A C*-algebra associated with dynamics on a graph of strings

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Abstract

An operator C*-algebra $E$ associated with a dynamical system on a metric graph is introduced. The system is governed by the wave equation and controlled from boundary vertices. Algebra $E$ is generated by the so-called eikonals, which are self-adjoint operators related with reachable sets of the system. Its structure is the main subject of the paper. We show that $E$ is a direct sum of "elementary blocks". Each block is an algebra of operators, which multiply $\mathbb{R}^n$-valued functions by continuous matrix-valued functions of special kind. The eikonal algebra is determined by the boundary inverse data. This shows promise of its possible applications to inverse problems.

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0 Introduction

About the paper

We introduce a self-adjoint (C*-t) operator algebra algebra $E$ associated with a dynamical system on a metric graph. The system is governed by the wave

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equation and controlled from the boundary vertices. Algebra $\mathcal{E}$ is generated by the so-called eikonals, which are the self-adjoint operators related with reachable sets of the system. A structure of reachable sets and the algebra is the main subject of the paper. We show that $\mathcal{E}$ is a direct sum of “elementary blocks”. Each block is an operator (sub)algebra, the operators multiplying $\mathbb{R}^n$-valued functions by continuous matrix-valued functions of special kind.

The eikonal algebra is determined by the boundary dynamical and/or spectral inverse data up to isometric isomorphism. It is an inspiring fact, which shows promise of its possible applications to inverse problems. In particular, one can hope to extract information about geometry of the graph from the algebra spectrum $\hat{\mathcal{E}}$. Such a technique works well on manifolds [4], [5], [6].

The paper develops an algebraic version of the boundary control method in inverse problems [1], [3]–[6]. Our approach reveals some new and hopefully prospective relations between inverse problems on graphs and $C^*$-algebras.

Content

In more detail, we deal with the dynamical system

$$
\begin{align*}
  u_{tt} - \Delta u &= 0 & \text{in } \Omega \times (0, T) \\
  u(\cdot, t) &\in \mathcal{K} & \text{for all } t \in [0, T] \\
  u|_{t=0} &= u_t|_{t=0} = 0 & \text{in } \Omega \\
  u &= f & \text{on } \Gamma \times [0, T],
\end{align*}
$$

where $\Omega$ is a finite compact metric graph, $\Gamma$ is the set of its boundary vertices; $\Delta$ is the Laplace operator in $\Omega$ defined on $\mathcal{K}$, which is a class of smooth functions satisfying the Kirchhoff conditions at the interior vertices; $T \leq \infty$; $f$ is a boundary control. A solution $u = u^f(x, t)$ describes a wave initiated at $\Gamma$ by the control $f$ and propagating into $\Omega$.

With the system one associates the reachable sets

$$
\mathcal{U}^s_\gamma = \{ u^f(\cdot, s) \mid f \in L_2(\Gamma \times [0, T]), \supp f \in \gamma \times [0, T] \} \quad \gamma \in \Gamma, \ 0 \leq s \leq T.
$$

Let $P^s_\gamma$ be the orthogonal projection in $L_2(\Omega)$ onto $\mathcal{U}^s_\gamma$. A self-adjoint operator

$$
E^T_\gamma = \int_0^T s \, dP^s_\gamma
$$
is called an eikonal (corresponding to the boundary vertex $\gamma$).

Choose a subset $\Sigma \subseteq \Gamma$. An eikonal algebra $E^T_\Sigma$ is defined as the minimal norm-closed $C^*$-subalgebra of the bounded operator algebra $\mathfrak{B}(L_2(\Omega))$, which contains all $E^T_\gamma$ as $\gamma \in \Sigma$.

We provide the characteristic description of the sets $U^s_\gamma$ and projections $P^s_\gamma$. As a result, we clarify how the eikonals $E^T_\gamma$ act. Thereafter, a structure of the eikonal algebra is revealed, and we arrive at the main result: the algebra is represented in the form of a finite direct sum

$$E^T_\Sigma = \bigoplus_j b^T_j,$$

where $b^T_j$ are the so-called block algebras. Each $b^T_j$ is isometrically isomorphic to a subalgebra $\tilde{b}^T_j \subset \mathfrak{B}(L_2([0, \delta_j]; \mathbb{R}^{M_j}))$ generated by the operators, which multiply elements of $L_2([0, \delta_j]; \mathbb{R}^{M_j})$ (vector-valued functions of $r \in [0, \delta_j]$) by the matrix-functions of the form $B^*_\gamma,j D^\gamma,j(r)B^\gamma,j$ ($\gamma \in \Sigma$). Here each $B^\gamma,j$ is a constant projecting matrix; $D^\gamma,j$ is a diagonal matrix, its diagonal elements being the linear functions of the form $T^\gamma,j \pm r$ with $T^\gamma,j \in (0, T]$. These functions are continuous, and, hence, we have

$$\tilde{b}^T_j \subset C([0, \delta_j]; \mathbb{M}^{M_j}),$$

where the latter is the algebra of continuous real $M_j \times M_j$ matrix valued functions on $[0, \delta_j]$.

**Comments**

- Algebra $E^T_\Sigma$ associated with a graph is a straightforward analog of the eikonal algebras associated with a Riemannian manifold: see [4], [6]. These algebras possess two principal features, which enable one to apply them to solving inverse problems on manifolds:

1. the eikonal algebra is determined (up to isometric isomorphism) by dynamical and/or spectral boundary inverse data

2. its spectrum is, roughly speaking, identical to the manifold.

By this, one can solve the problem of reconstruction of the manifold via its inverse data by the scheme [3–6]:

$$\text{data} \Rightarrow \text{relevant eikonal algebra } \mathfrak{E} \Rightarrow \text{its spectrum } \hat{\mathfrak{E}} \equiv \text{manifold.}$$
It is so effective application, which has motivated to extend this approach to inverse problems on graphs. The hope was that a graph seems to be a simpler object than a manifold of arbitrary dimension and topology.

Surprisingly, the latter turns out to be an illusion. First of all, in contrast to the eikonal algebras on manifolds\(^1\), the algebra \(\mathcal{E}_T^\Sigma\) is noncommutative. By this, in the general case, its spectrum \(\hat{\mathcal{E}}_T^\Sigma\) endowed with the Jacobson topology is a non-Hausdorff space. Hence, \(\hat{\mathcal{E}}_T^\Sigma\) is by no means identical to the (metric) graph \(\Omega\), so that property 2 fails.

However, property 1 does hold. Also, the known examples show that representation (0.1) and structure of the spectrum \(\hat{\mathcal{E}}_T^\Sigma\) reflect some features of the graph geometry. Therefore, an attempt to extract information on \(\Omega\) from the eikonal algebra and, eventually, to recover \(\Omega\) seems quite reasonable. Hopefully, our paper is a step towards this goal.

• In view of big volume of the paper, we omit the proofs of some technical propositions.

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1 Graph

1.1 Basic definitions

1.1.1 Standard star

Let \(I_j := (0, a_j) = \{s \in \mathbb{R} | 0 < s < a_j < \infty\}\), \(j = 1, \ldots, m\) be finite intervals, each interval being regarded as a subspace of the metric space \(\mathbb{R}\) with the distance \(|s - s'|\). The set \(S_m := \{0\} \cup I_1 \cup \cdots \cup I_m\) endowed with

\(^1\)somehow or other, these algebras are reduced to the algebra \(C(\Omega)\) of continuous functions
the metric
\[
\text{dist} (s, s') := \begin{cases} 
|s - s'| & s, s' \in I_j \\
 s + s' & s \in I_i, s' \in I_j, \ i \neq j \\
 s' & s = 0, s' \in I_j \\
s & s \in I_i, s' = 0 \\
 0 & s = s' = 0 
\end{cases}
\]

is called a (standard) \(m\)-star (see Fig. 1a). Note that a 2-star is evidently isometric to the interval \((-a_1, a_2)\).

### 1.1.2 Metric graph

A compact connected metric space \(\Omega\) with the metric \(\tau : \Omega \times \Omega \to [0, \infty)\) is said to be a \textit{homogeneous metric graph} if the following is fulfilled:

- \(\Omega = E \cup V \cup \Gamma\), where \(E = \{e_j\}_{j=1}^p\), \(e_j\) are the \textit{edges}; \(V = \{v_k\}_{k=1}^q\), \(v_k\) are the \textit{interior vertices}; \(\Gamma = \{\gamma_l\}_{l=1}^n\), \(\gamma_l\) are the \textit{boundary vertices}, \(\Gamma \neq \emptyset\)
each $e_j$ is isometric to a finite interval $\{s \in \mathbb{R} \mid a_j < s < b_j\}$

**Convention 1.** In what follows, we assume that the isometries (parametrizations) $\eta_j : e_j \to (a_j, b_j)$ are fixed and write $\tilde{y}(s) := (y \circ \eta_j^{-1})(s)$, $s \in (a_j, b_j)$ for a function $y = y(x)$ on $\Omega$ restricted to the edge $e_j$.

So, for $x, x' \in e_j$ one has $\tau(x, x') = |\eta_j(x) - \eta_j(x')|$.

- every $w \in V \cup \Gamma$ has a neighborhood (in $\Omega$) isometric to an $S_m$ with $m \geq 3$ or $m = 1$, the number $m = m(w)$ being called a multiplicity of $w$. The vertices of multiplicity $\geq 3$ constitute the set $V$. The vertices of multiplicity 1 form the boundary $\Gamma$.

As was noted above, 2-stars are isometric to intervals (edges). By this, they do not take part in further considerations.

For a point $x \in \Omega$, by $\Omega^r[x] := \{x' \in \Omega \mid \tau(x, x') < r\}$ we denote its metric neighborhood of radius $r > 0$. For a subset $A \subset \Omega$, we put $\Omega^r[A] := \{x \in \Omega \mid \tau(x, A) < r\}$.

Let the points $x \neq x'$ belong to a (parametrized) edge $e$. The set $\eta^{-1}(\min\{\eta(x), \eta(x')\}, \max\{\eta(x), \eta(x')\})) \subset e$ is called an interval and denoted by $[x, x']$.

### 1.1.3 Characteristic set

In the space-time $\Omega \times \mathbb{R}_+$, for a fixed $(x_0, t_0)$ define a characteristic cone

$$
\text{ch} \left( (x_0, t_0) \right) := \{ (x, t) \mid t - t_0 = \tau(x, x_0) \} ;
$$

for a subset $A \subset \Omega \times \mathbb{R}_+$ put

$$
\text{ch} [A] := \bigcup_{(x,t) \in A} \text{ch} \left( (x, t) \right) .
$$

A characteristic set $\text{Ch} \left( (x_0, t_0) \right)$ is introduced by the following recurrent procedure:

**Step 0:** put

$$
C^0[ (x_0, t_0) ] := \text{ch} \left( (x_0, t_0) \right)
$$

and

$$
W^0( x_0, t_0 ) := \{ (w, t) \in C^0[ (x_0, t_0) ] \mid w \in V \cup \Gamma \} ;
$$
Steps $j = 1, 2, \ldots$: put

$$C^j[(x_0, t_0)] := C^{j-1}[(x_0, t_0)] \cup \mathrm{ch} \left[ W^{j-1}(x_0, t_0) \right]$$

and

$$W^j(x_0, t_0) := \{(w, t) \in C^{j-1}[(x_0, t_0)] \mid w \in V \cup \Gamma \} ;$$

\[ \text{...............} \]

At last, define

$$\mathrm{Ch} [(x_0, t_0)] := \bigcup_{j=0}^{\infty} C^j[(x_0, t_0)] .$$

Note that $\mathrm{Ch} [(x_0, t_0)]$ can be also characterized as the minimal subset in $\Omega \times \mathbb{R}_+$ satisfying the conditions:

- $\mathrm{ch} [(x_0, t_0)] \subset \mathrm{Ch} [(x_0, t_0)]$
- if $w \in V \cup \Gamma$ and $t_w \in \mathbb{R}_+$ are such that $(w, t_w) \in \mathrm{Ch} [(x_0, t_0)]$ then $\mathrm{ch} [(w, t_w)] \subset \mathrm{Ch} [(x_0, t_0)].$

The characteristic set can be regarded as a space-time graph; Fig. 2 illustrates the case $x_0 = \gamma, \ t_0 = 0.$

Such a graph is also a metric space: it is endowed with the length element

$$dv^2 := d\tau^2 + dt^2 ,$$

i.e., for the close points $(x, t), (x', t') \in \mathrm{Ch} [(x_0, t_0)]$ one has $\nu ((x, t), (x', t')) = [\tau^2(x, x') + (t - t')^2]^\frac{1}{2}.$ For arbitrary points, the distance $\nu$ is defined as the length of the shortest curves lying in $\mathrm{Ch} [(x_0, t_0)]$ and connecting the points.

