2-Representations and Equivariant 2D Topological Field Theories

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December 24, 2008

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1 Introduction

This paper grew out of the author’s (continued) attempt to understand extended topological field theories, in particular, the so-called Baez-Dolan hypothesis, and possible generalizations of all that to Turaev’s equivariant topological field theories [19].

The idea of extending the classical definition of TFTs to include higher-categorical phenomena goes back to the early 90’s and was mostly motivated by Chern-Simons theory [7]. The subject has been developed significantly since then. We won’t attempt to give any rigorous definitions in this Introduction. Instead, we will explain what the content of this paper has to do with extended TFTs and how the aforementioned Baez-Dolan...
hypothesis motivates our results. The reader, familiar with the state of affairs in this subject, will certainly observe that we present the story in an oversimplified form. We refer the reader to \cite{8} for more thorough treatments.

As its title indicates, this paper deals with 2-dimensional TFTs only. So from now on, a “TFT” will mean a “2-dimensional TFT”. Let us also fix a ground field, $K$, once and for all.

According to M. Atiyah, a TFT is a rule that assigns a finite-dimensional vector space $C$ to an oriented circle $S^1$, the tensor power $C^\otimes n$ to the disjoint union of $n$ circles, and linear maps between the tensor powers to (isomorphism classes of) oriented 2-dimensional cobordisms between the unions of circles. The axioms that this assignment is required to satisfy are most conveniently expressed by saying that it is a monoidal functor from the symmetric monoidal category $\text{Cob}_2$, whose objects are closed oriented 1-manifolds and morphisms are 2-cobordisms, to the category of finite-dimensional vector spaces.

TFTs are well studied objects. It is a classical result that the vector space $C$ a TFT assigns to the circle carries a commutative Frobenius algebra structure, and conversely, any such algebra determines a TFT.

The definition of TFT can be refined as follows \cite{1}: An extended TFT is a rule that assigns small linear categories to oriented 0-manifolds, linear functors to oriented 1-cobordisms of 0-manifolds, and natural transformations to oriented 2-cobordisms with corners. As before, the assignment should give rise to a monoidal 2-functor from the symmetric monoidal 2-category $\text{Cob}^\text{ext}_2$, whose objects are 0-manifolds, 1-morphisms are 1-cobordisms, and 2-morphisms are 2-cobordisms between 1-cobordisms, to the 2-category of small linear categories. Notice that an extended TFT gives rise to a TFT in the usual sense.

The above definition can be spelled out for other “target” 2-categories. We will be interested in the 2-category $\text{Bim}_K$ whose objects are algebras, 1-morphisms are bimodules, and 2-morphisms are morphisms of bimodules. This 2-category is a first approximation to what should be called “the 2-category of non-commutative affine schemes over $K$”. This non-commutative geometric interpretation is quite relevant in this setting as we will see later.

There is, in fact, an analog for extended TFTs of the explicit description of the usual TFTs in terms of commutative Frobenius algebras. Namely, J. Baez and J. Dolan have conjectured \cite{1} that extended TFTs, valued in a 2-category $C$, should be completely determined by the objects they assign to an oriented point; moreover, they have observed that the image of the point should be dualizable in $C$ in a certain strong sense, and vice versa, every dualizable object in $C$ should give rise to an extended TFT.

A result of this sort has been proved recently by Hopkins and Lurie (see e.g. \cite{8}, Section 3). The result implies that in the case $C = \text{Bim}_K$ the extended TFTs are, roughly speaking, in one-to-one correspondence with symmetric Frobenius separable algebras, i.e. separable algebras equipped with a non-degenerate trace.

This result admits the following non-commutative geometric interpretation: essentially, it suggests that extended TFTs valued in the “right” 2-category of non-commutative schemes should be in one-to-one correspondence with non-commutative smooth com-

\footnote{From now on, by a trace we will understand a cyclically invariant functional, so we will omit the word “symmetric”.
}
pact Calabi-Yau spaces. Apparently, this more general statement also follows from the results of Hopkins and Lurie. This is in perfect agreement with the result of K. Costello establishing equivalence between a category of non-commutative Calabi-Yau spaces and a category of open-closed chain-level TFTs (see also [14]).

We believe that the idea of extended TFTs, as well as the language of non-commutative geometry, will prove useful beyond the above setting. For instance, one can try to apply these ideas to the so-called homotopy field theories [3, 4, 5, 16, 17, 19] in which manifolds and cobordisms are decorated by maps to a target space.

One of the first cases to look at is the case when the target is the classifying space of a finite group, $G$ [19]. In this case homotopy field theories (a.k.a. $G$-equivariant TFTs [15]) are symmetric monoidal functors from the category of principal $G$-bundles over manifolds, with morphisms being principal $G$-bundles over cobordisms, to the category of vector spaces. Equivariant TFTs provide an adequate language for describing the orbifolding procedure in the setting of TFTs with symmetries [12, 13, 15].

We will not attempt to define extended equivariant TFTs in this paper. Instead, we would like to emphasize the non-commutative geometric aspect of the sought-for theory. Namely, we believe that the right definition, whatever it is, should lead to the following statement: extended $G$-equivariant TFTs valued in the 2-category of non-commutative schemes are in one-to-one correspondence with non-commutative smooth compact Calabi-Yau spaces acted on by $G$. In particular, extended $G$-equivariant TFTs valued in $\mathbb{C} = \text{Bim}_K$ should be in one-to-one correspondence with Frobenius separable algebras equipped with a categorical $G$-action preserving the Frobenius structure. The meaning of the adjective “categorical” will be explained in the next section.

The aim of this paper is to present a piece of evidence in favor of the above point of view:

**Main result:** We explicitly construct a (non-extended) $G$-equivariant TFT from an arbitrary, not necessarily separable Frobenius algebra with a categorical $G$-action. In such a TFT, the connected components of cobordisms are required to have at least one incoming boundary component (cf. [6, 14]). When the algebra is separable, we get an honest equivariant TFT.

We note that the idea of producing equivariant TFTs from Frobenius algebras equipped with a $G$-action is not new: it arises naturally in the study of open-closed equivariant TFTs [15, Section 7]. The construction offered in the present paper is, in a sense, dual to the one presented in [15] (the meaning of “dual” will be explained in Section 6). Another difference is that we work in the more general setting of non-separable Frobenius algebras and twisted (=categorical) $G$-actions on them.

For the reader’s convenience, we introduce all the necessary definitions and formulate the results in a concise way in the first part of the paper. The proofs are collected in appendices in the remaining part. Throughout this paper, $G$ and $K$ stand for a finite group and a field, respectively.

**Acknowledgements.** I am grateful to K. Costello, D. Freed, and Y. Soibelman for various useful remarks on the results and the exposition. I would also like to thank the authors of [9, 18, 20] and especially [15] for inspiration.
2 Frobenius algebras with twisted $G$-action

Let us begin by recalling the notion of categorical representation of a group \([9]\).

A categorical representation of a group \(G\) is a category \(C\) together with the following data:

1. for each element \(g \in G\), an autoequivalence \(\rho(g)\) of \(C\);
2. for any pair of elements \(g, h \in G\), an isomorphism of functors
   \[c(g,h) : \rho(g) \circ \rho(h) \rightarrow \rho(gh);\]
3. an isomorphism of functors
   \[c(e) : \rho(e) \rightarrow \Id_C,\]
   where \(e\) is the unit element of \(G\).

The above autoequivalences and isomorphisms are required to satisfy the following conditions:

1. for any triple of elements \(g, h, k \in G\), the diagram
   \[
   \begin{array}{ccc}
   \rho(g) \circ \rho(h) \circ \rho(k) & \overset{\rho(g \circ (h,k))}{\longrightarrow} & \rho(g) \circ \rho(hk) \\
   c(g,h,k) \circ \rho(k) & \downarrow & c(g,h,k) \\
   \rho(gh) \circ \rho(k) & \overset{c(gh,k)}{\longrightarrow} & \rho(ghk)
   \end{array}
   \]
   commutes;

2. for any element \(g \in G\), the diagrams
   \[
   \begin{array}{ccc}
   \rho(g) \circ \rho(e) & \overset{\rho(g) \circ c(e)}{\longrightarrow} & \rho(g) \circ \Id_C \\
   c(e,g) & \downarrow & c(e,g) \\
   \rho(g) & \longrightarrow & \rho(g)
   \end{array}
   \quad \begin{array}{ccc}
   \rho(e) \circ \rho(g) & \overset{c(e) \circ \rho(g)}{\longrightarrow} & \Id_C \circ \rho(g) \\
   c(e,g) & \downarrow & c(e,g) \\
   \rho(g) & \longrightarrow & \rho(g)
   \end{array}
   \]
   commute.

From now on, we will only consider the case when the category \(C\) is \(K\)-linear and has one object. We will identify such a category with its endomorphism algebra. In this case, \(\rho(g)\) are automorphisms of \(C\), whereas the isomorphisms \(c(g,h)\) and \(c(e)\) are given by conjugation with some invertible elements of \(C\), which we denote by \(c_{g,h}\) and \(c_e\). Namely, the latter are defined by

\[
\rho(g) \cdot \rho(h) = \Ad(c_{g,h}) \cdot \rho(gh), \quad \rho(e) = \Ad(c_e),
\]
where $Ad(c)$ stands for the inner automorphism $c_1 \mapsto c \cdot c_1 \cdot c^{-1}$ and the equalities are understood as equalities of automorphisms of $C$. The above commutative diagrams boil down to the following equalities

$$c_{g,h}c_{gh,k} = g(c_{h,k})c_{g,h}, \quad c_{g,e} = g(c_e), \quad c_{c,g} = c_c,$$

where we write $g(c)$ instead of $\rho(g)(c)$.

