Binary Bose–Einstein condensates in a disordered time-dependent potential

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Received 10 October 2021, revised 9 December 2021
Accepted for publication 20 December 2021
Published 10 January 2022

Abstract
We study the non-equilibrium evolution of binary Bose–Einstein condensates in the presence of a weak random potential with Gaussian correlation function using the time-dependent perturbation theory. We apply this theory to construct a closed set of equations that highlight the role of the spectacular interplay between the disorder and the interspecies interactions in the time evolution of the density induced by disorder in each component. It is found that this latter increases with time favoring localization of both species. The time scale at which the theory remains valid depends on the respective system parameters. We show analytically and numerically that such a system supports a steady state that periodically changing during its time propagation. The obtained dynamical corrections indicate that disorder may transform the system into a stationary out-of-equilibrium states. Understanding this time evolution is pivotal for the realization of Floquet condensates.

Keywords: Bose–Bose mixture, time-dependent perturbation theory, condensate deformation, nonequilibrium steady state, time-periodic disorder potential

(Some figures may appear in colour only in the online journal)

1. Introduction

Ultracold Bose mixtures of atomic gases offer unprecedented control tools, opening promising new avenues for the investigation of Bose–Einstein condensate (BEC) in a disordered environment [1, 2]. Disordered binary BECs present rich physics not encountered in a single component condensate due the intriguing interplay of quantum fluctuations induced by intra- and interspecies interactions and disorder effects. Such dirty Bose mixtures could be regarded as a feasible simulator to analyze plethora of novel quantum phenomena such as supersolidity, and quantum glasses.

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Dilute Bose–Bose gases in a weak disorder potential has recently undergone a resurgence due to their fascinating properties (see e.g. [3–6]). One of the most amazing features arising from the presence of the disorder in Bose mixture quantum gases is the modification of the miscibility criterion and the dramatic phase separation between the two species [1].

The considerable interest in studying the dynamics of disordered BEC driven out-of-equilibrium by slow (adiabatic) or sudden (quenched) changes to system parameters such as the scattering length has been boosted by remarkable advances in the tunability of ultracold atomic gases [7–11]. Non-equilibrium evolution of BEC offers the unique opportunity to explore strongly correlated systems and transport in realistic physical systems. Chen et al [12] have shown that for a weakly non-equilibrium disordered Bose gas under a quantum quench in the interaction, the disorder can substantially destroy superfluidity more than the condensate leading to the...
so-called dynamical Bose glass. Moreover, it has been revealed that externally controlled spatiotemporal periodic drive constitutes an excellent platform for creating novel nonequilibrium states of quantum matter [13–15]. Recent study demonstrates that the condensate deformation is a signature of the non-equilibrium feature of steady states of a Bose gas in a temporally controlled weak disorder [16]. Quite recently, experimental realization of ultracold bosonic gases in dynamic disorder with controlled correlation time have been reported in reference [17], where the microscopic origin of friction and dissipation has been well illustrated. There has been also an extensive amount of work addressing dynamics of bosons in dissipation has been well illustrated. There has been also an

Motivated by the above experimental and theoretical works, we investigate in this paper the non-equilibrium evolution of homogeneous Bose–Bose mixtures subjected to a time-dependent disordered potential with Gaussian correlation function. To this end, we use the time-dependent perturbation theory. The perturbation method has proven to be a very useful and powerful tool to capture the main features of both equilibrium and non-equilibrium disordered single BECs [1, 16, 25–30].

We start with the equilibrium case and analyze some ground-state and thermodynamic aspects of disordered mixtures. We look in particular at how the interplay of a correlated disorder (i.e. non-zero correlation length) and interspecies interactions affect the condensates deformation and their equations of state (EoS). Our analysis reveal that the condensates remain robust when the disorder correlation length is larger than the healing length regardless of the strengths of the interspecies interaction and of the disorder potential. Comparison with our recent findings for Bose mixtures with white-noise potential [1] and against other predictions such as the Huang–Meng predictions [31, 32] suggests the importance of the disorder correlation length in the localization process.

Furthermore, we study the dynamical properties of two BECs in Gaussian disorder potential with time-periodic driving using the aforementioned time-dependent perturbation theory. Periodically-driven quantum systems have gained tremendous interest recently owing to the possibility of Floquet engineering (see for review [33]). In the field of ultra-cold quantum gases, this concept could offer powerful techniques to reach novel phase transitions (see [33–43] and references therein). We show that our theory predicts an oscillatory behavior of the condensates deformation during the time evolution. It is found in addition that disorder drives the growth of the disorder fraction with time, reducing the condensed fraction in each species. Therefore, the time scale for reaching a sizeable dynamical depletion in principle limits the validity of the present perturbation theory. Interestingly, a long time analysis predicts the existence of stationary states resembling to the Floquet states due to the combination of many-body effects, the drive frequency and disorder correlations.

