Excited states of odd-mass nuclei with different deformation-dependent mass coefficients

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Abstract

Experimental data indicate that the mass tensor of collective Bohr Hamiltonian cannot be considered as a constant but should be considered as a function of the collective coordinates. In this work our purpose is to investigate the properties of low-lying collective states of the odd nuclei \(^{173}\)Yb and \(^{163}\)Dy by using a new generalized version of the collective quadrupole Bohr Hamiltonian with deformation-dependent mass coefficients. The proposed new version of the Bohr Hamiltonian is solved for Davidson potential in \(\beta\) shape variable, while the \(\gamma\) potential is taken to be equal to the harmonic oscillator. The obtained results of the excitation energies and B(E2) reduced transition probabilities show an overall agreement with the experimental data. Moreover, we investigate the effect of the deformation dependent mass parameter on energy spectra and transition rates in both cases, namely: when the mass coefficients are different and when they are equal. Besides, we will show the positive effect of the present formalism on the moment of inertia.

1 Introduction

Quantum Phase Transitions \([1, 2, 3]\) in atomic nuclei within the Bohr-Mottelson Model (BMM) \([4, 5, 6]\) have attracted a considerable attention for describing the quadrupole collective excitations behavior in various deformed nuclei. In this context, the Bohr Hamiltonian involved in this model has a standard form of the kinetic energy term which contains one mass coefficient for all modes of excitation, namely: the \(\beta\) - and the \(\gamma\) - vibrations and the rotational motion, where \(\beta\) is the collective variable corresponding to nuclear deformation while \(\gamma\) denotes an angle measuring departure from axial symmetry \([7]\). This approximation is argued in terms of small oscillations around the equilibrium value. However, several authors have elaborated new approaches to generalize the usual form of the kinetic energy term of the Bohr Hamiltonian given in the intrinsic frame. One can cite two remarkable ways: The first one is the approach followed by Jolos and von Brentano \([8, 9, 10]\), in which they showed, for low lying collective states of well-deformed axially symmetric even-even nuclei, the necessity of the introduction of three different mass coefficients for each collective mode motion (ground state, \(\beta\) or \(\gamma\)). These latter are determined by the experimental data on B(E2)'s and the excitations energies. This approach was applied in Ref. \([11]\) with Davidson potential, and has shown a strong influence of different mass parameters, especially on the interband B(E2) rates. Also, the high-spin states of spectra and intraband B(E2) are affected by these differences. However, it is worth to notice here that the used energy formula in Ref. \([11]\) is inaccurate as we have already shown in \([12]\) for even-even nuclei and that we will show once again in the present work for odd-A nuclei. That is why our calculations are restricted to the two nuclei, namely \(^{173}\)Yb and \(^{163}\)Dy, which were
previously processed by Ermamatov et al. [11, 19] in order to correct the results that have already
been obtained with their erroneous formula on one hand, and to see the effect of DDM on these
calculations on the other hand.

The second one is the formalism proposed by Bonatsos et al. [13], allowing the mass to depend
on the nuclear deformation. This approach has been firstly achieved by applying Davidson
[15] and Kratzer [14] potentials to a huge number of γ-unstable and axially symmetric prolate
deformed nuclei. It has also been extended to a conjunction between the prolate γ-rigid and
γ-stable collective motions [15], and then tested on Davydov-Chaban Hamiltonian to describe
triaxial shape nuclei around γ = π/6 [16].

An investigation similar to the different mass parameters approach of Ref. [9] has been
extended in Ref. [17] for the collective single-particle ground state properties of deformed odd-A
nuclei such as 163,165Er [18] and 173Yb [11]. This latter approach has been improved in Ref.
[19] by adding the Coriolis interaction between the rotational and single-particle motion of the
odd nucleon. Its influence was tested by applications on the experimental data of 163Dy and
173Yb [19], while in the present study we consider the projection of the nuclear total angular
momentum onto the third axis K and that of the external nucleon Ω as conserved quantities
(i.e. K and Ω are good quantum numbers), which means that the Coriolis interaction does not
make a contribution.

The purpose of the present work is to investigate a new generalized version of the collective
quadrupole Bohr Hamiltonian with different deformation-dependent mass parameters, firstly
developed in [12]. We will then propose a combination of the first and second approach mentioned
above. Here, the Davidson and harmonic oscillator potentials are taken to characterize the β
and γ vibrations, respectively. This choice is dictated by the number of interesting works that
have been achieved with these potentials [11, 12, 13, 19, 20], which will allow a comparison of
our analytical results to those obtained by other authors.

The analytical expressions of spectra, wave functions and reduced E2 transition probabilities
are obtained by means of the asymptotic iteration method (AIM) [21]. Therefore, we test this
extended model to 173Yb and 163Dy nuclei, by evaluating the experimental observables like energy
spectra and B(E2) transition probabilities. Besides, we will study the effect of the deformation
mass parameter which is used either : when we consider three different mass parameters or
when only a global mass parameter. As mentioned in [11], the properties of the ground states of
odd-mass nuclei that have an angular momentum ≥ 5/2 can be studied with this model.

This paper is organized as follows : In Section 2 we present the Bohr Hamiltonian with
mass coefficients for the case of odd-mass nucleus, that we use in Section 3 in accordance with
def ormation-dependent mass formalism. Analytical expressions for the energy levels and excited-
state wave functions of the model are presented in Section 4, while the B(E2) transition proba-
bilities are given in Section 5. The numerical results for energy spectra and B(E2) are presented,
discussed, and compared with experimental data in Section 6, while Section 7 is devoted to our
conclusions. The formulas of special cases of energy spectrum are given in Appendix A, while
Appendix B is dedicated to collect the used formulas for the calculations of B(E2).

2 Bohr Hamiltonian with mass parameters

According to the approach of Jolos and von Brentano [9] for an odd-mass nucleus, for small
harmonic β and γ oscillations of a deformed nuclear surface with respect to the equilibrium
values β0 ≠ 0 and γ0 ≈ 0, the corresponding Hamiltonian with three different mass parameters,
can be written as

\[ H = H_{\text{vib}} + H_{\text{rot}} + H_{\text{int}}, \]  

where the operator describing the β and γ vibrations of the nuclear core surface is

\[
H_{\text{vib}} = -\frac{\hbar^2}{2} \left\{ \frac{1}{B_\beta} \frac{\partial^2}{\partial \beta^2} + \frac{2}{B_\gamma} \frac{1}{\partial \beta} \frac{\partial}{\partial \gamma} + \frac{1}{B_\gamma} \frac{1}{\partial \gamma} \left( \gamma \frac{\partial}{\partial \gamma} \right) - \frac{1}{B_\gamma} \frac{1}{4\beta^2} \times \left( \frac{1}{\gamma^2} + \frac{1}{3} \right) \left( L_3 - j_3 \right)^2 \right\} + V(\beta, \gamma),
\]
The nuclear rotational energy operator can conveniently be represented as

