COMPUTABLE BIAŁYNISKI-BIRULA DECOMPOSITION OF THE HILBERT SCHEME

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Abstract. We call the scheme parameterizing homogeneous ideals with fixed initial ideal the Gröbner scheme. We introduce a Białynicki-Birula decomposition of the Hilbert scheme $\text{Hilb}_n^P$ for any Hilbert polynomial $P$ such that the cells are the Gröbner schemes in set-theoretically. Then we obtain a computable homology formula for smooth Hilbert schemes. As a corollary of our argument, we show that the Gröbner scheme for a monomial ideal defining a smooth point in the Hilbert scheme is smooth.

1. Introduction

Let $k$ be a field. Our main target is the Hilbert scheme $\text{Hilb}_n^P$ parameterizing closed subschemes in $\mathbb{P}_k^n$ with Hilbert polynomial $P$. This paper is organized as follows.

Let $\prec$ be an arbitrary monomial order on $S = k[x_0, \ldots, x_n]$. We see a set-theoretically decomposition of the Hilbert scheme $\text{Hilb}_n^P$ into the loci of homogeneous ideals with fixed initial ideal (Proposition 3.3). We denote by $\text{Gröb}_J^\prec$ such a locus for a monomial ideal $J$ and we call $\text{Gröb}_J^\prec$ the Gröbner scheme. Namely, in set-theoretically,

$$\text{Gröb}_J^\prec = \{ I \subset S \mid I \text{ is a homogeneous ideal and } \text{in}_\prec I = J \}.$$  

We denote by $\mathcal{M}_{P,n}$ the set of monomial ideals appearing in the decomposition:

$$\text{Hilb}_n^P = \bigsqcup_{J \in \mathcal{M}_{P,n}} \text{Gröb}_J^\prec.$$  

This decomposition comes from the closed embedding into the Grassmannian. We show that this decomposition is computable (Proposition 3.4).

We see examples of decompositions of smooth Hilbert schemes (Example 3.1, Example 3.2, Example 7.1). Those examples give us symmetrical numbers. In fact, the numbers are the Betti numbers of the Hilbert scheme. A purpose of this paper is to explain this phenomenon.

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We show that the decomposition (∗) is the set-theoretically Białynicki-Birula decomposition [BB73, BB76] with respect to a $\mathbb{G}_m$-action compatible with $\prec$. Therefore we obtain a computable homology formula (Corollary 7.1). As a corollary of our argument, we show that the Gröbner scheme $\text{Gr}^J_\prec$ for $J \in \mathcal{M}_{P,n}$ is smooth if $\text{Hilb}^P_n$ is smooth at $\text{Proj} S/J \in \text{Hilb}^P_n(k)$ (Corollary 6.2). The same statement is known for segment monomial ideals which are monomial ideals with a combinatorial condition since the Gröbner scheme for a segment monomial ideal is an open subscheme of $\text{Hilb}^P_n$ as the marked family [CLMR11]. A remarkable point is that our theorem is provided in the context of geometries.

**Theorem 1.1.** If $\text{Hilb}^P_n$ is smooth at $\text{Proj} S/J \in \text{Hilb}^P_n(k)$ for $J \in \mathcal{M}_{P,n}$, then the Gröbner scheme $\text{Gr}^J_\prec$ is isomorphic to an affine space.

**Theorem 1.2.** Assume that $k$ is algebraically closed, $\text{char } k = 0$ and the Hilbert scheme $\text{Hilb}^P_n$ is smooth. Denote by $p(J)$ the dimension of the Zariski tangent space on $\text{Gr}^J_\prec$ at $J$. Then we have the following formula about the homology of $\text{Hilb}^P_n$:

$$H_m(\text{Hilb}^P_n, \mathbb{Z}) \cong \bigoplus_{J \in \mathcal{M}_{P,n}} H_{m-2p(J)}(\{J\}, \mathbb{Z}) \cong \bigoplus_{J \in \mathcal{M}_{P,n}} \mathbb{Z}$$

for any integer $m$ with $0 \leq m \leq 2 \dim \text{Hilb}^P_n$.

Sernesi construct a singular point of $\text{Hilb}^P_3$ defined by a monomial ideal [Ser06]. As an example of the decomposition of a non-smooth Hilbert scheme, we compute the decomposition of the Hilbert scheme $\text{Hilb}^P_3$ with respect to $\prec_{\text{lex}}$ and $\prec_{\text{rvlex}}$ (Example 6.1). Then, by applying Theorem 1.1, we find 18 singular points of $\text{Hilb}^P_3$ defined by monomial ideals.

We note background. The Gröbner scheme, or also called the Gröbner stratum, is introduced in [NS00]. The Gröbner scheme is an affine scheme of finite type over $k$ and has computable defining equations [Rob09, RT10, Led11]. Moreover, the Gröbner scheme $\text{Gr}^J_\prec$ is isomorphic to an affine space if $\text{Gr}^J_\prec$ is smooth at $J \in \text{Gr}^J_\prec(k)$ [Rob09, Corollary 3.7], [RT10, Corollary 3.6] (see also [FR09]). Precisely we define the Gröbner scheme $\text{Gr}^J_\prec$ as the scheme representing the following Gröbner functor:

$$\text{Gr}^J_\prec : (k\text{-Alg}) \to (\text{Set})$$

$$A \mapsto \left\{ G \subset A[x_0, \ldots, x_n] \left| G \text{ is a reduced Gröbner basis consisting of homogeneous polynomials, } \in_{\prec}(G) = J \otimes_k A \right. \right\}.$$
See [Wi07] for definition of reduced Gröbner bases that coefficients are in a ring. Sometime a property of Gröbner schemes is not compatible with the Hilbert scheme as schemes since a general ideal in $A[x_0, \ldots, x_n]$ may not have a reduced Gröbner basis. However, the Gröbner scheme $\text{Gr}^{\prec}_{\mathbb{A}^n} J$ is a locally closed subscheme of $\text{Hilb}^P_n$ if $J \in \mathcal{M}_{P, n}$ [LR13] Theorem 5.3.

The Gröbner deformation or the Gröbner degeneration, given in [Bay82], certainly exists for any element $G \in \text{Gr}^\prec_{\mathbb{A}^n} A$ in case $A$ is an arbitrary commutative ring. That is the flat family of closed subschemes $\{Y_t\}$ in $\mathbb{P}_{k}^n \times_k \text{Spec } A$ over $\mathbb{A}^1_k \times_k \text{Spec } A$ such that letting $I = \langle G \rangle$, general fibers are isomorphic to $\text{Proj } A[x_0, \ldots, x_n]/I$ and the special fiber at $\{0\} \times_k \text{Spec } A$ is isomorphic to $\text{Proj } A[x_0, \ldots, x_n]/(\text{in}_{\prec} I)$. The key point is that the Gröbner degeneration is provided as the orbit of $I$ with the limit in $\prec I$ by a $\mathbb{G}_m$-action on the polynomial ring on the polynomial ring.

