Flat Rotational Surface with Pointwise 1-type Gauss map in $E^4$

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Abstract

In this paper we study general rotational surfaces in the 4-dimensional Euclidean space $E^4$ and give a characterization of flat general rotation surface with pointwise 1-type Gauss map. Also, we show that a non-planar flat general rotation surface with pointwise 1-type Gauss map is a Lie group if and only if it is a Clifford torus.

Key words and Phrases: Rotation surface, Gauss map, Pointwise 1-type Gauss map, Euclidean space.

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1 Introduction

A submanifold $M$ of a Euclidean space $E^n$ is said to be of finite type if its position vector $x$ can be expressed as a finite sum of eigenvectors of the Laplacian $\Delta$ of $M$, that is, $x = x_0 + x_1 + \ldots x_k$, where $x_0$ is a constant map, $x_1, \ldots, x_k$ are non-constant maps such that $\Delta x_i = \lambda_i x_i$, $\lambda_i \in \mathbb{R}$, $i = 1, 2, \ldots, k$. If $\lambda_1, \lambda_2, \ldots, \lambda_k$ are all different, then $M$ is said to be of $k$–type. This definition was similarly extended to differentiable maps, in particular, to Gauss maps of submanifolds [6].

If a submanifold $M$ of a Euclidean space or pseudo-Euclidean space has 1-type Gauss map $G$, then $G$ satisfies $\Delta G = \lambda (G + C)$ for some $\lambda \in \mathbb{R}$ and some constant vector $C$. Chen and Piccinni made a general study on compact submanifolds of Euclidean spaces with finite type Gauss map and they proved that a compact hypersurface $M$ of $E^{n+1}$ has 1-type Gauss map if and only if $M$ is a hypersphere in $E^{n+1}$ [6].

However, the Laplacian of the Gauss map of some typical well known surfaces such as a helicoid, a catenoid and a right cone in Euclidean 3-space $E^3$ take a somewhat different form, namely,

$$\Delta G = f (G + C)$$ (1)

for some smooth function $f$ on $M$ and some constant vector $C$. A submanifold $M$ of a Euclidean space $E^n$ is said to have pointwise 1-type Gauss map if its Gauss...
map satisfies (1) for some smooth function $f$ on $M$ and some constant vector $C$. A submanifold with pointwise 1-type Gauss map is said to be of the first kind if the vector $C$ in (1) is zero vector. Otherwise, the pointwise 1-type Gauss map is said to be of the second kind.

Surfaces in Euclidean space and in pseudo-Euclidean space with pointwise 1-type Gauss map were recently studied in [7], [8], [10], [11], [12], [13], [14]. Also Dursun and Turgay in [9] gave all general rotational surfaces in $E^4$ with proper pointwise 1-type Gauss map of the first kind and classified minimal rotational surfaces with proper pointwise 1-type Gauss map of the second kind. Arslan et al. in [2] investigated rotational embedded surface with pointwise 1-type Gauss map. Arslan at el. in [3] gave necessary and sufficient conditions for Vranceanu rotation surface to have pointwise 1-type Gauss map. Yoon in [19] showed that flat Vranceanu rotation surface with pointwise 1-type Gauss map is a Clifford torus.

In this paper, we study general rotational surfaces in the 4-dimensional Euclidean space $E^4$ and give a characterization of flat general rotation with pointwise 1-type Gauss map. Also, we show that a non-planar flat general rotation surface with pointwise 1-type Gauss map is a Lie group if and only if it is a Clifford torus.

2 Preliminaries

Let $M$ be an oriented $n$-dimensional submanifold in $m$-dimensional Euclidean space $E^m$. Let $e_1,...,e_n,e_{n+1},...,e_m$ be an oriented local orthonormal frame in $E^m$ such that $e_1,...,e_n$ are tangent to $M$ and $e_{n+1},...,e_m$ normal to $M$. We use the following convention on the ranges of indices: $1 \leq i, j, k, \ldots \leq n$, $n+1 \leq r, s, t, \ldots \leq m$, $1 \leq A, B, C, \ldots \leq m$.

