Offshell quantum electrodynamics

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Abstract.
In this paper, we develop the quantum field theory of off-shell electromagnetism, and use it to calculate the Møller scattering cross-section. This calculation leads to qualitative deviations from the usual scattering cross-sections, which are, however, small effects, but may be visible at small angles near the forward direction.

1. Introduction

Covariant Quantum Mechanics

In 1941, Stueckelberg [1] (see also Fock [2]) proposed a Poincaré invariant Hamiltonian mechanics in which particle worldlines are traced out by the evolution of events according to an invariant parameter $\tau$. Stueckelberg’s purpose was to describe pair annihilation as the evolution of a single worldline, first forward and then backward in time. Since the Einstein time coordinate $x^0 = t$ does not increase monotonically as the system evolves in such a model (the basis of the Feynman-Stueckelberg interpretation of anti-particles [3]), it was necessary to replace $x^0$ with a new chronological parameter. The Poincaré invariant parameter $\tau$ is formally similar to the Galilean invariant time in Newtonian theory, and its introduction permits the adaptation of many techniques from nonrelativistic classical and quantum mechanics. In the symplectic mechanics which follows from this formulation, one writes classical Hamilton equations in the form

$$\frac{dx^\mu}{d\tau} = \frac{\partial K}{\partial p_\mu}$$

$$\frac{dp^\mu}{d\tau} = -\frac{\partial K}{\partial x_\mu}$$

(1)

where

$$\eta_{\mu\nu} = \text{diag}(-1, 1, 1, 1)$$

and

$$\mu, \nu = 0, \cdots, 3$$

(2)

and $K$ is the analog of the Hamiltonian which generates system evolution according to $\tau$. As in the nonrelativistic case, one makes the transition from classical to quantum mechanics by regarding the Hamiltonian as the Hermitian generator of unitary $\tau$ evolution and writing

$$i \frac{\partial}{\partial \tau} \psi_\tau(x) = K \psi_\tau(x)$$

(3)
as a covariant Schrödinger equation with first order $\tau$ evolution. The electromagnetic system

$$i\partial_\tau \psi(x, \tau) = \frac{1}{2m} \left[ p^\mu - eA^\mu(x) \right] \left[ p_\mu - eA_\mu(x) \right] \psi(x, \tau)$$

(4)

enjoys U(1) local gauge symmetry

$$\psi(x, \tau) \rightarrow \exp[ie\Lambda(x)] \psi(x, \tau) \quad A_\mu \rightarrow A_\mu + \partial_\mu \Lambda(x)$$

(5)

and global gauge symmetry leading to the conserved current

$$\partial_\mu j^\mu + \partial_\tau \rho = 0$$

(6)

where

$$\rho(x, \tau) = \left| \psi(x, \tau) \right|^2 \quad j^\mu = -\frac{i}{2M} \left\{ \psi^* (\partial^\mu - ieA^\mu) \psi - \psi (\partial^\mu + ieA^\mu) \psi^* \right\}$$

(7)

with an interpretation of $\rho(x, \tau)$ as the probability density at $\tau$ of finding the event at the spacetime point $x$. The non-zero divergence of the four-vector current $j^\mu(x, \tau)$ prevents its identification as the source of the field $A^\mu(x)$, but Stueckelberg observed that

$$\rho(x, \tau) \underset{\tau \rightarrow \pm \infty}{\longrightarrow} 0 \implies \partial_\mu j^\mu(x) = \partial_\mu \int_{-\infty}^{\infty} d\tau \ j^\mu(x, \tau) = 0.$$  

(8)

However, there is no guarantee that the particle worldline associated with the a posteriori current $J^\mu(x)$ will induce the field $A^\mu(x)$ that interacts with the particle instantaneously at $\tau$.

In order to obtain a well-posed theory, Sa’ad, Horwitz, and Arshansky [4] generalized (4) by introducing $\tau$-dependent potentials and a fifth gauge field. Writing $x^5 = \tau$ and adopting the index convention

$$\alpha, \beta, \gamma = 0, 1, 2, 3, 5\quad \lambda, \mu, \nu = 0, 1, 2, 3\quad i, j, k = 1, 2, 3$$

(9)

the Stueckelberg-Schrodinger equation

$$[i\partial_\tau + e_0a_5(x, \tau)] \psi(x, \tau) = \frac{1}{2M} \left[ p^\mu - e_0a^\mu(x, \tau) \right] \left[ p_\mu - e_0a_\mu(x, \tau) \right] \psi(x, \tau)$$

(10)

is locally gauge invariant under $\tau$-dependent gauge transformations

$$\psi \rightarrow e^{ie_0\Lambda(x, \tau)} \psi \quad a_\mu \rightarrow a_\mu + \partial_\mu \Lambda(x, \tau)$$

(11)

and leads to the conserved current

$$\partial_\mu j^\mu + \partial_\tau j^5 = 0$$

(12)

where

$$j^5 \equiv \rho = \left| \psi(x, \tau) \right|^2 \quad j^\mu = -\frac{i}{2M} \left\{ \psi^* (\partial^\mu - ie_0a^\mu) \psi - \psi (\partial^\mu + ie_0a^\mu) \psi^* \right\}.$$  

(13)

Equation (10) may be derived by variation of the action

$$S = \int d^4x d\tau \left\{ \psi^* (i\partial_\tau + e_0a_5) \psi - \frac{1}{2M} \psi^*(p^\mu - e_0a_\mu)(p_\mu - e_0a_\mu)\psi - \frac{\lambda}{4} f_{\alpha\beta\gamma} f^{\alpha\beta\gamma} \right\}$$

(14)
to which Sa’ad, et. al. added a kinetic term for the fields generalizing the Maxwell term from four to five dimensions. In raising the 5-index one chooses a value for \( \eta_{55} \) in
\[
\bar{f}_{\alpha\beta} f^{\alpha\beta} = f_{\mu\nu} f^{\mu\nu} + 2 f_{5\mu} f^{5\mu} = f_{\mu\nu} f^{\mu\nu} + 2 \eta_{55} f_{5\mu} f^{5\mu}
\]
corresponding to a formal metric
\[
\eta_{\alpha\beta} = \text{diag}(-1, 1, 1, 1, \sigma).
\]
It has been shown [5] that choosing \( \sigma = -1 \) leads to a consistent description of low energy classical Coulomb scattering. Varying the action (14) with respect to the gauge fields, the equations of motion are found to be
\[
\partial_\beta f^{\alpha\beta} = \frac{e_0}{\lambda} j^\alpha = e j^\alpha \quad \epsilon^{\alpha\beta\gamma\delta\epsilon} \partial_\delta f_{\beta\gamma} = 0
\]
where \( j^\alpha \) is given by (13). The fields satisfy the wave equation
\[
\partial_\alpha f^{\alpha\beta} = (\partial_\mu \partial^\mu + \partial_\tau \partial^\tau) f^{\beta\gamma} = (\partial_\mu \partial^\mu + \sigma \partial_\tau^2) f^{\beta\gamma} = -e (\partial^\alpha j^\beta - \partial^\beta j^\alpha)
\]
depending on the signature \( \sigma \) of \( \partial_\tau \). Under the boundary conditions
\[
\tau \to \pm \infty \quad \Rightarrow \quad a^{\delta}(x, \tau) \to 0 \quad j^5(x, \tau) \to 0
\]
the standard Maxwell theory is extracted as the equilibrium limit of (18) by integration over the worldline
\[
\partial_\beta f^{\alpha\beta}(x, \tau) = e j^\alpha (x, \tau) \quad \partial_\alpha [f_{\beta\gamma}] = 0 \quad \int d\tau \quad \partial_\mu F^{\mu\nu}(x) = e j^\mu(x) \quad \partial_\mu A^\mu(x) = \int d\tau a^\mu(x, \tau).
\]
The integration has been called concatenation [6] and as in [8] links the event current \( j^\mu(x, \tau) \) with the particle current \( j^\mu(x) \) defined on the entire particle worldline. It is seen from (10) and (21) that \( e_0 \) and \( \lambda \) must have dimensions of time, so that the dimensionless ratio \( e = e_0 / \lambda \) can be identified as the Maxwell charge. The microscopic \( \tau \)-dependent fields have been called pre-Maxwell fields. For further background on the classical theory underlying off-shell QED, see [7] and references contained therein.

**Off-Shell Quantum Electrodynamics**

In this paper, we develop the methods required to apply off-shell quantum electrodynamics as a scattering theory, and calculate certain elementary processes. In Section 2 we perform a canonical quantization of the interacting off-shell theory. As a parameterized evolution theory on an unconstrained phase space, it is natural to apply Jackiw’s method of quantizing an action that can be made linear in time derivatives, without introducing auxiliary fields [8]. In conventional quantum theories, the momentum is related to a \( t \)-derivative, so putting the action into first order form may break manifest Lorentz covariance. For the off-shell theory, the classical velocity is a \( \tau \)-derivative, and so Jackiw’s approach remains manifestly covariant throughout. A path integral quantization following this method was presented.
The results obtained are in complete agreement with those of Shnerb and Horwitz who used a bosonic gauge fixing method. Having identified the essential degrees of freedom, we perform the detailed Fourier expansion of the operators of the free fields, required in perturbation theory. In Section 3, we carry out a canonical quantization of the free off-shell matter field, and provide an expansion of the field operators in terms of annihilation operators in momentum space. We use these expansions to show that the vacuum expectation value of $\tau$-ordered fields is precisely the Stueckelberg-Schrödinger equation Green’s function, with Schwinger-Feynman boundary conditions. In Section 4, we carry out the canonical quantization of the five dimensional electromagnetic field and the expansion of field operators in momentum space. This field has three independent polarizations, one of which decouples in the Maxwell (zero mode) limit. We show that vacuum expectation value of products of two $\tau$-ordered fields is the Feynman Green’s function for the wave equation, and that it preserves the gauge condition. In Section 5, we develop the perturbation theory for the S-matrix in the interaction picture. In order to provide the link with the usual formulas for the perturbation expansion of Green’s functions, we adapt the LSZ reduction formulas to the off-shell theory. We then derive the Feynman rules. In Section 6, we derive the scattering cross-section in terms of the transition amplitude, and demonstrate the relationship of this expression to the usual cross-section. In Section 7, we analyze the Møller scattering amplitude in detail and examine the qualitative deviation from the corresponding amplitude calculated in conventional QED. In Section 8, we consider the problem of renormalization of the off-theory. We derive the Ward identity associated with current conservation and show that it connects the matter field propagators with both the 3-particle and the 4-particle vertex functions. We conclude that the off-shell theory is renormalizable when a cut-off is placed on the mass of the off-shell photons. We relate this cut-off to a modification of the electromagnetic action required in classical off-shell electrodynamics.

2. Canonical quantization of the interacting theory

Following we take the action for off-shell electromagnetism to be

$$S = \int d^4x d\tau \left\{ \psi^*(i\partial_\tau + e_0 a_5)\psi - \frac{1}{2M} \psi^*(-i\partial_\mu - e_0 a_\mu)(-i\partial^\mu - e_0 a^\mu)\psi - \frac{\lambda}{4} f_{\alpha\beta} f^{\alpha\beta} \right\}$$  \hspace{1cm} (22)

using the index conventions (9) and (16). Because of their support properties, 4-divergences as vanishing under spacetime integration while $\tau$-derivatives vanish independently under $\tau$-integration because of Riemann-Lebesgue oscillations (the asymptotic mass dependence of the fields $\exp\{-i(m^2/2M)\tau\}$ varies arbitrarily rapidly for large $\tau$). Thus, for the $\tau$-derivative terms in (22),

$$\frac{1}{2} [\psi^*(i\partial_\tau + e_0 a_5)\psi + \text{h.c.}] = i\psi^*\psi + e_0 a_5 \psi^*\psi + \partial_\tau (\psi^*\psi)$$  \hspace{1cm} (23)

where $\psi = \partial \psi/\partial \tau$. We will henceforth regard (22) as hermitian.

In Dirac’s method of quantization for gauge theories, we would form the Hamiltonian from (22), including a momentum $a_5$ conjugate to $a_5$ and a Lagrange multiplier to enforce the primary constraint $a_5 = 0$ (in [10], this is accomplished through the gauge fixing term). The secondary constraint (that the primary constraint commute with the Hamiltonian) would lead to the Gauss Law for the off-shell theory, which is the $a = 5$ term of the first of (17).

We follow a quantization scheme advocated by Jackiw in which one first eliminates the constraint from the Lagrangian, and then constructs the Hamiltonian from the unconstrained
degrees of freedom. In order to perform the transformation which eliminates the constrained degrees of freedom, it is necessary that the action be made linear in the time derivatives. Because the scalar \( \tau \) plays the role of time in the off-shell theory, Lorentz covariance is maintained throughout, in a manner consistent with the original spirit of the proper time method.

In Jackiw’s method, a choice of gauge is made implicitly by solving the constraints and introducing a decomposition of the fields consistent with that solution. The elimination of the longitudinal polarizations is carried out as an application of the Darboux theorem, diagonalizing the Hamiltonian. This procedure is evidently related to the quantization method proposed by Fermi in 1932 [12]. Applying Fermi’s method to (22), we would perform a gauge transformation which guarantees the condition \( \partial_\mu a^\mu = 0 \), and leads to the Gauss law for the component \( a_5 \). The presence of sources in the action would require the decomposition of the fields into longitudinal terms induced by the sources and transverse propagating terms with vanishing our-divergence. We could then argue that the longitudinal fields do not satisfy wave equations and need not be quantized; this would leave the unconstrained transverse fields in the action, and we would obtain a consistent canonical quantization. Elements of such a procedure may be recognized in the method presented below.

The kinetic term for the matter field is linear in \( \partial_\tau \) by construction. In order to put the kinetic term for the gauge field into explicitly canonical form, we rewrite \( f^{5\mu}_5 \) as \( f^{5\mu}_5 (\partial_\tau a_\mu - \partial_\mu a_5) \) and take the quantity \( f^{5\mu}_5 \) to be independent of the fields \( a_\mu \) (this is a variant of the first order Lagrangian form for the usual electromagnetic field). Expanding the electromagnetic term,

\[
f_{\alpha\beta}f^{\alpha\beta} = f_{\mu\nu}f^{\mu\nu} + 2\sigma f^{5\mu}_5 f^{5}_5 = f_{\mu\nu}f^{\mu\nu} + 2\sigma \left[ 2(\sigma\partial_\tau a^\mu - \partial^\mu a_5^2) f^{5\mu}_5 - f^{5\mu}_5 f^{5\mu}_5 \right]
\]

(24)

when we vary the action with respect to \( f^{5\mu}_5 \), we recover its relationship to the \( a^\mu \). Integrating by parts, we obtain

\[
(\sigma\partial_\tau a^\mu - \partial_\mu a_5^2) f^{5\mu}_5 = \sigma(\partial_\tau a^\mu) f^{5\mu}_5 + a^5 \partial_\mu f^{5\mu}_5 - \text{divergence}
\]

(25)

from which we drop the divergence and introduce the notation

\[
e^{\mu} = f^{5\mu}_5.
\]

(26)

Using (25) and (26), the action becomes

\[
S = \int d^4x d\tau \left[ i\psi^* \dot{\psi} - \lambda e_\mu \dot{a}^\mu - \frac{1}{2M} \psi^* (\lambda a_5^2 - \partial_\mu (\partial^\mu e_\mu) - \partial_\mu (\lambda a_5 e_\mu)) \right]
\]

\[
+ \frac{\lambda}{4} f_{\mu\nu} f^{\mu\nu} + \lambda \sigma e_\mu e_\mu + a_5 (e_0 \psi^* \psi - \lambda \partial_\mu e_\mu)
\]

(27)

where we have collected terms in \( a_5 \) and written \( \partial_\tau \psi = \dot{\psi} \). Following Jackiw, we regard (27) as an action for the conjugate pairs \( \{i\psi^*, \psi\} \) and \( \{e_\mu, a_\mu\} \), with \( a_5 \) playing the role of a Lagrange multiplier for the constraint

\[
\dot{e}_0 \psi^* \psi - \lambda \partial_\mu e_\mu = 0 \quad \Rightarrow \quad \partial_\mu e_\mu = \frac{e_0}{\lambda} \psi^* \psi = \epsilon_\rho,
\]

(28)

which is just the Gauss law for the off-shell theory. The constraint equation (28) can be solved through the decomposition

\[
e^{\mu} = (\epsilon_{\perp})^{\mu} + \epsilon^{\mu} [G\rho]
\]

