Optical Stochastic Cooling in Tevatron

Valeri Lebedev
Fermilab

Accelerator Physics seminar
Fermilab
June 1, 2010
Objectives

- Extension of Tevatron operation to 2014
- Are there luminosity upgrades?
- Can the Optical Stochastic Cooling (OSC) help?

Outline

- Tevatron luminosity and its evolution
- Requirements to the cooling
- Optical stochastic cooling principles
- Damping rates computation and optimization
- Optimization and efficiency of laser kick
- Requirements to the laser power
- Conclusions
**Tevatron Luminosity**

- All planned luminosity upgrades are completed in the spring of 2009
- From Run II start to 2009 the luminosity integral was doubling every 17 months
- Since 2009 average luminosity stays the same ~51 pb⁻¹/week
- The average luminosity is limited by the IBS
  - Larger beam brightness results in faster luminosity decay
- It is impossible to make significant (~2 times) average luminosity increase with one exception - The beam cooling in Tevatron
  - 10-20% is still possible (new tunes, larger intensity beams)

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Luminosity Evolution for Present Stores (Store 6950)

- About 10% of luminosity integral is lost due to beam-beam
- IBS is the main mechanism causing fast luminosity decrease
  - Presently, there are no means to reduce IBS in Tevatron
- About 40% of pbars are burned in luminosity
  - It is the second leading reason of luminosity decrease
Luminosity Evolution with Moderate Cooling

- Cooling rate is limited by $\zeta_{BB}$ of 0.02
- 1.58 times increase of luminosity integral
- 63% of pbars are burned in luminosity
- Much smaller luminosity variations
Luminosity Evolution with Aggressive Cooling

- Cooling rate is limited by peak luminosity of $4 \cdot 10^{32}$ and by $\xi_{BB} = 0.03$ for pbars
  - Requires tunes closer to half-integer (0.58 → 0.52)
- 1.96 times increase in average luminosity
  - 78% of pbars are burned in luminosity
Requirements to the Beam Cooling

- Cooling time has to be varied during the store independently for protons and pbar, and for transverse and longitudinal planes.
  - Beam overcooling results in:
    - Particle loss due to beam-beam (transverse overcooling).
    - Longitudinal instability (longitudinal overcooling).

- Simple estimate of required bandwidth based on \( \lambda = 2W/N \) results in ~200 GHz.
  - Well above bandwidth of normal stochastic cooling.
  - Only optical stochastic cooling has sufficient bandwidth.

- Cooling times (in amplitude):
  - Protons: \( L \) - 4.5 hour; \( \perp \) - 8 hour.
  - Antiprotons: \( L \) - 4.5 hour; \( \perp \) - 1.2 hour.

- Tevatron has considerable coupling and all transverse cooling can be applied in one plane.
  - It requires doubling the horizontal cooling decrement:
    - I.e. for protons \( \lambda_s = \lambda_x = 4.5 \) hour.
Optical Stochastic Cooling

- Suggested by Zolotorev, Zholents and Mikhailichenko (1994)
- Never tested experimentally
- OSC obeys the same principles as the microwave stochastic cooling, but exploits the superior bandwidth of optical amplifiers $\sim 10^{14}$ Hz
- Undulator can be used as pickup & kicker
- Pick-up and Kicker should be installed at locations with nonzero dispersion to have both $\perp$ and L cooling.

$\delta E \sim \sin(k \delta z)$

$\delta z$ is particle delay
MIT-Bates Proposal for Tevatron (2007)

OSC and Tevatron Luminosity

How to increase luminosity (peak and integrated) ?

- Peak luminosity increased 62% (180 → 292 µb⁻¹/s)
  \[ 1 \text{ µb}^{-1}/\text{s} = 10^{30} \text{ cm}^2 \text{ s}^{-1} \]
- Weekly integrated luminosity increased ~75% (25 pb⁻¹ → 45 pb⁻¹)
- Monthly integrated luminosity increased ~95% (85 pb⁻¹ → 167 pb⁻¹)
- One hour antiproton stacking record -- $2.3 \times 10^{10}$/hr
- Antiproton accumulation for one week -- $2800 \times 10^{10}$

\[
L = \frac{fN_p N_a}{2\pi (\epsilon_p + \alpha_p) \beta^*} H\left(\frac{\sigma_z}{\beta^*}\right)
\]

- $N$ = bunch intensity, $f$ = collision frequency
- $\epsilon = \text{transverse emittance (size)}, \sigma_z = \text{bunch length}$
- $H = \text{“hour glass” factor} \approx 1, \text{accounts for beam size over finite bunch length}$

OSC provides possible “damping” to the 1 TeV $p$ & $p\bar{p}$ beams.

Damping on:

$\epsilon_p, \epsilon_a, \sigma_z$

Could reduce

$N_p, N_a$ losses

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Bending angle and drift space set to get:
Path delay: $\Delta L = 10\text{mm} = 30\ \text{ps}$
$\Delta x = 55.7\ \text{cm}$
Ease magnet tolerances
MIT-Bates Proposal (continue)

Cooling $p$ and $p$bar Beams: Cooling Separately

- One cooling insertion for both $p$ and $p$bar.
- Special circumstance: two beams in one ring.

Timing two pump lasers to cool $p$ and $p$bar separately. For equal cooling time, cooling rate of each beam will be reduced to half.

Two Pump lasers

$P$ & $P$bar beam orbit (bypass center)

50.4 cm

0.24 nJ/pulse
0.41 mW (Avg.)

250 cm

Optical amplifier offset from bypass center

F. Wang
Fermilab. November 14, 2007

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## Cooling Estimates

|                           | Tevatron          | Bates            |
|---------------------------|-------------------|------------------|
| Gamma                     | 1045 (980 GeV)    | 587 (0.3 GeV)    |
| Bunch length (m)          | 0.57              | 0.025            |
| Particle/bunch            | 2.5E11            | 1.0E8            |
| Bunch number              | 36                | 12               |
| Laser $\lambda$ (\(\mu\)m) | 1.98              | 2.06             |
| Undulator period (m) / length (m) | 2.7/27            | 0.2/2            |
| Undulator parameter K     | 1.1               | 3.5              |
| Undulator radiation/pulse (pJ) | 222               | 0.13             |
| Average radiation power (\(\mu\)W) | 381               | 2.5              |
| Optical power limit (W)   | 20/200            | 5 (Not a limit)  |
| Optical power gain        | 4.84E4 / 5.25E5   | 1750             |
| Laser output/pulse (\(\mu\)J) | 11.6 / 116        | 0.23 nJ          |
| Damping time (hours)\(\times\)2 | 2\(\times\)2 / 0.6\(\times\)2 | 0.14 second      |
Questions to be Answered

- Do we have a fast way (2-3 years) of OSC implementation in Tevatron?
- What is the optimal optics and how to get to it?
- What is the optimal wiggler?
- What is the laser power?
Damping Rates

