G₂-structures for \( N = 1 \) supersymmetric AdS₄ solutions of \( M \)-theory

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Abstract
We study the \( N = 1 \) supersymmetric solutions of \( D = 11 \) supergravity obtained as a warped product of four-dimensional anti-de Sitter space with a seven-dimensional Riemannian manifold \( M \). Using the octonion bundle structure on \( M \) we reformulate the Killing spinor equations in terms of sections of the octonion bundle on \( M \). The solutions then define a single complexified G₂-structure on \( M \) or equivalently two real G₂-structures. We then study the torsion of these G₂-structures and the relationships between them.

Keywords: supergravity, flux compactification, differential geometry, G₂-structures, octonions, Killing spinors

1. Introduction

It is known that supersymmetric vacua of string theory and M-theory are determined by nowhere-vanishing spinors that satisfy a first-order differential equation, known as a Killing spinor equation. The precise form of the equation depends on particular details of the theory. In particular, in a theory with vanishing fluxes, the Killing spinor equation just reduces to an equation for a parallel spinor with respect to the Levi-Civita connection. Since existence of non-trivial parallel spinors is equivalent to a reduced holonomy group, this gives the key implication that if the full \( D \)-dimensional spacetime is a product of the 4-dimensional Minkowski space and correspondingly, a \((D - 4)\)-dimensional internal space, then the internal space needs to be a 6-dimensional Calabi–Yau manifold if \( D = 10 \) and must be a 7-dimensional manifold with holonomy contained in G₂ if \( D = 11 \). In terms of G-structures, this is equivalent to a torsion-free SU(3)-structure in 6 dimensions and a torsion-free G₂-structure in 7 dimensions. However, if there are non-trivial fluxes, the Killing spinor equation will have additional terms that are defined by the fluxes and the resulting Killing spinors will no longer be parallel. In the setting of a (warped) product \( M^4 \times M^{D-4} \) this will then induce a non-flat metric on \( M^4 \) and the spinors on the internal space will no longer define a torsion-free
$G$-structure, instead there will be some torsion components. Moreover, if quantum corrections are also considered, then the Killing spinor equation will also contain additional terms, which will also lead to corrections away from torsion-free structures. Since the early 1980’s there has been great progress in understanding the geometric implications of flux compactifications of different theories with different details, such as compactifications of type $I A$, type $IIB$, and heterotic supergravities for $D = 10$, and compactifications of $D = 11$ supergravities with different amounts of supersymmetry [1–23]. There have also been attempts in including higher-order corrections and approaches that do not necessarily factor Killing spinors into 4-dimensional and $(D − 4)$-dimensional pieces [24, 25]. Other approaches also involved higher derivative corrections [26–28]. There have also been multiple approaches using different aspects of generalized geometry [29–34].

In this paper we will primarily consider a warped product compactification of 11-dimensional $N = 1$ supergravity on $\text{AdS}_4 \times M$, where $M$ is a 7-dimensional space, however many of the results can also be adapted for a Minkowski background. Our emphasis is to gain a better understanding of the torsion of $G_2$-structures that are imposed by the supersymmetry constraints. The properties of $G_2$-structures in supergravity compactifications have been studied in both mathematical and physical literature, but for different details of the theory [1, 6, 9, 35–43]. Moreover, in the physics literature the results are usually expressed in terms of an $SU(3)$-structure or an $SU(2)$-structure on the 7-dimensional manifold [5, 9, 12, 15, 17, 18, 21]. While this approach has its merits since it more readily allows to compare 11-dimensional theories with 10-dimensional ones, it does not use some of the unique feature of 7-dimensional $G_2$-structures.

Seven-dimensional manifolds with a $G_2$-structure have been of great interest in differential geometry ever since Alfred Gray studied vector cross products on orientable manifolds in 1969 [44]. It turns out that a 2-fold vector cross product - that is, one that takes two vectors and outputs another one, exists only in 3 dimensions and in 7 dimensions. The 3-dimensional vector cross product is very well known in $\mathbb{R}^3$ and on a general oriented 3-manifold it comes from the volume 3-form, so it is a special case of a $(n − 1)$-fold vector cross product in a $n$-dimensional space, where it also comes from the volume form. In 7 dimensions, however, the vector cross product structure is even more special, since it is not part of a generic sequence. The 3-dimensional and 7-dimensional vector cross products do however have something in common since they are closely related to the normed division algebras - the quaternions and octonions, which are 4- and 8-dimensional, respectively. A good review of vector cross product geometries can also be found in [45]. On a 7-manifold, the group that preserves the vector cross product is precisely the 14-dimensional Lie group $G_2$ - this is the automorphism group of the octonion algebra. In order to be able to define a vector cross product globally on a 7-manifold, we need to introduce a $G_2$-structure, which is now a reduction of the frame bundle to $G_2$. A necessary and sufficient condition on the 7-manifold to admit a $G_2$-structure is that it is orientable and admits a spin structure. There are purely topological conditions of the vanishing of the first two Stiefel–Whitney classes [46, 47]. Once we have a 7-manifold that satisfies these conditions, any Riemannian metric $g$ will give rise to an $SO(7)$-structure, and this could then be reduced to a $G_2$-structure. By specifying a $G_2$-principal bundle, we are effectively also defining a $G_2$-invariant 3-form $\varphi$ that is compatible with $g$ and gives rise to the structure constants for the vector cross product. As it is well-known [48] for each metric $g$ there is a family of compatible $G_2$-structures. Pointwise, such a family is parametrized by $SO(7)/G_2 \cong \mathbb{R}P^7$.

Due to the close relationship between $G_2$ and octonions, it is natural to introduce an octonionic structure on a 7-manifold with a $G_2$-structure. In [49], the notion of an octonion bundle on a 7-manifold with $G_2$-structure has been introduced. A number of properties of $G_2$-structures are re-expressed in a very natural form using the octonion formalism, which we
review in section 3 of this paper. In many ways, the structure of the octonion bundle mirrors that of the spinor bundle on a 7-manifold. It is well known that a $G_2$-structure may be defined by a unit spinor on the manifold. Under the correspondence between the spinor bundle and the octonion bundle, the fixed spinor is then mapped to 1. A change of the unit spinor then corresponds to a transformation of the $G_2$-structure within the same metric class. As it is well known, the enveloping algebra of the octonions, i.e. the algebra of left multiplication maps by an octonion under composition, gives an isomorphism with the spinor bundle as Clifford modules [50]. However, the Clifford module structure is by definition associative, so this correspondence only captures part of the structure of the octonion bundle. The full non-associative structure of the octonion bundle cannot be seen in the spinor bundle, therefore it is expected that the octonion bundle carries more information than the spinor bundle, although the difference is subtle. In particular, while there is no natural binary operation on the spinor bundle, we can multiply octonions. In fact, Clifford multiplication of a vector and a spinor translates to multiplication of two octonions - therefore we are implicitly using the triality correspondence between vector and spinor representations which is unique to 7 dimensions [51]. This gives a non-associative division algebra structure, something that we generically do not have on spinor bundles. Moreover, as shown in [49], isometric $G_2$-structures, i.e. those that are compatible with the same metric, are readily described in terms of octonions and their torsion tensors are then re-interpreted as 1-forms with values in the imaginary octonion bundle. Since $N = 1$ supersymmetry in $D = 11$ is most readily described using a complex spinor on the 7-manifold, in section 3.2 we also introduce complexified octonions and corresponding complexified $G_2$-structures. Complexified octonions no longer form a division algebra and contain zero divisors.

Due to the close relationship between spinors and octonions in 7 dimensions, any spinorial expression on a 7-manifold with a $G_2$-structure can be recast in terms of octonions. In section 4 we rewrite the Killing spinor equation in 7 dimensions in terms of a complexified octonion section $Z$. It turns out that a key property of $Z$ is whether or not it is a zero divisor. On an AdS$_4$ background with non-vanishing 4-dimensional flux, it is shown that $Z$ is nowhere a zero divisor. On the other hand, if the 4-dimensional flux vanishes, there could be a locus where $Z$ is a zero divisor, however in any neighborhood, $Z$ will be not be everywhere a zero divisor. If the 4-dimensional spacetime is Minkowski, then $Z$ is either everywhere a zero divisor or is nowhere a zero divisor. In any case, wherever $Z$ is not a zero divisor, we are able to define a complexified $G_2$-structure corresponding to $Z$, and we show that its torsion lies in the class $1_C \oplus 7_C \oplus 27_C$ (here we are referring to the complex representations of $G_2$), with the component in $7_C$ moreover being an exact 1-form. It is also easy to see in this case that the Killing equation implies that the 7-dimensional flux 4-form $G$ has to satisfy the Bianchi identity $dG = 0$.

The complexified octonion section $Z$ decomposes as $X + iY$ in terms of $\mathbb{C}$-real octonions $X$ and $Y$, which define real $G_2$-structures $\varphi_X$ and $\varphi_Y$, respectively, both of which correspond to the same metric. The octonion section $W = XY^{-1}$ then defines the transformation from $\varphi_X$ to $\varphi_Y$. The property of $Z$ being a zero divisor is then equivalent to $|W|^2 = 1$ and the $\mathbb{C}$-real part $w_0$ of $W$ vanishing. Assumptions that are equivalent to these have sometimes been made in the literature, however $|W|^2$ and $w_0$ satisfy some particular differential equations and cannot be arbitrarily set to particular values. The vector field $w$ which is the $\mathbb{C}$-imaginary part of $W$, is precisely the vector field that has been used in the literature to reduce the $G_2$-structure on the internal 7-manifold to an $SU(3)$-structure. However, in section 4 we find that if the internal manifold is compact and the 4-dimensional spacetime is AdS$_4$, then $w$ cannot be nowhere-vanishing. Moreover, if the 4-dimensional flux is non-vanishing, then $w$ is proportional to
the gradient of the compactification warp factor, so in particular we can write \( w = h(\Delta) \, d\Delta \), with \( e^{2\phi} \) being the warp factor. Therefore, in these cases, \( SU(3) \)-structures or Sasakian structures on the 7-manifold would be degenerate and only valid locally, since they depend on a nowhere-vanishing vector field.

In section 4.2, we then use the external Killing spinor equations to obtain expressions for some of the components of the 4-form \( G \) in terms of \( W \) and the warp factor. This leads on to section 5, where the internal Killing spinor equations in terms of octonions are used to derive the torsion tensors \( T(\chi) \) and \( T(\psi) \) of the real \( G_2 \)-structures \( \varphi_X \) and \( \varphi_Y \). In particular, we find that if 4-dimensional flux is non-vanishing, then both \( T(\chi) \) and \( T(\psi) \) are in the generic torsion class \( 1 \oplus 7 \oplus 14 \oplus 27 \), however the 7 components are exact, so both \( T(\chi) \) and \( T(\psi) \) are conformally in the torsion class \( 1 \oplus 14 \oplus 27 \). If the 4-dimensional flux vanishes, then both \( T(\chi) \) and \( T(\psi) \) are in the torsion class \( 7 \oplus 14 \oplus 27 \), and moreover if \( w \) is not proportional to the gradient of warp factor, then all the torsion components are non-zero. In the case of a Minkowski background, the torsion is conformally in the class \( 14 \oplus 27 \). In the particular case of the Minkowski background with \( |W|^2 = 1 \) and \( w_0 = 0 \), our expression for the torsion reduces to the one obtained in [6], however, to the author’s knowledge, the torsion of a \( G_2 \)-structure obtained from a compactification to \( AdS_4 \) has not appeared previously in the literature. We then also work out the covariant derivative of \( W \). In the case when the 4-dimensional flux is non-zero, we find that, as functions of \( \Delta, h \) and \( w_0 \) satisfy coupled ODEs.

In section 6, we consider the integrability conditions for \( G_2 \)-structure torsion and outline how they relate to the equations of motion and Einstein’s equations. In general, the integrability conditions for Killing spinor equations in \( D = 11 \) have been considered in [52] where it has been shown that if the Bianchi identity and the equations of motion for the flux in \( D = 11 \) is satisfied, then the Killing spinor equations also imply Einstein’s equations. This result has also been used in [18] to conclude that in a \( N = 2 \) compactification of \( D = 11 \) supergravity to \( AdS_4 \) the Killing spinor equations imply the Bianchi identity, the equations of motion, and Einstein’s equations. Here we outline how to obtain all of this directly from the integrability conditions for the \( G_2 \)-structures \( \varphi_X \) and \( \varphi_Y \).

Conventions In this paper we will be using the following convention for Ricci and Riemann curvature:

\[
\text{Ric}_{ij} = g^{ik} \text{Riem}_{ikjl}.
\] (1.1)

Also, the convention for the orientation of a \( G_2 \)-structure will same as the one adopted by Bryant [48] and follows the author’s previous papers. In particular, this causes \( \psi = *\varphi \) to have an opposite sign compared to the works of Karigiannis, so many identities and definitions cited from [53–55] may have differing signs.

2. Supersymmetry

Consider the basic bosonic action of eleven-dimensional supergravity [56], which is supposed to describe low-energy \( M \)-theory:

\[
S = \frac{1}{2} \int \tilde{R} \text{vol} - \int \tilde{G} \wedge * \tilde{G} - \frac{1}{3} \int \tilde{C} \wedge \tilde{G} \wedge \tilde{G}
\] (2.1)

where \( \tilde{R} \) is the scalar curvature, \( \text{vol} \) is the volume, and \( * \) is the Hodge star of the metric \( \tilde{g} \) on the 11-dimensional spacetime; and \( \tilde{C} \) is a local 3-form potential with field strength
\( \mathcal{G} = dC \), so that \( d\mathcal{G} = 0 \). Note that additional higher-order terms may also be present in (2.1) [26, 28, 34]. From (2.1), the bosonic equations of motion are found to be

\[
\text{Ric} = \frac{1}{3} \left( \mathcal{G} \mathcal{G} - 2\mathcal{g} \right) \mathcal{G}^2 \]  

\[ d \ast \mathcal{G} = -\mathcal{G} \wedge \mathcal{G} \]  

(2.2a)

where \( (\mathcal{G} \mathcal{G})_{AB} = \mathcal{G}_{AMNP} \mathcal{G}_{BPNO} \) and \( |\mathcal{G}|^2 = \frac{1}{24} \mathcal{G}_{AMNP} \mathcal{G}^{AMNP} \) with \( \mathcal{g} \) used to contract indices. Note that we will be using Latin upper case letters for 11-dimensional indices, Latin lower case letters for 7-dimensional indices, and Greek letters for 4-dimensional indices. Moreover, the supersymmetry variations of the gravitino field \( \tilde{\Psi} \) give the following Killing equation:

\[
\delta \tilde{\Psi}_A = \left\{ \mathcal{\nabla}^S A + \frac{1}{144} \mathcal{G}_{BCDE} \left( \tilde{\gamma}^{BCDE} A - 8\tilde{\gamma}^{CDE} B \mathcal{g}_A \right) \right\} \tilde{\varepsilon} \]  

(2.3)

where all the checked objects are 11-dimensional: \( \mathcal{\nabla}^S \) is the spinorial Levi-Civita connection, \( \tilde{\gamma} \) are gamma matrices, and \( \tilde{\varepsilon} \) is a Majorana spinor. Nowhere-vanishing solutions to \( \delta \tilde{\Psi} = 0 \) are precisely the Killing spinors. A supersymmetric vacuum solution of 11-dimensional \( N = 1 \) supergravity would then be a triple \( (\mathcal{g}, \mathcal{G}, \tilde{\varepsilon}) \) where \( \mathcal{g} \) is metric and \( \mathcal{G} \) is a closed 4-form that satisfy equations (2.2) and \( \tilde{\varepsilon} \) is a nowhere-vanishing spinor that satisfies (2.3). Correspondingly, \( N = 2 \) supersymmetry would have two independent solutions to (2.3). Note that it is known [52] that if \( (\mathcal{g}, \mathcal{G}, \tilde{\varepsilon}) \) satisfy \( d\mathcal{G} = 0 \), (2.2b), and (2.3), then (2.2a) follows as an integrability condition.

Assuming a warped product compactification from 11 dimensions to 4, the 11-dimensional metric \( \mathcal{g} \) is written as

\[
\mathcal{g} = e^{2\Delta} (\eta \oplus \mathcal{g}) \]  

(2.4)

where \( \eta \) is the 4-dimensional Minkowski or AdS_4 metric and \( \mathcal{g} \) is a Riemannian metric on the 7-dimensional ‘internal manifold’ \( M \). The warp factor \( \Delta \) is assumed to be a real-valued smooth function on the internal space \( M \). Note that for convenience we will work with the ‘warped’ metric \( g = e^{2\Delta} \mathcal{g} \) on \( M \), so in fact our 11-dimensional metric will be taken to be

\[
\mathcal{g} = (e^{2\Delta} \eta) \oplus \mathcal{g} \]  

(2.5)

The 11-dimensional 4-form flux \( \mathcal{G} \) is decomposed as

\[
\mathcal{G} = 3\mu \text{vol}_4 + G \]  

(2.6)

where \( \mu \) is a real constant, \( \text{vol}_4 \) is the volume form of the metric \( \eta \) on the 4-dimensional spacetime and \( G \) is a 4-form on \( M \). Note that the condition \( d\mathcal{G} = 0 \) forces \( \mu \) to be constant and \( d\mathcal{G} = 0 \). With these ansätze for the metric and the 4-form, from the 11-dimensional equation of motion (2.2b) we obtain the following 7-dimensional equation

\[
d (e^{4\Delta} \ast G) = -6\mu \mathcal{G} \]  

(2.7)

and Einstein’s equation (2.2a) gives us

\[
\text{Ric}_{\alpha\beta} = \left( -12\mu^2 e^{-6\Delta} + \frac{2}{3} e^{2\Delta} |G|^2 \right) \eta_{\alpha\beta} \]  

(2.8a)

\[
\text{Ric}_{ab} = \frac{1}{3} (GG)_{ab} + \left( 6\mu^2 e^{-8\Delta} - \frac{2}{3} |G|^2 \right) g_{ab} \]  

(2.8b)
where now $|G|^2$ is with respect to warped metric $g$ on $M$. On the other hand, using a standard conformal transformation formula, Ric is related to the Ricci curvatures $\text{Ric}_4$ and $\text{Ric}$ of the unwarped metrics in 4 and 7 dimensions in the following way

$$\text{Ric} = \text{Ric}_4 + \text{Ric} - 9 \left( \tilde{\nabla} d\Delta - d\Delta d\Delta \right) - \left( \tilde{\nabla}^2 \Delta + 9 |d\Delta|^2 \right) (\eta + \tilde{g})$$

where $\tilde{\nabla}$ is the 7-dimensional Levi-Civita connection for the metric $\tilde{g}$. However, the Ricci curvature $\text{Ric}_4$ of the unwarped 4-dimensional metric $\eta$ is by assumption given by

$$\text{Ric}_4 = -12 |\lambda|^2 \eta$$

(2.9)

where $\lambda$ is a complex constant, which is of course 0 if $\eta$ is the Minkowski metric. Moreover, rewriting $\tilde{\nabla}$ and $\tilde{\text{Ric}}$ in terms of the warped metric $g$ on $M$, and separating $\text{Ric}$ into the 4-dimensional and 7-dimensional parts, we find

$$\text{Ric}_4 = -e^{2\Delta} \left( \nabla^2 \Delta + 4 |d\Delta|^2 + 12 |\lambda|^2 e^{-2\Delta} \right) \eta$$

(2.10a)

$$\text{Ric}_7 = \text{Ric} - 4 (\nabla d\Delta + d\Delta d\Delta).$$

(2.10b)

Now comparing with (2.8), we obtain the following equation

$$\nabla^2 \Delta = 12 \mu^2 e^{-8\Delta} + \frac{2}{3} |G|^2 - 4 |d\Delta|^2 - 12 |\lambda|^2 e^{-2\Delta}$$

(2.11a)

$$\text{Ric} = 4 (\nabla d\Delta + d\Delta d\Delta) + \frac{1}{3} GG + \left( 6\mu^2 e^{-8\Delta} - \frac{2}{3} |G|^2 \right) g.$$  

(2.11b)

So in particular, the 7-dimensional scalar curvature $R$ satisfies the following:

$$\frac{1}{4} R = \nabla^2 \Delta + |d\Delta|^2 + \frac{5}{6} |G|^2 + \frac{21}{2} \mu^2 e^{-8\Delta}.$$  

(2.12)

We can see that the condition (2.11a) forces the Ricci curvature of the 4-dimensional space-time to be negative if the internal space is compact. This is a well-known ‘no-go’ theorem from [57]. Also, (2.11a) implies that if $\Delta$ is constant (and without loss of generality can be set to 0), then

$$|G|^2 = 18 \left( |\lambda|^2 - \mu^2 \right).$$

(2.13)

In particular, this implies that in this case, on a Minkowski background, with $|\lambda| = 0$, both $G$ and $\mu$ have to vanish. On an AdS$_4$ background, this still gives the restrictive requirement that $|G|^2$ has to be constant. Moreover, from (2.12) we then see that this implies that the 7-dimensional scalar curvature has to be a positive constant.

The 11-dimensional gamma matrices $\hat{\gamma}$ are decomposed as

$$\hat{\gamma} = e^{\Delta} \tilde{\gamma} \otimes \mathbb{I} + \hat{\gamma}^{(5)} \otimes \gamma$$

(2.14)

where $\tilde{\gamma}$ are real 4-dimensional gamma matrices, $\hat{\gamma}^{(5)} = i\hat{\gamma}^{(1)} \hat{\gamma}^{(2)} \hat{\gamma}^{(3)} \hat{\gamma}^{(4)}$ is the four-dimensional chirality operator, and $\gamma$ are imaginary gamma matrices in 7 dimensions that satisfy the standard Clifford algebra identity:

$$\gamma_a \gamma_b + \gamma_b \gamma_a = 2g_{ab} \mathbb{I}.$$  

(2.15)

The 11-dimensional spinor $\hat{\bar{\epsilon}}$ is decomposed as
\[
\varepsilon = \varepsilon \otimes \theta + \varepsilon^* \otimes \theta^*
\]  
(2.16)

where \(\theta\) is a complex spinor on the 7-dimensional internal space, and \(\varepsilon\) is a 4-dimensional Weyl spinor that also satisfies the Killing spinor equation in 4 dimensions:

\[
\bar{\nabla}^S \varepsilon = \bar{\gamma} \lambda^* \varepsilon^*
\]  
(2.17)

where \(\bar{\nabla}^S\) is the 4-dimensional Levi-Civita connection on spinors with respect to the metric \(\eta\) and \(\lambda\) is a complex constant known as a superpotential or the gravitino mass. In particular, \(|\lambda|^2\) appears in the expression for the Ricci curvature (2.9).

