Localization and a generalization of 
MacDonald’s inner product

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Abstract

We find a limit formula for a generalization of MacDonald’s inner product in finitely many variables, using equivariant localization on the Grassmannian variety, and the main lemma from [2], which bounds the torus characters of the higher Čech cohomology groups. We show that the MacDonald inner product conjecture of type A follows from a special case, and the Pieri rules section of MacDonald’s book [12], making this limit suitable replacement for the norm squared of one, the usual normalizing constant.

1 Introduction

The MacDonald inner product formula of type A determines the norm squared of the MacDonald polynomial under the finite variable MacDonald inner product. It can be obtained up to a constant using the infinite variable theory, namely the Pieri rules section of MacDonald’s book. However, the normalizing constant, usually taken to be the norm squared of one, is difficult to find and is equivalent to Andrews’ q-Dyson conjecture. It has been been calculated from different angles by a number of authors as well as its deep extension to other root systems, see [1, 3, 5, 7, 8, 9, 10, 11, 13, 14].

In this paper, we replace the normalizing constant by a limit of norm squares, which we compute for a more general inner product. In section 4 we deduce the inner product formula from theorem 1 in the special case of the MacDonald inner product. This section assumes results about MacDonald polynomials in infinitely many variables from
Chapter VI of MacDonald’s book, which are in fact the same rules used to extract the inner product formula from the constant term formula.

The most interesting feature of this formula is that it comes from a geometric argument. We first identify the inner product with the character of the space of global sections of a virtual bundle on a torus equivariant Grassmannian variety. The main lemma of [2] shows that the higher Čech cohomology groups do not contribute when the bundle is twisted by a high power of the ample line bundle, so that the character of the space of global sections agrees with the equivariant Euler characteristic. We then use equivariant localization to find a formula for the Euler characteristic as a sum of factored rational functions. When the bundle is twisted by a high power of the ample line bundle, one term dominates the sum, leading to the desired formula.

2 Plethysm

Given a smooth complex projective variety $X$, let $K(X)$ denote its $K$-theory group over the complex numbers. We have the standard $\lambda$-ring operations defined on bundles by $\lambda^i([V]) = [\Lambda^i V]$, where $[V]$ is the induced class in $K$-theory. They also act on monomials in every variable in this paper by

$$\lambda^0(x) = 1, \quad \lambda^1(x) = x, \quad \lambda^i(x) = 0, \quad i \geq 2.$$ 

The generating function

$$\lambda(w A) = \sum_{i \geq 0} (-w)^i \lambda^i(A)$$

is the expansion about zero of a rational function of $w$. We define $\lambda(A)$ to be the analytic continuation of this function to $w = 1$, if it exists. For instance, if $x^I$ are formal monomials, we have

$$\lambda \left( \sum_I a_I x^I \right) = \prod_I (1 - x^I)^{a_I}, \quad a_I \in \mathbb{Z}.$$ 

We also have the dualizing operation $[E]^* = [E^*]$, which acts on monomials by $x^* = x^{-1}$.

Let $\Lambda_n$ denote the graded ring of symmetric polynomials in $n$ variables over $\mathbb{C}$ with inverse limit $\Lambda$, and let $M$ be a formal power series
with integer coefficients and constant term 1 in some set of variables. We define a general MacDonald type inner product on $\Lambda_n$ by

$$ (f, g)^M = \langle f(x)g(x^*)\Delta_M(x) \rangle, \quad x = \{x_1, \ldots, x_n\} $$

$$ \Delta_M = \lambda \left( M \sum_{i \neq j} x_i x_j^{-1} \right), \quad \langle F(x) \rangle = \frac{1}{n!} [x]^1 F(x). \quad (1) $$

Here $[x]^1$ denotes the constant term, which may be extracted termwise from the power series expansion of $\Delta_M(x)$ in the indeterminants of $M$, or by assigning those variables values of norm smaller than one, and defining

$$ [x]^1 = \int_{|x_1| = r} \frac{dx_1}{x_1} \cdots \int_{|x_n| = r} \frac{dx_n}{x_n}, \quad r > 1. \quad (2) $$

For instance, $M = 1$ produces the usual Hall inner product, while

$$ (f, g)^{t, q}_M = (f, g)^M, \quad M = \frac{1-t}{1-q} = (1-t)(1+q+q^2+\cdots), $$

is the finite variable MacDonald inner product as in [12].