- The optics design will be significantly simplified if the damping rates can be expressed through beta-functions, dispersions and their derivatives
- The sequence is
  - Express transfer matrices (6x6) through Twiss-parameters at kicker and pickup
  - Find eigen-values and eigen-vectors of the ring without cooling
  - Using perturbation theory find damping decrements
  - Determine the cooling range in amplitudes
    - Correction factors for the finite amplitude particles
Transfer Matrix Parameterization

- 
  Vertical plane is uncoupled and we omit it in further equations

- Matrix from point 1 to point 2

\[ M = \begin{bmatrix} M_{11} & M_{12} & 0 & M_{16} \\ M_{21} & M_{22} & 0 & M_{26} \\ M_{51} & M_{52} & 1 & M_{56} \\ 0 & 0 & 0 & 1 \end{bmatrix} , \quad x = \begin{bmatrix} x \\ \theta_x \\ s \\ \Delta p / p \end{bmatrix} \]

- \( M_{16} \) & \( M_{26} \) can be expressed through dispersion

\[ \begin{bmatrix} M_{11} & M_{12} & M_{16} \\ M_{21} & M_{22} & M_{26} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} D_1 \\ D'_1 \end{bmatrix} = \begin{bmatrix} D_2 \\ D'_2 \end{bmatrix} \Rightarrow \]

- Symplecticity ( \( M^T \mathbf{U} M = \mathbf{U} \) ) binds up \( M_{51}, M_{52} \) and \( M_{16}, M_{26} \) =>

- \( M_{56} \) is related to the partial slip factor, \( \eta_{1 \rightarrow 2} \)

\[ \Rightarrow \text{All matrix elements can be expressed through } \beta, \alpha, D, D', \eta_{1 \rightarrow 2} \]
Transfer Matrix Parameterization (continue)

- Partial momentum compaction and slip factor (from point 1 to point 2) are related to $M_{56}$

$$\Delta s_{1\to2} = 2\pi R \eta_1 \frac{\Delta p}{p} = M_{51} D_1 \frac{\Delta p}{p} + M_{52} D_1' \frac{\Delta p}{p} + M_{56} \frac{\Delta p}{p} + \frac{1}{\gamma^2} \frac{\Delta p}{p}$$

- Further we assume that $v = c$, i.e. $1/\gamma^2 = 0$ and $\eta_1 = -\alpha_{1\to2}$.

- That results in

$$\eta_1 = \frac{M_{51} D_1 + M_{52} D_1' + M_{56}}{2\pi R}$$

- Note that $M_{56}$ sign is positive if a particle with positive $\Delta p$ moves faster than the reference particle.
Damping Rates of Optical Stochastic Cooling

Longitudinal kick

\[
\frac{\delta p}{p} = \kappa \Delta s = \kappa \left( M_{151} x_1 + M_{152} \theta x_1 + M_{156} \frac{\Delta p}{p} \right)
\]

Or in the matrix form: \( \delta x_2 = M_c x_1 \)

\[
M_c = \kappa \begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
M_{151} & M_{152} & 0 & M_{156}
\end{bmatrix}
\]

Find the total ring matrix related to kicker

\[
M_{tot} x_2 = M_1 M_2 x_2 + \delta x_2 = M_1 M_2 x_2 + M_c x_1 = (M_1 M_2 + M_c M_2) x_2
\]

\[
M_{tot} = M + \Delta M_c
\]

where

\[
M = M_1 M_2 , \quad \Delta M = M_c M_2
\]

Perturbation theory yields that the tune shifts are:

\[
\delta Q_k = \frac{1}{4\pi} v_k^+ U M_c U M_1^T U v_k = \kappa \frac{1}{4\pi} v_k^+ \begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
M_{126} & -M_{116} & 0 & M_{156} \end{bmatrix} v_k
\]

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Damping Rates of Optical Stochastic Cooling (continue)

Expressing matrix elements and eigen-vectors through Twiss parameters one obtains the cooling rates

\[
\lambda_1 = -\frac{\kappa}{2} \left[ D_1 D_2 \frac{(1 + \alpha_1 \alpha_2) \sin \mu_1 + (\alpha_2 - \alpha_1) \cos \mu_1}{\sqrt{\beta_1 \beta_2}} - D'_1 D'_2 \sqrt{\frac{\beta_1}{\beta_2}} \left( \cos \mu_1 - \alpha_2 \cos \mu_1 \right) \right.
\]

\[
\left. + D_1 D'_2 \sqrt{\frac{\beta_2}{\beta_1}} \left( \cos \mu_1 + \alpha_1 \sin \mu_1 \right) + D'_1 D'_2 \sqrt{\frac{\beta_1 \beta_2}{\beta_1 \beta_2}} \sin \mu_1 \right]
\]

\[
\lambda_2 = -\frac{\kappa}{2} M_{156} - \lambda_1 = -\pi \kappa R \eta_1
\]

The bottom equation can be directly obtained from the definition of the partial slip factor.

The above equations yield that the sum of the decrements is

\[
\lambda_1 + \lambda_2 = -\frac{\kappa}{2} M_{156}
\]
Sample Lengthening on Pickup-to-Kicker Travel

- Zero length sample lengthens on its way from pickup-to-kicker

\[ \sigma_{\Delta s}^2 = \int \left( M_{151} x + M_{152} \theta_x + M_{156} \tilde{p} \right)^2 f(x, \theta_x, \tilde{p}) dx d\theta_x d\tilde{p}, \quad \tilde{p} = \frac{\Delta p}{p} \]

- Performing integration one obtains for Gaussian distribution

\[ \sigma_{\Delta s}^2 = \sigma_{\Delta s \varepsilon}^2 + \sigma_{\Delta s p}^2 \]

\[ \sigma_{\Delta s \varepsilon}^2 = \varepsilon \left( \beta_p M_{151}^2 - 2 \alpha_p M_{151} M_{152} + \gamma_p M_{152}^2 \right) \]

\[ \sigma_{\Delta s p}^2 = \sigma_p^2 \left( M_{151} D_p + M_{152} D'_p + M_{156} \right)^2 \]

- Both \( \Delta p/p \) and \( \varepsilon \) contribute to the lengthening

- Expressing matrix elements through Twiss parameters and assuming all derivatives (D & \( \beta \)) equal to zero\(^\dagger\) one obtains

\[ \sigma_{\Delta s}^2 = \varepsilon \left( \frac{D_k^2}{\beta_k} + \frac{D_p^2}{\beta_p} - \frac{2 D_k D_p}{\beta_k \beta_p} \cos \mu_1 \right) + \sigma_p^2 \left( M_{156} - \frac{D_k D_p}{\sqrt{\beta_k \beta_p}} \sin \mu_1 \right) \]

