Hadron Masses and Screening from AdS Wilson Loops

Andreas Karch, Emanuel Katz, and Neal Weiner
Department of Physics, University of Washington, Seattle, WA 98195, USA

We show that in strongly coupled $\mathcal{N} = 4$ SYM the binding energy of a heavy and a light quark is independent of the strength of the coupling constant. As a consequence we are able to show that in the presence of light quarks the analog of the QCD string can snap and color charges are screened. The resulting neutral mesons interact with each other only via pion exchange and we estimate the masses of those states.

I. INTRODUCTION

In QCD probably the most dramatic modification of the dynamics due to the presence of dynamical flavors is the string breaking: as one tries to separate two infinitely heavy test quarks (whose physics can be neatly captured by Wilson loops) the potential does not rise linearly with the distance forever, as it should in a confining theory. Instead, once the energy stored in the flux tube is big enough, the string snaps by pair producing a dynamical quark-antiquark pair. The interactions between the two resulting mesons rapidly die out with distance. The potential, as probed by the Wilson line, rises linearly for a while and then saturates at twice the mass of the meson, see Fig.1. One can consider the “probe limit” of finite number of flavors, $M$, and large number of colors, $N$, with $g^2 N$ fixed and hence $g^2 M \to 0$. Since the actual snapping process has an amplitude proportional to $g^2 M$, in the probe limit the transition becomes sharp. The potential is completely dominated by the lowest energy configuration.

In [1] it was shown that $\mathcal{N} = 4$ super Yang-Mills conformal field theory is dual to type IIB string theory on a background of $AdS_5 \times S_5$. Dynamical flavors with a finite mass can be included in the supergravity description of strongly coupled gauge theories via spacetime filling D-branes [2], at least in the probe limit. In this letter we will investigate to what extent the prescription of [2] can be used to study the behavior of the QCD string in the presence of dynamical quarks. In particular, we will find additional configurations where strings stretch between the boundary and the D7 branes, yielding a simple interpretation of string snapping in the string dual description.

For light quarks QCD is never truly confining in the sense of linearly rising potentials due to flux strings. However the main experimental fact is still true: colored quarks can never appear in isolation. They always pair produce a light quark-antiquark pair to form color neutral mesons. Note that latter can in principle also happen in a conformal theory. In a conformal theory the potential is forced to be Coulomb by dimensional analysis. So there is only a finite cost to separate two heavy quarks of mass $M$, and the total energy of the configuration asymptotes to $2M$ for large separation. However if the mass of the produced meson

$$m_{Meson} = M + m - E_{Bind}$$

is less than $M$, it is still energetically favorable for the string to snap, see Fig.2.

Note that for weakly coupled Coulomb potentials this never happens. There the strength of the Coulomb potential is proportional to the coupling constant, that is e.g. $g^2 N$ for $\mathcal{N} = 4$ SYM, and hence the binding energy goes like $E_{Bind} \sim (g^2 N)^2 m$ which at weak coupling

*Electronic address: karch@phys.washington.edu
†Electronic address: amikatz@phys.washington.edu
‡Electronic address: nealw@phys.washington.edu
is always less than \( m \). For example for QED the binding energy for the lightest meson (the hydrogen atom) formed out of the “heavy test quark” (the proton) and the “light dynamical quark” (the electron) is 13.6 eV, which is roughly a factor of 1/13 down from 0.5 MeV, the mass of the dynamical quark. So protons do not pop electrons out of the vacuum to shield their charge. However for strong coupling it seems that the binding energy can easily be as big or bigger than \( m \). Consequently, in a strongly coupled CFT the “Coulomb string” can snap, just like the “Confining string”. As in QCD, the charged heavy quarks can never be studied in isolation, since they’ll always pair produce dynamical quarks to form neutral mesons, color is screened.

In this letter we will use the AdS/CFT correspondence to show that that in strongly coupled \( \mathcal{N} = 4 \) SYM the binding energy at large \( g^2 N \) actually becomes independent of \( g^2 N \), it is an order one number times the mass of the light quark. The order one constant depends on the R-charge orientation of the quark. So in \( \mathcal{N} = 4 \) SYM the string can snap and the procedure of [3] is able to give a quantitative description of this effect. We expect the snapping of the string in a real confining theory to be described in precisely the same fashion.

