ASYMPTOTICS FOR A BIDIMENSIONAL RISK MODEL WITH TWO GEOMETRIC LÉVY PRICE PROCESSES

YANG YANG
Department of Statistics, Nanjing Audit University
Nanjing 211815, China

KAiyong WANG
School of Mathematics and Physics, Suzhou University of Science and Technology
Suzhou 215009, China

JIAJUN LIU
Department of Mathematical Sciences, Xi’an Jiaotong-Liverpool University
Suzhou 215123, China

ZHIMIN ZHANG*
College of Mathematics and Statistics, Chongqing University
Chongqing 401331, China

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Abstract. Consider a bidimensional risk model with two geometric Lévy price processes and dependent heavy-tailed claims, in which we allow arbitrary dependence structures between the two claim-number processes generated by two kinds of businesses, and between the two geometric Lévy price processes. Under the assumption that the claims have consistently varying tails, the asymptotics for the infinite-time and finite-time ruin probabilities are derived.

1. Introduction. Consider a bidimensional risk model with stochastic return and dependent claims, in which an insurance company operates two lines of business and is allowed to invest its wealth in financial assets. Each line is assumed to be exposed to some catastrophic risks like earthquakes, floods or terrorist attacks, which leads to some dependent claims in the same line of business. In addition, such risks may simultaneously affect the two lines of the company to some extent, therefore, some dependence structure may also exist between the two claim-number processes, but it is hard to know how closely dependent they are. For instance, in the businesses of car insurance and medical insurance by car accidents, a car accident may cause one claim for vehicle damage immediately and more than one medical claims for injures of both the driver and passengers in the subsequent periods. In such an example, the claim-number processes of the two businesses are neither independent nor identical, and it is difficult to measure the dependence between them.

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* Corresponding author: Zhimin Zhang.
Precisely speaking, in this model, for \( k = 1 \) and \( 2 \), the claim sizes \( X_i^{(k)} \), \( i \in \mathbb{N} \), of the \( k \)th business, arriving at successive epochs with inter-arrival times \( \theta_i^{(k)} \), \( i \in \mathbb{N} \), form a sequence of identically distributed but not necessarily independent random variables (r.v.s) with common distribution \( F_k = 1 - F_k \) on \( [0, \infty) \); the arrival times of the successive claims, \( \tau_i^{(k)} = \sum_{j=1}^{i} \theta_j^{(k)} \), \( i \in \mathbb{N} \), with \( \tau_0^{(k)} = 0 \), construct a counting process

\[
N_k(t) = \sup\{i \in \mathbb{N} : \tau_i^{(k)} \leq t\}, \quad t \geq 0,
\]

which represents the claim number of the \( k \)th business up to time \( t \), with a finite mean function \( \lambda_k(t) = EN_k(t) \); and the price process of the investment portfolio of the \( k \)th business is described as a geometric Lévy process \( e^{R_k(t)} \), \( t \in \mathbb{R}^+ \cup \{0\} \), with \( R_k(t) \), \( t \in \mathbb{R}^+ \cup \{0\} \), being a Lévy process, which starts from zero and has independent and stationary increments. This assumption on price processes is widely used in mathematical finance, see [10], [17], [18], [19] and [20], among others.

In our setting, assume that \( X_i^{(1)} \), \( i \in \mathbb{N} \), \( X_i^{(2)} \), \( i \in \mathbb{N} \), \( N(t) = (N_1(t), N_2(t))^T \), \( t \in \mathbb{R}^+ \cup \{0\} \) are mutually independent, but \( X_i^{(1)} \), \( i \in \mathbb{N} \), and \( X_i^{(2)} \), \( i \in \mathbb{N} \), follow some certain dependence structure, respectively, \( N_1(t) \) and \( N_2(t) \) are arbitrarily dependent, and \( R_1(t) \) and \( R_2(t) \) are also arbitrarily dependent. For \( k = 1 \) and \( 2 \), the discounted aggregate claims of the \( k \)th business up to time \( t \geq 0 \) can be expressed as

\[
D_k(t) = \sum_{i=1}^{N_k(t)} X_i^{(k)} e^{-R_k(\tau_i^{(k)})}.
\]

Let nonnegative \( x = (x_1, x_2)^T \) denote the initial surplus vector. In such a bidimensional setting, the discounted surplus process up to time \( t \geq 0 \), denoted by \( U(t) = (U_1(t), U_2(t))^T \) has the form

\[
U(t) = x + \int_0^t e^{-R(s)} C(ds) - D(t), \quad (1)
\]

where \( D(t) = (D_1(t), D_2(t))^T \) represents the bidimensional discounted aggregate claim process, and \( C(t) = (c_1(t), c_2(t))^T = (\int_0^t c_1(s)ds, \int_0^t c_2(s)ds)^T \) is the vector of the total premium accumulated up to time \( t \) and \( c_1(t) \), \( c_2(t) \) denote the density functions of premium income of the two businesses at time \( t \), respectively. Throughout the paper, we assume that the premium density functions \( c_k(t) \), \( k = 1, 2 \), are bounded, i.e., \( 0 \leq c_k(t) \leq M_k \) for some constant \( M_k > 0 \) and all \( t \geq 0 \).

In this paper, we are interested in the asymptotic behavior of ruin probabilities. Define the finite-time ruin probability within a finite time \( t > 0 \) and the infinite-time ruin probability corresponding to risk model (1) as

\[
\psi(x; t) = P\left(\inf_{0 \leq s \leq t} U_1(s) < 0, \inf_{0 \leq s \leq t} U_2(s) < 0 \bigg| U(0) = x\right),
\]

and

\[
\psi(x; \infty) = P\left(\inf_{s \geq 0} U_1(s) < 0, \inf_{s \geq 0} U_2(s) < 0 \bigg| U(0) = x\right).
\]

In the past decades, the investigation of multi-dimensional risk models has attracted a vast amount of attention due to their practical importance. The asymptotic behaviour of variously defined ruin probabilities for risk model (1) and its variants (e.g. with constant premium rate \( c > 0 \), i.e. \( c_1(t) = c_2(t) = c \) for all \( t \geq 0 \); with constant interest force \( r \geq 0 \), i.e. \( R_1(t) = R_2(t) = rt \) for all \( t \geq 0 \);
with the Poisson claim-number process; or with Brownian perturbations) has been widely investigated. Ruin for multi-dimensional heavy-tailed processes was initially studied by [7], who mainly focused on multivariate regularly varying random walks and provided sharp asymptotics for general ruin boundaries. [26] considered a bidimensional compound Poisson risk model with \( R_1(t) = R_2(t) = 0 \) for all \( t \geq 0 \) and a constant premium rate \( c > 0 \), and discussed various methods for evaluation of finite-time ruin probability. Moreover, [4] considered some nonstandard bidimensional risk models in which the two claim-number processes are totally independent. Another research trend is that two lines of businesses share a common claim-number process. [13] extended the renewal model by adding Brownian perturbation, and derived the asymptotics for finite-time ruin probability with subexponential claims. [5] considered an independent renewal model with no interest force and consistently varying tailed claims. Later, [22], [14] and [25] studied a dependent (delayed) renewal risk model with nonnegative constant interest force, in which the claim vectors \( (X^{(1)}_i, X^{(2)}_i)^T \), \( i \in \mathbb{N} \), are i.i.d. but each pair of the two components are however dependent. For more related results in this aspect, one can be referred to [8]. We remark that in most aforementioned works the assumptions on the two claim-number processes tend to two extremes: one is to assume that they are totally independent; the other is to assume that they are totally identical, i.e. the two lines of businesses share a common claim-number process. These two kinds of assumptions are both made for mathematical tractability rather than practical relevance. Hence one consideration of this paper is to allow arbitrary dependence between the two claim-number processes, which is motivated by [23].

Besides, most of the existing bidimensional risk models require a nonnegative constant interest force, which means that insurance companies invest their wealth only into a risk-free market, whereas in practical situations they are allowed to make risk-free and risky investments. In the one-dimensional setting, three recent interesting papers [20], [12] and [24] investigated the renewal risk model with stochastic return allowing the price process of the investment portfolio being a geometric Lévy process.

Following the work of [23] and [24], this paper aims to investigate a bidimensional dependent risk model with stochastic return in which the two sequences of claims \( X^{(1)}_i, i \in \mathbb{N}, \) and \( X^{(2)}_i, i \in \mathbb{N}, \) follow some certain dependence structure, respectively; the price processes of the investment portfolio of two businesses are two arbitrarily dependent geometric Lévy processes \( e^{R_1(t)} \) and \( e^{R_2(t)}, t \in \mathbb{R}^+ \cup \{0\} \); the two claim-number processes \( N_1(t) \) and \( N_2(t), t \in \mathbb{R}^+ \cup \{0\}, \) are arbitrarily dependent; and \( X^{(1)}_i, i \in \mathbb{N}, N(t), t \in \mathbb{R}^+ \cup \{0\} \) and \( R(t), t \in \mathbb{R}^+ \cup \{0\} \) are mutually independent.

The rest of the present paper is organized as follows. Section 2 contains the main results of this paper, after providing some preliminaries and introducing the dependence structures. In Sections 3 and 4, we give a series of lemmas and the proofs of the main results.

2. Preliminaries and main results.

2.1. Heavy-tailed distributions and dependence structures. We shall restrict the claim-size distribution to the class of heavy-tailed distributions, whose moment generating functions do not exist. An important class of heavy-tailed distributions is \( D \), which consists of all distributions with dominated variation. A
distribution \( V \) on \( \mathbb{R} \) belongs to the class \( \mathcal{D} \), if
\[
\limsup_{x \to \infty} \frac{V(xy)}{V(x)} < \infty,
\]
for any \( 0 < y < 1 \). A slightly smaller class is \( \mathcal{C} \) of consistently varying tailed distributions. A distribution \( V \) on \( \mathbb{R} \) belongs to the class \( \mathcal{L} \), if
\[
\limlimsup_{y ↑ 1} \frac{V(xy)}{V(x)} = 1.
\]
Closely related is a wider class \( \mathcal{L} \) of long-tailed distributions. A distribution \( V \) on \( \mathbb{R} \) belongs to the class \( \mathcal{L} \), if
\[
\lim_{x \to \infty} \frac{V(x + y)}{V(x)} = 1,
\]
for any \( y \in \mathbb{R} \). There are some other heavy-tailed subclasses, the class \( \mathcal{ERV} \) of distributions with extended regularly varying tails, and the class \( \mathcal{R} \) of distributions with regularly varying tails. A distribution \( V \) on \( \mathbb{R} \) belongs to the class \( \mathcal{ERV}(-\alpha, -\beta) \), if there are some constants \( 0 < \alpha \leq \beta < \infty \) such that
\[
y^{-\beta} \leq \liminf_{x \to \infty} \frac{V(xy)}{V(x)} \leq \limsup_{x \to \infty} \frac{V(xy)}{V(x)} \leq y^{-\alpha},
\]
for any \( y \geq 1 \). If \( \alpha = \beta \), \( V \) belongs to the class \( \mathcal{R}_{-\alpha} \). Clearly,
\[
\mathcal{R}_{-\alpha} \subset \mathcal{ERV}(-\alpha, -\beta) \subset \mathcal{C} \subset \mathcal{L} \cap \mathcal{D} \subset \mathcal{L}.
\]

For a distribution \( V \) on \([0, \infty)\), denote the upper and lower Matuszewska indices of \( V \), respectively, by
\[
J_V^+ = -\lim_{y \to \infty} \frac{\log \nu_+(y)}{\log y} \quad \text{with} \quad \nu_+(y) := \liminf_{x \to \infty} \frac{V(xy)}{V(x)} \quad \text{for} \ y > 1,
\]
\[
J_V^- = -\lim_{y \to \infty} \frac{\log \nu_-(y)}{\log y} \quad \text{with} \quad \nu_-(y) := \limsup_{x \to \infty} \frac{V(xy)}{V(x)} \quad \text{for} \ y > 1.
\]
Additionally, denote \( L_V = \lim_{y \downarrow 1} \nu_+(y) \) (clearly, \( 0 \leq L_V \leq 1 \)). The presented definitions yield that the following assertions are equivalent (for details, see [1]):

(i) \( V \in \mathcal{D} \), (ii) \( \nu_+(y) > 0 \) for some \( y > 1 \), (iii) \( L_V > 0 \), (iv) \( J_V^+ < \infty \).