\(^\dagger\) See complete expression in backup viewgraphs

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Cooling Range

- The cooling force depends on $\Delta s$ nonlinearly

$$\frac{\delta p}{p} = \frac{\Delta E_{\text{max}}}{E} \sin(k \delta s) = \frac{\Delta E_{\text{max}}}{E} \sin(a_x \sin(\psi_x) + a_p \sin(\psi_p))$$

where $a_x$ & $a_p$ are the lengthening amplitudes due to $\perp$ and $L$ motions measured in units of laser phase ($a = k \delta s$)

- The form-factor for damping rate of longitudinal cooling for particle with amplitudes $a_x$ & $a_p$

$$F_2(a_x, a_p) = \frac{2}{a_p} \int \sin(a_x \sin \psi_x + a_p \sin \psi_p) \sin \psi_p \frac{d\psi_x}{2\pi} \frac{d\psi_p}{2\pi}$$

$$F_2(a_x, a_p) = \frac{2}{a_p} J_0(a_x) J_1(a_p)$$

- Similar for transverse motion

$$F_1(a_x, a_p) = \frac{2}{a_x} J_0(a_p) J_1(a_x)$$

- Damping requires both lengthening amplitudes be smaller $\mu_0 \approx 2.405$


**Cooling of the Gaussian beam**

- Averaging the cooling form-factors for Gaussian distribution can be presented in the following form

  \[ F_{1G}(k\sigma_{\Delta\varepsilon}, k\sigma_{\Delta p}) = \frac{1}{2k^2\sigma_{\Delta\varepsilon}^2} \int_{-\infty}^{\infty} a_x^2 F_1(a_x, a_p) \exp \left( -\frac{a_x^2}{2k^2\sigma_{\Delta\varepsilon}^2} - \frac{a_p^2}{2k^2\sigma_{\Delta p}^2} \right) a_x da_x a_p da_p \]

  - Integration yields

  \[ F_{1G}(k\sigma_{\Delta\varepsilon}, k\sigma_{\Delta p}) = F_{2G}(k\sigma_{\Delta\varepsilon}, k\sigma_{\Delta p}) = \exp \left( -\frac{k^2\sigma_{\Delta p}^2}{2} \right) \exp \left( -\frac{k^2\sigma_{\Delta\varepsilon}^2}{2} \right) \]

- Good beam lifetime requires the cooling force to be positive for large amplitude particles

- Assuming that cooling becomes zero at 4\(\sigma\) for both planes

  \[ k\sigma_{\Delta p} = k\sigma_{\Delta\varepsilon} = \frac{\mu_0}{4} \approx 0.6 \]

  \[ \Rightarrow \text{Nonlinearity of cooling force results in the cooling force reduction by factor } F_{1G}(\mu_0/4, \mu_0/4) = F_{2G}(\mu_0/4, \mu_0/4) \approx 0.697 \]
Cooling Parameters Optimization

- Eqs. for the damping rates and the sample lengthening at pickup-to-kicker travel are simplified if \( \alpha_p = \alpha_k = D'_p = D'_k = 0 \)

\[
\lambda_1 = -\frac{\kappa}{2} \frac{D_1D_2}{\sqrt{\beta_1\beta_2}} \sin \mu_1 \\
\lambda_2 = -\frac{\kappa}{2} \left( M_{156} - \frac{D_1D_2}{\sqrt{\beta_1\beta_2}} \sin \mu_1 \right)
\]

\[
\sigma_{\Delta s}^2 = \varepsilon \left( \frac{D_k^2}{\beta_k} + \frac{D_p^2}{\beta_p} - \frac{2D_kD_p}{\sqrt{\beta_k\beta_p}} \cos \mu_1 \right) + \sigma_p^2 \left( M_{156} - \frac{D_kD_p}{\sqrt{\beta_k\beta_p}} \sin \mu_1 \right)
\]

- One can see that for fixed decrements a minimization of sample lengthening requires \( \frac{D_k^2}{\beta_k} = \frac{D_p^2}{\beta_p} \)

\[ \Rightarrow \quad \text{Ratio of cooling decrements bounds up } \frac{D^2}{\beta} \text{ and } M_{156} : \]

\[
\frac{D^2}{\beta} \sin \mu_1 = \frac{\lambda_1}{\lambda_1 + \lambda_2} M_{156}
\]

\[
\sigma_{\Delta s}^2 = 2\varepsilon M_{156} \frac{\lambda_1}{\lambda_1 + \lambda_2} \tan \frac{\mu_1}{2} + \sigma_p^2 M_{156}^2 \left( \frac{\lambda_2}{\lambda_1 + \lambda_2} \right)^2
\]
\textbf{Requirements}  \[ k \sigma_{\Delta \varepsilon} = k \sigma_{\Delta \phi} = \mu_0 / n_\sigma \xrightarrow{n_\sigma = 4} 0.601 \] yields

\[ 2 \varepsilon M_{156} \frac{\lambda_1}{\lambda_1 + \lambda_2} \tan \frac{\mu_1}{2} = \sigma_p^2 M_{156}^2 \left( \frac{\lambda_2}{\lambda_1 + \lambda_2} \right)^2 = \left( \frac{\lambda_w \mu_0}{2\pi n_\sigma} \right)^2, \quad \lambda_w = \frac{2\pi}{k} \]

The solution is

\[
\tan \frac{\mu_1}{2} = \frac{\mu_0}{n_\sigma} \frac{\sigma_p \lambda_w \lambda_2}{4\pi \varepsilon \lambda_1} \\
M_{156} = \frac{\mu_0}{n_\sigma} \frac{\lambda_w}{2\pi \sigma_p} \frac{\lambda_1 + \lambda_2}{\lambda_2} \\
D^2 / \beta = \frac{\varepsilon}{\sigma_p^2} \left[ \frac{\lambda_1^2}{\lambda_2^2} + \frac{\sigma_p^2}{4\varepsilon^2 n_\sigma^2 2\pi} \left( \frac{\mu_0 \lambda_w}{n_\sigma 2\pi} \right)^2 \right]
\]

\textbf{For} \( \lambda_w = 12 \ \mu m, \ v_{n95} = 20 \ \text{mm mrad}, \ \sigma_p = 1.2 \cdot 10^{-4}, \ n_\sigma = 4 \) and \( \lambda_1 = \lambda_2 \)

one obtains the optimal parameters

- \( \mu_{1,\text{opt}} / 2\pi = 6.88 \cdot 10^{-3} \)
- \( M_{156} = 1.91 \ \text{cm} \)
- \( D^2 / \beta = 22.1 \ \text{cm} \)
- \( ( \beta = 50 \ \text{m}, \ D = 3.3 \ \text{m} ) \)

\textbf{Tough requirements on the betatron phase advance} \( (\Delta \nu_1 \sim 10^{-3}) \)

- \textbf{Hardly possible for} \( \lambda_w = 2 \ \mu m \) \( (\Delta \nu_1 \sim 2 \cdot 10^{-4}) \)
Combinations of Optics Parameters for Optimal Cooling