### 1.2 Spaces and operators

#### 1.2.1 Derivatives

For an edge $e \in E$ parametrized by $\eta : e \to (a, b)$, a function $y$ on $\Omega$, and a point $x \in e,$ we define

$$\frac{dy}{de} (x) := \left. \frac{\tilde{y}}{ds} \right|_{s=\eta(x)} = \lim_{\eta(x') > \eta(x)} \frac{y(x') - y(x)}{\tau(x', x)}$$

(recall that $\tilde{y} := y \circ \eta^{-1}$).
Fix a vertex $w \in V \cup \Gamma$ and choose its neighborhood $\omega \subset \Omega$ isometric to $S_m$. We say an edge $e$ to be incident to $w$ if $\tau \ni w$ or, equivalently, if $e \cap \omega \neq \emptyset$. Note that $e \cap \omega$ can consist of two components (see Fig.1b) and settle that, in this case, each component is regarded as a single edge (of the subgraph $\omega$) incident to $w$.

For every $e$ incident to $w$, define an outward derivative
\[
\frac{dy}{de_+}(w) := \lim_{e \ni m \to w} \frac{y(m) - y(w)}{\tau(m, w)}.
\]
For an interior vertex $v \in V$ and a function $y$, define an outward flow
\[
\Pi_v[y] := \sum_{\tau \ni v} \frac{dy}{de_+}(v),
\]
the sum being taken over all edges incident to $v$ in a star neighborhood $\omega \ni v$. 

Figure 2: Characteristic set
1.2.2 Spaces

Introduce a (real) Hilbert space $\mathcal{H} := L_2(\Omega)$ of functions on $\Omega$ with the inner product

$$(y, u)_{\mathcal{H}} = \int_{\Omega} y u \, d\tau = \sum_{e \in E} \int_{e} y u \, d\tau := \sum_{e \in E} \int_{\eta(e)} \tilde{y}(s) \tilde{u}(s) \, ds.$$ 

By $C(\Omega) \subset \mathcal{H}$ we denote the class of functions continuous on $\Omega$.

We assign a function $y$ on $\Omega$ to a class $\mathcal{H}^2$ if $y \in C(\Omega)$ and $\tilde{y}|_{\eta(e)} \in H^2(\eta(e))$ for each $e \in E$.

Also, define the Kirchhoff class

$${\mathcal{K}} := \{y \in \mathcal{H}^2 \mid \Pi_v[y] = 0, \ v \in V\}. \quad (1.2)$$

1.2.3 Operator

The Laplace operator on the graph $\Delta : \mathcal{H} \to \mathcal{H}$, $\text{Dom} \, \Delta = \mathcal{K}$,

$$(\Delta y) \big|_e := \frac{d^2 y}{d e^2}, \quad e \in E \quad (1.3)$$

is well defined (does not depend on the parametrizations). It is a closed densely defined operator in $\mathcal{H}$.

2 Waves on graph

2.1 Dynamical system

An initial boundary value problem of the form

$$u_{tt} - \Delta u = 0 \quad \text{in } [\Omega \setminus (V \cup \Gamma)] \times (0, T) \quad (2.1)$$

$$u(\cdot, t) \in \mathcal{K} \quad \text{for all } t \in [0, T] \quad (2.2)$$

$$u|_{t=0} = u_t|_{t=0} = 0 \quad \text{in } \Omega \quad (2.3)$$

$$u = f \quad \text{on } \Gamma \times [0, T] \quad (2.4)$$

is referred to as a dynamical system associated with the graph $\Omega$. Here $T < \infty$; $f = f(\gamma, t)$ is a boundary control; the solution $u = u^f(x, t)$ describes a wave initiated at $\Gamma$ and propagating into $\Omega$.

$^2H^s(a, b)$ is the standard Sobolev space.
Note that, by definition (1.2), the condition (2.2) provides the Kirchhoff laws:

\[ u(\cdot, t) \in C(\Omega), \quad \Pi_v[u(\cdot, t)] = 0 \]  

for all \( t \geq 0 \) and \( v \in V \). Also, note that by (1.3), on each edge \( e \in E \) parametrized by \( \eta : e \to (a,b) \), the pull-back function \( \tilde{u}(\cdot, t) = u(\cdot, t) \circ \eta^{-1} \) satisfies the homogeneous string equation

\[ \tilde{u}_{tt} - \tilde{u}_{ss} = 0 \]  

in \((a,b) \times (0,T)\). (2.6)

It is the reason, by which we regard \( \Omega \) as a graph consisting of homogeneous strings. As is well known, for a \( C^2 \)-smooth (with respect to \( t \)) control \( f \) vanishing near \( t = 0 \) the problem has a unique classical solution \( u^f \). A space of controls \( F^T := L_2(\Gamma \times [0,T]) \) with the inner product

\[ (f,g)_{F^T} := \sum_{\gamma \in \Gamma} \int_0^T f(\gamma, t) g(\gamma, t) \, dt \]

is called an outer space of system (2.1)–(2.4). It contains the subspaces \( F^T_\gamma := \{ f \in F^T | \text{supp } f \subset \{\gamma\} \times [0,T] \} \) of controls, which act from single boundary vertices \( \gamma \in \Gamma \), so that

\[ F^T = \bigoplus_{\gamma \in \Gamma} F^T_\gamma \]  

(2.7)

holds. Each \( f \in F^T_\gamma \) is of the form \( f(\gamma', t) = \delta_\gamma(\gamma') \varphi(t) \) with \( \varphi \in L_2(0,T) \).

The space \( H \) is an inner space; the waves \( u^f(\cdot, t) \) are time–dependent elements of \( H \).

### 2.2 Fundamental solution

#### 2.2.1 Definition

Consider the system (2.1)–(2.4) with \( T = \infty \).

For \( \gamma, \gamma' \in \Gamma \), we denote

\[ \delta_\gamma(\gamma') := \begin{cases} 0, & \gamma' \neq \gamma \\ 1, & \gamma' = \gamma \end{cases} \]
let $\delta(t)$ be the Dirac delta-function of time.

Fix a boundary vertex $\gamma$. Taking the (generalized) control $f(\gamma', t) = \delta_\gamma(\gamma')\delta(t)$, one can define the generalized solution $u^{\delta_\gamma \delta}$ to (2.1)–(2.4). A possible way is to use a smooth regularization $\delta_{\varepsilon}(t) \to \delta(t)$ and then understand $u^{\delta_\gamma \delta}$ as a relevant limit of the classical solutions $u^{\delta_\gamma \delta_{\varepsilon}}$ as $\varepsilon \to 0$. Such a limit turns out to be a space-time distribution on $\Omega \times [0, T]$ of the class $C((0, T); H^{-1}(\Omega))$ (see, e.g., [2]).

The distribution $u^{\delta_\gamma \delta}$ is called a fundamental solution to (2.1)–(2.4) (corresponding to the given $\gamma$). It describes the wave initiated by an instantaneous source supported at $\gamma$. Consider its properties in more detail; all of them are well known. Recall that $\tau$ is the distance in $\Omega$.

2.2.2 First edge

Let $e$ be the edge incident to $\gamma$ and parametrised by $s = \eta(x) := \tau(x, \gamma) \in (0, \tau(\gamma, v))$. Let $v \in V$ be the second vertex incident to $e$. For times $0 < t \leq \tau(\gamma, v)$, by (2.6) one has

\begin{align*}
\tilde{u}_{tt} - \tilde{u}_{ss} &= 0 \quad \text{in } (0, \tau(\gamma, v)) \times (0, T) \\
\tilde{u}_{|t=0} &= \tilde{u}_{t|t=0} = 0 \quad \text{in } [0, \tau(\gamma, v)] \\
\tilde{u}_{|s=0} &= \delta(t), \quad 0 \leq t \leq \tau(\gamma, v),
\end{align*}

which implies $\tilde{u}(s, t) = \delta(t - s)$. This evidently leads to the representation

$$u^{\delta_\gamma \delta}(\cdot, t) = \delta_x(t)(\cdot), \quad 0 \leq t \leq \tau(\gamma, v), \quad (2.8)$$

where $x(t)$ belongs to $e$ and satisfies $\tau(x(t), \gamma) = t$, $\delta_p \in H^{-1}(\Omega)$ is the Dirac measure supported at $p \in \Omega$. It means that the $\delta$-singularity, which is injected into the graph from $\gamma$, moves along $e$ towards $v$ with velocity 1 (see Fig.3a).

2.2.3 Passing through interior vertex

At the moment $t = \tau(\gamma, v)$ the singularity reaches $v$ and then passes through $v$. A simple analysis using (2.5) and (2.6) provides

$$u^{\delta_\gamma \delta}(\cdot, t) = \sum_{e' \ni v} a(x_{e'}(t)) \delta_{x_{e'}(t)}(\cdot), \quad \tau(\gamma, v) < t \leq \tau(\gamma, v) + \varepsilon, \quad (2.9)$$
as $\varepsilon > 0$ is small (namely, $\varepsilon < \tau(v, (V \cup \Gamma) \setminus \{v\})$), where $x_{e'}(t)$ belongs to $e'$ and satisfies $\tau(x_{e'}(t), v) = t - \tau(\gamma, v)$. The function (amplitude) $a$ is

$$a(x) = \begin{cases} -\frac{m(v) - 2}{m(v)} & \text{as } x \in e \\ \frac{2}{m(v)} & \text{as } x \in e': e' \neq e, \overline{e'} \ni v. \end{cases} \quad (2.10)$$

Hence, in passing through $v$, the singularity splits onto $m(v)$ parts (singularities), the first one is reflected back into $e$, the others are injected into the other $m(v) - 1$ edges $e'$ incident to $v$. The reflected singularity has the negative amplitude. The process is illustrated by Fig.3b. Note that the 'conservation law'

$$-\frac{m(v) - 2}{m(v)} + (m(v) - 1) \frac{2}{m(v)} = 1, \quad (2.11)$$

is valid, so that the total amplitude after the passage through $v$ is equal to the amplitude of the incident singularity.

In what follows, we refer to (2.9)–(2.11) as a splitting rule.
2.2.4 Reflection from boundary

Let $\gamma' \in \Gamma$ be a boundary vertex nearest to $v$:

$$\tau(\gamma', v) = \min_{\gamma'' \in \Gamma} \tau(\gamma'', v)$$

(may be $\gamma = \gamma'$), so that $\tau(\gamma, \gamma') = \tau(\gamma, v) + \tau(v, \gamma')$ holds. Let $e'$ be the edge incident to $\gamma'$.

As $t \to \tau(\gamma, \gamma') - 0$, one of the singularities, which have appeared as a result of passing through $v$ (and, perhaps, through another vertices or reflected from $v$ back to $\gamma$), approaches to $\gamma'$ (see Fig.3c). Then this singularity is reflected from $\gamma'$. A simple analysis with the use of the condition $u_{\delta\gamma}(\gamma', t) = \delta_{\gamma}(\gamma')\delta(t) = 0, \ t > 0$ leads to the representation

$$u_{\delta,\delta}(\cdot, t) = \begin{cases} a\delta_{x(t)}(\cdot) & \text{as } t \in (\tau(\gamma, \gamma') - \varepsilon, \tau(\gamma, \gamma')) \\ -a\delta_{x(t)}(\cdot) & \text{as } t \in (\tau(\gamma, \gamma'), \tau(\gamma, \gamma') + \varepsilon) \end{cases},$$

(2.12)

where $x(t) \in e'$ satisfies $\tau(x(t), \gamma') = |t - \tau(\gamma, \gamma')|$, $a = \text{const} \neq 0$.

Thus, as a result of reflection from a boundary vertex, the singularity moves from it and changes the sign of the amplitude (see Fig.3d). This is a reflection rule.

The splitting and reflection rules, along with superposition principle (linearity of the system), uniquely determine the fundamental solution $u_{\delta,\delta}$ for all $t \geq 0$. Let us list some of its well-known properties.

