Complete Blow-up and Avalanche Formation for a Parabolic System with Non-Simultaneous Blow-up

Cristina Brändle
Departamento de Matemáticas
U. Carlos III de Madrid, 28911 Leganés, Spain
e-mail: cristina.brandle@uc3m.es

Fernando Quirós
Departamento de Matemáticas
U. Autónoma de Madrid, 28049 Madrid, Spain
e-mail: fernando.quiros@uam.es

Julio D. Rossi
Departamento de Matemática FCEyN
U. de Buenos Aires, 1428 Buenos Aires, Argentina
e-mail: jrossi@dm.uba.ar

Received in revised form 13 September 2009
Communicated by Herbert Amann

---

The author acknowledges the support of project MTM2008-06326-C02-01/02 (Spain) and ESF programme “Global and geometric aspects of nonlinear partial differential equations”

The author acknowledges the support of project MTM2008-06326-C02-01/02 (Spain) and ESF programme “Global and geometric aspects of nonlinear partial differential equations”

The author acknowledges the support of project ANPCyT PICT 5009, UBA X066 and CONICET (Argentina)
Abstract

We study the possibility of defining a nontrivial continuation after the blow-up time for a system of two heat equations with a nonlinear coupling at the boundary. It turns out that any possible continuation that verifies a maximum principle is identically infinity everywhere after the blow-up time; that is, both components blow up completely. We also analyze the propagation of the singularity to the whole space, the avalanche, when blow-up is non-simultaneous.

1991 Mathematics Subject Classification. 35B60, 35K60, 35K57.
Key words. Complete blow-up, parabolic system, nonlinear boundary conditions, avalanche

1 Introduction and main results

We consider solutions \((u, v)\) to two heat equations in the half line, \(\mathbb{R}^+ = (0, \infty)\),
\[
\begin{align*}
  u_t &= u_{xx}, \\
  v_t &= v_{xx},
\end{align*}
\]
with a nonlinear flux coupling at the boundary
\[
\begin{align*}
  -u_x(0,t) &= u^{p_{11}}(0,t)v^{p_{12}}(0,t), \\
  -v_x(0,t) &= u^{p_{21}}(0,t)v^{p_{22}}(0,t),
\end{align*}
\]
(1.2)
The initial data
\[
\begin{align*}
  u(x,0) &= u_0(x), \\
  v(x,0) &= v_0(x),
\end{align*}
\]
(1.3)
are assumed to be nonnegative, nontrivial, continuous, bounded and integrable. We will also assume that they are compatible with the boundary conditions, so that solutions may be (and will be) understood in a classical sense. In order to have a totally coupled system, we impose the condition \(p_{ij} > 0\) on the nonlinearities. We also require the monotonicity in time of the solution, \(u_t, v_t \geq 0\). This hypothesis, which is common in the literature, is satisfied if and only if \(u_0'' \geq 0\) and \(v_0'' \geq 0\). Under these conditions solutions decay as \(x \to \infty\) as long as they are defined.

The time \(T\) denotes the maximal existence time for the solution \((u,v)\). If it is infinite we say that the solution is global. If it is finite, we have
\[
\lim_{t \nearrow T} \{ \|u(\cdot,t)\|_\infty + \|v(\cdot,t)\|_\infty \} = \infty,
\]
and we say that the solution blows up. There are solutions of (1.1)–(1.3) which blow up if and only if the exponents \(p_{ij}\) satisfy any of the following conditions,
\[
p_{11} > 1, \quad p_{22} > 1, \quad p_{12}p_{21} > (1 - p_{11})(1 - p_{22}),
\]
see [16]. The study of blow-up due to reaction at the boundary, both for scalar problems and for systems (like the one under consideration here), has attracted a
Complete blow-up and avalanche formation

661

lot of attention in recent years, see for example the surveys [5], [6], [11] and the references therein.

The speed at which blow up takes place (the so called blow-up rate), that can be obtained as in [3], implies that for any \( x \neq 0 \) there is a constant \( K = K(x) \) such that \( \sup_{t \in (0,T)} \{ |u(x,t)| + |v(x,t)| \} \leq K \). Hence, \((u,v)\) blows up only at the origin. Thus, there may be a nontrivial extension of the solution for times \( t > T \) in some weak sense. If such a continuation exists, blow-up is said to be incomplete; otherwise, it is called complete. Complete blow-up was first studied for problems where there is a nonlinear reaction term in the equation, \( u_t = u_{xx} + f(u) \), see [2], [9], [10], [11], [12], [13], [18]. For the scalar version of the present problem complete blow-up is proved in [7], see also [17].

Our first aim is to study whether blow-up for problem (1.1)–(1.3) is complete or not. A natural way of obtaining a continuation consists of approximating the reaction nonlinearities in the boundary conditions by a sequence of functions that yield global in time solutions, and then pass to the limit in the approximations. Thus, we solve the heat equations (1.1) with initial data (1.3) and boundary conditions

\[
\begin{align*}
-\frac{\partial u}{\partial x}(0,t) &= f_n^{11}(u(0,t)) f_n^{12}(v(0,t)), \\
-\frac{\partial v}{\partial x}(0,t) &= f_n^{21}(u(0,t)) f_n^{22}(v(0,t)),
\end{align*}
\]

where

\[
f_n^{ij}(s) = \min\{s^{p_{ij}}, n^{p_{ij}}\},
\]

to obtain a globally defined solution \((u_n,v_n)\). Since the coupling functions \(f_n^{ij}\) increase with \(n\), the same is true for \(u_n\) and \(v_n\). Hence, one may attempt to extend the solution after \(T\) by taking the limit

\[
\lim_{n \to \infty} u_n = \overline{u}, \quad \lim_{n \to \infty} v_n = \overline{v}.
\]

The extension \((\overline{u}, \overline{v})\) obtained in this way is known in the literature as the proper solution, see [10], [11]. It is a minimal solution in the sense that any solution that satisfies a comparison principle must be above it. The next theorem shows that both components of the proper solution, and hence both components of any other reasonable extension, become infinite everywhere after the blow-up time; i.e., blow-up is always complete.

**Theorem 1.1** If a solution of (1.1)–(1.3) blows up at a finite time \(T\), it blows up completely. More precisely, for all \(x \in [0, \infty)\) the proper solution satisfies

\[
\begin{align*}
\overline{u}(x,t) &= u(x,t), & 0 < t < T, \\
\lim_{t \to T^-} u(x,t), & t = T, \\
\infty, & t > T, \\
\end{align*}
\]

\[
\begin{align*}
\overline{v}(x,t) &= v(x,t), & 0 < t < T, \\
\lim_{t \to T^-} v(x,t), & t = T, \\
\infty, & t > T.
\end{align*}
\]

Observe that blow-up is complete for both components even if one of them remains bounded up to the blow-up time, a possibility that is not excluded a priori. Indeed, for certain choices of the parameters \(p_{ij}\) there are initial data for which one
of the components of the system remains bounded while the other blows up. This phenomenon is commonly denoted as *non-simultaneous blow-up*. The possibility of non-simultaneous blow-up in nonlinear parabolic systems was first mentioned in [19], and has been studied more thoroughly later in [4], [15], [16] and [20]. For problem (1.1)–(1.3) this possibility was analyzed in [14], [16]: there exist solutions such that \( u \) blows up at time \( T \) while \( v \) remains bounded up to this time if and only if
\[ p_{11} > p_{21} + 1. \] (1.4)

Since blow-up takes place only at one point and there is complete blow-up, at \( t = T \) an instantaneous propagation of the blow-up singularity to the whole spatial domain takes place, what is called an *avalanche*, see [17], [18]. The avalanche may be regarded as a discontinuity at the blow-up time between the nontrivial blow-up profiles
\[ \pi(x, T^-) = \lim_{t \searrow T} u(x, t) \quad \text{and} \quad \pi(x, T^-) = \lim_{t \searrow T} v(x, t), \]
and the trivial values taken afterwards,
\[ \pi(x, T^+) = \infty = \pi(x, T^+) \quad \text{for all } x \in [0, \infty). \]

In the case of non-simultaneous blow-up, say \( u \) blows up while \( v \) remains bounded up to time \( T \), the discontinuity at \( t = T \) between \( \pi(x, T^-) \) (which is finite everywhere) and \( \pi(x, T^+) \) (which is infinite everywhere) is even more striking. In the sequel we will confine ourselves to this more appealing case, assuming that \( u \) blows up while \( v \) remains bounded up to \( t = T^- \). In particular, (1.4) holds and we have \( p_{11} > 1 \).

