Research Article

Universality Properties of a Double Series by the Generalized Walsh System

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We consider a question on existence of a double series by the generalized Walsh system, which is universal in weighted $L_1^\mu[0,1]^2$ spaces. In particular, we construct a weighted function $\mu(x,y)$ and a double series by generalized Walsh system of the form

$$\sum_{n,k=1}^{\infty} c_{n,k} \psi_n(x)\psi_k(y)$$

with the $\sum_{n,k=1}^{\infty} |c_{n,k}|^q<\infty$ for all $q>2$, which is universal in $L_1^\mu[0,1]^2$ concerning subseries with respect to convergence, in the sense of both spherical and rectangular partial sums.

1. Introduction

Let $X$ be a Banach space.

**Definition 1.** A series

$$\sum_{k=1}^{\infty} f_k, \quad f_k \in X$$

is said to be universal in $X$ with respect to rearrangements, if for any $f \in X$ the members of (1) can be rearranged so that the obtained series $\sum_{k=1}^{\infty} f_{\sigma(k)}$ converges to $f$ by norm of $X$.

**Definition 2.** The series (1) is said to be universal (in $X$) concerning subseries, if for any $f \in X$, it is possible to choose a subseries $\sum_{k=1}^{\infty} f_{n_k}$ from (1), which converges to the $f$ by norm of $X$.

Note that for one-dimensional case there are many papers that are devoted to the question on existence of various types of universal series in the sense of convergence almost everywhere and on a measure (see [1–10]).

Let $a \geq 2$ be a fixed integer and $\omega_n = e^{2\pi i / a}$. Recall the following definitions (see [11]).

The Rademacher system of order $a$ is defined inductively as follows. For $n = 0$ let

$$\varphi_0(x) = \omega_n^k \quad \text{if } x \in \left[ \frac{k}{a}, \frac{k+1}{a} \right), \quad k = 0, 1, \ldots, a-1,$$

and for $n \geq 1$ let

$$\varphi_n(x+1) = \varphi_n(x) = \varphi_0(a^n x).$$

The generalized Walsh system of order $a$ is defined by

$$\psi_0(x) = 1,$$

and if $n = \alpha_1 a^{n_1} + \cdots + \alpha_s a^{n_s}$, where $n_1 > \cdots > n_s$, $0 \leq \alpha_j < a$, $j = 1, 2, \ldots, s$, then

$$\psi_n(x) = \varphi_{n_1}^{\alpha_1}(x) \cdots \varphi_{n_s}^{\alpha_s}(x).$$

We denote the generalized Walsh system of order $a$ by $\Psi_a$. Not that $\Psi_2$ is the classical Walsh system. The basic properties of the generalized Walsh system of order $a$ have been obtained by Chrestenson, Fine, Watari, Young, Vilenkin, and others (see [11–16]).

In [6–9], the existence of universal one-dimensional series by trigonometric and the classical Walsh system with respect to rearrangements and subseries in some weighted...
space $L^1_{\mu}[0,1]$. Some results for two-dimensional case for the classical Walsh system were obtained in [10] and improved. In this paper we consider the universality properties of a double series by the generalized Walsh system.

2. Preliminary Notes

Now we list some properties of $\Psi_a$, $a \geq 2$, which will be useful later.

(i) Each $n$th Rademacher function has period $a^{-n}$ and

$$\varphi_n(x) = \text{const} \in \Omega_a = \{1, \omega, \omega^2, \ldots, \omega^{a-1}\}, \quad (6)$$

if $x \in \Delta^{(k)}_{n+1} = [k/a^{n+1}, (k+1)/a^{n+1})$, $k = 0, \ldots, 2^n - 1, n = 1, 2, \ldots$.

(ii) $$(\varphi_n(x))^k = (\varphi_n(x))^m, \quad \forall n, k \in \mathcal{N}, m = k \text{ (mod } a). \quad (7)$$

(iii) $\psi_n(x)$ is a finite product of the Rademacher functions with values in $\Omega_a$.

