ON THE JITTER RADIATION

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ABSTRACT

In a small-scale turbulent medium, when the nonrelativistic Larmor radius $R_L = mc^2/eB$ exceeds the correlation length $\lambda$ of the magnetic field, the magnetic Bremsstrahlung radiation of charged relativistic particles unavoidably proceeds to the so-called jitter radiation regime. The cooling timescale of parent particles is identical to the synchrotron cooling time, thus this radiation regime can be produced with very high efficiency in different astrophysical sources characterized by high turbulence. The jitter radiation has distinct spectral features shifted toward high energies, compared to synchrotron radiation. This effect makes the jitter mechanism an attractive broad-band gamma-ray production channel, which, in highly magnetized and turbulent environments, can compete or even dominate over other high-energy radiation mechanisms. In this paper, we present a novel study of the spectral properties of the jitter radiation performed within the framework of perturbation theory. The derived general expression for the spectral power of radiation is presented in a compact and convenient form for numerical calculations.

Key words: gamma rays: general – methods: analytical – radiation mechanisms: non-thermal – turbulence

Online-only material: color figures

1. INTRODUCTION

Charged particles moving in electric and magnetic fields experience effective energy losses via radiation. Because of high conductivity, the electric fields in astrophysical plasmas are typically screened, thus the radiation is dominated by interactions with the magnetic field due to the so-called magnetic Bremsstrahlung. This type of radiation is one of the major non-thermal radiation processes in astrophysics and operates with a high efficiency in a large variety of astrophysical environments. In the case of a regular magnetic field or a chaotic field characterized by large-scale fluctuations, we deal with synchrotron radiation. This process and its implications in astrophysics have been studied in great detail (see, e.g., Ginzburg & Syrovatskii 1969; Rybicki & Lightman 1979). In highly turbulent environments, namely, when the nonrelativistic Larmor radius $R_L = mc^2/eB$ does not exceed the characteristic scale of turbulence $\lambda$, the radiation proceeds in a significantly different regime. The astrophysical literature refers to the radiation as diffusive synchrotron radiation (Toptygin & Fleishman 1987, hereafter TF87), or jitter radiation (Medvedev 2000, hereafter M00). Hereafter, we will use the term “jitter radiation.”

The spectral features of jitter radiation substantially differ from those of synchrotron radiation. While the power of the synchrotron radiation of a monoenergetic particle $P_\omega$ is described with good accuracy as $P_\omega \propto \omega^{5/3} \exp \left[-\omega/\omega_c\right]$, where $\omega_c = 3\gamma^2eB/(2mc)$ is the characteristic synchrotron frequency, in the case of jitter radiation, the peak is shifted toward higher frequencies by a factor of $\alpha = R_L/\lambda$. Unless the distribution of emitting particles is strictly monodirectional, the power spectrum of jitter radiation below the maximum is flat, i.e., the spectral energy distribution (SED) is $P_\omega \propto \omega^\alpha$ beyond the cutoff energy $\epsilon_c$ it has a power-law behavior, $P_\omega \propto \omega^{1-\alpha}$, where $\alpha$ is the power-law index of the turbulence spectrum (TF87). Thus, instead of the typical exponential cutoff in the synchrotron spectrum, the jitter mechanism yields a power-law spectrum that can extend up to frequencies $\omega_{\epsilon_c}$ of order of $10^5$. This fact makes the jitter radiation of electrons an excellent high-energy gamma-ray production process in contrast to synchrotron radiation which, even in the case of extreme accelerators operating at the maximum possible rate allowed by classical electrodynamics (Aharonian et al. 2002), is limited by the maximum possible energy $\epsilon_{\epsilon_c} = h\omega_{\epsilon_c} = 9/4\alpha^{-1}mc^2 \sim 150$ MeV.

However, so far this remarkable feature of jitter radiation has practically been unexplored in regards to the interpretation of high-energy gamma-ray phenomena (see, however, the recent paper by Teraki & Takahara 2013). Instead, more emphasis has been placed on the energy interval below the cutoff. In particular, it is argued in M00 that the jitter radiation below the cutoff can result in harder spectra than synchrotron radiation, namely $P_\omega \propto \omega^\alpha$. However, the claimed energy dependence is closely related to the assumed geometry of the magnetic field. Namely, it can be achieved if the magnetic field has only one non-zero component, which can be realized only for a rather unrealistic configuration of the turbulent field (see Section 7 for details).

In the seminal paper on jitter radiation by TF87, it was realized that the spectral maximum of the jitter emission is located at higher frequencies than in the synchrotron regime, and that the high-energy part of the jitter spectrum could be described by a power law. Thus, even for the case of a monoenergetic particle distribution, one may expect a broken power-law spectrum. This should lead to the modification of the standard relations between spectral slopes, flux levels, and breaks found in synchrotron spectra. Possible applications of the jitter mechanism also has been discussed, basically in the low-energy range of cosmic electromagnetic radiation. In particular, it has been proposed that the jitter radiation can be responsible for the radio to optical (X-ray) spectra of some active galaxies and pulsar wind nebulae (TF87; Fleishman & Bietenholz 2007; Mao & Wang 2007).
The understimation of the potential of jitter radiation for
the production of high- and very high energy gamma-rays may
be related to the effect of weakening of the diffusive shock
acceleration process in the case of short-length scale turbulence.
A self-consistent consideration of the processes of particle
acceleration and emission in the framework of the diffusive
shock acceleration paradigm predicts a shift of the jitter radiation
peak toward low frequencies as compared to pure synchrotron
radiation (Derishev 2007; Kirk & Reville 2010). However,
if the inhomogeneities responsible for particle acceleration
and emission are different, e.g., when these processes occur
in spatially separated regions, the spectral maximum would be
shifted toward higher energies, thereby making the jitter
radiation a very effective high-energy gamma-ray production
mechanism. Therefore, the spectral features of this radiation
over the entire energy range deserve to be qualitatively studied
in detail.

To explore the process in a general form, we propose a
new approach based on perturbation theory. In terms of
addition assumptions, the proposed method is less demanding
compared to previous studies, and allows a precise control of the
applicability conditions for the derived solutions, e.g., the range
of the high-energy power-law extension beyond the spectral
maximum.

In this regard, we should note that in previous studies some
principal results have been obtained under specific, although
not always obvious assumptions. For example, M00 derived the
spectrum of radiation for the case of a very specific geometry of
the magnetic field fluctuations. In some other studies (see, e.g.,
Fleishman & Bietenholz 2007), the jitter radiation spectrum in
fact has not been strictly derived, but rather pre-defined through
its asymptotic behavior. Finally, some studies addressed the
case of anisotropic turbulence (Reynolds & Medvedev 2012),
however, the structure of the correlation tensor used was not
consistent with the fundamental requirement of $\nabla \cdot \mathbf{B} = 0$. We
discuss these concerns in detail in Section 7.

Finally, we should mention that significant progress has
recently been achieved through numerical computations based
on the particle-in-cell technique (see, e.g., Reville & Kirk
2010; Terak & Takahara 2011). This method has a great
potential to deal with quite complex distributions of emitting
particles. On the other hand, an analytical approach permits
a better understanding of and interpretation of the physics
behind the obtained results. In this regard, the two methods
are complementary and equally important.

The paper is organized as follows: in Section 2, the basic
results on the energy spectra, as well as the applicability limits
for the derived spectra, are presented. In Section 3, we con-
side the case of a chaotic magnetic field. Section 4 discusses
synchrotron radiation. In Section 5, we compare the radiation
properties of a chaotic magnetic field with conventional syn-
chrotron radiation. The case of anisotropic turbulence (under
the assumption of an isotropic distribution of emitting particles)
is considered in Section 6. Finally, we compare our results with
previous studies in Section 7 and summarize the main results in
Section 8.

2. PERTURBATION THEORY

The intensity and the energy distribution of radiation pro-
duced by a particle of a given charge $e$ depends only on its
trajectory. Let $\mathbf{r}(t)$ and $\mathbf{v}(t) = \dot{\mathbf{r}}(t)$ be the radius vector and the
velocity of the particle, respectively, at an instant $t$. Then, the en-
ergy spectrum of the radiation is described by Equation (14.65)
of Jackson (1998). For our purposes, it is convenient to present
the spectrum in the form:

$$
\frac{d\varepsilon_\text{iso}}{d\omega d\Omega} = \frac{e^2}{4\pi^2 c^3} \left| \int U(t) dt \right|^2
$$

where $d\varepsilon_\text{iso}$ is the energy radiated by a particle into a solid angle
$d\Omega$ over the frequency interval $d\omega$.

Here, the integrand

$$
U(t) = \frac{n \times [(n - \beta(t)) \times \mathbf{a}(t)]}{(1 - n \beta(t))^2} e^{i\Phi(t)}
$$

depends on the particle velocity $\mathbf{v}(t) = c\beta(t)$ and the ac-
celeration $\mathbf{a}(t) = c\dot{\beta}(t)$, as well as on the function $\Phi(t) = \omega t - n r(t)/c$, where $n$ is a unit vector toward the momentum of the radiated photon. The function $U^*(t)$ is the complex con-
jugation of $U(t)$. Note that Equation (1) is precise; it is derived
within the framework of classical electrodynamics through the
integration of the Maxwell equations in a vacuum.

It is convenient to introduce new variables of integration:
$t = (t_1 + t_2)/2$ and $\tau = t_1 - t_2$ (note that $dt_1 dt_2 = dt \, d\tau$). Then, Equation (1) can be written as:

$$
\frac{d\varepsilon_\text{iso}}{d\omega d\Omega} = \frac{e^2}{4\pi^2 c^3} \int U(t + \tau/2) U^*(t - \tau/2) dt d\tau.
$$

As is shown below, for a fixed value of $t$, the integrand rapidly decreases with increasing $|\tau|$. Also, the integrand is
characterized by a weak dependence on $t$. The integration of the integrand over $d\tau$ gives the spectral power of the emission at the moment $t$:

$$
P_{\text{iso}}(t) = \frac{e^2}{4\pi^2 c^3} \int_{-\infty}^{\infty} U(t + \tau/2) U^*(t - \tau/2) d\tau.
$$

Note that if one considers Equation (3) as a classical limit of the corresponding quantum relation, then the integrand in Equation (3) can be interpreted as the emission probability multiplied by the photon energy $\hbar\omega$ (see Berestetskii et al. 1989, Section 90).

The radiation detected by an observer is produced by the
ensemble of particles occupying a certain region of space. We
will consider the case of a chaotic magnetic field, assuming that
the statistically averaged (over time or space) magnetic field is
$\langle \mathbf{B} \rangle = 0$.

Let $\lambda$ be the correlation length of the magnetic field. If the
distance between two chosen points at $r_1$ and $r_2$ exceeds $\lambda$, then
the corresponding magnetic fields $\mathbf{B}_1$ and $\mathbf{B}_2$ can be treated as
statistically independent, thus the time-averaged product of these
fields $\langle \mathbf{B}_{1\tau} \mathbf{B}_{2\tau} \rangle = \langle \mathbf{B}_{1\tau} \rangle \langle \mathbf{B}_{2\tau} \rangle = 0$.