On the other hand, Białynicki-Birula introduces significant loci in a scheme $X$ with a $\mathbb{G}_m$-action. Nowadays these are called Białynicki-Birula cells or Białynicki-Birula schemes. For simplicity, we assume that $X$ is smooth projective over an algebraically closed field $k$ and has finite fixed points $X^\mathbb{G}_m = \{a_1, \ldots, a_r\}$. Therefore any orbit of $x \in X$ has a limit $\lim_{t \to 0} t \cdot x \in X^\mathbb{G}_m$ by the extension of the orbit morphism $\mathbb{G}_m \to X; t \mapsto t \cdot x$ to $t = 0$. The Białynicki-Birula scheme $X^+_t$ is defined as

$$X^+_t = \{x \in X \mid \lim_{t \to 0} t \cdot x = a_i\}.$$

The Białynicki-Birula’s theorem gives us that any $X^+_t$ is isomorphic to an affine space and $\{X^+_t\}$ gives a cell decomposition of $X$ [BB73, BB76]. Recently, for an arbitrary $X$ locally of finite type, the BB schemes have been defined and investigated in [Dri13, JS18]. Thanks to [Dri13, JS18], we combine Bayer’s degeneration and Białynicki-Birula’s idea on the Hilbert scheme $\text{Hilb}^P_n$.

**Theorem 1.3.** (Proposition 5.1, Theorem 5.1). There exists a $\mathbb{G}_m$-action on $\text{Hilb}^P_n$ such that the scheme of fixed points is $(\text{Hilb}^P_n)^{\mathbb{G}_m} = \{\text{Proj } S/J \mid J \in \mathcal{M}_{P, n}\}$ in set-theoretically and the BB scheme $(\text{Hilb}^P_n)^+_I$ for $\text{Proj } S/J$ is $\text{Gr}^\prec_{\mathbb{A}^n}$ in set-theoretically.

**Theorem 1.4.** (Theorem 6.3). If the BB scheme $(\text{Hilb}^P_n)^+_I$ is smooth at $\text{Proj } S/J \in (\text{Hilb}^P_n)^+_I(k)$, then the Gröbner scheme $\text{Gr}^\prec_{\mathbb{A}^n}$ is isomorphic to an affine space.

**Theorem 1.5.** (Proposition 6.2). If $\text{Hilb}^P_n$ is smooth at $\text{Proj } S/J \in \text{Hilb}^P_n(k)$, then the BB scheme $(\text{Hilb}^P_n)^+_I$ is smooth at $\text{Proj } S/J \in (\text{Hilb}^P_n)^+_I(k)$.

Such a combination already has been investigated for the Hilbert scheme $H^d_n = H^d(\mathbb{A}^n_k)$ of $d$ points in the affine space $\mathbb{A}^n_k$, but not been investigated for an arbitrary $\text{Hilb}^P_n$. Let us recall results on $H^d_n$. The BB scheme in $H^d_n$ with respect to weights on coordinates of $\mathbb{A}^n_k$ is described in [EL12] as the intersection of schemes determined by an argument of initial ideals. Moreover, [EL12] shows that the BB schemes included in a fiber of the Hilbert-Chow morphism corresponding to a coordinate with negative weight. These results seem to be similar and related to our construction of the decomposition ($\ast$). [Jel17] deals with the obstruction theory on the BB
2. Preliminaries and Notation

- Let $k$ be a field and $S = k[x] = k[x_0, \ldots, x_n]$ the polynomial ring over $k$ in $(n + 1)$ variables. We always fix a monomial order $\prec$ on $S$. We consider the ordinal degrees of polynomials in $S$. For a subset $A \subset S$, we denote by $A_r$ the homogeneous elements of $A$ with degree $r$ and denote by $\langle A \rangle$ the ideal generated by $A$ in $S$.
- For $\alpha = (\alpha_0, \ldots, \alpha_n) \in \mathbb{N}^{n+1}$, let $x^{\alpha} = x_0^{\alpha_0} \cdots x_n^{\alpha_n}$. Using this notation, we regard $\mathbb{N}^{n+1}$ as the set of monomials in $(n + 1)$ variables. The degree of $\alpha$ is $|\alpha| = \alpha_0 + \cdots + \alpha_n$. For a subset $A \subset \mathbb{N}^{n+1}$, let $A_r = \{ \alpha \in A \mid |\alpha| = r \}$. Let $e_i$ be the $i$-th canonical vector.
- For $k$-schemes $X$ and $Y$, let $X(Y) = \text{Hom}_k(Y, X)$. If $Y = \text{Spec } A$, we denote it by $X(A)$ instead.

The Hilbert scheme $\text{Hilb}^P_n$ is the scheme representing the following Hilbert functor:

$$\text{Hilb}^P_n : (k \text{-Alg}) \to (\text{Set}) \quad \begin{cases} A \mapsto \left\{ Y \subset \mathbb{P}^n_A \mid \text{Y is a closed subscheme in } \mathbb{P}^n_A \text{ flat over } \text{Spec } A, \\
\text{the Hilbert polynomials of all fibers on closed points of } \text{Spec } A \text{ are } P \right\} \end{cases}.$$

There exists a canonical morphism $\text{Gröbner}_f^I \to \text{Hilb}^P_n$ induced by the natural transformation

$$\text{Gröbner}_f^I \to \text{Hilb}^P_n \quad G \mapsto \text{Proj } A[x]/(G).$$

If we denote a morphism $\text{Gröbner}_f^I \to \text{Hilb}^P_n$, we always mean this morphism.

3. Computable decomposition of the Hilbert scheme into the Gröbner schemes

We recall the embedding of the Hilbert scheme into the Grassmannian. See [HS04, Mac07] or other references about Grothendieck’s construction of Hilbert schemes.

**Proposition 3.1.** [Vas98 Corollary B.5.1] Let $P$ be a Hilbert polynomial of a closed subscheme of a projective space. Then there exist integers $a_1 \geq a_2 \geq \cdots \geq a_r \geq 0$ such that

$$P(t) = \sum_{i} \binom{t + a_i - i + a}{a_i}.$$  

The number $r$ is called the Gotzmann number of $P$.

We call $I_{\leq r} = \oplus_{s \geq r} I_s$ the $r$-truncation of $I$. We say $I$ is $r$-truncated if $I_{\leq r} = I$. Moreover, we denote by $I_{\text{sat}}$ the saturation of $I$ by $\langle x_0, \ldots, x_n \rangle$. We say $I$ is saturated if $I_{\text{sat}} = I$. If $I$ is a homogeneous saturated ideal in $S$ such that the Hilbert polynomial of $S/I$ is $P$, then $\dim_k(S/I)_s = P(s)$ for any $s \geq r$. 
We have a closed embedding
\[ \text{Hilb}_n^P \rightarrow G \left( \binom{n+r}{r}, P(r), S_r \right), \]
where \( G \left( \binom{n+r}{r} - P(r), S_r \right) \) is the Grassmannian. Then we can describe
\[ \text{Hilb}_n^P(A) \cong \left\{ I \subset A[x] \mid I \text{ is the } r\text{-truncation of a saturated ideal} \right\} \]
defining an element of \( \text{Hilb}_n^P(A) \).