\[ H_{\text{rot}} = \frac{\hbar^2}{6B_{\text{rot}}\beta^2} \left( L^2 + j^2 - L_3^2 - j_3^2 \right), \] (3)

and the interaction operator which takes into account non-spherical part of the field of the core is given by the expression

\[ H_{\text{int}} = -\beta_0(T(r))(3j_3^2 - j^2), \] (4)

where \( B_{\text{rot}}, B_\beta \) and \( B_\gamma \) are three different mass coefficients for rotational, \( \beta \)-, \( \gamma \)- motion, respectively. \( L \) is the total angular momentum, where \( L_3 \) is the eigenvalue of the projection of angular momentum on the principal axis of nucleus, \( j \) and \( j_3 \) are the angular momentum operator of a single nucleon, and its projection. \( \beta_0 \) is the equilibrium value of the nuclear surface \( \beta \)-oscillations and \( T(r) \) is a function of the distance between the single nucleon and the center of the nuclear core, while \( \langle T \rangle \) is its average value over internal states of the external nucleon and zero nuclear surface oscillations [22, 23].

3 Connection between Deformation-Dependent mass and different parameters

By following the procedure in Ref. [8], we wish to construct a Bohr equation with three deformation dependent mass coefficients, in accordance with the DDM formalism described in Ref. [13]. So, the mass tensor of the collective Hamiltonian becomes

\[ B = \frac{\langle i | B_0 | i \rangle}{\langle \bar{f} | \bar{f} \rangle^2}, \] (5)

where \( i = \text{g.s.}, \beta \) or \( \gamma \) (g.s. is often replaced by \( \text{rot} \)) corresponding to three separable state bands of nuclei, namely: the ground state band, the \( \beta \) and \( \gamma \) vibrational bands, each one of these will have its own mass coefficient equal to its average value over the wave function of the considered state, such as: \( \langle \text{g.s.} | B_0 | \text{g.s.} \rangle \equiv B_{\text{rot}}, \langle \gamma | B_0 | \gamma \rangle \equiv B_\gamma \) and \( \langle \beta | B_0 | \beta \rangle \equiv B_\beta \) defined for each band. \( f \) is the deformation function depending only on the radial coordinate \( \beta \). Therefore, only the \( \beta \) part of the resulting equation will be affected.

The explicit equation reads as [12]

\[
\frac{\hbar^2}{2\langle i | B_0 | i \rangle} \left( -\sqrt{\tau} \frac{\partial}{\partial \beta} \beta \frac{\partial}{\partial \beta} \sqrt{\tau} - \frac{f^2}{3\tau \sin^3 \gamma} \frac{\partial}{\partial \gamma} \sin 3\gamma \frac{\partial}{\partial \gamma} \right) + \frac{f^2}{4\beta^2} \sum_{k=1,2,3} \frac{(Q_k - j_k)^2}{\sin^2(\gamma - \frac{2}{3}\pi k)} \Psi - f^2 \beta(T)(3j_3^2 - j^2)\Psi + V_{\text{eff}}\Psi = E\Psi, \] (6)

with,

\[ V_{\text{eff}} = V(\beta, \gamma) + \frac{\hbar^2}{2\langle i | B_0 | i \rangle} \left( \frac{1}{2}(1 - \delta - \lambda) f \nabla^2 f + \left( \frac{1}{2} - \delta \right) \left( \frac{1}{2} - \lambda \right) (\nabla f)^2 \right) , \] (7)

where \( \delta \) and \( \lambda \) are free parameters orignated from the construction procedure of the kinetic energy term within DDM formalism.

4 Energy spectrum and excited-state wave functions

Exact separation of variables \( \beta \) and \( \gamma \) can be achieved for potentials using the convenient form [12] \( V(\beta, \gamma) = U(\beta) + f^2W(\gamma)/\beta^2 \), in which the potential depending only on \( \gamma \) has a minimum around \( \gamma = 0 \). In the same context, the total wave function can be constructed as

\[ \Psi = F(\beta)\chi(\gamma)|LMjKm \rangle, \] (8)
where the rotational wave function $|LM_jKm\rangle$ has been expanded [19] [22] in terms of the Wigner $D(\theta,\phi)$-function of the Euler angles, and $\varphi(x_i)$ the eigenfunction of the single-particle states, in the following form

$$|LM_jKm\rangle = \sqrt{\frac{2L+1}{16\pi^2}} \left[ D_{LMK}^L(\theta_i)\varphi_{-2m}(x_i) + (-1)^{L-J} D_{LM-K}^L(\theta_i)\varphi_{+2m}(x_i) \right], \quad (9)$$

where $K$ the projection of the nuclear angular momentum on the third axis connected with the nucleus and $\Omega = K - 2m$ represents the projection of the angular momentum of the external nucleon on the same axis [22]. Note that $m$ should be an integer. As a result, Eq. (6) can be separated into three equations

$$\left[ \frac{\hbar^2}{2|B_0|} \left( -\sqrt{T} \frac{\partial}{\partial \beta} \beta^4 f^0 \frac{\partial}{\partial T} \sqrt{T} + \frac{f^2}{\partial^2} \Lambda + \frac{1}{4} (1 - \delta - \lambda) f \nabla^2 f + \left( \frac{1}{2} - \delta \right)(1/2 - \lambda)(\nabla f)^2 \right) + V(\beta) \right] F(\beta) = EF(\beta), \quad (10)$$

$$\left[ -\frac{\hbar^2}{2B_{\gamma}} \left( \sin 3\gamma \frac{\partial}{\partial \gamma} \sin 3\gamma \frac{\partial}{\partial \gamma} - \frac{1}{4} \left( \frac{1}{\gamma^2} + \frac{1}{3} \right) (L_j - j_3)^2 \right) + W(\gamma) \right] \chi_{n_\gamma|m}(\gamma) = \Lambda'(\gamma)n_{\gamma}|m|\langle \gamma), \quad (11)$$

$$\left[ \frac{\hbar^2}{6B_{rot}} \left( L_j^2 + j_3^2 - L_3^2 - j_3^2 \right) - \beta^3(T) \left( 3j_3^2 - 3j_3^2 \right) \right] LM_{jKm} = \bar{\Lambda}LM_{jKm}, \quad (12)$$

where $\Lambda = \Lambda' + 2\bar{\Lambda}$ is the parameter coming from the exact separation of variables, $\Lambda'$ is the eigenvalues corresponding to $\gamma$-vibrations, while $\bar{\Lambda}$ is the internal state energy corresponding to the rotational energy operator $H_{rot}$ and the interaction operator $H_{int}$. It is represented by the term

