On rectifiable spaces and its algebraical equivalents, topological algebraic systems and Mal’cev algebras

N. I. Sandu

Abstract

We investigate the rectifiable spaces, the Mal’cev algebras, the almost quasivarieties of topological algebraic systems and their free systems and others. It specifies and corrects the roughest mistakes, incorrect statements and nonsense of the introduced concepts connected with concepts listed before, which are available in numerous papers on topological algebraic systems, basically in papers of Academician Choban M. M. and his disciples.

Key words: rectifiable space, topological (left, right) loop, Mal’cev algebra, arity of operation, topological algebraic system, quasivariety, almost quasivariety, topological algebraic system with given defining topological space and given defining relations, topological free system of almost quasivariety.

Mathematics of subject classification: 08A05; 22A20; 54E35; 54A25; 08B05

By [57] a topological space $X$ is said to be rectifiable or a space with rectifiable diagonal (the terminology by [83], [84]) provided that there is a homeomorphism $\Phi : X \times X \rightarrow X \times X$ of $X \times X$ onto itself and an element $e \in X$ such that $\pi_1 \circ \Phi = \pi_1$ and for every $x \in X$ the equality $\Phi(x, x) = (x, e)$ is fulfilled, where $\pi_1 : X^2 \rightarrow X$ is the projection on the first coordinate.

As mentioned in [57], the notion of rectifiable space was introduced at the seminar of Prof. Arhangel’skii at the Moscow State University (see [83],...
A more general notion than rectifiable space is the notion of centralizable space introduced in [76] (see, also, [27], [28]): in the definition of centralizable space is not required for the homeomorphism $g : X \times X \to X \times X$ to be surjective. These definitions have a topological character. Rectifiable spaces turn out to be a good generalization of topological groups. In general, in papers, containing problems connected with rectifiable spaces, many known results from topological groups which are generalized by topological methods, are shown to remain valid for rectifiable spaces. But these papers have a shortcoming: they do not shown the algebraic structure, which the notion of rectifiable space corresponds to.

This paper elucidates the algebraic notions of quasigroup and loop and shows that a topological space $X$ is rectifiable if and only if $X$ is a topological right loop (implicitly this relation is used in [57], [68], [67]). Thus, in order to study rectifiable spaces the powerful methods of theory of topological quasigroups and loops will be applied (see, for example, [79]), which were earlier ignored. Then the results of rectifiable spaces become part of the theory of topological loops, and many proofs of these results become more simple. Unfortunately, in some works related to problems connected with rectifiable spaces [27], [28], [5], [4], [6], [7] serious errors are admitted. Using the fact that any rectifiable space is a topological right loop some of these errors become evident.

Others gross blunders, connected with study of rectifiable spaces, will be exposed and corrected below. In particular, a rectifiable spaces is a Mal’cev algebra. According to [70] a Mal’cev algebra is characterized as an algebra with permutable congruences. Proceeding from this we underlined on roughest mistakes and non-sense of some concepts, connected with the notion of Mal’cev algebra, considered in [4], [27], [28], [29], [30], [39], [40], [5], [6], [7], [8], [83], [84], [85] and others.

In Section 2 the notions of topological algebra of given defining relations (identities) and topological free algebra of a variety, considered in [71], are generalized for topological algebraic systems. It introduces the notion of almost quasivariety of topological algebraic systems of given signature as a class of systems which is closed with respect to Cartesian products, subsystems and contains an unitary system. Let $\mathcal{R}$ be a class of topological algebraic systems of fixed signature $\Omega$, which is closed with respect to Tychonoff topology and subsystems. It proves that every system of $\mathcal{R}$ can be given by defining topological space $X$ and by defining relations, which are a totality of quasiatomic formulas, if and only if $\mathcal{R}$ is almost quasivariety (Theorem 5).

2
Let $\mathfrak{K}$ be a class of topological algebraic systems of given signature with such a defining topological space that $\mathfrak{K}$ is closed with respect to Tychonoff topology and subsystems. In Section 3 is proved.

The class $\mathfrak{K}$ contains a $\mathfrak{K}$-free systems when and only when $\mathfrak{K}$ is an almost quasivariety. More specific, when and only when the non-trivial almost quasivarieties contain free topological algebraic systems $\mathcal{F}_m$ of any given rank $m \geq 1$ (Theorem 7).

Let $\mathfrak{K}$ be an almost quasivariety of topological algebraic systems of fixed signature $\Sigma$. Then:

1) an algebraic system $A$, free in relation to class $\mathfrak{K}$, is free in relation to any subclass $\mathcal{L} \subseteq \mathcal{K}$, and in relation to closure $HS \prod \mathcal{K}$;

2) all free $\mathfrak{K}$-systems of a given rank $M$ are topologically isomorphic among each other and any topological algebraic $\mathfrak{K}$-system topologically generated by a set of cardinality $m$ is an image of a continuous homomorphism of a free systems $\mathcal{F}_m(\mathfrak{K})$ of rank $m$;

3) a free basis of a free system $A = \mathcal{F}_m(A)$ of some class $\mathfrak{K}$ of topological algebraic system is a minimal generating set in $A$ (Theorem 8).

Using the listed above results in Section 5 underlines the roughest mistakes (some of them will be corrected) in definitions of classical algebraical notions, the non-sense of the introduced notions such as continuous signature, free algebras, various types of varieties, quasivarieties of topological algebraic system of continuous signature (for example, a class of algebras is called quasivariety if this class is closed with respect to Tychonoff products and subalgebras) and others, which make the basis of the monographs, the dissertations and numerous papers. Some of these works will be specified during the analysis.

1 Preliminaries. Topological algebraic systems, quasivarieties

For the general algebraic theory of quasigroups and loops see the works [15], [14], [79] and for general theory of topology see the works [63], [60]. But first, following [69] (see, also, [48], [70]) some facts from the theory of algebraic systems, particularly, $\Omega$-algebras, will be given.

Let $n \in N = \{0, 1, 2, \ldots\}$. A mapping $F_i, i \in I$, which maps every ordered sequence $x_1, \ldots, x_n$ of $n$ elements of the set $A$ to an unique defined
by element, \( F_i(x_1, \ldots, x_n) \), is called an \textit{n-ary operation}, defined on \( A \), and the integer \( n \) is called an \textit{arity} of operation \( F_i \). According to [69], an \textit{algebraic system} is called the object \( \mathcal{A} = < A, \Omega_F, \Omega_P > \), consisting of three sets: a non-empty basic set \( A \), a set of operations \( \Omega_F = \{ f_0, \ldots, f_k, \ldots \} \) defined on set \( A \), and a set of predicates \( \Omega_P = \{ p_0, \ldots, p_n, \ldots \} \), given on set \( A \). If \( \Omega_P = \emptyset \) and \( \Omega_F = \emptyset = \bigcup_{n \in N} \Omega_n \), where \( \Omega_n \) is the set of all operations of arity \( n \), then \( \mathcal{A} = < A, \emptyset > \), is called \textit{an algebra of signature} \( \Omega \), or \textit{\( \Omega \)-algebra}. The operations \( f_k \in \Omega_F \) and the predicates \( p_n \in \Omega_P \) are called \textit{basic} [48], [69].

Further, unless otherwise stipulated, we shall examine only an algebraic systems with basic operations of finite arity and follow, basically, terminology from [69].

A mapping of the basic set of algebraic system \( \mathcal{A} \) into the basic set of algebraic system \( \mathcal{B} \) is called a \textit{mapping} of algebraic system \( \mathcal{A} \) into algebraic system \( \mathcal{B} \). A mapping \( \varphi \) of algebraic system \( \mathcal{A} = < A, \{ f_i \}, \{ p_i \} > \) into algebraic system \( \mathcal{B} = < B, \{ g_i \}, \{ q_i \} > \) of the same type is called a \textit{homomorphism} of algebraic system \( \mathcal{A} \) into algebraic system \( \mathcal{B} \) if

\[
\varphi f_i(x_1, \ldots, x_{m_i}) = g_i(\varphi x_1, \ldots, \varphi x_{m_i}),
\]

\[
p_j(x_1, \ldots, x_{m_j}) \Rightarrow q_j(\varphi x_1, \ldots, \varphi x_{m_j}),
\]

for all \( x_1, \ldots, x_{m_r} \in A \), where \( r = i; j \). The relation \( \theta : x \theta y \Leftrightarrow \varphi x = \varphi y \) is called \textit{nuclear equivalence} of homomorphism \( \varphi \).

A homomorphism \( \varphi \) of system \( \mathcal{A} = < A, \{ f_i \}, \{ p_i \} > \) on system \( \mathcal{B} = < B, \{ g_i \}, \{ q_i \} > \) is called \textit{strong}, if for all elements \( b_1, \ldots, b_{m_j} \in B \) and every basic predicate \( q_j \) from \( q_j(b_1, \ldots, b_{m_j} = \{ \text{true} \} \) it follows the existence in \( A \) of inverse images \( a_1, \ldots, a_{m_j} \) of elements \( b_1, \ldots, b_{m_j} \) such that \( p_j(a_1, \ldots, a_{m_j} = \{ \text{true} \} \)

Let \( \mathcal{A} = < A, \Omega_F, \Omega_P > \) be an algebraic system. A relation \( p(x_1, \ldots, x_n) \) of set \( A \) is called \textit{stable} on algebraic system \( \mathcal{A} \) if for any \( m \)-ary operation \( f \) and for any elements \( a_{i1}, a_{i2}, \ldots, a_{in} \in A \) \((i = 1, 2, \ldots, m)\) from validity of relations \( p(a_{i1}, a_{i2}, \ldots, a_{im}) \) \((i = 1, 2, \ldots, m)\) the validity of relation \( p(f(a_{i1}, \ldots, a_{im}), \ldots, f(a_{in}, \ldots, a_{mn})) \) follows. A relation of equivalence \( \theta \) of set \( A \) is called \textit{congruence} of algebraic system \( \mathcal{A} \) if \( \theta \) is stable with respect to every basic operation of system \( \mathcal{A} \). The product of two congruences \( \theta, \xi \) on system \( \mathcal{A} \) is a congruence on \( \mathcal{A} \) if and only if \( \theta, \xi \) are permutable, i.e. \( \theta \xi = \xi \theta \) ([69, Corollary I.2.1]).

Let \( \theta \) be a congruence on algebraic system \( \mathcal{A} = < A, \Omega_F, \Omega_P > \). We denote \( [a] = \{ x \in A | x \theta a \} \) for \( a \in A \). On the totality \( \mathcal{A}/\theta \) of all classes of congruence
we define the operations
\[ f_n^*([a_1], \ldots, [a_n]) = [f_n(a_1, \ldots, a_n)] \] (2)
for \( f_n \in \Omega_F \) and the predicates, believing
\[ p_k^*([a_1], \ldots, [a_k]) \]
true if there exist such elements \( a'_j \theta a_j \) that \( p_k(a'_1, a'_k) \) is true for \( p_k \in \Omega_P \).

With respect to the introduced operations and predicates we get the quotient system \( A/\theta \). The canonical mapping \( \varphi : a \rightarrow [a] \) of system \( A \) on quotient system \( A/\theta \) is a strong homomorphism. The \( \Omega \)-algebras have no predicates, so for \( \Omega \)-algebras the notions of homomorphism and strong homomorphism coincide. Homomorphisms can exist for models, but are not strong.

Further up to Proposition 1, not breaking a generality, for an algebraic system \( A = \langle A, \Omega_F, \Omega_P \rangle \) let’s consider that \( \Omega_P = \emptyset \) and let’s consider the algebraic system \( A = \langle A, \Omega \rangle \) as \( \Omega \)-algebra \( A = \langle A, \Omega \rangle \).

For a set \( A \), let \( \mathcal{B}(A) \) denote the set of all its subsets. Any subset of \( \mathcal{B}(A) \) will be called system of subsets of set \( A \). A system \( \mathcal{L} \) of subsets of set \( A \) is called system of closures if \( \cap D \in \mathcal{L} \) for any subsystem \( D \subseteq \mathcal{L} \). If \( D = \emptyset \) then \( A \in \mathcal{L} \). Let \( X \) be a topological space and let \( \mathcal{T} \) be the system of all closed subsets. Then \( \mathcal{T} \) will be a system of closures with property: \( A \cup B \in \mathcal{T} \). Such a system is called topological. Conversely, if a topological system of closures \( \mathcal{T} \) on set \( X \) is given and \( \emptyset \in \mathcal{T} \), than one can define on \( X \) a topology with elements of \( \mathcal{T} \) as closed sets. In general, the received topology will not be separated.

A closure operator on set \( A \) is called a mapping \( J \) of \( \mathcal{B}(A) \) in to itself, having the following properties: 1) if \( X \subseteq Y \), then \( J(X) \subseteq J(Y) \); 2) \( X \subseteq J(X) \); 3) \( JJ(X) = J(X) \). Every system of closures \( \mathcal{L} \) on \( A \) defines a closure operator \( J \) on \( A \) by the rule \( J(X) = \cap \{ Y \in \mathcal{L} \mid Y \supseteq X \} \). Conversely, every closure operator \( J \) on \( A \) defines a system of closures \( \mathcal{L} = \{ X \subseteq A \mid J(X) = X \} \) and the so defined mapping is one-to-one [48, Theorem II. 1.1].

Let \( A = \langle A, \Omega \rangle \) be an \( \Omega \)-algebra and let \( (A_\lambda)_{\lambda \in \Lambda} \) be a family of subalgebras of \( A \). For every basic operation \( \omega \in \Omega \) any algebra \( A_\lambda \) is closed under \( \omega \), hence the intersection \( \cap A_\lambda \) also is closed under \( \omega \). Denote the corresponding closure operator by \( J_\Omega \). Consequently, \( J_\Omega(X) \) will be the intersection \( \cap A_\lambda \) of all subalgebras \( A_\lambda \) of \( A \) which contain the set \( X \subseteq A \). By definition \( J_\Omega(X) = \cap A_\lambda \) is called a subalgebra of \( \Omega \)-algebra \( A \) generated by set \( X \subseteq A \).
A closure operator $J$ on $A$ is called *algebraic* if for any $X \subseteq A$ and $a \in A$ the following holds: $a \in J(X)$ implies $a \in J(X_f)$ for some finite subset $X_f$ of set $X$. The system of closures is called *algebraic* if the corresponding closure operator is algebraic. A system of closures is algebraic if and only if it is inductive, i.e. each chain in $L$ has a least upper bound in $L$ [48, Theorem II.1.2].

A characteristic property of algebras considered widely in literature (see, for example, [69], [70], [71], [48]), is that they are defined by $(n,1)$-operations, where $n$ is a finite integer. Such a $(n,1)$-operations is called *finite-place operations*. Let's name them operations of finite arity. Further we investigate only such algebraic system $A = \langle A, \Omega_F, \Omega_P \rangle$ for which the basic operations in $\Omega_F$ are finite-place. The following is of crucial importance for such a requirement.

Let $A = \langle A, \Omega >$ be an $\Omega$-algebra with basic operations of finite arity and let $X \subseteq A$ be a subset of set $A$. By induction on $k$ we define the subset $X_k$ of $\Omega$-algebra $A$: $X_0 = X$; $X_{k+1} = \{ x \in A \text{ or } x = \omega(a) \text{ for some } a \in X^n_k \text{ and } \omega \in \Omega_n \}$. According to [48, Proposition II.5.1], [69, Theorem I.1.2] and [48, pag. 94] the operator $J_\Omega$ is algebraical and $\bigcup_{k=0}^{\infty} X_k = J_\Omega(X)$ is the subalgebra of $\Omega$-algebra $A$, algebraically generated by set $X$. We will called the set $X$ set of algebraic generators. Conversely, for any given algebraic system of closures $L$ of a set $A$ it is possible to define an $\Omega$-algebra $A = \langle A, \Omega >$ with basic operations in $\Omega$ of finite arity such that $B_\Omega(A) = L$, where $B_\Omega(A)$ denote the set of all subalgebras of $\Omega$-algebra $A = \langle A, \Omega >$ ([48, Theorem II.5.2]).

In [48, Theorem II.5.2] it is possible to choose the structure of $\Omega$-algebra by various ways. In particular, we shall state the statement [48, Theorem II.5.6] which shows the special role of unary and 0-ary operations for research of algebraic closure operator.

Let's note that in [69, pag. 56] the notion of algebraic system of closures, introduced in [48], coincides with notion of local totality, introduced in [69]. Remind the last notion. Let $A = \langle A, \Omega >$ be an $\Omega$-algebra and let $X \subseteq A$ be a subset of the set $A$. A totality $\mathfrak{B} = \{ A_\lambda | \lambda \in \Lambda \}$ of subsets $A_\lambda$ of set $A$ is called *locally in $X$* if any finite subset from $X$ contains in some set $A_\lambda$ from $\mathfrak{B}$. In such a case the set $B = \cup A_{\lambda \in \Lambda}$ is the subalgebra of $\Omega$-algebra $A = \langle A, \Omega >$ algebraically generated by algebraic generators $\{ x \in X | X \subseteq A \}$ ([69, Theorem 1.2.2]).

It is easy to see that the union $C \cup D$ of subalgebras $C, D$ of an $\Omega$-algebra $A$ with basic operations of finite arity can not be a subalgebra. However, the union of non-empty locally totality $\mathfrak{B}$ of subalgebras of any $\Omega$-algebra $A$ is
a subalgebra of Ω-algebra $A$ ([69, Theorem 1.2.2]). Moreover, the following statement ([48, Theorem II.5.6]) holds.

For an algebraic system of closures $L$ on set $A$ the following statements are equivalent:

(i) $L$ is a sublattice of lattice $B(A)$;

(ii) $L$ is a topologic system of closures;

(iii) there is such a domain of operations $\Omega$, whose the arity of operations has more unity (i.e. unary or 0-ary) and such a structure of $\Omega$-algebra on $A$ that $B_\Omega(A) = L$.

A polynomial of letters $x_1, \ldots, x_t$ is called the meaningful expression, consisting of these letters, brackets and symbols of basic operations of the algebraic system. Then from definition of sets $X_k$ (or $A_{\lambda \in \Lambda}$) it follows that if the system $A$ algebraically is generated by set $X$ then for all $a \in A$ there exist a basic operation $\omega \in \Omega$ of arity $n$ and such an elements $b_1, \ldots, b_n \in X$ that $a = \omega(b_1, \ldots, b_n)$. Hence any element $a \in A$ can be presented as polynomial of letters $x_1, \ldots, x_n \in X$. Then from the above-stated it follows

**Proposition 1. I.** For an $\Omega$-algebra $A = \langle A, \Omega \rangle$ and subset $X \subseteq A$ the following statements are equivalent:

a) $A$ is algebraically generated by set $X$, i.e. $X$ is a set of algebraic generators;

b) any system of closures of set $B(A)$ of all subsets of set $A$ is algebraic;

c) the closure operator $J_\Omega(X) = \bigcup_{k=0}^{\infty} X_k$, considered in [48, Proposition II.5.1], is algebraic;

d) the totality $\mathcal{B} = \{ A_{\lambda} \mid \lambda \in \Lambda \}$ of subsets $A_{\lambda}$ of set $A$, considered in [69, Theorem 1.2.2], is locally in $X$;

f) every element $a \in A$ can be presented with respect to basic operations from $\Omega$ as polynomial of variables $x_1, \ldots, x_r \in X$.

II). The equivalent statements a) - f) of item I) hold if any basic operation $\omega \in \Omega$ is a finite arity, i.e. has a form $\omega(x_1, \ldots, x_n) = x_{n+1}$ ([48, Proposition II.5.1], [69, Theorem I.2.2]). Conversely, if for an algebra with basic set $A$ hold the statements a) - f) of item I) then on set $A$ it is possible to define such an $\Omega$-algebra $A = \langle A, \Omega \rangle$ with basic operations in $\Omega$ of finite arity that $B_\Omega(A) = L$ ([48, Theorem II.5.2]).

From Proposition 1 it follows that the study of algebraical systems with basic operations of finite arity and algebraical systems with basic operations of non-finite arity requires different approaches. The expression "a subalgebra of $A$ is algebraically generated by set $X \subseteq A$" is meaningful if and only if the
basic operations in $\Omega_F$ are finite arity by item I). According to item II) only in such a case every element $a \in A$ can be presented with respect to the basic operations from $\Omega$ as polynomial of variables $x_1, \ldots, x_r \in X$. It is impossible to apply to the second case the algebraic closed operators as by item I) this notion characterizes the operations of finite arity.

Probably, there can not be a general way of researching the operations of non-finite arity and thus in every concrete case the achieved simplification would be significant. [48, pag. 69] stipulates that so far the study of operations with non-finite arity did not pay enough attention to mathematics. As shown above, for this it is possible to study the operations with non-finite arity with the help of topological spaces. Inversely, as in [71], any topological space could be considered as discrete algebras with systems of partial operations of infinite signature. But the properties of such partial operation are not expressed as identities, but as operations of limiting transitions.

Let $A = < A, \Omega_F, \Omega_P >$ be an algebraic system of signature $\Omega = \Omega_F \cup \Omega_P$, let $x_1, x_2, \ldots, x_k$ be object variables and let $f^{(n)}_m, m, n = 0, 1, \ldots$, be functional variables, where $m$ is the number of order, $n$ is the arity of variable. We define: 1) every word of form $x_i$ or $f_i^{(0)}$ is a term; 2) if $a_1, \ldots, a_n$ are terms, then $f_i^{(n)}(a_1, \ldots, a_n)$ is a term; 3) a word is a term if it is term by items 1), 2). The formulas of form 

$$P(f_1(x_1, \ldots, x_k), \ldots, f_n(x_1, \ldots, x_k)),$$

$$f(x_1, \ldots, x_k) = g(x_1, \ldots, x_k),$$

(4)

where $f, g, f_1, \ldots, f_n$ are some terms of signature $\Omega$, $P \in \Omega$, are called quasi-atomic from variables $x_1, x_2, \ldots, x_k$.

Let $S_1(x_1, \ldots, x_k), \ldots, S_{k+1}(x_1, \ldots, x_k)$ be some quasiatomic formulas of signature $\Omega$ (switching equality) from variables $x_1, \ldots, x_k$. A formula of form $(\forall x_1, \ldots, x_k)S_1(x_1, \ldots, x_k)$ is called identity and a formula of form $(\forall x_1, \ldots, x_k)(S_1 \lor \ldots \lor S_k \rightarrow S_{k+1})$ is called quasiidentity.

Particularly, if classes of algebras are considered, then $\Omega_P = \emptyset$ and the identities have the form $(\forall x_1, \ldots, x_k)(f(x_1, \ldots, x_k) = g(x_1, \ldots, x_k))$ and the quasiidentities have the form $(\forall x_1, \ldots, x_k)(f_1 = g_1, \lor, \ldots, \lor f_r = g_r \rightarrow f_{r+1} = g_{r+1})$.

A class $K$ of algebraic systems is called a variety (respect. a quasivariety) if such a totality $T$ of identities (respect. quasiidentities) of signature $\Omega$ exists that $K$ consists of those and only those algebraic systems of signature $\Omega$, where all formulae from $T$ hold true. The totality $T$ is called the defining set
of variety (respect. quasivariety) \( \mathcal{K} \). Since it is possible to regard any identity as quasiidentity, then any variety can be regarded as a quasivariety.

We give some notions and results from [69]. An algebraic system is called unitary if it is from one element and all its basic predicates have the value true. A class \( \mathcal{K} \) of algebraic systems is called hereditary if \( \mathcal{K} \) is closed with respect to the subsystems of its systems, and is called multiplicatively closed if \( \mathcal{K} \) is closed with respect to the Cartesian product.

**Proposition 2.** (Birkhoff’s Theorem). [69, Theorem VI.13.1]. A class \( \mathfrak{V} \) of algebraic systems of a fixed signature \( \Omega \) is a variety if and only if:

1. \( \mathfrak{V} \) is hereditary;
2. \( \mathfrak{V} \) is multiplicatively closed;
3. every homomorphic image of any algebraic system from \( \mathfrak{V} \) belongs to \( \mathfrak{V} \).

**Proposition 3** [69, Corollary V.11.3]. A class \( \mathfrak{Q} \) of algebraic systems of fixed signature \( \Omega \) is a quasivariety if and only if:

1. \( \mathfrak{Q} \) is ultraclosed;
2. \( \mathfrak{Q} \) is hereditary;
3. \( \mathfrak{Q} \) is multiplicatively closed;
4. \( \mathfrak{Q} \) contains an unitary system.

Remind that a formula is called closed if it does not contain any free subject variables. By [69, Theorem III.6.1] it is possible to define that a class \( \mathcal{K} \) of algebraic systems of signature \( \Omega \) is called axiomatizable if and only if such a totality of closed formulaes \( S \) exists that \( \mathcal{K} \) consists of those and only those systems of signature \( \Omega \) for which the formulae from \( S \) hold true. Every axiomatizable class of systems \( \mathcal{K} \) is ultraclosed [69, Corollary IV.8.10]. Moreover, an axiomatizable class of systems \( \mathcal{K} \) satisfies the conditions \( Q_2 \), \( Q_3 \) \( Q_4 \) if and only if the class \( \mathcal{K} \) is a quasivariety [69, Corollary V.11.7].

Recall that an algebraic system for which the basic set of elements is a topological space and the basic operations are continuous is called topological algebraic system. In paper [75] a variety \( \mathcal{K} \) is defined as class of topological algebraic systems, which satisfy some set of limiting identities \( \Sigma \). A variety \( \mathcal{K} \) is called primitive class if \( \Sigma \) consists of algebraical identities. For thus defined notions the analogues of the Offers of Propositions 2, 3 are proved. For a class \( \mathcal{K} \) to be a variety is it is necessary and sufficient for \( \mathcal{K} \) to be closed with respect to: closed subsystems of system from \( \mathcal{K} \); images of continuous homomorphisms: Tychonoff products. A variety \( \mathcal{M} \) of topological groups is a primitive class if and only if \( \mathcal{M} \) is closed with respect to ultraproduct.
Let us note that the condition \( Q_1 \) is essentially for the definition of quasi-variety. [69, pag. 295] gives a class of groups \( \mathcal{G} \) for which the conditions \( Q_2), Q_3) Q_4) \) hold, but \( \mathcal{G} \) is not a quasivariety. We also mention that from [69, Theorem V.11.4] it follows that the condition \( Q_4 \) is necessary in Proposition 3.

According to [69, Theorem V.11.5], a class of algebraic systems \( \mathcal{K} \) satisfies the conditions \( Q_2), Q_3) Q_4) \) if and only if the class \( \mathcal{K} \) is replica complete.

Let us remind this definition. Let \( \mathfrak{K} \) be a class of algebraic systems of fixed signature \( \Omega \) and let \( \mathcal{A} \) be some system of signature \( \Omega \), not necessarily belonging to a class \( \mathfrak{K} \). A homomorphism \( \alpha \) of system \( \mathcal{A} \) on some \( \mathfrak{K} \)-system \( \mathcal{A}_1 \) is called \( \mathfrak{K} \)-morphism if for every homomorphism \( \gamma \) of system \( \mathcal{A} \) in any system \( \mathfrak{K} \)-system \( \mathcal{B} \) such a homomorphism \( \varsigma \) of system \( \mathcal{A}_1 \) on system \( \mathcal{B} \) exists that \( \gamma = \varsigma \alpha \). Every \( \mathfrak{K} \)-morphism image of system \( \mathcal{A} \) is called replica of \( \mathcal{A} \) in class \( \mathfrak{K} \) (\( \mathfrak{K} \)-replica) and is denoted by \( \mathcal{A}_{\mathfrak{K}} \).

A class \( \mathfrak{K} \) of signature \( \Omega \) is called replica complete, if every algebraic system has a replica within it. According to [69, Theorem V.11.5] we have

**Lemma 1.** (Lemma-definition). A class \( \mathfrak{K} \) of algebraic systems of fixed signature \( \Omega \) is replica complete, if \( \mathfrak{K} \) satisfies the conditions \( Q_2), Q_3) Q_4) \), i.e. is hereditary, multiplicatively closed and contains an unitary system. According to Proposition 3 such class of algebraic systems \( \mathfrak{K} \) with listed properties is called an almost quasivariety.

From Proposition 3 it follows.

**Corollary 1.** An almost quasivariety \( \mathfrak{K} \) is a quasivariety if and only if the class \( \mathfrak{K} \) is axiomatizable.

Let \( \mathcal{A} = < \mathcal{A}, \Omega_F, \Omega_P > \) be an algebraic system generated by set \( X \). Every polynomial of letters \( x_1, \ldots, x_p \in X \) can be considered as a \( p \)-ary operation. Operations obtained with the help of polynomial are called derived.

Transformations of the set of algebraic system \( \mathcal{A} \), having the form \( x \rightarrow F(x) \), where \( F(x) \), is a polynomial of \( x \), are called translations of the system. The translation \( T \) is called reversible if such a translation \( S \) exists that \( ST = TS = E \), where \( E \) is the identical mapping. All reversible translations form the group of translations of given system.

**Theorem 1** (Mal'cev) [70]. All congruences on every algebraic system of some variety are permutable iff a polynomial \( \Psi(x, y, z) \) exists, satisfying the identities

\[
\Psi(x, x, z) = z, \quad \Psi(x, z, z) = z
\]
A biternary system of an algebraic system $\mathcal{A} = \langle A, \Omega_F, \Omega_P \rangle$ is a pair of ternary operations $\alpha, \beta : A \times A \times A \to A$ such that $\alpha(x, x, y) = y$, $\alpha(\beta(x, y, z), y, z) = x$, $\beta(\alpha(x, y, z), y, z) = x$ for all $x, y, z \in A$.

**Theorem 2** (Mal’cev) [70]. The groups of reversible translations of all systems of some variety is transitive iff derived ternary operations exist, relative to which the systems of a given class are biternary.

**Theorem 3.** The Theorems 1, 2 hold for any almost quasivariety class of algebraic systems of a given signature.

The proof of Theorem 3 almost literally repeats the proofs of Theorems 1 and 2. It is necessary only to use the Lemma 1 and the Proposition 1.

## 2 On almost quasivarieties of topological algebraic systems

A relation $\theta(x_1, \ldots, x_n)$ defined on topological space is called continuous, if for any sequence of elements $x_1, \ldots, x_n$ not satisfying the relation $\theta$ such a sequence of open set $U_i$, $x_i \in U_i$ ($i = 1, \ldots, n$) exists that for all $x'_i$ ($x'_i \in U_i$; $i = 1, \ldots, n$) the relation $\theta(x'_1, \ldots, x'_n)$ does not hold [70].

Let $\theta$ be an equivalence of a set $A$ and let $N$ be a subset of $A$. According to [70], the set $\cup_{n \in \mathbb{N}} \{a \in A | a \theta n\}$ is called a $\theta$-saturation of $N$. An equivalence $\theta$ of a topological space is called complete equivalence if any $\theta$-saturation of an open set is open.

Recall that an operation (mapping) $f(x_1, x_2, \ldots, x_n)$ of a topological space $G$ is continuous if for every neighborhood $V$ of point $f(x_1, x_2, \ldots, x_n)$ such neighborhoods $U_1, \ldots, U_n$ of points $x_1, x_2, \ldots, x_n$ exist that $f(U_1, \ldots, U_n) \subset V$. A homeomorphism is a continuous one-to-one mapping $\varphi$ of some topological space $X$ on some topological space $Y$, the inverse mapping $\varphi^{-1}$ is also continuous.

An algebraic system for which the basic set of elements is a topological space and the basic operations are continuous is called topological algebraic system. The totality of topological algebraic systems which form a variety (respect. quasivariety or almost quasivariety, or axiomatizable class) in algebraic sense is called a variety (respect. quasivariety or almost quasivariety, or axiomatizable class) of topological algebraic system. Replica of class of
topological algebraic systems is defined in natural way: all mappings in algebraic definition should be continuous.

Let $\mathcal{A} = < A, \{f_i\}, \{p_j\} >$, $\mathcal{B} = < B, \{g_i\}, \{q_j\} >$ be topological algebraic systems of the same type. A homomorphism $\varphi$ of $\mathcal{A}$ into $\mathcal{B}$ is defined by (1). If the mapping $\varphi$ of topological space $A$ into topological space $B$ is continuous, then $\varphi$ is called continuous homomorphism of topological algebraic systems.