2.2.5 Hydra

Return to the fundamental solution of (2.1)–(2.4) with $T = \infty$ and consider $u_{\delta,\delta}$ as a space-time distribution in $\Omega \times \mathbb{R}_+$. In what follows, its support

$$H_\gamma := \text{supp} u_{\delta,\delta}$$

plays important role and is called a hydra, the point $(\gamma, 0)$ being its root. The reason to introduce the characteristic set (see 1.1.3) is that it consists of the characteristic lines of the wave equation (2.1). As is seen from (2.8)–(2.12), singularities propagate along the characteristics that leads to the relation

$$H_\gamma \subseteq \text{Ch }[(\gamma, 0)].$$

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In particular, it shows that the space projections of singularities propagate along a homogeneous graph with velocity 1. This implies

$$\text{supp } u^{δ,δ}(\cdot,t) \subset \Omega(\gamma), \quad t > 0,$$

(2.13)

where the left hand side is understood as a time-dependent element of $H^{-1}(Ω)$. Also, note that the hydra is a connected set in $Ω \times [0,T]$: as is evident, any point $(x,t) \in H_γ$ is connected with the root $(γ,0)$ through a path in $H_γ$.

Let

$$\pi : H_γ \to Ω, \quad \pi((x,t)) := x, \quad ρ : H_γ \to \mathbb{R}_+, \quad ρ((x,t)) := t$$

be the space and time projection respectively. For $A \subset Ω$ and $B \subset \mathbb{R}_+$, denote

$$\pi^{-1}(A) := \{(x,t) \in H_γ | x \in A\}, \quad ρ^{-1}(B) := \{(x,t) \in H_γ | t \in B\}.$$

Choose an edge $e \in E$ parametrized by $η : e \to (a,b)$. Its pre-image $π^{-1}(e) \subset H_γ$ consists of the sets

$$\tilde{e}_j := \{((η^{-1}(s),t_j + σ_j (s-a)) | a < s < b), \quad σ_j = \pm 1, \quad j = 1,2,\ldots,$$

which are the edges of the hydra as a space-time graph. There is a part of the fundamental solution of the form

$$u^{δ,δ}_j(\cdot,t) = a_j δ_{x(t)}(\cdot), \quad t ∈ ρ(\tilde{e}_j)$$

(2.14)

supported on $\tilde{e}_j$, where $a_j = \text{const} \neq 0, \quad x(t) := η^{-1}(a + σ_j(t-t_j)).$ In dynamics, (2.14) describes the singularity moving along $e$ with velocity 1 as $t$ runs over the time interval $ρ(\tilde{e}_j)$, the sign $σ_j$ determining the direction of motion. The value of the amplitude $a_j$ is determined by the prehistory of $u^{δ,δ}_j$ as $t < \inf ρ(\tilde{e}_j)$ and can be derived from the splitting and reflection rules.

**Amplitude.** The aforesaid (see (2.8), (2.9), (2.12), (2.14)) enables one to endow the hydra with a function $a$ (amplitude) as follows. Take a point $(x,t) \in H_γ$ provided $x \in Ω\{V \cup Γ\}$, so that there is a graph edge $e \ni x$. Hence, there is a hydra edge $\tilde{e}_j \subset π^{-1}(e)$ such that $(x,t) \in \tilde{e}_j$ and (2.14) does hold. In the mean time, as is easy to see, there may be at most one more edge $\tilde{e}_i \in π^{-1}(e), \quad \tilde{e}_i \neq \tilde{e}_j$, which contains the given $(x,t)$ (so that $(x,t) = \tilde{e}_i \cap \tilde{e}_j$: see the point $p$ on Fig.4). Then,
Figure 4: Hydra and amplitude function

- (generic case) if $\tilde{e}_j$ is a unique edge, which contains $(x,t)$, we set $a(x,t) := a_j$

- (exclusive case) if there is the second $\tilde{e}_i \ni (x,t)$, we define $a(x,t) := a_i + a_j$

(on Fig.4, $a(p) = -\frac{4}{9} + \frac{1}{3} = -\frac{1}{3}$). To extend the amplitude to the whole hydra we add the following:

- for points $(\gamma', t) \in H_\gamma$ with $\gamma' \in \Gamma$, we put $a(\gamma', t) := 0$ as $t > 0$, and $a(\gamma', 0) := \delta_\gamma(\gamma')$ (Kronecker’s symbol)

- if $(v, t) \in H_\gamma$ and $v \in V$, we define $a(v, t) = a_1 + \cdots + a_p$, where $a_i$ are the amplitudes (2.14) on the hydra edges $\tilde{e}_i \subset H_\gamma \cap \{(x', t') | t' < t\}$ incident to $(v, t)$. Note that in the generic case such an $\tilde{e}_i$ is unique.

Corner points. Thus, the amplitude $a$ is a well-defined piece-wise constant function on $H_\gamma$. Moreover, as a function on the metric space-time graph (see (1.1)), it is piece-wise continuous, the continuity being broken only in some exceptional points. Namely, we say $(x, t) \in H_\gamma$ to be a corner point if either $x \in \Gamma \cup V$ or there are an edge $e \ni x$ and the hydra edges $\tilde{e}_i, \tilde{e}_j \subset \pi^{-1}(e)$ such that $(x, t) = \tilde{e}_i \cap \tilde{e}_j$ (see Fig.4).
2.3 Generalized solutions

Here we list some results on solutions of the problem (2.1)–(2.4), which can be easily derived from the above mentioned properties of the fundamental solution. Unless otherwise specified, we deal with $T < \infty$.

2.3.1 Definition

Fix a $\gamma \in \Gamma$. Let a control $f \in F^T_\gamma$ (see (2.7)) be such that $u^f$ is a classical solution. Then the Duhamel representation

$$u^f = u^\delta \ast f, \quad \text{in } \Omega \times [0, T]$$  (2.15)

(the convolution with respect to time) holds and motivates the following. For an $f \in F^T_\gamma$, we define a (generalized) solution to (2.1)–(2.4) by

$$u^f := \sum_{\gamma \in \Gamma} u^\delta \ast f, \quad \text{in } \Omega \times [0, T].$$  (2.16)

2.3.2 General properties

1. As can be shown, solution (2.16) belongs to the class $C([0, T]; \mathcal{H})$, i.e., is a continuous $\mathcal{H}$-valued function of time.

2. For $f \in F^T_\gamma$, relation (2.13) implies

$$\text{supp } u^f(\cdot, t) \subset \overline{\Omega^\gamma}, \quad t > 0,$$  (2.17)

which means that the waves propagate in $\Omega$ with velocity 1. Let $\Sigma \subseteq \Gamma$ be a set of boundary vertices and $f \in \bigoplus_{\gamma \in \Sigma} F^T_\gamma$. As a consequence of (2.17), we have the relation

$$\text{supp } u^f(\cdot, t) \subset \bigcup_{\gamma \in \Sigma} \overline{\Omega^\gamma} = \overline{\Omega^\Sigma}, \quad t > 0,$$  (2.18)

which is usually referred to as a finiteness of domain of influence.

3. For the rest of the paper we accept the following.

**Convention 2.** All functions depending on time $t \geq 0$ are extended to $t < 0$ by zero.
For $f \in \mathcal{F}^T$, denote by $f_s(\gamma, t) := f(\gamma, t - s)$ the delayed control. Since the graph and operator $\Delta$, which governs the evolution of system (2.21)–(2.24), do not depend on time, one has the relation (steady-state property)

$$u^f_s(\cdot, t) = u^f(\cdot, t - s).$$

(2.19)

2.3.3 Point-wise values of wave

Here we describe a ”mechanism”, which forms the values of waves $u^f$.

Fix a $\gamma \in \Gamma$. In $\Omega \times [0, T]$ define the truncated and delayed hydras

$$H^T_\gamma := \{ (x, t) \in H_\gamma \mid 0 \leq t \leq T \}, \quad H^{T, s}_\gamma := \{ (x, t + s) \in \Omega \times [0, T] \mid (x, t) \in H^T_\gamma \},$$

(2.20)

where $s \in (0, T)$ is a delay. Also, we put $H^{T, 0}_\gamma := H^T_\gamma$ and $H^{T, T}_\gamma := (\gamma, T)$. Each $H^{T, s}_\gamma$ is endowed with an amplitude function by

$$a^{T, s}(x, t) := a(x, t - s),$$

(2.21)

where $a$ is the amplitude on $H_\gamma$.

A set

$$\kappa^{T, s} := \{ x \in \Omega \mid (x, T) \in H^{T, s}_\gamma \} = \text{supp}\ u^{\delta, \delta}(\cdot, T - s) \quad (0 \leq s \leq T)$$

consists of finite number of points in $\Omega$, which we call heads of the hydra $H^{T, s}_\gamma$. The heads and delays are related by

$$\kappa^{T, s} = \pi \left( \rho^{-1}(T - s) \right).$$

(2.22)

The heads move into $\Omega$ as $s$ varies.

Fix a point $x \in \Omega$. We say that a hydra $H^{T, s}_\gamma$ influences on $x$ and write

$$H^{T, s}_\gamma \succ x$$

if $x \in \kappa^{T, s}$, i.e., $x$ is one of the heads of $H^{T, s}_\gamma$. The value $a^{T, s}(x, T)$ is referred to as amplitude of influence. As is easy to see, for any $x \in \Omega^T[\gamma]$ there is at least one hydra, which influences on $x$. The number of hydras influencing on $x$ is always finite and equal to $\#\rho(\pi^{-1}(x))$.

**Convention 3.** Here and in what follows, dealing with the truncated hydra $H^T_\gamma$, we understand $\pi^{-1}(A)$ as $\pi^{-1}(A) \cap H^T_\gamma$. 

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Take a control \( f \in \mathcal{F}_T^T \) of the form \( f(\gamma', t) = \delta_\gamma(\gamma') \varphi(t) \) with \( \varphi \in C[0, T] \). Fix an \( x \in \Omega \setminus \Gamma \). A structure of the fundamental solution and representation (2.15) easily imply that the value \( u^f(x, T) \) can be calculated by the following procedure:

- find all \( s \in [0, T) \) such that \( H^{T, s}_\gamma \triangleright x \) and determine the corresponding amplitudes \( a^{T, s}(x, T) \)

- find

\[
    u^f(x, T) = \sum_{s: H^{T, s}_\gamma \triangleright x} a^{T, s}(x, T) \varphi(s) = \sum_{\sigma \in \rho^{-1}(\pi^{-1}(x))} a(x, \sigma) \varphi(T - \sigma).
\]

(2.23)

For any \( f \in \mathcal{F}_T^T \) of the form \( f = \sum_{\gamma \in \Gamma} \delta_\gamma(\cdot) \varphi_\gamma \) with \( \varphi_\gamma \in C[0, T] \), relation (2.23) evidently implies

\[
    u^f(x, T) = \sum_{\gamma \in \Gamma} \sum_{s: H^{T, s}_\gamma \triangleright x} a^{T, s}_\gamma(x, T) \varphi_\gamma(s),
\]

(2.24)

where \( a^{T, s}_\gamma(x, T) \) are the amplitudes on hydras \( H^{T, s}_\gamma \). Also, representing \( u^f(\cdot, t) = u^{f_{T-t}}(\cdot, T) \) by (2.19), one can find the value of the wave for any intermediate \( t \in (0, T) \) via (2.23), (2.24).

## 2.4 Reachable sets

### 2.4.1 Definition

In dynamical system (2.1)–(2.4), a set of waves

\[
    \mathcal{U}^s := \{ u^f(\cdot, s) \mid f \in \mathcal{F}^T \} \quad (0 < s \leq T)
\]

is said to be reachable (from the boundary at the moment \( t = s \)). On graphs, \( \mathcal{U}^s \) is a closed subspace in \( \mathcal{H} \). Its structure is of principal importance for many applications, in particular to inverse problems: see [2] - [8], [12].

By (2.7), we have

\[
    \mathcal{U}^s = \sum_{\gamma \in \Gamma} \mathcal{U}^s_\gamma
\]

(algebraic sum), where

\[
    \mathcal{U}^s_\gamma := \{ u^f(\cdot, s) \mid f \in \mathcal{F}^T_\gamma \} \quad (0 < s \leq T)
\]
are the sets reachable from single boundary vertices.