Using the decomposition (2.16) as well as (2.17), the Killing equation (2.3) gives us two sets of equations on \(\theta\):

\[
0 = \lambda e^{-\Delta} \theta^* + \left[ \left( \mu e^{-4\Delta} i + \frac{1}{2} (\partial_i \Delta) \gamma^c \right) + \frac{1}{144} G_{bcde} \gamma^{bcde} \right] \theta
\]  
(2.18a)

\[
\nabla^a \theta = \left[ \frac{i}{2} \mu e^{-4\Delta} \gamma_a - \frac{1}{144} \left( G_{bcde} \gamma^{bcde}_a - 8 G_{abcd} \gamma^{bcd} \right) \right] \theta.
\]  
(2.18b)

Here the equation (2.18a) comes from the ‘external’ part of (2.3), i.e. when \(A = 0, \ldots, 3\), and (2.3) comes from the ‘internal’ part of (2.3), i.e. when \(A = 4, \ldots, 11\). Using standard gamma matrix identities, the equation (2.18b) can be rewritten as

\[
\nabla^a \theta = \left[ \frac{i}{2} \mu e^{-4\Delta} \gamma_a - \frac{1}{144} \gamma_a \left( G_{bcde} \gamma^{bcde} \right) + \frac{1}{12} G_{abcd} \gamma^{bcd} \right] \theta.
\]  
(2.19)

Since we require the solutions of (2.3) and hence (2.18) to be nowhere-vanishing, this implies that the 7-dimensional internal manifold \(M\) must admit a \(G_2\)-structure. In particular, given any metric \(g\) on \(M\), there will exist a family of \(G_2\)-structures that are compatible with \(g\). In terms of \(G_2\)-structures, the equations (2.18) then say that a \(G_2\)-structure with a particular torsion must exist within this metric class of \(G_2\)-structures. In [6] and [8], the expressions for the torsion of a particular member of the metric class were explicitly calculated in terms of the 4-form \(G\) under some assumptions on \(\theta_\pm\). In this paper, we will use the octonion bundle formalism developed in [49] to reformulate equations (2.18) in terms of octonions on \(M\) and will use that to derive properties of the torsion of the corresponding \(G_2\)-structures.

3. \(G_2\)-structures and octonion bundles

The 14-dimensional group \(G_2\) is the smallest of the five exceptional Lie groups and is closely related to the octonions, which is the noncommutative, nonassociative, 8-dimensional normed division algebra. In particular, \(G_2\) can be defined as the automorphism group of the octonion algebra. Given the octonion algebra \(\mathbb{O}\), there exists a unique orthogonal decomposition into a real part, that is isomorphic to \(\mathbb{R}\), and an imaginary (or pure) part, that is isomorphic to \(\mathbb{R}^7\):

\[
\mathbb{O} \cong \mathbb{R} \oplus \mathbb{R}^7.
\]  
(3.1)

Octonion multiplication to define a vector cross product \(\times\) on \(\mathbb{R}^7\). Given vectors \(u, v \in \mathbb{R}^7\), we regard them as octonions in \(\text{Im} \mathbb{O}\), multiply them together using octonion multiplication, and then project the result to \(\text{Im} \mathbb{O}\) to obtain a new vector in \(\mathbb{R}^7\):

\[
u \times v = \text{Im} (uv).
\]  
(3.2)
The subgroup of $GL(7,\mathbb{R})$ that preserves this vector cross product is then precisely the group $G_2$. A detailed account of the properties of the octonions and their relationship to exceptional Lie groups is given by John Baez in [51]. The structure constants of the vector cross product define a 3-form on $\mathbb{R}^7$, hence $G_2$ is alternatively defined as the subgroup of $GL(7,\mathbb{R})$ that preserves a particular 3-form $\varphi_0 \in \Lambda^3(\mathbb{R}^7)\dagger$ [58].

In general, given a $n$-dimensional manifold $M$, a $G$-structure on $M$ for some Lie subgroup $G$ of $GL(n,\mathbb{R})$ is a reduction of the frame bundle $F$ over $M$ to a principal subbundle $P$ with fibre $G$. A $G_2$-structure is then a reduction of the frame bundle on a 7-dimensional manifold $M$ to a $G_2$-principal subbundle. The obstructions for the existence of a $G_2$-structure are purely topological. Given a 7-dimensional smooth manifold that is both orientable ($w_1 = 0$) and admits a spin structure ($w_2 = 0$), there always exists a $G_2$-structure on it [44, 46, 47].

There is a 1-1 correspondence between $G_2$-structures on a 7-manifold and smooth 3-forms $\varphi$ for which the 7-form-valued bilinear form $B_{\varphi}$ as defined by (3.3) is positive definite (for more details, see [59] and the arXiv version of [60]).

$$B_{\varphi}(u,v) = \frac{1}{6} (u \lrcorner \varphi) \wedge (v \lrcorner \varphi) \wedge \varphi.$$  \hspace{1cm} (3.3)

Here the symbol $\lrcorner$ denotes contraction of a vector with the differential form:

$$(u \lrcorner \varphi)_{mn} = u^a \varphi_{amn}.$$  \hspace{1cm} (3.4)

Note that we will also use this symbol for contractions of differential forms using the metric.

A smooth 3-form $\varphi$ is said to be positive if $B_{\varphi}$ is the tensor product of a positive-definite bilinear form and a nowhere-vanishing 7-form. In this case, it defines a unique Riemannian metric $g_\varphi$ and volume form $\text{vol}_\varphi$ such that for vectors $u$ and $v$, the following holds

$$g_\varphi(u,v) \text{vol}_\varphi = \frac{1}{6} (u \lrcorner \varphi) \wedge (v \lrcorner \varphi) \wedge \varphi.$$  \hspace{1cm} (3.5)

An equivalent way of defining a positive 3-form $\varphi$, is to say that at every point, $\varphi$ is in the $GL(7,\mathbb{R})$-orbit of $\varphi_0$. It can be easily checked that the metric (3.5) for $\varphi = \varphi_0$ is in fact precisely the standard Euclidean metric $g_0$ on $\mathbb{R}^7$. Therefore, every $\varphi$ that is in the $GL(7,\mathbb{R})$-orbit of $\varphi_0$ has an associated Riemannian metric $g$, that is in the $GL(7,\mathbb{R})$-orbit of $g_0$. The only difference is that the stabilizer of $g_0$ (along with orientation) in this orbit is the group $SO(7)$, whereas the stabilizer of $\varphi_0$ is $G_2 \subset SO(7)$. This shows that positive 3-forms forms that correspond to the same metric, i.e. are isometric, are parametrized by $SO(7)/G_2 \cong \mathbb{RP}^7 \cong S^7/\mathbb{Z}_2$.

Therefore, on a Riemannian manifold, metric-compatible $G_2$-structures are parametrized by sections of an $\mathbb{RP}^7$-bundle, or alternatively, by sections of an $S^7$-bundle, with antipodal points identified.

**Definition 3.1.** If two $G_2$-structures $\varphi_1$ and $\varphi_2$ on $M$ have the same associated metric $g$, we say that $\varphi_1$ and $\varphi_2$ are in the same metric class.

Let $(M, g)$ be a smooth 7-dimensional Riemannian manifold, with $w_1 = w_2 = 0$. We know $M$ admits $G_2$-structures. In particular, let $\varphi$ be a $G_2$-structure for $M$ for which $g$ is the associated metric. We also use $g$ to define the Levi-Civita connection $\nabla$, and the Hodge star $\star$. In particular, $\star \varphi$ is a 4-form dual to $\varphi$, which we will denote by $\psi$. We will give a brief overview of the properties of octonion bundles as developed in [49].

**Definition 3.2.** The octonion bundle $\mathbb{O}M$ on $M$ is the rank 8 real vector bundle given by

$$\mathbb{O}M \cong \Lambda^0 \oplus TM$$  \hspace{1cm} (3.6)
where $A^0 \cong M \times \mathbb{R}$ is a trivial line bundle. At each point $p \in M$, $O_p M \cong \mathbb{R} \oplus T_p M$.

The definition 3.6 simply mimics the decomposition of octonions into real and imaginary parts. The bundle $O M$ is defined as a real bundle, but which will have additional structure as discussed below. Now let $A \in \Gamma (O M)$ be a section of the octonion bundle. We will call $A$ simply an octonion on $M$. From (3.6), $A$ has a real part $\text{Re} A$, which is a scalar function on $M$, as well as an imaginary part $\text{Im} A$ which is a vector field on $M$. For convenience, we may also write $A = (\text{Re} A, \text{Im} A)$. Also, let us define octonion conjugation given by $\bar{A} = (\text{Re} A, -\text{Im} A)$. Whenever there is a risk of ambiguity, we will use $\hat{1}$ to denote the generator of $\Gamma (\text{Re} O M)$.

Since $O M$ is defined as a tensor bundle, the Riemannian metric $g$ on $M$ induces a metric on $O M$. Let $A = (a, \alpha) \in \Gamma (O M)$. Then,

$$|A|^2 = \langle A, A \rangle = a^2 + g(\alpha, \alpha) = a^2 + |\alpha|^2.$$  \hspace{1cm} (3.7)

We will be using the same notation for the norm, metric and inner product for sections of $O M$ as for standard tensors on $M$. It will be clear from the context which is being used. If however we need to specify that only the octonion inner product is used, we will use the notation $\langle \cdot, \cdot \rangle_{O \cdot \cdot}$. The metric (3.7) ensures that the real and imaginary parts are orthogonal to each other.

**Definition 3.3.** Given the $G_2$-structure $\varphi$ on $M$, we define a vector cross product with respect to $\varphi$ on $M$. Let $\alpha$ and $\beta$ be two vector fields, then define

$$\langle \alpha \times^\varphi \beta, \gamma \rangle = \varphi(\alpha, \beta, \gamma)$$  \hspace{1cm} (3.8)

for any vector field $\gamma$ [44, 53].

In index notation, we can thus write

$$(\alpha \times^\varphi \beta)^a = \varphi_a^b \alpha^b \beta^c.$$  \hspace{1cm} (3.9)

Note that $\alpha \times^\varphi \beta = -\beta \times^\varphi \alpha$. If there is no ambiguity as to which $G_2$-structure is being used to define the cross product, we will simply denote it by $\times$, dropping the subscript.

Using the contraction identity for $\varphi$ [43, 48, 53]

$$\varphi_{abc} \varphi_{mn}^c = 8 \delta_{ab} \delta_{mn} - 8 \delta_{am} \delta_{bn} + \psi_{abmn}$$  \hspace{1cm} (3.10)

we obtain the following identity for the double cross product.

**Lemma 3.4.** Let $\alpha, \beta, \gamma$ be vector fields, then

$$\alpha \times (\beta \times \gamma) = \langle \alpha, \gamma \rangle \beta - \langle \alpha, \beta \rangle \gamma + \psi(\hat{1}, \alpha, \beta, \gamma)$$  \hspace{1cm} (3.11)

where $\hat{1}$ means that we raise the index using the inverse metric $g^{-1}$.

Using the inner product and the cross product, we can now define the octonion product on $O M$.

**Definition 3.5.** Let $A, B \in \Gamma (O M)$. Suppose $A = (a, \alpha)$ and $B = (b, \beta)$. Given the vector cross product (3.8) on $M$, we define the octonion product $A \circ^\varphi B$ with respect to $\varphi$ as follows:

$$A \circ^\varphi B = \begin{pmatrix} ab - \langle \alpha, \beta \rangle \\ a\beta + b\alpha + \alpha \times^\varphi \beta \end{pmatrix}$$  \hspace{1cm} (3.12)
If there is no ambiguity as to which $G_2$-structure is being used to define the octonion product, for convenience, we will simply write $AB$ to denote it. The octonion product behaves as expected with respect to conjugation - $\bar{AB} = B\bar{A}$ and the inner product $\langle AB, C \rangle = \langle B, AC \rangle$ [49].

A key property of the octonions is their non-associativity and given three octonions $A = (a, \alpha), B = (b, \beta), C = (c, \gamma)$, a measure of their non-associativity is their associator $[A, B, C]$ defined by

$$[A, B, C] = A (BC) - (AB) C = 2\psi (\overrightarrow{\mathcal{J}}, \alpha, \beta, \gamma). \quad (3.13)$$

We can use octonions to parametrize isometric $G_2$-structures is well-known. If $A = (a, \alpha)$ is any nowhere-vanishing octonion, then define

$$\sigma_A (\varphi) = \frac{1}{|A|^2} \left( \left(a^2 - |\alpha|^2\right) \varphi - 2a\alpha \wedge (\ast \varphi) + 2\alpha \wedge (\alpha \wedge \varphi) \right). \quad (3.14)$$

It is well-known [48] that given a fixed $G_2$-structure $\varphi$ that is associated with the metric $g$, then any other $G_2$-structure in this metric class is given by (3.14) for some octonion section $A$. It has been shown [49, theorem 4.8] that for any nowhere-vanishing octonions $U$ and $V$,

$$\sigma_U (\sigma_V \varphi) = \sigma_{UV} (\varphi) \quad (3.15)$$

Thus we can move freely between different $G_2$-structures in the same metric class, which we will refer to as a change of gauge. It has also been shown in [49] that the octonion product $A \circ_V B$ that corresponds to the $G_2$-structure $\sigma_V (\varphi)$ is given by

$$A \circ_V B := (AV) (V^{-1}B) = AB + [A, B, V] V^{-1}. \quad (3.16)$$

In particular, a change of gauge can be used to rewrite octonion expressions that involve multiple products. Consider the expression $A (BC)$. Then, assuming $C$ is non-vanishing, we have

$$A (BC) = (AB) C + [A, B, C] = (A \circ_C B)C. \quad (3.17)$$

Similarly,

$$(AB) C = A (BC) - [A, B, C] = A (BC - A^{-1} [A, B, C]) = A (B \circ_{A^{-1}} C). \quad (3.18)$$

Here we have used some properties of the associator from [49, lemma 3.9].

Also, note that the cross product $A \times_V B$ of two $\mathfrak{Q}$-imaginary octonions with respect to $\sigma_V (\varphi)$ can be obtained from (3.16) as

$$A \times_V B = \frac{1}{2} (A \circ_V B - B \circ_V A) = A \times B + [A, B, V] V^{-1}. \quad (3.19)$$

3.1. Torsion and octonion covariant derivative

Given a $G_2$-structure $\varphi$ with an associated metric $g$, we may use the metric to define the Levi-Civita connection $\nabla$. The intrinsic torsion of a $G_2$-structure is then defined by $\nabla \varphi$. Following [55, 61], we can write
\[ \nabla_a \varphi_{bcd} = 2 T_a^\epsilon \varphi_{\epsilon bcd} \] (3.20)

where \( T_{ab} \) is the full torsion tensor. Similarly, we can also write

\[ \nabla_a \psi_{bcde} = -8 T_a^b [\varphi_{bcde}] \].

We can also invert (3.20) to get an explicit expression for \( T^m_a \)

\[ T^m_a = \frac{1}{48} (\nabla_a \varphi_{bcd}) \psi^{mbcd}. \] (3.22)

This 2-tensor fully defines \( \nabla \varphi \) [61]. The torsion tensor \( T \) is defined here as in [49], but actually corresponds to \( \frac{1}{2} T \) in [61] and \( -\frac{1}{2} T \) in [55]. With respect to the octonion product, we get

\[ \nabla_X (AB) = (\nabla_X A) B + A (\nabla_X B) - [T_X, A, B]. \] (3.23)

In general we can obtain an orthogonal decomposition of \( T_{ab} \) according to representations of \( G_2 \) into torsion components:

\[ T = \tau_1 g + \tau_7 \varphi + \tau_{14} + \tau_{27} \] (3.24)

where \( \tau_1 \) is a function, and gives the 1 component of \( T \). We also have \( \tau_7 \), which is a 1-form and hence gives the 7 component, and, \( \tau_{14} \in \Lambda^1 \) gives the 14 component and \( \tau_{27} \) is traceless symmetric, giving the 27 component. As it was originally shown by Fernández and Gray [46], there are in fact a total of 16 torsion classes of \( G_2 \)-structures that arise depending on which of the components are non-zero. Moreover, as shown in [55], the torsion components \( \tau_i \) relate directly to the expression for \( d \varphi \) and \( d \psi \)

\[ d \varphi = 8 \tau_1 \psi - 6 \tau_7 \wedge \varphi + 8 \tau_8 (\tau_{27}) \] (3.25a)

\[ d \psi = -8 \tau_7 \wedge \psi - 4 \tau_{14} \] (3.25b)

where \( \tau_8 (h)_{abcd} = h_{[a}^m \psi_{m|bcd]} \) for any 2-tensor \( h \). In terms of the octonion bundle, we can think of \( T \) as an \( \text{Im} \mathbb{O} \)-valued 1-form. If we change gauge to a different \( G_2 \)-structure \( \varphi_V \), then the torsion transforms in the following way.

**Theorem 3.6 ([49, theorem 7.2]).** Let \( M \) be a smooth 7-dimensional manifold with a \( G_2 \)-structure \( (\varphi, g) \) with torsion \( T \in \Omega^1 (\text{Im} \mathbb{O} \mathcal{M}) \). For a nowhere-vanishing \( V \in \Gamma (\mathbb{O} \mathcal{M}) \), consider the \( G_2 \)-structure \( \sigma_V (\varphi) \). Then, the torsion \( T^V \) of \( \sigma_V (\varphi) \) is given by

\[ T^V = \text{Im} (\text{Ad}_V T - (\nabla V)^{-1}) \]

\[ = \text{Ad}_V T - (\nabla V)^{-1} + \frac{1}{2} \frac{1}{|V|^2} \left( d |V|^2 \right) \hat{1} \] (3.26)

Here \( \text{Ad}_V \) is the adjoint map \( \text{Ad}_V : \Gamma (\mathbb{O} \mathcal{M}) \rightarrow \Gamma (\mathbb{O} \mathcal{M}) \) given by \( \text{Ad}_V (A) = VAV^{-1} \). Then, as in [49] let us define a connection on \( \mathbb{O} \mathcal{M} \), using \( T \) as a ‘connection’ 1-form.

**Definition 3.7.** Define the octonion covariant derivative \( D \) such for any \( X \in \Gamma (TM) \),

\[ D_X : \Gamma (\mathbb{O} \mathcal{M}) \rightarrow \Gamma (\mathbb{O} \mathcal{M}) \]

given by

\[ D_X A = \nabla_X A - AT_X \] (3.27)

for any \( A \in \Gamma (\mathbb{O} \mathcal{M}) \).
Note that in particular, $D_X 1 = -T_X$. Important properties of this covariant derivative are that it satisfies a partial derivation property with respect to the octonion product and that it is metric compatible.

**Proposition 3.8 ([49, propositions 6.4 and 6.5]).** Suppose $A, B \in \Gamma (\mathcal{O}M)$ and $X \in \Gamma (TM)$, then

$$D_X (AB) = (\nabla_X A) B + A (D_X B)$$

(3.28)

and

$$\nabla_X (g(A, B)) = g(D_X A, B) + g(A, D_X B)$$

(3.29)

Moreover, $D_X$ is covariant with respect to change of gauge.

**Proposition 3.9.** Suppose $(\varphi, g)$ is a $G_2$-structure on a 7-manifold $M$, with torsion $T$ and corresponding octonion covariant derivative $D$. Suppose $V$ is a unit octonion section, and $\tilde{\varphi} = \sigma_V(\varphi)$ is the corresponding $G_2$-structure, that has torsion $\tilde{T}$, given by (3.26), and an octonion covariant derivative $\tilde{D}$. Then, for any octonion section $A$, we have

$$D (AV) = (\tilde{D}A)V$$

(3.30)

Note that if $V$ does not have unit norm, then we have an additional term:

$$D (AV) = (\tilde{D}A)V + \left( \frac{1}{2} \frac{1}{|V|^2} d |V|^2 \right) AV.$$  

(3.31)

In [49], it was also shown that the octonion bundle structure has a close relationship with the spinor bundle $S$ on $M$. It is well-known (e.g. [50, 51]) that octonions have a very close relationship with spinors in 7 dimensions, respectively. In particular, the enveloping algebra of the octonion algebra is isomorphic to the Clifford algebra in 7 dimensions. The (left) enveloping algebra of $\mathcal{O}$ consists of left multiplication maps $L_A : V \rightarrow AV$ for $A, V \in \mathcal{O}$, under composition [62]. Similarly, a right enveloping algebra may also be defined. Since the binary operation in the enveloping algebra is defined to be composition, it is associative. More concretely, for $\mathcal{O}$-imaginary octonions $A$ and $B$,

$$L_A L_B + L_B L_A = -2 \langle A, B \rangle \text{Id}$$

(3.32)

which is the defining identity for a Clifford algebra. We see that while the octonion algebra does give rise to the Clifford algebra, in the process we lose the nonassociative structure, and hence the octonion algebra has more structure than the corresponding Clifford algebra. Note that also due to non-associativity of the octonions, in general that $L_A L_B \neq L_B L_A$. In fact, we have the following relationship.

It is then well-known that a nowhere-vanishing spinor on $M$ defines a $G_2$-structure via a bilinear expression involving Clifford multiplication. In fact, given a unit norm spinor $\xi \in \Gamma (S)$, we may define

$$\varphi_\xi (\alpha, \beta, \gamma) = -\langle \xi, \alpha \cdot (\beta \cdot (\gamma \cdot \xi)) \rangle_S$$

(3.33)

where $\cdot$ denotes Clifford multiplication, $\alpha, \beta, \gamma$ are arbitrary vector fields and $\langle \cdot, \cdot \rangle_S$ is the inner product on the spinor bundle. Now suppose a $G_2$-structure $\varphi_\xi$ is defined by a unit norm spinor $\xi$ using (3.33). This choice of a reference $G_2$-structure then induces a correspondence between spinors and octonions. We can define the linear map $j_\xi : \Gamma (S) \rightarrow \Gamma (\mathcal{O}M)$ by
\[ j_\xi (\xi) = 1 \quad (3.34a) \]
\[ j_\xi (V \cdot \eta) = V \circ_{\varphi_\xi} j_\xi (\eta). \quad (3.34b) \]
As shown in [49] the map \( j_\xi \) is in fact pointwise an isomorphism of real vector spaces from spinors to octonions. Moreover, it is metric-preserving, and under this map we get a relationship between the spinor covariant derivative \( \nabla^S \) and the octonion covariant derivative \( D \). More precisely, for any \( \eta \in \Gamma (\mathcal{S}) \)
\[ j_\xi \left( \nabla^S X \eta \right) = D_X \left( j_\xi (\eta) \right) \quad (3.35) \]
where \( D \) is the octonion covariant derivative (3.27) with respect to the \( G_2 \)-structure \( \varphi_\xi \).

Therefore, the octonion bundle retains all of the information from the spinor bundle, but has an additional non-associative division algebra structure.

**Notation 3.10.** For convenience, given a vector field \( X \), we will denote \( \delta \left( X \right) \in \Gamma (\text{Im} \mathcal{O} M) \) by \( \hat{X} \).

