New examples of Weierstrass semigroups associated with a double covering of a curve on a Hirzebruch surface of degree one

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Received: 22 July 2021 / Accepted: 26 January 2022 / Published online: 11 February 2022 © The Managing Editors 2022

Abstract
Let $\varphi: \Sigma_1 \rightarrow \mathbb{P}^2$ be a blow up at a point on $\mathbb{P}^2$. Let $C$ be the proper transform of a smooth plane curve of degree $d \geq 4$ by $\varphi$, and let $P$ be a point on $C$. Let $\pi: \tilde{C} \rightarrow C$ be a double covering branched along the reduced divisor on $C$ obtained as the intersection of $C$ and a reduced divisor in $|−2K_{\Sigma_1}|$ containing $P$. In this paper, we investigate the Weierstrass semigroup $H(\tilde{P})$ at the ramification point $\tilde{P}$ of $\pi$ over $P$, in the case where the intersection multiplicity at $\varphi(P)$ of $\varphi(C)$ and the tangent line at $\varphi(P)$ of $\varphi(C)$ is $d − 1$.

Keywords Weierstrass semigroup · Double covering of a curve · Hirzebruch surface · Normalization of a curve

Mathematics Subject Classification 14J26 · 14H51 · 14H55

1 Introduction

We work over the complex number field $\mathbb{C}$. Let $C$ be a smooth projective curve, and let $P$ be a point on $C$. Then we call a natural number $n$ satisfying
\[
h^0(\mathcal{O}_C ((n − 1)P)) = h^0(\mathcal{O}_C (nP)),
\]
a gap at $P$ on $C$. Let $\mathbb{N}_0$ be the additive monoid consisting of non-negative integers. If we let $G(P)$ be the set of gaps at $P$ on $C$, the set $\mathbb{N}_0 \setminus G(P)$ forms an additive monoid. We call it the Weierstrass semigroup at $P$ on $C$, and denote it by $H(P)$. If the genus of $C$ is $g$, then the cardinality of $G(P)$ coincides with $g$ and it is called the genus of $H(P)$.
Assume that $C$ is a double covering of $\mathbb{P}^1$, that is, $C$ is hyperelliptic. Then the ramification points of it are Weierstrass points. Hence, in general, it is natural and interesting to consider the problem of whether the ramification points of a given double covering of a curve are Weierstrass.

Let $\pi : \tilde{C} \to C$ be a double covering, let $P$ be a branch point of it, and let $\tilde{P}$ be the ramification point of $\pi$ over $P$. Then we consider the Weierstrass semigroup $H(\tilde{P})$ called the double covering type. If the genus of $\tilde{C}$ is $g$ and $C = \mathbb{P}^1$, $H(\tilde{P})$ is generated by 2 and $2g + 1$. Komeda (2011) has determined the all possible Weierstrass semigroups $H(\tilde{P})$, in the case where the genus $g$ of $C$ is 3 and the genus of $\tilde{C}$ is $g \geq 9$. On the other hand, in the case where $C$ is a smooth plane curve of degree $d \geq 4$ and the intersection multiplicity at $P$ of $C$ and the tangent line at $P$ of $C$ is $d$ or $d - 1$, we have determined $H(\tilde{P})$, under the condition that the branch locus of $\pi$ is the reduced divisor on $C$ of degree $6d$ obtained as the intersection of $C$ and a reduced divisor on $\mathbb{P}^2$ of degree 6 (Watanabe 2013; Watanabe and Komeda 2015). If $\tilde{C}$ is smooth, that is, the branch divisor of $\pi$ is reduced, the computations of the Weierstrass semigroups at the ramification points are interesting from the point of view of the classification of Weierstrass points on a curve. However, in this paper, we will focus on the Weierstrass semigroups at the points on a curve obtained by the normalization of $\tilde{C}$, in the case where the branch divisor of $\pi$ is not reduced.

Let $\varphi : \Sigma_1 \to \mathbb{P}^2$ be a blow up at a point on $\mathbb{P}^2$. Then we call the surface $\Sigma_1$ a Hirzebruch surface. Let $E$ be the exceptional divisor of $\varphi$, and we denote by $L$ the total transform of a line on $\mathbb{P}^2$ by $\varphi$. Then we note that the Picard lattice $\text{Pic}(\Sigma_1)$ of $\Sigma_1$ is generated by the classes of $E$ and $L$. In the previous work, we had the following result.

