Chapter 7
Gravitational Radiation Reaction
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Abstract
In this chapter, we consider the radiation reaction to the motion of a point-like particle of mass $m$ and specific spin $S$ traveling on a curved background. Assuming $S = O(Gm)$ and $Gm \ll L$ where $L$ is the length scale of the background curvature, we divide the spacetime into two regions; the external region where the metric is approximated by the background metric plus perturbations due to a point-like particle and the internal region where the metric is approximated by that of a black hole plus perturbations due to the tidal effect of the background curvature, and use the technique of the matched asymptotic expansion to construct an approximate metric which is valid over the entire region. In this way, we avoid the divergent self-gravity at the position of the particle and derive the equations of motion from the consistency condition of the matching. The matching is done to the order necessary to include the effect of radiation reaction of $O(Gm)$ with respect to the background metric as well as the effect of spin-induced force. The reaction term of $O(Gm)$ is found to be completely due to tails of radiation, that is, due to curvature scattering of gravitational waves. In other words, the reaction force is found to depend on the entire history of the particle trajectory. Defining a regularized metric which consists of the background metric plus the tail part of the perturbed metric, we find the equations of motion reduce to the geodesic equation on this regularized metric, except for the spin-induced force which is locally expressed in terms of the curvature and spin tensors. Some implications of the result and future issues are briefly discussed.

§1. Introduction

The problem of radiation reaction has long been one of the fundamental theoretical issues in general relativity. Starting from the historical works of Eddington in his 1922 book\textsuperscript{1}, Chandrasekhar and Esposito\textsuperscript{2} discussed the radiation reaction of the self-gravitating fluid emphasizing the importance of the time asymmetric part of the metric appearing in the post-Newtonian expansion, and Burke and Thorne\textsuperscript{3} found that the leading contribution from the time asymmetric part can be compactly expressed in the form of a resistive potential.

The previous studies of radiation reaction\textsuperscript{2,3} were done under the assumption that the post-Newtonian expansion is valid. Here we consider this problem in the framework of linear perturbation theory in a general spacetime. A part of motivation is to give a rigid foundation of the method to solve the Einstein equations perturbatively as an expansion with respect to the perturbation caused by a point-like particle. Usually one adopts a point-like particle to represent a black hole or
neutron star in the linear perturbation studies as was done in Chapter 1. Then one
may pose several questions: Since the perturbed field diverges at the location of the
point-like particle, is the approximation scheme of linear perturbation still valid?
Does the point-like particle really represent a black hole or a neutron star? If it
represents a black hole, the center of it is inside the event horizon, and then in what
sense does ‘the motion of the particle’ make sense? Here we are going to clarify the
meaning of the particle trajectory and derive the equations of motion including the
effect of radiation reaction to the first non-trivial order.

Before starting the discussion of the gravitational radiation reaction, it is worth-
while to refer to the electromagnetic case in a fixed curved background spacetime
which was discussed by DeWitt and Brehme\(^4\). In the electromagnetic case, the total
energy momentum tensor composed of the particle and field contributions satisfies
the conservation law. The conservation law is integrated over the interior of a world
tube with an infinitesimal length surrounding the particle orbit. The part of the
integration which does not vanish in the limit of small tube radius is transformed
into the surface integrations over both ends of the tube and over the surface of the
tube by using the Gauss theorem. The integrations over the top and bottom of
the tube, respectively, give the definition of the particle momenta at both ends and
the difference between them represents the change of the momentum during this
infinitesimal time interval, which is to be equated with the momentum flow given by
the integration over the surface of the tube. In this way the equations of motion are
obtained.

In the case of gravitational radiation reaction, it is possible to construct a con-
served rank-two tensor defined on the background spacetime, composed of the matter
field and the metric perturbation\(^5\). However, there is an essential difference between
the electromagnetic and gravitational cases. In electromagnetism, we can consider
an extended charge distribution which is supported by a certain force other than
the electromagnetic field. Thus it is possible to assume that the charge and mass
distributions of a point-like particle are not distorted by the effect of the radiation re-
action. Therefore one may consistently assume that the momentum and the electric
current of the particle are proportional to the 4-velocity of the particle. Moreover
the electromagnetic charge \(e\) is not directly related to the energy momentum of the
particle which is proportional to the mass \(m\). Hence, even if the limit of zero particle
radius is taken, the divergent self-energy \((\propto e^2)\) can be renormalized into the mass.
In the case of gravitational radiation reaction, it is not possible to consider such an
ideal point-like particle because every force field universally couples with gravity.
Even worse, the role of \(e\) in electromagnetism is also attributed to \(m\). Thus a simple
renormalization scheme does not make any sense.

In order to deal with the gravitational case, we use the matched asymptotic ex-
pansion technique that has been studied by many authors (e.g., D’Eath\(^6\) and Thorne
and Hartle\(^7\)) in the context of the post-Minkowski (or post-Newtonian) approxima-
tion. We assume that the metric sufficiently far from the particle is approximated
by the perturbation on the background spacetime generated by a point-like particle.
We call this the external metric. We also assume that the internal metric which
describes the geometry around the particle is represented by a black hole metric of
small mass in the lowest order approximation. As the particle moves in the curved background, the internal metric suffers from the tidal distortion. Thus both internal and external metrics are constructed perturbatively. The expansion parameters for the internal and external metrics are, however, different. We call this construction of the metric in the internal region the internal scheme and that in the external region the external scheme. Assuming the existence of the matching region where both schemes are valid, the internal and external metrics are expanded there as double series with respect to the two expansion parameters. Then the terms in these series are labeled by two indices which denote the powers of the two expansion parameters. Equating them order by order, we obtain the matching condition, through which one scheme determines the boundary condition of the other and vice versa.

Using the matched asymptotic expansion to the first non-trivial orders of the expansion parameters, we present two different derivations of the equations of motion with the radiation reaction force of \( O(Gm) \); (1) by means of an explicit construction of the metric, and (2) by using the so-called laws of motion and precession.\(^7\)

As mentioned above, in constructing the internal metric, the tidal distortion of the geometry is taken into account by the perturbation of the black hole. In the method (1), we set the gauge condition in the internal metric so that \( J = 0 \) and 1 linear homogeneous perturbations of the black hole vanish since they are purely gauge degrees of freedom as long as both the mass and angular momentum of the black hole stay constant. Applying a limited class of coordinate transformations that keep the meaning of the center of the particle unambiguous, the internal metric is matched with the external one in the matching region. Then we find that for a given trajectory of the particle a consistent coordinate transformation does not always exist, and this consistency condition determines the equations of motion.

In the method (2), not all the metric components are evaluated in both schemes independently but we assume the existence of a coordinate transformation that gives a relation between the internal and external metrics. Once we know some metric components in one scheme, the counter parts in the other scheme are obtained from the matching condition. At this stage, the gauge condition is not fixed in a unique manner. The coordinate transformation between the internal metric and the external metric is chosen so that some of the metric components that are evaluated in both schemes are correctly matched in the matching region. Substituting the metric constructed in this way into the Einstein equations, we obtain the consistency condition. There is a convenient method to extract out the information about the equations of motion from the consistency condition. Namely to use the laws of motion and precession introduced by Thorne and Hartle.\(^5\) The laws of motion and precession are derived from the non-covariant but conserved form of the Einstein equations.

The resulting equations obtained from both derivations are the same, although the strategies are quite different. In the method (1), the metrics in both schemes are calculated independently by using the Einstein equations. The matching condition is used to obtain the consistency conditions, which in turn give the equations of motion. On the other hand, in the method (2), the matching condition is used to construct the metric. The consistency condition is derived by requiring that thus obtained metric satisfies the Einstein equations. The meaning of the matching condition in
deriving the equations of motion is clearer in the method (1) than in (2), but the method (2) is much simpler and straightforward than the method (1) as we shall see in the following.

The organization of this chapter is as follows. We use the terminology ‘a monopole (spinning) particle’ to refer to a particle which represents a Schwarzschild (Kerr) black hole. In section 2, the matched asymptotic expansion technique is explained in detail. In section 3, we discuss the metric perturbation in the external scheme. In section 4, the equations of motion for a monopole particle are derived by using the method (1). The method (1) is applied only to the case of a monopole particle because of the difficulty in constructing the perturbed metric of a Kerr black hole. The case for a spinning particle is considered in section 5 by using the method (2).

Throughout this chapter we assume that the background metric satisfies the vacuum Einstein equations\(^(*)\). Hence in the following calculations we use the fact that the background Ricci tensor vanishes;

\[
R_{\mu\nu} = 0. \tag{1.1}
\]

§2. Matched Asymptotic Expansion

The matched asymptotic expansion is a technique with which the same physical quantities derived in different zones by two different approximation schemes are matched in the overlapping region to obtain an approximate solution valid in the whole region. We first prepare the metrics in both internal and external zones by using different approximation schemes. The internal zone is the region where the self-gravity of the particle dominates while the external zone is the region where the background geometry dominates the full geometry.

In the internal zone, we assume that the metric can be described by that of a black hole plus perturbation. Namely, we assume that the particle is represented by a Schwarzschild/Kerr black hole in the lowest order of approximation. In the present case, the perturbation is caused by the tidal effect of the curvature of the spacetime in which the particle travels. As mentioned in Introduction, we call this construction of the metric the internal scheme. In order to make this scheme valid, the linear extension of the internal zone around the particle must be much smaller than the background curvature scale \(L\). We introduce the coordinate \(\{X^a\} = \{T, X^i\} \quad (a = 0, 1, 2, 3; \ i = 1, 2, 3)\) for the internal scheme and \(|X| := \sqrt{X^i X^i}\) is assumed to represent the physical distance scale\(^(**)\). Then the internal scheme is valid when

\[
|X| \ll L, \quad \tag{2.1}
\]

where \(L\) is the length scale of the background curvature.

\(^(*)\) The result is not altered even if we assume that the background spacetime is vacuum just around the particle.

\(^(**)\) In this chapter, we adopt the Minkowskian summation rule on \(a, b, \cdots\), and the Kronecker summation rule on \(i, j, \cdots\) over the repeated indices.
In the external zone, we expect that the metric is well approximated by the perturbation induced by a point source on a given background spacetime. We call this construction of the metric the external scheme. This approximation scheme is valid when the self-gravity of the particle is sufficiently weak, that is,

$$Gm \ll |X|,$$

where \((Gm)\) is the scale of Schwarzschild radius. As the point source is placed where the external scheme is invalid, there is no matter source in the external zone. Thus the external metric is given by a vacuum solution of the Einstein equations.

We require that the metrics obtained in both schemes be matched in the overlapping region of both zones, by considering a coordinate transformation between the internal and external metrics. Safely, we may assume the existence of the matching region as long as

$$Gm \ll L,$$

is satisfied. For definiteness, we set the matching radius at

$$|X| \sim (GmL)^{1/2},$$

in the spatial coordinates of the internal scheme, \(X^i\). Then writing down the metric in the internal scheme, we have two independent small parameters \(|X|/L\) and \(Gm/|X|\) in the matching region. The power expansion with respect to these two small parameters allows us to consider the matching order by order.

First we consider the expansion of the internal scheme. Recalling that the perturbation in the internal zone is induced by the external curvature which has a characteristic length scale \(L\), the metric can be expanded in powers of \(|X|/L\) as

$$\tilde{g}_{ab}(X) = \left. (0) H_{ab}(X) + \frac{1}{L} (1) H_{ab}(X) + \frac{1}{L^2} (2) H_{ab}(X) + \cdots \right|_{X^i},$$

where \(\left. (0) H_{ab}(X) \right|_{X^i}\) is the unperturbed black hole metric. We expect that \(\left. (1) H_{ab}(X) \right|_{X^i}\) will be given by the standard linear perturbation of the black hole. Later, we find that \(\left. (1) H_{ab}(X) \right|_{X^i}\) can be consistently set to zero, which is in accordance with the notion that the spacetime curvature is of \(O(1/L^2)\). Thus the standard black hole perturbation theory applies up to \(\left. (2) H_{ab}(X) \right|_{X^i}\). Further we expand the metric with respect to \(Gm/|X|\) which is also small at the matching radius:

$$\left. (0) H_{ab}(X) \right|_{X^i} = \eta_{ab} + Gm \left. (0) H_{ab}(X) + (Gm)^2 \left. (2) H_{ab}(X) + \cdots \right|_{X^i}, \right.$$

$$\left. \frac{1}{L} (1) H_{ab}(X) \right|_{X^i} = \frac{1}{L} \left. (0) H_{ab}(X) + \frac{Gm}{L} \left. (1) H_{ab}(X) + \frac{(Gm)^2}{L} \left. (2) H_{ab}(X) + \cdots \right|_{X^i}, \right.$$

$$\left. \frac{1}{L^2} (2) H_{ab}(X) \right|_{X^i} = \frac{1}{L^2} \left. (0) H_{ab}(X) + \frac{Gm}{L^2} \left. (1) H_{ab}(X) + \frac{(Gm)^2}{L^2} \left. (2) H_{ab}(X) + \cdots \right|_{X^i}.$$

Note that, from the definitions of the expansion parameters, the \(\binom{m}{n} H_{ab}\) component of the metric behaves as

$$\binom{m}{n} H_{ab} \sim |X|^{m-n}.$$
The explicit form of the coordinate transformation from the general coordinates of a background metric \( \{ x^\mu \} \) to the coordinates of the internal scheme \( \{ X^a \} \) will be discussed in section 3 for the method (1) and in section 5 for the method (2). Assuming the matching can be consistently done, the full metric in the external scheme \( \tilde{g}_{\mu\nu}(x) \) is written in terms of the internal coordinates as

\[
\tilde{g}_{ab}(X) dX^a dX^b = \bar{g}_{\mu\nu}(x) dx^\mu dx^\nu. \tag{2.8}
\]

