A CALABI OPERATOR FOR Riemannian Locally Symmetric Spaces

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Abstract. On a Riemannian manifold of constant curvature, the Calabi operator is a second order linear differential operator that provides local integrability conditions for the range of the Killing operator. We generalise this operator to provide linear second order local integrability conditions on Riemannian locally symmetric spaces, whenever this is possible. Specifically, we show that this generalised operator always works in the irreducible case and we identify precisely those products for which it fails.

0. Introduction

On any Riemannian manifold, the Killing operator is the linear first order differential operator $K : \wedge^1 \rightarrow \bigwedge^2 \wedge^1$ given by

$$X_a \mapsto \nabla_{(a}X_{b)} ,$$

where $\nabla_a$ is the metric connection and, following Penrose’s abstract index notation as we shall do throughout this article, round brackets on the indices mean to take the symmetric part. With its index raised using the metric, a vector field $X^a$ in the kernel of $K$ is precisely a Killing field. We define the Calabi operator to be a certain linear second order differential operator $C$ acting on $\bigwedge^2 \wedge^1$. Specifically, it is given by

$$h_{ab} \mapsto \nabla_{(a}h_{bd)} - \nabla_{(b}h_{ad)} + \nabla_{(b}h_{dc)} - \nabla_{(d}h_{ac)} - R_{ab}^e c h_{de} - R_{cd}^e a h_{be} ,$$

where $R_{abcd}$ is the Riemann curvature tensor (and square brackets on the indices mean to take the skew part). In [6], Calabi showed that if the Riemannian metric $g_{ab}$ is constant curvature, i.e.

$$R_{abcd} = \text{constant} \times (g_{ac}g_{bd} - g_{bc}g_{ad}) ,$$

then these two operators are the first two in a locally exact complex

$$\wedge^1 \xrightarrow{K} \bigwedge^2 \wedge^1 \xrightarrow{C} \cdots$$

of linear differential operators (the rest of which are first order). Nowadays [11, 13], this may be seen as an instance of the Bernstein-Gelfand-Gelfand complex in flat projective differential geometry (noting that, according to Beltrami’s Theorem [2], a Riemannian manifold is projectively flat if and only if it is constant curvature). In three-dimensional flat space, the operator $C$ was introduced by Saint-Venant in 1864 (as the integrability conditions for a strain in continuum mechanics to arise from a displacement [28]).

Our aim in this article is to construct, from the operator $C$, local integrability conditions for the range of the Killing operator. We choose to start with $C$ because it works in the constant curvature case [6] and because, as observed by Gasqui and Goldschmidt [17], it has the right symbol potentially to give second order constraints. We remark right away, however, that if we allow higher order constraints, then Gasqui and Goldschmidt [17] have

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already found a third order operator \cite{17} that provides local integrability conditions on the range of $\mathcal{K}$ for all Riemannian locally symmetric spaces (our methods easily reproduce \cite{17}, Théorème 7.2), as we briefly discuss in §6.5. We should also remark that we have not been able to find an elementary geometric origin for the operator $\mathcal{C}$, such as deformation of Riemannian curvature (and a detailed discussion of this point may be found in §6.4). In this article, $\mathcal{C}$ arises naïvely by prolonging the Killing equation, as discussed in \cite{11}. A more sophisticated origin for $\mathcal{C}$ is via parabolic geometry \cite{8} and, in particular, the second BGG operators of Hammerl, Somberg, Souček, and Šilhan \cite{20, §4} (see §6.6 for a brief discussion). The range of $\mathcal{K}$ may be interpreted as a gauge freedom in general relativity (as in \cite{27, (5.7.11)} and \cite{30, (C.2.17)}). This provides one motivation for our study. On a compact Riemannian manifold, this interpretation of the range of $\mathcal{K}$ is discussed in \cite[§12.21]{3} (and is used there in conjunction with the Ebin Slice Theorem).

The principal ingredient in our approach is the prolongation connection of \cite{11}. This was used already by Kostant \cite{24} and Geroch \cite[(B.2)]{19} in similar contexts and is part of the general theory in \cite{20}. Apart from algebraic considerations, such as arising from the Lie triple systems \cite{10} of locally symmetric spaces, our methods use only the general theory of connections and their curvature. As such, they are more widely applicable and we pursue a general theory in a follow-up article \cite{10}.

Only for constant curvature is it the case that $\mathcal{C} \circ \mathcal{K} = 0$. Nevertheless, in the locally symmetric case, i.e. $\nabla_a R_{bcde} = 0$, this composition is relatively simple and one can pick out just a part of $\mathcal{C}$, let us call it $\mathcal{L}$, so that $\mathcal{L} \circ \mathcal{K} = 0$. Under mild additional assumptions, we will show that this can be done in such a way that $\mathcal{L}$ provides local integrability conditions for the range of $\mathcal{K}$. More precisely, notice that the right hand side of \cite{11} is a tensor satisfying Riemann tensor symmetries and, indeed, with Young tableau notation \cite{16}, the Calabi complex on a constant curvature Riemannian manifold starts with

$$\bigwedge^1 = \square \to \square \to \square \to \square \to \cdots.$$  

The Riemann curvature tensor $R_{abcd}$ defines a homomorphism of vector bundles

$$\mathcal{R} : \bigwedge^2 = \square \to \square \quad \text{given by} \quad \mu_{cd} \mapsto 2R_{ab}^{ \ e}_{\ [c\mu\delta]e} + 2R_{cd}^{ \ e}_{\ [a\mu\nu]e},$$

and, in case that $\nabla_a R_{bcde} = 0$, the following diagram commutes.

\begin{center}
\begin{tikzcd}
\bigwedge^1 \ar[r, \mathcal{K}] \ar[d, \partial] & \square \\
\bigwedge^2 \ar[r, \mathcal{R}] & \square \ar[u, \mathcal{C}]
\end{tikzcd}
\end{center}

Therefore, if we form the quotient bundle

$$\bigboxplus \equiv \bigboxplus / \mathcal{R}(\bigwedge^2) \quad \text{(alternatively, take} \quad \bigboxplus \equiv \mathcal{R}(\bigwedge^2 \perp \subset \bigboxplus),$$

and write $\mathcal{L}$ for the composition

$$\square \to \square \to \square,$$

then we obtain a complex of linear differential operators

$$\square \to \square \to \square \to \square.$$

The main result of this article is that, in most cases, this complex is locally exact. The precise statement is as follows.
Theorem 1. Suppose $M$ is a Riemannian locally symmetric space. If we write $M$ as a product of irreducibles

$$M = M_1 \times M_2 \times \cdots \times M_k,$$

then the complex $\mathbf{(6)}$ is locally exact unless $M$ has at least one flat factor and at least one Hermitian factor, in which case $\mathbf{(6)}$ fails to be locally exact. (For example, the complex $\mathbf{(6)}$ is locally exact on $S^2 \times S^2$ or $S^3 \times S^1$ but not on $S^2 \times S^1$.)

In the statement of this theorem, we are supposing some preliminaries concerning the theory of Riemannian locally symmetric spaces, specifically that such spaces locally split as a product of irreducibles (those that split no further). The one-dimensional factors are flat. Otherwise, there is a rough divide into Hermitian and non-Hermitian types, where Hermitian (as typified by complex projective space with its Fubini-Study metric) is characterised by the existence of a non-zero 2-form $\omega_{bc}$ with $\nabla^a \omega_{bc} = 0$. Moreover, these irreducible factors are Einstein (with non-zero Einstein constant). Both observations are due to the following elementary fact.

Lemma 1. Any parallel symmetric 2-tensor $h_{ab}$ on an irreducible locally symmetric space is a constant multiple of the metric.

Proof. If $h_{ab}$ is parallel and symmetric, the corresponding endomorphism $h^b_c \equiv g^{ab} h_{ac}$ is also parallel and is diagonalisable over the reals with constant eigenvalues. If there were two distinct eigenvalues, the corresponding eigendistributions would be parallel, and this already contradicts irreducibility. □

Since the Ricci tensor of a locally symmetric space is parallel, this lemma implies that the metric is Einstein. Moreover, it is well-known that the Einstein constant has to be non-zero (for completeness, we prove this at the end of §4).

If the locally symmetric space admits a non-zero parallel 2-form $\omega_{ab}$, then $\omega_{ac} \omega^c_b$ is symmetric and parallel, hence a constant multiple of the metric. As $\omega_{ab}$ is skew, the corresponding endomorphism cannot have real eigenvalues and therefore the constant multiple has to be negative, in which case the metric is obliged to be Kähler with $\omega_{ab}$ a constant multiple of the Kähler form $J_{ab}$. Helgason’s classic text [21] provides further details on locally symmetric spaces and, indeed, a complete classification (due to Cartan). We shall not need this classification.