It also holds that \( V \in \mathcal{C} \) if and only if \( L_V = 1 \). In addition, for a distribution \( V \in \mathcal{D} \), the well-known Potter’s lemma gives that for any \( 0 < p_V < J_V^- \leq J_V^+ < p_V < \infty \), there exist two positive constants \( C_V \) and \( D_V \) such that
\[
\frac{1}{C_V} (y^{-p_V} \land y^{-p_V}) \leq \frac{V(xy)}{V(x)} \leq C_V (y^{-p_V} \lor y^{-p_V})
\]
holds for all \((x \land y) \geq D_V \), where \( a \lor b = \max\{a, b\} \) and \( a \land b = \min\{a, b\} \) for two real numbers \( a \) and \( b \), see Proposition 2.2.1 of [1]. Clearly, it can be derived that if \( V \in \mathcal{D} \), then for any \( p > J_V^+ \), as \( x \to \infty \),
\[
x^{-p} = \omega(V(x)),
\]
here \( f(x) = o(g(x)) \) means \( \lim_{x \to \infty} f(x)/g(x) = 0 \) for two positive functions \( f(x) \) and \( g(x) \).

Throughout the paper, all limit relationships hold for \( x_1 \land x_2 \to \infty \) unless otherwise stated. For two bivariate positive functions \( g_1(\cdot, \cdot) \) and \( g_2(\cdot, \cdot) \), we write \( g_1 < g_2 \) or \( g_2 \succ g_1 \) if \( \limsup g_1/g_2 \leq 1 \); write \( g_1 \sim g_2 \) if \( \lim g_1/g_2 = 1 \); write \( g_1 = \omega(g_2) \).
if \( \lim g_1 / g_2 = 0 \); write \( g_1 = O(g_2) \) if \( \limsup g_1 / g_2 < \infty \); and write \( g_1 \gg g_2 \) if \( 0 < \liminf g_1 / g_2 \leq \limsup g_1 / g_2 < \infty \). For an event \( A \) we denote its indicator function by \( \mathbb{I}(A) \).

Now we turn to the dependence structures modelling the claim sizes. A sequence of r.v.s \( \xi_n, n \in \mathbb{N} \), are said to be pairwise negatively quadrant dependent (NQD), if for any \( i \neq j \in \mathbb{N} \) and \( x_1, x_2 \in \mathbb{R} \),

\[
P(\xi_i > x_1, \xi_j > x_2) \leq P(\xi_i > x_1)P(\xi_j > x_2).
\]

The pairwise NQD structure was introduced by [11], and is weaker and more verifiable than the commonly used notions of the upper/lower negative dependence (see [2]), and the negative association (see [9]). A more general dependence structure, namely upper tail asymptotic independence (UTAI) structure, was proposed by [16]. A sequence of r.v.s \( \xi_n, n \in \mathbb{N} \), are said to be UTAI, if \( P(\xi_n > x) > 0 \) for all \( x \in \mathbb{R}, n \geq 1 \), and for any \( i \neq j \in \mathbb{N} \),

\[
\lim P(\xi_i > x_1 | \xi_j > x_2) = 0.
\]

Clearly, if a sequence of r.v.s are pairwise NQD, then they are also UTAI.

From Theorems 2.1 and 2.2 below we see that the asymptotic behavior of ruin probabilities is insensitive to the pairwise NQD or UTAI structure among heavy-tailed claims. A similar phenomenon in one-dimensional risk models was observed by [3], [15] and [24], among others.

2.2. Main results. Suppose that the two Lévy processes \( R_1(t) \) and \( R_2(t), t \in \mathbb{R}^+ \cup \{0\} \), are both right continuous with left limit. Let \( ER_k(1) > 0 \), \( k = 1, 2 \), so that \( R_k(t) \) drifts to \( \infty \) almost surely (a.s.) as \( t \to \infty \). For \( k = 1, 2 \), the Laplace exponent of the Lévy process \( R_k(t) \) is defined as

\[
\phi_k(z) = \log E e^{-z R_k(1)}, \quad z \in \mathbb{R}.
\]

If \( \phi_k(z) \) is finite, then for any \( t \geq 0 \),

\[
E e^{-z R_k(t)} = e^{t \phi_k(z)} < \infty,
\]

\( k = 1, 2 \), see, e.g., Proposition 3.14 of [6].

We are now ready to state the main results of this paper. The first one aims to investigate the asymptotic behavior of the infinite-time ruin probability with consistently varying tailed claims. We remark that real-valued \( R_k(t) \), \( k = 1, 2 \), allow the insurer to make risk-free and risky investments, whereas the nonnegative \( R_k(t) \)’s require the insurer to invest its wealth only in a risk-free market. Before introducing Theorem 2.1, we give a proposition below, which plays an important role in the proof of Theorem 2.1.

**Proposition 1.** Consider the renewal risk model (1) with i.i.d. inter-arrival times \( \theta_{i}^{(k)}, i \in \mathbb{N}, \) and \( F_k \in \mathcal{C}, J_k^+ > 0, k = 1, 2 \). Then, relation

\[
P(D_1(\infty) > x_1, D_2(\infty) > x_2)
\sim \int_0^\infty \int_0^\infty P \left( X_1^{(1)} e^{-R_1(s)} > x_1, X_1^{(2)} e^{-R_2(t)} > x_2 \right) d(EN_1(s) \cdot N_2(t)) \tag{4}
\]

holds, if either of the following conditions is satisfied:

1. for each \( k = 1, 2 \), \( X_i^{(k)}, i \in \mathbb{N} \), are pairwise NQD nonnegative r.v.s, and \( \phi_k(2p_{F_k}) < 0 \) for some \( p_{F_k} > J_k^+ \);

2. for each \( k = 1, 2 \), \( X_i^{(k)}, i \in \mathbb{N} \), are UTAI nonnegative r.v.s, and \( R_k(t) \geq 0 \) a.s. for any \( t \geq 0 \).
Clearly, if for each \( k = 1, 2 \), \( R_k(t) \geq 0 \) a.s. for any \( t \geq 0 \), then \( \phi_k(2p_{F_k}) < 0 \) for some \( p_{F_k} > J_{F_k}^+ \).

For simplicity, hereafter, for any \( 0 \leq T \leq \infty \), denote by
\[
\phi(x; T) = \int_0^T \int_0^T P(X_1^{(1)}e^{-R_1(s)} > x_1, X_1^{(2)}e^{-R_2(t)} > x_2) d(EN_1(s) \cdot N_2(t)).
\]

**Remark 1.** Under the conditions of Proposition 1, it holds that
\[
\phi(x; T) = \int_0^T \int_0^T P(X_1^{(1)}e^{-R_1(s)} > x_1, X_1^{(2)}e^{-R_2(t)} > x_2) d(EN_1(s) \cdot N_2(t)).
\]

**Theorem 2.1.** Under the conditions of Proposition 1, it holds that
\[
\psi(x; \infty) \sim \phi(x; \infty).
\]

The second result considers the more general bidimensional risk model, in which \( N_1(t) \) and \( N_2(t) \) are two nonnegative and inter-valued processes, however, the two Lévy processes \( R_1(t) \) and \( R_2(t) \) are assumed to be nonnegative. We aim to establish some asymptotic formulae for the tail probability of discounted aggregate claims and the finite-time ruin probability.

**Proposition 2.** Consider the risk model (1) with UTAI claim sizes \( X_i^{(k)} \), \( i \in \mathbb{N} \), and \( F_k \in \mathcal{L} \cap \mathcal{D} \), \( k = 1, 2 \). Assume that, for any \( t \geq 0 \), \( R_k(t) \geq 0 \) a.s., and \( E(N_k(t))^{2p_{F_k}+2} < \infty \), for some \( p_{F_k} > J_{F_k}^+ \), \( k = 1, 2 \). Then, for any fixed \( T > 0 \) such that \( EN_1(T)N_2(T) > 0 \), it holds that
\[
P(D_1(T) > x_1, D_2(T) > x_2) \sim \phi(x; T).
\]

**Theorem 2.2.** Under the conditions of Proposition 2, if \( F_k \in \mathcal{C} \), \( k = 1, 2 \), then for any fixed \( T > 0 \) such that \( EN_1(T)N_2(T) > 0 \), it holds that
\[
\psi(x; T) \sim \phi(x; T).
\]

In a special case that the nonnegative Lévy processes \( R_k(t) \), \( k = 1, 2 \), reduce to two deterministic linear functions with constant interest force \( r \geq 0 \), i.e., \( R_1(t) = R_2(t) = rt \) for all \( t \geq 0 \), Theorem 2.2 can be further refined.

**Corollary 1.** Under the conditions of Proposition 2, if \( R_1(t) = R_2(t) = rt \) for all \( t \geq 0 \) and some \( r \geq 0 \), \( k = 1, 2 \), then for any fixed \( T > 0 \) such that \( EN_1(T)N_2(T) > 0 \), relation (7) holds.

We remark that in Proposition 2, Theorem 2.2 and Corollary 1, the risk model is not necessarily assumed to be a renewal one, i.e., the inter-arrival times \( \theta_i^{(k)}, i \in \mathbb{N} \), are not necessarily i.i.d., and \( J_{F_k}^+ > 0 \) is also not needed, \( k = 1, 2 \).

**Remark 2.** For two counting processes \( N_1(t) \) and \( N_2(t) \) with their arrival times \( \tau_i^{(1)} \) and \( \tau_i^{(2)} \), \( i \in \mathbb{N} \), respectively, it holds that for any \( s, t \geq 0 \),
\[
EN_1(s) \cdot N_2(t) = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} P(\tau_i^{(1)} \leq s, \tau_j^{(2)} \leq t),
\]
by which, the right-hand sides of relations (5) and (6) can be rewritten, respectively,
\[
\phi(x; \infty) = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} P(X_1^{(1)}e^{-R_1(\tau_i^{(1)})} > x_1, X_1^{(2)}e^{-R_2(\tau_j^{(2)})} > x_2),
\]
and
\[
\phi(x; T) = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} P(X_1^{(1)}e^{-R_1(\tau_i^{(1)})} > x_1, X_1^{(2)}e^{-R_2(\tau_j^{(2)})} > x_2, \tau_i^{(1)} \leq T, \tau_j^{(2)} \leq T).
\]
3. Some lemmas.

Lemma 3.1. Consider the risk model (1) with \( F_k \in \mathcal{D} \) and \( J_{F_k}^+ > 0 \), \( k = 1, 2 \). Then, for each \( i, j \geq 1 \), there exists some constant \( C > 0 \) such that

\[
P \left( X_i^{(1)} e^{-R_1(r_1^{(i)})} > x_1, X_j^{(2)} e^{-R_2(r_2^{(j)})} > x_2 \right) > C \mathcal{F}_1(x_1) \mathcal{F}_2(x_2).
\]

The proof of this lemma is straightforward by using Fatou’s lemma.