If $\theta$ is a congruence on topological algebraic system $\mathcal{A} = < A, \{f_i\}, \{p_j\} >$, then by (2), (3) the quotient set $A/\theta$ turns into an algebraic system. We define the images of open sets of topological space $A$ under homomorphism $A \rightarrow A/\theta$ as open sets in $A/\theta$. Then the algebraic system $A/\theta$ turns into topological algebraic system and $A \rightarrow A/\theta$ will be a continuous homomorphism. If $\theta$ is a complete congruence, then $A \rightarrow A/\theta$ will be an open continuous homomorphism.

If $\theta$ is a complete congruence of topological algebraic system $\mathcal{A}$ then the quotient system $\mathcal{A}/\theta$ satisfies the axiom $T_2$, i.e. is a Hausdorff space. Conversely, let $\varphi$ be an open and continuous homomorphism of topological algebraic system $\mathcal{A}$ on topological algebraic system $\mathcal{B}$, which satisfies the axiom $T_2$. Then the nuclear congruence of $\varphi$ on $\mathcal{A}$ is complete and continuous and the mapping $\mathcal{A}/\theta \leftrightarrow \mathcal{B}$ is an open and continuous isomorphism (see, [70]).

Further all topological systems are assumed to have axiom $T_2$.

**Proposition 4.** A nuclear equivalence $\theta$ of every continuous homomorphism $\varphi$ of topological algebraic system $\mathcal{A}$ on topological algebraic system $\mathcal{B}$ of the same type is a congruence on $\mathcal{A}$ and the canonical mapping $\tau : \mathcal{A}/\theta \rightarrow \mathcal{B}$ is a continuous homomorphism. If the homomorphism $\varphi$ is open and continuous then the canonical mapping $\tau : \mathcal{A}/\theta \rightarrow \mathcal{B}$ is an open continuous homomorphism. If the homomorphism $\varphi$ is strong, open and continuous then the canonical mapping $\tau : \mathcal{A}/\theta \rightarrow \mathcal{B}$ is an open continuous isomorphism.

**Proof.** For algebraic systems the Proposition 4 coincides with Theorem I.2.1 from [69]. We consider the topological case and let the homomorphism $\varphi$ of $\mathcal{A}$ on $\mathcal{B}$ be continuous. As shown above, the homomorphism $\alpha$ of $\mathcal{A}$ on $\mathcal{A}/\theta$ is continuous.

Let $z$ be an element in $\varphi \mathcal{A} = \mathcal{B}$. Then such an elements $y \in \mathcal{A}/\theta$, $x \in \mathcal{A}$ exist that $\tau(y) = z$, $\alpha(x) = y$, $\varphi(x) = z$. Let $W$ be a neighborhood of $z$ in $\mathcal{B}$. From the definition of topology of $\mathcal{A}/\theta$ it follows that such a neighborhood $U$ of $x$ in $\mathcal{A}$ exists that $\varphi(U) \subseteq W$. The homomorphism $\alpha$ is open. Then $\alpha(U) = V$ is a neighborhood of $y$ in $\mathcal{A}/\theta$ and $\tau(V) \subseteq W$. Hence $\tau$ is a
continuous homomorphism.

Let now $\varphi$ be an open continuous homomorphism. Then the congruence $\theta$ is complete and the continuous homomorphism $\alpha : A \to A/\theta$ is open. Let $V$ be an open set in $A/\theta$. We denote $U = \alpha^{-1}(V)$. As $\theta$ is a complete congruence then $\alpha(U) = V$ and $U$ is an open set in $A$. The homomorphism $\varphi$ is open. Then the set $\varphi(U)$ is open in $B$. Hence the set $\tau(V) = \tau\alpha(U) = \varphi(U)$ is open in $B$ and the continuous homomorphism $\tau : A/\theta \leftrightarrow B$ is open. If the homomorphism $\varphi$ is strong, then by [69, Theorem I.2.1], the homomorphism of algebraic systems $\tau : A/\theta \leftrightarrow B$ is an isomorphism. This completes the proof of Proposition 4.

The Proposition 4 shows that the totality of all strong, open and continuous homomorphic images of a given algebraic system $A$ up to homeomorphism is settled by totality of all quotient systems on its various complete congruences. If $A$ is an $\Omega$-algebra, then a similar statement holds for open and continuous homomorphic images of $A$ and complete congruences of $A$.

The following Theorem 4 is proved in [70, Theorem 10] for a varieties of topological $\Omega$-algebras, i.e. for a varieties of topological algebraic systems $(A, \Omega_P, \Omega_P)$ with $\Omega_P = \emptyset$. The proof of Theorem 4 literally repeats the proof of Theorems 10 and 11 from [70]: for this purpose it is sufficient to use the Theorem 3 instead of Theorem 1.

**Theorem 4.** If the congruences are permutable on all systems of an almost quasivariety $\mathfrak{A}$ then all congruences are complete on topological systems of class $\mathfrak{A}$ and those and only those congruences are continuous on topological systems of class $\mathfrak{A}$, where adjacent classes are closed.

**Corollary 2.** Let $\mathfrak{R}$ be an almost quasivariety of topological algebraic systems with permutable congruences and let $A \in \mathfrak{R}$. Then any strong continuous homomorphism of $A$ is an open, strong and continuous homomorphism of $A$. If $A$ is an $\Omega$-algebra, then any continuous homomorphism of $A$ is an open continuous homomorphism of $A$.

**Remark 1.** Let $X$ and $Y$ be topological spaces, $X \times Y$ be their Cartesian product, and $f$ be a mapping of $X \times Y$ into a third topological space. The mapping $f$ is called **continuous on $x$** (respect. on $y$) if and only if for every $y \in Y$ (respect. $x \in X$) the function $f(\cdot, y)$, the value of which at point $x$ equals $f(x, y)$, is continuous. If the function $f$ is continuous, then it is continuous with respect to every variable. The opposite converts statement is not always true: there exist non-continuous functions, with respect to every
variable [63, pag. 143].

According to (4), let
\[ P_\lambda(f_{\lambda_1}(x_{\lambda_11}, \ldots, x_{\lambda_1k}), \ldots, f_{\lambda_n}(x_{\lambda_n1}, \ldots, x_{\lambda_nk})), \]
\[ f_\varsigma(x_{\varsigma 1}, \ldots, x_{\varsigma k}) = g_\varsigma(x_{\varsigma 1}, \ldots, x_{\varsigma k}), \quad (6) \]
where \( \varsigma, \lambda \in \Lambda \), \( f_\varsigma, g_\varsigma, f_{\lambda_1}, \ldots, f_{\lambda_n} \) are some terms of signature \( \Omega \), \( P_\lambda \in \Omega_P \), be a totality \( S \) of quasiatomic formulas of variables \( x_i \in X \) (\( i \in I \)). Let us note that every term of any topological algebraic system is a continuous operation.

By analogy with [71] we give.

**Definition 1.** Let \( \mathfrak{K} \) be a class of topological algebraic systems of given signature \( \Omega = \Omega_F \cup \Omega_P \), let \( X \) be a topological space and let \( S \) be a totality of formulas \( S \) of form (6) of signature \( \Omega \) of variables \( x_i \in X \) (\( i \in I \)). A topological algebraic system \( \mathcal{A} = < A, \Omega > \) of the class \( \mathfrak{K} \) with the given continuous mapping \( \sigma \) of topological space \( X \) in topological space \( A \) will called defined in \( \mathfrak{K} \) by defining space \( X \) and defining relations \( S \) if:

- \( F_1 \) in \( \mathcal{A} \) the formulas of \( S \) hold
  \[ P_\lambda(f_{\lambda_1}(x_{\lambda_11}, \ldots, x_{\lambda_1k}), \ldots, f_{\lambda_n}(x_{\lambda_n1}, \ldots, x_{\lambda_nk})), \]
  \[ f_\varsigma(x_{\varsigma 1}, \ldots, x_{\varsigma k}) = g_\varsigma(x_{\varsigma 1}, \ldots, x_{\varsigma k}); \]
  are fulfilled;

- \( F_2 \) the system \( \mathcal{A} \) is topologically generated by images of elements of \( X \), i.e. algebra \( < A, \Omega_F > \) does not contain a closed subalgebra, not equal to \( < A, \Omega_F > \) and containing \( \sigma X \);

- \( F_3 \) for every continuous mapping \( \gamma \) of the space \( X \) into any topological algebraic system \( \mathcal{C} \) of the class \( \mathfrak{K} \) under which the relations
  \[ P_\lambda(f_{\lambda_1}(\gamma x_{\lambda_11}, \ldots, \gamma x_{\lambda_1k}), \ldots, f_{\lambda_n}(\gamma x_{\lambda_n1}, \ldots, \gamma x_{\lambda_nk})), \]
  \[ f_\varsigma(\gamma x_{\varsigma 1}, \ldots, \gamma x_{\varsigma k}) = g_\varsigma(\gamma x_{\varsigma 1}, \ldots, \gamma x_{\varsigma k}); \]
  are satisfied there exists a continuous homomorphism \( \alpha \) of system \( \mathcal{A} \) in to \( \mathcal{C} \), compatible with mappings \( \sigma, \gamma \), i.e. such that \( \alpha \sigma(x) = \gamma(x) \) for all \( x \in X \).

The proof of following Lemma 2 is similar to [71, Theorem 2]. It is necessary only to use the Proposition 1, which describes the subsystem of algebraic system \( < A, \Omega > \), algebraically generated by a subset of set \( A \). Let us show the proof of Lemma 2 to ease the reading.

**Lemma 2.** The condition \( F_2 \) of Definition 1 is equivalent to condition:
The algebraic system $A$ is algebraically generated by the images of elements of $X$, i.e. the system $A$ is a totality of elements of set $A$, expressed as finite polynomials of images of elements from $X$ with respect to operations $\Omega_F$.

**Proof.** We denote by $A^*$ the totality of elements of $A$, which are expressed as finite $\Omega_F$-polynomials of images of elements of $X$. Leaving in $A^*$ the topology, which it has a subset of topological space $A$, we convert $A^*$ into a topological algebraic system $A^*$, where the space $X$ is reflected continuously in $A^*$ with the help of the same mapping $\sigma$ as in the given system $A$. By $F_3$), there is a continuous homomorphism $\tau$ of system $A$ into system $A^*$, coordinated with mappings $X \rightarrow A, X \rightarrow A^*$. As $A^*$ algebraically generate by images of elements of $X$ (Proposition 1), then $\tau$ will be the reflection $A$ on all system $A^*$, moreover, for $a \in A^*$ we have $\tau(a) = a$. As $A^*$ is dense â $A$, then for each $a \in A$ there is a directed set $\{a_\lambda\}$ of elements of $A^*$, converging to $a$. From continuously of $\tau$ it follows that $\tau(a) = \lim\{\tau(a_\lambda)\} = a$.

According to [60, Proposition 1.6.7] a topological space $X$ is Hausdorff if and only if any directed set in $X$ has no more than one limit point. Hence $A = A^*$, as required.

**Corollary 3.** If in some class $\mathcal{K}$ of topological algebraic systems of given signature, there exist systems $A, B$, having the same generalized topological space $X$ and the same totality of quasitomatic formulas as their defining relations, then the systems $A, B$ are images of continuous mappings of $X$ and are topologically isomorphic.

**Proof.** According to Definition 1 and Lemma 2 such continuous mappings of defining topological space $X$

$$x_i \rightarrow a_i \quad (a_i \in A, i \in I), \quad x_i \rightarrow b_i \quad (b_i \in B, i \in I) \tag{7}$$

exist that the elements $a_i$ algebraically generate $A$, the elements $b_i$ algebraically generate $B$ and in suitable continuous homomorphisms

$$\varphi : A \rightarrow B, \quad \psi : B \rightarrow A$$

we have $\varphi(a_i) = b_i, \psi(b_i) = a_i$, hence

$$\psi \varphi(a_i) = a_i, \quad \varphi \psi(b_i) = b_i(i \in I). \tag{8}$$

Let $f$ be a polynomial of signature $\Omega_F$. As the mappings $\varphi \psi$ and $\psi \varphi$ are continuous homomorphisms of $A$ and $B$ into themselves, then from (8) it
follows that
\[
\psi \varphi f(a_\lambda, \ldots, a_\gamma) = f(a_\lambda, \ldots, a_\gamma), \quad \varphi \psi f(b_\lambda, \ldots, b_\gamma) = f(b_\lambda, \ldots, b_\gamma),
\]
i.e. \(\varphi \psi\), \(\psi \varphi\) are identical mappings, \(\psi = \varphi^{-1}\) and \(\varphi\) are the continuous isomorphism of \(\mathcal{A}\) on \(\mathcal{B}\). Consequently, the mapping \(\varphi\) is a topological isomorphism of \(\mathcal{A}\) on \(\mathcal{B}\), moreover the image of any element of \(X\) in \(\mathcal{A}\) passes to the image of this element in \(\mathcal{B}\). This completes the Proof of Corollary 3.

From Corollary 3 it follows that if a topological space is a defining space for the same topological algebraic system then this system is defined unequivocally within the accuracy of topological isomorphism. Moreover, the following holds.

**Corollary 4.** Let \(\mathfrak{K}\) be a class of topological algebraic systems of fixed signature and let \(X, Y\) be the defining spaces with continuous mappings \(\alpha : X \to \mathcal{A}, \beta : Y \to \mathcal{B}\) for the systems \(\mathcal{A}, \mathcal{B} \in \mathfrak{K}\). If there is a continuous mapping \(\varphi\) of \(\mathcal{A}\) on \(\mathcal{B}\) then there exists a continuous epimorphism \(\xi : \mathcal{A} \to \mathcal{B}\), satisfying the condition \(\xi \alpha(x) = \beta \varphi(x), x \in X\).

**Proof.** We have that \(\beta \varphi : X \to \mathcal{B}\) and \(\beta \varphi\) is a continuous mapping as a product of continuous mappings. Further, by hypothesis \(\alpha : X \to \mathcal{A}\) and \(X\) is a defining space with continuous mapping for system \(\mathcal{A}\). Then from item \(F_3\) of Definition 1 the existence of continuous epimorphism \(\xi : \mathcal{A} \to \mathcal{B}\), satisfying the condition \(\xi \alpha(x) = \beta \varphi(x), x \in X\), follows, as required.

If in Definition 1 we fix the space \(X\) and change the class \(\mathfrak{K}\) then, in general, the system \(\mathcal{A}\) also changes. The following holds.

**Corollary 5.** Let a topological space \(X\) and a totality of quasiatomic formulas \(S\) define a system \(\mathcal{A}\) in class \(\mathfrak{K}\) of topological algebraic systems of given signature, and let us define a system \(\mathcal{B}\) in subclass \(\mathfrak{L} \subseteq \mathfrak{K}\). Then \(\mathcal{B}\) is a continuous homomorphic image of \(\mathcal{A}\).

**Proof.** By Definition 1 there exists a mapping (7) for which all formulas from \(S\) are true in \(\mathcal{A}\) and \(\mathcal{B}\), and the totalities of elements \(\{a_i\}, \{b_i\}\) algebraically generate these systems by hypothesis. From \(\mathcal{B} \in \mathfrak{L}\), it follows that \(\mathcal{B} \in \mathfrak{K}\), hence from statement \(F_3\) it follows the existence of such a continuous homomorphism \(\varphi : \mathcal{A} \to \mathcal{B}\) follows that \(\varphi(a_i) = b_i (i \in I)\). As the image \(\varphi(\mathcal{A})\) of system \(\mathcal{A}\) into \(\mathcal{B}\) contains the elements \(b_i\), which generate \(\mathcal{B}\), then \(\varphi\) is a continuous homomorphism of \(\mathcal{A}\) on \(\mathcal{B}\), as required.

**Corollary 6.** Let the sets \(S_1, S_2\) of defining relations on the same topological space \(X\) defines respective the systems \(\mathcal{A}\) and \(\mathcal{B}\) in a class \(\mathfrak{K}\) of topological algebraic systems of given signature. If all formulas from \(S_1\) are consequences
of the formulas of $S_2$ in $R$, particularly, if $S_1 \subseteq S_2$, then the system $B$ is an image of $A$ at a continuous homomorphism $\varphi : A \to B$. If the sets of formulas $S_1$ and $S_2$ are equivalent, then the systems $A$ and $B$ are topologically isomorphic.

**Proof.** According to Definition 1 and Lemma 2, there exist such continuous mappings $x_i \to a_i (a_i \in A, i \in I)$, $x_i \to b_i (b_i \in B, i \in I)$ of space $X$ that the elements $a_i$ generate algebraically $A$, the elements $b_i$ generate algebraically $B$, and the formulas from $S_1$ and $S_2$ for the specified values $x_i$ are true in $A$ and, respectively $B$. As the formulas from $S_1$ derive from those of $S_2$, then in $B$ all formulas from $S_1$ are true. According to item $F_3$ from here it follows that such a continuous homomorphism $\varphi$ of system $A$ into system $B$ exists, translating $a_i$ in $b_i$ ($i \in I$). Since the elements $b_i$ generate $B$, then $\varphi$ is the required continuous homomorphism of $A$ on $B$.

If the sets of formulas $S_1$ and $S_2$ are equivalent, then in addition to homomorphism $\varphi : A \to B$ there is a homomorphism $\psi : B \to A$ sending $b_i$ to $a_i$. Since the sets $\{a_i | i \in I\}$, $\{b_i | i \in I\}$ generate the systems $A$, $B$ and $\psi \varphi(a_i) = b_i$, $\varphi \psi(b_i) = a_i$, then $\psi = \varphi^{-1}$. Hence the systems $A$ and $B$ are topological isomorphic, as required.

Remind that a formula $r(x_{\lambda}, \ldots, x_{\gamma})$ is called topological consequence of totality of formulas $S$ in class $R$ if for every continuous mapping $x_i \to c_i$ of totality symbols $x_i$ of topological space $X$ in any $R$-system $C$, at which in $C$ all formulas from $S$ are true, the formula $r(c_{\lambda}, \ldots, c_{\gamma})$ is also true in $C$. Now we generalize [69, Theorem V.11.1], proved for algebraic systems.

**Proposition 5.** Let $R$ be a class of topological algebraic systems of given signature $\Omega = \Omega_F \cup \Omega_P$. A system $A = <A, \Omega > \in R$ is defined by a topological space $X$, a continuous mapping $\sigma$ of $X$ into the basic set $A$ of $A$ and a totality $S$ of quasitonic formulas of signature $\Omega$ of the form (6) if and only if the following conditions are met:

$f_1$) all formulas from $S$ hold in $A$ at continuous mapping $\sigma : x_i \to a_i$ ($x_i \in X$, $a_i \in A$, $i \in I$);

$f_2$) the system $A$ is generated topologically by images of elements of $X$, i.e. algebra $<A, \Omega_F >$ does not contain a closed subalgebra, not equal to $<A, \Omega_F >$ and containing $\sigma X$;

$f_3$) every formula from $S$, true in $A$ at continuous mapping $\sigma : x_i \to a_i$ ($i \in I$), is a topological consequence of totality of formulas $S$ in class $R$ hold true.

**Proof.** We use the Lemma 2 without reference. Sufficiency. Let the
system $\mathcal{A}$ satisfy the conditions $f_1)$, $f_2$) and $f_3)$. Clearly, $F_1)$, $F_2$) follow from $f_1) f_2)$. Let us show that $\mathcal{A}$ also have property $F_3)$. (Let us have a continuous mapping $\beta : X \to C$, $x_i \to c_i$ in some $\mathfrak{R}$-system $C =< C, \Omega >$, where all formulas from $S$ hold true. Let us consider on pair $< \mathcal{A}, C >$ the relation $\varphi$, consisting of every possible pairs of form

$$< f(a_\lambda, \ldots, a_\nu), f(c_\lambda, \ldots, c_\nu) >,$$

where $f$ is a term, we shall also show, that $\varphi$ is a continuous homomorphism of $\mathcal{A}$ in $C$. As the elements $a_i$ algebraically generate $\mathcal{A}$, then it is possible to present any element of system $\mathcal{A}$ as a polynomial $f((a_\lambda, \ldots, a_\nu))$. Consequently, the left part of relation $\varphi$ is $\mathcal{A}$. Further, if for a some terms $f$, $g$ we have

$$f(a_\lambda, \ldots, a_\nu) = g(a_\lambda, \ldots, a_\nu), \quad (10)$$

then by $f_3)$ the formula

$$f(x_\lambda, \ldots, x_\nu) = g(x_\lambda, \ldots, x_\nu) \quad (11)$$

is a topological consequence of totality of formulas $S$ in class $\mathfrak{R}$. But in system $C$ all formula from $S$ at mapping $x_i \to c_i$ are true. Therefore, the formula (11) also is true $C$, i.e.

$$f(c_\lambda, \ldots, c_\nu) = g(c_\lambda, \ldots, c_\nu). \quad (12)$$

holds in $C$. It means that $\varphi$ is a mapping from $\mathcal{A}$ on $C$. Any polynomial of topological algebraic system is a continuous function. As $\mathcal{A}$ and $C$ are a topological algebraic systems then from [60, Proposition 1.4.9] it follows that $\varphi$ is a continuous mapping.

It is similarly proved that if some relation of the form

$$p(f(a_\lambda, \ldots, a_\nu), \ldots, g(a_\lambda, \ldots, a_\nu)), \quad (13)$$

hold true in $\mathcal{A}$, where $p$ is a signature predicate, then the relation

$$p(f(c_\lambda, \ldots, c_\nu), \ldots, g(c_\lambda, \ldots, c_\nu)). \quad (14)$$

holds true in $C$.

From (12) and (14) it follows that the mapping $\varphi$ is a homomorphism. Remind, that a mapping of a topological algebraic system on another one is
called continuous homomorphism if it is continuous and is homomorphism in algebraic forms [70, pag. 12]. Hence $\varphi : A \to C$ is a continuous homomorphism.

As the pairs of form (9) belong to $\varphi$, than $< a_i, c_i > \in \varphi$, i.e. $\varphi(a_i) = c_i \ (i \in I)$, as required.

**Necessity.** Let the system $A$ satisfy the conditions $F_1, F_2, F_3$, and let for some terms $f, g, \ldots, h$ the relation (10) or respectively (13)) hold true in $A$. Then it is necessary to show that the formula (12) (or respectively (14)) in class $\mathfrak{K}$ is a topological consequence of the totality of formulas $S$, i.e. that in every $\mathfrak{K}$-system $C$ for any mapping $x_i \to c_i \ (c_i \in C, i \in I)$, where all formulas from $S$ hold true, the formula (12) (or respectively (14)) also holds. At the specified assumptions, there exists a continuous homomorphism of system $A$ in the system $C$ for which $\varphi(a_i) = c_i \ (i \in I)$ by $F_3)$. But under homomorphisms the quasiatomic formulas keep their validity, and consequently from the validity of (10), or (13) there follows the validity follow for (12), or respectively the formula (14). This completes the proof of Proposition 5.

We denote by $\mathfrak{R}$ a class of algebraic systems of signature $\Omega$ with defining relations $S$ and let $A_\alpha \ (\alpha \in \Sigma)$ be a system of topological algebraic systems of class $\mathfrak{R}$. Remind that the totality of all functions $h(\alpha)$ defined on $\Sigma$ with value in $\cup A_\alpha$, satisfying the requirements $h(\alpha) \in A_\alpha$, is the Cartesian product $H = \prod A_\alpha$.

In $H$ the Tychonoff topology is introduced, announcing open the Cartesian product of open subsets, chosen in a finite number of multiplied spaces, multiplied by others spaces, as well as unions of these products. Further, according to [69, section IV.7.5] quasiatomic formulas multiplicative are steady. We consider (6) and we assume by definition that $\varphi_i(h_1, \ldots, h_n) = h \ (h_i, h \in H)$, where $h(\alpha) = \varphi_i(h_1(\alpha), \ldots, h_n(\alpha)), \alpha \in \Sigma$. Thus, $H$ turns into a topological algebraic system of the same class $\mathfrak{R}$ as the given algebraic systems $A_\alpha$, which are called (direct) topological product of topological algebraic systems $A_\alpha ([71])$.

To research the almost quasivarieties of topological algebraic systems it will be necessary to use the following notions from [71]. Let $\mathfrak{R}$ be a class of topological algebraic systems of fixed signature $\Omega$ ($\mathfrak{R}$-systems) defined by a totality of quasiatomic formulas of signature $\Omega$ of the form (6). We will examine classes of topological algebraic systems, the topological spaces of which satisfy the conditions:

$K_1$) the class $\mathfrak{R}$ is closed under topological product;
$K_2$) every subsystem of $\mathfrak{A}$-system is an $\mathfrak{A}$-system.

Such classes, satisfying $K_1), K_2)$ are the classes of Hausdorff spaces, of regular spaces, of completely regular spaces, with functionally separated pairs of points etc. Moreover, to construct a class with the specified properties it is possible to take a certain set of topological spaces and to consider all those spaces, which can be obtained from operations, taken with the help of $K_1), K_2)$. Particularly, it is possible to take as initial spaces topological spaces with discrete topology. ++++ The received topological spaces are regular and completely discontinuous in the sense that for any finite set of points $x_1, \ldots, x_n$ of such space $X$ there is a splitting $X$ in $n$ pairwise disjoint closed sets, containing one of the specified points.

To describe the topological algebra $\mathcal{A}$, given by defining space $X$ and defining relations $S$ in class $\mathfrak{A}$ in [71] the following questions are considered:

A) Is $\mathcal{A}$ isomorphic to the abstract algebra $\overline{\mathcal{A}}$, given in class $\mathfrak{A}$ by the same defining relations $S$ and generating set $X$, as algebra $\mathcal{A}$ itself?

B) Does $\mathcal{A}$ contain a subspace $X$ topologically, i.e. is the mapping $\sigma$ of space $X$ in $\mathcal{A}$ a homeomorphism?

Sufficient conditions for a positive answer to questions A) and B) are contained in Remarks 1, 2. These remarks are transferred literally for topological algebraic systems. It follows directly from them that for topological algebraic systems generated by regular and completely discontinuous space $X$ the problem A) is solved positively for any $\mathfrak{A}$ and $S$, and problem B) is solved positive under the condition that the abstract algebraic system $\overline{\mathcal{A}}$ contain $X$. The last assertions hold true and for Hausdorff space $X$ as well.

The following Theorem 5 characterizes classes of topological algebraic systems, any system of which can be given by defining spaces and defining relations. Some part of proof of Theorem 5 is similar in form to the proofs of [71, Theorem 1], [69, Theorem V.11.5]. Given the importance of Theorem 5, and also to simplify the reading, the proof of Theorem 5 is given fully.

**Theorem 5.** Let $\mathfrak{A}$ be a class of topological algebraic systems of fixed signature $\Omega$. Then any system $\mathcal{A}$ of $\mathfrak{A}$ can be given by defining topological space $X$ and by defining relations $S$, i.e. by a totality $S$ of quasiatomic formulas of form (6), if and only if $\mathfrak{A}$ is an almost quasivariety. i.e. $\mathfrak{A}$ satisfies the conditions:

- $Q_2$) class $\mathfrak{A}$ is topologically hereditary, i.e. if $\mathcal{A} \in \mathfrak{A}$ and $\mathcal{B}$ is a subsystem of $\mathcal{A}$, then $\mathcal{B} \in \mathfrak{A}$;
- $Q_3$) class $\mathfrak{A}$ is topologically multiplicatively closed, i.e. the direct topologi-
Proof. Necessity. Let any system $A$ of a class $\mathcal{R}$ be given by defining topological space and defining relations. If we consider the topological algebraic system $A \in \mathcal{R}$ as algebraic system, then by Lemma 2 the algebraic system $A$ is with defining relations in the sense of [69, Theorem V.11.2.1]. Then from [69, Lemma V.11.3.4] it follows that the class $\mathcal{R}$ of algebraic systems is replica complete. Hence for a topological algebraic system $A$ considered as class of algebraic system we use the notions of $\mathcal{R}$-replica $A_{\mathcal{R}}$ and $\mathcal{R}$-morphism.

$Q_2$). Let $A$ be a topological algebraic system of class $\mathcal{R}$ defined by defining relations of form (6). Let $B$ be a subsystem of $A$. We denote $Y = \{x \in X | \sigma(x) \in B\}$. Leaving in $B$ the topology which it has as a subset of the topological space $A$ we convert $B$ into a topological system, where $Y$ is reflected continuously in $B$ with the help of narrowing $\tau$ of $\sigma$ on $B$. According to Lemma 2 the set $\{\sigma(x) | x \in X\}$ algebraically generate the system $A$, then the set $\{\tau(y) | y \in Y\}$ algebraically generate the system $B$. The continuity of $\sigma : X \rightarrow A$ induces the continuity of $\tau : Y \rightarrow B$ by [60, pag. 112]. Moreover, any element in $A$ is expressed in the form of a finite polynomial from images elements $X$ by item $F_2$) of Definition 1 and Lemma 2. Then from item $F_1$) it follows that the formulas from $S$ hold true in system $B$.

Let now $B_{\mathcal{R}}$ be the replica of system $B$ and let $\beta : B \rightarrow B_{\mathcal{R}}$ be the corresponding $\mathcal{R}$-morphism. On algebraic system $B_{\mathcal{R}}$ we introduce a topology regarding the images of open sets of topological algebraic system $B$ as open sets. Then $\beta$ is a continuous homomorphism, hence and $\beta \tau : Y \rightarrow B_{\mathcal{R}}$ is a continuous mapping.

From definition of replica $B_{\mathcal{R}}$ it follows that for identical embedding $\varepsilon : B \rightarrow A$ there should exist a continuous homomorphism $\varsigma : B_{\mathcal{R}} \rightarrow A$, satisfying the condition $\varepsilon = \varsigma \beta$. As $\beta$ is an epimorphism and $\varepsilon$ is the identical mapping of $B$ on itself, then from $\varepsilon = \varsigma \beta$ it follows that the mappings $\alpha, \varsigma$ mutually inverse, $\varsigma = \beta^{-1}$, therefore system $B$ is isomorphic to system $B_{\mathcal{R}}$, belonging to class $\mathcal{R}$.

Let us use the following statement: the quasitomic formulas maintain their validity in case of homomorphisms and in case of converting into subsystem of algebraic systems [69, pag. 278]. From here it follows that if a system $L$ of class $\mathcal{R}$ of topological algebraic systems of fixed signature satisfies the condition $f_3$) of Proposition 5 then the $\mathcal{R}$-morphism image of system
, i.e. the replica $L_R$ satisfies this condition $f_3 \rangle$. Moreover, using the definition of replica complete class and Lemma 2 we verify immediately that if the replica $L_R$ satisfies the condition $f_3 \rangle$, then any subsystem of $L_R$, which by [69, Theorem V.11.3.5] belongs to replica complete class $R$, also satisfies condition $f_3 \rangle$.

Now again we consider the topological algebraic system $A$ defined by defining topological space $X$, by continuous mapping $\sigma : X \to A$ and by set $S$ of defining relations and let $\alpha : A \to A_R$ be the corresponding $R$-morphism. On algebraic system $A_R$ we define the topology, induced by topology of topological system $A$ similarly for system $B$. Also we can show similarly that the mapping $\alpha \sigma : X \to A_R$ is continuous, $\alpha \to A_R$ is a continuous isomorphism and any element of $A_R$ is expressed in form of finite polynomial of variables $\alpha \sigma(x)$, $x \in X$. From here and the previous paragraph it follows that the system $A_R$ satisfies the conditions $f_1 \langle 0 - f_3 \rangle$ of Proposition 5 from which it follows that the topological algebraic system $A_R$ is defined by defining space $X$, by continuous mapping $\alpha \sigma : X \to A_R$ and by defining relations $S$. From Definition 1 it follows that $\alpha^{-1} : A_R \to A$ is a continuous homomorphism, correlated with continuous mappings $\alpha \sigma : X \to A_R$, $\sigma : X \to A$. We have shown above, that $\alpha : A \to A_R$ is a continuous isomorphism. Hence $\alpha : A \to A_R$ is a topological isomorphism. Now the topological isomorphism $\alpha$ induces the topological isomorphism $\beta : B \to B_R$ by narrowing the defining space $X$ for $A$ to defining space $Y$ for $B$.