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1. SOME GENERALITIES AND THE PROLONGATION CONNECTION

To begin, we need only assume that $g_{ab}$ is semi-Riemannian and locally symmetric, meaning that $\nabla_a R_{bcde} = 0$, where $\nabla_a$ is the Levi-Civita connection associated to $g_{ab}$ and $R_{abc}^d$ is the curvature tensor of $\nabla_a$, characterised by

$$(\nabla_a \nabla_b - \nabla_b \nabla_a)X^c = R_{abc}^d X^d.$$

For a general semi-Riemannian metric, it is easy to check that

$$h_{bc} = \nabla_b \sigma_c \iff \begin{bmatrix} h_{bc} \\ 2\nabla_c h_{db} \end{bmatrix} = \begin{bmatrix} \nabla_b \sigma_c - \mu_{bc} \\ \nabla_b \mu_{cd} - R_{cd}^e \sigma_e \end{bmatrix}, \text{ for some } \mu_{bc} = \mu_{[bc]}.$$
and we are, therefore, led to the prolongation connection:

\[ E \equiv \bigwedge^1 \oplus \bigwedge^2 \ni \left[ \begin{array}{c} \sigma_c \\ \mu_{cd} \end{array} \right] \xrightarrow{\nabla_b} \left[ \begin{array}{c} \nabla_b \sigma_c - \mu_{bc} \\ \nabla_b \mu_{cd} - R_{cde} \sigma_e \end{array} \right] \in \bigwedge^1 \otimes E \]

with curvature, in the locally symmetric case, given by

\[ (\nabla_a \nabla_b - \nabla_b \nabla_a) \left[ \begin{array}{c} \sigma_c \\ \mu_{cd} \end{array} \right] = \left[ \begin{array}{c} 0 \\ 2R_{ab}^{\ e}_{\ [cd]e} + 2R_{cde}^{\ f}_a R_{b]d}^{\ e} f_g \end{array} \right], \]

Notice that we are led to the homomorphism \( R : \bigwedge^2 \to \bigwedge^2 \subset \bigwedge^2 \otimes \bigwedge^2 \), appearing in the construction \( \mathcal{R} \) of the operator \( \mathcal{L} \) in \( \text{§8} \). We remark that the terminology ‘prolongation connection’ comes from ‘prolonging’ the Killing equation: see Proposition \( \text{§3} \).

By differentiating the condition \( \nabla_a R_{bc}^{\ d} e = 0 \), we find that

\[ R_{ab}^{\ e}_{\ [cd]e} f_g + R_{cde}^{\ f}_a R_{b]d}^{\ e} f_g = 0. \]

Indeed, a tensor with Riemann tensor symmetries \((R_{abcd} = R_{[ab][cd]} \text{ and } R_{[abc]d} = 0) \) and satisfying \( \text{(1)} \) is called a Lie triple system \( \text{[15, 21]} \). In any case, if we write \( K \subseteq \bigwedge^2 \) for the kernel of \( \mathcal{R} \), then \( R_{abcd} \) is actually a section of \( K \cap K \subseteq \bigwedge^2 \otimes \bigwedge^2 \). (Although we shall not need the general theory of locally symmetric spaces, we remark that the bundle \( K \) is an implicit feature of this theory \( \text{[15]} \). Specifically, the Lie algebra \( g \) of germs of Killing fields at a chosen basepoint \( p \), admits a Cartan decomposition \( \text{[15, §4]} \)

\[ g = \mathfrak{k} \oplus \mathfrak{p}, \]

where \( \mathfrak{k} \) denotes the subalgebra of Killing fields that vanish at \( p \). As a vector space, we may identify \( \mathfrak{p} \) with \( T_p M \). The Lie bracket \( [\mathfrak{k}, \mathfrak{p}] \subset \mathfrak{p} \) preserves the metric \( g_{ab} \) on \( T_p M \) and we therefore obtain a Lie algebra homomorphism \( \mathfrak{k} \to \bigwedge^2 \mathfrak{p}^* \), which turns out to be injective. The Cartan decomposition is sufficient to compute the Riemann curvature tensor \( \text{[15, Theorem 3]} \) and we find that the image of \( \mathfrak{k} \) in \( \bigwedge^2 \mathfrak{p}^* = \bigwedge^2_p K \) is \( K_p \). Observe that the subbundle \( K \subseteq \bigwedge^2 \) is preserved by the Levi-Civita connection. The formula \( \text{(8)} \) now shows that the subbundle

\[ F \equiv \bigwedge^1 K \subseteq \bigwedge^1 \bigwedge^2 = E \]

is preserved by the prolongation connection and, by construction, this subbundle is flat.

Restating the criterion \( \text{(7)} \) in terms of the prolongation connection, we find that

\[ h_{bc} = \nabla_b (\nabla_c) \Leftrightarrow \left[ \begin{array}{c} h_{bc} \\ 2\nabla_{[c} h_{d]b} \end{array} \right] = \nabla_b \left[ \begin{array}{c} \sigma_c \\ \mu_{cd} \end{array} \right], \quad \text{for some } \mu_{bc} = \mu_{[bc]}. \]

A straightforward calculation shows that, for \( h_{bc} \) symmetric,

\[ \nabla_a \left[ \begin{array}{c} h_{bc} \\ 2\nabla_{[c} h_{d]b} \end{array} \right] - \nabla_b \left[ \begin{array}{c} h_{ac} \\ 2\nabla_{[c} h_{d]a} \end{array} \right] = \left[ \begin{array}{c} 0 \\ (Ch)_{abcd} \end{array} \right], \]

where \( C : \bigwedge^2 \to \bigwedge^2 \subset \bigwedge^2 \otimes \bigwedge^2 \) is the operator \( \text{(1)} \). It is interesting to note, however, that this straightforward calculation gives several alternative formulæ for \( C \), for example

\[ h_{ab} \mapsto 2h_{c} \nabla_{[a} h_{b]d} - \nabla_d h_{[a} h_{b]c} - R_{ab}^{\ e}_{\ [cd]e} \]

(and this one will be useful later). In any case, from \( \text{(9)} \), we arrive at a necessary condition for \( h_{ab} \) to be in the range of the Killing operator, namely that \((Ch)_{abcd} \) be in the range of the homomorphism \( \mathcal{R} \). This is exactly the condition that was built into the operator \( \mathcal{L} \) in \( \text{§8} \) and reformulated there as \( \text{(6)} \) being a complex.
To summarise §1 in the prolongation connection (3) we have found a geometric formulation of the necessary condition to be in the range of the Killing operator, afforded by commutativity of the diagram (4). This reasoning pertains for any semi-Riemannian locally symmetric metric but now we ask whether this condition is locally sufficient, and to answer this question we shall now restrict attention to the Riemannian case.

2. The Riemannian locally symmetric case

We set $C ≡ K^⊥ ⊆ \mathcal{F}^2$ so that we have an orthogonal decomposition $\mathcal{F}^2 = K \oplus C$ preserved by $\nabla_a$. In particular, if $\mu_{cd} \in \Gamma(C)$, then $R_{ab}\epsilon^{c}_{[e\mu_{d]e}} \in \Gamma(K \otimes C)$. Now, by construction, the homomorphism

$$\mathcal{F}^2 \ni \mu_{cd} \mapsto 2R_{ab}\epsilon^{c}_{[e\mu_{d]e}} + 2R_{cd}\epsilon^{e}_{[a\mu_{b]e}} \in \mathcal{F}^2 \otimes \mathcal{F}^2$$

is injective but, since $2R_{ab}\epsilon^{c}_{[e\mu_{d]e}} \in \Gamma(K \otimes C)$, it follows that

$$C \rightarrow K \otimes C \subseteq \mathcal{F}^2 \otimes C \text{ given by } \mu_{cd} \mapsto 2R_{ab}\epsilon^{c}_{[e\mu_{d]e}}$$

is injective. We may feed these observations back into (7) in an attempt to find the range of the Killing operator. Looking back at (3), we may use the decomposition

$$E = \mathcal{F}^1 \oplus K \oplus C \subseteq \mathcal{F}^1 \otimes E$$

with curvature given by

$$\nabla_a \nabla_b \nabla_b - \nabla_b \nabla_a \mathcal{F}^2$$

$$\begin{bmatrix} \sigma_c \\ \lambda_{cd} \\ \theta_{cd} \end{bmatrix} \rightarrow \begin{bmatrix} \nabla_b \sigma_c - \lambda_{bc} - \theta_{bc} \\ \nabla_b \lambda_{cd} - R_{cd}\epsilon^{e}_{b\sigma_e} \\ \nabla_b \theta_{cd} \end{bmatrix} \in \mathcal{F}^1 \otimes E$$