Let $\nabla$ be the Levi-Civita connection of $E^m$ and $\nabla$ the induced connection on $M$. Let $\omega_A$ be the dual-1 form of $e_A$ defined by $\omega_A(e_B) = \delta_{AB}$. Also, the connection forms $\omega_{AB}$ are defined by

$$de_A = \sum_B \omega_{AB} e_B, \quad \omega_{AB} + \omega_{BA} = 0.$$ 

Then we have

$$\tilde{\nabla}_{e_k}^e_i = \sum_{j=1}^n \omega_{ij} (e_k) e_j + \sum_{r=n+1}^m h_{ik}^r e_r$$

and

$$\tilde{\nabla}_{e_k}^e_s = -A_s(e_k) + \sum_{r=n+1}^m \omega_{sr} (e_k) e_r, \quad \tilde{D}_{e_k}^e_s = \sum_{r=n+1}^m \omega_{sr} (e_k) e_r,$$

where $D$ is the normal connection, $h_{ik}^r$ the coefficients of the second fundamental form $h$ and $A_s$ the Weingarten map in the direction $e_s$. 

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For any real function \( f \) on \( M \) the Laplacian of \( f \) is defined by

\[
\Delta f = - \sum_i \left( \tilde{\nabla}_{e_i} \tilde{\nabla}_{e_i} f - \tilde{\nabla}_{\nabla_{e_i}} f \right). \tag{4}
\]

If we define a covariant differentiation \( \tilde{\nabla} h \) of the second fundamental form \( h \) on the direct sum of the tangent bundle and the normal bundle \( TM \oplus T^\perp M \) of \( M \) by

\[
(\tilde{\nabla}_X h)(Y, Z) = D_X h(Y, Z) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z)
\]

for any vector fields \( X, Y \) and \( Z \) tangent to \( M \). Then we have the Codazzi equation

\[
(\tilde{\nabla}_X h)(Y, Z) = (\tilde{\nabla}_Y h)(X, Z) \tag{5}
\]

and the Gauss equation is given by

\[
\langle R(X, Y) Z, W \rangle = \langle h(X, W), h(Y, Z) \rangle - \langle h(X, Z), h(Y, W) \rangle
\]

where the vectors \( X, Y, Z \) and \( W \) are tangent to \( M \) and \( R \) is the curvature tensor associated with \( \nabla \) and the curvature tensor \( R \) is defined by

\[
R(X, Y) Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z.
\]

Let us now define the Gauss map \( G \) of a submanifold \( M \) into \( G(n, m) \) in \( \wedge^n \mathbb{E}^m \), where \( G(n, m) \) is the Grassmannian manifold consisting of all oriented \( n \)-planes through the origin of \( \mathbb{E}^m \) and \( \wedge^n \mathbb{E}^m \) is the vector space obtained by the exterior product of \( n \) vectors in \( \mathbb{E}^m \). In a natural way, we can identify \( \wedge^n \mathbb{E}^m \) with some Euclidean space \( \mathbb{E}^N \) where \( N = \binom{m}{n} \). The map \( G : M \to G(n, m) \subset \mathbb{E}^N \) defined by \( G(p) = (e_1 \wedge ... \wedge e_n)(p) \) is called the Gauss map of \( M \), that is, a smooth map which carries a point \( p \) in \( M \) into the oriented \( n \)-plane through the origin of \( \mathbb{E}^m \) obtained from parallel translation of the tangent space of \( M \) at \( p \) in

Bicomplex number is defined by the basis \( \{1, i, j, ij\} \) where \( i, j, ij \) satisfy \( i^2 = -1, j^2 = -1, ij = ji \). Thus any bicomplex number \( x \) can be expressed as \( x = x_1 1 + x_2 i + x_3 j + x_4 ij \), \( \forall x_1, x_2, x_3, x_4 \in \mathbb{R} \). We denote the set of bicomplex numbers by \( C_2 \). For any \( x = x_1 1 + x_2 i + x_3 j + x_4 ij \) and \( y = y_1 1 + y_2 i + y_3 j + y_4 ij \) in \( C_2 \) the bicomplex number addition is defined by

\[
x + y = (x_1 + y_1) + (x_2 + y_2) i + (x_3 + y_3) j + (x_4 + y_4) ij.
\]