(29)
where
\[ \partial_\mu (\epsilon^\perp)_\mu = 0 \]  
(30)
and where \( G\rho \) is a shorthand for the functional
\[ [G\rho](x,\tau) = \int d^4 y \ G(x-y) \ \rho(y,\tau) \]  
(31)
in which we specify the Maxwell Green's function
\[ G(x-y) = \delta \left[ (x-y)^2 \right] \Rightarrow \Box G = 1. \]  
(32)
Performing a similar decomposition of \( a^\mu \),
\[ a^\mu = (a^\perp)^\mu + \partial^\mu [GA] \]  
\[ \partial_\mu (a^\perp)_\mu = 0 \]  
(33)
(by which we implicitly choose the gauge condition \( \partial_\mu a^\mu = \Lambda \)) the remaining terms of the theory are expressed as
\[ \dot{a}^\mu = (\dot{a}^\perp)^\mu + \partial^\mu [\dot{G}A] \]  
(34)
\[ f_\mu^\nu = \partial^\mu (a^\perp)^\nu - \partial^\nu (a^\perp)^\mu = (f^\perp)_\mu^\nu \]  
(35)
\[ -i\dot{\partial}^\mu - \partial_0 a^\mu = -i\dot{\partial}^\mu - \partial_0 (a^\perp)^\mu - e^\dot{\partial}^\mu [\dot{G}A] \]  
(36)
\[ \epsilon^\mu \epsilon_\mu = \left[ (e_\perp)^\mu + e\partial^\mu [G\rho] \right] \left[ (e_\perp)^\mu + e\partial^\mu [G\rho] \right] = (e_\perp)^\mu (e_\perp)^\mu - e^2 \rho [G\rho] + \text{divergence} \]  
(37)
\[ \epsilon^\mu \dot{a}^\mu = (e_\perp)^\mu (\dot{a}^\perp)^\mu - e\partial [G\dot{A}] \]  
(38)
In terms of this decomposition, the action becomes
\[ S = \int d^4 x d\tau \left\{ i\psi^* \dot{\psi} - \lambda (e_\perp)^\mu (\dot{a}^\perp)^\mu + \lambda e \rho [G\dot{A}] \right. \]
\[ - \frac{1}{2M} \psi^*(-i\dot{\partial}^\mu - \partial_0 (a^\perp)^\mu - e\partial^\mu [G\dot{A}]) (-i\dot{\partial}^\mu - \partial_0 (a^\perp)^\mu - e\partial^\mu [G\dot{A}]) \psi \]
\[ - \frac{\lambda}{4} (f^\perp)_\mu^\nu (f^\perp)^\mu_\nu + \frac{\lambda \sigma}{2} (e_\perp)^\mu (e_\perp)^\mu - \frac{\lambda \sigma}{2} e^2 \rho [G\rho] \} \]  
(39)
We now perform the gauge transformation
\[ \psi \rightarrow e^{ie_0[G\dot{A}]} \psi \]  
(40)
which entails
\[ \psi \rightarrow e^{ie_0[G\dot{A}]} [\psi + i\epsilon_0[G\dot{A}] \psi] \]  
\[ i\psi^* \psi \rightarrow i\psi^* \psi - e_0 \rho [G\dot{A}] \]  
(41)
and
\[ (-i\dot{\partial}^\mu - \partial_0 (a^\perp)^\mu - e\partial^\mu [G\dot{A}]) \psi \rightarrow (-i\dot{\partial}^\mu - \partial_0 (a^\perp)^\mu - e\partial^\mu [G\dot{A}]) e^{ie_0[G\dot{A}]} \psi \]
\[ = e^{ie_0[G\dot{A}]} (-i\dot{\partial}^\mu - \partial_0 (a^\perp)^\mu - e\partial^\mu [G\dot{A}] + e\partial^\mu [G\dot{A}]) \psi \]
\[ = e^{ie_0[G\dot{A}]} (-i\dot{\partial}^\mu - \partial_0 (a^\perp)^\mu) \psi \]  
(42)
This transforms the action to the form
\[ S = \int d^4 x d\tau \left\{ i\psi^* \dot{\psi} - \lambda (e_\perp)^\mu (\dot{a}^\perp)^\mu - \frac{1}{2M} \psi^*(-i\dot{\partial}^\mu - \partial_0 (a^\perp)^\mu)(-i\dot{\partial}^\mu - \partial_0 (a^\perp)^\mu) \psi \right. \]
\[ - \frac{\lambda}{4} (f^\perp)_\mu^\nu (f^\perp)^\mu_\nu + \frac{\lambda \sigma}{2} (e_\perp)^\mu (e_\perp)^\mu - \frac{\lambda \sigma}{2} e^2 \rho [G\rho] \} \]  
(43)
which is an unconstrained functional (only the unconstrained field degrees of freedom are present) and so may be canonically quantized. Henceforth, we may drop the subscript \( \perp \) from the quantized gauge field variables and assume the transversality conditions (30) and (33) for the field operators.

The conjugate momenta are found from (43) to be
\[
\pi_\psi = \frac{\partial \mathcal{L}}{\partial \dot{\psi}} = i \psi^* \quad \pi_{a_\mu} = \frac{\partial \mathcal{L}}{\partial \dot{a_\mu}} = -\lambda \epsilon^\mu
\]
from which we compute the Hamiltonian
\[
\mathcal{K} = \pi_\psi \dot{\psi} + \pi_{a_\mu} \dot{a_\mu} - \mathcal{L} = \frac{1}{2M} \left[ (i \partial_\mu - e_0 a_\mu) \dot{\psi}^* \right] \left[ (i \partial^\mu - e_0 a^\mu) \psi \right] + \frac{\lambda}{4} f_{\mu \nu} f^{\mu \nu} - \frac{\lambda \sigma}{2} \epsilon^\mu \epsilon_\mu + \frac{\lambda \sigma}{2} \epsilon^2 \rho \left[ G^\rho \right]
\]
which may be decomposed as
\[
\mathcal{K} = \mathcal{K}_{\text{photon}} + \mathcal{K}_{\text{matter}} + \mathcal{K}_{\text{interaction}}
\]
where
\[
\mathcal{K}_{\text{photon}} = \frac{\lambda}{4} f_{\mu \nu} f^{\mu \nu} - \frac{\lambda \sigma}{2} \epsilon^\mu \epsilon_\mu \quad \mathcal{K}_{\text{matter}} = \frac{1}{2M} \left( \partial_\mu \psi^* \right) \left( \partial^\mu \psi \right) \quad \mathcal{K}_{\text{interaction}} = \frac{ie_0}{2M} a_\mu \left( \psi^* \partial_\mu \psi - \psi \partial^\mu \psi^* \right) + \frac{e_0^2}{2M} a_\mu a^\mu |\psi|^2 + \frac{\lambda \sigma}{2} \epsilon^2 \rho \left[ G^\rho \right]
\]
The last term in (49) has the form of a c-number energy density which represents the mass-energy equivalent required to assemble the matter field (see also [10]). The coupling \( e_0 \epsilon = \lambda \epsilon^2 \) is characteristic of the classical Lorentz force due to a charge distribution. We demonstrate below that the remaining terms in (49), which lead to 3-particle and 4-particle interactions in the Feynman diagrams, are connected by the Ward identity for the conserved current (12).

3. Canonical quantization of the free spinless matter field

From the Hamiltonian density (48) the Hamiltonian operator is
\[
K = \int d^4x \mathcal{K}_{\text{matter}} = \frac{1}{2M} \int d^4x \left( \partial_\mu \psi^* \right) \left( \partial^\mu \psi \right).
\]
The fields must satisfy the canonical equal-\( \tau \) commutation relations
\[
[\psi(x, \tau), \pi_\psi(x', \tau)] = i\delta^4(x - x')
\]
which together with the first of (44) leads to
\[
[\psi(x, \tau), \psi^*(x', \tau)] = \delta^4(x - x').
\]
The field evolves dynamically according to the Heisenberg equation

\[ i \partial_\tau \psi = [\psi, K] \] (53)

and using equations (48), (50), (51), and (53), we find

\[
\begin{align*}
\partial_\tau \psi(x, \tau) &= \left[ \psi(x, \tau), \frac{1}{2M} \int d^4x' \left( \partial_\mu \psi^*(x', \tau) \right) \left( \partial^\mu \psi(x', \tau) \right) \right] \\
&= -\frac{1}{2M} \int d^4x' \left[ \psi(x, \tau), \psi^*(x', \tau) \right] \partial_\mu \partial^\mu \psi(x', \tau) \\
&= -\frac{1}{2M} \int d^4x' \delta^4(x - x') \partial_\mu \partial^\mu \psi(x', \tau) \\
&= -\frac{1}{2M} \partial_\mu \partial^\mu \psi(x, \tau)
\end{align*}
\] (54)

which is the Schrödinger equation (1) for the field operator \( \psi(x, \tau) \) in the absence of interaction. Because it is satisfied, we may perform the Fourier expansion

\[
\psi(x, \tau) = \int \frac{d^4k}{(2\pi)^4} b(k) e^{i(k \cdot x - \kappa \tau)} \quad \psi^*(x, \tau) = \int \frac{d^4k}{(2\pi)^4} b^*(k) e^{-i(k \cdot x - \kappa \tau)}
\] (55)

where from (54) we see that \( \kappa = k^2 / 2M \). From (51), we find that

\[
[b(k), b^*(k')] = (2\pi)^4 \delta^4(k - k') \quad [b(k), b(k')] = [b^*(k), b^*(k')] = 0.
\] (56)

In terms of the Fourier expansion, the Hamiltonian is given by

\[
K = \int d^4x \frac{1}{2M} \left[ \partial_\mu \psi^* \right] \left[ \partial^\mu \psi \right]
= \frac{1}{2M} \int d^4x \int \frac{d^4k}{(2\pi)^4} (-i\kappa_\mu) b^* \left( k \right) e^{-i(k \cdot x - \kappa \tau)} \int \frac{d^4k'}{(2\pi)^4} (i\kappa_\mu) b(k') e^{i(k' \cdot x - \kappa' \tau)}
= \frac{1}{2M} \int \frac{d^4k}{(2\pi)^4} \frac{d^4k'}{(2\pi)^4} b^* \left( k \right) b(k') (2\pi)^4 \delta^4(k - k') e^{i\kappa \tau} e^{-i\kappa' \tau} (k \cdot k')
= \int \frac{d^4k}{(2\pi)^4} \frac{k^2}{2M} b^*(k) b(k)
= \int \frac{d^4k}{(2\pi)^4} \frac{\kappa}{2M} b^*(k) b(k)
\] (57)

The ground state mass for this Hamiltonian vanishes even without normal ordering.

We write the Green's function \( G_i(x, \tau) \) for the Schrödinger equation (54) in the integral form,

\[
G(x, \tau) = \frac{1}{(2\pi)^3} \int_{C_i} d^4k e^{i(k \cdot x - \kappa \tau)}
\] (58)

where the contour of \( \kappa \) integration \( C_i \) determines the boundary conditions. If \( C_i \) includes the interval \((-\infty, \infty)\), then \( G_i(x, \tau) \) will satisfy the inhomogeneous equation

\[
(i \partial_\tau + \frac{1}{2M} \partial_\mu \partial^\mu) G_i(x, \tau) = -\delta^4(x) \delta(\tau).
\] (59)
We take the Green’s function for the matter field to be
\[
G(x, \tau) = \frac{1}{(2\pi)^4} \int d^4k \frac{e^{i(k \cdot x - \kappa \tau)}}{2\pi^2 k^2 - \kappa - ie} \tag{60}
\]
displacing the pole at \( \kappa = k^2 / 2M - ie \) into the lower half plane. The \( \kappa \)-integration vanishes for \( \tau < 0 \) (closing the contour in the upper half plane) and for \( \tau > 0 \) (closing in the lower half plane), we obtain
\[
\int d\kappa \frac{e^{-i\kappa \tau}}{2\pi^2 k^2 - \kappa - ie} = 2\pi i \text{Res} \frac{e^{-i\kappa \tau}}{2\pi^2 k^2 - \kappa - ie} \times \text{index of contour} = 2\pi ie^{-i(\frac{k^2}{2M} - ie)\tau} \theta(\tau). \tag{61}
\]
Thus, we find
\[
G(x, \tau) = \frac{i}{(2\pi)^4} \int d^4k e^{i(k \cdot x - \frac{k^2}{2M}\tau + ie\tau)} \theta(\tau) \tag{62}
\]
which exhibits retarded causality through \( \theta(\tau) \). We may also consider the Green’s function for the matter field as expressed through the vacuum expectation value of the \( \theta \) which exhibits retarded causality through \( \theta(\tau) \). Using the momentum expansions \( \frac{[55]}{55} \), we find that
\[
\langle 0 | T \psi(x_1, \tau_1) \psi^\dagger(x_2, \tau_2) | 0 \rangle = \\
= \theta(\tau_1 - \tau_2) \langle 0 | \psi(x_1, \tau_1) \psi^\dagger(x_2, \tau_2) | 0 \rangle + \theta(\tau_2 - \tau_1) \langle 0 | \psi^\dagger(x_2, \tau_2) \psi(x_1, \tau_1) | 0 \rangle \\
= \theta(\tau_1 - \tau_2) \int \frac{d^4k}{(2\pi)^4} \frac{d^4k'}{(2\pi)^4} e^{-i(k \cdot x_2 - \frac{k^2}{2M}\tau_2)} e^{i(k' \cdot x_1 - \frac{k'^2}{2M}\tau_1)} \langle 0 | b(k') b^\dagger(k) | 0 \rangle \\
= \theta(\tau_1 - \tau_2) \int \frac{d^4k}{(2\pi)^4} e^{i(k \cdot (x_1 - x_2) - \frac{k^2}{2M}(\tau_1 - \tau_2))} \\
= -iG(x_1 - x_2, \tau_1 - \tau_2) \tag{63}
\]
where we have used \( b(k) | 0 \rangle = 0 \) and \( \frac{[56]}{56} \). We thus verify that the Green’s function that solves the Schrödinger equation is equal to the vacuum expectation value of the \( \tau \)-ordered product of the operators, as required for the application of Wick’s theorem in perturbation theory.

In his 1951 calculation of vacuum polarization, Schwinger transformed the Dirac problem into a dynamical theory by representing the Green’s function as a parametric integral \( \frac{[13]}{13} \). Applying this method to the Klein-Gordon equation, we represent the Green’s function operator as
\[
G = \frac{1}{(p - eA)^2 + m^2 - ie} \tag{64}
\]
with spacetime representation
\[
G(x, x') = \langle x | G | x' \rangle = i \int_0^\infty ds e^{-i(m^2 - ie)s} \langle x | e^{-i(p - eA)^2s} | x' \rangle. \tag{65}
\]
The function
\[
G(x, x'; s) = \langle x(s) | x'(0) \rangle = \langle x | e^{-i(p - eA)^2s} | x' \rangle \tag{66}
\]
satisfies the Schrödinger equation \( \frac{[4]}{4} \) with the boundary condition
\[
\lim_{s \to 0} \langle x(s) | x'(0) \rangle = \delta^4(x - x') \tag{67}
\]
and following DeWitt [14] we regard \( (65) \) as defining the Green’s function for the field \( \psi(x, s) \) that solves \([4]\). In deriving the path integral for the Klein-Gordon equation, Feynman [15] a similar Schrödinger equation. He regarded the integration of the Green’s function with the weight \( e^{-im^2s} \) in \( (65) \) as the requirement (see also Nambu [16]) that asymptotic solutions of the be stationary eigenstates of the mass operator \( i\partial_s \). From this point of view, one picks the mass eigenvalue by extending the lower limit of integration in \( (65) \) from \( 0 \) to \(-\infty\) and adding the requirement that \( G(x, x'; s) = 0 \) for \( s < 0 \).

Returning to the off-shell Green’s function \( (61) \), we easily verify that the \( \tau \)-integral

\[
\int_{-\infty}^{\infty} d\tau e^{-i(m^2/2M)\tau} G(x, \tau) = \frac{i}{(2\pi)^4} \int_{0}^{\infty} d\tau e^{-i(m^2/2M)\tau} \int d^4k e^{i(k \cdot x - k^2/2\tau + i\epsilon)} \\
= i \int_{0}^{\infty} d\tau \int \frac{d^4k}{(2\pi)^4} e^{-i(m^2/2M + k^2/2m^2 - i\epsilon)\tau} e^{ik \cdot x} \\
= \int \frac{d^4k}{(2\pi)^4} \frac{1}{2M} (k^2 + m^2) - i\epsilon \\
= 2M \Delta_{\tau}(x) \tag{68}
\]

goes over to the Feynman propagator for a particle of mass \( m \) with an overall factor of \( 2M \). Thus, as noted by Feynman [15], the Feynman causality employed in standard QED emerges naturally from retarded causality in \( \tau \).