If we write

$$\begin{bmatrix} h_{bc} \\ 2\nabla_c h_{db} \end{bmatrix} \in \mathcal{F}^1 \otimes E \text{ as } \begin{bmatrix} \sigma_{bc} \\ \lambda_{bcd} \\ \theta_{bcd} \end{bmatrix} \in \mathcal{F}^1 \otimes \mathcal{F}^1$$

then, from (14) and (15), we see that $(Ch)_{abcd}$ being in the range of the homomorphism (3) implies that

$$\nabla_a \theta_{bcd} - \nabla_b \theta_{acd} = 2R_{ab}\epsilon^{c}_{[e\mu_{d]e}}$$

for some $\theta_{cd} \in \Gamma(C) \subseteq \Gamma(\mathcal{F}^2)$. We have already noted that (13) is injective. Therefore, the section $\theta_{cd} \in \Gamma(C)$ is uniquely determined and, if $\theta_{bcd} \in \mathcal{F}^2 \otimes C$ is to be in the range of the Levi-Civita connection $C \rightarrow \mathcal{F}^1 \otimes C$ (as is necessary from (14)), then the only possibility is that $\theta_{bcd} = \nabla_b \theta_{cd}$. Assuming this to be the case, we may now consider

$$\begin{bmatrix} \sigma_{bc} \\ \lambda_{bcd} \\ \theta_{bcd} \end{bmatrix} - \nabla_b \begin{bmatrix} 0 \\ 0 \\ \theta_{cd} \end{bmatrix} = \begin{bmatrix} \sigma_{bc} + \theta_{bc} \\ \lambda_{bcd} \\ 0 \end{bmatrix}.$$
Lemma 2. Suppose \((18)\) and \(\co \) coincides with \(R\). Rescale the metric so that \(\lambda < \) is vacuously satisfied and we have recovered Calabi’s result that (6) is locally exact for any signature, the homomorphism (3) vanishes. Consequently, the subbundle \(\mathcal{F}\) of any dimension or, indeed, any constant curvature metric is flat, the sequence

\[
\nabla \theta_{cde} - \nabla_e \theta_{bde} = R_{bc} f d \theta_{ef} - R_{bc} f e \theta_{df}
\]

and apply \(\nabla_a\), firstly noting that

\[
-\nabla_a \nabla_e \theta_{bde} = -\nabla_e \nabla_a \theta_{bde} + R_a c f d \theta_{jde} + R_a c f d \theta_{bef} + R_a c f e \theta_{bfj};
\]

to conclude that

\[
\nabla_a \nabla_e \theta_{bde} - \nabla_e \nabla_a \theta_{bde} + R_a c f d \theta_{jde} = R_{bc} f d \nabla_a \theta_{ef} + R_a c f d \theta_{bef} - R_a c f e \nabla_d \theta_{jef} - R_a c f e \theta_{bdf}.
\]

Skewing this equation over \(ab\) and using (16) to substitute for \(\nabla_c \nabla_{[a} \theta_{j]de}\) leads to

\[
R_{ab} f d X_{ce} f + R_{ca} f d X_{be} f + R_{bc} f d X_{ae} f - R_{ab} f e X_{def} - R_{ca} f e X_{bdf} - R_{bc} f e X_{aef} = 0,
\]

where \(X_{ce} \equiv \theta_{ce} f - \nabla_c \theta_{e} f\). According to Proposition 1, we have proved the following.

Proposition 2. Suppose \(M\) is a Riemannian locally symmetric space. In order to show that the complex (3) is locally exact, it suffices to show that if \(X_{bde} \in \Gamma(\Lambda^1 \otimes C)\) satisfies (17), then \(X_{bde} = 0\).

We are now in a position to prove Theorem 1 in case that \(M\) is irreducible.

3. The irreducible case

Although (17) is a seemingly powerful equation in \(\Lambda^3 \otimes \Lambda^2\), it is difficult to use in this form and, instead, we shall use only its trace, which yields an equation in \(\Lambda^2 \otimes \Lambda^1\), namely (bearing in mind that \(R_{abcd} \in \Gamma(K \otimes K)\) and that \(C\) is orthogonal to \(K\))

\[
R_{b} d X_{ate} - R_{a} d X_{bde} = R_{ab} ^{c}d X_{cde} - R_{ab} ^{e}e X_{col},
\]

where \(R_{b} ^{d} \equiv R_{ab} ^{ad}\) is the Ricci tensor. In the irreducible case, the metric is Einstein, i.e. \(R_{ab} = \lambda g_{ab}\) for some constant \(\lambda\). There are three cases according to the sign of \(\lambda\):

- \(\lambda > 0\) and \(M\) is said to be of compact type,
- \(\lambda < 0\) and \(M\) is said to be non-compact,
- \(\lambda = 0\), in which case \(M\) is flat (and one-dimensional).

For flat Euclidean space \(\mathbb{R}^p\) of any dimension or, indeed, any constant curvature metric of any signature, the homomorphism (3) vanishes. Consequently, the subbundle \(K \subseteq \Lambda^2\) coincides with \(\Lambda^2\) and \(C = K^{\perp}\) vanishes. In this case, the criterion of Proposition 2 is vacuously satisfied and we have recovered Calabi’s result that (6) is locally exact for constant curvature metrics. Hence, we may suppose \(\lambda \neq 0\) and, without loss of generality, rescale the metric so that \(R_{ab} = g_{ab}\) for compact type or \(R_{ab} = -g_{ab}\) for non-compact.

The following Lemma now suffices to establish Theorem 1 for irreducible \(M\).

Lemma 2. Suppose \(M\) is irreducible and \(X_{abc}\) is a section of \(\Lambda^1 \otimes C\). If \(X_{abc}\) satisfies (18) and \(R_{ab} = \pm g_{ab}\), then \(X_{abc} = 0\).
Proof. Suppose that $M$ is of compact type, i.e. $R_{ab} = g_{ab}$. If we set

$$Y_{abc} \equiv X_{[abc]},$$

then (18) now reads

$$2Y_{abe} = R_{ab}^{cd} X_{cde} - R_{ab}^{d} e X^{e} cd$$

and tracing over $be$ gives

$$2Y_{ab}^{b} = R_{a}^{bed} X_{cdb} - X^{c} _{ca}.$$

On the other hand, tracing $Y_{abc} = X_{[abc]}$ gives

$$2Y_{ab}^{b} = X^{b} _{ba}.$$

It follows that

$$4X^{c} _{ca} = 2R_{a}^{bed} X_{cdb} - R_{a}^{cd} X_{cdb},$$

where this last equality follows from the Bianchi symmetry $R_{a}^{[bcd]} = 0$. However, recall that (10) implies $R_{abcd} \in \Gamma(K \otimes K)$ whilst $X_{abc} \in \Gamma(\land^{1} \otimes C)$. Therefore $R_{a}^{cd} X_{cdb} = 0$. It follows that $X^{c} _{ca} = 0$ and, since $R_{ab}^{cd} X_{cde} = R_{ab}^{cd} Y_{cde}$, equation (18) now yields

$$R_{ab}^{cd} Y_{cde} = 2Y_{abe}.$$

We are, therefore, led to the operator

$$R : \land^{2} \to \land^{2}$$

given by $\omega_{ab} \mapsto R_{ab}^{cd} \omega_{cd}$

and its eigenvalues. The interchange symmetry for $R_{abcd}$ says that $R$ is symmetric and is, therefore, orthogonally diagonalisable. In the following section we shall show that, for locally symmetric irreducible compact type, normalised so that $R_{ab} = g_{ab}$, we have

$$R_{ab}^{cd} \omega_{cd} = \lambda \omega_{ab} \text{ for } \omega_{ab} \neq 0 \Rightarrow 0 \leq \lambda \leq 2$$

with $\lambda = 2$ only in the Hermitian case. Moreover, the 2-eigenspace in the Hermitian case is spanned by $J_{ab}$, the Kähler form. In particular, equation (20) implies that either $Y_{abc} = 0$ in the non-Hermitian case, or $Y_{abc} = J_{ab} \phi_{c}$ for some 1-form $\phi_{c}$ in the Hermitian case. But now, in the Hermitian case,

$$0 = Y_{ab}^{b} = J_{ab} \phi^{b}$$

and $J_{ab}$ being nondegenerate implies that $\phi_{c} = 0$. Thus, in all cases, we conclude that $Y_{abc} = 0$ and, therefore, that $X_{abc} = 0$, as required.

For non-compact type we may normalise the Ricci tensor $R_{ab} = -g_{ab}$, equation (20) is replaced by

$$R_{ab}^{cd} Y_{cde} = -2Y_{abe},$$

and $J_{ab}$ being nondegenerate implies that $\phi_{c} = 0$. Thus, in all cases, we conclude that $Y_{abc} = 0$ and, therefore, that $X_{abc} = 0$, as required.

4. Eigenvalues of the curvature operator

In this section we shall establish the result about the eigenvalues of the Riemann curvature as an operator on 2-forms that was needed in the proof of Lemma 2. This result may also be gleaned from [18, Table (39)] (cf. [4, 5, 25]). However, since we use a different normalisation for the metric and have a straightforward proof available, we present it here.
Theorem 2. Suppose that $M$ is an irreducible Riemannian locally symmetric space and consider the endomorphism (21). If $M$ is compact type, with Ricci tensor normalised so that $R_{ab} = g_{ab}$, then the eigenvalues of this endomorphism lie in the interval $[0, 2]$. Regarding the end points of this interval,

- $R_{ab}^{cd} \omega_{cd} = 0 \iff \omega_{ab} \in \Gamma(C)$;
- $R_{ab}^{cd} \omega_{cd} = 2 \omega_{ab}$ for $\omega_{ab} \neq 0 \iff M$ is Hermitian and $\omega_{ab}$ is a smooth multiple of the Kähler form, i.e. $\omega_{ab} = \lambda J_{ab}$ for some smooth function $\lambda$.

If $M$ is non-compact type, with Ricci tensor normalised so that $R_{ab} = -g_{ab}$, then the eigenvalues of (21) lie in the interval $[-2, 0]$ and

- $R_{ab}^{cd} \omega_{cd} = 0 \iff \omega_{ab} \in \Gamma(C)$;
- $R_{ab}^{cd} \omega_{cd} = -2 \omega_{ab}$ for $\omega_{ab} \neq 0 \iff M$ is Hermitian and $\omega_{ab}$ is a smooth multiple of the Kähler form, i.e. $\omega_{ab} = \lambda J_{ab}$ for some smooth function $\lambda$.