(iv) $$\psi_{a^n+j}(x) = \varphi_k(x) \cdot \psi_j(x), \quad \text{if } 0 \leq j \leq a^n - 1. \quad (8)$$

(v) $\Psi_a, a \geq 2$ is a complete orthonormal system in $L^2[0,1]$ and it is a basis in $L^p[0,1]$ for $p > 1$.

The rectangular and spherical partial sums of the double series

$$\sum_{k,v=1}^{\infty} c_{k,v} \psi_k(x) \psi_v(y), \quad (x, y) \in T = [0,1]^2 \quad (9)$$

will be denoted by

$$S_{n,m}(x,y) = \sum_{k=1}^{n} \sum_{v=1}^{m} c_{k,v} \psi_k(x) \psi_v(y), \quad (10)$$

$$S_R(x,y) = \sum_{\gamma^2+k^2 \leq R^2} c_{k,v} \psi_k(x) \psi_v(y).$$

If $g(x,y)$ is a continuous function on $T = [0,1]^2$, then we set

$$\|g(x,y)\|_{C} = \max_{(x,y) \in T} |g(x,y)|. \quad (11)$$

3. Main Results

Let us denote the generalized Walsh system of order $a$ by $\Psi_a$, $a \geq 2$. These are the main results of the paper.

Theorem 3. There exists a double series of the form

$$\sum_{k,v=1}^{\infty} c_{k,v} \psi_k(x) \psi_v(y) \quad \text{with} \quad \sum_{k=1}^{\infty} |c_{k,v}|^q < \infty \quad (12)$$

$$\forall q > 2$$

with the following property: for any number $\varepsilon > 0$ a weighted function $\mu(x,y)$ satisfying

$$0 < \mu(x,y) \leq 1, \quad \|\{(x,y) \in T : \mu(x,y) \neq 1\}\|_\varepsilon < \varepsilon \quad (13)$$

can be constructed so that the series (12) is universal in $L^1_{\mu}(T)$ concerning subseries with respect to convergence in the sense of both spherical and rectangular partial sums.

Theorem 4. There exists a double series of the form (12) with the following property: for any number $\varepsilon > 0$ a weighted function $\mu(x,y)$ with (13) can be constructed, so that the series (12) is universal in $L^1_{\mu}(T)$ concerning rearrangements with respect to convergence in the sense of both spherical and rectangular partial sums.

Repeating the reasoning of the proof of [17, Lemma 2] we will receive the following lemma.

Lemma 5. For any given numbers $0 < \varepsilon < 1$, $N_0 > 2$ ($N_0 \in \mathcal{N}$) and a step function

$$f(x) = \sum_{s=1}^{q} \psi_s \chi_{\Delta_s}(x), \quad (14)$$

where $\Delta_s$ is an interval of the form $\Delta_m = [(i-1)/2^m, i/2^m]$, $1 \leq i \leq 2^m$, there exist a measurable set $E \subset [0,1]$ and a polynomial $P(x)$ of the form

$$P(x) = \sum_{k=N_0}^{N} c_k \psi_k(x) \quad (15)$$

which satisfy the following conditions:

$$P(x) = f(x) \text{ on } E, \quad (16)$$

$$|E| > (1 - \varepsilon), \quad (17)$$

$$\sum_{k=N_0}^{N} |c_k|^{2+\varepsilon} < \varepsilon, \quad (18)$$

$$\max_{N_0 < m < N} \left[ \int_{[0,1]} \left| \sum_{k=N_0}^{m} c_k \psi_k(x) \right| \, dx \right] < \varepsilon + \int_{[0,1]} |f(x)| \, dx, \quad (19)$$

for every measurable subset $\varepsilon$ of $E$.