**Definition 2.1** A twisted algebra bundle on the classifying stack $BG$ is an algebra $C$ equipped with a categorical $G$-action as above.

Later on, we will need one more definition. Recall that a Frobenius algebra is a finite-dimensional (not necessarily commutative) unital algebra $C$ equipped with a trace $\theta : C \to K$ such that the pairing $(c_1, c_2) \mapsto \theta(c_1c_2)$ is non-degenerate.

**Definition 2.2** A twisted Frobenius algebra bundle on the classifying stack $BG$ is a Frobenius algebra $(C, \theta)$ equipped with a categorical $G$-action satisfying $\theta(g(c)) = \theta(c)$ for any $g \in G$ and $c \in C$.

### 3 Equivariant topological field theories

Turaev has shown [19] that the data of a $G$-equivariant TFT is equivalent to that of a crossed $G$-algebra, a $G$-equivariant analog of a commutative Frobenius algebra. In this section, we recall Turaev’s definition of crossed $G$-algebra. It will be convenient for us to first introduce some auxiliary notions.

**Definition 3.1** A vector bundle on the loop space $LBG$ is a $G$-graded finite-dimensional vector space $C = \bigoplus_g C_g$ together with a group homomorphism $G \to GL(C)$ such that $g(C_h) = C_{gh^{-1}}$ for all $g, h \in G$.

**Definition 3.2** A special vector bundle on $LBG$ is a vector bundle such that $g|_{C_g} = id$.

Now we are in position to define crossed $G$-algebras.

**Definition 3.3** A crossed $G$-algebra (or Turaev algebra in the terminology of [15]) is a special vector bundle $C = \bigoplus_g C_g$ on $LBG$ together with a $G$-invariant functional $\theta_e : C_e \to K$ and an algebra structure satisfying the following properties:

1. $G$ acts by automorphisms of the algebra $C$;
2. $C_g \cdot C_h \subset C_{gh}$;

---

2 Apparently, such objects should be called “$S^1$-equivariant bundles on $LBG$”. We decided to stick to the neutral term “special” to avoid the necessity of justifying our terminology.
(3) for all \( c' \in C_g \) and \( c'' \in C_h \),
\[
c' c'' = g(c'') c';
\]
(4) (Torus axiom) for all \( g, h \in G \) and \( c \in C_{hgh^{-1}g^{-1}} \),
\[
\text{Tr}_{C_h}(L'_c \cdot g) = \text{Tr}_{C_g}(h^{-1} \cdot L''_c),
\]
where \( L'_c : C_{ghg^{-1}} \to C_h \) and \( L''_c : C_g \to C_{hgh^{-1}} \) stand for the operators of left multiplication with \( c \).

(5*) \( C \) is unital;

(6*) \( \theta_e \) induces a non-degenerate pairing \( C_g \otimes C_g^{-1} \to K \).

We will also need a weaker notion which we call a weak crossed \( G \)-algebra. Weak crossed \( G \)-algebras correspond to equivariant TFTs in which the connected components of cobordisms are required to have at least one incoming boundary component.

**Definition 3.4** A weak crossed \( G \)-algebra is an object \( C \) satisfying all the requirements listed in Definition 3.3 except for (5*) and (6*); instead, \( C \) possesses the following extra structure: there is a coassociative coalgebra structure \( \Delta : C \to C \otimes C \) satisfying the following properties

(5) \( G \) acts by automorphisms of the coalgebra \( C \);

(6) \( \Delta \) respects the \( G \)-grading, i.e. \( \Delta(C_k) \subset \oplus_{gh=h} C_g \otimes C_h \) (we will denote the corresponding map \( C_{gh} \to C_g \otimes C_h \) by \( \Delta_{g,h} \));

(7) for all \( g, h \in G \), \( \Delta_{g,h} = \sigma(1 \otimes h) \Delta_{h^{-1}g,h^{-1}} \) (here \( \sigma \) is the transposition map);

(8) for any \( g \in G \), \( (\theta_e \otimes 1) \Delta_{g,e} = (1 \otimes \theta_e) \Delta_{g,e} = \text{id}_{C_g} \);

(9) \( \Delta \) is a morphism of \( C \)-bimodules.

Note that any crossed \( G \)-algebra possesses a weak crossed \( G \)-algebra structure: the maps \( \Delta_{g,h} \) come from the morphisms in the corresponding equivariant TFT defined by principal \( G \)-bundles on the genus 0 surface with one incoming and two outgoing boundaries.

Also observe that a weak crossed \( G \)-algebra is an honest crossed \( G \)-algebra iff it is unital. Indeed, if \( 1_C \) is the unit then for any \( c \in C_g \) we have
\[
(1 \otimes \theta_e)(1 \otimes m_{g^{-1}g})(\Delta_{g,g^{-1}}(1_C) \otimes c) \overset{(9)}{=} (1 \otimes \theta_e)(\Delta_{g,1}(c)) \overset{(8)}{=} c
\]
which implies triviality of the kernel of the pairing, defined by \( \theta_e \).
4 A $G$-equivariant version of the 0-th Hochschild homology

In this section, we will introduce a $G$-equivariant version of the 0-th Hochschild homology of an algebra.

Consider a twisted algebra bundle, i.e. an algebra $\mathcal{C}$ equipped with a categorical $G$-action (we will keep the notations from Section 2). For an element $h \in G$ set

$$\mathcal{C}_h = \text{span}\{c_1c_2 - c_2h(c_1) \mid c_1, c_2 \in \mathcal{C}\} \subset \mathcal{C}$$

and define

$$\mathcal{H}_0(C) = \oplus_g \mathcal{H}_0(C)_g, \quad \mathcal{H}_0(C)_g := \mathcal{C}/\mathcal{C}_g.$$ 

The space $\mathcal{H}_0(C)$ inherits a $G$-action from $\mathcal{C}$. In order to describe it, we need some auxiliary definitions and results.

For a pair of elements $g, h \in G$, define a linear map $T_h(g) : \mathcal{C} \to \mathcal{C}$ by the formula

$$T_h(g) : c \mapsto c^{-1}g^{-1}c^1g(c)g(h)c_{ghg^{-1}}.$$ 

**Proposition 4.1** For any $g, h \in G$, $T_h(g)(\mathcal{C}_h) = \mathcal{C}_{ghg^{-1}}$.

**Proof** can be found in Appendix A.

By the above proposition, we have linear maps

$$T_h(g) : \mathcal{H}_0(\mathcal{C})_h \to \mathcal{H}_0(\mathcal{C})_{ghg^{-1}}$$

induced by the maps $T_h(g)$. Define $T(g) : \mathcal{H}_0(\mathcal{C}) \to \mathcal{H}_0(\mathcal{C})$ by

$$T(g)|_{\mathcal{H}_0(\mathcal{C})_h} = T_h(g).$$ 

**Proposition 4.2** The operators $T(g)$ form a representation of $G$ in $\mathcal{H}_0(\mathcal{C})$. Moreover,

$$T(g)|_{\mathcal{H}_0(\mathcal{C})_g} = \text{id}.$$ 

**Proof** can be found in Appendix B.

Now we are ready to formulate

**Definition 4.3** The special vector bundle $\mathcal{H}_0(C)$ on $\mathcal{L}BG$ will be called the 0-th Hochschild homology bundle of $\mathcal{C}$.
5 An equivariant TFT structure on the 0-th Hochschild homology bundle

In this section, we equip the 0-th Hochschild homology bundle of a twisted Frobenius algebra bundle with a weak crossed $G$-algebra structure.

Let us fix a twisted Frobenius algebra bundle, i.e. a Frobenius algebra $C = (C, \theta)$ equipped with a categorical $G$-action such that $\theta$ is $G$-invariant (see Section 2). Let $\xi = \sum \xi'_i \otimes \xi''_i \in C \otimes C$ stand for the symmetric tensor inverse to the pairing defined by $\theta$:

$$c = \sum \xi'_i \theta(c \xi''_i), \quad \forall c \in C.$$

(In what follows, we omit the summation and write $\xi'_i \otimes \xi''_i$.)

For any pair of elements $g, h \in G$ define linear maps $m_{g,h} : C \otimes C \to C$ by

$$m_{g,h}(c' \otimes c'') = \xi'_i c' g(\xi''_i c'') c_{g,h}. \quad (5.1)$$

**Proposition 5.1** The maps $m_{g,h}$ descend to well-defined linear maps

$$m_{g,h} : \mathcal{H} \mathcal{H}_0(C)_g \otimes \mathcal{H} \mathcal{H}_0(C)_h \to \mathcal{H} \mathcal{H}_0(C)_{gh}.$$

Altogether, they define an associative algebra structure on $\mathcal{H} \mathcal{H}_0(C)$.

