On \(r\)-Edge-Connected \(r\)-Regular Bricks and Braces and Inscribability

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Abstract  A classical result due to Steinitz states that a graph is isomorphic to the graph of some 3-dimensional polytope \(P\) if and only if it is planar and 3-connected. If a graph \(G\) is isomorphic to the graph of a 3-dimensional polytope inscribed in a sphere, it is said to be of inscribable type. The problem of determining which graphs are of inscribable type dates back to 1832 and was open until Rivin proved a characterization in terms of the existence of a strictly feasible solution to a system of linear equations and inequalities which we call \(\text{sys}(G)\), which, surprisingly, also appears in the context of the Traveling Salesman Problem. Using such a characterization, various classes of graphs of inscribable type can be described. Dillencourt and Smith gave a characterization of 3-connected 3-regular planar graphs that are of inscribable and a linear-time algorithm for recognizing such graphs. In this paper, their results are generalized to \(r\)-edge-connected \(r\)-regular graphs for odd \(r \geq 3\) in the context of the existence of strictly feasible solutions to \(\text{sys}(G)\). An answer to an open question raised by D. Eppstein concerning the inscribability of 4-regular graphs is also given.

Keywords  Inscrribable, Polytope, Regular, Graph, Sphere

1 Background and main results

Given a 3-dimensional polytope \(P\), we define the graph of \(P\), denoted by \(G(P)\), to be the graph \((V, E)\) where \(V\) is the set of extreme points of \(P\) and \(uv \in E\) if and only if \(u\) and \(v\) are adjacent in \(P\).

Let \(G = (V, E)\) be an undirected simple graph. A classical result due to Steinitz [16] connecting graph theory to geometry is the following:

**Theorem 1.** \(G\) is isomorphic to the graph of some 3-dimensional polytope \(P\) in \(\mathbb{R}^3\) if and only if it is planar and 3-connected.

A 3-connected planar graph isomorphic to the graph of a 3-dimensional polytope inscribed in a sphere is said to be of inscribable type. Steinitz [17] gave examples of graphs that are not of inscribable type. The problem of determining which graphs are of inscribable type dates back to 1832 (see [15]) and was open until Hodgson et al. [10] announced the following in 1992:

**Theorem 2.** If \(G\) is 3-connected and planar, then \(G\) is of inscribable type if and only if there exists \(x \in \mathbb{R}^E\) satisfying:

\[
x(\delta(v)) = 2\pi \quad \forall v \in V,
\]

\[
x(\delta(S)) > 2\pi \quad \forall S \subset V, \ 2 \leq |S| \leq |V| - 2,
\]

\[
x_e > 0 \quad \forall e \in E,
\]

\[
x_e < \pi \quad \forall e \in E.
\]

(For \(S \subseteq V\), \(\delta(S)\) denotes the set of edges with one end-vertex in \(S\) and one end-vertex not in \(S\); \(\delta(\{v\})\) is abbreviated as \(\delta(v)\). For \(x \in \mathbb{R}^E\) and \(A \subseteq E\), \(x(A) := \sum_{e \in A} x_e\).)

Given a system of linear equations and inequalities, we call a solution to the system that satisfies all the
inequalities strictly a strictly feasible solution. Using this terminology and with $\text{sys}(G)$ denoting the system
\[
\begin{align*}
    x(\delta(v)) &= 2 \quad \forall v \in V, \\
x(\delta(S)) &\geq 2 \quad \forall S \subseteq V, \ 2 \leq |S| \leq |V| - 2, \\
x &\geq 0,
\end{align*}
\]

Theorem 2 can be rephrased as follows:

**Theorem 3.** If $G$ is 3-connected and planar, then $G$ is of inscribable type if and only if there exists a strictly feasible solution to $\text{sys}(G)$.

Note that constraints of the form $x_e \leq 1$ where $e \in E$ (corresponding to $x_e \leq \pi$ in Theorem 2) are superfluous because $x_{uv} \leq 1$ is implied by the constraints $x(\delta([u,v])) \geq 2$, $x(\delta(u)) = 2$, and $x(\delta(v)) = 2$ where $uv \in E$.

Rivin gave two proofs of Theorem 2. One uses hyperbolic geometry [13]. The other is an elementary proof using mathematical optimization [14].

Incidentally, $\text{sys}(G)$ also defines what is known as the subtour-elimination polytope of $G$ in connection with the Traveling Salesman Problem. This remarkable, albeit accidental, connection between the subtour-elimination polytope and the century-old geometry problem provided a motivation for studying strictly feasible solutions to the system $\text{sys}(G)$.

