Finsleroid gives rise to the angle-preserving connection

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Abstract

The Finslerian unit ball is called the *Finsleroid* if the covering indicatrix is a space of constant curvature. We prove that Finsler spaces with such indicatrices possess the remarkable property that the tangent spaces are conformally flat with the conformal factor of the power dependence on the Finsler metric function. It is amazing but the fact that in such spaces the notion of the two-vector angle defined by the geodesic arc on the indicatrix can readily be induced from the Riemannian space obtained upon the conformal transformation, which opens up the straightforward way to induce also the connection coefficients and the concomitant curvature tensor. Thus, we are successfully inducing the Levi-Civita connection from the Riemannian space into the Finsleroid space, obtaining the isometric connection. The resultant connection coefficients are not symmetric. However, the metricity condition holds fine, that is, the produced covariant derivative of the Finsleroid metric tensor vanishes identically. The particular case underlined by the axial Finsleroid of the $FF^{PD}_g$-type is explicitly evaluated in detail.

**Keywords:** Finsler metrics, connection, curvature, conformal properties.
1. Introduction and motivation

What matter makes the Levi-Civita connection canonical in the Riemannian geometry? The angle-parallelism property is the matter, namely that the angle of two vectors is left unchanged under the parallel transportation of the vectors with that connection. Can the property be put ahead also in the Finsler geometry?

In any tangent space $T_x M$ of a given Finsler space $F^N$ introduced on an $N$-dimensional differentiable manifold $M$ of points $x \in M$, the angle $\alpha_{\text{Finslerian}}(x, y_1, y_2)$ between two tangent vectors $y_1, y_2 \in T_x M$ can be defined by means of length of the respective geodesic arc $A(x, l_1, l_2)$ which joins the intersection points of the directions of vectors $y_1, y_2 \in T_x M$ with the indicatrix $I_x \subset T_x M$ supported by the point $x \in M$ and belongs to the indicatrix; $l_1 = y_1/F(x, y_1)$ and $l_2 = y_2/F(x, y_2)$ are respective unit vectors, and $F$ denotes the Finsler metric function of the space $F^N$. With such an angle, it can be hoped to prolong the Levi-Civita connection from the Riemannian geometry to the Finsler geometry to arrive at the Finsler notion of the parallel connection with just the same meaning as the connection notion that faces us in the Riemannian geometry. Namely, the parallel connection operates such that under the parallel transportation generated by the connection the vectors $y_1, y_2 \in T_x M$ are transported in the parallel way and the length of the arc $A(x, l_1, l_2)$ does not change. However, the hard analytical difficulties can arise that the dependence of $\alpha_{\text{Finslerian}}(x, y_1, y_2)$ on $y_1, y_2$ cannot be written in an explicit form, except for (rare?) particular cases of the space $F^N$. The desired prolongation may exist but not be explicit! Various interesting investigations of the connection and angle can be found in the Finsler geometry literature (see, for example, [1-9]).

Below, we come to the Finsleroid-produced parallel transports which do not change neither the norms of vectors nor the Finsleroid-produced angles between vectors. Therefore, they are isometric: the produced displacements of the tangent spaces along curves of the underlined manifold $M$ are isometries. In particular, they keep the Finsleroid indicatrices into the Finsleroid indicatrices. At the same time, in contrast to the Levi-Civita connection of the Riemannian geometry, the Finsleroid-produced parallel transports are not linear in general, namely, the connection coefficients processing that transports depend on tangent vectors $y \in M$ in a nonlinear way, except for possible particular cases. The entailed connection coefficients are not constructive from the Finslerian metric tensor and the first derivatives of the tensor — they are obtained from the parallel transportation of the two-vector angle.

We shall construct the angle with the help of the indicatrix arcs as follows.

The embedded position of the indicatrix $I_x \subset T_x M$ in the tangent Riemannian space $\{T_x M, g^{(F)}(x, y)\}$ (where $g^{(F)}(x, y)$ denotes the Finslerian metric tensor with $x$ considered fixed and $y$ used as being the variable) induces the Riemannian metric on the indicatrix through the well-known method (see, e.g., Section 5.8 in [1]) and in this sense makes the indicatrix a Riemannian space. Let $U_x$ be a simply connected and geodesically complete region on the indicatrix $I_x$ supported by a point $x \in M$. Any point pair $u_1, u_2 \in U_x$ can be joined by the respective arc $A(x, l_1, l_2) \subset I_x$ of the Riemannian geodesic line drawn on $U_x$. By identifying the length of the arc with the angle notion we arrive at the $\text{geodesic-arc angle}$ $\alpha_{\{x\}}(y_1, y_2)$, where $y_1, y_2 \in T_x M$ are two vectors issuing from the origin $0 \in T_x M$ and possessing the property that their direction rays $0y_1$ and $0y_2$ intersect the indicatrix at the point pair $u_1, u_2 \in U_x$. Denoting the respective Riemannian length of the geodesic arc $A(x, l_1, l_2)$ by $s$, we obtain

$$\alpha_{\{x\}}(y_1, y_2) = s.$$  \hspace{1cm} (1.1)
With this angle notion, we can naturally introduce the scalar product:

\[
< y_1, y_2 >_{(x)} = F(x, y_1)F(x, y_2)\alpha_{(x)}(y_1, y_2).
\]

We may use geodesic-arc angle to construct the geodesic-arc sector \( S_{(x)}(l_1, l_2) \subset T_x M \), where \( l_1, l_2 \in T_x M \) are two unit vectors issued from the origin \( 0 \in T_x M \) (and pointed to the indicatrix). The sector said is the surface swept by the unit vector \( l \in T_x M \) when its end runs along the geodesic arc from the point \( u_1 \) to the point \( u_2 \).

Let us call the arc-curve joining the end points \( u_1 \) and \( u_2 \) of the sector the arc-side of the sector. When a point moves along an arc-side parameterized by the geodesic length \( s \), we obviously have

\[
(ds)^2 = g_{mn}dl^m dl^n, \quad (1.2)
\]

which can be written as

\[
g_{mn} \frac{dl^m}{ds} \frac{dl^n}{ds} = 1. \quad (1.3)
\]

Each sector \( S_{(x)}(l_1, l_2) \) can naturally be coordinatized by the pair \( z^1, z^2 \) with

\[
z^1 = F, \quad z^2 = s, \quad (1.4)
\]

where \( F \) is the value of the Finslerian metric function, and \( s \) is the length of the geodesic arc which goes from the ray direction of the left vector \( l_1 \) to the consideration point. Let us consider the embedding \( y = y(z^1, z^2) \) of the sector \( S_{(x)}(l_1, l_2) \) in the tangent Riemannian space \( \{ T_x M, g^{(F)}(y, x) \} \). Projecting the tensor \( g^{(F)} \) on the sector gives rise to the intrinsic metric tensor, to be denoted by \( i_S \). The tensor \( i_S \) has the components: \( \{ i_{11}, i_{12} = i_{21}, i_{22} \} \).

In terms of a local coordinate system \( \{ x^m \} \), we have \( y^m = y^m(z^1, z^2) \) and

\[
i_{11} = g_{mn}y_1^m y_1^n, \quad i_{12} = g_{mn}y_1^m y_2^n, \quad i_{22} = g_{mn}y_2^m y_2^n, \quad (1.5)
\]

where \( y_1^m = \partial y^m / \partial z^1 \) and \( y_2^m = \partial y^m / \partial z^2 \), and \( g_{mn} \) are the Finslerian metric tensor components. The equality \( y^m = F y^m \) just entails that \( y_1^m = l^m \) and, therefore, \( i_{11} = 1 \).

Also, \( l_2 y_2^n = 0 \) (because \( l_2 \) is the gradient vector \( \partial F / \partial y^m \) and \( F \) is independent of \( s \)), which makes us conclude that \( i_{12} = 0 \). Finally, noting that in the case of an arc intersecting the ray at the distance \( F \) from the origin \( 0 \in T_x M \) the equality (1.3) must be modified to read \( F^2 g_{mn}(dy^m/ds)(dy^n/ds) = 1 \), we arrive at \( g_{mn}y_2^m y_2^n = 1 \).

Thus we have observed that the coordinates (1.4) introduce the orthogonal coordinate system on the geodesic-arc sectors and the intrinsic metric tensor of the geodesic-arc sector is Euclidean:

\[
i_{11} = 1, \quad i_{12} = 0, \quad i_{22} = 1. \quad (1.6)
\]

In this sense, each geodesic-arc sector is a Euclidean space.

This observation just entails that in any Finsler space the area \( ||S_{(x)}(l_1, l_2)|| \) of the geodesic-arc sector is presented by the formula

\[
||S_{(x)}(l_1, l_2)|| = \frac{1}{2} \left( \alpha_{(x)}(y_1, y_2) \right)^2 \quad (1.7)
\]

which is faithfully valid in all the Riemannian as well as Finslerian spaces.

Our underlined idea is to use the geodesic-arc angle to generate the notion of the angle-preserving connection. We shall always assume that the connection is metrical. Each respective Finsleroid-produced parallel transport along a curve of the underlined manifold \( M \) is an isometry for the involved angles and, therefore, for the involved geodesic-arc sectors.
We shall develop the idea by specifying the Finsler space as follows.

If an \( N \)-dimensional Finsler space \( \mathcal{F}^N \) is such that the indicatrices of the space possess the property of constant curvature (with respect to the Riemannian metric induced from the tangent Riemannian space \( \{ T_x M, g^{(F)}(x, y) \} \)), we call \( \mathcal{F}^N \) the Finsleroid-Finsler space, denote the fundamental metric function of the space by

\[
K = K(x, y),
\]

and apply the following definitions.

**Definition.** Within any tangent space \( T_x M \), the metric function \( K(x, y) \) produces the Finsleroid

\[
\mathcal{F}_{(x)} := \{ y \in \mathcal{F}_{(x)} : y \in T_x M, K(x, y) \leq 1 \}.
\]

**Definition.** The Finsleroid Indicatrix

\[\mathcal{I}\mathcal{F}_{(x)} \subset T_y M\]

is the boundary \( \partial \mathcal{F}_{(x)} \) of the Finsleroid, that is,

\[
\mathcal{IF}_{(x)} := \{ y \in \mathcal{I}\mathcal{F}_{(x)} : y \in T_y M, K(x, y) = 1 \}.
\]

The strong convexity and the positive homogeneity are assumed.

Let us also assume that the manifold \( M \) can be endowed with a Riemannian metric tensor, \( a_{mn}(x) \) in terms of local coordinates \( \{ x^m \} \). Then in addition to the Finsleroid indicatrix (1.11) we can bring to consideration the sphere obtainable from the associated Riemannian space

\[
\mathcal{R}^N := \{ M, a_{mn}(x) \},
\]

in accordance with

**Definition.** The sphere

\[
\mathcal{S}_{(x)} \subset T_x M
\]

is defined by

\[
\mathcal{S}_{(x)} := \{ y \in \mathcal{S}_{(x)} : y \in T_x M, a_{mn}(x)y^m y^n = 1 \}.
\]

Our input stipulation is that the Finsler space \( \mathcal{F}^N \) be conformally isomorphic to the Riemannian space:

\[
\mathcal{F}^N = \mathcal{X} \cdot \mathcal{R}^N
\]

in accordance with see (2.1)-(2.3). We shall see that such Finsler spaces are the Finsleroid-type spaces.

In Section 2 the basic theorems are sketched to enlighten the numerous remarkable properties which are shown by such Finsler spaces. It proves that the \( \mathcal{X} \)-transformation (1.15) must involve the conformal factor which is of a power dependence on the Finsler metric function, as shown by (2.5). In such Finsler spaces \( \mathcal{F}^N \), we have universally the ratio

\[
\frac{\text{Area of the Finsleroid geodesic-arc sector}}{\text{Area of the Riemannian geodesic-arc sector}} \bigg|_{x \in M} = \frac{1}{h^2(x)}
\]
(at any admissible data of the leg-vectors $y_1, y_2$ of the sector), where the denominator relates to the space $\mathcal{R}^N$ and $h^2(x)$ is the value of curvature of the indicatrix supported by point $x$ (see (2.6)).

By inducing the angle $\alpha_{\text{Finslerian}} = \kappa \cdot \alpha_{\text{Riemannian}}$ we obtain the remarkable equality

$$\alpha_{\text{Finsleroid space}} = \frac{1}{h} \alpha_{\text{Riemannian space}} \quad (1.17)$$

(see (2.6) and (2.7)).

At each point $x \in M$, the ratio $\text{AREA}_{\text{Finsleroid indicatrix}} / \text{VOLUME}_{\text{Finsleroid}}$ proves to be of the universal value in each dimension $N$ (is independent of $h$), so that the ratio is exactly the same as it holds in the Riemannian limit (that is, when $h = 1$). This property is lucidly described by the formulas (3.15)-(3.20).

In Sections 3 and 4 we explain how the sought connection coefficients and the curvature tensor are obtainable on the basis of the transformation (1.15), provided $h = \text{const}$. The connection coefficients involve the $\kappa$-transition functions, not being obtainable from the Finslerian metric tensor and the first derivatives of the tensor in any algebraic manner. The connection coefficients are not symmetric. They are non-linear in general regarding the $y$-dependence. However, the metricity, that is, the condition that the covariant derivative of the Finslerian metric tensor be the nought (see (2.12)), can well be fulfilled. The method is to postulate the transitivity of the covariant derivative under the $\kappa$-transformation. Analytically, the transitivity reads (4.9), and is fulfilled when the vanishing (4.4)-(4.5) is postulated. The angle (1.17) is preserved under respective covariant displacements:

$$d\alpha_{\text{Finsleroid space}} = 0, \quad \text{if} \quad h = \text{const}, \quad (1.18)$$

where $d = dx^i d_i$ and $d_i \alpha_{\text{Finsleroid}}$ symbolizes the left-hand part of (3.25).

In this way we are quite able to successfully construct in the Finsleroid space $\mathcal{F}^N$ the angle-preserving connection, to be denoted by $\mathcal{FC}$, by adhering faithfully at the method

$$\\{ \text{The Finsleroid-space connection } \mathcal{FC} \} = \kappa \cdot \{ \text{The Levi-Civita connection } \mathcal{LC} \}, \quad (1.19)$$

where $\mathcal{LC}$ is the canonical connection in the associated Riemannian space (1.13) (the connection coefficients of the $\mathcal{LC}$ are the Christoffel symbols $a^m_{\ ij} = a^m_{\ ij}(x)$ constructed from the Riemannian metric tensor $a_{mn}(x)$, in accordance with (3.22)).

In Section 5, we fix a tangent space and consider the tensor $k_{ij}$ obtainable from the tensor $g_{ij}$ by performing the conformal transformation. We observe that the property of vanishing the curvature tensor produced by $k_{ij}$ is arisen upon fulfilling a simple ODE, which can explicitly be solved to establish the theorem 2.2.

In Section 6, we confine the consideration to the $\mathcal{PF}^\mathcal{PD}$-space, in which the Finsleroid is of the axial type; $g$ denotes the characteristic parameter; the upperscript $\{ \mathcal{PD} \}$ means positive-definite. The connection coefficients are found explicitly. They depend on vectors $y$ in a non-linear way in dimensions $N \geq 3$. In the dimension $N = 2$, however, the connection is linear. The structure of the appeared curvature tensor $\rho_{k\ n\ ij}$ has been elucidated, resulting in the explicit representation (6.77)-(6.78). The square of the tensor is given by the simple formula (6.79).

Since the formula $h = \sqrt{1 - (1/4)g^2}$ is applicable in the $\mathcal{PF}^\mathcal{PD}$-Finsleroid space, from the universal law (1.16) we may conclude that
The angle measured by the lengths of geodesic arcs on the indicatrix was found for the $\mathcal{F}g$-space in the work \[7,8\]. The underlying idea was to derive the angular measure from the solutions to the respective geodesic equations. The solutions have been derived in simple explicit forms. This angle given by the formula (6.30) can also be obtained from the solutions to the respective geodesic equations. The solutions have been derived in Section 6. The involved preferred vector field $b(x)$ as well as the opposed vector prove to be the proper elements of the $\kappa$-transformation (see (6.28)).

Our evaluations will everywhere be of local nature. However, there exists a simple possibility to elucidate the global structure of the $\mathcal{F}g$-Finsleroid indicatrix. Indeed, in the $\mathcal{F}g$-Finsleroid space the desired $\kappa$-transformation (1.15) can be explicitly given by means of the substitution $\zeta^i = \zeta^i(x, y)$ indicated in (6.26). Inserting these $\zeta^i$ in the associated Riemannian metric $S(x, \zeta) = \sqrt{a_{mn}(x)\zeta^m\zeta^n}$ entails the equality (3.7), thereby producing the metric function $K(x, y)$ of the $\mathcal{F}g$-Finsleroid space. Let us consider the sphere $S^{[\{i\}} \subset T_xM$ in terms of the variables $\zeta$: $S^{[\{i\}} := \{\zeta \in S_{\{x\}}^{[\{i\}} : \zeta \in T_xM, S(x, \zeta) = 1\}$. The equality (3.7) manifests that the transformation (6.26) maps regions of the $\mathcal{F}g$-Finsleroid indicatrix in regions of the sphere $S^{[\{i\}}_{\{x\}}$. Also, the direction of the Finsleroid-axis vector $b^i$ as well as the direction of the opposed vector $-b^i$ are left invariant under the transformation (6.26) (see (6.28) and (6.29)).

Denoting by $\zeta^{(\text{North})}$, resp. by $\zeta^{(\text{South})}$, the point which is obtained in intersection of the direction of $b^i$, resp. by $-b^i$, with the sphere, we can consider the pointed spaces

$$S^{(+)}_{\{x\}} = S^{[\{i\}}_{\{x\}} \setminus \zeta^{(\text{South})}, \quad S^{(-)}_{\{x\}} = S^{[\{i\}}_{\{x\}} \setminus \zeta^{(\text{North})},$$

in which the south pole, resp. the north pole, is deleted. The regions $C^{(+)}_{g;\{x\}} = \kappa^{-1}S^{(+)}_{\{x\}}$ and $C^{(-)}_{g;\{x\}} = \kappa^{-1}S^{(-)}_{\{x\}}$ may be used to yield two covering charts for the indicatrix; they can be characterized by the angle ranges indicated in (6.22). Similarly to the Riemannian case proper, we need two charts to cover the indicatrix. In its sense and role, the south chart $C^{(-)}_{g;\{x\}}$ is entirely similar to the ordinary Euclidean chart obtainable by means of the so-called stereographic projection. Then it can readily be seen that the substitution $\zeta^i = \zeta^i(x, y)$ indicated in (6.26) acts diffeomorphically on each the chart:

$$C^{(+)}_{g;\{x\}} \overset{\kappa}{\longleftrightarrow} S^{(+)}_{\{x\}}, \quad C^{(-)}_{g;\{x\}} \overset{\kappa}{\longleftrightarrow} S^{(-)}_{\{x\}}.$$  

Thus, for the $\mathcal{F}g$-Finsleroid space the $\kappa$-transformation and, therefore, the representations of the connection coefficients and the curvature tensor obtained in Section 6, are meaningful globally regarding the $y$-dependence.

We are also entitled to say that the $\mathcal{F}g$-Finsleroid indicatrix is globally isometric to the Euclidean sphere of the radius $r = 1/h \equiv 1/\sqrt{1-(1/4)g^2}$.

Below when mentioning the $\mathcal{F}g$-Finsleroid space, we shall imply that we work on the upper regions $C^{(+)}_{g;\{x\}}$, unless otherwise stated explicitly. The development of extensions to regions of $C^{(-)}_{g;\{x\}}$ is, of course, a straightforward task.

In the Euclidean and Riemannian geometries, an important role is played by the spherical coordinates. Their use enables one to conveniently represent vectors, evaluate
squares and volumes, study curvature of surfaces, in many cases simplify consideration and solve rigorously equations, and also introduce and use various trigonometric functions. In the context of the $\mathcal{F}\mathcal{F}^P$-space theory, such coordinates can readily be arrived at. In the three-dimensional case, $N = 3$, they are given by (7.2), entailing the convenient representation (7.11) for vectors as well as the generalized trigonometric functions indicated in (7.12). The respective squared linear element $ds^2$ has been explicitly evaluated to read (7.14) which lucidly manifests the conformal-flat nature of tangent spaces as well as the validity of the key formulas (2.5)-(2.7). With the help of these coordinates, the equations for the arc $\mathcal{A}(x, l_1, l_2)$ can explicitly be integrated in the convenient form (7.26)-(7.27).

In Appendix A, the basic representations of objects of the $\mathcal{F}\mathcal{F}^P$-space are summarized. In Appendix B, the involved connection coefficients are evaluated. In Appendix C, many steps of calculation of the curvature tensor are presented. In Appendix D, we evaluate the coefficients which enter the transformation of the curvature tensor into the Riemannian space. In the last Appendix E, we show the explicit representation of components for the metric tensor of the $\mathcal{F}\mathcal{F}^P$-space in the fixed tangent space.

We are interested mainly in spaces of the dimension $N \geq 3$. The two-dimensional case has been studied in the preceding work [9].

2. Synopsis of main assertions

We start with the following idea of specifying the notion of a Finsler space.

INPUT STIPULATION. A Finsler space $\mathcal{F}^N$ is \textit{conformally isomorphic} to the Riemannian space $\mathcal{R}^N$:

$$\mathcal{F}^N = \mathcal{K} \cdot \mathcal{R}^N : \{g_{mn}(x,y)\} = \mathcal{K} \cdot \{t_{mn}(x,y)\} \quad \text{with} \quad t_{mn}(x,y) = k^2(x,y)a_{mn}(x), \quad (2.1)$$

where it is assumed that the applied $\mathcal{K}$-transformation does not influence any point $x \in M$ of the base manifold $M$. It is also natural to require that the $\mathcal{K}$-transformation sends unit vectors to unit vectors:

$$\mathcal{I}_x \mathcal{F} = \mathcal{K} \cdot \mathcal{S}_x. \quad (2.2)$$

It looks interesting to specify the conformal multiplier to be an algebraic function of the fundamental Finsler metric function $K = K(x,y)$ used in $\mathcal{F}^N$, so that

$$k = \mathcal{K}(x,K). \quad (2.3)$$

The smoothness of class $C^5$ regarding the $y$-dependence, and of class $C^4$ regarding the $x$-dependence, is necessary to require from the $\mathcal{K}$-transformations.

Under the above stipulation, the tangent spaces to the Finsler space $\mathcal{F}^N$ are conformally flat. At the same time, the space $\mathcal{F}^N$ is not conformal to any Riemannian space, unless $\partial k/\partial K = 0$. In terms of local coordinates, the stipulation (2.1) is described by the formulas (3.1)-(3.5).

In Finsler spaces $\mathcal{F}^N$ fulfilling the conditions (2.1)-(2.3) we can measure the angle by the conventional value $\alpha_{\text{Riemannian}}$ used in the Riemannian space $\mathcal{R}^N$ and obtain in $\mathcal{F}^N$ the induced angle

$$\alpha_{\text{Finslerian}} = \mathcal{K} \cdot \alpha_{\text{Riemannian}}. \quad (2.4)$$

Attentive calculations performed in Section 5 result in the following
Theorem 2.1. The claimed conditions (2.1)-(2.3) are realized if and only if the function $\varkappa$ is taken to be

$$\varkappa = \frac{1}{h}K^{1-h} \quad \text{with} \quad h = h(x),$$

where $h$ is a positive scalar.

The theorem is compatible with the homogeneity, namely we adopt

**Homogeneity condition.** Action of the $\varkappa$-transformation (2.1) on tangent vectors possesses the property of the positive homogeneity of degree $h$.

The formulas (3.3) and (3.4) proposed in Section 3 yield the analytical representation to the last condition.

Also, the following theorem can be obtained (see (5.6)).

**Theorem 2.2.** Under the conditions formulated in the preceding theorem, the indicatrix supported by a point $x \in M$ is of the constant curvature $h^2(x)$, such that

$$\mathcal{R}_{\text{Finsleroid Indicatrix}} = C(x) \quad \text{with} \quad C(x) = h^2(x).$$

The space of a constant curvature $C$ is realized on the sphere of the radius $r = 1/\sqrt{C}$.

Therefore, since we adhere at measuring the angle $\alpha$ by the geodesic arc-length on the indicatrix, from the above formulas (2.1)-(2.6) we are entitled to conclude the following.

**Theorem 2.3.** The simple property

$$\alpha_{\text{Finsleroid space}} = \frac{1}{\sqrt{C}}\alpha_{\text{Riemannian space}}$$

is valid for the angle.

To surely recognize the validity of this theorem, it is sufficient to take a glance on the equality (3.9) which represents infinitesimally the squared length of the geodesic arc on the indicatrix.

Inverting the theorem 2.2 can be justified by taking into account the derived formulas (5.4)-(5.8), namely the following theorem is fulfilled.

**Theorem 2.4.** If the indicatrix supported by a point $x \in M$ is a space of constant curvature, then the conformal property (2.1) holds at the point $x$.

The sought Finsleroid connection

$$\mathcal{F}C = \{N_j^k, D_{ij}^k\}$$

involves the coefficients $N_j^k = N_j^k(x,y)$ and $D_{ih}^k = D_{ih}^k(x,y)$ which are required to construct the operator

$$d_i = \frac{\partial}{\partial x^i} + N_i^k \frac{\partial}{\partial y^k},$$

and the covariant derivative $\mathcal{D}_i$ which action in the Finsleroid space $\mathcal{F}^N$ is exemplified in the conventional way:

$$\mathcal{D}_i w^m_n = d_i w^m_n + D_{ih}^n w^h_m - D_{im}^h w^n_h,$$
where \( w^n_m = w^n_m(x, y) \) is an arbitrary differentiable (1,1)-type tensor. In Section 4, we subject the connection \( \mathcal{FC} \) to the condition that the covariant derivative \( \mathcal{D}_i \) obeys the transitivity rule (4.9) and that the \( h \) entering (2.5) is independent of the points \( x \in M \), so that \( h = \text{const.} \) These conditions result in the following assertion.

**Theorem 2.5.** When \( h = \text{const.} \), the vanishing set

\[
\mathcal{D}_i K = 0, \quad \mathcal{D}_i y^j = 0, \quad \mathcal{D}_i y_j = 0,
\]

and the metricity

\[
\mathcal{D}_i g_{jn} = 0
\]

hold, when the relations

\[
D^k_{~in} = -\frac{\partial N^k_i}{\partial y^n}, \quad N^j_i = -D^j_{~ik}y^k
\]

are used and the coefficients \( N^k_i \) are constructed in accordance with the explicit formulas (3.30)-(3.32). The angle-preserving property (1.18) is entailed.

The curvature tensor can be explicited from the commutator of the covariant derivative (2.10), according to (4.12)-(4.18).

All the above formulas are explicitly (and brightly!) realized in the \( \mathcal{FF}PD \)-Finsleroid space (Section 6), in which the metric function can conveniently be introduced by the representation

\[
K(x, y) = \sqrt{B(x, y)} e^{-(1/2)g(x)\chi(x, y)},
\]

where the formulas (6.1)-(6.6) are of value. The scalar \( \chi \) thus appeared possesses the lucid geometrical meaning of the azimuthal angle measured from the direction assigned by the input vector \( b^i(x) \) (see (6.14)).

The metric function of the Finsleroid type has been first appeared in the paper [10], at but the Minkowskian level. Namely, the consideration in [10] was subjected to the following assumptions. (Z1) The sought metric function \( \tilde{K}(y) \) is positively homogeneous and smooth locally of at least class \( C^4 \). (Z2) The indicatrix of \( \tilde{K}(y) \) is a surface of revolution, say around the direction of the \( N \)-th component \( y^N \) of the tangent vector \( y \), in which case it is convenient to introduce the representation \( \tilde{K}(y) = y^NV(w) \), where the generating metric function \( V \) depends on a single argument \( w \). (Z3) The induced Riemannian curvature on the indicatrix is of a constant-curvature type. Treating the condition to be a differential equation to find the function \( V \), we can arrive after straightforward calculations (which are not short) to the ODE (which is non-linear and of the second order) which governs the \( V \). It is a bit surprising but the fact that the ODE can explicitly be resolved at a local level to specify the function \( V \), we can arrive after straightforward calculations (which are not short) to the ODE (which is non-linear and of the second order) which governs the \( V \). It is a bit surprising but the fact that the ODE can explicitly be resolved at a local level to specify the function \( V \). (Z4) The obtained function \( V = V(w) \) should obey the requirement that the entailed Finslerian metric tensor be positive definite. The final condition is (Z5): The indicatrix is closed and regular. The function \( \tilde{K}(y) \) obtainable in this way, after fulfilling all the conditions (Z1)-(Z5), is just the Minkowskian version of the \( \mathcal{FF}PD \)-Finsleroid metric function \( K \) given by the formulas (6.1)-(6.5).

