Stationary level surfaces and Liouville-type theorems
characterizing hyperplanes

Shigeru Sakaguchi

Abstract

We consider an entire graph $S : x_{N+1} = f(x), x \in \mathbb{R}^N$ in $\mathbb{R}^{N+1}$ of a continuous real function $f$ over $\mathbb{R}^N$ with $N \geq 1$. Let $\Omega$ be an unbounded domain in $\mathbb{R}^{N+1}$ with boundary $\partial \Omega = S$. Consider nonlinear diffusion equations of the form $\partial_t U = \Delta \phi(U)$ containing the heat equation $\partial_t U = \Delta U$. Let $U = U(X,t) = U(x,x_{N+1},t)$ be the solution of either the initial-boundary value problem over $\Omega$ where the initial value equals zero and the boundary value equals 1, or the Cauchy problem where the initial data is the characteristic function of the set $\mathbb{R}^{N+1} \setminus \Omega$. The problem we consider is to characterize $S$ in such a way that there exists a stationary level surface of $U$ in $\Omega$.

We introduce a new class $\mathcal{A}$ of entire graphs $S$ and, by using the sliding method due to Berestycki, Caffarelli, and Nirenberg, we show that $S \in \mathcal{A}$ must be a hyperplane if there exists a stationary level surface of $U$ in $\Omega$. This is an improvement of the previous result.

Next, we consider the heat equation in particular and we introduce the class $\mathcal{B}$ of entire graphs $S$ of functions $f$ such that each $\{|f(x) - f(y)| : |x - y| \leq 1\}$ is bounded. With the help of the theory of viscosity solutions, we show that $S \in \mathcal{B}$ must be a hyperplane if there exists a stationary isothermic surface of $U$ in $\Omega$. This is a considerable improvement of the previous result.

Related to the problem, we consider a class $\mathcal{W}$ of Weingarten hypersurfaces in $\mathbb{R}^{N+1}$ with $N \geq 1$. Then we show that, if $S$ belongs to $\mathcal{W}$ in the viscosity sense and $S$ satisfies some natural geometric condition, then $S \in \mathcal{B}$ must be a hyperplane. This is also a considerable improvement of the previous result.

Key words. nonlinear diffusion, heat equation, initial-boundary value problem, Cauchy problem, Liouville-type theorems, hyperplanes, stationary level surfaces, stationary isothermic surfaces, sliding method.

---

*This research was partially supported by a Grant-in-Aid for Scientific Research (B) (1 20340031) of Japan Society for the Promotion of Science.

†Department of Applied Mathematics, Graduate School of Engineering, Hiroshima University, Higashi-Hiroshima, 739-8527, Japan. (sakaguch@amath.hiroshima-u.ac.jp).
AMS subject classifications. Primary 35K55, 35K60, 35K05, 35K15, 35K20, 35J60, 53A07; Secondary 35J15, 53C21, 53C45.

1 Introduction

For $f \in C(\mathbb{R}^N)$ where $N \geq 1$, let $\Omega$ be a domain in $\mathbb{R}^{N+1}$ given by

$$\Omega = \{X = (x, x_{N+1}) \in \mathbb{R}^{N+1} : x_{N+1} > f(x)\}. \quad (1.1)$$

Throughout this paper we write $X = (x, x_{N+1}) \in \mathbb{R}^{N+1}$ for $x = (x_1, \ldots, x_N) \in \mathbb{R}^N$. Then we notice that $\partial \Omega = \partial (\mathbb{R}^{N+1} \setminus \overline{\Omega})$. Let $\phi : \mathbb{R} \to \mathbb{R}$ satisfy

$$\phi \in C^2(\mathbb{R}), \quad \phi(0) = 0, \quad \text{and} \quad 0 < \delta_1 \leq \phi'(s) \leq \delta_2 \quad \text{for} \quad s \in \mathbb{R}, \quad (1.2)$$

where $\delta_1, \delta_2$ are positive constants. Consider the unique bounded solution $U = U(X, t)$ of either the initial-boundary value problem:

$$\begin{align*}
    \partial_t U &= \Delta \phi(U) \quad \text{in} \quad \Omega \times (0, +\infty), \quad (1.3) \\
    U &= 1 \quad \text{on} \quad \partial \Omega \times (0, +\infty), \quad (1.4) \\
    U &= 0 \quad \text{on} \quad \Omega \times \{0\}, \quad (1.5)
\end{align*}$$

where $\Delta = \sum_{j=1}^{N+1} \frac{\partial^2}{\partial x_j^2}$, or the Cauchy problem:

$$\begin{align*}
    \partial_t U &= \Delta \phi(U) \quad \text{in} \quad \mathbb{R}^{N+1} \times (0, +\infty) \quad \text{and} \quad U = \chi_{\Omega^c} \quad \text{on} \quad \mathbb{R}^{N+1} \times \{0\}; \quad (1.6)
\end{align*}$$

where $\chi_{\Omega^c}$ denotes the characteristic function of the set $\Omega^c = \mathbb{R}^{N+1} \setminus \Omega$. Note that the uniqueness of the solution of either problem (1.3)-(1.5) or problem (1.6) follows from the comparison principle (see [MS3, Theorem A.1, p. 253]). We consider the solution $U \in C^{2,1}(\Omega \times (0, +\infty)) \cap L^\infty(\Omega \times (0, +\infty)) \cap C(\overline{\Omega} \times (0, +\infty))$ such that $U(\cdot, t) \to 0$ in $L^1_{\text{loc}}(\Omega)$ as $t \to 0^+$ for problem (1.3)-(1.5). For problem (1.6), we consider the solution $U \in C^{2,1}(\mathbb{R}^{N+1} \times (0, +\infty)) \cap L^\infty(\mathbb{R}^{N+1} \times (0, +\infty))$ such that $U(\cdot, t) \to \chi_{\Omega}(\cdot)$ in $L^1_{\text{loc}}(\mathbb{R}^{N+1})$ as $t \to 0^+$.

