POSITIVITY OF $\Delta$-GENERA FOR CONNECTED POLARIZED DEMI-NORMAL SCHEMES

JINGSHAN CHEN AND YONGCHANG CHEN

Abstract. In this paper, we show that the $\Delta$-genus $\Delta(X, \mathcal{L}) \geq 0$ for any connected polarized demi-normal scheme $(X, \mathcal{L})$ over an algebraically closed field $k$. As an application, we obtain $\Delta(X, I(K_X + \Lambda)) \geq 0$ for any KSBA stable log scheme $(X, \Lambda)$, where $I$ is the Cartier index of $K_X + \Lambda$. We also construct examples of KSBA stable log schemes with $I = 1$ and $\Delta(X, K_X + \Lambda) = 0$, which shows the inequality is sharp when $I = 1$.

1. Introduction

We work over an algebraically closed field $k$. A scheme $X$ is demi-normal if it is $S_2$ and at worst nodal at any generic point of codimension 1. The term "demi-normal" was coined by Kollár in [Kol13] to define KSBA stable (log) schemes. A KSBA stable log scheme $(X, \Lambda)$ is a demi-normal scheme $X$ with a boundary divisor $\Lambda$ such that it admits only slc singularities and $K_X + \Lambda$ is an ample $\mathbb{Q}$-Cartier divisor. A KSBA stable scheme is a KSBA stable log scheme with empty boundary. KSBA stable (log) schemes are the fundamental objects to construct the compactifications of moduli spaces of smooth varieties of general type.

Given a demi-normal scheme $X$, it is natural to consider a polarization, i.e., an ample invertible sheaf $\mathcal{L}$ on $X$. For a KSBA stable log scheme $(X, \Lambda)$, $I(K_X + \Lambda)$ is a polarization on it, where $I$ is the Cartier index of $K_X + \Lambda$. Fujita introduced an invariant, $\Delta$-genus $\Delta(X, \mathcal{L}) := (\mathcal{L})^{\dim X} - h^0(X, \mathcal{L}) + \dim X$ for polarized varieties. He showed that $\Delta(X, \mathcal{L}) \geq 0$ for any irreducible polarized variety (see [Fuj90, Theorem (1.4.2)]). In this paper, we show that

**Theorem 1.1** (see Theorem 4.6). For any connected polarized demi-normal scheme $(X, \mathcal{L})$, we have

$$\Delta(X, \mathcal{L}) \geq 0.$$  

As a result, we obtain $\Delta(X, I(K_X + \Lambda)) \geq 0$ for any KSBA stable log scheme $(X, \Lambda)$. In particular, when $(X, \Lambda)$ is Gorenstein, i.e., $I = 1$, we have $\Delta(X, K_X + \Lambda) \geq 0$, which is an inequality of Noether type. We remark that when $\dim X = 2$ and the boundary divisor $\Lambda$ is reduced, the inequality $\Delta(X, K_X + \Lambda) = (K_X + \Lambda)^2 - p_g(X, \Lambda) + 2 \geq 0$ is just the stable log Noether inequality established by Liu and Rollenske in [LR13]. We note that this paper is strongly inspired by Liu and Rollenske’s work.

We also characterize those connected polarized demi-normal schemes with $(X, \mathcal{L}) = 0$ in Theorem 4.7. They are trees of varieties $X_i$’s with $\Delta(X_i, \mathcal{L}|_{X_i}) = 0$ glued along

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hyperplanes. Then we construct examples of KSBA stable log scheme \((X, \Lambda)\) with \(I = 1\) and \(\Delta(X, K_X + \Lambda) = 0\) in Example 4.11, which implies the inequality \(\Delta(X, K_X + \Lambda) \geq 0\) is sharp for KSBA stable log schemes with \(I = 1\).

The paper is organised as follows. In §2, we state some facts about polarized demi-normal schemes. In §3, we review Fujita’s work on \(\Delta\)-genus of polarized varieties. In §4, we prove that \(\Delta(X, L) \geq 0\) for connected polarized demi-normal schemes.

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1.1. Notations and conventions. We work with schemes defined over an algebraically closed field \(k\). They are assumed to be proper and of finite type over \(k\).

