Realizability conditions for relativistic gases with a non-zero heat flux

Stefano Boccelli

Department of Mechanical Engineering, University of Ottawa, ON, Canada.

James G. McDonald

Department of Mechanical Engineering, University of Ottawa, ON, Canada.

Accepted for publication in Physics of Fluids, 2022

Abstract

This work introduces a limitation on the minimum value that can be assumed by the energy of a relativistic gas in the presence of a non-zero heat flux. Such a limitation arises from the non-negativity of the particle distribution function, and is found by solving the Hamburger moment problem. The resulting limitation is seen to recover the Taub inequality in the case of a zero heat flux, but is more strict if a non-zero heat flux is considered. These results imply that, in order for the distribution function to be non-negative, (i) the energy of a gas must be larger than a minimum threshold; (ii) the heat flux, on the other hand, has a maximum value determined by the energy and the pressure tensor; and (iii) there exists an upper limit for the adiabatic index $\Gamma$ of the relativistic equation of state, and that limit decreases in the presence of a heat flux and pressure anisotropy, asymptoting to a value $\Gamma = 1$. The latter point implies that the Synge equation of state is formally incompatible with a relativistic gas showing a heat flux, except in certain gas states.

1 Introduction

From kinetic theory arguments, Taub\(^1\) has shown that the kinetic energy density, $\rho \varepsilon$, and the hydrostatic pressure, $P$, of a relativistic gas must respect the inequality

$$ (\rho \varepsilon - 3P) \rho \varepsilon \geq \rho c^2, \quad (1) $$

where $\rho$ is the mass density of the gas in the local rest frame and $c$ is the speed of light. This inequality has proven crucial in the formulation of equations of state (EoS). For instance, the Taub inequality rules out the usage of an adiabatic index, $\Gamma$, equal to the classical value of $5/3$ for a monatomic gas, and implies instead that, in the relativistically hot gas limit, $\Gamma$ should decrease to $4/3$ (see for instance Mignone & McKinney\(^2\)). The Synge EoS\(^3\) embeds these limits naturally, but the presence of Bessel functions in its formulation, together with the need to invert it during numerical simulations in order to find primitive variables, has triggered the search for alternative equations of state that are at the same time accurate and reasonably easy to solve.\(^4,5\)

As discussed in Section 2, the relation between the energy and the pressure (and thus, ultimately, the EoS) depends on the shape of the phase-space particle distribution function, $f$. In general, all thermodynamic variables and macroscopic quantities can be obtained as statistical moments of $f$. Considering that $f$ is non-negative by definition, as it represents the number of particles per unit phase-space volume, one may expect that not all imaginable gas states (sets of macroscopic quantities) are possible, if the non-negativity of $f$ is to be preserved. For instance, states characterized by a negative density or a negative energy are impossible. However, less trivial conditions also exist and additional non-linear combinations of moments may also result to be impossible. Such conditions can be formalized by solving the Hamburger moment problem.\(^6\) In this work, we aim to do this for a single-component relativistic gas, and neglecting quantum effects.

In Section 2, we introduce the notation employed in this work, we define the moments of the particle distribution function and highlight their thermodynamic interpretation. Then, in Section 3, we consider the simple case of a gas made of particles that possess a single translational degree of freedom (particles in a spatially one-dimensional world). For such a gas, we obtain a set of necessary conditions that the moments need to respect in order to be realizable by a non-negative distribution function. The Taub inequality appears naturally as one such condition. In Section 4, we consider the implications of these conditions on the maximum allowable heat flux and on the admissible equations of state. Some material supporting the 1D (one-dimensional) problem, and a comparison of the Synge EoS with the results of this work, are presented in Appendix A. Finally, Section 5 extends the results to a more realistic gas with $N$ translational degrees of freedom ($N = 3$ being the typical case).
2 Notation and relativistic kinetic description

In this work, uppercase indices are used to indicate four-vectors, and lowercase indices indicate their spatial part. For instance, the space-time coordinates are defined by the four-vector \( x^A = (ct, x^a) \), where \( c \) is the speed of light. The momentum four-vector of a particle is written as \( p^A = (p^0, p^a) \), and its contraction is \( p^A p_A = m^2 c^2 \), with \( m \) the particle mass.

The derivations discussed in this work simplify significantly if one considers particles with a single momentum component, \( N = 1 \). This corresponds to limiting the spatial index to \( a = x \), and the metric tensor in that case is written as \( \eta^{AA} = \text{diag}(1, -1) \). In this work, we start by considering results in this simplified scenario, and then extend them to the typical case of \( N = 3 \) spatial dimensions, with \( \eta^{AA} = \text{diag}(1, -1, -1, -1) \).