To obtain the radiation spectrum, the integrand in Equation (4)
should be averaged over all possible configurations of the
magnetic field. It is convenient to perform this procedure in
the framework of perturbation theory. The acceleration of a
particle is proportional to the strength of the magnetic field $\mathbf{B}$:

$$
\mathbf{a} = e(\beta \times \mathbf{B})/(m\gamma).$$

In the first approximation, all other relevant
parameters can be treated as in the absence of the magnetic
The applicability of this approach is discussed below. This approximation results in:

\[ P_{\text{no}}(t) = \frac{e^2}{4\pi^2c^3(1-n\beta)^2} \times \int_{-\infty}^{\infty} \left[ a_+ \left(na_-\right) - \frac{(na_+)(na_-)}{\gamma^2(1-n\beta)^2} \right] e^{i\omega(1-n\beta)\tau} d\tau, \tag{5} \]

where \( a_\pm = a(\pm \tau/2) \) and \( \beta = \beta(t) \); \( \gamma \equiv 1/\sqrt{1-\beta^2} \) is the particle Lorentz factor. The derivation of Equation (5) was performed using the formula for the standard double vector product: \( a \times (b \times c) = b(a(c) - c(ab)) \) and taking into account that in the magnetic field the acceleration and velocity vectors are orthogonal. Equation (5) represents the first non-vanishing term in the expansion of the emission spectrum in powers of the magnetic field.

Our ultimate aim is to derive the emission spectrum integrated over the emission angles of photons and averaged over the magnetic field fluctuations. It is convenient to select the \( z \) axis to be parallel to the particle velocity \( \beta \) and to start with averaging over the azimuthal angle \( \phi \) with respect to the direction of the particle velocity \( \beta \). Then, the scalar product of the vectors \( n \) and \( \beta \) does not depend on the azimuthal angle \( \phi \); \( (n\beta) = \beta \cos \theta \). Given that \( a_\pm \perp \beta \), one obtains:

\[ \langle (na_+)(na_-) \rangle = \frac{1}{2} \left( a_+a_- \right) \sin^2 \theta. \tag{6} \]

After averaging Equation (5) over \( \phi \), we obtain:

\[ P_{\text{no}}(t) = \frac{e^2}{4\pi^2c^3(1-n\beta)^2} \left( 1 - \frac{\sin^2 \theta}{2\gamma^2(1-n\beta)^2} \right) \times \int_{-\infty}^{\infty} \left( a_+a_- \right) e^{i\omega(1-n\beta)\tau} d\tau. \tag{7} \]

In Equation (7) the charge velocity, \( \beta = \beta(t) \), is treated as a constant (independent of \( \tau \)). Averaging over the magnetic field configurations results in the appearance of a correlation function, \( \langle a_+a_- \rangle \), under the integral. Note that for \( \beta = \text{const} \), the acceleration and magnetic field have the same statistical properties. In particular, \( \langle a(t) \rangle = 0 \) and \( \langle a(t+\tau/2)a(t-\tau/2) \rangle = \langle a(t+\tau/2) \rangle \langle a(t-\tau/2) \rangle = 0 \) if the distance between the corresponding points exceeds \( \lambda \) (i.e., if \( c\beta \tau > \lambda \)). This feature of \( \langle a_+a_- \rangle \) is illustrated in Figure 1.

Since the radiation of ultrarelativistic particles is strongly beamed toward the direction of motion (\( \theta \sim 1/\gamma \)), we will consider only the region of small angles, \( \theta \ll 1 \). This approximation allows for significant simplifications of the calculations, which result in:

\[ P_{\text{no}}(t) = \frac{e^2}{\pi^2c^3} \frac{\gamma^2(1+\gamma^2\tilde{\omega}^4)}{(1+\gamma^2\tilde{\omega}^2)^2} \int_{-\infty}^{\infty} \langle a_+a_- \rangle e^{i\tilde{\omega}r} d\tau, \tag{8} \]

where

\[ \tilde{\omega} = \frac{\omega}{2\gamma^2} \left( 1 + \gamma^2\theta^2 \right). \tag{9} \]

Since the above results are derived within the framework of perturbation approach, it is important to study the range of applicability of Equation (8).

The integrand in Equation (8) rapidly decreases over the range of \( |\tau| \gg \lambda/c \). Therefore, the expression obtained describes correctly the emission power if the terms neglected in the derivation of Equation (8) are small when \( |\tau| \ll \lambda/c \). In the precise Equation (4), the denominator contains a term \( d_s \equiv 1 - n\beta(t \mp \tau/2) \). For small values of \( \tau \), we have \( d_s \approx (1-n\beta) \mp n\beta \tau/2 \). For ultrarelativistic particles, the angle \( \theta \) between \( n \) and \( \beta \) is small (\( \sim 1/\gamma \)), thus, given the orthogonality of \( \tilde{\beta} \) and \( \beta \), the last term in \( d_s \) can be estimated as \( eB\lambda/\gamma m^2c^2 \). Since this term was neglected in the derivation of Equation (8), it must be small compared to the first term in \( d_s \), which is estimated to be \( \sim 1/\gamma^2 \).

Thus, the range of applicability is determined by the condition:

\[ \frac{eB\lambda}{m^2c^2} \ll 1. \tag{10} \]

Furthermore, the exponential term in Equation (4) contains a function, \( i\Delta \equiv i(\Phi(t+\tau/2) - \Phi(t-\tau/2)) \). The Taylor expansion of the function \( \Delta \) gives \( \Delta = \omega(1 - n\beta)\tau - \omega n\beta \tau^3/24 \). In the derivation of Equation (5), only the first term in this expansion was kept, therefore the applicability can be reduced to the condition of neglecting the second term. Since the function \( \Delta \) is in the exponent, the condition is \( \omega n\beta \tau (\lambda/c)^3 \ll 1 \). The module of the particle velocity in the magnetic field remains constant, \( \tilde{\beta}^2 = \text{const} \), thus \( (1/2)(d^2/d\tau^2)\tilde{\beta}^2 = (\tilde{\beta}^2 + \tilde{\beta}) = \tilde{\beta} \). Since the emitted photons and the particle velocities are nearly parallel, \( n \approx \beta \), the term \( (n\beta) \) can be estimated as \( (n\beta) \approx (\tilde{\beta}^2) = (\tilde{\beta}^2) \). This gives the second condition of applicability of Equation (8): \( \omega \tilde{\beta} \gamma^2(\lambda/c)^3 \ll 1 \). By expressing the acceleration \( \tilde{\beta} \) through the magnetic field strength, the condition of applicability of Equation (8) can be written in the form:

\[ \omega \ll \frac{m^2c^2\gamma^2}{eB^2\lambda^3}. \tag{11} \]

Note that for a homogeneous magnetic field \( \lambda = \infty \), therefore the standard synchrotron spectrum cannot be derived in the framework of perturbation theory.

The conditions of the applicability of the perturbation approach, given by Equations (10) and (11), can be interpreted in the following way. If a charged particle travels in a region filled with magnetic field and traverses a path \( \lambda \), which is short compared with the trajectory curvature \( R \), then the particle is deflected by an angle \( \delta \theta \sim \lambda/R \). The first condition given by Equation (10) implies that \( \delta \theta \ll 1/\gamma \). Concerning the second condition given by Equation (11), it is equivalent to the requirement that the segment of the trajectory of length of order \( \sim \lambda \) can be treated as a straight line.
The characteristic frequency of the radiation in this regime, \( \omega_j \), can be estimated from first principles. Namely, while the emission is formed during the time interval \( \delta t_{\text{rad}} \sim \lambda/c \), it is registered during \( \delta t_{\text{obs}} = \delta t_{\text{rad}}/(1 - nB) \sim \delta t_{\text{rad}}/\gamma^2 \). Thus, the characteristic frequency is estimated as (see, e.g., Landau & Lifshitz 1975):

\[
\omega_j = \frac{1}{\delta t_{\text{obs}}} = \frac{c\gamma^2}{\lambda}.
\] (12)

Note that this frequency \( \omega_j \) is independent of the magnetic field strength \( B \).

It is interesting to compare the characteristic frequencies at which the bulk of the radiation is produced in highly turbulent and homogeneous magnetic fields corresponding to the jitter and synchrotron radiation regimes. The characteristic synchrotron frequency can be expressed through the nonrelativistic Larmor radius \( R_L = mc^2/eB \):

\[
\omega_c = \frac{3c\gamma^2}{2R_L} = \frac{3}{2} \frac{\lambda}{R_L} \omega_j.
\] (13)

It is convenient to also express Equations (10) and (11) in terms of \( R_L \):

\[
\frac{\lambda}{R_L} \ll 1 , \quad \omega \ll \omega_j \left( \frac{R_L}{\lambda} \right)^2 .
\] (14)

When these conditions are satisfied, the ratio \( \omega_j/\omega_c \sim R_L/\lambda \gg 1 \), i.e., the characteristic energy of photons emitted by charged particles in a highly turbulent magnetic field may significantly exceed, by a factor of \( R_L/\lambda \), the characteristic energy of synchrotron photons emitted by the same particles in a regular magnetic field of the same strength.

Finally, one should mention another constraint on the applicability of Equation (8) related to plasma effects. The basic Equation (2) describes emission in a vacuum neglecting the impact of the surrounding plasma. In the frequency range \( \omega \gg \omega_p \), plasma can be treated as a medium with a dielectric permittivity \( \varepsilon(\omega) = 1 - \omega_p^2/\omega^2 \), where:

\[
\omega_p = \sqrt{\frac{4\pi e^2 n_e}{m_e}}
\] (15)

is the plasma frequency (\( n_e, m_e \) and \( e \) are the number density, mass and charge of electrons, respectively). At \( \varepsilon(\omega) \approx 1 \), the term \( (1 - nB) \approx ((1/\gamma^2) + \theta^2)/2 \) in all the above derived formulae should be replaced by the term corrected for the dielectric permittivity: \( 1 - \sqrt{\varepsilon nB} \approx ((1/\gamma^2) + (\omega_p^2/\omega^2) + \theta^2)/2 \). Thus, the influence of the medium can be ignored for sufficiently high frequencies, \( \omega \gg \omega_p \gamma \). Note that the particle Lorentz factor \( \gamma \) and the plasma frequency enter the equation in the form of the combination \( 1/\gamma^2 + (\omega_p^2/\omega^2) \). Therefore, the influence of the medium can be taken into account if in all the above equations we replace the particle Lorentz factor \( \gamma \) with:

\[
\gamma^*(\omega) = \frac{\gamma}{\sqrt{1 + (\omega_p \gamma/\omega)^2}}
\] (16)

(see, e.g., Ginzburg & Syrovatskii 1969; Fleishman 2006b). However, as long as we are interested in the high frequency range \( \omega \gg \omega_p \gamma \), for the sake of simplicity we will ignore the difference between \( \gamma \) and \( \gamma^* \), unless we state otherwise.

