On the Liouville-Arnold integrable flows related with quantum algebras and their Poissonian representations

A. M. Samoilenko*, Y.A. Prykarpatsky**, D.L. Blackmore*** and A. K. Prykarpatsky**

*) Institute of Mathematics at the National Academy of Sciences, 3 Tereshchenkivska Str., Kyiv 00601 Ukraine
**) Dept. of Applied Mathematics at the AGH University of Science and Technology, 30 Mickiewicz Al. Bl. A4, 30059 Krakow, Poland and Dept. of Nonlinear Math. Analysis at the Institute of APMM of the Nat. Acad. of Sciences, Lviv,79601(email: prykanat@cyberagl.com, pryk.anat@ua.fm)
***) Dept. of Mathem. Studies at the NJIT, University Heights, New Jersey 07102 USA

Abstract. Based on the structure of Casimir elements associated with general Hopf algebras there are constructed Liouville-Arnold integrable flows related with naturally induced Poisson structures on arbitrary co-algebra and their deformations. Some interesting special cases including the oscillatory Heisenberg-Weil algebra related co-algebra structures and adjoin with them integrable Hamiltonian systems are considered.

1 Hopf algebras and co-algebras: main definitions

Consider a Hopf algebra \( A \) over \( \mathbb{C} \) endowed with two special homomorphisms called coproduct \( \Delta : A \to A \otimes A \) and counit \( \varepsilon : A \to \mathbb{C} \), as well an antihomomorphism (antipode) \( \nu : A \to A \), such that for any \( a \in A \)

\[
(id \otimes \Delta)(\Delta(a)) = (\Delta \otimes id)(\Delta(a)), \quad (id \otimes \varepsilon)(\Delta(a)) = (\varepsilon \otimes id)(\Delta(a)) = a,
\]

\[
m((id \otimes \nu)(\Delta(a))) = m((\nu \otimes id)(\Delta(a))) = \varepsilon(a)I,
\]

where \( m : A \otimes A \to A \) is the usual multiplication mapping, that is for any \( a, b \in A \) \( m(a \otimes b) = ab \). The conditions (1.1) were introduced by Hopf [1] in a cohomological context. Since most of the Hopf algebras properties depend on the coproduct operation \( \Delta : A \to A \otimes A \) and related with it Casimir elements, below we shall dwell mainly on the objects called co-algebras endowed with this coproduct.

The most interesting examples of co-algebras are provided by the universal enveloping algebras \( U(\mathcal{G}) \) of Lie algebras \( \mathcal{G} \). If, for instance, a Lie algebra \( \mathcal{G} \) possesses generators \( X_i \in \mathcal{G}, i = 1, \ldots, n = \dim \mathcal{G} \), the corresponding enveloping
algebra $U(\mathcal{G})$ can be naturally endowed with a Hopf algebra structure by defining
\[\Delta(X_i) = I \otimes X_i + X_i \otimes I, \quad \Delta(I) = I \otimes I,\]
\[\varepsilon(X_i) = -X_i, \quad \nu(I) = -I.\] (1.2)

These mappings acting only on the generators of $\mathcal{G}$ are straightforwardly extended to any monomial in $U(\mathcal{G})$ by means of the homomorphism condition $\Delta(XY) = \Delta(X)\Delta(Y)$ for any $X, Y \in \mathcal{G} \subset U(\mathcal{G})$. In general, an element $Y \in U(\mathcal{G})$ of a Hopf algebra such that $\Delta(Y) = I \otimes Y + Y \otimes I$ is called primitive, and the known Friedrichs theorem [2] ensures, that in $U(\mathcal{G})$ the only primitive elements are exactly generators $X_i \in \mathcal{G}$, $i = 1, n$.

On the other hand, the homomorphism condition for the coproduct $\Delta : A \to A \otimes A$ implies the compatibility of the coproduct with the Lie algebra commutator structure:
\[\left[\Delta(X_i), \Delta(X_j)\right]_{A \otimes A} = \Delta([X_i, X_j]_A)\] (1.3)
for any $X_i, X_j \in \mathcal{G}$, $i, j = 1, n$. Since the Drinfeld report [3] the co-algebras defined above are also often called "quantum" groups due to their importance [4] in studying many two-dimensional quantum models of modern field theory and statistical physics.