The instantaneous propagation of the singularity to the whole domain has a counterpart when we consider finite, but large, values of \( n \): the propagation of \( k \)-level sets for \( k \) large. Our next aim is to explain the evolution of such level sets. We will devote special attention to \( k = n^\gamma \). Let us remark that the study of the approximate problems for large \( n \) is in many cases (combustion, chemistry) more realistic than the blow-up problem, which is a mathematical idealization.

Given any function \( w \) such that \( w_t \geq 0, w(0, 0) \leq k \), we denote by \( t_w(k) \) the first time when \( w(0, t) = k \). Hence, \( t_{u_n}(n) \) and \( t_{v_n}(n) \) are the times at which truncations start taking place. In the next result we estimate these times.

**Theorem 1.2** Let \( n \) be large enough. Then,
\[ T - t_{u_n}(n) \sim n^{-2(p_{11} - 1)}, \quad t_{v_n}(n) - T \sim \begin{cases} n^{-2p_{21}}, & p_{22} > 1, \\ n^{-2p_{21} \log n}, & p_{22} = 1, \\ n^{-2p_{21} + 2(1 - p_{22})}, & p_{22} < 1. \end{cases} \]

By \( f \sim g \) we mean that there exist constants \( c_1, c_2 > 0 \) such that \( c_1 f \leq g \leq c_2 f \).

Since \( p_{11} > 1 \), \( t_{u_n}(n) \uparrow T \) as \( n \to \infty \). However, the behaviour of \( t_{v_n}(n) \) depends on the involved exponents. It may happen that \( t_{v_n}(n) \to \infty \) as \( n \to \infty \) (this occurs if \( p_{21} + p_{22} < 1 \)). This does not contradict the complete blow-up result for \( \pi \), since the size of \( v_n \) for \( t > T \) can be very large for \( t \approx T \) without reaching the level \( n \) before a long time after \( T \). This is indeed the case, as shown next.
Theorem 1.3 Let $n$ be large enough and let $0 < \gamma < 1$. Then,

$$T - t_u(n\gamma) \sim n^{-2\gamma(p_{11} - 1)},$$  \hfill (1.5)

$$t_u(n\gamma) - T \sim \begin{cases} 
  n^{-2p_{21}}, & p_{22} > 1, \\
  n^{-2p_{21}} \log n, & p_{22} = 1, \\
  n^{-2p_{21} + 2\gamma(1 - p_{22})}, & p_{22} < 1.
\end{cases}$$

When $p_{22} \geq 1$, the order of magnitude of the quantities $t_u(n\gamma) - T$ does not depend on $\gamma$. Are they equal up to leading order? The answer is given in the next theorem: they are equal to leading order when $p_{22} > 1$ and differ by a constant coefficient depending on $\gamma$ in the main term when $p_{22} = 1$.

Theorem 1.4 Let $n$ be large enough and let $0 < \gamma \leq 1$.

(i) If $p_{22} > 1$, then $t_u(n) - t_u(n\gamma) \sim n^{2(\gamma(p_{22} - 1) - p_{21})}$.

(ii) If $p_{22} = 1$, then $\lim_{n \to \infty} \frac{n^{2p_{21}}}{\log n} (t_u(n) - t_u(n\gamma)) = 1 - \gamma$.

Next, we want to describe how the level $n$ starts to propagate to the interior of the domain. To this aim, we define

$$v_T = \lim_{t \to T} v(0, t),$$

look at times which are close to $t_u(n)$, and scale $u_n$ by a factor $n$, so that we get something of order one. If we want the new dependent variable to be a solution to the heat equation, we also have to scale the space variable. The following result shows how the $n$-level set of $u_n$ evolves for times $t \approx T$ close to the origin, $x \approx 0$.

Theorem 1.5 There exists a nontrivial function $\Phi : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$ such that

$$\lim_{n \to \infty} n^{-1} u_n(n^{-2(p_{11} - 1)} y, t_{u_n}(n) + n^{-2(p_{11} - 1)} \tau) = \Phi(y, \tau),$$

$$\lim_{n \to \infty} v_n(n^{-2(p_{11} - 1)} y, t_{u_n}(n) + n^{-2(p_{11} - 1)} \tau) = v_T,$$

uniformly on compact subsets of $\mathbb{R}^+ \times \mathbb{R}$. The limit $\Phi$ depends on the initial data only through $v_T$.

Theorem 1.5 implies that the $n$-level set of $u_n$, written in terms of $(x, t)$ is given approximately by the 1-level set of the limit function $\Phi$ written in $(y, \tau)$ variables,

$$\{(x, t) : u_n(x, t) = n \approx \{(y, \tau) : \Phi(y, \tau) = 1\}.$$

These sets are related through the transformation

$$x = n^{-(p_{11} - 1)} y, \quad t = t_{u_n}(n) + n^{-2(p_{11} - 1)} \tau.$$

A monotonicity argument shows that the 1-level set of $\Phi$ can be expressed as the graph of some increasing function $g$. Since $t_{u_n}(n) \approx T$, this in turn implies that the $n$-level set of $u_n$ is given approximately by

$$\{(x, t) : x = n^{-(p_{11} - 1)} g(n^{2(p_{11} - 1)}(t - T))\}.$$
Remark 1.1 The function $\Phi$ that describes the onset of the avalanche for $u$ is a solution to the heat equation with inwards flux $-\Phi_y(0, \tau) = \int_0^1 (\Phi(0, \tau)) v_T^{p_{22}}$. It reaches the truncation level 1 at some finite time. Hence, its long time behaviour is given by a self-similar solution of the heat equation in the half-line with inwards flux equal to $v_T^{p_{22}}$ at the boundary; i.e.,

$$w(y, \tau) \sim v_T^{p_{22}} \tau^{1/2} H \left( \frac{y}{\tau^{1/2}} \right) \text{ as } \tau \to \infty,$$

where the profile $H$ solves $H''(\xi) + \frac{1}{2} \xi H'(\xi) - \frac{1}{2} H(\xi) = 0$, $-H'(0) = 1$, see [17] for the details. Using this asymptotic behaviour we see that the $k$-level set of $u_n$ behaves as

$$\{(y, \tau) : y = \tau^{1/2} H^{-1} \left( \frac{k}{v_T^{p_{22}}} \tau^{1/2} \right) \}.$$

We have described the onset of the avalanche for the $u$ variable. However, since the obtained profile for $v$ is the constant $v_T$, there is some lack of information concerning the spatial shape of the $v$ variable. To obtain a more precise description of the behaviour of $v_n$ we have to look at times which are a little bit larger.

Theorem 1.6 There exists a nontrivial function $\Psi : \mathbb{R}_+ \times [0, \tau_0) \mapsto \mathbb{R}$ such that

$$\lim_{n \to \infty} v_n(n^{-p_{22}} y, T + n^{-2p_{21}} \tau) = \Psi(y, \tau),$$

uniformly on compact subsets of $\mathbb{R}_+ \times [0, \tau_0)$. The time $\tau_0$ verifies $0 < \tau_0 < \infty$ if $p_{22} > 1$, $\tau_0 = \infty$ if $p_{22} \leq 1$. The limit $\Psi$ depends on the initial data only through $v_T$.

The behaviour of the $n$-level set of $v_n$ in terms of the graph of some increasing function follows arguing as we did before for $u_n$. We omit the details.

Remark 1.2 If $p_{22} > 1$ the limit function $\Psi$ blows up at the threshold time $\tau = \tau_0$.

Remark 1.3 The fact that $\tau_0$ has to be finite for $p_{22} > 1$ and not for $p_{22} \leq 1$ could have been guessed from Theorem 1.2.

Theorems 1.5 and 1.6 provide information near the origin close to the blow-up time. To completely describe the avalanche, we also give the behaviour of both components, $u_n$ and $v_n$, for fixed $(x, t)$, with $t > T$.