By (2.19), to study $U_s^\gamma$ is to study $U_T^\gamma$, and we deal mainly with the latter set. Its structure will be described in detail. However, the description requires certain preliminary considerations in 2.4.2 – 2.4.4.

2.4.2 Lattices and determination set

We say two different points $l' = (x', t')$, $l'' = (x'', t'')$ of $H_T^\gamma$ to be neighbors and write $l' \simeq l''$, if either $x' = x''$ or $t' = t''$ (equivalently: either $\pi(l') = \pi(l'')$ or $\rho(l') = \rho(l'')$). We write $l' \cong l''$ if there are $l_k$ such that $l' \simeq l_1 \simeq l_2 \cdots \simeq l_p \simeq l''$. As is easy to check, $\cong$ is an equivalence on the hydra. For an $l \in H_T^\gamma$, its equivalence class is called a lattice and denoted by $L[l]$.\(^3\)

For a $B \subset H_T^\gamma$ we set

$$L[B] := \bigcup_{l \in B} L[l] \subset H_T^\gamma.$$  

We omit simple proofs of the following facts, which can be derived from the above-accepted definitions:

- for any $B \subset H_T^\gamma$, one has $\pi^{-1}(\pi(L[B])) = \rho^{-1}(\rho(L[B])) = L[B]$
- the operation $L : B \mapsto L[B]$ satisfies the Kuratovski axioms:
  - (extensiveness) $L[B] \supset B$,
  - (idempotency) $L[L[B]] = L[B]$,
  - (additivity) $L[B' \cup B''] = L[B'] \cup L[B'']$

and, hence, is a topological closure. More precisely, there is a unique topology on the hydra, in which the closure coincides with $L$ (see, e.g., [11]).

- each $L[l]$ is a finite set; it is a closure of the single point set $\{l\}$ in the above mentioned topology. Any point $l' \in L[l]$ determines the whole set $L[l]$.

\(^3\)Here we regard $L[l]$ as a subset of $H_T^\gamma$ but not as an element of the factor-set $H_T^\gamma / \cong$.  

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Figure 5: Lattices and sets $\Lambda_T^\gamma[x]$.

For a point $x \in \overline{\Omega^T[\gamma]} \setminus \Gamma$, define its **determination set** by

$$\Lambda_T^\gamma[x] := \pi \left( L[\pi^{-1}(x)] \right) \subset \Omega. \quad (2.25)$$

The **alternating property** holds: for $x \neq x'$, one has either $\Lambda_T^\gamma[x] = \Lambda_T^\gamma[x']$ or $\Lambda_T^\gamma[x] \cap \Lambda_T^\gamma[x'] = \emptyset$.

Determination set $\Lambda_T^\gamma[x]$ consists of the heads of the delayed hydrams $H_{T, s_i}^\gamma$, which satisfy $T - s_i \in \rho(\mathcal{L}[\pi^{-1}(x)])$. It is the hydrams, which enter in representation (2.23).

On Fig.5,

- $L[\pi^{-1}(p)]$ is the point of $H_T^\gamma$ above $p$, $\Lambda_T^\gamma[p] = \{p\}$
- $L[\pi^{-1}(q_m)]$ is the grey points on $H_T^\gamma$, $\Lambda_T^\gamma[q_m] = \{q_1, q_2, q_3, q_4, q_5\}$
- $L[\pi^{-1}(r_m)]$ is the black points on $H_T^\gamma$, $\Lambda_T^\gamma[r_m] = \{r_1, r_2, r_3\}$.

**2.4.3 Amplitude vectors**

Return to representation (2.23) and modify it as follows.
Let
\[ \rho \left( \mathcal{L} \left[ \pi^{-1}(x) \right] \right) = \{ t_i \}_{i=1}^N, \quad 0 \leq t_1 < t_2 < \cdots < t_N \leq T, \]
so that \( N = \sharp \rho \left( \mathcal{L} \left[ \pi^{-1}(x) \right] \right) \). Introduce the functions \( \alpha^{T,t_i} : \Lambda_T^T[x] \to \mathbb{R}, \)
\[ \alpha^{T,t_i}(h) := \begin{cases} a(h,t_i) & \text{if } (h,t_i) \in H_T^T \gamma \\ 0 & \text{otherwise} \end{cases} \] (2.26)
and call them amplitude vectors.

So, the set \( \Lambda_T^T[x] \) is endowed with amplitude vectors \( \alpha^{T,t_1}, \ldots, \alpha^{T,t_N} \). Let \( l_2(\Lambda_T^T[x]) \) be Euclidean space of functions on \( \Lambda_T^T[x] \) with the standard product
\[ \langle \alpha, \beta \rangle = \sum_{h \in \Lambda^T[x]} \alpha(h) \beta(h). \]
By \( \mathcal{A}^T[x] \) we denote the subspace in \( l_2(\Lambda_T^T[x]) \) generated by amplitude vectors:
\[ \mathcal{A}^T[x] := \text{span} \{ \alpha^{T,t_1}, \ldots, \alpha^{T,t_N} \}. \]

Now, we are able to clarify the meaning of definition (2.25) and the term "determination set". In accordance with (2.23), given \( f = \delta_\gamma \varphi \), for all \( x' \in \Lambda_T^T[x] \) the values \( u^f(x',T) \) are determined by hydras \( H_T^T, \varphi \), whose heads constitute \( \Lambda_T^T[x] \). Moreover, each value is a combination of components of the amplitude vectors \( \alpha^{T,t_1}, \ldots, \alpha^{T,t_N} \). Hence, one can regard the set of values \( \{ u^f(x',T) \mid x' \in \Lambda_T^T[x] \} \) as an element of \( l_2(\Lambda_T^T[x]) \), whereas (2.23) can be written in the form
\[ u^f(\cdot, T)|_{\Lambda_T^T[x]} = \sum_{i=1}^N \alpha^{T,t_i}(T - t_i) \varphi(t_i) \in \mathcal{A}^T[x]. \] (2.27)

Also, one has to keep in mind the dependence \( t_i = t_i(x), \ N = N(x) \).

2.4.4 Partition \( \Pi_T^\gamma \)
Recall that \( \Lambda_T^T[x] \) is defined for points \( x \in \Omega^T[\gamma] \setminus \Gamma \). For a while, let \( x \) belong to the open set \( \Omega^T[\gamma] \setminus [\Gamma \cup V] \); hence, there is an edge \( e \ni x \).
Under the conditions, which we are going to specify now, small variations of position of \( x \) on \( e \) lead to small variations of the set \( \Lambda_T^e \) in \( \Omega \), which do not change its "staff". Namely, the number

\[
M := 2^{\# \Lambda_T^e} = \dim l_2(\Lambda_T^e)
\]

the number \( N \) of amplitude vectors \( \alpha^{T,i} \), and values of the vectors remain the same (do not depend on \( x \)). The question arises: What are the bounds for such variations? In this section the answer is given.

**Critical points.** Recall that the corner points on the complete hydra \( H_\gamma \) are introduced at the end of 2.2.5. Dealing with \( H_T^e \), it is also convenient to assign its top \( \{(x,t) \in H_T^e \mid t = T\} = \rho^{-1}(T) \) to corner points. So, we say a space-time point \( (x,t) \in H_T^e \) to be a **corner point** if it is a corner point of \( H_\gamma \) or belongs to the top of \( H_T^e \). The set of corner points is denoted by Corn\( H_T^e \).

Note that, for any vertex \( w \in [V \cup \Gamma] \cap \Omega_T^e \), its pre-image \( \pi^{-1}(w) \) consists of corner points.

Introduce the lattice \( L[\text{Corn } H_T^e] \), which is a finite set of points on \( H_T^e \). This lattice divides hydra \( H_T^e \) so that the set \( H_T^e \setminus L[\text{Corn } H_T^e] \) consists of a finite number of open space-time intervals, which do not contain corner points. By the latter, on these intervals the amplitude \( a(\cdot) \) takes constant values.

Points of the set

\[
\Theta_T^\gamma := \pi \left( L[\text{Corn } H_T^e] \right) \subset \Omega_T^\gamma
\]

are called **critical**.

Critical points divide neighborhood \( \overline{\Omega_T^\gamma} \) into parts. Namely, the set \( \overline{\Omega_T^\gamma} \setminus \Theta_T^\gamma \) is a collection of open intervals, each interval lying into an edge of \( \Omega \), whereas the critical points are the endpoints of these intervals. We refer to this collection as a **partition** \( \Pi_T^\gamma \).

**Families and cells.** Intervals of partition \( \Pi_T^\gamma \) are joined in the families as follows. Let \( c, c' \in e \) be critical points such that the interval \( \omega = ]c,c'[ \subset e \) contains no critical points. This means that \( \omega \in \Pi_T^e \). As one can easily see, the preimage \( \pi^{-1}(\omega) \) consists of a finite number of connected components. Each component is a (space-time) interval on \( H_T^e \) of the same (space-time) length \( \sqrt{2} \tau(c,c') \) (see \( [\square] \)), the interval being free of corner points. By the latter, the same is valid for the lattice \( L[\pi^{-1}(\omega)] \): it is also a finite collection of open intervals on \( H_T^e \) of length \( \sqrt{2} \tau(c,c') \), which do not contain corner points.
As a consequence, the set
\[ \Phi := \pi \left( \mathcal{L} \left[ \pi^{-1}(\omega) \right] \right) \supset \omega \] (2.29)
turns out to be a finite collection of open intervals \( \omega_1, \omega_2, \ldots, \omega_M \subset \overline{\Omega_T[\gamma]} \setminus \Theta_T^\gamma \) (with \( \omega \) among them) of the same length:
\[ \Phi = \bigcup_{l=1}^{M} \omega_m, \quad \text{diam } \omega_m = \tau(c, c') =: \delta_{\Phi}. \] (2.30)

We say this collection to be a \textit{family}, intervals \( \omega_m \) are called \textit{cells} of \( \Phi \).

Comparing definitions (2.25) and (2.29), (2.30), one can easily conclude the following. For any \( x \in \Phi \), the set \( \Lambda_T^\gamma[x] \subset \Phi \) consists of the points \( x_1, \ldots, x_M \), each cell \( \omega_m \) containing one (and only one) point \( x_m \). Hence, we have
\[ \# \Lambda_T^\gamma[x] = \dim \mathcal{L}(\Lambda_T^\gamma[x]) = M, \quad \Phi = \bigcup_{x \in \omega} \Lambda_T^\gamma[x] \] (2.31)
as \( x \) varies in any cell \( \omega \subset \Phi \).

Starting with another interval \( \omega' \not\subset \Phi \) bounded by critical points, we get another family \( \Phi' \), which has no mutual cells or points with \( \Phi \). Going on this way, we get a finite set of families \( \Phi^1, \Phi^2, \ldots, \Phi^J \) and conclude that partition \( \Pi_T^\gamma \) corresponds to the representation
\[ \overline{\Omega_T[\gamma]} \setminus \Theta_T^\gamma = \bigcup_{j=1}^{J} \Phi^j = \bigcup_{j=1}^{J} \bigcup_{m=1}^{M_j} \omega_m^{(j)} \] (2.32)
in the form of disjoint sums,
\[ \text{diam } \omega_1^{(j)} = \cdots = \text{diam } \omega_{M_j}^{(j)} = \delta_{\Phi^j} \]
being valid.

On Fig.6,

- the set \( \text{Corn } H_T^\gamma \) is the black points with holes, the lattice \( \mathcal{L} \left[ \text{Corn } H_T^\gamma \right] \)
is the grey points along with the corner points
- the critical points set is \( \Theta_T^\gamma = \bigcup_{k=1}^{10} c_k \), \( c_1 = \gamma, c_5 = v, c_T = \gamma' \)
Figure 6: Partition $\Pi^T_\gamma [x]$
• the families and cells are

\[ \Phi^1 = \omega^{(1)} = ]c_1, c_2[ \quad (\text{dotted line}), \]

\[ \Phi^2 = \bigcup_{m=1}^{5} \omega^{(2)} = ]c_2, c_3[ \cup ]c_3, c_4[ \cup ]c_6, c_7[ \cup ]c_8, c_9[ \cup ]c_9, c_{10}[ \quad (\text{grey intervals}), \]

\[ \Phi^3 = \bigcup_{m=1}^{3} \omega^{(3)} = ]c_4, c_5[ \cup ]c_5, c_6[ \cup ]c_5, c_7[ \cup ]c_9[ \quad (\text{black intervals}). \]

**Variations and bounds.** Return to the question on the bounds at the beginning of 2.4.5.

Take a cell \( \omega = ]c, c'[ \subset \Phi = \bigcup_{i=1}^{M} \omega_m \) and choose an \( x \in \omega \). The determination set \( \Lambda_T^\gamma[\bigcup_r x] \) consists of the points \( x_1, \ldots, x_M \) (among them), \( x_m \in \omega_m \). Varying \( x \), one varies the set \( \Lambda_T^\gamma[\bigcup_r x] \) (the points \( x_m \)).