Generally, as the external metric can be expanded by \( Gm/|X| \), we write it as

\[
\tilde{g}_{ab}(X) = g_{ab}(X) + Gm(1)h_{ab}(X) + (Gm)^2(2)h_{ab}(X) + \cdots. \tag{2.9}
\]

Then \( Gm(1)h_{ab}(X) \) can be recognized as the linear perturbation on the background \( g_{ab}(X) \). Further we expand it with respect to \( |X|/L \) as

\[
g_{ab}(X) = \left( 0 \right) h_{ab}(X) + \frac{1}{L} \left( 1 \right) h_{ab}(X) + \frac{1}{L^2} \left( 2 \right) h_{ab}(X) + \cdots,
\]

\[
Gm(1)h_{ab}(X) = Gm(0)h_{ab}(X) + \frac{Gm(1)}{L} h_{ab}(X) + \frac{Gm(2)}{L^2} h_{ab}(X) + \cdots,
\]

\[
(Gm)^2(2)h_{ab}(X) = (Gm)^2(0)h_{ab}(X) + \frac{(Gm)^2(1)}{L} h_{ab}(X) + \frac{(Gm)^2(2)}{L^2} h_{ab}(X) + \cdots. \tag{2.10}
\]

As before,

\[
\left( m \atop n \right) h_{ab} \sim |X|^{m-n}. \tag{2.11}
\]

For brevity, we call \( \left( m \atop n \right) h_{ab} \) or \( \left( m \atop n \right) H_{ab} \) the \( (m,n) \)-component and the matching condition for them as the \( (m,n) \)-matching. In the matching region \( (|X| \sim (GmL)^{1/2}) \), the \( (m,n) \)-component is of \( O \left( (Gm/L)^{(m+n)/2} \right) \). The matching condition requires that all the corresponding terms in Eqs. (2.6) and (2.10) should be identical. Then the matching condition is given by equating the terms of the same power in \( |X| \) in both schemes to desired accuracy. Thus the condition for the \( (m,n) \) matching is

\[
\sum_{m' - n' = m - n} \frac{(Gm)^n'}{L^{m'}} \left( m' \atop n' \right) h_{ab} = \sum_{m' - n' = m - n} \frac{(Gm)^n'}{L^{m'}} \left( m' \atop n' \right) H_{ab} + O \left( \frac{(Gm)^{n+1}}{L^{m+1}} |X|^{(m-n)} \right). \tag{2.12}
\]

§3. External Scheme

As we assume that the gravitational radius of the particle, \( Gm \), is small compared with the length scale of the background curvature, \( L \), we approximate \( \delta g_{\mu\nu} \) by the linear perturbation induced by a point-like particle, \( h_{\mu\nu} \), in the whole spacetime region except for the vicinity of the world line of the particle. The calculation is performed in an analogous manner to the case of the scalar and vector perturbations developed by DeWitt and Brehme.\footnote{DeWitt and Brehme.}
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We take a Green function approach to study the linear perturbation of the metric generated by a point source. In order to calculate the tensor Green function in a background covariant manner, we begin with introducing the concept of bi-tensors.

3.1. Bi-tensor formalism

Bi-tensors are tensors which depend on two distinct spacetime points, say, \( x \) and \( z \), so that they can have two types of indices. The simplest example is given by a direct product of tensors at the points \( x \) and \( z \) as

\[
A^{\mu\alpha}(x, z) = B^\mu(x)C^\alpha(z).
\]

In what follows, we use \( x \) for a field point and \( z \) for a point on the particle trajectory, and assign the letters \( \alpha, \beta, \gamma, \delta, \epsilon, \zeta, \eta \) for the tensor indices of \( z \) and \( \mu, \nu, \xi, \rho, \sigma \) for \( x \).

Basic bi-tensors used in our calculations are half the squared geodetic interval \( \sigma(x, z) \),

\[
\sigma(x, z) = \frac{1}{2}g^{\mu\nu}(x)\sigma_{,\mu}(x, z)\sigma_{,\nu}(x, z) = \frac{1}{2}g^{\alpha\beta}(z)\sigma_{,\alpha}(x, z)\sigma_{,\beta}(x, z),
\]

and the geodetic parallel displacement bi-vector,

\[
\bar{g}_{\mu\alpha;\nu}(x, z)g^{\mu\sigma}(x)\sigma_{,\sigma}(x, z) = 0, \quad \bar{g}_{\mu\alpha;\beta}(x, z)g^{\beta\gamma}(z)\sigma_{,\gamma}(x, z) = 0,
\]

\[
\lim_{x \to z}\bar{g}_{\mu\alpha}(x, z) = \delta^\mu_\alpha.
\]

These are used to expand bi-tensors around the orbit of a particle. For example, we have

\[
A^\alpha(x, z) = \lim_{x' \to z}\left( A^\alpha(x', z) - \sigma_{,\mu'}(x, x')A^{\alpha;\mu'}(x', z) + O(\epsilon^2) \right),
\]

\[
B^\mu(x) = \bar{g}_\mu^\alpha(x, z)\left( B^\alpha(z) - \sigma_{,\beta}(x, z)B^{\alpha;\beta}(z) + O(\epsilon^2) \right),
\]

for a small geodetic interval between \( x \) and \( z \), where \( \epsilon = \sqrt{2|\sigma(x, z)|} \). These relations can be verified by taking the \( x \to z \) limit of their repeated derivatives.

By evaluating the repeated derivatives of Eqs. (3.2) and (3.3) in the coincidence limit \( x \to z \), we obtain some useful formulas for expansion in \( \epsilon \):

\[
\sigma_{,\alpha\beta}(x, z) = g_{\alpha\beta}(z) - \frac{1}{3}R^\gamma_{\alpha\beta\delta}(z)\sigma_{,\gamma}(x, z)\sigma_{,\delta}(x, z) + O(\epsilon^3),
\]

\[
\sigma_{,\mu\beta}(x, z) = -\bar{g}_\mu^\alpha(x, z)\left( g_{\alpha\beta}(z) + \frac{1}{6}R_{\alpha\beta\gamma\delta}(z)\sigma^{\gamma}(x, z)\sigma^{\delta}(x, z) \right) + O(\epsilon^3),
\]

\[
\bar{g}_{\mu\alpha;\beta}(x, z) = -\frac{1}{2}g^{\mu\gamma}(x, z)R^\alpha_{\gamma\beta\delta}(z)\sigma^{\delta}(x, z) + O(\epsilon^2),
\]

\[
\bar{g}_{\mu\alpha;\nu}(x, z) = -\frac{1}{2}g^{\mu\beta}(x, z)g^\nu_\gamma(x, z)R^\alpha_{\beta\gamma\delta}(z)\sigma^{\delta}(x, z) + O(\epsilon^2).
\]
We also introduce the van Vleck-Morette determinant, $\Delta(x, z)$:

$$\Delta(x, z) := |\delta^{\alpha\mu}(z, x)\sigma_{\mu\beta}(x, z)|,$$

(3.9)

which appears in the expression of the Green function later.

### 3.2. Tensor Green function

We consider the linearized Einstein equations. We introduce the trace-reversed metric perturbation,

$$\psi_{\mu\nu}(x) = h_{\mu\nu}(x) - \frac{1}{2}g_{\mu\nu}(x)h(x),$$

(3.10)

and set the harmonic gauge condition,

$$\psi^\mu_\nu,\nu(x) = 0,$$

(3.11)

where $h(x)$ and $\psi(x)$ are the trace of $h_{\mu\nu}(x)$ and that of $\psi_{\mu\nu}(x)$, respectively, and the semicolon means the covariant derivative with respect to the background metric. In this gauge, the linearized Einstein equations become

$$-\frac{1}{2}\psi_{\mu\nu,\xi\xi}(x) - R_{\xi\nu \rho}(x)\psi_{\xi\rho}(x) = 8\pi G T_{\mu\nu}(x).$$

(3.12)

Thus we define the tensor Green function $G^{\mu\nu\alpha\beta}(x, z)$ which satisfies

$$G^{\mu\nu\alpha\beta,\xi\xi}(x, z) + 2R_{\xi\nu \rho}(x)G^{\xi\rho\alpha\beta}(x, z)$$

$$= -2\tilde{g}^{\alpha}(\mu(x, z))\tilde{g}^{\beta}(\nu(x, z))\delta^{(4)}(z - x)\sqrt{-g},$$

(3.13)

where $g$ is the determinant of the metric $g_{\mu\nu}(x)$.

First we consider the elementary solution $G^{\mu\nu\alpha\beta}_*(x, z)$ which satisfies Eq. (3.13) except at the $\sigma(x, z) \to 0$ limit and takes the Hadamard form,

$$G^{\mu\nu\alpha\beta}_*(x, z) = \frac{1}{(2\pi)^2} \left( \frac{u^{\mu\nu\alpha\beta}(x, z)}{\sigma(x, z)} + w^{\mu\nu\alpha\beta}(x, z) \log |\sigma(x, z)| \right)$$

$$+ w^{\mu\nu\alpha\beta}(x, z).$$

(3.14)

The bi-tensors $u^{\mu\nu\alpha\beta}(x, z), w^{\mu\nu\alpha\beta}(x, z)$ and $w^{\mu\nu\alpha\beta}(x, z)$ are regular in the $\sigma(x, z) \to 0$ limit and $u^{\mu\nu\alpha\beta}(x, z)$ satisfies the normalization condition,

$$\lim_{x \to z} u^{\mu\nu\alpha\beta}(x, z) = \lim_{x \to z} 2\tilde{g}^{\alpha}(\mu(x, z))\tilde{g}^{\beta}(x, z).$$

(3.15)

If we put the form (3.14) into the left hand side of Eq. (3.13), the terms can be classified into three parts. One is the terms which contain the factor $1/\sigma^2(x, z)$ manifestly and another is the terms which contain $\log |\sigma(x, z)|$. The remaining terms have no
singular behavior at the \( \sigma(x, z) \to 0 \) limit. Since the form (3.14) is redundant, we can set these three sets to vanish separately:

\[
2u^{\mu \nu \alpha \beta}(x, z) - \frac{\Delta \xi(x, z)}{\Delta(x, z)} u^{\mu \nu \alpha \beta}(x, z) \sigma_\xi(x, z) = 0, \tag{3.16}
\]

\[
u^{\mu \nu \alpha \beta} \xi(x, z) + 2R^{\mu \nu \rho \sigma}(x, z) = 0, \tag{3.17}
\]

\[
2v^{\mu \nu \alpha \beta}(x, z) + 2u^{\mu \nu \alpha \beta} \xi(x, z) - \frac{\Delta \xi(x, z)}{\Delta(x, z)} \nu^{\mu \nu \alpha \beta}(x, z) \sigma_\xi(x, z) + 2R^{\mu \nu \rho \sigma}(x, z)
\]

\[
\xi(x, z) \Delta(x, z) = 0. \tag{3.18}
\]

Equation (3.16) is solved with the normalization (3.15) as

\[
u^{\mu \nu \alpha \beta}(x, z) = 2\tilde{g}^{\alpha \beta}(x, z) \sqrt{\Delta(x, z)}. \tag{3.19}
\]

The bi-tensors \( v^{\mu \nu \alpha \beta}(x, z) \) and \( w^{\mu \nu \alpha \beta}(x, z) \) are to be determined by solving Eqs. (3.17) and (3.18). The bi-tensor \( w^{\mu \nu \alpha \beta}(x, z) \) is not needed but the bi-tensor \( v^{\mu \nu \alpha \beta}(x, z) \) plays an important role in the following discussion. Although it is difficult to find the solution of \( v^{\mu \nu \alpha \beta}(x, z) \) in an arbitrary background spacetime, its explicit form is not required for the succeeding discussions. However it is important to note that \( v^{\mu \nu \alpha \beta}(x, z) \) is uniquely determined. The reason is as follows. From Eq. (3.17) one finds it satisfies a hyperbolic equation. Hence the problem is if its Cauchy data are unique or not. First we note the coincidence limit of Eq. (3.18), which gives

\[
\lim_{x \to z} v^{\mu \nu \alpha \beta}(x, z) = \lim_{x \to z} 2\tilde{g}^{\alpha \beta}(z, x) \tilde{g}^{\alpha \beta}(z, x) \Delta^2 \xi(z, x).
\]

Then taking the null limit \( \sigma(x, z) \to 0 \) of Eq. (3.18), we obtain the first order differential equation for \( v^{\mu \nu \alpha \beta}(x, z) \) which can be solved along a null geodesic. Thus this equation with the boundary condition (3.20) uniquely determines \( v^{\mu \nu \alpha \beta}(x, z) \) on the light cone emanating from \( z \). Therefore the hyperbolic equation (3.17) has a unique solution. We also mention that \( v^{\mu \nu \alpha \beta}(x, z) \) is divergence free,

\[
\nu^{\mu \nu \alpha \beta}_{\mu}(x, z) = 0. \tag{3.21}
\]

To see this we note the harmonic gauge condition on the Green function requires

\[
\lim_{\sigma \to 0} v^{\mu \nu \alpha \beta} \sigma_{\nu}(x, z) = 0. \tag{3.22}
\]

We also see that the equation for \( v^{\mu \nu \alpha \beta}_{\mu}(x, z) \) follows from Eq. (3.17),

\[
\left[ v^{\mu \nu \alpha \beta}_{\mu}(x, z) \right] \Delta \xi(x, z) = 0, \tag{3.23}
\]

where we have used the fact \( R^{\mu \nu \rho \sigma} = 0 \), which is proved by contracting the Bianchi identities for the vacuum case. Thus we conclude that Eq. (3.21) holds everywhere.
The Feynman propagator $G_F^{\mu\nu\alpha\beta}(x, z)$ can be derived from the elementary solution $G_s^{\mu\nu\alpha\beta}(x, z)$ by the $i\epsilon$-prescription.

