Fractional ideals and integration with respect to the generalised Euler characteristic

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Abstract Let \( b \) be a fractional ideal of a one-dimensional Cohen–Macaulay local ring containing a perfect field. This paper is devoted to the study of the motivic Poincaré series defined by different filtrations associated with \( b \) in the form of Euler integrals with respect to the generalised Euler characteristic; in particular, the functional equations of the Poincaré series are also described.

Keywords Fractional ideal · Poincaré series · One-dimensional local ring · Motivic integration · Functional equation

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1 Introduction

(1.1) Canonical ideals over one-dimensional Cohen–Macaulay rings were profusely studied by Kunz and Herzog, e.g., \([10,12]\), giving remarkable characterisations of the Gorenstein property, such as the formula bringing conductor and delta-invariant...
together, or the relation between being Gorenstein and the symmetry property of
the semigroup of values of the ring. This idea was developed by Delgado [8] for
complex curve singularities with several branches, and in a more general framework
by Campillo, Delgado and Kiyek [7]; they also introduced a Poincaré series \( P(t) \) for
certain ideal filtration associated to the singularity, and deduced its functional equation
for Gorenstein rings. Later on, Campillo, Delgado and Gusein-Zade showed that this
Poincaré series coincides with the integral over the projectivisation of the ring with
respect to the Euler characteristic (cf. [4, 5]). A motivic approach to \( P(t) \) was also
introduced just by taking in the integral the generalized Euler characteristic instead,
see e.g., Campillo, Delgado and Gusein-Zade [6], or the recent results by the author
[13].

(1.2) Recently, some connections with number-theoretical local zeta functions were
found by Delgado and the author [9]; see also Zúñiga and the author [14]. The paper [9]
introduces integrals with respect to the generalized Euler characteristic over fractional
ideals of the ring; this goes back to some constructions involving fractional ideals by
Stöhr [15,16]. The aim of the present work is to investigate those motivic integrals
over fractional ideals: they turn out to be Poincaré series related to the “classical”
series \( P(t) \) mentioned above. We also formulate their functional equations in the
absence of the Gorenstein condition; in particular we obtain functional equations for
the generalized Poincaré series defined in [6].

(1.3) The paper goes as follows. Section 2 is an introduction to the notion of canonical
ideal. We define the dual of a fractional ideal and characterise the self-dual ideals;
in fact, this phenomenon is a generalisation of well-known properties of Gorenstein
rings, cf. Theorem (2.11). Section 3 is devoted to the study of an analogue of the value
semigroup \( S(O) \) of a one-dimensional Cohen–Macaulay local ring \( O \) for a fractional
ideal \( b \): the resulting set \( S(b) \) is no longer a semigroup, but it has structure of module
over \( S(O) \). We also give a notion of the symmetry of \( S(b) \), and we characterise it
by means of the statements in Sect. (2), cf. Proposition (3.8). In Sect. 4 we define a
Poincaré series of motivic nature for the fractional ideal \( b \), cf. Definition (4.10); we
prove its rationality and also its functional equations in absence of the Gorenstein
condition, see Theorem (4.19). Finally, in Sect. 5 we investigate the set \( \tilde{S}(b) \), i.e., the
analogue of the extended semigroup \( \tilde{S}(O) \) of the ring \( O \) for a fractional ideal \( b \)—it has
again structure of \( \tilde{S}(O) \)-module—, as well as an alternative motivic Poincaré series
associated with \( b \), and its corresponding functional equations, cf. Proposition (5.5),
Theorem (5.8).

(1.4) Set \( \mathbb{N}_0 := \{0, 1, 2, \ldots \} \) and \( I_0 := \{1, 2, \ldots, r\} \subset \mathbb{N}_0 \). We will denote by \( R^\times \) the
set of units of a ring \( R \). In particular, if \( R \) is a field, we have \( R^\times = R \smallsetminus \{0\} \). Recall that a
fractional ideal of \( R \) is an \( R \)-submodule \( a \neq (0) \) of the total ring of fractions of \( R \) such
that \( a R \subseteq R \) for some regular \( a \in R \). Notice also that the fractional ideals we are going
to deal with are always regular. For a general reference on the topics discussed here
we refer the reader to Campillo, Delgado, and Gusein-Zade [3], Campillo, Delgado,
and Kiyek [7], Herzog and Kunz [10], and the book of Kiyek and Vicente [11].
2 Duality and fractional ideals

(2.1) Let \( O \) be a one-dimensional Cohen–Macaulay local ring containing a perfect field \( k \) with maximal ideal \( m \). Let \( \overline{O} \) be its integral closure with respect to its total ring of fractions \( K \). Let us assume that \( O \) is analytically unramified (or, equivalently, that \( O \) is a finitely generated \( O \)-module; see Kiyek and Vicente [11, Theorem II.(3.22)]) and that the degree \( \rho := [O/m : k] \) is finite. Let \( \delta := \dim_k (\overline{O}/O) \) be the \( \delta \)-invariant of the ring \( O \).

The ring \( \overline{O} \) decomposes into a finite intersection of Manis valuation rings, let us say \( \overline{O} = V_1 \cap \ldots \cap V_r \). We will denote by \( v_i : K \to \mathbb{Z}_\infty \) the discrete Manis valuation associated with \( V_i \) for every \( i = 1, \ldots, r \). Define \( \overline{v} : K \to (\mathbb{Z}_\infty)^r \) by \( \overline{v}(z) = (v_1(z), \ldots, v_r(z)) \) for every \( z \in K \).

If \( m(V_i) \) denotes the maximal ideal of \( V_i \) for every \( i \in I_0 \), then the ideals \( m_i := m(V_i) \cap \overline{O} \) are principal, regular and maximal (cf. [11, Theorem II.(2.11)]) so that \( m_i = \tau_i \overline{O} \) for every \( i \in I_0 \). If \( k_i := V_i/m(V_i) = \overline{O}/m_i \) for every \( i \in I_0 \), then the extension degrees

\[ d_i := [k_i : k], \quad i \in I_0 \]

are finite (because \( \overline{O} \) is a finitely generated \( O \)-module). In case of \( d_i = 1 \) for all \( i \in I_0 \), the ring \( O \) is said to be residually rational. Write also \( d := d_1 + \cdots + d_r \).

(2.2) Let \( a \) be a regular fractional ideal of \( O \). It is easily seen that \( a \) can be written uniquely as a product \( a = m_1^{v_1} \cdots m_r^{v_r} \) for \( \overline{v} := (v_1, \ldots, v_r) \in \mathbb{Z}^r \). Set \( m^\overline{v} := m_1^{v_1} \cdots m_r^{v_r} \) and \( \overline{v}(a) := v_1 + \cdots + v_r \).

(2.3) Definition A fractional ideal \( c \) of \( O \) is called canonical if it satisfies the following two properties:

(a) \( c \cdot K = K \) (i.e., \( c \) is a regular ideal of \( O \)).
(b) For any regular fractional ideal \( a \) of \( O \) one has that

\[ a = c : (c : a). \]

(2.4) Since \( O \) is a one-dimensional local ring and \( \overline{O} \) is a finitely generated \( O \)-module, a canonical ideal of \( O \) does always exist (as a consequence of [10, Satz 2.9, p. 22] and [10, Korollar 2.12, p. 24]).

(2.5) Let us fix a canonical ideal \( c \) of \( O \). For every regular fractional ideal \( a \) of \( O \), we shall write \( a^* := (c : a) \). The ideal \( a^* \) will be called the dual (ideal) of \( a \). The ideal \( a \) is said to be self-dual if \( a = a^* \). Notice that it is straightforward to see that \( (a^*)^* = a \) by (2.3).

(2.6) Note also that, if the ring \( O \) is self-dual, then it is a canonical ideal, and therefore it satisfies the Gorenstein property. In fact, assuming \( O = c : O \), one sees immediately that \( c : c = c^* = O \) after [10, Bemerkung 2.5, p.19], and therefore \( c = O^* \). Finally, [10,
Korollar 3.4, p. 27] implies the Gorenstein condition. Thus the ring \( \mathcal{O} \) is a canonical ideal of itself if it is self-dual.

(2.7) Set \( f := (\mathcal{O} : \mathcal{O}) \); it will be called the conductor ideal of \( \mathcal{O} \) in \( \mathcal{O} \), and we can easily check that it is the biggest ideal of both \( \mathcal{O} \) and \( \mathcal{O} \) at the same time. Let us define \( \nu(f) := \gamma \) and

\[
\gamma^b := \nu((b : \mathcal{O}) : b) = \nu(b : \mathcal{O}) - \nu(b \cdot \mathcal{O}).
\]

Notice that \( \gamma^\mathcal{O} = \gamma \).

