Global well-posedness of 2D Euler-\(\alpha\) equation in exterior domain

Xiaoguang You\(^1\), Aibin Zang\(^2,\ast,\ast\ast\) and Yin Li\(^3,\ast\ast\ast\)

1 School of Mathematics, Northwest University, Xi’an 710069, People’s Republic of China
2 School of Mathematics and Computer Science & The Center of Applied Mathematics, Yichun University, Yichun, Jiangxi, 336000, People’s Republic of China
3 Faculty of Education, Shaoguan University, 512005, Shaoguan, People’s Republic of China

E-mail: wiliam_you@aliyun.com, zangab05@126.com and liyin2009521@163.com

Received 18 December 2021, revised 22 August 2022
Accepted for publication 26 September 2022
Published 13 October 2022

Abstract
After casting Euler-\(\alpha\) equations into vorticity-stream function formula, we obtain some very useful estimates from the properties of the vorticity formula in exterior domain. Basing on these estimates, one can have got the global existence and uniqueness of the solutions to Euler-\(\alpha\) equations in 2D exterior domain provided that the initial data is regular enough.

Keywords: exterior domain, Dirichlet boundary conditions, vorticity-stream function, Euler-\(\alpha\) equations
Mathematics Subject Classification numbers: 76B03, 76A05, 35G16.

1. Introduction
Let \(\Omega \subset \mathbb{R}^2\) be a bounded, simply connected domain with \(C^\infty\) Jordan boundary \(\Gamma\). We consider Euler-\(\alpha\) equations in exterior domain \(\Omega = \mathbb{R}^2 \setminus \overline{\varnothing}\) given by:

\[
\begin{align*}
\partial_t v + u \cdot \nabla v + \sum_{j=1}^{2} v_j \nabla u_j + \nabla p &= 0 & \text{in } \Omega \times (0, \infty), \\
\text{div } u &= 0 & \text{in } \Omega \times (0, \infty)
\end{align*}
\]

(1.1)

(1.2)

with initial data \(u_0\), where \(v = u - \alpha^2 \Delta u\) and \(p\) is the pressure. Above \(u\) is called the filtered velocity, while \(v\) is the unfiltered velocity and \(\nabla \perp \cdot v\) (noted by \(q\)) is the unfiltered vorticity.

\(^\ast\) Authors to whom any correspondence should be addressed.
\(^\ast\ast\) Recommended by Dr Karima Khushnutdinova.
\(^\ast\ast\ast\) These authors have contributed equality to this work.
Recall that $\alpha$ is a parameter having the dimensions of a length, and when Euler-$\alpha$ (1.1) and (1.2) is used in large Eddy simulations, this parameter should be related to the smallest resolved scale, we can refer [1]. Euler-$\alpha$ equations are viewed as a conservative generalization of incompressible Euler equations. Generally, Euler-$\alpha$ equations go back to at least three independent developments [12].

First, Euler-$\alpha$ equations arise as a generalization into several spatial dimensions of the scalar Fobas–Fuchssteiner–Camassa–Holm (FFCH) equation

$$u_t - u_{xxx} + 2\kappa u_x = -3uu_x + 2uu_{xx} + uu_{xxx},$$

which models unidirectional surface wave in shallow water. The FFCH equation can be realised as a geodesic equation on the diffeomorphism group of the circle equipped with a metric that arises from $H^\ell$ metric on its Lie algebra of vector field. It is this property that can be used to obtain Euler-$\alpha$ equations, an idea introduced by Holm et al [8, 9].

Second, Euler-$\alpha$ equations are the equations of motion of an inviscid non-Newtonian fluid of second grade. In this interpretation, $\alpha^2$ is a material constant expressing the elastic response of the fluid. Due to essential material frame indifference and observer objectivity on Rivlin–Ericksen law, Markovitz and Coleman [16] and Nolland Truesdell [19] obtain Euler-$\alpha$ equations without viscosity in second grade fluid.

Third, Euler-$\alpha$ model can be seen as the continuous analog of a computational vortex blob method for two-dimensional hydrodynamics, we can refer the review by Leonard [13].

There has been substantial work on the well-posedness for Euler-$\alpha$ equations. Busuioc [4] proved the global existence and uniqueness of the solutions for Euler-$\alpha$ equations in $\mathbb{R}^2$ and local well-posedness in $\mathbb{R}^3$. The second author has obtained the global existence for two-dimensional period domain and the local existence in some uniform time with respect to the parameter $\alpha$ for three-dimensional period domain in [20]. In bounded domains with Dirichlet boundary conditions, namely

$$u = 0 \text{ on } \Gamma.$$  (1.3)

Shkoller [18] showed that Euler-$\alpha$ equations are well-posed by transferring the problem from Eulerian to the Lagrangian setting. Moreover, the solution is global for 2D case. In Eulerian setting, well-posedness for 2D case was proved in [14], by Banach fixed point theorem. In half plane, the global existence of weak solutions were established in [3] in the space of Radon measures.

It is an important issue to investigate the well-posedness of Euler-$\alpha$ flow around an obstacle. This work is inspired by the singular limits for Euler-$\alpha$ equations to Euler equations as $\alpha \to 0$ or both $\alpha$ and the radius of obstacle go to zero. The latter is analogy to the results of [11] for the viscous incompressible flow around a small obstacle. To arrive these limits, we must firstly discuss on the well-posedness problem of Euler-$\alpha$ equations in the exterior domain.

Therefore, we focus on investigating the global existence and uniqueness of the system (1.1) and (1.2) with initial velocity $u_0$, the boundary conditions (1.3) and $u$ decays to zero at infinity, that is

$$u \to 0 \text{ as } |x| \to +\infty.$$  (1.4)

To solve the exterior problem, one can meet with two main difficulties: one is from the nonlinearities with high order derivatives; other is from the unboundedness of the domain.
To overcome these difficulties, we will cast the system (1.1) and (1.2) into the following vorticity-stream function formula

\[
\begin{align*}
\partial_t q + \mathbf{u} \cdot \nabla q &= 0 & \text{in } \Omega \times [0, \infty), \\
\Delta \psi &= q & \text{in } \Omega \times [0, \infty), \\
\mathbf{u} - \alpha^2 \Delta \mathbf{u} + \nabla p &= \nabla^\perp \psi & \text{in } \Omega \times [0, \infty)
\end{align*}
\]  

(1.5)  
(1.6)  
(1.7)

here \( \psi \) is called the stream function of the unfiltered velocity \( \mathbf{v} \equiv \mathbf{u} - \alpha^2 \Delta \mathbf{u} \) with the divergence-free vector field \( \mathbf{u} \). It is easy to see that the system (1.5)–(1.7) is an elliptic-hyperbolic coupled system. To tackle this problem, we must linearise the system (1.5)–(1.7). In fact, by solving the Poisson equation (1.6) for the stream function \( \psi \) with \( q \), we can obtain the uniform bound of high order derivatives of \( \psi \). Applying the Galdi’s estimate [6] to the Stokes type equation (1.7), we are thus able to find the uniform estimates of high order derivatives for the velocity \( \mathbf{u} \).

However, since the domain is unbounded, it is nontrivial to establish a priori estimates of low order derivatives of \( \mathbf{u} \) from (1.5)–(1.7). To solve this issue, we assume temporarily that the initial unfiltered vorticity \( q_0 = \nabla^\perp \cdot (\mathbf{u}_0 - \alpha^2 \Delta \mathbf{u}_0) \) has a compact support. Observing that \( q \) satisfies a transport equation, one shows that \( q \), in finite time interval, still has a compact support. Then by the estimates of the Poisson equation, we can get the uniform bound for the stream function \( \psi \). Therefore, by Banach fixed point theorem, we can find a unique triple \( (\mathbf{u}, \psi, q) \) satisfies the system (1.5)–(1.7) provided that \( q_0 \) has a compact support.

Finally, using the previous uniform estimates of \( \mathbf{u} \) and \( \psi \) which are independent of the support of \( q_0 \) and the system (1.1) and (1.2), we can immediately obtain the uniform estimates of lower order derivatives of \( \mathbf{u} \), which help to eliminate the assumption for compact support of \( q_0 \).

The article is organised as follows. In section 2, we will introduce some notations and preliminaries. In section 3, we will present some technical lemmas and their proofs. In section 4, we will state and prove main result. In last section, we will give some comments and discussions.

2. Notations and preliminaries

In this section, we introduce notations and present preliminary results. We use the notation \( H^s(\Omega) \) for the usual \( L^2 \)-based Sobolev spaces of order \( s \). \( C_0^\infty(\Omega) \) represents the space of smooth functions with infinitely many derivatives, compactly supported in \( \Omega \), and \( H_0^1(\Omega) \) the closure of \( C_0^\infty(\Omega) \) under the \( H^1 \)-norm. For the sake of simplicity, \( H^s(\Omega) \) (respectively \( H_0^1(\Omega) \)) stands for vector space (\( H^s(\Omega) \))^2 (respectively \( (H_0^1(\Omega))^2 \)). Let \( \Omega_L := \Omega \cap B(0, L) \) and \( \Omega_L^c := \Omega \setminus B(0, L) \), where \( B(0, L) \) is the disk centred at origin with radius \( L \). By the way, the unit disk centred at origin in \( \mathbb{R}^2 \) is denoted by \( D \). Let \( \mathcal{A} \) be an arbitrary set of \( \mathbb{R}^2 \), \( \delta(\mathcal{A}) \) represents the diameter of \( \mathcal{A} \), that is

\[
\delta(\mathcal{A}) := \sup_{x, y \in \mathcal{A}} |x - y|.
\]

We also make use of the following notations

\[
\begin{align*}
\mathcal{D} &:= \{ \mathbf{u} \in (C_0^\infty(\Omega))^2; \text{div} \mathbf{u} = 0 \}, \\
\mathcal{V} &:= \{ \mathbf{u} \in H_0^1(\Omega); \text{div} \mathbf{u} = 0 \quad \text{in } \Omega \}.
\end{align*}
\]
\[
L^2_\sigma := \{ u \in (L^2(\Omega))^2; \text{div } u = 0, u \cdot \nu |_{\Gamma} = 0 \},
\]
\[
X^2_{\text{har}} := \{ h \in (L^2(\Omega))^2; \text{div } h = 0, \text{rot } h = 0, h \cdot \nu |_{\Gamma} = 0 \},
\]

here \( \nu \) is the normal vector to \( \Gamma \). The homogeneous Sobolev space will be used several times, which was defined as
\[
\dot{H} := \left\{ u \in L^2_{\text{loc}}(\Omega); \int_{\Omega} |\nabla u|^2 \, dx < \infty \right\},
\]
while \( \dot{H} \) stands for vector space \( (\dot{H})^2 \).

We then present some well-known results which will be used in this paper. The following lemma, in the matter of transport equation, can be found in [5].

**Lemma 2.1.** Let \( p \in [1, \infty) \), \( q_0 \in L^p(\Omega) \), \( u \in L^\infty([0, T]; V) \) with \( T > 0 \) fixed, then the transport equation:
\[
\begin{cases}
\partial_t q + u \cdot \nabla q = 0, \\
q|_{t=0} = q_0
\end{cases}
\]
has a unique weak solution \( q \in C([0, T]; L^p(\Omega)) \), and satisfying:
\[
\sup_{t \in [0, T]} \| q(t) \|_{L^p(\Omega)} \leq \| q_0 \|_{L^p(\Omega)},
\]
(2.3)

**Remark 2.1.** Assume that \( \delta(\text{supp } q_0) < \infty \) and \( u \in L^1([0, T]; V \cap H^2(\Omega)) \), then we have
\[
\delta(\text{supp } q(t)) < \delta(\text{supp } q_0) + C \int_0^t \| u(s) \|_{H^2(\Omega)} \, ds,
\]
(2.4)

for \( t \in [0, T] \), where \( C \) is a constant depends only on \( \Omega \).

Indeed, let \( X \in C([0, T]; (C(\Omega))^2) \) be the unique weak solution of the following ordinary differential equations
\[
\begin{cases}
\frac{dX(t, \alpha)}{dt} = u(X(t, \alpha), t) & \text{in } \Omega \times [0, T] \\
X(0, \alpha) = \alpha & \forall \alpha \text{ in } \Omega,
\end{cases}
\]

since \( q(x, t) \) satisfies the transport equation (2.1), we have
\[
q(X(t, \alpha), t) = q_0(\alpha),
\]
then it is easy to see (2.4) holds.

