\infty + \lambda_{n, 3} \left\{ |\hat{\Delta} \mathbf{S}(\hat{\Sigma})|_1 \right. \\
\left. - |\hat{\Delta} \mathbf{S}(\hat{\Sigma})_c|_1 + |\beta^*_L|_1 \right\} \\
\leq \left( 8 \sqrt{\frac{T}{n}} q_2 R_{q_2} |\hat{\Delta}|_2 + 4 R_{q_2} T^{1 - q_2} \right) \lambda_{n, 3} \\
\leq c_1 \sqrt{\frac{T}{n}} q_2 R_{q_2} |\hat{\Delta}|_2 \lambda_{n, 3} + c_2 \delta \\
\leq \max \left\{ c_1 R_{q_2}^\frac{1}{2} \kappa_L^{\frac{q_2}{2}} (\lambda_{n, 3})^{\frac{1 - q_2}{2}} |\hat{\Delta}|_2, c_2 \delta \right\}
\]
where the third and fourth inequalities follow from our choices of $\hat{\Sigma} = \frac{\lambda_{n, 3}}{\kappa_L}$ and $\delta \asymp R_{q_2} \lambda_{n, 3} T^{1 - q_2}$. Now we proceed by cases. If
\[
\max \left\{ c_1 R_{q_2}^\frac{1}{2} \kappa_L^{\frac{q_2}{2}} (\lambda_{n, 3})^{\frac{1 - q_2}{2}} |\hat{\Delta}|_2, c_2 \delta \right\} = c_1 R_{q_2}^\frac{1}{2} \kappa_L^{\frac{q_2}{2}} (\lambda_{n, 3})^{\frac{1 - q_2}{2}} |\hat{\Delta}|_2,
\]
so that under the condition
\[
R_{q_2} T^{1 - q_2} \left( \frac{\log p}{n} + T_1 + T_2 + T_3 \right) = O(\kappa_L),
\]
and provided $c > 0$ is sufficiently large, we have
\[
|\hat{\Delta}|_2 \geq c \kappa_{L}^{-1 + \frac{q_2}{2}} R_{q_2}^\frac{1}{2} (\lambda_{n, 3})^{1 - \frac{q_2}{2}} \geq \delta^* \tag{A.3}
\]
where
\[
\delta^* = O \left( \max \left\{ T_1^{\frac{1}{2} - q_2}, T_2^{\frac{1}{2} - q_2}, T_3^{\frac{1}{2} - q_2}, T_4^{2 - q_2} \right\} R_{q_2} \right).
\]
Consequently, (A.1) and (A.3) together imply that
\[
\hat{\Delta} \in \mathbb{K}(\hat{\Delta}, S_T) := \mathcal{C}(S_T; q_2, 3) \cap \{ \Delta \in \mathbb{R}^p : |\Delta|_2 \geq \delta^* \} \tag{A.4}
\]
where $\mathcal{C}(S_T; 3) := \{ \Delta \in \mathbb{R}^p : |\Delta S_T|_1 \leq 3 |\Delta S_T^c|_1 + 4 |\beta^*_L|_1 \}$. By Lemma A.2 and Lemma A.3 (or Lemma A.5), the random matrix $\hat{\Gamma} = \hat{\beta}^T \hat{\Sigma}$ satisfies the RE condition (2.1) over
\[
\mathcal{C}(S_T; 3) \cap \{ \Delta \in \mathbb{R}^p : |\Delta|_2 \geq \delta^* \},
\]
therefore, we have
\[
\kappa_{L} |\hat{\Delta}|^2_2 \leq \frac{1}{2n} |\hat{\beta}^T \hat{\Delta}|^2 \leq c_1 R_{q_2}^\frac{1}{2} \kappa_L^{\frac{q_2}{2}} (\lambda_{n, 3})^{1 - \frac{q_2}{2}} |\hat{\Delta}|_2
\]
so the claim follows. On the other hand, if
\[
\max \left\{ c_1 R_{q_2}^\frac{1}{2} \kappa_L^{\frac{q_2}{2}} (\lambda_{n, 3})^{1 - \frac{q_2}{2}} |\hat{\Delta}|_2, c_2 \delta \right\} = \delta,
\]
then
\[
|\hat{\Delta}|_2 \leq c \kappa_{L}^{-1 + \frac{q_2}{2}} R_{q_2}^\frac{1}{2} (\lambda_{n, 3})^{1 - \frac{q_2}{2}}
\]
so again the claim follows.

To prove the case of $q_2 = 0$, simply choose $\delta = 0$ and $S_T = \mathcal{J}(\beta^*)$ and the claim follows trivially from
we have the RE condition (2.1) of $\hat{\Delta}$ into condition (A.5) from Lemma A.3 and condition (A.9) from Lemma A.5 respectively yields
\[ |\hat{\Delta}| \lesssim R_{q_2} (\lambda_{n,3})^{1-q_2}. \]

**Remark.** Inequality (A.2) implies that $|\hat{\Delta}| \lesssim R_{q_2} (\lambda_{n,3})^{1-q_2}$.

**Lemma A.2:** Define the thresholded subset
\[ S_{\hat{\Delta}} := \{ j \in \{1, 2, ..., p \} : |\beta_j^*| > \tau \}. \]
Under the assumptions in Theorem 4.1 (or Theorem 4.2) and the choice $\tau = \frac{\lambda_{n,3}}{\kappa L}$, if
\[ R_{q_2} \hat{\Delta} \left( \frac{\log p}{n} + T_1 + T_2 + T_3 \right) = O(\kappa L), \]
the RE condition (2.1) of $\frac{\hat{\Delta}}{n}$ holds over the set
\[ C(S_{\hat{\Delta}}; q_2, 3) \cap \{ \Delta \in \mathbb{R}^p : |\Delta|_2 \geq \delta^* \} \]
where
\[ \delta^* = O \left( \max \left\{ T_1^{3/2}, T_2^{3/2}, T_3^{3/2}, T_4^{3/2} \right\} R_{q_2} \right). \]

**Proof.** The argument is similar to what is used in the proof of Lemma 2 from Negahban, et. al (2010). For any $\Delta \in C(S_{\hat{\Delta}}; q_2, 3)$, we have
\[ |\Delta| \leq 4|\Delta|_{S_{\hat{\Delta}}} + 4|\beta_{S_{\hat{\Delta}}}^*| \leq 4\sqrt{|S_{\hat{\Delta}}|} |\Delta|_2 + 4R_{q_2} \hat{\Delta}^{1-q_2} \leq 4\sqrt{R_{q_2} \hat{\Delta}^{1-q_2}} |\Delta|_2 + 4R_{q_2} \hat{\Delta}^{1-q_2}, \]
where we have used the bound in (A.2) from the proof of Lemma A.1. Therefore, for any vector $\Delta \in C(S_{\hat{\Delta}}; q_2, 3)$ and the choice $\tau = \frac{\lambda_{n,3}}{\kappa L}$, substituting the upper bound $4\sqrt{R_{q_2} \hat{\Delta}^{1-q_2}} |\Delta|_2 + 4R_{q_2} \hat{\Delta}^{1-q_2}$ on $|\Delta|_1$ into condition (A.5) from Lemma A.3 and condition (A.9) from Lemma A.5 respectively yields
\[ \left| \Delta \otimes \hat{\Delta}^T \hat{\Delta} \right| \geq |\Delta|_2 \left\{ c_{\kappa L} - b_0 R_{q_2} \hat{\Delta}^{1-q_2} \left( \frac{\log p}{n} + T_1 + T_2 + T_3 \right) \right\} - b_1 R_{q_2} \hat{\Delta}^{2-2q_2} \left( \frac{\log p}{n} + T_1 + T_2 + T_3 \right), \]
for some positive constants $b_0$ and $b_1$. With the choice of
\[ \delta^* = O \left( \max \left\{ T_1^{3/2}, T_2^{3/2}, T_3^{3/2}, T_4^{3/2} \right\} R_{q_2} \right), \]
if
\[ R_{q_2} \hat{\Delta} \left( \frac{\log p}{n} + T_1 + T_2 + T_3 \right) = O(\kappa L), \]
we have
\[ \left| \Delta \otimes \hat{\Delta}^T \hat{\Delta} \right| \geq c'_{\kappa L} \left\{ |\Delta|_2 - \frac{|\Delta|_2^2}{2} \right\} = c''_{\kappa L} |\Delta|_2 \]
for any $\Delta$ such that $|\Delta|_2 \geq \delta^*$.
A.2 Theorem 4.1

Lemma A.1 implies that the $l_2$-consistency of $\hat{\beta}_{HSEL}$ requires verifications of two conditions: (i) $\hat{\Gamma} = \hat{v}^T \hat{v}$ satisfies the RE condition (2.1) with $\gamma = 3$, and (ii) the term $|\frac{1}{n} \hat{v}^T \hat{e}|_\infty \lesssim \phi(k_1, k_2, d, p, n)$ with high probability. This is done via Lemmas A.3 and A.4. For the proofs for Lemmas A.3 and A.4, let the local complexity measure $G_n(r_j; F_j)$ be defined over the set

$$\Omega(r_j; F_j) = \{ f : f \in F_j \mid |f_{\theta^*}|_n \leq r_j \}$$

where $|f_{\theta^*}|_n := \sqrt{\frac{1}{n} \sum_{i=1}^n (f(\hat{w}_i^T \theta)^*)^2}$ and $j = 0, ..., p$. Recall the notations $\mathbb{E}(x_{ij} | w_i^T \theta) := m_j(w_i^T \theta)$, $\mathbb{E}(y_{k2} | w_i^T \theta) := m_0(w_i^T \theta)$, $\mathbb{E}(y_{k2} | w_i^T \theta) := m_0(w_i^T \theta)$. To avoid cluttering, write $\{\hat{m}_j(w_i^T \theta)\}^n_{i=1} := \hat{m}_j(\theta)$, $\{\hat{m}_j(w_i^T \theta)\}^n_{i=1} := \hat{m}_j(\theta)$, and $\{m_j(w_i^T \theta)\}^n_{i=1} := m_j(\theta)$; these definitions are somewhat abuse of notation but keep in mind that the individual component $\hat{m}_j(\cdot)$ of $m_j(\cdot)$, $\hat{m}_j(\cdot)$ of $\hat{m}_j(\cdot)$, and $m_j(\cdot)$ of $m_j(\cdot)$ are functions from $\mathbb{R}$ to $\mathbb{R}$. Also recall $T_j' := T_j^* \vee T_j$, $T_j' := \sup_{f \in F_j} \frac{1}{n} \sum_{i=1}^n (f(\hat{w}_i^T \hat{\theta}) - f(\hat{w}_i^T \theta^*))^2$, $T_j := \sup_{f \in F_j} \frac{1}{n} \sum_{i=1}^n (f(\hat{w}_i^T \hat{\theta}) - f(\hat{w}_i^T \theta^*))^2$, and the following definitions for Theorem 4.1:

$$\begin{align*}
T_1 &= \max_{j \in \{0, ..., p\}} \left( T_j' \vee \sqrt{T_j} \right) \\
T_2 &= \max_{j \in \{0, ..., p\}} \lambda_{nj}^2 \\
T_3 &= \max_{j \in \{0, ..., p\}} \inf_{m_j \in F_j} \left( |\hat{m}_j(\hat{\theta}) - m_j(\hat{\theta})|^2 + |\hat{m}_j(\theta^*) - m_j(\theta^*)|_n \right) \\
T_4 &= \sigma_v \sigma_q \sqrt{\frac{|\log p|}{n}}.
\end{align*}$$

To facilitate the presentation, the proofs for Lemmas A.3 and A.4 work through the case $t_{nj} = r_{nj}$. Inspecting these proofs suggests that with minor notation changes, the case where $t_{nj} \geq r_{nj}$ can be analyzed with almost the same argument because for any star-shaped function class $\bar{F}_j := \{ f = f' - f'' : f', f'' \in F_j \}$, the function $t \mapsto \frac{G_n(t; F_j)}{t}$ is non-decreasing on $(0, \infty)$.

Lemma A.3 (RE condition): Define $\hat{\Delta} = \hat{\beta}_{HSEL} - \beta^*$. Let $r_{nj} > 0$ be the smallest positive quantity satisfying the critical inequality

$$G_n(r_{nj}; F_j) \leq \frac{r_{nj}^2}{\sigma_{v_j}}.$$

Under Assumptions 2.1, 2.3, 4.1-4.3, we have

$$|\Delta^T \hat{v}^T \hat{v} | \geq \kappa_1 |\Delta|_2^2 - \kappa_2 \frac{\log p}{n} |\Delta|_1^2 - c (T_1 + T_2 + T_3) |\Delta|_1^2$$

(A.5)

with probability at least $1 - c' \sum_{j=1}^p \exp \left(-n G_j \frac{r_{nj}^2}{\sigma_{v_j}} \right)$, where $\kappa_1 = \frac{\kappa_4}{2}$, $\kappa_2 = c_0 \kappa_L \max \{ \frac{\sigma_v^4}{\kappa_L^4}, 1 \}$, $\sigma_v = \max_j \sigma_{v_j}$.

Proof. We have

$$|\Delta^T \frac{\hat{v}^T \hat{v}}{n} | + |\Delta^T \left( \frac{\hat{v}^T \hat{v} - \hat{v}^T \hat{v}}{n} \right) | \Delta | \geq |\Delta^T \frac{\hat{v}^T \hat{v}}{n} |.$$
which implies
\[
\left| \frac{\Delta^T v^T v}{n} \Delta - \Delta^T \left( \frac{v^Tv - \hat{v}^T \hat{v}}{n} \right) \Delta \right| \geq \left| \Delta^T v^T v \Delta - \left( \left| v^T (\hat{v} - v) \right|_\infty + \left| (\hat{v} - v)^T \hat{v} \right|_\infty \right) \right| |\Delta|^2_1
\]
\[
- \left| \frac{v^T (\hat{v} - v)}{n} \right|_\infty |\Delta|^2_1 - \left| \frac{(\hat{v} - v)^T (\hat{v} - v)}{n} \right|_\infty |\Delta|^2_1.
\]

To bound the term \( |\frac{v^T (\hat{v} - v)}{n}|_\infty \), let us first fix \((j, j')\) and bound the \((j, j')\) element of the matrix \( \frac{v^T (\hat{v} - v)}{n} \).

Notice that
\[
\left| \frac{1}{n} v^T_j (\hat{v}_j - v_j) \right| = \left| \frac{1}{n} \sum_{i=1}^n v_{ij} (\hat{v}_{ij} - v_{ij}) \right|
\]
\[
= \left| \frac{1}{n} \sum_{i=1}^n v_{ij} \left( \hat{m}_{j'}(w_i^T \hat{\theta}) - m_{j'}(w_i^T \hat{\theta}) + m_{j'}(w_i^T \hat{\theta}) - m_{j'}(w_i^T \theta^*) \right) \right|
\]
\[
\leq \left| \frac{1}{n} \sum_{i=1}^n v_{ij} \left[ \hat{m}_{j'}(w_i^T \hat{\theta}) - m_{j'}(w_i^T \hat{\theta}) \right] \right| + \sqrt{\frac{1}{n} \sum_{i=1}^n v^2_{ij} T^*_{j'}},
\]
where \( T^*_{j'} := \sup_{f \in F^*_{j'}} \frac{1}{n} \sum_{i=1}^n \left[ f(w_i^T \hat{\theta}) - f(w_i^T \theta^*) \right]^2 \).

For the term \( \sqrt{\frac{1}{n} \sum_{i=1}^n v^2_{ij}} \), by Lemma B.1, we have
\[
\mathbb{P} \left[ \max_{j=1, \ldots, p} \left\{ \frac{1}{n} \sum_{i=1}^n v^2_{ij} \right\} \geq \sigma^2_{v_j} + t \right] \leq 2 \exp(-c \min \left\{ \frac{nt^2}{4\sigma^4_{v_j}}, \frac{nt}{2\sigma^2_{v_j}} \right\} + \log p).
\]

Hence, setting \( t = \sigma^2_{v} := \max_j \sigma^2_{v_j} \) and under the condition \( n \gtrsim \log p \), we have
\[
\mathbb{P} \left[ \max_{j=1, \ldots, p} \left\{ \frac{1}{n} \sum_{i=1}^n v^2_{ij} \right\} \geq 2\sigma^2_{v} \right]\leq 2 \exp(-c' n).
\]

Let \( \hat{\Delta}_{ij'}(\theta) := \hat{m}_{j'}(w_i^T \theta) - m_{j'}(w_i^T \theta) \), \( \tilde{\Delta}_{ij'}(\theta) := \hat{m}_{j'}(w_i^T \theta) - \hat{m}_{j'}(w_i^T \theta) \), and \( \tilde{\Delta}_{ij'}(\theta) := \hat{m}_{j'}(w_i^T \theta) - m_{j'}(w_i^T \theta) \). Note that \( \hat{m}_{j'} \in F^*_{j'} \) and \( \hat{\Delta}_{ij'} \in F_{j'} \). For the term
\[
\left| \frac{1}{n} \sum_{i=1}^n v_{ij} \left[ \hat{m}_{j'}(w_i^T \hat{\theta}) - m_{j'}(w_i^T \hat{\theta}) \right] \right|,
\]
we have
\[
\left| \frac{1}{n} \sum_{i=1}^{n} v_{ij} \hat{\Delta}_{ij}'(\hat{\theta}) \right| \leq \left| \frac{1}{n} \sum_{i=1}^{n} v_{ij} \hat{\Delta}_{ij}'(\theta^*) \right| + \left( \frac{1}{n} \sum_{i=1}^{n} v_{ij}^2 \right) \left| \frac{1}{n} \sum_{i=1}^{n} \left[ \hat{\Delta}_{ij}'(\hat{\theta}) - \hat{\Delta}_{ij}'(\theta^*) \right]^2 \right|
\]
\[
\leq \left| \frac{1}{n} \sum_{i=1}^{n} v_{ij} \hat{\Delta}_{ij}'(\theta^*) \right| + \left( \frac{2}{n} \sum_{i=1}^{n} v_{ij}^2 \right) \left| T_{j}' + T_{\tilde{j}} \right|
\]
\[
\leq \left| \frac{1}{n} \sum_{i=1}^{n} v_{ij} \hat{\Delta}_{ij}'(\theta^*) \right| + \sqrt{\frac{1}{n} \sum_{i=1}^{n} v_{ij}^2} \left| \hat{m}_{j}'(\theta^*) - m_{j}'(\theta^*) \right| \left| n \right| + \sqrt{\frac{1}{n} \sum_{i=1}^{n} v_{ij}^2} \left( n \Omega \right)
\].

To upper bound the term \( \frac{1}{n} \sum_{i=1}^{n} v_{ij} \hat{\Delta}_{ij}'(\theta^*) \), we argue similarly as in Lemma A.11. First, note that by Lemma A.11 and the triangle inequality, we have
\[
\left| \hat{\Delta}_{ij}'(\theta^*) \right| \leq \left| \hat{m}_{j}'(\hat{\theta}) - m_{j}'(\hat{\theta}) \right| + \left| \hat{m}_{j}'(\hat{\theta}) - m_{j}'(\theta^*) \right| + 2\sqrt{T_{\tilde{j}}}
\]
\[
\leq c' \left\{ \sqrt{T_{\tilde{j}}} + \left( \sigma^2_{\tilde{j}} \right) \hat{w}_{\tilde{j}}^2 \right\} + \left| \hat{m}_{j}'(\hat{\theta}) - m_{j}'(\theta^*) \right| + r_{\tilde{j}} := \hat{r}_{\tilde{j}}.
\]

Setting \( u = \hat{r}_{\tilde{j}} \) in \( \Omega(u; F_j) \) from Lemma B.3 and following the argument in the proof for Lemma B.3, we obtain
\[
\max_{j} \left\{ \frac{1}{n} \sum_{i=1}^{n} v_{ij} \hat{\Delta}_{ij}'(\theta^*) \right\} \leq 2 \max_{j} \hat{r}_{\tilde{j}}^2
\]
with probability at least \( 1 - c \sum_{j=1}^{p} \exp \left( -nC_{\tilde{j}}^2 \hat{r}_{\tilde{j}}^2 \right) \). Hence, we obtain
\[
\max_{j, j'} \left\{ \left| \frac{1}{n} \sum_{i=1}^{n} v_{ij} \left[ \hat{m}_{j}'(w_i^T \hat{\theta}) - m_{j}'(w_i^T \hat{\theta}) \right] \right| \right\} \leq c \max_{j} \left\{ \hat{r}_{\tilde{j}}^2 + \sigma_v \left| m_{j}'(\theta^*) - m_{j}'(\theta^*) \right| + \sqrt{T_{\tilde{j}}} \right\}.
\]

Consequently, we have
\[
\left| \frac{(\hat{\theta} - \theta)^T}{n} \left( \hat{\theta} - \hat{\theta} \right) \right|_{\infty} \leq c \max_{j} \left\{ \hat{r}_{\tilde{j}}^2 + \sigma_v \left| m_{j}'(\theta^*) - m_{j}'(\theta^*) \right| + \sqrt{T_{\tilde{j}}} \right\}.
\]

To bound the term \( \left| \frac{(\hat{\theta} - \theta)^T}{n} \left( \hat{\theta} - \theta \right) \right|_{\infty} \), note that we have
\[
\left| \frac{(\hat{\theta} - \theta)^T}{n} \left( \hat{\theta} - \theta \right) \right|_{\infty} \leq 2 \max_{j, j'} \left( \frac{1}{n} \sum_{i=1}^{n} \left[ \hat{m}_{j}'(w_i^T \hat{\theta}) - m_{j}'(w_i^T \hat{\theta}) \right] \right) \left( \frac{1}{n} \sum_{i=1}^{n} \left[ \hat{m}_{j}'(w_i^T \hat{\theta}) - m_{j}'(w_i^T \theta^*) \right] \right)^2
\]
\[
+ 2 \max_{j, j'} \left( \frac{1}{n} \sum_{i=1}^{n} \left[ m_{j}'(w_i^T \theta) - m_{j}'(w_i^T \theta^*) \right] \right) \left( \frac{1}{n} \sum_{i=1}^{n} \left[ m_{j}'(w_i^T \theta) - m_{j}'(w_i^T \theta^*) \right] \right)^2
\]
\[
\leq c \max_{j} \left( T_{j} + \left| \hat{m}_{j}(\hat{\theta}) - m_{j}(\hat{\theta}) \right| \right)^2 \left| n \right| + r_{\tilde{j}}^2
\]
with probability at least \( 1 - c' \sum_{j=1}^{p} \exp \left( -nC_{\tilde{j}}^2 \hat{r}_{\tilde{j}}^2 \right) \).
Putting everything together and applying Lemma B.2, we have
\[
\left| \Delta'T\hat{\nu}'T\hat{\Delta} \right| \geq \kappa_1|\Delta'2 - \kappa_2 \frac{\log p}{n} |\Delta'2 | \nonumber
- b(\sigma_v) \max_j \left\{ \frac{2}{n} + \inf_{\hat{m}_j} \left( |\hat{m}_j(\hat{\theta}) - m_j(\hat{\theta})|^2_n + |\hat{m}_j(\theta^*) - m_j(\theta^*)|^2_n \right) + \sqrt{T_j' \vee T_j'^2} \right\} |\Delta'1 |
\]
where \( \kappa_1 = \frac{\kappa L}{\kappa L} \), \( \kappa_2 = c_0 \kappa L \max \left\{ \frac{\sigma_v^2}{\kappa L}, 1 \right\}, \sigma_v = \max_j \sigma_{v_j} \), and a known function \( b(\sigma_v) \) depending only on \( \sigma_v, \Box \).

Lemma A.4 (Upper bound on \( \frac{1}{n} \hat{\nu}T \epsilon_{\infty} \)): Let \( r_{nj} > 0 \) be the smallest positive quantity satisfying the critical inequality
\[
G_n (r_{nj}; F_j) \leq \frac{r_{nj}^2}{\sigma_{v_j}}.
\]
Under Assumptions 2.1 and 4.1-4.3, we have
\[
\left| \frac{\hat{\nu}T \epsilon}{n} \right| \leq |\beta^*|_1 b(\sigma_v, \sigma_{v_j}) (T_1 + T_2 + T_3) + T_4,
\]
with probability at least \( 1 - c_1 \exp ( -c_2 \log p ) - c_3 \sum_{j=0}^p \exp \left( -nG_{j, r_{nj}}^2 \right) \).

Proof. Recall that
\[
e = \left[ \hat{E} \left( x|w\hat{\theta} \right) - E \left( x|w\hat{\theta} \right) + E \left( x|w\theta^* \right) \right] \beta^*
- \left[ \hat{E} \left( y_2|w\hat{\theta} \right) - E \left( y_2|w\hat{\theta} \right) + E \left( y_2|w\theta^* \right) \right] + \eta.
\]
and
\[
\hat{\nu} = x - \hat{E} \left( x|w\hat{\theta} \right)
= x - E \left( x|w\theta^* \right) + E \left( x|w\theta^* \right) - E \left( x|w\hat{\theta} \right) + E \left( x|w\hat{\theta} \right) - \hat{E} \left( x|w\hat{\theta} \right).
\]
Recall \( v = x - E \left( x|w\theta^* \right) \). Let us introduce the following notations \( m_j(w_i^T \hat{\theta}) - m_j(w_i^T \theta^*) := T''_j \) and \( \hat{m}_j(w_i^T \hat{\theta}) - \hat{m}_j(w_i^T \hat{\theta}) := \hat{\Delta}_j(w_i^T \hat{\theta}) \), and
\[
T_1 = \max_{j,j'=1, ..., p} \left| \frac{1}{n} \sum_{i=1}^n \left[ v_{ij} - T''_{ij} - \hat{\Delta}_j(w_i^T \hat{\theta}) \right] \left[ T''_{ij} + \hat{\Delta}_j(w_i^T \hat{\theta}) \right] \right| \beta^* |_1 \ (A.6)
\]
\[
T_2 = \max_{j=1, ..., p} \left| \frac{1}{n} \sum_{i=1}^n \left[ v_{ij} - T''_{ij} - \hat{\Delta}_j(w_i^T \hat{\theta}) \right] \left[ T''_{ij} + \hat{\Delta}_j(w_i^T \hat{\theta}) \right] \right| \eta \ (A.7)
\]
\[
T_3 = \max_{j=1, ..., p} \left| \frac{1}{n} \sum_{i=1}^n \left[ v_{ij} - T''_{ij} - \hat{\Delta}_j(w_i^T \hat{\theta}) \right] \eta \right| \ (A.8)
\]
Expanding the products in (A.6)-(A.8), applying the Cauchy-Schwarz inequality on some of the terms in
the expansion, we obtain the following inequalities

\[
\frac{1}{n} \sum_{i=1}^{n} v_{ij} T_{ij}'' \leq \sqrt{\frac{1}{n} \sum_{i=1}^{n} v_{ij}^2} \sqrt{\frac{1}{n} \sum_{i=1}^{n} T_{ij}''^2} \\
\frac{1}{n} \sum_{i=1}^{n} v_{ij} \hat{\Delta}_{ij}'(\hat{\theta}) \leq \sqrt{\frac{1}{n} \sum_{i=1}^{n} v_{ij}^2} \sqrt{\frac{1}{n} \sum_{i=1}^{n} \left(\hat{\Delta}_{ij}'(w_i^T \theta^*) - \hat{\Delta}_{ij}(w_i^T \theta^*)\right)^2} \\
\frac{1}{n} \sum_{i=1}^{n} T_{ij}'' \hat{\Delta}_{ij}'(\hat{\theta}) \leq \sqrt{\frac{1}{n} \sum_{i=1}^{n} T_{ij}''^2} \sqrt{\frac{1}{n} \sum_{i=1}^{n} \hat{\Delta}_{ij}'(w_i^T \hat{\theta})} \\
\frac{1}{n} \sum_{i=1}^{n} \hat{\Delta}_{ij}(\hat{\theta}) \hat{\Delta}_{ij}'(\hat{\theta}) \leq \sqrt{\frac{1}{n} \sum_{i=1}^{n} \hat{\Delta}_{ij}'(w_i^T \hat{\theta})} \sqrt{\frac{1}{n} \sum_{i=1}^{n} \hat{\Delta}_{ij}(w_i^T \hat{\theta})} \\
\frac{1}{n} \sum_{i=1}^{n} T_{ij}' \eta_i \leq \sqrt{\frac{1}{n} \sum_{i=1}^{n} \eta_i^2} \sqrt{\frac{1}{n} \sum_{i=1}^{n} T_{ij}''^2} \\
\frac{1}{n} \sum_{i=1}^{n} \hat{\Delta}_{ij}(\hat{\theta}) \eta_i \leq \sqrt{\frac{1}{n} \sum_{i=1}^{n} \eta_i^2} \sqrt{\frac{1}{n} \sum_{i=1}^{n} \left(\hat{\Delta}_{ij}(w_i^T \hat{\theta}) - \hat{\Delta}_{ij}(w_i^T \theta^*)\right)^2}.
\]

To upper bound \( \left\| \frac{v_{ij} T_{ij}'}{n} \right\|_{\infty} \), we need to control the RHS inequalities listed above as well as the additional term \( \max_j \left\{ \frac{1}{n} \sum_{i=1}^{n} v_{ij} \eta_i \right\} \).

