Relativistic theory of surficial Love numbers

Philippe Landry and Eric Poisson

Department of Physics, University of Guelph, Guelph, Ontario, N1G 2W1, Canada

(Dated: April 27, 2014)

A relativistic theory of surficial Love numbers, which characterize the surface deformation of a body subjected to tidal forces, was initiated by Damour and Nagar. We revisit this effort in order to extend it, clarify some of its aspects, and simplify its computational implementation. First, we refine the definition of surficial Love numbers proposed by Damour and Nagar, and formulate it directly in terms of the deformed curvature of the body’s surface, a meaningful geometrical quantity. Second, we develop a unified theory of surficial Love numbers that applies equally well to material bodies and black holes. Third, we derive a compactness-dependent relation between the surficial and (electric-type) gravitational Love numbers of a perfect-fluid body, and show that it reduces to the familiar Newtonian relation when the compactness is small. And fourth, we simplify the tasks associated with the practical computation of the surficial and gravitational Love numbers for a material body.

I. INTRODUCTION AND SUMMARY

In a famous 1911 treatise [1], the mathematician and geophysicist A.E.H. Love introduced dimensionless quantities that characterize the deformation of an (otherwise spherical) astronomical body under the application of tidal forces exerted by remote bodies. The gravitational Love numbers $k_\ell$ describe the deformation of the body’s gravitational potential, as measured by the $\ell$-pole moment of its mass distribution. The surficial Love numbers $h_\ell$ describe the deformation of the body’s surface, also expanded in multipole moments.

The gravitational and surficial Love numbers provide complementary information regarding the tidal deformation. Because $k_\ell$ enters the description of the gravitational field outside the body, it affects the orbital motion of satellites and other bodies, and the gravitational Love numbers are of interest to astronomers. Because $h_\ell$ is involved in the description of the surface, the surficial Love numbers are of interest to geophysicists. A measurement of either set of Love numbers reveals useful information about the body’s internal structure and composition. For a body consisting of a perfect fluid, the Love numbers are linked by the universal relation $h_\ell = 1 + 2k_\ell$.

These notions have recently been ported to general relativity. The tidal deformation of neutron stars has been a topic of active interest since Flanagan and Hinderer [2, 3] pointed out that tidal effects can have measurable consequences on the gravitational waves emitted by a binary neutron star in the late stages of its orbital evolution. This implies that the gravitational waves carry information regarding the internal structure of each body, from which one can extract useful constraints on the equation of state of dense nuclear matter. Their initial study was followed up with more detailed analyses [4–15] which conclude that tidal effects might be accessible to measurement by the current generation of gravitational-wave detectors.

These observations have motivated the formulation of a proper relativistic theory of tidal deformations, featuring the precise definition of relativistic Love numbers. The gravitational Love numbers $k_\ell$ were promoted to general relativity by Damour and Nagar [16] and Binnington and Poisson [17], who showed that they actually come in two guises: an electric-type variety associated with the gravito-electric part of the tidal gravitational field, and a magnetic-type variety associated with the gravito-magnetic interaction. The gravitational Love numbers were computed for relativistic polytropes [16, 17], and for realistic neutron-star models constructed from tabulated equations of state [4, 8, 10, 15]. The gravitational Love numbers of a black hole were shown to be precisely zero [17].

The gravitational Love numbers of neutron stars have been implicated in a set of relations — the $I$-Love-$Q$ relations [19–24] — involving the moment of inertia $I$, the Love number $k_2^I$, and the quadrupole moment $Q$ of a neutron star. While each quantity depends on the star’s internal structure and composition, particular relations between these quantities display a remarkable independence relative to the details of internal structure.

The surficial Love numbers $h_\ell$ were also promoted to general relativity by Damour and Nagar [16]. In this paper we revisit this effort in order to extend it, clarify some of its aspects, and simplify its computational implementation.

Our first goal is to provide a refined definition of the surficial Love numbers as coordinate-invariant quantities in general relativity. In the original definition proposed by Damour and Nagar, the Love numbers are related to the coordinate deformation $\delta R$ of the body’s surface under the application of a tidal force. This definition is imported from the Newtonian theory, and it is made geometrically meaningful by embedding the two-dimensional surface in a three-dimensional, Euclidean space, so that $\delta R$ can be related to the curvature of this surface. Here we proceed differently. Instead of introducing a fictitious Euclidean space in the relativistic theory, we begin by formulating an alternative definition of the Newtonian Love numbers in terms of a curvature perturbation $\delta R$ instead of a surface displacement $\delta R$. The relation between $\delta R$ and $\delta R$ is still obtained by embedding the body’s surface in a Euclidean...
space, but this is now done in the Newtonian theory, for which the Euclidean manifold is a key foundational element. The new definition for the Newtonian love numbers is then promoted to general relativity by preserving the Newtonian relation between $\delta R$ and the tidal field. Our relativistic definition is equivalent to the one proposed by Damour and Nagar, but we feel that it is more compelling from a geometrical point of view. In essence, our approach is to geometrize the Newtonian definition by relying on the pre-existing Euclidean manifold, and to promote this definition to general relativity; their approach is to promote the coordinate definition first, and then to geometrize it with the help of a fictitious Euclidean space introduced as an additional structure.

Our second goal is to formulate a unified theory of relativistic surficial Love numbers that applies equally well to material bodies and (nonrotating) black holes. In their original work, Damour and Nagar noticed that the surficial Love numbers of a black hole, which were previously calculated by Damour and Lecian [25], can be recovered in the limit when the compactness $M/R$ of a material body approaches $1/2$; $M$ is the body’s mass, and $R$ is its areal radius. Because the limit cannot be physically attained by material bodies in hydrostatic equilibrium, the meaning of this coincidence was left unclear in the original work. We aim to provide clarification by showing that our geometrical definition for the surficial Love numbers applies to both material bodies and black holes, and that the equation relating $h_\ell$ to the metric perturbations takes the same form in both cases.

