Spontaneous CP violation and symplectic modular symmetry in Calabi-Yau compactifications

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Abstract

We explore the geometrical origin of CP and the spontaneous CP violation in Calabi-Yau compactifications. We find that the CP symmetry is identified with an outer automorphism of the symplectic modular group in the large complex structure regime of Calabi-Yau threefolds, thereby enlarging the symplectic modular group to their semidirect product group. The spontaneous CP violation is realized by the introduction of fluxes, whose effective action is invariant under CP as well as the discrete $\mathbb{Z}_2$ symmetry or $\mathbb{Z}_4$ R-symmetry. We explicitly demonstrate the spontaneous CP violation on a specific Calabi-Yau threefold.

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1 Introduction

The origin of CP violation is one of important issues to study in particle physics. Indeed, it has been studied extensively in various scenarios. CP violation may originate from an underlying theory such as superstring theory.

As discussed in Refs. [1–3], one can embed the four-dimensional (4D) CP symmetry into proper Lorentz symmetry in higher dimensional theory, e.g., ten-dimensional (10D) proper Lorentz symmetry in superstring theory on six-dimensional (6D) Calabi-Yau (CY) manifolds. That is, as the 10D proper Lorentz transformation, one performs simultaneously the 4D space-time orientation and orientation reversing transformation of CY threefolds. The latter 6D transformation can be achieved by \( z_i \rightarrow -z_i^* \) with \( z_i, \ i = 1, 2, 3 \), being the complex coordinates, corresponding to the negative determinant in the transformation of the 6D CY manifolds. (See also for CP symmetry on 6D orbifold compactifications [4].) The 6D transformation reverses the sign of the volume form as well as the Kähler form of CY threefolds. To clarify the nature of CP symmetry, let us consider Majorana-Weyl spinor representation of 10D Lorentz group \( SO(1,9) \) which decomposes as \( 16 = (2, 4) \oplus (2', \bar{4}) \) under the 4D and the 6D Lorentz groups \( SO(1,3) \times SO(6) \). Here, 2 and 2’ denote the left- and right-handed spinor representations of \( SL(2,\mathbb{C}) \), and 4 and \( \bar{4} \) represent the positive- and negative-chirality spinor representations of \( SO(6) \simeq SU(4) \), respectively. Then, simultaneous transformations of 4D parity and 6D orientation reversing exchange \( (2, 4) \) with \( (2', \bar{4}) \). For example, in the context of standard embedding in the heterotic string theory, fundamental and anti-fundamental representations of \( E_6, 27 \) and \( 27' \), correspond to 4 and \( \bar{4} \), respectively. Hence, they are exchanged under such simultaneous transformations. Such a CP transformation is identified with an outer automorphism transformation of gauge symmetries [5] and each of 4D and 6D Lorentz symmetries [6].
In Type II D-brane models, massless fermions localize on D-branes wrapping certain cycles of the CY threefold. When we consider the orientation reversing transformation on the internal cycles wrapped by D-branes, one can also embed the 4D CP into the higher-dimensional Lorentz transformation on the worldvolume of D-branes on which CP exchanges left-handed massless matters with their complex conjugates.

The torus compactification as well as toroidal orbifold compactifications is one of simplest compactifications. The 2D torus has two independent one-cycles. We have the degree of freedom to change bases of these cycles. That is the $SL(2, \mathbb{Z})$ transformation, which is called as the modular symmetry. The modular group is the geometrical symmetry, and also transforms zero-modes corresponding to quarks and leptons, that is, the flavor symmetry. (See for the modular symmetry in magnetized D-brane models [7–9] and the classification of its subgroup [10].) When there are 4D models having non-trivial flavor symmetries, the 4D CP symmetry can be enlarged to a generalized CP symmetry, which includes outer automorphisms of flavor symmetries [12–14]. Such a generalized CP symmetry is also extended to the modular symmetry on the torus compactification, where CP symmetry is identified with the outer automorphism of $SL(2, \mathbb{Z})$ [15, 16].

The size and shape of a compact space is described by moduli parameters, which are vacuum expectation values of moduli fields. Their vacuum expectation values can be determined at the potential minima of moduli fields. The complex structure moduli also transform under the CP symmetry, $z_i \rightarrow -z_i^*$. Thus, the CP symmetry can be spontaneously broken through the moduli stabilization. For example, the spontaneous CP-violation was studied by assuming non-perturbative moduli potentials [17–20]. The three-form flux background is of controllable ways to stabilize the complex structure moduli as well as the axio-dilaton [21]. Indeed, in Ref. [23], spontaneous CP-violation was studied within the framework of Type IIB superstring theory on toroidal orbifold compactifications with three-form flux background. The CP-invariant superpotential, which is induced by three-form fluxes, is either even or odd polynomial functions of the complex structure moduli and the axio-dilaton. The potential minima were examined, but the spontaneous CP-violation can not be realized in the torus compactifications.

Our purpose in this paper is to reveal the geometrical origin of CP and search for the spontaneous CP violation in CY compactifications, with an emphasis on the effective action of the complex structure moduli. As discussed in Ref. [1], the orientation reversing isometry of CY threefolds induces anti-holomorphic transformations for the complex structure, which restrict the form of low-energy effective action. We find that CP symmetry is identified with the outer automorphism of the symplectic modular group of generic CY threefolds in the large complex structure regime. That is a natural extension of the discussion in toroidal backgrounds, where the CP symmetry is also identified with the outer automorphism of $SL(2, \mathbb{Z})$ modular group on the 2D torus.

Furthermore, we discuss the spontaneous CP violation in the CY flux compactifications. Flux compactifications in the string theory have a potential to break CP spontaneously. In a similar discussion with the toroidal background [23], the flux-induced superpotential is restricted to be odd or even polynomials with respect to the moduli fields, having the discrete

\[1\text{Recently, the modular flavor symmetries are studied extensively from the phenomenological viewpoint [11].}\]
$\mathbb{Z}_2$ symmetry or $\mathbb{Z}_4$ R-symmetry from the field theoretical point of view. It turns out that the CP-conserving vacua exist in the large complex structure regime of generic CY threefolds. The spontaneous CP violation occurs in some class of CY threefolds whose prepotentials have a structure different from the toroidal one.

This paper is organized as follows. In Sec. 2, we show a relationship between the CP symmetry and the symplectic modular symmetry. Sec. 3 is devoted to the construction of the flux-induced effective action in a CP-invariant way. Concrete CY flux compactifications are analyzed in Sec. 4, in which the CP-breaking and -conserving vacua are demonstrated. Finally, we conclude our results in Sec. 5.

## 2 CP and symplectic modular symmetries

In this section, we focus on the complex structure moduli of CY threefolds $\mathcal{M}$, whose effective action is described by the Kähler potential in the reduced Planck mass unit $M_{Pl} = 1$,

$$K_{cs} = - \ln \left[ -i \int_{\mathcal{M}} \Omega \wedge \bar{\Omega} \right],$$

where $\Omega$ denotes a holomorphic three-form of CY threefolds determining the complex structure of $\mathcal{M}$. We examine the CP invariance of the holomorphic three-form $\Omega$ as well as its relation to the symplectic modular symmetry on $\mathcal{M}$.

Recalling that $i\Omega \wedge \bar{\Omega}$ is represented by the volume form of CY threefolds $dV$, namely $i\Omega \wedge \bar{\Omega} = ||\Omega||^2 dV$ with $||\Omega||^2$ being a scalar, the orientation reversing transformation changes the sign of the volume form leading $i\Omega \wedge \bar{\Omega} \rightarrow -i\Omega \wedge \bar{\Omega}$. It results in the transformation of $\Omega$ under the orientation reversing

$$\Omega \rightarrow -\bar{\Omega}.$$ (2)

Note that in the local coordinates of CY threefolds $\{z_i\}$, the holomorphic three-form is given by $\Omega = dz_1 \wedge dz_2 \wedge dz_3$. Hence, the orientation reversing transformation $z_i \rightarrow -\bar{z}_i$ gives rise to Eq. (2).