For short, we denote by \( \mathbf{I}_{P,n} \) the above condition: \( \mathbf{I}_{P,n} \equiv I \) is the \( r\text{-truncation} \) of a saturated ideal defining an element of \( \text{Hilb}_n^P \). The image of \( \text{Hilb}_n^P(k) \) in \( \text{Hom}_k(\text{Spec } k, G \left( \binom{n+r}{r} - P(r), S_r \right) ) \) is the set of subspaces \( V \subset S_r \) such that \( \dim_k V = \binom{n+r}{r} - P(r) \) and \( \dim_k S_1 \cdot V = \binom{n+r+1}{r+1} - P(r+1) \). Therefore the condition \( \mathbf{I}_{P,n} \) is equivalent to the following condition: \( I \) is generated by \( I_r, \dim_k I_r = \binom{n+r}{r} - P(r) \) and \( \dim_k I_{r+1} = \binom{n+r+1}{r+1} - P(r+1) \).

We introduce a decomposition of the Hilbert scheme into the Gröbner schemes.

**Lemma 3.1.** A homogeneous ideal \( I \) in \( S \) satisfies \( \mathbf{I}_{P,n} \) if and only if the initial ideal \( J = \text{in}_< J \) satisfies \( \mathbf{I}_{P,n} \).

**Proof.** Assume that \( I \) satisfies \( \mathbf{I}_{P,n} \). For any \( s \geq r \), we have \( \dim_k(S/J)_s = \dim_k(S/I)_s = P(s) \). The Hilbert polynomial of \( \text{Proj } S/J \) in \( \mathbb{P}^n_k \) is also \( P \), so \( \dim_k(S/J \text{sat})_s = P(s) \). Thus \( (J \text{sat})_{s+r} = J_{s+r} = \text{in}_< (I_{s+r}) = J \). Conversely, assume that \( J \) satisfies \( \mathbf{I}_{P,n} \). Then there exists a saturated monomial ideal \( J' \) such that \( J = J'_{s+r} \) and \( \text{Proj } S/J' \in \text{Hilb}_n^P(k) \). Put \( I' = I_{\text{sat}} \). Then for any \( s \geq r \), we have \( \dim_k(S/I)_s = \dim_k(S/J)_s = \dim_k(S/J')_s = \dim_k(S/I')_s = P(s) \). Therefore we obtain \( I = I'_{s+r} \). \( \square \)

**Proposition 3.2.** (\cite{LR16} Theorem 5.3) The morphism \( \text{Gröb}_<^J \rightarrow \text{Hilb}_n^P \) is a locally closed immersion if \( J \) satisfies \( \mathbf{I}_{P,n} \).

**Proposition 3.3.** Let \( \mathcal{M}_{P,n} \) be the set of monomial ideals satisfying \( \mathbf{I}_{P,n} \). Then for any field extension \( k \subset K \),
\[ \text{Hilb}_n^P(K) = \coprod_{J \in \mathcal{M}_{P,n}} \text{Gröb}_<^J(K). \]
Namely, in set-theoretically,
\[ \text{Hilb}_n^P = \coprod_{J \in \mathcal{M}_{P,n}} \text{Gröb}_<^J. \]

**Proof.** This is easy from
\[ \text{Gröb}_<^J(K) \cong \{ I \subset K[x] \mid I \text{ is a homogeneous ideal with in}_< I = J \otimes_k K \} \]
and Lemma 3.1. \( \square \)

Let us see that the set \( \mathcal{M}_{P,n} \) is computable.

**Definition 3.1.** (\cite{Led11}) A subset \( \Delta \) in \( \mathbb{N}^{n+1} \) is a standard set if \( \alpha + \beta \in \Delta \) implies \( \alpha, \beta \in \Delta \) for any \( \alpha, \beta \in \mathbb{N}^{n+1} \). There is a one-to-one correspondence between the set of standard sets in \( \mathbb{N}^{n+1} \) and the set of monomial ideals in
S given by $\Delta \mapsto J_\Delta = \{x^\alpha \mid \alpha \in \mathbb{N}^{n+1} \setminus \Delta\}$. For a standard set $\Delta$, we define the set of corners
\[ \mathcal{C}(\Delta) = \{\alpha \in \mathbb{N}^{n+1} \setminus \Delta \mid \forall i \alpha - e_i \notin \mathbb{N}^{n+1} \setminus \Delta\}. \]
The set of corners $\mathcal{C}(\Delta)$ corresponds to the minimal generators of $J_\Delta$.

**Proposition 3.4.** Let $\Delta$ be a standard set. The monomial ideal $J_\Delta$ is an element of $\mathcal{M}_{P,n}$ if and only if the set of corners $\mathcal{C}(\Delta)$ satisfies
\[
\begin{align*}
(1) & \quad \mathcal{C}(\Delta) \subset (\mathbb{N}^{n+1})_r, \\
(2) & \quad \#(\mathcal{C}(\Delta)) = \binom{n+r}{r} - P(r), \\
(3) & \quad \#(\{\alpha + e_i \mid \alpha \in \mathcal{C}(\Delta), i = 0, \ldots, n\}) = \binom{n+r+1}{r+1} - P(r+1).
\end{align*}
\]
Therefore the set-theoretical decomposition
\[
\text{Hilb}_n^P = \coprod_{J \in \mathcal{M}_{P,n}} \text{Gröb}_J \prec \sim = \mathbb{A}^m_k
\]
is computable.

**Example 3.1.** We compute an example of the decomposition in Proposition 3.3. We consider the Hilbert scheme of $d$ points in $\mathbb{P}^2_k$. The Hilbert scheme $\text{Hilb}^d_2$ is smooth and its dimension is $2d$ [Har10]. Using the same argument of [Har10] (i.e. using an obstruction theory on the Gröbner scheme), we obtain that the Gröbner scheme $\text{Gröb}_J \prec \sim$ is isomorphic to an affine space $\mathbb{A}^m_k$ for any $J \in \mathcal{M}_{d,2}$. We make a table of the numbers of $J \in \mathcal{M}_{d,2}$ such that $\text{Gröb}_J \prec \sim \sim = \mathbb{A}^m_k$. In fact, the numbers are the Betti numbers of the Hilbert schemes [ES87].