Using Theorem 3, one can show that various classes of graphs are of inscribable type. For instance, Dillencourt and Smith [3] showed that, among others, 4-connected planar graphs and planar graphs obtained from 4-connected planar graphs by removing one vertex are of inscribable type. In [2], they gave necessary and sufficient conditions for a 3-connected 3-regular planar graph to be of inscribable type and a linear-time algorithm for recognizing such a graph. In particular, they showed the following:

**Theorem 4.** If $G$ is a 3-edge-connected 3-regular planar graph, then $G$ is of inscribable type if and only if $G$ is more-than-1-tough or $G$ is bipartite and has a 4-connected dual.

In this paper, we extend the above result to the following:

**Theorem 5.** If $G$ is an $r$-edge-connected $r$-regular graph where $r \geq 3$ is an odd integer, then $\text{sys}(G)$ has a strictly feasible solution if and only if $G$ is more-than-1-tough or $G$ is bipartite and has no non-trivial $r$-edge cut.

To see that Theorem 4 follows directly from Theorem 5, note that the dual of a 3-edge-connected 3-regular planar graph $G$ is a maximal planar graph with more than four vertices. Such a maximal planar graph is 4-connected if and only if it does not have a separating triangle (see [9]), or equivalently, $G$ has no non-trivial 3-edge-cut.

It turns out that Theorem 5 is equivalent to the following:

**Theorem 6.** If $G$ is an $r$-edge-connected $r$-regular graph where $r \geq 3$ is an odd integer, then $\text{sys}(G)$ has a strictly feasible solution if and only if $G$ is a brick or a brace.

Observe that every $r$-edge-connected $r$-regular graph where $r \geq 3$ is odd is necessarily 3-connected. The above equivalent formulation suggests connections with results in perfect matchings for $r$-edge-connected $r$-regular graphs. (Bricks and braces are the fundamental objects in the tight cut decomposition of a matching-covered graph, a procedure described in a landmark paper by Lovász [11] in the study of the matching lattice.) In addition to proving Theorems 5 and 6, we also give a characterization of when such graphs are bricks and braces; the characterization generalizes a notion introduced by Dillencourt and Smith for 3-connected 3-regular planar graphs.

Before we end this section, we remark that a simple $r$-regular graph cannot be planar for $r \geq 6$. In addition, a simple $r$-regular planar graph cannot be bipartite for $r \geq 4$. Thus, specializing Theorem 6 to 5-edge-connected 5-regular planar graphs gives:

**Theorem 7.** A 5-edge-connected 5-regular simple planar graph is of inscribable type if and only if it is a brick.

We mention in passing that 5-edge-connected 5-regular simple bricks do exist and so they give a previously unrecognized class of graphs of inscribable type. An example of such a graph can be found in [1].

## 2 Notation and definitions

Unless otherwise stated, graphs are assumed to be undirected and loopless but they may contain parallel edges. The vector with all entries equal to 1 is denoted by $\mathbf{e}$. Let $S$ be a finite set. If $T \subseteq S$, then the vector $x \in \mathbb{R}^S$ with $x_e = 1$ if $e \in T$ and $x_e = 0$ if $e \notin T$ is called the incidence vector of $T$. A family $\mathcal{F}$ of sets is called nested if for any non-disjoint distinct members $S$ and $T$ of $\mathcal{F}$, either $S \subset T$ or $T \subset S$. 
Let $G = (V, E)$ be a graph. The number of components of $G$ is denoted by $\omega(G)$. $G$ is said to be 1-tough if $|S| \geq \omega(G - S)$ for every subset $S$ of $V$ with $\omega(G - S) > 1$, and is more-than-1-tough if the inequality is strict.

For a subset $S$ of $V$, the graph induced by $S$ is denoted by $G[S]$.

If $S$ is a proper subset of $V$ with $|S| > 1$, we let $G \times S$ denote the (possibly not simple) graph obtained from $G$ by contracting $S$, that is, removing all the vertices in $S$ and all the edges incident with a vertex in $S$ from $G$ and adding a new vertex called $S$ and edges $uS$ for every edge $us \in E$ where $s \in S$ and $u \notin S$. The new vertex is called a pseudo-vertex of $G \times S$.

For $S, T \subset V$, define $\gamma(S, T)$ to be the set of edges incident with a vertex in $S$ and a vertex in $T$. Let $S \subset V$ with $0 < |S| < |V|$. $N(S)$ denotes the set $\{v \in V \setminus S : v$ is adjacent to some vertex in $S\}$. $N(\{v\})$ is abbreviated as $N(v)$. Define $\delta(S) := \gamma(S, N(S))$. $\delta(\{v\})$ is abbreviated as $\delta(v)$.