Then applying this Lemma we get the next one.
Lemma 6. For any numbers $\gamma \neq 0$, $0 < \delta < 1$, $N > 1$ and for any square $\Delta = \Delta_1 \times \Delta_2 \subset T$, there exists a measurable set $E \subset T$ and a polynomial $P(x, y)$ of the form
\begin{equation}
P(x, y) = \sum_{k,s=N}^{M} c_{k,s} \psi_k(x) \cdot \psi_s(y),
\end{equation}
with the following properties:

1. $|E| > 1 - \delta$,
2. $\sum_{k,s=N}^{M} |c_{k,s}|^{2+\delta} < \delta$,
3. $P(x, y) = \gamma \cdot \chi_\Delta(x, y)$ for $(x, y) \in E$,
4. \begin{equation}
\max_{N \leq R \leq M} \left[ \left| \int \int E \left| \sum_{k,s=N}^{M} c_{k,s} \psi_k(x) \cdot \psi_s(y) \right| dx \, dy \right| \right] \\
\leq 16 \cdot |\gamma| \cdot |\Delta|,
\end{equation}
for every measurable subset $e$ of $E$.

Proof. We apply Lemma 5, setting $f(x) = \gamma \cdot \chi_\Delta(x)$, $N_0 = N$, $\varepsilon = \frac{\delta}{2}$.

Then we can define a measurable set $E_1 \subset [0, 1]$ and a polynomial $P_1(x)$ of the form
\begin{equation}
P_1(x) = \sum_{k=N}^{N_1} a_k \psi_k(x)
\end{equation}
which satisfy the following conditions:

1. $P_1(x) = \gamma \cdot \chi_{\Delta_1}(x)$ for $x \in E_1$,
2. \begin{equation}
|E_1| > 1 - \frac{\delta}{2},
\end{equation}

By applying Lemma 5 again, setting $f(y) = \chi_{\Delta_2}(y)$, $N_0 = M_0$, $\varepsilon = \frac{\delta}{2}$,

Then we can define a measurable set $E_2 \subset [0, 1]$ and a polynomial $P_2(y)$ of the form
\begin{equation}
P_2(y) = \sum_{s=M_0}^{M} b_s \psi_s(y)
\end{equation}
which satisfy the following conditions:

1. $P_2(y) = \chi_{\Delta_2}(y)$ for $y \in E_2$,
2. \begin{equation}
|E_2| > 1 - \frac{\delta}{2},
\end{equation}

Set
\begin{equation}
E = E_1 \times E_2,
\end{equation}
\begin{equation}
P(x, y) = P_1(x) \cdot P_2(y) = \sum_{k,s=N}^{M} c_{k,s} \psi_k(x) \cdot \psi_s(y),
\end{equation}
where
\begin{equation}
c_{k,s} = a_k \cdot b_s, \quad \text{if } N \leq k \leq N_1, \quad M_0 \leq s \leq M,
\end{equation}
\begin{equation}
c_{k,s} = 0, \quad \text{for other } k, s.
\end{equation}
By (1)–(3), (10)–(30), and (38), (39), we obtain

\[ |E| > 1 - \delta, \]

\[ \sum_{k=1}^{N_1} |\epsilon_{k,x}|^{2+\delta} = \sum_{k=1}^{N_1} |\epsilon_{k,x}|^{2+\delta} \cdot \sum_{x=M_0}^{M} |\epsilon_{k,x}|^{2+\delta} < \delta, \quad (40) \]

\[ P(x, y) = y \cdot \chi_{\Delta}(x, y) \quad \text{for} \ (x, y) \in E. \]

Thus, the statements (1)–(3) of Lemma 6 are satisfied. Now we will check the fulfillment of statement (4).

Let \( N_0^2 + M_0^2 < R^2 < N_1^2 + M_1^2 \), then for some \( m_0 > M_0 \) we have \( m_0 < R \sim m_0 + 1 \) and from (31) it follows that \( R^2 - N_0^2 > (m_0 - 1)^2 \).

