The Park–Pham Theorem
with Optimal Convergence Rate

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Abstract

Park and Pham’s recent proof of the Kahn–Kalai conjecture was a major breakthrough in the field of graph and hypergraph thresholds. Their result gives an upper bound on the threshold at which a probabilistic construction has a $1 - \epsilon$ chance of achieving a given monotone property. While their bound in other parameters is optimal up to constant factors for any fixed $\epsilon$, it does not have the optimal dependence on $\epsilon$ as $\epsilon \to 0$. In this short paper, we prove a version of the Park–Pham Theorem with optimal $\epsilon$-dependence.

Mathematics Subject Classifications: 05D05, 05D40

1 Introduction

One of the most fundamental tasks in probabilistic combinatorics is finding the thresholds for graph and hypergraph properties. At what $p$ should you expect $G(n, p)$ to be more likely than not to contain a triangle? A Hamiltonian cycle? A common first attempt is to lower bound this $p$ by the first moment method. The Park–Pham Theorem essentially says that applying the first moment method on some structure that is necessary for your desired graph to appear is always within a logarithmic factor of the true threshold.

Let $H$ be a hypergraph on a finite vertex set $X$. The upward closure of $H$ is

$$
\langle H \rangle = \{ R \subseteq X : \exists S \in H \text{ s.t. } S \subseteq R \},
$$

that is, the subsets of $X$ that contain a hyperedge in $H$. A hypergraph $G$ undercovers $H$ if $H \subseteq \langle G \rangle$, that is, every hyperedge in $H$ contains a hyperedge in $G$.

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Definition 1 ([5], Section 1). Let $q \in (0, 1)$. $H$ is $q$-small if there is a hypergraph $G$ such that $H \subseteq (G)$ and $\sum_{R \in G} q^{\#R} \leq \frac{1}{2}$.

Let $X_p$ denote a subset of $X$ where each element is included independently with probability $p$, let $p_c(H)$ be the probability $p_c$ such that $\mathbb{P}(\exists S \in H$ s.t. $S \subseteq X_{p_c}) = \frac{1}{2}$ (if $H$ is non-trivial, this must exist and be unique by monotonicity), let $\ell(H)$ be the size of the largest hyperedge of $H$, and let $q(H)$ be the maximum $q$ such that $H$ is $q$-small.

To apply the Park–Pham Theorem, $X$ should be the set of objects that are being selected independently at random, for example, the edges of $G(n, p)$ or hyperedges of a random hypergraph, and $H$ is the graph or hypergraph property you want to find the threshold of, for example, hyperedges consisting of minimal edge sets which form a triangle or Hamiltonian cycle. $\ell(H)$ is the maximal number of elements that make up one instance of that property, for example, $n$ for Hamiltonian cycles and 3 for triangles. $q(H)$ is often relatively easy to compute for structured $H$. Our main goal is to find $p_c(H)$. One motivation for the definition of $q$-small is that $q(H) \leq p_c(H)$ by first moment method on the probability of some $R \in G$ being contained in $X_q$, a necessary condition for some $S \in H$ to be contained in $X_q$. The Park–Pham Theorem shows that $q(H)$ gives rise to an upper bound, as well as the lower bound, on $p_c(H)$.

Theorem 2 (Kahn–Kalai Conjecture, now Park–Pham Theorem). For any hypergraph $H$,

$$p_c(H) \leq 8q(H) \log(2\ell(H)).$$

In this theorem and throughout the paper, all logarithms are base 2. The Park–Pham Theorem (with arbitrary constant) was conjectured by Kahn and Kalai [5], who called it “extremely strong” and showed many applications of it, and was proven by Park and Pham [7], building off previous work [1, 3]. Theorem 2 will be proven in Section 2. Our proof achieves a significantly lower constant and avoids some of the complications of the original Park–Pham proof. We use similar techniques to Rao [9].