The rest of the paper is organized as follows. In section 2, we review the various formulas describing equilibrium Bose mixtures for arbitrary disorder potential. We then restrict ourselves to symmetric mixture with Gaussian-correlated disorder. The results for asymmetric mixtures are shown in appendix. Section 3 is the main section of this paper. We introduce the time-dependent perturbative theory for disordered Bose mixtures. We calculate the time-dependent condensate deformation due to the disorder. Many appealing issues are also discussed. Finally, in section 4 we present conclusions and lessons learned.

2. Equilibrium mixture

Consider weakly interacting homogeneous two-component BECs with equal masses, \( m_1 = m_2 = m \), subjected to a weak random potential \( U(\mathbf{r}) \), labeling the components by the index \( j \). At zero temperature, the system is described by the coupled Gross–Pitaevskii equations (GPE):

\[
\mu_j \Phi_j = \left[ -\frac{\hbar^2}{2m} \nabla^2 + U + g_j |\Phi_j|^2 + g_{12} |\Phi_j|^2 \right] \Phi_j, \quad (1)
\]

where \( \Phi_j \) is wavefunction of each condensate, \( j = 3 − j \), \( \mu_j \) is the chemical potential of each condensate, \( g_j = 4\pi\hbar^2 a_j/m \) and \( g_{12} = g_{21} = 4\pi\hbar^2 a_{12}/m \) with \( a_j \) and \( a_{12} \) being the intraspecies and the interspecies scattering lengths, respectively. The disorder potential is assumed to have vanishing ensemble averages \( \langle U(\mathbf{r}) \rangle = 0 \) and a finite correlation of the form \( \langle U(\mathbf{r})U(\mathbf{r}') \rangle = R(\mathbf{r} − \mathbf{r}') \). In what follows, we restrict ourselves to the case of Gaussian correlation with the Fourier transform [28, 44–47]

\[
R(\mathbf{k}) = R_0 e^{-\sigma^2 k^2/2}, \quad (2)
\]

where \( R_0 \) is the disorder strength with dimension (energy) \( (\text{length})^3 \) and \( \sigma \) is the disorder correlation length.

For weak disorder, equation (1) can be solved using straightforward perturbation theory in powers of \( U \) using the expansion [1]

\[
\Phi_j = \Phi^{(0)}_j + \Phi^{(1)}_j(\mathbf{r}) + \Phi^{(2)}_j(\mathbf{r}) + \cdots, \quad j = 1, 2 \quad (3)
\]

where the index \( i \) in the real valued functions \( \Phi^{(i)}(\mathbf{r}) \) signals the \( i \)th order contribution with respect to the disorder potential. They can be determined by inserting the perturbation series (3) into the equation (1) and by collecting the terms up to \( U^2 \).

2.1. Glassy fraction

The condensate fluctuations (or condensate deformation) due to disorder, known as glassy fraction, can be given as the variance of the wavefunction \( n^G_{\mathbf{k}} = \langle \Phi^{(12)}_j(\mathbf{r}) \rangle + \cdots \). Working in momentum space, the glassy fraction reads [1]:

\[
n^G_{\mathbf{k}} = n J \int \frac{d\mathbf{k}}{(2\pi)^3} R(\mathbf{k}) \left[ \frac{E_k + 2a_j (g_j - g_{12})}{\xi_k} \right]^2, \quad (4)
\]

where \( \xi_k = (E_k + 2a_j n_j) \right) (E_k + 2g_{12} n_j) - 4g_{12}^2 n_j n_{12} \) and \( R(\mathbf{k}) \) is the disorder correlation.
Let us assume a symmetric mixture where $a_1 = a_2 = a$ and $n_1 = n_2 = n$. Performing the integral (4), the glassy fraction turns out to be given:

$$\frac{n_R^{\text{eq}}}{n} = \frac{\xi_+}{\ell_L} f \left( \frac{\sigma}{\xi_+} \right),$$

where the disorder function reads

$$f \left( \frac{\sigma}{\xi_+} \right) = \sqrt{2} e^{\frac{\sigma^2}{2x_+^2}} \left( \frac{2\sigma^2}{x_+^2} + 1 \right) \text{erfc} \left( \frac{\sigma}{\sqrt{2\pi}x_+} \right) - \frac{2\sqrt{2\pi}\sigma}{\sqrt{\pi}x_+},$$

where $\text{erfc}(x)$ is the complementary error function, $\xi_+ = \xi/\sqrt{\delta a_+}$ is the extended healing length for a symmetric mixture with short-range interactions, $\xi = \hbar/\sqrt{\Delta m}$, $\delta a_+ = 1 + a_{12}/a$, and $\ell_L = 4\pi h^2/\langle m^2 R_0 \rangle$ accounts for the Larkin length which is associated with the pinning energy due to the disorder [16, 48, 49]. For $\sigma \to 0$, $f(0) = \sqrt{2}$, thus, the results of binary BECs with a weak delta-correlated disorder are recovered [1]. For $\sigma \to 0$ and $a_{12} = 0$, one can reproduce the seminal Huang–Meng findings for a dirty single BEC [31].