Consequently taking relations (4), (40), and (38), (39) for any measurable set \( e \subset E \) (\( e = e_1 \times e_2, \ e_1 \subset E_1, \ e_2 \subset E_2 \)) we obtain

\[ \int \int \left| \sum_{k=1}^{N_1} a_{k} \psi_{k} (x) \cdot \psi_{y} (y) \right| dx \, dy \]

\[ \leq \int \int \left| \sum_{k=1}^{N_1} \sum_{x=M_0}^{M} b_{k} \psi_{y} (y) \right| dx \, dy \]

\[ + \max_{N \leq n \leq N_1} \left[ \int \int \left| \sum_{k=1}^{N_1} a_{k} \psi_{k} (x) \cdot \psi_{y} (y) \right| dx \, dy \right] \]

\[ \leq 12 \cdot |\Delta| \cdot \left| \Delta_{\Delta} \right|. \]

Similarly, for \( N \leq n \leq N_1 \), \( M_0 \leq m \leq M \), we get

\[ \int \int \left| \sum_{k=1}^{N_1} a_{k} \psi_{k} (x) \cdot \psi_{y} (y) \right| dx \, dy \leq 4 \cdot |\Delta| \cdot \left| \Delta_{\Delta} \right|. \quad (42) \]

Lemma 6 is proved. \( \square \)

**Lemma 7.** For any numbers \( \varepsilon > 0, N > 1 \) and a step function

\[ f(x, y) = \sum_{\gamma=1}^{\gamma_0} y_{\gamma} \cdot \chi_{\Delta_{\gamma}}(x, y), \quad (43) \]

there exists a measurable set \( E \subset T \) and a polynomial \( P(x, y) \) of the form

\[ P(x, y) = \sum_{k=1}^{M} c_{k} \psi_{k} (x) \cdot \psi_{y} (y), \quad (44) \]

which satisfy the following conditions:

\[ (1^0) \quad P(x, y) = f(x, y) \quad \text{for} \ (x, y) \in E, \]

\[ (2^0) \quad |E| > 1 - \varepsilon, \quad (45) \]

\[ (3^0) \quad \sum_{k=1}^{M} |c_{k}|^{2+\varepsilon} < \varepsilon, \quad (46) \]

\[ (4^0) \quad \max \left[ \int \int \left| \sum_{k=1}^{N_1} a_{k} \psi_{k} (x) \cdot \psi_{y} (y) \right| dx \, dy \right] \]

\[ + \max_{\sqrt{2N} \leq R \leq \sqrt{2M}} \left[ \int \int \left| \sum_{k=1}^{N_1} a_{k} \psi_{k} (x) \right| \cdot \left| \psi_{y} (y) \right| \cdot dx \, dy \right] \]

\[ \leq 2 \cdot \int \int |f(x, y)| \cdot dx \, dy + \varepsilon, \quad (48) \]

for every measurable subset \( e \) of \( E \).

**Proof.** Without any loss of generality, we assume that

\[ \max_{1 \leq \gamma \leq \gamma_0} \left| \gamma_{e} \right| \cdot \left| \Delta_{\Delta} \right| < \frac{\varepsilon}{32}. \quad (49) \]

\( \Delta_{\Delta}, 1 \leq \gamma \leq \gamma_0 \) are the constancy rectangular domain of \( f(x, y) \), that is, where the function \( f(x, y) \) is constant.

Given an integer \( 1 \leq \gamma \leq \gamma_0 \), by applying Lemma 6 with \( \delta = \varepsilon/16\gamma_0 \), we find that there exists a measurable set \( E_{\gamma} \subset T \) and a polynomial \( P_{\gamma}(x, y) \) of the form

\[ P_{\gamma}(x, y) = \sum_{k=1}^{M} c_{k}^{(\gamma)} \psi_{k} (x) \cdot \psi_{y} (y) \quad (50) \]

with the following properties:

\[ |E_{\gamma}| > 1 - \frac{\varepsilon}{2}, \quad (51) \]

\[ \sum_{k=1}^{M} |c_{k}^{(\gamma)}|^{2+\varepsilon} < \frac{\varepsilon}{\gamma_0}, \quad (52) \]