**Theorem 1.1** [Watanabe (2019), Theorem 1.2] Let $C$ be a smooth curve on $\Sigma_1$ which is linearly equivalent to the divisor $dL - E$ with $d \geq 4$. Let $\pi : \tilde{C} \to C$ be a double covering of $C$ with a branch point $P$, $\pi^{-1}(P) = \{\tilde{P}\}$, and assume that there exists an effective divisor $T_P \in |L|$ such that $T_P|_C = dP$. Moreover, let $M_d$ be the following condition.

$M_d$: There exists a double covering $\tilde{\pi} : X \to \Sigma_1$ branched along a reduced divisor belonging to $| - 2K_{\Sigma_1}|$ such that $\tilde{C} \subset X$ and $\tilde{\pi}|_{\tilde{C}} = \pi$.

Then, we have the following results.

(a) If $P \in E$, then the following conditions are equivalent.

(i) $M_d$ holds.

(ii) $H(\tilde{P}) = 2H(P) + (6d - 3)N_0$.

(b) If $P \notin E$ and $M_d$ holds, then $H(\tilde{P}) = 2H(P) + (6d - 1)N_0 + (2d^2 + 1)N_0$.

In Theorem 1.1, if the double covering $\pi$ satisfies the condition $M_d$, $\tilde{C}$ is the normalization of the double covering of $\varphi(C)$ branched along the divisor obtained as the restriction to $\varphi(C)$ of a reduced divisor on $\mathbb{P}^2$ of degree 6 which has a singularity at $\varphi(E)$. However, conversely, it is difficult to consider the problem of whether $\pi$ satisfies the condition $M_d$, even if the Weierstrass semigroup $H(\tilde{P})$ at the ramification point $\tilde{P}$ of $\pi$ over the branch point $P$ is given. In this paper, we will investigate the Weierstrass semigroup $H(\tilde{P})$, in the case where $C$ is the proper transform of a smooth plane curve of degree $d \geq 4$ by $\varphi$, and the intersection multiplicity at $\varphi(P)$ of $\varphi(C)$.
and the tangent line at $\varphi(P)$ of $\varphi(C)$ is $d - 1$. Moreover, we consider a necessary and sufficient condition for a double covering $\pi$ of a smooth curve on a Hirzebruch surface to satisfy the condition $M_d$. Our main theorem is the following.

**Theorem 1.2** Let $L$, $E$, and $C$ be as in Theorem 1.1. Let $\pi : \tilde{C} \rightarrow C$ be a double covering which has a branch point $P$ on $C$. We set $\pi^{-1}(P) = \{ \tilde{P} \}$. Assume that there exist $T_P \in |L|$, a point $Q \in C$ with $Q \neq P$, and $T_Q \in |L|$ such that $T_P|_C = (d - 1)P + Q$ and $T_Q|_C = dQ$, and let $M_d$ be the following condition.

$M_d$: There exists a double covering $\tilde{\pi} : X \rightarrow \Sigma_1$ branched along a reduced divisor belonging to $|-2K\Sigma_1|$ such that $\tilde{C} \subset X$ and $\tilde{\pi}|_{\tilde{C}} = \pi$. Then we get the following results.

(a) If $Q \in E$, then the following conditions are equivalent.

(i) $\pi$ satisfies $M_d$.

(ii) $H(\tilde{P}) = 2H(P) + (8d - 9)N_0 + \cdots + (8d - 9 + 2r(d - 2))N_0$

+ $\cdots + (8d - 9 + 2(d - 3)(d - 2))N_0$.

(b) If $P \in E$, then we get the following assertion.

(i) If $\pi$ satisfies $M_d$, then $H(\tilde{P}) = 2H(P) + (8d - 11)N_0$

+ $\cdots + (8d - 11 + 2r(d - 2))N_0 + \cdots + (8d - 11 + 2(d - 4)(d - 2))N_0$.