The behavior of the function (6) is shown in figure 1(a). As expected, the function $f$ is decreasing with the disorder correlation length $\sigma/\xi$ regardless of the strength of interspecies interactions indicating that the condensate depletion due to the disorder effects is suppressed for $\sigma \gg \xi$. One might explain this delocalization as the results of a screening of disorder by the interspecies interaction. The same situation takes place in single dirty dipolar and nondipolar BECs [28, 30, 44, 45, 50].

For fixed $\sigma$, $n_R^{\text{eq}}$ is decreasing with the interspecies interactions $a_{12}/a$.

The validity of the present perturbation theory requires to have the condensate depletion due to the disorder much smaller than the total density $n_R^{\text{eq}} \ll n$. This implies that to have a weak disorder potential, the following condition must be fulfilled

$$\ell_L \gg \xi_+ f \left( \frac{\sigma}{\xi_+} \right).$$

Equation (7) is a natural extension of the result of [16].

### 2.2. Equation of state

The EoS in second-order of the disorder strength is given by [1]

$$\mu_j = g_j m_j + g_{12} m_2^2 \int \frac{dk}{(2\pi)^3} \frac{R(k)}{(g_j m_j^2 - g_{12} m_2^2)^2} E_k$$

$$+ \left\{ 4 g_j^2 g_{12} (E_k + g_j m_j^2) (E_k + 2 g_j m_j^2) + 4 g_j g_{12} m_2^2 (E_k + 2 g_j m_j^2) \right\}.$$

In equation (8), higher order terms in $g_{12}$ were omitted. Upon calculating the integral (8), for the spacial case of $m_1 = m_2 = n$, and $a_1 = a_2 = a$, we find for the EoS

$$\mu = ng \left[ \delta a_+ + \frac{\xi}{\ell_L} h \left( \delta a_+ - \frac{\sigma}{\xi} \right) \right],$$

where the disorder functions read

$$h \left( \frac{\delta a_+ - \sigma}{\xi} \right) = \frac{\sqrt{2}}{\delta a_+} \left\{ \left( \frac{\sigma}{\xi} \right) \left( \frac{\delta a_+ - \delta a_-}{2\delta a_+ - \delta a_-} \right)^2 + \frac{\sqrt{\pi} \delta a_+}{(\delta a_+ - \delta a_-)} \left( A_+ - A_- \right) + \frac{1}{\delta a_+} \left( \frac{2}{\xi_+} \right) (\delta a_+ - \delta a_-) + \frac{\sqrt{\pi} \delta a_+}{(\delta a_+ - \delta a_-)} (B_+ - B_-) \right\},$$

where

$$A_+ = \frac{e^{\sigma/\xi_+^2}}{(\delta a_+)^2} \text{erfc} \left( \frac{\sigma}{\xi_+} \right) \left[ 7 + \left( \frac{\sigma}{\xi_+} \right)^2 - 9 \delta a_+ - \frac{\delta a_+ - \delta a_-}{4} \left( 6 + \left( \frac{\sigma}{\xi_+} \right)^2 + 5 \left( \frac{\sigma}{\xi_+} \right)^2 \right) \right],$$

$$B_+ = \sqrt{\delta a_+} \left[ 3 + 2 \left( \frac{\sigma}{\xi_+} \right)^2 \right] e^{\sigma/\xi_+^2} \text{erfc} \left( \frac{\sigma}{\xi_+} \right), \quad \xi_+ = \xi/\delta a_-, \quad \text{and} \quad \delta a_- = 1 - a_{12}/a.$$ In the limit $\sigma \to 0$, one has:

$$h_{\sigma \to 0} = \frac{2(\sqrt{\delta a_+} - \sqrt{\delta a_-}) - 3 \left( \frac{a_{12}}{a} \right)^2 (\sqrt{\delta a_+} - 5 \sqrt{\delta a_-}) - 12 \left( \frac{a_{12}}{a} \right)^3 (\sqrt{\delta a_+} + \sqrt{\delta a_-}) + (a_{12}/a) (13 \sqrt{\delta a_+} + \sqrt{\delta a_-})}{\sqrt{2} \sqrt{\delta a_+} \left( \frac{1}{a_{12}/a} \right) \left( 1 - \left( \frac{a_{12}}{a} \right)^2 \right)^{3/2}}.$$
2.3. Sound velocity