In relativistic kinetic theory, macroscopic variables are obtained as moments of the distribution function (or “phase density”), \( f(x^A, p^a) \), integrated over the \( N \)-dimensional momentum space.\(^7\) The first two moments of \( f \) are the particle four-flow and the energy-momentum tensor. The former reads
\[
N^A = c \int_{-\infty}^{+\infty} p^A f \, dp,
\]
where \( dp = dp^4 / p^0 \) is the invariant infinitesimal momentum element. In the typical case of particles with three momentum components, \( dp = dp^3 / p^0 \) and the integral in Eq. (2) is thus triple. The energy-momentum tensor is obtained as
\[
T^{AB} = c \int_{-\infty}^{+\infty} p^A p^B f \, dp.
\]
In this work, we employ the Eckart decomposition, such that the heat flux is associated to the energy-momentum tensor. Also, the quantities are considered in the Lorentz rest frame (“LRF”, indicated by a subscript \( R \)), co-moving with the gas. Under these premises, the moments assume a simple thermodynamic meaning.\(^7\) The particle four-flow becomes
\[
N^A = (cn, 0),
\]
with \( n \) the number density of the gas (\( \rho = nm \) being the mass density in the rest frame), and where the boldface font is used to denote the spatial component of the four-vector. The energy-momentum tensor in the LRF is
\[
T^{ab}_R = \begin{bmatrix} T^{00}_R & T^{0a}_R \\ T^{a0}_R & T^{aa}_R \end{bmatrix} = \begin{bmatrix} \rho c^2 & q^a / c \\ q^a / c & p^{aa} \end{bmatrix}.
\]

In the LRF, the purely time component \( T^{00}_R \) represents the energy density of the gas, \( \rho c^2 \), that decomposes as \( \rho c^2 = \rho c^2 + \rho \mathcal{E} \), giving the rest energy and the kinetic energy respectively. The purely spatial components, \( T^{aa}_R \), represent the \( n \)-dimensional pressure tensor, \( p^{aa} \). Additionally, we define the hydrostatic pressure as the (scaled) trace of the pressure tensor, \( P := -P^{aa} / N \). For an equilibrium gas, the pressure tensor is diagonal and reads \( p^{aa} = -P \eta^{aa} \). However, this condition is not necessary in the present work, where we retain instead its general form. The mixed time-space components \( T^{0a}_R \) represent the flux of energy, and in the LRF are equal to the heat flux, \( q^a \), scaled by the speed of light. The contraction of the energy-momentum tensor is \( (T^{AB}_R)_R = \rho \mathcal{E} - NP \).

As discussed by Dreyer & Weiss,\(^8\) in the classical limit, the relativistic thermodynamic variables \( \rho \mathcal{E}, P^{ab} \) and \( q^a \) recover the traditional definitions of classical kinetic theory\(^9\) (subscript “clas”) within a factor of \( \mathcal{O}(c^2) \). Symbolically denoting the classical limit as “\( c \to \infty \)”, we have:
\[
\lim_{c \to \infty} \rho \mathcal{E} = \int \frac{m^2}{2} f_{\text{clas}}(v) \, dv = (\rho \mathcal{E})_{\text{clas}},
\]
\[
\lim_{c \to \infty} P^{ab} = \int m v^a v^b f_{\text{clas}}(v) \, dv = P^{ab}_{\text{clas}},
\]
\[
\lim_{c \to \infty} q^a = \int \frac{m^2}{2} v f_{\text{clas}}(v) \, dv = q^a_{\text{clas}},
\]
where \( f_{\text{clas}}(v) \) is the classical velocity distribution function, and where the particle velocity \( v \) appears in place of the peculiar velocity because we have assumed to be in the LRF.

3 Realizability conditions in \( N = 1 \) spatial dimensions

We consider here a gas composed of particles that possess a single spatial momentum component, \( p^A = (p^0, p^x) \). In the following, we refer to such a gas as spatially one-dimensional, or “1D”. While not necessarily physically realistic, this case simplifies drastically the formulation and allows to gain a deeper understanding of the problem. The general “ND” case is considered in Section 5.

The distribution function, \( f \), introduced in Section 1 represents the number of particles per unit of phase-space volume, and is thus non-negative by definition. As known, this non-negativity causes the density, energy and pressure to be non-negative as well. However, non-negativity also introduces further and less obvious constraints, as not all combinations of moments are possible. The problem of finding the states (sets of moments) that are compatible with a non-negative distribution function is known as the Hamburger moment problem.\(^6\) This problem has been frequently employed in classical gas dynamics,\(^10\) where the states that are compatible with a non-negative distribution function are said to be “physically realizable”. Our aim here is to extend this to a relativistic gas.

First, we shall compose an array \( \mathbf{M} = (1, p^0, p^x) \), and use it to build the matrix \( \mathbf{M}^T \mathbf{M} \). If we compute moments of the distribution function using such a matrix as a weight, we obtain
\[
\mathbf{Y} = c \int \mathbf{M}^T \mathbf{M} f \, dp = c \int \begin{bmatrix} 1 & p^0 & p^x \\ p^0 & p^0 & p^0 p^x \\ p^x & p^0 p^x & p^x p^x \end{bmatrix} f \, dp.
\]

Most of these moments have been defined in Eqs. (4) and (5), while the first entry can be written as:
\[
Y_{1,1} = c \int f \, dp = c \int \frac{p^A p_A}{m^2 c^2} f \, dp = \frac{T^{AA}}{m^2 c^2}.
\]
In the LRF, these moments are

\[
Y_{kr} = \begin{bmatrix}
\frac{\rho e - P}{m c^2} & nc & 0 \\
nc & \rho e & q/e_c \\
0 & q/e_c & P
\end{bmatrix}.
\]  

(9)

As mentioned, in this 1D case, we have defined the hydrostatic pressure as \( P = \rho c^2 \). Also, we have omitted the superscript on the heat flux, meaning \( q = q^* \).