(ii) Assume that $Q$ is a branch point of $\pi$, and let $\tilde{Q}$ be the ramification point over $Q$. If $2d^2 - 3 \notin H(\tilde{P})$ and $6d - 1 \in H(\tilde{Q})$, then $\pi$ satisfies $M_d$.

(c) If $P \notin E$, $Q \notin E$, and $\pi$ satisfies $M_d$, then $H(\tilde{P}) = 2H(P) + (8d - 9)N_0$

+ $\cdots + (8d - 9 + 2r(d - 2))N_0 + \cdots + (8d - 9 + 2(d - 4)(d - 2))N_0 + (2d^2 - 1)N_0$.

In Theorem 1.2(b), if $H(\tilde{P})$ is the Weierstrass semigroup as in (i), then $2d^2 - 3 \notin H(\tilde{P})$. This means that the assertion of (ii) gives the converse assertion of (i), under the condition that $H(\tilde{Q}) = 2H(Q) + (6d - 1)N_0 + (2d^2 + 1)N_0$.

**Notations and Conventions** A curve and a surface are smooth and projective. For a curve or a surface $Y$, we denote a canonical divisor of $Y$ by $K_Y$. If $C$ is a curve on a surface $Y$, by the adjunction formula, $K_C = (K_Y + C)|_C$. For a curve $C$, we denote the genus of $C$ by $g(C)$. For a divisor $D$ on a curve or a surface, we denote by $|D|$ the linear system defined by it. If two divisors $D_1$ and $D_2$ belong to the same linear system, we will write $D_1 \sim D_2$. For two divisors $D_1$ and $D_2$ on a surface $Y$ and $R \in \text{Supp } D_1 \cap \text{Supp } D_2$, we denote by $I_R(D_1 \cap D_2)$ the intersection multiplicity at $R$ of $D_1$ and $D_2$. We call the minimum degree of pencils on a curve $C$ the **gonality** of $C$.

A submonoid $H \subset \mathbb{N}_0$ is called a **numerical semigroup** if the set $\mathbb{N}_0 \setminus H$ is a finite set. The **genus** of a numerical semigroup $H$ is defined by the cardinality of $\mathbb{N}_0 \setminus H$, and it is denoted by $g(H)$. For a numerical semigroup $\tilde{H}$, we set

$$d_2(\tilde{H}) := \left\{ \frac{h}{2} \mid h \text{ is even and } h \in \tilde{H} \right\},$$

which is a numerical semigroup.
2 Proof of Theorem 1.2

In this section, we will give a proof of Theorem 1.2 and some examples of it. Assume that the notation is as in Theorem 1.2. Then C is the proper transform of a smooth curve of degree $d \geq 4$ by the blow up $\varphi : \Sigma \to \mathbb{P}^2$ at a point on $\mathbb{P}^2$. Moreover, it is well known that $d_2(H(\tilde{P})) = H(P)$ (cf. Torres (1994)). Hence, the results on the computation of the Weierstrass semigroup at a Weierstrass point on a smooth plane curve are often used to compute the Weierstrass semigroup $H(\tilde{P})$. First of all, we recall the following useful result.

Lemma 2.1 [cf. Kang and Kim (2007) Tables 3 and 4]. Let $C$ be a plane curve of degree $d \geq 4$, and let $P \in C$. Then, we have the following results.

(i) If $I_P(C \cap T_P) = d$, then $H(P) = d\mathbb{N}_0 + (d-1)\mathbb{N}_0$.

(ii) If there exists a point $Q \in C$ with $T_P|_C = (d-1)P + Q$ and $I_Q(C \cap T_Q) = d$, then

$$H(P) = (d-1)\mathbb{N}_0 + \cdots + (d-1 + r(d-2)) \mathbb{N}_0 + \cdots + \left(d-1 + (d-2)^2\right) \mathbb{N}_0.$$ 

Here, we denote the tangent line of $C$ at $R$ by $T_R$.

