Probing the Scale of New Physics in the $ZZ\gamma$ Coupling at $e^+e^-$ Colliders

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Abstract:
The $ZZ\gamma$ triple neutral gauge couplings are absent in the Standard Model (SM) at the tree level. They receive no contributions from dimension-6 effective operators, but can arise from effective operators of dimension-8. We study the scale of new physics associated with such dimension-8 operators that can be probed by measuring the reaction $e^+e^- \rightarrow Z\gamma$, followed by $Z \rightarrow \ell\bar{\ell}, \nu\bar{\nu}$ decays, at future $e^+e^-$ colliders including the ILC, CEPC, FCC-ee and CLIC. We demonstrate how angular distributions of the final-state mono-photon and leptons can play a key role in suppressing SM backgrounds. We further show that using electron/positron beam polarizations can significantly improve the signal sensitivities. We find that the dimension-8 new physics scale can be probed up to the multi-TeV region at such lepton colliders.

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1. Introduction

At the time of writing, there is no confirmed evidence for phenomena in accelerator experiments that require new physics beyond the Standard Model (SM) [1], pending clarifications of the apparent discrepancy between the SM prediction and the experimental value of the anomalous magnetic moment of the muon, and of the apparent anomalies in $b$-hadron decays into strange and charmed particles. It is therefore plausible to assume that the SM particles have the same dimension-4 interactions as in the SM, and seek to characterize possible deviations from SM predictions in terms of higher-dimensional effective operators constructed out of SM fields, whose contributions are suppressed by some power of an underlying new physics scale $\Lambda \gg 100$ GeV [2].

This Standard Model Effective Field Theory (SMEFT) approach has mainly been applied with the assumption that only dimension-6 SMEFT operators [3] contribute
to the experimental observables under study [4]. With this restriction, global SMEFT analyses [5] have been made of the available data from the LHC and other accelerators, and the sensitivities of experiments at possible future accelerators to the scales of new physics in dimension-6 operators have also been estimated [5–7]. However, there are some instances in which dimension-6 contributions are absent, and the first SMEFT operators to which experimental measurements are sensitive are those of higher dimensions [8]. Examples where dimension-8 operators dominate include light-by-light scattering [9], $\gamma\gamma \rightarrow \gamma\gamma$, which has recently been measured for the first time in heavy-ion collisions at the LHC [10], and $gg \rightarrow \gamma\gamma$ scattering [11], which is constrained by ATLAS measurements of events with isolated diphotons in $pp$ collisions at the LHC [12]. We note also that the effect of dimension-8 operators on Higgs observables was discussed in [13].

Another promising way to probe directly dimension-8 operators is via the $ZZ\gamma$ and $Z\gamma\gamma$ triple neutral gauge couplings (nTGCs) [14, 15]. These couplings are absent in the SM and receive no dimension-6 contributions [2, 3]. Within the SMEFT approach, the first contributions arise from effective operators of dimension 8. These operators involve the Higgs doublet and hence their origin is tied to the spontaneous symmetry breaking. Therefore, probing these neutral TGCs opens up an important window on the physics of the Higgs boson and electroweak symmetry breaking. We study here how these dimension-8 operators can be probed via the reaction $e^+e^- \rightarrow Z\gamma$ (with $Z \rightarrow \ell^+\ell^-, \nu\bar{\nu}$ decays) at future $e^+e^-$ colliders including the ILC [16], CEPC [17], FCC-ee [18] and CLIC [19], offering one of the rare direct windows on the new physics at dimension-8. (The test of nTGCs at the FCC-hh via future $pp(100\text{TeV})$ collisions was also considered recently [20].)

Our analysis framework is described in Section 2. We first discuss in Section 2.1 how the neutral triple-gauge couplings $ZV\gamma$ ($V = Z, \gamma$) can be generated by effective dimension-8 operators, and then present cross sections for $e^+e^- \rightarrow Z\gamma$ production in the different $Z$ polarization states $Z_{T,L}$ in Section 2.2. Since the SM produces $Z_{T,\gamma}$ final states copiously, with the vector bosons emerging preferentially in the forward and backward directions, we can make use of angular distributions in the $e^+e^-$ centre-of-mass frame and $Z$ decay frame to separate the SM contribution to $Z\gamma$ final states and distinguish $Z_{L}$ from $Z_{T}$ via their decays into dileptons $\ell^+\ell^-$. We study angular observables in Section 3, where the angular distributions are presented in Section 3.1 and their uses for isolating and analyzing new physics contributions are discussed in Section 3.2, with the focus on $O(\Lambda^{-4})$ contributions in Section 3.2.1 and on $O(\Lambda^{-8})$ contributions in Section 3.2.2. A systematical analysis of the sensitivities to $\Lambda$ by measurements at different $e^+e^-$ collider energies $\sqrt{s}$ from 250 GeV to 5 TeV is presented in Section 3.2.3. We present a refined analysis in Section 3.3 by including additional non-resonant SM backgrounds. In Section 4, we analyze the probe of new physics scale via the invisible decay channel $Z \rightarrow \nu\bar{\nu}$, which we then combine with the sensitivity of the dilepton channels $Z\rightarrow \ell^+\ell^-$. Furthermore, we study the improved sensitivity in
Section 5 obtainable by using the $e^+\bar{e}$ beam polarizations. Finally, we summarize our conclusions in Section 6. The $5\sigma$ sensitivity to $\Lambda$ may reach into the multi-TeV range, depending on the $e^+e^-$ collision energy, even after taking into account the fact that in many new physics scenarios the SMEFT approach may be valid only when $\Lambda \gtrsim \sqrt{s}$ or $\Lambda \gtrsim \sqrt{s}/2$. Thus, the reaction $e^+e^-\rightarrow Z\gamma$ may provide a unique and interesting probe of new physics in $e^+e^-$ collisions.

2. Neutral Triple-Gauge Couplings and $e^+e^-\rightarrow Z\gamma$ Production

In this Section, we first discuss the neutral triple-gauge couplings $ZV\gamma$ ($V = Z, \gamma$), and the corresponding dimension-8 effective operators as their unique lowest-order gauge-invariant formulations in the SMEFT. We then analyze the scattering amplitudes for $e^+e^-\rightarrow Z\gamma$, considering separately the transverse and longitudinal polarizations of the final-state $Z$ bosons.

2.1. $ZV\gamma$ Coupling from Dimension-8 Operator

The neutral triple gauge couplings (nTGCs) $ZV\gamma$ ($V = Z, \gamma$) vanish at tree level in the SM and do not receive contributions from any dimension-6 effective operators. However, at the dimension-8 level there are four CP-conserving effective operators that can contribute to the nTGCs [15],

$$\Delta L(\text{dim-8}) = \sum_{j=1}^{4} \frac{c_j}{\Lambda_j^4} \mathcal{O}_j = \sum_{j=1}^{4} \frac{\text{sign}(c_j)}{\Lambda_j^4} \mathcal{O}_j,$$

(2.1)

where the dimensionless coefficients $c_j$ may be $\mathcal{O}(1)$, with signs $\text{sign}(c_j) = \pm$, and the corresponding cutoff scales are $\Lambda_j \equiv \Lambda/|c_j|^{1/4}$. The four dimension-8 CP-even effective operators $\mathcal{O}_j$ contributing to the nTGCs may be written as

$$\mathcal{O}_{\tilde{B}W} = i H^\dagger \tilde{B}_{\mu\nu} W^{\mu\nu} \{ D_\rho, D^\rho \} H + \text{h.c.},$$

(2.2a)

$$\mathcal{O}_{BW} = i H^\dagger B_{\mu\nu} \tilde{W}^{\mu\nu} \{ D_\rho, D^\rho \} H + \text{h.c.},$$

(2.2b)

$$\mathcal{O}_{\tilde{W}W} = i H^\dagger \tilde{W}_{\mu\nu} W^{\mu\nu} \{ D_\rho, D^\rho \} H + \text{h.c.},$$

(2.2c)

$$\mathcal{O}_{\tilde{B}B} = i H^\dagger \tilde{B}_{\mu\nu} B^{\mu\nu} \{ D_\rho, D^\rho \} H + \text{h.c.},$$

(2.2d)

where $H$ denotes the SM Higgs doublet. The above operators are Hermitian and we take their coefficients $c_j$ to be real for the present study, as we assume CP conservation.

We define the dual field strengths $\tilde{B}_{\mu\nu} \equiv \epsilon_{\mu\nu\alpha\beta} B^{\alpha\beta}$ and $\tilde{W}_{\mu\nu} \equiv \epsilon_{\mu\nu\alpha\beta} W^{\alpha\beta}$, and $W_{\mu\nu} \equiv W_{\mu\nu}^{\alpha} \epsilon^{\alpha} / 2$ corresponds to the third component of the weak gauge group SU(2)$_W$. Among the above operators, one can use the equations of motion (EOM) and integration by parts to show that $\mathcal{O}_{BW}$ is equivalent to $\mathcal{O}_{\tilde{B}W}$ up to operators with more currents, or more field-strength tensors, or with quartic gauge boson couplings. Moreover, the operators $\mathcal{O}_{\tilde{W}W}$ and $\mathcal{O}_{\tilde{B}B}$ do not contribute to $ZV\gamma$ coupling for on-shell $Z$ and $\gamma$. Thus,
there is only one independent CP-conserving dimension-8 operator to be considered in our nTGC study. We choose $\mathcal{O}_{BW}$ for our analysis, and denote the corresponding cutoff scale $\Lambda_{\bar{B}W} = \Lambda$, for simplicity.

We note that all the dimension-8 operators in Eq.(2.2) involve Higgs doublets and the induced nTGCs vanish as the Higgs vacuum expectation value $\langle H \rangle \to 0$, so they originate from the spontaneous electroweak symmetry breaking (EWSB). Hence, probing the nTGCs given by dimension-8 operators (2.2) is in fact a probe of new physics associated with the spontaneous EWSB. One could write down dimension-8 operators with three gauge-field-strength tensors and two covariant derivatives (but without any Higgs doublet) that contribute to the nTGC. For instance, the following pure gauge operator can contribute to nTGC:

$$g\mathcal{O}_{\bar{B}WW} = \bar{B}_{\mu\nu} W^{a\mu\nu} (D_\rho D_\lambda W^{a\rho\lambda} + D_\nu D^\lambda W^{a\rho\lambda}) .$$ (2.3)

However, the equations of motion (EOM) can be used to convert such operators into operators with two Higgs doublets [cf. Eq.(2.2)] plus extra operators involving the gauge current of left-handed fermions [15]. In this connection, we note that the EOM of the gauge field $W^a_{\mu}$ is given by

$$D^\nu W^a_{\mu\nu} = i g [H^\dagger T^a D_\mu H - (D_\mu H)^\dagger T^a H] + g \bar{\psi}_L T^a \gamma_\mu \psi_L ,$$ (2.4)

where $T^a = \tau^a/2$ and $\psi_L$ denotes the left-handed weak doublet fermions (leptons or quarks). The summation over the fermion flavor indices is implied in the last term of Eq.(2.4). Thus, for the pure gauge operator (2.3), we can make use of the EOM (2.4) and re-express the new dimension-8 operator (2.3) as follows:

$$\mathcal{O}_{\bar{B}WW} = \mathcal{O}_{BW} + \bar{B}_{\mu\nu} W^{a\mu\nu} [D_\rho (\bar{\psi}_L T^a \gamma_\rho \psi_L) + D_\nu (\bar{\psi}_L T^a \gamma_\nu \psi_L)] ,$$ (2.5)

where $\mathcal{O}_{BW}$ on the right-hand-side (RHS) is just the original dimension-8 operator (2.2a).

We have explicitly verified that for the process $e^- e^+ \to Z \gamma$ with on-shell final states, the contributions from the above dimension-8 pure gauge operator $\mathcal{O}_{\bar{B}WW}$ still vanishes in the limit $\langle H \rangle \to 0$, because the extra fermionic contact contribution is proportional to $M_Z^2 \propto \langle H \rangle^2$. Hence, the crucial point is that the nTGC, as they are absent in the SM and at the level of dimension-6 operators, can only originate from the new physics generating the dimension-8 operators, whose contributions vanish in the limit $\langle H \rangle \to 0$, and thus are tied to the spontaneous EWSB. This explains why testing the nTGC can provide an important window for probing the new physics associated with the spontaneous EWSB.

The dimension-8 operator $\mathcal{O}_{BW}$ yields the following effective $Z \gamma Z^*$ coupling in momentum space:

$$i \Gamma^{\mu\nu}_{Z\gamma Z^*}(q_1, q_2, q_3) = \text{sign}(c_j) \frac{v M_Z (q_3^2 - M_Z^2)}{\Lambda^4} \varepsilon^{\mu\nu\alpha\beta} q_{2\beta} ,$$ (2.6)
where \( v/\sqrt{2} = \langle H \rangle \) is the Higgs vacuum expectation value. However, \( \mathcal{O}_{BW} \) does not contribute to the \( Z \gamma \gamma^* \) coupling for on-shell gauge bosons \( Z \) and \( \gamma \). Moreover, there is no \( \gamma \gamma \gamma^* \) triple photon coupling with two photons on-shell. This fact is consistent with our observation that the nTGC is tied to the Higgs VEV and spontaneous EWSB.

### 2.2. \( Z\gamma \) Production at \( e^+e^- \) Colliders

The SM contributes to the production process \( e^-(p_1)e^+(p_2) \to Z(q_1)\gamma(q_2) \), via \( t- \) and \( u- \) channel exchange diagrams at tree level. In general, the final-state \( Z \) boson may have either longitudinal or transverse polarizations.

Working in the centre-of-mass (c.m.) frame of the \( e^+e^- \) collider and neglecting the electron mass, we denote the momenta of the initial- and final-state particles as follows:

\[
\begin{align*}
p_1 &= E_1(1,0,0,1), \\
p_2 &= E_1(1,0,0,-1), \\
q_1 &= (E_Z, q \sin \theta, 0, q \cos \theta), \\
q_2 &= q(1, -\sin \theta, 0, -\cos \theta),
\end{align*}
\]

(2.7a)

(2.7b)

where the electron (positron) energy \( E_1 = \frac{1}{2} \sqrt{s} \), the momentum \( q = \frac{1}{2 \sqrt{s}} (s - M_Z^2) \), and the \( Z \) boson energy \( E_Z = \sqrt{q^2 + M_Z^2} \). The squared scattering amplitudes for the SM contributions to final states with the different \( Z \) polarizations take the following forms:

\[
\begin{align*}
|T_{sm}|^2[Z_L\gamma_T] &= e^4 \left( 8 s_W^4 - 4 s_W^2 + 1 \right) \frac{M_Z^2 s}{c_W^2 s_W^2 (s - M_Z^2)^2}, \\
|T_{sm}|^2[Z_T\gamma_T] &= e^4 \left( 8 s_W^4 - 4 s_W^2 + 1 \right) \frac{(1 + \cos^2 \theta) (s^2 + M_Z^2)}{2 s_W^2 c_W^2 \sin^2 \theta (s - M_Z^2)^2},
\end{align*}
\]

(2.8a)

(2.8b)

where we have averaged over the initial-state spins, and used the notations \((s_W, c_W) = (\sin \theta_W, \cos \theta_W)\) with \( \theta_W \) being the weak mixing angle. We have verified that the above formulae agree with the previous results in the literature [15].

We see from the above equations that the squared amplitude for a final-state longitudinal weak boson \( Z_L \) is suppressed by \( 1/s \) in the high-energy region \( s \gg M_Z^2 \). This behaviour can be understood via the equivalence theorem [22], which connects the longitudinal scattering amplitude to the corresponding Goldstone boson amplitude at high energies,

\[
T[Z_L\gamma_T] = T[\pi^0\gamma_T] + O(M_Z/\sqrt{s}),
\]

(2.9)

where \( \pi^0 \) is the would-be Goldstone boson absorbed by the longitudinally-polarized \( Z \) via the Higgs mechanism of the SM. Since the SM does not contain any tree-level \( ZV\gamma \) and \( \pi^0V\gamma \) (\( V = Z, \gamma \)) triple couplings, at tree level the production processes \( e^+e^- \to Z_L\gamma_T \) and \( e^+e^- \to \pi^0\gamma_T \) must proceed through the \( t- \) channel electron-exchange process. Since the electron Yukawa coupling \( y_e = \sqrt{2} m_e/v = O(10^{-6}) \) is very small and can be neglected for practical purposes, we have for the SM contributions

\[
T_{sm}[\pi^0\gamma_T] \approx 0, \quad |T_{sm}[Z_L\gamma_T]|^2 = O(M_Z^2/s).
\]

(2.10)
This explains the high-energy behavior of Eq.(2.8a).

We note also that, for the final state with a transverse weak boson \(Z_T\), Eq.(2.8b) exhibits a collinear divergence at \(\theta = 0, \pi\) due to our neglect of the electron mass \(m_e \approx 0\). In the following analysis we implement a lower cut on the transverse momentum of the final state photon: \(P_T^0 = q \sin \theta > P_T^0\) to remove the collinear divergence, corresponding to a lower cut on the scattering angle \(\theta > \delta = \arcsin(P_T^0/q)\). For \(\theta \neq 0, \pi\), Eq.(2.8b) gives the asymptotic behavior, \(\mathcal{T}_{\text{sm}}[Z_T \gamma_T] = O(s^0)\), in the high-energy regime \(s \gg M_Z^2\), as expected. This completes the explanation why production of the transversely polarized final state \(Z_T \gamma_T\) dominates over that of the longitudinal final state \(Z_L \gamma_T\).

