Weights sharing the same eigenvalue

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Dedicated to Professor Wataru Takahashi, with esteem and friendship, on his seventieth birthday

Abstract: Here is the simplest particular case of our main result: let \( f : \mathbb{R} \to \mathbb{R} \) be a function of class \( C^1 \), with \( \sup_{\mathbb{R}} f' > 0 \), such that

\[
\lim_{|\xi| \to +\infty} \frac{f(\xi)}{\xi} = 0.
\]

Then, for each \( \lambda > \frac{\pi^2}{\sup_{\mathbb{R}} f} \), the set of all \( u \in H^1_0([0,1]) \) for which the problem

\[
\begin{cases}
-\v'' = \lambda f'(u(x))v & \text{in } ]0,1[ \\
v(0) = v(1) = 0
\end{cases}
\]

has a non-zero solution is closed and not \( \sigma \)-compact in \( H^1_0([0,1]) \).

Key words: Eigenvalue; weight; Dirichlet problem; \( \sigma \)-compact.

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1. Introduction

Let \( \Omega \subset \mathbb{R}^n \) be a smooth bounded domain. We consider the Sobolev space \( H^1_0(\Omega) \) endowed with the scalar product

\[
\langle u, v \rangle = \int_\Omega \nabla u(x) \nabla v(x) dx
\]

and the induced norm

\[
\|u\| = \left( \int_\Omega |\nabla u(x)|^2 dx \right)^{\frac{1}{2}}.
\]

We are interested in pairs \((\lambda, \beta)\), where \( \lambda \) is a positive number and \( \beta \) is a measurable function, such that the linear problem

\[
\begin{cases}
-\Delta v = \lambda \beta(x)v & \text{in } \Omega \\
v_{|\partial\Omega} = 0
\end{cases}
\]

has a non-zero weak solution, that is to say a \( v \in H^1_0(\Omega) \setminus \{0\} \) such that

\[
\int_\Omega \nabla v(x) \nabla w(x) dx = \lambda \int_\Omega \beta(x)v(x)w(x) dx
\]

for all \( w \in H^1_0(\Omega) \).

If this happens, we say that \( \lambda \) is an eigenvalue related to the weight \( \beta \).
While, the structure of the set of all eigenvalues related to a fixed weight is well understood, it seems that much less is known about the structure of the set of all weights $\beta$ for which a fixed positive number $\lambda$ turns out to be an eigenvalue related to $\beta$.

In this very short note, we intend to give a contribution along the latter direction.

More precisely, we identify a quite general class of continuous functions $g : \mathbb{R} \to \mathbb{R}$ such that, for each $\lambda$ in a suitable interval, the set of all $u \in H^1_0(\Omega)$ for which $\lambda$ is an eigenvalue related to the weight $g(u(\cdot))$ is closed and not $\sigma$-compact in $H^1_0(\Omega)$.

2. Results

Let us recall that a set in a topological space is said to be $\sigma$-compact if it is the union of an at most countable family of compact sets.

For each $\alpha \in L^\infty(\Omega) \setminus \{0\}$, with $\alpha \geq 0$, we denote by $\lambda_\alpha$ the first eigenvalue of the problem

$$
\begin{cases}
-\Delta v = \lambda \alpha(x)v & \text{in } \Omega \\
v|_{\partial \Omega} = 0
\end{cases}
$$

Let us recall that

$$
\lambda_\alpha = \min_{v \in H^1_0(\Omega) \setminus \{0\}} \frac{\|v\|^2}{\int_\Omega \alpha(x)|v(x)|^2\,dx}.
$$

With the conventions $\frac{1}{\infty} = 0$, $\frac{1}{0} = +\infty$, here is the statement of the result introduced above:

**THEOREM 1.** - Let $f : \mathbb{R} \to \mathbb{R}$ be a function of class $C^1$ such that

$$
\max \left\{ 0, 2 \limsup_{|\xi| \to +\infty} \frac{\int_0^\xi f(t)\,dt}{\xi^2}, \limsup_{|\xi| \to +\infty} \frac{f(\xi)}{\xi} \right\} < \sup_{\mathbb{R}} f'.
$$

Moreover, if $n \geq 2$, assume that

$$
\sup_{\xi \in \mathbb{R}} \frac{|f'(\xi)|}{1 + |\xi|^q} < +\infty
$$

for some $q > 0$, with $q < \frac{4}{n-2}$ if $n \geq 3$.