Parametrize

\[ \omega \ni x = x(r), \quad r = \tau(x, c) \in (0, \delta_\Phi); \quad (2.33) \]

simultaneously, all \( x_m(r) \in \Lambda_T^\gamma[\bigcup_r x] \) turn out to be also parametrized. As \( r \) varies from 0 to \( \delta_\Phi \), each \( x_m(r) \) runs over \( \omega_m = ]c_m, c'_m[ \) (from \( c_m \) to \( c'_m \) or in the opposite direction) and sweeps the cell \( \omega_m \). Correspondingly, \( \Lambda_T^\gamma[\bigcup_r x] \) varies continuously on the graph and sweeps the given family \( \Phi \).

An important fact is that, in process of such varying, the amplitude vectors \( \alpha_{T, t_1(x(r))}, \ldots, \alpha_{T, t_N(x(r))} \in l_2(\Lambda_T^\gamma[\bigcup_r x]) \) do not vary. This follows from definition (2.26): the points \( (x_m(r), t_i(x(r))) \) of the "horizontal layer" \( \rho^{-1}(t_i) \) move along the hydra but do not leave the intervals of \( H_T^\gamma \), on which there are no corner points, and hence amplitude \( a \) takes constant values (does not depend on \( r \)):

\[ \alpha_{T, t_i(x(r))}(x_m(r)) = a(x_m(r), t_i(x(r))) = \text{const} =: \alpha_{m, i}^T \quad (2.34) \]

as \( i = 1, \ldots, N; \ m = 1, \ldots, M \). Therefore, it is natural to associate amplitude vectors not with the set \( \Lambda_T^\gamma[\bigcup_r x] \) but the given family \( \Phi \ni x \) and regard them as piece-wise constant functions on \( \Phi \) defined by

\[ \alpha_{T, i}^x(x) := \alpha_{m, i}^T \quad \text{as } x \in \omega_m \subset \Phi. \]

We do it in what follows.
If \( x \) passes through a critical point \( c \) and enters a cell of another family, the picture of amplitude vectors changes. So, it is the set \( \Theta^T_\gamma \), which provides the bounds for variations, which do not disturb \( I_2(\Lambda^T_\gamma[\gamma]) \) and \( \mathcal{A}^T[\gamma] \).

Varying \( T \), one varies the set of critical points. Some of them (in particular, the vertices \( V \cup \Gamma \)) do not change the position in \( \Omega \), the others are moving along the graph with velocity 1. By this, the set varies continuously in the following sense: for a given \( T \), there is a positive \( \varepsilon_0 \) such that

\[
\Theta^{T+\varepsilon}_\gamma \subset \Omega^{\varepsilon}[\Theta^T_\gamma] 
\]

holds for all \( \varepsilon \in (-\varepsilon_0, \varepsilon_0) \).

### 2.4.5 Local and global structure of wave

Return to (2.27) and recall that, in such a general representation, the amplitude vectors and delays are determined by position of \( x \):

\[
u^f(\cdot, T)|_{\Lambda^T_\gamma[\gamma]} = \sum_{i=1}^{N(x)} \alpha^{T,t_i(x)}(x) \varphi(T - t_i(x)).
\]

Now, choose an \( x \) in a cell \( \omega \) of a family \( \Phi = \bigcup_{m=1}^M \omega_m \) and parametrize by (2.33): \( x_m(r) \in \omega_m, \ r \in (0, \delta_\Phi) \). Taking into account (2.34), we arrive at basic representation of the wave \( u^f \) on the family \( \Phi \):

\[
u^f(x_m(r), T) = \sum_{i=1}^N \alpha^{T,t_i(x)}_m \psi_i(r), \quad r \in (0, \delta_\Phi), \ m = 1, \ldots, M,
\]

where \( \psi_i(r) := \varphi(T - t_i(x(r))) \).

By (2.32) and (2.36), we represent the wave everywhere in \( \Omega^T[\gamma] \) except of critical points, i.e., almost everywhere on the graph. Such a representation clarifies a local structure of waves in the cells of families.

Recall that we deal with a control of the form \( f(\gamma', t) = \delta_\gamma(\gamma')\varphi(t) \) with \( \varphi \in C[0, T] \). However, one can extend the representation to all controls \( f \in F^T_\gamma \) just by taking \( \psi_i \in L_2(0, \delta_\Phi) \) in (2.36).

Representation (2.36) provides a characteristic description of waves on families. A function \( y \in \mathcal{H} = L_2(\Omega) \), \( \supp y \subset \Phi \) is a wave (i.e., \( y \in \mathcal{U}^T_\gamma \) does hold) if and only if \( y \) can be represented in the form of the right hand
side of (2.36) with \( \psi_i \in L_2(0, \delta_\Phi) \). Functions \( \psi_i \) play the role of independent function parameters, which determine a wave supported in \( \Phi \).

Taking into account (2.32), we get a global characteristic description of the reachable set \( U^T_\gamma \): to be a wave a function \( y \) supported in \( \Omega^T_\gamma \) has to admit the representation (2.36) on each \( \Phi \subset \Omega^T_\gamma \).

### 2.4.6 Projection \( P^T_\gamma \)

Let \( P^T_\gamma \) be the (orthogonal) projection in \( \mathcal{H} = L_2(\Omega) \) onto the reachable subspace \( U^T_\gamma \). Here we provide a constructive description of this projection by the use of representation (2.36).

For a subset \( B \subset \Omega \), by \( \chi_B \) we denote its indicator (the characteristic function) and introduce the subspace

\[
\mathcal{H}\langle B \rangle := \chi_B \mathcal{H} = \{ \chi_B y \mid y \in \mathcal{H} \}
\]

of functions supported on \( B \). In accordance with (2.32) and results of 2.4.5, one has

\[
\mathcal{H}\langle \Omega^T_\gamma \rangle = \bigoplus_{\Phi \in \Pi^T_\gamma} \mathcal{H}\langle \Phi \rangle , \quad U^T_\gamma = \bigoplus_{\Phi \in \Pi^T_\gamma} U^T_\gamma \langle \Phi \rangle ,
\]

where \( U^T_\gamma \langle \Phi \rangle \subset \mathcal{H}\langle \Phi \rangle \) is the subspace of waves supported in \( \Phi \) and represented by (2.36). Therefore,

\[
P^T_\gamma = \sum_{\Phi \in \Pi^T_\gamma} Q_\Phi ,
\]

where \( Q_\Phi \) project in \( \mathcal{H}\langle \Phi \rangle \) onto subspaces \( U^T_\gamma \langle \Phi \rangle \). Hence, to characterize \( P^T_\gamma \) is to describe projections \( Q_\Phi \).

Parametrize \( \Phi \) by (2.33) and introduce an isometry \( U \) by

\[
\mathcal{H}\langle \Phi \rangle \ni y \xmapsto{U} \begin{pmatrix} y(x_1(r)) \\ \vdots \\ y(x_M(r)) \end{pmatrix} \bigg|_{r \in (0, \delta_\Phi)} \in L_2((0, \delta_\Phi); \mathbb{R}^M) .
\]

Since \( \psi_i \) in (2.36) can be arbitrary, this representation implies

\[
U U^T_\gamma \langle \Phi \rangle = L_2((0, \delta_\Phi); A_\Phi) , \quad A_\Phi := \text{span} \left\{ \begin{pmatrix} \alpha^{T,1}_1 \\ \vdots \\ \alpha^{T,1}_M \\ \alpha^{T,N}_1 \\ \vdots \\ \alpha^{T,N}_M \end{pmatrix} \right\} \subset \mathbb{R}^M .
\]
A substantial fact is that the vectors, which span $A\Phi$, are constant (do not depend on $r$). Let $p\Phi = \{p_{mm'}^{mm'}\}_{m,m'=1,...,M}$ be the (matrix) projection in $R^M$ onto $A\Phi$. Projection $\tilde{Q}\Phi$ in the space $L_2((0,\delta\Phi); R^M)$ onto its subspace $L_2((0,\delta\Phi); A\Phi)$ acts point wise by the rule

$$(\tilde{Q}\Phi v)(r) = p\Phi \begin{pmatrix} v_1(r) \\ \vdots \\ v_M(r) \end{pmatrix}, \quad r \in (0,\delta\Phi).$$

In the mean time, one has $Q\Phi = U^*\tilde{Q}\Phi U$. Summarizing, we arrive at the representation

$$(Q\Phi y)(x_m(r)) = \sum_{m'=1}^{M} p_{mm'}^{mm'} y(x_{m'}(r)) , \quad r \in (0,\delta\Phi), \quad m = 1,...,M \quad (2.40)$$

with the constant matrix $p\Phi$, which characterizes the action of $Q\Phi$.

**System $\tilde{\beta}_\gamma^T$** The latter representation can be written in more detail as follows.

Redesign the system of amplitude vectors $\{\alpha^{T,1},\ldots,\alpha^{T,N}\}$ by the Schmidt procedure:

$$\beta^{T,i} := \begin{cases} \frac{\alpha^{T,i} - \sum_{j=1}^{i-1} \langle \alpha^{T,i}, \beta^{T,j} \rangle \beta^{T,j}}{\|\alpha^{T,i} - \sum_{j=1}^{i-1} \langle \alpha^{T,i}, \beta^{T,j} \rangle \beta^{T,j}\|} & \text{if } \alpha^{T,i} \notin \text{span } \{\alpha^{T,1},\ldots,\alpha^{T,i-1}\} \\ 0 & \text{otherwise} \end{cases} \quad (2.41)$$

and get a system $\tilde{\beta}^T := \{\beta^{T,1},\ldots,\beta^{T,N}\}$. Its nonzero elements satisfy $\langle \beta^{T,i}, \beta^{T,j} \rangle = \delta_{ij}$, and span $\tilde{\beta}^T = A^T[x]$ holds.

By analogy with original vectors $\alpha^{T,i}$, it is convenient to regard new amplitude vectors as piece-wise constant functions on the family $\Phi$:

$$\beta^{T,i}(x) := \beta^{T,i}_m, \quad x \in \omega_m \subset \Phi. \quad (2.42)$$

Expressing the projection matrix $p\Phi$ via system $\beta^{T,1},\ldots,\beta^{T,N}$ in (2.40), one can represent the action of $Q\Phi$ in the following final form

$$(Q\Phi y)(x) = \begin{cases} \sum_{i=1}^{N} \langle y|_{\Lambda_{x}[x]}, \beta^{T,i} \rangle \beta^{T,i}(x) & x \in \Phi, \\ 0 & x \in \Omega \setminus \Phi \end{cases} \quad (2.43)$$
which is valid for any \( y \in \mathcal{H} \).

At last, recalling (2.38), we conclude that \( P^T_\gamma \) is characterized.

Note in addition that representations (2.38), (2.43) provide a look at controllability of a graph. Recall a version of the boundary control problem: given \( y \in \mathcal{H} \) to find \( f \in \mathcal{F}^T_\gamma \) such that \( u^f(\cdot, T) = y \) holds. Controllability from \( \gamma \) means that \( \mathcal{U}^T_\gamma = \mathcal{H} \), i.e., this problem is well solvable. In our terms, the latter is equivalent to the relations \( A^T_\Phi = \mathbb{R}^{M(\Phi)}, \Phi \in \Pi^T_\gamma \).

**Dependence on \( T \).** Varying \( T \), one varies the neighborhood \( \Omega^T[\gamma] \) filled with waves, reachable set \( \mathcal{U}^T_\gamma \) and projection \( P^T_\gamma \). As is evident, \( \mathcal{U}^T_\gamma \) is increasing in \( \mathcal{H} \) as \( T \) grows. The following arguments show that \( P^T_\gamma \) is varied continuously.