$D_1 = D_2 = D$, $D^2/\beta = 22.1$ cm, $\delta v_1 = 6.88 \cdot 10^{-3}$

| $\nu_1 = \mu_1 / 2\pi$ | $M_{156}$ [cm] |
|------------------------|----------------|
| $n- \delta v_1$        | -1.91          |
| $n+ \delta v_1$        | 1.91           |

$D_1 = -D_2 = D$, $D^2/\beta = 22$ cm, $\delta v_1 = 6.88 \cdot 10^{-3}$

| $\nu_1 = \mu_1 / 2\pi$ | $M_{156}$ [cm] |
|------------------------|----------------|
| $n+1/2 - \delta v_1$   | -1.91          |
| $n+1/2 + \delta v_1$   | 1.91           |
Requirements to the System Stability

The major limitations on system stability come from:

- Relative change path length for the beam and the light

  - Cooling force: \( F(a, \delta a) \approx \frac{2}{a} J_1(a) \cos(\delta a), \quad \delta a = k \delta L \)
    
    - Reduces the force but does not change cooling acceptance
    
    \( \Rightarrow k \delta L < 0.5 \), i.e. \( \delta L \sim 1 \, \mu m \) (\( \lambda_w=12 \, \mu m \), 10% force reduction)

- No additional requirements for high frequency jitter

- Changes of cooling rates due to optics variations

  \[
  \lambda_1 = -\frac{\kappa}{2} \left( D_2 M_{1,6} - D'_2 M'_{1,6} \right) \\
  \lambda_2 = -\frac{\kappa}{2} \left( -D_2 M_{12,6} + D'_2 M'_{1,6} + M_{15,6} \right)
  \]

- External (changes in kicker dispersion)
  
  - \( \Delta D/D < 5-10\% \) - Is not expected to be a problem

- Internal (pickup-to-kicker transport matrix)
  
  - Looks extremely sensitive: \( \Delta v_1 \sim 10^{-3} \) is required
  
  - Additional insight is needed
**Longitudinal Kick by E.-M. Wave**

- Electric field of e.-m. wave focused at \( z=0 \) to the rms size \( \sigma_\perp \)

\[
E_x(x, y, z, t) = \text{Re} \left( E_0 e^{i(\omega t - k_z z)} \frac{\sigma_\perp^2}{\sigma^2(z)} \exp \left( -\frac{1}{2} \frac{x^2 + y^2}{\sigma^2(z)} \right) \right)
\]

\[
E_y(x, y, z, t) = 0
\]

\[
E_z(x, y, z, t) = \text{Re} \left( iE_0 e^{i(\omega t - k_z z)} \frac{\sigma_\perp^2 x}{k\sigma^4(z)} \exp \left( -\frac{1}{2} \frac{x^2 + y^2}{\sigma^2(z)} \right) \right)
\]

\[
E_0 = \sqrt{\frac{8P}{c\sigma_\perp^2}}, \quad \sigma^2(z) = \sigma_\perp^2 - i \frac{z}{k}, \quad k = \frac{2\pi}{\lambda_w}
\]

- The beam is deflected in the x-plane by wiggler magnetic field
  
  - That results in the beam energy change \( \Delta E = e \int (E \cdot v) dt \)

\[
\Delta E = eE_0 \int \text{Re} \left[ \left( \frac{dx}{dz} \frac{\sigma_\perp^2}{\sigma^2(z)} + i \frac{\sigma_\perp^2 x}{k\sigma^4(z)} \right) \exp \left( -\frac{1}{2} \frac{x^2 + y^2}{\sigma^2(z)} + ik \left( \frac{z}{2\gamma^2} + \frac{1}{2} \int_0^z \left( \frac{dx}{dz'} \right)^2 dz' \right) + i\psi \right) \right] dz
\]

where \( \psi \) is the accelerating phase (\( \Delta E = 0 \) for \( \psi = 0 \))

and \( \frac{1}{2} \int_0^z \left( \frac{dx}{dz'} \right)^2 dz' \) represents the path length difference between light and beam introduced by wiggler (relative to wiggler center)
**Energy Kick in Dipole Wiggler**

- Wiggler consists of positive and negative dipoles which are immediately followed by dipole of the same field for further separation of beams
  - Dipole length, $\sigma_{\perp}$ and the beam centroid offset are adjusted to maximize the kick
  - $\sigma_{\perp}$ is much larger than the beam transverse size
- Because of tighter light focusing the kick in a dipole is only marginally lower than in the 3 dipole wiggler
Energy Kick in Dipole Wiggler

- Both $E_x$ and $E_z$ fields contribute to the kick
  - That allows one to get additional kick in the case of single dipole
- Kick in 4 T dipole is 64% of the 5 dipole 2T wiggler
  - Length of 5 dipoles is 27.5 m
  - The total length of 5 dipole system determined by beam separation is ~40 m
- Taking into account available space and comparatively high kick efficiency in a dipole as well as other limitations it looks possible to use a standard Tevatron dipole instead of wiggler
**Energy Kick in Helical Wiggler**

- Helical dipole suggest $\sqrt{2}$ times better kicker efficiency
  - Circular polarized light
- For large number of periods ($n_{wgl} \gg 1$) the kicker strength is:
  \[
  \frac{\Delta E_{\text{max}}}{e} \approx \sqrt{8.837 n_{wgl} P Z_0 \frac{K_u^2}{1 + K_u^2}}
  \]
  where $K_u = \frac{2\pi}{\lambda_{wgl}} \frac{eB}{mc^2}$, $Z_0=377 \ \Omega$

- The waist size is growing with kicker length $-\sigma_\perp \approx \sqrt{0.946 L \lambda_w}$
- The kicker is less effective than formula prediction for small $n_{wgl}$
  - $\rho_{wgl} \sim \sigma_\perp$
  - Negative contribution of $E_z$

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*M. Zolotorev*

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Comparison of Different Wiggler Types

- For large wiggler period the wiggler consisting of dipoles is easier to make than a usual harmonic wiggler
  - Little loss in efficiency is compensated by shorter length
- Helical dipole wiggler is $\sim \sqrt{2}$ time more efficient

Comparison of wiggler parameters for $\lambda_w = 12 \, \mu$m and different wigglers (2.5 wiggles each)
**Longitudinal Damping Rate**

- Long. cooling decrement is proportional to the kick amplitude ($\Delta E_{\text{max}}$) excited by a single particle
  - Requirement to have the cooling range of $\sigma n_\sigma$ times yields

$$\lambda_2 = \frac{1}{2} \int_0^\infty \frac{\Delta E_{\text{max}}}{c \sigma_p n_\sigma} F_{2G} \left( \frac{\mu_0}{n_\sigma}, \frac{\mu_0}{n_\sigma} \right)$$