Then, for each $\alpha \in L^\infty(\Omega) \setminus \{0\}$, with $\alpha \geq 0$, and for every $\lambda$ satisfying

$$
\frac{\lambda_\alpha}{\sup_{\mathbb{R}} f'} < \lambda < \frac{\lambda_\alpha}{\max \left\{ 0, 2 \limsup_{|\xi| \to +\infty} \frac{\int_0^\xi f(t)\,dt}{\xi^2}, \limsup_{|\xi| \to +\infty} \frac{f(\xi)}{\xi} \right\}},
$$

the set of all $u \in H^1_0(\Omega)$ for which the problem

$$
\begin{cases}
-\Delta v = \lambda \alpha f'(u(x))v & \text{in } \Omega \\
v|_{\partial \Omega} = 0
\end{cases}
$$

has a non-zero weak solution is closed and not $\sigma$-compact in $H^1_0(\Omega)$.

**REMARK 1.** - It is worth noticing that the linear hull of any closed and not $\sigma$-compact set in $H^1_0(\Omega)$ is infinite-dimensional. This comes from the fact that any closed set in a finite-dimensional normed space is $\sigma$-compact.
The key tool we use to prove Theorem 1 is Theorem 2 below whose proof, in turn, is entirely based on the following particular case of a result recently established in [1]:

**Theorem A** ([1], Theorem 2) Let \((X, \langle \cdot, \cdot \rangle)\) be an infinite-dimensional real Hilbert space and let \(I : X \to \mathbb{R}\) be a sequentially weakly lower semicontinuous, not convex functional of class \(C^2\) such that \(I'\) is closed and \(\lim_{\|x\| \to +\infty}(I(x) + \langle z, x \rangle) = +\infty\) for all \(z \in X\).

Then, the set

\[ \{ x \in X : I''(x) \text{ is not invertible} \} \]

is closed and not \(\sigma\)-compact.

**Theorem 2.** Let \((X, \langle \cdot, \cdot \rangle)\) be an infinite-dimensional real Hilbert space, and let \(J : X \to \mathbb{R}\) be a functional of class \(C^2\), with compact derivative. For each \(\lambda \in \mathbb{R}\), put

\[ A_\lambda = \{ x \in X : y = \lambda J''(x)(y) \text{ for some } y \in X \setminus \{0\} \}. \]

Assume that

\[
\max \left\{ 0, 2 \limsup_{\|x\| \to +\infty} \frac{\langle J'(x), x \rangle}{\|x\|^2}, \limsup_{\|x\| \to +\infty} \frac{\langle J''(x)(y), y \rangle}{\|y\|^2} \right\} < \sup_{(x, y) \in X \times (X \setminus \{0\})} \frac{\langle J''(x)(y), y \rangle}{\|y\|^2}.
\]

Then, for every \(\lambda\) satisfying

\[
\inf_{\{(x, y) \in X \times X : \langle J''(x)(y), y \rangle > 0\}} \frac{\|y\|^2}{\langle J''(x)(y), y \rangle} < \frac{1}{\max \left\{ 0, 2 \limsup_{\|x\| \to +\infty} \frac{J(x)}{\|x\|^2}, \limsup_{\|x\| \to +\infty} \frac{J'(x)}{\|x\|^2} \right\}},
\]

the set \(A_\lambda\) is closed and not \(\sigma\)-compact.

**Proof.** Fix \(\lambda\) satisfying

\[
\inf_{\{(x, y) \in X \times X : \langle J''(x)(y), y \rangle > 0\}} \frac{\|y\|^2}{\langle J''(x)(y), y \rangle} < \frac{1}{\max \left\{ 0, 2 \limsup_{\|x\| \to +\infty} \frac{J(x)}{\|x\|^2}, \limsup_{\|x\| \to +\infty} \frac{J'(x)}{\|x\|^2} \right\}}.
\]

For each \(x \in X\), put

\[ I_\lambda(x) = \frac{1}{2} \|x\|^2 - \lambda J(x). \]

Clearly, for some \((x, y) \in X \times X\), with \(\langle J''(x)(y), y \rangle > 0\), we have

\[ \langle y - \lambda J''(x)(y), y \rangle < 0 \]

and so, since

\[ I''_\lambda(x)(y) = y - \lambda J''(x)(y), \]

by a classical characterization (Theorem 2.1.11 of [2]), the functional \(I_\lambda\) is not convex. Now, let us show that

\[ \lim_{\|x\| \to +\infty} \|x - \lambda J'(x)\| = +\infty. \quad (1) \]

