UNCOUNTABLY MANY PLANAR EMBEDDINGS OF
UNIMODAL INVERSE LIMIT SPACES

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Abstract. For a point $x$ in the inverse limit space $X$ with a single unimodal bonding map we construct, with the use of symbolic dynamics, a planar embedding such that $x$ is accessible. It follows that there are uncountably many non-equivalent planar embeddings of $X$.

1. Introduction. Inverse limit spaces can be often used as a model to construct attractors of some planar diffeomorphisms, see for example [25, 26, 4]. One of the simplest examples is the Knaster continuum (bucket handle), which is the attractor of Smale’s horseshoe map and can be modeled as the inverse limit space of the full tent map $T_2(x) := \min\{2x, 2(1-x)\}$ for $x \in [0,1]$, see [1]. The topological unfolding of the Smale’s horseshoe has been a topic of ongoing interest, related to the pruning front conjecture for the Hénon family developed by Cvitanović et al. in [12, 13] and demonstrated more recently with the work of Boyland, de Carvalho & Hall [6] and Mendoza [20].

Inverse limit spaces with unimodal bonding maps $T : [0,1] \to [0,1]$ (from now on denoted by $X$) are chainable. The study of embeddings of chainable continua dates back to 1951 when Bing proved in [5] that every chainable continuum can be embedded in the plane. However, his proof does not offer any insight what such embeddings look like. The first explicit class of embeddings of $X$ was given by Brucks & Diamond in [8]. Later, Bruin [10] extended this result showing that the embedding of $X$ can be made such that the shift-homeomorphism extends to a

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Lipschitz map on $\mathbb{R}^2$. Both mentioned results are using symbolic dynamics as the main tool in the description of $X$.

Locally, inverse limit spaces of unimodal maps roughly resemble Cantor sets of arcs. However, this is not true in general. In [2] Barge, Brucks & Diamond proved that in the tent family $T_s$ for a dense $G_\delta$ set of slopes $s \in [\sqrt{2}, 2]$, every open set of the inverse limit space $\lim \{[s(1-s^2), s^2], T_s\}$ not only contains a homeomorphic copy of the space itself but also homeomorphic copies of every inverse limit space of a tent map. Thus it would be interesting to see what kind of embeddings in the plane of complicated $X$ are possible.

Philip Boyland posed the following questions on the Continuum Theory and Dynamical Systems Workshop in Vienna in July 2015:

Can a complicated $X$ be embedded in $\mathbb{R}^2$ in multiple ways? For example, do there exist embeddings in $\mathbb{R}^2$ of $X$ that are non-equivalent to the standard embeddings constructed in [8] and [10]?

For a special case of the full tent map these two questions were already answered in the affirmative by Mayer [18], Mahavier [17], Schwartz [24] and Dębski & Tymchatyn [14].

Let $K$ be a continuum. A composant $U \subset K$ of a point $x \in U$ is the union of all proper subcontinua of $K$ containing point $x$. We call $K$ indecomposable if it cannot be expressed as the union of two proper subcontinua. In this case, $K$ has uncountably many composants and every composant is dense in $K$.

There are two fixed points of the map $T$: 0 and $r$. Denote the composant of $(\ldots, 0, 0)$ by $C$. We can decompose $X = C \cup X'$, where $X'$ is the core of $X$ and $C$ compactifies on $X'$, see e.g. [16]. Let $R$ be the arc-component of $(\ldots, r, r)$; if $X'$ is indecomposable then $R$ is indeed the composant of $(\ldots, r, r)$ within $X'$. The standard embeddings given in [8] and [10] make $C$ and $R$ respectively accessible.

**Definition 1.1.** A point $a \in X \subset \mathbb{R}^2$ is accessible (i.e., from the complement of $X$) if there exists an arc $A = [x, y] \subset \mathbb{R}^2$ such that $a = x$ and $A \cap X = \{a\}$. We say that a composant $U \subset X'$ or $U = C$ is accessible, if $U$ contains an accessible point.

In the special case of the full tent map, i.e., with the Knaster continuum $X$ as the inverse limit space, Mayer constructed uncountably many non-equivalent embeddings in [13]. Later, Mahavier showed in [17] that for every composant $U \subset X$, there exists a homeomorphism $h : X \to \mathbb{R}^2$ such that each point of $h(U)$ is accessible. Schwartz extended Mahavier’s result and proved that embeddings of $X$ which do not make $C$ or $R$ accessible are non-equivalent to the standard embeddings.

**Definition 1.2.** Denote two planar embeddings of $X$ by $g_1 : X \to E_1 \subset \mathbb{R}^2$ and $g_2 : X \to E_2 \subset \mathbb{R}^2$. We say that $g_1$ and $g_2$ are equivalent embeddings if there exists a homeomorphism $h : E_1 \to E_2$ which can be extended to the homeomorphism of the plane.

In this paper we make use of symbolic dynamics description of $X$ introduced in [8] and [10] and answer the questions of Boyland in the affirmative. We construct embeddings of $X$ by selecting (the itinerary of) an accessible point. For every unimodal map of positive topological entropy, we obtain uncountably many embeddings by making an arbitrary point from $X$ accessible.

**Theorem 1.3.** For every point $a \in X$ there exists an embedding of $X$ in the plane such that the image of $a$ is accessible.
Corollary 1. Let \( T \) be a unimodal interval map with \( h_{\text{top}}(T) > 0 \). Then there are uncountably many non-equivalent embeddings of \( X \) in the plane.

The short outline of the paper is as follows. Section 2 provides the basic set-up as introduced in [8] and [10]. Next, we construct specific representations of \( X \) equipped with the product metric. In Section 3, we prove that the representations given in Section 3 are indeed embeddings and prove the main results.

2. Preliminaries. Let \( \mathbb{N} := \{1, 2, 3, \ldots \} \) and \( \mathbb{N}_0 := \{0\} \cup \mathbb{N} \). Let \( T : [0, 1] \to [0, 1] \) be a unimodal map fixing 0 and let \( c \) denote the critical point of \( T \). We define the inverse limit space with the bonding map \( T \) by

\[
X := \{(\ldots, x_{-2}, x_{-1}, x_0) : T(x_{-(i+1)}) = x_i, i \in \mathbb{N}_0 \} \subset [0, 1]^{\mathbb{N}}
\]

equipped with the product metric

\[
d((\ldots, x_{-2}, x_{-1}, x_0), (\ldots, y_{-2}, y_{-1}, y_0)) := \sum_{i \leq 0} 2^i |x_i - y_i|.
\]

This makes \( X \) a continuum, i.e., a compact and connected metric space. Define the shift homeomorphism \( \sigma : X \to X, \sigma((\ldots, x_{-2}, x_{-1}, x_0)) := (\ldots, x_{-2}, x_{-1}, x_0, T(x_0)) \) and the projection map as \( \pi_n : X \to [0, 1], \pi_n((\ldots, x_{-2}, x_{-1}, x_0)) := x_{-n}, \) for every \( n \in \mathbb{N}_0 \).