By the strong comparison principle, we know that

$$0 < U < 1 \quad \text{and} \quad \frac{\partial U}{\partial x_{N+1}} < 0 \quad \text{either in} \quad \Omega \times (0, +\infty) \quad \text{or in} \quad \mathbb{R}^{N+1} \times (0, +\infty). \quad (1.7)$$

The profile of $U$ as $t \to 0^+$ is controlled by the function $\Phi$ defined by

$$\Phi(s) = \int_1^s \frac{\phi'(\xi)}{\xi} d\xi \quad \text{for} \quad s > 0. \quad (1.8)$$
In fact, in \(\text{MS3, Theorem 2.1 and Remark 2.2, p. 239}\) (see also \(\text{MS1, Theorem 1.1 and Theorem 4.1, p. 940 and p. 947}\)) it is shown that, if \(U\) is the solution of either problem (1.3)-(1.5) or problem (1.6), then

\[ -4t\Phi(U(X,t)) \to d(X)^2 \text{ as } t \to 0^+ \text{ uniformly on every compact subset of } \Omega. \quad (1.9) \]

Here, \(d = d(X)\) is the distance function:

\[ d(X) = \text{dist}(X, \partial\Omega) \text{ for } X = (x, x_{N+1}) \in \Omega. \quad (1.10) \]

Formula (1.9) is regarded as a nonlinear version of one obtained by Varadhan \(\text{[Va]}\).

A hypersurface \(\Gamma\) in \(\Omega\) is said to be a stationary level surface of \(U\) (stationary isothermic surface of \(U\) when \(\phi(s) \equiv s\)) if at each time \(t\) the solution \(U\) remains constant on \(\Gamma\) (a constant depending on \(t\)). Hence it follows from (1.9) that there exists \(R > 0\) such that

\[ d(X) = R \text{ for every } X \in \Gamma, \quad (1.11) \]

provided \(\Gamma\) is a stationary level surface of \(U\). The following theorem characterizes the boundary \(\partial\Omega\) in such a way that \(U\) has a stationary level surface \(\Gamma\) in \(\Omega\).

**Theorem 1.1** Let \(U\) be the solution of either problem (1.3)-(1.5) or problem (1.6). Assume that there exists a basis \(\{y^1, y^2, \ldots, y^N\} \subset \mathbb{R}^N\) such that for every \(j = 1, \ldots, N\) the function \(f(x+y^j) - f(x)\) has either a maximum or a minimum in \(\mathbb{R}^N\). Suppose that \(U\) has a stationary level surface \(\Gamma\) in \(\Omega\). Then \(f\) is affine and \(\partial\Omega\) must be a hyperplane.

**Remark 1.2** In order to prove Theorem 1.1, we shall also use the sliding method due to Berestycki, Caffarelli, and Nirenberg \(\text{[BCN]}\). In \(\text{MS3, Theorem 2.3 and Remark 2.4, p. 240}\), instead of the assumption on \(f\), it is assumed that for each \(y \in \mathbb{R}^N\) there exists \(h(y) \in \mathbb{R}\) such that

\[ \lim_{|x| \to \infty} [f(x+y) - f(x)] = h(y), \quad (1.12) \]

which implies the assumption on \(f\) in Theorem 1.1. The condition (1.12) is a modified version of \(\text{[BCN] (7.2), p. 1108]}\), in which \(h(y)\) is supposed identically zero. When \(N = 1\), \(f(x) = ax + b + \sin x (a, b \in \mathbb{R})\) satisfies the assumption on \(f\) in Theorem 1.1 but it does not satisfy (1.12) provided \(\frac{b}{2\pi}\) is not an integer. Another \(f(x) = ax + b + \sin x \tan^{-1} x (a, b \in \mathbb{R})\) does not satisfy the assumption, but it is Lipschitz continuous on \(\mathbb{R}\).

Let us consider the case where \(\phi(s) \equiv s\), that is, that of the heat equation, in particular. The following theorem characterizes the boundary \(\partial\Omega\) in such a way that the caloric function \(U\) has a stationary isothermic surface in \(\Omega\).
Theorem 1.3  Let $\phi(s) \equiv s$ and let $U$ be the solution of either problem (1.3)-(1.5) or problem (1.6). Assume that $U$ has a stationary isothermic surface $\Gamma$ in $\Omega$. Then $f$ is affine and $\partial \Omega$ must be a hyperplane, if either $N \leq 2$ or $\{|f(x) - f(y)| : |x - y| \leq 1\}$ is bounded.

Remark 1.4  When $f$ is Lipschitz continuous in $\mathbb{R}^N$ and $\Omega$ satisfies the uniform exterior sphere condition, this theorem was proved in [MS2, Theorem 1.1 (ii), p. 1113]. By combining [MS4, Lemma 3.1] with [S, Theorem 1.1, p. 887], we see that the assumption that $\Omega$ satisfies the uniform exterior sphere condition is not needed. Also, the Lipschitz continuity of $f$ can be replaced by the uniform continuity of $f$, because of Professor Hitoshi Ishii’s suggestion. Namely, by essentially the same proof as in [S], it can be shown that [S, Theorem 1.1, p. 887] holds even if the Lipschitz continuity is replaced by the uniform continuity. Here, the advantage of Theorem 1.3 is that we do not need to assume any uniform continuity of $f$.

Let $F = F(s)$ be a $C^1$ symmetric and concave function on the positive cone $\Lambda$ given by

$$\Lambda = \{s = (s_1, \cdots, s_N) \in \mathbb{R}^N : \min_{1 \leq j \leq N} s_j > 0\},$$

where $N \geq 1$. Assume that $F$ satisfies

$$F_{s_j} = \frac{\partial F}{\partial s_j} > 0 \quad \text{for all } j = 1, \cdots, N \text{ in } \Lambda. \quad (1.13)$$

Define $G = G(s)$ by

$$G(s) = F(1/s_1, \cdots, 1/s_N) \quad \text{for } s \in \Lambda. \quad (1.14)$$

Assume that $G$ is convex in $\Lambda$. Such a class of functions $F$ is dealt with in [A], [S]. Related to Theorems 1.1 and 1.3 for $f \in C(\mathbb{R}^N)$ we consider the domain $\Omega$ given by (1.1). Consider the entire graph $\partial \Omega = \{(x, f(x)) \in \mathbb{R}^{N+1} : x \in \mathbb{R}^N\}$ in $\mathbb{R}^{N+1}$ of $f$. Let $\kappa_1(x), \cdots, \kappa_N(x)$ be the principal curvatures of $\partial \Omega$ with respect to the upward unit normal vector to $\partial \Omega$ at $(x, f(x))$ for $x \in \mathbb{R}^N$. For each $R > 0$, we introduce a function $g \in C(\mathbb{R}^N)$ defined by

$$g(x) = \sup_{|x-y| \leq R} \left\{f(y) + \sqrt{R^2 - |x-y|^2}\right\} \quad \text{for every } x \in \mathbb{R}^N. \quad (1.15)$$