Thus, from the standpoint of the indicatrix geometry, the \( \mathcal{FF}PD \)-space occupies a unique position in the class of the Finsler spaces specified by the condition that the Finsler metric function \( K = K(x, y) \) be of the functional dependence

\[
K = \Phi(g(x), b_i(x), a_{ij}(x), y),
\]
where \( g(x) \) is a scalar, \( b_i(x) \) is a covariant vector field, and \( a_{ij}(x) \) is a Riemannian metric tensor. No Finsler metric function allowing for the representation (2.15) can meet the requirements that the entailed indicatrix is of constant curvature and the smoothness class \( C^k \) is attained at \( k \geq 3 \) regarding the global \( y \)-dependence. Admitting the class \( C^2 \) results uniquely in the \( FFFg^{PD} \)-Finsleroid metric function \( K \) given by the formulas (6.1)-(6.6). This function \( K \) when considered on the \( b \)-slit tangent bundle \( T_bM := TM \setminus 0 \setminus b \setminus -b \) is smooth of the class \( C^\infty \) regarding the global \( y \)-dependence.

Owing to the theorems 2.1 and 2.2, the \( FFFg^{PD} \)-space also occupies the unique position when in the above chain \((Z_1) \- (Z_3)\) the condition \((Z_3)\) is replaced by the stipulation (2.1)-(2.3) with prescription of the dependence (2.15).

In the \( FFFg^{PD} \)-space, the method of Sections 3 and 4 proves to produce the explicit and simple representations for the respective connection coefficients and curvature tensor, on assuming \( h = \text{const} \) (which entails \( g = \text{const} \) because of (6.5)). The success has been predetermined by the possibility to write down the explicit coefficients (6.26) of the \( \mathcal{K} \)-transformation. The involved preferred vector field \( b'(x) \) proves to be the proper element of the \( FFFg^{PD} \)-space \( \mathcal{K} \)-transformation. The metricity (2.12) holds fine. The obtained formulas (6.30) and (6.60)-(6.62) straightforwardly entail the vanishing (1.18).

The \( FFFg^{PD} \)-space connection coefficients (6.49) involve the fraction \( 1/q \), where \( q = \sqrt{r_{mn}y^my^n} \) with \( r_{mn} = a_{mn} - b_mb_n \). Since the input 1-form \( b \) is of the unit norm \(||b|| = 1\), the scalar \( q \) is zero when \( y = b \). Therefore, we may apply the coefficients on but the \( b \)-slit tangent bundle \( T_bM := TM \setminus 0 \setminus b \setminus -b \) (obtained by deleting out in \( TM \setminus 0 \) all the directions which point along, or oppose, the directions given rise to by the 1-form \( b \)), on which the coefficients are smooth of the class \( C^\infty \) regarding the \( y \)-dependence.

As we are entitled to conclude from the right-hand part of the representation (6.49) of the \( FFFg^{PD} \)-space connection coefficients \( D^k_{mn} \), the coefficients are not equal to the Riemannian Christoffel symbols \( a^k_{mn} \) of the space \( R^N \), unless we meet the vanishing \( \nabla_nb_m = 0 \), where \( \nabla \) stands for the Riemannian covariant derivative in the space \( R^N \). The last vanishing means geometrically that the vector field \( b_m(x) \) is parallel in the space \( R^N \), in which case the coefficients \( D^k_{mn} \) are equivalent to the coefficients \( a^k_{mn} \). So, we are entitled to set forth the following assertion.

**Theorem 2.6.** When the involved vector field \( b_m(x) \) is parallel in the associated Riemannian space \( R^N \), the \( FFFg^{PD} \)-space connection obtained is equivalent to the Levi-Civita connection in the space \( R^N \).

Otherwise the connection coefficients \( D^k_{mn} \) are nonlinear (regarding the \( y \)-dependence) in any dimension \( N \geq 3 \).

In the dimension \( N = 2 \) we always have \( \eta^k_m = 0 \) (see (6.50) and (6.51)) and, therefore, the connection coefficients \( D^k_{mn} \) are independent of \( y \) (see (6.55)), which in turn entails the independence of the tensor (6.75) of \( y \). Thus we can formulate the following remarkable result.

**Theorem 2.7.** The \( FFFg^{PD} \)-space connection obtained is linear in the dimension \( N = 2 \).

In the \( FFFg^{PD} \)-space of the dimension \( N = 3 \), the arc \( A(x, l_1, l_2) \) can be described by means of dependence of the azimuthal angle \( \chi \) (see (6.4) and (6.14)) and the polar angle \( \phi \) on the arc-length parameter \( s \) (defined by (1.2)). Due attentive consideration performed in Section 7 leads to the following assertion.
Theorem 2.8. In the \( \mathcal{F} \mathcal{F}_g^{PD} \)-space of the dimension \( N = 3 \), the geodesic equation for the arc \( \mathcal{A}(x,l_1,l_2) \) can be completely integrated, yielding the following explicit dependence:

\[
\chi(s) = \frac{1}{h} \arccos \left( \sqrt{1 - h^2 \tilde{C}^2 \cos(h(s - \bar{s}))} \right)
\]  

(2.16)

and

\[
\phi(s) = \tilde{\phi} - \frac{\pi}{2} + \arctan \left( \frac{1}{h\tilde{C}} \tan(h(s - \bar{s})) \right), \quad \text{if } \tilde{C} \neq 0; \quad \phi = \tilde{\phi}, \quad \text{if } \tilde{C} = 0
\]  

(2.17)

where \( \tilde{C}, \bar{s}, \tilde{\phi} \) are integration constants.

Using this dependence, we obtain the following theorem.

Theorem 2.9. The behaviour of the unit vector \( l^i \) along the arc \( \mathcal{A}(x,l_1,l_2) \) of the \( (N = 3) \)-dimensional space \( \mathcal{F} \mathcal{F}_g^{PD} \) is governed by the expansion

\[
l^i(s) = k_1(s)l_1^i + k_2(s)l_2^i + k_3(s)b^i,
\]  

(2.18)

which in addition to the pair \( l_1^i, l_2^i \) involves the Finsleroid-axis vector \( b^i \).

The coefficients \( k_1(s), k_2(s), k_3(s) \) are given explicitly by means of the formulas (7.43)-(7.46).

3. Preliminary observations

Let the desired \( \varkappa \)-transformation (2.1) of a Finsleroid space \( \mathcal{F}^N \) be realized over the tangent vectors by means of a convenient diffeomorphic transformation

\[
y = \varkappa \cdot \zeta : \quad y^i = y^i(x, \zeta).
\]  

(3.1)

Denote the inverse by

\[
\zeta = \varkappa^{-1} \cdot y : \quad \zeta^i = \zeta^i(x, y).
\]  

(3.2)

In (3.1), as well as in (3.2), it is implied that \( y \in T_x M \) and \( \zeta \in T_x M \) with the same point \( x \in M \) of support. The homogeneity condition formulated below theorem 2.1 takes on the explicit form

\[
\zeta^i(x, \gamma y) = \gamma^h \zeta^i(x, y), \quad \gamma > 0, \; \forall y,
\]  

(3.3)

which entails the identity

\[
y^a \zeta^i_n = h \zeta^i_n,
\]  

(3.4)

where \( \zeta^i_n = \partial \zeta^i / \partial y^a \). The transformation (2.1) can be written in the tensorial form

\[
g_{mn}(x, y) = \varkappa^2 \zeta_m^i(x, y) \zeta_n^j(x, y) a_{ij}(x).
\]  

(3.5)

From (3.4) and (3.5) it just ensues that the Finsleroid metric function \( K(x, y) = \sqrt{g_{mn} y^m y^n} \) and the Riemannian metric function \( S(x, \zeta) = \sqrt{a_{mn}(x) \zeta^m \zeta^n} \) are connected by means of the relation

\[
K = h \varkappa S.
\]  

(3.6)

Owing to \( \varkappa = (1/h) K^{1-h} \) (see (2.5)), from (3.6) we can obtain the remarkable equality

\[
(K(x, y))^{h(x)} = S(x, \zeta).
\]  

(3.7)
The indicatrix property (2.2) is a direct implication of the formulas (3.7) and
\[
l = \mathbf{x} \cdot L : \quad l^i = y^i(x, L); \quad L = \mathbf{x}^{-1} \cdot l : \quad L^i = \zeta^i(x, l)
\] (3.8)
(see (3.1) and (3.2)), where \(l^i = y^i/K(x, y)\) and \(L^i = \zeta^i/(S(x, \zeta)\) are the respective unit vectors which possess the properties \(K(x, l) = 1\) and \(S(x, L) = 1\). From (2.5) and (3.5) it follows that
\[
g_{mn}(x, l)dl^m dl^n = \frac{1}{h^2(x)}a_{ij}(x)dl^i dt^j.
\] (3.9)

No support vector enters here the right-hand part.

Any two nonzero tangent vectors \(y_1, y_2 \in T_x M\) in a fixed tangent space \(T_x M\) form the Finsleroid-space angle
\[
\alpha_{\{x\}}(y_1, y_2) = \frac{1}{h(x)} \arccos \lambda,
\] (3.10)
where the scalar
\[
\lambda = \frac{a_{mn}(x) \zeta^m_1 \zeta^m_2}{\sqrt{S^2(x, \zeta_1)} / \sqrt{S^2(x, \zeta_2)}}, \quad \text{with} \quad \zeta^m_1 = \zeta^m(x, y_1) \quad \text{and} \quad \zeta^m_2 = \zeta^m(x, y_2),
\] (3.11)
is of the entire Riemannian meaning in the space \(\mathcal{R}^N\). These representations (3.10) and (3.11) realize the claimed relation (2.7), with making the choice \(\sqrt{\mathcal{C}} = h\) in accordance with (2.6).

Let the Finsleroid indicatrix \(\mathcal{I}_F\{x\}\) supported by a fixed point \(x \in M\) be parameterized by means of a convenient variable set \(u^a\) (for instance, we can take \(u^a = \zeta^a/\zeta^N\) in regions with \(\zeta^N \neq 0\), or \(u^a = y^a/y^N\), whenever \(y^N \neq 0\). The indices \(a, b, c, d, e\) will be specified over the range \((1, \ldots, N - 1)\). Using the parametrical representation \(l^i = l^i(u^a)\) of the indicatrix, where \(l^i\) are unit vectors (possessing the property \(K(l) = 1\)), we can construct the induced metric tensor
\[
i_{ab}(u^c) = g_{mn} t^m_a t^n_b
\] (3.12)
on the indicatrix by the help of the projection factors \(t^m_a = \partial l^m / \partial u^a\) (the method was described in detail in Section 5.8 of [1]). Applying (3.5) yields the equality
\[
i_{ab} = \frac{1}{h^2(x)} \tilde{i}_{ab},
\] (3.13)
where \(\tilde{i}_{ab}(u^c) = a_{mn} \tilde{t}^m_a \tilde{t}^n_b\) with \(\tilde{t}^m_a = t^j_a \zeta^m_j\) is the Riemannian version of the indicatrix induced metric tensor, obtainable when one puts \(h = 1\). We have taken into account the fact that the conformal factor \(\mathbf{x}\), having been proposed by (2.5), equals \(1/h(x)\) on the indicatrix. From this standpoint, it is easy to make the search into the curvature of the indicatrix. Indeed, since \(h\) is independent of the vectors \(y\), the associated Christoffel symbols
\[
i_{a}^c b = \frac{1}{2} \delta^{ce} \left( \frac{\partial i_{ea}}{\partial u^b} + \frac{\partial i_{eb}}{\partial u^a} - \frac{\partial i_{ab}}{\partial u^e} \right), \quad \tilde{i}_{a}^c b = \frac{1}{2} \delta^{ce} \left( \tilde{\partial} i_{ea} + \tilde{\partial} i_{eb} - \tilde{\partial} i_{ab} \right)
\]
are equivalent: \(i_{a}^c b = \tilde{i}_{a}^c b\). Therefore, the indicatrix curvature tensor
\[
I_a^c b d = \frac{\partial i_a^c b}{\partial u^d} - \frac{\partial i_a^c d}{\partial u^b} + i_a^e b d c e - i_a^e d c e b
\]
is identical to the tensor $\tilde{I}_{ab}^{cd}$ constructible by the same rule from the tensor $\tilde{I}_{ab}^{cd}$, that is, $I^{c}_{ab} = \tilde{I}_{ab}^{cd}$. Let us now consider the tensors $I_{cdab} = i_{ce}I^{e}_{ab} \tilde{I}_{acbd}$ and $\tilde{I}_{cdab} = \tilde{i}_{ce}I^{e}_{ab} \tilde{I}_{acbd}$. Since $I_{cdab} = -(\tilde{i}_{ce}i_{ad} - \tilde{i}_{cd}i_{ab})$ is the ordinary case characteristic of the Riemannian geometry (which reflects the fact that the curvature of the unit sphere is equal to 1), we get

$$I_{cdab} = -\left(\tilde{i}_{ce}i_{ad} - \tilde{i}_{cd}i_{ab}\right)/\hbar^2$$

and, then, arrive at the representation

$$I_{acbd} = -\left(\tilde{i}_{ce}i_{ad} - \tilde{i}_{cd}i_{ab}\right) (3.14)$$

which manifests that the indicatrix curvature is constant and equals $\hbar^2$ (in compliance with (2.6)).

Also, from (3.13) we have $\det(i_{ab}) = \hbar^{-2(N-1)} \det(\tilde{i}_{ab})$. The area of the indicatrix is the volume $\int \sqrt{\det(i_{ab})} du^1...du^{N-1}$ of the internal indicatrix space, where integration is performed over all the space. Then we are entitled to conclude that in any dimension $N$ and at each point $x \in M$ the ratio

$$\frac{\text{AREA}_{\text{Finsleroid Indicatrix}}}{\text{AREA}_{\text{Euclidean Unit Sphere}}} = \frac{1}{\hbar^{N-1}}$$

(3.15)

is valid.

The tensor $i_{ab}(u^c)$ is defined on the indicatrix. We can, however, extend the meaning of the parameters $u^a$ by homothety to obtain the scalars $u^a(y)$ defined at any point of the Finsleroid, using the zero-degree homogeneity $u^a(ky) = u^a(y), \ k > 0, \forall y$. With their help we can obtain the tensor $i^*_{ab}(y) = i_{ab}(u(y))$ meaningful at any point of the Finsleroid and construct the extended tensor $f_{AB} = f_{AB}(u^0, u^a)$ with $u^0 = \ln K$ as follows:

$$f_{AB} = e^{2u^0} f^*_{AB}, \quad \text{with} \quad f^*_{ab} = i^*_{ab}, \quad f^*_{a0} = 0, \quad f^*_{00} = 1, \quad (3.16)$$

where the sets $u^A = \{u^0, u^a\}$ and $u^B = \{u^0, u^b\}$ have been used. In terms of the parametrization $y^m = y^m(u^A)$ thus arisen, it can readily be seen that the tensor $f_{AB}$ is the covariant transform of the Finslerian metric tensor $g_{mn}$:

$$f_{AB} = g_{mn} \frac{\partial y^m}{\partial u^A} \frac{\partial y^n}{\partial u^B}.$$  

(3.17)

So, the volume $\int \sqrt{\det(g_{mn})} d^N y$ of the Finsleroid when written in terms of the coordinates $u^A$ is given by the integral

$$\int \sqrt{\det(f_{AB})} du^1...du^{N-1} du^0 = \int e^{Nu^0} du^0 \int \sqrt{\det(f^*_{ab})} du^1...du^{N-1},$$

in which $u^0 \in (-\infty, 0)$, so that the integral is equal to the $1/N$ multiplied by the area of the indicatrix. Whence in addition to the law (3.15) we have the ratio

$$\frac{\text{VOLUME}_{\text{Finsleroid}}}{\text{VOLUME}_{\text{Euclidean Unit Ball}}} = \frac{1}{\hbar^{N-1}}.$$  

(3.18)

Therefore, at each point $x \in M$ the following law is valid:

$$\frac{\text{AREA}_{\text{Finsleroid Indicatrix}}}{\text{VOLUME}_{\text{Finsleroid}}} = \frac{\text{AREA}_{\text{Euclidean Unit Sphere}}}{\text{VOLUME}_{\text{Euclidean Unit Ball}}}, \quad \text{when} \quad N \geq 3,$$

(3.19)

and
\[
\frac{\text{LENGTH}_{\text{Finsleroid Indicatrix}}}{\text{AREA}_{\text{Finsleroid}}} = \frac{\text{LENGTH}_{\text{Euclidean Unit Circle}}}{\text{AREA}_{\text{Euclidean Unit Circle}}} = \frac{2\pi}{\frac{h}{\hbar}} = 2, \quad \text{when} \quad N = 2. \tag{3.20}
\]

With the help of the derivative coefficients
\[
\zeta_n^i = \frac{\partial \zeta^i}{\partial y^n}, \quad \zeta_{nk}^i = \frac{\partial \zeta^i}{\partial y^k}, \quad y_m^i = \frac{\partial y^i}{\partial \zeta^m}, \quad y_{mh}^i = \frac{\partial y_{m}^i}{\partial \zeta^h}, \tag{3.21}
\]
it is possible to develop a direct method to induce the connection in the Finsleroid space \( F^N \) from the Riemannian space \( R^N \). To this end we can naturally use in \( R^N \) the Levi-Civita connection
\[
\mathcal{LC} = \{ L^m_j, L^m_{ij} \} : \quad L^m_j = -L^m_{ij}\zeta^i, \quad L^m_{ij} = a^m_{ij}, \tag{3.22}
\]
with \( a^m_{ij} = a^m_{ij}(x) \) standing for the Christoffel symbols constructed from the Riemannian metric tensor \( a_{mn}(x) \).

First of all, we need the coefficients \( N^k_i(x, y) \) to construct the operator \( d_i \) indicated in (2.9). It proves fruitful to obtain the coefficients by means of the map
\[
\{ N^k_i \} = \mathbf{\kappa} \cdot \{ L^k_i \}. \tag{3.23}
\]

Namely, starting with the fundamental property of the Levi-Civita connection that the Riemannian angle is preserving under the parallel displacements, which in terms of our notation can be written as
\[
\left( \frac{\partial}{\partial x^i} + L^k_i(x, \zeta_1) \frac{\partial}{\partial \zeta^k_1} + L^k_i(x, \zeta_2) \frac{\partial}{\partial \zeta^k_2} \right) \alpha_{\text{Riemannian space}}(x, \zeta_1, \zeta_2) = 0, \tag{3.24}
\]
we want to have the similar vanishing in the Finsleroid space \( F^N \):
\[
\left( \frac{\partial}{\partial x^i} + N^k_i(x, y_1) \frac{\partial}{\partial y^k_1} + N^k_i(x, y_2) \frac{\partial}{\partial y^k_2} \right) \alpha_{(x,y)}(y_1, y_2) = 0, \tag{3.25}
\]
assuming also that the vanishing
\[
\left( \frac{\partial}{\partial x^i} + N^k_i(x, y) \frac{\partial}{\partial y^k} \right) K(x, y) = 0 \tag{3.26}
\]
arises after performing the \( \mathbf{\kappa} \)-transformation of the Riemannian vanishing
\[
\left( \frac{\partial}{\partial x^i} + L^k_i(x, \zeta) \frac{\partial}{\partial \zeta^k} \right) S(x, \zeta) = 0. \tag{3.27}
\]

With an arbitrary differentiable scalar \( w(x, y) \), we consider the \( \mathbf{\kappa} \)-transform
\[
W(x, \zeta) = w(x, y), \quad \text{which entails} \quad \frac{\partial W}{\partial \zeta^m} = y^m_n \frac{\partial w}{\partial y^k}, \tag{3.28}
\]
and postulate that the \( \mathbf{\kappa} \)-transformation is covariantly transitive, so that
\[
\left( \frac{\partial}{\partial x^i} + N^k_i(x, y) \frac{\partial}{\partial y^k} \right) w(x, y) = \left( \frac{\partial}{\partial x^i} + L^k_i(x, \zeta) \frac{\partial}{\partial \zeta^k} \right) W(x, \zeta). \tag{3.29}
\]
Since the field $w$ is arbitrary, the last equality is fulfilled if and only if
\[ N^m_n = \frac{\partial y^m(x, \zeta)}{\partial x^n} + y^m_i L_i^n. \] (3.30)
This is the representation which is required to realize the map (3.23).

Since the equality (3.30) can be written in the form
\[ \frac{\partial \zeta^i}{\partial x^n} + N^m_n \zeta^i_m + a^i_{kn} \zeta^k = 0, \] (3.31)
we have
\[ N^m_n = -y^m_i \left( \frac{\partial \zeta^i}{\partial x^n} + a^i_{kn} \zeta^k \right). \] (3.32)

It can readily be noted that the transitivity property (3.29) can straightforwardly be extended to scalars dependent on two vectors. Namely, if
\[ W(x, \zeta_1, \zeta_2) = w(x, y_1, y_2), \] (3.33)
then
\[ \left( \frac{\partial}{\partial x^i} + N^k_i \frac{\partial}{\partial y^k_1} + N^{k_2} \frac{\partial}{\partial y^k_2} \right) w(x, y_1, y_2) = \left( \frac{\partial}{\partial x^i} + L^k_1 \frac{\partial}{\partial \zeta^i_1} + L^k_2 \frac{\partial}{\partial \zeta^i_2} \right) W(x, \zeta_1, \zeta_2), \] (3.34)
where $N^k_i = N^k_i(x, y_1)$, $N^{k_2} = N^{k_2}(x, y_2)$, $L^k_1 = L^k_1(x, \zeta_1)$, $L^k_2 = L^k_2(x, \zeta_2)$. The equality (3.34) is verified by using (3.31) or (3.32). When this implication is applied to the equality (3.10), the Riemannian vanishing (3.24) just entails the Finsleroid-space vanishing (3.25), whenever $h = const$.

Differentiating (3.5) with respect to $y^k$ yields the representation
\[ 2C_{mnk} = (1 - h) \frac{2}{K} l_k g_{mn} + \zeta^2 (\zeta^i_{mnk} \zeta^j + \zeta^i_{mnk} \zeta^j) a_{ij} \] (3.35)
for the Cartan tensor. Contracting this by $y^n$ results in the equality
\[ \zeta^2 h \zeta^i_{mnk} \zeta^j a_{ij} = (1 - h)(h_{km} - l_k l_m), \] (3.36)
where the vanishing $C_{mnk} y^n = 0$ and the homogeneity identity (3.4) have been taken into account.

From (3.11) it follows that
\[ \frac{\partial \lambda}{\partial x^i} = \frac{a_{mn,i} \zeta^m_n \zeta^i_2}{\sqrt{S^2(x, \zeta_1)} \sqrt{S^2(x, \zeta_2)}} + \frac{1}{\sqrt{S^2(x, \zeta_1)} \sqrt{S^2(x, \zeta_2)}} a_{mn} \left( \frac{\partial \zeta^m_2}{\partial x^i} \zeta^i_2 + \zeta^m_1 \frac{\partial \zeta^i_2}{\partial x^i} \right) \]
\[ - \frac{1}{2} \frac{1}{S^2(x, \zeta_1)} \left( a_{mn,i} \zeta^m_1 \zeta^i_1 + 2a_{mn} \frac{\partial \zeta^m_1}{\partial x^i} \zeta^i_1 \right) + \frac{1}{S^2(x, \zeta_2)} \left( a_{mn,i} \zeta^m_2 \zeta^i_2 + 2a_{mn} \frac{\partial \zeta^m_2}{\partial x^i} \zeta^i_2 \right), \] (3.37)
where $a_{mn,i} = \partial a_{mn}/\partial x^i$, and
\[ \frac{\partial \lambda}{\partial y^1_i} = \frac{a_{mn,i} \zeta^m_1 \zeta^i_2}{\sqrt{S^2(x, \zeta_1)} \sqrt{S^2(x, \zeta_2)}}, \quad \frac{\partial \lambda}{\partial y^2_i} = \frac{a_{mn,i} \zeta^m_1 \zeta^i_2}{\sqrt{S^2(x, \zeta_2)} \sqrt{S^2(x, \zeta_1)}}, \] (3.38)
With (3.32) we find
\[ N^k_{1i} \frac{\partial \lambda}{\partial y^k_i} = - \left( \frac{\partial \zeta^m_1}{\partial x^i} + a^m_{ti} \zeta^1_t \right) \left[ \frac{a_{mn} \zeta^n_2}{\sqrt{S^2(x, \zeta_1)}} \sqrt{S^2(x, \zeta_2)} - \frac{a_{mn} \zeta^n_1}{S^2(x, \zeta_1)} \lambda \right] \]  
(3.39)
and
\[ N^k_{2i} \frac{\partial \lambda}{\partial y^k_i} = - \left( \frac{\partial \zeta^m_2}{\partial x^i} + a^m_{ti} \zeta^2_t \right) \left[ \frac{a_{mn} \zeta^n_1}{\sqrt{S^2(x, \zeta_1)}} \sqrt{S^2(x, \zeta_2)} - \frac{a_{mn} \zeta^n_2}{S^2(x, \zeta_2)} \lambda \right], \]  
(3.40)
where the identity \( y^k_i \zeta^m_k = \delta^m_j \) has been taken into account.

The formulas (3.37), (3.39), and (3.40) just entail the vanishing
\[ \frac{\partial \lambda}{\partial x^i} + N^k_{1i} \frac{\partial \lambda}{\partial y^k_i} + N^k_{2i} \frac{\partial \lambda}{\partial y^k_i} = 0. \]  
(3.41)
Thus, from (3.10) we may conclude that whenever \( h = \text{const} \) the angle preservation (1.18) holds fine.

4. Entailed connection coefficients and curvature tensor

Let us trace the validity of the theorem 2.5 and the involved formulas (2.11)-(2.13). Since \( D_iK = d_iK \), the vanishing \( d_iK = 0 \) indicated in (2.11) ensues from (3.7) and (3.27).

The second vanishing in (2.11) is tantamount to \( N^j_{ik} = -D^j_{ik} \) (because of \( \partial y^j_i / \partial x^i = 0 \)).

The third equality entered (2.11) reads
\[ \frac{\partial y^j_i}{\partial x^i} + N^k_{1i} \frac{\partial \lambda}{\partial y^k_i} - D^h_{ij} y^h_i = 0. \]  
(4.1)
Let us differentiate this equality with respect to \( y^n \). We obtain
\[ d_i g_{jn} + \frac{\partial N^k_{1i}}{\partial y^n} g_{jk} - D^h_{ij} y^h_i - y^h_i \frac{\partial D^h_{ij}}{\partial y^n} = 0. \]  
(4.2)
By making the choice \( D^k_{in} = -\partial N^k_{1i} / \partial y^n \) we obtain from (4.2) the metricity \( D_i g_{jn} = 0 \), if
\[ y^h_i \frac{\partial D^h_{ij}}{\partial y^n} = 0. \]  
(4.3)

From (3.30) and \( D^k_{in} = -\partial N^k_{1i} / \partial y^n \) it follows that
\[ \frac{\partial y^n_i}{\partial x^i} + L^t_i y^n_k + D^n_{is} y^s_k - L^h_{ik} y^h_i = 0. \]  
(4.4)
Since \( y^k_i \zeta^m_j = \delta^m_j \), the previous identity can be written as
\[ \frac{\partial \zeta^m_n}{\partial x^i} + N^k_{1i} \zeta^m_h + L^s_{it} \zeta^s_m - D^h_{im} \zeta^s_h = 0. \]  
(4.5)

Can the last vanishing be materialized?