**Remark 2.2.** Let \( s \geq 1 \), \( q_0 \in H^s(\Omega) \) and \( u \in L^\infty([0, T]; H^{s+2}(\Omega)) \), then it follows that for all \( t \in [0, T] \)
\[
\| q(t) \|_{H^s(\Omega)} \leq C \| q_0 \|_{H^s(\Omega)},
\]
(2.5)

where \( C \) depends on \( \| u \|_{L^\infty([0, T]; H^{s+2}(\Omega))} \). We begin to check it by using approximation arguments.

First, let us extend both \( u \) and \( q_0 \) to the whole plane such that
\[
\| u(t, \cdot) \|_{H^{s+2}(\mathbb{R}^2)} \leq C \| u(t, \cdot) \|_{H^{s+2}(\Omega)}, \| q_0 \|_{H^s(\mathbb{R}^2)} \leq C \| q_0 \|_{H^s(\Omega)},
\]
(2.6)
where \( t \in [0, T] \). It should be noted \( u \) may not be divergence free in \( \mathbb{R}^2 \setminus \Omega \).

We then start to construct approximate equations. Let \( \eta \) be any smooth function that
\[
\eta \in C^\infty_0 (\mathbb{R}^2), \quad \eta > 0, \quad \int_{\mathbb{R}^2} \eta = 1,
\]
we define the mollification \( J_\varepsilon v \) of functions \( v \in L^2(\mathbb{R}^2) \) by
\[
(J_\varepsilon v)(x) = \varepsilon^{-2} \int_{\mathbb{R}^2} \eta \left( \frac{x-y}{\varepsilon} \right) v(y) dy, \quad \varepsilon > 0.
\]

There are several well-known properties (see for example [15]):
(a) For all \( v \in L^2(\mathbb{R}^2) \), \( J_\varepsilon v \) is smooth and belongs to \( H^k(\mathbb{R}^2) \) for arbitrary \( k \in \mathbb{N} \).
(b) Mollifiers commute with distribution derivatives,
\[
D^\alpha J_\varepsilon v = J_\varepsilon D^\alpha v.
\]
(c) For \( k \in \mathbb{N} \) and \( v \in H^k(\mathbb{R}^2) \), \( J_\varepsilon v \) converges to \( v \) in \( H^k(\mathbb{R}^2) \).
(d) For all \( u, v \in L^2(\mathbb{R}^2) \),
\[
\int_{\mathbb{R}^2} (J_\varepsilon u)v = \int_{\mathbb{R}^2} u(J_\varepsilon v).
\]

The approximate equations of the transport equations are defined as:
\[
\begin{cases}
\partial_t q^\varepsilon + J_\varepsilon [u \cdot \nabla (J_\varepsilon q^\varepsilon)] = 0 \text{ in } [0, T] \times \mathbb{R}^2, \\
q^\varepsilon|_{t=0} = q_0 \text{ in } \mathbb{R}^2.
\end{cases}
\]

We can use Cauchy–Lipschitz–Picard theorem to obtain a unique solution \( q^\varepsilon \in \text{Lip}([0, T]; H^s(\mathbb{R}^2)) \) of equations (2.11) and (2.12).

We then focus on obtaining the inequality (2.5). Indeed, differentiating equation (2.11) \( \alpha \) times, we obtain:
\[
\partial_t D^\alpha q^\varepsilon + J_\varepsilon [u \cdot \nabla (J_\varepsilon D^\alpha q^\varepsilon)] + \sum_{\beta \leq \alpha, \beta + \gamma = \alpha} J_\varepsilon [D^\beta u \cdot \nabla J_\varepsilon (D^\gamma q^\varepsilon)] = 0,
\]
where we have used the properties of Mollifiers. We then multiply the above equation by \( D^\alpha q^\varepsilon \), integrate over \( \mathbb{R}^2 \) and sum up from \( \alpha = 0 \) to \( |\alpha| = s \), if follows
\[
\frac{d}{dt} \| q^\varepsilon(t) \|_{H^s(\mathbb{R}^2)}^2 \leq C \| u \|_{L^\infty([0, T]; H^{s+2}(\mathbb{R}^2))} \| q^\varepsilon(t) \|_{H^s(\mathbb{R}^2)}^2
\]
for \( t \in [0, T] \). Thanks to Grönwall inequality, we conclude that
\[
\| q^\varepsilon(t) \|_{H^s(\Omega)} \leq C \| q_0 \|_{H^s(\mathbb{R}^2)} \leq C \| q_0 \|_{H^s(\Omega)}.
\]

By virtue of Banach–Alaoglu theorem, there exists a subsequence of \( q^\varepsilon \) converges weak-star to some \( q \) in \( L^\infty([0, T]; H^s(\Omega)) \), and \( q \) satisfies inequality (2.5). It is left to show \( q \) is exactly the weak solution of the original transport equations (2.1) and (2.2). In other words, we need to check that
\[
J_\varepsilon [u \cdot \nabla (J_\varepsilon q^\varepsilon)] \to u \cdot \nabla q^\varepsilon \text{ in distribution sense.}
\]
Indeed, for arbitrary test function $\phi \in C^\infty_0([0,T) \times \Omega)$, we have:

$$
\int_0^T \int_\Omega J_\varepsilon [u \cdot \nabla (J_\varepsilon q^\varepsilon)] \phi = \int_0^T \int_\Omega [u \cdot \nabla (J_\varepsilon q^\varepsilon)] J_\varepsilon \phi = -\int_0^T \int_\Omega [u_\cdot \nabla (J_\varepsilon \phi)] q^\varepsilon
$$

(2.15)

holds for $\varepsilon > 0$ small enough. Let $\varepsilon \to 0$, we immediately obtain (2.14).

As we know, a smooth irrotational vector field in simple connected domain is a gradient field of some scalar function. Although two-dimensional exterior domain is not simple connected, the same conclusion holds true from the following lemma in suitable function space.

**Lemma 2.2 (See [7]).** The following equation

$$
\begin{align*}
\nabla \cdot u &= 0 \quad \text{in } \Omega \\
\nabla^\perp \cdot u &= 0 \quad \text{in } \Omega \\
u \cdot \nu &= 0 \quad \text{on } \Gamma
\end{align*}
$$

only has zero solution in $L^2(\Omega)$ space, in other words, $X^{2}_{\text{curl}}(\Omega) = \{0\}$.

**Remark 2.3.** Suppose that $u \in (L^2(\Omega))^2$ is irrotational, then there exists a scalar function $p \in L^2_{\text{loc}}(\Omega)$ such that $u = \nabla p$. Indeed, from the classical Helmholtz decomposition, we have

$$u = v + \nabla p$$

(2.16)

for some $v \in L^2_\sigma$ and $p \in H$. Recalling that $u$ is irrotational, it follows $\nabla^\perp \cdot v = 0$, lemma 2.2 then implies $v \equiv 0$.

Observing that the equation (1.7) is the Stokes equation. In the following lemma, we state some well-known results about Stokes equations in exterior domain.

**Lemma 2.3 (See [2, 6]).** Let $\varphi \in L^2(\Omega) \cap H^1(\Omega)$, $\lambda > 0$, the following Stokes equation

$$
\begin{align*}
&\begin{cases}
u(x) - \lambda \Delta u(x) + \nabla p = \varphi(x) &\text{in } \Omega \\
u = 0 &\text{on } \Gamma
\end{cases} \quad (2.17) \\
u = 0 &\text{on } \Gamma
\end{align*}
$$

has a unique solution $u \in H^3(\Omega) \cap V$ with the estimates

$$
\|u\|_{H^3(\Omega)} \leq C \|\varphi\|_{H^1(\Omega)},
\|D^{k+1} u\|_{L^2(\Omega)} \leq C \left(\|\nabla \varphi\|_{H^{k+1}(\Omega)} + \|u\|_{L^2(\Omega)}\right), \quad k \geq 0
$$

(2.19)

(2.20)

where $C$ is a constant depends only on $\Omega$ and $\lambda$.

We then introduce the Calderón–Zygmund theorem, which would be used several times. By integral transform with kernel $K$ of a function $f$, we mean that the function $\Psi$ is defined as:

$$
\Psi(x) = \int_\Omega K(x, y) f(y) dy.
$$

(2.21)
In this article, we consider the kernel $K$ of the form
$$K(x, y) = \frac{k(x, y)}{|y|^2},$$  
(2.22)
where $k(x, y)$ is a regular function in $\mathbb{R}^2 \times (\mathbb{R}^2 - \{0\})$. We say the kernel $K$ is singular if the following assumptions hold:
(a) For any $x, y \in \mathbb{R}^2 \times (\mathbb{R}^2 - \{0\})$ and every $\alpha > 0$
$$k(x, \alpha y) = k(x, y).$$  
(2.23)
(b) For every $x \in \mathbb{R}^2$, $k(x, y)$ is integrable on the circle $|y| = 1$ and
$$\int_{|y|=1} k(x, y)dy = 0.$$  
(2.24)
(c) There exists $C > 0$, such that
$$\text{ess sup}_{x \in \mathbb{R}^2 \ |y| = 1} |k(x, y)| \leq C.$$  
(2.25)

For integral transforms defined by (2.21), we have the following fundamental result due to Calderón and Zygmund (which could be found in [6]).

**Lemma 2.4.** Assume $K(x, y)$ is a singular kernel, $f \in L^2(\mathbb{R}^2)$, then the P.V. convolution integral
$$\Psi(x) = \lim_{\varepsilon \to 0} \int_{|x-y| \geq \varepsilon} K(x, x-y) f(y)dy$$  
(2.26)
exists for almost all $x \in \mathbb{R}^2$. Moreover
$$\|\Psi\|_{L^2(\Omega)} \leq C\|f\|_{L^2(\Omega)}.$$  
(2.27)

### 3. Technical lemmas and their proofs

In this section, we will state and prove technical lemmas that are helpful to investigate the existence of Euler-\(\alpha\) equations in exterior domain.

Firstly, we will discuss on the Poisson equation with Dirichlet boundary conditions:
$$\begin{cases}
\Delta \psi = q & \text{in } \Omega, \\
\psi = 0 & \text{on } \Gamma.
\end{cases}$$  
(3.1)
(3.2)
Comparing with the case in bounded domain, we could not obtain the following estimate in exterior domain $\Omega$:
$$\|\psi\|_{H^1(\Omega)} \leq C\|q\|_{L^2(\Omega)}.$$  
(3.3)
However, there are several results in homogeneous Sobolev spaces (see [17]).

Since $\mathbb{R}^2 \setminus \Omega$ is a bounded, open and simple connected domain with $C^\infty$ Jordan boundary, in [10] it has been shown that there exists a smooth biholomorphism $T : \Omega \mapsto \mathbb{R}^2 \setminus \Omega$, extending smoothly up to the boundary, mapping $\Gamma$ to $\partial \mathbb{D}$.
Additionally, one follows that in \[10\]
\[
\|DT\|_{L^\infty} \leq C \quad \text{and} \quad \|D_xT^{-1}\|_{L^\infty} \leq C,
\]
Moreover, the authors of [10] have given an explicit formula for the Green’s function \(G_\Omega\) of the Laplacian operator with homogeneous Dirichlet boundary conditions in \(\Omega\), namely

\[
G_\Omega(x,y) = \frac{1}{2\pi} \ln \frac{|T(x) - T(y)|}{|T(x) - T(y)^*||T(y)|},
\]
where \(\eta^* = \frac{\eta}{|\eta|^2}\) for \(\eta \in D_c\). We can now establish the following lemma.

**Lemma 3.1.** Let \(R > 0\) be fixed. Suppose \(q \in L^2(\Omega)\) and \(\text{supp } q \subset B(0,R)\), then the Poisson equations (3.1) and (3.2) has a unique solution \(\psi \in \dot{H}\), which can be written explicitly as

\[
\psi(x) = \int_\Omega G_\Omega(x,y)q(y)dy,
\]
and \(\psi\) obeys

\[
\|\nabla\psi\|_{L^2(\Omega)} \leq CR\left(\|q\|_{L^2(\Omega)} + \|q\|_{L^1(\Omega)}\right),
\]
\[
\|D^2\psi\|_{L^2(\Omega)} \leq C(R\|q\|_{L^2(\Omega)} + \|q\|_{L^1(\Omega)}),
\]
where \(C\) is a constant only depends on \(\Omega\). Moreover, if \(q \in H^s(\Omega)\) for \(s \in \mathbb{N}\), it follows

\[
\|D^{s+2}\psi\|_{L^2(\Omega)} \leq C\left(\|q\|_{H^s(\Omega)} + \|\nabla\psi\|_{L^2(\Omega)}\right),
\]
where \(C\) also only depends on \(\Omega\).

