Hermitian Positive Semidefinite Matrices Whose Entries Are 0 Or 1 in Modulus

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Abstract
We show that a matrix is a Hermitian positive semidefinite matrix whose nonzero entries have modulus 1 if and only if it similar to a direct sum of all 1's matrices and a 0 matrix via a unitary monomial similarity. In particular, the only such nonsingular matrix is the identity matrix and the only such irreducible matrix is similar to an all 1's matrix by means of a unitary diagonal similarity. Our results extend earlier results of Jain and Snyder for the case in which the nonzero entries (actually) equal 1. Our methods of proof, which rely on the so called principal submatrix rank property, differ from the approach used by Jain and Snyder.

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1 Introduction

In this note we characterize the set of all Hermitian positive semidefinite matrices $A$ whose entries have modulus 1 or 0. We comment that the real matrices in this set whose diagonal entries are all nonzero, and hence necessarily 1, belong to the set of correlation matrices (see Horn and Johnson [1, p.400]).

Our characterization here is motivated by a surprising result due to Jain and Snyder [5] which can be stated as follows:

**THEOREM 1.1** (Jain and Snyder [5, Theorems 2 and 3]) If $A$ is any positive semidefinite $(0,1)$-matrix, then $A$ is permutationally similar to a direct sum of matrices each of which is either an all $1$’s matrix or a zero matrix. In particular, if $A$ is (also) irreducible, then $A$ is the $n \times n$ all $1$’s matrix.

Jain’s and Snyder’s proof of Theorem 1.1 rests on their observation that any positive semidefinite $(0,1)$-matrix has a Cholesky factorization with a $(0,1)$-Cholesky factor. The proof of our generalization of Theorem 1.1 relies on the following property which is possessed by Hermitian positive semidefinite matrices and from which results on Cholesky factorizations follow:

**DEFINITION 1.2** A matrix $A \in \mathbb{C}^{n \times n}$ is said to have the principal submatrix rank property (PSRP) if the following conditions holds:

(i) The column space determined by every set of rows of $A$ is equal to the column space of the principal submatrix lying in these rows.

(ii) The row space determined by every set of columns of $A$ is equal to the row space of the principal submatrix lying in these columns.

It is known (cf. Hershkowitz and Schneider [2, 3] and Johnson [4]) that a positive semidefinite Hermitian matrix has PSRP. We give the following simple proof for the sake of completeness:

Since a permutation similarity applied to a positive semidefinite Hermitian matrix again yields a positive semidefinite Hermitian matrix, it is enough to show that the row space determined by the first $k$ columns of a
positive semidefinite Hermitian matrix $A$ is equal to the row space determined by the leading principal minor $B$ of $A$ of size $k$. This is equivalent to showing the following: If $v^T = [w, 0]^T$, where $w$ is of length $k$, and $Bw = 0$ then $Av = 0$. But, for such $v$, we have $v^*Av = w^*Bw = 0$, and the result follows since $A$ is positive semidefinite.

2 Main Result

To facilitate our extension of Theorem 1.1, we introduce the following notion:

**DEFINITION 2.1** A matrix $P \in \mathbb{C}^{n,n}$ is called a unitary monomial matrix if $P = QD$, where $Q$ is a permutation matrix and $D$ is a diagonal matrix all of whose diagonal entries are of modulus 1.

We are now ready to state the main result of this note:

**THEOREM 2.2** A matrix $A \in \mathbb{C}^{n,n}$ is Hermitian positive semidefinite and all its entries have modulus 1 or 0 if and only if $A$ is similar, by means of a unitary monomial matrix, to a direct sum of matrices each of which is either an all 1’s matrix or a zero matrix.

**Proof:** The proof of the “if” part is obvious, so we proceed to prove the “only if” part. This is done by induction on $n$, the size of $A$. The result is trivial if $n = 1$. So let $n > 1$ and assume theorem holds for matrices of all sizes less than $n$.

If $A$ is reducible, then $A$ is permutation similar to a direct sum of positive semidefinite Hermitian matrices of size less than $n$ whose nonzero entries are all of modulus 1. By our inductive assumption each direct summand is unitarily monomially similar to the direct sum of all 1’s matrices and a 0 matrix, and hence the same is true for $A$.