Since CY threefolds are described by the special geometry\footnote{Although the following discussion also holds for the anti-holomorphic involution $\Omega \rightarrow \bar{\Omega}$ corresponding to $z_i \rightarrow z_i$ in the local patch, we adopt the orientation reversing in Eq. (2) without loss of generality.}, the holomorphic three-form is expanded in the symplectic basis. When we denote $(A^I, B_I)$ ($I, J = 0, 1, \cdots, h^{2,1}(\mathcal{M})$) a canonical homology basis for $H_3(\mathcal{M}, \mathbb{Z})$, the dual cohomology basis $(\alpha_I, \beta^I)$ is defined such that

$$\int_{A^J} \alpha_I = - \int_{B^I} \beta^J = \int_{\mathcal{M}} \alpha_I \wedge \beta^J = \delta^I_J, \quad \int_{\mathcal{M}} \alpha_I \wedge \alpha_J = \int_{\mathcal{M}} \beta^I \wedge \beta^J = 0.$$ (3)

In terms of the real three-form basis $(\alpha_I, \beta^I)$, the holomorphic three-form can be expanded as

$$\Omega = X^I \alpha_I - F_I \beta^I,$$ (4)
with \( \mathcal{F}_I \equiv \partial_I \mathcal{F} \). Here, \( \mathcal{F} \) is the prepotential as a function of homogeneous coordinates \( X^I \) on the moduli space. Note that we have rescaling degrees of freedom on \( X^I \).

Let us analyze the CP transformation of \( \Omega \) in the symplectic basis in more detail. The transformation of \( \Omega \) in Eq. (2) is satisfied when

\[
X^0 \alpha_0 \rightarrow -X^0 \alpha_0, \quad X^i \alpha_i \rightarrow -X^i \alpha_i, \quad \mathcal{F}_0 \beta^0 \rightarrow -\mathcal{F}_0 \beta^0, \quad \mathcal{F}_i \beta^i \rightarrow -\mathcal{F}_i \beta^i.
\]  

(5)

We define the flat coordinates \( u^i = X^i/X^0, \; i = 1, 2, \ldots, h^{2,1} \). The orientation reversing transformation requires that the complex structure moduli should transform \( u^i \rightarrow \pm \bar{u}^i \). In the following discussion, we adopt \( u^i \rightarrow -\bar{u}^i \), restricting ourselves to the \( \text{Im} u^i > 0 \) plane\(^4\) and focus on the large complex structure regime. Note that the space of harmonic forms splits under the CP transformation into even and odd eigenspaces, for instance,

\[
H^3(\mathcal{M}) = H^3_{\text{CP}}(\mathcal{M}) + H^3_{\text{CP}}(\mathcal{M}),
\]

(6)

for the third cohomology group. The CP-even and -odd bases are elements of

\[
H^3_{\text{CP}}(\mathcal{M}) = \{ \alpha^{(\text{CP})}_M, \beta^{(\text{CP})}_M \}, \quad H^3_{\text{CP}}(\mathcal{M}) = \{ \tilde{\alpha}^{(\text{CP})}_M, \tilde{\beta}^{(\text{CP})}_M \},
\]

(7)

with \( M, \tilde{M} = 0,1,\ldots,h^{2,1} \), whose intersections satisfy

\[
\int_{\mathcal{M}} \alpha^{(\text{CP})}_N \wedge \beta^{(\text{CP})}_M = \delta^M_N, \quad \int_{\mathcal{M}} \tilde{\alpha}^{(\text{CP})}_N \wedge \tilde{\beta}^{(\text{CP})}_M = \delta^{\tilde{M}}_N,
\]

(8)

otherwise 0. The definition of CP-even/odd bases are analogous to Type IIA orientifolds with O6-planes \([26]\) and also consistent with orientifold projection in Type IIB string theory with O3/O7-planes \([27]\) as discussed in Sec. 3.1. These structures can be seen in the local patch parametrized by \( z_i = x_i + i y_i \). Indeed, CP-even and odd bases under the orientation reversing \( \{ x_i \rightarrow -x_i, y_i \rightarrow y_i \} \) are locally given by \( \{ dx_i \wedge dx_j \wedge dy_k, dy_i \wedge dy_j \wedge dy_k \} \) and \( \{ dx_i \wedge dx_j \wedge dx_k, dx_i \wedge dy_j \wedge dy_k \} \), \( i \neq j \neq k \), taking into account Eq. (8), respectively. Then, we allow two types of CP transformations of \( \{ X^I, \mathcal{F}_I \} \) and three-form basis to realize Eq. (3):

- \( \{ \alpha^{(\text{CP})}_i, \beta^{(\text{CP})} \} \in H^3_{\text{CP}}(\mathcal{M}), \; \{ \tilde{\alpha}^{(\text{CP})}_0, \tilde{\beta}^{(\text{CP})} \} \in H^3_{\text{CP}}(\mathcal{M}) \) (\( i = 1, 2, \ldots, h^{2,1} \))

\[
X^0 \rightarrow +X^0, \quad X^i \rightarrow -X^i, \quad \mathcal{F}_0 \rightarrow -\mathcal{F}_0, \quad \mathcal{F}_i \rightarrow +\mathcal{F}_i,
\]

\[
\alpha_0 \rightarrow -\alpha_0, \quad \alpha_i \rightarrow +\alpha_i, \quad \beta^0 \rightarrow +\beta^0, \quad \beta^i \rightarrow -\beta^i,
\]

(9)

- \( \{ \alpha^{(\text{CP})}_i, \beta^{(\text{CP})} \} \in H^3_{\text{CP}}(\mathcal{M}), \; \{ \tilde{\alpha}^{(\text{CP})}_0, \tilde{\beta}^{(\text{CP})} \} \in H^3_{\text{CP}}(\mathcal{M}) \) (\( i = 1, 2, \ldots, h^{2,1} \))

\[
X^0 \rightarrow -X^0, \quad X^i \rightarrow +X^i, \quad \mathcal{F}_0 \rightarrow +\mathcal{F}_0, \quad \mathcal{F}_i \rightarrow -\mathcal{F}_i,
\]

\[
\alpha_0 \rightarrow +\alpha_0, \quad \alpha_i \rightarrow -\alpha_i, \quad \beta^0 \rightarrow -\beta^0, \quad \beta^i \rightarrow +\beta^i.
\]

(10)

\(^4\)That is a generalization of the upper half plane of the complex plane, realized in the \( SL(2,\mathbb{Z}) \) moduli space of the torus.
Recall that we restrict ourselves to the large complex structure regime.

However, it is difficult to achieve these CP transformations for a generic form of the prepotential. In the large complex structure regime, the prepotential \( F(X) = (X^0)^2 F(u) \) is indeed expanded as

\[
F(u) = \frac{1}{3!} \kappa_{ijk} u^i u^j u^k + \frac{1}{2!} \kappa_{ij} u^i u^j + \kappa_i u^i + \frac{1}{2} \kappa_0,
\]

up to geometrical corrections [28]. Here, the coefficients \( \kappa_{ijk}, \kappa_{ij}, \kappa_i, \kappa_0 \) are the topological quantities determined by the CY data, i.e.

\[
\kappa_{ijk} = \int_{\tilde{M}} J_i \wedge J_j \wedge J_k, \quad \kappa_{ij} = \frac{1}{2} \int_{\tilde{M}} J_i \wedge J_j^2, \quad \kappa_i = -\frac{1}{24} \int_{\tilde{M}} c_2(\tilde{M}) \wedge J_i, \quad \kappa_0 = -\frac{\zeta(3) \chi(\tilde{M})}{(2\pi i)^3}.
\]

(12)

These classical topological quantities are calculated on the mirror CY threefold \( \tilde{M} \), where the \((1,1)\)-forms are denoted by \( J_i \), and the second Chern class and the Euler characteristic of \( M \) are represented by \( c_2(\tilde{M}) \) and \( \chi(\tilde{M}) \), respectively.

To satisfy the CP transformations (9) and (10) taking into account \( u^i \to -\bar{u}^i \), the prepotential is restricted to be a cubic type, i.e.

\[
F_{\text{cubic}} = \frac{1}{3!} \kappa_{ijk} u^i u^j u^k.
\]

(13)

The linear term in the prepotential is also allowed under the CP transformation, but we focus on a strict large complex structure regime of generic CY threefolds in the following analysis.

