On the Light dilaton in the Large $N$ Tri-critical $O(N)$ Model

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The leading order of the large $N$ limit of the $O(N)$ symmetric phi-six theory in three dimensions has a phase which exhibits spontaneous breaking of scale symmetry accompanied by a massless dilaton which is a Goldstone boson. At the next-to-leading order in large $N$, the phi-six coupling has a beta function of order $1/N$ and it is expected that the dilaton acquires a small mass, proportional to the beta function and the condensate. In this note, we show that this “light dilaton” is actually a tachyon. This indicates an instability of the phase of the theory with spontaneously broken approximate scale invariance.

FIG. 1: $N \times$ the beta function of large $N$ regime of $g^2(\bar{\phi}^2)^3$ theory in three dimensions. The infrared fixed point is $g^2_{IR} = 0$ and the ultra-violet fixed point occurs at $g^2_{UV} = 192$. The critical coupling where in the infinite $N$ limit scale symmetry breaking occurs is $g^2 = (4\pi)^2 \approx 158$.
will be the interplay between the loss of tune-ability of the coupling constant when it has a renormalization group flow and dynamical breaking of scale invariance which is driven by strong coupling dynamics and occurs at a specific fixed point. Our central conclusion will be that the “light dilaton” of this theory is actually a tachyon. This indicates an instability of the phase of the theory with spontaneously broken approximate scale invariance. Our computation is in complete perturbative control, at least in the context of a renormalization group improved large N expansion, when N is large enough. We note that, potential instability, based on the fact that this is ultimately a theory with a cubic potential was pointed out by Gudmundsdottr et. al. 22.

The composite operator effective action was originally computed by Townsend [14] and our results are in agreement with his where they overlap, the main difference being that he studied O(N) symmetry breaking whereas we study the the massive phase which occurs near the tricritical point. The O(N) vector field will be denoted \( \vec{\phi}(x) \). We will find it convenient to describe the theory by two variables, the composite field \( \chi(x) = \frac{1}{N}\vec{\phi}^2 \) and an auxiliary field \( M(x) \), whose expectation value is proportional to the \( \phi \)-field mass. Both \( \chi(x) \) and \( M(x) \) have classical dimension one and \( \frac{M(x)}{\chi(x)} \) is dimensionless. Whenever \( <M(x)> \) is not zero, the \( \vec{\phi} \)-field is massive and it does not obtain an expectation value. We will find that, at the leading and next-to leading order in the 1/N expansion, the renormalized background field effective action is

\[
S = N \int d^3x \left\{ \frac{\chi^3(x)}{6} \left( g^2(M(x)) - g^{*2} \frac{(M(x)}{\chi(x)}) \right) + \frac{\partial M(x) \cdot \partial M(x)}{96\pi|M(x)|} + \ldots \right\}
\]

where \( g^2(M) \) is the running coupling at scale \( M \) and \( g^{*2}(x) \), where \( x = \frac{\chi}{N} \), is the scale invariant part of the non-derivative terms in the effective action, containing contributions of order one and of order \( \frac{1}{N} \). The ellipses denote contributions of order \( \frac{1}{N^2} \) or higher of any type and terms with more than two derivatives. Although \( \chi \) is nominally a positive operator, an infinite normal ordering constant has been subtracted from it so that it can now be either positive or negative. The couplings have been tuned so that terms proportional to \( (\vec{\phi})^2 \) or \( \vec{\phi}^3 \) are absent.

To use the background field effective action (1) we should first solve the equations which determine its extrema,

\[
\frac{\delta S}{\delta \chi(x)} = 0, \quad \frac{\delta S}{\delta M(x)} = 0
\]

Solutions of these equations are the classical fields which we shall denote by \( M_0 \) and \( \chi_0 \). If there are more than one solution (there will not be in our example), we should choose the solution where \( S \), when evaluated on the solution, has the smallest real part. The expansion of the action in (1) in derivatives assumes that \( M_0 \) and \( \chi_0 \) are non-zero and that they are slowly varying functions, sufficiently so that the expansion in their derivatives is accurate. \((M_0 \text{ and } \chi_0 \text{ are usually constants.})\). Then, in order to compute a one-particle irreducible correlation function of the fields \( \chi(x) \) and \( M(x) \), we take functional derivatives of the background field action \( S \) by the variables \( \chi(x) \) and \( M(x) \), and we subsequently evaluate the resulting functions “on-shell” by setting \( \chi(x) \) and \( M(x) \) to \( \chi_0 \) and \( M_0 \), respectively. This yields the renormalized, connected, one-particle-irreducible multi-point correlation functions of the quantum fields \( \chi(x) \) and \( M(x) \).