3. DEALING WITH A CHAOTIC MAGNETIC FIELD

The integrand of Equation (8) contains the term

\[
(a, a_-) = \frac{e^2}{m^2\gamma^2} (\beta \times B_\perp)(\beta \times B_-)
\]

\[
= \frac{e^2\beta^2}{m^2\gamma^2} (\delta_{\rho\sigma} - v_{\rho}v_{\sigma}) B_{\rho+}B_{-\sigma},
\] (17)

which should be averaged over different configurations of the magnetic field. Here, \( v = \beta/|\beta| \) is a unit velocity vector. The magnetic field values \( B_\perp \) and \( B_- \) correspond to the points where the charged particle is located at times \( t \pm \tau/2 \), i.e., \( B_\perp = B(r(t) \pm \beta(t)\tau/2, t \pm \tau/2) \). The statistical averaging of this expression results in appearance of the correlation function:

\[
K_{\rho\sigma} \equiv \langle B_\rho(r_1, t_1)B_\sigma(r_2, t_2) \rangle ,
\] (18)

which is a second-order tensor. Here, under the statistical averaging, we suppose a general standard procedure; it could be a space-time homogenization or an integration over an ensemble of field configurations (see, e.g., Section 118 of Landau & Lifshitz 1980). Here, we assume that the field is statistically homogeneous and stationary. These assumptions imply that the correlation function depends only on the difference of the coordinates \( (r_1 - r_2) \) and the times \( (t_1 - t_2) \), i.e., \( K_{\rho\sigma} = K_{\rho\sigma}(r_1 - r_2, t_1 - t_2) \). In this case, \( (B^2) = K_{\rho\rho}(0) = \text{const.} \)

It is convenient to present the correlation function \( K_{\rho\sigma} \) in the form of a Fourier integral:

\[
K_{\rho\sigma}(r, t) = \int \tilde{K}_{\rho\sigma}(q, \chi) e^{i(qr - \chi t)} d^3q dq \frac{d\chi}{2\pi} .
\] (19)

Since the magnetic field is divergence free \( (\nabla \cdot B = 0) \), \( K_{\rho\sigma} \) should satisfy the following conditions:

\[
\partial K_{\rho\sigma}/\partial x_\rho = 0 , \quad \partial K_{\rho\sigma}/\partial x_\sigma = 0 ,
\] (20)

which, for the Fourier transform \( \tilde{K}_{\rho\sigma} \), take the form (the transversality condition):

\[
\tilde{K}_{\rho\sigma}(q, \chi) = 0 , \quad \tilde{K}_{\rho\sigma}(q, \chi) = 0 .
\] (21)

While in Section 7 we will briefly discuss the different tensor structures of the correlation function, here we consider the case of isotropic turbulence. This results in the following form of the correlation function (see, e.g., Fleishman 2006a):

\[
\tilde{K}_{\rho\sigma}(q, \chi) = \frac{1}{2} \left( \delta_{\rho\sigma} - \frac{q_\rho q_\sigma}{q^2} \right) \Psi(|q|, \chi)(B^2) .
\] (22)

Here, the constant factor \( (B^2) \) is introduced that allows \( \Psi \) to meet the normalization condition:

\[
\int \Psi(q, \chi) \frac{d^3q}{(2\pi)^3} \frac{d\chi}{2\pi} = \frac{1}{4\pi^3} \int d\chi \int dq q^2 \Psi(q, \chi) = 1 .
\] (23)

The tensor structure given by Equation (22) obviously satisfies the transversality condition of Equation (21).

The averaged values of \( (a, a_-) \) can be expressed through the correlation function as

\[
\langle a, a_- \rangle = \frac{e^2}{m^2\gamma^2} (\delta_{\rho\sigma} - v_{\rho}v_{\sigma}) K_{\rho\sigma}(\epsilon\beta \tau, \tau) .
\] (24)
Here, we took into account that $r_s - r_c = c\beta \tau$, and replaced $\beta^2$ with 1 in the numerator. From Equations (19) and (22), we find

$$\int_{-\infty}^{\infty} \langle a, a_- \rangle e^{i\omega t} d\tau = \frac{e^2}{2m^2\gamma^2} \langle B^2 \rangle$$

$$\times \int \left(1 + \frac{(vq)^2}{q^2}\right) \Psi(q, \omega) e^{i(q\beta - \omega + \omega_0)q} d^3q \frac{d\lambda}{2\pi^3} d\tau. \quad (25)$$

After substituting the vector $v$ for the velocity $\beta$, and integrating over $d\tau$, we recover a $\delta$-function $2\pi \delta(\mathbf{qv} - \omega + \omega_0)$. Thus, the integral over $d\lambda$ can be computed analytically:

$$\int_{-\infty}^{\infty} \langle a, a_- \rangle e^{i\omega t} d\tau = \frac{e^2}{2m^2\gamma^2} \langle B^2 \rangle$$

$$\times \int \left(1 + \frac{(vq)^2}{q^2}\right) \Psi(q, \omega) d^3q \frac{d\lambda}{2\pi^3}. \quad (26)$$

Thus, we arrive at the following expression for the energy and angular distribution of the radiation per unit time

$$P_{\omega}(t) = \frac{e^4}{2\pi^2m^2c^3} \langle B^2 \rangle (1 + \gamma^2\beta^4)$$

$$\times \int \left(1 + \frac{(vq)^2}{q^2}\right) \Psi(q, \omega) d^3q \frac{d\lambda}{2\pi^3}. \quad (27)$$

where $\omega_0$ is determined by Equation (9).

Let us consider now the case of steady turbulence, i.e., when the correlation function given by Equation (18) is time-independent. Then, the Fourier image of the correlation function contains a $\delta$-function, $\Psi(q, \omega) = \Psi(q) 2\pi \delta(\omega)$, and the normalization condition (23) becomes

$$\int \Psi(q) d^3q = \frac{\langle B^2 \rangle}{2\pi^3} \int_0^\infty \Psi(q) q^2 dq = 1. \quad (28)$$

Note that the function $\Psi$ determines the spectrum of the energy density of the stochastic magnetic field, since

$$\frac{\langle B^2 \rangle}{8\pi} = \frac{\langle B^2 \rangle}{16\pi} \int_0^\infty \Psi(q) q^2 dq. \quad (29)$$

In the case of stationary turbulence, a $\delta$-functional factor, $2\pi \delta(\omega + c\mathbf{qv})$, appears in the integrand of Equation (27). This makes the integration over $d\Omega_q$ (note that $d^3q = q^2 dq d\Omega_q$) rather trivial:

$$P_{\omega}(t) = \frac{e^4}{4\pi^3m^2c^3} \langle B^2 \rangle (1 + \gamma^2\beta^4)$$

$$\times \int \left(1 + \frac{\omega^2}{c^2q^2}\right) \Psi(q) q^2 dq. \quad (30)$$

Now we can conduct an analytical integration over the photons’ directions, i.e., over the solid angle $\Omega$. The major contribution to the integral comes from a range of small angles $\theta \lesssim 1/\gamma$; the contribution from large angles, $\theta \gg 1/\gamma$, is negligibly small. Thus, applying the standard approach for calculations of radiation of ultra-relativistic particles, one can adopt $d\Omega = 2\pi \theta d\theta$ and perform integration over $\theta$ from zero to infinity. It is also convenient to introduce a new integration variable, $\xi = \gamma^2\beta^2$, and change the order of integration over $\xi$ and $q$. After performing a trivial integration over $\xi$, we arrive at:

$$P_{\omega}(t) = \frac{e^4\langle B^2 \rangle}{6\pi^2m^2c^4} \int_0^\infty \Psi(q) q^2 dq \times$$

$$\times \int_{\omega/(2c\gamma^2)}^{\infty} u(\xi) \Psi(q) q^2 d\xi. \quad (31)$$

In the range of integration over $dq$, the variable $\xi$ changes from 1 to $\infty$, while the function $u(\xi)$ increases monotonically from $u(1) = 0$ to $u(\infty) = 1$. Adopting $\xi$ as the integration variable, Equation (31) can be presented in a form convenient for numerical computations:

$$P_{\omega}(t) = \frac{e^4\langle B^2 \rangle}{6\pi^2m^2c^4} \int_1^\infty u(\xi) \left(\frac{\omega\xi}{2c\gamma^2}\right)^2 \Psi\left(\frac{\omega\xi}{2c\gamma^2}\right) \frac{d\xi}{\xi}. \quad (32)$$

Equation (31) is an integral function that depends on the turbulence spectrum. However, it obeys some general properties not affected by the turbulence. In particular, it follows from Equation (31) that independent of $\Psi(q)$, the radiation spectrum $P_{\omega}(t)$ is a monotonically decreasing function of $\omega$. This feature becomes obvious after differentiating over $\omega$:

$$\frac{dP_{\omega}}{d\omega} \sim \int_{\omega/(2c\gamma^2)}^{\infty} \frac{d\xi}{\xi} \frac{du(\xi)}{d\omega} \Psi(q) q^2 d\xi. \quad (34)$$

Here, we take into account the fact that the contribution to the derivative from the lower integration limit is null (given that $u(1) = 0$). Since $du(\xi)/d\omega > 0$, $\Psi(q) \geq 0$, and $d\xi/d\omega < 0$, the integrand is negative, and integration results in $\frac{dP_{\omega}}{d\omega} < 0$. Thus, this function achieves its maximum value at $\omega = 0$, i.e.,

$$P_{\omega}(t) \leq P_0 = \frac{e^4\langle B^2 \rangle}{6\pi^2m^2c^4} \int_0^\infty \Psi(q) q^2 dq. \quad (35)$$

Of course, this estimate is meaningful only if the integral on the right side of the equation converges.

Note that the photon energy and the particle Lorentz factor enter into Equation (31) only in the form of the ratio $\omega/\gamma^2$. Thus, the spectrum $P_{\omega}$ is, in fact, a function of one argument $\omega/\gamma$ (for intermediate calculations we drop, for simplicity, the argument $t$):

$$P_{\omega} \equiv \tilde{P}\left(\frac{\omega}{\gamma}\right) = \tilde{P}\left(\frac{\lambda \omega}{c^2\gamma^2}\right). \quad (36)$$

Here, $\tilde{P}(\omega/\gamma)$ is a monotonically decreasing function. In the absence of other characteristic frequencies in the physical setup except for $\omega_j$, in the range of $\omega \ll \omega_j$ the function $\tilde{P}$ is nearly constant, $\tilde{P} \approx P_0$. However, at very small frequencies, the surrounding plasma may significantly change the behavior
of $P_\omega$. The substitution of $\gamma^*$ for $\gamma$ (in accordance with Equation (16)) leads to Equation (31), in which $\xi$ should be replaced by $\xi_\ast = 2q_c \gamma^2 / \omega$.

The derivative

$$\frac{d\xi_\ast}{d\omega} = -2q_c \gamma^2 \frac{\omega^2 - \gamma^2 \omega_p^2}{(\omega^2 + \gamma^2 \omega_p^2)^2}$$  \hspace{1cm} (37)

has a positive sign when $\omega < \gamma \omega_p$, and becomes negative when $\omega > \gamma \omega_p$. Therefore, independent of the choice of the spectrum of turbulence $\Psi(q)$, the emission intensity increases with frequency in the range of $\omega < \gamma \omega_p$ and decreases when $\omega > \gamma \omega_p$, while the maximum is reached at $\omega = \gamma \omega_p$. Then, instead of Equation (36), we have:

$$P_\omega = \bar{P} \frac{\lambda}{\gamma^2} \left(\omega + \frac{\gamma^2 \omega_p^2}{\omega}\right).$$  \hspace{1cm} (38)

The argument of this function has a minimum at $\omega = \gamma \omega_p$, and consequently the function achieves its maximum at this frequency. However, we should note that in the case of convergence of the integral in Equation (32), this maximum would be practically invisible. To demonstrate the behavior of $P_\omega$ at small frequencies, we show in Figure 2 calculations for three different turbulence spectra $\Psi$ presented in the following specific form

$$\Psi(q) = \frac{A_{\omega_1}}{q^{2\alpha_1} (1 + \lambda^2 q^2)^{1 - \alpha_1}/2}.$$  \hspace{1cm} (39)

Here, according to Equation (28), the normalization constant is:

$$A_{\omega_1} = 4\pi^{3/2} \lambda^{1 - \alpha_1} \frac{\Gamma(1 - \alpha_1/2)}{\Gamma((1 - \alpha_1)/2)},$$  \hspace{1cm} (40)

where $\Gamma(z)$ is the gamma function. The results of calculations in Figure 2 correspond to three different values of $\alpha_1$: $\alpha_1 = -1$, 0, and 1/2. It can be seen that while for $\alpha_1 = 0$ or $1/2$ the integral in Equation (32) diverges and the maximum of $P_\omega$ is clearly seen at $\gamma \omega_p$, for the value of $\alpha_1 = -1$ the emission intensity is characterized by a broad plateau without any distinct maximum.