**Remark 3.1.** Since \(q\) is compactly supported on \(\Omega\), it is clear that the \(L^1(\Omega)\)-norm of \(q\) is bounded by its \(L^2(\Omega)\)-norm. Here, the \(L^1(\Omega)\)-norms of \(q\) are added, for showing that the coefficients of the inequalities (3.7) and (3.8) linearly depend on \(R\).

**Proof of lemma 3.1.** Let us consider uniqueness first. Suppose \(\psi_1, \psi_2 \in \dot{H}\) are both solutions to equations (3.1) and (3.2). Set \(w = \nabla^\perp(\psi_1 - \psi_2)\), it follows

\[
\nabla \cdot w = 0 \quad \text{in } \Omega,
\]
\[
\nabla^\perp \cdot w = 0 \quad \text{in } \Omega,
\]
\[
w \cdot \nu = 0 \quad \text{on } \Gamma,
\]
where \(\nu\) is the unit normal vector to \(\Gamma\). From lemma 2.2, we know that \(w = 0\). Observing that \(\psi_1 \equiv \psi_2 \equiv 0\) on \(\Gamma\), we conclude that \(\psi_1 \equiv \psi_2\) in \(\Omega\).

From [10], one follows that the expression (3.6) is a solution of equations (3.1) and (3.2). We now check that \(\psi \in \dot{H}\) with the inequalities (3.7) and (3.8). For convenience, zero extension of \(q\) on \(\mathbb{R}^2\setminus \Omega\) is still denoted by \(q\). We will use frequently the following general relation, for arbitrary vectors \(a, b \in \mathbb{R}^2, a \neq 0, b \neq 0\),

\[
\frac{|a|}{|a|^2} - \frac{|b|}{|b|^2} = \frac{|a - b|}{|a| |b|}.
\]
Indeed,
\[
\left( \frac{a}{|a|^2} - \frac{b}{|b|^2} \right)^2 = \frac{1}{|a|^2} + \frac{1}{|b|^2} - \frac{2}{|a|^2|b|^2} (|a|^2 + |b|^2 - 2a \cdot b) \\
= \frac{1}{|a|^2|b|^2} (|a|^2 + |b|^2 - 2a \cdot b) \\
= \frac{1}{|a|^2|b|^2} (a - b)^2.
\] (3.11)

Taking \( \bar{R} := \max \{1, \|D_x T\|_{L^\infty}, \|D_y T\|_{L^\infty}^{-1} \} R \), and then noting that \( \Omega \equiv \{ x \in \mathbb{R}^2 \mid |T(x)| > 1 \} \), by (3.10), it follows:
\[
\| \nabla \psi \|^2_{L^2(\Omega)} = \frac{1}{4\pi^2} \int_{\Omega} \left| \int_{\Omega} \frac{|T(x) - T(y)|}{|T(x) - (T(y))'|} \frac{q(y)dy}{|T(x) - T(y)|} \right|^2 dx \\
\leq \frac{1}{4\pi^2} \int_{|T(x)| \leq 2R} \left| \int_{\Omega} \frac{D_x(|T(x) - T(y)|)}{|T(x) - T(y)|} q(y)dy \right|^2 dx \\
+ \frac{1}{4\pi^2} \int_{|T(x)| < 2R} \left| \int_{\Omega} \frac{D_y(|T(x) - T(y)|)}{|T(x) - (T(y))'|} \frac{q(z)dy}{|T(x) - T(y)|} \right|^2 \frac{q(y)dy}{|T(x) - T(y)|} \\
+ \frac{1}{4\pi^2} \int_{|T(x)| > 2R} \left| \int_{\Omega} \frac{|D_x T(x)|}{|T(x) - T(y)|} |T(x) - (T(y))'| \frac{q(y)dy}{|T(x) - T(y)|} \right|^2 dx \\
= : I_1 + I_2 + I_3. 
\] (3.12)

To estimate the term \( I_1 \), we introduce a truncation function \( \chi \), which equals 0 in \( \mathbb{D} \), while equals 1 in \( \mathbb{D}^c \). By (3.4), and recalling \( q \) is supported on \( B(0, R) \), we deduce:
\[
I_1 = \frac{1}{4\pi^2} \int_{|T(x)| \leq 2R} \left| \int_{\Omega} \frac{D_x(|T(x) - T(y)|)}{|T(x) - T(y)|} q(y)dy \right|^2 dx \\
\leq C \int_{|T(x)| \leq 2R} \left| \int_{\mathbb{R}^2} \frac{|q(y)|}{|T(x) - T(y)|} dy \right| dx \\
\leq C \int_{\mathbb{R}^2} \left| \int_{|T(x) - T(y)| < 3R} |T(x) - T(y)|^{-1} |q(y)|dy \right| dx \\
\leq C \int_{\mathbb{R}^2} \left| \int_{|T(x) - T(y)| < 3R} |T(x) - T(y)|^{-1} \left( \frac{|T(x) - T(y)|}{3R} \right) |q(y)|dy \right| dx,
\] (3.13)

applying the transformation of variables
\[
\eta = T(x) \quad \text{and} \quad \xi = T(y)
\]
and Young’s convolution inequality, it follows:
\[
I_1 \leq C \int_{\mathbb{R}^2} \left| \int_{\mathbb{R}^2} |\eta - \xi|^{-1} \left( \frac{\eta - \xi}{CR} \right) |q(T^{-1}(\xi))|d\xi \right|^2 d\eta \\
\leq CR^2 \|q\|^2_{L^2(\Omega)}. 
\] (3.14)
For the second term \( I_2 \), observing that \( \Omega \equiv \{ x \in \mathbb{R}^2 | |T(x)| > 1 \} \equiv \{ x \in \mathbb{R}^2 | (T(x))^* < 1 \} \), we find

\[
I_2 = \frac{1}{4\pi^2} \int_{1 \leq |T(x)| \leq 2R} \left| \int_{|T(x)| > 1} \left( \frac{|D_x(T(x) - (T(y))^*)|}{|T(x) - (T(y))^*|} \right) q(y) dy \right|^2 dx
\]

\[
\leq \frac{\|DT\|_{L^\infty}}{4\pi^2} \int_{1 \leq |T(x)| \leq 2R} \left( \int_{|T(x)| > 1} \left\| \frac{|q(y)|}{|T(x) - (T(y))^*|} \right\|^2 dy \right) dx
\]

\[
= I_{21} + I_{22},
\]

(3.15)

Let us make change of the variables again

\[
\eta = T(x) \quad \text{and} \quad \xi = (T(y))^*,
\]

and note that \( dy = |DT^{-1}(\xi^*)| \frac{d\xi}{\xi} \), we then get

\[
I_{21} \leq C \int_{1 \leq |\eta| \leq \frac{1}{2}} \left( \int_{|\xi| < 1 / (|\eta| - |\xi|)} \left\| \frac{|q(T^{-1}(\xi^*))|}{|\eta - \xi|} \right\|^2 |d\xi| |d\eta|,
\]

(3.17)

and then by Young’s convolution inequality again, the above inequality implies

\[
I_{21} \leq C \|q\|_{L^2(\Omega)}^2.
\]

(3.18)

The bound of the term \( I_{22} \) is easily obtained, indeed,

\[
I_{22} = \frac{\|DT\|_{L^\infty}}{4\pi^2} \int_{1 \leq |T(x)| \leq 2R} \left( \int_{|T(x)| > 1} \left\| \frac{|q(y)|}{|T(x) - (T(y))^*|} \right\|^2 dy \right) dx
\]

\[
\leq C \int_{|T(x)| \leq 2R} \left| \int_{\Omega} |q(y)| dy \right|^2 dx
\]

\[
\leq CR^2 \|q\|_{L^2(\Omega)}^2.
\]

(3.19)

From the estimates of \( I_{21} \) and \( I_{22} \), it follows immediately

\[
I_2 \leq CR^2 \left( \|q\|_{L^2(\Omega)}^2 + \|q\|_{L^2(\Omega)}^2 \right).
\]

(3.20)

Now we focus on the term \( I_3 \). As \( q \) is supported on \( B(0, R) \), it follows

\[
I_3 = \frac{1}{4\pi^2} \int_{|T(x)| > 2R} \left| \int_{\Omega} \left( \frac{|D_x(T(x) - (T(y))^*)|}{|T(x) - (T(y))^*|} \right) q(y) dy \right|^2 dx
\]

\[
\leq \frac{1}{4\pi^2} \int_{|T(x)| > 2R} \left( \int_{\Omega} (\hat{R} + 1) \cdot \frac{|D_x(T(x)|}{|T(x)|^2} \cdot q(y) dy \right)^2 dx \leq C \|q\|_{L^2(\Omega)}^2.
\]

(3.21)
where we have taken account of (3.4). Collecting the estimates of $I_1, I_2$ and $I_3$, one arrives at the inequality (3.7).

It remains to check the estimate of $\|D^2\psi\|_{L^2}$. Noting again that $\Omega$ is a bounded by $|\Omega| = |Q|$. Due to (3.4), the term $J_1$ is bounded by

$$
J_1 = \frac{1}{4\pi^2} \int_{\Omega} \left| \lim_{\varepsilon \to 0} \int_{|T(x) - T(y)| = \varepsilon} \partial_{\eta} \left( \frac{|T(x) - T(y)|}{|T(x) - T(y)|} \right) \nu_j(x) dS(x) \right|^2 |q(y)|^2 dy
$$

$$
\leq \frac{1}{4\pi^2} \int_{\Omega} \left| \lim_{\varepsilon \to 0} \int_{|T(x) - T(y)| = \varepsilon} \frac{|DT(x)|}{|T(x) - T(y)|} dS(x) \right|^2 |q(y)|^2 dy
$$

$$
\leq C |q|^2_{L^2(\Omega)}. 
$$

From the transformation of variables

$$
\eta = T(x) \quad \text{and} \quad \xi = T(y),
$$

we then have checked the term $J_2$ as follows

$$
J_2 = \frac{1}{4\pi} \int_{\Omega} \int_{\Omega} K(\eta, \eta - \xi) Q_2(\xi) d\xi \left\| \partial_{\xi} \left( \frac{|\xi|}{|\xi|} \right) D_1 T_1(T^{-1}(\eta)) D_1 T_1(T^{-1}(\eta)) |DT^{-1}(\eta)| \right\|^2 d\eta
$$

$$
\leq J_{21} + J_{22}
$$

with

$$
K(\eta, \xi) := \partial_{\xi} \left( \frac{|\xi|}{|\xi|} \right) D_1 T_1(T^{-1}(\eta)) D_1 T_1(T^{-1}(\eta)) |DT^{-1}(\eta)|,
$$

$$
N(\eta, \xi) := \partial_{\xi} \left( \frac{|\xi|}{|\xi|} \right) \partial_{\eta} T_1(T^{-1}(\eta)) |DT^{-1}(\eta)|,
$$

$$
Q_2(\xi) := q(T^{-1}(\xi)) |DT^{-1}(\xi)|.
$$

One is able to check that $K$ is a singular kernel, from lemma 2.4, one follows

$$
J_{21} \leq C |Q_2|^2_{L^2(\Omega)} \leq C |q|^2_{L^2(\Omega)}. 
$$
Recalling that $D_{ij}T(x)$ is bounded in $\Omega$ and satisfies $D_{ij}T(x) = O(|x|^{-3})$ for $|x| \to \infty$, we arrive at

$$J_{22} = \frac{1}{4\pi^2} \int_{\Omega} \left| \int_{\mathbb{R}^n} \frac{1}{|\eta - \zeta|} Q_2(\zeta) d\zeta \right|^2 d\eta$$

$$\leq C \left( \int_{|\eta| > 2R} \left| \int_{|\xi| > 2R} \frac{1}{|\eta - \xi|} Q_2(\zeta) d\zeta \right|^2 d\eta + \int_{|\eta| < 2R} \left| \int_{|\xi| < 2R} \frac{1}{|\eta - \xi|} Q_2(\zeta) d\zeta \right|^2 d\eta \right).$$