For example, the connected two-point correlation functions are found by inverting the one-particle irreducible two-point functions which are obtained as functional second derivatives of the effective action. They are thus given by

\[
\left[ \langle \chi\chi \rangle - \langle \chi \rangle \langle \chi \rangle \right] (M_\chi - M_0 \chi_0) = \frac{\partial^2 S}{\partial \chi^2} - \frac{\partial^2 S}{\partial M \partial \chi} \frac{\partial^2 S}{\partial \chi \partial M}^{-1} \frac{\partial^2 S}{\partial M^2} \right]_{M_\chi=M_0,\chi_0}
\]

For example, we obtain the composite operator correlation function

\[
\frac{1}{N} \phi^2(x) \frac{\phi^2(y)}{N} - \frac{1}{N} \phi^2(x) - \frac{1}{N} \phi^2(y)
\]

\[
= \frac{1}{N} \int \frac{d^3p}{(2\pi)^3} e^{ip(x-y)} \left[ \frac{48\pi\chi^2_0/M_0}{p^2 + \frac{24\pi\chi_0^2}{M_0}} \beta(g^2(M_0)) \right]
\]

Let us review a few interesting features of our results:

1. We are putatively working in the leading and next-to-leading orders of the large \( N \) expansion. The quantities in brackets in (1) are of order one and of order \( \frac{1}{N} \). The running coupling constant, \( g^2(M) \), on the other hand, is the solution of the renormalization group equation using the beta function which is of order \( \frac{1}{N} \). If expanded in \( \frac{1}{N} \), it contains all orders of \( \frac{1}{M} \), multiplied by powers of logarithms of the mass scale ratio. This “sum of leading logarithms” is needed in order to accommodate possible very small or very large values of the condensate, \( M \sim \mu \exp(N \ldots) \).

2. At this order in the large \( N \) expansion, the only renormalization group function entering the effective action (1) is the running coupling constant \( g^2(M) \) which is to be evaluated at the scale determined by the condensate.

3. The generic features of the result in equation (3) do not depend much on the details of the function \( g^2(x) \) in equation (1). It relies only on the fact that its leading contribution at large \( N \) is independent of \( N \) and that it is scale invariant,
that is, it is a function of only the dimensionless ratio $\frac{m^2}{\chi}$ and $g^2$. Validity of the derivative expansion also requires non-zero $\chi_0$ and $M_0$ as the classical solutions. When evaluated on the solutions of the equations of motion (2), $g^2 = (4\pi)^2 + O(\frac{1}{N})$. At leading order in large $N$, $g^2 = 4\pi$, the value of the coupling at the Bardeen-Moshe-Bander fixed point.

4. When $N$ goes to infinity, the beta function vanishes and $g^2(M)$ becomes $M$-independent and tuneable. Consequently, at this limit, the action (1) has a term of first order phase transitions where the potential is equal to $\frac{r}{g^2}(\phi^2)^3$. The signature of the dilaton is the presence of the tachyonic mass $m_\phi = \frac{\chi_0}{\sqrt{2}}$. The tachyonic mass indicates that the phase that we are describing unstable to fluctuations. However, the values $\chi_0$ and $M_0$ which solve the equations of motion turn out to have opposite signs. That opposite sign results in the mass squared in (4) being $m^2 = \frac{1}{6} \chi^2(\phi^3)^3$. The latter vanishes at infinite $N$, leaving the dilaton massless in that limit.

5. The signature of the dilaton is the presence of the pole in the correlation function of $\langle \phi^2(x)\phi^2(y) \rangle$ in equation (3). Note that the mass of this pole is proportional to the beta function, $\beta(g^2(M))$. The latter vanishes at infinite $N$, leaving the dilaton massless in that limit.

6. For large but finite $N$, the pole in the two-point function (3) occurs at $-g^2 = 24\pi\chi_0^3 M_0 \beta(g^2(M_0))$

The beta-function is positive, $\beta(g^2(M_0)) > 0$. However, the values $\chi_0$ and $M_0$ which solve the equations of motion turn out to have opposite signs. That opposite sign results in the mass squared in the pole in the propagator in (3) having a negative sign and the dilaton has become a tachyon. The tachyonic mass indicates that the phase that we are describing unstable to fluctuations.