4. Quantization of the free gauge field

We begin with the Hamiltonian obtained from \( (47) \) as

\[
K = \int d^4x K_{\text{photon}} = \int d^4x \left[ \frac{\lambda}{4} f_{\mu\nu}^2 - \frac{\lambda\sigma}{2} e^\mu e_\mu \right]. \tag{69}
\]

In order to evaluate the canonical commutation relations for the transverse fields, we first consider the projection operator

\[
\delta_{\perp}^{\mu\nu}(x - y) = \int \frac{d^4k}{(2\pi)^4} \left[ g^{\mu\nu} - \frac{k^\mu k^\nu}{k^2} \right] e^{ik \cdot (x - y)} = \int \frac{d^4k}{(2\pi)^4} \delta^{\mu\nu}(k) e^{ik \cdot (x - y)} \tag{70}
\]

which projects onto the transverse part of a vector function. For a function \( f_\mu(x, \tau) \) with Fourier transform \( f_\mu(k, \tau) \),

\[
[\delta_{\perp}^{\mu\nu} f_\mu](x, \tau) = \int d^4y \delta_{\perp}^{\mu\nu}(x - y) f_\mu(y, \tau) = \int d^4k \left[ g^{\mu\nu} - \frac{k^\mu k^\nu}{k^2} \right] f_\mu(k, \tau) e^{ik \cdot x} \tag{71}
\]

which clearly satisfies \( \partial_\tau f_\mu(x, \tau) = 0 \). We represent the transverse fields as

\[
a^{\mu}_\perp(x, \tau) = \left[ \delta_{\perp}^{\mu\nu} a_\nu \right](x, \tau) \quad \epsilon^{\mu}_\perp(x, \tau) = \left[ \delta_{\perp}^{\mu\nu} \epsilon_\nu \right](x, \tau). \tag{72}
\]

Canonical quantization requires the equal-\( \tau \) commutation relations for the field components

\[
[a^{\mu}(x, \tau), \pi_\nu(y, \tau)] = ig^{\mu\nu} \delta^4(x - y) \tag{73}
\]
which becomes
\[
[a^\mu(x, \tau), e^\nu(y, \tau)] = -\frac{i}{\lambda} \delta^{\mu\nu}\delta^4(x - y)
\] (74)

using (44) for \(\pi_{\alpha}\). We find for the transverse field components
\[
[a^\mu_\perp(x, \tau), e^\nu_\perp(y, \tau)] = \left[ \left[ \delta^{\mu\nu}_\perp a^\mu \right](x, \tau), \left[ \delta^{\nu\sigma}_\perp e^\sigma(y, \tau) \right] \right] = -\frac{i}{\lambda} \delta^{\mu\sigma}_\perp \delta^4_\perp \delta^4(x - y)
\]

leading to
\[
= -\frac{i}{\lambda} \delta^{\mu\nu}_\perp(x - y)
\] (75)

and so drop the subscript \(\perp\) from the field variables, using (75) as the canonical commutation relation. This relation insures that the Gauss Law constraint (28) commutes with the other variables of the theory, and so, with the Hamiltonian (69).

The Heisenberg equations for the fields \(a^\mu\) are
\[
i\partial_\tau a^\mu(x, \tau) = [a^\mu(x, \tau), \int d^4y \frac{\lambda}{4} f^\mu f^\nu - \frac{\lambda\sigma}{2} e^\rho \epsilon_\rho]
\]

leading to
\[
\partial_\tau a^\mu(x, \tau) = \sigma e^\mu(x, \tau) = \sigma f^5^\mu = f^5^\mu
\] (76)
in agreement with the classical definition of \(f^5^\mu\) in the gauge \(a_5 = 0\). The Heisenberg equation for the fields \(e^\mu\) are
\[
i\partial_\tau e^\mu(x, \tau) = [e^\mu(x, \tau), \int d^4y \frac{\lambda}{4} f^\mu f^\nu]
\]

leading to
\[
= \frac{\lambda}{4} \int d^4y \frac{\lambda}{4} [e^\mu(x, \tau), (\partial_\lambda a_\nu - \partial_\nu a_\lambda)] f^\lambda^\mu + f^\nu^\lambda e^\mu(x, \tau), (\partial_\lambda a_\nu - \partial_\nu a_\lambda)]
\]

leading to
\[
= -i\partial_\lambda f^\lambda^\mu(x, \tau).
\] (77)

Since \(e^\mu = f^5^\mu\) (77) can be written
\[
\partial_\lambda f^\lambda^\mu + \partial_\tau f^5^\mu = 0
\] (78)

and combined with (50) to write
\[
\partial_\alpha f^\alpha^\beta = 0
\] (79)

for \(\alpha, \beta = 0, \ldots, 3, 5\). These are half of the pre-Maxwell field equations; the other pre-Maxwell equations follow from the definition of \(f^{\mu\nu}\) and (76). By virtue of (53), (78) may be written in the form
\[
0 = \partial_\tau f^5^\mu + (\partial_\lambda (\partial_\lambda a^\mu - \partial_\mu a_\lambda)) = \partial_\tau f^5^\mu + \partial_\lambda (\partial_\lambda a^\mu - \partial_\mu a_\lambda) = \partial_\tau f^5^\mu + \partial_\lambda (\partial_\lambda a^\mu).
\] (80)

Combining (80) with (76), we obtain the wave equation
\[
0 = \partial_\lambda (\partial_\lambda a^\mu + \partial_\tau e^\mu = \partial_\lambda (\partial_\lambda a^\mu + \partial_\tau (\sigma \partial_\tau a^\mu) = (\partial_\lambda \partial_\lambda + \sigma \partial_\tau^2) a^\mu.
\] (81)
Therefore, we may perform a Fourier expansion of the field operator,

$$a^\mu(x,\tau) = \sum_{s=\text{polarizations}} \int \frac{d^4k}{2\kappa} \left[ \varepsilon_s^\mu a(k,s) e^{i(k\cdot x + \sigma \kappa \tau)} + \varepsilon_s^{\mu*} a^*(k,s) e^{-i(k\cdot x + \sigma \kappa \tau)} \right]$$  \hspace{1cm} (82)

where the five-dimensional mass shell condition is $$\kappa = \sqrt{-\sigma k^2}$$. For free fields, the condition $$-\sigma k^2 > 0$$ is required to prevent divergence at $$\tau \to \pm \infty$$, however this condition need not apply for virtual photons. Using (76), $$\varepsilon^\mu$$ is given by

$$\varepsilon^\mu(x,\tau) = i \sum_{s=\text{polarizations}} \int \frac{d^4k}{2\kappa} \left[ \varepsilon_s^\mu a(k,s) e^{i(k\cdot x + \sigma \kappa \tau)} - \varepsilon_s^{\mu*} a^*(k,s) e^{-i(k\cdot x + \sigma \kappa \tau)} \right].$$  \hspace{1cm} (83)

In momentum space (33) becomes $k_\mu a^\mu = 0$, so that there will be three independent polarizations (see also [17]). To choose the polarizations in a covariant way, we begin by choosing an arbitrary timelike vector $n^\mu$ ($n^2 = -1$). There are then two orthonormal spacelike vectors, $\varepsilon_1$ and $\varepsilon_2$, which satisfy the conditions

$$n \cdot \varepsilon_1 = n \cdot \varepsilon_2 = k \cdot \varepsilon_1 = k \cdot \varepsilon_2 = 0$$  \hspace{1cm} (84)

and these may be constructed in the following way. Since $n^2 = -1$,

$$(n^2 - (n^0)^2) = -1 \Rightarrow n^0 = \pm \sqrt{(n^2)^2 + 1}$$  \hspace{1cm} (85)

so that taking

$$\varepsilon_0^1 = \frac{n \cdot \varepsilon_{1,2}}{\sqrt{(n^2)^2 + 1}}$$  \hspace{1cm} (86)

guarantees that $n \cdot \varepsilon_{1,2} = 0$. We now require that

$$0 = k \cdot \varepsilon_{1,2} = \bar{k} \cdot \varepsilon_{1,2} - (k^0) \varepsilon_1^0 = \left[ \bar{k} - \frac{k^0}{\sqrt{(n^2)^2 + 1}} n \right] \cdot \varepsilon_{1,2}.$$  \hspace{1cm} (87)

Since two orthogonal 3-vectors can be found which satisfy (87), this establishes that (84) can also be satisfied. We normalize $\varepsilon_{1,2}$ so that

$$\varepsilon_1 \cdot \varepsilon_2 = 0 \hspace{1cm} (\varepsilon_1)^2 = (\varepsilon_2)^2 = 1$$  \hspace{1cm} (88)

and form the third polarization, which we will denote $\varepsilon_5$, by taking a linear combination of $n$ and $k$ to guarantee linear independence from $\varepsilon_{1,2}$. Writing

$$\varepsilon_5 = A k + B n$$  \hspace{1cm} (89)

we apply (83) and find that

$$0 = k \cdot \varepsilon_5 = k \cdot (A k + B n) = A k^2 + B (k \cdot n)$$  \hspace{1cm} (90)

We may take

$$A = -B \frac{k \cdot n}{k^2} = B \frac{k \cdot n}{\kappa^2}$$  \hspace{1cm} (91)
so that
\[ \varepsilon_5 = B \left[ n + \sigma \frac{k \cdot n}{k^2} k \right] \]  
(92)

and
\[ (\varepsilon_5)^2 = B^2 \left[ n^2 + 2\sigma \frac{(k \cdot n)^2}{k^2} + \frac{k^2}{k^2} \right] = B^2 \frac{\sigma}{k^2} [k^2 + (k \cdot n)^2] = B^2 \frac{\sigma}{k^2} [k + n(k \cdot n)]^2 \]  
(93)

The expression \( k + n(k \cdot n) \) is just \( k \perp \) and its square is positive definite (expanding as a quadratic in \( k^0 \), one sees that there are no roots). We thus find that choosing
\[ B = \frac{\kappa}{|k + n(k \cdot n)|} \]  
(94)

(92) becomes
\[ (\varepsilon_5)^2 = \sigma. \]  
(95)

In the simple case that \( n = (1, 0, 0, 0) \), we may take as \( \varepsilon_1 \) and \( \varepsilon_2 \) the vectors
\[ \varepsilon_{1,2} = \begin{bmatrix} 0 \\ \tilde{e}_{1,2} \end{bmatrix} \]  
(96)

where
\[ \tilde{k} \cdot \tilde{e}_{1,2} = 0 \quad \tilde{e}_1 \cdot \tilde{e}_2 = 0 \quad \tilde{e}_1 \cdot \tilde{e}_1 = \tilde{e}_2 \cdot \tilde{e}_2 = 1 \]  
(97)

Thus, the set \( \{ \tilde{e}_1, \tilde{e}_2, \tilde{k} = \frac{k}{|k|} \} \) forms an orthonormal basis for the 3-space in this frame. We find for the third polarization \( \varepsilon_5 \),
\[ \varepsilon_5 = \frac{1}{\kappa} \begin{bmatrix} |\tilde{k}| \\ k^0 \tilde{k} \end{bmatrix}. \]  
(98)

Notice that for \( k^2 \to 0, \kappa \to 0 \) and \( \varepsilon_5 \cdot \varepsilon_5 \to 0/0 \), so we will treat the lightlike case as the limit of the massive case, in which we take \( \kappa \to 0 \) at the end of all other computations.

In order to evaluate the commutation relations among the operators \( a(k, s) \) and \( a^*(k, s) \), we must first invert the Fourier expansion of \( a^p(x, \tau) \). To do this, we write the orthogonality relations among the polarization vectors in the form
\[ \varepsilon_s \cdot \varepsilon_{s'} = g_{ss'} = g(s) \delta_{ss'} \quad \text{for } s, s' = 1, 2, 5 \]  
(99)

where
\[ g(s) = \begin{cases} 1, & \text{if } s = s' = 1, 2 \\ \sigma, & \text{if } s = s' = 5 \end{cases} \]  
(100)

so that we have from (82)
\[
\int d^4x e^{-i(k \cdot x + \sigma \tau)} \varepsilon_s \cdot a(x, \tau) = \\
= \sum_{s'} \int d^4x \frac{d^4k'}{2k'} \varepsilon_s \cdot \varepsilon_{s'} \left[ a(k', s') e^{i(k' \cdot x - \sigma(k' \cdot \tau))} + a^*(k', s') e^{-i(k' \cdot x + \sigma(k' \cdot \tau))} \right] \\
= \frac{(2\pi)^4}{2k} g(s) \left[ a(k, s) + a^*(k, s) e^{-2i\sigma \tau} \right] 
\]  
(101)
Similarly,

\[\int d^4x e^{-i(k \cdot x + \sigma x)} \varepsilon_s \cdot \hat{a}(x, \tau) = \sum_{s'} \int d^4x' \frac{d^4k'}{2k'} \varepsilon_s \cdot \varepsilon_{s'}(i\sigma k') \]

\[a(k', s') e^{i(x' \cdot (k' - \kappa s' + \kappa - s) \tau)} - a^*(k', s') e^{-i(x' \cdot (k + k') + \kappa s + k' s') \tau} = \frac{i\sigma}{2} (2\pi)^4 g(s) \left[ a(k, s) - a^*(k, s)e^{-2i\sigma \tau} \right] \]

(102)

Combining (101) and (102), we obtain

\[i\sigma g(s)(2\pi)^4 a(k, s) = \int d^4x e^{-i(k \cdot x + \sigma x)} \varepsilon_s \cdot \left[ \hat{a}(x, \tau) + i\sigma \kappa a(x, \tau) \right] \]

\[= \int d^4x e^{-i(k \cdot x + \sigma x)} \varepsilon_s \cdot \left[ \hat{a}_\tau a(x, \tau) - \hat{a}_x a(x, \tau) \right] \]

\[= \int d^4x e^{-i(k \cdot x + \sigma x)} \varepsilon_s \cdot \left[ \hat{a}_\tau a(x, \tau) \right] \]

so that

\[a(k, s) = \frac{-i\sigma g(s)}{(2\pi)^4} \int d^4x e^{-i(k \cdot x + \sigma x)} \varepsilon_s \cdot \left[ \hat{a}_\tau a(x, \tau) \right] \]

(103)

and

\[a^*(k, s) = \frac{i\sigma g(s)}{(2\pi)^4} \int d^4x e^{i(k \cdot x + \sigma x)} \varepsilon_s \cdot \left[ \hat{a}_\tau a(x, \tau) \right] \]

(105)

where we have used \(1/g(s) = g(s)\).

Expressions (104) and (105) permit us to evaluate the commutators

\[\left[ a(k, s), a^*(k', s') \right] = \frac{g(s)g(s')}{(2\pi)^8} \left\{ \int d^4x \int d^4x' e^{-i(k \cdot x + \sigma x)} e^{i(k' \cdot x' + \sigma x')} \varepsilon_s(k) \cdot \left[ \hat{a}_\tau a(x, \tau) \right], \varepsilon_s'(k') \cdot \left[ \hat{a}_\tau a(x', \tau') \right] \right\} \]

(106)

where

\[\left[ \hat{a}_\tau a^\mu(x, \tau), \hat{a}_\tau a^\nu(x', \tau') \right] = \left[ \left( \hat{a}_\tau - \hat{a}_\tau \right) a^\mu(x, \tau), \left( \hat{a}_\tau - \hat{a}_\tau \right) a^\nu(x', \tau') \right] \]

\[= \left[ \hat{a}_\tau a^\mu(x, \tau), \hat{a}_\tau a^\nu(x', \tau') \right] + \hat{a}_\tau \hat{a}_\tau \left[ a^\mu(x, \tau), a^\nu(x', \tau') \right] \]

(107)

so

\[\left[ a(k, s), a^*(k', s') \right] = \frac{g(s)g(s')}{(2\pi)^8} \varepsilon_s(k) \varepsilon_s'(k') \int d^4x \int d^4x' e^{-i(k \cdot x + \sigma x)} e^{i(k' \cdot x' + \sigma x')} \left\{ \left[ a^\mu(x, \tau), a^\nu(x', \tau') \right] + i\kappa \left[ a^\mu(x, \tau), a^\nu(x', \tau') \right] \right\} \]

\[\{ a^\mu(x, \tau), a^\nu(x', \tau') \} = i\sigma \delta^\mu_\nu (x - x'), \]

(108)

Evaluating the commutators at equal-\(\tau\) and using (74) and (76) to establish

\[a^\mu(x, \tau), a^\nu(x', \tau') = \frac{i\sigma}{\Lambda} \delta^\mu_\nu (x - x'), \]

(109)
one obtains
\[ [a(k, s), a^* (k', s')] = \frac{g(s)g(s')}{(2\pi)^8} \sum_{\epsilon, s} \epsilon_s \epsilon_s' \int d^4x \int d^4x' e^{-i(k \cdot x + s \epsilon x)} e^{i(k' \cdot x + s' \epsilon' x')} \left\{ -i \sigma \kappa' \left( \frac{i \epsilon}{\lambda} \delta_{\perp} (x - x') \right) + i \sigma \kappa \left( \frac{-i \epsilon}{\lambda} \delta_{\perp} (x - x') \right) \right\} \]
\[ = \frac{g(s)g(s')}{(2\pi)^8} \sum_{s, s'} \epsilon_s \epsilon_s' \int d^4x \int d^4x' e^{i(x - (k' - s \epsilon') + s \epsilon x)} \left[ \frac{\kappa + \kappa'}{\lambda} \right] \]
\[ = \frac{2\pi}{\lambda} \frac{g(s)}{(2\pi)^4} \delta_{ss'} \delta^4(k - k') \] (110)

where we have used \( c^2 = 1 \), \( g(s)^2 = 1 \), (84), and (99).