Proof of Theorem 1.2. Let $\tilde{\pi} : X \to \Sigma$ be a double covering of $\Sigma$ as in the condition $M_d$. Since $K_\Sigma \sim -3L + E$ and $\tilde{C}$ is smooth, $C$ intersects transversely the branch locus of $\tilde{\pi}$ at $6d - 2$ distinct smooth points of it. Let $\eta : \tilde{X} \to X$ be a minimal resolution of $X$. Since $\tilde{C}$ is smooth, it does not contain any singular point of $X$. Hence, $\eta^{-1}(\tilde{C}) = \tilde{C}$. Since the exceptional divisor of $\eta$ does not intersect $\eta^{-1}(\tilde{C})$, we have $K_{\tilde{C}} \sim \tilde{C}$. From now on we set $D := (\tilde{\pi} \circ \eta)^*D$, for a divisor $D$ on $\Sigma$. 

(a) (i) $\implies$ (ii) We classify divisors $D$ on $\Sigma$ which are linearly equivalent to $C$ and such that $D|_C$ is effective to investigate the set of gaps $G(\tilde{P})$ at $\tilde{P}$. Let $L_P \in |L|$ be a divisor such that $P \in L_P$ and $L_P \neq T_P$. For $0 \leq s \leq d$ and $0 \leq t \leq \min(d-2, d-s)$, we set $D_1 = sT_P + tL_P + (d-s-t)T_Q - E$. Moreover, we set $D_2 = (d+1)T_P - T_Q - E$. Since $I_P(\tilde{D}_1 \cap \tilde{C}) = 2s(d-1) + 2t$, we have $2s(d-1)+2t+1 \in \mathbb{N}_0 \setminus H(\tilde{P}).$ Since $K_{\tilde{C}} = \tilde{D}_2|_{\tilde{C}} = (2d^2-2)\tilde{P}, 2d^2-1 \in \mathbb{N}_0 \setminus H(\tilde{P}).$ We set $G = \{ h \in \mathbb{N}_0 \setminus H(\tilde{P}) \mid h \text{ is odd} \}. Then we have

$$G = \{ 2s(d-1) + 2t + 1 \mid 0 \leq s \leq d, \ 0 \leq t \leq \min\{d-2, d-s\} \} \cup \left\{ 2d^2 - 1 \right\} .$$

Indeed, by easy computation, the cardinality of the set on the right hand side is $d^2 + 3d - 2$, and it coincides with $g(\tilde{C}) - g(C)$. On the other hand, since $d_2(H(\tilde{P})) = H(P)$, the cardinality of the set of even gaps at $\tilde{P}$ is $g(C)$ which is the genus of $H(P)$. If $2s(d-1) + 2(d-s) + 3 \leq 2(s+1)(d-1) - 1$, then $s \geq 3$. Hence, the minimum odd number of $H(\tilde{P})$ is $8d - 9$. We have $2H(P) + (8d - 9)\mathbb{N}_0 \subset H(\tilde{P})$. Let $n$ be an odd number satisfying $n \in H(\tilde{P}) \setminus (2H(P) + (8d - 9)\mathbb{N}_0)$. Any integer $m \geq 2g(C)$ belongs to $H(P)$, and hence, $8d - 9 + 2m \in 2H(P) + (8d - 9)\mathbb{N}_0$. Therefore, $n \leq 8d - 9 + 2(2g(C) - 1) = 2d^2 + 2d - 7.$
Assume that \(2d^2 + 1 \leq n \leq 2d^2 + 2d - 7\). Then there exists an odd number \(k\) such that \(3 \leq k \leq 2d - 5\) and \(n = 2(d + 2)(d - 1) - k\). We set
\[
l = \frac{2d - 3 - k}{2}\quad\text{and}\quad m = \frac{k - 1}{2}.
\]
Then since \(m \leq (d - 2)l\), there exist integers \(r_1, \ldots, r_l\) such that \(m = \sum_{i=1}^{l} r_i\) and \(0 \leq r_i \leq d - 2\) for each \(1 \leq i \leq l\). Hence, we have
\[n - (8d - 9) = 2l(d - 1) + 2m(d - 2) \in 2H(P) .\]
This is a contradiction.