- A variety is an integral scheme of finite type over \(k\).
- We assume that a normal singular variety always admits a resolution of singularities.
- By abuse of notation, we sometimes do not distinguish between a Cartier divisor \(D\) and its associated invertible sheaf \(\mathcal{O}_X(D)\).
- We use \(\equiv\) to denote linear equivalence relation of divisors.
- If \(D\) is a Cartier divisor on \(X\), then we denote by \(\Phi_{|D|} : X \dashrightarrow \mathbb{P} := |D|^*\) the rational map defined by the linear system \(|D|\).
- A hyperplane \(H\) on a scheme \(X\) with respect to \(\mathcal{O}_X(1)\) is a subscheme isomorphic to \(\mathbb{P}^{\dim X - 1}\) such that \(\mathcal{O}_X(1)|_H \cong \mathcal{O}_{\mathbb{P}^{\dim X - 1}}(1)\).
- We use \(\text{Bs}|L|\) to denote the set-theoretic intersection of all the members of \(|L|\).
- We use \((L)^n\) to denote the \(n\)-th self-intersection of the associated Cartier divisor of \(L\) in the Chow ring.

2. Demi-normal schemes

A scheme \(X\) is **demi-normal** if it is \(S_2\) and at worst nodal at any generic point of codimension 1. The schemes we consider are always assumed to be connected.

Let \(X\) be a demi-normal scheme and let \(\pi : \tilde{X} \to X\) denote its normalization morphism. The conductor ideal \(\text{Hom}_{\mathcal{O}_X}(\pi_* \mathcal{O}_{\tilde{X}}, \mathcal{O}_X)\) is an ideal sheaf on both \(X\) and \(\tilde{X}\) and hence defines subschemes \(D \subset X\) and \(\tilde{D} \subset \tilde{X}\), both reduced and of pure codimension 1; we often refer to \(D\) as the non-normal locus of \(X\).

A **polarized demi-normal scheme** is a pair \((X, L)\), where \(X\) is a proper demi-normal scheme and \(L\) is an ample invertible sheaf on \(X\).

**Definition 2.1.** The **\(\Delta\)-genus** of a polarized demi-normal scheme \((X, L)\) is defined as

\[
\Delta(X, L) := (L)^{\dim X} - h^0(X, L) + \dim X.
\]

**Remark 2.2.** Fujita defined \(\Delta\)-genus for polarized varieties (cf. Definition 3.6). We generalize it to polarized demi-normal schemes.
Let $(X, \mathcal{L})$ be a connected polarized proper demi-normal scheme. Assume that $X$ can be decomposed into two connected components, i.e., $X = X_1 \cup X_2$, where $X_1$ and $X_2$ are connected. The subscheme $C := X_1 \cap X_2$ is contained in the non-normal locus $D$ of $X$ and is of codimension 1. We call $C$ the connecting subscheme of $X_1$ and $X_2$. We have the Mayer-Vietoris exact sequence:

$$0 \to \mathcal{L} \to \mathcal{L}|_{X_1} \oplus \mathcal{L}|_{X_2} \to \mathcal{L}|_C \to 0.$$ 

Taking the associated long exact sequence, we obtain

$$0 \to H^0(X, \mathcal{L}) \to H^0(X_1, \mathcal{L}|_{X_1}) \oplus H^0(X_2, \mathcal{L}|_{X_2}) \xrightarrow{\phi} H^0(C, \mathcal{L}|_C).\tag{2.1}$$

The morphism $\phi$ is defined by $(R_{X_1 \to C}, -R_{X_2 \to C})$, where $R_{X_1 \to C}$ is the restriction map. Denote $\dim \mathcal{R}_{X_1 \to C}$ by $r_{X_1 \to C}(\mathcal{L}|_{X_1})$ or $r_{X_1 \to C}$ for simplicity. We have

$$h^0(X, \mathcal{L}) = h^0(X_1, \mathcal{L}|_{X_1}) + h^0(X_2, \mathcal{L}|_{X_2}) - \dim \mathcal{R}_{X_1 \to C}\tag{2.2}$$

In this section we recall some definitions and results of Fujita from [Fuj90, Chapter 1]. Note that the ground field $k$ is an algebraically closed field $k$ of characteristic $\geq 0$ in [Fuj90].

Let $X$ be a variety over $k$ of dimension $n$ and $\mathcal{L}$ is ample invertible sheaf on it. Such a pair $(X, \mathcal{L})$ is called a polarized variety.

**Definition 3.1.** ([Fuj90, (1.2.0)]) An element $D \in |\mathcal{L}|$ is called a rung of $(X, \mathcal{L})$ if $D$ is reduced and irreducible as a subscheme of $X$.

**Remark 3.2.** If $D$ is a rung of $(X, \mathcal{L})$, then $(D, \mathcal{L}|_D)$ is a polarized variety of dimension $n - 1$.

Let $\chi(t\mathcal{L})$ be the Euler-Poincaré characteristic of $\mathcal{L}^t$, which is a polynomial in $t$ of degree $n$. We put

$$\chi(t\mathcal{L}) = \sum_{j=0}^{n} \chi_j(X, \mathcal{L}) \frac{t^j}{j!},$$

where $t^j = t(t + 1)...(t + j - 1)$ for $j \geq 1$ and $t^0 = 1$.