Given the commutation relations for the operators \( a(k, s) \) and \( a^* (k, s) \), we may evaluate the photon propagator as the vacuum expectation value of the \( \tau \)-ordered product of the fields. Thus,
\[ \langle 0 | a^\mu (x, \tau) a^\nu (x', \tau') | 0 \rangle = \theta (\tau - \tau') \langle 0 | a^\mu (x, \tau) a^\nu (x', \tau') | 0 \rangle + \theta (\tau' - \tau) \langle 0 | a^\nu (x', \tau') a^\mu (x, \tau) | 0 \rangle \] (111)

where we use (82) to expand
\[ \langle 0 | a^\mu (x, \tau) a^\nu (x', \tau') | 0 \rangle = \sum_{s, s'} \int \frac{d^4k}{2\pi} \frac{d^4k'}{2\pi} \epsilon_s^\mu \epsilon_{s'}^\nu e^{i(k \cdot x + s \epsilon x)} \left[ \frac{a^\epsilon (k, s) + a^\epsilon (k', s') + a^\epsilon (k' - s' \epsilon') + a^\epsilon (k, s')}{\lambda} \right] \]
\[ = \sum_{s, s'} \int \frac{d^4k}{2\pi} \frac{d^4k'}{2\pi} \epsilon_s^\mu \epsilon_{s'}^\nu e^{i(k \cdot x + s \epsilon x)} \left[ \frac{a^\epsilon (k, s) + a^\epsilon (k', s') + a^\epsilon (k' - s' \epsilon') + a^\epsilon (k, s')}{\lambda} \right] \]
\[ \times \left[ \frac{2\pi}{\lambda} \frac{g(s)}{(2\pi)^4} \delta_{ss'} \delta^4(x - x') \right] \]
\[ = \sum_{s} \frac{g(s)}{(2\pi)^4} \int \frac{d^4k}{2\pi} \epsilon_s^\mu \epsilon_s^\nu e^{i(k \cdot x - x')} \left[ \theta (\tau - \tau') e^{i\sigma (\tau - \tau')} + \theta (\tau' - \tau) e^{i\sigma (\tau - \tau')} \right] \] (112)

so that
\[ \langle 0 | T a^\mu (x, \tau) a^\nu (x', \tau') | 0 \rangle = \sum_{s} \frac{g(s)}{(2\pi)^4} \int \frac{d^4k}{2\pi} \epsilon_s^\mu \epsilon_s^\nu e^{i(k \cdot x - x')} \left[ \theta (\tau - \tau') e^{i\sigma (\tau - \tau')} + \theta (\tau' - \tau) e^{i\sigma (\tau - \tau')} \right] \] (113)

We recognize the
\[ \frac{1}{2\pi} \left[ \theta (\tau - \tau') e^{i\sigma (\tau - \tau')} + \theta (\tau' - \tau) e^{i\sigma (\tau - \tau')} \right] = - \frac{i}{2\pi} \int \frac{dK}{k^2 + \sigma K^2 - i\epsilon} \] (144)

which we may use to put the Green’s function (the vacuum expectation value of the \( \tau \)-ordered product of the fields) in the form of the Feynman propagator for the five-dimensional field:
\[ d^\nu (x - x', \tau - \tau') = i \langle 0 | T a^\mu (x, \tau) a^\nu (x', \tau') | 0 \rangle \]
\[ = \int \frac{d^4k}{(2\pi)^4} \sum_{s} \frac{g(s)}{\lambda} \epsilon_s^\mu \epsilon_s^\nu e^{i(k \cdot (x - x') + s \epsilon (x - x'))} \frac{1}{k^2 + \sigma K^2 - i\epsilon} \]. (115)
In order to evaluate the sum over polarizations in (115), we must consider the cases $\sigma = \pm 1$ separately. For the case that $\sigma = 1$, we must satisfy $k^2 = -\kappa^2 < 0$, and we may take
\[
k = \lim_{\alpha \to 0} \left( \sqrt{\kappa^2 + \alpha^2}, 0, 0, \alpha \right) = (\kappa, 0, 0, 0).
\]
Choosing
\[
n = (1, 0, 0, 0) \quad \epsilon_1 = (0, 1, 0, 0) \quad \epsilon_2 = (0, 0, 1, 0)
\]
we find from (98) and (116) that
\[
\epsilon_5 = (\vec{k}, k^0 k) \frac{1}{\sqrt{-k^2}} = \lim_{\alpha \to 0} \frac{1}{\sqrt{\kappa^2 + \alpha^2}}(\alpha, 0, \sqrt{\kappa^2 + \alpha^2}) = (0, 0, 1).
\]
The completeness relation may then be written in the form
\[
g^{\mu\nu} = -\frac{1}{\kappa^2} k^\mu k^\nu + (\epsilon_1)^\mu (\epsilon_1)^\nu + (\epsilon_2)^\mu (\epsilon_2)^\nu + (\epsilon_5)^\mu (\epsilon_5)^\nu
\]
which we may rearrange to write
\[
\sum_{s=1,2,5} g(s)(\epsilon_s)^\mu (\epsilon_s)^\nu = \sum_{s=1,2,5} (\epsilon_s)^\mu (\epsilon_s)^\nu = g^{\mu\nu} + \frac{1}{\kappa^2} k^\mu k^\nu = g^{\mu\nu} - \frac{1}{\kappa^2} k^\mu k^\nu
\]
where we have used $g(1, 2) = 1$ and $g(5) = \sigma = 1$.
For the case that $\sigma = -1$, we have $k^2 = \kappa^2 > 0$ so we may take
\[
k = (0, 0, 0, \kappa).
\]
and using (98) we find that
\[
\epsilon_5 = (\vec{k}, k^0 k) \frac{1}{\sqrt{-k^2}} = (1, 0, 0, 0).
\]
In this case the completeness relation is
\[
g^{\mu\nu} = - (\epsilon_5)^\mu (\epsilon_5)^\nu + (\epsilon_1)^\mu (\epsilon_1)^\nu + (\epsilon_2)^\mu (\epsilon_2)^\nu + \frac{1}{\kappa^2} k^\mu k^\nu
\]
and it may be rearranged as
\[
\sum_{s=1,2,5} g(s)(\epsilon_s)^\mu (\epsilon_s)^\nu = \sum_{s=1,2} (\epsilon_s)^\mu (\epsilon_s)^\nu - (\epsilon_5)^\mu (\epsilon_5)^\nu = g^{\mu\nu} - \frac{1}{\kappa^2} k^\mu k^\nu = g^{\mu\nu} - \frac{1}{\kappa^2} k^\mu k^\nu
\]
where we have used $g(1, 2) = 1$ and $g(5) = \sigma = -1$. Notice that (124) and (120) are identical and are consistent with the transversality requirements on the polarization states. Using this expression and observing that the sum over polarization states gives the projection operator $P^{\mu\nu}(k)$ defined in (70), equation (75) may be verified explicitly.

To evaluate the Hamiltonian in the momentum representation,
\[
\frac{\lambda}{4} f_{\mu\nu} f^{\mu\nu} = \frac{\lambda}{4} (\partial_\mu a_\nu - \partial_\nu a_\mu)(\partial^\mu a^\nu - \partial^\nu a^\mu) = -\frac{\lambda}{2} a_\mu \Box a^\mu + \text{divergence}
\]
and
\[-\frac{\lambda\sigma}{2} e^{\mu} e_\mu = -\frac{\lambda\sigma}{2} (\sigma \partial_\tau a_\mu) (\sigma \partial_\tau a^\mu) = -\frac{\lambda\sigma}{2} [\partial_\tau (a^{\mu} \partial_\tau a_\mu) - a_\mu \partial_\tau^2 a^{\mu}] = \frac{\lambda\sigma}{2} a_\mu \partial_\tau^2 a^{\mu}\] (126)
where we drop divergences and total \(\tau\)-derivatives. The Hamiltonian becomes
\[K = \frac{\lambda}{2} \int d^4x a_\mu [-\Box + \sigma \partial_\tau^2] a^{\mu} = \lambda\sigma \int d^4x a_\mu \partial_\tau^2 a^{\mu}\] (127)
where we have used the wave equation (81). Using the momentum expansion (82) and the
\[\langle \psi \rangle\] theory. The time independence of \(\langle \psi \rangle\) imposed unitarity on \(U(\tau)\) which is easily
shown to be generated by the Hamiltonian operator through \(U = e^{-iK\tau}\), such that \(\langle \psi(\tau)\rangle\).

5. Perturbation theory

Defining Heisenberg states \(\langle \psi \rangle\) as \(\tau\)-independent and the corresponding Schrödinger states through
\[\langle \psi(\tau) \rangle = U(\tau) \langle \psi(0) \rangle = U(\tau) \langle \psi \rangle\] (132)
the familiar \(t\)-dependent perturbation theory becomes a manifestly covariant \(\tau\)-dependent
theory. The time independence of \(\langle \psi(\tau) \rangle \langle \psi(\tau) \rangle\) imposes unitarity on \(U(\tau)\) which is easily
shown to be generated by the Hamiltonian operator through \(U = e^{-iK\tau}\), such that \(\langle \psi(\tau)\rangle\)
satisfies the Schrödinger equation. Writing $K = K_0 + K_I$ where $K_0$ is the free Hamiltonian and $K_I$ represents the interaction, we are led to the interaction picture [18] in which

$$i\partial_\tau |\phi(\tau)\rangle_I = e^{iK_0\tau}Ke^{-iK_0\tau} |\phi(\tau)\rangle_I = K_I(\tau) |\phi(\tau)\rangle_I.$$  \hspace{1cm} (133)

Equation (133) has the formal solution

$$|\phi(\tau)\rangle_I = V(\tau, \tau')|\phi(\tau')\rangle_I = Te^{-i\int_\tau^{\tau'}d\tau'' K_I(\tau'')}|\phi(\tau')\rangle_I$$ \hspace{1cm} (134)

in which the symbol $T$ refers to $\tau$-ordering. The S-matrix is given by

$$S = \lim_{\tau' \rightarrow \infty} Te^{-i\int_\tau^{\tau'}d\tau'' K_I(\tau'')} = Te^{-i\int d^4x d\tau K_I}.$$ \hspace{1cm} (135)

Scattering amplitudes may be computed from the interaction picture states as

$$W_{i \rightarrow f} = \langle f|S|i \rangle.$$ \hspace{1cm} (136)

Since the initial and final states are defined when there is no interaction, the asymptotic interaction picture states are identical with the asymptotic Heisenberg states. Moreover, since the Heisenberg picture Hamiltonian is formed of bilinear combinations of the fields without $\tau$-derivatives, the interacting Hamiltonian in (135) has the same form as the Heisenberg picture Hamiltonian.

**Reduction Formulas**

In order to use Wick’s theorem for perturbation expansions of the S-matrix, [3], we must adapt the LSZ reduction formulas [3] to the present theory. Consider the amplitude

$$\langle k' \text{ out}|k \text{ in} \rangle = \langle k' \text{ out}|b^*_m(k)|0 \rangle$$ \hspace{1cm} (137)

From (55) we find that

$$b^*(k) = \int d^4x e^{i(k\cdot x - \tau \cdot \kappa)} \psi^*(x, \tau) \quad b(k) = \int d^4x e^{i(k\cdot x)} \psi(x, \tau)$$ \hspace{1cm} (138)

which we may insert in (137) to obtain

$$\langle k' \text{ out}|k \text{ in} \rangle = \int d^4x \langle k' \text{ out}|\psi^*_m(x, \tau)|0 \rangle e^{i(k\cdot x - \tau \cdot \kappa)}$$ \hspace{1cm} (139)

which is valid for arbitrary $\tau_{in}$. Now, consider the integral

$$\int_{\tau_i}^{\tau_f} d\tau \int d^4x \partial_\tau \left\{ \langle k' \text{ out}|\psi^*(x, \tau)|0 \rangle e^{i(k\cdot x - \tau \cdot \kappa)} \right\} =$$

$$= \int d^4x e^{i(k\cdot x - \tau \cdot \kappa)} \langle k' \text{ out}|\psi^*(x, \tau_f) - \psi^*(x, \tau_i)|0 \rangle$$

$$= \langle k' \text{ out}|b^*_m(k)|0 \rangle - \int d^4x e^{i(k\cdot x - \tau \cdot \kappa)} \langle k' \text{ out}|\psi^*_m(x, \tau_i)|0 \rangle,$$ \hspace{1cm} (140)
which we may rearrange as

\[
\langle k' \text{ out}|k \text{ in} \rangle = - \int \! \! d^4 x d \tau d \tau' \ \left\{ e^{i(k' \cdot x - k' \cdot \tau)} \langle k' \text{ out}|\psi^*(x, \tau)|0\rangle + \langle k' \text{ out}|b_{\text{out}}^*(k)|0\rangle \right\}
\]

\[
= - \int \! \! d^4 x d \tau e^{i(k' \cdot x - k' \cdot \tau)} \setminus \left\{ -i k + \partial \tau \right\} \langle k' \text{ out}|\psi^*(x, \tau)|0\rangle + \langle k' \text{ out}|b_{\text{out}}^*(k)|0\rangle
\]

\[
= - \int \! \! d^4 x d \tau e^{i(k' \cdot x - k' \cdot \tau)} \setminus \left\{ \frac{i k^2}{2M} + \partial \tau \right\} \langle k' \text{ out}|\psi^*(x, \tau)|0\rangle + \langle k' \text{ out}|b_{\text{out}}^*(k)|0\rangle
\]

\[
= - \int \! \! d^4 x d \tau e^{i(k' \cdot x - k' \cdot \tau)} \setminus \left\{ \frac{i}{2M} \square + \partial \tau \right\} \langle k' \text{ out}|\psi^*(x, \tau)|0\rangle + \langle k' \text{ out}|b_{\text{out}}^*(k)|0\rangle
\]

\[
= i \int \! \! d^4 x \! d \tau e^{i(k' \cdot x - k' \cdot \tau)} \setminus \left\{ i \partial \tau - \frac{1}{2M} \square \right\} \langle k' \text{ out}|\psi^*(x, \tau)|0\rangle + \langle k' \text{ out}|b_{\text{out}}^*(k)|0\rangle
\]

(141)

where we performed two integrations by parts in the second to last step. The last term is a “disconnected term”; it corresponds to the non-scattering path and makes no contribution to the scattering amplitude. Now consider the integral

\[
\int_{T_f}^{T_i} \! d \tau' \int \! \! d^4 x' \partial \tau' \left[ \langle 0|\Psi(x', \tau')\psi^*(x, \tau)|0\rangle e^{-i(k' \cdot x' - k' \cdot \tau')} \right]
\]

\[
= \int \! \! d^4 x' e^{-i(k' \cdot x' - k' \cdot \tau')} \langle 0|\Psi(x', \tau')\psi^*(x, \tau)|0\rangle \bigg|_{T_f}^{T_i}
\]

\[
= \langle 0|b_{\text{out}}(k')\psi^*(x, \tau)|0\rangle - \langle 0|\psi^*(x, \tau)b_{\text{in}}(k')|0\rangle
\]

(142)

in which the second term in (142) vanishes. For the second term in (141), we use (142) to similarly expand as

\[
\langle k' \text{ out}|\psi^*(x, \tau)|0\rangle = \langle 0|b_{\text{out}}(k')\psi^*(x, \tau)|0\rangle
\]

\[
= \int \! \! d^4 x' e^{i(k' \cdot x' - k' \cdot \tau')} \langle 0|\psi_{\text{out}}(x', \tau')\psi^*(x, \tau)|0\rangle
\]

\[
= \int \! \! d^4 x' d \tau' \partial \tau' \left\{ e^{-i(k' \cdot x' - k' \cdot \tau')} \langle 0|\Psi(x', \tau')\psi^*(x, \tau)|0\rangle \right\} +
\]

\[
\langle 0|\psi^*(x, \tau)b_{\text{in}}(k')|0\rangle
\]

(143)

and again the second term vanishes. Finally, we arrive at

\[
\langle k' \text{ out}|k \text{ in} \rangle = i^2 \int \! \! d^4 x \! d^4 x' d \tau d \tau' e^{i(k' \cdot x - k' \cdot \tau)} e^{-i(k' \cdot x' - k' \cdot \tau')}
\]

\[
\left[ i \partial \tau' + \frac{1}{2M} \square \right] \left[ i \partial \tau - \frac{1}{2M} \square \right] \langle 0|\Psi(x', \tau')\psi^*(x, \tau)|0\rangle
\]

(144)

By continuing this process, we can write

\[
\langle k_1' \cdots k_n' \text{ out}|k_1 \cdots k_m \text{ in} \rangle = i^{n + m} \int \! \! d^5 x_1 \cdots d^5 x_m d^5 x_1' \cdots d^5 x_n'
\]

\[
e^{i(k_1' \cdot x_1 + \cdots + k_m' \cdot x_m)} \approx^{i(k_1' \cdot x_1' + \cdots + k_m' \cdot x_m')}
\]

\[
\left[ i \partial \tau_1' + \frac{1}{2M} \square_1 \right] \cdots \left[ i \partial \tau_n' + \frac{1}{2M} \square_n \right] \left[ i \partial \tau_1 - \frac{1}{2M} \square_1 \right] \cdots \left[ i \partial \tau_m - \frac{1}{2M} \square_m \right]
\]

\[
\langle 0|\Psi(x_1', \tau_1') \cdots \psi(x_n', \tau_n')\psi^*(x_1, \tau_1) \cdots \psi^*(x_m, \tau_m)|0\rangle
\]

(145)

where we denote $d^5 x = d^4 x d \tau$. 
For the gauge field, the reduction formula follows closely the development for the usual Maxwell case. We begin with

\[
\langle \beta, k, s \text{ out} | \alpha \text{ in} \rangle = \langle \beta \text{ out} | a(k, s) | \alpha \text{ in} \rangle = \frac{-i \sigma g(s)}{(2\pi)^4} \int d^4x e^{-i(k \cdot x + \sigma \epsilon(x, \tau))} \langle \beta \text{ out} | \epsilon_s \cdot \partial_\tau a(x, \tau) | \alpha \text{ in} \rangle
\]