Then we have

$$\{(x, g(x)) \in \mathbb{R}^{N+1} : x \in \mathbb{R}^N\} = \{X \in \mathbb{R}^{N+1} : d(X) = R\} = \Gamma. \quad (1.16)$$

Moreover, let us introduce a function $f^* \in C(\mathbb{R}^N)$ defined by

$$f^*(x) = \inf_{|x-y| \leq R} \left\{g(y) - \sqrt{R^2 - |x-y|^2}\right\} \quad \text{for every } x \in \mathbb{R}^N. \quad (1.17)$$

4
Then, by setting
\[ D = \{ X = (x, x_{N+1}) \in \mathbb{R}^{N+1} : x_{N+1} > g(x) \}, \]  
we notice the following:
\[ \{(x, f^*(x)) \in \mathbb{R}^{N+1} : x \in \mathbb{R}^N\} = \{ X \in \mathbb{R}^{N+1} : \text{dist}(X, D) = R\}, \]  
\[ f(x) \leq f^*(x) \quad \text{for every } x \in \mathbb{R}^N. \]  

The third theorem gives a Liouville-type theorem for some Weingarten hypersurfaces in the viscosity sense.

**Theorem 1.5** Suppose that there exist two real constants \( R > 0 \) and \( c \) such that \( f \in C(\mathbb{R}^N) \) satisfies in the viscosity sense
\[ F(1 - R\kappa_1, \ldots, 1 - R\kappa_N) = c \quad \text{in} \quad \mathbb{R}^N, \]  
and moreover suppose that the equality holds in (1.20), that is,
\[ f(x) = f^*(x) \quad \text{for every } x \in \mathbb{R}^N, \]  
where \( f^* = f^*(x) \) is defined by (1.17). Then, \( c = F(1, \ldots, 1) \) and \( f \) is an affine function, that is, \( \partial \Omega \) must be a hyperplane, provided \( \{|f(x) - f(y)| : |x - y| \leq 1\} \) is bounded.

**Remark 1.6** The case where \( F(s) = \left( \prod_{j=1}^N s_j \right)^{1/N} \) or \( F(s) = \sum_{j=1}^N \log s_j \) is related to Theorem 1.3. The assumption (1.22), that is,
\[ f(x) = \inf_{|x-y| \leq R} \left\{ g(y) - \sqrt{R^2 - |x-y|^2} \right\} \quad \text{for every } x \in \mathbb{R}^N, \]  
implies that
\[ \max_{1 \leq j \leq N} \kappa_j \leq \frac{1}{R} \quad \text{in} \quad \mathbb{R}^N \]  
holds in the viscosity sense, because (1.22) yields that for every point \( X \in \partial \Omega \) there exists an open ball \( B_R(Y) \) with radius \( R \) and centered at \( Y \in \Gamma \) satisfying
\[ X \in \partial B_R(Y) \quad \text{and} \quad B_R(Y) \subset \Omega. \]  
(1.25)

(1.24) is one of main assumptions of [5, Theorem 1.1, p. 887]. Namely, boundedness of \( \{|f(x) - f(y)| : |x - y| \leq 1\} \) is much weaker than Lipschitz continuity of \( f \), but (1.22) is stronger than (1.24). Also, (1.22) is satisfied by every classical \( C^2 \) solution \( f \) of (1.21) having the strict inequality in (1.24), because of the implicit function theorem.
The present paper is organized as follows. In Section 2, we prove Theorem 1.1 by using the sliding method due to Berestycki, Caffarelli, and Nirenberg [BCN]. In Section 3, we prove Theorem 1.3 with the aid of the theory of viscosity solutions. We follow the proof of [S, Theorem 1.1, p. 887] basically, but we here need a key lemma (see Lemma 3.4) which gives new gradient estimates for \( f \) and \( g \), because we do not assume any uniform continuity of \( f \). Section 4 is devoted to a proof of Theorem 1.5, where gradient estimates for \( f \) and \( g \) are replaced by Lipschitz constant estimates for \( f \) and \( g \) (see Lemma 4.2). In Section 5, we give a Bernstein-type theorem for some \( C^2 \) Weingarten hypersurfaces (see Theorem 5.1) as a remark on Theorem 1.5.

2 Proof of Theorem 1.1

Since \( \Gamma \) is a stationary level surface of \( U \), it follows from (1.9), (1.7) and the implicit function theorem that there exist a number \( R > 0 \) and a function \( g \in C^2(\mathbb{R}^N) \) such that both (1.15) and (1.16) hold.

Conversely, let \( \nu(y) \) denote the upward unit normal vector to \( \Gamma \) at \( (y, g(y)) \in \Gamma \). The facts that \( g \) is smooth, \( \partial \Omega \) is a graph, and \( (y, g(y)) - R\nu(y) \in \partial \Omega \) for every \( y \in \mathbb{R}^N \), imply that (1.22), (1.17), and (1.19) hold, namely, both (1.23) and (1.19) where \( f^\ast \) is replaced by \( f \) hold. Hence, we have in particular

\[
\partial \Omega = \{(x, f(x)) \in \mathbb{R}^{N+1} : x \in \mathbb{R}^N \} = \{X \in \mathbb{R}^{N+1} : \text{dist}(X, \overline{D}) = R\}, \tag{2.1}
\]

where \( D \) is given by (1.18). Thus, it follows from (2.1) that for every \( X \in \partial \Omega \) there exists \( Y \in \Gamma \) satisfying

\[
X \in \partial B_R(Y) \quad \text{and} \quad B_R(Y) \subset \Omega. \tag{2.2}
\]

Choose \( j \) arbitrarily. By the assumption of Theorem 1.1, the function \( f(x + y^j) - f(x) \) has either a maximum or a minimum in \( \mathbb{R}^N \). Since the proof below is similar, say \( f(x + y^j) - f(x) \) has a maximum \( M \) in \( \mathbb{R}^N \). Then there exists \( x_0 \in \mathbb{R}^N \) such that

\[
f(x + y^j) - f(x) \leq M = f(x_0 + y^j) - f(x_0) \quad \text{for every } x \in \mathbb{R}^N. \tag{2.3}
\]

Let us use the sliding method due to Berestycki, Caffarelli, and Nirenberg [BCN]. We set

\[
\Omega_{y^j, M} = \{(x, x_{N+1}) \in \mathbb{R}^{N+1} : (x + y^j, x_{N+1} + M) \in \Omega\}.
\]