Let us realize the action of the \( \kappa \)-transformation (3.1)-(3.2) on tensors by the help of the transitivity rule, that is,
\[ \{ w^n_m(x, y) \} = \kappa \cdot \{ W^n_m(x, \zeta) \} : w^n_m = y^n_h \zeta^j_m W^h_j, \]  
(4.6)
and define the covariant derivative $\nabla$ in $\mathcal{R}^N$ according to the conventional Riemannian rule:

$$\nabla_i W^m = \frac{\partial W^m}{\partial x^i} + L^h_i \frac{\partial W^m}{\partial s^h} + L^h_h W^m - L^h_m W^h = 0$$

(4.7)

and

$$\nabla_i S = 0, \quad \nabla_i \zeta^j = 0, \quad \nabla_i a_{mn} = 0.$$  

(4.8)

Due to (4.4) and (4.5), we have the transitivity property

$$D_i w^m = y^n_+ s^j_\zeta W^h_j.$$  

(4.9)

Applying the rule (4.9) to the transformation (3.5) of the metric tensor yields

$$g_{mn} d_i \left( \frac{1}{\kappa^2} \right) + \frac{1}{\kappa^2} D_i g_{mn} = 0.$$  

(4.10)

Thus, the metricity condition $D_i g_{jn} = 0$ holds if and only if $d_i \kappa = 0.$

Applying (2.5) and the vanishing $D_i K = 0$ to (4.10) makes us conclude that the following assertion is valid.

**Theorem 4.1.** Under the input stipulation (2.1)-(2.3), the covariant derivative $D_i$ obtained through the transitivity (4.9) fulfills the metricity condition $D_i g_{jn} = 0$ if and only if

$$\frac{\partial h}{\partial x^i} = 0 : \quad h = \text{const.}$$

(4.11)

The last condition entails the vanishing (4.3); in the $\mathcal{F}_F^P$-Finsleroid space the validity of this implication can explicitly be verified with the help of the representation (6.55) derived in Section 6.

Commuting the covariant derivative (2.10) yields the equality

$$[D_i D_j - D_j D_i] w^k = M^k_{ij} \frac{\partial w^k}{\partial y^h} - E^k_{ij} w^h + E^i_{nj} w^h$$

(4.12)

with the tensors

$$M^n_{ij} := d_i N^n_j - d_j N^n_i$$

(4.13)

and

$$E^n_{ij} := d_i D^n_{jk} - d_j D^n_{ik} + D^n_{jk} D^n_{im} - D^n_{ik} D^n_{jm}.$$  

(4.14)

If the choice $D^n_{in} = -N^n_{in}$ is made (see (2.13)), the tensor (4.13) can be written in the form

$$M^n_{ij} = \frac{\partial N^n_j}{\partial x^i} - \frac{\partial N^n_i}{\partial x^j} - N^n_h D^n_{j} + N^n_j D^n_{ih}.$$  

(4.15)

By applying the commutation rule (4.12) to the particular choices $\{K, y^n, y_k, g_{nk}\},$ we obtain the identities

$$y_n M^n_{ij} = 0, \quad y_k E^n_{kij} = -M^n_{ij}, \quad y_n E^n_{kij} = M_{kij},$$

(4.16)

and

$$E_{nmij} + E_{nmi} = 2 C_{mnh} M^n_{ij} \quad \text{with} \quad C_{mn} = \frac{1}{2} \frac{\partial g_{mn}}{\partial y^h}.$$  

(4.17)
Differentiating (4.15) with respect to $y^k$ and using the equality $N^j_i = -D^j_{ik}y^k$ (see (2.13)) yield

$$E^m_{ik} = -\frac{\partial M^m_{ij}}{\partial y^k}.$$  \hfill (4.18)

The cyclic identity

$$D_k M^m_{ij} + D_j M^m_{ki} + D_i M^m_{jk} = 0$$  \hfill (4.19)

is valid, where

$$D_k M^m_{ij} = \frac{\partial M^m_{ij}}{\partial x^k} + N^m_k \frac{\partial M^m_{ij}}{\partial y^m} + D^m_{ki} M^l_{ij} - \alpha^s_{ki} M^m_{sj} - \alpha^s_{kj} M^m_{is}.$$  \hfill (4.20)

It proves pertinent to replace in the commutator (4.12) the partial derivative $\partial w^m_k/\partial y^k$ by the definition

$$S_h w^m_k = \frac{\partial w^m_k}{\partial y^n} + C^m_{hk} w^h_k - C^m_{hk} w^m_n$$  \hfill (4.21)

which has the meaning of the covariant derivative in the tangent Riemannian space $\mathcal{R}_{\{x\}}$. With the curvature tensor

$$\rho^m_{ij} = E^m_{ij} - M^h_{ij} C^m_{hk},$$  \hfill (4.22)

the commutator (4.12) takes on the form

$$(D_i D_j - D_j D_i) w^m_k = M^h_{ij} S_h w^m_k - \rho^h_{ik} w^m_k + \rho^m_{in} w^h_{nk}. $$  \hfill (4.23)

The skew-symmetry

$$\rho_{mnij} = -\rho_{nmij}$$  \hfill (4.24)

holds (cf. (4.17)).

5. Specifying the conformal multiplier

In a fixed tangent space endowed with a Finslerian metric tensor $g_{ij}$ produced by a Finslerian metric function $F$, we may consider the conformal transform

$$k_{ij} = e^{2\psi} g_{ij}, \quad \psi = \psi(F),$$  \hfill (5.1)

where $\psi = \psi(F)$ is a smooth function, and construct from $k_{ij}$ the Christoffel symbols $\kappa_{ij}$, which yields $\kappa_{ij} = \psi' l^i l^j + \psi' l^i \delta^j - \psi' l^j \delta^i - \psi' g_{ij} + C_{ij}$, where the prime means differentiation with respect to $F$ and $C_{ij} = (1/2) \partial g_{ij}/\partial y^n$. We directly derive the equalities

$$\left(\partial \psi / \partial y^m - \partial \psi / \partial y^i \right) \delta_m^i = \left(\psi'^m l^i l^j + \psi' \frac{1}{F} h^m_{ij}\right) \delta_m^i - \left(\psi'^m l^i l^j + \psi' \frac{1}{F} h^m_{ij}\right) g_{ij} - 2C^m_{lm} C^j_{ij} - [jm]$$

and

$$k^t_{ij} k^m_{tm} - k^t_{im} k^m_{tj} = \psi' l^i l^j l^m - \psi' g_{ij} \psi' / h^m + C^m_{lm} C^j_{ij} - [jm],$$

together with

$$\left(\partial \psi / \partial y^m - \partial \psi / \partial y^i \right) + k^t_{ij} k^m_{tm} - k^t_{im} k^m_{tj} + C^m_{lm} C^j_{ij} - C^m_{tj} C^j_{im}$$

$$= - \left(\psi'^m l^i l^j + \psi' \frac{1}{F} h^m_{ij}\right) \delta_m^i - \left(\psi'^m l^i l^j + \psi' \frac{1}{F} h^m_{ij}\right) g_{ij} + \psi' l^i l^j \delta_m^i - \psi' g_{ij} \psi' / h^m - [jm].$$
The notation \([jm]\) symbolizes the skew-symmetric terms. We introduce the associated tensor

\[
L_i^m_{mj} = \frac{\partial k^n_{im}}{\partial y^m} + \frac{\partial k^n_{im}}{\partial y^j} + k^{ij}k^{im} - k^{im}k^n_{tj}
\]  
(5.2)

and the indicatrix curvature tensor

\[
\hat{R}_i^m_{mj} = C^n_{tm}C^t_{mj} - C^n_{tj}C^t_{im},
\]  
(5.3)

obtaining

\[
L_i^m_{mj} + \hat{R}_i^m_{mj} = -(\psi'' - \psi'\psi')l_i(l_jh^n_{m} - l_mh^n_{j}) - \psi''l_n(l_mh_{ij} - l_jh_{im})
\]

\[
+ \psi' \frac{1}{F}(h_{im}\delta^m_j - h_{ij}\delta^m_m) + \psi' \frac{1}{F}(h_{ij}g_{im} - h_{im}g_{ij}) + \psi'\psi'(g_{im}h^n_{j} - g_{ij}h^n_{m})
\]

and

\[
e^{-2\psi}L_{inmj} + \hat{R}_{inmj} = -(\psi'' - \psi'\psi')l_i(l_jh_{nm} - l_mh_{nj}) - \psi''l_n(l_mh_{ij} - l_jh_{im})
\]

\[
+ \psi' \frac{1}{F}(h_{im}g_{nj} - h_{ij}g_{nm}) + \psi' \frac{1}{F}(h_{nj}g_{im} - h_{nm}g_{ij}) + \psi'\psi'(g_{im}h^n_{j} - g_{ij}h^n_{nm}),
\]

where \(L_{inmj} = k_{nk}L_i^k_{mj}\) and \(\hat{R}_{inmj} = g_{nk}\hat{R}_i^k_{mj}\). Using the equality \(h_{ij} = g_{ij} - l_il_j\) yields

\[
e^{-2\psi}L_{inmj} + \hat{R}_{inmj} = -\left(\psi'' + \psi'\frac{1}{F}\right)\left[l_i(l_jh_{nm} - l_mh_{nj}) - l_n(l_jh_{im} - l_mh_{ij})\right]
\]

\[
- \left(\psi'\frac{2}{F} + \psi'\psi'\right)(h_{ij}h_{nm} - h_{im}h_{nj}).
\]  
(5.4)

If the curvature tensor \(L_i^m_{mj}\) obtained under the conformal transformation (5.1) vanishes identically, then because of the known Finslerian identities \(y'C_{ijk} = 0\) and \(y'h_{ij} = 0\) the equality (5.4) would entail the equation \(\psi''F + \psi' = 0\), which solution is

\[
e^\psi = c_1 F^{c_2},
\]  
(5.5)

where \(c_1 > 0\) and \(c_2\) are integration constants. The result (5.5) permits writing the conformal multiplier (2.3) in the form \(\zeta = (1/c_1)F^{1-h}\), where we have identified \(c_1\) with \(h - 1\). This entails that the tensor \(C_m^i(x,y)C_n^j(x,y)a_{ij}(x)\) appeared in the right-hand part of the Finslerian metric tensor representation (3.5) is positively homogeneous of degree \(2h - 2\). The last observation is in agreement with the homogeneity condition (3.3), whence we have \(c_1 = h\). Therefore, the theorem 2.1 of Section 2 is valid.

If we put \(L_i^m_{mj} = 0\) and insert (2.5) into the right-hand part of (5.4), we obtain for the indicatrix curvature tensor (5.3) the representation

\[
F^2\hat{R}_{inmj} = (1 - h^2)(h_{ij}h_{nm} - h_{im}h_{nj})
\]  
(5.6)
which says us that the indicatrix is of the constant curvature $h^2$. Thus, the theorem 2.2 of Section 2 is fulfilled.

Inversely, let the indicatrix be of constant curvature, so that

$$F^2 R_{inmj} = (1 - C)(h_{ij}h_{nm} - h_{in}h_{mj}), \quad (5.7)$$

where $C > 0$ is the curvature value (and $C$ is independent of the tangent vectors $y$).

Inserting (5.7) in (5.4), performing the conformal transformation (5.1), and applying (5.5) with the choice of the exponent $c_2$ according to the condition

$$ (1 + c_2)^2 = C, \quad (5.8)$$

from (5.4) we obtain the vanishing $L_{i}^{n}m_{j} = 0$, which means that the space is conformally flat. Therefore, the theorem 2.4 of Section 2 is also valid.

6. Performing the choice of the $\mathcal{F}\mathcal{F}_{g}^{PD}$-Finsleroid space

Let us assume that in addition to a Riemannian metric $\sqrt{a_{ij}(x)y^{i}y^{j}}$ the manifold $M$ admits a non-vanishing 1-form $b = b_{i}(x)y^{i}$ of the unit length:

$$a_{ij}(x)b^{i}(x)b^{j}(x) = 1, \quad (6.1)$$

where $b^{i}(x) = a^{ij}(x)b_{j}(x)$. The tensor $a^{ij}(x)$ is reciprocal to $a_{ij}(x)$, so that $a_{ij}a^{jm} = \delta_{i}^{m}$, where $\delta_{i}^{m}$ stands for the Kronecker symbol. The Finsleroid space is specified in accordance with the condition that the metric function $K(x, y)$ is (2.14) with

$$B = b^2 + gbq + q^2 \equiv A^2 + h^2q^2 \quad \text{with} \quad A = b + \frac{1}{2}gq, \quad (6.2)$$

where

$$q = \sqrt{r_{mn}y^{m}y^{n}} \quad \text{and} \quad r_{mn} = a_{mn} - b_{m}b_{n}, \quad (6.3)$$

so that

$$a_{ij}(x)y^{i}y^{j} = b^2 + q^2. \quad (6.4)$$

The scalar $g(x)$ obtained through

$$h(x) = \sqrt{1 - \frac{g^2(x)}{4}}, \quad \text{with} \quad -2 < g(x) < 2, \quad (6.5)$$

plays the role of the characteristic parameter. The variable $\chi$ entering the exponential representation (2.14) of the $\mathcal{F}\mathcal{F}_{g}^{PD}$-Finsleroid metric function $K$ is given as it follows:

$$\chi = \frac{1}{h}\left(-\arctan \frac{G}{2} + \arctan \frac{L}{hb}\right), \quad \text{if} \quad b \geq 0; \quad \chi = \frac{1}{h}\left(\pi - \arctan \frac{G}{2} + \arctan \frac{L}{hb}\right), \quad \text{if} \quad b \leq 0, \quad (6.6)$$

with the function $L = q + (g/2)b$ fulfilling the identity

$$L^2 + h^2b^2 = B. \quad (6.7)$$

The definition range

$$0 \leq \chi \leq \frac{1}{h}\pi$$
is of value to describe all the tangent space. The normalization in (6.6) is such that
\[ \chi \big|_{y=b} = 0. \]  
(6.8)

The quantity (6.6) can conveniently be written as
\[ \chi = \frac{1}{h} f \]  
(6.9)

with the function
\[ f = \arccos \frac{A(x,y)}{\sqrt{B(x,y)}} \]  
(6.10)

ranging as follows:
\[ 0 \leq f \leq \pi. \]  
(6.11)

The Finsleroid-axis vector \( b^i \) relates to the value \( f = 0 \), and the opposed vector \( -b^i \) relates to the value \( f = \pi \):
\[ f = 0 \sim y = b; \quad f = \pi \sim y = -b. \]  
(6.12)

It is frequently convenient to represent the function \( K \) in the form
\[ K = \sqrt{B} J, \quad \text{with } J = e^{-\frac{1}{2}g^x}. \]  
(6.13)

The normalization is such that
\[ K(x, b(x)) = 1 \]  
(6.14)

(notice that \( q = 0 \) at \( y^i = b^i \)). The positive (not absolute) homogeneity holds: \( K(x, \gamma y) = \gamma K(x, y) \) for any \( \gamma > 0 \) and all admissible \( (x,y) \).

Under these conditions, we call \( K(x,y) \) the \( \mathcal{FF}^{PD}_{g} \)-Finsleroid metric function, obtaining the \( \mathcal{FF}^{PD}_{g} \)-Finsleroid
\[ \mathcal{FF}^{PD}_{g} := \{M; a_{ij}(x); b_{i}(x); g(x); K(x,y)\}. \]  
(6.15)

Definition. Within any tangent space \( T_x M \), the metric function \( K(x,y) \) produces the \( \mathcal{FF}^{PD}_{g} \)-Finsleroid
\[ \mathcal{FF}^{PD}_{g;\{x\}} := \{y \in \mathcal{FF}^{PD}_{g;\{x\}} : y \in T_x M, K(x,y) \leq 1\}. \]  
(6.16)

Definition. The \( \mathcal{FF}^{PD}_{g} \)-Indicatrix \( \mathcal{I}^{PD}_{g;\{x\}} \subset T_x M \) is the boundary of the \( \mathcal{FF}^{PD}_{g} \)-Finsleroid, that is,
\[ \mathcal{I}^{PD}_{g;\{x\}} := \{y \in \mathcal{FF}^{PD}_{g;\{x\}} : y \in T_x M, K(x,y) = 1\}. \]  
(6.17)

Definition. The scalar \( g(x) \) is called the Finsleroid charge. The 1-form \( b = b_{i}(x)y^{i} \) is called the Finsleroid-axis 1-form.

The entailed components \( y_{i} := (1/2)\partial K^{2}/\partial y^{i} \) of the covariant tangent vector \( \hat{y} = \{y_{i}\} \) can be found in the simple form
\[ y_{i} = (u_{i} + gqb_{i})J^{2}, \]  
(6.18)
where \( u_i = a_{ij} y^j \).

By making due inspection into the formulas (6.6)-(6.12) it proves convenient to separate the tangent space \( T_xM \) into the unification

\[
T_xM = T_{g;\{x\}}^{(+)} \cup T_{g;\{x\}}^{(-)}
\]

(6.19)
of the regions

\[
T_{g;\{x\}}^{(+)} := \{ y \in T_{g;\{x\}}^{(+)} : y \in T_xM, \ 0 \leq f < h\pi \}
\]

(6.20)

and

\[
T_{g;\{x\}}^{(-)} := \{ y \in T_{g;\{x\}}^{(-)} : y \in T_xM, \ (1 - h)\pi < f \leq \pi \},
\]

(6.21)

which depend on value of \( g \). We have the range correspondence

\[
0 \leq f < h\pi \sim \chi \in [0, \pi), \quad \text{and} \quad (1 - h)\pi < f \leq \pi \sim \chi \in \left(\frac{1 - h}{h} \pi, \frac{1}{h} \pi\right].
\]

(6.22)
The directions involving the Finsleroid axis vector \( y = b \) belong to the region (6.20), and the opposed cases belong to the region (6.21), that is,

\[
b(x) \in T_{g;\{x\}}^{(+)} \quad \text{and} \quad -b(x) \in T_{g;\{x\}}^{(-)}.
\]

(6.23)
The intersections

\[
C_{g;\{x\}}^{(+)} = T_{g;\{x\}}^{(+)} \cap \mathcal{I}F^{PD}_{g;\{x\}} \quad \text{and} \quad C_{g;\{x\}}^{(-)} = T_{g;\{x\}}^{(-)} \cap \mathcal{I}F^{PD}_{g;\{x\}}
\]

(6.24)
yield two covering charts for the indicatrix (6.17).

It will be noted that, in contrast to \( b^i \), the opposed vector \(-b^i\) is not unit. Therefore, we introduce the normalized vector

\[
b^{(-)i} = -b^i e^{\frac{1}{2} g\pi} \]

(6.25)

which is unit: it can readily be seen that

\[
K(x, b^{(-)}(x)) = 1.
\]

In this space the \( \kappa \)-transformation (3.2) can be realized in the explicit and simple form

\[
\zeta^i = \left[ hv^i + (b + \frac{1}{2} gq)b^i \right] \frac{J}{\kappa h}, \]

(6.26)

where \( v^i = y^i - bb^i \) and \( \kappa = (1/h)K^{1-h} \). Obviously, the right-hand part in (6.26) possesses the homogeneity properties (3.3)-(3.4) of degree \( h \). Simple calculation shows that

\[
det(\zeta_m^i) = \left( \frac{J}{\kappa} \right)^N.
\]

(6.27)

Since \( v^i = q = 0 \) at \( y^i = \pm b^i \), from (6.26) it follows that

\[
\zeta^i(x, b) = b^i(x), \quad \zeta^i(x, b^{(-)}) = b^{(-)i}(x),
\]

(6.28)

where the equality \( K(x, b) = 1 \) (see (6.14)) has been taken into account. Therefore, the involved preferred vector field \( b'(x) \) as well as the opposed field \( b^{(-)i}(x) \) are the proper elements of the \( \mathcal{IF}^{PD} \)-space \( \kappa \)-transformation (6.26), that is,

\[
\{ b'(x) \} \overset{\kappa}{\rightarrow} \{ b^i(x) \}, \quad \{ b^{(-)i}(x) \} \overset{\kappa}{\rightarrow} \{ b^{(-)i}(x) \}.
\]

(6.29)
When the substitution (6.26) is applied, from (3.10)-(3.11) we obtain the $\mathcal{F}_g^{PD}$-angle
\[ \alpha_{\{x\}}(y_1, y_2) = \frac{1}{h} \arccos \lambda \quad \text{with} \quad \lambda = \frac{A(x, y_1) A(x, y_2) + h^2 v_{12}}{\sqrt{B(x, y_1)} \sqrt{B(x, y_2)}}, \] (6.30)
where $v_{12} = r_{mn}(x) y_1^m y_2^n$.

If, fixing a point $x$, we consider the angle $\alpha_{\{x\}}(y, b)$ formed by a vector $y \in T_xM$ with the input characteristic vector $b^i(x)$, from (6.30) we get the respective value to be
\[ \alpha_{\{x\}}(y, b) = \frac{1}{h} \arccos \lambda \equiv \chi \] (6.31)
(notice that $q = 0$ and $A = 1$ whenever $y = b$). In terms of the variables $\zeta^i$, the last formula reads
\[ \alpha_{\{x\}}(y, b) = \frac{1}{h} \arccos \frac{\zeta^m b_m(x)}{\sqrt{S^2(x, \zeta)}} \] (6.32)

We have
\[ A = \sqrt{B} \cos(h\chi), \quad qh = \sqrt{B} \sin(h\chi), \] (6.33)
so that the transformation (6.26) can clearly be written in terms of the angle $\chi$:
\[ \zeta^i = \left( \frac{v^i}{q} \sin(h\chi) + b^i \cos(h\chi) \right) K^h. \] (6.34)

From (6.32) we can conclude that
\[ \alpha_{\{x\}}(b^{-1}, b) = \frac{1}{h} \pi \] (6.35)
which is more than $\pi$ whenever $g \neq 0$.

Let us verify that the transformation (6.26) obeys the input stipulation (2.1). Differentiating (6.26) leads to
\[ \zeta^m_n = E^m_n + \frac{1}{N} \zeta^m C_n - \frac{1}{\zeta} \zeta_n \zeta^m \quad \text{with} \quad E^m_n = \left[ h(\delta^m_n - b_n b^m) + \left( b_n + \frac{1}{2q} q v_n \right) b^m \right] J \frac{1}{\zeta h}, \] (6.36)
where
\[ \zeta_n = \partial \zeta / \partial y^n = (1 - h)y_n \zeta / K^2, \] (6.37)
and we have used the equality $\partial \ln J / \partial y^n = (1/N) C_n$ (which is a direct implication of the formulas (A.6) and $C_n = \partial \ln(\sqrt{\det(g_{ij})}) / \partial y^n$). It is useful to take into account that
\[ E^m_n y^n = \zeta^m, \quad E^m_n b^n = b^m J \frac{1}{\zeta h}. \]

Noting also the vanishing
\[ -\frac{\zeta^i \zeta^j}{N^2} C_m C^m - \frac{(1 - h)^2}{K^2} \zeta^i \zeta^j + \frac{1 - h}{K^2} \left( \zeta^i y^m E^j_m + \zeta^j y^m E^i_m \right) = 0, \]
where the equality $K^2 C_m C^m = N^2 g^{i/2}/4$ has been applied (see (A.7) in Appendix A), and using the contravariant components $g^{mn}$ written in (A.12) of Appendix A, we get
\[ g^{mn} \zeta^i \zeta^j_n = \frac{1}{N}(\zeta^i C^m E^j_m + \zeta^j C^m E^i_m) \]
\[ a^{mn} E^i_m E^j_n = \left[ a^{ij} - \frac{1}{2q} g v^i b^j \right] \frac{1}{\mathcal{H}^2} \]

Here,

\[ a^{mn} E^i_m E^j_n = h \left[ h(a^{ij} - b^i b^j) + \frac{1}{2q} g v^i b^j \right] \frac{J^2}{\mathcal{H}^2} + \left[ 1 - \frac{1}{2q} gb \right] b^i b^j \frac{1}{\mathcal{H}^2} + b^i g \zeta^j \frac{1}{\mathcal{H}^2}. \]

We can write

\[ h^2 g_{mn} \xi^i \xi^j = -\frac{g}{qK^2} (b + gq) \xi^i \xi^j h^2 + \frac{1}{2q} b^i \xi^j h + \frac{1}{\mathcal{H}^2} J \frac{1}{\mathcal{H}^2}, \]

or

\[ h^2 g_{mn} = -\frac{1}{2q} \xi^i \xi^j h^2 + h^2 a^{ij} \frac{1}{\mathcal{H}^2} + \frac{1}{2q} b^i \xi^j h + \left[ 1 - \frac{1}{2q} gb \right] b^i b^j \frac{1}{\mathcal{H}^2}, \]

so that

\[ g_{mn} \xi^i \xi^j = \frac{1}{\mathcal{H}^2} a^{ij}. \]

The metric tensor transformation (6.38) can be inverted to read (3.5). Thus the verification is complete.

Several interesting relations can be found. First of all, constructing the function

\[ S^2 = a_{ij} \xi^i \xi^j \]

with the help of the choice (6.26), applying (6.2) and the identity \( v^i b_i = 0 \),

we obtain the useful equality

\[ \left( \frac{J}{\mathcal{H}} \right)^2 = S^2(x, \xi) B(x, y). \]  

(6.39)

Also, from (6.26) it follows that

\[ \zeta^i b_i = \left( b + \frac{1}{2q} gq \right) \frac{J}{\mathcal{H}}, \quad \zeta^i - (\zeta^m b_n) b^i = \frac{J}{\mathcal{H}} h^i, \]

and

\[ \sqrt{r_{mn} \zeta^m \zeta^n} = h q \frac{J}{\mathcal{H}}, \quad b = \left( \zeta^m b_n - \frac{1}{2h} g \sqrt{r_{mn} \zeta^m \zeta^n} \right) \frac{h x}{J}. \]  

(6.40)

From (6.39) and (6.13) we can obtain the equality

\[ h \mathcal{H} = \left( S^2(x, \xi) \right)^{(1 - h)/(2h)}, \]  

(6.41)

\[ h \mathcal{H} = \left( S^2(x, \xi) \right)^{(1 - h)/(2h)}. \]  

(6.42)
which is equivalent to (3.6). According to (6.13), (6.32), and (6.33), we have

$$\frac{1}{J} = e^{\frac{1}{2} g \chi} \quad \text{with} \quad \chi = \frac{1}{h} \arccos \left( \frac{\zeta^n b_n(x)}{\sqrt{S^2(x, \zeta)}} \right)$$

(6.43)

The indicated formulas allow us to write down the explicit form of the inverse to the transformation (6.26), namely we find

$$y^i = y^i(x, \zeta) \quad \text{with} \quad y^i = b b^i + \frac{1}{h} \left( \zeta^i - (\zeta^n b_n) b^i \right) \frac{h \chi}{J},$$

(6.44)

where $b$ can be taken from (6.41). It is possible to find straightforwardly the coefficients

$$y^i_i = \frac{1}{h} K^2 \frac{1}{S^2} \zeta_j,$$

(6.45)

where $\zeta_j = a_{jk} \zeta^k$ and $T = 1/\sqrt{r_{mn} \zeta^m \zeta^n}$, which entails the useful identities

$$y^i_i = \frac{1}{h} K^2 \frac{1}{S^2} \zeta^i, \quad \frac{1}{h} S^2 \zeta_j \zeta_j = y_n.$$

(6.46)

In the rest of this section we assume that

$$\frac{\partial g}{\partial x^i} = 0 : \quad g = \text{const} \quad \text{and} \quad h = \text{const}.$$

(6.47)

The right-hand part in (6.26) is such that

$$\frac{\partial \zeta^i}{\partial x^n} = \frac{\partial \zeta^i}{\partial b_j} \partial_n b_j + \frac{\partial \zeta^i}{\partial a_m j} \partial_n a_m.$$

Under these conditions, straightforward calculations with the help of the representation (3.32) result in

$$N^k_n = -\left( (1 - h)b + \frac{1}{2} gq \right) \frac{1}{h} a^{kj} \nabla_n b_j - \left( \frac{g}{2q} v^k - (1 - h)b^k \right) \frac{1}{h} y^i \nabla_n b_j - a^{nj} y^j,$$

(6.48)

where $\nabla_n b_j = \partial b_j / \partial x^n - b_k a^k b_j$. Evaluating the coefficients $D^k_{nm} = -\partial N^k_n / \partial y^m$ (see (2.13)) yields

$$D^k_{nm} = \left( (1 - h)b_m + \frac{g}{2q} v_m \right) \frac{1}{h} a^{kj} \nabla_n b_j + \frac{g}{2q h} \eta_m y^i \nabla_n b_j + \left( \frac{g}{2q} v^k - (1 - h)b^k \right) \frac{1}{h} \nabla_n b_m + a^k n,$$

(6.49)

where

$$\eta_m = \frac{1}{q^2} v^k v_m \equiv a_{mn} \eta^k, \quad \eta^k = a^k n - \frac{1}{q^2} v^k v^n.$$

(6.50)

This tensor obeys the nullification

$$y_k \eta^k_m = b_k \eta^k_m = 0.$$

(6.51)

We can alternatively write (6.48) as follows:

$$N^k_n = \left[ b a^{kj} - b^k y^j - \frac{1}{h} \left( b + \frac{1}{2} gq \right) \eta^k + \frac{1}{h} \left( b^k - \frac{1}{q^2} (b + gq)v^k \right) y^j \right] \nabla_n b_j - a^{nj} y^j,$$

(6.52)
\[ N^k_n = \left[ \left( b - \frac{1}{h} \left( b + \frac{1}{2} g q \right) \right) \eta^{kj} + \left( \frac{1}{q^2} v^k \left( b - \frac{1}{h} \left( b + g q \right) \right) + \left( \frac{1}{h} \right) b^k \right) y^j \right] \nabla_n b_j - a^{k}_{nj} y^j. \] (6.53)

By using \( y_k = (u_k + g q b_k) K^2 / B \) (see (A.3) in Appendix A), we obtain
\[ y_k N^k_n = -g q J^2 y^j \nabla_n b_j - y_k a^{k}_{nj} y^j. \] (6.54)

The subsequent differentiation of the coefficients (6.49) results in
\[ \frac{\partial D^k_{nm}}{\partial y^i} = \frac{g}{2 q h} \eta_{mo} \eta^{kj} \nabla_n b_j - \frac{g}{2 q^2 h} (\eta^k_{\mu} v_{i} + \eta^k_{\nu} v_{m}) y^j \nabla_n b_j + \frac{g}{2 q h} (\eta^k_{\mu} \nabla_n b_i + \eta^k_{\nu} \nabla_n b_m). \] (6.55)

Owing to the identity (6.51), the vanishing \( y_k \partial D^k_{nm} / \partial y^i = 0 \) (see (4.3)) holds true.