(2.8) Theorem (Gorenstein; Apéry; Samuel; Herzog, Kunz) We have

\[
2 \dim_k(\mathcal{O}/f) \leq \dim_k(\mathcal{O}/f^\ast).
\]

Moreover, the equality holds if and only if the ring \( \mathcal{O} \) is Gorenstein.

(2.9) The rest of the section is devoted to generalise Theorem (2.8) to any fractional ideal of \( \mathcal{O} \). The first obvious task is to search for a candidate to substitute the conductor ideal in the formula preserving the dimensions above. This is not difficult: The ideal \( b : \mathcal{O} \) is the biggest fractional \( \mathcal{O} \)-ideal contained in \( b \). Notice also that \( (b : \mathcal{O})^* = b^* \cdot \mathcal{O} \) and \( (b \cdot \mathcal{O})^* = (b^* : \mathcal{O}) \) (cf. Stöhr [16, Prop. 4.1]). The following fact is also remarkable (see again [16]):

(2.10) Lemma We have

\[
\dim_k(b^* \cdot \mathcal{O}/b^*) = \dim_k(b/b : \mathcal{O}).
\]

(2.11) Theorem Let \( b \) be a fractional ideal of \( \mathcal{O} \). The equality

\[
2 \dim_k(b/b : \mathcal{O}) = \dim_k(b \cdot \mathcal{O}/b : \mathcal{O})
\]

holds if and only if \( b \) is self-dual.

Proof Without loss of generality, let us assume that \( b \subseteq \mathcal{O} \subseteq \mathcal{C} \subseteq \mathcal{O} \). It is easily seen that \( b^* = c : b \supseteq b \). It holds indeed \( b^* \supseteq c^* = \mathcal{O} \supseteq b \), hence

\[
\text{(†)}
\]

By looking at the diagram (†) we deduce

\[
\dim_k(b^* \cdot \mathcal{O}/b : \mathcal{O}) = \dim_k(b^* \cdot \mathcal{O}/b \cdot \mathcal{O}) + \dim_k(b \cdot \mathcal{O}/b : \mathcal{O})
\]

\[
= \dim_k(b^* \cdot \mathcal{O}/b \cdot \mathcal{O}) + \dim_k(b \cdot \mathcal{O}/b) + \dim_k(b/b : \mathcal{O}).
\]
If \( b \) is self–dual, then \( \dim_k(b \cdot \mathcal{O}/b) = \dim_k(b/b : \mathcal{O}) \) by Lemma (2.10), and a substitution above let us finish. Conversely, assuming the following equalities hold:

\[
\dim_k(b \cdot \mathcal{O}/b : \mathcal{O}) \stackrel{(1)}{=} 2 \dim_k(b/b : \mathcal{O}) \stackrel{(2)}{=} 2 \dim_k(b^* \cdot \mathcal{O}/b^*);
\]

again looking at (†) we get

\[
\dim_k(b^* \cdot \mathcal{O}/b : \mathcal{O}) = \dim_k(b^* \cdot \mathcal{O}/b : \mathcal{O}) + \dim_k(b \cdot \mathcal{O}/b : \mathcal{O})
\]

\[
= \dim_k(b^* \cdot \mathcal{O}/b : \mathcal{O}) + \dim_k(b \cdot \mathcal{O}/b : \mathcal{O})
\]

\[
\stackrel{(1)}{=} \dim_k(b^* \cdot \mathcal{O}/b : \mathcal{O}) + 2 \dim_k(b/b : \mathcal{O})
\]

\[
\stackrel{(2)}{=} \dim_k(b^* \cdot \mathcal{O}/b : \mathcal{O}) + \dim_k(b/b : \mathcal{O}) + \dim_k(b^* \cdot \mathcal{O}/b^*);
\]

and also

\[
\dim_k(b^* \cdot \mathcal{O}/b : \mathcal{O}) \stackrel{(3)}{=} \dim_k(b^* \cdot \mathcal{O}/b^*) + \dim_k(b^*/b) + \dim_k(b/b : \mathcal{O}).
\]

Substituting and plugging the last equality into (3) one gets

\[
\dim_k(b^* \cdot \mathcal{O}/b : \mathcal{O}) = \dim_k(b^*/b);
\]

notice also that \((b : \mathcal{O})^* \neq (b^* : \mathcal{O})^* = (b^*: \mathcal{O})^* = (b^* : \mathcal{O})\), and then \(\dim_k(b^*/b) = 0\), i.e. \(b^* = b\), hence the ideal \(b\) is self–dual. \(\square\)

(2.12) Remark From Lemma (2.10) and Theorem (2.11) it follows: The ideal \(b\) is self–dual if and only if

\[
2 \dim_k(b \cdot \mathcal{O}/b) = 2 \dim_k(b/b : \mathcal{O}) = \dim_k(b \cdot \mathcal{O}/b : \mathcal{O}).
\]

(2.13) We need to introduce here some notations to be used in the paper. Let \(v := (v_1, \ldots, v_r)\) and \(w := (w_1, \ldots, w_r)\) be vectors in \(\mathbb{Z}^r\). We will write \(v \geq w\) if and only if \(v_i \geq w_i\) for every \(i \in I_0 := \{0, 0, \ldots, 0\} \in \mathbb{Z}^r\) and \(1 := (1, 1, \ldots, 1) \in \mathbb{Z}^r\). Moreover, for every subset \(I \subseteq I_0 = \{1, \ldots, r\}\), let \(z_I\) be the number of elements in \(I\), and let \(z_I\) be the element of \(\mathbb{Z}^r\) whose \(i\)th component is equal to 1 resp. 0 if \(i \in I\) resp. \(i \notin I\). For any \(v \in \mathbb{Z}^r\) and any fractional ideal \(b\) in \(\mathcal{O}\), we define the set

\[
J^b(v) := \{z \in b \setminus \{0\} \mid v(z) \geq v\},
\]

with \(v(z) := (v_1(z), \ldots, v_r(z))\), cf. (2.1). They are ideals of \(b\) defining a multi–index filtration \(\{J^b(v)\}\), as \(J^b(v) \supseteq J^b(w)\) if \(w \geq v\).

Let us define the \(k\)–vector space \(C^b(v, i) := J^b(v)/J^b(v + 1_{\{i\}})\), and its dimension \(c^b(v, i) := \dim_k(C^b(v, i))\), for \(i \in I_0\); we write also \(C^b(v) := J^b(v)/J^b(v + 1)\), as well as \(c^b(v) := \dim_k(C^b(v))\). Since \(\mathcal{O}\) is Cohen–Macaulay, \(c^b(v) < \infty\) for every \(v \in \mathbb{Z}^r\); thus the filtration is finitely determined, i.e., for any \(v \in \mathbb{Z}^r\) there exists \(N \in \mathbb{Z}\) such that \(J^b(v) \supseteq m^N\); this means that every subspace \(J^b(v)\) of \(\mathcal{O}\) has
finite codimension \( c^b(\psi) \) (cf. [6, p. 194]). Notice that, for every \( i = 1, \ldots, r \) one has \( 0 \leq c^b(\psi, i) \leq d_i \), and if \( \psi \geq \gamma^b \), then \( c^b(\psi, i) = d_i \) for every \( i \in I_0 \) (The proof of these facts follows much more [7]).