**Example 3.2.** Let us consider the case $P = t + 1$, $n = 3$. Then the Hilbert scheme $\text{Hilb}^{t+1}_3$ parameterizes lines in $\mathbb{P}^3_k$ and isomorphic to the Grassmannian $G(1,3)$. The numbers of $J \in \mathcal{M}_{t+1,3}$ are on Table 3. The Betti numbers of $G(1,3)$ is computed by determining Schubert cycles in $G(1,3)$ [Ehr34]. The numbers on Table 3 are just the Betti numbers of $G(1,3)$.

| $d \setminus m$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|-----------------|---|---|---|---|---|---|---|---|---|---|----|
| 1               | 1 | 1 | 1 |   |   |   |   |   |   |   |    |
| 2               | 1 | 2 | 3 | 2 | 1 |   |   |   |   |   |    |
| 3               | 1 | 2 | 5 | 6 | 5 | 2 | 1 |   |   |   |    |
| 4               | 1 | 2 | 6 | 10| 13| 10| 6 | 2 | 1 |   |    |
| 5               | 1 | 2 | 6 | 12| 21| 24| 21| 12| 6 | 2 | 1  |

**Table 2.** The numbers of $J \in \mathcal{M}_{d,2}$ such that $\text{Gröb}_J \prec \sim \sim = \mathbb{A}^m_k$

| $m$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|-----|---|---|---|---|---|---|---|---|---|---|----|
| 1   | 1 | 1 | 1 | 1 | 1 |   |   |   |   |   |    |
| 2   | 1 | 1 | 2 | 1 |   |   |   |   |   |   |    |

**Table 3.** The numbers of $J \in \mathcal{M}_{t+1,3}$ such that $\text{Gröb}_J \prec \sim \sim = \mathbb{A}^m_k$
The Betti numbers of Hilb$_d^2$ is determined in [ESS7] by computing a 
Bialynicki-Birula decomposition of Hilb$_d^2$ [BB73, BB76]. The next purpose 
is to show that the decomposition

$$\text{Hilb}_n^P = \prod_{J \in M_{P,n}} \text{Gröber}_J$$

is just the Bialynicki-Birula decomposition with respect to a 
$G_m$-action on Hilb$_n^P$.

4. $G_m$-ACTION ON THE HILBERT SCHEME COMPATIBLE WITH A 
MONOMIAL ORDER

**Proposition 4.1.** (Bay82 Proposition 1.8) Let $\prec$ be a monomial order on $S$, and let $A$ be a finite subset of $\mathbb{N}^{n+1}$. Then there exists a vector $\omega \in \mathbb{N}^{n+1}$ such that for any $\alpha, \beta \in A$, $\alpha \prec \beta$ if and only if $\omega \cdot \alpha < \omega \cdot \beta$. Here $\omega \cdot \alpha$ is the ordinary inner product $\omega_0\alpha_0 + \cdots + \omega_n\alpha_n$.

We fix a vector $\omega \in \mathbb{N}^{n+1}$ given by Proposition 4.1 for the finite subset 
$((\mathbb{N}^{n+1})_r$. This vector $\omega$ implies a $G_m$-action on the Gröber scheme $\text{Gröber}_J$ 
for each $J \in M_{P,n}$ as follows.

**Proposition 4.2.** ([RT10, Led11]) The Gröber scheme $\text{Gröber}_J$ for $J = J_\Delta \in M_{P,n}$ is isomorphic to a closed subscheme of $\text{Spec} k[T_{\alpha,\beta} \mid \alpha \in \mathcal{C}(\Delta), \beta \in \Delta_r, \alpha > \beta]$. We define a grading on $R = k[T_{\alpha,\beta} \mid \alpha \in \mathcal{C}(\Delta), \beta \in \Delta_r, \alpha > \beta]$ such that $\text{deg}(T_{\alpha,\beta}) = \omega \cdot \alpha - \omega \cdot \beta$ and attach a $G_m$-action on $R$ from this grading. Then the Gröber scheme $\text{Gröber}_J$ is $G_m$-invariant in $\text{Spec} R$.

The vector $\omega$ defines $G_m$-actions on $S$ and on $S_r$ as a negative grading 
t $\cdot x^\alpha = t^{-\omega \cdot \alpha} x^\alpha$. Therefore there exist $G_m$-actions on the Hilbert scheme 
Hilb$_n^P$ and on the Grassmnanian $G((n+1)/r) - P(r), S_r$ respectively such that 
Hilb$_n^P \rightarrow G((n+1)/r) - P(r), S_r$ is $G_m$-equivariant. Moreover, we also obtain 
a $G_m$-action on the projective space $P = \mathbb{P}(\wedge (n+1)/r) - P(r), S_r$ such that the 
Plücker embedding $G((n+1)/r) - P(r), S_r) \rightarrow P$ is $G_m$-equivariant.

**Proposition 4.3.** If $J \in M_{P,n}$, then the morphism $\text{Gröber}_J \rightarrow \text{Hilb}_n^P$ is a 
$G_m$-equivariant morphism.

**Proof.** For each reduced Gröber basis

$$G = \left\{ g_\alpha = x^\alpha - \sum_{\beta \in \Delta} a_{\alpha,\beta} x^\beta \bigg| \alpha \in \mathcal{C}(\Delta) \right\} \in \text{Gröber}_J(A),$$

we have

$$t \cdot g_\alpha = t^{-\omega \cdot \alpha} x^\alpha - \sum_{\beta \in \Delta} t^{-\omega} a_{\alpha,\beta} x^\beta \quad (t \in A^x)$$

under the $G_m$-action on $A[x]$. Let $I$ be the ideal generated by $G$, and let 
$Y = \text{Proj} A[x]/I$. Then $t \cdot I = \{ t \cdot f \mid f \in I \}$ is generated by the set 
$\{ x^\alpha - \sum_{\beta \in \Delta} t^\omega a_{\alpha,\beta} x^\beta \mid \alpha \in \mathcal{C}(\Delta) \}$. We have $(t \cdot I)_{\geq r} = t \cdot I_{\geq r}$ for any 
integer $r \geq 0$. Since $I = (I_{\text{sat}})_{\geq r}$, we have

$$t \cdot Y = \text{Proj} A[x]/(t \cdot I_{\text{sat}}) = \text{Proj} A[x]/(t \cdot I_{\text{sat}})_{\geq r} = \text{Proj} A[x]/(t \cdot I).$$
Thus the morphism $\text{Gröb}_J^r \to \text{Hilb}_n^P$ is a $\mathbb{G}_m$-equivariant morphism. □

From now on, we always attach the $\mathbb{G}_m$-action on $\text{Hilb}_n^P$ introduced in the above for given monomial order $\prec$.

5. Białynicki-Birula schemes in the Hilbert scheme

Let $X$ be a scheme locally of finite type over $k$ with a $\mathbb{G}_m$-action. For any $k$-algebra $A$, we attach a $\mathbb{G}_m$-action on $\text{Spec} \ A$ as the projection $\mathbb{G}_m \times_k \text{Spec} \ A \to \text{Spec} \ A$. We also attach the trivial $\mathbb{G}_m$-action on $A_k^1 \times_k \text{Spec} \ A$ induced from the canonical $\mathbb{G}_m$-action on $A_k^1$.