Take a small \( \Delta T > 0 \). The lattice \( \mathcal{L}[\rho^{-1}([T - \Delta T, T])] \subset H^T_\gamma \) is also "small": it consists of a final set of (closed) intervals on the hydra, the total length of the intervals vanishing as \( \Delta T \to 0 \). The same holds for the intervals in \( \Omega \), which constitute the set \( \pi(\mathcal{L}[\rho^{-1}([T - \Delta T, T])] \subset \Pi^T_\gamma \).

As is easy to see, for a point \( x \notin \pi(\mathcal{L}[\rho^{-1}([T - \Delta T, T])] \) one has \( \Lambda_T^T[x] = \Lambda^T - \Delta T[x] \), whereas the amplitude vectors, which take part in projecting (2.36), are the same: \( \alpha^T, i = \alpha^T - \Delta T, i \). Therefore, for any function \( y \in \mathcal{H} \) we have \( (P^T_\gamma y)(x) = (P^T - \Delta T y)(x) \). Hence, the difference \( P^T_\gamma y - P^T - \Delta T y \) has to be supported on the complement to such points:

\[
\text{supp} \left( P^T_\gamma - P^T - \Delta T \right)y \subset \pi(\mathcal{L}[\rho^{-1}([T - \Delta T, T])]) \subset \overline{\Omega^T [\Theta^T_\gamma]}.
\] (2.44)

As \( \Delta T \to 0 \), the neighborhood \( \Omega^T [\Theta^T_\gamma] \) shrinks to the finite set \( \Theta^T_\gamma \) that implies \( \| (P^T_\gamma - P^T - \Delta T) y \| \to 0 \), i.e., \( P^T - \Delta T \to P^T_\gamma \) in the strong operator topology in \( \mathcal{H} \).

Quite analogous arguments with regard to property (2.35) show that \( P^T + \Delta T \to P^T_\gamma \) as \( \Delta T \to 0 \). Hence, \( \{P^T_\gamma\}_{T \geq 0} \) is an increasing continuous family of projections in \( \mathcal{H} \).

### 3 Eikonal algebra

#### 3.1 Single eikonal

Fix \( \gamma \in \Gamma \) and \( T > 0 \). Let \( \Xi = \{\xi_k\}_{k=0}^K, 0 = \xi_0 < \xi_1 < \cdots < \xi_K = T \) be a partition of \([0, T]\) of the range \( r(\Xi) = \max(\xi_k - \xi_{k-1}) \); denote \( \Delta P^T_\gamma := \cdots \)
$P_\gamma^{\xi_k} - P_\gamma^{\xi_{k-1}}$. With each boundary vertex $\gamma \in \Gamma$ we associate a bounded self-adjoint operator in $\mathcal{H}$ of the form

$$E_\gamma^T := \int_0^T \xi \, dP^\xi_\gamma = \lim_{r(\Xi) \to 0} \sum_{k=1}^K \xi_k \Delta P^{\xi_k}_\gamma$$

(see, e.g., [9]) and call it an \textit{eikonal}. Our nearest purpose is to describe how it acts.

### 3.1.1 Small $T$

Begin with the case $T \leq \tau(\gamma, V)$. In outer space $\mathcal{F}^T$, choose a control $f(\gamma', t) = \delta_\gamma(\gamma')\varphi(t)$. By (2.8) and (2.15), one has

$$u^f(x, t) = \varphi(t - \tau(x, \gamma))$$

(recall the Convention [2]). Therefore, the reachable sets are

$$U_\gamma^\xi = \{ \varphi(\xi - \tau(\cdot, \gamma)) \mid \varphi \in L_2(0, T) \} = \mathcal{H}(\Omega^{\xi}[\gamma]), \quad 0 \leq \xi \leq T.$$  

Correspondingly, projection $P_\gamma^{\xi}$ in $\mathcal{H}$ onto $U_\gamma^\xi$ cuts off functions on the part $\Omega^{\xi}[\gamma]$ of the edge $e$ incident to $\gamma$, i.e., multiplies functions by the indicator $\chi_{\Omega^{\xi}[\gamma]}$. Therefore, for a $y \in \mathcal{H}$, the summands in (3.1) are

$$\left( \xi_k \Delta P^{\xi_k}_\gamma y \right)(x) = \begin{cases} 
\xi_k \, y(x) \approx \tau(x, \gamma) y(x) & \text{for } x \in \Omega^{\xi_k}[\gamma] \setminus \Omega^{\xi_{k-1}}[\gamma] \\
0 & \text{for other } x \in \Omega
\end{cases}$$

(\approx \text{means that } \tau(x, \gamma) = \xi_k + O(r(\Xi)) \text{ for } x \in \Omega^{\xi_k}[\gamma] \setminus \Omega^{\xi_{k-1}}[\gamma]). \text{ Summing up the terms and passing to the limit as } r(\Xi) \to 0, \text{ we easily obtain}

$$\left( E_\gamma^T y \right)(x) = \begin{cases} 
\tau(x, \gamma) \, y(x) & \text{for } x \in \Omega^T[\gamma] \\
0 & \text{for other } x \in \Omega
\end{cases}.$$ 

Thus, for small enough $T$’s, the eikonal cuts off functions on $\Omega^T[\gamma]$ and multiplies by the distance to $\gamma$. 

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3.1.2 Functions \( \tau_i \)

Let \( T > 0 \) be arbitrary. Choose a family \( \Phi \in \Pi_T \). The set

\[
\rho \left( \mathcal{L} \left[ \pi^{-1}(\Phi) \right] \right) = \bigcup_{i=1}^{N} I_i
\]

consists of \( N = N(\Phi) \) disjoined open intervals \( I_i \subset (0, T) \) of the same length \( \delta_\Phi \). Note that the segments \( T_i \) may intersect at the endpoints. On Fig.6, there is \( \rho \left( \mathcal{L} \left[ \pi^{-1}(\Phi^2) \right] \right) = \bigcup_{i=1}^{4} I_i \), where \( I_1 = (T_1, T_2), I_2 = (T_2, T_3), I_3 = (T_4, T_5), I_4 = (T_5, T_6) \) (see also Fig.7).

With representation (3.2) one associates \( N \) functions on \( \Phi \) of the form

\[
\tau_i(x) := \begin{cases} 
I_i \cap \rho \left( \pi^{-1}(x) \right) & \text{if } I_i \cap \rho \left( \pi^{-1}(x) \right) \neq \emptyset \\
0 & \text{otherwise}
\end{cases} = \begin{cases} 
t & \text{if } (x, t) \in \rho^{-1}(I_i) \\
0 & \text{otherwise}
\end{cases}
\]

which take values in the corresponding intervals \( I_i \). They are of clear geometric meaning in terms of the distance on the hydra: if \( \tau_i(x) \neq 0 \) then \( (x, \tau_i(x)) \in H_T^T \) and

\[
\tau_i(x) = \frac{1}{\sqrt{2}} \nu \left( (x, \tau_i(x)), (\gamma, 0) \right)
\]

holds (see (1.1)).

On Fig.7, the functions \( \tau_1, \tau_2, \tau_3, \tau_4 \), which correspond to family \( \Phi^2 \) (see the grey part of \( \Omega \) on Fig.6), are shown. The space between the supports and graphs of the functions is shaded.

3.1.3 Arbitrary \( T \)

Fix a \( T > \tau(\gamma, V) \). Let \( y \in \mathcal{H} \) be such that \( \text{supp } y \subset \Phi \in \Pi_T \) (i.e., \( y \in \mathcal{H}(\Phi) \); see (2.37)). Choose a \( \xi \in (0, T] \) and a small \( \Delta \xi > 0 \).

Assume that

\[
(\xi - \Delta \xi, \xi) \subset [0, T] \setminus \bigcup_{i=1}^{N} I_i
\]
Figure 7: Functions $\tau_i$
In such a case one has $\pi \left( \mathcal{L} \left[ \rho^{-1} \left( (\xi - \Delta \xi, \xi) \right) \right] \right) \cap \Phi = \emptyset$. Therefore, by (2.44), the supports of the functions $(P^\xi_\gamma - P^\xi_\gamma - \Delta \xi) y$ and $y$ do not intersect, and we have $(P^\xi_\gamma y - P^\xi_\gamma - \Delta \xi, y, y) = 0$, i.e., $\|\Delta P^\xi_\gamma y\| = 0$, where $\Delta P^\xi_\gamma = P^\xi_\gamma - P^\xi_\gamma - \Delta \xi$. By the latter, the interval $(\xi - \Delta \xi, \xi)$ contributes nothing to the integral, which determines $E^T_y$.

By the aforesaid and with regard to continuity of $P^\xi_\gamma$, for $y \in \mathcal{H}(\Phi)$ the integral (3.1) can be taken over the intervals $I_i$ only:

$$E^T_y = \int_0^T \xi \, dP^\xi_\gamma y = \sum_{i=1}^N \int_{I_i} \xi \, dP^\xi_\gamma y. \quad (3.5)$$

The summands are pairwise orthogonal since the subspace $P^\xi_\gamma \mathcal{H}$ is increasing as $\xi$ grows.

Let $\Phi$ and $y$ be the same as before. For what follows, it is convenient to renumber the endpoints of the intervals so that $I_i = (T_i, T_i + \delta_y)$. The amplitude vectors $\beta^{T,1}$ are regarded as piece-wise constant functions on $\Phi$.

Now, assume that $(\xi - \Delta \xi, \xi) \subset I_1$. In this case, the only amplitude vector, which contributes to the values of $P^\xi_\gamma y$ on $\Phi$, is $\beta^{T,1}$. Therefore, in accordance with (2.43), (2.44) we have

$$(\Delta P^\xi_\gamma y) (x) = \Delta \chi^\xi(x) \langle y |_{\Lambda^T_\gamma[x]} \beta^{T,1} \rangle \beta^{T,1} (x), \quad x \in \Phi,$$

where $\Delta \chi^\xi$ is the indicator of the set $\pi \left( \mathcal{L} \left[ \rho^{-1} \left( (\xi - \Delta \xi, \xi) \right) \right] \right) \subset \Phi$. Correspondingly,

$$(\xi \Delta P^\xi_\gamma y) (x) \approx \tau_1(x) \Delta \chi^\xi(x) \langle y |_{\Lambda^T_\gamma[x]} \beta^{T,1} \rangle \beta^{T,1} (x), \quad x \in \Phi,$$

where $\tau_i(x)$ are introduced by (3.3). Summing up the terms of this form, one can easily justify the limit passage as $r(\Xi) \to 0$ and get the equality

$$\left( \int_{T_1}^{T_1 + \delta_y} \xi \, dP^\xi_\gamma y \right) (x) = \tau_1(x) \langle y |_{\Lambda^T_\gamma[x]} \beta^{T,1} \rangle \beta^{T,1} (x), \quad x \in \Phi.$$

Assume that $(\xi - \Delta \xi, \xi) \subset I_2$. In this case, the amplitude vectors, which contribute to the values of $P^\xi_\gamma y$ on $\Phi$, are $\beta^{T,1}$ and $\beta^{T,2}$. In the mean time, (2.42) and (2.43) imply $P^{T_1 + \delta_y} \beta^{T,1} = \beta^{T,1}$ that leads to

$$(\Delta P^\xi_\gamma y, \beta^{T,1})_{\mathcal{H}(\Phi)} = (\Delta P^\xi_\gamma y, P^{T_1 + \delta_y} \beta^{T,1})_{\mathcal{H}(\Phi)} = 0,$$

$$([P^\xi_\gamma P^{T_1 + \delta_y} - P^\xi_\Delta \xi P^{T_1 + \delta_y} y, \beta^{T,1}]_{\mathcal{H}(\Phi)} = 0.$$
by monotonicity of \( P_\gamma^\xi \). Hence, \( \Delta P_\gamma^\xi y \) has to be proportional to \( \beta^{T,2} \) and we easily get
\[
(\Delta P_\gamma^\xi y) (x) = \Delta \chi^\xi(x) \langle y|_{\Lambda_\gamma^T[x]}, \beta^{T,2} \rangle \beta^{T,2}(x), \quad x \in \Phi.
\]
Correspondingly,
\[
(\xi \Delta P_\gamma^\xi y) (x) \approx \tau_2(x) \Delta \chi^\xi(x) \langle y|_{\Lambda_\gamma^T[x]}, \beta^{T,2} \rangle \beta^{T,2}(x), \quad x \in \Phi.
\]
Summing up such terms and passing to the limit, we obtain
\[
\left( \int_{T_2}^{T_2+\delta_\Phi} \xi dP_\gamma^\xi y \right) (x) = \tau_2(x) \langle y|_{\Lambda_\gamma^T[x]}, \beta^{T,2} \rangle \beta^{T,2}(x), \quad x \in \Phi.
\]
Continuing in the same way, with regard to (3.5) we arrive at the representation
\[
(E^T_\gamma y) (x) = \sum_{i=1}^N \tau_i(x) \langle y|_{\Lambda_\gamma^T[x]}, \beta^{T,i} \rangle \beta^{T,i}(x), \quad x \in \Phi. \quad (3.6)
\]
So, the eikonal projects functions \( y \in \mathcal{H}(\Phi) \) on amplitude vectors and multiplies by relevant distances.