(2.14) The Gorenstein condition on the ring \( \mathcal{O} \) was proven to be equivalent to the following equality (cf. [7, Corollary (3.7)]):

\[
c^\mathcal{O}(\psi) + c^\mathcal{O}(\gamma - \psi - 1) = d, \quad \text{for every } \psi \in \mathbb{Z}^r.
\]

Our purpose now is to state the analogue of this result for the case of a fractional ideal. The proof is adapted from [7]. We show first:

(2.15) Lemma \( \text{Let } i \in I_0. \text{ We have } c^b(\gamma^b - 1_{[i]}), i < d_i. \)

Proof Write \( \gamma^b = (\gamma^b_1, \ldots, \gamma^b_r) \) and \( \tau^b := (\tau^b_1, \ldots, \tau^b_r) \) (see (2.1) for the definition of \( \tau \)). Let \( i \in I_0 \). Let us consider the \( k \)-linear map \( \phi_i : C^b(\gamma^b - 1_{[i]}) \rightarrow (b \cdot \mathcal{O}/b : \mathcal{O}) \) given by

\[
z \bmod J^b(\gamma^b - 1_{[i]}) \mapsto z \cdot \tau^b_i - (\gamma^b_i - 1) \bmod (b : \mathcal{O}).
\]

This map is clearly injective, so we have to prove that \( \phi \) is not an epimorphism. It is easily seen that

\[
\tau^b_i - 1_{[i]} \notin (b : \mathcal{O}) : (b \cdot \mathcal{O}) = (b : \mathcal{O}) : b.
\]

Hence there exists \( \xi \in b \cdot \mathcal{O} \) such that \( \xi \cdot \tau^b_i - 1_{[i]} \notin b \). Let \( x_i := \xi \cdot \tau^b_i - \gamma^b_i \). Notice that \( x_i \in b \cdot \mathcal{O} \). Furthermore, if \( z \in J^b(\gamma^b - 1_{[i]}) \) then \( \xi \cdot \tau^b_i - 1_{[i]} - z \notin b : \mathcal{O} = b \cdot \mathcal{O} \) by definition of \( J^b(\gamma^b - 1_{[i]}) \); thus \( v_i(\xi \cdot \tau^b_i - 1_{[i]} - z) = \gamma^b_i - 1 + v_i(b : \mathcal{O}) \) and so

\[
v_i(x_i - \xi \cdot \tau^b_i - (\gamma^b_i - 1)) = v_i(\xi \cdot \tau^b_i - 1_{[i]} \cdot \tau^b_i - \gamma^b_i - z - \tau^b_i - 1_{[i]} \cdot \gamma^b_i) = v_i(b : \mathcal{O}),
\]

which proves the non–surjectivity of \( \phi_i \).

(2.16) Proposition \( \text{Let } b \text{ be a fractional ideal. For every } \psi \in \mathbb{Z}^r \text{ and for every } i \in I_0 \text{ one has:}\)

\[
c^b(\psi, i) + c^b(\gamma^b - \psi - 1_{[i]}, i) \leq d_i.
\]

Proof For every \( \psi \in \mathbb{Z}^r \) and every \( i \in I_0 \), the map \( \eta_{\psi, i} : C^b(\psi, i) \rightarrow k_i \) defined by \( z \bmod J^b(\psi + 1_{[i]}) \mapsto z \tau^b_i - v_i \bmod m_i \) is a \( k \)-monomorphism, hence \( c^b(\psi, i) \leq d_i \). Take now \( \omega = (w_1, \ldots, w_r) \in \mathbb{Z}^r \) with \( \psi \leq \omega \). Thus for every \( i \in I_0 \), the inclusion \( J^b(\omega) \rightarrow J^b(\psi) \) induces a \( k \)-homomorphism \( \varphi_{\omega, \psi, i} : C^b(\omega, i) \rightarrow C^b(\psi, i) \).
\( i \in I_0 \) with \( v_i = w_i \) we have \( \eta_{w,i} = \eta_{v,i} \circ \varphi_{w,v,i} \) and \( \varphi_{w,v,i} \) is injective. Notice also that for every \( p, m \in \mathbb{Z}^r \) the following inclusion holds:

\[
J^b(v) J^C(w) + J^b(w) J^C(v) \subseteq J^b(v + w);
\]

(*)

note that, for \( a \in J^b(v), b \in J^b(w), i \in I_0 \) we have

\[
\eta_{v,i}(a \mod J^b(v + 1_{[i]})) \cdot \eta_{w,i}(b \mod J^b(w + 1_{[i]})) = \eta_{w+i,v,i}(ab \mod J^b(v + w + 1_{[i]})).
\]

Let \( H_i \subseteq k_i \) be a 1-codimensional subspace of the \( k \)-vector space \( k_i \) containing the image \( \text{im}(\eta_{v,b - 1_{[i]},i}) \) (this is possible by Lemma (2.15)); consider the \( k \)-bilinear pairing \( k_i \times k_i \to k_i \to k_i/H_i \) defined by \( (a, b) \to a \cdot b \mod H_i \) (see [7, (3.5)]), which is non-degenerate (multiplying by scalars of \( k_i \) is a \( k \)-automorphism on \( k_i \)). Because of (*) we have

\[
J^b(v) J^C(\gamma^b - v - 1_{[i]})) + J^b(\gamma^b - v - 1_{[i]}) J^C(v) \subseteq \gamma^b - 1_{[i]}),
\]

therefore \( \text{im}(\eta_{v,i}) \) lies in the orthogonal complement of \( \text{im}(\eta_{v,b - u - 1_{[i]},i}) \), hence

\[
c^b(v, i) = d_i - c^b(\gamma^b - v - 1_{[i]}, i).
\]

\( \square \)

It remains to show that the equality of Proposition (2.16) holds if and only if \( b \) is self-dual. This follows by the same method as in the proof of [7, Theorem (3.6)], just by applying the characterisation of the self-dual fractional ideals provided by Theorem (2.11) instead of using Theorem (2.8).

(2.17) Lemma Let \( \{\overline{v}(p)\}_{0 \leq p \leq h} \) be a strictly increasing sequence in \( \mathbb{Z}^r \) such that \( \overline{v}(0) = 0, \overline{v}(h) = \gamma^b \), and for every \( p \in \{1, \ldots, h\} \) there exists \( i(p) \in I_0 \) satisfying \( \overline{v}(p) - \overline{v}(p-1) = 1_{[i(p)]} \). Then \( b \) is self-dual if and only if

\[
c^b(\overline{v}(p), i(p + 1)) + c^b(\gamma^b - \overline{v}(p) - 1_{[i(p+1)]}, i(p + 1)) = d_i(p+1)
\]

for every \( p \in \{0, \ldots, h-1\} \).

Proof Define the vectors \( \overline{w}(p) := \gamma^b - \overline{v}(p) \in \mathbb{Z}^r \) for every \( p \in \{0, \ldots, h\} \). We have

\[
\overline{v}(p) + \overline{w}(p+1) = \gamma^b - 1_{[i(p+1)]}, \overline{v}(p) - \overline{w}(p+1) = 1_{[i(p+1)]}
\]

and

\[
\begin{align*}
\overline{b} &= J^b(\overline{v}(0)) \supseteq J^b(\overline{v}(1)) \supseteq \ldots \supseteq J^b(\overline{v}(h)) = b : \overline{O} \\
\overline{b} : \overline{O} &= J^b(\overline{w}(0)) \subseteq J^b(\overline{w}(1)) \subseteq \ldots \subseteq J^b(\overline{w}(h)) = b.
\end{align*}
\]
Therefore
\[ \sum_{p=0}^{h-1} c^b(w^{(p)}, i(p + 1)) = \dim_k(b/b : \mathcal{O}) = \sum_{p=0}^{h-1} c^b(w^{(p+1)}, i(p + 1)), \]
and then
\[ 2 \dim_k(b/b : \mathcal{O}) = \sum_{p=0}^{h-1} c^b(w^{(p)}, i(p + 1)) + c^b(w^{(p)} - 1_{[i(p+1)]}, i(p + 1)). \]

By Proposition (2.16) this expression is smaller than or equal to
\[ \sum_{p=0}^{h-1} d_i(p+1) \leq \sum_{p=0}^{r} \gamma_i^b d_i = \dim_k(\mathcal{O}/(b : \mathcal{O}) : b) = \dim_k(b : b : \mathcal{O}), \]
where the latter two identities follow from the Chinese Remainder Theorem (see also the book of Campillo and Castellanos [2, 2.1.7], or [16, p. 851]) resp. from the equalities
\[ (b : \mathcal{O}) : b = (b : \mathcal{O}) : (b : \mathcal{O}) = (b : \mathcal{O}) \cdot (b : \mathcal{O})^{-1} \]
in [16, p. 853]. The application of Theorem (2.11) allows us to conclude. The converse follows in the same manner as in the proof of [7, (3.6)], part (d).

\[(2.18) \text{ Theorem} \] Let \( b \) be a fractional \( \mathcal{O} \)-ideal. The following statements are equivalent:

1. For every \( v \in \mathbb{Z}^r \) we have \( c^b(v, i) + c^b(b^b - v - 1_{[i]}), i \) \((\ast)\) for all \( i \in I_0 \).
2. \( b \) is self–dual.

\[\text{Proof} \] If the equality \((\ast)\) holds for every \( v \in \mathbb{Z}^r \) and for every \( i \in I_0 \), then one can choose a strictly increasing sequence as in Lemma (2.17), and we obtain that
\[ 2 \dim_k(b/b : \mathcal{O}) = \dim_k(\mathcal{O}/b : \mathcal{O}), \]
which by Theorem (2.11) implies the statement.