The multiplication of a bicomplex number \( x = x_1 1 + x_2 i + x_3 j + x_4 ij \) by a real scalar \( \lambda \) is given by

\[
\lambda x = \lambda x_1 1 + \lambda x_2 i + \lambda x_3 j + \lambda x_4 ij.
\]
With this addition and scalar multiplication, \( C_2 \) is a real vector space. Bicomplex number product, denoted by \( \times \), over the set of bicomplex numbers \( C_2 \) is given by

\[
x \times y = (x_1 y_1 - x_2 y_2 - x_3 y_3 + x_4 y_4) + (x_1 y_2 + x_2 y_1 - x_3 y_4 - x_4 y_3) i + (x_1 y_3 + x_3 y_1 - x_2 y_4 - x_4 y_2) j + (x_1 y_4 + x_4 y_1 + x_2 y_3 + x_3 y_2) i j.
\]

Vector space \( C_2 \) together with the bicomplex product \( \times \) is a real algebra. Since the bicomplex algebra is associative, it can be considered in terms of matrices. Consider the set of matrices

\[
Q = \left\{ \begin{pmatrix} x_1 & -x_2 & -x_3 & x_4 \\ x_2 & x_1 & -x_4 & -x_3 \\ x_3 & -x_4 & x_1 & -x_2 \\ x_4 & x_3 & x_2 & x_1 \end{pmatrix} ; \quad x_i \in \mathbb{R}, \quad 1 \leq i \leq 4 \right\}.
\]

The set \( Q \) together with matrix addition and scalar matrix multiplication is a real vector space. Furthermore, the vector space together with matrix product is an algebra \([15]\).

The transformation

\[
g : C_2 \to Q
\]

given by

\[
g(x = x_1 1 + x_2 i + x_3 j + x_4 i j) = \begin{pmatrix} x_1 & -x_2 & -x_3 & x_4 \\ x_2 & x_1 & -x_4 & -x_3 \\ x_3 & -x_4 & x_1 & -x_2 \\ x_4 & x_3 & x_2 & x_1 \end{pmatrix}
\]
is one to one and onto. Moreover \( \forall x, y \in C_2 \) and \( \lambda \in \mathbb{R} \), we have

\[
\begin{align*}
g(x + y) &= g(x) + g(y) \\
g(\lambda x) &= \lambda g(x) \\
g(xy) &= g(x) g(y).
\end{align*}
\]

Thus the algebras \( C_2 \) and \( Q \) are isomorphic. Let \( x \in C_2 \). Then \( x \) can be expressed as \( x = (x_1 + x_2 i) + (x_3 + x_4 i) j \). In this case, there is three different conjugations for bicomplex numbers as follows:

\[
x^{t_1} = [(x_1 + x_2 i) + (x_3 + x_4 i) j]^{t_1} = (x_1 - x_2 i) + (x_3 - x_4 i) j \\
x^{t_2} = [(x_1 + x_2 i) + (x_3 + x_4 i) j]^{t_2} = (x_1 + x_2 i) - (x_3 + x_4 i) j \\
x^{t_3} = [(x_1 + x_2 i) + (x_3 + x_4 i) j]^{t_3} = (x_1 - x_2 i) - (x_3 - x_4 i) j
\]
3 Flat Rotation Surfaces with Pointwise 1-Type Gauss Map in $E^4$

In this section, we consider the flat rotation surfaces with pointwise 1-type Gauss map in Euclidean 4-space. Let consider the equation of the general rotation surface given in [16].