Corrections to the sound velocity of each component due to the disorder fluctuations are given by [46]:

\[ c_j^2 = \frac{n_j}{m_j} \frac{\partial \mu_j}{\partial n_j}. \]

In the case of a balanced mixture \( a_1 = a_2 = a \) and \( n_1 = n_2 = n \), we find after a straightforward calculation:

\[ c_j^2 = 1 + \frac{\xi}{\xi_L} S \left( \delta a_{\pm}, \sigma / \xi \right), \]

where \( c_{co} = \sqrt{gn/m} \) is the zeroth-order sound velocity and the disorder function \( S(\delta a_{\pm}, \sigma / \xi) \) is given as:

\[ S \left( \delta a_{\pm}, \sigma / \xi \right) = \frac{1}{2} \left( h + 2\gamma \frac{\partial h}{\partial n} \right), \]

which has practically the same behavior as the function \( h \) as is displayed figure 1(c).

3. Non-equilibrium evolution

We consider two weakly interacting ultracold Bose gases, subjected to a weak random potential \( U(r, t) = u(r)F(t) \), such that \( F(0) = 0 \) and \( 0 \leq F(t) \leq 1 \). The system evolution at \( t \geq 0 \) is described by the coupled time-dependent GPEs which can be written as:

\[ i\hbar \frac{\partial}{\partial t} \Phi_j(r, t) = \left( -\frac{\hbar^2}{2m} \nabla^2 + u(r)F(t) - \mu_{0j} \right) \Phi_j(r, t) + g_j |\Phi_j(r, t)|^2 \Phi_j(r, t). \]

For sufficiently small \( u(r) \), the system can be treated perturbatively. Therefore, we can write the wavefunctions as:

\[ \Phi_j(r, t) = \Phi_j^{(0)}(r) + \Phi_j^{(1)}(r, t) + \Phi_j^{(2)}(r, t) + \cdots, \]

where \( \Phi_j^{(0)}(r) \) are the equilibrium solutions of equation (1) at \( t = 0 \), and \( \Phi_j^{(0)}(r, 0) = 0 \) for \( \alpha > 1 \). The particle densities \( n_j = |\Phi_j(r)|^2 \) determine the chemical potentials of the system in its equilibrium ground-state:

\[ \mu_{0j} = g_j \rho_j + g_{12} \rho_{12}. \]

The condensate deformation due to the disorder potential, can be given by

\[ n_R(t) = \langle |\Phi_j(r, t)|^2 \rangle - \langle |\Phi_j(r, t)|^2 \rangle. \]

Using the perturbative expansion (15) up to second-order, and assuming that the disorder have a vanishing ensemble averages the deformation of each BEC becomes:

\[ n_R(t) = \langle |\Phi_j^{(1)}(r, t)|^2 \rangle. \]

The first-order coupled equations follow from equation (14) read:

\[ i\hbar \frac{\partial}{\partial t} \Phi_j^{(1)}(r, t) = \left( -\frac{\hbar^2}{2m} \nabla^2 + g_j \rho_j \right) \Phi_j^{(1)}(r, t) + g_{12} \sqrt{\rho_{12}} \left( \Phi_j^{(1)}(r, t) + \Phi_j^{(1)*}(r, t) \right). \]

In Fourier space equation (19) turn out to be given as:

\[ \left( \hbar \omega_k + g_j \rho_j - i\hbar \right) \Phi_j^{(1)}(k, s) + g_j \rho_j \Phi_j^{(1)*}(k, s), + \right) \Phi_j^{(1)}(k, s) = -\sqrt{\rho_{12}} \mu(k) f(s), \]

\[ \left( \hbar \omega_k + g_j \rho_j + i\hbar \right) \Phi_j^{(1)}(k, s) + g_j \rho_j \Phi_j^{(1)*}(k, s), + \right) \Phi_j^{(1)}(k, s) = -\sqrt{\rho_{12}} \mu(k) f(s), \]

\[ \left( \hbar \omega_k + g_j \rho_j + i\hbar \right) \Phi_j^{(1)}(k, s) + g_j \rho_j \Phi_j^{(1)*}(k, s), + \right) \Phi_j^{(1)}(k, s) = -\sqrt{\rho_{12}} \mu(k) f(s), \]

\[ \left( \hbar \omega_k + g_j \rho_j + i\hbar \right) \Phi_j^{(1)}(k, s) + g_j \rho_j \Phi_j^{(1)*}(k, s), + \right) \Phi_j^{(1)}(k, s) = -\sqrt{\rho_{12}} \mu(k) f(s), \]

\[ \left( \hbar \omega_k + g_j \rho_j + i\hbar \right) \Phi_j^{(1)}(k, s) + g_j \rho_j \Phi_j^{(1)*}(k, s), + \right) \Phi_j^{(1)}(k, s) = -\sqrt{\rho_{12}} \mu(k) f(s), \]
where $\hbar \omega_k = E_k$ is the kinetic energy. Here we used the Laplace transform $F(s) = \int_0^\infty dt F(t) e^{-st}$. The solution of equations (21) and (22) reads

$$\Phi^{(1)}_j(k, s) = -\sqrt{\pi \rho(k)} \frac{\omega_k}{i} \left[ \frac{2 g_{12} n_j \omega_k - h (\Omega_{k_+}^2 + s^2)}{4 g_{12} n_j \omega_k - h (\Omega_{k_+}^2 + s^2)} \right]$$