**Definition 3.3.** ([Fuj90, (1.2.1)]) The degree of $(X, \mathcal{L})$ is defined as $d(X, \mathcal{L}) := \chi_0(X, \mathcal{L})$, which equals $(\mathcal{L})^n$ by Riemann Roch theorem.

The sectional genus of $(X, \mathcal{L})$ is defined as $g(X, \mathcal{L}) := 1 - \chi_{n-1}(X, \mathcal{L})$.

**Remark 3.4.** If $\dim X = 1$, $g(X, \mathcal{L}) = h^1(X, \mathcal{O}_X)$ is just the arithmetic genus of the curve $X$. If $\dim X \geq 2$ and $X$ is non-singular, by Riemann Roch theorem we have $g(X, \mathcal{L}) = (K_X + (n - 1)\mathcal{L}) \cdot (\mathcal{L})^{n-1}/2 + 1$.

**Proposition 3.5.** ([Fuj90, (1.2.1)])

Let $D$ be a rung of $(X, \mathcal{L})$. Then

$$\chi_r(D, \mathcal{L}|_D) = \chi_{r+1}(X, \mathcal{L}) \text{ for } r \geq 0.$$  

In particular, $g(D, \mathcal{L}|_D) = g(X, \mathcal{L})$. 

**Definition 3.6.** (Fujita’s $\Delta$-genus, cf. §Fuj90, (1.2.2))

The $\Delta$-genus of a polarized variety $(X, L)$ is defined as

$$\Delta(X, L) := d(X, L) - h^0(X, L) + \dim X = (L)^\dim X - h^0(X, L) + \dim X.$$  

**Definition 3.7.** ([Fuj90, (1.3.1)])

A sequence $D_1 \subset D_2 \subset \ldots \subset D_n \subset X$ of subvarieties of $X$ is called a ladder of $(X, L)$ if $D_j$ is a rung of $(D_{j+1}, L_{j+1})$ for each $j \geq 1$, where $L_{j+1} = L(D_{j+1}).$

**Proposition 3.8.** ([Fuj90, Proposition (1.3.4)])

Let $(X, L)$ be a polarized variety with $g(X, L) = 0$. Suppose that $(X, L)$ has a ladder $\{D_j\}$ such that each rung $D_j$ (including $X$ itself) is a normal variety.

Then $\Delta(X, L) = 0$.

**Theorem 3.9.** ([Fuj90, Theorem (1.3.5)])

Let $(X, L)$ be a polarized variety having a ladder. Assume that $g(X, L) \geq \Delta(X, L)$.

Then $Bs|L| = \emptyset$, if $d(X, L) \geq 2\Delta(X, L)$.

**Theorem 3.10.** ([Fuj90, Theorem (1.4.2)])

Let $(X, L)$ be a polarized variety, then $\Delta(X, L) \geq \dim Bs|L| + 1$, where $Bs|L|$ is the base locus of $|L|$ and $\dim \emptyset$ is defined to be $-1$.

In particular,

$$\Delta(X, L) \geq 0.$$  

Moreover, $Bs|L| = \emptyset$ if $\Delta(X, L) = 0$.

**Theorem 3.11.** ([Fuj90, Corollary (1.4.12)])

Let $(X, L)$ be a polarized variety with $\Delta(X, L) = 0$. Then $L$ is very ample, and $X$ is normal.

**Proposition 3.12.** ([Fuj90, (1.4.13)])

Let $(X, L)$ be a polarized variety with $\dim X = n$. Suppose that $\dim \im \Phi_{|L|} = n$, $B := Bs|L|$ is finite and $X$ has only Cohen Macaulay singularities at each point of $B$. Suppose in addition that $d(X, L) \geq 2\Delta(X, L) - 1$ if $\char k > 0$.

Then $(X, L)$ has a ladder.

**Definition 3.13.** ([Fuj90, (1.5.12) (5.13)], or [Har77, Ex I.5.12])

Let $M$ be a subvariety of $\mathbb{P}$ and $L$ be a linear subspace of $\mathbb{P}$ such that $M \cap L = \emptyset$. A **generalized cone** over $M$ with axis $L$ is defined as

$$M \ast L := \bigcup_{x \in M, y \in L} x \ast y,$$

where $x \ast y$ is a line passing through $x$ and $y$.

**Theorem 3.14.** ([Fuj90, Theorem (1.5.10), (1.5.15)])

Let $(X, L)$ be a polarized variety with $\Delta(X, L) = 0$ and $n = \dim X \geq 2$.