\[ P_{\gamma}(x, y) = \gamma_{e} \cdot \chi_{\Delta_{\gamma}}(x, y) \quad \text{for} \ (x, y) \in E_{\gamma}, \quad (53) \]
In view of the conditions (51)–(54) and the equality
\[ P(x, y) = f(x, y) \] on \( E \), for any measurable set \( e \subset E \) we obtain
\[
\left\| \int_{e} \sum_{k=1}^{M} c_{k} \psi_{k}(x) \cdot \psi_{s}(y) \, dx \, dy \right\| 
\leq \int_{e} \left\| \sum_{k=1}^{M} P_{r}(x, y) \right\| \, dx \, dy 
+ \int_{e} \left\| \sum_{k=1}^{M} c_{k} \psi_{k}(x) \cdot \psi_{s}(y) \right\| \, dx \, dy 
\leq \int_{e} |f(x, y)| \, dx \, dy + \frac{\varepsilon}{2}.
\] Similarly, for any \( e \subset E \) we have
\[
\max_{N \leq R \leq M} \left[ \int_{e} \left\| \sum_{k=1}^{M} c_{k} \psi_{k}(x) \cdot \psi_{s}(y) \right\| \, dx \, dy \right] 
\leq \int_{e} |f(x, y)| \, dx \, dy + \frac{\varepsilon}{2}.
\] Lemma 7 is proved. \( \square \)

4. Proofs of the Theorems

Theorem 3 is proved similarly [10, Theorem 3], but for maintenance of integrity of this paper, here we will give the proof.

**Proof of Theorem 3.** Let
\[
\{f_{s}(x, y)\}^{\infty}_{s=1}, \quad (x, y) \in T
\] be a sequence of all step functions, values, and constancy interval endpoints which are rational numbers. Applying Lemma 7 consecutively, we can find a sequence \( \{E_{s}\}^{\infty}_{s=1} \) of sets and a sequence of polynomials
\[
P_{s}(x, y) = \sum_{k=1}^{N_{s}-1} c_{k}^{-1} \psi_{k}(x) \psi_{s}(y),
\] which satisfy the following conditions:

\[
P_{s}(x, y) = f_{s}(x, y), \quad (x, y) \in E_{s}, \quad \left| E_{s} \right| > 1 - 2^{-2(s+1)}, \quad E_{s} \subset T,
\] and
\[
\sum_{k=1}^{N_{s}-1} \left| c_{k}^{-1} \right|^{2s+2s} < 2^{-2s},
\]
\[\max_{N_{s+1} \leq k \leq N_s} \left[ \int_{\mathbb{R}^2} \left| \sum_{k, v = N_{s+1}}^{\infty} c_{k,v}(x) \cdot \psi_v(y) \right| dx \, dy \right]\]

\[+ \max_{\sqrt{2N_{s+1}} \leq R \leq \sqrt{2N_s}} \left[ \int_{\Omega} \left| \sum_{k = N_{s+1}}^{N_s-1} c_{k,s}(x) \cdot \psi_s(y) \right| dx \, dy \right]\]  

(69)

\[\leq 2 \cdot \int_{\Omega} |f_s(x, y)| \, dx \, dy + 2^{-2(s+1)},\]

for every measurable subset \(\varepsilon\) of \(E_s\).

Denote

\[\sum_{k, v = 1}^{\infty} c_{k,v}(x) \cdot \psi_v(y) = \sum_{v = 1}^{\infty} \left[ \sum_{k = N_{s+1}}^{N_s-1} c_{k,v}(x) \cdot \psi_v(y) \right],\]  

(70)

where

\[c_{k,v} = c_{k,v}^{(s)} \quad \text{for } N_{s+1} \leq k, \ v < N_s, \ s = 1, 2, \ldots.\]  

(71)

For an arbitrary number \(\varepsilon > 0\) we set

\[\Omega_n = \bigcap_{n = 1}^{\infty} E_s, \quad n = 1, 2, \ldots,\]

\[E = \Omega_n = \bigcap_{s = n_0}^{\infty} E_s, \quad n_0 = \left[ \log_{1/2} \varepsilon \right] + 1,\]  

(72)

\[B = \bigcup_{n = n_0}^{\infty} \Omega_n = \Omega_n \cup \left( \bigcup_{s = n_0}^{\infty} \Omega_s \setminus \Omega_{n-1} \right).\]

It is obvious (see (67) and (72)) that \(|B| = 1\) and \(|E| > 1 - \varepsilon\).