Lemma 3.2. Under the conditions of Lemma 3.1, if \( \phi_k(2 p_{F_k}) < 0 \) for some \( p_{F_k} > J_{F_k}^+ \), \( k = 1, 2 \), then for each \( i, j \geq 1 \), and any \( 0 < z_1 < 1 \), \( 0 < z_2 < 1 \),

\[
P \left( X_i^{(1)} e^{-R_1(r_1^{(i)})} > x_1 z_1, X_j^{(2)} e^{-R_2(r_2^{(j)})} > x_2 z_2 \right)
\]

\[
< (\mathcal{F}_1(z_1^{-1}) \mathcal{F}_2(z_2^{-1})) P \left( X_i^{(1)} e^{-R_1(r_1^{(i)})} > x_1, X_j^{(2)} e^{-R_2(r_2^{(j)})} > x_2 \right). \tag{11}
\]

Proof. Let \( l(x) \) be an increasing function such that \( l(x) \uparrow \infty \) and \( x^s / l(x) \to \infty \) as \( x \to \infty \) for any \( s > 0 \). By Markov’s inequality, \( \phi_1(2 p_{F_1}) < 0 \) for \( p_{F_1} > J_{F_1}^{+} \) and (3), we have

\[
P \left( e^{-R_1(r_1^{(i)})} > \frac{x_1}{l(x_1)} \right) \leq x_1^{-2 p_{F_1}} (l(x_1))^{2 p_{F_1}} E e^{r_1^{(i)} \phi_1(2 p_{F_1})} = o((\mathcal{F}_1(x_1))^2). \tag{12}
\]

Similarly,

\[
P \left( e^{-R_2(r_2^{(j)})} > \frac{x_2}{l(x_2)} \right) = o((\mathcal{F}_2(x_2))^2), \tag{13}
\]

Let \( \Delta \in (0, 1) \) be a sufficiently small constant such that

\[
E(e^{-p_{F_1} R_1(r_1^{(i)})} \wedge e^{-p_{F_1} R_1(r_1^{(i)})}) E(e^{-R_2(r_2^{(j)})} > \Delta) > 0,
\]

\[
E(e^{-p_{F_2} R_2(r_2^{(j)})} \wedge e^{-p_{F_2} R_2(r_2^{(j)})}) E(e^{-R_1(r_1^{(i)})} > \Delta) > 0
\]

and

\[
P(e^{-R_1(r_1^{(i)})} > \Delta, e^{-R_2(r_2^{(j)})} > \Delta) > 0.
\]

For any \( 0 < z_1 < 1 \) and \( 0 < z_2 < 1 \),

\[
I := \frac{P \left( X_i^{(1)} e^{-R_1(r_1^{(i)})} > x_1 z_1, X_j^{(2)} e^{-R_2(r_2^{(j)})} > x_2 z_2 \right)}{P \left( X_i^{(1)} e^{-R_1(r_1^{(i)})} > x_1, X_j^{(2)} e^{-R_2(r_2^{(j)})} > x_2 \right)}
\]

\[
\leq \int_{\frac{r_1^{(1)}}{x_1}}^{\frac{r_1^{(1)}}{x_1}} \int_{\frac{r_2^{(2)}}{x_2}}^{\frac{r_2^{(2)}}{x_2}} \mathcal{F}_1 \left( \frac{z_1}{u} \right) \mathcal{F}_2 \left( \frac{z_2}{v} \right) P \left( e^{-R_1(r_1^{(i)})} \in du, e^{-R_2(r_2^{(j)})} \in dv \right)
\]

\[
+ \int_{\frac{r_1^{(1)}}{x_1}}^{\frac{r_1^{(1)}}{x_1}} \int_{\frac{r_2^{(2)}}{x_2}}^{\infty} \mathcal{F}_1 \left( \frac{z_1}{u} \right) \mathcal{F}_2 \left( \frac{z_2}{v} \right) P \left( e^{-R_1(r_1^{(i)})} \in du, e^{-R_2(r_2^{(j)})} \in dv \right)
\]

\[
+ \int_{\frac{r_1^{(1)}}{x_1}}^{\frac{r_1^{(1)}}{x_1}} \int_{\frac{r_2^{(2)}}{x_2}}^{\infty} \mathcal{F}_1 \left( \frac{z_1}{u} \right) \mathcal{F}_2 \left( \frac{z_2}{v} \right) P \left( e^{-R_1(r_1^{(i)})} \in du, e^{-R_2(r_2^{(j)})} \in dv \right)
\]

\[
= I_1 + I_2 + I_3 + I_4. \tag{14}
\]
By \( F_k \in \mathcal{D}, \ k = 1, 2, \)

\[
\limsup I_1 \leq \limsup \sup_{z \geq (x_1)} \frac{F_1'(zz_1)}{F_1(z)} \cdot \sup_{z \geq (x_2)} \frac{F_2'(zz_2)}{F_2(z)} = (F_1'(z_1)F_2'(z_2))^{-1}. \tag{15}
\]

By Fatou’s lemma, (2), Hölder’s inequality, (13) and \( F_2 \in \mathcal{D}, \) we have that

\[
\limsup I_2 \leq C_{F_1}^2 \limsup_{x_2 \to \infty} \int_0^\infty \int_0^\infty F_2\left(\frac{z}{x_2}\right) \frac{F_2\left(\frac{z}{x_2}\right)}{F_2(z)} \left( e^{-R_1(\tau_1^{(1)})} + e^{-R_1(\tau_1^{(2)})} \right) \left( e^{-R_2(\tau_2^{(1)})} + e^{-R_2(\tau_2^{(2)})} \right) \, du \, dv.
\]

\[
\leq C_{F_1}^2 \int_0^\infty \int_0^\infty \left( e^{-pF_1 R_1(\tau_1^{(1)})} + e^{-pF_1 R_1(\tau_1^{(2)})} \right) \left( e^{-R_2(\tau_2^{(1)})} + e^{-R_2(\tau_2^{(2)})} \right) \frac{x_2}{F_2\left(\frac{z}{x_2}\right)} \, du \, dv.
\]

\[
\leq C_{F_1}^2 \cdot \limsup_{x_2 \to \infty} \frac{P(e^{-R_2(\tau_2^{(2)})} > \frac{x_2}{F_2(\Delta)})}{F_2\left(\frac{z}{x_2}\right)} = 0, \tag{16}
\]

where in the last step we used the fact

\[
E \left( e^{-pF_1 R_1(\tau_1^{(1)})} + e^{-pF_1 R_1(\tau_1^{(2)})} \right)^2 \leq 2 \left( E e^{-2pF_1 R_1(\tau_1^{(1)})} + E e^{-2pF_1 R_1(\tau_1^{(2)})} \right)
\]

\[
\leq 2 \left( E e^{\tau_1^{(1)} \phi_1(2pF_1)} + E e^{\tau_1^{(2)} \phi_1(2pF_1)} \right) \phi_1(2pF_1) < \infty, \tag{17}
\]

by Jensen’s inequality and \( \phi_1(2pF_1) < 0. \) The same arguments can be used to estimate \( I_3. \) For \( I_4, \) by Hölder’s inequality, (12), (13) and \( F_k \in \mathcal{D}, \ k = 1, 2, \) we have that

\[
\limsup I_4 \leq \limsup \frac{P\left( e^{-R_1(\tau_1^{(1)})} > \frac{x_1}{F_1(z)}, e^{-R_2(\tau_2^{(2)})} > \frac{x_2}{F_2(z)} \right)}{F_1\left(\frac{z}{x_1}\right)F_2\left(\frac{z}{x_2}\right)} P\left( e^{-R_1(\tau_1^{(1)})} > \Delta, e^{-R_2(\tau_2^{(2)})} > \Delta \right)
\]

\[
\leq \limsup \frac{P\left( e^{-R_1(\tau_1^{(1)})} > \frac{x_1}{F_1(z)}, e^{-R_2(\tau_2^{(2)})} > \frac{x_2}{F_2(z)} \right)\left( e^{-R_2(\tau_2^{(2)})} > \frac{x_2}{F_2(\Delta)} \right) \frac{x_1}{F_1(\Delta)}}{F_1\left(\frac{z}{x_1}\right)F_2\left(\frac{z}{x_2}\right)} \frac{x_2}{F_2(\Delta)} P\left( e^{-R_1(\tau_1^{(1)})} > \Delta, e^{-R_2(\tau_2^{(2)})} > \Delta \right)
\]

\[
= 0. \tag{18}
\]

Plugging (15)–(18) into (14) gives the desired relation (11). \( \square \)

**Lemma 3.3.** Under the conditions of Lemma 3.2, if the inter-arrival times \( \theta_k^{(i)}, \ i \in \mathbb{N}, \) are i.i.d. nonnegative r.v.s, \( k = 1, 2, \) then
\[
\lim_{n_0 \to \infty} \limsup_{n \to \infty} \frac{P\left(\sum_{i=n_0}^{\infty} X_i^{(1)} e^{-R_1(r^{(1)})} > x_1, \sum_{j=1}^{\infty} X_j^{(2)} e^{-R_2(r^{(2)})} > x_2\right)}{F_1(x_1) F_2(x_2)}
= \lim_{n_0 \to \infty} \limsup_{n \to \infty} \sum_{i=n_0}^{\infty} \sum_{j=1}^{\infty} \frac{P\left(X_i^{(1)} e^{-R_1(r^{(1)})} > x_1, X_j^{(2)} e^{-R_2(r^{(2)})} > x_2\right)}{F_1(x_1) F_2(x_2)}
= 0.
\] (19)

**Proof.** As the proof of Theorem 1 of Chen and Ng (2007), choose a large constant \(A\) such that \(\sum_{i=1}^{\infty} i^{-2} < A\). Thus, for any \(n_0 \geq 1\),

\[
I := P\left(\sum_{i=n_0}^{\infty} X_i^{(1)} e^{-R_1(r^{(1)})} > x_1, \sum_{j=1}^{\infty} X_j^{(2)} e^{-R_2(r^{(2)})} > x_2\right)
\leq P\left(\sum_{i=n_0}^{\infty} X_i^{(1)} e^{-R_1(r^{(1)})} > \sum_{i=n_0}^{\infty} \frac{x_1}{i^2A}, \sum_{j=1}^{\infty} X_j^{(2)} e^{-R_2(r^{(2)})} > \sum_{j=1}^{\infty} \frac{x_2}{j^2A}\right)
\leq \sum_{i=n_0}^{\infty} \sum_{j=1}^{\infty} P\left(X_i^{(1)} e^{-R_1(r^{(1)})} > \frac{x_1}{i^2A}, X_j^{(2)} e^{-R_2(r^{(2)})} > \frac{x_2}{j^2A}\right).
\]

Split the last relation into four parts

\[
\sum_{i=n_0}^{\infty} \sum_{j=1}^{\infty} \left(\int_0^{\frac{x_1}{i^2A}} \int_0^{\frac{x_2}{j^2A}} + \int_0^{\frac{x_1}{i^2A}} \int_{\frac{x_2}{j^2A}}^{\infty} + \int_{\frac{x_1}{i^2A}}^{\infty} \int_0^{\frac{x_2}{j^2A}} + \int_0^{\infty} \int_{\frac{x_1}{i^2A}}^{\frac{x_2}{j^2A}}\right) F_1\left(\frac{x_1}{i^2A}\right) F_2\left(\frac{x_2}{j^2A}\right) P\left(e^{-R_1(r^{(1)})} e^{-R_2(r^{(2)})} \in du, dv\right)
=: I_1 + I_2 + I_3 + I_4,
\]

where \(D_{F_1}\) and \(D_{F_2}\) were defined in (2). Since \(\theta_i^{(k)}\), \(i \in \mathbb{N}\), are i.i.d. r.v.s, \(k = 1, 2\), by (2), Hölder’s inequality, (17) and \(C_e\) inequality, we have that