Since $Q_2$ is supported on $B(0, R)$, by Young’s convolution inequality, we deduce

$$J_{22} \leq C \left( \int_{|\eta| > 2R} \eta^{-8} \left[ \int_{|\xi| > 2R} Q_2(\zeta) d\zeta \right]^2 d\eta + \int_{|\eta| < 2R} \frac{1}{|\eta - \xi|} Q_2(\zeta) d\zeta \right)^2 d\eta$$

$$\leq C(R^2\|q\|_{L^2(\Omega)}^2 + \|q\|_{L^2(\Omega)}^2).$$

From the estimates of $J_{21}$ and $J_{22}$, one yields

$$J_2 \leq C(R^2\|q\|_{L^2(\Omega)}^2 + \|q\|_{L^2(\Omega)}^2).$$

As

$$\eta = T(x) \quad \text{and} \quad \xi = (T(y))^*,$$

we have

$$J_3 = \frac{1}{4\pi^2} \int_{\Omega} \left( \frac{\partial_n |T(x) - T(y)|}{|T(x) - T(y)|} \right) g(y) d\eta \right|^2 d\eta$$

$$= \int_{\Omega} \left( \frac{K(\eta, \eta - \zeta) Q_3(\zeta) d\zeta}{|T(x) - T(y)|} \right)^2 d\eta + \int_{\Omega} \left[ \frac{N(\eta, \eta - \zeta) Q_3(\zeta) d\zeta}{|T(x) - T(y)|} \right]^2 d\eta$$

$$\leq \frac{1}{4\pi^2} \int_{\Omega} \int_{|\xi| < 1} K(\eta, \eta - \zeta) Q_3(\zeta) d\zeta \right|^2 d\eta + \frac{1}{4\pi^2} \int_{\Omega} \left[ \frac{N(\eta, \eta - \zeta) Q_3(\zeta) d\zeta}{|T(x) - T(y)|} \right]^2 d\eta$$

$$+ \frac{1}{4\pi^2} \int_{\Omega} \int_{|\xi| < 1} N(\eta, \eta - \zeta) Q_3(\zeta) d\zeta \right|^2 d\eta$$

$$= J_{31} + J_{32} + J_{33} + J_{34}.$$ 

with

$$Q_3(\xi) = \frac{|DT^{-1}(\xi^*) q(T^{-1}(\xi^*))|}{|\xi|^3}.$$ 

By lemma 2.4, we obtain

$$J_{31} = \frac{1}{4\pi^2} \int_{\Omega} \int_{|\xi| < 1} K(\eta, \eta - \zeta) Q_3(\zeta) d\zeta \right|^2 d\eta$$

$$\leq C \int_{|\xi| < 1} |Q_3(\zeta)|^2 d\zeta \leq C\|q\|_{L^2(\Omega)}^2.$$
Let us now proceed the term $J_{32}$ as

$$
J_{32} = \frac{1}{4\pi^2} \int_{\mathbb{R}^d} \left| \int_{|\xi| < \frac{1}{4}} K(\eta, \eta - \xi)Q_3(\xi) \, d\xi \right|^2 \, d\eta
$$

\begin{align*}
\leq C & \int_{\mathbb{R}^d} \left| \int_{|\xi| < \frac{1}{4}} \frac{1}{|\eta - \xi|^2} |Q_3(\xi)| \, d\xi \right|^2 \, d\eta \\
\leq C & \int_{\mathbb{R}^d} |Q_3(\xi)|^2 \left| \int_{|\eta| > \frac{1}{4}} \frac{1}{|\eta|^2} \, d\eta \right|^2 \\
\leq C & \|q\|_{L^2(\Omega)}^2. \tag{3.32}
\end{align*}

We observe that the following fact for $J_{33}$

\begin{align*}
J_{33} &= \frac{1}{4\pi^2} \int_{\mathbb{R}^d} \left| \int_{|\xi| < \frac{1}{4}} N(\eta, \eta - \xi)Q_3(\xi) \, d\xi \right|^2 \, d\eta \\
&\leq C \int_{|\eta| < 2R} \left| \int_{|\xi| < \frac{1}{4}} N(\eta, \eta - \xi)Q_3(\xi) \, d\xi \right|^2 \, d\eta \\
&\quad + C \int_{|\eta| > 2R} \left| \int_{|\xi| < \frac{1}{4}} N(\eta, \eta - \xi)Q_3(\xi) \, d\xi \right|^2 \, d\eta \\
&\leq C \int_{|\eta| < 2R} \left| \int_{|\xi| < \frac{1}{4}} \frac{1}{|\eta - \xi|} Q_3(\xi) \, d\xi \right|^2 \, d\eta \\
&\quad + C \int_{|\eta| > 2R} \left| \int_{|\xi| < \frac{1}{4}} \frac{1}{|\eta - \xi||\eta|^2} Q_3(\xi) \, d\xi \right|^2 \, d\eta. \tag{3.33}
\end{align*}

By Young’s convolution inequality to the first term of the above inequality, we conclude that

$$
J_{33} \leq C \|q\|_{L^2(\Omega)}^2 + R^2 \|q\|_{L^2(\Omega)}^2. \tag{3.34}
$$

For $J_{34}$, it turns out to show

\begin{align*}
J_{34} &= \frac{1}{4\pi^2} \int_{\mathbb{R}^d} \left| \int_{|\xi| < \frac{1}{4}} N(\eta, \eta - \xi)Q_3(\xi) \, d\xi \right|^2 \, d\eta \\
&\leq C \int_{|\eta| < 2R} \left| \int_{|\xi| < \frac{1}{4}} \frac{1}{|\eta - \xi|} Q_3(\xi) \, d\xi \right|^2 \, d\eta \\
&\leq C \|q\|_{L^2(\Omega)}^2. \tag{3.35}
\end{align*}

Collecting the estimates of $J_{31}$, $J_{32}$, $J_{33}$ and $J_{34}$, we find

$$
J_3 \leq C \left( \|q\|_{L^2(\Omega)}^2 + R^2 \|q\|_{L^2(\Omega)}^2 \right). \tag{3.36}
$$
All these estimates for $J_1, J_2$ and $J_3$ imply
\[ \|D^2 \psi\|_{L^2(\Omega)} \leq C \left( \|q\|_{L^1(\Omega)} + R\|q\|_{L^2(\Omega)} \right), \tag{3.37} \]
which is exactly the inequality (3.8).

At last, we check that the inequality (3.9) holds. Taking $L > 0$ such that $\mathbb{R}^2 \setminus B(0, L/2) \subset B(0, L/2)$, and then choosing $\phi \in C^\infty(\mathbb{R}^2)$ which is a nonnegative function such that $\phi(x) = 0$ for $|x| < L/2$ and $\phi(x) = 1$ for $|x| > L$, one finds that $\phi \psi$ obeys
\[ \Delta(\psi) = \phi q + 2 \nabla \phi \cdot \nabla \psi + \Delta \phi \psi \quad \text{in } \mathbb{R}^2, \tag{3.38} \]
therefore, it follows
\[ \|D^{r+2} \psi\|_{L^2(\Omega^r)} \leq \|D^{r+2} (\phi \psi)\|_{L^2(\mathbb{R}^2)} \leq C \left( \|q\|_{H^r(\Omega)} + \|\psi\|_{H^{r+1}(\Omega)} \right), \tag{3.39} \]
Noting that $\psi = 0$ on $\Gamma'$, by Sobolev interpolation inequality and Poincaré inequality, we follow
\[ \|D^{r+2} \psi\|_{L^2(\Omega^r)} \leq C \left( \|q\|_{H^r(\Omega)} + \|\nabla \psi\|_{L^2(\Omega)} + \|D^{r+1} \psi\|_{L^2(\Omega)} \right). \tag{3.40} \]
Now it is sufficient to consider the Poisson equation in bounded domain $\Omega_L$. Let us denote $\Pi_{L, 2L} = \{x \in \mathbb{R}^2 | L < |x| < 2L\}$, we arrive at
\[ \|D^{r+2} \psi\|_{L^2(\Omega_{L, 2L})} \leq C \left( \|q\|_{H^{r+1}(\Omega_{L, 2L})} + \|\psi\|_{H^{r+2} \frac{1}{2}(\mathbb{R}^2 \setminus B(0, L))} \right) \leq C \left( \|q\|_{H^r(\Omega)} + \|\psi\|_{H^{r+1}(\Omega_{L, 2L})} \right). \tag{3.41} \]
In view of Gagliardo–Nirenberg inequality, it follows that
\[ \|\psi\|_{H^{r+1}(\Omega_{L, 2L})} \leq C \left( \|\psi\|_{L^2(\Omega_{L, 2L})} + \|D^{r+2} \psi\|_{L^2(\Omega')} \right), \tag{3.42} \]
then by Poincaré inequality, (3.41) and (3.42) immediately bring out
\[ \|\psi\|_{H^{r+1}(\Omega_{L, 2L})} \leq C \left( \|q\|_{H^r(\Omega)} + \|\nabla \psi\|_{L^2(\Omega)} + \|D^{r+2} \psi\|_{L^2(\Omega')} \right). \tag{3.43} \]
From (3.40) and (3.43), we obtain that
\[ \|D^{r+2} \psi\|_{L^2(\Omega)} \leq C \left( \|q\|_{H^r(\Omega)} + \|D^{r+1} \psi\|_{L^2(\Omega)} + \|\nabla \psi\|_{L^2(\Omega')} \right), \tag{3.44} \]
consequently, we can deduce (3.9) by induction.

As we know, in simple connected domain of $\mathbb{R}^2$, a divergence free vector field $\mathbf{u}$ is related to a stream function $\psi$, i.e. $\mathbf{u} = \nabla \times \psi$. For exterior domain of $\mathbb{R}^2$, although it is not simple connected, we still have similar result.

**Lemma 3.2.** Let the vector field $\mathbf{u} \in \mathbb{L}^2 \cap H^1(\Omega)$, then there exists a scalar function $\psi \in \dot{H}$ such that $\mathbf{u} \equiv \nabla \times \psi$.

**Proof.** As above, the Green function of the Laplacian in $\Omega$ is given by (3.5), and we now show that the following function
\[ \psi(x) := \int_{\Omega} G_\Omega(x, y) \nabla^\perp y \cdot \mathbf{u}(y) \, dy \tag{3.45} \]
is the stream function of \( u \) with \( u \equiv \nabla^\perp \psi \). Let \( w = u - \nabla^\perp \psi \). In view of lemma 2.2, it suffices to verify that \( w \) satisfies

(a) \( \nabla \cdot w = 0 \) in \( \Omega \);
(b) \( \nabla^\perp \cdot w = 0 \) in \( \Omega \);
(c) \( w \cdot \nu \equiv 0 \) on \( \Gamma \);
(d) \( w \in L^2(\Omega) \).

Where \( \nu \) is the normal vector to \( \Gamma \). It is easy to check that condition (a)–(c) hold. We here only verify the condition (d), equivalently, we need to check \( \nabla \psi \in L^2(\Omega) \). Indeed, \( \forall \phi \in C_0^\infty(\Omega) \), by integrating by parts, it follows

\[
(\partial_t \psi, \phi)_{L^2} = -\langle \psi, \partial_t \phi \rangle_{L^2} \quad (3.46)
\]

\[
= -\int_\Omega \int G_\Omega(y,x) \nabla^\perp y \cdot u(y) \, dy \, dx
\]

\[
= \int_\Omega \int \left[ \nabla^\perp y \cdot G_\Omega(y,x) \right] \cdot u(y) \, dy \, dx
\]

\[
= \int_\Omega \int \left[ \partial_y \phi(x) \left[ \nabla^\perp y \cdot G_\Omega(y,x) \right] \right] \cdot u(y) \, dy
\]

\[
= \frac{1}{4\pi^2} \int_\Omega \left[ \lim_{\ell \to 0} \int_{|T(x)-T(y)|=\ell} \nu_y \phi(x) \left[ \nabla^\perp y \ln |T(y) - T(x)| \right] \, dS(x) \right] \cdot u(y) \, dy \quad (3.47)
\]

\[
= \frac{1}{4\pi^2} \int_\Omega \int \left[ \phi(x) \left[ \nabla^\perp y \ln |T(y) - T(x)| \right] \right] \cdot u(y) \, dy
\]

\[
= \frac{1}{4\pi^2} \int_\Omega \int \left[ \phi(x) \left[ \nabla^\perp y \ln |T(y) - T(x)| \right] \right] \cdot u(y) \, dy
\]

\[
=: L_1 + L_2 + L_3,
\]

where the property that \( G(x,y) = G(y,x) \) in \( \Omega \) was used. From \( ||DT||_{L^\infty} < \infty \) and Hölder inequality, the term \( L_1 \) can be bounded

\[
|L_1| = \frac{1}{4\pi^2} \int_\Omega \int \left[ \lim_{\ell \to 0} \int_{|T(x)-T(y)|=\ell} \nu_y \phi(x) \left[ \nabla^\perp y \ln |T(y) - T(x)| \right] \, dS(x) \right] \cdot u(y) \, dy
\]

\[
\leq \frac{1}{4\pi^2} \int_\Omega \int \left[ \lim_{\ell \to 0} \int_{|T(x)-T(y)|=\ell} \phi(x) \frac{|DT(y)|}{|T(y) - T(x)|} \, dS(x) \right] u(y) \, dy
\]

\[
\leq C \int_\Omega |\phi(y)| |u(y)| \, dy \leq C ||\phi||_{L^2(\Omega)} ||u||_{L^2(\Omega)}
\]