Assume that \(2d^2 - 2d + 3 \leq n \leq 2d^2 - 3\). Then there exists an odd number \(k\) such that \(1 \leq k \leq 2d - 5\) and \(n = 2(d^2 - 1) - k\). Since
\[n - (8d - 9) = (2d - 5 - k)(d - 1) + (k - 1)(d - 2) \notin 2H(P),\]
we have \(k = 2d - 5\). Hence, we have \(n = 8d - 9 + 2(d - 3)(d - 2) = 2d^2 - 2d + 3\).
Assume that there exists an integer \(s\) satisfying
\[3 \leq s \leq d - 1\quad\text{and}\quad 2s(d - 1) + 2(d - s) + 3 \leq n \leq 2(s + 1)(d - 1) - 1 .\]
Then there exists an odd number \(k\) with \(1 \leq k \leq 2s - 5\) and \(n = 2(s + 1)(d - 1) - k\). Since \(n - (8d - 9) = (2s - 5 - k)(d - 1) + (k - 1)(d - 2) \notin 2H(P)\), we have \(k = 2s - 5\). This means that there exists an integer \(r\) with \(0 \leq r \leq d - 4\) and \(n = 8d - 9 + 2r(d - 2)\). Therefore, we have the assertion.

(ii) \implies (i) \ Let \(B\) be the branch divisor of \(\pi\). Then by the Hurwitz formula, there exists an effective divisor \(D\) of degree \(3d - 1\) on \(C\) with \(B \sim 2D\). We set \(D' = -K_{\Sigma_1}\cdot C - D\). Since \(T_P|_{C} \sim T_Q|_{C}\), we have \((d - 1)P \sim (d - 1)Q\). Since \(C\) is linearly equivalent to \(dT_P - E\) as a divisor on \(\Sigma_1\), if we set \(F = K_C + D' + P\), we have \(F \sim d^2P - D\). Since \(\text{deg}(D') = 0\), we show that \(h^0(O_C(-(D')) > 0\) to prove that \(B \in |-2K_{\Sigma_1}|. \) Assume that \(h^0(O_C(-D')) = 0\). Since \(K_C + F - P = -D\), we have \(h^0(O_C(K_C - F + P)) = h^0(O_C(K_C - F)) = 0\). This means that \(h^0(O_C(d^2P - D)) = h^0(O_C((d^2 - 1)P - D)) + 1\). On the other hand, since \(d^2 \in H(P)\), we have \(h^0(O_C(d^2P)) = h^0(O_C((d^2 - 1)P)) + 1\). Since
\[h^0(O_C(2d^2P)) = h^0(O_C(d^2P)) + h^0(O_C(d^2P - D))\]
and \(h^0(O_C((2d^2 - 2)P)) = h^0(O_C((d^2 - 1)P) + h^0(O_C((d^2 - 1)P - D))\), we have \(h^0(O_C(2d^2P)) = h^0(O_C((2d^2 - 2)P)) + 2\). This contradicts the fact that \(2d^2 - 1 \notin H(P)\). Hence, \(\pi\) satisfies \(M_d\).

(b) (i) By the same way as above, we determine the set of gaps \(G(\tilde{P})\) at \(\tilde{P}\). Let \(L_P\) be as above. For \(0 \leq s \leq d\) and \(0 \leq t \leq \min(d - 2, d - s)\) with \((s, t) \neq (0, 0)\), we set
Let $n \leq \min\{d-2, d-s\}$, and $(s, t) \neq (0, 0) \cup \{2d^2 - 2d + 1, 2d^2 - 3\}$.

If $2s(d-1)+2(d-s)+1 \leq 2(s+1)(d-1)-3$, then $s \geq 3$. Hence, the minimum odd number of $H(\tilde{P})$ is $8d - 11$. Therefore, we have $2H(P) + (8d - 11)N_0 \subset H(\tilde{P})$. Let $n$ be an odd number such that $n \in H(\tilde{P}) \setminus (2H(P) + (8d - 11)N_0)$. Then we have $n \leq 8d - 11 + 2(2g(C) - 1) = 2d^2 + 2d - 9$. We note that there exists an integer $s$ satisfying $3 \leq s \leq d - 1$, and $2s(d-1) + 2(d-s) + 1 \leq n \leq (s+1)(d-1)-3$. In fact, if we assume that $2d^2 - 2d + 3 \leq n \leq 2d^2 - 5$, there exists an odd number $k$ such that $3 \leq k \leq 2d - 5$ and $n = 2(d^2 - 1) - k$. Hence, we have