- In optimum the long. damping rate does not depend on details of beam optics

- For Gaussian dependence of laser gain on $f$ the energy in a single particle pulse is related to the peak power and the FWHM bandwidth (power) as:

$$\int P(t)dt = \sqrt{\frac{\ln(2)}{\pi}} \frac{P_{\text{peak}}}{\Delta f_{\text{FWHM}}}$$

- RHIC proposal (2004), $\lambda_w=12$ $\mu$m, $((\Delta f/f)_{\text{FWHM}}=6\%$
**Longitudinal Damping Rate (2)**

- For beam with $n_b$ bunches and $N_p$ particles/bunch the average laser power is

\[
P_{\text{laser}} = n_b N_p f_0 \sqrt{\frac{\ln(2)}{\pi}} \frac{P_{\text{peak}}}{\Delta f_{\text{FWHM}}} = \frac{n_b N_p f_0}{\Delta f_{\text{FWHM}}} \sqrt{\frac{\ln(2)}{\pi}} \left( \frac{\Delta E_{\text{max}}}{G_{\text{kick}}} \right)^2
\]

where $G_{\text{kick}}$ is the kicker efficiency determined by the equation for monochromatic wave $\Delta E_{\text{max}} = G_{\text{kick}} \sqrt{P}$

⇒ For helical dipole with large number of wiggles

\[
P_{\text{laser}} = 1.26 \left( \frac{1}{n_{\text{wgl}} \left( \frac{\Delta f}{f} \right)_{\text{FWHM}}} \right) \frac{1 + K_u^2}{K_u^2} \frac{n_b N_p \lambda_w^2 \lambda_w \left( cp\sigma_p / e \right)^2}{cf_0 Z_0}
\]

\[
K_u \gg 1, n_{\text{wgl}} \left( \frac{\Delta f}{f} \right)_{\text{FWHM}} \approx \frac{n_b N_p \lambda_w^2 \lambda_w \left( cp\sigma_p / e \right)^2}{cf_0 Z_0}
\]

- Number of wiggles is limited by bandwidth: $n_{\text{wgl}} \leq 1 / (\Delta f / f)$
- For efficient kick the undulator parameter $K_u \geq 2$
  - For larger magnetic field the kicker is shorter for same $n_{\text{wgl}}$
- In optimal setup $\perp$ cooling does not require additional power
  - but requires an optimized optics
**Possible Choice of OSC Parameters**

Damping time 4.5 hour, $N_p=3\cdot10^{11}$, $n_b=36$, $\sigma_p=1.2\cdot10^{-4}$, $\lambda_2^{-1}=4.5$ hour

⇒ Amplitude of single particle kick, $\Delta E_{\text{max}}=0.66$ eV

| Wave length [$\mu$m] | Wiggler type/n_{wgl} | B [T] | Total length [m] | $G_{kicker}$ [eV/√W] | $\Delta f/f_{\text{FWHM}}$ % | P [W] |
|---------------------|---------------------|-------|------------------|------------------------|----------------------------|-------|
| 12                  | Tevatron dipole/(N/A) | 4     | N/A              | 26                     | 6                          | 125   |
| 6                   |                      |       |                  | 18                     | 6                          | 133   |
| 2                   |                      |       |                  | 14                     |                            | 71    |
| 12                  | Helical dipole/2.5   | 2     | 40               | 56                     | 6                          | 28    |
|                     | Helical dipole/8     | 8     | 44               | 132                    | 6                          | 5     |
| 6                   | Helical dipole/7     | 6     | 38               | 110                    | 6                          | 3.5   |
| 2                   | Helical dipole/12    | 6     | 36               | 116                    | 6                          | 1.05  |

♦ Peak optical amplifier power is ~100 times larger than the average one
♦ Bandwidth is limited by optical amplifier
Discussion

- **OSC would double the average Tevatron luminosity**
- **Cooling installation requires a modification of beam optics**
  - CO straight is available
  - New optics implies
    - new quad circuits
    - may be new quads
    - shuffling existing and/or installation of new dipoles
    - Installation of wigglers?
  - Considerable work
    - Fractional tunes should stay the same
    - Helices should not be affected
- **Antiproton beam has less particles but requires faster cooling**
  - That results in approximately the same power requirements for optics amplifier but its larger gain
2 \( \mu \)m wavelength

- 2 \( \mu \)m parametric optical amplifier is feasible (MIT-Bates)
  - 20-100 W (pumped by Nd:YAG laser)
- Can be used with Tevatron dipoles being pickups and kickers (no wigglers), 70 W amplifier per beam
  - 2T helical wiggler (~20 m) requires ~12 W amplifier per beam
- Optics stability and path length control are questionable
  - We will continue to look into optics issues

12 \( \mu \)m wavelength

- Looks good for control of optics and the path length
- Parametric optical amplifier pumped by 2-nd harmonic of \( CO_2 \) laser
  - Was not demonstrated yet
    - Attempt for RHIC was not quite successful
  - 5-10 W looks reasonable request
    - But R&D is required to prove feasibility
- Requires ~6-8 T helical wiggler (>4 years)

There is no fast way (2-3 years) to introduce OSC in Tevatron
- looks possible for 5-6 years
This Work Results and Plans for Further Studies

**Done**
- Better understanding of beam optics issues for OSC
  - Formulation of requirements for optimal beam optics
  - Understanding of cooling range
- Better understanding of kicker efficiency
  - Helical undulator allows to reduce its length and/or laser power

**Future work**
- Look into realistic Tevatron optics
- Study its sensitivity
  - Is the 2 μm wavelength possible?
    ⇒ If yes then the fast scenario can work with 60 W amplifier (No wigglers, pickup and kicker are in dipoles)
- Making experiment in Bates would be extremely helpful but ?
Backup Viewgraphs
Damping Rates of Optical Stochastic Cooling

Transfer Matrix Parameterization

- Vertical degree of freedom is uncoupled and we will omit it in further consideration

\[
\mathbf{M} = \begin{bmatrix}
M_{11} & M_{12} & 0 & M_{16} \\
M_{21} & M_{22} & 0 & M_{26} \\
M_{51} & M_{52} & 1 & M_{56} \\
0 & 0 & 0 & 1
\end{bmatrix}, \quad \mathbf{X} = \begin{bmatrix}
x \\
\theta_x \\
s \\
\Delta p / p
\end{bmatrix}
\]

- \( M_{16} \) & \( M_{26} \) can be expressed through dispersion

\[
\begin{bmatrix}
M_{11} & M_{12} & M_{16} \\
M_{21} & M_{22} & M_{26} \\
0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
D_1 \\
D'_1 \\
1
\end{bmatrix} = \begin{bmatrix}
D_2 \\
D'_2 \\
1
\end{bmatrix}
\]