$$\frac{B_0}{\hbar^2} = \frac{B_{\beta}}{6B_{rot}} \left( L(L+1) + j(j+1) - K^2 - (K-2m)^2 \right) - \frac{1}{6\xi} \left( 3(K-2m)^2 - j(j+1) \right), \quad (13)$$

with $\xi = \frac{\hbar^2}{6B_{rot}}$. Concerning the $\gamma$-angular part Eq. (11), the potential $W(\gamma)$ is assumed to be harmonic oscillator around $\gamma = 0$ as in Ref. [12], namely $W(\gamma) = \frac{1}{2}(\beta_0^2C_\gamma)^2$, where $C_\gamma$ is a free parameter. It can be seen that Eq. (11) is similar to the $\gamma$ part of the differential equation in Ref. [12]. The corresponding analytical solution of Eq. (11) is given with eigenvalues [12]

$$\frac{B_{\beta}}{\hbar^2} \Lambda' = \frac{2B_{\beta}}{g} B_\gamma \left( 1 + n_{\gamma} \right) + \frac{m^2 B_{\beta}}{3} B_\gamma, \quad (14)$$

and the corresponding eigenfunctions, are obtained in terms of the Laguerre polynomials as

$$\chi_{n_\gamma|m}(\gamma) = N_{n_\gamma|m} \gamma^{|m|} e^{-\frac{\gamma^2}{2g}} L_{n_\gamma}^{|m|} \left( \frac{\gamma^2}{g} \right), \quad (15)$$

with $g = \frac{1}{B_\gamma} \frac{\hbar}{\sqrt{B_0C_\gamma}}$. $\tilde{n}_\gamma = \frac{n_{\gamma}-|m|}{2}$, $n_{\gamma}$ is the quantum number related to $\gamma$-oscillations and $N_{n_\gamma|m}$ is a normalization constant determined from the normalization condition. This leads to

$$N_{n_\gamma|m} = \left( \frac{2}{\beta} \frac{1}{\Gamma(|m| + \tilde{n}_\gamma + 1)} \right)^{1/2}, \quad (16)$$

Concerning the $\beta$-oscillation states of deformed odd-A nuclei, they are determined by the solution of radial equation [10] with Davidson potential, $V(\beta) = V_0 \left( \frac{\beta}{\beta_0} - \frac{\beta_0}{\beta} \right)^2$, where $V_0$ represents the
depth of the minimum, located at $\beta_0$ and $f(\beta)$ the deformation function. According to specific form of Davidson potential we are going to consider for the deformation function the special form $f(\beta) = 1 + a\beta^2$, with $a \ll 1$. In fact, for each potential an appropriate deformation function will be used. With these last considerations, the resulting radial equation becomes the same as the $\beta$ part of the differential equation given in Sect. VI of Ref. [12]. Note that the only difference between them resides in $\Lambda$ the eigenvalue of the exact separation of variables, depending on the nature of the nucleus.

Thus, the energy spectrum of the radial equation is determined by the following expression: [12]

$$E_{n,\beta,L|m|} = \frac{\hbar^2}{2B_\beta} \left( k_0 + \frac{a}{2}(2 + \frac{B_\beta}{B_\gamma} + 2p + 2q + pq) + 2a(2 + p + q)n_\beta + 4an_\beta^2 \right), \quad (17)$$

where $n_\beta$ is the principal quantum number of $\beta$ vibrations, and

$$q \equiv q_{n,\beta}(L,|m|) = \sqrt{1 + 4k_{-2}},$$

$$p = p_{n,\beta}(L,|m|) = \sqrt{\frac{B_\beta}{B_\gamma} - 3 + 4\frac{k_2}{a^2}}, \quad (18)$$

$$k_2 = \frac{a}{2}\left[ (1 + \frac{B_\beta}{B_\gamma}) (6\frac{B_\beta}{B_\gamma} + (1 - 2\delta)(1 - 2\lambda) + 5(1 - \delta - \lambda)) + \frac{2B_\beta}{\hbar^2}\Lambda \right] + \frac{2g_\beta}{\beta_0^2},$$

$$k_0 = \frac{a}{2}\left[ (1 + \frac{B_\beta}{B_\gamma}) (8\frac{B_\beta}{B_\gamma} + 5(1 - \delta - \lambda)) + \frac{4B_\beta}{\hbar^2}\Lambda \right] - \frac{4g_\beta}{\beta_0^2},$$

$$k_{-2} = \frac{B_\beta}{B_\gamma} (1 + \frac{B_\beta}{B_\gamma}) + \frac{B_\beta}{\hbar^2}\Lambda + 2g_\beta, \quad (19)$$

where $g_\beta = \frac{B_\beta V_{0,\beta^2}}{\hbar^2}$, the excitation energies $[17]$ do not depend on $j$ because $j,\Omega$, and $K$ are conserved [11], so they depend on five quantum numbers, namely: $n_\beta, n_\gamma, L, K$, and $m$, and nine parameters: $g, g_\beta, \xi, B_\beta/B_\gamma, B_\beta/B_\text{rot}$ ratio of the mass coefficients, $a$ the deformation mass parameter, $\beta_0$ the minimum of the potential and the free parameters $\delta$ and $\lambda$ coming from the DDM formalism. In the numerical results Section, a comparison to the experiment will be carried out by fitting the theoretical spectra to experimental data. Finally, it will be shown that the predicted energy levels turn out to be independent of the choice made for $\delta$ and $\lambda$.

In the limit cases of the energy spectrum, our general formula $[17]$ can well reproduce three special cases, namely: the first without mass coefficients i.e. if we assume $B_\beta = B_\gamma = B_\text{rot} = 1$, the second in the limit of no dependence of the mass on the deformation, i.e. $a = 0$ and the third standard case, when $B_\beta = B_\gamma = B_\text{rot}$ and $a = 0$. All these special cases are carried out in the Appendix $[A]$ with their energy spectrum expressions.