The coefficients (6.48) show the properties
\[ u_k N^k_n = -\frac{1}{h} g q y^j \nabla_n b_j - u_k a^{k}_{nj} y^j, \quad b_k N^k_n = \frac{1}{h} (1 - h) y^j \nabla_n b_j - b_k a^{k}_{nj} y^j, \] (6.56)
and
\[ d_n b \equiv \frac{\partial b}{\partial x^n} + b_k N^k_n = \frac{1}{h} y^j \nabla_n b_j, \quad d_n q \equiv \frac{\partial q}{\partial x^n} + \frac{1}{q} v_k N^k_n = -\frac{1}{h q} (b + g q) y^j \nabla_n b_j, \] (6.57)

together with
\[ d_n \left( \frac{q}{b} \right) = -\frac{1}{b^2 q h} B y^j \nabla_n b_j, \quad d_n B = -\frac{g}{q h} B y^j \nabla_n b_j, \quad d_n B = -\frac{2 q + g b}{b^2 q h} B y^j \nabla_n b_j. \] (6.58)

With the formulas (6.48)-(6.58) it is possible to verify directly the validity of the desired vanishing set
\[ \frac{\partial K}{\partial x^n} + N^m_n l_m = 0 \] (6.59)
(see (3.26)),
\[ \frac{\partial y^j}{\partial x^n} + N^m_n g_{mj} - D^m_{nj} y_m = 0 \] (6.60)
(see (4.1)), and
\[ \frac{\partial g_{ij}}{\partial x^n} + 2 N^m_n C_{mji} - D^m_{nj} g_{mi} - D^m_{ni} g_{mj} = 0 \] (6.61)
(see (4.2)-(4.3)).

With the help of (6.54) and (6.58) the coefficients (6.53) can be transformed to
\[ N^k_n = -l^k \frac{\partial K}{\partial x^n} + \left[ \left( b - \frac{1}{h} \left( b + \frac{1}{2} g q \right) \right) \mathcal{H}^{kj} \frac{K^2}{B} + \left( \frac{1}{h q} - \frac{1}{q} + \frac{g b}{B} \right) K m^k y^j \right] \nabla_n b_j - h^k_{ji} a^{i}_{nj} y^j \] (6.62)
(see Appendix B), where \( m^k = (2 / N g) \Lambda^k \) and \( \mathcal{H}^{kj} = h^{kj} - m^k m^j = (K^2 / B) \eta^{kj} \) (see (A.30)-(A.35) in Appendix A).

The entailed coefficients \( N^k_{nm} = \partial N^k_n / \partial y^m \) are found as follows:
\[ N^k_{nm} = -l^k g q \frac{K^2}{K} \nabla_n b_m + \frac{1}{K} \left[ q - \frac{1}{h} q + \frac{1}{2 h} g (b + q q) \right] \mathcal{H}^{kj} m^m_n \frac{K^2}{B} \nabla_n b_j \]
\[ + \left( b - \frac{1}{h} \left( b + \frac{1}{2} gq \right) \right) \frac{K}{B} \left( \mathcal{H}^{k\lambda} l_m - l^k \mathcal{H}^k_m - l^\lambda \mathcal{H}^{\lambda}_m + \frac{b \mathcal{H}^k_i + b \mathcal{H}^{\lambda}_m}{q} \right) \nabla_n b_j \]

\[ + K \frac{b}{q^2} \left( \frac{1}{h} - 1 \right) m_m m^k \mathcal{H}^k_m \nabla_n b_j - \frac{K}{q} \left( \frac{1}{h} - 1 \right) m_m l^k \mathcal{H}^{\lambda}_m \nabla_n b_j \]

\[ - K \left( \frac{1}{h} (b + gq) - \frac{b}{q} \right) \frac{1}{q} \mathcal{H}^{k\lambda} m \nabla_n b_j + \left( \frac{1}{h} - \frac{1}{q} + \frac{gb}{B} \right) K m^k \nabla_n b_m - a^{k\lambda}_{nm} \]

(see (B.5) in Appendix B). They fulfill the equality \( N^k_{nm} = -D^k_{nm} \) with \( D^k_{nm} \) given by (6.49).

Moreover, with the coefficients \( N^k_i \) given by (6.48) we get straightforwardly the vanishing

\[ \frac{\partial \lambda(x, y_1, y_2)}{\partial x^i} + N^k_i (x, y_1) \frac{\partial \lambda(x, y_1, y_2)}{\partial y^k_1} + N^k_i (x, y_2) \frac{\partial \lambda(x, y_1, y_2)}{\partial y^k_2} = 0, \quad \text{when} \ h = \text{const}, \]

(6.63) where \( \lambda \) is the scalar indicated in (6.30). To verify the statement, it is worth deriving the equality

\[ \frac{\partial \lambda}{\partial y^k_1} = h^2 \frac{B_1 v_{2k} + q_1^2 b_k A_2 - b_1 A_2 v_{1k} - v_{12} \left( h^2 v_{1k} + \frac{1}{2} q \frac{1}{q_1} v_{1k} \right) A_1}{B_1 \sqrt{B_1} \sqrt{B_2}} \]

(6.64) with the counterpart

\[ \frac{\partial \lambda}{\partial y^k_2} = h^2 \frac{B_2 v_{1k} + q_2^2 b_k A_1 - b_2 A_1 v_{2k} - v_{12} \left( h^2 v_{2k} + \frac{1}{2} q \frac{1}{q_2} v_{2k} \right) A_2}{B_2 \sqrt{B_2} \sqrt{B_1}}, \]

(6.65) where \( A_1 = A(x, y_1), A_2 = A(x, y_2), B_1 = B(x, y_1), B_2 = B(x, y_2), q_1 = q(x, y_1), q_2 = q(x, y_2), b_1 = b(x, y_1), b_2 = b(x, y_2), \) together with \( v_{1i} = r_{1i} y_1^i \) and \( v_{2i} = r_{2i} y_2^i \). Plugging these derivatives in (6.63) results in the claimed vanishing after attentive couplepage reductions.

It will be noted that

\[ b^k \frac{\partial \lambda}{\partial y^k_1} = h^2 \frac{q_1^2 A_2 - v_{12} A_1}{B_1 \sqrt{B_1} \sqrt{B_2}}, \quad b^k \frac{\partial \lambda}{\partial y^k_2} = h^2 \frac{q_2^2 A_1 - v_{12} A_2}{B_2 \sqrt{B_2} \sqrt{B_1}}. \]

From (6.30) we have also

\[ \frac{\partial \lambda}{\partial g} = - \frac{1}{2} \left( \frac{b_1 q_1}{B_1} + \frac{b_2 q_2}{B_2} \right) \lambda + \frac{q_1 A_2 + q_2 A_1 - g v_{12}}{2 \sqrt{B_1} \sqrt{B_2}}, \]

or

\[ \frac{\partial \lambda}{\partial g} = \frac{1}{2 \sqrt{B_1} \sqrt{B_2}} \left[ \frac{q_1^2 A_2}{B_1} \sigma_1 + \frac{q_2^2 A_1}{B_2} \sigma_2 - v_{12} \left( \frac{A_1}{B_1} \sigma_1 + \frac{A_2}{B_2} \sigma_2 \right) \right], \]
\[ \sigma_1 = \frac{g}{2} A_1 + h^2 q_1 \equiv q_1 + \frac{g}{2} b_1, \quad \sigma_2 = \frac{g}{2} A_2 + h^2 q_2 \equiv q_2 + \frac{g}{2} b_2. \]  

(6.66)

There arises the equality
\[
\frac{\partial \lambda}{\partial g} = \frac{1}{2h^2} \left[ \sigma_1 b^k \frac{\partial \lambda}{\partial y_1^k} + \sigma_2 b^k \frac{\partial \lambda}{\partial y_2^k} \right].
\]

(6.67)

Using the formula (A.26) of Appendix A, we arrive at
\[
\frac{\partial \lambda}{\partial g} = \frac{1}{h^2} \left[ \mu_1 C_{ik} \frac{\partial \lambda}{\partial y_1^k} + \mu_2 C_{ik} \frac{\partial \lambda}{\partial y_2^k} \right],
\]

(6.68)

where
\[
\mu_1 = q_1 K_1^2 \sigma_1, \quad \mu_2 = q_2 K_2^2 \sigma_2.
\]

(6.69)

The associated Riemannian curvature tensor is constructed as follows:
\[
a_{ik}^{\ nkm} = \frac{\partial a_{ik}^{\ nm}}{\partial x^k} - \frac{\partial a_{ik}^{\ nk}}{\partial x^m} + a_{nm} a_{ik}^{\ um} - a_{nk} a_{ik}^{\ um}.
\]

(6.70)

The evaluation of the tensor (4.15) from the coefficients (6.48) gives us
\[
M_{nij} = \left[ \left( 1 - h \right) b + \frac{1}{2} q a^n t + \left( \frac{g}{2q} v^n - (1 - h) b^n \right) y^t \right] \frac{1}{h} b_i a_{ij}^t - a_{nij}^t y^t,
\]

(6.71)

or
\[
M_{nij} = \left[ \left( 1 - h \right) b + \frac{1}{2} q a^n t + \frac{1}{h} b a_{tij} + y^t a_{tij} \right] \frac{K^2}{B} \mathcal{H}_n - \frac{1}{q} K m_n \frac{1}{h} y^t b a_{tij}
\]

(6.72)

(see (A.45)), which entails the equalities
\[
\frac{1}{N} C_n M_{nij} = -\frac{g}{2qh} b_n a_{nij} y^t, \quad A_{knm} M_{nij} = -K \mathcal{H}_n \frac{q}{2qh} b_m a_{tij} + \frac{1}{N} (A_k M_{nij} + A_n M_{kij})
\]

(6.73)

((A.8) has been used and the tensor \( \mathcal{H}_{kn} \) has been defined in (A.30)), and
\[
M_{nij} = -y_i^m \zeta_h a_{nij}^t.
\]

(6.74)

The tensor (4.14) is found to read
\[
E_{k}^{\ nij} = -\left[ \left( 1 - h \right) b_k + \frac{1}{2} q t_k \right] a^n t + \frac{g}{2q} \eta^n y^t + \left( \frac{g}{2q} v^n - (1 - h) b^n \right) \delta_k^n \frac{1}{h} b m a_{tij} + a_{nij}^k,
\]

(6.75)

which entails
\[
E_{k}^{\ nij} = y_i^m \zeta_h M_{nij} + y_i^m a_{nij}^m \zeta_h.
\]

(6.76)

Obeying the identities (4.16)-(4.18) can straightforwardly be verified.

The following explicit representation for the curvature tensor (4.22) can be proposed:
\[
\rho_{knij} = -\frac{1}{K} (l_k M_{nij} - l_n M_{kij}) + \left( m_k H_n^t - m_n H_k^t \right) P_{tij} + H_k^t H_n a_{tij} \frac{K^2}{B}
\]

(6.77)
with
\[ P_{ti} = \left[ -\left[hq^2 + b\left(b + \frac{1}{2}gq\right)\right] b^t a_{ti} + \left(b + \frac{1}{2}gq\right) v^t a_{ti} \right] \frac{K}{qB}, \]
or
\[ P_{ti} = \left[ -hq^2 b^t a_{ti} + \left(b + \frac{1}{2}gq\right) v^t a_{ti} \right] \frac{K}{qB} \] (6.78)
(see (C.14)-(C.17) in Appendix C).

With \( \rho_{knij} \) we associate the tensor
\[ \rho^{knij} = g^{pk} g^{qm} a^{mi} a^{nj} \rho_{pqmn}. \]

We can straightforwardly obtain the contraction
\[ \rho^{knij} \rho_{knij} = \delta^{kl} a^{nlij} \zeta^i a^{nij} \zeta^h a^{nhij} \] (6.79)
(see Appendix C).

We can also find that the tensor
\[ M_{ni} = g_{nm} M^m_{ij} \] (6.80)
possesses the simple representation
\[ \frac{B}{K^2} M_{ni} = \left((1 - h)b + \frac{1}{2}gq\right) b_n a^t_{ni} - \left(\frac{g}{2q} v_n + (1 - h)b_n\right) \frac{1}{h} y^t a^t_{ni} - a^t_{ni} y^t. \] (6.81)
The identity \( y^n M_{ni} = 0 \) holds. Squaring this tensor leads to the quadratic expressions
\[ \frac{B}{K^2} M^{nij} M_{ni} = \left(\frac{1}{h} \left((1 - h)b + \frac{g}{2q}\right) b_n a^{nij} - a^{nij} y^h\right) \left(\frac{1}{h} \left((1 - h)b + \frac{g}{2q}\right) b^t a^t_{ni} - a^t_{ni} y^t\right) \] (6.82)
and
\[ M^{nij} M_{ni} = \zeta^i a^t_{ni} \zeta^h a^{nhij} \] (6.83)
(as shown in Appendix A); here, \( \zeta^2 = K^2/h^2 S^2 \) in accordance with (3.6).

Using (6.74) together with
\[ y_{k}^{n} M^{m}_{ij} = C_{kn} M^{m}_{ij} + (1 - h) \frac{1}{K^2} \left(y^{n} g_{km} M^{m}_{ij} - y^{n} M^{m}_{ij}\right) \] (6.84)
(see (D.4) in Appendix D) reduces the curvature tensor (4.22) to the sum
\[ \rho^{n}_{i} = y^{n} a_{ni} ^{m} \zeta^{h} + (1 - h) \frac{1}{K^2} \left(y^{n} g_{km} M^{m}_{ij} - y^{n} M^{m}_{ij}\right). \] (6.85)

Make the transform
\[ y_{k}^{l} \zeta^{n}_{i} \rho^{t}_{ij} = a^{n}_{ij} + (1 - h) \frac{1}{K^2} \left(h \zeta^{n} \left(\frac{K^2}{h S^2} a^t_{ki} M^{i}_{lj} - \frac{1}{h} \frac{K^2}{S^2} \zeta^k M^{l}_{ij}\right)\right), \]
where (3.4)-(3.6) and (6.46) have been taken into account. Using (6.74) leads to
\[ y_{k}^{l} \zeta^{n}_{i} \rho^{t}_{ij} = a^{n}_{ij} + (1 - h) \frac{1}{h S^2} (\delta^{n}_{i} \zeta - \zeta a_{kl}) \zeta^{h} a^{t}_{ij}. \] (6.86)
Now we contract

\[ \rho^{tij} \rho^{tlij} = y^{p} c^{k} \rho^{p} g^{ij} y^{k} c^{l} \rho^{tij}, \]

so that

\[ \rho^{tij} \rho^{tlij} = a^{n} k^{ij} \left[ (1 - h) \frac{1}{hS^{2}} \delta_{t}^{n} \zeta_{k} - \zeta_{n} a_{kl} \right] \zeta^{h} a_{h}^{lij} \]

\[ = a^{n} k^{ij} \left[ a^{n}_{kl} (1 - h) \frac{1}{hS^{2}} \delta_{t}^{n} \zeta_{k} - \zeta_{n} a_{kl} \right] \zeta^{h} a_{h}^{lij} \]

\[ + (1 - h) \frac{1}{hS^{2}} \zeta_{n} \zeta^{s} a_{s}^{lij} \left[ a^{n}_{kl} (1 - h) \frac{1}{hS^{2}} \delta_{t}^{n} \zeta_{k} - \zeta_{n} a_{kl} \right] \zeta^{h} a_{h}^{lij} \]

\[ - (1 - h) \frac{1}{hS^{2}} \zeta_{k} \zeta^{s} a_{s}^{lij} \left[ a^{n}_{kl} (1 - h) \frac{1}{hS^{2}} \delta_{t}^{n} \zeta_{k} - \zeta_{n} a_{kl} \right] \zeta^{h} a_{h}^{lij} \]

or

\[ \rho^{tij} \rho^{tlij} = a^{n} k^{ij} a^{n}_{kl} + (1 - h) \frac{1}{hS^{2}} \zeta_{n} \zeta^{s} a_{s}^{lij} \left[ 2 a^{n}_{kl} (1 - h) \frac{1}{hS^{2}} \zeta_{n} a_{kl} \right] \zeta^{h} a_{h}^{lij} \]

\[ - (1 - h) \frac{1}{hS^{2}} \zeta_{k} \zeta^{s} a_{s}^{lij} \left[ 2 a^{n}_{kl} (1 - h) \frac{1}{hS^{2}} \zeta_{n} a_{kl} \right] \zeta^{h} a_{h}^{lij} \]

\[ = a^{n} k^{ij} a^{n}_{kl} - 2(1 - h) \frac{1}{hS^{2}} \zeta_{n} \zeta^{s} a_{s}^{lij} \left[ 2 \zeta^{k} a_{kl} (1 - h) \frac{1}{hS^{2}} \zeta_{n} a_{kl} \right] \zeta^{h} a_{h}^{lij}. \]

The result

\[ \rho^{tij} \rho^{tlij} = a^{n} k^{ij} a^{n}_{kl} - 2(1 - h^{2}) \frac{1}{h^{2} S^{2}} a^{n} a^{lij} \zeta^{k} a_{kl} \]

is equivalent to (6.79).

Let us introduce the object \( Y^{n}_{k}(x, y) = y^{n}_{k}(x, \zeta) \). We get

\[ \frac{\partial Y^{n}_{k}}{\partial x^{i}} = \frac{\partial y^{n}_{k}}{\partial x^{i}} + y^{n}_{k} \frac{\partial \zeta^{s}}{\partial x^{i}}. \]

Using here (4.4) and (3.31) leads to

\[ \frac{\partial Y^{n}_{k}}{\partial x^{i}} = -L_{i}^{n} y^{n}_{k} - D^{n}_{is} y^{s}_{k} + L^{i}_{k} y^{n}_{h} - y^{n}_{k} \left( N^{i}_{k} \zeta^{s} + a^{s} \zeta^{uk} \right). \]

Therefore, from the representation \( M^{n}_{k} = -Y^{n}_{k} \zeta^{h} a_{h}^{iji} \) (see (6.74)) we find the partial derivative

\[ \frac{\partial M^{n}_{k}}{\partial x^{i}} = \left[ L_{i}^{n} y^{n}_{th} - L^{i}_{k} y^{n}_{h} + y^{n}_{ls} \left( N^{r}_{k} \zeta^{s} + a^{s} \zeta^{uk} \right) \right] \zeta^{h} a_{h}^{lij}. \]
\[ +Y^t_n \left( N^a_k \zeta^h_k + a^h_{uk} \zeta^u_k \right) a_h^{\ i\ j} - Y^t_n \zeta^h_k \frac{\partial a_h^{\ i\ j}}{\partial x^k}. \]

The covariant derivative \((4.20)\) can now be written in the form
\[
D_k M^n_{ij} = \left[ L^u_i y^m_{tu} - L^u_{kt} y^m_u + y^m_{ts} \left( N^r_k \zeta^s_r + a^s_{uk} \zeta^u_k \right) \right] \zeta^h_k a_h^{\ i\ j} + Y^t_n \left( N^a_k \zeta^h_k + a^h_{uk} \zeta^u_k \right) a_h^{\ i\ j}
\]

\[-Y^t_n \zeta^h_k \frac{\partial a_h^{\ i\ j}}{\partial x^k} - N^m_k \left[ y^n_{tsm} \zeta^h_k a_h^{\ i\ j} + Y^m_n \zeta^h_k a_h^{\ i\ j} \right] - a^s_{ki} M^n_{sj} - a^s_{kj} M^n_{is}. \]

Cancelling here similar terms leaves us with
\[
D_k M^n_{ij} = \left( L^u_i y^m_{tu} - L^u_{kt} y^m_u + y^m_{ts} a^s_{uk} \zeta^u_k \right) \zeta^h_k a_h^{\ i\ j}
\]

\[+y^n_{ts} a^h_{uk} \zeta^u_k a_h^{\ i\ j} - y^n_{ts} \zeta^h_k \frac{\partial a_h^{\ i\ j}}{\partial x^k} - a^s_{ki} M^n_{sj} - a^s_{kj} M^n_{is}. \]

Recollecting the equalities \(L^m_j = -L^m_{ij} \zeta^i\) and \(L^m_{ij} = a^m_{ij}\) indicated in \((3.22)\), we obtain simply
\[
D_k M^n_{ij} = -y^n_{ts} \zeta^h_k \nabla_k a_h^{\ i\ j}, \quad (6.87)
\]

where
\[
\nabla_k a_h^{\ i\ j} = \frac{\partial a_h^{\ i\ j}}{\partial x^k} + a^t_{ku} a_h^{\ u\ i\ j} - a^n_{kh} a_u^{\ i\ j} - a^n_{ki} a_h^{\ u\ j} - a^n_{kj} a_h^{\ u\ i}. \quad (6.88)
\]

is the Riemannian covariant derivative of the Riemannian curvature tensor.

The cyclic identity \((4.19)\) proves to be a direct implication of the known Riemannian identity
\[
\nabla_k a^{\ i\ j} + \nabla_j a^{\ n\ ki} + \nabla_i a^{\ n\ jk} = 0. \quad (6.89)
\]

Using the equality
\[
g_{nm} y^m_i = \varepsilon^2 \zeta^m_n a_{ij} \quad (6.90)
\]

(ensued from \((3.5)\)), we can obtain the tensor \((6.80)\) to read
\[
M_{nij} = -\varepsilon^2 \zeta^h_k \zeta^m_n a_{nij} \quad (6.91)
\]

and juxtapose to \((6.85)\) the tensor \(\rho_{nij} = g_{nm} \rho_k^{\ m\ ij}\) which is
\[
\rho_{nij} = T_k^{\ hm} a_{nij}, \quad (6.92)
\]

where
\[
T_k^{\ hm} = \varepsilon^2 \left[ \frac{1}{2} \left( c^h_k \zeta^m_n - \zeta^m_k \zeta^h_n \right) + (1 - h) \frac{1}{K^2} (y^h_k \zeta^m_n - y^m_k \zeta^h_n) \right]. \quad (6.93)
\]

Since
\[
D_l T_k^{\ hm} = 0, \quad (6.94)
\]

we have
\[
D_l \rho_{nij} = T_k^{\ hm} \nabla_l a_{nij}, \quad (6.95)
\]

together with the cyclic identity
\[
D_l \rho_{nij} + D_j \rho_{knl} + D_t \rho_{kj} = 0. \quad (6.96)
\]
7. $\mathcal{F}\mathcal{F}_g^{PD}$-space coordinates and angles

We now fix the tangent space (in accordance with Appendix E) and choose the three-dimensional case

$$N = 3, \quad R^p = \{R^1, R^2, R^3\}. \quad (7.1)$$

It is convenient to relabel the coordinates $R^p$ as follows:

$$R^1 = x, \quad R^2 = y, \quad R^3 = z. \quad (7.2)$$

We get

$$q = \sqrt{x^2 + y^2}, \quad B = x^2 + y^2 + z^2 + gzq. \quad (7.3)$$

In terms of such coordinates, the metric tensor components $g_{pq}$ can be obtained from the list (E.6)-(E.7). The result reads

$$g_{11} = \left(1 - \frac{gBq}{z}x^2\right)J^2, \quad g_{22} = \left(1 - \frac{gBq}{z}y^2\right)J^2, \quad g_{33} = \left(1 + \frac{gq}{B}(z + gz)\right)J^2, \quad (7.4)$$

$$g_{12} = -\frac{gBq}{z}xyJ^2, \quad g_{13} = \frac{gqxy}{B}J^2, \quad g_{23} = \frac{gqy}{B}J^2. \quad (7.5)$$

From the formulas (E.8)-(E.9) it follows that

$$g_{11} = \left(1 + \frac{gBq}{z}x^2\right)\frac{1}{J^2}, \quad g_{22} = \left(1 + \frac{gBq}{z}y^2\right)\frac{1}{J^2}, \quad g_{33} = \left(1 - \frac{gq}{B}z\right)\frac{1}{J^2}, \quad (7.6)$$

$$g_{12} = \frac{gBq}{z}xy\frac{1}{J^2}, \quad g_{13} = -\frac{gq}{B}x\frac{1}{J^2}, \quad g_{23} = -\frac{gq}{B}y\frac{1}{J^2}. \quad (7.7)$$

The $\mathcal{F}\mathcal{F}_g^{PD}$-space coordinates $\{z^p\}$ are given by

$$z^1 = K, \quad z^2 = \phi, \quad z^3 = \chi \equiv \frac{1}{\hbar}f, \quad (7.8)$$

where $K$ is the Finsleroid metric function (6.13), $\phi$ is the polar angle in the $R^1 \times R^2$-plane, and $\chi$ plays the role of the Finsleroid azimuthal angle measured from the direction of the input vector $b^i$ (see (6.32)). The indices $p, q, ...$ will be specified over the range 1,2,3. For the vector $\{R^p\}$ we construct the representation

$$R^p = R^p(g; z^q) \quad (7.9)$$

which possesses the invariance property

$$K(g; R^p(g; z^q)) = z^1. \quad (7.10)$$

The representations

$$R^1 = K \sin \chi \cos \phi, \quad R^2 = K \sin \chi \sin \phi, \quad R^3 = K \cos \chi, \quad (7.11)$$

with

$$\sin \chi = \frac{1}{\hbar} \sin f, \quad \cos \chi = \frac{1}{\hbar} \left(\cos f - \frac{G}{2} \sin f\right). \quad (7.12)$$
can readily be arrived at, entailing

\[ q = K \sin \chi, \quad b + \frac{1}{2} g q = \frac{K}{J} \cos f, \quad b = K \cos \chi. \quad (7.13) \]

The arisen functions \( \sin \chi \) and \( \cos \chi \) can be interpreted as the required extensions of the trigonometric functions to the \( \mathcal{F}_g^{PD} \)-space.

The squared linear element \( ds^2 = g_{rs}(R)dR^rdR^s \) is found to be of the diagonal form

\[ (ds)^2 = (dz^1)^2 + (z^1)^2 \left[ (d\chi)^2 + \frac{1}{h^2} \sin^2(h\chi)(d\phi)^2 \right]. \quad (7.14) \]

On the other hand, when the components (7.11) are inserted in the \( \mathcal{F}_g^{PD} \)-angle (6.30), the following result is obtained:

\[ \alpha_{(x)}(y_1, y_2) = \frac{1}{h} \arccos \tau_{12}, \quad \text{with} \quad \tau_{12} = \cos(f_2 - f_1) - (1 - \cos(\phi_2 - \phi_1)) \sin f_1 \sin f_2. \quad (7.15) \]

This \( \tau_{12} \) does not involve any support vector \( y \). By developing here the infinitesimal version, putting \( \chi_1 = \chi, \chi_2 = \chi + d\chi, \phi_1 = \phi, \phi_2 = \phi + d\phi \), we come to the infinitesimal angle \( d\alpha = d\alpha_{(x)} \) which square reads

\[ (d\alpha)^2 = (d\chi)^2 + \frac{1}{h^2} \sin^2(h\chi)(d\phi)^2. \quad (7.16) \]

By comparing (7.14) with (7.16) we conclude that

\[ (ds)^2 = (dz^1)^2 + (z^1)^2(d\alpha)^2. \quad (7.17) \]

This formula is remarkable because showing us frankly that \( d\alpha \) is the infinitesimal arc-length on the indicatrix (keeping in mind that \( z^1 = 1 \) holds along the indicatrix).