Before analyzing the CP-invariant effective potential, we discuss the relation between the CP symmetry and the symplectic modular symmetry of CY threefolds, described by the special geometry. Given the holomorphic three-form \( \Omega \) expanded on the basis of the symplectic basis \( (\alpha_I, \beta^I) \) in \( H^3(M, \mathbb{Z}) \) as in Eq. [4], the symplectic basis transforms under the symplectic modular symmetry as

\[
\begin{pmatrix} \alpha \\ \beta \end{pmatrix} \to \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix},
\]

(14)

with

\[
\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in Sp(2h^{2,1} + 2; \mathbb{Z}).
\]

(15)

That corresponds to the group of \( (2h^{2,1} + 2) \times (2h^{2,1} + 2) \) matrices preserving the symplectic matrix:

\[
\Sigma = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.
\]

(16)
Under the symplectic transformations, the so-called period integrals of the holomorphic three-form $\Omega$ on the three-cycles $\{A^I, B_I\}$ transform as
\[
\Pi = \left( \int_A \Omega, \int_B \Omega \right) = \left( \vec{X}, \vec{F} \right) \rightarrow \left( d \begin{array}{c} X \\ \bar{F} \end{array} \right) = \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \left( \vec{X}, \vec{F} \right),
\]
with $\vec{X} = \{X^I\}$ and $\vec{F} = \{F_I\}$. These transformations are reflected by the fact that holomorphic three-form itself is invariant under the symplectic transformation. Let us introduce the period matrix
\[
F_{IJ} = \frac{\partial^2}{\partial u^I \partial u^J} F = \partial_I F_J,
\]
which transforms under the symplectic transformation as
\[
F_{IJ} \rightarrow \left((aF + b)(cF + d)^{-1}\right)_{IJ},
\]
analogous to the period matrices of Riemann surfaces with genus $g$. Correspondingly, when we take the gauge $X^0 = 1$, the complex structure moduli form a vector-valued modular form of $Sp(2h^{2,1} + 2; \mathbb{Z})$,
\[
u \rightarrow (cF + d)u,
\]
with $u^0 = X^0 = 1$. We recall that the Kähler potential
\[
K_{cs} = -\ln \left[ -i\Pi^\dagger \cdot \Sigma \cdot \Pi \right],
\]
is indeed invariant under the symplectic transformation of the period integrals [17]. Such a symplectic modular symmetry arises from the fact that the CY moduli space is described by the special geometry.

On the toroidal background $T^{2n}$, the geometrical space group is the $SL(2n, \mathbb{Z})$ modular group. For example, the CP transformation on $T^2$ is identified with the outer automorphism of $SL(2, \mathbb{Z})$ [15][16]. It is interesting to ask whether CP is embedded into the symplectic modular transformation on CY threefolds. The CP transformation we discussed corresponds to the following transformation for the period integrals,
\[
\Pi = \left( \begin{array}{c} X^0 \\ X^i \\ F_0 \\ F_i \end{array} \right) \rightarrow \pm \left( \begin{array}{c} X^0 \\ -X^i \\ -F_0 \\ F_i \end{array} \right) = \pm \left( \begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{array} \right) \Pi.
\]

We find that the matrix
\[
CP = \pm \left( \begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{array} \right) \notin Sp(2h^{2,1} + 2; \mathbb{Z}),
\]
is not the element of $Sp(2h^{2,1} + 2; \mathbb{Z})$, due to the property $CP^T \cdot \Sigma \cdot CP = \Sigma^T \neq \Sigma$. We have two options to define the CP transformation as in Eq. (23), but both of them share the common properties. Hence, we consider the case with the positive sign in Eq. (23) without loss of generality. Under CP and symplectic modular transformation $\gamma$, we obtain

$$\Pi \xrightarrow{CP} CP\Pi \xrightarrow{\gamma} CP \cdot \gamma\Pi \xrightarrow{CP^{-1}} CP \cdot \gamma \cdot CP^{-1}\Pi,$$

namely

$$CP \cdot \gamma \cdot CP^{-1} = \begin{pmatrix} \hat{\sigma}^3 & 0 \\ 0 & -\hat{\sigma}^3 \end{pmatrix} \begin{pmatrix} d & c \\ b & a \end{pmatrix} \begin{pmatrix} \hat{\sigma}^3 & 0 \\ 0 & -\hat{\sigma}^3 \end{pmatrix} = \begin{pmatrix} \hat{\sigma}^3 d\hat{\sigma}^3 & -\hat{\sigma}^3 c\hat{\sigma}^3 \\ -\hat{\sigma}^3 b\hat{\sigma}^3 & \hat{\sigma}^3 a\hat{\sigma}^3 \end{pmatrix} = \begin{pmatrix} d & -c \\ -b & a \end{pmatrix},$$

with

$$\hat{\sigma}^3 \equiv \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$  

In this way, we can define the outer automorphism $Q$:

$$\gamma = \begin{pmatrix} d & c \\ b & a \end{pmatrix} \rightarrow Q(\gamma) \equiv CP \cdot \gamma \cdot CP^{-1} = \begin{pmatrix} d & -c \\ -b & a \end{pmatrix},$$

satisfying

$$Q(\gamma_1)Q(\gamma_2) = CP \cdot \gamma_1 \cdot CP^{-1}CP \cdot \gamma_2 \cdot CP^{-1} = Q(\gamma_1 \gamma_2),$$

and no group element $\hat{\gamma} \in Sp(2h^{2,1} + 2, \mathbb{Z})$ exists such that $Q(\gamma) = \hat{\gamma}^{-1}\gamma\hat{\gamma}$. In this respect, we argue that CP has a geometrical origin in the CY moduli space, namely the outer automorphism of the $Sp(2h^{2,1} + 2, \mathbb{Z})$ modular group. The whole group is described by the semidirect product group $Sp(2h^{2,1} + 2, \mathbb{Z}) \rtimes CP$, since there exists a group homomorphism from CP to the automorphism group of $Sp(2h^{2,1} + 2, \mathbb{Z})$ and $Sp(2h^{2,1} + 2, \mathbb{Z})$ is a normal subgroup of $Sp(2h^{2,1} + 2, \mathbb{Z}) \rtimes CP$. That is a natural extension of the $SL(2, \mathbb{Z}) \simeq Sp(2, \mathbb{Z})$ modular group known in the $T^2$ toroidal background \cite{15,16}, where the whole group is given by the “generalized modular group” $GL(2, \mathbb{Z}) \simeq SL(2, \mathbb{Z}) \rtimes CP$.

\footnote{We can treat an element of $Sp(2h^{2,1} + 2, \mathbb{Z}) \rtimes CP$ as a pair $(g, h)$ where $g \in Sp(2h^{2,1} + 2, \mathbb{Z})$, $h \in CP \simeq \mathbb{Z}_2$, and a product between two pairs is represented as $(g_1, h_1)(g_2, h_2) = (g_1 h_1^{-1} g_2 h_1, h_1 h_2)$ with respect to the outer automorphism. Nevertheless, in the toroidal case, it is enough to consider a set of elements $(g, h)(h \neq 1)$ as a negative determinant part of $GL(2, \mathbb{Z})$, so we simply use $GL(2, \mathbb{Z})$ instead of $SL(2, \mathbb{Z}) \rtimes CP$ thanks to the isomorphism. However, because of the explicit form of $\hat{\sigma}^3$, we cannot think $Sp(2h^{2,1} + 2, \mathbb{Z}) \rtimes CP$ as the same with the toroidal case for odd $h^{2,1}$, where CP has a positive determinant. For even $h^{2,1}$, CP has a negative determinant then it could be similar to the toroidal case that we can distinguish new elements of the whole group $Sp(2h^{2,1} + 2, \mathbb{Z}) \rtimes CP$ that are not in original $Sp(2h^{2,1} + 2, \mathbb{Z})$ by their signs of determinants.}
3 CP-invariant flux compactifications and discrete symmetry

In this section, we introduce the fluxes into the previous effective action to determine the size of CP-violation, namely the vacuum expectation values of moduli fields. We mainly focus on the effective action of Type IIB flux compactifications on CY orientifolds with $O3/O7$-planes, but it is straightforward to extend our analysis to T-dual Type IIA flux compactifications as well as the heterotic string theory in the large volume/complex structure and weak coupling regime. After examining the relation between CP and orientifold actions in section 3.1, we first derive the CP-invariant flux compactifications and next discuss the discrete $\mathbb{Z}_2$ or $\mathbb{Z}_4$ symmetry appearing in the effective action in section 3.2.

3.1 CP and orientifold actions

Before studying the relation between the CP transformation and orientifold projection in the context of Type II string theory, we remind the CP transformation of the Kähler form in the context of heterotic string theory. The orientation reversing in the 6D space is characterized by the transformation of the holomorphic three-form $\Omega$ and the Kähler form $J$:

$$\Omega \rightarrow -\bar{\Omega}, \quad J \rightarrow -J.$$  \hspace{1cm} (29)

Since we examined the CP transformation of the holomorphic three-form in the previous section, we focus on the CP transformation of the Kähler form on CY threefolds $\mathcal{M}$ in more detail.