(23)

$$\Phi^{(1)v}_j(k, s) = -\sqrt{\pi \rho(k)} \frac{\omega_k}{i} \left[ \frac{2 g_{12} n_j \omega_k - h (\Omega_{k_+}^2 + s^2)}{4 g_{12} n_j \omega_k - h (\Omega_{k_+}^2 + s^2)} \right]$$

(24)

where $\hbar \Omega_{ks} = \sqrt{\hbar \omega_k (\omega_k - 2 g_{12} n_j)}$ is the standard dispersion relation for a single BEC. Using the inverse Laplace transform we get:

$$\Phi^{(1)}_j(k, t) = -\sqrt{\pi \rho(k)} \frac{\omega_k}{i} \int_0^t dt' K_j(k, t - t') f(t'),$$

(25)

$$\Phi^{(1)v}_j(k, t) = -\sqrt{\pi \rho(k)} \frac{\omega_k}{i} \int_0^t dt' K'_j(k, t - t') f(t'),$$

where

$$K_j(k, t) = \frac{K_{j,1}(k, t) - K_{j,2}(k, t)}{(\Omega_{k_+}^2 - \Omega_{k_-}^2)},$$

(26)

and

$$K_{j,1}(k, t) = \frac{i \cos(\Omega_{k_+} t) + \frac{\omega_j}{\Omega_{k_+}} \sin(\Omega_{k_+} t)}{\Omega_{k_+}^2 - \Omega_{k_-}^2} \times \left[ \frac{\Omega_{k_+}^2 - \Omega_{k_-}^2}{2 g_{12} n_j \omega_j} \right].$$

For a symmetric mixture where $a_1 = a_2 = a$ and $n_1 = n_2 = n$, the Bogoliubov spectrum (27) reduces to the following dimensionless form: $\Omega_{k_\pm} = \Omega_k (\kappa \xi)^2 + 2(1 \pm a_{12}/a)$, where $\Omega_k = n g / h$ is the inverse characteristic mean-field time scale. It is clearly seen that for $a_{12}/a > 1$, the spectrum $\Omega_{k_+}$ may become imaginary and thus, the mixture would be destabilized. Inserting $\Omega_{k_\pm}$ and equation (26) into equation (28), and keeping in mind that terms associated with $\Omega_{k_-}$ cancel, one finds for $n_R(t)$:

$$n_R(t) = \frac{n}{4 h^2} \int \frac{dk}{(2\pi)^2} \frac{R(k)}{\Omega_{k_+}^4 - (\Omega_{k_+}^2 + \omega^2)^2} \times \left\{ -\omega_k^2 \Omega_{k_+}^2 \left[ (1 - \Omega_{k_+}^2) \sin(\omega t) - \omega \sin(\Omega_{k_+} t) \right]^2 
+ \omega_k^2 \left[ \omega^2 (1 + \Omega_{k_+}^2) + \Omega_{k_+}^2 \cos(\omega t) - \omega^2 \cos(\Omega_{k_+} t) \right]^2 \right\}. \quad (30)$$

This equation tells us that once the depletion due to disorder is known at $t = 0$, both the condensed density $n - n_R$ and $n_R$ can be calculated in some rather simple way at time $t > 0$.

For a given $k$ and $\omega > 0$, the density (30) is peaked ‘resonant’ at frequencies $\Omega_{k_+} = \omega$ (i.e. when the external frequency nearly matches the eigenfrequencies). In terms of momenta this condition yields

$$k = k_{res} = \frac{m c_s \sqrt{1 + a_{12}/a}}{\hbar \sqrt{2 - 2(1 + \hbar^2 c_s^4/2a_{12}/a)^2}}.$$  

For $k \geq k_{res}$, the disorder depletion becomes very large, $n_R(t)/n \gg 1$, indicating that the perturbation theory is no longer valid in such an unstable regime despite the fact that the disorder is naively weak. Therefore, the intuitive stability criterion reads $k < k_{res}$.