If $X$ is smooth, then $(X, L)$ is isomorphic to

1) $(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1))$,
2) $(C, \mathcal{O}_C(1))$, where $C$ is a hyperquadric in $\mathbb{P}^{n+1}$ and $\mathcal{O}_C(1) = \mathcal{O}_{\mathbb{P}^{n+1}}(1)|_C$,
3) $(\mathbb{P}(E), \mathcal{O}(1))$, where $\mathbb{P}(E)$ is the scroll of a vector bundle on $\mathbb{P}^1$ which is a direct sum of line bundles of positive degrees and $\mathcal{O}(1)$ is the tautological line bundle, or
4) $(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(2))$ (the Veronese surface).
If $X$ is singular, then $(X, \mathcal{L})$ is a generalized cone over a smooth variety $M$ with $\Delta(M, \mathcal{L}_M) = 0$.

**Remark 3.15.** The four classes above in the smooth case are disjoint except for the case $n = d(X, \mathcal{L}) = 2$ where type 2) and type 3) coincide.

**Lemma 3.16.** Let $(X, \mathcal{L})$ be a normal polarized variety with $\dim X = 2$ and $\Delta(X, \mathcal{L}) = 1$. If $|\mathcal{L}|$ is not composed with a pencil, i.e., $\dim \Phi_{|\mathcal{L}|}(X) = 2$, then $|\mathcal{L}|$ is base-point-free.

**Proof.** Since $h^0(X, \mathcal{L}) \geq 3$, we may assume $(\mathcal{L})^2 \geq 2$. Note that $\text{Bs}|\mathcal{L}|$ is finite since $1 = \Delta(X, \mathcal{L}) \geq \dim \text{Bs}|\mathcal{L}| + 1$ by Theorem 3.10. By Prop 3.12, $(X, \mathcal{L})$ has a ladder $D \in |\mathcal{L}|$. The sectional genus $g := g(X, \mathcal{L}) = g(D, \mathcal{L}|_D) = h^1(D, \mathcal{O}_D) \geq 0$.

First we show that $g \neq 0$. Otherwise, $D$ is a smooth rational curve. By Prop 3.8, we have $\Delta(X, \mathcal{L}) = 0$, a contradiction. Therefore, $g > 0$, and then $|\mathcal{L}|$ is base-point-free by Thm 3.9.

\[ \square \]

4. REDUCIBLE POLARIZED DEMI-NORMAL SCHEMES

Let $X$ be a connected scheme, $\mathcal{L}$ be an invertible sheaf on $X$, and $C$ be a reduced subscheme. Denote by $\mathcal{R}_{X \to C, \mathcal{L}}$ the restriction map $H^0(X, \mathcal{L}) \to H^0(C, \mathcal{L}|_C)$ and $r_{X \to C}(\mathcal{L}) := \dim \mathcal{R}_{X \to C, \mathcal{L}}$. We also use $\mathcal{R}_{X \to C}$ to denote $\mathcal{R}_{X \to C, \mathcal{L}}$ for simplicity if $\mathcal{L}$ is clear from the context.

**Lemma 4.1.** Let $X$ be a proper pure-dimensional connected $S_2$ scheme, let $\mathcal{L}$ be an invertible sheaf with $\dim \text{Bs}|\mathcal{L}| < \dim X - 1$ and let $C$ be a reduced subscheme of codimension one on $X$.

Then

$$r_{X \to C}(\mathcal{L}) = \dim < \Phi_{|\mathcal{L}|}(C) > + 1,$$

where $< \Phi_{|\mathcal{L}|}(C) >$ is the projective subspace of $|\mathcal{L}|^*$ spanned by $\Phi_{|\mathcal{L}|}(C)$.

**Proof.** Denote $\mathcal{P} := |\mathcal{L}|^*$. Since $\dim \text{Bs}|\mathcal{L}| < \dim X - 1 = \dim C$, $\Phi_{|\mathcal{L}|}$ is well-defined on $C^* := C - \text{Bs}|\mathcal{L}|$. Thus, $\Phi_{|\mathcal{L}|}(C) := \Phi_{|\mathcal{L}|}(C^*)$ is well-defined. We have the following commutative diagram:

\[
\begin{array}{cccc}
H^0(\mathcal{P}, \mathcal{O}_\mathcal{P}(1)) & \xrightarrow{\mathcal{R}_{\mathcal{P} \to \Phi_{|\mathcal{L}|}(C)}} & H^0(\Phi_{|\mathcal{L}|}(C), \mathcal{O}_\mathcal{P}(1)_{|\Phi_{|\mathcal{L}|}(C)}) & \xrightarrow{\Phi_{|\mathcal{L}|}^*} \\
\Phi_{|\mathcal{L}|} & \xrightarrow{\mathcal{R}_{X \to C}} & H^0(X, \mathcal{L}) & \xrightarrow{\mathcal{R}_{X \to C}} H^0(C, \mathcal{L}|_C)
\end{array}
\]

Therefore $r_{X \to C}(\mathcal{L}) = r_{\mathcal{P} \to \Phi_{|\mathcal{L}|}(C)}(\mathcal{O}_\mathcal{P}(1)) = \dim < \Phi_{|\mathcal{L}|}(C) > + 1$. \[ \square \]

**Theorem 4.2.** Let $(X, \mathcal{L})$ be a normal polarized variety and let $C$ be a reduced subscheme of codimension 1.