Under Assumptions 4.2 and 4.3, and the condition \( n \geq \log p \), applying Lemma B.1 yields

\[
\mathbb{P} \left[ \max_j \left\{ \frac{1}{n} \sum_{i=1}^{n} v_{ij}^2 \right\} \geq 2 \sigma_v^2 \right] \leq 2 \exp(-c' n), \\
\mathbb{P} \left[ \frac{1}{n} \sum_{i=1}^{n} \eta_i^2 \geq 2 \sigma_{\eta}^2 \right] \leq 2 \exp(-c'' \log p), \\
\mathbb{P} \left[ \max_j \left\{ \frac{1}{n} \sum_{i=1}^{n} v_{ij} \eta_i \right\} \geq c_0 \sigma_v \sigma_{\eta} \sqrt{\frac{\log p}{n}} \right] \leq 2 \exp(-c'' \log p).
\]

An upper bound on the term \( \sqrt{\frac{1}{n} \sum_{i=1}^{n} \hat{\Delta}_{ij}'(w_i^T \hat{\theta})} \) is derived in Lemma A.11, which yields

\[
\max_j \left\{ \sqrt{\frac{1}{n} \sum_{i=1}^{n} \hat{\Delta}_{ij}'(w_i^T \hat{\theta})} \right\} \leq c \max_j \left\{ \sqrt{T_j' + \left( \sigma_v^2 T_j' \right)^2 + \left| \tilde{m}_j(\hat{\theta}) - m_j(\hat{\theta}) \right|} \right\},
\]

with probability at least \( 1 - c' \sum_{j=0}^{p} \exp\left(-n C_{\eta}^2 r_{n_j}^2 \right) \). An upper bound on the term \( \sqrt{\frac{1}{n} \sum_{i=1}^{n} T_{ij}''^2} \) is simply \( \max_j \sqrt{T_j'} \) by recalling the definition of \( T_j' \).

To upper bound the term \( \left\| \frac{1}{n} \sum_{i=1}^{n} v_{ij} \hat{\Delta}_{ij}'(w_i^T \theta^*) \right\| \), recall from the proof for Lemma A.3, we have
where for the proofs for Lemmas A.5 and A.6, let the local complexity

\[ 1 \]

with probability at least \( 1 - c' \sum_{j=1}^{p} \exp \left( -nC_j^* r_{nj}^2 \right) \). Applying the same argument from above to control the term \( \left| \frac{1}{n} \sum_{i=1}^{n} \eta_i \Delta_{ij}^{*}(\theta^*) \right| \) yields

\[ \max_{j', j} \left| \frac{1}{n} \sum_{i=1}^{n} v_{ij} \Delta_{ij}^{*}(w_i^T \theta^*) \right| \leq c \max_{j} \left\{ \hat{r}^2_{nj} + \sigma_v \left| \tilde{m}_j^{*}(\theta^*) - m_j^{*}(\theta^*) \right| \right\} + \sqrt{\sum_{j} \sigma^2 T_j} \]

with probability at least \( 1 - c'' \sum_{j=1}^{p} \exp \left( -nC_j^* r_{nj}^2 \right) \).

Putting all the pieces together, we have

\[ \left| \frac{\hat{\beta}^T e}{n} \right|_{\infty} \leq |\beta^*|_1 b(\sigma_v, \sigma_\eta) (T_1 + T_2 + T_3) + T_4 \]

with probability at least \( 1 - c_1 \exp \left( -c_2 \log p \right) - c_3 \sum_{j=1}^{p} \exp \left( -nC_j^* r_{nj}^2 \right) \). \( \square \)

Now, by applying Lemma A.1 and setting \( \lambda_{n, 3} \geq |\beta^*|_1 b(\sigma_v, \sigma_\eta) (T_1 + T_2 + T_3) + T_4 \) and combining with Lemma A.3, we obtain

\[ |\hat{\beta}_{HSEL} - \beta^*|_2 \leq \frac{c'' R_{\beta}^2}{\kappa L^{1-\frac{\omega_2}{2}}} \left| |\beta^*|_1 b(\sigma_v, \sigma_\eta) (T_1 + T_2 + T_3) + T_4 \right|^{1-\frac{\omega_2}{2}}, \quad q_2 \in [0, 1] \]

with probability at least \( 1 - c_1 \exp \left( -c_2 \log p \right) - c_3 \sum_{j=1}^{p} \exp \left( -nC_j^* r_{nj}^2 \right) \).

**A.3 Theorem 4.2**

As in the proofs for Theorem 4.1, the proofs for Lemmas A.5 and A.6 work through the case \( t_{nj} = r_{nj} \). The case where \( t_{nj} \geq r_{nj} \) can be analyzed with almost the same argument because for any star-shaped function class \( \bar{F}_j := \left\{ f = f' - f'' : f', f'' \in F_j \right\} \), the function \( t \mapsto \mathcal{G}_n(t; F_j) \) is non-decreasing on \((0, \infty)\). For the proofs for Lemmas A.5 and A.6, let the *local complexity* measure \( \mathcal{G}_n(r_j; F_j) \) be defined over the set

\[ \Omega(r_j; F_j) = \left\{ f : f \in F_j \quad |f_{\theta^*}| \leq r_j, \quad |f|_{F_j} \leq 1 \right\} \]

where \( |f_{\theta^*}| : = \sqrt{\frac{1}{n} \sum_{i=1}^{n} \left[ f(w_i^T \theta^*) \right]^2} \) and \( j = 0, \ldots, p \). Recall the following definitions for Theorem 4.2:

\[ T_1 = \max_{j \in \{0, \ldots, p\}} \left( T_j' \vee \sqrt{T_j'} \right) \]

\[ T_2 = \max_{j \in \{0, \ldots, p\}} R_{nj}^2 \]

\[ T_3 = \max_{j \in \{0, \ldots, p\}} \inf_{\tilde{m}_j \in F_j, |\tilde{m}_j|_{F_j} \leq \tilde{R}_j} \left( \left| \tilde{m}_j(\hat{\theta}) - m_j(\hat{\theta}) \right|^2 \right) \]

\[ T_4 = \sigma_v \sigma_\eta \sqrt{\frac{\log p}{n}} \]

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Lemma A.5 (RE condition): Let \( r_{nj} > 0 \) be the smallest positive quantity satisfying the critical inequality
\[
\mathcal{G}_n (r_{nj}; \mathcal{F}_j) \leq \frac{\bar{R}_j r_{nj}^2}{\sigma_{v_j}},
\]
where \( \bar{R}_j > 0 \) is a user-defined radius. Under Assumptions 2.1, 2.3, 4.1-4.3, if we choose the second-stage regularization parameter \( \lambda_{nj,2} = (2 + \varsigma)r_{nj}^2 \) for \( j = 1, \ldots, p \) where \( \varsigma \) is a small positive constant, then
\[
\left| \frac{\Delta^T \hat{\nu}^T \hat{\nu}}{n} \right| \geq \kappa_1 |\Delta|^2 - \kappa_2 \frac{\log p}{n} |\Delta|^2 - c (T_1 + T_2 + T_3) |\Delta|^2
\]
(A.9)
with probability at least \( 1 - c' \sum_{j=1}^p \exp \left( -n C_j^* \bar{R}_j^2 r_{nj}^2 \right) \).

Proof. The proof is almost identical to that of Lemma A.3 except that in upper bounding the term
\[
\frac{1}{n} \sum_{i=1}^n \left( \hat{m}_j (w_i^T \hat{\theta}) - m_j (w_i^T \hat{\theta}) \right)^2
\]
we apply Lemma A.12 with the second-stage regularization parameter \( \lambda_{nj,2} = (2 + \varsigma)r_{nj}^2 \) where \( \varsigma \) is a small positive constant for \( j = 1, \ldots, p \). Using similar argument as in the proof for Lemma A.3 yields
\[
\frac{1}{n} \sum_{i=1}^n \left[ \hat{m}_j (w_i^T \hat{\theta}) - m_j (w_i^T \hat{\theta}) \right]^2 \leq c_0 \left( \hat{m}_j (\theta) - m_j (\hat{\theta}) \right)^2_n + c_1 \bar{R}_j^2 r_{nj}^2 + c_2 T_j' + c_3 \sqrt{\sigma_{v_j}^2 T_j'}
\]
with probability at least \( 1 - c \exp \left( -n C_j^* \bar{R}_j^2 r_{nj}^2 \right) \). The second-stage regularization parameter \( \lambda_{nj,2} \) is chosen such that \( \lambda_{nj,2} \geq 2r_{nj}^2 \) for \( j = 1, \ldots, p \). Using similar argument as in the proof for Lemma A.3 yields
\[
\frac{|\nu^T (\hat{\nu} - v)|}{\infty} \leq c \max_j \left\{ \inf_{\hat{\nu}_j \in \mathcal{F}_j, |\hat{\nu}_j|_{\infty} \leq \bar{R}_j} \left( \left| \hat{m}_j (\hat{\theta}) - m_j (\hat{\theta}) \right|_n^2 + \left| \hat{m}_j (\theta^*) - m_j (\theta^*) \right|_n \right) + \bar{R}_j^2 r_{nj}^2 + T_j' + \sqrt{\left( \sigma_{v_j}^2 T_j' \right)} \right\}
\]
with probability at least \( 1 - c \sum_{j=1}^p \exp \left( -n C_j^* \bar{R}_j^2 r_{nj}^2 \right) \).

For the term \( \left| \frac{(\hat{\nu} - v)^T (\hat{\nu} - v)}{\infty} \right| \), we have
\[
\left| \frac{(\hat{\nu} - v)^T (\hat{\nu} - v)}{\infty} \right| \leq c' \max_j \left[ \left| \hat{m}_j (\hat{\theta}) - m_j (\hat{\theta}) \right|_n^2 + \bar{R}_j^2 r_{nj}^2 + T_j' + \sqrt{\sigma_{v_j}^2 T_j'} \right]
\]
with probability at least \( 1 - c \sum_{j=1}^p \exp \left( -n C_j^* \bar{R}_j^2 r_{nj}^2 \right) \).

Putting everything together and applying Lemma B.2, we have
\[
\left| \Delta^T \hat{\nu}^T \hat{\nu} \Delta \right| \geq \kappa_1 |\Delta|^2 - \kappa_2 \frac{\log p}{n} |\Delta|^2 - b(\sigma_{v}) \max_j \left\{ \bar{R}_j^2 r_{nj}^2 + \inf_{\hat{\nu}_j \in \mathcal{F}_j} \left( \left| \hat{m}_j (\hat{\theta}) - m_j (\hat{\theta}) \right|_n^2 + \left| \hat{m}_j (\theta^*) - m_j (\theta^*) \right|_n \right) + \sqrt{T_j' \vee T_j''} \right\} |\Delta|^2
\]
(A.10)
where \( \kappa_1 = \frac{\sigma_{v}^2}{\kappa_L} \), \( \kappa_2 = c_0 \kappa_L \max \left\{ \frac{\sigma_{v}^2}{\kappa_L}, 1 \right\} \), and \( \sigma_{v} = \max_j \sigma_{v_j} \), and a known function \( b(\sigma_{v}) \) depending only on \( \sigma_{v} \). \( \Box \)
**Lemma A.6** (Upper bound on $\frac{1}{n} \hat{v}^T e_\infty$): Let $r_{nj} > 0$ be the smallest positive quantity satisfying the critical inequality
\[
\mathcal{G}_n (r_{nj}; F_j) \leq \frac{R_j r_{nj}^2}{\sigma_{v_j}},
\]
and $R_j > 0$ is a user-defined radius. Under Assumptions 2.1 and 4.1-4.3 and if we choose the second-stage regularization parameter $\lambda_{nj,2} = (2 + \varsigma) r_{nj}^2$ where $\varsigma$ is a small positive constant for $j = 1, ..., p$, we have
\[
\frac{1}{n} \sum_{i=1}^{n} \left| \hat{v}^T e \right|_\infty \leq |\beta^*_1|_1 b(\sigma_v, \sigma_\eta) (T_1 + T_2 + T_3) + T_4,
\]
with probability at least $1 - c_1 \exp (-c_2 \log p) - c_3 \sum_{j=1}^{p} \exp \left(-n C_j^* R_j^2 r_{nj}^2 \right)$.

**Proof.** The proof is almost identical to that of Lemma A.4 except that in upper bounding the term $\sqrt{\frac{1}{n} \sum_{i=1}^{n} \left[ \hat{m}_j (w_i^T \hat{\theta}) - m_j (w_i^T \hat{\theta}) \right]^2}$, we apply Lemma A.12 just as what we have done in the proof for Lemma A.5. □

Consequently, by applying Lemma A.1 and setting $\lambda_{n,3} \geq |\beta^*_1|_1 b(\sigma_v, \sigma_\eta) (T_1 + T_2 + T_3) + T_4$ and combining with Lemma A.5, we obtain
\[
|\hat{\beta}_{HSEL} - \beta^*|_2 \leq \frac{c'}{\kappa_L^2} \left[ |\beta^*_1|_1 b(\sigma_v, \sigma_\eta) (T_1 + T_2 + T_3) + T_4 \right]^{1 - \frac{q_2}{2}}, \quad q_2 \in (0, 1]
\]
with probability at least $1 - c'_1 \exp (-c'_2 \log p) - c'_3 \sum_{j=1}^{p} \exp \left(-n C_j^* \bar{R}_j^2 r_{nj}^2 \right)$.

**A.4 Corollary 4.3**

**Proof.** This follows from the previous proofs for the RE condition and Lemma A.1. □

**A.5 Proposition 4.1 and Corollaries 4.4-4.6**

**Proof of Proposition 4.1.** Note that under the assumption $m_j(\cdot) \in F_j$ for $j = 0, ..., p$, we have $\inf \left| \hat{m}_j (\hat{\theta}) - m_j (\hat{\theta}) \right|^2_n = 0$. When $F_j$ is $L$–Lipschitz, under the Lipschitz regression (22),
\[
T'_j = \frac{1}{n} \sum_{i=1}^{n} L^2 \left[ w_i^T \hat{\theta} - w_i^T \theta^* \right]^2 := L^2 B'
\]
for every $j = 0, ..., p$. Now let us compute the critical radius $r_{nj}$ that satisfies
\[
\mathcal{G}_n (r_{nj}; F_j) \leq \frac{r_{nj}^2}{\sigma_{v_j}},
\]
where $\mathcal{F}_j$ is the space of $L-$Lipschitz functions. The metric entropy of this class (in the sup-norm) scales as $\log N_\infty(t; \mathcal{F}_j) \asymp \frac{1}{t}$ by Lemma B.11. Consequently, applying Lemma B.9 yields

$$
\frac{1}{\sqrt{n}} \int_0^{r_{nj}} \sqrt{\log N_n(t; \mathcal{B}_n(r_{nj}; \mathcal{H}_j))} dt \leq \frac{1}{\sqrt{n}} \int_0^{r_{nj}} \sqrt{\log N_\infty(t; \text{sym}(\mathcal{H}_j))} dt
$$

$$
\leq \frac{c}{\sqrt{n}} \int_0^{r_{nj}} \left( \frac{|\theta^*|_1}{t} \right)^{\frac{1}{2}} dt
$$

$$
= \frac{c}{\sqrt{n}} \sqrt{|\theta^*|_1 t_{nj}}.
$$

Thus, it suffices to choose $r_{nj} > 0$ such that

$$
\frac{\sqrt{r_{nj}}}{\sqrt{n}} \times \frac{r_{nj}^2}{\sigma_{v_j}}
$$

$$
\implies r_{nj}^2 \times \left( \frac{|\theta^*|_1 \sigma_{v_j}^2}{n} \right)^{\frac{2}{3}}.
$$

\[ \Box \]

Proof of Corollaries 4.4-4.6. In proving the two corollaries, we follow the proof for Theorem 4.1. We prove the case where $\beta^*$ and $\theta^*$ are exactly sparse and the approximate sparse case of $\beta^*$ and $\theta^*$ follows the same argument. We have

$$
\max_{j=0, \ldots, p} T'_j \leq \frac{1}{n} \sum_{i=1}^n L^2 \left[ w_i^T \hat{\theta} - w_i^T \theta^* \right]^2 = L^2 B'
$$

Now, depending on the assumptions on $w_i$, we have

$$
B' = \frac{1}{n} \sum_{i=1}^n \left[ w_i^T \hat{\theta} - w_i^T \theta^* \right]^2 \leq c \frac{\kappa_w^2 \sigma_w^2 \alpha_u k_1 \log d}{(\kappa_L^w)^2 n} \quad \text{(Corollary 4.4)}
$$

$$
B' = \frac{1}{n} \sum_{i=1}^n \left[ w_i^T \hat{\theta} - w_i^T \theta^* \right]^2 \leq c \frac{|\theta^*|_1}{\alpha_l} \sqrt{k_1 \log d \frac{1}{n}} \quad \text{(Corollary 4.5)}
$$

with probability at least $1 - O \left( \frac{1}{d} \right)$. Putting the pieces together and applying Theorem 4.1, choosing $t_{nj}^2 \geq r_{nj}^2$ in $T_2$ such that $|\beta^*|_1 T_2$ is at most

$$
O \left( \sqrt{\frac{\log p}{n}} \vee \left( |\beta^*|_1 \sqrt{\frac{k_1 \log d}{n}} \right) \right).
$$

and $\lambda_{n,3}$ in Corollaries 4.4 and 4.5, if $\sqrt{\frac{2 \log p}{|\beta^*|_1} \vee \sqrt{n k_1 \log d}} \lesssim \log p$, and

$$
\kappa_2 \frac{k_2 \log p}{n} + c k_2 \sigma_v \left[ \left( \frac{k_1 \sigma_v^2}{n} \right)^{\frac{2}{3}} \right] + k_2 B' + k_2 \sqrt{\sigma_v^2 B'} = O(\kappa_1),
$$

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then, with probability at least $1 - O\left(\frac{1}{d^{\theta}}\right)$, the error satisfies the bound

$$
|\hat{\beta}_{HSEL} - \beta^*|_2 \leq \frac{c_1 \sqrt{k_2}}{\kappa_L} \left[ \left( \sigma_v \sigma_\eta \sqrt{\frac{\log p}{n}} \right) \vee \left( L \sqrt{\frac{\kappa_L^w \sigma_v^2 \alpha_u}{(\kappa_L^w)^2}} \cdot b(\sigma_v, \sigma_\eta) \cdot |\beta^*|_1 \sqrt{\frac{k_1 \log d}{n}} \right) \right]
$$

(Corollary 4.4)

and

$$
|\hat{\beta}_{HSEL} - \beta^*|_2 \leq \frac{c_2 \sqrt{k_2}}{\kappa_L} \left[ \left( \sigma_v \sigma_\eta \sqrt{\frac{\log p}{n}} \right) \vee \left( L \sqrt{\frac{|\theta^*|_{1,b} (\sigma_v, \sigma_\eta) \cdot |\beta^*|_1 \left( \frac{k_1 \log d}{n} \right)^{\frac{3}{4}}} \right) \right]
$$

(Corollary 4.5).

□

A.6 Proposition 4.2

Proof. First, by Corollary 5 in Negahban, et. al (2012) for the logit model and Lemma A.7 for the probit model, we have, for $\lambda_{n,1} \asymp c_\sigma \sigma_w \sqrt{\alpha_u} \sqrt{\frac{\log d}{n}}$,

$$
|\hat{\theta} - \theta^*|^2 \leq \frac{c_1 \sigma_v^2 \alpha_u k_1 \log d}{(\kappa_L^w)^2} \frac{n}{\lambda_{n,1}}
$$

with probability at least $1 - O\left(\frac{1}{d^{\theta}}\right)$. Let $\Delta = \hat{\theta} - \theta^*$. Under Assumption 4.6, Lemma B.2 implies

$$
\frac{|w\Delta|_2^2}{n} \leq \frac{3\kappa_L^w}{2} |\Delta|_1^2 + \frac{\alpha^{\prime} \log d}{n} |\Delta|_1^2.
$$

Combining the inequality above with Lemma 1 in Negahban, et. al which shows that $|\Delta|_1 \leq 4 \sqrt{k_1} |\Delta|_2$ yields the desired result. □

Lemma A.7: Suppose the number of regressors $d(=d_n)$ can grow with and exceed the sample size $n$ and the number of non-zero components in $\theta^*$ is at most $k_1(=k_{1n})$ and $k_1$ can increase to infinity with $n$ but slowly compared to $n$. Let Assumptions 2.1 and 4.6 (the lower bound part) hold. If $\hat{\theta}$ solves program (13) with $\lambda_{n,1} \geq c_\sigma \sigma_w \sqrt{\alpha_u} \sqrt{\frac{\log d}{n}}$ and $n \gtrsim k_1 \log d$, then,

$$
|\hat{\theta} - \theta^*|_2 \leq \frac{\sqrt{k_1}}{\kappa_L^w} \left( \sqrt{\frac{\alpha_u \sigma_v^2 \log d}{n}} \vee \lambda_{n,1} \right)
$$

with probability at least $1 - O\left(\frac{1}{d^{\theta}}\right)$.

Proof. Recall

$$
L_n(\theta) = \left\{ -\frac{1}{n} \sum_{i=1}^{n} y_i \phi_1(w_i^T \theta) + \frac{1}{n} \sum_{i=1}^{n} \phi_2(w_i^T \theta) + \lambda_{n,1} |\theta|_1 \right\},
$$
where

\[
\phi_1(w_i^T \theta) = \log \frac{\Phi(w_i^T \theta)}{1 - \Phi(w_i^T \theta)},
\]
\[
\phi_2(w_i^T \theta) = -\log [1 - \Phi(w_i^T \theta)] = -\log \frac{1}{1 + \exp(\phi_1(w_i^T \theta))}
\]

for the probit model. Define the following quantity

\[
\delta L_n(\theta^*, \Delta) := L_n(\theta^* + \Delta) - L_n(\theta^*) - \langle \nabla L_n(\theta^*), \Delta \rangle.
\]

To prove Lemma A.7, by Theorem 1 in Negahban, et. al (2012), it suffices to show that Step 1: \( \delta L_n(\theta^*, \hat{\Delta}) \) satisfies the RSC condition (2.4) where \( \hat{\Delta} = \hat{\theta} - \theta^* \); and Step 2: with high probability, we have

\[
|\nabla L_n(\theta^*)|_\infty \leq \sqrt{\frac{\alpha_n \sigma_n^2 \log d}{n}}.
\]

**Step 1:** We need to show that \( \delta L_n(\theta^*, \hat{\Delta}) \) satisfies the RSC condition where \( \hat{\Delta} = \hat{\theta} - \theta^* \). Define \( \rho_{i, \theta} = -y_i \phi_1(w_i^T \theta) + \phi_2(w_i^T \theta), \hat{Q}_1(\theta) = E_{n \rho_{i, \theta}}, \) and \( Q_1(\theta) = E_{\rho_{i, \theta}} \). By the mean value theorem, we have

\[
L_n(\theta^*, \hat{\Delta}) = \hat{Q}_1(\hat{\theta}) - \hat{Q}_1(\theta^*) = \frac{1}{n} \sum_{i=1}^n \rho''_{i, \hat{\theta}} \left[ w_i^T (\hat{\theta} - \theta^*) \right]^2,
\]

where \( \rho''_{i, \hat{\theta}} \) denotes the second derivative of \( \rho_{i, \hat{\theta}} \) and \( \hat{\theta} \) is some intermediate value.

Proposition 1 in Loh and Wainwright (2013) implies that there exists a constant \( \alpha > 0 \), depending on the probit model and \( (\sigma_w, \Sigma_w) \), such that for all vectors \( \theta_2 \in B_2(3) \cap B_1(R) \)

\[
\hat{Q}_1(\theta_2) - \hat{Q}_1(\theta_1) \geq \left\{ \begin{array}{ll}
\frac{T}{2} |\Delta|^2 - \frac{c^2 \sigma_n^2}{2} \log p |\Delta|^2 & \text{for all } |\Delta|^2 \leq 3, \\
3 \frac{\alpha}{2} |\Delta|^2 - 3c \sigma_w \sqrt{\frac{\log p}{n}} |\Delta|_1 & \text{for all } |\Delta|_2 \geq 3,
\end{array} \right.
\]

with probability at least \( 1 - \alpha \exp(-c_2 n) \) and \( \Delta = \theta_2 - \theta_1 \). Combining this result with Lemma 1 in Negahban, et. al (2012) which shows that \( |\hat{\Delta}|_1 \leq 4 \sqrt{\alpha_1} |\hat{\Delta}|_2 \) yields the RSC of \( \delta L_n(\theta^*, \hat{\Delta}) \).

**Step 2:** In the following, we show that \( |\nabla L_n(\theta^*)|_\infty \leq \sqrt{\frac{\alpha_n \sigma_n^2 \log d}{n}} \) with high probability. For a fixed index \( j = 1, ..., d \), we begin by establishing an upper bound on

\[
\frac{1}{n} \sum_{i=1}^n V_{ij} := \frac{1}{n} \sum_{i=1}^n \left[ w_{ij} y_i \phi'_1(w_i^T \theta) - w_{ij} \phi'_2(w_i^T \theta) \right]
\]

where \( \phi'_1(u) = \frac{\partial \phi_1}{\partial u} \) and \( \phi'_2(u) = \frac{\partial \phi_2}{\partial u} \). Let us condition on \( \{w_i\}_{i=1}^n \), so that \( y_{1i} \) is drawn from the exponential family with parameter \( \phi_1(w_i^T \theta) = \log \frac{\Phi(w_i^T \theta)}{1 - \Phi(w_i^T \theta)} \) for the probit model. For any \( t \in \mathbb{R} \), compute
the cumulant function

$$\log \mathbb{E} [\exp(t V_{ij}) \mid w_i] = \log \left\{ \mathbb{E} \left[ \exp \left( t w_{ij} y_{1i} \phi'_i(w_i^T \theta^*) \right) \mid w_i \right] \exp \left( -t w_{ij} \phi'_i(w_i^T \theta^*) \right) \right\}$$

$$= \phi_2 \left( \phi'(w_i^T \theta^*) t w_{ij} + \phi_1(w_i^T \theta^*) - \phi_2(\phi_1(w_i^T \theta^*)) - \phi'_2(\phi_1(w_i^T \theta^*)) (t w_{ij}) \right)$$

$$= \frac{t^2}{2} w_{ij}^2 \left[ \phi''(\phi_1(w_i^T \theta^*) + v_i \phi'_1(w_i^T \theta^*) t w_{ij}) \right]$$

for some $v_i \in [0, 1]$. Since this upper bound holds for each $i = 1, \ldots, n$, we obtain

$$\frac{1}{n} \sum_{i=1}^{n} \log \mathbb{E} [\exp(t V_{ij}) \mid w_i] \leq \frac{t^2}{2} \left\{ \frac{1}{n} \sum_{i=1}^{n} w_{ij}^2 \left[ \phi''(\phi_1(w_i^T \theta^*) + v_i \phi'_1(w_i^T \theta^*) t w_{ij}) \right] \right\}$$

$$\leq \frac{t^2 \alpha_u}{2} \left\{ \frac{1}{n} \sum_{i=1}^{n} w_{ij}^2 \right\}$$

where the inequality follows since $\phi''(u) \leq \alpha_u$ for some $\alpha_u > 0$ and all $u \in \mathbb{R}$. Note that for each $j = 1, \ldots, d$, the variables $\left\{ w_{ij}^2 - \mathbb{E} \left[ w_{ij}^2 \right] \right\}_{i=1}^{n}$ are i.i.d. zero-mean and sub-Gaussian with parameter at most $\sigma_w$. Consequently, we have $\mathbb{E} \left[ w_{ij}^2 \right] \leq \sigma_w^2$. Since the squared variables are sub-exponential, by Lemma B.1, we have the tail bound

$$\mathbb{P} \left[ \frac{1}{n} \sum_{i=1}^{n} w_{ij}^2 \geq 2 \sigma_w^2 \right] \leq 2 \exp \left( -c_1 n \right).$$

Define the event $\Xi = \left\{ \max_{j=1,\ldots,d} \frac{1}{n} \sum_{i=1}^{n} w_{ij}^2 \leq 2 \sigma_w^2 \right\}$. An application of union bound yields

$$\mathbb{P} [\Xi^c] \leq 2 \exp \left( -c_1 n + \log d \right) \leq 2 \exp(-c_1 n)$$

under the condition $n \gtrsim \log d$. Conditioning on the event $\Xi$, we obtain

$$\frac{1}{n} \sum_{i=1}^{n} \log \mathbb{E} [\exp(t V_{ij}) \mid w_i] \leq t^2 \alpha_u \sigma_w^2, \quad \text{for each } j = 1, \ldots, d.$$

An application of the Chernoff bound and the union bound yields

$$\mathbb{P} \left[ \max_{j=1,\ldots,d} \left| \frac{1}{n} \sum_{i=1}^{n} V_{ij} \right| \geq \varepsilon \mid \Xi \right] \leq 2 \exp \left( -c_1 n \frac{\varepsilon^2}{\alpha_u \sigma_w^2} + \log d \right).$$

Setting $\varepsilon = \sqrt{\alpha_u \sigma_w} \log \frac{d}{n}$ and combining with the bound on $\mathbb{P} [\Xi^c]$ yields

$$\mathbb{P} \left[ \max_{j=1,\ldots,d} \left| \frac{1}{n} \sum_{i=1}^{n} V_{ij} \right| \geq \sqrt{\alpha_u \sigma_w} \log \frac{d}{n} \right] \leq \mathbb{P} [\Xi^c] + \mathbb{P} \left[ \max_{j=1,\ldots,d} \left| \frac{1}{n} \sum_{i=1}^{n} V_{ij} \right| \geq \varepsilon \mid \Xi \right]$$

$$\leq c_2 \exp \left( -c_3 n \frac{\varepsilon^2}{\alpha_u \sigma_w} + \log d \right) \leq c_4 \exp(-c_5 \log d),$$

where the last inequality follows since $p > n$ is the regime of our interest.
Now put the pieces together, we obtain

$$|\nabla L_n(\theta^*)|_\infty \leq \sqrt{\alpha u \sigma_w^2 \log d \over n}$$

with probability at least $1 - c' \exp \left(-c'' \log d\right)$.