Proof. Recall from (10) that $R_{abcd}$ is a section of $K \circ K \subseteq \wedge^2 \circ \wedge^2$. Therefore, as an endomorphism of $\wedge^2$, it is self-adjoint and annihilates $C \equiv K^\perp$. This endomorphism therefore preserves the orthogonal decomposition $\wedge^2 = K \oplus C$. Thus, we may restrict $\omega_{ab} \mapsto R_{ab}^{cd} \omega_{cd}$ to $K \subseteq \Lambda^2$ and, supposing that $R_{ab} = g_{ab}$, we are required to show that its eigenvalues (necessarily real) lie in the interval $(0, 2]$ and to prove the stated consequences of having an eigenvalue equal to 2. Continuing to suppose that $R_{ab} = g_{ab}$, tracing (10) over the indices $dg$ yields

\begin{equation}
R_{abcd} = \frac{1}{2} R_{ab}^{ef} R_{cdef} + R_{a}^{ef} c R_{b}^{ef} d - R_{b}^{ef} c R_{ae}^{ef} .
\end{equation}

Therefore, if $R_{ab}^{cd} \omega_{cd} = \lambda \omega_{ab}$, then

\begin{equation}
\lambda \omega_{ab} = \frac{1}{2} \lambda^2 \omega_{ab} + 2 R_{a}^{ef} c R_{be}^{fd} \omega_{cd},
\end{equation}

whence

\begin{equation}
(\lambda - \frac{1}{2} \lambda^2) \omega_{ab} = 2 \omega_{ab}^2 R_{a}^{ef} c R_{be}^{fd} \omega_{cd} = 2 \omega_{a}^{[b} R_{a}^{ef} c R_{be}^{[f d} \omega_{c]d}.
\end{equation}

Recall that we are restricting to the case where $\omega_{ab}$ lies in $K$, which, by definition, implies that

\begin{equation}
R_{be}^{[f d} \omega_{c]d} = - R_{fe}^{[b d} \omega_{c]d}.
\end{equation}

and we conclude that

\begin{equation}
(\lambda - \frac{1}{2} \lambda^2) \| \omega_{ab} \|^2 = (\lambda - \frac{1}{2} \lambda^2) \omega_{ab} \| \omega_{ab} \| = - 2 \omega_{a}^{[b} R_{a}^{ef} c R_{be}^{[f d} \omega_{c]d} = 2 \| R_{fe}^{[b d} \omega_{c]d} \|^2.
\end{equation}

It follows that $\lambda - \frac{1}{2} \lambda^2 \geq 0$, in other words that $\lambda \in [0, 2]$. Furthermore, if $\lambda = 0$ or $\lambda = 2$, then $R_{a}^{ef} c \omega_{d} = 0$. Tracing this equation over $bc$ gives

\begin{equation}
\omega_{ab} = R_{a}^{ef} c \omega_{de} = R_{a}^{be} d \omega_{be} = \frac{1}{2} R_{ad}^{be} \omega_{be}, \quad (\text{forcing } \lambda = 2),
\end{equation}

this last equality by the Bianchi symmetry in the form $R_{a[be]} = \frac{1}{2} R_{dae}$. Hence, the possibility that $\lambda = 0$ for $\omega_{ab}$ a section of $K$ is eliminated. We are left with $\lambda \in (0, 2]$, with $\lambda = 2$ implying that $\omega_{ab}$ is a section of the bundle

\begin{equation}
\{ X_{ab} \in \wedge^2 \mid R_{ab}^{ef} [c X_{d]} = 0 \},
\end{equation}

which, by local symmetry, is parallel and manifestly flat. In this case, therefore, this bundle admits a non-zero parallel section, a 2-form $\omega_{ab}$. As noted just after Lemma 1, this forces $M$ to be Hermitian and $\omega_{ab}$ to be a constant multiple of $J_{ab}$, the Kähler form. We conclude the bundle (23) is rank one, and that $\omega_{ab}$ itself is a smooth multiple of $J_{ab}$.

For irreducible non-compact type normalised so that $R_{ab} = -g_{ab}$, the same argument applies mutatis mutandis. Details are left to the reader.
Remark. A simple variation on this argument proves the classical result that, if a locally symmetric Riemannian metric is Ricci-flat, then it is flat. Specifically, tracing (10) over the indices $dg$ in this case yields

$$0 = \frac{1}{2} R_{ab} e^f R_{cdef} + R_a e^f R_{befd} - R_b e^f R_{acfd}.$$ 

Now, if $\omega_{ab}$ is a section of $K$, it follows that

$$0 = \frac{1}{2} \| R_{abcd} \omega_{cd} \|^2 + 2 \| R_{e[a} d^b \omega_{e]d} \|^2.$$ 

and, therefore, that $R_{ab} \omega_{cd} = 0$. In particular, since $R_{abcd}$ is a section of $K \otimes K$, we conclude that $\| R_{abcd} \|^2 = R_{ab} \omega_{cd} = 0$ so $R_{abcd} = 0$, as required.

5. Products

We may now modify our proof of Lemma 2 and hence of Theorem 1 in the irreducible case, so that it applies to a product

$$M = M_1 \times M_2 \times \cdots \times M_k,$$

of irreducible Riemannian locally symmetric spaces, none of which is flat. Since the Ricci curvature of such a product is block diagonal, with each block being the Ricci curvature of an individual factor, a simple way of saying that there are no flat factors is to say that the Ricci tensor $R_{ab}$ of the whole is nondegenerate. Thus, according to Proposition 2, in order to prove Theorem 1 in this case, it suffices to establish the following.

Lemma 3. Suppose $X_{abc}$ is a section of $\wedge^1 \otimes C$ and satisfies (18). If $R_{ab}$ is nondegenerate, then $X_{abc} = 0$.

Proof. If we set $Y_{abc} \equiv R_{[a} d X_{b]cd}$, then (18) reads

$$2 Y_{abc} = R_{ab} d X_{cde} - R_{ab} e X_{cd}.$$ 

Notice that $Y_{abc} \equiv R_{[a} d X_{b]cd} \Rightarrow 2 Y_{ab} b = R_{a} d X_{cd}$ whereas (21) implies

$$2 Y_{ab} b = R_{a} b X_{cde} - R_{a} d X_{cd}.$$ 

As in our proof of Lemma 2, it follows that $R_{a} d X_{cd} = 0$ and, therefore, since $R_{ab}$ is nonsingular, that $X_{cd} = 0$. Equation (24) now reads

$$2 Y_{abc} = R_{ab} d X_{cde}.$$ 

Let us write $S_{ab}$ for the inverse of $R_{ab}$ and define

$$S_{abcd} \equiv R_{abc} e S_{de}.$$ 

Tracing (10) over $dg$ shows that $S_{abcd}$ satisfies Riemann tensor symmetries

$$S_{abcd} = S_{[ab][cd]} \quad \text{and} \quad S_{[abc]d} = 0$$

and (25) becomes

$$S_{ab} e^d Y_{cde} = 2 Y_{abe}$$

for the symmetric endomorphism $S_{ab} e^d$ of $\wedge^2$. An advantage of $S_{ab} e^d$ over $R_{ab} e^d$, however, is that this tensor does not see a constant rescaling of the metric. In fact, all of the following

$$\nabla_a, \quad R_{ab} c d, \quad R_{ab}, \quad S_{ab}, \quad \text{and} \quad S_{ab} e^d S_{cde}$$

are preserved under $g_{ab} \mapsto \text{constant} \times g_{ab}$. Consequently, for products without flat factors, Theorem 2 now implies, factor-by-factor, that all eigenvalues of $S : \wedge^2 \rightarrow \wedge^2$ given by $\omega_{ab} \mapsto S_{ab} e^d \omega_{cd}$
are real and lie in the range $[0, 2]$. Furthermore, eigenvalue 2 is only attained on Hermitian factors, in which case $\omega_{ab}$ must be a multiple of the Kähler form on each such factor. From (26) and $Y_{ab} = \frac{1}{2} R_{d}^{\ \ a} X_{cd} = 0$ it follows that $Y_{abc} = 0$ and hence that $X_{abc} = 0$. □

The prolongation connection (8) not only controls the range of the Killing operator, as in (7), but also its kernel, as follows.

**Proposition 3.** There is an isomorphism

$$\left\{ \begin{bmatrix} \sigma_b \\ \mu_{bc} \end{bmatrix} \in \Gamma(E) \left| \nabla_{a} \begin{bmatrix} \sigma_b \\ \mu_{bc} \end{bmatrix} = 0 \right. \right\} \cong \{ \sigma_b \in \Gamma(\Lambda^1) \mid \nabla_{(a} \sigma_b) = 0 \}.$$

**Proof.** Clearly,

$$\nabla_{(a} \sigma_b) = 0 \iff \nabla_{a} \sigma_b = \mu_{ab} \text{, for some } \mu_{ab} \in \Gamma(\Lambda^2).$$

Hence, it suffices to show that, if $\nabla_{a} \sigma_b = \mu_{ab}$, then $\nabla_{a} \mu_{bc} = R_{bc}^{\ \ d} \sigma_d$. To see this, observe that

$$\nabla_{a} \mu_{bc} = \nabla_{c} \mu_{ba} - \nabla_{b} \mu_{ca} = \nabla_{c} \nabla_{b} \sigma_a - \nabla_{b} \nabla_{c} \sigma_a = R_{bc}^{\ \ d} \sigma_d,$$

as required. □

This proposition leads us to consider just the second line of the prolongation connection,

$$\nabla_{a} \mu_{bc} = R_{bc}^{\ \ d} \sigma_d,$$

noting that if $R_{ab}$ is non-singular then $\nabla_{c} \mu_{bc} = R_{bc}^{\ \ d} \sigma_d$ shows that $\sigma_b$ is determined by $\mu_{bc}$. We ask if $\sigma_b$ is then necessarily a Killing field.