$$n - (8d - 11) = (2d - 3 - k)(d - 1) + (k - 3)(d - 2) \in 2H(P).$$

This is a contradiction. If we assume that $2d^2 - 1 \leq n \leq 2d^2 + 2d - 9$, then there exists an odd number $k$ such that $5 \leq k \leq 2d - 3$ and $n = 2(d + 2)(d - 1) - k$. Hence, we have

$$n - (8d - 11) = (2d - 1 - k)(d - 1) + (k - 3)(d - 2) \in 2H(P).$$

This is a contradiction. Therefore, there exists an odd number $k$ such that $3 \leq k \leq 2s - 3$ and $n = 2(s+1)(d-1) - k$.

Since $n - (8d - 11) = (2s - 3 - k)(d - 1) + (k - 3)(d - 2) \notin 2H(P)$, we have $k = 2s - 3$. Therefore, there exists an integer $r$ such that $0 \leq r \leq d - 4$ and $n = 8d - 11 + 2r(d - 2)$. Hence, we have the assertion.

(ii) Let $D$ be a divisor with $B \sim 2D$ for the branch divisor $B$ of $\pi$. We set $D' = -K_{\Sigma_1}|C - D - Q$. Since $E \cap C = \{P\}$, by the same reason as above, if we set $F = K_C + D + P$, we have $F \sim (d^2 - 1)P - D$. Then we have $h^0(O_C(-D')) > 0$. Assume that $h^0(O_C(-D')) = 0$. Then $h^0(O_C(K_C - F)) = 0$. This means that $h^0(O_C((d^2 - 1)P - D)) = h^0(O_C((d^2 - 2)P - D)) + 1$. On the other hand, since $d^2 - 1 \in H(P)$, we have $h^0(O_C((d^2 - 1)P)) = h^0(O_C((d^2 - 2)P)) + 1$. Since $h^0(O_C((2d^2 - 2)\tilde{P})) = h^0(O_C((d^2 - 1)P)) + h^0(O_C((d^2 - 1)P - D))$ and $h^0(O_C((2d^2 - 4)\tilde{P})) = h^0(O_C((d^2 - 2)P)) + h^0(O_C((d^2 - 2)P - D))$, we have $h^0(O_C((2d^2 - 2)\tilde{P})) = h^0(O_C((2d^2 - 4)\tilde{P})) + 2$. However, this contradicts the assumption that $2d^2 - 3 \notin H(\tilde{P})$. Since $\deg(-D') = 1$, there exists a point $\tilde{Q}'$ on $C$ belonging to $|-D'|$. Then we have

$$D \sim -K_{\Sigma_1}|C - Q + \tilde{Q}' .$$  (2.1)
On the other hand, since $T_Q|_C = dQ$, by Lemma 2.1 (i), we have $3d \in H(Q)$. Hence, we have $6d \in H(\tilde{Q})$. Moreover, by the assumption that $6d - 1 \in H(\tilde{Q})$, we have $h^0(\mathcal{O}_{\tilde{C}}(6d\tilde{Q})) = h^0(\mathcal{O}_{\tilde{C}}((6d - 2)\tilde{Q})) + 2$. Here, we set $D'' \sim 3dQ - D$. Since $h^0(\mathcal{O}_{\tilde{C}}(6d\tilde{Q})) = h^0(\mathcal{O}_{\tilde{C}}(3d\tilde{Q})) + h^0(\mathcal{O}_{\tilde{C}}(D''))$ and

$$h^0\left(\mathcal{O}_{\tilde{C}}\left((6d - 2)\tilde{Q}\right)\right) = h^0\left(\mathcal{O}_{\tilde{C}}\left((3d - 1)\tilde{Q}\right)\right) + h^0\left(\mathcal{O}_{\tilde{C}}\left(D'' - \tilde{Q}\right)\right),$$

we have

$$h^0\left(\mathcal{O}_{\tilde{C}}\left(D''\right)\right) = h^0\left(\mathcal{O}_{\tilde{C}}\left(D'' - \tilde{Q}\right)\right) + 1 > 0. \quad (2.2)$$