The \( M^T M \) matrix is symmetric, by construction. Also, it is easily verified to be positive semi-definite (PSD), with eigen-values \( \lambda_1, 2 = 0 \), and \( \lambda_3 = 1 + p^0 \rho^0 + p^0 p^* > 0 \). Since it is PSD, all its principal minors are non-negative. Considering that \( dP = dp/p^0 \) is positive, one obtains the following result: if \( f \) is non-negative, then the matrix \( Y \) is PSD, and all its principal minors are non-negative. This translates into a set of necessary conditions for the moments that appear in Eq. (9). A gas state (a set of moments) that does not respect such conditions cannot be represented by a non-negative distribution function \( f \) and is thus kinetically impossible.

Considering the principal minors of \( Y_{kr} \), obtained by removing either zero, one or two rows and columns, one obtains seven different conditions for the moments. Among these, the following three conditions are interesting and constitute the core of this work:

\[
\begin{align*}
\rho e - P & \geq 0 & (C_i), \\
\rho e - P & |\rho e| - \rho^2 c^4 \geq 0 & (C_{ii}), \\
\rho e - P & | \rho e - q^2 \rho e / P c^2 | - \rho^2 c^4 \geq 0 & (C_{iii}).
\end{align*}
\]  

(10)

We refer to these conditions as \( C_i, C_{ii} \) and \( C_{iii} \). The first condition states that the total energy (rest energy plus kinetic energy) must be larger than the pressure. The second condition is well known and was previously obtained by Taub from different arguments. This form of condition \( C_{ii} \) differs from the original Taub’s inequality by a factor 3 multiplying the pressure. This factor does not appear here since we are considering a 1D gas, but is recovered in Section 5. The third condition \( C_{iii} \) appears as a generalization of Taub’s result, and includes the effect of a non-zero heat flux on realizability. All three conditions must be satisfied for the distribution function to be non-negative.

The PSD requirement for \( Y_{kr} \) gives four other conditions, that are however less interesting. Indeed, two of these only require the pressure and the energy density to be non-negative quantities. As for the two remaining conditions, one is a duplicate of \( C_i \) (provided that the pressure is positive), while the other reads

\[
\rho e P - q^2 / \rho e^2 \geq 0.
\]  

(11)

This condition is satisfied automatically by the previous Conditions (i-iii) and does not bring any additional information. Therefore, the only conditions that we need to analyze in this work are Conditions (i-iii).

Finally, note that relations analogous to Conditions (i-iii) for higher-order moments can be obtained by considering additional entries in the vector \( M \). For instance, one could consider \( M = (1, p^0, p^*, p^0 p^*, p^0 p^* p^*, \cdots) \). However, this goes beyond the scope of the present work.

In the following section, we analyze the region of moment space where all conditions are satisfied. The implications of these conditions are then discussed in Section 4.

### 3.1 Realizability boundary in the \((\rho e)_*, -q_*\) space

It is convenient to recast the conditions in Eq. (10) in a non-dimensional form. One has different possibilities, such as scaling the quantities by powers of the speed of light. However, such a constant scaling would not let any self-similarity features emerge. Instead, we proceed as follows: if we consider the roots of the third condition, \( C_{iii} \), we have

\[
(\rho e)^{(iii)}_{\pm} = \frac{1}{2} \left[ P + \frac{q^2}{P c^2} \right] \pm \frac{1}{2} \sqrt{p^2 + 4\rho^2 c^4 + \frac{q^2}{c^2} - \rho^2 c^2}.
\]  

(12)

The dimensionless term \( (q^2 / P c^2 - 2) \) under the square root suggests that we employ a non-dimensionalization of the heat flux based on the pressure. In particular, if we divide Eq. (12) by \( P \), we obtain the following dimensionless groups (subscript “*”):

\[
q_* = q/(Pc), \quad (\rho e)_* = \rho e/P, \quad \Theta = P/(Pc^2).
\]  

(13)

As customary, the symbol \( \Theta \) is used here to denote the pressure–density ratio, that expresses the non-dimensional gas temperature. This non-dimensionalization happens to be very convenient, as it

- Reduces the number of unknowns, removing the density from the picture;
- Provides an automatic scaling for the heat flux, allowing us to compare the results obtained for different values of \( \Theta \).

Conditions (i), (ii) and (iii) in dimensionless form read:

\[
\begin{align*}
(\rho e)_* - 1 & \geq 0 & (C_i), \\
(\rho e)_* - 1 - 1/\Theta^2 & \geq 0 & (C_{ii}), \\
(\rho e)_* - 1 - q_*^2 & - 1/\Theta^2 & \geq 0 & (C_{iii}).
\end{align*}
\]  

(14)

All three conditions of Eq. (10) need to be satisfied in order to have a non-negative distribution function. After solving for the energy, these conditions are shown separately in Fig. 1, where we consider the special case of \( \Theta = 1 \). Notice that Fig. 1 shows Conditions (i-iii) employing the kinetic energy \( (\rho e)_* \), instead of the total energy \( (\rho e)_* \).

Conditions \( C_i \) and \( C_{ii} \) do not contain any information on the heat flux, and thus appear as straight lines in moment space. From Fig. 1 and Eq. (14), we notice that \( C_{iii} \) recovers exactly the Taub inequality, in the special case of \( q_* = 0 \). Considering
all three conditions, the most stringent one is constituted by the positive root of $C_{iii}$, that reads

$$\langle \rho \varepsilon \rangle_{iii}^{*} = \frac{1}{2} \left[ 1 + q_{*}^2 + \sqrt{1 + \frac{4}{\Theta^2} + q_{*}^2 (q_{*}^2 - 2)} \right] - \frac{1}{\Theta}. \quad (15)$$

This condition is denoted here as the “realizability boundary”, since only states with an energy $\langle \rho \varepsilon \rangle_{iii}^{*} \geq \langle \rho \varepsilon \rangle_{iii}^{*}$ can be realized by a non-negative distribution function. This boundary is shown in Fig. 2 for various values of $\Theta$. Thermodynamic equilibrium (the Maxwell-Jüttner distribution) is represented by a single point on that space, that is always above the realizability line and has a value of $q_{*} = 0$. In Fig. 2, equilibrium is shown by black circles, for varying values of $\Theta$. A discussion of the 1D Maxwell-Jüttner distribution is given in Appendix A. All other points in the $\langle \rho \varepsilon \rangle_{iii}^{*}, q_{*}$ space represent different non-equilibrium states.