In this paper, we will denote a representation of a one-dimensional complex torus $\mathbb{C}^* = \{z\}$ and its character in $\mathbb{Z}[z^{\pm 1}]$ by the same letter. The $\lambda$-ring notation allows us to define an evaluation homomorphism $f \mapsto f(A)$ for any virtual character $A$ by its action on generators $A \to \mathbb{Z}[z^{\pm 1}], \quad e_i \mapsto \lambda^i(A), \quad A \in \mathbb{Z}[z^{\pm 1}],$

where $e_i$ are the elementary symmetric functions. Let us set

$$ \Omega(A) = \lambda(Ax)^{-1} = \sum_{\mu} u_{\mu}(A)v_{\mu} $$

for any dual bases $u_{\mu}, v_{\mu}$ of $\Lambda$. In particular, $\Omega = \Omega(1)$ is the sum of the complete symmetric polynomials. We also extend the dimension map to any virtual character $A \in \mathbb{Z}[z^{\pm 1}]$ by

$$ \dim(A) = [z^a]A, \quad a \in \mathbb{Z}, \quad \dim(A) = \sum_a \dim(A), $$

where $[z^a]$ denotes the coefficient of $z^a$. 

3
3 The main theorem

Given a one dimensional complex torus $T = \{z\}$ and a finite dimensional representation $Z$, let $X$ denote the Grassmannian of subspaces of $Z$ of codimension $n$. We label a point in this variety by its $n$-dimensional quotient space $U = Z/V$, rather than the space itself. Let $\mathcal{U}$ denote the $n$-dimensional tautological quotient bundle on $X$ whose fiber over $U \in X$ is $U$ itself. Let $K_T(X)$ denote the equivariant $K$-theory group with respect to the induced action of $T$ on $X$.

The facts we will need about this variety are summarized below.

1. We have an algebra homomorphism defined on generators by
   $$\Lambda_n \otimes \Lambda_n \to K_T(X), \quad e_i \otimes e_j \mapsto \Lambda^i(\mathcal{U}) \otimes \Lambda^j(\mathcal{U})^*,$$
   where $e_i$ is the $i$th elementary symmetric function. We denote the image of $f \otimes g$ by $f(\mathcal{U})g(\mathcal{U}^*)$.

2. We have a linear map $\chi^0 : \Lambda_n \otimes \Lambda_n \to \mathbb{C}[t^{\pm 1}]$ defined on Schur positive elements by
   $$\chi^0_X(f(\mathcal{U})g(\mathcal{U}^*)) = H^0_X(f(\mathcal{U})g(\mathcal{U}^*)) \in \mathbb{Z}[T].$$
   The following formula follows from Borel-Weil-Bott and also \cite{5}.
   $$\chi^0_X(f(\mathcal{U})g(\mathcal{U}^*)) = \langle f(x)g(x^*)\lambda(x^*Z)^{-1}\Delta(x) \rangle. \quad (3)$$
   The fractional term refers to its expansion about $x = \infty$.

3. We also have the equivariant Euler characteristic
   $$\chi : K_T(X) \to \mathbb{C}[t^{\pm 1}], \quad \chi([\mathcal{E}]) = \sum_{i \geq 0} (-1)^i H^i_X(\mathcal{E}).$$
   If the torus fixed points $U \in X^T$ are isolated, then they are indexed by the $n$ element subsets of the weight set. The character of the cotangent space to such a point is
   $$T^*_U X = \text{Hom}(V,U)^* = U^*V.$$
   The $K$-theoretic localization formula in this context says that
   $$\chi(f(\mathcal{U})g(\mathcal{U}^*)) = \sum_{U \in X^T} f(U)g(U^*)\lambda(U^*V)^{-1}. \quad (4)$$
   See \cite{4} for a reference.
Now let $A, B, C$ be power series in $\mathbb{Z}((z))$, and define
\[ E_U = A U + B U^* + C U^* \in K_T(X) \otimes \mathbb{C}[[z]]. \]

The following is the main lemma from [2].