$$G_F^{\mu\nu\alpha\beta}(x, z) = \frac{1}{(2\pi)^2} \left( \frac{u^{\mu\nu\alpha\beta}(x, z)}{\sigma(x, z) + i\epsilon} + v^{\mu\nu\alpha\beta}(x, z) \log(\sigma(x, z) + i\epsilon) + w^{\mu\nu\alpha\beta}(x, z) \right).$$  (3.24)

The imaginary part of the Feynman propagator $G_F^{\mu\nu\alpha\beta}(x, z)$ gives the symmetric Green function $\bar{G}^{\mu\nu\alpha\beta}(x, z)$, from which we can obtain the retarded Green function $G_{\text{Ret}}^{\mu\nu\alpha\beta}(x, z)$, and the advanced Green function $G_{\text{Adv}}^{\mu\nu\alpha\beta}(x, z)$ as

$$\bar{G}^{\mu\nu\alpha\beta}(x, z) = -\frac{1}{2} \text{Im} \left[ G_F^{\mu\nu\alpha\beta}(x, z) \right] = \frac{1}{8\pi} \left[ u^{\mu\nu\alpha\beta}(x, z) \delta(\sigma(x, z)) - v^{\mu\nu\alpha\beta}(x, z) \theta(-\sigma(x, z)) \right],$$  (3.25)

$$G_{\text{Ret}}^{\mu\nu\alpha\beta}(x, z) = 2\theta[\Sigma(x), z] \bar{G}^{\mu\nu\alpha\beta}(x, z)$$  (3.26)

$$G_{\text{Adv}}^{\mu\nu\alpha\beta}(x, z) = 2\theta[z, \Sigma(x)] \bar{G}^{\mu\nu\alpha\beta}(x, z)$$  (3.27)

where $\Sigma(x)$ is an arbitrary space-like hypersurface containing $x$, and $\theta[\Sigma(x), z] = 1 - \theta[z, \Sigma(x)]$ is equal to 1 when $z$ lies in the past of $\Sigma(x)$ and vanishes when $z$ lies in the future.

### 3.3. Metric perturbation

Using the above obtained retarded Green function, we compute the trace-reversed metric perturbation $\psi^{\mu\nu}(x)$ induced by a point-like particle. We assume the energy-momentum tensor of the form,

$$T^{\mu\nu} = T_{\text{(mono)}}^{\mu\nu} + T_{\text{(spin)}}^{\mu\nu},$$  (3.28)

$$T_{\text{(mono)}}^{\mu\nu}(x) = m \int dT v^\mu(x, T)v^\nu(x, T) \frac{\delta^{(4)}(x - z(T))}{\sqrt{-g}},$$  (3.29)

$$T_{\text{(spin)}}^{\mu\nu} = -m \int dT \nabla_\xi \left( S^{\xi(\mu)(T)} v^{\nu}(x, T) \frac{\delta^{(4)}(x - z(T))}{\sqrt{-g}} \right),$$  (3.30)

$$v^\mu(x, T) = \bar{g}^\mu_\alpha(x, z(T)) \dot{z}^\alpha(T),$$  (3.31)

$$S^{\mu\nu}(x, T) = \bar{g}^{\mu}_{\alpha}(x, z(T)) \bar{g}^{\nu}_{\beta}(x, z(T)) S^{\alpha\beta}(T),$$  (3.32)

where $\dot{z}^\alpha(T) = d z^\alpha / dT$, $m$ is the mass of the particle and $S^{\alpha\beta}(T)$ is an antisymmetric tensor representing the specific spin of the particle per unit mass. We call it the spin tensor of the particle and assume that it satisfies the center of mass condition,

$$S^{\alpha\beta}(T) \dot{z}^\beta(T) = 0.$$  (3.33)

In Chapter 1, section 11, we have given the energy-momentum tensor of a spinning test particle. There the four-velocity of the orbit $v^\alpha = \dot{z}^\alpha$ is distinguished from

\footnote{Note that $S_{\alpha\beta}$ there corresponds to $mS_{\alpha\beta}$ here.}
the specific four-momentum of the particle \( u^\alpha = p^\alpha/m \). The difference is \( O(S^2/L^2) \) where \( S \) is the magnitude of the spin tensor \( S := \sqrt{S_{\alpha\beta}S^{\alpha\beta}/2} \). Here we ignore this difference because of the following reason. Since the particle is assumed to represent a black hole, \( m \) will be identified with the black hole mass and \( S \) with the Kerr spin parameter \( a \). Therefore \( S \) is assumed to be of order \( Gm \), hence the difference between \( v^\alpha \) and \( u^\alpha \) is \( O((Gm/L)^2) \). Since we are interested in the radiation reaction of \( O(Gm/L^2) \) to the equations of motion, we may consistently neglect this difference.

At this point, we must comment on the reason why we may assume the point-like particle for the source. Even in the linear perturbation, in order to generate a general gravitational field in the external zone, we need to consider a source with arbitrary higher multipole moments. However, the \( \ell \)-th moment of the gravitational field will be \( O((Gm/|X|)^{\ell+1}) \) in the matching region if the particle represents a black hole. As we shall see in the following discussions, we find it is not necessary to consider the matchings at \( O((Gm)^3/|X|^3) \) or higher in order to derive the equations of motion with the reaction force of \( O(Gm/L^2) \). Hence the moments higher than the spin can be consistently neglected.

We should also note that the metric perturbation induced by \( T_{(\text{spin})}^{\mu\nu} \) is \( O((Gm)^2) \) for \( S = O(Gm) \). At first glance, one might think that this implies the necessity of the second order perturbation theory if we are to incorporate the spin effect of the particle in the expansion with respect to \( Gm \) in a consistent way. However, provided that the construction of the metric by the matched asymptotic expansion is consistent, the second order perturbation theory turns out to be unnecessary. In fact, we shall find that the spin-induced metric perturbation of \( O((Gm)^2) \) gives rise to the leading order spin-curvature coupling term of \( O(Gm/L^2) \) in the equations of motion, while the spin-independent metric perturbation of \( O((Gm)^2) \) does not contribute to the reaction force term at \( O(Gm/L^2) \).

Without any further approximation, the metric perturbation due to the point-like particle becomes

\[
\psi^{\mu\nu}(x) = 2Gm \left[ \frac{1}{\bar{\sigma}(x,z(T))} u^{\mu\nu} \alpha\beta(x,z(T)) \dot{z}^\alpha(T) \dot{z}^\beta(T) \\
+ \frac{\bar{\sigma}(x,z(T))}{\bar{\sigma}^3(x,z(T))} u^{\mu\nu} \alpha\beta(x,z(T)) \sigma_{\gamma\delta}(x,z(T)) S^{\gamma\alpha}(T) \dot{z}^\beta(T) \\
+ \frac{1}{\bar{\sigma}(x,z(T))} u^{\mu\nu} \alpha\beta\gamma(x,z(T)) S^{\gamma\alpha}(T) \dot{z}^\beta(T) \\
- \frac{1}{\bar{\sigma}^2(x,z(T))} \frac{d}{dT} \left( u^{\mu\nu} \alpha\beta(x,z(T)) \sigma_{\gamma\delta}(x,z(T)) S^{\gamma\alpha}(T) \dot{z}^\beta(T) \right) \\
+ \frac{1}{\bar{\sigma}(x,z(T))} u^{\mu\nu} \alpha\beta(x,z(T)) \sigma_{\gamma\delta}(x,z(T)) S^{\gamma\alpha}(T) \dot{z}^\beta(T) \right]_{T=T_{\text{Ret}}(x)}
\]

\(^{\ast}) \text{A distributional form of the energy-momentum tensor with arbitrary higher multipole moments was discussed by Dixon.}\)
\[- \int_{-\infty}^{T_{\text{Ret}}(x)} dT \left( v_{\mu\nu}^{\alpha\beta}(x, z(T)) \dot{z}^\alpha(T) \dot{z}^\beta(T) \\
+ v_{\mu\nu}^{\alpha\beta\gamma}(x, z(T)) S^\gamma_{\alpha\beta}(T) \dot{z}^\beta(T) \right), \]

(3.34)

where \( T_{\text{Ret}}(x) \) is the retarded time of the particle and is a scalar function which is determined by

\[ \sigma(x, z(T_{\text{Ret}})) = 0, \quad \theta(\Sigma(x), z(T_{\text{Ret}})) = 1. \]

(3.35)

Since the retarded time \( T_{\text{Ret}}(x) \) is not convenient for specifying the field point \( x \) around the particle trajectory in the following computations, we introduce a new specification of \( x \) as follows. We foliate the spacetime with spacelike 3-surfaces perpendicular to the particle trajectory. Specifically, the 3-surfaces are defined as a one-parameter family of \( T \) by the relation, \( \sigma_{\alpha}(x, z(T)) \dot{z}^\alpha(T) = 0 \). We denote the value of \( T \) of the 3-surface containing the point \( x \) by \( T_{x} \). That is

\[ \sigma_{\alpha}(x, z(T_{x})) \dot{z}^\alpha(T_{x}) = 0, \]

(3.36)

where we have introduced the notation,

\[ Q_{\alpha}(x, z(T_{x})) := Q_{\alpha}(x, z)\big|_{z=z(T_{x})}, \]
\[ Q_{\mu}(x, z(T_{x})) := Q_{\mu}(x, z)\big|_{z=z(T_{x})}. \]

(3.37)

Note that

\[ [Q(x, z(T_{x}))]_{\alpha\mu} = Q_{\alpha}(x, z(T_{x})) + Q_{\alpha}(x, z(T_{x})) \dot{z}^\alpha(T_{x}) T_{x\mu}. \]

(3.38)

We use \( \sigma_{\alpha}(x, z(T_{x})) \) to distinguish the spatial points on the same 3-surface, and denote the spatial distance from \( z(T_{x}) \) to \( x \) by

\[ \epsilon(x) := \sqrt{2\sigma(x, z(T_{x})).} \]

(3.39)

In the matching region, we have

\[ Gm \ll \epsilon(x) \ll L. \]

(3.40)

To obtain the external metric in the matching region, we first consider the \( \epsilon \)-expansion of the time retardation, \( \delta_{\text{Ret}}(x) \),

\[ \delta_{\text{Ret}}(x) := T_{\text{Ret}}(x) - T_{x}. \]

(3.41)

It is given by expanding Eq. (3.36) as

\[ 0 = [\sigma(x, z(T))]_{\tau=T_{\text{Ret}}(x)} = \sigma(x, z(T_{x})) + \dot{\sigma}(x, z(T_{x}))\delta_{\text{Ret}}(x) + \frac{1}{2} \ddot{\sigma}(x, z(T_{x}))\delta_{\text{Ret}}^2(x) + \frac{1}{3!} \dddot{\sigma}(x, z(T_{x}))\delta_{\text{Ret}}^3(x) + \frac{1}{4!} \ddddot{\sigma}(x, z(T_{x}))\delta_{\text{Ret}}^4(x) + O(\epsilon^5). \]

(3.42)
Using Eqs. (3.6), (3.36), and the normalization condition, \((dz/dT)^2 = -1 + O(Gm/L)\), which will be proved to be consistent later, each term in the above is computed as

\[
\sigma(x,z(T_x)) = \frac{1}{2} \epsilon^2(x), \tag{3.43}
\]

\[
\dot{\sigma}(x,z(T_x)) = \sigma_\alpha(x,z(T_x)) \dot{\sigma}^\alpha(T_x) = 0, \tag{3.44}
\]

\[
\ddot{\sigma}(x,z(T_x)) = -\kappa^2(x)
\]

\[
= \sigma_{\alpha \beta}(x,z(T_x)) \dot{\sigma}^\alpha(T_x) \dot{\sigma}^\beta(T_x) + \sigma_\alpha(x,z(T_x)) \ddot{\sigma}^\alpha(T_x)
\]

\[
= \left( g_{\alpha \beta}(z(T_x)) - \frac{1}{3} R_{\alpha \beta}^\gamma \delta(z(T_x)) \sigma_{\gamma}(x,z(T_x)) \sigma_{\delta}(x,z(T_x)) \right) \dot{\sigma}^\alpha(T_x) \dot{\sigma}^\beta(T_x)
\]

\[
+ \sigma_\alpha(x,z(T_x)) \ddot{\sigma}^\alpha(T_x) + O(\epsilon^3), \tag{3.45}
\]

\[
\dddot{\sigma}(x,z(T_x)) = \sigma_{\alpha}(x,z(T_x)) \dddot{\sigma}^\alpha(T_x) + O(\epsilon^2), \tag{3.46}
\]

\[
\ddot{\sigma}(x,z(T_x)) = -g_{\alpha \beta}(z(T_x)) \ddot{\sigma}^\alpha(T_x) \ddot{\sigma}^\beta(T_x) + O(\epsilon), \tag{3.47}
\]

where we have introduced \(\kappa(x)\) to denote \(\sqrt{-\sigma(x,z(T_x))}\). From these, we obtain

\[
\delta_{\text{Rel}}(x) = -\epsilon(x) \kappa^{-1}(x) \left( 1 - \frac{1}{6} \epsilon(x) \kappa^{-3}(x) \dddot{\sigma}^\alpha(T_x) \sigma_\alpha(x,z(T_x)) \right)
\]

\[- \frac{1}{24} \epsilon^2(x) \kappa^{-4}(x) \ddot{\sigma}^2(T_x) + O(\epsilon^4). \tag{3.48}
\]