Analogously as in [7, (3.7)] one shows:

\[(2.19) \text{ Corollary} \] Let \( v \in \mathbb{Z}^r \). Then
\[ c^b(v) + c^b(b^b - v - 1) \leq d = \sum_{i=1}^{r} d_i. \]
Moreover, \( b \) is self–dual if and only if the equality holds for every \( v \in \mathbb{Z}^r \).

3 The value ideal of a fractional ideal

Let \( b \) be a fractional ideal of \( \mathcal{O} \). Consider the set
\[ S(b) := \{ v(z) - v(b \cdot \mathcal{O}) \mid z \in b \setminus \{0\} \}. \]
This is a subsemigroup of \( \mathbb{Z}^r \) which is in fact a module over the semigroup \( S(\mathcal{O}) \)—for this reason we will speak about \textit{semimodule}—and it only depends on the ideal class of \( b \) in the ideal class semigroup of \( \mathcal{O} \).

(3.1) For every \( \nu = (v_1, \ldots, v_r) \in \mathbb{Z}^r \) and for every \( i \in I_0 \), we define

\[
\Delta_i(\nu) := \{ (\sigma_1, \ldots, \sigma_r) \in S(b) \mid \sigma_i = v_i \text{ and } \sigma_j > v_i \text{ for } j \in I_0, j \neq i \}.
\]

Moreover we set \( \Delta^*_i(\nu) \) to be

\[
\{(\sigma_1, \ldots, \sigma_r) \in S(b) \mid \sigma_i = v_i; \sigma_s \geq v_s \text{ for all } s \leq j; \sigma_t > v_t \text{ for all } t > j, t \neq i \}.
\]

and

\[
\Delta(\nu) := \bigcup_{i=1}^{r} \Delta_i(\nu).
\]

The elements of \( S(b) \) are related with the filtrations \( \{J^b(\nu)\} \) as follows:

(3.2) Lemma Let \( \mathcal{O} \) be residually rational such that \( k \) is an infinite field. Let be the vector \( \nu = (v_1, \ldots, v_r) \in \mathbb{Z}^r \). Then \( \nu \in S(b) \) if and only if \( J^b(\nu)/J^b(\nu + 1_{[i]} \mathcal{O}) \neq 0 \) for every \( i \in I_0 \). Moreover, \( c^b(\nu, i) = 1 \) if \( \Delta^*_i(\nu) \neq \emptyset \).

Proof If \( \nu \in S(b) \), then there exists \( \omega = (w_1, \ldots, w_r) \in S(b) \) such that \( w_i = v_i \) and \( w_j \geq v_j \) for every \( j \in I_0, j \neq i \). Thus \( J^b(\nu)/J^b(\nu + 1_{[i]} \mathcal{O}) \neq 0 \) for every \( i \in I_0 \). Conversely, assume that \( J^b(\nu)/J^b(\nu + 1_{[i]} \mathcal{O}) \neq 0 \) for every \( i \in I_0 \), and choose an element \( z_i \in J^b(\nu)/J^b(\nu + 1_{[i]} \mathcal{O}) \). Since \( k \) is infinite and \( k = k_i \) for every \( i \in I_0 \), there exist elements \( a_1, \ldots, a_r \in \mathcal{O} \) such that \( v_i(a_1z_1 + \cdots + a_rz_r) = v_i \) for every \( i \in I_0 \), i.e., \( \nu(a_1z_1 + \cdots + a_rz_r) = \nu \). For the last assertion, consider \( a, b \in J^b(\nu)/J^b(\nu + 1_{[i]} \mathcal{O}), u \in \mathcal{O}^X \) such that \( v_j(ua + b) > v_i \) and \( v_j(ua + b) \geq \min\{v_j(a), v_j(b)\} \) for \( j \neq i \). This implies \( c^b(\nu, i) \leq 1 \) and by the first assertion this must be exactly 1.

(3.3) Notice that if \( b = \mathcal{O} \) then \( S(\mathcal{O}) \) is the value semigroup of the ring \( \mathcal{O} \). Recall that \( \gamma^b = \nu((b : \mathcal{O}) : b) = \nu(b : \mathcal{O}) - \nu(b \cdot \mathcal{O}) \), and \( \gamma^\mathcal{O} = \gamma \) if \( b = \mathcal{O} \). The next two lemmas show that \( \gamma^b \) plays the role of the conductor of the set \( S(b) \).

(3.4) Lemma For every fractional ideal \( b \) in \( \mathcal{O} \), there exists \( m \in \mathbb{Z}^r \) such that \( J^b(\nu) \subseteq b \) for all \( \nu \geq m \).

Proof First of all, notice that \( (\mathcal{O} : \mathcal{O}) = \{ x \in \mathcal{O} \mid \nu(x) \geq \gamma \} \). Then \( x \in \mathcal{O} \) for all \( x \in \mathcal{O} \) and \( z \in b \), hence \( (\mathcal{O} : \mathcal{O}) \subseteq z^{-1}b \) for all \( z \in b \) and therefore we have the equality \( z \cdot (\mathcal{O} : \mathcal{O}) = \{ x \mid \nu(x) \geq \gamma + \nu(z) \} \subseteq b \) for all \( z \in b \). Thus \( J^b(\nu) \subseteq b \) for every \( \nu \geq m \).

(3.5) Lemma We have:

(a) \( 0 \leq \gamma^b \leq \nu((\mathcal{O} : \mathcal{O})) = \gamma \).
(b) If $b : O = \overline{O}$, then $\gamma^b := \min\{n \mid m^n \subseteq b\}$.

**Proof** From [16, Lemma 3.1], one has the inclusions

$$(O : \overline{O}) \subseteq (b : \overline{O}) : (b : O) \subseteq \overline{O}$$

and (a) follows. (b) is deduced from the fact that $(b : \overline{O}) : b = (b : \overline{O}) : (b : O)$.

(3.6) In the rest of the section, the ring $O$ is assumed to be residually rational.

(3.7) **Definition** The $S(O)$-module $S(b)$ is said to be symmetric if there exists $\tau \in \mathbb{Z}^r$ such that, for every $v \in \mathbb{Z}^r$, $v \in S(b)$ if and only if $\Delta(\tau - v) = \emptyset$.

(3.8) **Proposition** Let $O$ be residually rational. We have

1. $\Delta(\gamma^b - v - 1) = \emptyset$ for every $v \in S(b)$.
2. If $S(b)$ is symmetric then $b$ is self-dual.
3. Suppose, in addition, that $k$ is an infinite field; if $b$ is a self-dual fractional ideal, then $S(b)$ is symmetric.

**Proof** (1) Let $v \in S(b)$. Then we have $J^b(v)/J^b(v + 1_{\{i\}}) \neq 0$ for every $i \in I_0$, therefore $J^b(\gamma^b - v - 1_{\{i\}})/J^b(\gamma^b - v) \neq 0$ for every $i \in I_0$ and so $\Delta(\gamma^b - v - 1)$ is empty. (2) Let $v = (v_1, \ldots, v_r) \in \mathbb{Z}^r$ and take $i \in I_0$. If $J^b(v)/J^b(v + 1_{\{i\}}) \neq 0$ then we must have that $J^b(\gamma^b - v - 1_{\{i\}})/J^b(\gamma^b - v) = 0$. Let us consider the case in which $J^b(v)/J^b(v + 1_{\{i\}}) = 0$. There exists a vector $w = (w_1, \ldots, w_r) \in \mathbb{Z}^r$ with $\Delta(w) = \emptyset, w_i = v_i$ and $w_j < v_j$ for every $j \in I_0, j \neq i$. Since $S(b)$ is symmetric, $\gamma^b - w - 1 \in S(b)$. Now $\gamma_j^b - w_j - 1 \geq \gamma^b - v_j$ for every $j \in I_0$ with $j \neq i$, and $\gamma_i^b - w_i - 1 = \gamma_i^b - v_i - 1$, hence for any regular element $z \in O$ satisfying $\nu(z) = \gamma^b - w - 1$ it follows that $z \in J^b(\gamma^b - v - 1_{\{i\}}), z \notin J^b(\gamma^b - v)$, and therefore $J^b(\gamma^b - v - 1_{\{i\}})/J^b(\gamma^b - v) \neq 0$. Theorem (2.18) implies that $b$ is self-dual. (3) Assume that $k$ is infinite. Let $v \in \mathbb{Z}^r$ and assume that $\Delta(\gamma^b - v - 1) = \emptyset$. Then the vector space $J^b(\gamma^b - v - 1_{\{i\}})/J^b(\gamma^b - v) = 0$ for every $i \in I_0$, hence $J^b(v)/J^b(v + 1_{\{i\}}) \neq 0$ for every $i \in I_0$ by Theorem (2.18) and therefore $z \in S(b)$ by Lemma (3.2). Thus $S(b)$ is symmetric.