That yields

\[
M_{16} = D_2 - M_{11}D_1 - M_{12}D'_1
\]

\[
M_{26} = D'_2 - M_{21}D_1 - M_{22}D'_1
\]

\[
M_{11} = \sqrt{\frac{\beta_2}{\beta_1}} (\cos \mu + \alpha_1 \sin \mu)
\]

\[
M_{22} = \sqrt{\frac{\beta_1}{\beta_2}} (\cos \mu - \alpha_2 \sin \mu)
\]

\[
M_{12} = \sqrt{\beta_1 \beta_2} \sin \mu
\]

\[
M_{21} = \frac{\alpha_1 - \alpha_2}{\sqrt{\beta_1 \beta_2}} \cos \mu - \frac{1 + \alpha_1 \alpha_2}{\sqrt{\beta_1 \beta_2}} \sin \mu
\]
Transfer Matrix Parameterization (continue)

- **Symplecticity** \( \mathbf{M}^T \mathbf{U} \mathbf{M} = \mathbf{U} \) binds up \( M_{51}, M_{52} \) and \( M_{16}, M_{26} \)

- That yields
  \[
  M_{51} = M_{21}M_{16} - M_{11}M_{26}
  \]
  \[
  M_{52} = M_{22}M_{16} - M_{12}M_{26}
  \]

- Finally one can write
  \[
  M_{16} = D_2 - D_1 \sqrt{\frac{\beta_2}{\beta_1}} \left( \cos \mu + \alpha_1 \sin \mu \right) - D'_1 \sqrt{\beta_1 \beta_2} \sin \mu
  \]
  \[
  M_{26} = D_1 \left( \frac{1 + \alpha_1 \alpha_2}{\sqrt{\beta_1 \beta_2}} \sin \mu + \frac{\alpha_2 - \alpha_1}{\sqrt{\beta_1 \beta_2}} \cos \mu \right) + D'_1 \sqrt{\frac{\beta_1}{\beta_2}} \left( \cos \mu - \alpha_2 \sin \mu \right)
  \]
  \[
  M_{51} = -D_2 \left( \frac{1 + \alpha_1 \alpha_2}{\sqrt{\beta_1 \beta_2}} \sin \mu + \frac{\alpha_2 - \alpha_1}{\sqrt{\beta_1 \beta_2}} \cos \mu \right) + D'_2 \sqrt{\frac{\beta_2}{\beta_1}} \left( \cos \mu + \alpha_1 \sin \mu \right)
  \]
  \[
  M_{52} = -D_1 + D_2 \sqrt{\frac{\beta_1}{\beta_2}} \left( \cos \mu - \alpha_2 \sin \mu \right) - D'_2 \sqrt{\beta_1 \beta_2} \sin \mu
  \]

- In the first order the orbit lengthening due to betatron motion is equal to zero if \( D_1 = D'_1 = D_2 = D'_2 = 0 \)
Transfer Matrix Parameterization (continue)

- Partial momentum compaction and slip factor (from point 1 to point 2) are related to $M_{56}$

$$\Delta s_{1\rightarrow 2} \equiv 2\pi R \eta_1 \frac{\Delta p}{p} = M_{51} D_1 \frac{\Delta p}{p} + M_{52} D_1' \frac{\Delta p}{p} + M_{56} \frac{\Delta p}{p} + \frac{1}{\gamma^2} \frac{\Delta p}{p}$$

- Further we assume that $v = c$, $v = c$, i.e. $1/\gamma^2 = 0$ and $\eta_1 = \alpha_{1\rightarrow 2}$.

- That results in $\eta_1 = \frac{M_{51} D_1 + M_{52} D_1' + M_{56}}{2\pi R}$ or

$$M_{56} = 2\pi R \eta_1 + D_1 D_2 \left( \frac{1 + \alpha_1 \alpha_2}{\sqrt{\beta_1 \beta_2}} \sin \mu + \frac{\alpha_2 - \alpha_1}{\sqrt{\beta_1 \beta_2}} \cos \mu \right) + D_1 D_2' \sqrt{\frac{\beta_2}{\beta_1}} \left( \cos \mu + \alpha_1 \sin \mu \right)$$

$$- D_1' D_2 \sqrt{\frac{\beta_1}{\beta_2}} \left( \cos \mu - \alpha_2 \sin \mu \right) + D_1' D_2' \sqrt{\beta_1 \beta_2} \sin \mu$$

- Thus, the entire transfer matrix from a point 1 to a point 2 can be expressed through the $\beta$-functions, dispersions and their derivatives at these points and the partial slip factor.
Parameterization of the Entire Ring Transfer Matrix

Formulas for the entire ring look more compact

\[
\begin{align*}
M_{11} &= \cos \mu + \alpha \sin \mu \\
M_{21} &= -\frac{1 + \alpha^2}{\beta} \sin \mu \\
M_{12} &= \beta \sin \mu \\
M_{22} &= \cos \mu - \alpha \sin \mu \\
M_{16} &= D(1 - \cos \mu - \alpha \sin \mu) - D'\beta \sin \mu \\
M_{26} &= D \frac{1 + \alpha^2}{\beta} \sin \mu + D'(1 - \cos \mu + \alpha \sin \mu) \\
M_{51} &= -D \frac{1 + \alpha^2}{\beta} \sin \mu + D'(1 - \cos \mu - \alpha \sin \mu) \\
M_{52} &= -D(1 - \cos \mu + \alpha \sin \mu) - D'\beta \sin \mu \\
M_{56} &= 2\pi R\alpha_{1\to2} + D^2 \frac{1 + \alpha^2}{\beta} \sin \mu + 2DD'\alpha \sin \mu + D'^2 \beta \sin \mu
\end{align*}
\]

\[
\begin{bmatrix}
M_{11} & M_{12} & 0 & M_{16} \\
M_{21} & M_{22} & 0 & M_{26} \\
M_{51} & M_{52} & 1 & M_{56} \\
0 & 0 & 0 & 1
\end{bmatrix}, \quad
\begin{bmatrix}
x \\
\theta_s \\
\Delta p / p
\end{bmatrix}
\]
Damping Rates of Optical Stochastic Cooling

Longitudinal kick

\[
\frac{\delta p}{p} = \kappa \Delta L = \kappa \left( M_{151} x_1 + M_{152} \theta x_1 + M_{156} \frac{\Delta p}{p} \right)
\]

Or in the matrix form: \( \delta X = M_c X_1 \)

\[
M_c = \kappa \begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
M_{151} & M_{152} & 0 & M_{156}
\end{bmatrix}
\]

Total ring matrix related to kicker (Ring&RF&damper)

\[
M_{tot} X_2 = M_1 M_2 X_2 + \delta X_2 = M_1 M_2 X_2 + M_c X_1 = (M_1 M_2 + M_c M_2) X_2
\]

\[ \Rightarrow \quad M_{tot} = M + \Delta M_c \]

where

\[
M = M_1 M_2 , \quad \Delta M = M_c M_2
\]
Damping Rates of Optical Stochastic Cooling (continue)