It was also observed (see for instance [4]), that the standard co-algebra structure (1.2) of the universal enveloping algebra $U(\mathcal{G})$ can be nontrivially extended making use of some of its infinitesimal deformations saving the co-associativity (1.3) of the deformed coproduct $\Delta : U_z(\mathcal{G}) \to U_z(\mathcal{G}) \otimes U_z(\mathcal{G})$ with $U_z(\mathcal{G})$ being the corresponding universal enveloping algebra deformation by means of a parameter $z \in \mathbb{C}$, such that $\lim_{z \to 0} U_z(\mathcal{G}) = U(\mathcal{G})$ subject to some natural topology on $U_z(\mathcal{G})$.

2 Casimir elements and their special properties

Take any Casimir element $C \in U_z(\mathcal{G})$, that is an element satisfying the condition $[C, U_z(\mathcal{G})] = 0$, and consider the action on it of the coproduct mapping $\Delta$:
\[\Delta(C) = C(\{\Delta(X)\}),\] (2.1)
where we put, by definition, $C := C(\{X\})$ with a set $\{X\} \subset \mathcal{G}$. It is a trivial consequence that for $A := U_z(\mathcal{G})$
\[\left[\Delta(C), \Delta(X_i)\right]_{A \otimes A} = \Delta([C, X_i]_A) = 0\] (2.2)
for any $X_i \in \mathcal{G}$, $i = 1, n$.

Define now recurrently the following $N$-th coproduct $\Delta^{(N)} : A \to A^{(N+1)}$ for any $N \in \mathbb{Z}_+$, where $\Delta^{(2)} := \Delta$ and $\Delta^{(1)} := id$ and
\[\Delta^{(N)} := ((id \otimes)^{N-2} \Delta) \cdot \Delta^{(N-1)},\] (2.3)
or as
\[ \Delta^{(N)} := (\Delta \otimes (id \otimes id^{N-2}) \otimes id) \cdot \Delta^{(N-1)}. \] (2.4)

One can straightforwardly verify that
\[ \Delta^{(N)} := (\Delta^{(m)} \otimes \Delta^{(N-m)}) \cdot \Delta \] (2.5)

for any \( m = 0, N \), and the mapping \( \Delta^{(N)} : \mathcal{A} \rightarrow \otimes^{(N+1)} \mathcal{A} \) is an algebras homomorphism, that is
\[ \left[ \Delta^{(N)}(X), \Delta^{(N)}(Y) \right]_{\otimes^{(N+1)} \mathcal{A}} = \Delta^{(N)}([X,Y]_{\mathcal{A}}) \] (2.6)

for any \( X, Y \in \mathcal{A} \). In a particular case if \( \mathcal{A} = U(\mathcal{G}) \), the following exact expression
\[ \Delta^{(N)}(X) = X \otimes id^{N-1} \otimes id + id \otimes X \otimes id^{N-1} \otimes id + ... \] (2.7)

holds for any \( X \in \mathcal{G} \).

### 3 Poisson co-algebras and their realizations

As is well known [5],[6], a Poisson algebra \( \mathcal{P} \) is a vector space endowed with a commutative multiplication and a Lie bracket \{.,.\} including a derivation on \( \mathcal{P} \) in the form
\[ \{a, bc\} = b\{a, c\} + \{a, b\}c \] (3.1)

for any \( a, b \) and \( c \in \mathcal{P} \). If \( \mathcal{P} \) and \( \mathcal{Q} \) are Poisson algebras one can naturally define the following Poisson structure on \( \mathcal{P} \otimes \mathcal{Q} : \)
\[ \{a \otimes b, c \otimes d\} \mathcal{P} \otimes \mathcal{Q} = \{a, c\} \mathcal{P} \otimes (bd) + (ac) \otimes \{b, d\} \mathcal{Q} \] (3.2)

for any \( a, c \in \mathcal{P} \) and \( b, d \in \mathcal{Q} \). We shall also say that \( (\mathcal{P}; \Delta) \) is a Poisson co-algebra if \( \mathcal{P} \) is a Poisson algebra and \( \Delta : \mathcal{P} \rightarrow \mathcal{P} \otimes \mathcal{P} \) is a Poisson algebras homomorphism, that is
\[ \{\Delta(a), \Delta(b)\}_{\mathcal{P} \otimes \mathcal{P}} = \Delta(\{a, b\}_{\mathcal{P}}) \] (3.3)

for any \( a, b \in \mathcal{P} \).