In the construction of planar embeddings of spaces \( X \) we recall a well-known symbolic description introduced in [8]. The space \( X \) will be represented by the quotient space \( \Sigma_{\text{adm}}/\sim \), where \( \Sigma_{\text{adm}} \subseteq \{0, 1\}^\mathbb{Z} \) is equipped with the product topology. We first need to recall the kneading theory for unimodal maps. To every \( x \in [0, 1] \) we assign its itinerary:

\[
I(x) := \nu_0(x)\nu_1(x)\ldots,
\]

where

\[
\nu_i(x) := \begin{cases} 
0, & T^i(x) \in [0, c), \\
*, & T^i(x) = c, \\
1, & T^i(x) \in (c, 1].
\end{cases}
\]

Note that if \( \nu_i(x) = * \) for some \( i \in \mathbb{N}_0 \), then \( \nu_{i+1}(x)\nu_{i+2}(x)\ldots = I(T(c)) \). The sequence \( \nu := I(T(c)) \) is called the kneading sequence of \( T \) and is denoted by \( \nu = c_1c_2\ldots \), where \( c_i := \nu_i(T(c)) \in \{0, *, 1\} \) for every \( i \in \mathbb{N} \). Observe that if \( * \) appears in the kneading sequence, then \( c \) is periodic under \( T \), i.e., there exists \( n > 0 \) such that \( T^n(c) = c \) and the kneading sequence is of the form \( \nu = (c_1\ldots c_{n-1}*)^\infty \). In this case we adjust the kneading sequence by taking the smallest of \( (c_1\ldots c_{n-1}0)^\infty \) and \( (c_1\ldots c_{n-1}1)^\infty \) in the parity-lexicographical ordering defined below.

By \( \#_1(a_1\ldots a_n) \) we denote the number of ones in a finite word \( a_1\ldots a_n \in \{0, 1\}^n \); it can be either even or odd.

Choose \( t = t_1t_2\ldots \in \{0, 1\}^\mathbb{N} \) and \( s = s_1s_2\ldots \in \{0, 1\}^\mathbb{N} \) such that \( s \neq t \). Take the smallest \( k \in \mathbb{N} \) such that \( s_k \neq t_k \). Then the parity-lexicographical ordering is defined as

\[
s < t \iff \begin{cases} 
s_k < t_k \text{ and } \#_1(s_1\ldots s_{k-1}) \text{ is even}, \text{ or} \\
 s_k > t_k \text{ and } \#_1(s_1\ldots s_{k-1}) \text{ is odd}.
\end{cases}
\]

This ordering is also well-defined on \( \{0, *, 1\}^\mathbb{N} \) once we define \( 0 < * < 1 \).
Thus if $(c_1 \ldots c_{n-1}0)^\infty \prec (c_1 \ldots c_{n-1}1)^\infty$ we modify $\nu = (c_1 \ldots c_{n-1}0)^\infty$, otherwise $\nu = (c_1 \ldots c_{n-1}1)^\infty$.

**Example.** If $c$ is periodic of period 3 then the kneading sequence for $T$ is $\nu = (10*)^\infty$. Since $101 \ldots < 100 \ldots$ in parity-lexicographical ordering, we modify $\nu = (101)^\infty$.

In the same way we modify itinerary of an arbitrary point $x \in [0,1]$. If $\nu_i(x) = *$ and $i$ is the smallest positive integer with this property then we replace $\nu_{i+1}(x)\nu_{i+2}(x) \ldots$ with the modified kneading sequence. Thus $*$ can appear only once in the modified itinerary of an arbitrary point $x \in [0,1]$. From now onwards we assume that the itineraries of points from $[0,1]$ are modified.

It is a well-known fact (see [21]) that a kneading sequence completely characterizes the dynamics of unimodal map in the sense of the following proposition:

**Proposition 1.** If a sequence $s_0s_1 \ldots \in \{0,*,1\}^\mathbb{N}$ is the itinerary of a point $x \in [T^2(c),T(c)]$, then

$$I(T^2(c)) \preceq s_k s_{k+1} \ldots \preceq \nu = I(T(c)), \text{ for every } k \in \mathbb{N}_0. \quad (1)$$

Conversely, assume $s_0s_1 \ldots \in \{0,*,1\}^\mathbb{N}$ satisfies (1). If there exists $j \in \mathbb{N}_0$ such that $s_{j+1}s_{j+2} \ldots = \nu$, and $j$ is minimal with this property, assume additionally that $s_j = *$. Then $s_0s_1 \ldots$ is realized as the itinerary of some $x \in [T^2(c),T(c)]$.

**Definition 2.1.** We say that a sequence $s_0s_1 \ldots \in \{0,*,1\}^\mathbb{N}$ is admissible if it is realized as the itinerary of some $x \in [0,1]$.

**Remark 1.** Note that the Proposition [1] gives conditions on admissible itineraries of points $x \in [T^2(c),T(c)]$. For points $y \in [0,T^2(c)] \cup (T(c),1]$ admissible itineraries are exactly $0^\infty, 1^\infty, 0^i s_0 s_1 \ldots, 10^{i-1} s_0 s_1 \ldots$ where $s_0 s_1 \ldots \in \{0,*,1\}^\mathbb{N}$ is the itinerary of the point $T^j (y)$ which satisfies the conditions of Proposition [1] for $j := \min\{i \in \mathbb{N} : T^j (y) \in [T^2(c),T(c)]\}$.

Next we expand the above construction to $X$. Take $x = (\ldots,x_{-2},x_{-1},x_0) \in X$. Define the itinerary of $x$ as a two-sided infinite sequence

$$I(x) := \ldots\nu_{-2}(x)\nu_{-1}(x)\nu_0(x)\nu_1(x)\ldots \in \{0,*,1\}^\mathbb{Z},$$

where $\nu_0(x)\nu_1(x)\ldots = I(x_0)$ and

$$\nu_i(x) = \begin{cases} 0, & x_i \in [0,c), \\
* , & x_i = c, \\
1, & x_i \in (c,1], \end{cases}$$

for all $i < 0$.

We make the same modifications as above. If $*$ appears for the first time at $\nu_k(x)$ for some $k \in \mathbb{Z}$, then $\nu_{k+1}(x)\nu_{k+2}(x) \ldots = \nu$. If there is no such minimal $k$, then the kneading sequence is periodic with a period $n \in \mathbb{N}$, $\nu = (c_1c_2 \ldots c_{n-1}*)^\infty$ and the itinerary of $x$ is of the form $(c_1c_2 \ldots c_{n-1}*)^\mathbb{Z}$. Replace $(c_1c_2 \ldots c_{n-1}*)^\mathbb{Z}$ with the modified itinerary $(c_1c_2 \ldots c_{n-1}c_n)^\mathbb{Z}$, where $\nu = (c_1 \ldots c_{n-1}c_n)^\infty$. In this way $*$ can appear at most once in every itinerary. Now we are ready to identify the inverse limit space with a quotient of a space of two-sided sequences consisting of two symbols.