**Proof** can be found in Appendix C.

Now for any pair of elements $g, h \in G$ define linear maps $\Delta_{g,h} : C \to C \otimes C$ by

$$\Delta_{g,h}(c) = c g_{g,h}^{-1}(\xi'_i) \otimes \xi''_i. \quad (5.2)$$

**Proposition 5.2** The maps $\Delta_{g,h}$ descend to well-defined linear maps

$$\Delta_{g,h} : \mathcal{H} \mathcal{H}_0(C)_{gh} \to \mathcal{H} \mathcal{H}_0(C)_g \otimes \mathcal{H} \mathcal{H}_0(C)_h.$$

Altogether, they define a coassociative coalgebra structure on $\mathcal{H} \mathcal{H}_0(C)$.

**Proof** can be found in Appendix C.

Let us formulate the main result of this paper:

**Theorem 5.3** For any twisted Frobenius algebra bundle $C = (C, \theta)$ on $BG$, the 0-th Hochschild homology bundle $\mathcal{H} \mathcal{H}_0(C)$, equipped with the $G$-action (4.1), the multiplication (5.1), the comultiplication (5.2), and the functional

$$\theta_e : \mathcal{H} \mathcal{H}_0(C)_e \to K, \quad \theta_e(c) = \theta(cc_e) \quad (5.3)$$

is a weak crossed $G$-algebra.
Proof can be found in Appendix [10].

Using the observation, mentioned at the end of Section 8, one can show that \( \mathcal{H} \mathcal{H}_0(\mathcal{C}) \) is an honest crossed \( G \)-algebra iff the central element \( \xi'_i \xi''_i \) is invertible. Indeed, in this case \( (\xi'_i \xi''_i c_e)^{-1} \) is the unit of \( \mathcal{H} \mathcal{H}_0(\mathcal{C}) \), as one can see from the following computation:

\[
m_{g,e}((\xi'_i \xi''_i c_e)^{-1} \otimes c) = \xi'_i ((\xi'_i \xi''_i c_e)^{-1}) c_{e,g} = \xi'_i(\xi'_i \xi''_i)^{-1} \xi'_i c = c,
\]

\[
m_{g,e}(c \otimes (\xi'_i \xi''_i c_e)^{-1}) = \xi'_i cg((\xi'_i \xi''_i c_e)^{-1}) c_{g,e} = \xi'_i cg(\xi'_i \xi''_i)^{-1} g(c_e)^{-1} c_{g,e}
\]

\[
\xi'_i cg((\xi'_i \xi''_i c_e)^{-1}) \quad \text{modulo } c_g.
\]

6 Concluding remarks

We already mentioned in the Introduction that the content of this paper is closely related to that of Section 7 of [15]. In the first part of this section we will expand a little bit on how our result compares with those obtained in [15]. The second part is devoted to a simple observation relating equivariant TFTs to generalized group characters [10][11][9].

In Section 7.3 of [15], the authors describe a construction, which goes back to [19], that associates a \( G \)-equivariant TFT with a finite \( G \)-space, \( X \), equipped with a \( G \)-invariant “volume form” and a \( B \)-field. The latter is an element of the second cohomology group \( H^2(G, A(X)^\times) \) of \( G \) with values in the abelian group \( A(X)^\times \) of invertible functions on \( X \). Such data – a \( G \)-space with an invariant trace and a \( B \)-field – give rise to a Frobenius algebra bundle on \( BG \) in the sense of the present paper. Notice that the “fiber” of this bundle is a commutative semisimple algebra. Reversing the logic, one can say that our result is about non-commutative \( G \)-spaces.

Actually, a special case of non-commutative \( G \)-spaces is mentioned already in [15]. Namely, in Section 7.4, where the authors introduce and study open-closed equivariant TFTs, they describe an explicit construction of equivariant TFTs from Frobenius semisimple categories with \( G \)-action (see Theorem 10 in loc. cit.). However, by a category with \( G \)-action the authors understand a category enriched in the category of representations of \( G \). It means, in particular, that one has an honest, non-twisted action of \( G \) on the morphism spaces. On a final note, our construction of equivariant TFTs differs from the one described in [15] even in the cases when both constructions are applicable. In our approach, the underlying space of the equivariant TFT is a \( G \)-equivariant version of the 0-th Hochschild homology of an algebra whereas in [15] it is a \( G \)-equivariant version of the center (= the 0-th Hochschild cohomology). The reason we have decided to work with the equivariant Hochschild homology is that the latter, we believe, is more suitable for the purpose of constructing equivariant extended TFTs.

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3This is so when \( \mathcal{C} \) is separable.
4The elements \( c_{g,h} \) should then be viewed as defining a \( B \)-field which belongs to some “non-abelian cohomology group” \( H^2(G, C^\times) \).
5Observe that this is precisely the categorical trace in the sense of [2][2].
We would like to conclude this section with the following curious (although straightforward) observation. In their recent work [9], N. Ganter and M. Kapranov have emphasized the importance of categorical representations of groups in the study of generalized equivariant cohomology theories and related subjects. In particular, they have noticed that the so-called 2-class functions on finite groups studied in [10, 11] arise naturally as the 2-characters of 2-representations of the groups. Namely, one defines the categorical character of an arbitrary 2-representation which is a vector bundle $T_r = \oplus_g T_r g$ on $L BG$ in the sense of Definition 3.1; then the 2-character is the “character” of the categorical character:

$$\chi(g,h) = \text{Tr}(g : T_r h \rightarrow T_r h)$$

where $gh = hg$. Clearly, $\chi(kgk^{-1},khk^{-1}) = \chi(g,h)$ for any $k \in G$ which is what being a 2-class function means.

The point we would like to stress is that by the above mentioned results of [15] the categorical traces of 2-representations of a finite group $G$ in 2-vector spaces can be upgraded to $G$-equivariant TFTs. This implies that the 2-characters are modular invariant in the following sense: for any $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z})$ one has

$$\chi(g^a h^b, g^c h^d) = \chi(g,h)$$

for any pair of commuting elements $g, h \in G$. Indeed, it is enough to show that $\chi$ is preserved by the generators $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ of $SL(2,\mathbb{Z})$, i.e.

$$\chi(gh,h) = \chi(g,h), \quad \chi(h^{-1},g) = \chi(g,h).$$

The first equality follows from the fact that $T_r$ is a special vector bundle, i.e. $h|_{T_r g} = id$; the second equality is a consequence of the torus axiom.
\[\text{A Proof of Proposition 4.1}\]

Let us compute \(T_h(g)(c_1c_2 - c_2h(c_1))\):
\[
T_h(g)(c_1c_2 - c_2h(c_1)) = c_e^{-1}_{g} (g(c_1)g(c_2) - g(c_2)g(h(c_1))) c_{g,h} c_{gh,g^{-1}}^{-1}
\]
\[
= c_e^{-1}_{g} (g(c_1)g(c_2) - g(c_2)Ad(c_{g,h})(g(h(c_1)))) c_{g,h} c_{gh,g^{-1}}^{-1}
\]
\[
= c_e^{-1}_{g} (g(c_1)g(c_2) - g(c_2)Ad(c_{g,h})(c_{g,h}^{-1})(c_1)) c_{g,h} c_{gh,g^{-1}}^{-1}
\]
\[
= c_e^{-1}_{g} (c''Ad(c_{g,h})(c_{g,h}^{-1})(c_1)) c_{g,h} c_{gh,g^{-1}}^{-1}
\]
\[
= c_e^{-1}_{g} c''(g(c_1)g(c_2) - g(c_2)Ad(c_{g,h})(c_{g,h}^{-1})(c_1)) c_{g,h} c_{gh,g^{-1}}^{-1}
\]
\[
= c_e^{-1}_{g} c''(g(c_1)g(c_2) - g(c_2)c_{g,h}c_{gh,g^{-1}c_{gh,g^{-1}}^{-1}}(c_1)) c_{g,h} c_{gh,g^{-1}}^{-1}
\]
where \(c' = g(c_1)\) and \(c'' = g(c_2)\). Observe that, modulo \(C_{gh,g^{-1}}\), the first summand in (A.1) is equal to
\[
c''c_{g,h}c_{gh,g^{-1}}(c_1) c_{g,h} c_{gh,g^{-1}}^{-1}(c_1)
\]
which, in its turn, is equal to
\[
c''c_{g,h}c_{gh,g^{-1}}^{-1}(c_1)
\]
due to (2.1). On the other hand, modulo \(C_{gh,g^{-1}}\), the second summand in (A.1) is equal to
\[
c''c_{g,h}c_{gh,g^{-1}}^{-1}(c_1) c_{g,h} c_{gh,g^{-1}}^{-1}(c_1)
\]
and, by (2.1), this is the same thing as
\[
c''c_{g,h}c_{gh,g^{-1}}^{-1}(c_1)
\]
Thus, modulo \(C_{gh,g^{-1}}\), the expression (A.1) is equal to
\[
c'(c''c_{g,h}c_{gh,g^{-1}}^{-1}(c_1) - c''c_{g,h}c_{gh,g^{-1}}^{-1}(c_1)) \equiv 0.
\]
The proposition is proved.

\[\text{B Proof of Proposition 4.2}\]