A set of edges $A$ is called a cut of $G$ if $A = \delta(S)$ for some $S \subset V$; $S$ and $V \setminus S$ are called the shores of the cut $A$ if $G$ is connected. A shore $S$ is called a proper shore if $|S| \leq |V| - 2$. Cuts of the form $\delta(v)$ for some vertex $v$ are trivial cuts. All other cuts are non-trivial.

We denote the set of cuts of $G$ by $C(G)$. Two cuts $\delta(S)$ and $\delta(T)$ are said to cross if the four sets $S \cap T$, $S \setminus T$, $T \setminus S$, and $V \setminus (S \cup T)$ are all non-empty. Two cuts that do not cross are said to be non-crossing.

A subset $M$ of $E$ is a matching of $G$ if no two edges in $M$ are incident with the same vertex. If every vertex is an end-vertex of some edge in $M$, then $M$ is a perfect matching. $G$ is called matching-covered if for every edge $e \in E$, there exists a perfect matching that contains $e$. The following characterization is due to Tutte [18].

**Theorem 8.** $G$ has a perfect matching if and only if for every $S \subset V$, odd($G - S$) $\leq |S|$. (Here, odd($H$) denotes the number of components of $H$ having an odd number of vertices.)

Let PM($G$) denote the convex hull of incidence vectors of perfect matchings of $G$. An important result in matching theory is the following:

**Theorem 9.** (Edmonds [4]) PM($G$) is the set of all $x \in \mathbb{R}^E$ satisfying

$$
x(\delta(v)) = 1 \quad \forall \ v \in V,$$

$$x(\delta(S)) \geq 1 \quad \forall \ S \subset V, 3 \leq |S| \leq \frac{|V|}{2}, |S| \text{ is odd} \ x \geq 0.$$

An immediate consequence of the above theorem is the following:

**Corollary 10.** If PM($G$) is non-empty, then $G$ has a perfect matching. Furthermore, if there exists $x \in$ PM($G$) with $x > 0$, then $G$ is matching-covered.

A cut $A \subset C(G)$ is said to be tight if every perfect matching of $G$ contains exactly one edge in $A$. $G$ is said to be bicritical if $G - \{u, v\}$ has a perfect matching for every pair $u, v \in V$. A graph is called a brick if it is 3-connected, bicritical, and has at least four vertices. A bipartite graph $G$ with bipartition $(U, W)$ is called a brace if $G$ is matching-covered with at least four vertices and for all distinct $u, u' \in U$ and $w, w' \in W$, $G - \{u, v, u', w'\}$ has a perfect matching. It can be shown that a bipartite graph $G$ with bipartition $(U, W)$ and $|U| = |W| \geq 2$ is a brace if and only if $|N(X)| \geq |X| + 2$ and for every subset $X \subseteq U$ with $1 \leq |X| \leq |U| - 2$. Bricks and braces are fundamental objects in the study of matchings. (See for instance [12], [5] and [11].) Using the result [5] that each tight cut in a brick is trivial, Lovász [11] showed:

**Theorem 11.** A matching-covered graph has no non-trivial tight cuts if and only if it is either a brick or a brace.

The set of solutions to sys($G$) is denoted by SEP($G$). $G$ is said to be feasible if SEP($G$) is non-empty. A cut $A$ of a feasible graph $G$ is said to be constricted if $x(A) = 2$ for all $x \in$ SEP($G$). It is not difficult to show the following:

**Proposition 12.** Feasible graphs are 1-tough.

## 3 Proofs of Theorems 5 and 6

Throughout this section, $G = (V, E)$ denotes an $r$-edge-connected $r$-regular graph where $r \geq 3$ is an odd integer. (We allow $G$ to have parallel edges.) The next result gives a connection between the subtour-elimination polytope and the perfect matching polytope of $G$.

**Proposition 13.** dim(PM($G$)) = dim(SEP($G$)). Furthermore, a non-trivial cut of $G$ is tight if and only if it is constricted.