To explore the emission spectra over the frequency ranges $\gamma \omega_p \ll \omega \ll \omega_j$ and $\omega \gg \omega_p$, let us assume that the turbulence spectrum has a broken power-law form:

$$\Psi(q) = \begin{cases} \lambda^3 \left(\frac{q}{q_1}\right)^{2\alpha_1} & q < \frac{1}{\gamma}, \\ \lambda^3 \left(\frac{q}{q_2}\right)^{2\alpha_2} & q \gg \frac{1}{\gamma}, \end{cases}$$  \hspace{1cm} (41)

where $q_1$ and $q_2$ are constants of the order of $1/\lambda$ and the factor $\lambda^3$ is introduced for dimensional consistency. The condition for the convergence of the integral in Equation (28) implies the following conditions on the upper and lower limits:

$$\alpha_1 < 1, \quad \alpha_2 > 1.$$  \hspace{1cm} (42)

Depending on the value of $\alpha_1$, there are two different cases related to the convergence of the integral in Equation (32). If the integral is converging at the lower limit (i.e., $\alpha_1 < 0$), we have the case discussed above. Let us consider now the range of $0 < \alpha_1 < 1$. Then, for the frequency interval $\gamma \omega_p \ll \omega \ll \omega_j$, we have:

$$P_\omega = \frac{e^{4} \lambda^3 q_1^2 (B_1^2)}{2\pi^2 m^2 c^4} \left(\frac{2c\gamma^2 q_1}{\omega}\right)^{\alpha_1 - 1} \frac{C_1}{\alpha_1},$$  \hspace{1cm} (43)

where

$$C_1 = \frac{4 + 3\alpha_1 + \alpha_2^2}{(3 + \alpha_1)(2 + \alpha_2)^2}. \hspace{1cm} (44)$$

At lower frequencies, $\omega_p \ll \omega \ll \gamma \omega_p$,

$$P_\omega = \frac{e^{4} \lambda^3 q_1^2 (B_1^2)}{6\pi^2 m^2 c^4} \left(\frac{2c\gamma^2 q_2}{\omega}\right)^{\alpha_2} \frac{C_2}{\alpha_2},$$  \hspace{1cm} (45)

where

$$C_2 = \frac{1}{\alpha_2} + \frac{3}{1 + \alpha_2} - \frac{3}{(2 + \alpha_2)^2} - \frac{4}{3 + \alpha_2}. \hspace{1cm} (47)$$

The energy loss rate of a charged particle due to radiation in a magnetic field is given by the classical formula (see, e.g., Landau & Lifshitz 1975):

$$I = \frac{2e^4 \gamma^2}{3m^2 c^3} (\beta \times B)^2 = \frac{2e^4 \beta^2 v^2}{3m^2 c^3} B^2 \sin^2 \chi,$$  \hspace{1cm} (48)

where $\chi$ is the angle between the particle velocity and the direction of the magnetic field. By averaging $I$, first over directions and then over the strength of the magnetic field, and taking into account that $\langle \sin^2 \chi \rangle = (2/3)$, one finds:

$$I = \frac{4e^4 \gamma^2}{9m^2 c^3} \langle B^2 \rangle.$$  \hspace{1cm} (49)
frequencies: $\int_0^\infty P_\omega d\omega = 1$. Nevertheless, it is worthwhile to perform such computations; they serve as a good test for the consistency of the results.

From Equation (33), we find:

$$\int_0^\infty P_\omega d\omega = \frac{e^4 \langle B^2 \rangle}{6\pi^2 m^2 c^4} \int_1^\infty \frac{d\xi}{\xi} u(\xi) \int_0^\infty \left( \frac{\omega \xi}{2cy^2} \right)^2 \Psi \left( \frac{\omega \xi}{2cy^2} \right) d\omega. \tag{50}$$

After the substitution of the new variable $\omega = q \times (2cy^2)/\xi$, integration over $\omega$ leads to Equation (28), and we obtain:

$$\int_0^\infty P_\omega d\omega = \frac{2e^4 y^2 \langle B^2 \rangle}{3m^2 c^3} \int_1^\infty \frac{d\xi}{\xi^2} u(\xi). \tag{51}$$

The remaining integral is equal to $2/3$, so the direct integration of the emission spectrum leads to Equation (49). This interesting result, when the integration of the approximate Equation (33) provides a *precise* expression for energy losses, has a quite natural explanation: while Equation (33) contains the first (quadratic) term in the expansion of the spectrum over the magnetic field strength $B$, the precise expression for the energy losses given by Equation (49) is proportional to the second power of $B$. Also, we note that the energy losses are independent of the spectrum of turbulence $\Psi$.

In a similar way, one can find the angular distribution of the emission after integrating over frequencies; therefore, we simply write down the final result:

$$dI_\theta = \frac{4e^4 y^4 \langle B^2 \rangle}{3\pi^2 m^2 c^3} \frac{1 + y^2 \theta^2}{(1 + y^2 \theta^2)^2} d\Omega. \tag{52}$$

Here, $dI_\theta$ is the energy emitted into a solid angle $d\Omega$ per unit time. This angular distribution also does not depend on the turbulence spectrum $\Psi$.

### 4. LARGE-SCALE TURBULENCE

In the case of a large-scale magnetic field turbulence, $\lambda \gg R_L$, the conditions imposed by Equation (14) are violated; therefore, the results of the previous section are not valid anymore. On the other hand, the radiation spectrum formed in the regime $\lambda \gg R_L$ can be derived analytically. In this case, the particle deflection angle exceeds $1/\gamma$ and the radiation spectrum, $P_\omega$, is determined by the instantaneous or the trajectory (or the instant value of the magnetic field). Thus, the result should be similar to the spectrum of synchrotron radiation in a homogeneous magnetic field (Schwinger 1949; Ginzburg & Syrovatskii 1969; Landau & Lifshitz 1975). If the charged particle moves perpendicularly to the magnetic field, the emission spectrum is determined as:

$$P_\omega(t) = \frac{\sqrt{3} e^2}{2\pi R_L} F(x), \tag{53}$$

where

$$F(x) = x \int_x^\infty K_{5/3}(u) du. \tag{54}$$

Here, $K_{5/3}(u)$ is the modified Bessel function, $x = \omega_0/\omega_c$, and $\omega_c$ is determined by Equation (13). If the charged particle moves at an angle $\chi$ to the magnetic field, in Equation (53) $B$ should be substituted by the perpendicular component of the field, $B_\perp \equiv B \sin \chi$ (Ginzburg & Syrovatskii 1969).

If the magnetic field is turbulent, then the spectrum $P_\omega(t)$ should be averaged over the directions of the field, i.e., integrated over the pitch angle $\chi$. This process results in the following expression (Crusius & Schlickeiser 1986):

$$P_\omega(t) = \frac{\sqrt{3} e^2}{2\pi R_L} G(x), \tag{55}$$

where

$$G(x) = \frac{x}{20} \left( W_{0,5}(x) W_{0,5}(x) - W_{1,5}(x) W_{1,5}(x) \right). \tag{56}$$

Here, $W_{p,q}(x)$ is the Whittaker function.

The function $G(x)$ can be presented in a more convenient form:

$$G(x) = \frac{x}{20} [(8 + 3x^2)(\kappa_{1/3})^2 + x \kappa_{2/3} (2\kappa_{2/3} - 3x\kappa_{2/3})], \tag{57}$$

via the familiar Bessel functions $\kappa_{1/3} = K_{1/3}(x/2)$ and $\kappa_{2/3} = K_{2/3}(x/2)$ (Aharonian et al. 2010). $G(x)$ has a simple asymptotic behavior both at low and high frequencies:

$$G(x) \approx \begin{cases} \frac{\kappa_{1/3}}{2} (\Gamma(1/3))^2 x^{1/3}, & x \ll 1, \\ \frac{\pi}{2} e^{-x}, & x \gg 1. \end{cases} \tag{58}$$

Although differences between the spectra of synchrotron radiation in homogeneous and (large-scale) chaotic fields, i.e., between the functions $F(x)$ and $G(x)$, are not dramatic, they are still not small enough to be neglected in calculations (Aharonian et al. 2010). In particular, these functions achieve their maximum, $\max(F) = 0.918$ and $\max(G) = 0.713$, at different points, $x = 0.286$ and $x = 0.229$, respectively. Obviously, we should expect similar differences for the spectral energy distributions described by the functions $x F(x)$ and $x G(x)$. Namely, $\max(x F) = 0.693$ is achieved at $x = 1.33$ and $\max(x G) = 0.444$ is achieved at $x = 1.15$.

Finally, we note that function $G(x)$ can be approximated by a simple analytical expression,

$$G(x) = \frac{1.808 x^{1/3}}{\sqrt{1 + 3.4 x^{2/3}}} 1 + 2.21 x^{2/3} + 0.347 x^{4/3} e^{-x}, \tag{59}$$

which provides better than 0.2% accuracy (Aharonian et al. 2010). Thus, this approximation can be safely used, instead of the precise Equation (57), in detailed calculations of radiation in environments with large-scale turbulent magnetic fields.

When deriving Equation (55), we assumed that the magnetic field $B$ was oriented chaotically, but that its absolute value, $|B|$, was fixed. However, in a turbulent medium, the spatial variation of the field strength can be quite significant; therefore we have to average the results also over the absolute value of the field (see, e.g., Bykov et al. 2012). Let us introduce the distribution function

$$w(B) dB = h_0(b) dB/B_0, \tag{60}$$

where $B_0 \equiv \sqrt{\langle B^2 \rangle}$ and $b = B/B_0$. By definition, $w(B) dB$ is the probability that the strength of the magnetic field is in the interval $(B, B + dB)$. We will consider three different distributions with the function $h_0(b)$ presented in the following forms:

$$h_0(b) = \delta(b - 1), \tag{61}$$
significantly broadened compared to the radiation spectrum is somewhat shifted toward higher energies, depending on the distribution of the strengths of the magnetic fields. For the variance of these distributions, it is convenient to compare the averaged spectra for the same value of \( \langle B^2 \rangle \):

\[
\langle P \rangle \equiv \int_0^\infty P \omega w(B) dB = \frac{1}{\tau_0} R_b(x)/\omega_0 ,
\]

where \( \omega_0 = 3eB_0\gamma^2/(2mc) \), \( x = \omega/\omega_0 \), and \( \int_0^\infty R_b(x) dx = 1 \).

In Figure 3, we show the SED of synchrotron radiation, \( xR_b(x) \), calculated for the magnetic field distributions given by Equations (61)–(63).

One can see that the spectrum of synchrotron radiation is somewhat shifted toward higher energies, depending on the distribution of the magnetic field. More importantly, it is significantly broadened compared to the radiation spectrum relevant to the \( \delta \)-function distribution of magnetic field \( \eta_0 \). One can show (see Appendix A) that if \( w(B) \) is characterized by a power-law asymptotic dependence: \( w(B) \propto B^{-\sigma} \) for \( B \rightarrow \infty \), then the spectrum of synchrotron radiation also has a power-law asymptotic dependence, namely \( \langle P \rangle \propto \omega^{-\sigma+2} \) for \( \omega \gg \omega_0 \). Note that the power-law index should exceed \( \sigma > 3 \) in order to ensure the convergence of \( \langle B^2 \rangle \).

It is important to emphasize that the broadening and the shifting of the spectrum of synchrotron radiation in a large-scale turbulent field has a quite different origin and should not be confused with the effects related to the jitter regime of radiation in a small-scale turbulent field. In this regard, we should note that the broadening of synchrotron radiation has been “observed” in the numerical simulations of Teraki & Takahara (2011). However, the authors most likely misinterpreted the obtained spectral feature and relegated it to an intermediate regime between the synchrotron and jitter regimes. However, we believe that this component of radiation revealed in their simulations has a standard synchrotron origin, but may simply be broadened because of the distribution of the strengths of the magnetic fields.