Let us apply \(\mathcal{T}(g_1)\mathcal{T}(g_2)\) to an element \(c \in \mathcal{HH}_0(C)_h\):
\[
\mathcal{T}(g_1)\mathcal{T}(g_2)(c) = \mathcal{T}(g_1)(c) c_{g_1,h} c_{g_2,h} c_{g_1g_2,h}^{-1}
\]
\[
= c_{g_1,h}^{-1}_{g_2,hs_2^{-1}} g_1(c_{g_1,h}^{-1}_{g_2,hs_2^{-1}} g_2(c)) c_{g_1g_2,h} c_{g_1g_2,h}^{-1}
\]
\[
= c_{g_1,h}^{-1}_{g_2,hs_2^{-1}} g_1(c_{g_1,h}^{-1}_{g_2,hs_2^{-1}} g_2(c)) c_{g_1g_2,h} c_{g_1g_2,h}^{-1}
\]
\[
= c_{g_1,h}^{-1}_{g_2,hs_2^{-1}} g_1(c_{g_1,h}^{-1}_{g_2,hs_2^{-1}} g_2(c)) c_{g_1g_2,h} c_{g_1g_2,h}^{-1}
\]
\[
= c_{g_1,h}^{-1}_{g_2,hs_2^{-1}} g_1(c_{g_1,h}^{-1}_{g_2,hs_2^{-1}} g_2(c)) c_{g_1g_2,h} c_{g_1g_2,h}^{-1}
\]
On the other hand,
\[
\mathcal{T}(g_1 g_2)(c) = c_{g_1,h}^{-1}_{g_2,hs_2^{-1}} g_1 g_2(c) c_{g_1g_2,h} c_{g_1g_2,h}^{-1}
\]
We have to show that \((B.1)\) coincides with \((B.2)\) modulo \(C_{g_1g_2h_{g_2}^{-1}g_1^{-1}}\). We will use the relation \(c_1c_2 = c_2g_1g_2h_{g_2}^{-1}g_1^{-1}(c_1)\) to “move” everything that is on the left of \(g_1g_2(c)\) in \((B.1)\) and \((B.2)\) to the right side of the expressions.

Thus, modulo \(C_{g_1g_2h_{g_2}^{-1}g_1^{-1}}\), the expressions \((B.1)\) and \((B.2)\) are equal to

\[
g_1g_2(c)g_{s_1s_2}^{-1}c_{g_2h_{g_2}^{-1}}g_1(c_{g_2h_{g_2}^{-1}})c_{g_1g_2h_{g_2}^{-1}g_1^{-1}}g_1g_2h_{g_2}^{-1}g_1^{-1}(c_{g_1g_2h_{g_2}^{-1}g_1^{-1}})g_1(c_{g_1g_2h_{g_2}^{-1}g_1^{-1}})c_{g_1g_2h_{g_2}^{-1}g_1^{-1}}
\]

and

\[
g_1g_2(c)g_{s_1s_2}^{-1}c_{g_1g_2h_{g_2}^{-1}g_1^{-1}}g_1g_2h_{g_2}^{-1}g_1^{-1}(c_{g_1g_2h_{g_2}^{-1}g_1^{-1}})g_1(c_{g_1g_2h_{g_2}^{-1}g_1^{-1}})c_{g_1g_2h_{g_2}^{-1}g_1^{-1}}
\]

respectively. Therefore, it is enough to show that

\[
c_{s_1s_2}^{-1}g_1(c_{g_2h_{g_2}^{-1}})g_1(c_{g_2h_{g_2}^{-1}})c_{g_1g_2h_{g_2}^{-1}g_1^{-1}}g_1g_2h_{g_2}^{-1}g_1^{-1}(c_{g_1g_2h_{g_2}^{-1}g_1^{-1}})g_1(c_{g_1g_2h_{g_2}^{-1}g_1^{-1}})c_{g_1g_2h_{g_2}^{-1}g_1^{-1}}
\]

\[
= c_{s_1s_2}^{-1}c_{g_1g_2h_{g_2}^{-1}g_1^{-1}}g_1g_2h_{g_2}^{-1}g_1^{-1}(c_{g_1g_2h_{g_2}^{-1}g_1^{-1}})g_1(c_{g_1g_2h_{g_2}^{-1}g_1^{-1}})c_{g_1g_2h_{g_2}^{-1}g_1^{-1}}
\]

Observe that by \((2.1)\), the left hand side of the latter equality can be simplified as follows (we will underline the places in the formulas that are about to be changed):

\[
c_{s_1s_2}^{-1}g_1(c_{g_2h_{g_2}^{-1}})g_1(c_{g_2h_{g_2}^{-1}})c_{g_1g_2h_{g_2}^{-1}g_1^{-1}}g_1g_2h_{g_2}^{-1}g_1^{-1}(c_{g_1g_2h_{g_2}^{-1}g_1^{-1}})g_1(c_{g_1g_2h_{g_2}^{-1}g_1^{-1}})c_{g_1g_2h_{g_2}^{-1}g_1^{-1}}
\]

\[
= c_{s_1s_2}^{-1}g_1(c_{g_2h_{g_2}^{-1}})g_1(c_{g_2h_{g_2}^{-1}})c_{g_1g_2h_{g_2}^{-1}g_1^{-1}}g_1g_2h_{g_2}^{-1}g_1^{-1}(c_{g_1g_2h_{g_2}^{-1}g_1^{-1}})g_1(c_{g_1g_2h_{g_2}^{-1}g_1^{-1}})c_{g_1g_2h_{g_2}^{-1}g_1^{-1}}
\]

so now we have to prove that

\[
c_{s_1s_2}^{-1}g_1(c_{g_2h_{g_2}^{-1}})g_1(c_{g_2h_{g_2}^{-1}})c_{g_1g_2h_{g_2}^{-1}g_1^{-1}}g_1g_2h_{g_2}^{-1}g_1^{-1}(c_{g_1g_2h_{g_2}^{-1}g_1^{-1}})g_1(c_{g_1g_2h_{g_2}^{-1}g_1^{-1}})c_{g_1g_2h_{g_2}^{-1}g_1^{-1}}
\]

\[
= c_{s_1s_2}^{-1}c_{g_1g_2h_{g_2}^{-1}g_1^{-1}}g_1g_2h_{g_2}^{-1}g_1^{-1}(c_{g_1g_2h_{g_2}^{-1}g_1^{-1}})g_1(c_{g_1g_2h_{g_2}^{-1}g_1^{-1}})c_{g_1g_2h_{g_2}^{-1}g_1^{-1}}
\]

By \((2.1)\), the right hand side of the latter equality can be simplified as follows

\[
c_{s_1s_2}^{-1}g_1(c_{g_2h_{g_2}^{-1}})g_1(c_{g_2h_{g_2}^{-1}})c_{g_1g_2h_{g_2}^{-1}g_1^{-1}}g_1g_2h_{g_2}^{-1}g_1^{-1}(c_{g_1g_2h_{g_2}^{-1}g_1^{-1}})g_1(c_{g_1g_2h_{g_2}^{-1}g_1^{-1}})c_{g_1g_2h_{g_2}^{-1}g_1^{-1}}
\]

\[
= c_{s_1s_2}^{-1}c_{g_1g_2h_{g_2}^{-1}g_1^{-1}}g_1g_2h_{g_2}^{-1}g_1^{-1}(c_{g_1g_2h_{g_2}^{-1}g_1^{-1}})g_1(c_{g_1g_2h_{g_2}^{-1}g_1^{-1}})c_{g_1g_2h_{g_2}^{-1}g_1^{-1}}
\]

and now we will be proving that

\[
c_{s_1s_2}^{-1}g_1(c_{g_2h_{g_2}^{-1}})g_1(c_{g_2h_{g_2}^{-1}})c_{g_1g_2h_{g_2}^{-1}g_1^{-1}}g_1g_2h_{g_2}^{-1}g_1^{-1}(c_{g_1g_2h_{g_2}^{-1}g_1^{-1}})g_1(c_{g_1g_2h_{g_2}^{-1}g_1^{-1}})c_{g_1g_2h_{g_2}^{-1}g_1^{-1}}
\]

\[
= c_{s_1s_2}^{-1}c_{g_1g_2h_{g_2}^{-1}g_1^{-1}}g_1g_2h_{g_2}^{-1}g_1^{-1}(c_{g_1g_2h_{g_2}^{-1}g_1^{-1}})g_1(c_{g_1g_2h_{g_2}^{-1}g_1^{-1}})c_{g_1g_2h_{g_2}^{-1}g_1^{-1}}
\]

Let us simplify the left-hand side further:

\[
c_{s_1s_2}^{-1}g_1(c_{g_2h_{g_2}^{-1}})g_1(c_{g_2h_{g_2}^{-1}})c_{g_1g_2h_{g_2}^{-1}g_1^{-1}}g_1g_2h_{g_2}^{-1}g_1^{-1}(c_{g_1g_2h_{g_2}^{-1}g_1^{-1}})g_1(c_{g_1g_2h_{g_2}^{-1}g_1^{-1}})c_{g_1g_2h_{g_2}^{-1}g_1^{-1}}
\]
Proof of Proposition 5.1. 

modulo \(c\) follows immediately from its definition: for any \(c\) follows immediately from its definition: for any 

so (B.3) is equivalent to 

\[
\begin{align*}
\mathcal{C}_1 \text{ or } \mathcal{C}_1 
\end{align*}
\]

Consider the left-hand side of the latter equality: 

Thus, (B.4) is equivalent 

which is nothing but (2.1) written in a different way. The first part of the proposition is proved.

To prove the second part, we need to show that for \(c \in \mathcal{H}_0(C)_g\)

modulo \(C_g\). One has

\[
\begin{align*}
\mathcal{C}_1 \text{ of } \mathcal{C}_1 
\end{align*}
\]

The proposition is proved completely.