Let $\Sigma := \{0,1\}^\mathbb{Z}$ be the space of two-sided sequences equipped with the metric

$$d((s_i)_{i \in \mathbb{Z}},(t_i)_{i \in \mathbb{Z}}) := \sum_{i \in \mathbb{Z}} \frac{|s_i - t_i|}{2^{|i|}},$$
for \((s_i)_{i \in \mathbb{Z}}, (t_i)_{i \in \mathbb{Z}} \in \Sigma\). We define the shift homeomorphism \(\sigma : \Sigma \to \Sigma\) as
\[
\sigma((\ldots s_{-2}s_{-1}s_0s_{-1} \ldots)) := \ldots s_{-2}s_{-1}s_0s_{-1} \ldots
\]
By \(\Sigma_{adm} \subseteq \Sigma\) we denote all \(s \in \Sigma\) such that either
(a) \(s_k s_{k+1} \ldots\) is admissible for every \(k \in \mathbb{Z}\), or
(b) there exists \(k \in \mathbb{Z}\) such that \(s_{k+1}s_{k+2} \ldots = \nu\) and \(s_{k-i} \ldots s_{k-1} * s_{k+1}s_{k+2} \ldots\)
is admissible for every \(i \in \mathbb{N}\).

We abuse notation and call the two-sided sequences in \(\Sigma_{adm}\) also admissible.

Let us define an equivalence relation on the space \(\Sigma_{adm}\). For sequences \(s = (s_i)_{i \in \mathbb{Z}}, t = (t_i)_{i \in \mathbb{Z}} \in \Sigma_{adm}\) we define the relation
\[
s \sim t \iff \begin{cases} 
\text{either } s_i = t_i \text{ for every } i \in \mathbb{Z}, \\
or \text{ if there exists } k \in \mathbb{Z} \text{ such that } s_i = t_i \text{ for all } i \neq k \text{ but } s_k \neq t_k \\
\text{and } s_{k+1}s_{k+2} \ldots = t_{k+1}t_{k+2} \ldots = \nu.
\end{cases}
\]

It is not difficult to see that this is indeed an equivalence relation on the space \(\Sigma_{adm}\). Furthermore, every itinerary is identified with at most one different itinerary and the quotient space \(\Sigma_{adm}/\sim\) of \(\Sigma_{adm}\) is well defined. It was also shown that \(\Sigma_{adm}/\sim\) is homeomorphic to \(X\). So in order to embed \(X\) in the plane it is enough to embed \(\Sigma_{adm}/\sim\) in the plane. For all observations in this paragraph we refer to the paper [8] of Brucks & Diamond (Lemmas 2.2-2.4 and Theorem 2.5).

An arc is a homeomorphic image of an interval \([a, b] \subset \mathbb{R}\). A key fact for constructing embeddings is that \(X\) is the union of basic arcs defined below. Let \(\overleftarrow{s} = \ldots s_{-2}s_{-1} \in \{0, 1\}^{-\mathbb{N}}\) be an admissible left-infinite sequence (i.e., every finite subword is admissible). The subset
\[
A(\overleftarrow{s}) := \{x \in X : \nu_i(x) = s_i, \forall i < 0\} \subset X
\]
is called a basic arc. Note that \(\pi_0 : A(\overleftarrow{s}) \to [0, 1]\) is injective. In [10] Lemma 1 it was observed that \(A(\overleftarrow{s})\) is indeed an arc (possibly degenerate). For every basic arc we define two quantities as follows:
\[
\tau_L(\overleftarrow{s}) := \sup\{n > 1 : s_{-n} = c_{1}c_{2} \ldots c_{n-1}, \#(c_{1} \ldots c_{n-1}) \text{ odd}\},
\]
\[
\tau_R(\overleftarrow{s}) := \sup\{n \geq 1 : s_{-n} = c_{1}c_{2} \ldots c_{n-1}, \#(c_{1} \ldots c_{n-1}) \text{ even}\}.
\]

**Remark 2.** For \(n = 1\), \(c_{1}c_{2} \ldots c_{n-1} = 0\) and \(\#(\emptyset)\) is even. Thus \(\tau_R(\overleftarrow{s}) = 1\) if and only if \(s_{-n} = c_{1}c_{2} \ldots c_{n-1} \neq c_{1}c_{2} \ldots c_{n-1}\) for all \(n > 1\).

These definitions first appeared in [10] in order to study the number of endpoints of inverse limit spaces \(X\). We now adapt two lemmas from [10] that we will use later in the paper.

**Lemma 2.2.** ([10], Lemma 2) Let \(\overleftarrow{s} \in \{0, 1\}^{-\mathbb{N}}\) be admissible with \(\tau_L(\overleftarrow{s}), \tau_R(\overleftarrow{s}) < \infty\). Then
\[
\pi_0(A(\overleftarrow{s})) = \{T^{\tau_L(\overleftarrow{s})}(c), T^{\tau_R(\overleftarrow{s})}(c)\}.
\]

If \(\overleftarrow{t} \in \{0, 1\}^{-\mathbb{N}}\) is another admissible left-infinite sequence such that \(s_i = t_i\) for all \(i < 0\) except for \(i = -\tau_R(\overleftarrow{s}) = -\tau_R(\overleftarrow{t})\) (or \(i = -\tau_L(\overleftarrow{s}) = -\tau_L(\overleftarrow{t})\)), then \(A(\overleftarrow{s})\) and \(A(\overleftarrow{t})\) have a common boundary point.

**Lemma 2.3.** ([10], Lemma 3) If \(\overleftarrow{s} \in \{0, 1\}^{-\mathbb{N}}\) is admissible, then
\[
\sup \pi_0(A(\overleftarrow{s})) = \inf\{T^n(c) : s_{-n} = c_{1} \ldots c_{n-1}, n \geq 1, \#(c_{1} \ldots c_{n-1}) \text{ even}\}, \quad \inf \pi_0(A(\overleftarrow{s})) = \sup\{T^n(c) : s_{-n} = c_{1} \ldots c_{n-1}, n \geq 1, \#(c_{1} \ldots c_{n-1}) \text{ odd}\}.
\]
Example. Take the unimodal map with the kneading sequence $\nu = (101)^\infty$. Then $\overset{<}{s} = (011)^\infty010.$ and $\overset{<}{t} = (011)^\infty110.$ are admissible, $\tau_L(\overset{<}{s}) = \tau_L(\overset{<}{t}) = 3$, $\tau_R(\overset{<}{s}) = \tau_R(\overset{<}{t}) = 1$ and $s_i = t_i$ for all $i < 0$ except for $i = -3 = -\tau_L(\overset{<}{s}) = -\tau_L(\overset{<}{t})$. By Lemma 2.3, $\pi_0(A(\overset{<}{s})) = \pi_0(A(\overset{<}{t})) = [T^3(c), T(c)]$, and by Lemma 2.2, $A(\overset{<}{s})$ and $A(\overset{<}{t})$ have a common boundary point which is projected to $T^3(c)$, see Figure 1. Note that in this example both $\tau_L$ and $\tau_R$ agree for $\overset{<}{s}$ and $\overset{<}{t}$, which need not be the case in general.