In the following, we will present our derivation of equation (1) and the simple computation leading from equation (1) to (3). We consider the Euclidean quantum field theory which has $N$ real scalar fields $\phi = (\phi^1, \phi^2, \ldots, \phi^N)$ and $O(N)$ symmetry in three space-time dimensions. The classical Landau-Ginzburg potential is given by

$$V(\phi^2) = \frac{r}{2} \phi^2 + \frac{u}{4} (\phi^2)^2 + \frac{g^2}{6} (\phi^2)^3 \quad (4)$$

where

$$\phi^2 = \sum_{a=1}^{N} \phi^a \phi^a$$

When $u > 0$, there is a line of second order phase transitions at $r = 0$ as depicted in FIG. 2. When $u < 0$ there is a line of first order phase transitions. These lines of transitions terminate at the tri-critical point $O$ where $u = r = 0$. The classical level this phase structure persists for all positive values of $g^2$ and the $g^2(\phi^2)^3$ coupling is exactly marginal.

To examine fluctuations, we consider the Euclidean functional integral

$$Z[j] = \int [d\phi] e^{-\int d^3x L(\phi, j)} \quad (5)$$

with the Lagrangian density

$$L = \frac{1}{2} \partial_\mu \phi^a \partial^\mu \phi^a + NV(\phi^2/N) - j \cdot \phi \quad (6)$$

where $\mu = 1, 2, 3$ and repeated indices are summed and we have introduced a source $j(x)$ in order to use the functional integral as a generating functional for correlators of $\phi(x)$. In order to study the large $N$ limit, we introduce two auxiliary fields by inserting

$$1 = \int_{-\infty}^{\infty} [d\chi(x)] \delta(\chi(x) - \phi^2/N) \quad (7)$$

$$= \int_{-\infty}^{\infty} [d\chi(x)] \int_{-\infty}^{\infty} [dm^2(x)] e^{\frac{1}{2} m^2(Nx - \phi^2)} \quad (8)$$

into the functional integral (5). This introduces two new fields $\chi(x)$ and $m^2(x)$ and it will allow us to integrate out the scalar field $\phi(x)$. We must be careful to note that the integration for $m^2$ is on the imaginary axis. We will find out and explicit form for the scale $M^2$ that was present in (1) as a function of $m^2$. With these additional fields, the Lagrangian becomes

$$L = \frac{1}{2} \partial_\mu \phi^a \partial^\mu \phi^a + \frac{m^2}{2} \phi^2 - N \frac{m^2}{2} \chi + NV(\chi) - j \cdot \phi \quad (9)$$
The $\vec{\phi}$-fields now appear in a quadratic form and we integrate them exactly to get an effective action

$$S[m^2, \chi, j] = \frac{N}{2} \text{Tr} \ln(-\partial^2 + m^2)$$

$$+ \int \left( N V(\chi) - N \frac{m^2}{2} \chi - \frac{1}{2} j^2 \frac{1}{\partial^2 + m^2} \right).$$

(10)

To find the partition function, it remains to integrate $\chi$ and $m^2$,

$$Z[j] = \int [dm^2 d\chi] e^{-S[m^2, \chi, j]}$$

(11)

This would yield a generating functional where functional derivatives with respect to $j$ give the correlation functions of the $\vec{\phi}$-fields.

We will study the region of the phase diagram where the O(N) symmetry is not spontaneously broken. Instead, there will be a condensates $\langle \chi(x) \rangle$ and $\langle m^2(x) \rangle$ which will result in a mass gap for the $\vec{\phi}$-field. To begin, it is instructive to put the source $j(x)$ to zero and to write the effective action in [10] in an expansion in derivatives of the variable $\chi(x)$ and $m^2(x)$,

$$S[N] = \int \left\{ \frac{\Lambda m^2}{4\pi^2} \frac{|m|^3}{12\pi} + V(\chi) - \frac{m^2 \chi}{2} + \frac{\partial m.\partial m}{96\pi|m|} + \ldots \right\}$$

(12)

where $\Lambda$ is the ultra-violet cut-off and the ellipses represent terms with more than two derivatives of $m$. We have dropped a constant term that is $m^2$ and $\chi$ independent. The effective action in [12] has an ultra-violet divergent $\Lambda$-dependent term which must be removed by renormalization. We can renormalize the expression by introducing counter-terms. This is accomplished by replacing $V(\chi)$ by $V(\chi - \frac{\Lambda}{\partial x})$. Then, after a field translation, $\chi(x) = \tilde{\chi}(x) + \frac{\Lambda}{\partial x}$, the cut-off dependent term cancels from [12]. Although $\chi(x)$ was originally a positive field, $\tilde{\chi}(x)$ may be positive or negative. We hereafter drop the tilde from $\tilde{\chi}$. The second, third and fourth terms in [12] are the effective potential for $m$ and $\chi$ at the leading order in the large $N$ expansion. In the remainder of this paper we will choose the potential $V(\chi)$ to be the specific dimension-three operator

$$V(\chi) = \frac{g^2}{6} \chi^3(x) + \text{counterterms}$$

(13)

where the counterterms will be needed to cancel divergences at higher orders in $\frac{1}{N}$. With this choice, the field theory is scale invariant at the classical level and, since there are no logarithms in [12], it remains scale invariant at the quantum level in the leading order in the large $N$ expansion.