**Proof.** Clearly, $\frac{1}{2}$ SEP($G$) $\subseteq$ PM($G$). Hence, dim(SEP($G$)) $\leq$ dim(PM($G$)).

We now show that dim(SEP($G$)) $\geq$ dim(PM($G$)). Define the affine function $f : \mathbb{R}^E \rightarrow \mathbb{R}^E$ by $f(x) = \frac{1}{r} x + \frac{2r - 1}{r^2} e$. Let $M$ be any perfect matching of $G$. Mathematics and Statistics 1(3): 135-143, 2013 137
Let \( \hat{x} = f(\chi^M) \) where \( \chi^M \) denotes the incidence vector of \( M \). Then for any vertex \( v \in V \),
\[
\hat{x}(\delta(v)) = \frac{1}{r} + \sum_{e \in \delta(v)} \frac{2r - 1}{r^2} = \frac{1}{r} + \frac{r(2r - 1)}{r^2} = 2.
\]
Consider \( S \subset V \) such that \( 1 < |S| < |V| \).
If \( |S| \) is odd, then \( |\delta(S) \cap M| \geq 1 \) and \( |\delta(S)| \geq r \).
Hence,
\[
\hat{x}(\delta(S)) \geq \frac{1}{r} + \sum_{e \in \delta(S)} \frac{2r - 1}{r^2} = \frac{1}{r} + \frac{r(2r - 1)}{r^2} = 2.
\]
If \( |S| \) is even, then \( |\delta(S)| \) is even and so \( |\delta(S)| \geq r + 1 \).
Hence,
\[
\hat{x}(\delta(S)) \geq \sum_{e \in \delta(S)} \frac{2r - 1}{r^2} \geq \frac{(r + 1)(2r - 1)}{r^2} > 2.
\]
Hence, \( \hat{x} \in \text{SEP}(G) \). It follows that \( f(\text{PM}(G)) \subseteq \text{SEP}(G) \). As \( f \) is bijective, \( \dim(f(\text{PM}(G))) = \dim(\text{PM}(G)) \). Therefore, \( \dim(\text{PM}(G)) \leq \dim(\text{SEP}(G)) \).
This proves the first part of the theorem.
We now prove the second part. Let \( C \) be a non-trivial cut. Suppose \( \hat{x} \in \text{PM}(G) \) is such that \( \hat{x}(C) > 1 \). Let \( \hat{y} = f(\hat{x}) \). Then \( \hat{y}(C) > 2 \) and \( \hat{y} \in \text{SEP}(G) \), implying that \( C \) is not a constricted cut. Suppose \( \hat{x} \in \text{SEP}(G) \) is such that \( \hat{x}(C) > 2 \). Then \( \frac{1}{2} \hat{x}(C) > 1 \). Since \( \frac{1}{2} \hat{x} \in \text{PM}(G) \), \( C \) is not a tight cut. The result now follows.

**Proof of Theorem 6.** Since \( x = \frac{3}{2}e \) is a solution to \( \text{sys}(G) \) with \( x > 0 \), \( \text{sys}(G) \) has a strictly feasible solution if and only if \( G \) has no non-trivial constricted cut. By the second part of Proposition 13, \( G \) has no non-trivial constricted cut if and only if \( G \) has no non-trivial tight cut. The result now follows from Theorem 11 because \( G \) is matching-covered by Corollary 10 as \( \frac{1}{2} \in \text{PM}(G) \).
4 A note on 4-regular graphs

So far, the results that have been discussed concern r-regular graphs where r is odd. When r is even, the situation is somewhat unclear and a characterization of all 3-connected 4-regular planar graphs of inscribable type using simple graph-theoretical terms is not yet known. For example, with regards to 4-regular planar graphs, Eppstein [6] raised the following question: Is a more-than-1-tough 3-connected 4-regular planar graph of inscribable type? The answer is ‘no’ and the graph depicted in Figure 1 is more-than-1-tough but is not of inscribable type. The technical details for showing this fact can be found in Section 6.2.1 of [1].

![Figure 1. A more-than-1-tough 3-connected 4-regular planar graph](image)

However, we do have the following positive result:

**Theorem 17.** Let $G = (V, E)$ be a 3-connected 4-regular planar graph. If each non-trivial 4-edge cut is a matching of $G$, then $\text{sys}(G)$ has a strictly feasible solution.

We establish a number of lemmas before proving the result. We first define a useless edge. An edge $e$ is said to be useless if $x_e = 0$ for all $x \in \text{SEP}(G)$. Hence, $\text{sys}(G)$ has a strictly feasible solution if and only if $G$ has no useless edge and no non-trivial constricted cut.