\[
I_1 \leq C_{F_1} C_{F_2} F_1(x_1) F_2(x_2) \sum_{i=n_0}^{\infty} \sum_{j=1}^{\infty} \int_0^{\frac{x_1}{i^2A}} \int_0^{\frac{x_2}{j^2A}} \left((i^2Au)^{P_{F_1}} + (i^2Au)^{P_{F_2}}\right) P\left(e^{-R_1(r^{(1)})} e^{-R_2(r^{(2)})} \in du, dv\right)
\leq C_{F_1} C_{F_2} A^{P_{F_1} + P_{F_2}} F_1(x_1) F_2(x_2) \sum_{i=n_0}^{\infty} \sum_{j=1}^{\infty} i^{2P_{F_1}} j^{2P_{F_2}}
\times \left(E\left(e^{-P_{F_1} R_1(r^{(1)})} + e^{-P_{F_1} R_1(r^{(1)})}\right) e^{-P_{F_2} R_2(r^{(2)})} e^{-P_{F_2} R_2(r^{(2)})}\right)
\leq C_{F_1} C_{F_2} A^{P_{F_1} + P_{F_2}} F_1(x_1) F_2(x_2) \sum_{i=n_0}^{\infty} \sum_{j=1}^{\infty} i^{2P_{F_1}} j^{2P_{F_2}}
\times \left(E\left(e^{-P_{F_1} R_1(r^{(1)})} + e^{-P_{F_1} R_1(r^{(1)})}\right) E\left(e^{-P_{F_2} R_2(r^{(2)})} e^{-P_{F_2} R_2(r^{(2)})}\right)\right)^{\frac{1}{2}}
\leq 2C_{F_1} C_{F_2} A^{P_{F_1} + P_{F_2}} F_1(x_1) F_2(x_2)
\]
\[
\times \sum_{i=n_0}^{\infty} i^{2p_{F_1}} \left( E e^{\tau_1(1) \phi_1(2p_{F_1})} + \left( E e^{\tau_1(1) \phi_1(2p_{F_1})} \right)^{\frac{1}{2}} \right) \\
\times \sum_{j=1}^{\infty} j^{2p_{F_2}} \left( E e^{\tau_2(2) \phi_2(2p_{F_2})} + \left( E e^{\tau_2(2) \phi_2(2p_{F_2})} \right)^{\frac{1}{2}} \right) \\
\leq 2C_{F_1} C_{F_2} A^{p_{F_1}+p_{F_2}} F_1(x_1) F_2(x_2) \\
\times \sum_{i=n_0}^{\infty} i^{2p_{F_1}} \left( E e^{\theta_1(1) \phi_1(2p_{F_1})} + \left( E e^{\theta_1(1) \phi_1(2p_{F_1})} \right)^{\frac{1}{2}} \right) \\
\times \sum_{j=1}^{\infty} j^{2p_{F_2}} \left( E e^{\theta_2(2) \phi_2(2p_{F_2})} + \left( E e^{\theta_2(2) \phi_2(2p_{F_2})} \right)^{\frac{1}{2}} \right),
\]
which, together with \( \phi_k(2p_{F_k}) < 0, k = 1, 2 \), yields that
\[
\lim_{n_0 \to -\infty} \limsup_{F_1(x_1) F_2(x_2)} \frac{I_k}{\phi_k(x_1) \phi_k(x_2)} = 0
\]
holds for \( k = 1 \). Again by (2), Markov’s inequality, Hölder’s inequality and \( C_r \) inequality,
\[
I_2 \leq C_{F_1} F_1(x_1) \sum_{i=n_0}^{\infty} \sum_{j=1}^{\infty} \int_{-x_2}^{x_2} \int_{-x_2}^{x_2} \left( (t^2 A u)^{p_{F_1}} + (t^2 A u)^{p_{F_2}} \right) \\
\times (x_2^{-1} j^{2D_{F_1} A})^p F_2 P \left( e^{-R_1(\tau_1(1))} e^{-R_2(\tau_2(2))} \right) \\
\leq C_{F_1} A^{p_{F_1}+p_{F_2}} D_{F_1} F_1(x_1) x_2^{-p_{F_2}} \\
\times \sum_{i=n_0}^{\infty} \sum_{j=1}^{\infty} i^{2p_{F_1}} j^{2p_{F_2}} E \left( e^{-p_{F_1} R_1(\tau_1(1))} + e^{-p_{F_1} R_1(\tau_1(1))} e^{-p_{F_2} R_2(\tau_2(2))} \right) \\
\leq C_{F_1} A^{p_{F_1}+p_{F_2}} D_{F_1}(x_1) x_2^{-p_{F_2}} \\
\times \sum_{i=n_0}^{\infty} i^{2p_{F_1}} \left( E e^{\theta_1(1) \phi_1(2p_{F_1})} \right)^{\frac{1}{2}} + \left( E e^{\theta_1(1) \phi_1(2p_{F_1})} \right)^{\frac{1}{2}} \\
\times \sum_{j=1}^{\infty} j^{2p_{F_2}} \left( E e^{\theta_2(2) \phi_2(2p_{F_2})} \right)^{\frac{1}{2}},
\]
which, combining \( \phi_k(2p_{F_k}) < 0, k = 1, 2 \), and by (3), leads to \( (20) \) holding for \( k = 2 \).
Along the similar line, we can prove that \( (20) \) also holds for \( k = 3, 4 \). Therefore, the first claim of \( (19) \) follows.

The second claim of \( (19) \) can be dealt with in the same method by noting \( (20) \).

**Lemma 3.4.** Under the conditions of Proposition 1,
\[
P \left( \sum_{i=1}^{n} X_i^{(1)} e^{-R_1(\tau_1^{(1)})} > x_1, \sum_{j=1}^{n} X_j^{(2)} e^{-R_2(\tau_2^{(2)})} > x_2 \right)
\]
\[ \sim \sum_{i=1}^{n} \sum_{j=1}^{n} P \left( X_i^{(1)} e^{-R_1(\tau_i^{(1)})} > x_1, X_j^{(2)} e^{-R_2(\tau_j^{(2)})} > x_2 \right) \tag{21} \]

holds for any fixed \( n \in \mathbb{N} \).

**Proof.** Clearly, relation (21) holds for \( n = 1 \). Hereafter, assume that \( n \geq 2 \). We first deal with the upper bound. For any \( 0 < \epsilon < 1 \),

\begin{align*}
I & := P \left( \sum_{i=1}^{n} X_i^{(1)} e^{-R_1(\tau_i^{(1)})} > x_1, \sum_{j=1}^{n} X_j^{(2)} e^{-R_2(\tau_j^{(2)})} > x_2 \right) \\
& \leq P \left( \bigcup_{i=1}^{n} \left\{ X_i^{(1)} e^{-R_1(\tau_i^{(1)})} > (1 - \epsilon) x_1 \right\} : \bigcup_{j=1}^{n} \left\{ X_j^{(2)} e^{-R_2(\tau_j^{(2)})} > (1 - \epsilon) x_2 \right\} \right) \\
& \quad + P \left( \sum_{i=1}^{n} X_i^{(1)} e^{-R_1(\tau_i^{(1)})} > x_1, \sum_{j=1}^{n} X_j^{(2)} e^{-R_2(\tau_j^{(2)})} > x_2, \right. \\
& \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \left\{ \bigcap_{i=1}^{n} \left\{ X_i^{(1)} e^{-R_1(\tau_i^{(1)})} \leq (1 - \epsilon) x_1 \right\} \right\} \\
& \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \left\{ \bigcap_{j=1}^{n} \left\{ X_j^{(2)} e^{-R_2(\tau_j^{(2)})} \leq (1 - \epsilon) x_2 \right\} \right\} \\
& =: I_1 + I_2 + I_3. \tag{22} \end{align*}

By Lemma 3.2 and Remark 1, we have that

\[ I_1 \leq \sum_{i=1}^{n} \sum_{j=1}^{n} P \left( X_i^{(1)} e^{-R_1(\tau_i^{(1)})} > (1 - \epsilon) x_1, X_j^{(2)} e^{-R_2(\tau_j^{(2)})} > (1 - \epsilon) x_2 \right) \]

\[ < \left( \frac{1}{\mathbb{F}_{1+1}((1 - \epsilon)^{-1})} \mathbb{F}_{2+1}((1 - \epsilon)^{-1}) \right)^{-1} \]

\[ \times \sum_{i=1}^{n} \sum_{j=1}^{n} P \left( X_i^{(1)} e^{-R_1(\tau_i^{(1)})} > x_1, X_j^{(2)} e^{-R_2(\tau_j^{(2)})} > x_2 \right). \tag{23} \]

Now we turn to \( I_2 \). Clearly,

\begin{align*}
I_2 & \leq \sum_{i=1}^{n} \sum_{j=1}^{n} P \left( X_i^{(1)} e^{-R_1(\tau_i^{(1)})} > \frac{x_1}{n}, X_j^{(2)} e^{-R_2(\tau_j^{(2)})} > \frac{x_2}{n}, \right. \\
& \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \left\{ \sum_{k=1, k \neq i}^{n} X_k^{(1)} e^{-R_1(\tau_k^{(1)})} > \epsilon x_1 \right\} \\
& \leq \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{k=1, k \neq i}^{n} P \left( X_i^{(1)} e^{-R_1(\tau_i^{(1)})} > \frac{x_1}{n}, X_j^{(2)} e^{-R_2(\tau_j^{(2)})} > \frac{x_2}{n}, \right. \\
& \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \left. X_k^{(1)} e^{-R_1(\tau_k^{(1)})} > \frac{\epsilon x_1}{n} \right) \tag{24}.
\end{align*}
In case (1), since \( X^{(k)}_i, \, i \in \mathbb{N} \), are pairwise NQD, \( k = 1, 2 \), we have that
\[
I_2 \leq \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{k=1, k \neq i}^{n} \int_0^\infty \int_0^\infty \int_0^\infty P \left( X^{(1)}_i > \frac{x_1}{nu}, X^{(1)}_k > \frac{x_2}{nt} \right) P \left( X^{(2)}_j > \frac{x_2}{nv} \right)
\]
\[
P \left( e^{-R_1(\tau^{(1)}_i)} \in du, e^{-R_2(\tau^{(2)}_j)} \in dv, e^{-R_1(\tau^{(1)}_k)} \in dt \right)
\]
\[
\leq \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{k=1, k \neq i}^{n} \left( \int \frac{c_{F_1}}{nD_{F_2}} \int \frac{c_{F_2}}{nD_{F_2}} + \int \frac{c_{F_1}}{nD_{F_2}} \int \frac{c_{F_2}}{nD_{F_2}} \right)
\]
\[
+ \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{k=1, k \neq i}^{n} \int_0^\infty X_1 \left( \frac{x_1}{nu} \right) F_1 \left( \frac{x_2}{nv} \right) F_1 \left( \frac{c_{F_1}}{nt} \right)
\]
\[
P \left( e^{-R_1(\tau^{(1)}_i)} \in du, e^{-R_2(\tau^{(2)}_j)} \in dv, e^{-R_1(\tau^{(1)}_k)} \in dt \right)
\]
\[
=: I_{21} + I_{22} + I_{23} + I_{24},
\]
where \( D_{F_1} \) and \( D_{F_2} \) were defined in (2). By (2),
\[
\frac{I_{21}}{F_1(x_1)F_2(x_2)} \leq C_{F_1} C_{F_2} \sum_{i,j,k}^{n} \int_0^\infty \int_0^\infty \left( (nu)^{p_{F_1}} \vee (nu)^{p_{F_1}} \right) \left( (nv)^{p_{F_2}} \vee (nv)^{p_{F_2}} \right)
\]
\[
\times F_1 \left( \frac{c_{F_1}}{nt} \right) P \left( e^{-R_1(\tau^{(1)}_i)} \in du, e^{-R_2(\tau^{(2)}_j)} \in dv, e^{-R_1(\tau^{(1)}_k)} \in dt \right)
\]
\[
\leq C_{F_1} C_{F_2} n^{p_{F_1} + p_{F_2}} \sum_{i,j,k}^{n} \int_0^\infty \int_0^\infty \left( e^{-p_{F_1} R_1(\tau^{(1)}_i)} + e^{-p_{F_2} R_1(\tau^{(1)}_i)} \right)
\]
\[
\times \left( e^{-p_{F_2} R_2(\tau^{(2)}_j)} + e^{-p_{F_2} R_2(\tau^{(2)}_j)} \right) \mathbb{I} \left( X^{(1)}_k > \frac{x_2}{nt} \right)
\]
\[
\rightarrow 0,
\]
where the last limit follows from
\[
E \left( e^{-p_{F_1} R_1(\tau^{(1)}_i)} + e^{-p_{F_2} R_1(\tau^{(1)}_i)} \right) \left( e^{-p_{F_2} R_2(\tau^{(2)}_j)} + e^{-p_{F_2} R_2(\tau^{(2)}_j)} \right) < \infty,
\]
by H"{o}lder’s inequality and (17). For \( I_{22} \), again by (2), H"{o}lder’s inequality, (17) and (13), we have that
\[
\frac{I_{22}}{F_1(x_1)F_2(x_2)} \leq \frac{n^{p_{F_1} + 1} C_{F_1}}{F_1(x_2)} \sum_{i,j,k}^{n} \int_0^\infty \int_0^\infty \left( (nu)^{p_{F_1}} \vee (nu)^{p_{F_1}} \right)
\]
\[
P \left( e^{-R_1(\tau^{(1)}_i)} \in du, e^{-R_2(\tau^{(2)}_j)} \in dv \right)
\]
\[
\leq \frac{n^{p_{F_2} + 1} C_{F_1}}{F_2(x_2)} \sum_{i,j,k}^{n} \sum_{i,j,k}^{n} \left( e^{-p_{F_1} R_1(\tau^{(1)}_i)} + e^{-p_{F_2} R_1(\tau^{(1)}_i)} \right) \mathbb{I} \left( e^{-R_2(\tau^{(2)}_j)} > \frac{x_2}{nt} \right)
\]
\[
\leq \frac{n^{p_{F_2} + 1} C_{F_1}}{F_2(x_2)} \sum_{i,j,k}^{n} \sum_{i,j,k}^{n} \left( e^{-p_{F_1} R_1(\tau^{(1)}_i)} + e^{-p_{F_2} R_1(\tau^{(1)}_i)} \right)^2
The above estimates lead to
\[ \psi_{\mathcal{F}_1(X_1)\mathcal{F}_2(X_2)}(x) \]  
which, by Lemma 3.1, implies that
\[ \underline{I}_{24} \leq n \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{P\left( e^{-R_1(\tau_i^{(1)})} > \frac{x_1}{nD_{F_1}}, e^{-R_2(\tau_j^{(2)})} > \frac{x_2}{nD_{F_2}} \right)}{\mathcal{F}_1(X_1)\mathcal{F}_2(X_2)} \]
\[ \leq n \sum_{i=1}^{n} \sum_{j=1}^{n} \left( P\left( e^{-R_1(\tau_i^{(1)})} > \frac{x_1}{nD_{F_1}} \right) P\left( e^{-R_2(\tau_j^{(2)})} > \frac{x_2}{nD_{F_2}} \right) \right)^{\frac{1}{2}} \]
\[ \to 0. \]