The scheme of fixed points [Fog73] is defined as the subscheme $X^{\mathbb{G}_m}$ such that for any $k$-algebra $A$,

$$X^{\mathbb{G}_m}(A) = \{ \varphi \in X(A) \mid \varphi \text{ is } \mathbb{G}_m\text{-equivariant} \}.$$

The scheme of fixed points exists and it is a closed subscheme of $X$ [Dri13, Proposition 1.2.2].

We define the scheme of attractors in $X$ as the scheme $X^+$ such that for any $k$-algebra $A$,

$$X^+(A) \equiv \{ \varphi : A_k^1 \times_k \text{Spec} \ A \to X \mid \varphi \text{ is } \mathbb{G}_m\text{-equivariant} \}.$$

The scheme of attractors exists and it is locally of finite type over $k$ [Dri13, Corollary 1.4.3], [JS18 Theorem 6.17].

**Proposition 5.1.** The scheme of fixed points of the Hilbert scheme $\text{Hilb}_n^P$ satisfies $\text{Hilb}_n^P(\mathbb{G}_m) = \{ \text{Proj} \ K[x]/(J \otimes_k K) \mid J \in \mathcal{M}_{P,n} \}$ for any field extension $k \subset K$. In particular, we have $X^{\mathbb{G}_m} = \{ \text{Proj} \ S/J \mid J \in \mathcal{M}_{P,n} \}$ in set-theoretically.

**Proof.** Let $I$ be a homogeneous ideal that is the $r$-truncation of a saturated ideal defining an element of $\text{Hilb}_n^P(K)$. Then there exists a monomial ideal $J \in \mathcal{M}_{P,n}$ such that $\text{in}_J I = J \otimes_k K$. Let $\Delta$ be the standard set attached to $J$. The reduced Gröbner basis of $t \cdot I$ ($t \in K \setminus \{0\}$) is in the following form:

$$G = \left\{ x^\alpha - \sum_{\beta \in \Delta} t^{\omega - \alpha \cdot \omega} a_{\alpha,\beta} x^\beta \mid \alpha \in \mathcal{C}(\Delta) \right\}.$$

Therefore $\text{Proj} \ K[x]/I$ is a fixed point if and only if $a_{\alpha,\beta} = 0$ for any $\alpha \in \mathcal{C}(\Delta)$ and $\beta \in \Delta$ since $\text{Gröb}_J^r \to \text{Hilb}_n^P$ is $\mathbb{G}_m$-equivariant (Proposition 4.3). □

We obtain canonical maps by taking restrictions to 1:

$$i_X : X^+(A) \to X(A)$$

$$\varphi \mapsto \varphi | \{1\} \times_k \text{Spec} \ A.$$

If $X$ is separated, then this map is an injection for each $A$ [Dri13 Proposition 1.4.11]. We also obtain maps by taking restrictions to 0:

$$\pi_X : X^+(A) \to X^{\mathbb{G}_m}(A)$$

$$\varphi \mapsto \varphi | \{0\} \times_k \text{Spec} \ A.$$

This morphism $\pi_X : X^+ \to X^{\mathbb{G}_m}$ is $\mathbb{G}_m$-equivariant and affine of finite type [JS18 Theorem 6.17].
We describe the connected components of $X^{G_m}$ by $F_1, \ldots, F_r$. The Białynicki-Birula schemes are defined as the preimages of components under $\pi_X$. More precisely, the Białynicki-Birula scheme $X_i^+$ is the subscheme of $X^+$ such that

$$X_i^+(A) = \{ \varphi \in X^+(A) \mid \pi_X(\varphi) \in F_i(A) \}. $$

The right side set is the sections of the Białynicki-Birula functor. For short, we call these by BB scheme and BB functor respectively. 

**Theorem 5.1.** Let $J$ be an element of $\mathcal{M}_{P,n}$. Then for any field extension $k \subset K$, we have 

$$\mathcal{B}B_J^J(K) = \mathcal{G}r\mathcal{b}_J^J(K) $$

in the Hilbert functor $\mathcal{H}ilb_{P^n}(K)$. Namely, $\mathcal{B}B^J_\omega = Gr\mathcal{b}^J_\omega$ in set-theoretically.

**Proof.** Taking Gröbner degenerations [Bay82 Proposition 2.12], $Gr\mathcal{b}^J_\omega$ is a subscheme of $BB^J_\omega$. Then we obtain $Gr\mathcal{b}^J_\omega(K) \subset BB^J_\omega(K)$. Conversely, for any $\varphi \in BB^J_\omega(K)$, put $Y = \varphi|_{\{1\}} \in \mathcal{H}ilb^J_{n}(K)$ and assume that $Y \in Gr\mathcal{b}^J_{\omega,h}(K)$ with $J' \in \mathcal{M}_{P,n}$. Then taking the Gröbner degeneration of $Y$, there exists a $G_m$-equivariant morphism $\psi : A^k_1 \to \mathcal{H}ilb_{n}^P$ such that $\psi|_{\{1\}} = Y$ and $\psi|_{\{0\}} = \text{Proj} K[x]/J$. Since $(\mathcal{H}ilb^J_{P^n})+ \to \mathcal{H}ilb^J_{n}$ is monomorphism, we obtain $\varphi = \psi$. Hence $J = J'$. □

6. Smoothness

Let $X$ still be a scheme locally of finite type over $k$ with a $G_m$-action. We recall the following Białynicki-Birula’s result.

**Theorem 6.1.** (BB73, BB76, see also Dri13, JS18) Let $X$ be a smooth projective scheme over an algebraically closed field $k$ with a $G_m$-action. We assume that $X^{G_m}$ is 0-dimensional. Then there exist closed subschemes $Z_0 \supset \cdots \supset Z_q$ such that

- $Z_0 = X$ and $Z_q = \emptyset$,
- each $Z_i \setminus Z_{i+1}$ is a BB scheme in $X$,
- any BB scheme is isomorphic to an affine space over $k$.

Therefore $X$ has a cell decomposition.

Here we say that a sequence of closed subschemes $Z_0 \supset \cdots \supset Z_q$ is a cell decomposition of $X$ [Ful98] if

- $Z_0 = X$ and $Z_q = \emptyset$,
- each $Z_i \setminus Z_{i+1}$ is the disjoint sum of schemes isomorphic to affine spaces.

The above Białynicki-Birula’s result is generalized as follows.

**Theorem 6.2.** (JS18 Corollary 7.3). Suppose that $X$ is smooth over $k$. Then $\pi_X : X^+ \to X^{G_m}$ is an affine fiber bundle. Moreover, both $X^{G_m}$ and $X^+$ are smooth.