**Lemma A.** Suppose the following conditions hold:

a) $A \in \mathbb{Z}_{\geq 0}[z^{\pm 1}]$.

b) $B \in \mathbb{Z}_{\geq 0}((z))$, and $\dim_a(B) \leq p(a)$ for some polynomial $p(a)$.

c) $C \in (z)\mathbb{Z}[z]$ is a polynomial with constant term zero.

d) For any weights $a, b \in \mathbb{Z}$ with $\dim_a(Z) > 0$, $\dim_b(C) < 0$, we have
\[ \dim_{a+b}(B) \geq \dim_a(Z) - \dim_b(C) - 1. \]

Then we have that
\[ \sum_{i \geq 0} (-w)^i (\chi - \chi^0) \left( \det(U)^n \lambda^i (E_U) \right) \in \mathbb{C}((z))[[w]] \quad (5) \]

is the $w$-expansion of a meromorphic function $c_m(z, w)$, and the leading exponent of the expansion of $c_m(z, w)$ in the $z$ direction is bounded below by $mk + c$, where $k$ is the largest torus weight of $Z$, and $c$ is some constant.

Remark. This does not mean that the coefficient $[w^i]c_m(z, w)$ has high leading degree. In fact, Serre duality implies that these terms should contribute large negative powers of $z$, and would therefore dominate the power series.

Since $\Lambda$ is spanned by expressions of the form $\lambda(Ax)$ for a general $A$, we may insert a factor of $f(U)$ into (5), for arbitrary $f \in \Lambda$. Then apply (3) and (4) to get
\[ c_m(z, w) = \langle e_m f(x) \lambda(w E_x) \lambda(Z x^*)^{-1} \Delta(x) \rangle - \sum_{U \in X^T} e_m^n f(U) \lambda(w E_U) \lambda(T_U^* X)^{-1}. \quad (6) \]

**Theorem 1.** Let $M \in \mathbb{Z}[z]$ be a polynomial with constant term 1, and let
\[ E = MU_n U_n^* - U_n^*, \quad U_n = 1 + \cdots + z^{n-1}. \]

Suppose that $\dim_a(M) \geq -1$ for all $a \in \mathbb{Z}$, and $\dim_0(E) = 0$, so that the right hand side below is defined and nonzero. Then
\[ \lim_{m \to \infty} z^{-m/2} (f e_n^m, \Omega)_M^t = \lambda(1 - M)^n f(U_n) \lambda(E). \quad (7) \]
Proof. Let
\[A = 0, \quad B = z + z^2 + \cdots,\]
\[C = M - 1, \quad Z = 1 + \cdots + z^k,\]
which satisfy the conditions of the lemma. For large \(m\), the sum in (6) is dominated by \(U = U_n\). By the lemma we have
\[
\lim_{m \to \infty} z^{-m(n/2)} \langle c_m f(x) \lambda(w \mathcal{E}_x) \lambda(Zx^*)^{-1} \Delta(x) \rangle = f(U_n) \lambda(w \mathcal{E}_{U_n}) \lambda(T_{U_n}^n X)^{-1}, \quad k > \left(\frac{n}{2}\right).
\]
To complete the proof, interpret the above as an equality of rational functions in \(z, w\), and set \(w = 1\). Then take the limit as \(k\) tends to infinity, and notice that
\[
\Delta(x) \lambda(Cx^*) = \lambda(C)^n \Delta_M(x).
\]

Remark. We must take the limit over \(m\) before setting \(w = 1\), because there are examples of fixed points \(U \in X^T\) such that \(\lambda(w \mathcal{E}_U)\) has a pole at \(w = 1\). This is the reason we use the main lemma from [2] rather than the main theorem.

4 The inner product formula

Corollary 1. We have
\[
(P_\mu, P_\mu)'_{q, t} = \lambda \left(\frac{t - q}{1 - q} + \frac{q - t^n}{1 - q} t^{-\rho}\right)
\]
\[
P_\mu(t^\rho) \lim_{m \to \infty} a_{\mu + m^n}(q, t)^{-1}, \quad (8)
\]
where \(a_\mu(q, t)\) is the coefficient of \(P_\mu\) in the expansion of \(\Omega\) in the Macdonald basis, and
\[
\mu + m^n = [\mu_1 + m, \ldots, \mu_n + m], \quad u^\rho = 1 + \cdots + u^{n-1}.
\]

Proof. We first claim that equation (7) holds at
\[
M = (1 - t)(1 + \cdots + q^{N-1}), \quad z = t.
\]
It is not hard to see that the condition of the theorem holds at \( q = t^k \) for \( k \) larger than \( n \), whence the claim holds for that specialization.

