Comment on “Can oscillating neutrino states be formulated universally?”

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Abstract We comment on the recent paper by A. Tureanu (Eur. Phys. J. C 80: 68 (2020).) We show that definition of oscillating neutrino states proposed in that work can be derived in a particular case of the flavor Fock space approach, when mass vacuum is chosen as the physical vacuum. We discuss some problems of such an approach, which appears to be mathematically inconsistent and physically not acceptable.

1 Introduction

Although many facts on neutrino oscillation physics are nowadays well established both theoretically [1, 2] and experimentally [3], the discussion on the correct treatment of flavor states in quantum field theory (QFT) permeated the last thirty years. The main approaches can be grouped as: external wave-packets approach [4], the weak process states approach [5] and Blasone-Vitiello (BV), or flavor Fock space approach [6, 7]. In the first one, flavor neutrino states are not explicitly constructed, because neutrinos are only regarded as internal lines of macroscopic Feynman diagrams. In the second one it is assumed that flavor neutrino states can be expanded as linear combination of mass eigenstates, where the amplitudes of such expansion depends on the production/detection process under consideration. In the third one, an Hilbert space of flavor states is explicitly constructed by means of a Bogoliubov (canonical) transformation. This flavor Fock space is not the same as the Fock space of mass eigenstates, and flavor and mass representations turns out to be unitarily inequivalent representations of canonical anticommutation relations [8].

In Ref. [9] it was proposed a universal, i.e. process independent, definition of oscillating neutrino states. This construction was based on the self-consistent method of Umezawa, Takahashi and Kamefuchi [10], where an appropriate boundary condition on the Hamiltonian fixes parameters of a Bogoliubov transformation, which maps all inequivalent choices for the physical Fock space. The authors identifies the mass neutrino Fock space (see Ref. [6]), as the physical one and consequently defines oscillating flavor states associated to flavor fields, by means of a Bogoliubov transformation which implements a mass shift. Clearly, within this method, the choice of the boundary condition plays a crucial rôle [11].

In this paper we show that the definition of oscillating neutrino states of Ref. [9] can be obtained in a particular case of the BV approach, when the mass vacuum (the state annihilated by mass-annihilation operators) is taken as physical vacuum and a particular choice of some parameters of the model is performed. We also propose some arguments which show that these choices are physically inconsistent.

2 Quantization of flavor fields in massless representation

Let consider the Lagrangian density
\[ \mathcal{L}(x) = \bar{\nu}(x) \left( i \gamma^\mu \partial_\mu - M_\nu \right) \nu(x), \]
with
\[ \nu(x) = \begin{bmatrix} \nu_e(x) \\ \nu_\mu(x) \end{bmatrix}, \quad M_\nu = \begin{bmatrix} m_e & m_{e\mu} \\ m_{e\mu} & m_\mu \end{bmatrix}. \]
The field equations are then:
\[ \left( i \gamma^\mu \partial_\mu - M \right) \nu(x) = 0. \]
These can be rewritten as two independent Dirac equations thanks to the mixing transformation:
\[ \nu_e(x) = \nu_1(x) \cos \theta + \nu_2(x) \sin \theta, \]
\[ \nu_\mu(x) = -\nu_1(x) \sin \theta + \nu_2(x) \cos \theta, \]
with \( \tan 2\theta = 2m_{\mu}/(m_{\mu} - m_{e}) \). \( \nu_e \) and \( \nu_{\mu} \) are called flavor fields while \( \nu_1 \) and \( \nu_2 \) are the mass fields, satisfying free Dirac equations:

\[
(i\gamma^\mu \partial_\mu - m_j) \nu_j(x) = 0, \quad j = 1, 2. \tag{6}
\]

The masses \( m_1 \) and \( m_2 \) are related to the original parameters as:

\[
m_e = m_1 \cos^2 \theta + m_2 \sin^2 \theta, \tag{7}
\]

\[
m_\mu = m_1 \sin^2 \theta + m_2 \cos^2 \theta. \tag{8}
\]