We have above shown, that the replica $B_R$ is a topological algebraic system defined by defining space $X$ and by defining relations $S$. Then $B_R$ satisfies the condition $f_3 \rangle$ of Proposition 5. We have also shown above that in such a case the topological system $A_R$ also satisfies the condition $f_3 \rangle$, as well as $B = \beta^{-1}(B_R)$, since $\beta$ is a topological isomorphism. We have shown above, that $B$ satisfies the conditions $f_1 \rangle$, $f_2 \rangle$ and from Proposition 5 it follows that the topological system $B$ will be defined by defining space $Y$, continuous mapping $\beta : Y \to B$ and defining relations $S$. The item $Q_2 \rangle$ is proved.

$Q_3 \rangle$. Like in the previous case we consider that $R$ is a replica complete class of algebraic systems. Let $A = \prod_{i \in I} A_i$ be a Cartesian product, where $A_i \in R$. By [69, Theorem V.11.3.5] $A \in R$. By hypothesis $A_i$ is a topological algebraic system defined by defining space $X_i$, continuous mapping $\sigma : X_i \to A_i$ and defining relations $S_i$. The system $A$ together with Tychonoff topology induced by topologies of $A_i$ is also a topological system.

Let us prove that $A$ is a topological algebraic system given by defining space $X = \prod_{i \in I} X_i$ (Cartesian product of spaces $X_i$), by defining relations
\[ S = \cap_{i \in I} S_i \] and by mapping \( \sigma = \prod_{i \in I} \sigma_i \), using the items \( f_1 \) - \( f_3 \) of Proposition 5. The Cartesian product \( \prod_{i \in I} \sigma_i \) of mappings \( \sigma_i \) is a mapping \( \sigma : X \to \mathcal{A} \) of space \( X \) in space \( \mathcal{A} \) whose each point \( x = \{x_i\} \in \prod_{i \in I} X_i \) corresponds the point \( \sigma(x) = \{\sigma_i(x_i)\} \in \prod_{i \in I} \mathcal{A}_i \). Any mapping \( \sigma_i \) is continuous, then by [60, Corollary 2.3.5] the mapping \( \sigma : X \to \mathcal{A} \) is also continuous.

According to item \( f_2 \) any set \( \sigma_i(X_i) \) topologically generate the system \( \mathcal{A}_i \), i.e. the space \( \sigma_i(X_i) \) is dense in space \( \mathcal{A}_i \). Then from [60, Proposition 2.3.6] it follows that the space \( \sigma(X) \) is dense in space \( \mathcal{A} \), i.e. \( \sigma_i(X) \) topologically generate the system \( \mathcal{A} \). Hence the condition \( f_2 \) for system \( \mathcal{A} \) holds true.

We have shown above that the algebraic system \( \mathcal{A} \) belongs to replica complete class \( \mathcal{K} \). Then similarly to system \( \mathcal{B} \) from item \( Q_2 \) we prove that the topological system \( \mathcal{A} \) satisfies the condition \( f_3 \). Clearly, \( \mathcal{A} \) satisfies the relations \( S = \cap_{i \in I} S_i \). Hence \( \mathcal{A} \) satisfies the condition \( f_1 \) for \( S \). Consequently, by Proposition 4 the topological system \( \mathcal{A} \) satisfies the condition \( Q_3 \).

\( Q_4 \). Let us consider the unitary system \( \mathcal{A}_e \) as \( \mathcal{A} \). The unitary system satisfy any quasitomac formula of form (6) and any homomorphic image of unitary system can be only an unitary system. Then \( \mathcal{A}_e \in \mathcal{K} \). Hence the unitary system \( \mathcal{A}_e \) is given by defining topological space \( X \), consisting of one point, and continuous mapping \( \sigma : X \to \mathcal{A}_e, \sigma(X) = e \). The condition \( Q_4 \) holds.

The necessity of Theorem 5 is proved. The sufficiency follows from the next statement.

**Theorem 6.** Let \( \mathcal{K} \) be an almost quasivariety of topological algebraic systems of given signature \( \Omega \). Then for any topological space \( X \), linked with \( \mathcal{K} \) by conditions \( K_1 \), \( K_2 \), and for any totality \( S \) of quasitomac formulas of signature \( \Omega \) of form (6) there exists a topological algebraic system \( \mathcal{A} \) with properties \( F_1 \), \( F_2 \), \( F_3 \) and is defined by these properties univocally with accuracy of topological isomorphism over the image of space \( X \) in it.

**Proof.** Let us consider the property \( F_2' \) instead of \( F_2 \) by Lemma 2. Let \( X = \{x_i| i \in I\} \) and let \( m = \max([\Omega],[I],[\aleph_0]) \). According to [69, pag. 163] any algebraic system of class \( \mathcal{K} \), which is generated by a set of cardinality \( \leq |I| \) has itself the cardinality \( \leq m \).

We denote by \( M \) the totality of every possible pairwise topological non-isomorphic \( \mathcal{K} \)-systems of cardinality \( \leq m \). According to [69, page 280], the cardinality of \( M \) is no more than a number depending only on \( m \) and \( |\Omega|\).

We consider the totality of all \( \mathcal{K} \)-systems from \( M \) which satisfy conditions \( F_1 \), \( F_2 \). We denote by \( \sigma_\lambda \) that continuous mapping of \( X \) into \( \mathcal{A}_\lambda \), which is
considered in $F_1, F_2$). Then with respect to mappings
\[ \sigma_\lambda : x_i \rightarrow a^\lambda_i \quad (a^\lambda_i \in A_\lambda, A_\lambda \in M) \] (15)
all formulas from $S$ hold in $A_\lambda$. The system of mappings $\sigma_\lambda$ is non-empty, since the almost quasivariety $\mathcal{K}$ contains the unitary system $A_e = < e, \Omega >$ (the condition $Q_4$)) in which not only formulas from $S$ hold at mapping $x_i \rightarrow e$, but also all quasiatomic formulas in general.

We consider the Cartesian product $B = \prod_{\lambda \in J} A_\lambda$. As all its factors belong to the class $\mathcal{K}$, which is multiplicatively closed by condition $Q_3$, then $B \in \mathcal{K}$. Moreover, $B$ satisfies the condition $K_1$. Hence, $B$ is a topological space with topology induced by topology of space $X$ under continuous mapping $\sigma : x_i \rightarrow B$.

By [60, Proposition 2.3.6] a continuous homomorphism $\sigma : x_i \rightarrow B (i \in I)$ exists for the totality of continuous homomorphisms (15). According to [69, Section III.7.5] the quasiatomic formulas are multiplicatively stable. The formulas from $S$ are quasiatomic and hold true in factors $A_\lambda$ at continuous mappings $\sigma_\lambda$. Hence, the formulas from $S$ hold at continuous mappings $\sigma$.

We put $\sigma(x_i) = a_i$ and denote by $A$ the subsystem of system $B$, generated in $B$ by elements $a_i (i \in I)$. As the formulas from $S$ do not contain a quantifier, then from their validity in system $B$ at mappings $\sigma : x_i \rightarrow a_i$ follows their validity in subsystem $A$. We want to show that the generating symbols $x_i$ and the formulas from $S$ generate the very system $A$ in class $\mathcal{K}$. Indeed, $A$ is a subsystem of $B$, hence by $Q_2$ belongs to $\mathcal{K}$. The elements $a_i$ generate $A$ and under continuous mapping $\sigma : x_i \rightarrow a_i$ the formulas from $S$ hold in $A$.

It remains to show that the mapping $\sigma$ satisfies the condition $F_3$) from Definition 1. Indeed, let a topological algebraic $\mathcal{K}$-system $C$ and the mapping $x_i \rightarrow c_i (c_i \in C, i \in I)$ be given. We denote by $D$ the subsystem generated in $C$ by elements $c_i$. The system $D$ belongs to $\mathcal{K}$ and under the mapping $x_i \rightarrow c_i$ the formulas from $S$ hold in $D$. It is necessary to find a continuous homomorphism $\gamma : A \rightarrow D$ at which $\gamma(a_i) = c_i (i \in I)$.

The elements $c_i$ generate $D$ and the cardinality of $D$ does not exceed $|I|$. Hence with the accuracy of a topological isomorphism the system $D$ coincides with some system $A_j$ and the mapping $x_i \rightarrow c_i$ coincides with mapping $\sigma_j : x_i \rightarrow a^j_i$, and it is sufficient to find a continuous homomorphism $\gamma : A \rightarrow A_j$ with condition $\gamma(a_i) = a^j_i (i \in I)$.

The continuous homomorphism $\sigma : x_i \rightarrow B (i \in I)$ is induced by mappings
(15), hence for projection $\pi_j$ we have $\pi_j : A \to A_j$, $\pi_j \sigma_i = \sigma_i^j$. Hence $\pi_j$ is the required continuous homomorphism of $A$ into $A_j$.

Uniqueness of system $A$ follows from Corollary 3. This completes the proof of Theorem 6.

Let $\mathfrak{K}$ be a class of topological algebraic systems defined by a totality of quasiatomic formula. In general, the class $\mathfrak{K}$ is not closed with respect to continuous homomorphic images. However it takes place

**Corollary 7.** Let $X$ be such a topological space that a class $\mathfrak{K}$ of topological algebraic systems of fixed signature over $X$ satisfies the conditions $K_1)$, $K_2)$. Then any set of generating symbols $\{x_i \in X | i \in I\}$ and any totality of quasiatomic formulas from these symbols determine the appropriate $\mathfrak{K}$-system if and only if $\mathfrak{K}$ is a almost quasivariety.

**Proof.** Sufficiency follows from Theorem 6, and necessity follows from necessity of Theorem 5.

**Corollary 8.** An axiomatizable class $\mathfrak{K}$ of topological algebraic systems is an almost quasivariety if and only if $\mathfrak{K}$ is a quasivariety.

**Proof.** Every axiomatizable class of systems $\mathcal{K}$ is ultraclosed [69, Corollary IV.8.10]. Then the proof of this statement follows from Proposition 3 and Lemma 1.

For the class $\mathfrak{K}$ we denote by symbol $\prod \mathfrak{K}$ the class of all homeomorphic copies of topological products of topological $\mathfrak{K}$-systems, by $S\mathfrak{K}$ the class of all subsystems of $\mathfrak{K}$-systems and by $\mathfrak{K}_e$ the class, obtained by connection to $\mathfrak{K}$ of an unitary system.

**Proposition 6.** For a class of topological algebraic systems $\mathfrak{K}$ the class $S\prod \mathfrak{K}_e$ is a minimal almost quasivariety, containing in itself the class $\mathfrak{K}$.

**Proof.** The minimality of $S\prod \mathfrak{K}_e$ is obvious, since the almost quasivariety contains the unitary system, the Cartesian products and the subsystems of $\mathfrak{K}$-systems. Therefore it is necessary to show that $S\prod \mathfrak{K}_e$ is an almost quasivariety. This class contains the unitary system. If $\mathcal{A}$ is a subsystem of the product $\prod \mathcal{A}_i$ ($i \in I, \mathcal{A}_i \in \mathfrak{K}_e$), then each of its subsystems is a subsystem of the same product. Hence the class $S\prod \mathfrak{K}_e$ is hereditary. At last, let a sequence $\{\mathcal{A}_i | i \in I\}$ of systems from $S\prod \mathfrak{A}_e$ be given. By condition,

$$\mathcal{A}_i \subseteq \prod B_{\lambda} \quad (\lambda \in K_i) \quad (B_{\lambda} \in \mathfrak{K}_e; K_i \cap K_j = \emptyset, i \neq j).$$

But in this case, according to [69, section 1.2.5], there is an embedding
\[ \prod A_i \subseteq \prod B_{\lambda} \cong \prod B_\nu \quad (\nu \in \cup K_i), \]
from which it follows that \( \prod A_i \in S \prod \mathcal{K}_e \).

From Proposition 6 it follows that the class \( S \prod \mathcal{K} \) is a minimal topological almost quasivariety which contains the class \( \mathcal{K}_e \). Let us call \( S \prod \mathcal{K}_e \) the closure of class \( \mathcal{K} \) with respect to topological almost quasivarieties.

**Corollary 9.** Suppose that up to topological isomorphism the class \( \mathcal{K} \) consists only of one topological algebraic system \( A \). Then the minimal almost quasivariety of topological algebraic systems, containing \( A \), consists of unitary system and topological isomorphic copies of subsystems of Cartesian degrees of system \( A \).

**Corollary 10.** If a class \( \mathcal{K} \) of topological algebraic system is axiomatizable, then the class \( S \prod \mathcal{K}_e \) is a quasivariety.

**Proof.** If the unitary system is joined with the axiomatizable class \( \mathcal{K} \) the axiomatizable class \( \mathcal{K}_e \) is obtained. By Corollary 8 \( S \prod \mathcal{K}_e \) is a quasivariety, as required.

The class \( (A) \) of topological isomorphic copies of finite topological algebraic system \( A \) is axiomatizable. Then from Corollary 8 it follows

**Corollary 11.** Let \( A \) be a finite topological algebraic system. Then \( S \prod (A)_e \) is a quasivariety.

Really, the class \( (A) \) of topological isomorphic copies of finite system \( A \) is axiomatizable. Then Corollary 11 follows from Proposition 6.

**Remark 2.** The Definition 1, Lemma 2, Corollaries 3 – 6, Proposition 5 have sense according to Theorem 5 only for almost quasivarieties of topological algebraic systems \( \mathcal{K} \), defined by sets of quasitomonic formulas. Since \( A, C \in \mathcal{K} \) in item \( F_3 \) of Definition 1, then its continuous homomorphism \( \gamma : A \rightarrow C \) is strong as is mentioned in the beginning of Section 1. Moreover, all continuous homomorphisms considered in proofs of listed above statements, and also in Theorems 5 and 6, Proposition 6, Corollaries 7 – 11 are strong.

Let now \( \varphi \) be a strong continuous homomorphism of a topological algebraic system \( A \) of a topological almost quasivarieties of topological algebraic systems \( \mathcal{K} \) and let \( \theta \) be the congruence on \( A \) induced by \( \varphi \). The congruence
$\theta$ will be complete and continuous. In general, the quotient system $\mathcal{A}/\theta$ will not be a topological algebraic system. But from Corollary 2 it follows

**Proposition 7.** Let $\mathcal{R}$ be an almost quasivariety of topological algebraic systems with permutable congruences and let $\mathcal{A} \in \mathcal{R}$. Then any strong continuous homomorphism of $\mathcal{A}$ is open, strong and continuous and the quotient system $\mathcal{A}/\theta$ is a topological algebraic system.

**Remark 3.** Till now we examined any Hausdorff topological spaces and we use a characteristic property of such spaces only to prove Lemma 2. On page 176 of [69] it is mentioned that the Lemma 2 holds true for such topological space that the conditions $K_1$) and $K_2$) hold. Hence all statements of Section 2 will stay in force, if instead of Hausdorff spaces to examine a topological spaces with properties of capture subsystems of topological systems and closed under direct topological product.

**Corollary 12.** If a topological space $X$ is defining space for some topological algebraic system, then it is necessary for $X$ to be completely regular (Tychonoff) space.

3 On free systems of almost quasivarieties of topological algebraic systems.

This section is the continuation of Section 2. Similarly to Section 2, in this section any considered homomorphism of topological algebraic system is strong according to Remark 2 and according to Remark 3 the considered topological spaces are or Hausdorff, or such spaces that satisfy the conditions $K_1)$, $K_2$). In both cases these spaces are completely regular (Tychonoff) by Corollary 12.

We shall follow the concepts from [69, section V.12.2]. Let $\mathcal{K}$ be a class of topological algebraic systems of fixed signature $\Omega$ and let $\mathcal{A} = \langle A, \Omega \rangle$ be a topological $\mathcal{K}$-system with such a continuous mapping $\sigma : X \to A$ that $\sigma(X)$ topologically generate $\mathcal{A}$. A non-empty totality of elements $S$ from $A$ is called independent in $\mathcal{A}$ with respect to class $\mathcal{R}$ ($\mathcal{R}$-independent) if every continuous mapping of $S$ into any $\mathcal{R}$-system $\mathcal{B}$ can be continued up to continuous homomorphism of $S$ in $\mathcal{B}$, where $S$ denotes the subsystem of $\mathcal{A}$ generated topologically by elements $S \subseteq \sigma(X)$. The class $\mathcal{K}$ satisfies the condition $K_2)$. Then by Lemma 2 the topological $\mathcal{K}$-system $S$ is generated
algebraically by set $S$.

**Proposition 8.** If pairwise distinct elements $a_1, \ldots, a_n$ of topological algebraic system $A$ are $K$-independent and $A$ satisfies some quasiatomic relation

$$p(f_1(a_1, \ldots, a_n), \ldots, f_s(a_1, \ldots, a_n)) = \{\text{true}\},$$

where $p \in \{\Omega, =\}$, and $f_i(x_1, \ldots, x_n)$ are some terms of signature $\Omega$ not necessarily containing all variable $x_1, \ldots, x_n \in X$, than the identity

$$(\forall x_1, \ldots, x_n)p(f_1(x_1, \ldots, x_n), \ldots, f_s(x_1, \ldots, x_n)) = \{\text{true}\}$$

holds true in class $K$. Conversely, let for a certain set $S$ of elements of topological algebraic system $A$ from the validity of a quasiatomic relation of form (16) for some pairwise distinct elements $a_1, \ldots, a_n \in S$ follows the validity of the identity (17) in $K$. Then the set $S$ is $K$-independent.

**Proof.** Let $a_1, \ldots, a_n \in A$ be distinct $K$-independent elements, $S_1 = \{a_1, \ldots, a_n\}$ and let $b_1, \ldots, b_n$ be some (not necessarily distinct) elements of a $K$-system $B = \langle B, \Omega \rangle$ with continuous mapping $\eta : X \to B$. Let $b_1 = \eta(x_1), \ldots, b_n = \eta(x_n)$. From the $K$-independence of elements $a_1, \ldots, a_n$ it follows that the mapping $\alpha : a_i \to b_i$ ($i = 1, \ldots, n$) can be continued up to a homomorphism $\beta : \overline{S_1} \to \mathcal{B}$. From here and (16) it follows that

$$\{\text{true}\} = p(f_1(\beta(a_1), \ldots, \beta(a_n)), \ldots, f_s(\beta(a_1), \ldots, \beta(a_n)) = p(f_1(b_1, \ldots, b_n), \ldots, f_s(b_1, \ldots, b_n)) = p(\eta(x_1), \ldots, \eta(x_n)), \ldots, f_s(\eta(x_1), \ldots, \eta(x_n))).$$

The elements $b_1 = \eta(x_1), \ldots, b_n = \eta(x_n) \in B$, where $x_1, \ldots, x_n \in X$, are arbitrary. From here it follows that (17) is an identity of class $K$.

Conversely, identity (17) follows in $K$ from any relation of form (16) for an arbitrary pairwise distinct elements $a_1, \ldots, a_n$ of a some set $S$ of elements in $A$. We take some mapping $\alpha$ of $S$ in an arbitrary $K$-system $B$. It is necessary to continue $\alpha$ up to homomorphism $\overline{S}$ in $\mathcal{B}$. Above we have shown that each element $a \in \overline{S}$ can be presented as

$$a = f(a_1, \ldots, a_n) \quad (a_i \in S, a_i \neq a_j; i \neq j, i, j = 1, \ldots, n),$$

where $f(x_1, \ldots, x_n)$ is a term of signature $\Omega$. Let us introduce the mapping

$$\varphi : f(a_1, \ldots, a_n) \to f(\alpha(a_1), \ldots, \alpha(a_n)).$$
Now show that the mapping $\varphi$ is the unique continuous homomorphism of $\overline{S}$ into $B$, i.e. that the validity of relation

$$p(f_1(\alpha(a_1), \ldots, \alpha(a_n)), \ldots, f_s(\alpha(a_1), \ldots, \alpha(a_n)))$$

(18)

follows from the validity of relation of form (16).

But it is obvious, since the relation (18) follows from identity (17) at mapping $x_i \rightarrow \alpha(a_i) \ (i = 1, \ldots, n)$. The continuity of homomorphism $\varphi$ follows from the definition of identity (17) (see, for example, item $F_1$)). This completes the proof of Proposition 8.

**Corollary 13.** If each finite subset of an infinite set $S$ of elements of topological algebraic system $A$ is $\mathcal{R}$-independent, then the set $S$ is also $\mathcal{R}$-independent.

Really, every quasiatomic relation of form (16) between the elements of $S$ contains only a finite subset of elements from $S$. Then identities (17) follow from $\mathcal{R}$-independence $\mathcal{R}$, as required.

In the definition of $\mathcal{R}$-independence not the system $A$ itself is important, but and only subsystem $\overline{S}$, so the $\mathcal{R}$-independence of set $S$ in system $A$ is equivalent to $\mathcal{R}$-independence of $S$ in any subsystem of system $A$. Also, the $\mathcal{R}$-independence of any subset of $S$ follows from $\mathcal{R}$-independence of $S$.

Literally repeating the proof of Theorem V.12.3 from [69] (it is necessary to apply only the Proposition 8) we prove the following Corollary 13 in which it is underlined that a $\mathcal{R}$-independent set of elements is independent not only with respect to subclass $L \subseteq \mathcal{R}$, but is independent with respect to an wider class. Remind, that for any class $L$ symbols $SL$, $\prod L$, $HL$ denote the classes of all subsystem of $L$-systems, topological isomorphic copies of topological products and, respectively, continuous homomorphic images.

**Corollary 14.** If a set $S$ of elements in topological algebraic system $A$ is independent with respect to a class $\mathcal{R}$, then $S$ is independent also with respect to class $HS \prod A$ and with respect to any subclass $L \subseteq \mathcal{R}$.

The topological $\mathcal{R}$-system $A$ will be called free with respect to class $\mathcal{R}$, if in $A$ there exists a set $S$ of elements, independent from and topologically generating $A$. The totality $S$ with this property is called a $\mathcal{R}$-free basis of system $A$. A system $A$ is called free system of rang $m$ in class $\mathcal{R}$ (denoted by $F_m(A)$), if $A \in \mathcal{R}$ and a $\mathcal{R}$-free basis of cardinality $m$ exists in $A$.

Let $\mathcal{R}$ be a class of topological algebraic systems of fixed signature $\Omega$. Comparing with each other the definition of free system with respect to class
\( \mathfrak{K} \) and the system described by Proposition 5 given in class \( \mathfrak{K} \) by defining relations (Definition 1) we ascertain.

**Lemma 3.** Let \( \mathcal{A} = \langle A, \Omega \rangle \) be a \( \mathfrak{K} \)-system, given by topological space \( X \), by continuous mapping \( \sigma : X \to A \) and by defining relations \( \Delta \). Then:

1. Any \( \mathfrak{K} \)-independent set \( S \) of elements of system \( \mathcal{A} \) generate a subsystem \( \overline{S} \), free with respect to class \( \mathfrak{K} \). The set \( S \) is a \( \mathfrak{K} \)-free basis for system \( \overline{S} \):

2. The system \( \mathcal{A} \) is free with \( \mathfrak{K} \)-free basis \( \sigma(X) \) when and only when the set of defining relations \( \Delta \) is empty.

It is worth mentioning that the notion of free system was introduced earlier in [71] in form of item 2).

The classes of systems, consisting only from one-element systems is called trivial. In trivial classes only free systems of rank 1 can exist. Thus the system \( \mathcal{A} \) will be free, if in it each formula of a form \( p(a, \ldots, a) \) (\( p \in \Omega_P, a \in \mathcal{A} \)) is true then and only then, when it holds true in each system of this class.

**Theorem 7.** A non-trivial class \( \mathfrak{K} \) of topological algebraic systems of given signature \( \Omega \) contains a \( \mathfrak{K} \)-free systems when and only when \( \mathfrak{K} \) is an almost quasivariety. More specific, when and only when the non-trivial almost quasivarieties contain free topological algebraic systems \( F_m \) of any given rank \( m \geq 1 \).

**Proof.** The first part follows from item 2) of Lemma 3 and Theorem 5. Let us prove the second part. We use Lemma 3 and Proposition 5. Let \( \sigma : X \to A \) be the continuous mapping from Proposition 5. Assume that \( |\sigma(X)| = m \). By Proposition 5 the totality of elements \( \sigma(x_i), x_i \in X \), topologically generate the system \( \mathcal{A} \) and is \( \mathfrak{K} \)-independent. According to Lemma 3 to complete the proof we will show that the cardinality of set of all elements \( \sigma x_i \) is equal to \( m \), i.e. that \( \sigma x_i \neq \sigma x_j \) for \( i \neq j \). But it is were the case that \( i \neq j \) and \( \sigma x_i = \sigma x_j \), then from the independence of elements \( x_i, \sigma x_j \) it follows that in class \( \mathfrak{K} \) the identity \( x = y \) would be true and class \( \mathfrak{K} \) would be trivial. The Theorem 7 is proved.

**Corollary 15.** A non-trivial class \( \mathfrak{K} \) of topological algebraic systems of given signature \( \Omega \) contains an axiomatizable \( \mathfrak{K} \)-free systems when and only when \( \mathfrak{K} \) is a quasivariety. More specific, when and only when the non-trivial quasivarieties contain an axiomatizable free topological algebraic systems \( F_m \) of any given rank \( m \geq 1 \).

Corollary 10 also Follows from Theorem 7.

A generating set of elements of a system \( \mathcal{A} \) is called minimal, if any of its
proportion subsets does not generate \( \mathcal{A} \).

**Theorem 8.** Let \( \mathcal{K} \) be an almost quasivariety of topological algebraic systems of fixed signature \( \Sigma \). Then:

1) an algebraic system \( \mathcal{A} \), free in relation to class \( \mathcal{K} \), is free in relation to any subclass \( \mathcal{L} \subseteq \mathcal{K} \), and in relation to closure \( HS \prod K \);

2) all free \( \mathcal{K} \)-systems of a given rank \( M \) are topologically isomorphic among each other and any topological algebraic \( \mathcal{K} \)-system topologically generated by a set of cardinality \( m \) is an image of a continuous homomorphism of a free systems \( F_m(\mathcal{K}) \) of rank \( m \);

3) a free basis of a free system \( \mathcal{A} = F_m(\mathcal{A}) \) of some class \( \mathcal{K} \) of topological algebraic system is a minimal generating set in \( \mathcal{A} \).

**Proof.** The item 1) and the first part of item 2) follows from Lemma 3, Corollaries 3, 14 and established after Corollary 13 the properties of \( \mathcal{K} \)-independent sets of elements. We prove the second part of item 2).

Let \( S \) be a \( \mathcal{K} \)-free basis of \( \mathcal{A} \) with cardinality \( |S| = m \) and let \( \mathcal{B} \) be a topological algebraic \( \mathcal{K} \)-system topologically generated by a set \( U \) of cardinality \( m \). We consider a mapping \( \alpha \) of set \( S \) on \( U \). By hypothesis \( \alpha \) may be continued to continuous homomorphism \( \varphi : \mathcal{A} \rightarrow \mathcal{B} \). Since \( \varphi(\mathcal{A}) \subseteq U \) and \( U \) generate \( \mathcal{B} \), then \( \varphi : \mathcal{A} \rightarrow \mathcal{B} \) is a continuous homomorphism.

3) We admit on the contrary that there is a proper part \( S_1 \) of basis \( S \) which generates \( \mathcal{A} \). Then by Lemma 2 for each element \( a \in S \setminus S_1 \) a representation of the form

\[
a = f(a_1, \ldots, a_n)
\]

exists, where \( f(x_1, \ldots, x_n) \) is some term of signature \( \Omega \) and \( a_1, \ldots, a_n \) are different elements from \( S_1 \). From (19) it follows that the identity \( x = f(x_1, \ldots, x_n) \) holds true in class \( \mathcal{K} \) by Proposition 8. As \( \mathcal{A} \in \mathcal{K} \), then this identity also holds true in \( \mathcal{A} \). Putting here \( x_1 = a_1, \ldots, x_n = a_n, x = a, a_1 \) we obtain \( a = a_1 \), that is impossible. This completes the Proof of Theorem 8.

A free system of a class of all systems of signature \( \Omega \) is called **absolutely free**. From Theorems 7, 8 it follows that a free system of rank \( m \) of arbitrary class \( \mathcal{K} \) is an image of absolutely free system of rank \( m \) with respect to continuous endomorphism.

Let \( \mathcal{K} \) be an almost quasivariety of topological algebraic systems of fixed signature. If \( \mathcal{A} \in \mathcal{K} \) has an infinite set of topological generators then all sets of such generators have the same cardinality (proved similarly to analogical statement for algebraic systems [69, pag. 318]). Moreover, if the set of topological generators of \( \mathcal{A} \) has the cardinality \( m \) then each of its topological
isomorphic system has minimal set of topological generators of cardinality $m$. Comparing these statements with item 3) of Theorem 8 we get the analog of Fujiwara Theorem [69, pag. 318].

**Corollary 16.** If in some almost quasivariety of topological algebraic systems of fixed signature $\mathcal{K}$ a free systems $F_m, F_n$ of various ranks $m, n$ are topological isomorphic, then $m, n$ are finite.

Corollary 16 is also specified by analogy with [69, pag. 320].

**Corollary 17.** If an almost quasivariety of topological algebraic systems of fixed signature $\mathcal{K}$ contains a finite non-unitary system then all free systems of various ranks of $\mathcal{K}$ are topologically non-isomorphic.

Let $\mathcal{K}$ be a class of topological algebraic systems of fixed signature and let $F_n(\mathcal{K})$ be a free system of finite rank $n$. We suppose that $\mathcal{K} = \{F_n\}$. According to Lemma 3 a set $f_1, \ldots, f_n \in F_n$ which topologically generate $F_n$ will called free, if any continuous mapping $f_i \to a_i \in F_n (i = 1, \ldots, n)$ can be continued until the continuous homomorphism of $F_n$ in itself. The following holds.

**Corollary 18.** In free system $F_n$ of finite rank $n$ any set of $n$ elements which topologically generate $F_n$ is free when and only when any continuous epimorphism of $F$ on itself is a continuous isomorphism.

**Proof.** Let $\varphi : F_n \to F_n$ be a continuous epimorphism which is not an isomorphism and let $\{f_1, \ldots, f_n\} \subseteq F_n$ be a free generating set. Then $\{\varphi f_1, \ldots, \varphi f_n\}$ topologically generate $F_n$. If the latter was free then by Corollary 3 there would have existed a continuous isomorphism $\psi : F_n \to F_n$ turning $\{f_1, \ldots, f_n\} \to \{\varphi f_1, \ldots, \varphi f_n\}$ respectively. Since the homomorphisms $\varphi, \psi$ coincide on the generating set $\{f_1, \ldots, f_n\}$, then they should coincide on $F_n$ as well, i.e. $\varphi = \psi$, which contradicts the assumption.

Conversely, let any set from $n$ elements be free in $F_n$ and let $\varphi : F_n \to F_n$ be a continuous epimorphism. Then the elements $\varphi f_1, \ldots, \varphi f_n$ topologically generate $F_n$ and therefore are free in $F_n$. That is why there should exist a continuous isomorphism $\psi : F_n \to F_n$ turning $\{f_1, \ldots, f_n\}$ into $\{\varphi f_1, \ldots, \varphi f_n\}$ respectively. Since the homomorphisms $\varphi, \psi$ coincide on the generating set $\{f_1, \ldots, f_n\}$, the should coincide on $F_n$ as well, i.e. $\varphi = \psi$, as required.

**Remark 4.** We present next [71, pag. 178]. All spaces are considered that satisfy the conditions of Lemma 2 of present paper. Let the topological algebra $A$ be finitely generated by a subset $X$ of $A$. We denote by $S$ the totality of all relations between elements from $X$ in algebra $A$ and consider
the algebra $B$ with generating space $X$ and defining relations $S$. By [71, Remark 2] the algebra $B$ contains topologically $X$ and the continuous homomorphism, which is the continuation of identical mapping of $X$ on $X$, will be an algebraical isomorphism between $B$ and $A$. Therefore algebra $B$ can be viewed as the same algebra $A$, only with a different topology. This topology is called free regarding to $X$. From item 2) of Lemma 3 of present paper it follows that the identical mapping of algebra $A$ with free topology regarding to $X$ on algebra $A$ with other topology will be continuous. This property is characteristic for free topology.

4 Topological and paratopological quasigroups

The theory of quasigroups and loops emerged in 1920’s – 1930’s ([19], [52], [80], [?]) as a specification of the notion of groupoid.