Theorem 1.7 Let

$$\gamma_1 = p_{11} + p_{22}, \quad \gamma_2 = p_{21} + p_{22} \quad \text{if } p_{21} + p_{22} > 1,$$

$$\gamma_1 = p_{11} + \frac{p_{12} p_{21}}{1 - p_{22}}, \quad \gamma_2 = \frac{p_{21}}{1 - p_{22}} \quad \text{if } p_{21} + p_{22} \leq 1.$$

There exist profiles $\varphi$ and $\psi$ such that

$$\lim_{n \to \infty} n^{-\gamma_1} u_n(x, t) = \varphi(x, t), \quad \lim_{n \to \infty} n^{-\gamma_2} v_n(x, t) = \psi(x, t),$$

uniformly on compact subsets of $[0, \infty) \times (T, \infty)$. The limits $\varphi$, $\psi$ are independent of the initial data.
The quantity $p_{21} + p_{22}$ measures the combined strength of the two reaction factors in the boundary condition for $v$. If it is above 1, reaction is big, and $v_n$ reaches the $n$-level quickly, in a time $t_{v_n}(n) \to T$ as $n \to \infty$. Hence, for any time $t > T$ all truncations have taken place for $n$ large. Thus, $u_n$ and $v_n$ are both solutions of the heat equation with inwards boundary fluxes given respectively by $n^{p_{11} + p_{12}}$ and $n^{p_{21} + p_{22}}$. What the theorem says is that we have to scale precisely by these powers of $n$ if we want to obtain a non-trivial limit. Consider now $p_{21} + p_{22} < 1$. Given any fixed time $t$, the truncation of $v_n$ takes place afterwards for all $n$ large enough. In this case the exponent $\gamma_2$ which gives the size of $v_n$ coincides with the critical value of $\gamma$ below which $t_{v_n}(n^\gamma)$ goes to $T$. The adequate value of $\gamma_1$ follows by inserting the size of $v_n$ in the boundary condition for $u_n$.

Our last step consists of considering the asymptotic behaviour of $u_n$ and $v_n$ as $t \to \infty$. Since we are considering $t > t_{v_n}(n)$, both truncations have taken place. Again, $u_n$ and $v_n$ are solutions of the heat equation with inwards boundary fluxes given respectively by $n^{p_{11} + p_{12}}$ and $n^{p_{21} + p_{22}}$. As is well known, solutions to this problem with integrable initial datum converge, uniformly on compact subsets, as $t$ goes to infinity to the solution of the same problem with zero initial data, see for instance [17].

**Remark 1.4** The nontrivial limit profiles that appear in Theorems 1.5, 1.6, and 1.7 are solutions of explicit problems, see Sections 4 and 5.

**Organization of the paper.** In Section 2 we prove complete blow-up for both components, Theorem 1.1. The estimates for $T - t_{u_n}(n^\gamma)$ and $t_{v_n}(n^\gamma) - T$ that give the truncations times are gathered in Section 3. Section 4 is devoted to proving Theorems 1.5 and 1.6, which describe the onset of the avalanche. In Section 5, we prove Theorem 1.7, which gives the behaviour of $(u_n, v_n)$ for fixed $(x, t)$, $t > T$. Finally, we devote an appendix to the analysis of certain self-similar profiles that are used in the proofs.

Throughout the paper $C, c$ denote constants, independent of $n$, which may be different in different occurrences.

## 2 Complete blow-up

**Proof of Theorem 1.1.** We can assume without loss of generality that $u$ blows up at time $t = T$. If this is not the case, then $v$ has to blow up, and the same proof applies interchanging the roles of $u$ and $v$. The component $v$ may blow up or not, since we have not assumed a priori that blow-up is non-simultaneous.

For times before the blow-up time, $t < T$, $(u, v)$ is bounded. Hence, if $n$ is large enough, $(u, v)$ solves the truncated problem up to time $t$ and therefore $u_n(x, t) = u(x, t)$ and $v_n(x, t) = v(x, t)$.

To study what happens exactly at $t = T$ we use the monotonicity with respect to $t$ and a comparison argument. For $t < T$ we have

$$\lim_{t \searrow T} u(x, t) \geq u_n(x, T) \geq u_n(x, t).$$
Taking limits as \( n \to \infty \), and using that \( \lim_{n \to \infty} u_n(x, t) = u(x, t) \) for \( t < T \), and then letting \( t \nearrow T \), we get

\[
\lim_{t \nearrow T} u(x, t) \geq \lim_{n \to \infty} u_n(x, T) \geq \lim_{t \nearrow T} u(x, t).
\]

The same applies to \( v \).

To end the proof we have to study the behaviour for \( t > T \). From the previous step we know that \( \lim_{t \nearrow T} u(0, t) = \lim_{n \to \infty} u_n(0, T) \). Since \( u_n \) is increasing in time and \( u \) blows up at the origin,

\[ \infty = \lim_{t \nearrow T} u(0, t) = \lim_{n \to \infty} u_n(0, T) \leq \lim_{n \to \infty} u_n(0, t), \tag{2.1} \]

and we conclude that the proper solution, \( v = \lim_{n \to \infty} u_n \), is identically infinite at the origin for \( t > T \). In order to propagate the singularity to the whole interval for both components we use the representation formula obtained from the heat kernel. Let \( \Gamma \) be the fundamental solution of the heat equation in \( \mathbb{R}_+ \times (0, \infty) \), namely

\[
\Gamma(x, t) = \frac{1}{(\pi t)^{1/2}} \exp\left(-\frac{x^2}{4t}\right).
\]

For \( x \in \mathbb{R}_+ \) we have

\[
u_n(x, t) = \int_{\mathbb{R}_+} u_n(y, 0) \Gamma(x - y, t) \, dy - \int_0^t \frac{\partial u_n}{\partial x}(0, \tau) \Gamma(x, t - \tau) \, d\tau
- \int_0^t u_n(0, \tau) \frac{\partial \Gamma}{\partial x}(x, t - \tau) \, d\tau.
\]

Since \( \Gamma \) and the boundary flux, \(-u_n(x, 0, \tau)\), are both nonnegative, we can bound \( u_n(x, t) \) from below by

\[
u_n(x, t) \geq -\int_0^t u_n(0, \tau) \frac{\partial \Gamma}{\partial x}(x, t - \tau) \, d\tau.
\]

From (2.1) we have that \( u_n(0, t) \geq M \) if \( n \) is large enough, how large depending on \( M \). Hence, for any \( 0 < \delta < t - T \),

\[
u_n(x, t) \geq -M \int_T^{T+\delta} \frac{\partial \Gamma}{\partial x}(x, t - \tau) \, d\tau.
\]

We conclude that

\[
\lim_{n \to \infty} u_n(x, t) = \infty, \quad t > T, \quad x \in \mathbb{R}_+.
\]

This proves complete blow-up for \( u \).
Complete blow-up and avalanche formation

To obtain complete blow-up for \( v \) we also use the representation formula. Since \( v_n \) grows due to the influence of \( u_n \) through the boundary flux, we keep the flux term in the bound from below:

\[
v_n(x,t) \geq \int_0^t f_n^{21}(u_n(0,\tau))f_n^{22}(v_n(0,\tau))\Gamma(x,t-\tau)\,d\tau.
\]

From the monotonicity of the solutions we have that \( v_n(0,t) \geq v_n(0,0) = v(0,0) = c > 0 \). Therefore, if \( n \) is large enough and for any \( 0 < \delta < t - T \),

\[
v_n(x,t) \geq M^{p_{21}}e^{p_{22}} \int_T^{T+\delta} \Gamma(x,t-\tau)\,d\tau.
\]

This implies complete blow-up for \( v \), i.e., \( \lim_{n \to \infty} v_n(x,t) = \infty \), for \( t > T \) and \( x \in \mathbb{R}_+ \).

3 Time estimates

When blow-up is non-simultaneous, the \( u \) component touches the level \( n \) before the \( v \) component, which is bounded up to time \( T \). Since the solutions of the truncated problems coincide with those of the non-truncated one until some component reaches the level \( n \), we have \( t_{u_n}(n) < T < t_{v_n}(n) \) for any \( \gamma > 0 \) and large \( n \). The aim of this section is to estimate the differences \( T - t_{u_n}(n) \) and \( t_{v_n}(n) - T \) for different values of \( \gamma \in (0,1] \).

In the sequel we use the notation

\[
\alpha = \frac{p_{21}}{(p_{22} - 1)}, \quad \beta = \frac{1}{2(p_{22} - 1)}
\]

for a couple of exponents that will appear frequently.

Proofs of Theorems 1.2 and 1.3. We first obtain an estimate for \( t_{u_n}(n) \), which will then be used to estimate \( t_{v_n}(n) \).