With the help of Eq. (3.48), we then obtain the expansion of various terms in Eq. (3.34). We have

\[
\left[ \frac{1}{\sigma(x,z(T))} \right]_{T=T_{\text{Rel}(x)}} = \frac{1}{\epsilon(x) \kappa(x)} \left( 1 - \frac{1}{3} \epsilon(x) \dddot{\sigma}^\alpha(T_x) \sigma_\alpha(x,z(T_x)) - \frac{1}{8} \epsilon^2(x) \ddot{\sigma}^2(T_x) + O(\epsilon^3) \right). \tag{3.49}
\]

In order to obtain the expansion of \(u^{\mu \alpha \beta}(x,z)\) given by Eq. (3.19), we also need the following expansions:

\[
\left[ \Delta^{1/2}(x,z(T)) \right]_{T=T_{\text{Rel}(x)}} = 1 + O(\epsilon^3), \tag{3.50}
\]

\[
\left[ \bar{g}_{\mu \alpha}(x,z(T)) \right]_{T=T_{\text{Rel}(x)}} = \bar{g}_{\mu \alpha}(x,z(T_x)) - \frac{1}{2} \bar{g}_{\mu \beta}(x,z(T_x)) R_{\alpha \beta \gamma \delta}(z(T_x)) \sigma^\gamma(x,z(T_x)) \dot{\sigma}^\delta(T_x) \epsilon(x)
\]

\[
+ O(\epsilon^3), \tag{3.51}
\]

\[
\left[ \ddot{\sigma}^\alpha(T) \right]_{T=T_{\text{Rel}(x)}} = \ddot{\sigma}^\alpha(T_x) - \epsilon(x) \kappa^{-1}(x) \dddot{\sigma}^\alpha(T_x) + \frac{1}{2} \epsilon^2(x) \ddot{\sigma}^\alpha(T_x) + O(\epsilon^4). \tag{3.52}
\]

In the above expressions there appear higher derivatives of \(\dot{\sigma}\), such as \(\ddot{\sigma}\) and \(\dddot{\sigma}\), where a dot means the covariant derivative \(\partial/dT\) along the trajectory of the particle.
Since we are considering the case in which the radiation reaction force is $O(Gm/L^2)$, it is reasonable to assume these derivatives are smaller by a factor of $O(1/T_r)$, i.e.,

$$\frac{D^{n+1}z(T)}{dT^{n+1}} \sim \frac{1}{T_r L^{n-1}} < O\left(\frac{\epsilon(x)}{L^{n+1}}\right) \quad (n \geq 1),$$  \hspace{1cm} (3.53)

where $T_r = O(L^2/(Gm))$ is the reaction time scale. We shall find that this is consistent with the equations of motion in the end.

Keeping this fact in mind, and using Eqs. (3.49) $\sim$ (3.52), we obtain the $\epsilon$-expansion of the trace-reversed metric perturbation, Eq. (3.34), as

$$\psi^{\mu\nu} = \psi^{\mu\nu}_{(mono)} + \psi^{\mu\nu}_{(spin)} + \psi^{\mu\nu}_{(tail)};$$  \hspace{1cm} (3.54)

where

$$\psi^{\mu\nu}_{(mono)}(x) = 2Gm\bar{g}^{\alpha}_{\alpha\beta}\left(\frac{2}{\epsilon}\kappa^{-1}z^\alpha z^\beta \right.$$

$$\left. - 4\dot{z}^{(\alpha}z^{\beta)} + 2\dot{z}\gamma\sigma^{\alpha\beta\gamma\epsilon}R_{\gamma\delta\epsilon}^{\alpha\beta} - 2\epsilon R_{\gamma\delta}^{\alpha\beta}\dot{z}^\gamma z^\delta + O(\epsilon^3)\right), \hspace{1cm} (3.55)$$

$$\psi^{\mu\nu}_{(spin)}(x) = -4Gm\bar{g}^{\alpha\mu}_{\alpha\beta}\left(\frac{1}{\epsilon^3}z^{(\alpha}S^{\beta)\gamma}\sigma_{\gamma\epsilon} + O((Gm)\epsilon^0)\right), \hspace{1cm} (3.56)$$

$$\psi^{\mu\nu}_{(tail)}(x) = 2Gm\bar{g}^{\alpha\mu}_{\alpha\beta}\left( - \int_{-\infty}^{T_x} dT' \left(v^{\alpha\beta}_{\alpha\alpha'}\left(z(T_x), z(T')\right)\dot{z}^{\alpha'}(T')\dot{z}^{\beta'}(T') \right.$$

$$\left. + v^{\alpha\beta}_{\alpha\alpha'}\left(z(T_x), z(T')\right)S^{\gamma\alpha'}(T')\dot{z}^{\gamma'}(T') \right)$$

$$\left. + \sigma_{\gamma\epsilon} \int_{-\infty}^{T_x} dT' \left(v^{\alpha\beta}_{\alpha\alpha'}\left(z(T_x), z(T')\right)\dot{z}^{\alpha'}(T')\dot{z}^{\beta'}(T') \right.$$

$$\left. + v^{\alpha\beta}_{\alpha\alpha'}\left(z(T_x), z(T')\right)S^{\gamma\alpha'}(T')\dot{z}^{\gamma'}(T') \right) + O(\epsilon^2)\right); \hspace{1cm} (3.57)$$

where $\bar{g}^{\beta}_{\alpha}(x, z(T_x))$. The part $\psi^{\mu\nu}_{(tail)}$ is called the tail term because it is not due to the direct light cone propagation of waves but due to multiple curvature scattering of waves as described by the $v^{\mu\nu\alpha\beta}(x, z)$ term in the Green function.

### 3.4. Transformation to the internal coordinates

In order to write down the external metric in terms of the internal coordinates, we consider a coordinate transformation from $x$ to $\{X^a\}$ given in the form,

$$\sigma_{\alpha}(x, z(T)) = -F_{\alpha}(T, X).$$  \hspace{1cm} (3.58)

We restrict our consideration on a coordinate transformation which satisfies the following requirements. We assume $X^i = 0$ corresponds to the center of the particle,
x^\alpha = z^\alpha(T), hence \( F_\alpha = 0 \) at \( X^i = 0 \). We also assume that the right hand side of Eq. (3.58) can be expanded in positive powers of \( X^i \) as

\[
F_\alpha(T, X) = f_{\alpha i}(T)X^i + \frac{1}{2}f_{\alpha ij}(T)X^iX^j + \frac{1}{3!}f_{\alpha ijk}(T)X^iX^jX^k + \cdots. (3.59)
\]

Although it is possible that there appear more complicated terms such as \( X^iX^j/|X| \), we simply ignore such kinds of terms. We shall find it is consistent within the order of the approximation to which we are going to develop our consideration. Here \( f_{\alpha i\cdots i_n}(T) \) is totally symmetric for \( i_1 \cdots i_n \) and is at most of \( O(L^{-(n-1)}) \). Using Eqs. (3.6) and (3.7), the total derivative of Eq. (3.58) gives the important relation,

\[
\bar{g}^\alpha_{\mu}(z(T), x)dx^\mu = \left( \frac{dz^\alpha}{dT}(T) + \frac{Df_{\alpha i}}{dT}(T)X^i + \frac{1}{2}\frac{Df_{\alpha ij}}{dT}(T)X^iX^j \right. \\
- \frac{1}{2}R^\alpha_{\beta\gamma\delta}(z(T))f_{i}(T)\frac{dz^\gamma}{dT}(T)f_{j}(T)X^iX^j \\
+ \left( f_{\alpha i} + f_{\alpha ij}(T)X^j + \frac{1}{2}f_{\alpha ijk}(T)X^jX^k \right) \\
- \frac{1}{6}R^\alpha_{\beta\gamma\delta}(z(T))f_{j}(T)f_{i}(T)f_{k}(T)X^jX^k) \\
+ O(|X|^3) \\
(3.60)
\]

In the following sections, we write down the external metric in terms of the internal coordinates in the matching region to obtain the equations of motion.

§4. Equations of motion for a monopole particle

In this section, we adopt the method (1) mentioned in section 1 to derive the equations of motion. We restrict our consideration to the case of a monopole particle, which is necessary because we use a well-established method to decompose the metric in the internal scheme by the tensor harmonics. The tensor harmonics are classified by the total angular momentum, \( J \), reflecting the spherical symmetry of the Schwarzschild black hole.

In the internal scheme, the monopole mode \( (J = 0) \) corresponds to the mass perturbation. Thus we may set this mode to zero since it is natural to suppose that the change of mass due to the radiation reaction is negligible. The dipole modes \( (J = 1) \) are related to the translation and rotation. The translation modes are purely gauge and thus we set them to zero to fix the center of the black hole. As we are considering a non-rotating black hole, we also set the notational modes to zero. In general, the higher modes contain gauge degrees of freedom as well as the physical ones. However, for these higher modes, we do not give any principle to fix the gauge for the moment.
Before the explicit computation of the \( \binom{m}{n} \) matching condition, we briefly review the construction of the scalar and vector harmonics in terms of the symmetric trace-free (STF) tensor\( \mathcal{H} \).

4.1. **Spherical harmonics expansion**

We introduce the notation,

\[
A_{\langle i_1 i_2 \cdots i_\ell \rangle},
\]

(4.1)

to represent the totally symmetric trace-free part of \( A_{i_1 i_2 \cdots i_\ell} \). More explicitly in the cases of \( \ell = 2, 3 \),

\[
A_{\langle ij \rangle} = A_{(ij)} - \frac{1}{3} \delta_{ij} A_{kk},
\]

\[
A_{\langle ijk \rangle} = A_{(ijk)} - \frac{1}{5} \left( \delta_{ij} A_{(kmm)} + \delta_{jk} A_{(imm)} + \delta_{ki} A_{(jmm)} \right).
\]

(4.2)

The spherical harmonics expansion of a scalar function \( A \) on the unit-sphere can be written as

\[
A = \sum_{\ell=0}^{\infty} A_{\langle i_1 i_2 \cdots i_\ell \rangle} n_{\langle i_1 i_2 \cdots i_\ell \rangle},
\]

(4.3)

where \( n^i = X^i/|X| \). In this case, the order \( \ell \), which is associated with the angular dependence, is equivalent to the total angular momentum, \( J \). Thus the \( J \) mode of the \( (TT) \)-component of the metric perturbation is totally determined by its angular dependence. Namely, the terms in the \( (TT) \)-component of the metric perturbation which contain

\[
1, \quad n^i, \quad n_{\langle i n j \rangle},
\]

(4.4)

correspond to the \( J = 0, 1, 2 \) modes, respectively.

Next we consider the expansion of a vector field \( A_i \),

\[
A_i = \sum_{\ell=0}^{\infty} A_{i\langle i_1 i_2 \cdots i_\ell \rangle} n_{\langle i_1 i_2 \cdots i_\ell \rangle}.
\]

(4.5)

In this case the term of the \( \ell \)-th order in the angular dependence is decomposed into \( J = \ell + 1, \ell \) and \( \ell - 1 \). This is done by using the Clebsch-Gordan reduction formula\( \mathcal{H} \),

\[
U_i T_{i_1 i_2 \cdots i_\ell} = R_{i\langle i_1 i_2 \cdots i_\ell \rangle}^{(\ell + 1)} + \frac{\ell}{\ell + 1} \delta_{ji} R_{i_1 i_2 \cdots i_{\ell-1}}^{(0)} + \frac{2\ell - 1}{2\ell + 1} \delta_{ij} R_{i_1 i_2 \cdots i_{\ell-1}}^{(-)}
\]

(4.6)

where \( T_{i_1 i_2 \cdots i_\ell} \) is a STF tensor of order \( \ell \) and

\[
R_{i_1 i_2 \cdots i_{\ell+1}}^{(\ell)} := U_{i\langle i_{\ell+1} \rangle} T_{i_1 i_2 \cdots i_\ell},
\]

\[
R_{i_1 i_2 \cdots i_\ell}^{(0)} := U_j T_k_{i\langle i_1 i_2 \cdots i_{\ell-1} \rangle} \epsilon_{ij} \epsilon_{jk},
\]

\[
R_{i_1 i_2 \cdots i_{\ell-1}}^{(-)} := U_j T_k_{j_1 j_2 \cdots i_{\ell-1}}.
\]

(4.7)

We perform the decomposition explicitly for \( \ell \leq 2 \) here. For \( \ell = 0 \), there exists no \( J = 0 \) mode and it trivially corresponds to the \( J = 1 \) mode. For \( \ell = 1 \), the
decomposition is performed as
\[ A_{ij}n^j = \left[ A_{(ij)} - \frac{1}{3} \delta_{ij} A_{kk} \right] + A_{[ij]} + \frac{1}{3} \delta_{ij} A_{kk} \]n^j, \tag{4.8}
and the first, second and third terms in the square brackets correspond to the \( J = 2 \),1 and 0 modes, respectively. For \( \ell = 2 \), we obtain the decomposition formula as
\[ A_{i<jk>} n^{<j} n^{k>} = \left[ A_{ijkl} + \frac{2}{3} \epsilon_{i<j} B^{(2)}_{k>m} + \frac{3}{5} \delta_{i<j} B^{(1)}_{k} \right] n^{<j} n^{k>}, \tag{4.9}\]
where
\[ B^{(2)}_{ij} = \frac{1}{2} (A_{i<jk>} \epsilon_{jkm} + A_{k<mi>} \epsilon_{ikm}), \] \[ B^{(1)}_{k} = A_{i<jk>} \delta_{ij}, \tag{4.10}\]
and the first, second and third terms correspond to the \( J = 3, 2 \) and 1 modes, respectively.