II. SUPERGRAVITY EVALUATION OF THE WILSON LOOPS

It was proposed in [3, 4] that Wilson loops of the CFT can be described in AdS by

\[
\langle W(C) \rangle \sim \lim_{\Phi \to \infty} e^{-(S_\Phi - i\Phi)},
\]

where \( S \) is the proper area of a fundamental string worldsheet which lies on the loop \( C \) on the boundary of AdS. \( l \) is the total length of the Wilson loop and \( \Phi \) is the mass of the W boson. For us, the appropriate quantity will be a suitable Legendre transform \( \bar{S} \), as we will discuss shortly. The heavy quark is constructed by taking one brane and sending it to infinity. As this brane goes to infinity, its dynamics decouple and strings stretching between it and the boundary act as infinitely massive sources of color charge.

The energy of the first configuration shown in Fig.3 was calculated in [3] to be

\[
E = - \frac{2(2g_\Lambda^2 N)^{1/2}}{\pi L} \left(1 - l^2(\theta_i, \phi_i)^{1/2}I_1(l(\theta_i, \phi_i))\right),
\]

where

\[
I_1(l) = \frac{1}{(1-l^2)^{1/2}} \times
\]

\[
\left(2-l^2\right)E \left(\frac{\pi}{2} \sqrt{\frac{1-l^2}{2-l^2}} - F \left(\frac{\pi}{2} \sqrt{\frac{1-l^2}{2-l^2}}\right)\right),
\]

and \( L \) is the separation between the heavy quarks, and \( F, E \) are elliptic integrals of the first and second kind,

![Diagram of Wilson loops](image)

FIG. 3: The two classical configurations contributing to the expectation value of the Wilson line in the presence of dynamical quarks.

respectively. The quantity \( l \) is defined by in terms of the angular separation of the two heavy quarks by

\[
\frac{\Delta \theta}{2} = \frac{l}{\sqrt{2-\Delta l^2}} F\left(\frac{\pi}{2} \sqrt{\frac{1-\Delta l^2}{2-\Delta l^2}}\right).
\]

In the presence of dynamical light quarks, described by the presence of the D7 branes, there is a second set of consistent boundary conditions. The Wilson loop will be given by the sum of the exponentials of the actions associated with each configuration.

\[
< W(C) > = \sum_i e^{-\bar{S}_i},
\]

where \( \bar{S} \) is the suitably regulated action. For distances \( L \ll m^{-1} \) we expect the first configuration to dominate the Wilson loop, while for \( L \gg m^{-1} \) we expect the meson pair configuration to dominate.

Actually the action of the second configuration can be assembled out of three contributions. The energy of the two string pieces and the energy of the flux that connects the two string endpoints. In the field theory these three contributions correspond to the mass of the two heavy-light mesons and the potential energy between them.

A. Heavy-Light Binding Energy

The string will be fixed in the 3-dimensional space coordinates as we vary \( \sigma \), and will lie along a great circle, so that its position in \( S_\sigma \) can be parameterized by one coordinate \( \theta \). The metric is then

\[
\alpha' \left( \frac{U^2}{R^2} dt^2 + \frac{R^2}{U^2} dU^2 + R^2 d\Theta^2 \right).
\]

We want time independent solutions, so we set \( \tau = t \) and since the string will not double-back in \( U \), we can set \( \sigma = U \). The induced metric on the worldsheet is then

\[
\alpha' \left( \frac{U^2}{R^2} dt^2 + \frac{R^2}{U^2} dU^2 + \left(\frac{\partial \Theta}{\partial U}\right)^2 R^2 \right)
\]

The resulting action is

\[
S = \frac{T}{2\pi} \int dU \left( 1 + U^2 \left(\frac{\partial \Theta}{\partial U}\right)^2 \right)^{1/2}.
\]
Notice that this action is completely independent of $R$.

Interestingly, this is just the metric for a string in flat space! It is convenient to change coordinates to those where the D7 brane lies along the one-parameter family $(0, 2\pi m, z)$ where $m$ specifies the mass of the quarks.