$$\varphi(t, s) = \begin{pmatrix} \cos mt & -\sin mt & 0 & 0 \\ \sin mt & \cos mt & 0 & 0 \\ 0 & 0 & \cos nt & -\sin nt \\ 0 & 0 & \sin nt & \cos nt \end{pmatrix} \begin{pmatrix} \alpha_1(s) \\ \alpha_2(s) \\ \alpha_3(s) \\ \alpha_4(s) \end{pmatrix},$$

where $\alpha(s) = (\alpha_1(s), \alpha_2(s), \alpha_3(s), \alpha_4(s))$ is a regular smooth curve in $E^4$ on an open interval $I$ in $\mathbb{R}$ and $m, n$ are some real numbers which are the rates of the rotation in fixed planes of the rotation. If we choose the meridian curve $\alpha$ as $\alpha(s) = (x(s), 0, y(s), 0)$ is unit speed curve and the rates of the rotation $m$ and $n$ as $m = n = 1$, we obtain the surface as follows:

$$M : X(s, t) = (x(s) \cos t, x(s) \sin t, y(s) \cos t, y(s) \sin t)$$

(7)

Let $M$ be a general rotation surface in $E^4$ given by (7). We consider the following orthonormal moving frame $\{e_1, e_2, e_3, e_4\}$ on $M$ such that $e_1, e_2$ are tangent to $M$ and $e_3, e_4$ are normal to $M$:

$$e_1 = \frac{1}{\sqrt{x^2(s) + y^2(s)}} (-x(s) \sin t, x(s) \cos t, -y(s) \sin t, y(s) \cos t)$$

$$e_2 = (x'(s) \cos t, x'(s) \sin t, y'(s) \cos t, y'(s) \sin t)$$

$$e_3 = (-y'(s) \cos t, -y'(s) \sin t, x'(s) \cos t, x'(s) \sin t)$$

$$e_4 = \frac{1}{\sqrt{x^2(s) + y^2(s)}} (-y(s) \sin t, y(s) \cos t, x(s) \sin t, -x(s) \cos t)$$

where $e_1 = \frac{1}{\sqrt{x^2(s) + y^2(s)}} \frac{\partial}{\partial t}$ and $e_2 = \frac{\partial}{\partial s}$. Then we have the dual 1-forms as:

$$\omega_1 = \sqrt{x^2(s) + y^2(s)} dt \quad \text{and} \quad \omega_2 = ds$$

By a direct computation we have components of the second fundamental form and the connection forms as:

$$h_{11}^3 = b(s), \ h_{12}^3 = 0, \ h_{22}^3 = c(s),$$

$$h_{11}^4 = 0, \ h_{12}^4 = -b(s), \ h_{22}^4 = 0,$$

$$\omega_{12} = -a(s) \omega_1, \ \omega_{13} = b(s) \omega_1, \ \omega_{14} = -b(s) \omega_2$$

$$\omega_{23} = c(s) \omega_2, \ \omega_{24} = -b(s) \omega_1, \ \omega_{34} = -a(s) \omega_1.$$
By covariant differentiation with respect to $e_1$ and $e_2$ a straightforward calculation gives:

\[
\begin{align*}
\tilde{\nabla}_{e_1} e_1 &= -a(s)e_2 + b(s)e_3, \\
\tilde{\nabla}_{e_2} e_1 &= -b(s)e_4, \\
\tilde{\nabla}_{e_1} e_2 &= a(s)e_1 - b(s)e_4, \\
\tilde{\nabla}_{e_2} e_2 &= c(s)e_3, \\
\tilde{\nabla}_{e_1} e_3 &= -b(s)e_1 - a(s)e_4, \\
\tilde{\nabla}_{e_2} e_3 &= -c(s)e_2, \\
\tilde{\nabla}_{e_1} e_4 &= b(s)e_2 + a(s)e_3, \\
\tilde{\nabla}_{e_2} e_4 &= b(s)e_1,
\end{align*}
\]

where

\[
\begin{align*}
a(s) &= \frac{x(s)x'(s) + y(s)y'(s)}{x^2(s) + y^2(s)}, \\
b(s) &= \frac{x(s)y'(s) - x'(s)y(s)}{x^2(s) + y^2(s)}, \\
c(s) &= x'(s)y'' - x''y'(s).
\end{align*}
\]

The Gaussian curvature is obtained by

\[K = \det \left( h_{ij}^3 \right) + \det \left( h_{ij}^4 \right) = b(s)c(s) - b^2(s).\]  \hspace{1cm} (12)

If the surface $M$ is flat, from (12) we get

\[b(s)c(s) - b^2(s) = 0.\]  \hspace{1cm} (13)

Furthermore, by using the equations of Gauss and Codazzi after some computation we obtain

\[a'(s) + a^2(s) = b^2(s) - b(s)c(s)\]  \hspace{1cm} (14)

and

\[b'(s) = -2a(s)b(s) + a(s)c(s),\]  \hspace{1cm} (15)

respectively.