We omit the general discussion on the expansion of the tensor field and we shall give a specific argument when necessary.

4.2. Geodesics; \((0)_0\) and \((0)_1\) matching

We begin with the \((0)_0\) and \((0)_1\) matchings which are, respectively, of \( O(\langle Gm/L \rangle^0) \) and of \( O(\langle Gm/L \rangle^{1/2}) \) in the matching region. First we consider the external scheme. In these matchings the external metric is the background itself. Here, the necessary order of expansion in \( |X| \) is \( O(\langle X \rangle) \). We note
\[ g_{\mu\nu}(x) dx^\mu dx^\nu = g^{(0)}_{\alpha\beta}(z) \bar{g}^\alpha_\mu(z,x) \bar{g}^\beta_\nu(z,x) dx^\mu dx^\nu. \tag{4.11}\]
Then from Eq. (3.60), we get
\[ g_{\mu\nu}(x) dx^\mu dx^\nu = \left( \left( \frac{dz}{dT} \right)^2 (T) + 2 \frac{dz^\alpha}{dT}(T) \frac{Df_{\alpha i}(T)}{dT}(T) X^i \right) dT^2 + 2 \left( \frac{dz^\alpha}{dT}(T) f_{\alpha i}(T) + \frac{dz^\alpha}{dT}(T) f_{\alpha ij}(T) X^j + f_{\alpha i}(T) \frac{Df_{\alpha i}(T)}{dT}(T) X^i \right) dT dX^i + \left( f_{\alpha i}(T) f_{\alpha j}(T) + 2 f_{\alpha i}(T) f_{\alpha j k}(T) X^k \right) dX^i dX^j + O \left( \frac{|X|^2}{L^2} \right). \tag{4.12}\]
Comparing the above equation with Eq. (2.10) and looking at the dependence on \( X \), one can readily extract out \((0)_0 h_{ab}\) and \((0)_1 h_{ab}\) to the lowest order in \( Gm/L \).

Next we consider the internal scheme. The \((0)_0\)-component is trivially given by the flat Minkowski metric. Hence the \((0)_0\) matching becomes
\[ -1 = \left( \frac{dz}{dT} \right)^2 (T) + O \left( \frac{Gm}{L} \right), \quad \text{\((TT)\)-component,}\tag{4.13}\]
\[ 0 = \frac{dz^\alpha}{dT}(T) f_{\alpha i}(T) + O \left( \frac{Gm}{L} \right), \quad \text{\((Ti)\)-component,}\tag{4.14}\]
\[ \delta_{ij} = f_{\alpha i}(T) f_{\alpha j}(T) + O \left( \frac{Gm}{L} \right), \quad \text{\((ij)\)-component.}\tag{4.15}\]
Equations (4.14) and (4.15) indicate that \( f^{\alpha i}(T) \) are spatial triad basis along the orbit, i.e.,
\[
f^{\alpha k}(T)f^{\beta k}(T) = g^{\alpha \beta}(z(T)) + \frac{dz^\alpha}{dT}(T)\frac{dz^\beta}{dT}(T) + O\left(\frac{Gm}{L}\right).
\] (4.16)

To know the \((1_0)\)-component of the internal scheme, it is better to consider all the \((1_n)\)-components at the same time. Namely we consider the linear perturbation of the black hole \( (1)_H^{ab} \). For this purpose, we consider the harmonic decomposition of linear perturbation as discussed in subsection 4.1. Since the time scale associated with the perturbation should be of the order of the background curvature scale \( L \), it is much larger than the matching radius \( (GmL)^{1/2} \). Therefore the perturbation may be regarded as static. It is known that all the physical static perturbations regular on the black hole horizon behave as \( \sim |X|^J \) asymptotically where \( J \) is the angular momentum eigenvalue. However, in \( (1)_n^{H ab} \), there exists no term which behaves as \( \sim |X|^m \) \((m \geq 2)\). Hence, except for gauge degrees of freedom, \( (1)_n^{H ab} \) contain only \( J = 0, 1 \) modes. As mentioned before, we set the perturbation of \( J = 0, 1 \) modes to zero. Thus we conclude that we may set
\[
(1)_n^{H ab} = 0,
\] (4.17)
for all \( n \). This is the gauge condition we adopt for the internal scheme at \( O(1/L) \).

In particular this condition gives the \((1_0)\) matching as
\[
0 = 2\frac{dz^\alpha}{dT}(T)\frac{df^{\alpha i}(T)}{dT}X^i + O\left(\frac{Gm}{L}\frac{|X|}{L}\right), \quad (TT)\text{-component},
\] (4.18)
\[
0 = \frac{dz^\alpha}{dT}(T)f^{\alpha i_j}(T)X^j + f^{\alpha i}(T)\frac{df^{\alpha i_j}}{dT}(T)X^j
+ O\left(\frac{Gm}{L}\frac{|X|}{L}\right), \quad (Ti)\text{-component},
\] (4.19)
\[
0 = 2f^{\alpha(1)}(T)f^{\alpha 2}(T)X^k + O\left(\frac{Gm}{L}\frac{|X|}{L}\right), \quad (ij)\text{-component}.
\] (4.20)

Then the covariant \( T \)-derivative of Eq. (4.13) and that of Eq. (4.14) with Eq. (4.18) result in the background geodetic motion,
\[
\frac{D}{dT}\left(\frac{dz^\alpha}{dT}\right)(T) = O\left(\frac{Gm}{L}\frac{1}{L}\right).
\] (4.21)

One can see from Eq. (4.13) that the internal time coordinate \( T \) becomes a proper time of the orbit in the lowest order in \( Gm/L \). In the same manner, Eq. (4.18) and the covariant \( T \)-derivative of Eq. (4.15) with \((ij)\)-antisymmetric part of Eq. (4.19) give the geodetic parallel transport of the triad \( f^{\alpha i}(T) \),
\[
\frac{D}{dT}f^{\alpha i}(T) = O\left(\frac{Gm}{L}\frac{1}{L}\right).
\] (4.22)

Further, from Eqs. (4.19) and (4.20), we can see
\[
f^{\alpha i}(T) = O\left(\frac{Gm}{L}\frac{1}{L}\right).
\] (4.23)
4.3. **Hypersurface condition; \((\gamma)\) matching**

We now proceed to the \((\gamma)\) matching, in which the external metric is still given by the background but there appear non-trivial perturbations in the internal scheme. Although it is of \(O(Gm/L)\) in the matching region and \(O((Gm/L)^{1/2})\) higher than the remaining \((\eta)\)-component, we consider it first for the reason which will be clarified below.

In order to obtain \((2)_{(0)} h_{ab}\), we expand the external metric in terms of the internal coordinates up to \(O(|X|^2)\), i.e., we have to go one order higher than Eq. (4.12). Then the \((\gamma)\) matching becomes

\[
\begin{align*}
\frac{1}{L^2} (2)_{(0)} H_{TT} &= -R_{\alpha\beta\gamma\delta}(z(T)) \frac{dz^\alpha}{dT}(T) f^\beta_i(T) \frac{dz^\gamma}{dT}(T) f^\delta_j(T) X^i X^j \\
&\quad + O \left( \frac{Gm}{L} \frac{|X|^2}{L^2} \right), \quad (TT)\text{-component,} \\
\frac{1}{L^2} (2)_{(0)} H_{Ti} &= \frac{1}{2} \frac{dz^\alpha}{dT}(T) f_{\alpha jk}(T) X^j X^k \\
&\quad - \frac{2}{3} R_{\alpha\beta\gamma\delta}(z(T)) \frac{dz^\alpha}{dT}(T) f^\beta_j(T) f^\gamma_i(T) f^\delta_k(T) X^j X^k \\
&\quad + O \left( \frac{Gm}{L} \frac{|X|^2}{L^2} \right), \quad (Ti)\text{-component,} \\
\frac{1}{L^2} (2)_{(0)} H_{ij} &= f_{\alpha(i(T)} f^\alpha_{j)kl}(T) X^k X^l \\
&\quad - \frac{1}{3} R_{\alpha\beta\gamma\delta}(z(T)) f^\alpha_i(T) f^\beta_k(T) f^\gamma_j(T) f^\delta_l(T) X^k X^l \\
&\quad + O \left( \frac{Gm}{L} \frac{|X|^2}{L^2} \right), \quad (ij)\text{-component,}
\end{align*}
\]

where Eqs. (4.22) and (4.23) have been used to simplify the expressions. Since we have set \((1)_{(n)} H_{ab} = 0\), the first non-trivial perturbations of the internal metric appear in \((2)_{(n)} H_{ab}\). Hence they describe the linear perturbation of the black hole metric in the internal scheme. Then we have to fix the gauge condition for this perturbation to perform the matching. For \((2)_{(0)} H_{ab}\), since the physical perturbation contained in it is quadrupolar, we fix the gauge so that all the \(J\) modes except \(J = 2\) are zero. Then the \((\gamma)\) matching becomes as follows.

First consider the \((TT)\)-component of the metric. The right hand side of Eq. (4.24) may contain \(J = 0, 2\) modes. The \(J = 0\) mode, however, vanishes because of the background Ricci flatness. Hence this matching just determines the physical perturbation in the \((TT)\)-component.

As for the \((Ti)\)-component, the right hand side of Eq. (4.25) may contain \(J = 1, 2, 3\) modes. As before, the \(J = 2\) mode just determines the physical perturbation of the \((Ti)\)-component. So we put \(J = 0, 3\) modes to zero. However, they are found to be absent in the second term of Eq. (4.25). To see this we first decompose its
angular dependence,

\[ \frac{dz^\alpha}{dT} R_{\alpha\beta\gamma\delta} f^\gamma_i \left( f^\beta_{<j} f^\delta_{k>} X^{<j} X^{k>} + \frac{1}{3} f^\beta_k f^\delta_k |X|^2 \right). \]  

(4.27)

Using Eq. (4.16) and the fact that the Ricci tensor vanishes, the second term in the parentheses is rewritten as

\[ \frac{1}{3} \frac{dz^\alpha}{dT} R_{\alpha\beta\gamma\delta} f^\gamma_i \frac{dz^\beta}{dT} \frac{dz^\delta}{dT} |X|^2, \]

(4.28)

and is found to be zero due to the symmetry of the Riemann tensor. The first term in the parentheses of Eq. (4.27) is decomposed further with the aid of the formulas (4.10) and (4.9) as

\[ \frac{dz^\alpha}{dT} R_{\alpha\beta\gamma\delta} \left( f^\gamma_{<i} f^\beta_{j} f^\delta_{k>} + 2 \frac{2}{3} \epsilon_{mi<j} F_{k>}^{(2)} f_{m}^{\gamma\delta} + 3 \frac{3}{5} \delta_{i<j} F_{k>}^{(1)} f_{\gamma\beta\delta} \right) X^{<j} X^{k>}, \]

(4.29)

where

\[ F_{ij}^{(2)} f_{\gamma\beta\delta} := \frac{1}{2} \left( f^\gamma_{m} f^\beta_{<n} f^\delta_{j>} \epsilon_{jmn} + f^\gamma_{m} f^\beta_{<n} f^\delta_{j>} \epsilon_{imn} \right), \]

\[ F_{i}^{(1)} f_{\gamma\beta\delta} := \frac{1}{2} \left( f^\gamma_{k} f^\beta_{i} f^\delta_{k} + f^\gamma_{k} f^\beta_{k} f^\delta_{i} \right) - \frac{1}{3} f^\gamma_{i} f^\beta_{j} f^\delta_{k}. \]  

(4.30)

It is easy to see that the first and third terms in the parentheses of Eq. (4.29) vanish due to the symmetry of the Riemann tensor and the Ricci flatness. Thus only the \( J = 2 \) mode remains in the second term in the right hand side of Eq. (4.25).

Decomposing the first term in the right hand side of Eq. (4.25) in a similar manner, we find it contains \( J = 1, 3 \) modes as well as \( J = 2 \) mode. Putting the \( J = 1 \) mode to zero, we obtain

\[ \frac{dz^\alpha}{dT} (T) f_{aikk}(T) = O \left( \frac{Gm}{L^2} \frac{1}{L^2} \right). \]  

(4.31)

Putting the \( J = 3 \) mode to zero gives

\[ \frac{1}{2} \frac{dz^\alpha}{dT} (T) f_{a<ijk>(T)} X^j X^k = O \left( \frac{Gm}{L} \frac{|X|^2}{L} \right). \]  

(4.32)

Then combining this with Eq. (4.31), we find

\[ \frac{dz^\alpha}{dT} (T) f_{aijk}(T) = O \left( \frac{Gm}{L} \frac{1}{L^2} \right). \]  

(4.33)

From Eqs. (4.14), (4.23) and (4.33), we find

\[ \frac{dz^\alpha}{dT} (T) \sigma_{\alpha} (x(T, X), z(T)) = \frac{dz^\alpha}{dT} (T) F_{\alpha}(T, X) = O \left( \frac{|X|^4}{L^4} L \right), \]  

(4.34)

to the lowest order in \( Gm/L \). Comparing this with the hypersurface condition of \( T_x \), Eq. (3.36), one finds that the \( T = \) constant hypersurface differs from
T_x = constant hypersurface only by \( O(\epsilon^4) = O(|X|^4) \). It then follows that all the calculations done in section 3 remain valid even if we replace Eq. (3.36) with

\[
\sigma_{,\alpha}(x, z(T_x)) \dot{z}^\alpha(T_x) = O(\epsilon^4/L^3).
\] (4.35)

Thus \( T \) can be identified \( T_x \) to the lowest order in \( Gm/L \). The reason why we have done the \( (\frac{2}{0}) \) matching prior to the remaining \( (\frac{1}{1}) \) matching is to establish this equivalence of \( T \) and \( T_x \).