For the next few lemmas, let $(P)$ denote the linear programming problem:

\[
\begin{align*}
\text{max} & \quad 0 \\
\text{subject to} & \quad x(\delta(v)) = 2 \quad \forall v \in V(G) \\
& \quad -x(A) \leq -2 \quad \forall A \in C(G) \\
& \quad x \geq 0
\end{align*}
\]

and let $(D)$ denote the dual of $(P)$:

\[
\begin{align*}
\min & \quad 2 \sum_{v \in V(G)} z_v - 2 \sum_{A \in C(G)} y_A \\
\text{subject to} & \quad z_u + z_v - \sum_{A \in C(G) : uv \in A} y_A \geq 0 \quad \forall uv \in E(G) \\
& \quad y \geq 0.
\end{align*}
\]

Let $\text{sys}'(G)$ denote the set of constraints in $(D)$.

The next lemma gives a sufficient condition for a cut to be constricted and an edge to be useless.

**Lemma 18.** Let $G$ be a feasible graph. If there exist $\bar{y} \in \mathbb{R}^{C(G)}_+$ and $\bar{z} \in \mathbb{R}^{V(G)}$ feasible for $\text{sys}'(G)$ such that $\sum_{A \in C(G)} \bar{y}_A = \sum_{v \in V(G)} \bar{z}_v$, then all the cuts in $\{A : \bar{y}_A > 0\}$ are constricted and all the edges in $\{uv : \bar{z}_u + \bar{z}_v - \sum_{A \in C(G) : uv \in A} \bar{y}_A > 0\}$ are useless.

**Proof.** As $G$ is feasible, $(P)$ has an optimal solution. The result now follows from complementary slackness. \[\square\]

**Lemma 19.** Let $G$ be a feasible graph. Then there exist $\bar{y} \in \mathbb{R}^{C(G)}_+$ and $\bar{z} \in \mathbb{R}^{V(G)}$ feasible for $\text{sys}'(G)$ such that the following hold: $\sum_{A \in C(G)} \bar{y}_A = \sum_{v \in V(G)} \bar{z}_v$, a cut $A$ is constricted if and only if $\bar{y}_A > 0$, and an edge $uv$ is useless if and only if $\bar{z}_u + \bar{z}_v - \sum_{A \in C(G) : uv \in A} \bar{y}_A > 0$.

**Proof.** Since $G$ is feasible, $(P)$ has an optimal solution. By strict complementarity for linear programming, there exist an optimal solution $\bar{y}, \bar{z}$ such that a cut $A$ is constricted if and only if $\bar{y}_A > 0$, and an edge $uv$ is useless if and only if $\bar{z}_u + \bar{z}_v - \sum_{A \in C(G) : uv \in A} \bar{y}_A > 0$. As the optimal value is 0, we have $2 \sum_{v \in V(G)} \bar{z}_v - 2 \sum_{A \in C(G)} \bar{y}_A = 0$, giving

\[\sum_{v \in V(G)} \bar{z}_v = \sum_{A \in C(G)} \bar{y}_A.\] \[\square\]

Next, we obtain a refinement of Lemma 19 using the notion of uncrossing. Let $\bar{y}, \bar{z}$ be integral and feasible for $\text{sys}'(G)$. Let $\mathcal{A}(\bar{y})$ denote the set $\{A \in C(G) : \bar{y}_A > 0\}$.

Let $\delta(S)$ and $\delta(T)$ be crossing cuts in $\mathcal{A}(\bar{y})$. By uncrossing $\delta(S)$ and $\delta(T)$, we mean applying the following modifications to $\bar{y}, \bar{z}$: Let $\rho = \min(\bar{y}_S(\delta(S)), \bar{y}_T(\delta(T)))$. Decrease $\bar{y}_S(\delta(S))$ and $\bar{y}_T(\delta(T))$ by $\rho$. If $S \cap T$ or $V(G) \setminus (S \cup T)$ is
equal to \{v\} for some \(v \in V(G)\), then decrease \(z_v\) by \(\rho\); otherwise, increase \(\bar{y}_{(S \cap T)}\) by \(\rho\).

This technique of uncrossing is quite common in combinatorics. (See for instance Chapter 4 of [7].) The next result is a specialization of the technique for the purposes of the current paper. The idea of the proof is similar to the idea used in the proof of Claim 1 of Theorem 4.7 in [5].

**Lemma 20.** Given an integral pair \(\bar{y}, \bar{z}\) feasible for \(sys'(G)\), one can obtain, by performing a finite number of uncrossings, an integral pair \(y', z'\) feasible for \(sys'(G)\) such that 
\[
\sum_{A \in C(G)} \bar{y}_A - \sum_{v \in V(G)} \bar{z}_v = \sum_{A \in C(G)} y'_A - \sum_{v \in V(G)} z'_v
\]
and \(\{A \in C(G) : \bar{y}_A > 0\}\) is a non-crossing family of cuts.