The contributions of the dimension-8 operator include \(O(\Lambda^{-4})\) and \(O(\Lambda^{-8})\) terms. The terms of \(O(\Lambda^{-4})\) arises from the interference between the dimension-8 operator contribution and the SM contribution, and takes the forms

\[
2 \text{Re} \left( \mathcal{T}_{\text{sm}}^{(8)} \mathcal{T}_{\text{sm}}^{(8)} \right) [Z_L \gamma_T] = \pm \frac{e^2 (1 - 4s_W^2)}{2s_W c_W} \frac{M_Z^2 s}{\Lambda^4}, \tag{2.11a}
\]

\[
2 \text{Re} \left( \mathcal{T}_{\text{sm}}^{(8)} \mathcal{T}_{\text{sm}}^{(8)} \right) [Z_T \gamma_T] = \pm \frac{e^2 (1 - 4s_W^2)}{2s_W c_W} \frac{M_Z^2 s}{\Lambda^4}, \tag{2.11b}
\]

which are consistent with results in the literature [15]. [Here the \(\pm\) signs of the \(O(\Lambda^{-4})\) term correspond to the two possible signs of a given dimension-8 operator, \(\text{sign}(c_j) = \pm\), as shown in Eq.(2.1).] We see that the contribution to the \(Z_L \gamma_T\) production channel is enhanced relative to that of the \(Z_T \gamma_T\) production channel by a factor of \(s/M_Z^2\) at \(O(\Lambda^{-4})\).

The \(O(\Lambda^{-8})\) terms originate from the pure dimension-8 contributions:

\[
|\mathcal{T}_{\text{sm}}^{(8)}|^2 [Z_L \gamma_T] = \frac{(8s_W^4 - 4s_W^2 + 1)(\cos 2\theta + 3)}{32} \frac{M_Z^2 (s - M_Z^2)^2 s}{\Lambda^8}, \tag{2.12a}
\]

\[
|\mathcal{T}_{\text{sm}}^{(8)}|^2 [Z_T \gamma_T] = \frac{(8s_W^4 - 4s_W^2 + 1) \sin^2 \theta}{8} \frac{M_Z^2 (s - M_Z^2)^2}{\Lambda^8}. \tag{2.12b}
\]

The energy dependence in the above formulas can be directly understood by power counting,

\[
\mathcal{T}_{\text{(8)}}[Z_L \gamma_T] \sim \frac{M_Z s^3}{\Lambda^4}, \tag{2.13a}
\]

\[
\mathcal{T}_{\text{(8)}}[Z_T \gamma_T] \sim \frac{M_Z^2 s}{\Lambda^4}, \tag{2.13b}
\]

which explains the asymptotic high-energy behaviors in Eq.(2.12) when \(s \gg M_Z^2\). We see that at \(O(\Lambda^{-8})\) the \(Z_L \gamma_T\) production channel dominates over the \(Z_T \gamma_T\) production channel at high energies \(s \gg M_Z^2\).

We can understand further the asymptotic behavior of the interference terms (2.11) for \(s \gg M_Z^2\). In the case of the final state \(Z_L \gamma_T\), since we have \(\mathcal{T}_{\text{sm}}[Z_L \gamma_T] \sim \frac{M_Z}{\sqrt{s}}\) [Eq.(2.10)] and \(\mathcal{T}_{\text{(8)}}[Z_L \gamma_T] \sim \frac{M_Z s^3}{\Lambda^4}\) [Eq.(2.13a)], we find that their interference term
behaves as \( \mathcal{T}_{\text{sm}}(8) [Z_L \gamma_T] \sim \frac{M_Z^2 s}{4 \Lambda^4} \). This explains nicely the asymptotic behavior of Eq.\( (2.11a) \). However, for the final state \( Z_T \gamma_T \), using the naive power counting from Eqs.\( (2.8b) \) and \( (2.13b) \) we infer the asymptotic behaviors, \( \mathcal{T}_{\text{sm}}[Z_T \gamma_T] \sim s^0 \) and \( \mathcal{T}(8)[Z_T \gamma_T] \sim \frac{M_Z^2 s}{4 \Lambda^4} \). Combining these would lead to the following behavior for their interference: \( \mathcal{T}_{\text{sm}}(8)[Z_T \gamma_T] \sim \frac{M_Z^2 s}{4 \Lambda^4} \). However, this naive power counting contradicts Eq.\( (2.11b) \), where we see that \( \mathcal{T}_{\text{sm}}(8)[Z_T \gamma_T] \sim \frac{M_Z^2 s^0}{4 \Lambda^4} \). Naive power counting fails in this case for a nontrivial reason, which is connected to the special structure of the helicity amplitude \( \mathcal{T}(8)[Z_T \gamma_T] \). We see from Eqs.\( (A.5a) \) and \( (A.6) \) of Appendix A.1 that the off-diagonal helicity amplitudes \( \mathcal{T}(8)[Z_T \gamma_T] \) with \( \lambda \lambda' = +-, -, + \) vanish because of the antisymmetric tensor \( \epsilon^\mu\nu\alpha\beta \) contained in the \( Z\gamma Z^* \) vertex [Eq.\( (2.6) \)]. Hence, the energy dependence of \( \mathcal{T}_{\text{sm}}(8)[Z_T \gamma_T] \) is determined by the diagonal helicity amplitudes with \( \lambda \lambda' = ++, -- \). The SM amplitude \( \mathcal{T}_{\text{sm}}[Z_T \gamma_T] \) has a negative power of energy \( \propto s^{-1} \) in its diagonal helicity amplitudes as shown in Eq.\( (A.4a) \). This explains neatly the high-energy behavior \( \mathcal{T}_{\text{sm}}(8)[Z_T \gamma_T] \sim \frac{M_Z^2 s^0}{4 \Lambda^4} \), in agreement with Eq.\( (2.11b) \).

3. Probing New Physics in the \( ZV\gamma \) Coupling at \( e^+e^- \) Colliders

In this section, we first analyze the kinematical structure of the reaction \( e^+e^- \rightarrow Z\gamma \) followed by \( Z \) decays into pairs of charged leptons \( \ell^\pm \). We then propose suitable kinematical cuts to suppress effectively the SM backgrounds, and derive the optimal sensitivity reach for the scale of the new physics in the \( ZV\gamma \) coupling. In Section 3.1, we analyze the angular observables for \( Z\gamma \) production with \( Z \rightarrow \ell^+\ell^- \), and then study probes of the new physics contributions at \( \mathcal{O}(\Lambda^{-4}) \) in Section 3.2.1 and at \( \mathcal{O}(\Lambda^{-8}) \) in Section 3.2.2, making use of angular observables to suppress the SM backgrounds for the specific \( e^+e^- \) collision energy \( \sqrt{s} = 3 \) TeV. Then, we extend the analysis to other collider energies \( \sqrt{s} = (250, 500, 1000, 5000) \) GeV in Section 3.2.3, showing the increase in sensitivity obtainable from increasing the collider energy. Finally, in Section 3.3, we present a more complete background analysis including additional non-resonant SM backgrounds with the same final state \( \ell^-\ell^+\gamma \) (but \( \ell^-\ell^+ \) not coming from \( Z \) decay).

3.1. Analysis of Angular Observables

In this subsection, we analyze the kinematical observables for the reaction \( e^+e^- \rightarrow Z\gamma \) followed by the leptonic decays \( Z \rightarrow \ell^+\ell^- \). We illustrate the kinematics in Fig. 1, where the scattering plane is determined by the incident \( e^-e^+ \) and the outgoing \( Z\gamma \) in the collision frame (with scattering angle \( \theta \)), and the directions of the final-state leptons \( \ell^-\ell^+ \) determine the decay plane. We denote the angle between the two planes as \( \phi \) in the laboratory frame (which is equal to \( \phi_\ast \) in the \( Z \) rest frame).

In order to study the leptonic final states \( Z(q_1) \rightarrow \ell^-(k_1)\ell^+(k_2) \), we denote the lepton momenta as follows in the \( Z \) rest frame:

\[
k_1 = \frac{M_Z}{2} \{(1, \sin\theta_\ast \cos\phi_\ast, \sin\theta_\ast \sin\phi_\ast, \cos\theta_\ast)\} ,
\]

(3.1a)
Figure 1. Illustration of the kinematical structure of the reaction $e^+e^- \rightarrow Z\gamma$ followed by the leptonic decay $Z \rightarrow \ell^+\ell^-$. 

\[ k_2 = \frac{M_Z}{2} \left( 1, -\sin\theta_*\cos\phi_* , -\sin\theta_*\sin\phi_* , -\cos\theta_* \right). \]  

Here the positive $z_*$ direction in the $Z$ rest frame is chosen to be opposite to the final-state photon direction in the laboratory frame, and $\theta_*$ denotes the angle between the positive $z_*$ direction and the $\ell^-$ direction in the $Z$ rest frame. When boosted back to the $e^-e^+$ collision frame (laboratory frame), the angle $\theta_*$ changes but the azimuthal angle $\phi_*$ is invariant. This is why the angle $\phi_*$ is equal to the angle $\phi$ between the scattering plane (defined by the incoming $e^-e^+$ directions and the outgoing $Z\gamma$ directions) and $Z$ decay plane (defined by the outgoing $\ell^-$ and $\ell^+$ directions) in the $e^-e^+$ collision frame.

Imposing a lower cut on the scattering angle in the laboratory frame $\theta > \delta$ (where $\delta \ll 1$) will correspond to a lower cut on the transverse momentum of the final-state photon $P_T^\gamma > q \sin \delta$. With this lower cut, we find the following total cross section for $Z\gamma$ production:

\[
\sigma(Z\gamma) = \frac{e^4(1-4s_W^2+8s_W^4)[-(s-M_Z^2)^2-2(s^2+M_Z^4)\ln(\sin^2\delta)]}{32\pi s_W^2 c_W^2 (s-M_Z^2)s^2} \\
\pm \frac{e^2(1-4s_W^2)M_Z^2(s-M_Z^2)(s+M_Z^2)}{32\pi s_W c_W A^4 s^2} \\
+ \frac{(1-4s_W^2+8s_W^4)M_Z^2(s+M_Z^2)(s-M_Z^2)^3}{192\pi A^8 s^2} + O(\delta),
\]

the two possible signs of a given dimension-8 operator, $\text{sign}(c_j) = \pm$, as shown in Eq.(2.1).

We compute numerically the exact cross sections for $e^+e^- \rightarrow Z\gamma$,\footnote{Since the leptonic vector coupling of $Z$ boson is proportional to $(1-4s_W^2)$, it is sensitive to the value of $s_W^2$. Here we use the $\overline{\text{MS}}$ value $s_W^2 = 0.23122 \pm 0.00003 \ (\mu = M_Z)$ [21].} as a function of
the new physics scale $\Lambda$ and for different collider energies. imposing a photon transverse momentum cut $P_T^\gamma = q \sin \delta$ with $\delta > 0.2$:

\[
\sqrt{s} = 250\text{GeV}, \quad \sigma(Z\gamma) = \left[ 7749 \pm 8.90 \left( \frac{0.5\text{TeV}}{\Lambda} \right)^4 + 1.98 \left( \frac{0.5\text{TeV}}{\Lambda} \right)^8 \right] \text{fb}, \quad (3.3a)
\]
\[
\sqrt{s} = 500\text{GeV}, \quad \sigma(Z\gamma) = \left[ 1624 \pm 1.38 \left( \frac{0.8\text{TeV}}{\Lambda} \right)^4 + 0.929 \left( \frac{0.8\text{TeV}}{\Lambda} \right)^8 \right] \text{fb}, \quad (3.3b)
\]
\[
\sqrt{s} = 1\text{TeV}, \quad \sigma(Z\gamma) = \left[ 390 \pm 0.566 \left( \frac{\text{TeV}}{\Lambda} \right)^4 + 2.62 \left( \frac{\text{TeV}}{\Lambda} \right)^8 \right] \text{fb}, \quad (3.3c)
\]
\[
\sqrt{s} = 3\text{TeV}, \quad \sigma(Z\gamma) = \left[ 42.9 \pm 0.0354 \left( \frac{2\text{TeV}}{\Lambda} \right)^4 + 0.843 \left( \frac{2\text{TeV}}{\Lambda} \right)^8 \right] \text{fb}, \quad (3.3d)
\]
\[
\sqrt{s} = 5\text{TeV}, \quad \sigma(Z\gamma) = \left[ 15.4 \pm 0.0145 \left( \frac{2.5\text{TeV}}{\Lambda} \right)^4 + 1.09 \left( \frac{2.5\text{TeV}}{\Lambda} \right)^8 \right] \text{fb}. \quad (3.3e)
\]

As we show in Sec. 3.2-3.3 (cf. Table 2), the sensitivity reach of $\Lambda$ in each case is such that on the right-hand-side of the corresponding formula above, the ratio inside each $[\cdots]$ is $O(1)$. Thus, we see from Eq.(3.3) that, for the relevant sensitivity reaches of $\Lambda$, the contributions of the dimension-8 operator are always much smaller than the SM contributions, so the perturbation expansion is valid. Also, Eq.(3.3) shows that for $\sqrt{s} < 1\text{ TeV}$ the $O(\Lambda^{-4})$ contribution is dominant, whereas for $\sqrt{s} \gtrsim 1\text{ TeV}$, the $O(\Lambda^{-8})$ contribution becomes dominant. This is because the $O(\Lambda^{-8})$ contributions have higher energy dependence than the $O(\Lambda^{-4})$ contributions, as shown in Eqs.(2.11)-(2.12).

The total cross section for $e^+e^- \rightarrow Z\gamma \rightarrow \ell^+\ell^-\gamma$ is given by the product

\[
\sigma(\ell^+\ell^-\gamma) = \sigma(Z\gamma) \times \text{Br}(\ell^+\ell^-). \quad (3.4)
\]

The differential cross section is a function of the three kinematical angles ($\theta$, $\theta_*$, $\phi_*$), and is computed from the helicity amplitudes (A.12)-(A.13) in Appendix-A.2. We define the normalized angular distribution function as

\[
f^j_\xi = \frac{d\sigma_j}{\sigma_j d\xi},
\]

where $\xi = \theta, \theta_*, \phi_*$, and $\sigma_j$ (with $j = 0, 1, 2$) represents the SM contribution ($\sigma_0$), the $O(\Lambda^{-4})$ contribution ($\sigma_1$), and the $O(\Lambda^{-8})$ contribution ($\sigma_2$), respectively.

We find the following normalized polar angular distribution functions $f^j_\theta$ and $f^j_\theta$,

\[
f^0_\theta = \frac{\csc\theta \left[ 3s^2 + 2\cos2\theta(s-M_Z^2)^2 + 2M_Z^2 s + 3M_Z^4 \right]}{4 \left[ (s-M_Z^2)^2 + 2(s^2+M_Z^4)\ln(\sin\frac{\phi_*}{2}) \right]}, \quad (3.6a)
\]
\[
f^1_\theta = \frac{1}{2} \sin \theta, \quad (3.6b)
\]
\[
f^2_\theta = \frac{3 \sin\theta \left[ 3s + \cos2\theta(s-2M_Z^2) + 2M_Z^2 \right]}{16(s+M_Z^2)}; \quad (3.6c)
\]
Figure 2. Normalized angular distributions in the polar scattering angle $\theta$ in the laboratory frame for different collision energies, $\sqrt{s} = (250 \text{ GeV}, 500 \text{ GeV}, 1 \text{ TeV}, 3 \text{ TeV})$. In each plot, the black, red and blue curves denote the contributions from the SM, the $O(\Lambda^{-4})$ and $O(\Lambda^{-8})$ terms, respectively. We use a polar angle cut $\delta = 0.2$ for illustration.

and

\begin{align}
 f_0^0 &= \frac{3 \sin \theta_* (3 + \cos 2 \theta_*)}{16} + \frac{3 \sin \theta_* (1 + 3 \cos 2 \theta_*) M_Z^2 s}{8 [(s - M_Z^2)^2 + 2 (s^2 + M_Z^4) \ln (\sin^2 \frac{\theta_*}{2})]} + O(\delta), \\
 f_0^1 &= \frac{3 \sin \theta_* (2s - \cos 2 \theta_*) (2s - M_Z^2) + 3 M_Z^2}{16(s + M_Z^2)} + O(\delta), \\
 f_0^2 &= \frac{3 \sin \theta_* (2s - \cos 2 \theta_*) (2s - M_Z^2) + 3 M_Z^2}{16(s + M_Z^2)} + O(\delta).
\end{align}

Then we compute the normalized azimuthal angular distribution functions $f^j_{\phi_*}$ as follows,

\begin{align}
 f^0_{\phi_*} &= \frac{1}{2 \pi} + \frac{3 \pi^2 (c_L^2 - c_R^2)^2 M_Z \sqrt{s} (s + M_Z^2) \cos \phi_* - 8 (c_L^2 + c_R^2)^2 M_Z^2 s \cos 2 \phi_*}{16 \pi (c_L^2 + c_R^2)^2 [(s - M_Z^2)^2 + 2 (s^2 + M_Z^4) \ln (\sin^2 \frac{\theta_*}{2})]} + O(\delta), \\
 f^1_{\phi_*} &= \frac{1}{2 \pi} - \frac{9 \pi^2 \sqrt{s} (s + M_Z^2) \cos \phi_* - 32 M_Z s \cos 2 \phi_*}{128 \pi M_Z (s + M_Z^2)} + O(\delta).
\end{align}
Figure 3. Normalized angular distribution in the polar angle $\theta_*$ in the $Z$ decay frame for different collision energies, $\sqrt{s} = (250 \text{ GeV}, 500 \text{ GeV}, 1 \text{ TeV}, 3 \text{ TeV})$. In each plot, the black, red and blue curves denote the contributions from the SM, the $\mathcal{O}(\Lambda^{-4})$ and $\mathcal{O}(\Lambda^{-8})$ terms, respectively, where the red and blue curves exactly overlap. We use a laboratory polar angle cut $\delta = 0.2$ for illustration.

\[
\phi_* = \frac{1}{2\pi} - \frac{9\pi(c_L^2 - c_R^2)^2 M_Z \sqrt{s} \cos \theta_*}{128(c_L^2 + c_R^2)^2 (s + M_Z^2)} + \mathcal{O}(\delta), \tag{3.8c}
\]

where the coefficients $(c_L, c_R) = (s_W^2 - \frac{1}{2}, s_W^2)$ are the gauge couplings of the $Z$ boson to the (left, right)-handed leptons. Here we have again chosen a lower cutoff $\delta \ll 1$ on the polar scattering angle $\theta$, which corresponds to a lower cut on the transverse momentum of the final state photon, $P_T^\gamma > q \sin \delta$.