Turning to the \( (ij) \)-component, it may contain \( J = 0 \sim 4 \) modes. we first note that the second term of Eq. (4.26) contains only \( J = 2 \) mode. This can be seen as follows. First, we define the spatial triad components of the Riemann tensor by

\[
R_{ijkl} := R_{\alpha\beta\gamma\delta} f^\alpha_i f^\beta_j f^\gamma_k f^\delta_l.
\] (4.36)

Introducing a symmetric tensor defined by

\[
\mathcal{R}_{ij} = \frac{1}{4} \epsilon_{ikm} \epsilon_{jns} R_{kmns},
\] (4.37)

we can express \( R^{ikjm} \) in terms of \( \mathcal{R}_{ij} \) as

\[
R_{ikjm} = \epsilon^{nij} \epsilon^{skm} \mathcal{R}_{ns}.
\] (4.38)

Then the symmetric tensor \( \mathcal{R}_{ij} \) is decomposed into STF tensors as

\[
\mathcal{R}_{ij} = \mathcal{R}_{<ij>} + \frac{1}{3} \delta_{ij} \mathcal{R}_{kk}.
\] (4.39)

Counting the number of indices, we find that the first and second terms in Eq. (4.39) correspond to \( J = 2 \) and 0 modes, respectively. However, again owing to the symmetry of the Riemann tensor and the Ricci flatness, the \( J = 0 \) mode vanishes and only the \( J = 2 \) mode remains. Therefore the gauge condition for the \( (ij) \)-component implies

\[
\left[ f_{\alpha\beta}(T) f^{\alpha} \right]_{j \neq 2} = O \left( \frac{Gm}{L \ L^2} \right),
\] (4.40)

where \([\cdots]_{j \neq 2}\) means the \( J \neq 2 \) parts of the quantity. This will be used in the \( (\frac{1}{1}) \) matching below.

### 4.4. External perturbation; \( (\frac{1}{1}) \) matching

Now we proceed to the first non-trivial order in \( Gm/|X| \). For this purpose, we must develop the external scheme. However, since the time slicing by the internal time coordinate \( T \) is now identical to that by \( T_x \) in the lowest order in \( Gm/L \), we can use the previously obtained formula (3.54) for the external metric perturbation.

Among the matchings which becomes of \( O((Gm/L)^{1/2}) \) in the matching region, there remains the \( (\frac{1}{1}) \) matching. This matching relates the masses of the particle in both schemes. Since this matching is independent of \( L \), we may regard the background external metric as if it were flat. As is well-known, the linear perturbation induced by a point-like particle of mass \( m \) in the flat background spacetime is exactly
equal to the asymptotic metric of a Schwarzschild black hole of mass \( m \) in the linear order in \( m \). This fact indicates that the matching gives a consistency condition at this order.

In order to directly check the consistency, we rewrite Eq. (3.54) in terms of the internal coordinates. Since \( (0)_{(1)}h_{ab} \sim |X|^{-1} \), we have only to consider the \( \psi_{(\text{mono})}^{\mu\nu} \) term of Eq. (3.54). Using Eqs. (2.8), (3.60) and the fact that \( \epsilon = \sqrt{F_{\alpha}(T,X)F^{\alpha}(T,X)} \), we find

\[
Gm_{(1)}(0)h_{ab}(X)dX^adX^b = Gm \left( \frac{2}{|X|}dT^2 + \frac{2}{|X|}dX^idX^i \right),
\]

which corresponds to the asymptotic form of the Schwarzschild black hole of mass \( m \) in the harmonic coordinates.

4.5. Radiation reaction; (\( 1 \)) and (\( 2 \)) matchings

There are many components which become of \( O(Gm/L) \) and \( O((Gm/L)^{3/2}) \) in the matching region. However, we are interested in the leading order correction to the equations of motion with respect to \( Gm/L \) and we found in the (\( 0 \)) and (\( 1 \)) matchings that in the lowest order the terms which behave as \( \sim |X|^0 \) or \( |X|^1 \) determines the motion of the particle. Therefore we consider the (\( 1 \)) and (\( 2 \)) matchings here.

In order to perform the (\( 1 \)) and (\( 2 \)) matchings, the calculation we have done to obtain Eq. (4.41) must be extended to the linear order in \( |X| \). Then the (\( 1 \)) matching equations are found as

\[
\begin{align*}
\frac{Gm_{(1)}}{L^2}H_{TT} &= \left\{ \left( \frac{dz}{dT} \right)^2(T) + 1 \right\} + Gm \frac{dz}{dT}(T)\frac{dz}{dT}(T)\Theta_{\alpha\beta}(T) \\
&\quad + O \left( \left( \frac{Gm}{L} \right)^2 \right), \quad (TT)-\text{component}, \quad (4.42) \\
\frac{Gm_{(1)}}{L}H_{Ti} &= \frac{dz}{dT}(T)\delta_{ai}(T) + Gm \frac{dz}{dT}(T)\delta_{i}\Theta_{\alpha\beta}(T) \\
&\quad + O \left( \left( \frac{Gm}{L} \right)^2 \right), \quad (Ti)-\text{component}, \quad (4.43) \\
\frac{Gm_{(1)}}{L}H_{ij} &= \{ f^{\alpha}_i(T)\delta_{aj}(T) - \delta_{ij} \} + Gmf^{\alpha}_i(T)f^{\beta}_j(T)\Theta_{\alpha\beta}(T) \\
&\quad + O \left( \left( \frac{Gm}{L} \right)^2 \right), \quad (ij)-\text{component}, \quad (4.44)
\end{align*}
\]

and the (\( 2 \)) matching as

\[
\begin{align*}
\frac{Gm_{(2)}}{L^2}H_{TT} &= 2\frac{dz}{dT}(T)\frac{DF_{\alpha i}(T)X^i}{dT} \\
&\quad + Gm \left\{ \frac{dz}{dT}(T)\frac{dz}{dT}(T)f^{i\gamma}(T)\Theta_{\alpha\beta\gamma}(T)X^i \\
&\quad - \frac{1}{3|X|^3}f_{\alpha}(T)f^{\alpha}_jkl(T)X^iX^jX^kX^l \right\}
\end{align*}
\]
\[ + \frac{5}{3|X|} R_{\alpha\beta\gamma\delta}(z(T)) \frac{dz^\alpha}{dT}(T) f^{\beta i}(T) \frac{dz^\gamma}{dT}(T) f^{\delta j}(T) X^i X^j \]
\[ + O \left( \left( \frac{Gm}{L} \right)^2 \frac{|X|}{L} \right), \quad (TT)\)-component, \quad (4.45) \]
\[ \frac{Gm(2)}{L^2} H_{Ti} = \frac{dz^\alpha}{dT}(T) f_{\alpha i}(T) X^j + \frac{dz^\alpha}{dT}(T) D f_{\alpha i}(T) X^j \]
\[ + Gm \left\{ \frac{dz^\alpha}{dT}(T) f^{\beta i}(T) f^{\gamma j}(T) \Theta_{\alpha\beta\gamma}(T) X^j \right. \]
\[ + 2 R_{\alpha\beta\gamma\delta}(z(T)) \frac{dz^\alpha}{dT}(T) f^{\beta i}(T) \frac{dz^\gamma}{dT}(T) f^{\delta j}(T) X^j \]
\[ + \frac{2}{3|X|} R_{\alpha\beta\gamma\delta}(z(T)) \frac{dz^\alpha}{dT}(T) f^{\beta j}(T) \Theta_{\alpha\beta\gamma\delta}(T) X^j X^k \left. \right\} \]
\[ + O \left( \left( \frac{Gm}{L} \right)^2 \frac{|X|}{L} \right), \quad (Ti)\)-component, \quad (4.46) \]

where

\[ Gm \Theta_{\alpha\beta}(T) := h_{(tail)\alpha\beta}(z(T)), \]
\[ Gm \Theta_{\alpha\beta\gamma}(T) := h_{(tail)\alpha\beta\gamma}(z(T)), \quad (4.47) \]

with

\[ h_{(tail)\mu\nu}(x) := \psi_{(tail)\mu\nu}(x) - \frac{1}{2} g_{\mu\nu}(x) \psi_{(tail)}(x). \quad (4.48) \]

Note that \( h_{(tail)\mu\nu}(x) \) is the metric perturbation due to \( v_{\mu\nu\alpha\beta}(x, z) \) in the Green function. The \((ij)\)-component of the \((1)\) matching is not presented here since it will not be used in the following discussion.

As we have discussed in subsection 4.2, we require \( (1) H_{ab} = 0 \). Thus the right hand sides of Eqs. (4.42), (4.43) and (4.44) must vanish. As for \( (2) H_{ab} \), following the discussion in subsection 4.3, we set all the modes except \( J = 2 \) to zero. Inspection of the right hand sides of Eqs. (4.45) and (4.46) reveals that the terms involving the Riemann tensor are in the same forms as those appeared in Eqs. (4.24), (4.25) and (4.26). Hence they contain only \( J = 2 \) modes and do not give any matching condition. Furthermore, from Eq. (4.40), all the modes except \( J = 2 \) contained in the term involving \( f^\alpha_{\ jkl} \) in Eq. (4.45) vanish at the lowest order in \( Gm/L \). Hence we only have to consider the remaining terms in Eqs. (4.45) and (4.46). The \( J = 1 \) modes are extracted out to give

\[ 0 = 2 \frac{dz^\alpha}{dT}(T) D f_{\alpha i}(T) + Gm \frac{dz^\alpha}{dT}(T) \frac{dz^\beta}{dT}(T) f^{\gamma i}(T) \Theta_{\alpha\beta\gamma}(T) \]
\[ + O \left( \left( \frac{Gm}{L} \right)^2 \frac{1}{L} \right), \quad (TT)\)-component, \quad (4.49) \]
\[ 0 = f_{\alpha i}(T) \frac{Df_{\beta j}}{dT}(T) + Gm \Theta_{\alpha \beta \gamma}(T) \frac{dz^\alpha}{dT}(T)f^\beta_i(T)f^\gamma_j(T) + O \left( \left( \frac{Gm}{L} \right)^2 \frac{1}{L} \right), \quad (Ti)\text{-component.} \tag{4.50} \]

The \( J = 0 \) mode is absent in the \((TT)\)-component, while that in the \((Ti)\)-component exists but it just gives the equation which determines \((dz^\alpha/dT)f_{\alpha ii}\) to the first order in \(Gm/L\).

Taking the covariant \( T \)-derivative of Eqs. (4.42) and (4.43) and using Eq. (4.49), we obtain the equations of motion with the \(O(Gm/L^2)\) correction due to the radiation reaction,

\[ D \frac{dz^\alpha}{dT}(T) = -\frac{Gm}{2} \left( \Theta_{\alpha \beta \gamma}(T) + \Theta_{\alpha \gamma \beta}(T) - \Theta_{\beta \gamma \alpha}(T) \right) \frac{dz^\beta}{dT}(T) \frac{dz^\gamma}{dT}(T) + O \left( \left( \frac{Gm}{L} \right)^2 \frac{1}{L} \right). \tag{4.51} \]

Similarly the \(O(Gm/L^2)\) correction to the evolution equations of the ‘triad’ basis, \( f_{\alpha i}(T) \), are obtained from the covariant \( T \)-derivative of Eq. (4.44), and Eqs. (4.49) and (4.50). The result is

\[ D \frac{f_{\alpha i}}{dT}(T) = -\frac{Gm}{2} \left( \Theta_{\alpha \beta \gamma}(T) + \Theta_{\alpha \gamma \beta}(T) - \Theta_{\beta \gamma \alpha}(T) \right) f^\beta_i(T) \frac{dz^\gamma}{dT}(T) + O \left( \left( \frac{Gm}{L} \right)^2 \frac{1}{L} \right). \tag{4.52} \]

Since the internal time coordinate \( T \) is not properly normalized in the external metric, we define the proper time, \( \tau = \tau(T) \), such that \( (dz/d\tau)^2 = -1 \). It is easy to see that we should choose

\[ \frac{d\tau}{dT} = 1 + \frac{Gm}{2} \Theta_{\alpha \beta}(T) \frac{dz^\alpha}{d\tau}(T) \frac{dz^\beta}{d\tau}(T) + O \left( \left( \frac{Gm}{L} \right)^2 \right). \tag{4.53} \]

Since the second term on the right hand side of this equation is proportional to the small perturbation induced by the particle, it is guaranteed to stay small even after a long time interval compared with the reaction time scale \( T_r = O \left( (Gm/L)^{-1}L \right) \). Then Eq. (4.51) becomes

\[ D \frac{dz^\alpha}{d\tau}(\tau) = -\frac{Gm}{2} \left( \frac{dz^\alpha}{d\tau} \frac{dz^\beta}{d\tau} \frac{dz^\gamma}{d\tau} \frac{dz^\delta}{d\tau} + 2g^{\alpha \beta}(z) \frac{dz^\gamma}{d\tau} \frac{dz^\delta}{d\tau} - g^{\alpha \delta}(z) \frac{dz^\beta}{d\tau} \frac{dz^\delta}{d\tau} \right)(\tau) \Theta_{\beta \gamma \delta}(\tau) + O \left( \left( \frac{Gm}{L} \right)^2 \frac{1}{L} \right). \tag{4.54} \]

Also, the triad basis are not properly normalized in the external metric. Thus we define \( e_{\alpha i}(\tau) \) as

\[ e_{\alpha i}(\tau)e^\alpha_j(\tau) = \delta_{ij}, \tag{4.55} \]
where \( s_{ij} \) is of \( O(Gm/L) \) and recalling Eq. (4.43) the last term is added so as to satisfy the orthonormal condition,
\[
e_{\alpha i}(\tau)(dz^\alpha/d\tau)(\tau) = 0.
\] (4.57)

From Eq. (4.55) we find
\[
s_{ij} = -\frac{Gm}{2} \Theta_{\alpha\beta}(\tau) f^\alpha_i(\tau) f^\beta_j(\tau) + O\left(\left(\frac{Gm}{L}\right)^2\right).
\] (4.58)

Again the correction terms in \( e^\alpha_i \) are guaranteed to stay small. Then the evolution equations of the normalized triad \( e^\alpha_i(\tau) \) become
\[
\frac{D}{d\tau} e^\alpha_i(\tau) = -\frac{Gm}{2} \left(\frac{dz^\alpha}{d\tau} \frac{dz^\beta}{d\tau} e^\gamma_i + g^{\alpha\beta}(z) \frac{dz^\gamma}{d\tau} e^\delta_i - g^{\alpha\delta}(z) e^\beta_j \frac{dz^\gamma}{d\tau}\right)(\tau) \Theta_{\beta\gamma\delta}(\tau) + O\left(\left(\frac{Gm}{L}\right)^2 \frac{1}{L}\right).
\] (4.59)

§5. Equations of motion for a spinning particle

In this section, we consider the equations of motion for a spinning particle. Different from the Schwarzschild case, we cannot make use of the mode decomposition by the spherical harmonics since the background in the internal scheme does not have the spherical symmetry. Therefore, it is quite unclear for us how to fix the gauge in the internal scheme, and hence we cannot derive the equations of motion by the consistency condition of matching.