In order to substantiate the relevance of the above time-dependent perturbative mean-field approach for laboratory experiments, we consider the $^{87}\text{Rb} - ^{87}\text{Rb}$ mixture in two different internal states, but our theory can be readily generalized to other mixtures. The scattering lengths and the densities are chosen to $a_1 = a_2 = a = 95.44 a_0$ [52] with $a_0$ being the Bohr radius, and $n_1 = n_2 = n = 10^{22} \text{m}^{-3}$, respectively, which are sufficient to ensure that the system meets the requirement of weakly interacting gas, $\sqrt{n}/a \sim 10^{-2} \ll 1$. The interspecies scattering length $a_{12}$ which can be adjusted via Feshbach resonance is selected in such a way that the phase-separated condition is fulfilled throughout the dynamics. The disorder strength is fixed to be $R = 0.5$ which gives $n_R/n \lesssim 1\%$, ensuring the sufficient criterion for the weak disorder regime. We employ various disorder driving frequencies and correlation lengths.

The numerical solutions of equation (30) is shown in figure 2. One can clearly identify two phases of evolution: in phase I, $\Omega_{k_\pm} \lesssim 4$, although the density $n_R(t)$ is somehow low, the dynamics follows an exponential growth due to the considerable effect of both quasiparticles and disorder onto the two BECs. In region II, $\Omega_{k_\pm} > 4$, we observe that as the disorder evolves in time, the glassy fraction increases and exhibits
by a vertical dotted line.

The sine function signaling that the system being completely depleted at long time. This can be attributed to the motion of the Gaussian disorder which may create elementary excitations (i.e. the Bogoliubov phonons at low momenta/free particle in the high-energy regime) leading to enhance the disorder depletion. Therefore, whatever the strength and the frequency of the disordered potential, the condensates are completely depleted at long time. This can be attributed to the time-dependent glassy fraction (see figure 2(b)). Furthermore, as the healing length decreases, the chemical potential rises and the density of two BECs decreases, the chemical potential rises and the density of two BECs (i.e. the Bogoliubov phonons at low momenta/free particle in the high-energy regime) leading to enhance the excitations which could be an indicator of the existence of stationary Floquet condensates [33, 34]. Frankly speaking, a qualitative analysis of these Floquet states in the presence of such periodic perturbations requires further thoughts.

In the long-time limit \( t = (2j + 1)\pi/\omega \), equation (30) reduces to

\[
\eta_R(\omega) = \frac{n}{4\hbar^2} \int \frac{dk}{(2\pi)^3} \frac{R(k)}{k^2 \left( \Omega_{k+}^2 - \omega^2 \right)} + \left\{ 4\omega^2 \left[ \frac{1}{2} \left( \frac{2j + 1}{2\omega} \Omega_{k+} \right) - \Omega_{k+}^2 \right] \right\}.
\]

This equation shows that the time-dependence cancels meaning that the state hardly moves over such a time interval. The leading term in equation (31) represents the equilibrium disorder fluctuations, while the subleading terms are of dynamical origin revealing the non-equilibrium feature of the mixture as is shown in figure 4 (dotted and dashed lines). It is clear that the stationary depletion due to the disorder diverges from the equilibrium one in particular for a small correlation length regardless of the values of the interspecies interactions.

On the other hand, after the adiabatic introduction of the disorder followed by the adiabatic switch-off i.e. \( t = \pi/\omega \) and for \( \omega \rightarrow 0 \), the stationary depletion (31) becomes close to the equilibrium state, namely \( \eta_R = (n/8\pi^3\hbar^2) \int dk [R(k)\omega^2]/\Omega_{k+}^2 \) as is depicted in figure 4 (dotted and solid lines). Conversely for \( \omega \rightarrow \infty \), one has from equation (31), \( \eta_R(\omega) = \pi^{1/2}(\xi/\ell) (a_0/4\hbar^2)^3 (\xi/\ell)^3 \). Such a condensate deformation is a signature of the nonequilibrium

\[\text{Figure 2. (a) Time evolution of the disorder fraction from equation (30) for different values of } a_{12}/a. \text{ Parameters are: } a/a_0 = 95.44, n = 10^{11} \text{ m}^{-3}, R' = 0.5, \omega/\Omega_0 = 1, (a) } \sigma/\xi = 0.2, \text{ and (b) } \sigma/\xi = 2. \text{ Here } R' = R_0/g^2n. \text{ There are two different regimes: I and II delineated by a vertical dotted line.} \]
Figure 3. (a) Disorder fraction $n_R(t)/n$ from equation (30) as a function of $\Omega_0 t$ and $\sigma/\xi$ for $a_{12}/a = 0.5$ and $\omega/\Omega_0 = 0.3$. (b) $n_R(t)/n$ as a function of $\Omega_0 t$ and $a_{12}/a$ for $\sigma/\xi = 0.5$ and $\omega/\Omega_0 = 0.3$. (c) $n_R(t)/n$ as a function of $\Omega_0 t$ and $\omega/\Omega_0$ for $\sigma/\xi = 1.8$ and $a_{12}/a = 0.5$. Parameters are the same as in figure 2.