We parameterize the position of our heavy quark brane by $(x_0, y_0, z_0)$, which we will eventually take to infinity. In this case, the path from the heavy quark brane to the light quark brane will be given by

$$
(x(t), y(t), z(t)) = (x_0, y_0 - 2\pi m, 0)t + (0, 2\pi m, z_0)
$$

For order one angles, the screening length (beyond which the binding energy of the system is

$$
E = \frac{S}{T} = \frac{\sqrt{2}}{2\pi} \frac{\tilde{A}}{A - P_1 Y^i},
$$

where

$$
P_1 = \frac{\delta A}{\delta \phi Y^i}.
$$

Setting $\sigma = t$ we find

$$
\tilde{A} = \sqrt{x_0^2 + (y_0 - 2\pi m)^2} - \frac{x_0^2 + y_0(y_0 - 2\pi m)}{\sqrt{x_0^2 + (y_0 - 2\pi m)^2}},
$$

$$
= \frac{2\pi m(2\pi m - y_0)}{\sqrt{x_0^2 + (y_0 - 2\pi m)^2}}.
$$

We parameterize the position of the distant brane by

$$(x_0, y_0, z_0) = M(\sin \theta \sin \phi, \sin \theta \cos \phi, \cos \theta).$$

We will see shortly that the parameter $\phi$ controls the strength of the heavy quark-light quark interaction. In the $M \to \infty$ limit we find

$$
\tilde{A} = -m \cos \phi.
$$

The binding energy of the system is

$$
E_{bind} = 2m(\cos^2(\phi_1/2) + \cos^2(\phi_2/2))
$$

For order one angles, the screening length (beyond which the meson configuration will dominate), is given by

$$
L_{sc} \sim \frac{2\sqrt{g_s M^4 N}}{m}.
$$

For comparison one can also study the case of a probe D3 brane instead of the probe D7 brane. This corresponds to adding dynamical W-bosons instead of the dynamical hypermultiplets: the dual gauge theory is $N = 4 \, SU(N + M)$ broken down to $SU(N) \times U(M)$. For $M \ll N$ the probe approximation can be used, even though in this case it is not necessary since the full metric is known. In the probe limit a very similar calculation shows that in this case the binding energy is independent of $g_s^2 N$ at large $g_s^2 N$ as well and just given by $\cos(\alpha)m$,

where $\alpha$ is the angle between the heavy quark and the dynamical W-boson on the $S^5$. The action we have to minimize is still given by (9), so again we are interested in the geometry of straight lines in flat 6d space. The stack of $N$ D3 branes is sitting at the origin. We choose our coordinates such that the probe flavor brane is sitting at $x = 2\pi m$ with all other coordinates being zero, and the distant brane at $(x, y) = M(\cos(\alpha), \sin(\alpha))$. That is there are 3 special points in $R^6$ and we pick $x, y$ to be the cartesian coordinates of the plane defined by those 3 points. The corresponding area is

$$
\tilde{A} = \frac{\sqrt{(M \cos(\alpha) - 2\pi m)^2 + M^2 \sin^2(\alpha) - M}}{2\pi} = -\cos(\alpha)m
$$

and hence the binding energy becomes

$$
E = 2m(\cos^2(\alpha_1/2) + \cos^2(\alpha_2/2)).
$$

Even though the bound state energy is comparable in the two cases, the physics in the end is quite different, since the meson-meson interactions differ significantly.

B. Meson - Meson potential

In order to capture the contribution of the meson-meson potential to the Wilson loop, we have to calculate the contribution to the action due to the flux lines connecting the ends of the open strings. Fig. 4 displays the flux for the two cases of D7 brane and D3 brane.

In the case of the D3 brane, the flux lines are confined to a Poincare slice and just behave like flux lines in flat space. They give rise to a Coulomb interaction, due to the exchange of the massless gauge fields living on the D3 brane. What happens in the dual field theory is that our mesons are charged under the $U(M)$ gauge fields in $SU(N + M) \to SU(N) \times U(M)$. The dynamical W-bosons after all transform in the bifundamental representation of $SU(N) \times SU(M)$. The $SU(N)$ charge of the

![Diagram](image-url)
heavy quarks gets screened as before once the binding energy becomes bigger than the mass of the W, that is for appropriate choices of $\alpha$, the relative R-charge orientation. Instead of the crossover from Coulomb to constant behavior, we get a crossover from the strong Coulomb of the $SU(N)$ gauge group (with order one potentials independent of $g^2 N$) to constant plus weak Coulomb (with potentials of order $g^2 M$) due to the massless $U(M)$ gauge boson exchange.