Our third goal is to derive a simple universal relation between the surficial and (electric-type) gravitational Love numbers of a perfect-fluid body; such a relation was not noticed in the original work by Damour and Nagar. The relation is

$$h_\ell = \Gamma_1 + 2\Gamma_2 k_\ell^3,$$

and it is universal in the sense that the coefficients $\Gamma_1$ and $\Gamma_2$ are functions of the compactness $M/R$ only; they are independent of all other details of internal structure. They are given by

$$\Gamma_1 = \frac{\ell + 1}{\ell - 1} (1 - M/R) F(-\ell, -\ell; -2\ell; 2M/R) - \frac{2}{\ell - 1} F(-\ell, -\ell - 1; -2\ell; 2M/R),$$

$$\Gamma_2 = \frac{\ell}{\ell + 2} (1 - M/R) F(\ell + 1, \ell + 1; 2\ell + 2; 2M/R) + \frac{2}{\ell + 2} F(\ell + 1, \ell + 2\ell + 2; 2M/R),$$

where $F(a, b; c; z)$ is the hypergeometric function. A relationship of the form of Eq. (1.1) was first written down by Yagi [23], but the expression provided here is substantially simpler. The figures reveal that $\Gamma_1$ is a decreasing function

![FIG. 1. Plot of the coefficient $\Gamma_1$ as a function of compactness $M/R$, for selected values of $\ell$.](image-url)
FIG. 2. Plot of the coefficient $\Gamma_2$ as a function of compactness $M/R$, for selected values of $\ell$.

of $M/R$, while $\Gamma_2$ is an increasing function. When $M/R \ll 1$ the functions can be approximated by

$$\Gamma_1 = 1 - (\ell + 1)(M/R) + \frac{\ell(\ell + 1)(\ell^2 - 2\ell + 2)}{(\ell - 1)(2\ell - 1)}(M/R)^2 + \cdots, \quad (1.3a)$$

$$\Gamma_2 = 1 + \ell(M/R) + \frac{\ell(\ell + 1)(\ell^2 + 4\ell + 5)}{(\ell + 2)(2\ell + 3)}(M/R)^2 + \cdots. \quad (1.3b)$$

In the limit $M/R \to 0$ we recover the Newtonian relation, $h_\ell = 1 + 2k_\ell^2$.

Because it is issued from the unified theory, the relation of Eq. (1.1) can be applied directly to black holes, for which $M/R = 1/2$ and $k_\ell^2 = 0$. Evaluation of the hypergeometric functions reveals that the surficial Love numbers of a nonrotating black hole are given by

$$h_\ell = \frac{\ell + 1}{2(\ell - 1)(2\ell)!} \ell^2. \quad (1.4)$$

This agrees with the result of Damour and Lecian [23].

Our fourth goal is to simplify the tasks associated with the practical computation of the (electric-type and magnetic-type) gravitational Love numbers for a material body governed by an arbitrary equation of state, in the hope to facilitate future investigations. These simplified procedures can also benefit the computation of surficial Love numbers, which are obtained from Eq. (1.1).

The paper is organized as follows. We begin in Sec. II with a review of Love numbers in the Newtonian theory of tidal deformations. To prepare the way for a relativistic definition of surficial Love numbers, we translate the traditional Newtonian definition, formulated in terms of a displacement of the surface, to an equivalent definition involving the perturbation of its intrinsic (Gaussian) curvature. This Newtonian definition is promoted to general relativity in Sec. III. In the following sections we endeavor to calculate the surficial Love numbers for perfect-fluid bodies and black holes. In Sec. IV we describe the perturbed hydrostatic equilibrium of a tidally deformed body, and establish that the surficial Love numbers are gauge-invariant quantities. In Sec. V we show that the main results obtained in Sec. IV apply just as well to a tidally deformed black hole, in spite of the fact that the surface of a material body and the event horizon of a black hole are physically very different. This observation implies that the surficial Love numbers of all compact bodies can be defined in a unified way. In Sec. VI we derive the relation of Eq. (1.1), and in Sec. VII we establish Eq. (1.4). In Sec. VIII we review and refine the method devised by Damour
and Nagar [16] to calculate the (electric-type and magnetic-type) gravitational Love numbers, a necessary input into the computation of the surficial Love numbers.

In this paper we complete the task, initiated by Damour and Nagar, of providing Love numbers with a proper relativistic formulation. The physical significance of gravitational Love numbers has been clearly demonstrated in the work reviewed above, and Tsang et al. have begun an exploration of the importance of surficial Love numbers in astrophysical processes involving neutron stars [26]. Our interest in carrying out this work, however, stems mostly from the observation that the Newtonian theory of tidal deformations and dynamics has achieved a great degree of perfection in the last three hundred years, in addition to delivering a host of highly accurate results. We aspire to the same degree of completeness and perfection for the relativistic theory.

II. LOVE NUMBERS IN NEWTONIAN THEORY

(Our treatment below follows Secs. 2.4 and 2.5 of Ref. [27], except for a change of normalization for the tidal moments.)

We consider, within the context of Newtonian gravity, a spherical body of mass $M$ and radius $R$ that is slightly deformed by tidal forces exerted by remote bodies. Working in the moving (and noninertial) reference frame of the body, the external potential is conveniently expressed as a Taylor expansion about the body’s center-of-mass. We have

$$U_{\text{ext}}(\mathbf{x}) = -\sum_{\ell=2}^{\infty} \frac{1}{(\ell-1)!} r^\ell \mathcal{E}_\ell \Omega^L,$$

where $\mathbf{x}$ is the displacement from the center-of-mass, $r := |\mathbf{x}|$, $\Omega := \mathbf{x}/r = [\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta]$, $L := a_1 a_2 \cdots a_\ell$ is a multi-index containing a number $\ell$ of individual indices, and $\Omega^L := \Omega^{a_1} \Omega^{a_2} \cdots \Omega^{a_\ell}$ is a string of $\ell$ angular vectors; summation over a repeated multi-index implies summation over all individual indices. The tensors

$$\mathcal{E}_\ell := -\frac{1}{(\ell-2)!} \partial_\ell U_{\text{ext}}(0),$$

where $\partial_\ell := \partial_{a_1} \partial_{a_2} \cdots \partial_{a_\ell}$, are tidal moments associated with the external field. The tensors $\mathcal{E}_\ell$ are symmetric in all their indices, and they are also completely tracefree (a contraction of any pair of indices vanishes) by virtue of the fact that the external potential satisfies Laplace’s equation $\nabla^2 U_{\text{ext}} = 0$ in the vicinity of the reference body. A factor of $(\ell-2)!$ was inserted in Eq. (2.2) to make contact with the relativistic description of the tidal environment, to be introduced below.