As a side remark, we note that if the $Z$ boson were a stable particle, one could in principle measure its polarization directly to extract the new physics signal of the dimension-8 operator. However, since the $Z$ decays rapidly into fermion pairs, the contributions of the out-going longitudinal $Z_L$ and transverse $Z_T$ intermediate states interfere in the angular distributions of the fermions produced in $Z_L$ and $Z_T$ decays. Such interference effects appear as the $\phi_*$ angular dependence in Eq.(3.8).

Using the above results, we present numerical results for the normalized angular distribution functions of $\theta, \theta_*$, and $\phi_*$, in Fig. 2, Fig. 3, and Fig. 4, respectively. In each fig-

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Figure 4. Normalized angular distribution in the azimuthal angle $\phi_*$ for different collision energies, $\sqrt{s} = (250 \text{ GeV}, 500 \text{ GeV}, 1 \text{ TeV}, 3 \text{ TeV})$. In each plot, the black, red and blue curves denote the contributions from the SM, the $\mathcal{O}(\Lambda^{-4})$ and $\mathcal{O}(\Lambda^{-8})$ terms, respectively, where the blue and black curves nearly overlap. We use a laboratory polar angle cut $\delta = 0.2$ for illustration.

It is of interest to examine the behaviours of the angular distribution functions $f_\xi^j$ in the high-energy limit $s \gg M_Z^2$. For all the functions $f_\theta^j$ and $f_{\phi_*}^j$, the coefficients of all trigonometric functions approach constants. This is why Figs. 2 and 3 show that the distributions in $\theta$ and $\theta_*$ are not sensitive to the collision energy $\sqrt{s}$, as we vary the collision energy $\sqrt{s} = (250 \text{ GeV}, 500 \text{ GeV}, 1 \text{ TeV}, 3 \text{ TeV})$ in the four plots. For the angular functions $f_\theta^0$ and $f_\theta^2$, the coefficients of $\cos \phi_*$ are suppressed by $M_Z/\sqrt{s}$, so they approach the constant term $1/2\pi$ for $s \gg M_Z^2$. This is why in Fig. 4 the angular functions $f_\phi^0$ and $f_\phi^2$ (shown as the black and blue curves) appear fairly flat and largely
overlap each other. In contrast, for the angular function \( f_{\phi^*} \), the coefficient of \( \cos \phi^* \) is enhanced by an energy factor \( \sqrt{s}/M_Z \), and can be much larger than the constant term. For \( s \gg M_Z^2 \), we can approximate Eq.(3.8b) in the following form:

\[
f_{\phi^*}^1 = \frac{1}{2\pi} \left( \frac{1}{2} + \cos^2 \phi^* \right) - \frac{9\pi \sqrt{s}}{128M_Z} \cos \phi^* + \mathcal{O}(M_Z^2/s, \delta) \simeq 0.159(0.5 + \cos^2 \phi^*) - 0.606 \left( \frac{\sqrt{s}}{250 \text{ GeV}} \right) \cos \phi^* + \mathcal{O}(M_Z^2/s, \delta).
\] (3.9)

We see that the \( \cos \phi^* \) term dominates \( f_{\phi^*}^1 \) for \( \sqrt{s} > 250 \text{ GeV} \). This also explains why in Fig. 4 the magnitude of the angular function \( f_{\phi^*}^1 \) (red curve) grows almost linearly with the collision energy \( \sqrt{s} \), and has its maximum at \( \phi^* = \pi \) and minima at \( \phi^* = 0, 2\pi \).

In the laboratory frame, the opening angle \( \Delta \theta_{\ell\ell} \) between the two outgoing leptons from \( Z \) decay is a function of \( \theta^* \). For \( \sqrt{s} \sim M_Z \), we expect \( \Delta \theta_{\ell\ell} \sim \pi \), while for \( s \gg M_Z^2 \), we have \( \Delta \theta_{\ell\ell} \to 0 \).

3.2. Probing the New Physics Scale in the \( ZZ\gamma \) Coupling

In this subsection, we analyze how to probe the new physics contributions at \( \mathcal{O}(\Lambda^{-4}) \) (Sec. 3.2.1) and at \( \mathcal{O}(\Lambda^{-8}) \) (Sec. 3.2.2) for the \( e^+e^- \) collision energy \( \sqrt{s} = 3 \text{ TeV} \). We demonstrate that making use of the angular observables can suppress the SM backgrounds efficiently.

3.2.1. Analysis of the \( \mathcal{O}(\Lambda^{-4}) \) Contribution

In order to analyze the sensitivity to the new physics scale \( \Lambda \) considering the SM and \( \mathcal{O}(\Lambda^{-4}) \) contributions, motivated by Fig. 4, we divide the range of \( \phi^* \) into two regions — regions (a) and (b). Region (a) includes the ranges \([0, \frac{\pi}{2}] \cup [\frac{3\pi}{2}, 2\pi]\) and region (b) is the range \( (\frac{\pi}{2}, \frac{3\pi}{2}) \). The sum of the areas of regions (a) and (b) \( (S_a + S_b) = 1 \), because the angular function is normalized to unity. However, the difference \( |S_a - S_b| \) is much larger than 1 for the angular function \( f_{\phi^*}^1 \), while \( |S_a - S_b| \) is subject to a strong cancellation in the SM contribution \( f_{\phi^*}^0 \). We can make use of this feature to suppress the SM background and enhance significantly the \( \mathcal{O}(\Lambda^{-4}) \) signal at the same time. To this end, we define the functions

\[
\mathcal{O}_j \equiv |\sigma_j| \left( \int_{\frac{3\pi}{2}}^{\frac{\pi}{2}} - \int_{0}^{\frac{\pi}{2}} - \int_{\frac{3\pi}{2}}^{2\pi} \right) f_{\phi^*}^j d\phi^*, \quad (3.10)
\]

for \( j = 0, 1, 2 \). Furthermore, Fig. 4 shows that the \( \mathcal{O}(\Lambda^{-8}) \) distribution \( f_{\phi^*}^2 \) is fairly flat and largely overlaps with \( f_{\phi^*}^0 \) of the SM. Thus, the \( \mathcal{O}(\Lambda^{-8}) \) contributions to \( |S_a - S_b| \) also cancel strongly and become negligible. Hence, for the signal analysis, here we only need to consider the \( \mathcal{O}(\Lambda^{-4}) \) contributions.

We define the signal and background event numbers as follows:

\[
S = \mathcal{O}^1 \times \mathcal{L} \times \epsilon, \quad (3.11a)
\]
\[ B = \mathcal{O}^0 \times \mathcal{L} \times \epsilon, \quad (3.11b) \]

where \( \mathcal{L} \) denotes the luminosity and \( \epsilon \) is the detection efficiency. With these definitions, we find that the SM background \( B \) is quite small due to the large cancellation between regions (a) and (b), whereas its statistical error \( \Delta_B \) is not so small:

\[ B = N_a - N_b, \quad (3.12a) \]

\[ \Delta_B = \sqrt{\Delta_a^2 + \Delta_b^2} = \sqrt{N_0^a + N_0^b} = \sqrt{\sigma_0 \times \mathcal{L} \times \epsilon}, \quad (3.12b) \]

where \( N_0^a \) and \( N_0^b \) are the SM event numbers in regions (a) and (b), respectively. We estimate the signal significance by

\[ Z_4 = \frac{S}{\Delta_B} = \frac{\mathcal{O}^1(Z\gamma)}{\sqrt{\sigma^0(Z\gamma)}} \times \sqrt{\text{Br}(Z \rightarrow \ell\ell)} \times \mathcal{L} \times \epsilon. \quad (3.13) \]

We note that \( \mathcal{O}^1 \) and \( \sigma^0 \) are functions of the angular cut \( \delta \) (corresponding to the photon transverse momentum cut \( P_T^\gamma > q \sin \delta \)). Fig. 4 shows that the magnitude \( |f_{\phi_*}^1| \) is very small around \( \phi_* = \frac{\pi}{2}, \frac{3\pi}{2} \) since it is dominated by the \( \cos \phi_* \) term as in Eq.(3.9). So we may cut off the nearby area to reduce the SM backgrounds. For this, we introduce a cut parameter \( 0 < \phi_c < \frac{\pi}{2} \), using which region (a) reduces to \([0, \phi_c] \cup [2\pi - \phi_c, 2\pi)\) and region (b) becomes \([\pi - \phi_c, \pi + \phi_c]\). We then compute the corresponding signal observable \( \mathcal{O}^1_c \) and the background fluctuation \( \sqrt{\sigma^0_c} \). In Fig. 4, the angular function \( f_{\phi_*}^0 \) appears rather flat, so we obtain a simple expression for \( \sigma^0_c \), as follows:

\[ \mathcal{O}^1_c = |\sigma_1| \left( \int_{\pi-\phi_c}^{\phi_c} - \int_{0}^{\phi_c} - \int_{2\pi-\phi_c}^{2\pi} \right) f_{\phi_*}^1 \ d\phi_* \quad (3.14a) \]

\[ \approx \frac{3\alpha(1 - 4s_W^2)M_Z(s - M_Z^2)[3(\pi - 2\delta)(s + M_Z^2) - (s - 3M_Z^2)\sin2\delta] \sin \phi_c}{256 s_W c_W \Lambda^4 s^2}, \]

\[ \sigma^0_c \approx \frac{2\phi_c}{\pi} \sigma^0 \quad (3.14b) \]

\[ = \frac{\alpha^2(1 - 4s_W^2 + 8s_W^4)[- \cos \delta(s - M_Z^2)^2 + 2(s^2 + M_Z^4)\ln(\cot \frac{\phi_c}{2})]}{c_W s_W (s - M_Z^2)s^2} \phi_c, \]

where \( \alpha = e^2/4\pi \) is the fine structure constant.

For our analysis, we choose the values of \( \phi_c = \phi^m_c \) and \( \delta = \delta^m \) such that the signal significance \( Z_4 = (\mathcal{O}^1_c/\sqrt{\sigma^0_c}) \sqrt{\text{Br}(Z \rightarrow \ell\ell)} \times \mathcal{L} \times \epsilon \) is maximized. Thus, \( \phi_c = \phi^m_c \) corresponds to the maximum of the function \( \sin \phi_c/\phi_c \), and we derive \( \phi_c^m \approx 1.17 \), which is independent of the collision energy \( \sqrt{s} \). The value of \( \delta^m \) required to obtain the maximal significance of \( Z_4 \propto (\mathcal{O}^1_c/\sqrt{\sigma^0_c}) \) depends on the collision energy \( \sqrt{s} \): at high energies \( s \gg M_Z^2 \), and we find that \( \delta^m \approx 0.329 \).

We present in Fig. 5 the signal significance obtained in this way for the collision energy \( \sqrt{s} = 3 \text{ TeV} \). We input the total leptonic branching fraction \( \text{Br}(Z \rightarrow \ell^- \ell^+) \approx \).
0.10, and assume an integrated luminosity $\mathcal{L} = 2 \text{ab}^{-1}$ and an ideal detection efficiency $\epsilon = 1$ for simplicity. In Fig. 5(a), we depict the significance $Z_4$ as a function of the angular cut $\delta$, which exhibits the maximum at $\delta_m \simeq 0.33$ for $\Lambda = 1 \text{TeV}$, as expected. Thus, under the angular cuts $(\phi_c, \delta) = (\phi^m_c, \delta_m)$, we derive

$$
(\sigma^0_c, \mathcal{O}^1_c) \simeq \left( 23.1, 11.1 \left( \frac{\text{TeV}}{\Lambda} \right)^4 \right) \text{fb}. \tag{3.15}
$$

For illustration, we may then use Eq.(3.13) to estimate the signal significance as follows:

$$
Z_4 \simeq 32.7 \left( \frac{\text{TeV}}{\Lambda} \right)^4 \simeq 5.0 \left( \frac{1.60 \text{TeV}}{\Lambda} \right)^4, \tag{3.16}
$$

which is plotted in Fig. 5(b). From this, we find that the probe of the new physics scale can reach $\Lambda = (2.0, 1.8, 1.6) \text{TeV}$ at $(2\sigma, 3\sigma, 5\sigma)$ level, respectively.

We note that the practical detection efficiency would be smaller than 100%, so the actual sensitivity may be somewhat weaker. But, as we show later in Eqs.(3.25) and (3.29) of Sec.3.3, the sensitivity reach for $\Lambda$ has rather weak dependences on the integrated luminosity and detection efficiency, namely, $\Lambda \propto (\mathcal{L} \epsilon)^{\frac{1}{8}}$ at $\mathcal{O}(\Lambda^{-4})$ and $\Lambda \propto (\mathcal{L} \epsilon)^{\frac{1}{16}}$ at $\mathcal{O}(\Lambda^{-8})$. Hence, increasing $\mathcal{L}$ or $\epsilon$ only has minor effect on the sensitivity reach of the new physics scale $\Lambda$. In contrast, raising the collision energy $\sqrt{s}$ can do more to improve the sensitivity reach of $\Lambda$ because $\Lambda \propto (\sqrt{s})^{\frac{1}{2}}$ at $\mathcal{O}(\Lambda^{-4})$ and $\Lambda \propto (\sqrt{s})^{\frac{5}{8}}$ at $\mathcal{O}(\Lambda^{-8})$.

**3.2.2. Analysis Including the $\mathcal{O}(\Lambda^{-8})$ Contribution**

In this subsection, we include in the analysis the contribution of $\mathcal{O}(\Lambda^{-8})$. Since the $\mathcal{O}(\Lambda^{-8})$ term has a higher power of energy dependence, it may have better sensitivity.
when the collision energy is higher, e.g., $\sqrt{s} = 3$ TeV, even though it is suppressed by $\Lambda^{-8}$.

We see from Fig. 4 that both the distributions $f^0_{\phi^*$ and $f^2_{\phi^*$ are rather flat, and thus insensitive to the $\mathcal{O}(\Lambda^{-8})$ contribution. Hence, in order to enhance the signal sensitivity to the $\mathcal{O}(\Lambda^{-8})$ contribution, we study instead the distributions in $\theta$ and $\theta^*$. For this, we choose the region $\theta \in [\delta, \pi - \delta]$ and $\theta^* \in [\delta^*, \pi - \delta_*]$. With the angular cuts ($\delta, \delta^*$), we compute the SM contribution $\sigma^0_c(Z\gamma)$, the $\mathcal{O}(\Lambda^{-4})$ contribution $\sigma^1_c(Z\gamma)$, and the $\mathcal{O}(\Lambda^{-8})$ contribution $\sigma^2_c(Z\gamma)$ as follows:

\[
\sigma^0_c(Z\gamma) = \frac{e^4 (8s^4_W - 4s^2_W + 1)}{32\pi s^2_W c^2_W (s - M^2_Z)} \times \frac{1}{16} \left[ 4 \cos \delta (9 \cos \delta_s - \cos 3\delta_s) M^2_Z s \right. \\
\left. - (15 \cos \delta_s + \cos 3\delta_s) s^2 + M^2_Z \right] \left( \cos \delta + 2 \ln \tan \frac{\delta}{2} \right),
\]

\[
\sigma^1_c(Z\gamma) = \pm \frac{e^2 (1 - 4s^2_W) M^2_Z (s - M^2_Z)}{32\pi s_W c_W \Lambda^4 s^2} \times \frac{1}{8} \left[ 2(5 - \cos 2\delta_s) s + (\cos 2\delta_s + 7) M^2_Z \right] \cos \delta \cos \delta_s,
\]

\[
\sigma^2_c(Z\gamma) = \frac{(8s^4_W - 4s^2_W + 1) M^2_Z (s - M^2_Z)^3}{192\pi\Lambda^8 s^2} \times \frac{\cos \delta}{64} \times \left[ (7 + \cos 2\delta)(9 \cos \delta_s - \cos 3\delta_s) s + (5 - \cos 2\delta)(15 \cos \delta_s + \cos 3\delta_s) M^2_Z \right],
\]

where the $\pm$ signs of the $\mathcal{O}(\Lambda^{-4})$ term correspond to the two possible signs of a given dimension-8 operator, $\text{sign}(c_j) = \pm$, as shown in Eq.(2.1). If we take the limit ($\delta, \delta^* \to 0$) in the above formulas, we find that they reduce consistently to Eq.(3.3), as expected.