Figure 4. Disorder fraction from equation (31) as a function of $\sigma/\xi$. Parameters are: $a/a_0 = 95.44$, $n = 10^{21}$ m$^{-3}$, $K' = 0.5$ and $a_{12}/a = 1$.

The property of steady states of BEC in a time-dependent disorder potential.

4. Conclusions

We investigated the non-equilibrium evolution of binary BECs subjected to a time-dependent random potential with Gaussian correlation function. To this end, we applied the time-dependent perturbative mean-field theory which allows us to reveal the spectacular interplay of the disorder and the interspecies interactions in the non-equilibrium regime. The theory assumes weak interactions and weak disorder, hence it remains valid provided the depletion remains small throughout the full subsequent dynamics.

We first shed new light on our understanding of equilibrium process and also established some useful formulas for the glassy fraction and the EoS for both symmetric and asymmetric (see appendix) Bose mixtures. We pointed out that for a large disorder correlation length, the localization of bosons is suppressed owing to the screening of the random potential by the interaction.

In the case of Gaussian disorder potentials with time-periodic driving, we showed that the complex combination of atomic interactions, the disorder potential, and time-periodic perturbations may uncover new phenomena in dirty Bose mixtures. The disorder fluctuations grow with time and exhibit an oscillating character, its magnitude strongly depends on the system parameters. To date, there is no experimental work confirming this oscillating character of the condensate deformation. We found that at short time such an evolution follows an exponential growth law while at larger times where the theory fails, the dynamics is characterized by another law. Among the main results emerging from our study is that even though the disorder is naively weak, it could have dramatic effects on the localization of atoms during the time evolution of the system. The present analysis revealed also the occurrence of a stationary state due to the crucial role played by the drive frequency in the limit of large disorder correlation length. We conjecture the existence of specific parameters enabling one to transform dynamic BECs into Floquet condensates.

Our time-dependent results can readily be extended to the case of trapped mixtures under the assumption that the disorder correlation length should be much smaller than the spatial extent of the atomic cloud [16, 45]. We believe that our predictions open up new perspectives for an experimental demonstration of the peculiar interplay between the disorder and interaction in the non-equilibrium regime.

Acknowledgments

We acknowledge Axel Pelster for fruitful discussions and useful comments about the paper.

Data availability statement

No new data were created or analysed in this study.
Appendix. Asymmetric mixtures

In this appendix we extend our results to the case of asymmetric disordered Bose mixtures.

The time-independent glassy density can be obtained from equation (4)

\[ n_{\text{g}}^{\text{BM}} = n_{\text{HM}} f_j(\Delta, \sigma), \]

where \( n_{\text{HM}} = 2\pi R^\prime n_j \sqrt{n_{\text{eq}}}/\pi \) is the HM result for equilibrium deformation of each BEC [1] with \( R^\prime_j = R_0/g_j^2 n_j \) being a dimensionless disorder strength, \( \sigma_j = \sigma/j \xi_j \), \( \Delta = g_{j2}/g_j^2 \), and \( \xi_j = \hbar/\sqrt{m_j g_j} \) is the healing length of each component. The disorder functions of the glassy fraction \( f_j(\Delta, \sigma_j) \) of equation (32) can be written as:

\[
\begin{align*}
  f_j(\Delta, \sigma_j) &= \frac{\sqrt{2/\pi}}{\sqrt{\beta_j \alpha_j (n_j - \beta_j)}} \left\{ -2\sigma_j \sqrt{\alpha_j (\alpha_j - \beta_j)} \right. \\
  &\times \left[ 2\nu_j^2 + \alpha_j^2 + \beta_j^2 - 2\nu_j (\alpha_j + \beta_j) \right] + \sqrt{\pi} e^{\nu_j^2} \\
  &\times \left[ 3\nu_j \alpha_j \beta_j + \sqrt{\beta_j (\alpha_j, \beta_j)} \right] (1 - 4\sigma_j^2 \nu_j) \\
  &+ \sqrt{\pi} \nu_j^2 \left[ -2\nu_j (\alpha_j + \beta_j) - 2\nu_j^2 (1 + \sigma_j^2 \nu_j) + \beta_j^2 \right] \\
  &+ \left[ 2\nu_j (\alpha_j + \beta_j) - 2\sigma_j^2 \nu_j (\alpha_j - \beta_j) \right] \left[ 1 - \text{erfc} \left( \frac{\sqrt{\beta_j}}{\sqrt{\alpha_j}} \right) \right] \\
  &+ \sqrt{\pi} e^{\nu_j^2} \sqrt{\beta_j (\alpha_j - \nu_j) \left[ \nu_j (\alpha_j + (3 + 2\sigma_j^2 (\alpha_j - \beta_j)) \beta_j) \right]} \\
  &+ \beta_j [-5 \alpha_j + \beta_j - 2\sigma_j^2 \beta_j (\alpha_j - \beta_j)] \left[ 1 - \text{erfc} \left( \frac{\sqrt{\beta_j}}{\sqrt{\alpha_j}} \right) \right] \\
  &+ \sqrt{\pi} e^{\nu_j^2} \sqrt{\beta_j (\alpha_j - \nu_j) \left[ \nu_j (\alpha_j + 2\sigma_j^2 \alpha_j (\alpha_j - \beta_j) - 5\beta_j) \right]} \\
  &+ \nu_j \beta_j \alpha_j (3 - 2\sigma_j^2 (\alpha_j - \beta_j)) \text{erfc} \left( \frac{\sqrt{\beta_j}}{\sqrt{\alpha_j}} \right),
\end{align*}
\]