The decomposition of the external potential in symmetric-tracefree (STF) tensors is mathematically equivalent to a decomposition in spherical harmonics (see Sec. 1.5 of Ref. [27]). It can indeed be shown that the STF tensor $\Omega^{(L)}$ (with the angular brackets indicating the operation of trace removal on all indices) is in a one-to-one correspondence with the spherical harmonics of degree $\ell$. As a consequence we may write

$$\mathcal{E}_\ell \Omega^L = \sum_{\ell=-m}^{\ell} \mathcal{E}_{m}^{(\ell)} Y_m(\theta, \phi),$$

with the $2\ell + 1$ coefficients $c_m^{(\ell)}$ providing a packaging of the $2\ell + 1$ independent components of $\mathcal{E}_\ell$.

We consider situations for which the time dependence contained in the tidal moments is sufficiently slow that the tidal forces never take the body out of equilibrium. In such situations, the external time scale (comparable to $\sqrt{a^3/(GM)}$, in which $a$ is the inter-body distance) is much longer than the internal time scale (comparable to $\sqrt{R^3/(GM)}$) attached to physical processes taking place inside the body. In this regime the relaxation to a new equilibrium state following a slow change in the tidal environment (caused by the orbital motion of the remote bodies) can be idealized as taking place instantaneously. The time dependence of $\mathcal{E}_\ell$ is therefore irrelevant to the internal dynamics, and it can be ignored for the purposes of calculating the body’s deformation; although the tidal moments carry a parametric dependence upon time, they can still be treated as constants in our developments. This defines the regime of static tides.

The tidal forces produced by the remote bodies create a slight deformation of the reference body. One way of measuring the deformation is through the multipole moments of its mass distribution, defined by

$$I^L := \int \rho r^L \Omega^L \, d^3x.$$ 

These would all vanish in spherical symmetry, but in the presence of tidal forces we have instead

$$I_L = \frac{2(\ell-2)!}{(2\ell-1)!} k_\ell R^{2\ell+1} \mathcal{E}_\ell,$$
where \( k_\ell \), the gravitational Love numbers, are dimensionless and scale-free measures of the deformation; these quantities depend on the details of internal structure, and are computed on the basis of a detailed description of the perturbed interior. The total gravitational potential can then be written as \( U = M/r + \delta U \), where

\[
\delta U = -\sum_{\ell=2}^{\infty} \frac{1}{(\ell-1)!} \left[ 1 + 2k_\ell (R/r)^{2\ell+1} \right]^{\ell} \mathcal{E}_L \Omega^L,
\]

(2.5)

with the first set of terms (growing as \( r^\ell \)) representing the external potential, and the second set (decaying as \( r^{-(\ell+1)} \)) representing the body’s response.

Another way of measuring the tidal deformation is to examine the body’s surface. In the absence of a perturbation the surface is spherical and described by the equation \( r = R \). In the presence of a deformation we have instead \( r = R + \delta R \), with

\[
\delta R = -\sum_{\ell=2}^{\infty} \frac{h_\ell}{(\ell-1)!} \frac{R^{\ell+2}}{M} \mathcal{E}_L \Omega^L,
\]

(2.6)

where \( h_\ell \), the surficial Love numbers, are other dimensionless and scale-free measures of the deformation. These also depend on the details of internal structure. When the body consists of a perfect fluid, the body’s surface is necessarily an equipotential, and evaluating \( U \) at \( r = R + \delta R \) implies that the Love numbers are related by

\[
h_\ell = 1 + 2k_\ell.
\]

(2.7)

The relation is violated when the body’s interior is not well modeled as a fluid; this happens, for example, when there is a solid mantle surrounding a fluid core.

Because Eq. (2.6) is formulated in terms of a coordinate displacement, it is not immediately amenable to a relativistic generalization, which should provide \( h_\ell \) with a coordinate-invariant meaning. It is easy, however, to turn Eq. (2.6) into a geometrical statement that can be promoted to general relativity. The key is to describe the deformation not in terms of the surface displacement, but equivalently in terms of the perturbation in the surface’s intrinsic curvature.

To effect this translation we examine the intrinsic geometry of a surface \( r = R + \delta R \) in flat space. Working with the angular coordinates \( \theta^A = (\theta, \phi) \), we find that to first order in the displacement, the metric on the deformed surface is given by

\[
ds^2 = R^2 (1 + 2F) d\Omega^2,
\]

(2.8)

in which \( \Omega_{AB} := \text{diag}[1, \sin^2 \theta] \), the metric on a round two-sphere of unit radius, is given a deformation \( \delta \Omega_{AB} \). It is easy to show that to first order in the deformation, the Ricci scalar of the two-dimensional surface is given by

\[
\mathcal{R} = \frac{1}{R^2} (2 + \delta \mathcal{R}),
\]

(2.9)

with

\[
\delta \mathcal{R} = \left[ D^A D^B - (D^2 + 1) \Omega^{AB} \right] \delta \Omega_{AB},
\]

(2.10)

where \( D_A \) is the covariant derivative compatible with \( \Omega_{AB} \) (so that \( D_A \Omega_{BC} = 0 \)), \( D^A := \Omega^{AB} D_B \) with \( \Omega^{AB} \) the matrix inverse to \( \Omega_{AB} \), and \( D^2 := \Omega^{AB} D_A D_B \) is the Laplacian operator on the unit two-sphere. Notice that in Eq. (2.10), all tensorial operations are performed on a unit two-sphere with metric \( \Omega_{AB} \) instead of a two-sphere of radius \( R \).