![Figure 1](image.png)

**Figure 1.** Example of two basic arcs having a boundary point in common.

3. Representation in the plane. This section is the first step towards embedding $X$ in the plane so that an arbitrary point $a \in X$ becomes accessible. We denote the symbolic representation of $a$ by $\ldots l_{-2l_{-1}}l_0l_1 \ldots := \overrightarrow{I}(a)$, so $a \in A(\ldots l_{-2l_{-1}})$. We present the following ordering on $\{0, 1\}^\infty$ depending on some $L = \ldots l_{-2l_{-1}}$, and we work with this ordering from now onwards.

**Definition 3.1.** Let $\overset{<}{s}, \overset{<}{t} \in \{0, 1\}^\infty$ and let $k \in \mathbb{N}$ be the smallest natural number such that $s_{-k} \neq t_{-k}$. Then

$$s_{-k} \overset{L}{<} t_{-k} \iff \begin{cases} t_{-k} = l_{-k} \text{ and } \#_1(s_{-(k-1)} \ldots s_{-1}) = \#_1(l_{-(k-1)} \ldots l_{-1}) \text{ even}, \\ s_{-k} = l_{-k} \text{ and } \#_1(s_{-(k-1)} \ldots s_{-1}) = \#_1(l_{-(k-1)} \ldots l_{-1}) \text{ odd}. \end{cases}$$

Note that such ordering is well-defined and the left infinite tail $L$ is the largest sequence.

**Lemma 3.2.** Assume $\overset{<}{s} \overset{L}{<} \overset{<}{u} \overset{L}{<} \overset{<}{t}$ and assume that $s_{-n} \ldots s_{-1} = t_{-n} \ldots t_{-1}$. Then also $u_{-n} \ldots u_{-1} = s_{-n} \ldots s_{-1} = t_{-n} \ldots t_{-1}$.

**Proof.** If $n = 1$ the statement follows easily so let us assume that $n \geq 2$. Assume that there exists $k < n$ such that $u_{-k} \neq s_{-k}$ and take $k$ the smallest natural number with this property. Assume without loss of generality that $(-1)^{\#_1(s_{-(k-1)} \ldots s_{-1})} = (-1)^{\#_1(l_{-(k-1)} \ldots l_{-1})}$ (the proof follows similarly when the parities are different). Since $\overset{<}{s} \overset{L}{<} \overset{<}{u}$ it follows that $u_{-k} = l_{-k}$. Also, $\overset{<}{u} \overset{L}{<} \overset{<}{t}$ gives $t_{-k} = l_{-k}$. Since $u_{-k} \neq t_{-k}$, we get a contradiction. \(\square\)

Let $C \subset [0, 1]$ be the middle-third Cantor set,

$$C := [0, 1] \setminus \bigcup_{m=1}^{\infty} \bigcup_{k=0}^{3m-1} \left( \frac{3k+1}{3m}, \frac{3k+2}{3m} \right).$$

Points in $C$ are coded by the left-infinite sequences of zeros and ones. We embed basic arcs in the plane as horizontal lines along the Cantor set and then join
corresponding endpoints with semi-circles as in Figure 1. The ordering has to be defined in a way that semi-circles neither cross horizontal lines nor each other.

**Example.** For \( L = 1^\infty \), points in \( C \) are coded as in Figure 2 (a). Note that this is the same ordering as in the paper by Bruin [10]. The ordering obtained by \( L = 0^\infty \) is the ordering from the paper by Brucks & Diamond [8]. In Figure 2 (b) points in Cantor set are coded with respect to \( L = \ldots 101 \).

|   | 111. |   | 101. |   |
|---|------|---|------|---|
|   | 011. |   | 001. |   |
| 1. |      |   |      | 1.|
|   | 001. |   | 011. |   |
|   | 101. |   |       |   |

|   | 100. |   | 110. |   |
|---|------|---|------|---|
|   | 000. |   | 000. |   |
| 0. |      |   |      | 0.|
|   | 010. |   |       |   |
|   | 110. |   |       |   |

(a) ![Figure 2](image)

(b) ![Figure 2](image)

**Figure 2.** Coding the Cantor set with respect to (a) \( L = \ldots 111 \) and (b) \( L = \ldots 101 \).

From now onwards, we assume that \( \bar{s} = \{0,1\}^\infty \) is an admissible left-infinite sequence. Define \( \psi_L : \{0,1\}^\infty \rightarrow C \) as

\[
\psi_L(\bar{s}) := \sum_{i=1}^{\infty} (-1)^{#_1(l_{-i-1})-#_1(s_{-i-1}...s_{-1})} 3^{-i} + \frac{1}{2},
\]

and we let \( C_{adm} := \{ \psi_L(\bar{s}) : \bar{s} \text{ admissible left-infinite sequence} \} \) be the subset of “admissible vertical coordinates”. Note that \( \psi_L(L) = 1 \) is the largest point in \( C_{adm} \).

From now onwards let \( d_e \) denote the Euclidean distance in \( \mathbb{R}^2 \).

**Remark 3.** Note that if \( \bar{s}, \bar{t} \in \{0,1\}^\infty \) are such that \( s_{-n}...s_{-1} = t_{-n}...t_{-1} \), then \( d_e(\psi_L(\bar{s}), \psi_L(\bar{t})) \leq 3^{-n} \). If \( s_{-n} \neq t_{-n} \), then \( d_e(\psi_L(\bar{s}), \psi_L(\bar{t})) \geq 3^{-n} \).

Now we represent \( X \) as the quotient space of the subset of \( I \times C_{adm} \) for \( I := [0,1] \). To every point \( x = (\ldots, x_{-2}, x_{-1}, x_0) \in X \) we will assign either a point or two points in \( I \times C_{adm} \) by rule (2) below. From now on, write \( \ldots s_{-3}s_{-2}s_{-1} := \)
...ν − 2(α)ν − 2(x)ν − 1(x). Let \( \varphi : X \rightarrow I \times C_{adm} \) be defined in the following way:

\[
\varphi(x) := \begin{cases} 
(x_0, \psi_L((s_i)_{i < 0})), & \text{if } s_i \neq * \text{ for every } i < 0, \\
(x_0, p) \cup (x_0, q), & \text{if } s_i = * \text{ for some } i < 0,
\end{cases}
\]

(2)

where

\[
\begin{align*}
p &= \psi_L(\ldots s_{-(i+1)}0s_{-(i-1)} \ldots s_{-1}), \\
q &= \psi_L(\ldots s_{-(i+1)}1s_{-(i-1)} \ldots s_{-1}).
\end{align*}
\]

Set \( Y := \varphi(X) \subset I \times C_{adm} \). The next step is to identify points in \( Y \) in the same way as they are identified in the symbolic representation of \( X \). For \( a, b \in Y \):

\[ a \sim b \text{ if there exists } x \in X \text{ such that } a, b \in \varphi(x). \]

If \( a \neq b \sim a \) we write \( \tilde{a} := \tilde{b} \). If \( \tilde{a} = \tilde{b} \) and \( x \in X \) is such that \( a, b \in \varphi(x) \) and \( s_{-i} = * \) we say that \( a \) and \( b \) are joined at level \( i \).