By using (4) and (8) and straight-forward computation the Laplacian $\Delta G$ of the Gauss map $G$ can be expressed as

\[
\Delta G = (3b^2(s) + c^2(s)) (e_1 \wedge e_2) + (2a(s)b(s) - a(s)c(s) - c'(s)) (e_1 \wedge e_3) + (-3a(s)b(s) - b'(s)) (e_2 \wedge e_4) + (2b^2(s) - 2b(s)c(s)) (e_3 \wedge e_4).\]  \hspace{1cm} (16)

**Remark 1.** Similar computations to above computations is given for tensor product surfaces in [4] and for general rotational surface in [9].
Now we investigate the flat rotation surface with the pointwise 1-type Gauss map. From (13), we obtain that $b(s) = 0$ or $b(s) = c(s)$. We assume that $b(s) \neq c(s)$. Then $b(s)$ is equal to zero and (15) implies that $a(s)c(s) = 0$. Since $b(s) \neq c(s)$, it implies that $c(s)$ is not equal to zero. Then we obtain as $a(s) = 0$. In that case, by using (9) and (10) we obtain that $\alpha(s) = (x(s), 0, y(s), 0)$ is a constant vector. This is a contradiction. Therefore $b(s) = c(s)$ for all $s$. From (14), we get

$$a'(s) + a^2(s) = 0$$

whose the trivial solution and non-trivial solution

$$a(s) = 0$$

and

$$a(s) = \frac{1}{s + c},$$

respectively. We assume that $a(s) = 0$. By (15) $b = b_0$ is a constant and so is $c$. In that case by using (9), (10) and (11), $x$ and $y$ satisfy the following differential equations

$$x^2(s) + y^2(s) = \lambda^2 \quad \lambda \text{ is a non-zero constant},$$

(18)

$$x(s)y'(s) - x'(s)y(s) = b_0\lambda^2,$$

(19)

$$x'(s)y'' - x''y'(s) = b_0.$$  

(20)

From (18) we may put

$$x(s) = \lambda \cos \theta(s), \quad y(s) = \lambda \sin \theta(s),$$

(21)

where $\theta(s)$ is some angle function. Differentiating (21) with respect to $s$, we have

$$x'(s) = -\theta'(s)y(s) \quad \text{and} \quad y'(s) = \theta'(s)x(s).$$

(22)

By substituting (21) and (22) into (19), we get

$$\theta(s) = b_0s + d, \quad d = \text{const.}$$

And since the curve $\alpha$ is a unit speed curve, we have

$$b_0^2\lambda^2 = 1.$$  

Then we can write components of the curve $\alpha$ as:

$$x(s) = \lambda \cos (b_0s + d) \quad \text{and} \quad y(s) = \lambda \sin (b_0s + d), \quad b_0^2\lambda^2 = 1.$$  

On the other hand, by using (16) we can rewrite the Laplacian of the Gauss map $G$ with $a(s) = 0$ and $b = c = b_0$ as follows:

$$\Delta G = 4b_0^2(e_1 \wedge e_2).$$
that is, the flat surface $M$ is pointwise 1-type Gauss map with the function $f = 4b_0^2$ and $C = 0$. Even if it is a pointwise 1-type Gauss map of the first kind.