The relevant radial eigenfunctions of Eq. $[10]$ are found in Ref. [12] to be

$$R(t) = N_{n_\beta} 2^{-(1 + \frac{B_\beta}{\pi^2})/2 - (q + p)/4} a^{-(1 + q)/4} (1 - t)^{(1 + 2\frac{B_\beta}{\pi^2} + p)/4} (1 + t)^{(q + 1)/4} P_{n_\beta}^{(q/2,p/2)}(t),$$

$$t = -1 + a\beta^2, \quad (20)$$

$P_{n_\beta}^{(q/2,p/2)}(t)$ denotes the Jacobi polynomials $[24]$, while the normalization coefficient $N_{n_\beta}$ is given by

$$N_{n_\beta} = \left( 2a^{q/2 + 1} n_\beta! \right)^{1/2} \left[ \frac{\Gamma(n_\beta + \frac{q + p}{2} + 1)\Gamma(2n_\beta + \frac{q + p}{2} + 1 + \frac{B_\beta}{\pi^2})}{\Gamma(n_\beta + \frac{q}{2} + 1)\Gamma(n_\beta + \frac{B_\beta}{\pi^2} + \frac{p}{2} + 1)\Gamma(2n_\beta + \frac{q + p}{2} + 1)} \right]^{1/2}. \quad (21)$$

The reduction of the present wave functions Eq. $[20]$ and the normalization constant Eq. $[21]$ to the form they have in $B_\beta = B_\gamma = B_\text{rot} = 1$ limit are in agreement with Eq. $(108)$ and Eq. $(112)$ of Ref. [13], respectively, when the latter are simplified to consider only even-mass nuclei and conserved $K$. On the other hand, in the special case of no dependence of the mass on the deformation $a \to 0$, the excited-state wave functions are found in Ref [12] to be

$$R(\beta) = N_{n_\beta} \beta^{1/2} e^{-\beta^2} L_{n_\beta}^{1/2}(2\beta^2), \quad (22)$$
where \( b = \sqrt{\frac{\alpha}{2\gamma}} \), \( L \) denotes the Laguerre polynomials and \( N_{\beta} \) is a normalization coefficient reduced to the form

\[
N_{\beta} = \left[ 2(2b)^{\frac{1}{2}} q^{+1} \frac{n_{\beta}!}{\Gamma(n_{\beta} + \frac{1}{2} q + 1)} \right]^{\frac{1}{2}},
\]

(23)

5 B(E2) transition probabilities

The B(E2) transition rates from an initial to a final state are given by \[25\],

\[
B(E2; L_i K_i \rightarrow L_f K_f) = \frac{5}{16\pi} \frac{|\langle L_f K_f | T(E2) | L_i K_i \rangle|^2}{2L_i + 1},
\]

(24)

and the reduced matrix element can be obtained by using the Wigner-Eckart theorem \[25\],

\[
\langle L_f M_f K_f | T(E2) | L_i M_i K_i \rangle = \frac{(L_i 2L_f | M_i M_M_f)}{\sqrt{2L_f + 1}} \langle L_f K_f | T(E2) | L_i K_i \rangle.
\]

(25)

The final result \[26\] reads

\[
B(E2; n_{\beta} L_n K | m \rightarrow n'_{\beta} L' n'_K K' | m') = \frac{5}{16\pi} \langle L, K, 2, K' - K | L', K' \rangle^2 I_{n_{\beta} L_n K L'}^2 \gamma \sin^4 \gamma \chi_{n_{\gamma}', |m|, n_{\gamma}' |m'|},
\]

(26)

with

\[
I_{n_{\beta} L_n K L'} = \int \beta F_{n_{\beta} L_n}(\beta) F_{n'_K L'}(\beta) \beta^{2+2\frac{\alpha}{\gamma}} d\beta = \int \beta R_{n_{\beta} L_n}(\beta) R_{n'_K L'}(\beta) d\beta,
\]

(27)

where \( C_{n_{\gamma}, |m|, n_{\gamma}' |m'|} \) contains the integral over \( \gamma \). For \( \Delta m = 0 \) corresponding to transitions (g.s. \( \rightarrow \) g.s.), (\( \gamma \rightarrow \gamma \)), (\( \beta \rightarrow \beta \)) and (\( \beta \rightarrow g.s. \)), the \( \gamma \)-integral part reduces to the orthonormality condition of the \( \gamma \)-wave functions: \( C_{n_{\gamma}, |m|, n_{\gamma}' |m'|} = \delta_{n_{\gamma}, n_{\gamma}'} \delta_{m, m'} \). For \( |\Delta m| = 1 \) corresponding to transitions (\( \gamma \rightarrow g.s. \)), (\( \gamma \rightarrow \beta \)), this integral takes rather the form.

\[
C_{n_{\gamma}, |m|, n_{\gamma}' |m'|} = \int \sin \gamma \chi_{n_{\gamma}, |m|} \chi_{n_{\gamma}' |m'|} |\sin 3\gamma| d\gamma.
\]

(28)

In the next sections, all values of B(E2) are calculated in units of \( B(E2; 9_{g.s.} \rightarrow 5_{g.s.}) \).

6 Numerical results of energy and B(E2) ratios and discussion

Before starting our calculations of energy spectra and transition rates for the two deformed \( ^{173}\text{Yb} \) and \( ^{163}\text{Dy} \) nuclei, in the special case without DDM formalism (i.e. \( a = 0 \)), we have determined the optimal values of the free parameters \( B_\beta / B_\gamma \), \( B_\beta / B_{\text{rot}} \), \( g \) and \( g_\beta \) by fitting the energy formula \[32\] (Appendix A) on the available experimental data, except the value of \( \xi \) that takes into account the interaction of the extra nucleon with the core, which is chosen as \( \xi = 1 \), because, we have considered the quantum numbers \( K \) and \( \Omega \) are conserved. Then the parameter \( \xi \) will not change the results significantly. These parameters are adjusted to reproduce the experimental data by applying a least-squares fitting procedure for each considered nucleus. For this purpose we have minimized the root mean square (r.m.s) deviation between the theoretical values and the experimental data via the following quantity factor.

| nucleus    | \( g \)  | \( g_\beta \) | \( B_\beta / B_\gamma \) | \( B_\beta / B_{\text{rot}} \) | \( g(B_\beta = B_\gamma = B_{\text{rot}}) \) | \( g_\beta(B_\beta = B_\gamma = B_{\text{rot}}) \) |
|------------|---------|--------------|-----------------|-----------------|-----------------|-----------------|
| \( ^{173}\text{Yb} \) | 0.0094  | 1033.26      | 1.3008          | 7.5             | 0.0464          | 0.3643          |
| \( ^{163}\text{Dy} \) | 0.003   | 235.44       | 2.3731          | 7.9063          | 0.0628          | 0.5853          |

Table 1: The values of free parameters fitted to experimental data by using Eqs. \[32\] and \[35\] in the case where \( B_\beta \neq B_\gamma \neq B_{\text{rot}} \) and \( B_\beta = B_\gamma = B_{\text{rot}} \), respectively.
where $n$ denotes the number of considered levels, $E_i(exp)$ and $E_i^{th}$ represent the experimental and theoretical energies of the $i$-th level, respectively. $E(7/2_{g.s.})$ is the g.s. band head energy.

All bands (i.e., ground state, $\beta$ and $\gamma$) are labelled by the quantum numbers $n_\beta, n_\gamma, m, L^\pi, K$ and $j$, such as the ground state band (g.s.) is characterized by $n_\beta=0, n_\gamma=0, m=0$ , the $\beta$-band by $n_\beta=1, n_\gamma=0, m=0$ , and the $\gamma$-band by $n_\beta=0, n_\gamma=1, m=1$, while the appropriate value of the angular momentum of the external nucleon is $j=7/2$. In fact, the shell-model calculations achieved in Ref. [19] predict that the major contribution to the ground state structure of considered nuclei comes from the neutron $f_{7/2}$ orbital. The value of $K$ is fixed to 5/2 for both nuclei in accord with [11].