A quasigroup (respect. right quasigroup or left quasigroup) is a non-empty set $Q$ together with a binary operation $Q \times Q \rightarrow Q; (x, y) \rightarrow xy$ such that the equations $ax = c$ and $yb = c$ (respect. $ax = b$ or $xa = b$) have unique solutions.

A loop (respect. right loop) $L$ is a quasigroup with a base point, or distinguished element, $e \in L$ (respect. $f \in L$) satisfying the equations $ea = ae = a$ (respect. $af = a$) for all $a \in L$.

The notions of quasigroups and loops are defined with the help of only one binary groupoid operation. These definitions of quasigroups and loops will be called groupoid definitions and the quasigroups and loops defined in such a way will be temporarily called GD-quasigroups and respectively, GD-loops.

In order to apply the universal algebraic techniques, one must use the universal algebraic description of quasigroups as algebras $(Q, \cdot, /, \\backslash)$ with three binary operations, multiplication $\cdot$, left division $\\backslash$, and right division $/$, satisfying the identities

$$\frac{x \cdot y}{y} = x, \quad \frac{x}{y} \cdot y = x, \quad x \backslash (x \cdot y) = y, \quad x \cdot (x \\backslash y) = y. \quad (20)$$

Quasigroups defined by “equations” in this way are sometimes referred to as equasigroups, to distinguish this point of view from the one often taken that quasigroups $(Q, \cdot)$ are a special kind of groupoid. If one adds a nullary (=constant) operation $L \rightarrow L; x \rightarrow e, ae = a = ea \quad \forall a \in L$, to the operations of a equasigroup $(L, \cdot, /, \\backslash)$ then $(L, \cdot, /, \\backslash, e)$ is called eloop.
A non-empty set $G$ together with two binary operations $(\cdot), (\setminus)$ and nullary operation $f$ will be called right eloop with right unit $f \in G$ if

$$x \setminus (x \cdot y) = y, \quad x \cdot (x \setminus y) = y, \quad xf = x \quad (21)$$

for all $x, y \in G$.

Similarly, a non-empty set $G$ together with two binary operations $(\cdot), (/)$ and nullary operation $l$ will be called left eloop with left unit $l \in G$ if

$$(x \cdot y)/y = x, \quad (x/y) \cdot y = x, \quad lx = x \quad (22)$$

for all $x, y \in G$.

**Proposition 9.** Every equasigroup (respect. eloop either right eloop, or left eloop) $Q$ is a GD-quasigroup (respect. GD-loop either right GD-loop, or left GD-loop).

**Proof.** For equasigroup $Q$ we consider the equation $ax = b$, where $a, b \in Q$. Then by $(20)$ $a \setminus (ax) = a \setminus b, x = a \setminus b$. If $ab = ac$, where $a, b, c \in Q$, then $a \setminus (ab) = a \setminus (ac), b = c$. Hence the equation $ax = b$ has an unique solution. Similarly it is proved that the equation $xa = b$ has an unique solution. This completes the proof of Proposition 9.

Recall that a topological group is a group $G$ with such a (Hausdorff) topology that the binary operations $G \times G \rightarrow G, (x, y) \rightarrow xy, x^{-1}, xy^{-1}$ are continuous. A paratopological group $G$ is a group $G$ with such a topology that the product maps of $G \times G$ into $G, (x, y) \rightarrow xy$ are jointly continuous.

Now similarly to the definition of topological algebraic system, and also similarly to the notions of topological group and paratopological group we define the following.

A topological quasigroup (respect. topological right quasigroup or topological left quasigroup) is a equasigroup (respect. right eloop or left eloop) $Q$ together with such a topology on $Q$ that the binary operations $Q \times Q \rightarrow Q; (x, y) \rightarrow xy, x/y, x \setminus y$ (respect. $xy, x \setminus y$ or $xy, /$) are continuous. A topological loop (respect. topological right loop or topological left loop) is a eloop (respect. right eloop or left eloop) $L$ which, in addition, is a topological quasigroup (respect. topological right quasigroup or topological left quasigroup).

A paratopological quasigroup (respect. paratopological loop either paratopological right quasigroup, or paratopological left quasigroup, or paratopological right loop, or paratopological left loop) will be called a GD-quasigroup (respect. GD-loop either right GD-quasigroup, or left GD-quasigroup, or right
GD-loop, or left GD-loop) $Q$ together with such a topology on $Q$ that the binary operation $Q \times Q \to Q; (x, y) \to xy$ is continuous.

**Remark 5.** The proofs of the following assertions almost literally repeat the proofs of Theorems 7, 8, 9 from [70].

a). The space of topological right quasigroup (respect. topological left quasigroup) is regular (compare with [79, Proposition IX.1.15]).

b). If a topological right loop (respect. topological left loop) $L$ is generated by elements of some connected neighborhood of right unit (respect. left unit), then the space $L$ is connected.

c) Any convex topological right loop (respect. topological left loop) is generated by elements of any neighborhood of right unit (respect. left unit).

**Lemma 4.** If $A$ is an open set of a topological right quasigroup $(Q, \cdot, \backslash)$ (respect. topological left quasigroup $(Q, \cdot, /)$) and $g \in Q$, then the sets $gA$, $g \backslash A$ (respect. $Ag$, $A/g$) are open.

**Proof.** We prove only the first assertion, the others are proved similarly. If $b = ga$, $a \in A$, then $g \backslash b = g \backslash (ga)$, $a = g \backslash b$ by (20). The set $A$ is open and the operation $(\backslash)$ is continuous by definition of topological right quasigroup. Then there exists such a neighborhood $U$ of element $b$ that $g \backslash U \subseteq A$. By (20) $g \cdot (g \backslash U) \subseteq g \cdot A$, $U \subseteq g$. Hence for any element $b \in gA$ the existence of such an open set $U$ that $b \in U \subseteq gA$. Then from the definition of open set it follows that the set $gA$ is open, as required.

If $S$ is a set with a binary operation $S \times S \to S; (x, y) \to x \ast y$, then for $a \in S$, the mappings $L_{(a)}(a), R_{(a)}(a) : S \to S$, defined by $L_{(a)}(a)x = a \ast x$ and $R_{(a)}(a)x = x \ast a$, are called the left translations, respectively, right translations of $S$.

Let $(Q, \cdot, \backslash)$ be a topological right quasigroup, $a, x \in Q$ and $E$ denote the identical mapping on $Q$. From the definitions (20) (or (21)) we get $L_{(\cdot)}(a)L_{(\backslash)}(a)x = x$, $L_{(\backslash)}(a)L_{(\cdot)}(a)x = x$, $L_{(\cdot)}(a)L_{(\backslash)}(a) = E$, $L_{(\backslash)}(a)L_{(\cdot)}(a) = E$. Hence the translations $L_{(\cdot)}(a), L_{(\backslash)}(a)$ are reversible. From Proposition 9 it follows that the translations $L_{(\cdot)}(a), L_{(\backslash)}(a)$ are one-to-one. Write the last equalities as: $L_{(\cdot)}^{-1}(a) = L_{(\backslash)}(a), L_{(\backslash)}^{-1}(a) = L_{(\cdot)}(a)$. The operations $(\cdot), (\backslash)$, which define a topological right quasigroup, are continuous. Every translation of an algebraic systems is a continuous mapping [70]. Then from the last equalities it follows that the translations $L_{(\cdot)}(a), L_{(\backslash)}(a)$ are homeomorphic. From Lemma 4 it follows that the translations $L_{(\cdot)}(a), L_{(\backslash)}(a)$ are open.
According to definition ([79]), let $LM(Q)$ denote the left multiplication group of the right quasigroup $(Q,\cdot,\setminus)$, i.e. the group generated by all translations $L_{(\cdot)}(a)$ of $(Q,\cdot,\setminus)$. By equality $L_{(\cdot)}^{-1}(a) = L_{(\setminus)}(a)$ the group $LM(Q)$ coincides with the semigroup generated by all translations $L_{(\cdot)}(a)$, $L_{(\setminus)}(a)$. According to [70] an element of semigroup $LM(Q)$ will be called elementary left translation of right quasigroup $(Q,\cdot,\setminus)$. The product of homogeneous or open mappings is a homogeneous or open mapping. Hence we proved

**Theorem 9.** The left multiplication group $LM(Q)$ of a right equasigroup $(Q,\cdot,\setminus)$ is transitive on $Q$ and any elementary left translation in $LM(Q)$ is homeomorphic and open.

Using definition (20) or (22) it can be proved

**Theorem 10.** The right multiplication group $RM(Q)$ of a left equasigroup $(Q,\cdot,/)$ is transitive on $Q$ and any elementary right translation in $RM(Q)$ is homeomorphic and open.

A topological space is called homogeneous iff its group of homeomorphisms operates transitively, i.e. iff for any pair $(x,y)$ of points there is a homeomorphism $\varphi$ of the space with $\varphi(x) = y$ [79]. Then from Theorems 9, 10 it follows

**Corollary 19.** The underlying space of any topological right quasigroup (respect. topological left quasigroup) is homogeneous.

From Theorems 2, 9 and 10 it follows

**Corollary 20.** There exists derived ternary operations $\alpha(x,y,z)$, $\beta(x,y,z)$ of a equasigroup $(Q,\cdot,/\setminus)$ with respect to which the set $Q$ is a biternary system.

We consider a biternary system with ternary operations $\alpha, \beta$. From definition of ternary system it follows that $x = \beta(\alpha(x, x, y), x, y) = \beta(y, x, y)$, $x = \beta(y, x, y)$. We put $\Psi(x, y, z) = \beta((\alpha(x, y, a), z, a)$. Using the definition of ternary system and $x = \beta(y, x, y)$ it is easy to see that the polynomial $\Psi(x, y, z)$ satisfies the identities (5).

We also mention that for equasigroups $(Q,\cdot,/,\setminus)$ the identities (5) are satisfied by polynomial $W(x, y, z) = (x(a/y))(a/z)$, where $a$ is a fixed element in $Q$, (Mal’cev), and by polynomial $W(x, y, z) = x \cdot (y/z)$ (respect. $W(x, y, z) = (x/y)\cdot z$) for right loops $(L,\cdot,\setminus, f)$ (respect. left loops $(L,\cdot,\setminus, l)$)
Then from Theorem 1 it follows

**Proposition 10.** Let \((Q, \cdot, \backslash, /)\) (respect. \((Q, \cdot, \backslash, f)\) or \((Q, \cdot, /, l)\)) be a quasigroup (respect. right eloop or left eloop). Then the polynomial \(W(x, y, z) = (x(a/y))\backslash(a/z)\), where \(a\) is a fixed element in \(Q\) (respect. \(W(x, y, z) = x \cdot (y\backslash z)\) or \(W(x, y, z) = (x/y) \cdot z\)) of quasigroup \((Q, \cdot, \backslash, /)\) (respect. right eloops \((Q, \cdot, \backslash, f)\) or left eloops \((Q, \cdot, /, l)\)) satisfies the identities (5) and the congruences of quasigroups (respect. right eloops or left eloops) are permutable.

According to (20) - (22) the homomorphic image of a left quasigroup (respect. right quasigroup either left eloop, or right eloop) is a left quasigroup (respect. right quasigroup either left eloop, or right eloop).

**Corollary 21.** Let \(\varphi : Q \rightarrow \overline{Q}\) be a homomorphism of a topological quasigroup (respect. topological left loop or topological right loop), let \(\theta\) be the congruence corresponding to the homomorphism \(\varphi\) and let for element \(a \in Q\)

\[K_a = \{x \in Q | x \theta a\}\]

denote the class of congruence \(\theta\). Then:

1) \(\varphi\) is an open (closed) mapping;

2) for every open (closed) set \(A \subseteq Q\) the union of all congruence classes intersecting \(A\) is open (closed) in \(Q\);

3) if \(\varphi\) is a homomorphism "on"\,\, then \(\varphi\) is a quotient homomorphism;

4) the isomorphism \(Q/\theta \rightarrow \overline{Q}\) is a homeomorphism.

6) if \(a \cdot b = c\) (or \(a \backslash b = c\)) for elements \(a, b, c\) in topological right quasigroup \((Q, \cdot, \backslash)\) then \(a \cdot K_b = K_c\) (or \(a \backslash K_b = K_c\)), \(K_a \cdot b \subseteq K_c\) (or \(K_a \backslash b \subseteq K_c\)) and the spaces \(K_b, K_c\) are homeomorphic.

**Proof.** The items 1) - 4) follow from Proposition 4, Theorem 10 in [70] and Propositions 2.4.3, 2.4.4 in [60].

5). We prove the assertion only for operation \((\cdot)\) as for \((\backslash)\) it is analogical. Clearly, if \(a \in K_b\) then \(K_a = K_b\) and conversely. Then, obviously, \(K_a \cdot b \subseteq K_c\).

We prove that \(a \cdot K_b = K_c\). Indeed, if \(y \in K_{ab}\), then \(y \theta ab\). Let \(y = az\), then by (21) \(az \theta ab, a \backslash (az) \theta a \backslash (ab), z \theta b\) and \(y = az \in a K_b\). Hence \(K_{ab} \subseteq a K_b\). Let now \(x \in a K_b\). Then \(x = ab_1\), where \(b_1 \in K_b\). Hence \(b_1 \theta b, ab_1 \theta ab, x \in K_{ab}, a K_b \subseteq K_{ab}\). Consequently, \(a K_b = K_{ab}\), as required.

**Corollary 22.** Let \(\varphi : Q \rightarrow \overline{Q}\) be a continuous homomorphism of a locally compact left quasigroups (respect. locally compact right quasigroups) \(Q, \overline{Q}\), satisfying the second axiom of countability, and let \(\theta\) be the congruence corresponding to the homomorphism \(\varphi\). Then the similar assertions to 1) -
5) of Corollary 21 hold.

It follows from Theorems 9, 10, Theorem 12 in [70] and [60, Propositions 2.4.3, 2.4.4].

**Proposition 11.** The underlying space of any topological quasigroup is rectifiable.

**Proof.** Let \((Q, \cdot, \backslash, \rangle)\) be a topological quasigroup. The right translations \(R_{\langle}(x), R_{\rangle}(x), x \in Q\), are homeomorphisms of the space \(Q\) according to Theorem 10. We use (20). For \(x, y \in Q\) we have \(R_{\langle}^{-1}(x) = R_{\rangle}(x), R_{\rangle}((x \backslash y)\langle x) = y\). Let \(a\) be a fixed element in \(Q\). We put \(\Phi(x, y) = (x, R_{\langle}(x \backslash a)(y)), \Phi^{-1}(x, y) = (x, R_{\rangle}(x \backslash a)(y))\). It is easy to see that \(\Phi\) is a homeomorphism of \(Q \times Q\) onto itself and \(\Phi(x, x) = (x, a)\). Then from the definition it follows that the space \(Q\) is rectifiable, as required.

For compact quasigroups the Proposition 11 is proved in [84, pag. 1112].

The definitions of Mal’cev space and of Mal’cev operation can be found in the assertion (8) below. In [83] (see, also, [84]) it is proved that compact Mal’cev spaces are Dugundji (Theorem 1) and if \(X\) is a countable Mal’cev space, then the Stone-\v{C}ech compactification \(\beta X\) of \(X\) is Dugundji (Theorem 2). It is known that if \(\beta X\) is Dugundji, then \(X\) is pseudocompact. Conversely, if \(X\) is a pseudocompact rectifiable space and satisfies the Suslin property, then \(\beta X\) is compact Dugundji (Theorem 3). Any countably Mal’cev space satisfies the Suslin property (Theorem 8 in [84]).

Except enumerated facts the papers [83], [84], [57] contain many other results related to rectifiable spaces, Mal’cev spaces which with the help of the Theorem 10 and the Propositions 10, 12 are easy transferred on quasigroup structures. Particularly, from the listed results and [57, Corollary 2.2, Theorem 3.2] it follows

**Proposition 12.** Let \(Q\) be the topological space of a topological quasigroup either topological left loop, or topological right loop. Then:

1) if \(Q\) is a compact space then \(Q\) is Dugundji;
2) if \(Q\) is a countably space, then the Stone-\v{C}ech compactification \(\beta Q\) of \(Q\) is Dugundji;
3) if \(Q\) is a pseudocompact space and satisfies the Suslin property, then \(\beta X\) is compact Dugundji;
4) the space \(Q\) is regular;
5) if \(Q\) is a countable space, then \(Q\) is metrizable.

In [79, Proposition IX.1.15 (Salzmann 1757)] it is proved that the under-
lying space of a topological quasigroup is regular.

**Remark 6.** In [84, Corollary 9] it is proved that any compact quasigroup is Dugundji (see item 1) of Proposition 11). It is also marked that this result is due to Choban, published in: Choban M. M. The structure of locally compact algebras//Bacu internat. topol. conf. (3—9 of October 1787 á.): Tez. Ch. 2. Bacu: Communist, 1787. P. 334. 56. In [84, Corollary 6] it is proved that any compact Mal’cev space is Dugundji. It is marked (pag. 1094, 1109) that this result is due to Choban, but is not indicated where they published.

Further, in [57, pag. 110] it is mentioned that the Theorem 3.2 (stated above) is a generalization of Choban’s theorem, who proved under the same conditions that $X$ is a Moore space. Again, it is not mentioned where this result is published. Other cases, where in detriment to authors results are unmerited assigned to Choban, will be analyzed in assertion (9). The item 4) of Proposition 12 is similar to [70, Theorem 7].

According to Theorems 9, 10 the left and right translations in any topological quasigroup are homeomorphisms. But there are loops defined on topological spaces in which multiplication is jointly continuous and translations fail to be homeomorphisms. Hence such loops are paratopological loops, and not topological loops.

**Example.** ([79, Example IX.1.5]). Let $L$ be the real Hilbert space $l^2$ of all square summate real sequences and define for $x = (x_n)_{n \in \mathbb{N}}$ a product

$$xy = (((x - n + y_n)/(1 + (n - 1)|x_1||y_1|)))_{n \in \mathbb{N}}.$$ 

Then $L$ is a commutative loop and $L \times L \to L; \quad (x, y) \mapsto xy$ is continuous, but translation with $(1, 0, 0, \ldots)$ is not open. Hence $L$ is not a topological loop.

If $L_1$ is the subset of all sequences $x = (x_n)_{n \in \mathbb{N}}$ with $x_n = 0$ for $n > 1$ and $L_2$ is the subset of all $x$ with $x_1 = 0$, then $L_2$ is a normal subloop with $L/L_2 \cong L_1 \cong R$, and $L_2$ is an abelian topological group isomorphic to the additive group of $l^2$. Furthermore, $L = L_1L_2$ and $L_1 \cap L_2 = \{0\}$.

**Remark 7.** In some works, particularly [84], pag. 1095) the notion of topological quasigroup is defined as a space $X$ with such continuous operation $(x, y) \to xy$ that the equations $ax = b, ya = b$ have unique solutions for any $a, b \in X$. According to [70] this definition is not correct. This is the definition of paratopological quasigroup. In [84] there are examined only compact paratopological quasigroups. According to the following assertion ([79], Proposition IX.1.6) this does not influence the truth of other results.
Let \( (Q, \cdot) \) be a paratopological quasigroup. Then \( Q \) is a topological quasigroup if at least one of the following conditions is satisfied: (i) the space \( Q \) is compact, (ii) the space \( Q \) is locally compact and locally connected, and the mappings \( Q \to Q; x \mapsto ax, xa \) are open for all \( q \in Q \).

**Lemma 5.** Let \( (Q, \cdot, \backslash, /) \) be a paratopological quasigroup and let the set \( aH \) (respect. \( Ha \)) be open for every open set \( H \) of \( Q \) and every \( a \in Q \). Then the operation \( (\backslash) \) (respect. \( (/) \)) is continuous.

**Proof.** Let \( M \) be an arbitrary open set of \( Q \). Then the set \( K = \bigcup_{m \in M} (m \cdot H) \) is open. Let \( k \in K, m \in M, h \in H \), i.e. \( k = m \cdot h \). According to (20) we have \( m \backslash k = m \backslash (m \cdot h) = h, k/h = (m \cdot h)/h = m \), i.e. \( m \backslash k = h, k/h = m \). For the first equality we have: for all open set \( H \) such an open sets \( M, K \) exists that \( M \backslash K = H \). This means that the operation \( (\backslash) \) is continuous. Similarly, the operation \( (/) \) is continuous, as required.

**Corollary 23.** If \( G \) is a paratopological group, then from conditions \( a \in G, H \) is an open set it does not result that the sets \( a \cdot H, Ha \) are open.

**Proposition 13.** Let \( \mathcal{P}K \) denote the class of all paratopological loops. Then:

1) the class \( \mathcal{P}K \) contains paratopological loops which are not topological algebraic systems;

2) the class \( \mathcal{P}K \) is not a variety: in such a case it is not possible to consider free paratopological loops of \( \mathcal{P}K \);

3) the class \( \mathcal{P}K \) contains paratopological loops with non-permutable congruences;

4) every paratopological loop admits such a structure of ternary operation \([x, y, z]\) that it satisfies the identities \([x, x, y] = y, [y, x, x] = y\). But in the class \( \mathcal{P}K \) there are paratopological loops \( (L, \cdot) \), which do not contain polynomials \( \Psi(x, y, z) \), satisfying identities (5), i.e. the GD-loops \( (L, \cdot) \) do not satisfy the Theorem 1;

5) on paratopological loops \( (L, \cdot) \) from item 3) it is not possible to define a structure of biternary algebra.

**Proof.** 1) The definition of GD-loop does not satisfy the definition of algebraic system.

Further, we use GD-loops constructed in [49], [82]. But this GD-loops may be considered as paratopological loops (for example, with discrete topology or anti-discrete topology).
2) By Birkhoff Theorem, a class of algebraic systems is a variety if and only if this class is closed with respect to taking subsystems, direct products and homomorphic images (see, the Proposition 2). [49] proves that for any groupoid with division (not necessary eloop) $M$ there exists a such $GD$-loop $F$ homomorphic to $M$ that the corresponding congruence is permutable with all congruences on $F$. Hence, the class $\mathcal{PK}$ is not closed with respect to homomorphic images. Consequently, the class $\mathcal{PK}$ is not a variety. Further, it is known that any variety is characterized by its free objects.

3) [82] offers an example of an $GD$-quasigroup with a pair of non-permutable congruences and this method can be used to construct a similar example for $GD$-loops [49].

4) Let $(Q, \cdot, e)$ be a right $GD$-loop with the right unit $e$. Then the equality $a \cdot x = b$ has an unique solution $x_0$ for any $a, b \in Q$. We denote $A(a, b) = x_0$. Then $x \cdot A(x, y) = y$ for any $x, y \in Q$. Further, from $x \cdot A(x, x) = x$, $x \cdot e = x$ it follows that $A(x, x) = e$ for all $x \in Q$. We put $[x, y, z] = x \cdot A(y, z)$. Then $[x, x, y] = y, y, x, x] = y \cdot A(x, x) = y \cdot e = y$.

In the proof of Theorem 1 the free algebras of varieties are used essentially. But it is impossible to prove the item 4) directly using Theorem 1 since the item 2) does not permit it.

Let $(L, \cdot)$ be the $GD$-loop satisfying the item 3). We assume that the loop $(L, \cdot)$ contains a polynomials $\Psi(x, y, z)$, satisfying identities $\Psi(x, x, z) = z$, $\Psi(x, z, z) = z$. Let $\theta_1, \theta_2$ be such congruences of $(L, \cdot)$ that $\theta_1 \theta_2 \neq \theta_2 \theta_1$. Then such elements $a, b \in L$ exist that $a \equiv b(\theta_1 \theta_2)$, but $a \not\equiv b(\theta_2 \theta_1)$. From $a \theta_1 \theta_2 b$ it follows that $a \theta_1 c, c \theta_2 b$ for some $c \in L$. Then

$$\Psi(a, c, b) \equiv \Psi(a, a, b)(\theta_1), \quad \Psi(a, c, b) \equiv \Psi(a, c, c)(\theta_2)$$

or

$$\Psi(a, c, b) \equiv b(\theta_1), \quad \Psi(a, c, b) \equiv a(\theta_2).$$

From here it follows that $\theta_1 \theta_2 = \theta_2 \theta_1$. We get a contradiction, as $\theta_1 \theta_2 \neq \theta_2 \theta_1$. Hence item 4) holds.

5) Let us define a ternary system $(L, \alpha, \beta)$ on $GD$-loop $(L, \cdot)$ from item 3). Then the polynomial $\Psi(x, y, z) = \beta(\alpha(x, y, a), z, a)$, where $a$ is a fixed element in $L$, satisfies the identities (5). Contradiction with item 3). This completes the proof of Proposition 13.

Now, let us denote by $\mathcal{TK}$ the class of all topological quasigroups (respect. topological right quasigroups either topological left quasigroups, or topological loops, or topological right loops, or topological left loops) and denote
by \( \mathcal{PK} \) the class of all paratopological quasigroups (respect. paratopological right quasigroups or paratopological left quasigroups, or paratopological loops, or paratopological right loops, or paratopological left loops). The translations with respect to the basic operations of topological algebraic systems are continuous. Clearly, the Proposition 13 holds for any class \( \mathcal{PK} \). Then from Propositions 17, 13 and Example it follows.

**Corollary 24.** The inclusion \( \mathcal{TK} \subset \mathcal{PK} \) is strong.

## 5 Correlation of rectifiable spaces and topological right loops

We shall use the notion of rectifiable space from [57], given at the beginning of our work. Other information about rectifiable space are given in the next section in the analyzes of paper [27], [28], [4].

Let \( A \) be a topological space. We denote by \( \pi_1, \pi_2 : A \times A \mapsto A \) the projections of first and second coordinates of Cartesian product \( A \times A \). A mapping \( \Phi : A \times A \mapsto A \times A \) will be called \( l \)-mapping if \( \pi_2 \circ \Phi = \pi_2 \) and will called \( r \)-mapping if \( \pi_1 \circ \Phi = \pi_1 \). Clearly, if \( \Phi : A \times A \rightarrow A \times A \) is a homeomorphism (respect. one-to-one mapping) of \( A \times A \) onto itself, then the inverse mapping \( \Phi^{-1} \) is a homeomorphism (respect. one-to-one mapping) of \( A \times A \) onto itself. We will follow the scheme of the proof of Proposition 2.1 from [57].

**Theorem 11.** Let \( A \) be a topological space and let \( \Phi, \Psi : A \times A \mapsto A \times A \) be a surjective homeomorphisms. We denote \((\cdot) = \pi_2 \circ \Psi, (\ast) = \pi_1 \circ \Phi, (\backslash) = \pi_2 \circ \Psi^{-1}, (\div) = \pi_1 \circ \Phi^{-1} \). Then:

1) a space \( A \) is a topological left quasigroup with respect to multiplication \((\cdot)\) and right division \((\div)\) iff \( \Phi \) is a \( l \)-mapping;

2) a space \( A \) is a topological right quasigroup with respect to multiplication \((\cdot)\) and left division \((\backslash)\) iff \( \Psi \) is a \( r \)-mapping;

3) a space \( A \) is a topological quasigroup with respect to multiplication \((\cdot)\), left division \((\backslash)\) and right division \((\div)\) iff \( \Phi \) is a \( l \)-mapping, \( \Psi \) is a \( r \)-mapping and \( \pi_2 \circ \Psi^{-1} = \pi_1 \circ \Phi^{-1} \);

4) a topological left quasigroup \((A, \div)\), defined by homeomorphism \( \Phi \), is a topological left loop with a left unit element \( f \in A \) if for any \( x \in A \) the equality \( \Phi^{-1}(x, x) = (f, x) \) is fulfilled;

5) a topological right quasigroup \((A, \backslash)\), defined by homeomorphism \( \Psi \), is
a topological right loop with a right unit element \( e \in A \) if for any \( x \in A \) the equality \( \Psi^{-1}(x, x) = (x, e) \) is fulfilled;

6) a topological quasigroup \((A, /, \setminus)\), defined by homeomorphisms \( \Phi, \Psi \), is a topological loop with a unit element \( e \in A \) if for any \( x \in A \) the equalities \( \Psi^{-1}(x, x) = (x, e) \) and \( \Phi^{-1}(x, x) = (e, x) \) are fulfilled.

**Proof.** 1, 2). The mappings \( \Phi, \Phi^{-1}, \Psi, \Psi^{-1} \) on Cartesian product are homeomorphisms. Then they are continuous. Particularly, according to Remark 1 the projections to the coordinates \((\cdot) = \pi_2 \circ \Psi, (\ast) = \pi_1 \circ \Phi, (\setminus) = \pi_2 \circ \Psi^{-1}, (/) = \pi_1 \circ \Phi^{-1}\), are continuous.

Let \( \pi_2 \circ \Phi = \pi_2 \) for item 1) or \( \pi_1 \circ \Psi = \pi_1 \) for item 2). Then \((y \ast x)/x) = \pi_1 \circ \Phi^{-1}(\pi_1 \circ \Phi(y, x), x) = \pi_1 \circ \Phi^{-1} \circ \Phi(y, x) = \pi_1(y, x) = y\) and similarly for the other three identities \((y/x) \ast x = y, x \cdot (x\setminus y) = y, x \setminus (x \cdot y) = y\). Hence by definitions, \((A, \ast, /)\) is a topological left quasigroup and \((A, \cdot, \setminus)\) is a topological right quasigroup.

Conversely, let \((A, \ast, /)\) be a topological left quasigroup, and \((A, \cdot, \setminus)\) be a topological right quasigroup. We put \( \Phi(y, x) = (y \ast x, x) \Phi^{-1}(y, x) = (y/x, x)\), for left quasigroup and \( \Psi(x, y) = (x, x \cdot y), \Psi^{-1}(x, y) = (x, x\setminus y)\) for right quasigroup. Further we shall consider only the right quasigroup as for the left quasigroup reasonings are similar.

We have \( \Psi \circ \Psi^{-1}(x, y) = \Psi(x, x \setminus y) = (x, x(x\setminus y)) = (x, y)\) and \( \Psi^{-1} \circ \Psi(x, y) = (x, y)\). Hence \( \Psi \circ \Psi^{-1} = \Psi^{-1} \circ \Psi = E\), where \( E \) is the identical mapping. Then from here and the definition of right quasigroup it follows that \( \Psi, \Psi^{-1} \) are an one-to-one mappings of \( A \times A \) onto itself.

For any fixed element \( x \in A \) we have \( \Psi(\{x\} \times A) = \{x\} \times A\). By hypothesis the operation \((\cdot)\) is continuous, hence the mapping \( y \to \pi \circ \Psi(x, y)\) is continuous. Then from the definition of topology of Cartesian product it follows that the mapping \( \Psi : A \times A \to A \times A\) is continuous. It is similarly proved that the mappings \( \Psi^{-1}, \Phi, \Phi^{-1}\) are continuous.

3) From \( \pi_2 \circ \Psi = \pi_1 \circ \Phi \) it follows that \((\cdot) = (\ast)\). Then the item 3) follows from items 1, 2).

4), 5), 6). From \((/\setminus) = \pi_1 \circ \Phi^{-1}\) and \( \Phi^{-1}(x, x) = (f, x)\) it follows that \( x/x = f \). Then by (3) \((x/x) \ast x = f \ast x, x = f \ast x\). Hence \( f \) is a left unit element of topological left loop \((A, \ast, /)\). Similarly it is proved that from \((\setminus) = \pi_2 \circ \Psi^{-1}\) and \( \Psi^{-1}(x, x) = (x, e)\) it follows that \( e \) is a right unit element of right loop \((A, \cdot, \setminus)\). This completes the proof of Theorem 11.

The topological structures, considered in Theorem 11, are topological algebraic systems. Hence for its investigation the powerful methods of theory
of topological algebraic systems [70], [71] will be applied, as well as of theory
of topological quasigroups and loops [79]. However, if in Theorem 11 the
condition that the mappings Φ, Ψ are homeomorphisms is weakened to the
condition that the mappings Φ, Ψ are only continuous mappings, then the
obtained topological structure is no longer a topological algebraic system.