Now \( \nu_1 \) and \( \nu_2 \) can be expanded as

\[
\nu_j(x) = \frac{1}{\sqrt{\sum_k \lambda^k}} \left[ \alpha^j k, \theta \right] e^{-i\omega_k x} + \nu^{*-j} k, \theta \left[ \beta^{*j} k, \theta \right] e^{i\omega_k x}, \quad j = 1, 2. \tag{9}
\]

We can try to expand flavor fields in a similar way:

\[
\nu_\sigma(x) = \frac{1}{\sqrt{\sum_k \lambda^k}} \left[ \alpha^j k, \sigma \left( t \right) \right] e^{-i\omega_k x} + \nu^{*-j} \left[ \beta^{*j} k, \sigma \left( t \right) \right] e^{i\omega_k x}, \quad \sigma = e, \mu, \tag{10}
\]

where \( \alpha^j k, \sigma \equiv \sqrt{\lambda^k + m_\sigma^2} \) and \( m_\sigma \) are mass parameters which have to be specified. From Eqs.(4),(9),(10), one can deduce the relations among flavor and the mass creation and annihilation operators [12]:

\[
\begin{bmatrix}
\alpha^j_{\sigma} \\
\beta^{*j} k, \sigma
\end{bmatrix}
\]

\[
= \begin{bmatrix}
c_\theta \rho^k \lambda_1 & i c_\theta \lambda_1 & s_\theta \rho^k \lambda_2 & i s_\theta \lambda_2 \\
0 & c_\theta \lambda_1 & s_\theta \rho^k_1 & i s_\theta \rho^k_2 \\
-0 & c_\theta \lambda_2 & s_\theta \rho^k_2 & i s_\theta \rho^k_1 \\
-0 & c_\theta \lambda_2 & s_\theta \rho^k_2 & i s_\theta \rho^k_1
\end{bmatrix}
\begin{bmatrix}
\alpha^j_{\sigma} \\
\beta^{*j} k, \sigma
\end{bmatrix}.
\]

Here \( \rho^k_1 \equiv |\rho^k_1| e^{i(\omega_1 x - \omega_1 b)}, \lambda^k_1 \equiv |\lambda^k_1| e^{i(\omega_1 x + \omega_1 b)}, \theta \equiv \cos \theta, s_\theta \equiv \sin \theta \) and

\[
|\rho^k_1| \equiv \cos \frac{\omega_1 - \omega_2}{2}, \quad |\lambda^k_1| \equiv \sin \frac{\omega_1 - \omega_2}{2}, \quad \omega_2 \equiv \cos^{-1} \frac{k}{m_\sigma}, \quad m_a, m_b = m_1, m_2, \mu_e, \mu_\mu. \tag{12}
\]

In Ref. [6] the choice \( \mu_e = m_1 \) and \( \mu_\mu = m_2 \) was performed. Other choices were considered in literature [13]. Although this point was criticized [15], these parameters cannot be arbitrary and have to related with \( m_1 \) and \( m_2 \) by physical considerations [14].

We show that definition of oscillating neutrino states of Ref. [9] comes from the choice \( \mu_e = \mu_\mu = 0 \). The transformation (11) can be written, in that case, as:

\[
\begin{bmatrix}
\alpha^j_{\epsilon} \\
\beta^{*j} k, \epsilon
\end{bmatrix}
\]

\[
= \begin{bmatrix}
c_\theta \rho^k_1 & -i c_\theta \lambda_1 & s_\theta \rho^k_2 & -i s_\theta \lambda_2 \\
-s_\theta \rho^k_1 & c_\theta \lambda_1 & s_\theta \rho^k_2 & -i s_\theta \lambda_2 \\
i s_\theta \rho^k_1 & c_\theta \lambda_2 & s_\theta \rho^k_2 & -i s_\theta \lambda_2 \\
i s_\theta \rho^k_1 & c_\theta \lambda_2 & s_\theta \rho^k_2 & -i s_\theta \lambda_2
\end{bmatrix}
\begin{bmatrix}
\alpha^j_{\epsilon} \\
\beta^{*j} k, \epsilon
\end{bmatrix}.
\]