So assume that $A$ is irreducible. We shall first show that there exists $(0,1)$-matrix $E = D^{-1}P^{-1}APD$, where $D$ is diagonal and $P = (p_{i,j})$ is a unitary monomial matrix with $p_{n,n} = 1$. Further all elements of the last row and column of $E$ are 1.

No diagonal element of $A$ is 0, for then the corresponding row and column would also be 0. Since the leading submatrix $B$ of size $n - 1$ of $A$
is positive semidefinite, it follows there is a unitary monomial similarity of $B$ such that the resulting matrix is a direct sum of all 1’s matrices and a 0 matrix. We extend this similarity to a unitary monomial similarity of $A$ which leaves the last row and column of $A$ in place. We thus obtain a matrix $C = P^{-1}AP$, where $p_{n,n} = 1$, such that all nonzero elements in the last row and column of $C$ are of modulus 1 and the leading submatrix of size $n - 1$ of $C$ is a direct sum of all 1’s matrices and a 0 matrix. We partition the last row and column of $C$ in conformity with this direct sum. Since $C$ is positive semidefinite, it follows by PSRP that each subvector of the last row and column determined by this partition is a multiple of the all 1’s vector of the appropriate size by a number of modulus 1 or by 0. But if one of the last column is a 0 multiple of the all 1’s vector, then $C$ reducible. Hence each subvector of the last column is a multiple of an all 1’s vector by a number of modulus 1. Let $D$ be the unitary diagonal matrix whose diagonal entries coincide with the last column of $C$. Since $D$ has equal entries corresponding to the blocks of $B$, it follows that $E = D^{-1}CD$ is a $(0, 1)$–matrix whose last row and column consists of 1’s.

Our proof shows that the last row and column of $A$ have no 0 entries. By applying permutation similarities to $A$ and repeating the above construction, we deduce that the same is true of every row and column. Hence the matrix $E$ we have obtained is the all 1’s matrix.

\[ \square \]

Theorem 2.2 has several corollaries:

**COROLLARY 2.3** A matrix $A \in \mathbb{C}^{n,n}$ is a positive definite Hermitian matrix whose nonzero entries are all of modulus 1 if and only if $A$ is the identity matrix.

**COROLLARY 2.4** A matrix $A \in \mathbb{C}^{n,n}$ is an irreducible positive semidefinite Hermitian matrix whose nonzero entries are all of modulus 1 if and only if $A$ can be transformed to the all 1’s matrix by a unitary diagonal similarity.

**COROLLARY 2.5** Let $A \in \mathbb{C}^{n,n}$ be a positive semidefinite Hermitian matrix whose nonzero entries are all of modulus 1. Then there is an LU factorization of $A$ with $L$ nonsingular where $L$ and $U$ are similar to $(0, 1)$–matrices via the same unitary diagonal matrix.
Proof: If $C$ is the $k \times k$ block of all 1’s then it admits the factorization $C = L_1 U_1$, where the first column of $L_1$ consists of 1’s, the diagonal entries of $L_1$ are 1, the first row of $U_1$ consists of 1’s and all other elements of $L_1$ and $U_1$ are 0. Now, if $C$ is a direct sum of such blocks and a 0 matrix, it easily follows that $C$ admits an LU factorization where $L_2$ is a lower triangular nonsingular $(0,1)$–matrix and $U_2$ is an upper triangular $(0,1)$–matrix. If $B$ is permutation similar to $C$ then we can find a permutation matrix $P$ which does not change the order of the order of rows and columns in any given block and for which $B = P^T C P$. Then $L_3 = P^T L_2 P$ is a lower triangular nonsingular $(0,1)$–matrix and $U_3 = P^T U_2 P$ is an upper triangular $(0,1)$–matrix. If $A = D^* B D$, where $D$ is a unitary diagonal matrix, it follows that $L = D^* L_3 D$ and $U = D^* U_3 D$ satisfy the conditions of the corollary. The conclusion of the corollary now follow by applying Theorem 2.2. \(\blacksquare\)

In a very similar way to the proof of the above corollary, we can prove the following corollary:

**COROLLARY 2.6** Let $A \in \mathbb{C}^{n \times n}$ be a positive semidefinite Hermitian matrix whose nonzero entries are all of modulus 1. Then there is an Cholesky $LL^*$ factorization of $A$ where $L$ is similar to $(0,1)$–matrices via a unitary diagonal matrix.

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