It is useful to note here that any Lie algebra \( \mathcal{G} \) generates naturally a Poisson co-algebra \( (\mathcal{P}; \Delta) \) by defining a Poisson bracket on \( \mathcal{P} \) by means of the following expression: for any \( a, b \in \mathcal{P} \)
\[ \{a, b\}_{\mathcal{P}} := \langle \text{grad}, \vartheta \text{grad} b \rangle. \] (3.4)

Here \( \mathcal{P} \simeq C^\infty(\mathbb{R}^n; \mathbb{R}) \) is a space of smooth mappings linked with a base variables of the Lie algebra \( \mathcal{G} \), \( n = \dim \mathcal{G} \), and the implictic [6] matrix \( \vartheta : T^*(\mathcal{P}) \rightarrow T(\mathcal{P}) \) is given as
\[ \vartheta(x) = \sum_{k=1}^{n} c_{ij}^k x_k : i, j = 1, \cdots, n, \] (3.5)
where \( c^k_{ij}, i, j, k = 1, n \), are the corresponding structure constants of the Lie algebra \( \mathcal{G} \) and \( x \in \mathbb{R}^n \) are the corresponding linked coordinates. It is easy to check that the coproduct (1.2) is a Poisson algebras homomorphism between \( \mathcal{P} \) and \( \mathcal{P} \otimes \mathcal{P} \). If one can find a "quantum" deformation \( U_z(\mathcal{G}) \), then the corresponding Poisson co-algebra \( \mathcal{P}_z \) can be constructed making use of the naturally deformed implicative matrix \( \vartheta_z \): 

\[
\begin{bmatrix}
\tilde{X}_2, \tilde{X}_1 \\
\tilde{X}_3, \tilde{X}_1 \\
\end{bmatrix} = \begin{bmatrix}
\tilde{X}_3, \tilde{X}_2, \tilde{X}_3 = -\tilde{X}_1 \\
\tilde{X}_1 \\
\end{bmatrix} = \frac{1}{z} \sinh(z \tilde{x}_2),
\]

(3.6)

where at \( z = 0 \) elements \( \tilde{X}_i \big|_{z=0} = X_i \in \mathfrak{so}(2,1), i = 1, 3 \), compile a base of generators of the Lie algebra \( \mathfrak{so}(2,1) \). Then, based on expressions (3.6) one can easily construct the corresponding Poisson co-algebra \( \mathcal{P}_z \), endowed with the implicative matrix

\[
\vartheta_z(\tilde{x}) = \begin{pmatrix}
0 & -\tilde{x}_3 & -\frac{1}{z} \sinh(z \tilde{x}_2) \\
\tilde{x}_3 & -\tilde{x}_1 \\
\frac{1}{z} \sinh(z \tilde{x}_2) & \tilde{x}_1 & 0
\end{pmatrix}
\]

(3.7)

for any point \( \tilde{x} \in \mathbb{R}^3 \), linked naturally with the deformed generators \( \tilde{X}_i, i = 1, 3 \), taken above. Since the corresponding coproduct on \( U_z(\mathfrak{so}(2,1)) \) acts on this deformed base of generators as

\[
\Delta(\tilde{X}_2) = I \otimes \tilde{X}_2 + \tilde{X}_2 \otimes I,
\]

\[
\Delta(\tilde{X}_1) = \exp\left(-\frac{z}{2} \tilde{X}_2\right) \otimes \tilde{X}_1 + \exp\left(\frac{z}{2} \tilde{X}_2\right) \otimes \tilde{X}_1,
\]

\[
\Delta(\tilde{X}_2) = \exp\left(-\frac{z}{2} \tilde{X}_2\right) \otimes \tilde{X}_3 + \exp\left(\frac{z}{2} \tilde{X}_2\right) \otimes \tilde{X}_3,
\]

satisfying the main homomorphism property for the whole deformed universal enveloping algebra \( U_z(\mathfrak{so}(2,1)) \).

Consider now some realization of the deformed generators \( \tilde{X}_i \in U_z(\mathcal{G}), i = 1, n \), that is a homomorphism mapping \( D_z : \mathcal{U}_z(\mathcal{G}) \to \mathcal{P}(M) \), such that

\[
D_z(\tilde{X}_i) = \tilde{e}_i,
\]

(3.9)

\( i = 1, n \), are some elements of a Poisson manifold \( \mathcal{P}(M) \) realized as a space of functions on a finite-dimensional manifold \( M \), satisfying the deformed commutator relationships

\[
\{ \tilde{e}_i, \tilde{e}_j \}_{\mathcal{P}(M)} = \vartheta_{z,ij}(\tilde{e}),
\]