The Park–Pham Theorem we prove is equivalent to saying that for $q > q(H)$, $\ell = \ell(H)$, and $p = 8q \log(2\ell(H))$, we have $\mathbb{P}(\exists S \in H$ s.t. $S \subseteq X_p) > \frac{1}{2}$. But you may want to know more than just when $G(n, p)$ has a 50/50 chance of containing a triangle or Hamiltonian cycle; you may want to know when it has a .999 chance of containing these. So a natural question is to replace $\frac{1}{2}$ with $1 - \epsilon$ for any $\epsilon > 0$. Park and Pham also proved an $O(q(H) \log(\ell(H)))$ upper bound for any fixed $\epsilon$, but as it was not their focus, their dependence on $\epsilon$ is exponentially worse than the dependence we give. The following theorem is our main result:

Theorem 3. Let $H$ be a hypergraph that is not $q$-small and let $\epsilon \in (0, 1)$. Let $p = 48q \log \left( \frac{\ell(H)}{\epsilon} \right)$. Then $\mathbb{P}(\exists S \in H$ s.t. $S \subseteq X_p) > 1 - \epsilon$.

This bound is optimal up to constant factors, that is, has the optimal dependence on all of $\ell$, $\epsilon$, and $q$. We will prove Theorem 3 in Section 3, and then will relate it to other work in Section 4.
2 Proof of the Park–Pham Theorem

Let $H$ be a $\ell$-bounded hypergraph, that is, $|S| \leq \ell$ for every $S \in H$. Given a set $W$, and $S \in H$, let $T(S, W)$ be $S' \setminus W$ for $S' = \arg\min_{S' \in H, S' \subseteq W} |S' \setminus W|$ (break ties arbitrarily). Note that for a given $W$, we have that $\{T(S, W) : S \in H\}$ undercovers $H$, as for every $S \in H$, we have $T(S, W) \subseteq S$. If $H$ is not $q$-small, then $\{T(S, W) : S \in H\}$ is also not $q$-small, as any $G$ undercovering $\{T(S, W) : S \in H\}$ also undercovers $H$.

**Proposition 4** ([7], Lemma 2.1). Let $H$ be any $\ell$-bounded hypergraph (which may or may not be $q$-small!) and $L > 1$. Let $1 \leq t \leq \ell$ and $\mathcal{U}_t(H, W) = \{T(S, W) : S \in H, |T(S, W)| = t\}$. Let $W$ be chosen uniformly at random from $\binom{X}{Lq|X|}$. Then

$$\mathbb{E}_W \sum_{U \in \mathcal{U}_t(H, W)} q^t < L^{-t} \binom{\ell}{t}.$$

**Proof of Proposition 4.** We will follow the proof of Park and Pham [7]. It is equivalent for us to show that $\sum_{W \in \binom{X}{Lq|X|}} |\mathcal{U}_t(H, W)| < \binom{|X|}{Lq|X|} L^{-t} q^{-t} \binom{\ell}{t}$. To achieve an upper bound on the number of $T = T(S, W)$, it suffices to give a procedure for uniquely specifying any valid $(W, T)$ pair (where the $T$ is $T(S, W)$ for that $W$ and some $S \in H$).

First, fix a universal “tiebreaker” function $\chi : \langle H \rangle \to H$ such that $\chi(Y) \subseteq Y$ for all $Y \subseteq \langle H \rangle$.

Now, specify $Z = W \sqcup T$. Note that these two sets are disjoint by definition, so this has size exactly $Lq|X| + t$ and we have at most

$$\binom{|X|}{Lq|X| + t} = \binom{|X|}{Lq|X|} \prod_{i=1}^{t} \frac{|X| - Lq|X| - i + 1}{Lq|X| + i} \leq \binom{|X|}{Lq|X|} (Lq)^{-t}$$

valid choices.

Now, we claim that $T \subseteq \chi(Z)$. We know $Z \in \langle H \rangle$ since $S \subseteq Z$, so $\chi(Z) \subseteq Z = W \sqcup T$. If $T \not\subseteq \chi(Z)$, we could not have that $T$ was the minimizer, as we could have instead taken $S' = \chi(Z)$ and then $\chi(Z) \setminus W \subseteq T$.

We can thus specify $T$ (with $|T| = t$) as a subset of $\chi(Z)$ (with $|\chi(Z)| \leq \ell$ since $\chi(Z) \in H$), so there are at most $\binom{|X|}{t}$ choices for $T$.