**Proposition 14.** Let \( A \) be a topological space and let \( \Phi, \Psi : A \times A \mapsto A \times A \) be continuous one-to-one surjective mappings. We denote \((\cdot) = \pi_2 \circ \Psi, (\ast) = \pi_1 \circ \Phi, (\backslash) = \pi_2 \circ \Psi^{-1}, (\div) = \pi_1 \circ \Phi^{-1}\). Then:

1) \( A \) is a paratopological left quasigroup with respect to multiplication \((\cdot)\) and right division \((\div)\) iff \( \Phi \) is a \( l \)-mapping;

2) \( A \) is a paratopological right quasigroup with respect to multiplication \((\cdot)\) and left division \((\backslash)\) iff \( \Psi \) is a \( r \)-mapping;

3) \( A \) is a paratopological quasigroup with respect to multiplication \((\cdot)\), left division \((\backslash)\) and right division \((\div)\) iff \( \Phi \) is a \( l \)-mapping, \( \Psi \) is a \( r \)-mapping and \( \pi_2 \circ \Psi^{-1} = \pi_1 \circ \Phi^{-1} \);

4) a paratopological left quasigroup \((A, \backslash)\), defined by homeomorphism \( \Phi \), is a paratopological left loop with a left unit element \( f \in A \) if for any \( x \in A \) the equality \( \Phi^{-1}(x, x) = (f, x) \) is fulfilled;

5) a paratopological right quasigroup \((A, \backslash)\), defined by homeomorphism \( \Psi \), is a paratopological right loop with right unit element \( e \in A \) if for any \( x \in A \) the equality \( \Psi^{-1}(x, x) = (x, e) \) is fulfilled;

6) a paratopological quasigroup \((A, /, \backslash)\), defined by homeomorphisms \( \Phi, \Psi \), is a paratopological loop with a unit element \( e \in A \) if for any \( x \in A \) the equalities \( \Psi^{-1}(x, x) = (x, e) \) and \( \Phi^{-1}(x, x) = (e, x) \) are fulfilled.

**Proof.** The proof of Proposition 14 is contained in the proof of Theorem 11.

**Remark 8.** The topological structures considered in item 5) of Theorem 11 coincide with the notion of rectifiable space, introduced in [83], [57]. This structure also coincides with the notion of homogeneous space, introduced in [27], [28].

Let \((X, \mathcal{T})\) be a topological space. A subset of \( X \) is \( \mathcal{T} \)-open if it is an union of elements of \( \mathcal{T} \). We define on the set \( Y \) the topology \( \mathcal{A} \) induced by the topology \( \mathcal{T} \) as the family of intersections of elements of \( \mathcal{T} \) with the set \( Y \). The space \((Y, \mathcal{A})\) is called subspace of space \((X, \mathcal{T})\). Hence the set \( U \) belongs to the induced topology \( \mathcal{A} \) if and only if \( U = V \cap Y \) for some \( \mathcal{T} \)-open set \( V \). If the set \( Y \) is \( \mathcal{T} \)-open then every \( \mathcal{A} \)-open set of \( Y \) is \( \mathcal{T} \)-open as an intersection of \( \mathcal{T} \)-open
Lemma 6. Let \((Y, \mathcal{A})\) be a subspace of a topological space \((X, \mathcal{T})\) and let the set \(Y\) be \(\mathcal{T}\)-open. Then a subset of \(Y\) is \(\mathcal{T}\)-open if and only if it is \(\mathcal{A}\)-open.

Corollary 25. Let a subspace \(A\) of a topological space \((X, \mathcal{T})\) be a rectifiable space. Then \(A\) is open.

Proof. According to item 5) of Theorem 11 the rectifiable space \(A\) admits a structure of topological right loop with right unit \((A, \cdot, \backslash, e)\) with topology \(A\) induced by topology \(\mathcal{T}\).

Let \(U(e)\) be a neighborhood of the unit \(e\) in \((X, \mathcal{T})\). By Lemma 6 \(U'(e) = U(e) \cap A\) is open in \(A\). Let \(a \in A\). By Lemma 4 \(aU'(e)\) is open in \((A, \mathcal{A})\). Then by Lemma 6 \(aU'(e)\) is open in \((A, \mathcal{T})\). As \(e \in U'(e)\), then \(a \in U'(e)\). Hence \(a \in aU'(e) \subseteq X\). This means that \(A\) is open, as required.

Proposition 15. If a non-empty subspace \(A\) of a compact space \(X\) is a rectifiable space, then \(A\) is a compact space. Moreover, the set \(A\) is open, regular, Hausdorff and, consequently, normal.

Proof. By Corollary 25 the set \(A\) is open. Then the set \(B = X \setminus A\) is closed and \(A \cap B = \emptyset\). Any closed subspace of a compact space is a compact space [60]. Then \(B\) is a compact space. Let \(\{U_i|i = 1, \ldots, r\}\) be a finite open cover of \(B\). According to the definition of induced topology on a subspace, for any \(i\) let \(U_i = B \cap U'_i\), where \(U'_i\) is an open set of \(X\). Then \(A \cup \{U'_i|i = 1, \ldots, r\}\) is a open cover of \(X\).

Let \(\{V_j|j \in J\}\) be a cover of \(A\). As \(A\) is an open set then by Lemma 6 \(V_j\) is an open set of \(X\) for any \(j \in J\). Hence \(\{V_j|j \in J\} \cup \{U'_i|i = 1, \ldots, r\}\) is an open cover of \(X\). According to the definition of compact space, from open cover \(\{V_j|j \in J\} \cup \{U'_i|i = 1, \ldots, r\}\) of \(X\) we pull out a finite open cover \(\{V_{j_i}|j_i = 1, \ldots, s\} \cup \{U'_i|i = 1, \ldots, r\}\). As \(A \cap B = \emptyset\), then \(\{V_{j_i}|j_i = 1, \ldots, s\}\) is a finite open cover for \(A\). Consequently, \(A\) is a compact space.

Now we prove that the space \(A\) is regular. Indeed, every rectifiable space \(X\) is regular [57] (see, also, [70, Theorem 7]). Then, obviously, \(X\) is a Hausdorff space. Every compact (in other terminology bicompact) Hausdorff space is normal. This completes the proof of Proposition 15.

From Proposition 15 it follows.

Corollary 26. Let a subspace \(A\) of a compact space \(X\) admit a structure of topological right loop. Then the set \(A\) is compact, normal, regular, Hausdorff and open in \(X\).
6 Algebraical analysis of some papers

I have acquired minimal knowledge of topology through my scientific activity. However, I have decided to publish this paper with the purpose to correct the mistakes found out in papers [4], [23], published in the journal: Buletinul Academiei de Științe a Republicii Moldova, Matematica. In paper [4] are available references on papers [27], [28]: [4] ⇒ [27], [28]. Further, [27], [28] ⇒ [29]. Initially, I planned to analyze papers [4], [27], [28], [29] (the analysis is presented below). But my plan changed cardinaliy after I saw the materials of the 20th conference of applied and industrial mathematics, dedicated to Academician Mitrofan M. Ciobanu, 2012: www.romai.ro/conferinte. But my opinion was affected the most by papers [4], [27], [28], [29] and other works, for which references are available. As a result, I saw an awful reality, a catastrophe. It appeared that the works of Academician Ciobanu M. M. and his followers, specifically monographs [53], [39], [40], [21], [41], [25] and dissertation [22], [58], [44], [73], [26], contain senseless gross errors. These works are anti-scientific, contain a lot of deceptive statements and lies. Obviously, the authors are unaware of the elementary concepts of algebras and topology.

To stay within the limits of this paper we will analyze only several works, paying special attention to the algebraic nature of these works. The necessary theoretical basis is presented in Sections 1 – 5.

Continuous signature. Quasivarities. Free topological algebraic systems

We will start with paper [29]. Ideologically, it is connected to papers [71], and also [69]. It refers many times to questions related to the results from [69], [71], though no such referrals can be found in [29]. The fundamental paper [71] contains the basis of topological algebraic systems, in particular, of free topological algebraic systems of given variety (primitive class) of such systems. In [69], [71] any algebraic system $G$ is defined only with operations of form $(g_1, g_2, \ldots, g_n) \to g_{n+1}$, $n = 0, 1, \ldots$, where $g_1, \ldots, g_{n+1} \in G$, i.e. $(n, 1)$-operations, where $n$ is a finite integer. [29] considers $(n, m)$-operations with $n < \alpha$, $m < \beta$, where $\alpha, \beta$ are any fixed ordinal numbers. However, difference between [29] and [69], [71] is really catastrophic. Paper [29] could be compared with a world of absurd statements, both in terms of definitions, competence, and logical and mathematical judgement. Excuse my lack of modesty, but I think, that a mentally healthy person cannot even imagine such things, even less publish them. i

We bring some notions and results from paper [29, §1]. Fix a non-empty set $G$ and an ordinal numbers $n$, $m$. An unequivocal mapping
ω\( (n, m) : G^n \rightarrow G^m \) is called \((n, m)\)-operation on \(G\). We put \(Ω^n_m = \{ ω(n, m) \}\), \(Ω = ∪Ω^n_m\), \(P = ∪P_q\), where \(P_q = \{ p | p \text{ is a } q\text{-ary predicate on } G \}\), and let \(α_S, β_S, γ_S \) will be such a least ordinal numbers that \(n < α_S, m < β_S, q < γ_S\).

The object \( < G, Ω, P > \) is called algebraic system of signature \(S = Ω ∪ P\) (Definition 1).

The Definition 1 of algebraic systems has drawbacks, which as will further result in roughest mistakes. The author does not know that for rare cases in algebra only finite-place relations, functions, words, terms, formulas, operations, predicates are considered (see, for example, [69, pag. 42, 138, 141, 167]). For this look at the comment after Proposition 1 of present paper. Definition 1 for ordinal numbers \(n, m, q, α_S, β_S, γ_S\) does not impose even restrictions pag. 108, third line above): they can be any ordinal numbers. It is not clear for what reasons in the last §1 algebraic systems are investigated, defined by \((1, m)\)-operations, where \(m\) is any ordinal number. Let’s specify some drawbacks (items 1a), 1b)) of Definition 1.

Remind, that according to [70], [71] an algebraic system \( \mathcal{A} = < A, Ω_F, Ω_P > \) is called topological algebraic system if the basic set \(A\) is a topological space \((A, T)\) and the basic operations of \(Ω_F\) are continuous. From definition of topological algebraic system follows that the study of topological algebraic system \( \mathcal{A} \) consists of separate study of the algebraic system \( \mathcal{A} \), or separate study of topological space \((A, T)\), or separate study of interrelation of algebraic system \( \mathcal{A} \) and topological space \((A, T)\).

1a). Let \( \mathcal{A} = < A, Ω_F, Ω_P > \) be an algebraic system and let \( ω \) be a basic \((n, 1)\)-operation with \(n ≥ ∞\). Then by Proposition 1 of present paper the expression "subsystem \( < H > \) of system \( \mathcal{A} \) algebraically generated by set \( \emptyset \neq H ⊆ A \)" is not correct, is false. It is possible only to assert that the subsystem \( < H > \) is generated by set \(H\). In general, according to the definition, \( < H > \) is the intersection of all subsystems of \( \mathcal{A} \) which contain \(H\) and for \( < H > \) it is impossible to apply the Proposition 1 of present paper. Therefore, if for basic \((n, m)\)-operations of algebraic system \( \mathcal{A} \) not to put any restrictions on ordinal numbers \(n, m\) (can be \(n ≥ ∞ \text{ or } m > 1\)), then the statement "the elements in \( < H > \) are expressed through elements in \(H\) by means of basic and derived operations" is without sense.

1b) Let \( \mathcal{A} = < A, Ω_F, Ω_P > \) be a topological algebraic system and let \( ω ∈ Ω^n_m \). From the definition of topological algebraic system it follows that a topology \( T \) is defined on set \(A\) and the basic \((n, m)\)-operation \( ω \) is continuous. Let \(m > 1\). From last condition it follows that the given topology \( T \) of \(A\) can be a topology on degree \(A^m\). It takes place when \( T \) is a Tychonoff topology.
Hence, the condition $m > 1$ for basic operations attracts restriction for given topology $T$.

We pass to §2. Let $A = \langle A, \Omega_F, \Omega_P \rangle$ be an algebraic system of signature $S = \Omega_F \cup \Omega_P$. Let’s present the excerpts. Let some topologies $\tau(n, m)$ be given on set $\Omega^m_n$. In such a case we say that signature $\Omega$ is continuous. If the topologies on $\Omega^m_n$ are discrete, then $\Omega$ is called a discrete signature. Define the mappings $\psi_{(G, n, m)} : \Omega^m_n \times G^n \to G^m$, where $\psi_{(G, n, m)}(\omega, g) = \omega(g)$ for all $\omega \in \Omega$, $g \in A^n$, and $\varphi_{(G, q)} : P_q \times G^q \to L$, where $L$ is a complete lattices, $\varphi_{(G, q)}(p, a) = p(a)$ for all $p \in P_q$, $a \in G^q$. An algebraical system $G$ together with the topology, given on its $T$ and given topologies $\tau(n, m)$ on $\Omega^m_n$ is called a topological system of continuous signature $\Omega$, if all mappings $\psi_{(G, n, m)}, \varphi_{(G, q)}$ are continuous (Definition 3.)

Let’s show that the notion of continuous signature from Definition 3 is fictitious. It does not have any value and cannot bring any value added for the research of topological algebraic systems. As a confirmation hereof, we present some gaps from from the definition of continuous signature.

1c). A necessary condition for the topology $\tau(n, m)$ to be non-discreetly is the infinity of set $\Omega^m_n$. The sum $\Omega = \cup\{\Omega^m_n\}$ in Definition 3 is discrete (for an obvious confirmation to this see [28]). Therefore, there is no connection between topologies $\tau(n, m)$ by the definition of discrete sum of topological spaces.

1d). Without loss of generality we shall consider that $P = \emptyset$. We denote by $f$ the mapping $\psi_{(G, n, m)} : \Omega^m_n \times G^n \to G^m$ and assume that $f$ is continuous. We consider $\Omega$ as a set of symbols of basic operations. By Remark 1 of present paper for a fixed operation $\omega \in \Omega^m_n$ the mapping $\mu : \Omega^m_n \to G^m : \omega \to h = \omega(g)$ for a fixed $g \in G^n$ is continuous. Similarly, for a fixed $\omega \in \Omega^m_n$ the mapping $\nu : g \to h = \omega(g)$, $g \in G^n, h \in G^m$ is continuous. It means that the topology $\tau(n, m)$ on $\Omega^m_n$ does not bear any topological and algebraic information of topological system $< G, \Omega >$, i.e. does not change the topological and algebraical structure of $< G, \Omega >$, does not change the topological and algebraical structure of objects of any class $K$ of topological algebraic systems of signature $\Omega$. Consequently, the topology $T$ on $G$ does not depend on topologies $\tau(n, m)$ on $\Omega^m_n$. Finally, if the mappings $\mu, \nu$ are continuous then by Remark 1 of present paper the mapping $f$ should not be necessarily continuous. Hence the definition of topology $\tau(n, m)$ on set $\Omega^m_n$ does not depend on the topology of algebraic system $< G, \Omega >$ and on the requirement for the continuity of mapping $\psi_{(G, n, m)} : \Omega^m_n \times G^n \to G^m$.

1e). Consequently, without prejudice to research of classes of topologi-
cal algebraic systems $\mathcal{K}$, also separate systems of $\mathcal{K}$ expressed in the form ". . . system of continuous signature . . ." the words "of continuous signature" should be omitted. In such a case almost all results of paper (for example, the sections 4, 7, 8 and others) become trivial or without sense. The reasoning for introducing notion of "continuous signature" is not clear. But if necessary, it is possible to define the topology $\tau(n, m)$ considering $\Omega^m_n$ as a set of symbols of basic operations.

Further, we quote by [29, pag. 110, 111]. Fix a class of topological systems of continuous signature. Let's allocate the following properties, which can have the class $\mathcal{K}$.

Condition 1. Closed with respect to subsystems.
Condition 2. Closed with respect to Tychonoff products.
Condition 3. All objects from $\mathcal{K}$ satisfy a given set of topological, algebraical properties $Q$ (the set $Q$ can be $\{\emptyset\}$ or the separation axioms $T_0$, $T_1$, $T_2$, $T_3$, or the requirement of a regularity, or the requirement of a completely regularity and others).
Condition 4. If $(G, \tau) \in \mathcal{K}$, $\tau_d$ is the discrete topology and $(G, \tau_d)$ is a topological system of continuous signature, then $(G, \tau_d) \in \mathcal{K}$.
Condition 5. Closed with respect to continuous homomorphic images with property $Q$.
Condition 6. If $(G, \tau_1) \in \mathcal{K}$ and with respect to topology $\tau$ the pair $(G, \tau)$ is a topological system with properties $Q$ of continuous signature $S$, then $(G, \tau) \in \mathcal{K}$.

Definition 4. A class $\mathcal{K}$ is called:
1. Topological $Q$-quasivariety, if the conditions 1m - 4m are satisfied.
2. Topological $Q$-prequasivariety, if the conditions 1m - 5m are satisfied.
3. Topological $Q$-variety, if the conditions 1m - 6m are satisfied.
4. Topological complete $Q$-quasivariety, if the conditions 1m - 4m and 6m are satisfied.

Definition 5. A class $\mathcal{K}$ of algebraical systems of signature $S = \Omega \cup P$ is called quasivariety, if
1. The class $\mathcal{K}$ is closed with respect to subsystems;
2. The class $\mathcal{K}$ is closed with respect to Cartesian product.

The notions defined in the following excerpt are a basic object of research not only in paper [29], but also in many other papers, therefore we analyze this excerpt in more details. But at first remind that according to [70], [71] an algebraic system for which the basic set of elements is a topological space (one and only one space) and the basic operations are continuous is called
topological algebraic system. The totality of topological algebraic systems (defined above by one and only by one topological space) which make a variety (respect. quasivariety) in algebraic sense is called a variety of topological algebraic system (respect. a quasivariety of topological algebraic system).

1f). Let’s show that the defined notions are not correct, without sense, a total chaos, and do not correspond to the classical notions. We give reason for it.

1f1). The conditions 1m, 2m coincide with conditions $K_1), K_2) (Section 2)$ by [60, Proposition 2.3.1]. In condition 4m the relation $(G, \tau_1) \in R$ is false. As marked in the comment below conditions $K_1), K_2) (Section 2)$ the discrete topology does not satisfy conditions $K_1), K_2)$. It generates topological spaces which are regular and completely discontinuous in the sense that for any finite set of points $x_1, \ldots, x_n$ of such space $X$ there exists a splitting $X$ on $n$ in pairs not crossing closed sets, containing one of the specified points. These topological spaces satisfy conditions $K_1), K_2)$.

1f2). Even not considering item 1d) the topologies $\tau_1$ and $\tau$ from the conditions are "almost" independent by Remark 1 of present paper. Hence the topology $\tau$ from relation $(G, \tau) \in R$ is "almost" arbitrary. To what extent such topologies $\tau$ are available and how are they connected with conditions 1m, 2m?

1g). By item 1f) the notions from Definition 4 are not correct, false. Let’s show that a similar situation takes place and the definition of quasivariety (Definition 5).

1g1). In the literature, [48], [69] and others (see Section 2) the notion of quasivariety (or quasiprimitive class) is defined by means of quasiidentities. An quasiidentity is a certain set of quasiatomic formulas, of $\forall$-formulas. Any quasiatomic formula is a finite totality of free variables of basic operations of algebraic system. Any basic operation of algebraic system is a $(n, 1)$-operation, where $n$ is a finite integer. Definition 4 examines $(n, m)$-operations with arbitrary numbers $n, m$, that is a roughest mistake.

1g2). The definition of quasivariety from Proposition 5 is incorrect. Besides conditions 1g, 2g it is necessary to add the conditions $Q_1), Q_4) of Proposition 3 from Section 2. The condition $Q_4)$: the class $R$ contains an unitary system is a characteristic property for quasivarieties by Corollary 8. The condition $Q_1)$: the class $R$ is ultraclosed is also necessary in the definition of quasivariety. There are classes $R$ with conditions $Q_2), Q_3), Q_4$ which are not quasivarieties ([69, pag. 272]).

1g3). Further, the notion of Cartesian product does not characterize fully
the quasivariety ([69, pag. 193]. The notion of filter product corrects this drawback. For example, by [69, Corollary V.11.3] a class \( K \) of algebraic systems is a quasivariety if and only if \( K \): (i) is closed with respect to filter products; (ii) is closed with respect to captures of subsystems; (iii) contains an unitary system.

1h). The section 3 "Free algebraical systems" of [29, pag. 112 - 118] begins with the following definitions, which are similar to the Definitions in [71].

Definition 7 (resp. Definition 8). We fix a non-trivial class \( \mathcal{K} \) of topological systems of continuous signature \( \Omega \) and non-empty set (respect. space) \( X \). The pair \((F^a_{\mathcal{K}}(X), i_X)\) (resp. \((F_{\mathcal{K}}(X), \delta_X)\)), when \( F^a_{\mathcal{K}}(X), F_{\mathcal{K}}(X) \in \mathcal{K} \) and \( i_X : X \to F^a_{\mathcal{K}}(X) \) (resp. \( \delta_X : X \to F_{\mathcal{K}}(X) \)) is unequivocal mapping (resp. continuous mapping), is called algebraically free system of set (respect. space) \( X \), if it satisfies the conditions:

1a (resp. 1t). The set \( i_X(X) \) (resp. \( \delta_X(X) \)) algebraically generate the system \( F^a_{\mathcal{K}}(X) \) (resp. \( F_{\mathcal{K}}(X) \));

2a (resp. 2t). For every unequivocal mapping \( f : X \to G \), when \( G \in \mathcal{K} \), it is exists a homomorphism \( \hat{f} : F^a_{\mathcal{K}}(X) \to G \) (respect. continuous homomorphism \( \hat{f} : F_{\mathcal{K}}(X) \to G \)) such that \( f(x) = \hat{f}(i_X(x)) \) (resp. \( f(x) = \hat{f}(\delta_X(x)) \)) for all \( x \in X \).

At the first sight the introduced notions (and their properties proved below) are an essential generalization of classical notion of topological algebra of a given variety of topological algebras with given generating topological space and with given generating relations, and of free objects of varieties of topological algebras (and their properties), which were introduced and investigated in detail by A. I. Mal’cev in [71]. Actually the introduced notions and their properties from [29] are a senseless imitation of [71] (which are not mentioned in [29]), are a complete absurdity. We will give arguments to support this. But first we note that a complete generalization of Mal’cev’s notions and results is presented in Sections 2, 3.

1h1). The condition "of continuous signature" should be ignored by virtue of item 1e).

1i). With such a wording the item 1a (resp. 1t) is incorrect: it is impossible to use the expression "algebraically generated" by item 1a) and Proposition 1. It is meaningful if and only if the basic operations of topological systems in \( \mathcal{K} \) are \((n, 1)\)-operations with finite integer \( n \). With such restrictions on basic operations the Definition 7 (resp. Definition 8) is a special case (together with Lemma 2) of Definition 1 (see, also [71]) provided that the
item 1t it is replaced by the condition: the set $\delta_X$ topologically generate the system $F_K(X)$.

Below Definition 7 (resp. Definition 8) one may find it follows something improbable, a reasoning without sense, which does not follow any mathematical logic. To present Properties 1 - 15, their corollaries, and definitions that describe the systems from Definitions 7, 8 in a case when the class of topological spaces $\mathcal{K}$ satisfies only the conditions 1m, 2m, let us present some of them. Let us have a non-trivial class $\mathcal{K}$ of topological systems of continuous signature $S = \Omega \cup P$ which satisfies the conditions 1m, 2m. Then the following hold.

Property 1. For each non-empty space $X$ exists an unique algebraically free topological system $(F^a_K(X), i_X)$ of space $X$ in the class $\mathcal{K}$ and the mapping $i_X$ which to apply one-to-one $X$ in $(F^a_K(X)$.

Property 2. For each non-empty space $X$ exists an unique topologically free topological system $(F_K(X), \delta_X)$ of space $X$ in class $\mathcal{K}$.

Property 3. For each non-empty space $X$ exists an unique continuous homomorphism $g_X : F^a_K(X) \rightarrow F_K(X)$ such that $i_X \circ g_X = \delta_X$.

Property 4. If the space $X$ is discrete then the homomorphism $g_X$ is a topological isomorphism.

The following fact shows, that the conditions 1m, 2m are not only sufficient for existence of free systems, but also almost necessary.

Property 13. Let $\mathcal{K}$ be a quasivariety of algebraic systems of signature $S$. Then for each non-empty space $X$ exists an unique algebraically free topological system $(F^a_K(X)^a, i_X)$. If $\mathcal{K}$ contains a not one-element system, then $i_X$ one-to-one apply $X$ in $(F^a_K(X)$.

Proof. Similarly of proof of Property 1.

To prove Properties 1 - 4, 13 the following Proposition 2 is used essentially.

Proposition 2. Let $G$ be an algebraic system of signature $\Omega = \cup \Omega^M_n$ with $n < \alpha_S$, $m < \beta_S$. Let, further, $A(Y)$ be a subsystem of system $G$ generated by non-empty set $Y \subseteq G$. Then $|A(Y)| \leq |\Omega \cup Y|^\tau + 2^\tau$, where $\tau = \max\{|\alpha_S|, |\beta_S|, \kappa_0\}$.

Proof. Denote by $\tau^+$ the first ordinal number more than $\tau$. If $\omega \in \Omega^m_n$ and $g \in G^n$ then $\omega(g) = \{g_\xi| \xi < m\} \subseteq G^m$ and put $B(\omega(g)) = \{g_\xi| \xi < m\} \subseteq G$. For any non-empty set $Z \subseteq G$ denote $\Omega^m_n(Z) = \cup\{B(\omega(g))|g \in G^n \subseteq G^m\}$ and $\omega in \Omega^m_n\}$. Let’s observe, that $|\Omega^m_n(Z)| \leq |\Omega^m_n \times (Z^m)| \cdot |m| = |\Omega^m_n| \cdot |Z^m| \cdot |m|$. Hence $|\Omega(Z)| \leq |\Omega \cdot |Z^\tau| \cdot \tau$. Put $A_0(Y) = Y$ and $A_\alpha(Y) = \Omega(\cup\{A_\xi(Y)| \xi < \alpha\})$ for all $\alpha < \tau^+$. Then for every $\alpha < \tau^+$ we have $|A_\alpha(Y)| \leq |\Omega \cup Z|^\tau + 2^\tau$. As $A(Y) = \cup\{A_\alpha(Y)| \alpha < \tau^+\}$,
that proof is completed.

1j). With such wordings the Proposition 2 and its proof are a roughest absurdity, which cannot be explained by any means. The nature of $\aleph_0$ is not clear. It is known that $2^{\aleph_0}$ is the continuum. Then what kind of restrictions $\leq$ are mentioned in Proposition 2? The role of number $\tau^+$ is also not clear. Statement: "As $A(Y) = \cup\{A_\alpha(Y)|\alpha < \tau^+\}$, that proof is completed." contains severe errors. Remind that the subsystem $A(Y)$ is the intersection of all subsystem of $G$ which contain the set $Y$. If $m > 1$, for example $m = 5$, for any $n, m$-operation $\omega \in \Omega$, then it is not clear how to express every element of $A(Y)$ through elements of $Y$ with the help of operations $\omega$, i.e. that equality $A(Y) = \cup\{A_\alpha(Y)|\alpha < \tau^+\}$ is correct. According to Proposition 1 of present paper this equality holds if and only if all operations in $\Omega$ have a form $(n, 1)$-operation with finite integer $n$. In such case the statement of Proposition 2 is known: $|A(L)| \leq \max\{|\Omega|, |Y|, \aleph_0\} [69, Example 2, pag. 163].

On proof of Properties 1 - 4, 13. Schematically the proof is without sense, imitating the proof of Theorem 2 from [71](see also the Theorem 6), though no reference is made in the paper about it. We present and will analyze the proof of Properties 1 - 4. First, let’s note that the proof is erroneous as it used the incorrect Proposition 2.

Proof of Properties 1 - 4. Uniqueness to within natural isomorphism easily follows from Definition 7. Let’s prove existence of algebraically free system. Put $\tau = \max\{|X|, |\Omega|, |\alpha_S|, |\beta_S|, \aleph_0\}$. By Proposition 2 we can consider( only a systems of cardinality $\leq 2^\tau$. The totality $K_\tau = \{G \in K||G| \leq 2^\tau\}$ is a set. Then is set and the totality $\{f_\xi : X \rightarrow G_\xi|\xi \in A\}$ all mappings of set $X$ in systems $G_\xi$ from $K_\tau$. We consider the diagonal product $i_X \rightarrow G = \prod G_\xi|\xi \in A$. Let $F^K_a(X)$ is the subsystem of system $G$ generated by set $i_X(X)$. Using projections $\hat{f}_\xi = \pi_\xi : F^K_a(X) \rightarrow G_\xi$ and Proposition 2 easily to show that $(F^K_a(X), i_X)$ is an algebraically free system of set $X$ in class $K$. One-to-one unambiguity of mapping $i_X$ follows from a condition of not trivialities of class $K$. To it the property 1 is proved. Let now $X$ is a topological space. Put $B = \{\alpha \in A|f_\alpha : X \rightarrow G_\alpha$ is continuous}. Considering diagonal product $\delta_X = \prod\{f_\alpha|\alpha \in B\} : X \rightarrow H = \prod\{G_\alpha|\alpha \in B\}$, we similarly let’s construct a topologically free system $F^K_a(X), \delta_X)$ of space $X$ in class $K$. If the space $X$ is discrete, then $A = B$, hence $F^K_a(X) = F^K(X)$. It proves the following three properties 2 - 4.

By item 1j) the proof of Properties 1 - 4 is erroneous as it used the Proposition 2. Assume that the Proposition 2 holds. It is known that $2^{\aleph_0}$ is
continuum. We can assume that $\tau > \aleph_0$. It is not clear why totality $\mathcal{K}_\tau$ is a set. Let's assume that this is correct. We consider the Cartesian product $G = \prod G_\xi | \xi \in A\}$, where $A = \{\xi | G_\xi \in \mathcal{K}_\tau\}$. Assume that all systems $G_\xi$ are non-empty. There is a question: will the Cartesian product be non-empty? If the totality of indexes is finite, then the answer to this question is positive. Non-emptiness of set $G_\xi$ means that in each set $G_\xi$ it is possible "to choose" at least one element. Having made such a choice for every $\xi$ we will receive an element of Cartesian product. If the totality $A$ is infinite, then it is necessary to make an infinite number of choices. But it is an axiom of choice. Hence, for $\prod G_\xi | \xi \in A\} \neq \emptyset$ it is necessary to require that it should be possible to index the totality $A$. Only in such a case it is possible to speak about existence of system $F^\alpha_\mathcal{K}(X)$. To have a free system $F^\alpha_\mathcal{K}(X)$ it is necessary to require that the totality of all projections $p_\xi : F^\alpha_\mathcal{K}(X) \to G_\xi$ is complete [69, Theorem I.2.2].

According to the above stated we conclude.

1k). The entire material from Section 3 of [29], where the notions of algebraically (or topologically) free topological systems of various classes of topological systems are defined and the various properties of such systems are investigated, particularly their existence, are erroneous, incorrect, and without any sense. Similar questions are investigated in detail in Section 3 and Section 2 of this paper.

1l). Properties 1 - 4 prove the existence and uniqueness of the free systems from Definitions 7, 8. These properties are the basic tool for all further researches of this paper. By item 1k), and 1e), 1f), 1g), 1h), 1i) we conclude that the entire content of paper [29], including numerous generalizations of known results, is erroneous and senseless, consisting of statements assembled mechanically as proofs. The given paper [29] is not mathematical. I even do not know how to name it. To confirm these statements we will analyze some more excerpts from the given paper.

In "Introduction" it is mentioned that the main purpose of paper [29] is the decision of problem on representation of topological systems in form of quotient homomorphic images of zero dimensional systems from same class. This problem is completely solved for discrete signature. (As corollary it follows that amplifies the result from [2] and given problem is positively solved for topological universal algebras of discrete signature, for topological semigroups, for topological quasigroups and loops, for topological rings, and also for topological groups with multioperators, where on set of multioperators to consider the discrete topology. It is possible to consider $\Phi$-algebras also
with set of differentiated $\Delta$, where $\Phi$, $\Delta$ are discrete (see, pag. 132). If the signature is not discrete, that follows from the Theorem 5, then the given problem has a negative decision even in rather well known varieties (see, item 1q). For topological groups this problem is solved completely in [2]. The proof scheme from [29] is similar to the proof scheme from [2]. Except for the concepts and some results taken from [2], it is absolutely impossible to regard as authentic the solution of the problem stated in [29]. Let’s specify only a few rough mistakes in comparison with [2].