The obtained metric (7.14) is of the conformally flat type

\[ (ds)^2 = \kappa^2(ds)^2 \bigg|_{\text{Euclidean}} \quad \text{with} \quad \kappa = \frac{1}{h} K^{1-h}. \quad (7.18) \]

To verify this assertion, it is appropriate to use the substitution \( z^1 = e^{\sigma} \) in (7.14), which yields

\[ (ds)^2 = e^{2\sigma} \left[ (d\chi)^2 + \frac{1}{h^2} \sin^2(h\chi)(d\phi)^2 + (d\sigma)^2 \right]. \quad (7.19) \]

Making the coordinate transformation

\[ \rho = e^{h\sigma} \sin(h\chi), \quad \tau = e^{h\sigma} \cos(h\chi) \]

leads to (7.18) with

\[ \left. (ds)^2 \right|_{\text{Euclidean}} = (d\rho)^2 + \rho^2 \sin^2(h\chi)(d\phi)^2 + (d\tau)^2 \]

and

\[ \kappa = \frac{1}{h} (\rho^2 + \tau^2)^{(1-h)/2h.} \]

The observations can be summarized by formulating the following
**Proposition.** Given an $\mathcal{F}_g^{PD}$-space of the dimension $N = 3$. In terms of the Finsleroid coordinates (7.2), which directly extend the spherical coordinates applied conventionally in the tangent spaces to the three-dimensional Riemannian space, the induced metric on the indicatrix is of the diagonal representation (7.14). The metric is of the conformally flat type as shown by (7.18).

According to (7.17), the length element on the indicatrix is given by the representation

$$ds|_{\mathcal{F}_g^{PD}-indicatrix} = \sqrt{d\chi^2 + \frac{1}{h^2} \sin^2(h\chi) d\phi^2},$$

(7.21)

which can be used to find the geodesics which are the solutions of the Euler-Lagrange equation written by the help of the Lagrangian $L = \sqrt{(d\chi/dt)^2 + (1/h^2) \sin^2(h\chi) (d\phi/dt)^2}$, where $t$ is an appropriate parameter. Since $\phi$ is a cyclic coordinate, we have

$$\frac{1}{h^2} \sin^2(h\chi) \phi' = \tilde{C},$$

(7.22)

where $\tilde{C}$ is a constant, thereafter we get

$$\chi'' = h^3 \tilde{C}^2 \frac{\cos(h\chi)}{\sin^3(h\chi)},$$

(7.23)

and

$$\chi' = \sqrt{1 - h^2 \tilde{C}^2 \frac{1}{\sin^2(h\chi)}}.$$  

(7.24)

The prime $'$ means differentiation with respect to the parameter $s$ defined by (7.21). It follows that

$$\left(\frac{1}{h} \sin f\chi\right)' = \cos f\chi' \chi' + \frac{1}{h} \sin f\chi'' = \cos f.$$

(7.25)

The equation (7.24) can readily be integrated, yielding the explicit dependence

$$\chi(s) = \frac{1}{h} \arccos \left(\sqrt{1 - h^2 \tilde{C}^2 \cos(h(s - \tilde{s}))}\right),$$

(7.26)

where $\tilde{s}$ is an integration constant. So,

$$\cos(h\chi) = \sqrt{1 - h^2 \tilde{C}^2 \cos(h(s - \tilde{s}))}, \quad \sin(h\chi) = \sqrt{1 - \left(1 - h^2 \tilde{C}^2\right) \cos^2(h(s - \tilde{s}))},$$

(7.27)

and from (7.22) we get

$$\phi' = h\tilde{C} \frac{h}{1 - \left(1 - h^2 \tilde{C}^2\right) \cos^2(h(s - \tilde{s}))}.$$  

(7.28)

Integrating yields explicitly

$$\phi(s) = \arctan \left(\frac{1}{h\tilde{C}} \tan(h(s - \tilde{s}))\right), \quad \text{if } \tilde{C} \neq 0; \quad \phi = \frac{\pi}{2}, \quad \text{if } \tilde{C} = 0,$$

(7.29)
from which we have
\[
\cos \phi = \frac{\tilde{h} \tilde{C}}{\sqrt{1 - \left(1 - h^2 \tilde{C}^2 \right) \cos^2 (h(s - \tilde{s}))}}, \quad \sin \phi = \frac{1 - h^2 \tilde{C}^2 \sin (h(s - \tilde{s}))}{\sqrt{1 - \left(1 - h^2 \tilde{C}^2 \right) \cos^2 (h(s - \tilde{s}))}}.
\]  
(7.30)

With these representations, we are able to obtain from the formulas (7.11)-(7.13) the explicit behavior of the unit vector components \( l^1 = R^1 / K \), \( l^2 = R^2 / K \), and \( l^3 = R^3 / K \) along the geodesic arc. The result reads
\[
e^{-\frac{i}{2}g\chi(s)} l^1(s) = \tilde{C} \sin (h s),
\]  
(7.31)
\[
e^{-\frac{i}{2}g\chi(s)} l^2(s) = \frac{1}{h} \sqrt{1 - h^2 \tilde{C}^2} \sin (h s),
\]  
(7.32)
and
\[
e^{-\frac{i}{2}g\chi(s)} l^3(s) = \sqrt{1 - h^2 \tilde{C}^2 \cos (h s) - \frac{G}{2} \sqrt{1 - \left(1 - h^2 \tilde{C}^2 \right) \cos^2 (h s)}},
\]  
(7.33)

where \( \chi(s) \) is the function (7.26) and we have put \( \tilde{s} = 0 \).

Given a geodesic-arc \( A(x, l_1, l_2) \). Let the left-side vector \( l_1 \) correspond to \( s = s_1 \), and the right-side vector \( l_2 \) relate to a value \( s_2 > s_1 \). From (7.31)-(7.33) we obtain
\[
e^{-\frac{i}{2}g\chi_1} l^1_1 = \tilde{C} \sin (h s_1),
\]  
(7.34)
\[
e^{-\frac{i}{2}g\chi_1} l^2_1 = \frac{1}{h} \sqrt{1 - h^2 \tilde{C}^2} \sin (h s_1),
\]  
(7.35)
\[
e^{-\frac{i}{2}g\chi_1} l^3_1 = \sqrt{1 - h^2 \tilde{C}^2 \cos (h s_1) - \frac{G}{2} \sqrt{1 - \left(1 - h^2 \tilde{C}^2 \right) \cos^2 (h s_1)}},
\]  
(7.36)
where
\[
\chi_1 = \frac{1}{h} \arccos \left( \sqrt{1 - h^2 \tilde{C}^2 \cos (h s_1)} \right),
\]  
(7.37)
and
\[
e^{-\frac{i}{2}g\chi_2} l^1_2 = \tilde{C} \sin (h s_2),
\]  
(7.38)
\[
e^{-\frac{i}{2}g\chi_2} l^2_2 = \frac{1}{h} \sqrt{1 - h^2 \tilde{C}^2} \sin (h s_2),
\]  
(7.39)
\[
e^{-\frac{i}{2}g\chi_2} l^3_2 = \sqrt{1 - h^2 \tilde{C}^2 \cos (h s_2) - \frac{G}{2} \sqrt{1 - \left(1 - h^2 \tilde{C}^2 \right) \cos^2 (h s_2)}},
\]  
(7.40)
where
\[
\chi_2 = \frac{1}{h} \arccos \left( \sqrt{1 - h^2 \tilde{C}^2 \cos (h s_2)} \right),
\]  
(7.41)

With the last formulas, the representations (7.31)-(7.33) can be written as follows:
\[
l^1(s) = k_1 l^1_1 + k_2 l^1_2, \quad l^2(s) = k_1 l^2_1 + k_2 l^2_2, \quad l^3(s) = k_1 l^3_1 + k_2 l^3_2 + k_3,
\]  
(7.42)
where
\[
k_1 = \left[ \frac{\sin(h(s - s_1))}{\sin(h(s_2 - s_1))} \right] \sin(h(s_1)) + \left[ \frac{\sin(h(s_2 + s_1))}{\sin(h(s_2 - s_1))} \right] \sin(h(s_2)) e^{\frac{1}{2}g(\chi(s) - \chi_1)},
\]
\[
k_2 = \left[ \frac{\sin(h(s_2 - s))}{\sin(h(s_2 - s_1))} \right] \sin(h(s_1)) - \left[ \frac{\sin(h(s_2 + s_1))}{\sin(h(s_2 - s_1))} \right] \sin(h(s_2)) e^{\frac{1}{2}g(\chi(s) - \chi_2)},
\]
\[
k_3 = \frac{g}{2h} Y,
\]
and
\[
Y = -\sqrt{1 - \left(1 - h^2\tilde{C}^2\right)\cos^2(hs)} e^{\frac{1}{2}g\chi(s)} + k_1 \sqrt{1 - \left(1 - h^2\tilde{C}^2\right)\cos^2(hs_1)} e^{\frac{1}{2}g\chi_1}
+ k_2 \sqrt{1 - \left(1 - h^2\tilde{C}^2\right)\cos^2(hs_2)} e^{\frac{1}{2}g\chi_2}.
\]

Thus we have arrived at the vector representation
\[
l^p = k_1 l_1^p + k_2 l_2^p + k_3 \delta_3^p
\]
With respect to arbitrary local coordinates \(x^i\), we eventually obtain the expansion
\[
l^i(s) = k_1(s) l_1^i + k_2(s) l_2^i + k_3(s) b^i,
\]
which does involve the vector \(b^i\) in addition to \(l_1^i, l_2^i\).

The function (7.46) possesses the property
\[
\tilde{C} = 0 \quad \Rightarrow \quad Y = 0
\]
(at any value of \(g\)).

**Appendix A: Involved \(\mathcal{F}_g^{PD}\)-notions**

By \(K\) we denote the metric function obtainable from the formulas (2.14) and (6.1)–(6.6).

**Definition.** Within any tangent space \(T_xM\), the function \(K(x, y)\) produces the \(\mathcal{F}_g^{PD}\)-Finsleroid
\[
\mathcal{F}_g^{PD}(x) := \{y \in \mathcal{F}_g^{PD} : y \in T_xM, K(x, y) \leq 1\}.
\]

**Definition.** The \(\mathcal{F}_g^{PD}\)-Indicatrix \(\mathcal{I}_g^{PD}(x) \subset T_xM\) is the boundary of the \(\mathcal{F}_g^{PD}\)-Finsleroid, that is,
\[
\mathcal{I}_g^{PD}(x) := \{y \in \mathcal{F}_g^{PD} : y \in T_xM, K(x, y) = 1\}.
\]

**Definition.** The scalar \(g(x)\) is called the Finsleroid charge. The 1-form \(b = b_i(x)y^i\) is called the Finsleroid-axis 1-form.
We can explicitly extract from the function $K$ the distinguished Finslerian tensors, and first of all the covariant tangent vector $\hat{y} = \{y_i\}$ from $y_i := (1/2)\partial K^2 / \partial y^i$, obtaining
\[
y_i = (u_i + gq b_i) \frac{K^2}{B}, \tag{A.3}
\]
where $u_i = a_{ij} y^j$. After that, we can find the Finslerian metric tensor $\{g_{ij}\}$ together with the contravariant tensor $\{g^{ij}\}$ defined by the reciprocity conditions $g_{ij} g^{jk} = \delta_i^k$, and the angular metric tensor $\{h_{ij}\}$, by making use of the following conventional Finslerian rules in succession:
\[
g_{ij} := \frac{1}{2} \frac{\partial^2 K^2}{\partial y^i \partial y^j} = \frac{\partial y_i}{\partial y^j}, \quad h_{ij} := g_{ij} - y_i y_j \frac{1}{K^2},
\]
thereafter the Cartan tensor
\[
A_{ijk} := \frac{K}{2} \frac{\partial g_{ij}}{\partial y^k} \tag{A.4}
\]
and the contraction
\[
A_i := g^{jk} A_{ijk} = K \frac{\partial \ln(\sqrt{\det(g_{mn})})}{\partial y^i} \tag{A.5}
\]
can readily be evaluated.

It can straightforwardly be verified that
\[
\det(g_{ij}) = \left( \frac{K^2}{B} \right)^N \det(a_{ij}) > 0. \tag{A.6}
\]
Contracting the components $A_i$ and $A^i$ yields the formula
\[
A^i A_i = \frac{N^2 g^2}{4} \tag{A.7}
\]
and evaluating the Cartan tensor results in the lucid representation
\[
A_{ijk} = \frac{1}{N} \left[ A_i h_{jk} + A_j h_{ik} + A_k h_{ij} - \frac{4}{N^2 g^2} A_i A_j A_k \right]. \tag{A.8}
\]

If we insert (A.8) into the indicatrix curvature tensor (5.3), we obtain the representation (5.6) which manifests that in the $\mathcal{FF}^{PD}_g$-space the indicatrix is of constant positive curvature (in compliance with (2.6)).

We use the Riemannian covariant derivative
\[
\nabla_i b_j := \partial_i b_j - b_{k a}^k a_{ij} \tag{A.9}
\]
where
\[
a_{ij} := \frac{1}{2} a^{kn} \partial_i a_{mn} + \partial_n a_{ij} - \partial_i a_{nj} \tag{A.10}
\]
are the associated Riemannian Christoffel symbols.

The associated Riemannian metric tensor $a_{ij}$ has the meaning
\[
a_{ij} = g_{ij} \bigg|_{g=0}.
\]

The following explicit representation is obtained:
\[
g_{ij} = \left[ a_{ij} + \frac{q}{B} \left( (gq^2 - \frac{bS^2}{q}) b_i b_j - \frac{b}{q} u_i u_j + \frac{S^2}{q} (b_i u_j + b_j u_i) \right) \right] \frac{K^2}{B}. \tag{A.11}
\]
The reciprocal components \((g^{ij}) = (g_{ij})^{-1}\) read
\[
g^{ij} = \left[ a^{ij} + \frac{g}{q} \left( b^{ib^{j'}} - b^{i}y^{j'} - b^{j}y^{i} \right) + \frac{g}{Bq} (b + gq)y^{i}y^{j} \right] \frac{B}{K^2}. \tag{A.12}
\]

In many cases it is convenient to use the variables
\[
v^i := y^i - bb^i, \quad v_m := u_m - bb_m = r_{mn} y^n \equiv r_{mn} v^n \equiv a_{mn} v^n,
\tag{A.13}
\]
where \(r_{mn} = a_{mn} - b_m b_n\). Notice that
\[
\begin{align*}
r^i_n &:= a^imr_{mn} = \delta^n_i - b^i b_n = \frac{\partial v^i}{\partial y^n}, \tag{A.14} \\
v_i b^i = v^i b_i = 0, \quad r_{ij} b^j = r^i_j b^j = b_i r^i_j = 0, \quad u_i v^i = v_i y^i = q^2, \tag{A.15} \\
q &= \sqrt{r_{ij} v^i v^j}, \tag{A.16}
\end{align*}
\]
and
\[
\begin{align*}
\frac{\partial b}{\partial y^i} &= b_i, \quad \frac{\partial q}{\partial y^i} = \frac{v_i}{q}, \quad \frac{\partial (b/q)}{\partial y^i} = \frac{2B}{NKq^2} A_i. \tag{A.17}
\end{align*}
\]

In terms of the variables (A.13) we obtain the representations
\[
y_i = \left( v_i + (b + gq)b_i \right) \frac{K^2}{B}, \tag{A.18}
\]
\[
g_{ij} = \left[ a_{ij} + \frac{g}{B} \left( q(b + gq)b_ib_j + q(b_i v_j + b_j v_i) - b^i v^j \right) \right] \frac{K^2}{B}, \tag{A.19}
\]
and
\[
g^{ij} = \left[ a^{ij} + \frac{g}{B} \left( -bq b^i b^j - q(b^i v^j + b^j v^i) + (b + gq) \frac{v^i v^j}{q} \right) \right] \frac{B}{K^2} \tag{A.20}
\]
which are alternative to (A.11)–(A.12).

We have
\[
y_i b^i = (b + gq) \frac{K^2}{B}, \quad g_{ij} b^j = (b_i + gq y_i \frac{y_i}{K^2}) \frac{K^2}{B}, \tag{A.21}
\]
\[
g_{ij} v^j = (S^2 v_i + gq^2 b_i) \frac{K^2}{B^2}, \tag{A.22}
\]
\[
g^{ij} a_{ij} = \frac{NB + gq^2}{K^2}, \quad h_{ij} b^j = (b_i - b y_i \frac{y_i}{K^2}) \frac{K^2}{B}. \tag{A.23}
\]

By the help of the formulas (A.5) and (A.12) we find
\[
A_i = \frac{NK}{2} g \frac{1}{q} (b_i - \frac{b}{K^2} y_i), \tag{A.24}
\]
or
\[
A_i = \frac{NK}{2} g \frac{1}{qB} (q^2 b_i - b v_i), \tag{A.25}
\]
and
\[ A^i = \frac{N}{2} g^2 \frac{1}{qK} \left[ Bb^i - (b + gq)y^i \right], \] (A.26)
or
\[ A^i = \frac{N}{2} g^2 \frac{1}{qK} \left[ q^2 b^i - (b + gq)v^i \right], \] (A.27)
together with
\[ A_i b^i = \frac{N}{2} gq \frac{K}{B}, \quad A^i b_i = \frac{N}{2} gq \frac{1}{K}. \] (A.28)
These formulas are convenient to verify the contraction (A.7) and the algebraic structure (A.8).

Since
\[ \frac{v^i v^j}{q} \to 0 \quad \text{when} \quad v^i \to 0 \] (notice (A.16)) the components \( g_{ij} \) and \( g^{ij} \) given by (A.11) and (A.12) are smooth on all the slit tangent bundle. However, the components of the Cartan tensor are singular at \( v^i = 0 \), as this is apparent from the above formulas (A.24)–(A.28) in which the pole singularity takes place at \( q = 0 \). Therefore, on the slit tangent bundle the \( \mathcal{F}_g^{PD} \)-space is smooth of the class \( C^2 \) and not of the class \( C^3 \).

Also,
\[ A_{ij} := K \partial A_i / \partial y^j + l_i A_j = -\frac{N}{2} gb \frac{q}{K} \mathcal{H}_{ij} + \frac{2}{N} A_i A_j \] (A.29)
with the tensor
\[ \mathcal{H}_{ij} = h_{ij} - \frac{A_i A_j}{A_n A^n}. \] (A.30)

It can readily be verified that
\[ g^{ij} \mathcal{H}_{ij} = N - 2, \] (A.31)
\[ g^{mn} \mathcal{H}_{im} \mathcal{H}_{jn} = \mathcal{H}_{ij}, \] (A.32)
\[ \mathcal{H}_{ij} = \left( r_{ij} - \frac{1}{q^2} v^i v^j \right) \frac{K^2}{B} = \eta_{ij} \frac{K^2}{B} \quad \text{and} \quad \mathcal{H}_{i}^{\ j} := g^{jm} \mathcal{H}_{ni} = r_{ij} - \frac{1}{q^2} v^i v^j \equiv \eta_{ij}^{\ j}. \] (A.33)
The last tensor fulfills obviously the identities
\[ \mathcal{H}_{ij} y^j = 0, \quad \mathcal{H}_{ij} b^j = 0, \] (A.34)
which in turn entails
\[ \mathcal{H}_{ij} A^j = 0 \] (A.35)
because \( A^i \) are linear combinations of \( y^i \) and \( b^i \) (see (A.26)). We also have
\[ K \left( \frac{\partial \mathcal{H}_{ij}}{\partial y^k} - \frac{\partial \mathcal{H}_{kj}}{\partial y^i} \right) = l_k \mathcal{H}_{ij} - l_i \mathcal{H}_{kj} - \frac{1}{A_n A^n} \frac{Ngb}{2q} (A_k \mathcal{H}_{ij} - A_i \mathcal{H}_{kj}) \] (A.36)
with the tensor
\[ \frac{\partial (q^2 \mathcal{H}_{ij})}{\partial y^k} - \frac{\partial (q^2 \mathcal{H}_{kj})}{\partial y^i} = 3 \begin{bmatrix} (\delta_i^j - b_i b^j)(a_{kn} y^n - b b_k) - (\delta_k^j - b_k b^j)(a_{in} y^n - b b_i) \end{bmatrix} = 3 (\mathcal{H}_{i}^{\ j} v_k - \mathcal{H}_{k}^{\ j} v_i). \] (A.37)
The structure (A.8) of the $\mathcal{F}\mathcal{F}_g^{PD}$-space Cartan tensor is such that

$$A_k A_i^k = \frac{1}{N} (A_i A_j + h_{ij} A_k A^k) = \frac{1}{N} (2A_i A_j + \mathcal{H}_{ij} A_k A^k),$$

(A.38)

so that the tensor

$$\tau_{ij} := A_{ij} - A_k A_i^k = -\frac{N g(2b + gq)}{q} \mathcal{H}_{ij},$$

(A.39)

obeys the identities

$$\tau_{ij} b^j = b_j \tau_i^j = \tau_{ij} A^j = 0.$$ (A.40)

The tensor

$$\tau_{ijmn} := K \partial A_{jmn}/\partial y^i - A_{ij}^h A_{hmn} - A_{im}^h A_{hnj} - A_{in}^h A_{hjm} + l_j A_{imn} + l_m A_{ijn} + l_n A_{ijm},$$

(A.41)

can be expressed as follows:

$$\tau_{ijmn} = -\frac{g(2b + gq)}{4q} (\mathcal{H}_{ij} \mathcal{H}_{mn} + \mathcal{H}_{im} \mathcal{H}_{jn} + \mathcal{H}_{in} \mathcal{H}_{jm}),$$

(A.42)

showing the total symmetry in all four indices and the properties

$$\tau_{ij} = g^{mn} \tau_{ijmn}$$

and

$$g^i \tau_{ijmn} = 0, \quad A^i \tau_{ijmn} = 0, \quad b^i \tau_{ijmn} = 0.$$ (A.43)

Evaluations frequently involve the vector $m_i = (2/Ng)A_i$ which possesses the properties

$$g^{ij} m_i m_j = 1, \quad y^i m_i = 0.$$

From (A.24) it follows that

$$m_i = K \frac{1}{q} (b_i - \frac{b}{K^2} y_i).$$

(A.44)

The equality

$$K \frac{\partial m_i}{\partial y^m} = -m_n l_i + g m_n m_i - \frac{b}{q} \mathcal{H}_{in},$$

(A.45)

holds. The contravariant components $m^i$ can be taken from (A.27):

$$m^i = \frac{1}{qK} \left[ q^2 b^i - (b + gq)v^i \right],$$

(A.46)

entailing

$$K \frac{\partial m^i}{\partial y^m} = -m_n l^i - g m^i m_n - \frac{1}{q}(b + gq) \mathcal{H}_{m}^i.$$ (A.47)

It is also valid that

$$K \frac{\partial \mathcal{H}_{kj}}{\partial y^m} = -g m_m \mathcal{H}_{kj} - l^k \mathcal{H}_{ij} - l^j \mathcal{H}_k + \frac{b}{q} (m^j \mathcal{H}_m^k + m^k \mathcal{H}_m^j)$$

(A.48)

and

$$K \frac{\partial \left( \mathcal{H}_{kj} \frac{K^2}{B} \right)}{\partial y^m} = \left[ -l^k \mathcal{H}_m^j - l^j \mathcal{H}_m^k + \frac{b}{q} (m^j \mathcal{H}_m^k + m^k \mathcal{H}_m^j) \right] \frac{K^2}{B}.$$ (A.49)
If we introduce the covariant derivative $S$ operative in the tangent Riemannian spaces, such that

$$ KS_m m^i = K \frac{\partial m^i}{\partial y^m} + A_m^i m^i, \quad KS_m m_i = K \frac{\partial m_i}{\partial y^m} - A_{m i}^i m_t, $$

we obtain

$$ KS_m m^i = -m_m^i - gm_m^i m_m - \frac{1}{q} (b + gq) H^i_m + \frac{1}{N} m^t \left[ A^t h_m + A_t h^t_m + A_m h^i_t - \frac{4}{N^2 g^2} A^t A_t A_m \right], $$

so that

$$ KS_m m_i = -m_m^i - \frac{1}{q} \left( b + \frac{1}{2} gq \right) H^i_m. \quad (A.50) $$

Also, with the definition

$$ KS_m H^{kj} = K \frac{\partial H^{kj}}{\partial y^m} + A^k_m H^{ij} + A^i_m H^{jk}, \quad (A.51) $$

we get

$$ KS_m H^{kj} = -l^k H^i_m - l^i H^k_m + \frac{1}{q} \left( b + \frac{1}{2} gq \right) (m^i H^k_m + m^k H^i_m). \quad (A.52) $$

The equality $\partial K^2 / \partial g = \bar{M} K^2$ holds with

$$ \bar{M} = -\frac{1}{h^3} f + \frac{1}{2 h B} q^2 + \frac{1}{h^2 B} b q. \quad (A.53) $$

In obtaining this formula we have used the derivatives

$$ \frac{\partial h}{\partial g} = -\frac{1}{4} G, \quad \frac{\partial G}{\partial g} = \frac{1}{h^3}, \quad \frac{\partial \left( \frac{G}{h} \right)}{\partial g} = \frac{1}{h^2} \left( 1 + \frac{g^2}{4} + \frac{1}{4} G q + \frac{1}{2 h} b \right). \quad (A.54) $$

Therefore,

$$ \partial_n^* K = \bar{M} K^2 \frac{\partial g}{\partial x^n}. \quad (A.55) $$

It follows that

$$ \frac{\partial \bar{M}}{\partial y^h} = \frac{2 b^4}{B^2} \frac{\partial b}{\partial y^m} = \frac{4 q^2 X}{g B K} A_h \quad (A.56) $$

and

$$ \partial_{n t}^* = \frac{\partial \left( \partial_n^* K \right)}{\partial y^m}. \quad (A.57) $$

The substitution

$$ a^{nt} = \frac{K^2}{B} H^{nt} + b^n b^t + \frac{1}{q^2} v^n v^t $$

transforms the tensor (6.71) to

$$ M^n_{ij} = \left( (1 - h) b + \frac{1}{2} gq \right) \frac{K^2}{B} H^{nt} \frac{1}{h} b_i a^t_{ij} - a^n_{i j} y^t.
\[
+ \left[ (1 - h)b \left( b^n y^t + \frac{1}{q^2} v^n v^t \right) + \frac{1}{2} gq \left( b^n b^t + \frac{1}{q^2} v^n v^t \right) + \left( \frac{g}{2q} v^n - (1 - h)b^n \right) y^t \right] \frac{1}{h} b_i a^t_{ij}
\]

\[
= \left( (1 - h)b + \frac{1}{2} gq \right) \frac{K^2}{B} \mathcal{H}^{n\mu} \frac{1}{h} b_i a^t_{ij} + \left[ (1 - h) \left( \frac{b}{q^2} v^n - b^n \right) + \frac{g}{q} v^n \right] \frac{1}{h} y^t b_i a^t_{ij} - a_i^n y^t.
\]

Applying
\[
y_i = (u_i + gb_i) \frac{K^2}{B}, \quad b^n = \frac{1}{B} \left( K q m^n + (b + gq)y^n \right)
\]
together with
\[
a_i^n y^t = h_i^n a_i^p y^t + l^n a^t_{ij} y^t, \quad l_i a^t_{ij} y^t = gq B b_i a^t_{ij} y^t
\]
leads to
\[
M^n_{ij} = \left( (1 - h)b + \frac{1}{2} gq \right) \frac{K^2}{B} \mathcal{H}^{n\mu} \frac{1}{h} b_i a^t_{ij} + \left[ (1 - h) \left( \frac{b}{q^2} v^n - b^n \right) + \frac{g}{q} v^n \right] \frac{1}{h} y^t b_i a^t_{ij}
\]

\[
- h_i^n a_i^p y^t - l^n gq B b_i a^t_{ij} y^t,
\]
or
\[
M^n_{ij} = \left( (1 - h)b + \frac{1}{2} gq \right) \frac{K^2}{B} \mathcal{H}^{n\mu} \frac{1}{h} b_i a^t_{ij} - \left[ (1 - h) \frac{B}{q} + hgb \right] \frac{1}{B} K m^n \frac{1}{h} y^t b_i a^t_{ij}
\]

\[
+ \left[ (1 - h) \left( \frac{b}{q^2} v^n - \frac{B - gbq}{q^2} \frac{1}{B} (b + gq)y^n \right) + gq y^n \frac{1}{B} \right] \frac{1}{h} y^t b_i a^t_{ij}
\]

\[
- h_i^n a_i^p y^t - l^n gq B b_i a^t_{ij} y^t,
\]
which is
\[
M^n_{ij} = \left( (1 - h)b + \frac{1}{2} gq \right) \frac{K^2}{B} \mathcal{H}^{n\mu} \frac{1}{h} b_i a^t_{ij} - \left[ (1 - h) \frac{B}{q} + hgb \right] \frac{1}{B} K m^n \frac{1}{h} y^t b_i a^t_{ij}
\]

\[
+ \left[ (1 - h) \left( \frac{b}{q^2} v^n - \frac{1}{q^2} (b + gq)y^n + \frac{g}{q} \frac{1}{B} (B - q^2)y^n \right) + gq y^n \frac{1}{B} \right] \frac{1}{h} y^t b_i a^t_{ij}
\]
\[-h^a_t a^i_{tj} y^t - l^m_i K B b_t a^l_{ij} y^t,\]

so that

\[
M^a_{ij} = \left( (1 - h)b + \frac{1}{2} gq \right) \frac{K^2}{B} \mathcal{H}^{nt} \frac{1}{h} b_t a^l_{ij} - \left[ B \frac{S^2}{q} - h \right] \frac{1}{h} K m^1_i y^t b_t a^l_{ij}
\]

\[-\mathcal{H}^t a^l_{tij} y^t - m^m_i a^l_{ij} y^t.\]

The result is

\[
M^a_{ij} = \left( (1 - h)b + \frac{1}{2} gq \right) \frac{K^2}{B} \mathcal{H}^{nt} \frac{1}{h} b_t a^l_{ij} - \mathcal{H}^t a^l_{tij} y^t - \frac{1}{q} K m^1_i y^t b_t a^l_{ij}. \tag{A.58}
\]

Lowering here the index \(n\) leads to the representation (6.72).