In the large $N$ limit, we can use the saddle-point technique to evaluate the remaining functional integral [11]. The saddle points are field configurations which solve the equations of motion derived from the effective action [10]. We will use the notation $\chi_0$ and $m_0$ to denote fields which satisfy the equations of motion. When the fields are constant, the saddle points are extrema of the renormalized effective potential obtained from [12]. The potential in [12] has a line of extrema, located at $m_0 = -4\pi\chi_0$ and these extrema exist only when the coupling constant $g$ is set to the Bardeen-Moshe-Bander fixed point at $g = g^* = 4\pi$. To see this, consider the classical action (10).

$$\Delta(\chi, m) = \langle \frac{\chi^2}{\partial^2 + m^2} \rangle - \langle \chi \rangle^2$$

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(14)

and ask whether there is now a solution for $\chi$. When $g > g^*$, this expression has no extrema and for $g < g^*$ there is no spontaneous symmetry breaking ($\chi = 0$). However at $g = g^*$, the potential is flat and any (negative) constant $\chi_0$ is a solution.

To find the effective action to the next-to-leading order in large $N$, we use the background field technique. To implement this technique, we do the substitution

$$\chi \rightarrow \chi + \delta \chi , \ m^2 \rightarrow m^2 + i\delta m^2$$

(15)

and, following the recipe in [23], we drop the linear terms in $\delta \chi$ and $\delta m^2$. Then, the action expanded to quadratic order is

$$S = \frac{N}{2} \text{Tr} \ln(-\partial^2 + m^2) - \int N \frac{\Lambda m^2}{4\pi^2}$$

$$+ \int \left( N V(\chi) - N \frac{m^2}{2} \chi - \frac{1}{2} j^2 \frac{1}{\partial^2 + m^2} \right)$$

$$+ \frac{N}{2} \int \left[ \delta \chi, \delta m^2 \right] \left[ \partial \chi \right] \left[ \frac{i}{2} \frac{\delta m^2}{\partial^2 + m^2} \right] \times \ldots$$

(16)

where

$$\Delta(\chi, m) = \langle x | \frac{1}{\partial^2 + m^2} \langle y | \frac{1}{\partial^2 + m^2} | x \rangle \rangle$$

and

$$\mathcal{J} [x, y; j] = \frac{1}{N} \int dwdz j^\alpha(x) j^\beta(z) \cdot \langle w | \frac{1}{\partial^2 + m^2} | x \rangle \langle x | \frac{1}{\partial^2 + m^2} \langle y | \frac{1}{\partial^2 + m^2} | z \rangle$$

(18)

When $m^2$ is a constant,

$$\Delta(\chi, m) = \int \frac{d^d p}{(2\pi)^d} e^{i p (x - y)} \Delta(p)$$

$$\Delta(p) = \frac{1}{4\pi} \arctan \frac{p}{2|m|}$$

(19)
Before we proceed, we can use the action \[ S = N^2 \frac{1}{2} \phi^2 + \frac{1}{6} \phi^6 \] to study the spectrum of fluctuations in the infinite \( N \) limit. For this purpose, we invert the quadratic form in \[ S \] and find the propagator

\[
\langle \frac{1}{N} \phi^2 \phi^2 \rangle - \langle \frac{1}{N} \phi^2 \phi^2 \rangle = \langle \delta \chi \delta \chi \rangle = \frac{2}{N} \Delta(p) \left( 1 + 2N^2(\chi) \Delta(p) \right) = \frac{2}{N} \frac{\Delta(p)}{1 + 2N^2(\chi) \Delta(p)} \approx \frac{2}{N} \frac{\Delta(p)}{1 + \frac{2m^2}{p} \arctan \left( \frac{p}{m} \right)} \approx 3m \frac{1}{N \pi p^2}
\]

(20)

where, in the last equality, we have put the condensate on shell and the coupling constant equal to the fixed point value, \( 4\pi \). The last expression reproduces the sum of bubble diagrams which would be expected from studying the Feynman diagrams for this correlation function. The massless pole is due to the dilaton which is a Goldstone boson for spontaneous breaking of the scale symmetry which is exact at this order in the large \( N \) expansion. We can see that this massless pole is the only pole by studying the denominator of \( \Delta(p) \).

\[
1 - \frac{2m^2}{p} \arctan \left( \frac{p}{m} \right) = \int_0^1 dx \frac{x^2}{\pi x^2 + 2m^2}
\]

(21)

in the complex \(-p^2\)-plane. It is easiest to see from the integral representation of the function that the only zero is at \(-p^2 = 0\). There is also a cut singularity on the positive \(-p^2\)-axis beginning at \(4m^2\) due to intermediate \(\phi\)-particle pairs.