In our specific case we have \( \delta \Omega_{AB} = 2F \Omega_{AB} \), and the curvature perturbation reduces to \( \delta \mathcal{R} = -2(D^2 + 2)F \). With \( F \) given by Eq. (2.6), substitution of the spherical-harmonic decomposition of Eq. (2.3) produces

\[
\delta \mathcal{R} = -2 \sum_{\ell=2}^{\infty} \frac{\ell + 2}{\ell} h_\ell \frac{R^{\ell+1}}{M} \mathcal{E}_L \Omega^L,
\]

(2.11)

an alternative definition for the surficial Love numbers. To arrive at this result we invoked the eigenvalue equation for spherical harmonics, \( D^2 Y_{\ell m} = -\ell (\ell + 1) Y_{\ell m} \).
III. SURFICIAL LOVE NUMBERS IN GENERAL RELATIVITY

With a suitable reinterpretation of $\delta \mathcal{R}$ on the left-hand side, and $\mathcal{E}_L$ on the right-hand side, Eq. (2.11) can be promoted to a precise definition of surficial Love numbers in general relativity. As we shall show, this definition possesses the essential property that the Love numbers are coordinate-invariant quantities. Zhang [28] provided a characterization of a tidal environment in general relativity with two sets of tidal moments $\mathcal{E}_L$ and $\mathcal{B}_L$. These are defined in terms of the asymptotic behavior of the spacetime’s Weyl tensor far away from the reference body (where $M \ll r \ll a$). Zhang’s expression for $g_{tt}$, specialized to a weak-field situation, is compatible with Eq. (2.1) with the normalization of the tidal moments specified in Eq. (2.2). Zhang’s construction, however, allows us to define the tidal moments $\mathcal{E}_L$ in full general relativity, and therefore to provide a meaningful interpretation for the right-hand side of Eq. (2.11). The second set of tidal moments, $\mathcal{B}_L$, does not enter in the definition of surficial Love numbers in general relativity.

The boundary of a material body defines a three-dimensional hypersurface in curved spacetime, the body’s world tube. In the regime of static tides described previously, this hypersurface can be foliated by $t = \text{constant}$ slices of the background spacetime, and each leaf represents a closed two-surface, the body’s boundary at each moment of time. This two-surface possesses a well-defined intrinsic geometry, and a well-defined Ricci scalar that can be expressed as in Eq. (2.9). This construction supplies $\delta \mathcal{R}$ with a precise meaning in general relativity, and an interpretation for the left-hand side of Eq. (2.11).

A precise meaning can also be assigned to $\delta \mathcal{R}$ in the case of a deformed black hole. The tidal deformation of the event horizon of a nonrotating black hole was examined in great generality by Vega, Poisson, and Massey [29]. Making use of the horizon’s null generators to establish a coordinate system on the horizon, the horizon’s intrinsic geometry is described in terms of a degenerate metric $\gamma_{AB}$ that is explicitly two-dimensional. The Ricci curvature associated with this metric can be expressed as in Eq. (2.9) with $\mathcal{R} = 2M$, thereby providing a precise definition for $\delta \mathcal{R}$ in the case of a black hole.

With $\mathcal{E}_L$ and $\delta \mathcal{R}$ thus defined in general relativity (for both material bodies and black holes), Eq. (2.11) provides $h_t$ with a proper general-relativistic definition. Because the tidal moments and surface curvature are both defined in a coordinate-invariant manner, this property is necessarily shared by the surficial Love numbers.

In the following we will endeavor to express $\delta \mathcal{R}$ in terms of the perturbation of the four-dimensional spacetime metric, and to find a relation between $h_t$ and $k_t^i$ that generalizes Eq. (2.7) to general relativity.

IV. PERTURBED HYDROSTATIC EQUILIBRIUM

We begin with a computation of $\delta \mathcal{R}$ in the case of a material body; the case of a black hole will be considered next. The first step is to locate the deformed surface of the body, which requires an understanding of the hydrostatic equilibrium of the perturbed configuration.

The background spacetime of the unperturbed body has a static and spherically-symmetric metric given by

$$ds^2 = -e^{2\psi} dt^2 + f^{-1} dr^2 + r^2 d\Omega^2$$  \hspace{1cm} (4.1)

with $f := 1 - 2m/r$, in which $\psi$ and $m$ depend on the radial coordinate $r$. The metric is a solution to the Einstein field equations with a perfect-fluid energy-momentum tensor

$$T^{\alpha\beta} = (\mu + p)u^{\alpha}u^{\beta} + pg^{\alpha\beta},$$  \hspace{1cm} (4.2)

in which $u^\alpha$ is the fluid’s velocity field, $p$ is the pressure, and $\mu = \rho + \epsilon$ is the total energy density, decomposed into a rest-mass density $\rho$ and a density of internal (thermodynamic) energy $\epsilon$. For a static configuration the only nonvanishing component of the velocity vector is $u^t = e^{-\psi}$, and the configuration is determined by the field equations

$$m' = 4\pi r^2 \mu, \quad \psi' = \frac{m + 4\pi r^3 p}{r^2 f},$$  \hspace{1cm} (4.3)

in which a prime indicates differentiation with respect to $r$. In the vacuum exterior the field equations produce the Schwarzschild solution $m = M = \text{constant}$ and $e^{2\psi} = 1 - 2M/r$. The body’s surface is situated at $r = R$, as determined by the condition $p(R) = 0$.

The equation of hydrostatic equilibrium, $(\mu + p)a_\alpha + \partial_\alpha p = 0$, in which $a_\alpha := u^\beta \nabla_\beta u_\alpha$ is the fluid’s covariant acceleration (with $a_\epsilon = \psi'$ as its only nonvanishing component) reduces to

$$p' = -(\mu + p)\psi' = -\frac{(\mu + p)(m + 4\pi r^3 p)}{r^2 f};$$  \hspace{1cm} (4.4)
this is the famous TOV (Tolman-Oppenheimer-Volkov) equation.

The perturbed spacetime has a metric $g_{\alpha\beta} + p_{\alpha\beta}$, and the perturbed fluid has an energy density $\mu + \delta\mu$, a pressure $p + \delta p$, a velocity $u^\alpha + \delta u^\alpha$, and an acceleration $a_\alpha + \delta a_\alpha$. The perturbed configuration satisfies the equation of hydrostatic equilibrium

$$(\mu + p)\delta a_\alpha + (\delta\mu + \delta p)a_\alpha + \partial_\alpha \delta p = 0, \quad (4.5)$$

and for a static configuration we find that $\delta a_t = 0$, $\delta a_r = -\frac{1}{2} \partial_r (e^{-2\psi} p_{tt})$, and $\delta a_A = -\frac{1}{2} \partial_A (e^{-2\psi} p_{tt})$. The radial component of Eq. (4.5) is

$$-\frac{1}{2}(\mu + p) \partial_r (e^{-2\psi} p_{tt}) + \psi' (\delta\mu + \delta p) + \partial_r \delta p = 0, \quad (4.6)$$

while the angular components are

$$\partial_A \left[ -\frac{1}{2}(\mu + p) e^{-2\psi} p_{tt} + \delta p \right] = 0. \quad (4.7)$$