(3.9) **Corollary** (Campillo, Delgado, Kiyek) Let $k$ be an infinite field. The value semigroup $S(O)$ is symmetric if and only if the ring $O$ is Gorenstein.

4 **Generalised Poincaré series of a fractional ideal**

(4.1) Let $b$ be a fractional ideal in $O$. The multi-index filtration $\{J^b(v)\}$ (cf. (2.13)) defines a Laurent series

$$L(b, t_1, \ldots, t_r) := \sum_{v \in \mathbb{Z}^r} \dim_k \left( \frac{J^b(v)}{J^b(v + 1)} \right) \cdot t^v \in \mathbb{Z}[[t_1, \ldots, t_r, t_1^{-1}, \ldots, t_r^{-1}]].$$
where $t^\nu := t_1^{v_1} \cdots t_r^{v_r}$. We will write $L(b, t)$ instead of $L(b, t_1, \ldots, t_r)$ if the number of variables is clear from the context.

**4.2** There is a priori no fixed way to choose a suitable coefficient in $L(b, t)$. We may consider the following spaces:

1. $J^b(v)/J^b(v + 1)$;
2. $J^b(v)/J^b(v + 1) \cup J^b(v + 1)$;
3. $J^b(v) \cup J^b(v + 1)$.

Filtration (1) is related to the semigroup of values of the ring, (2) defines the Poincaré series in terms of the extended semigroup of the ring, and (3) introduces the Poincaré series as an integral with respect to the Euler characteristic. Exactly this last point of view makes clear the association between the dimension of a vector space and the Euler characteristic with compact support $\chi$ of its projectivisation, namely

$$\dim_k \left( J^O(v)/J^O(v + 1) \right) = \chi \left( \mathbb{P} \left( J^O(v)/J^O(v + 1) \right) \right).$$

**4.3** We can also choose other measures than $\chi$, for instance the so-called generalised Euler characteristic $\chi_g$. It is a sort of motivic Euler characteristic which makes use of the notion of Grothendieck ring. The Grothendieck ring $K_0(V_k)$ is defined to be the free Abelian group on isomorphism classes $[X]$ of quasi-projective schemes $X$ of finite type over $k$ subject to the following relations:

1. $[X_1] = [X_2]$ if $X_1 \cong X_2$ for $X_1, X_2 \in V_k$;
2. $[X] = [X \setminus Z] + [Z]$ for a closed subscheme $Z$ of $X \in V_k$; and taking the fibre product as multiplication:
3. $[X_1] \cdot [X_2] = [X_1 \times_k X_2]$ for $X_1, X_2 \in V_k$.

**4.4** Let $k[T]$ be the polynomial ring in one indeterminate $T$ over the field $k$. The affine scheme $\text{Spec}(k[T])$ over $k$ is the affine line over $k$, which will be denoted by $A^1_k$. The class of the affine line in $K_0(V_k)$, denoted by $\mathbb{L}$, is called the Lefschetz class of $K_0(V_k)$.

**4.5** Let $p$ be a non-negative integer and let $J^p_O$ be the space of $p$-jets over $O$, which is a finite-dimensional $k$-vector space of dimension $d(p)$. Let us consider its projectivisation $\mathbb{P} J^p_O$ and let us adjoin one point to this (that is, $\mathbb{P}^* J^p_O = \mathbb{P} J^p_O \cup \{*\}$ with $*$ representing the added point) in order to have a well-defined map $\pi_p : \mathbb{P} O \to \mathbb{P}^* J^p_O$. A subset $X \subset \mathbb{P} O$ is said to be cylindric if there exists a constructible subset $Y \subset \mathbb{P} J^p_O \subset \mathbb{P}^* J^p_O$ such that $X = \pi_p^{-1}(Y)$.

**4.6** The generalised Euler characteristic $\chi_g(X)$ of a cylindric subset $X = \pi_p^{-1}(Y)$ is the element $[Y] \cdot \mathbb{L}^{-d(p)}$ in the ring $K_0(V_k)_{(\mathbb{L})}$, where $Y$ is a constructible subset of $\mathbb{P} O$. Note that $\chi_g(X)$ is well-defined, because if $X = \pi_q^{-1}(Y')$, $Y' \subset \mathbb{P} J^q_O$ and $p \geq q$, then $Y$ is a locally trivial fibration over $Y'$ and therefore $[Y] = [Y'] \cdot \mathbb{L}^{d(p) - d(q)}$.

**4.7** As in the paper of Delgado and the author [9], we can extend these definitions to subsets of $K$ (in particular to fractional ideals): A subset $X \subset K$ is called cylindric
if there exists a non-zero divisor element \( z \in \mathcal{O} \) such that the set \( zX \) is a subset of \( \mathcal{O} \) and is cylindric. In this situation, the generalised Euler characteristic is

\[
\chi_g(X) := \frac{\chi_g(zX)}{\chi_g(z\mathcal{O})}.
\]

Let \( a \subseteq \mathcal{O} \) be an ideal of \( \mathcal{O} \). Since \( a \) is \( m \)-primary, we have \( m^{p+1} + 1 \subseteq a \). Let \( a \) be the ideal \( a/m^{p+1} \) of \( \mathcal{O}/m^{p+1} \) so that \( \pi^{-1}(a) = a \). As \( \mathcal{O}/m^{p+1} \) is a finite-dimensional \( k \)-vector space, the ideal \( a \) is constructible. Then \( a \) is cylindric and we get

\[
\chi_g(a) = \left[ \frac{a}{m^{p+1}} \right] \cdot \mathbb{L}^{-d(p)}
= \mathbb{L}^{\dim_k(a/m^{p+1}) - d(p)}
= \mathbb{L}^{\deg(a)},
\]

were \( \deg(a) \) denotes the degree of \( a \); the degree of a fractional \( \mathcal{O} \)-ideal is defined by the following two properties: (1) \( \deg(\mathcal{O}) := 0 \); (2) \( \deg(a) - \deg(b) = \dim(a/b) \) for every two fractional \( \mathcal{O} \)-ideals \( a, b \) whenever \( a \supseteq b \). In particular, \( \chi_g(m^{p+1}) = \mathbb{L}^{-d(p)} \).

(4.8) Let \( G \) be an abelian group with countable many values. Let \( X \) be a cylindric subset of \( \mathcal{K} \). A function \( \psi : X \rightarrow G \) is called cylindric if the set \( \psi^{-1}(a) \subseteq \mathcal{K} \) is cylindric for all \( a \in G \setminus \{0\} \). The integral of \( \psi \) over \( X \) with respect to the generalised Euler characteristic is

\[
\int_X \psi \, d\chi_g := \sum_{a \in G \setminus \{0\}} \chi_g(\psi^{-1}(a)) \cdot a,
\]

if this sum makes sense in \( K_0(V_k)_\mathcal{L} \otimes \mathbb{Z} G \); in such a case, the function \( \psi \) is said to be integrable.

(4.9) Remark Let \( \psi : \mathbb{P}X \rightarrow G \) be a cylindric function of \( X \). Let us denote by \( \psi' : X \rightarrow G \) the function induced by \( \psi \) on \( X \) with \( \psi'(0) = 0 \). Then the function \( \psi' \) is cylindric if and only if \( \psi \) is cylindric; in this case we have

\[
(\mathbb{L} - 1) \int_{\mathbb{P}X} \psi \, d\chi_g = \int_X \psi' \, d\chi_g
\]

(cf. [9, (2.7)]).

We define now the generalised Poincaré series of the projectivisation of the fractional ideal \( b \):

(4.10) Definition The generalised Poincaré series of a multi-index filtration given by the ideals \( J(v) \) is the integral

\[
P_g(b, \mathcal{L}, \mathcal{L}) := \int_{\mathbb{P}b} t^{v(z)} \, d\chi_g \in K_0(V_k)_\mathcal{L} [t_1, \ldots, t_r],
\]

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where \( t^v(z) := t_1^{v_1(z)} \cdot \ldots \cdot t_r^{v_r(z)} \) is considered as a (cylindric) function on \( \mathbb{P} \mathcal{O} \) with values in \( \mathbb{Z}[t_1, \ldots, t_r] \) (the entry \( v_i(z) \) is supposed to be 0 as soon as \( v_i(z) = \infty \) for \( i \in I_0 \)).