We define a function \(\mu(x, y)\) in the following way:

\[\mu(x, y) = \begin{cases} 1, & \text{for } (x, y) \in E \cup (T \setminus B); \\ \mu_n, & \text{for } (x, y) \in \Omega_n \setminus \Omega_{n-1}, \ n \geq n_0 + 1, \end{cases}\]  

(73)

where

\[\mu_n = \left[ 2^{2n}, \prod_{k = 1}^{n} \Omega_k \right]^{-1},\]

\[h_s = \left\| f_s \right\|_C + \max_{N_{s+1} \leq k \leq N_s} \left\| \sum_{k, v = N_{s+1}}^{\infty} c_{k,v}(x) \cdot \psi_v(y) \right\|_C\]

\[+ \max_{\sqrt{2N_{s+1}} \leq R \leq \sqrt{2N_s}} \left\| \sum_{k = N_{s+1}}^{N_s-1} c_{k,s}(x) \cdot \psi_s(y) \right\|_C + 1.\]  

(74)

From (68) and (70)–(74) we obtain the following:

(A) \(0 < \mu(x, y) \leq 1, \ \mu(x, y)\) is a measurable function and

\[\left| \{(x, y) \in T : \mu(x, y) \neq 1\} \right| < \varepsilon.\]  

(75)

(B) Consider \(\sum_{k = 1}^{\infty} |c_{k,s}|^q < \infty\) for all \(q > 2\).

Hence, obviously we have (see (68) and (70))

\[\lim_{\min \{k, v\} \to \infty} c_{k,v} = 0.\]  

(76)

It follows from (72)–(74) that for all \(s \geq n_0\) and \(N_{s+1} \leq \bar{n}, \ m < N_s\)

\[\int_{T \cap \Omega_s} \left| \sum_{k, v = N_{s+1}}^{\infty} c_{k,v}(x) \cdot \psi_v(y) \right| \mu(x, y) \, dx \, dy\]

\[= \sum_{n = n_1}^{\infty} \left[ \int_{T \cap \Omega_n} \left| \sum_{k, v = N_{s+1}}^{\infty} c_{k,v}(x) \cdot \psi_v(y) \right| \mu(x, y) \, dx \, dy \right]\]

\[\leq 2^{-2n} \sum_{n = n_1}^{\infty} \left[ \int_{T} \left| \sum_{k, v = N_{s+1}}^{\infty} c_{k,v}(x) \cdot \psi_v(y) \right| h_s^1 \, dx \, dy \right]\]

\[< \frac{1}{3} \cdot 2^{-2s}.\]  

(77)

Analogously for all \(s \geq n_0\) and \(\sqrt{2N_{s+1}} \leq R \leq \sqrt{2N_s}\) we have

\[\int_{T \cap \Omega_s} \left| \sum_{k, v = N_{s+1}}^{\infty} c_{k,v}(x) \cdot \psi_v(y) \right| \mu(x, y) \, dx \, dy \]

\[< \frac{1}{3} \cdot 2^{-2s}.\]  

(78)

By (65) and (72)–(74) for all \(s \geq n_0\) we have

\[\int_{T} |P_s(x, y) - f_s(x, y)| \mu(x, y) \, dx \, dy\]

\[= \int_{T \cap \Omega_s} |P_s(x, y) - f_s(x, y)| \mu(x, y) \, dx \, dy\]

\[+ \int_{T \setminus \Omega_s} |P_s(x, y) - f_s(x, y)| \mu(x, y) \, dx \, dy\]

\[= \sum_{n = n_1}^{\infty} \left[ \int_{T \cap \Omega_n} |P_s(x, y) - f_s(x, y)| \mu_n \, dx \, dy \right]\]