It is also possible to write

\[
\frac{B}{K^2} M_{nij} = \left( (1 - h)b + \frac{1}{2} gq \right) \frac{1}{h} b_n a^l_{nij} - \frac{1}{q^2} v_n y^t \left( (1 - h)b + \frac{1}{2} gq \right) \frac{1}{h} b_t a^l_{tij}
\]

\[-a_{tnij} y^t + b_n b_t a_{tnij} y^t - \frac{b}{q^2} v_n b_t a^l_{tij} y^t - \frac{B}{K^2} \frac{1}{q} K \frac{K}{qB} (q^2 b_n - b v_n) \frac{1}{h} y^t b_t a^l_{tij},\]

which can be simplified to read

\[
\frac{B}{K^2} M_{nij} = \left( (1 - h)b + \frac{1}{2} gq \right) \frac{1}{h} b_n a^l_{nij} - \left( \frac{q^2}{2 q} v_n + (1 - h)b_n \right) \frac{1}{h} y^t b_t a^l_{tij} - a_{tnij} y^t. \tag{A.59}
\]

Also, considering the contraction

\[
\frac{B}{K^2} M_{nij} M_{nij} = \]

\[
\left( (1 - h)b + \frac{1}{2} gq \right) \frac{1}{h} b_n a^l_{nij} \left[ (1 - h)b + \frac{1}{2} gq \right] \frac{1}{h} b_n a^l_{nij} - \left( \frac{q^2}{2 q} v_n + (1 - h)b_n \right) \frac{1}{h} y^t b_t a^l_{tij} - a_{tnij} y^t\]

\[+ \frac{q}{2 q} y^t b_t a^l_{tij} \left[ (1 - h)b + \frac{1}{2} gq \right] \frac{1}{h} b_n a^l_{nij} - \left( \frac{q^2}{2 q} v_n + (1 - h)b_n \right) \frac{1}{h} y^t b_t a^l_{tij}\]

\[+ \left( \frac{g}{2 q} b + 1 - h \right) y^t b_t a^l_{tij} \left[ (1 - h)b + \frac{1}{2} gq \right] \frac{1}{h} b_n a^l_{nij} + a_{tnij} y^t.\]
leads to (6.82).

We can start also with (A.58), observing that

\[ M_{nij} = \left( (1-h)b + \frac{q}{2} gq \right) \frac{K^2}{B} H_{n}^{h} b_{n} a_{n}^{t}_{i j} - \mathcal{H}_{p}^{n} a_{n}^{p}_{i j} y^{h} - \frac{1}{q} K m_{n}^{h} b_{n} a_{n}^{t}_{i j} \right] \]

\[ = \left( (1-h)b + \frac{1}{2} gq \right) \frac{K^2}{B} b_{n} a_{n}^{t}_{i j} - \mathcal{H}_{n}^{h} a_{t}^{t}_{i j} y^{h} - \frac{1}{q} K m_{n}^{h} b_{t} a_{t}^{t}_{i j} \]

\[ - \frac{1}{h^2} \left( (1-h)b + \frac{1}{2} gq \right) K^2 b_{p} a_{p}^{t}_{i j} \left( a^{t}_{h} - \frac{1}{q^2} y^{t}_{h} y^{h} \right) b_{t} a_{t}^{t}_{i j} \]

\[ -2 \frac{1}{h} \left( (1-h)b + \frac{1}{2} gq \right) K^2 b_{p} a_{p}^{t}_{i j} \left( \delta^{h}_{t} + \frac{1}{q^2} b y^{h}_{t} b_{t} \right) a_{t}^{t}_{i j} y^{t} \]

\[ + \frac{K^2}{B^2} \left( a^{t}_{p} - b_{p} b_{t} - \frac{1}{q^2} b^{2} b_{p} b_{t} \right) a_{t}^{t}_{i j} y^{t} + \frac{K^2}{B^2} g^{2} y^{h} b_{p} a_{h}^{p}_{i j} y^{t} b_{t} a_{t}^{t}_{i j} \]

\[ \left[ \left( (1-h)b + \frac{1}{2} gq \right) b_{n} a_{n}^{t}_{i j} - \frac{2}{h} \left( (1-h)b + \frac{1}{2} gq \right) y^{h} a_{h}^{n}_{i j} b_{i}^{i}_{n} + y^{h} a_{h}^{n}_{i j} a_{n}^{t}_{i j} y^{t} \right] \frac{K^2}{B} = E \]

with
\[
E = - \left[ \frac{1}{\hbar^2} \left( (1-h)b + \frac{1}{2}qg \right) \right]^2 \frac{K^2}{B} b_p a_h p_{ij} \frac{1}{q^2} + \frac{2}{\hbar} \left( (1-h)b + \frac{1}{2}qg \right) \frac{K^2}{B} b_p a_h p_{ij} \frac{b}{q^2} y^\prime b_l a^l_{ij} y^\prime
\]

\[
+ a_h p_{ij} y^h b \left( \frac{1}{q^2} b_p b_1 \right) a^l_{ij} y^\prime + \frac{1}{q^2} K^2 \frac{1}{\hbar^2} y^h b_p a_h p_{ij} y^\prime b_l a^l_{ij} \left( \frac{b + \frac{1}{2}qg}{K^2} \right)^2 = 0.
\]

Thus we obtain (6.82) from new standpoint.

The contraction can be written in the concise form

\[
M^{ij} M_{nj} = \frac{K^2}{h^2 B} \left[ h v^l + (b + \frac{g}{2} q) b^l \right] a^{ni} \left[ h v^h + (b + \frac{g}{2} q) b^h \right] a_{hnj}.
\]

With (6.26) and (6.39), we have

\[
\zeta^i = \left[ h v^i + (b + \frac{1}{2} qg) b^i \right] \frac{S}{\sqrt{B}},
\]

obtaining the simple representation

\[
M^{ij} M_{nj} = \frac{K^2}{h^2} \zeta^l a^{ni} \zeta^h a_{hnj}
\]

which is equivalent to (6.83); \( \zeta^2 = K^2/h^2 S^2 \) in accordance with (3.6).

Also, it is possible to get

\[
E_{n,ij} + \frac{1}{K} l_k M^{n,ij} - \frac{1}{K} l^n M_{kij} = \left[ - \left( (1-h)q - \frac{g}{2}(b + qg) \right) b^l - h K m^l \right] m_k \frac{K}{B} \mathcal{H}^n t^l a_{ltij}
\]

\[
+ \left( \frac{g}{2} b + q \right) m^n K \frac{1}{\hbar} \mathcal{H}^l k b_{ltij} - K \frac{1}{B} q m^n \mathcal{H}^l k b_{ltij}
\]

\[
- \frac{g}{2q} \mathcal{H}^l k b_{ltij} y^\prime + \mathcal{H}^n \mathcal{H}^l k a^l_{ij} + m^n m_i \mathcal{H}^l k a^l_{ij} - g m^n m_k \frac{1}{q} b_{ltij} y^\prime
\]

(A.60)

and

\[
B E_{knij} E^{knij} = \frac{B}{K^2} (K E_{knij} + l_k M_{nij} - l^n M_{kij}) (K E^{knij} + l^k M^{nij} - l^n M^{kij}) + 2 \frac{B}{K^2} M_{kij} M^{kij},
\]

(A.61)

together with

\[
E_{knij} E^{knij} = a_{knij} a^{knij} + \frac{g^2}{h^2 q^2} \left( (N - 2) \frac{1}{4} + 1 \right) b^l y^\prime a^l_{ij} y^h b^l a^l_{ij}
\]

\[
+ \frac{g^2}{B} \left[ \left( b - \frac{1}{h} \left( b + \frac{1}{2} qg \right) \right) b_h a^{nhij} - a^h_{nj} y^h \right] \left[ \left( b - \frac{1}{h} \left( b + \frac{1}{2} qg \right) \right) b_l a^l_{nj} - a_{ntij} y^l \right]
\]

(A.62)
(see Appendix C).

Let us verify the formulas (A.45) and (A.47).

Upon differentiating (A.44) we directly obtain the equality

$$\frac{\partial m_i}{\partial y^n} = \frac{1}{K} l_n m_i - \frac{1}{q^2} v_n m_i - \frac{1}{q} b_n l_i + \frac{b}{qK} l_n l_i - \frac{b}{qK} h_{iin},$$

in which the substitutions

$$b_n = \frac{q}{K} m_n + \frac{b}{K} l_n, \quad (A.63)$$

$$b v_n = q^2 b_n - \frac{1}{K} qB m_n = q^2 \frac{q}{K} m_n + q^2 \frac{b}{K} l_n - \frac{1}{K} qB m_n, \quad (A.64)$$

and

$$v_n = q^2 \frac{1}{K} l_n - \frac{1}{K} q(b + gq)m_n \quad (A.65)$$
can conveniently be used. We obtain

$$\frac{\partial m_i}{\partial y^n} = \frac{1}{K} l_n m_i - \frac{1}{q^2} v_n m_i + \frac{2}{q} v_n b^i - \frac{1}{q} \left(b_n + \frac{q}{q} v_n\right) v^i - \frac{1}{q} (b + gq) \left(\mathcal{H}_n^i + \frac{1}{q^2} v^i v_n\right)$$

$$= \frac{1}{q} (b + gq) m_n m_i - m_n l_i - \frac{b}{q} h_{iin},$$

so that the formula (A.45) is valid.

Also, we differentiate (A.46) and apply $b^i = (K q m^i + (b + gq) y^i) / B$ together with $v^i = q (-K b m^i + q y^i) / B$. We obtain

$$K \frac{\partial m^i}{\partial y^n} = -l_n m^i - \frac{1}{q^2} v_n m^i + \frac{2}{q} v_n b^i - \frac{1}{q} \left(b_n + \frac{q}{q} v_n\right) v^i - \frac{1}{q} (b + gq) \left(\mathcal{H}_n^i + \frac{1}{q^2} v^i v_n\right)$$

$$= -l_n m^i - \frac{K}{q^2} v_n m^i + \frac{2}{q} v_n \frac{K}{B} [qm^i + (b + gq) l^i] - \frac{1}{q} \left(\frac{q}{K} m_n + \frac{b}{K} l_n + \frac{q}{q} v_n\right) v^i$$

$$- \frac{1}{q} (b + gq) \frac{1}{q^2} v^i v_n - \frac{1}{q} (b + gq) \mathcal{H}_n^i$$

$$= -l_n m^i - \frac{1}{q^2} [q^2 l_n - q(b + gq)m_n] m^i$$

$$+ \frac{2}{q} q^2 \frac{1}{K} l_n \frac{K}{B} [qm^i + (b + gq) l^i] - \frac{2}{q} \frac{1}{q} q(b + gq)m_n \frac{K}{B} [qm^i + (b + gq) l^i] - \frac{1}{q} \left(\frac{q}{K} m_n + \frac{b}{K} l_n\right) v^i$$

$$- \frac{1}{q} (b + 2gq) \frac{1}{q^2} v^i q^2 \frac{1}{K} l_n + \frac{1}{q} (b + 2gq) \frac{1}{q^2} v^i \frac{1}{K} q(b + gq)m_n - \frac{1}{q} (b + gq) \mathcal{H}_n^i.$$
\[ = -2l_n m^i + \frac{1}{q^2}q(b + gq)m_n m^i \]
\[ + \frac{2}{q^2}l_n \frac{1}{B} [gm^i + (b + gq)l^i] - \frac{2}{q} \frac{1}{K} q(b + gq)m_n \frac{K}{B} [qm^i + (b + gq)l^i] - \frac{1}{K} m_n v^i \]
\[ - \frac{2}{q} (b + gq)v^i l_n + \frac{1}{q} (b + 2gq) \frac{1}{q^2} v^i \frac{1}{K} q(b + gq)m_n - \frac{1}{q} (b + gq)\mathcal{H}_n^i. \]
Making here natural reductions leads to
\[ K \frac{\partial m^i}{\partial y^n} = -2l_n m^i + \frac{1}{q} (b + gq)m_n m^i + \frac{2}{B} q^2 l_n m^i - \frac{2}{B} (b + gq)m_n [qm^i + (b + gq)l^i] \]
\[ - \frac{1}{K} m_n v^i + \frac{2}{q} (b + gq) \frac{q}{B} bm^i l_n + \frac{1}{q} (b + 2gq) \frac{1}{q^2} v^i \frac{1}{K} q(b + gq)m_n - \frac{1}{q} (b + gq)\mathcal{H}_n^i \]
\[ = -m_n l^i + \frac{1}{q} (b + gq)m_n m^i - \frac{1}{q} (b + 2gq)m_n m^i - \frac{1}{q} (b + gq)\mathcal{H}_n^i. \]
In this way the validity of (A.47) is straightforwardly verified.

Finally, considering the equality
\[ K \frac{\partial \mathcal{H}^{kj}}{\partial y^m} = K \frac{\partial (h^{kj} - m^k m^j)}{\partial y^m} = -2A_{-m}^k (l^k h_m^i + l_j h_m^k) \]
\[ + m^j \left[ m_m l^k + gm^k m_m + \frac{1}{q} (b + gq)\mathcal{H}^k_m \right] + m^k \left[ m_m l^j + gm^j m_m + \frac{1}{q} (b + gq)\mathcal{H}^j_m \right], \]
which is simplified to read
\[ K \frac{\partial \mathcal{H}^{kj}}{\partial y^m} = -gm_m \mathcal{H}^{ik} - gm^i \mathcal{H}^k_m - gm^k \mathcal{H}^i_m - 2gm^k m^j m_m - l^k (\mathcal{H}^j_m + m^j m_m) - l^j (\mathcal{H}^k_m + m^k m_m) \]
\[ + m^j \left[ m_m l^k + gm^k m_m + \frac{1}{q} (b + gq)\mathcal{H}^k_m \right] + m^k \left[ m_m l^j + gm^j m_m + \frac{1}{q} (b + gq)\mathcal{H}^j_m \right], \]
we can readily conclude that the formulas (A.48) and (A.49) are true.

**Appendix B: Representations for connection coefficients**

With (6.53) and (6.54) we evaluate the sum
\[ N^k_n + l^k \frac{\partial K}{\partial x^m} = N^k_n - l^k N^m_n l_m = \frac{gg}{B} y^k y^j \nabla_n b_j \]
\begin{align*}
+ \left[ \left( b - \frac{1}{h} \left( b + \frac{1}{2} gg \right) \right) \eta^{kj} + \left( \frac{1}{q^2} \nu^k \left( b - \frac{1}{h} (b + gg) \right) + \left( \frac{1}{h} - 1 \right) b^k \right) y^j \right] \nabla_n b_j - h_t^i a^t_{nj} y^j \\
= \left[ \left( b - \frac{1}{h} \left( b + \frac{1}{2} gg \right) \right) \eta^{kj} + \frac{1}{h q^2} \left[-(b + gg) \nu^k + q^2 b^k \right] y^j + \left( \frac{b}{q^2} \nu^k - b^k \right) y^j \right] \nabla_n b_j - h_t^i a^t_{nj} y^j \\
+ \frac{gg}{B} y^k y^j \nabla_n b_j,
\end{align*}
coming to the representation

\begin{equation}
N_k^n = -l^k \frac{\partial K}{\partial x^n} + \left[ \left( b - \frac{1}{h} \left( b + \frac{1}{2} gg \right) \right) \eta^{kj} + \left( \frac{1}{h q} - \frac{b^2 + q^2}{q B} \right) K m^k y^j \right] \nabla_n b_j - h_t^i a^t_{nj} y^j
\tag{B.1}
\end{equation}
which is obviously equivalent to (6.62).

Let us differentiate (6.62) with respect to $y^m$:

\begin{align*}
N_k^n = -\frac{1}{K} h_m^k \frac{\partial K}{\partial x^n} - l^k \frac{\partial l_m}{\partial x^n} + \left( b_m - \frac{1}{h} \left( b_m + \frac{1}{2q} g v_m \right) \right) H^{kj} \frac{K^2}{B} \nabla_n b_j \\
+ \left( b - \frac{1}{h} \left( b + \frac{1}{2} gg \right) \right) \left( -l^k H_m^j - \nu^i H_m^k + \frac{b}{q} (m^i H_m^k + m^k H_m^i) \right) \frac{K}{B} \nabla_n b_j \\
+ Z_m m^k y^i \nabla_n b_j + \left( \frac{1}{h q} - \frac{1}{q} + \frac{gb}{B} \right) \left[ -m_m l^k - g m^k m_m - \frac{1}{q} (b + gg) H_m^i \right] y^j \nabla_n b_j \\
+ \left( \frac{1}{h q} - \frac{1}{q} + \frac{gb}{B} \right) K m^k \nabla_n b_m - h_t^i a^t_{nm} + \frac{1}{K} (l^k h_m + l_t h_m^k) a^t_{nj} y^j,
\end{align*}
where

\begin{align*}
Z_m = K \left[ \frac{v_m h - 1}{q^2 h} + \frac{gb_m}{B} - \frac{gb}{B^2} \left( 2 b b_m + g q b_m + g b \frac{1}{q} v_m + 2 v_m \right) \right] + l_m \left( \frac{1}{h q} - \frac{1}{q} + \frac{gb}{B} \right).
\end{align*}

Apply

\begin{align*}
K b_m = q m_m + b l_m, \quad K v_m = q^2 l_m - q (b + gg) m_m,
\end{align*}
which yields

\begin{align*}
Z_m = -\frac{1}{h q^3} (q^2 l_m - q (b + gg) m_m) + \frac{1}{q^2} (q^2 l_m - q (b + gg) m_m) + \frac{g (q m_m + b l_m)}{B}
\end{align*}
\[-\frac{gb}{B^2} \left( b(2qm_m + 2bl_m) + gq(qm_m + bl_m) + gb(ql_m - (b + gq)m_m) + 2(q^2l_m - q(b + gq)m_m) \right)\]

\[+ l_m \left( \frac{1}{hq} - \frac{1}{q} + \frac{gb}{B} \right),\]

or

\[Z_m = -\frac{1}{hq^3} \left[ q^2l_m - q(b + gq)m_m \right] + \frac{1}{q^3} \left[ q^2l_m - q(b + gq)m_m \right] + \frac{(q + gb)m_m - bl_m}{B}\]

\[+ l_m \left( \frac{1}{hq} - \frac{1}{q} + \frac{gb}{B} \right).\]

So we have

\[Z_m = \frac{1}{hq^2}(b + gq) \left[ \frac{K^2}{B} l^i \nabla_n b_j + y_k a^k_{nj} l^j \right] - l^k \frac{\partial l_m}{\partial x^n} \]

\[+ \left( qm_m + bl_m - \frac{1}{h} \left( qm_m + bl_m + \frac{1}{2q} \left( g(q^2l_m - q(b + gq)m_m) \right) \right) \right) \frac{K}{B} l^k \nabla_n b_j \]

\[+ \left( b - \frac{1}{h} \left( b + \frac{1}{2} gq \right) \right) \left( -l^k \mathcal{H}^j_m - l^j \mathcal{H}^k_m + \frac{b}{q} \left( q^2 \mathcal{H}^j_m + m^k \mathcal{H}^i_m \right) \right) \frac{K}{B} \nabla_n b_j \]

\[+ \left( \frac{1}{hq^2}(b + gq) - \frac{1}{q^2} (b + gq) + g \frac{q + gb}{B} \right) m_m m^k y^j \nabla_n b_j \]

\[+ \left( \frac{1}{hq} - \frac{1}{q} + \frac{gb}{B} \right) \left[ -m_m l^k - gm^k m_m - \frac{1}{q} (b + gq) \mathcal{H}^k_m \right] y^j \nabla_n b_j \]

\[+ \left( \frac{1}{hq} - \frac{1}{q} + \frac{gb}{B} \right) Km^k \nabla_n b_m - h^k_m a^t_{nm} + \frac{1}{K} (l^k h_{tm} + l_t h^k_m) a^t_{nj} y^j.\]

Cancelling similar terms leads to

\[N^k_{nm} = -g \frac{q}{K} \frac{K^2}{B} l^i \nabla_n b_j - l^k \left( \frac{\partial l_m}{\partial x^n} - h_{tm} a^t_{nj} l^j \right).\]
The equality
\[
\frac{\partial K}{\partial x^n} = gq K^2 B l^j \nabla_n b_j + y t a^t_{nm}.
\] (B.3)

(we assume \( g = \text{const} \)) ensuing from (6.54) and (6.59) can readily be differentiated with respect to \( y^m \), yielding

\[
\frac{\partial l_m}{\partial x^n} - h t m a^t_{nj} l^j = g [q l_m - (b + gq) m] \frac{K}{B} l^j \nabla_n b_j + \frac{gq K}{B} h^j_m \nabla n b_j + g^2 q m m \frac{K^2}{B} l^j \nabla_n b_j + l t a^t_{nm},
\]
or

\[
\frac{\partial l_m}{\partial x^n} - h t m a^t_{nj} l^j = -g b m_m \frac{K}{B} l^j \nabla_n b_j + g q K^2 \frac{B}{B} \nabla_n b m + l t a^t_{nm}.
\] (B.4)

The representation (B.3) takes on the form

\[
N^k_{nm} = -g q \frac{K}{K} \frac{K^2}{B} l^j \nabla_n b_j - l^k \left[ -g b m_m \frac{K}{B} l^j \nabla_n b_j + g q K^2 \frac{B}{B} \nabla_n b m \right]
\]

\[
+ \frac{1}{K} \left[ q - \frac{1}{h} q + \frac{1}{2h} g (b + gq) \right] \frac{K^2}{B} \nabla_n b_j
\]

\[
+ \left( b - \frac{1}{h} \left( b + \frac{1}{2} gq \right) \right) \frac{K}{B} \left( \frac{K^2}{B} \nabla_n b_j - \frac{b}{q} \left( m \nabla_n b_j - \frac{1}{q} (b + gq) \right) \nabla_n b_j \right)
\]

\[
+ K \left[ \frac{1}{B} \frac{B}{B} - \frac{1}{B} \frac{B}{B} \right] m m^k l^j \nabla_n b_j - K \left[ \frac{1}{B} \frac{B}{B} \nabla_n b_j + \left( \frac{1}{B} \frac{B}{B} + \frac{g B}{B} \right) K m^k \nabla_n b m - a^k_{nm},
\]
or

\[ N_{nm}^k = -t^K g \frac{q}{K} \frac{K^2}{B} \nabla_n b_m + \frac{1}{K} \left[ \frac{1}{2h} g(b + gg) \right] K^{kj} m_m \frac{K^2}{B} \nabla_n b_j \]

\[ + \left( b - \frac{1}{h} \left( b + \frac{1}{2} g q \right) \right) \frac{K}{B} \left( \nabla_j l_m - t^K H_m^j - B m^K H_m^j + b m^K H_m^j \right) \nabla_n b_j \]

\[ + \frac{K}{q} \left[ \frac{1}{h q^2} b - \frac{1}{q^2} b \right] m_m m^K l^K \nabla_n b_j - \frac{K}{q} \left( \frac{1}{h} - 1 \right) m_m l^K l^K \nabla_n b_j \]

\[-K \left( \frac{1}{h q} (b + g q) - \frac{b}{q} \right) \frac{1}{q} H_m^j \nabla_n b_j + \left( \frac{1}{h q} - \frac{1}{q} + \frac{g b}{B} \right) K m^K \nabla_n b_m - \alpha^K_{nm}. \quad (B.5) \]

By the help of the equalities

\[ B \frac{b}{q^2} b^i = K \frac{b}{q} - m^i + \frac{B - q^2}{q^2} y^i, \quad \frac{b}{q} m^j - \nu = \frac{B}{K} \frac{b}{q^2} b^j - \frac{B}{q^2} v^j \]

we come to the representation

\[ N_{nm}^k = -t^K g \frac{q}{K} \frac{K^2}{B} \nabla_n b_m + \frac{1}{K} \left[ \frac{1}{2h} g(b + gg) \right] K^{kj} m_m \frac{K^2}{B} \nabla_n b_j \]

\[ + \left( b - \frac{1}{h} \left( b + \frac{1}{2} g q \right) \right) \frac{K}{B} \left( \nabla_j l_m - t^K H_m^j + \frac{b}{q} m^K H_m^j \right) \nabla_n b_j + \frac{b}{q^2} \frac{1}{h} - m_m m^K l^K \nabla_n b_j \]

\[-K \left( \frac{1}{h q} (b + g q) - \frac{b}{q} \right) \frac{1}{q} H_m^j \nabla_n b_j + \left( \frac{1}{h q} - \frac{1}{q} + \frac{g b}{B} \right) K m^K \nabla_n b_m - \alpha^K_{nm} \]

which is reduced to read

\[ N_{nm}^k = -t^K g \frac{q}{K} \frac{K^2}{B} \nabla_n b_m + \frac{1}{K} \left[ \frac{1}{2h} g(b + gg) \right] K^{kj} m_m \frac{K^2}{B} \nabla_n b_j \]

\[ + \left( b - \frac{1}{h} \left( b + \frac{1}{2} g q \right) \right) \frac{K}{B} \left( \nabla_j l_m - t^K H_m^j + \frac{b}{q} m^K H_m^j \right) \nabla_n b_j \]

\[ + \frac{K}{q^2} \left( \frac{1}{h} - 1 \right) m_m m^K l^K \nabla_n b_j - \frac{K}{q} \left( \frac{1}{h} - 1 \right) m_m l^K l^K \nabla_n b_j - \frac{g}{2 q h} H_m^j \nabla_n b_j \]
\[ + \left( \frac{1}{hq} - \frac{1}{q} + \frac{gb}{B} \right) K m^k \nabla_n b_m - a^k_{nm}. \]  \hspace{1cm} (B.6)