**Proof.** For \(y \in Z(C(G))\), let \(M(y)\) denote 
\[
\sum_{A \in C(G)} \sum_{B \in C(G)} \pi_y(A, B)
\]
where
\[
\pi_y(A, B) = \begin{cases} 
\bar{y}_{AB} & \text{if } A, B \text{ cross;}
\bar{y}_{AB} & \text{otherwise.}
\end{cases}
\]

Let \(A(\bar{y})\) denote \(\{A \in C(G) : \bar{y}_A > 0\}\). If \(M(\bar{y}) = 0\), then \(A(\bar{y})\) is a non-crossing family of cuts and we are done. Suppose that \(M(\bar{y}) > 0\). Then there exist \(S, T \subset V(G)\) such that \(\delta(S), \delta(T) \in A(\bar{y})\) cross. Pick any such pair \(S, T\). Let \(A = \delta(S)\) and \(B = \delta(T)\). Uncross \(A\) and \(B\) to obtain \(y', z'\). It is not difficult to see that \(y', z'\) are still feasible for \(sys'(G)\) and 
\[
\sum_{A \in C(G)} \bar{y}_A - \sum_{v \in V(G)} \bar{z}_v = \sum_{A \in C(G)} y'_A - \sum_{v \in V(G)} z'_v
\]
for a cut \(C \in C(G)\), let \(K(C)\) denote the multiset of cuts \(D \in C(G)\) such that \(y_{(S \cap T)}\) is the number of times \(D\) appears in \(K(C)\) given \(y_{\bar{D}}\). Since \(y'_{\bar{D}} = y_{\bar{D}}\) for all \(D \notin \{A, B, \delta(S \cap T), \delta(S \cup T)\}\), we have
\[
M(y') \leq M(y) + \rho(|K(\delta(S \cap T))| + |K(\delta(S \cup T))|) - \rho(|K(A)| + |K(B)|) \leq 0.
\]

Note that any cut that crosses both \(\delta(S \cap T)\) and \(\delta(S \cup T)\) also crosses both \(A\) and \(B\). And any cut that crosses neither \(A\) nor \(B\) also cross neither \(\delta(S \cap T)\) nor \(\delta(S \cup T)\). It follows that
\[
\rho(|K(\delta(S \cap T))| + |K(\delta(S \cup T))|) - \rho(|K(A)| + |K(B)|) \leq 0.
\]
However, this inequality is strict since \(B \in K(A)\) but \(B \notin K(\delta(S \cap T)) \cup K(\delta(S \cup T))\). Hence, \(M(y') < M(y)\) and we set \(\bar{y} \rightarrow y'\) and repeat the process. As \(M(\bar{y})\) is integral whenever \(y\) is integral, each uncrossing reduces \(M(\bar{y})\) by an integral amount until it reaches 0.

**Lemma 21.** For any 3-connected feasible graph \(G\), \(sys(G)\) has no strictly feasible solution if and only if there exist \(\bar{y} \in R_+^{C(G)}\) and \(\bar{z} \in R^{V(G)}\) feasible for \(sys'(G)\) such that the following hold: 
\[
\sum_{A \in C(G)} \bar{y}_A = \sum_{v \in V(G)} \bar{z}_v,
\]
the set \(\{A \in C(G) : \bar{y}_A > 0\}\) is non-crossing, and that \(\bar{y}_A > 0\) for some \(A \in C(G)\) or \(\bar{z}_u + \bar{z}_v - \sum_{A \in C(G)} \bar{y}_A > 0\) for some \(uv \in E(G)\). (Here, \("or\" is not exclusive.)

**Proof.** Sufficiency follows from Lemma 18.

To prove necessity, suppose that \(sys(G)\) has no strictly feasible solution. Then there exists either a constricted cut \(C \in C(G)\) or a useless edge \(e \in E(G)\). By Lemma 19, there exist an optimal solution \(\bar{y}, \bar{z}\) for \(D\) such that \(\bar{y}_A > 0\) for every non-trivial constricted cut \(A\) and \(\bar{z}_u + \bar{z}_v - \sum_{A \in C(G)} \bar{y}_A > 0\) for every useless edge \(uv\). Since the coefficients in \(D\) are integral and the constraints of \(D\) are homogeneous with optimal value equal to zero, we may assume that \(\bar{y}\) and \(\bar{z}\) are integral. By Lemma 20, we may assume \(\{A \in C(G) : \bar{y}_A > 0\}\) is a family of non-crossing cuts after uncrossing pairs of crossing cuts, if any.