In components, the imaginary part of \( \delta \) is simply represented by the Kronecker delta:
\[ \delta_i = (0, \delta^i) \quad (3.37) \]

Below are some properties of \( \delta \)

**Lemma 3.11 ([49]).** Suppose \( \delta \in \Omega^1 (\text{Im} \mathcal{O} M) \) is defined by (3.37) on a 7-manifold \( M \) with \( G_2 \)-structure \( \varphi \) and metric \( g \). It then satisfies the following properties, where octonion multiplication is with respect to \( \varphi \)
\[ \delta \delta_j = \left( -g_{ij}, \varphi_j \right) \]
\[ \delta_j \left( \delta_k \delta_l \right) = \left( -\varphi_{jk}, \varphi_{lk} - \delta^a_{jk} g_{kl} + \delta^a_{lk} g_{kj} - \delta^a_{kl} g_{jk} \right) \]
\[ \text{For any } A = (a_0, \alpha) \in \Gamma (\mathcal{O} M), \]
\[ \delta A = \left( \begin{array}{c} -\alpha \\ a_0 \delta - \alpha \varphi \end{array} \right) \quad (3.38) \]

From lemma 3.11 we see that left multiplication by \( \delta \) gives a representation of the Clifford algebra. In particular, \( \delta \) satisfy the Clifford algebra identity (3.32)
\[ \delta_a \delta_b + \delta_b \delta_a = -2g_{ab} \text{Id}. \quad (3.39) \]

In fact, using the map \( j_\xi \), we can relate the Clifford algebra representation on spinors in terms of imaginary gamma matrices to \( \delta \):
\[ j_\xi (\gamma_a) = i \delta_a. \quad (3.40) \]

We can now define the octonion Dirac operator \( \mathcal{D} \) using \( \delta \) and the octonion covariant derivative \( D \) (3.27). Let \( A \in \Gamma (\mathcal{O} M) \), then define \( \mathcal{D} A \) as
\[ \mathcal{D} A = \delta \left( D_i A \right). \quad (3.41) \]
This operator is precisely what we obtain by applying the map $j_{\xi}$ to the standard Dirac operator on the spinor bundle.

A very useful property of $\mathcal{D}$ is that it gives the $\tau_1$ and $\tau_7$ components of the torsion [49]. Suppose $V$ is a nowhere-vanishing octonion section, and suppose $\tilde{\varphi} = \sigma_V (\varphi)$ has torsion tensor $\tilde{T}$ with 1-dimensional and 7-dimensional components $\tilde{\tau}_1$ and $\tilde{\tau}_7$. Then,

$$\mathcal{D} V = \left( \frac{1}{|\nabla|} \nabla |V| - 6 \tilde{\tau}_7 \right) V.$$  \hspace{1cm} (3.42)

In particular,

$$\mathcal{D} 1 = \left( \begin{array}{c} 7 \tau_1 \\ -6 \tau_7 \end{array} \right)$$  \hspace{1cm} (3.43)

where $\tau_1$ and $\tau_7$ are the corresponding components of $T$—the torsion tensor of the $G_2$-structure $\varphi$.

Let us also introduce some additional notation that will be useful later on.

**Notation 3.12.** Let $A \in \Omega^1 (\mathbb{O} M)$, Define the transpose $A'$ of $A$ as

$$(A')_i = (\text{Re} A)_i \hat{1} + \langle \delta_i, A_j \rangle \delta^j.$$  \hspace{1cm} (3.44)

Thus, $A'$ is an $\mathbb{O} M$-valued 1-form with the same $\mathbb{O}$-real part as $A$, but the $\mathbb{O}$-imaginary part is transposed. Similarly, define

$$(\text{Sym} A)_i = \frac{1}{2} (A + A') = (\text{Re} A)_i \hat{1} + \frac{1}{2} (\langle \delta_i, A_j \rangle \delta^j + (\text{Im} A)_i)$$  \hspace{1cm} (3.45)

and

$$(\text{Skew} A)_i = \frac{1}{2} (A - A') = \frac{1}{2} (\langle \delta_i, A_j \rangle \delta^j - (\text{Im} A)_i)$$  \hspace{1cm} (3.46)

A useful property of the transpose that follows from (3.38) is the following

$$(\delta A)' = A \delta.$$  \hspace{1cm} (3.47)

### 3.2. Complexified octonions

Recall that the Lie group $G_2$ also has a non-compact complex form $G_2^C$, which is a complexification of the compact real form $G_2$. In particular, $G_2^C$ is a subgroup of $SO (7, \mathbb{C})$ - the complexified special orthogonal group. The group $G_2^C$ then preserves the complexified vector product on $\mathbb{C}^7$ and is the automorphism group of the complexified octonion algebra - the bioctonions [51, 63]. We define the *bioctonion algebra* $\mathbb{O}_C$ as the complexification $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{O}$ of the real octonion algebra $\mathbb{O}$. This can be regarded as an octonion algebra over the field $\mathbb{C}$. In fact, it is known that any octonion algebra over $\mathbb{C}$ is isomorphic to $\mathbb{O}_C$ [63]. Any $Z \in \mathbb{O}_C$ can be uniquely written as $Z = X + iY$, where $X, Y \in \mathbb{O}$. Since $\mathbb{C}$ is the base field for this algebra, the imaginary unit $i$ commutes with octonions. To avoid ambiguity we will make a distinction between the complex real and imaginary parts and the octonion real and imaginary parts, so whenever there could be ambiguity, we will refer to $\mathbb{C}$-real and $\mathbb{C}$-imaginary parts or $\mathbb{O}$-real and $\mathbb{O}$-imaginary parts of a complex octonion. Also, we will make a distinction between complex conjugation, which will be denoted by $^*$ and
octonionic conjugation that will be denoted by "\(^\ast\). That is, for \(Z = X + iY \in \mathbb{O}_C\), where \(X, Y\) are \(\mathbb{C}\)-real octonions, we have
\[
\bar{Z} = X + iY
\]
\[
Z^\ast = X - iY. 
\]  
(3.48a)

Let us also linearly extend the inner product \(\langle \cdot, \cdot \rangle\) from \(\mathbb{O}\) to a complex-valued bilinear form on \(\mathbb{O}_C\). So if \(Z_1 = X_1 + iY_1\) and \(Z_2 = X_2 + iY_2\), we define
\[
\langle Z_1, Z_2 \rangle = \langle X_1 + iY_1, X_2 + iY_2 \rangle = \langle X_1, X_2 \rangle + i\langle Y_1, Y_2 \rangle + i\langle X_1, Y_2 \rangle + i\langle Y_1, X_2 \rangle.
\]  
(3.49)

Furthermore, define the complex-valued quadratic form \(N\) on \(\mathbb{O}_C\) given by
\[
N(Z) = ZZ = \langle Z, Z \rangle = \left( |X|^2 - |Y|^2 \right) + 2i \langle X, Y \rangle
\]  
(3.50)

for any \(Z = X + iY \in \mathbb{O}_C\). Note that while \(ZZ\) gives a \(\mathbb{O}\)-real expression, taking \(ZZ^\ast\) is in general not \(\mathbb{O}\)-real:
\[
ZZ^\ast = \left( |X|^2 + |Y|^2 \right) + i(YX - XY).
\]  
(3.51)

In fact, it is curious that \(\text{Re}_\mathbb{C} (ZZ^\ast) = \text{Re}_\mathbb{O} (ZZ^\ast)\) and \(\text{Im}_\mathbb{C} (ZZ^\ast) = \text{Im}_\mathbb{O} (ZZ^\ast)\).

It is easy to check that \(N\) gives \(\mathbb{O}_C\) the structure of a composition algebra, that is, for any \(Z_1, Z_2 \in \mathbb{O}_C\),
\[
N(Z_1 Z_2) = N(Z_1) N(Z_2).
\]  
(3.52)

This quadratic form is moreover isotropic, that is there exist non-zero \(Z \in \mathbb{O}_C\) for which \(N(Z) = 0\). Therefore, \(\mathbb{O}_C\) is a split composition algebra. Indeed, \(N(Z) = 0\) if and only if \(Z = X + iY\) with \(X, Y \in \mathbb{O}\) such that
\[
|X| = |Y| \quad \text{and} \quad \langle X, Y \rangle = 0.
\]  
(3.53)

In fact a complex octonion \(Z \in \mathbb{O}_C\) is a zero divisor if and only if \(N(Z) = 0\). Suppose now \(Z \in \mathbb{O}_C\) is not a zero divisor. We can then define the inverse of \(Z\) in a standard way:
\[
Z^{-1} = \frac{Z}{N(Z)}.
\]  
(3.54)

Given a real manifold \(M\), we may always complexify the tangent bundle as \(TM \otimes \mathbb{C}\), and then if \(M\) has a real Riemannian metric and a choice of orientation, it can be uniquely linearly extended to \(TM \otimes \mathbb{C}\), thus defining a \(SO(n, \mathbb{C})\)-structure. Similarly, given a real \(G_2\)-structure \(\varphi\) on a 7-manifold \(M\) we can extend it to \(TM \otimes \mathbb{C}\) and this will define a reduction of the \(SO(n, \mathbb{C})\)-structure to a \(G_2^\mathbb{C}\)-structure. The corresponding octonion bundle can now be also complexified, to give the biotyonion bundle \(\mathbb{O}_C M\), with \(g\) defining the quadratic form \(N\) on \(\mathbb{O}_C M\).

However note that now, given a \(Z \in \Gamma (\mathbb{O}_C M)\) that is nowhere a zero divisor, we may define a complex-valued 3-form \(\varphi_Z\) by
\[
\varphi_Z = \sigma_Z (\varphi).
\]  
(3.55)

Similarly as for real octonions, this will still be compatible with \(g\), but will define a new \(G_2^\mathbb{C}\)-structure. This is now invariant under \(Z \mapsto fZ\) for any nowhere-vanishing \(\mathbb{C}\)-valued
function \( f \), and hence this explicitly shows that isometric \( G_2^C \)-structures are pointwise parametrized by \( \mathbb{CP}^1 \cong SO(n, \mathbb{C}) / G_2^C \).

It is easy to see that all the standard \( G_2 \)-structure identities are satisfied by \( \varphi_Z \) and its corresponding 4-form \( \psi_Z \). Using \( \varphi_Z \) we can define a new product structure on \( \mathcal{O}_\mathbb{C}M \) in the same way as (3.16):

\[
A \circ_Z B := (AZ) (Z^{-1}B) = AB + [A, B, Z] Z^{-1}.
\]

The product \( \circ_Z \) for complex \( Z \) satisfies the same properties as for \( \mathbb{C} \)-real \( Z \), namely an important property is the following:

**Lemma 3.13.** Suppose \( A, B, Z \in \Gamma (\mathcal{O}_\mathbb{C}M) \), such that \( Z \) is nowhere a zero divisor, then

\[
A (BZ) = (A \circ_Z B) Z
\]

The irreducible representations of \( G_2^C \) are just the complexifications of the corresponding representations of the real form of \( G_2 \), so we still have the same decompositions of (complexified) differential forms. We will denote the relevant complexified representations with a superscript \( \mathbb{C} \).

Finally, we can also define the torsion \( T^{(Z)} \) of \( \varphi^{(Z)} \) as in (3.26)

\[
T^{(Z)} = - (DZ) Z^{-1} + \frac{1}{2} N(Z) d(N(Z)) \hat{\mathbb{I}}.
\]

With respect to representations of \( G_2^C \), this will have a similar decomposition into components as the torsion of a real \( G_2 \)-structure, with \( d\varphi_Z \) and \( d\psi_Z \) satisfying a complexified version of (3.25):

\[
d\varphi_Z = 8 \tau^{(Z)}_1 \psi_Z - 6 \tau^{(Z)}_7 \wedge \varphi_Z + 8 \iota_Z \left( \frac{(Z)}{27} \right)
\]

\[
d\psi_Z = -8 \tau^{(Z)}_1 \wedge \psi_Z - 4 \star \tau^{(Z)}_{14}
\]

where \( \tau^{(Z)}_1, \tau^{(Z)}_7, \tau^{(Z)}_{14}, \) and \( \tau^{(Z)}_{27} \) are the components of \( T^{(Z)} \) in representations \( 1^\mathbb{C}, 7^\mathbb{C}, 14^\mathbb{C}, \) and \( 27^\mathbb{C} \), respectively, with respect to the \( G_2^C \)-structure \( \varphi_Z \).

4. Supersymmetry equations in terms of octonions

Recall that the supersymmetry equations in 7 dimensions for a spinor \( \theta \) are given by

\[
0 = \lambda e^{-\Delta} \theta^* + \left[ \mu e^{-4\Delta} + \frac{1}{2} (\partial_i \Delta) \gamma^i \right] + \frac{1}{144} G_{bcde} \gamma^{bcde} \theta
\]

\[
\nabla^S_a \theta = \left[ i \frac{1}{2} e^{-4\Delta} \gamma_a - \frac{1}{144} \gamma_a \left( G_{bcde} \gamma^{bcde} \right) + \frac{1}{12} G_{abcd} \gamma^{abcd} \right] \theta.
\]

Suppose we have a fixed \( G_2 \)-structure \( \varphi \) that is defined by a unit spinor \( \xi \). Then, using the ideas in section 3.1, we can apply the map \( j \xi \) to the above equations, in order to obtain a set of equations for a complex octonion \( Z = j \xi (\theta) \). We will suppose that \( Z = X + iY \).

\[
0 = \lambda e^{-\Delta} Z^* + i \left( \mu e^{-4\Delta} + \frac{1}{2} (\partial_i \Delta) \delta^i \right) Z + \frac{1}{144} G_{bcde} \delta^b \left( \delta^c \left( \delta^d (\delta^e Z) \right) \right)
\]
\[ D_a Z = -\frac{1}{2} \mu e^{-4\Delta} \delta_a Z - \frac{i}{144} G_{bcde} \delta_a \left( \delta^b \left( \delta^c \left( \delta^d (Z) \right) \right) \right) - \frac{i}{12} G_{abcd} \left( \delta^b \left( \delta^c \left( \delta^d (Z) \right) \right) \right). \] (4.2b)

Let \( M = \mu e^{-4\Delta} + \frac{1}{2} \left( \partial \Delta \right) = \left( \mu e^{-4\Delta} \right) \), and also define the following operators on \( \Gamma (\mathcal{O}_C M) \)

\[ -\frac{1}{144} G_{bcde} \delta^b \left( \delta^c \left( \delta^d (\delta^e Z) \right) \right) = G^4 (A) \] (4.3a)

\[ -\frac{1}{144} G_{abcd} \left( \delta^b \left( \delta^c \left( \delta^d (A) \right) \right) \right) = G^3 (A) \] (4.3b)

for any octonion \( A \). These operators satisfy some properties that are an easy consequence of the Clifford algebra identity (3.39).

**Lemma 4.1.** The operators \( G^4 (A) \) and \( G^3 (A) \) satisfy the following properties for any (bi)octonion sections \( A \) and \( B \)

\[ \delta_a \left( G^4 (A) \right) = G^4 (\delta_a A) - 8G^3_a (A) \] (4.4a)

\[ \langle G^4 (A), B \rangle = \langle A, G^4 (B) \rangle \] (4.4b)

\[ \langle G^3_a (A), B \rangle = \langle A, G^3_a (B) \rangle \] (4.4c)

\[ \langle \delta_a G^4 (A), B \rangle = \langle \delta_a A, G^4 (B) \rangle - 8 \langle A, G^3_a (B) \rangle \] (4.4d)

\[ \langle \delta_a G^4 (A), A \rangle = -4 \langle A, G^3_a (A) \rangle \] (4.4e)

Now we can rewrite (4.2) more succinctly as

\[ G^4 (Z) = iMZ + \lambda e^{-\Delta} Z^* \] (4.5a)

\[ D_a Z = -\frac{1}{2} \mu e^{-4\Delta} \delta_a Z + i\delta_a G^4 (Z) + 12iG^3_a (Z). \] (4.5b)

Recall that given a \( G_2 \)-structure \( \varphi \), we can decompose the 4-form \( G \) according to the representations of \( G_2 \) as

\[ G = Q_0 \psi + Q_1 \wedge \varphi_X + t_\psi (Q_2) \] (4.6)

\[ = t_\psi \left( Q_0 g - \frac{4}{3} Q_1 \varphi Z + 4Q_2 \right) \] (4.7)

where \( Q_0 \) is a scalar function that represents the 1-dimensional component of \( G \), \( Q_1 \) is a vector that represents the 7-dimensional component, and \( Q_2 \) is a symmetric 2-tensor that represents the 27-dimensional component. Then, a straightforward calculation, gives us \( G^4 (1) \) and \( G^3 (1) \) in terms of these components.
Lemma 4.2. In terms of the decomposition (4.6) of the 4-form $G$, the quantities $G^4$ and $G^3$ are given by

$$G^4 (1) = \left( \frac{7}{3} Q_0 \right)$$

(4.8a)

$$G^3 (1) = \left( -\frac{1}{6} Q_0 \delta + \frac{1}{8} Q_1 \gamma \varphi + \frac{i}{12} Q_2 \right)$$

(4.8b)

Suppose $Z$ is not a zero divisor, that is, suppose we are working in a neighborhood where $N (Z) \neq 0$. We will justify this assumption later on. In that case, we can define a complexified $G_2$-structure $\varphi_Z = \sigma_Z (\varphi)$. Similarly to (4.6), with respect to $\varphi_Z$ we can decompose the 4-form $G$ according to the representations of $G_2$ as

$$G = Q_0 (Z) \psi_Z + Q_1 (Z) \wedge \varphi_Z + \iota_{\psi_Z} \left( Q_2 (Z) \right)$$

(4.9)

where the components $Q_i (Z)$ will in general now be complex-valued, even though $G$ is of course still $C$-real. Applying lemma 3.13, The equation (4.5b) can be rewritten with respect to $\varphi_Z$ as

$$DZ = -\frac{1}{2} \mu e^{-4 \Delta} \delta Z + \iota \left( \delta \circ_Z G^4 (Z) (1) \right) Z + 12i \left( G^3 (Z) (1) \right) Z,$$

where $G^4 (Z)$ and $G^3 (Z)$ are the operators $G^4$ and $G^3$ with respect to the $G_2$-structure $\varphi_Z$.

Now applying lemma 4.2 to $G^4 (Z) (1)$ and $G^3 (Z) (1)$ we see that

$$DZ = \iota \left( \frac{i}{2} \mu e^{-4 \Delta} + \frac{1}{6} Q_0 (Z) \right) \delta + \iota \left( Q_1 (Z) \gamma \varphi_Z - Q_2 (Z) \right) Z.$$

(4.11)

Taking the inner product of (4.11) with $Z$, we obtain

$$Q_1 (Z) = -\frac{3}{8 N (Z)} \frac{1}{6} \delta (N (Z))$$

(4.12)

and moreover, from (3.58) we find that the torsion of $\varphi_Z$ is given by

$$T (Z) = \left( \frac{1}{2} \mu e^{-4 \Delta} + \frac{5}{6} i Q_0 (Z) \right) g - \iota \left( \frac{1}{3} Q_1 (Z) \gamma \varphi_Z + i Q_2 (Z) \right).$$

(4.13)

Hence, the internal equation (4.5b) gives us the following information regarding the components of $T (Z)$:

$$\tau_1 (Z) = \frac{1}{2} \mu e^{-4 \Delta} + \frac{5}{6} i Q_0 (Z)$$

(4.14a)

$$\tau_7 (Z) = -\frac{1}{3} Q_1 (Z) = -\frac{1}{8 N (Z)} \delta (N (Z))$$

(4.14b)

$$\tau_{14} (Z) = 0$$

(4.14c)

$$\tau_{27} (Z) = i Q_2 (Z).$$

(4.14d)
In particular, the 14-dimensional component of the torsion vanishes and \( \tau_7^{(Z)} \) is an exact 1-form. Note that the \( G_2 \)-structure \( \varphi_Z \), and hence its torsion, depend on \( Z \) only projectively, that is they are invariant under scalings \( Z \rightarrow \kappa Z \) for any nowhere-vanishing complex-valued functions \( \kappa \). On the other hand, the original equation (4.2b) is only invariant under constant scalings. The expression (4.12) then determines \( N(Z) \) up to a constant factor.

Using (4.7) we can write the 4-form \( G \) in terms of the linear operator \( \iota_{\psi_Z} \) as

\[
G = \iota_{\psi_Z} \left( Q_0^{(Z)} \hat{g} - \frac{4}{3} Q_1^{(Z)} \omega \varphi_Z + 4 Q_2^{(Z)} \right)
\]

and therefore, using (4.13),

\[
G = -4i \iota_{\psi_Z} \left( T^{(Z)} \right) + \left( 2i \mu e^{-4\Delta} - \frac{7}{3} Q_0^{(Z)} \right) \psi_Z,
\]

but from the definition (3.20) of the torsion,

\[
d \varphi_Z = 8 \iota_{\psi_Z} \left( T^{(Z)} \right).
\]

However, in terms of \( \tau_1^{(Z)} \),

\[
2i \mu e^{-4\Delta} - \frac{7}{3} Q_0^{(Z)} = \frac{14}{5} \iota_{\tau_1^{(Z)}} + \frac{3}{5} i \mu e^{-4\Delta}.
\]

Therefore,

\[
G = -\frac{1}{2} id \varphi_Z + \frac{i}{5} \left( 14 \tau_1^{(Z)} + 3 \mu e^{-4\Delta} \right) \psi_Z.
\]

We can summarize our findings so far.

**Theorem 4.3.** Let \( (\varphi, g) \) be a real \( G_2 \)-structure on a 7-manifold \( \hat{M} \). Then, a non-zero divisor section \( Z \in \Gamma(\mathbb{C}M) \) satisfies the equation (4.5b) for a 4-form \( G \), real constant \( \mu \), and a warp parameter \( \Delta \), if the torsion of the complexified \( G_2 \)-structure \( \varphi_Z = \sigma_Z(\varphi) \) satisfies (4.14) and \( N(Z) \) satisfies (4.12). Conversely, if \( \varphi_Z \) is in the torsion class \( 1^C \oplus 7^C \oplus 27^C \), with \( \tau_1^{(Z)} = -\frac{1}{8} \frac{1}{N(Z)} d N(Z) \), and the 4-form \( G = -\frac{1}{2} id \varphi_Z + \frac{i}{5} \left( 14 \tau_1^{(Z)} + 3 \mu e^{-4\Delta} \right) \psi_Z \) is \( \mathbb{C} \)-real, then \( Z \) satisfies the equation (4.5b) with 4-form \( G \), constant \( \mu \), and warp parameter \( \Delta \).