To study the next order in the large \( N \) expansion, we do the Gaussian integral over the fluctuations in \( S \) to get the effective action

\[
S = \frac{N}{2} \text{Tr} \ln(-\partial^2 + m^2)
\]

\[
+ \int \left( N^2 V(\chi) - m^2 \frac{1}{2} x^2 - \frac{1}{2N} \frac{1}{\partial^2 + m^2} \right)
\]

\[
+ \frac{1}{2} \ln \det \left[ -i/2 \chi - \Delta/2 + J \right] + \ldots
\]

(22)

where the ellipses stand for corrections of order \(1/N\) and higher. When we assume that the the source \( j \) and the classical fields \( m^2 \) and \( \chi \) are constants, we obtain the effective action evaluated on constant fields,

\[
S = N \int \left\{ -\frac{1}{12\pi} \left[ m^2 \right]^2 + V(\chi) - \frac{m^2}{2} \chi - \frac{j^2}{2m^2} \right\}
\]

\[
+ \frac{1}{2N} \int \frac{d^3p}{(2\pi)^3} \ln \left[ 1 + 2V''(\chi) \left( \Delta(p) + \frac{j^2}{2m^2(p^2 + m^2)} \right) \right]
\]

\[
+ \ldots
\]

(23)

Corrections represented by the ellipses in the last line of \( \text{(23)} \) are functions of \(1/N^2\) or higher order with \( m^2, \chi \) and \( j \) and terms with derivatives of \( m^2, \chi \) and \( j \).

The first line in \( \text{(25)} \) is the leading order in large \( N \) and the second line is the next-to-leading order. The integral in the next-to-leading order is ultra-violet divergent and renormalization is required. The linear term in the effective action in a Taylor expansion in \( j^2/N \) is

\[
- \frac{j^2}{2N} \left[ 1 \frac{1}{m^2} - \frac{4\pi^2}{N^2} \frac{1}{p^2 + m^2} \right.
\]

\[
\left. + \frac{1}{2N} \int \frac{d^3p}{(2\pi)^3} \frac{1}{p^2 + m^2} \right] + \ldots
\]

(24)

where the parameter \( M \) is proportional to the renormalized mass of the \( \phi \)-field. At this order in the large \( N \) expansion, the \( \phi \)-field wave-function renormalization is finite. The prime on the integration in the third line means that the first two divergent terms which are written before have been subtracted, resulting in a finite integral. (These divergent terms are the first two terms in a Taylor expansion of the integral in \( g^2 \).) Keeping \( M \) finite as the ultra-violet cut-off is scaled to infinity requires that we take \( m^2 \) to be a divergent function of \( M^2 \).

\[
m^2 = M^2 - \frac{4\pi^2}{N^2} \frac{g^2 \chi}{\Lambda^2} \left( \frac{M}{4\pi} - \frac{1}{g^2 M^2} \frac{1}{\xi_1 M} \right)
\]

\[
+ \frac{1}{2N} \int \frac{d^3p}{(2\pi)^3} \ln \left[ 1 + 2g^2 \chi \left( \frac{\Lambda}{M} + \frac{j^2}{M^2} \right) \right] + \ldots
\]

(25)

where \( \xi_1 \) parameterizes the finite part of the logarithmically divergent integral. The effective action is

\[
S = N \int \left\{ -\frac{1}{12\pi} \left[ M^3 + V(\chi) - \frac{M^2}{2} \chi - \frac{j^2}{2M^2} \right]
\]

\[
+ \left[ \frac{\chi}{2} + \frac{M}{8\pi} \right] \left[ \frac{4\pi^2}{N^2} \frac{g^2 \chi}{\Lambda^2} - \frac{M}{4\pi} \right] - \frac{g^2 \chi}{2N^2} \ln \frac{\Lambda}{\xi_1 M}
\]

\[
+ \frac{4\pi^2}{N^2} \frac{g^2 \chi}{\Lambda^2} \left[ \frac{1}{(2\pi)^3} \frac{1}{p^2 + M^2} \right. \left. + \frac{1}{2N} \int \frac{d^3p}{(2\pi)^3} \ln \left[ 1 + 2g^2 \chi \left( \Delta(p) + \frac{j^2}{M^2(p^2 + M^2)} \right) \right] \right]+ \ldots
\]