The next purpose is to apply the above theorems to our Hilbert scheme and Gröbner schemes.
Lemma 6.1. Let $A$ and $B$ be Noetherian local $k$-algebras with residue field $k$. Assume that $B$ is regular and there exists a $k$-morphism $\varphi : B \to A$ such that $\varphi$ induces bijections $l.\text{Hom}_k(A, K) \to l.\text{Hom}_k(B, K)$ for any field extension $k \subset K$. Here we denote by $l.\text{Hom}_k$ the local ring $k$-morphisms. Then $\dim A \geq \dim B$.

Proof. Let $\hat{A}$ and $\hat{B}$ be the completion of $A$ and $B$ respectively. Then $\hat{B} \cong k[[z_1, \ldots, z_m]]$ for some $m \in \mathbb{N}$ by Cohen’s structure theorem. Let $K_i$ be the fraction field of $k[[z_1, \ldots, z_i]]$. There exist canonical morphisms $\psi_i : \hat{B} \to K_i$ and $\eta_i : K_{i+1} \to K_i$. Since $\varphi^* : l.\text{Hom}_k(A, K) \to l.\text{Hom}_k(B, K)$ is bijective, $\varphi^* : l.\text{Hom}_k(\hat{A}, K_i) \to l.\text{Hom}_k(\hat{B}, K_i)$ is also bijective. Then there uniquely exists a morphism $\rho_i : \hat{A} \to K_i$ such that $\psi_i = \rho_i \circ \varphi$. Since the diagram

$$
\begin{array}{ccc}
\text{l.\text{Hom}_k(\hat{A}, K_{i+1})} & \longrightarrow & \text{l.\text{Hom}_k(\hat{B}, K_{i+1})} \\
\downarrow & & \downarrow \\
\text{l.\text{Hom}_k(\hat{A}, K_i)} & \longrightarrow & \text{l.\text{Hom}_k(\hat{B}, K_i)}
\end{array}
$$

is commutative, the diagram

$$
\begin{array}{ccc}
\hat{A} & \xrightarrow{\rho_{i+1}} & K_{i+1} \\
\downarrow & \nearrow \eta_i & \downarrow \\
\rho_i & & K_i
\end{array}
$$

is also commutative. Then $\text{Ker} \rho_i \subset \text{Ker} \rho_{i+1}$ for each $i$ with $0 \leq i \leq m$. We have $\varphi(z_{i+1}) \in \text{Ker} \rho_{i+1} \setminus \text{Ker} \rho_i$, thus the sequence $\text{Ker} \rho_0 \subset \text{Ker} \rho_1 \subset \cdots \subset \text{Ker} \rho_m$ is a strictly ascending chain of prime ideals. Therefore $\dim A = \dim A \geq m = \dim B$. □

Theorem 6.3. For any $J \in \mathcal{M}_{P,n}$, if $BB^J_\omega$ is smooth at $\text{Proj} S/J \in BB^{J}(k)$, then $\text{Gröb}^J_\omega$ is isomorphic to an affine space.

Proof. Let $\mathcal{O}_B$ and $\mathcal{O}_G$ be the local rings of $BB^J_\omega$ at $\text{Proj} S/J \in BB^{J}(k)$ and of $\text{Gröb}^J_\omega$ at $J \in \text{Gröb}^{J}(k)$ respectively. Since the morphism $\text{Gröb}^J_\omega \to BB^J_\omega$ maps $\text{Proj} S/J$ to $J$, there exists a morphism $\text{Spec} \mathcal{O}_G \to \text{Spec} \mathcal{O}_B$ that implies a bijective $l.\text{Hom}_k(\mathcal{O}_B, K) \to l.\text{Hom}_k(\mathcal{O}_G, K)$ for any field extension $k \subset K$ (Theorem 5.1). Then $\dim \mathcal{O}_G \geq \dim \mathcal{O}_B$ by Lemma 6.1. Let $T_G$ be the Zariski tangent space on $\text{Gröb}^J_\omega$ at $J$ and $T_B$ the Zariski tangent space on $BB^J_\omega$ at $\text{Proj} S/J$. We claim that the $k$-linear map $T_G \to T_B$ induced by $\text{Gröb}^J_\omega \to BB^J_\omega$ is injective. Indeed, we can regard $T_G$ and $T_B$ as the subsets of $\text{Hom}_k(\text{Spec} k[\epsilon]/(\epsilon^2), \text{Gröb}^J_\omega)$ and $\text{Hom}_k(\text{Spec} k[\epsilon]/(\epsilon^2), BB^J_\omega)$ respectively [Har77], and the morphism $\text{Gröb}^J_\omega \to BB^J_\omega$ is monomorphism since $J \in \mathcal{M}_{P,n}$. In fact, there exists a closed embedding $\text{Gröb}^J_\omega \to T_G$ as schemes [FR09, RT10]. Therefore by $\dim \mathcal{O}_G \geq \dim_k T_G \leq \dim_k T_B = \dim \mathcal{O}_B \leq \dim \mathcal{O}_G$, the closed embedding $\text{Gröb}^J_\omega \to T_G$ is an isomorphism. □

Corollary 6.1. Assume that $\text{Hilb}^P_n$ is smooth over $k$. Then the Gröbner scheme $\text{Gröb}^J_\omega$ is isomorphic to an affine space for any $J \in \mathcal{M}_{P,n}$. 

We localize the assumption of Corollary 6.1. Namely, we show that Gröbner schemes are isomorphic to an affine space if Hilb
P
 is smooth at Proj S/J ∈ Hilb
P
(k).

Proposition 6.1. ([JS18 Proposition 5.2]). Let f : X → Y be a \( \mathbb{G}_m \)-equivariant morphism. If f is an open immersion, then the induced morphism \( f^+ : X^+ → Y^+ \) is also an open immersion.

Proposition 6.2. Let \( x ∈ X^{\mathbb{G}_m}(k) \). Assume that \( \dim X^{\mathbb{G}_m} = 0 \) and X is smooth at x. Then the BB scheme \( X^+_x \) for x is smooth at x.

Proof. Let U be the smooth locus of X. Then U is \( \mathbb{G}_m \)-invariant, smooth and open in X. From Proposition 6.1 U + is open in X +. Then the BB scheme \( U^+_x \) is also open in the BB scheme \( X^+_x \). Therefore \( X^+_x \) is smooth at x by Theorem 6.2.

Therefore we obtain the following by Theorem 6.3 and Proposition 6.2.

Corollary 6.2. For any \( J ∈ \mathcal{M}_{\mathbb{P}, n} \), if the Hilbert scheme Hilb
P
 is smooth at Proj S/J ∈ Hilb
P
(k), then the Gröbner scheme Gröbner \( J \) is isomorphic to an affine space.

The converse is not true by the following example.