From (4.17), (4.14b), and (3.25b), we see that \( dG = 0 \) if and only if

\[
d \left( \left( 14 \tau_1^{(Z)} + 3 \mu e^{-4\Delta} \right) N(Z) \right) = 0.
\]

So far we have only used equation (4.5b). Using (4.5a) will give us additional information. First, let us take the inner product of (4.5a) with the \( \mathbb{C} \)-conjugate \( Z^* = X - i Y \). Then, we get

\[
\left\langle G^{(4)}(Z), Z^* \right\rangle = i \left\langle MZ, Z^* \right\rangle + \left\langle \lambda e^{-\Delta} Z^*, Z^* \right\rangle.
\]

However since \( G \) is a real 4-form, the left hand side of (4.19) is \( \mathbb{C} \)-real, so the \( \mathbb{C} \)-imaginary part of the right hand side must vanish. Now, \( \left\langle MZ, Z^* \right\rangle = \left\langle M, Z^* \bar{Z} \right\rangle \) and \( M \) is \( \mathbb{C} \)-real. Moreover, from (3.51) we see that that the \( \mathbb{C} \)-real part of \( Z^* \bar{Z} \) is \( |X|^2 + |Y|^2 \), which is \( \mathbb{C} \)-real, so we find

\[
\mu e^{-4\Delta} \left( |X|^2 + |Y|^2 \right) + \text{Im}_\mathbb{C} \left( \lambda e^{-\Delta} N(Z)^* \right) = 0.
\]

In particular, we can rewrite (4.5a) as
Now from (3.51) we have

\[ Z^\ast Z = \left( |X|^2 + |Y|^2 \right) - i (Y \bar{X} - X Y) = \left( |X|^2 + |Y|^2 \right) - 2i \text{Im}_\Theta (Y \bar{X}). \]

Therefore, together with (4.8a) and (4.21) gives us

\[
\left( \frac{\partial}{\partial r} \frac{Q(Z)}{r^2} \right) = \left( \frac{\mu e^{-\Delta}}{4} + \frac{\lambda e^{-\Delta}}{N(Z)} \left( |X|^2 + |Y|^2 \right) \right) \frac{1}{2} i \partial \Delta - \frac{2 \lambda |e^{-\Delta} N(Z)|}{N(Z)} \text{Im}_\Theta (Y \bar{X}).
\]

(4.22)

In particular, comparing with (4.12), we obtain

\[
d N(Z) = -2 \left( d \Delta \right) N(Z) + 8 \lambda e^{-\Delta} \text{Im}_\Theta (Y \bar{X}).
\]

(4.23)

We obtained (4.23) by switching to the Z-frame, and hence implicitly assuming that \( N(Z) \) is non-zero. However, (4.23) can also be obtained by directly considering \( d (Z, \bar{Z}) \). From (4.23) we can also find that

\[
d |N(Z)|^2 = -4 \left( d \Delta \right) |N(Z)|^2 + 16 \text{Re}(\lambda^* N(Z)) e^{-\Delta} \text{Im}_\Theta (Y \bar{X}).
\]

(4.24)

Note that if \( \lambda \neq 0 \), (4.23) can be written as

\[
d \left( e^{2\Delta} \lambda^* N(Z) \right) = 8 |\lambda|^2 e^{\Delta} \text{Im}_\Theta (Y \bar{X}).
\]

(4.25)

Since the right-hand side of (4.25) is \( \mathbb{C} \)-real, this then shows that

\[
d \left( \text{Im}_\mathbb{C} \left( e^{2\Delta} \lambda^* N(Z) \right) \right) = 0.
\]

Together with (4.20) this then shows that if \( \mu \neq 0 \), then \( |X|^2 + |Y|^2 \) \( e^{-\Delta} \) is constant. Without loss of generality we can then say that

\[
|X|^2 + |Y|^2 = e^{\Delta}
\]

(4.26a)

\[
\text{Im}_\mathbb{C} \left( e^{-\Delta} \lambda^* N(Z) \right) = \mu.
\]

(4.26b)

If \( \mu = 0 \), then (4.26a) will not follow from (4.20), but we can still reach the same conclusion by considering \( \nabla \langle Z, Z^\ast \rangle \). This is shown in lemma A.1 in the appendix. Comparing, for example, with equation (2.16) in [18], if we set \( \chi^+ = \chi^- \) to reduce to \( N = 1 \) supersymmetry, then (4.26a) is equivalent to the equation \( \bar{\chi} + \chi = 1 \) and \( N(Z) \) is equivalent to the quantity \( S \).

Differentiating (4.25) one more time (and assuming \( \lambda \neq 0 \)), we obtain

\[
d \left( e^{\Delta} \text{Im}_\Theta (Y \bar{X}) \right) = 0.
\]

(4.27)

However if we divide (4.23) by \( N(Z) \), the left-hand side will still be closed, so differentiating again we also get

\[
d \left( \frac{e^{-\Delta} \lambda N(Z)^*}{|N(Z)|^2} \text{Im}_\Theta (Y \bar{X}) \right) = 0.
\]

Taking the \( \mathbb{C} \)-imaginary part using (4.26b) gives us
\[
\frac{\mu}{|N(Z)|^2} \text{Im} \Omega (Y \bar{X}) = 0.
\]

Using (4.24), this then tells us that
\[
\mu d (e^{4\Delta} \text{Im} \Omega (Y \bar{X})) = 0. \tag{4.28}
\]

Comparing with (4.27), we see that if both \( \lambda \) and \( \mu \) non-zero, then \( d (\text{Im} \Omega (Y \bar{X})) = 0 \) and moreover, if \( d\Delta \neq 0 \), then
\[
\text{Im} \Omega (Y \bar{X}) = f (\Delta) d\Delta
\]
for some real-valued function \( f (\Delta) \). Hence, if \( \lambda \) and \( \mu \) are nonzero, we see that \( \text{Im} \Omega (Y \bar{X}) \) and \( d\Delta \) are everywhere linearly dependent. In this case, from (4.23) we see that \( N (Z) \) is also a function of \( \Delta \) such that
\[
\frac{dN (Z)}{d\Delta} = -2N + 8 \lambda e^{-\Delta} f (\Delta). \tag{4.30}
\]

If we write \( N (Z) = \rho e^{i\theta} \) for some positive real-valued function \( \rho \) and a real-valued function \( \theta \), then from (4.23), (4.26b), and (4.24) we obtain
\[
d \left( e^{2\Delta} \rho \right) = 8 (\lambda_1 \cos \theta + \lambda_2 \sin \theta) e^\Delta \text{Im} \Omega (Y \bar{X}) \tag{4.31a}
\]
\[
d\theta = -\frac{8\mu}{\rho^2} \text{Im} \Omega (Y \bar{X}). \tag{4.31b}
\]

In particular, this shows that \( \mu = 0 \) implies a constant phase factor \( \theta \) for \( N (Z) \). By a change of basis of Killing spinors on AdS4, we may assume in that case that \( N (Z) = \rho \) and \( \lambda = \lambda_1 \).

**Remark 4.4.** For \( \lambda \neq 0 \), (4.26b) also shows that if \( \mu \neq 0 \), then \( N (Z) \) is nowhere-vanishing. If \( \mu = 0 \), then \( N (Z) \) may vanish at a point. However in (4.23) suppose \( \lambda \) is nonzero and \( N (Z) = 0 \) at a point. Then, at that point \( \text{Im} \Omega (Y \bar{X}) = 0 \) (otherwise \( X = Y = 0 \)) and hence \( dN (Z) \neq 0 \). Therefore, \( N (Z) \) cannot be identically zero in a neighborhood. So for nonzero \( \lambda \), we conclude that \( N (Z) \neq 0 \) almost everywhere. On the other hand, if \( \lambda = 0 \), i.e. if the 4-dimensional space is Minkowski then (4.26b) forces \( \mu = 0 \) and (4.23) shows that \( N (Z) = ke^{-2\Delta} \) for some \( k = k_1 + ik_2 \in \mathbb{C} \), and is thus either nowhere-vanishing or zero everywhere. The case where the 4-dimensional space is Minkowski and \( N (Z) = 0 \) everywhere is precisely the case that was considered in detail in [6]. Therefore, in all cases except this one, we may assume that \( N (Z) \neq 0 \) at least locally. We will also assume that \( \lambda \neq 0 \), unless specified otherwise. Whenever \( N (Z) \neq 0 \), from (4.31b) we have \( e^\theta \) constant in the case \( \mu = 0 \). However, by continuity of \( dN (Z) \), we find that \( e^\theta \) has to be constant everywhere. Hence, we can still apply a change of basis of spinors to assume that \( N (Z) = \rho \) (with \( \rho \geq 0 \)) and \( \lambda = \lambda_1 \).

Now using (4.22) and (4.26a) we can rewrite \( r_1^{(Z)} \) as
\[
r_1^{(Z)} = -\frac{3}{14} \mu e^{-4\Delta} + \frac{5}{7} \lambda i \frac{\bar{N}(Z)}{N(Z)} \tag{4.32}
\]
and the expression (4.17) for \( G \) can now be rewritten as
\[
G = -\frac{1}{2} \frac{i}{2} \bar{\phi} Z - \frac{2\lambda}{N(Z)} \psi Z. \tag{4.33}
\]
This now allows to verify that \( dG = 0 \), since (4.32) gives us
\[
\left(14\gamma_1^{(Z)} + 3\mu e^{-4\Delta}\right) N(Z) = 10\lambda i,
\]
and hence (4.18) is satisfied.

If \( \mu = 0 \), we will not be able define \( \varphi_Z \) and \( T^{(Z)} \) whenever \( N(Z) = 0 \). However in neighborhoods where \( N(Z) \neq 0 \) we can still do this and obtain \( dG = 0 \). On the other hand, if \( \lambda \neq 0 \), we know \( N(Z) \) cannot vanish on any open set, hence by continuity must have \( dG = 0 \) everywhere. We can now extend theorem 4.3 with conditions to satisfy equation (4.5a).

**Theorem 4.5.** Let \((\varphi, g)\) be a real \( G_2 \)-structure on a 7-manifold \( \mathcal{M} \). Then, a non-zero divisor section \( Z = X + iY \in \Gamma(\mathcal{O}_C \mathcal{M}) \) satisfies equations (4.5a) and (4.5b) for a closed 4-form \( G \), real constant \( \mu \), complex constant \( \lambda \), and a warp parameter \( \Delta \), if the torsion of the complexified \( G_2 \)-structure \( \varphi_Z = \sigma_Z(\varphi) \) satisfies the conditions of theorem 4.3 and moreover \( Z \) satisfies

\[
d(e^{2\Delta} N(Z)) \equiv 8\lambda e^{\Delta} \text{Im}_\mathbb{C}(YX) \tag{4.35a}
\]

\[
\text{Im}_\mathbb{C}(e^{-\Delta} \lambda^* N(Z)) = \mu \tag{4.35b}
\]

\[
|X|^2 + |Y|^2 = e^\Delta \tag{4.35c}
\]

\[
d(e^\Delta \text{Im}_\mathbb{C}(YX)) = 0 \tag{4.35d}
\]

\[
\text{Im}_\mathbb{C}(YX) = f(\Delta) d\Delta \text{ if } \mu \neq 0 \tag{4.35e}
\]

for some real-valued function \( f(\Delta) \). Conversely, if \( \varphi_Z \) is in the torsion class \( 1^C \oplus 7^C \oplus 27^C \), with \( \gamma_1^{(Z)} = -\frac{1}{8} \frac{1}{N(Z)^2} d(N(Z)) \), the 4-form \( G = -\frac{1}{2} i d \varphi_Z + \frac{i}{2} \left(14\gamma_1^{(Z)} + 3\mu e^{-4\Delta}\right) \psi_Z \) is \( \mathbb{C} \)-real, and \( Z \) satisfies (4.35a), then it satisfies the equations (4.5a) and (4.5b) with the 4-form \( G \), constant \( \mu \) defined by (4.35b), constant \( \lambda \) defined by (4.34), and warp parameter \( \Delta \) defined by (4.35c).

### 4.1. Decomposition into real and imaginary parts

To more concretely understand what are the properties of the corresponding real \( G_2 \)-structures, we need to decompose everything into \( \mathbb{C} \)-real and \( \mathbb{C} \)-imaginary parts. However for convenience, without loss of generality, we may change the reference \( G_2 \)-structure to \( \varphi_X := \sigma_X(\varphi) \), setting

\[
A = ZX^{-1} = 1 + iW \tag{4.36}
\]

where we defined \( W = w_0 + \bar{w} := YX^{-1} \). Then, using (3.31),

\[
DZ = D(AX) = \left(D^{(X)}A\right)X + (d \ln |X|)AX
\]

and

\[
i\delta G^{(4)}(Z) + 12i G^{(3)}(Z) = \left(i\delta \circ_X G^{(4)(X)}(A) + 12i G^{(3)(X)}(A)\right)X
\]

where \( D^{(X)} \), \( \circ_X \), \( G^{(3)(X)} \), and \( G^{(4)(X)} \) denote quantities with respect to \( \varphi_X \). From now on, we will drop the \( X \) subscripts and superscripts, since we will take \( \varphi_X \) to be the standard \( G_2 \)-structure. Overall, the equations (4.5) are equivalent to
\[ G^{(4)}(A) = iMA + \lambda e^{-\Delta}A^* \]  
(4.37a)

\[ DA = -\frac{1}{2} \mu e^{-4\Delta} \delta A + i\delta G^{(4)}(A) + 12iG^{(3)}(A) - (d \ln |X|) A. \]  
(4.37b)

Note that the choice of the \( G_2 \)-structure as \( \varphi_X \) is somewhat arbitrary - we could have alternatively chosen to work with \( \varphi_Y \) and then defined \( B = iZY^{-1} = (1 - iW^{-1}). \) The corresponding equations for \( B \) would then be

\[ G^{(4)}(B) = iMB - \lambda e^{-\Delta}B^* \]  
(4.38a)

\[ D_aB = -\frac{1}{2} \mu e^{-4\Delta} \delta_aB + i\delta_aG^{(4)}(B) + 12iG^{(3)}(B) - (d \ln |Y|) B. \]  
(4.38b)

Therefore, to obtain corresponding results for \( Y \), we just need to perform the transformation \( \{ X \rightarrow Y, W \rightarrow -W, \lambda \rightarrow -\lambda \}. \)

(4.39)

From (A.9) and (4.23), we find that

\[ d |X|^2 = \frac{1}{2} \left( 3 |Y|^2 - |X|^2 \right) d\Delta + 4\lambda_1 e^{-\Delta} \text{Im}_\Omega (YX) \]  
(4.40a)

\[ d |Y|^2 = \frac{1}{2} \left( 3 |X|^2 - |Y|^2 \right) d\Delta - 4\lambda_1 e^{-\Delta} \text{Im}_\Omega (YX). \]  
(4.40b)

However, \( \text{Im}_\Omega (YX) = |X|^2 w \), thus,

\[ d \ln |X| = \frac{1}{4} \left( 3 |W|^2 - 1 \right) d\Delta + 2\lambda_1 e^{-\Delta} w \]  
(4.41a)

\[ d \ln |Y| = |W|^2 \left( \frac{1}{4} \left( 3 - |W|^2 \right) d\Delta - 2\lambda_1 e^{-\Delta} w \right) \]  
(4.41b)

and hence,

\[ d |W|^2 = \frac{3}{2} \left( 1 - |W|^4 \right) d\Delta - 4\lambda_1 e^{-\Delta} \left( 1 + |W|^2 \right) w. \]  
(4.42)

From (4.42), we see that necessary conditions for \( |W|^2 \equiv 1 \) are \( \lambda_1 = 0 \) or \( w = 0. \) Now, (4.42) can be used to rewrite (4.41a) and (4.41b) as

\[ 2d \ln |X| = d\Delta - d \left( \ln \left( 1 + |W|^2 \right) \right) \]  
(4.43a)

\[ 2d \ln |Y| = d\Delta + d \ln \frac{|W|^2}{1 + |W|^2}. \]  
(4.43b)

Together with (4.26a) and \( |W|^2 = \frac{|Y|^2}{|X|^2} \) these equations imply

\[ |X|^2 = \frac{e^\Delta}{1 + |W|^2} \]  
(4.44a)
\[ |Y|^2 = \frac{e^{\Delta} |W|^2}{1 + |W|^2}. \] (4.44b)

Using the \( \mathbb{C} \)-imaginary part of (4.23) and the fact that \( w_0 = \frac{(X,Y)}{|X|^2} \), we obtain an expression for \( dw_0 \):

\[ dw_0 = -\frac{3}{2} \left( 1 + |W|^2 \right) w_0 d\Delta - 4 \left( w_0 \lambda_1 - \lambda_2 \right) e^{-\Delta} w. \] (4.45)

This shows that a necessary condition for \( w_0 = 0 \) is \( \lambda_2 = 0 \). Note that if \( w = 0 \), then \( |W|^2 = w_0^2 \), so we cannot have \( w_0 = 0 \). In fact, it is easy to see from (4.42) and (4.45) that \( w = 0 \) is only consistent when \( d\Delta = 0 \), and hence \( w_0 \) is also constant. This case is known to reduce to the standard Freund-Rubin solution on \( S^3 \) [8, 15], so we will not consider it. Hence, we will assume that \( w \) does not vanish identically. Also, recall that if \( \mu \neq 0 \), \( N \) (Z), and hence \( |W|^2 \) and \( w_0 \) are functions of \( \Delta \), and \( w = |X|^2 \) \( \text{Im}_0 \) \( (YX) \) is a multiple of \( d\Delta \). Since \( |X|^2 \) is then also a function of \( \Delta \), we will write \( w = h(\Delta) d\Delta \) whenever \( \mu \neq 0 \).

If \( w_0 \neq 0 \) and \( |W|^2 \neq 1 \), we can use (4.42) to rewrite (4.45) as

\[ d \left( \ln \frac{w_0}{1 - |W|^2} \right) = 4 \left( \frac{\lambda_2}{w_0} - \frac{2\lambda_1 w_0}{1 - |W|^2} \right) e^{-\Delta} w. \] (4.46)

However, from (4.26b) and (4.44a),

\[ 2w_0 \lambda_1 - \lambda_2 \left( 1 - |W|^2 \right) = \mu e^{-3\Delta} \left( 1 + |W|^2 \right). \] (4.47)

Hence, if \( \lambda_2 \) and \( \mu \) are non-zero, we see that

\[ |W|^2 = \frac{\lambda_2 + \mu e^{-3\Delta} - 2\lambda_1 w_0}{\lambda_2 - \mu e^{-3\Delta}}. \] (4.48)

**Remark 4.6.** From (4.47), we see that if \( w_0 \equiv 0 \), and hence \( \lambda_2 = 0 \) (from (4.45)), then we must also have \( \mu = 0 \). As we noted in remark 4.4, we can assume the converse is also true, that is if \( \mu = 0 \), then \( \lambda_2 = 0 \), and \( N \) (Z) is real, which means \( w_0 = 0 \). Similarly, if \( |W|^2 \equiv 1 \), and hence \( \lambda_1 = 0 \), then we must also have \( \mu = 0 \). It should be emphasized that (4.47) is a very important relation between \( w_0 \) and \( |W|^2 \) that we will use over and over again.

If \( |W|^2 \neq 1 \) and \( w_0 \neq 0 \), then dividing (4.47) by \( w_0 \) and \( 1 - |W|^2 \), and differentiating gives us

\[ d \left( \ln \frac{w_0}{1 - |W|^2} \right) = -\frac{4\mu e^{-3\Delta}}{w_0} \frac{1 + |W|^2}{1 - |W|^2} w. \] (4.49)

By applying \( d \) to (4.49) and using (4.42), we find that in the case when \( w_0 \neq 0 \) and \( |W|^2 \neq 1 \),

\[ d \left( \frac{4\mu e^{-\Delta}}{w_0} w \right) = 0. \] (4.50)

In the special case where \( \lambda_1 = 0 \), then (4.42) immediately gives

\[ |W|^2 = \frac{1 - k_1 e^{-3\Delta}}{1 + k_1 e^{-3\Delta}}. \] (4.51)
for some constant $k_1$. Note that for non-zero $\lambda_2$ and $\mu$, this is equivalent to (4.48) with $k_1 = -\frac{\mu}{\lambda_2}$.

If on the other hand, $\lambda_2 = 0$, then from (4.46) and (4.42), we find that for some constant $k_2$,

$$w_0 = k_2 \left(1 + |W|^2\right) e^{-3\Delta}$$

(4.52)

and then if $\lambda_1$ and $\mu$ are non-zero, then this is equivalent to (4.48) with $k_2 = -\frac{\mu}{\lambda_2}$.

In the Minkowski case, when $\lambda_1 = \lambda_2 = 0$, combining (4.51) and (4.52) gives us

$$w_0 = \frac{2k_1 k_2 e^{-3\Delta}}{1 + k_1 e^{-\Delta}}.$$  

(4.53)

We know that if $\mu \neq 0$, we can write $w = h(\Delta) d\Delta$ for some real function $h$. Thus,

$$|W|^2 = h^2 |d\Delta|^2 + w_0^2,$$

and from (4.48), we see that

$$h^2 |d\Delta|^2 = \frac{\lambda_2 (1 - w_0^2) + \mu e^{-3\Delta} (1 + w_0^2) - 2\lambda_1 w_0}{\lambda_2 - \mu e^{-3\Delta}}$$

(4.54)

where $w_0$ satisfies the equation

$$\frac{dw_0}{d\Delta} = \frac{3w_0 (w_0 \lambda_1 - \lambda_2)}{\lambda_2 - \mu e^{-3\Delta}} - 4 (w_0 \lambda_1 - \lambda_2) e^{-\Delta} h$$

(4.55)

which we obtained from (4.45).

### 4.2. External equation

Recall that we have

$$A = 1 + iW.$$  

(4.56)

So that taking the $C$-real and $C$-imaginary parts of the external equation (4.37a), we obtain two equations

$$G^{(4)}(1) = -MW + e^{-\Delta} (\lambda_1 + \lambda_2 W)$$

(4.57a)

$$G^{(4)}(W) = M + e^{-\Delta} (\lambda_2 - \lambda_1 W)$$

(4.57b)

where $G^{(4)}(1)$ and $G^{(4)}(W)$ are now defined with respect to the $G_2$-structure $\varphi_X$. Let us now decompose the 4-form $G$ with respect to $\varphi_X$ as:

$$G = Q_0 \psi_X + Q_1 \wedge \varphi_X + \iota_{\psi_X} (Q_2).$$

(4.58)

Taking $A = W = w_0 + \hat{w}$ and then $A = 1$, the equations (4.57) become

$$\left(\begin{array}{c} \frac{\partial}{\partial \Delta} Q_0 \\ \frac{\partial}{\partial \Delta} Q_1 \end{array}\right) = \left(\begin{array}{c} 2e^{-\Delta} \left(\lambda_2 - \mu e^{-3\Delta}\right) \\ -\partial \Delta \end{array}\right) W + 2\lambda_1 e^{-\Delta} \left(\begin{array}{c} 2e^{-\Delta} \left(\lambda_2 w_0 + \lambda_1 - \mu w_0 e^{-3\Delta}\right) + (\partial \Delta, w) \\ 2e^{-\Delta} \left(\lambda_2 - \mu e^{-3\Delta}\right) w - w_0 \partial \Delta - \partial \Delta \times w \end{array}\right)$$

(4.59a)
\[
\left( \frac{3}{4} w_0 Q_0 + \frac{1}{4} \langle Q_1, w \rangle \right) - \frac{1}{2} \left( \frac{2}{3} w_0 - \frac{1}{3} Q_0 + \frac{2}{3} Q_2 \right) (w) = \left( \frac{2 e^{-\Delta} \left( \lambda_2 + \mu e^{-3\Delta} \right)}{\partial \Delta} \right) - 2 \lambda_1 e^{-\Delta} W \\
= \left( \frac{2 e^{-\Delta} \left( \lambda_2 - \lambda_1 w_0 + \mu e^{-3\Delta} \right)}{\partial \Delta} - 2 \lambda_1 e^{-\Delta} w \right). 
\]
(4.59b)

From this, we find the following relationships

\[
Q_0 = \frac{3}{7} \langle \partial \Delta, w \rangle + \frac{6}{7} e^{-\Delta} \left( \lambda_2 w_0 + \lambda_1 - \mu w_0 e^{-3\Delta} \right) 
\]
(4.60a)

\[
\langle Q_1, w \rangle = \frac{7}{4} w_0 Q_0 + \frac{3}{2} e^{-\Delta} \left( \lambda_2 + \mu e^{-3\Delta} - \lambda_1 w_0 \right) 
\]
(4.60b)

\[
Q_1 = -\frac{3}{4} w_0 \partial \Delta - \frac{3}{4} \partial \Delta \times w + \frac{3}{2} e^{-\Delta} \left( \lambda_2 - \mu e^{-3\Delta} \right) w 
\]
(4.60c)

\[
Q_2 (w) = w_0 Q_1 - \frac{1}{4} \left( Q_0 - 6 \lambda_1 e^{-\Delta} \right) w - \frac{3}{4} \partial \Delta. 
\]
(4.60d)

Note that the expressions (4.60c) and (4.60b) are in fact compatible thanks to the relation (4.47).