(146)

where \( \alpha \) and \( \beta \) are the other quantum numbers for the states. We may again use

\[
\int^{\tau_f}_{\tau_i} d\tau \int d^4x \partial_\tau g(x, \tau) = \int d^4x [g(x, \tau_f) - g(x, \tau_i)]
\]

(147)

to write

\[
\int^{\tau_f}_{\tau_i} d\tau \int d^4x \partial_\tau g(x, \tau) = \frac{-i \sigma g(s)}{(2\pi)^4} \int d^4x e^{-i(k \cdot x + \sigma \epsilon(x, \tau))} \left\{ \langle \beta \text{ out} | \epsilon_s \cdot \partial_\tau a(x, \tau_f) | \alpha \text{ in} \rangle - \langle \beta \text{ out} | \epsilon_s \cdot \partial_\tau a(x, \tau_i) | \alpha \text{ in} \rangle \right\}
\]

\[
= \frac{-i \sigma g(s)}{(2\pi)^4} \int d^4x e^{-i(k \cdot x + \sigma \epsilon(x, \tau))} \langle \beta \text{ out} | \epsilon_s \cdot \partial_\tau a(x, \tau_f) | \alpha \text{ in} \rangle
\]

(148)

where we have used the fact that \( a(x, \tau_i) | 0 \text{ in} \rangle = 0 \). By expanding \( \partial_\tau \), using the wave equation (81), and performing two integrations by parts, we arrive at

\[
\langle \beta, k, s \text{ out} | \alpha \text{ in} \rangle = \frac{-i g(s)}{(2\pi)^4} \int d^3x e^{-i(k \cdot x + \sigma \epsilon(x, \tau))} \left[ \Box + \sigma \partial^2_\tau \right] \langle \beta \text{ out} | \epsilon_s \cdot a(x, \tau) | \alpha \text{ in} \rangle
\]

(149)

A second application of this procedure enables us to write

\[
\langle \beta, k, s \text{ out} | \alpha', s' \text{ in} \rangle = \frac{(-i)^2 g(s) g(s')}{(2\pi)^8} \int d^3x d^3x' e^{-i(k \cdot x + \sigma \epsilon(x, \tau))} \epsilon(k' \cdot x' + \sigma \epsilon(x', \tau')}
\]

\[
[\Box + \sigma \partial^2_\tau] \left[ \Box + \sigma \partial^2_\tau' \right] \langle \beta \text{ out} | \epsilon_s \cdot a(x, \tau) \epsilon_{s'} \cdot a(x', \tau') | \alpha \text{ in} \rangle
\]

(150)

**Feynman Rules**

In order to write the Feynman rules for the interacting off-shell theory, may use the general expression for Green’s functions, which may be derived from the path integral [19] or by operator methods [3],

\[
G^{(n)}(x_1, \tau_1, \cdots, x_n, \tau_n) = \langle 0 | T \phi(x_1, \tau_1) \cdots \phi(x_n, \tau_n) e^{i \int d^4y \tau L_{\text{int}}} | 0 \rangle
\]

(151)

in which vacuum-vacuum diagrams are excluded from (151), and \( \phi \) represents the non-interacting free fields of the theory, which permits us to use Wick’s theorem and the free propagators (63) and (115). In (151), we adopt the conventional notation for Green’s functions in field theory; in relation to the non-interaction propagators calculated above, these are

\[
G^{(2)}(x, \tau) = \langle 0 | T \psi(x, \tau) \psi^*(0) | 0 \rangle_{\text{tree}} = -iG(x, \tau)
\]

\[
d^{(2)}_{\mu\nu}(x, \tau) = \langle 0 | T a_\mu(x, \tau) a_\nu(0) | 0 \rangle_{\text{tree}} = -id_{\mu\nu}(x, \tau)
\]

(152)
where \(G(x, \tau)\) and \(d_{\mu} (x, \tau)\) are defined in (60) and (115).

Using (49) for the interacting Hamiltonian, we have

\[
\mathcal{L}_{\text{int}} = -\mathcal{K}_{\text{interaction}} = -\frac{ie_0}{2M} a_\mu (\psi^* \partial^\mu \psi - (\partial^\mu \psi^*) \psi) - \frac{e_1^2}{2M} a_\mu a^\mu |\psi|^2. \tag{153}
\]

The two terms in (153) correspond to the two basic diagrams

\[\text{Diagram 1} \quad \text{Diagram 2}\]

and we may calculate the vertex factors separately. For the first diagram, we have

\[
G^{(3)}_\mu (x_1, \tau_1, x_2, \tau_2, x_3, \tau_3) = \langle 0 | T \psi^* (x_1, \tau_1) \psi (x_2, \tau_2) a_\mu (x_3, \tau_3) e^{i \int d^4 y d\tau a_\mu (y, \tau) \mu (y, \tau)} | 0_{\text{tree}} \rangle \tag{154}
\]

where only the tree-level diagram is considered in (154) and \(j^\mu\) is the vector part of the free matter field current (13),

\[
j^\mu = -\frac{i}{2M} (\psi^* \partial^\mu \psi - (\partial^\mu \psi^*) \psi) \tag{155}
\]

Expanding the exponential in (154), we find that the tree-level term is given by

\[
G^{(3)}_\mu (x_1, \tau_1, x_2, \tau_2, x_3, \tau_3) = \\
= \langle 0 | T \psi^* (x_1, \tau_1) \psi (x_2, \tau_2) a_\mu (x_3, \tau_3) e^{i \int d^4 y d\tau a_\mu (y, \tau) j^\mu (y, \tau)} | 0 \rangle \\
= \frac{e_0}{2M} \langle 0 | T \psi^* (x_1, \tau_1) \psi (x_2, \tau_2) a_\mu (x_3, \tau_3) e^{i \int d^4 y d\tau a_\mu (y, \tau) \psi^* (y, \tau) \overleftrightarrow{\partial^\nu} \psi (y, \tau)} | 0 \rangle \\
= \frac{e_0}{2M} \int d^4 y d\tau \langle 0 | T a_\mu (x_3, \tau_3) a^\mu (y, \tau) | 0 \rangle \langle 0 | T \psi^* (x_1, \tau_1) \psi (x_2, \tau_2) \psi^* (y, \tau) \overleftrightarrow{\partial^\nu} \psi (y, \tau) | 0 \rangle \\
= \frac{e_0}{2M} \int d^4 y d\tau \langle 0 | T a_\mu (x_3, \tau_3) a^\mu (y, \tau) | 0 \rangle \\
\left[ \langle 0 | T \psi^* (x_2, \tau_2) \psi^* (y, \tau) | 0 \rangle \overleftrightarrow{\partial^\nu} \langle 0 | T \psi^* (x_1, \tau_1) \psi (y, \tau) | 0 \rangle \right] \\
= \frac{e_0}{2M} \int d^4 y d\tau \left[ -i d^\rho_\mu (x_3 - y, \tau - \tau_3) \right] \\
\left[ (-i G (x_2 - y, \tau_2 - \tau)) \overleftrightarrow{\partial^\nu} (-i G (y - x_1, \tau - \tau_1)) \right] \tag{156}
\]

We now use the Fourier expansions for the free propagators to write

\[
G^{(3)}_\mu (x_1, \tau_1, x_2, \tau_2, x_3, \tau_3) = \frac{i e_0}{2M} \int d^4 y d\tau \frac{d^3 k d^3 p d^3 p'}{(2\pi)^{15} \lambda} \frac{e^{i \langle k \cdot (x_3 - y) + \sigma (\tau - \tau_3) \rangle}}{k^2 + \sigma \kappa^2 - i \epsilon} \\
\left[ e^{i \langle p^\nu (y - x_1) - P^\mu (\tau - \tau_1) \rangle} \overleftrightarrow{\partial^\nu} e^{i \langle p^\nu (y - x_2) \rangle - P^\mu (\tau - \tau_2) \rangle} \right] \\
\left[ e^{i \langle p^\nu (x_1 - P^\mu) \rangle} - e^{-i \langle p^\nu (x_1 - P^\mu) \rangle} \right] \\
= \frac{i e_0}{2M} \int d^4 y d\tau \frac{d^3 k d^3 p d^3 p'}{(2\pi)^{15} \lambda} \frac{e^{i \langle k \cdot (x_3 - y) + \sigma (\tau - \tau_3) \rangle}}{k^2 + \sigma \kappa^2 - i \epsilon} \\
\left[ e^{i \langle p^\nu (y - x_1) - P^\mu (\tau - \tau_1) \rangle} \overleftrightarrow{\partial^\nu} e^{i \langle p^\nu (y - x_2) \rangle - P^\mu (\tau - \tau_2) \rangle} \right] \\
\left[ e^{i \langle p^\nu (x_1 - P^\mu) \rangle} - e^{-i \langle p^\nu (x_1 - P^\mu) \rangle} \right] 
\]
\[
G^{(3)}_{\mu}(p, p', k) = \frac{e_0}{2M} P^\nu_\mu(k) i(p + p') \nu \left(2\pi\right)^5 \delta^4(p - p' - k) \delta(P - P' + \sigma\kappa) 
\]

which we identify as the product of the three propagators and the vertex factor for the interaction

\[
e_0 \frac{i(p + p') \nu}{2M} \left(2\pi\right)^5 \delta^4(p - p' - k) \delta(P - P' + \sigma\kappa)
\]

in which

\[
\kappa = \sqrt{-\sigma k^2} \quad P = \frac{p^2}{2M} \quad P' = \frac{p'^2}{2M}
\]
We may summarize the Feynman rules for the momentum space Green's functions as follows:

\[ G(\mu\nu)(p) = -\frac{i\hbar^2}{M\lambda^2} \int \frac{d^5k'd^5k d^5p d^5p'}{(2\pi)^{20}} (2\pi)^4 \delta^4(k - k' - p' + p) \delta(\sigma\kappa - \sigma' + P' - P) \mathcal{P}_\mu(k') e^{-i k' \cdot x} \mathcal{P}_\nu(k) e^{i P' \cdot x} \]

Notice that a factor of 2 appears in the fourth line, because there are two ways to contract the photon operators. Transforming to the momentum space Green's function, with diagram

\[ \begin{array}{c}
  k, \nu \\
  \downarrow \\
  p \\
  \downarrow \\
  k', \nu' \\
  \to \\
  p' \\
\end{array} \]

we find

\[ G^{(4)}_{\mu\nu}(p, p', k, k') = -\frac{i\hbar^2}{M}\lambda^2 (2\pi)^4 \delta^4(k - k' - p' + p) \delta(\sigma\kappa - \sigma' + P' - P) \mathcal{P}_\mu(k') \mathcal{P}_\nu(k) \]

\[ \frac{-i \hbar^2}{M} (2\pi)^4 \delta^4(k - k' - p' + p) \delta(\sigma\kappa - \sigma' + P' - P) \]

which we recognize as the product of the four propagators and the vertex factor

\[ \frac{-i \hbar^2}{M} (2\pi)^4 \delta^4(k - k' - p' + p) \delta(\sigma\kappa - \sigma' + P' - P) \]

We may summarize the Feynman rules for the momentum space Green's functions as follows:

(i) For each matter field propagator, draw a directed line associated with the factor

\[ \frac{1}{(2\pi)^5} \frac{-i}{2\pi} \frac{1}{p^2 - P - i\epsilon} \]

(ii) For each photon propagator, draw a photon line associated with the factor

\[ \frac{1}{\lambda} \mathcal{P}_{\mu\nu} \frac{-i}{k^2 + \sigma\kappa^2 - i\epsilon} \]

(iii) For the three-particle interaction, write the vertex factor

\[ \frac{\epsilon_0}{2M} (p + p')^\nu (2\pi)^5 \delta^4(p - p' - k) \delta(P' - P' + \sigma\kappa) \]

(iv) For the four-particle interaction, write the vertex factor

\[ \frac{-i\hbar^2}{M} (2\pi)^5 \delta^4(k - k' - p' + p) \delta(\sigma\kappa - \sigma' + P' - P) \]
To obtain the Feynman rules for the S-matrix elements, we use the reduction formulas together with the free propagators for the incoming and outgoing particles. From (145) for the matter fields, we see that inserting a propagator will precisely cancel the derivative operator, so that in calculating S-matrix elements, the incoming and outgoing propagators are replaced by 1. Using (149) for the photons, we see that the derivative operator will cancel the factor \(-i/\left(k^2 + \sigma \kappa^2 - i\epsilon \right)\), but not the factor \(\lambda\). So in the rules for the S-matrix elements we replace the incoming and outgoing photon propagators with the factor

\[
\pm \frac{ig(s)}{(2\pi)^4} \epsilon^\mu(k, s)
\]  

where the incoming photon takes the sign \(-\) and the outgoing photon takes the sign \(+\).

### 6. Scattering cross-sections

In this section, we calculate the relationship of the scattering cross-section to the transition amplitudes calculated from the S-matrix expansions discussed in the previous section. In particular, we specialize the five-dimensional quantum theory to the case of scattering. The development here generally follows the presentation given in [3], with elements taken from [20] and [21].

The initial state will be represented by constructing wave packets of the form

\[
|\Psi_p\rangle = \int \frac{d^4 q}{(2\pi)^4} \tilde{\psi}_p(q) |q\rangle = \int \frac{d^4 q}{(2\pi)^4} \tilde{\psi}(q - p) |q\rangle
\]  

in which we take \(\tilde{\psi}_p(q) = \tilde{\psi}(q - p)\) to be a narrow distribution centered around the momentum \(p\). These wave packets correspond to linear superpositions of solutions to the quantum mechanical eigenvalue equation

\[
K \psi_q = \kappa_q \psi_q
\]  

where

\[
K = \frac{p^2}{2M} + V \quad \text{and} \quad \kappa_q = \frac{q^2}{2M} + \delta \kappa_q.
\]

The wavefunctions may be written in the form

\[
\Psi_p(x) = \int \frac{d^4 q}{(2\pi)^4} \tilde{\psi}(q - p) e^{i q \cdot x} = e^{i p \cdot x} \int \frac{d^4 \rho}{(2\pi)^4} \tilde{\psi}(\rho) e^{i \rho \cdot x} = e^{i p \cdot x} G(x)
\]  

in which \(G(x)\) is a slowly varying, localized function of \(x\). The \(\tau\)-dependent wavefunctions are

\[
\Psi(x, \tau) = e^{-i K \tau} \Psi_p(x) = \int \frac{d^4 q}{(2\pi)^4} \tilde{\psi}(q - p) e^{i \left(q \cdot x - \kappa_q \tau\right)}.
\]

Since the momentum distribution is narrowly centered around \(p\), the wavepacket in (169), may be expanded in \(q = p + \rho\). To do this note that

\[
\kappa_q = \kappa_p + \rho^\mu \partial_{\rho^\mu} \kappa_p + \frac{1}{2} \rho^\mu \rho^\nu \partial_{\rho^\mu} \partial_{\rho^\nu} \kappa_p + \cdots = \kappa_p + \rho \cdot u + o(\rho^2)
\]
where \( u^\nu = \partial \kappa / \partial p_\nu \) is the group 4-velocity. Then, to first order in \( \rho \),

\[
\Psi(x, \tau) = \int \frac{d^4 \rho}{(2\pi)^4} e^{i[(p+p)\cdot x - (\kappa + p \cdot u)\cdot \tau]} \tilde{\psi}(\rho) = e^{i[p \cdot x - \kappa \cdot \tau]} \int \frac{d^4 \rho}{(2\pi)^4} \tilde{\psi}(\rho) e^{i[p \cdot x - \kappa \cdot \tau]} = e^{i[p \cdot x - \kappa \cdot \tau]} G(x - u\tau)
\]

and the slowly varying function \( G(x - u\tau) \) describes the Ehrenfest motion of the wavepacket. The event density corresponding to the wavefunction in (171) is given by the integral of the current

\[
f^4(x, \tau) = \overline{\Psi}_p \Psi_p = |G(x - u\tau)|^2.
\]

In scattering, the initial state is a product of the target state and the beam state, and is represented by a wavepacket of the form

\[
|i\rangle = |\Psi_{\text{target}}\rangle |\Psi_{\text{beam}}\rangle = \int \frac{d^4 q_T}{(2\pi)^4} \frac{d^4 q_B}{(2\pi)^4} \bar{\psi}_T(q_T - p_T) \bar{\psi}_B(q_B - p_B) |q_T q_B\rangle.
\]

Writing the transition matrix \( T \) in terms of the scattering matrix \( S \)

\[
S = 1 + iT
\]

we find for the initial state defined in (173),

\[
\langle f|\bar{T}|i\rangle = \int \frac{d^4 q_T}{(2\pi)^4} \frac{d^4 q_B}{(2\pi)^4} \bar{\psi}_T(q_T - p_T) \bar{\psi}_B(q_B - p_B) \langle f|\bar{T}|q_T q_B\rangle.
\]