Parametrize \( \Phi \) by (2.33), (2.39): \( x_m(r) \in \omega_m \subset \Phi, \; m = 1, 2, \ldots, M, \; 0 < r < \delta_\Phi \). Denote
\[
\bar{y}(r) := \begin{pmatrix} y(x_1(r)) \\ \vdots \\ y(x_M(r)) \end{pmatrix}, \quad B_\Phi := \begin{pmatrix} \beta_1^{T,1} & \cdots & \beta_M^{T,1} \\ \beta_1^{T,2} & \cdots & \beta_M^{T,2} \\ \vdots & \cdots & \vdots \\ \beta_1^{T,N} & \cdots & \beta_M^{T,N} \end{pmatrix},
\]
\[
D_\Phi(r) = \{ \tau_i(r) \delta_{ij} \}_{i,j=1}^N,
\]
where \( \tau_i(r) := \tau_i(x(r)) \) is either \( T_i + r \) or \( T_i + \delta_\Phi - r \) (see (3.3)). Note that \( B_\Phi^* B_\Phi \) is the matrix of the projection \( p_\Phi \) in \( \mathbb{R}^M \) onto \( \Lambda_\Phi = \text{span} \{ \beta^{T,1}, \ldots, \beta^{T,N} \} \).

In this notation, (3.6) takes the form
\[
(\overrightarrow{E_\gamma y})(r) = [B_\Phi^* D_\Phi(r) B_\Phi] \bar{y}(r), \quad r \in (0, \delta_\Phi). \quad (3.7)
\]
Recalling the decomposition \((2.37)\), we conclude that it reduces the eikonal and the representation

\[
E_T^\gamma = \bigoplus_{\Phi \in \Pi_T^\gamma} E_T^\gamma \chi_\Phi
\]

holds, where \(\chi_\Phi\) is understood as an operator in \(\mathcal{H}\) multiplying by the indicator. Each part \(E_T^\gamma \chi_\Phi\) acts in \(\mathcal{H}(\Phi)\) by \((3.6)\) (by \((3.7)\) in the parametrized form).

Note in addition that \((3.7)\) and \((3.8)\) in fact provide a canonical representation (diagonalization) of the eikonal in the sense of the Spectral Theorem for self-adjoint operators: see, e.g., \([9]\). Also, one can easily see that its spectrum \(\sigma(E_T^\gamma)\) is ordinary (of multiplicity 1) and absolutely continuous, \(\sigma(E_T^\gamma) = [0, T]\).

### 3.2 Algebra \(\mathcal{E}_\Sigma^T\)

#### 3.2.1 Definition

Let \(\Sigma \subseteq \Gamma\) be a subset of boundary vertices. For controls

\[f \in \mathcal{F}_\Sigma^T := \bigoplus_{\gamma \in \Sigma} \mathcal{F}_\gamma^T,\]

the waves \(u^f(\cdot, T)\) are supported in the metric neighborhood \(\Omega_T[\Sigma] \subset \Omega\) (see \((2.18)\)). These waves constitute a reachable set

\[\mathcal{U}_\Sigma^T = \bigoplus_{\gamma \in \Sigma} \mathcal{U}_\gamma^T \subset \mathcal{H}(\Omega_T[\Sigma])\]

(algebraic sum).

Let \(\mathcal{B}(\mathcal{H})\) be the normed algebra of bounded operators in \(\mathcal{H}\). With each \(\gamma \in \Sigma\) one associates the eikonal \(E_T^\gamma \in \mathcal{B}(\mathcal{H})\). By \(\mathcal{E}_\Sigma^T\) we denote the \(C^*\)-subalgebra of \(\mathcal{B}(\mathcal{H})\) generated by eikonals \(\{E_T^\gamma\}_{\gamma \in \Sigma}\), i.e., the minimal norm-closed \(C^*\)-subalgebra in \(\mathcal{B}(\mathcal{H})\), which contains all these eikonals \([10], [13]\).

Our paper is written for the sake of introducing algebra \(\mathcal{E}_\Sigma^T\). It is defined by perfect analogy with the eikonal algebras associated with Riemannian manifolds: see \([4], [6]\). In the rest of the paper, we clarify a structure of \(\mathcal{E}_\Sigma^T\).
3.2.2 Partition $\Pi^T_\Sigma$

The set
\[ H^T_\Sigma := \bigcup_{\gamma \in \Sigma} H^T_\gamma \subset \Omega^T[\Sigma] \times [0, T] \]
is also said to be a \textit{hydra}. It is also a space-time graph.

The analogs of the objects, which are related with $H^T_\gamma$, are introduced for $H^T_\Sigma$ as follows.

- The projections $\pi : (x, t) \mapsto x$ and $\rho : (x, t) \mapsto t$ are now understood as the maps from $H^T_\Sigma$ to $\Omega$ and $[0, T]$ respectively.

- An \textit{amplitude} on $H^T_\Sigma$ is
\[ a(x, t) := \sum_{\gamma \in \Sigma: (x, t) \in H^T_\gamma} a_\gamma(x, t), \]
where $a_\gamma$ is the amplitude on $H^T_\gamma$.

- The equivalence $l' \cong l''$ and lattices $\mathcal{L}[B]$ are defined as in 2.4.2, just replacing $H^T_\gamma$ by $H^T_\Sigma$.

- The set $\text{Corn } H^T_\Sigma$ of \textit{corner points} is defined as in 2.4.4, replacing $H^T_\gamma$ by $H^T_\Sigma$. The evident relation
\[ \text{Corn } H^T_\Sigma \supset \bigcup_{\gamma \in \Sigma} \text{Corn } H^T_\gamma \]
holds. However, the latter sum can be smaller than $\text{Corn } H^T_\Sigma$ because additional corner points on $H^T_\Sigma$ do appear owing to intersection of the space-time edges of $H^T_\gamma$ with edges of $H^T_{\gamma'}$ for different $\gamma, \gamma' \in \Sigma$. It happens as $T > \frac{1}{2} \tau(\gamma, \gamma')$.

- \textit{Critical points} are introduced in the same way as (2.28): they constitute a finite set
\[ \Theta^T_\Sigma := \pi \left( \mathcal{L} \left[ \text{Corn } H^T_\Sigma \right] \right) \supset \bigcup_{\gamma \in \Sigma} \Theta^T_\gamma. \]
• Fix an \( x \in \overline{\Omega^T[\Sigma] \setminus \Theta^T_\Sigma} \); let \( \gamma, \gamma' \ni x \) be an open interval in the graph between the critical points \( \gamma, \gamma' \), which contains no critical points. By analogy with (2.25), define a determination set

\[
\Lambda^T_\Sigma[x] := \pi (\mathcal{L} [\pi^{-1}(x)]) \ni x
\]

(here \( \mathcal{L} \) is a lattice on \( \Sigma^T \)). As is evident, one has \( \Lambda^T_\Sigma[x] \supset \Lambda^T_{\Sigma^T}[x] \) as \( \gamma \in \Sigma \). If \( x \) runs over \( \gamma, \gamma' \) from \( \gamma \) to \( \gamma' \), the set \( \Lambda^T_\Sigma[x] \) sweeps a family \( \Phi = \bigcup_{m=1}^{M} \omega_m \) of open intervals \( \omega_m \) (cells) of the same length \( \delta_\Phi \).

By the aforesaid and analogy with (2.32), we have the representation

\[
\overline{\Omega^T[\Sigma] \setminus \Theta^T_\Sigma} = \bigcup_{j=1}^{J} \bigcup_{i=1}^{M_j} \omega_m^{(j)} \quad (3.9)
\]
in the form of disjoint sums, where \( \text{diam} \omega_1^{(j)} = \cdots = \text{diam} \omega_M^{(j)} = \delta_\Phi \). This representation is referred to as a partition \( \Pi^T_\Sigma \).

Fig.8 illustrates the case \( \Sigma = \{\gamma, \gamma'\} \) for \( T = \tau(\gamma, \gamma') + \varepsilon \):

• Corn \( H^T_\Sigma \) is the black points with small holes at the center, \( \mathcal{L}[\text{Corn} H^T_\Sigma] \) is Corn \( H^T_\Sigma \) plus the grey points

• the critical points \( \Theta^T_\Sigma = \bigcup_{k=1}^{16} c_k \), \( c_1 = \gamma, c_5 = v, c_{13} = \gamma' \) are denoted by \( c_k \equiv k \)

• the families and cells are

\[
\Phi^1 = \bigcup_{m=1}^{2} \omega_m^{(1)} \text{ (dashed)}, \quad \Phi^2 = \bigcup_{m=1}^{8} \omega_m^{(2)} \text{ (grey)}, \quad \Phi^3 = \bigcup_{m=1}^{5} \omega_m^{(3)} \text{ (black)}
\]

• the neighborhoods \( \Omega^T[\gamma] \) and \( \Omega^T[\gamma'] \) filled with waves are contoured with the dashed lines.

3.2.3 Functions \( \tau^T_\gamma, \Phi \) and system \( \tilde{\beta}^T_\gamma, \Phi \)

Choose a family \( \Phi = \bigcup_{m=1}^{M} \omega_m \), which is an element of the partition \( \Pi^T_\Sigma \). Quite analogously to (3.2), the set

\[
\rho (\mathcal{L} [\pi^{-1}(\Phi)]) = \bigcup_{i=1}^{N} I_i \quad (3.10)
\]

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Figure 8: Partition $\Pi_{\Sigma}^T$
consists of \( N = N(\Phi) \) disjoined open intervals \( I_i = (T_i, T_i + \delta_\Phi) \subset (0, T) \) of the same length \( \delta_\Phi \). We number them so that \( 0 \leq T_1 < T_2 < \ldots T_N < T_N + \delta_\Phi \leq T \).

Fix a \( \gamma \in \Sigma \). With representation (3.10) one associates \( N \) functions on \( \Phi \) of the form

\[
\tau_{i}^{\gamma, \Phi}(x) := \begin{cases} 
I_i \cap \rho \left( \pi^{-1}(x) \cap H_{T}^T \right) & \text{if } I_i \cap \rho \left( \pi^{-1}(x) \cap H_{T}^T \right) \neq \emptyset \\
0 & \text{otherwise}
\end{cases}
= \begin{cases} 
t & \text{if } (x, t) \in \rho^{-1}(I_i) \cap H_{T}^T \\
0 & \text{otherwise}
\end{cases}.
\]

(3.11)

As is easy to recognize, these functions are just a version of the functions (3.3), the version being relevant to partition \( \Pi^T_{\Sigma} \).