with \( \rho^k_1 \equiv \cos \frac{\omega_1}{2} e^{-i\omega_1 t} \) and \( \lambda^k_1 \equiv \sin \frac{\omega_1}{2} e^{i\omega_1 t} \). Then neutrino flavor (oscillating) states can be defined as:

\[
|\nu^j_{\epsilon}, \sigma \rangle_m \equiv |\alpha^j_{\epsilon, \sigma}(0)|0\rangle_{1,2}. \tag{14}
\]

where \( |0\rangle_{1,2} \) is the mass vacuum, which is annihilated by \( \alpha^j_{\epsilon, \sigma}, \beta^{*j} k, \epsilon \). Explicitly:

\[
\begin{bmatrix}
|\nu^j_{\epsilon}, \sigma \rangle_m \\
|\nu^{*-j} k, \sigma \rangle_m
\end{bmatrix}
\]

\[
= \begin{bmatrix}
c_\theta |\rho^k_1| & i s_\theta |\rho^k_2| & |\nu^j_{\epsilon, \sigma} \rangle \\
-s_\theta |\rho^k_1| & c_\theta |\rho^k_2| & |\nu^{*-j} k, \sigma \rangle
\end{bmatrix}.
\]

\[
|\nu^j_{\epsilon, \sigma} \rangle = |\alpha^j_{\epsilon, \sigma}(0)|0\rangle_{1,2}. \tag{15}
\]

\[
|\nu^{*-j} k, \sigma \rangle = \beta^{*j} k, \sigma \sin \frac{\omega_1 t}{2} \cos \frac{\omega_2 t}{2}
\]

In the next section we will discuss this choice of mass parameters. The choice of mass vacuum as physical vacuum could seem to be the most natural one, because it diagonalizes the Hamiltonian associated to the Lagrangian (1), which is the usual boundary condition to fix physical Fock space [8, 10]. However, this is true only if we assume that physical fields behave as free fields, far from the interaction region [8]. In the case of mixing the situation is much more delicate [11]. As we will discuss in Section 4 this choice leads to lepton number violation in the production/detection vertex.

### 3 A limit case

When \( m_{\mu} = 0 \), i.e. \( \theta = 0 \), then \( m_e = m_1 \) and \( m_\mu = m_2 \). The Lagrangian (1) reduces to:

\[
\mathcal{L}(x) = \sum_\sigma \nu_\sigma(x) \left( i\gamma^\mu \partial_\mu - m_\sigma \right) \nu_\sigma(x). \tag{16}
\]

In that case, the fields are expanded as

\[
\nu_\sigma(x) = \frac{1}{\sqrt{\sum_k \lambda^k}} \left[ \alpha^j k, \sigma \right] e^{-i\omega_k x} + \nu^{*-j} \left[ \beta^{*j} k, \sigma \right] e^{i\omega_k x}, \quad \sigma = e, \mu,
\]

\[
|\nu^j_{\epsilon, \sigma} \rangle = |\alpha^j_{\epsilon, \sigma}(0)|0\rangle_{1,2}. \tag{17}
\]

with \( \alpha^j_{\epsilon, \sigma} \equiv \sqrt{\lambda^2 + m_\sigma^2} \). If we set \( \theta = 0 \) in Eq.(13), we get

\[
\begin{bmatrix}
\alpha^j_{\epsilon} \\
\beta^{*j} k, \epsilon
\end{bmatrix}
\]

\[
= \begin{bmatrix}
\rho^k_{1} & 0 & 0 & 0 \\
0 & -i \lambda^k_{2} & 0 & 0 \\
0 & 0 & \rho^k_{2} & 0 \\
0 & 0 & 0 & -i \lambda^k_{2}
\end{bmatrix}
\begin{bmatrix}
\alpha^j_{\epsilon} \\
\beta^{*j} k, \epsilon
\end{bmatrix}.
\]