Then we have

\[
f(x + y^j) - M \leq f(x) \quad \text{for every } x \in \mathbb{R}^N,
\]

\[
\Omega_{y^j, M} \supset \Omega \quad \text{and} \quad (x_0, f(x_0)) \in \partial \Omega \cap \partial \Omega_{y^j, M}.
\]
Suppose that $\Omega_{y,M} \not\subset \Omega$. Then, by the strong comparison principle we have
\[ U(x + y^j, x_{N+1} + M, t) < U(X, t) \quad \text{for every} \quad (X, t) = (x, x_{N+1}, t) \in \Omega \times (0, +\infty). \quad (2.4) \]
On the other hand, since $(x_0, f(x_0)) \in \partial \Omega \cap \partial \Omega_{y,M}$ and $\Omega_{y,M} \supset \Omega$, it follows from (2.2) that there exists $Y_0 = (y_0, g(y_0)) \in \Gamma$ satisfying
\[ (x_0, f(x_0)) \in \partial B_R(Y_0) \quad \text{and} \quad B_R(Y_0) \subset \Omega \subset \Omega_{y,M}. \]
Hence, since $\Gamma = \{ X \in \mathbb{R}^{N+1} : d(X) = R \}$ and $\Gamma$ is a stationary level surface of $U$, we have
\[ U(y_0 + y^j, g(y_0) + M, t) = U(Y_0, t) \quad \text{for every} \quad t > 0, \]
which contradicts (2.4). Thus, we get $\Omega_{y,M} = \Omega$, that is,
\[ f(x + y^j) - M = f(x) \quad \text{for every} \quad x \in \mathbb{R}^N. \]
Therefore we conclude that there exist $a_1, \ldots, a_N \in \mathbb{R}$ satisfying
\[ f(x + y^j) = f(x) + a_j \quad \text{for every} \quad x \in \mathbb{R}^N \quad \text{and for} \quad j = 1, \ldots, N, \quad (2.5) \]
since $j$ is chosen arbitrarily. Since $f$ is continuous on $\mathbb{R}^N$ and $\{y^1, y^2, \ldots, y^N\}$ is a basis of $\mathbb{R}^N$, we can solve (2.5) as a system of functional equations and conclude that $f(x)$ is determined by its values on $E = \{ \sum_{j=1}^N \beta_j y^j \in \mathbb{R}^N : 0 \leq \beta_j < 1, \ j = 1, \ldots, N \}$. Indeed, if $x = \sum_{j=1}^N (r_j + \beta_j) y^j$ for $r = (r_1, \ldots, r_N) \in \mathbb{Z}^N$ and $\beta = (\beta_1, \ldots, \beta_N) \in [0, 1)^N$, then $f(x) = f(\sum_{j=1}^N \beta_j y^j) + \sum_{j=1}^N r_j a_j$. Moreover, this property of $f$ implies that for every $y \in \mathbb{R}^N$ the function $f(x + y) - f(x)$ has either a maximum or a minimum on $\mathbb{R}^N$. Thus, by employing the sliding method again, we get
\[ f(x + y) - f(x) = f(z + y) - f(z) \quad \text{for every} \quad x, y, z \in \mathbb{R}^N. \quad (2.6) \]
Since $f$ is continuous on $\mathbb{R}^N$, we solve (2.6) as a system of functional equations and conclude that $f$ is affine. This completes the proof of Theorem 1.1.

### 3 Proof of Theorem 1.3

Note that $U$ is real analytic in $x$, since $U$ satisfies the heat equation. Since $\Gamma$ is a stationary isothermic surface of $U$, it follows from (1.7) and the implicit function theorem that $\Gamma$ is the graph of a real analytic function $g = g(x)$ for $x \in \mathbb{R}^N$. Let us first quote an important lemma from [MS4, Lemma 3.1]. We can use this lemma, since $\partial \Omega = \partial (\mathbb{R}^{N+1} \setminus \overline{\Omega})$, $\Gamma$ is
already real analytic and $\Gamma = \partial D$ where $D$ is given by (1.18). The interior cone condition of $D$ in the lemma) with respect to $\Gamma$ is of course satisfied, but in [MS4] it is used only to show that $\Gamma$ is smooth.

**Lemma 3.1 ([MS4])** The following assertions hold:

1. There exists a number $R > 0$ such that $d(X) = R$ for every $X \in \Gamma$;
2. $\Gamma$ is a real analytic hypersurface;
3. $\partial \Omega$ is also a real analytic hypersurface, such that the mapping $\partial \Omega \ni (x, f(x)) \mapsto Y(x, f(x)) \equiv (x, f(x)) + R\nu(x) \in \Gamma$, where $\nu(x)$ is the upward unit normal vector to $\partial \Omega$ at $(x, f(x)) \in \partial \Omega$, is a diffeomorphism; in particular, $\partial \Omega$ and $\Gamma$ are parallel hypersurfaces at distance $R$;
4. it holds that
   $$\max_{1 \leq j \leq N} \kappa_j(x) < \frac{1}{R} \quad \text{for every } x \in \mathbb{R}^N,$$
   where $\kappa_1(x), \ldots, \kappa_N(x)$ are the principal curvatures of $\partial \Omega$ at $(x, f(x)) \in \partial \Omega$ with respect to the upward unit normal vector to $\partial \Omega$;
5. there exists a number $c > 0$ such that
   $$\prod_{j=1}^{N} (1 - R\kappa_j(x)) = c \quad \text{for every } x \in \mathbb{R}^N.$$ (3.2)

Note that (1) follows from (1.9) and (2) follows simply from the implicit function theorem. When $N = 1$, (5) implies the conclusion of Theorem 1.3 since the curvature of the curve $\partial \Omega$ is constant. Let $N \geq 2$. With the aid of Lemma 3.1 applying [S, Lemmas 4.2 and 4.3, p. 891 and p. 892] to $F(s) = \left(\prod_{j=1}^{N} s_j\right)^{1/N}$ yields the following lemma.

**Lemma 3.2** $c = 1$ and $H_{\partial \Omega} \leq 0 \leq H_\Gamma$ in $\mathbb{R}^N$, where $H_{\partial \Omega}$ (resp. $H_\Gamma$) denotes the mean curvature of $\partial \Omega$ (resp. $\Gamma$) with respect to the upward unit normal vector to $\partial \Omega$ (resp. $\Gamma$).

When $N = 2$, by setting
$$\Gamma^* = \left\{X \in \Omega : d(X) = \frac{R}{2}\right\},$$ (3.3)
the fact that $c = 1$ implies that $\Gamma^*$ is an entire minimal graph over $\mathbb{R}^2$. Therefore, by the Bernstein’s theorem for the minimal surface equation, $\Gamma^*$ must be a hyperplane as in
(See [GT, Gi] for the Bernstein’s theorem, and for more general setting see also Theorem 5.1 in Section 5 in the present paper.) Thus it remains to consider the case where \( N \geq 3 \) and \( \{ |f(x) - f(y)| : |x - y| \leq 1 \} \) is bounded.