Combining Step 1 and Step 2 and applying Theorem 1 in Negahban, et. al (2012) yields the desired result. □

**Remark.** Theorem 1 in Negahban, et. al (2012) can be applied to obtain the upper bound in Proposition 4.2 for the case of approximate sparsity $q_1 \in (0, 1]$.

### A.7 Proposition 4.3

**Proof.** Recall from the proof for Proposition 4.2:

$$Q_1(\hat{\theta}) - Q_1(\theta^*) = {1 \over n} \sum_{i=1}^n \rho''_{i, \hat{\theta}} \left[ w_i^T (\hat{\theta} - \theta^*) \right]^2,$$

where $\rho''_{i, \hat{\theta}}$ denotes the second derivative of $\rho_{i, \hat{\theta}}$ and $\hat{\theta}$ is some intermediate value. Assumption 4.4 implies $\rho''_{i, \hat{\theta}} \geq \alpha_l > 0$, so we further have

$$1 \over n \sum_{i=1}^n \left[ w_i^T (\hat{\theta} - \theta^*) \right]^2 \leq \frac{1}{\alpha_l} \left[ \hat{Q}_1(\hat{\theta}) - \hat{Q}_1(\theta^*) \right]. \quad (A.11)$$

Hence, it remains to upper bound $\hat{Q}_1(\hat{\theta}) - \hat{Q}_1(\theta^*)$. Write

$$\hat{Q}_1(\hat{\theta}) - \hat{Q}_1(\theta^*) = \hat{Q}_1(\hat{\theta}) - Q_1(\hat{\theta}) - \left[ \hat{Q}_1(\theta^*) - Q_1(\theta^*) \right] + \left[ Q_1(\hat{\theta}) - Q_1(\theta^*) \right]$$

$$= \Lambda_{\hat{\theta}} - \Lambda_{\theta^*} + E_{\hat{\theta}}, \quad (A.12)$$

where we define $\hat{Q}_1(\hat{\theta}) - Q_1(\hat{\theta}) - \left[ \hat{Q}_1(\theta^*) - Q_1(\theta^*) \right] := \Lambda_{\hat{\theta}} - \Lambda_{\theta^*}$ and the excess risk $Q_1(\hat{\theta}) - Q_1(\theta^*) := E_{\hat{\theta}}$.

Furthermore, by the definition of $\hat{\theta}$, we have $\hat{Q}_1(\hat{\theta}) + \lambda_{n,1} \left| \hat{\theta} \right|_1 \geq \hat{Q}_1(\theta^*) + \lambda_{n,1} \left| \theta^* \right|_1$, which yields the following *basic inequality*

$$E_{\hat{\theta}} + \lambda_{n,1} \left| \hat{\theta} \right|_1 \leq - \left[ \Lambda_{\hat{\theta}} - \Lambda_{\theta^*} \right] + \lambda_{n,1} \left| \theta^* \right|_1 + E_{\theta^*}$$

Under our model setup, $E_{\theta^*} = 0$. This implies

$$E_{\hat{\theta}} \leq \left| \Lambda_{\hat{\theta}} - \Lambda_{\theta^*} \right| + \lambda_{n,1} \left| \hat{\theta} - \theta^* \right|_1,$$

by the triangle inequality. To control for $\Lambda_{\hat{\theta}} - \Lambda_{\theta^*}$ and $E_{\hat{\theta}}$, we follow Lemma 6.7 in Bühlmann and Van de Geer (2011) (alternatively, see Theorems 2.1 or 2.2 in Van de Geer, 2008) to obtain the desired upper bound on $\hat{Q}_1(\hat{\theta}) - \hat{Q}_1(\theta^*)$ in (A.12). □
A.8 Theorem 4.7

The proof for Theorem 4.7 hinges on an intermediate result that shows the “mutual incoherence” assumption on $\mathbb{E}[v_1^Tv_1]$ (the population version of $\frac{1}{n}v^Tv$) guarantees that, with high probability, analogous conditions hold for the estimated quantity $\frac{1}{n}\hat{v}^T\hat{v}$. This result is established in Lemma A.13. The main proof for Theorem 4.7 is based on a construction called Primal-Dual Witness (PDW) method developed by Wainwright (2009). This method constructs a pair $(\hat{\beta}, \hat{\mu})$. When this procedure succeeds, the constructed pair is primal-dual optimal, and acts as a witness for the fact that the Lasso has a unique optimal solution with the correct signed support. The procedure is described in the following.

1. Set $\hat{\beta}_{\mathcal{H}_t} = 0$.

2. Obtain $(\hat{\beta}_{\mathcal{H}_t}, \hat{\mu}_{\mathcal{H}_t})$ by solving the oracle subproblem

$$
\hat{\beta}_{\mathcal{H}_t} \in \arg \min_{\beta_{\mathcal{H}_t} \in \mathbb{R}^k} \left\{ \frac{1}{2n} |\hat{v}_0 - \hat{v}_{\mathcal{H}_t}\beta_{\mathcal{H}_t}|_2^2 + \lambda_n,3|\beta_{\mathcal{H}_t}|_1 \right\},
$$

and choose $\hat{\mu}_{\mathcal{H}_t} \in \partial|\hat{\beta}_{\mathcal{H}_t}|_1$, where $\partial|\hat{\beta}_{\mathcal{H}_t}|_1$ denotes the set of subgradients at $\hat{\beta}_{\mathcal{H}_t}$ for the function $| \cdot |_1 : \mathbb{R}^k \to \mathbb{R}$.

3. Solve for $\hat{\mu}_{\mathcal{H}_t}$ via the zero-subgradient equation

$$
\frac{1}{n} \hat{v}_0 - \hat{\beta} + \lambda_n,3 \hat{\mu} = 0,
$$

and check whether or not the strict dual feasibility condition $|\hat{\mu}_{\mathcal{H}_t}|_\infty < 1$ holds.

The proof includes four parts. Lemma A.8 guarantees the uniqueness of the optimal solution of the two-stage Lasso procedure, $\hat{\beta}_{\mathcal{H}_t}$, and shows that $\hat{\beta}_{\mathcal{H}_t} = (\hat{\beta}_{\mathcal{H}_t}, 0)$ where $\hat{\beta}_{\mathcal{H}_t}$ is the solution obtained in step 2 of the PDW construction. Based on this uniqueness claim, one can then talk unambiguously about the support of the two-stage Lasso estimate. Lemma A.9 proves part (a) of Theorem 4.7 by verifying the strict dual feasibility condition in step 3 of PDW and ensures that $\hat{\beta}_{\mathcal{H}_t}$ is uniformly close to $\beta_{\mathcal{H}_t}$ in the $l_\infty$-norm. Part (b) of Theorem 4.7 is a consequence of this uniform norm bound in Lemma A.9: as long as the minimum value of $|\beta_j|$ over $j \in J(\beta^*)$ is not too small, then the multi-stage estimator does not falsely exclude elements that are in the support of $\beta^*$ with high probability. To simplify the notations in the following analysis, let $J(\beta^*) = K$, $J(\beta^*)_c := K^c$, $\Sigma_{K^c K} := \mathbb{E} [v_1^T v_1 K_{1:K}]$, $\Sigma_{K^c K} := \frac{1}{n} v_1^T v_K$, and $\Sigma_{K^c K} := \frac{1}{n} v_1^T v_K$. Similarly, let $\Sigma_{KK} := \mathbb{E} [v_1^T v_1 K_{1:K}]$, $\Sigma_{KK} := \frac{1}{n} v_1^T v_K$, and $\Sigma_{KK} := \frac{1}{n} v_1^T v_K$.

Lemma A.8: If the PDW construction succeeds and if $\lambda_{\min} \left( \mathbb{E} [v_1^T v_1, J(\beta^*)] \right) \geq C_{\min} > 0$, then the vector $(\hat{\beta}_K, 0) \in \mathbb{R}^p$ is the unique optimal solution of the Lasso.

Proof. The proof for Lemma A.8 adopts the proof for Lemma 1 from Chapter 6.4.2 of Wainwright (2015). If the PDW construction succeeds, then $\hat{\beta} = (\hat{\beta}_K, 0)$ is an optimal solution with associated subgradient vector $\hat{\mu} \in \mathbb{R}^p$ satisfying $|\hat{\mu}_K|_\infty < 1$, and $|\hat{\mu}_K|_1 = |\hat{\beta}|_1$. Suppose $\hat{\beta}$ is another optimal solution. Letting

$$
F(\beta) = \frac{1}{2n} |v_0 - \hat{\beta}|_2^2,
$$

then $F(\hat{\beta}) + \lambda_n,3 \left( \hat{\mu}, \hat{\beta} \right) = F(\hat{\beta}) + \lambda_n,3 |\hat{\beta}|_1$, and hence

$$
F(\beta) - \lambda_n,3 \left( \hat{\mu}, \hat{\beta} - \beta \right) = F(\beta) + \lambda_n,3 \left( |\beta|_1 - \left| \hat{\mu}, \hat{\beta} \right|_1 \right).
$$

However, by the zero-subgradient conditions for optimality, we have

---

1Given a convex function $g : \mathbb{R}^p \to \mathbb{R}$, $\mu \in \mathbb{R}^p$ is a subgradient at $\beta$, denoted by $\mu \in \partial g(\beta)$, if $g(\beta + \Delta) \geq g(\beta) + \langle \mu, \Delta \rangle$ for all $\Delta \in \mathbb{R}^p$. When $g(\beta) = |\beta|_1$, notice that $\mu \in \partial |\beta|_1$ if and only if $\mu_j = \text{sign}(\beta_j)$ for all $j = 1,...,p$, where sign(0) is
\( \lambda_{n,3} \hat{\mu} = -\nabla F(\hat{\beta}) \), which implies that \( F(\hat{\beta}) + \langle \nabla F(\hat{\beta}), \hat{\beta} - \bar{\beta} \rangle - F(\bar{\beta}) = \lambda_{n,3} \left( |\bar{\beta}|_1 - \langle \hat{\mu}, \bar{\beta} \rangle \right) \). By convexity of \( F \), the left-hand side is non-positive, which implies that \( |\bar{\beta}|_1 \leq \langle \hat{\mu}, \bar{\beta} \rangle \). But since we also have \( \langle \hat{\mu}, \bar{\beta} \rangle \leq |\hat{\mu}|_\infty |\bar{\beta}|_1 \), we must have \( |\bar{\beta}|_1 = \langle \hat{\mu}, \bar{\beta} \rangle \). Since \( |\hat{\mu}_{K^c}|_\infty < 1 \), this equality can only occur if \( \bar{\beta}_j = 0 \) for all \( j \in K^c \). Thus, all optimal solutions are supported only on \( K \), and hence can be obtained by solving the oracle subproblem in the PDW procedure. Given \( \lambda_{\min} \left( \mathbb{E} \left[ v_{1,J(\beta^* \top J)} v_{1,J(\hat{\beta}^*)} \right] \right) \geq C_{\min} > 0 \), this subproblem is strictly convex, and hence it has a unique minimizer. □

**Lemma A.9:** Suppose the assumptions in Theorem 4.7 hold. With the choice of the regularization parameter

\[
\lambda_{n,3} \geq \frac{8(2 - \frac{2}{3})}{\phi} \left[ \left( \sigma_v \sigma_y \sqrt{\log p} \right) \vee \left( \frac{L |\beta^*|_1}{\sqrt{w, \beta^*}} b (\sigma_v, \sigma_y) \sqrt{\frac{k_1 \log d}{n}} \right) \right]
\]

and under the condition \( n \geq (k_3^3 \log p) \vee (k_3^2 k_1 \log d) \), \( \sqrt{\frac{n \log p}{|\beta^*|_1}} \vee \sqrt{n k_1 \log d} \geq \log p \), and \( \frac{\log d}{n} = o(1) \), we have \( |\hat{\mu}_{K^c}|_\infty \leq 1 - \frac{\phi}{8} \) with probability at least \( 1 - c_1 \exp(-c_2 \log(p \land d)) \). Furthermore,

\[
|\hat{\beta}_K - \beta^*_K|_\infty \leq \frac{c \sqrt{k_2}}{\lambda_{\min}(\Sigma_{KK})} \left[ \left( \sigma_v \sigma_y \sqrt{\log p} \right) \vee \left( \frac{L |\beta^*|_1}{\sqrt{w, \beta^*}} b (\sigma_v, \sigma_y) \sqrt{\frac{k_1 \log d}{n}} \right) \right]
\]

with probability at least \( 1 - c_1 \exp(-c_2 \log(p \land d)) \).

**Proof.** By construction, the sub-vectors \( \hat{\beta}_K, \hat{\mu}_K, \) and \( \hat{\mu}_{K^c} \) satisfy the zero-subgradient condition in the PDW construction. Recall the definition of \( e \) from Lemma A.1. With the fact that \( \hat{\beta}_{K^c} = \beta^*_{K^c} = 0 \), we have

\[
\frac{1}{n} \hat{v}_K^T \hat{v}_K \left( \hat{\beta}_K - \beta^*_K \right) + \frac{1}{n} \hat{v}_K^T e + \lambda_{n,3} \mu_K = 0,
\]

\[
\frac{1}{n} \hat{v}_{K^c}^T \hat{v}_{K^c} \left( \hat{\beta}_K - \beta^*_K \right) + \frac{1}{n} \hat{v}_{K^c}^T e + \lambda_{n,3} \mu_{K^c} = 0.
\]

From the equations above, by solving for the vector \( \hat{\mu}_{K^c} \in \mathbb{R}^{p-k_2} \), we obtain

\[
\hat{\mu}_{K^c} = -\frac{1}{n \lambda_{n,3}} \hat{v}_{K^c}^T \hat{v}_K \left( \hat{\beta}_K - \beta^*_K \right) - \frac{\hat{v}_{K^c}^T e}{n \lambda_{n,3}},
\]

\[
\hat{\beta}_K - \beta^*_K = -\left( \frac{1}{n} \hat{v}_K^T \hat{v}_K \right)^{-1} \hat{v}_K^T e - \lambda_{n,3} \left( \frac{\hat{v}_K^T \hat{v}_K}{n} \right)^{-1} \hat{\mu}_K,
\]

which yields

\[
\hat{\mu}_{K^c} = \left( \Sigma_{K^c K} \Sigma_{K K}^{-1} \right) \hat{\mu}_K + \left( \hat{v}_{K^c}^T e \right) \frac{1}{n \lambda_{n,3}} - \left( \Sigma_{K^c K} \Sigma_{K K}^{-1} \right) \left( \frac{\hat{v}_K^T \hat{v}_K}{n} \right)^{-1} \hat{\mu}_K,
\]

By the triangle inequality, we have

\[
|\hat{\mu}_{K^c}|_\infty \leq \left\| \Sigma_{K^c K} \Sigma_{K K}^{-1} \right\|_\infty + \left| \hat{v}_{K^c}^T e \right|_\infty \frac{1}{n \lambda_{n,3}} + \left\| \Sigma_{K^c K} \Sigma_{K K}^{-1} \right\|_\infty \left| \hat{v}_K^T \hat{v}_K e \right|_\infty \frac{1}{n \lambda_{n,3}},
\]

allowed to be any number in \([-1, 1] \).
where the fact that $|\hat{\mu}_K|_\infty \leq 1$ is used in the inequality above. By Lemma A.13, we have $\|\Sigma_{K^cK}\Sigma_{K^cK}^{-1}\|_\infty \leq 1 - \frac{\phi}{4}$ with probability at least $1 - c \exp(-\log(p \land d))$. Hence,

\[
|\hat{\mu}_{K^c}|_\infty \leq 1 - \frac{\phi}{4} + \|\Sigma_{K^cK}\Sigma_{K^cK}^{-1}\|_\infty \|\hat{v}_K^n\|_\infty \leq 1 - \frac{\phi}{4} + \left(2 - \frac{\phi}{4}\right) \|\hat{v}_K^n\|_\infty.
\]

Therefore, it suffices to show that $\left(2 - \frac{\phi}{4}\right) \|\hat{v}_K^n\|_\infty \leq \frac{\phi}{8}$ with high probability. This result is established in Lemma A.16. Thus, we have $|\hat{\mu}_{K^c}|_\infty \leq 1 - \frac{\phi}{8}$ with high probability.

It remains to establish a bound on the $l_\infty$-norm of the error $\hat{\beta}_K - \beta^*_K$. By the triangle inequality, we have

\[
|\hat{\beta}_K - \beta^*_K|_\infty \leq \left\|\left(\frac{\hat{v}_K}{n}\right)\right\|_\infty + \left\|\left(\frac{\hat{v}_K}{n}\right)^{-1}\right\|_\infty + \left\|\left(\frac{\hat{v}_K}{n}\right)^{-1}\right\|_\infty \|\hat{v}_K^n\|_\infty + \lambda_{n,3} \left\|\left(\frac{\hat{v}_K}{n}\right)^{-1}\right\|_\infty.
\]

Using bound (A.35) from Lemma A.15, we have

\[
\left\|\left(\frac{\hat{v}_K}{n}\right)^{-1}\right\|_\infty \leq \frac{2\sqrt{k_2}}{\lambda_{\min}(\Sigma_{K^cK})} \leq \frac{4\sqrt{k_2}}{\lambda_{\min}(\Sigma_{K^cK})}.
\]

From the proof for Corollary 4.4, we have, with probability at least $1 - c_1 \exp(-c_2 \log(p \land d))$,

\[
\left|\frac{1}{n}\hat{v}_K^n\right|_\infty \lesssim \left(\sigma_v\sigma_\eta \sqrt{\frac{\log p}{n}}\right) \sqrt{L |\beta^*_1|_1 \sqrt{T_{w,\theta\gamma}} b(\sigma_v, \sigma_\eta) \sqrt{k_1 \log d}}.
\]

Putting everything together, with the choice of $\lambda_{n,3}$, we obtain

\[
|\hat{\beta}_K - \beta^*_K|_\infty \leq \frac{c\sqrt{k_2}}{\lambda_{\min}(\Sigma_{K^cK})} \left(\sigma_v\sigma_\eta \sqrt{\frac{\log p}{n}}\right) \sqrt{L |\beta^*_1|_1 \sqrt{T_{w,\theta\gamma}} b(\sigma_v, \sigma_\eta) \sqrt{k_1 \log d}}.
\]

with probability at least $1 - c_1 \exp(-c_2 \log(p \land d))$, as claimed. □

**A.9 Theorem 4.8**

**Proof.** Let $\Delta := D^{-1}(\hat{\beta} - \beta^*)$ and $\Psi_n := \frac{1}{n} D \hat{v}_K^n \hat{v}_K D$ and we have

\[
|\Psi_n \Delta| = \left|\frac{1}{n} D \hat{v}_K^n(\hat{v}_0 - \hat{v}_K) - \frac{1}{n} D \hat{v}_K^n(\hat{v}_0 - \hat{v}_K^*)\right|.
\]

Consequently,

\[
|\Psi_n \Delta|_\infty \leq \left|\frac{1}{n} D \hat{v}_K^n(\hat{v}_0 - \hat{v}_K^*)\right|_\infty + \left|\frac{1}{n} D \hat{v}_K^n(\hat{v}_0 - \hat{v}_K^*)\right|_\infty \leq \hat{\sigma} \xi + \left|\frac{1}{n} D \hat{v}_K^n e\right|_\infty,
\]

with probability at least $1 - c_1 \exp(-c_2 \log(p \land d))$, as claimed. □
recalling $e := \hat{v}_0 - \hat{v}\beta^*$ defined in Lemma A.1. As in the nonpivotal case, to upper bound $\left| D\hat{v}^T \right|_\infty$, we control the terms except $\max_j \left\{ \frac{1}{n} \sum_{i=1}^n v_{ij}\eta_i \right\} \in (A.6)-(A.8)$ with the argument from the proofs for Lemma A.4 and Corollary 4.4. To control for the term $\max_j \left\{ \frac{1}{n} \sum_{i=1}^n v_{ij}\eta_i \right\}$, we use an argument similar to Gautier and Tsybakov (2011). Let $Q(\beta) := \frac{1}{n} |v_0 - v_\beta|^2$ and define the event

$$\mathcal{E} = \left\{ \left. \left| \frac{1}{n} Dv^T \eta \right|_\infty \leq \sqrt{Q(\beta^*)}\xi \right\} \right.$$ .

Since $Q(\beta^*) = \frac{1}{n} \sum_{i=1}^n \eta_i^2$, we have

$$\mathbb{P}(\mathcal{E}^C) \leq \sum_{j=1}^p \mathbb{P}

\left( \left. \left| \frac{1}{n} \frac{\sum_{i=1}^n v_{ij}\eta_i}{\sqrt{\sum_{i=1}^n (v_{ij}\eta_i)^2}} \right| \geq \xi \right) \right)

\leq \sum_{j=1}^p \mathbb{P}

\left( \left. \left| \frac{\sum_{i=1}^n v_{ij}\eta_i}{\sqrt{\sum_{i=1}^n (v_{ij}\eta_i)^2}} \right| \geq \sqrt{n}\xi \right) \right).

By Lemma B.16, for all $j = 1, \ldots, p$,

$$\mathbb{P}

\left( \left. \left| \frac{\sum_{i=1}^n v_{ij}\eta_i}{\sqrt{\sum_{i=1}^n (v_{ij}\eta_i)^2}} \right| \geq \sqrt{n}\xi \right) \right) \leq 2 \left( 1 - \Phi \left( \sqrt{n}\xi \right) \right) + 2a_0 \frac{(1 + \sqrt{n}\xi)^{1+\delta'}}{p^{a^2-1}b^{2+\delta'}}.

Thus, the event $\mathcal{E}$ holds with probability at least $1 - \alpha$ where

$$\alpha = 2L \left( 1 - \Phi \left( \sqrt{n}\xi \right) \right) + 2a_0 \frac{(1 + \sqrt{n}\xi)^{1+\delta'}}{p^{a^2-1}b^{2+\delta'}}.

Now, let $\hat{Q}(\beta) := \frac{1}{n} |\hat{v}_0 - \hat{v}\beta|^2$. The following shows that, conditioning on the event $\mathcal{E}$ and by appropriately choosing $\xi$, we have

$$\left| \frac{1}{n} D\hat{v}^T (\hat{v}_0 - \hat{v}\beta^*) \right|_\infty \leq \hat{Q}(\beta^*)\xi,

namely, $\left( \beta^*, \hat{Q}(\beta^*) \right) \in A$. First, notice that we have

$$\left| \frac{1}{n} D\hat{v}^T (\hat{v}_0 - \hat{v}\beta^*) \right|_\infty \leq \left| \frac{1}{n} Dv^T (v_0 - v_\beta^*) \right|_\infty + \left| \frac{1}{n} D\hat{v}^T (\hat{v}_0 - \hat{v}\beta^*) - \frac{1}{n} Dv^T (v_0 - v\beta^*) \right|_\infty

\leq \left| \frac{1}{n} Dv^T (v_0 - v_\beta^*) \right|_\infty + 2 \left| \frac{1}{n} Dv^T (\hat{v}_0 - v_0) \right|_\infty + \left| \frac{1}{n} D(\hat{v} - v)^T (\hat{v}_0 - v_0) \right|_\infty

+ 2 \left| \frac{1}{n} Dv^T (\hat{v} - v)\beta^* \right|_\infty + \left| \frac{1}{n} D(\hat{v} - v)^T (\hat{v} - v)\beta^* \right|_\infty

\leq \sqrt{Q(\beta^*)}\xi + \frac{Lb(\sigma_v)\sqrt{B}}{\min_{j=1,\ldots,p} v_{j*}} |\beta^*|_1

\leq \sqrt{\hat{Q}(\beta^*)}\xi + \left| \sqrt{\hat{Q}(\beta^*)} - \sqrt{Q(\beta^*)} \right| \xi + \frac{Lb(\sigma_v)\sqrt{B}}{\min_{j=1,\ldots,p} v_{j*}} |\beta^*|_1
where the third inequality follows from argument similar to the proofs of the upper bounds for Lemma A.3 and Corollary 4.4, and \( b(\sigma_v) \) is a positive constant only depending on \( \sigma_v \). We now control for the term \( \sqrt{Q(\beta^*)} - \sqrt{Q(\beta^*)} \). Note that

\[
\left| \hat{Q}(\beta^*) - Q(\beta^*) \right| \leq \frac{1}{n} \sum_{i=1}^{n} (\hat{v}_{i0}^2 - v_{i0}^2) + \frac{2}{n} \sum_{i=1}^{n} (\hat{v}_{i0} \hat{v}_i - v_{i0} v_i) \beta^* + \frac{1}{n} \sum_{i=1}^{n} \beta^{*T} (\hat{v}_i^T - v_i^T) \beta^*
\]

\[
\leq \frac{1}{n} \sum_{i=1}^{n} (\hat{v}_{i0}^2 - v_{i0}^2) + \max_{j=1,\ldots,p} \left| \frac{2}{n} \sum_{i=1}^{n} (\hat{v}_{i0} \hat{v}_{ij} - v_{i0} v_{ij}) \right| \beta^* \| \beta^* \|
\]

\[
+ \max_{j,j'=1,\ldots,p} \left| \frac{2}{n} \sum_{i=1}^{n} (\hat{v}_i^T \hat{v}_{ij'} - v_i^T v_{ij'}) \right| \beta^* \| \beta^* \|
\]

Following from argument similar to the proofs of the upper bounds for Lemma A.3 and Corollary 4.4, we can upper bound the terms

\[
\max_{j=1,\ldots,p} \left| \frac{2}{n} \sum_{i=1}^{n} (\hat{v}_{i0} \hat{v}_{ij} - v_{i0} v_{ij}) \right| \beta^* \| \beta^* \|
\]

\[
\max_{j,j'=1,\ldots,p} \left| \frac{2}{n} \sum_{i=1}^{n} (\hat{v}_i^T \hat{v}_{ij'} - v_i^T v_{ij'}) \right| \beta^* \| \beta^* \|
\]

which yields

\[
\left| \hat{Q}(\beta^*) - Q(\beta^*) \right| = O \left( \left( \frac{1}{n} \right)^{\frac{3}{2}} \| \beta^* \| + \left( \frac{\log d}{n} \right)^{\frac{1}{2}} \| \beta^* \| \right).
\]

Now, notice that if \(|a_1 - a_2| \leq a_3\), then \(|\sqrt{|a_1|} - \sqrt{|a_2|}| \leq \sqrt{|a_3|}\), we have

\[
\left| \sqrt{\hat{Q}(\beta^*)} - \sqrt{Q(\beta^*)} \right| = O \left( \left( \frac{1}{n} \right)^{\frac{3}{4}} \| \beta^* \|^{\frac{1}{2}} + \left( \frac{\log d}{n} \right)^{\frac{1}{4}} \| \beta^* \|^{\frac{1}{2}} \right),
\]

and therefore,

\[
\left| \sqrt{\hat{Q}(\beta^*)} - \sqrt{Q(\beta^*)} \right| \xi \leq c \sqrt{\hat{Q}(\beta^*)} \xi,
\]

for \( c > 1 \). Consequently, if we choose

\[
\xi \geq a \max \left\{ c \sqrt{\frac{\log p}{n}}, \frac{Lb(\sigma_v)\sqrt{B}}{\min_{j=1,\ldots,p} v_{ij*}} \left( \hat{Q}(\beta^*) \right)^{-\frac{1}{2}} \| \beta^* \| \right\}
\]

we have

\[
\left| \frac{1}{n} D \hat{v}^T (\hat{v}_0 - \hat{\beta} \beta^*) \right| \leq \sqrt{\hat{Q}(\beta^*)} \xi
\]

with probability at least \( 1 - O \left( \frac{1}{p^{\frac{1}{2}}} \right) - \alpha \). On the other hand, \((\hat{\beta}, \hat{\sigma})\) minimizes the criterion \( |D^{-1} \hat{\beta}|_1 + C \sigma \) on the same set \( A \). Thus, we have

\[
|D^{-1} \hat{\beta}|_1 + C \hat{\sigma} \leq |D^{-1} \beta^*|_1 + C \sqrt{\hat{Q}(\beta^*)},
\]