The transition probability becomes

\[
|\langle f|T|i\rangle|^2 = \int \frac{d^4 q_T}{(2\pi)^4} \frac{d^4 q_T'}{(2\pi)^4} \frac{d^4 q_B}{(2\pi)^4} \frac{d^4 q_B'}{(2\pi)^4} \bar{\psi}_T(q_T - p_T) \bar{\psi}_T(q_T' - p_T) \bar{\psi}_B(q_B - p_B) \bar{\psi}_B(q_B' - p_B) \langle f|\bar{T}|q_T q_B\rangle \langle f|\bar{T}|q_T' q_B'\rangle \delta(\kappa_f - \kappa_i) \delta^4(p_f - p_i) \delta(\kappa_f' - \kappa_i') \delta^4(p_f' - p_i').
\]

where

\[
p_i = q_T + q_B = p_T + p_B + \rho_T + \rho_B
\]

\[
p_i' = q_T' + q_B' = p_T + p_B + \rho_T' + \rho_B'
\]

\[
\kappa_i = \kappa_T + \kappa_B = \frac{q_T^2}{2M_T} + \frac{q_B^2}{2M_B}
\]

We assume that the interaction occurs close to the central momenta, so that

\[
\langle f|\bar{T}|q_T q_B\rangle \langle f|\bar{T}|q_T' q_B'\rangle \delta(\kappa_f - \kappa_i) \delta^4(p_f - p_i) \approx |\langle f|\bar{T}|p_T p_B\rangle|^2 = |\tau_f|^2.
\]
We may rewrite the $\delta$-functions, replacing $q = p + \rho$ for each momentum:

$$\delta^4(p_f - p_i) \delta^4(p_f - p_i') = \delta^4(p_f - p_T - \rho_T - p_B - \rho_B) \delta^4(p_f - p_T - \rho_T' - p_B - \rho_B')$$

Similarly,

$$\delta(\kappa_f - \kappa_i) \delta(\kappa_f - \kappa_i') \simeq \delta(\kappa_{p_f} - \kappa_{p_T} - \kappa_{p_B} - \kappa_{p_B'}) \delta(\kappa_{p_T} + \kappa_{p_B} - \kappa_{p_T'} - \kappa_{p_B'}) \delta(\kappa_{p_B} - \kappa_{p_B'} - \kappa_{p_T} - \kappa_{p_T'}) \tag{182}$$

so that, since $d^4q = d^4p$, (176) becomes

$$\sqrt{\langle f | T | i \rangle^2} = (2\pi)^{10} \delta(\kappa_{p_f} - \kappa_{p_T} - \kappa_{p_B} - \kappa_{p_B'}) \delta^4(p_f - p_T - p_B) \langle T_{fi} \rangle^2$$

$$\int d^4\rho_T \, d^4\rho_T' \, d^4\rho_B \, d^4\rho_B' \, d^4\rho_{i} \, \delta^4(\rho_T - \rho_T' + \rho_B - \rho_B')$$

$$\delta(\kappa_{p_T} + \kappa_{p_B} - \kappa_{p_T'} - \kappa_{p_B'}) \tag{183}$$

Notice that since $\kappa_{p} \simeq \rho \cdot u$,

$$\kappa_{p_T} - \kappa_{p_T'} + \kappa_{p_B} - \kappa_{p_B'} \simeq \rho_T \cdot u_T - \rho_T' \cdot u_T + \rho_B \cdot u_B - \rho_B' \cdot u_B$$

$$= (\rho_T - \rho_T') \cdot u_T + (\rho_B - \rho_B') \cdot u_B$$

$$= (u_B - u_T) \cdot (\rho_B - \rho_B') + u_T \cdot (\rho_B - \rho_B' + \rho_T - \rho_T')$$

$$= (u_B - u_T) \cdot (\rho_B - \rho_B') \tag{184}$$

where we have used (181) in the last line of (184). To continue, we write integral representations for the $\delta$-functions. Thus,

$$\delta^4(\rho_T - \rho_T' + \rho_B - \rho_B') = \frac{1}{(2\pi)^4} \int d^4x e^{i \mathbf{x} \cdot \mathbf{p}}$$

and

$$\delta((u_B - u_T) \cdot (\rho_B - \rho_B')) = \int \frac{d\alpha}{2\pi} e^{i \mathbf{a} \cdot (u_B - u_T) \cdot (\rho_B - \rho_B')} \tag{185}$$

Inserting (184), (185) and (186) into (183), we have

$$\sqrt{\langle f | T | i \rangle^2} = (2\pi)^{10} \delta(\kappa_{p_f} - \kappa_{p_T} - \kappa_{p_B} - \kappa_{p_B'}) \delta^4(p_f - p_T - p_B) \langle T_{fi} \rangle^2$$

$$\int d^4\rho_T \, d^4\rho_T' \, d^4\rho_B \, d^4\rho_B' \, d^4\rho_{i} \, \delta^4(\rho_T - \rho_T' + \rho_B - \rho_B') e^{i \mathbf{x} \cdot \mathbf{p}}$$

$$\delta(\kappa_{p_T} + \kappa_{p_B} - \kappa_{p_T'} - \kappa_{p_B'}) \delta^4(p_f - p_T - p_B) \langle T_{fi} \rangle^2$$

$$\int d^4\rho_T \, d^4\rho_B \, d^4\rho_{i} \, \delta(\kappa_{p_T} + \kappa_{p_B} - \kappa_{p_T'} - \kappa_{p_B'}) \delta^4(p_f - p_T - p_B) \langle T_{fi} \rangle^2$$

$$\int d^4x \, d\alpha \, |G_T(x)|^2 \left| G_B \left( x + \alpha (u_B - u_T) \right) \right|^2 \tag{187}$$

The transition probability per unit spacetime volume is then

$$\frac{d}{dVdT} |\langle f | T | i \rangle|^2 = (2\pi)^{5} \delta(\kappa_{p_f} - \kappa_{p_T} - \kappa_{p_B}) \delta^4(p_f - p_T - p_B) \langle T_{fi} \rangle^2$$

$$\int d\alpha \, |G_T(x)|^2 \left| G_B \left( x + \alpha (u_B - u_T) \right) \right|^2$$

$$= (2\pi)^{5} \delta(\kappa_{p_f} - \kappa_{p_T} - \kappa_{p_B}) \delta^4(p_f - p_T - p_B) \langle T_{fi} \rangle^2 |G_T(x)|^2$$

$$\int d\alpha \, |G_B (\alpha (u_B - u_T))|^2 \tag{188}$$
We take the target-beam axis to be along the z-axis. Then, the relative group 4-velocity becomes

\[ u_B - u_T = (u_B^0 - u_T^0, 0, 0, u_B^3 - u_T^3) \]  
(189)

and so

\[ \int d\alpha |G_B(\alpha(u_B - u_T))|^2 = \int d\alpha |G_B(\alpha(u_B^0 - u_T^0), 0, 0, \alpha(u_B^3 - u_T^3))|^2. \]  
(190)

Making the change of variable \( \xi = \alpha(u_B^3 - u_T^3) \), this becomes

\[ \int d\alpha |G_B(\alpha(u_B - u_T))|^2 = \frac{1}{(u_B^3 - u_T^3)} \int d\xi |G_B\left(\frac{\xi}{\alpha^0}, 0, 0, \xi\right)|^2 \]  
(191)

where

\[ \nu = \frac{u_B^3 - u_T^3}{u_B^0 - u_T^0} \]  
(192)

corresponds to the relative speed (\( \sim dx/dt \)) of the target and beam events. By (172), and since \( G(x - u\tau) \) satisfies

\[ \partial_\tau G(x - u\tau) = -u^{\mu}\partial_\chi G(x - u\tau). \]  
(193)

we may take \( j^\mu \) to be

\[ j^\mu = u^{\mu}G(x - u\tau) \]  
(194)

In order to understand the integral over \( \xi \) in (191), we notice that for \( u^3 = u_B^3 - u_T^3 > 0 \), the relative motion will cross from \( x^3 < 0 \) to \( x^3 > 0 \) during the scattering. We define by \( N^+(\tau) \) the proportion of all events for which \( x^3 > 0 \), given by

\[ N^+(\tau) = \int_0^\infty dx^3 \int dt dx_\perp j^4(x, \tau) \]  
(195)

where \( x_\perp \) refers to the 1-2 plane. The rate at which events cross \( x^3 = 0 \) is then given by

\[
\frac{d}{d\tau} N^+(\tau) = \int_0^\infty dx^3 \int dt dx_\perp \partial_\tau j^4(x, \tau) \\
= \int_0^\infty dx^3 \int_0^\infty dt \int dx_\perp \partial_\mu j^4(x, \tau) \\
= \int_0^\infty dx^3 \int dt dx_\perp \partial j^4(x, \tau) \]  
(196)

where in the last line, we used \( j^4(x, \tau) \to 0 \) as \( x^\mu \to \pm \infty \). Evaluating the integral on \( dx^3 \), we find

\[
\frac{d}{d\tau} N^+(\tau) = -u^3 \int dt dx_\perp j^4(t, x_\perp, x^3, \tau)|^2 \bigg|\bigg|_{x^3=0} = u^3 \int dt dx_\perp j^4(t, x_\perp, 0, \tau). \]  
(197)

In terms of (196) and (172), the total number of events which cross \( x^3 = 0 \) for all \( \tau \) is given by

\[
N^+(\infty) - N^-(\infty) = \int dt \frac{d}{d\tau} N^+(\tau) \\
= \int dt u^3 \int dt dx_\perp j^4(t, x_\perp, 0, \tau) \\
= \int dt u^3 \int dt dx_\perp |G(t, x_\perp, 0, u_0, u_0, u^3)\|^2 \\
= \int dt u^3 \int dt dx_\perp |G(t - u_\tau, x_\perp, -u^3\tau)|^2. \]  
(198)
Making the change of variables $\xi = u^3 \tau$ puts \ref{eq:198} into the form

$$N^+(\infty) - N^+(-\infty) = \int d\xi \int dtd^2x_\perp |G\left(t - \frac{u_0}{u^3\xi}, x_\perp, -\xi\right)|^2 = \int d\xi \int dtd^2x_\perp |G\left(t + \frac{1}{\eta}\xi, x_\perp, \xi\right)|^2. \quad (199)$$

in which $v$ is given by \ref{eq:192}. The total number of events which cross $x^3 = 0$, per unit area (in the 1-2 plane) per unit time (which defines the 3-flux $F^{(3)}$ of the beam) is just,

$$F^{(3)} = \int d\xi \left|G\left(\frac{1}{\eta}\xi, 0, 0, \xi\right)\right|^2. \quad (200)$$

Comparison with \ref{eq:191} shows that

$$\int d\alpha \left|G_B\left(\alpha(u_B - u_T)\right)\right|^2 = \frac{1}{|\vec{u}_B - \vec{u}_T|} F^{(3)} \quad (201)$$

Therefore, we may rewrite \ref{eq:188} as

$$\frac{d}{dVdT} |\langle f|T|i \rangle|^2 = (2\pi)^5 \delta(\kappa_{p_f} - \kappa_{p_T} - \kappa_{p_B}) \delta^4(\vec{p}_f - \vec{p}_T - \vec{p}_B) |T_{fi}|^2 |G_T(x)|^2 \frac{1}{|\vec{u}_B - \vec{u}_T|} F^{(3)}. \quad (202)$$

The scattering 3-cross-section is defined through

$$\frac{d}{dVdT} |\langle f|T|i \rangle|^2 = d\sigma^{(3)} \times \text{beam flux} \times \text{target density} \quad (203)$$

and so, we find that

$$d\sigma^{(3)} = (2\pi)^5 \delta(\kappa_{p_f} - \kappa_{p_T} - \kappa_{p_B}) \delta^4(\vec{p}_f - \vec{p}_T - \vec{p}_B) |\langle f|T|p_Tp_B \rangle|^2 \frac{1}{|\vec{u}_B - \vec{u}_T|}. \quad (204)$$

**The Process** $B + T \rightarrow 1 + 2$

We will find it convenient to treat the case of two particle final states in relative coordinates, and we make the following definitions

$$P = \frac{1}{2}(p_T + p_B) \quad p = p_T - p_B \quad (205)$$

$$P' = \frac{1}{2}(p_1 + p_2) \quad p' = p_1 - p_2 \quad (206)$$

which have the inverse relations

$$p_T = P + \frac{1}{2}p \quad p_B = P - \frac{1}{2}p \quad (207)$$

$$p_1 = P' + \frac{1}{2}p' \quad p_2 = P' - \frac{1}{2}p' \quad (208)$$
In the center of mass system, we have that
\[ \vec{p}_B + \vec{p}_T = \vec{p}_1 + \vec{p}_2 = 0 \] (209)
so that
\[ P = \frac{1}{2} \left[ \frac{E(\vec{p}_B) + E(\vec{p}_T)}{\vec{p}_B + \vec{p}_T} \right] = \left[ \frac{\frac{1}{2}[E(\vec{p}_B) + E(\vec{p}_T)]}{0} \right] = \left[ \frac{1}{2} \sqrt{s} \right] \] (210)
where
\[ s = -(p_B + p_T)^2 \] (211)
is the usual Mandelstam parameter in this metric (and so \( P^2 = -s/4 \)), and
\[ E(\vec{p}) = \sqrt{(\vec{p})^2 + m^2}. \] (212)
Similarly,
\[ p' = p_1 - p_2 = \left[ \frac{E(p_1) - E(p_2)}{2\vec{p}_1} \right] \] (213)
and
\[ p_1^2 = p^2 + \frac{1}{4}(p')^2 + P \cdot p' = -\frac{1}{4} s + \frac{1}{4}(p')^2 - \frac{1}{2} \sqrt{s} \left[ E(p_1) - E(p_2) \right] \]
\[ p_2^2 = p^2 + \frac{1}{4}(p')^2 - P \cdot p' = -\frac{1}{4} s + \frac{1}{4}(p')^2 + \frac{1}{2} \sqrt{s} \left[ E(p_1) - E(p_2) \right], \] (214)
Since the relative momentum is spacelike, we may introduce the parameterization
\[ p' = \rho \left[ \begin{array}{c} \sinh \beta \\ \cosh \beta \hat{n} \end{array} \right] \] (215)
so that (214) becomes
\[ p_1^2 = -\frac{1}{4} (s - \rho^2 + 2\sqrt{s} \rho \sinh \beta) \quad p_2^2 = -\frac{1}{4} (s - \rho^2 - 2\sqrt{s} \rho \sinh \beta). \] (216)
We may see the utility of this approach in the conventional description of relativistic scattering, where the cross-section has the O(3,1) invariant measure
\[ dR_2 = \delta^4(p_1 + p_2 - p_T - p_B) \delta(p_1^2 + m_1^2) \delta(p_2^2 + m_2^2) \frac{d^4p_1 \ d^4p_2}{(2\pi)^4 (2\pi)^4} \] (217)
which in relative coordinates becomes
\[ dR_2 = \frac{1}{2(2\pi)^8} \delta^4(P - P') \delta(p_1^2 + m_1^2) \delta(p_2^2 + m_2^2) d^4P' d^4p' \]
\[ = \frac{1}{2(2\pi)^8} \delta(p_1^2 + m_1^2) \delta(p_2^2 + m_2^2) d^4p' \]
\[ = \frac{1}{2(2\pi)^8} \cdot 4 \cdot 4 \cdot \delta(s - \rho^2 + 2\sqrt{s} \rho \sinh \beta - 4m_1^2) \delta(s - \rho^2 - 2\sqrt{s} \rho \sinh \beta - 4m_2^2) \cdot \rho^3 \cosh^2 \beta d\rho d\beta d\Omega \]
\[ = \frac{4}{(2\pi)^8} \rho^3 \cosh^2 \beta d\rho d\beta d\Omega \delta(\rho^2 - s + (m_1^2 + m_2^2)) \delta(s - \rho^2 - 2\sqrt{s} \rho \sinh \beta - 4m_2^2) \]
\[ = \frac{\rho}{(2\pi)^8 \sqrt{s}} \cosh^2 \beta d\rho d\beta d\Omega \delta \left[ \rho - \sqrt{s - 2(m_1^2 + m_2^2)} \right] \delta \left[ \sinh \beta - \frac{m_1^2 - m_2^2}{\sqrt{s} \rho} \right] \] (218)
Changing variables as
\[ \rho \cosh^2 \beta d\beta = \rho \cosh \beta d(\sinh \beta) \] (219)
and recognizing \( \rho \cosh \beta \) as \( |\vec{p}'| = 2|\vec{p}_f| \), this becomes,
\[ dR_2 = \frac{2|\vec{p}_f|}{\sqrt{s}} d\Omega \] (220)
with
\[ \rho = \sqrt{s - 2(m_1^2 + m_2^2)} \] (221)
so that
\[ |\vec{p}_f| = \frac{\rho}{2} \sqrt{1 + \left[ \frac{m_1^2 - m_2^2}{\sqrt{s} \rho} \right]^2} = \frac{1}{2} \frac{\sqrt{s^2 - 2s(m_1^2 + m_2^2) + (m_1^2 - m_2^2)^2}}{\sqrt{s}} = \frac{\lambda^1(s, m_1^2, m_2^2)}{2\sqrt{s}} \] (222)
in agreement with the usual derivation [3].