Estimates for \( t_{u_n}(n) \):

Since \( t_{u_n}(n) \leq t_{u_n}(n) < T \) for \( \gamma \leq 1 \), then \( u_n(x,t) = u(x,t) \) for \( t \leq t_{u_n}(n) \). Now, we note that the blow-up rate of \( u \) is given by

\[
\max_x u(x,t) = u(0,t) \sim (T - t)^{-\frac{1}{p_{21} - 1}}.
\]

(3.1)

Remark that this blow-up rate is the same that holds for a single equation. In fact, since we are assuming that blow-up is non-simultaneous, we have that \( v^{p_{22}}(0,t) \) remains bounded up to \( T \) and hence the results for a single equation can be applied, see [8] and [16].

This blow-up rate provides the estimate for \( t_{u_n}(n) \). Indeed, from (3.1) we get

\[
n^{\gamma} \sim (T - t_{u_n}(n))^{-\frac{1}{p_{21} - 1}},
\]

which is equivalent to (1.5).
Estimates for $t_{v_n}(n^n)$. We look at the problem satisfied by $v_n$ for times larger than $t_{u_n}(n)$. For these times the nonlinearity that involves the $u$ variable is truncated. Hence, $v_n$ is a solution to

$$
\begin{cases}
(v_n)_t = (v_n)_{xx}, & (x,t) \in \mathbb{R}_+ \times (t_{u_n}(n), \infty), \\
-(v_n)_x(0,t) = n^{p_{21}} f_{n}^{22}(v_n(0,t)), & t \in (t_{u_n}(n), \infty), \\
v_n(x,t_{u_n}(n)) = v(x,t_{u_n}(n)), & x \in \mathbb{R}_+.
\end{cases}
$$

(3.2)

Since blow-up is non-simultaneous, $v$ is bounded up to time $T$. Hence the initial datum in (3.2) is also bounded, $v(x,t_{u_n}(n)) < v_T$. The idea is to compare $v_n$ with suitable modifications, $z_{\pm}$, of a self-similar solution, $z$, of

$$
z_t = z_{xx}, \quad -z_x(0,t) = \ell n^{p_{21}} z_{p_{22}}(0,t), \quad \ell > 0.
$$

(3.3)

The function $z$ has an explicit formula, so that $t_{z_{\pm}}(n^n)$ can be computed.

Let $\phi = \phi(\xi)$ be the unique positive and bounded solution of

$$
\phi''(\xi) - \frac{1}{2} \xi \phi'(\xi) + \beta \phi(\xi) = 0, \quad \xi > 0 \quad -\phi'(0) = \phi^{p_{22}}(0).
$$

It has an explicit formula that shows in particular that it decays to zero at infinity, see [8]. The function

$$
z(x,t) = \ell^{-2p_{22}n^{-\alpha}} (\hat{T} - t)^{-\beta} \phi(x(\hat{T} - t)^{-1/2})
$$

is a self-similar solution to (3.3).

To obtain an estimate from above for $t_{v_n}(n^n) - T$, we consider $z_- = z - \delta_-$ with $\ell = 2^{-p_{22}} \hat{T} = \hat{T}_- := K^{-1/2} n^{-2p_{21}}$ for some positive constants $K$ and $\delta_-$, both independent of $n$, to be determined later, and use it as a subsolution to (3.2) in $[t_{u_n}(n), t_{z_-}(n)]$.

Since $z_-$ verifies the heat equation, it is enough to check that

$$
z_-(x,0) \leq v_n(x,t_{u_n}(n)) = v(x,t_{u_n}(n))
$$

and

$$
-(z_-)_x(0,t) \leq n^{p_{21}} z_{p_{22}}(0,t).
$$

The condition on the initial data can be rewritten as

$$
2^{2\beta p_{22}} K \phi(x(K^{p_{22}^{-1} n^{p_{21}}}) - \delta_- \leq v(x,t_{u_n}(n)).
$$

(3.4)

The condition at the boundary is equivalent to $\delta_- \leq z(0,t)/2$. Since $z(0,t) \geq 0$, it is fulfilled if $\delta_- \leq \frac{1}{2} z(0,0)$, i.e.,

$$
\delta_- \leq 2^{2\beta p_{22}^{-1}} K \phi(0).
$$

(3.5)
Let us check that there are values of $\delta_-$ and $K$ such that (3.4) and (3.5) hold simultaneously. Let $x > 0$ be a point such that $v_0(x) > 0$. Notice that, for $n$ large, $v_0(x) \leq \min_{x \in [0, x]} u_n(x, t_{u_n}(n))$. Since $\phi$ is decreasing, the choice

$$2^{2p_{22}} K \phi(0) - \delta_- = v_0(x)$$

guarantees (3.4) (at least for $x \in [0, x]$). With this relation between $K$ and $\delta_-$, (3.5) holds if $\delta_- \leq v_0(x)$.

It is straightforward to see that (3.4) also holds for $x \geq x$. Indeed, since $\phi$ is decreasing and tends to 0 at infinity, we have

$$2^{2p_{22}} \phi(K^{p_{22} - 1} x^n P_{21}) - \delta_- \leq 2^{2p_{22}} \phi(K^{p_{22} - 1} x^n P_{21}) - \delta_- < 0 \quad \text{for } n \text{ large},$$

from where (3.4) follows.

Summing up,

$$z_-(x, t) \leq v_n(x, t + t_{u_n}(n)), \quad 0 < t < t_{z_-}(n).$$

Since $\gamma \leq 1$, $t_{z_-}(n^\gamma) \leq t_{z_-}(n)$, and hence $n^\gamma \leq v_n(x, t_{z_-}(n^\gamma) + t_{u_n}(n))$, which implies

$$t_{v_n}(n^\gamma) \leq t_{u_n}(n) + t_{z_-}(n^\gamma).$$

Since, $z_-$ blows up at $t = \hat{T}_-$, then $t_{z_-}(n^\gamma) \leq \hat{T}_-$, from where we finally get that

$$t_{v_n}(n^\gamma) - T \leq t_{u_n}(n) - T + \hat{T}_- \leq \hat{T}_- = K^{-1/\beta} n^{-2/p_{21}}.$$

To obtain an estimate from below, we consider the function $z_+(x, t) = z(x, t) + \delta_+$ with $\ell = 2^{p_{22}}, \tilde{T} = \tilde{T}_+ := k^{-1/\beta} n^{-2/p_{21}}, k, \delta_+ > 0$. Let us prove that it is a supersolution to

$$z_+ = z_{xx}, \quad -z_+(0, t) = n^{p_{21}} z_{x}^{p_{22}}(0, t),$$

for $x > 0, t \in (0, \tilde{T}_+)$. It is enough to see that $-(z_+)(0, t) \geq n^{p_{21}} z_{x}^{p_{22}}(0, t)$, which is equivalent to $z(0, t) \geq \delta_+$. The latter inequality follows easily from the monotonicity of $z(0, t)$ if we take $k = 2^{2p_{22}} \delta_+^{-1}(0)$.

On the other hand, since blow-up is non-simultaneous, $v_n(x, t_{u_n}(n)) \leq v_T$. If we choose $\delta_+ = v_T$, we get $z_+(x, 0) \geq v_n(x, t_{u_n}(n))$. Since $v_n$ is a subsolution to problem (3.6) for $t \geq t_{u_n}(n)$, comparison yields $z_+(x, t) \geq v_n(x, t + t_{u_n}(n))$, $t \in (0, \tilde{T}_+)$. Therefore, $t_{v_n}(n^\gamma) \geq t_{u_n}(n) + t_{z_+}(n^\gamma)$ and, thus, using (1.5), we have

$$t_{v_n}(n^\gamma) - T \geq -(T - t_{u_n}(n)) + t_{z_+}(n^\gamma) \geq -C n^{-2(p_{21} - 1)} + t_{z_+}(n^\gamma).$$

(3.7)

Since $n^\gamma = z_+(0, t_{z_+}(n^\gamma)) = \ell^{-2\beta} n^{-\alpha}(k^{-1/\beta} n^{-2/p_{21}} - t_{z_+}(n^\gamma)) - \beta \phi(0)$, we get

$$t_{z_+}(n^\gamma) \geq k^{-1/\beta} n^{-2/p_{21}} - C n^{-\alpha/\beta} n^{-\gamma/\beta} = n^{-2/p_{21}}(k^{-1/\beta} - C n^{-\gamma/\beta}) \geq C n^{-2/p_{21}}$$

for $n$ large. This estimate, together with (3.7) and the non-simultaneous blow-up condition on the exponents, yields the desired lower bound,

$$t_{v_n}(n^\gamma) - T \geq c n^{-2/p_{21}}.$$
Again we compare $v_n$ with $z_{\pm} = z \pm \delta_{\pm}$, where now $z$ is a forwards self-similar solution to (3.3),

$$z(x, t) = t^{-2\beta} n^{-\alpha} (\bar{T} + t)^{-\beta} \phi(x(\bar{T} + t)^{-1/2}).$$

The profile $\phi$ is the unique bounded positive solution to

$$\phi''(\xi) + \frac{\xi}{2} \phi'(\xi) + \frac{1}{2(p_{22} - 1)} \phi(\xi) = 0, \quad \xi > 0, \quad -\phi'(0) = \phi^{p_{22}}(0). \quad (3.8)$$

The existence and uniqueness of such a profile is proved, using Kummer’s functions, in the Appendix.