Integrating the second equation and inserting the result within the first allows us to express $\delta\mu$ and $\delta p$ in terms of the metric perturbation $p_{tt}$. We obtain

$$\delta\mu = -r \mu' F, \quad \delta p = -rp' F, \quad (4.8)$$

in which $F$ is defined by

$$r\psi' F := \frac{1}{2} e^{-2\psi} p_{tt}. \quad (4.9)$$

The solutions (4.8) to the equation of hydrostatic equilibrium provide $F$ with a clear physical meaning. Suppose that $r = r_0$ describes a surface of constant density $\mu = \mu_0$ in the spherical, unperturbed configuration. Suppose also that $r = r_0 + \delta r$ describes the deformed surface $\mu = \mu_0 + \delta\mu$ in the perturbed configuration, and that we wish to find the displacement $\delta r$. We have $\mu_0 = \mu(r_0 + \delta r) + \delta\mu(r_0) = \mu_0 + \mu'(r_0) \delta r + \delta\mu(r_0)$, and inserting Eq. (4.8) produces $\delta r = r_0 F(r_0)$. This calculation reveals that a spherical surface $\mu = \text{constant}$ at radius $r$ is deformed to a surface $r + \delta r$ by the perturbation, with $\delta r = rF$. The same statement applies to a surface $p = \text{constant}$, and it applies in particular to $p = 0$, which marks the boundary of the fluid configuration — the body’s surface. This implies that the deformation of the body’s surface is described by

$$\delta R \over R = F(R, \theta^A), \quad (4.10)$$

with $F$ defined by Eq. (4.9). Taking into account that $e^{2\psi} = f$ and $\psi' = M/(R^2 f)$ on the boundary, we arrive at

$$F(R, \theta^A) = \frac{R}{2M} p_{tt}(R, \theta^A), \quad (4.11)$$

an explicit expression for $\delta R/R$ in terms of the metric perturbation.

The computation of $\delta R$ can now be completed. It is easy to show that to first order in the perturbation, the induced metric on a two-surface $t = \text{constant}$, $r = R(1 + F)$ in a spacetime with metric $g_{\alpha\beta} + p_{\alpha\beta}$ is given by Eq. (2.8) with

$$\delta \Omega_{AB} = 2F \Omega_{AB} + R^{-2} p_{AB}, \quad (4.12)$$

in which $F$ and $p_{AB}$ are evaluated at $r = R$ and expressed as functions of the angular coordinates $\theta^A$. Inserting this within Eq. (2.10) gives

$$\delta R = -2(D^2 + 2)F + \frac{1}{R^2} \left[ D^A D^B - (D^2 + 1) \Omega^{AB} \right] p_{AB}, \quad (4.13)$$

where we adopt the same conventions regarding the raising of angular indices as described below Eq. (2.10).

At this stage it is helpful to introduce a decomposition of $p_{\alpha\beta}$ into tensorial spherical harmonics. The relevant fields are (we adopt the notation of Ref. [30])

$$p_{tt} = \sum_{\ell m} h_{\ell m}^{t m} Y_{\ell m}, \quad (4.14a)$$

$$p_{AB} = r^2 \sum_{\ell m} \left( K_{\ell m}^{t m} \Omega_{AB} Y_{\ell m} + G_{\ell m}^{t m} \Omega_{AB}^{t m} \right) + \sum_{\ell m} h_{\ell m}^{t m} X_{AB}^{\ell m}, \quad (4.14b)$$
in which \( h^{\ell m}, K^{\ell m}, G^{\ell m}, h_2^{\ell m} \) depend on \( r \) only, and
\[
Y_{AB}^{\ell m} = \left[ D_A D_B + \frac{1}{2} \ell (\ell + 1) \Omega_{AB} \right] Y^{\ell m}, \tag{4.15a}
\]
\[
X_{AB}^{\ell m} = -\frac{1}{2} (\varepsilon_A^C D_B + \varepsilon_B^C D_A) D_C Y^{\ell m} \tag{4.15b}
\]
are (tracefree) tensorial harmonics of even and odd parity, respectively; \( \varepsilon_{AB} \) is the Levi-Civita tensor on the unit two-sphere, with components \( \varepsilon_{\theta \phi} = -\varepsilon_{\phi \theta} = \sin \theta \). Making the substitutions in Eq. (4.13), and taking into account Eq. (4.11), we arrive at
\[
\delta R = \sum_{\ell m} (\ell - 1)(\ell + 2) \left[ \frac{R}{M} h_{tt}^{\ell m} + K^{\ell m} + \frac{1}{2} \ell (\ell + 1) G^{\ell m} \right] Y^{\ell m} \tag{4.16}
\]
after evaluating all perturbation fields at \( r = R \), and after simplification of the angular operations on the spherical harmonics. These manipulations make use of \( D^2 Y^{\ell m} = -\ell (\ell + 1) Y^{\ell m} \), \( D^A D^B Y^{\ell m}_{AB} = \frac{1}{2} (\ell - 1) \ell (\ell + 1)(\ell + 2) Y^{\ell m} \), and \( D^A D^B X_{AB} = 0 \). Notice that \( \delta R \) involves only the even-parity fields \( h_{tt}^{\ell m}, K^{\ell m}, \) and \( G^{\ell m} \); there is no contribution from the odd-parity field \( h_2^{\ell m} \). This was to be expected, because \( \delta R \) is a scalar that can only be constructed from even-parity perturbations.