Instead, we here apply the laws of motion and precession discussed by Thorne and Hartle. As noted in section §, assuming the consistency between the internal and external schemes, we can make use of the matching condition to obtain the internal metric from the knowledge of the external metric. The problem to derive the equations of motion for a spinning particle was discussed by Thorne and Hartle and the spin-induced force was derived. The discussion given below is an extension of Ref. § in the sense that we take into account the effect of radiation reaction to the motion. Both derivations of the radiation reaction and the spin-induced force are discussed in a unified manner.

5.1. Laws of motion and precession

The laws of motion and precession are derived from the integral identities given in terms of the Landau-Lifshitz pseudo-tensor, \( t^\alpha_\beta_{L-L} \), and the Landau-Lifshitz super-potential, \( H^\alpha_\beta_{L-L} \). The Einstein equations can be put into the form,
\[
H^\alpha_\gamma^\beta_\delta_{L-L} = 16\pi G(-g)\left(T^\alpha_\beta + t^\alpha_\beta_{L-L}\right),
\] (5.1)
where

\[ H_{L-L}^{\alpha\gamma\delta\beta} = g^{\alpha\beta} g^{\gamma\delta} - g^{\alpha\delta} g^{\gamma\beta}, \quad (5.2) \]

\[ (-g) t_{L-L}^{\alpha\beta} = \frac{1}{16\pi} \left\{ g^{\alpha\beta} g^{\gamma\delta} - g^{\alpha\gamma} g^{\beta\delta} + \frac{1}{2} g^{\alpha\beta} g^{\gamma\delta} \zeta, \zeta - \frac{1}{8} \left( 2 g^{\alpha\gamma} g^{\beta\delta} + g^{\alpha\beta} g^{\gamma\delta} \right) \right\}, \quad (5.3) \]

\[ g^{\alpha\beta} := (-g)^{1/2} g^{\alpha\beta}, \quad (5.4) \]

and a comma denotes the ordinary derivative. By construction, the following conservation laws are satisfied:

\[ \left( (-g) \left( T^{\alpha\beta} + t_{L-L}^{\alpha\beta} \right) \right)_\beta = 0. \quad (5.5) \]

Suppose that the internal metric around a Kerr black hole is calculated for a given trajectory of the particle. In terms of the internal metric we define

\[ P^a(T, r) := \frac{1}{16\pi G} \int_{|X| = r} d^2 S_j H_{L-L}^{a0j}, \quad (5.6) \]

\[ J^{ij}(T, r) := \frac{1}{16\pi G} \int_{|X| = r} d^2 S_k \left( X^i H_{L-L}^{j0k} - X^j H_{L-L}^{i0k} \right) + H_{L-L}^{ijk0} - H_{L-L}^{jik0}, \quad (5.7) \]

where \( d^2 S_j \) is the surface element of a two-sphere at \( |X| = r \). Then by using the Einstein equations (5.1), we have the following integral identities:

\[ \frac{d}{dT} P^a(T, r) = \int_{|X| = r} d^2 S_j (-g) t_{L-L}^{a0j}(X), \quad (5.8) \]

\[ \frac{d}{dT} J^{ij}(T, r) = \int_{|X| = r} d^2 S_k \left( X^i (-g) t_{L-L}^{jk}(X) - X^j (-g) t_{L-L}^{ik}(X) \right). \quad (5.9) \]

These are called the laws of motion and precession. By explicitly evaluating the right hand sides of Eqs. (5.6), (5.7), (5.8) and (5.9), and eliminating \( P^a(T, r) \) and \( J^{ij}(T, r) \) from the resulting equations, one obtains the equations of motion.

5.2. Use of the matched asymptotic expansion

In the present method, we construct the external metric and use the matching conditions to obtain the necessary components of the internal metric. The \( (0_n)^n \)-components of the internal metric are assumed to be given by the metric of a Kerr black hole. Since we do not construct the internal metric independently, there exists no a priori requirement for thus obtained internal metric to satisfy some specific gauge condition. Hence the transformation from the external coordinates to the internal ones can be rather arbitrarily chosen. Here, we make use of the knowledge we have obtained in section 4 and we choose the coordinate conditions as follows.
We assume that the external metric is generated by the point-like source, Eq. (3.28), and calculate the external metric in the matching region as in the previous section. In order to do so, the hypersurfaces of $T = \text{constant}$ and $T_x = \text{constant}$ should be identical to each other as given by Eq. (4.35). To satisfy this requirement, we adopt the coordinate transformation from $x$ to $X$ in the form,

$$\sigma, \alpha(x, z(T)) + f, \alpha(T) X^i = O((Gm)^2/L).$$  \hspace{1cm} (5.10)

This is satisfied by setting

$$L f, \alpha_{ij} = \frac{L^2 f, \alpha_{ijk}}{2} = O\left(\frac{(Gm)}{L}\right),$$  \hspace{1cm} (5.11)

in Eq. (3.59). Note that, in the case of a monopole particle discussed in section 4, the conditions that are required to guarantee Eq. (4.35) are obtained from the $(0^2)$-matchings $(n = 0, 1, 2)$. On the contrary, here we impose the conditions (5.11) by hand to guarantee Eq. (4.35).

Furthermore, to determine the internal metric from the matching conditions, we set the $(1^n)$-components of the internal metric to zero:

$$^{(1)}_{(n)} H_{ab} = 0 \hspace{0.5cm} (n = 0, 1, 2, \ldots).$$  \hspace{1cm} (5.12)

In the case of a monopole particle, we have found we can impose these conditions. However, in the present case, since we have imposed the coordinate condition (5.10) by hand, it is not clear if a similar argument can be made to justify these conditions. Nevertheless, at least for $n = 0, 1$, we should be able to require the conditions (5.12). This is because the spin of the black hole appears at $O\left(\left(\frac{(Gm)}{L}\right)^2\right)$ or higher in the internal metric, hence the discussion we gave in the case of a monopole particle should be equally applicable to the $(1^0)$ and $(1^1)$-components of the metric. In fact, we see below that the conditions (5.12) for $n = 0, 1$ consistently determine the internal metric in the local rest frame by matching.

First consider the background metric in the internal scheme. For convenience we define the trace-reversed $(m^n)$-components of the metric with respect to the flat Minkowski space:

$$^{(m)}_{(n)} \bar{H}_{ab} = \left(\begin{array}{c} m \\ n \end{array}\right) H_{ab} - \frac{1}{2} \eta_{ab} \eta^{cd} \left(\begin{array}{c} m \\ n \end{array}\right) H_{cd}$$  \hspace{1cm} (5.13)

Expanding the Kerr metric with respect to $Gm$, the $(0^n)$-components of the metric in the harmonic coordinates are found as

$$^{(0)}_{(n)} H_{ab} = \eta_{ab},$$  \hspace{1cm} (5.14)

$$Gm^{(0)}_{(1)} \bar{H}_{TT} = \frac{4Gm}{|X|},$$  \hspace{1cm} (5.15)

$$Gm^{(0)}_{(1)} \bar{H}_{Ti} = 0,$$  \hspace{1cm} (5.16)

$$Gm^{(0)}_{(1)} \bar{H}_{ij} = 0,$$  \hspace{1cm} (5.17)
where $S_{ij}$ is the specific spin tensor in the local rest frame of the black hole. Then calculating the $(00)$ and $(01)$-components of the external metric in the matching region, we find they are consistent with Eqs. (5.14) ∼ (5.17) provided that $\dot{z}(T)$ and $f_i^\alpha(T)$ satisfy the lowest order orthonormal conditions, Eqs. (4.13), (4.14) and (4.15). Further, the spin contribution to the metric, Eq. (5.19), can be reproduced from the external metric with the source (3.28) by the identification,

$$S_{\alpha\beta}(T) = S_{ij}f_i^\alpha(T)f_j^\beta + O((Gm)^2/L).$$

(5.21)

This fact indicates the consistency of using the point-particle energy momentum tensor (3.28) in the perturbation analysis.

Keeping in mind the imposed conditions (5.11) and (5.12), the calculation of the $(10)$-components of the external metric in the matching region gives

$$\frac{dz^\alpha}{dT}(T)\frac{Df_{ai}}{dT}(T) = O(Gm/L^2), \quad f_i^\alpha(T)\frac{Df_{aj}}{dT}(T) = O(Gm/L^2).$$

(5.22)

As before, these equations imply that $f_i^\alpha(T)$ is parallel transported along the particle trajectory at the lowest order. Similarly, the calculation of the $(11)$-components of the external metric gives the same conditions as we have found in the previous section (see Eqs. (4.42) ∼ (4.44)):

$$\dot{z}^2(T) = -1 - \frac{Gm}{2}\bar{\Theta}_{\alpha\beta}(T) \left( g^{\alpha\beta}(z(T)) + 2\dot{z}^\alpha(T)\dot{z}^\beta(T) \right) + O((Gm)^2/L^2),$$

(5.23)

$$\dot{z}^\alpha(T)f_i^\alpha(T) = -Gm\bar{\Theta}_{\alpha\beta}(T)\dot{z}^\alpha(T)f_i^\beta(T) + O((Gm)^2/L^2),$$

(5.24)

$$f_i^\alpha(T)f_{ai}^\alpha(T) = \delta_{ij} - \frac{Gm}{2}\bar{\Theta}_{\alpha\beta}(T) \left( -\delta_{ij}g^{\alpha\beta}(z(T)) + 2f_i^\alpha(T)f_j^\beta(T) \right) + O((Gm)^2/L^2),$$

(5.25)

where we have introduced

$$\bar{\Theta}_{\alpha\beta} = \Theta_{\alpha\beta} - 1/2g_{\alpha\beta}\Theta^{\delta} = \frac{1}{Gm}\bar{\psi}_{(tail)\alpha\beta}.$$

(5.26)

These equations may be viewed as a coordinate condition on the internal time $T$. Clearly there is no inconsistency in them.

Computation of the rest of components of the internal metric which are needed to derive the equations of motion is straightforward. The results are

$$\frac{1}{L^2}(2)\bar{H}_{TT} = -\frac{2}{3}R_{\alpha\beta\gamma\delta}(z(T))\dot{z}^\alpha(T)X^\beta(T)\dot{z}^\gamma(T)X^\delta(T).$$
\[\frac{1}{L^2} \hat{H}_{TT} = \frac{2}{3} \rho_{\alpha\beta\gamma}(z(T)) f^\alpha_i(T) f^\beta_j(T) + \delta_{ij} \dot{z}^\alpha(T) \dot{z}^\gamma(T) + O(Gm^2 |X|/L^3), \]

\[\frac{1}{L^2} \frac{d}{dT} \dot{J} = m S_{\alpha\beta}(T) f^{\alpha\beta} + O(G^2 m^3/L) + (r\text{-dependent terms}). \]

Equation (5.9) vanishes: \[\frac{d}{dT} \dot{J} = O(G^2 m^3/L^2) + (r\text{-dependent terms}). \]
Since the spatial triad are geodetic parallel transported in the background geometry to the leading order, Eqs. (5-34) and (5-35) result in
\[ \frac{D}{dT}S^{\alpha\beta}(T) = O\left(\frac{(Gm)^2}{L^2}\right). \]  
(5-36)
Thus in the test particle limit \( m \to 0 \) the spin tensor is parallel transported along the particle trajectory in the background geometry.