C Proof of Propositions 5.1 and 5.2

Proof of Propositions 5.1 First of all, let us point out the following property of \(\xi\), which follows immediately from its definition: for any \(c \in C\), we have

To prove that \(m_{gh}\) descends to a well-defined map from \(\mathcal{H}_0(C)_g \otimes \mathcal{H}_0(C)_h\) to \(\mathcal{H}_0(C)_{gh}\) we need to show that \(m_{gh}(C_g \otimes C_h) \subset C_{gh}\) and \(m_{gh}(C_\otimes C_h) \subset C_{gh}\).

To prove the first inclusion, let us substitute \(c_1c_2 - c_2g(c_1)\) for \(c\) in the expression \(\xi_{1}^\prime \xi_{1}'' (\xi_1' c') \xi_{1}'' \) for \(c\) in the expression 

\[
\begin{align*}
\mathcal{C}_1 \text{ of } \mathcal{C}_1 
\end{align*}
\]
\[ = \xi' c_2 g(\xi'') c_{g,h} - \xi' c_2 g(c_1 \xi'') c_{g,h} \quad \text{by (C.1)} \]

Now let us substitute \( c_1 c_2 - c_2 h(c_1) \) for \( c'' \) in \( \xi' c' g(\xi'') c_{g,h} \) and show that the result belongs to \( C_{gh} \):

\[
\xi' c' g(\xi''(c_1 c_2 - c_2 h(c_1))) c_{g,h} = \xi' c' g(\xi'' c_{g,h}) c_{g,h} - \xi' c_2 g(\xi''(c_1 c_2)) c_{g,h} = 0.
\]

The latter equality follows for (C.1), applied to \( \xi \) and \( \xi'' \).

Now let us substitute \( c_1 c_2 - c_2 h(c_1) \) for \( c'' \) in \( \xi' c' g(\xi'') c_{g,h} \) and show that the result belongs to \( C_{gh} \):

\[
\xi' c' g(\xi''(c_1 c_2 - c_2 h(c_1))) c_{g,h} = \xi' c' g(\xi''(h(c_1))) c_{g,h} = c_1 c_2 \xi' c' g(\xi'') c_{g,h} - c_2 \xi' c' g(\xi'(h(c_1))) c_{g,h}
\]

The first part of Proposition 5.1 is proved.

Now let us prove associativity of the maps \( m_{g,h} \). We need to show that for any \( g, h, k \in G \) and \( c', c'', c''' \in C \)

\[
\xi' c' g(\xi'' c') c_{g,h} = \xi' c' g(h(\xi'' c')) c_{g,h} = \xi' c' g(\xi'' c') c_{g,h} c_{g,h} k
\]

modulo \( C_{gh} \). In fact, we will show that the equality holds in \( C \). Indeed, the left-hand side equals

\[
\xi' c' g(\xi'' c') c_{g,h} c_{g,h} k = \xi' c' g(\xi'' c') c_{g,h} c_{g,h} k
\]

and the right-hand side equals

\[
\xi' c' g(\xi'' c') c_{g,h} c_{g,h} k = \xi' c' g(\xi'' c') c_{g,h} c_{g,h} k
\]

so by (2.1) we need to prove that

\[
\xi' c' g(\xi'' c') c_{g,h} c_{g,h} k = \xi' c' g(\xi'' c') c_{g,h} c_{g,h} k
\]

or, equivalently,

\[
\xi' c' g(\xi'' c') c_{g,h} c_{g,h} k = \xi' c' g(\xi'' c') c_{g,h} c_{g,h} k
\]

The latter equality follows for (C.1), applied to \( \xi' \) in the right-hand side. Proposition 5.1 is proved completely.

**Proof of Propositions 5.2** To prove that \( \Delta_{g,h} \) descends to a map from \( \mathcal{H} \mathcal{H}_0(C)_{gh} \) to \( \mathcal{H} \mathcal{H}_0(C)_{g} \otimes \mathcal{H} \mathcal{H}_0(C)_{h} \), we need to show that \( \Delta_{g,h}(g_{gh}) \subset C_{g} \otimes C + C \otimes C_{h} \). Let us substitute \( c_1 c_2 - c_2 h(c_1) \) for \( c \in C_{gh} \) and \( \xi_i' \) for \( \xi' 

\[
(c_1 c_2 - c_2 h(c_1)) c^{-1} g(\xi_i') \otimes \xi'' = c_1 c_2 c^{-1} g(\xi_i') \otimes \xi'' - c_2 h(c_1) c^{-1} g(\xi_i') \otimes \xi''
\]

\[
= c_1 c_2 c^{-1} g(\xi_i') \otimes \xi'' - c_2 c^{-1} g(\xi_i') c^{-1} g(\xi_i') \otimes \xi''
\]

\[
= c_1 c_2 c^{-1} g(\xi_i') \otimes \xi'' - c_2 c^{-1} g(h(c_1)) \xi_i' \otimes \xi''
\]

\[
= c_1 c_2 c^{-1} g(h(c_1)) \xi_i' \otimes \xi'' - c_2 c^{-1} g(h(c_1)) \xi_i' \otimes \xi''
\]

Modulo \( C_{g} \otimes C \), the latter expression equals

\[
\xi' c_1 c_2 c^{-1} g(\xi_i') \otimes \xi'' - h(c_1) \xi' c_2 c^{-1} g(\xi_i') \otimes \xi''
\]
Therefore, it is enough to show that
\[ G \]
Proof of the multiplication and the comultiplication. As before, to make our computations
The first part of the proposition is proved.
To prove coassociativity, observe that
\[ (\Delta_{g,h} \otimes 1) \Delta_{gh,k}(c) = c^{-1} g_{gh,k} g h(\xi'_i) c^{-1} g_{gh,k} g h(\xi'_i) \otimes \xi''_i, \]
\[ (1 \otimes \Delta_{h,k}) \Delta_{g,hk}(c) = c^{-1} g_{hk,k} g_{h,k} h(\xi'_i) \otimes \xi''_k \]
by (C.1)
\[ \]
Therefore, it is enough to show that
\[ c^{-1} g_{gh,k} g h(\xi'_i) c^{-1} g_{gh,k} g h(\xi'_i) = c^{-1} g_{gh,k} g h(\xi'_i) c^{-1} g_{gh,k} g h(\xi'_i). \]
or, equivalently,
\[ c^{-1} g_{gh,k} g h(\xi'_i) c^{-1} g_{gh,k} g h(\xi'_i) = c^{-1} g_{gh,k} g h(\xi'_i) c^{-1} g_{gh,k} g h(\xi'_i). \]
This is an immediate consequence of (2.1).

D Proof of Theorem 5.3

We need to prove the G-invariance of \( \theta_c \) and properties (1), (3), (4), (5), (7), (8), and (9)
from Definitions 3.3 and 3.4 (properties (2) and (6) are satisfied by construction of the multiplication and the comultiplication). As before, to make following our computations easier, we will sometimes underline those places in formulas that are about to be changed.

Proof of the G-invariance of \( \theta_c \). For any \( c \in \mathcal{H} \mathcal{H}_0(C)_c \) and \( g \in G \)
\[ \theta_c(T_c(g)(c)) = \theta(c^{-1} g_{g,1} g c_g c_{g,1} c_c) = \theta(c^{-1} g_{g,1} g c_g c_{g,1} c_c) = \theta(g(c g c_c)) = \theta(c c_c) = \theta_c(c). \]

Proof of property (1). Let \( c' \in \mathcal{H} \mathcal{H}_0(C)_h \), \( c'' \in \mathcal{H} \mathcal{H}_0(C)_k \), and \( g \in G \). We need to show that
\[ m^{-1}_{ghg^{-1}} (T_h(g)(c') \otimes T_k(g)(c'')) = T_{hk}(g) (m_{h,k}(c' \otimes c'')). \]
We have
\[ m^{-1}_{ghg^{-1}} (T_h(g)(c') \otimes T_k(g)(c'')) = m^{-1}_{ghg^{-1}} ((c^{-1} g_{g,1} g c_g c_{g,1} c_c) \otimes (c^{-1} g_{g,1} g c_g c_{g,1} c_c)) \]
\[ = \xi'_e c^{-1} g_{g,1} g c_g c_{g,1} g h^{-1} (\xi''_e c^{-1} g_{g,1} g c_g c_{g,1} g h^{-1} (c''_e c^{-1} g_{g,1} g c_g c_{g,1} g h^{-1} c_{ghg^{-1}}) \]
\[ = \xi'_e c^{-1} g_{g,1} g c_g c_{g,1} g h^{-1} (\xi''_e c^{-1} g_{g,1} g c_g c_{g,1} g h^{-1} (c''_e c^{-1} g_{g,1} g c_g c_{g,1} g h^{-1} c_{ghg^{-1}}) \]
\[ \times c^{-1} g_{ghg^{-1}} (c_{g,k} g h^{-1} (c_{g,k} g h^{-1} c_{g,hg^{-1}})). \]
On the other hand,

\[ T_{hk}(g)(m_{h,k}(c' \otimes c'')) = T_{hk}(g)(\xi'_i c'h(\xi''_i c'')) c_{h,k} = c_{e}^{-1} g(\xi'_i c'h(\xi''_i c'')) c_{h,k} c_{gh_k g^{-1}} \]

We need to show that

\[ \xi'_i \otimes g^{-1}(\xi''_i) = Ad(c_{e}^{-1}) Ad(c_{g^{-1}} g(\xi'_i)) \otimes \xi''_i. \]  