We may now prove that \( \delta R \) is a gauge-invariant quantity. Under a small coordinate transformation \( x'^\alpha = x'^\alpha + \Xi^\alpha \) the metric perturbation transforms as \( p'^{\alpha \beta} = p^{\alpha \beta} - \nabla_\alpha \Xi_\beta - \nabla_\beta \Xi_\alpha \). Adopting the decompositions \( \Xi = \sum_{\ell m} \xi^{\ell m} Y^{\ell m} \) and \( \Xi_A = \sum_{\ell m} \xi^{\ell m} D_A Y^{\ell m} \), we find that the perturbation fields change according to
\[
\begin{align*}
\tilde{h}_{tt}^{\ell m} &= h_{tt}^{\ell m} + 2 e^{2\psi} \psi' \xi^r, \tag{4.17a} \\
\tilde{K}^{\ell m} &= K^{\ell m} - \frac{2}{r} \xi^r + \frac{\ell (\ell + 1)}{r^2} \xi, \tag{4.17b} \\
\tilde{G}^{\ell m} &= G^{\ell m} - \frac{2}{r^2} \xi. \tag{4.17c}
\end{align*}
\]
Evaluating this at \( r = R \) and making the substitutions in Eq. (4.16) reveals that \( \delta R^{\text{new}} = \delta R^{\text{old}} \); the curvature perturbation is indeed gauge invariant. This conclusion was expected, given that the surface curvature is a meaningful, coordinate-independent, geometrical quantity.

V. CURVATURE PERTURBATION OF A BLACK HOLE

Equation (4.16) describes the curvature perturbation of the deformed surface of a material body. This will be related to the tidal moments \( E_L \) in Sec. VI but before we proceed we calculate \( \delta R \) for the event horizon of a deformed black hole.

Most of the work was carried out by Vega, Poisson, and Massey [29], who examined the more general case of dynamical tides; here we specialize their results to the regime of static tides. A general expression for \( \delta R \) is given in their Eq. (3.45), in terms of quantities defined in Eqs. (3.37) and (3.38); they have
\[
\delta R = \sum_{\ell m} (\ell - 1)(\ell + 2) \left[ 2 h^{\ell m} + K^{\ell m} + \frac{1}{2} \ell (\ell + 1) G^{\ell m} \right] Y^{\ell m}, \tag{5.1}
\]
with \( K^{\ell m} \) and \( G^{\ell m} \) evaluated at \( r = 2M \) as functions of \( \nu \), and
\[
h^{\ell m}(\nu) := \kappa \int_0^{\infty} e^{-\kappa(v'-\nu)} h_{tt}^{\ell m}(\nu', 2M) \, dv'; \tag{5.2}
\]
here \( \nu \) is the Eddington-Finkelstein advanced time defined by \( dv := dt + f^{-1} \, dr \), which is well behaved on the event horizon, and \( \kappa := 1/(4M) \) is the horizon’s surface gravity. In the regime of static tides the time scale associated with variations in \( h_{tt}^{\ell m} \) is very long compared with \( \kappa^{-1} \), and the integral of Eq. (5.2) can be approximated by \( h^{\ell m} = h_{tt}^{\ell m} \). Making the substitution in Eq. (5.1) returns
\[
\delta R = \sum_{\ell m} (\ell - 1)(\ell + 2) \left[ 2 h_{tt}^{\ell m} + K^{\ell m} + \frac{1}{2} \ell (\ell + 1) G^{\ell m} \right] Y^{\ell m}, \tag{5.3}
\]
in which all perturbation fields are evaluated at \( r = 2M \).

Equation (5.3) can be compared with Eq. (4.10), and this reveals that the black-hole result can be obtained as a special case of the material-body result by setting \( R = 2M \). This is a remarkable outcome, given that the surface of a material body and the event horizon of a black hole are physically extremely different. Nevertheless, we have established that the surface deformation of material bodies and black holes can be described meaningfully in a unified manner, in terms of their surface curvature.

VI. RELATION BETWEEN SURFICIAL AND GRAVITATIONAL LOVE NUMBERS

Because the perturbation fields \( h_{tt}^m \), \( K_{tt}^m \), and \( G_{tt}^m \) are continuous at \( r = R \), the curvature perturbation of Eq. (4.16) can be evaluated by relying on the external solutions of the perturbation equations instead of the internal solutions. And because \( \delta \mathcal{R} \) is gauge invariant, the computation can be carried out in any convenient gauge; we adopt the simplest choice of the Regge-Wheeler gauge. The solutions to the vacuum external problem are well known [31], and can be directly imported here.

Specifically we adapt the discussion contained in Sec. III of Binnington and Poisson [17], which was framed in the light-cone gauge, to the Regge-Wheeler gauge. We find that the external fields are given by

\[
\begin{align*}
&h_{tt}^m = -\frac{2}{(\ell - 1)\ell} r^\ell f^2 \left[ A_1 + 2k_{\ell}^b (R/r)^{2\ell + 1} B_1 \right] \mathcal{E}_m^{(\ell)}, \\
&K_{tt}^m = -\frac{2}{(\ell - 1)\ell} r^\ell \left[ A_2 + 2k_{\ell}^b (R/r)^{2\ell + 1} B_2 \right] \mathcal{E}_m^{(\ell)}, \\
&G_{tt}^m = 0,
\end{align*}
\]

where \( f = 1 - 2M/r \), \( k_{\ell}^b \) are the electric-type gravitational Love numbers, and

\[
\begin{align*}
&A_1 = F(-\ell, -\ell + 2; -2\ell; 2M/r), \\
&B_1 = F(\ell + 1, \ell + 3; 2\ell + 2; 2M/r), \\
&A_2 = \frac{\ell + 1}{\ell - 1} F(-\ell, -\ell; -2\ell; 2M/r) - \frac{2}{\ell - 1} F(-\ell, -\ell - 1; -2\ell; 2M/r), \\
&B_2 = \frac{2}{\ell + 2} F(\ell + 1, \ell + 1; 2\ell + 2; 2M/r) + \frac{2}{\ell + 2} F(\ell + 1, \ell; 2\ell + 2; 2M/r)
\end{align*}
\]

are functions of \( 2M/r \) expressed in terms of hypergeometric functions.