It now suffices to show that after the uncrossings, we do not end up with \(\bar{y} = 0\) and \(\bar{z}_u + \bar{z}_v - \sum_{uv \in A} \bar{y}_A = 0\) for all \(uv \in E(G)\). The case when \(G\) has a useless edge \(uv\) is easy since uncrossings could not decrease the value of \(\bar{z}_u + \bar{z}_v - \sum_{uv \in A} \bar{y}_A\), which initially was greater than zero. So, suppose that \(G\) has no useless edge. Then \(G\) has at least one non-trivial constricted cut. We claim that uncrossing leaves at least one cut in \(\{A \in C(G) : \bar{y}_A > 0\}\). Suppose that at some point, we uncrossed \(\delta(S)\) and \(\delta(T)\) where \(S \cap T = \{v\}\) and \(V(\overline{S \cup T}) = \{v\}\), the only type of uncrossing that does not increase \(\bar{y}_A\) for some \(A \in C(G)\). Since \(G\) has no useless edge, there exists \(\bar{x} \in SEP(G)\) such that \(\bar{x} > 0\). Then \(4 = \bar{x}(\delta(S)) + \bar{x}(\delta(T)) = \bar{x}(\delta(S \cap T)) + \bar{x}(\delta(S \cup T)) + 2\bar{x}(\gamma(S \setminus T, S)) \geq 4\). It follows that \(\gamma(S \setminus T, S) = 0\). But this means \(G - \{v, e\}\) is disconnected, contradicting that \(G\) is 3-connected. Hence, each time we perform uncrossing, there is at least one non-trivial cut \(A\) such that \(\bar{y}_A > 0\).

For a set \(S\), \(2^S\) denotes the set of all subsets of \(S\). The following easy result is rather useful.

**Lemma 22.** Let \(G = (V, E)\) be a connected graph. If \(A\) is a non-crossing family of cuts of \(G\), then there exists a nested family \(S(A) \subset 2^V\) that contains precisely one proper shore of each cut in \(A\).

**Proof.** For each cut \(A \in A\), pick a shore that has at most half the number of vertices in the graph and put
Lemma 23. Let $G$ be a feasible graph. If $\delta(S)$ is a non-trivial constricted cut of $G$, then $G[S]$ and $G[V \setminus S]$ are connected.

Proof. Suppose that the statement is false. Without loss of generality, we may assume that $G[S]$ is not connected. Let $T$ and $U$ be non-empty proper subsets of $S$ such that $S = T \cup U$, $T \cap U = \emptyset$, and there is no edge in $G[S]$ joining a vertex in $T$ and a vertex in $U$. Then, for any $x \in \text{SEP}(G)$,

$$x(\delta(S)) = x(\delta(T)) + x(\delta(U)) \geq 2 + 2 = 4,$$

contradicting that $\delta(S)$ is constricted. □

The next result appears in Grünbaum [8]. The proof of the lemma is included here for the sake of completeness.

Lemma 24. Let $G = (V, E)$ be a connected simple plane graph with at most one vertex of degree two. Then there are at least six degree-three vertices and degree-three faces in total.

Proof. Let $f$ denote the number of faces. Let $n_i$ denote the number of vertices of degree $i$. Let $f_i$ denote the number of faces having $i$ edges on its boundary. Observe that $\sum_{i \geq 2} in_i = 2|E| = \sum_{i \geq 3} if_i$. By Euler’s formula, $|V| - |E| + f = 2$. Hence,

$$6 = (4\sum_{i \geq 2} n_i - 4|E| + 4\sum_{i \geq 3} f_i) - 2 \leq (4\sum_{i \geq 2} n_i - 2|E|) + (4\sum_{i \geq 3} f_i - 2|E|) - 2 \leq 2n_2 + n_3 + f_3 - 2 \leq n_3 + f_3$$

as desired. □

From Lemma 24, one deduces that

**Lemma 25.** If $G = (V, E)$ is a 4-regular planar graph, then $G$ cannot be bipartite.

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**Proof of Theorem 17.** Suppose that $\text{sys}(G)$ has no strictly feasible solution. Since $\frac{1}{2} \hat{v} \in \text{SEP}(G)$, $G$ is feasible and has no useless edge. Therefore, $G$ must have a non-trivial constricted cut. By Lemma 21, there exist $\hat{y} \in \mathbb{R}^{C(G)}_+$ and $\hat{z} \in \mathbb{R}^V$ feasible for $\text{sys}'(G)$ such that

$$\sum_{A \in C(G)} \hat{y}_A = \sum_{u \in V} \hat{z}_u = 0$$

as $A(\hat{y})$ is non-crossing, by Lemma 22, there exists a nested family $S$ of subsets of $V$ containing exactly one shore of each cut in $A(\hat{y})$. Because $G$ is feasible, by Lemma 23, $G[S]$ is connected for all $S \in S$.