(4.11) Remark Notice that if \( b = \mathcal{O} \), then \( P_g(\mathcal{O}, t, \mathbb{L}) \) is the generalised Poincaré series of a filtration \( J(v) \) over the projectivisation of the ring \( \mathcal{O} \) introduced by Campillo, Delgado, and Gusein-Zade in [6, Section 2, p. 198] for the case of the ring \( \mathcal{O}_{W,0} \) of functions on a germ \((W,0)\) of a complex analytic variety.

(4.12) Define

\[
L_g(b, t, \mathbb{L}) := \sum_{v \in \mathbb{Z}^r} \left( \mathbb{L}^{\deg(J^b(v))} - \mathbb{L}^{\deg(J^b(v+1))} \right) \cdot t^v,
\]

then we get

(4.13) Lemma

\[
P_g(b, t, \mathbb{L}) = \prod_{i=1}^r \frac{(t_i - 1) L_g(\mathcal{O}, b; t)}{t_1 \cdot \ldots \cdot t_r - 1}.
\]

Proof The result may be proved in much the same way as in [6, Proposition 2]. \( \Box \)

We describe now the functional equations for the series \( P_g(b, t, \mathbb{L}) \). First of all, we state the following two results, due to Stöhr [15,16].

(4.14) Lemma Let \( a, b \) be fractional ideals of \( \mathcal{O} \) such that \( a \supseteq b \). We have

\[
\dim_k (b^* \cap a/b \cap a^*) = \dim_k (a/b).
\]

(4.15) Lemma Let \( a \) be a fractional ideal of \( \overline{\mathcal{O}} \). The following assertions hold:

(a) For each \( i \in I_0 \), let \( t_i \) be a generator of the ideal \( m_i \). The fractional ideals \( a \) of \( \mathcal{O} \) are of the form \( \tau^{-v} \cdot b \), where \( \tau^v := \tau_1^{v_1} \cdots \tau_r^{v_r} \) for some \( v = (v_1, \ldots, v_r) \in \mathbb{Z}^r \) and being \( b \) a fractional ideal of \( \mathcal{O} \) such that \( b \cdot \overline{\mathcal{O}} = \overline{\mathcal{O}} \).

(b) There exists some \( v \in \mathbb{Z}^r \) such that \( a = J^K(v) \) and \( a^* = J^K(-v) \).

(c) For some \( v \in \mathbb{Z}^r \), we have \( \deg(a) = \delta - v \cdot d \).

The next proposition relates the degree of the ideal \( J^{b^*}(v) \) and the value \( \gamma^b \).

(4.16) Proposition For every \( v \in \mathbb{Z}^r \), we have

\[
\deg(J^{b^*}(v)) = \deg(J^b(\gamma^b - v)) + \dim_k (b/(b : \overline{\mathcal{O}})) - v \cdot d
\]

\[
= \deg(J^b(\gamma^b - v)) + \dim_k (b^* : \overline{\mathcal{O}}/b^*) - v \cdot d,
\]

where \( v \cdot d := v_1 d_1 + \cdots + v_r d_r \).
Proof The second equality holds by Lemma (2.10). Moreover, since \( J^b(v) = b \cap J^K(v) \) for \( v \in \mathbb{Z}' \), by Lemmas (4.14) and (4.15) it follows that

\[
\text{deg} \left( b^* \cap J^K(v) \right) = \dim_k \left( b/b : \mathcal{O} \right) + \text{deg} \left( J^K(-v) \cdot \mathcal{O}^* \cap b \right) - v \cdot d.
\]

The definition of \( \gamma^b \) allows us to conclude. \( \Box \)

For every \( v \in \mathbb{Z}' \), set \( \ell(v) := \dim_k (\mathcal{O} / J^\mathcal{O}(v)) \). As a consequence of Proposition (4.16) we obtain the following result due to Zúñiga and the author [14, Lemma 9]:

(4.17) Corollary (Moyano-Fernández, Zúñiga) The ring \( \mathcal{O} \) is self-dual if and only if

\[
\ell(\gamma - v) - \ell(v) = \delta - v \cdot d
\]

for every \( v \in \mathbb{Z}' \). In particular, the ring \( \mathcal{O} \) is Gorenstein.

Proof It is just to apply Proposition (4.16) to \( b = \mathcal{O} \). Notice that \( \mathcal{O} \) is Gorenstein (cf. (2.6)). \( \Box \)

(4.18) Remark Notice that \( \delta - d = \ell(\gamma - 1) - \rho \), if the ring is Gorenstein. It follows from Corollary (4.17), because \( \delta - 1 \cdot d = \ell(\gamma - 1) - \ell(1) \) and \( \ell(1) = \rho \).

Proposition (4.16) allows us to describe the functional equations for the generalised Poincaré series:

(4.19) Theorem

\[
L_g(b, \mathbb{L}^{d_1} t_1, \ldots, \mathbb{L}^{d_r} t_r, \mathbb{L}) = \mathbb{L}^{\dim \left( b^* / b^* \right)} - d \cdot t^{\gamma^b} - 1 \cdot L_g(b^*, \mathbb{L}^{-1}, \mathbb{L}).
\]

Proof Let \( A(b, t) := \sum_{v \in \mathbb{Z}'} \mathbb{L}^{\deg(j^b(v))} \cdot t^v \). Then

\[
L_g(b, t, \mathbb{L}) = \sum_{v \in \mathbb{Z}'} \left( \mathbb{L}^{\deg(j^b(v))} - \mathbb{L}^{\deg(j^b(v + 1))} \right) \cdot t^v = (1 - t^{-1}) \cdot A(b, t).
\]

Using Proposition (4.16) it is easily seen that

\[
A(b, \mathbb{L}^{d_1} t_1, \ldots, \mathbb{L}^{d_r} t_r) = \mathbb{L}^{\dim \left( b^* / b^* \right)} \cdot t^{\gamma^b} \cdot A(b^*, t_1^{-1}, \ldots, t_r^{-1}).
\]

Moreover, taking the inverse \( t^{-1} \) of \( t \), we deduce the equality

\[
L_g(b^*, t^{-1}, \mathbb{L}) = (1 - t) \cdot A(b^*, t^{-1}), \quad (\dagger)
\]
and therefore we obtain

\[
L_g (b, \mathbb{L}^{d_1} t_1, \ldots, \mathbb{L}^{d_r} t_r, \mathbb{L}) = \frac{d}{d \cdot t} \cdot A (b, \mathbb{L}^{d_1} t_1, \ldots, \mathbb{L}^{d_r} t_r) \\
= \frac{d}{d \cdot t} \cdot \mathbb{L}^{\dim (b^* \mathcal{O} / b^*)} \cdot \mathcal{L}^{b} \cdot A (b^*, t_1^{-1}, \ldots, t_r^{-1}) \\
= \frac{d}{d \cdot t} \cdot \mathbb{L}^{\dim (b^* \mathcal{O} / b^*)} \cdot \mathcal{L}^{b} \cdot L_g (b^*, t_1^{-1}, \mathbb{L}) \\
= \frac{d}{d \cdot t} \cdot \mathbb{L}^{\dim (b^* \mathcal{O} / b^*)} \cdot \mathcal{L}^{b} \cdot L_g (b^*, t_1^{-1}, \mathbb{L}).
\]

\(\square\)

4.20 Corollary

\[
P_g (b, \mathbb{L}^{d_1} t_1, \ldots, \mathbb{L}^{d_r} t_r, \mathbb{L}) = \mathbb{L}^{\dim (b^* \mathcal{O} / b^*)} \cdot \mathcal{L}^{b} \cdot L_g (b^*, t_1^{-1}, \mathbb{L}).
\]

4.21 Corollary If \(b = \mathcal{O}\), then we have

\[
P_g (\mathcal{O}, \mathbb{L}^{d_1} t_1, \ldots, \mathbb{L}^{d_r} t_r, \mathbb{L}) = \mathbb{L}^{\dim (\mathcal{O}^* \mathcal{O} / \mathcal{O}^*)} \cdot \mathcal{L}^{b} \cdot L_g (\mathcal{O}^*, t_1^{-1}, \mathbb{L}).
\]

Furthermore, if \(\mathcal{O}\) is self-dual, then we obtain

\[
P_g (\mathcal{O}, \mathbb{L}^{d_1} t_1, \ldots, \mathbb{L}^{d_r} t_r, \mathbb{L}) = \mathbb{L}^{\dim (\mathcal{O}^* \mathcal{O} / \mathcal{O}^*)} \cdot \mathcal{L}^{b} \cdot L_g (\mathcal{O}^*, t_1^{-1}, \mathbb{L}).
\]

4.22 Corollary If \(\mathcal{O}\) is both residually rational and Gorenstein, then we have

\[
P_g (\mathcal{O}, \mathbb{L}^{d_1} t_1, \ldots, \mathbb{L}^{d_r} t_r, \mathbb{L}) = \mathbb{L}^{\dim (\mathcal{O}^* \mathcal{O} / \mathcal{O}^*)} \cdot \mathcal{L}^{b} \cdot L_g (\mathcal{O}^*, t_1^{-1}, \mathbb{L}).
\]

5 Extended generalised semimodule Poincaré series

Campillo, Delgado and Gusein-Zade introduced in [3] the notion of extended semigroup of a germ of complex plane curve singularity. We want to extend it for a fractional ideal \(b\) of \(\mathcal{O}\).