Perturbation theory yields that the eigen-value correction is \([HB2008]\):

\[
\delta\lambda_k = \frac{i}{2} v_k^+ U \Delta M v_k = \frac{i}{2} v_k^+ U M_c M_1^{-1} (M_1 M_2) v_k = \frac{i}{2} \lambda_k v_k^+ U M_c M_1^{-1} v_k
\]

Corresponding tune shift is:

\[
\delta Q_k = \frac{i}{2\pi} \frac{\delta \lambda_k}{\lambda_k} = -\frac{1}{4\pi} v_k^+ U M_c M_1^{-1} v_k
\]

Symplecticity relates the transfer matrix and its inverse:

\[
M_1^{-1} = -UM_1^T U
\]

\[\Rightarrow \]

\[
\delta Q_k = \frac{1}{4\pi} v_k^+ U M_c U M_1^T U v_k
\]

Performing matrix multiplication and taking into account that symplecticity binds up \(M_{51}, M_{52}\) and \(M_{16}, M_{26}\) one finally obtains:

\[
\delta Q_k = \frac{\kappa}{4\pi} v_k^+ \begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
M_{126} & -M_{116} & 0 & M_{156} \\
0 & 0 & 0 & 0
\end{bmatrix} v_k
\]
Eigen-vectors and Damping Decrement (Mode 1)

- There are two eigen-vectors
  - One related to the betatron motion $v_1$
  - And one related to the synchrotron motion $v_2$

- They are normalized as: $v_k^+ U v_k = -2i$

- If the synchrotron tune and dispersion in RF cavities are small, the effect of RF can be neglected in the computation of $v_1$
  - In this case $\lambda_1 = e^{-i\mu}$ and
    the eigen-vector related to the kicker position is

\[
\begin{bmatrix}
\sqrt{\beta_2} \\
-(i + \alpha_2) / \sqrt{\beta_2} \\
v_{13} \\
0
\end{bmatrix}, \quad Mv_k = \lambda_k v_k, \quad M = \\
\begin{bmatrix}
M_{11} & M_{12} & 0 & M_{16} \\
M_{21} & M_{22} & 0 & M_{26} \\
M_{51} & M_{52} & 1 & M_{56} \\
0 & 0 & 0 & 1
\end{bmatrix}
\]

The first 2 components are the same as for uncoupled case. The third component has to be found from the third equation

\[
=> v_{13} = -\frac{iD_2 (1 - i\alpha_2) + D'_2 \beta_2}{\sqrt{\beta_2}}
\]
Corresponding damping rate is

\[ \lambda_1 = -2\pi \text{Im} \delta Q_1 \]

\[ \lambda_1 = -\frac{\kappa}{2} \text{Im} \left( \begin{pmatrix} \sqrt{\beta_2} \\ -(i + \alpha_2)/\sqrt{\beta_2} \\ v_{13} \\ 0 \end{pmatrix}^+ \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ M_{126} & -M_{116} & 0 & M_{156} \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{pmatrix} \sqrt{\beta_2} \\ -(i + \alpha_2)/\sqrt{\beta_2} \\ v_{13} \\ 0 \end{pmatrix} \right) \]

\[ \lambda_1 = -\frac{\kappa}{2} \left( D_2 M_{12,6} - D_2' M_{1,6} \right) \]

That yields

\[ \lambda_1 = -\frac{\kappa}{2} \left[ D_1 D_2 \frac{(1 + \alpha_1 \alpha_2) \sin \mu_1 + (\alpha_2 - \alpha_1) \cos \mu_1}{\sqrt{\beta_1 \beta_2}} - D_1' D_2 \sqrt{\frac{\beta_1}{\beta_2}} (\cos \mu_1 - \alpha_2 \cos \mu_1) \right. \]

\[ + D_1 D_2' \sqrt{\beta_2 \beta_1} (\cos \mu_1 + \alpha_1 \sin \mu_1) + D_1' D_2' \sqrt{\beta_1 \beta_2} \sin \mu_1 \left] \right. \]

Expressing it through the partial slip factor one gets

\[ \lambda_1 = -\frac{\kappa}{2} (M_{56} - 2\pi R \eta_1) \]
Eigen-vectors and Damping Decrement (Mode 2)

To find the second eigen-vector we will ignore the second order effects of betatron motion on the longitudinal dynamics.

- The linearized RF kick is
  \[ \frac{\delta p}{p} = -\Phi_s s \]

- Simple calculations yield for the eigen value \( \lambda_1 = e^{-i\mu_s} \)

  where the synchrotron tune \( \mu_s = \sqrt{2\pi R \eta \Phi_s} \)

- Corresponding eigen-vector related to the kicker position is
  \[
  v_1 = \begin{pmatrix}
  -iD_2 / \sqrt{\beta_s} \\
  -iD'_2 / \sqrt{\beta_s} \\
  \sqrt{\beta_s} \\
  -i / \sqrt{\beta_s}
  \end{pmatrix}
  \]

  where the longitudinal beta-function \( \beta_s = 2\pi R \eta / \mu_s \)
Corresponding damping rate is

\[ \lambda_2 = -2\pi \text{Im} \delta Q_2 \]

\[
\begin{align*}
\lambda_2 &= -\frac{\kappa}{2} \text{Im} \\
&= -\frac{\kappa}{2}
\begin{pmatrix}
-iD_2 / \sqrt{\beta_s} \\
-iD'_2 / \sqrt{\beta_s} \\
\sqrt{\beta_s} \\
-i / \sqrt{\beta_s}
\end{pmatrix}
\begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
M_{126} & -M_{116} & 0 & M_{156} \\
0 & 0 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
-iD_2 / \sqrt{\beta_s} \\
-iD'_2 / \sqrt{\beta_s} \\
\sqrt{\beta_s} \\
-i / \sqrt{\beta_s}
\end{pmatrix}
\end{align*}
\]

\[
= -\frac{\kappa}{2} \left( M_{156} - D_2 M_{126} + D'_2 M_{116} \right)
\]

Expressing the matrix elements through Twiss parameters one obtains

\[ \lambda_2 = -\frac{\kappa}{2} M_{156} - \lambda_1 = -\pi \kappa R \eta_1 \]

The last expression can be directly obtained from the definition of the partial slip factor

The above equation yields the sum of the decrements is

\[ \lambda_1 + \lambda_2 = -\frac{\kappa}{2} M_{156} \]
Damping Rates for Smooth Lattice Approximation

- For zero derivatives of beta-function and dispersion at pickup and kicker one obtains

\[
\lambda_1 = -\frac{\kappa}{2} \frac{D_1 D_2}{\sqrt{\beta_1 \beta_2}} \sin \mu_1
\]

\[
\lambda_2 = -\frac{\kappa}{2} \left[ M_{156} - \frac{D_1 D_2}{\sqrt{\beta_1 \beta_2}} \sin \mu_1 \right]
\]