5. ENERGY SPECTRA OF RADIATION IN THE JITTER AND SYNCHROTRON REGIMES

In this section, we compare spectra of synchrotron and jitter radiation produced in two different large- and small-scale turbulent magnetic fields but with the same value \( \langle B^2 \rangle \); thus, the total radiation power given by Equation (49) is the same for the radiation in both regimes. For comparison, it is convenient to introduce the normalized emission intensity:

\[
R(x) dx = P_\omega d\omega/\tau_0 , \quad x = \omega/\omega_0 .
\]

Obviously, the following condition holds: \( \int_0^\infty R(x) dx = 1 \). For calculations, we have to assume a turbulence spectrum and a distribution of the magnetic field strengths. For the sake of simplicity, we consider below the case of chaotic synchrotron emission, i.e., we adopt a field distribution corresponding to Equation (61). Then, the function \( R \) depends only on the magnetic field:

\[
R(x) = \frac{27\sqrt{3}}{16\pi} \frac{G(x)}{x}. \tag{68}
\]

We note, however, that \( R \) as a function of the argument \( x \) does not depend on \( \langle B^2 \rangle \). In this section, for calculations of jitter radiation, we consider the turbulence spectrum to be a one-parameter family of functions:

\[
\Psi(q) = \frac{1}{(1 + \lambda^2 q^2)^{\sigma/2}}. \tag{69}
\]

The normalization constant, \( A_\sigma \), is obtained from Equation (28):

\[
A_\sigma = \frac{8\pi^{3/2} \Gamma(1+\alpha/2)}{\Gamma((\alpha-1)/2)}. \tag{70}
\]

The spectrum presented in the form of Equation (69) is characterized by a power-law dependence for \( q \gg \lambda^{-1} \). Although the spectra of turbulence, which can be generated in astrophysical environments, remain an open question, they are usually approximated as power laws. This assumption is justified by a few fundamental considerations. In particular, power-law spectra of turbulence with spectral indices of 5/3 and 3/2 appear in hydrodynamical (Kolmogorov 1941) and magnetohydrodynamical (Iroshnikov 1963; Kraichnan 1965) turbulent media.

Note that the asymptotic form of Equation (69) is consistent with Equation (41) for \( \alpha_1 = -2 \) and \( \alpha_2 = \alpha \). In Figure 4, we show the normalized spectral energy distributions of the synchrotron and jitter radiation, \( xR(x) \) produced by particles of fixed energy \( ymc^2 \). The spectra are plotted as a function of \( x = \epsilon/\epsilon_c = h\omega/h\omega_c \), for three different indices characterizing the turbulence, \( \alpha = 2, 5/3, \) and \( 3/2 \).

For a rather broad range of values, from 3/2 to 3, the presentation of the turbulence spectrum in the form of Equation (69)
allows for simple analytical approximations for the radiation power:

$$P_\omega d\omega = I f(x_j) d\omega/\omega_j ,$$  \hspace{1cm} (71)

where $x_j = \omega_j/\omega$, and

$$f(x_j) = C_\alpha (1 + 0.22 x_j + 0.43 x_j^3)^{-\alpha/2} .$$  \hspace{1cm} (72)

The coefficient $C_\alpha$ is determined from the normalization $\int_0^\infty f(x_j) dx_j = 1$. Comparisons with exact numerical calculations show that the precision of this approximation is better than 7%.

Figure 4 shows the basic spectral features of the jitter radiation. The SED peaks at an energy comparable to the maximum energy of synchrotron radiation at $\epsilon = 1.155 \epsilon_c$, although it is shifted by a factor of $2/3 R_L/\lambda$. Below the maximum, $x R(x) \propto x$, i.e., the SED increases with energy slower than the SED of synchrotron radiation ($x R(x) \propto x^{4/3}$). Moreover, while the synchrotron spectrum has a quite sharp (exponential) cutoff beyond $x \sim 1$, the SED of jitter radiation after the break at $x \sim R_L/\lambda$ continues as a power law, $x R(x) \propto x^{1-\alpha}$ up to $x \sim (R_L/\lambda)^{3/2}$.

In astrophysical environments, acceleration of particles typically leads to broad energy distributions. Below, we compare the synchrotron and jitter radiation for different distributions of accelerated particles $N(\gamma)$:

$$P(\omega) = \int_0^\infty P_\omega N(\gamma) d\gamma .$$  \hspace{1cm} (73)

Here, we assume that the energy distribution of all particles occupies a certain energy interval $(\gamma_{\text{min}}, \gamma_{\text{max}})$. Outside this interval, the function $N$ is null.\footnote{We would like to indicate the non-physical lower limit in the integral in Equation (73). However, this form convenient for the integration representation, is correct as long as the function $N$ is taken to be zero outside of the physically meaningful region.} For the jitter radiation, using Equation (33) and introducing a new dimensionless function $\Psi_\gamma(\lambda q) = \Psi(q)/\lambda^3$, as well as substituting the integration variable $\gamma$ for $\eta = \lambda \omega \xi/(2c\gamma^2)$, we obtain:

$$P(\omega) = \frac{e^4 \lambda (B^2)}{12 \pi^2 m_c c^4} \int_1^\infty d\xi u(\xi) \left( \frac{\lambda \omega \xi}{2c\xi} \right)^{-(\mu+1)/2} \int_0^\infty d\eta \Psi_\gamma(\eta) \eta^{(\mu+1)/2} .$$  \hspace{1cm} (74)

Let us now assume that the relativistic charged particles have a power-law distribution, $N(\gamma) = N_0 \gamma^{-\mu}$. It can be shown that for the range of the power-law index, $1 < \mu < 2\alpha + 1$, particles of energy $\gamma \sim (\lambda \omega/c)^{1/2}$ are the main contributors to Equation (73). Therefore, for the energy interval $\omega \gg c/\lambda$, Equation (74) can be integrated over $d\eta$ over the limits from 0 to $\infty$:

$$P(\omega) = \frac{e^4 \lambda (B^2)}{12 \pi^2 m_c c^4} \left( \frac{2 \omega}{\lambda \omega} \right)^{(\mu-1)/2} \int_1^\infty d\xi u(\xi) \xi^{-(\mu+1)/2} \int_0^\infty d\eta \Psi_\gamma(\eta) \eta^{(\mu+1)/2} .$$  \hspace{1cm} (75)

The reason for the power-law dependence of the spectra $P(\omega) \sim \omega^{-(\mu-1)/2}$ is the same as in the case of the synchrotron radiation: $\omega$ and $\gamma$ enter into $P_\omega$ in a combined form $\omega/\gamma^2$ (for a discussion of the case of synchrotron radiation, see Rybicki & Lightman 1979). Thus, for a power-law particle distribution, the synchrotron and jitter mechanisms lead to the same type of energy spectra. Therefore, the ratio of the emission intensities due to these two processes:

$$r = \frac{P_{\text{synchr}}(\omega)}{P_{\text{jitter}}(\omega)} = C(\mu, \alpha) \left( \frac{R_L}{\lambda} \right)^{(\mu-3)/2} ,$$  \hspace{1cm} (76)

does not depend on the photon energy $\omega$. Interestingly, the index of $\mu = 3$ appears to be special, in the sense that independent of the turbulence spectrum, the result $r = 1$. This result, in particular, can be seen at low energies in Figure 5. Note that although the energy losses due to the synchrotron and jitter mechanisms in the large and small turbulent fields are equal (for the same mean magnetic field), formally for $\mu > 3$ a larger energy is radiated due to the jitter mechanism ($r > 1$), and vice versa, $r < 1$ for $\mu < 3$. This apparent inconsistency is related to the assumption of pure power-law particle distribution. However, for a realistic distribution of particles with a high-energy cutoff, the spectral shape of the synchrotron and jitter radiation differ significantly. In particular, for power-law distributions with an exponential cutoff, given in the rather general form

$$N \sim \gamma^{-\mu} \exp\left(-\gamma/\gamma_{\text{cut}}\right)^\beta ,$$  \hspace{1cm} (77)

in the high-energy limit, the shapes of the synchrotron and jitter radiations spectra differ significantly (see Figure 5). While the synchrotron component decreases exponentially beyond the maximum (Lefa et al. 2012; also see Fritz 1989; Zirakashvili & Aharonian 2007),

$$P(\omega) \propto \exp\left[ -\frac{\beta + 2}{2} \left( \frac{2 \omega}{\omega_{\text{cut}}} \right)^{\beta/(\beta+2)} \right] ,$$  \hspace{1cm} (78)
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6. ANISOTROPIC TURBULENCE

If the distribution of the charged particles is isotropic, the analytical solutions derived in the previous sections can be generalized to the case of a correlation tensor with an arbitrary angular structure. Indeed, similar to Equation (27), the radiation power can be expressed as:

$$ P_{\text{iso}}(\omega) = \frac{e^4}{\pi^2 m^2 c^3} \frac{\gamma^2}{(1 + \gamma^2 \theta^2)^4} \left( \delta_{\rho \sigma} - v_{\rho}v_{\sigma} \right) \int \hat{K}_{\rho \sigma}(q, \chi) \frac{d^3 q}{(2\pi)^3} \frac{d\chi}{2\pi}, $$

with

$$ \omega_{\text{cut}} = \frac{3eB}{2mc} \frac{\gamma^2}{\gamma_{\text{cut}}}, \tag{79} $$

the jitter emission spectrum beyond the break around \( \omega_{\text{cut}}(R_L/\lambda) \) has a long power-law tail, \( P(\omega) \propto \omega^{-\beta} \), independent of the shape of the particle distribution in the cutoff region (i.e., the value of \( \beta \)). In this regard, this is a unique feature of the jitter radiation that provides direct and model-independent information about the spectrum of turbulence. As long as the condition of small-scale turbulence is satisfied \((\lambda < R_L)\), we should expect radiation with a characteristic broken power law-type spectrum. While the photon index at low energies is directly related to the spectral index of the relativistic particles, \( \Gamma = (\mu + 1)/2 \), or, in the case of a low-energy cutoff or a very hard particle spectrum below the cutoff energy (e.g., in the case of a Maxwellian type distribution; see Figure 6), \( \Gamma = 1 \), the spectrum after the break depends only on the spectrum of turbulence. In an environment with large-scale turbulence, the picture is just the opposite. The radiation proceeds in the synchrotron regime and therefore is not sensitive to the details of the turbulence. On the other hand, the synchrotron radiation carries information about the overall spectrum of parent particles, including the most important (from the point of view of the acceleration theory) cutoff region.

In Figure 6, we show a comparison of the synchrotron and jitter radiation for the case of Maxwellian distribution of emitting particles.

To obtain the radiation spectrum, this equation should be integrated over the photon emitting angles and averaged over the directions of the velocities of the emitting particles, \( v \equiv \beta/|\beta| \).

In order to simplify calculations, let us introduce the following intermediary tensor:

$$ T_{\rho \sigma} = \frac{1}{2} \int (\delta_{\rho \sigma} - v_{\rho}v_{\sigma}) \gamma^2 (1 + \gamma^4 \theta^4) \times \delta(\bar{\omega} + cqv - \chi) d\Omega d\Omega_{q}, \tag{81} $$

where \( d\Omega \) and \( d\Omega_{q} \) are the solid angles related to the directions of the momenta of the emitted photon and the emitting particle, respectively. Note that Equation (81) contains all the “directional” terms. Thus, the radiation spectral power can be expressed as:

$$ P_{\omega}(t) = \frac{e^4}{\pi^2 m^2 c^3} \int T_{\rho \sigma} \hat{K}_{\rho \sigma}(q, \chi) \frac{d^3 q}{(2\pi)^3} \frac{d\chi}{2\pi}. \tag{82} $$

The correlation tensor \( K_{\rho \sigma} \) and its Fourier transformation \( \hat{K}_{\rho \sigma} \) are defined in Section 3.

According to Equation (81), the tensor \( T_{\rho \sigma} \) is a symmetric tensor of the second order; it depends only on the vector \( q \).