\[< \sum_{n = n_1}^{\infty} 2^{-2n} \left[ \int_{T} \left( |f_s(x, y)| + \sum_{k, v = N_{s+1}}^{N_s-1} c_{k,v}(x) \cdot \psi_v(y) \right) h_s^1 \, dx \, dy \right]\]

\[< \frac{1}{3} \cdot 2^{-2s} < 2^{-2s}.\]
By (69) and (72–77) for all \( \bar{n}, \bar{m} < N_i \) and \( s \geq n_0 + 1 \) we obtain

\[
\int_T \left| \int_{[k,\bar{n}]} \sum_{i=1}^{m} c_k^{(i)} \psi_k(x) \cdot \psi_s(y) \right| \mu(x, y) \, dx \, dy < 2^{-2s} \tag{84}
\]

\[
\int_T \left| \int_{[k,\bar{n}]} \sum_{i=1}^{m} c_k^{(i)} \psi_k(x) \cdot \psi_s(y) \right| \mu(x, y) \, dx \, dy < 2^{-2s} \tag{85}
\]

\[
\int_T \left| \int_{[k,\bar{n}]} \sum_{i=1}^{m} c_k^{(i)} \psi_k(x) \cdot \psi_s(y) \right| \mu(x, y) \, dx \, dy < 2^{-2s} \tag{86}
\]

\[
\int_T \left| \int_{[k,\bar{n}]} \sum_{i=1}^{m} c_k^{(i)} \psi_k(x) \cdot \psi_s(y) \right| \mu(x, y) \, dx \, dy < 2^{-2s} \tag{87}
\]

\[
\int_T \left| \int_{[k,\bar{n}]} \sum_{i=1}^{m} c_k^{(i)} \psi_k(x) \cdot \psi_s(y) \right| \mu(x, y) \, dx \, dy < 2^{-2s} \tag{88}
\]

\[
\int_T \left| \int_{[k,\bar{n}]} \sum_{i=1}^{m} c_k^{(i)} \psi_k(x) \cdot \psi_s(y) \right| \mu(x, y) \, dx \, dy < 2^{-2s} \tag{89}
\]

\[
\int_T \left| \int_{[k,\bar{n}]} \sum_{i=1}^{m} c_k^{(i)} \psi_k(x) \cdot \psi_s(y) \right| \mu(x, y) \, dx \, dy < 2^{-2s} \tag{90}
\]

\[
\int_T \left| \int_{[k,\bar{n}]} \sum_{i=1}^{m} c_k^{(i)} \psi_k(x) \cdot \psi_s(y) \right| \mu(x, y) \, dx \, dy < 2^{-2s} \tag{91}
\]

Hence, we have

From (80) and (83) we get

\[
\int_T \left| \int_{[k,\bar{n}]} \sum_{i=1}^{m} c_k^{(i)} \psi_k(x) \cdot \psi_s(y) \right| \mu(x, y) \, dx \, dy < 2^{-2s} \tag{92}
\]

Assume that numbers \( n_1 < n_2 < \cdots < n_{q-1} \) are chosen in such a way that the following condition is satisfied:

\[
\int_T \left| \int_{[k,\bar{n}]} \sum_{i=1}^{m} c_k^{(i)} \psi_k(x) \cdot \psi_s(y) \right| \mu(x, y) \, dx \, dy < 2 \cdot 2^{-2j}, \tag{93}
\]

\[
1 \leq j \leq q - 1. \tag{94}
\]

Now we choose a function \( f_{n_q}(x, y) \) from the sequence (64) such that

\[
\int_T \left| \int_{[k,\bar{n}]} \sum_{i=1}^{m} c_k^{(i)} \psi_k(x) \cdot \psi_s(y) \right| \mu(x, y) \, dx \, dy < 2 \cdot 2^{-q}, \tag{95}
\]

This with (86) implies

\[
\int_T \left| \int_{[k,\bar{n}]} \sum_{i=1}^{m} c_k^{(i)} \psi_k(x) \cdot \psi_s(y) \right| \mu(x, y) \, dx \, dy < 2 \cdot 2^{-q}, \tag{96}
\]