(3.10)

where, by definition, expressions \( [\tilde{X}_i, \tilde{X}_j] = \vartheta_{z,ij}(\tilde{X}), i, j = 1, n \), generate a Poisson co-algebra structure on the function space \( \mathcal{P}_z := \mathcal{P}_z(\mathcal{G}) \) linked with a given Lie algebra \( \mathcal{G} \). Making use of the homomorphism property (3.3) for the
coproduct mapping $\Delta : P_z(G) \rightarrow P_z(G) \otimes P_z(G)$, one finds that for all $i, j = \overline{1,n}$

$$\{\Delta(x_i), \Delta(x_j)\}_{P_z(G) \otimes P_z(G)} = \Delta(\{x_i, x_j\}_{P_z(G)}) = \vartheta_{z,ij}(\Delta(x))$$ (3.11)

and for the corresponding coproduct $\Delta : P(M) \rightarrow P(M) \otimes P(M)$ one gets similarly

$$\{\Delta(\tilde{e}_i), \Delta(\tilde{e}_j)\}_{P(M) \otimes P(M)} = \Delta(\{\tilde{e}_i, \tilde{e}_j\}_{P(M)}) = \vartheta_{z,ij}(\Delta(\tilde{e}))$$ (3.12)

where $\{\ldots\}_{P(M)}$ is some, eventually, canonical Poisson structure on a finite-dimensional manifold $M$.

Let $q \in M$ be a point of $M$ and consider its coordinates as elements of $P(M)$. Then one can define the following elements

$$q_j := (I \otimes)^{j-1} q(\otimes I)^{N-j} \in \otimes^N P(M),$$ (3.13)

where $j = \overline{1,N}$ by means of which one can construct the corresponding $N$-tuple realization of the Poisson co-algebra structure (3.12) as follows:

$$\{\tilde{e}_i^{(N)}, \tilde{e}_j^{(N)}\}_{\otimes^N P(M)} = \vartheta_{z,ij}(\tilde{e}^{(N)}),$$ (3.14)

with $i, j = \overline{1,n}$ and

$$(N) \otimes D_z(\Delta^{(N-1)}(\tilde{e}_i) := \tilde{e}_i^{(N)}(q_1, q_2, \ldots, q_N).$$ (3.15)

For instance, for the $U_z(\mathfrak{so}(2,1))$ case (3.6), one can take [7] the realization Poisson manifold $P(M) = P(\mathbb{R}^2)$ with the standard canonical Heisenberg-Weil Poissonian structure on it:

$$\{q, q\}_{P(\mathbb{R}^2)} = 0 = \{p, p\}_{P(\mathbb{R}^2)}, \quad \{p, q\}_{P(\mathbb{R}^2)} = 1,$$ (3.16)

where $(q, p) \in \mathbb{R}^2$. Then expressions (3.15) for $N = 2$ give rise to the following relationships

$$\tilde{e}_1^{(2)}(q_1, q_2, p_1, p_2) := (D_z \otimes D_z)\Delta(\tilde{X}_1) = 2\sinh(\tilde{z}p_1)\cos q_1 \exp(\tilde{z}p_2) + 2\exp(-\tilde{z}p_1)\frac{\sinh(\tilde{z}p_2)}{\tilde{z}} \cos q_2,$$

$$\tilde{e}_2^{(2)}(q_1, q_2, p_1, p_2) := (D_z \otimes D_z)\Delta(\tilde{X}_2) = p_1 + p_2,$$ (3.17)

where elements $(q_1, q_2, p_1, p_2) \in P(\mathbb{R}^2) \otimes P(\mathbb{R}^2)$ satisfy the induced by (3.16) Heisenberg-Weil commutator relations:

$$\{q_i, q_j\}_{P(\mathbb{R}^2) \otimes P(\mathbb{R}^2)} = 0 = \{p_i, p_j\}_{P(\mathbb{R}^2) \otimes P(\mathbb{R}^2)}, \quad \{p_i, q_j\}_{P(\mathbb{R}^2) \otimes P(\mathbb{R}^2)} = \delta_{ij}$$ (3.18)

for any $i, j = \overline{1,2}$.
4 Casimir elements and the Heisenberg-Weil algebra related algebraic structures

Consider any Casimir element $\tilde{C} \in U_z(G)$ related with an $\mathbb{R} \ni z$--deformed Lie algebra $G$ structure in the form

\[ [\tilde{X}_i, \tilde{X}_j] = \vartheta_{z,ij}(\tilde{X}), \] (4.1)

where $i, j = 1, n$, $n = \dim G$, and, by definition, $[\tilde{C}, \tilde{X}_i] = 0$. The following general lemma holds.