Arguing as for the case $p_{22} > 1$, we arrive at

$$t_{v_n}(n^\gamma) - T \leq -(T - t_{u_n}(n)) + t_{z_-}(n^\gamma), \quad t_{v_n}(n^\gamma) - T \geq -(T - t_{u_n}(n)) + t_{z_+}(n^\gamma). \quad (3.9)$$

Now we estimate $t_{z_\pm}(n^\gamma)$ using the explicit expressions for $z_{\pm}$ to get

$$t_{z_-}(n^\gamma) \leq -K^{-1/\beta} n^{-2p_{22}} + Cn^{-2(\gamma(p_{22} - 1) + p_{22})} \leq Cn^{-2(\gamma(p_{22} - 1) + p_{22})},$$

$$t_{z_+}(n^\gamma) \geq -K^{-1/\beta} n^{-2p_{22}} + Cn^{-2(\gamma(p_{22} - 1) + p_{22})} \geq Cn^{-2(\gamma(p_{22} - 1) + p_{22})},$$

where we have used the condition $p_{22} < 1$ to obtain the last inequality in the second line. Introducing this into (3.9), we arrive at

$$t_{v_n}(n^\gamma) - T \sim -n^{2(\gamma(p_{22} + 1))} + n^{2(1 - p_{22}) \gamma - 2p_{22}} \sim n^{2(1 - p_{22}) \gamma - 2p_{22}}.$$

Now $z(x, t) = Ke^{a t - bz}$, with $a = b^2 = \ell^2 n^{2p_{22}}$. Arguing as in the previous cases, we take first $\ell = 1/2$, $K = K := \delta_- + v_0(x)$ and $\delta_- \leq v_0(x)$, to obtain $z_{\pm}(x, t) \leq v_n(x, t + t_{u_n}(n))$. On the other hand, if $\ell = 2$, $K = K_+ := \delta_+$ and $\delta_+ = v_0$, we get $z_{\pm}(x, t) \geq v_n(x, t + t_{u_n}(n))$.

Thus, to conclude we just need to estimate $t_{z_{\pm}}(n^\gamma)$. Since $n^\gamma = K_+ e^{a t - nz(n^\gamma) + \delta_\pm}$, taking logarithms we arrive at

$$t_{z_-}(n^\gamma) \leq Cn^{-2p_{22}} \log n, \quad t_{z_+}(n^\gamma) \geq Cn^{-2p_{22}} \log n,$$

from where we obtain $t_{v_n}(n^\gamma) - T \sim n^{-2p_{22}} \log n$. \hfill \Box

**Proof of Theorem 1.4.** We use the same technique as in the proof of theorems 1.2 and 1.3, but taking as initial time $t_{v_n}(n^\gamma)$. Let us make a brief sketch for $p_{22} > 1$: $v_n$ is a solution of

$$\begin{cases}
(v_n)_t = (v_n)_{xx}, & (x, t) \in \mathbb{R}_+ \times (t_{v_n}(n^\gamma), \infty), \\
-(v_n)_x(0, t) = n^{p_{22}} f_n^{22}(v_n(0, t)), & t \in (t_{v_n}(n^\gamma), \infty), \\
v_n(x, t_{v_n}(n^\gamma)) \sim h(x) n^\gamma, & x \in \mathbb{R}_+, 
\end{cases}$$
where $h$ is a continuous, nonincreasing function that decays to zero as $x \to \infty$ and such that $h(0) = 1$. The main difference with respect to Theorem 1.2 is that now $\hat{T}_\pm := \kappa_\pm n^{-\frac{1}{2(p-1)}}$. Comparison yields

$$t_{z_+}(n) \leq t_{v_n}(n) - t_{v_n}(n^\gamma) \leq t_{z_-}(n).$$

As before, we can estimate $t_{z_+}(n)$,

$$t_{z_-}(n) \leq \hat{T}_- \leq Cn^{-\frac{2\gamma}{1+\gamma}}, \quad t_{z_+}(n) \geq C(\hat{T}_+ - n^{-\frac{1}{2(p-1)}}) \geq Cn^{-\frac{2\gamma}{1+\gamma}},$$

where we have used that $\gamma < 1$. Therefore, $t_{v_n}(n) - t_{v_n}(n^\gamma) \sim n^{-\frac{2\gamma}{1+\gamma}}$.

\[4\text{ Onset of the avalanche}\]

The aim of this section is to prove Theorems 1.5 and 1.6. In order to study the onset of the avalanche for the component $u$, we first need to know its behaviour at time $t = T^-$. It turns out that it is given by a self-similar profile.

**Lemma 4.1** Let $p > 1$ and let $u$ be a solution to

$$\begin{cases}
  u_t = u_{xx}, & (x, t) \in \mathbb{R}_+ \times (0, T), \\
  -u_x(0, t) = g(t) u^p(0, t), & t \in (0, T), \\
  u(x, 0) = u_0(x), & x \in \mathbb{R}_+, 
\end{cases}$$

where $g$ is monotone increasing and bounded and $u$ blows up at time $T$. Then,

$$\lim_{t \searrow T} (T - t)^{-\frac{1}{2(p-1)}} u(y(T - t)^{1/2}, t) = g(T)^{\frac{1}{p-1}} \zeta(y)$$

uniformly on compact intervals $0 \leq y \leq C$, where $\zeta = \zeta(y)$ is the unique bounded, positive solution of

$$\zeta''(y) + \frac{y}{2} \zeta'(y) + \frac{1}{2(p-1)} \zeta(y) = 0, \quad y > 0, \quad -\zeta'(0) = \zeta(0)^p.$$

**Proof.** As we have already mentioned, the blow-up rate of $u$ is given by

$$c(T - t)^{-\frac{1}{2(p-1)}} \leq \|u(\cdot, t)\|_{\infty} \leq C(T - t)^{-\frac{1}{2(p-1)}},$$

see [8], [16].

If we write the function $u$ in self-similar variables, we get that the rescaled function

$$w(y, \tau) = (T - t)^{\frac{1}{2(p-1)}} u(y(T - t)^{1/2}, t), \quad \tau = -\log(T - t),$$
is a bounded solution of

\begin{align*}
  w_\tau &= w_{yy} - \frac{1}{2} y w_y - \frac{1}{2(p-1)} w, \quad (y, \tau) \in \mathbb{R}_+ \times (-\log T, \infty), \\
  -w_y(0, \tau) &= g(T - e^{-\tau}) w^p(0, \tau), \quad \tau \in (-\log T, \infty), \\
  w(y, -\log T) &= T^{\frac{1}{2(p-1)}} u_0(y T^{1/2}), \quad y \in \mathbb{R}_+.
\end{align*}

If \( g \) is a constant, \( g = k \), then \( w \) converges as \( \tau \to \infty \), uniformly on sets of the form \( 0 \leq y \leq C \), to a steady state, which is a multiple of the profile \( \zeta \),

\[
  \lim_{\tau \to \infty} w(y, \tau) = k^{1/\tau^{p-1}} \zeta(y),
\]

see [8]. For a general monotone \( g \), a comparison argument yields that for every \( \eta > 0 \)

\[
  (g(T - \eta))^{1/\tau^{p-1}} \zeta(y) \leq \liminf_{\tau \to \infty} w(y, \tau) \leq \limsup_{\tau \to \infty} w(y, \tau) \leq (g(T))^{1/\tau^{p-1}} \zeta(y),
\]

from where (4.1) follows. \( \square \)