Then

$$\Delta(X, \mathcal{L}) + r_{X \to C}(\mathcal{L}) - \dim X \geq 0.$$

**Proof.** Denote $n := \dim X$. The case $n = 1$ is trivial, so we may assume $n \geq 2$.

We only need to consider the case $\Delta(X, \mathcal{L}) < n$. Therefore we may assume $\dim \text{Bs}|\mathcal{L}| \leq n - 2$ by Thm 3.10. We may assume further $h^0(X, \mathcal{L}) \geq 2$ (otherwise, $\Delta(X, \mathcal{L}) + r_{X \to C}(\mathcal{L}) - \dim X \geq \Delta(X, \mathcal{L}) - n = (\mathcal{L})^n - h^0(X, \mathcal{L}) \geq 0$).
Let $\pi: \tilde{X} \to X$ be a composition of minimal resolution of singularities and blowups such that $\pi^*|L| = |M| + F$, where $F$ is the fixed part and $|M|$ is the movable part which is base-point-free.

Let $W$ be the image of $\tilde{X}$ under $\Phi|_{|M|}$ and $\phi$ be the induced map of $\Phi|_{|M|}$ onto $W$. We have the following commutative diagram:

\[
\begin{array}{ccc}
\tilde{X} & \xrightarrow{\pi} & X \\
\downarrow{\phi} & & \downarrow{\Phi|_{|M|}} \\
W & \xrightarrow{\Phi|_{|L|}} & \mathbb{P}^{h^0(X,L)-1}.
\end{array}
\]

We claim that $(|L|)^n \geq \deg W$. First we have $\pi_* (F \cdot |L|) = \pi_*(F) \cdot L = 0$, since $\pi(F) = \text{Bs}(|L|)$ is of codimension at least 2. If $\dim W = \dim X = n$, $(|L|)^n = (\pi^*|L|)^n = M^n + |M|^{n-1} \cdot F \geq M^n = \deg \phi \deg W \geq \deg W$. If $m := \dim W < \dim X$, $(|L|)^n = M^n \cdot (\pi^*|L|)^{n-m} = (\deg W \cdot Z) \cdot (\pi^*|L|)^{n-m} \geq \deg W$, where $Z$ is a general fiber of $\phi$. Hence the claim is true.

Therefore, $\Delta(X,L) = (|L|)^n - h^0(X,L) + n \geq \deg W - h^0(W,\mathcal{O}_W(1)) + n = \Delta(W,\mathcal{O}_W(1)) + n - \dim W$. Hence $\Delta(X,L) + r_{X\to C}(\mathcal{L}) - n \geq \Delta(W,\mathcal{O}_W(1)) + r_{X\to C}(\mathcal{L}) - \dim W$. Next we show that $r_{X\to C}(\mathcal{L}) \geq \dim W$.

We claim that the image $\Phi|_C(C)$ of $C$ is of codimension equal or less than 1 in $W$. For the case $n = 2$ and $|L|$ is composed with a pencil, i.e., $\dim \Phi|_C(X) = 1$, the claim is trivial. For the case $n = 2$ and $|L|$ is not composed with a pencil, $|L|$ is base-point-free by Lemma 3.16. Hence $\Phi|_C(C)$ is a morphism and it contracts no curve as $|L|$ is ample. Therefore $\Phi|_C(C)$ is of codimension 1 in $W$. For the case $n \geq 3$, suppose for a contradiction that $\Phi|_C(C)$ is of codimension greater than 1 in $W$. Thus we have $\Phi|_C(C)$ and then $\phi(C) = 0$, where $C$ is the strict transformation of $C$ in $\tilde{X}$. We would have $0 < C \cdot (|L|)^{n-1} = C \cdot (\pi^*|L|)^{n-1} = C \cdot M \cdot (\pi^*|L|)^{n-2} + C \cdot F \cdot (\pi^*|L|)^{n-2} = 0$, a contradiction. Thus we have proven the claim.