On the other hand, (3) of Lemma 3.1 gives us the following geometric property.

**Lemma 3.3** The following two assertions hold:

(i) For every \( Y \in \Gamma \) there exists \( X \in \partial \Omega \) such that \( Y \in \partial B_R(X) \) and \( B_R(X) \subset \mathbb{R}^{N+1} \setminus \overline{D} \).

(ii) For every \( X \in \partial \Omega \) there exists \( Y \in \Gamma \) such that \( X \in \partial B_R(Y) \) and \( B_R(Y) \subset \Omega \).

Recall that \( f \) and \( g \) have the relationship, (1.15) and (1.23). Since \( \{ |f(x) - f(y)| : |x - y| \leq 1 \} \) is bounded, we see that \( \{ |g(x) - g(y)| : |x - y| \leq 1 \} \) is also bounded. By Lemma 3.2 we have

\[
\mathcal{M}(f) \leq 0 \leq \mathcal{M}(g) \equiv \text{div} \left( \frac{\nabla g}{\sqrt{1 + |\nabla g|^2}} \right) \text{ in } \mathbb{R}^N. \quad (3.4)
\]

Let \( B_n = \{ x \in \mathbb{R}^N : |x| < n \} \) for \( n \in \mathbb{N} \). Then, by [GT, Theorem 16.9, pp. 407–408], for each \( n \in \mathbb{N} \), there exist two functions \( f_n, g_n \in C^2(B_n) \cap C(\overline{B_n}) \) solving

\[
\mathcal{M}(f_n) = \mathcal{M}(g_n) = 0 \text{ in } B_n, \\
f_n = f \text{ and } g_n = g \text{ on } \partial B_n.
\]

Hence it follows from the comparison principle that for each \( n \in \mathbb{N} \) there exists \( z_n \in \partial B_n \) such that

\[
f_{n+1} \leq f_n \leq f < g \leq g_n \leq g_{n+1} \text{ and } g_n - f_n \leq g(z_n) - f(z_n) \text{ in } B_n. \quad (3.5)
\]

Since \( \{ |f(x) - f(y)| : |x - y| \leq 1 \} \) is bounded, it follows from (1.15) that \( g - f \) is bounded in \( \mathbb{R}^N \) and hence with the aid of (3.5) there exists a constant \( C_* > 0 \) satisfying

\[
g - C_* \leq f_n \leq f \text{ and } g \leq g_n \leq f + C_* \text{ in } B_n \text{ for every } n \in \mathbb{N}. \quad (3.6)
\]

Thus, since both \( \{ |f(x) - f(y)| : |x - y| \leq 1 \} \) and \( \{ |g(x) - g(y)| : |x - y| \leq 1 \} \) are bounded, by using the interior estimates for the minimal surface equation (see [GT, Corollary 16.7, p. 407]) with the aid of (3.6) and the monotonicity with \( n \in \mathbb{N} \), we proceed as in [S, pp. 893–894] to see that there exist two functions \( f_\infty, g_\infty \in C^\infty(\mathbb{R}^N) \) satisfying

\[
\mathcal{M}(f_\infty) = \mathcal{M}(g_\infty) = 0 \text{ in } \mathbb{R}^N, \\
|\nabla f_\infty| \text{ and } |\nabla g_\infty| \text{ are bounded on } \mathbb{R}^N, \\
f_n \to f_\infty \text{ and } g_n \to g_\infty \text{ as } n \to \infty \text{ uniformly on every compact set in } \mathbb{R}^N.
\]

9
Then it follows from Moser’s theorem \cite[Corollary, p. 591]{Mo} that both \(f_\infty\) and \(g_\infty\) are affine and hence the graph of \(f_\infty\) is parallel to that of \(g_\infty\) because \(f_\infty \leq g_\infty\) in \(\mathbb{R}^N\). Thus there exists \(\eta \in \mathbb{R}^N\) satisfying

\[
f_\infty(x) = \eta \cdot x + f_\infty(0) \quad \text{and} \quad g_\infty(x) = \eta \cdot x + g_\infty(0) \quad \text{for every} \quad x \in \mathbb{R}^N. \tag{3.7}
\]

Moreover we have

\[
f_\infty \leq f < g \leq g_\infty \quad \text{in} \quad \mathbb{R}^N, \tag{3.8}
\]

\[
f(z_n) - f_\infty(z_n) \quad \text{and} \quad g_\infty(z_n) - g(z_n) \quad \to 0 \quad \text{as} \quad n \to \infty. \tag{3.9}
\]

Indeed, (3.8) follows from (3.5). Observe that for each \(n \in \mathbb{N}\)

\[
g_n(0) - f_n(0) \leq g(z_n) - f(z_n) \leq g_{n+1}(z_n) - f_{n+1}(z_n)
\]

\[
\leq g(z_{n+1}) - f(z_{n+1}) \leq g_\infty(z_{n+1}) - f_\infty(z_{n+1}) = g_\infty(0) - f_\infty(0).
\]

Hence letting \(n \to \infty\) yields that \(g(z_n) - f(z_n) \to g_\infty(0) - f_\infty(0)\) as \(n \to \infty\). Thus as \(n \to \infty\)

\[
(f(z_n) - f_\infty(z_n)) + (g_\infty(z_n) - g(z_n)) = (g_\infty(0) - f_\infty(0)) - (g(z_n) - f(z_n)) \to 0,
\]

which gives (3.9).

It suffices to show that \(f \equiv f_\infty\) and \(g \equiv g_\infty\). Lemma 3.3 yields the following key lemma.