5.1 Let us preserve notations as in (2.1). Let \(\mathcal{O}\) assume to have a perfect coefficient field \(K\). Since \(\mathcal{O}/m\) is perfect, the finite field extension \(\mathcal{O}/m \hookrightarrow \mathcal{O}/m_i = k_i\) is separable, and so there exists a unique coefficient field \(K_i \supseteq K\) of \(V_i\) which is isomorphic
to $\mathcal{O}/m_i$ for every $i \in I_0$. Notice that the separability is an essential condition, as Abhyankar showed in [1].

(5.2) If $\tau_i$ is an indeterminate over $K_i$, then one can identify $V_i \cong K_i[\tau_i]$ and the discrete Manis valuation $v_i$ with the order function with respect to $\tau_i$ in $K_i[\tau_i]$ for every $i \in I_0$. Hence we may summarize:

\[
\mathcal{O} \cong K[\tau] \longrightarrow \mathcal{O} \cong K_1[\tau_1] \times \cdots \times K_r[\tau_r] \\
\mathcal{O}/m \quad \longrightarrow \quad \mathcal{O}/m_i =: k_i \cong K_i
\]

(5.3) Consider the morphism

\[
b : b \longrightarrow K_1[\tau_1] \times \cdots \times K_r[\tau_r] \\
g \mapsto (g(\tau_1), \ldots, g(\tau_r)),
\]

where $g(\tau_i) = a_i \tau_i^{v_i} + \cdots$ and $a_i \neq 0$ for all $i \in I_0$. Consider the image of $g \in J^b(\nu) \subseteq b$ in $K_1[\tau_1] \times \cdots \times K_r[\tau_r]$. If $(a_1, \ldots, a_r) \neq 0$, then $g \in J^b(\nu)$; otherwise $g \in J^b(\nu)$ for $\nu > \nu$. This yields a natural linear mapping $j_{\nu} : J^b(\nu) \to K_1^\times \times \cdots \times K_r^\times$ such that $j_{\nu}(g) = (a_1, \ldots, a_r)$ whose kernel is just the subspace $J^b(\nu + 1)$.

(5.4) For every $\nu \in \mathbb{Z}^r$ and every $i \in I_0$, consider the $\mathcal{O}$-module $C^b(\nu, i) = J^b(\nu)/J^b(\nu + 1)$, cf. (2.13). Since $m_i \cap \mathcal{O} = m$, the $\mathcal{O}$-module $C^b(\nu, i)$ is annihilated by $m$ so that $C^b(\nu, i)$ naturally gets a structure of $k_i$-vector space of dimension $c^b(\nu, i) := \dim_{k_i} C^b(\nu, i) \leq d_i$. On the other hand, consider $C^b(\nu_i, i) := J^b(\nu_i)/J^b(\nu_i + 1)$ so that $C^b(\nu, i) \hookrightarrow C^b(\nu, i)$ for every $i \in I_0$; in this way we obtain

\[
C^b(\nu) \hookrightarrow C^b(\nu, 1) \times \cdots \times C^b(\nu, r) \\
\cong K_1^\times \times \cdots \times K_r^\times \hookrightarrow C^b(\nu_1, 1) \times \cdots \times C^b(\nu_r, r).
\]

Let $\nu \in \mathbb{Z}^r$. The purpose is to consider the initial forms of those elements in $\mathcal{O}$ having value exactly $\nu$, and to this aim it is useful to study the vector space $C^b(\nu)$ by removing the coordinate hyperplanes. In doing so we define

\[
F^b_\nu := C^b(\nu) \cap \left( (C^b(\nu, 1) \setminus \{0\}) \times \cdots \times (C^b(\nu, r) \setminus \{0\}) \right).
\]

This is in fact equivalent to

\[
F^b_\nu = C^b(\nu) \cap (K_1^\times \times \cdots \times K_r^\times),
\]

see (5.3).
(5.5) If we attach $F^b_v$ to each element of the semimodule $S(b)$ we obtain the extended semimodule associated to $b$:

$$\hat{S}(b) := \bigcup_{v \in S(b)} F^b_v \times \{v\}.$$  

The spaces $F^b_v$ are called fibres of the extended semimodule $\hat{S}(b)$.

For a fixed value $v$, the extended semimodule measures which initial forms reach that level. Note that, if $b = O$, then $\hat{S}(O)$ is the extended semigroup of the ring $O$ introduced by Campillo, Delgado and Gusein-Zade in [3].

Moreover, for $v \in S$ the fibre $F^b_v$ is a finite dimensional central hyperplane arrangement in $\text{Im } j_v \cong J^b(v)/J^b(v + 1)$ (see [3, Theorem 1]):

$$F^b_v = J^b(v)/J^b(v + 1) \setminus \bigcup_{i=1}^{r} J^b(v + 1_{(i)})/J^b(v + 1);$$

in particular, $F^v_v$ is not a vector subspace itself.

(5.6) The fibre $F^b_v$ admits a free $K^\times$-action—namely, multiplication by a nonzero element of $K$—and therefore the projective arrangement $\mathbb{P}F^b_v := F^b_v / K^\times$ can be defined. This allows us to consider the projectivisation of the extended semimodule as

$$\mathbb{P}\hat{S}(b) := \bigcup_{v \in S(b)} \mathbb{P}F^b_v \times \{v\},$$

which is also a graded semimodule in a natural sense. For $v \in \hat{S}(b)$, the space $\mathbb{P}F^b_v$ is the complement to an arrangement of projective hyperplanes in a projective space $\mathbb{P}\left(J^b(v)/J^b(v + 1)\right)$. So it makes sense to consider the Laurent series

$$\chi_g(\mathbb{P}\hat{S}(b)) := \sum_{v \in \mathbb{Z}^r} \chi_g(\mathbb{P}F^b_v) \cdot t^v.$$  

On the other hand, one can also define the extended generalised semimodule Poincaré series of a filtration $\{J^b(v)\}$ defined by $\underline{v}(z) = (v_1(z), \ldots, v_r(z))$, for $z \in b$, as

$$\hat{P}_g(b, \underline{t}, \underline{L}, \{v_i\}) := \int_{\mathbb{P}\hat{S}(b)} L^v d\chi_g$$

(we will write $\hat{P}_g(b, \underline{t}, \underline{L})$ instead of $\hat{P}_g(b, \underline{t}, \underline{L}, \{v_i\})$ when the filtration is clear from the context). Notice that if $b = O$, then $\hat{P}_g(O, \underline{t}, \underline{L})$ coincides with the generalised semigroup Poincaré series defined in [6, p. 507].