Note that \( \varphi : X \rightarrow Y/\sim \) is a well-defined map. Equip \( Y \) with the Euclidean topology and \( Y/\sim \) with the standard quotient topology. Let \( \pi_C : I \times C \rightarrow C \) and \( \pi_I : I \times C \rightarrow I \) denote the natural projections. The next proposition is an analogue of Proposition 4 from [10]. We prove it here for the sake of completeness.

**Proposition 2.** The map \( \varphi : X \rightarrow Y/\sim \) is a homeomorphism.

**Proof.** We first prove that \( Y/\sim \) is a Hausdorff space and because \( X \) is compact it is enough to check that \( \varphi \) is a continuous bijection to obtain a homeomorphism between \( X \) and \( Y/\sim \), see e.g. Theorem 26.6 in [22].

Take \( x \neq y \in Y \) such that \( x \neq y \). First assume that \( |\pi_I(x) − \pi_I(y)| = 0 \). Let \( \delta := \min\{|\pi_C(x) − \pi_C(y)|, |\pi_C(y) − \pi_C(x)|\} \). Then take \( \{z : |\pi_C(x) − \pi_C(z)| < \delta/3\} \) and \( \{z : |\pi_C(y) − \pi_C(z)| < \delta/3\} \) for open neighbourhoods of \( x \) and \( y \) respectively and they are disjoint. Now assume that \( \varepsilon := |\pi_I(x) − \pi_I(y)| > 0 \). Then \( \{z : |\pi_I(x) − \pi_I(z)| < \varepsilon/3\} \) are disjoint open neighbourhoods for \( x \) and \( y \) respectively, so \( Y/\sim \) is indeed a Hausdorff space.

Now we prove that \( \varphi \) is continuous. It is enough to prove that for \( a \in X \) and a sequence \( (x_n)_{n \in \mathbb{N}} \subset X \) such that \( \lim_{n \to \infty} x_n = a \) it holds that \( \lim_{n \to \infty} \varphi(x_n) = \varphi(a) \). Assume that \( \lim_{n \to \infty} x_n = a \). Thus for every \( M \in \mathbb{N} \) there exists \( N \in \mathbb{N} \) such that for every \( n \geq N \) it follows that \( \nu_M(x_n) \ldots \nu_M(x_0) = \nu_M(a) \ldots \nu_M(a) \). We need to show that for every open \( \varphi(a) \in U \subset Y/\sim \) there exists \( N' \in \mathbb{N} \) such that for every \( n \geq N' \) it holds that \( \varphi(x_n) \in U \). Let us fix an open set \( U \supset \varphi(a) \). If for \( x = (\ldots, x_{−1}, x_0) \in X \) there exists \( i \in \mathbb{N} \) such that \( \nu_{−i}(x) = * \) then we set \( \varphi(x) = \varphi'(x) \cup \varphi''(x) \) where \( \varphi'(x) := (x_0, \psi_L(\ldots \nu_{−i+1}(x)0\nu_{−i+1}(x) \ldots \nu_{−2}(x)) \ldots \nu_{−2}(x)) \) and \( \varphi''(x) := (x_0, \psi_L(\ldots \nu_{−i+1}(x)1\nu_{−i+1}(x) \ldots \nu_{−2}(x)) \ldots \nu_{−2}(x)) \).

**Case I.** For every \( i \in \mathbb{N} \), \( \nu_{−i}(a) \neq * \). If there exists \( K \in \mathbb{N} \) such that for every \( n \geq K \) it follows that \( \nu_{−j}(x_n) \neq * \) for every \( j \in \mathbb{N} \), then there is \( N' \geq K \) such that \( \varphi(x_n) \subset U \) for every \( n \geq N' \).

Assume that there exists an increasing sequence \( (n_i)_{i \in \mathbb{N}} \subset \mathbb{N} \) such that \( \nu_{−j}(x_{n_i}) = * \) for some \( j \in \mathbb{N} \). Then there exists an open set \( U_1 \cup U_2 \subset Y \) such that \( \varphi'(x_{n_i}) \subset U_1 \) and \( \varphi''(x_{n_i}) \subset U_2 \) and \( \varphi^{-1}(U) = U_1 \cup U_2 \) for every \( i \in \mathbb{N} \). Because \( x_n \to a \) as \( n \to \infty \), by the definition of \( \varphi \) it follows that \( \varphi'(x_n) \to \varphi(a) \) and \( \varphi''(x_n) \to \varphi(a) \).
as \( i \to \infty \). Thus we again conclude that there exists \( N' \in \mathbb{N} \) such that for every \( n \geq N' \) it follows that \( \phi(x^n) \in U \).

**Case II.** Let \( K \in \mathbb{N} \) be such that \( \nu_K(a) = * \) and thus \( \phi(a) = \phi'(a) \cup \phi''(a) \). Take \( M > K \) so that \( \nu_M(a) = \nu_M(x^n) \) for every \( n \geq N \), and so \( \phi(x^n) = \phi'(x^n) \cup \phi''(x^n) \). Thus there exist open sets \( U_1, U_2 \subset Y \) such that \( \phi'(a) \in U_1, \phi''(a) \in U_2 \) and \( \phi^{-1}(U) = U_1 \cup U_2 \). It follows that there exists \( N' > N \) such that for every \( n > N' \) it holds that \( \phi'(x^n) \in U_1 \) and \( \phi''(x^n) \in U_2 \) and thus \( \phi(x^n) \in U \).

Now we are ready to represent \( X \) in the plane. This is still not an embedding but it is the first step towards it. Connect identified points in \( \mathcal{I} \times \mathcal{C}_{adm} \) with semi-circles. Suppose \( a \neq b \in Y \) are joined at level \( n \). By Lemma 2.2, points \( a \) and \( b \) are both endpoints of basic arcs in \( \mathcal{I} \times \mathcal{C}_{adm} \) and are both right or left endpoints. If \( \#_{1}(c_1 \ldots c_{n-1}) \) is even (odd), \( a \) and \( b \) are right (left) endpoints and we join them with a semi-circle on the right (left), see Figure 1.