**Lemma 3.4 (gradient estimates)** There exist three constants \(\varepsilon_0 > 0, \delta_0 > 0, \text{ and } C_0 > 0\) such that

\[
(1) \quad \text{if} \quad z \in \mathbb{R}^N \quad \text{and} \quad (0 \leq) g_\infty(z) - g(z) \leq \varepsilon_0, \quad \text{then} \quad \sup_{|y-z| \leq \delta_0} |\nabla g(y)| \leq C_0.
\]

\[
(2) \quad \text{if} \quad z \in \mathbb{R}^N \quad \text{and} \quad (0 \leq) f(z) - f_\infty(z) \leq \varepsilon_0, \quad \text{then} \quad \sup_{|x-z| \leq \delta_0} |\nabla f(x)| \leq C_0.
\]

**Proof.** (i) of Lemma 3.3 yields (1) and (ii) of Lemma 3.3 yields (2), respectively. Let us show (1). Recall that \(g_\infty\) is affine and \(\nabla g_\infty \equiv \eta\). Denote by \(\mathcal{H}\) the hyperplane given by the graph of \(g_\infty\). Then \(
\frac{(-\eta, 1)}{\sqrt{1 + |\eta|^2}}\) is the upward unit normal vector to \(\mathcal{H}\). By (i) of Lemma 3.3 for every \(Y = (y, g(y)) \in \Gamma\) there exists \(X = (x, f(x)) \in \partial \Omega\) such that the ball \(B_R(X)\) touching \(\Gamma\) from below at \(Y \in \Gamma\) must be below \(\mathcal{H}\). Hence,

\[
\text{if} \quad Y \quad \text{is sufficiently close to} \quad \mathcal{H}, \quad \text{then} \quad \frac{Y - X}{R} \quad \text{is sufficiently close to} \quad \frac{(-\eta, 1)}{\sqrt{1 + |\eta|^2}}. \tag{3.10}
\]
Namely, for every \( \mu > 0 \) there exists \( \lambda > 0 \) such that, if \((0 \leq) g_\infty(y) - g(y) \leq \lambda\), then
\[
\left| \frac{y - x}{R} - \frac{-\eta}{\sqrt{1 + |\eta|^2}} \right|^2 + \left( \frac{g(y) - f(x)}{R} - \frac{1}{\sqrt{1 + |\eta|^2}} \right)^2 < \mu^2. \tag{3.11}
\]
Of course, at the touching point \( Y \), \( \nabla g(y) \) equals the gradient of \( f(x) + \sqrt{R^2 - |y - x|^2} \) with respect to \( y \), that is,
\[
\nabla g(y) = -\frac{y - x}{\sqrt{R^2 - |y - x|^2}}. \tag{3.12}
\]
On the other hand, if a point \((z, g(z)) \in \Gamma \) is sufficiently close to \( \mathcal{H} \), then by (3.10) there exists a uniform neighborhood \( \mathcal{N}_z \) of \( z \) in \( \mathbb{R}^N \) such that every point \( Y = (y, g(y)) \in \Gamma \) with \( y \in \mathcal{N}_z \) is sufficiently close to \( \mathcal{H} \). Namely, for every \( \lambda > 0 \) there exist \( \varepsilon > 0 \) and \( \delta > 0 \) such that, if \((0 \leq) g_\infty(z) - g(z) \leq \varepsilon \) and \(|y - z| < \delta\), then \((0 \leq) g_\infty(y) - g(y) \leq \lambda\). Thus, combining this fact with (3.11) and (3.12) yields (1). (2) is similar. \( \blacksquare \)

The last lemma is

**Lemma 3.5** The following two assertions hold:

(i) \( g_\infty(x + z_n) - g(x + z_n) \to 0 \) as \( n \to \infty \) uniformly on every compact set in \( \mathbb{R}^N \).

(ii) \( f(x + z_n) - f_\infty(x + z_n) \to 0 \) as \( n \to \infty \) uniformly on every compact set in \( \mathbb{R}^N \).

This lemma implies the conclusion of Theorem 1.3. Indeed, in view of (3.8) and Lemma 3.3, Lemma 3.5 yields that the graphs of \( g_\infty \) and \( f_\infty \) are parallel hyperplanes at distance \( R \). This means that \( f \equiv f_\infty \) and \( g \equiv g_\infty \). Thus it remains to prove Lemma 3.5.

**Proof of Lemma 3.5**. Since (ii) is similar to (i), let us show (i). Set
\[ G_n(x) = g(x + z_n) - g(z_n) \text{ for } x \in \mathbb{R}^N \text{ and } n \in \mathbb{N}. \]
Then \( G_n(0) = 0 \) for every \( n \in \mathbb{N} \). Since by (3.9) \( g_\infty(z_n) - g(z_n) \to 0 \) as \( n \to \infty \), it follows from (1) of Lemma 3.3 that there exists \( N_0 \in \mathbb{N} \) such that \( \{G_n : n \geq N_0\} \) is equicontinuous and bounded on \( \overline{B_{\delta_0}(0)} \subset \mathbb{R}^N \). Arzela-Ascoli theorem gives us that there exist a subsequence \( \{G_n'\} \) and a function \( G_\infty \in C(\overline{B_{\delta_0}(0)}) \) such that
\[ G_n' \to G_\infty \text{ as } n \to \infty \text{ uniformly on } \overline{B_{\delta_0}(0)}. \tag{3.13} \]
Notice that \( G_\infty(0) = 0 \). Since \( \mathcal{M}(G_n) \geq 0 \) in \( \mathbb{R}^N \) by (3.4), we have that \( \mathcal{M}(G_\infty) \geq 0 \) in \( B_{\delta_0}(0) \) in the viscosity sense. Observe that
\[
G_n'(x) = g(x + z_n') - g(z_n') \leq g_\infty(x + z_n') - g(z_n')
= \{g_\infty(x + z_n') - g_\infty(z_n')\} + \{g_\infty(z_n') - g(z_n')\} = \eta \cdot x + \{g_\infty(z_n') - g(z_n')\}.
\]
Hence, by (3.9) and (3.13), letting \( n' \to \infty \) yields

\[
G_\infty(x) \leq \eta \cdot x \quad \text{in } B_{\delta_0}(0). \tag{3.14}
\]

Therefore, since \( \mathcal{M}(\eta \cdot x) = 0 \leq \mathcal{M}(G_\infty) \) in \( B_{\delta_0}(0) \) in the viscosity sense and \( \eta \cdot 0 = 0 = G_\infty(0) \), by the strong comparison principle of Giga and Ohnuma \([GO]\) Theorem 3.1, p. 173] we see that

\[
G_\infty(x) \equiv \eta \cdot x \quad \text{in } B_{\delta_0}(0).
\]

Thus \( G_\infty \) is uniquely determined independently of the choice of the subsequence and therefore from (3.13) we conclude that

\[
G_n(x) \to \eta \cdot x \quad \text{as } n \to \infty \text{ uniformly on } B_{\delta_0}(0). \tag{3.15}
\]

Then, since

\[
g_\infty(x + z_n) - g(x + z_n) = \{g_\infty(x + z_n) - g_\infty(z_n)\} - G_n(x) + \{g_\infty(z_n) - g(z_n)\} = \eta \cdot x - G_n(x) + \{g_\infty(z_n) - g(z_n)\},
\]

we get from (3.9) and (3.15)

\[
g_\infty(x + z_n) - g(x + z_n) \to 0 \quad \text{as } n \to \infty \text{ uniformly on } B_{\delta_0}(0). \tag{3.16}
\]

Moreover, by using (1) of Lemma 3.4 again for any point \( z \in \partial B_{\delta_0}(0) \) and repeating the same argument as above, we see that (3.16) holds even if \( B_{\delta_0}(0) \) is replaced by \( B_{3\delta_0}(0) \).