(26)

As before, the prime on the integral in the fourth line indicates that the term of order \( j^2/N \) and the divergent terms which are written in the fifth line (these are the first, second and third order terms in a Taylor expansion in \( g^2 \)) have been subtracted to render the integral finite. The terms that have been introduced by the mass renormalization are proportional to \( \left( \frac{\chi}{2} + \frac{M}{8\pi} \right) \) which vanishes on-shell. Here, we will first renormalize the effective action off-shell and then later on we will put the variables on-shell. We will find that the action is both on-shell and
off-shell renormalizable. The divergent terms in the fifth line are
\[
\frac{1}{N} \int \left[ V'' \Delta - (V'' \Delta)^2 + \frac{4}{3} V'' \Delta^3 \right] = \frac{g^2 \chi}{N} \left( \frac{\Lambda}{2\pi^2} - \frac{M}{4\pi} \right)^2
\]

\[
- \frac{g^4 \chi^2}{N} \frac{2 \xi}{\pi^2} - 4M \ln \frac{\Lambda}{\pi \xi} + \frac{g^6 \chi^2}{3 \cdot 2^8 \pi^2} \ln \frac{\Lambda}{M \xi_3}
\]
where \( \xi_2 \) and \( \xi_3 \) are constants which parameterize the finite parts of divergent integrals.

Putting these in the effective action, we find a miraculous cancellation. All divergent terms with a power of \( M \) in the numerator cancel. The remaining divergent terms can be canceled by counter-terms added to \( V(\chi) \) alone. What remains is
\[
S = N \int \left\{ -\frac{1}{12\pi} M^4 + \frac{g^2}{6} \chi^3 - \frac{M^2}{2} \chi - j^2/N - \frac{M^3}{2M^2} \right\} + \left( \chi + \frac{M}{4\pi} \right) \frac{g^2 \chi M^2}{2\pi N^2} - \frac{M^2}{16\pi^2 N^2} \ln \xi_2
\]
\[
+ \left( \chi + \frac{M}{4\pi} \right) \frac{g^2 \chi M^2}{2\pi N} \int \frac{dp}{p^2 + 1} + \frac{1}{g^2 M \arctan(p/2)} - \frac{M^3}{4\pi^2 N} \int \frac{dp^2}{p^2} \ln \left[ 1 + \frac{g^2 \chi}{2\pi^2} \left( \arctan(p/2) / 2\pi^2 + \frac{4j^2/NM^6}{(p^2 + 1)} \right) \right]
\]
\[
- \frac{g^4 \chi^3}{4\pi^2 N^2} \ln \frac{\mu}{\xi_1 M} + \frac{g^6 \chi^3}{3 \cdot 2^8 \pi^2 N} \ln \frac{\mu}{M \xi_3} + \ldots \right\} \quad (27)
\]

We shall set \( j^2 = 0 \) and seek solutions of the equations of motion
\[
\frac{\delta S}{\delta M} = 0 \quad (28)
\]
\[
\frac{\delta S}{\delta \chi} = 0 \quad (29)
\]

There are three important lessons to be learned from the form of the effective action \((27)\).

1. First of all, to this order \( 1/N \) the theory is off-shell renormalizable. The effective action that we have computed can be used to find the renormalized correlation functions of \( \phi, \chi, \) and \( \mu \) fields where all external lines have vanishing momenta.

2. The second lesson is that scale invariance is indeed violated at next-to-leading order in large \( N \), by the last two, logarithmic terms in \((27)\). From those terms we can find the beta function for the \( \beta^2/(\phi^2)^3 \) interaction. The effective action is a physical quantity, the volume times the energy of the theory when the fields are constrained to have certain expectation values. As such, it should not depend on the renormalization scale \( \mu \). This is so if \( g \) depends on \( \mu \) in such a way that the action does not depend on \( \mu \). This yields
\[
\frac{\partial}{\partial \mu} \left( g^2(\mu) - \frac{1}{N} \left[ \frac{3g^4(\mu)}{2\pi^2} - \frac{g^6(\mu)}{2^7\pi^2} \right] \ln \frac{\mu}{M} \right) = 0
\]
\[
\beta(g) = \mu \frac{d}{d\mu} [g^2(\mu)] = \frac{1}{N} \left( \frac{3g^4}{2\pi^2} - \frac{g^6}{2^7\pi^2} \right) + \ldots \quad (30)
\]
where the ellipses denote contributions of order \( 1/N \) and higher. This result matches the large \( N \) limit of the known perturbative beta-function \([17]\).