The Kalb-Ramond two-form ($B$) and the Kähler form ($J$) are expanded by the basis of $H^{1,1}(\mathcal{M}, \mathbb{R})$,

$$B = \sum_{a=1}^{h^{1,1}} b^a(x)w_a, \quad J = \sum_{a=1}^{h^{1,1}} t^a(x)w_a,$$  \hspace{1cm} (30)

with $h^{1,1}$ being the dimension of $H^{1,1}(\mathcal{M}, \mathbb{R})$. Here, 4D Kähler axions $b^a$ and volume moduli associated with two-cycles $t^a$ (saxions) are 4D CP-odd and -even fields, respectively. Recalling that the space of harmonic forms splits under the CP transformation into even and odd eigenspaces, $H^2(\mathcal{M}) = H^2_{+\mathbb{C}}(\mathcal{M}) \oplus H^2_{-\mathbb{C}}(\mathcal{M})$, the orientation reversing of the Kähler form \( (29) \) requires that the bases $w_a$ are CP-odd bases, i.e.,

$$w_l \in H^{1,1-\mathbb{C}}(\mathcal{M}),$$  \hspace{1cm} (31)

where $l = 1, 2, ..., h^{1,1}_{-\mathbb{C}}$ with $h^{1,1}_{-\mathbb{C}}$ being the dimension of $H^{1,1}_{-\mathbb{C}}(\mathcal{M})$, transforming as $w_l \rightarrow -w_l$. Thus, the simultaneous transformation of 4D parity and 6D orientation reversing (i.e., CP transformation) of complexified Kähler form $J_c = B + iJ$ is provided by the anti-holomorphic transformation:

$$J_c \rightarrow \bar{J}_c.$$  \hspace{1cm} (32)
Let us move to Type IIA/IIB CY orientifolds, where a part of complex structure moduli and Kähler moduli also determines the strong CP phase and Cabbibo-Kobayashi-Maskawa phase. (See for more details, e.g., Ref. [22].) It indicates that the 4D CP transformation is understood from the 10D point of view, i.e., simultaneous transformations of 4D parity and 6D orientation reversing, as discussed before. In subsequent discussions, we examine the relation between the CP transformation and orientifold projections, with an emphasis on Type IIA orientifolds with O6-planes and Type IIB orientifolds with O3/O7-planes whose geometric $\mathbb{Z}_2$ action is represented by $\mathcal{R}$. (For the details about the orientifold projections, we refer to Refs. [26,27].)

1. Type IIA orientifolds with O6-planes: $\mathcal{R}\Omega = e^{2i\theta}\bar{\Omega}$, $\mathcal{R}J = -J$

The constant phase $\theta$ can be absorbed into the definition of holomorphic three-form. In particular, $\theta = \pi/2$ case with $z_i \to -\bar{z}_i$ is the same with the CP transformation in our notation. As stated before, the phase of CP transformation $\Omega \to \pm\bar{\Omega}$ is a matter of conventions. Indeed, $\Omega \to \bar{\Omega}$ is achieved under the coordinate transformation $z_i \to \bar{z}_i$. In this way, the orientifold action is consistent with the 6D orientation reversing by identifying orientifold-even (-odd) basis with the CP-even (-odd) basis in the space of harmonic forms.

2. Type IIB orientifolds with O3/O7-planes: $\mathcal{R}\Omega = -\Omega$, $\mathcal{R}J = J$

We first analyze the orientifold action of the holomorphic three-form, expanded in the orientifold-odd bases of $H_2^{\perp}(\mathcal{M},\mathbb{C})$ i.e., $\{\alpha_{\kappa}^{(O)}, \beta^{(O)\kappa}\}$,

$$\Omega = X^\kappa \alpha_{\kappa}^{(O)} - F_n^l \beta^{(O)\kappa},$$

(33)

where $\kappa = 0, 1, ..., h_2^{\perp}$ being the hodge number. In the local patch, orientifold-odd bases $\{\alpha_{\kappa}\}$ are spanned by $\{dx_i \wedge dx_j \wedge dx_k, dx_i \wedge dx_j \wedge dy_k, dx_i \wedge dy_j \wedge dy_k, dy_i \wedge dy_j \wedge dy_k\}$ with $i \neq j \neq k$, satisfying $\{\alpha_{\kappa}^{(O)}\} \to -\{\alpha_{\kappa}^{(O)}\}$ under $z_i \to -\bar{z}_i$ ($x_i \to -x_i, y_i \to -y_i$). The CP-odd and -even bases classified by $z_i \to -\bar{z}_i$ ($x_i \to -x_i, y_i \to y_i$) are also spanned by a part of the orientifold-odd bases, from the fact that their bases in the local coordinates are given by $\{dx_i \wedge dx_j \wedge dx_k, dx_i \wedge dy_j \wedge dy_k\}$ and $\{dx_i \wedge dx_j \wedge dy_k, dy_i \wedge dy_j \wedge dy_k\}$ with $i \neq j \neq k$, respectively. Hence, the CP invariant action is consistent with the orientifold action, since CP-odd/even bases are more restrictive than the orientifold-odd bases. In this way, the form of the prepotential is restricted to be a cubic type, although the generic form is allowed in Type IIB CY orientifolds with O3/O7-planes.

Next, we examine the orientifold action with an emphasis on Kähler form $J$ and Ramond-Ramond four-form $C_4$ consisting of the Kähler moduli\[9\]. It was known that they are expanded in the orientifold-even basis $\{w_i^{(O)}\}$ of $H_+^{1,1}(\mathcal{M},\mathbb{R})$,

$$J = t^i w_i^{(O)}, \quad C_4 = \rho^l \bar{w}_l^{(O)},$$

(34)

where $\{\bar{w}_l^{(O)}\}$ denote the hodge dual of $\{w_l^{(O)}\}$, $l = 1, 2, ..., h_+^{1,1}$, with $h_+^{1,1}$ being the dimension of $H_+^{1,1}(\mathcal{M},\mathbb{R})$. Note that the orientifold-even Kähler moduli are given by the

\[9\] We omit the contribution from the orientifold-odd moduli, for simplicity.
integral of \( C_4 + iJ \wedge J \) over the four-cycle,

\[
T^l = \rho^l + i\kappa_{lmn} t^m t^n, \tag{35}
\]

with \( \kappa_{lmn} \) being the triple intersection numbers. As analyzed in the context of heterotic string, the CP-transformation of the Kähler form requires that the (1,1)-forms are elements of \( H^{1,1}_{\text{(CP)}}(\mathcal{M}) \), and both \( J \) and \( C_4 \) are expanded in the same CP-odd bases due to the existence of supersymmetry,

\[
J = t^l w_l^{(\text{CP})}, \quad C_4 = \rho^l \tilde{w}_l^{(\text{CP})}. \tag{36}
\]

It leads to the anti-holomorphic transformation of the Kähler moduli

\[
T^l \rightarrow -\bar{T}^l, \tag{37}
\]

since the RR axions \( \rho^l \) are 4D CP-odd fields, namely the axions, similar to the Kalb-Ramond B-field.

Let us check the consistency between the bases split by the orientifold and CP actions. Note that the basis of \( H^{1,1}_{\text{e}}(\mathcal{M}, \mathbb{R}) \) is locally given in \( w_l = w_i^j d\bar{z}_i \wedge dz_j \), which transforms as \( w_l \rightarrow w_l \) under the orientifold action \( z_i \rightarrow -z_i \) and \( w_l \rightarrow -w_l \) under the CP action \( z_i \rightarrow -\bar{z}_i \). Thus, the orientifold-even and CP-odd bases are consistent with each other. So far, we have focused on the CP transformations of moduli fields relevant to flux compactifications, but it is straightforward to analyze the CP transformation of other supergravity fields in a similar way.

### 3.2 Flux compactifications

We study the flux compactification in Type IIB string on CY orientifolds \( \mathcal{M} \). In addition to the effective action of the complex structure moduli shown in Sec. 2, the moduli effective action includes the axio-dilaton and the Kähler moduli,

\[
K = -\ln(i(\bar{S} - S)) - \ln\left[-i \int_\mathcal{M} \Omega \wedge \bar{\Omega} \right] - 2 \ln \mathcal{V},
\]

\[
W = \int_\mathcal{M} G_3 \wedge \Omega, \tag{38}
\]

where \( \mathcal{V} \) represents for the CY volume in the Einstein frame, depending on the Kähler moduli \( T^l \) of \( \mathcal{M} \). Here, we introduce the flux-induced superpotential \[21\] generated by a linear combination of Ramond-Ramond (RR) three-form flux \( F_3 \) and Neveu-Schwarz (NS) three-form flux \( H_3 \), namely \( G_3 = F_3 - SH_3 \) as a function of the axio-dilaton \( S \). Since the CY volume is invariant under the orientation reversing transformation, kinetic terms of the Kähler moduli as well as the complex structure moduli are CP-invariant quantities. This is because both of them originate from the Einstein-Hilbert action in 10D supergravity action, which is invariant under the 10D proper Lorentz transformation. Since the axio-dilaton \( S \) is a 4D CP-odd field,

\[
S \rightarrow -\bar{S}, \tag{39}
\]
the kinetic term of $S$ is also invariant under the CP transformation. However, the three-form fluxes generically break CP as well as the 6D Lorentz symmetry. Hence, not all the flux quanta are allowed in the CP-invariant effective action. We classify the possible pattern of CP-invariant three-form fluxes inserted on three-cycles of CY threefolds.