1m). The expressions "continuous signature", "discrete signature" should be ignored by item 1e).

1n). The problem is solved for classes of topological systems for which the conditions 1m, 2m are correct. Thus these are supposed mistakes of items 1f), 1g).

1o). The following concept is used to solve the problem in [2]. It is known (see Corollary 2 of present paper), that any continuous homomorphism of topological group on topological group is open. For factor-system of topological system the property of open homomorphism is not always correct as the conditions of a Consequence 2 are not always fair.

1p). In [2] in quality of group $G$ from item 1f is considered (the free topological group described below Definition 1 of present paper. For classes of topological systems in [29] is considered the topologically free topological system from Definitions 7, 8. The admitted mistakes are specified in the analysis of validity of Properties 1 - 4.

1q). Really, the class $\mathcal{K}_l$ of all linear topological spaces form a topological variety of topological algebras with continuous signature. But in $\mathcal{K}_l$ only one-pointed spaces are null-dimensional. This is incorrect. The author confuses elementary notions: the dimension of linear space and the dimension of topological space. Example: the subset of $n$-dimensional Euclidean space, consisting from all points with real coordinates, is a null-dimensional topological space as this subset does not contain intervals.

Now we pass to analysis of §4 and let’s show complete incompetence of the author with respect to theory of identities and quasiidentities of algebraic systems. Like other sections of analyzed paper, the §4 consists only of senseless notions and statements. To back this we will use the following excerpt. Fix a continuous signature and an algebraic system $G$ of signature $S$. Let $\mathfrak{S}(S, i)$ denote the totality of all algebraic systems of signature $S$ which satisfies the separated axiom $T_i$. Fix $i \in \{-1, 0, 1, 2, 3, 3\frac{1}{2}, r, \rho\}$. The class $\mathfrak{S}(\emptyset, i)$ consists of all non-empty $T_i$-spaces. For any non-empty space $X$ we denote
by \((X/i, g_{iX})\) such a \(T_i\)-spaces \(X/i\) and continuous mapping \(g_{iX} : X \to X/i\) that for every continuous mapping \(f : X \to Y\) in \(T_i\)-space \(Y\) a continuous mapping \(f_i : X/i \to Y\) exists that \(f = f_i \circ g_{iX}\). Actually, the pair \((X/i, g_{iX})\) is a free topological system of space \(X\) in class \(\mathfrak{R}(\emptyset, i)\). For class \(\mathfrak{R}(\emptyset, i)\) the signature is empty. We consider the signature \(S = \Omega \cup P\). If \(\omega, \mu \in \Omega^\beta_\alpha\), then the identity of form \(\omega = \mu\) is called a simple identity. (Absolutely unclear: the identity from what? Actually, \(\omega, \mu \in \Omega^\beta_\alpha\) are fixed elements of the set of symbols of basic operations of fixed signature \(S = \Omega \cup P\).)

If \(\omega \in \Omega^\beta_\alpha\), \(\mu \in \Omega^\beta_\lambda\) and let the set \(A\) and the mappings \(f : A \to \prod(\alpha), g : A \to \prod(\lambda)\) be given, then for any \(x = \{x_i\} \in G^\alpha\) and \(y = \{y_\eta\} \in G^\lambda\) we have \(\omega(x) = \mu(y)\) as only \(x_{f(a)} = y_{f(a)}\) for all \(a \in A\). Instead of basic operations it is possible to take a derived operations, then under \(\beta_S \leq 2\) and \(P = \emptyset\) thus all identities can be received. Let a quasivariety \(\mathfrak{K}\) of algebraic systems of signature \(S\) be given. Through \(J(\mathfrak{K})\) and \(qJ(\mathfrak{K})\) denote the totality of all identities and quasiidentities which define respectively the class \(\mathfrak{K}\) (uttermost absurdity: according to definition of quasivarieties they are not defined with the help of identities). Through \(Q\mathfrak{K}\) denote the totality of all systems from \(\mathfrak{K}\), for which topologies with properties \(Q\) can be defined. Let’s say that \(Q\mathfrak{K}\) is topologically defined by identities \(J(\mathfrak{K})\) and quasiidentities \(qJ(\mathfrak{K})\). The identities \(J(Q\mathfrak{K}) \setminus J(\mathfrak{K})\) and the quasiidentities \(qJ(Q\mathfrak{K}) \setminus qJ(\mathfrak{K})\) are called relatives identities and quasiidentities of topological type. The identities \(J(Q\mathfrak{K}) \setminus J(\mathfrak{K})\) and the quasiidentities \(qJ(Q\mathfrak{K}(S, i)) \setminus qJ(\mathfrak{R}(S, i))\) are called absolute identities and quasiidentities of topological type. Let \(J\) denote the set of all simple identities. Put \(J_\alpha^\beta = \{\omega = \mu \in J | \omega, \mu \in \Omega^\beta_\alpha\}\). On space \(\Omega^\beta_\alpha\) the identities \(J_\alpha^\beta\) generate a partition of classes of equivalence. The space of partitions in factor topology we denote by \(\Omega^\beta_\alpha/J\) and by \(\pi_{\alpha\beta} : \Omega^\beta_\alpha \to \Omega^\beta_\alpha/J\) we denote the projection. By \(\mathfrak{K}(S, J, i)\) denote all topological systems from \(\mathfrak{K}(S, i)\) on which the identities \(J\) are true.

Further these senseless notions are used to describe the free topological system \(F_{\mathfrak{R}(S,J,i)}\) (statements 1 - 5). Moreover, in Example 7 even is given constructive construction such free systems and a maximal (in some sense) topologies on such free systems. We cannot even speak about its validity. According to item 1l), such free systems do not exist.

To raise the paper’s prestige the author uses the following methods (at least in the author’s opinion).

1r). In the beginning of paper there is a statement: I use this opportunity...
to express my gratitude to professor A. V. Arhangel’skii, the conversations with whom have inspired the author to write the paper.

1s). The author claims that the $X_\lambda$-topologies construction procedure, offered by A. I. Mal’cev [71, pag. 448] served as a basis for §4. This is far from reality.

1t). In some of its papers [50], [51], [40], [10], [45], [47] the author changes the name of the journal, where the given paper [29] is published. Specifically: M. M. Choban, On the theory of topological algebraic systems, Trans. Amer. Math. Soc., 48, 1986, 115 - 159. But this is not true. Paper [29] was published in "Trans. Moscov. Math. Soc."

1t₁). Moreover, various monographs[?], [20], [25], dissertations [44], [73], [26], [58] have references to non-existent works, which were not published in the indicated prestigious journals [65], [46], [42], [43], [24] and others.

1u). We analyzed only some excerpts from paper [29]. The situation of the remaining part (Sections 4 - 13) is much worse: absurd notions and results. Thus, on the basis of the above-mentioned it is possible to ascertain that paper [29], which consists only of unclear proofs and severe errors, does not represent any scientific interest. Unfortunately, such statements are not enough to describe the paper of Academician Choban M. M. [29]. The absurd and senseless notions and results, as well as absurd and senseless proofs, are still used as basic objects and tools of research for many papers, monographs and dissertations. We listed some of them: [39], [40], [9], [10], [32], [53], [25], [26], [55], [45], [21], [58]. Naturally, they are erroneous. Let us analyze briefly some of them.

With some ”peculiarities” paper [55] introduced and studied the same notions and results as paper [29].

Let $G^A$ denote the totality of indexed sequence $\{x_\alpha|\alpha \in A\}$ of elements of $G$. We consider a fixed set of indexes and a non-empty set $G$. Every mapping $\omega : G^A \to G$ is called $A$-ary operation on $G$. Often the set $A$ is identified with set of ordinal numbers which are less than some given number $\alpha$. In such case $A$-ary operation is called $\alpha$-ary operation.

Let is given the set $B \supset A$. Then the $A$-operation $\omega$ it is possible to considered as $B$-operation, if to identify $\omega$ with operation $\omega_1 : G^B \to G$ such that for each point $\{x_\beta|\beta \in B\} \in G^B$ $\omega_1(\{x_\beta|\beta \in B\}) = \omega(\{x_\beta|\beta \in A\})$. It allows to consider any given set of operations $\Omega = \{\omega_\gamma|\gamma \in \Gamma\}$ above the same set of indexes $A$. If $\omega_\gamma$ is an $A_\gamma$-ary operation, then let’s assume $A = \cup\{A_\gamma|\gamma \in \Gamma\}$ and we shall consider that all operations $\omega_\gamma$ are $A$-ary (pag. 29).
2a). Severe error. By equating the finite arity of operation with non-finite arity one of the basic properties of operation with finite arity, specified in Proposition 1 of that paper, is lost. Moreover, according to Proposition 1 it is necessary to consider only operations of finite arity.

If on Ω is given a topology, then Ω is called continuous system of operations. If is given a some set G together with mapping Ψ : Ω × G^A → G, that speak, that is given an universal algebra G with system of A-ary operations Ω or is given an Ω-algebra. If thus Ω is a continuous system of operations, G is a topological space it the mapping Ψ : Ω × G^A → G is continuous with respect to topology of Tychonoff product in Ω × G^A, then G is called continuous Ω-algebra.

2b). According to item 1e) the notion of algebra of continuous signature is fictitious, it can be ignored.

Besides the concept of free topological algebra with continuous Signature, paper [55] investigates the following basic notions. A non-empty class K of Ω-algebras is called prevariety if:

A1) K is closed with respect to direct product;
A2) K is closed with respect to taking of subalgebras.

A non-empty class K of continuous Ω-algebras is called T_i-prevariety, if K satisfies the conditions 1m, 2m, 3m, 6m, considered above for paper [29]. Similarly to Definition 7, it defines the notion of free algebra for algebraical prevariety. The notion of free topological algebra for T_i-prevariety coincides with Definition 8. As mentioned in item 1u) the notions of prevariety and its free algebras, of T_i-prevariety and its free topological algebras are a basic object of researches of many papers. Therefore, we analyze the evidence of existence and uniqueness of such free algebras. We mention only that these proofs are an unsuccessful imitation of Mal’cev’s proofs [71, Theorem 1], [69, Theorem V.11.4], and also of Theorem 6 from that paper.

We present it literally.

LEMMA 1. Let G be a Ω-algebra, X ⊂ G X ≠ 0. Then X generate in G a subalgebra G_1, for which |G_1| ≤ (|X ∪ Ω| + 2)^|∆| + ℵ₀.

PROOF. Let us note previously Ω^* = Ω ∪ {ω*}, where ω* is the identical operation. For every subset M ⊂ G let’s assume Ω^* = ω* ∪ (ω(M^A)|ω ∈ Ω}. Then we have |ω^*(M)| ≤ (|Ω ∪ M| + 2)^|∆| + 1. Let λ₀ be the first ordinal of cardinality 2^|∆| + ℵ₀. We put X₀ = X and X_λ = Ω^* × (X_α|α < λ}) for all λ < λ₀. Let G_1 = ∪{X_λ|λ < λ₀}. Clearly that the cardinality of G_1 does not exceed (|X ∪ Ω| + 2)^|∆| + ℵ₀. It is necessary to show, that G_1 is a subalgebra of algebra G. Really, if {x_α|α ∈ A} ∈ G^A_1, that will be such an element λ₁ ≤ λ₀
that \( \{x_\alpha | \alpha \in A \} \in X_{\lambda_1} \). Then we have \( \Omega(\{x_\alpha | \alpha \in A \}) \subset X_{\lambda_1+1} \subset G_1 \). The lemma is proved.

2c). Gross mistake by Proposition 1 from that paper: \( G_1 \) is not a subalgebra of algebra \( G \). It is not clear what \( \Delta \) means. For more details see item 1j).

Lemma 1 is used essentially in the proof of Theorem 1.

**THEOREM 1.** What were not \( T_i \)-prevariety \( K \) with continuous system of \( A \)-ary operations \( \Omega \) and the topological space \( X \) the free topological algebra \( (F_K(X), \sigma) \) exists and is unique.

**PROOF.** The uniqueness is proved similarly of [29, Property 1] to expound above. Let us prove now the existence of \( F_K(X) \). We consider the totality \( B \) of all \( G \in K \), for which \( |G| \leq (|X \cup \Omega| + 2)2^{\aleph_1} + \aleph_0 \). Obviously, \( B \) forms a not empty set, if to consider identical the topologically isomorphic algebras from \( K \). Therefore the family all continuous mappings \( F = \{g : X \to G | G \in B \} \) is also non-empty set. We consider the diagonal product \( \sigma = \Delta F : X \to \Pi\{G | G \in B \} \). The set \( \sigma X \) generate in \( \Pi\{G | G \in B \} \) a some subalgebra \( F \). Then \( (F, \sigma) = (F_K(X), \sigma) \) is the free topological algebra. The Theorem is proved.

**THEOREM 2.** Let \( K \) be an algebraic prevariety of \( \Omega \)-algebras. Then for any non-empty set \( X \) the free algebra \( F^a_K(X) \) it is exists and is unique (to within to isomorphism above \( \sigma^a_X \).

**PROOF.** The existence and the uniqueness of algebra \( (F^a_K(X), \sigma^a) \) is established similarly of existence and of uniqueness of algebra \( (F_K(X), \sigma) \) from Theorem 1.

2d). It is not clear how to estimate the authenticity of proofs of Lemma 1 and Theorems 1, 2. It is possible to use items 1j), 1k), 1l). I think that the reader will agree with me, that the proofs of Lemma 1 and Theorems 1, 2 are a real non-sense, a delirium. Hence, the existence and uniqueness of free algebras \( (F_K(X), \sigma), (F^a_K(X), \sigma) \) are not proved. It is impossible to prove the existence of such free algebras operating with the definitions of prevariety and operations from Proposition 1 and Theorems 6, 7 of that paper.

On page 33 is defined the notion of point finite arity operation. Actually, it is the notion of operation \( \omega \) of finite arity \( I_\omega \).

Further, let us present the excerpt, page 33.

Let \( K \) be a \( T_i \)-prevariety. If for any \( \omega \in \Omega \) it is exist a finite set \( B \) and a neighborhood \( O_\omega \) of point \( \omega \in \Omega \) such that \( I_{\omega'} \subset B \) for all \( \omega' \in O_\omega \), then \( K \) is called locally finite arity. On analogy shall speak about point finite arity and locally finite arity systems of operations \( \Omega \).
Let $G \in K$ and let the mapping $\Psi : \Omega \times G^A$ be continuous by the definition of algebra with continuous signature. According to item 1d) the topology $\mathcal{T}$ on $\Omega$ does not change the topological and algebraical structure of topological algebra. If to consider $\mathcal{T}$ as anti-discrete topology, then from the definition of locally finite arity systems it follows that this notion coincides with the notion of system with basic operations of finite arity.

2e). The notion of point finite arity $\Omega$-operation coincides with notion of $\Omega$-operation with finite arity of basic operation from $\Omega$, and the notion of locally finite arity topological $\Omega$-algebra coincides with notion of topological $\Omega$-algebra with finite arity of basic operation from $\Omega$.

The authors claim on page 33 that bringing all operations to the same arity plays a positive role for the proof of many facts. The notion of locally finite arity of prevariety is a basis for this paper’s theorems and results. Is necessary to add to this the notions of $T_i$-prevariety, the notions of algebraic and topological free algebras of $T_i$-prevariety. The paper generalizes many known results, corrects the incorrect results of other authors (see, items 2h), 2i)) and also gives valuable indications how to investigate the topological algebras (see item 2j)). We are limited to the volume of this paper, therefore we will not analyze the above-mentioned. However, I think that it is enough to ascertain items 2a) - 2e).

2f). The work [55] consists only of erroneous and senseless statements. It is not a scientific, but rather an anti-scientific paper. To confirm the last statement, we will present some excerpts (pag. 36).

Let is given a $T_i$-prevariety $\mathcal{K}$ with continuous system of $A$-ary operations $\Omega$. For every topological space $X$ it is exist a free topological algebra $(F_\mathcal{K}(X), \sigma)$ (Theorem 1) and a free algebra $(F_\mathcal{K}^a(X), \sigma)$ (Theorem 2). The topology on $F_\mathcal{K}(X)$ denote by $\mathcal{T}_F$. There are following questions.

Question 1. How depends the topology $\mathcal{T}_F$ from topology of space $X$?

Question 2. Under what conditions the mapping $\sigma$ is a homeomorphic embedding?

Question 3. Whether is the mapping $i_X : F_\mathcal{K}^a(X) \to F_\mathcal{K}(X)$ an algebraic isomorphism?

These questions were formulated A. I. Mal’cev [71] for case, when $\Omega$ is a finite discrete space.

2g). The statement is senseless.

2h). As will be shown further, answers to questions 2 and 3 are positive for a wide class of prevarieties and spaces, in particular, and case of completely regular space $X$. Meanwhile [18] presents a negative answer to question 3
even for the case $X$ and $T_2$-prevariety $\mathcal{K}$ (the completely regular space and $T_2$-space are the same.) However a careful analysis of the statements from [17] shows that the author interprets incorrectly the problem... The author [18] has a wrong approach to the A. I. Mal’cev problem, which resulted in other mistakes. Some of the author’s statements about $\beta$-classes are in particular wrong. It follows from the results obtained in paper (§6) that any non-trivial prevariety is a $\beta$-class in the sense of A. I. Mal’cev [71]. Therefore this concept, from our point of view, to examine is inexpedient, as a case non-trivial prevarieties does not represent interest.

2i). (page 51). REMARK. Theorems 7 and 8 generate the lemma and its consequence, formulated in [71, page 186]. The basic result of M. S. Burghin [18] is false.

2j). (page 39). REMARK. It is possible, following directly A. I. Mal’cev [71], to construct sequence enclosed topologies $X_\lambda$ using polynomials, though the last definition, when all operations from $\Omega$ are $A$-ary, is very cumbersome. We define the initial topology $\mathcal{T}_0$ otherwise, not using polynomials. It is possible to show that the sequence of topology, constructed by us, contains the sequence of A. I. Mal’cev as a co-final sequence. In particular, we receive the initial topology $X_0$ at a certain step $\lambda_0$.

2k). Excuse my immodesty, but such ingenious ideas are impossible to comment.

2l). The authors express their gratitude to the seminar leader, professor V. I. Arnautov, and all its participants for valuable remarks made during the discussion of this work.

Item 2l) is similar to item 1f). The authenticity of 2l) follows from items 2a) - 2k). Item 2l) also refers to other papers, published by Academician Choban M. M. in journal ”Matematicheskie issledovania” [34], [35], [35]. We present literally the excerptions a) – h) from [34] and analyze them briefly.

a). A space $X$ is zero-dimensional if $\dim_\beta X = 0$. (pag. 121)

3a). This statement is not correct. [60, example 6.2.20] presents a zero-dimensional space $X$ with $\dim_\beta X \neq 0$. This excerpt is copied mechanically from another paper. This statement is correct for non-empty class of locally compact paracompacts, for non-empty class of compacts [60, Theorem 6.2.9, Corollary 6.2.10].

3b). A topological space $X$ is called zero-dimensional, $\dim X = 0$, if $X$ is a $T_1$-space and posed a basis from open-closed sets. Any discrete space is zero-dimensional. A topological space $X$ is zero-dimensional if and only if $\dim X = \text{ind} X = 0$ [60, pag. 562].
b). Let us consider a non-empty class $\mathcal{K}$ of topological groups and emphasize some properties, which can have such class:

I. Closed with respect to direct topological products.
II. Closed with respect to subgroups.
III. Closed with respect to imagines of continuous homomorphisms "on".
IV. If $G \in \mathcal{K}$ and with respect to topology $\tau$ the pair $(G, \tau)$ is a topological group, then $(G, \tau) \in \mathcal{K}$.

ú). A class $\mathcal{K}$ with properties I - III will called variety. A class $\mathcal{K}$ with properties I - IV will called primitive class according to [70, pag. 425], [71, pag. 23].

The reference to Mal’cev’s papers is inaccurate. In these papers a primitive class of algebraic systems is defined as totality of all algebraic systems having same basic operations of finite arity and all elements which satisfy some system of identities of form $f = g$, where $f, g$ are a polynomials. Clearly, that the identities of form $f = g$ are quasiatomic formulas.

In [69, pag. 269] a variety of algebraic systems is defined as totality of all algebraic systems having the same basic operations of finite arity and satisfying the same system of quasiatomic formulas of form $f = g$. Further, by [70], [71] a topological primitive class is called a totality of topological algebraic systems, which make a primitive class in algebraically sense.

3c). The notion of topological primitive class defined in item a) is incorrect. The reference from item b) to Mal’cev’s papers is inaccurate. The notions of topological (algebraical) primitive class and topological (algebraical) variety of algebraic systems, defined by A. I. Mal’cev, coincide. By [69, Theorem VI.13.1] or Proposition 2 of that paper, these notions for topological groups coincide with a class of topological groups that satisfy conditions I, II, III of item a).

d). Let $\mathcal{K}$ be a non-empty class of topological groups. By $\mathcal{M}(\mathcal{K})$ we denote least variety, containing the class $\mathcal{K}$ and by $\mathcal{P}(\mathcal{K})$ we denote least primitive class, containing the class $\mathcal{K}$. Always $\mathcal{M}(\mathcal{K}) \subseteq \mathcal{P}(\mathcal{K})$. The classes $\mathcal{M}(\mathcal{K})$, $\mathcal{P}(\mathcal{K})$ are supplied with same "set" of abstract groups, but each group in class $\mathcal{P}(\mathcal{K})$ is supplied with the large number of topologies.

**LEMMA 2.** Let $\mathcal{K}$ be a non-empty class of bicompact groups. Then: 1. $\mathcal{P}(\mathcal{K}) \neq \mathcal{M}(\mathcal{K})$. ...

**PROOF.** In $\mathcal{M}(\mathcal{K})$ it exists a non-finite group $G$. Let $\tau$ denote the discrete topology on $G$. Then $(G, \tau) \in \mathcal{P}(\mathcal{K})$. Lemma is proved.

We presented a fragment of Lemma 2 with proof to show the sense of condition IV for introduced notion of primitive class. Takes place
3d). In definition of primitive class the condition IV is senseless. It contradicts conditions I, II, III. Does not exist neither class $\mathcal{K}$ of topological groups such that the primitive class $\mathcal{P}(\mathcal{K})$ exists.

Really, we assume that $\mathcal{M}(\mathcal{K}) \subset \mathcal{P}(\mathcal{K})$. Let $(G, \tau_1) \in \mathcal{M}(\mathcal{K})$ be a non-discrete topological group and let $\tau$ denote the discrete topology on $G$. Similarly to Lemma 2 $(G, \tau) \in \mathcal{P}(\mathcal{K})$ by condition IV. Let $(G, \tau)_F$ denote the free discrete group with free generators of variety of groups in Mal’cev sense, generated by $(G, \tau)$ [69, Corollary VI.13.4]. The congruences of groups are permutable. Then by [71, Theorem 3] the identical mapping of set $G$ induces a homeomorphism between discrete space $(G, \tau)_F$ and non-discrete space $(G, \tau_1)$. We get a contradiction. Hence the primitive class $\mathcal{P}(\mathcal{K})$ does not exist.

3e). The basic object of researches in Lemmas 4, 7, Theorems 1, 2, Propositions 1, 2, Corollaries 5, 6, 7 is the primitive class of topological groups. Hence, by item 3d) the listed statements are erroneous, without sense.

e) DEFINITION 2. We consider a class of topological groups $\mathcal{K}$ and a set $X$. The group $F \in \mathcal{K}$ free topological group of space $X$ with respect to class $\mathcal{K}$, if:

1. $X$ is a subspace of space $F$.
2. $X$ topologically generate $F$.
3. For any continuous mapping "on" $\varphi : X \to G$, where $G \in \mathcal{K}$, it is exists a continuous mapping $\tilde{\varphi} : F \to G$ such that $\tilde{\varphi}/X = \varphi$.

If for a space $X$ in a class $\mathcal{K}$ a free topological group exists, then this group we denote by $F_{\mathcal{K}}(X)$.

From Lemma 3 it follows

COROLLARY 1. Let $\mathcal{K}$ be a non-empty class of zero-dimensional bicom pact groups and $\mathcal{M} = \mathcal{M}(\mathcal{K})$. For a space $X$ the group $F_{\mathcal{M}}(X)$ exists if and only if $\text{ind} X = 0$.

Thus, the object $F_{\mathcal{K}}(X)$ does not exists in any class $\mathcal{K}$, even in any variety. With other party, the Świerczkowski’s Theorem [81] assert that object $F_{\mathcal{K}}(X)$ exists for any completely regular set $X$, if $\mathcal{K}$ is a primitive class. Hence, the topologically notion of primitive class more flexibly, than notion of variety. In some works on topological algebras these notions are identified, what it is no correct (see, [17]). It can to result and to erroneous results. For example, the Theorems 1 - 4 and them corollaries from [18] are erroneous.

3f). The reference of Świerczkowski’s Theorem [81] is inaccurate by item 3d).

According to Theorem 8 of this paper and item 3b), the Corollary 1 is also
senseless. Hence, the assertion that the Theorems 1 - 4 and their corollaries from [18] are erroneous is not correct (see, also, items 2i), 2k)).

3g). Similarly to Corollary 1, according to Theorem 8 of this paper and item 3b), the following excerpt (page 133) is also a gross mistake. Let a class $\mathcal{K}$ satisfies to conditions I, II of item b), i.e. is a prevariety. In such case for any space $X$ exists $F_K(X)$ as soon as ind$X = 0$. Theorems 5, 6, 7 are presented and they claim that they can be proved with the help of the presented excerpt and methods from §2, consisting only of mistakes. Hence, the Theorems 5, 6, 7 are not proved, actually they are erroneous.

d). (pages 125, 126). The Theorem 1 is more general, than results from Bel’nov’s work [13]. In particular, from Theorem 1 it follows

COROLLARY 2. Every metrizable topological group is a quotient group of some zero-dimensional metrizable topological group.

e). The Corollary 2 gives the complete answer to one A. V. Arhangel’skii question [1] and independently is proved by Bel’nov V. K [13].

f). Corollary 7 gives an answer to other question from Arhangel’skii’s work [1] (page 127).

3i). As noted in item 3e), Theorems 1, 2 and Propositions 1, 2 are inaccurate. Therefore, their Corollaries 2 – 7 are also inaccurate. Hence, the Arhangel’skii questions from items e), f) are not solved.

3j). On a deceit, on senseless of item e) has got the Professor A. V. Arhangel’skii. Paper [2, page 1037] quotes the false statement e).

h). The Corollaries 8, 9, 11, 12 generalize many statements of Michael’s paper [72], of Gleason’s paper [56], of Bartle’s and Grawes’s work [12], of Kenderov’s paper [64] ... The Corollary 13 generalizes many statements of Hofman’s and Mostert’s monograph [62] and known Iwasawa Theorem [59]. I think that the comments to items g), h) are clear.
3k). To result from items a) – h), 3a) – 3j) to ascertain, that the item 2f) concern and to papers [34], [35], [36]. The paper [34] consist only from erroneous nonsense statements. [34] is an anti-scientific paper.

4a). The senseless and incorrect notions and results from the above analyzed works [29], [55] about operations and arity of operations (items 1a), 1b), 2a)), about continuous signature (items 1m), 1e), 1h), 2b)), about various definitions of quasivarieties (items 1e1) – 1g3)), about existence and the interrelations of various definitions of free topological algebraic systems (items 1h) – 1l), 2b) – 2d)) and many others are used in other works, which implies incorrect and many times senseless results of these works. We present only some examples. But before that let us present some well-known facts.

4a1). We present the following literally.

THEOREM 3 ([71]). Let \( A \) be a topological algebra with defining space \( X \) and with defining relations \( S \) in primitive class (variety), defined by identities \( V \) and let \( B \) be a topological algebra with defining space \( Y \) and with defining relations \( T \) in primitive class, defined by identities \( W \). We suppose that such a continuous mapping \( \tau \) of space \( X \) on space \( Y \) is given that every equality from \( S \), written for the corresponding elements \( Y \), is contained in \( \tilde{N} \), and let every identity of \( S \) be contained in \( T \). Then the mapping \( \tau \) can be continued to continuous mapping \( \tau^* \) of algebra \( A \) on algebra \( B \). This homomorphism will be open if in the class, defined by identities \( V \) the congruences are permutable, \( B \) topologically containing \( Y \) the mapping \( \tau \) of \( X \) on \( Y \) is homeomorphism.

4a2). Every topological algebra of primitive class \( \mathfrak{K} \) is an image of continuous homomorphism of free algebra of class \( \mathfrak{K} \) according to Theorem 3. But to construct a topological quotient algebra the homomorphism should be not only continuous, but also open. One such sufficient condition is presented in Theorem 3. But in general this is not correct. [71, pag. 177] contains an example of a continuous homomorphism of free semigroup which is not open.

4b). We analyze briefly papers [31], [32]. These papers consider the topological algebras of continuous signature and no limitations are imposed on the arity of basic operations. In such a case Proposition 1 and Lemma 2 of the given paper cannot be applied. How could we speak about the soundness of these papers’ results if the author Academician Choban M. M. is unaware of the basic notions of algebras, which can be found in any manual and monograph on algebra [69], [48], [66]: terms, polynomials, identities, varieties, replica ([31, pag. 185, Definition 5.5], [32, paragraph 2.4; 3.1], signature, arity ([31, Problems 4, 4], direct and Cartesian products ([32, pag. 14] and many others. We will present in form of excerpts from [31].
The discrete sum $E = \bigoplus \{ E_n : n \in N = \{0, 1, 2, \ldots\} \}$ of topological spaces $\{ E_n : n \in N \}$ is called the continuous signature.

**Definition 1.1** An $E$-algebra or a universal algebra of the signature $E$ is a family $\{ G, e_{nG} : n \in N \}$ for which:

1. $G$ is a nonempty set.
2. $e_{nG} : E_n \times G^n \to G$ is a mapping for every $n \in N$.

Let $i \in \{-1, 0, 1, 2, 3, 5\}$. If a topological $E$-algebra $G$ is a $T_i$-space, then $G$ is called $T_i$-$E$-algebra.

If $p \in E_0$ and $G$ is an $E$-algebra, then $e_{0G}(\{p\} \times G^0) = 1_p$. If $n \in N$ and $q \in E_n$, then $q : G^n \to G$, where $q(x_1, \ldots, x_n) = e_{nG}(q, (x_1, \ldots, x_n))$, is an operation of type $n$ on $G$. (Correct: operation of arity $n$).

4c). The set of terms $T(E)$ is the smallest class of the operations on the $E$-algebras such that:

1. $E \subseteq T(E)$ and $e_G \in T(E)$, where $e_G(x) = x$ for every $x \in G$.
2. If $n > 0$, $e \in E_n$ and $u_1, \ldots, u_n \in T(E)$, then $e(u_1, \ldots, u_n) \in T(E)$.

Let $1 \leq m < n$, $N_m = \{1, 2, \ldots, m\}$ and $h : N_n \to N_m$ (onto) be a mapping. The operation $\omega : G_n \to G$ of type $n$ and $h$ generates the operation $\psi : G_m \to G$, where $\psi(x_1, \ldots, x_m) = \omega(x_{h(1)}, \ldots, x_{h(n)})$, and $\psi$ is called an $h$-permutation of the operation $\omega$. (It is not allowed to change the arity of basic operation, i.e. the signature of given algebra).

4c). The set of the polynomials $P(E)$ or of the derived operations is the smallest class of operations on $E$-algebras such that:

1. $T(E) \subseteq P(E)$.
2. If $f \in P(E)$ and $g$ is an $h$-permutation of $f$, then $g \in P(E)$.

4c). If $\omega$ and $\psi$ are polynomials of types $n$ and $m$, then the form $\omega(x_1, \ldots, x_n) = \psi(y_1, \ldots, y_m)$ is called an identity on the class of $E$-algebras.

4c). Any comments are irrelevant. Similarly to item 1q), the presented definitions are a delirium, senseless statements.