This process specified $T$ and the disjoint union of $T$ and $W$, so we have also specified $W$, and thus have given a way to specify every possible $(W, T(S, W))$ pair, with at most $\binom{|X|}{t} (Lq)^{-t} \binom{\ell}{t}$ possible choices.

**Proof of Theorem 2 from Proposition 4.** We will iterate the process of replacing each $S$ by $T(S, W)$. Start with $H_0 = H$, $X_0 = X$, and $\ell_0 = \ell(H)$. We will choose $W_i$ to be a uniformly random set in $\binom{X_i}{Lq|X_i|}$. Set

$$\mathcal{C}_i = \cup_{t_i=1}^{\ell_i/2} \mathcal{U}_t(H_i, W_i) \quad \text{and} \quad H_{i+1} = \{T(S, W_i) : S \in H_i, |T(S, W_i)| \leq \ell_i/2\}.$$ 

Set $\ell_{i+1} = \lceil \ell_i/2 \rceil$. Note that $H_{i+1}$ is a $\ell_{i+1}$-bounded hypergraph on $X_{i+1} = X \setminus \cup_{j=0}^{i} W_j$. 

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Now, we repeat until we reach an $i = I$ where $\ell_i + 1 < 1$, which then gives $H_i = \emptyset$ or $H_{i+1} = \{\emptyset\}$. As $\ell_i \leq 2^{-i} t$, we have $I \leq \lfloor \log(\ell) \rfloor$. Let $\mathcal{U} = \cup_{i=0}^{I} \mathcal{C}_i$ and $W = \cup_{i=0}^{I} W_i$. Now, we claim that either there is some $S \in \mathcal{H}$ such that $S \subseteq W$ (in which case $H = \emptyset$ and we have succeeded), or else $H_i = \emptyset$ and $\mathcal{U}$ undercovers $H$. This is true because if you trace any $S_0 = S$ through $S_{i+1} = T(S_i, W_i)$, there is some $0 \leq i \leq I$ at which we have either $T(S_i, W_i) \in \mathcal{C}_i$, which gives that this $\mathcal{C}_i$ undercovers $S$; or $T(S_i, W_i) = \emptyset$, which means that $S_i \subseteq W_i$, and thus there exists an $S' \in H$ that is in $\cup_{j=0}^{i-1} W_j$, as $S_i = S' \cup_{j=0}^{i-1} W_j$ for some $S' \in H$.

Therefore, to show that there is a high probability of some $S \in \mathcal{H}$ being in $W$, it suffices to show there is a low probability of $\mathcal{U}$ undercovering $H$. For each $1 \leq t \leq \ell$, let $i(t)$ be the highest $i$ such that $\ell_i \geq t$. Note that sets of size $t$ are only added to $\mathcal{U}$ at one step of our process, into $\mathcal{C}_{i(t)}$. In other words, $U_t = \cup_{i(t+1)} \cup_{i(t)}$ and $U_t = \cup_{i=1}^{t} \mathcal{V}_t$. Then

$$E_H = \sum_{U \in \mathcal{U}} q_{1} = \sum_{t=1}^{\ell} E_{U \in \mathcal{U}} 8^{-t/4} \left( \begin{array}{c} 2t - 1 \\ t \end{array} \right) < \sum_{t=1}^{4} 8^{-t} \left( \begin{array}{c} 2t - 1 \\ t \end{array} \right) + \sum_{t=5}^{\infty} 8^{-t} 2^{2t-1} = \frac{819}{4096} + \frac{1}{32} < \frac{1}{4}.$$

If $\mathcal{U}$ undercovers $H$, then $\sum_{U \in \mathcal{U}} q_{1} > \frac{1}{2}$. By Markov’s Inequality, $P(\sum_{U \in \mathcal{U}} q_{1} > \frac{1}{2}) < \frac{1}{2}$. Using for the first time that $H$ is $q$-small, this means that with probability more than half, $\mathcal{U}$ does not undercover $H$ and thus some $S \in \mathcal{H}$ has $S \subseteq W$.

$W$ is then a uniformly random set of size $\sum_{t=0}^{\ell} 8q_{1} X_{t} \leq \sum_{t=0}^{\lfloor \log(\ell) \rfloor} 8q_{1} X = p_{1} X$. If we make $W = X_{p}$ rather than a random set of size $p_{1} X$, Theorem 2 still holds: we can freely add elements to $X$ that are not in any hyperedge in $H$ and take the limit as the number of these points goes to infinity.