Making the substitutions in Eq. (4.16) and incorporating Eq. (2.3) returns

\[
\delta \mathcal{R} = -2 \sum_{\ell} \frac{\ell + 2}{\ell} \left[ f^2 (A_1 + 2k_{\ell}^b B_1) + \frac{M}{R} (A_2 + 2k_{\ell}^b B_2) \right] R^{\ell + 1} \frac{M}{r} \mathcal{E}_\ell \Omega^L.
\]

Comparing with Eq. (2.11) reveals that the quantity within square brackets can be identified with \( h_{\ell} \). After simplification we arrive at

\[
h_{\ell} = \Gamma_1 + 2\Gamma_2 k_{\ell}^e
\]

with

\[
\begin{align*}
\Gamma_1 &= \frac{\ell + 1}{\ell - 1} (1 - M/R) F(-\ell, -\ell; -2\ell; 2M/R) - \frac{2}{\ell - 1} F(-\ell, -\ell - 1; -2\ell; 2M/R), \\
\Gamma_2 &= \frac{\ell}{\ell + 2} (1 - M/R) F(\ell + 1, \ell + 1; 2\ell + 2; 2M/R) + \frac{2}{\ell + 2} F(\ell + 1, \ell; 2\ell + 2; 2M/R).
\end{align*}
\]

To obtain these expressions we made use of standard relations among contiguous hypergeometric functions. The properties of \( \Gamma_1 \) and \( \Gamma_2 \) were described in Sec. [1].

Equation (6.4) provides a complete solution to the problem of determining the surficial Love numbers of material bodies and black holes in general relativity. A computation of \( h_{\ell} \) is exceedingly simple once \( k_{\ell}^e \) and \( M/R \) have been obtained for a selected body.
VII. SURFICIAL LOVE NUMBERS OF BLACK HOLES

To compute \( h_\ell \) for black holes we rely on the fact that \( k^el = 0 \) and evaluate Eq. (6.4) explicitly. This requires the evaluation of hypergeometric functions at \( 2M/R = 1 \), which is achieved with the Chu-Vandermonde identity [Eq. (15.4.24) of Ref. [32]]. \( F(-\ell, -\ell; -\ell; 1) = (c - b)\ell/(c)\ell \), where \( (a)\ell = a(a + 1)\cdots(a + \ell - 1) \) is the Pochhammer symbol. With simple manipulations we find that \( F(-\ell, -\ell; -2\ell; 1) = \ell!^2/(2\ell)! \), \( F(-\ell, -\ell - 1; -2\ell; 1) = 0 \), and substitution within Eq. (6.4) yields

\[
    h_\ell = \frac{\ell + 1}{2(\ell - 1)(2\ell)!}. \tag{7.1}
\]

Equation (7.1) agrees with the result of Damour and Lecian [25].

VIII. COMPUTATION OF GRAVITATIONAL LOVE NUMBERS

In this section we summarize and simplify the recipe concocted by Damour and Nagar [10] to calculate the gravitational Love numbers of a perfect-fluid body in general relativity. Since most of this material can be extracted from Damour and Nagar and Binnington and Poisson [17], we provide very few derivations.

The electric-type Love numbers \( k^el \) are determined by integrating the equations governing the even-parity perturbations of a spherical body. After incorporating the solutions to the equation of hydrostatic equilibrium obtained in Sec. [V] performing a decomposition in spherical harmonics, and adopting the Regge-Wheeler gauge, the field equations for the metric perturbation inside the body give rise to a decoupled differential equation for \( h^elm \),

\[
    r^2h'' + Arh'_m - Bh_m = 0, \tag{8.1}
\]

with

\[
    A = \frac{2}{\ell} \left[ 1 - 3m/r - 2\pi r^2(\mu + 3p) \right], \tag{8.2a}
\]

\[
    B = \frac{1}{\ell} \left[ (\ell + 1) - 4\pi r^2(\mu + p)(3 + dp/dp) \right]. \tag{8.2b}
\]

This equation is integrated from \( r = 0 \), near which the solution behaves as \( h_m \propto r^\ell \). The internal solution is matched to the external solution (6.1) at \( r = R \), and the matching returns \( k^el \).

An efficient way to implement this procedure is to introduce the logarithmic derivative \( \eta := r\ell'h'/h_m \) and to recast Eq. (8.1) as

\[
    r\eta' + \eta(\eta - 1) + A\eta - B = 0. \tag{8.3}
\]

The main advantage of this formulation is that it is better conditioned for numerical integration. It also eliminates the need to determine the overall constant that multiplies \( h_m \) in the original formulation. This equation also is integrated from \( r = 0 \), at which \( \eta = \ell \), and the main outcome of the computation is \( \eta_h := \eta(r = R) \), the surface value of the logarithmic derivative.

Calculating \( r\ell'h'/h_m \) from Eq. (6.1) reveals that the matching condition at \( r = R \) is

\[
    \eta_h = \ell + \frac{4M}{R - 2M} + \frac{RA'_1 + 2k^el[RB'_1 - (2\ell + 1)B_1]}{A_1 + 2k^elB_1}, \tag{8.4}
\]

in which \( A_1 \) and \( B_1 \) are the hypergeometric functions introduced in Eq. (6.2). Solving for \( k^el \) gives

\[
    2k^el = \frac{RA'_1 - [\eta_h - \ell - 4M/(R - 2M)]A_1}{[\eta_h + \ell + 1 - 4M/(R - 2M)]B_1 - RB'_1}. \tag{8.5}
\]

With \( \eta_h \) obtained from the integration of Eq. (8.3), the electric-type Love number \( k^el \) follows after evaluating a couple of hypergeometric functions and their derivatives at \( r = R \).

A similar recipe can be formulated for the magnetic-type Love numbers \( k^mag \). Here the decoupled equation for the internal perturbation involves the coefficients \( h^elm \) of the decomposition of \( p_{\ell A} \) in odd-parity vectorial harmonics. It reads

\[
    r^2h'' - Prh'_m - Qh_m = 0, \tag{8.6}
\]
with
\[ P = \frac{4\pi r^2}{f}(\mu + p), \quad (8.7a) \]
\[ Q = \frac{1}{f}\left[\ell(\ell + 1) - 4m/r + 8\pi r^2(\mu + p)\right]. \quad (8.7b) \]

This equation is integrated from \( r = 0 \), near which the solution behaves as \( h_i \propto r^{\ell+1} \). The internal solution is matched across \( r = R \) to the external solution
\[ h_i^{\text{em}} = \frac{2}{3(\ell+1)}r^{\ell+1}\left[A_3 - 2\frac{\ell+1}{\ell}k_{\text{mag}}^{\ell}(R/r)^{2\ell+1}B_3\right]B_m^{(\ell)}, \quad (8.8) \]
where \( k_{\text{mag}}^{\ell} \) are the magnetic-type gravitational Love numbers,
\[ A_3 = F(-\ell + 1, -\ell - 2; -2\ell; 2M/r), \quad (8.9a) \]
\[ B_3 = F(\ell - 1, \ell + 2; 2\ell + 2; 2M/r), \quad (8.9b) \]
and \( B_m^{(\ell)} \) is the spherical-harmonic packaging of the tidal moments \( B_L \), defined as in Eq. 4.3.