In the Minkowski case, together with \( w_0 = 0 \), these simplify to the same relations as equation (3.12) in [6] once the slightly different definitions of \( Q_0, Q_1, Q_2 \) are taken into account. In the AdS\(_4\) case, we also have simplifications either \( \mu = \lambda_2 = w_0 = 0 \) or \( \mu \neq 0 \) and \( w = h \partial \Delta \). Hence, in the case

\[
Q_0 = \frac{3}{7} \langle \partial \Delta, w \rangle + \frac{6}{7} \lambda_1 e^{-\Delta} 
\]
(4.61a)

\[
\langle Q_1, w \rangle = 0 
\]
(4.61b)

\[
Q_1 = -\frac{3}{4} \partial \Delta \times w 
\]
(4.61c)

\[
Q_2 (w) = -\frac{1}{4} \left( Q_0 - 6 \lambda_1 e^{-\Delta} \right) w - \frac{3}{4} \partial \Delta 
\]
(4.61d)

and in the second case

\[
Q_0 = \frac{3}{7} h |\partial \Delta|^2 + \frac{6}{7} e^{-\Delta} \left( \lambda_2 w_0 + \lambda_1 - \mu w_0 e^{-3\Delta} \right) 
\]
(4.62a)

\[
Q_1 = -\frac{3}{4} \left( 2 h e^{-\Delta} \left( \mu e^{-3\Delta} - \lambda_2 \right) + w_0 \right) \partial \Delta 
\]
(4.62b)

\[
Q_2 (w) = -\frac{3}{28} \left( |\partial \Delta|^2 h^2 + 2 h e^{-\Delta} \left( 7 \mu w_0 e^{-3\Delta} - \mu e^{-3\Delta} - 6 \lambda_2 w_0 - 6 \lambda_1 \right) + 7 \left( w_0^2 + 1 \right) \right) \partial \Delta. 
\]
(4.62c)
5. Torsion

Now consider the internal equation (4.37b). Expanding $D\mathcal{M}$ (4.37b), we can equivalently rewrite it as

$$AT^{(X)} = \nabla A + \frac{1}{2} \mu e^{-4\Delta} \delta A - i\delta G^{(4)} (A) - 12iG^{(3)} (A) + (d \ln |X|) A. \quad (5.1)$$

So that now taking the \mathcal{C}-real and \mathcal{C}-imaginary parts of (5.1) gives us two \mathcal{C}-real equations

$$T^{(X)} = \frac{1}{2} \mu e^{-4\Delta} \delta + \delta G^{(4)} (W) + 12G^{(3)} (W) + (d \ln |X|) \tilde{I} \quad (5.2a)$$

$$WT^{(X)} = \frac{1}{2} \mu e^{-4\Delta} \delta W + \nabla W - \delta G^{(4)} (1) - 12G^{(3)} (1) + (d \ln |X|) W. \quad (5.2b)$$

As shown lemma A.3 in the appendix, for an octonion $A = a + \alpha$, we have

$$\delta \left( G^{(4)} (A) \right) + 12G^{(3)} (A) = F (A) - \left( \hat{Q}_2 \hat{A} \right)'$$

where

$$F (A) = \left( -\frac{1}{7} a_0 Q_1 - \frac{1}{7} \alpha Q_0 + \frac{2}{7} Q_2 (\alpha) - Q_1 \times \alpha \right) - \left( \frac{1}{7} Q_0 \alpha + \frac{1}{7} a_0 Q_1 - \frac{2}{7} Q_2 (\alpha) \right) \lambda \varphi \chi - \alpha Q_1$$

and in particular,

$$F (1) = \left( -\frac{1}{7} Q_0 \delta + \frac{2}{7} Q_1 \lambda \varphi \chi \right). \quad (5.5)$$

Note that in (5.4), and in similar expressions hereafter, we are implicitly using tensor products - so, $\alpha Q_1 \in \Omega^1 (\mathcal{M}) \otimes \Gamma (\text{Im} \mathcal{M})$ and the order therefore matters. We also suppress indices for brevity.

We can then use the relationships (4.60) that we obtained from the external equation, to simplify the expression for $F (W)$. In particular, we also obtain

$$Q_1 \times w = \frac{3}{4} w_0 \varphi \chi + \frac{3}{4} (\varphi \chi \times w) \times w$$

$$= \frac{3}{4} w_0 \varphi \chi + \frac{3}{4} \varphi \chi w - \frac{3}{4} w^2 \partial \Delta$$

$$= w_0 Q_1 - \frac{1}{4} (7Q_0 - 6\lambda_1 e^{-\Delta}) w + \frac{3}{4} W^2 \partial \Delta \quad (5.6)$$

and hence,

$$F (W) = -\frac{1}{4} \left( w_0 Q_0 e^{\Delta} + 2 |W|^2 \lambda_2 - 2 |W|^2 \mu e^{-3\Delta} + 2w_0 \lambda_1 \right) e^{-\Delta} \delta$$

$$+ \left( -\frac{1}{2} \left( 3 |W|^2 + 2 \right) d \Delta + w_0 Q_1 - \frac{1}{4} \left( Q_0 + 2\lambda_1 e^{-\Delta} \right) w \right) \frac{1}{2} (\partial \Delta + Q_0 \varphi \chi - 2\lambda_1 e^{-\Delta} w)$$

$$= \left( -\frac{1}{4} (Q_0 - 2\lambda_1 e^{-\Delta}) \delta \right) \tilde{W} + \left( \frac{1}{2} (3 |W|^2 + 2) d \Delta - \lambda_1 e^{-\Delta} w \right)$$

$$\frac{1}{2} \partial \Delta \lambda \varphi \chi - \frac{1}{2} (\lambda_2 + \mu e^{-3\Delta}) e^{-\Delta} \delta \quad (5.7)$$
where we have also used the expression for $Q_2 (w)$ from (4.60) the relationship (4.47) between $\lambda_1, \lambda_2$ and $\mu e^{-4\Delta}$. It is important to note that $F (1)$ and $F (W)$ do not contain any dependence on $Q_2$. Hence, overall, also using (5.7) and (4.41a), the expression (5.2a) for the torsion $T^{(X)}$ can be rewritten succinctly as

$$T^{(X)} = \frac{1}{2} \mu e^{-4\Delta} \delta + F (W) - \left( \hat{Q}_1 \hat{W} \right)' + (d \ln |X|) \hat{1} \hat{P}$$

$$= - \left( \hat{Q} W \right)' + \left( \frac{1}{4} \hat{\partial} \Delta - \frac{1}{2} \hat{\partial} \Delta \cdot \hat{\varphi}_X \right) + \frac{1}{2} e^{-\Delta} \left( \lambda_1 w \right)$$

$$= - \left( \hat{Q} W \right)' - \frac{1}{4} \hat{\partial} \Delta + \left( \frac{1}{2} \lambda_1 e^{-\Delta} w - \frac{1}{2} \lambda_2 e^{-\Delta} \delta \right)$$

(5.8a)

where we set

$$\hat{Q} = \left( \frac{1}{4} (Q_0 - 2 \lambda_1 e^{-\Delta}) \delta + Q_2 \right).$$

(5.9)

From (5.8a), we get

$$\left( T^{(X)} \right)' = - \hat{Q} W + \frac{1}{4} \hat{\partial} \Delta + \left( \frac{1}{2} \lambda_1 e^{-\Delta} w - \frac{1}{2} \lambda_2 e^{-\Delta} \delta \right).$$

(5.10)

The equation (5.2b) can be rewritten as

$$W T^{(X)} = \frac{1}{2} \mu e^{-4\Delta} \delta W + \nabla W - F (1) + \hat{Q}_2 + (d \ln |X|) W$$

$$= \nabla W + \left( \frac{1}{2} \mu e^{-4\Delta} \delta + d \ln |X| \right) W + \left( \frac{1}{4} \hat{Q}_1 - \frac{1}{2} Q_1 \cdot \hat{\varphi}_X + Q_2 \right).$$

(5.11a)

We can also write out (5.8a) explicitly as

$$T^{(X)} = - \frac{1}{4} \left( w_0 Q_0 - 2 w_0 \lambda_1 e^{-\Delta} + 2 \lambda_2 e^{-\Delta} \right) \delta + \frac{1}{4} (\hat{\partial} \Delta + Q_0 w - 2 \lambda_1 e^{-\Delta} w) \cdot \hat{\varphi}_X$$

$$- w \hat{Q}_1 - w_0 \hat{Q}_2 + (Q_2 \cdot \hat{\varphi} X)'.$$

(5.12)

This expression can be rewritten depending on whether $\mu$ vanishes or not. If $\mu = \lambda_2 = w_0 = 0$, then

$$T^{(X)} = \frac{1}{4} \left( \hat{\partial} \Delta + Q_0 w - 2 \lambda_1 e^{-\Delta} w \right) \cdot \hat{\varphi}_X - w \hat{Q}_1 + (Q_2 \cdot \hat{\varphi} X)'.$$

(5.13)

If $\mu \neq 0$ and $w = h (\Delta) \hat{\partial} \Delta$, then using (4.62),

$$T^{(X)} = - \frac{1}{4} \left( w_0 Q_0 - 2 w_0 \lambda_1 e^{-\Delta} + 2 \lambda_2 e^{-\Delta} \right) \delta + \frac{1}{4} \left( 1 + Q_0 h - 2 \lambda_1 e^{-\Delta} h \right) \hat{\partial} \Delta \cdot \hat{\varphi}_X$$

$$+ \frac{3}{4} \left( w_0 h - 2 e^{-\Delta} h \left( \lambda_2 - \mu e^{-3\Delta} \right) \right) \hat{\partial} \Delta \cdot \hat{\varphi}_X - w_0 Q_1 + h (Q_2 \cdot \hat{\varphi} X)'.$$

(5.14)

From the expression for the torsion (5.8a) we can immediately work out the 1-dimensional and 7-dimensional components of $T^{(X)}$. From (3.43), we find that
\[ \delta^a \left( T^a_X \right)' = \left( -\frac{7}{16} \tau_1 \right) = \left( -\frac{7}{6} \tau_1 \right) \]

and then, using the property \( \delta^a \delta_a = -7 \), we get

\[ \delta^a \left( T^a_X \right)' = \delta^a \left( \hat{Q}_a \hat{W} \right) - \frac{9}{4} \hat{\partial} \Delta + \lambda_1 e^{-\Delta} \hat{w} + \frac{7}{2} e^{-\Delta} \lambda_2 \]

\[ = \left( \delta^a \hat{Q}_a \right) \hat{W} + \left[ \delta^a, \hat{Q}_a, \hat{W} \right] - \frac{9}{4} \hat{\partial} \Delta + \lambda_1 e^{-\Delta} \hat{w} + \frac{7}{2} \lambda_2 e^{-\Delta}. \]

In components, \( \left[ \delta^a, \hat{Q}_a, \hat{W} \right]^c = -2\psi c_{mpq} Q^{mp} w^p = 0 \), since \( Q^{mp} \) is symmetric. Moreover, using (4.59a),

\[ \delta^a \hat{Q}_a = \left( \frac{7}{2} Q_0 \right) - \frac{7}{2} \lambda_1 e^{-\Delta} \]

\[ = \frac{3}{4} \left( 2 e^{-\Delta} \left( \lambda_2 - \mu e^{-3\Delta} \right) \right) \hat{W} - 2 \lambda_1 e^{-\Delta}. \]

Overall,

\[ \delta^a \left( T^a_X \right)' = \left( \frac{3}{2} \left( \lambda_2 - \mu e^{-3\Delta} \right) e^{-\Delta} \left| W \right|^2 - 2 \lambda_1 e^{-\Delta} w_0 + \frac{7}{2} \lambda_2 e^{-\Delta} w \right) \]

\[ - \frac{3}{4} \left( \left| W \right|^2 + 3 \right) \hat{\partial} \Delta + 3 \lambda_1 e^{-\Delta} w. \]

Using (4.47) to simplify the \( \hat{Q} \)-real part, we obtain

\[ \tau_1^{(X)} = -\frac{1}{14} \left( 1 + \left| W \right|^2 \right) e^{-\Delta} \left( 5 \lambda_2 - \frac{5}{2} \left| W \right|^2 + 2 \mu e^{-3\Delta} \right) \]  \hspace{1cm} (5.15a)

\[ \tau_2^{(X)} = \frac{1}{8} \left( \left| W \right|^2 + 3 \right) d\Delta = \frac{1}{2} \lambda_1 e^{-\Delta} w. \]  \hspace{1cm} (5.15b)

If \( \mu = \lambda_2 = w_0 = 0 \), we see that \( \tau_1^{(X)} = 0 \), and thus the torsion is in the class \( 7 \oplus 14 \oplus 27 \).

On the other hand, if \( \mu \neq 0 \) and \( w = h(\Delta) \hat{\partial} \Delta \), then since \( \left| W \right|^2 \) is then also a function of \( \Delta \), we find that \( \tau_2^{(X)} \) is an exact 1-form. This is notable since it shows that the torsion (5.12) is \textit{conformally} in the class \( 1 \oplus 14 \oplus 27 \). As it is well-known [61, 64], an exact \( \tau_2^{(X)} \) torsion component may be removed by a suitable conformal transformation of the metric (and hence the \( G_2 \)-structure), with the remaining torsion classes just being scaled. In general, however, we can use (4.42) to rewrite \( \tau_2^{(X)} \) as

\[ \tau_2^{(X)} = \frac{1}{16} \left( 5 \left| W \right|^2 + 3 \right) \hat{\partial} \Delta + \frac{1}{8} d \left( \ln \left( 1 + \left| W \right|^2 \right) \right). \]  \hspace{1cm} (5.16)

Thus \( \tau_2^{(X)} \) is closed if and only if \( d \left| W \right|^2 \wedge d\Delta = 0 \), or from (4.42) equivalently if

\[ \lambda_1 w \wedge d\Delta = 0 \]

that is, if either \( \lambda_1 = 0 \) or \( w \) and \( \hat{\partial} \Delta \) are linearly dependent. The latter case is already discussed above. In the case when \( \lambda_1 = 0 \), the above simplify to

\[ \tau_1^{(X)} = \frac{2}{7} \mu e^{-\Delta} \left( 10 - 3 k_1 e^{-3\Delta} \right) \]  \hspace{1cm} (5.17a)
\[ \tau^{(X)}_{7} = \frac{1}{8} \left( |W|^2 + 3 \right) d\Delta. \]  
\hspace{1cm} (5.17b)

However recall that we have the expression (4.51) for $|W|^2$ in terms of $\Delta$. Integrating that, we find that
\[ \tau^{(X)}_{7} = \frac{1}{12} \ln \left( |W|^2 + 3 \right) d\Delta. \]  
\hspace{1cm} (5.18)

Thus, $\tau^{(X)}_{7}$ is again exact. In the case of a Minkowski background, when $\lambda_2$ and $\mu$ also vanish, $\tau^{(X)}_{1}$ is then 0. Hence in that case, $\varphi_X$ is conformally equivalent to a $G_2$-structure in the class $14 \oplus 27$.

5.1. Torsion of $\varphi_Y$

The equation (5.11b) can be reformulated to give $T^{(Y)}$ - since this is torsion of the $G_2$-structure $\varphi_Y = \sigma_W (\varphi_X)$. Indeed, we can rewrite it as
\[ DW = \left( -\frac{5}{6} Q_0 \delta + \frac{1}{3} Q_1 \varphi_X - Q_2 \right) d\log |X| + \left( \frac{1}{2} \mu e^{-4\Delta} \right) W \]  
\hspace{1cm} (5.19)

and thus,
\[ T^{(Y)} = -(DW) W^{-1} + d\log |W| \]
\[ = -\left( -\frac{5}{6} Q_0 \delta + \frac{1}{3} Q_1 \varphi_X - Q_2 \right) W^{-1} + \left( \frac{1}{2} \mu e^{-4\Delta} \right) W + (d\log |Y|) \hat{1} \]  
\hspace{1cm} (5.20)

since $|W| = |Y| |X|^{-1}$. Now note that
\[ \left( -\frac{5}{6} Q_0 \delta + \frac{1}{3} Q_1 \varphi_X - Q_2 \right) = -\hat{Q} - \left( -\frac{5}{12} Q_0 \delta + \frac{1}{3} Q_1 \varphi_X \right) - \frac{1}{2} \lambda_1 e^{-\Delta} \delta. \]

But,
\[ \left( -\frac{5}{12} Q_0 \delta + \frac{1}{3} Q_1 \varphi_X \right) = -\delta \left( \frac{7}{12} Q_0 \right) \]
\[ = -\frac{1}{4} \delta \left( \left( 2e^{-\Delta} (\lambda_2 - \mu e^{-3\Delta}) \right) W + 2e^{-\Delta} \lambda_1 \right). \]

where we have also used (4.59). Hence,
\[ \left( -\frac{5}{6} Q_0 \delta + \frac{1}{3} Q_1 \varphi_X - Q_2 \right) = -\hat{Q} - \lambda_1 e^{-\Delta} \delta - \frac{1}{4} \delta \left( \left( 2e^{-\Delta} (\lambda_2 - \mu e^{-3\Delta}) \right) W \right). \]

Therefore,
\[ T^{(Y)} = \hat{Q} W^{-1} - \frac{1}{4} \delta \left( \left( \delta W \right) \right) W^{-1} + \lambda_1 e^{-\Delta} \left( \delta W^{-1} \right) + \frac{1}{2} \lambda_2 e^{-\Delta} \delta + (d\log |Y|) \hat{1}. \]  
\hspace{1cm} (5.21)

Using (3.17), note that
Thus, we conclude that
\[ T^{(Y)} = \hat{Q} W^{-1} - \frac{1}{4} \delta \circ_{Y} (\hat{\partial} \Delta) + \lambda_1 e^{-\Delta} (\delta W^{-1}) + \frac{1}{2} \lambda_2 e^{-\Delta} \delta + (\text{d ln } |Y|) \hat{1}. \] (5.22)

Using the same procedure as for \( T^{(X)} \), we can find the components \( \tau_1^{(Y)} \) and \( \tau_7^{(Y)} \):
\[ \tau_1^{(Y)} = \frac{1}{7} e^{-\Delta} \left( 5 w_0 \lambda_1 |W|^{-2} + 5 \lambda_2 - \frac{3}{2} \mu e^{-3\Delta} \right) \] (5.23a)
\[ \tau_7^{(Y)} = \frac{1}{8} \left(|W|^{-2} + 3\right) \partial \Delta + \frac{1}{2} \lambda_1 e^{-\Delta} w |W|^{-2}. \] (5.23b)

Comparing (5.22) with \( T^{(X)} \), we find
\[ \left( T^{(X)} \right)^2 + |W|^2 T^{(Y)} = \frac{1}{4} \left( |W|^2 \partial \Delta \partial \partial \phi_X - \partial \Delta \partial \partial \phi_Y \right) + \lambda_1 e^{-\Delta} w \partial \partial \phi_X + \frac{1}{2} \left( |W|^2 + 1 \right) \mu e^{-4\Delta} \delta \] (5.24)
where we have also used (4.47). Since the right-hand side of this expression has a vanishing traceless symmetric part, it follows that
\[ \tau_{27}^{(X)} + \tau_{27}^{(Y)} = 0. \] (5.25)

On the other hand, recall that
\[ \pi_7 T^{(X)} = \frac{1}{8} \left( |W|^2 + 3 \right) \partial \Delta \partial \partial \phi_X - \frac{1}{2} \lambda_1 e^{-\Delta} w \partial \partial \phi_X \] (5.26a)
\[ |W|^2 \pi_7 T^{(Y)} = \frac{1}{8} \left( 1 + 3 |W|^2 \right) \partial \Delta \partial \partial \phi_Y + \frac{1}{2} \lambda_1 e^{-\Delta} w \partial \partial \phi_Y. \] (5.26b)

Thus, taking the skew-symmetric part of (5.24), subtracting the appropriate 7-dimensional components, and noting that \( w \partial \partial \phi_X = w \partial \partial \phi_Y \), we get
\[ -\pi_4 T^{(X)} + |W|^2 \pi_4 T^{(Y)} = \frac{1}{8} \left( |W|^2 + 1 \right) \left( \partial \Delta \partial \partial \phi_X - \partial \Delta \partial \partial \phi_Y \right). \]

Going back to the octonion description again, we see that
\[ \partial \Delta \partial \partial \phi_X - \partial \Delta \partial \partial \phi_Y = \delta \circ_{Y} (\hat{\partial} \Delta) - \delta \circ_{X} (\hat{\partial} \Delta) = \left[ \delta, \hat{\partial} \Delta, W \right] W^{-1} \] (5.27)

where we have used (3.16) and (4.59), as well as properties of the associator. Hence, overall,
\[ |W|^2 \pi_4 T^{(Y)} - \pi_4 T^{(X)} = \frac{1}{8} \left( |W|^2 + 1 \right) \left[ \delta, \hat{\partial} \Delta, W \right] W^{-1}. \] (5.28)

Using (5.27) we can then rewrite the skew-symmetric part of (5.24) as
\[ \text{Skew} \left( \left( T^{(X)} \right)^t + |W|^2 T^{(Y)} \right) = \frac{1}{4} |W|^2 \left( \partial \Delta, \varphi_Y - \partial \Delta, \varphi_X \right) + \frac{1}{4} \left( |W|^2 - 1 \right) \partial \Delta, \varphi_X + \lambda_1 e^{-\Delta} \iota_w \varphi_X \]
\[ = - \frac{1}{4} \left[ \delta, \partial \Delta, w \right] \iota_w + \left( \frac{1}{4} \left( |W|^2 - 1 \right) \partial \Delta + \lambda_1 e^{-\Delta} \right) \iota_w \varphi_X. \]

We can summarize our findings in a theorem.