By Lemma 4.1 we have $r_{X\to C}(\mathcal{L}) = \dim(\Phi|_C(C)) + 1$. It is obvious that $\dim(\Phi|_C(C)) \geq \dim(\Phi|_C(C))$. Therefore $r_{X\to C}(\mathcal{L}) \geq \dim(\Phi|_C(C)) + 1 \geq \dim W$.

We conclude that $\Delta(X,L) + r_{X\to C}(\mathcal{L}) - n \geq \Delta(W,\mathcal{O}_W(1)) \geq 0$. \hfill \Box

Corollary 4.3. Let $(X,L)$ and $C$ be as in Thm 4.2. Assume further $\Delta(X,L) = 0$ and $r_{X\to C}(\mathcal{L}) = \dim X$. Then $C$ is a hyperplane of $X$ with respect to $L$.

Proof. First $\Delta(X,L) = 0$ implies that $L$ is very ample by Thm 3.11. As a result, $|L|_C$ is very ample as well. By Lemma 4.1, $r_{X\to C}(\mathcal{L}) = \dim X$ implies that $\dim(\Phi|_C(C)) = \dim X - 1$. Therefore, $\langle \Phi|_C(C) \rangle \cong \mathbb{P}^{\dim X - 1} \subset \mathbb{P}^{h^0(X,L)-1}$. However, since $\Phi|_C(C)$ is an embedding, $\dim(\Phi|_C(C)) = \dim C = \dim X - 1$. Therefore, $\dim(\Phi|_C(C)) = \dim(\Phi|_C(C))$, which implies that $\Phi|_C(C) = \langle \Phi|_C(C) \rangle \cong \mathbb{P}^{\dim X - 1}$. Therefore, $C \cong \Phi|_C(C) \cong \mathbb{P}^{\dim X - 1}$ and $L|_C \cong \mathcal{O}_{\mathbb{P}^{\dim X - 1}}(1)$. Hence, $C$ is a hyperplane of $X$ with respect to $L$. \hfill \Box

Theorem 4.4. Let $(X,L)$ be an irreducible polarized non-normal variety. Then $\Delta(X,L) \geq 1$.

Moreover, if $C$ is a reduced subscheme of codimension 1 which does not contain any component of the non-normal locus, then we have $\Delta(X,L) + r_{X\to C}(\mathcal{L}) - \dim X \geq 0$. 


Proof. The first statement is a direct consequence of Thm 3.10 and Thm 3.11.

Let \( \pi : \overline{X} \to X \) be the normalization morphism and let \( \overline{C} \) be the proper transformation of \( C \) in \( \overline{X} \). Since \( H^0(X, \mathcal{L}) \cong \pi^*H^0(X, \mathcal{L}) \subset H^0(\overline{X}, \pi^*\mathcal{L}) \) and \((\mathcal{L})^{\dim X} = (\pi^*\mathcal{L})^{\dim \overline{X}}\), we have \( \Delta(X, \mathcal{L}) \geq \Delta(\overline{X}, \pi^*\mathcal{L}) \geq 0 \).

Next we have the following commutative diagram:

\[
\begin{array}{ccc}
0 & \to & H^0(\overline{X}, \pi^*\mathcal{L} - \overline{C}) \\
& \downarrow{\pi^*} & \downarrow{\pi^*} \\
0 & \to & H^0(X, \mathcal{L} - C) \\
\end{array}
\]

\[
\begin{array}{ccc}
& & \mathcal{R}_{\overline{X} \to \overline{C}} \to H^0(\overline{C}, \pi^*\mathcal{L}|_{\overline{C}}) \\
0 & \to & H^0(\overline{X}, \p \mathcal{R}_{X \to \mathcal{L}} \to H^0(C, \mathcal{L}|_{\overline{C}}).
\end{array}
\]

Hence \( h^0(X, \mathcal{L}) - r_{X \to C}(\mathcal{L}) = h^0(X, \mathcal{L} - C) \leq h^0(\overline{X}, \pi^*\mathcal{L} - \overline{C}) = h^0(\overline{X}, \pi^*\mathcal{L}) - r_{\overline{X} \to \overline{C}}(\pi^*\mathcal{L}). \) Therefore \( \Delta(X, \mathcal{L}) + r_{X \to C}(\mathcal{L}) - \dim X \geq \Delta(\overline{X}, \pi^*\mathcal{L}) + r_{\overline{X} \to \overline{C}}(\pi^*\mathcal{L}) - \dim \overline{X} \geq 0 \), where the second inequality follows from Thm 4.2. \( \square \)

Lemma 4.5. Let \((X, \mathcal{L})\) be a connected polarized demi-normal scheme. Assume \(X = X_1 \cup X_2\) where \(X_1\) is connected and \(X_2\) is irreducible. Then

\[ \Delta(X, \mathcal{L}) \geq \Delta(X_1, \mathcal{L}|_{X_1}). \]