(D.1)

Proof of property (3). Let \( c' \in \mathcal{H}_0(C)_g, c'' \in \mathcal{H}_0(C)_h, \) and \( g \in G. \) We need to show that

\[ m_{g,h}(c' \otimes c'') = m_{gh^{-1},g}(T_h(g)(c'')) \otimes c' \]

modulo \( C_{gh}. \) We have

\[ m_{gh^{-1},g}(T_h(g)(c'') \otimes c') = m_{gh^{-1},g}(c_{e}^{-1} g(\xi''_i c'')) c_{g,h} c_{gh,g^{-1}} \otimes c' \]

\[ = \xi'_i c_{e}^{-1} g(\xi''_i c'') c_{gh,h^{-1}} g^{-1}(\xi''_i c') c_{gh,g^{-1}} \]

\[ = \xi'_i c_{e}^{-1} g(\xi''_i c'') c_{gh,h^{-1}} g^{-1}(\xi''_i c') c_{gh,g^{-1}} \]

Modulo \( C_{gh}, \) the latter expression equals

\[ g^{-1}(\xi''_i c') c_{g^{-1},g} c_{e} \xi'_i c_{e}^{-1} g(\xi''_i) c_{g,h}. \]
We need to show that for any $g$, $h_k \in G$ and $c \in \mathcal{H}_0(C)_{hk}$
\[
\Delta_{gkg^{-1}} \circ (T_{hk}(g)(c)) = (T_h(g) \otimes T_k(g)) \Delta_{h,k}(c).
\]

We have
\[
(T_h(g) \otimes T_k(g)) \Delta_{h,k}(c) = (T_h(g) \otimes T_k(g))(c c^{-1} h(\xi''_i) \otimes \xi''_k)
\]
\[
= c^{-1} c^{-1} g(c^{-1} h(\xi'_i)) c g_{c\otimes g^{-1}} \otimes c^{-1} c^{-1} g(\xi''_i) c g_{g\otimes g^{-1}}
\]
\[
= c^{-1} c^{-1} g(c^{-1} h(\xi'_i)) c g_{c\otimes g^{-1}} \otimes c^{-1} c^{-1} g(\xi''_i) c g_{g\otimes g^{-1}}
\]
\[
= c^{-1} c^{-1} g(c^{-1} h(\xi'_i)) c g_{c\otimes g^{-1}} \otimes c^{-1} c^{-1} g(\xi''_i) c g_{g\otimes g^{-1}}
\]

The latter expression equals $m_{g,h}(c' \otimes c'')$ since $\xi$ is symmetric.

**Proof of property (5).** We need to show that for any $g$, $h_k \in G$ and $c \in \mathcal{H}_0(C)_{hk}$
\[
\Delta_{gkg^{-1}} \circ (T_{hk}(g)(c)) = (T_h(g) \otimes T_k(g)) \Delta_{h,k}(c).
\]

We have
\[
(T_h(g) \otimes T_k(g)) \Delta_{h,k}(c) = (T_h(g) \otimes T_k(g))(c c^{-1} h(\xi'_i) \otimes \xi''_k)
\]
\[
= c^{-1} c^{-1} g(c^{-1} h(\xi'_i)) c g_{c\otimes g^{-1}} \otimes c^{-1} c^{-1} g(\xi''_i) c g_{g\otimes g^{-1}}
\]
\[
= c^{-1} c^{-1} g(c^{-1} h(\xi'_i)) c g_{c\otimes g^{-1}} \otimes c^{-1} c^{-1} g(\xi''_i) c g_{g\otimes g^{-1}}
\]
\[
= c^{-1} c^{-1} g(c^{-1} h(\xi'_i)) c g_{c\otimes g^{-1}} \otimes c^{-1} c^{-1} g(\xi''_i) c g_{g\otimes g^{-1}}
\]

On the other hand,
\[
\Delta_{gkg^{-1}} \circ (T_{hk}(g)(c)) = \Delta_{gkg^{-1}} \circ (c c^{-1} g(\xi'_i) c g_{c\otimes g^{-1}})
\]
\[
= c^{-1} c^{-1} g(c^{-1} h(\xi'_i)) c g_{c\otimes g^{-1}} \otimes c^{-1} c^{-1} g(\xi''_i) c g_{g\otimes g^{-1}}
\]
Thus, we need to show that
\[
g(c_{h,k}^{-1})c_{g,h}c_{ghg^{-1},g}g^{-1}(c_{g,k}c_{gkh,g}^{-1}) = c_{g,h,k}c_{gkh,g}^{-1}c_{gkh,g}^{-1},
\]
or, equivalently,
\[
c_{g,h}c_{ghg^{-1},g}g^{-1}(c_{g,k})g^{-1}(c_{gkh,g}^{-1})c_{ghg^{-1},g}g^{-1} = g(c_{h,k})c_{g,h,k}c_{gkh,g}^{-1}.
\]
Let us transform both hand sides of the latter equality simultaneously. First of all, (2.1), applied to the following parts of the expressions
\[
c_{g,h}c_{ghg^{-1},g}g^{-1}(c_{g,k})g^{-1}(c_{gkh,g}^{-1})c_{ghg^{-1},g}g^{-1} = g(c_{h,k})c_{g,h,k}c_{gkh,g}^{-1},
\]
gives us
\[
c_{g,h}c_{ghg^{-1},g}g^{-1}(c_{g,k})c_{ghg^{-1},g}c_{gkh,g}^{-1} = c_{g,h,k}c_{gkh,k}c_{gkh,g}^{-1},
\]
which is equivalent to
\[
c_{ghg^{-1},g}g^{-1}(c_{g,k})c_{gkh,g}^{-1} = c_{g,h,k}.
\]
The latter is equivalent to (2.1).

**Proof of property (7).** Let \( c \in \mathcal{H}_{0}(C)_{gh} \). We need to show that
\[
\Delta_{g,h}(c) = \sigma(1 \otimes T_{h^{-1}gh}(h))(\Delta_{h,h^{-1}gh}(c))
\]
in \( \mathcal{H}_{0}(C)_{g} \otimes \mathcal{H}_{0}(C)_{h} \). We have
\[
(1 \otimes T_{h^{-1}gh}(h))(\Delta_{h,h^{-1}gh}(c)) = (1 \otimes T_{h^{-1}gh}(h))(c_{h,h^{-1}gh}^{-1}h(\xi_{l}')) \otimes \xi_{l}'')
\]
\[
= (1 \otimes T_{h^{-1}gh}(h))(c_{h,h^{-1}gh}^{-1}h(\xi_{l}')) \otimes c_{e}^{-1}c_{h^{-1}gh}^{-1}h(\xi_{l}'')c_{h,h^{-1}gh}c_{gh,h^{-1}}.
\]
Now observe that (2.1) is equivalent
\[
h(\xi_{l}') \otimes h(\xi_{l}'') = Ad(c_{h,h^{-1}})Ad(c_{e})(\xi_{l}') \otimes Ad(c_{h,h^{-1}})Ad(c_{e})(\xi_{l}'') \text{ by (2.1)}.
\]
Therefore,
\[
(1 \otimes T_{h^{-1}gh}(h))(\Delta_{h,h^{-1}gh}(c)) = c_{h,h^{-1}gh}^{-1}\xi_{l}' \otimes c_{e}^{-1}c_{h^{-1}gh}^{-1}\xi_{l}''c_{h,h^{-1}gh}c_{gh,h^{-1}}
\]
\[
\text{by (2.1)}
\]
\[
\xi_{l}' \otimes c_{e}^{-1}c_{h^{-1}gh}^{-1}\xi_{l}''c_{h,h^{-1}gh}c_{h,h^{-1}gh}c_{gh,h^{-1}}.
\]
Thus,
\[
\sigma(1 \otimes T_{h^{-1}gh}(h))(\Delta_{h,h^{-1}gh}(c)) = c_{e}^{-1}c_{h,h^{-1}}\xi_{l}'c_{h,h^{-1}gh}c_{h,h^{-1}gh}c_{gh,h^{-1}} \otimes \xi_{l}''.
\]
Modulo \( C_{g} \otimes C \), the latter expression equals
\[
c_{h,h^{-1}gh}c_{h,h^{-1}gh}c_{gh,h^{-1}}g(c_{e}^{-1}c_{h,h^{-1}}\xi_{l}') \otimes \xi_{l}''' = c_{g,h^{-1}gh}g(c_{e}^{-1}c_{h,h^{-1}}\xi_{l}') \otimes \xi_{l}''.
\]
To complete the proof, observe that
\[
c_{g_{h,h^{-1}}}(c^{-1}_{h_{h^{-1}}}c_1) \otimes \xi''_i = c_{g_{h,h^{-1}}}(c^{-1}_{h_{h^{-1}}}c_1) \otimes \xi''_i = \Delta_{g,h}(c).
\]

**Proof of property (8).** Let \( g \in G \) and \( c \in H \mathcal{H}_0(\mathcal{C})_g \). Then
\[
(\theta \otimes 1)\Delta_{e,g}(c) = (\theta \otimes 1) (c c_{e,g}^{-1}(\xi'_i) \otimes \xi''_i) \quad \text{by (2.1) and (2.3)}
\]
\[= \theta(c^{-1}_c(\xi'_i) c_c) \xi''_i = \theta(c \xi'_i) \xi''_i = c.
\]
Similarly,
\[
(1 \otimes \theta)\Delta_{g,e}(c) = (1 \otimes \theta) (c c_{g,e}^{-1}(\xi'_i) \otimes \xi''_i) = c c_{g,e}^{-1}(\xi'_i) \theta(\xi''_i) = c c_{g,e}^{-1}(\xi'_i) c_c = c.
\]