where \( \text{erfc}(x) \) is the complementary error function. \( \alpha_j = 1 + \bar{\mu}_j + \sqrt{(1 + \bar{\mu}_j)^2 - 4\bar{\mu}_j (\Delta - 1)/\Delta} \), \( \beta_j = 1 + \bar{\mu}_j - \sqrt{(1 + \bar{\mu}_j)^2 - 4\bar{\mu}_j (\Delta - 1)/\Delta} \), and \( \nu_j = 2\bar{\mu}_j (1 - g_{j2}/g_j^2) \). For \( \sigma \to 0 \), the disorder functions \( f_j \) tend to the delta-function obtained in our recent work [1]. In the regime \( \sigma \to 0 \) and \( \Delta \to \infty \) (or \( g_{j2} \to 0 \), equivalently), one has \( f_j(\infty, 0) = 1 \). This immediately leads to reproduce the Huang and Meng result [31] for the single component disorder fraction. If \( \sigma \to 0 \) and \( \Delta \to 1 \) (or \( g_{j2} \to \sqrt{8}\Gamma^2 \), i.e. in the vicinity of the phase separation), the functions \( f_j(1, 0) \) are diverging and become complex for \( \Delta < 1 \).

For our simulations we consider the above \( ^{87}\text{Rb}\ldots ^{87}\text{Rb} \) mixture in two different internal states with the following parameters: \( a_1 = 100.4 \, a_0 \), \( a_2 = 95.44 \, a_0 \,[52] \), and \( n_1 = n_2 = 10^{21} \, \text{m}^{-3} \).

The behavior of the disorder functions \( f_j \) versus of the interspecies interactions and the disorder correlation length is displayed in figure 5. We see that for \( \sigma < \xi_j \), the effects of the disorder fluctuations are important. Whereas they vanish in the case of \( \sigma > \xi_j \) for any value of interspecies interactions leading to arrest the localization in both species. For fixed \( \sigma < \xi_j \), the disorder functions \( f_j \) rise with \( a_{j2} \) and diverge near the vicinity of the transition between the miscible and immiscible phases (\( a_{j2} \approx 98 \, a_0 \)) [1].

One should stress that in the case of an asymmetric mixture, the criterion validity of the perturbation theory can be established from equation (32). In such a situation, there exists always a critical disorder strength \( R_j^\prime \) above which a transition from miscible to immiscible phase occurs [1].

The chemical potential of each component can be obtained from integral (8), which gives

\[
\mu_j = g_j n_j + g_{j2} n_j + g_j n_{\text{BM}} h_j(\Delta, \sigma_j),
\]

The disorder functions \( h_j(\Delta, \sigma_j) \) associated with the EoS read:

\[
h_j(\Delta, \sigma_j) = \left( \frac{\Delta}{\Delta - 1} \right) \left[ H_j(\Delta, \sigma_j) + \frac{g_{j2} n_j}{g_j^2} h_j(\Delta, \sigma_j) \right],
\]

where
Figure 6. Behavior of the disorder functions $h_1$ as a function of the interspecies interaction strength $a_{12}$ and the disorder correlation length for $^{87}\text{Rb}-^{87}\text{Rb}$ mixture. Parameters are the same as in figure 5.

\begin{equation}
H_1(\Delta, \sigma_j) = \frac{4\sqrt{2/\pi}}{(\alpha_j - \beta_j)^3} \left\{ 2\sigma_j(\alpha_j - \beta_j) \right\}
\end{equation}

and

\begin{equation}
H_2(\Delta, \sigma_j) = \frac{2\sqrt{2/\pi}}{(\alpha_j \beta_j)^3(\alpha_j - \beta_j)^3} \left\{ -2\sigma_j \sqrt{\alpha_j \beta_j} (\alpha_j - \beta_j) \right\}
\end{equation}

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