3 Proof of Theorem 3

In this section, we will prove the main theorem of this paper, which we recall below:

Theorem 5 (3). Let $H$ be a $\ell$-bounded hypergraph that is not $q$-small and let $\epsilon \in (0, 1)$. Let $p = 48q \log \left( \frac{1}{\ell} \right)$. Then $P(\exists S \in H \text{ s.t. } S \subseteq X_{p}) > 1 - \epsilon$.

As a warm-up to the proof of Theorem 3, we first note that we can quickly get logarithmic $\epsilon$-dependence if we allow a product of $\log(\ell)$ and $\log(1/\ell)$ instead of a sum:

Proposition 6. Let $H$ be a $\ell$-bounded hypergraph that is not $q$-small and let $\epsilon \in (0, 1)$. Let $p = 8q \log(2\ell) \log \left( \frac{1}{\ell} \right)$. Then $P(\exists S \in H \text{ s.t. } S \subseteq X_{p}) > 1 - \epsilon$.

Proof. Note that if some set $W$ does not contain a hyperedge in $H$, then $H' = \{S \setminus W : S \in H \}$ undercovers $H$, and is thus is also not $q$-small. So let $H_0 = H$ and $X_0 = X$. For all $1 \leq i \leq \lfloor \log(\frac{1}{\ell}) \rfloor$, we take $W_i = (X_i)_{8q \log(2\ell)}$, take $X_{i+1} = X_i \setminus W_i$, and take $H_{i+1} = \{S \setminus W : S \in H_i \}$. At each step, either we have some $S \in H_i$ such that $S \subseteq X_i$ or
\(H_{i+1}\) is \(\ell\)-bounded and not \(q\)-small. Thus, at each step, if we have not yet succeeded, we have probability \(> \frac{1}{2}\) of \(W_i\) containing a hyperedge in \(H_i\) by Theorem 2. So after \(\lceil \log \left( \frac{1}{\ell} \right) \rceil\) steps, we have that \(W = \cup_{i=1}^{\lceil \log(1/\ell) \rceil} W_i\) has more than a \(1 - \epsilon\) chance of containing some hyperedge in \(H\). We have \(W \sim X_1 - (1 - 8q \log(2\ell))^{\lceil \log(1/\ell) \rceil}\) and \(1 - (1 - 8q \log(2\ell))^{\lceil \log(1/\ell) \rceil} < p,\) so this is also true for \(W \sim X_p\).

However, we will see in the next section that Proposition 6 is not the bound we want. What the above proof does give us is the important idea that, in this setting, we can repeat a random trial where success is more likely than failure until it succeeds.

**Proof of Theorem 3.** As in the proof of Theorem 2, we will iterate the process of replacing each \(S\) by \(T(S,W),\) starting with \(H_0 = H,\) \(X_0 = X,\) and \(\ell_0 = \ell(H)\). We choose \(W_i\) uniformly at random from \(\left( \frac{X_i}{8q|X_i|} \right)\) and let \(C_i = \cup_{t=\lfloor \ell_i/2 \rfloor + 1}^{\ell_i/2} U_t(H_i,W_i).\) The main difference now is that we will have a “success” and a “failure” criteria at each stage. By Proposition 4, we know that

\[
\mathbb{E} \sum_{U \in C_i} q^{|U|} = \mathbb{E} \sum_{t=\lfloor \ell_i/2 \rfloor + 1}^{\ell_i} \sum_{U \in \mathcal{U}_t} q^t < \sum_{t=\lfloor \ell_i/2 \rfloor + 1}^{\ell_i} 8^{-t} \binom{\ell_i}{t}.
\]

We consider step \(i\) a “failure” if

\[
\sum_{U \in C_i} q^{|U|} > 2 \left( \sum_{t=\lfloor \ell_i/2 \rfloor + 1}^{\ell_i} 8^{-t} \binom{\ell_i}{t} \right).
\]