3. The third important feature of the effective action in \((27)\) is that the argument of the logarithms in the \( \mu \)-dependent terms contains only \( M \) and \( \mu \), and not \( \chi \). Moreover, its coefficient contains only \( \chi^3 \) and does not depend on \( M \). As a result of this structure, the equation of motion for \( M \), \((28)\), does not depend on \( \mu \), and it is therefore scale invariant. If we set \( j^2 = 0 \), \( \frac{2}{3} S = 0 \), \( \beta^2/\phi^2 \) is a homogeneous function of \( \chi \) and \( M \) and it is therefore solved by \( M = \alpha \chi \) whence it gives an equation for \( \alpha \). That equation is solved by \( \alpha = -4\pi + 6\delta \alpha \) where \( \delta \alpha \sim 1/N \).

We can write the effective action in the form
\[
\frac{S}{\xi} = \int \frac{3}{g^2} \left[ g^2(\mu) - g^2(\chi, g) - \beta(g(\mu)) \ln \frac{\mu}{M} + \ldots \right] \quad (31)
\]
where
\[
g^2(x, g) = \frac{1}{2\pi} x^3 + 3x^2 + O \left( \frac{1}{N} \right) \quad (32)
\]
and the \( g \)-dependence is only in the higher orders in \( 1/N \) and can be substituted for its leading order \( g = 4\pi \). Also, on-shell,
\[
g^2(x_0, g) = (4\pi)^2 + O \left( \frac{1}{N} \right) \quad (33)
\]
In \(-\chi^3/6 \cdot g^2\), we have gathered all of the terms in the effective action \((27)\) except those proportional to \( \ln \frac{\mu}{\pi} \) in the last line and the \( \frac{2}{3} \chi^3 \) term in the first line. In the last equality, we have used the leading order solution of the equation \( \delta S/\delta M = 0 \), which is \( M = \frac{\mu}{\chi} = -4\pi + O(1/N) \) in \( g^2 \).

Then minimum of \((31)\) occurs at
\[
M = \mu \exp \left( \frac{g^2 - g^2}{\beta(g)} - \frac{1}{3} \right) \quad (34)
\]
where we use equation \((33)\) for \( g^* \). This solution is non-perturbative, both in the sense that, since \( \beta \sim 1/\xi \), it does not have a Taylor expansion in \( 1/N \), and in the sense that, when it is substituted into the effective action, the logarithm produces a factor of \( 1/\beta \) \( \sim N \) which invalidates the large \( N \) expansion. In higher orders, powers of \( \ln \frac{\mu}{\pi} \) will produce factors of \( N \) which can cancel their large \( N \) suppression. This is similar to the phenomenon in the scalar field theory example in Coleman and Weinberg’s work \([23]\) on dynamical symmetry breaking. There, they used the renormalization group to re-sum higher order logarithmic terms to obtain a more accurate result. When
they did, the extremum went away - there was no longer a symmetry breaking solution. In the present case, we will be more fortunate. What allows us to find a solution is the presence of $g^*$ in the action. To begin, we will use the renormalization group to sum the leading logarithms of perturbation theory to all orders. In this particular case, it is very simple. We replace the combination which occurs in the effective action,

$$g^2(\mu) - \beta(g(\mu)) \ln \frac{\mu}{M},$$

(35)

by the running coupling at scale $M$, $g^2(M)$, which is defined by integrating the beta function

$$\int \frac{g^2(M)}{g^2(\mu)} \frac{d\mu}{\beta(g)} = \ln \frac{M}{\mu}$$

(36)

The result of the integral, $g^2(M)$, has a $1/N$ expansion and the leading terms reproduce (35). The corrections have higher orders in $1/N \ln M$. The renormalization group improved potential energy of the effective action is then the one given in equation (1), which we recopy here for the reader’s convenience,

$$S = N \int d^3x \left\{ \frac{\lambda^2(x)}{6} \left( g^2(M(x)) - g^2 \left( \frac{M(x)}{\chi(x)} \right) \right) + \frac{\partial M(x) \cdot \partial M(x)}{96\pi |M(x)|} + \ldots \right\}$$

We will now study the states of the theory using this effective action. The equations of motion are,

$$0 = \frac{\delta S}{\delta \chi(x)} = \frac{\lambda_0}{2} \left( g^2(M_0) - g^{*2} \right) + \frac{\chi_0 M_0}{6} g^{*2}$$

(37)

$$0 = \frac{\delta S}{\delta M(x)} = \frac{\chi_0}{6} \left( \frac{1}{M_0} \beta - \frac{1}{\chi_0} g^{*2} \right)$$

(38)

where $g^{*2}$ is a derivative of $g^2$ by its argument $M/\chi$. $M_0$ and $\chi_0$ are the solutions of these equations. We have dropped the derivative terms since we assume that the solutions will be constant fields.