Example 6.1. In [Ser06], Sernesi shows that the Hilbert scheme Hilb
3
\( 2+2 \) is singular at a monomial scheme. To find other singular points, let us compute our decomposition of Hilb
3
\( 2+2 \) with respect to the reverse lexicographic order \( ≺_{reverse} \) on \( k[x, y, z, w] \) such that \( x ≻ y ≻ z ≻ w \). Then we obtain:

- \#(\mathcal{M}_{2+2, 3}) = 159.
- The 144 monomial ideals in \( \mathcal{M}_{2+2, 3} \) define smooth Gröbner schemes.
- The dimensions are in the Table 4.

- The following 15 monomial ideals in \( \mathcal{M}_{2+2, 3} \) define singular Gröbner schemes:

\[
\begin{align*}
J_1 &= \langle w^3, zw^2, yw^2, yzw, y^2 w, y^2 z, y^3, xw^2, xyw, xyz, xy^2, x^2 y \rangle, \\
J_2 &= \langle w^3, zw^2, yw^2, xw^2, xzw, xz^2, xzw, xzw, xz^2, x^2 y, x^3 \rangle, \\
J_3 &= \langle w^3, zw^2, yw^2, xw^2, xzw, xyw, xzw, xyz, xy^2, x^2 w, x^2 z, x^2 y, x^3 \rangle, \\
J_4 &= \langle zw^2, z^2 w, yzw, zw^2, xzw, xzw, xzw, xzw, xzw, x^2 w, x^2 z, x^2 y, x^3 \rangle, \\
J_5 &= \langle z^2 w, z^3, yzw, yzw, yzw, x^2 w, y^2 z, y^3, xzw, xz^2, xyz, xy^2, x^2 w, x^2 z \rangle, \\
J_6 &= \langle z^2 w, z^3, yzw, yzw, yzw, x^2 z, xzw, xzw, xzw, xzw, x^2 w, x^2 z, x^2 y, x^3 \rangle, \\
J_7 &= \langle z^2 w, z^3, yzw, yzw, yzw, x^2 z, xzw, xzw, xzw, xzw, x^2 w, x^2 z, x^2 y, x^3 \rangle, \\
J_8 &= \langle z^2 w, z^3, yzw, yzw, yzw, x^2 z, xzw, xzw, xzw, xzw, x^2 w, x^2 z, x^2 y, x^3 \rangle, \\
J_9 &= \langle z^2 w, z^3, yzw, yzw, yzw, x^2 z, xzw, xzw, xzw, xzw, x^2 w, x^2 z, x^2 y, x^3 \rangle, \\
J_{10} &= \langle yw^2, yzw, y^2 w, y^2 z, xw^2, xzw, xzw, xzw, xzw, x^2 w, x^2 y \rangle, \\
J_{11} &= \langle yw^2, yzw, y^2 w, xw^2, xzw, xzw, xzw, xzw, xzw, x^2 w, x^2 z, x^2 y, x^3 \rangle, \\
J_{12} &= \langle yw^2, yzw, y^2 w, y^2 z, y^3, xzw, xzw, xzw, xzw, x^2 w, x^2 z, x^2 y, x^3 \rangle, \\
J_{13} &= \langle yw^2, yzw, y^2 w, y^2 z, y^3, xzw, xzw, xzw, xzw, x^2 w, x^2 z, x^2 y, x^3 \rangle, \\
J_{14} &= \langle yw^2, yzw, y^2 w, xw^2, xzw, xzw, xzw, xzw, xzw, x^2 w, x^2 z, x^2 y, x^3 \rangle, \\
J_{15} &= \langle yw^2, yzw, y^2 w, xw^2, xzw, xzw, xzw, xzw, xzw, x^2 w, x^2 z, x^2 y, x^3 \rangle.
\end{align*}
\]
Therefore $\operatorname{Hilb}_3^{2t+2}$ includes the 15 singular points defined by the above 15 monomial ideals.

Let us change the monomial order to the lexicographic order $\prec = \prec_{\text{lex}}$. Then:

- The 143 monomial ideals in $\mathcal{M}_{2t+2,3}$ define smooth Gröbner schemes. The dimensions are in the Table 4.
- The following 16 monomial ideals in $\mathcal{M}_{2t+2,3}$ define singular Gröbner schemes:
  - $J_1, J_3, J_4, J_5, J_6, J_7, J_9, J_{10}, J_{11}, J_{12}, J_{14}, J_{15}$ and
  - $J_{16} = \langle w^3, zw^2, yzw, y^2 z, xy^2, xzw, xyw, y^2 w, x^2 w \rangle$,
  - $J_{17} = \langle z^2 w, z^3, yz^2, xw^2, xzw, xz^2, xyw, xyz, x^2 w, x^2 z, x^2 y, x^3 \rangle$,
  - $J_{18} = \langle y^2 w, y^2 z, y^3, xw^2, xzw, xyw, xyz, xy^2, x^2 w, x^2 z, x^2 y, x^3 \rangle$.

The consequence is that $\operatorname{Hilb}_3^{2t+2}$ includes the 18 singular points defined the above 18 monomial ideals.

One may suppose that this method covers all singular points in $\operatorname{Hilb}_n^P$ defined monomial ideals by running monomial order $\prec$ enough. However, we do not have investigated it yet.

| $m$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 |
|-----|---|---|---|---|---|---|---|---|---|---|----|----|
| $\prec_{\text{rulex}}$ | 1 | 3 | 8 | 18 | 23 | 24 | 25 | 20 | 14 | 6 | 2 | 0 |
| $\prec_{\text{lex}}$ | 1 | 3 | 9 | 17 | 22 | 24 | 23 | 19 | 15 | 6 | 3 | 1 |

Table 4. The numbers of $J \in \mathcal{M}_{2t+2,3}$ such that $\text{Gröb}_J \cong \mathbb{A}_k^m$. 


7. Homology formula

We attach a \(G_m\)-action to \(P^N_k = \text{Proj} \ k[z_0, \ldots, z_N]\) such that \(t \cdot [z_0, \ldots, z_N] = [t^{u_0} z_0, \ldots, t^{u_N} z_N]\). Let \(X\) be a \(k\)-scheme with a \(G_m\)-equivariant embedding into \(P^N_k\), as like as the Plücker embedding \(\text{Hilb}^P_n \hookrightarrow P\). In this setting, Białynicki-Birula shows that the family of the BB schemes \(\{X^+_i\}\) is filtrable \([BB76, \text{Theorem 3}]\) (note that the proof only uses the existence of a \(G_m\)-equivariant embedding). Here we say that \(\{X^+_i\}\) is filtrable if there exists a sequence of closed subschemes \(Z_0 \supset \cdots \supset Z_q\) such that

- \(Z_0 = X\), \(Z_q = \emptyset\),
- each \(Z_i \setminus Z_{i+1}\) is a BB scheme.

We obtain the followings by applying \([BB76, \text{Theorem 3}]\) to the Hilbert scheme.

**Proposition 7.1.** The family of the BB schemes \(\{\text{BB}(J^\omega)\}\) in the Hilbert scheme \(\text{Hilb}^P_n\) is filtrable.