Similarly, we have \( I_{23} = o(\mathcal{F}_1(X_1)\mathcal{F}_2(X_2)) \). For \( I_{24} \), by Hölder’s inequality and (12), (13), we have that
\[ \underline{I}_{24} \leq n \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{P\left( e^{-R_1(\tau_i^{(1)})} > \frac{x_1}{nD_{F_1}}, e^{-R_2(\tau_j^{(2)})} > \frac{x_2}{nD_{F_2}} \right)}{\mathcal{F}_1(X_1)\mathcal{F}_2(X_2)} \]
\[ \leq n \sum_{i=1}^{n} \sum_{j=1}^{n} \left( P\left( e^{-R_1(\tau_i^{(1)})} > \frac{x_1}{nD_{F_1}} \right) P\left( e^{-R_2(\tau_j^{(2)})} > \frac{x_2}{nD_{F_2}} \right) \right)^{\frac{1}{2}} \]
\[ \to 0. \]

The above estimates lead to
\[ I_2 = o(\mathcal{F}_1(X_1)\mathcal{F}_2(X_2)), \]
which, by Lemma 3.1, implies that
\[ I_2 = o(1) \sum_{i=1}^{n} \sum_{j=1}^{n} P\left( X_i^{(1)} e^{-R_1(\tau_i^{(1)})} > x_1, X_j^{(2)} e^{-R_2(\tau_j^{(2)})} > x_2 \right). \]  

In case (2), since \( R_k(t) \geq 0 \) a.s. for any \( t \geq 0 \) and \( X_i^{(k)}, i \in \mathbb{N} \), are UTAI, \( k = 1, 2 \), by (24) we have that
\[ I_2 \leq \sum_{i=1}^{n} \sum_{j=1}^{n} P\left( e^{-R_1(\tau_i^{(1)})} > \frac{x_1}{n}, X_j^{(2)} e^{-R_2(\tau_j^{(2)})} > \frac{x_2}{n} \right) \]
\[ \leq o(1) \sum_{i=1}^{n} \sum_{j=1}^{n} P\left( X_i^{(1)} e^{-R_1(\tau_i^{(1)})} > \frac{x_1}{n}, X_j^{(2)} e^{-R_2(\tau_j^{(2)})} > \frac{x_2}{n} \right) \]
\[ = o(1) \sum_{i=1}^{n} \sum_{j=1}^{n} P\left( X_i^{(1)} e^{-R_1(\tau_i^{(1)})} > x_1, X_j^{(2)} e^{-R_2(\tau_j^{(2)})} > x_2 \right), \]
where in the last step we used Lemma 3.2 and Remark 1. This coincides with relation (25). In the same way,
\[ I_3 = o(1) \sum_{i=1}^{n} \sum_{j=1}^{n} P\left( X_i^{(1)} e^{-R_1(\tau_i^{(1)})} > x_1, X_j^{(2)} e^{-R_2(\tau_j^{(2)})} > x_2 \right). \]  

Plugging (23), (25) and (26) into (22), and by \( F_k \in \mathcal{C} \), \( k = 1, 2 \), we obtain that
\[ \limsup_{\epsilon \to 0} \frac{1}{\sum_{i=1}^{n} \sum_{j=1}^{n} P\left( X_i^{(1)} e^{-R_1(\tau_i^{(1)})} > x_1, X_j^{(2)} e^{-R_2(\tau_j^{(2)})} > x_2 \right)} \leq 1. \]
For the lower bound of (21), by Bofforoni’s inequality,

\[
I \geq P \left( \bigcup_{i=1}^{n} \left\{ X_i^{(1)} e^{-R_1(\tau_i^{(1)})} > x_1 \right\} \right) \\
+ P \left( \bigcup_{j=1}^{n} \left\{ X_j^{(2)} e^{-R_2(\tau_j^{(2)})} > x_2 \right\} \right) \\
- \sum_{i=1}^{n} \sum_{j=1}^{n} P \left( X_i^{(1)} e^{-R_1(\tau_i^{(1)})} > x_1, X_j^{(2)} e^{-R_2(\tau_j^{(2)})} > x_2, X_k^{(1)} e^{-R_1(\tau_k^{(1)})} > x_1 \right) \\
- \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{k=1, k \neq j}^{n} P \left( X_i^{(1)} e^{-R_1(\tau_i^{(1)})} > x_1, X_j^{(2)} e^{-R_2(\tau_j^{(2)})} > x_2, X_k^{(2)} e^{-R_2(\tau_k^{(2)})} > x_2 \right) \\
= \sum_{i=1}^{n} \sum_{j=1}^{n} P \left( X_i^{(1)} e^{-R_1(\tau_i^{(1)})} > x_1, X_j^{(2)} e^{-R_2(\tau_j^{(2)})} > x_2 \right) - I_4 - I_5.
\]

As done in (24), we can obtain that in both cases,

\[
I_k = o(1) \sum_{i=1}^{n} \sum_{j=1}^{n} P \left( X_i^{(1)} e^{-R_1(\tau_i^{(1)})} > x_1, X_j^{(2)} e^{-R_2(\tau_j^{(2)})} > x_2 \right),
\]

\( k = 4, 5 \). The above two estimates give that

\[
\lim \inf \frac{I}{\sum_{i=1}^{n} \sum_{j=1}^{n} P \left( X_i^{(1)} e^{-R_1(\tau_i^{(1)})} > x_1, X_j^{(2)} e^{-R_2(\tau_j^{(2)})} > x_2 \right)} \geq 1.
\]

Therefore, the desired relation (21) follows.

The last lemma plays an important role in the proof of Proposition 2. We remark that such a lemma is a bidimensional version of (3.6) of [24], and our lemma allows that \( R_1(t) \) and \( R_2(t) \) can be arbitrarily dependent.

**Lemma 3.5.** Under the conditions of Proposition 2, for any fixed integer \( N \), it holds that uniformly for all \( 0 \leq s_i \leq T \), \( 0 \leq t_j \leq T \), \( 1 \leq i \leq m \), \( 1 \leq j \leq n \leq N \),

\[
P \left( \sum_{i=1}^{m} X_i^{(1)} e^{-R_1(s_i)} > x_1, \sum_{j=1}^{n} X_j^{(2)} e^{-R_2(t_j)} > x_2 \right) \\
\sim \sum_{i=1}^{m} \sum_{j=1}^{n} P \left( X_i^{(1)} e^{-R_1(s_i)} > x_1, X_j^{(2)} e^{-R_2(t_j)} > x_2 \right).
\]

**Proof.** It is sufficient to show that, for some \( \Delta > 0 \), which can be arbitrarily small, there exists some large \( z_i > 0 \), depending only on \( F_k \), \( k = 1, 2 \), \( \Delta \) and \( N \), such that for all \( 0 \leq s_i \leq T \), \( 0 \leq t_j \leq T \), \( 1 \leq i \leq m \), \( 1 \leq j \leq n \leq N \), and all
Indeed, $P \left( \sup_{t \in [0,T]} R_k(t) \leq \infty, \sup_{t \in [0,T]} R_2(t) < \infty \right) = 1$, which implies that for any $0 < \epsilon < 1$, there exists a sufficiently small $0 < \delta(\epsilon) < 1$, such that

$$P \left( \sup_{t \in [0,T]} R_1(t) < -\log \delta, \sup_{t \in [0,T]} R_2(t) < -\log \delta \right) \geq 1 - \epsilon. \quad (28)$$

By $F_k \in D$, $k = 1, 2$, for any fixed constant $0 < a < 1$, there exist some $C_k(a) > 0$ and large $z_1(a) > 0$, such that for all $x_1 \wedge x_2 \geq z_1(a)$,

$$F_k(ax_k) \leq C_k(a)F_k(x_k), \quad (29)$$

$k = 1, 2$. Thus, by (28) and (29), we have that for all $0 \leq s_i, s_j \leq T, 0 \leq t_j \leq T, 1 \leq i \leq m \leq N, 1 \leq j \leq n \leq N$, and all $x_1 \wedge x_2 \geq \delta z_1(\delta)$,

$$P \left( X_i^{(1)} e^{-R_1(s_i)} > x_1, X_j^{(2)} e^{-R_2(t_j)} > x_2 \right) \geq \int_\delta^1 \int_\delta^1 F_1 \left( \frac{x_1}{u} \right) F_2 \left( \frac{x_2}{v} \right) P \left( e^{-R_1(s_i)} \in du, e^{-R_2(t_j)} \in dv \right) \geq \int_\delta^1 \int_\delta^1 F_1 \left( \frac{x_1}{\delta} \right) F_2 \left( \frac{x_2}{\delta} \right) P \left( e^{-R_1(s_i)} > \delta, e^{-R_2(t_j)} > \delta \right) \geq C_1^{-1}(\delta)C_2^{-1}(\delta)F_1(x_1)F_2(x_2)P \left( \sup_{t \in [0,T]} R_1(s_i) < -\log \delta, \sup_{t \in [0,T]} R_2(t_j) < -\log \delta \right) \geq (1 - \epsilon)C_1^{-1}(\delta)C_2^{-1}(\delta)F_1(x_1)F_2(x_2). \quad (30)$$