### 3.2 Ultrarelativistic and classical limits

For $\Theta \to \infty$ (ultrarelativistic limit), the realizability boundary is given by the line

$$\lim_{\Theta \to \infty} \langle \rho \varepsilon \rangle_{iii}^{*} = \frac{1}{2} \left[ 1 + q_{*}^2 + \sqrt{1 + \frac{4}{\Theta^2} + q_{*}^2 (q_{*}^2 - 2)} \right], \quad (17)$$

that is a parabola truncated at its bottom, and can be rewritten as

$$\lim_{\Theta \to \infty} \langle \rho \varepsilon \rangle_{iii}^{*} = \begin{cases} q_{*}^2 & \text{for } |q_{*}| \geq 1, \\ 1 & \text{for } -1 < q_{*} < 1. \end{cases} \quad (18)$$

In the case of a relativistically cold gas (“$\Theta \to 0$”), the realizability boundary becomes instead

$$\lim_{\Theta \to 0} \langle \rho \varepsilon \rangle_{iii}^{*} = \frac{1}{2} (q_{*}^2 + 1). \quad (19)$$

Figure 2: Realizability boundary from Eq. (16) for different values of $\Theta$. From the bottom: $\Theta = 0, 0.25, 1, 5$ and $\Theta \to \infty$. Black circles at $q_{*} = 0$ denote the equilibrium Maxwell-Jüttner distribution at these values of $\Theta$.

Notice that, wherease this limit appears in Fig. 2, that does not mean that a classical gas actually does reach all such states. Indeed, for recovering the classical behaviour, it is not sufficient to compute the limit of $\Theta \to 0$, but one also needs to consider that $q_{*} \to 0$. This is easily understood if one considers that the natural scaling for the heat flux of a classical gas is $\rho v_{th}^2$, with $v_{th}$ the thermal velocity. Therefore, as the classical regime is approached, we have

$$\lim_{\Theta \to 0, q_{*} \to 0} \rho v_{th}^2 \to 0, \quad (20)$$

where the ratio $P/\rho$ for a classical gas is proportional to the thermal velocity squared. Therefore, the classical limit of $C_{iii}$ is properly obtained by setting both $\Theta \to 0$ and $q_{*} \to 0$,

$$\lim_{\Theta \to 0, q_{*} \to 0} \langle \rho \varepsilon \rangle_{iii}^{*} = \frac{1}{2}. \quad (21)$$
or, in dimensional form,
\[
\lim_{\Theta \to 0, q_* \to 0} (\rho \varepsilon)^{(iii)} = P/2, \tag{22}
\]
that corresponds to the only possible value of the energy for a classical gas with a single translational degree of freedom (adiabatic constant \(\Gamma = 3\)). In other words, Condition \(C_{iii}\) reduces to well known principles, in the classical regime.

\section{Implications of Condition (iii)}

In this section, we remark the implications of \(C_{iii}\). First, as already discussed, \(C_{iii}\) can be interpreted as a lower limit on the energy, \(\rho \varepsilon\). Second, \(C_{iii}\) can be seen as a limit on the heat flux. Finally, \(C_{iii}\) can be seen to have an effect on the allowable equations of state.

\subsection{Maximum allowable heat flux}

By solving \(C_{iii}\) for the heat flux, we obtain that a 1D relativistic gas with a given energy, \(\rho \varepsilon\), and pressure, \(P\), can support at most a maximum heat flux, such that \(|q| < q_{\text{max}}\). From Eq. (14), we get
\[
q_{\text{max}} = \sqrt{\frac{(\rho \varepsilon)_+ [(\rho \varepsilon)_+ - 1] - 1/\Theta^2}{(\rho \varepsilon)_+ - 1}}. \tag{23}
\]
Note that the total energy \((\rho \varepsilon)_+ = (\rho \varepsilon)_+ + 1/\Theta\) is used in Eq. (23). This limit can prove useful for assessing the kinetic compliance of heat flux closures and models.\(^{13–15}\)

\subsection{Allowable Equations of State}

The Taub inequality is known to pose a limitation on the physically allowable equations of state.\(^{2}\) \(C_{iii}\) generalizes this by introducing the effect of the heat flux. After introducing an index \(\Gamma\) (that does not need to be constant), \(C_{iii}\) reads
\[
P \frac{\Gamma - 1}{\Theta} = \rho \varepsilon \geq (\rho \varepsilon)^{(iii)} = \frac{P}{\Gamma^{(iii)} - 1}, \tag{24}
\]
or
\[
\Gamma \leq \Gamma^{(iii)} = 1 + \frac{1}{(\rho \varepsilon)_+^{(iii)}}. \tag{25}
\]
This gives an upper limit to the allowable values for \(\Gamma\). Note that, since the energy \((\rho \varepsilon)_+^{(iii)}\) depends on both the dimensionless temperature, \(\Theta\), and the heat flux, then \(\Gamma^{(iii)} = \Gamma^{(iii)}(\Theta, q_*)\). This is shown in Fig. 3. As expected, for \(q_* \to 0\), \(\Gamma^{(iii)}\) recovers the Taub inequality, with limits of \(\Gamma = 3\) and 2 for a cold and hot gas respectively, as also discussed in Appendix A.