**Proof of Theorem 1.5.** Since we have \( u_n(0, t_{u_n}(n)) = n \) and \( v_n(0, t_{u_n}(n)) \leq K \), the change of variables

\[
  \begin{align*}
  w_n(y, \tau) &= n^{-1} u_n(n^{-p_{11}^{-1}} y, t_{u_n}(n) + n^{-2(p_{11}^{-1})} \tau), \\
  z_n(y, \tau) &= v_n(n^{-p_{11}^{-1}} y, t_{u_n}(n) + n^{-2(p_{11}^{-1})} \tau),
  \end{align*}
\]

scales both components to order one for \( \tau = 0 \). We obtain a system of two heat equations

\[
  \begin{cases}
  (w_n)_\tau = (w_n)_{yy}, \\
  (z_n)_\tau = (z_n)_{yy}, \\
  (y, \tau) \in \mathbb{R}_+ \times (-n^{-2(p_{11}^{-1})} t_{u_n}(n), \infty),
  \end{cases}
\]

coupled through the boundary conditions,

\[
  \begin{cases}
  -(w_n)_y(0, \tau) = f_1^{11}(w_n(0, \tau)) f_1^{12}(z_n(0, \tau)), \\
  -(z_n)_y(0, \tau) = n^{-p_{11}+p_{21}} f_1^{21}(w_n(0, \tau)) f_2^{22}(z_n(0, \tau)), \\
  \tau \in (-n^{-2(p_{11}^{-1})} t_{u_n}(n), \infty).
  \end{cases}
\]

The supersolutions \( z_+ \) used in the proof of Theorem 1.2, rewritten in terms of the new variables \((y, \tau)\), guarantee that the functions \( z_n \) are uniformly bounded (independently of \( n \)) on compact sets. Hence, we obtain \(- (w_n)_y(0, \tau) \leq C \). Let \( \omega = \omega(y, \tau) \) be the (self-similar) solution to the heat equation with zero initial data and constant boundary flux, \(-\omega_y(0, \tau) = C \). Since \( w_n(y, -n^{-2(p_{11}^{-1})} t_{u_n}(n)) \leq u_0(0) \), we have

\[
  w_n(y, \tau) \leq u_0(0) + \omega(y, \tau).
\]

Thus, the functions \( w_n \) are uniformly bounded on compact sets. Therefore, by standard regularity theory, we get local uniform bounds for \( w_n \) and \( z_n \) in \( C^{2+\epsilon, 1+\epsilon/2} \),
which allow to pass to the limit in the equations and boundary conditions along subsequences.

Let \( Z \) be the limit of \( z_n \) along some subsequence. Since \( p_{11} > 1 + p_{21} \), then \( Z = Z(y, \tau) \) is a solution of the heat equation in \( \mathbb{R}_+ \times \mathbb{R} \) with zero normal derivative. Hopf’s lemma implies then that \( Z \) is constant. Passing to the limit for any fixed \( y \) and \( \tau \leq 0 \), we finally get that \( Z \equiv v_T \), which moreover implies the uniqueness of the limit and hence that convergence is not restricted to subsequences.

Let \( \Phi \) be the limit of \( w_n \) along some subsequence. Then,

\[
\begin{align*}
\Phi_{\tau} &= \Phi_{yy}, \\
-\Phi_y(0, \tau) &= f_1^{1/2}(\Phi(0, \tau))v_T^{p_{12}},
\end{align*}
\]

\((y, \tau) \in \mathbb{R}_+ \times \mathbb{R}, \quad \tau \in \mathbb{R}.
\]

This problem does not have uniqueness, because it is invariant under time translations. In order to characterize the limit completely we need some extra information. This information comes from the behaviour of the component \( u \) as \( t \nearrow T \).

From Lemma 4.1 we know that the blow-up profile is self-similar,

\[
\lim_{t \nearrow T}(T - t)^{\frac{1}{2(p_{11} - 1)}} u(y(T - t)^{1/2}, t) = v_T^{p_{12}} \zeta(y),
\]

uniformly on compact intervals \( 0 \leq y \leq C \). The blow-up rate guarantees that, for \( t < T \),

\[
u(x, t) \leq C(T - t)^{-\frac{1}{2(p_{11} - 1)}}
\]

for some constant \( C \). Hence, if \( t \) is such that \( C(T - t)^{-1/(2(p_{11} - 1))} \leq n \) (this is equivalent to \( t \leq T - Cn^{-2(p_{11} - 1)} \)), we have \( u(x, t) \leq n \). Thus,

\[
u_n(\cdot, n^{-2(p_{11} - 1)} \tau + t_n(n)) = u(\cdot, n^{-2(p_{11} - 1)} \tau + t_n(n))
\]

for \( \tau \leq \tau_0 := C_1 - C2(p_{11} - 1) \), where \( C_1 > 0 \) is a constant such that

\[
(T - t_n(n))n^{2(p_{11} - 1)} \geq C_1,
\]

see Theorem 1.2.

For any \( \tau \leq \tau_0 \), the self-similar asymptotic behaviour written in terms of \( w_n \) reads

\[
\Phi(y, \tau) = \lim_{n \to \infty} w_n(y, \tau)
= \lim_{n \to \infty} \frac{1}{n} u(n^{-2(p_{11} - 1)}y, t_n(n)) + n^{-2(p_{11} - 1)} \tau)
= \lim_{t \nearrow T} \left( T - t \right)^{\frac{1}{2(p_{11} - 1)}} u(yt^{-1/2}(T - t)^{1/2}, t)
= v_T^{p_{12}} (-\tau)^{-\frac{1}{2(p_{11} - 1)}} \zeta(y(-\tau)^{-1/2}).
\]

This gives the uniqueness of the limit \( \Phi \) and, hence, eliminates the restriction of convergence to subsequences. \( \square \)
Proof of Theorem 1.6. Let \( z_n(y, \tau) = v_n(n^{-p_2}y, T + n^{-2p_2} \tau) \). The function \( z_n \) satisfies
\[
\begin{aligned}
(z_n)_\tau &= (z_n)_{yy}, & (y, \tau) \in \mathbb{R}_+ \times \mathbb{R}_+, \\
-(z_n)_y(0, \tau) &= f_n^{22}(z_n(0, \tau)), & \tau \in \mathbb{R}_+, \\
z_n(y, 0) &= v_n(n^{-p_2}y, T), & y \in \mathbb{R}_+.
\end{aligned}
\] (4.2)

Let \( \Psi \) be the unique solution of the heat equation with initial datum \( \Psi(y, 0) = v_T \) and boundary flux \( -\Psi_y(0, \tau) = \Psi^{p_2}(0, \tau) \). Let \( \tau_0 \) be the maximal existence time of \( \Psi \). Notice that \( \tau_0 \) is finite if and only if \( p_2 > 1 \). Since \( \Psi \) is a supersolution to (4.2), comparison yields \( z_n(y, \tau) \leq \Psi(y, \tau) \) for \( \tau < \tau_0 \). This fact provides a uniform bound for \( z_n \) (independently of \( n \)) for \( \tau \) in compact subsets of \([0, \tau_0] \). This implies local uniform bounds for \( z_n \) in \( C^{2+\epsilon,1+\epsilon/2} \), and, hence, convergence along subsequences. The limit turns out to be precisely the function \( \Psi \). The uniqueness of the possible limits implies again that convergence is not restricted to subsequences. \( \square \)

5 Avalanche

Proof of Theorem 1.7. We prove first the result for \( v_n \). This result is used later in the study of the limit for the other component.

Avalanche for \( v_n \). Let \( z_n(x, t) = n^{-\gamma_2}v_n(x, t) \). For times \( \tau \geq t_{u_n}(n) \), the truncation for \( u_n \) has already taken place. Hence \( z_n \) is a solution to the heat equation in \( \mathbb{R}_+ \times (\tau, \infty) \) with initial data \( z_n(x, \tau) = n^{-\gamma_2}v_n(x, \tau) \), and boundary flux
\[
-(z_n)_x(0, \tau) = n^{p_2 - \gamma_2}f_n^{22}(v_n(0, t)), \quad t \in (\tau, \infty).
\] (5.1)

The idea is to choose \( \gamma_2 \) so that in the limit we get a nontrivial boundary flux condition. This choice depends on the value of \( f_n^{22}(v_n(0, t)) \), which is determined by the behaviour of \( t_{u_n}(n) \) as \( n \) tends to infinity.