**Proof of property (9).** Let \( g, h, k \in G, c' \in \mathcal{H}_0(\mathcal{C})_g, c'' \in \mathcal{H}_0(\mathcal{C})_{hk} \). Firstly, we need to prove that \( \Delta \) is a morphism of left \( \mathcal{H}_0(\mathcal{C}) \)-modules, i.e.
\[
\Delta_{g,h,k}(m_{g,hk}(c' \otimes c'')) = (m_{g,h} \otimes 1)(c' \otimes \Delta_{h,k}(c''))
\]
We have
\[
\Delta_{g,h,k}(m_{g,hk}(c' \otimes c'')) = \Delta_{g,h,k}(\xi'_i c' g(\xi''_i) c_{g,hk}) = \xi'_i c' g(\xi''_i) c_{g,hk} = \xi'_i c' g(\xi''_i) c_{g,hk}.
\]
On the other hand,
\[
(m_{g,h} \otimes 1)(c' \otimes \Delta_{h,k}(c'')) = (m_{g,h} \otimes 1)(c' \otimes c'' c_{g,hk}^{-1} h(\xi'_i) \otimes \xi''_i) = \xi'_i c' g(\xi''_i) c_{g,hk}^{-1} h(\xi'_i) c_{g,hk} \otimes \xi''_i.
\]

Secondly, we need to prove that \( \Delta \) is a morphism of right \( \mathcal{H}_0(\mathcal{C}) \)-modules, i.e.
\[
\Delta_{h,k,g}(m_{hk,g}(c'' \otimes c')) = (1 \otimes m_{k,g})(\Delta_{h,k}(c'') \otimes c')
\]
in \( \mathcal{H}_0(\mathcal{C})_h \otimes \mathcal{H}_0(\mathcal{C})_{kg} \). We have
\[
\Delta_{h,k,g}(m_{hk,g}(c'' \otimes c')) = \Delta_{h,k,g}(\xi'_i c'' h k(\xi''_i) c_{hk,g}) = \xi'_i c'' h k(\xi''_i) c_{hk,g}^{-1} h(\xi'_i) \otimes \xi''_i.
\]
On the other hand,
\[
(1 \otimes m_{k,g})(\Delta_{h,k}(c'') \otimes c') = (1 \otimes m_{k,g})(c'' c_{h,k}^{-1} h(\xi'_i) \otimes \xi''_i) = c'' c_{h,k}^{-1} h(\xi'_i) \otimes \xi''_i \quad \text{by (2.1)}
\]
\[= c'' h k(\xi''_i) c_{hk,g} \xi''_i \otimes \xi''_i \quad \text{by (2.1)}
\]
\[= c'' h k(\xi''_i) c_{hk,g} \xi''_i \otimes \xi''_i \quad \text{by (2.1)}
\]

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Thus, we see that
\[
\Delta_{h,k^g}(m_{h,k^g}(c'' \otimes c')) - (1 \otimes m_{k,g})(\Delta_{h,k}(c'') \otimes c')
\]
\[= \xi'_h \left( c'' h(\xi''_{c'}) c_{h,g}^{-1} h(\xi'_h) \right) \otimes \xi''_c - \left( c'' h(\xi''_{c'}) c_{h,g}^{-1} h(\xi'_h) \right) h(\xi'_h) \otimes \xi''_c \in C_h \otimes C.
\]

**Proof of property (4) (the torus axiom).** First of all, let us reformulate the torus axiom. Let \( C \) be an object satisfying all the axioms of a weak crossed \( G \)-algebra except for the torus axiom. For \( g, h \in G \), let \( m_{g,h} \) stand for the multiplication map \( C_g \otimes C_h \rightarrow C_{gh} \).

Fix \( g_1, g_2 \in C, c \in C_{g_1}, c' \in C_{g_2} \). Then
\[
(1 \otimes \theta_c)(1 \otimes m_{g_2^{-1}, g_2})(\Delta_{g_1 g_2^{-1} g_2^{-1}}(c) \otimes c') \overset{by \ (8)}{=} (1 \otimes \theta_c)(\Delta_{g_1 g_2^{-1} g_2^{-1}}(m_{g_1, g_2}(c \otimes c')) ) \overset{by \ (7)}{=} m_{g_1, g_2}(c \otimes c').
\]
If we set \( \Delta_{g_1 g_2^{-1} g_2^{-1}}(c) = \sum c^{(1)} \otimes c^{(2)} \) then the above computation means that under the canonical isomorphism
\[
\text{Hom}(C_{g_2}, C_{g_1 g_2}) \cong C_{g_1 g_2} \otimes C_{g_2},
\]
the operator \( m_{g_1, g_2}(c \otimes \bullet) \) corresponds to the element
\[
c^{(1)} \otimes \theta_c(m_{g_2^{-1}, g_2}(c^{(2)} \otimes \bullet)).
\]
Thus, if \( T : C_{g_1 g_2} \rightarrow C_{g_2} \) is a linear operator then
\[
\text{Tr}_{C_{g_1 g_2}}(m_{g_1, g_2}(c \otimes \bullet) \cdot T) = \text{Tr}_{C_{g_1 g_2}}(c^{(1)} \otimes \theta_c(m_{g_2^{-1}, g_2}(c^{(2)} \otimes T(\bullet)))) = \theta_c(m_{g_2^{-1}, g_2}(c^{(2)} \otimes T(c^{(1)}))) = \theta_c(m_{g_2^{-1}, g_2} \cdot \sigma \cdot (T \otimes 1) \cdot \Delta_{g_1 g_2^{-1} g_2^{-1}}(c)) \overset{(D.4)}{=}
\]
and, similarly,
\[
\text{Tr}_{C_{g_2}}(T \cdot m_{g_1, g_2}(c \otimes \bullet)) = \theta_c(m_{g_2^{-1}, g_2}(c^{(2)} \otimes \bullet))) = \theta_c(m_{g_2^{-1}, g_2}(c^{(2)} \otimes T(c^{(1)}))) = \theta_c(m_{g_2^{-1}, g_2} \cdot \sigma \cdot (T \otimes 1) \cdot \Delta_{g_1 g_2^{-1} g_2^{-1}}(c)). \overset{(D.5)}{=}
\]
Now we are ready to reformulate the torus axiom. Let us fix \( g, h \in G \) and \( c \in C_{h g h^{-1} g^{-1}} \). Then, by (D.4)
\[
\text{Tr}_c(m_{h g h^{-1} g^{-1}, g h g^{-1}}(c \otimes \bullet) \cdot g) = \theta_c(m_{g h^{-1} g^{-1}, g h g^{-1}} \cdot \sigma \cdot (g \otimes 1) \cdot \Delta_{h, g h^{-1} g^{-1}}(c)) \overset{by \ G \text{-inv.}}{=}
\]
\[
\theta_c(g^{-1} \cdot m_{g h^{-1} g^{-1}, g h g^{-1}} \cdot \sigma \cdot (g \otimes 1) \cdot \Delta_{h, g h^{-1} g^{-1}}(c)) \overset{by \ (1)}{=}
\]
\[
\theta_c(m_{h^{-1}, h} \cdot (g^{-1} \otimes g^{-1}) \cdot \sigma \cdot (g \otimes 1) \cdot \Delta_{h, g h^{-1} g^{-1}}(c)) = \theta_c(m_{h^{-1}, h} \cdot \sigma \cdot (1 \otimes g^{-1}) \cdot \Delta_{h, g h^{-1} g^{-1}}(c)) \overset{by \ (3)}{=}
\]
\[
\theta_c(m_{h h^{-1}, 1} \cdot (1 \otimes g^{-1}) \cdot \Delta_{h, g h^{-1} g^{-1}}(c)).
\]
On the other hand, by (D.5)
\[
\text{Tr}_c(h^{-1} \cdot m_{h g h^{-1} g^{-1}, g}(c \otimes \bullet)) = \theta_c(m_{g^{-1}, g} \cdot \sigma \cdot (h^{-1} \otimes 1) \cdot \Delta_{h g h^{-1} g^{-1}}(c))
\]

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On the other hand,

\[ \theta_c(m_{g, h^{-1}} \cdot (h^{-1} \otimes 1) \cdot \Delta_{hgh^{-1}, g^{-1}}(c)) \]

by \( G \)-inv.

\[ = \theta_c(h \cdot m_{g, h^{-1}} \cdot (h^{-1} \otimes 1) \cdot \Delta_{hgh^{-1}, g^{-1}}(c)) \]

by \((1)\)

\[ = \theta_c(m_{hgh^{-1}, hg^{-1}h^{-1}} \cdot (1 \otimes h) \cdot \Delta_{hgh^{-1}, g^{-1}}(c)). \]

Therefore, the torus axiom is equivalent to the following statement: for any \( g, h, c \in \mathcal{C}_{hgh^{-1}g^{-1}} \)

\[ \theta_c(m_{h, h^{-1}} \cdot (1 \otimes g^{-1}) \cdot \Delta_{hgh^{-1}g^{-1}}(c)) = \theta_c(m_{ngh^{-1}, hg^{-1}h^{-1}} \cdot (1 \otimes h) \cdot \Delta_{hgh^{-1}, g^{-1}}(c)). \tag{D.6} \]

We will show that \((D.6)\) is satisfied in \( \mathcal{H}_0(\mathcal{C}) \).