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which implies
\[
|\Delta_{J(\beta^*)c}|_1 = \sum_{j \in J(\beta^*)c} |v_j, \hat{\beta}_j| \\
\leq \sum_{j \in J(\beta^*)c} \left( |v_j, \beta_j^*| - |v_j, \hat{\beta}_j| \right) + C \left( \sqrt{\hat{Q}(\beta^*)} - \sqrt{\hat{Q}(\hat{\beta})} \right) \\
\leq |\Delta_{J(\beta^*)}|_1 + C \left( \sqrt{\hat{Q}(\beta^*)} - \sqrt{\hat{Q}(\hat{\beta})} \right) \\
\leq |\Delta_{J(\beta^*)}|_1 + C \left( \frac{\frac{1}{n} \sum_{i=1}^n \hat{v}_{ij} e_i}{\sqrt{\frac{1}{n} \sum_{i=1}^n e_i^2}} \right) \|\Delta\|_1 \\
\leq |\Delta_{J(\beta^*)}|_1 + C |\Delta|_1,
\]

where the third inequality follows from the convexity of $\beta \mapsto \sqrt{\hat{Q}(\beta)}$ and the last inequality follows from the concavity of $\sqrt{\cdot}$ and the Cauchy-Schwarz inequality. Consequently, we have the following cone condition:
\[
|\Delta_{J(\beta^*)c}|_1 \leq \frac{1 + C}{1 - C} |\Delta_{J(\beta^*)}|_1.
\]

We now finish upper bounding $|\Psi_n \Delta|_\infty$. Recall that
\[
|\Psi_n \Delta|_\infty \leq \hat{\sigma} \xi + \left| \frac{1}{n} D \hat{v}^T e \right|_\infty \\
\leq \hat{\sigma} \xi + \sqrt{\hat{Q}(\beta^*)} \xi + \frac{Lb(\sigma_v) \sqrt{\hat{B}}}{\min_{j=1,\ldots,p} v_{j*}} |\beta^*|_1 \\
\leq \xi \left( 2\hat{\sigma} + \sqrt{\hat{Q}(\beta^*)} - \sqrt{\hat{Q}(\hat{\beta})} \right) + \xi \left( \sqrt{\hat{Q}(\hat{\beta})} - \sqrt{\hat{Q}(\hat{\beta})} \right) \\
+ \frac{Lb(\sigma_v) \sqrt{\hat{B}}}{\min_{j=1,\ldots,p} v_{j*}} |\beta^*|_1.
\]

We first upper bound the term $\sqrt{\hat{Q}(\beta^*)} - \sqrt{\hat{Q}(\hat{\beta})}$. Arguing as in the proof for the cone condition, we have
\[
\sqrt{\hat{Q}(\beta^*)} - \sqrt{\hat{Q}(\hat{\beta})} \leq \left| \frac{\frac{1}{n} \sum_{i=1}^n v_{ij} \eta_i}{\sqrt{\frac{1}{n} \sum_{i=1}^n \eta_i^2}} \right| \\
\leq \max_{j \in 1,\ldots,p} \left| \frac{\frac{1}{n} \sum_{i=1}^n v_{ij} \eta_i}{\sqrt{\frac{1}{n} \sum_{i=1}^n v_{ij}^2 \eta_i^2}} \right| |\Delta|_1 \\
\leq \xi |\Delta|_1.
\]
To upper bound the term $\sqrt{Q(\hat{\beta})} - \sqrt{Q(\beta)}$, we argue as in the proof for upper bounding $|\sqrt{Q(\beta^*)} - \sqrt{Q(\beta^*)}|$ from the above and obtain

$$\sqrt{Q(\hat{\beta})} - \sqrt{Q(\beta)} = O\left(\left(\frac{1}{n}\right)^{\frac{1}{2}} |\hat{\beta}|_1^{\frac{1}{2}} + \left(\frac{\log d}{n}\right)^{\frac{1}{4}} |\beta^*|_1^{\frac{1}{4}}\right),$$

and therefore,

$$\xi \left(\sqrt{Q(\hat{\beta})} - \sqrt{Q(\beta)}\right) < \xi \hat{\sigma}.$$

Combining the pieces together, we obtain

$$|\Psi_n \Delta|_\infty \leq 2\xi \hat{\sigma} + \xi^2 |\Delta|_1 + \frac{Lb(\sigma_v)\sqrt{B'}}{\min_{j=1,\ldots,p} v_{j*}} |\beta^*|_1$$

$$\leq \xi \left(2\hat{\sigma} + \xi |\Psi_n \Delta|_\infty\right) + \frac{Lb(\sigma_v)\sqrt{B'}}{\min_{j=1,\ldots,p} v_{j*}} |\beta^*|_1$$

where the second inequality uses the definition of the sensitivity $\kappa_{J(\beta^*)}$. The above yields

$$|\Psi_n \Delta|_\infty \leq \left[\frac{Lb(\sigma_v)\sqrt{B'}}{\min_{j=1,\ldots,p} v_{j*}} |\beta^*|_1 + 2\xi \hat{\sigma}\right] \left[1 - \frac{\xi^2}{\kappa_{J(\beta^*)}}\right]^{-1}.$$

Now, we can apply the cone condition to obtain

$$\left|D^{-1}(\hat{\beta} - \beta^*)\right|_2 \leq \frac{1}{\kappa_{J(\beta^*)}} \left[\frac{Lb(\sigma_v)\sqrt{B'}}{\min_{j=1,\ldots,p} v_{j*}} |\hat{\beta}|_1 + 2\xi \hat{\sigma}\right] \left[1 - \frac{\xi^2}{\kappa_{J(\beta^*)}}\right]^{-1}.$$

Some further algebraic manipulation shows

$$\left|D^{-1}(\hat{\beta} - \beta^*)\right|_2 \leq \frac{1}{\kappa_{J(\beta^*)}} \left[\frac{Lb(\sigma_v)\sqrt{B'}}{\min_{j=1,\ldots,p} v_{j*}} |\hat{\beta}|_1 + 2\xi \hat{\sigma}\right] \left[1 - \frac{\xi^2}{\kappa_{J(\beta^*)}}\right]^{-1}$$

$$+ \frac{1}{\kappa_{J(\beta^*)}} \left[\frac{Lb(\sigma_v)\sqrt{B'}}{\min_{j=1,\ldots,p} v_{j*}} \sqrt{k_2 D^{-1}(\hat{\beta} - \beta^*)}_2\right] \left[1 - \frac{\xi^2}{\kappa_{J(\beta^*)}}\right]^{-1}$$

which implies

$$\left|D^{-1}(\hat{\beta} - \beta^*)\right|_2 \leq \frac{1}{\kappa_{J(\beta^*)}} \left[\frac{Lb(\sigma_v)\sqrt{B'}}{\min_{j=1,\ldots,p} v_{j*}} |\hat{\beta}|_1 + 2\xi \hat{\sigma}\right] \left[1 - \frac{\xi^2}{\kappa_{J(\beta^*)}}\right]^{-1}$$

$$\cdot \left[1 - \frac{1}{\kappa_{J(\beta^*)}} \left[\frac{Lb(\sigma_v)\sqrt{k_2 B'}}{\min_{j=1,\ldots,p} v_{j*}}\right] \left[1 - \frac{\xi^2}{\kappa_{J(\beta^*)}}\right]^{-1}\right]^{-1}.$$
Simple calculations yield, for all $j = 1, \ldots, p$,

$$\left| \hat{\beta}_j - \beta^*_j \right| \leq \frac{1}{v_j \kappa^*_J(\beta^*)} \left[ \frac{Lb(\sigma_v) \sqrt{B}}{\min_{j=1,\ldots,p} v_j^*} \left| \hat{\beta} \right|_1 + 2\xi \hat{\sigma} \right] \left( \frac{1 - \xi^2}{\kappa^*_J(\beta^*)} \right)^{-1}$$

$$\cdot \left[ \frac{1}{\kappa^*_J(\beta^*)} \left[ \frac{Lb(\sigma_v) \sqrt{k_2 B}}{(\min_{j=1,\ldots,p} v_j^*)^2} \right] \left( \frac{1 - \xi^2}{\kappa^*_J(\beta^*)} \right)^{-1} \right].$$

To establish the upper bound on $\hat{\sigma}$, note that the optimality of $(\hat{\beta}, \hat{\sigma})$ implies

$$C \hat{\sigma} \leq \left| \Delta_{J(\beta^*)} \right|_1 + C \sqrt{Q(\beta^*)}$$

$$\leq \frac{\|\Psi_n \Delta\|_\infty}{\kappa^*_J(\beta^*),J(\beta^*)} + C \sqrt{Q(\beta^*)}.$$

\[ \square \]

### A.10 Theorems 4.9 and 4.10

**Proof.** The estimator of $g(w^T \theta^*)$ in Theorems 4.9 and 4.10 takes on the following form

$$\hat{g} \left( y_{2i} | w_i^T \hat{\theta} \right) - \hat{g} \left( x_i | w_i^T \hat{\theta} \right)^T \hat{\beta} := \hat{g}(w_i^T \hat{\theta}),$$

where $\hat{\beta} = \hat{\beta}_{HSEL}$. We now derive an upper bound for

$$\left| \hat{g}(\hat{\theta}) - g(\theta^*) \right|_2 := \sqrt{\frac{1}{n} \sum_{i=1}^{n} \left[ \hat{g}(w_i^T \hat{\theta}) - g(w_i^T \theta^*) \right]^2}$$

Note that we can write

$$\hat{g}(w_i^T \hat{\theta}) - g(w_i^T \theta^*)$$

$$= \left[ \hat{E} \left( y_{2i} | w_i^T \hat{\theta} \right) - \hat{E} \left( y_{2i} | w_i^T \theta^* \right) \right] - \left[ \hat{E} \left( x_i | w_i^T \hat{\theta} \right)^T \hat{\beta} - \hat{E} \left( x_i | w_i^T \theta^* \right)^T \beta^* \right]$$

$$= \left[ \hat{E} \left( y_{2i} | w_i^T \hat{\theta} \right) - \hat{E} \left( y_{2i} | w_i^T \theta^* \right) \right] - \left[ \hat{E} \left( x_i | w_i^T \hat{\theta} \right) - \hat{E} \left( x_i | w_i^T \theta^* \right) \right]^T \hat{\beta}$$

$$- \left[ \hat{E} \left( x_i | w_i^T \hat{\theta} \right) - \hat{E} \left( x_i | w_i^T \theta^* \right) \right]^T \hat{\beta} - \hat{E} \left( x_i | w_i^T \theta^* \right)^T (\hat{\beta} - \beta^*).$$
By an elementary inequality and Hölder’s inequality, we have
\[
\left[\hat{g}(w_i^T \hat{\theta}) - g(w_i^T \theta^*)\right]^2 \\
\leq 4 \left[\hat{E}(y_{2i} \mid w_i^T \hat{\theta}) - E(y_{2i} \mid w_i^T \theta^*)\right]^2 + 4 \left[\hat{E}\left(x_i \mid w_i^T \hat{\theta}\right) - E\left(x_i \mid w_i^T \hat{\theta}\right)\right]^T \beta + 4 \left[\hat{E}\left(x_i \mid w_i^T \theta^*\right)\right]^T (\hat{\beta} - \beta^*)^2 \\
+ 4 \left[\hat{E}\left(x_i \mid w_i^T \hat{\theta}\right) - E\left(x_i \mid w_i^T \hat{\theta}\right)\right]^2 + 4 \left[\hat{E}\left(x_i \mid w_i^T \hat{\theta}\right) - E\left(x_i \mid w_i^T \hat{\theta}\right)\right]^2 + 4 \left[\hat{E}\left(x_i \mid w_i^T \theta^*\right)\right]^2 (\hat{\beta} - \beta^*)^2 \\
+ 4 \left[\hat{E}\left(x_i \mid w_i^T \hat{\theta}\right) - E\left(x_i \mid w_i^T \hat{\theta}\right)\right]^2 \left|\hat{\beta} - \beta^*\right|^2 \\
+ 8 \max_j \left[\hat{E}\left(x_{ij} \mid w_i^T \hat{\theta}\right) - E\left(x_{ij} \mid w_i^T \hat{\theta}\right)\right]^2 \left|\beta^*_{1j}\right| + 8 \max_j \left[\hat{E}\left(x_{ij} \mid w_i^T \hat{\theta}\right) - E\left(x_{ij} \mid w_i^T \hat{\theta}\right)\right]^2 \left|\hat{\beta} - \beta^*\right|^2 \\
+ 8 \max_j \left[\hat{E}\left(x_{ij} \mid w_i^T \hat{\theta}\right) - E\left(x_{ij} \mid w_i^T \hat{\theta}\right)\right]^2 \left|\beta^*_{1j}\right| + 8 \max_j \left[\hat{E}\left(x_{ij} \mid w_i^T \hat{\theta}\right) - E\left(x_{ij} \mid w_i^T \hat{\theta}\right)\right]^2 \left|\hat{\beta} - \beta^*\right|^2 \\
+ 4 \max_j \left[\hat{E}\left(x_{ij} \mid w_i^T \theta^*\right)\right]^2 \left|\hat{\beta} - \beta^*\right|^2.
\]

Applying previous results yields the desired result. By a standard integration over the tail probability (in the exponential form), the bound above with high probability can be converted to a bound in expectation as stated in Theorem 4.8. □

### A.11 Theorems 4.11 and 4.12

**Proof.** We prove the case where \(\beta^*\) and \(\theta^*\) are exactly sparse and the approximate sparse case of \(\beta^*\) and \(\theta^*\) follows the same argument (see the remark at the end of this section). The estimator of \(g(w_i^T \theta^*)\) in Theorem 4.11 is obtained by solving the following program

\[
\hat{g} \in \arg \min_{f \in F} \frac{1}{n} \sum_{i=1}^{n} \left(y_{2i} - x_{iT} \hat{\beta} - f(w_i^T \hat{\theta})\right)^2
\]

where \(\hat{\beta} = \hat{\beta}_{HSEL}\). We derive an upper bound for

\[
\left|\hat{g}(\hat{\theta}) - g(\theta^*)\right|^2 := \frac{1}{n} \sum_{i=1}^{n} \left[\hat{g}(w_i^T \hat{\theta}) - g(w_i^T \theta^*)\right]^2.
\]

Since \(\hat{g}\) is optimal and \(g\) is feasible, we have the basic inequality

\[
\frac{1}{n} \sum_{i=1}^{n} \left(y_{2i} - x_{iT} \hat{\beta} - \hat{g}(w_i^T \hat{\theta})\right)^2 \leq \frac{1}{n} \sum_{i=1}^{n} \left(y_{2i} - x_{iT} \hat{\beta} - g(w_i^T \hat{\theta})\right)^2.
\]
Some algebra leads to the equivalent expression
\[
\frac{1}{2} \left| \tilde{g}(\hat{\theta}) - g(\hat{\theta}) \right|_n^2 \leq \frac{1}{n} \sum_{i=1}^{n} \left[ x_i^T (\beta^* - \hat{\beta}) \right] \left[ \tilde{g}(w_i^T \hat{\theta}) - g(w_i^T \hat{\theta}) \right] + \frac{1}{n} \sum_{i=1}^{n} \left[ g(w_i^T \theta^*) - g(w_i^T \hat{\theta}) \right] \left[ \tilde{g}(w_i^T \hat{\theta}) - g(w_i^T \hat{\theta}) \right] + \frac{1}{n} \sum_{i=1}^{n} \eta_i \left[ \tilde{g}(w_i^T \theta^*) - g(w_i^T \theta^*) \right] + \frac{1}{n} \sum_{i=1}^{n} \eta_i \left[ \tilde{g}(w_i^T \hat{\theta}) - g(w_i^T \theta^*) \right] - \frac{1}{n} \sum_{i=1}^{n} \eta_i \left[ g(w_i^T \hat{\theta}) - g(w_i^T \theta^*) \right].
\]

Now, by the Fenchel-Young inequality, we have
\[
\frac{1}{n} \sum_{i=1}^{n} \left[ x_i^T (\beta^* - \hat{\beta}) \right] \left[ \tilde{g}(w_i^T \hat{\theta}) - g(w_i^T \hat{\theta}) \right] + \frac{1}{n} \sum_{i=1}^{n} \left[ g(w_i^T \theta^*) - g(w_i^T \hat{\theta}) \right] \left[ \tilde{g}(w_i^T \hat{\theta}) - g(w_i^T \hat{\theta}) \right] \leq \frac{2}{n} \sum_{i=1}^{n} \left[ x_i^T (\beta^* - \hat{\beta}) \right]^2 + \frac{1}{8n} \sum_{i=1}^{n} \left[ \tilde{g}(w_i^T \hat{\theta}) - g(w_i^T \hat{\theta}) \right]^2 + \frac{2}{n} \sum_{i=1}^{n} \left[ g(w_i^T \theta^*) - g(w_i^T \hat{\theta}) \right]^2 + \frac{1}{n} \sum_{i=1}^{n} \left[ g(w_i^T \hat{\theta}) - g(w_i^T \theta^*) \right]^2.
\]

Consequently, we have
\[
\frac{1}{4} \left| \tilde{g}(\hat{\theta}) - g(\hat{\theta}) \right|_n^2 \leq \frac{2}{n} \sum_{i=1}^{n} \left[ x_i^T (\beta^* - \hat{\beta}) \right]^2 + \frac{2}{n} \sum_{i=1}^{n} \left[ g(w_i^T \theta^*) - g(w_i^T \hat{\theta}) \right]^2 + \frac{1}{n} \sum_{i=1}^{n} \eta_i \left[ \tilde{g}(w_i^T \theta^*) - g(w_i^T \theta^*) \right] + \frac{1}{n} \sum_{i=1}^{n} \eta_i \left[ \tilde{g}(w_i^T \hat{\theta}) - g(w_i^T \hat{\theta}) \right] - \frac{1}{n} \sum_{i=1}^{n} \eta_i \left[ g(w_i^T \hat{\theta}) - g(w_i^T \theta^*) \right].
\]

First, recall the upper bound on $\left| \hat{\beta} - \beta^* \right|^2$ from Corollary 4.4. Let $\Delta = \hat{\beta} - \beta^*$. Applying the inequality in Lemma B.2 yields
\[
\left| \frac{x \Delta}{n} \right|^2 \leq \frac{3\kappa \bar{d}}{2} \left| \Delta \right|_2^2 + \alpha \frac{1}{n} \log \frac{d}{n} \left| \Delta \right|_1^2,
\]
and combining with the fact that $\left| \Delta \right|_1 \leq 4\kappa \sqrt{\bar{d}} \left| \Delta \right|_2$ shown previously yields an upper bound on the term $\frac{1}{n} \sum_{i=1}^{n} \left[ x_i^T (\beta^* - \hat{\beta}) \right]^2$.

Now, by the $L$–Lipschitz condition on $g(\cdot)$, we have
\[
\frac{1}{n} \sum_{i=1}^{n} \left[ g(w_i^T \theta^*) - g(w_i^T \hat{\theta}) \right]^2 \leq \frac{L^2}{n} \sum_{i=1}^{n} \left[ w_i^T \hat{\theta} - w_i^T \theta^* \right]^2 \leq \frac{L^2}{n} \sum_{i=1}^{n} \left[ w_i^T \hat{\theta} - g(w_i^T \hat{\theta}) \right]^2 \leq \frac{L^2}{n} \sum_{i=1}^{n} \left[ w_i^T \hat{\theta} - g(w_i^T \theta^*) \right]^2,
\]
which is upper bounded by Proposition 4.2.
For the term $\frac{1}{n} \sum_{i=1}^{n} \eta_i \left[ \hat{g}(w_i^T \theta^*) - g(w_i^T \theta^*) \right]$, we follow the argument as in the proof for Corollary 4.4. Using the fact that $\mathbb{E} \left[ \eta_i | w_i \right] = 0$, we obtain

$$\frac{1}{n} \sum_{i=1}^{n} \eta_i \left[ \hat{g}(w_i^T \theta^*) - g(w_i^T \theta^*) \right] \leq c \left( \frac{k_2 \sigma_n^2}{n} \right)^{\frac{1}{2}}.$$ 

It remains to upper bound the terms $\frac{1}{n} \sum_{i=1}^{n} \eta_i \left[ \hat{g}(w_i^T \theta^*) - \hat{g}(w_i^T \theta^*) \right]$ and $\frac{1}{n} \sum_{i=1}^{n} \eta_i \left[ g(w_i^T \theta) - g(w_i^T \theta^*) \right]$. By the $\tilde{L}$–Lipschitz assumption on $g(\cdot)$ and $\hat{g}(\cdot)$, we have

$$\frac{1}{n} \sum_{i=1}^{n} \eta_i \left[ g(w_i^T \theta^*) - g(w_i^T \theta) \right] \leq \frac{\tilde{L}}{n} \sum_{i=1}^{n} \eta_i \left[ \theta_i^\ast - \theta_i \right],$$

$$\frac{1}{n} \sum_{i=1}^{n} \eta_i \left[ \hat{g}(w_i^T \theta^*) - \hat{g}(w_i^T \theta) \right] \leq \frac{\tilde{L}}{n} \sum_{i=1}^{n} \eta_i \left[ \theta_i^\ast - \theta_i \right].$$

We provide upper bounds on the RHS terms in the inequalities above. For $r_1, r_2 > 0$, define the set

$$S(r_1, r_2) := \left\{ \delta \in \mathbb{R}^d \mid |\delta|_1 \leq r_1, \ |\delta|_2 \leq r_2 \right\},$$

and the random variables $U_n = U_n(r_1, r_2)$ given by

$$U_n := \sup_{\delta \in S(r_1, r_2)} \frac{1}{n} \left| w^T \delta \right|.$$ 

For a given $t \in (0, 1)$ to be chosen, let us upper bound the minimal cardinality of a set that covers $S(r_1, r_2)$ up to $r_2 t$–accuracy in $l_2$–norm. By Lemma B.10, we can find such a covering set $\{\delta^1, \ldots, \delta^N\} \subset S(r_1, r_2)$ with cardinality $N = N(r_1, r_2, t)$ that is upper bounded as

$$\log N(r_1, r_2, t) \leq c_0 r_2^2 \left( \frac{1}{t} \right)^2 \log d.$$ 

Consequently, for each $\delta \in S(r_1, r_2)$, we may find some $\delta^i$ such that $|\delta - \delta^i|_2 \leq r_2 t$. By the triangle inequality, we then have

$$\frac{1}{n} \left| \eta^T w \delta \right| \leq \frac{1}{n} \left| \eta^T w \delta^i \right| + \frac{1}{n} \left| \eta^T w (\delta - \delta^i) \right| \leq \frac{1}{n} \left| \eta^T w \delta^i \right| + \frac{\|w\|_2}{\sqrt{n}} \frac{|w(\delta - \delta^i)|_2}{\sqrt{n}}.$$ 

Given the assumptions on $w$, we have $\frac{|w(\delta - \delta^i)|_2}{\sqrt{n}} \leq \sqrt{\kappa_U^w} |\delta - \delta^i|_2 \leq \sqrt{\kappa_U^w} r_2 t$. Moreover, by Lemma B.1, we have $\frac{|w|}{\sqrt{n}} \leq 2\sigma_n$ with probability $1 - c_1 \exp(-c_2 n)$. Putting together the pieces, we obtain

$$\frac{1}{n} \left| \eta^T w \delta \right| \leq \frac{1}{n} \left| \eta^T w \delta^i \right| + 2 \sqrt{\kappa_U^w} \sigma_n r_2 t$$

with probability $1 - c_1 \exp(-c_2 n)$. Taking the supremum over $\delta$ on both sides yields

$$U_n \leq \max_{i=1, \ldots, n} \frac{1}{n} \left| \eta^T w \delta^i \right| + 2 \sqrt{\kappa_U^w} \sigma_n r_2 t.$$ 

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It remains to bound the finite maximum over the covering set. Note that conditioning on \( \{w_i\}_{i=1}^n \), each variate \( \frac{1}{n} \eta_i^T w \delta_i \) is zero-mean (by the fact that \( \mathbb{E}[\eta_i | w_i] = 0 \)) sub-Gaussian with parameter \( \sigma^2_{w_i^2} r_i^2 \). Under the assumptions on \( \delta^i \) and \( w \), by the property of sub-Gaussian variables (Lemma B.17), we conclude that

\[
U_n \leq \sigma r_2 \sqrt{\kappa_U} \left\{ \sqrt{\frac{3 \log N(r_1, r_2, t)}{n}} + 2t \right\}.
\]

Now, by Proposition 4.2, we can set \( r_1 = c' k_1 \sqrt{\frac{\log d}{n}} \) and \( r_2 = c'' \sqrt{\frac{k_1 \log d}{n}} \). By choosing \( t \propto \sqrt{\frac{k_1 \log d}{n}} \) so that \( \sqrt{\frac{3 \log N(r_1, r_2, t)}{n}} = 2t \), we obtain

\[
U_n \leq c \sigma r_1 \sqrt{k_1 \log d}.
\]

Consequently, we conclude that,

\[
\frac{1}{n} \sum_{i=1}^n \eta_i \left[ g(w_i^T \theta^*) - g(w_i^T \hat{\theta}) \right] \leq c \sigma r_1 \sqrt{k_1 \log d},
\]

\[
\frac{1}{n} \sum_{i=1}^n \eta_i \left[ \tilde{g}(w_i^T \theta^*) - \tilde{g}(w_i^T \hat{\theta}) \right] \leq c \sigma r_1 \sqrt{k_1 \log d}.
\]

Putting the pieces together, we have,

\[
\left| \tilde{g}(\hat{\theta}) - g(\theta^*) \right|_n \leq \left| \tilde{g}(\hat{\theta}) - g(\hat{\theta}) \right|_n + \left| g(\hat{\theta}) - g(\theta^*) \right|_n
\]

\[
\leq c' b' \max \left\{ \sqrt{\frac{k_2 \log p}{n}}, \left| \beta^* \right|_1 \sqrt{\frac{k_1 k_2 \log d}{n}}, \left( \frac{k_1}{n} \right)^{\frac{1}{2}} \right\}.
\]

where \( b' \) is some constant depending only on the model-specific structure and independent of \( n, d, p, k_1, \) and \( k_2 \). By a standard integration over the tail probability, the bound above with high probability can be converted to a bound in expectation as stated in Theorem 4.11. □

**Remark.** To prove for the general sparsity case on \( \beta^* \) and \( \theta^* (q_1, q_2 \in [0, 1]) \), we replace \( r_1 = c' k_1 \sqrt{\frac{\log d}{n}} \) and \( r_2 = c'' \sqrt{\frac{k_1 \log d}{n}} \) with \( r_1 = c' R_{q_2} \left( \sqrt{\frac{\log d}{n}} \right)^{1-q_2} \) and \( r_2 = c'' R_{q_2}^{\frac{1}{2}} \left( \sqrt{\frac{\log d}{n}} \right)^{1-q_2} \).

**A.12 Theorem 4.13**

Define \( \mathcal{H} = \mathcal{B}_{q_2}^p (R_{q_2}) \times \mathcal{F} \circ \mathcal{B}_{q_1}^d (R_{q_1}) \) where \( q_1, q_2 \in [0, 1] \). Let \( \tilde{x} := (x, \tilde{f}_0) \in \mathbb{R}^{n \times (p+1)} \) where \( \tilde{f}_0 := (f(w_1^T \theta), \ldots, f(w_n^T \theta))^T \) with \( \theta \in \mathcal{B}_{q_1}^d (R_{q_1}) \) and \( f(\cdot) \in \mathcal{F} \). The lower bounds derived in the following involve the set defined by intersecting the kernel of \( \tilde{x} \) with \( \mathcal{B}_{q_2}^p (R_{q_2}) \), which we denote \( \mathcal{N}_{q_2}(\tilde{x}) := \text{Ker}(\tilde{x}) \cap \mathcal{B}_{q_2}^p (R_{q_2}) \) where

\[
\text{Ker}(\tilde{x}) := \left\{ \beta : \tilde{x} \begin{pmatrix} \beta \\ 1 \end{pmatrix} = 0 \right\},
\]

and define the \( \mathcal{B}_{q_2}^p (R_{q_2}) \)-kernel diameter in the \( l_2 \)-norm

\[
\text{diam}(\mathcal{N}_{q_2}(\tilde{x})) := \max_{\beta \in \mathcal{N}_{q_2}(\tilde{x})} |\beta|_2.
\]

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The following Lemma controls for the term $\text{diam} \left( N_{q_2}(\hat{x}) \right)$.