Returning to the off-shell theory, we first consider scattering of indistinguishable particles, which means that we may take the invariant mass parameter \( M \) to be the same for all events. The invariant measure for the matter field is given by
\[
dR_2 = \delta^4(p_1 + p_2 - p_T - p_B) \delta \left( \frac{(p_1)^2}{2M} + \frac{(p_2)^2}{2M} - \frac{(p_T)^2}{2M} - \frac{(p_B)^2}{2M} \right) d^4p_1 d^4p_2 \]
\[ = \frac{1}{2(2\pi)^8} \delta^4(p - p') d^4p' \cdot 2M \delta ((p_1)^2 + (p_2)^2 - (p_T)^2) d^4p' \]
\[ = \frac{2M}{(2\pi)^8} \delta^4(p^2 - p'^2) d^4p' \] (223)
where we have used
\[ (p_1)^2 + (p_2)^2 = p^2 + \frac{1}{4} p'^2 + p' \cdot p' + p'^2 + \frac{1}{4} p'^2 = 2p'^2 + \frac{1}{2} p'^2 \] (224)
\[ (p_B)^2 + (p_T)^2 = 2p^2 + \frac{1}{2} p^2 \] (225)
Now using the parameterization in (215), we find
\[ dR_2 = \frac{2M}{(2\pi)^8} \delta^4(p^2 - \rho^2) \rho^3 \cosh^2 \beta d\rho d\beta d\Omega = \frac{M}{(2\pi)^8} \delta \left( \sqrt{p^2 - \rho} \right) \left( 2|\vec{p}_f| \right)^2 d\rho d\beta d\Omega \] (226)
where the absolute value of
\[ |\vec{p}_f| = \vec{p}_1 - \vec{p}_2 = \frac{1}{2} \rho \cosh \beta \delta \] (227)
is undetermined in the off-shell theory, because the value of \( \beta \) is not fixed by conservation of mass-momentum. But notice that \( \rho \) is determined by
\[ \rho^2 = p^2 = (p_B - p_T)^2 = (p_B)^2 + (p_T)^2 - 2p_B \cdot p_T = s - 2[(m_B)^2 + (m_T)^2] \] (228)
which is identical to the usual on-shell value given in (221). Combining (226) with (201), we have for the scattering cross-section (see also [18])
\[ d\sigma^{(3)} = \frac{(2\pi)^5}{(2\pi)^8} \frac{|\langle p_1p_2 | T | p_Tp_B \rangle|^2}{|\vec{u}_B - \vec{u}_T|} \frac{4M|\vec{p}_f|^2}{4M|\vec{p}_f|^2} d\beta d\Omega. \] (229)
Since \( p = M\mu \) for the asymptotic free matter field, we may write this in the form

\[
d\sigma^{(3)} = \frac{1}{(2\pi)^3} \frac{M^2}{|\vec{p}_f|^2} \frac{|\vec{p}_f|^2}{|\vec{p}_i|^2} |\langle p_1 p_2 | T | p_T p_B \rangle|^2 d\beta d\Omega.
\] (230)

where \( \vec{p}_i = \vec{p}_T - \vec{p}_B \), and we must keep in mind that \( \vec{p}_f \) is dependent on \( \beta \).

We denote by \( \Delta m^2 = (m_1)^2 - (m_2)^2 \), and notice that from (216),

\[
\Delta m^2 = (p_2)^2 - (p_1)^2 = \sqrt{s} \rho \sinh \beta
\] (231)

so that

\[
d(\Delta m^2) = \sqrt{s} \rho \cosh \beta d\beta = \sqrt{s^2} |\vec{p}_f|^2 d\beta.
\] (232)

Therefore, the cross-section can be written as

\[
d\sigma^{(3)} = \frac{1}{(2\pi)^3} \frac{M^2}{|\vec{p}_f|^2} \frac{|\vec{p}_f|^2}{|\vec{p}_i|^2} |\langle p_1 p_2 | T | p_T p_B \rangle|^2 d\Omega d(\Delta m^2).
\] (233)

We may compare the cross-section with the comparable expression in the usual on-shell theory, which can be written in the center of mass as [3]

\[
\frac{d\sigma^{(2)}}{d\Omega} = \frac{1}{64\pi^2} \frac{1}{s} \frac{|\vec{p}_f|^2}{|\vec{p}_i|^2} |\langle p_1 p_2 | T | p_T p_B \rangle|^2.
\] (234)

Putting (233) into a form similar to (234), we have

\[
\frac{d\sigma^{(3)}}{d\Omega} = \frac{1}{64\pi^2} \frac{1}{s} \frac{|\vec{p}_f|^2}{|\vec{p}_i|^2} |\langle p_1 p_2 | T | p_T p_B \rangle|^2 \frac{4M^2 \sqrt{s}}{\pi} d(\Delta m^2).
\] (235)

Examining the Feynman rules for the single photon interaction in Section 6, we see that the usual vertex factor of \( e \) is replaced by \( e_\theta/2M = \lambda e/2M \) and photon propagator has an extra factor of \( 1/\lambda \), so that the squared transition amplitude will have a factor of \( \lambda^2/16M^4 \) in relation to the usual on-shell case. Combining this factor with the extra factor in (235), we have an overall extra factor of

\[
\frac{\lambda^2}{16M^4} d(\frac{4M^2 \sqrt{s}}{\pi} \Delta m^2) = d(\frac{\sqrt{s} \lambda^2}{4\pi M^2} \Delta m^2)
\] (236)

which we see has the expected units of length. We may understand this factor in the following way: During the scattering, the event will propagate a distance \( \Delta x \sim (p/M)\Delta \tau \), so that the uncertainty relation tells us that

\[
\Delta \tau \sim \frac{\Delta x}{(p/M)} = \frac{M}{p} \Delta x \sim \frac{1}{2\Delta p} \frac{M}{p}
\] (237)

So from \( (\vec{p})^2 = E^2 - m^2 \), we find that

\[
\vec{p} \cdot d\vec{p} \sim m dm \sim \frac{1}{2} \Delta(m^2)
\] (238)

and combining with (237),

\[
\Delta \tau \sim \frac{M}{\Delta(m^2)}
\] (239)
Shnerb and Horwitz [10] have shown that for $\Delta \tau > \lambda$, the photon-current interaction becomes uncorrelated, so we may take $\lambda \sim \Delta \tau$. Thus, the factor in (236) becomes

$$
d (\frac{\sqrt{s} \Delta^2}{4\pi M^2} (m^2)^3) \sim d(\frac{\sqrt{s} (\Delta \tau)^2}{4\pi M} \Delta (m^2)) = d(\frac{\sqrt{s} (\Delta \tau)}{4\pi M})
$$

$$
\sim d(\Delta \tau \frac{E}{4\pi M}) \sim \frac{1}{4\pi} d\tau \sim dt_{\text{shift}},
$$

(240)

where we understand $dt_{\text{shift}}$ as the change in the relative time coordinate of the two particles during the scattering. Thus, the usual on-shell scattering cross-section corresponds to a specific $dt_{\text{int}}$, and we may compare the off-shell cross-sections with the usual results through the expression.

$$
\frac{d\sigma^{(3)}}{d\Omega dt_{\text{int}}} = \frac{1}{64\pi^2 s} \left| \frac{p_f}{p_i} \right|^2 |\langle p_1 p_2 | T | p_T p_B \rangle|^2
$$

(241)

where $T = (\frac{\lambda}{4\pi \mu^2})^2 T$.

7. Møller scattering

We consider the scattering of two identical scalar particles to first order. The diagrams which contribute are

\[ \begin{array}{ccc}
  p_1 & \quad & p_3 \\
  k & \quad & \tilde{p}_1 \\
  p_2 & \quad & p_4 \\
\end{array} \]

\[ \begin{array}{ccc}
  p_1 & \quad & p_3 \\
  k & \quad & \tilde{p}_1 \\
  p_2 & \quad & p_4 \\
\end{array} \]

which contribute

$$
\langle 3 4 | iT | 1 2 \rangle = \int d^4 q \kappa_q (1)^4 \frac{i\epsilon_0}{2M} (p_1 + p_2)^{\mu} (2\pi)^5 \delta^4 (p_1 + q - p_3) \delta (\kappa_{p_1} - \kappa_{p_3} - \sigma \kappa_q)
$$

$$
\times \frac{i\epsilon_0}{2M} (p_2 + p_4)^{\mu} (2\pi)^5 \delta^4 (p_2 - q - p_4) \delta (\kappa_{p_2} - \kappa_{p_4} + \sigma \kappa_q)
$$

$$
\times \frac{1}{\lambda \lambda^2 + \sigma \kappa_q^2 - i\epsilon} T_{\mu \nu} + (3, 4) \leftrightarrow (4, 3)
$$

$$
\times \left\{ (p_1 + p_3)^{\mu} (p_2 + p_4)^{\nu} \left[ g_{\mu \nu} + \frac{(p_1 + p_3)^\mu (p_2 - p_4)^\nu}{(p_1 - p_3)^2} \right] \right\}
$$

$$
\times \frac{1}{(p_1 - p_3)^2 + \sigma (\kappa_{p_1} - \kappa_{p_3})^2 - i\epsilon} + (3, 4) \leftrightarrow (4, 3)
$$

(242)

where $\kappa_p = \frac{p^2}{2M}$. From (174) which defines $T$, we have

$$
\langle 3 4 | T | 1 2 \rangle = (2\pi)^5 \frac{\epsilon_0 \epsilon}{(2M)^2} \left\{ \left[ (p_1 + p_3) \cdot (p_2 + p_4) + \frac{(p_1^2 - p_3^2)(p_2^2 - p_4^2)}{(p_1 - p_3)^2} \right] \right\}
$$

$$
\times \frac{1}{(p_1 - p_3)^2 + \sigma (\kappa_{p_1} - \kappa_{p_3})^2 - i\epsilon} + (3, 4) \leftrightarrow (4, 3)
$$

(243)
At this stage it is convenient to introduce the Mandelstam parameters

\[ t = -(p_1 - p_3)^2 = -\frac{1}{4}(p - p')^2 \]

\[ u = -(p_1 - p_4)^2 = -\frac{1}{4}(p + p')^2 \] (244)

which complement the definition of \( s \) in (211), and where \( p, p' \) refer to the relative coordinates defined in (205) and (206). As in the on-shell case, we have

\[ s + t + u = - [p_1^2 + p_2^2 + p_3^2 + p_4^2 + 2(p_2 - p_3 - p_4) \cdot p_1] \]

\[ = - [p_1^2 + p_2^2 + p_3^2 + p_4^2] \]

\[ = m_1^2 + m_2^2 + m_3^2 + m_4^2. \] (245)

Similarly, since \( p_1^2 + p_2^2 = p_3^2 + p_4^2 \) is guaranteed by the ‘fifth’ \( \delta \)-function, but the masses are not in general invariant, we have

\[ (p_1 + p_2)^2 = (p_3 + p_4)^2 \Rightarrow p_1 \cdot p_2 = p_3 \cdot p_4 = -\frac{1}{2}(s + p_1^2 + p_2^2) = -\frac{1}{2}(s + p_3^2 + p_4^2) \]

\[ p_1 \cdot p_3 = \frac{1}{2}(p_1^2 + p_3^2 + t) \neq p_2 \cdot p_4 = \frac{1}{2}(p_2^2 + p_4^2 + t) \]

\[ p_1 \cdot p_4 = \frac{1}{2}(p_1^2 + p_4^2 + u) \neq p_2 \cdot p_3 = \frac{1}{2}(p_2^2 + p_3^2 + u) \] (246)

Using these relations among the Mandelstam parameters, we may rewrite the terms of (243) in the form,

\[ (p_1 + p_3) \cdot (p_2 + p_4) = -s + u \]

\[ (p_1 + p_4) \cdot (p_2 + p_3) = -s + t \] (247)

so that

\[ \langle 3 4 | T | 1 2 \rangle = (2\pi)^5 \frac{e_0 e}{(2M)^2} \left\{ \frac{s - u - (p_1^2 - p_2^2)^2}{t - \sigma(\kappa_{p_1} - \kappa_{p_3})^2} + \frac{s - t - (p_1^2 - p_2^2)^2}{u - \sigma(\kappa_{p_1} - \kappa_{p_4})^2} \right\}. \]

(248)

We notice that for the case of on-shell scattering \( (m_1^2 = m_2^2) \) the amplitude becomes

\[ \langle 3 4 | T | 1 2 \rangle = (2\pi)^5 \frac{e_0 e}{(2M)^2} \left\{ \frac{s - u}{t} + \frac{s - t}{u} \right\} \] (249)

which is the familiar result for identical Klein-Gordon particles in scalar QED [3].

We consider particles in their center of mass, so that \( m_1 = m_2 = m \) and \( E(p_1) = E(p_2) \). Then

\[ p = p_1 - p_2 = (0, 2\vec{p}_1) \] (250)

and \( |p| = |p'| = \rho = 2|\vec{p}_1| \). In terms of the parameterization (215), the \( \beta \) of the incoming system is just zero. The relative momentum of the outgoing system is then

\[ p' = p_3 - p_4 = 2|\vec{p}_1|(\sinh \beta, \cosh \beta \hat{n}) \] (251)

and so the Mandelstam parameters become

\[ t = -\frac{1}{4}(p - p')^2 = -\frac{1}{4}(2\rho^2 - 2\rho^2 \cosh \beta \cos \theta) = -2|\vec{p}_1|^2(1 - \cosh \beta \cos \theta) \] (252)
and
\[ u = -\frac{1}{4}(p + p')^2 = -2|\vec{p}_1|^2(1 + \cosh \beta \cos \theta). \] (253)
These expressions agree with the usual on-shell expressions for \( t \) and \( u \) when \( \beta = 0 \). Since \( p_1^2 = p_2^2 \), we may write
\[ p_1^2 + p_2^2 = 2p_1^2 = p_3^2 + p_4^2 \] (254)
so that
\[ p_1^2 - p_3^2 = \frac{1}{2}(p_3^2 + p_4^2) - p_3^2 = \frac{1}{2}(p_3 - p_4) \cdot (p_3 + p_4) = -\frac{1}{2}p' \cdot P' = \frac{1}{4}\sqrt{s}p \sinh \beta \] (255)
where we have used \( P' = P, (210) \) and \( (215) \). The amplitude in \( (248) \) contains the term
\[ \frac{(p_1^2 - p_3^2)^2}{t} = -\frac{1}{2}sp^2 \sinh^2 \beta = -\frac{s \sinh^2 \beta}{8(1 - \cosh \beta \cos \theta)} \] (256)
and the denominator
\[ t - \sigma(\kappa_{p_1} - \kappa_{p_3})^2 = t - \frac{\sigma}{4M^2}(p_1^2 - p_3^2)^2 = -\frac{\sigma}{2} \left( 1 - \cosh \beta \cos \theta \right) \pm \frac{\sigma s}{32M^2} \sinh^2 \beta \] (257)
Therefore while on-shell scattering has a single forward direction pole at \( \cos \theta = 1 \Rightarrow t = 0 \), the off-shell scattering amplitude has two forward direction poles:
\[ t = 0 \Rightarrow \cos \theta = \frac{1}{\cosh \beta} \] (258)
\[ t - \sigma(\kappa_{p_1} - \kappa_{p_3})^2 = 0 \Rightarrow \cos \theta = \frac{1}{\cosh \beta} \left[ 1 + \frac{\sigma s}{32M^2} \sinh^2 \beta \right]. \] (259)
The pole structure for a 1% mass shift is shown in Figure 1.

![Figure 1 — Møller cross-section \( \frac{d\sigma}{d\Omega} \) versus angle for 1% mass exchange](image)

The appearance of two distinct poles in Møller scattering would be a consequence of \( \Delta m^2 \neq 0 \) and provide an experimental signature for off-shell phenomena.
8. Renormalization

Unlike conventional relativistic quantum field theories, the off-shell matter field corresponds
to an underlying evolution mechanics, and the field undergoes retarded propagation from \( \tau_1 \)
to \( \tau_2 > \tau_1 \). Since a closed loop would require the field to propagate first from \( \tau_1 \) to \( \tau_2 \) and then
from \( \tau_2 \) to \( \tau_1 \), there can be no matter field loops in off-shell QED. The absence of matter field
loops leads us to expect that the charge \( e_0 \) will not be renormalized, and as we demonstrate
below, this follows from the absence of wave function renormalization for the photon.

The Vector Ward Identity

The Ward identity in on-shell (Klein-Gordon) scalar quantum electrodynamics expresses the
symmetry associated with the conservation of the four-current as a relationship between the
vertex function of the 3-particle interaction and the single particle propagators. Since the Ward
identity is preserved at all orders of perturbation theory, it leads to the universality of charge
renormalization.

In off-shell quantum electrodynamics, the conservation of current is expressed as a vanishing
five-divergence \(^{(12)}\). Recalling equation \(^{(153)}\)

\[
\mathcal{L}_{\text{int}} = -\frac{i e_0}{2M} a_\mu (\psi^* \partial^\mu \psi - (\partial^\mu \psi^*) \psi) - \frac{e_0^2}{2M} a_\mu a^\mu |\psi|^2 ,
\]

we may express the interaction in terms of the current as

\[
\mathcal{L}_{\text{int}} = e_0 a_\mu j^\mu - \frac{e_0^2}{2M} a_\mu a^\mu j^5
\]

where \((j^\mu, j^5)\) is the current for the free matter field. Since the entire interaction Lagrangian is
in the form of photon \(\times\) current, we shall see that the Ward identity is a relationship among the
single particle propagators and the vertex functions of both the 3-particle interaction and the
4-particle interaction.