The CP invariance of the 4D effective action requires $W \rightarrow e^{i\gamma} \bar{W}$ for the superpotential, where $\gamma$ denotes a complex phase. Taking into account the CP transformation of $\Omega$ in Eq. (2), the three-form flux $G_3$ transforms into

$$G_3 \rightarrow -e^{i\gamma} \bar{G}_3,$$

from which the integral flux quanta require $\gamma = 0$ or $\pi$.

Recalling that the axio-dilaton $S$ is a 4D CP-odd field, RR and NS three-form fluxes should transform as

$$\left\{ \begin{array}{ll}
\gamma = 0, & F_3 \rightarrow -F_3, \quad H_3 \rightarrow H_3 \\
\gamma = \pi, & F_3 \rightarrow F_3, \quad H_3 \rightarrow -H_3 
\end{array} \right. \quad (41)$$

Hence, the expansion of the three-form fluxes on the symplectic basis is categorized into two classes:

- **$\gamma = 0$**
  $$F_3 = f^i_0 \alpha_0 + f_i \beta^i,$$
  $$H_3 = h^i \alpha_i + h_0 \beta^0. \quad (42)$$

- **$\gamma = \pi$**
  $$F_3 = f^i \alpha_i + f_0 \beta^0,$$
  $$H_3 = h^0 \alpha_0 + h_i \beta^i. \quad (43)$$

As a result, we obtain two classes of 4D CP-invariant effective action in the large complex structure regime of CY threefolds. In both classes, the Kähler potential is described by

$$K_{cs} = -\ln \left[ -i \int_M \Omega \wedge \bar{\Omega} \right] = -\ln \left[ -i (\bar{u}^I F_I - u^I \bar{F}_I) \right]$$

$$= -\ln \left[ \frac{i}{3!} \kappa_{ijk} (u^i - \bar{u}^i)(u^j - \bar{u}^j)(u^k - \bar{u}^k) \right], \quad (44)$$

where we employ Eq. (13)\(^{10}\) with $i = 1, 2, \cdots, h_{-1}^{2,1}$. However, for the superpotential we have two options:

\(^{10}\)Here and in what follows, we adopt the cubic-type prepotential under the gauge $X^0 = 1$, and the dimension of the complex structure moduli space is given by $h_{-1}^{2,1}$. 

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\( \cdot \gamma = 0 \)

\[
W = -f_i u^i - f^0 (2F - u^i \partial_i F) - S (-h_0 - h^i \partial_i F)
= -f_i u^i - f^0 \left( -\frac{1}{6} \kappa_{ijk} u^i u^j u^k \right) + h_0 S + h^i S \left( \frac{1}{2} \kappa_{ijk} u^i u^j u^k \right). \tag{45}
\]

\( \cdot \gamma = \pi \)

\[
W = -f_0 - f^i \partial_i F - S \left[ -h_i u^i - h^0 (2F - u^i \partial_i F) \right]
= -f_0 - f^i \left( \frac{1}{2} \kappa_{ijk} u^i u^j u^k \right) + h_i u^i S + h^0 S \left( -\frac{1}{6} \kappa_{ijk} u^i u^j u^k \right). \tag{46}
\]

Hence, the CP-invariant superpotential is restricted to be either odd or even polynomials with respect to the moduli fields. Note that these fluxes induce the D3-brane charge:

\[
N_{\text{flux}} = \int H_3 \wedge F_3 = \begin{cases} 
-f^0 h_0 + \sum_i f_i h^i, & (\gamma = 0) \\
 f_0 h^0 - \sum_i f^i h_i, & (\gamma = \pi)
\end{cases}, \tag{47}
\]

which should be canceled by mobile D3-branes and orientifold contributions.

Finally, we comment on the (accidental) symmetry which arises in the CP-invariant effective action from the field theoretical point of view. The superpotential of even degree with respect to the moduli fields in Eq. (46) is invariant under the discrete \( \mathbb{Z}_2 \) symmetry (not related to the discrete R-symmetry)

\[
u^i \rightarrow -\nu^i, \quad S \rightarrow -S. \tag{48}
\]

For the superpotential with the odd degree of moduli fields, one can assign the R-charge 2 for all the moduli fields due to the existence of linear terms. However, the presence of cubic terms breaks the continuous R-symmetry and it results in the discrete \( \mathbb{Z}_4 \) R-symmetry in the above superpotential. Note that in the superpotential of both odd and even degrees, the Kähler potential is also invariant under the discrete \( \mathbb{Z}_2 \) and \( \mathbb{Z}_4 \) symmetries, taking into account transformations of the axio-dilaton and the complex structure moduli at the same time.

Let us consider the generic flux-induced superpotential having both odd and even degrees of polynomials with respect to the moduli fields:

\[
W = -f_i u^i - f^0 (2F - u^i \partial_i F) - S (-h_0 - h^i \partial_i F) - f_0 - f^i \partial_i F - S \left[ -h_i u^i - h^0 (2F - u^i \partial_i F) \right]. \tag{49}
\]

When we impose the discrete \( \mathbb{Z}_2 \) or \( \mathbb{Z}_4 \) symmetry for moduli fields in the supersymmetric effective action, the generic flux-induced superpotential (49) in the large complex structure regime has a CP-invariance categorized by two classes, i.e. Eqs. (45) and (46). Hence, the necessary and sufficient condition to possess CP in the moduli effective action is the existence of discrete \( \mathbb{Z}_2 \) or \( \mathbb{Z}_4 \) symmetry and the supersymmetry (SUSY). Recalling that the spontaneous CP violation is realized by the non-zero vacuum expectation values of axionic fields Re\( u^i \) and Re\( S \), the discrete \( \mathbb{Z}_2 \) or \( \mathbb{Z}_4 \) symmetry is spontaneously broken at the CP-breaking vacua.
4 CP-conserving and -breaking vacua

We are ready to analyze the vacuum structure of CP-invariant effective action. We first discuss the existence of CP-conserving vacua in the large complex structure regime of generic CY threefolds in Sec. 4.1. Next, we deal with concrete CY threefolds in the large complex structure regime to clarify whether the spontaneous CP violation is realized or not. It turns out that for the prepotential having a similar structure of factorizable tori in Secs. 4.2 and 4.3, it is difficult to achieve the spontaneous CP violation. When the structure of prepotential deviates from the toroidal one, one can find the CP-breaking vacua as discussed in detail in Sec. 4.4. Here, we choose a specific CY such as the degree 18 hypersurface in a weighted projective space $\mathbb{CP}_{11169}$.

4.1 CP-conserving vacua in generic CY

Before analyzing the CP-breaking minimum of the scalar potential, we investigate the CP-conserving minima satisfying the SUSY condition,

$$ D_I W = (\partial_I + K_I) W = 0, \quad (50) $$

where $I = \{S, u^i\}$ runs over the axio-dilaton $S$ and all the complex structure moduli $u^i$, and $K_I = \partial_I K$. Since the effective action in the large volume regime possesses the no-scale structure for the Kähler moduli, we focus on the dynamics of the axio-dilaton and the complex structure moduli in the following analysis.