In this range of parameters \( t_{u_n}(n) \rightarrow \infty \), see Theorem 1.2. Hence, for any fixed time \( \bar{t} > T \) and \( n \) large, \( t_{u_n}(n) > \bar{t} \). Thus, \( u_n \) does not reach level \( n \) before \( \bar{t} \), and equation (5.1) becomes, taking \( \tau = t_{u_n}(n) \) and \( \gamma_2 = \frac{p_2}{1 - p_2} \),
\[
-(z_n)_x(0, \bar{t}) = n^{p_2 - \gamma_2 + \gamma_2 p_2}z_n^{22}(0, \bar{t}) = z_n^{p_2}(0, \bar{t}), \quad \bar{t} \in (t_{u_n}(n), \bar{t}).
\]

Using a comparison argument we have, for \( t \in [t_{u_n}(n), \bar{t}] \) and \( x \in \mathbb{R}_+ \),
\[
\psi_n(x, t) \leq z_n(x, t) \leq \varepsilon + \psi_n(x, t),
\]
where \( \psi_n \) is the unique solution of the heat equation in \( \mathbb{R}_+ \times (t_{u_n}(n), \infty) \) that is positive for all \( t \geq t_{u_n}(n) \), has boundary flux \( -(\psi_n)_x(0, t) = \psi_n^{p_2}(0, t) \) and initial datum \( \psi_n(x, t_{u_n}(n)) = 0 \). Letting first \( n \rightarrow \infty \) and then \( \varepsilon \rightarrow 0 \), \( z_n \) converges uniformly on compact subsets of \( \mathbb{R}_+ \times (T, \infty) \) (notice that \( \bar{t} \) is arbitrary) to \( \psi \), the unique solution to
\[
\begin{align*}
\psi_x &= \psi_{xx}, & (x, t) \in \mathbb{R}_+ \times (T, \infty), \\
\psi(x, T) &= 0, & x \in \mathbb{R}_+.
\end{align*}
\] (5.2)
that is positive for all \( t > T \) and has boundary flux

\[-\psi_x(0, t) = \psi^{\mathcal{P}22}(0, t), \quad t \in (T, \infty).\]

This function has a self similar form, \( \psi(x, t) = (t - T)^{-1/(2(p_{22}-1)})\phi((t - T)^{-1/2}) \), where the profile \( \phi = \phi(\xi) \) is the unique bounded positive solution to (3.8).

\( p_{21} + p_{22} = 1 \) In this case \( c + T \leq t_{v_n}(n) \leq C + T \), see Theorem 1.2, and we cannot determine if the truncation takes place before or after \( \bar{\xi} \). Anyway, we take \( \gamma_2 = 1 \), so that (5.1) becomes \(- (z_n)_x(0, t) \leq f^{22}_n(z_n(0, t)) \). The limit function \( \psi \) is now the unique solution to (5.2) which is positive for \( t > T \) and has boundary flux

\[-\psi_x(0, t) = f^{22}(\psi(0, t)), \quad t \in (T, \infty).\]

It coincides with the one of the previous case until the time \( \bar{t} \) where it reaches level one, \( \psi(0, \bar{t}) = 1 \). From that time on it behaves like the solution of the heat equation with boundary flux equal to one and initial datum \( \psi(x, \bar{t}) \).

\( p_{21} + p_{22} > 1 \) Since \( t_{v_n}(n) \rightarrow T \), for large values of \( n \) we have that \( t_{v_n}(n) < \bar{t} \), the truncation takes place, and (5.1) becomes

\[- (z_n)_x(0, t) = n^{p_{21}-\gamma_2+p_{22}} = 1, \quad t \in (t_{v_n}(n), \bar{t}),\]

if \( \tau = t_{v_n}(n) \) and \( \gamma_2 = p_{21} + p_{22} \). This choice of \( \gamma_2 \) implies that \( z(x, t_{v_n}(n)) \leq n^{1-\gamma_2} \rightarrow 0 \) as \( n \rightarrow \infty \). Using a comparison argument as before, we have that \( z_n \) converges uniformly on compact subsets of \([0, \infty) \times (T, \infty)\) to \( \psi \), the unique nonnegative solution of (5.2) with boundary data

\[-\psi_x(0, t) = 1, \quad t \in (T, \infty).\]

Once more \( \psi \) has a self-similar structure, \( \psi(x, t) = (t - T)^{1/2} \phi((t - T)^{-1/2}) \). The profile \( \phi \), which is the unique bounded positive solution to

\[\phi''(\xi) + \frac{1}{2}\xi\phi'(\xi) - \frac{1}{2}\phi(\xi) = 0, \quad \xi > 0, \quad -\phi'(0) = 1,\]

can be computed explicitly.

**Avalanche for \( u_n \).** We define \( w_n(x, t) = n^{-\gamma_1}u_n(x, t) \), which is a solution to

\[
\begin{align*}
(w_n)_t &= (w_n)_{xx}, & (x, t) &\in \mathbb{R}_+ \times (t_{u_n}(n), \infty), \\
-(w_n)_x(0, t) &= n^{p_{11}-\gamma_1}f^{12}_n(v_n(0, t)), & t &\in (t_{u_n}(n), \infty), \\
w_n(x, t_{u_n}(n)) &= n^{1-\gamma_1}u_n(x, t_{u_n}(n)) \leq n^{1-\gamma_1}, & x &\in \mathbb{R}_+, 
\end{align*}
\]

with \( \gamma_1 \) a constant that will be chosen so that in the limit we get a nontrivial flux boundary condition.

\( p_{21} + p_{22} < 1 \) Since in this case \( f^{12}_n(v_n(0, t)) = n^{\gamma_2 p_{12} z_n^{p_{12}}(0, t)} \) for \( n \) large, if we take \( \gamma_1 = p_{11} + \gamma_2 p_{12} \), we get the flux condition \(- (w_n)_x(0, t) = z_n^{p_{12}}(0, t) \). As we
already know that \( z_n(0, t) \to \psi(0, t) \) uniformly as \( t \in (T, \infty) \), we get, as in the proof of Theorem 1.5, that the functions \( w_n \) are uniformly bounded in \( C^{2+\epsilon, 1+\epsilon/2} \). Hence, we may pass to the limit along any subsequence and obtain that the limit, \( \varphi \), is a solution to

\[
\begin{align*}
\varphi_t &= \varphi_{xx}, & (x, t) &\in \mathbb{R}_+ \times (T, \infty), \\
\varphi(x, T) &= 0, & x &\in \mathbb{R}_+,
\end{align*}
\]

(5.3)

with boundary flux \( -\varphi_x(0, t) = \psi^{p_{12}}(0, t) \).

If we take \( \gamma_2 = 1 \) and \( \gamma_1 = p_{11} + p_{12} \), the boundary condition becomes \( -(w_n)_x(0, t) = f_1^{1/2}(z_n(0, t)) \). Since the inwards flux can be bounded independently of \( n \) and the same is true for the initial data (at \( t = t_{un}(n) \)), the functions \( w_n \) are uniformly bounded in \( C^{2+\epsilon, 1+\epsilon/2} \). The limit is now a solution to (5.3) with boundary flux \( -\varphi_x(0, t) = f_1^{1/2}(\psi(0, t)) \).

For large \( n \) the flux condition becomes \( -(w_n)_x(0, t) = n^{p_{11} - \gamma_1 + p_{12}} \). Taking \( \gamma_1 = p_{11} + p_{12} \), we have \( -(w_n)_x(0, t) = 1 \), which ensures compactness. The limit is in this case a solution to (5.3) with boundary condition \( -\varphi_x(0, t) = 1 \). \( \square \)

Appendix: Self-similar solutions

The proof of Theorem 1.2 relies strongly on the existence of self-similar solutions to the heat equation satisfying the boundary condition \( -u_x(0, t) = u^p(0, t) \). When \( p > 1 \) there is a unique self-similar solution (backwards in time)

\[
B(x, t) = (T - t)^{-1/(2(p-1))} \phi_B(x(T - t)^{-1/2})
\]

such that the profile \( \phi_B \), which does not depend on \( T \), is strictly positive and bounded [8]. This profile, which is explicit, decreases to zero at infinity.

When \( p < 1 \), there exists a self-similar solution of the form (forwards in time)

\[
F(x, t) = (T + t)^{-\beta} \phi_F(x(T + t)^{-1/2}).
\]

We have not found in the literature a proof of the existence of the corresponding profile \( \phi_F \). Though it is not very difficult, we include it here for the sake of completeness.

In order for \( F \) to be the desired self-similar solution, the profile \( \phi_F \) has to solve the stationary problem

\[
\phi''(\xi) + \frac{\xi}{2} \phi'(-\xi) + \beta \phi(\xi) = 0, \quad \xi > 0, \quad -\phi'(0) = \phi^p(0),
\]

(5.1)

where \( \beta = -1/2(1-p) \). In our next lemma we study this latter problem.