We can then estimate the corresponding signal significance to be

\[
Z_8 = \frac{S}{\Delta_B} = \frac{|\sigma^1_c(Z\gamma) + \sigma^2_c(Z\gamma)|}{\sigma^0_c(Z\gamma)} \times \sqrt{\text{Br}(Z \to \ell\ell)} \times L \times \epsilon .
\]

For the collision energy $\sqrt{s} = 3$ TeV, we find that the $\mathcal{O}(\Lambda^{-8})$ term dominates. To obtain the maximal signal significance, we derive the corresponding values of the angular cuts ($\delta, \delta^*$) = ($\delta_m, \delta^*_m$), which are ($\delta_m, \delta^*_m$) $\simeq$ (0.623, 0.820). Inputting $\text{Br}(Z \to \ell\ell)$ $\simeq$ 0.10 and choosing $\sqrt{s} = 3$ TeV, $L = 2 \text{ ab}^{-1}$ and $\epsilon = 1$, we compute the cross section for $Z\gamma$ production:

\[
\sigma(Z\gamma) = \left[ 10.1 \pm 0.0251 \left( \frac{2\text{TeV}}{\Lambda} \right)^4 + 0.554 \left( \frac{2\text{TeV}}{\Lambda} \right)^8 \right] \text{fb}.
\]

Thus, from Eq.(3.18) we estimate the signal significance

\[
Z_8 \simeq \left| \pm 1.79 \left( \frac{\text{TeV}}{\Lambda} \right)^4 + 631 \left( \frac{\text{TeV}}{\Lambda} \right)^8 \right| \simeq \pm 0.112 \left( \frac{2\text{TeV}}{\Lambda} \right)^4 + 2.46 \left( \frac{2\text{TeV}}{\Lambda} \right)^8 .
\]

\[\text{2}\]We note that the interference of higher-dimensional operators with the SM may appear at the same order for high energy scales. However, such interference terms could in general be distinguished by different angular dependences.
Finally, we may combine \( Z_4 \) and \( Z_8 \) to achieve a better sensitivity reach to the new physics scale \( \Lambda \):

\[
Z = \sqrt{Z_4^2 + Z_8^2},
\]

which is depicted by the red curve in Fig. 6. In this way, we find that the new physics scale can be probed up to \( \Lambda \simeq (2.2, 2.0, 1.9) \) TeV at the (2\( \sigma \), 3\( \sigma \), 5\( \sigma \)) levels, respectively. These numbers apply for both \( \pm \) signs in Eq.(3.20), since we find that the case of minus sign in Eq.(3.20) only causes a tiny difference in the \( \Lambda \) bound by less than 1%. Hence, the \( \mathcal{O}(\Lambda^{-4}) \) term in Eq.(3.20) has negligible effect for the collider energy \( \sqrt{s} = 3 \) TeV. As we will show in Sec.3.2.3, this feature applies to all cases with \( \sqrt{s} \gtrsim 1 \) TeV.

### 3.2.3. Analysis of Different Collision Energies

In this subsection, we further extend our analysis of \( \sqrt{s} = 3 \) TeV case to different collision energies \( \sqrt{s} = (250, 500, 1000, 5000) \) GeV, in each case with a sample integrated luminosity \( \mathcal{L} = 2 \) ab\(^{-1}\).

Using the same method as we presented in Sec.3.2.1-3.2.2, we analyze the SM backgrounds and signal contributions for different collider energies. For each given collider energy \( \sqrt{s} \), we derive the optimal angular cuts for realizing the maximal signal significance \( Z_4 \) and \( Z_8 \). Namely, for the analysis of \( Z_4 \), we use the angular cuts \((\delta_m, \phi_c^m)\) for angles \((\theta, \phi_c)\); while for the analysis of \( Z_8 \), we use the angular cuts \((\delta_m, \delta_{s_m})\) for angles \((\theta, \theta_s)\). We summarize the optimal angular cuts for different collider energies in Table 1. As we noted below Eq.(3.14), the dominant contribution to the signal significance \( Z_4 \) depends on the cut \( \phi_c \) only through a simple function \( \sin \phi_c/\sqrt{\phi_c} \),
\[ \sqrt{s} \text{ (GeV)} \quad 250 \quad 500 \quad 1000 \quad 3000 \quad 5000 \]

| \( \mathcal{Z}_4 \): (\( \delta_m, \phi^m_c \)) |
|---|
| (0.368, 1.17) (0.340, 1.17) (0.332, 1.17) (0.329, 1.17) |
| \( \mathcal{Z}_8 \): (\( \delta_m, \delta^*_m \)) |
| (0.608, 0.692) (0.616, 0.790) (0.621, 0.814) (0.623, 0.820) (0.623, 0.821) |

Table 1. Summary of the optimal angular cuts for realizing the maximal signal significance.

For the signal significance \( \mathcal{Z}_4 \), we impose the cuts (\( \delta_m, \phi^m_c \)) on the angular distributions of (\( \theta, \phi^*_c \)) whereas, for the signal significance \( \mathcal{Z}_8 \), we set the cuts (\( \delta_m, \delta^*_m \)) on the angular distributions of (\( \theta, \theta^*_c \)).

which does not depend on energy \( \sqrt{s} \) and reaches its maximum at \( \phi^m_c \simeq 1.17 \). This is why the optimal cut \( \phi^m_c \) is nearly independent of the collider energy \( \sqrt{s} \), as shown in Table 1.

With the optimal kinematical cuts in Table 1, we compute the SM contributions and the \( \mathcal{O}(\Lambda^{-4}) \) contributions for different collider energies,

\[ \sqrt{s} = 250 \text{ GeV}, \quad (\sigma^0_c, \mathcal{O}^1_c) = \left( 3936, 0.913 \left( \frac{\text{TeV}}{\Lambda} \right)^4 \right) \text{ fb}, \quad (3.22a) \]
\[ \sqrt{s} = 500 \text{ GeV}, \quad (\sigma^0_c, \mathcal{O}^1_c) = \left( 860, 1.85 \left( \frac{\text{TeV}}{\Lambda} \right)^4 \right) \text{ fb}, \quad (3.22b) \]
\[ \sqrt{s} = 1 \text{ TeV}, \quad (\sigma^0_c, \mathcal{O}^1_c) = \left( 209, 3.71 \left( \frac{\text{TeV}}{\Lambda} \right)^4 \right) \text{ fb}, \quad (3.22c) \]
\[ \sqrt{s} = 3 \text{ TeV}, \quad (\sigma^0_c, \mathcal{O}^1_c) = \left( 23.1, 11.1 \left( \frac{\text{TeV}}{\Lambda} \right)^4 \right) \text{ fb}, \quad (3.22d) \]
\[ \sqrt{s} = 5 \text{ TeV}, \quad (\sigma^0_c, \mathcal{O}^1_c) = \left( 8.30, 18.5 \left( \frac{\text{TeV}}{\Lambda} \right)^4 \right) \text{ fb}, \quad (3.22e) \]

where we include the case of \( \sqrt{s} = 3 \text{ TeV} \) from Eq.(3.15) for comparison.

With these, we derive the following signal significances at each collision energy, for the leptonic branching fraction \( \text{Br}(Z \to \ell\ell) \simeq 0.10 \) and an integrated luminosity \( \mathcal{L} = 2 \text{ ab}^{-1} \):

\[ \sqrt{s} = 250 \text{ GeV}, \quad Z_4 = 3.29 \left( \frac{0.5\text{TeV}}{\Lambda} \right)^4 \times \sqrt{\epsilon}, \quad (3.23a) \]
\[ \sqrt{s} = 500 \text{ GeV}, \quad Z_4 = 2.18 \left( \frac{0.8\text{TeV}}{\Lambda} \right)^4 \times \sqrt{\epsilon}, \quad (3.23b) \]
\[ \sqrt{s} = 1 \text{ TeV}, \quad Z_4 = 3.62 \left( \frac{\text{TeV}}{\Lambda} \right)^4 \times \sqrt{\epsilon}, \quad (3.23c) \]
\[ \sqrt{s} = 3 \text{ TeV}, \quad Z_4 \simeq 2.05 \left( \frac{2\text{TeV}}{\Lambda} \right)^4 \times \sqrt{\epsilon}, \quad (3.23d) \]
\[ \sqrt{s} = 5 \text{ TeV}, \quad Z_4 = 2.33 \left( \frac{2.5\text{TeV}}{\Lambda} \right)^4 \times \sqrt{\epsilon}. \quad (3.23e) \]
We note that the signal significance is nearly proportional to the squared centre-of-mass collision energy \((\sqrt{s})^2\). At high energies \(s \gg M_Z^2\), we have

\[
Z_4 \propto \frac{M_Z}{\Lambda^4} \sqrt{\mathcal{L} \times \epsilon}.
\]

Thus, for a given significance \(Z_4\), the corresponding reach of the new physics scale \(\Lambda\) is

\[
\Lambda \propto \left( \frac{M_Z}{Z_4} \sqrt{\mathcal{L} \times \epsilon} \right)^{\frac{1}{4}} \times (\sqrt{s})^{\frac{1}{2}}.
\]

We see that the collision energy \(\sqrt{s}\) has the most sensitive effect on the reach of the new physics scale \(\Lambda\). For instance, raising the collision energy from \(\sqrt{s} = 250\text{ GeV}\) to \(\sqrt{s} = 3\text{ TeV}\), the reach of the new physics scale is improved by a significant factor \(\Lambda(3\text{ TeV})/\Lambda(250\text{ GeV}) \approx 3.46\). On the other hand, \(\Lambda\) has a rather weak dependence on the significance, \(\Lambda \propto Z_4^{-\frac{1}{4}}\), so the 5\(\sigma\) reach is only slightly weaker than the 2\(\sigma\) reach: \(\Lambda(5\sigma)/\Lambda(2\sigma) \approx 1/1.26\). Furthermore, we note that \(\Lambda\) depends much more weakly on the integrated luminosity and the detection efficiency, \(\Lambda \propto \mathcal{L} \times \epsilon^{\frac{1}{8}}\). For instance, increasing the integrated luminosity from \(L = 2\text{ ab}^{-1}\) to \(L = 6\text{ ab}^{-1}\), would enhance the reach of the new physics scale by only a factor of \(\Lambda(6\text{ ab}^{-1})/\Lambda(2\text{ ab}^{-1}) \approx 1.15\). Also, if the detection efficiency is increased from \(\epsilon = 40\%\) to \(\epsilon = 90\%\), the reach of the new physics scale would be only slightly extended by a factor of \(\Lambda(90\%)/\Lambda(40\%) \approx 1.11\).

Next, extending Section 3.2.2 to different collision energies, we include contributions up to \(\mathcal{O}(\Lambda^{-8})\) in a similar manner. We apply the optimal kinematical cuts as in Table 1 and compute the cross sections of \(e^+e^- \to Z\gamma\) as follows:

\[
\sqrt{s} = 250\text{ GeV}, \quad \sigma(Z\gamma) = \left[ 2427 \pm 6.62 \left( \frac{0.5\text{TeV}}{\Lambda} \right)^4 + 1.39 \left( \frac{0.5\text{TeV}}{\Lambda} \right)^8 \right] \text{fb}, \quad (3.26a)
\]

\[
\sqrt{s} = 500\text{ GeV}, \quad \sigma(Z\gamma) = \left[ 417 \pm 0.996 \left( \frac{0.8\text{TeV}}{\Lambda} \right)^4 + 0.624 \left( \frac{0.8\text{TeV}}{\Lambda} \right)^8 \right] \text{fb}, \quad (3.26b)
\]

\[
\sqrt{s} = 1\text{ TeV}, \quad \sigma(Z\gamma) = \left[ 94.0 \pm 0.404 \left( \frac{\text{TeV}}{\Lambda} \right)^4 + 1.73 \left( \frac{\text{TeV}}{\Lambda} \right)^8 \right] \text{fb}, \quad (3.26c)
\]

\[
\sqrt{s} = 3\text{ TeV}, \quad \sigma(Z\gamma) = \left[ 10.1 \pm 0.0252 \left( \frac{2\text{TeV}}{\Lambda} \right)^4 + 0.554 \left( \frac{2\text{TeV}}{\Lambda} \right)^8 \right] \text{fb}, \quad (3.26d)
\]

\[
\sqrt{s} = 5\text{ TeV}, \quad \sigma(Z\gamma) = \left[ 3.63 \pm 0.0103 \left( \frac{2.5\text{TeV}}{\Lambda} \right)^4 + 0.718 \left( \frac{2.5\text{TeV}}{\Lambda} \right)^8 \right] \text{fb}, \quad (3.26e)
\]

where for comparison we have also included the result from Eq.(3.19) for the case of \(\sqrt{s} = 3\text{ TeV}\). The above can be compared to the cross sections (3.3) with only a preliminary angular cut \(\delta > 0.2\) (corresponding to a lower cut on the photon transverse momentum \(P_T^\gamma = q \sin \delta\)). We see that under the final angular cuts on \((\theta, \theta_*, \phi_*)\) the
SM contribution is substantially reduced in each case, whereas the signal contributions at $\mathcal{O}(\Lambda^{-4})$ and $\mathcal{O}(\Lambda^{-8})$ are little changed.

Using the above, we analyze the signal significance up to $\mathcal{O}(\Lambda^{-8})$, for different collider energies. With inputs of the leptonic branching fraction $\text{Br}(Z\to\ell\ell) \simeq 0.10$ and an integrated luminosity $\mathcal{L} = 2 \text{ ab}^{-1}$, we arrive at

\[ \sqrt{s} = 250 \text{ GeV}, \quad Z_8 = \pm 1.90 \left( \frac{0.5 \text{ TeV}}{\Lambda} \right)^4 \pm 0.400 \left( \frac{0.5 \text{ TeV}}{\Lambda} \right)^8 \times \sqrt{\epsilon}, \quad (3.27a) \]

\[ \sqrt{s} = 500 \text{ GeV}, \quad Z_8 = \pm 0.689 \left( \frac{0.8 \text{ TeV}}{\Lambda} \right)^4 \pm 0.432 \left( \frac{0.8 \text{ TeV}}{\Lambda} \right)^8 \times \sqrt{\epsilon}, \quad (3.27b) \]

\[ \sqrt{s} = 1 \text{ TeV}, \quad Z_8 = \pm 0.589 \left( \frac{1 \text{ TeV}}{\Lambda} \right)^4 \pm 2.53 \left( \frac{1 \text{ TeV}}{\Lambda} \right)^8 \times \sqrt{\epsilon}, \quad (3.27c) \]

\[ \sqrt{s} = 3 \text{ TeV}, \quad Z_8 = \pm 0.112 \left( \frac{2 \text{ TeV}}{\Lambda} \right)^4 \pm 2.46 \left( \frac{2 \text{ TeV}}{\Lambda} \right)^8 \times \sqrt{\epsilon}, \quad (3.27d) \]

\[ \sqrt{s} = 5 \text{ TeV}, \quad Z_8 = \pm 0.0764 \left( \frac{2.5 \text{ TeV}}{\Lambda} \right)^4 \pm 5.32 \left( \frac{2.5 \text{ TeV}}{\Lambda} \right)^8 \times \sqrt{\epsilon}. \quad (3.27e) \]

From the above, we note that for the relevant reach of $\Lambda$, the $\mathcal{O}(\Lambda^{-4})$ terms give the dominant contributions for collision energies $\sqrt{s} < 1 \text{ TeV}$, while the $\mathcal{O}(\Lambda^{-8})$ terms become dominant for $\sqrt{s} \gtrsim 1 \text{ TeV}$. When $\mathcal{O}(\Lambda^{-8})$ becomes dominant at high energies, we have $\Lambda \propto Z_8^{-\frac{1}{2}}$. In such cases, the reach in $\Lambda$ becomes rather insensitive to the significance $Z_8$. For instance, at high energies $\sqrt{s} \gtrsim 1 \text{ TeV}$, we have $\Lambda(5\sigma)/\Lambda(2\sigma) \simeq 1/1.12$ for $Z_8$, whereas we previously found $\Lambda(5\sigma)/\Lambda(2\sigma) \simeq 1/1.26$ for $Z_4$.

At high energies $s \gtrsim (1 \text{ TeV})^2 \gg M_Z^2$, the $\mathcal{O}(\Lambda^{-8})$ terms become dominant, so we have the approximate relation

\[ Z_8 \propto \frac{M_Z^2(\sqrt{s})^5}{\Lambda^8} \sqrt{\mathcal{L} \times \epsilon}, \quad (3.28) \]

and hence

\[ \Lambda \propto \left( \frac{M_Z^2 \sqrt{\mathcal{L} \epsilon}}{Z_8} \right)^{\frac{1}{2}} (\sqrt{s})^{\frac{5}{2}}. \quad (3.29) \]

We see from Eqs.(3.25) and (3.29) that the sensitivity to $\Lambda$ increases with the collision energy with the power $(\sqrt{s})^{\frac{5}{2}}$ or $(\sqrt{s})^{\frac{3}{2}}$, a relatively slow rate of increase. We note also that the sensitivity to the new physics scale $\Lambda$ is rather insensitive to the integrated luminosity $\mathcal{L}$ and the detection efficiency $\epsilon$, owing to their small power-law dependence $(\mathcal{L} \epsilon)^{\frac{1}{16}}$.