For one side, as done in (27), since $R_k(t) \geq 0$ for all $t \geq 0, k = 1, 2$, we have that for all $0 \leq s_i \leq T, 0 \leq t_j \leq T, 1 \leq i \leq m \leq N, 1 \leq j \leq n \leq N$, and all $x_1 > 0, x_2 > 0$,

$$I := P \left( \sum_{i=1}^m X_i^{(1)} e^{-R_1(s_i)} > x_1, \sum_{j=1}^n X_j^{(2)} e^{-R_2(t_j)} > x_2 \right) \geq \sum_{i=1}^m \sum_{j=1}^n P \left( X_i^{(1)} e^{-R_1(s_i)} > x_1, X_j^{(2)} e^{-R_2(t_j)} > x_2 \right) - \sum_{i=1}^m \sum_{j=1}^n P \left( X_i^{(1)} > x_1, X_j^{(2)} > x_2, X_k^{(1)} > x_1 \right) - \sum_{i=1}^m \sum_{j=1}^n P \left( X_i^{(1)} > x_1, X_j^{(2)} > x_2, X_k^{(2)} > x_2 \right) =: \sum_{i=1}^m \sum_{j=1}^n P \left( X_i^{(1)} e^{-R_1(s_i)} > x_1, X_j^{(2)} e^{-R_2(t_j)} > x_2 \right) - I_1 - I_2. \quad (31)$$
By $F_k \in \mathcal{L}$, $k = 1, 2$, there exists a positive, infinitely increasing and slowly varying function $l(x_1)$ such that $l(x_1)/x_1 \to 0$ and for any fixed constant $K > 0$,

$$F_k(x_k - K l(x_k)) \sim F_k(x_k),$$

which implies that for the above $\epsilon > 0$, there exists some large $z_2 \geq \delta z_1(\delta)$, depending only on $F_k$, $k = 1, 2$, and $\epsilon$, such that for all $x_1 \wedge x_2 \geq z_2$,

$$F_k(x_k - l(x_k)) \leq F_k \left( x_k - \frac{l(x_k)}{\delta} \right) \leq (1 + \epsilon) F_k(x_k), \quad (32)$$

$k = 1, 2$. Since $X^{(k)}_i, i \in \mathbb{N}$, are UTAI r.v.s, and by $F_k \in \mathcal{D}$, for the above $\epsilon > 0$, there exists some large $z_3 \geq z_2$, depending only on $F_k$, $k = 1, 2$, and $\epsilon$, $N$, such that for all $0 \leq i < N, 1 \leq i \leq N$ and $x_1 \wedge x_2 \geq z_3$,

$$P \left( X^{(k)}_i > x_k, X^{(k)}_j > x_k \right) \leq P \left( X^{(k)}_i > \frac{l(x_k)}{N}, X^{(k)}_j > \frac{x_k}{N} \right) \leq \epsilon F_k(x_k), \quad (33)$$

$k = 1, 2$, which, together with (30), implies that for all $0 \leq s_i \leq T, 0 \leq t_j \leq T, 1 \leq i < m \leq N, 1 \leq j \leq n \leq N$, and all $x_1 \wedge x_2 \geq z_3$,

$$I_k \leq N^3 \delta F(x_1) F(x_2)$$

$k = 1, 2$. Combining (31) and (34) yields that for all $0 \leq s_i \leq T, 0 \leq t_j \leq T, 1 \leq i < m \leq N, 1 \leq j \leq n \leq N$, and all $x_1 \wedge x_2 \geq z_3$,

$$I \geq (1 - \Delta_1) \sum_{i=1}^{m} \sum_{j=1}^{n} P \left( X^{(1)}_i e^{-R_i(s_i)} > x_1, X^{(2)}_j e^{-R_j(t_j)} > x_2 \right), \quad (35)$$

where $\Delta_1 = 2N^3 \delta (1 - \epsilon)^{-1} C_1(\delta) C_2(\delta) \in (0, 1)$, by letting $\epsilon$ to be sufficiently small.

For another side, as done in (22) and (24), we have that for all $0 \leq s_i \leq T, 0 \leq t_j \leq T, 1 \leq i < m \leq N, 1 \leq j \leq n \leq N$, and all $x_1 > 0, x_2 > 0$,

$$I \leq \sum_{i=1}^{m} \sum_{j=1}^{n} P \left( X^{(1)}_i e^{-R_i(s_i)} > x_1 - l(x_1), X^{(2)}_j e^{-R_j(t_j)} > x_2 - l(x_2) \right)$$

$$+ \sum_{i=1}^{m} \sum_{j=1}^{n} \sum_{k=1, k \neq i}^{m} P \left( X^{(1)}_i > \frac{x_1}{m}, X^{(2)}_j > \frac{x_2}{n}, X^{(1)}_k > \frac{l(x_1)}{m} \right)$$

$$+ \sum_{i=1}^{m} \sum_{j=1}^{n} \sum_{k=1, k \neq j}^{n} P \left( X^{(1)}_i > \frac{x_1}{m}, X^{(2)}_j > \frac{x_2}{n}, X^{(2)}_k > \frac{l(x_2)}{n} \right)$$

$$=: \ I_3 + I_4 + I_5. \quad (36)$$

By (33), (29) and (30), there exists some large $z_0 \geq z_3 \vee z_1(N^{-1})$, depending only on $F_k, k = 1, 2$, and $\epsilon$, $N$, such that for all $0 \leq s_i \leq T, 0 \leq t_j \leq T, 1 \leq i < m \leq N, 1 \leq j \leq n \leq N$, and all $x_1 \wedge x_2 \geq z_0$,

$$I_4 + I_5 \leq N^3 \left( \epsilon F(x_1) F(x_2) \left( \frac{x_2}{N} \right) + \epsilon F(x_2) F(x_1) \left( \frac{x_1}{N} \right) \right)$$

$$\leq N^3 \epsilon \left( C_1(N^{-1}) + C_2(N^{-1}) \right) F(x_1) F(x_2)$$

$$\leq N^3 \epsilon \left( C_1(N^{-1}) + C_2(N^{-1}) \right) \left( 1 - \epsilon \right)^{-1} C_1(\delta) C_2(\delta)$$

$$P \left( X^{(1)}_i e^{-R_i(s_i)} > x_1, X^{(2)}_j e^{-R_j(t_j)} > x_2 \right). \quad (37)$$
For all $0 \leq s_i \leq T$, $0 \leq t_j \leq T$, $1 \leq i \leq m \leq N$, $1 \leq j \leq n \leq N$,

$$J := \Pr \left( X_i^{(1)} e^{-R_1(s_i)} > x_1 - l(x_1), X_j^{(2)} e^{-R_2(t_j)} > x_2 - l(x_2) \right)$$

$$\leq \left( \int_{\delta}^{1} \int_{0}^{1} \int_{0}^{\delta} \int_{0}^{1} \int_{0}^{\delta} \right) \left( \frac{x_1 - l(x_1)}{\delta} \right) \left( \frac{x_2 - l(x_2)}{\delta} \right) \Pr \left( e^{-R_1(s_i)} \in du, e^{-R_2(t_j)} \in dv \right)$$

$$=: J_1 + J_2 + J_3. \quad (38)$$

Since $l(\cdot)$ is infinitely increasing, by (32) we have that for all $0 \leq s_i \leq T$, $0 \leq t_j \leq T$, $1 \leq i \leq m \leq N$, $1 \leq j \leq n \leq N$, and all $x_1 \wedge x_2 \geq z_0$,

$$J_1 \leq \int_{\delta}^{1} \int_{0}^{1} \left( \frac{x_1}{\delta} - \frac{l(x_1)}{\delta} \right) \left( \frac{x_2}{\delta} - \frac{l(x_2)}{\delta} \right) \Pr \left( e^{-R_1(s_i)} \in du, e^{-R_2(t_j)} \in dv \right)$$

$$\leq (1 + \epsilon)^2 \int_{\delta}^{1} \int_{0}^{1} \left( \frac{x_1}{\delta} \right) \left( \frac{x_2}{\delta} \right) \Pr \left( e^{-R_1(s_i)} \in du, e^{-R_2(t_j)} \in dv \right)$$

$$\leq (1 + \epsilon)^2 \Pr \left( X_i^{(1)} e^{-R_1(s_i)} > x_1, X_j^{(2)} e^{-R_2(t_j)} > x_2 \right). \quad (39)$$

By (32), (30) and (28), we have that for all $0 \leq s_i \leq T$, $0 \leq t_j \leq T$, $1 \leq i \leq m \leq N$, $1 \leq j \leq n \leq N$, and all $x_1 \wedge x_2 \geq z_0$,

$$J_2 + J_3 \leq \int_{\delta}^{1} \int_{0}^{1} \left( \frac{x_1}{\delta} \right) \left( \frac{x_2}{\delta} \right) \Pr \left( e^{-R_1(s_i)} \leq \delta, e^{-R_2(t_j)} \leq \delta \right)$$

$$\leq (1 + \epsilon)^2 \int_{\delta}^{1} \int_{0}^{1} \left( \frac{x_1}{\delta} \right) \left( \frac{x_2}{\delta} \right) \Pr \left( R_1(s_i) \geq -\log \delta, R_2(t_j) \geq -\log \delta \right)$$

$$\leq \frac{2\epsilon(1 + \epsilon)^2}{1 - \epsilon} C_1(\delta) C_2(\delta) \Pr \left( X_i^{(1)} e^{-R_1(s_i)} > x_1, X_j^{(2)} e^{-R_2(t_j)} > x_2 \right). \quad (40)$$

Plugging (39) and (40) into (38) yields that for all $0 \leq s_i \leq T$, $0 \leq t_j \leq T$, $1 \leq i \leq m \leq N$, $1 \leq j \leq n \leq N$, and all $x_1 \wedge x_2 \geq z_0$,

$$J \leq \left( 1 + \epsilon^2 + 2\epsilon + \frac{2\epsilon(1 + \epsilon)^2}{1 - \epsilon} C_1(\delta) C_2(\delta) \right) \Pr \left( X_i^{(1)} e^{-R_1(s_i)} > x_1, X_j^{(2)} e^{-R_2(t_j)} > x_2 \right),$$

which implies that for all $0 \leq s_i \leq T$, $0 \leq t_j \leq T$, $1 \leq i \leq m \leq N$, $1 \leq j \leq n \leq N$, and all $x_1 \wedge x_2 \geq z_0$,

$$I_3 \leq \left( 1 + \epsilon^2 + 2\epsilon + \frac{2\epsilon(1 + \epsilon)^2}{1 - \epsilon} C_1(\delta) C_2(\delta) \right) \times \sum_{i=1}^{m} \sum_{j=1}^{n} \Pr \left( X_i^{(1)} e^{-R_1(s_i)} > x_1, X_j^{(2)} e^{-R_2(t_j)} > x_2 \right). \quad (41)$$

Thus, we obtain from (36), (37) and (41) that for all $0 \leq s_i \leq T$, $0 \leq t_j \leq T$, $1 \leq i \leq m \leq N$, $1 \leq j \leq n \leq N$, and all $x_1 \wedge x_2 \geq z_0$,

$$I \leq (1 + \Delta_2) \sum_{i=1}^{m} \sum_{j=1}^{n} \Pr \left( X_i^{(1)} e^{-R_1(s_i)} > x_1, X_j^{(2)} e^{-R_2(t_j)} > x_2 \right). \quad (42)$$
where $\Delta_2 = \epsilon^2 + 2\epsilon + (1-\epsilon)^{-1} C_1(\delta) C_2(\delta) (2(1+\epsilon)^2 + N^3(C_1(N^{-1}) + C_2(N^{-1}))) > 0$.