**Lemma A.10**: If Assumptions 4.9-4.10 hold for any $q_2 \in [0, 1]$, then the $B^p_{q_2}(R_{q_2})$-kernel diameter in $l_2$-norm is upper bounded as

$$\text{diam} \left( N_{q_2}(\hat{x}) \right) \leq \frac{f_l(R_{q_2}, q_2, n, p)}{\kappa_l}.$$  

**Proof.** This proof is similar to the proof for Lemma 1 in Raskutti, et. al (2011). Under Assumption 4.10, we are guaranteed that there is no measurable function $f(w_i^T \theta)$ such that $x_i^T \lambda = f(w_i^T \theta)$ when $y_{1i} = 1$ for $\lambda \in B^p_{q_2}(R_{q_2})$. On the other hand, if

$$\text{diam} \left( N_{q_2}(\hat{x}) \right) > \frac{f_l(R_{q_2}, q_2, n, p)}{\kappa_l},$$

then there must exist some $\beta \in B^p_{q_2}(R_{q_2})$ with $x \beta = 0$ and $|\beta|_2 > \frac{f_l(R_{q_2}, q_2, n, p)}{\kappa_l}$. We then have

$$0 = \frac{1}{\sqrt{n}} |x \beta|_2 < \kappa_l |\beta|_2 - f_l(R_{q_2}, q_2, n, p),$$

which contradicts the lower bound condition on $\frac{1}{\sqrt{n}} |x \beta|_2$ in Assumption 4.9. □

**Proof.** Let $M = M_2(\delta_n; \mathcal{H})$ be the cardinality of a maximal packing of $\mathcal{H}$ in the $L^2(\mathbb{P})$-metric, say with elements $\{h^1, ..., h^M\}$. Also let the random vector $H \in \mathbb{R}^{p+1}$ be uniformly distributed over the packing set $\{h^1, ..., h^M\}$. A standard argument yields a lower bound on the minimax $L^2(\mathbb{P})$-risk in terms of the error in a multi-way hypothesis testing problem: in particular, we have

$$\min_{\hat{h}} \max_{h \in \mathcal{H}} \mathbb{E} \left[ |\hat{h} - h|_2^2 \right] \geq \frac{1}{4} \frac{\sigma_n^2}{\delta_n^2} \min_{\hat{h}} \mathbb{P} \left[ \hat{h} \neq H \right] \quad \text{(A.13)}$$

where the estimator $\hat{h}$ takes values in the packing set. Fano’s inequality implies the lower bound

$$\mathbb{P} \left[ \hat{h} \neq H \right] \geq 1 - \frac{I(H; y_2) + \log 2}{\log M_2(\delta_n; \mathcal{H})}$$

where $I(H; y_2)$ is the mutual information between random parameters $H$ in the packing set and the observation vector $y_2 \in \mathbb{R}^n$.

Let us first upper bound the mutual information; this is done via the procedure of Yang and Barron (1999), which is based on covering the model space $\mathcal{P}(\mathcal{H}) := \{\mathbb{P}_h : h \in \mathcal{H}\}$ under the square-root Kullback-Leibler divergence. Following the argument of Yang and Barron (1999), the mutual information is upper bounded as

$$I(H; y_2) \leq \log N(\varepsilon_n; \mathcal{P}(\mathcal{H})) + \frac{\sigma_n^2}{\varepsilon_n^2} (\kappa_{1u} \vee \kappa_{2u}) \varepsilon_n^2$$

where $N(\varepsilon_n; \mathcal{P}(\mathcal{H}))$ is the covering number of $\mathcal{P}(\mathcal{H})$ under the KL divergence. Combining this upper bound with the Fano lower bound above yields

$$\mathbb{P} \left[ \hat{h} \neq H \right] \geq 1 - \frac{\log N(\varepsilon_n; \mathcal{P}(\mathcal{H})) + \frac{\sigma_n^2}{\varepsilon_n^2} (\kappa_{1u} \vee \kappa_{2u}) \varepsilon_n^2 + \log 2}{\log M_2(\delta_n; \mathcal{H})}.$$  

(A.14)
An upper bound on $\log N(\varepsilon; \mathcal{P}(\mathcal{H}))$ can be obtained using the following argument. For the Gaussian models considered here, the KL divergence takes the form

$$D(\mathbb{P}_h \parallel \mathbb{P}_{h'}) = \frac{1}{2\sigma_h^2} \left| x(\beta - \beta') + f_\theta - f'_\theta \right|^2_2$$

$$\leq \frac{1}{2\sigma_h^2} \left| x(\beta - \beta') \right|^2_2 + \frac{n}{2\sigma_h^2} \left| f_\theta - f'_\theta \right|^2_n$$

$$\leq \frac{1}{2\sigma_h^2} \left| x(\beta - \beta') \right|^2_2 + \frac{2n}{\sigma_h^2} \left| f_\theta - f'_\theta \right|^2_\infty$$

$$\leq \frac{n\kappa_{k_2}}{\sigma_h^2} \left| \beta - \beta' \right|^2_2 + \frac{2n}{\sigma_h^2} \left| f_\theta - f'_\theta \right|^2_\infty + \frac{2nL^2\kappa_{k_1u}}{\sigma_h^2} \left| \theta - \theta' \right|^2_2$$

where the last inequality uses the $\bar{L}$--Lipschitz assumption on $f' \in \mathcal{F}$ and the upper-RE condition on $w$ and $x$ as well as the fact that $|f - g|_n \leq |f - g|_{\infty}$.

Now, applying the upper bound on the covering number of $\mathcal{B}_0(k_1)$ in the $l_2$--metric provided by Lemma B.14, we conclude that there exists a set $\{\theta^1, \ldots, \theta^{N_1}\}$ such that for all $\theta \in \mathcal{B}_0(k_1)$, there exists some index $i$ with $|\theta - \theta^i|_2^2 \leq \varepsilon_n^2$ and $\log N_2(\varepsilon_n; \mathcal{B}_0(k_1)) \leq k_1 \log d + k_1 \log \frac{\sqrt{k_1}}{\varepsilon_n}$; similarly, there also exists a set $\{\beta^1, \ldots, \beta^{N_2}\}$ such that for all $\beta \in \mathcal{B}_0(k_2)$, there exists some index $j$ with $|\beta - \beta^j|_2^2 \leq \varepsilon_n^2$ and $\log N_2(\varepsilon_n; \mathcal{B}_0(k_2)) \leq k_2 \log p + k_2 \log \frac{\sqrt{k_2}}{\varepsilon_n}$. Applying the upper bound on the covering number of the $\bar{L}$--Lipschitz function class $\mathcal{F}$ in the sup-norm provided by Lemma B.11, we conclude that there exists a set $\{f_{\theta^1}, \ldots, f_{\theta^{N_1}}\}$ such that for all $f \in \mathcal{F}$ and $\theta \in \{\theta^1, \ldots, \theta^{N_1}\}$, there exists some index $l$ with $\sup_{w, \theta} |f(w_l \theta) - f_l(w_l \theta)| \leq \varepsilon_n$ and $\log N_{\infty}(\varepsilon_n; \mathcal{F}) \leq \frac{2k_1}{\varepsilon_n}$ for all $\theta \in \{\theta^1, \ldots, \theta^{N_1}\}$. It is not hard to see that a covering set of $\mathcal{P}(\mathcal{H})$ under the KL divergence can be formed by

$$\left\{\theta^1, \ldots, \theta^{N_1}\right\} \times \{\beta^1, \ldots, \beta^{N_2}\} \times \left\{f_{\theta^1}, \ldots, f_{\theta^{N_1}} : \theta \in \{\theta^1, \ldots, \theta^{N_1}\}\right\}$$

and

$$\log N(\varepsilon_n; \mathcal{P}(\mathcal{H})) \leq \log \left[ \left( N_2(\varepsilon_n; \mathcal{B}_0(k_1)) \right)^2 \cdot N_2(\varepsilon_n; \mathcal{B}_0(k_2)) \cdot N_{\infty}(\varepsilon_n; \mathcal{F}) \right]$$

$$\leq c \left\{ k_1 \log d + k_1 \log \frac{\sqrt{k_1}}{\varepsilon_n} + k_2 \log p + k_2 \log \frac{\sqrt{k_2}}{\varepsilon_n} + \frac{k_1}{\varepsilon_n} \right\}.$$  \hspace{1cm} \text{(A.15)}

It remains to lower bound $\log M_2(\delta_n; \mathcal{H})$; we do so by constructing a minimal covering set of $\mathcal{H} = \mathcal{B}_0(k_2) \times \mathcal{F} \circ \mathcal{B}_0(k_1)$ in the $L^2(\mathbb{P})$--metric namely, $\log N_2(\delta_n; \mathcal{H})$. Note that

$$\left| h - h' \right|_{L^2(\mathbb{P})}^2 = \left| \beta - \beta' \right|^2_2 + \left| f(w_l \theta) - f_l(w_l \theta') \right|^2_2$$

$$\leq \left| \beta - \beta' \right|^2_2 + 2 \left| f(w_l \theta) - f_l(w_l \theta) \right|^2_2 + 2 \left| f_l(w_l \theta) - f_l(w_l \theta') \right|^2_2$$

$$\leq \left| \beta - \beta' \right|^2_2 + 2 \left| f(w_l \theta) - f_l(w_l \theta) \right|^2_2 + 2\kappa_{k_1u}L^2 \left| \theta - \theta' \right|^2_2$$

where the last inequality uses the $\bar{L}$--Lipschitz assumption on $f' \in \mathcal{F}$ and the population upper-RE condition on $w$. Using the similar argument from above, we conclude from the first inequality of Lemma B.12 and Lemma B.15 that there exists a set $\{\theta^1, \ldots, \theta^{N_1}\}$ such that for all $\theta \in \mathcal{B}_0(k_1)$, there exists some index $i$ with $|\theta - \theta^i|_2^2 \leq \delta_n^2$ and $\log N_2(\delta_n; \mathcal{B}_0(k_1)) \geq \log M_2(2\delta_n; \mathcal{B}_0(k_1)) \geq \frac{k_1}{2} \log \frac{\delta_n}{\varepsilon_n}$; similarly, there
also exists a set \( \{ \beta^1, ..., \beta^{N_2} \} \) such that for all \( \beta \in B^0_{\delta}(k_2) \), there exists some index \( j \) with \( |\beta - \beta^j|^2 \leq \delta_n^2 \) and \( \log N_2(\delta_n; B^0_{\delta}(k_2)) \geq \log M_2(2\delta_n; B^0_{\delta}(k_2)) \geq \frac{k_2}{2} \log \frac{d - k_2}{k_2/2}. \) Furthermore, by Lemma B.11 and the first inequality of Lemma B.12, we conclude that there exists a set \( \{ f^j_\theta, ... , f^{N_3}_\theta \} \) such that for all \( f \in F \) and \( \theta \in \{ \theta^1, ..., \theta^{N_3} \} \), there exists some index \( l \) with \( \mathbb{E} [f(w^T \theta) - f^l(w^T \theta)]^2 \leq \delta_n^2 \) and \( \log N_2(\delta_n; F_\theta) \geq \log M_2(2\delta_n; F_\theta) \geq \frac{k_2}{\delta_n} \) for all \( \theta \in \{ \theta^1, ..., \theta^{N_3} \} \). It is not hard to see that a covering set of \( \mathcal{H} \) under the \( L^2(\mathbb{P}) \)-metric can be formed by

\[ \{ \theta^1, ..., \theta^{N_1} \} \times \{ \beta^1, ..., \beta^{N_2} \} \times \{ f^j_\theta, ... , f^{N_3}_\theta : \theta \in \{ \theta^1, ..., \theta^{N_3} \} \} \]

and

\[
\log N_2(\delta_n; \mathcal{H}) \geq \log \left[ \left( M_2(2\delta_n; B^0_{\delta}(k_1)) \right)^2 \cdot M_2(2\delta_n; B^0_{\delta}(k_2)) \cdot M_2(2\delta_n; F) \right] \\
\geq c' \left\{ \frac{k_1}{2} \log \frac{d - k_1}{k_1/2} + \frac{k_2}{2} \log \frac{d - k_2}{k_2/2} + \frac{k_1}{\delta_n} \right\}.
\]

Now applying the second inequality of Lemma B.12 yields

\[
\log M(\delta_n; \mathcal{H}) \geq \log N(\delta_n; \mathcal{H}) \\
\geq c' \left\{ \frac{k_1}{2} \log \frac{d - k_1}{k_1/2} + \frac{k_2}{2} \log \frac{d - k_2}{k_2/2} + \frac{k_1}{\delta_n} \right\}.
\]

(A.16)

Substituting (A.15) and (A.16) into (A.14) yields

\[
\mathbb{P} \left[ \tilde{h} \neq H \right] \geq 1 - \frac{c' \left\{ \frac{k_1}{2} \log \frac{d - k_1}{k_1/2} + \frac{k_2}{2} \log \frac{d - k_2}{k_2/2} + \frac{k_1}{\delta_n} \right\}}{c' \left\{ \frac{k_1}{2} \log \frac{d - k_1}{k_1/2} + \frac{k_2}{2} \log \frac{d - k_2}{k_2/2} + \frac{k_1}{\delta_n} \right\}}.
\]

Under the condition \( \varepsilon \geq \frac{1}{d} \), setting

\[
\frac{c^2_n}{\sigma^2_q}(\kappa_{1u} \lor \kappa_{2u})\varepsilon_n^2 = c' \left\{ \frac{k_1}{2} \log \frac{d - k_1}{k_1/2} + \frac{k_2}{2} \log \frac{d - k_2}{k_2/2} + \frac{k_1}{\delta_n} \right\},
\]

and \( 4\frac{c^2_n}{\sigma^2_q}(\kappa_{1u} \lor \kappa_{2u})\varepsilon_n^2 = c'' \left\{ \frac{k_1}{2} \log \frac{d - k_1}{k_1/2} + \frac{k_2}{2} \log \frac{d - k_2}{k_2/2} + \frac{k_1}{\delta_n} \right\} \)

yields the following

\[
\mathbb{P} \left[ \tilde{h} \neq H \right] \geq \frac{1}{4},
\]

and \( \delta_n^2 \asymp \varepsilon_n^2 \times \max \left\{ \frac{k_1}{n}, \frac{k_2}{n}, \left( \frac{k_1}{n} \right)^{\frac{3}{2}} \right\} \).

It can be easily verified that under the scaling of \( \varepsilon_n \), we are guaranteed to have \( \varepsilon_n \geq \frac{1}{d} \).

Consequently, substituting the pieces above into (A.13) yields

\[
\min_{\tilde{h}} \max_{h \in \mathcal{H}} \mathbb{E} \left[ \tilde{h} - h \right]^2 \geq \max \left\{ \frac{k_1}{n}, \frac{k_2}{n}, \left( \frac{k_1}{n} \right)^{\frac{3}{2}} \right\}.
\]
\[ \mathbb{P} \left[ \tilde{h} \neq H \right] \geq \frac{1}{4}, \]

and \( \delta_n^2 \propto \frac{\varepsilon_n^2}{n} \times \max \left\{ R_{q_1} \left( \frac{\log d}{n} \right)^{\frac{2q_1}{2q_1 - 1}}, R_{q_2} \left( \frac{\log p}{n} \right)^{\frac{2q_2}{2q_2 - 1}}, \left( \frac{R}{n} \right)^{\frac{2}{3}} \right\}. \)

Consequently, substituting the pieces above into (A.13) yields

\[ \min_h \max_{\hat{h} \in \mathcal{H}} \mathbb{E} \left[ \left| \hat{h} - h \right|_2^2 \right] \propto \max \left\{ R_{q_1} \left( \frac{\log d}{n} \right)^{\frac{2q_1}{2q_1 - 2}}, R_{q_2} \left( \frac{\log p}{n} \right)^{\frac{2q_2}{2q_2 - 2}}, \left( \frac{R}{n} \right)^{\frac{2}{3}} \right\}. \]

\[ \square \]

### A.13 Theorem 4.14

**Proof.** The proof for Theorem 4.14 is almost identical to the proof for Theorem 4.13 except that instead of Lemma B.15, we apply Lemma B.10 to lower bound the packing number of \( \mathcal{B}_d^q(R_{q_1}) \) and \( \mathcal{B}_d^p(R_{q_2}) \) in the \( l_2 \)-metric where \( q_1, q_2 \in (0, 1] \). These steps give us

\[
\mathbb{P} \left[ \tilde{h} \neq H \right] \geq \frac{1}{2}, \quad \text{and} \quad \tilde{h} = \left[ \begin{array}{c} R_{q_1} \left( \frac{\log d}{n} \right)^{\frac{2q_1}{2q_1 - 1}} \log d \end{array} \right] + l \left[ \begin{array}{c} R_{q_1} \left( \frac{\log d}{n} \right)^{\frac{2q_1}{2q_1 - 1}} \log d \end{array} \right] + c' \frac{R}{\varepsilon_n} + c'' \frac{\varepsilon_n^2}{\sigma_n^2} (\kappa_1 \lor \kappa_2) \varepsilon_n^2 + \log 2 \]

yields the following

\[
\mathbb{P} \left[ \tilde{h} \neq H \right] \geq \frac{1}{4}, \quad \text{and} \quad \delta_n^2 \propto \frac{\varepsilon_n^2}{n} \times \max \left\{ R_{q_1} \left( \frac{\log d}{n} \right)^{\frac{2q_1}{2q_1 - 1}}, R_{q_2} \left( \frac{\log p}{n} \right)^{\frac{2q_2}{2q_2 - 1}}, \left( \frac{R}{n} \right)^{\frac{2}{3}} \right\}. \]

Consequently, substituting the pieces above into (A.13) yields

\[
\min_h \max_{\hat{h} \in \mathcal{H}} \mathbb{E} \left[ \left| \hat{h} - h \right|_2^2 \right] \propto \max \left\{ R_{q_1} \left( \frac{\log d}{n} \right)^{\frac{2q_1}{2q_1 - 2}}, R_{q_2} \left( \frac{\log p}{n} \right)^{\frac{2q_2}{2q_2 - 2}}, \left( \frac{R}{n} \right)^{\frac{2}{3}} \right\}. \]

\[ \square \]

### A.14 Upper bounds on the second-stage estimators

The following lemma upper bounds the term \( \frac{1}{n} \sum_{i=1}^n \left( \hat{m}_j(w_i^T \theta) - m_j(w_i^T \theta^*) \right)^2 \) (for \( j = 0, ..., p \)). Recall \( y_i = z_{i0} \) and \( x_{ij} = z_{ij} \) for \( j = 1, ..., p \). From Section 3 we have

\[
z_{ij} = \mathbb{E} \left( z_{ij} | w_i^T \theta^* \right) + v_{ij} := m_j(w_i^T \theta^*) + v_{ij} \quad \text{for} \quad j = 0, ..., p
\]

which can be rewritten as

\[
z_{ij} = m_j(w_i^T \theta) + v_{ij} - \left( m_j(w_i^T \theta^*) - m_j(w_i^T \theta^*) \right).
\]
To avoid notation clustering, write \( \{\hat{m}_j(w_i^T \theta)\}_{i=1}^n := \hat{m}_j(\theta) \). Recall the set
\[
\mathcal{F}_j := \{ f = f' - f'' : f', f'' \in \mathcal{F}_j \},
\]
and
\[
\Omega(r_j; \mathcal{F}_j) = \{ f : f \in \mathcal{F}_j, |f_{\theta^*}|_n \leq r_j \},
\]
where \( |f_{\theta^*}|_n := \sqrt{\frac{1}{n} \sum_{i=1}^n [f(w_i^T \theta^*)]^2} \), and the conditional \textit{local complexity}
\[
\mathcal{G}_n(r_j; \mathcal{F}_j) := \mathbb{E}_{v_j} \left[ \sup_{f \in \Omega(r_j; \mathcal{F}_j)} \left\{ \frac{1}{n} \sum_{i=1}^n v_{ij} f(w_i^T \theta^* \right\} |w_i^T \theta^* \right\},
\]
where \( \{v_{ij}\}_{i=1}^n \) for \( j = 0, \ldots, p \) are \textit{i.i.d.} variates that satisfy Assumption 4.2. In terms of this complexity measure (A.17), we have the following result.

\textbf{Lemma A.11:} Let \( r_{nj} > 0 \) be the smallest positive quantity satisfying the \textit{critical inequality}
\[
\mathcal{G}_n(r_{nj}; \mathcal{F}_j) \leq \frac{r_{nj}^2}{\sigma_{v_j}} \tag{A.18}
\]
Then there are universal positive constants \((c_1, c_2)\) such that for any \( t \geq r_{nj} \) and \( \hat{m}_j \in \mathcal{F}_j \), the non-parametric least squares estimate \( \hat{m}_j \) via program (11) satisfies
\[
|\hat{m}_j(\hat{\theta}) - m_j(\hat{\theta})|^2_2 \leq 7 |\hat{m}_j(\hat{\theta}) - m_j(\hat{\theta})|^2_2 + 128 tr_{nj} + 4 \sqrt{32 \sigma_{v_j}^2 T_j^r + 16 T_j^r}
\]
with probability at least \( 1 - c_1 \exp\left(-nC_{\theta}^* tr_{nj}\right) \).

\textbf{Remark.} The constants in the error bound is not optimal and can be improved.

\textbf{Proof.} Since \( \hat{m}_j \) is optimal and \( \hat{m}_j \) is feasible for the program (11), we have
\[
\frac{1}{2} \sum_{i=1}^n \left( z_{ij} - \hat{m}_j(w_i^T \hat{\theta}) \right)^2 \leq \frac{1}{2} \sum_{i=1}^n \left( z_{ij} - \hat{m}_j(w_i^T \hat{\theta}) \right)^2.
\]
Define \( \hat{\Delta}_{ij}(\theta) = (\hat{m}_j - m_j) \circ (w_i^T \theta), \{\hat{\Delta}_{ij}(\theta)\}_{i=1}^n = \hat{\Delta}_j(\theta), \bar{\Delta}_{ij}(\theta) = (\hat{m}_j - m_j) \circ (w_i^T \theta), \{\bar{\Delta}_{ij}(\theta)\}_{i=1}^n = \bar{\Delta}_j(\theta) \). Performing some algebra yields the \textit{basic inequality}
\[
|\hat{\Delta}_j(\hat{\theta})|^2_n \leq |\bar{\Delta}_j(\hat{\theta})|^2_n + 2 \left| \frac{1}{n} \sum_{i=1}^n (v_{ij} + T_{ij}^r) \Delta_{ij}(\hat{\theta}) \right|
\]
Furthermore, we have
\[
\frac{1}{n} \left| \sum_{i=1}^n T_{ij}^r \hat{\Delta}_{ij}(\hat{\theta}) \right| \leq \frac{1}{n} \left| \sum_{i=1}^n T_{ij}^r \bar{\Delta}_{ij}(\hat{\theta}) \right| + \frac{1}{n} \left| \sum_{i=1}^n T_{ij}^r \bar{\Delta}_{ij}(\hat{\theta}) \right|
\]
\[
\leq \frac{2}{n} \left| \sum_{i=1}^n T_{ij}^r \right| + \frac{1}{4n} \left| \sum_{i=1}^n \hat{\Delta}_{ij}^2(\hat{\theta}) \right| + \frac{1}{4n} \left| \sum_{i=1}^n \bar{\Delta}_{ij}^2(\hat{\theta}) \right|,
\]
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where the first inequality follows from the triangle inequality and the second inequality follows from the Fenchel-Young inequality. Consequently, we have

\[
\frac{1}{2} |\Delta_j(\theta)|_n^2 \leq \frac{3}{2} |\Delta_j(\hat{\theta})|_n^2 + 2 \left| \frac{1}{n} \sum_{i=1}^{n} v_i \bar{\Delta}_{ij}(\hat{\theta}) \right| + 4T'_j, \tag{A.19}
\]

recalling the definition of \( T'_j \).

We need to upper bound the stochastic process \( \frac{1}{n} \sum_{i=1}^{n} v_i \bar{\Delta}_{ij}(\hat{\theta}) \). Now, note that by the elementary inequalities, we have

\[
\frac{1}{n} \left| \sum_{i=1}^{n} v_i \bar{\Delta}_{ij}(\hat{\theta}) \right| \leq \frac{1}{n} \left| \sum_{i=1}^{n} v_i \left[ \bar{\Delta}_{ij}(\hat{\theta}) - \bar{\Delta}_{ij}(\theta^*) \right] \right| + \frac{1}{n} \left| \sum_{i=1}^{n} v_i \bar{\Delta}_{ij}(\theta^*) \right| \\
\leq \sqrt{\frac{1}{n} \sum_{i=1}^{n} v_i^2} \sqrt{\frac{1}{n} \sum_{i=1}^{n} \left[ \bar{\Delta}_{ij}(\hat{\theta}) - \bar{\Delta}_{ij}(\theta^*) \right]^2} + \frac{1}{n} \left| \sum_{i=1}^{n} v_i \bar{\Delta}_{ij}(\theta^*) \right| \\
\leq \sqrt{\frac{1}{n} \sum_{i=1}^{n} v_i^2} \sqrt{4 \sup_{f \in \mathcal{F}_j} \frac{1}{n} \sum_{i=1}^{n} \left[ f(w_i^T \hat{\theta}) - f(w_i^T \theta^*) \right]^2} + \frac{1}{n} \left| \sum_{i=1}^{n} v_i \bar{\Delta}_{ij}(\theta^*) \right| \\
\leq \sqrt{\frac{1}{n} \sum_{i=1}^{n} v_i^2} \sqrt{4T_j} + \frac{1}{n} \left| \sum_{i=1}^{n} v_i \bar{\Delta}_{ij}(\theta^*) \right|
\]

For the term \( \sqrt{\frac{1}{n} \sum_{i=1}^{n} v_i^2} \), by Lemma B.1, we have

\[
P \left[ \frac{1}{n} \sum_{i=1}^{n} v_i^2 \geq 2\sigma_{v_i}^2 \right] \leq 2 \exp(-cn).
\]

For the term \( \frac{1}{n} \left| \sum_{i=1}^{n} v_i \bar{\Delta}_{ij}(\theta^*) \right| \), notice that the error function \( \bar{\Delta}_j \) belongs to the set \( \mathcal{F}_j \). If \( |\bar{\Delta}_j(\theta^*)|_n \geq u = \sqrt{tr_{nj}} \), an upper bound on the term \( \frac{1}{n} \left| \sum_{i=1}^{n} v_i \bar{\Delta}_{ij}(\theta^*) \right| \) can be obtained by Lemma B.3:

\[
\frac{1}{n} \left| \sum_{i=1}^{n} v_i \bar{\Delta}_{ij}(\theta^*) \right| \leq \sup_{f(\theta^*) \in \Omega(\bar{\Delta}_j; \mathcal{F}_j)} \left| \frac{1}{n} \sum_{i=1}^{n} v_i f(w_i^T \theta^*) \right| \\
\leq 2 \left| \Delta_j(\theta^*) \right|_n \sqrt{tr_{nj}},
\]

with probability at least \( 1 - \exp \left( -nC_j^* tr_{nj} \right) \), where \( r_{nj} > 0 \) is the smallest positive quantity satisfying the critical inequality

\[
\mathcal{G}_n (r_{nj}; \mathcal{F}_j) \leq \frac{r_{nj}^2}{\sigma_{v_j}}.
\]

Applying the triangle inequality and the Fenchel-Young inequality twice yields

\[
|\Delta_j(\theta^*)|_n \sqrt{tr_{nj}} \leq 8tr_{nj} + \frac{1}{16} \left( |\bar{\Delta}_j(\theta^*)|_n^2 + |\bar{\Delta}_j(\theta^*)|_n^2 \right).
\]
Putting the pieces together, we obtain
\[
\frac{1}{n} \left| \tilde{\Delta}_j(\hat{\theta}) \right|^2 \leq \frac{7}{4} \left| \bar{\Delta}_j(\hat{\theta}) \right|^2 + 32 \text{tr}_{nj} + \sqrt{32\sigma^2_{\bar{\Delta}} T'_j} + 4T'_j
\]
where
\[
T'_j = \sup_{f \in F_j} \frac{1}{n} \sum_{i=1}^n \left[ f(w_i^T \hat{\theta}) - f(w_i^T \theta^*) \right]^2.
\]
If \( \left| \bar{\Delta}_j(\theta^*) \right|_n < \sqrt{\text{tr}_{nj}} \), then the claim follows trivially by triangle inequality and the Fenchel-Young inequality.\( \square \)

**Lemma A.12:** For a user-defined radius \( \bar{R}_j > 0 \), let \( r_{nj} > 0 \) be the smallest positive quantity satisfying the critical inequality
\[
G_n(r_{nj}, F_j) \leq \frac{R_j^2 r_{nj}^2}{\sigma_{v_j}}, \tag{A.20}
\]
where \( \Omega(r_j; F_j) = \{ f : f \in F_j, |f|_{F_j} \leq r_j \} \) for \( j = 0, ..., p \). Suppose that we solve program (12) with \( \lambda_{nj,2} \geq 2^2 r_{nj}^2 \). Let \( \hat{m}_j \in F_j \) with \( |\hat{m}_j|_{F_j} \leq \bar{R}_j \). Then there are universal positive constants \( (c_0, c_1, c_2, c_3) \) such that the non-parametric least squares estimate \( \hat{m}_j \) via (12) satisfies
\[
\left| \hat{m}_j(\hat{\theta}) - m_j(\hat{\theta}) \right|_n^2 \leq c_0 \left| \hat{m}_j(\hat{\theta}) - m_j(\hat{\theta}) \right|_n^2 + c_1 \bar{R}_j^2 \left\{ r_{nj}^2 + \lambda_{nj,2} \right\} + c_2 T'_j + c_3 \sqrt{\sigma^2_{\bar{\Delta}} T'_j}
\]
with probability at least \( 1 - c \exp \left( -nC_j^* \bar{R}_j^2 r_{nj}^2 \right) \).