**Lemma 1** Let $(U_z(G); \Delta)$ be a co-algebra with generators satisfying (4.1) and $\tilde{C} \in U_z(G)$ be its Casimir element; then

\[ [\Delta^{(m)}(\tilde{C}), \Delta^{(N)}(\tilde{X}_i)]_{(N+1)} U_z(G) = 0 \] (4.2)

for any $i = 1, n$ and $m = 1, N$.

As a simple corollary of this Lemma one finds from (4.2) that

\[ [\Delta^{(m)}(\tilde{C}), \Delta^{(N)}(\tilde{C})]_{(N+1)} U_z(G) = 0 \]

for any $k, m \in \mathbb{Z}_+$. Consider now some realization (3.9) of our deformed Poisson co-algebra structure (4.1) and check that the expression

\[ [\Delta^{(m)}(\tilde{C}), \Delta^{(N)}(\tilde{C})]_{(N+1)} P(M) = 0 \] (4.3)

too for any $m = 1, N$, $N \in \mathbb{Z}_+$, if $C(\tilde{e}) \in I(\mathcal{P}(M))$, that is $\{C(\tilde{e}), q\}_{\mathcal{P}(M)} = 0$ for any $q \in M$. Since

\[ \mathcal{H}^{(N)}(q) := \Delta^{(N+1)}(\tilde{H}(\tilde{e})) \] (4.4)

are in general, smooth functions on $\mathcal{P}(M)$, which can be used as Hamilton ones subject to the Poisson structure on $\mathcal{P}(M)$, the expressions (4.4) mean nothing else that functions

\[ \gamma^{(m)}(q) := \Delta^{(N)}(C(\tilde{e})) \] (4.5)

are their invariants, that is

\[ \{\gamma^{(m)}(q), \mathcal{H}^{(N)}(q)\}_{(N+1)} \mathcal{P}(M) = 0 \] (4.6)

for any $m = 1, N$. Thereby, the functions (4.4) and (4.5) generate under some additional but natural conditions a hierarchy of a priori Liouville-Arnold integrable Hamiltonian flows on the Poisson manifold $\mathcal{P}(M)$.
Consider now a case when a Poisson manifold $\mathcal{P}(M)$ and its co-algebra deformation $\mathcal{P}_{z}(\mathcal{G})$. Thus for any coordinate points $x_i \in \mathcal{P}(\mathcal{G})$, $i = 1, n$, the following relationships

$$\{x_i, x_j\} = \sum_{k=1}^{n} c_{ij}^{k} x_k := \vartheta_{ij}(x)$$ (4.7)

define a Poisson structure on $\mathcal{P}(\mathcal{G})$, related with the corresponding Lie algebra structure of $\mathcal{G}$, and there exists a representation (3.9), such that elements $\tilde{\epsilon}_i := D_z(\tilde{X}_i) = \tilde{\epsilon}_i(x)$ satisfy the relationships $\{\tilde{\epsilon}_i, \tilde{\epsilon}_j\}_{\mathcal{P}_{z}(\mathcal{G})} = \tilde{\vartheta}_{z,ij}(\tilde{\epsilon})$ for any $i = 1, n$, with the limiting conditions

$$\lim_{z \to 0} \vartheta_{z,ij}(\tilde{\epsilon}) = \sum_{k=1}^{n} c_{ij}^{k} x_k, \quad \lim_{z \to 0} \tilde{\epsilon}_i(x) = x_i$$ (4.8)

for any $i, j = 1, n$ being held. For instance, take the Poisson co-algebra $\mathcal{P}_{z}(so(2, 1))$ for which there exists a realization (3.9) in the following form:

$$\tilde{\epsilon}_1 := D_z(\tilde{X}_1) = \frac{\sinh(\frac{z}{2} x_2)}{x_2} x_1, \quad \tilde{\epsilon}_2 := D_z(\tilde{X}_2) = x_2,$$ (4.9)

$$\tilde{\epsilon}_3 := D_z(\tilde{X}_3) = \frac{\sinh(\frac{z}{2} x_2)}{x_2} x_3,$$

where $x_i \in \mathcal{P}(so(2, 1))$, $i = 1, 3$, satisfy the $so(2, 1)$–commutator relationships

$$\{x_2, x_1\}_{\mathcal{P}(so(2, 1))} = x_3, \quad \{x_2, x_3\}_{\mathcal{P}(so(2, 1))} = -x_1,$$ (4.10)

$$\{x_3, x_1\}_{\mathcal{P}(so(2, 1))} = x_2,$$

with the coproduct operator $\Delta : \mathcal{U}_{z}(so(2, 1)) \to \mathcal{U}_{z}(so(2, 1)) \otimes \mathcal{U}_{z}(so(2, 1))$ being given by (3.8). It is easy to check that conditions (4.7) and (4.8) hold.