**Theorem 5.1.** Let \( M \) be a 7-dimensional manifold that admits \( G_2 \)-structures. Then, for a given metric \( g \), an arbitrary \( G_2 \)-structure \( \varphi \) that is compatible with \( g \), a 4-form \( G \), a real function \( \Delta \), and real constants \( \lambda_1, \lambda_2, \mu \), there exists a solution \( Z = X + iY \) to the equations (4.2) if and only if all the following conditions are satisfied:

1. Given \( W = (w_0, w) = YX^{-1} \), the quantities \( w_0 \) and \( |W|^2 \) satisfy the following equations

\[
\begin{align*}
2w_0 \lambda_1 - \lambda_2 (1 - |W|^2) &= \mu e^{-3\Delta} \left( 1 + |W|^2 \right) \\
\frac{d}{d|W|^2} \left( -\lambda_2 e^{-3\Delta} \right) &= 2 \left( 1 - |W|^2 \right) \Delta - 4 \lambda_1 e^{-\Delta} w
\end{align*}
\]

2. The components \( Q_0, Q_1, Q_2 \) of \( G \) with respect to \( \varphi_X \) satisfy

\[
\begin{align*}
\frac{Q_2}{(Q_0)^2} &= w_0 Q_1 - \frac{1}{2} \left( \frac{Q_0}{6} - 6 \lambda_1 e^{-\Delta} \right) w - \frac{1}{2} \partial \Delta \\
\frac{Q_0}{Q_1} &= 2 \left( 2 \lambda_2 - \mu e^{-3\Delta} \right) W + 2 \lambda_1 e^{-\Delta}
\end{align*}
\]

3. The torsion \( T^{(X)} \) of \( \varphi_X \) is given by

\[
\left( T^{(X)} \right)^t = \hat{Q} W + \frac{1}{4} \delta \left( \partial \Delta \right) + \left( \frac{\lambda_1 e^{-\Delta} w - \frac{1}{2} \partial \Delta}{-\frac{1}{2} \lambda_2 e^{-\Delta} \delta} \right) \iota_w \varphi_X
\]

4. The components of the torsion tensor \( T^{(Y)} \) of \( \varphi_Y \) are related to the components of \( T^{(X)} \) via the following relations

\[
\begin{align*}
\tau_{1}^{(X)} + |W|^2 \tau_{1}^{(Y)} &= \frac{1}{4} \mu e^{-4\Delta} \left( 1 + |W|^2 \right) \\
\tau_{1}^{(X)} + |W|^2 \tau_{1}^{(Y)} &= \frac{1}{4} \left( 1 + |W|^2 \right) \partial \Delta \\
\tau_{14}^{(X)} - |W|^2 \tau_{14}^{(Y)} &= -\frac{1}{2} \left( |W|^2 + 1 \right) \left[ \delta, \partial \Delta, W \right] W^{-1} \\
\tau_{27}^{(X)} + |W|^2 \tau_{27}^{(Y)} &= 0
\end{align*}
\]

Moreover, we can see that if \( \partial \Delta \) and \( w \) are linearly independent (which is possible if \( \mu = 0 \)), then both \( T^{(X)} \) and \( T^{(Y)} \) have all torsion components nonzero except the 1-dimensional components. From (4.61c), we see that the condition of \( \partial \Delta \) and \( w \) being not multiples of one another is equivalent to \( Q_1 \) being nonzero.

**Theorem 5.2.** Let \( \mu = 0 \), \( |x| \neq 0 \) and suppose \( \partial \Delta \) and \( w \) are not linearly dependent everywhere. Then \( T^{(X)} \) and \( T^{(Y)} \) have the same torsion type \( 7 \oplus 14 \oplus 27 \) with all components non-zero.

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Proof. Since $\partial \Delta$ and $w$ are not linearly dependent everywhere, we may assume that at some point, $\partial \Delta$ is not proportional to $w$ and in particular, $\partial \Delta \neq 0$. Since $\mu = 0$, then we know $\tau_{1}^{(x)} = \tau_{1}^{(y)} = 0$. To show that $T^{(x)}$ and $T^{(y)}$ have the same torsion type it is now sufficient to show using (5.30) that if $\tau_{i}^{(y)} = 0$ then $\tau_{i}^{(x)} = 0$ for any $i = 7, 14, 27$. The converse will follow by a similar argument. From (5.30) this is immediately obvious for the 27-dimensional component. From (5.15 b) and (5.23 b) it also follows immediately that each of $\tau_{7}^{(x)}$ and $\tau_{7}^{(y)}$ vanish only if $\partial \Delta$ is proportional to $w$. Thus, if $\partial \Delta$ is not proportional to $w$, then both $\tau_{7}^{(x)}$ and $\tau_{7}^{(y)}$ are non-zero.

Suppose now $\tau_{14}^{(y)} = 0$, then from (5.30),

$$\tau_{14}^{(x)} = -\frac{1}{8} \left( |W|^2 + 1 \right) \left[ \delta, \partial \Delta, W \right] W^{-1}.$$  

If $\partial \Delta$ is not a multiple of $w$, then $\left[ \delta, \partial \Delta, W \right] \neq 0$. However, we can rewrite this expression for $\tau_{14}^{(x)}$ as

$$\left( \tau_{14}^{(x)} \right)_{ab} = -\frac{1}{4} \left( |W|^{-2} + 1 \right) \psi_{abcd} \left( \partial \Delta \times w \right)^e w^d. \quad (5.31)$$

With respect to $\varphi_X$, the right-hand side will always have a $\Lambda^2_{14}$ component that is proportional to $\pi_7 \left( \left( \partial \Delta \times w \right) \wedge w \right)$. Since $w$ is not identically zero, this component is non-zero if and only if $\partial \Delta$ is not a multiple of $w$. Hence, we get a contradiction, since the left-hand side of (5.31) is in $\Lambda^2_{14}$ and the right-hand side has a nonvanishing component in $\Lambda^2_{7}$. So, $\tau_{14}^{(y)}$ must be non-zero and a similar argument will show that $\tau_{14}^{(x)}$ must also be non-zero.

To show that the 27-dimensional component must be non-zero, from (5.12) we can write down $\tau_{27}^{(x)}$ in the case $\mu = \lambda_2 = w_0 = 0$:

$$\tau_{27}^{(x)} = \text{Sym} \left( Q_2 \times w - w Q_1 \right). \quad (5.32)$$

From (5.32), we can work out $w_{,J} \tau_{27}^{(x)}$, which we simplify using (4.60):

$$w, \tau_{27}^{(x)} = \frac{3}{8} \left( 1 - |W|^2 \right) w \times \partial \Delta. \quad (5.33)$$

Clearly, if $\tau_{27}^{(x)} = 0$ then (5.33) also vanishes, however under our hypothesis, $w$ and $\partial \Delta$ are linearly independent. Hence, the only way for (5.33) to vanish identically is if $|W|^2 \equiv 1$, and hence $\lambda_1 = 0$, but we have $|\lambda| \neq 0$, hence $\tau_{27}^{(x)}$ is non-zero. 

5.2. Derivative of $W$

Instead of using (5.37) to calculate the torsion of $T^{(y)}$, we may instead use it to give a differential equation for $W$: 
\[ \nabla W = W^{(\nabla)} + \left( -\frac{i}{2} Q_0 \delta + \frac{i}{3} Q_1 \omega_0 - Q_2 \right) - \left( \frac{1}{2} i \mu e^{-4A} \delta + \delta \ln |x| \right) W \]
\[= - W \left( \delta \nabla \right)^{\prime} - \frac{1}{4} W \left( \delta \left( \partial \Delta \right) \right) + W \left( \frac{1}{2} \lambda_1 e^{-\Delta w} \partial \Delta - \frac{1}{2} \lambda_2 e^{-\Delta} \right) \]
\[+ \left( -\frac{i}{2} Q_0 \delta + \frac{i}{3} Q_1 \omega_0 - Q_2 \right) - \left( \frac{1}{2} i \mu e^{-4A} \delta + \delta \ln |x| \right) W. \quad (5.34) \]

The symmetric part of (5.34) gives a differential equation for \( w_0 \), however it can be shown to recover (4.45). However, the imaginary part gives a differential equation for the vector field \( w \) (and equivalently, its dual 1-form, which we will also denote by \( w \)). Overall,
\[ \nabla w = -\frac{1}{12} \left( \left( 3 |W|^2 + 17 \right) Q_0 - 6 \left( 1 + |W|^2 \right) \lambda_1 e^{-\Delta} + 12 \mu w_0 e^{-4A} \right) g \]
\[+ \frac{1}{4} \left( 2 + 3 |W|^2 \right) w (\partial \Delta) - \frac{3}{4} \left( |W|^2 + 1 \right) (\partial \Delta) w \]
\[+ w_0 Q_1 w - \frac{1}{4} \left( 7 Q_0 - 2 \lambda_1 e^{-\Delta} \right) w w - Q_2 - \text{Im}_\theta \left( W \left( \hat{Q}_1 \right) \right). \quad (5.35) \]

Using lemma A.4 from the appendix, we see that
\[ \text{Skew} \left( W \left( \hat{Q}_1 \right) \right) = \left( Q_2 \right) w - w \left( Q_2 \right) \]
\[= w_0 (Q_1 w - w Q_1) + \frac{3}{4} w (\partial \Delta) - \frac{3}{4} (\partial \Delta) w. \]

Hence, taking the skew-symmetric part of (5.35), we find that
\[ dw = - \frac{1}{12} \left( 3 |W|^2 + 1 \right) d \Delta \wedge w. \quad (5.36) \]

Note that this is equivalent to (4.27). If \( \mu \neq 0 \), then we know \( w \) is proportional to \( d \Delta \), hence we simply obtain \( dw = 0 \), as expected. If \( \mu = 0 \), then for \( |W|^2 = 1 \), then we can rewrite it as
\[ d \left( e^{2\Delta} w \right) = 0. \quad (5.37) \]

This exactly recovers equation (3.28) from [6]. However, whenever \( |W|^2 \neq 1 \), using (4.42), we obtain
\[ d \left( \left( |W|^2 - 1 \right)^{-\frac{3}{4}} \left( |W|^2 + 1 \right)^{-\frac{3}{4}} w \right) = 0. \quad (5.38) \]

For the symmetric part of \( \nabla w \), we obtain
\[ \text{Sym} \nabla w = - \frac{1}{12} \left( \left( 3 |W|^2 + 17 \right) Q_0 - 6 \left( 1 + |W|^2 \right) \lambda_1 e^{-\Delta} + 12 \mu w_0 e^{-4A} \right) g \]
\[+ \frac{1}{4} \left( 7 Q_0 - 2 \lambda_1 e^{-\Delta} \right) w w - Q_2 \]
\[+ \text{Sym} \left( w_0 Q_1 w - \frac{1}{4} w (\partial \Delta) - \text{Im}_\theta \left( W \left( \hat{Q}_1 \right) \right) \right). \quad (5.39) \]
Explicitly, in coordinates, this gives

\[
\nabla_{(a w_b) -} = -\frac{1}{12} \left( (3 |W|^2 + 17) Q_0 - 6 \left( 1 + |W|^2 \right) \lambda_1 e^{-\Delta} + 12 \mu w_0 e^{-4\Delta} \right) g_{ab} \\
- \frac{1}{2} (3 Q_0 + 2 \lambda_1 e^{-\Delta}) w_a w_b - (Q_2)_{ab} (w_0^2 + 1) \\
+ \frac{1}{2} w_{(a} (\partial_{b) \Delta) - 2 w_0 (Q_2)^m_w \phi_{b,mn} - \varphi_{amn} \varphi_{bqp} (Q_2)^m_p w^n w^q}. \tag{5.40}
\]

In particular, if \( \mu = w_0 = \lambda_2 = 0 \), this simplifies to

\[
\nabla_{(a w_b) -} = -\frac{1}{28} \left( (3 |W|^2 + 17) (\partial \Delta, w) + 4 \left( 5 - 2 |W|^2 \right) \lambda_1 e^{-\Delta} \right) g_{ab} \\
- \frac{1}{14} (3 (\partial \Delta, w) + 20 \lambda_1 e^{-\Delta}) w_a w_b - (Q_2)_{ab} + \frac{1}{2} w_{(a} (\partial_{b) \Delta) \\
- \varphi_{amn} \varphi_{bqp} (Q_2)^m_p w^n w^q}. \tag{5.41}
\]

where we have also used (4.61). Furthermore, in the Minkowski case, together with \(|W|^2 = 1\), we get

\[
\nabla_{(a w_b) -} = -\frac{5}{3} Q_0 g_{ab} - \frac{3}{2} Q_0 w_a w_b - (Q_2)_{ab} + \frac{1}{2} w_{(a} (\partial_{b) \Delta) - \varphi_{amn} \varphi_{bqp} (Q_2)^m_p w^n w^q}. \tag{5.42}
\]

This has some additional terms compared to the corresponding expression (3.29) in [6], however it agrees with [8].

Taking the trace in (5.40), using (4.47), and then using the expression (4.60a) for \( Q_0 \), we obtain

\[
\text{div } w = -\frac{3}{2} (|W|^2 + 3) (\partial \Delta, w) + \left( 4 w_0^2 + |W|^2 - 5 \right) \lambda_1 e^{-\Delta} - 10 w_0 \lambda_2 e^{-\Delta}. \tag{5.43}
\]

We can also use the equation (5.34) to express \( Q_2 \) in terms of \( W \) and \( \partial \Delta \). First note that

\[
\dot{W} Q_2 = -\dot{Q}_2 \dot{W} + 2 Q_2 (w) + 2 w_0 Q_2
\]

and hence multiplying (5.34) by \( W \) on the left, and rearranging, we obtain

\[
\dot{Q}_2 W - |W|^2 (\dot{Q}_2 W) - 2 w_0 \dot{Q}_2 = W \nabla W - |W|^2 F(W) - WF(1) + 2 Q_2 (w) + \frac{1}{2} \mu e^{-4\Delta} \left( W \delta W - |W|^2 \delta \right). \tag{5.44}
\]

Note that due to (4.60d), the right-hand side of (5.44) has no dependence on \( Q_2 \). When \( W^2 \neq -1 \), using lemma A.5 we can solve equation (5.44) for \( Q_2 \) to obtain

\[
\dot{Q}_2 = -\frac{K + |W|^2 K'}{(1 + |W|^2) |W|^2} \left( \dot{W} + |W|^2 \dot{W} \right), \tag{5.45}
\]

where \( K = \bar{W} \nabla W - |W|^2 F(W) - WF(1) + 2 Q_2 (w) + \frac{1}{2} \mu e^{-4\Delta} \left( W \delta W - |W|^2 \delta \right) \). The condition \( W^2 \neq -1 \) is precisely equivalent to saying that the complex octonion \( A = 1 + iW \) (or equivalently, \( Z = X + iY \)) is not a zero divisor. We know this is true when \( \mu \neq 0 \) and for nonzero \( \lambda \), this is true at least locally even if \( \mu = 0 \), so we may assume that \( W^2 \neq -1 \).

The expression (5.45) gives us a solution for \( Q_2 \) that only involves \( \nabla w, d\Delta, w_0, |W|^2 \), and \( \Delta \). Furthermore, we know that if \( \mu \neq 0 \), then \( W = h (\Delta) d\Delta \), and hence all the other relevant quantities
are also functions of \( \Delta \). Therefore, in this case, \( Q_2 \) also becomes a function only of \( \Delta \) and its derivative. Moreover, from (5.8\alpha) we see that in this case, the torsion is also expressed solely in terms of \( \Delta \) and \( d\Delta \).

In the case \( \mu \neq 0 \), (5.43) can tell us more about the function \( h \), since

\[
\text{div } w = \text{div } (h\Delta) = \frac{dh}{d\Delta} |d\Delta|^2 + h \nabla^2 \Delta.
\]

Thus, overall,

\[
\frac{dh}{d\Delta} |d\Delta|^2 = -\frac{3}{2} \left( w_0^2 + h^2 |d\Delta|^2 + \frac{1}{3} \right) h |d\Delta|^2 + \left( 5w_0^2 + h^2 |d\Delta|^2 - 5 \right) \lambda_1 e^{-\Delta} - 10w_0 \lambda_2 e^{-\Delta} - h \nabla^2 \Delta
\]

\[
(5.46)
\]

where we have also used \( |W|^2 = w_0^2 + h^2 |d\Delta|^2 \). As we will see in the next section, integrability conditions for the \( G_2 \)-structures \( T(X) \) and \( T(Y) \) imply Einstein’s equations, and in particular, (2.11a), which also us to rewrite \( \nabla^2 \Delta \) in terms of \( |G|^2 \). Hence, (5.46) becomes

\[
\frac{dh}{d\Delta} |d\Delta|^2 = -\frac{3}{2} \left( w_0^2 + h^2 |d\Delta|^2 + \frac{1}{3} \right) h |d\Delta|^2 + \left( 5w_0^2 + h^2 |d\Delta|^2 - 5 \right) \lambda_1 e^{-\Delta} - 10w_0 \lambda_2 e^{-\Delta} - 12 \left( \frac{1}{18} |G|^2 + \mu^2 e^{-8\Delta} - |\lambda|^2 e^{-2\Delta} \right) h
\]

\[
(5.47)
\]

where \( h \) and \( |d\Delta|^2 \) are related by (4.54) and \( w_0 \) satisfies the ODE (4.55). In particular, this shows that \( |G|^2 \) is also a function of \( \Delta \). Now, whenever \( |d\Delta|^2 \neq 0 \), using (4.54) we can eliminate \( |d\Delta|^2 \) from (5.47) and obtain \( \frac{dh}{d\Delta} \). Overall, \( h \) and \( w_0 \) satisfy the following system of ODEs

\[
\frac{dh}{d\Delta} = \frac{12 \left( \lambda_1 - \mu e^{-3\Delta} \right) \left( \frac{1}{18} |G|^2 + \mu^2 e^{-8\Delta} - |\lambda|^2 e^{-2\Delta} \right) h^3}{w_0^2 \left( \lambda_2 - \mu e^{-3\Delta} \right) + 2w_0 \lambda_1 - \left( \lambda_1 + \mu e^{-3\Delta} \right) - 2e^{-\Delta} \left( 2\lambda_1 \left( \lambda_2 - \mu e^{-3\Delta} \right) w_0^2 - \left( \lambda_1^2 + 5\lambda_2^2 - 5\lambda_2 \mu e^{-3\Delta} \right) w_0 - 2\lambda_1 \lambda_2 + 3\lambda_1 \mu e^{-3\Delta} \right) h^2}{w_0^2 \left( \lambda_2 - \mu e^{-3\Delta} \right) + 2w_0 \lambda_1 - \left( \lambda_2 + \mu e^{-3\Delta} \right) + \frac{3\lambda_1 w_0 - 2\lambda_2 - \mu e^{-3\Delta}}{\left( \lambda_2 - \mu e^{-3\Delta} \right)} h}
\]

\[
(5.48a)
\]

\[
\frac{dw_0}{d\Delta} = \frac{3w_0 \left( w_0 \lambda_1 - \lambda_2 \right)}{\lambda_2 - \mu e^{-3\Delta}} - 4 \left( w_0 \lambda_1 - \lambda_2 \right) e^{-\Delta} h.
\]

(5.48b)

The dependence on \( |G|^2 \) in (5.48a) could be substituted for a dependence on \( |Q_2|^2 \), and then from (5.45), we could substitute that for some expression in terms of \( \nabla d\Delta \). However, that still does not give something that could be solved explicitly.

### 6. Integrability conditions

From [55, 61, 65] we know that the torsion of a \( G_2 \)-structure satisfies the following integrability condition:

\[
\frac{1}{4} \text{Riem}_{ij}^\alpha \gamma \varphi_i^\alpha \gamma = \nabla_i T_j^\alpha - \nabla_j T_i^\alpha + 2T_i^\beta T_j^\gamma \varphi_i^\alpha \gamma.
\]

(6.1)
In [49], the right-hand side of this expression was interpreted as the \( \text{Im} \mathbb{O}M \)-valued 2-form \( d\Pi T \), where \( d\Pi \) is the octonionic covariant exterior derivative obtained from \( D (3.27) \). The left-hand side expression has been denoted in [55] as \( \pi_7 \text{Riem} \) since we are effectively taking a projection to \( \Lambda_2^7 \) of one of the 2-form pieces of \( \text{Riem} \). Hence, (6.1) is equivalently written in terms of octonions as

\[
d\Pi T = \frac{1}{4} \pi_7 \text{Riem}.
\]

(6.2)

The quantity \( d\Pi T \) is pointwise in the \( (7 \oplus 14) \otimes 7 \) representation of \( G_2 \), which, as shown in [66], decomposes in terms irreducible representations as

\[
(7 \oplus 14) \otimes 7 \cong (1 \oplus 7 \oplus 14 \oplus 27) \oplus (7 \oplus 27 \oplus 64).
\]

(6.3)

We can consider different projections of \( d\Pi T \) to project out the component in \( 64 \). In particular, let

\[
(T_1)_{ab} = (d\Pi T)_{acd} \varphi^{cd}_{\ b},
\]

(6.4a)

\[
(T_2)_{ab} = (d\Pi T)_{cda} \varphi^{cd}_{\ b}.
\]

(6.4b)

From (6.1), these conditions then give

\[
T_1 = \frac{1}{2} \text{Ric}
\]

(6.5a)

\[
T_2 = \frac{1}{4} \text{Ric}^*
\]

(6.5b)

where \( (\text{Ric}^*)_{ab} = \text{Riem}_{\ mnpq} \varphi^m_{\ a} \varphi^n_{\ b} \varphi^p_{\ q} \) is the *-Ricci curvature as defined in [65]. From (6.5a) we can then obtain an expression for \( \text{Ric} \) in terms of \( T \) and its derivatives which has originally been shown by Bryant in [48]. In particular, both \( \text{Ric} \) and \( \text{Ric}^* \) are symmetric and their traces give the scalar curvature \( R \). Together, they generate the \( 1 \oplus 27 \oplus 27 \) part in (6.3). The skew-symmetric parts of \( T_1 \) and \( T_2 \) therefore vanish - the two \( 7 \) components generate the \( 1 \oplus 7 \oplus 7 \) part in (6.3), and the two \( 14 \) components are actually proportional and correspondingly give the \( 14 \) component in (6.3). An equivalent way to obtain the \( 7 \oplus 7 \oplus 14 \) components of the integrability condition is from \( d^2 \varphi = 0 \) and \( d^2 \psi = 0 \). The expression for the scalar curvature can be easily written out explicitly in terms of components of the torsion:

\[
\frac{1}{4} R = 42 \tau_1^2 + 30 |\tau_7|^2 - |\tau_{14}|^2 - |\tau_{27}|^2 + 6 \text{ div } \tau_7
\]

(6.6)

or equivalently,

\[
\frac{1}{4} R = 49 \tau_1^2 + 36 |\tau_7|^2 - |T|^2 + 6 \text{ div } \tau_7.
\]

(6.7)

In our case, we have two \( G_2 \)-structure \( \varphi_X \) and \( \varphi_Y \) with torsion tensors \( T^{(X)} \) and \( T^{(Y)} \) given by (5.12) and (5.22), respectively. Each of these torsion tensors satisfies their corresponding integrability conditions (6.1). However, since they correspond to the same metric, and thus the same Ricci curvature, we get \( T^{(X)}_1 = T^{(Y)}_1 \). It also turns out that \( 7 \oplus 7 \oplus 14 \) components of (6.1) for \( T^{(X)} \) and \( T^{(Y)} \) are equivalent, and together give the condition \( dG = 0 \). Moreover, whenever \( w_0 \neq 0 \) and \( |W|^2 \neq 1 \) (which holds almost everywhere if \( \mu \neq 0 \)), we will find that \( T^{(X)}_1 - T^{(Y)}_1 = 0 \) gives precisely the equation of motion (2.7) and the 1-dimensional component is moreover equivalent to the 4-dimensional Einstein’s equation (2.11a).
Let us first work out $R$ using both $T^{(X)}$ and $T^{(Y)}$. To work out $R$, from (6.7) we need to know $(\tau_1^{(X)})^2$, $(\tau_2^{(X)})^2$, $|T^{(X)}|^2$, and $\text{div} \tau_1^{(X)}$. The first two are immediately obtained from expressions (5.15a) and (5.15b). Then, $|T^{(X)}|^2 = |T^{(Y)}|^2$ can be worked out from the expression (5.10) for $|T^{(X)}|^2$. After some manipulations, we obtain

\[
|T^{(X)}|^2 = |W|^2 |Q_1|^2 - \frac{21}{8} |W|^2 Q_0 \left( Q_0 - \frac{4}{3} \lambda_1 e^{-\Delta} \right) + \frac{3}{16} \left( 3 |W|^4 + 2 |W|^2 + 1 \right) |d\Delta|^2
- \frac{3}{2} |W|^2 \lambda_2^2 e^{-2\Delta} + \frac{1}{2} \left( 5 |W|^4 + 4 |W|^2 + 2 \right) \lambda_2^2 e^{-2\Delta} - \frac{1}{2} \left( 10 |W|^4 + 5 |W|^2 + 1 \right) \mu \lambda_2 e^{-5\Delta}
+ \frac{1}{4} \left( 10 |W|^4 + 2 |W|^2 + 1 \right) \mu^2 e^{-8\Delta}.
\]

Hence, we can use (6.9) to rewrite (6.8) as

\[
|T^{(X)}|^2 = \frac{1}{2} |W|^2 |G|^2 + \frac{3}{16} (1 - 3 |W|^2) \left( 1 - |W|^2 \right) |d\Delta|^2 - 2 |W|^2 \lambda_1 e^{-\Delta} \langle d\Delta, w \rangle
- 3 |W|^2 \lambda_2^2 e^{-2\Delta} + \left( |W|^4 - |W|^2 + 1 \right) \lambda_2^2 e^{-2\Delta} - \frac{1}{2} \left( 1 + |W|^2 \right) \left( 1 + 4 |W|^2 \right) \mu \lambda_2 e^{-5\Delta}
+ \frac{1}{4} \left( 4 |W|^4 + 14 |W|^2 + 1 \right) \mu^2 e^{-8\Delta}.
\]

where we have also used (4.60a) to eliminate $Q_0$ and (4.47) to eliminate $w_0$.