\[
|\nu^j_{\epsilon} \rangle = |\alpha^j_{\epsilon}(0)|0\rangle_{1,2}. \tag{18}
\]
This is not compatible with the expansion (17). The only physical admissible representations are evidently those for which \( \lim_{\theta \to 0} \mu_e = m_1 \), \( \lim_{\theta \to 0} \mu_\mu = m_2 \) or \( \lim_{\theta \to 0} \mu_\nu = m_3 \). This can be also inferred looking at the limit of neutrino states (14):

\[
\lim_{\theta \to 0} |\nu_{k,\sigma}\rangle_m = |\rho^k| |\nu_{k,\sigma}'\rangle,
\]

with \((\sigma, j) = (e, 1), (\mu, 2)\). This gives the expected result (i.e. the normalized mass state) only if \(|\rho^k| = 1\), i.e. in the trivial case \(m_3 \to 0\) considered in Ref. [9].

### 4 Lepton number violation

Let us consider a weak decay \( W^+ \to e^+ + \nu_e \), which can be regarded as production or detection process. The relevant part of the effective Standard Model Lagrangian\(^1\) (after spontaneous symmetry breaking) is \( \mathcal{L}_w = \mathcal{L}_0 + \mathcal{L}_{\text{int}} \) with

\[
\mathcal{L}_0 = \bar{\nu} (i\gamma^\mu \partial^\mu - M_\nu) \nu + \bar{\nu} (i\gamma^\mu \partial^\mu - M_\nu) \nu, \\
\mathcal{L}_{\text{int}} = \frac{g}{2\sqrt{2}} \left[ W^\mu_\nu \nu^\nu (1 - \gamma^5) l + h.c. \right],
\]

where \( l = [e \mu]^T\), and

\[
M_l = \begin{bmatrix} m_e & 0 \\ 0 & m_\mu \end{bmatrix}.
\]

The Lagrangian (1) is just the neutrino part of \( \mathcal{L}_0 \), describing neutrino propagation. The entire Lagrangian \( \mathcal{L}_w \) is invariant under the global \( U(1) \) transformations \( \nu \to e^{i\delta} \nu \) and \( l \to e^{i\phi} l \) leading to the conservation of the total flavor charge \( Q^{\text{tot}} \) corresponding to the lepton-number conservation [16]. This can be written in terms of the flavor charges for neutrinos and charged leptons

\[
Q^{\text{tot}} = \sum_{\sigma = e, \mu} Q^{\text{tot}}_\sigma(t), \\
Q^{\text{tot}}_\sigma(t) = Q_{\nu\sigma}(t) + Q_{\ell\sigma},
\]

with

\[
Q_{\sigma} = \int d^3x : l'_\alpha(x) l_\alpha(x) :,
\]

\[
Q_{\nu\sigma}(t) = \int d^3x : \nu'_\sigma(x) \bar{\nu}_\sigma(x) :,
\]

\(\sigma = e, \mu\).

Note that \([\mathcal{L}_w(x, t), Q^{\text{tot}}_\sigma(t)] \neq 0\). However, by observing that \([\mathcal{L}_{\text{int}}(x, t), Q^{\text{tot}}_\sigma(t)] = 0\), we see that a neutrino flavor state is well defined in the production vertex as an eigenstate of the corresponding flavor charge [17]. This corresponds to the fact that flavor of a neutrino is defined by the flavor of the associated charged lepton [18], i.e. when neutrinos are produced and detected “carry identity cards” [19], i.e. a definite flavor and “can surreptitiously change them if given the right opportunity” [19].