The next example is related with the co-algebra $\mathcal{U}_{z}(\pi(1, 1))$ of the Poincare algebra $\pi(1, 1)$ for which the following non-deformed relationships

$$[X_1, X_2] = X_3, \quad [X_1, X_3] = X_2, \quad [X_3, X_2] = 0$$ (4.11)

hold. The corresponding coproduct $\Delta : \mathcal{U}_{z}(\pi(1, 1)) \to \mathcal{U}_{z}(\pi(1, 1)) \otimes \mathcal{U}_{z}(\pi(1, 1))$ is given by the Woronowicz [8] expressions

$$\Delta(\tilde{X}_1) = I \otimes \tilde{X}_1 + \tilde{X}_1 \otimes I,$$ (4.12)

$$\Delta(\tilde{X}_2) = \exp\left(-\frac{z}{2} \tilde{X}_1\right) \otimes \tilde{X}_1 + \tilde{X}_1 \otimes \exp\left(\frac{z}{2} \tilde{X}_1\right),$$

$$\Delta(\tilde{X}_3) = \exp\left(-\frac{z}{2} \tilde{X}_1\right) \otimes \tilde{X}_3 + \tilde{X}_3 \otimes \exp\left(\frac{z}{2} \tilde{X}_1\right),$$

where $z \in \mathbb{R}$ is a parameter. Under the deformed expressions (4.12) the elements $\tilde{X}_j \in \mathcal{U}_{z}(\pi(1, 1))$, $j = 1, 3$, satisfy still undeformed commutator relationships, that is $\vartheta_{z,ij}(\tilde{X}) = \tilde{\vartheta}_{ij}(X)|_{X \to \tilde{X}}$ for any $z \in \mathbb{R}$, $i, j = 1, 3$, being given by
co-algebra, for which \( x \in \mathcal{P}(1,1) \), as follows:

\[
\{x_1, x_2\}_{\mathcal{P}(\pi(1,1))} = x_3, \quad \{x_1, x_3\}_{\mathcal{P}(\pi(1,1))} = x_2, \quad \{x_2, x_3\}_{\mathcal{P}(\pi(1,1))} = 0
\]

\( (4.13) \)

h holds. Moreover, since \( C = x_2^2 - x_3^2 \in I(\mathcal{P}(\pi(1,1))) \), that is \( \{C, x_i\}_{\mathcal{P}(\pi(1,1))} = 0 \) for any \( i = 1, 3 \), on can construct, making use of (4.4) and (4.5), integrable Hamiltonian systems on \( \mathcal{P}(\pi(1,1)) \). The same one can do subject to the discussed above Poisson co-algebra \( \mathcal{P}_z(so(2,1)) \) realized by means of the Poisson manifold \( \mathcal{P}(so(2,1)) \), taking into account that the following element \( C = x_1^2 - x_2^2 - x_3^2 \in I(\mathcal{P}(so(2,1))) \) is a Casimir one.

Now we will consider a special extended Heisenberg-Weil co-algebra \( \mathcal{U}_z(h_4) \), called still the oscillator co-algebra. The undeformed Lie algebra \( h_4 \) commutator relationships take the form:

\[
[n,a_+] = a_+, \quad [n,a_-] = -a_-, \quad [a_-, a_+] = m, \quad [m,a_i] = 0,
\]

\( (4.14) \)

where \( \{n, a_\pm, m\} \subset h_4 \) compile a basis of \( h_4 \), \( \dim h_4 = 4 \). The Poisson co-algebra \( \mathcal{P}(h_4) \) is naturally endowed with the Poisson structure like (4.14) and admits its realization (3.9) on the Poisson manifold \( \mathcal{P}(\mathbb{R}^2) \). Namely, on \( \mathcal{P}(\mathbb{R}^2) \) one has

\[
e_\pm = D(a_\pm) = \sqrt{\nu} \exp(\mp q), \quad e_1 = D(m) = 1, \quad e_0 = D(n) = p,
\]

\( (4.15) \)

where \( (q, p) \in \mathbb{R}^2 \) and the Poisson structure on \( \mathcal{P}(\mathbb{R}^2) \) is canonical, that is the same as (3.16).