Proof. Let \( C := X_1 \cap X_2 \) be the connecting subscheme. By (2.2), we have

\[
\Delta(X, \mathcal{L}) \geq \Delta(X_1, \mathcal{L}|_{X_1}) + \Delta(X_2, \mathcal{L}|_{X_2}) + \max\{r_{X_1 \to C}(\mathcal{L}|_{X_1}), r_{X_2 \to C}(\mathcal{L}|_{X_2})\} - \dim X \\
\geq \Delta(X_1, \mathcal{L}|_{X_1}) + \Delta(X_2, \mathcal{L}|_{X_2}) + r_{X_2 \to C}(\mathcal{L}|_{X_2}) - \dim X.
\]

By Thm 4.2 and Thm 4.4, \(\Delta(X_2, \mathcal{L}|_{X_2}) + r_{X_2 \to C}(\mathcal{L}|_{X_2}) - \dim X \geq 0\) whenever \(X_2\) is normal or not. Therefore \( \Delta(X, \mathcal{L}) \geq \Delta(X_1, \mathcal{L}|_{X_1}). \) \( \square \)

As a result, we obtain the following theorem

**Theorem 4.6.** Let \((X, \mathcal{L})\) be a connected polarized demi-normal scheme. Then

\[ \Delta(X, \mathcal{L}) \geq 0. \]

We say that a connected demi-normal scheme \(X\) is a **tree of subschemes** \(X_i\), if \(X = \bigcup X_i\) and \(X_i \cap X_j\) is either an irreducible subscheme of codimension one or a subscheme of codimension \(\geq 2\) for \(i \neq j\). This is equivalent to say that the dual graph of codimension \(\leq 1\) points of \(X\) is a tree.

Let \(X = \bigcup X_i\) be a connected demi-normal scheme, and let \(C_{ij} := X_i \cap X_j\) be the connecting subscheme of \(X_i\) and \(X_j\). Let \(C := \sum_{\text{codim} \, C_{ij} = 1} C_{ij}\). Normalizing \(X\) along \(C\), we obtain \((\overline{X} := \bigcup X_i, \overline{C} := \sum C_{ij}) = \bigcup\{X_i, \sum_{j} C_{ij}\}\). Denote by \(\nu: \overline{X} \to X\) the normalization morphism along \(C\). The morphism \(\nu\) induces an isomorphism \(\phi_{ij}: \overline{C}_{ij} \cong \overline{C}_{ji}\). Conversely, \(X\) can be viewed as being constructed from \(\bigcup \{X_i, \sum C_{ij}\}\) by gluing \(X\) along \(\overline{C}\) via \(\phi_{ij}\)'s (see [Kol13, chapter 5]).

**Theorem 4.7.** Let \((X, \mathcal{L})\) be a connected polarized demi-normal scheme with \(\Delta(X, \mathcal{L}) = 0\).

Then \(X\) is a tree of subschemes \(X_i\)'s with \(\Delta(X_i, \mathcal{L}|_{X_i}) = 0\) glued along hyperplanes.

**Proof.** By Lemma 4.5, we see that \(\Delta(X_i, \mathcal{L}|_{X_i}) = 0\) for each \((X_i, \mathcal{L}|_{X_i})\).

Assume that \(C_{ij} := X_i \cap X_j\) is a subscheme of codimension one. By Lemma 4.5, we have \(\Delta(X_i \cup X_j, \mathcal{L}|_{X_i \cup X_j}) = \Delta(X_i, \mathcal{L}|_{X_i}) = \Delta(X_j, \mathcal{L}|_{X_j}) = 0\). By the proof of
Lemma 4.5, we have \( r_{X_i \rightarrow C_{ij}}(\mathcal{L}|_{X_i}) = r_{X_j \rightarrow C_{ij}}(\mathcal{L}|_{X_j}) = \dim X \). By Cor 4.3, we see that \( C_{ij} \) is a hyperplane of \( X_i \) (resp. \( X_j \)) with respect to \( \mathcal{L}|_{X_i} \) (resp. \( \mathcal{L}|_{X_j} \)). Therefore, the statement follows.