**Proposition 3.** Every semi-circle defined above crosses neither \( Y \) nor another semi-circle.

**Proof.**

**Case I.** Assume that there is a semi-circle oriented to the right which intersects an arc \( A \) in \( Y \). (See Figure 3)

![Figure 3. Case I in the proof of Proposition 3](image)

Translated to symbolics, this means that there exist \( n \in \mathbb{N} \) and sequences \( \pi u \prec_L \pi l \pi c \pi v \) such that \( s_{-(n-1)} \ldots s_{-1} = t_{-(n-1)} \ldots t_{-1} = c_1 \ldots c_{n-1}, \ s_{-n} \neq t_{-n} \) and \( \#_{1}(c_1 \ldots c_{n-1}) \) is even. By Lemma 3.2, \( u_{-(n-1)} \ldots u_{-1} = c_1 \ldots c_{n-1} \). By Lemma 2.3 it follows that \( \sup \{ \pi_{I}(A) \} \leq T^n(c) \), and thus an intersection between the arc \( A \) and a semi-circle cannot occur.

**Case II.** Assume that we have a crossing of two semi-circles which project to the same point in \( \mathcal{I} \). (See Figure 4)

![Figure 4. Case II in the proof of Proposition 3](image)
Assume that there exist \( n \in \mathbb{N} \) and \( \frac{n}{m} \prec_L \frac{n}{n} \prec_L \frac{1}{m} \prec_L \frac{t}{u} \) such that \( s_i = t_i \) for all \( i < 0 \) except for \( i = -n \) and \( s_{-n} = t_{-n-1} \ldots t_{-1} = c_1 \ldots c_{n-1} \) and \( u_i = v_i \) for all \( i < 0 \) except for \( i = -n \) and \( u_{-n} = v_{-n-1} \ldots v_{-1} = c_1 \ldots c_{n-1} \). If \( s_{-n} = v_{-n} \), then by Lemma 3.2 also \( t_{-n} = u_{-n} = s_{-n} = v_{-n} \) which contradicts the assumption. It follows that \( s_{-n} \neq v_{-n} \), because \( \frac{t}{u} \) and \( \frac{t}{s} \) are respectively connected by a right semi-circle. Assume without loss of generality that \( v_{-n} = 1 \) and \( s_{-n} = 0 \). This gives \( t_{-n} = 1 \) and \( u_{-n} = 0 \).

Take the smallest integer \( m > n \) such that \( u_{-m} \neq t_{-m} \); this \( m \) is also the smallest integer such that \( u_{-m} \neq s_{-m} \). By the previous paragraph, \( (-1)\#_1(s_{-(m-1)} \ldots s_{-1}) = (-1)\#_1(u_{-(m-1)} \ldots u_{-1}) \neq (-1)\#_1(t_{-(m-1)} \ldots t_{-1}) = (-1)\#_1(v_{-(m-1)} \ldots v_{-1}) \), the possibilities for \( s_{-m} \), \( u_{-m}, t_{-m}, v_{-m} \) are (depending on the parities of ones): (1) \( s_{-m} = 0, u_{-m} = 1, t_{-m} = 1, v_{-m} = 0 \), or (2) \( s_{-m} = 1, u_{-m} = 0, t_{-m} = 0, v_{-m} = 1 \). Both cases lead to a contradiction with \( s_{-m} = t_{-m} \) and \( u_{-m} = v_{-m} \).

Thus our ordering gives a representation \( Y \cup \{ \text{semi-circles} \} \) of \( X \) in the plane. Figure 5 and Figure 6 give two examples of these planar representations.

4. Embeddings. In this section we show that representations of \( X \) constructed in the previous section are indeed embeddings.

**Lemma 4.1.** Let \( U \subset \mathbb{R}^2 \) be homeomorphic to the open unit disk, and let \( W \subset \mathbb{R} \) be a closed set such that \( W \cap U \) for some closed interval \( J \). There exists a continuous function \( f : \mathbb{R}^2 \to \mathbb{R}^2 \) such that \( f(\{w\} \times J) \) is a point for every \( w \in W \), \( f(\{w\} \times J) \neq f(\{w'\} \times J) \) for every \( w \neq w' \in W \), \( f|_{U \setminus W \times J} \) is injective and \( f|_{\mathbb{R}^2 \setminus U} \) is the identity.

**Proof.** Without loss of generality we can take \( U := (-1,2) \times (-1,1) \), \( J := [-1/2,1/2] \) and \( \min(W) = 0 \), \( \max(W) = 1 \), see Figure 7.

For every \( a \in [0,2] \) we define a continuous function \( g(a,\cdot) : [-1,1] \to [-1,1] \) as

\[
g(a,x) := \begin{cases} 
(2-a)x + 1 - a, & x \in [-1,-1/2] \\
ax, & x \in [-1/2,1/2] \\
(2-a)x + a - 1, & x \in [1/2,1].
\end{cases}
\]

Note that \( g(a,\cdot) \) is injective for every \( a \in [0,2] \), \( g(0,x) = 0 \) for all \( x \in [-1/2,1/2] \), and \( g(1,x) = x \) for all \( x \in [-1,1] \).

Define \( f : [-1,2] \times [-1,1] \to [-1,2] \times [-1,1] \) as

\[
f(x,y) := (x, g(d_e(x,W), y)),
\]

where \( d_e(x,W) = \inf_{w \in W} d_e(x,w) \). Note that \( x \mapsto d_e(x,W) \) is continuous, so \( \hat{f} \) is continuous. Also, \( f(w,y) = (w, g(0,y)) = (w,0) \) for \( (w,y) \in W \times J \) and \( \hat{f} \) is injective otherwise. Also note that \( \hat{f} \) is the identity on the boundary of \( [-1,2] \times [-1,1] \), so \( \hat{f} \) can be extended continuously to the map \( f : \mathbb{R}^2 \to \mathbb{R}^2 \) such that \( f|_{\mathbb{R}^2 \setminus U} \) is the identity.

Define \( W_n \subset \mathbb{R}^2 \) to be the set consisting of all semi-circles that join pairs of points at level \( n \). Note that there exists a set \( W \subset \mathbb{R} \) such that \( W_n \) is homeomorphic to \( W \times J \). Observe that \( W \) is closed. Indeed, if for a sequence \( (s^k)_{k \in \mathbb{N}} \subset \{0,1\}^N \) there exists \( m \in \mathbb{N} \) such that \( \tau_k(s^k) = m \) for every \( k \in \mathbb{N} \) and \( \lim_{k \to \infty} s^k = \hat{s} \), then \( \tau_k(\hat{s}) = m \). The analogous argument holds for \( \tau_L \).