**Lemma 5.1** There is a unique (strictly) positive bounded solution of (A.1). It is given by

\[
\phi(\xi) = De^{-\xi^2/4} U \left( \frac{1}{2} - \beta, \frac{1}{2}, \frac{\xi^2}{4} \right),
\]
where

\[
U(a, b; x) = \frac{1}{\Gamma(a)} \int_0^\infty e^{-xt} t^{a-1} (1 + t)^{b-a-1} \, dt, \quad D = \frac{1}{2\Gamma(1/2)} \left( \frac{\Gamma(1 - \beta)}{\Gamma(1/2 - \beta)} \right)^{2\beta}.
\]

Moreover, \( \phi'(\xi) < 0 \) for \( \xi \geq 0 \) and

\[
\phi(\xi) = De^{-\xi^2/4}(\xi^2/4)^{-\frac{1}{2}}(1 + O(|\xi|^2)) \quad \text{as} \quad \xi \to \infty.
\]

**Proof.** In what follows we will use several well-known results concerning special functions. They can be found for example in [1]. We write the equation in (A.1) in terms of a new variable defined by \( z(-\xi^2/4) = \phi(\xi) \). Then \( z = z(\eta) \) solves, for \( \eta < 0 \), the degenerate hypergeometric equation

\[
\eta z''(\eta) + \left( \frac{1}{2} - \eta \right) z'(\eta) - \beta z(\eta) = 0,
\]

whose general solution is

\[
z(\eta) = C_1 M \left( \beta, \frac{1}{2}; \eta \right) + C_2 \eta^{1/2} M \left( \beta + \frac{1}{2}, \frac{3}{2}; \eta \right),
\]

where \( M \) is a Kummer’s series (valid for \( b \neq 0, -1, -2, \ldots \))

\[
M(a, b; x) = 1 + \sum_{k=1}^\infty \frac{(a)_k x^k}{(b)_k k!}, \quad (a)_k = a(a + 1) \cdots (a + k - 1), \quad (a)_0 = 1.
\]

Since \( M(a, b; x) = e^x M(b - a, b; -x) \), the general solution \( z \) can be rewritten as

\[
z(\eta) = e^\eta \left( C_1 M \left( \frac{1}{2} - \beta, \frac{1}{2}; -\eta \right) + C_2 \eta^{1/2} M \left( 1 - \beta, \frac{3}{2}; -\eta \right) \right).
\]

It turns out that the function \( U \) given in (A.2) can be expressed in terms of \( M \),

\[
U(a, b; x) = \frac{\Gamma(b)}{\Gamma(a)} M(a, b; x) + \frac{\Gamma(1 - b)}{\Gamma(a)} x^{1-b} M(a - b + 1, 2 - b; x).
\]

Hence, as is standard, it is possible (and convenient) to express the solution of the degenerate hypergeometric equation as a linear combination of \( M \) and \( U \),

\[
z(\eta) = e^\eta \left( C_1 M \left( \frac{1}{2} - \beta, \frac{1}{2}; -\eta \right) + C_2 U \left( \frac{1}{2} - \beta, \frac{1}{2}; -\eta \right) \right).
\]

Since

\[
M(a, b; x) = \frac{\Gamma(b)}{\Gamma(a)} e^x x^{a-b} (1 + O(|x|^{-1})) \quad \text{as} \quad x \to \infty,
\]

\[
U(a, b; x) = x^{-a} (1 + O(|x|^{-1}))
\]

as \( x \to \infty \).
we conclude that, in order to have a bounded solution we need to take $C_1 = 0$.

Now we use that
\[
U(a, b; 0) = \frac{\Gamma(1-b)}{\Gamma(a-b+1)},
\]
\[
U'(a, b; x) = -aU(a+1, b+1; x),
\]
\[
U(a, b; x) = \frac{\Gamma(b-1)}{\Gamma(a)}x^{1-b} + O(1) \quad \text{as } x \to 0, \quad b \in (1, 2),
\]
in order to compute the unique number $D$ for which $\phi$ satisfies the boundary condition. The rest of the properties are then immediate. \qed

Acknowledgement

The authors want to thank the referee for the careful reading of this manuscript.

References

[1] M. Abramowitz and I. A. Stegun, “Handbook of Mathematical Functions With Formulas, Graphs, and Mathematical Tables”, National Bureau of Standards Applied Mathematics Series, 55, Washington, D.C., 1964.

[2] P. Baras and L. Cohen, Complete blow-up after $T_{\text{max}}$ for the solution of a semilinear heat equation, J. Funct. Anal. 71 (1987), no. 1, 142–174.

[3] C. Brändle, F. Quirós and J. D. Rossi, A complete classification of simultaneous blow-up rates, Appl. Math. Lett. 19 (2006), no. 7, 607–611.

[4] C. Brändle, F. Quirós and J. D. Rossi, Non-simultaneous blow-up for a quasilinear parabolic system with reaction at the boundary, Commun. Pure Appl. Anal. 4 (2005), no. 3, 523–526.

[5] M. Chlebík and M. Fila, Some recent results on blow-up on the boundary for the heat equation, Evolution equations: existence, regularity and singularities (Warsaw, 1998), 61–71, Banach Center Publ., vol. 52. Polish Acad. Sci. Warsaw, 2000.

[6] M. Fila and J. Filo, Blow-up on the boundary: a survey, Singularities and differential equations (Warsaw, 1993) 67–78, Banach Center Publ., vol. 33. Polish Acad. Sci. Warsaw, 1996.

[7] M. Fila and J-S. Guo, Complete blow-up and incomplete quenching for the heat equation with a nonlinear boundary condition, Nonlinear Anal. 48 (2002), no. 7, Ser. A: Theory Methods, 995–1002.

[8] M. Fila and P. Quittner, The blow-up rate for the heat equation with a nonlinear boundary condition, Math. Meth. Appl. Sci. 14 (1991), 197–205.
[9] V. A. Galaktionov and J. L. Vázquez, *Necessary and sufficient conditions for complete blow-up and extinction for one-dimensional quasilinear heat equations*, Arch. Rational Mech. Anal. **129** (1995), no. 3, 225–244.

[10] V. A. Galaktionov and J. L. Vázquez, *Continuation of blowup solutions of nonlinear heat equations in several space dimensions*, Comm. Pure Appl. Math. **50** (1997), no. 1, 1–67.

[11] V. A. Galaktionov and J. L. Vázquez, *The problem of blow-up in nonlinear parabolic equations*, Current developments in partial differential equations (Temuco, 1999), Discrete Contin. Dyn. Syst. **8** (2002), no. 2, 399–433.

[12] A. A. Lacey and D. Tzanetis, *Complete blow-up for a semilinear diffusion equation with a sufficiently large initial condition*, IMA J. Appl. Math. **41** (1988), no. 3, 207–215.

[13] Y. Martel, *Complete blow up and global behaviour of solutions of $u_t - \Delta u = g(u)$*, Ann. Inst. H. Poincaré. Anal. Non Linéaire **15** (1998), no. 6, 687–723.

[14] J. P. Pinasco and J. D. Rossi, *Simultaneous versus non-simultaneous blow-up*, New Zealand J. Math. **29** (2000), no. 1, 55–59.

[15] F. Quirós and J. D. Rossi, *Non-simultaneous blow-up in a semilinear parabolic system*, Z. Angew. Math. Phys. **52** (2001), no. 2, 342–346.

[16] F. Quirós and J. D. Rossi, *Non-simultaneous blow-up in a nonlinear parabolic system*, Adv. Nonlinear Stud. **3** (2003), no. 3, 397–418.

[17] F. Quirós, J. D. Rossi and J. L. Vázquez, *Complete blow-up and thermal avalanche for heat equations with nonlinear boundary conditions*, Comm. Partial Differential Equations **27** (2002), no. 1-2, 395–424.

[18] F. Quirós, J. D. Rossi and J. L. Vázquez, *Thermal avalanche for blowup solutions of semilinear heat equations*, Comm. Pure Appl. Math. **57** (2004), no. 1, 59–98.

[19] A. A. Samarskii, V. A. Galaktionov, S. P. Kurdyumov and A. P. Mikhailov, “Blow-up in Quasilinear Parabolic Equations”, de Gruyter Expositions in Mathematics, vol. 19, Walter de Gruyter & Co. Berlin, 1995. Translated from the 1987 Russian original by Michael Grinfeld and revised by the authors.

[20] P. Souplet and S. Tayachi, *Optimal condition for non-simultaneous blow-up in a reaction-diffusion system*, J. Math. Soc. Japan **56** (2004), no. 2, 571–584.