Therefore, \( T_{\rho \sigma} \) has the following structure:

$$ T_{\rho \sigma} = F_1 \delta_{\rho \sigma} + F_2 q_{\rho} q_{\sigma}, \tag{83} $$

where \( F_1 \) and \( F_2 \) are scalar functions. The convolution of tensors \( T_{\rho \sigma} \) and \( \hat{K}_{\rho \sigma} \), taking into account the transversality condition of Equation (21), gives

$$ T_{\rho \sigma} \hat{K}_{\rho \sigma} = F_1 \hat{K}_{pp}. \tag{84} $$

This expression determines the integrand in Equation (82).

To obtain the scalar function \( F_1 \), the following relations can be used:

$$ T_{pp} = 3F_1 + q^2 F_2, \quad q_{\rho} q_{\sigma} T_{\rho \sigma} = q^2 F_1 + q^4 F_2, \tag{85} $$

which give

$$ F_1 = \frac{1}{2} (T_{pp} - q_{\rho} q_{\sigma} T_{\rho \sigma}/q^2). \tag{86} $$
Using Equation (81), one finds
\[ F_1 = \frac{1}{4} \int \left( 1 + \frac{q^2}{q^2} \right) y^2 \left( 1 + y^2 \right)^4 \delta (\omega + c q v - \xi) d\Omega d\Omega_\nu, \]  
which, after integration, can be presented in the form (see Appendix B):
\[ F_1 = \frac{\pi^2}{3 q c} U (\xi, \kappa/(q c)) . \]  
Here, \( U \) is determined by Equation (B5).

The trace of the correlation tensor can be represented as
\[ \Psi_\rho(q, \kappa) = \langle B^2 \rangle \Psi(q, \kappa) , \]  
where \( \Psi(q, \kappa) \) satisfies the normalization condition:
\[ \int \Psi(q, \kappa) \frac{d^3 q}{(2\pi)^3} \frac{d\kappa}{2\pi} = 1 . \]  
Thus, the radiation power can be represented as
\[ P_\omega(t) = \frac{\epsilon^4 B^2}{3 m^2 c^2} \int \frac{1}{q} U(\xi, \kappa/(q c)) \Psi(q, \kappa) \frac{d^3 q}{(2\pi)^3} \frac{d\kappa}{2\pi} . \]  

If the correlation function \( K_\rho(q, \kappa) \) does not depend on time, i.e., \( \Psi(q, \kappa) = 2\pi \delta(\kappa) \Psi(q) \), then the integration over \( \kappa \) is trivial, leading to
\[ P_\omega(t) = \frac{\epsilon^4 B^2}{24\pi^3 m^3 c^4} \int u(\xi) \Psi(q) \Theta(\xi - 1) \frac{d^3 q}{q} . \]  

In the derivation of this equation, we took into account the fact that
\[ U(\xi, 0) = u(\xi) \Theta(\xi - 1) . \]  
Here, \( u(\xi) \) is defined by Equation (32) and \( \Theta(\xi) \) is the Heaviside step function (i.e., \( \Theta(\xi) = 1 \) if \( \xi > 0 \) and \( \Theta(\xi) = 0 \) if \( \xi < 0 \)).

Obviously, in the case of isotropic turbulence, the general Equation (92) should be equal to with Equation (31). Moreover, Equation (31) can describe even the case of anisotropic turbulence, if one substitutes the function \( \Psi(q) \) with the spectrum of turbulence averaged over the directions of the vector \( q \). \( \Psi(q) \equiv \langle \Psi(q) \rangle \). This implies that the averaged radiation power does not depend on the structure of the correlation tensor. In particular, the monotonic decrease of the intensity given by Equation (35), is also observed in the case of anisotropic turbulence.

Although this conclusion was derived under the assumption of an isotropic distribution of emitting particles, in fact the obtained result is valid also for the case of an anisotropic particle distribution when the change in the particle density is small for typical “angular shifts” of the value of \( 1/\gamma \). Thus, as long as the emission is considered to be in the ultrarelativistic regime, this assumption can be broken only in the case of highly collimated particle beams (which are likely inconsistent with the underlying assumption of the turbulent magnetic field). Also, we note that if we deal with a strongly anisotropic distribution of particles, the radiation does depend on the structure of the correlation tensor; therefore, it is important to define the condition tensor correctly (see Section 7).

7. COMPARISONS WITH PREVIOUS RESULTS

In recent years, a large number of studies have been devoted to calculations of radiation (the magnetic Bremsstrahlung) generated by charged particles in small-scale turbulent magnetic fields. However, to our knowledge, the general expression for the radiation spectrum described by Equation (31) is derived for the first time in this paper. Also, in previous studies a few additional conditions were assumed, which now, however, appear redundant and may actually not be needed at all in the framework of our approach. This redundancy not only superficially confines the applicability of the obtained results, but also introduces some confusion in the analysis and comparison of different radiation regimes. Finally, some solutions and related conclusions derived in this paper do not agree with the results of previous studies. Therefore, we present below a short overview of a few important papers on the topic, compare their main results with our study, and outline the key differences between the approaches that might cause, in our view, these discrepancies.

There are two basic theoretical approaches to studying radiation in random magnetic fields. The first is based on the seminal paper by TF87, where a kinetic equation was derived for the probability of different particle trajectories in a chaotic magnetic field (see Equation (12) of TF87) and an approximate solution was found to this equation (see also Fleishman 2006a for a simplified description of the approach of TF87). However, the introduced simplifications significantly limit the applicability of this approach and do not allow a self-consistent treatment of the problem. More specifically, we discuss these issues below.

The second approach is based on perturbation theory (M00; Medvedev 2006; Fleishman 2006b). In all these papers, the authors start from an expression for the emission produced by a particle deflected by a small angle in a magnetic field localized in a compact region of space (see Landau & Lifshitz 1975, Section 7). This expression can be written as:
\[ \frac{d\mathcal{E}_\omega}{d\omega} = \frac{e^2 \omega}{2\pi^2 c} \int_{\omega/(2\gamma^2)}^{\infty} \frac{|\mathbf{a}_\omega|^2}{\omega^2} \left( 1 - \frac{\omega}{\omega'_y} + \frac{\omega^2}{2\omega'_y^2} \right) d\omega' , \]  
where \( \mathbf{a}_\omega = \int_{-\infty}^{\infty} \mathbf{a}(t) e^{i\omega t} dt \) is the Fourier component of the acceleration. However, if the magnetic field occupies a large volume, then even in the case of a chaotic magnetic field, the particle deflection will be (unavoidably) large (because of multiple scatterings over the emission correlation length). Therefore, the solution based on this expression has a rather limited applicability compared to the practical realizations of the chaotic magnetic field.

The approached employed in our study is also based on perturbation theory, but it is valid when the particle deflection is small compared with the typical magnetic field correlation length, or, equivalently, if \( R_L \gg \lambda \).

Note that the latter approximation was also implicitly used by TF87 (see Equation (11) of TF87) when deriving the kinetic equation. So, even in the case of a precise solution of this equation, the results cannot be expanded beyond the parameter region described by the perturbation theory approach presented in our paper. Moreover, since the derived kinetic equation appeared to be too complex to be treated analytically, a few further simplifications have been introduced to obtain an analytical solution. In particular, the original Equation (12) of TF87 was replaced by Equation (17) of TF87, which indeed could be equivalent to the original one if in the right-hand side of
this equation the function \( q(\omega, \theta) \) determined by Equation (15) of TF87 was used. In Equation (17) of TF87, \( \omega \) enters as a parameter. Therefore, in solving this equation, \( \omega \) can be taken to be a constant, and the function \( q \) can be treated as a function of one variable \( \theta(q) \). However, since in this case the equation does not have a solution, the authors replaced the function \( q(\omega, \theta) \) by an empirical function \( q(\omega) \). This simplification allows for an analytical solution, but since it concerns the term with the highest derivative in the equation, the uncertainties imposed by this substitution cannot be evaluated and, correspondingly, the limits of applicability remain highly unknown. Note that the empirical function \( q(\omega) \) itself determines the radiation spectrum in the absence of the regular component of the magnetic field.

However, within the framework of TF87, this function, strictly speaking, is not derived. Instead, based on arguments of the asymptotic behavior, these authors proposed the form \( q(\omega) = q(\omega, \theta = \theta_*), \) where \( \theta_*^2 = (a - 1)/\gamma^2 \) and the value of the parameter \( a \) were determined “from the requirement that at high frequencies, where the perturbation expansion (the method of equivalent photons, see Appendix) is valid, the present method yields the same result as the perturbation expansion” (TF87, p. 218).

Let us now consider the results of Fleishman & Bietenholz (2007), in which the approach of TF87 was applied to the case of the random magnetic field without a regular component. To make the comparison transparent and less bulky, we discuss the results for a fixed value of the index of the turbulence spectrum \( \alpha = 2 \) and ignore the impact of the surrounding medium (i.e., we assume that \( \alpha_p = 0 \)). For this specific case, the spectrum obtained in Fleishman & Bietenholz (2007) can be expressed as:

\[
P_\omega = \frac{8e^2\gamma^2}{3\pi c} q(\omega) \Phi(s). \tag{95}
\]

Here

\[
q(\omega) = \frac{\omega^2_s \omega_0}{(a\omega/2)^2 + (\omega_0^2 \gamma^2)^2}, \tag{96}
\]

\[
\Phi(s) = \frac{24s^2}{\pi} \int_0^\infty \frac{dt}{t} e^{-2st} \sin(2st) \left( \coth t - \frac{1}{t} \right), \tag{97}
\]

where

\[
s = \frac{1}{8\gamma^2} \left( \frac{\omega}{q(\omega)} \right)^{1/2}, \quad \omega_s = \frac{c}{R_L}, \quad \omega_0 = \frac{c}{\lambda}. \tag{98}
\]

\( \Phi(s) \) has the following asymptotic limits (Fleishman & Bietenholz 2007):

\[
\Phi(s) \approx 1, \quad \text{if } s \gg 1; \quad \Phi(s) \approx 6s, \quad \text{if } s \ll 1. \tag{99}
\]

At \( \omega \sim \omega_1 = \omega_0 \gamma^2 \), the \( s \) parameter is large: \( s \sim \omega_0 / \omega_s = R_L / \lambda \gg 1 \). Thus, one can use the asymptotic limit for \( \Phi(s) = 1 \). Then, Equation (95) can be expressed in a simple form:

\[
P_\omega = \frac{8e^2\gamma^2}{3\pi c} q(\omega). \tag{100}
\]

Apparently, the function \( q(\omega) \) determines the shape of the radiation spectrum. However, this function has not been derived either by Fleishman & Bietenholz (2007) or by Fleishman (2006a). We can only guess that the authors used the simplified form of Equation (39) of TF87 (after the removal of the bulky complex term from that equation).

Fleishman & Bietenholz (2007) performed numerical calculations of the radiation spectra also for the case of a strong, random magnetic field, i.e., in the regime \( \lambda \gg R_L \). In the asymptotic limit of \( \lambda \gg R_L \), the spectrum can be obtained analytically; in this regime, we deal with the standard synchrotron spectrum described by Equation (55). On the other hand, Equation (99) with the asymptotic limit of \( \Psi(s) \) for \( s \ll 1 \) from Equation (99) differs significantly from Equation (55). In our view, the reason for this discrepancy is that the basic kinetic equation in the theory of TF87 is derived under the assumption of \( \lambda \ll R_L \) (see Equation (9) of TF87). Thus, this approach cannot be applied to the regime \( \lambda \gg R_L \).