**Lemma 4.2.** There exist open sets \( U_n \subset \mathbb{R}^2 \) such that \( W_n \subset U_n \) and \( \text{diam}(U_n) \to 0 \) as \( n \to \infty \).
Proof. We define the set $G_n := \{ \psi_L(\vec{s}) : \vec{s} \in \{0, 1\}^{-N} \text{ admissible, } \tau_R(\vec{s}) = n \text{ or } \tau_L(\vec{s}) = n \}$ for every $n \in \mathbb{N}$ and let $A_n$ be the smallest interval in $[0, 1]$ containing $G_n$. Note that $A_n$ is closed and $\text{diam}(A_n) \leq 3^{-n}$.
Figure 6. The planar representation of the same arc as in Figure 5 in $X$ with the corresponding kneading sequence $\nu = 100110010 \ldots$. The ordering on basic arcs is such that the basic arc coded by $L = (101)\infty$ is the largest.
Let $M_n$ denote the midpoint of $A_n$. If $\#(c_1 \ldots c_n)$ is odd, let
\[
V'_n = \left\{ (x, y) \in \mathbb{R}^2 : (x - T^n(c))^2 + (y - M_n)^2 \leq \left( \frac{\operatorname{diam}(A_n)}{2} \right)^2, \ x \leq T^n(c) \right\}
\]
be the closed left semi-disc centered around $(T^n(c), M_n)$. Similarly, if $\#(c_1 \ldots c_n)$ is even, let
\[
V'_n = \left\{ (x, y) \in \mathbb{R}^2 : (x - T^n(c))^2 + (y - M_n)^2 \leq \left( \frac{\operatorname{diam}(A_n)}{2} \right)^2, \ x \geq T^n(c) \right\}.
\]
be the closed right semi-disc centered around $(T^n(c), M_n)$. Note that $W_n \subset V'_n$, $\operatorname{diam}(V'_n) \leq 3^{-n}$ and that $d_w(A_n, \psi_L(t^n)) > 3^{-n}$ for all $\psi_L(t^n) \notin A_n$. Let $V_n$ be the $\frac{\operatorname{diam}(A_n)}{2 \cdot 3}$-neighbourhood of $V'_n$, that is,
\[
V_n = \left\{ x \in \mathbb{R}^2 : \text{there exists } y \in V'_n \text{ such that } d_w(x, y) < \frac{\operatorname{diam}(A_n)}{2 \cdot 3} \right\},
\]
see Figure 7. For every $n \in \mathbb{N}$, the open set
\[
U_n := V_n \setminus \bigcup_{i>n} \{ V_i : V_i \cap \partial V_n \neq \emptyset \}
\]
contains $W_n$ and $\operatorname{diam}(U_n) \leq \operatorname{diam}(V_n) \to 0$ as $n \to \infty$.

Now define a continuous function $f_n : \mathbb{R}^2 \to \mathbb{R}^2$ as in Lemma 4.1 replacing $U$ with $U_n$ and $W$ with $W_n$. Let $F_n : \mathbb{R}^2 \to \mathbb{R}^2$ be defined as $F_n := f_{\frac{\sqrt{2} + \ldots + \sqrt{2}}{3}}$ for every $n \in \mathbb{N}$. We need to show that $F := \lim_{n \to \infty} F_n$ exists and is continuous. It is enough to show the following:

**Lemma 4.3.** Sequence $(F_n)_{n \in \mathbb{N}}$ is uniformly Cauchy.

**Proof.** Take $n < m \in \mathbb{N}$ and note that $\sup_{x \in \mathbb{R}^2} d_w(F_m(x), F_n(x)) = \sup_{x \in \mathbb{R}^2} d_w(f_m \circ \ldots \circ f_{n+1} \circ F_n(x), F_n(x)) \to 0$ as $n, m \to \infty$ since for every $n \in \mathbb{N}$, and all $U_{k_i}$ such that $U_{k_i} \subset U_n$, by the uniform continuity of $f_n$ it follows that $\operatorname{diam}(F_n(U_{k_i})) \to 0$ as $i \to \infty$.

Denote by $Z := Y \cup \{ \text{semi-circles} \} \subset \mathbb{R}^2$. We want to argue that $F(Z) \subset \mathbb{R}^2$ is homeomorphic to $Y/\sim$. Since $F : Z \to F(Z)$ is continuous, it follows from [23, Theorem 3.21], that $\{ F^{-1}(y) : y \in F(Z) \}$ is a decomposition of $Z$ homeomorphic to $F(Z)$. Note that this decomposition is exactly $Y/\sim$. 

\[\text{Figure 7. Set-up in Lemma 4.1.}\]
Figure 8. Sets constructed in the proof of Lemma 4.1.

Figure 9. Point \( a = (a_0, \psi_L(L)) \) is accessible.

**Proof of Theorem 1.3.** Let the symbolic representation of \( a = (\ldots, a_{-2}, a_{-1}, a_0) \in X \) be given by \( I(a) = \ldots l_{-2}l_{-1}l_0l_1 \ldots \). Consider the planar representation \( Z \) of \( X \) obtained by the ordering on \( C \) making \( L = \ldots l_{-2}l_{-1} \) the largest. The point \( a \) is represented as \( (a_0, 1) \). Take the arc \( A = \{(a_0, t + 1), t \in [0, 1]\} \) which is a vertical interval in the plane (see Figure 9). Note that \( A \cap Z = \{a\} \). Then \( F(A) \) is an arc such that \( F(A) \cap F(Z) = \{F(a)\} \) which concludes the proof.

Recall that the space \( X' = \lim_{\leftarrow}([T^2(c), T(c)], T) \) is called the core inverse limit space.

**Lemma 4.4.** Let \( T \) be a unimodal map such that \( s := \exp(h_{\text{top}}(T)) > \sqrt{2} \). Then the core inverse limit space \( X' \) is indecomposable.

**Proof.** The map \( T \) is semiconjugate to the tent map \( T_s \), and if \( T \) is locally eventually onto (leo), then the semiconjugacy \( h \) is in fact a conjugacy. In this case, \( X' \) is indecomposable, see [16]. We give the argument if \( T \) is not leo (which also works in the general case). Let \( p \) be the orientation reversing fixed point of \( T \), so \( h(p) = s^{s+1} \) is the fixed point of \( T_s \). Let \( J \ni p \) be a neighbourhood such that \( h(J) \) is a non-degenerate neighbourhood of \( s^{s+1} \). Since \( s > \sqrt{2} \), \( T_s \) is leo, so there is \( N \in \mathbb{N} \) such that \( T^N(J_-) = T^N(J_+) = [T^2(c), T(c)] \) for both components \( J_\pm \) of \( J \setminus \{p\} \).
Suppose now by contradiction that $X' = A \cup B$ for some proper subcontinua $A$ and $B$ of $X'$. Hence there exists $n_0 \in \mathbb{N}$ such that the projections $\pi_{n_0}(A) \neq [T^2(c), T(c)] \neq \pi_{n_0}(B)$. Take $n_1 = n_0 + N$. Since $\pi_{n_1}(A)$ and $\pi_{n_1}(B)$ are intervals and $\pi_{n_1}(A) \cup \pi_{n_1}(B) = [T^2(c), T(c)]$, at least one of them, say $\pi_{n_1}(A)$, contains at least one of $T_-$ or $T_+$. But then $\pi_{n_0}(A) \supset T^N(T^-) \cap T^N(T_+) = [T^2(c), T(c)]$, contradicting the definition of $n_0$. This completes the proof. \hfill $\Box$

**Proof of Corollary 1.** First assume that $\exp(h_{top}(T)) > \sqrt{2}$. By Lemma 4.4, $X'$ is indecomposable and thus it has uncountably many pairwise disjoint composants which are dense in $X'$. By Proposition 2 from [9] every subcontinuum $H \subset X'$ contains a ray which is dense in $H$. Therefore, every composant $\mathcal{U}$ of $X'$ contains a non-degenerate basic arc in $X'$. We embed $X$ so that this non-degenerate basic arc is the largest. Such embedding of $X$ makes a non-degenerate arc of $\mathcal{U}$ accessible.