(3.48)

by making change of variable, the second term \( L_2 \) can be written by

\[
L_2 = -\frac{1}{4\pi^2} \int_\Omega \int \left[ \phi(x) \left[ \partial_y \nabla^\perp y \ln |T(y) - T(x)| \right] \right] \cdot u \, dy
\]

\[
= -\frac{1}{4\pi^2} \int_\Omega \int \left[ \phi(x) \partial_y \frac{\nabla^\perp y |T(y) - T(x)|}{|T(y) - T(x)|} \right] \cdot u \, dy
\]

\[
= \frac{1}{4\pi^2} \int_{D^c} \int_{D^c} \frac{\eta - T(x)}{\xi - T(y)} K_{kl}(\xi - \eta) \Phi_{ik}(\eta) U_i(\xi) \, d\eta \, d\xi
\]
with
\[ K_{kl}(\eta) := -\partial_{\eta_k} \left[ \frac{\partial_{\eta_l} |\eta|}{|\eta|} \right] \]
\[ \Phi_{kl}(\eta) := \phi(T^{-1}(\eta))(\partial_{T_k}(T^{-1}(\eta))|DT^{-1}(\eta)|) \]
\[ U_l(\xi) := (\nabla^\perp T_l(T^{-1}(\xi)) \cdot u(T^{-1}(\xi))|DT^{-1}(\xi)|. \]
(3.50)

As \( K_{kl} \) is singular kernel, from lemma 2.4, we obtain
\[ |L_2| \leq C \| \Phi \|_{L^2(D)} \| U \|_{L^2(D)}. \]

Taking account of property (3.4), the above inequality then implies
\[ |L_2| \leq C \| \phi \|_{L^2(D)} \| u \|_{L^2(D)}. \]
(3.51)

Similarly, we change the term \( L_3 \) into the following formula
\[ L_3 = -\frac{1}{4\pi^2} \int_D \left[ \int_D \phi(x) \left[ \partial_{\tilde{T}_k} \nabla_y^\perp \ln |T(y) - (T(x))^*| \right] dx \right] \cdot u(y) dy \]
\[ = -\frac{1}{4\pi^2} \int_D \left[ \int_D \phi(x) \partial_{\tilde{T}_k} \nabla_y^\perp \frac{|T(y) - (T(x))^*|}{|T(y) - T(x)|} dx \right] \cdot u(y) dy \]
\[ \frac{\eta=(T(x))^*}{\xi=T(y)} - \frac{1}{4\pi^2} \int_D \int_D K_{kl}(\xi - \eta) \Theta_{ik}(\eta) U_l(\xi) d\eta d\xi \]
(3.52)

where \( K \) and \( U \) are defined in (3.50), while \( \Theta \) is defined as
\[ \Theta_{ik}(\eta) = \phi(T^{-1}(\eta^*)) \frac{|DT^{-1}(\eta^*)|}{|\eta|^2} \left( \partial_{T_k}(T^{-1}(\eta^*)) - 2|\eta|^2 \eta^*_k \eta^*_j \partial_{T_j}(T^{-1}(\eta^*)) \right). \]
(3.53)

Again by lemma 2.4 and (3.4), we arrive at
\[ |L_3| \leq C \| \Theta \|_{L^2(D)} \| U \|_{L^2(D)} \]
\[ \leq C \| \Theta \|_{L^2(D)} \| u \|_{L^2(D)}. \]
(3.54)

We now show that \( \Theta \in L^2(D) \), indeed,
\[ \| \Theta \|_{L^2(D)}^2 = \int_D |\Theta(\eta)|^2 d\eta \]
\[ = \int_D \left| \phi(T^{-1}(\eta^*)) \frac{|DT^{-1}(\eta^*)|}{|\eta|^2} \left( \partial_{T_k}(T^{-1}(\eta^*)) - 2|\eta|^2 \eta^*_k \eta^*_j \partial_{T_j}(T^{-1}(\eta^*)) \right) \right|^2 d\eta \]
\[ \leq C \int_D \frac{|\phi(T^{-1}(\eta^*))|^2}{|\eta|^2} d\eta. \]
(3.55)
By variable substitution, it follows
\[
\int_{\mathbb{D}} \frac{|\phi(T^{-1}(\eta^*))|^2}{|\eta|^4} \, d\eta = \frac{\eta = (T(x))^*}{\int_{\Omega} |\phi(x)|^2 \frac{1}{|T(x)|^4} |DT(x)| \, dx} \leq C \int_{\Omega} |\phi(x)|^2 \, dx.
\] (3.56)

By (3.54)–(3.56), we thus obtain
\[
|L_3| \leq C \|\phi\|_{L^2(\Omega)} \|u\|_{L^2(\Omega)}.
\] (3.57)

Collecting (3.48), (3.51) and (3.57), we conclude that
\[
|\partial_i \psi, \phi|_{L^2(\Omega)} \leq C \|\phi\|_{L^2(\Omega)} \|u\|_{L^2(\Omega)}.
\] (3.58)

Therefore the condition (d) holds, in virtue of lemma 2.2, we have \(u \equiv \nabla^\perp \psi\).

In this article, we will first consider the special case that the initial unfiltered vorticity is compactly supported. The following lemma helps to construct such an approximate sequence for more general initial data \(u_0\).

**Lemma 3.3.** Assume that \(u \in V \cap H^s(\Omega), s \geq 1\). Then there exists a approximate sequence \(\{u^a\} \subset V \cap H^s(\Omega)\) such that \(u^a\) is compactly supported and converges to \(u\) in \(H^s(\Omega)\) strongly.

**Proof.** Without loss of generality, we assume \(\mathbb{R}^2 \setminus \Pi \subset B(0,1)\). Now let \(\phi \in C^\infty_0(\mathbb{R}^2)\) be a non-negative function such that
(a) \(0 \leq \phi \leq 1\) in \(\mathbb{R}^2\);
(b) \(\phi \equiv 1\) in \(B(0,1)\);
(c) \(\phi \equiv 0\) in \(\mathbb{R}^2 \setminus B(0,2)\).

We will use the following notations in the following context
\[
\phi^a(x) := \phi\left(\frac{x}{a}\right),
\Pi_{n,2n} := \{x \in \mathbb{R}^2 | n < |x| < 2n\}.
\] (3.59)

Let \(\psi\) be the stream function of \(u\) constructed in lemma 3.2, we define the approximate sequence \(\{u^a\}\) as
\[
u^a := \nabla^\perp (\phi^a(\psi - a_n)),
\] (3.60)

with
\[
a_n = \frac{1}{|\Pi_{n,2n}|} \int_{\Pi_{n,2n}} \psi(x) \, dx.
\] (3.61)

Noting that \(\phi\) is compactly supported, it follows \(u^a\) is compactly supported. It suffices to prove that \(u^a\) converges to \(u\) in \(H^s\). Since
\[ \|u^n - u\|_{H^s(\Omega)}^2 = \sum_{|\alpha| \leq s} \|D^\alpha u^n - D^\alpha u\|_{L^2}^2 \]
\[ = \sum_{|\alpha| \leq s} \|D^\alpha (\nabla^\perp (\phi^\alpha (\psi - a_n))) - D^\alpha u\|_{L^2}^2 \]
\[ \leq \sum_{|\alpha| \leq s} \|D^\alpha u\|_{L^2}^2 + \sum_{1 \leq |\alpha| \leq s, |\beta| \leq 1} C_{\alpha, \beta} \|D^\alpha \phi^\beta (\psi - a_n)\|_{L^2(\Omega)}^2, \]
\[ (3.62) \]

it is obvious to see that the first term of the above inequality converges to zero as \( n \to \infty \). For the second one, we treat \( \beta = 0 \) and \( \beta \neq 0 \) respectively. In the case \( \beta = (\beta_1, \beta_2) \neq 0 \), it follows
\[ \|D^\alpha \phi^\beta D^\beta (\psi - a_n)\|_{L^2(\Omega)}^2 \]
\[ \leq \frac{C_\alpha}{n^{|\alpha|}} \|u\|_{H(\Omega)}^2. \]
\[ (3.63) \]
\[ (3.64) \]

As \( \alpha > 0 \), we deduce that
\[ \|D^\alpha \phi^\beta D^\beta (\psi - a_n)\|_{L^2(\Omega)}^2 \to 0 \quad \text{as} \quad n \to \infty. \]
\[ (3.65) \]

In the case \( \beta = 0 \), by Poincaré inequality, it follows
\[ \|D^\alpha \phi^\beta (\psi - a_n)\|_{L^2(\Omega)}^2 = \int_\Omega |D^\alpha \phi^\beta (\psi(x) - a_n)|^2 dx \]
\[ \leq \frac{C_\alpha}{n^{|\alpha|}} \int_{|x| \leq 2n} |\psi(x) - a_n|^2 dx \]
\[ \leq \frac{C_\alpha}{n^{|\alpha|}} \int_{1 \leq |y| \leq 2} |\psi(y)|^2 dy \]
\[ \leq \frac{C_\alpha}{n^{|\alpha|}} \int_{1 \leq |y| \leq 2} |u(y)|^2 dy \]
\[ \leq \frac{C_\alpha}{n^{|\alpha|}} \int_{n \leq |x| \leq 2n} |u(x)|^2 dx. \]
\[ (3.66) \]

Since \( u \in H^s(\Omega) \) and \( |\alpha| \geq 1 \), the above inequality implies that
\[ \|D^\alpha \phi^\beta (\psi - a_n)\|_{L^2(\Omega)}^2 \to 0 \quad \text{as} \quad n \to \infty. \]
\[ (3.67) \]

By (3.65) and (3.67), it follows that \( u^n \) converges to \( u \) in \( H^s(\Omega) \) strongly. \( \square \)
4. Global existence theorems to Euler-$\alpha$ equations in 2D exterior domain

In this whole section, without loss of generality, we set $\alpha = 1$, and the equations (1.1) and (1.2) with initial data and boundary conditions (1.3) and (1.4) are being rewritten as

\[
\begin{aligned}
\partial_t v + u \cdot \nabla v + (\nabla u)^\ell \cdot v + \nabla p &= 0 \quad \text{in } \Omega \times (0, T), \\
\text{div } u &= 0 \quad \text{in } \Omega \times [0, T], \\
u|_{t=0} &= u_0 \quad \text{in } \Omega, \\
u(x, t) \to 0 &= \forall t \in [0, T), |x| \to \infty,
\end{aligned}
\]

where $v = u - \Delta u$, $u_0$ is the initial filtered velocity, and $T > 0$ is arbitrary. Let us firstly consider the special case that the initial unfiltered vorticity $q_0$ is compactly supported, we have the following proposition.

**Proposition 4.1.** Assume that any $T > 0$ and the initial filtered velocity $u_0 \in H^3(\Omega) \cap V$ with the initial unfiltered vorticity $q_0$ is compactly supported, then the equations (4.1)–(4.5) has a unique weak solution $u \in L^\infty(0, T; H^3(\Omega)) \cap C([0, T]; V)$ in the following sense, for any $\phi \in C^\infty([0, T]; D)$, the identity

\[
\int_0^t (u(t), \phi)_{L^2(\Omega)} - \int_0^t (u \cdot \nabla v + (\nabla u)^\ell \cdot v, \phi)_{L^2(\Omega)} = \int_0^t (v, \phi_t)_{L^2(\Omega)}
\]

holds for $t \in [0, T]$, where $v = u - \Delta u$.

**Proof.** Let $R_0 > 0$ such that $\text{supp } q_0 \subset B(0, R_0)$. Let $T_0 > 0$ be determined later, we will construct a map $\mathcal{F}$ from $C([0, T_0]; V)$ to itself, and then we will show $\mathcal{F}$ is a contraction map for a short time interval such that we can use Banach fixed point theorem to prove the local well-posedness of equations (4.1)–(4.5), finally we will extend the solution to any time interval.