If we write this expression in the form

\[ N^k_{nm} = -l^k g q \frac{K^2}{K} \nabla_n b_m + \frac{1}{K} \left[ q - \frac{1}{h} q + \frac{1}{2h} g(b + gq) \right] \mathcal{H}^{kj} m_m \frac{K^2}{B} \nabla_n b_j \]

\[ + \left( b - \frac{1}{h} \left( b + \frac{1}{2} g q \right) \right) \frac{K}{B} \left( \mathcal{H}^{kj} m_m - l^k \mathcal{H}^j_m + \frac{b}{q} m^k \mathcal{H}^j_m \right) \nabla_n b_j \]

\[ + K \left[ \frac{1}{hq^2} b - \frac{1}{q^2} b \right] m_m m^k l^j \nabla_n b_j - \frac{K}{q} \left( \frac{1}{h} q - 1 \right) m_m l^k l^j \nabla_n b_j - g \frac{1}{2qh} \mathcal{H}^{kj} y^j \nabla_n b_j \]

\[ + \left( \frac{1}{hq} - \frac{1}{q} + \frac{gb}{B} \right) \left( K m^k - \frac{q}{b} y^k \right) \nabla_n b_m + \left( \frac{1}{hq} - \frac{1}{q} \right) \frac{g}{b} y^k \nabla_n b_m - a^k_{nm}. \]

and use

\[ \mathcal{H}_{ij} = \left( r_{ij} - \frac{1}{q^2} v_i v_j \right) \frac{K^2}{B} \quad \text{and} \quad \mathcal{H}^i_j = r^i_j - \frac{1}{q^2} v_i v^j, \]

we obtain

\[ N^k_{nm} = \frac{1}{K} \left[ q - \frac{1}{h} q + \frac{1}{2h} g(b + gq) \right] \left( a^{kj} - \frac{1}{q^2} v^k y^j \right) m_m \nabla_n b_j \]

\[ + \left( b - \frac{1}{h} \left( b + \frac{1}{2} g q \right) \right) \frac{1}{B} \left( \left( a^{kj} - \frac{1}{q^2} v^k y^j \right) v_m + \left( a^{kj} - \frac{1}{q^2} v^k y^j \right) (b + gq) b_m \right) \nabla_n b_j \]

\[ + \left( b - \frac{1}{h} \left( b + \frac{1}{2} g q \right) \right) \frac{1}{B} \left( -l^k + \frac{b}{q} m^k \right) \nabla_n b_m \]

\[ + \frac{1}{q^2} \left( b - \frac{1}{h} \left( b + \frac{1}{2} g q \right) \right) \frac{1}{B} \left( y^k - \frac{b}{q} K^k \right) m_m y^j \nabla_n b_j \]

\[ + K \left[ \frac{1}{hq^2} b - \frac{1}{q^2} b \right] m_m m^k l^j \nabla_n b_j - \frac{K}{q} \left( \frac{1}{h} q - 1 \right) m_m l^k l^j \nabla_n b_j - g \frac{1}{2qh} \mathcal{H}^{kj} y^j \nabla_n b_j \]

\[ + \left( \frac{1}{hq} - \frac{1}{q} + \frac{gb}{B} \right) \left( K m^k - \frac{q}{b} y^k \right) \nabla_n b_m + \left( \frac{1}{hq} - \frac{1}{q} \right) \frac{q}{b} y^k \nabla_n b_m - a^k_{nm}. \]
Taking into account the formula

\[ m_m = \frac{K}{qB}(q^2b_m - bv_m), \]

we arrive at

\[ N^k_{nm} = \left[q - \frac{1}{h}q + \frac{1}{2h}g(b + qg) - q - \frac{1}{h}q + \frac{1}{2h}g(b + qg)\right] \frac{1}{qB}b_m a^{kj} \nabla_n b_j \]

\[ - \left[q - \frac{1}{h}q + \frac{1}{2h}g(b + qg)\right] \frac{1}{q^2}v^k v_m b_m v^j \nabla_n b_j + \left[q - \frac{1}{h}q + \frac{1}{2h}g(b + qg)\right] \frac{1}{q^2}v^k \frac{1}{qB}v_m b_m v^j \nabla_n b_j \]

\[ + \left(b - \frac{1}{h}\right) b + \frac{1}{2}gq \right) \frac{1}{B} \left(v_m + (b + qg) b_m\right) a^{kj} \nabla_n b_j \]

\[ - \left(b - \frac{1}{h}\right) b + \frac{1}{2}gq \right) \frac{1}{B} q^2 v^k v_m y^j \nabla_n b_j - \left(b - \frac{1}{h}\right) b + \frac{1}{2}gq \right) \frac{1}{B} q^2 v^k (b + qg) b_m y^j \nabla_n b_j \]

\[ + \frac{1}{h} \frac{q_k}{2gq} \frac{K}{B} \left( -l^k + \frac{b}{q} m^k \right) \nabla_n b_m \]

\[ + \frac{1}{q^2} \left(b - \frac{1}{h}\right) b + \frac{1}{2}gq \right) \frac{1}{B} \left(y^k - \frac{b}{q} K m^k \right) v_m y^j \nabla_n b_j \]

\[ - \frac{q}{b} \left[\frac{1}{h}\frac{b}{q} - \frac{1}{q^2}\right] m_m \left(y^k - \frac{b}{q} K m^k \right) v^j \nabla_n b_j \]

\[ + \frac{q}{b} \left[\frac{1}{h}\frac{b}{q} - \frac{1}{q^2}\right] m_m y^k l^j \nabla_n b_j - \frac{K}{q} \left(\frac{1}{h} - 1\right) m_m l^k l^j \nabla_n b_j - \frac{1}{2q} \frac{1}{B} \nabla_n b_m \]

\[ + \left(\frac{1}{h} - \frac{b}{hqB} (b + qg) - \frac{q}{B}\right) \left(K m^k - \frac{b}{q} y^k \right) \nabla_n b_m + \left(\frac{1}{h} - \frac{1}{q}\right) \frac{1}{hq} \frac{y^k}{b} \nabla_n b_m - a^{kn}_{nm}. \]

Noting also

\[ y^k - \frac{b}{q} K m^k = \frac{B}{q^2} v^k, \]

we observe that

\[ N^k_{nm} = \left(1 - \frac{1}{h}\right) b_m a^{kj} \nabla_n b_j - \frac{q}{2h} v_m a^{kj} \nabla_n b_j \]
\[-\left(q - \frac{1}{h} q\right) v^k \frac{1}{qB} b_m y^j \nabla_n b_j + \left[q - \frac{1}{h} q + \frac{1}{2h} g(b + gq)\right] \frac{1}{q^2} v^k \frac{1}{qB} b_m y^j \nabla_n b_j \]

\[ - \left(b - \frac{1}{h} \left(b + \frac{1}{2} gq\right)\right) \frac{1}{B} \frac{1}{q^2} v^k v_m y^j \nabla_n b_j - \left(b - \frac{1}{h} b\right) \frac{K}{B} \frac{1}{q^2} v^k (b + gq) b_m y^j \nabla_n b_j \]

\[ - \frac{1}{h} 2 q \frac{1}{q^2} v^k \nabla_n b_m + \frac{1}{q^2} \left(b - \frac{1}{h} \left(b + \frac{1}{2} gq\right)\right) \frac{1}{q^2} v^k v_m y^j \nabla_n b_j \]

\[ - \frac{q}{b} \left[ \frac{1}{h q^2} b - \frac{1}{q^2} \right] m_m B \frac{1}{q^2} v^k l^i \nabla_n b_j - g \frac{1}{2 q h} \mathcal{H}^k_m y^j \nabla_n b_j + \left(1 - \frac{1}{h}\right) b^k \nabla_n b_m - a^k_{nm}. \]

These coefficients fulfill the equality $N^k_{nm} = -D^k_{nm}$ with $D^k_{nm}$ given by (6.49).

**Appendix C: Evaluation of curvature tensor**

The substitution

\[ a^{nt} = \frac{K^2}{B} \mathcal{H}^{nt} + b^n b^t + \frac{1}{q^2} v^n v^t \]

changes the representation (6.75) to the form

\[ E^t_k n_{ij} = -\left( (1 - h) b_k + \frac{1}{2 q} v_k \right) \frac{K^2}{B} \mathcal{H}^{nt} \frac{1}{h} b_n a_t^m_{ij} \]

\[ - \left[ \left( (1 - h) b_k + \frac{1}{2 q} v_k \right) \frac{1}{q^2} v^n y^t + \frac{g}{2 q} \eta^l y^t + \left( \frac{g}{2 q} v^n - (1 - h) b^n \right) \delta^t_k \right] \frac{1}{h} b_n a_t^m_{ij} + a_k^{n ij}. \]

Noting also the expansion

\[ \delta^t_k = \mathcal{H}^t_k + b_k b^t + \frac{1}{q^2} v_k v^t, \]

we obtain

\[ E^t_k n_{ij} = -\left( (1 - h) b_k + \frac{1}{2 q} v_k \right) \frac{K^2}{B} \mathcal{H}^{nt} \frac{1}{h} b_n a_t^m_{ij} - \left( \frac{g}{2 q} v^n - (1 - h) b^n \right) \delta^t_k \frac{1}{h} b_n a_t^m_{ij} \]

\[ - \left[ \left( (1 - h) b_k + \frac{1}{2 q} v_k \right) \frac{1}{q^2} v^n y^t + \frac{g}{2 q} \eta^l y^t + \left( \frac{g}{2 q} v^n - (1 - h) b^n \right) \frac{1}{q^2} v_k y^t \right] \frac{1}{h} b_n a_t^m_{ij} \]

\[ + h^t_i a_k^{t ij} + l^n i_k a_i^{t ij} \]
Using here

In this way we come to

leads to

and apply (A.63)–(A.65). We obtain

and apply (A.63)–(A.65). We obtain

Using here

leads to

Therefore, the term

can be traced to be

In this way we come to

\[ b_k v^n - b^n v_k = l^n (q m_k + b l_k) - \frac{1}{B} \left( (B - gbq) l_k - gq^2 m_k \right) b^n \]

\[ = l^n (q m_k + b l_k) - \frac{1}{B} \left( (B - gbq) l_k - gq^2 m_k \right) \frac{1}{B} \left( (B - gbq) l_k - gq^2 m_k \right) \frac{1}{B} (b + gq) l^n. \]
or

\[ b_k v^n - b^n v_k = ql^n m_k - ((B - gbq) l_k - gq^2 m_k) \frac{1}{B} qm^n - \frac{gq^3}{B} l_k l^n + gq^2 m_k \frac{1}{B} (b + gq) l^n. \]  \hspace{1cm} (C.1)

It will be noted also that

\[ l_i a_t^l_{ij} y^t = gq \frac{K}{B} b_i a_t^l_{ij} y^t. \]  \hspace{1cm} (C.2)

Thus we can write

\[ E_k^n_{ij} = - \left( 1 - h \right) b_k + \frac{1}{2} q v_k \frac{K^2}{B} \mathcal{H}^n \frac{1}{h} b_m a^m_{ij} - \left( \frac{g}{2q} v^n - (1 - h) b^n \right) \mathcal{H}^n \frac{1}{h} b_m a^m_{ij} \]

\[ - \frac{g}{2q} \mathcal{H}_k^n y^t \frac{1}{h} b_m a^m_{ij} - \left[ l^n m_k + g m_k m^n - l_k m^n \right] \frac{1}{q} b_m a^m_{ij} y^t \]

\[ + \left[ q l^n m_k - ((B - gbq) l_k - gq^2 m_k) \frac{1}{B} qm^n - \frac{gq^3}{B} l_k l^n + gq^2 m_k \frac{1}{B} (b + gq) l^n \right] \frac{1}{q^2} b_m a^m_{ij} y^t \]

+ \left[ h^n h^p a^l_{ij} + h^n l_k l^p a^l_{ij} + l^n l_i h^p a^l_{ij} + l^n l_k g q \frac{1}{B} b_i a^l_{ij} y^t. \right]

Applying the equalities

\[ y_i = (u_i + g q b_i) \frac{K^2}{B}, \hspace{1cm} b^n = \frac{1}{B} [K q m^n + (b + g q) y^n] \]

to

\[ l_i K q m^t a^l_{ij} = l_i \left[ B b^t - (b + g q) y^t \right] a^l_{ij} \]

yields

\[ l_i B q m^t a^l_{ij} = (u_i + g q b_i) \left[ B b^t - (b + g q) y^t \right] a^l_{ij} = \left[ B b^t u_i - g q b_i (b + g q) y^t \right] a^l_{ij}, \]

so that

\[ l_i m^t a^l_{ij} = - \frac{1}{B q} \left[ B + g q (b + g q) \right] b_i a^l_{ij} y^t. \]  \hspace{1cm} (C.3)

So we have

\[ E_k^n_{ij} = - \left( 1 - h \right) b_k + \frac{1}{2} q v_k \frac{K^2}{B} \mathcal{H}^n \frac{1}{h} b_m a^m_{ij} - \left( \frac{g}{2q} v^n - (1 - h) b^n \right) \mathcal{H}^n \frac{1}{h} b_m a^m_{ij} \]

\[ - \frac{g}{2q} \mathcal{H}_k^n y^t \frac{1}{h} b_m a^m_{ij} - \left[ l^n m_k + g m_k m^n - l_k m^n \right] \frac{1}{q} b_m a^m_{ij} y^t \]

\[ + \left[ q l^n m_k - ((B - gbq) l_k - gq^2 m_k) \frac{1}{B} qm^n + gq^2 m_k \frac{1}{B} (b + gq) l^n \right] \frac{1}{q^2} b_m a^m_{ij} y^t \]
\[ + h_t^n h_l a_t^l m_{ij} + h_l^n t_l a_t^l m_{ij} + l^n l_i H_k a_t^l m_{ij} - \frac{1}{B^q} \left[ B + gq(b + gq) \right] l^n m_k b_{i j} = 0 \]

\[ = - \left( (1 - h) k + \frac{1}{2 q} v^k \right) K^2 B H_t^m a_t^l m_{ij} - \left( \frac{g}{2 q} v^n - (1 - h) b^n \right) H_k^l b_{ij} \]

\[ - \frac{g}{2 q} H_k^l b_m a_{ij} - \left[ l^n m_k + g m_k m^n - l_k m^n \right] \frac{1}{q} b_m a_{ij} = 0 \]

\[ + H_k^l l_l a_t^l m_{ij} + l^n l_i H_k a_t^l b_{ij} = 0 \]

The next step is to consider the term

\[ m_l K m a_t^l m_{ij} = m_l \left[ B b^l - (b + gq) y^l \right] a_t^l m_{ij} \]

and use \((1/K) q B m_l = q^2 b_l - bv_l = S^2 b_l - bu_l\), obtaining

\[ q B_m q m a_t^l m_{ij} = (S^2 b_l - bu_l) \left[ B b^l - (b + gq) y^l \right] a_t^l m_{ij} = - \left[ bu_l B b^l + S^2 b_l (b + gq) y^l \right] a_t^l m_{ij}, \]

so that

\[ m_l m a_t^l m_{ij} = - \frac{gq}{B} b_{ij} a_t^l m_{ij}. \]  \( \text{C.4} \)

The studied tensor takes now the form

\[ E_k^n m_{ij} = - \left( (1 - h) k + \frac{1}{2 q} v^k \right) K^2 B H_t^m a_t^l m_{ij} - \left( \frac{g}{2 q} v^n - (1 - h) b^n \right) H_k^l b_{ij} \]

\[ - \frac{g}{2 q} H_k^l b_m a_{ij} = 0 \]

\[ + H_k^l l_l a_t^l m_{ij} + H_k^l m_k m^t a_t^l m_{ij} + m^n m_l H_k a_t^l b_{ij} = 0 \]

\[ = - \left( (1 - h) k + \frac{1}{2 q} v^k \right) K^2 B H_t^m a_t^l m_{ij} - \left[ l^n m_k + g m_k m^n - l_k m^n \right] \frac{1}{q} b_{ij} a_t^l m_{ij} \]

\[ - \left[ \frac{g}{2 q} v^n - \left( \frac{g}{2 q} b + 1 \right) K^1_B \left[ g m^n + (b + gq) l^n \right] + h K^1_B \left[ g m^n + (b + gq) l^n \right] \right] H_k^l b_{ij} a_t^l m_{ij} \]
Here, we can eventually write
\[ E_{k}^{n,ij} = \left( (1 - h)(qm_{k} + bl_{k}) + \frac{g}{2}(qK_{k} - (b + gq)m_{k}) \right) \frac{K}{B} \mathcal{H}^{nt} \frac{1}{h} b_{1} a_{t}^{l} a_{t}^{l} y^{t} \]

Recollecting
\[ b_{k} = \frac{1}{K}(qm_{k} + bl_{k}), \quad bv_{k} = \frac{1}{K}q \left( qK_{k} - (B - q^{2})m_{k} \right), \]

we can eventually write
\[ E_{k}^{n,ij} = \left( (1 - h)(qm_{k} + bl_{k}) + \frac{g}{2}(qK_{k} - (b + gq)m_{k}) \right) \frac{K}{B} \mathcal{H}^{nt} \frac{1}{h} b_{1} a_{t}^{l} a_{t}^{l} y^{t} \]

- \frac{g}{2q} bK_{1} \mathcal{H}^{t} \mathcal{H}^{m,ij} \mathcal{H}^{nt} b_{1} a_{t}^{l} a_{t}^{l} y^{t} \]

\[ + \mathcal{H}^{nt} l_{k} l_{i} a_{t}^{l} a_{t}^{l} + l_{m} l_{i} \mathcal{H}^{nt} a_{t}^{l} a_{t}^{l} + \mathcal{H}^{nt} m_{k} m_{i} a_{t}^{l} a_{t}^{l} + m_{n} m_{i} \mathcal{H}^{nt} a_{t}^{l} a_{t}^{l} \]

\[ - \left[ l_{m} m_{k} m_{n} - l_{k} m_{n} \right] \frac{1}{q} \frac{1}{h} b_{1} a_{t}^{l} a_{t}^{l} y^{t}. \] (C.5)

Here,
\[ \mathcal{H}^{nt} a_{t}^{l} a_{t}^{l} = \frac{K^{2}}{B} \mathcal{H}^{nt} a_{t}^{l} a_{t}^{l}. \]

The tensor
\[ E_{k}^{n,ij} + \frac{1}{K} l_{k} M_{n,ij} - \frac{1}{K} l_{m} M_{kij} = \left( (1 - h)q - \frac{g}{2}(b + gq) \right) m_{k} \frac{K}{B} \mathcal{H}^{nt} \frac{1}{h} b_{1} a_{t}^{l} a_{t}^{l} y^{t} \]

- \frac{g}{2m} b_{m} m_{n} - q m_{n} - (b + gq) l_{m} \frac{1}{B} \mathcal{H}^{nt} \frac{1}{h} b_{1} a_{t}^{l} a_{t}^{l} y^{t} \]

\[ - \frac{K}{B} \left[ q m_{n} + (b + gq) l_{m} \right] \mathcal{H}^{nt} b_{1} a_{t}^{l} a_{t}^{l} y^{t} - \frac{g}{2q} \mathcal{H}^{nt} b_{1} a_{t}^{l} a_{t}^{l} y^{t} + l_{m} \left( u_{i} + gq b_{i} \right) \frac{K}{B} \mathcal{H}^{nt} a_{t}^{l} a_{t}^{l} \]

\[ + \mathcal{H}^{nt} \mathcal{H}^{nt} a_{t}^{l} a_{t}^{l} a_{t}^{l} a_{t}^{l} + \mathcal{H}^{nt} m_{k} m_{i} a_{t}^{l} a_{t}^{l} + m_{n} m_{i} \mathcal{H}^{nt} a_{t}^{l} a_{t}^{l} - \left( l_{m} + gq m_{n} \right) m_{k} \frac{1}{q} \frac{1}{h} b_{1} a_{t}^{l} a_{t}^{l} y^{t} \]
\[
E_{k}^{i|j} + \frac{1}{K}l_{k}M_{n}^{i|j} - \frac{1}{K}l_{n}M_{k}^{i|j} = - \left[ \frac{1}{h} \left( q - \frac{g}{2} (b + gq) \right) b' - \frac{1}{q} \left[ \left( B - q^{2} \right) b' - (b + gq) y' \right] \right] m_{k} \frac{K}{B} \mathcal{H}^{a_{t}l_{ij}}
\]

\[
+ \left[ \frac{g}{2} b + q^{2} \right] b_{l} + h (b^{2} b_{l} - b_{u_{l}}) \right] m_{n}^{K} \frac{1}{B q} \mathcal{H}_{l_{ij}}^{a_{t}l_{ij}}
\]

\[
- \frac{g}{2 q} \mathcal{H}_{l_{ij}}^{b_{t}a_{t}l_{ij}y'} + \mathcal{H}_{l_{ij}}^{b_{t}a_{t}l_{ij}y'} + m_{n}^{K} m_{k} \frac{1}{q h} b_{t} a_{t}^{l_{ij}y'}. \quad (C.6)
\]

The contraction

\[
\frac{B}{K^{2}} h^{2} (K E_{km}^{ij} + l_{k} M_{n}^{ij} - l_{n} M_{k}^{ij}) (K E^{k_{n}ij} + l^{k} M^{nij} - l^{n} M^{k_{ij}})
\]

\[
= \left[ \left( q - \frac{g}{2} (b + gq) + \frac{1}{q} (B - q^{2}) \right) b' - \frac{1}{q} (b + gq) y' \right] a_{t}^{ij} \times
\]

\[
\left[ \left( q - \frac{g}{2} (b + gq) + \frac{1}{q} (B - q^{2}) \right) b^{h} - \frac{1}{q} (b + gq) y^{h} \right] a_{t}^{h_{ij}}
\]
\(-\frac{S^2}{q^2} \frac{1}{q^2} h^2 (b + gq)^2 b' y^t a_{t \cdot ij} y^h b^s a_{shij}\)

\[ + 2 \frac{b}{q^2} \left( \left( q - \frac{g}{2} (b + gq) \right) b' + h \frac{1}{q} (B - q^2) b' \right) a_{t \cdot ij} \left( \left( q - \frac{g}{2} (b + gq) \right) b^h + h \frac{1}{q} (B - q^2) b^h \right) a_{shij} \]

\[-\frac{1}{q^2} y^t y^s \left( \left( q - \frac{g}{2} (b + gq) \right) b' + h \frac{1}{q} (B - q^2) b' \right) a_{t \cdot ij} \left( \left( q - \frac{g}{2} (b + gq) \right) b^h + h \frac{1}{q} (B - q^2) b^h \right) a_{shij} \]

\[ + \frac{1}{q^2} \left( \left( \frac{g}{2} bq + q^2 + hb^2 \right) b' - hby^t \right) a_{t \cdot ij} \left( \left( \frac{g}{2} bq + q^2 + hb^2 \right) b^h - hby^h \right) a_{hij} \]

\[-\frac{1}{q^3} \frac{S^2}{q^2} h^2 b^2 b' y^t a_{u \cdot ij} y^h b^s a_{shij} + 2h \frac{1}{q^2} \frac{b^2}{q^2} \left( \frac{g}{2} bq + q^2 + hb^2 \right) b' y^t a_{u \cdot ij} y^h b^s a_{shij} \]

\[-\frac{1}{q^3} \frac{1}{q^2} y^t y^s \left( \frac{g}{2} bq + q^2 + hb^2 \right) b' a_{u \cdot ij} \left( \frac{g}{2} bq + q^2 + hb^2 \right) b^h a_{shij} \]

\[+(N - 2) \frac{g^2}{4q^2} b_i a_t \cdot ij y^t b_h a_s \cdot ij y^h B + g^2 \frac{1}{q^2} b_i a_t \cdot ij y^t b_h a_s \cdot ij y^h B + Bh^2 \mathcal{H}_h \mathcal{H}^s a_t \cdot ij a_s \cdot ij \]

can be written in the simple form

\[(KE_{kni} + l_k M_{nij} - l_n M_{kij}) (KE^{kni} + l^k M^{nij} - l^n M^{kij}) = KE_{kni} (KE^{kni} + l^k M^{nij} - l^n M^{kij}) \]

\[= K^2 E_{kni} E^{kni} - 2M_{kij} M^{kij}. \quad \text{(C.7)} \]

Therefore,

\[BE_{kni} E^{kni} = \frac{B}{K^2} (KA_{kni} + l_k M_{nij} - l_n M_{kij}) (KA^{kni} + l^k M^{nij} - l^n M^{kij}) + 2 \frac{B}{K^2} M_{kij} M^{kij}. \quad \text{(C.8)} \]

The term \(M_{kij} M^{kij}\) can be taken from (6.82).

The contraction (C.8) can be written in the alternative form

\[Bh^2 E_{kni} E^{kni} = Bh^2 a_{lh} a_{t} \cdot ij a_s \cdot ij \]
\[ + \frac{1}{q^2} T' a_{ij} b^h a'_{hij} - 2h \frac{1}{q^2} (Z - 2hbB) b' a_{ij} y^h a'_{hij} + \frac{1}{q^2} h^2 [g^2 q^2 - 2q^2] y' a_{ij} y^h a'_{hij} \]

\[ + g^2 \frac{B}{q^2} \left( (N - 2) \frac{1}{4} + 1 \right) b' y' a_{ij} y h a'_{hij} \]

\[ + 2 \left( (1 - h)b + \frac{g}{2} q \right) b_h a^{n_{hij}} - h a_{h_{ij}} y^h \left( (1 - h)b + \frac{g}{2} q \right) b_{ij} a_{hij} - ha_{h_{ij}} y^h \right). \quad (C.9) \]

Here,

\[ T = \left( \frac{g}{2} bq + q^2 + hb^2 \right)^2 + \left( q^2 - \frac{g}{2} q(b + gq) + h(B - q^2) \right)^2 - 2h^2 b^2 B - 2h^2 q^2 B \]

\[ = \left( \frac{g}{2} bq + q^2 \right)^2 + 2 \left( \frac{g}{2} bq + q^2 \right) hb^2 + h^2 b^4 \]

\[ + \left( q^2 - \frac{g}{2} q(b + gq) \right)^2 + 2 \left( q^2 - \frac{g}{2} q(b + gq) \right) hb(b + gq) + h^2 b^2 (b^2 + 2gbq + g^2 q^2) \]

\[ - 2h^2 b^2 (b^2 + gbq + q^2) - 2h^2 q^2 (b^2 + gbq + q^2), \]

so that

\[ T + 2 \left( (1 - h)b + \frac{g}{2} q \right)^2 q^2 = 2(1 - 2h + h^2)b^2 q^2 + 2gbq(1 - h)q^2 + 2(1 - h^2)q^4 \]

\[ + \left( \frac{g}{2} bq + q^2 \right)^2 + 2q^2 hb^2 \]

\[ + \left( q^2 - \frac{g}{2} q(b + gq) \right)^2 + 2 \left( q^2 - \frac{g}{2} q(gq) \right) hbq + h^2 g^2 b^2 q^2 - 2h^2 b^2 q^2 \]

\[ - 2h^2 q^2 (b^2 + gbq + q^2) \]

\[ = 4b^2 q^2 + 2gbq^2 + 4(1 - h^2)q^4 - g^2 q^4 + \frac{g^2}{4} q^2 (2gbq + g^2 q^2) \]

\[ - g^2 q^2 b(b + gq) + h^2 g^2 b^2 q^2 - 2h^2 b^2 q^2 - 2h^2 q^2 (b^2 + gbq). \]

Thus we have simply
\[
\frac{1}{q^2} \left[ T + 2 \left( (1 - h) b + \frac{g}{2} q \right)^2 \right] = g^2 \left( -\frac{1}{2} (2b + gq) + hb \right)^2. \tag{C.10}
\]

Another coefficient is
\[
Z = \left( q^2 - \frac{g}{2} (b + gq) + h (B - q^2) \right)b + \left( q^2 - \frac{g}{2} (b + gq) + h (B - q^2) \right)gq + b \left( \frac{g}{2} bq + q^2 + hb^2 \right).
\]

We can follow the steps
\[
Z - 2q^2 \left( (1 - h) b + \frac{g}{2} q \right) = -2q^2 (1 - h) b + \left( q^2 - \frac{g}{2} (b + gq) + h (B - q^2) \right) b
\]
\[+ \left( -\frac{g}{2} q (b + gq) + h (B - q^2) \right) gq + b \left( \frac{g}{2} bq + q^2 + hb^2 \right)
\]
\[= 2q^2 hb - \frac{g}{2} q b (b + gq) + hb^2 (b + gq) + \left( -\frac{g}{2} q + hb \right) gq (b + gq) + b \left( \frac{g}{2} bq + hb^2 \right)
\]
\[= -\frac{g^2}{2} b q^2 - \frac{g^2}{2} (b + gq) q^2 + g^2 b q^2 + 2hbB,
\]
obtaining
\[
\frac{1}{q^2} \left[ Z - 2q^2 \left( (1 - h) b + \frac{g}{2} q \right) - 2hbB \right] = -\frac{g^2}{2} (2b + gq) + g^2 hb. \tag{C.11}
\]

The contraction becomes eventually
\[
E_{knij} E^{knij} = a_{knij} a^{knij} + g^2 \frac{1}{h^2 q^2} \left( (N - 2) \frac{1}{4} + 1 \right) b^l y^l a^m_{ij} y^l h^n a_{nij}
\]
\[+ \frac{g^2}{B} \left[ \left( b - \frac{1}{h} \left( b + \frac{1}{2} gq \right) \right) b_l a^{nhij} - a^{n}_{hij} y^l \right] \left[ \left( b - \frac{1}{h} \left( b + \frac{1}{2} gq \right) \right) b_l a_{nij}^l - a_{nij} y^l \right]. \tag{C.12}
\]

Now we are to verify the representation (6.77)–(6.78).