Now we assume that $a(s) = \frac{1}{s+c}$. Since $b(s)$ is equal to $c(s)$, from (15) we get

$$b'(s) = -a(s)b(s)$$

or we can write

$$b'(s) = -\frac{b(s)}{s+c},$$

whose the solution

$$b(s) = \mu a(s), \quad \mu \text{ is a constant.}$$

By using (16) we can rewrite the Laplacian of the Gauss map $G$ with $c(s) = b(s) = \mu a(s)$ as:

$$\Delta G = \left(4\mu^2a^2(s)\right) (e_1 \wedge e_2) + 2\mu a^2(s) (e_1 \wedge e_3) - 2\mu a^2(s) (e_2 \wedge e_4).$$

(23)

We suppose that the flat rotational surface has pointwise 1-type Gauss map. From (1) and (22), we get

$$4\mu^2 a^2(s) = f + f \langle C, e_1 \wedge e_2 \rangle$$

(24)

$$2\mu a^2(s) = f \langle C, e_1 \wedge e_3 \rangle$$

(25)

$$-2\mu a^2(s) = f \langle C, e_2 \wedge e_4 \rangle$$

(26)

Then, we have

$$\langle C, e_1 \wedge e_4 \rangle = 0, \quad \langle C, e_2 \wedge e_3 \rangle = 0, \quad \langle C, e_3 \wedge e_4 \rangle = 0$$

(27)

By using (25) and (26) we obtain

$$\langle C, e_1 \wedge e_3 \rangle + \langle C, e_2 \wedge e_4 \rangle = 0$$

(28)

By differentiating the first equation in (27) with respect to $e_1$ and by using (8), the third equation in (27) and (28), we get

$$2a(s) \langle C, e_1 \wedge e_3 \rangle + \mu a(s) \langle C, e_1 \wedge e_2 \rangle = 0$$

(29)

Combining (24), (25) and (29) we then have

$$f = 4 \left(a^2(s) + \mu^2 a^2(s)\right)$$

(30)

that is, a smooth function $f$ depends only on $s$. By differentiating $f$ with respect to $s$ and by using the equality $a'(s) = -a^2(s)$, we get

$$f' = -2a(s)f$$

(31)
By differentiating (25) with respect to $s$ and by using (8), (24), the third equation in (27), (30), (31) and the equality $a'(s) = -a^2(s)$, we have

$$\mu a^3 = 0$$

Since $a(s) \neq 0$, it follows that $\mu = 0$. Then we obtain that $b = c = 0$. Then the surface $M$ is a part of plane.

Thus we can give the following theorem and corollary.

**Theorem 1.** Let $M$ be the flat rotation surface given by the parametrization (7). Then $M$ has pointwise 1-type Gauss map if and only if $M$ is either totally geodesic or parametrized by

$$X(s,t) = \begin{pmatrix} \lambda \cos(b_0s + d) \cos t, \lambda \cos(b_0s + d) \sin t, \\ \lambda \sin(b_0s + d) \cos t, \lambda \sin(b_0s + d) \sin t \end{pmatrix}, \quad b_0^2\lambda^2 = 1$$

(32)

where $b_0$, $\lambda$ and $d$ are real constants.

**Corollary 1.** Let $M$ be flat rotation surface given by the parametrization (7). If $M$ has pointwise 1-type Gauss map then the Gauss map $G$ on $M$ is of 1-type.

### 4 The general rotation surface and Lie group

In this section, we determine the profile curve of the general rotation surface which has a group structure with the bicomplex number product.

Let the hyperquadric $P$ be given by

$$P = \{x = (x_1, x_2, x_3, x_4) \neq 0; \quad x_1x_4 = x_2x_3\}.$$

We consider $P$ as the set of bicomplex number

$$P = \{x = x_11 + x_2i + x_3j + x_4ij; \quad x_1x_4 = x_2x_3, \quad x \neq 0\}.$$

The components of $P$ are easily obtained by representing bicomplex number multiplication in matrix form.

$$\tilde{P} = \left\{ M_x = \begin{pmatrix} x_1 & -x_2 & -x_3 & x_4 \\ x_2 & x_1 & -x_4 & -x_3 \\ x_3 & -x_4 & x_1 & -x_2 \\ x_4 & x_3 & x_2 & x_1 \end{pmatrix}; \quad x_1x_4 = x_2x_3, \quad x \neq 0 \right\}.$$

**Theorem 2.** The set of $P$ together with the bicomplex number product is a Lie group.
Proof. \( \tilde{P} \) is a differentiable manifold and at the same time a group with group operation given by matrix multiplication. The group function

\[ . : \tilde{P} \times \tilde{P} \to \tilde{P} \]

defined by \((x, y) \to x.y\) is differentiable. So \((P, .)\) can be made a Lie group so that \( g \) is a isomorphism \([15] \). \hfill \Box

Remark 2. The surface \( M \) given by the parametrization (7) is a subset of \( P \).