Since $\deg(D'') = 1$, there exists a point $Q'$ on $C$ belonging to $|D''|$. Since, by the linear equivalence (2.1),

$$3dQ - Q'' \sim D \sim (3T_Q - E)|_C - Q + Q' = 3dQ - P - Q + Q',$$

we have $P + Q = Q' + Q''$. Since $C$ is isomorphic to the smooth plane curve $\varphi(C)$ via $\varphi$, the gonality of $C$ is $d - 1$ and the divisor $(T_P - E)|_C = (d - 2)P + Q$ gives a gonality pencil on $C$. Hence, $\dim|P + Q| = 0$. This means that $P + Q = Q' + Q''$. By the equality (2.2), we have $Q'' \neq Q'$, and hence, we have $Q' = Q$ and $Q'' = P$. Therefore, we have $D \sim -K_{\Sigma_1}|_{\tilde{C}}$ and hence, $\pi$ satisfies $M_d$.

(c) We set $C \cap E = \{R\}$. Let $L_{Q,R}, L_{P,R} \in |L|$ be divisors such that $Q, R \in L_{Q,R}$ and $P, R \in L_{P,R}$. For integers $s$ and $t$ satisfying $0 \leq s \leq d, 0 \leq t \leq \min\{d - 2, d - s\}$, and $(s, t) \neq (d, 0)$ we set

$$D_1 = sT_P + tL_{P,R} + (d - s - t)L_{Q,R} - E.$$

Moreover, we set $D_2 = L_{P,R} + dT_P - T_Q - E$ and $D_3 = L_{Q,R} + dT_P - T_Q - E$. Since $I_{\tilde{P}}(\tilde{D}_1 \cap \tilde{C}) = 2s(d - 1) + 2t$, we have $2s(d - 1) + 2t + 1 \in \mathbb{N}_0 \setminus H(\tilde{P})$. Since $I_{\tilde{P}}(\tilde{D}_2 \cap \tilde{C}) = 2d(d - 1) + 2$, we have $2d^2 - 2d + 3 \in \mathbb{N}_0 \setminus H(\tilde{P})$. Since $I_{\tilde{P}}(\tilde{D}_3 \cap \tilde{C}) = 2d(d - 1)$, we have $2d(d - 1) + 1 \in \mathbb{N}_0 \setminus H(\tilde{P})$. We set $G = \{h \in \mathbb{N}_0 \setminus H(\tilde{P}) \mid h \text{ is odd}\}$. Then, by the same reason as above, we have

$$G = \{2s(d - 1) + 2t + 1 \mid 0 \leq s \leq d, \ 0 \leq t \leq \min\{d - 2, d - s\}\} \cup \left\{2d^2 - 2d + 3\right\}.$$

If $2s(d - 1) + 2(d - s) + 3 \leq 2(s + 1)(d - 1) - 1$, then $s \geq 3$. Hence, the minimum odd number of $H(\tilde{P})$ is $8d - 9$. We have $2H(P) + (8d - 9)\mathbb{N}_0 \subset H(\tilde{P})$. Let $n$ be an odd number satisfying $n \in H(\tilde{P}) \setminus (2H(P) + (8d - 9)\mathbb{N}_0)$. Then we have $n \leq 8d - 9 + 2(2g(C) - 1) = 2d^2 + 2d - 7$. By the same reason as in the proof of (a), the case that $2d^2 + 1 \leq n \leq 2d^2 + 2d - 7$ does not occur.

Assume that $2d^2 - 2d + 5 \leq n \leq 2d^2 - 1$. Then there exists an odd number $k$ such that $-1 \leq k \leq 2d - 7$ and $n = (d^2 - 1) - k$. Since

$$n - (8d - 9) = (2d - 5 - k)(d - 1) + (k - 1)(d - 2) \notin 2H(P),$$
we have \( k = -1 \). Hence, we have \( n = 2d^2 - 1 \).

Assume that there exists an integer \( s \) satisfying

\[
3 \leq s \leq d - 1 \text{ and } 2s(d - 1) + 2(d - s) + 3 \leq n \leq 2(s + 1)(d - 1) - 1.
\]

By the same reason as in the proof of (a), there exists an integer \( r \) such that \( 0 \leq r \leq d - 4 \) and \( n = 8d - 9 + 2r(d - 2) \). Therefore, we have the assertion. \( \square \)

We can construct an example for each case as in Theorem 1.2.