The superpotential and its covariant derivatives transform under the CP transformations $\{S \rightarrow -\bar{S}, u^i \rightarrow -\bar{u}^i\}$ as

- $\gamma = 0$
  $$ W \rightarrow \overline{W}, \quad D_s W \rightarrow \overline{D_s W}, \quad D_i W \rightarrow \overline{D_i W}, \quad (51) $$

- $\gamma = \pi$
  $$ W \rightarrow -\overline{W}, \quad D_s W \rightarrow -\overline{D_s W}, \quad D_i W \rightarrow -\overline{D_i W}, \quad (52) $$

where we use $K_i = -K_i$ and $K_i$ is a CP-even function due to the axionic shift symmetries. Hence, for both $\gamma = 0$ and $\pi$ cases, the SUSY minimum $D_I W = 0$ has a symmetry, $\langle \text{Re} S \rangle \rightarrow -\langle \text{Re} S \rangle$ and $\langle \text{Re} u^i \rangle \rightarrow -\langle \text{Re} u^i \rangle$. It suggests that CP-invariant moduli effective action may have the CP-conserving vacua in generic CY flux compactifications. Indeed, among the $h^{2,1}_c + 1$ number of SUSY conditions $D_I W = 0$, the CP-conserving vacuum $\text{Re} S = \text{Re} u^i = 0$ is a solution for half of them, namely $\text{Re}(D_I W) = 0$ and $\text{Im}(D_I W) = 0$ for $\gamma = \pi$ and $\gamma = 0$, respectively. By introducing

$$ X^i \equiv \kappa_{ijk} \text{Re} u^j \text{Im} u^k, \quad Y^i \equiv \kappa_{ijk} \text{Re} u^j \text{Re} u^k, \quad Z^i \equiv \kappa_{ijk} \text{Im} u^j \text{Im} u^k, \quad (53) $$

one can explicitly check that $\text{Re} S = \text{Re} u^i = 0$ lead to
\[ \gamma = 0 \]

\[ \text{Im}(D_S W) = h_i X^i + \text{Im}K_S \text{Re} W = 0, \]
\[ \text{Im}(D_I W) = f^0 X^i + \text{Re}S\kappa_{ijk}h^j \text{Im}u^k + \text{Im}S\kappa_{ijk}h^j \text{Re}u^k + \text{Im}K_I \text{Re} W = 0, \quad (54) \]

with

\[ \text{Re} W = -f_i \text{Re}u^i + \frac{f^0}{6} \left( Y^i - 3X^i \right) \text{Re}u^i + h_0 \text{Re}S + \frac{h^i}{2} \text{Re}S \left( Y^i - Z^i \right) - \text{Im}Sh^i X^i. \quad (55) \]

\[ \gamma = \pi \]

\[ \text{Re}(D_S W) = h_i \text{Re}u^i - \frac{h^0}{6} \left( Y^i - 3Z^i \right) \text{Re}u^i - \text{Im}K_S \text{Im}W = 0, \]
\[ \text{Re}(D_I W) = -\kappa_{ijk}f^j \text{Re}u^k + \text{Re}S \left( h_i - \frac{h^0}{2} Y^i + \frac{h^0}{2} Z^i \right) + h^0 \text{Im}S X^i - \text{Im}K_I \text{Im}W = 0, \quad (56) \]

with

\[ \text{Im} W = -f^i X^i + \text{Re}S \left( h_i - \frac{h^0}{2} Y^i + \frac{h^0}{6} Z^i \right) \text{Im}u^i + \text{Im}S \left( h_i - \frac{h^0}{6} Y^i + \frac{h^0}{2} Z^i \right) \text{Re}u^i. \quad (57) \]

Note that \( \text{Im}K_I \) are functions of imaginary parts of moduli fields.

In this way, half of the SUSY conditions \( D_I W = 0 \) are satisfied by \( \text{Re} S = \text{Re}u^i = 0 \), and the imaginary parts are determined by the remaining SUSY conditions for both \( \gamma = 0 \) and \( \gamma = \pi \) cases. In the following, we explicitly check the existence of CP-conserving vacua and search for the CP-breaking vacua on concrete CY threefolds.

### 4.2 One modulus case: \( \mathbb{CP}^{1111}_5 \)

We begin with the CY threefold with a single modulus, especially the mirror dual of the quintic \( \mathbb{CP}^{1111}_5 \) in Ref. [30], defined by the degree 5 hypersurface in a projective space \( \mathbb{CP}^{1111}_5 \).

To realize the CP-invariant moduli potential, we restrict ourselves to the large complex structure regime \( \text{Im}U > 1 \), where we denote the single complex structure modulus by \( U \). Given the triple intersection number \( \kappa_{UUU} = 5 \), the Kähler potential is given by

\[ K = -\ln(i(S - \bar{S})) - \ln \left[ \frac{5i}{6} (U - \bar{U})^2 \right], \quad (58) \]

and the superpotential is categorized by two classes:

\[ W = \begin{cases} 
-f_U U + \frac{f^0}{6} U^3 + h_0 S + \frac{5h^U}{6} S U^2 & (\gamma = 0) \\
-f_U - \frac{f^U}{2} U^2 + h_U S - \frac{5h^U}{3} S U^3 & (\gamma = \pi) 
\end{cases}, \quad (59) \]

with \( \{ f^0, f_O, f_U, f^U, h_0, h^U, h_U \} \) being flux quanta.

By solving SUSY conditions \( D_I W = 0 \) for \( S \) and \( U \), we find the CP-conserving minima:
\[ \gamma = 0 \]
\[ \text{Re} U = \text{Re} S = 0, \quad \text{Im} U = \sqrt{\frac{2}{5}} \left( -\frac{3f_0 h_0}{f^0 h_0} \right)^{1/4}, \quad \text{Im} S = \sqrt{\frac{2}{5}} \left( -\frac{f^0}{3h_0} \right)^{1/4} \left( \frac{f_U}{h_U} \right)^{3/4}, \]

\[ (60) \]

\[ \gamma = \pi \]
\[ \text{Re} U = \text{Re} S = 0, \quad \text{Im} U = \sqrt{2} \left( -\frac{3f_0 h_U}{5f^0 h_0} \right)^{1/4}, \quad \text{Im} S = \sqrt{\frac{1}{2}} \left( \frac{f_0}{5h_0} \right)^{1/4} \left( -\frac{f_U}{h_U} \right)^{3/4}, \]

\[ (61) \]

and four classes of degenerate CP-conserving and -breaking minima:

(i) \[ |U|^2 = \frac{6f_U}{5f^0} = -\frac{2h_0}{5h_U}, \quad |S|^2 = \left( \frac{f_0}{h_U} \right)^2 |U|^2, \quad \text{Im} S = \frac{5(f^0)^2(\text{Im} U)^3}{8f_U h_U + 15f^0 h_U (\text{Im} U)^2}, \]

(ii) \[ |U|^2 = -\frac{2f_U}{5f^0} = \frac{6h_0}{5h_U}, \quad |S|^2 = \left( \frac{f_U}{h_U} \right)^2 |U|^2, \quad \text{Im} S = \frac{5(f_U h_U)^3}{2(\text{Re} U)^2 - 30h_0 h_U (\text{Re} U)^2}, \quad f_0 = -\frac{f_U h_U}{15h_0}, \]

(iii) \[ |U|^2 = \frac{6h_U}{5h^0}, \quad |S|^2 = -\frac{f_U}{2h_U} U, \quad f_0 = -\frac{3f_U h_U}{5h_0}, \]

(iv) \[ |U|^2 = -\frac{2h_U}{5h^0}, \quad |S|^2 = -\frac{(f_U)^2}{10h^0 h_U}, \quad \text{Im} S = \frac{5f_U h_U (\text{Im} U)^3}{4(h_U)^2 - 30h_0 h_U (\text{Re} U)^2}, \quad f_0 = -\frac{f_U h_U}{15h_0}, \]

\[ (63) \]

for \( \gamma = 0 \) and

\( \gamma = \pi \), respectively. Hence, we cannot realize the spontaneous CP-violation in the large complex structure regime. This is because the prepotential has a structure similar to one of the toroidal background with the overall complex structure modulus, where it was pointed out in Ref. \[23\] that the spontaneous CP violation is difficult to achieve. This argument also holds for other one-parameter CY threefolds in the large complex structure regime by changing the value of triple intersection number, for instance the mirror dual of \( \mathbb{C}P_{1112} \)[6], \( \mathbb{C}P_{1114} \)[8] and \( \mathbb{C}P_{1125} \)[10] defined on a single polynomial in an ambient weighted projective space.