By Markov’s inequality, success is more likely than failure at every step. We always set \(X_{i+1} = X_i \setminus W_i.\) If step \(i\) fails, then we keep \(H_{i+1} = \{S \setminus W_i : S \in H_i\}\) (or for that matter, \(H_{i+1} = \{T(S,W_i) : S \in H_i\}\) and \(\ell_{i+1} = \ell_i,\) essentially keeping the same hypergraph and retrying. If step \(i\) succeeds, then as before we set \(H_{i+1} = \{T(S,W_i) : S \in H_i, |T(S,W_i)| \leq \ell_i/2\}\) and \(\ell_{i+1} = \lfloor \ell_i/2 \rfloor.\) In either case, \(H_i\) is a \(\ell_{i+1}\)-bounded hypergraph on \(X \setminus \cup_{j=0}^{\ell_i} W_j,\) so our claim of success on step \(i + 1\) being more likely than failure still holds. If \(H_i\) only contains the empty set, we set \(\ell_{i+1} = 0\) and simply do nothing for all remaining \(i\) (these steps can be considered successes).

We repeat this for \(I = \lceil \log \left( \frac{\ell}{\ell} \right) \rceil\) steps. Let

\[
\mathcal{U} = \bigcup_{1 \leq i \leq I \text{ step } i \text{ succeeded}} C_i.
\]

Again, letting \(U(t)\) to be the highest \(i\) such that \(\ell_i \geq t,\) we have that \(\mathcal{U}_t = \mathcal{U}_t(H_{i(t)},W_{i(t)})\) and \(\mathcal{U} = \cup_{t=0}^{\ell(H)} \mathcal{U}_t.\) Now, by our success criteria and our proof of Theorem 2, we know for sure that

\[
\sum_{U \in \mathcal{U}} q^{|U|} \leq \sum_{t=1}^{\ell} 2^{t-1} \binom{\ell}{t} 8^{-t} \binom{\ell}{t} < \frac{1}{2},
\]

so as \(H\) is \(q\)-small, \(\mathcal{U}\) for sure does not undercover \(H.\)
If \( \ell_{\ell+1} < 1 \), then this means that \( \bigcup_{i=1}^{\ell} W_i \) contains a hyperedge in \( H \). If we have had at least \( \lfloor \log(\ell) \rfloor + 1 \) successes, then we do have \( \ell_{\ell+1} < 1 \). We have had \( 6|\log(\frac{\ell}{\epsilon})| \) steps, each of which had a greater than \( \frac{1}{2} \) probability of succeeding (regardless of the success or failure of previous steps). Let \( X \) be our number of successes, which we then know is stochastically dominated by \( Y \sim Bin(I, \frac{1}{2}) \). Standard Chernoff bounds give that

\[
\mathbb{P}(Y \leq (1-\delta)\mathbb{E}Y) \leq \left( \frac{e^{-\delta}}{(1-\delta)^{1-\delta}} \right)^{\mathbb{E}Y}.
\]

Note that \( \frac{|\log(\ell)|}{\mathbb{E}Y} = \frac{|\log(\ell)|}{3|\log(\ell/\epsilon)|} \leq \frac{1}{3} \) as \( \epsilon \leq 1 \). So here,

\[
\mathbb{P}(\exists S \in H \text{ s.t. } S \subseteq \bigcup_{i=1}^{\ell} W_i) \leq \mathbb{P}(\ell_{\ell+1} \geq 1) = \mathbb{P}(X \leq |\log(\ell)|)
\leq \mathbb{P}(Y \leq |\log(\ell)|) \leq \mathbb{P}(Y \leq (1 - 2/3)\mathbb{E}Y)
\leq \left( \frac{e^{-2/3}}{(1/3)^{1/3}} \right)^{|\log(\ell/\epsilon)|} \leq \left( \frac{1}{2} \right)^{|\log(\ell/\epsilon)| - 1} = \frac{2\epsilon}{\ell} \leq \epsilon
\]

for \( \ell \geq 2 \). Then \( |\bigcup_{i=1}^{\ell} W_i| \leq 8Iq|X| \leq 48q\log(\frac{\ell}{\epsilon})|X| \), and once again we can take \( W = X_{48q \log(\ell/\epsilon)} \) instead by adding points to \( X \) not in any hyperedge in \( H \). \( \square \)