The situation becomes more interesting in the case of the D7 branes. Here the interactions between the heavy-light mesons are due to pion exchange, that is exchange of light-light mesons. The pions are massive and hence the interaction is exponentially suppressed. The masses of the pions are the masses of the KK-modes of the worldvolume gauge field on the D7 brane. For zero mass quarks, the D7 brane is $AdS_5$ filling and one has a continuum of states. For finite mass quarks the D7 brane ends and the KK-spectrum becomes discrete with a mass gap. This is analogous to what happens to the graviton in a confining gauge theory: The continuous spectrum of the $AdS_5$ background becomes a discrete spectrum with a mass gap, the modes corresponding to the glueballs of the gauge theory in that case. To calculate the energy in the flux explicitly, one calculates the Green’s function for the gauge field on the D-brane. This can be done in a mode expansion, which makes it obvious that also from the bulk perspective the meson-meson potential is dominated by the lightest KK-mode of the gauge field.

In order to find the mass of the lightest KK-mode we would have to solve Maxwell’s equation on the curved D7-brane worldvolume. In order to at least determine the coupling constant dependence we can consider a somewhat simpler toy model: instead of the position dependent tension of the D7 brane, we approximate it with a simple box shape:

$$T(U) \propto \left( \sqrt{1 - (2\pi m / U)^2} \right)^3 \rightarrow \begin{cases} 1 & \text{for } U > 2\pi m \\ 0 & \text{for } U \leq 2\pi m \end{cases}$$

(19)

This is merely a study of gauge fields in $AdS$ with an IR cutoff which has been studied already \[6, 7, 8\]. Regularity at the UV boundary requires that only the $J_1$ Bessel function contributes. We can consider states with either Neumann and Dirichlet boundary conditions on this IR brane which then demands that mass of the $n$-th mode satisfies:

$$J_1(m_n / U)|_{U=2\pi m} = 0,$$

(20)

or

$$J'_1(m_n / U)|_{U=2\pi m} = 0.$$  

(21)

The masses of the pions are order one numbers times the mass of the quarks. This is not surprising: just like the heavy-light binding energy the light-light binding energy is an order 1 number times the mass of the quarks $m$. Like the heavy-light binding energy it never becomes equal or bigger than $2m$, which would correspond to massless or tachyonic pions.

III. CONCLUSIONS

One can exploit the AdS/CFT correspondence to gain great insight into questions related to the qualitative behavior of QCD. We have seen that the inclusion of dynamical flavors into N=4 SYM gives a simple interpretation of QCD string snapping in the gravity dual. Although there is no confinement in these sense of area law Wilson loops, all asymptotic states are color neutral and color charge is completely screened. This furthers the hope that many more interesting features of QCD can be understood using string duals.

Acknowledgments: We would like to thank Hirosi Ooguri, Matt Strassler and Juan Maldacena for helpful comments. This work was partially supported by the DOE under contract DE-FGO3-96-ER40956.

[1] J. M. Maldacena, Adv. Theor. Math. Phys. 2, 231 (1998), hep-th/9711200.
[2] A. Karch and E. Katz, JHEP 06, 043 (2002), hep-th/0205236.
[3] J. M. Maldacena, Phys. Rev. Lett. 80, 4859 (1998), hep-th/9803002.
[4] S.-J. Rey and J. Yee, Eur. Phys. J. C22, 379 (2001), hep-th/9803001.
[5] N. Drukker, D. J. Gross, and H. Ooguri, Phys. Rev. D60, 125006 (1999), hep-th/9904191.
[6] H. Davoudiasl, J. L. Hewett, and T. G. Rizzo, Phys. Lett. B473, 43 (2000), hep-ph/9911262.
[7] A. Pomarol, Phys. Lett. B486, 153 (2000), hep-ph/9911294.
[8] N. Kaloper, E. Silverstein, and L. Susskind, JHEP 05, 031 (2001), hep-th/0006192.