Interestingly, for a particular choice of fluxes, CP is embedded into \( SL(2,\mathbb{Z})_S \) duality group of the axio-dilaton and/or the modular symmetry of the complex structure moduli. For instance, the vacuum expectation value of the axio-dilaton in the solution (i) is given by

\[ |S|^2 = \frac{2f^0 f_U}{15(h_U)^2} = 1 \]

by setting \( f^0 f_U = 15(h_U)^2/2 \). The \( S \)-transformation of the \( SL(2,\mathbb{Z})_S \) duality group at the vacuum \[ (64) \]

\[ S \rightarrow -1/S = -\bar{S} \]

\[ (65) \]

corresponds to the CP transformation in Eq. \[ (59) \]. In this respect, CP is unified into the duality group for a particular choice of fluxes.
4.3 Two moduli case: $\mathbb{CP}_{11222}[8]$

The next example is the CY threefold with two complex structure moduli labelled by $u^1$ and $u^2$. In particular, we deal with the mirror dual of CY threefold defined by the degree 8 hypersurface in a weighted projective space $\mathbb{CP}_{11222}$ studied in Refs. [31,32], where the triple intersection numbers are specified by

$$\kappa_{111} = 8, \quad \kappa_{112} = 4,$$

and otherwise 0. By restricting ourselves to the large complex structure regime $\{\text{Im} u^1, \text{Im} u^2 > 1\}$, the Kähler potential is given by

$$K = - \ln(i(S - \bar{S})) - \ln \left[ \frac{i}{6} (8(u^1 - \bar{u}^1)^3 + 12(u^1 - \bar{u}^1)(u^2 - \bar{u}^2)) \right],$$

and the superpotential is categorized by two classes:

- $\gamma = 0$

$$W = - f_1 u^1 - f_2 u^2 + \frac{f^0}{6}(u^1)^2(8u^1 + 12u^2) + h_0 S + \frac{h_1}{2} S (8(u^1)^2 + 8u^1 u^2) + \frac{h_2}{2} S (8u^1 u^2),$$

- $\gamma = \pi$

$$W = - f_0 - \frac{f^1}{2}(8(u^1)^2 + 8u^1 u^2) - \frac{f^2}{2}(8u^1 u^2) + (h_1 u^1 + h_2 u^2) S - \frac{h_0}{6} S (8(u^1)^3 + 12(u^1)^2 u^2),$$

where we denote the flux quanta $\{f^0, f_0, f_{1,2}, f_{1,2}^1, h_0, h_{1,2}, h_{1,2}, h_{1,2}\}$.

By solving SUSY conditions $D_I W = 0$ with $I = S, u^1, u^2$, we find CP-conserving solutions:

- $\gamma = 0$

$$\text{Re} u^1 = \text{Re} u^2 = \text{Re} S = 0,$$

$$\text{Im} u^1 = 2^{-3/4} \left( \frac{3f_2 h_0}{f^0(2h_2^2 - h_1)} \right)^{1/4}, \quad \text{Im} u^2 = - \frac{h_0}{2f^0} \frac{\text{Im} S}{(\text{Im} u^1)^2} - \frac{2}{3} \text{Im} u^1,$$

$$\text{Im} S = 2^{-1} \left( \frac{f^0(2f_2 - 3f_1)}{h_0(h_1^2 + h_2^2)} \right)^{1/2} \left( \frac{f_2 h_0}{6f^0(2h_2^2 - h_1)} \right)^{1/4}. \quad (70)$$

- $\gamma = \pi$

$$\text{Re} u^1 = \text{Re} u^2 = \text{Re} S = 0,$$

$$\text{Im} u^1 = 2^{-3/4} \left( \frac{3f_0 h_2}{h^0(2f_2^2 - f_1)} \right)^{1/4}, \quad \text{Im} u^2 = - \frac{3h_1 + 2h_2}{12(f_1^2 + f_2^2)} \text{Im} S - \frac{2}{3} \text{Im} u^1,$$

$$\text{Im} S = 2 \left( \frac{f^1 + f^2}{h^0 h_2(2h_2^2 - 3h_1)} \right)^{1/2} \left( 6f_0 h^0 h_2(2f_2^2 - f_1) \right)^{1/4}. \quad (71)$$
On the other hand, we rely on the numerical search to find the CP-breaking vacua in both \( \gamma = 0 \) and \( \pi \) cases\(^{11}\). Numerical search for the randomly generated \( 2 \times 10^7 \) dataset of fluxes within \(-30 \leq \{ f^0, f_0, f_{1,2}, f_{1,2}^1, h_0, h_0^1, h_1, h_1 \} \leq 30 \) leading to \( 0 \leq N_{\text{flux}} \leq 150 \) allows only \( 5.8 \times 10^3 \) and 87 stable CP-conserving vacua for \( \gamma = 0 \) and \( \gamma = \pi \), respectively. The reason why the CP-breaking vacuum is absent is that the Kähler potential and the superpotential have a similar structure with the toroidal one due to the torus-type prepotential. Indeed, when we redefine the modulus field as \( 12u^2 = -8u^1 + u \), the prepotential is given by

\[
F(u) = \frac{1}{6} (8(u_1)^3 + 12(u_1)^2u_2) = \frac{1}{6} (u_1)^2u.
\]

(72)

Because of the moduli redefinition, the prepotential has a similar structure to the factorizable \( T^6 \) torus by identifying the two complex structure moduli with the identical one.

### 4.4 CP-breaking vacua on \( \mathbb{CP}_{11169}^{18} \)

Finally, we analyze the different two-parameter CY threefold, especially the mirror dual of the degree 18 hypersurface in a weighted projective space \( \mathbb{CP}_{11169} \) studied in Ref. \[33\], where the complex structure moduli are labelled by \( u^1 \) and \( u^2 \). We can also consider the original CY threefold as follows\(^{12}\). Taking into account a \( G = Z_6 \times Z_{18} \) discrete symmetry of this CY, the complex structure moduli space parametrized by \( u^1 \) and \( u^2 \) is invariant under this action. Other non-invariant complex structure moduli can be fixed at the fixed points under \( G \), thanks to the three-form fluxes along the \( G \)-invariant three-forms. Hence, the period integrals in the mirror dual of the CY are the same with the original two-parameter CY threefold.

To realize the CP-invariant moduli potential, we further restrict ourselves to the large complex structure regime \( \{ \text{Im}u^1, \text{Im}u^2 > 1 \} \). Given the non-vanishing triple intersection numbers \( \kappa_{111} = 9, \kappa_{112} = 3 \) and \( \kappa_{122} = 1 \), the Kähler potential is given by

\[
K = -\ln(i(\bar{S} - S)) - \ln \left[ \frac{i}{6} (9(u^1 - \bar{u}^1)^3 + 9(u^1 - \bar{u}^1)^2(u^2 - \bar{u}^2) + 3(u^1 - \bar{u}^1)(u^2 - \bar{u}^2)^2) \right],
\]

(73)

and the superpotential is categorized by two classes:

\[
W = \begin{cases} 
-f_1u^1 - f_2u^2 + \left( \frac{h_0}{6} (9(u^1)^3 + 9(u^1)^2u^2 + 3u^1(u^2)^2) 
+ h_0 S + \frac{h_1}{2} S (9(u^1)^2 + 6u^1u^2 + (u^2)^2) + \frac{h_2}{2} S (3(u^1)^2 + 2u^1u^2) \right) & (\gamma = 0) \\
-f_0 - \left( \frac{h_0^1}{2} (9(u^1)^2 + 6u^1u^2 + (u^2)^2) - \frac{h_0^2}{2} (3(u^1)^2 + 2u^1u^2) \right) + (h_1u^1 + h_2u^2) S - \frac{h_0^1}{6} S (9(u^1)^3 + 9(u^1)^2u^2 + 3u^1(u^2)^2) & (\gamma = \pi)
\end{cases}
\]

(74)

where we denote the flux quanta \( \{ f^0, f_0, f_{1,2}, f_{1,2}^1, h_0, h_0^1, h_1, h_1 \} \). These integral flux quanta are constrained by the tadpole cancellation condition \[35\]

\[
0 \leq N_{\text{flux}} \leq 138.
\]

(75)

\(^{11}\)Specifically, we used “FindRoot” of Mathematica (v12.0) to solve SUSY conditions \( D_iW = 0 \) with randomly generated fluxes and initial values of moduli fields.