**Remark 4.8.** Polarized varieties \((X_i, \mathcal{L}_i)\) with \( \Delta(X_i, \mathcal{L}_i) = 0 \) have been classified in Thm 3.14. We are able to describe hyperplanes \( C_{ij} \) on \((X_i, \mathcal{L}_i)\). In order to construct a connected polarized demi-normal scheme \((X, \mathcal{L})\) from \((X_i, \mathcal{L}_i, C_{ij})'s\), we may first embed \( X_i \) to \( \mathbb{P}_i \) via \( \Phi_{\mathcal{L}_i} \) and then embed \( \mathbb{P}_i's \) into a big projective space \( \mathbb{P} \). The hyperplanes \( C_{ij}'s \) are linear subspaces in \( \mathbb{P} \) of the same dimension, so they are isomorphic to each other. Thus, we can glue \( X_i's \) along \( C_{ij}'s \) in \( \mathbb{P} \) and obtain the desired \( X \) and \( \mathcal{L} = \mathcal{O}_X(1) \).

**Corollary 4.9.** Let \((X, \Lambda)\) be a connected KSBA stable log scheme with Cartier index \( I \). Then

\[
\Delta(X, I(K_X + \Lambda)) \geq 0.
\]

**Remark 4.10.** When \( I = 1 \), \( \dim X = 2 \) and \( \Lambda \) is reduced, \( \Delta(X, K_X + \Lambda) = (K_X + \Lambda)^2 - p_g(X, \Lambda) + 2 \geq 0 \) is just the stable log Noether inequality in [LR13].

In the following, we will construct a KSBA stable log scheme with Cartier index \( I = 1 \) and \( \Delta(X, K_X + \Lambda) = 0 \). To avoid repetition, we only give examples with \( 2m + 1 \) irreducible components. The construction of KSBA stable log scheme with \( 2m \) irreducible components is similar. These examples demonstrate that when \( I = 1 \), the inequality \( \Delta(X, K_X + \Lambda) \geq 0 \) is sharp.

**Example 4.11.** Let \( B = H_1 + ... + H_{n+2} \) be a snc divisor consisting of hyperplanes on \( \mathbb{P}^n \). We have \( K_{\mathbb{P}^n} + B \sim \mathcal{O}_{\mathbb{P}^n}(1) \) and \( (\mathbb{P}^n, B) \) is lc.

Let

\[
\left( \overline{X_1}, \overline{D_{1,2} + \Lambda_1} \right) \cong (\mathbb{P}^n, H_1 + (H_2 + ... + H_{n+2})),
\]

\[
\left( \overline{X_k}, \overline{D_{k,k-1} + D_{k,k+1} + \Lambda_k} \right) \cong (\mathbb{P}^n, H_1 + H_2 + (H_3 + ... + H_{n+2})), \quad k = 2, ..., 2m,
\]

\[
\left( \overline{X_k}, \overline{D_{k,k-1} + D_{k,k+1} + \Lambda_k} \right) \cong (\mathbb{P}^n, H_2 + H_1 + (H_3 + ... + H_{n+2})), \quad k = 3, ..., 2m - 1,
\]

and

\[
\left( \overline{X_{2m+1}}, \overline{D_{2m+1,2m+2} + \Lambda_{2m+1}} \right) \cong (\mathbb{P}^n, H_2 + (H_1 + ... + H_{n+2})).
\]

We use the method in Remark 4.10 to construct the polarized demi-normal scheme. We glue \( X_{2i-1} \) and \( X_{2i} \) by the isomorphism \( \overline{D_{2i-1,2i}} \cong H_1 \cong \overline{D_{2i,2i-1}} \) induced from \( \overline{X_{2i-1}} \cong \mathbb{P}^n \cong \overline{X_{2i}}, \quad i = 1, ..., m \). Similarly, we glue \( X_{2i} \) and \( X_{2i+1} \) by the isomorphism \( \overline{D_{2i,2i+1}} \cong H_2 \cong \overline{D_{2i+1,2i}} \), \( i = 1, ..., m \). Then we obtain a demi-normal log scheme \( X \) and a boundary divisor \( \Lambda \) which is the image of \( \sum \overline{\Lambda}_i \). The divisor \( K_X + \Lambda \) is a polarization on \( X \) with \( \Delta(X, K_X + \Lambda) = 0 \). Note that \( (K_X + \Lambda)|_{X_i} \cong \mathcal{O}_{\mathbb{P}^n}(1) \). By Thm 5.38 in [Kol13] the singularities of the log scheme \( (X, \Lambda) \) are slc. Thus \( (X, \Lambda) \) is a KSBA stable log scheme with Cartier index one and \( \Delta(X, K_X + \Lambda) = 0 \). Moreover, \( (K_X + \Lambda)^n = 2m + 1 \). See Figure 1 for the \( n = 2 \) case.

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Figure 1. $n = 2$ case.

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SCHOOL OF MATHEMATICS AND STATISTICS, HUBEI MINZU UNIVERSITY
Email address: chjingsh@hbmzu.edu.cn

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF HOUSTON
Email address: ychen224@cougar.net.uh.edu