**Lemma 4.** Suppose $M$ is a Riemannian locally symmetric space with neither Hermitian nor flat factors. If $\mu_{cd}$ is 2-form on $M$ so that

$$\nabla_{b} \mu_{cd} = R_{cd}^{\ e} \sigma_{e}, \text{ for some (uniquely determined) 1-form } \sigma_{e} \text{ on } M,$$

then $\nabla_{b} \sigma_{c} = \mu_{bc}$.

**Proof.** In terms of the prolongation connection, we are given that

$$\nabla_{b} \begin{bmatrix} \sigma_{c} \\ \mu_{cd} \end{bmatrix} = \begin{bmatrix} \phi_{bc} \\ 0 \end{bmatrix}, \text{ for some tensor } \phi_{bc},$$

and from its curvature (9) we deduce that

$$\begin{bmatrix} \nabla_{a} \phi_{bc} \\ -R_{cd}^{\ e} \phi_{[a} \phi_{b]e} \end{bmatrix} = \begin{bmatrix} 0 \\ R_{ab}^{\ c} \mu_{d]e} + R_{cd}^{\ e} \mu_{[a} \phi_{b]e} \end{bmatrix}.$$

In particular, we see that $R_{cd}^{\ e} \phi_{[a} \phi_{b]e}$ is in the range of $\mathcal{R} : \Lambda^2 \rightarrow \Lambda^2 \otimes \Lambda^2$ and, therefore, in the range of $\mathcal{R} : C \rightarrow \Lambda^2 \otimes \Lambda^2$. Hence,

$$R_{cd}^{\ e} \phi_{[a} \phi_{b]e} = R_{cd}^{\ e} \mu_{[a} \phi_{b]e} = R_{ab}^{\ c} \omega_{d]e}, \text{ for some } \omega_{ab} \in \Gamma(C).$$

However, the tensor on the right hand side of this equation lies in $K \otimes C$, whereas the tensor on the left lies in $\Lambda^2 \otimes K$. Therefore, both sides vanish and, in particular, we deduce that

$$R_{cd}^{\ e} \phi_{[a} \phi_{b]e} = 0. \text{ (27)}$$

Tracing this equation over $bc$ gives

$$R_{d}^{\ e} \phi_{ae} = R_{d}^{\ b} \phi_{be}. \text{ (28)}$$
Now recall, as in our proof of Lemma \ref{lem:1}, that if we write $S_{ab}$ for the inverse of $R_{ab}$, then $S_{abcd} \equiv R_{abc}^d S_{de}$ satisfies Riemann tensor symmetries. Thus, applying $S$ to \eqref{eq:29} gives

$$\phi_{af} = S^f_d R_{da}^b \phi_{be} = S^f_d a \phi_{be},$$

which decomposes into symmetric and skew parts

$$\phi_{(af)} = S^f_d a \phi_{(be)} \quad \text{and} \quad \phi_{[af]} = S^f_d a \phi_{[be]} = \frac{1}{2} S^f_d a \phi_{[be]},$$

this final equality from the Bianchi symmetry in the form $S_{[f}^d a] = \frac{1}{2} S_{a}^d b e$. However, as observed in the proof of Lemma \ref{lem:1}, since $M$ has no Hermitian factors, the eigenvalues of $S_{ab}^{cd}$ lie in the interval $[0, 2]$. It follows that $\phi_{ab}$ is symmetric. Let $\psi_{ab} \equiv \phi_a^c \phi_{bc}$. Since $\phi_{ab}$ is symmetric so is $\psi_{ab}$ and, from \eqref{eq:27}, we find that

$$R_{ab} e^f (\psi d)_e = 0.$$  

When the Ricci tensor is nondegenerate, a 1-form annihilated by the curvature necessarily vanishes:

$$R_{ab} d^f \theta_d = 0 \Rightarrow R_{a}^d \theta_d = 0 \Rightarrow \theta_a = 0.$$  

This implies that a symmetric 2-form on a Riemannian product with nondegenerate Ricci tensor and satisfying \eqref{eq:29} can have no ‘cross terms,’ i.e. must be block diagonal. Therefore, without loss of generality, we may suppose for the rest of this proof that $M$ is irreducible and that $R_{ab} = \pm g_{ab}$, as usual. Having done this, equation \eqref{eq:29} says that the natural action of curvature annihilates the symmetric form $\psi_{de}$ and since $R_{ab}^{ec}$ is covariantly constant the same is true for $\nabla_a \psi_{de}$. As in the proof of Theorem \ref{thm:2}, now using Lemma \ref{lem:1} this implies that $\psi_{ab} = \lambda g_{ab}$ for some smooth function $\lambda \geq 0$ and it remains to show that $\lambda \equiv 0$. Let us record our conclusion so far

$$\phi_a^c \phi_{bc} = \lambda g_{ab}$$

and now deal with the case $R_{ab} = g_{ab}$. Equation \eqref{eq:28} becomes $R_{abcd} \phi^{bc} = \phi_{ab}$ and \eqref{eq:22} therefore yields

$$4 \phi_{ad} \phi_{ad} = \phi_{bd} \phi_{ad} R_{a}^{ef} R_{bcde} + 2 \phi_{bd} \phi_{bc} R_{a}^{ef} R_{b}^{ef}_{e} R_{be}^{ef}_{d}.$$  

But \eqref{eq:27} also implies, with \eqref{eq:31}, that

$$\phi_{bd} \phi_{ad} R_{a}^{ef} R_{bcde} = \phi_{bd} \phi_{ad} R_{a}^{ef} R_{bcde} = \lambda \|R_{abcd}\|^2.$$  

We also know that $\phi_{ad} \phi_{ad} = \lambda \delta_{ad} = \lambda n$, where $n$ is the dimension of $M$, so \eqref{eq:32} becomes

$$4 \lambda n = \lambda \|R_{abcd}\|^2 + 2 \phi_{bd} \phi_{bc} R_{a}^{ef} c R_{be}^{ef}_{d}.$$  

To sort out the last term, again we use \eqref{eq:27} to conclude that

$$0 = R^{def}_{a \phi_{bf}} R_{a \phi_{df}} \phi_{d} = R^{def}_{a \phi_{bf}} \phi_{d} = \phi_{ad} \phi_{bc} R_{a}^{ef} c R_{be}^{ef}_{d}$$

and, therefore,

$$2 \phi_{ad} \phi_{bd} R_{a}^{ef} c R_{be}^{ef}_{d} = 2 \phi_{bd} \phi_{bc} R_{a}^{ef} c R_{be}^{ef}_{d} = \phi_{ad} \phi_{bc} R_{a}^{ef} c R_{be}^{ef}_{d} = \phi_{ad} \phi_{bc} R_{a}^{ef} c R_{be}^{ef}_{d},$$

which we have already found to be $\lambda \|R_{abcd}\|^2$. We conclude that $4 \lambda n = 2 \lambda \|R_{abcd}\|^2$ and hence, unless $\lambda \equiv 0$, that $\|R_{abcd}\|^2 = 2 n$. We also know that $R_{ab}^{ab} = \delta_{ab} = n$. In summary, for the endomorphism $R_{ab}^{cd}$ of $\wedge^2$, we have found that

- all eigenvalues are in the range $[0, 2]$;
- the sum of the eigenvalues is $n$;
- the sum of the squares of the eigenvalues is $2n$.

This is a contradiction. The case $R_{ab} = -g_{ab}$ follows in a similar fashion.
To complete the proof Theorem 11 it remains to consider Riemannian locally symmetric products of the form
\[ M \times \mathbb{R}^p, \]
where \( M \) has no flat factors and \( \mathbb{R}^p \) is equipped with its standard metric. We need to show that if \( M \) has no Hermitian factors, then (13) is locally exact but, if \( M \) has a Hermitian factor, then (13) fails to be locally exact.

To approach these final cases, let us simply write out the complex (13) on a Riemannian product \( M \times \mathbb{R}^p \). For this purpose we may borrow some notation from complex geometry and write
\[ \Lambda^1 = \Lambda^{1,0} \oplus \Lambda^{0,1}, \]
where \( \Lambda^{1,0} \) denotes the pull-back to \( M \times \mathbb{R}^p \) of the cotangent bundle on \( M \) and \( \Lambda^{0,1} \) denotes the pull-back of the cotangent bundle on \( \mathbb{R}^p \). We obtain an induced splitting of symmetric forms on \( M \times \mathbb{R}^p \) according to ‘type’:
\[ \bigodot^2 \Lambda^1 = \bigodot^2 \Lambda^{1,0} \oplus \Lambda^{1,1} \oplus \bigodot^2 \Lambda^{0,1}, \quad \text{where } \Lambda^{1,1} = \Lambda^{1,0} \otimes \Lambda^{0,1}. \]

If we also also use the ‘barred and unbarred’ indices from complex geometry, the Killing operator becomes
\[ \begin{bmatrix} X_b \\ \xi_b \end{bmatrix} \mapsto \begin{bmatrix} \nabla_{(a}X_{b)} \\ \nabla_a \xi_a + \partial_a X_a \\ \partial_a \xi_b \end{bmatrix}, \]
where \( \nabla_a \) is the metric connection on \( M \) and \( \partial_a \) is the standard coordinate derivative on \( \mathbb{R}^p \), both pulled back to \( M \times \mathbb{R}^p \) in the obvious way. Notice that \( \nabla_a \partial_a = \partial_b \nabla_a \) acting upon any tensor field. The Calabi operator \( C : \boxtimes \rightarrow \boxplus \) breaks up into irreducibles
\[ \Lambda^{1,0} \rightarrow \Lambda^{1,0} \otimes \Lambda^{0,1} \rightarrow \Lambda^{1,0} \otimes \Lambda^{0,1} \rightarrow \Lambda^{1,0} \otimes \Lambda^{0,1} \rightarrow \Lambda^{1,0} \otimes \Lambda^{0,1} \rightarrow \Lambda^{0,1}, \]
where, for example, \( \boxtimes \Lambda^{1,0} \) denotes the symmetric tensor product functor applied to \( \Lambda^{1,0} \).