Finally, we compute from Eqs.(3.23) and (3.27), the combined significance, $Z = \sqrt{Z_4^2 + Z_8^2}$, for each collider energy. With these, in Table 2 we present the corresponding combined sensitivity reaches to the new physics scale $\Lambda$ at different $e^+e^-$ collider energies. In the last row of this Table, the two numbers in the parentheses correspond
| $\sqrt{s}$ (GeV) | 250  | 500  | 1000 | 3000 | 5000 |
|----------------|------|------|------|------|------|
| $\Lambda_{2\sigma}$ (TeV) | 0.59(0.58) | 0.84(0.82) | 1.2  | 2.2  | 2.9  |
| $\Lambda_{5\sigma}$ (TeV)  | 0.48(0.46) | 0.68(0.65) | 1.0  | 1.9  | 2.6  |

Table 2. Combined sensitivity reaches to the new physics scale $\Lambda$ at the 2σ and 5σ levels, for different collider energies. Here the two numbers in the parentheses correspond to the case of the dimension-8 operator whose coefficient has a minus sign, while in all other entries the effects due to the coefficient having a minus sign are negligible. For illustration, we have input a fixed representative integrated luminosity $\mathcal{L} = 2$ ab$^{-1}$ and an ideal detection efficiency $\epsilon = 100\%$.

to the case of the dimension-8 operator whose coefficient has a minus sign, whereas in all other entries the effects due to the coefficient having a minus sign are negligible.

Before concluding this Section, we mention that we have performed a numerical Monte Carlo simulation based on the analytical formula (3.14). We used for this purpose CUDAlink in Mathematica, so as to exploit the CUDA parallel computing architecture on Graphical Processing Units (GPUs), which can generate millions of events in seconds. We have computed the probability density function of $\theta, \theta^*$ and $\phi^*$ for the case of $\sqrt{s} = 3$ TeV and $\Lambda = 2$ TeV. Eq.(3.3d) shows that the SM contribution dominates the total cross section. According to Eq.(3.15), we have $\mathcal{O}_1^c/\sigma_0^c \simeq 0.03$. For comparison, our Monte Carlo simulation yielded the following event counts: $|N_a - N_b| = 3054$ and $N_a + N_b = 104752$, corresponding to $|N_a - N_b|/(N_a + N_b) \simeq 0.029$. This agrees well with the ratio $\mathcal{O}_1^c/\sigma_0^c \simeq 0.03$ inferred from our analytical formula (3.14), serving as a consistency check on our Eq.(3.15). Our Monte Carlo simulation package may be used to generate other distributions and quantities that may be of interest for experiments.3

3.3. Non-Resonant Backgrounds

In the analyses so far, we have considered $e^-e^+ \rightarrow Z\gamma$ production with on-shell $Z$ decays ($Z \rightarrow \ell^{-}\ell^+$). This means that we have considered only the signal contribution Fig. 7(a) and the irreducible background Fig. 7(b). There are other (non-resonant) SM backgrounds with the same final state of $f\bar{f}\gamma$ ($f = \ell, \nu$), but having very different topology where $\gamma$ is either radiated from the final-state fermions [Fig. 7(c)-(d)], or from a $t$-channel $W$ boson [Fig. 7(e)] in the case of $Z \rightarrow \nu\bar{\nu}$ decay (which will be studied in Sec. 4). These backgrounds may give visible but small contributions after proper kinematic cuts.

For the backgrounds with an $e^-e^+\gamma$ final state, we have type (b) (with 4 diagrams), type (c) (with 4 diagrams) and type (d) (with 8 diagrams), for a total of 16 diagrams. For the other $f\bar{f}\gamma$ final states with $f = \mu, \tau$, we have type (b) (with 4 diagrams) and type (c) (with 8 diagrams), which amount to 12 diagrams. For the final state $\nu\bar{\nu}\gamma$,

---

3Further details may be obtained from RQX.
Figure 7. Five types of Feynman diagrams which contribute to the process \( e^- e^+ \rightarrow f \bar{f} \gamma \) (with \( f = \ell, \nu \)). Type (a) is our signal with a \( Z^* Z \gamma \) vertex solely from the dimension-8 operator (2.2a), and types (b),(c),(d),(e) are the SM backgrounds. The type (b) process \( e^- e^+ \rightarrow V \gamma \rightarrow \gamma f \bar{f} \) (with \( V = Z, \gamma^* \)) gives an irreducible background (and there is a similar \( u \)-channel diagram). Type (c) is \( s \)-channel gauge boson exchange \((V = Z^*, \gamma^*)\) with final-state \( \gamma \) radiation. Type (d) is \( t \)-channel \( V \) exchange \((V = Z^*, \gamma^*)\) with the final-state \( \gamma \) radiated from \( e^\pm \) (in the final or initial state). Type (e) is for the \( \nu \bar{\nu} \gamma \) final state with \( t \)-channel \( W^* \)-exchange and the \( \gamma \) radiated from either the \( W^* \) or the initial-state \( e^\pm \).

we have the SM backgrounds from type (b)(with 2 diagrams) and type (e) (with 3 diagrams). For each of these diagrams, there are \( 2^3 = 8 \) helicity combinations. Rather than writing explicitly the cross sections for all these combinations and calculating them analytically, we have computed the cross section numerically using a Monte Carlo method. The relative accuracy of numerical Monte Carlo integration is \( O(1/\sqrt{N}) \), where \( N \) is the number of samples, and one needs a large sample to obtain precise results. We use FeynArts [24] to generate all the background diagrams and then compute them by FeynCalc [25, 26]. Finally, we convert the expressions to \( C \) form and use CUDAlink for numerical integrations to compute the cross sections and other observables. All steps are done in Mathematica and the numerical computation speed is \( O(10^7) \) diagrams/s.

The diagrams of Fig. 7(c)-(d) have additional soft and collinear divergences, which can be removed by imposing lower cuts on the photon transverse momentum \( P_T^\gamma > 0.2 P_T^e \) and on the lepton-photon invariant mass \( M(\ell \gamma) > 0.1 \sqrt{s} \). We further require \( |M(\ell \ell) - M_Z| < 10 \text{GeV} \) so as to be close to the \( Z \) boson mass-shell. Applying these cuts together with those in Table 1, we first compute the observables for the reaction channel \( e^- e^+ \rightarrow \gamma e^- e^+ \) by including the additional backgrounds in Fig. 7(c)-(d),

\[
\sqrt{s} = 250 \text{GeV}, \quad (\sigma_{e^0 e^+}^0, \sigma_{e^e}^{1e}) = \left( 141, 0.0256 \left( \frac{\text{TeV}}{\Lambda} \right)^4 \right) \text{fb}, \quad (3.30a)
\]

\[
\text{We to perform the numerical integrations using GPU parallel computing, which is faster than standard CPU computing by a factor of } O(10^3). \text{ Further details may be obtained from RQX.} \]
\[
\sqrt{s} = 500 \text{ GeV}, \quad (\sigma^{0e}_c, \mathcal{O}^{1e}_c) = \left(26.6, 0.0524 \left(\frac{\text{TeV}}{\Lambda}\right)^4\right) \text{ fb}, \quad (3.30b)
\]
\[
\sqrt{s} = 1 \text{ TeV}, \quad (\sigma^{0e}_c, \mathcal{O}^{1e}_c) = \left(6.15, 0.109 \left(\frac{\text{TeV}}{\Lambda}\right)^4\right) \text{ fb}, \quad (3.30c)
\]
\[
\sqrt{s} = 3 \text{ TeV}, \quad (\sigma^{0e}_c, \mathcal{O}^{1e}_c) = \left(0.691, 0.340 \left(\frac{\text{TeV}}{\Lambda}\right)^4\right) \text{ fb}, \quad (3.30d)
\]
\[
\sqrt{s} = 5 \text{ TeV}, \quad (\sigma^{0e}_c, \mathcal{O}^{1e}_c) = \left(0.250, 0.567 \left(\frac{\text{TeV}}{\Lambda}\right)^4\right) \text{ fb}. \quad (3.30e)
\]

We then derive the following signal significance \(Z^e_4\) at each collision energy, assuming an integrated luminosity \(L = 2 \text{ ab}^{-1}\):

\[
\sqrt{s} = 250 \text{ GeV}, \quad Z^e_4 = 1.54 \left(\frac{0.5 \text{ TeV}}{\Lambda}\right)^4 \times \sqrt{\epsilon}, \quad (3.31a)
\]
\[
\sqrt{s} = 500 \text{ GeV}, \quad Z^e_4 = 1.11 \left(\frac{0.8 \text{ TeV}}{\Lambda}\right)^4 \times \sqrt{\epsilon}, \quad (3.31b)
\]
\[
\sqrt{s} = 1 \text{ TeV}, \quad Z^e_4 = 1.97 \left(\frac{\text{TeV}}{\Lambda}\right)^4 \times \sqrt{\epsilon}, \quad (3.31c)
\]
\[
\sqrt{s} = 3 \text{ TeV}, \quad Z^e_4 = 1.14 \left(\frac{2 \text{TeV}}{\Lambda}\right)^4 \times \sqrt{\epsilon}, \quad (3.31d)
\]
\[
\sqrt{s} = 5 \text{ TeV}, \quad Z^e_4 = 1.30 \left(\frac{2.5 \text{TeV}}{\Lambda}\right)^4 \times \sqrt{\epsilon}. \quad (3.31e)
\]

Next, we extend the analysis of Sec. 3.2.2 by including the \(O(\Lambda^{-8})\) contributions and the additional backgrounds in Fig. 7(c)-(d). Thus, we arrive at

\[
\sqrt{s} = 250 \text{ GeV}, \quad \sigma(ee\gamma) = \left[85.0 \pm 0.20 \left(\frac{0.5 \text{ TeV}}{\Lambda}\right)^4 + 0.0418 \left(\frac{0.5 \text{ TeV}}{\Lambda}\right)^8\right] \text{ fb}, \quad (3.32a)
\]
\[
\sqrt{s} = 500 \text{ GeV}, \quad \sigma(ee\gamma) = \left[13.6 \pm 0.32 \left(\frac{0.8 \text{ TeV}}{\Lambda}\right)^4 + 0.0192 \left(\frac{0.8 \text{ TeV}}{\Lambda}\right)^8\right] \text{ fb}, \quad (3.32b)
\]
\[
\sqrt{s} = 1 \text{ TeV}, \quad \sigma(ee\gamma) = \left[3.03 \pm 0.13 \left(\frac{\text{TeV}}{\Lambda}\right)^4 + 0.0536 \left(\frac{\text{TeV}}{\Lambda}\right)^8\right] \text{ fb}, \quad (3.32c)
\]
\[
\sqrt{s} = 3 \text{ TeV}, \quad \sigma(ee\gamma) = \left[0.325 \pm 0.0008 \left(\frac{2 \text{TeV}}{\Lambda}\right)^4 + 0.0172 \left(\frac{2 \text{TeV}}{\Lambda}\right)^8\right] \text{ fb}, \quad (3.32d)
\]
\[
\sqrt{s} = 5 \text{ TeV}, \quad \sigma(ee\gamma) = \left[0.116 \pm 0.0004 \left(\frac{2.5 \text{TeV}}{\Lambda}\right)^4 + 0.0222 \left(\frac{2.5 \text{TeV}}{\Lambda}\right)^8\right] \text{ fb}. \quad (3.32e)
\]

With these, we derive the signal significance \(Z^e_8\) for the \(ee\gamma\) channel,

\[
\sqrt{s} = 250 \text{ GeV}, \quad Z^e_8 = \left|\pm 0.96 \left(\frac{0.5 \text{ TeV}}{\Lambda}\right)^4 + 0.203 \left(\frac{0.5 \text{ TeV}}{\Lambda}\right)^8\right| \times \sqrt{\epsilon}, \quad (3.33a)
\]
\[ \sqrt{s} = 500 \text{ GeV}, \quad Z^\mu_8 = |\pm 0.38 \left( \frac{0.8 \text{TeV}}{\Lambda} \right)^4 + 0.232 \left( \frac{0.8 \text{TeV}}{\Lambda} \right)^8 | \times \sqrt{\epsilon}, \quad (3.33b) \]

\[ \sqrt{s} = 1 \text{ TeV}, \quad Z^\mu_8 = |\pm 0.31 \left( \frac{\text{TeV}}{\Lambda} \right)^4 + 1.38 \left( \frac{\text{TeV}}{\Lambda} \right)^8 | \times \sqrt{\epsilon}, \quad (3.33c) \]

\[ \sqrt{s} = 3 \text{ TeV}, \quad Z^\mu_8 = |\pm 0.06 \left( \frac{2 \text{TeV}}{\Lambda} \right)^4 + 1.35 \left( \frac{2 \text{TeV}}{\Lambda} \right)^8 | \times \sqrt{\epsilon}, \quad (3.33d) \]

\[ \sqrt{s} = 5 \text{ TeV}, \quad Z^\mu_8 = |\pm 0.03 \left( \frac{2.5 \text{TeV}}{\Lambda} \right)^4 + 2.90 \left( \frac{2.5 \text{TeV}}{\Lambda} \right)^8 | \times \sqrt{\epsilon}. \quad (3.33e) \]

Then, we analyze the reaction channel \( e^- e^+ \rightarrow \gamma \mu^- \mu^+ \) under the same cuts and including the additional backgrounds as in Fig. 7(c). With these we obtain the following,

\[ \sqrt{s} = 250 \text{ GeV}, \quad \sigma_{0\mu}^{\gamma\mu} = \left( 112, 0.0256 \left( \frac{\text{TeV}}{\Lambda} \right)^4 \right) \text{ fb}, \quad (3.34a) \]

\[ \sqrt{s} = 500 \text{ GeV}, \quad \sigma_{0\mu}^{\gamma\mu} = \left( 24.1, 0.0522 \left( \frac{\text{TeV}}{\Lambda} \right)^4 \right) \text{ fb}, \quad (3.34b) \]

\[ \sqrt{s} = 1 \text{ TeV}, \quad \sigma_{0\mu}^{\gamma\mu} = \left( 6.00, 0.109 \left( \frac{\text{TeV}}{\Lambda} \right)^4 \right) \text{ fb}, \quad (3.34c) \]

\[ \sqrt{s} = 3 \text{ TeV}, \quad \sigma_{0\mu}^{\gamma\mu} = \left( 0.687, 0.340 \left( \frac{\text{TeV}}{\Lambda} \right)^4 \right) \text{ fb}, \quad (3.34d) \]

\[ \sqrt{s} = 5 \text{ TeV}, \quad \sigma_{0\mu}^{\gamma\mu} = \left( 0.250, 0.567 \left( \frac{\text{TeV}}{\Lambda} \right)^4 \right) \text{ fb}. \quad (3.34e) \]

Thus, we derive the signal significance \( Z^\mu_4 \) at each collision energy and with an integrated luminosity \( L = 2 \text{ ab}^{-1} \),

\[ \sqrt{s} = 250 \text{ GeV}, \quad Z^\mu_4 = 1.74 \left( \frac{0.5 \text{TeV}}{\Lambda} \right)^4 \times \sqrt{\epsilon}, \quad (3.35a) \]

\[ \sqrt{s} = 500 \text{ GeV}, \quad Z^\mu_4 = 1.16 \left( \frac{0.8 \text{TeV}}{\Lambda} \right)^4 \times \sqrt{\epsilon}, \quad (3.35b) \]

\[ \sqrt{s} = 1 \text{ TeV}, \quad Z^\mu_4 = 1.99 \left( \frac{\text{TeV}}{\Lambda} \right)^4 \times \sqrt{\epsilon}, \quad (3.35c) \]

\[ \sqrt{s} = 3 \text{ TeV}, \quad Z^\mu_4 = 1.14 \left( \frac{2 \text{TeV}}{\Lambda} \right)^4 \times \sqrt{\epsilon}, \quad (3.35d) \]

\[ \sqrt{s} = 5 \text{ TeV}, \quad Z^\mu_4 = 1.30 \left( \frac{2.5 \text{TeV}}{\Lambda} \right)^4 \times \sqrt{\epsilon}. \quad (3.35e) \]

Next, similar to Eq.(3.32) for the \( ee\gamma \) channel, we compute the cross sections of the \( \mu\mu\gamma \) channel, including the \( O(\Lambda^{-8}) \) contributions and the additional backgrounds in Fig. 7(c). Thus, we arrive at

\[ \sqrt{s} = 250 \text{ GeV}, \quad \sigma(\mu\mu\gamma) = 75.8 \pm 0.20 \left( \frac{0.5 \text{TeV}}{\Lambda} \right)^4 + 0.0418 \left( \frac{0.5 \text{TeV}}{\Lambda} \right)^8 \text{ fb}, \quad (3.36a) \]
| $\sqrt{s}$ (GeV) | 250   | 500   | 1000  | 3000  | 5000  |
|------------------|-------|-------|-------|-------|-------|
| $\Lambda_\ell^2$ (TeV) | 0.57(0.56) | 0.82(0.80) | 1.2 | 2.1 | 2.9 |
| $\Lambda_\ell^5$ (TeV) | 0.46(0.44) | 0.67(0.64) | 0.98(0.95) | 1.9 | 2.5 |

Table 3. Sensitivity reaches of the new physics scale $\Lambda$ from the $\ell^-\ell^+\gamma$ channel, including additional backgrounds [Fig.7(c)-(d)], at the $2\sigma$ and $5\sigma$ levels, for different collider energies. The numbers in the parentheses correspond to the case of the dimension-8 operator whose coefficient is negative, while in the other entries the sensitivities for the two signs of the coefficient are indistinguishable. These results were obtained assuming a fixed representative integrated luminosity $L = 2\text{ab}^{-1}$ and an ideal detection efficiency $\epsilon = 100\%$.