**Proof.** Introduce the shorthand \( \bar{\sigma}_{v_j} = \frac{\sigma_{v_j}}{\bar{R}_j} \) and we work with an equivalent model with noise variance \( \left( \frac{\sigma_{v_j}}{\bar{R}_j} \right)^2 \) and the rescaled approximation error \( \left| \hat{m}_j(\hat{\theta}) - m_j(\hat{\theta}) \right|_n^2 \). Note that the final error then should be multiplied by \( \bar{R}_j^2 \). Now, since \( \hat{m}_j \) is optimal and \( \hat{m}_j \) is feasible for program (12), we have
\[
\frac{1}{2} \sum_{i=1}^n \left( z_{ij} - \hat{m}_j(w_i^T \hat{\theta}) \right)^2 + \lambda_{nj,2} |\hat{m}_j|^2_{F_j} \leq \frac{1}{2} \sum_{i=1}^n \left( z_{ij} - \hat{m}_j(w_i^T \hat{\theta}) \right)^2 + \lambda_{nj,2} |\hat{m}_j|^2_{F_j}.
\]
Recall the definitions \( \hat{\Delta}_{ij}(\theta) := (\hat{m}_j - m_j) \circ (w_i^T \theta) \), \( \bar{\Delta}_{ij}(\theta) := (\hat{m}_j - m_j) \circ (w_i^T \theta) \), \( \bar{\Delta}_{ij}(\theta) := (\hat{m}_j - m_j) \circ (w_i^T \theta) \), \( \{ \hat{\Delta}_{ij}(\theta) \}_{i=1}^n \) := \( \hat{\Delta}_{ij}(\theta) \), \( \{ \bar{\Delta}_{ij}(\theta) \}_{i=1}^n \) := \( \bar{\Delta}_{ij}(\theta) \), and \( \{ \Delta_{ij}(\theta) \}_{i=1}^n \) := \( \Delta_{ij}(\theta) \). Performing some algebra yields the modified basic inequality
\[
\frac{1}{2} \left| \hat{\Delta}_j(\hat{\theta}) \right|_n^2 \leq \frac{1}{2} \left| \bar{\Delta}_j(\hat{\theta}) \right|_n^2 + \frac{1}{n} \sum_{i=1}^n (v_{ij} + T_{ij}) \bar{\Delta}_{ij}(\hat{\theta}) \left| \bar{\Delta}_{ij}(\hat{\theta}) \right| + \lambda_{nj,2} \left\{ |\hat{m}_j|^2_{F_j} - |\hat{m}_j|^2_{F_j} \right\}
\]
\[
\leq \frac{1}{2} \left| \hat{\Delta}_j(\hat{\theta}) \right|_n^2 + \frac{1}{n} \sum_{i=1}^n v_{ij} \hat{\Delta}_{ij}(\hat{\theta}) \left| \hat{\Delta}_{ij}(\hat{\theta}) \right| + \frac{1}{n} \sum_{i=1}^n T_{ij} \bar{\Delta}_{ij}(\hat{\theta}) \left| \bar{\Delta}_{ij}(\hat{\theta}) \right| + \lambda_{nj,2} \left\{ |\hat{m}_j|^2_{F_j} - |\hat{m}_j|^2_{F_j} \right\}. \tag{A.21}
\]
Furthermore, we have
\[
\frac{1}{n} \left| \sum_{i=1}^{n} T_{ij} \Delta_{ij}(\hat{\theta}) \right| \leq \frac{1}{n} \left| \sum_{i=1}^{n} T_{ij} \Delta_{ij}(\hat{\theta}) \right| + \frac{1}{n} \left| \sum_{i=1}^{n} T_{ij} \Delta_{ij}(\hat{\theta}) \right| \\
\leq \frac{2}{n} \left| \sum_{i=1}^{n} T_{ij}^2 \right| + \frac{1}{4n} \left| \sum_{i=1}^{n} \Delta_{ij}^2(\hat{\theta}) \right| + \frac{1}{4n} \left| \sum_{i=1}^{n} \Delta_{ij}^2(\hat{\theta}) \right|
\]
where the first inequality uses the triangle inequality and the second uses the Fenchel-Young inequality. Substituting the inequality into (A.21), we obtain
\[
\frac{1}{4} \left| \Delta_j(\hat{\theta}) \right|^2 \leq \frac{3}{4} \left| \Delta_j(\hat{\theta}) \right|^2 + \frac{1}{n} \left| \sum_{i=1}^{n} v_{ij} \Delta_{ij}(\hat{\theta}) \right| + \frac{2}{n} \left| \sum_{i=1}^{n} T_{ij}^2 \right| + \lambda_{nj,2} \left\{ |\hat{m}_j|_{F_j}^2 - |m_j|_{F_j}^2 \right\}.
\]
(A.22)

The remainder of the proof can be divided into two cases.

**Case (i):** $|\hat{m}_j|_{F_j} \leq 2$. In this case, (A.22) implies that
\[
\frac{1}{4} \left| \Delta_j(\hat{\theta}) \right|^2 \leq \frac{3}{4} \left| \Delta_j(\hat{\theta}) \right|^2 + \frac{1}{n} \left| \sum_{i=1}^{n} v_{ij} \Delta_{ij}(\hat{\theta}) \right| + \frac{2}{n} \left| \sum_{i=1}^{n} T_{ij}^2 \right| + 9\lambda_{nj,2},
\]
using the fact that $|\hat{m}_j|_{F_j} \leq 1$. Since $|\hat{m}_j|_{F_j} \leq 2$, we have $|\Delta_j|_{F_j} \leq 3$ and hence
\[
|\hat{m}_j|_{F_j}^2 - |m_j|_{F_j}^2 = \left( |\hat{m}_j|_{F_j} + |m_j|_{F_j} \right) \left( |\hat{m}_j|_{F_j} - |m_j|_{F_j} \right) \leq 9
\]
by the triangle inequality. We need to upper bound the stochastic process $\frac{1}{n} \left| \sum_{i=1}^{n} v_{ij} \Delta_{ij}(\hat{\theta}) \right|$. 
\[
\frac{1}{n} \left| \sum_{i=1}^{n} v_{ij} \Delta_{ij}(\hat{\theta}) \right| \leq \frac{1}{n} \left| \sum_{i=1}^{n} v_{ij} \left[ \Delta_{ij}(\hat{\theta}) - \Delta_{ij}(\theta^*) \right] \right| + \frac{1}{n} \left| \sum_{i=1}^{n} v_{ij} \Delta_{ij}(\theta^*) \right| \\
\leq \sqrt{\frac{1}{n} \sum_{i=1}^{n} v_{ij}^2 \left| \sum_{i=1}^{n} \left[ \Delta_{ij}(\hat{\theta}) - \Delta_{ij}(\theta^*) \right] \right|^2 + \frac{1}{n} \left| \sum_{i=1}^{n} v_{ij} \Delta_{ij}(\theta^*) \right|} \\
\leq \sqrt{\frac{1}{n} \sum_{i=1}^{n} v_{ij}^2 \left| \sum_{i=1}^{n} \left[ f(w_i^T \hat{\theta}) - f(w_i^T \theta^*) \right] \right|^2 + \frac{1}{n} \left| \sum_{i=1}^{n} v_{ij} \Delta_{ij}(\theta^*) \right|} \\
\leq \sqrt{\frac{1}{n} \sum_{i=1}^{n} v_{ij}^2 \left| 4T_j^2 + \frac{1}{n} \left| \sum_{i=1}^{n} v_{ij} \Delta_{ij}(\theta^*) \right| \right|}.
\]
Now, applying Lemma B.4 yields
\[
\frac{1}{n} \left| \sum_{i=1}^{n} v_{ij} \Delta_{ij}(\theta^*) \right| \leq 6r_{nj} \left| \Delta_j(\theta^*) \right|_n + \frac{1}{32} \left| \Delta_j(\theta^*) \right|_n^2
\]
(A.23)
with probability at least \(1 - c \exp \left( -nC_j^2 \tilde{R}_j^2 \rho_j^2 \right)\). Consequently,

\[
\frac{1}{4} \left| \tilde{\Delta}_j(\hat{\theta}) \right|_n^2 \leq \frac{3}{4} \left| \tilde{\Delta}_j(\hat{\theta}) \right|_n^2 + 6r_nj \left| \tilde{\Delta}_j(\hat{\theta}) \right|_n + \frac{1}{32} \left| \tilde{\Delta}_j(\hat{\theta}) \right|_n^2 + \frac{2}{n} \left| \sum_{i=1}^n T_{ij}^2 \right| + 9\lambda_{nj,2}
\]

\[
\leq \frac{3}{4} \left| \tilde{\Delta}_j(\hat{\theta}) \right|_n^2 + 216r_nj^2 + \frac{1}{12} \left| \tilde{\Delta}_j(\hat{\theta}) \right|_n^2 + \frac{2}{n} \left| \sum_{i=1}^n T_{ij}^2 \right| + 9\lambda_{nj,2},
\]

where the second inequality follows by the triangle inequality, and the Fenchel-Young inequality. This inequality above implies

\[
\frac{5}{48} \left| \Delta_j(\hat{\theta}) \right|_n^2 \leq \frac{43}{48} \left| \Delta_j(\hat{\theta}) \right|_n^2 + 216r_nj^2 + \frac{2}{n} \left| \sum_{i=1}^n T_{ij}^2 \right| + 9\lambda_{nj,2}.
\]

**Case (ii):** \(|\tilde{m}_j|_{\mathcal{F}_j} > 2 > 1 \geq |\tilde{m}_j|_{\mathcal{F}_j}\). In this case, we have

\[
|\tilde{m}_j|_{\mathcal{F}_j}^2 - |\tilde{m}_j|_{\mathcal{F}_j} = \left( |\tilde{m}_j|_{\mathcal{F}_j} + |\tilde{m}_j|_{\mathcal{F}_j} \right) \left( \tilde{m}_j|_{\mathcal{F}_j} - |\tilde{m}_j|_{\mathcal{F}_j} \right)
\]

\[
|\tilde{m}_j|_{\mathcal{F}_j} - |\tilde{m}_j|_{\mathcal{F}_j} \leq |\tilde{m}_j|_{\mathcal{F}_j} - |\tilde{m}_j|_{\mathcal{F}_j}.
\]

since \(\tilde{m}_j|_{\mathcal{F}_j} - \tilde{m}_j|_{\mathcal{F}_j} < 0\). Writing \(\hat{m}_j = \tilde{m}_j + \tilde{\Delta}_j\) and applying the triangle inequality \(|\hat{m}_j|_{\mathcal{F}_j} \geq |\tilde{\Delta}_j|_{\mathcal{F}_j} - |\tilde{m}_j|_{\mathcal{F}_j}\) yields

\[
\lambda_{nj,2} \left\{ |\tilde{m}_j|_{\mathcal{F}_j}^2 - |\tilde{m}_j|_{\mathcal{F}_j} \right\} \leq \lambda_{nj,2} \left\{ 2 \tilde{m}_j|_{\mathcal{F}_j} - |\tilde{\Delta}_j|_{\mathcal{F}_j} \right\}
\]

\[
\leq 2\lambda_{nj,2} - \lambda_{nj,2} |\tilde{\Delta}_j|_{\mathcal{F}_j}.
\]

Substituting this upper bound into (A.22) yields

\[
\frac{1}{4} \left| \Delta_j(\hat{\theta}) \right|_n^2 \leq \frac{3}{4} \left| \Delta_j(\hat{\theta}) \right|_n^2 + \frac{1}{n} \left| \sum_{i=1}^n v_{ij} \tilde{\Delta}_{ij}(\hat{\theta}) \right| + \left| \frac{2}{n} \sum_{i=1}^n T_{ij}^2 \right| + 2\lambda_{nj,2} - \lambda_{nj,2} |\tilde{\Delta}_j|_{\mathcal{F}_j}
\]

\[
+ \frac{2}{n} \left| \sum_{i=1}^n T_{ij}^2 \right| + 2\lambda_{nj,2} - \lambda_{nj,2} |\tilde{\Delta}_j|_{\mathcal{F}_j}
\]

\[
(A.24)
\]

Lemma B.5 upper bounds the stochastic component \(\frac{1}{n} \left| \sum_{i=1}^n v_{ij} \tilde{\Delta}_{ij}(\hat{\theta}) \right|\) in inequality (A.24): There are universal positive constants \((c_1, c_2)\) such that for all \(|\Delta_j|_{\mathcal{F}_j} \geq 1\),

\[
\frac{1}{n} \left| \sum_{i=1}^n v_{ij} \tilde{\Delta}_{ij}(\hat{\theta}) \right| \leq 2r_nj \left| \Delta_j(\hat{\theta}) \right|_n + 2r_nj^2 |\Delta_j|_{\mathcal{F}_j} + \frac{1}{16} |\Delta_j(\hat{\theta})|^2_n
\]

\[
(A.25)
\]

with probability at least \(1 - c \exp \left( -nC_j^2 \tilde{R}_j^2 \rho_j^2 \right)\). We now complete the proof using inequality (A.25).

Since \(|\tilde{m}_j|_{\mathcal{F}_j} > 2 > 1 \geq |\tilde{m}_j|_{\mathcal{F}_j}\), the triangle inequality implies that \(|\tilde{\Delta}_j|_{\mathcal{F}_j} \geq |\tilde{m}_j|_{\mathcal{F}_j} - |\tilde{m}_j|_{\mathcal{F}_j} > 1\), so that
inequality (A.25) can be applied. Consequently,

\[
\frac{1}{4} |\Delta_j(\hat{\theta})|^2_n \leq \frac{3}{4} |\Delta_j(\hat{\theta})|^2_n + 2r_{n_j} |\Delta_j(\hat{\theta})|^n + (2r_{n_j} - \lambda_{n_j,2}) |\Delta_j|^r_n
\]

\[
+ \frac{1}{16} |\Delta_j(\hat{\theta})|^2_n + \frac{1}{16} \sum_{i=1}^{n} v_{ij}^2 \sqrt{4T_j} + \frac{2}{n} \sum_{i=1}^{n} T_{ij}^2 + 2\lambda_{n_j,2}
\]

\[
\leq \frac{3}{4} |\Delta_j(\hat{\theta})|^2_n + 32r_{n_j}^2 + \frac{1}{16} |\Delta_j(\hat{\theta})|^2_n + \frac{1}{16} |\Delta_j(\hat{\theta})|^2_n
\]

\[
+ \frac{1}{8} |\Delta_j(\hat{\theta})|^2_n + \frac{1}{8} |\Delta_j(\hat{\theta})|^2_n + \sqrt{4T_j} + \frac{2}{n} \sum_{i=1}^{n} T_{ij}^2 + 2\lambda_{n_j,2},
\]

where the second inequality follows by the choice of \(\lambda_{n_j,2}\), the triangle inequality, and the Fenchel-Young inequality. This inequality above implies

\[
\frac{1}{16} |\Delta_j(\hat{\theta})|^2_n \leq 15 \frac{1}{16} |\Delta_j(\hat{\theta})|^2_n + 32r_{n_j}^2 + \frac{1}{n} \sum_{i=1}^{n} v_{ij}^2 \sqrt{4T_j} + \frac{2}{n} \sum_{i=1}^{n} T_{ij}^2 + 2\lambda_{n_j,2}.
\]

\[\square\]

**A.15 Lemmas for Theorem 4.7**

**Lemma A.13**: If the assumptions in Corollary 4.4 and Assumption 4.7 hold, then under the conditions \(n \gtrsim (k_2^3 \log p) \lor (k_2^2 k_1 \log d)\), \(\sqrt{n \log p} \lor \sqrt{n k_1 \log d} \sim \log p\), and \(\sqrt{k_1 \log d} = o(1)\), the sample matrix \(\frac{1}{n} \hat{\theta}^T \hat{v}\) satisfies an analogous version of the “mutual incoherence” assumption with high probability, i.e.,

\[
P \left[ \left\| \frac{1}{n} \hat{v}_K^T \hat{v}_K \left( \frac{1}{n} \hat{v}_K^T \hat{v}_K \right)^{-1} \right\|_{\infty} \geq 1 - \frac{\phi}{4} \right] \leq O \left( \frac{1}{p \land d} \right).
\]

**Proof.** We use the following decomposition similar to the method used in Ravikumar, et al. (2010)

\[
\Sigma_{K^c} \Sigma_{K^c}^{-1} - \Sigma_{K^c} \Sigma_{K^c}^{-1} = R_1 + R_2 + R_3 + R_4 + R_5 + R_6,
\]

where

\[
R_1 = \Sigma_{K^c} \Sigma_{K^c}^{-1} \Sigma_{K^c}^{-1} - \Sigma_{K^c} \Sigma_{K^c}^{-1} \Sigma_{K^c}^{-1},
\]

\[
R_2 = \Sigma_{K^c} \Sigma_{K^c}^{-1} \Sigma_{K^c}^{-1} - \Sigma_{K^c} \Sigma_{K^c}^{-1} \Sigma_{K^c}^{-1},
\]

\[
R_3 = \Sigma_{K^c} \Sigma_{K^c}^{-1} \Sigma_{K^c}^{-1} - \Sigma_{K^c} \Sigma_{K^c}^{-1} \Sigma_{K^c}^{-1},
\]

\[
R_4 = \Sigma_{K^c} \Sigma_{K^c}^{-1} \Sigma_{K^c}^{-1} - \Sigma_{K^c} \Sigma_{K^c}^{-1} \Sigma_{K^c}^{-1},
\]

\[
R_5 = \Sigma_{K^c} \Sigma_{K^c}^{-1} \Sigma_{K^c}^{-1} - \Sigma_{K^c} \Sigma_{K^c}^{-1} \Sigma_{K^c}^{-1},
\]

\[
R_6 = \Sigma_{K^c} \Sigma_{K^c}^{-1} \Sigma_{K^c}^{-1} - \Sigma_{K^c} \Sigma_{K^c}^{-1} \Sigma_{K^c}^{-1}.
\]

By Assumption 4.7, we have

\[
\|\Sigma_{K^c} \Sigma_{K^c}^{-1} \|_{\infty} \leq 1 - \phi.
\]

It suffices to show that \(\|R_i\|_{\infty} \leq \frac{\phi}{5}\) for \(i = 1, \ldots, 3\) and \(\|R_i\|_{\infty} \leq \frac{\phi}{12}\) for \(i = 4, \ldots, 6\).
For the first term $R_1$, we have

$$R_1 = -\Sigma K^* K \Sigma_{KK}^{-1} (\hat{\Sigma}_{KK} - \Sigma_{KK}) \hat{\Sigma}_{KK}^{-1},$$

Using the sub-multiplicative property $||AB||_\infty \leq ||A||_\infty ||B||_\infty$ and the elementary inequality $||A||_\infty \leq \sqrt{m}||A||_2$ for any symmetric matrix $A \in \mathbb{R}^{m \times m}$, we can bound $R_1$ as follows:

$$||R_1||_\infty \leq ||\Sigma K^* K \Sigma_{KK}^{-1}||_\infty \left\| \hat{\Sigma}_{KK} - \Sigma_{KK} \right\|_\infty \left\| \hat{\Sigma}_{KK}^{-1} \right\|_\infty \leq (1 - \phi) \left\| \hat{\Sigma}_{KK} - \Sigma_{KK} \right\|_\infty \sqrt{k_2} \left\| \hat{\Sigma}_{KK}^{-1} \right\|_2,$$

where the last inequality follows from Assumption 4.7. Using bound (A.32) from the proof for Lemma A.14, we have

$$\left\| \hat{\Sigma}_{KK}^{-1} \right\|_2 \leq \frac{2}{\lambda_{\min}(\Sigma_{KK})}$$

with probability at least $1 - c_1 \exp(-c_2 n)$. Next, applying bound (A.27) from Lemma A.14 with $\varepsilon = \frac{\phi \lambda_{\min}(\Sigma_{KK})}{12(1 - \phi) \sqrt{k_2}}$, we have

$$\mathbb{P}\left[ \left\| \hat{\Sigma}_{KK} - \Sigma_{KK} \right\|_\infty \geq \frac{\phi \lambda_{\min}(\Sigma_{KK})}{12(1 - \phi) \sqrt{k_2}} \right] \leq 2 \exp(-b \frac{n}{k_2^3} + 2 \log k_2).$$

Then, we are guaranteed that

$$\mathbb{P}\left[ ||R_1||_\infty \geq \frac{\phi}{6} \right] \leq 2 \exp(-b \frac{n}{k_2^3} + 2 \log k_2).$$

For the second term $R_2$, we first write

$$||R_2||_\infty \leq \sqrt{k_2} \left\| \Sigma_{KK}^{-1} \right\|_2 \left\| \hat{\Sigma}_{K^* K} - \Sigma_{K^* K} \right\|_\infty \leq \frac{\sqrt{k_2}}{\lambda_{\min}(\Sigma_{KK})} \left\| \hat{\Sigma}_{K^* K} - \Sigma_{K^* K} \right\|_\infty.$$

An application of bound (A.26) from Lemma A.14 with $\varepsilon = \frac{\phi \lambda_{\min}(\Sigma_{KK})}{\sqrt{k_2}}$ to bound the term $\left\| \hat{\Sigma}_{K^* K} - \Sigma_{K^* K} \right\|_\infty$ yields

$$\mathbb{P}[||R_2||_\infty \geq \frac{\phi}{6}] \leq 2 \exp(-b \frac{n}{k_2^3} + \log(p - k_2) + \log k_2).$$

For the third term $R_3$, by applying bounds (A.26) from Lemma A.14 with $\varepsilon = \frac{\phi \lambda_{\min}(\Sigma_{KK})}{\sqrt{k_2}}$ to bound the term $\left\| \hat{\Sigma}_{K^* K} - \Sigma_{K^* K} \right\|_\infty$ and (A.28) from Lemma A.14 to bound the term $\left\| \hat{\Sigma}_{KK}^{-1} - \Sigma_{KK}^{-1} \right\|_\infty$, we have

$$\mathbb{P}[||R_3||_\infty \geq \frac{\phi}{6}] \leq 2 \exp(-b \frac{n}{k_2^3} + \log(p - k_2) + \log k_2).$$

Putting everything together, we conclude that

$$\mathbb{P}[||\hat{\Sigma}_{K^* K} \hat{\Sigma}_{K^* K}^{-1}||_\infty \geq 1 - \frac{\phi}{2}] \leq c \exp(-b \frac{n}{k_2^3} + 2 \log p).$$
For the fourth term $R_4$, we have, with probability at least $1 - c \exp(-bn \min\{\frac{1}{k_2}, \frac{1}{k_2^2}\} + 2 \log p)$,

$$||R_4||_\infty \leq ||\hat{\Sigma}_{K^c, K} \hat{\Sigma}_{K}^{-1}||_\infty \left(||\hat{\Sigma}_{KK} - \hat{\Sigma}_{K^c, K}||_\infty ||\hat{\Sigma}_{K}^{-1}||_\infty \right) \leq (1 - \frac{\phi}{2}) ||\hat{\Sigma}_{KK} - \hat{\Sigma}_{K^c, K}||_\infty \sqrt{k_2} ||\hat{\Sigma}_{K}^{-1}||_2,$$

where the last inequality follows from the bound on $||\hat{\Sigma}_{K^c, K} \hat{\Sigma}_{K}^{-1}||_\infty$ established previously. Using bounds (A.35) from the proof for Lemma A.15, we have

$$||\hat{\Sigma}_{K}^{-1}||_2 \leq \frac{4}{\lambda_{\min}(\Sigma_{KK})}$$

with probability at least $1 - c_1 \exp(-c_2 \log (p \lor d))$. Next, applying Lemma A.15 with $\varepsilon = \frac{\phi \lambda_{\min}(\Sigma_{KK})}{48(1 - \frac{d}{2})\sqrt{k_2}}$ to bound the term $||\hat{\Sigma}_{KK} - \hat{\Sigma}_{K^c, K}||_\infty$ yields,

$$\mathbb{P}[||R_4||_\infty \geq \frac{\phi}{12}] \leq cK^2 \exp\left(-\frac{c_1 n}{k_2}\right) + c_2 \exp(-c_3 \log d)$$

For the fifth term $R_5$, using bound (A.32) from the proof for Lemma A.14, we have

$$||R_5||_\infty \leq \sqrt{k_2} ||\hat{\Sigma}_{K}^{-1}||_2 ||\hat{\Sigma}_{K^c, K}||_\infty \leq \frac{2\sqrt{k_2}}{\lambda_{\min}(\Sigma_{KK})} ||\hat{\Sigma}_{K^c, K} - \hat{\Sigma}_{K}||_\infty.$$

Applying Lemma A.15 with $\varepsilon = \frac{\phi \lambda_{\min}(\Sigma_{KK})}{24\sqrt{k_2}}$ to bound the term $||\hat{\Sigma}_{K^c, K} - \hat{\Sigma}_{K}||_\infty$ yields

$$\mathbb{P}[||R_5||_\infty \geq \frac{\phi}{12}] \leq c(p - k_2)k_2 \exp\left(-\frac{c_1 n}{k_2}\right) + c_2 \exp(-c_3 \log d)$$

For the sixth term $R_6$, by applying Lemma A.15 to bound the terms $||\hat{\Sigma}_{K^c, K} - \hat{\Sigma}_{K}||_\infty$ and $||\hat{\Sigma}_{K}^{-1} - \hat{\Sigma}_{K^c, K}^{-1}||_\infty$ respectively, with $\varepsilon = \frac{\phi \lambda_{\min}(\Sigma_{KK})}{8}$, we are guaranteed that

$$\mathbb{P}[||R_6||_\infty \geq \frac{\phi}{12}] \leq c(p - k_2)k_2 \exp\left(-\frac{c_1 n}{k_2}\right) + c_2 \exp(-c_3 \log d)$$

Under the conditions $n \gtrsim (k_2^3 \log p) \lor (k_2^2 k_1 \log d) \lor \sqrt{\frac{n \log p}{|\beta|^2}} \lor \sqrt{n k_1 \log d} \gtrsim \log p$, and $\sqrt{\frac{k_1 \log d}{n}} = o(1)$, putting the bounds on $R_1 - R_6$ together, we conclude that

$$\mathbb{P}[||\hat{\Sigma}_{K^c, K} \hat{\Sigma}_{K}^{-1}||_\infty \geq 1 - \frac{\phi}{4}] \leq O\left(\frac{1}{p \lor d}\right).$$

\Box

**Lemma A.14:** Suppose Assumptions 2.1 and 4.2 hold. For any $\varepsilon > 0$ and constant $c$, we have
To prove the last bound (A.28), write
\[ P \left\{ \left\| \hat{\Sigma}_{K^cK} - \Sigma_{K^cK} \right\|_\infty \geq \varepsilon \right\} \leq (p-k_2)k_2 \cdot 2 \exp(-cn \min\{ \frac{\varepsilon^2}{4k_2^2\sigma_v^4}, \frac{\varepsilon}{2k_2^2\sigma_v^2} \}), \] (A.26)
\[ P \left\{ \left\| \hat{\Sigma}_{KK} - \Sigma_{KK} \right\|_\infty \geq \varepsilon \right\} \leq k_2^2 \cdot 2 \exp(-cn \min\{ \frac{\varepsilon^2}{4k_2^2\sigma_v^4}, \frac{\varepsilon}{2k_2^2\sigma_v^2} \}). \] (A.27)

Furthermore, under the scaling \( n \gtrsim k_2 \log p \), for constants \( b_1 \) and \( b_2 \), we have
\[ \left\| \hat{\Sigma}_{KK}^{-1} - \Sigma_{KK}^{-1} \right\|_\infty \leq \frac{1}{\lambda_{\min}(\Sigma_{KK})}, \] (A.28)
with probability at least \( 1 - c_1 \exp(-c_2n \min\{ \frac{\lambda_{\min}(\Sigma_{KK})}{4k_2^2\sigma_v^4}, \frac{\lambda_{\min}(\Sigma_{KK})}{2k_2^2\sigma_v^2} \}) \).