Thus, repeating this argument as many times as one wants yields conclusion (i).

\[\square\]

**Remark 3.6** For the proof of Theorem 1.5 we give a remark for the case where \( N = 1 \).

Even when \( N = 1 \), all the lemmas 3.2 - 3.5 hold true. Indeed, when \( N = 1 \), \( \mathcal{M}(g) = g''(1 + (g')^2)^{-\frac{3}{2}} \) in (3.4). Hence the graphs of \( f_n \) and \( g_n \) are line segments and without using Moser’s theorem we can get two affine functions \( f_\infty \) and \( g_\infty \) in (3.7).

### 4 Proof of Theorem 1.5

We follow the proof of Theorem 1.3. By [S] Lemmas 4.2 and 4.3, p. 891 and p. 892], we have instead of Lemma 3.2

**Lemma 4.1** \( c = F(1, \cdots , 1) \) and \( H_{\partial \Omega} \leq 0 \leq H_\Gamma \) in \( \mathbb{R}^N \) in the viscosity sense, where \( H_{\partial \Omega} \) (resp. \( H_\Gamma \)) denotes the mean curvature of \( \partial \Omega \) (resp. \( \Gamma \)) with respect to the upward unit normal vector to \( \partial \Omega \) (resp. \( \Gamma \)).
Also, in view of (1.15) and (1.23) coming from (1.22), we see that Lemma 3.3 also holds. Then proceeding as in the proof of Theorem 1.3 yields two affine functions $f_\infty$ and $g_\infty$ satisfying (3.7), (3.8), and (3.9). Hence, it suffices to show that $f \equiv f_\infty$ and $g \equiv g_\infty$. Lemma 3.3 yields the following key lemma instead of Lemma 3.4.

**Lemma 4.2 (Lipschitz constant estimates)** There exist three constants $\varepsilon_0 > 0, \delta_0 > 0$, and $C_0 > 0$ such that

1. if $z \in \mathbb{R}^N$ and $(0 \leq) g_\infty(z) - g(z) \leq \varepsilon_0$, then $\sup_{x, y \in B_0(z), x \neq y} \frac{|g(x) - g(y)|}{|x - y|} \leq C_0$.

2. if $z \in \mathbb{R}^N$ and $(0 \leq) f(z) - f_\infty(z) \leq \varepsilon_0$, then $\sup_{x, y \in B_0(z), x \neq y} \frac{|f(x) - f(y)|}{|x - y|} \leq C_0$.

**Proof.** We adjust the proof of Lemma 3.4 to this situation. (i) of Lemma 3.3 yields (1) and (ii) of Lemma 3.3 yields (2), respectively. Let us show (1). Recall that $g_\infty$ is affine and $\nabla g_\infty \equiv \eta$. Denote by $H$ the hyperplane given by the graph of $g_\infty$. Then $\frac{(-\eta, \mathbf{1})}{\sqrt{1 + |\eta|^2}}$ is the upward unit normal vector to $H$. By (i) of Lemma 3.3 for every $Y = (y, g(y)) \in \Gamma$ there exists $X = (x, f(x)) \in \partial \Omega$ such that the ball $B_R(X)$ touching $\Gamma$ from below at $Y \in \Gamma$ must be below $H$. Hence,

$$\text{if } Y \text{ is sufficiently close to } H, \text{ then } \frac{Y - X}{R} \text{ is sufficiently close to } \frac{(-\eta, \mathbf{1})}{\sqrt{1 + |\eta|^2}}. \tag{4.1}$$

Namely, for every $\mu > 0$ there exists $\lambda > 0$ such that, if $(0 \leq) g_\infty(y) - g(y) \leq \lambda$, then

$$\left| \frac{y - x}{R} - \frac{-\eta}{\sqrt{1 + |\eta|^2}} \right|^2 + \left( \frac{g(y) - f(x)}{R} - \frac{1}{\sqrt{1 + |\eta|^2}} \right)^2 < \mu^2. \tag{4.2}$$

On the other hand, if a point $(z, g(z)) \in \Gamma$ is sufficiently close to $H$, then by (4.1) there exists a uniform neighborhood $\mathcal{N}_z$ of $z$ in $\mathbb{R}^N$ such that every point $Y = (y, g(y)) \in \Gamma$ with $y \in \mathcal{N}_z$ is sufficiently close to $H$. Namely, for every $\lambda > 0$ there exist $\varepsilon > 0$ and $\delta > 0$ such that, if $(0 \leq) g_\infty(z) - g(z) \leq \varepsilon$ and $|y - z| < \delta$, then $(0 \leq) g_\infty(y) - g(y) \leq \lambda$.

Moreover, in view of (1.11), by choosing $\frac{\mathbf{2}}{\theta} > \theta > 0$ sufficiently small and introducing a cone $V$ defined by

$$V = \{ \Xi = (\xi, \xi_{N+1}) \in \mathbb{R}^{N+1} : \xi_{N+1} > |\xi| \cos \theta \},$$

we see that, if $Y \in \Gamma$ is sufficiently close to $H$, then $V + Y = \{ \Xi + Y : \Xi \in V \} \subset D$, where $V + Y$ is a cone with vertex $Y$. Here $D$ is given by (1.18). Indeed, if $V + Y \not\subset D$, then there exists another point $\hat{Y}(\neq Y) \in \Gamma \cap (V + Y)$. However, in view of (4.1), a ball $B_R(\hat{X})$
touching \( \Gamma \) from below at \( \tilde{Y} \) might contain \( Y \) since \( \theta > 0 \) is small. This is a contradiction. Namely, if \( Y = (y, g(y)) \in \Gamma \) with \( y \in \mathcal{N}^2 \), then, with the aid of (i) of Lemma 3.4 we must have

\[
V + Y \subset D, \quad B_R(X) \subset \mathbb{R}^{N+1} \setminus \overline{D}, \quad \text{and} \quad Y \in \partial(V + Y) \cap \partial B_R(X).
\]

This gives (1). (2) is similar. \( \square \)

Hence, by using Lemma 4.2 instead of Lemma 3.4 we can proceed as in the proof of Theorem 1.3 to see that Lemma 3.5 also holds. Therefore, (3.8), Lemma 3.3 and Lemma 3.5 yield the conclusion of Theorem 1.5.