Now let us work out $\text{div} \tau_1^{(X)}$. From (5.15b), we have

\[
\text{div} \tau_1^{(X)} = \frac{1}{8} \left( 3 + |W|^2 \right) \nabla^2 \Delta + \frac{1}{8} \left( d |W|^2, d\Delta \right) + \frac{1}{2} \lambda_1 e^{-\Delta} \langle d\Delta, w \rangle - \frac{1}{2} \lambda_1 e^{-\Delta} \text{div} w.
\]

Using the expression (5.43) for $\text{div} w$ and (4.42) for $d |W|^2$, as well as (4.60a) and (4.47) again, we obtain

\[
\text{div} \tau_1^{(X)} = \frac{1}{8} \left( 3 + |W|^2 \right) \nabla^2 \Delta + \frac{3}{16} \left( 1 - |W|^2 \right) |d\Delta|^2 + \frac{1}{4} \left( 9 + |W|^2 \right) \lambda_1 e^{-\Delta} \langle d\Delta, w \rangle
+ \frac{1}{2} \left( 5 - |W|^2 \right) \lambda_2^2 e^{-2\Delta} + \frac{1}{2} \left( 4 + |W|^2 \right) \left( 1 - |W|^2 \right) \lambda_2^2 e^{-2\Delta}
+ \frac{1}{2} \left( 3 + 2 |W|^2 \right) \left( 1 + |W|^2 \right) \mu \lambda_2 e^{-5\Delta} - \frac{1}{2} \left( 1 + |W|^2 \right)^2 \mu^2 e^{-8\Delta}.
\]

Overall, combining everything, we conclude that

\[
R = -2 |W|^2 |G|^2 + 3 (3 + |W|^2) \nabla^2 \Delta + 12 \left( 2 + |W|^2 \right) |d\Delta|^2 + 12 \left( 5 + 3 |W|^2 \right) (\lambda_1^2 + \lambda_2^2) e^{-2\Delta}
- 18 \left( 1 + 2 |W|^2 \right) \mu^2 e^{-8\Delta}.
\]
Now recall that since the original choice of the $G_2$-structure $\varphi_X$ or $\varphi_Y$ was arbitrary, our expressions are invariant under the transformation \(X \rightarrow Y, W \rightarrow -W^{-1}, \lambda \rightarrow -\lambda\). In \((6.12)\), this means that it must be invariant under the transformation $|W|^2 \rightarrow |W|^{-2}$, since $[G]^2$ and other quantities are independent of the choice of the $G_2$-structure. Hence the corresponding corresponding expression for $R$ that we would have obtained from $T^{(V)}$ is

\[
R = -2 |W|^{-2} [G]^2 + 3 \left( 3 + |W|^{-2} \right) \nabla^2 \Delta + 12 \left( 2 + |W|^{-2} \right) |d\Delta|^2 + 12 \left( 5 + 3 |W|^{-2} \right) (\lambda_1^2 + \lambda_2^2) e^{-2\Delta} - 18 \left( 1 + 2 |W|^{-2} \right) \mu^2 e^{-8\Delta}.
\]

\[\text{(6.13)}\]

Subtracting \((6.12)\) from \((6.13)\), we obtain

\[
\left( |W|^{-2} - |W|^2 \right) \left( -2 |G|^2 + 3 \nabla^2 \Delta + 12 |d\Delta|^2 + 36 (\lambda_1^2 + \lambda_2^2) e^{-2\Delta} - 36 \mu^2 e^{-8\Delta} \right) = 0.
\]

\[\text{(6.14)}\]

Therefore, we see that if $|W|^2 \neq 1$, equality of \((6.12)\) and \((6.13)\) is precisely equivalent to equation \((2.11a)\), which is the 4-dimensional Einstein’s equation in supergravity. As we have argued previously, when $\mu \neq 0$, $|W|^2$ cannot be equal to 1 in a neighborhood, therefore, if \((2.11a)\) holds where $|W|^2 \neq 1$, by continuity it will have to hold where $|W|^2 = 1$. If on the other hand, $\mu = 0$, we already assume that $w_0 = 0$, so $|W|^2 = 1$ implies that the complex octonion $Z = X + iY$ is a zero divisor. However, if $\lambda \neq 0$, again we know that $Z$ cannot be a zero divisor everywhere in a neighborhood, so again, we can conclude by continuity that \((2.11a)\) holds everywhere.

Now using \((2.11a)\) to eliminate $|\lambda|^2$ from \((6.12)\), we precisely obtain \((2.12)\), which is the scalar curvature expression obtained from the 7-dimensional Einstein’s equation. Therefore, the two integrability conditions that lie in the I representation give us

\[
\frac{1}{4} R = \nabla^2 \Delta + |d\Delta|^2 + \frac{5}{6} |G|^2 + \frac{21}{2} \mu^2 e^{-8\Delta}
\]

\[\text{(6.15a)}\]

\[
\nabla^2 \Delta = 12 \mu^2 e^{-8\Delta} + \frac{2}{3} |G|^2 - 4 |d\Delta|^2 - 12 |\lambda|^2 e^{-2\Delta}.
\]

\[\text{(6.15b)}\]

We may now attempt to obtain further conditions by calculating the Ricci tensor using $T^{(X)}$ and $T^{(V)}$, and then equating. These are long computations that have been completed in Maple, but below we outline the general approach. To express Ric in terms of octonions, let us first extend the Dirac operator \((3.41)\) to $\Omega^1(\mathbb{O}M)$, so that if $P = (P_0, P) \in \Omega^1(\mathbb{O}M)$,

\[
\delta^i (D_\alpha \hat{P}_a) = \delta^i \left( D_\alpha \hat{P}_a \right).
\]

\[\text{(6.16)}\]

Similarly, define

\[
\nabla \hat{P}_a = \delta^i \left( \nabla \hat{P}_a \right) = \left( \nabla^\alpha (P_0)_a + \nabla^i P_a \delta^\alpha \right) - (\nabla^i P_a)_a + \text{curl} (P^i).
\]

\[\text{(6.17)}\]

We can rewrite \((6.16)\) as

\[
\delta^i \left( D_\alpha \hat{P}_a \right) = \delta^i \left( D_\alpha ((P_0)_a + \delta^i P_a) \right)
\]

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and hence

$$\bar{D}\bar{P} = \bar{\nabla}P + P_0 (\bar{D}1) - P_{a\bar{j}}\delta^i (\delta^j T_i) .$$  \hspace{1cm} (6.18)

Using these definitions, we have the following.

**Lemma 6.1.** Given a $G_2$-structure $\varphi$ with torsion $T$, the Ricci curvature of the corresponding metric is given by

$$\frac{1}{2} \tilde{\text{Ric}}_c = -\delta^i \left( (d_D T)_{ij} \right) = -\bar{D}T - D\bar{D}1$$  \hspace{1cm} (6.19)

where $\tilde{\text{Ric}}_c = \text{Ric}_{ab} \delta^b$ is an $\text{Im} \mathbb{O}$-valued 1-form.

**Proof.** Using (6.2), we have

$$\delta^i \left( (d_D T)_{ij} \right) = \frac{1}{4} \delta^i \left( (\pi_7 \text{Riem})_{ij} \right) = \frac{1}{4} \left( \delta^i \times (\pi_7 \text{Riem})_{ij} \right).$$

However,

$$\left( \delta^i, (\pi_7 \text{Riem})_{ij} \right) = \delta^\alpha R_{ij\gamma} \varphi^\beta_\alpha = R_{ij\gamma} \varphi^\alpha = 0,$$

$$\left( \delta^i \times (\pi_7 \text{Riem})_{ij} \right)^\alpha = \delta^\beta R_{jm\alpha} \varphi^m_\gamma \varphi^\alpha \gamma = R_{jm\alpha} (\psi^{m\alpha\gamma} + g^{m\alpha} g^{n\gamma} - g^{m\gamma} g^{n\alpha}) = -2 \text{Ric}^c_j .$$

Hence, we get the expression (6.19).

On the other hand,

$$\delta^i \left( (d_D T)_{ij} \right) = \delta^i (D_j T_i - D_i T_j) = \bar{D}T_i - \delta^i (D_j T_i) = D\bar{D}1 .$$

Recall from (3.43) that $\bar{D}1 = \begin{pmatrix} 7\tau_1 \\ -6\tau_7 \end{pmatrix}$, so only the $\bar{D}T$ term in (6.19) involves derivatives of $\tau_{14}$ and $\tau_7$. The expression (6.19) in fact contains all integrability conditions except one of the 27 components (related to $\text{Ric}^*$) and the 64 component. In fact, taking the $\mathbb{O}$-real part of (6.19) gives a 7 component, taking the projections to $\Lambda_7^2$ and $\Lambda_{14}^2$ of the $\mathbb{O}$-imaginary part gives another 7 component and the 14 component, respectively. Of course, the symmetric part of the $\mathbb{O}$-imaginary part of (6.19) gives the $1 \otimes 27$ components of the integrability conditions (scalar curvature and the traceless Ricci tensor).

Now consider a $G_2$-structure $\varphi_V$ for some nowhere-vanishing octonion section $V$. Then, (6.2) becomes
\[
\frac{1}{4} \pi_7^{(V)} \text{Riem} = d_{\mathcal{D}^{(V)}} T^{(V)}
\]
\[
= d_{\mathcal{D}^{(V)}} \left( -(DV) V^{-1} + d \ln |V| \right)
\]
\[
= - d_D^2 V^{-1} + (d \ln |V|) \wedge (DV) V^{-1} + d \ln |V| \wedge T^{(V)}
\]
\[
= - d_D^2 V^{-1}.
\]

Thus,
\[
(d^2_D V) = - \frac{1}{4} \left( \pi_7^{(V)} \text{Riem} \right) V
\]

where \(T^{(V)}\) is the torsion of \(\varphi_V\) and we have used (3.26). Also, using (6.19) for \(T^{(V)}\), we have
\[
\frac{1}{2} \widetilde{\text{Ric}} = - \delta' \circ \varphi \left( d_{\mathcal{D}^{(V)}} T^{(V)} \right) = - \delta' \circ \varphi \left( d_D \left( T^{(V)} \right) V^{-1} - (\partial_i \ln |V|) T_i^{(V)} + (\partial_j \ln |V|) T_j^{(V)} \right)
\]
\[
= - \left( \delta' \left( d_D \left( T^{(V)} \right) \right) V^{-1} + \left( \left( \partial_i \ln |V| \right) \left( T_i^{(V)} \right) V^{-1} + (\partial_j \ln |V|) \mathcal{D}^{(V)} \right) V^{-1}
\]
\[
= - \left( \delta' \left( d_D \left( T^{(V)} \right) \right) V^{-1} + \left( \left( \partial_i \ln |V| \right) \left( T_i^{(V)} \right) V^{-1} + (\partial_j \ln |V|) \mathcal{D}^{(V)} \right) V^{-1}
\]

where we have also used (3.17) and (3.31). Now moreover,
\[
\delta' \left( d_D \left( T^{(V)} \right) \right) = \delta' \left( d_T \left( T^{(V)} \right) \right) - D_j \left( T_j^{(V)} \right)
\]
\[
= \mathcal{D} \left( T_j^{(V)} \right) V^{-1} + D_j \left( \mathcal{D}^{(V)} \right) V^{-1}
\]

Hence, we find that
\[
\frac{1}{2} \widetilde{\text{Ric}} = - \left( \mathcal{D} \left( T^{(V)} \right) \right) V^{-1} - D \left( \left( \mathcal{D}^{(V)} \right) \right) V^{-1}
\]
\[
+ \left( \left( \partial_i \ln |V| \right) \left( T_i^{(V)} \right) V^{-1} + (\partial_j \ln |V|) \mathcal{D}^{(V)} \right) V^{-1} \quad \text{(6.20)}
\]

and then by comparing with (6.19), which we can also transpose since \(\text{Ric}\) is symmetric, we conclude the following.

**Lemma 6.2.** Suppose \(\varphi\) and \(\varphi_V = \sigma_V(\varphi)\) are isometric G2-structures with torsion \(T\) and \(T^{(V)}\), respectively. Then,
\[
(\mathcal{D}T) V - \mathcal{D} \left( T^{(V)} \right) V = D \left( \left( \mathcal{D}^{(V)} \right) \right) V - (D \mathcal{D} 1) V
\]
\[
= D \left( \left( \partial_i \ln |V| \right) \left( T_i^{(V)} \right) V^{-1} - (\partial_j \ln |V|) \mathcal{D}^{(V)} \right) V^{-1} \quad \text{(6.21)}
\]

and equivalently,
\( (\mathcal{D}T)^i V - \mathcal{D} (T^1 V) = D \left( \left( \mathcal{D}^V 1 \right) V \right) - (D \mathcal{D} 1)^i V \\
- \left( \left( \partial \ln |V| \right) \left( T^i V \right) \right) V^{-1} - (\partial \ln |V|) \mathcal{D}^V 1 \right) \\
\tag{6.22} \]}

We will use lemma 6.2 to our torsions \( T^{(X)} \) and \( T^{(Y)} \), however, recall from (5.8a) that our expression for \( T^{(X)} \) involves the transpose of an octonion product, so in lemma 6.3 below, we find how to work out \( \mathcal{D} \) of such expressions.

**Lemma 6.3.** Suppose \( \hat{P} \) is an \( \mathcal{O} \mathcal{M} \)-valued 1-form. Then, for any \( V \in \Gamma (\mathcal{O} \mathcal{M}) \),

\[
\mathcal{D} \left( \left( \hat{P} V \right)^i \right) = \left( \text{curl } \hat{P} \right)^i - \left( \text{curl } \hat{P} \right)^i + \delta_a \left( \text{div } \hat{P} \right) \hat{V} \right) i \\
+ \mathcal{D} \left( \hat{P}_a, V \right) + \left( \hat{P}_j, \delta_a (\nabla_i V) \right) \delta^j \delta^i - \left( \hat{P}_j, [T_i, \delta_a, V] \right) \delta^j. \\
\tag{6.23} \]

**Proof.** We have

\[
\mathcal{D} \left( \left( \hat{P} V \right)^i \right) = \delta^i \left( D_i \left( \left( \hat{P} V \right)^i \right) \right) \\
= D_i \left( \delta^i \left( \hat{P} V \right)^i \right). \\
\]

However,

\[
\delta^i \left( \hat{P} V \right)^i = \left( \hat{P}_a, V \right) \delta^i + \left( \hat{P}_j, \delta_a \hat{V} \right) \delta^j \delta^i. \\
\]

Then, using (3.28) and (3.29), we get

\[
D_i \left( \delta^i \left( \hat{P} V \right)^i \right) = \mathcal{D} \left( \hat{P}_a, V \right) + \left( \nabla_i \hat{P}_j, \delta_a \hat{V} \right) \delta^j \delta^i \\
+ \left( \hat{P}_j, \delta_a (\nabla_i V) \right) \delta^j \delta^i - \left( \hat{P}_j, [T_i, \delta_a, V] \right) \delta^j \delta^i \\
- \left( \hat{P}_j, \delta_a \hat{V} \right) \delta^j (\delta^i T_i) \\
= - \left( \left( \text{div } \hat{P} \right) V, \delta_a \right) + \left( \left( \text{curl } \hat{P} \right)^{i}_{\kappa} V, \delta_a \right) \delta^k \\
+ \mathcal{D} \left( \hat{P}_a, V \right) + \left( \hat{P}_j, \delta_a (\nabla_i V) \right) \delta^j \delta^i \\
- \left( \hat{P}_j, \delta_a \hat{V} \right) \delta^j (\delta^i T_i) - \left( \hat{P}_j, [T_i, \delta_a, V] \right) \delta^j \delta^i \\
\]

from which we obtain (6.23). \( \square \)

Now recall from (5.8a) and (5.21) that we can write \( T^{(X)} \) and \( T^{(Y)} \) as:

\[
T^{(X)} = - \left( \mathcal{Q} W \right)^i + \left( K^{(X)}_i \right)^i \tag{6.24a} \]
\[
T^{(Y)} W = \hat{Q} + K_1^{(Y)} \tag{6.24b}
\]

where
\[
\left( K_1^{(X)} \right)' = -\frac{1}{4} \delta \left( -\frac{1}{2} \lambda_2 e^{-\Delta} \partial_\Delta \right) + \left( \lambda_1 e^{-\Delta} W - \partial_\Delta \right) \hat{I} \tag{6.25a}
\]

\[
K_1^{(Y)} = -\frac{1}{4} \left( \delta \left( \partial_\Delta \right) W \right) + \lambda_1 e^{-\Delta} \delta + \frac{1}{2} \lambda_2 e^{-\Delta} \delta W + (d \ln |Y|) W \tag{6.25b}
\]

\[
\hat{Q} = \left( \frac{1}{2} (Q_0 - 2 \lambda_1 e^{-\Delta}) \delta + Q_2 \right). \tag{6.25c}
\]

Now from (6.23),
\[
\hat{D} T^{(X)} = -\hat{D} \left( \left( \hat{Q} W \right)' \right) + \hat{D} \left( K_1^{(X)} \right)' \nonumber
\]
\[
= -\left( \left( \text{curl} \hat{Q} \right)' W \right)' + \left( \left( \text{curl} \hat{Q} \right)' + \delta_\alpha \left( \text{div} \hat{Q} \right) \right) W \hat{I}
\]
\[
+ \hat{D} \left( \hat{Q}_a W \right) - \left( \hat{Q}_a, \delta_\alpha \left( \nabla W \right) \right) \delta^i \delta^j
\]
\[
+ \left( \hat{Q}_a, \delta_\alpha W \right) \delta^i \left( \delta^j T_i^{(X)} \right) + \left( \hat{Q}_a, \left[ T_i^{(X)} , \delta_\alpha W \right] \right) \delta^i \delta^j.
\]
\[
+ \hat{D} \left( \left( K_1^{(X)} \right)' \right)_a , 1 \right) + \left( \text{curl} K_1^{(X)} \right)_a
\]
\[
- \left( \left( \text{curl} K_1^{(X)} \right)' + \delta_\alpha \left( \text{div} K_1^{(X)} \right), 1 \right) \hat{I}
\]
\[
- \left( \left( K_1^{(X)} \right)' , \delta_\alpha T_i \right) \delta^i \delta^j
\]
\[
= \left( \left( \text{curl} \hat{Q}_a \right)' , \nabla_i (Q_2)_a \right) \phi_a^{ij} \varphi_m \nonumber
\]
\[
= \nabla_i Q_2 \left( w \right) \phi_a^{ij} \varphi_m - \varphi_a^{ij} \left( Q_2 \right)_m \nabla_i \varphi_m; \nonumber
\]

but we know from (4.60d) and (5.37) that \( Q_2 \left( w \right) \) and \( \nabla w \) are also given as polynomial expressions in our set of basic variables. Therefore, let us rewrite (6.26) as
\[
\hat{D} T^{(X)} = -\left( \text{curl} \hat{Q}_a \right)' W + \left( \delta \left( \text{div} \hat{Q}_a \right) \right) W \hat{I} + K_2^{(X)} \tag{6.27}
\]

where \( K_2^{(X)} \) contains no derivatives of \( Q_2 \). On the other hand, using (6.17),
\[ \mathcal{D} \left( T^{(Y)} W \right) = \mathcal{D} \hat{Q} + \mathcal{D} k_2^{(Y)} \]
\[ = \nabla \hat{Q} - \hat{Q}_1 (\mathcal{D} 1) - \left( \text{Im} \hat{Q} \right)_{\alpha j} \delta_{i} \left( \delta/|T^{(X)}| \right) \]
\[ = \text{curl} \hat{Q}_2 - \left( \text{div} \hat{Q}_2 \right) \hat{1} + K_2^{(Y)} \]

where \( k_2^{(X)} \) also contains no derivatives of \( Q_2 \). Overall, from (6.22) we have,
\[ \mathcal{D} \left( T^{(Y)} W \right) - \left( \mathcal{D} T^{(X)} \right) W = (D \mathcal{D} 1)^{T} W - D \left( (\mathcal{D} W) 1 \right) W \]
\[ + \left( \left( \partial \ln |W| \right) \left( T_{j}^{(Y)} W \right) \right) W^{-1} + (\partial \ln |W|) \mathcal{D}^{(W)} 1 - K_2^{(Y)} + K_2^{(X)} W \]

and hence,
\[ \text{curl} \hat{Q}_2 + \left( \text{curl} \hat{Q}_2 \right)^{T} |W|^2 - \left( \text{div} \hat{Q}_2 \right) \hat{1} + \left( \text{div} \hat{Q}_2 \right) W, \delta) W = K_3 \]

where
\[ K_3 = (D \mathcal{D} 1)^{T} W - D \left( (\mathcal{D} W) 1 \right) W \]
\[ + \left( \left( \partial \ln |W| \right) \left( T_{j}^{(Y)} W \right) \right) W^{-1} + (\partial \ln |W|) \mathcal{D}^{(W)} 1 - K_2^{(Y)} + K_2^{(X)} W \]

and also does not depend on derivatives of \( Q_2 \). Taking the real part of (6.30), we find an expression for \( \text{div} \hat{Q}_2 \):
\[ \text{div} \hat{Q}_2 = - \frac{1}{6} GG (w) - \frac{3}{2} \left( 3 + |W|^2 \right) Q_2 (d \Delta) - 2 \left( 2 w_0 \lambda_1 - \lambda_2 \right) e^{-\Delta} Q_1 - \frac{3}{28} d (w, d \Delta) \]
\[ - \frac{3}{56} \lambda_2 (3 |W|^2 + 65) (w, d \Delta) - 108 w_0 \mu e^{-4 \Delta} + \left( 154 + 120 w_0 \lambda_2 - 50 |W|^2 \lambda_1 \right) e^{-\Delta} \]
\[ + \frac{1}{14} w (7 |G|^2 + 12 w_0 \lambda_1 (\lambda_2 - \mu e^{-3 \Delta}) e^{-2 \Delta} + (126 \lambda_1^2 - 6 \lambda_2^2) e^{-2 \Delta} + 12 \lambda_2 \mu e^{-5 \Delta}) \].