**Step 1** Construct the map $\mathcal{F} : C([0, T_0]; V) \to C(0, T_0]; V)$. The domain of $\mathcal{F}$ is defined as $D(\mathcal{F}) := \{ u \in C([0, T_0]; V \cap H^3(\Omega) \cap C([0, T_0]; V)| \| u(t) \|_{H^3} \leq M, u|_{t=0} = u_0 \}$, where $M > 0$ is a constant only depends on the initial data $u_0$ and will be determined later. Let $u \in D(\mathcal{F})$, we can find that there exists a unique triple $(q, \psi, \tilde{u})$ such that

\[
\begin{aligned}
q &\in C([0, T_0]; L^2(\Omega) \cap L^4(\Omega)) \\
\psi &\in C([0, T_0]; H^1) \\
\tilde{u} &\in C([0, T_0]; H^3(\Omega) \cap V),
\end{aligned}
\]

and satisfies the following equations

\[
\begin{aligned}
\partial_t q + u \cdot \nabla q &= 0 \quad \text{in } \Omega \times [0, T_0], \\
q|_{t=0} &= q_0 \quad \text{in } \Omega, \\
\Delta_x \psi(x, t) &= q(x, t) \quad \text{in } \Omega \times [0, T_0], \\
\psi(x, t) &= 0 \quad \text{on } \Gamma \times [0, T_0], \\
\tilde{u}(x, t) + A\tilde{u}(x, t) &= \nabla^\perp \psi(x, t) \quad \text{in } \Omega \times [0, T_0], \\
\tilde{u}(x, t) &= 0 \quad \text{on } \Gamma \times [0, T_0].
\end{aligned}
\]
Indeed, from lemma 2.1, we know that the transport equations (4.8) and (4.9) admits a unique weak solution \( q \in C([0, T_0]; L^2(\Omega) \cap L^1(\Omega)) \) with the following estimate
\[
\sup_{t \in [0, T_0]} \|q(t)\|_{L^2(\Omega)} + \|q(t)\|_{L^1(\Omega)} \leq (\|q_0\|_{L^2(\Omega)} + \|q_0\|_{L^1(\Omega)}).
\]
Moreover, concerning remark 2.1, we observe that the diameter of the support of \( q(t) \) obeys
\[
\delta(\mathrm{supp} \ q(t)) < R_0 + \int_0^t \|u(\cdot, s)\|_{H^1(\Omega)} ds,
\]
for \( t \in [0, T_0] \). In the light of lemma 3.1, we know that the Poisson equations (4.10) and (4.11) has a unique solution \( \psi \in C([0, T_0]; \hat{H}) \) such that \( \nabla \psi \) belongs to \( C([0, T_0]; H^1(\Omega)) \) and satisfies the following property
\[
\|\nabla \psi(t)\|_{H^1(\Omega)} \leq C\delta(\mathrm{supp} \ q(t))(\|q(t)\|_{L^2(\Omega)} + \|q(t)\|_{L^1(\Omega)}), \quad \forall t \in [0, T_0].
\]
Furthermore, by lemma 2.3, we observe that the Stokes equations (4.12) and (4.13) has a unique solution \( \tilde{u} \in C([0, T_0]; H^1(\Omega) \cap V) \) such that for all \( t \in [0, T_0] \), the following estimate holds
\[
\|\tilde{u}(t)\|_{H^1(\Omega)} \leq C\|\nabla \psi(t)\|_{H^1(\Omega)}.
\]
The contraction map is then defined by \( \mathcal{F}[u] := \tilde{u} \).

**Step 2** We shall now determine the parameters \( T_0 \) and \( M \) to ensure that \( \tilde{u} \in D(\mathcal{F}) \). Collecting (4.15)–(4.17), it follows that:
\[
\|\tilde{u}(t)\|_{H^1(\Omega)} \leq C\left(R_0 + \int_0^t \|u(\cdot, s)\|_{H^1(\Omega)} ds\right)(\|q_0\|_{L^2(\Omega)} + \|q_0\|_{L^1(\Omega)})
\]
for \( t \in [0, T_0] \). Setting \( M := \max\left(2CR_0(\|q_0\|_{L^2(\Omega)} + \|q_0\|_{L^1(\Omega)}), \|u_0\|_{H^1(\Omega)}\right) \) and \( T_0 := \frac{R_0}{M} \), observing that \( u \in D(\mathcal{F}) \), we have
\[
\|\tilde{u}(t)\|_{H^1(\Omega)} \leq 2CR_0(\|q_0\|_{L^2(\Omega)} + \|q_0\|_{L^1(\Omega)})
\]
\[
\leq M.
\]
To ensure that \( \tilde{u} \in D(\mathcal{F}) \), it remains to check that \( \tilde{u}_{|t=0} = u_0 \). From equations (4.8)–(4.13), we deduce that
\[
\nabla \cdot (\tilde{u} + A\tilde{u})|_{t=0} = \Delta \psi|_{t=0} = q_0 = \nabla \cdot (u_0 + Au_0).
\]
Since both \( u \) and \( \tilde{u} \) are divergence-free vectors, we at once obtain
\[
\nabla \cdot (\tilde{u} + A\tilde{u})|_{t=0} = \nabla \cdot (u_0 + Au_0) = 0.
\]
Collecting the above two equalities, in view of lemma 2.2, we thus have
\[
(\tilde{u}_{|t=0} - u_0) + A(\tilde{u}_{|t=0} - u_0) = 0,
\]
which, in turn, secures \( \tilde{u}_{|t=0} = u_0 \) by lemma 2.3.

**Step 3** We have to prove that the map \( \mathcal{F} \) is a contraction. We set
\[
Q(t) := \tilde{v}(t) - \tilde{v}(0) + \int_0^t [u(s) \cdot \nabla \tilde{v}(s) - (\nabla \tilde{v})(s)u(s)] ds \quad \forall t \in [0, T_0],
\]
and observe that
\[
\frac{d}{dt} \|Q(t)\|^2_{L^2(\Omega)} \leq C\|Q(t)\|_{L^2(\Omega)}.
\]
Integrating on \( [0, T_0] \) and using the identity (4.10), we have
\[
\int_0^{T_0} \frac{d}{dt} \|Q(t)\|^2_{L^2(\Omega)} dt \leq C\int_0^{T_0} \|Q(t)\|_{L^2(\Omega)} dt.
\]
Using the Gronwall’s inequality, we conclude that
\[
\|Q(t)\|^2_{L^2(\Omega)} \leq \left(\int_0^{T_0} \|Q(s)\|^2_{L^2(\Omega)} ds\right)e^{CT_0} \leq C\int_0^{T_0} \|Q(s)\|^2_{L^2(\Omega)} ds.
\]
Hence, we have
\[
\|Q(t)\|_{L^2(\Omega)} \leq C\|Q(0)\|_{L^2(\Omega)} e^{CT_0} \leq CM,
\]
where $\tilde{\mathbf{v}} = \tilde{\mathbf{u}} - \Delta \tilde{\mathbf{u}}$. Observe that $\mathbf{u}, \tilde{\mathbf{u}} \in C([0, T_0]; H^3(\Omega) \cap V)$, it follows $Q(t) \in C([0, T_0]; (L^2(\Omega))^2)$. Taking account of (4.8), we also know that

$$\nabla_t \cdot Q(x, t) = \int_0^t [\partial_t q(x, s) + \mathbf{u} \cdot \nabla q(x, s)] ds = 0. \quad (4.24)$$

From lemma 2.2, we deduce that

$$\mathbb{P}Q(t) = 0, \quad (4.25)$$

for $t \in [0, T_0]$. Here $\mathbb{P}$ is the Helmholtz projection in exterior domain (see [7]). Differentiate equation (4.25) with respect to $t$, we obtain:

$$\partial_t (\tilde{\mathbf{u}} + A\tilde{\mathbf{u}}) = \mathbb{P}(-\mathbf{u} \cdot \nabla \tilde{\mathbf{v}} + (\nabla \tilde{\mathbf{v}})^t \mathbf{u}), \quad (4.26)$$

where $A \equiv -\mathbb{P} \Delta$ is the Stokes operator in exterior domain (see [2]). From equation (4.26), we see that $\mathbf{u} + A\mathbf{u}$ belongs to $C^1([0, T_0]; (L^2(\Omega))^2)$. By lemma 2.3, it follows that $(\mathbf{u}, \tilde{\mathbf{u}})$ satisfies the following equations

$$\begin{aligned}
\partial_t \tilde{\mathbf{v}} + \mathbf{u} \cdot \nabla \tilde{\mathbf{v}} - (\nabla \tilde{\mathbf{v}})^t \mathbf{u} + \nabla \tilde{\mathbf{v}} &= 0 \quad &\text{in } \Omega \times (0, T_0), \\
\text{div } \tilde{\mathbf{u}} &= 0 \quad &\text{in } \Omega \times [0, T_0], \\
\tilde{\mathbf{u}}|_{t=0} &= \mathbf{u}_0 \quad &\text{in } \Omega, \\
\tilde{\mathbf{u}}(x, t) &\rightarrow 0 \quad &\forall t \in [0, T_0], |x| \rightarrow \infty. 
\end{aligned} \quad (4.27-4.30)$$

We now focus on the energy estimates from these equations to show $\mathcal{F}$ is a contraction. Assume that $\mathbf{u}_1^1, \mathbf{u}_2^1 \in C([0, T_0]; V \cap H^3(\Omega))$, we set

$$\begin{aligned}
\tilde{\mathbf{u}}_1^1 &= \mathcal{F}[\mathbf{u}_1^1], \\
\tilde{\mathbf{u}}_2^1 &= \mathcal{F}[\mathbf{u}_2^1], \\
\tilde{\mathbf{v}}_1^1 &= \tilde{\mathbf{v}}_1 - \Delta \tilde{\mathbf{u}}_1^1, \\
\tilde{\mathbf{v}}_2^1 &= \tilde{\mathbf{u}}_1^1 - \Delta \tilde{\mathbf{u}}_2^1, \\
\mathbf{W} &= \mathbf{u}_1^1 - \mathbf{u}_2^1, \\
\mathbf{S} &= \tilde{\mathbf{u}}_1^1 - \tilde{\mathbf{u}}_2^1,
\end{aligned} \quad (4.32)$$

then $\tilde{\mathbf{v}}_1^1 - \tilde{\mathbf{v}}_2^1 = \mathbf{S} - \Delta \mathbf{S}$. Subtracting the equation (4.26) for $\tilde{\mathbf{u}}_2^1$ from the one for $\tilde{\mathbf{u}}_1^1$, it follows

$$\partial_t (\mathbf{S} + A\mathbf{S}) + \mathbb{P} \left( \mathbf{u}_1^1 \cdot \nabla \tilde{\mathbf{v}}_1^1 - \mathbf{u}_2^1 \cdot \nabla \tilde{\mathbf{v}}_2^1 + \sum_{j=1}^{2} \mathbf{u}_j^1 \cdot \nabla \tilde{\mathbf{v}}_j^1 - \sum_{j=1}^{2} \mathbf{u}_j^1 \cdot \nabla \tilde{\mathbf{v}}_j^2 \right) = 0. \quad (4.33)$$