All projectivisations $\mathbb{P}F^b_v$ of the fibres $F^b_v$ (i.e., all connected components of $\mathbb{P}\hat{S}(b)$) are complements to arrangements of projective subspaces in finite dimensional projective spaces. We define

$$\hat{L}_g(b, \underline{t}, \underline{L}) := \sum_{v \in \mathbb{Z}^r} \left[ \mathbb{P}(J^b(v)/J^b(v + 1)) \right] \cdot t^v.$$
(5.7) Proposition

\[ \chi_g(\mathbb{P}\hat{S}(b)) = \hat{P}_g(b, l, L) = \frac{\hat{L}_g(b, l, L) \cdot \prod_{i=1}^{r} (t_i - 1)}{t_1 \cdots t_r - 1}. \]

Proof Let \( I \subseteq I_0 \) and set \( L_I := \{(a_1, \ldots, a_r) \in K^r \mid a_i = 0 \text{ for } i \in I\} \). Then

\[ \chi_g(\mathbb{P}\hat{F}_b) = \chi_g\left( \mathbb{P}\hat{J}_b(v) / J^b(v + 1) \right) - \chi_g\left( \mathbb{P}\hat{J}_b(v) / J^b(v + 1) \cap L_I \right) \]

\[ = \sum_{I \subseteq I_0} (-1)^{\#I} \chi_g\left( \mathbb{P}\hat{J}_b(v) / J^b(v + 1) \cap L_I \right) \]

\[ = \sum_{I \subseteq I_0} (-1)^{\#I} \left[ \mathbb{P}\hat{J}_b(v) / J^b(v + 1) \cap L_I \right] \]

Therefore

\[ (t_1 \cdots t_r - 1) \chi_g(\mathbb{P}\hat{F}_b) = \sum_{v \in \mathbb{Z}^r} \sum_{I \subseteq I_0} (-1)^{\#I} \left[ \mathbb{P}\hat{J}_b(v + 1 - I) / J^b(v) \right] \cdot t_v \]

\[ - \sum_{v \in \mathbb{Z}^r} \sum_{I \subseteq I_0} (-1)^{\#I} \left[ \mathbb{P}\hat{J}_b(v + 1 - I) / J^b(v + 1) \right] \cdot t_v \]

\[ = \sum_{v \in \mathbb{Z}^r} \sum_{I \subseteq I_0} (-1)^{\#I} \left[ \mathbb{P}\hat{J}_b(v + 1 - I) / J^b(v + 1) \right] \cdot t_v. \]

\((*)\)

The coefficient of \( t_v \) in the polynomial

\[ \left( \sum_{v \in \mathbb{Z}^r} \left[ \mathbb{P}\hat{J}_b(v) / J^b(v + 1) \right] \right) \cdot \prod_{i=1}^{r} (t_i - 1) \]

is equal to

\[ \sum_{I \subseteq I_0} (-1)^{\#I} \left[ \mathbb{P}\hat{J}_b(v - 1 + I) / J^b(v + 1) \right], \]

and the latter formula coincides with \((*)\). \(\square\)
(5.8) Remark Since \( L - 1 \) is invertible in \( K_0(\nu_k)_L \), the extended generalised semi-module Poincaré series can be rewritten as
\[
\hat{P}_g(b, t, L) = \prod_{i=1}^r \frac{(t_i - 1)}{t_1 \ldots t_r - 1} \cdot \sum_{v \in \mathbb{Z}'} \frac{L^b(v) - 1}{L - 1} \cdot t_v.
\]

Observe the following equality:

(5.9) Proposition For every \( v \in \mathbb{Z}' \), we have
\[
c^b^*(v) = d - c^b(\gamma^b - v - 1).
\]

Proof From Proposition (4.16) we deduce the equalities
\[
c^b^*(v) = \deg(J^b^*(v)) - \deg(J^b^*(v + 1))
= \dim(b/b : \emptyset) - v \cdot d + \deg(J^b(\gamma^b - v))
- \dim(b/b : \emptyset) - (v + 1) \cdot d - \deg(J^b(\gamma^b - v - 1))
= d - c^b(\gamma^b - v - 1).
\]

Proposition (5.9) allows us to describe functional equations for the series \( \hat{L}_g(b, t, L) \) and the Poincaré series \( \hat{P}_g(b, t, L) \), which constitutes the closing result of the paper.

(5.10) Theorem
\[
t^{\gamma^b - 1} \hat{L}_g(b, t^{-1}, L) = -L^{d-1} \hat{L}_g(b^*, t, L^{-1}).
\]
\[
t^{\gamma^b - 1} \hat{P}_g(b, t^{-1}, L) = (-1)^r L^{d-1} \hat{P}_g(b^*, t, L^{-1}).
\]

Proof By Remark (5.8) we have \( \hat{L}_g(b, t, L) = \sum_{v \in \mathbb{Z}'} \frac{L^{c(v)} - 1}{L - 1} \cdot t_v \). It suffices to take \( \hat{L}_g(b, t, L) = \sum_{v \in \mathbb{Z}'} \frac{L^{c(v)} - 1}{L - 1} \cdot t_v \). By Proposition (5.9) it holds \( c^b^*(v) + c^b(\gamma - v - 1) = d \) and we have
\[
t^{\gamma^b - 1} \hat{L}_g(b, t^{-1}, L) = L^d \cdot \sum_{v \in \mathbb{Z}'} \frac{L^{c^b^*(v)} - 1}{L - 1} \cdot t_v.
\]

On the other hand, we have
\[
\hat{L}_g(b^*, t, L^{-1}) = \sum_{v \in \mathbb{Z}'} \frac{L^{c^b^*(v)} - 1}{L - 1} \cdot t_v
= -L \cdot \sum_{v \in \mathbb{Z}'} \frac{L^{c^b^*(v)} - 1}{L - 1} \cdot t_v.
\]
Therefore

\[-\mathbb{L}^{d-1} \cdot \widetilde{L}_g(b^*, L, \mathbb{L}^{-1}) = L^{b-1} \cdot \widetilde{L}_g(b, L^{-1}, \mathbb{L}).\]

Taking now into account that

\[
\hat{P}_g(b, L, \mathbb{L}) = \frac{(t_1 - 1) \cdots (t_r - 1)}{t_1 \cdots t_r - 1} \cdot \widetilde{L}_g(b, L^{-1}, \mathbb{L}),
\]

and the simple relation

\[
(-1)^{r-1} \cdot \frac{(t_1 - 1) \cdots (t_r - 1)}{t_1 \cdots t_r - 1} = \frac{(1 - t_1) \cdots (1 - t_r)}{1 - t_1 \cdots t_r},
\]

we get

\[
\hat{P}_g(b, L^{-1}, \mathbb{L}) = \frac{(t_1^{-1} - 1) \cdots (t_r^{-1} - 1)}{t_1^{-1} \cdots t_r^{-1} - 1} \cdot \widetilde{L}_g(b, L^{-1}, \mathbb{L})
= \frac{(1 - t_1) \cdots (1 - t_r)}{1 - t_1 \cdots t_r} \cdot \widetilde{L}_g(b, L^{-1}, \mathbb{L})
= (-1)^{r-1} \cdot \frac{(t_1 - 1) \cdots (t_r - 1)}{t_1 \cdots t_r - 1} \cdot \widetilde{L}_g(b, L^{-1}, \mathbb{L}).
\]

Because of the equality

\[
\hat{P}_g(b^*, L, \mathbb{L}^{-1}) = \frac{(t_1 - 1) \cdots (t_r - 1)}{t_1 \cdots t_r - 1} \cdot \widetilde{L}_g(b^*, L, \mathbb{L}^{-1}),
\]

we have

\[
\mathbb{L}^{\gamma^* - 1} \cdot \hat{P}_g(b, L^{-1}, \mathbb{L}) = (-1)^{r-1} \cdot \frac{(t_1 - 1) \cdots (t_r - 1)}{t_1 \cdots t_r - 1} \cdot \mathbb{L}^{\gamma^* - 1} \cdot \widetilde{L}_g(b, L^{-1}, \mathbb{L})
= (-1)^{r-1} \cdot \frac{(t_1 - 1) \cdots (t_r - 1)}{t_1 \cdots t_r - 1} \cdot (-\mathbb{L})^{d-1} \cdot \widetilde{L}_g(b^*, L, \mathbb{L}^{-1})
= (-1)^{r} \cdot \frac{(t_1 - 1) \cdots (t_r - 1)}{t_1 \cdots t_r - 1} \cdot \mathbb{L}^{d-1} \cdot \widetilde{L}_g(b^*, L, \mathbb{L}^{-1})
= (-1)^{r} \cdot \mathbb{L}^{d-1} \cdot \hat{P}_g(b^*, L, \mathbb{L}^{-1}),
\]

and we are done. \(\square\)

(5.11) Corollary If \(b = O\) and \(O\) is Gorenstein, then we have

\[
\mathbb{L}^{\gamma^* - 1} \hat{L}_g(O, L^{-1}, \mathbb{L}) = -\mathbb{L}^{d-1} \hat{L}_g(O, L, \mathbb{L}^{-1}).
\]

\[
\mathbb{L}^{\gamma^* - 1} \hat{P}_g(O, L^{-1}, \mathbb{L}) = (-1)^r \mathbb{L}^{d-1} \hat{P}_g(O, L, \mathbb{L}^{-1}).
\]
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