\[ \sqrt{s} = 500 \text{ GeV}, \quad \sigma(\mu\mu\gamma) = \left[ 13.2 \pm 0.31 \left( \frac{0.8\text{TeV}}{\Lambda} \right)^4 + 0.0192 \left( \frac{0.8\text{TeV}}{\Lambda} \right)^8 \right] \text{fb}, \quad (3.36b) \]
\[ \sqrt{s} = 1 \text{ TeV}, \quad \sigma(\mu\mu\gamma) = \left[ 3.02 \pm 0.013 \left( \frac{\text{TeV}}{\Lambda} \right)^4 + 0.0536 \left( \frac{\text{TeV}}{\Lambda} \right)^8 \right] \text{fb}, \quad (3.36c) \]
\[ \sqrt{s} = 3 \text{ TeV}, \quad \sigma(\mu\mu\gamma) = \left[ 0.325 \pm 0.0008 \left( \frac{2\text{TeV}}{\Lambda} \right)^4 + 0.0172 \left( \frac{2\text{TeV}}{\Lambda} \right)^8 \right] \text{fb}, \quad (3.36d) \]
\[ \sqrt{s} = 5 \text{ TeV}, \quad \sigma(\mu\mu\gamma) = \left[ 0.116 \pm 0.0004 \left( \frac{2.5\text{TeV}}{\Lambda} \right)^4 + 0.0222 \left( \frac{2.5\text{TeV}}{\Lambda} \right)^8 \right] \text{fb}. \quad (3.36e) \]

With these, we obtain the signal significance $Z_8^\mu$ for the $\mu\mu\gamma$ channel,

\[ \sqrt{s} = 250 \text{ GeV}, \quad Z_8^\mu = \left| \pm 1.0 \left( \frac{0.5\text{TeV}}{\Lambda} \right)^4 + 0.215 \left( \frac{0.5\text{TeV}}{\Lambda} \right)^8 \right| \times \sqrt{\epsilon}, \quad (3.37a) \]
\[ \sqrt{s} = 500 \text{ GeV}, \quad Z_8^\mu = \left| \pm 0.39 \left( \frac{0.8\text{TeV}}{\Lambda} \right)^4 + 0.236 \left( \frac{0.8\text{TeV}}{\Lambda} \right)^8 \right| \times \sqrt{\epsilon}, \quad (3.37b) \]
\[ \sqrt{s} = 1 \text{ TeV}, \quad Z_8^\mu = \left| \pm 0.32 \left( \frac{\text{TeV}}{\Lambda} \right)^4 + 1.38 \left( \frac{\text{TeV}}{\Lambda} \right)^8 \right| \times \sqrt{\epsilon}, \quad (3.37c) \]
\[ \sqrt{s} = 3 \text{ TeV}, \quad Z_8^\mu = \left| \pm 0.06 \left( \frac{2\text{TeV}}{\Lambda} \right)^4 + 1.35 \left( \frac{2\text{TeV}}{\Lambda} \right)^8 \right| \times \sqrt{\epsilon}, \quad (3.37d) \]
\[ \sqrt{s} = 5 \text{ TeV}, \quad Z_8^\mu = \left| \pm 0.03 \left( \frac{2.5\text{TeV}}{\Lambda} \right)^4 + 2.90 \left( \frac{2.5\text{TeV}}{\Lambda} \right)^8 \right| \times \sqrt{\epsilon}. \quad (3.37e) \]

The analysis of the $\tau\tau\gamma$ channel is the same as that of the $\mu\mu\gamma$ channel, since the $\mu$ and $\tau$ masses are negligible compared to the collision energy $\sqrt{s}$ and the $Z$ boson mass $M_Z$. Finally, we obtain the combined signal significance:

\[ Z_{\ell\ell} = \sqrt{(Z_4^\ell)^2 + (Z_4^\mu)^2 + (Z_4^\tau)^2 + (Z_6^\ell)^2 + (Z_8^\mu)^2 + (Z_8^\tau)^2}, \quad (3.38) \]
where $Z_4^+ \simeq Z_4^-$ and $Z_8^+ \simeq Z_8^-$. By requiring the signal significance $Z_{\ell\ell} = 2, 5$, we derive the 2σ and 5σ bounds on the corresponding new physics scale $\Lambda = \Lambda_{\ell\ell}^{2\sigma}, \Lambda_{\ell\ell}^{5\sigma}$. We present these bounds in Table 3. In comparison with Table 2, we see that the refinements on the $\Lambda$ bounds are rather minor, so the results of Sec. 3.2 are little affected. This is because the additional non-resonant backgrounds in Fig. 7 can be sufficiently suppressed by kinematic cuts on the photon transverse momentum and the invariant mass of lepton pair. Furthermore, Eqs. (3.25) and (3.29) indicate the relation $\Lambda \propto Z_4^{-1/4} \propto (\sigma_0^c)^{1/8}$ when the $\Lambda^{-4}$ contribution dominates the signal, and the relation $\Lambda \propto Z_8^{-1/8} \propto (\sigma_0^c)^{1/16}$ when the $\Lambda^{-8}$ contribution dominates the signal, which are insensitive to the change of the background cross section $\sigma_0^c$.

Finally, we present in Fig. 8 reaches for the new physics scale $\Lambda$ as functions of the $e^+e^-$ collision energy $\sqrt{s}$. In Fig. 8(a), we show the $\Lambda$ reaches for the signal significances ($Z_4$, $Z_8$) at 2σ and 5σ levels, respectively. Then, in Fig. 8(b), we depict the combined sensitivity $Z = \sqrt{Z_4^2 + Z_8^2} = (2, 3, 5)\sigma$, shown as the (red, green, blue) curves. For reference, we also show two lines $\Lambda = \sqrt{s}$ and $\Lambda = \sqrt{s}/2$ in each plot, since the effective field theory description may be expected to hold when $\Lambda > \sqrt{s}$ or $\Lambda > \sqrt{s}/2$.

5In the effective theory approach, the exact relation between the cutoff scale $\Lambda$ and the mass $M$ of the lowest underlying new state $X$ is unknown, and one expects $M/\Lambda = O(1)$. If the new state $X$ could only be produced in pairs in the $e^+e^-$ collisions (e.g., the production of dark matter particles), $\sqrt{s}/2$ would be the appropriate comparison scale for $M$ when considering applicability of the effective field theory.
Figure 9. Normalized angular distributions for the scattering angle $\theta$ in the lab frame for different collision energies, $\sqrt{s} = (250 \text{ GeV}, 500 \text{ GeV}, 1 \text{ TeV}, 3 \text{ TeV})$. In each plot, the black curve denotes the SM contribution and the red and blue curves present the contributions from new physics with two sample values of $\Lambda$.

4. Analysis of Invisible Decay Channels $Z \rightarrow \nu \bar{\nu}$

In this section, we analyze $e^- e^+ \rightarrow Z \gamma$ production followed by the invisible decays $Z \rightarrow \nu \bar{\nu}$. Then, we combine its sensitivity with that of the leptonic channels $Z \rightarrow \ell^- \ell^+$ presented in Sec. 3.

In the case of the invisible decay channels $Z \rightarrow \nu \bar{\nu}$, we can only apply the angular cut on the scattering angle of the final state mono-photon, $\theta > \delta_m$, which corresponds to a cut on the photon transverse momentum $P_T^\gamma = q \sin \theta > q \sin \delta_m$. The angular distributions of $\theta$ are presented in Fig. 9. We estimate the following optimal cuts on the scattering angle of photon:

$$\theta > \delta_m = (0.633, 0.626, 0.624, 0.623, 0.623),$$

(4.1)

for various collider energies $\sqrt{s} = (250, 500, 1000, 3000, 5000) \text{ GeV}$, respectively. We see that the optimal angular cut $\theta > \delta_m$ is not very sensitive to the variation of collider energy $\sqrt{s}$. Using the cut $\theta > \delta_m$, we derive the following $Z \gamma$ production cross sections
With these, we derive the signal significance $\sqrt{s}$ being more sensitive for significances of these two types of channels are comparable, sample new physics scale $\Lambda$, as shown in the second row. We see that the signal channels $Z\sqrt{s}$ sensitive for $\sqrt{s}$ in the first row), for the dilepton channels

$$ \sqrt{s} = 250 \text{ GeV}, \quad \sigma(Z\gamma) = \left[ 3236 \pm 7.25 \left( \frac{0.5 \text{ TeV}}{\Lambda} \right)^4 + 1.53 \left( \frac{0.5 \text{ TeV}}{\Lambda} \right)^8 \right] \text{ fb}, \quad (4.2a) $$

$$ \sqrt{s} = 500 \text{ GeV}, \quad \sigma(Z\gamma) = \left[ 656 \pm 1.13 \left( \frac{0.8 \text{ TeV}}{\Lambda} \right)^4 + 0.709 \left( \frac{0.8 \text{ TeV}}{\Lambda} \right)^8 \right] \text{ fb}, \quad (4.2b) $$

$$ \sqrt{s} = 1 \text{ TeV}, \quad \sigma(Z\gamma) = \left[ 156 \pm 0.465 \left( \frac{\text{TeV}}{\Lambda} \right)^4 + 2.00 \left( \frac{\text{TeV}}{\Lambda} \right)^8 \right] \text{ fb}, \quad (4.2c) $$

$$ \sqrt{s} = 3 \text{ TeV}, \quad \sigma(Z\gamma) = \left[ 17.1 \pm 0.0291 \left( \frac{2 \text{ TeV}}{\Lambda} \right)^4 + 0.640 \left( \frac{2 \text{ TeV}}{\Lambda} \right)^8 \right] \text{ fb}, \quad (4.2d) $$

$$ \sqrt{s} = 5 \text{ TeV}, \quad \sigma(Z\gamma) = \left[ 6.15 \pm 0.0119 \left( \frac{2.5 \text{ TeV}}{\Lambda} \right)^4 + 0.830 \left( \frac{2.5 \text{ TeV}}{\Lambda} \right)^8 \right] \text{ fb}. \quad (4.2e) $$

With these, we derive the signal significance $Z_8$ as follows,

$$ \sqrt{s} = 250 \text{ GeV}, \quad Z_{8\nu\bar{\nu}} = \left| \pm 2.55 \left( \frac{0.5 \text{ TeV}}{\Lambda} \right)^4 + 0.538 \left( \frac{0.5 \text{ TeV}}{\Lambda} \right)^8 \right| \times \sqrt{\epsilon}, \quad (4.3a) $$

$$ \sqrt{s} = 500 \text{ GeV}, \quad Z_{8\nu\bar{\nu}} = \left| \pm 0.884 \left( \frac{0.8 \text{ TeV}}{\Lambda} \right)^4 + 0.554 \left( \frac{0.8 \text{ TeV}}{\Lambda} \right)^8 \right| \times \sqrt{\epsilon}, \quad (4.3b) $$

$$ \sqrt{s} = 1 \text{ TeV}, \quad Z_{8\nu\bar{\nu}} = \left| \pm 0.744 \left( \frac{\text{TeV}}{\Lambda} \right)^4 + 3.20 \left( \frac{\text{TeV}}{\Lambda} \right)^8 \right| \times \sqrt{\epsilon}, \quad (4.3c) $$

$$ \sqrt{s} = 3 \text{ TeV}, \quad Z_{8\nu\bar{\nu}} = \left| \pm 0.141 \left( \frac{2 \text{ TeV}}{\Lambda} \right)^4 + 3.10 \left( \frac{2 \text{ TeV}}{\Lambda} \right)^8 \right| \times \sqrt{\epsilon}, \quad (4.3d) $$

$$ \sqrt{s} = 5 \text{ TeV}, \quad Z_{8\nu\bar{\nu}} = \left| \pm 0.0961 \left( \frac{2.5 \text{ TeV}}{\Lambda} \right)^4 + 6.69 \left( \frac{2.5 \text{ TeV}}{\Lambda} \right)^8 \right| \times \sqrt{\epsilon}. \quad (4.3e) $$

In Table 4, we present the signal significances at the different collider energies (shown in the first row), for the dilepton channels $Z \to \ell^- \ell^+$ (third row) and for the invisible channels $Z \to \nu \bar{\nu}$ (fourth row). For each collider energy $\sqrt{s}$, we input the relevant sample new physics scale $\Lambda$, as shown in the second row. We see that the signal significances of these two types of channels are comparable, with the dilepton channels being more sensitive for $\sqrt{s} \lesssim 1.5$ TeV, whereas the invisible channels become more sensitive for $\sqrt{s} \gtrsim 1.5$ TeV. We present the combined signal significance $Z_{\text{combined}} = \sqrt{Z_{8\nu\bar{\nu}}^2 + Z_{8\nu\bar{\nu}}^2}$, in the last row for each given collision energy. This shows that in each case the combined sensitivity is enhanced over the individual channels by a sizeable factor of about $1.3 - 1.4$. As previously, the numbers in the parentheses of Table 4 correspond to the case of the dimension-8 operator with a negative coefficient. For illustration, we have assumed a fixed representative integrated luminosity $\mathcal{L} = 2 \text{ ab}^{-1}$ and an ideal detection efficiency $\epsilon = 100\%$.

---

6The dilepton channel results shown here are based on the analysis of Sec. 3.3.
Table 4. Signal significances for the dilepton channels (3rd row) and invisible channels (4th row) at different collider energies (shown in the 1st row). For each collider energy $\sqrt{s}$, we assume a representative new physics scale $\Lambda$ (shown in the 2nd row). The combined signal significance $\mathcal{Z}_{\text{combined}}$ for each collision energy is presented in the last row. Here the numbers in the parentheses correspond to the case of the dimension-8 operator whose coefficient is negative. For illustration, we have assumed a fixed representative integrated luminosity $L = 2 \text{ ab}^{-1}$ and an ideal detection efficiency $\epsilon = 100\%$.

| $\sqrt{s}$ (GeV) | 250 | 500 | 1000 | 3000 | 5000 |
|-----------------|-----|-----|------|------|------|
| $\Lambda$ (TeV) | 0.5 | 0.7 | 1.0  | 1.9  | 2.6  |
| $\mathcal{Z}_{\ell\ell}$ | 3.6(3.2) | 4.1(3.4) | 4.5(3.9) | 4.4(4.2) | 4.2(4.1) |
| $\mathcal{Z}_{8,\nu\bar{\nu}}$ | 3.1(2.0) | 3.1(0.10) | 3.9(2.5) | 4.8(4.5) | 5.0(4.8) |
| $\mathcal{Z}_{\text{combined}}$ | 4.7(3.8) | 5.2(3.4) | 6.0(4.6) | 6.5(6.1) | 6.5(6.3) |

Table 5. Sensitivity reaches for the new physics scale $\Lambda$ from the $e^-e^+\rightarrow \nu\bar{\nu}\gamma$ channel, and from combining both $\ell^-\ell^+\gamma$ and $\nu\bar{\nu}\gamma$ channels, at the $2\sigma$ and $5\sigma$ levels, for different collider energies. Here again the numbers in parentheses correspond to the case of the dimension-8 operator whose coefficient has a minus sign, while in all other entries the differences between the two signs of the coefficient are negligible. For illustration, we have assumed a fixed representative integrated luminosity $L = 2 \text{ ab}^{-1}$ and an ideal detection efficiency $\epsilon = 100\%$.

| $\sqrt{s}$ (GeV) | 250 | 500 | 1000 | 3000 | 5000 |
|-----------------|-----|-----|------|------|------|
| $\Lambda^{2\sigma}_{\ell\ell}$ (TeV) | 0.57(0.56) | 0.82(0.80) | 1.2 | 2.1 | 2.9 |
| $\Lambda^{5\sigma}_{\ell\ell}$ (TeV) | 0.46(0.44) | 0.67(0.64) | 0.98(0.95) | 1.9 | 2.5 |
| $\Lambda^{2\sigma}_{\nu\nu}$ (TeV) | 0.55(0.32) | 0.75(0.62) | 1.1 | 2.1 | 2.9 |
| $\Lambda^{5\sigma}_{\nu\nu}$ (TeV) | 0.45(0.32) | 0.65(0.57) | 0.97(0.93) | 1.9 | 2.6 |
| $\Lambda^{2\sigma}_{\ell\nu,\text{comb}}$ (TeV) | 0.61(0.59) | 0.85(0.80) | 1.2 | 2.3 | 3.0 |
| $\Lambda^{5\sigma}_{\ell\nu,\text{comb}}$ (TeV) | 0.49(0.46) | 0.70(0.64) | 1.0 | 2.0 | 2.7 |

By requiring $Z_8 = 2$ and $Z_8 = 5$ in Eq.(4.3), we derive the reaches for the new physics scale $\Lambda$ at the $2\sigma$ and $5\sigma$ levels, denoted as $\Lambda^{2\sigma}_{\ell\ell}$ and $\Lambda^{5\sigma}_{\ell\ell}$, respectively. We summarize the findings in Table 5, as shown in the fourth and fifth rows. For comparison, we also list the new physics reaches $\Lambda^{2\sigma}_{\nu\nu}$ and $\Lambda^{5\sigma}_{\nu\nu}$ (second and third rows of Table 5) from the dilepton channels of Table 3. Then, we derive the combined sensitivity reaches of the new physics scale $\Lambda$ from both the dilepton channels and invisible channels, which are presented in the sixth and seventh rows of the current Table 5, denoted as $\Lambda^{2\sigma}_{\ell\nu,\text{comb}}$ and $\Lambda^{5\sigma}_{\ell\nu,\text{comb}}$. We see that the combined bounds $\Lambda^{2\sigma}_{\ell\nu,\text{comb}}$ and $\Lambda^{5\sigma}_{\ell\nu,\text{comb}}$ are only slightly enhanced compared to the analysis using the $Z \rightarrow \ell^+\ell^-$ channel alone,
which can be understood by noting that the new physics scale $\Lambda$ is rather insensitive to the significance $Z$. This is because Eqs.(3.25) and (3.29) show, $\Lambda \propto Z^{-\frac{3}{4}}$ (when the $\Lambda^{-4}$ contribution dominates the signal) and $\Lambda \propto Z_{s}^{-\frac{1}{3}}$ (when the $\Lambda^{-8}$ contribution dominates the signal).