\(^{12}\)For more details, see, e.g. Ref. \[33\].
By solving SUSY conditions $D_IW = 0$ with $I = S, u^1, u^2$, we find that CP-conserving solutions are realized to satisfy

- $\gamma = 0$
  
  $\text{Re}u^I = \text{Re}u^2 = \text{Re}S = 0,$
  
  $h_0 = -\frac{\text{Im}u^1(3(\text{Im}u^1)^2 + 3\text{Im}u^1\text{Im}u^2 + (\text{Im}u^2)^2)}{2\text{Im}S},$
  
  $\frac{f_0}{\text{Im}u^1} = -\frac{3f_2(\text{Im}u^1 + \text{Im}u^2)(3\text{Im}u^1 + \text{Im}u^2) - f_1(3(\text{Im}u^1)^2 + 6\text{Im}u^1\text{Im}u^2 + 2(\text{Im}u^2)^2)}{\text{Im}u^2(3\text{Im}u^1 + \text{Im}u^2)(3(\text{Im}u^1)^2 + 3\text{Im}u^1\text{Im}u^2 + (\text{Im}u^2)^2)\text{Im}S},$
  
  $h_1 = -\frac{3f_1(\text{Im}u^1)^2(\text{Im}u^1 + \text{Im}u^2) - f_2(9(\text{Im}u^1)^3 + 9(\text{Im}u^1)^2\text{Im}u^2 + 3\text{Im}u^1(\text{Im}u^2)^2 + (\text{Im}u^2)^3)}{\text{Im}u^1\text{Im}u^2(3(\text{Im}u^1)^2 + 3\text{Im}u^1\text{Im}u^2 + (\text{Im}u^2)^2)\text{Im}S}.$

- $\gamma = \pi$
  
  $\text{Re}u^I = \text{Re}u^2 = \text{Re}S = 0,$
  
  $f_0 = \frac{\text{Im}u^1\text{Im}S(3(\text{Im}u^1)^2 + 3\text{Im}u^1\text{Im}u^2 + (\text{Im}u^2)^2)}{2},$
  
  $\frac{f_1}{\text{Im}u^1} = -\frac{3f_2\text{Im}u^1(\text{Im}u^1 + \text{Im}u^2)(3\text{Im}u^1 + \text{Im}u^2) + 2h_1\text{Im}S(3(\text{Im}u^1)^2 + 3\text{Im}u^1\text{Im}u^2 + (\text{Im}u^2)^2)}{(3\text{Im}u^1 + \text{Im}u^2)(9(\text{Im}u^1)^3 + 9(\text{Im}u^1)^2\text{Im}u^2 + 3\text{Im}u^1(\text{Im}u^2)^2 + (\text{Im}u^2)^3)},$
  
  $h_2 = -\frac{f_2\text{Im}u^2(3(\text{Im}u^1)^2 + 3\text{Im}u^1\text{Im}u^2 + (\text{Im}u^2)^2) + 3h_1\text{Im}u^1(\text{Im}u^1 + \text{Im}u^2)\text{Im}S}{(9(\text{Im}u^1)^3 + 9(\text{Im}u^1)^2\text{Im}u^2 + 3\text{Im}u^1(\text{Im}u^2)^2 + (\text{Im}u^2)^3)\text{Im}S}.$

(76)

On the other hand, we numerically searched for the CP-breaking vacua under the set of randomly generated fluxes satisfying the tadpole cancellation condition (75). For the even polynomial case, one cannot find the CP-breaking vacua for 4 \times 10^7 dataset of fluxes within $-30 \leq \{f_0, f_1, h_0, h_{1,2}\} \leq 30$, whereas there exist CP-breaking vacua in the superpotential with odd degrees. Indeed, the numerical search under the randomly generated 2 \times 10^7 dataset of fluxes within $-30 \leq \{f_0, f_{1,2}, h_0, h_{1,2}\} \leq 30$ leads to 559 stable CP-breaking vacua in the large complex structure regime $\text{Im}u^1 > 1$. For the benchmark dataset of fluxes:

$(f_0, f_1, f_2, h_0, h^1, h^2) = (-1, 25, -23, 22, 1, -3),$

leading to $N_{\text{flux}} = 116$, the vacuum expectation values of moduli fields are evaluated as

$\langle \text{Re}u^1 \rangle \simeq 1.86, \quad \langle \text{Re}u^2 \rangle \simeq -2.70, \quad \langle \text{Re}S \rangle \simeq 4.32,$

$\langle \text{Im}u^1 \rangle \simeq 3.01, \quad \langle \text{Im}u^2 \rangle \simeq 4.15, \quad \langle \text{Im}S \rangle \simeq 4.81,$

(79)

\[13\] We used the same numerical method in Sec. 4.3 but the result could be highly dependent on initial values. Hence, this non-existence of CP-breaking vacua for the even polynomial case could not be a general statement but may capture some tendency.
showing that CP is spontaneously broken due to the nonvanishing values of axionic fields.\cite{13}

Note that masses squared of the moduli fields are positive as shown in the descendent order,

\[ \mathcal{V}^{-2}(9.72, 5.90, 4.73, 2.38, 5.29 \times 10^{-1}, 6.60 \times 10^{-2}). \]

Hence, CP is spontaneously broken in this class of flux compactification. The realization of spontaneous CP violation depends on the structure of the prepotential and the functional form of the superpotential. The reason why the CP-breaking vacua are absent in the choice of the superpotential with even degrees is unclear owing to the fact that the SUSY conditions are non-linear functions of the axionic fields as in Eqs. (54) and (56). A profound understanding of the origin of CP-breaking vacua will be reached by studying other CY flux compactifications, which will be investigated in future work. So far, we have focused on the stabilization of the complex structure moduli and the axio-dilaton. The stabilization of Kähler moduli would be realized by non-perturbative effects such as D-brane instanton effects. Combining the stabilization of Kähler moduli with CP-breaking flux compactifications will also be important future work.

5 Conclusions

We have revealed the geometrical origin of CP embedded into the 10D proper Lorentz transformation with an emphasis on the complex structure of compact 6D spaces, in particular the CY threefolds. We find that the anti-holomorphic involution of the complex structure is regarded as the anti-holomorphic involution of period integrals on CY threefolds with the large complex structure. Consequently, the anti-holomorphic involution of the period integrals corresponds to the outer automorphism of $Sp(2h^{2,1} + 2, \mathbb{Z})$ symplectic modular group, rather than the element of $Sp(2h^{2,1} + 2, \mathbb{Z})$ in the complex structure moduli space. The moduli group is then enlarged into $Sp(2h^{2,1} + 2, \mathbb{Z}) \times CP$. That is a natural extension of the known toroidal cases, where the CP symmetry is regarded as the outer automorphism of the $SL(2, \mathbb{Z})$ modular group \cite{15,16}.

The CP violation is strongly correlated with the dynamics of the moduli fields, whose vacuum expectation values determine the size of CP violation. It is an important issue to check whether the CP symmetry is spontaneously broken in the string landscape. Indeed, it was stated in Ref. \cite{23} that the spontaneous CP violation is difficult to realize on toroidal backgrounds. To resolve this issue, we examined CY flux compactifications. For concreteness, we deal with Type IIB flux compactifications, where we turn on three-form fluxes on CY three-cycles. Imposing CP invariance on the moduli effective action requires the restricted choices of RR and NSNS flux quanta in the large complex structure regime of CY threefolds. It results in the flux-induced superpotential consisting of either odd or even polynomials with respect to the moduli fields. The CP-invariant superpotential possesses the discrete $\mathbb{Z}_2$ symmetry or $\mathbb{Z}_4$.

\footnote{We deal with the cubic-type prepotential realized in the large complex structure and weak coupling regime to check whether the CP symmetry is spontaneously broken. Even if we include the sub-leading terms to the prepotential as calculated in Ref. \cite{32}, the moduli fields are still around these CP-breaking vacua, but the CP symmetry is explicitly broken by the sub-leading effects.}
R-symmetry from the field theoretical point of view. We analyzed the vacuum structure of CP-invariant effective action. It turned out that CP-conserving vacua appear in generic CY flux compactifications in the large complex structure regime. To check the existence of CP-breaking vacua, we work with some concrete CY threefolds. It indicates that the prepotential having the toroidal structure does not lead to the spontaneous CP violation. For a particular choice of fluxes, CP is embedded into the duality symmetry of the axio-dilaton and/or the modular symmetry of the complex structure moduli. When the structure of the prepotential differs from the toroidal one, the spontaneous CP violation can be achieved on the CY, illustrated on $\mathbb{CP}^{11169}$.

In this paper, we examined the geometrical origin of CP and its violation in the complex structure moduli space of CY threefolds, but phenomenologically it is required to consider the CP violation in the matter sector. From our numerical search within $2 \times 10^7$ data set of fluxes, it turned out that the CP symmetry was spontaneously broken at only 559 number of vacua, at which the vacuum expectation values of axions as well as saxions are determined. Remarkably, these moduli fields appear in gauge kinetic functions and Yukawa couplings of certain D-branes in Type IIB string theory, thereby one can predict the size of strong CP phase as well as the Cabbibo-Kobayashi-Maskawa phase from such a small corner of string landscape. Furthermore, it is interesting to examine more examples to reveal the nature of spontaneous CP violation in more broad classes of the string theory, which leaves us for future work. Also, it was stated in Ref. [36] that the CP and the flavor symmetries can be unified in the common group. We will report the analysis of CP and the flavor on curved CY manifolds in a separate paper.

**Note added**

After finishing this work, Ref. [37] appeared, where symplectic modular symmetries were studied from the phenomenological viewpoints.

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