The obtained optimal parameters for the two considered nuclei are given in Table 1 in both cases, namely: the case where the mass parameters are different and the case where they are equal within and outside the DDM formalism. Together with the energy formula Eq. (32) (Appendix A) and the obtained parameters, we have evaluated the energy spectra of the considered nuclei as well as the corresponding transition rates. But, here we have to bear in mind that our formula Eq. (32) (Appendix A) is a correction to the one previously obtained by Ermamatov et al. [11]. So, the obtained values outside the DDM formalism ($a=0$) are just the results which should be obtained by the authors [11].

The DDM formalism has been introduced in the present work in order to see its impact in both above-cited cases. For that, we recalculated the energy ratios with the more elaborated formula given in Eq. (17). Such an expression contains two supplementary parameters, namely $a$ and $\beta_0$. The optimal values of both parameters are evaluated through r.m.s fits of energy levels by making use of Eq. (29) for each band of each nucleus. The obtained values are summarized in Table 2 and Table 3 for $^{173}$Yb and $^{163}$Dy nuclei, respectively.

From Table 4, where the energy ratios $E(L_{g.s.})/E(7/2_{g.s.}^+), E(7/2_{g.s.}^+)/E(7/2_{g.s.}^+)$ and $E(L_\gamma)/E(7/2_{g.s.}^+)$ of $^{173}$Yb are presented, one can see that outside the DDM formalism ($a=0$), the obtained results with different mass parameters are more precise than those obtained with equal ones. Also, the introduction of the DDM parameter ($a \neq 0$) has improved the obtained results in both cases, namely: $B_\beta \neq B_\gamma \neq B_{rot}$ and $B_\beta = B_\gamma = B_{rot}$ but with keeping the prevalence of the first one. From the same Table, one can observe that all level bands are more sensitive to the effect of the DDM parameter $a$ particularly in the case of equal mass parameters, while when the mass parameters are different, the $\gamma$-band has not been influenced by the DDM parameter.

In Table 5, we present the calculated energy ratios normalized to the first g.s excited level $7/2^-$ for $^{164}$Dy in the g.s. and $\beta$-bands only because the experimental data are not available in the $\gamma$-band. Here again, the obtained results with $B_\beta \neq B_\gamma \neq B_{rot}$ are better than those corresponding to $B_\beta = B_\gamma = B_{rot}$ and the effect of the DDM formalism is more pronounced in the later case.

Similarly, we have calculated the intraband $B(E2; L^\pi_{g.s.} \rightarrow L^\pi_{g.s.})$ and interband $B(E2; L_\beta \rightarrow L^\pi_{g.s.})$ and $B(E2; L_\gamma \rightarrow L^\pi_{g.s.})$ transition probabilities given in Eqs. (26), (27) and (28) (Appendix B), respectively, normalized to transition rate from the first excited level in the g.s. band $B(E2; 9/2_{g.s.} \rightarrow 5/2_{g.s.})$ for the same nuclei in both cases $B_\beta \neq B_\gamma \neq B_{rot}$ and $B_\beta = B_\gamma = B_{rot}$ within and without the DDM formalism. For each nucleus, the parameters obtained by fitting the spectra have been used. The results are shown in Table 6 and 7. Only theoretical calculations of interband transitions rates from the $\beta$ and $\gamma$ bands to g.s. band are presented since there are no experimental values for them as yet. Concerning the intraband transitions within the g.s. band, it is clearly shown that our results in the case of $B_\beta \neq B_\gamma \neq B_{rot}$ are better than those with $B_\beta = B_\gamma = B_{rot}$ and at higher spins the intraband E2 transition probabilities are affected by these differences of mass parameters and showed an overall agreement with the experimental data. However, one can remark that the deformation mass parameter has no effect in the case of different mass coefficients, while for equal mass parameters, its effect is pronounced particularly for interband transitions.

In addition to the above obtained results, here we have to notice the double effect of both formalisms, namely: the DDM and the different mass parameters on the variation of the moment of inertia. It is clear that in the case of the Bohr Hamiltonian with different
deformation-dependent mass parameters as seen from Eq. (6), the moment of inertia is defined by \( (B_{rot} \beta^2 / B_{\beta} f(\beta)^2) \sin^2(\gamma - 2\pi k/3) \). The effect of the function \( B_{rot} \beta^2 / B_{\beta}(1 + a\beta^2)^2 \) on the moment of inertia is shown in Fig. 1 for \( B_{rot} \neq B_{\beta} \) and \( B_{rot} = B_{\beta} \) within (i.e. \( a \neq 0 \)) and without (i.e. \( a = 0 \)) the DDM formalism. It is apparent that the increase of the moment of inertia is slowed down by the function of deformation \( f(\beta) \) (i.e. when \( a \neq 0 \)) and even more in the presence of different mass coefficients (i.e. when \( B_{rot} \neq B_{\beta} \)). Then, the present approach, reduces significantly the rate of increase of the moment of inertia, removing a main important drawback [30] of the model.

7 CONCLUSION

In this paper, we have studied two deformed odd-mass nuclei, \(^{173}\)Yb and \(^{163}\)Dy, in the framework of the Bohr Hamiltonian with deformation-dependent mass coefficients using Davidson potential in \( \beta \) shape and the harmonic oscillator in \( \gamma \) potential. Analytical expressions have been obtained for the excited-state energies, wave functions and E2 transition probabilities. Such formulas are the corrected ones for those previously obtained in [11]. Energy levels of the g.s., \( \beta \) and \( \gamma \) bands with \( K = 5/2 \) as well as the interband transitions rates from the \( \beta \) and \( \gamma \) bands to g.s. band and intraband transitions within g.s. band, were calculated in both cases \( B_{\beta} \neq B_{\gamma} \neq B_{rot} \) and \( B_{\beta} = B_{\gamma} = B_{rot} \) within and without the DDM formalism and compared to available experimental data. Some predictions for transition rates are made where the experimental data are not available. Also, we have studied the effect of the deformation mass parameter on energy spectra and transition rates in both above-cited cases. Moreover, we have shown the importance of the mass parameter to be introduced in numerical calculations, unlike what has been done by other authors who have neglected the important role played by this parameter in such calculations.
| $B_\beta \neq B_\gamma \neq B_{\text{rot}}$ | $a=0$ | DDM | $B_\beta = B_\gamma = B_{\text{rot}}$ | $a=0$ | DDM |
|---|---|---|---|---|---|
| $g.s$ | | | | | |
| $\sigma$ | 0.0399 | 0.0253 | 1.2419 | 0.0235 | |
| $a$ | 0.001 | 0.0881 | | | |
| $\beta_0$ | 6.94 | 1.029 | | | |
| $\beta$ | | | | | |
| $\sigma$ | 0.6839 | 0.6837 | 1.1682 | 1.1682 | |
| $a$ | 0.1724 | 1.94.10^{-11} | | | |
| $\beta_0$ | 0.0913 | 3.8091 | | | |
| $\gamma$ | | | | | |
| $\sigma$ | 0.347 | 0.347 | 1.01 | 0.9828 | |
| $a$ | 1.14.10^{-7} | | | | |
| $\beta_0$ | 0.3703 | 1.8036 | | | |
| $\sigma_{\text{total}}$ | 0.4135 | 0.4134 | 1.0974 | 1.0009 | |
| $a$ | 0.0999 | 0.0037 | | | |
| $\beta_0$ | 0.3251 | 1.3531 | | | |