In particular, there are four parts to \( C \) applied to \( \Lambda^{1,0} \otimes \Lambda^{0,1} \), specifically
\[ \nabla_c \nabla_{[a}h_{b]}^\gamma - \frac{1}{2} R_{ab}^d c h_d^\gamma \in \Gamma(\boxplus \Lambda^{1,0} \otimes \Lambda^{0,1}), \]
\[ \nabla_{(a} \partial^{(a}h_{b)}^\gamma \in \Gamma(\boxtimes \Lambda^{1,0} \otimes \boxtimes \Lambda^{0,1}), \]
\[ \nabla_{[a} \partial^{[a}h_{b]}^\gamma \in \Gamma(\boxtimes \Lambda^{2,0} \otimes \Lambda^{0,2}), \]
\[ \partial^c \partial^{[a}h_{b]}^\gamma \in \Gamma(\Lambda^{1,0} \otimes \boxplus \Lambda^{0,1}), \]
where we have used (12) to obtain the first operator. Only this first operator is quotiented by \( \mathcal{R} : \Lambda^2 \rightarrow \boxplus \) in passing to the operator \( \mathcal{L} \) in (13), specifically to obtain the compatibility condition
\[ \nabla_c \nabla_{[a}h_{b]}^\gamma = R_{ab}^d c \kappa_{db} \quad \text{for some } \kappa_{db} \in \Gamma(\Lambda^{1,1}). \]

**Proposition 4.** Suppose \( M \) is locally symmetric with nondegenerate Ricci tensor but having a Hermitian factor. Then (13) fails to be locally exact on the Riemannian product \( M \times \mathbb{R}^p \), where \( \mathbb{R}^p \) has the standard flat metric.
Proof. Choose a non-trivial covariantly constant 1-form \( \theta_b \) on \( \mathbb{R}^p \) and on \( M \) let us choose a 2-form \( J_{ab} \) by pulling back the Kähler form from a Hermitian factor. Being closed, we may locally choose a 1-form \( \phi_a \) on \( M \) such that \( \nabla_{[a} \phi_{b]} = J_{ab} \). Let us use the same notation \( \phi_a \) for the pullback of this form to \( M \times \mathbb{R}^p \). Also pull back \( \theta_b \) to the product and consider \( h_{kb} \equiv \phi_a \theta_b \in \Gamma(\wedge^{1,1}) \) as a symmetric 2-form there. As \( \partial_b \theta_b = 0 \), the last three operators from (37) annihilate \( h_{kb} \). Also, as \( \nabla_c J_{ab} = 0 \), the compatibility condition (35) holds with \( \kappa_{kb} = 0 \). But \( h_{kb} \) cannot be written as

\[
\nabla_b \xi_b + \partial_b X_b \quad \text{for} \quad X_b \in \Gamma(\wedge^{1,0}) \quad \text{such that} \quad \nabla_{(a} X_{b)} = 0 \quad \text{and} \quad \xi_b \in \Gamma(\wedge^{0,1})
\]

as would be required by (35) to be in the range of \( K \) since, if this were the case, then

\[
0 = \nabla_{[a} \nabla_b \xi_b = \nabla_{[a} h_{b)b} - \partial_b \nabla_{[a} X_b] \quad \Rightarrow \quad \theta_b J_{ab} = \partial_b \nabla_{[a} X_b] = \partial_b \nabla_{a} X_b.
\]

Then, as \( J_{ab} \) is covariant constant, we now find that

\[
0 = \theta_b \nabla_{[c} J_{a]b} = \partial_b \nabla_{[c} \nabla_{a]} X_b = \frac{1}{2} R_{ac} \nabla_{d} \partial_b X_d
\]

and hence that \( \partial_b X_d = 0 \). It follows that \( J_{ab} = 0 \), a contradiction. \( \square \)

The following proposition completes the proof of Theorem 1.

**Proposition 5.** Suppose \( M \) is a Riemannian locally symmetric space with nondegenerate Ricci tensor and no Hermitian factors. Then (3) is locally exact on the Riemannian product \( M \times \mathbb{R}^p \), where \( \mathbb{R}^p \) has the standard flat metric.

**Proof.** We already know that (3) is locally exact on \( M \) and on \( \mathbb{R}^p \). Therefore, looking at (35), (36), (37), and (33), we are required to show that if \( h_{kb} \in \Gamma(\wedge^{1,1}) \) satisfies

\[
(39) \quad \nabla_c \nabla_{[a} h_{b)b} = R_{ab} \nabla_{[c} \kappa_{a]b}, \quad \nabla_{(a} \partial_{[b} h_{b)} = 0, \quad \nabla_{[a} \partial_{[b} h_{b]} = 0, \quad \text{and} \quad \partial_{[a} \partial_{b]} h_{b]} = 0,
\]

then locally we may find \( X_b \in \Gamma(\wedge^{1,0}) \) and \( \xi_b \in \Gamma(\wedge^{0,1}) \) such that

\[
(40) \quad \nabla_{(a} X_{b)} = 0, \quad \nabla_{a} \xi_{a} + \partial_{a} X_{a} = h_{a\bar{a}}, \quad \text{and} \quad \partial_{a} \xi_{b} = 0.
\]

From the first equation of (39), Lemma 4 tells us that \( \nabla_{a} \kappa_{b} = \nabla_{[a} h_{b]b} \) and the third equation from (39) implies that \( \nabla_{a} \partial_{[b} h_{b]} = 0 \). Therefore \( R_{ab} \nabla_{[c} \kappa_{a]b} = 0 \). From (30), it follows that \( \partial_{[a} \kappa_{b]} = 0 \). Thus, as a closed 1-form along the fibres of \( M \times \mathbb{R}^p \to M \), we may integrate to find \( X_b \) such that \( \partial_b X_b = \kappa_{b} \) and, by differentiating under the integral sign, it follows that \( \nabla_{(a} X_{b)} = 0 \), which is the first requirement of (40). If we introduce \( \psi_{a\bar{a}} \equiv h_{a\bar{a}} - \kappa_{a\bar{a}}, \) then

\[
(41) \quad \nabla_{a} \psi_{b} = \nabla_{a} h_{b} - \nabla_{a} \kappa_{b} = \nabla_{a} h_{b} - \nabla_{[a} h_{b)]b = \nabla_{(a} h_{b) }b,
\]

so

\[
\nabla_{a} \partial_{b} \psi_{b} = \nabla_{(a} \partial_{b} h_{b) } = \nabla_{(a} \partial_{b} h_{b) } = \nabla_{(a} \partial_{b} h_{b)},
\]

this last equality from the second equation of (39). Using, once more, the third equation from (35), we now have

\[
\nabla_{a} \partial_{b} \psi_{b} = \nabla_{(a} \partial_{b} h_{b) } = \nabla_{(a} \partial_{b} h_{b)} = \nabla_{a} \partial_{[b} h_{b]},
\]

and (30) implies that

\[
(42) \quad \partial_{b} \psi_{b} = \partial_{[b} h_{b]}.
\]

Having found \( X_b \) and introduced \( \psi_{a\bar{a}} = h_{a\bar{a}} - \kappa_{a\bar{a}} \), in order to satisfy (40) it remains to find \( \xi_b \), such that

\[
\nabla_{b} \xi_b = \psi_{b\bar{a}} \quad \text{and} \quad \partial_{b} \xi_{b} = 0.
\]
From (41) and (42) it follows that
\[ \nabla_{[a} \psi_{b]} = 0 \quad \text{and} \quad \partial^{[a} \psi_{b]} = 0 \]
so \( \xi_b \) can be obtained by integrating along the fibres of \( M \times \mathbb{R}^p \rightarrow \mathbb{R}^p \).

Notice that, in this proof, we apparently omitted to use the last constraint of (39), namely that \( \partial^a \partial^{[a} h_{b]} = 0 \). In fact, as in the proof of Proposition 3, this constraint follows by prolongation from (42) so nothing is lost by this apparent omission.