- Smooth lattice approximation additionally yields

\[
\beta = \frac{R}{v}, \quad D = \frac{R}{v^2}, \quad \mu_1 = \nu \frac{L_{pk}}{R}, \quad \eta_1 = -\frac{L_{pk}}{2\pi v^2 R}, \quad M_{156} = -\frac{L_{pk}}{v^2} + \frac{R}{v^3} \sin \left( \nu \frac{L_{pk}}{R} \right),
\]

where \( L_{pk} \) is the pickup-to-kicker path length, and \( \nu \) is the betatron tune

\[
\lambda_1 = -\frac{\kappa}{2} \frac{R}{v^3} \sin \left( \nu \frac{L_{pk}}{R} \right)
\]

\[
\lambda_2 = \frac{\kappa}{2} \frac{L_{pk}}{v^2}
\]
Comparison to Zholents-Zolotorev result

Eqs. (A9) and (A11) in the paper Appendix can be rewritten in the following simplified form

$$\lambda_1 = \frac{\kappa}{2} \left( D_2 M_{151}^{-1} + D'_{2} M_{152}^{-1} \right)$$

$$\lambda_2 = -\frac{\kappa}{2} \left( D_2 M_{151}^{-1} + D'_{2} M_{152}^{-1} + M_{156}^{-1} \right)$$

The inverse of the matrix is

$$M_1^{-1} = -U M_1^T U = \begin{bmatrix} M_{122} & -M_{112} & 0 & M_{152} \\ -M_{121} & M_{111} & 0 & M_{151} \\ M_{126} & M_{116} & 1 & -M_{156} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Substituting expressions for matrix elements into above Eqs. for decrements one arrives to the same results
Sample Lengthening on Pickup-to-Kicker Travel

- Zero length sample lengthens on its way from pickup-to-kicker

\[ \sigma_{\Delta L}^2 = \int \left( M_{151} x + M_{152} \theta_x + M_{156} \tilde{\eta} \right)^2 f(x, \theta_x, \tilde{\eta}) dx d\theta_x d\tilde{\eta}, \quad \tilde{\eta} = \frac{\Delta p}{p} \]

where for Gaussian distribution

\[ f(x, \theta_x, \tilde{\eta}) = \frac{1}{\sqrt{2\pi} 2\pi \sigma_p} \exp \left( -\gamma_p (x - D_p \tilde{\eta})^2 + 2\alpha_p (\theta_x - D'_p \tilde{\eta})(x - D_p \tilde{\eta}) + \beta_p (x - D_p \tilde{\eta}) - \frac{\tilde{\eta}^2}{2\sigma_p^2} \right), \quad \gamma_p = \frac{1 + \alpha_p^2}{\beta_p} \]

- Performing integration one obtains

\[ \sigma_{\Delta L}^2 = \varepsilon \left( \beta_p M_{151}^2 - 2\alpha_p M_{151} M_{152} + \gamma_p M_{152}^2 \right) + \sigma_p^2 \left( M_{151} D_p + M_{152} D'_p + M_{156} \right)^2 \]

- Expressing matrix elements through Twiss parameters yields

\[ \sigma_{\Delta L}^2 = \varepsilon F_\varepsilon + \sigma_p^2 \left( 2\pi R \alpha_{1\rightarrow 2} \right)^2 \]

\[ F_\varepsilon = D_p^2 \gamma_p + D_k^2 \gamma_k - \frac{2D_p D_k}{\sqrt{\beta_p \beta_k}} \left( (1 + \alpha_p \alpha_k) \cos \mu_1 + (\alpha_p - \alpha_k) \sin \mu_1 \right) + D'_p^2 \beta_p + D'_k^2 \beta_k + 2D_p D'_p \alpha_p + \]

\[ 2D_p D'_p \alpha_p + 2D_p D'_k \sqrt{\beta_k \beta_p} (\sin \mu_1 - \alpha_p \cos \mu_1) - 2D_k D'_p \sqrt{\beta_p \beta_k} (\sin \mu_1 + \alpha_k \cos \mu_1) - 2D'_k D'_p \sqrt{\beta_p \beta_k} \cos \mu_1 \]
For zero derivatives it yields

$$\sigma_{\Delta L}^2 = \varepsilon \left( \frac{D_k^2}{\beta_k} + \frac{D_p^2}{\beta_p} - \frac{2D_k D_p}{\sqrt{\beta_k \beta_p}} \cos \mu_1 \right) + \sigma_p^2 \left( M_{156} - \frac{D_k D_p}{\sqrt{\beta_k \beta_p}} \sin \mu_1 \right)$$
Estimate of Energy Kick in Helical Wiggler

- Assuming that $\rho_\perp \ll \sigma_\perp$ the kick amplitude is
  \[
  \frac{\Delta E}{e} = \sqrt{\frac{4P}{c\sigma_\perp^2}} \theta_0 2 \int_0^{L/2} \frac{\sigma_\perp^2 dz}{\sigma_\perp^2 - iz/k} = 4\sqrt{\frac{P}{c}} \theta_0 k \sigma_\perp \sinh\left( \frac{L}{2k \sigma_\perp^2} \right)
  \]

- The function $x\sinh\left(\frac{1}{x^2}\right)$ achieves its maximum at $x = c_0 \approx 0.54884$

  $\Rightarrow$ Maximum kick of 
  \[
  \frac{\Delta E}{e} \bigg|_{opt} = 4c_0 \sqrt{2} \sinh\left( \frac{1}{c_0^2} \right) \sqrt{\frac{P}{c}} \theta_0 \sqrt{kL} \]
  is achieved at $\sigma_\perp = \sqrt{\frac{c_0^2 \ L}{2k}}$

- Taking into account that $4\pi / c = Z_0$ and $kL = 2\pi n_{wgl}$ we obtain
  \[
  \frac{\Delta E}{e} \bigg|_{opt} = 2c_0 \sinh\left( \frac{1}{c_0^2} \right) \theta_0 \sqrt{PZ_0 n_{wgl}}
  \]

- The condition of resonance is: $k\left(1/(2\gamma^2) + \theta_0^2/2\right) = k_{wgl}$, where the particle angle (relative to wave direction) is $\theta_0 = \frac{1}{k_{wgl} R_L}$, $R_L = \frac{pc}{eB_0}$

- That yields
  \[
  \frac{\Delta E}{e} \bigg|_{opt} = c_0 \sinh\left( \frac{1}{c_0^2} \right) \sqrt{\frac{8PZ_0 n_{wgl} K_u^2}{1 + K_u^2}} \approx \sqrt{\frac{8.837 PZ_0 n_{wgl} K_u^2}{1 + K_u^2}} \quad , \quad K_u = \frac{eB_0}{pck_{wgl}}
  \]
References
HB2008 - V. Lebedev, A. Burov, “Coupling and its Effects on Beam Dynamics”, HB-2008