To establish the claim, it suffices to prove that both sides are rational functions of \( q, t \), because rational functions that agree at \( q = t^k \) for infinitely many values of \( k \) must be equal. Let \( F_m(q,t) \) denote the quantity inside the limit in the left hand side. Using the contour description of the inner product \([2]\), and Cauchy’s residue formula, we see that there exists \( G(q, t, x, y) \) such that

\[
F_m(q, t) = G(q, t, q^m, t^m), \quad G(q, t, x, y) \in \mathbb{C}(q, t)[x, y],
\]

when \( m \) is large enough that there are no singularities at \( x_i = 0 \). The limit is obtained by setting \( x = y = 0 \), proving that the left side is a rational function of \( q, t \). The right hand side is obviously also such a function.

Now let \( f = P_\mu \) and take the limit over \( N \). The lemma follows because the MacDonald polynomials are orthogonal, and

\[
e_{m}^n P_\mu(x) = P_{\mu + m^n}(x).
\]

We will now recover the inner product formula. Given a Young diagram \( \mu \), and a square \( s \in \mu \), let \( a(s), l(s), a'(s), l'(s) \) denote the arm, leg, coarm, and coleg lengths. Let

\[
c_\mu(q, t) = \prod_{s \in \mu} (1 - q^{a(s)} t^{l(s)}), \quad c'_\mu(q, t) = \prod_{s \in \mu} (1 - q^{a(s)+1} t^{l(s)}),
\]

so that

\[
(P_\mu, P_\mu)_{q,t} = \frac{c'_\mu(q, t)}{c_\mu(q, t)},
\]

as in chapter VI, section 6 of MacDonald’s book \([12]\).

**Lemma 2.** The coefficients from corollary \([4]\) are given by

\[
a_\mu(s) = c'_\mu(q, t)^{-1} \prod_{s \in \mu} (t^{l'(s)} - q^{a'(s)+1}).
\]

**Proof.** We have

\[
(f, \Omega)_{q,t} = \left(f, \Omega \left( \frac{1-q}{1-t} \right) \right) = \varepsilon_{q,t}(f),
\]

7
where $\varepsilon_{u,t}$ is the homomorphism defined on the power sum generators of $\Lambda$ by

$$
\varepsilon_{u,t}(p_j) = \frac{1 - u^j}{1 - t^j}.
$$

Equation (6.17), chapter VI, of [12] says that

$$
\varepsilon_{u,t}(P_\mu) = c_\mu(q, t)^{-1} \prod_{s \in \nu} \left( t^l(s) - q^{\alpha(s)}u \right).
$$

(10)

The answer follows by setting $f = P_\mu$, and using (9).

\[\square\]

**Corollary 3.** We have the inner product formula

$$
(P_\mu, P_\mu)'_{q,t} = \lambda \left( \frac{t - q}{1 - q} \sum_{1 \leq i < j \leq n} q^{\mu_i - \mu_j} t^{j-i-1} \right).
$$

(11)

**Proof.** Let us take the limit in equation (8). We will suppress the limit symbols, and instead suppose that $\nu = \mu + m^n$ for very large $m$. Using equation (10) to evaluate $P_\mu(t^\rho)$, we have

$$
(P_\mu, P_\mu)'_{q,t} = \lambda \left( \frac{n t - q}{1 - q} t^{-\rho} + \frac{1 - t}{1 - q} t^\rho t^{-\rho} \right)
$$

$$
\frac{c_\nu(q,t)}{c_\nu(q,t)} \prod_{s \in \nu \setminus \nu} \left( t^l(s) - q^{\alpha(s)} \right) = \lambda(A),
$$

where

$$
A = n \frac{t - q}{1 - q} t^{-\rho} + \frac{1 - t^n}{1 - q} t^\rho t^{-\rho} +
$$

$$
\sum_{s \in \nu} \left( t^{-l(s)} q^{\alpha(s)} (t^n - q) \right) +
$$

$$
\sum_{s \in \nu} q^{\alpha(s)+1} l(s) - q^{\alpha(s)} l(s)+1 =
$$

$$
\frac{n t - q}{1 - q} + (q - t) \sum_{s \in \nu} q^{\alpha(s)} l(s) =
$$

$$
(t - q) \left( \frac{n}{1 - q} - \sum_{s \in \nu} q^{\alpha(s)} l(s) \right) =
$$
\[(t - q)(1 - t) \sum_{k \geq 0} c_k(t)q^k,\]

\[c_k(t) = \sum_{s \in \nu, a(s) = k} (1 + \cdots + t^{l(s) - 1}).\]

Since \(m\) is large, there is a unique box \(s\) in each row \(i\) with arm length \(k\). It is straightforward to match terms under this correspondence with equation (11), completing the proof.

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