Consider an element \( c \in \mathcal{H}_0(\mathcal{C})_{hgh^{-1}g^{-1}}. \) Then

\[ \theta_c(m_{h, h^{-1}} \cdot (1 \otimes T_{gh^{-1}g^{-1}}(g^{-1})) \cdot \Delta_{hgh^{-1}g^{-1}}(c)) \]

by \((3)\)

\[ = \theta_c(m_{h, h^{-1}} \cdot (1 \otimes T_{gh^{-1}g^{-1}}(g^{-1}))(c_{hgh^{-1}g^{-1}} \cdot h(\xi''_c \otimes 1))) \]

\[ = \theta_c(m_{h, h^{-1}}(c_{hgh^{-1}g^{-1}} \cdot h(\xi''_c \otimes 1)) \cdot (c_{hgh^{-1}g^{-1}} \cdot h(\xi''_c \otimes 1)) \cdot (c_{hgh^{-1}g^{-1}} \cdot h(\xi''_c \otimes 1))) \]

\[ = \theta_c(c_{hgh^{-1}g^{-1}} \cdot h(\xi''_c) \cdot h(\xi''_c) \cdot h(\xi''_c) \cdot h(\xi''_c)) \]

by \((1)\)

\[ \theta_c(c_{hgh^{-1}g^{-1}} \cdot h(\xi''_c) \cdot h(\xi''_c) \cdot h(\xi''_c) \cdot h(\xi''_c)) \]

On the other hand,

\[ \theta_c(m_{hgh^{-1}, hg^{-1}h^{-1}} \cdot (1 \otimes T_{g^{-1}}(h)) \cdot \Delta_{hgh^{-1}, g^{-1}}(c)) \]

\[ = \theta_c(m_{hgh^{-1}, hg^{-1}h^{-1}} \cdot (1 \otimes T_{g^{-1}}(h))(c_{hgh^{-1}, g^{-1}} \cdot hgh^{-1}((\xi''_c \otimes 1))) \]

\[ = \theta_c(m_{hgh^{-1}, hg^{-1}h^{-1}}(c_{hgh^{-1}, g^{-1}} \cdot hgh^{-1}((\xi''_c \otimes 1))) \cdot (c_{hgh^{-1}, g^{-1}} \cdot hgh^{-1}((\xi''_c \otimes 1)) \cdot (c_{hgh^{-1}, g^{-1}} \cdot hgh^{-1}((\xi''_c \otimes 1)))) \]

\[ = \theta_c(c_{hgh^{-1}, g^{-1}} \cdot hgh^{-1}((\xi''_c \otimes 1)) \cdot hgh^{-1}((\xi''_c \otimes 1)) \cdot hgh^{-1}((\xi''_c \otimes 1)) \cdot hgh^{-1}((\xi''_c \otimes 1))) \]

\[ \times hgh^{-1}(c_{hgh^{-1}, g^{-1}}) \cdot hgh^{-1}(c_{hgh^{-1}, g^{-1}}) \cdot hgh^{-1}(c_{hgh^{-1}, g^{-1}}) \cdot hgh^{-1}(c_{hgh^{-1}, g^{-1}}) \]

\[ = \theta_c(c_{hgh^{-1}, g^{-1}} \cdot hgh^{-1}((\xi''_c \otimes 1)) \cdot hgh^{-1}((\xi''_c \otimes 1)) \cdot hgh^{-1}((\xi''_c \otimes 1)) \cdot hgh^{-1}((\xi''_c \otimes 1))) \]

\[ \times hgh^{-1}(c_{hgh^{-1}, g^{-1}}) \cdot hgh^{-1}(c_{hgh^{-1}, g^{-1}}) \cdot hgh^{-1}(c_{hgh^{-1}, g^{-1}}) \cdot hgh^{-1}(c_{hgh^{-1}, g^{-1}}) \]

\[ = \theta_c(c_{hgh^{-1}, g^{-1}} \cdot hgh^{-1}((\xi''_c \otimes 1)) \cdot hgh^{-1}((\xi''_c \otimes 1)) \cdot hgh^{-1}((\xi''_c \otimes 1)) \cdot hgh^{-1}((\xi''_c \otimes 1))) \]

\[ = \theta(c_{hgh^{-1}, g^{-1}} \cdot hgh^{-1}((\xi''_c \otimes 1)) \cdot hgh^{-1}((\xi''_c \otimes 1)) \cdot hgh^{-1}((\xi''_c \otimes 1)) \cdot hgh^{-1}((\xi''_c \otimes 1))) \].
By (D.7) and (D.8), it suffices to show that
\[ c_{hgh^{-1},g}^{-1}h(\xi'_i)h(\xi''_f) c_{h^{-1},g}^{-1}(\xi''_f) c_{hgh^{-1},g}^{-1}(\xi'') c_{gh^{-1},g}c_{e\xi'_j} \]
\[ = c_{hgh^{-1},g}^{-1}h(\xi'_i)h(\xi''_f) c_{h^{-1},g}^{-1}(\xi''_f) c_{hgh^{-1},g}^{-1}(\xi'') c_{gh^{-1},g}c_{e\xi'_j}. \]  
(D.9)

Let us apply $h^{-1}$ to both hand sides of the latter equality. We will start with the left-hand side:
\[ h^{-1}(c_{hgh^{-1},g}^{-1})c_{h^{-1},g} c_{e\xi'_j} c_{e} c_{gh^{-1},g}^{-1}(\xi''_f) c_{h^{-1},g}^{-1}(\xi''_f) c_{gh^{-1},g}c_{e\xi'_j} \times \]
\[ c_{h^{-1},g}^{-1}h^{-1}(c_{gh^{-1},g}^{-1})h^{-1}(c_{e\xi''_f}c_{e\xi''_f}) \]
\[ \text{by } (D.1) \]
\[ c_{h^{-1},g}^{-1}h^{-1}(c_{gh^{-1},g}^{-1})h^{-1}(c_{e\xi''_f}c_{e\xi''_f}) \]
\[ \text{by } (C.1) \]
\[ = c_{h^{-1},g}^{-1}(\xi''_f) c_{e} c_{gh^{-1},g}^{-1}(\xi''_f) c_{h^{-1},g}^{-1}(\xi''_f) c_{gh^{-1},g}c_{e\xi'_j} h^{-1}(\xi'_i) \]
\[ \text{by } (C.1) \]
\[ = c_{h^{-1},g}^{-1}(\xi''_f) c_{e} c_{gh^{-1},g}^{-1}(\xi''_f) c_{h^{-1},g}^{-1}(\xi''_f) c_{gh^{-1},g}c_{e\xi'_j} h^{-1}(\xi'_i) \]
\[ \text{by } (D.9) \]

Now we will apply $h^{-1}$ to the right-hand side of (D.9):
\[ h^{-1}(c_{hgh^{-1},g}^{-1})c_{h^{-1},g}^{-1}(\xi'_i) h^{-1}(\xi''_f) c_{h^{-1},g}^{-1}(\xi''_f) c_{h^{-1},g}^{-1}(\xi''_f) c_{h^{-1},g}^{-1}(\xi''_f) c_{gh^{-1},g}c_{e\xi'_j} \times \]
\[ c_{h^{-1},g}^{-1}h^{-1}(c_{gh^{-1},g}^{-1})h^{-1}(c_{e\xi''_f}c_{e\xi''_f}) \]
\[ \text{by } (D.1) \]
\[ c_{h^{-1},g}^{-1}h^{-1}(c_{gh^{-1},g}^{-1})h^{-1}(c_{e\xi''_f}c_{e\xi''_f}) \]
\[ \text{by } (C.1) \]
\[ = c_{h^{-1},g}^{-1}(\xi''_f) c_{e} c_{gh^{-1},g}^{-1}(\xi''_f) c_{h^{-1},g}^{-1}(\xi''_f) c_{gh^{-1},g}c_{e\xi'_j} h^{-1}(\xi'_i) \]
\[ \text{by } (C.1) \]
\[ = c_{h^{-1},g}^{-1}(\xi''_f) c_{e} c_{gh^{-1},g}^{-1}(\xi''_f) c_{h^{-1},g}^{-1}(\xi''_f) c_{gh^{-1},g}c_{e\xi'_j} h^{-1}(\xi'_i) \]
\[ \text{by } (D.9) \]
Now let us compare the latter expression with (D.10): clearly, to finish proving the torus axiom, it remains to show that

\[ c^{-1} g_{g,h^{-1}} (c^{-1} g_{g,h^{-1}}) = c^{-1} g_{g,h^{-1}}. \]

By (2.1), the right-hand side equals \( c^{-1} g_{g,h^{-1}} (c^{-1} g_{g,h^{-1}}) \), so the equality reduces to

\[ c^{-1} g_{g,h^{-1}} (c^{-1} g_{g,h^{-1}}) = c^{-1} g_{g,h^{-1}}. \]

or, equivalently,

\[ g_{g,h^{-1}} (c^{-1} g_{g,h^{-1}}) = c^{-1} g_{g,h^{-1}}. \]

The latter equality follows from (2.1). The torus axiom is proved.

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