**Proof.** Denote the element \((j', j)\) of the matrix difference \( \hat{\Sigma}_{K^cK} - \Sigma_{K^cK} \) by \( \omega_{j', j} \). By the definition of the \( l_\infty \) matrix norm, we have
\[ P \left\{ \left\| \hat{\Sigma}_{K^cK} - \Sigma_{K^cK} \right\|_\infty \geq \varepsilon \right\} = P \left\{ \max_{j' \in K^c} \sum_{j \in K} |\omega_{j', j}| \geq \varepsilon \right\} \leq (p-k_2)P \left\{ \sum_{j \in K} |\omega_{j', j}| \geq \varepsilon \right\} \leq (p-k_2)P \left\{ \exists j \in K \mid |\omega_{j', j}| \geq \frac{\varepsilon}{k_2} \right\} \leq (p-k_2)k_2P \left\{ |\omega_{j', j}| \geq \frac{\varepsilon}{k_2} \right\} \leq (p-k_2)k_2 \cdot 2 \exp(-cn \min\{ \frac{\varepsilon^2}{k_2^2\sigma_v^4}, \frac{\varepsilon}{k_2^2\sigma_v^2} \}), \]
where the last inequality follows the deviation bound for sub-exponential random variables, i.e., Lemma B.1. Bound (A.27) can be obtained in a similar way except that the pre-factor \( (p-k_2) \) is replaced by \( k_2 \).

To prove the last bound (A.28), write
\[ \left\| \hat{\Sigma}_{KK}^{-1} - \Sigma_{KK}^{-1} \right\|_\infty = \left\| \Sigma_{KK}^{-1} \left[ \Sigma_{KK} - \hat{\Sigma}_{KK} \right] \hat{\Sigma}_{KK}^{-1} \right\|_\infty = \sqrt{k_2} \left\| \Sigma_{KK}^{-1} \left[ \Sigma_{KK} - \hat{\Sigma}_{KK} \right] \hat{\Sigma}_{KK}^{-1} \right\|_2 \leq \sqrt{k_2} \left\| \Sigma_{KK}^{-1} \right\|_2 \left\| \Sigma_{KK} - \hat{\Sigma}_{KK} \right\|_2 \left\| \hat{\Sigma}_{KK}^{-1} \right\|_2 \leq \frac{\sqrt{k_2}}{\lambda_{\min}(\Sigma_{KK})} \left\| \Sigma_{KK} - \hat{\Sigma}_{KK} \right\|_2 \left\| \hat{\Sigma}_{KK}^{-1} \right\|_2. \] (A.29)

To bound the term \( \left\| \Sigma_{KK} - \hat{\Sigma}_{KK} \right\|_2 \) in (A.29), applying Lemma B.1 with \( v^T v = \hat{\Sigma}_{KK} \) and \( t = \frac{\lambda_{\min}(\Sigma_{KK})}{2\sqrt{k_2}} \) yields
\[ \left\| \hat{\Sigma}_{KK} - \Sigma_{KK} \right\|_2 \leq \frac{\lambda_{\min}(\Sigma_{KK})}{2\sqrt{k_2}}, \]
with probability at least \( 1 - c_1 \exp(-c_2n \min\{ \frac{\lambda_{\min}(\Sigma_{KK})}{4k_2^2\sigma_v^4}, \frac{\lambda_{\min}(\Sigma_{KK})}{2k_2^2\sigma_v^2} \}) \).
To bound the term \( \|\hat{\Sigma}^{-1}_{KK}\|_2 \) in [A.29], note that we can write

\[
\lambda_{\min}(\Sigma_{KK}) = \min_{\|h'\|_2=1} h'^T \Sigma_{KK} h' \\
= \min_{\|h'\|_2=1} \left[ h'^T \hat{\Sigma}_{KK} h' + h'^T (\Sigma_{KK} - \hat{\Sigma}_{KK}) h' \right] \\
\leq h'^T \hat{\Sigma}_{KK} h + h'^T (\Sigma_{KK} - \hat{\Sigma}_{KK}) h
\]

(A.30)

where \( h \in \mathbb{R}^{k_2} \) is a unit-norm minimal eigenvector of \( \hat{\Sigma}_{KK} \). Applying Lemma B.1 yields

\[
\left| h'^T (\Sigma_{KK} - \hat{\Sigma}_{KK}) h \right| \leq \frac{\lambda_{\min}(\Sigma_{KK})}{2}
\]

with probability at least \( 1 - c_1 \exp(-c_2 n) \). Therefore,

\[
\lambda_{\min}(\Sigma_{KK}) \leq \lambda_{\min}(\hat{\Sigma}_{KK}) + \frac{\lambda_{\min}(\Sigma_{KK})}{2}
\]

\( \implies \lambda_{\min}(\hat{\Sigma}_{KK}) \geq \frac{\lambda_{\min}(\Sigma_{KK})}{2} \),

(A.31)

and consequently,

\[
\|\hat{\Sigma}^{-1}_{KK}\| \leq \frac{2}{\lambda_{\min}(\Sigma_{KK})}.
\]

(A.32)

Putting everything together, we have

\[
\|\hat{\Sigma}^{-1}_{KK} - \Sigma^{-1}_{KK}\|_\infty \leq \frac{\sqrt{2}}{\lambda_{\min}(\Sigma_{KK})} \cdot \frac{\lambda_{\min}(\Sigma_{KK})}{2} \cdot \frac{2}{\lambda_{\min}(\Sigma_{KK})} = \frac{1}{\lambda_{\min}(\Sigma_{KK})}.
\]

with probability at least \( 1 - c_1 \exp\left(-c_2 n \min\left\{ \frac{\lambda_{\min}(\Sigma_{KK})}{4\sigma^2_{\epsilon}}, \frac{\lambda_{\min}(\Sigma_{KK})}{2\sigma^2_{\epsilon}} \right\} \right) \). \( \square \)

**Lemma A.15:** Suppose the assumptions in Lemma A.13 hold. For any \( \varepsilon > 0 \), under the condition \( n \gtrsim (k_2^3 \log p) \lor (k_2^2 k_1 \log d) \lor \sqrt{\frac{n \log p}{k_2}} \lor \sqrt{n k_1 \log d} \gtrsim \log p \), and \( \sqrt{\frac{k_1 \log d}{n}} = o(1) \), we have

\[
\mathbb{P}\left\{ \left\| \hat{\Sigma}^{\epsilon,K} - \Sigma^{\epsilon,K} \right\|_\infty \geq \varepsilon \right\} \leq c(p - k_2) k_2 \cdot \exp\left( -\frac{n \varepsilon}{c_1 k_2} \right) + c_2 \exp(-c_3 \log d)
\]

\[
\mathbb{P}\left\{ \left\| \hat{\Sigma}^{K,K} - \Sigma^{K,K} \right\|_\infty \geq \varepsilon \right\} \leq c' k_2^2 \cdot \exp\left( -\frac{n \varepsilon}{c_1 k_2} \right) + c_2 \exp(-c_3 \log d)
\]

Furthermore, we have

\[
\|\hat{\Sigma}^{-1}_{KK} - \Sigma^{-1}_{KK}\|_\infty \leq \frac{8}{\lambda_{\min}(\Sigma_{KK})} \quad \text{with probability at least } 1 - c_1 \exp(-c_2 \log(p \lor d)).
\]

(A.33)

**Proof.** Denote the element \((j', j)\) of the matrix difference \( \hat{\Sigma}^{\epsilon,K} - \Sigma^{\epsilon,K} \) by \( \omega_{j',j} \). Using the same argument as in the proof for Lemma A.14, we have

\[
\mathbb{P}\left\{ \left\| \hat{\Sigma}^{\epsilon,K} - \Sigma^{\epsilon,K} \right\|_\infty \geq \varepsilon \right\} \leq (p - k_2) k_2 \mathbb{P}\left\{ |\omega_{j',j}| \geq \frac{\varepsilon}{k_2} \right\}.
\]
Following the derivation of the upper bounds on $\frac{(\tilde{x} - \tilde{\mu})^T \tilde{\Sigma} \tilde{x}}{n}$ and $\frac{(\tilde{x} - \tilde{\mu})^T (\tilde{\Sigma} - \tilde{\mu})}{n}$ in the proofs for Lemma A.3 and Corollary 4.4 and the identity
\[
\frac{1}{n} \left( \Sigma_{K^eK} - \hat{\Sigma}_{K^eK} \right) = \frac{1}{n} v^T_K (\hat{\nu}_K - v_K) + \frac{1}{n} (\hat{v}_K - v_K^e)^T v_K + \frac{1}{n} (\hat{v}_K - v_K^e)^T (\hat{\nu}_K - v_K),
\]
we notice that to upper bound $|\omega_j^T|$, it suffices to upper bound $3 \cdot \left| \frac{1}{n} v^T_j (\hat{\nu}_j - v_j) \right|$. From the proofs for Lemma A.3 and Corollary 4.4, we have
\[
\left| \frac{1}{n} v^T_j (\hat{\nu}_j - v_j) \right| \leq b \left( u^2 + \sqrt{\frac{k_1 \log d}{n}} \right),
\]
with probability at least $1 - c_1 \exp \left( -nC^* u^2 \right) - O \left( \frac{1}{d} \right)$. If $n \geq k_2^2 k_1 \log d$, setting $u = \sqrt{\frac{n}{k_2}}$ for any $\varepsilon > 0$ yields
\[
P \left[ |\omega_j^T| \geq \frac{\varepsilon}{k_2} \right] \leq c \exp \left( -\frac{n \varepsilon}{c_1 k_2} \right) + c_2 \exp(-c_3 \log d).
\]
Therefore,
\[
P \left\{ \left\| \hat{\Sigma}_{K^eK} - \hat{\Sigma}_{K^eK} \right\| \geq \varepsilon \right\} \leq c(p - k_2) k_2 \cdot \exp \left( -\frac{n \varepsilon}{c_1 k_2} \right) + c_2 \exp(-c_3 \log d).
\]
The second bound in Lemma A.15 can be obtained in a similar way except that the pre-factor $(p - k_2)$ is replaced by $k_2$.

To prove the third bound in Lemma A.15, by applying the same argument as in the proof for Lemma A.14, we have
\[
\left\| \hat{\Sigma}_{K^eK} - \hat{\Sigma}_{K^eK} \right\|_\infty \leq \frac{\sqrt{k_2}}{\lambda_{\min}(\hat{\Sigma}_{K^eK})} \left\| \hat{\Sigma}_{K^eK} - \hat{\Sigma}_{K^eK} \right\|_2 \left\| \hat{\Sigma}_{K^eK} \right\|_2 \leq \frac{2\sqrt{k_2}}{\lambda_{\min}(\hat{\Sigma}_{K^eK})} \left\| \hat{\Sigma}_{K^eK} - \hat{\Sigma}_{K^eK} \right\|_2 \left\| \hat{\Sigma}_{K^eK} \right\|_2,
\]
where the last inequality comes from bound (A.31).

For the term $\left\| \hat{\Sigma}_{K^eK} - \hat{\Sigma}_{K^eK} \right\|_2$, setting $\varepsilon = \frac{\lambda_{\min}(\hat{\Sigma}_{K^eK})}{\sqrt{k_2}}$ in the previous upper bound on $\left\| \hat{\Sigma}_{K^eK} - \hat{\Sigma}_{K^eK} \right\|_\infty$ yields
\[
\left\| \hat{\Sigma}_{K^eK} - \hat{\Sigma}_{K^eK} \right\|_2 \leq \left\| \hat{\Sigma}_{K^eK} - \hat{\Sigma}_{K^eK} \right\|_\infty \leq \frac{\lambda_{\min}(\hat{\Sigma}_{K^eK})}{\sqrt{k_2}}, \tag{A.34}
\]
with probability at least $1 - c k_2^2 \cdot \exp \left( -\frac{n \varepsilon}{c_1 k_2} \right) - c_2 \exp(-c_3 \log d)$.

To bound the term $\left\| \hat{\Sigma}_{K^eK} \right\|_2$, again we have,
\[
\lambda_{\min}(\hat{\Sigma}_{K^eK}) \leq h^T \hat{\Sigma}_{K^eK} h + h^T (\hat{\Sigma}_{K^eK} - \hat{\Sigma}_{K^eK}) h \leq h^T \hat{\Sigma}_{K^eK} h + k_2 \left\| \hat{\Sigma}_{K^eK} - \hat{\Sigma}_{K^eK} \right\|_\infty \leq h^T \hat{\Sigma}_{K^eK} h + b' \left\{ \left( \frac{1}{n} \right)^{\frac{3}{2}} + \sqrt{\frac{k_1 \log d}{n}} \right\},
\]
where \( h \in \mathbb{R}^{k_2} \) is a unit-norm minimal eigenvector of \( \tilde{\Sigma}_{KK} \). The last inequality follows from the bounds on 
\[
\left| \frac{(\hat{v}-v)^T v}{n} \right|_{\infty} \quad \text{and} \quad \left| \frac{(\hat{v}-v)^T (\hat{v}-v)}{n} \right|_{\infty}
\]
from the proofs for Lemma A.3 and Corollary 4.4. Therefore, if \( \sqrt{\frac{k_1 \log d}{n}} = o(1) \), then we have
\[
\lambda_{\min}(\tilde{\Sigma}_{KK}) \geq \frac{\lambda_{\min}(\hat{\Sigma}_{KK})}{2}
\]
\[
\Rightarrow \left\| \tilde{\Sigma}_{KK}^{-1} \right\|_2 \leq \frac{2}{\lambda_{\min}(\tilde{\Sigma}_{KK})} \leq \frac{4}{\lambda_{\min}(\tilde{\Sigma}_{KK})}
\]
where the last inequality follows from bound (A.31) in the proof for Lemma A.14. Putting everything together, we have
\[
\left\| \tilde{\Sigma}_{KK}^{-1} - \hat{\Sigma}_{KK}^{-1} \right\|_{\infty} \leq \frac{2\sqrt{k_2}}{\lambda_{\min}(\tilde{\Sigma}_{KK})} \frac{\lambda_{\min}(\Sigma_{KK})}{\sqrt{k_2}} \frac{4}{\lambda_{\min}(\Sigma_{KK})} = \frac{8}{\lambda_{\min}(\Sigma_{KK})},
\]
with probability at least \( 1 - O \left( \frac{1}{d \wedge p} \right) \).

**Lemma A.16:** Suppose the conditions in Corollary 4.4 hold. With the choice of the tuning parameter
\[
\lambda_{n,3} \geq c \frac{8(2 - \frac{\phi}{4})}{\phi} \left[ \left( \sigma_v \sigma_\eta \sqrt{\frac{\log p}{n}} \right) \vee \left( L |\beta^*_1| \sqrt{Y_{w,\beta^*} (\sigma_v, \sigma_\eta)} \sqrt{\frac{k_1 \log d}{n}} \right) \right]
\]
for some sufficiently large constant \( c > 0 \), under the conditions \( n \gtrsim (k_2 \log p) \vee (k_2^2 k_1 \log d) \) and \( \sqrt{n \log \frac{p}{|\beta^*_1|}} \vee \sqrt{n k_1 \log d} \gtrsim \log p \), then, we have
\[
\left( 2 - \frac{\phi}{4} \right) \left| \frac{1}{n \lambda_{n,3}} \tilde{v}^T e \right|_{\infty} \leq \frac{\phi}{8},
\]
with probability at least \( 1 - c_1 \exp(-c_2 \log (d \wedge p)) \).

**Proof.** Recall from the proofs for Lemma A.4 and Corollary 4.4 on \( \left| \frac{1}{n \lambda_{n,3}} \tilde{v}^T e \right|_{\infty} \), so as long as
\[
\lambda_{n,3} \geq c \frac{8(2 - \frac{\phi}{4})}{\phi} \left[ \left( \sigma_v \sigma_\eta \sqrt{\frac{\log p}{n}} \right) \vee \left( L |\beta^*_1| \sqrt{Y_{w,\beta^*} (\sigma_v, \sigma_\eta)} \sqrt{\frac{k_1 \log d}{n}} \right) \right]
\]
for some sufficiently large constant \( c > 0 \), under the conditions \( n \gtrsim (k_2 \log p) \vee (k_2^2 k_1 \log d) \) and \( \sqrt{n \log \frac{p}{|\beta^*_1|}} \vee \sqrt{n k_1 \log d} \gtrsim \log p \), we have
\[
\left( 2 - \frac{\phi}{4} \right) \left| \frac{1}{n \lambda_{n,3}} \tilde{v}^T e \right|_{\infty} \leq \frac{\phi}{8},
\]
with probability at least \( 1 - c_1 \exp(-c_2 \log (d \wedge p)) \). □
B Appendix II: Technical lemmas and the proofs

**Lemma B.1:** Let $X \in \mathbb{R}^{n \times p_1}$ be a zero-mean sub-Gaussian matrix with parameters $(\Sigma_X, \sigma_X^2)$. For any fixed (unit) vector $\Delta \in \mathbb{R}^{p_1}$, we have

$$
P \left[ \frac{|X\Delta|^2}{n} - E\left[ \frac{|X\Delta|^2}{n} \right] \geq \varepsilon \right] \leq 2 \exp \left( -c_0 n \min \left\{ \frac{\varepsilon^2}{\sigma_X^4}, \frac{\varepsilon}{\sigma_X^2} \right\} \right).
$$

Moreover, if $Y \in \mathbb{R}^{n \times p_2}$ is a zero-mean sub-Gaussian matrix with parameters $(\Sigma_Y, \sigma_Y^2)$, then

$$
P \left[ \frac{Y^T X}{n} - \text{cov}(Y_i, X_i) \right] \leq 6 \exp \left( -c_1 n \min \left\{ \frac{\varepsilon^2}{\sigma_X \sigma_Y}, \frac{\varepsilon}{\sigma_X \sigma_Y} \right\} + \log p_1 + \log p_2 \right),
$$

where $X_i$ and $Y_i$ are the $i^{th}$ rows of $X$ and $Y$, respectively.

**Remark.** This lemma is Lemma 14 in Loh and Wainwright (2012). □

**Lemma B.2:** Let $X \in \mathbb{R}^{n \times p_1}$ be a sub-Gaussian matrix with parameters $(\Sigma_X, \sigma_X^2)$. We have

$$
\frac{|X\Delta|^2}{n} \geq \frac{\alpha}{2} |\Delta|^2 - \alpha' \frac{\log p_1}{n} |\Delta|^2, \quad \text{for all } \Delta \in \mathbb{R}^{p_1}
$$

and

$$
\frac{|X\Delta|^2}{n} \leq \frac{3\bar{\alpha}}{2} |\Delta|^2 + \alpha' \frac{\log p_1}{n} |\Delta|^2, \quad \text{for all } \Delta \in \mathbb{R}^{p_1}
$$

with probability at least $1 - c_1 \exp(-c_2 n)$, where $\underline{\alpha}$, $\bar{\alpha}$, and $\alpha'$ only depend on $\Sigma_X$ and $\sigma_X$.

**Remark.** This lemma is Lemma 13 in Loh and Wainwright (2012). The choice of $\underline{\alpha}$, $\bar{\alpha}$, and $\alpha'$ depends on the problems. □

**Lemma B.3:** For all $u \geq r_{nj}$, we have

$$
P \left[ A_j(u) \mid \{ w_i^T \theta^* \}_{i=1}^n \right] \leq \exp \left( -n C_j^* u^2 \right)
$$

where

$$
A_j(u) := \begin{cases} 
\exists f \in \mathcal{F}_j \cap \{|f|_{n+} \geq u \} : \frac{1}{n} \sum_{i=1}^n v_{ij} f(w_i^T \theta^*) \geq 2 |f|_{n+} u 
\end{cases},
$$

and

$$
C_j^* = \frac{\gamma v_j \wedge (B_{v_j}^2 \lor B_n^2)^{-1}}{\sigma_{v_j}^2 \lor \sigma_n^2} \quad \text{(B.1)}
$$

when $v_j$ is either (i) sub-Gaussian with *strongly log-concave* distribution for some $\gamma v_j > 0$; or, (ii) a bounded vector such that for every $i = 1, ..., n$, $v_{ij}$ is supported on the interval $(a'_{v_j}, a''_{v_j})$ with $B_{v_j} := a''_{v_j} - a'_{v_j}$; or, (iii) a mixture of (i) and (ii) in terms of its probability measure.

**Proof.** This proof is a modification of the proofs for Corollary 8.3 in van de Geer (2000) and Lemma 13.2
in Wainwright (2015). Suppose that there exists some \( f \in \tilde{F}_j \) with \( |f_{\theta^*}|_n \geq u \) such that

\[
\frac{1}{n} \left| \sum_{i=1}^{n} v_{ij} f(w_i^T \theta^*) \right| \geq 2 |f_{\theta^*}|_n u. \tag{B.2}
\]

Defining the function \( \tilde{f}_{\theta^*} := \frac{1}{|f_{\theta^*}|_n} f_{\theta^*} \), observe that \( |\tilde{f}_{\theta^*}|_n = u \). Since \( f \in \tilde{F}_j \) and \( u \leq |f_{\theta^*}|_n \) by construction, the star-shaped assumption implies that \( \tilde{f} \in \tilde{F}_j \). Consequently, we have shown that if there exists a function \( \tilde{f} \) satisfying inequality (B.2), which occurs whenever the event \( A_j(u) \) is true, then there exists a function \( \tilde{f} \in \tilde{F}_j \) with \( |\tilde{f}_{\theta^*}|_n = u \) such that

\[
\frac{1}{n} \left| \sum_{i=1}^{n} v_{ij} \tilde{f}(w_i^T \theta^*) \right| = \frac{u}{|f_{\theta^*}|_n} \frac{1}{n} \left| \sum_{i=1}^{n} v_{ij} f(w_i^T \theta^*) \right| \geq 2u^2.
\]

Summarizing then, we have established the inequality

\[
P \left[ A_j(u) \mid \{ w_i^T \theta^* \}_{i=1}^{n} \right] \leq P \left[ A_{nj}(u) \geq 2u^2 \mid \{ w_i^T \theta^* \}_{i=1}^{n} \right],
\]

where

\[
A_{nj}(u) := \sup_{f(\theta^*) \in \Omega(u; F_j)} \frac{1}{n} \left| \sum_{i=1}^{n} v_{ij} \tilde{f}(w_i^T \theta^*) \right|
\]

where

\[
\Omega(u; F_j) = \{ f : f \in \tilde{F}_j, |f_{\theta^*}|_n \leq u \}.
\]

By construction, \( E(v_{ij}|w_i^T \theta^*) = 0 \) for all \( i = 1, \ldots, n \). Consequently, conditioning on \( \{ w_i^T \theta^* \}_{i=1}^{n} \), for each fixed \( \tilde{f} \), the variable \( \frac{1}{n} \sum_{i=1}^{n} v_{ij} \tilde{f}(w_i^T \theta^*) \) is zero-mean sub-Gaussian, so that the variable \( A_{nj}(u) \) is the supremum of a sub-Gaussian process. If we view this supremum as a function of a standardized sub-Gaussian vector, then by standard maximal inequality (e.g., Corollary 8.3 in van de Geer, 2000), we have

\[
P \left[ A_{nj}(u) \geq E_v \left[ A_{nj}(u) \mid \{ w_i^T \theta^* \}_{i=1}^{n} \right] + b \right] \leq \exp \left( -\frac{nC_j^s u^2}{2u^2} \right).
\]

Consequently, for any \( b = u^2 \),

\[
P \left[ A_{nj}(u) \geq E_v \left[ A_{nj}(u) \mid \{ w_i^T \theta^* \}_{i=1}^{n} \right] + u^2 \right] \leq \exp \left( -nC_j^s u^2 \right). \tag{B.3}
\]

Now let us look at special cases that yield sharp constant \( C_j^s \).

**Special case (i):** If \( v_j \) is sub-Gaussian with strongly log-concave distribution for some \( \gamma_{v_j} > 0 \), then by Cauchy-Schwarz inequality, it can be verified that the associated Lipschitz constant is at most \( \frac{\sigma_{v_j} u}{\sqrt{n}} \). Consequently, by Lemma B.6, for any \( b > 0 \), we have the sub-Gaussian tail bound

\[
P \left[ A_{nj}(u) \geq E_v \left[ A_{nj}(u) \mid \{ w_i^T \theta^* \}_{i=1}^{n} \right] + b \right] \leq \exp \left( -\frac{n\gamma_{v_j} b^2}{2u^2 \sigma_{v_j}^2} \right).
\]
Setting $b = u^2$ yields
\[
\mathbb{P} \left[ A_{nj}(u) \geq \mathbb{E}_{v_j} \left[ A_{nj}(u) \mid \left\{ w_i^T \theta^* \right\}_{i=1}^n \right] + u^2 \mid \left\{ w_i^T \theta^* \right\}_{i=1}^n \right] \leq \exp \left( -\frac{n \gamma_{v_j} u^2}{2 \sigma_{v_j}^2} \right).
\]

**Special case (ii):** If $v_j$ is a bounded vector such that for every $i = 1, ..., n$, $v_{ij}$ is supported on the interval $(a_{v_j}', a_{v_j}'')$ with $B_{v_j} := a_{v_j}'' - a_{v_j}'$, then by Lemma B.7, following similar arguments as above, we have
\[
\mathbb{P} \left[ A_{nj}(u) \geq \mathbb{E}_{v_j} \left[ A_{nj}(u) \mid \left\{ w_i^T \theta^* \right\}_{i=1}^n \right] + u^2 \mid \left\{ w_i^T \theta^* \right\}_{i=1}^n \right] \leq \exp \left( -\frac{nu^2}{2 \sigma_{v_j}^2 B_{v_j}^2} \right).
\]

Finally, by definition of $A_{nj}(u)$ and $G_n(u; F_j)$, we have $\mathbb{E}_{v_j} \left[ A_{nj}(u) \mid \left\{ w_i^T \theta^* \right\}_{i=1}^n \right] = \sigma_{v_j} G_n(u; F_j)$. By Lemma B.8, the function $t \mapsto \frac{G_n(t; F_j)}{\sigma_{v_j}}$ is non-decreasing, and since $u \geq r_{nj}$ by assumption, we have
\[
\sigma_{v_j} G_n(u; F_j) \leq \sigma_{v_j} G_n(r_{nj}; F_j) \leq r_{nj},
\]
where the last inequality uses the definition of $r_{nj}$. Putting everything together, we have shown that $\mathbb{E}_{v_j} \left[ A_{nj}(u) \mid \left\{ w_i^T \theta^* \right\}_{i=1}^n \right] \leq ur_{nj}$. Combined with the tail bound (B.3), respectively, we obtain
\[
\mathbb{P} \left[ A_{nj}(u) \geq 2u^2 \mid \left\{ w_i^T \theta^* \right\}_{i=1}^n \right] \leq \mathbb{P} \left[ A_{nj}(u) \geq ur_{nj} + u^2 \mid \left\{ w_i^T \theta^* \right\}_{i=1}^n \right] \leq \exp \left( -n C_j^* u^2 \right)
\]
where the inequality uses the fact that $u^2 \geq ur_{nj}$. □

**Lemma B.4:** There are universal positive constants $(c_1, c_2)$ such that for all $\Delta_j \in \left\{ f \in \tilde{F}_j : |f|_{\tilde{F}_j} \leq 3, |f_{\theta^*}|_n \geq u \right\},$
\[
\frac{1}{n} \left| \sum_{i=1}^n v_{ij} \Delta_j (w_i^T \theta^*) \right| \leq 6r_{nj} |\Delta_j(\theta^*)|_n + \frac{1}{32} |\Delta_j(\theta^*)|_n^2
\]
with probability at least $1 - c_1 \exp \left( -n C_j^* r_{nj}^2 \right)$ and $C_j^*$ follows (B.1) under the same special cases in Lemma B.3.