Multiplying the above equation by $\mathbf{S}$ and integrating over $\Omega \times [0, t]$ for any $t \in (0, T_0]$, we obtain, after integrating by parts,
it follows then taking account of (4.19) and using Hölder inequality and Gagliardo–Nirenberg inequality, it follows

\[ \frac{1}{2} \left( \|S(t)\|_{L^2}^2 + \| \nabla S(t) \|_{L^2}^2 \right) = \int_0^t \int_{\Omega} S \cdot [(W \cdot \nabla)\tilde{u}^1 + (u^2 \cdot \nabla)(S - \Delta S)] \, dx \, ds \]

\[ + \int_0^t \int_{\Omega} S \cdot \left[ \sum_{j=1}^2 u_j \nabla(S_j - \Delta S_j) + \sum_{j=1}^2 W_j \nabla \tilde{u}^j_1 \right] \, dx \, ds \]

\[ =: K_1 + K_2, \quad (4.34) \]

where we have used the property that \( \tilde{u}^2 |_{t=0} = \tilde{u}_1^0 = u_0 \). As usual, noting that \( (S, u^2 \cdot \nabla S) = 0 \) and integrating by parts, we deduce:

\[ K_1 = \int_0^t \int_{\Omega} S \cdot [(W \cdot \nabla)\tilde{u}^1 + (u^2 \cdot \nabla)(S - \Delta S)] \, dx \, ds \]

\[ = \int_0^t \int_{\Omega} S \cdot [(W \cdot \nabla)\tilde{u}^1] \, dx \, ds - \int_0^t \int_{\Omega} S \cdot [(u^2 \cdot \nabla)(\Delta S)] \, dx \, ds \]

\[ = \int_0^t \int_{\Omega} S \cdot [(W \cdot \nabla)\tilde{u}^1] \, dx \, ds + \int_0^t \int_{\Omega} \Delta S \cdot [(u^2 \cdot \nabla)S] \, dx \, ds \]

\[ = \int_0^t \int_{\Omega} S \cdot [(W \cdot \nabla)\tilde{u}^1] \, dx \, ds - \int_0^t \int_{\Omega} \sum_{k,i} [\partial_i u_k^2 \partial_k S + u_k^2 \partial_k \partial_k S] \partial_i S \, dx \, ds \]

\[ = \int_0^t \int_{\Omega} S \cdot [(W \cdot \nabla)\tilde{u}^1] \, dx \, ds - \int_0^t \int_{\Omega} \sum_{k,i} [\partial_i u_k^2 \partial_k S] \partial_i S \, dx \, ds, \quad (4.35) \]

then taking account of (4.19) and using Hölder inequality and Gagliardo–Nirenberg inequality, it follows

\[ |K_1| \leq \int_0^t \left[ \|S\|_{L^2} \|W\|_{L^2} \|\nabla \tilde{u}^1\|_{L^2} + \|\nabla u^2\|_{L^\infty} \|\nabla S\|_{L^2}^2 \right] \, ds \]

\[ \leq CM \int_0^t \left[ \|S\|_{W^1(\Omega)}^2 + \|W\|_{W^1(\Omega)}^2 \right] \, ds. \quad (4.36) \]

We now estimate the second term \( K_2 \), which is similar to \( K_1 \). Indeed, by integration by parts, \( K_2 \) can be rewritten as:

\[ K_2 = \int_0^t \int_{\Omega} S \cdot [u_j \nabla(S_j - \Delta S_j) + W_j \nabla \tilde{u}^j_1] \, dx \, ds \]

\[ = \int_0^t \int_{\Omega} \left[ Su_j \partial_i S_j - Su_j \partial_i (\Delta S_j) + S_i W_j \partial_i \tilde{u}^j_1 \right] \, dx \, ds \]

\[ = \int_0^t \int_{\Omega} \left[ Su_j \partial_i S_j + S_j \partial_j u_j \Delta S_j + S_i W_j \partial^j_1 \right] \, dx \, ds \]

\[ = \int_0^t \int_{\Omega} \left[ Su_j \partial_i S_j - \partial_i S_j \partial_j S_j - \partial_i u_j \partial_j S_j + \partial_i \tilde{u}^j_1 S_j W_j \right] \, dx \, ds, \quad (4.37) \]
again using inequality (4.19), Hölder inequality and Gagliardo–Nirenberg inequality, it is easy to deduce that

\[ |K_2| \leq \int_0^t \left[ \|S\|_{L^2} \|u^1\|_{L^\infty} \|\nabla S\|_{L^2} + \|\nabla^2 u^1\|_{L^4} \|S\|_{L^4} \|\nabla S\|_{L^2} \right] ds \\
+ \int_0^t \left[ \|\nabla u^1\|_{L^\infty} \|\nabla S\|_{L^2}^2 + \|S\|_{L^4} \|W\|_{L^4} \|\nabla u^1\|_{L^2} \right] ds \\
\leq CM \int_0^t \left( \|S\|^2_{H^1(\Omega)} + \|W\|^2_{H^1(\Omega)} \right) ds. \tag{4.38} \]

In view of (4.34), (4.36) and (4.38), it shows that for all \( t \in [0, T_0] \)

\[ \frac{1}{2} \|S(t)\|^2_{H^1(\Omega)} \leq CM \int_0^t \left( \|S\|^2_{H^1(\Omega)} + \|W\|^2_{H^1(\Omega)} \right) ds \tag{4.39} \]

where \( M \) is defined in step 2 and \( C \) only depends on \( \Omega \). Thanks to Grönwall’s inequality, we have

\[ \|S\|^2_{H^1(\Omega)}(t) \leq CM \int_0^t \|W\|^2_{H^1(\Omega)}(s) \exp\{CM(t-s)\} ds. \tag{4.40} \]

For arbitrary \( h \in (0, T_0] \), the above inequality implies

\[ \sup_{t \in [0,h]} \|S\|^2_{H^1(\Omega)}(t) \leq (e^{CM} - 1) \sup_{t \in [0,h]} \|W\|^2_{H^1(\Omega)}(t). \tag{4.41} \]

Consequently, if choose \( h \in (0, T_0] \) small enough, so that \( e^{CM} - 1 < 1 \), then we have that \( F \) is a contraction mapping with respect to the \( H^1(\Omega) \) norm, for short time interval \([0, h]\).

**Step 4** Local existence. By Banach fixed point theorem, we could conclude there exists a unique fixed point \( u \in C([0,h]; V) \). This fixed point is also the limit of the fixed point iteration, with \( u^0 := u_0 \) and \( u^n := F[u^{n-1}] \). From (4.19), we know that \( u^n \) is uniformly bounded in \( H^3(\Omega) \), thanks to Banach–Alaoglu theorem we have that there exists a subsequence \( u^{n_k} \) converges, weak-star to \( u \) in \( L^\infty([0,h]; H^3(\Omega)) \). It is easy to see that \( u \in C([0,T]; H^3(\Omega)) \). Indeed, since \( F[u] \in C([0,T]; H^3(\Omega) \cap V) \) and \( u \) is the fixed point, we have \( u := F[u] \in C([0,T]; H^3(\Omega) \cap V) \).

**Step 5** Extending the solution to any time interval. Since \( u \) is the limit of \( u^{n_k} \), it follows from (4.18) that

\[ \|u(\cdot,t)\|_{H^1} \leq C \left( R_0 + \int_0^t \|u(\cdot,s)\|_{H^2(\Omega)} ds \right) (\|q_0\|_{L^1} + \|q_0\|_{L^2}) \\
\leq C \left( R_0 + \int_0^t \|u(\cdot,s)\|_{H^1(\Omega)} ds \right) \tag{4.42} \]

for \( t \in [0,h] \) and \( C \) is independent of time \( t \). Using Grönwall’s inequality, we obtain:

\[ \|u\|_{L^\infty([0,h];H^1(\Omega))} \leq CR_0 \exp(Ch), \tag{4.43} \]

which implies the solution can be extended to any time interval. \( \square \)

**Remark 4.1.** By combining (2.5) and the arguments in the proof of proposition 4.1, one is able to obtain the following estimate provided that \( u_0 \in H^1(\Omega) \) for \( s > 3 \),
\[ \|u\|_{L^\infty([0,T];H^s(\Omega))} \leq C\|u_0\|_{H^s(\Omega)} \quad (4.44) \]

where \( C \) depends on \( T \) and \( H^s(\Omega) \)-norm of \( u_0 \).

With the above technical lemmas and proposition, we finally can obtain the main result in this paper.

**Main theorem.** Assume that the initial filtered velocity \( u_0 \in H^s(\Omega) \cap V \) (\( s \geq 3 \)), then for arbitrary \( T > 0 \), the equations \((4.1)-(4.5)\) admit a unique weak solution \( u \in L^\infty(0,T;H^s(\Omega)) \cap C([0,T];V) \) in the following sense, for any \( \phi \in C^\infty([0,T];T) \), the identity

\[
(u(t), \phi)_2^2 + (\nabla u(t), \nabla \phi)_2^2 - (u_0, \phi)_2^2 - (\nabla u_0, \nabla \phi)_2^2 = \int_0^t (v \cdot \nabla v + (\nabla u)^t \cdot v, \phi)_2^2 \quad (4.45)
\]

holds for \( t \in [0,T] \), where \( v = u - \Delta u \). Moreover,

\[
\|u(t)\|_{H^s(\Omega)} \leq C\|u_0\|_{H^s(\Omega)} \quad (4.46)
\]

where \( C \) depends on \( \Omega, T \) and \( \|u_0\|_{H^s(\Omega)} \) for \( s > 3 \), while depends only on \( \Omega \) for \( s = 3 \).

**Proof.** Let \( T > 0 \) be arbitrary. Assume that \( u_0 \in V \cap H^s(\Omega) \), \( s \geq 3 \) and choose \( \{u^n_0\} \) be the approximate sequence constructed in lemma 3.3. For any fixed \( n \in \mathbb{N} \), we know from proposition 4.1 that there exists a unique solution \( u^n \) to equations \((4.1)-(4.5)\) with initial value \( u^n_0 \).

It is crucial to prove that the limit of \( u^n \) in the suitable space is the solution \( u \) of the formula \((4.6)\) with the initial data \( u_0 \). Fortunately, we observe that it is able to obtain the uniform bound for the lower order partial derivatives of \( u^n \) from equation \((4.1)\) and the one for the high order derivatives from equations \((4.8)-(4.13)\) respectively.

Beginning to the estimates of low order derivatives, it is easy to obtain this bound from energy estimates for the equation \((4.1)\) as follows,

\[
\frac{1}{2} \frac{d}{dt} \left( \|u^n\|_{L^2(\Omega)}^2 + \|\nabla u^n\|_{L^2(\Omega)}^2 \right) = \int_\Omega \nabla u^n \cdot \nabla u^n - \Delta u^n u^n = \int_\Omega \nabla u^n \cdot \nabla u^n - \|u^n\|_{L^2(\Omega)}^2 + (u^n - \Delta u^n) u^n + \Delta u^n u^n
\]

\[
\int_\Omega \frac{d}{dt} \left( \frac{1}{2} u^n \right)^2 - \int_\Omega u^n \Delta u^n u^n
\]

\[
\int_\Omega (u^n) \partial_i u^n u^n - \int_\Omega \Delta u^n \partial_i u^n u^n
\]

\[
= - \int_\Omega u^n \Delta u^n u^n + \Delta u^n \partial_i u^n u^n
\]

\[
= 0, \quad (4.47)
\]

where we used the condition \( \nabla \cdot u^n = 0 \). It follows that

\[
\sup_{t \in [0,T)} \|u^n(t)\|_{H^s(\Omega)} \leq C\|u^n_0\|_{H^s(\Omega)} \leq C\|u_0\|_{H^s(\Omega)}, \quad (4.48)
\]

It is subtle to obtain the uniform estimates of the high order derivatives. As in the proof in proposition 4.1, the solution \( u^n \) satisfies \( \mathcal{F}[u^n] = u^n \), by \((4.8)-(4.13)\) we know that there exists
the triple \((u^n, q^n, \psi^n)\) obeys the following system

\[
\begin{aligned}
\partial_t q^n + u^n \cdot \nabla q^n &= 0 \quad \text{in } \Omega \times [0, T], \quad (4.49) \\
q^n|_{t=0} &= q_0^n \quad \text{in } \Omega, \quad (4.50) \\
\Delta \psi^n(x, t) &= q^n(x, t) \quad \text{in } \Omega \times [0, T], \quad (4.51) \\
\psi^n(x, t) &= 0 \quad \text{on } \Gamma \times [0, T], \quad (4.52) \\
u^n(x, t) - \Delta u^n(x, t) + \nabla p^n &= \nabla \cdot \psi^n(x, t) \quad \text{in } \Omega \times [0, T], \quad (4.53) \\
u^n(x, t) &= 0 \quad \text{on } \Gamma \times [0, T]. \quad (4.54)
\end{aligned}
\]

Since \((u^n, \nabla \cdot \psi^n)\) satisfies the stationary Stokes equations, by lemma 2.3, we obtain

\[
\|D^{k+3}u^n(\cdot, t)\|_{L^2(\Omega)} \leq C(\|D^k\psi^n(\cdot, t)\|_{H^{k+1}(\Omega)} + \|u^n(\cdot, t)\|_{L^2(\Omega)}) \quad (4.55)
\]

for \(t \in [0, T]\) and \(k \geq 0\). Collecting the inequality (3.9) and the above inequality, it follows

\[
\|D^{k+3}u^n(\cdot, t)\|_{L^2(\Omega)} \leq C(\|q^n(\cdot, t)\|_{H^k(\Omega)} + \|\nabla \psi^n\|_{L^2(\Omega)} + \|u^n(\cdot, t)\|_{L^2(\Omega)}).
\]

(4.56)

Noting that \(\|\nabla \cdot \psi^n\|_{L^2(\Omega)} \leq \|u^n\|_{H^1(\Omega)}\), by Sobolev interpolation inequality, we then arrive at

\[
\|D^{k+3}u^n(\cdot, t)\|_{L^2(\Omega)} \leq C(\|q^n(\cdot, t)\|_{H^k(\Omega)} + \|u^n(\cdot, t)\|_{L^2(\Omega)}).
\]

(4.57)

for \(t \in [0, T]\) and \(k \geq 0\). Let us consider the special case that \(k = 0\), by the above inequality, (2.3) and (4.48), we can see that

\[
\|u^n\|_{L^\infty(0, T); H^k(\Omega)} \leq C(\|q_0^n\|_{L^2(\Omega)} + \|u_0^n\|_{L^2(\Omega)}) \leq C\|u_0^n\|_{H^1(\Omega)} \leq C\|u_0\|_{H^1(\Omega)}.
\]

(4.58)

Taking into account the above two inequalities and (2.5), we finally deduce by induction that

\[
\|u^n\|_{L^\infty(0, T); H^k(\Omega)} \leq C\|u_0\|_{H^1(\Omega)},
\]

(4.59)

where \(C\) depends on \(T\) and \(H^1(\Omega)\)-norm of \(u_0\) for \(s \geq 4\). By virtue of Banach–Alaoglu theorem, we find that there exists a subsequence \(u^{n_k}\) converges weak-star to some \(u \in L^\infty([0, T]; H^1(\Omega))\).