Table 2: $\beta_0$ and $a$ the position of the minimum of Davidson potential and the deformation dependence of the mass parameter respectively for the $^{173}\text{Yb}$ nucleus, while $\sigma$ is the quality measure Eq[29].

| $B_\beta \neq B_\gamma \neq B_{\text{rot}}$ | $a=0$ | DDM | $B_\beta = B_\gamma = B_{\text{rot}}$ | $a=0$ | DDM |
|---|---|---|---|---|---|
| $g.s$ | | | | | |
| $\sigma$ | 0.0191 | 0.0191 | 1.4517 | 0.0362 | |
| $a$ | 0.0002 | 0.0041 | | | |
| $\beta_0$ | 0.0527 | 5.5363 | | | |
| $\beta$ | | | | | |
| $\sigma$ | 0.2462 | 0.2462 | 1.0349 | 0.2283 | |
| $a$ | 0.00011 | 0.004 | | | |
| $\beta_0$ | 0.0614 | 2.082 | | | |
| $\sigma_{\text{total}}$ | 0.1113 | 0.1113 | 1.2997 | 0.7706 | |
| $a$ | 0.00005 | 0.0072 | | | |
| $\beta_0$ | 1.0456 | 2.2852 | | | |

Table 3: $\beta_0$ and $a$ the position of the minimum of Davidson potential and the deformation dependence of the mass parameter respectively for the $^{163}\text{Dy}$ nucleus, while $\sigma$ is the quality measure Eq[29].
| L     | Exp. [29] | a=0 DDM | a=0 DDM |
|-------|-----------|----------|----------|
| g.s.  |           |          |          |
| 9/2^- | 2.27 2.28 | 2.28     | 2.25 2.28|
| 11/2^-| 3.84 3.84 | 3.84     | 3.74 3.83|
| 13/2^-| 5.68 5.67 | 5.67     | 5.43 5.66|
| 15/2^-| 7.78 7.76 | 7.77     | 7.32 7.76|
| 17/2^-| 10.07 10.12| 10.13   | 9.37 10.12|
| 19/2^-| 12.76 12.73| 12.75   | 11.58 12.74|
| 21/2^-| 15.63 15.58| 15.63   | 13.91 15.62|
| 23/2^-| 18.75 18.68| 18.74   | 16.37 18.76|

| L     | Exp. [29] | a=0 DDM | a=0 DDM |
|-------|-----------|----------|----------|
| β 9/2^-| 13.46 13.44| 13.42   | 14.40 14.40|
| 11/2^-| 14.75 15.00| 14.98   | 15.89 15.89|
| 13/2^-| 16.38 16.83| 16.81   | 17.58 17.58|
| 15/2^-| 19.48 18.92| 18.90   | 19.47 19.47|
| 17/2^-| 20.73 21.28| 21.26   | 21.52 21.52|
| 19/2^-| 23.27 23.89| 23.87   | 23.73 23.73|
| 21/2^-| 27.99 26.74| 26.73   | 26.06 26.06|

| L     | Exp. [29] | a=0 DDM | a=0 DDM |
|-------|-----------|----------|----------|
| γ 9/2^-| 18.59 18.69| 18.69   | 19.94 20.09|
| 11/2^-| 20.61 20.16| 20.16   | 21.01 21.18|
| 13/2^-| 22.22 21.88| 21.88   | 22.26 22.45|
| 15/2^-| 23.40 23.86| 23.86   | 23.67 23.88|
| 17/2^-| 25.85 26.09| 26.09   | 25.23 25.47|
| 19/2^-| 28.56 28.56| 28.56   | 26.93 27.21|

Table 4: Comparison of the theoretical predictions of energy levels of the ground state band, the β and γ bands normalized to the energy of the first excited state $E(7/2^-_{g.s.})$ using the parameters given in Tables 1 and 2 for $^{173}$Yb for this work with experimental values taken from Ref [29].
Table 5: Comparison of the theoretical predictions of energy levels of the ground state and $\beta$ bands normalized to the energy of the first excited state $E(7/2_{g.s.}^-)$ using the parameters given in Tables 1 and 3 for $^{163}$Dy with experimental values taken from Ref [29].

| L   | Exp. | $a = 0$ | $a = 0$ | DDM |
|-----|------|---------|---------|-----|
| g.s |      |         |         |     |
| 9/2^- | 2.28 | 2.28    | 2.24    | 2.28 |
| 11/2^- | 3.84 | 3.83    | 3.71    | 3.83 |
| 13/2^- | 5.66 | 5.66    | 5.37    | 5.64 |
| 15/2^- | 7.75 | 7.74    | 7.20    | 7.72 |
| 17/2^- | 10.13 | 10.08  | 9.18    | 10.06 |
| 19/2^- | 12.68 | 12.67  | 11.28   | 12.65 |
| 21/2^- | 15.49 | 15.51  | 13.50   | 15.50 |
| 23/2^- | 18.56 | 18.57  | 15.82   | 18.60 |