Indeed, for each \(x \in X \setminus \{0\}\), we have

\[
\|x - \lambda J'(x)\| = \sup_{\|y\| = 1} (x + \lambda J'(x), y) \geq \left\langle x - \lambda J'(x), \frac{x}{\|x\|} \right\rangle \geq \|x\| \left(1 - \lambda \frac{\langle J'(x), x \rangle}{\|x\|^2}\right). \quad (2)
\]

On the other hand, we also have

\[
\liminf_{\|x\| \to +\infty} \left(1 - \lambda \frac{\langle J'(x), x \rangle}{\|x\|^2}\right) = 1 - \lambda \limsup_{\|x\| \to +\infty} \frac{\langle J'(x), x \rangle}{\|x\|^2} > 0. \quad (3)
\]
So, (1) is a direct consequence of (2) and (3). Furthermore, for each \( z \in X \), since

\[
I_\lambda(x) + \langle z, x \rangle = \|x\|^2 \left( \frac{1}{2} - \lambda \frac{J(x)}{\|x\|^2} + \frac{\langle z, x \rangle}{\|x\|^2} \right)
\]

and

\[
\liminf_{\|x\| \to +\infty} \left( \frac{1}{2} - \lambda \frac{J(x)}{\|x\|^2} + \frac{\langle z, x \rangle}{\|x\|^2} \right) = \frac{1}{2} - \lambda \limsup_{\|x\| \to +\infty} \frac{J(x)}{\|x\|^2} > 0,
\]

we have

\[
\lim_{\|x\| \to +\infty} (I_\lambda(x) + \langle z, x \rangle) = +\infty.
\]

Since \( J' \) is compact, on the one hand, \( J \) is sequentially weakly continuous ([4], Corollary 41.9) and, on the other hand, in view of (1), the operator \( I'_\lambda \) is closed ([3], Example 4.43). The compactness of \( J' \) also implies that, for each \( x \in X \), the operator \( J''(x) \) is compact ([3], Proposition 7.33) and so, for each \( \lambda \in \mathbb{R} \), the operator \( y \to y - \lambda J''(x)(y) \) is injective if and only if it is surjective ([3], Example 8.16). At this point, the fact that \( A_\lambda \) is closed and not \( \sigma \)-compact follows directly from Theorem A which can be applied to the functional \( I_\lambda \). \( \triangle \)

**Proof of Theorem 1.** Fix \( \alpha \in L^\infty(\Omega) \setminus \{0\} \), with \( \alpha \geq 0 \). For each \( u \in H_0^1(\Omega) \), put

\[
J_f(u) = \int_\Omega \alpha(x)F(u(x))dx,
\]

where

\[
F(\xi) = \int_0^\xi f(t)dt.
\]

Our assumptions ensure that the functional \( J_f \) is of class \( C^2 \) in \( H_0^1(\Omega) \), and we have

\[
\langle J'_f(u), v \rangle = \int_\Omega \alpha(x)f'(u(x))v(x)dx,
\]

\[
\langle J''_f(u)(v), w \rangle = \int_\Omega \alpha(x)f''(u(x))v(x)w(x)dx
\]

for all \( u, v, w \in H_0^1(\Omega) \). Moreover, \( J_f \) is compact. Fix \( \nu > \limsup_{\|\xi\| \to +\infty} \frac{F(\xi)}{\xi} \). Then, for a suitable constant \( c > 0 \), we have

\[
F(\xi) \leq \nu \xi^2 + c
\]

for all \( \xi \in \mathbb{R} \). Hence, for each \( u \in H_0^1(\Omega) \), we obtain

\[
J_f(u) \leq \nu \int_\Omega \alpha(x)|u(x)|^2dx + c \int_\Omega \alpha(x)dx \leq \nu \lambda_\alpha^{-1}\|u\|^2 + c \int_\Omega \alpha(x)dx.
\]

This clearly implies that

\[
\limsup_{\|u\| \to +\infty} \frac{J_f(u)}{\|u\|^2} \leq \lambda_\alpha^{-1} \limsup_{\|\xi\| \to +\infty} \frac{F(\xi)}{\xi}.
\]

In the same way, we obtain

\[
\limsup_{\|u\| \to +\infty} \frac{\langle J'_f(u), u \rangle}{\|u\|^2} \leq \lambda_\alpha^{-1} \limsup_{\|\xi\| \to +\infty} \frac{f(\xi)}{\xi}.
\]