Therefore, by (35) and (42), choose $\Delta = \Delta_1 \vee \Delta_2 > 0$, which can be arbitrarily small, then relation (27) holds for all $0 \leq s_i \leq T$, $0 \leq t_j \leq T$, $1 \leq i \leq m \leq N$, $1 \leq j \leq n \leq N$, and all $x_1 \wedge x_2 \geq z_0$. \hfill \Box

4. The proofs of main results.

4.1. Proof of Proposition 1. For any $\epsilon > 0$, by Lemmas 3.3 and 3.1, there exists some large integer $n_0$ such that

$$P \left( \sum_{i=n_0}^{\infty} X_i^{(1)} e^{-R_1(\tau_i^{(1)})} > x_1, \sum_{j=1}^{\infty} X_j^{(2)} e^{-R_2(\tau_j^{(2)})} > x_2 \right)$$

$$\vee P \left( \sum_{i=n_0}^{\infty} \sum_{j=1}^{\infty} P \left( X_i^{(1)} e^{-R_1(\tau_i^{(1)})} > x_1, X_j^{(2)} e^{-R_2(\tau_j^{(2)})} > x_2 \right) \right)$$

$$\vee P \left( \sum_{i=1}^{\infty} X_i^{(1)} e^{-R_1(\tau_i^{(1)})} > x_1, \sum_{j=n_0}^{\infty} X_j^{(2)} e^{-R_2(\tau_j^{(2)})} > x_2 \right)$$

$$\vee P \left( \sum_{i=1}^{\infty} \sum_{j=n_0}^{\infty} P \left( X_i^{(1)} e^{-R_1(\tau_i^{(1)})} > x_1, X_j^{(2)} e^{-R_2(\tau_j^{(2)})} > x_2 \right) \right)$$

$$< \epsilon P \left( X_1^{(1)} e^{-R_1(\tau_1^{(1)})} > x_1, X_1^{(2)} e^{-R_2(\tau_1^{(2)})} > x_2 \right). \quad (43)$$

For the upper bound, we split $P(D_1(\infty) > x_1, D_2(\infty) > x_2)$ into four parts. For any $0 < \delta < 1$,

$I := P(D_1(\infty) > x_1, D_2(\infty) > x_2)$

$$\leq P \left( \sum_{i=1}^{n_0-1} X_i^{(1)} e^{-R_1(\tau_i^{(1)})} > (1-\delta)x_1, \sum_{j=1}^{n_0-1} X_j^{(2)} e^{-R_2(\tau_j^{(2)})} > (1-\delta)x_2 \right)$$

$$+ P \left( \sum_{i=1}^{n_0-1} X_i^{(1)} e^{-R_1(\tau_i^{(1)})} > (1-\delta)x_1, \sum_{j=n_0}^{\infty} X_j^{(2)} e^{-R_2(\tau_j^{(2)})} > \delta x_2 \right)$$

$$+ P \left( \sum_{i=n_0}^{\infty} X_i^{(1)} e^{-R_1(\tau_i^{(1)})} > \delta x_1, \sum_{j=1}^{n_0-1} X_j^{(2)} e^{-R_2(\tau_j^{(2)})} > (1-\delta)x_2 \right)$$

$$+ P \left( \sum_{i=n_0}^{\infty} X_i^{(1)} e^{-R_1(\tau_i^{(1)})} > \delta x_1, \sum_{j=n_0}^{\infty} X_j^{(2)} e^{-R_2(\tau_j^{(2)})} > \delta x_2 \right)$$

$$=: I_1 + I_2 + I_3 + I_4. \quad (44)$$

By Lemmas 3.4 and 3.2, we have that

$I_1 \sim \sum_{i=1}^{n_0-1} \sum_{j=1}^{n_0-1} P \left( X_i^{(1)} e^{-R_1(\tau_i^{(1)})} > (1-\delta)x_1, X_j^{(2)} e^{-R_2(\tau_j^{(2)})} > (1-\delta)x_2 \right)$

$$< (F_{1*}(1-\delta)^{-1}F_{2*}(1-\delta)^{-1})^{-1}$$
\[
\sum_{i=1}^{n_0-1} \sum_{j=1}^{n_0-1} P\left(X_i^{(1)} e^{-R_1(\tau_i^{(1)})} > x_1, X_j^{(2)} e^{-R_2(\tau_j^{(2)})} > x_2\right).
\] (45)

By (43) and Lemma 3.2, we have that
\[
I_2 < \epsilon P\left(X_i^{(1)} e^{-R_1(\tau_i^{(1)})} > (1 - \delta)x_1, X_j^{(2)} e^{-R_2(\tau_j^{(2)})} > \delta x_2\right)
\] < \epsilon \left(\frac{F_{1\alpha}}{(1 - \delta)^{-1}F_{2\alpha}(\delta^{-1})} - 1\right)^{-1} \times P\left(X_i^{(1)} e^{-R_1(\tau_i^{(1)})} > x_1, X_j^{(2)} e^{-R_2(\tau_j^{(2)})} > x_2\right).
\] (46)

Similarly to (46), again by (43) and Lemma 3.2,
\[
I_3 + I_4 < \epsilon \left(\frac{F_{1\alpha}((1 - \delta)^{-1})F_{2\alpha}((1 - \delta)^{-1}) - 1}{(1 - \delta)^{-1}F_{2\alpha}((1 - \delta)^{-1}) - 1} \times P\left(X_i^{(1)} e^{-R_1(\tau_i^{(1)})} > x_1, X_j^{(2)} e^{-R_2(\tau_j^{(2)})} > x_2\right)\right)
\] (47)

Plugging (45)–(47) into (44) yields that
\[
I < \left(\frac{F_{1\alpha}((1 - \delta)^{-1})F_{2\alpha}((1 - \delta)^{-1}) - 1}{(1 - \delta)^{-1}F_{2\alpha}((1 - \delta)^{-1}) - 1} \times \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} P\left(X_i^{(1)} e^{-R_1(\tau_i^{(1)})} > x_1, X_j^{(2)} e^{-R_2(\tau_j^{(2)})} > x_2\right)\right),
\]
which tends to \(\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} P\left(X_i^{(1)} e^{-R_1(\tau_i^{(1)})} > x_1, X_j^{(2)} e^{-R_2(\tau_j^{(2)})} > x_2\right)\) by first letting \(\epsilon \downarrow 0\) then \(\delta \downarrow 0\), and noting \(F_k \in C, k = 1, 2\).

For the lower bound, by Lemma 3.4 and (43), it holds that
\[
I > \sum_{i=1}^{n_0-1} \sum_{j=1}^{n_0-1} P\left(X_i^{(1)} e^{-R_1(\tau_i^{(1)})} > x_1, X_j^{(2)} e^{-R_2(\tau_j^{(2)})} > x_2\right)
\]
\[\geq \left(\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} - \sum_{i=n_0}^{\infty} \sum_{j=1}^{\infty} \sum_{i=1}^{n_0} \sum_{j=n_0}^{\infty} \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} - \sum_{i=n_0}^{\infty} \sum_{j=1}^{\infty} \sum_{i=1}^{\infty} \sum_{j=n_0}^{\infty} P\left(X_i^{(1)} e^{-R_1(\tau_i^{(1)})} > x_1, X_j^{(2)} e^{-R_2(\tau_j^{(2)})} > x_2\right)\right),
\]
which tends to \(\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} P\left(X_i^{(1)} e^{-R_1(\tau_i^{(1)})} > x_1, X_j^{(2)} e^{-R_2(\tau_j^{(2)})} > x_2\right)\) by letting \(\epsilon \downarrow 0\). This completes the proof of Proposition 1.

Similarly to the proof of Proposition 1, we can obtain the corresponding result for the one-dimensional version. We remark that the following lemma can also be proven by slightly modification of the proof of Theorem 2.2 of [24].

**Lemma 4.1.** Consider the one-dimensional renewal risk model with i.i.d. inter-arrival times \(\theta_i^{(1)}, i \in \mathbb{N}\), and \(F_1 \in C, F_1^- > 0\). Then, relation
\[
P(D_1(\infty) > x_1) \sim \sum_{i=1}^{\infty} P\left(X_i^{(1)} e^{-R_1(\tau_i^{(1)})} > x_1\right)
\]
holds, if either of the following conditions is satisfied:
Proof of Theorem 2.1. The upper bound of $\psi(x; \infty)$ is trivial. Indeed, by Proposition 1 it holds that

$$\psi(x; \infty) \leq P(D_1(\infty) > x_1, D_2(\infty) > x_2) \sim \varphi(x; \infty).$$  \hfill (48)

We mainly estimate the lower bound of (5). For any $\epsilon > 0$ and the integer $n_0$ defined in (43), we have that

$$\psi(x; \infty) \geq P(D_1(\infty) > x_1, D_2(\infty) > x_2)$$

$$\geq P(D_1(\infty) > (1 + \epsilon)x_1, M_1 \leq \epsilon x_1, D_2(\infty) > (1 + \epsilon)x_2, M_2 \leq \epsilon x_2)$$

$$\geq P(D_1(\infty) > (1 + \epsilon)x_1, D_2(\infty) > (1 + \epsilon)x_2) - P(D_1(\infty) > (1 + \epsilon)x_1) P(M_2 \leq \epsilon x_2)$$

$$- P(D_2(\infty) > (1 + \epsilon)x_2) P(M_1 \leq \epsilon x_1)$$

$$=: I_1 - I_2 - I_3,$$ \hfill (49)

where $Z_k = \int_0^\infty e^{-R_k(s)} ds$ and $M_k > 0$ is the upper bound of the intensity of premium payments of the $k$th business, $k = 1, 2$. By Lemmas 3.4, 3.2 and (43), we have that

$$I_1 \geq \sum_{i=1}^{n_0-1} \sum_{j=1}^{n_0-1} P\left(X_{i}^{(1)} e^{-R_1(\tau_i^{(1)})} > (1 + \epsilon)x_1, X_{j}^{(2)} e^{-R_2(\tau_j^{(2)})} > (1 + \epsilon)x_2\right)$$

$$\geq \prod_{i=1}^{n_0-1} \prod_{j=1}^{n_0-1} P\left(X_{i}^{(1)} e^{-R_1(\tau_i^{(1)})} > x_1, X_{j}^{(2)} e^{-R_2(\tau_j^{(2)})} > x_2\right)$$

Similarly to (17), by Lemma 4.1, (2), Markov’s inequality, (3), Jensen’s inequality and the conditions of Theorem 2.1, we have that

$$P(D_1(\infty) > x_1) \sim \sum_{i=1}^{\infty} P\left(X_{i}^{(1)} e^{-R_1(\tau_i^{(1)})} > x_1\right)$$

$$\leq \sum_{i=1}^{\infty} \left( \int_0^{\frac{x_1}{D_{R_1}}} \frac{X_1}{u} P\left(e^{-R_1(\tau_i^{(1)})} \leq du \right) + P\left(e^{-R_1(\tau_i^{(1)})} > \frac{x_1}{D_{R_1}}\right) \right)$$

$$\leq \sum_{i=1}^{\infty} \left( C_{R_1} E\left(e^{-pF_{R_1}(\tau_i^{(1)})} + e^{-pF_{R_1}^\prime(\tau_i^{(1)})}\right) + D_{pF_{R_1}(\tau_i^{(1)})}\right)$$

$$= (1 + o(1)) C_{R_1} E\left(e^{-pF_{R_1}(\tau_i^{(1)})} + e^{-pF_{R_1}^\prime(\tau_i^{(1)})}\right)$$

$$\leq (1 + o(1)) \sum_{i=1}^{\infty} \left( E\left(e^{\theta_1(2pF_{R_1})}\right) \right) + \sum_{i=1}^{\infty} \left( E\left(e^{\phi_1(2pF_{R_1})}\right) \right)$$
which, by Lemma 3.1, leads to

\[ P(M_2Z_2 > \epsilon x_2) \leq \left( \frac{M_2}{\epsilon} \right)^{2p_{F_2}} E Z_2^{2p_{F_2}} \cdot x_2^{-2p_{F_2}} = o(F_2(x_2)). \]

The above two estimates yield that

\[ I_2 = o(F_1(x_1)F_2(x_2)), \]

which, by Lemma 3.1, leads to

\[ I_2 = o(1)^{\infty} \sum_{i=1}^{\infty} P \left( X_i^{(1)} e^{-R_1(\tau_i^{(1)})} > x_1, X_j^{(2)} e^{-R_2(\tau_j^{(2)})} > x_2 \right). \]  

(51)

\[ I_3 \] can be dealt in the same way. Therefore, the lower bound of (5) follows from (49)–(51), by letting \( \epsilon \downarrow 0 \). This completes the proof of Theorem 2.1.