We next consider Eqs. (5-6) and (5-8). Equation (5-6) has a dimension of \((mass)^1\) and we extract out the terms of \( O(m) \) and \( O(Gm^2/L) \). We find that there will be linear contributions from \( (\bar{l}^1)\)-, \( (\bar{l}^0)\)-components of the metric, and bilinear contributions from pairs of \( (\bar{l}^1)\)– and \( (\bar{l}^0)\)-components of the metric. We obtain
\[ P^{0}(T, r) = m + O(G^2m^3/L^2) + (r\text{-dependent terms}), \]  
(5-37)
\[ P^{i}(T, r) = O(G^2m^3/L^2) + (r\text{-dependent terms}). \]  
(5-38)
Eq. (5-8) has a dimension of \((mass)/(length)\) and we consider the terms of \( O(m/L) \) and \( O(Gm^2/L^2) \). There will be bilinear contributions from pairs of the \( (\bar{l}^0)\)- and \( (\bar{l}^2)\)-components and pairs of the \( (\bar{l}^0)\)– and \( (\bar{l}^2)\)-components of the metric. We find the former pairs give the spin-induced force and the latter pairs give the radiation reaction force. A straightforward computation results in
\[ \frac{d}{dT}P^{0}(T, r) = O(G^2m^3/L^3) + (r\text{-dependent terms}), \]  
(5-39)
\[ \frac{d}{dT}P^{i}(T, r) = -\frac{m}{2}R_{\alpha\beta\gamma\delta}(z(T))f^{\alpha i}(T)\dot{z}^{\beta}(T)S^{\gamma\delta}(T) \]
\[ + \frac{Gm^2}{4} \Theta_{\alpha\beta\gamma}(T) f^{\gamma i}(T) \left( 2\dot{z}^{\alpha}(T)\dot{z}^{\beta}(T) + g^{\alpha\beta}(z(T)) \right) \]
\[ + m\dot{z}^{\alpha}(T)D_{\alpha i}\frac{d}{dT}f_{\alpha i}(T) \]
\[ + O(G^2m^3/L^3) + (r\text{-dependent terms}). \]  
(5-40)
Taking the \( T \)-derivative of Eqs. (5-23) and (5-24), we obtain the equations of motion,
\[ \frac{D}{dT}\dot{z}^{\alpha}(T) = -\frac{Gm}{2} \Theta_{\alpha\beta\gamma\delta}(T) \left( 2\dot{z}^{\beta}(T)g^{\alpha\gamma}(z(T))\dot{z}^{\delta}(T) - \dot{z}^{\beta}(T)\dot{z}^{\gamma}(T)g^{\alpha\delta}(z(T)) \right) \]
\[ - \frac{1}{2} R^{\alpha\beta\gamma\delta}(z(T))\dot{z}^{\beta}(T)S^{\gamma\delta}(T) + O(G^2m^2/L^3). \]  
(5-41)
Introducing the proper time \( \tau \) of the orbit,
\[ \frac{d\tau}{dT} = 1 + \frac{Gm}{2} \Theta_{\alpha\beta\gamma\delta}(T)\dot{z}^{\alpha}(T)\dot{z}^{\beta}(T), \]  
(5-42)
we finally arrive at
\[ \frac{D}{d\tau} \frac{dz^{\alpha}}{d\tau}(\tau) = -\frac{Gm}{2} \Theta_{\alpha\beta\gamma\delta}(\tau) \left( \frac{dz^{\alpha}}{d\tau} \frac{dz^{\beta}}{d\tau} \frac{dz^{\gamma}}{d\tau} \frac{dz^{\delta}}{d\tau} + 2 \frac{dz^{\beta}}{d\tau}g^{\alpha\gamma}(\tau) \frac{dz^{\delta}}{d\tau} - \frac{dz^{\beta}}{d\tau} \frac{dz^{\gamma}}{d\tau} g^{\alpha\delta}(\tau) \right) \]
\[ - \frac{1}{2} R^{\alpha\beta\gamma\delta}(\tau) \frac{dz^{\beta}}{d\tau} S^{\gamma\delta}(\tau) + O(G^2m^2/L^3), \]  
(5-43)
where \( Q(\tau) = Q(z(\tau)) \). One finds that the result is exactly equal to Eq. (4.54) except for the spin-curvature coupling term.

In the case of a monopole particle discussed in the previous section, the (\( q_1 \)) matching gave two conditions (4.49) and (4.50). The latter condition was crucial to obtain the \( O(Gm/L^2) \) correction terms in the evolution equations of \( f^\alpha_i \). In the present analysis, we do not have the counterpart of this condition. This indicates that the gauge condition relating with the notational mode must be specified to determine \( Df_{\alpha i}/dT \).

§6. Discussion

Let us first discuss the physical meaning of the equations of motion obtained in the preceding two sections. For simplicity, we consider the case of a monopole particle. We divide the perturbed metric in the external scheme into the two:

\[
h_{\mu\nu}(x) = h_{\mu\nu}^{(\text{mono})}(x) + h_{\mu\nu}^{(\text{tail})}(x),
\]

where \( h_{\mu\nu}^{(\text{tail})}(x) \) is the part due to the \( v_{\mu\nu\alpha\beta} \) in the Green function while \( h_{\mu\nu}^{(\text{mono})} \) is due to the \( u_{\mu\nu\alpha\beta} \) term (see Eq. (3.34)). The singular behavior of the perturbed metric in the coincidence limit \( x \rightarrow z \) is totally due to \( h_{\mu\nu}^{(\text{mono})}(x) \). Thus, we introduce the regularized perturbed metric as

\[
\tilde{g}_{\mu\nu}^{(\text{reg})}(x) := g_{\mu\nu}(x) + h_{\mu\nu}^{(\text{tail})}(x),
\]

which has no singular behavior any more. Then we find the equations of motion (4.51) and the evolution equations of the triad basis (4.52) coincide with the geodesic equation and the geodetic parallel transport equation, respectively, on the regularized spacetime with the metric \( \tilde{g}_{\mu\nu}^{(\text{reg})} \). To see this let us consider the parallel transport of a vector \( A^\alpha \) along a geodesic \( x^\alpha = z^\alpha(\tilde{\tau}) \) in this spacetime. It is given by

\[
\frac{\tilde{D}}{d\tilde{\tau}} A^\alpha := \frac{D}{d\tilde{\tau}} A^\alpha + \delta\Gamma_{\mu(\text{reg})}^{\alpha\beta\gamma} A^\beta \frac{dz^\gamma}{d\tilde{\tau}} = 0,
\]

to the linear order in \( h_{\mu\nu}^{(\text{tail})} \), where

\[
\delta\Gamma_{\mu(\text{reg})}^{\alpha\beta\gamma} := \frac{1}{2} \left( h_{\mu(\text{tail})}^{\alpha\beta;\gamma} + h_{\mu(\text{tail})}^{\alpha;\beta\gamma} - h_{\mu(\text{tail})}^{\beta\gamma;\alpha} \right).
\]

Then one recovers Eqs. (4.51) and (4.52) by identifying \( \tilde{\tau} \) with \( T \) and replacing \( A^\alpha \) with \( dz^\alpha/dT \) or \( f^\alpha_i \). In the case of a spinning particle, there exists an additional force in the equations of motion (5.41) due to the coupling of the spin and the background curvature.

The result for the monopole particle seems analogous to that in the electromagnetic case\[\text{[1]}\], except that the instantaneous reaction force which is proportional to higher derivatives of the particle velocity is absent in the present case. This is because the particle traces a geodesic in the lowest order approximation. If an external force field exists, the assumption of the geodetic motion in the lowest order breaks...
down and furthermore the contribution of the external force field to the energy momentum tensor must be taken into account. Since this fact makes the problem too complicated, it is beyond the scope of the present discussion.

Now let us consider how to construct $\tilde{g}_{\langle \text{reg} \rangle \mu \nu}$, in the case of a monopole particle. Unfortunately, we do not have any satisfactory formalism that can be applied to such a calculation, even for a specific background spacetime such as the Kerr geometry, mainly due to the difficulty in evaluating the bi-tensor $v_{\rho \sigma \alpha \beta}(x, z)$. Here we just give a few primitive discussions on this matter.

Basically, there seems to be two approaches for calculating $\tilde{g}_{\langle \text{reg} \rangle \mu \nu}$ (or equivalently $h_{\langle \text{tail} \rangle \mu \nu}$). The first one is to calculate $h_{\langle \text{tail} \rangle \mu \nu}$ directly. The second one is to calculate $h_{\mu \nu} = h_{\langle \text{mono} \rangle \mu \nu} + h_{\langle \text{tail} \rangle \mu \nu}$ and subtract $h_{\langle \text{mono} \rangle \mu \nu}$ from it. In the following, we discuss only the first approach. As for the second approach, we have nothing to mention here, but this direction of research may be fruitful.

By definition, $h_{\langle \text{mono} \rangle \mu \nu}$ evaluated on the particle trajectory is independent of the past history of the particle. Therefore if we consider the metric defined by

$$h_{\mu \nu}^{(\Delta \tau)}(x) = Gm \left( \delta_{\mu}^{\rho} \delta_{\nu}^{\sigma} - \frac{1}{2} g_{\mu \nu} g^{\rho \sigma}(x) \right) \int_{-\infty}^{\tau_x - \Delta \tau} d\tau' G_{\rho \sigma \alpha \beta}(x, z(\tau')) \dot{z}^{\alpha}(\tau') \dot{z}^{\beta}(\tau'),$$

for any finite $\Delta \tau > 0$, it will not contain $h_{\langle \text{mono} \rangle \mu \nu}$ when it is evaluated on the particle trajectory. The difference between $h_{\mu \nu}^{(\Delta \tau)}$ and $h_{\langle \text{tail} \rangle \mu \nu}$ comes from the integral over a small interval,

$$\sim Gm \int_{\tau_x - \Delta \tau}^{\tau_x} d\tau' v_{\rho \sigma \alpha \beta}(x, z(\tau')) \dot{z}^{\alpha}(\tau') \dot{z}^{\beta}(\tau').$$

Since $v_{\rho \sigma \alpha \beta}(x, z)$ is regular in the coincidence limit $x \to z$, this integral will be negligible for a sufficiently small $\Delta \tau$. Thus $\lim_{\Delta \tau \to 0} h_{\mu \nu}^{(\Delta \tau)}$ will give $h_{\langle \text{tail} \rangle \mu \nu}$.

In the case of the electromagnetic (vector) Green function, a calculation along the above strategy was performed by DeWitt and DeWitt by assuming the background gravitational field is weak so that its metric is given by the small perturbation on the Minkowski metric,

$$g_{\mu \nu} = \eta_{\mu \nu} + h_{\mu \nu}^{(b)}.$$  \hspace{1cm} (6.7)

DeWitt and DeWitt calculated the relevant part of the Green function perturbatively to the first order in $h_{\mu \nu}^{(b)}$ by using the Minkowski Green function.

Here we should mention one important fact. We have obtained the equations of motion with the correction term of $O(Gm/L^2)$. Although we use the terminology ‘radiation reaction’ to describe it, it is not appropriate in a narrow sense because the correction term may well contain something more than just the usual effect of radiation reaction. In fact, in the electromagnetic case, the existence of the effect which is termed as ‘the induced polarization force on the background spacetime’ is reported by several authors. Furthermore, a calculation analogous to that done

\footnote{There is a possibility that the future light cone emanating from $z$ crosses the particle trajectory again. Since inclusion of this possibility makes the problem too complicated, we do not consider it here.}
by DeWitt and DeWitt\textsuperscript{[1]} for the electromagnetic case was done by Carmeli\textsuperscript{[3]} for the gravitational case and it was shown that the tail part correctly reproduces the lowest order post-Newtonian corrections to the equations of motion. However, no such calculation has been done for the background with strong gravity, such as a black hole spacetime. It seems difficult to develop DeWitt and DeWitt’s approach to higher orders in $h^{(b)}_{\mu\nu}$. It is a challenging issue to formulate a systematic method to evaluate the tail part of the metric when the background gravity is strong and clarify its physical content.

Turning back to the effect of the gravitational radiation reaction, we should make one additional comment. There has been some proposals to obtain the radiation reaction force in a quite different manner. Among others is the use of the radiative Green function (a half of the retarded minus advanced Green functions) in the case of a Kerr background proposed by Gal’tsov\textsuperscript{[4]}. As easily seen from the results in section\textsuperscript{[2]} the use of the radiative Green function instead of the retarded one results in the replacement of $\psi_{(\text{tail})\mu\nu}(x)$ by $\psi_{\mu\nu}^{\text{Rad}}(v)$, which is defined by

$$
\psi_{\mu\nu}^{\text{Rad}}(v) := -Gm \int_{-\infty}^{+\infty} d\tau' v_{\beta'\gamma'\alpha'}(x, z(\tau')) z^{\alpha'}(\tau') z^{\beta'}(\tau').
$$

(6.8)

Gal’tsov proved that the back reaction force computed using the radiative Green function gives the loss rates of the energy and the $z$-component of the angular momentum of the particle in quasi-periodic orbits which correctly balance with the emission rates of the corresponding quantities by gravitational radiation. However, we do not think that this fact indicates the correctness of the prescription, even if we restrict it to the case of a Kerr background, because those constants of motion are special ones which reflect the existence of the corresponding Killing vector fields. For such quantities, there may be some symmetry in the structure of the Green function which makes the use of the radiative Green function valid. However, it is doubtful that the radiative Green function correctly describes the radiation reaction effect on the Carter constant.

Finally we make a couple of comments on the implications of our results. It is important to note that the particle does not have to be a black hole since the detailed internal boundary condition was not used to determine the metric in the internal scheme. The resulting equations of motion should be equally applicable to any compact body such as a neutron star. The essential assumption here is that the only length scale associated with the particle is $Gm$. In this sense, we have shown the strong equivalence principle to the first order in $Gm$.

We also note that our results strongly support, if not rigorously justify, the so-called black hole perturbation approach. In the black hole perturbation approach, one calculates the gravitational radiation from a particle orbiting a black hole with the assumption that the particle is a point-like object with the energy momentum tensor described by the delta function. Although this approach has been fruitful, there has been always skepticism about the validity of the delta functional source. What we have shown in this chapter is the consistency of using the delta function in the source energy momentum tensor within the order of matched asymptotic expansion we have examined.
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