**Proof.** To establish bound (B.4), we first consider over the ball
\[
\left\{ \Delta_j \in \tilde{F}_j : |\Delta_j|_{\tilde{F}_j} \leq 3 \right\} \cap \left\{ \Delta_j \in \tilde{F}_j : |\Delta_j(\theta^*)|_n \leq u \right\}
\]
for some fixed radius $u \geq r_{nj}$. Later in the proof we extend the bound to one that is uniform in $|\Delta_j(\theta^*)|_n$ via a peeling argument. Define the random variable
\[
A_{nj}'(u) := \sup_{\Delta_j \in \Omega'(u; F_j)} \frac{1}{n} \left| \sum_{i=1}^n v_{ij} \Delta_j (w_i^T \theta^*) \right|
\]
where $\Omega'(u; F_j) = \left\{ \Delta_j \in \tilde{F}_j : |\Delta_j|_{\tilde{F}_j} \leq 3, |\Delta_j(\theta^*)|_n \leq u \right\}$. Following the same argument in Lemma B.3, we obtain
\[
\mathbb{P} \left[ A_{nj}'(u) \geq \mathbb{E}_{v_j} \left[ A_{nj}'(u) \mid \left\{ w_i^T \theta^* \right\}_{i=1}^n \right] + b \mid \left\{ w_i^T \theta^* \right\}_{i=1}^n \right] \leq \exp \left( -\frac{n C_j^* u^2}{u^2} \right) \quad \text{(B.5)}
\]

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We first derive a bound for \( u = r_{nj} \). By definition of \( A'_{nj}(u) \) and \( \mathcal{G}_n (r_{nj}; \mathcal{F}_j) \) and the critical radius \( r_{nj} \), we have
\[
\mathbb{E}_{v_{nj}} \left[ A'_{nj}(r_{nj}) \mid \{w_i^T \theta^*\}_{i=1}^n \right] = \tilde{\sigma}_{v_{nj}} \mathcal{G}_n (r_{nj}; \mathcal{F}_j) \leq 3r_{nj}^2.
\]
Setting \( b = 3r_{nj}^2 \) in the tail bound (B.5) yields
\[
\mathbb{P} \left[ A'_{nj}(r_{nj}) \geq 6r_{nj}^2 \mid \{w_i^T \theta^*\}_{i=1}^n \right] \leq \exp \left( -nC'_j r_{nj}^2 \right).
\]
On the other hand, for any \( u > r_{nj} \), we have
\[
\mathbb{E}_{v_{nj}} \left[ A'_{nj}(u) \mid \{w_i^T \theta^*\}_{i=1}^n \right] = \tilde{\sigma}_{v_{nj}} \mathcal{G}_n (u; \mathbb{B}_{\mathcal{F}_j}(3)) = \frac{\tilde{\sigma}_{v_{nj}} \mathcal{G}_n (u; \mathbb{B}_{\mathcal{F}_j}(3))}{u} \leq \frac{\tilde{\sigma}_{v_{nj}} \mathcal{G}_n (r_{nj}; \mathbb{B}_{\mathcal{F}_j}(3))}{r_{nj}} \leq 3ur_{nj},
\]
where the first inequality follows from Lemma B.8 and the second inequality follows from the critical radius \( r_{nj} \). Setting \( b = \frac{u^2}{128} \) in the tail bound (B.5) yields
\[
\mathbb{P} \left[ A'_{nj}(u) \geq 3ur_{nj} + \frac{u^2}{128} \mid \{w_i^T \theta^*\}_{i=1}^n \right] \leq \exp \left( -nC'_j u^2 \right). \tag{B.6}
\]
It remains to prove the bound (B.4) via a peeling argument. Let \( \mathcal{E} \) denote the event that the bound (B.4) is violated for some function \( \Delta_j \) with \( \|\Delta_j\|_{\mathcal{F}_j} \leq 3 \). For real numbers \( 0 \leq a_1 < a_2 \), let \( \mathcal{E}(a_1, a_2) \) denote the event that it is violated for some function such that \( |\Delta_j(\theta^*)|_n \in [a_1, a_2] \), and \( |\Delta_j|_{\mathcal{F}_j} \leq 3 \). For \( m = 0, 1, 2, \ldots \), define \( t_m = 2^mr_{nj} \). We then have the decomposition \( \mathcal{E} = \mathcal{E}(0, t_0) \cup (\cup_{m=0}^\infty \mathcal{E}(t_m, t_{m+1})) \) and hence by union bound,
\[
\mathbb{P} [\mathcal{E}] \leq \mathbb{P} [\mathcal{E}(0, t_0)] + \sum_{m=0}^\infty \mathbb{P} [\mathcal{E}(t_m, t_{m+1})]. \tag{B.7}
\]
The final step is to bound each of the terms in this summation. Since \( t_0 = r_{nj} \), we have
\[
\mathbb{P} [\mathcal{E}(0, t_0)] \leq \mathbb{P} \left[ A'_{nj}(r_{nj}) \geq 6r_{nj}^2 \mid \{w_i^T \theta^*\}_{i=1}^n \right] \leq \exp \left( -nC'_j r_{nj}^2 \right). \tag{B.8}
\]
On the other hand, suppose that \( \mathcal{E}(t_m, t_{m+1}) \) holds, meaning that there exists some function \( \Delta_j \) with \( |\Delta_j|_{\mathcal{F}_j} \leq 3 \) and \( |\Delta_j(\theta^*)|_n \leq t_{m+1} \) such that
\[
\frac{1}{n} \left| \sum_{i=1}^{n} v_{ij} \Delta_j (w_i^T \theta^*) \right| > 6r_{nj} |\Delta_j(\theta^*)|_n + \frac{1}{32} |\Delta_j(\theta^*)|_n^2
\]
\[
> 6r_{nj} t_m + \frac{1}{32} t_m^2
\]
\[
= 3r_{nj} t_{m+1} + \frac{1}{128} t_{m+1}^2,
\]
where the second inequality follows since \( |\Delta_j(\theta^*)|_n \geq t_{m} \); and the third inequality follows since \( t_{m+1} = 2t_m \). This lower bound implies that
\[
A'_{nj}(t_{m+1}) \geq 3r_{nj} t_{m+1} + \frac{1}{128} t_{m+1}^2,
\]
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hence the bound (B.6) implies that

\[ \mathbb{P} [ \mathcal{E}(t_m, t_{m+1}) ] \leq \exp \left( - \frac{nt^2_{m+1}}{c_2 \bar{\sigma}_{v_j}} \right) = \exp \left( - \frac{n2^{2(m+1)r^2_{n_j}}}{c_2 \bar{\sigma}_{v_j}} \right). \]

Combining this tail bound with our earlier bound (B.8), and substituting into the union bound (B.7) yields

\[ \mathbb{P} [ \mathcal{E} ] \leq \exp \left( - \frac{n\gamma_{r_{n_j}}}{c_2 \bar{\sigma}_{v_j}} \right) + \sum_{m=0}^{\infty} \exp \left( - \frac{n2^{2(m+1)r^2_{n_j}}}{c_2 \bar{\sigma}_{v_j}} \right) \leq c_1 \exp \left( - \frac{n\gamma_{r_{n_j}}}{c_2 \bar{\sigma}_{v_j}^2} \right). \]

**Lemma B.5:** There are universal positive constants \((c_1, c_2)\) such that for all \(\Delta_j \in \{ f \in \mathcal{F}_j : |f|_{\mathcal{F}_j} \geq 1 \},\)

\[ \frac{1}{n} \left| \sum_{i=1}^{n} v_{ij} \Delta_{ij}(\theta^*) \right| \leq 2r_{n_j} |\Delta_j(\theta^*)|_n + 2r^2_{n_j} |\Delta_j|_{\mathcal{F}_j} + \frac{1}{16} |\Delta_j(\theta^*)|_n^2 \]

with probability at least \(1 - c_1 \exp \left( -nC^*_j \gamma_{r_{n_j}} \right)\) and \(C^*_j\) follows (B.1) under the same special cases in Lemma B.3.

**Proof.** This proof is a modification of the proofs for Corollary 8.3 in van de Geer (2000) and Lemma 13.2 in Wainwright (2015). It suffices to show that

\[ \frac{1}{n} \left| \sum_{i=1}^{n} v_{ij} f(w_i^T \theta^*) \right| \leq 2r_{n_j} |f_{\theta^*}|_n + 2r^2_{n_j} + \frac{1}{32} |f_{\theta^*}|_n^2, \quad \text{for all } |f|_{\mathcal{F}_j} = 1. \quad (B.9) \]

To see this, noting that for any function \(\Delta\) with \(|\Delta|_{\mathcal{F}_j} \geq \frac{1}{2}\), we can define \(f = \frac{\Delta(\theta^*)}{|\Delta(\theta^*)|_{\mathcal{F}_j}}\). Substituting this definition and then multiplying both sides of inequality (B.9) by \(|\Delta(\theta^*)|_{\mathcal{F}_j}\), we obtain

\[ \frac{1}{n} \left| \sum_{i=1}^{n} v_{ij} \Delta_{ij}(\theta^*) \right| \leq 2r_{n_j} |\Delta_j(\theta^*)|_{\mathcal{F}_j} + 2r^2_{n_j} |\Delta_j|_{\mathcal{F}_j} + \frac{1}{32} |\Delta_j(\theta^*)|_{\mathcal{F}_j}^2 \leq 2r_{n_j} |\Delta_j(\theta^*)|_{\mathcal{F}_j} + 2r^2_{n_j} |\Delta_j(\theta^*)|_{\mathcal{F}_j} + \frac{1}{16} |\Delta_j(\theta^*)|_{\mathcal{F}_j}^2, \]

where the second inequality follows by the fact that \(|\Delta|_{\mathcal{F}_j} \geq \frac{1}{2}\).

To establish the bound (B.9), we first consider it over the ball \(|f_{\theta^*}|_n \leq u\) for some fixed radius \(u \geq r_{n_j}\). Later in the proof we extend bound (B.9) to one that is uniform in \(|f_{\theta^*}|_n\) via a peeling argument. Define the random variable

\[ A''_{n_j}(u) := \sup_{f(\theta) \in \Omega''(u; \mathcal{F}_j)} \frac{1}{n} \left| \sum_{i=1}^{n} v_{ij} f(w_i^T \theta^*) \right|, \]

where \(\Omega''(u; \mathcal{F}_j) = \{ f \in \mathcal{F}_j : |f|_{\mathcal{F}_j} \leq 1, |f_{\theta^*}|_n \leq u \}\).

The next part of the proof is almost identical to that in the proof for Lemma B.4 except that we replace \(A''_{n_j}(u)\) with \(A''_{n_j}(u)\) and \(\Omega''(u; \mathcal{F}_j)\) with \(\Omega''(u; \mathcal{F}_j)\). We first derive a bound for \(u = r_{n_j}\). By definition of \(A''_{n_j}(u)\) and \(G_n(r_{n_j}; \mathcal{F}_j)\) and the critical radius \(r_{n_j}\), we have

\[ \mathbb{E}_{v_j} \left[ A''_{n_j}(r_{n_j}) \big| \{ w_i^T \theta^* \}_{i=1}^n \right] = \bar{\sigma}_{v_j} G_n (r_{n_j}; \mathbb{B}_{\mathcal{F}_j}(1)) \leq r^2_{n_j}. \]
Setting \( b = r_{nj}^2 \) in the tail bound (B.5) with \( A_{nj}^\sigma(u) \) replaced by \( A_{nj}^{\sigma'}(u) \) and \( \Omega'(u; F_j) \) replaced by \( \Omega''(u; F_j) \) yields

\[
P \left[ A_{nj}^{\sigma'}(r_{nj}) \geq 2r_{nj}^2 | \{ w_i^T \theta^* \}_{i=1}^n \right] \leq \exp \left( -nC_j^2 r_{nj}^2 \right). \tag{B.10}
\]

On the other hand, for any \( u > r_{nj} \), we have

\[
E_{v_j} \left[ A_{nj}^{\sigma'}(u) | \{ w_i^T \theta^* \}_{i=1}^n \right] = \sigma_{v_j} \mathcal{G}_n \left( u; B_{F_j}(1) \right) = u \frac{\sigma_{v_j} \mathcal{G}_n \left( u; B_{F_j}(1) \right)}{u} \leq u \frac{\sigma_{v_j} \mathcal{G}_n \left( r_{nj}; B_{F_j}(1) \right)}{r_{nj}} \leq ur_{nj},
\]

where the first inequality follows from Lemma B.8 and the second inequality follows from the critical radius \( r_{nj} \). Setting \( b = \frac{u^2}{128} \) in the tail bound (B.5) with \( A_{nj}^\sigma(u) \) replaced by \( A_{nj}^{\sigma'}(u) \) and \( \Omega'(u; F_j) \) replaced by \( \Omega''(u; F_j) \) yields

\[
P \left[ A_{nj}^{\sigma'}(u) \geq ur_{nj} + \frac{u^2}{128} | \{ w_i^T \theta^* \}_{i=1}^n \right] \leq \exp \left( -nC_j^2 u^2 \right). \tag{B.11}
\]

It remains to prove the bound (B.9) via a peeling argument. Let \( \mathcal{E}' \) denote the event that the bound (B.9) is violated for some function \( f \) with \( |f|_{F_j} = 1 \). For real numbers \( 0 \leq a_1 < a_2 \), let \( \mathcal{E}'(a_1, a_2) \) denote the event that it is violated for some function such that \( |f|_{F_j} \in [a_1, a_2] \), and \( |f|_{F_j} = 1 \). For \( m = 0, 1, 2, \ldots \), define \( t_m = 2^{m} r_{nj} \). We then have the decomposition \( \mathcal{E}' = \mathcal{E}'(0, t_0) \cup \bigcup_{m=0}^\infty \mathcal{E}'(t_m, t_{m+1}) \) and hence by union bound,

\[
P \left[ \mathcal{E}' \right] \leq P \left[ \mathcal{E}'(0, t_0) \right] + \sum_{m=0}^\infty P \left[ \mathcal{E}'(t_m, t_{m+1}) \right]. \tag{B.12}
\]

The final step is to bound each of the terms in this summation. Since \( t_0 = r_{nj} \), we have

\[
P \left[ \mathcal{E}'(0, t_0) \right] \leq P \left[ A_{nj}^\sigma(r_{nj}) \geq 2r_{nj}^2 | \{ w_i^T \theta^* \}_{i=1}^n \right] \leq \exp \left( -nC_j^2 r_{nj}^2 \right). \tag{B.13}
\]

On the other hand, suppose that \( \mathcal{E}'(t_m, t_{m+1}) \) holds, meaning that there exists some function \( f \) with \( |f|_{F_j} = 1 \) and \( |f|_{F_j} \leq t_{m+1} \) such that

\[
\frac{1}{n} \sum_{i=1}^n v_{ij} f(w_i^T \theta^*) \geq 2r_{nj} |f|_{F_j} + 2r_{nj}^2 + \frac{1}{32} |f|_{F_j}^2 \\
\geq 2r_{nj} t_m + 2r_{nj}^2 + \frac{1}{32} t_m^2 \\
= r_{nj} t_{m+1} + 2r_{nj}^2 + \frac{1}{128} t_{m+1}^2,
\]

where the second inequality follows since \( |f|_{F_j} \geq t_m \); and the third inequality follows since \( t_{m+1} = 2t_m \). This lower bound implies that

\[
A_{nj}^{\sigma'}(t_{m+1}) \geq r_{nj} t_{m+1} + \frac{1}{128} t_{m+1}^2,
\]

hence the bound (B.11) implies that

\[
P \left[ \mathcal{E}'(t_m, t_{m+1}) \right] \leq \exp \left( -\frac{t_{m+1}^2}{c_2 \sigma_{v_j}} \right) = \exp \left( -\frac{n^2 (m+1) r_{nj}^2}{c_2 \sigma_{v_j}} \right).
\]

Combining this tail bound with our earlier bound (B.13), and substituting into the union bound (B.12)
yields
\[ \mathbb{P}[E'] \leq \exp \left(-\frac{n\gamma_{v_j}r_{nj}^2}{2\sigma_{v_j}}\right) + \sum_{m=0}^{\infty} \exp \left(-\frac{n\gamma_{v_j}2^{2(m+1)}r_{nj}^2}{c_2\sigma_{v_j}}\right) \leq c_1 \exp \left(-\frac{n\gamma_{v_j}r_{nj}^2}{c_2\sigma_{v_j}}\right). \]
\[ \square \]

**Lemma B.6:** Suppose the variable \( X \) has strongly log-concave distribution with parameter \( \gamma > 0 \). Then for any function \( f : \mathbb{R}^n \to \mathbb{R} \) that is \( L \)-Lipschitz with respect to Euclidean norm, we have
\[ \mathbb{P}[|f(X) - \mathbb{E}[f(X)]| \geq t] \leq 2 \exp \left(-\frac{\gamma t^2}{2L^2}\right). \]

**Remark:** This result was initiated by Maurey (1991) and further developed by Bobkov and Ledoux (2000).

**Definition B.1** (Separate convex functions). A function \( f : \mathbb{R}^n \to \mathbb{R} \) is separately convex if, for each \( j \in \{ 1, 2, ..., n \} \), the co-ordinate function \( f_j : \mathbb{R} \to \mathbb{R} \) defined by varying only the \( j^{th} \) co-ordinate, is a convex function of \( x_j \).

**Lemma B.7:** Let \( \{X_i\}_{i=1}^n \) be independent random variables, each supported on the interval \([a, b]\), and let \( f : \mathbb{R}^n \to \mathbb{R} \) be separately convex, and \( L \)-Lipschitz with respect to the Euclidean norm. Then for all \( t > 0 \), we have
\[ \mathbb{P}[f(X) - \mathbb{E}[f(X)] \geq t] \leq 2 \exp \left(-\frac{t^2}{2L^2(b-a)^2}\right). \]

**Remark:** The proof can be found in Wainwright (2015), which is based on Ledoux (1996).

**Lemma B.8:** For any star-shaped function class \( F \), the function \( t \mapsto \frac{G_0(t; F)}{\min\{|t|, \sigma/2\sqrt{n}\}} \) is non-decreasing on the interval \((0, \infty)\).

**Proof:** This is Lemma 13.1 from Wainwright (2015).

**Definition B.2** (Covering and packing numbers). Consider a metric space consisting of a set \( \mathcal{X} \) and a metric \( \rho : \mathcal{X} \times \mathcal{X} \to \mathbb{R}_+ \).

1. An \( t \)-covering of \( \mathcal{X} \) in the metric \( \rho \) is a collection \( \{\beta^1, ..., \beta^N\} \subset \mathcal{X} \) such that for all \( \beta \in \mathcal{X} \), there exists some \( i \in \{1, ..., N\} \) with \( \rho(\beta, \beta^i) \leq t \). The \( t \)-covering number \( N(t; \mathcal{X}, \rho) \) is the cardinality of the smallest \( t \)-covering.

2. An \( t' \)-packing of \( \mathcal{X} \) in the metric \( \rho \) is a collection \( \{\beta^1, ..., \beta^N\} \subset \mathcal{X} \) such that \( \rho(\beta^i, \beta^{j}) \geq t \) for all \( i \neq j \). The \( t \)-packing number \( M(t' ; \mathcal{X}, \rho) \) is the cardinality of the largest \( t' \)-covering.

**Lemma B.9:** Let \( N_n(t; \mathbb{B}_n(r_n; \mathcal{F})) \) denote the \( t \)-covering number of the set
\[ \mathbb{B}_n(r_n; \mathcal{F}) = \{ f \in 2\mathcal{F} : |f|_n \leq r_n \} \]
in the empirical \( L^2(\mathbb{P}_n) \) norm. Then the critical radius condition (A.18) holds for any \( r_n \in (0, \sigma] \) such that
\[ \frac{32}{\sqrt{n}} \int_{\frac{r_n}{2\sigma}}^{r_n} \sqrt{\log N_n(t; \mathbb{B}_n(r_n; \mathcal{F}))} dt \leq \frac{r_n^2}{\sigma}. \]
Remark. This result is established by van der Vaart and Wellner (1996), van de Geer (2000), Barlett and Mendelson (2002), Koltchinski (2006), Wainwright (2015), etc.

**Lemma B.10:** For $q \in (0, 1]$, let

$$\theta \in B^d(R_q) := \left\{ \theta' \in \mathbb{R}^d : \left| \theta' \right|_1^q = \sum_{j=1}^{d} |\theta'_j|^q \leq R_q \right\}$$

and $N_2(t; B^d(R_q))$ be the $t$–covering number of the set $B^d(R_q)$ in the $l_2$–norm. Then there is a universal constant $c$ such that

$$\log N_2(t; B^d(R_q)) \leq c R_q^{\frac{2}{2-q}} \left( \frac{1}{t} \right)^{\frac{2q}{2-q}} \log d \quad \text{for all } t \in (0, R_q^\frac{1}{2})$$

Conversely, suppose in addition that $t < 1$ and $t^2 \geq c'' \left( \frac{\log d}{d} \right)^{\frac{2-q}{q}}$ for some fixed $\kappa \in (0, 1)$. Then there is a constant $c' \leq c$ such that

$$\log N_2(t; B^d(R_q)) \geq c' R_q^{\frac{2}{2-q}} \left( \frac{1}{t} \right)^{\frac{2q}{2-q}} \log d.$$

**Proof.** The result is obtained by inverting known results on (dyadic) entropy numbers of $l_q$–balls as in Raskutii, Wainwright, and Yu (2011). For a $d$–dimensional $l_q$–ball with $q \in (0, p)$, it is known (e.g., Schütt, 1984; Guedon and E. Litvak, 2000; Kühn, 2001) that for all integers $k \in [\log d, d]$, the dyadic entropy numbers $t_k$ of the $d$–dimensional $l_q$–ball of unit radius $B^d_q(1)$ with respect to the $l_p$–norm scale as

$$t_k \left( l_q, \left| \cdot \right|_p \right) = C_{q,p} \left[ \log \left( 1 + \frac{d}{k} \right) \right]^{\frac{1}{q} - \frac{1}{p}}. $$

Moreover, for $k \in [1, \log d]$, we have $t_k(l_q) \leq C_{q,p}$. We first establish the upper bound on the metric entropy for $q \in (0, 1]$ and $p = 2$. Since $d \geq 2$, we have

$$t_k \left( l_1, \left| \cdot \right|_2 \right) \leq C_{q,2} \left( \frac{\log \left( 1 + \frac{d}{k} \right)}{k} \right)^{\frac{1}{q} - \frac{1}{2}} \leq C_{q,2} \left( \frac{\log d}{k} \right)^{\frac{1}{q} - \frac{1}{2}}. $$

Inverting this inequality for $k = \log N_2(t; B^d(R_q))$ and allowing for a ball radius $R_q$ yields

$$\log N_2(t; B^d(R_q)) \leq c R_q^{\frac{2}{2-q}} \left( \frac{1}{t} \right)^{\frac{2q}{2-q}} \log d. \quad \text{(B.14)}$$

For the lower bound on the metric entropy, which requires the existence of some fixed $\kappa \in (0, 1)$ such that $k \leq d^{1-\kappa}$. Under this assumption, we have $1 + \frac{d}{k} \geq \frac{d}{k} \geq d^\kappa$, and hence

$$C_{q,2} \left( \frac{\log \left( 1 + \frac{d}{k} \right)}{k} \right)^{\frac{1}{q} - \frac{1}{2}} \geq C_{q,2} \left( \frac{\kappa \log d}{k} \right)^{\frac{1}{q} - \frac{1}{2}}. $$

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Accounting for the radius $R_q$ yields

$$\log N_2(t; \mathcal{B}^d(R_q)) \geq \kappa C_{q, 2} R_q^\frac{2}{2-q} \left( \frac{1}{t} \right)^{\frac{2q}{2-q}} \log d.$$ 

Finally, let us check that our assumptions on $k$ needed to perform the inversion are ensured by the conditions that we have imposed on $t$. The condition $k \geq \log d$ is ensured by setting $t < 1$. Turning to the condition $k \leq d^{1-\kappa}$, from bound [B.14], it suffices to choose $t$ such that $\left( \frac{C_{q, 2}}{4} \right)^{\frac{2q}{2-q}} \log d \leq d^{1-\kappa}$. This condition is ensured by enforcing the bound $t^2 \geq c'' \left( \frac{\log d}{d^q} \right)^{\frac{2-\kappa}{q}}$. □

**Lemma B.11:** Consider the class of smooth functions

$$\mathcal{F}_L := \left\{ f : [0, 1] \to \mathbb{R} : f(0) = 0, \left| f(x) - f(x') \right| \leq L \left| x - x' \right| \quad \forall x, x' \in [0, 1] \right\}.$$ 

We have

$$\log N_{\infty}(t; \mathcal{F}_L) \asymp \frac{L}{t},$$

and

$$\log N_2(t; \mathcal{F}_L) \asymp \frac{L}{t},$$

where $N_{\infty}(t; \mathcal{F}_L)$ is the $t$–covering number of the set $\mathcal{F}_L$ in the sup-norm and $N_2(t; \mathcal{F}_L)$ is the $t$–covering number of the set $\mathcal{F}_L$ in the $L^2(\mathbb{P})$–norm.

**Remark.** The proof for $\log N_{\infty}(t; \mathcal{F}_L)$ is a well-known result due to Kolmogorov and Tikhomirov (1961). To proof for $\log N_2(t; \mathcal{F}_L)$ is an application of Varshamov-Gilbert lemma. □

**Lemma B.12:** For all $\varepsilon > 0$, the packing and covering number of a set $\mathbb{T}$ with respect to a metric $\rho$ are related as follows:

$$M(2\varepsilon; \mathbb{T}, \rho) \leq N(\varepsilon; \mathbb{T}, \rho) \leq M(\varepsilon; \mathbb{T}, \rho).$$

**Proof.** To prove the first inequality, let $\{\theta^1, ..., \theta^{2M}\}$ be a maximal $2\varepsilon$–packing, so that $\rho(\theta^i, \theta^j) > 2\varepsilon$ for all $i \neq j$. Given a $\varepsilon$–cover, let $\gamma$ be an arbitrary element. For any distinct pair $i \neq j$, we have

$$2\varepsilon < \rho(\theta^i, \theta^j) \leq \rho(\theta^i, \gamma) + \rho(\gamma, \theta^j),$$

so that it is not possible to have $\rho(\gamma, \theta^i) \leq \varepsilon$ and $\rho(\gamma, \theta^j) \leq \varepsilon$ simultaneously. Conversely, we must have at least one element in the cover for each element of the packing set, showing that $M(2\varepsilon; \mathbb{T}, \rho) \leq N(\varepsilon; \mathbb{T}, \rho)$. Turning to the second inequality, let $\{\theta^1, ..., \theta^{2M}\}$ be a maximal $\varepsilon$–packing. Maximality implies that for any other $\theta$ not already in the packing set, we must have $\rho(\theta, \theta^i) \leq \varepsilon$ for some $\theta^i$. Thus, the given set forms a $\varepsilon$–cover, showing that $N(\varepsilon; \mathbb{T}, \rho) \leq M(\varepsilon; \mathbb{T}, \rho)$. □

**Lemma B.13:** Let $k \geq 1$. There exists a subset $W$ of $\{0, 1\}^k$ with $|W| > e^{k/8}$ such that the Hamming distance,

$$d(\tau, \tau') := \sum_{j=1}^{k} \mathbb{I}\{\tau_i \neq \tau'_i\} > \frac{k}{4}$$

for all $\tau, \tau' \in W$ with $\tau \neq \tau'$. 54
Remark. This is Varshamov-Gilbert Lemma.

Lemma B.14: For a given radius \( r > 0 \), define the set

\[
S(s, r) := \{ \delta \in \mathbb{R}^d \mid |\delta|_0 \leq 2s, |\delta|_2 \leq r \},
\]

there is a covering set \( \{\delta^1, ..., \delta^N\} \in S(s, r) \) with cardinality \( N = N(s, r, \epsilon) \) such that

\[
\log N(s, r, \epsilon) \leq \log \left( \frac{d}{2s} \right) + 2s \log \left( \frac{1}{\epsilon} \right).
\]

Remark. This result is established in Matousek (2002).

Lemma B.15: There exists a subset \( \tilde{M} \subset M \) with cardinality \( |\tilde{M}| \geq \exp\left( s^2 \log d - \frac{s^2}{2} \right) \) such that \( \rho_M(z, z') \geq \frac{s}{2} \) for all \( z, z' \in \tilde{M} \).

Remark. This is Lemma 5 from Raskutii, Wainwright, and Yu (2011).

Lemma B.16: Let \( X_1, ..., X_n \) be independent random variables such that, for every \( i \), \( \mathbb{E}(X_i) = 0 \) and \( 0 < \mathbb{E}\left(|X_i|^{2+\delta}\right) < \infty \) for some \( 0 < \delta < 1 \). Set \( S_n = \sum_{i=1}^n X_i, B_n^2 = \sum_{i=1}^n \mathbb{E}(X_i^2), V_n^2 = \sum_{i=1}^n X_i^2, \) \( L_{n,\delta} = \sum_{i=1}^n \mathbb{E}\left(|X_i|^{2+\delta}\right), b_{n,\delta} = \frac{B_n}{L_{n,\delta}^{1/2}} \). Then for all \( 0 \leq t \leq b_{n,\delta} \) and an absolute constant \( a_0 > 0 \)

\[
\mathbb{P}\left[ \frac{S_n}{V_n} \geq t \right] - [1 - \Phi(t)] \leq a_0 (1 + t)^{1+\delta} \frac{\epsilon}{b_{n,\delta}^{2+\delta}}.
\]

Remark. This is formula 2.11 from Jing, Shao, and Wang (2003) on moderate deviations for self-normalized sums.

Lemma B.17: Let \( X_k, k \geq 1 \) be i.i.d. sub-Gaussian variables with parameters at most \( \sigma \) and \( Z := \max_{k=1, ..., N} |X_k| \). For some integer \( N \geq 10 \), we have \( \mathbb{E}[Z] \leq 3\sigma \sqrt{N} \).

Remark. This is a well-known result on sub-Gaussian maxima (e.g., van der Vaart and Wellner, 1996).