**Corollary 7.1.** Assume that \(k\) is algebraically closed, \(\text{char} \ k = 0\) and the Hilbert scheme \(\text{Hilb}^P_n\) is smooth. Denote by \(p(J)\) the dimension of the Zariski tangent space of \(\text{Gröb}^J \approx\) at \(J\). Then we have the following formula about the homology of \(\text{Hilb}^P_n\):

\[
H_m(\text{Hilb}^P_n, Z) \cong \bigoplus_{J \in \mathcal{M}_{P,n}} H_{m-2p(J)(\{J\}, Z)} \cong \bigoplus_{J \in \mathcal{M}_{P,n}} Z
\]

for any integer \(m\) with \(0 \leq m \leq 2 \dim \text{Hilb}^P_n\).

**Proof.** From the hypothesis, \(\text{BB}(J)\) is isomorphic to the affine space \(A^m_k\) for any \(J \in \mathcal{M}_{P,n}\). Therefore we obtain the above formula by \([BBCM02, \text{II. Theorem 4.4, Corollary 4.15}]\). \(\square\)

**Example 7.1.** Let \(P = 2t + 1\) and \(n = 3\). Then the Hilbert scheme \(\text{Hilb}^{2t+1}_3\) is smooth and the dimension is 8. The numbers of monomial ideals \(J \in \mathcal{M}_{2t+1,3}\) are in Table 5. Therefore if \(k\) is an algebraically closed field with char \(k = 0\), the homologies \(H_m = H_m(\text{Hilb}^{2t+1}_3, Z)\) are the followings:

\[
H_0 = \mathbb{Z}, H_2 = \mathbb{Z}^2, H_4 = \mathbb{Z}^3, H_6 = \mathbb{Z}^4,
H_8 = \mathbb{Z}^4, H_{10} = \mathbb{Z}^4, H_{12} = \mathbb{Z}^3, H_{14} = \mathbb{Z}^2, H_{16} = \mathbb{Z}.
H_1 = H_3 = \cdots = H_{15} = 0.
\]

| \(m\) | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|------|---|---|---|---|---|---|---|---|---|----|----|
|      | 1 | 2 | 3 | 4 | 4 | 4 | 4 | 3 | 2 | 1 |    |

**Table 5.** The numbers of \(J \in \mathcal{M}_{2t+1,3}\) such that \(\text{Gröb}^J \approx \mathbb{A}^m_k\)**
REFERENCES

[Bay82] David Allen Bayer. The Division Algorithm and the Hilbert Scheme. ProQuest LLC, Ann Arbor, MI, 1982. Thesis (Ph.D.)–Harvard University.

[BB73] A. Białynicki-Birula. Some theorems on actions of algebraic groups. Ann. of Math. (2), 98:480–497, 1973.

[BB76] A. Białynicki-Birula. Some properties of the decompositions of algebraic varieties determined by actions of a torus. Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys., 24(9):667–674, 1976.

[BBCM02] A. Białynicki-Birula, J. B. Carrell, and W. M. McGovern. Algebraic quotients. Torus actions and cohomology. The adjoint representation and the adjoint action, volume 131 of Encyclopaedia of Mathematical Sciences. Springer-Verlag, Berlin, 2002. Invariant Theory and Algebraic Transformation Groups, II.

[CLMR11] Francesca Cioffi, Paolo Lella, Maria Grazia Marinari, and Margherita Roggero. Segments and Hilbert schemes of points. Discrete Math., 311(20):2238–2252, 2011.

[Dri13] Vladimir Drinfeld. On algebraic spaces with an action of \( g_m \). arXiv preprint arXiv:1308.2604, 2013.

[Ehr34] Charles Ehresmann. Sur la topologie de certains espaces homogènes. Ann. of Math. (2), 35(2):396–443, 1934.

[EL12] Laurent Evain and Mathias Lederer. Białynicki-birula schemes in higher dimensional hilbert schemes of points. arXiv preprint arXiv:1209.2026, 2012.

[ES87] Geir Ellingsrud and Stein Arild Stroème. On the homology of the Hilbert scheme of points in the plane. Invent. Math., 87(2):343–352, 1987.

[Fog73] John Fogarty. Fixed point schemes. American Journal of Mathematics, 95(1):35–51, 1973.

[FR09] Giorgio Ferrarese and Margherita Roggero. Homogeneous varieties for Hilbert schemes. Int. J. Algebra, 3(9-12):547–557, 2009.

[Ful98] William Fulton. Intersection theory, volume 2 of Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]. Springer-Verlag, Berlin, second edition, 1998.

[Har77] Robin Hartshorne. Algebraic geometry. Springer-Verlag, New York-Heidelberg, 1977. Graduate Texts in Mathematics, No. 52.

[Har10] Robin Hartshorne. Deformation theory, volume 257 of Graduate Texts in Mathematics. Springer, New York, 2010.

[HS04] Mark Haiman and Bernd Sturmfels. Multigraded Hilbert schemes. J. Algebra, 134(2):725–769, 2004.

[Jel17] Joachim Jelisiejew. Elementary components of hilbert schemes. arXiv preprint arXiv:1710.06124, 2017.

[JS18] Joachim Jelisiejew and Lukasz Sienkiewicz. Białynicki-birula decomposition for reductive groups. arXiv preprint arXiv:1805.11558, 2018.

[Led11] Mathias Lederer. Gröbner strata in the Hilbert scheme of points. J. Commut. Algebra, 3(3):349–404, 2011.

[LR16] Paolo Lella and Margherita Roggero. On the functoriality of marked families. J. Commut. Algebra, 8(3):367–410, 2016.

[Mac07] Diane Maclagan. Notes on hilbert schemes. https://homepages.warwick.ac.uk/staff/D.Maclagan/, 2007.

[NS00] R. Notari and M. L. Spreafico. A stratification of Hilbert schemes by initial ideals and applications. Manuscripta Math., 101(4):429–448, 2000.

[Rob09] L. Robbiano. On border basis and Gröbner basis schemes. Collect. Math., 60(1):11–25, 2009.

[RT10] Margherita Roggero and Lea Terracini. Ideals with an assigned initial ideals. Int. Math. Forum, 5(53-56):2731–2750, 2010.

[Ser06] Edoardo Sernesi. Deformations of algebraic schemes, volume 334 of Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]. Springer-Verlag, Berlin, 2006.
[Vas98] Wolmer V. Vasconcelos. *Computational methods in commutative algebra and algebraic geometry*, volume 2 of *Algorithms and Computation in Mathematics*. Springer-Verlag, Berlin, 1998. With chapters by David Eisenbud, Daniel R. Grayson, Jürgen Herzog and Michael Stillman.

[Wib07] Michael Wibmer. Gröbner bases for families of affine or projective schemes. *J. Symbolic Comput.*, 42(8):803–834, 2007.

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