6. Concluding remarks

6.1. Some key examples. By design, Theorem 1 includes the round sphere \( S^n \). In this case, the homomorphism \( \mathcal{R} \) in (5) vanishes, whence \( K = \wedge^2 \) and (11) reads
\[ h_{ab} \mapsto (\nabla_{(a} \nabla_{c)} + g_{ac}) h_{bd} - (\nabla_{(b} \nabla_{c)} + g_{bc}) h_{ad} - (\nabla_{(a} \nabla_{d)} + g_{ad}) h_{bc} + (\nabla_{(b} \nabla_{d)} + g_{bd}) h_{ac} \]
on the unit sphere (where \( R_{abcd} = g_{ac}g_{bd} - g_{bc}g_{ad} \)), as may be gleaned from the Lagrangian atlases of [6] or found explicitly as [17, Théorème 6.1].

For a product of round spheres \( S^p \times S^q \), we may decompose bundles according to type as in (55) and it is easy to check that \( K = \wedge^2 \otimes \wedge^0 \) and, therefore, that \( C = \wedge^{1,1} \).

Theorem 1 says that (6) is exact on all these products save for \( S^2 \times S^1 \) (and \( S^1 \times S^2 \)).

For complex projective space \( \mathbb{CP}_n \) with its Fubini-Study metric and Kähler form \( J_{ab} \), we find that
\[ K = \{ \mu_{cd} \mid J_a^c J_b^d \mu_{cd} = \mu_{ab} \} \quad \text{and} \quad C = \{ \mu_{cd} \mid J_a^c J_b^d \mu_{cd} = -\mu_{ab} \}, \]
bundles with familiar complexifications, namely \( \wedge^{1,1} \) and \( \wedge^{2,0} \otimes \wedge^{0,2} \), respectively. As an SU\((n)\)-bundle, the image \( C \hookrightarrow \mathfrak{su} \) is irreducible (it is \( \mathcal{W}_0 \) in the full decomposition of \( \mathfrak{su} \) into 10 irreducible bundles given in [29]). This raises the possibility of removing further SU\((n)\)-subbundles from \( \mathfrak{su} \) whilst keeping local exactness of the resulting complex. One such option has already been considered in [14], specifically the complex
\[
\wedge^1 = \Box \xrightarrow{K} \mathfrak{su} \rightarrow \mathfrak{su}_+, 
\]
where this last bundle comprises Riemann tensors that are trace-free with respect to \( J_{ab} \).

In [12] it is shown that (43) is globally exact on \( \mathbb{CP}_n \), whilst in [14] it is shown that the local cohomology is a constant sheaf with fibre \( \mathfrak{su}(n+1) \). In fact, we have removed three SU\((n)\)-irreducibles to obtain (43):
\[
\mathfrak{su} = \mathfrak{su}_+ \oplus \mathcal{R}(C) \oplus \wedge_{+}^{1,1} \oplus \wedge^0.
\]
Replacing just \( \wedge^0 \) is sufficient to restore local exactness as follows.

**Theorem 3.** On complex projective space with its Fubini-Study metric, the complex
\[
\wedge^1 = \Box \xrightarrow{K} \mathfrak{su} \rightarrow \mathfrak{su}_+ \oplus \wedge^0
\]
is locally exact.

**Proof.** In [14] it is shown that the local kernel of \( \mathfrak{su} \rightarrow \mathfrak{su}_+ \) comprises tensors of the form
\[ h_{ab} = \nabla_{(a} X_{b)} + Y_{(a} \phi_{b)}, \]
where \( \nabla_{(a} Y_{b)} = 0 \) and \( \nabla_{(a} \phi_{b)} = J_{ab} \). Looking ahead to (6.3) for \( \mathfrak{su} \rightarrow \wedge^0 \), we may as well take the differential operator (17) and compute that, in this case,
\[ \nabla_{(a} X_{b)} + Y_{(a} \phi_{b)} \mapsto 6 J^{ab} \nabla_a Y_b. \]
Hence, being in the kernel of this operator forces $J^k \nabla_b Y_c = 0$. From Proposition 3 it follows that, if we set $\mu_{ab} = \nabla_a Y_b$, then $\nabla_a \mu_{bc} = R_{bc} \, d \, Y_d$ and now
\[
0 = \nabla_a (J^b \mu_{bc}) = J^b \nabla_a \mu_{bc} = J^b R_{bc} \, d \, Y_d = 2 J^d \, d \, Y_d.
\]
We conclude that $Y_a = 0$ and the proof is complete. \hfill \Box

6.2. The Riemann curvature operator. Critical to our proof of Theorem 1 was some control of the eigenvalues of the Riemann curvature tensor when viewed as a symmetric endomorphism (21) of the 2-forms. It is worthwhile noting what are these eigenvalues for the sphere and complex projective space. The normalisation $R_{ab} = g_{ab}$ implies

- for $S^n$: $R_{abcd} = \frac{1}{n-1} (g_{ac} g_{bd} - g_{bc} g_{ad})$
- for $\mathbb{C}P_n$: $R_{abcd} = \frac{1}{2(n+1)} (g_{ac} g_{bd} - g_{bc} g_{ad} + J_{ac} J_{bd} - J_{bc} J_{ad} + 2 J_{ab} J_{cd})$,

where $J_{ab}$ is the Kähler form on $\mathbb{C}P_n$. Thus, on $S^n$ there is just one eigenspace, having eigenvalue $2/(n - 1)$. On $\mathbb{C}P_n$, we find

- $\wedge^{2,0} \oplus \wedge^{0,2}$ with eigenvalue 0,
- $\wedge^{1,1}$ with eigenvalue $2/(n + 1)$,
- $\langle J_{ab} \rangle$ with eigenvalue 2.

Complex projective space provides a good illustration of Theorem 2. It is essential that we are in the Riemannian setting in order to conclude that all eigenvalues of $\omega_{ab} \mapsto R_{abcd} \omega_{cd}$ are real. This and several other steps break down in the Lorentzian setting. For example, it is no longer true that a Ricci flat locally symmetric space need be flat. We return to the Lorentzian case in a separate article [10].

6.3. Warped products. The methods in this article also apply to metrics beyond the locally symmetric realm. For example, following Khavkine [22], we may ask about the range of the Killing operator on Friedmann-Lemaitre-Robertson-Walker (FLRW) metrics, namely warped products

\begin{equation}
\Omega^2(t) g_{ab} \pm dt^2 \quad \text{on} \quad M \times \mathbb{R},
\end{equation}

where $\Omega : \mathbb{R} \to \mathbb{R}_{>0}$ is a smooth function and $g_{ab}$ is Riemannian constant curvature. In general, a warped product (44) with the negative sign in front of $dt^2$ and $g_{ab}$ an arbitrary Riemannian metric is known as a Generalised Robertson-Walker (GRW) spacetime [20]. In the spirit of this article, however, we shall restrict to the spatially locally symmetric case, i.e. where the Riemannian metric $g_{ab}$ is locally symmetric.

In two dimensions, for example, let us consider a metric of the form
\begin{equation}
\Omega^2(t) dx^2 + dt^2 \quad \text{on} \quad \mathbb{R} \times \mathbb{R}.
\end{equation}

Splitting $\mathcal{O}^2 \wedge^1$ according to type (31), the Killing operator is
\[
X \, dx + \xi \, dt \mapsto \begin{bmatrix} X_x + \Omega^2 \Upsilon \xi \\ \xi_x + X_t - 2 \Upsilon X \\ \xi_t \end{bmatrix},
\]
where $\Upsilon \equiv \Omega^{-1} \Upsilon_t$. The Khavkine operator is then
\begin{equation}
\begin{bmatrix} p \\ q \\ r \end{bmatrix} \mapsto \begin{bmatrix} J_{xx} - \Omega^2 \Upsilon J_t - \Omega^2 \Upsilon' J - q_x + p_t - 2 \Upsilon p \\ J_t - r \\ J_r \end{bmatrix},
\end{equation}
where, supposing that $\Upsilon'' + 2 \Upsilon \Upsilon' \neq 0$,
\[
J \equiv \frac{pu - 2 \Upsilon p_t - 2 \Upsilon' p - q_x + r_{xx} - \Omega^2 (\Upsilon r_t + 2 \Upsilon' r + \Omega^2 r)}{\Omega^2 (\Upsilon'' + 2 \Upsilon \Upsilon')}.
\]
Proposition 6. The sequence
\[ \wedge^1 \xrightarrow{K} \bigcirc^2 \wedge^1 \xrightarrow{\text{Khavkine}} \wedge^0 \oplus \wedge^0 \]
is a locally exact complex on \( \mathbb{R} \times \mathbb{R} \).

Proof. A computation shows that the composition
\[ \wedge^1 \xrightarrow{K} \bigcirc^2 \wedge^1 \xrightarrow{\mathcal{J}} \wedge^0 \]
sends \( X \, dx + \xi \, dt \) to \( \xi \). Therefore, the first component \( J_t - r \) of the Khavkine operator forces \( r = \xi_t \). This effectively eliminates \( \xi \) from \( X \, dx + \xi \, dt \) and the second component of the Khavkine operator is exactly what is needed as a consequence of
\[ X \, dx \mapsto -\nabla^a \nabla^b X_{ab} - 2\Upsilon X \]
defining a flat connection. \( \square \)

Notice that the integrability operator (46) is of fourth order. The key to Khavkine’s construction is the following observation.

Lemma 5. For a general semi-Riemannian metric, the composition
\[ \wedge^1 \xrightarrow{\sigma} \bigcirc^2 \wedge^1 \xrightarrow{\mathcal{J}} \wedge^0 \]
is a homomorphism sending \( \sigma_b \) to \(-\nabla^b R)\sigma_b \), where \( R \) is the scalar curvature.