In terms of flavor creation and annihilation operators, flavor charges are given by (11):

\[
Q_{\nu\sigma}(t) = \int d^3k \left( \alpha^{\dagger}_k \sigma(t) \alpha_k \sigma(t) - \beta^{\dagger}_k \sigma(t) \beta_k \sigma(t) \right).
\]

This is true for any choice of \( \mu_e \) and \( \mu_\mu \). In particular, it holds true in the case (13). A flavor eigenstate at \( t = 0 \) (e.g. production time), can be thus constructed as:

\[
|\nu_{k,\sigma}\rangle = \alpha^{\dagger}_k \sigma(0)|0\rangle_{e,\mu},
\]

where \(|0\rangle_{e,\mu}\) is named flavor vacuum and it satisfies

\[
\alpha_k \sigma(0)|0\rangle_{e,\mu} = \beta_k \sigma(0)|0\rangle_{e,\mu} = 0.
\]

\(\nu_{k,\sigma}\rangle\) is an eigenstate of flavor charges at \( t = 0 \) and it is a Perelomov \( SU(2) \) coherent state. It is also evident that \(|\nu_{k,\sigma}\rangle\) are not eigenstates of flavor charges, because mass vacuum \(|0\rangle_{1,2}\) is not. Therefore, the neutrino states (14) spoil the defining property of flavor states\(^2\). Moreover, the oscillation probability, as defined in Ref. [9]

\[
P_{\nu\sigma \to \nu\rho}(t) = |\langle m |\nu_{k,\sigma}\rangle |^2 |\langle m |\nu_{k,\rho}\rangle|^2, \\
\rho \neq \sigma.
\]

with the time evolved (Schrödinger representation) neutrino state given by \(|\nu_{k,\sigma}\rangle |m \equiv \alpha^{\dagger}_k \sigma(t)|0\rangle_{1,2}\), leads to absurd consequences as:

\[
P_{\nu\sigma \to \nu\rho}(0) = \left( c_\sigma^2 |\rho^k|^2 + s_\sigma^2 |\rho^k|^2 \right)^2 \neq 1.
\]

In other words, there is a non-zero probability that an electronic neutrino at \( t = 0 \) were not an electronic neutrino at the same time. The problem arises because \(|\nu_{k,\rho}\rangle_m\) and \(|\nu_{k,\rho}\rangle_m\) are not orthogonal. This is justified in Ref. [9] because “The lack of orthogonality appears to be the price for coherence”, but this cannot be true: as a counterexample, the exact flavor states \(|\nu_{k,\sigma}\rangle\) and \(|\nu_{k,\mu}\rangle\) are orthogonal to each other and belong to a basis in the flavor Hilbert space.

More generally, one should take into account the fact that oscillating neutrinos cannot be viewed as asymptotically stable particles, and this is epitomized by the flavor-energy uncertainty relations [20], which pose a lower bound on their energy variance measurement. It is also important to remark that the choice of flavor vacuum permits to recover a QFT oscillation formula [22], which agrees with the one derived in relativistic quantum mechanics, taking into account contributions of both positive and negative frequency modes [23]. Moreover, it was recently proved [24] that a covariant extension of such a formula is Lorentz invariant\(^3\).

\(^1\)Strictly speaking this is a minimal extension of Standard Model Lagrangian, where non-zero off-diagonal terms are added to neutrino mass matrix. Moreover, we limit to the two-flavor case, although the following considerations are very general.

\(^2\)A similar problem was encountered also in Ref. [6], where neutrino states were defined as in Eq. (14), but choosing a different mass parametrization \((\mu_e = m_1, \mu_\mu = m_2)\).

\(^3\)This result was achieved for scalar particles oscillations, but it can be generalized to the fermion case.
5 Conclusion

In this paper we have shown that the definition of flavor neutrino states introduced in Ref. [9], can be seen as a particular case of the flavor Fock space formalism, once $\mu_e = \mu_\mu = 0$ and the mass vacuum is taken as physical vacuum. Both these choices are actually discussed: the first one brings to a discontinuity with the case $\theta = 0$ (no-mixing), while the second one leads to a violation of lepton number conservation in the production/detection vertex, at tree level. We thus conclude that the formalism of Ref. [9] leads to unphysical results.

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