Proposition 1. Let \( M \) be a rotation surface given by the parametrization (7). If \( x(s) \) and \( y(s) \) satisfy the following equations then \( M \) is a Lie subgroup of \( P \).

\[
\begin{align*}
x(s_1)x(s_2) - y(s_1)y(s_2) &= x(s_1 + s_2) & (33) \\
x(s_1)y(s_2) + x(s_2)y(s_1) &= y(s_1 + s_2) & (34) \\
\frac{x(s)}{x^2(s) + y^2(s)} &= x(-s) & (35) \\
-\frac{y(s)}{x^2(s) + y^2(s)} &= y(-s) & (36)
\end{align*}
\]

Proof. Let \( \alpha(s) = (x(s), 0, y(s), 0) \) be a profile curve of the rotation surface given by the parametrization (7) such that \( x(s) \) and \( y(s) \) satisfy the equations (33), (34), (35) and (36). In that case we obtain that the inverse of \( X(s, t) \) is \( X(-s, -t) \) and \( X(s_1, t_1) \times X(s_2, t_2) = X(s_1 + s_2, t_1 + t_2) \). This completes the proof. \hfill \Box

Proposition 2. Let \( \alpha(s) = (x(s), 0, y(s), 0) \) be a profile curve of the rotation surface given by the parametrization (7) such that \( x(s) \) and \( y(s) \) satisfy the equation \( x^2(s) + y^2(s) = \lambda^2 \), where \( \lambda \) is a non-zero constant. If \( M \) is a subgroup of \( P \) then the profile curve \( \alpha \) is a unit circle.

Proof. We assume that \( x(s) \) and \( y(s) \) satisfy the equation \( x^2(s) + y^2(s) = \lambda^2 \). Then we can put

\[
x(s) = \lambda \cos \theta(s) \quad \text{and} \quad y(s) = \lambda \sin \theta(s)
\]

(37)

where \( \lambda \) is a real constant and \( \theta(s) \) is a smooth function. Since \( M \) is a group, there exists one and only inverse of all elements on \( M \). In that case the inverse of \( X(s, t) \) is given by

\[
X^{-1}(s, t) = \begin{pmatrix}
\frac{x(s)}{x^2(s) + y^2(s)} \cos(-t) & \frac{x(s)}{y(s)} \sin(-t) \\
-\frac{y(s)}{x^2(s) + y^2(s)} \cos(-t) & -\frac{y(s)}{x^2(s) + y^2(s)} \sin(-t)
\end{pmatrix}
\]
where
\[
\begin{align*}
\frac{x(s)}{x^2(s) + y^2(s)} &= x(f(s)), \\
\frac{y(s)}{x^2(s) + y^2(s)} &= y(f(s)), \text{ f is a smooth function.} \tag{38}
\end{align*}
\]

By using (38), we get
\[
\begin{align*}
x(s) &= \lambda^2 x(f(s)) \tag{39} \\
y(s) &= -\lambda^2 y(f(s)) \tag{40}
\end{align*}
\]

By summing of the squares on both sides in (39) and (40) and by using (37), we obtain that
\[
\lambda^2 = 1.
\]
This completes the proof.

**Proposition 3.** Let \( \alpha(s) = (x(s), 0, y(s), 0) \) be a profile curve of the rotation surface given by the parametrization (7) such that \( x(s) \) and \( y(s) \) is given by \( x(s) = \lambda \cos \theta(s) \) and \( y(s) = \lambda \sin \theta(s) \). Then if \( \lambda = 1 \) and \( \theta \) is a linear function then \( M \) is a Lie subgroup of \( P \).