Hence and from (65) and (79)–(81) we obtain

\[
\int_T \left| \int_{[k,\bar{n}]} \sum_{i=1}^{m} c_k^{(i)} \psi_k(x) \cdot \psi_s(y) \right| \mu(x, y) \, dx \, dy < 2 \cdot 2^{-q}, \tag{97}
\]

where

\[
P_{n_q}(x, y) = \sum_{k, \bar{n} = k, \bar{n} \leq N_i} \epsilon_k^{(n_q)} \psi_k(x) \psi_s(y), \tag{98}
\]

\[
\max_{N_{q-1} \leq \bar{n} < N_i} \left[ \int_T \left| \int_{[k,\bar{n}]} \sum_{i=1}^{m} c_k^{(i)} \psi_k(x) \cdot \psi_s(y) \right| \mu(x, y) \, dx \, dy \right] < 19 \cdot 2^{-2q}. \tag{99}
\]
Analogously we have
\[
\max_{\sqrt{2} N_{n_q-1} \leq R \leq \sqrt{2} N_{n_q}} \left[ \int_T \left| \sum_{k=1}^{N_{n_q-1}} c_{n_q}^{(m)} \psi_k (x) \right| \mu (x, y) \, dx \, dy \right].
\]
(92)
\[
< 19 \cdot 2^{-2q}.
\]
In quality subseries of the theorem we will take
\[
\sum_{q=1}^{\infty} P_{n_q} (x, y) = \sum_{q=1}^{\infty} \left[ \sum_{k=1}^{N_{n_q-1}} c_{n_q}^{(m)} \psi_k (x) \psi_q (y) \right].
\]
(93)
From (87) and (88) we have
\[
\int_T \left| f (x, y) - \sum_{s=1}^{d} P_{n_s} (x, y) \right| \mu (x, y) \, dx \, dy \leq \int_T \left| f (x, y) - \sum_{s=1}^{q-1} P_{n_s} (x, y) \right| \mu (x, y) \, dx \, dy \times \mu (x, y) \, dx \, dy \nonumber \nonumber + \int_T \left| f_{n_q} (x, y) - P_{n_q} (x, y) \right| \mu (x, y) \, dx \, dy \nonumber \nonumber < 2 \cdot 2^{-2q}.
\]
(94)
Let \(\overline{n}\) and \(\overline{m}\) be arbitrary natural numbers. Then for some natural number \(q\) we have
\[
N_{n_q-1} \leq \min \{|\overline{n}|, |\overline{m}|\} < N_{n_q}.
\]
(95)
Taking into account (89) and (93) for rectangular partial sums \(S_{\overline{n}, \overline{m}} (x, y)\) of (91) we get
\[
\int_T \left| S_{\overline{n}, \overline{m}} (x, y) - f (x, y) \right| \mu (x, y) \, dx \, dy \leq \int_T \left| f (x, y) - \sum_{s=1}^{\overline{n}} P_{n_s} (x, y) \right| \mu (x, y) \, dx \, dy
\]
\[
+ \max_{N_{n_q-1} \leq R \leq \sqrt{2} N_{n_q}} \left[ \int_T \left| \sum_{k=1}^{\overline{m}} c_{n_q}^{(m)} \psi_k (x) \psi_q (y) \right| \mu (x, y) \, dx \, dy \right]
\]
\[
< 21 \cdot 2^{-2q}.
\]
(96)
Analogously for \(\sqrt{2} N_{n_{q-1}} \leq R \leq \sqrt{2} N_{n_q}\) we have
\[
\int_T \left| S_R (x, y) - f (x, y) \right| \mu (x, y) \, dx \, dy < 21 \cdot 2^{-2q},
\]
(97)
where \(S_R (x, y)\) is the spherical partial sums of (91).
From (96) and (97) we conclude that the series (70) is universal in \(L^1 (T)\) concerning subseries with respect to convergence by both spherical and rectangular partial sums (see Definition 2).

Theorem 3 is proved.

\[\Box\]

Remark 8. We can prove Theorem 4 by the same method used in the proof of Theorem 3.

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