The equivalent formulation in terms of \( \kappa := rh_i^2/h_i^2 \) is
\[ r\kappa' + (\kappa - 1) - P\kappa - Q = 0, \quad (8.10) \]
and the equation is integrated from \( r = 0 \) with \( \kappa = \ell + 1 \) to \( r = R \) at which \( \kappa = \kappa_s \). In this case the matching condition at \( r = R \) produces
\[ 2\frac{\ell+1}{\ell}k_{\text{mag}}^{\ell} = \frac{RA'_3 - (\kappa_s - \ell - 1)A_3}{RB'_3 - (\kappa_s + \ell)B_3}. \quad (8.11) \]

ACKNOWLEDGMENTS

We are grateful for discussions with Kent Yagi. This work was supported by the Natural Sciences and Engineering Research Council of Canada.

[1] A. E. H. Love, *Some problems of geodynamics* (Cornell University Library, Ithaca, USA, 1911).
[2] E. E. Flanagan and T. Hinderer, *Constraining neutron star tidal Love numbers with gravitational wave detectors*, Phys. Rev. D 77, 021502(R) (2008), arXiv:0709.1915.
[3] T. Hinderer, *Tidal Love numbers of neutron stars*, Astrophys. J. 677, 1216 (2008), erratum: Astrophys. J. 697, 964 (2009), arXiv:0711.2420.
[4] T. Hinderer, B. D. Lackey, R. N. Lang, and J. S. Read, *Tidal deformability of neutron stars with realistic equations of state and their gravitational wave signatures in binary inspiral*, Phys. Rev. D 81, 123016 (2010), arXiv:0911.3535.
[5] L. Baiotti, T. Damour, B. Giacomazzo, A. Nagar, and L. Rezzolla, *Analytic modeling of tidal effects in the relativistic inspiral of binary neutron stars*, Phys. Rev. Lett. 105, 261101 (2010), arXiv:1009.0521.
[6] L. Baiotti, T. Damour, B. Giacomazzo, A. Nagar, and L. Rezzolla, *Accurate numerical simulations of inspiralling binary neutron stars and their comparison with effective-one-body analytical models*, Phys. Rev. D 84, 024017 (2011), arXiv:1103.3874.
[7] J. Vines, E. E. Flanagan, and T. Hinderer, *Post-1-Newtonian tidal effects in the gravitational waveform from binary inspirals*, Phys. Rev. D 83, 084051 (2011), arXiv:1101.1673.
[8] F. Pannarale, L. Rezzolla, F. Ohme, and J. S. Read, *Will black hole-neutron star binary inspirals tell us about the neutron star equation of state?*, Phys. Rev. D 84, 104017 (2011), arXiv:1103.3526.
[9] B. D. Lackey, K. Kyutoku, M. Shibata, P. R. Brady, and J. L. Friedman, *Extracting equation of state parameters from black hole-neutron star mergers. I. Nonspinning black holes*, Phys. Rev. D 85, 044061 (2012), arXiv:1109.3402.
[10] T. Damour, A. Nagar, and L. Villain, *Measurability of the tidal polarizability of neutron stars in late-inspiral gravitational-wave signals*, Phys. Rev. D 85, 123007 (2012), arXiv:1203.4352.
[11] J. S. Read, L. Baiotti, J. D. E. Creighton, J. L. Friedman, B. Giacomazzo, K. Kyutoku, C. Markakis, L. Rezzolla, M. Shibata, and K. Taniguchi, *Matter effects on binary neutron star waveforms*, Phys. Rev. D 88, 044042 (2013), arXiv:1306.4065.
[12] J. E. Vines and E. E. Flanagan, *First-post-Newtonian quadrupole tidal interactions in binary systems*, Phys. Rev. D 88, 024046 (2013), arXiv:1009.4919.
B. D. Lackey, K. Kyutoku, M. Shibata, P. R. Brady, and J. L. Friedman, *Extracting equation of state parameters from black hole–neutron star mergers: Aligned-spin black holes and a preliminary waveform model*, Phys. Rev. D **89**, 043009 (2014), arXiv:1303.6298.

M. Favata, *Systematic parameter errors in inspiraling neutron star binaries*, Phys. Rev. Lett. **112**, 101101 (2014), arXiv:1310.8288.

K. Yagi and N. Yunes, *Love number can be hard to measure*, Phys. Rev. D **89**, 021303 (2014), arXiv:1310.8358.

T. Damour and A. Nagar, *Relativistic tidal properties of neutron stars*, Phys. Rev. D **80**, 084035 (2009), arXiv:0906.0096.

T. Binnington and E. Poisson, *Relativistic theory of tidal Love numbers*, Phys. Rev. D **80**, 084018 (2009), arXiv:0906.1366.

S. Postnikov, M. Prakash, and J. M. Lattimer, *Tidal Love numbers of neutron and self-bound quark stars*, Phys. Rev. D **82**, 024016 (2010), arXiv:1004.5098.

K. Yagi and N. Yunes, *I-Love-Q*, Science **341**, 365 (2013), arXiv:1302.4499.

K. Yagi and N. Yunes, *I-Love-Q relations in neutron stars and their applications to astrophysics, gravitational waves, and fundamental physics*, Phys. Rev. D **88**, 023009 (2013), arXiv:1303.1528.

T. Damour and O. M. Lecian, *On the gravitational polarizability of black holes*, Phys. Rev. D **80**, 044017 (2009), arXiv:0906.3003.

D. Tsang, J. S. Read, T. Hinderer, A. L. Piro, and R. Bondarescu, *Resonant shattering of neutron star crusts*, Phys. Rev. Lett. **108**, 011102 (2012).

E. Poisson and C. M. Will, *Gravity: Newtonian, Post-Newtonian, Relativistic* (Cambridge University Press, Cambridge, England, 2014).

F. W. J. Olver, D. W. Lozier, R. F. Boisvert, and C. W. Clark, *NIST Handbook of Mathematical Functions* (Cambridge University Press, Cambridge, England, 2010).