Using (6.32), we rewrite (6.30) as
\[ |W|^2 \left( \text{curl} \hat{Q}_2 \right)^{T} + \text{curl} \hat{Q}_2 = K_4 \]

where \( K_4 \) does not contain derivatives of \( Q_2 \). Taking the transpose of (6.33), and solving for \( \text{curl} \hat{Q}_2 \) we get
\[ \left( 1 - |W|^2 \right)^{T} \left( \text{curl} \hat{Q}_2 \right) = K_4 - |W|^2 K_4 \].

Thus we see that if \( |W|^2 \neq 1 \), we obtain a solution for \( \text{curl} \hat{Q}_2 \). In general, it’s a very long expression. In the case when \( \mu \neq 0 \), and hence \( w = \hbar d \Delta \), some of the terms cancel, so it’s a bit shorter to write down. Schematically, in this case we have
\[ \text{curl} \hat{Q}_2 = w_0 \varphi \varphi Q_2 Q_2 + 2 w_0 Q_2 Q_2 - h \varphi \psi (d \Delta) Q_2 Q_2 + 2 h \varphi (d \Delta) Q_2 Q_2 \]
\[ + k_1 \varphi (d \Delta) \varphi (d \Delta) Q_2 + k_2 (Q_2 \times (d \Delta)) + k_3 (Q_2 \times (d \Delta))^T \]
\[ - \frac{3}{4} w_0 \nabla d \Delta + \frac{3}{4} h (\nabla d \Delta \times d \Delta)^T + k_4 Q_2 + k_5 d \left( |d \Delta|^2 \right) \varphi \varphi \]
\[ + k_6 (d \Delta) (d \Delta) + k_7 (d \Delta) \varphi \varphi + k_8 g \]
where \( k_1, \ldots, k_8 \) are scalar coefficients involving \( h, |W|^2, w_0, |d\Delta|^2, e^{-\Delta}, \mu, \lambda_1, \lambda_2 \).

Now using the expressions for \( \text{div} Q_2 \) and \( \text{curl} Q_2 \) in (6.19) or (6.20), we can obtain (after some manipulations) the expression for \( \text{Ric} (2.11b) \) that comes from Einstein’s equations. Moreover, it can be seen that the expression for \( \text{div} Q_2 \) and \( \text{curl} Q_2 \), together with (2.11b) and (6.15b), do in fact imply (6.19) and (6.20). Hence we do not obtain any other conditions.

Also, schematically, we can write

\[
* dG \sim \text{Skew} (\text{curl} Q_2) + \text{terms without derivatives of } Q_2 \quad (6.36a)
\]

\[
d*G \sim \iota_\psi (\text{Sym} (\text{curl} Q_2)) + (\text{div} Q_2) \wedge \varphi + \text{terms without derivatives of } Q_2. \quad (6.36b)
\]

It can be shown that with the above expressions for \( \text{curl} Q_2 \) and \( \text{div} Q_2 \), together with (6.15b), we do get \( dG = 0 \) and \( d*G \) satisfies the equation of motion (2.7). Moreover, since (6.36) also uniquely determine \( \text{curl} Q_2 \) and \( \text{div} Q_2 \), together with Einstein’s equations (2.11b) and (6.15b), they are equivalent to the integrability conditions (6.19) and (6.20).

7. Concluding remarks

The main results in this paper were to reformulate \( N = 1 \) Killing spinor equations for compactifications of 11-dimensional supergravity on \( \text{AdS}_4 \) in terms of the torsion of \( G_2 \)-structures on 7-manifolds. We obtain either a single complexified \( G_2 \)-structure that corresponds to complex spinor that solves the 7-dimensional Killing spinor equation, or correspondingly 2 real \( G_2 \)-structures. To obtain these results we used the octonion bundle structure that was originally developed in [49], which is a structure that is specific to 7 dimensions. In the process, we also found that on a compact 7-manifold, the defining vector field \( w \) that relates the two \( \mathbb{C} \)-real octonions (or equivalently, spinors) is necessarily vanishing at some point, and in fact, if \( \mu \neq 0 \), then the 1-form that corresponds to \( w \) is exact, and we can write \( w = h (\Delta) d\Delta \) for some function of \( \Delta \) (where \( \Delta \) is a such that \( e^{2\Delta} \) is the warp factor). This shows that the \( SU(3) \)-structure defined by \( w \) is not globally defined, and is hence only local. This is in sharp contrast with the \( N = 2 \) case studied in [18], where there is always a nowhere-vanishing vector field on the 7-manifold. In the \( \mu \neq 0 \) case, it is important to understand what the function \( h \) can be, since the torsion of both \( G_2 \)-structures that correspond to solutions of the Killing spinor equation is expressible in terms of \( h \) and the derivatives of \( \Delta \). Equations (5.48) gives a system of ODEs that \( h \) and \( w_0 \) must satisfy whenever \( d\Delta \neq 0 \). This system also depends on \( |G|^2 \). Alternatively, using (6.15a) and (6.15b) together, we could instead rewrite it using the scalar curvature \( R \). It would be interesting however to see if these equations could be solved explicitly in some particular cases.

In order to obtain the torsion of the complexified \( G_2 \)-structure we expressed the Killing spinor equation solutions as a complexified octonion section \( Z \). This had very different properties depending on whether it could be a zero divisor or not. It is interesting to understand the physical significance of the submanifold where \( Z \) is a zero divisor. In the \( N = 2 \) supersymmetry case, we would need to have 4 \( \mathbb{C} \)-real octonion sections, which could be combined together to give a quaternionic octonion, i.e. objects that pointwise lie in \( \mathbb{H} \otimes \mathbb{O} \). This should then define some kind of a quaternionic \( G_2 \)-structure. However, even from a pure algebraic point of view it would be challenging to understand this, because this is not as simple as just changing the underlying field from \( \mathbb{R} \) to \( \mathbb{C} \), as is the case for complexified \( G_2 \)-structures. However such an approach could give an alternative point of view to the results in [18].
Appendix. Proofs of identities

Lemma A.1. Suppose $Z = X + iY$ satisfies the equations (4.5), then
\[ |X|^2 + |Y|^2 = k_1 e^{\Delta} \] \hspace{1cm} (A.1)
for an arbitrary real constant $k$.

Proof. Let us consider the derivative of $\langle Z, Z^* \rangle$:
\[
\nabla \langle Z, Z^* \rangle = \langle DZ, Z^* \rangle + \langle Z, DZ^* \rangle
\]
\[=
\left\langle \frac{1}{2} i \mu e^{-4\Delta} \delta Z + i \delta G^{(4)}(Z) + 12iG^{(3)}(Z), Z^* \right\rangle
\]
\[+ \left\langle Z, \frac{1}{2} i e^{-4\Delta} \delta Z^* - i \delta G^{(4)}(Z^*) - 12iG^{(3)}(Z^*) \right\rangle. \] \hspace{1cm} (A.2)

However from the properties of $G^{(3)}$ and $G^{(4)}$ from lemma 4.1, we get
\[
\left\langle Z, i \delta G^{(4)}(Z^*) \right\rangle = -i \left\langle \delta G^{(4)}(Z), Z^* \right\rangle - 8i \left\langle G^{(3)}(Z), Z^* \right\rangle
\]
\[\left\langle Z, 12iG^{(3)}(Z^*) \right\rangle = \left\langle 12iG^{(3)}(Z), Z^* \right\rangle.\]

Also,
\[
\langle \delta Z, Z^* \rangle + \langle Z, \delta Z^* \rangle = \langle \delta, Z^*Z + ZZ^* \rangle
\]
\[= \langle \delta, 2 \text{Re}_\mathbb{C}(Z^*Z) \rangle = 0
\]
where we have used the property $\text{Re}_\mathbb{C}(Z^*Z) = \text{Re}_\mathbb{C}(Z^*Z)$ from (3.51). Thus, (A.2) becomes
\[
\nabla \langle Z, Z^* \rangle = 2i \langle \delta G^{(4)}(Z), Z^* \rangle + 8i \langle G^{(3)}(Z), Z^* \rangle
\]
\[= -2 \langle \delta (MZ - i\lambda Z^*), Z^* \rangle + 8i \langle G^{(3)}(Z), Z^* \rangle
\]
\[= 2 \langle M, \delta Z^* \rangle Z + 8i \langle G^{(3)}(Z), Z^* \rangle. \] \hspace{1cm} (A.3)

where we used the property $\langle \delta Z^*, Z^* \rangle = \langle \delta, Z^*Z^* \rangle = 0$. Decomposing (4.5a) into $\mathbb{C}$-real and $\mathbb{C}$-imaginary parts gives us
\[
G^{(4)}(X) = -MY + \lambda_1 e^{-\Delta} X + \lambda_2 e^{-\Delta} Y \] \hspace{1cm} (A.4a)
\[
G^{(4)}(Y) = MX + \lambda_2 e^{-\Delta} X - \lambda_1 e^{-\Delta} Y. \] \hspace{1cm} (A.4b)

Using this decomposition, we get
\[
8 \langle G^{(3)}(Z), Z^* \rangle = 8 \langle G^{(3)}(X), X \rangle + 8 \langle G^{(3)}(Y), Y \rangle
\]
\[= -2 \langle \delta G^{(4)}(X), X \rangle - 2 \langle \delta G^{(4)}(Y), Y \rangle
\]
\[= 2 \langle \delta (MY - \lambda_1 X - \lambda_2 Y), X \rangle - 2 \langle \delta (MX + \lambda_2 X - \lambda_1 Y), Y \rangle. \] \hspace{1cm} (A.5)
\[ 2 \langle \delta (MY), X \rangle - 2 \langle \delta (MX), Y \rangle - 2 \lambda_2 \langle \delta Y, X \rangle - 2 \lambda_2 \langle \delta X, Y \rangle = 0 \]  \tag{A.6} \\
\[ -2 \langle M, (\delta X) \rangle Y + 2 \langle M, (\delta Y) \rangle X. \]  \tag{A.7} \\
\]
Also note that
\[
(\delta Z^*) Z = (\delta (X - iY)) (X + iY) \\
= \delta \left( |X|^2 + |Y|^2 \right) + i \left( ((\delta X) Y) - (\delta Y) X \right).  \tag{A.8} \\
\]
Thus, substituting (A.5) and (A.8) into (A.3), we may conclude that
\[
\nabla \langle Z, Z^* \rangle = 2 \langle M, \delta \rangle \left( |X|^2 + |Y|^2 \right) = \partial \Delta \left( |X|^2 + |Y|^2 \right)  \tag{A.9} \\
\]
and hence, (A.1).

**Lemma A.2.** Let \( A = a_0 + \hat{\alpha} \) be a (bi)octonion section and let \( G \) be a 4-form with a decomposition with respect to a \( G_2 \)-structure \( \varphi \) given by (4.6). Then, \( G^{(4)}(A) \) satisfies the following relation
\[
G^{(4)}(A) = -\left( \frac{7}{3} a_0 Q_0 + \frac{1}{3} \langle Q_1, \alpha \rangle \right) - \left( \frac{2}{3} a_0 Q_1 - \frac{1}{6} Q_0 \alpha - \frac{1}{3} Q_2 (\alpha) \right) \tag{A.10} \\
\]

**Proof.** We have
\[
G^{(4)}(A) = a_0 G^{(4)}(1) + \alpha' G^{(4)}(\delta) \\
= a_0 G^{(4)}(1) + \alpha' \left( \delta G^{(4)}(1) + 8 G^{(3)}(1) \right) \tag{A.11} \\
= AG^{(4)}(1) + 8 \alpha' G^{(3)}(1) \\
\]
where we have used properties of \( G^{(4)} \) from lemma 4.1. Now, using lemma 4.2 gives us (A.10).

**Lemma A.3.** Let \( A = a_0 + \hat{\alpha} \) be a (bi)octonion section, then the quantities \( G^{(4)}(A) \) and \( G^{(3)}(A) \) satisfy the following relation
\[
\delta \left( G^{(4)}(A) \right) + 12 G^{(3)}(A) = F(A) + \left( \hat{Q}_2 \hat{A} \right)^T  \tag{A.12} \\
\]
where
\[
F(A) = \left( \frac{1}{3} \langle Q_1, \alpha \rangle + \frac{1}{2} a_0 Q_1 \right) \delta + \left( \frac{1}{2} a_0 \alpha - \frac{1}{3} a_0 Q_1 + \frac{1}{3} Q_2 (\alpha) \right) \cdot \varphi + \alpha Q_1 \right) \tag{A.13} \\
\]

**Proof.** The expression (A.10) gives us a relation that expresses \( G^{(4)}(A) \) in terms of \( A \) and components of \( G \) with respect to \( \varphi \). Similarly, let us re-express \( G^{(3)}(A) \). Using (3.32), we can show that
\[ G_a^{(3)}(\delta_b) = -\delta_b G_a^{(3)} - 6G_{ab}^{(2)} \]  
\[ G_a^{(3)}(A) = a_0 G_a^{(3)} + \alpha^G a^{(3)}(\delta_i) \]
\[ = a_0 G_a^{(3)} - \alpha G_a^{(3)} + 6 \left( \alpha G^{(2)} \right)_a \]
\[ = \tilde{A} G_a^{(3)} + 6 \left( \alpha G^{(2)} \right)_a. \]  
(A.15)

Note however, that if we compare \( G^{(3)} \) and \( \delta G^{(4)} \) in (4.8), we find
\[ G_a^{(3)} = -\frac{1}{8} \left( \delta_a G^{(4)} + \frac{2}{3} (Q_1)_a - \frac{1}{6} Q_0 \delta_a - \frac{2}{3} \left( \tilde{Q}_2 \right)_a \right) + 6 \left( \alpha G^{(2)} \right)_a. \]  
(A.16)

It’s not difficult to work out \( (\alpha G^{(2)}) \) explicitly. In fact, note that \( G^{(2)} \) is \( \mathcal{O} \)-imaginary, since \( \delta^i \delta^d \) has a symmetric \( \mathcal{O} \)-real part. So,
\[ (\alpha G^{(2)}) = \frac{1}{144} \alpha^b G_{bad} \phi^{bd}. \]

Just plugging in the expression (4.6) for \( G \), we find
\[ \alpha G^{(2)} = \frac{1}{72} (2Q_0 \alpha - Q_1 \times \alpha + Q_2 (\alpha)) \phi \]
\[ + \frac{1}{36} (Q_1, \alpha) \delta + \frac{1}{72} (Q_2 \times (Q_2 + Q_1 \alpha)) + \alpha Q_1 + \tilde{Q}_2 \alpha \]
\[ = \frac{1}{36} (Q_1, \alpha) \delta + \frac{1}{72} (2Q_0 \alpha + Q_2 (\alpha)) \phi - \frac{1}{72} \psi (Q_1, \alpha) \]
\[ - \frac{1}{72} (Q_2 \times (Q_2 + Q_1 \alpha)) - \frac{1}{36} Q_1 \alpha. \]

Now, overall,
\[ \delta G^{(4)}(A) + 12G_a^{(3)}(A) = \delta_i \left( G^{(4)}(A) \right) - \frac{3}{2} \delta^i \left( \delta_a G^{(4)} \right) - (Q_1)_a \tilde{A} + \frac{1}{4} Q_0 \left( \tilde{A} \delta_a \right) + 6 \left( \tilde{Q}_2 \right)_a \]
\[ + (Q_2, \alpha) \delta_\alpha + (2Q_0 \alpha + Q_2 (\alpha)) \phi - \psi (Q_1, \alpha) - 2 (Q_1)_a \alpha \]
\[ - (Q_2 \times (Q_2 + Q_1 \alpha))_a. \]

Note however that
\[ \tilde{A} \tilde{Q}_2 = \left( \frac{Q_2 (\alpha)}{a_0 Q_2 + Q_2 \times \alpha} \right) \]
and similarly,
\[ \hat{Q}_2 \bar{A} = \left( \frac{Q_2}{a_0 Q_2 - Q_2 \times \alpha} \right). \]

Hence,
\[
\left( \hat{Q}_2 \bar{A} \right)' = \left( \frac{Q_2}{a_0 Q_2 - (Q_2 \times \alpha)'} \right) = \bar{A} \hat{Q}_2 - \left( \frac{0}{Q_2 \times \alpha + (Q_2 \times \alpha)'} \right).
\]

Therefore, we can write
\[
\delta_a G^{(4)}(A) + 12 G^{(3)}_a(A) = \delta_a \left( G^{(4)}(A) \right) - \frac{3}{2} \bar{A} \left( \delta_a G^{(4)} \right) - \left( \Omega_1 \right)_a \hat{A} + \frac{1}{4} Q_0 (\hat{A} \delta_a)
\]
\[+ 2 \left( Q_1, \alpha \right) \delta_a + 2 Q_0 (Q_1 \alpha) \iota_{\varphi_a - \psi (Q_1, \alpha)} - 2 (Q_1, \alpha).
\]

Now using the expression (A.10) for \( G^{(4)}(A) \), we find,
\[
\delta_a \left( G^{(4)}(A) \right) = \left( - \left( 2 a_0 Q_1 - \frac{3}{2} Q_0 \alpha - \frac{3}{2} Q_2 \right) \delta_a + \left( \hat{Q}_2 \bar{A} \right)' \right).
\]

Using (4.8a) for \( G^{(4)}(1) \) we also get
\[
\delta \left( G^{(4)}(1) \right) = \left( - \frac{1}{2} Q_0 \delta + Q_1 \iota_{\varphi} \right)
\]
and hence,
\[
\frac{3}{2} \bar{A} \left( \delta G^{(4)}(1) \right) = \left( \frac{a_0}{- \alpha} \right) \left( - \frac{1}{2} Q_0 \delta + Q_1 \iota_{\varphi} \right)
\]
\[= \left( - \frac{7}{2} a_0 Q_0 + (Q_1, \alpha) \right) \delta + (a_0 Q_1 - \frac{3}{2} Q_0 \alpha - \frac{3}{2} Q_2) \iota_{\varphi - \psi (Q_1, \alpha) - \alpha Q_1 - Q_1 \alpha}. \]

Thus, combining everything, we do indeed obtain (A.13).

\[ \square \]

**Lemma A.4.** Suppose \( \hat{P} \) is a symmetric Im \( \mathcal{O} \)-valued 1-form and \( W = w_0 + \hat{w} \in \Gamma(\mathcal{O} M) \).

Then,
\[
\text{Skew} \left( \text{Im} \left( W \left( \hat{P} \bar{W} \right)' \right) \right) = \frac{1}{2} P(w) \wedge w \quad \text{(A.17)}
\]

**Proof.** Using the definition (3.44) of the transpose for sections of \( \Omega^1(\mathcal{O} M) \), we have
\[
\left(\hat{P}W\right)^t_a = P\left(\overline{w}\right)_a + \left\langle \hat{P} \overline{W}, \delta_a \right\rangle \delta^b = P\left(\overline{w}\right)_a + P^{bc} \left\langle \delta_b \overline{W}, \delta_a \right\rangle \delta_b.
\]

Hence,
\[
W \left(\hat{P}W\right)^t = P\left(\overline{w}\right)_a W + P^{bc} \left\langle \delta_b \overline{W}, \delta_a \right\rangle W \delta_b.
\]

So,
\[
\text{Im} \left( W \left(\hat{P}W\right)^t \right)_{ad} = P\left(\overline{w}\right)_a \langle \delta_d \overline{W}, \delta_a \rangle = P\left(\overline{w}\right)_a w_d + P^{bc} \left\langle \delta_c \overline{W}, \delta_a \right\rangle \langle \delta_b \overline{W}, \delta_d \rangle.
\]

Thus, skew-symmetrizing, we get
\[
\text{Im} \left( W \left(\hat{P}W\right)^t \right)_{[ad]} = P\left(\overline{w}\right)_{[a} w_{d]}.
\]

**Lemma A.5.** Suppose \( \hat{P} \) is a symmetric \( \text{Im} \mathbb{O} \)-valued 1-form and \( W \in \Gamma (\mathbb{O}M) \). Then, for any \( \text{Im} \mathbb{O} \)-valued 1-form \( K \), the equation

\[
\hat{P}W - |W|^2 \left(\hat{P}W\right)^t - 2w_0 \hat{P} = K
\]

(A.18)

has a unique solution if and only if \( W^2 \neq -1 \). In that case, the solution is

\[
\hat{P} = -\frac{\left( K + |W|^2 K^T \right) \left( \overline{W} + |W|^2 W \right)}{\left( 1 + |W|^2 \right) |W|^2 \left( 1 + |W|^2 \right)^2}
\]

(A.19)

**Proof.** Take the transpose of (A.18), then we get

\[
- |W|^2 \left( \hat{P}W \right)^T + \left( \hat{P}W \right)^T - 2w_0 \hat{P} = K^T.
\]

(A.20)

Then, multiplying (A.20) by \( |W|^2 \) and adding it to (A.18), we eliminate \( \left( \hat{P}W \right)^T \), and obtain

\[
\left( 1 - |W|^2 \right) \left( \hat{P}W \right) - 2w_0 \left( 1 + |W|^2 \right) \hat{P} = K + |W|^2 K^T
\]

and thus,

\[
\hat{P} \left( \left( 1 - |W|^2 \right) W - 2w_0 \right) = \frac{K + |W|^2 K^T}{1 + |W|^2}
\]

However

\[
\left( 1 - |W|^2 \right) \overline{W} - 2w_0 = - \left( W + |W|^2 \overline{W} \right)
\]
and

\[ W + |W|^2 \overline{W} = W \left( 1 + |W|^2 \overline{W} W^{-1} \right) = W \left( 1 + \overline{W}^2 \right). \]

Hence,

\[ \left| W + |W|^2 \overline{W} \right|^2 = |W|^2 \left| 1 + W^2 \right|^2. \]

Therefore indeed, there is a unique solution if and only if \( W^2 \neq -1 \), and that solution is given by (A.19).

\[ \blacksquare \]

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