**Proof.** We assume that \( \lambda = 1 \) and \( \theta \) is a linear function. Then we can write
\[
\begin{align*}
x(s) &= \cos \eta s \\
y(s) &= \sin \eta s
\end{align*}
\]
and in that case \( x(s) \) and \( y(s) \) satisfy the equations (33), (34), (35) and (36). Thus from Proposition (2) \( M \) is a subgroup of \( P \). Also, it is a submanifold of \( P \).

**Proposition 4.** Let \( \alpha(s) = (x(s), 0, y(s), 0) \) be a profile curve of the rotation surface given by the parametrization (7) such that \( x(s) \) and \( y(s) \) is given by \( x(s) = u(s) \cos \theta(s) \) and \( y(s) = u(s) \sin \theta(s) \). Then if \( u : (\mathbb{R}, +) \to (\mathbb{R}^+, .) \) is a group homomorphism and \( \theta \) is a linear function then \( M \) is a Lie subgroup of \( P \).

**Proof.** Let \( x(s) \) and \( y(s) \) be given by \( x(s) = u(s) \cos \theta(s) \) and \( y(s) = u(s) \sin \theta(s) \) and let \( u : (\mathbb{R}, +) \to (\mathbb{R}^+, .) \) be a group homomorphism and \( \theta \) be a linear function. In that case \( x(s) \) and \( y(s) \) satisfy the equations (33), (34), (35) and (36). Thus from Proposition (1) \( M \) is a subgroup of \( P \). Also, it is a submanifold of \( P \). So it is a Lie subgroup of \( P \).

**Corollary 2.** Let \( \alpha(s) = (x(s), 0, y(s), 0) \) be a profile curve of the rotation surface given by the parametrization (7) such that \( x(s) \) and \( y(s) \) is given by \( x(s) = \lambda \cos \theta(s) \) and \( y(s) = \lambda \sin \theta(s) \) for \( \theta \) linear function. If \( M \) is a Lie subgroup then \( \lambda = 1 \).

**Proof.** We assume that \( M \) is a group and \( \lambda \neq 1 \). From Proposition (1) we obtain that \( \lambda = -1 \). On the other hand, for \( \lambda = -1 \) and \( \theta \) linear function the closure property is not satisfied on \( M \). This is a contradiction. Then we obtain that \( \lambda = 1 \).
Remark 3. Let $M$ be a Vranceanu surface. If the surface $M$ is flat then it is given by

$$X(s, t) = (e^{ks} \cos s \cos t, e^{ks} \cos s \sin t, e^{ks} \sin s \cos t, e^{ks} \sin s \sin t)$$

where $k$ is a real constant. In that case we can say that flat Vranceanu surface is a Lie subgroup of $P$ with bicomplex multiplication. Also, flat Vranceanu surface with pointwise 1-type Gauss map is Clifford torus and it is given by

$$X(s, t) = (\cos s \cos t, \cos s \sin t, \sin s \cos t \sin s \sin t)$$

and Clifford Torus is a Lie subgroup of $P$ with bicomplex multiplication. See for more details [1].

Theorem 3. Let $M$ be non-planar flat rotation surface with pointwise 1-type Gauss map given by the parametrization (32) with $d = 2k\pi$. Then $M$ is a Lie group with bicomplex multiplication if and only if it is a Clifford torus.

Proof. We assume that $M$ is a Lie group with bicomplex multiplication then from Corollary (2) we get that $\lambda = 1$. Since $b_0^2 \lambda^2 = 1$, it follows that $b_0 = \varepsilon$, where $\varepsilon = \pm 1$. In that case the surface $M$ is given by

$$X(s, t) = (\cos \varepsilon s \cos t, \cos \varepsilon s \sin t, \sin \varepsilon s \cos t \sin \varepsilon s \sin t)$$

and $M$ is a Clifford torus, that is, the product of two plane circle with the same radius.

Conversely, Clifford torus is a flat rotational surface with pointwise 1-type Gauss map the surface which can be obtained by the parametrization (32) and it is a Lie group with bicomplex multiplication. This completes the proof. \qed

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