Proof. The curvature of the prolongation connection acquires an extra term (beyond the locally symmetric formula (9)):
\[ (\nabla_a \nabla_b - \nabla_b \nabla_a) \begin{bmatrix} \sigma_c \\ \mu_{cd} \end{bmatrix} = \begin{bmatrix} 0 \\ 2R_{ab}^c [\epsilon_d \mu]_c + 2R_{cd}^f [\epsilon_a \mu]_c - (\nabla^e R_{abcd})\sigma_e \end{bmatrix} \]
and the conclusion follows from (11). \( \square \)

Alternatively, one can compute the composition \( \bigcirc \rightarrow \bigcirc \rightarrow \wedge^0 \) to be
\[ h_{ab} \mapsto 2 \left[ \Delta h_{a}^a - \nabla^a \nabla^b h_{ab} + R^{ab} h_{ab} \right] \]
and check that, when \( h_{ab} = \nabla^{(a} \sigma^{b)} \), this formula gives \(-\nabla^b R)\sigma_b \). For a warped product with the metric on \( M \) being locally symmetric, it is clear that the scalar curvature \( R \) is a function of \( t \) alone. Therefore, supposing that \( \partial R/\partial t \neq 0 \), we may define
\[ J(h_{ab}) \equiv -2 \frac{\Delta h_{a}^a - \nabla^a \nabla^b h_{ab} + R^{ab} h_{ab}}{\partial R/\partial t} \]
and, in case that \( h_{ab} = \nabla^{(a} \sigma^{b)} \), conclude from Lemma 5 that
\[ J(h_{ab}) = \frac{\partial}{\partial t} \nabla^{a} \sigma_{b}, \]
which was the first step in our proof of Proposition 6. More generally, it is straightforward to combine this conclusion with the prolongation connection and Calabi operator on \( M \) to manufacture a complex
\[ \wedge^1 \xrightarrow{K} \bigcirc^2 \wedge^1 \xrightarrow{\mathcal{J}} \wedge^0 \oplus \wedge^0 \oplus \wedge^1, \]
which is locally exact unless $M$ has at least one flat factor and at least one Hermitian factor (as in Theorem 1). The details, in the constant curvature case, are in [22].

Also treated in [22] is the 4-dimensional Schwarzschild metric and higher dimensional versions thereof, warped products $S^m \times N$ for which the warping factor $\Omega^2$ multiplying the round sphere metric is annihilated by a Killing field on the 2-dimensional Lorentzian manifold $N$. Finally, although not a warped product, the Kerr metric is treated in [1].

6.4. The deformation operator. Apart from (1), there is another natural differential operator $\Box \to \Box^2$, obtained by deforming the metric and observing the change in the Riemann curvature tensor. Specifically, if $\tilde{g}_{ab} = g_{ab} + \epsilon h_{ab}$, then we find that

\begin{equation}
(48) \quad \tilde{R}_{abcd} = R_{abcd} - \frac{\epsilon}{2} \left[ \nabla_a (\nabla_c h_{bd} - \nabla_b (\nabla_c h_{ad} - \nabla_a (\nabla_d h_{bc}) + \nabla_b (\nabla_d h_{ac})) \right] + O(\epsilon^2)
\end{equation}

and, initially, one might be disturbed by the resulting operator $h_{ab} \mapsto [ \ldots ]_{abcd}$ having the opposite sign from (1) in front of the curvature terms. To reconcile this discrepancy, let us consider the deformation of scalar curvature

\begin{equation}
(49) \quad \tilde{R} = \tilde{g}^{ac} g^{bd} \tilde{R}_{abcd} = (g^{ac} - \epsilon h^{ac})(g^{bd} - \epsilon h^{bd})(R_{abcd} - \frac{\epsilon}{2} [ \ldots ]_{abcd}) + O(\epsilon^2),
\end{equation}

which simplifies as

\begin{equation}
(50) \quad \tilde{R} = R - \epsilon [\Delta h]^a - \nabla^a \nabla^b h_{ab} + R_{ab} h_{ab}] + O(\epsilon^2)
\end{equation}

and we have obtained the operator (17), exactly as expected. The Killing operator is directly related to the Lie derivative of the metric: for any vector field $X^a$,

\begin{equation}
(51) \quad \mathcal{L}_X g_{ab} = 2 \nabla_a (X_b).
\end{equation}

That the association of scalar curvature to a metric is coordinate-free is also expressible in terms of Lie derivatives,

\begin{equation}
(51) \quad \tilde{g}_{ab} = g_{ab} + \epsilon \mathcal{L}_X g_{ab} \implies \tilde{R} = R + \epsilon \mathcal{L}_X R + O(\epsilon^2) = R + \epsilon X^e \nabla_e R + O(\epsilon^2).
\end{equation}

Comparing (51), (21), and (49) gives a geometric proof of Lemma 5. Indeed, this is the interpretation of the operator (17) given by Khavkine [22].

An alternative potential reconciliation, already suggested by Gasqui and Goldschmidt [17, p. 207] (see also [23, §2.2]), is to consider the operator

\begin{equation}
D_g : \bigwedge^2 \rightarrow \bigwedge^2
\end{equation}

obtained by deforming the Riemann curvature viewed as an operator $R_{ab}^{cd} : \bigwedge^2 \to \bigwedge^2$, as in (21), and then lowering the last two indices with the undeformed metric $g_{ab}$. From (18) we obtain

\begin{equation}
(48) \quad h_{ab} \mapsto \frac{1}{2} \left[ \nabla_a (\nabla_c h_{bd} - \nabla_b (\nabla_c h_{ad} - \nabla_a (\nabla_d h_{bc}) + \nabla_b (\nabla_d h_{ac}))
\right]
\end{equation}

Unfortunately, this does not generally satisfy the interchange symmetry to end up in $\mathbb{H}$. Indeed, elementary representation theory shows that this happens if and only if $g_{ab}$ is constant curvature [2]. Whilst symmetrising over $ab \leftrightarrow cd$ forces the result into $\mathbb{H}$

\begin{equation}
\bigwedge^2 \rightarrow \bigwedge^2 \otimes \bigwedge^2
\end{equation}
and gives \((-\frac{1}{2}X)\) the Calabi operator, this observation does not appear to lead anywhere. About as much as one can say, as already observed by Gasqui-Goldschmidt and others, is that in the constant curvature case, it is clear that the composition
\[
\bigwedge^1 \xrightarrow{\mathcal{L}} \bigwedge^2 \xrightarrow{D_\mathcal{G} \otimes \bigwedge^2}
\]
vanishes since \(\mathcal{L}_X(\delta_a^c \delta_b^d - \delta_a^e \delta_b^d) = 0\) for any vector field \(X\).

6.5. Third order compatibility. The differential operator \(\mathcal{L}\) in the complex (11) is second order and, as detailed in Theorem [11], is often sufficient locally to characterise the range of the Killing operator. That is not to say that a higher order differential operator might not more easily provide such integrability conditions. Indeed, the deformation operator of §6.4 suggests that one might augment the Calabi operator with
\[
\begin{array}{c}
\begin{array}{c}
\mathcal{C} \\
\mathcal{C} \oplus
\end{array}
\end{array}
\]
where \(\widetilde{\mathcal{C}}\) deforms \(\nabla_a R_{bcde}\) (with symmetries in \(\bigoplus\) by the Bianchi identity). This option is pursued by Gasqui and Goldschmidt who demonstrate [17, Théorème 7.2] that the resulting third order integrability conditions are sufficient locally to characterise the range of the Killing operator on all Riemannian locally symmetric spaces. It is straightforward to obtain this result by augmenting the curvature (9) of the prolongation connection. Details may be found in our follow-up article [10].

6.6. Projective differential geometry. Although hidden in our presentation so far, several aspects of our constructions are projectively invariant. In particular, the Killing and Calabi operators are the first two operators in a Bernstein-Gelfand-Gelfand sequence [7, 9, 20] from projective differential geometry. From this point of view, the prolongation connection (8) should be rewritten as
\[
\left[
\begin{array}{c}
\sigma_c \\
\mu_{cd}
\end{array}
\right] \xrightarrow{\nabla_b} \left[
\begin{array}{c}
\nabla_b \sigma_c - \mu_{bc} \\
\nabla_d \mu_{cd} + 2P_{[bc][d]}[\sigma_c]
\end{array}
\right] - P_{cd} \sigma_c,
\]
where \(P_{ab} \equiv \frac{1}{n-1}R_{ab}\) is the projective Schouten tensor and \(W_{ab}^c d\) is the projective Weyl tensor. Correspondingly, the Calabi operator (1) should be rewritten as
\[
\begin{aligned}
h_{ab} &\mapsto (\nabla_c (\nabla_a \nabla_c + P_{ac}) + \nabla_{bd} h_{bd} - (\nabla_b (\nabla_c + P_{bc}) h_{ad} - (\nabla_a \nabla_d + P_{ad}) h_{bc} + (\nabla_b \nabla_d + P_{bd}) h_{ac} \\
&- W_{ab}^{\phantom{a}c \phantom{d}e} [h_{d|e} - W_{cd}^{\phantom{c}e|a} h_{b|e}] ,
\end{aligned}
\]
each line of which is projectively invariant (acting on symmetric tensors \(h_{ab}\) of weight 2).

Finally, we suspect that Beltrami’s theorem [2] should generalise to yield a projectively invariant characterisation of locally symmetric metrics.

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