The fact that Equation (95) is not applicable to the case of \( \lambda \gg R_L \) can be also illustrated by the computation of the total power emitted by a particle. Let us consider the ratio of the radiated and lost energies by the relativistic charged particle:

\[
\rho = \int_0^\infty P_\omega(d\omega) / I. \tag{101}
\]

Here, \( I \) and \( P_\omega \) are determined by Equations (49) and (95), respectively. For a particle emitting in a vacuum, the condition \( \rho = 1 \) should be satisfied. In the asymptotic case of \( \lambda \ll R_L \), one can use Equation (100) and demonstrate that for \( a = 2 \) we indeed have \( \rho = 1 \). However, in the limit of \( \lambda \gg R_L \),

\[
\rho = 5.6 \times p^{2/3}, \tag{102}
\]

i.e., the condition \( \rho = 1 \) is violated.

Thus, we can conclude that in the case when the non-chaotic magnetic field is zero, the approach developed by TF87 has a very limited applicability. Namely, one can derive the spectrum in the form of Equation (100) with a function \( q(\omega) \) constrained by its asymptotic behavior only.

Now let us compare our results with the studies based, like our work, on perturbation theory.

In M00, a specific geometry of interaction was postulated. Namely, it was assumed that the particle moves along the axis \( x \) and that the magnetic field has only a \( y \) component, \( B_y \). Therefore, the acceleration is parallel to the \( z \)-axis: \( a(t) = e/(m\gamma^2) B_y vt, 0, 0 \equiv (e/m\gamma^2) B(t) \). Although the radiation power obtained in Section 3 was derived under the assumption of turbulence homogeneity, and thus is not applicable to the case considered by M00, it is straightforward to apply our approach to this case as well. Namely, accepting the definition of the spectral power given by Equation (4), one can average over the magnetic field configurations in Equation (94). These calculations give:

\[
P_\omega(t) = \frac{e^4}{2\pi m^2 \gamma^2 c^3} \int_0^\infty \frac{d\omega'}{\omega'^2} \left( 1 - \frac{\omega}{\omega' \gamma^2} + \frac{\omega^2}{2\omega'^2 \gamma^4} \right) d\omega'. \tag{104}
\]

where

\[
(B^2)_{\omega} = \int_{-\infty}^{\infty} (B(t + \tau/2)) B(t - \tau/2) e^{i\omega \tau} d\tau. \tag{105}
\]
is the Fourier component of the magnetic field correlation function; generally this component may depend not only on $\omega$, but also on $t$. Obviously,

$$\langle B^2(t) \rangle = \int_0^\infty \frac{d\omega}{\pi} \tilde{B}^2(\omega).$$

Equation (104) describes the radiation power for the geometry postulated for an arbitrary spectrum of turbulence. A specific spectrum of turbulence was considered by M00 for the derivation of their Equation (17). The interesting feature of the latter is that in the limit of $\omega \to 0$, the spectrum $P_\omega \sim \omega$ contains an abrupt cutoff, $P_\omega = 0$ at $\omega > 2\omega_0$. However, we should note that these spectral features do not in general characterize the jitter radiation (as it can be seen from Equation (104)), but simply are the consequence of the choice of a specific turbulence spectrum and/or interaction geometry (see also Fleishman 2006b). Indeed, if one adopts a different turbulence spectrum, e.g., $(\tilde{B}^2(\omega) = (\omega^2 + \omega_0^2)^{-\alpha}$, then for any positive value of the index $\alpha$, the spectrum is a monotonically decreasing function of $\omega$. Moreover, if the stochastic field has both $y$- and $z$-components, and the correlation function is azimuthally symmetric with respect to the $x$-axis, then even for the power-law spectrum of turbulence, adopted by M00, the spectrum is not expected to be linear in the limit of small $\omega$.

The treatment of radiation in a chaotic magnetic field always leads to the appearance of the correlation tensor, $\tilde{K}_{\rho\sigma}$ (see Equation (19) of this paper and Equation (12) of Fleishman 2006b). However, often the structure of this tensor, $\tilde{K}_{\rho\sigma}(q, \lambda)$, is wrongly postulated. If we consider a homogeneous environment without preferred directions, then we deal with only two second-order tensors: $\delta_{\rho\sigma}$ and $q_\rho q_\sigma$. Therefore, the correlation tensor should have the following structure: $\tilde{K}_{\rho\sigma} = c_1 \delta_{\rho\sigma} + c_2 q_\rho q_\sigma$, where $c_1$ and $c_2$ are two scalar functions. The transversality condition implies $c_1 + c_2 q^2 = 0$, thus $\tilde{K}_{\rho\sigma}$ has to be proportional to $(\delta_{\rho\sigma} - q_\rho q_\sigma / q^2)$, as used in Equation (22). However, in the case of the existence of a distinct direction, $s$, e.g., normal to the shock front, the tensor structure becomes more complex:

$$\tilde{K}_{\rho\sigma} = c_1 \delta_{\rho\sigma} + c_2 q_\rho q_\sigma + c_3 s_\rho s_\sigma + c_4 q_\rho s_\sigma + c_5 s_\rho q_\sigma.$$

The transversality condition imposes three constraints on the scalar functions $c_i$, thus the correlation tensor $\tilde{K}_{\rho\sigma}$ must have the following structure:

$$\tilde{K}_{\rho\sigma} = \Psi_1 \left( \delta_{\rho\sigma} - \frac{q_\rho q_\sigma}{q^2} \right) + \Psi_2 \left( s_\rho - q_\rho \frac{(s q)^2}{q^2} \right) \left( s_\sigma - q_\sigma \frac{(s q)^2}{q^2} \right).$$

where the functions $\Psi_{1,2}$ depend on three arguments: $|q|$, $(sq)$, and $\lambda$. In a gyrotropic medium, the correlation function may contain an additional term: $\epsilon_{\rho\sigma\tau} q_\tau \Psi_3$, where $\epsilon_{\rho\sigma\tau}$ is the Levi-Civita tensor and $\Psi_3$ is a complex function following from the general theory of fluctuations (see, e.g., Landau & Lifshitz 1980, Section 122). Note, however, that this term does not contribute to the emission power, since in Equation (24) the tensor $\tilde{K}_{\rho\sigma}$ is convolved with a symmetric tensor.

For the additional assumption that the magnetic field is perpendicular to the direction $s$, the correlation function should satisfy the equations $\tilde{K}_{\rho\sigma} s_\rho = 0$ and $\tilde{K}_{\rho\sigma} s_\sigma = 0$. In this case, the functions $\Psi_1$ are linked to the functions $\Psi_2$ via the relation

$$\Psi_1 + (1 - (sq)^2 / q^2) \Psi_2 = 0,$$

and the correlation function $\tilde{K}_{\rho\sigma}$ is determined just by one scalar function.

However, in some studies dealing with anisotropic turbulence, different tensor structures have been proposed for the correlation function: $\tilde{K}_{\rho\sigma} \propto (\delta_{\rho\sigma} - s_\rho s_\sigma)$—we can refer, for example, to Equation (8) in Medvedev (2006), Equation (18) in Fleishman (2006b), Equation (10) in Medvedev et al. (2011), Equation (11) in Reynolds & Medvedev (2012). This correlation function does not satisfy the transversality condition, i.e., the considered magnetic field is not divergence free, $\nabla B \neq 0$. Apparently, this is a wrong result, therefore the results obtained under the assumption of $\tilde{K}_{\rho\sigma} \propto (\delta_{\rho\sigma} - s_\rho s_\sigma)$ should be revised.

It is important to note that certain mathematical operation often used for computation of emission in a chaotic magnetic field lack mathematical strictness (also see the discussions in Fleishman 2006b; Medvedev 2005). In particular, this concerns the involvement of the field harmonics, $B_\lambda$, which imply that the Fourier transformation can be applied to the stochastic magnetic field. This assumption hardly can be justified or disproved from first principles, however, this assumption may lead to a rather controversial expression for the Fourier image of the correlation function. For example, the following structure has been obtained for the correlation tensor: $\tilde{K}_{\rho\sigma} \propto \tilde{B}_\rho \tilde{B}_\sigma$ (see footnote 2 and Equation (5) in Fleishman 2006b; Medvedev 2006, respectively), which, however, contradicts the general tensor structure given by Equation (108) (see also Landau & Lifshitz 1980, Section 122).

Finally, we note that in the framework of our approach, no assumptions regarding the properties of the stochastic magnetic field are required. Instead, we assumed that the Fourier transformation can be applied to the magnetic field correlation function, which is a significantly less demanding assumption.

8. DISCUSSION AND SUMMARY

The so-called jitter radiation mechanism represents a version of magnetic Bremsstrahlung of relativistically charged particles in a turbulent magnetic field. This regime of radiation can be realized with an efficiency as high as the “standard” synchrotron radiation but with quite a distinct energy spectrum strongly shifted toward higher energies. This result makes the jitter radiation an attractive gamma-ray production channel in highly turbulent astrophysical environments.

In this paper, we present a novel study of the spectral properties of the jitter radiation performed within the framework of perturbation theory in the regime of so-called small-scale turbulence, when the coherence length of the field is significantly smaller than the nonrelativistic Larmor radius, $\lambda \ll R_L = m c^2 / eB$, or

$$\lambda \ll 1.7 \times 10^3 (m / m_e) (B / 1 \text{ G})^{-1} \text{ cm}.$$

Here, $B$ is the average magnetic field and $m$ is the mass of the radiating charged particle. It is remarkable that the condition for the realization of the jitter regime does not depend on the particle energy but only on its mass. For example, for electrons, the condition imposed by Equation (110) implies a turbulence
scale as small as 100 km in young supernova remnants, less than 10 m in the gamma-ray production regions of blazars, and 1 cm in gamma-ray bursts (GRBs), assuming typical values of the magnetic field in these objects of about 100 μG, 1 G, and 1 kG, respectively. For protons, these conditions are relaxed by three orders of magnitude. However, the magnetic Bremsstrahlung of protons is a much slower process compared to that of electrons. It becomes adequately effective only at very high proton energies and in the presence of large magnetic fields. These conditions in turn imply tight conditions on the turbulence scale.

Whether turbulent fields can be generated on scales imposed by Equation (110) is a non-trivial issue, the discussion of which is beyond the scope of this paper. Here, we focused merely on the study of radiation properties and performed calculations under the assumption that Equation (110) is (by definition) fulfilled. We derived an expression for the spectral power of radiation presented in a general but rather compact form convenient for numerical calculations.

The jitter radiation has a simple spectral form. Its SED for a monoenergetic particle distribution is shown in Figure 4 together with the SED of synchrotron radiation. Both SEDs have pronounced maximums separated from each other by a factor of \(R_L/\lambda\). Obviously, when the jitter regime is realized, the maximum of its SED is shifted toward higher energies (the position of the peak in the synchrotron SED is at an energy \(\approx 1.15\mu \omega_c\)). Unless one introduces strong assumptions regarding the turbulence spectrum and/or its geometry, the low-energy part of the spectrum has a standard photon index \(\Gamma = 1\). It is hard but still softer than the spectrum of synchrotron radiation, which has \(\Gamma = 4/3\). The jitter and synchrotron spectra are very different beyond their respective maximums. While the standard synchrotron spectrum cuts off quite sharply (exponentially) just after the maximum, the spectrum of the jitter radiation continues as a power law until the energy \(\sim (R_L/\lambda)^2\omega_c\) with a photon index \(\Gamma = \alpha + 1\), where \(\alpha\) is the power-law index of the turbulence spectrum (in the framework of perturbation theory, the spectral shape of radiation above this limit cannot be calculated). Remarkably, this part of the spectrum is not sensitive to the details of the energy distribution of the particles, but depends only on the position of the cutoff in the particle distribution. The latter determines the start of the power-law tail that should be (by definition) quite long since \(R_L \gg \lambda\). For example, if the ratio \(R_L/\lambda\) exceeds 10, the power-law tail of the jitter radiation, which mimics the turbulence spectrum, would span over more than two energy decades after the maximum. Below the maximum, the jitter radiation does depend on the particle distribution. In particular, if the relativistic particles have a power-law distribution with an index \(\mu\), the spectrum of the jitter radiation is also a power law with a photon index \(\Gamma = (\mu + 1)/2\); i.e., exactly the same as in the case of synchrotron radiation.