Closely related with the relationships (4.14) there is a generalized \( \mathcal{U}_z(su(2)) \) co-algebra, for which

\[
[x_3, x_\pm] = \pm x_\pm, \quad [y_\pm, \cdot] = 0, \quad [x_+, x_-] = y_+ \sin(2zx_3) + y_- \cos(2zx_3) \frac{1}{\sin z},
\]

\( (4.16) \)

where \( z \in \mathbb{C} \) is an arbitrary parameter. The co-algebra structure is given now as follows:

\[
\Delta(x_\pm) = c^\pm_{1(2)} e^{izx_3} \otimes x_\pm + x_\pm \otimes c^\pm_{2(1)} e^{-izx_3},
\]

\( (4.17) \)
with \( c_i^\pm \in \mathcal{U}(su(2)), i = \overline{1,2} \), being fixed elements. One can check that the corresponding to (4.16) Poisson structure on \( \mathcal{P}_2(su(2)) \) can be realized by means of the canonical Poisson structure on the phase space \( \mathcal{P}(\mathbb{R}^2) \) as follows:

\[
[q, p] = i, \quad D_z(x_3) = q, \quad D_z(x_\mp) = e^{\mp ip} g_z(q), \quad (4.18)
\]

\[
g_z(q) = (k + \sin[z(q - q)])(y_+ \sin[(q + s + 1)] + y_- \cos[z(q + s + 1)])^{1/2} \frac{1}{\sin z},
\]

where \( k, s \in \mathbb{C} \) are constant parameters. Thereby making use of (4.5) and (4.6), one can construct a new class of Liouville integrable Hamiltonian flows.

\section{The Heisenberg-Weil co-algebra structure and related integrable flows}

Consider the Heisenberg-Weil algebra commutator relationships (4.14) and related with them the following homogenous quadratic forms

\[
\begin{aligned}
&x_1x_2 - x_2x_1 - \alpha x_3^2 = 0, \\
x_1x_3 - x_3x_1 = 0, & x_2x_3 - x_3x_2 = 0
\end{aligned}
\]

where \( \alpha \in \mathbb{C} \), \( x_i \in A, i = \overline{1,3} \), are some elements of a free associative algebra \( A \). The quadratic algebra \( A/R(x) \) can be deformed via

\[
\begin{aligned}
x_1x_2 - z_1x_2x_1 - \alpha x_3^2 = 0, \\
x_1x_3 - z_2x_3x_1 = 0, & x_2x_3 - z_2^{-1}x_3x_2 = 0
\end{aligned}
\]

where \( z_1, z_2 \in \mathbb{C} \setminus \{0\} \) are some parameters.

Let \( V \) be the vector space of columns \( X := (x_1, x_2, x_3)^T \) and define the following action

\[
h_T : V \to (V \otimes V^*) \otimes V,
\]

where, by definition, for any \( X \in V \)

\[
h_T(X) = T \otimes X.
\]

It is easy to check that conditions (5.2) will be satisfied if the following relations [9]

\[
T_{11}T_{33} = T_{33}T_{11}, \quad T_{12}T_{33} = z_2^{-2}T_{33}T_{12}, \quad T_{21}T_{33} = z_1^2T_{33}T_{21},
\]

\[
T_{22}T_{33} = T_{33}T_{22}, \quad T_{31}T_{33} = z_2T_{33}T_{31}, \quad T_{32}T_{33} = z_1^{-1}T_{33}T_{32},
\]

\[
T_{11}T_{12} = z_1T_{11}T_{11}, \quad T_{21}T_{22} = z_1T_{21}T_{22}, \quad T_{21}T_{33} = z_2T_{33}T_{21}, \quad z_2T_{11}T_{32} - z_2T_{32}T_{11} =
\]

\[
= z_1z_2T_{12}T_{31} - T_{31}T_{12}, \quad T_{21}T_{32} - z_1z_2T_{32}T_{21} =
\]

\[
= z_1T_{22}T_{31} - z_2T_{31}T_{22}, \quad T_{11}T_{22} - T_{22}T_{11} =
\]

\[
= z_1T_{12}T_{21} - z_1^{-1}T_{21}T_{12}, \quad (T_{11}T_{22} - z_1T_{12}T_{21}) =
\]

\[
= \alpha T_{33}^2 - T_{31}T_{32} + z_1T_{32}T_{31}
\]
Hold. Put now for further convenience \( z_1 = z_2^2 := z^2 \in \mathbb{C} \) and compute the "quantum" determinant \( D(T) \) of the matrix \( T : (A/R_z(x))^3 \rightarrow (A/R_z(x))^3 \):

\[
D(T) = (T_{11}T_{22} - z^{-2}T_{21}T_{12})T_{33}. \tag{5.6}
\]