| $\beta$ |      |         |         |     |
|---------|------|---------|---------|-----|
| 5/2^-   | 9.70 | 9.93    | 10.76   | 9.90 |
| 7/2^-   | 10.91 | 10.93   | 11.76   | 10.93 |
| 9/2^-   | 12.47 | 12.21   | 13.01   | 12.22 |
| 11/2^-  | 13.77 |         | 14.47   | 13.75 |
| 13/2^-  | 15.59 |         | 16.13   | 15.49 |
| 15/2^-  | 17.68 |         | 17.96   | 17.44 |
| 17/2^-  | 20.02 |         | 19.94   | 19.56 |
| 19/2^-  | 22.61 |         | 22.04   | 21.85 |
| 21/2^-  | 25.44 |         | 24.26   | 24.28 |
| 23/2^-  | 28.50 |         | 26.58   | 26.85 |
| 25/2^-  | 31.79 |         | 28.98   | 29.55 |
\[
\begin{array}{c|c|c|c|c|c}
\hline
B_\beta \neq B_\gamma \neq B_{rot} & B_{\beta} & B_{\gamma} = B_{rot} \\
\hline
B(E2; L_{g.s.} \rightarrow L_{g.s.}) & \text{Exp. [29]} & a = 0 & DDM & a = 0 & DDM \\
\hline
11/2^- \rightarrow 7/2^- & 2.03 & 1.70 & 1.74 & 1.72 \\
13/2^- \rightarrow 9/2^- & 2.06 & 2.18 & 2.27 & 2.23 \\
15/2^- \rightarrow 11/2^- & 2.31 & 2.51 & 2.68 & 2.60 \\
17/2^- \rightarrow 13/2^- & 2.93 & 2.76 & 3.02 & 2.89 \\
21/2^- \rightarrow 17/2^- & 3.21 & 3.10 & 3.60 & 3.33 \\
23/2^- \rightarrow 19/2^- & 3.26 & 3.23 & 3.86 & 3.51 \\
25/2^- \rightarrow 21/2^- & 3.37 & 3.34 & 4.11 & 3.66 \\
\hline
\end{array}
\]

\[
\begin{array}{c|c|c|c|c|c}
\hline
B(E2; L_{\beta} \rightarrow L_{g.s.}) & B(E2; L_{\gamma} \rightarrow L_{g.s.}) \times 10^3 \\
\hline
9/2^- \rightarrow 5/2^- & 1.74 & 11.81 & 11.81 \\
13/2^- \rightarrow 9/2^- & 1.66 & 14.47 & 14.47 \\
17/2^- \rightarrow 13/2^- & 0.55 & 9.78 & 9.78 \\
9/2^- \rightarrow 9/2^- & 0.76 & 4.65 & 4.65 \\
13/2^- \rightarrow 13/2^- & 4.82 & 29.55 & 29.55 \\
17/2^- \rightarrow 17/2^- & 7.45 & 45.81 & 45.81 \\
5/2^- \rightarrow 9/2^- & 16.63 & 16.61 & 97.81 & 97.81 \\
9/2^- \rightarrow 13/2^- & 39.24 & 39.21 & 220.25 & 220.25 \\
13/2^- \rightarrow 17/2^- & 56.40 & 56.35 & 296.76 & 296.77 \\
\hline
\end{array}
\]

\[
\begin{array}{c|c|c|c|c|c}
\hline
B(E2; L_{\beta} \rightarrow L_{\gamma}) & B(E2; L_{\gamma} \rightarrow L_{g.s.}) \times 10^3 \\
\hline
9/2^- \rightarrow 5/2^- & 9.19 & 41.03 & 41.30 \\
9/2^- \rightarrow 9/2^- & 1.42 & 6.79 & 6.82 \\
9/2^- \rightarrow 13/2^- & 28.84 & 150.41 & 150.92 \\
11/2^- \rightarrow 9/2^- & 21.66 & 102.63 & 103.23 \\
11/2^- \rightarrow 13/2^- & 19.95 & 103.17 & 103.57 \\
13/2^- \rightarrow 9/2^- & 19.91 & 93.00 & 93.61 \\
13/2^- \rightarrow 13/2^- & 9.10 & 46.53 & 46.75 \\
13/2^- \rightarrow 17/2^- & 33.70 & 188.55 & 189.04 \\
15/2^- \rightarrow 13/2^- & 13.29 & 67.06 & 67.42 \\
\hline
\end{array}
\]

Table 6: Comparison of the theoretical predictions of B(E2) in units of \(B(E2; 9/2_{g.s.} \rightarrow 5/2_{g.s.})\) using the parameters given in Tables 1 and 2 for \(^{173}\text{Yb}\) in this work with experimental values [29].
\[ B_\beta \neq B_\gamma \neq B_{\text{rot}} \]

| \( B(E2; L'_{g.s.} \rightarrow L_{g.s.}) \) | Exp. [29] | \( a = 0 \) DDM | \( a = 0 \) DDM |
|------------------------------------------|----------|-----------------|-----------------|
| \( 11/2^- \rightarrow 7/2^- \)          | 1.63     | 1.71            | 1.75            |
|                                          |          |                 | 1.73            |
| \( 13/2^- \rightarrow 9/2^- \)          | 2.04     | 2.18            | 2.30            |
|                                          |          |                 | 2.25            |
| \( 15/2^- \rightarrow 11/2^- \)         | 2.80     | 2.52            | 2.73            |
|                                          |          |                 | 2.63            |
| \( 17/2^- \rightarrow 13/2^- \)         | 2.44     | 2.77            | 3.10            |
|                                          |          |                 | 2.94            |
| \( 19/2^- \rightarrow 15/2^- \)         | 2.6      | 2.96            | 3.43            |
|                                          |          |                 | 3.20            |
| \( 21/2^- \rightarrow 17/2^- \)         | 2.44     | 3.12            | 3.73            |
|                                          |          |                 | 3.41            |
| \( 23/2^- \rightarrow 19/2^- \)         | 2.28     | 3.26            | 4.03            |
|                                          |          |                 | 3.61            |

| \( B(E2; L'_{g.s.} \rightarrow L_{g.s.}) \) \( \times 10^3 \) | \( 9/2^- \rightarrow 5/2^- \) | 1.55 | 11.17 | 15.02 |
|---------------------------------------------------------------|--------------------------------|------|-------|-------|
| \( 13/2^- \rightarrow 9/2^- \)                               | 1.17                           | 12.36| 19.70 |       |
| \( 9/2^- \rightarrow 9/2^- \)                                 | 0.81                           | 5.16 | 5.85  |       |
| \( 13/2^- \rightarrow 13/2^- \)                               | 5.13                           | 5.12 | 32.83 | 37.49 |
| \( 17/2^- \rightarrow 17/2^- \)                               | 7.93                           | 50.91| 58.62 |       |
| \( 9/2^- \rightarrow 13/2^- \)                                | 45.67                          | 45.66| 264.08| 271.78|
| \( 13/2^- \rightarrow 17/2^- \)                               | 66.47                          | 66.45| 354.9 | 360.08|

Table 7: Comparison of the theoretical predictions of \( B(E2) \) in units of \( B(E2; 9/2^-_{g.s.} \rightarrow 5/2^-_{g.s.}) \) using the parameters given in Tables 1 and 3 for \( ^{163}\text{Dy} \) in this work with experimental values [29].
A Special cases of energy spectrum

Special case 1: Without mass coefficients
If we assume \( B_\beta = B_\gamma = B_{\text{rot}} \) and \( \bar{h} = 1 \), we get from Eq. (19)

\[
\begin{align*}
    k_2 &= a^2 \left[ 5(1 - \delta - \lambda) + (1 - 2\delta)(1 - 2\lambda) + 6 + \Lambda \right] + \frac{2V_0}{\beta_0^2}, \\
    k_0 &= a \left[ 5(1 - \delta - \lambda) + 8 + 2\Lambda \right] - 4V_0, \\
    k_{-2} &= 2 + \Lambda + 2V_0\beta_0^2.
\end{align*}
\]

(30)

Consequently, the energy spectrum formula Eq. (17) is identical to Eq. (82) of Ref. [13] obtained by means of supersymmetric quantum mechanical method (SUSYQM) [27, 28]. The slight difference between our coefficients \( k_2, k_0 \) and \( k_{-2} \) and those of Ref. [13] comes from the adopted expression of Davidson potential.