Condition \(C_{iii}\) can be seen to have four asymptotic behaviors, denoted in Fig. 3 as regions (a), (b), (c) and (d). In region (a), where \((\Theta, q_*) \to (0, 0)\), the classical limit is recovered. Moreover, the gradient of \(\Gamma^{(iii)}\) is zero in this region, as \(C_{iii}\) loses all dependence on the heat flux. Region (b) is characterized by relativistic particle velocities, and by heat fluxes ranging from \(q_* \to 0\) (symmetric distribution function, Taub limit) to \(q_* = 1\). In this region, the heat flux appears to play no role. This reflects the shape of the ultrarelativistic limit of Condition \(C_{iii}\) in moment space (see Fig. 2), that shows a kink at \(q_* = 1\).

As the heat flux is increased, \(\Gamma^{(iii)}\) asymptotes to a value of 1. This is expected, since large values of the heat flux, \(q_*\), can be realized only if the energy also increases (see Fig. 2). In the limit of \(q_* \to \infty\), and for a given and finite pressure, \(P\), this can only be realized if the denominator of Eq. (24) goes to zero, and thus \(\Gamma \to 1\). In Region (c), the gas is characterized by a fully relativistic distribution function with strong asymmetry.

Region (d) instead is more subtle. In this region, the gas has a low temperature and thus may appear to be classical, yet the heat flux is relativistically significant. Considering again Fig. 2, for a cold gas, a non-zero heat flux requires that the energy, \(\rho \varepsilon\), is significantly super-Maxwellian. This apparent paradox (low temperature yet high energy) could be realized for instance by distribution functions composed of two populations of particles, such as bump-on-tail distributions. One may think of a non-relativistic cold gas, crossed by a very dim beam of relativistic particles. If the beam is sufficiently rarefied with respect to the bulk population, the overall temperature \(\Theta = P/\rho c^2\) remains low. Yet, the overall energy is significantly increased by the high-velocity beam. Moreover, the heat flux might also be significantly affected by the high-velocity particles (asymmetric distribution). An analogous effect could be played by asymmetric and non-Maxwellian tails.

The decrease of \(\Gamma^{(iii)}\) implies that the Synge EoS\(^{3}\) becomes unphysical in the presence of a significant heat flux. This is shown in Appendix A. In the ultrarelativistic regime, the Synge EoS happens to be still valid as long as \(q_* \leq 1\), but it breaks \(C_{iii}\) immediately after. Instead, for small values of \(\Theta\),
even small values of the heat flux are such that the Synge EoS lies above \( \Gamma^{(iii)} \). This violation is not dramatic if \( q_s \) is small (for \( q_s = 0.1 \), the discrepancy is below the 1%) but deviations become significant soon after.

The question of what equation of state is the most suitable for such relativistic non-equilibrium cases is not trivial and ultimately depends on the specific system of equations to be solved (e.g. traditional hydrodynamics or higher-order methods\(^{16-21} \)). However, one may decide to employ directly the limiting condition, \( \Gamma^{(iii)} \), as an equation of state, for the lack of a better model. An analogous choice was previously employed by Mignone et al.,\(^{22} \) in the context of an equilibrium gas with \( q_s = 0 \).

5 Realizability conditions in \( N \) spatial dimensions

The results of the previous sections extend easily to the case of \( N \) spatial dimensions. All that one has to do is to introduce additional spatial momentum components into the array \( M \). For instance, in the case of \( N = 3 \) dimensions, one has \( M = \sqrt{c} \left( 1, p^0, p^1, p^2, p^3 \right) \) and the matrix \( \mathcal{Y}_R \) reads

\[
\mathcal{Y}_R = \left[ \begin{array}{cccc}
(T^4_{\mathcal{A}})_{\mathcal{R}} & nc & 0 & 0 \\
nc & \rho e & q^1/c & q^2/c & q^3/c \\
0 & q^1/c & p^{xx} & p^{xy} & p^{xz} \\
0 & q^2/c & p^{xy} & p^{yy} & p^{yz} \\
0 & q^3/c & p^{xz} & p^{yz} & p^{zz}
\end{array} \right].
\]  

with \( (T^4_{\mathcal{A}})_{\mathcal{R}} = (\rho e - NP)/m^2 c^2 \). The definitions of \( M \) and \( \mathcal{Y}_R \) extend trivially to different values of \( N \); the expressions shown in the remaining of this section are general and hold for every value of \( N \geq 1 \). In particular, the results of the previous section can be recovered by considering one single spatial component and setting \( N = 1 \). The case \( N = 2 \) can be of relevance for the study of solid-state configurations, where particles are bounded on a two-dimensional surface\(^{23,24} \), while \( N = 3 \) represents a typical gas.