Choose $T \in S$ such that there exists a proper subset of $T$ that is in $S$ and for any proper subset $R$ of $T$ that is in $S$, there is no proper subset of $R$ that is in $S$. If no such $T$ exists, let $T = V$.

Let $S' = \{S \in S : S \subseteq T\}$. Observe that the elements in $S'$ are pairwise disjoint. Consider the graph $H$ obtained from $G[T]$ by contracting each $S \in S'$. Note that $H$ is connected and planar. We will show that $H$ is simple and non-bipartite.

To show that $H$ is simple, we first prove the following claim:

**Claim.** Let $S \in S'$. Then for every edge $uv$ such that $u \in S$ and $v \notin S$, $\hat{z}_u = 0$ and $\hat{z}_v > 0$.

An immediate consequence of this claim is that if $w$ is a neighbour of a pseudo-vertex in $H$, then $w$ is not a pseudo-vertex and $\hat{z}_w > 0$.

To prove the claim, note that as $G$ has no useless edge, by Lemma 18, for all $pq \in E$ such that $p, q \in S$, we have $\hat{z}_p + \hat{z}_q - \sum_{A \in C(G), pq \in A} \hat{y}_A = 0$, giving $\hat{z}_p + \hat{z}_q = 0$ as $\hat{y}_A = 0$ for all $A \in C(G)$ such that $pq \in A$. By Lemma 24, $G[S]$ contains a triangle as $G[S]$ is connected and planar and has no more than 4 vertices of degree at most 3. Hence, $\hat{z}_p = 0$ for all $p \in S$, giving $\hat{z}_w = 0$. As $\text{sys}'(G)$ contains the constraint $\hat{z}_u + \hat{z}_v - \sum_{A \in C(G), uv \in A} \hat{y}_A \geq 0$, having $\hat{y}_{\delta(S)} > 0$ and $\hat{z}_u = 0$ implies that $\hat{z}_v > 0$. This completes the proof of the claim.

From this claim, one can see that $\delta(S_1)$ and $\delta(S_2)$ are disjoint for any distinct $S_1, S_2 \in S'$. Thus contracting
each element of $S'$ does not create parallel edges. So $H$ is simple. Also, the set of pseudo-vertices in $H$ is independent.

To show that $H$ is non-bipartite, first suppose that $T = V$. In this case, $H$ is simple, connected, planar, and 4-regular and therefore is non-bipartite by Lemma 25. Otherwise, $H$ has exactly four vertices of degree three and no vertex of degree two. By Lemma 24, $H$ has a triangle and therefore is non-bipartite.

Let $v$ be a neighbour of a pseudo-vertex in $H$. By the claim above, $v$ is not a pseudo-vertex and $\hat{z}_v > 0$. Let $X = \{v \in T : \hat{z}_v > 0\}$. Then, $X$ is an independent set in $H$ because, by Lemma 18, $\hat{z}_u + \hat{z}_v = 0$ for all $uv \in E$ such that $u, v \in T \setminus \bigcup_{S \in S'} S$ as $G$ has no useless edge. In addition, $H$ being connected implies that for every vertex $v$ in $H$ that is not a pseudo-vertex, $\hat{z}_v \neq 0$. If we let $Y$ be the set containing all the vertices $v \in T$ with $\hat{z}_v < 0$ and the pseudo-vertices, then $X$ and $Y$ partition the set of vertices of $H$. Clearly, $Y$ is an independent set in $H$. Hence, $H$ is bipartite with bipartition $(X, Y)$, contradicting that $H$ is non-bipartite.

Note that Theorem 17 does not hold if the condition to be planar is dropped. The graph depicted in Figure 3 is a 3-connected 4-regular graph $G$ whose non-trivial 4-edge cuts are matchings of $G$ but $\text{sys}(G)$ has no strictly feasible solution. Note that the graph is not more-than-1-tough and it has a non-trivial constricted cut. However, every non-trivial 4-edge cut of the graph is a matching. One might ask what happens if we restrict our attention to more-than-1-tough 4-regular graphs. We do not know the answer and so we have the following problem:

**Problem 26.** Let $G$ be a more-than-1-tough 4-regular graph. If every non-trivial 4-edge cut of $G$ is a matching, must $\text{sys}(G)$ have a strictly feasible solution?

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