Assume that $g_1 : X \to E_1$ and $g_2 : X \to E_2$ are equivalent embeddings, so there exists a homeomorphism $h : \mathbb{R}^2 \to \mathbb{R}^2$ such that $h(E_1) = E_2$. It was proven in [9] that every homeomorphism $h : X \to X$ is pseudo-isotopic to $\sigma^R$ for some $R \in \mathbb{Z}$. This means that $h$ permutes the composants of $X'$ in the same way as $\sigma^R$ does. We apply this to $h = g_2 \circ h \circ g_1^{-1}$. Thus if $p \in E_1$ is an accessible point, then $q := g_2 \circ h \circ g_1^{-1}(p)$ is also accessible, and $g_2 \circ \sigma^R \circ g_1^{-1}(p)$ belongs to the same composant of $X'$ as $q$. Hence $h$ maps a composant of $X'$ with an accessible non-degenerate arc to a composant of $X'$ with an accessible non-degenerate arc. Because $h$ is pseudo-isotopic to $\sigma^R$ there are at most countably many different choices for composant $h(\mathcal{U})$.

From the prime end theory established by Carathéodory [11] and others, it follows that there is a 1–1 correspondence between the composants of $X'$ containing more than one accessible point and pairwise disjoint non-degenerate open intervals in the circle of prime ends (see [13] 7). Because there are at most countably many pairwise disjoint non-degenerate open arcs in the circle of prime ends, it follows that $X'$ has at most countably many composants with an accessible non-degenerate arc. See also Mazurkiewicz [19]. This finishes the proof when $\exp(h_{top}(T)) > \sqrt{2}$.

Now assume that $\sqrt{2} \geq \exp(h_{top}(T)) > 1$. The core $X'$ is decomposable and there exists an indecomposable subcontinuum of $X'$ which is homeomorphic to the inverse limit space of a unimodal map with entropy greater than $\log \sqrt{2}$. It follows from the arguments above that we can embed this indecomposable subcontinuum in uncountably many non-equivalent ways; therefore we obtain uncountably many non-equivalent embeddings of $X$.

**Remark 4.** The results from the prime end theory or the Mazurkiewicz’ result are perhaps too strong tools to apply in the proof of Corollary 1. However, to complete the argument we need to know which points are accessible besides those in the arc which is made the largest. Frequently, the only accessible points are the points from the arc-component of a point from the largest basic arc, but there are many exceptions. Determining which points are accessible is a nontrivial matter, to which we plan to come back in a forthcoming paper.

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**REFERENCES**

[1] M. Barge, [Horseshoe maps and inverse limits](Pacific J. Math., 121 (1986), 29–39).
[2] M. Barge, K. Brucks and B. Diamond, Self-similarity of inverse limits of tent maps, Proc. Amer. Math. Soc., 124 (1996), 3563–3570.
[2] M. Barge, H. Bruin and S. Štimac, The Ingram conjecture, Geom. Topol., 16 (2012), 2481–2516.
[4] M. Barge and S. Holte, Nearly one-dimensional Hénon attractors and inverse limits Nonlinearity, 8 (1995), 29–42.
[5] R. H. Bing, Snake-like continua, Duke Math. J., 18 (1951), 653–663.
[6] P. Boyland, A. de Carvalho and T. Hall, Inverse limits as attractors in parametrized families, Bull. Lond. Math. Soc., 45 (2013), 1075–1085.
[7] B. Brechner, On stable homeomorphisms and imbeddings of the pseudo-arc, Illinois Journal of Mathematics, 22 (1978), 630–661.
[8] K. Brucks and B. Diamond, A symbolic representation of inverse limit spaces for a class of unimodal maps, Continuum Theory and Dynamical Systems, Lecture Notes in Pure Appl. Math., 170 (1995), 207–226.
[9] K. Brucks and H. Bruin, Subcontinua of inverse limit spaces of unimodal maps, Fund. Math., 160 (1999), 219–246.
[10] H. Bruin, Planar embeddings of inverse limit spaces of unimodal maps Topology Appl., 96 (1999), 191–208.
[11] C. Carathéodory, Über die Begrenzung einfach zusammenhängender Gebiete, Math. Ann., 73 (1913), 323–370.
[12] P. Cvitanović, Periodic orbits as the skeleton of classical and quantum chaos, Phys. D., 51 (1991), 138–151.
[13] P. Cvitanović, G. Gunaratne and I. Procaccia, Topological and metric properties of Hénon-type strange attractors, Phys. Rev. A (3), 38 (1988), 1503–1520.
[14] W. Dębski and E. Tymchatyn, A note on accessible composites in Knaster continua, Houston J. Math., 19 (1993), 435–442.
[15] D. Iliadis, An investigation of plane continua via Carathéodory prime ends, (Russian)Dokl. Akad. Nauk SSSR, 204 (1972), 828–832.
[16] W. T. Ingram and W. S. Mahavier, “Inverse Limits: From Continua to Chaos” Developments in Mathematics, vol. 25, Springer, New York, 2012.
[17] W. S. Mahavier, Embeddings of simple indecomposable continua in the plane, Topology Proc., 14 (1989), 131–140.
[18] J. C. Mayer, Inequivalent embeddings and prime ends, Topology Proc., 8 (1983), 99–159.
[19] S. Mazurkiewicz, Un théorème sur l’accessibilité des continus indécomposables, Fund. Math., 14 (1929), 271–276.
[20] V. Mendoza, Proof of the pruning front conjecture for certain Hénon parameters Nonlinearity, 26 (2013), 679–690.
[21] J. Milnor and W. Thurston, On Iterated maps of the interval in “Dynamical Systems” (College Park, MD, 1986–87), Lecture Notes in Math., 1342, Springer, Berlin, 1988., 465–563.
[22] J. R. Munkres, Topology, Second Edition, Prentice-Hall, Inc., Englewood Cliffs, New Jersey, 1975.
[23] S. B. Nadler, Continuum Theory: An Introduction, Marcel Dekker, Inc., New York, 1992.
[24] S. P. Schwartz, Some Planar Embeddings of Chainable Continua Can be Expressed as Inverse Limit Spaces, PhD. Thesis, Montana State University, 1992.
[25] R. F. Williams, One-dimensional non-wandering sets Topology, 6 (1967), 473–487.
[26] R. F. Williams, Classification of one dimensional attractors in “Global Analysis” (Proc. Sympos. Pure Math., Vol. XIV, Berkeley, Calif. 1968), Amer. Math. Soc., Providence, R.I., (1970), 341–361.

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