As before, realizable states make the matrix \( \mathcal{Y}_R \) PSD. The first two upper-left sub-matrices result in conditions that are completely analogous to the 1D case, except that a factor \( N \) multiplies the pressure. In dimensionless form:

\[
\begin{cases}
(\rho e) - N \geq 0 \\
(\rho e) - N |(\rho e)| - 1/\Theta^2 \geq 0
\end{cases} \quad (C_i),
\]

\[
(\rho e) - N |(\rho e)| - 1/\Theta^2 \geq 0 \quad (C_{ii}).
\]  

As before, the second condition is the Taub inequality. The dimensionless temperature, \( \Theta \), was defined using the hydrostatic pressure, \( P \), and therefore still reads \( \Theta = P/\rho c^2 \). The next condition introduces the heat flux component \( q^1 \) and reads

\[
[\rho e - NP] \left[ \rho e p^{xx} - \frac{q^1}{c^2} \right] - \rho^2 c^4 p^{xx} \geq 0.
\]  

This introduces the need to non-dimensionalize the components of the \( N \)-dimensional pressure tensor. We define the scaled quantities \( \rho e^{\text{ab}} = \rho^{\text{ab}}/P \). With this definition, we can write

\[
\left[ (\rho e) - N \right] \left[ (\rho e) - \frac{q^1}{P^{xx}} \right] - \frac{1}{\Theta^2} \geq 0. \tag{29}
\]

Analogous conditions can be obtained for the remaining entries of the heat flux vector \( q^i \), just by swapping the rows and columns of the matrix \( \mathcal{Y}_R \). However, we neglect them here and consider instead the full condition, obtained from non-negativity of the determinant of the full matrix \( \mathcal{Y}_R \), that we write as

\[
\det \mathcal{Y}_R = \frac{\rho e - NP}{m^2 c^2} \det \left[ \begin{array}{ccc}
\rho e & q^1/c & p^{ab} \\
q^1/c & p^{xx} & p^{xy} \\
p^{xy} & p^{yy} & p^{yz} \\
p^{xz} & p^{yz} & p^{zz}
\end{array} \right] - n^2 c^2 \det [p^{ab}] \geq 0,
\]  

where the first determinant can be computed by exploiting the block structure of that sub-matrix, following

\[
\det \left[ \begin{array}{cc}
\mathcal{A} & \mathcal{B} \mathcal{C} \\
\mathcal{B} & \mathcal{C}^{-1} \mathcal{B} \mathcal{C}
\end{array} \right] = \det(\mathcal{C}) \left( \mathcal{A} - \mathcal{B} \mathcal{C}^{-1} \mathcal{B} \right),
\]

where, using matrix notation, \( \mathcal{A} = \rho e \), \( \mathcal{B} = q/c = q^i/c \) and \( \mathcal{C} = \mathcal{P} = \rho^{ab} \). The quantity \( \det(p^{ab}) \) eventually cancels out being positive, and the highest-order condition (that we simply denote again as \( C_{iii} \) for clarity) ultimately reads

\[
(\rho e - NP) \left( \rho e - \frac{1}{c^2} q^1 (P^{xx})^{-1} q \right) - \rho^2 c^4 \geq 0 \quad (C_{iii}),
\]  

where we have switched to matrix notation. Notice that \( C_{iii} \) includes the previous conditions on the heat flux components as sub-cases. This condition is non-dimensionalized by dividing by \( P^2 \), giving

\[
[\rho e] - N \left[ (\rho e) - \chi^2 \right] - \frac{1}{\Theta^2} \geq 0,
\]  

where we have introduced the following shorthand to simplify the notation:

\[
\chi = \sqrt{q^1 (P^{xx})^{-1} q}.
\]  

Solving Eq. (33) for the total energy and then computing the kinetic energy \( \rho e = \rho e - \rho c^2 \), Condition \( C_{iii} \) for an \( N \)-dimensional gas reads

\[
(\rho e)^{(iii)} = \frac{1}{2} \left[ N + \chi^2 + \sqrt{N^2 + \frac{4}{\Theta^2} + \chi^2 (\chi^2 - 2N)} \right] - \frac{1}{\Theta}.
\]  

As for the 1D case, this is a lower threshold for the acceptable energies, that translates into an upper limit for \( \Gamma \), and we have that \( \Gamma^{(iii)} = 1 + 1/(\rho e) \). The upper limit for \( N = 3 \) is shown in Fig. 4. For a zero heat flux, this expression recovers the known values of \( 5/3 \) and \( 4/3 \) for a relativistically cold and hot gas respectively. As discussed for the 1D case, \( \Gamma^{(iii)} \to 1 \) for relativistically large heat fluxes.

Notice that the parameter \( \chi \) embeds both the heat flux and the details of the pressure tensor. If asymmetries or anisotropies in the pressure tensor are present, these do have an effect on the realizability. In other words, besides the heat flux, models for the gas shear stresses and/or viscosity also need to take into account the mentioned limits.
the equilibrium Maxwell-Jüttner distribution, from which the
expected in the presence of a non-zero heat flux, since
instance, the classically valid adiabatic index
effects and limitations on the allowable equations of state: for
As is well known, the temperature introduces relativistic ef-
critical simulations.
also have important effects on viscous hydrodynamic numer-
whenever a sufficiently high heat flux is present. This may
This result rules out the use of the Synge equation of state
ogous influence, introducing additional relativistic effects.
show that the heat flux and pressure anisotropy have an anal-
known to become unphysical for relativistically hot gases. We
In this work, we show that the non-negativity of the particle
improving moments distribution function of a relativistic
tion of the energy-momentum tensor, \( T^{\mu\nu} \), as well as the heat flux and the pressure tensor anisotropies. The Taub inequality is re-
covered as a special case, in the limit of a zero heat flux. The
discussed condition appears to be more strict than the Taub
inequality, and can be interpreted as either:

- A lower limit on the energy, for a prescribed value of the
  pressure and heat flux;

- An upper limit on the heat flux, if the pressure and en-
  ergy are given;

- A limit on the realizable equations of state (EoS).