4.3. Proof of Proposition 2. For some sufficiently large integer \( N \), we split the joint tail probability \( P(D_1(T) > x_1, D_2(T) > x_2) \) into three parts:

\[ I := P(D_1(T) > x_1, D_2(T) > x_2) = \left( \sum_{m=1}^{\infty} \sum_{n=N+1}^{N} + \sum_{m=N+1}^{\infty} \sum_{n=1}^{N} + \sum_{m=1}^{N} \sum_{n=1}^{N} \right) \]

\[ P \left( \sum_{i=1}^{m} X_i^{(1)} e^{-R_1(\tau_i^{(1)})} > x_1, \sum_{j=1}^{n} X_j^{(2)} e^{-R_2(\tau_j^{(2)})} > x_2, N_1(T) = m, N_2(T) = n \right) =: I_1 + I_2 + I_3. \]  

(52)

The first term \( I_1 \) can be further separated as

\[ I_1 \leq \sum_{m=1}^{\infty} \sum_{n=N+1}^{\infty} P \left( \sum_{i=1}^{m} X_i^{(1)} > x_1 \right) P \left( \sum_{j=1}^{n} X_j^{(2)} > x_2 \right) \times P(N_1(T) = m, N_2(T) = n) \]

\[ = \left( \sum_{1 \leq m \leq \frac{x_1}{F_1}} \sum_{N < n \leq \frac{x_2}{F_2}} + \sum_{1 \leq m \leq \frac{x_1}{F_1}} \sum_{n > \frac{x_2}{F_2}} + \sum_{m > \frac{x_1}{F_1}} \sum_{1 \leq n \leq \frac{x_2}{F_2}} + \sum_{m > \frac{x_1}{F_1}} \sum_{n > \frac{x_2}{F_2}} \right) \]

\[ P \left( \sum_{i=1}^{m} X_i^{(1)} > x_1 \right) P \left( \sum_{j=1}^{n} X_j^{(2)} > x_2 \right) P(N_1(T) = m, N_2(T) = n) \]

\[ =: I_{11} + I_{12} + I_{13} + I_{14}. \]  

(53)

By (2) and Hölder’s inequality, we have that

\[ I_{11} \leq \sum_{1 \leq m \leq \frac{x_1}{F_1}} \sum_{N < n \leq \frac{x_2}{F_2}} mnF_1 \left( \frac{x_1}{m} \right) F_2 \left( \frac{x_2}{n} \right) P(N_1(T) = m, N_2(T) = n) \]

\[ \leq C_{F_1} C_{F_2} F_1(x_1) F_2(x_2) E ((N_1(T))^{p_{F_1}+1} \cdot (N_2(T))^{p_{F_2}+1} 1_{(N_2(T) > N)}) \]

\[ \leq C_{F_1} C_{F_2} F_1(x_1) F_2(x_2) \]

\[ \times (E(N_1(T))^{2p_{F_1}+2} \cdot E(N_2(T))^{2p_{F_2}+2} 1_{(N_2(T) > N)})^{1/2}. \]  

(54)
Again by (2), H"older’s inequality and Markov’s inequality, we have that
\[ I_{12} \leq C_{F_1} \overline{F}_1(x_1) \sum_{m=1}^{\infty} m^{p_F_1+1} P \left( N_1(T) = m, N_2(T) > \frac{x_2}{D_{F_2}} \right) \]
\[ \leq C_{F_1} \overline{F}_1(x_1) \left( E(N_1(T))^{2p_{F_1}+2} \cdot P \left( N_2(T) > \frac{x_2}{D_{F_2}} \right) \right)^{\frac{1}{2}} \]
\[ \leq C_{F_1} D_{F_2}^{-p_{F_2}} \overline{F}_1(x_1) x_2^{-p_{F_2}} \left( E(N_1(T))^{2p_{F_1}+2} \cdot E(N_2(T))^{2p_{F_2}} \right)^{\frac{1}{2}}. \quad (55) \]

Similarly to (54) and (55), we have that
\[ I_{13} \leq D_{F_1}^{p_{F_1}} C_{F_2} x_1^{-p_{F_1}} \overline{F}_2(x_2) \]
\[ \times \left( E(N_1(T))^{2p_{F_1}} \cdot E(N_2(T))^{2p_{F_2}+2}(N_2(T) > N) \right)^{\frac{1}{2}}, \quad (56) \]
and
\[ I_{14} \leq D_{F_1}^{p_{F_1}} D_{F_2}^{p_{F_2}} x_1^{-p_{F_1}} x_2^{-p_{F_2}} \left( E(N_1(T))^{2p_{F_1}} \cdot E(N_2(T))^{2p_{F_2}} \right)^{\frac{1}{2}}. \quad (57) \]

In addition, since the positive Lévy process (i.e., a subordinator) has nondecreasing paths, we obtain from (10) and (8) that
\[ \varphi(x; T) \geq P(X_1^{(1)} e^{-R_1(T)} > x_1, X_1^{(2)} e^{-R_2(T)} > x_2) E N_1(T) N_2(T) \]
\[ = \overline{F}_1(x_1) \overline{F}_2(x_2), \quad (58) \]
where the second step follows from (30), $E N_1(T) N_2(T) > 0$ and the fact
\[ \lim_{N \to \infty} \frac{P(X_1^{(1)} e^{-R_1(T)} > x_1, X_1^{(2)} e^{-R_2(T)} > x_2)}{\overline{F}_1(x_1) \overline{F}_2(x_2)} \leq 1. \]

Combining (53)-(58) and by $F_k \in \mathcal{D}$, $k = 1, 2$, we obtain that
\[ \lim_{N \to \infty} \limsup_{x \to \infty} \frac{I_1}{\varphi(x; T)} = 0, \quad (59) \]
by $E(N_k(T))^{2p_{F_k}+2} < \infty$, $k = 1, 2$, and (3). We can use the similar method to obtain
\[ \lim_{N \to \infty} \limsup_{x \to \infty} \frac{I_2}{\varphi(x; T)} = 0. \quad (60) \]

Now we turn to $I_3$. Let $H(s_1, \ldots, s_{m+1}, t_1, \ldots, t_{n+1})$ be the joint distribution function of the random vector $(\tau_1^{(1)}, \ldots, \tau_m^{(1)}, \tau_1^{(2)}, \ldots, \tau_{n+1}^{(2)})$, $m \geq 1, n \geq 1$. Clearly, for $1 \leq m \leq N$ and $1 \leq n \leq N$,
\[ P \left( \sum_{i=1}^{m} X_i^{(1)} e^{-R_1(\tau_i^{(1)})} > x_1, \sum_{j=1}^{n} X_j^{(2)} e^{-R_2(\tau_j^{(2)})} > x_2, N_1(T) = m, N_2(T) = n \right) \]
\[ = \int \cdots \int P \left( \sum_{i=1}^{m} X_i^{(1)} e^{-R_1(s_i)} > x_1, \sum_{j=1}^{n} X_j^{(2)} e^{-R_2(t_j)} > x_2 \right) H(ds_1, \ldots, ds_{m+1}, dt_1, \ldots, dt_{n+1}) \]
By (10) we have that by Proposition 2, i.e.,

\[ 4.4. \]

for all \( i \)

\[ \sum_{i=1}^{m} \sum_{j=1}^{n} \int_{0 \leq s_1 \leq \cdots \leq s_m \leq t} \int_{0 \leq t_1 \leq \cdots \leq t_m \leq T} P \left( X_i^{(1)} e^{-R_1(s_i)} > x_1, X_j^{(2)} e^{-R_2(t_j)} > x_2 \right) \]

\[ = \sum_{i=1}^{m} \sum_{j=1}^{n} P \left( X_i^{(1)} e^{-R_1(s_i)} > x_1, X_j^{(2)} e^{-R_2(t_j)} > x_2, N_1(T) = m, N_2(T) = n \right), \]

where in the second step we used Lemma 3.5. Thus, we have that

\[ I_3 \sim \left( \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} - \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \right) \sum_{i=1}^{m} \sum_{j=1}^{n} P \left( X_i^{(1)} e^{-R_1(s_i)} > x_1, X_j^{(2)} e^{-R_2(t_j)} > x_2, N_1(T) = m, N_2(T) = n \right) =: I_{31} - I_{32}. \]

By (10) we have that

\[ I_{31} = \varphi(x; T). \]

As for \( I_{32} \), by Hölder’s inequality,

\[ I_{32} \leq F_1(x_1) F_2(x_2) \]

\[ \times (EN_1(T) \cdot N_2(T) \| N_2(T) > N \) + EN_1(T) \| N_1(T) > N \) \cdot N_2(T)) \]

\[ \leq F_1(x_1) F_2(x_2) \left( (E(N_1(T))^2 \cdot E(N_2(T))^2 \| N_2(T) > N \) \right)^{1/2} \]

\[ + \left( E(N_1(T))^2 \| N_1(T) > N \) \cdot E(N_2(T))^2 \right)^{1/2}. \]

Then, by (58) and \( E(N_k(T))^2 \leq (E(N_k(T))^{2p_{k+2}} + \frac{2^p}{n_k+1} \leq \infty, k = 1, 2 \), we have that

\[ \lim_{N \to \infty} \limsup \frac{I_{32}}{\varphi(x; T)} = 0. \]

Therefore, the desired relation (6) follows from (52) and (59)–(63). This completes the proof of the Proposition 2.

4.4. Proof of Theorem 2.2. As done in (48), the upper bound of \( \psi(x; T) \) is clear by Proposition 2, i.e.,

\[ \psi(x; T) \leq P(D_1(T) > x_1, D_2(T) > x_2) \sim \varphi(x; T). \]

We now turn to the lower bound for \( \psi(x; T) \). For any \( 0 < \epsilon < 1 \), since the positive Lévy process \( R_k(t) \) has nondecreasing paths, and using the fact that \( 0 \leq c_k(t) \leq M_k \) for all \( t \geq 0, k = 1, 2 \), according to Proposition 2, we have that for sufficiently large \( x_1 \) and \( x_2 \),

\[ \psi(x; T) \geq P(D_1(T) > x_1 + M_1 T, D_2(T) > x_2 + M_2 T) \]

\[ \geq P(D_1(T) > (1 + \epsilon)x_1, D_2(T) > (1 + \epsilon)x_2) \]

\[ \sim \varphi((1 + \epsilon)x; T) \]

\[ \geq \int_{0}^{T} \int_{0}^{T} \int_{0}^{1} \int_{0}^{1} F_1 \left( \frac{(1 + \epsilon)x_1}{u} \right) F_2 \left( \frac{(1 + \epsilon)x_2}{v} \right) \]
\[
P\left(e^{-R_k(s)} \in du, e^{-R_k(t)} \in dv\right) d(EN_1(s) \cdot N_2(t)) \\
\geq \inf_{u \in (0, 1]} \frac{F_1(1+\epsilon u)}{F_1(\frac{x_1}{u})}, \quad \inf_{v \in (0, 1]} \frac{F_2(1+\epsilon v)}{F_2(\frac{x_2}{v})} \cdot \varphi(x; T) \\
> \frac{F_1(x + 1+\epsilon)}{F_2(x + 1+\epsilon)} \cdot \varphi(x; T). \tag{64}
\]

Noting that \(F_k(x + 1+\epsilon) \to 1\) as \(\epsilon \downarrow 0\) because of \(F_k \in C, k = 1, 2\), the desired lower bound follows from (64). This completes the proof of Theorem 2.2.

4.5. **Proof of Corollary 1.** The proof of Corollary 1 is similar to that of Theorem 2.2 by reconsidering (64) as

\[
\psi(x; T) \geq P(D_1(T) > x_1 + M_1T, D_2(T) > x_2 + M_2T) \\
\sim \varphi((x_1 + M_1T, x_2 + M_2T)^T; T) \\
= \int_0^T \int_0^T \frac{F_1(1+\epsilon u)}{F_1(\frac{x_1}{u})} \frac{F_2(1+\epsilon v)}{F_2(\frac{x_2}{v})} d(EN_1(s) \cdot N_2(t)) \\
\geq \int_0^T \int_0^T \frac{F_1(x_1 + 1+\epsilon u + M_1T e^{rT})}{F_1(x_1 e^{rT})} \frac{F_2(x_2 + 1+\epsilon v + M_2T e^{rT})}{F_2(x_2 e^{rT})} d(EN_1(s) \cdot N_2(t)) \\
\geq \frac{F_1(x_1 + 1+\epsilon u + M_1T e^{rT})}{F_1(x_1 e^{rT})} \cdot \frac{F_2(x_2 + 1+\epsilon v + M_2T e^{rT})}{F_2(x_2 e^{rT})} \cdot \varphi(x; T) \\
\sim \varphi(x; T),
\]

by noting \(R_k(t) = rt\) for all \(t \geq 0\) and \(F_k \in C, k = 1, 2\). This completes the proof of Corollary 1.

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