We begin with the three-point Green’s function associated with the diagram

\[
\begin{array}{ccc}
q, Q, \mu & & \\
p, P & \rightarrow & p', P' \\
\end{array}
\]

where \(p' = q + p\) and \(P' = P - \sigma Q\). To all orders in perturbation theory, the vertex function
(the amputated Green’s function) is given by \( \Gamma_\mu^{(3)} (p, P; q, Q) \), where

\[
G^{(3)} (p, P; q, Q) = G^{(2)} (p, P) G^{(2)} (p', P') d^{\mu\nu} (q, Q) \Gamma_\mu^{(3)} (p, P; q, Q).
\]

To calculate the vertex function, we may write \[^{(22)}\]

\[
G_\mu^{(3)} (p, P; q, Q) = \mathcal{F} \left\{ \langle 0 | T a_\mu (x_1, \tau_1) \psi^* (x_2, \tau_2) \psi (x_3, \tau_3) | 0 \rangle \right\}
\]

\[
= \mathcal{F} \left\{ \langle 0 | T a_\mu (x_1, \tau_1) \psi^* (x_2, \tau_2) \psi (x_3, \tau_3) e^{ie_0 \int dx_4 dx_5 a_\mu (x_4)} | 0 \rangle \rangle_{\text{free}} \right\}
\]
where $\mathcal{F}$ represents the Fourier transform. The vertex function becomes,

$$
\Gamma^{(3)}_\mu(p, P; q, Q) = \frac{1}{G^{(2)}(p)G^{(2)}(p')} \int d^4x d^4x' d^4\tau d^4\tau' e^{-i[p-x-P\tau]e^{-i[q-x+\sigma Q\tau]}}
\langle 0| T j_\mu(x, \tau) \psi(x', \tau') \psi^*(0) | 0 \rangle
$$

(264)

where we have used translation invariance of the Green’s functions to shift one of the field points to zero. Contracting with $\theta^\mu$, we obtain

$$
q^\mu \Gamma^{(3)}_\mu(p, P; q, Q) = e_0 \frac{1}{G^{(2)}(p)G^{(2)}(p')} \int d^4x d^4x' d^4\tau d^4\tau' \left[\left(-\partial_\tau^\mu\right) e^{-i[p-x'-P\tau']e^{-i[q-x+\sigma Q\tau]}}\right]
\langle 0| T j_\mu(x, \tau) \psi(x', \tau') \psi^*(0) | 0 \rangle
$$

(265)

where we have performed one integration by parts. Using current conservation, we may make the replacement

$$
\partial_\tau j_\mu(x, \tau) = -\partial_\tau \tilde{\psi}^*(x, \tau) .
$$

(266)

Since the products are $\tau$-ordered, we must carefully differentiate the implied $\theta$-functions to find

$$
\partial_\tau \langle 0| T \tilde{\psi}^*(x, \tau) \psi(x', \tau') \psi^*(0) | 0 \rangle = \left\langle 0| T \partial_\tau \tilde{\psi}^*(x, \tau) \psi(x', \tau') \psi^*(0) | 0 \rangle + \delta(\tau - \tau') \left\langle 0| T \left[j^\delta(x, \tau), \psi(x', \tau') \right] \psi^*(0) | 0 \rangle + \delta(\tau) \left\langle 0| T \left[j^\delta(x, \tau), \psi^*(0) \right] \psi(x', \tau') | 0 \rangle\right.
$$

(267)

$$
- \delta(\tau - \tau') \delta^4(x-x') \left\langle 0| T \psi(x, \tau) \psi^*(0) | 0 \rangle + \delta(\tau) \delta^4(x) \langle 0| T \psi(x', \tau') \psi^*(0) | 0 \rangle\right\rangle
$$

where we have used the commutation relations (52). Using (266) and (267) in (265), we find

$$
q^\mu \Gamma^{(3)}_\mu(p, P; q, Q) = e_0 \frac{1}{G^{(2)}(p)G^{(2)}(p')} \int d^4x d^4x' d^4\tau d^4\tau' \left[ e^{-i[p-x'-P\tau']e^{-i[q-x+\sigma Q\tau]}} \right]
$$

$$
\left[ -\partial_\tau \langle 0| T \tilde{\psi}^*(x, \tau) \psi(x', \tau') \psi^*(0) | 0 \rangle - \delta(\tau - \tau') \delta^4(x-x') \langle 0| T \psi(x, \tau) \psi^*(0) | 0 \rangle + \delta(\tau) \delta^4(x) \langle 0| T \psi(x', \tau') \psi^*(0) | 0 \rangle\right]
$$

(268)
where we have performed one integration by parts in the last line. Now, carrying out the integrations, we find that

\[
\int d^4 x d\tau d^4 x' d\tau' \left[ e^{-i[p' - P'\tau]} e^{-i[q - x + \sigma Q\tau]} \right] \delta(\tau - \tau') \delta^4(x - x') G^{(2)}(x, \tau) = \int d^4 x d\tau e^{-i[(q + p) - (P - \sigma Q)\tau]} G^{(2)}(x, \tau) = G^{(2)}(q + p, P - \sigma Q) = G^{(2)}(p', P')
\]

(269)

and

\[
\int d^4 x d\tau d^4 x' d\tau' \left[ e^{-i[p' - P'\tau]} e^{-i[q - x + \sigma Q\tau]} \right] \delta(\tau) \delta^4(x) G^{(2)}(x', \tau') = G^{(2)}(p, P) \quad \text{(270)}
\]

We notice that the remaining Green’s function is proportional to the vertex function for the 4-particle interaction,

\[
\Gamma^{(4)}(p, P; q, Q) = \frac{1}{G^{(2)}(p)G^{(2)}(P')} \int d^4 x d\tau d^4 x' d\tau' e^{-i[p' - P'\tau]} e^{-i[q - x + \sigma Q\tau]} 
\]

\[
\langle 0 | T \left( -i \frac{e_0^2}{2M} \bar{\psi}(x, \tau) \psi(x', \tau') \psi^*(0) | 0 \rangle.
\]

(271)

Thus, we find

\[
q^\mu \Gamma^{(3)}_\mu(p, P; q, Q) = \frac{e_0}{G^{(2)}(p)G^{(2)}(p')} \int d^4 x d\tau d^4 x' d\tau' \left[ e^{-i[p' - P'\tau]} e^{-i[q - x + \sigma Q\tau]} \right] 
\]

\[
(-i\sigma Q) \langle 0 | T \bar{\psi}(x, \tau) \psi(x', \tau') \psi^*(0) | 0 \rangle + \frac{e_0}{G^{(2)}(p', P')} - \frac{e_0}{G^{(2)}(p, P)}
\]

\[
= e_0(-i\sigma Q)(-\frac{2M}{ie_0^2})\Gamma^{(4)}(p, P; q, Q, k, k') + \frac{e_0}{G^{(2)}(p', P')} - \frac{e_0}{G^{(2)}(p, P)}
\]

\[
= \frac{\sigma Q}{e_0/(2M)} \Gamma^{(4)}(p, P; q, Q, k, k') + \frac{e_0}{G^{(2)}(p', P')} - \frac{e_0}{G^{(2)}(p, P)}
\]

(272)

and the Ward identity takes the form,

\[
e_0 q^\mu \Gamma^{(3)}_\mu(p, P; q, Q) - \sigma Q 2M\Gamma^{(4)}(p, P; q, Q, k, k') = e_0^2 \left[ \frac{1}{G^{(2)}(p', P')} - \frac{1}{G^{(2)}(p, P)} \right].
\]

(273)

As expected, in the case that \( Q = 0 \), (273) reduces to the Ward identity for on-shell Klein-Gordon scalar QED [22]. At the tree level, where

\[
\Gamma^{(4)}(p, P; q, Q, k, k') = \frac{-ie_0^2}{2M} G^{(2)}(p, P) = \frac{-i}{2M - p} \quad \Gamma^{(3)}_\mu(p, P; q, Q) = i\frac{e_0}{2M}(p + p')_\mu
\]

(274)

we verify that (273) is satisfied:

\[
e_0(p' - p)_\mu \cdot \frac{ie_0^2}{2M}(p' + p)_\mu - \sigma Q M \cdot \frac{-ie_0^2}{M} - e_0^2 \left[ i\left(\frac{p'^2}{2M} - P'\right) - i\left(\frac{p^2}{2M} - P\right) \right]
\]

\[
= i\frac{e_0^2}{2M} \left( p'^2 - p^2 \right) + ie_0^2(p - P') - e_0^2 \left[ i\left(\frac{p'^2}{2M} - P'\right) - i\left(\frac{p^2}{2M} - P\right) \right] = 0
\]

(275)
Since the invariance of the Lagrangian is not changed by multiplying each invariant term by a constant, the most general gauge invariant form of the Lagrangian of (22), written in terms of renormalized quantities is given by

\[ \mathcal{L} = Z_2 \psi^* \left( i \partial_\tau + \frac{Z_1}{Z_2} e_0 a_5 \right) \psi - \frac{1}{2Z_4 M} Z_2 \psi^* \left( -i \partial_\mu - \frac{Z_1}{Z_2} e_0 a_\mu \right) \left( -i \partial^\nu - \frac{Z_1}{Z_2} e_0 a^\nu \right) \psi - \frac{\lambda}{4} \left[ Z_3 f_{\mu \nu} f^{\mu \nu} + 2 Z_5 f_{5 \nu} f^{5 \nu} \right] \]

where we have written the bare field operators in terms of the renormalized field operators as

\[ f_{\mu \nu}^B = Z_3 f_{\mu \nu} \quad a_\mu^B = Z_3^{1/2} a_\mu \quad f_{5 \nu}^B = Z_5 f_{5 \nu} \quad a_5^B = Z_5^{1/2} a_5 \]

Any renormalization of the coupling \( \lambda \) may be absorbed into the wave function renormalizations \( Z_3 \) and \( Z_5 \). Consistency requires that

\[ \left( \frac{Z_1}{Z_2} \right) e_0 a_5 = \left( \frac{Z_1}{Z_2} \right) e_0 Z_3^{1/2} a_5 = e_0 a_5^B \quad \text{and} \quad \left( \frac{Z_1}{Z_2} \right) e_0 a_\mu = \left( \frac{Z_1}{Z_2} \right) e_0 Z_3^{1/2} a_\mu = e_0 a_\mu^B \]

so we must have

\[ Z_5 = Z_3 \quad e_0 = \frac{Z_2 Z_3^{1/2}}{Z_1} e_0^B . \]

We may write the bare Green’s functions in terms of the renormalized Green’s functions as

\[ \Gamma^{(3)}_{\mu B} = \frac{1}{Z_3 Z_2} \Gamma^{(3)}_{\mu} \]

\[ \Gamma^{(4)}_{B} = \frac{1}{Z_3 Z_2} \Gamma^{(4)} \]

Since the Ward identity must be valid for the renormalized quantities as well as the unrenormalized quantities, we may compare

\[ e_0 q^\mu \Gamma^{(3)}_{\mu} (p, P, q, Q) = - \sigma Q 2 M \Gamma^{(4)} (p, P; q, Q, k, k') = e_0^2 \left[ \frac{1}{G^{(2)}(p', P')} - \frac{1}{G^{(2)}(p, P)} \right] \]

\[ \frac{Z_1 Z_3^{1/2}}{Z_2} e_0^B q^\mu Z_3^{1/2} Z_2 \Gamma^{(3)}_{\mu B} (p, P, q, Q) = - \sigma Q 2 Z_4 M_B Z_2 Z_3 \Gamma^{(4)}_{B} (p, P; q, Q, k, k') = \left( \frac{Z_1 Z_3^{1/2}}{Z_2} e_0^B \right)^2 Z_2 \left[ \frac{1}{G^{(2)}_B (p', P')} - \frac{1}{G^{(2)}_B (p, P)} \right] \]

38
The remaining renormalization factor is

\[ Z_1Z_3e_0^B q^\mu \Gamma^{(3)}_{\mu B}(q, P', Q, Q) = Z_2Z_3Z_4\sigma Q 2M_B\Gamma^{(4)}_{B}(p, P; q, Q, k, k') = \]

\[ \frac{Z_2^2Z_3(Z_1 e_0^B)^2}{Z_2^2} \left[ \frac{1}{G_B^{(2)}(p', P')} - \frac{1}{G^{(2)}(p, P)_B} \right] \]

\[ e_0^B q^\mu \Gamma^{(3)}_{\mu B}(p, P; q, Q) = \frac{Z_2Z_4}{Z_1}\sigma Q 2M_B\Gamma^{(4)}_{B}(p, P; q, Q, k, k') = \]

\[ \frac{Z_4(Z_1 e_0^B)^2}{Z_2} \left[ \frac{1}{G_B^{(2)}(p', P')} - \frac{1}{G^{(2)}(p, P)_B} \right] \]

(283)

and find that

\[ Z_1 \equiv Z_2, \quad Z_4 \equiv 1. \]  

(284)

Notice that although the charge \( e_0 \) appears linearly and quadratically in the Ward identity, (283) makes no restriction on \( Z_3 \), which, from is seen to determine the charge renormalization by inserting \( Z_1 = Z_2 \) into (283). Thus, the appearance of \( \Gamma^{(4)} \) in the Ward identity does not change the universality of charge renormalization found in on-shell QED since the conserved five-current is a consequence of gauge invariance. Nevertheless, since there are no matter field loops in off-shell QED, there are no possible contributions to photon renormalization, and we take \( Z_3 \equiv 1 \). Therefore, the charge \( e_0 \) is not renormalized.

**Renormalizability**

The remaining renormalization factor is \( Z_2 \), which derives from the renormalization of the matter field by photon loops. These photon loops (matter field self-energy diagrams) also contribute to the mass renormalization of the matter field, however, the mass term \( (\psi^\ast i\partial_\tau \psi) \) absorbs these contributions. Nevertheless, by examining the primitive divergent self-energy diagrams of successively higher order, it may be seen that in order to make the theory counter-term renormalizable, a cut-off must be applied to the integrations over mass in loop diagrams. For example, at second order, the self-energy diagram with two overlapping photon loops is given by

\[ G_2^{(2)}(p) = G_0^{(2)}(p)G_0^{(2)}(p) \int d^4qdQd^4q'dQ' \left( (2\pi)^5 \frac{i\epsilon_0}{2M} \right)^4 (2p - q)_\mu(2p - q - 2q')_\nu d^{\mu\nu}(q) \]

\[ (2p - 2q - q')_\lambda(2p - q')_\sigma d^{\lambda\sigma}(q')G_0^{(2)}(p - q)G_0^{(2)}(p - q')G_0^{(2)}(p - q - q') \]  

(285)

which contains the following term proportional to \( p^4 \)

\[ G_0^{(2)}(p)G_0^{(2)}(p) \left( (2\pi)^5 \frac{i\epsilon_0}{2M} \right)^4 16p^4 \int d^4qdQd^4q'dQ' \]

\[ \frac{1}{\lambda^2 q^2 + \sigma Q^2} \frac{1}{q^2 + \sigma Q^2} \frac{2M}{(p - q)^2 - 2M(P + \sigma Q)} \frac{2M}{(p - q')^2 - 2M(P + \sigma Q')} \]  

(286)

and diverges if the mass integrations on \( dQ \) and \( dQ' \) are taken to infinity. Since this term is proportional to \( p^4 \) it may not be renormalized by a counter term of a form which appears in the original Lagrangian. This divergence may be controlled by inserting a mass cut-off into the photon propagator as

\[ \frac{1}{\lambda} P^{\mu\nu} \frac{-i}{k^2 + \sigma k^2 - i\epsilon} \rightarrow \frac{1}{\lambda} P^{\mu\nu} \frac{-i}{k^2 + \sigma k^2 - i\epsilon} \frac{1}{1 + \lambda^2 k^2} \]

(287)
which can be accomplished by adding the gauge-invariant scalar term
\[ -\frac{\lambda^3}{4} \left[ \partial_\tau f^{a\beta}(x,\tau) \right] \left[ \partial_\tau f_{a\beta}(x,\tau) \right] \] (288)
to kinetic term for the electromagnetic field. The action for the field is then
\[ S_{em} = -\frac{\lambda}{4} \int d^4x \, d\tau \, ds \, f^{a\beta}(x,\tau) \Phi(\tau-s) f_{a\beta}(x,s) \] (289)
where
\[ \Phi(\tau) = \delta(\tau) - \lambda^2 \delta''(\tau) = \frac{1}{2\pi} \int dk \left[ 1 + (\lambda k)^2 \right] e^{-ik\tau}. \] (290)
This cut-off is also required at the classical level [23] in order to smooth the event current as
\[ \partial_\beta f^{a\beta}(x,\tau) = ej^a(x,\tau) = e \int ds \, \varphi(\tau-s) f^a(x,s) \] (291)
where
\[ \int_{-\infty}^{\infty} ds \, \Phi(\tau-s) \varphi(s) = \delta(\tau) \to \varphi(\tau) = \int dk \frac{e^{-ik\tau}}{2\pi \left[ 1 + (\lambda k)^2 \right]} = \frac{1}{2\lambda} e^{-|\tau|/\lambda}. \] (292)
The set of divergent diagrams in the resulting theory is just the subset of the divergent diagrams in the on-shell theory which contain no matter field loops. Therefore, with the mass integrations made finite, the off-shell theory is seen to be renormalizable by the same arguments given by Rohrlich [24] for the on-shell theory. Unlike a momentum cut-off, the mass cut-off does not affect the invariances of the original theory.

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