Equations (37) and (38) imply

$$g^{*2} = \frac{\chi_0}{M_0} \beta (g^2(M_0))$$

(39)

$$g^2(M_0) - g^{*2} = -\frac{1}{3} \beta (g^2(M_0))$$

(40)

Equation (39) is an algebraic equation containing terms of order one and of order $1/N$ and the variables $\frac{M_0}{\chi_0}$ and $g^2$. $g^2$ appears only in the terms of order $1/N$ and it can therefore be regarded as a constant, and set to $4\pi$. In the leading order, equation (39) has the solution $\frac{M_0}{\chi_0} = -4\pi + O(1/N)$ and the order $1/N$ terms are easily computable.

We recall that $g^{*2}$ has a similar structure to equation (39), it contains terms of order one and of order $1/N$ and the variables $\frac{M_0}{\chi_0}$ and $g^2$, and $g^2$ appears only in the terms of order $1/N$. We can then plug the solution for $\frac{M_0}{\chi_0}$ which we discussed in the above paragraph into $g^*$ to obtain a $1/N$ corrected expression for it.

Then we use the corrected $g^{*2}$ in equation (40). The solution of equation (40) is a mass scale, that is, the value of the mass scale where the running coupling solves the equation. This yields the value of the condensate $M_0$ and the above considerations then determine $\chi_0 = -\frac{1}{4\pi} M_0 + O(\frac{1}{N})$. Due to the order $1/N$ violation of scale invariance, and unlike the scale invariant infinite $N$ limit, the values of these condensates are no longer arbitrary, but they are fixed by the value of the running coupling constant at some reference scale.

We substitute the solution into the effective action and then obtain

$$S_{\text{on-shell}} = N \int \left\{ \frac{M_0^3}{18(4\pi)^2} \beta (g(M_0)) + \ldots \right\}$$

(41)

where the ellipses are terms of order $1/N$ and higher.

To examine the fate of the dilaton, we return to the action (1) and we consider the fluctuation matrix about the solution that we have found,

$$\frac{1}{N} \delta^2 S = \frac{\chi_0}{3} \beta - \frac{M_0^2}{6\chi_0} g^{*2}$$

(42)

$$\frac{1}{N} \delta^2 S = \frac{\chi_0}{6M_0} \beta + \frac{M_0}{6} g^{*2}$$

(43)

$$\frac{1}{N} \delta^2 S = -\frac{\chi_0}{6M_0} \beta - \frac{\chi_0}{6} g^{*2} + \frac{M_0^2}{32\pi |M_0|}$$

(44)

where we have used equations (39) and (40) to simplify the right-hand-sides.

We can determine determinant of the fluctuation matrix and find,

$$\frac{1}{N^2} \det \left[ \frac{\delta^2 S}{\delta \chi^2} \frac{\delta^2 S}{\delta M^2} \frac{\delta^2 S}{\delta M^2} \frac{\delta^2 S}{\delta M^2} \right] = \frac{M_0^2}{32\pi^2} \beta - \frac{p^2}{4}$$

(45)

where we have used the fact that,

$$g^{*2} = \frac{1}{2\pi} \frac{M^3}{\chi^3} + \frac{3M^2}{\chi^2} = (4\pi)^2 + \ldots$$

(46)

and,

$$g^{*2} = \frac{3 M}{\pi \chi^2} + 6 + \ldots = -6 + \ldots$$

(47)

The determinant of the the fluctuation matrix is proportional to the inverse propagator of the $\chi$- and $M$-fields. The beta function is positive over the interesting range of $g^2$ (see FIG. 1). Clearly, from equation (45), we see that these excitations are tachyonic with mass given by,

$$m_{\text{dilaton}} = -\frac{3M_0^2}{8\pi^2 \beta}.$$
We conclude our paper by summarizing our results. We found that, although at leading order in $1/N$, phi-six theory in three dimensions exhibits spontaneous scale symmetry breaking accompanied by a massless dilaton, at the next-to-leading order in $1/N$, dilaton acquires a tachyonic mass and the spontaneously broken phase is therefore unstable. Our background field technique found the perturbative beta function of the $O(N)$ symmetric $\phi^2(\phi^2)^3$ theory. The result agreed with the beta function originally found by Pisarski [17]. Our result is that the tri-critical behaviour which is described by the massless $O(N)$ symmetric $\phi^2(\phi^2)^3$ theory is stable over a larger range of coupling constants that it is commonly thought to be. Of course, our analysis applies only if $N$ is not infinite, but if it is large enough that our large $N$ expansion is accurate.

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