As is well known, the temperature introduces relativistic ef-
fected and limitations on the allowable equations of state: for
instance, the classically valid adiabatic index \( \Gamma = 5/3 \) is
known to become unphysical for relativistically hot gases. We
show that the heat flux and pressure anisotropy have an anal-
ogous influence, introducing additional relativistic effects.
This result rules out the use of the Synge equation of state
whenever a sufficiently high heat flux is present. This may also
have important effects on viscous hydrodynamic numerical simulations.25–29

It should be noted that inaccuracies in the Synge EoS may
be expected in the presence of a non-zero heat flux, since
this would imply that the distribution function deviates from
the equilibrium Maxwell-Jüttner distribution, from which the
Synge EoS itself is built. However, besides possible inaccur-
cacies, we show that this EoS is incompatible with a positive
distribution function (and thus with kinetic theory) for a range
of non-equilibrium states. The actual region of realizability of
this EoS is reported in the Appendix and appears to be largest
in the ultra-relativistic limit. On the other hand, for a cold
gas, even small values of the heat flux are such that the Synge
EoS violates the realizability condition.

Data availability statement

Data sharing is not applicable to this article as no new data
were created or analyzed in this study.

Acknowledgments

We wish to thank L. P. Quartapelle and I. Hawke for the en-
lightening conversations. Funding for this project was pro-
vided by the Natural Sciences and Engineering Research
Council of Canada through grant number RGPAS-2020-

A Maxwell-Jüttner distribution for \( N = 1 \) and
its equation of state

The Maxwell-Jüttner (MJ) distribution function for a gas
with \( N = 1 \) momentum spatial components is easily ob-
tained following the derivations by Cercignani & Kremer7
and Dreyer,30 together with the following properties of the
modified Bessel functions of the second type, \( K_v(x) \):

\[
K_{v+1}(x) - K_{v-1}(x) = \frac{2v}{x} K_v(x). \tag{36}
\]

Ultimately, the 1D Maxwell-Jüttner distribution is obtained as

\[
f_{1D}^{\text{MJ}}(x) = \frac{n}{2mcK_1(1/\Theta)} \exp \left( -\frac{\gamma(p_R^0)}{\Theta} \right), \tag{37}
\]

where \( \gamma(p_R^0) \) is the Lorentz factor, \( p_R^0 \) is the spatial component of the particle momentum evaluated in the Lorentz rest frame, and
\( \Theta = P/\rho c^2 \) as for the 3D case. With respect to the typical
3D MJ distribution, in the 1D case one has Bessel functions of lower order and no factor \( \Theta \) appears at the denominator.

The equation of state can be obtained by considering the contrac-
tion of the energy-momentum tensor, \( T^{\mu\nu} A_\mu \), and by evaluating
the energy and the pressure from a direct integration of the
distribution in Eq. (37). Ultimately, the 1D EoS for an ideal
gas at equilibrium takes the form

\[
\rho e - P = \rho c^2 \left[ \frac{K_0(\zeta)}{K_1(\zeta)} - 1 \right], \tag{38}
\]

with \( \zeta = 1/\Theta \), and thus

\[
\Gamma = \left[ 1 + \frac{\rho c^2}{P} \left( \frac{K_0(\zeta)}{K_1(\zeta)} - 1 \right) \right]^{-1} + 1. \tag{39}
\]
This equation takes the expected value of 3 for a classical gas and 2 in the ultrarelativistic limit. By repeating the same derivation for a 3D gas, one obtains the typical 3D Maxwell-Jüttner distribution function and the Synge equation of state, with Bessel functions $K_2$ and $K_3$. Figure 5 shows the Synge EoS for a one-dimensional gas, from Eq. (39). In the figure, the Synge EoS is plotted in the $\Theta - q_*$ plane, in order to compare it with the realizability limit, $\Gamma^{(iii)}$, obtained in this work. Since the Synge EoS is obtained at equilibrium, it does not depend on the heat flux, and its gradient along the axis $q_*$ is zero. It can be seen that the Synge EoS:

- Respects the Taub inequality (Fig. 5 for $q_* \rightarrow 0$);
- Respects condition $C_{iii}$ only in a limited part of the $\Theta - q_*$ plane.

![Figure 5: Synge equation of state, $\Gamma_{\text{Synge}}$, realizability limit, $\Gamma^{(iii)}$, and Taub inequality, $\Gamma_{\text{Taub}}$, for a gas in $N = 1$ spatial dimensions. The white dashed line delimits the region where the Synge EOS is incompatible with a positive distribution function ($\Gamma_{\text{Synge}} > \Gamma^{(iii)}$).]

References

[1] AH Taub. Relativistic rankine-hugoniot equations. *Physical Review*, 74(3):328, 1948.

[2] A Mignone and Jonathan C McKinney. Equation of state in relativistic magnetohydrodynamics: variable versus constant adiabatic index. *Monthly Notices of the Royal Astronomical Society*, 378(3):1118–1130, 2007.

[3] John Lighton Synge. *The relativistic gas*. North-Holland publishing company, 1957.