5 Concluding remarks

When \( N = 2 \), we have a Bernstein-type theorem for some \( C^2 \) Weingarten hypersurfaces related to Theorem 1.3.

**Theorem 5.1** Suppose that there exist two real constants \( R > 0 \) and \( c \) such that \( f \in C^2(\mathbb{R}^2) \) satisfies

\[
F(1 - R\kappa_1, 1 - R\kappa_2) = c \quad \text{and} \quad \max_{1 \leq j \leq 2} \kappa_j(x) < \frac{1}{R} \quad \text{in} \quad \mathbb{R}^2.
\]

Then, \( c = F(1, 1) \) and \( f \) is an affine function, that is, \( \partial \Omega \) must be a hyperplane.

**Proof.** Here we have Lemma 4.1. We consider \( \Gamma^* \) defined by (3.3) as in Section 3. Then \( \partial \Omega, \Gamma^* \), and \( \Gamma \) are parallel hypersurfaces. Denote by \( \kappa_1^*(Z), \kappa_2^*(Z) \) the principal curvatures of \( \Gamma^* \) with respect to the upward unit normal vector \( \nu^*(Z) \) to \( \Gamma^* \) at \( Z \in \Gamma^* \), and denote by \( \hat{\kappa}_1(Y), \hat{\kappa}_2(Y) \) the principal curvatures of \( \Gamma \) with respect to the upward unit normal vector at \( Y = Z + \frac{R}{2}\nu^*(Z) \in \Gamma \). Also, here for the principal curvatures of \( \partial \Omega \) we use the notation \( \kappa_1(X), \kappa_2(X) \) instead of \( \kappa_1(x), \kappa_2(x) \) with \( (x, f(x)) = X = Z - \frac{R}{2}\nu^*(Z) \in \partial \Omega \).

These principal curvatures have the following relationship:

\[
\kappa_j(X) = \frac{\kappa_j^*(Z)}{1 + \frac{R}{2}\kappa_j^*(Z)} \quad \text{and} \quad \hat{\kappa}_j(Y) = \frac{\kappa_j^*(Z)}{1 - \frac{R}{2}\kappa_j^*(Z)} \quad \text{for each} \quad j = 1, 2.
\]

Since \( \max_{1 \leq j \leq 2} \kappa_j(X) < \frac{1}{R} \) and \( 1 - R\kappa_j(X) = \frac{1}{1 + R\kappa_j(Y)} \), we see that

\[
-\frac{2}{R} < \kappa_j^*(Z) < \frac{2}{R} \quad \text{for each} \quad j = 1, 2.
\]

(5.2)

On the other hand, by Lemma 4.1 we have

\[
\sum_{j=1}^{2} \frac{\kappa_j^*(Z)}{1 + \frac{R}{2}\kappa_j^*(Z)} \leq 0 \leq \sum_{j=1}^{2} \frac{\kappa_j^*(Z)}{1 - \frac{R}{2}\kappa_j^*(Z)}.
\]
This gives
\[ \kappa_1^* + \kappa_2^* + R\kappa_1^* \kappa_2^* \leq 0 \leq \kappa_1^* + \kappa_2^* - R\kappa_1^* \kappa_2^*, \]
and hence
\[ \kappa_1^* \kappa_2^* \leq 0 \quad \text{and} \quad R\kappa_1^* \kappa_2^* \leq \kappa_1^* + \kappa_2^* \leq -R\kappa_1^* \kappa_2^*. \]
Then, with the aid of (5.2), we conclude that
\[ (\kappa_1^*)^2 + (\kappa_2^*)^2 \leq 2 \cdot (-3) \kappa_1^* \kappa_2^*. \]
Hence the Gauss map of \( \Gamma^* \) is \((-3, 0)\)-quasiconformal on \( \mathbb{R}^2 \) (see [GT, (16.88), p. 424]) and hence by [GT] Corollary 16.19, p. 429 \( \Gamma^* \) must be a hyperplane.

References

[A] B. Andrews, Pinching estimates and motion of hypersurfaces by curvature functions, J. Reine Angew. Math. 608 (2007), 17–33.

[BCN] H. Berestycki, L. A. Caffarelli, and L. Nirenberg, Monotonicity for elliptic equations in unbounded Lipschitz domains, Comm. Pure Appl. Math. 50 (1997), 1089–1111.

[GO] Y. Giga and M. Ohnuma, On strong comparison principle for semicontinuous viscosity solutions of some nonlinear elliptic equations, Int. J. Pure Appl. Math. 22 (2005), 165–184.

[GT] D. Gilbarg and N. S. Trudinger, Elliptic Partial Differential Equations of Second Order, (Second Edition.), Springer-Verlag, Berlin, Heidelberg, New York, Tokyo, 1983.

[Gi] E. Giusti, Minimal Surfaces and Functions of Bounded Variations, Birkhäuser, Boston, Basel, Stuttgart, 1984.

[MS1] R. Magnanini and S. Sakaguchi, Nonlinear diffusion with a bounded stationary level surface, Ann. Inst. Henri Poincaré - (C) Anal. Non Linéaire 27 (2010), 937–952.

[MS2] R. Magnanini and S. Sakaguchi, Stationary isothermic surfaces and some characterizations of the hyperplane in the \( N \)-dimensional Euclidean space, J. Differential Equations 248 (2010), 1112–1119.

[MS3] R. Magnanini and S. Sakaguchi, Interaction between nonlinear diffusion and geometry of domain, J. Differential Equations 252 (2012), 236–257.

[MS4] R. Magnanini and S. Sakaguchi, Matzoh ball soup revisited: the boundary regularity issue, Math. Methods Applied Sciences, in press.

[Mo] J. Moser, On Harnack’s theorem for elliptic differential equations, Comm. Pure Appl. Math. 14 (1961), 577–591.
[S] S. Sakaguchi, A Liouville-type theorem for some Weingarten hypersurfaces, Discrete and Continuous Dynamical Systems - Series S, 4 (2011), 887–895.

[Va] S. R. S. Varadhan, On the behavior of the fundamental solution of the heat equation with variable coefficients, Comm. Pure Appl. Math. 20 (1967), 431–455.