Now, fix a function \( \tilde{v} \in H_0^1(\Omega) \), with \( \|\tilde{v}\| = 1 \), such that

\[
\int_\Omega \alpha(x)|\tilde{v}(x)|^2dx = \lambda_\alpha^{-1}.
\]
Fix also $\epsilon > 0, \tilde{\xi} \in \mathbb{R}$, with $f'(\tilde{\xi}) > 0$, and a closed set $C \subseteq \Omega$ so that

$$\int_C \alpha(x)|\tilde{v}(x)|^2 dx > \int_\Omega \alpha(x)|\tilde{v}(x)|^2 dx - \epsilon$$

and

$$\int_{\Omega \setminus C} \alpha(x)|\tilde{v}(x)|^2 dx < \frac{\epsilon}{\sup_{[-|\tilde{\xi}|,|\tilde{\xi}|]}|f'|}.$$ 

Finally, fix a function $\tilde{u} \in H_0^1(\Omega)$ such that

$$\tilde{u}(x) = \tilde{\xi}$$

for all $\xi \in C$ and

$$|\tilde{u}(x)| \leq |\tilde{\xi}|$$

for all $\xi \in \Omega$. Then, we have

$$f'(\tilde{\xi}) \left(\int_\Omega \alpha(x)|\tilde{v}(x)|^2 dx - \epsilon\right) < f'(\tilde{\xi}) \int_\Omega \alpha(x)|\tilde{v}(x)|^2 dx = \int_\Omega \alpha(x)f'(\tilde{u}(x))|\tilde{v}(x)|^2 dx - \int_{\Omega \setminus C} \alpha(x)f'(\tilde{u}(x))|\tilde{v}(x)|^2 dx$$

$$\leq \sup_{(u,v) \in H_0^1(\Omega) \times (H_0^1(\Omega) \setminus \{0\})} \frac{\int_\Omega \alpha(x)f'(u(x))|v(x)|^2 dx}{\|v\|^2} + \epsilon.$$ 

Since $\tilde{\xi}$ and $\epsilon$ are arbitrary, we then infer that

$$\lambda_\alpha^{-1} \sup_{\mathbb{R}} f' \leq \sup_{(u,v) \in H_0^1(\Omega) \times (H_0^1(\Omega) \setminus \{0\})} \frac{\int_\Omega \alpha(x)f'(u(x))|v(x)|^2 dx}{\|v\|^2}.$$ 

Consequently, putting (4), (5) and (6) together, we obtain

$$\max \left\{0, 2 \limsup_{\|u\| \to +\infty} \frac{J_f(u)}{\|u\|^2}, \frac{\langle J'_f(u), u \rangle}{\|u\|^2} \right\} \leq \lambda_\alpha^{-1} \max \left\{0, 2 \limsup_{|\xi| \to +\infty} \frac{J_f(\xi)}{\xi^2}, \limsup_{|\xi| \to +\infty} \frac{J_f(\xi)}{\xi} \right\}$$

$$< \lambda_\alpha^{-1} \sup_{\mathbb{R}} f' \leq \sup_{(u,v) \in H_0^1(\Omega) \times (H_0^1(\Omega) \setminus \{0\})} \frac{\int_\Omega \alpha(x)f'(u(x))|v(x)|^2 dx}{\|v\|^2}.$$ 

Therefore, we can apply Theorem 2 taking $X = H_0^1(\Omega)$ and $J = J_f$. Therefore, for every $\lambda$ satisfying

$$\frac{\lambda_\alpha}{\sup_{\mathbb{R}} f'} < \lambda < \max \left\{0, 2 \limsup_{|\xi| \to +\infty} \frac{J_f(\xi)}{\xi^2}, \limsup_{|\xi| \to +\infty} \frac{f'(\xi)}{\xi} \right\},$$

the set $A_\lambda$ (defined in Theorem 2) is closed and not $\sigma$-compact in $H_0^1(\Omega)$. But, clearly, a $u \in H_0^1(\Omega)$ belongs to $A_\lambda$ if and only if the problem

$$\begin{cases} -\Delta v = \lambda \alpha(x)f'(u(x))v & \text{in } \Omega \\ v_{|\partial\Omega} = 0 \end{cases}$$

has a non-zero weak solution, and the proof is complete. \(\triangle\)

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