Remark here that the determinant (5.6) is not central, that is

\[
D^{-1}T_{11} = T_{11}D^{-1}, \quad D^{-1}T_{12} = z^{-6}T_{12}D^{-1}, \tag{5.7}
\]

\[
D^{-1}T_{33} = T_{33}D^{-1}, \quad D^{-1}T_{31} = T_{31}D^{-1},
\]

\[
D^{-1}T_{22} = T_{22}D^{-1}, \quad z^{-3}D^{-1}T_{21} = T_{21}D^{-1},
\]

\[
D^{-1}T_{32} = z^{-3}T_{32}D^{-1}.
\]

Taking into account properties (5.5) - (5.7), one can construct the Heisenberg-Weil related co-algebra \( U_z(\hbar) \) being a Hopf algebra with the following coproduct \( \Delta \), counit \( \varepsilon \) and antipode \( \nu \):

\[
\Delta(T) = T \otimes T, \quad \Delta(D^{-1}) = D^{-1} \otimes D^{-1}, \tag{5.8}
\]

\[
\varepsilon(T) = I, \quad \varepsilon(D^{-1}) = I, \quad \nu(T) = T^{-1}, \quad \nu(D) = D^{-1}.
\]

Based now on relationships (5.5), one can easily construct the Poisson tensor

\[
\{\Delta(\tilde{T}), \Delta(\tilde{T})\}_{P_z(\hbar) \otimes P_z(\hbar)} = \Delta(\{\tilde{T}, \tilde{T}\}_{P_z(\hbar)}) := \vartheta_z(\Delta(\tilde{T})), \tag{5.9}
\]

subject to which all of functionals (4.5) will be commuting to each other, and moreover, will be Casimir ones. Choosing some appropriate Hamiltonian function \( H(N) := \Delta^{(N-1)}(H(\tilde{T})) \) for \( N \in \mathbb{Z}_+ \) one makes it possible to present a priori nontrivial integrable Hamiltonian systems. On the other hand, the co-algebra \( U_z(\hbar) \) built by (5.7) and (5.8) possesses the following fundamental \( R \)-matrix [4] property:

\[
\mathcal{R}(z)(T \otimes I)(I \otimes T) = (I \otimes T)(T \otimes I)\mathcal{R}(z) \tag{5.10}
\]

for some complex-valued matrix \( \mathcal{R}(z) \in Aut(\mathbb{C}^3 \otimes \mathbb{C}^3), \ z \in \mathbb{C} \). The latter, as is well known [4], gives rise to a regular procedure of constructing an infinite hierarchy of Liouville-integrable operator (quantum) Hamiltonian systems on related quantum Poissonian phase spaces. On their special cases interesting for applications we plan to go on in another place.

References

[1] Hopf H. Noncommutative associative algebraic structures. Annales Mathem., 42, N1, p.22-52 (1941)

[2] Postnikov M. Lie groups and Lie algebras. M.: Mir Publishers, 1982

[3] Drinfeld V.G. Quantum Groups. Proceedings of the International Congress of Mathematicians, MRSI Berkeley, p. 798 -812 (1986)
[4] Korepin V., Bogoliubov N. and Izergin A. Quantum Inverse Scattering Method and Correlation Functions. Cambridge University Press, 1993

[5] Perelomov F. Integrable systems of classical mechanics and Lie algebras. Birkhauser Publ., 1990

[6] Prykarpatsky A.K. and Mykytyuk I.V. Algebraic integrability of nonlinear dynamical systems on manifolds: classical and quantum aspects. Kluwer Acad. Publ., the Netherlands, 1998

[7] Ballesteros A. and Ragnisco O. A systematic construction of completely integrable Hamiltonian flows from co-algebras.//solv-int/9802008-6 Feb 1998, 26 p.

[8] Woronowicz S.L. Communications Math. Phys., 149,p. 637-652 (1992)

[9] Bertrand J. and Irac-Astaud M. Invariance quantum groups of the deformed oscillator algebra. J. Phys. A: Math.&Gen.,30, 2021-2026 (1997)