[4] Dongsu Ryu, Indranil Chattopadhyay, and Eunwoo Choi. Equation of state in numerical relativistic hydrodynamics. *The Astrophysical Journal Supplement Series*, 166(1):410, 2006.

[5] Indranil Chattopadhyay and Dongsu Ryu. Effects of fluid composition on spherical flows around black holes. *The Astrophysical Journal*, 694(1):492, 2009.

[6] Hans Ludwig Hamburger. Hermitian transformations of deficiency-index (1, 1), jacobi matrices and undetermined moment problems. *American Journal of Mathematics*, 66(4):489–522, 1944.

[7] Carlo Cercignani and Gilberto Medeiros Kremer. Relativistic boltzmann equation. In *The Relativistic Boltzmann Equation: Theory and Applications*, pages 31–63. Springer, 2002.

[8] Wolfgang Dreyer and Wolf Weiss. The classical limit of relativistic extended thermodinamics. In *Annales de l’IHP Physique théorique*, volume 45, pages 401–418, 1986.

[9] JH Ferziger and HG Kaper. Mathematical theory of transport processes in gases. *American Journal of Physics*, 41(4):601–603, 1973.

[10] James McDonald and Manuel Torrilhon. Affordable robust moment closures for cfd based on the maximum-entropy hierarchy. *Journal of Computational Physics*, 251:500–523, 2013.

[11] John E Prussing. The principal minor test for semidefinite matrices. *Journal of Guidance, Control, and Dynamics*, 9(1):121–122, 1986.

[12] A Mignone and G Bodo. An hllc riemann solver for relativistic flows—i. hydrodynamics. *Monthly Notices of the Royal Astronomical Society*, 364(1):126–136, 2005.

[13] AL García-Perciante, AR Méndez, and E Escobar-Aguilar. Heat flux for a relativistic dilute bidimensional gas. *Journal of Statistical Physics*, 167(1):123–134, 2017.

[14] AR Méndez, AL García-Perciante, and G Chacón-Acosta. Dissipative properties of degenerate relativistic gases: The complete kernel calculation in a $(d+1)$ flat space-time. *Journal of Statistical Physics*, 186(3):1–26, 2022.

[15] A Gabbana, D Simeoni, S Succi, and R Tripeceone. Relativistic dissipation obeys chapman-enskog asymptotics: Analytical and numerical evidence as a basis for accurate kinetic simulations. *Physical Review E*, 99(5):052126, 2019.

[16] Harold Grad. On the kinetic theory of rarefied gases. *Communications on pure and applied mathematics*, 2(4):331–407, 1949.

[17] C David Levermore. Moment closure hierarchies for kinetic theories. *Journal of statistical Physics*, 83(5):1021–1065, 1996.
[18] A Gabbana, Miller Mendoza, Sauro Succi, and Raffaele Tripiccione. Kinetic approach to relativistic dissipation. *Physical Review E*, 96(2):023305, 2017.

[19] Victor E Ambruş and Robert Blaga. High-order quadrature-based lattice boltzmann models for the flow of ultrarelativistic rarefied gases. *Physical Review C*, 98(3):035201, 2018.

[20] Sebastiano Pennisi and Tommaso Ruggeri. Relativistic extended thermodynamics of rarefied polyatomic gas. *Annals of Physics*, 377:414–445, 2017.

[21] Henning Struchtrup. Projected moments in relativistic kinetic theory. *Physica A: Statistical Mechanics and its Applications*, 253(1-4):555–593, 1998.

[22] A Mignone, T Plewa, and G Bodo. The piecewise parabolic method for multidimensional relativistic fluid dynamics. *The Astrophysical Journal Supplement Series*, 160(1):199, 2005.

[23] Kostya S Novoselov, Andre K Geim, Sergei Vladimirovich Morozov, Dingde Jiang, Michail I Katsnelson, IVa Grigorieva, SVb Dubonos, and andAA Firsov. Two-dimensional gas of massless dirac fermions in graphene. *Nature*, 438(7065):197–200, 2005.

[24] Mark Watson. Relativistic wind farm effect: Possibly turbulent flow of a charged, massless relativistic fluid in graphene. *Physics of Fluids*, In press 2022.

[25] A Gabbana, S Plumari, G Galesi, V Greco, D Simonetti, S Succi, and R Tripiccione. Dissipative hydrodynamics of relativistic shock waves in a quark gluon plasma: Comparing and benchmarking alternate numerical methods. *Physical Review C*, 101(6):064904, 2020.

[26] Michail Chabanov, Luciano Rezzolla, and Dirk H Rischke. General-relativistic hydrodynamics of non-perfect fluids: 3+ 1 conservative formulation and application to viscous black hole accretion. *Monthly Notices of the Royal Astronomical Society*, 505(4):5910–5940, 2021.

[27] Ryosuke Yano and Kojiro Suzuki. Kinetic analysis of thermally relativistic flow with dissipation. *Physical Review D*, 83(2):023517, 2011.

[28] Shisheng Wang. Extensions to the Navier-Stokes equations. *Physics of Fluids*, 34(5):053106, 2022.

[29] PK Sahu. Magnetogasdynamic exponential shock wave in a self-gravitating, rotational axisymmetric non-ideal gas under the influence of heat-conduction and radiation heat-flux. *Ricerche di Matematica*, pages 1–37, 2021.

[30] W Dreyer. Maximisation of entropy for relativistic and degenerate gases in non-equilibrium. *Kinetic theory and extended thermodynamics*, pages 107–123, 1987.