4d). Sections 1, 2 contain the definitions of $T_i$-quasivariety, of $T_i$-variety, of complete $T_i$-quasivariety, of complete $T_i$-variety of topological $T_i$-$E$-algebras similarly to item 1e1.

4d). For example, a class $\mathfrak{K}$ of topological $T_i$-$E$-algebras is called a $T_i$-quasivariety if $\mathfrak{K}$ is closed with respect to subalgebras and Tychonoff product.

4d). It contains the definitions of topologically free algebra $(F(X, \mathfrak{K}), i_X)$ of a space $X$ in a $T_i$-quasivariety $\mathfrak{K}$ and of algebraically free algebra $(F^a(X, \mathfrak{K}), j_X)$ of a space $X$ in a $T_i$-quasivariety $\mathfrak{K}$ as in item 1g3), which are presented in item 6c). Almost all results of paper [31] are related to free algebras or the free algebras are a crucial instrument to prove the statements.
of the analyzed paper. Item 6d1) shows that this notions are not corrects, senseless. Hence, all results from the analyzed paper [31] are false, many of them are absurd.

We present the following literally.

4e1). **Definition 5.5.** Let \( E \) be a continuous signature. The class \( K \) of \( E \)-algebras is called a Mal’cev class if there exists a polynomial \( p(x, y, z) \) such that the equations \( x = p(y, y, x) = p(x, y, y) \) hold identically in \( K \). The polynomial \( p(x, y, z) \) is called a Mal’cev polynomial.

4e2). **Lemma 5.8 (Mal’cev [70] for finite \( E \)).** Let \( g : A \to B \) be a homomorphism of a topological \( E \)-algebra \( A \) with the Mal’cev polynomial onto an \( E \)-algebra \( B \). Then the \( E \)-algebra \( B \) equipped with the quotient topology is a topological \( E \)-algebra and the quotient mapping \( g \) is open.

Proof. Let \( p(x, y, z) \) be a Mal’cev polynomial on the algebra \( A \) and \( V \) be an open set in \( A \). We prove that \( U = g^{-1}(g(V)) \) is open in \( A \) (this statement was proved by Mal’cev [70]. Fix a point \( x \in U \). Then \( g(x) = g(y) \) for some \( y \in V \). 

4e3). The Definition 5.5 is unclear, it does not correspond to the generally accepted notion of Mal’cev algebra (see Theorem 1 of this paper) according to item 4c4) about the senselessness of notion of polynomial.

The expression \( x = p(y, y, x) = p(x, y, y) \) are identities, rather than equations.

The referral to [70] is the Theorem 10 from [70]. It cannot be used in the proof of Lemma 5.8 according to item 4e3). Hence, Lemma 5.8 is not proved. From here it follows that all results of Sections 5, 6 are false.

4e4). Items 4a), 4b) – 4e3) confirm only partially the following statement: paper [31] consist only of erroneous, senseless statements and introduced notions. [31] is an anti-scientific paper.

5a). Moreover, the results from [54], [65], presented in Sections 5, 6 from [31] (see item 4c8)) are false. Consequently, the results of works [54], [65] are false.

Now let us analyze paper [32].

6a. Section 2 contains the same senseless notions as in [31]: algebra of continuous signature (\( E \)-algebra), various notions of quasivariety, polynomials, terms, identities, free algebras. The list for the analyzed paper is complemented with the following notions: constants (see [69, pag. 335], supports of elements, replica, saturated classes of algebras and others. We will present them literally.

6b. Let \( D \subseteq E \) and \( D_n = D \cap E_n \) for any \( n \in N \). Then \( D \) is a continuous
signature and every topological $E$-algebra is a topological $D$-algebra, where $d_{nA} = e_{nA}|D_n \times A^n$ for any $n \in N$. Moreover, every $E$-homomorphism $f : A \to B$ of $E$-algebras $A$ and $B$ is a $D$-homomorphism, too.

6c. Definition 2.2.1. Fix $i \in \{-1; 0; 1; 2; 3; 3, 5\}$, a $T_i$-quasivariety $K$ of topological $E$-algebras and a topological $D$-algebra $X$. Then:

(T) A couple $(F(X, K, D), i_X)$ is called a topological $T_i$-free $E$-algebra of the $D$-algebra $X$ in the class $K$ if the following conditions hold:

((1)) $(F(X, K, D) \in K$ and $i_X : X \to (F(X, K, D)$ is a continuous $D$-homomorphism;

(2) the set $i_X(X)$ generates $(F(X, K, D)$.

(3) for each $D$-homomorphism $f : X \to G \in K$ there exists a continuous $E$-homomorphism $\overline{f} : (F(X, K, D) \to G$ such that $f(x) = \overline{f}(i_X(x))$ or every $x \in X$.

(A) A couple $(F^a(X, K, D), j_X)$ is called a $T_i$-free $E$-algebra of the $D$-algebra $X$ in the class $K$ if the following conditions hold:

((4)) $(F(X, K, D) \in K$ and $i_X : X \to (F(X, K, D)$ is a $D$-homomorphism;

(5) the set $j_X(X)$ generates $(F^a(X, K, D)$.

(6) for each $D$-homomorphism $f : X \to G \in K$ there exists a continuous $E$-homomorphism $\overline{f} : (F^a(X, K, D) \to G$ such that $f(x) = \overline{f}(i_X(x))$ or every $x \in X$.

6d. Theorem 2.2.3 Let $K$ be a $T_i$-quasivariety of topological $E$-algebras. Then for every topological $D$-algebra $X$ there exist:

- a unique $D$-free algebra $(F^a(X, K, D), j_X)$;
- a unique topologically $D$-free algebra $(F^a(X, K, D), i_X)$
- a unique continuous homomorphism $k_X : F^a(X, K, D) \to F(X, K, D)$ and $i_X = k_X \cdot j_X$.

Proof. Let $\tau$ be an infinite cardinal and $\tau \geq \text{max}\{|X|, |E|\}$. Then the collection $\{f_\xi : X \to G_\xi : \xi \in L\}$ of all $D$-homomorphism of $X$ into the $E$-algebra $G \in K$ of the cardinality $\leq \tau$ is a non-empty set. We consider the diagonal product $j_X : X \to G = \prod \{G_\xi : \xi \in L\}$, where $j_X(x) = (f_\xi : x) : \xi \in L\}$, and denote by $F^a(X, K, D)$ the $E$-subalgebra of $G$ generated by the set $j_X(X)$. If $f : X \to A \in K$ is a $D$-homomorphism, then for some $\xi \in L$ we have $G_\xi = d(E; f(X))$ denote the $E$-subalgebra generated by $f(X)$ and $f_\xi$. The natural projection $\overline{f} : F^a(X, K, D) \to G_\xi \subseteq A$ is the homomorphism generated by $f$.

If $M = \{\xi \in L : f_\xi$ is continuous}, then we consider the diagonal product $i_X : X \to G' = \prod \{G_\xi : \xi \in L\}$ and $(X, K, D)$ is the $E$-subalgebra of $G'$
generated by the set $i_X(X)$. For some $\xi \in M$ we have $F(X, K, D) = G_\xi$ and $i_X = f_\xi$. The proof is complete.

6d). The uniqueness of $D$-free algebra $(F^a(X, K, D), j_X)$ and topologically $D$-free algebra $(F^a(X, K, D), i_X)$ is not proved at all. Proofs of similar statements can be found in [71] and Corollary 3 of this paper. The proof of the third statement is obviously wrong. The very first sentence of Proof is not correct. The Proof is meaningful only in case when $|\prod\{G_\xi : \xi \in L\}| \leq \tau$.

6c). From items 6a) – 6d) it follows that paper [32] can be appraised in the same manner as item 4c) for [31], i.e. [32] it is an anti-scientific paper.

7a). Moreover, in [32] is mentioned that [37], [38] were examinees the same questions about free objects, various types of equivalence that [?]. From here we conclude that the results from [37] and [38] are false.

Homogeneous algebra, rectifiable space, Mal’cev operation

In terms of scientific importance the paper [28], published in journal: Serbica, Bulgaricae mathematicae publications, does not differ from the previous paper [29]. It, in general, consists of easily noticed gross errors and incorrect or senseless statements.

In the beginning of the paper, following [29], presents the definitions of topological algebra of continuous signature (the basic operations have a finite arity), calls $E$-algebra, of quasivariety $K$ of $E$-algebras as class of all $E$-algebras of given continuous signature with conditions 1m. 2m. For quasivariety $K$ presents the Definition 8 and the Property 2.

Further, for a quasivariety $K$ of topological $E$-algebras present Theorems 01, 02 as analogical to Theorems 2, 3 from [70], proved for primitive classes (varieties) of algebras. Claims that the proofs of Theorems 2, 3 for primitive classes of algebras are valid for Theorems 01, 02. But it is not true. Really, the existence of free algebras for primitive classes is used essentially to prove Theorems 2 and 3. However, for quasivarieties it is not true. It follows from Theorem 7 of this paper, if to ignore the requirement of continuous signature by item 1e).

The paper contains only three statements with proof, specifically Theorems 1.1, 2.1, and 3.1. Theorem 1.1 investigates such regular $\sigma$-compact $E$-algebras for which the signature $E$ is a $\sigma$-compact space. By item 1e) the requirement of signature can be ignored. Then, Theorem 1.1 and its proof literally coincides with the appropriate result from [74].

Further, let us consider the excerpt from [28].

Theorem 2.1. Let $K$ be a quasivariety of topological $E$-algebras and let the free algebra $F(D, K)$ be discrete. Then the following conditions are equivalent:
1. Any algebra \( A \in \mathcal{K} \) has permutable congruences;
2. There exists a term \( p(x, y, z) \) such that the equations \( p(x, y, y) = p(y, y, x) = x \) hold identically in \( \mathcal{K} \);
3. Any congruence in an algebra of \( \mathcal{K} \) is open.

We ignore the requirement about quasivariety and about \( E \)-algebra, because, as mentioned above, they are fictitious. Let’s assume that \( \mathcal{K} \) is a primitive class of algebraic systems. Then the equivalence 1) \( \iff \) 2) is the famous Mal’cev’s Theorem, the Theorem 1. The implication 1) \( \implies \) 3) is not less remarkable Mal’cev’s result, the Theorem 10 from [70] (the congruence, considered in Theorem 10 are called complete congruences, but not open congruences). See also Theorem 4 and Corollary 2 of this paper.

Let the implication 3) \( \implies \) 1) hold. Then by [60, Proposition 2.4.9] to equivalent conditions 1), 2), 3) of Theorem 2.1 it is possible to add and equivalent conditions 1), 2), 3) of Corollary 3, and also the items 4), 5) of Corollary 3. In result the Mal’cev’s results mentioned above is very much amplified. But, naturally, it not so, it is utopia. From here, it is even possible to see by intuition that the implication 3) \( \implies \) 1) is false. It is also possible to see that the author of paper [28] has not understood the algebraical and topological essences of Mal’cev Theorems.

Actually, without going into too many details we will note that the proof of implication 3) \( \implies \) 1) of Theorem 2.1 is false. Particularly, to prove this implication the author uses essentially Properties 1, 2 from [29], i.e. the existence of free topological algebras in any quasivariety. As shown during the analysis of paper [29], Properties 1 and 2 cannot be regarded as proved. More specific, their proofs consist of a set of gross errors, see items 1j) - 1l). Further, after Theorem 2.1 one may find erroneous statement presented without any proof as Corollaries 2.1, 2.2, 2.3 and Remarks 2.2, 2.4. It is claimed that they can be proved with the help of the false Theorem 2.1.

Before starting the analysis of Theorem 3.1 we will present an excerpt from [28].

\[
\text{If for the } E\text{-algebra } A \text{ there exist two polynomials } p(x, y) \text{ and } q(x, y) \text{ with equations } p(x, x) = p(y, y), p(x, q(x, y)) = y, \text{ then } A \text{ is called a homogeneous algebra}^{(5)}. \quad \text{We have } q(x, p(x, y)) = y.^{(2)} \quad \text{Let } \ldots P_a(x) = p(a, x), Q_a(x) = q(a, x) \ldots \text{Then } \ldots P_a \text{ and } Q_a \text{ are homeomorphisms and } Q_a^{-1} = P_a^{(3)}. \quad \text{An algebra } A \text{ is homogenous iff } A \text{ is a biternary algebra}^{(4)}. \ldots
\]

\[
\text{The mapping } g : X \times X \to X \times X \text{ is a centralizing mapping with center } a \in X \text{ if } g \text{ is a homeomorphism}^{(5)}, g(x, x) = (x, a) \text{ and } \{x\} \times X =
\]
g(\{x\} \times X) for every \(x \in X\). The space \(X\) is called a centralizable space if there exists a centralizable mapping for \(X\) (see [76]).

Theorem 3.1 The space \(X\) is a centralizable space iff for some two continuous binary operations \(X\) is a homogeneous algebra.

Proof. Let for the continuous binary operations \(p\) and \(q\) the space \(X\) be a homogeneous algebra. Then \(h(x, y) = (x, p(x, y))\) is a centralizing mapping with centre \(p(x, x)\). Let \(h\) be a centralizing mapping with centre \(a \in X\). We put \(g(x, y) = y, p(x, y) = g(h(x, y))\) and \(q(x, y) = g(h^{-1}(x, y))\). Then \(p(x, x) = a\), and \(q(x, p(x, y)) = y\).

(1) The \(E\)-algebras are considered only here. By item 1e) a question arises: what is the role of the continuous signature of \(E\) in the definition of homogeneous algebras? Further, an elementary remark: the notions of equation and identity are totally different. The expressions \(p(x, x) = p(y, y), p(x, q(x, y)) = y\) are not equations, but identities.

According to item 5) of Theorem 11 the identities \(p(x, x) = p(y, y), p(x, g(x, y)) = y, q(x, p(x, y)) = y\) transform a non-empty space \(A\) into a topological right loop \((A, \cdot, \backslash, e)\), where \(e = p(x, x), x \cdot y = q(x, y), x \backslash y = p(x, y)\). The solution of equation \(a \cdot x = b\) follows from identity \(p(x, g(x, y)) = y\) and its uniqueness follows from identity \(q(x, p(x, y)) = y\). For a groupoid \((A, \cdot)\) the existence and the uniqueness of solution are independents (see, [15], [14], [79]). Then on space \(A\) the identity \(q(x, p(x, y)) = y\) does not follow from identity \(p(x, g(x, y)) = y\) and the identity \(p(x, q(x, y)) = y\) does not follow from identity \(q(x, p(x, y)) = y\). Consequently, these identities defined on topological space are independents. Hence the assertion (2) is false. In this case the definition of homogeneous algebra from this paper is not correct with respect to the corresponding definitions from [83], [84], [57].

Under definition (1) of homogeneous algebra the assertion (3) is false. But under the corrected definition the assertion (3) is not an obvious fact. For the proof of (3) under such a condition see the comment on assertion (19), presented below.

The assertion (4) is false. For more details see the comment on assertion (11).

(5), (6). As is mentioned in the beginning of this paper the unique distinction between classes of centralizable spaces and rectifiable spaces is that for rectifiable spaces in the definition of centralizable spaces it is necessary to require that the homeomorphism \(g : X \times X \rightarrow X \times X\) be surjective. Otherwise the operation \(q(x, y) = g(h^{-1}(x, y))\) is not determined for all \(x, y \in X\).

The assertion (7) is not an obvious fact, but not proofs are given for (7).
Moreover, \((7)\) does not correspond to definition \((1)\).

Further, the following are included without any proof

Lemma 3.1. Let \(f : A \to B\) be a homomorphism of the homogeneous topological \(E\)-algebra \(A\) onto the \(E\)-algebra \(B\). Then for every \(x, y \in B\) the spaces \(f^{-1}(x)\) and \(f^{-1}(y)\) are homeomorphic.

Lemma 3.1 holds when \(A, B\) are topological groups. According to Theorems 9 and 10, it holds when \(A, B\) are topological loops. But Lemma 3.1 is false when \(A, B\) are homogeneous topological \(E\)-algebras. We ignore the senseless statement that \(A, B\) are \(E\)-algebras, i.e. we ignore the condition of continuous signature. According to \((1)\) and \((9)\) below let’s consider that the notion of homogeneous algebra is defined correctly. In this case by item 5) of Theorem 11 \(A, B\) are topological right loops.

Recall that a congruence is called correct, if it is defined unequivocally by any adjacent class ([70]). Let \(x, y\) be arbitrary elements in topological right algebra \((A, \cdot, \setminus, f)\). If \((A, \cdot, \setminus, f)\) is not a left loop then from the definition of right loop it follows that there exist such elements \(x, y\) in \(A\) that \(T(x) \neq y\) for any translation \(T (A, \cdot, \setminus, f)\). It means that \((A, \cdot, \setminus, f)\) does not satisfy a necessary condition for a congruence of algebraic system to be correct [70, pag. 10]. Hence Lemma 3.1 is false. This statement also follows from item 6) of Corollary 21.

From the above-stated it follows that the false Lemma 3.1 was written as follows. The expression ”topological group” was replaced mechanically with expression ”homogeneous topological \(E\)-algebra” in true version of Lemma 3.1 for topological groups.

The following can be noticed from the above-mentioned comments.

a). The listed properties (without proofs) for homogeneous algebras defined by \((1)\) are false. Theorem 3 is false for centralizable spaces. As result, the proof of Theorem 3 represents a chaotic set of senseless and unrelated statements.

b). The listed properties, except for (4), and Theorem 3 hold for the modified definitions of homogeneous algebras and of centralizable spaces (called rectifiable spaces) from [83], [84], [57]. The first two papers were published before the analyzed paper was submitted for publication.

c). Unless knowing beforehand the results of papers [83], [84], [57], it is not possible to understand and verify the soundness of the statements from the excerpt, in particular Theorem 3 and its proof.

d). In the beginning of the article it was mentioned that, according to [22], the notion of rectifiable space was introduced at the seminar of Prof.
Arhangel’skii. I believe that the above-mentioned excerpt is an unsuccessful compilation of statements, collected during the mentioned seminar but not understood by the author himself.

e). The above excerpt, in particular Theorem 3 and its proof, cannot be regarded as a mathematical research.

The content of the remaining part of the given paper covers the entire spectrum of topological science, and probably, the entire set of topological notions. It presents plenty of remarks and open questions, which are compiled by the following principles. In text of the original, mainly on topological groups, the expression "topological group" is replaced by "homogeneous topological $E$-algebra" (for example, the Lemma 3.1) and it is stated that they are proved by same methods as the original, using not only the incorrect, but as shown above, the senseless Theorem 2.1 and false Lemma 3.1.

Other principle underlying the listed remarks and open questions is the incompetence of the author and groundlessness introduction of new concepts.

Let’s confirm this by excerpts.

≪Let $E = E_3 = \{p\}$. We denote by $V$ the class of all topological $E$-algebras with Mal’cev equations $p(x, y, y) = p(y, y, x) = x$. By $W_1$ we denote the class of all algebras $A \in V$ with the equation $p(y, x, y) = x$. Further, let $W_2$ designate the class of all algebras $A \in V$ with the equation $p(x, y, x) = x$ and $W = W_1 \cup W_2$.≫

There is not any connection between the definitions of classes $V, W_1, W_2, W$ and topology of topological $E$-algebras. Further, Mal’cev identities is the correct form, and not Mal’cev equations. The author has not understood the meaning of Theorem 1 (or Theorem 3), the importance of which is shown in Theorem 10 of [70] (or Theorem 4, Corollary 2). There is not any connection between Mal’cev equations of $E$-algebras and ternary operation $\Psi(x, y, z)$ of Theorems 1, 3. It follows from item 4) of Proposition 13. Therefore it is inadmissible to name the identities $p(x, y, y) = p(y, y, x) = x$ as Mal’cev equations. Consequently, if one follows this principle, then it is possible to introduce and investigate classes of $E$-algebras, defined by arbitrary operations and arbitrary identities, which obviously is not correct.

As a consequence of the above-stated, we list the remarks and open questions that are incorrect. Let us present several examples.

≪Proposition 4.2. If $\mathcal{K} \subset W$ is a quasivariety, then $\mathcal{K} \subset W_1$ or $\mathcal{K} \subset W_2$.≫. Without sense and trivial assertion, at than here requirement about quasivariety. The proposition is false. This follows from following example. Let $A$ be an $E$-algebra consisting from one element. Then $A \in \mathcal{K}$, $A \in W$,
A \in W_1, A \in W_2.

≪Remark 4.3. If X is a zero-dimensional metrizable space, then: 1. \(X \in W_1\) for some \(E_{3X}\); 2. \(X \in W_2\) for some \(E_{3X}\).≫. The incompetence follows from not knowledge of definition of zero-dimensional space. No any connection between it notion and definitions of \(W_1, W_2\).

≪Question 4.1. Let \(A \in V\) be a pseudocompact space. Is \(A^2\) a pseudocompact space?≫. According to Proposition 2 \(A \in A^2\). The requirements \(A \in V\) and the pseudocompactly of space are independents. The answer to this question is negative, as the family of all pseudocompact spaces is not closed with respect to product of spaces [60, pag. 746].

≪Question 4.2. Let \(A\) be a pseudocompact homogeneous algebra. Is \(A^2\) a pseudocompact space?≫.

≪Question 4.3. Let \(A \in V\) be a compact space. Is it true that \(\dim A = \text{ind} A?≫\) Rash question. Conditions \(A \in V\) and compact space are independents. An example of compact space \(X\) for which \(\dim X \neq \text{ind} X\), see [60, pag. 612]. For every metrizable compact spaces \(X\) \(\dim X = \text{ind} X\), see [60, pag. 600].

Now we pass to analysis the paper [4]. Theorems 2, 3, 4, 5, 9, Proposition 9, and Corollary 6 have the following form:

Suppose that \(B\) is a compact Hausdorff space . . . Suppose further that \(B = X \cup Y\), where \(X, Y\) are non-locally compact rectifiable spaces. Then . . .

Proof. Clearly, \(Y\) and \(X\) are non-empty, since they are not locally compact. Hence, \(B\) is non-empty. . . .

According to Proposition 15 the spaces \(X, Y\) are compact. Then, obviously, they are locally compact (see, [63, pag. 196]). Consequently, Theorems 2, 3, 4, 5, 9, Proposition 9 and Corollary 6 are trivial, they hold only for \(B = \emptyset\). But even without this remark the proofs of these theorems are erroneous. Let’s note only that the proofs use essentially the Dichotomy Theorem for remainders of rectifiable spaces from [5]. When analyzing paper [5] we will show that in [5] this theorem is not proved.

The expression ≪It is obvious that if a \(k\)-gentle paratopological group is a \(k\)-space, then this paratopological group is a topological group≫ from the proof of Proposition 2 is not clear (and is incorrect).

Based on the above-mentioned we conclude that all statements from sections 2, 3 (Theorems 2, 3, 4, 5, Propositions 1, 2, Example 7) are either trivial, or senseless and their proofs are erroneous. Let’s also note that in the beginning of section 2 a definition of rectifiable space is presented from papers [27], [28]. But it is not correct. These papers do not define the rectifiable
space; moreover, the term "rectifiable space" cannot be even found in these papers (see, the analysis of these papers above). The circumstances how this definition appeared are presented in the beginning of this article.

Further, section 4 begins with the following excerpt.

"A Mal’cev operation on a space \(X\) is a continuous mapping \(\mu : X^3 \to X\) such that \(\mu(x, x, z) = z\) and \(\mu(x, y, y) = x\), for all \(x, y, z \in X\). A space is called a Mal’cev space if it admits a Mal’cev operation (see [27], [28], [31], [70], [84]) (8).

"A homogeneous algebra on a space \(G\) is a pair of binary continuous operations \(p, q : G \times G \to G\) such that \(p(x, x) = p(y, y)\), and \(p(x, q(x, x)) = y\), \((x, p(x, y)) = y\) for all \(x, y \in G\) (9). If the above conditions are satisfied, then "the ternary operation \(\mu(x, x, z) = q(x, p(y, z))\) (10) is a Mal’cev operation (see [9, 10]).

"(In [9, 10] (see also [19]) it was proved that for an arbitrary space \(G\) the following conditions are equivalent:

1) \(G\) is a rectifiable space;
2) \(G\) is homeomorphic to a homogeneous algebra;
3) There exists a structure of a biternary algebra on \(G\).

A structure of a topological quasigroup on a space \(G\) is a triplet of "binary continuous operations \(p, l, r : G \times G \times G \to G\) such that \(p(x, l(x, y)) = p(r(y, x), x) = l(x, p(x, y)) = l(r(x, y), x) = r(p(y, x), x) = r(x, l(y, x)) = y, for all \(x, y \in G\) (12). If there exists an element \(e \in G\) such that \(p(e, x) = x\) for any \(x \in G\), then we say that \(G\) is a topological loop and \(e\) is the identity of \(G\). "Any topological quasigroup admits the structure of a topological loop (see [70]) (13). "If \(e \in G\) and \(p(e, x) = x\) for any \(x \in G\), then \(x + y = p(y, x)\) and \(x \Delta y = r(y, x)\) is a structure of a homogeneous algebra (14). "If \((G, \Delta)\) is a topological group with the neutral element \(e\), then the mapping \(\varphi(x, y) = (x, x^{-1} \Delta y)\) is a rectification on the space \(G\) with the neutral element \(e\), and the mappings \(p(x, y) = x^{-1} \Delta y\) and \(q(x, y) = x \Delta y\) form a structure of homogeneous algebra on \(G\). Therefore, "every topological quasigroup is a rectifiable space (16) (15).

(8) The need to study the Mal’cev operation and Mal’cev space follows from Mal’cev’s paper [70], which puts the foundation of topological algebraic systems. The cited literature specifies several papers [27], [28], [31], [70], [84] that study (use) these notions. We can add papers [83], [85], [57], as well as [4], [6], [7], [8], where various generalizations are introduced. However, some peculiarities arise during their consideration. The definitions of Mal’cev operation and Mal’cev space are too general and it is impossible to apply to
them the fundamental results from [70].

Theorem 1 characterizes an algebraic systems with permutable congruences. For this purpose on such algebraic systems a ternary operation $\Psi(x, y, z)$ is defined with identities (5), i.e. (a Mal’cev operation $\Psi(x, y, z)$ is defined. But in item 4) of Proposition 13 a loop is indicated, which admits a Mal’cev operation with non-permutable congruences. Such a situation occurs because in Theorem 1 the ternary operation should be a polynomial (from the basic operations) of the given algebraic system.

To avoid such misunderstandings some authors (see, for example, [78]) do the following. Define a Mal’cev algebraic system as algebraic system with permutable congruences. Then for such Mal’cev algebraic systems it is possible to apply Theorem 1. Moreover, for the Mal’cev algebraic systems defined in such a way it is possible to apply the important Theorem 10 from [70] (see also the Theorem 4 of this paper): every quotient homomorphism of Mal’cev algebraic system is open.

Some authors, [83], [84], apply Theorem 10 in the following way: for Mal’cev operation $[,]$, defined generally, they consider only those homomorphisms $\varphi$ of Mal’cev space $(A, [, ,])$ which are consistent with Mal’cev operation $[,]$: $\varphi[x, y, z] = [\varphi x, \varphi y, \varphi z]$ for all $x, y, z \in A$. Then from here (or Theorem 1 and [70, Theorem 10]), Proposition 10 and the equivalence $1) \iff 2)$ from (11) below we get.

Every homomorphism of a topological right loop (respect. homogeneous algebra) is open.

(9). The notion of homogeneous algebra, presented here, differs from the notion of homogeneous algebra from [28]. I believe that the term ”homogeneous algebra” is not successful: it should be concretized. Though obviously there is no connection between the notion of homogeneous algebra and the notion of homogeneous space, I think that the term ”homogeneous algebra” appeared under the influence of the notion of strongly homogeneous space from [84] (or something similar). Recall, that for a compact $K$ by Aut $K$ we denote the group of all autohomeomorphisms of space $K$, provided with compact open topology. A compact $K$ is called strongly homogeneous if there exists a continuous mapping $(x, y) \rightarrow h_{x, y}$ from $K^2$ into Aut $K$ such that $h_{x, y}(x) = y$ for all $x, y \in K$.

According to (23) any homogeneous algebra coincides with a topological right loop. By Corollary 1 the underling space of any topological right loop if homogeneous. Consequently, the underling space of any homogeneous algebra is topologically homogeneous. Moreover, from item 5) of Theorem 11 and
Proposition 15 from [84] it follows: a compact $K$ is a homogeneous algebra if $K$ is a strongly homogeneous space.

(10). At the first glance it seems that the senseless expression $\mu(x, x, z) = q(x, p(y, z))$ is a typo. But similar senseless statements can be found and in other works of the author (see, for example, [5],[55]).

(11). Strange things occur in connection with the proof of the equivalence of conditions 1) and 2). The authors of this paper [4] specify that this equivalence is proved in [28]. But according to assertions (6) - (7) the proof of equivalence of conditions 1), 2) is and absurdity, it cannot be called a mathematical proof. It is a set of basically unproved and unrelated statements. As is mentioned in (5) even the notion of rectifiable space is not correct.

The equivalence of conditions 1), 2) is proved in [83], [84], [57]. These works give various correct definition of rectifiable space. Clearly, the authors of paper [4] (at least M. M. Choban) were aware of these papers, but do not make any references to them and assign this result to M. M. Choban. On the other hand, it is weird and surprising that the authors of [83], [84], [57] assign this result to M. M. Choban, but fail to indicate where is was published.

The term "rectifiable space" and the equivalence of items 1), 2) are copied from [57], but this is not mentioned in the article. To hide this fact item 2) is written in such a "confusing" form. The following would be more correct: On underling space $G$ of rectifiable space $G$ there exist a structure of a homogeneous algebra.

The authors do not understand (or are not capable to understand) the meaning and importance of Mal’cev’s Theorem 2. The notion of biternary algebra is introduced in Theorem 2 as a characteristic notion (if and only if) for primitive classes of algebraic systems, is introduced as derived operation of algebraic systems. The implication 3) $\to$ 2) is false. This follows from item 5) of Proposition 5.

(12). According to (22) any quasigroup $(G, p, r, l)$ is defined by identities $p(x, l(x, y)) = y, l(x, p(x, y)) = y, p(r(y, x), x) = y, r(p(y, x), x) = y$ with respect to multiplication $p$, left division $l$ and right division $r$. The remaining identities $l(r(x, y), x) = r(x, l(y, x)) = y$ follow from the first and it is not clear why they are given. There exists six parastrophs with respect to the basic operation ($\cdot$) for any quasigroup $(Q, \cdot, \backslash, /)$ (by our terminology - equasigroup) [15], [14]. Then the given list can be complemented with many identities. It seems that the authors are not familiar with definition of equasigroup.

(13). This expression is too general, it does not contain any information. Unlike [70], it does not indicate the relation between quasigroup operation
and loop operation. We concretize\(^{(13)}\) for topological quasigroups. Recall that if \(Q_1\) and \(Q_2\) are topological quasigroups then we say that \(Q_1\) and \(Q_2\) are \textit{isotopic} if there are homeomorphisms \(f, g, h : Q_1 \to Q_2\) such that \(f(xy) = g(x)h(y)\). The triple \((f, g, h)\) is called an \textit{isotopy from} \(Q_1\) \textit{to} \(Q_2\). The concept of isotopy is one of basic tools in theory of topological projective planes.

Then from [16, pag. 5] (it follows one from forms of expression\(^{(7)}\). Let \(Q\) be a topological quasigroup and \(e\) any fixed element of \(Q\). If we take any homeomorphism \(f : Q \to Q\) with \(f(e) = ee\) (e. g. the one defined by \(f(x) = L(\langle ee \rangle/e)x = \langle ee \rangle/e \cdot x\)), then the underlying space of \(Q\) together with the multiplication \(x \circ y = f^{-1}(\langle f(e)/e \rangle \cdot f(y))\) define a topological loop \(L\) with unit \(e\); and the triple \((f, R(e)^{-1}f, L(e)^{-1}f)\) is an isotopy from \(Q\) to \(L\).

\(^{(14)}\) Unclear and senseless assertion. According to item 5) of Theorem 11 any homogeneous algebra is a topological right loop with right unit. Then instead of the erroneous \(p(e, x) = x\) more correct would be \(p(x, e) = x\). It is not clear why operations (+), (·) were introduced. Probably the authors wanted to show the affinity of homogeneous algebras with classical notions of algebras, rings.