5. Improvements from $e^{\mp}$ Beam Polarizations

In this section we extend our analysis to include the effects of initial-state electron/positron polarizations, and demonstrate how the sensitivity reaches of physics scale $\Lambda$ can be improved.

The leading contribution to the differential cross section at $\mathcal{O}(\Lambda^{-4})$ is proportional to

$$
\Re\left[\mathcal{T}(0 \pm) \mathcal{T}^*(\mp \pm)\right] \sin \theta \sin \theta_s
$$

(5.1)

where $(e_L, e_R) = (c_L \delta_{\sigma,-\frac{1}{2}}, c_R \delta_{\sigma,\frac{1}{2}})$ are the $Z$ gauge couplings to the (left, right)-handed electrons (with the index $s = \mp \frac{1}{2}$ denoting the initial-state electron helicities). The final-state $Z$ boson decays into leptons $\ell^{-}\ell^{+}$ with couplings $(f_L, f_R) = (c_L \delta_{\sigma,-\frac{1}{2}}, c_R \delta_{\sigma,\frac{1}{2}})$, where $\sigma$ denotes the helicity of the massless lepton $\ell^{\mp}$ and $(c_L, c_R) = (s_{W}^{2} - \frac{1}{2}, s_{W}^{2})$ give the $Z$ gauge couplings to the (left, right)-handed leptons. We note from the right-hand-side (RHS) of Eq.(5.1) that for unpolarized initial states $e^{\mp}$ the observable $\mathcal{O}_e^1$ is suppressed by the coupling factor $\propto f_{L}^{2} - f_{R}^{2} \propto \frac{1}{4} - \sin^{2}\theta_{W}$ in the first term, and the second term is suppressed by the coupling factor $e_{L}^{2} - e_{R}^{2}$ plus the factors $\cos \theta \cos \theta_s$ which can be either positive or negative. If the initial-state $e^{\mp}$ are polarized, we can largely remove the suppressions in the second term of the RHS of Eq.(5.1), since the coupling factor $e_{L}^{2} - e_{R}^{2}$ is replaced by $e_{L}^{2} (or e_{R}^{2})$ in the fully-polarized case, and the factors $\cos \theta \cos \theta_s$ can be made positive by defining $\mathcal{O}_e^1$ appropriately.

In the ideal case of a fully left-polarized $e^{-}$ beam, we can redefine $\mathcal{O}_e^1$ as follows:

$$
\mathcal{O}_e^1 = \left|\sigma_1 \int d\theta d\theta_s d\phi_s dM_s f^L \text{sign}(\cos \theta) \text{sign}(\cos \theta_s) \text{sign}(\cos \phi_s)\right|,
$$

(5.2a)

$$
f_j = \frac{d^4 \sigma_j}{\sigma_j d\theta d\theta_s d\phi_s dM_s},
$$

(5.2b)

As can readily be seen, in this case the first term on the RHS of Eq.(5.1) gives zero contribution to the observable $\mathcal{O}_e^1$. Thus, $\mathcal{O}_e^1$ is dominated by the leading contributions of the second term on the RHS of Eq.(5.1), and is proportional to $e_{L}^{2}(c_{L}^{2} + c_{R}^{2})$ rather than $(c_{L}^{2} + c_{R}^{2})(c_{L}^{2} - c_{R}^{2})$. Thus, at different collider energies, we can derive the following signal significance $Z_{e}^\pm$ for the final state $e^{-}e^{+}\gamma$,

$$
\sqrt{s} = 250 \text{ GeV}, \quad Z_{e}^\pm = 4.46 \left(\frac{0.5 \text{ TeV}^{-4}}{\Lambda}\right) \times \sqrt{\epsilon},
$$

(5.3a)
\[ \sqrt{s} = 500 \text{ GeV}, \quad \mathcal{Z}_4^e = 3.64 \left( \frac{0.8 \text{TeV}}{\Lambda} \right)^4 \times \sqrt{\epsilon}, \quad (5.3b) \]
\[ \sqrt{s} = 1 \text{ TeV}, \quad \mathcal{Z}_4^e = 6.40 \left( \frac{\text{TeV}^4}{\Lambda} \right) \times \sqrt{\epsilon}, \quad (5.3c) \]
\[ \sqrt{s} = 3 \text{ TeV}, \quad \mathcal{Z}_4^e = 3.80 \left( \frac{2 \text{TeV}^4}{\Lambda} \right) \times \sqrt{\epsilon}, \quad (5.3d) \]
\[ \sqrt{s} = 5 \text{ TeV}, \quad \mathcal{Z}_4^e = 4.32 \left( \frac{2.5 \text{TeV}^4}{\Lambda} \right) \times \sqrt{\epsilon}; \quad (5.3e) \]

for the final state \( \mu^- \mu^+ \gamma \),
\[ \sqrt{s} = 250 \text{ GeV}, \quad \mathcal{Z}_4^\mu = 4.86 \left( \frac{0.5 \text{TeV}^4}{\Lambda} \right) \times \sqrt{\epsilon}, \quad (5.4a) \]
\[ \sqrt{s} = 500 \text{ GeV}, \quad \mathcal{Z}_4^\mu = 3.78 \left( \frac{0.8 \text{TeV}^4}{\Lambda} \right) \times \sqrt{\epsilon}, \quad (5.4b) \]
\[ \sqrt{s} = 1 \text{ TeV}, \quad \mathcal{Z}_4^\mu = 6.47 \left( \frac{\text{TeV}^4}{\Lambda} \right) \times \sqrt{\epsilon}, \quad (5.4c) \]
\[ \sqrt{s} = 3 \text{ TeV}, \quad \mathcal{Z}_4^\mu = 3.80 \left( \frac{2 \text{TeV}^4}{\Lambda} \right) \times \sqrt{\epsilon}, \quad (5.4d) \]
\[ \sqrt{s} = 5 \text{ TeV}, \quad \mathcal{Z}_4^\mu = 4.33 \left( \frac{2.5 \text{TeV}^4}{\Lambda} \right) \times \sqrt{\epsilon}; \quad (5.4e) \]

and for the \( \tau^- \tau^+ \gamma \) final state we have \( \mathcal{Z}_4^\tau \simeq \mathcal{Z}_4^\mu \).

In reality, the \( e^\pm \) beams could only be partially polarized. Let \( P_L^e \) (\( P_R^e \)) denote the left (right) polarization of the electrons (positrons) in the beam. We then have the following relations between the observable \( \mathcal{O}_c^1 \) with partial and full polarizations:
\[ \mathcal{O}_c^1(P_L^e, P_R^e) \simeq \left| \frac{c_L^2 P_L^e P_R^e - c_R^2 (1 - P_L^e)(1 - P_R^e)}{c_L^2} \right| \mathcal{O}_c^1(1, 1), \quad (5.5a) \]
\[ \sigma_c^0(P_L^e, P_R^e) \simeq \left| \frac{c_L^2 P_L^e P_R^e + c_R^2 (1 - P_L^e)(1 - P_R^e)}{c_L^2} \right| \sigma_c^0(1, 1), \quad (5.5b) \]
\[ \mathcal{Z}_4(P_L^e, P_R^e) \simeq \left| \frac{c_L^2 P_L^e P_R^e - c_R^2 (1 - P_L^e)(1 - P_R^e)}{c_L} \right| \mathcal{Z}_4(1, 1), \quad (5.5c) \]

where the signal significance of the fully polarized case, \( \mathcal{Z}_4(1, 1) = \sqrt{(\mathcal{Z}_4^e)^2 + (\mathcal{Z}_4^\mu)^2 + (\mathcal{Z}_4^\tau)^2} \), with \( (\mathcal{Z}_4^e, \mathcal{Z}_4^\mu, \mathcal{Z}_4^\tau) \) given by Eqs.(5.3)-(5.4). For instance, assuming the polarizations \( (P_L^e, P_R^e) = (0.9, 0.65) \), we derive
\[ \mathcal{Z}_4(P_L^e, P_R^e) \simeq 0.715 \mathcal{Z}_4(1, 1). \quad (5.6) \]

We note that the \( e^\pm \) polarization possibilities have been well studied for the linear colliders ILC [16] and CLIC [19], whereas the longitudinal polarization is harder to realize at the circular colliders CEPC [17] and FCC-e$e^-$ [18].

We can derive the following relations between the the cross sections \( (\sigma_0, \sigma_1, \sigma_2) \) for
For illustration, we assume a representative integrated luminosity coefficient, while in all other entries the effects of the sign of the coefficient are negligible. The numbers in parentheses correspond to the case of the dimension-8 operator with negative channels and their combinations, for polarized $e^\pm$ beams with $(P_L^e, P_R^e) = (0.9, 0.65)$. The bounds are shown at the 2$\sigma$ and 5$\sigma$ levels, and for different collider energies. As previously, for the signal significance $Z_8$, we denote its $\mathcal{O}(\Lambda^{-4})$ contribution as $Z_8^{(4)}$ and its $\mathcal{O}(\Lambda^{-8})$ contribution as $Z_8^{(8)}$, with $Z_8 = Z_8^{(4)} + Z_8^{(8)}$. We obtain the following signal significances $Z_8$ with partial polarizations $(P_L^e, P_R^e)$:

\[
Z_8(P_L^e, P_R^e) = Z_8^{(4)}(P_L^e, P_R^e) + Z_8^{(8)}(P_L^e, P_R^e),
\]

\[
Z_8^{(4)}(P_L^e, P_R^e) = 2\sqrt{\frac{c_L^2 + c_R^2}{c_L^2 - c_R^2}} \frac{c_L^2 P_L^e P_R^e - c_R^2 (1 - P_L^e)(1 - P_R^e)}{\sqrt{c_L^2 P_L^e P_R^e + c_R^2 (1 - P_L^e)(1 - P_R^e)}} Z_8^{(4)}(0.5, 0.5),
\]

\[
Z_8^{(8)}(P_L^e, P_R^e) = 2\sqrt{\frac{c_L^2 P_L^e P_R^e + c_R^2 (1 - P_L^e)(1 - P_R^e)}{c_L^2 + c_R^2}} Z_8^{(8)}(0.5, 0.5).
\]

For instance, with $e^\pm$ beam polarizations $(P_L^e, P_R^e) = (0.9, 0.65)$, we find the following relations:

\[
Z_8(P_L^e, P_R^e) \simeq 7.18Z_8^{(4)}(0.5, 0.5) + 1.19Z_8^{(8)}(0.5, 0.5),
\]

Table 6. Sensitivity reaches of the new physics scale $\Lambda$ via $e^-e^+\rightarrow \ell^-\ell^+\gamma$ and $e^-e^+\rightarrow \nu\bar{\nu}\gamma$ channels and their combinations, for polarized $e^\pm$ beams with $(P_L^e, P_R^e) = (0.9, 0.65)$. The bounds are shown at the 2$\sigma$ and 5$\sigma$ levels, and for different collider energies. As previously, the numbers in parentheses correspond to the case of the dimension-8 operator with negative coefficient, while in all other entries the effects of the sign of the coefficient are negligible. For illustration, we assume a representative integrated luminosity $\mathcal{L} = 2\,\text{ab}^{-1}$ and an ideal detection efficiency $\epsilon = 100\%$. The partially polarized and unpolarized cases:

\[
\sigma_0(P_L^e, P_R^e) = \frac{c_L^2 P_L^e P_R^e + c_R^2 (1 - P_L^e)(1 - P_R^e)}{0.5^2(c_L^2 + c_R^2)} \sigma_0(0.5, 0.5),
\]

\[
\sigma_1(P_L^e, P_R^e) = \frac{c_L^2 P_L^e P_R^e - c_R^2 (1 - P_L^e)(1 - P_R^e)}{0.5^2(c_L^2 - c_R^2)} \sigma_1(0.5, 0.5),
\]

\[
\sigma_2(P_L^e, P_R^e) = \frac{c_L^2 P_L^e P_R^e + c_R^2 (1 - P_L^e)(1 - P_R^e)}{0.5^2(c_L^2 + c_R^2)} \sigma_2(0.5, 0.5),
\]
where the contributions $Z^8_8(0.5,0.5)$ and $Z^8_8(0.5,0.5)$ correspond to the unpolarized case, as computed in Sec. 3-4.

From Eqs. (5.6)(5.3)(5.4) and Eqs. (5.9), we compute the sensitivity reaches of the new physics scale $\Lambda$ via $e^-e^+\rightarrow \ell^-\ell^+\gamma$ and $e^-e^+\rightarrow \nu\bar{\nu}\gamma$ channels, for polarized electron/positron beams with $(P^e_L, P^e_R) = (0.9, 0.65)$. These results are summarized in Table 6. Here the numbers in the parentheses correspond to the case of the dimension-8 operator with negative coefficient, while in all other entries the differences for opposite signs of the coefficient are negligible. For illustration, we have assumed a representative integrated luminosity $\mathcal{L} = 2 \text{ab}^{-1}$ and an ideal detection efficiency $\epsilon = 100\%$.

We present in Table 6 the $2\sigma$ and $5\sigma$ bounds on $\Lambda$ for different collider energies. The limits from the $\ell^-\ell^+\gamma$ channel are shown in the 2nd and 3rd rows, while the 4th and 5th rows give the limits in the $\nu\bar{\nu}\gamma$ channel. Finally, we derive the combined limits of $\ell^-\ell^+\gamma$ and $\nu\bar{\nu}\gamma$ channels, as shown in the 6th and 7th rows. Comparing the results in Table 6 with those in the previous Table 5, we see that for collider energies $\sqrt{s} = 250 - 1000 \text{ GeV}$, the $e^\pm$ beam polarization can enhance the sensitivity reaches of $\Lambda$ significantly, by about $(50-88)\%$, whereas for $\sqrt{s} = 3 - 5 \text{ TeV}$, the polarization effects are much milder, yielding an enhancement of around $(5-13)\%$.

We present in Fig. 10 the sensitivity reaches for the new physics scale $\Lambda$ as functions of the collision energy $\sqrt{s}$, comparing our results for the unpolarized and polarized cases in plots (a) and (b), respectively, assuming $(P^e_L, P^e_R) = (90\%, 65\%)$ for the polarized electron and positron beams in the plot (b). In each plot, we show the
limits \( Z = (2, 3, 5)\sigma \) by the (red, green, blue) curves, where the signal significance \( Z = \sqrt{Z_{\ell\ell}^2 + Z_{\nu\nu}^2} \) combines both \( \ell^-\ell^+\gamma \) and \( \nu\bar{\nu}\gamma \) channels. We see that electron/positron beam polarizations can improve significantly the sensitivity reaches for the new physics scale. For reference, we also draw the lines \( \Lambda = \sqrt{s} \) and \( \Lambda = \sqrt{s}/2 \) in each plot.

6. Conclusions

As we have discussed in this work, the reaction \( e^+e^- \rightarrow Z\gamma \) provides a rare opportunity to probe an effective dimension-8 operator in the SMEFT. The \( ZV\gamma \) vertices \( (V = Z, \gamma) \) have no tree-level SM contributions, and nor do they receive any contributions from dimension-6 operators, opening up the possibility of probing the new physics scale associated with one particular dimension-8 operator. Such dimension-8 operators invoke the Higgs doublets and are tied to the Higgs boson and the spontaneous electroweak symmetry breaking. We have presented a general analysis of the angular distributions for \( Z\gamma \) production in the lab frame and for \( Z \rightarrow \ell^+\ell^- \) in the \( Z \) rest frame to identify particular angular distributions and cuts that maximize the statistical sensitivity to the possible new physics scale \( \Lambda \), either including only the \( \mathcal{O}(\Lambda^{-4}) \) contributions that interfere with the SM contributions, or including together the \( \mathcal{O}(\Lambda^{-8}) \) contributions of the dimension-8 operator.

As seen in Fig. 8(b) and Tables 2-3, the prospective sensitivities to \( \Lambda \) extend into the multi-TeV range. As one would expect from the energy dependences of the dimension-8 contributions to the cross section for \( e^+e^- \rightarrow Z\gamma \), the prospective sensitivities increase with the collision energies. However, since we assume a constant integrated luminosity, the sensitivities increase more slowly than \( \sqrt{s} \). The sensitivities at the (2\( \sigma \), 3\( \sigma \), 5\( \sigma \)) levels of significances are not greatly different, as discussed in the text and seen by comparing the (red, green, blue) curves in Fig. 8(b).