Furthermore, we assume that \(u^n\) converges to \(u\) in \(C([0, T]; H^1(\Omega))\) strongly. Firstly, subtracting the equation (4.1) for \(u^{n_k}\) from the one for \(u^n\), respectively, then multiplying by \((u^n - u^{n_k})\) and integrating over \(\Omega\), we have that

\[
\frac{1}{2} \frac{d}{dt} \|u^n - u^{n_k}\|_{H^1(\Omega)}^2 = \int_\Omega [(u^n - u^{n_k}) \cdot \nabla v^n] \cdot (u^n - u^{n_k})
\]

\[
+ \int_\Omega [u^{n_k} \cdot \nabla (v^n - v^{n_k})] \cdot (u^n - u^{n_k})
\]

\[
+ \int_\Omega [(\nabla u^n - \nabla u^{n_k}) \cdot v^n] \cdot (u^n - u^{n_k})
\]

\[
+ \int_\Omega [(\nabla u^{n_k}) \cdot (v^n - v^{n_k})] \cdot (u^n - u^{n_k})
\]

\[
=: M_1 + M_2 + M_3 + M_4.
\]

(4.60)
It is easy to observe that \( M_1 + M_3 = 0 \), clearly, integrating by parts, we find

\[
M_3 = \int_\Omega \left[ (\nabla u^n - \nabla u^m) \cdot (u^n - u^m) \right]
= \int_\Omega \delta_j(u^n_j - u^m_j)u^n_j(u^n_j - u^m_j)
= -M_1. \tag{4.61}
\]

Using integration by parts, we obtain more favourable formula for \( M_2 \)

\[
M_2 = \int_\Omega \left[ u^m \cdot (\nabla (v^n - v^m)) \cdot (u^n - u^m) \right]
= -\int_\Omega \left[ u^m \cdot \nabla (u^n - u^m) \right] \cdot (v^n - v^m)
= -\int_\Omega \left[ u^m \cdot \nabla (u^n - u^m) \right] \cdot (u^n - u^m) + \int_\Omega \left[ u^m \cdot \nabla (u^n - u^m) \right] \cdot (\Delta u^n - \Delta u^m)
= -\int_\Omega \left[ u^m \cdot \nabla (u^n - u^m) \right] \cdot (u^n - u^m) - \int_\Omega \left[ \partial_j u^n_j \partial_j (u^n_j - u^m_j) \right] \cdot \partial_i (u^n_i - u^m_i), \tag{4.62}
\]

with the aid of Hölder inequality and Sobolev embedding theorem, we arrive at

\[
M_2 \leq \| u^m \|_{L^\infty} \| u^n - u^m \|^2_{H^1(\Omega)} + \| \nabla u^m \|_{L^\infty} \| u^n - u^m \|^2_{H^1(\Omega)}
\leq C \| u^m \|_{H^1(\Omega)} \| u^n - u^m \|^2_{H^1(\Omega)}. \tag{4.63}
\]

It remains to check the estimate of the term \( M_4 \). Similarly, using integration by parts, \( M_4 \) can be rewritten as follows

\[
M_4 = \int_\Omega \left[ (\nabla u^n)^j \cdot (v^n - v^m) \right] \cdot (u^n - u^m)
= \int_\Omega \delta_j u^n_j (v^n_j - v^m_j)(u^n_j - u^m_j)
= \int_\Omega \delta_j u^n_j (u^n_j - u^m_j)(u^n_j - u^m_j) - \int_\Omega \delta_j u^n_j (\Delta u^n_j - \Delta u^m_j)(u^n_j - u^m_j)
= \int_\Omega \delta_j u^n_j (u^n_j - u^m_j)(u^n_j - u^m_j) + \int_\Omega \delta_i \delta_j u^n_i \delta_j (u^n_j - u^m_j)(u^n_j - u^m_j)
+ \int_\Omega \delta_i \delta_j (u^n_j - u^m_j)(u^n_i - u^m_i), \tag{4.64}
\]

then applying Hölder inequality and Gagliardo–Nirenberg inequality, we deduce, in particular

\[
M_4 \leq \| \nabla u^n \|_{L^\infty} \| u^n - u^m \|^2_{L^2} + \| D^2 u^n \|_{L^4} \| \nabla u^n - \nabla u^m \|_{L^2} \| u^n - u^m \|_{L^4}
+ \| \nabla u^m \|_{L^\infty} \| \nabla u^n - \nabla u^m \|^2_{L^2}
\leq C \| u^n \|_{H^1(\Omega)} \| u^n - u^m \|^2_{H^1(\Omega)}. \tag{4.65}
\]
The previous estimates about $M_1, M_2, M_3$ and $M_4$ then imply that
\[
\frac{1}{2} \frac{d}{dt} \|u^n - u^m\|_{H^1(\Omega)}^2 \leq C \|u^n - u^m\|_{H^1(\Omega)},
\] (4.66)
thanks to Grönwall’s inequality, we obtain
\[
\sup_{t \in [0, T]} \|u^n - u^m\|_{H^1(\Omega)}(t) \leq e^{CT} \|u_0^n - u_0^m\|_{H^1(\Omega)},
\] (4.67)

Consequently, we can conclude that $u^n$ converges strongly in $C([0, T]; H^1(\Omega))$ to $u$ with $u|_{t=0} = u_0$.

Since $u^n$ converges weak-star (respectively, strongly) to $u$ in $L^\infty([0, T]; H^s(\Omega))$ (respectively, $C([0, T]; H^s(\Omega))$), and then it is easy to check that $u$ is a solenoidal vector, we thus know that $u$ satisfies the equations (4.1)–(4.5) in the sense of (4.45). □

5. Discussions and comments

As the solution to Euler-α equations in main theorem satisfies the formula (4.45), we find it is not necessary to require the regularity of initial data in $H^3(\Omega) \cap V$. It is a natural problem that one would like to look for the weak solution satisfying the formula (4.45) provided that the initial data are less regularity, for instance, $u_0$ belongs to $H^s(\Omega) \cap V$ for some positive number $s < 3$. This is subtle to find such solution, because it is crucial not to obtain the estimates for the unfiltered vorticity $q$ when initial data are less regular.

Further more, it is an interesting problem to hunt for the solutions to 3D Euler-α equations in exterior domain, which exist in some uniform time interval with respect to $\alpha$. It is very difficult to do this, since in 3D case, the equations for the unfiltered vorticity $q$ are the transport equations given by
\[
\partial_t q + u \cdot \nabla q = q \cdot \nabla u,
\]
which is different from the formula (4.8), and that one has no idea to get the desired uniform estimates for high order derivatives. However, we can expect to obtain the limit problems of the 3D Euler-α equations past an obstacle as both $\alpha$ and the diameter of the obstacle go to zero.

Acknowledgments

The work of Aibin Zang was supported in part by the National Natural Science Foundation of China (Grant Nos. 11771382, 12061080). This work of Yin Li is partially supported by the Natural Science Foundation of Guangdong Province (Nos. 2019A1515011320, 2021A1515010292, 2022A1515011358), the project of ordinary universities of Guangdong Province (No. 2020KTSCX134,2020KCXTD024).

ORCID IDs

Xiaoguang You © https://orcid.org/0000-0002-0834-9712
Aibin Zang © https://orcid.org/0000-0002-5115-8372
Yin Li © https://orcid.org/0000-0003-2751-6958
References

[1] Berselli L C, Iliescu T and Layton W J 2006 Mathematics of Large Eddy Simulation of Turbulent Flows (Berlin: Springer)
[2] Borchers W and Varnhorn W 1993 On the boundedness of the Stokes semigroup in two-dimensional exterior domains Math. Z. 213 275–99
[3] Busuioc A V, Iftimie D, Lopes Filho M C and Nussenzveig Lopes H J 2020 The limit $\alpha \to 0$ of the $\alpha$-Euler equations in the half-plane with no-slip boundary conditions and vortex sheet initial data SIAM J. Math. Anal. 52 5257–86
[4] Busuioc V 1999 On second grade fluids with vanishing viscosity C. R. Acad. Sci., Paris I 328 1241–6
[5] DiPerna R J and Lions P L 1989 Ordinary differential equations, transport theory and Sobolev spaces Invent. Math. 98 511–47
[6] Galdi G 2011 An Introduction to the Mathematical Theory of the Navier-Stokes Equations: Steady-State Problems (New York: Springer) pp 128–30
[7] Hieber M, Kozono H, Seyfert A, Shimizu S and Yanagisawa T 2021 The Helmholtz–Weyl decomposition of $L^p$ vector fields for two dimensional exterior domains J. Geom. Anal. 31 5146–65
[8] Holm D D, Marsden J E and Ratiu T S 1998 The Euler–Poincaré equations and semidirect products with applications to continuum theories Adv. Math. 137 1–81
[9] Holm D D, Marsden J E and Ratiu T S 1998 Euler–Poincaré models of ideal fluids with nonlinear dispersion Phys. Rev. Lett. 80 4173–6
[10] Iftimie D, Lopes Filho M C and Nussenzveig Lopes H J 2003 Two dimensional incompressible ideal flow around a small obstacle Commun. PDE 28 349–79
[11] Iftimie D, Lopes Filho M C and Nussenzveig Lopes H J 2009 Incompressible flow around a small obstacle and the vanishing viscosity limit Commun. Math. Phys. 287 99–115
[12] Kouranbaeva S and Oliver M 2000 Global well-posedness for the averaged Euler equations in two dimensions Physica D 138 197–209
[13] Leonard A 1980 Vortex methods for flow simulation J. Comput. Phys. 37 289–335
[14] Lopes Filho M C, Nussenzveig Lopes H J, Titi E S and Zang A 2015 Convergence of the 2D Euler-$\alpha$ to Euler equations in the Dirichlet case: indifference to boundary layers Physica D 292–293 51–61
[15] Majda A J, Bertozzi A L and Ogawa A 2002 Vorticity and incompressible flow. Cambridge texts in applied mathematics Appl. Mech. Rev. 55 98–9
[16] Markowitz H and Coleman B D 1964 Incompressible second-order fluids Adv. Appl. Mech. 8 69–101
[17] Samrowski T and Varnhorn W 2004 The Poisson equation in homogeneous Sobolev spaces Int. J. Math. Math. Sci. 2004 1909–21
[18] Shkoller S 2000 Analysis on groups of diffeomorphisms of manifolds with boundary and the averaged motion of a fluid J. Differ. Geom. 55 145–91
[19] Truesdell C and Noll W 2004 The Non-Linear Field Theories of Mechanics (Berlin: Springer) pp 1–579
[20] Zang A 2018 Uniform time of existence of the smooth solution for 3D Euler-$\alpha$ equations with periodic boundary conditions Math. Models Methods Appl. Sci. 28 1881–97