Special case 2: No dependence of the mass on the deformation
If \( a = 0 \), i.e., the dependence of the mass on the deformation is canceled, then one has from Eq. (19)

\[
\begin{align*}
    k_2 &= \frac{2g_\beta}{\beta_0^3}, \\
    k_0 &= -\frac{4g_\beta}{\beta_0^3}, \\
    k_{-2} &= \frac{B_\beta}{B_\gamma} \left( 1 + \frac{B_\beta}{B_\gamma} \right) + \frac{B_\beta}{\bar{h}^2} \Lambda + 2g_\beta.
\end{align*}
\]

(31)

In this case, the energy spectrum formula reads

\[
E_{n, \beta n, L|m|} = \sqrt{2 \left( \frac{V_0^2}{g_\beta} \left[ 1 + 2n_\beta + \frac{1}{2} q_{\beta n} (L, |m|) - \sqrt{\frac{V_0^2}{g_\beta}} \right] \right)}.
\]

(32)

with

\[
\frac{1}{2} q_{\beta n} (L, |m|) = \sqrt{\frac{1}{4} + \frac{B_\beta}{B_\gamma} \left( 1 + \frac{B_\beta}{B_\gamma} \right) + \frac{B_\beta}{\bar{h}^2} \Lambda + 2g_\beta},
\]

(33)

and

\[
\begin{align*}
    \frac{B_\beta}{\bar{h}^2} \Lambda &= \frac{2}{g} \frac{B_\beta}{B_\gamma} (1 + n_\gamma) + \frac{m^2}{3} \frac{B_\beta}{B_\gamma} + \frac{1}{3} \frac{B_\beta}{B_{\text{rot}}} \left( L(L + 1) \\
    &+ j(j + 1) - K^2 - (K - 2m)^2 \right) \\
    &- \frac{1}{3\xi} \left( 3(K - 2m)^2 - j(j + 1) \right),
\end{align*}
\]

(34)

note that Eq. (32) represents the correct formula of the energy spectrum, compared to Eq. (11) given in Ref. [11], where the mass parameter term is missed in the analog formula of Eq. (33).

Special case 3: Standard case
In the case of \( a = 0 \) and \( B_\beta = B_\gamma = B_{\text{rot}} \), our formula Eq. (17) becomes

\[
E_{n, \beta n, L|m|} = \sqrt{2 \left( \frac{V_0^2}{g_\beta} \left[ 1 + 2n_\beta + \sqrt{\frac{9}{4} + \Lambda + 2g_\beta} \right] - 2V_0 \right)}.
\]

(35)

the Eq. (35) represents the correct formula of the energy spectrum, compared to Eq. (13) given in Ref. [17].
B Formulas used for the calculations of the B(E2) transitions

In this Appendix we present the expressions used for calculations of the transitions probabilities B(E2):

\[
\frac{B(E2; L^{'}, \text{g.s.} \rightarrow L^{'}, \text{g.s.})}{B(E2; \frac{9}{2}\text{g.s.} \rightarrow \frac{7}{2}\text{g.s.})} = \left( \frac{(L^{'}, 2K0|LK)}{(L, 2\frac{2}{2}|L, 2\frac{2}{2})} \right)^2 \\
\times \left( \frac{\Gamma[0.25(q_0(L^{'}, 0) + q_0(L, 0)) + 1.5]}{\Gamma[0.25(q_0(\frac{9}{2}, 0) + q_0(\frac{7}{2}, 0)) + 1.5]} \right)^2 \\
\times \left( \frac{\Gamma[0.5q_0(\frac{9}{2}, 0) + 1][\Gamma[0.5q_0(\frac{7}{2}, 0) + 1]}{\Gamma[0.5q_0(L^{'}, 0) + 1][\Gamma[0.5q_0(L, 0) + 1]} \right),
\]

\[
\frac{B(E2; L^{'}, \text{g.s.} \rightarrow L^{'}, \text{g.s.})}{B(E2; \frac{9}{2}\text{g.s.} \rightarrow \frac{7}{2}\text{g.s.})} = \frac{1}{4} \times \left( \frac{(L^{'}, 2K0|LK)}{(L, 2\frac{2}{2}|L, 2\frac{2}{2})} \right)^2 \\
\times \left( \frac{\Gamma[0.25(q_0(L^{'}, 0) + q_0(L, 0)) + 1.5]}{\Gamma[0.25(q_0(\frac{9}{2}, 0) + q_0(\frac{7}{2}, 0)) + 1.5]} \right)^2 \\
\times \left( \frac{\Gamma[0.5q_0(\frac{9}{2}, 0) + 1][\Gamma[0.5q_0(\frac{7}{2}, 0) + 1]}{\Gamma[0.5q_0(L^{'}, 0) + 1][\Gamma[0.5q_0(L, 0) + 1]} \right) \\
\times (0.5q_0(L^{'}, 0) - 0.5q_0(L, 0) - 1)^2 \\
\times 0.5q_0(L^{'}, 0) + 1
\]

\[
\frac{B(E2; L^{'}, \text{g.s.} \rightarrow L^{'}, \text{g.s.})}{B(E2; \frac{9}{2}\text{g.s.} \rightarrow \frac{7}{2}\text{g.s.})} = \frac{g}{4} \times \left( \frac{(L^{'}, 2K0|LK)}{(L, 2\frac{2}{2}|L, 2\frac{2}{2})} \right)^2 \\
\times \left( \frac{\Gamma[0.25(q_1(L^{'}, 1) + q_0(L, 0)) + 1.5]}{\Gamma[0.25(q_0(\frac{9}{2}, 0) + q_0(\frac{7}{2}, 0)) + 1.5]} \right)^2 \\
\times \left( \frac{\Gamma[0.5q_0(\frac{9}{2}, 0) + 1][\Gamma[0.5q_0(\frac{7}{2}, 0) + 1]}{\Gamma[0.5q_1(L^{'}, 1) + 1][\Gamma[0.5q_0(L, 0) + 1]} \right),
\]

where \((L^{'}, 2K0|LK)\) is Clebsch-Gordan coefficient.
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