We have also drawn in Fig. 8 the two reference lines \( \Lambda = \sqrt{s} \) and \( \Lambda = \sqrt{s}/2 \). In general, one would expect the SMEFT approach to be suitable only for energy scales that are small compared to \( \Lambda \). However, the way that we have defined \( \Lambda \) in Eq.(2.1) of this paper corresponds to the true new physics scale \( \tilde{\Lambda} \) only if the unknown coefficient \( c_j \) has a magnitude of unity. If, on the other hand, the true magnitude of \( c_j \gg 1 \), the true new physics scale \( \tilde{\Lambda} \) could be sizably larger than the value of \( \Lambda \) extracted from our analysis, and the SMEFT approach would have broader applicability.

We have studied the effect of including the reaction \( e^-e^+ \rightarrow Z\gamma \) with \( Z \rightarrow \nu\bar{\nu} \) in Sec. 4. We have presented the sensitivities of this channel in Table 4 and Table 5, and derived the combined new physics reaches for both the leptonic and invisible channels \( \ell^-\ell^+\gamma \) and \( \nu\bar{\nu}\gamma \). We found that the sensitivity of the invisible channel is comparable to that of the lepton channel (cf. Table 4). Then, we demonstrated in Sec. 5 that including the electron/proton beam polarizations can also improve significantly the signal sensitivities. We have presented our findings for the polarized case in Fig. 10(b), to be compared with the unpolarized case in Fig. 10(a). We have summarized the 2\( \sigma \)
and 5σ bounds on the new physics scale Λ in Table 6, including the combined reaches of both leptonic and invisible channels.

It is interesting to compare the sensitivity to the dimension-8 coefficient found here with that found previously in studies of the dimension-8 operator contributions to light-by-light scattering and the process $gg \to \gamma\gamma$ at the LHC. The former is sensitive to a dimension-8 scale that is $\mathcal{O}(100)$ GeV [9], whereas the latter is sensitive to a dimension-8 scale that is $\mathcal{O}(1)$ TeV[11]. The dimension-8 operators studied in those analyses contain gauge fields only and differ from what we studied here, hence they probe very different aspects of dimension-8 physics. However, it is encouraging that we have found in this work that future $e^+e^-$ colliders (such as the ILC, CEPC, FCC-ee, and CLIC) may be able to provide very competitive sensitive probes of the scale of new physics. We therefore encourage further detailed studies of the reaction $e^+e^- \to Z\gamma$ by our experimental colleagues.

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Appendix
A. Helicity Amplitudes for $Z\gamma$ Production with Z Decays

In this Appendix we present the helicity amplitudes for the production process $e^-e^+\to Z(q_1,\epsilon_\lambda)\gamma(q_2,\epsilon_{\lambda'})$, and then we include leptonic Z decays. These results are used in the analyses of Sections 2 and 3 in the main text.

A.1. Helicity Amplitudes for $Z\gamma$ Production

The helicity amplitudes for $e^-(p_1)e^+(p_2) \to Z(q_1,\epsilon_\lambda)\gamma(q_2,\epsilon_{\lambda'})$ can be written as

$$ T^{s's'}_{\lambda\lambda'} = \bar{v}^{s'}(p_2) \left[ \frac{e^2}{s_W c_W} \left( \frac{f_{\lambda'}^*(q_2)(q_1-p_1)}{t} f_{\lambda}^*(q_1) + \frac{f_{\lambda}^*(q_1)(q_2-p_1)}{u} f_{\lambda'}^*(q_2) \right) + \frac{i2M_Z^2}{\Lambda^4} \epsilon^{\mu\nu\alpha\beta} \gamma_\mu \epsilon_{\lambda,\nu}(q_1) \epsilon_{\lambda',\alpha}(q_2) q_{2\beta} (c_LP_L+c_RP_R) u^s(p_1), \right. $$

(A.1)
where we have used the standard spinor notations \( u^s(p_1) \) and \( \bar{v}^s(p_2) \) [23] for the initial-state \( e^- \) and \( e^+ \), and \((\epsilon_\lambda, \epsilon_\lambda')\) denote the polarization vectors of the final-state gauge bosons \((Z, \gamma)\). In the above, \( P_{L,R} = \frac{1}{2}(1 \mp \gamma_5) \) are the chirality projection operators, and the coefficients \((c_L, c_R) = (s_W^2 - \frac{1}{2}, s_W^2)\) arise from the (left-, right)-handed gauge couplings of electrons to the \( Z \) boson. In the above, we have used the Mandelstam variables \( t = (p_1 - q_1)^2 = -\frac{1}{2}(s-M_Z^2)(1-\cos\theta) \) and \( u = (p_1 - q_2)^2 = -\frac{1}{2}(s-M_Z^2)(1+\cos\theta) \).

In Eq.(2.7) we defined the momenta of the final-state particles \( Z(q_1)\gamma(q_2) \) as \( q_1 = (E_Z, q \sin\theta, 0, q \cos\theta) \) and \( q_2 = q(1, -\sin\theta, 0, -\cos\theta) \). Then, we can express the three polarization vectors \( \epsilon_\lambda(q_1, \theta) \) of the \( Z \) boson as follows:

\[
\epsilon^Z_\pm(\theta) = \frac{1}{\sqrt{2}} (0, \mp \cos\theta, -i, \pm \sin\theta), \quad (A.2a)
\]

\[
\epsilon^Z_0(q_1, \theta) = \frac{1}{M_Z} (q_1, E_Z \sin\theta, 0, E_Z \cos\theta), \quad (A.2b)
\]

where \( E_Z = \sqrt{q_1^2 + M_Z^2} \). The final-state photon has two transverse polarization vectors that are similar to those of the \( Z \) boson,

\[
\epsilon^\gamma_\pm(\theta) = \epsilon^Z_\pm(\theta + \pi) = \epsilon^Z_\pm(\theta), \quad (A.3)
\]

The first two terms in Eq.(A.1) arise from the SM contributions via the \( t- \) and \( u- \) channel exchanges, respectively, while the third term is contributed by the dimension-8 operator. For the final-state \( \lambda(\lambda')(\lambda'') \) helicity combinations \( \lambda \lambda' = (--, --, +-, ++) \) and \( \lambda \lambda'' = (0-, 0+) \), we compute the SM contributions to the scattering amplitudes as follows:

\[
T_{ss'}^{\text{sm}}(\begin{array}{c} - - + \\ + + - \\ -- + \end{array}) = \frac{2e^2}{s_W c_W (s-M_Z^2)} \left[ (e_L \cot \theta R - e_R \tan \theta R) M_Z^2 (e_L \cot \theta T + e_R \tan \theta T) s \right]
\]

\[
T_{ss'}^{\text{sm}}(0-, 0+) = \frac{2\sqrt{2}(e_L + e_R)e^2 M_Z \sqrt{s}}{s_W c_W (s-M_Z^2)}(1, -1), \quad (A.4b)
\]

where \((e_L, e_R) = (c_L \delta_{s, -\frac{1}{2}}, c_R \delta_{s, \frac{1}{2}})\), with the subscript index \( s = \pm \frac{1}{2} \) denoting the initial-state electron helicities. For the massless initial-state \( e^- \) and \( e^+ \), we have \( s = -s' \).

We note that, in Eq.(A.4a), the identical-helicity amplitudes \( T_{ss'}^{\text{sm}}(\pm \pm) \) in the diagonal entries are proportional to \( M_Z^2 \) (unlike the off-diagonal entries, which are \( \propto s \)). This is expected because the identical-helicity amplitudes should vanish exactly in the massless limit \( M_Z \to 0 \), after ignoring the tiny electron mass, as in the pair-annihilation process \( e^- e^+ \to \gamma \gamma \) in QED [23]. Hence the non-zero amplitudes have asymptotic behaviors \( T_{ss'}^{\text{sm}}(\pm \pm) \propto M_Z^2 / s \).

Next, we compute the corresponding helicity amplitudes from the new physics contributions of the dimension-8 operator, which are as follows:

\[
T_{ss'}^{(8)}(\begin{array}{c} - - + \\ + + - \\ -- + \end{array}) = \frac{(e_L + e_R) \sin \theta M_Z^2 (s-M_Z^2)}{\Lambda^4} \left( \begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array} \right), \quad (A.5a)
\]
\[
\mathcal{T}_{ss'}^{\ell L}(0-,0+) = \frac{\sqrt{2}M_Z(s-M_Z^2)\sqrt{s}}{\Lambda^4} \left( e_L \sin \frac{\theta}{2} - e_R \cos \frac{\theta}{2}, e_R \sin \frac{\theta}{2} - e_L \cos \frac{\theta}{2} \right). \quad (A.5b)
\]

We note that in Eq.(A.5a) the off-diagonal amplitudes \( T_{ss'}^{+\ell}(--+) \) and \( T_{ss'}^{+\ell}(+++) \) vanish exactly. This can be understood by noting that the \( Z\gamma\gamma \) vertex [cf. Eqs.(2.6) and (A.1)] contains the rank-4 antisymmetric tensor \( \epsilon^{\mu\nu\lambda\sigma} \), which contracts with the \( Z \) and \( \gamma \) polarization vectors \( e^Z_{\lambda,\nu}(\theta) \) and \( e^{\gamma}_{\lambda,\nu}(\theta) \) as in Eq.(A.1). For the off-diagonal \( Z\gamma \) helicity combinations \( \lambda\lambda' = +-, -+ \), we deduce
\[
\epsilon^{\mu\nu\lambda\mu} e^Z_{\pm,\nu}(\theta) e^{\gamma}_{\pm,\nu}(\theta) = 0,
\]
according to Eq.(A.3).

### A.2. Helicity Amplitudes Including Z Decays

In this Appendix we incorporate the leptonic decays of the \( Z \) boson. We first consider \( Z \) decay in its rest frame, \( Z \rightarrow \ell^-(k_1)\ell^+(k_2) \), where the final-state leptons have momenta:
\[
k_1 = k(1, \sin \theta, \cos \phi, \sin \phi, \cos \theta),
\]
\[
k_2 = k(1, -\sin \theta, \cos \phi, -\sin \phi, -\cos \theta), \quad (A.7)
\]
where the leptons are treated as effectively massless and \( k = |\vec{k}| \simeq \frac{1}{2}M_Z \). In the \( Z \) rest frame, the massless lepton spinors are defined as follows,
\[
u_+(k_1) = \sqrt{2k} \left( 0, 0, e^{-i\phi} \cos \frac{\theta}{2}, e^{i\phi} \sin \frac{\theta}{2} \right),
\]
\[
u_-(k_1) = \sqrt{2k} \left( -e^{i\phi} \sin \frac{\theta}{2}, e^{-i\phi} \cos \frac{\theta}{2}, 0, 0 \right),
\]
\[
u_+(k_2) = \sqrt{2k} \left( e^{-i\phi} \cos \frac{\theta}{2}, e^{i\phi} \sin \frac{\theta}{2}, 0, 0 \right),
\]
\[
u_-(k_2) = \sqrt{2k} \left( 0, 0, e^{-i\phi} \sin \frac{\theta}{2}, -e^{i\phi} \cos \frac{\theta}{2} \right), \quad (A.8)
\]
where \( u_+ \) (\( u_- \)) correspond to spin-up (-down) and \( v_+ \) (\( v_- \)) correspond to spin-down (-up) along their directions of motion in Eq.(A.7).

Then, we write down the left-handed and right-handed spinor currents in the \( Z \) boson rest frame,
\[
\overline{C}_{L}^\mu = \bar{\nu}_L \gamma^\mu u_L = M_Z(0, -\cos \theta \cos \phi - i \sin \phi, -\cos \theta \sin \phi + i \cos \phi, \sin \theta), \quad (A.9a)
\]
\[
\overline{C}_{R}^\mu = \bar{\nu}_R \gamma^\mu u_R = M_Z(0, -\cos \theta \cos \phi + i \sin \phi, -\cos \theta \sin \phi - i \cos \phi, \sin \theta), \quad (A.9b)
\]
where \((u_L, u_R) = (u_-, u_+)\) and \((v_L, v_R) = (v_+, v_-)\). After making a Lorentz boost \( \hat{L} \) back to the laboratory frame (i.e., the c.m. frame of the \( Z\gamma \) pair) and rotating the axis \( z^* \) back to the axis \( z \) by the rotation \( \hat{R} \), we have new currents \( C_{L,R}^\mu = \hat{R}\hat{L}\overline{C}_{L,R}^\mu \) in...
the lab frame. The Lorentz boost \( \hat{L} \) acts on the \((0, 3)\) components, with \( \hat{L}_{03} = \hat{L}_{33} = \gamma \) and \( \hat{L}_{03} = \hat{L}_{30} = \gamma \beta \), where \((\beta, \gamma) = (p_Z/E_Z, E_Z/M_Z)\). The rotation matrix \( \hat{R} \) acts on the \((1, 3)\) components, with elements \( \hat{R}_{11} = \hat{R}_{33} = \cos \theta \) and \( \hat{R}_{13} = -\hat{R}_{31} = \sin \theta \). Thus, we can derive the currents \( C_{L,R}^\mu \) and express them in terms of \( Z \) boson polarization vectors,

\[
\begin{align*}
C_L^\mu &= M_Z \left( \sin \theta_s \epsilon_0^Z - \sqrt{2} \sin^2 \frac{\theta_s}{2} e^{-i\phi} \epsilon_+^Z - \sqrt{2} \cos^2 \frac{\theta_s}{2} e^{i\phi} \epsilon_-^Z \right), \\
C_R^\mu &= M_Z \left( \sin \theta_s \epsilon_0^Z + \sqrt{2} \cos^2 \frac{\theta_s}{2} e^{-i\phi} \epsilon_+^Z + \sqrt{2} \sin^2 \frac{\theta_s}{2} e^{i\phi} \epsilon_-^Z \right),
\end{align*}
\]

(A.10a)

(A.10b)

Next, we can obtain the amplitude for \( e^- e^+ \to \ell^- \ell^+ \gamma \) by replacing the \( Z \) polarization vector \( \epsilon^{\mu}(q_1) \) in Eq.(A.1) with \( C_L^\mu(q_1) \mathcal{D}_Z \), where \( \mathcal{D}_Z = 1/(q_1^2 - M_Z^2 + i M_Z \Gamma) \) is from the \( Z \) propagator. Since \( q_{1\mu} C_L(q_1) = 0 \), we can drop the \( q_1^\mu \) term in the \( Z \)-propagator. Then, we derive the \( \ell^- \ell^+ \gamma \) amplitude as follows,

\[
\mathcal{T}_{\sigma\sigma'}(\ell\bar{\ell}) = \frac{e f_{LR} \mathcal{D}_Z}{s_W c_W} \vec{v}(p_2) \left[ e^2 \left( f_{L}^*(q_2)(\phi_1 - \phi_1) C_{L,R}^\mu(q_1) + C_{L,R}^\mu(q_1)(\phi_2 - \phi_2) f_{L}^*(q_2) \right) \right] \left( c_L P_L + c_R P_R \right) u^\dagger(p_1),
\]

(A.11)

where \((\sigma, \sigma', \lambda)\) denote the helicities of the final-state particles \((\ell^-, \ell^+, \gamma)\) with \( \sigma = -\sigma' \) for massless leptons, and we have defined the coefficients \((f_L, f_R) = (c_L \delta_{\sigma,-\frac{1}{2}}, c_R \delta_{\sigma,\frac{1}{2}})\).

Substituting Eq.(A.10) into Eq.(A.11), we can express the amplitude (A.1) in terms of the helicity amplitudes (A.4)-(A.5) of Appendix A.1,

\[
\mathcal{T}_{\sigma\sigma'}(\ell\bar{\ell}) = \frac{e M_Z \mathcal{D}_Z}{s_W c_W} \left[ \sqrt{2} e^{i\phi} \left( f_R^* \cos^2 \frac{\theta_s}{2} - f_L^* \sin^2 \frac{\theta_s}{2} \right) \mathcal{T}_{ss'}^{\tau}(+\lambda) \right. \\
\left. + \sqrt{2} e^{-i\phi} \left( f_R^* \sin^2 \frac{\theta_s}{2} - f_L^* \cos^2 \frac{\theta_s}{2} \right) \mathcal{T}_{ss'}^{\tau}(-\lambda) \right] + (f_R^* + f_L^*) \sin \theta_s \mathcal{T}_{ss'}^{L}(0\lambda),
\]

(A.12)

where \( \mathcal{T}_{ss'}^{\tau}(\pm\lambda) \) and \( \mathcal{T}_{ss'}^{L}(0\lambda) \) are the on-shell helicity amplitudes of \( e^- e^+ \to Z \gamma \),

\[
\begin{align*}
\mathcal{T}_{ss'}^{\tau}(\pm\lambda) &= \mathcal{T}_{sm}^{ss';\tau}(\pm\lambda) + \mathcal{T}_{sm}^{ss';T}(\pm\lambda), \\
\mathcal{T}_{ss'}^{L}(0\lambda) &= \mathcal{T}_{sm}^{ss';L}(0\lambda) + \mathcal{T}_{sm}^{ss';T}(0\lambda),
\end{align*}
\]

(A.13)

which sum up the contributions from both the SM and the dimension-8 operator as derived in Eqs.(A.4)-(A.5) of Appendix A.1. From Eq.(A.12), we see that the full cross section for \( e^- e^+ \to \ell^- \ell^+ \gamma \) depends on the angle \( \phi_s \), due to the interference between the terms with different \( Z \) boson helicities \( \lambda' = +, -, 0 \). Eq.(A.12) also exhibits the \( \theta_s \) dependence associated with each \( Z \) boson helicity. We have used Eqs.(A.12)-(A.13) in the analysis of angular observables in Section 3.
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