In this paper, we do not aim to discuss the astrophysical implications of jitter radiation; these deserve separate consideration. Instead, we rather comment on some interesting features that make this mechanism quite unique amongst other high-energy radiation processes.

First of all, the slight dependence (or rather independence) of the high-energy spectral tail on the distribution of parent relativistic particles is quite unusual and apparently does not have an analog in astrophysics. Moreover, the fact that the spectral shape of this tail simply mimics the spectrum of turbulence opens a unique opportunity for a straightforward probe of the spectrum of small-scale turbulence by measuring the characteristic high-energy electromagnetic radiation and identifying it with the jitter mechanism.

While the case of the injection of relativistic electrons into a highly turbulent medium, where the condition of Equation (110) is satisfied, guarantees production of radiation in the jitter regime, the questions of the detection of this radiation depends on the total energetics in the relativistic particles. Given the usually (very) high efficiency of jitter radiation, and the typical parameters characterizing the nonthermal astrophysical sources of both Galactic and extragalactic origin, the production of detectable fluxes of jitter radiation in the X-ray and/or gamma-ray bands could be readily realized in standard acceleration and radiation scenarios.

The identification of the origin of the radiation is the second critical issue. Fortunately, the jitter radiation does provide us with distinct features which can be used to identify its nature. In particular, for a standard power-law injection distribution of electrons with a high-energy cutoff given, for example, by the form of Equation (77) with \(\mu = 2\), and assuming a Kolmogorov-type spectrum of turbulence, \(\alpha = 5/3\), we expect a gamma-ray spectrum that can be described as a broken power law. The high-energy part of the spectrum above the break is expected to have a photon index of \(\Gamma_1 = \alpha + 1 \approx 2.7\), while the low-energy part (below the break) would have \(\Gamma_1 = (\mu + 1)/2 = 1.5\) or \(\Gamma_1 = (\mu + 2)/2 = 2\) for the slow and fast cooling regimes, respectively. This corresponds to the change of the spectral index by \(\Delta \Gamma = 1.2\) or 0.7 depending on the cooling regime. Such a behavior differs significantly from the standard synchrotron cooling break with \(\Delta \Gamma = 0.5\). In the case of a low-energy cutoff in the electron spectrum, which is often required to fit the data, e.g., the spectra of gamma-ray blazars, we should expect another break below which the photon index would be \(\Gamma_1 = 1\). Therefore, in the case of the detection of a non-standard broken power-law spectrum, especially when the high-energy power-law tail has a photon index close to 2.5 and extends well beyond the break, the jitter mechanism can be treated as the process responsible for the observed spectral features (see also Fleishman & Bietenholz 2007).

Despite all the attractive properties of synchrotron radiation of ultrarelativistic electrons, the synchrotron spectrum usually terminates before reaching the gamma-ray domain. Even in extreme accelerators, it cannot extend beyond the so-called synchrotron limit of \(\sim 100\) MeV, unless being additionally Doppler boosted in sources with relativistic Doppler factors. This can be the case, for example, in the recently discovered flares of the Crab Nebula (see, e.g., Buehler et al. 2012; Striani et al. 2013, and references therein) or the multi-GeV counterparts of GRBs (Abdo et al. 2009). On the other hand, the jitter mechanism may offer another possibility for the extension of the spectrum well beyond the synchrotron limit. We should note in the case when the conditions of Equation (110) are fulfilled, the appearance of jitter radiation is not only unavoidable, but its spectrum could extend to high or even very high energies. A more basic issue in this regard is the challenge of the formation of turbulence on very small scales, e.g., through Weibel-type instabilities.

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APPENDIX A
HIGH-ENERGY ASYMPTOTICS OF SYNCHROTRON RADIATION

At high frequencies, the intensity of synchrotron radiation decreases exponentially. \( G(x) \propto \exp(-2\omega mc^2eB^2/\gamma^2) = \exp(-\omega/\omega_0) \) (see Equation (58)). Therefore, at \( \omega \gg \omega_0 \), the contribution to the radiation spectrum given by Equation (66) is dominated by regions characterized by large magnetic fields: \( b \gg 1 \). Let us assume that for \( b \gg 1 \), the magnetic field strength is distributed as a power law: \( w \propto B^{-d} \).

It is convenient to start the calculations of the spectrum from Equation (53), where the magnetic field strength is replaced by \( B \rightarrow B \sin \chi \), i.e., before integration over the directions of the magnetic field. This substitution gives:

\[
P_\omega = \frac{3e^2}{2\pi R_L^0} x \int_0^\infty K_{5/3}(u) du ,
\]

where \( x = \omega/\omega_0 \), \( R_L^0 = mc^2/eB_0 \), and \( b = B/B_0 \). To average over the strength of the magnetic field strength, one should multiply Equation (A1) by the distribution of the magnetic field, \( A b^{-d} db \), and then integrate over \( db \). Let us introduce a new variable \( b \rightarrow \xi = x/b \sin \chi \), and change the order of integration. Then, after a rather simple analytical integration over \( d\xi \), we obtain:

\[
\int P_\omega w(b) db = A \frac{3e^2}{2\pi R_L^0} \int_0^\infty u^{-\sigma-1} K_{5/3}(u) du .
\]

The remaining integral over \( du \) can be expressed in terms of Gamma functions:

\[
\int_0^\infty u^{-\sigma-1} K_{5/3}(u) du = 2^{\sigma-2} \Gamma \left( \frac{\sigma}{2} + \frac{5}{6} \right) \Gamma \left( \frac{\sigma}{2} - \frac{5}{6} \right) .
\]

The remaining integral over pitch angles also leads to an expression containing Gamma functions:

\[
\langle \sin \chi \rangle^{-\gamma} = \frac{\sqrt{\pi} \Gamma \left( \frac{\sigma}{2} + \frac{1}{2} \right)}{2 \Gamma \left( \frac{\sigma}{2} + 1 \right)} .
\]

Thus, in the limit of large frequencies \( \omega \gg \omega_0 \), the spectrum of synchrotron radiation averaged over the direction and strength of the magnetic field is described by a power-law function:

\[
\langle P_\omega \rangle = A \frac{e^2}{R_L^0} C_\sigma x^{-\sigma+2} ,
\]

where

\[
C_\sigma = \sqrt{\frac{3}{2}} \frac{2^\sigma-4}{\sqrt{\pi} (\sigma-1)} \frac{\Gamma \left( \frac{\sigma}{2} + \frac{5}{6} \right) \Gamma \left( \frac{\sigma}{2} - \frac{5}{6} \right)}{\Gamma \left( \frac{\sigma}{2} + 1 \right)} .
\]

Note that if the magnetic field is formally distributed as a pure power law, \( B \rightarrow B^{-\alpha} \), Equation (A5) gives a precise solution for the radiation spectrum. In this case, \( A = AB_0^{-1-\sigma} \), and therefore \( \langle P_\omega \rangle \) appears to be independent of \( B_0 \).

To understand the condition for applicability of Equation (A5), let us estimate the correction terms to this equation for a specific distributions of the magnetic field. Let us assume, for example, the following distribution:

\[
w(B) dB = \frac{A b^2}{(1 + b^2)^{1+\gamma/2}} .
\]

In the limit \( b \gg 1 \), the first two terms of the series are

\[
\frac{A b^2}{(1 + b^2)^{1+\gamma/2}} \approx A \left( \frac{1}{b^\gamma} - \frac{1 + \sigma/2}{b^{\sigma+2}} \right) .
\]

Correspondingly,

\[
\langle P_\omega \rangle = A \frac{e^2}{R_L^0} C_\sigma x^{-\sigma+2} - (1 + \sigma/2) C_{\sigma+2} x^{-\sigma} .
\]

The ratio of these two terms can be expressed as

\[
r = \frac{(\sigma - 1)(9\sigma^2 - 25)}{18x^2} \approx \frac{\sigma^3}{2x^2} ,
\]

which implies that one can neglect the second term when \( x \gtrsim \sigma^{3/2} \). This can be treated as the condition for the applicability of Equation (A5).

APPENDIX B
THE CASE OF ANISOTROPIC TURBULENCE

Here, we present some intermediate calculations required for the derivation of Equation (88) from Equation (87). To compute the integrals, it is convenient to introduce the following new variables:

\[
\xi = \gamma^2 \theta^2 , \quad x = \cos \theta ,
\]

where \( \theta \) is the angle between the vectors \( \nu \) and \( \xi \). The integration over the azimuthal angle is trivial: it gives \( d\Omega d\Omega_\alpha = (2\pi^2/\gamma^3) dx d\xi \). Then, the integration of Equation (87) results in

\[
F_1 = \frac{\pi^2}{2q} \int_0^1 d\zeta \int_0^1 dx (1 + x^2)^{1+\varepsilon^2} \delta(\omega + cqx - \varepsilon) .
\]

For the upper limit of integration over \( \xi \) we adopt \( \infty \), which is valid only in the ultrarelativistic regime (see also the discussion after Equation (30)). The argument of the \( \delta \)-function becomes zero for \( x = x_0 = (\kappa - \omega)/cq \). The integral becomes zero if \( x_0 \) lies beyond the integration interval, \( x_0 > 1 \) or \( x_0 < -1 \). For the value of \( x_0 \) within the integration range, i.e., \( |x_0| < 1 \), we obtain:

\[
F_1 = \frac{\pi^2}{2q} \int_{\kappa_1}^{\kappa_2} d\xi (1 + x_0^2)^{1+\varepsilon^2} (1 + \varepsilon)^2 ,
\]

where the lower and upper integration limits, \( \kappa_1, \kappa_2 \), are determined by the conditions \( |x_0| = 1 \) and \( \kappa > 0 \).

It is convenient to express these limits as \( \kappa_1 = \max(0, \kappa(\kappa - 1) - 1) \) and \( \kappa_2 = \kappa(\kappa + 1) < 1 \) (\( \kappa_2 \) should be positive), where

\[
\kappa = \frac{\chi}{qc} , \quad \xi = \frac{2qc\gamma^2}{\omega} .
\]
This allows the derivation of Equation (88) via analytical integration:

\[
U(\xi, \kappa) = \Theta(\xi(\kappa + 1) - 1) \{ U_1 \Theta(1 - \xi(\kappa - 1)) + U_2 \Theta(\xi(\kappa - 1) - 1) \},
\]  

(B5)

where

\[
U_1 = \frac{1}{\xi^3} (\xi(\kappa + 1) - 1) \left( \frac{\xi^2(\kappa - 1) + 4\xi + 2\xi^2 - 2\xi + 1}{\kappa + 1} \right)
- \frac{\xi - 1}{(\kappa + 1)^2} + \frac{2}{(\kappa + 1)^3} - \frac{3}{\xi^2} (\xi + 1) \ln(\xi + 1),
\]

(B6)

and

\[
U_2 = \frac{2}{\xi^3(\kappa^2 - 1)^3} \left( 4 + 3\kappa^6\xi^2 + 3\xi\kappa^5 - 6\kappa^2\kappa^4 - 12\kappa^3 \right)
+ (3\kappa^2 + 4)\kappa^2 + 9\xi\kappa) - \frac{3}{\xi^2} (\xi + 1) \ln \left( \frac{\kappa + 1}{\kappa - 1} \right).
\]

(B7)

Equation (B5) implies that the function \( U(\xi, \kappa) \) has non-zero values only if \( \kappa > (1/\xi) - 1 \). The two terms in Equation (B5), \( U_{1,2} \), provide a non-zero contribution of \((1/\xi) - 1 < \kappa < (1/\xi) + 1 \) and \( \kappa > (1/\xi) + 1 \), respectively. The continuous function \( U \) has a break at \( \kappa = (1/\xi) + 1 \).

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