\(^{(15)}\) Ingenious logic. From the absolutely illogical and unrelated statements (copied from different sources) the authors guessed and actually generalized the following result from [84]: every compact topological quasigroup is rectifiable. The statement\(^{(16)}\) holds. This follows from Proposition 11.

Propositions 4, 5, and Corollary 8, which use the notion of rectifiable space and the connected to it homogeneous algebra, Mal’cev space, are not on the above-mentioned list of senseless and false statements: Theorems 2, 3, 4, 5, Propositions 1, 2, and Example 7. But Propositions 4, 5, and Corollary 8 could be added on that list. As a confirmation of that we will analyze Proposition 13 and its proof.

Proposition 5. \(\ll\)Let \(X\) be a subalgebra of a homogeneous algebra \(G\). If the space \(G\) is regular\(^{(17)}\) and Lindelöf, and the space \(X\) is of pointwise countable type and is dense in \(G\), then there exist a separable metrizable homogeneous algebra \(G'\) and a homomorphism \(g : G \to G'\) such that \(X = g^{-1}(g(X))\) and the mapping \(g\) is open and perfect. In particular, it follows that \(X\) is a Lindelöf \(p\)-space.

Proof. \(\ll\)By the assumptions, there is a pair of binary continuous operations \(p, q : G \times G \to G\) on the space \(G\) such that:

\(p(x, x) = p(y, y)\), and \(p(x, q(x, y)) = y, q(x, p(x, y)) = y\) for all \(x, y \in G\);

\(p(x, y) \in X\) and \(q(x, y) \in X\) for all \(x, y \in X\).
We put \( e = p(x, x) \). If \( a \in G \), then \( p_a(x) = p(a, x) \) and \( q_a(x) = q(a, x) \) for any \( x \in G \). We have \( q_a^{-1} = p_a \) and \( q_a(e) = a \). Thus, \( p_a \) and \( q_a \) are homeomorphisms. Moreover, \( p_a(X) = q_a(X) = X \) for each \( a \in X \). \( \Rightarrow ^{(19)} \Rightarrow ^{(18)} \)

\( \Leftarrow \) Let \( F \) be a non-empty compact subspace of \( X \) with a countable base of open neighborhoods in \( X \). \( \Rightarrow ^{(21)} \) We can assume that \( e \in F \). \( \Rightarrow \) Since \( X \) is dense in \( G \), the set \( F \) also has a countable base of open neighborhoods in the space \( G \). Therefore, \( X \) and \( G \) are \( p \)-spaces \( \Rightarrow ^{(9)} \) (see [6], Proposition 2.1). \( \Rightarrow ^{(20)} \)

\( \Leftarrow \) Since \( F \) is a compact \( G_δ \)-subset of the Lindelöf algebra \( G \), there exist a separable metrizable homogeneous algebra \( G' \) and a homomorphism \( g : G \to G' \) such that \( F = g^{-1}(g(F)) \) and \( g \) is a perfect mapping [28]. \( \Rightarrow ^{(23)} \) The quotient homomorphism of a Mal’cev algebra is an open mapping [28]. \( \Rightarrow ^{(24)} \) Thus, the mapping \( g \) is open. \( \Rightarrow ^{(25)} \) We can assume that \( e' = g(e) \) and \( p(z, z) = e' \) for any \( z \in G' \).

\( \Leftarrow \) Let \( b \in g(X) \subseteq G' \). Fix \( a \in X \cap g^{-1}(b) \). If \( H = g^{-1}(e') \), then \( H \subseteq F \subseteq X \) and \( q_a(H) = g^{-1}(qb(e')) \subseteq X \). Thus, \( g^{-1}(b) = g^{-1}(qb(e')) \subseteq X \). Therefore, \( g^{-1}(g(X)) = X \). The proof is complete. \( \Rightarrow ^{(26)} \)

(17). The contents of Proposition 5 is ill-considered. From the equivalence \( 1) \Leftrightarrow 2) \) of (11) and [57, Corollary 2.2] it follows that the underlying space of every homogeneous algebra is regular. Then the expression ”regular” from (17) is unnecessary.

(18), (19). The text (18) is copied from [28], but with corrected definition of homogeneous algebra. Unlike [28] the statements from (19) hold, but they are not proved, only facts are presented. This can be explained by item d) above. Let us present evidence. We use item 5) of Theorem 11, i.e. let’s be assume that the homogeneous algebra \((G, q, p, e)\) is a topological right loop with right unit \( e \). From \( q(a, p(a, x)) = x \) it follows that \( q_a p_a = E \), where \( E \) denote the identical mapping. But from \( q_a p_a = E \) it does not follow that \( q_a^{-1} = p_a \). This equality and the assertion ”\( p_a \) and \( q_a \) are homeomorphisms” follow from Theorem 9. The equalities \( p_a(X) = q_a(X) = X \) follow from the property that \((X, q, p, e)\) is a right loop. We mentioned that (19) is not used further in the proof of Proposition 5.

(20). The implication [5, Proposition 2.1] \( \Rightarrow ^{(20)} \) is false, does not correspond to reality.

(21). It is necessary to show that space \( X \) contains a non-empty subspace \( F \), at least from the following reasons. Every topological group that contains a non-empty compact subspace with a countable base of open neighborhoods.
is a paracompact $p$-space [3, pag. 122].

(9) The excerpt is copied from [3, end of proof of Theorem 2.18].

(23). Another deceptive statement. It is not clear how (23) follows from
the results of [28]. Moreover, we have shown above that the results of [28]
are either false, or not proved. Indeed, (23) follows from the theorem of V. V.
Filippov on preservation of the class of $p$-space by perfect mappings [61] (see,
[3, pag. 120]).

(24). In [28] are not considered Mal’cev algebras as defined in (8). Hence,
the assertion (24) is not proved, either. In such a case, assertion (25) is also
not proved. For more details see (8).

(26). In this excerpt only the words ”Let, fix, then, ..., the proof is com-
plete.” are correct. All the rest are delirium, senseless assertions. For ex-
ample, ”Fix $a \in X \cap g^{-1}(b)$.”. Obviously the case $a \in g^{-1}(b) \setminus X$
should be considered, as well. Further, ”If $H = g^{-1}(e')$, then $H \subseteq F \subseteq X$
and $q_a(H) = g^{-1}(qb(e')) \subseteq X$. Thus, $g^{-1}(b) = g^{-1}(q_a(e')) \subseteq X$. Therefore,
$g^{-1}(g(X)) = X.””. The role of inclusions $H \subseteq F \subseteq X$ is not clear in the proof
of (26). In general, $q_a(H) \subseteq g^{-1}(qb(e'))$. It is not clear, why $g^{-1}(qb(e')) \subseteq X$. It
is easier to understand all this if to consider the classes $K_a = g^{-1}$, $a \in G$, of
congruence, induced by homomorphism $g$, and to take into account that $K_a$
is a complete class of congruence, as well as item 6) of Corollary 3 and the
analysis of [28, Lemma 3.1], presented above.

(27). Paper [23] is published in the same issue of the scientific journal
”Buletinul Academiei de Științe a Republicii Moldova” as [4]. With the ac-
curacy of translation the paper [23] is copied word by word from monograph
[20] (for example, sections 4, 5 of [23] Coincide literally with §§ 4.14, 4.15
respectively. This is not mentioned in [23]. It is also not shown that the re-
sults of monograph [20] coincide exactly, including the paragraph numbering,
with the results of monograph [21] (with some minor differences in Chapter
7).

I got interested in other papers that consider the rectifiable spaces. Let
us start with [5]. The main result of this paper is the Theorem 3.1, known as
Dichotomy Theorem.

**Theorem 3.1** For every rectifiable space $G$, any remainder of $G$ in a
compactification $bG$ is either pseudocompact or Lindelöf.

A crucial moment in proof of this theorem is the Corollary 2.8, which
is a direct corollary of Proposition 2.7. The latter is a generalization of
Proposition 2.6. Let us present it with its proof.

**Proposition 2.6.** Let $p, q : G \times G \to G$ be a structure of homogeneous
algebra on a space $G$, $F$ and $\Phi$ be two non-empty compact subsets of $G$, $H = p(F \times \Phi)$ and $L = q(F \times \Phi)$. Suppose further that $\mathfrak{B}_1$ is a base (a $\pi$-base) of $G$ at $F$ and that $\mathfrak{B}_2$ is a base (a $\pi$-base) of $G$ at $F$. Then:

- $\{p(F \times U) : U \in \mathfrak{B}_2\}$ and $\{p(V \times U) : V \in \mathfrak{B}_1, U \in \mathfrak{B}_1\}$ are bases (\$\pi$-bases, respectively) of $G$ at $H$; and

- $\{q(F \times U) : U \in \mathfrak{B}_2\}$ and $\{q(V \times U) : V \in \mathfrak{B}_1, U \in \mathfrak{B}_1\}$ are bases (\$\pi$-bases, respectively) of $G$ at $LH$.

**Proof.** Let $F_1$ and $F_2$ be two compact subsets of $G$, $U$ be an open subset of $G$ and $m(F_1 \times F_2) \subseteq U$ (where $m$ is either $p$ or $q$). Then, by Wallace theorem [11, Theorem 3.2.10], there exist open subsets $U_1$ and $U_2$ of $G$ such that $F_1 \subseteq U_1, F_2 \subseteq U_2$ and $m(U_1 \times U_2) \subseteq U$. \(\Box\)

Let us now analyze the next Section "4. On paratopological groups and dissentive spaces" from paper [5], which starts with the definition: a group $G$ T is called: – a paratopological group if the multiplication $(x, y) \to x \cdot y$ is a continuous mapping of $G \times G$ onto $G$. Recall that every topological group $(G, \cdot, -1, e)$ is an associative topological loop $(G, \cdot, \backslash, /, e)$. From here it follows that

* if $G$ is a paratopological group (respect. topological right loop) then from conditions $a \in G$, $H$ is an open set does not result that the set $a \cdot H$ is open. On the contrary $G$ is a topological group (respect. topological loop).

Further one may find a genuine nightmare (Proposition 4.1, Corollaries 4.2, 4.3, 4.5, and Theorem 4.4). To justify this we present the proof of Proposition 4.1, which has a crucial importance for the proof of other statements.

**Proposition 4.1.** Let $G$ be a paratopological group. Then:

1. If there exists a non-empty compact subset of $G$ of countable character in $G$, then $G$ is a space of countable type.

2. If there exists a non-empty compact subset of $G$ of countable $\delta$-character, then $G$ is a space of countable $\delta$-type.

**Proof.** Let $\{U_n : n \in \omega\}$ be a sequence of open subsets of $G$, $F$ be a non-empty compact subset of $G$, $F \subseteq \text{cl}_X U_n, U_{n+1} \subseteq U_n$ for each $n \in \omega$. We may assume that $e \in F$.

Fix a compact subset $L$ of $G$. Put $\Phi = L \cdot H$ and $V_n = L \cdot H_n$ for any $n \in \omega$. Then $\Phi$ is a compact subset of $G$, $L \subseteq \Phi \subseteq \text{cl}_X V_n$, and $V_n = \cup\{Q_a(U_n) = a \cdot U_n : a \in L\}$ is open in $G$ for each $n \in \omega$.

Let $\{U_n : n \in \omega\}$ be a $\pi$-base of $G$ at $F$. Then, by Proposition 2.6, $\{V_n : n \in \omega\}$ is a $\pi$-base of $G$ at $\Phi$. Thus, $\Phi$ is a compact subset of countable $\delta$-character in $G$. Statement 2 is proved.

Let $\{U_n : n \in \omega\}$ be a base of $G$ at $F$. Then $\{V_n : n \in \omega\}$ is a base of $G$.
at Φ. Thus, Φ is a compact subset of countable character in G. Statement 1 is proved.

This is not a mathematical proof. Actually, this could be regarded as a compilation of unclear and unrelated denotations and topological notions. The meaning of denotations $cl_X U_n$, $H$, $H_n$, $Q_a(U_n)$ is not explained. It is not explained why Φ is a compact set of G. It is stated that $V_n = \cup\{Q_a(U_n) = a \cdot U_n : a \in L\}$ is open in G. But this contradicts the statement presented before Proposition 4.1.

Moreover, according to item 5) of Theorem 11 the Proposition 2.6 is proved for topological right loops. For such loops the operation $(\setminus)$ is continuous. Then Proposition 2.6 cannot be used for paratopological groups.

In the beginning of the analyzed paper [5] it is mentioned that it will use the terminology from [60]. Let us present the following notions and results from [60]. Let $\{X_s\}_{s \in S}$ be a family of topological spaces such that

$$X_s \cap X_{s'} = \emptyset \quad \text{for} \quad s \neq s'.$$  \tag{14}$$

Let $X = \bigcup_{s \in S} X_s$. We define the set $U \subseteq X$ open in $X$ if and only if $U \cap X_s$ is an open set in $X_s$ for each $s \in S$. The family of all such defined open sets is a topology on $X$. The set $X$ together with this topology is called sum of spaces $\{X_s\}_{s \in S}$ and is designated $\bigoplus_{s \in S} X_s$ or $X_1 \bigoplus X_2 \bigoplus \ldots \bigoplus X_n$ if $S = \{1, 2, \ldots, n\}$.

It is also possible to define the sum $\bigoplus_{s \in S} X_s$ for the family of topological spaces $\{X_s\}$ without the condition (14), but in [60] it is proved that the sum defined in such a way is homeomorphic to the sum of topological spaces with condition (14). An analogical notion is considered in [70], the sum $\bigoplus_{s \in S} X_s$ without condition (14) is called topological union of spaces $X_s$ and the sum $\bigoplus_{s \in S} X_s$ with condition (14) is called free topological union of spaces $X_s$. We mentioned also that the notion of singleton set is used to define the topology of topological algebras with the help of limit of partial algebraical operations with infinite of arguments.

We present three excerpts from Section 4 of [5] for consideration.

(*) A Mal’cev operation on a space $X$ is a continuous mapping $\mu : X^3 \to X$ such that $\mu(x, x, z) = z$ and $\mu(x, y, y) = x$, for all $x, y, z \in X$. A space is called a Mal’cev space if it admits a Mal’cev operation (see [8–10,14,15,19]). Let $\{X_\alpha : \alpha \in A\}$ be a non-empty family of non-empty Mal’cev spaces. Assume that the set $A$ is well ordered. Denote by $\mu_\alpha$ the Mal’cev operation on $X_\alpha$. Let $X$ be the discrete sum of the spaces $\{X_\alpha : \alpha \in A\}$. If $\alpha \in A$ and $x, y, z \in X_\alpha$,
then $\mu(x, y, z) = \mu_\alpha(x, y, z)$. Let $x, y, z \in X$ and \(\{x, y, z\} \neq \emptyset\) for each $\alpha \in A$. Then $\{\mu(x, y, z)\} = \{x, y, z\} \cap X_\alpha$, where $\alpha$ is the first element $\beta$ of $A$ for which $\{x, y, z\} \cap X_\beta$ is a singleton set. Then $\mu$ is a Mal’cev operation on $X$. Thus, the discrete sum of Mal’cev spaces is a Mal’cev space.

(**) A dissentive operation (a dissentor) on a space $X$ is a continuous mapping $\mu : X^3 \to X$ satisfying the following conditions:
- $\mu(x, x, y) = y$ for all $x, y \in X$;
- for every open set $U$ of $X$ and all $b, c \in X$ the set $\mu(U, b, c) = \{\mu(x, b, c) : x \in U\}$ is open in $X$.

A space is dissentive if it admits a dissentive operation. Every rectifiable space is dissentive. If $p, q : G \times G \to G$ is a structure of homogeneous algebra on a space $G$, then $\mu(x, y, z) = p(x, q(y, z))$ is a Mal’cev dissentive operation.

(***) Example 5.18. Let $G_1$ and $G_2$ be two disjoint topological groups with the following properties:
1. $G_1$ is metrizable and is not locally compact;
2. $G_2$ is pseudocompact and is not compact.

Then the free topological sum $G = G_1 \oplus G_2$ is a space with the following properties.

**Property 1.** $G$ is a Mal’cev non-dissentive space. In particular, $G$ is not rectifiable.

**Property 2.** Every remainder of $G$ is not pseudocompact.

**Property 3.** Every remainder of $G$ is not Lindelöf.

Therefore, the Dichotomy Theorem cannot be extended to all Mal’cev spaces.

The proof of paragraph (*) is a mockery of the readers. It is not clear what the discrete sum of topological spaces means.

Assume that the Mal’cev operation $\mu$ on discrete sum $X$ is defined, though it is not so. Indeed, let $x_\alpha \in X_\alpha$, $x_\beta \in X_\beta$, $x_\gamma \in X_\gamma$ and let $x_\alpha < x_\beta < x_\gamma$. From $\{x_\alpha, x_\beta, x_\gamma\} \cap X_\alpha = \{x_\alpha\}$ it follows that $\mu(x_\beta, x_\alpha, x_\gamma) = x_\alpha$. From here it follows that $\mu(x, y, z)$ is not a Mal’cev operation. Moreover, unlike the sum of topological spaces, defined above, there isn’t any connection between the topology of discrete sum $X$ and the topologies of spaces $\{X_\alpha : \alpha \in A\}$. Now let us present excerpts from [5].

Senseless properties and incorrect conclusion. The main topological object of research of this paper is the remainder of a space $X$ (for example, Theorem 3.1). This is the subspace $bX \setminus X$ of a Hausdorff compactification $bX$ of $X$. However, taking into account the Example 5.18 it seems that the
authors are unaware of the definition of Hausdorff compactification. Example 5.13 also seems weird, stating that Theorems 3.1, 3.4 do not generalize to homogeneous spaces, though in excerpt (4) it is stated that the rectifiable spaces and homogeneous spaces coincide topologically.

Further, let us move on to the next excerpt.

It is clear that statement "Every rectifiable space is dissentive" aims at showing the importance of introducing the notion of "dissentive operation". But this statement, included in the paper without any proof, is incorrect. Indeed, according to Theorem 11 every homogeneous algebra \((G,p,q)\) is a topological right loop \((G,\cdot,\backslash,e)\), which is not a left loop. If \(p = (\cdot), q = (\backslash)\) then \(\mu(x,y,z) = x \cdot (y\backslash z)\). Let \(H\) be an open set of \(G\) and fix the element \(y, z \in G\). According to definition of dissentive operation the set \(\mu(H, y, z)\) should be open, i.e. the set \(H \cdot (y\backslash z)\) should be open. This contradicts the statement *. In a similar way we get a contradiction for \(p = (\cdot), p = (\backslash)\).

The following question arises: was it necessary to introduce a new term "dissentive operation"? For a space \(X\) this definition is equivalent to assertion: for any \(a, b \in X\) the continuous mapping \(\varphi_{ab}(x) = \mu(x, a, b)\) is open and \(\varphi_{ab}(a) = b\).

The last assertion without condition \(\varphi_{ab}(a) = b\) is changed and new terms, such as \(o\)-homogeneous, \(i\)-homogeneous, \(c\)-homogeneous, \(d\)-homogeneous are introduced in paper [6]. Besides, other new terms as \(c\)-sequence, \(A\)-sieve, densely sieve-complete, densely \(q\)-complete, densely fan-complete and others are introduced. These new notions, in tandem with plenty of topological terms, used both with and without sense, make it impossible to verify the authenticity of the paper's results. Papers [7], [8] are written in the same manner.

**Note:** Part 2 of this work will follow soon, where we will examine the other papers, in particular: [45], [25], [47].

**References**

[1] Arhangel’skii A. V. *On mappings, nonconnected with topological groups.* DAN SSSR, 6, 181(1968), 1303 – 1306 (In Russian).

[2] Arhangel’skii A. V. *Any topological group is a quotient group of zero-dimensional topological group.* DAN SSSR, 5, 258(1981), 1037 – 1040 (in Russian).
[3] Arhangel’skii A. V. *Two types of remainders of topological groups*, Comment. Math. Univ. Carolin., 49.1(2008), 119 – 126.

[4] Arhangel’skii A. V., Choban M. M. *Some addition theorems for rectifiable spaces*. Buletinul Academiei de Științe a Republicii Moldova, Matematica, 2(60), 2011, 60 – 69.

[5] Arhangel’skii A. V., Choban M. M. *Remainders of rectifiable spaces*. Topology and Appl., 157(2010), 789 – 799.

[6] Arhangel’skii A. V., Choban M. M., Mihaylova E. P. *About homogeneous spaces and conditions of completeness of spaces*. Math. and Education in Math., 41 (2012), 129–133.

[7] Arhangel’skii A. V., Choban M. M., Mihaylova E. P. *About homogeneous spaces and Baire property in remainders*. Math. and Education in Math., 41 (2012), 134–138.

[8] Arhangel’skii A. V., Choban M. M., Mihaylova E. P. *About topological groups and the Baire property in remainders*. Math. and Education in Math., 41 (2012), 139–142.

[9] Banakh Taras, Hrynev Olena. *Free topological universal algebraal and absolute neighborhood retracts*. Buletinul Academiei de Științe a Republicii Moldova, Matematica, 1(65), 2011, 50 – 59: arXiv:1002.3352v1[math GN].

[10] Taras Banakh, Mitrofan Choban, Igor Guran, Igor Protasov. *Some Open Problems in Topological Algebra*. arXiv: 1202 1619v1[math GN].

[11] Bates G. E., Kiokemeister F. A note of homomorphic mappings of quasigroups into multiplicative systems. Bull. Amer. Math. Soc., 54(1948), 1180 – 1185.

[12] Bartle R., Grawes L. M. *Mappings between function spaces*. Trans. Amer. Math. Soc., 72(1952), 400 – 413.

[13] Bel’nov V. K. *On zero-dimensional topological groups*. DAN SSSR, 4, 226(1976), 749 – 752 (In Russian).

[14] Belousov V. D. *Fondations of the theory of quasigroups and loops*. Moscow, Nauka, 1967.
[15] R. H. Bruck. A survey of binary systems. Berlin-Göttingen-Heidelberg, Springer-Verlag, 1958.

[16] Burghin M. S. Free topological groups and universal algebras. DAN SSSR, ser. matemat., 1, 204(1972), 9 – 11 (In Russian).

[17] Burghin M. S. Topological algebras with continuous systems of operations. DAN SSSR, ser. matemat., 3, 213(1973), 505 – 508 (In Russian).

[18] Burghin M. S. Free algebras with continuous systems of operations. Uspehi matematicheskikh nauk, 3, 35(1980), 147 – 151 (In Russian).

[19] Burstin C., Mayer W. Distributive Gruppen von endlicher Ordnung. J. Reine und Angew. Math., 1929, 160, 111 – 130.

[20] Calmutchii L., Cioban M. Compact extensions of topological spaces. Chişinău, 2009 (in Romanian).

[21] Calmutchi L. L. Algebraic and functional methods in the theory of extensions of topological spaces. Piteşti, 2007.

[22] Calmutchi L. L. Algebraic and functional methods in the theory of extensions of topological spaces. Thesis for a Habilitat Doctors Degree, Chişinău, 2007.

[23] Calmutchi L. I., Choban M. M. On Wallman compactifications of $T_0$-spaces and related questions. Buletinul Academiei de Științe a Republicii Moldova, Matematica, 2(60), 2011, 102 – 111.

[24] Calmutchii L. I., Choban M.M. On the Nagata theorem on lattice of semcontinuous functions. Sibirskii Matem. Journ., 2(30)(1989, 185 – 191 (in Russian)

[25] Chiriac L. L. Topological Algebraic Systems, Chişinău, 2009.

[26] Chiriac L. L. Topological Algebraic Systems, Thesis for a Habilitat Doctors Degree, Chişinău, 2011.

[27] Choban M. M. On topological homogeneous algebras. Interim Reports of the Prague Topolog. Symposium, 2(1987), 25 – 26.
[28] Choban M. M. *The structure of locally compact algebras*. Serbica, Bulgaricae Math. Publ., 18(1992), 129 – 137.

[29] Choban M. M. *On the theory of topological algebraic systems*. Trudy Moscov. Matem. Ob-va, 48(1985), 106 – 149.

[30] Choban M. M. *Algebras and some questions of the theory of maps*, Proc. Fifth Prague Topol. Symposium 1981, (1983) 86 - 97.

[31] Choban M. M. *Some topics in topological algebra*. Topology and Appl., 54(1993), 183 – 202.

[32] Choban M. M. *Algebraical equivalences of of topological spaces*. Buletinul Academiei de Științe a Republicii Moldova, Matematica, 1(2001), 12 – 36.

[33] Choban M. M. *Stable metrics on universal algebras*. Uspekhi Mat. Nauk, 5(251), 41(1986), 201 – 202 (in Russian).

[34] Choban M. M. *On some questions of theory of topological groups*. Știința. General algebra and discrete geometry, Știința, 1980, 120 – 135 (In Russian).

[35] Choban M. M. *Reduced theorems on existence of continuous sections. Sections over subsets of quotient spaces of topological groups*. Mat. issled., 8, 4, Știința, 1973, 111–156 (In Russian).

[36] Choban M. M. *Topological construction of subsets of topological groups and it quotient spaces*. In coll. Topological lattices and algebraical systems. Știința, Știința, 1977, 117–163 (In Russian).

[37] Choban M. M. *The theory of stable metrics*. Math. Balkanica, 2(1988), 357 – 373.

[38] Choban M. M. *General conditions of the existence of free objects*. Acta Comment. Univ. Tartuenis, 836(1989), 157 – 171 (in Russian).

[39] Choban M. M. *Universal Topological Algebras*, Edition House of the University of Oradia, 1999.

[40] Cioban M. M. *Topological Algebras. Problems*, Chișinău, 2006.
[41] Choban M. M., Calmuțchi L. L. *Compact extensions of topological spaces*. Chișinău, 2009.

[42] Choban M.M., Kiriyak L.L. On applying uniform structures to study of free topological algebras. Sibirskii Matem. J., 33, 5, 1992, 159 – 172. (English: Trans. Siberian Mathematical Journal, Springer New York, 0037- 4466, 1573-9260, Volume 33, Number 5, 1992, 10.1007/BF00970997, p. 891-904).

[43] Choban M.M., Kiriyak L.L. *Compact subsets of free algebras with topologies and equivalence of space*. Hadronic Journal, Volume 25, Number 5, USA, October 2002, p. 609-631.

[44] Ciobanu I. D. *On construction of algebraical structures over a compactifications of topological algebras*. Thesis for a Doctors Degree, Chișinău, 2011.

[45] Choban M. M., Ciobanu Ina. *Compactness and free topological algebras*, ROMAI J., 3, 2(2007), 55 – 85.

[46] Choban M. M., Ciobanu I. D. *On totally boubded universal algebras*. Creative Mathematics and. Informatics, 21, no.2, 2012, 151 – 165.

[47] Choban M. M., Chiriac L. L. *Selected problems and results of topological algebra*. ROMAI J., 9, 1*2013), 1 – 25.

[48] Cohn P. M. *Universal algebra*, New York, Harper and Row, 1965.

[49] Cowell W. R. *Concerning a class of permutable congruence relations on loops*. Proc. Amer. Math. Soc., 7, 4(1956), 583 – 588.

[50] *Curriculum vitae*. www romai.ro/documente/poze/Anunturi-informatii/CiobanMM

[51] *Curriculum vitae*. www romai.ro/conferinte-romai/caim-en.html

[52] Dörnte W. *Untersuchungen ü einen veralgemeinerten Gruppenbegriff*. Math. Z., 1928, 29, 1 = 19.

[53] Dumitrascu S. S. *Topologies on Universal Algebras*, Edition House of the University of Oradia, 1995.
[54] Dumitrascu S. S. *On the problem of A. I. Mal’cev*. Proceedings Fifth All-Union Symposium of the Theory of Rings and Algebras (Academiya Nauk SSSR, 1982), 49 – 50 (in Russian).

[55] Dumitrascu S. S., Choban M. M. *On free topological algebras with continuous signature*, Matem. issledovania, Shtiinta (Chishinev), 65(1982), 27 – 53 (Russian).

[56] Gleason A. M. *Spaces with a compact Lie group of transformations*. Proc. Amer. Math. Soc., 1(1950), 35 – 43.

[57] Gul’ko A. S. *Rectifiable spaces*. Topology and Appl., 68(1996), 107 – 112.

[58] Ipaté D. M. *General Problem on approximation of continuous mappings of topological spaces*. Thesis for a Habilitat Doctors Degree, Chișinău, 2007.

[59] Iwasawa K. *On some types of topological groups*. Ann. of Math., 50(1949), 507 – 558.

[60] Engelking R. *General Topology*, PWN, Verszawa, 1977.

[61] Filippov V. v. *On perfect images of paracompact p-spaces*, Soviet. Math. Docl. 176(1967), 533 – 536.

[62] Hofmann K. H. *Non-associative Topological Algebra*. Tulane University Lecture Note, 1961.

[63] Kelley John L. *General Topology*, Moskow, Nauka, 1981 (in Russian).

[64] Kenderov P. *On topological groups*. DAN SSSR, 4, 194(19 70), 760 – 762 (In Russian).

[65] Kiriak L. L. *On the topology of free topological algebras with Mal’cev condition and k-spaces*. Izv. Akad. Nauk SSR Moldova, ser.mat. 3(1990), 7 – 13.

[66] Kurosh A.G. *Lectures on general algebra*. Gos. izdatel’stvo fiz-mat. literature, Moscow, 1962, (in Russian).
[67] Lin F., Liu C., Lin S. A note on rectifiable spaces, Topology Appl., doi:10.1016/j.topol.2012.02.002.

[68] F. Lin F., Shen R. On rectifiable spaces and paratopological groups, Topology Appl., 158(2011), 597–610.

[69] Mal’cev A. I. Algebraic systems, Moskow, Nauka, 1970 (in Russian).

[70] Mal’cev A. I. To general theory of algebraic systems, Mat. Sb., 35(1954), 3 – 20 (in Russian). (English translation: Trans. Amer, Math. Soc., 27(1963), 125 – 148).

[71] Mal’cev A. I. Free topological algebras, Izv. AN. SSSR, ser. mat., 21, 2(1957), 171 – 198 (in Russian).

[72] Michael E. Convex structures and continuous selections. Canad. J. Math., 4, 11(1959), 556 – 575.

[73] Pavel Dorin. Almost periodic mappings on topological spaces. Thesis for a Doctors Degree, Chișinău, 2010.

[74] Prodanov I. Division of points in separable bicom pact universal algebras, Matematika i matematicheskie obrazovanie, 1981 186 - 189.

[75] Protasov I. V., Sidorchuk A. D. On varieties of algebraic systems. DAN SSSR, ser. matemat., 256(1981), 1314 – 1318 (In Russian).

[76] Shapirovkii Â. L. Special types of embeddings in Tychonoff cubes, Coll. math. soc. Janos Bolyai. 23. Topology. Budapest, 1978, 1055 –1086.

[77] Smith J. D. H. Mal’cev Varieties. Lecture Notes in Mathematics, v. 554, 1976, 21,

[78] Smith J. D. H. Mal’cev Varieties. Lecture Notes in Mathematics, Vol. 554, Springer-Verlag, Berlin – New York, 1076.

[79] Chein O., Pflugfelder H.O., Smith J.D.H. Quasigroups and Loops: Theory and applications. Berlin, Helderman Verlag, 1990.

[80] Suschkewitsch A.K. On a generalization of the associative law. Trans. Amer. Math. Soc., 1929, 31, 204 – 214.
[81] Swierczkowski S. *Topologies in free algebras*. Proc. London Math. Soc., 3, 14, 55(1968), 566 – 576.

[82] Thurston H. A. *Noncommuting quasigroup congruences*. Proc. Amer. Math. Soc., 3(1952), 363 – 372.

[83] Uspenskii V. V. *The Mal’tsev operation on countably compact spaces*, Comment. Math. Univ. Carolin. 30 (1989) 395 – 402.

[84] Uspenskii V. V. *Topological groups and Dugundji compacta*, Mat. Sb. 180(1989), no. 8, 1092–1118 (Russian); English transl. in: Math. USSR-Sb. 67(1990), no. 2, 555–580.

[85] Uspenskii V. V. *On continuous images of Lindelöf topological groups*. DAN USSR, 4, 285(1985). 824—827 (in Russian).

Nicolae I. Sandu,
Tiraspol State University of Moldova,
Chisinău, R. Moldova
sandumn@yahoo.com