**Example 2.1.** Let \( B_0 \subset \mathbb{P}^2 \) be a reduced and irreducible divisor of degree 6 defined by an equation \( f(x, y, z) = 0 \). Assume that \( B_0 \) has a double point \( R \), and \( B_0 \) does not have other singularity. Let \( X_0 \) be the hypersurface in \( \mathbb{P}(1, 1, 1, 3) \) defined by the equation \( w^2 = f(x, y, z) \). Let \( X \) be the blow up at the point on \( X_0 \) corresponding to \( R \). If \( \varphi : \Sigma_1 \rightarrow \mathbb{P}^2 \) is the blow up at \( R \) and \( B \) is the proper transform of \( B_0 \) by \( \varphi \), \( X \) is obtained as the double covering \( \tilde{\pi} : X \rightarrow \Sigma_1 \) of \( \Sigma_1 \) branched along \( B \).

Let \( C_0 \subset \mathbb{P}^2 \) be the plane curve defined by the equation \( yz^{d-1} + x^{d-1}z + y^d = 0 \). Assume that \( R \in C_0 \) and \( C_0 \) intersects transversely \( B_0 \) at other smooth points of \( B_0 \). Let \( C \) be the proper transform of \( C_0 \) by \( \varphi \). Moreover, we set \( \varphi^{-1}(R) = E \). We note that if we let \( L \in |\varphi^*O_{\mathbb{P}^2}(1)| \) be a divisor, then \( C \in |dL - E| \). We set \( \tilde{C} := \tilde{\pi}^{-1}(C) \).

Then \( \tilde{\pi} \) induces the double covering \( \pi : \tilde{C} \rightarrow C \) branched along the divisor \( B \cap C \) on \( C \). We set \( \varphi^{-1}(0 : 0 : 1) \cap C = \{P\} \). Assume that \( P \) is a branch point of \( \pi \) and we set \( \pi^{-1}(P) = \{\tilde{P}\} \).

(a) We consider the case where \( R = (1 : 0 : 0) \). We set \( E \cap C = \{Q\} \). Since \( T_R|C_0 = dR \) and \( T_{\varphi(P)}|C_0 = (d - 1)\varphi(P) + R \), if we set \( T_Q = \varphi^{-1}(T_R) \) and \( T_P = \varphi^{-1}(T_{\varphi(P)}) \), we have \( T_Q|C = dQ \) and \( T_P|C = (d - 1)P + Q \). Here, \( T_R \) and \( T_{\varphi(P)} \) are the tangent lines of \( C_0 \) at \( R \) and \( \varphi(P) \), respectively. Hence, \( H(\tilde{P}) \) coincides with the Weierstrass semigroup as in Theorem 1.2 (a) (ii).

(b) We consider the case where \( R = (0 : 0 : 1) \). Then \( P \in E \). We note that if \( I_R(C_0 \cap B_0) = 3 \), this case can occur. We set \( \varphi^{-1}(1 : 0 : 0) = \{Q\} \). Since \( T_R|C_0 = (d - 1)R + \varphi(Q) \) and \( T_{\varphi(Q)}|C_0 = d\varphi(Q) \), if we set \( T_P = \varphi^{-1}(T_R) \) and \( T_Q = \varphi^{-1}(T_{\varphi(Q)}) \), we have \( T_P|C = (d - 1)P + Q \) and \( T_Q|C = dQ \). Hence, \( H(\tilde{P}) \) coincides with the Weierstrass semigroup as in Theorem 1.2 (b) (i).

(c) We consider the case where \( R \neq (1 : 0 : 0) \) and \( R \neq (0 : 0 : 1) \). If we set \( \varphi^{-1}(1 : 0 : 0) = \{Q\} \), \( T_P = \varphi^{-1}(T_{\varphi(P)}) \), and \( T_Q = \varphi^{-1}(T_{\varphi(Q)}) \), by the same reason as above, \( H(\tilde{P}) \) coincides with the Weierstrass semigroup as in Theorem 1.2 (c).

**Acknowledgements** I would like to thank Jiryo Komeda and the referee for the helpful comments concerning this work.

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