Using (A.60) together with (6.73) yields
\[
\rho_k^{\mu_{ij}} + \frac{1}{K} l_k M^{\mu_{ij}} - \frac{1}{K} l^\mu M_{kij} = \left[ -\left( (1 - h) q - \frac{g}{2} (b + gq) \right) b^l - h K m^i \right] m_k \frac{K}{B} \mathcal{H}^{\mu_{ij}} a_{n_{ij}}
\]
\[+ \left( \frac{g}{2} b + q \right) m^n K \frac{1}{B} \mathcal{H}^{\mu_{ij}} b_{n_{ij}} - \frac{1}{K} q m^n \mathcal{H}^{\mu_{ij}} b_{n_{ij}}.
\]
\[-\frac{g}{2q} \mathcal{H}_k \frac{1}{h} b_i a_{t}^{i,j} y' + \mathcal{H}_k \mathcal{H}_k a_t^{i,j} + m^m m_i \mathcal{H}_k a_t^{i,j} - \frac{1}{q} \frac{1}{K} \frac{1}{N} (A_k m_{i,j} + A^a M_{kij}) \]

\[= -\frac{1}{K} g m_k H_t H_k b_i a_{t}^{i,j} + \mathcal{H}_k \mathcal{H}_k a_t^{i,j} + m^m m_i \mathcal{H}_k a_t^{i,j} - \frac{1}{q} \frac{1}{K} \frac{1}{N} (A_k m_{i,j} + A^a M_{kij}) \]

where (6.72) has been applied. Reducing similar terms leaves us with

\[\rho_{k,ij} + \frac{1}{K} l_k m^m_{i,j} - \frac{1}{K} l^m M_{kij} = \left[ \frac{g}{2} g b' - h K m^l \right] m_k \frac{K}{B} \mathcal{H}_k \frac{1}{h} a_{t}^{i,j} \]

\[+ \frac{1}{h} (1 - h) q K \frac{1}{B} (\mathcal{H}_k m^a - \mathcal{H}_k m_k) b_i a_{t}^{i,j} + \mathcal{H}_k \mathcal{H}_k a_t^{i,j} + m^m m_i \mathcal{H}_k a_t^{i,j} \]

\[-\frac{1}{K} \frac{1}{2} m_k \left[ (1 - h) b + \frac{1}{2} g q \right] \frac{K^2}{B} \mathcal{H}_k \frac{1}{h} b_i a_{t}^{i,j} - \mathcal{H}_k \mathcal{H}_k a_t^{i,j} \]

\[-\frac{1}{K} \frac{1}{2} m^a \left[ (1 - h) b + \frac{1}{2} g q \right] \frac{K^2}{B} \mathcal{H}_k \frac{1}{h} b_i a_{t}^{i,j} - \mathcal{H}_k \mathcal{H}_k a_t^{i,j} \]

Here it is convenient to apply the relation

\[m_i a^{i} - \frac{K}{B} \left[ \frac{q^2 + b^2}{B} K m^l + \frac{gL}{B} y^l \right] \]

which comes from the chain

\[m_i a^{i} = \frac{K}{qB} (q^2 b' - b v') = \frac{K}{qB} \left[ \frac{q^2}{B} \left[ K q m^l + (b + gq) y^l \right] - b \frac{q}{B} \left[ -K b m^l + q y^l \right] \right]. \]
By lowering the index, we obtain

\[
\rho_{k_{ij}} + \frac{1}{K} l_{k_{M_{ij}}} - \frac{1}{K} l_{n_{M_{kij}}} = \left[ \frac{g}{2} q_{b'} - h K m' \right] m_k K \frac{1}{h} a_{diij}
\]

\[
+ \frac{1}{h} (1 - h) q K \frac{1}{B} (H^l_{k_{M_{n}} - n_{M_{k}}} b_{a_{diij}}
\]

\[
+ H^l_{n_{M_{k}}} \frac{K^2}{B} + m_n K \left[ \frac{g^2 + b^2}{B} K m' + \frac{g q^2}{B} y' \right] H^l_{a_{diij}}
\]

\[
- \frac{1}{K^2} m_k \left[ (-hb + \frac{1}{2} gq) H^l_{n_{M_{k}}} - H^l_{a_{diij} y'} \right] \frac{K^2}{B}
\]

\[
- \frac{1}{K^2} m_n \left[ (-hb + \frac{1}{2} gq) H^l_{n_{M_{k}}} - H^l_{a_{diij} y'} \right] \frac{K^2}{B}.
\]

Using the equality \( B_b' = \left[ K q m' + (b + g q) y' \right] \) leads to the following result after a short simplification:

\[
\rho_{k_{ij}} = - \frac{1}{K} (l_{k_{M_{ij}}} - l_{n_{M_{kij}}} + (m_k H^l_{n_{M_{k}}} - m_n H^l_{n_{M_{k}}} P_{ij} + H^l_{a_{diij} K^2} \frac{B}{B})
\]

with

\[
P_{ij} = - \frac{1}{h} (1 - h) q K \frac{1}{B} b_{a_{diij}} - \frac{K^2}{B^2} \left( B - \frac{1}{2} g b q \right) m a_{diij}
\]

\[
- \frac{g}{2} q^2 y' a_{diij} \frac{K}{B^2} + \frac{1}{h} \frac{g^2 q}{4} (b + g q) y' a_{diij} \frac{K}{B^2} + \frac{1}{h} \frac{g^2 q^2}{4} m a_{diij} \frac{K^2}{B^2}.
\]

(C.15)

Inserting the vector

\[
m' = \frac{1}{K q} [B b' - (b + g q) y']
\]

leads to

\[
P_{ij} = - \frac{1}{h} (1 - h) q K \frac{1}{B} b_{a_{diij}} - \frac{K}{B^2} \left( B - \frac{1}{2} g b q \right) \frac{1}{q} [B b' - (b + g q) y'] a_{diij}
\]

\[
- \frac{g}{2} q^2 y' a_{diij} \frac{K}{B^2} + \frac{1}{h} \frac{g^2 q}{4} (b + g q) y' a_{diij} \frac{K}{B^2} + \frac{1}{h} \frac{g^2 q^2}{4} \frac{1}{q} [B b' - (b + g q) y'] a_{diij} \frac{K}{B^2},
\]

(C.16)

so that

\[
P_{ij} = - \left[ h q^2 + b \left( b + \frac{1}{2} g q \right) \right] \frac{K}{q B} b_{a_{diij}} + \frac{K}{B^2} \left( B - \frac{1}{2} g b q \right) \frac{1}{q} (b + g q) y' a_{diij} - \frac{g}{2} q^2 y' a_{diij} \frac{K}{B^2},
\]
which can be simplified to read

\[ P_{tij} = \left[ -\left[ hq^2 + b \left( b + \frac{1}{2} gq \right) \right] b^l a_{tij} + \left( b + \frac{1}{2} gq \right) y^l a_{tij} \right] \frac{K}{qB^2}. \]  

(C.17)

The representation (6.77)–(6.78) is valid.

Let us find

\[ \frac{qB}{K} \eta^{nt} P_n^{ij} = \left[ -\left[ hq^2 + b \left( b + \frac{1}{2} gq \right) \right] b^l + \left( b + \frac{1}{2} gq \right) y^l \right] a^t_{tij} \]

\[
- b^t \left( b + \frac{1}{2} gq \right) y^l b^h a_{hij} - \frac{1}{q^2} b^t \left[ -\left[ hq^2 + b \left( b + \frac{1}{2} gq \right) \right] b^l + \left( b + \frac{1}{2} gq \right) y^l \right] v^l a_{hij}
\]

= \left[ -\left[ hq^2 + b \left( b + \frac{1}{2} gq \right) \right] b^l + \left( b + \frac{1}{2} gq \right) y^l \right] a^t_{tij} - \left[ hv^t + \left( b + \frac{1}{2} gq \right) b^t \right] y^l b^h a_{hij}

and

\[ \left( \frac{qB}{K} \right)^2 \eta^{nt} P_n^{ij} P_{tij} = hq^2 b^l a_{tij} \left[ hq^2 + b \left( b + \frac{1}{2} gq \right) \right] b^h a^t_{hij} \]

\[ - \left( b + \frac{1}{2} gq \right) v^l a_{tij} \left[ hq^2 + b \left( b + \frac{1}{2} gq \right) \right] b^h a^t_{hij} \]

\[ + \left[ -hq^2 b^l a_{tij} + \left( b + \frac{1}{2} gq \right) v^l a_{tij} \right] \left( b + \frac{1}{2} gq \right) v^h a^t_{hij} \]

\[ + \left[ -hq^2 b^l a_{tij} + \left( b + \frac{1}{2} gq \right) v^l a_{tij} \right] \left( b + \frac{1}{2} gq \right) b^h a^t_{hij} \]

\[ - \left[ -hq^2 b^l a_{tij} + \left( b + \frac{1}{2} gq \right) v^l a_{tij} \right] \left[ hv^t + \left( b + \frac{1}{2} gq \right) b^t \right] v^h b^u a_{uhij}. \]

So we have

\[ \left( \frac{qB}{K} \right)^2 \eta^{nt} P_n^{ij} P_{tij} = -B b^l b^u a_{uij} v^l v^l a_{tij} \]

\[ + h^2 q^4 b^l a_{tij} b^h a^t_{hij} + \left[ 2hq^2 b^l a_{tij} + \left( b + \frac{1}{2} gq \right) v^l a_{tij} \right] \left( b + \frac{1}{2} gq \right) v^h a^t_{hij}. \]  

(C.18)

From (C.14) it follows that
\[
\rho_{knij} = 2 \frac{1}{K^2} M^{nij} M_{nij} + 2 H^{uv} P_{uji} H_{nij} + H^{kuv} H^{nv} a_{uvij} \frac{K^2}{B} H_{k} H_{n} a_{lij} \frac{K^2}{B}
\]

\[
= 2 \frac{1}{K^2} M^{nij} M_{nij} + 2 \eta^{nt} P_{uji} B \frac{K^2}{K^2_{ij}} + \eta^{tu} \eta^{lu} a_{uvij} a_{lij}
\]

\[
= 2 \frac{1}{K^2} M^{nij} M_{nij} - 2 \frac{1}{q^2} b^v b_{uv} a_{uvij} v^l a_{lij} + a^{lu} a_{uvij} \left( a^{lv} - b^v b^v - \frac{1}{q^2} v^l v^j \right) a_{lij}
\]

\[
+ 2 \frac{h^2 q^2 b^v a_{lij} b^h a^t_{nij}}{B q^2} \left[-2 h q^2 b^{l} a_{lij} \left( b + \frac{1}{2} g q \right) + \left(b + \frac{1}{2} g q \right)^2 v^l a_{lij} \right] v^h a^{t}_{nij}
\]

\[-b^t u a_{uvij} \left( a^{lv} - \frac{1}{q^2} v^l v^j \right) a_{lij} - \frac{1}{q^2} b^v v^l a_{uvij} \left(a^{lv} - b^v b^v \right) a_{lij},
\]

or

\[
\rho_{knij} = a^{knij} a_{knij} + 2 \frac{1}{K^2} M^{nij} M_{nij} - \frac{2}{B} \left(b + \frac{1}{2} g q \right)^2 b^{t} a_{lij} b^h a^{t}_{nij}
\]

\[-h \frac{2}{B} b^{t} a_{lij} \left(b + \frac{1}{2} g q \right) v^h a^{t}_{nij} - h \frac{2}{B} \left[b^{t} a_{lij} \left(b + \frac{1}{2} g q \right) + h v^l a_{lij} \right] v^h a^{t}_{nij}
\]

\[= a^{knij} a_{knij} + 2 \frac{1}{K^2} M^{nij} M_{nij} - \frac{2}{B} \left[h v^l + \left(b + \frac{g q}{2} \right) b^{l} \right] a^{nij}_{t} \left[h v^h + \left(b + \frac{g q}{2} \right) b^{h} \right] a_{hni}.
\]

Inserting here (6.82), we find

\[\rho_{knij} a^{knij} + 2 \left( \frac{1}{h^2} - 1 \right) B \left[h v^l + \left(b + \frac{g q}{2} \right) b^{l} \right] a^{nij}_{t} \left[h v^h + \left(b + \frac{g q}{2} \right) b^{h} \right] a_{hni}.
\]

Using

\[\zeta^i = \left[ h v^i + \left(b + \frac{1}{2} g q \right) b^{l} \right] \frac{S}{\sqrt{B}}
\]

(see (6.26) and (6.39)), we arrive at the representation

\[\rho_{knij} a^{knij} + \frac{2}{S^2} \left( \frac{1}{h^2} - 1 \right) \zeta^l a^{nij}_{t} \zeta^h a_{hni}.
\]

which is equivalent to (6.87).
Appendix D: Important coefficients

In processing the involved calculations it is useful to take into account the equalities

\[ y_m^a h_{ij}^m \zeta_k^h = - \left( \frac{1}{N} C_k - \frac{1}{K^2} (1 - h) y_k \right) M_{ij}^n + y_m^a h_{ij}^m E_k^h \]  

(D.1)

(\zeta_k^h \text{ and } E_k^h \text{ are indicated in (6.36))},

\[ \zeta_{km}^h = \frac{\partial \zeta_k^h}{\partial y_m} = \frac{1}{2q} g \eta_{km} h^b J \frac{1}{\varepsilon h} + \frac{1}{N} E_k^h C_m - (1 - h) y_m \frac{1}{K^2} E_k^h \]

\[ + \frac{1}{N} \zeta_{km} C_k - (1 - h) y_k \frac{1}{K^2} \zeta_m^h + \frac{1}{N} \zeta_{km} \frac{1}{K^2} \zeta_m^h - (1 - h) y_{km} \frac{1}{K^2} \zeta_m^h + 2(1 - h) y_k y_m \frac{1}{K^4} \zeta_m^h, \]  

(D.2)

and

\[ y_{hm}^n \zeta_{km}^h = \frac{1}{N} \delta_m^n C_k - (1 - h) y_k \frac{1}{K^2} \delta_m^n + \frac{1}{N} C_m J^2 \eta_{km} + \frac{1}{N} h_k^n C_m - (1 - h) y_m \frac{1}{K^2} \left( \delta_m^n - \frac{1}{h} \frac{1}{N} C_k y_n \right) \]

\[ - y_m \frac{1}{K^2} (1 - h) y_k \frac{1}{K^2} y_n - \frac{1}{h} y_n \frac{1}{K^2} y_m C_k + (1 - h) y_n g_{km} \frac{1}{K^2} + 2 \frac{1}{h} y_n (1 - h) y_k y_m \frac{1}{K^4}, \]  

(D.3)

together with

\[ y_{hm}^n \zeta_{km}^h M_{ij}^m = \left( \frac{C_k}{N} - (1 - h) \frac{y_k}{K^2} \right) M_{ij}^m + \left( \frac{C_m}{N} J^2 \eta_{km} + h_k^n \frac{C_m}{N} + \frac{y_n}{K^2} (1 - h) g_{km} \right) M_{ij}^m \]

\[ = \frac{1}{N} C_k M_{ij}^m - (1 - h) y_k \frac{1}{K^2} M_{ij}^m + \frac{1}{N} h_k^n C_m M_{ij}^m + \frac{1}{N} C_m g_{km} M_{ij}^m + y_n (1 - h) g_{km} \frac{1}{K^2} M_{ij}^m. \]

With the help of the formula (A.8) of Appendix A, the last representation can be written merely as

\[ y_{hm}^n \zeta_{km}^h M_{ij}^m = C_{km} M_{ij}^m + (1 - h) \frac{1}{K^2} (y_n g_{km} M_{ij}^m - y_k M_{ij}^m). \]  

(D.4)

Also,

\[ \frac{1}{J^2} g_{kl} M_{ij}^l = a_{k,ij}^m v_m + \frac{1}{h} \left[ b + \frac{1}{2} g q \right] b_m a_{k,ij}^m - \frac{1}{h} \left[ \frac{g}{2q} v_k + (1 - h) b_k \right] b_m y_t a_{k,ij}^m. \]  

(D.5)

If we use the representation (6.36) and apply the formulas (A.24), (A.25), (A.29), (A.30), and (A.33), we obtain

\[ \zeta_{mij}^n = \left( \frac{1}{N} C_n - (1 - h) \frac{1}{K^2} y_n \right) \zeta_{mij}^n + \left( \frac{1}{N} C_j - (1 - h) \frac{1}{K^2} y_j \right) \zeta_{mij}^n + \frac{g}{2q} \eta_{nj} b^n J \frac{1}{\varepsilon h}. \]
\[-\left( \frac{1}{N} C_n - (1 - h) \frac{1}{K^2} y_n \right) \frac{1}{N} C_j \zeta^m + \frac{1}{K^2} (1 - h) \left( \frac{1}{N} C_n - (1 - h) \frac{1}{K^2} y_n \right) y_j \zeta^m \]

\[+ \left( -\frac{1}{K} \frac{1}{N} C_j - \frac{1}{K} \frac{1}{N} C_n - \frac{1}{B^2} \frac{gb}{q} \eta_{nj} + \frac{2}{N N} C_n C_j \right) \zeta^m - (1 - h)(g_{nj} - 2l_{nj}) \frac{1}{K^2} \zeta^m. \]

Simplifying yields the representation

\[\zeta^m_\eta = \left( \frac{1}{N} C_n - (1 - h) \frac{1}{K^2} y_n \right) \zeta^m_j + \left( \frac{1}{N} C_j - (1 - h) \frac{1}{K^2} y_j \right) \zeta^m_n + \frac{g}{2q} \eta_{nj} b^m J \frac{1}{\zeta h} \]

\[\frac{-h}{K^2} \frac{1}{N} (y_n C_j + y_j C_n) \zeta^m + h(1 - h) \frac{1}{K^2} y_n y_j \zeta^m \]

\[+ \left( -\frac{1}{B^2} \frac{1}{q} \eta_{nj} + \frac{1}{N N} C_n C_j \right) \zeta^m_j - (1 - h) h_{nj} \frac{1}{K^2} \zeta^m, \]

from which it follows that

\[y_j^k \zeta^m_n y_h^m = \left( \frac{1}{N} C_n y_h^m - \frac{1}{h S^2} \zeta^m_h \right) \delta^m_j + \left( \frac{1}{N} C_n y_j^m - \frac{1}{h S^2} \zeta^m_j \right) \delta^m_n + \frac{g}{2q} b^m J \frac{1}{\zeta h} \eta_{nj} y_h^m y_j^k \]

\[-\frac{1}{N} C_k y_j^k \frac{1}{S^2} \zeta^m_h - \frac{1}{N} C_k y_h^k \frac{1}{S^2} \zeta^m_j + h(1 - h) \frac{1}{h^2 S^4} \zeta^m_h \zeta^m_j \]

\[\frac{-\frac{1}{B^2} \frac{1}{q} \eta_{nk} y_h^k y_j^k \zeta^m + \frac{1}{N} C_n C_k y_h^k y_j^k \zeta^m - (1 - h) \eta_{nk} \frac{1}{B} y_h^k y_j^k \zeta^m - (1 - h) \frac{2}{gg} \frac{1}{N} C_n C_k y_h^k y_j^k \zeta^m}. \]

Eventually, we arrive at

\[y_j^k \zeta^m_n y_h^m = Q_h \delta^m_j + Q_j \delta^m_n + \frac{g}{2q} b^m J \frac{1}{\zeta h} \eta_{nj} + h(1 - h) \frac{1}{h^2 S^4} \zeta^m_h \zeta^m_j \]

\[-M_j \frac{1}{S^2} \zeta^m_h - M_h \frac{1}{S^2} \zeta^m_j - \frac{1}{B^2} \frac{1}{q} \eta_{nj} \frac{1}{J} \frac{2}{\zeta^m_j} + M_h M_j \zeta^m \]

\[-(1 - h) \eta_{nk} \frac{1}{B} y_h^k y_j^k \zeta^m - (1 - h) \frac{2}{gg} \frac{1}{N} C_n C_k y_h^k y_j^k \zeta^m. \]

(D.7)

where

\[Q_h = M_h - (1 - h) \frac{1}{h S^2} \zeta_h \]

with

\[M_h = \left[ \frac{1}{N} C_h + \frac{gq}{2B} \left( (h - 1) b_h - \frac{gT}{2} (\zeta_h - \zeta^l b_h) \right) \right] \frac{1}{J}. \]
Using the equalities
\[
\frac{1}{N}C_h = \frac{1}{h^2}g \frac{1}{2q}B (hq^2 b_h - bh v_h) = \frac{1}{h^2}g \frac{1}{2q}B [hq^2 + b(b + \frac{1}{2} gq) b_h - g \frac{1}{2q} B \zeta_h \frac{\kappa}{J}]
\]
yields
\[
M_h = \left[ -g \frac{1}{2q} B \zeta_h \frac{\kappa}{J} + \frac{g q}{2 B} \left( h b_h - \frac{g T}{2} \zeta_h + \frac{1}{h} \left( b + \frac{1}{2} gq \right)^2 \right) \right] \frac{\kappa}{J}
\]
\[
= \left[ -g \frac{1}{2q} B \zeta_h \frac{\kappa}{J} + \frac{g q}{2 B} \left( \frac{B}{h} b_h - \frac{\zeta_h}{J} \frac{g}{2q} \right) \right] \frac{\kappa}{J},
\]
or
\[
M_h = -g \frac{1}{2q B} \left( b + \frac{1}{2} gq \right) \zeta_h \frac{\kappa}{J} + \frac{g}{2q B h} \frac{\zeta_h}{J}.\]
Here we can apply (6.39), which yields
\[
M_h = -\frac{g}{2q B} (b + \frac{1}{2} gq) \zeta_h \frac{\kappa}{J} + \frac{g}{2q B h} \frac{\zeta_h}{J}.
\]
(D.8)

By means of the transition rule (4.6) the tensor \( E_{knij} \) can be transformed into the tensor \( L_{knij} := y_{(k} \zeta^n_{l]} E_{tlij} \) of the Riemannian space \( \mathcal{R}^N \), which yields
\[
L_{knij} = a_{knij} - \gamma_{nklt} \zeta^n a_{htij} \quad (D.9)
\]
with the coefficients \( \gamma_{nklt} = y_{(k} \zeta^n_{l]} \zeta^n \) given by (D.7).

The coefficients (6.45) can be transformed as follows:
\[
y_{(k} = \left[ \left( b_i - \frac{g}{2q} v_i \right) b^k + \frac{1}{h} \left( \delta_i^k - b_i b^k \right) \right] \frac{h \zeta}{J}
\]
\[+ \frac{1}{B} \left\{ \frac{1 - h}{h} \left[ hv_i + (b + \frac{1}{2} gq) b_i \right] + \frac{g q}{2h^2} (b + \frac{1}{2} gq) \left[ hv_i + (b + \frac{1}{2} gq) b_i \right] - \frac{g}{2h^2} \frac{b_i}{\zeta_h} \right\} \frac{h \zeta}{J} y_i^k,
\]
getting
\[
y_{(k} = \left[ \left( b_i - \frac{g}{2q} v_i \right) b^k + \frac{1}{h} \left( \delta_i^k - b_i b^k \right) \right] \frac{h \zeta}{J}
\]
\[+ \frac{1}{B} \left[ (h - h^2) q + \frac{g}{2} (b + \frac{1}{2} gq) \right] \frac{h \zeta}{J} v_i y_i^k + \frac{1}{B} \left( \frac{1}{h} \left( b + \frac{1}{2} gq \right) - b - gq \right) \frac{h \zeta}{J} b_i y_i^k \quad (D.10)
\]
or
\[
y_{(m} = \delta_m^m \frac{\zeta}{J} - v_m C_m^{\frac{J^2}{N}} \frac{\zeta}{J} - (1 - h)b_m C_m^{\frac{2q J^2}{gN}} \frac{\zeta}{J} - \frac{1}{B} \left( (1 - h)v_m + \frac{g}{2} b_m \right) \frac{\zeta}{J} y_m^m \quad (D.11)
\]
which entails
\[
y^k_i b^j = \left[ b^k + \frac{1}{B} \left( \frac{1}{h} \left( b + \frac{1}{2} gq \right) - b - gq \right) y^k \right] \frac{h\zeta}{J}.
\]  \quad \text{(D.12)}

**Appendix E: Fixed tangent space of the } \mathcal{FF}_{FD} \text{-space**}

Let us introduce the orthonormal frame \( h^p_i (x) \) of the input Riemannian metric tensor \( a_{ij} (x) \):

\[
a_{ij} = e_{pq} h^p_i h^q_j,
\]  \quad \text{(E.1)}

where \( \{ e_{pq} \} \) is the Euclidean diagonal: \( e_{pq} = \text{diagonal}(+...+) \); the indices \( p, q, ... \) will be specified on the range \( 1, ..., N \); and the indices \( a, b, ... \) on the range \( 1, ..., N - 1 \). Denote by \( h^i_p \) the reciprocal frame, so that \( h^j_p h^p_i = \delta^j_i \). At any fixed point \( x \), we can represent the tangent vectors \( y \) by their frame-components:

\[
R^p = h^p_i y^i,
\]  \quad \text{(E.2)}

and use the components

\[
g_{pq} = h^i_p h^j_q g_{ij}
\]  \quad \text{(E.3)}

of the Finslerian metric tensor \( g_{ij} \).

In the \( \mathcal{FF}_{FD} \)-space, it is convenient to specify the frame such that the \( N \)-th component \( h^N_i (x) \) becomes collinear to the input vector field \( b_i (x) \). Under these conditions, the 1-form \( b \) reads merely \( b = z \) and we have \( b^p = \{ 0, 0, ..., 1 \} \) and \( b_p = \{ 0, 0, ..., 1 \} \). We obtain the decomposition \( R^p = \{ R^a, R^N \} \), together with \( q^2 = e_{ad} R^a R^d \). Also the notation

\[
R^N = z
\]  \quad \text{(E.4)}

will be used.

In any fixed tangent space \( T_x M \) we can obtain the covariant components \( R_p = h^i_p y_i \) through the definition

\[
R_p := \frac{1}{2} \frac{\partial K^2 (g; R)}{\partial R^p}.
\]

With the help of (6.1)-(6.5) we find

\[
R_a = e_{ab} R^b J^2, \quad R_N = (z + gq) J^2.
\]  \quad \text{(E.5)}

For the respective Finsleroid metric tensor components

\[
g_{pq} (g; R) := \frac{1}{2} \frac{\partial^2 K^2 (g; R)}{\partial R^p \partial R^q} = \frac{\partial R_p (g; R)}{\partial R^p}
\]

we obtain

\[
g_{NN} (g; R) = [(z + gq)^2 + q^2] J^2, \quad g_{Na} (g; R) = gq e_{ab} R^b J^2,
\]  \quad \text{(E.6)}

\[
g_{ab} (g; R) = J^2 e_{ab} - g e_{ad} R^d e_{be} R^e z \frac{q}{J^2} J^2.
\]  \quad \text{(E.7)}

The components of the inverted metric tensor read

\[
g^{NN} (g; R) = (z^2 + q^2) \frac{1}{K^2}, \quad g^{Na} (g; R) = -gq R^a \frac{1}{K^2},
\]  \quad \text{(E.8)}
\[ g^{ab}(g; R) = \frac{B}{K^2}e^{ab} + g(z + gq)\frac{R^a R^b}{q} \frac{1}{K^2}. \]  

(E.9)

It can readily be verified that

\[ \det(g_{pq}) = J^{2N} > 0. \]  

(E.10)

The above formulas are valid at an arbitrary dimension \( N \geq 2 \).

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