For the family \( \Phi \in \Pi^T_{\Sigma} \), a relevant version of the system \( \tilde{\beta}^T \) (see (2.41)) is constructed as follows:

- For each \( i = 1, \ldots, N \), take a \( t_i \in I_i \) and choose an \( x \in \pi \left( \rho^{-1}(t_i) \right) \subset \Lambda^T_{\Sigma}[x] = \{ x_1, \ldots, x_M \} \), where \( x_m \in \omega_m \). Define a vector \( \alpha^i \in l^2(\Lambda^T_{\Sigma}[x]) \) by

\[
\alpha^i(x_m) := \begin{cases} 
a_\gamma(x_m, t_i) & \text{as } (x_m, t_i) \in H_{T}^T \\
0 & \text{otherwise}
\end{cases}.
\]

- Redesign the system \( \alpha^1, \ldots, \alpha^N \) by the Schmidt process (see (2.41)) and get the system \( \tilde{\beta}^T_{\gamma, \Phi} := \{ \beta^{T,1}_{\gamma, \Phi}, \ldots, \beta^{T,N}_{\gamma, \Phi} \} \). The amplitude subspace is

\[
A^{T}_{\gamma, \Phi}[x] := \text{span} \tilde{\beta}^T_{\gamma, \Phi} \subset l^2(\Lambda^T_{\Sigma}[x])
\]

Also, with each vector \( \beta^{T,i}_{\gamma, \Phi} \) we associate a piece-wise constant function

\[
\beta^{T,i}_{\gamma, \Phi}(x) := (\beta^{T,i}_{\gamma, \Phi})_m, \quad x \in \omega_m \subset \Phi
\]

(3.12)

3.2.4 Projections and eikonals

Recall that \( P^T_\gamma \) projects in \( \mathcal{H} \) onto \( \mathcal{U}^T_\gamma \). Repeating the arguments, which have led to representations (2.38) and (2.43), one can modify them to the following
form relevant to the complete hydra $H^T_{\Sigma}$:

$$ P^T_{\gamma} = \bigoplus_{\Phi \in \Pi^T_{\Sigma}} Q^\gamma_{\Phi}, \quad (3.13) $$

where

$$ (Q^\gamma_{\Phi}y)(x) = \begin{cases} 
\sum_{i=1}^{N(\Phi)} \left< y|_{A^\gamma_{\Phi}[x]}, \beta^{T,i}_{\gamma,\Phi}\right> \beta^{T,i}_{\gamma,\Phi}(x), & x \in \Phi \\
0, & x \in \Omega \setminus \Phi 
\end{cases}. \quad (3.14) $$

Quite analogously, a relevant version of representations (3.6) and (3.8) takes the form

$$ E^T_{\gamma} = \bigoplus_{\Phi \in \Pi^T_{\Sigma}} E^T_{\gamma,\Phi}, \quad (3.15) $$

where

$$ (E^T_{\gamma,\Phi}y)(x) = \begin{cases} 
\sum_{i=1}^{N(\Phi)} \tau^\gamma_{\gamma,\Phi}(x) \left< y|_{A^\gamma_{\Phi}[x]}, \beta^{T,i}_{\gamma,\Phi}\right> \beta^{T,i}_{\gamma,\Phi}(x), & x \in \Phi \\
0, & x \in \Omega \setminus \Phi 
\end{cases}. \quad (3.16) $$

Fix a $\gamma \in \Sigma$ and choose a family $\Phi = \bigcup_{m=1}^{M} \omega_m \in \Pi^T_{\Sigma}$; recall that the number $N$ is defined in (3.10). Parametrize $\Phi$ by (2.33), (2.39): $x_m(r) \in \omega_m$, $m = 1, 2, \ldots, M$, $0 < r < \delta_{\Phi}$. Denote

$$ \bar{y}(r) := \begin{pmatrix} y(x_1(r)) \\ \vdots \\ y(x_M(r)) \end{pmatrix}, \quad B_{\gamma,\Phi} := \begin{pmatrix} (\beta^{T,1}_{\gamma,\Phi})_1 & \cdots & (\beta^{T,1}_{\gamma,\Phi})_M \\ (\beta^{T,2}_{\gamma,\Phi})_1 & \cdots & (\beta^{T,2}_{\gamma,\Phi})_M \\ \vdots & \cdots & \vdots \\ (\beta^{T,N}_{\gamma,\Phi})_1 & \cdots & (\beta^{T,N}_{\gamma,\Phi})_M \end{pmatrix}, $$

$$ D_{\gamma,\Phi}(r) = \{D_{\gamma,\Phi}^{ij}(r)\}_{i,j=1}^{N} : \quad D_{\gamma,\Phi}^{ij}(r) = \tau^\gamma_{\gamma,\Phi}(r) \delta_{ij}, $$

where $\tau^\gamma_{\gamma,\Phi}(r) := \tau^\gamma_{\gamma,\Phi}(x(r))$ is either $T_i + r$ or $T_i + \delta_{\Phi} - r$. Note that $B^*_{\gamma,\Phi}B_{\gamma,\Phi}$ is the matrix of the projection $p_{\gamma,\Phi}$ in $\mathbb{R}^M$ onto $A^T_{\gamma,\Phi} = \text{span} \beta^{T}_{\gamma,\Phi}$. In this notation, the first line in the right hand side of (3.16) is

$$ \overrightarrow{(E^T_{\gamma}y)}(r) = [B^*_{\gamma,\Phi}D_{\gamma,\Phi}(r)B_{\gamma,\Phi}] \bar{y}(r), \quad r \in (0, \delta_{\Phi}) \quad (\gamma \in \Sigma), \quad (3.17) $$

which is just a relevant form of (3.7).

A key feature of representations (3.16) and (3.17) is the following. They represent eikonals $E^T_{\gamma}$ in the form, which is common to all the vertices $\gamma \in \Sigma$ and available for any family $\Phi$ of partition $\Pi^T_{\Sigma}$ of the complete hydra $H^T_{\Sigma}$. 

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3.2.5 Block algebras. Structure of \( E^T_\Sigma \).

In accordance with (3.9), we have the decomposition

\[
\mathcal{H}(\Omega^T[\Sigma]) = \bigoplus_{\Phi \in \Pi^T_\Sigma} \mathcal{H}(\Phi),
\]

which reduces each eikonal \( E^T_\gamma \) for \( \gamma \in \Sigma \):

\[
E^T_\gamma = \bigoplus_{\Phi \in \Pi^T_\Sigma} E^T_\gamma \big|_{\mathcal{H}(\Phi)}
\]

(compare with (2.37) and (3.8)). Each part \( E^T_\gamma \big|_{\mathcal{H}(\Phi)} \) acts in the subspace \( \mathcal{H}(\Phi) \) by (3.16) or, equivalently, by (3.17) in the parametrized form.

Let \( b^T_\Phi \subset \mathcal{B}(\mathcal{H}(\Phi)) \) be the \( \mathcal{C}^* \)-subalgebra generated by the system \( \{ E^T_\gamma \big|_{\mathcal{H}(\Phi)} \mid \gamma \in \Sigma \} \) of the eikonal parts. We say \( b^T_\Phi \) to be a block algebra.

By (3.17), each \( b^T_\Phi \) is isometrically isomorphic to the subalgebra \( \tilde{b}^T_\Phi \subset \mathcal{B}(L_2(\text{[0, } \delta_\Phi]; \mathbb{R}^{M(\Phi)})) \) generated by the operators, which multiply elements (vector-functions) \( \vec{y}(\cdot) \) by the matrix-functions \( B^*_\gamma,\Phi D_{\gamma,\Phi}(\cdot)B_{\gamma,\Phi} \) (\( \gamma \in \Sigma \)). These functions are continuous \(^4\) and hence we have

\[
\tilde{b}^T_\Phi \subset C(\text{[0, } \delta_\Phi]; \mathbb{M}^{M(\Phi)})
\]

where the latter is the algebra of continuous real \( M(\Phi) \times M(\Phi) \)-matrix valued functions on \( [0, \delta_\Phi] \).

Just summarizing these considerations, we arrive at the main result of the paper: decomposition (3.18) reduces eikonal algebra \( E^T_\Sigma \), and the representation

\[
E^T_\Sigma = \bigoplus_{\Phi \in \Pi^T_\Sigma} b^T_\Phi
\]

holds.

3.2.6 Noncommutativity

As we noted in Introduction, algebra \( E^T_\Sigma \) is noncommutative. The reason is that the matrix projections \( p_{\gamma,\Phi} \) (in \( \mathbb{R}^{M(\Phi)} \) onto \( A^T_{\gamma,\Phi} \)) corresponding to
different $\gamma \in \Sigma$ do not have to commute. As a result, eikonal parts $E_T^\gamma |_{H(\Phi)}$ and $E_T^{\gamma'} |_{H(\Phi)}$ do not commute as $\gamma \neq \gamma'$, so that the block-algebra $b_T^\Phi$ turns out to be noncommutative.

Moreover, in a sense, the eikonal algebra on a graph is strongly noncommutative. We mean the following. In the Maxwell dynamical system on a Riemannian manifold, a straightforward analog of $E_T^\Sigma$ is also a noncommutative algebra but its factor over the ideal of compact operators turns out to be commutative [6] (and, moreover, isometric to the algebra $C(\Omega)$). This may be referred to as a weak noncommutativity. On graphs, it is definitely not the case: simple examples show that no factorization eliminates noncommutativity of a generic block-algebra.

### 3.2.7 Spectrum

A spectrum $\hat{\mathcal{E}}_\Sigma^T$ of the algebra $\mathcal{E}_\Sigma^T$ is the set of its primitive ideals endowed with the Jacobson topology (see [10], [13]). By (3.21), one has

$$\hat{\mathcal{E}}_\Sigma^T = \bigcup_{\Phi \in \Pi_T^\Sigma} \hat{b}_\Phi^T, \tag{3.22}$$

so that to study a structure of $\hat{\mathcal{E}}_\Sigma^T$ is to analyze $\hat{b}_\Phi^T$. In the general case of arbitrary graph, such an analysis is an open and difficult problem. Here we discuss some "experimental material" provided by examples on simple graphs.

In the known examples, in accordance with (3.20), one encounters

$$b_\Phi^T = \{ b \in C ([0, \delta]; \mathbb{M}^M) \mid L_0 [b(0)] = 0, L_1 [b(\delta)] = 0 \}, \tag{3.23}$$

where $L_0 [b(0)] = 0$ and $L_1 [b(\delta)] = 0$ are the "linear-type" conditions, which can be (or not be) imposed on elements $b$. For instance, at $r = 0$ there may be

$$1 - \sum_{i=1}^M b_{ij}(0) = 1 - \sum_{j=1}^k b_{ij}(0) = 0. \tag{3.24}$$

As is seen from (3.23), the "massive" part of the spectrum $\hat{b}_\Phi^T$ (and $\hat{\mathcal{E}}_\Sigma^T$: see (3.22)) consists of the ideals of the form

$$\mathcal{I}_{r_0} = \{ b \in b_\Phi^T \mid b(r_0) = 0 \}, \quad r_0 \in (0, \delta).$$

---

5 The discussion is short since we plan to devote a separate paper to these examples.
In the mean time, noncommutativity implies that the (topologized) spectrum \( \hat{b}_\phi^T \) is not necessarily a Hausdorff space: it may contain the clusters. We say a subset \( c \subset \hat{b}_\phi^T \) to be a cluster, if its points are not separable, i.e., for any point \( p \in c \) and arbitrary neighborhood \( U \ni p \) one has \( c \subset U \). In examples, clusters do appear as a consequence of Kirschhoff laws (2.5). By them, the amplitude vectors, which correspond to different families \( \Phi, \Phi' \) are not quite independent. It is the dependence, which implies the conditions like (3.24). In the known examples, the clusters in \( \hat{E}_\Sigma^T \) do appear if \( \Omega[\gamma] \cap \Omega[\gamma'] \ni v \) occurs for the different \( \gamma, \gamma' \in \Sigma \) and an interior vertex \( v \in V \).

### 3.3 Open questions

- By the latter, it would be reasonable to suggest that the number of interior vertices \( n_V \) and the number of clusters \( n_c \) are related through an inequality: presumably, \( n_V \geq n_c \) holds. Since \( n_c \) is a topological invariant of spectrum \( \hat{E}_\Sigma^T \), this relation could be helpful in inverse problems on graphs, in which the inverse data do determine the eikonal algebra \( E_\Sigma^T \) up to an isometric isomorphism (see [3]–[7]). Therefore, the data determine the spectrum \( \hat{E}_\Sigma^T \) up to a homeomorphism, and the external observer, which possesses the data, can hope for getting information about the graph from the spectrum.

- An intriguing question is whether another geometric characteristics of the graph (number of edges and cycles, multiplicity of interior vertices, etc) are related with topological invariants of the spectrum \( \hat{E}_\Sigma^T \). A prospective (but rather far) goal is to recover the graph from the boundary inverse data via its eikonal algebra.

Hopefully, our approach relates inverse problems on graphs with C*-algebras. The answers on the above posed questions could confirm a productivity of these relations.

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