4 Why Theorem 3 is the “Optimal Convergence Rate”

Now that we have proven our main theorem, we will explain how it relates to prior work. Park–Pham’s paper implied Theorem 3 but with a bound of \( p = O(q(\log(\ell) + \epsilon^{-c})) \) for some \( c \approx 2 \) \cite{ParkPham}, an exponentially worse \( \epsilon \)-dependence than our bound of \( p = 48q \log(\frac{\ell}{\epsilon}) \). Our Theorem 3 can be rephrased as follows:

**Corollary 7.** Let \( H \) be a \( \ell \)-bounded hypergraph that is not \( q \)-small and choose any \( c \geq 1 \). Let \( p = 48cq \log(2\ell) \). Then \( \mathbb{P}(\exists S \in H \text{ s.t. } S \subseteq X_p) > 1 - \ell^{-c} \).

**Proof.** At the end of the proof of Theorem 3 our failure probability was \( \frac{2\epsilon}{\ell} \). Therefore, we can simply set \( \epsilon = \frac{1}{2\ell^{1-c}} \), which is at most 1 as required. \( \square \)

The above corollary is important because it matches the “with high probability” statements of prior work, that is, the probability of \( X_p \) containing a hyperedge in \( H \) goes to 1 as \( \ell \) goes to infinity. \( \ell^{-c} \) can be thought of as the convergence rate of this probability to 1. The Park–Pham bounds earlier gave for \( p = O(q \log(\ell)) \) a convergence rate of \( (\log(\ell))^{-c} \) for some \( c \approx \frac{1}{2} \) \cite{ParkPham}, so as before this is an exponential improvement.

One motivation for achieving the bound in Theorem 3 was to generalize the similar bound that was achieved under the “fractional expectation-threshold” or “\( \kappa \)-spread” framework by Rao \cite{Rao}, improving the \( \epsilon \)-dependence of previous work \cite{Krivelevich, AlonNazarov}. 

**Definition 8 (\cite{AlonKrivelevich}, Definition 6.6).** \( H \) is \( \kappa \)-spread if for all \( Y \subseteq X \),

\[
|\{ S \in H : S \subseteq Y \}| \leq \kappa^{-|Y|}|H|.
\]
Theorem 9 ([8], Lemma 4). Let $H$ be a $\ell$-bounded hypergraph that is $\kappa$-spread and let $\epsilon \in (0, 1)$. There exists a universal constant $\beta$ such that for $p = \frac{\kappa}{\kappa} \log \left( \frac{1}{\ell} \right)$, we have that $P(\exists S \in H \text{ s.t. } S \subseteq X_p) > 1 - \epsilon$.

Our Theorem 3 is a generalization of Rao’s Theorem 9 due to the following basic lemma:

Lemma 10 ([10], Propositions 6.2, 6.7). If $H$ is $\kappa$-spread, $H$ is not $\frac{1}{\kappa}$-small.

Proof. Let $G$ such that $H \subseteq \langle G \rangle$. For all $r \in \mathbb{N}$, let $n_r = |\{R \in G : |R| = r\}|$ and let $c_r$ be the number of hyperedges $S \in H$ such that $\exists R \in G$ s.t. $R \subseteq S$ and $|R| = r$. Because $H$ is $\kappa$-spread, $c_r \leq n_r |H|^{\kappa - r}$ for all $r \in \mathbb{N}$. However, as every hyperedge in $H$ contains some hyperedge in $G$, $\sum_{r \in \mathbb{N}} c_r \geq |H| \Rightarrow \sum_{r \in \mathbb{N}} \frac{c_r}{|H|} \geq 1$. Putting these inequalities together, $\sum_{R \subseteq G} \left( \frac{1}{|H|} \right)^{|R|} = \sum_{r \in \mathbb{N}} n_r |H|^{\kappa - r} \geq 1 > \frac{1}{2}$. As this is true for any $G$ such that $H \subseteq \langle G \rangle$, we must have that $H$ is not $\frac{1}{\kappa}$-small. \hfill \Box

Rao’s result, and thus ours, is asymptotically optimal (that is, optimal except for the constant) in $\ell$, $\epsilon$, and $\kappa$ (or $q$):

Proposition 11 ([2], Lemma 4; adapted from [1]). Let $\epsilon \in (0, \frac{1}{2})$ and $\kappa, \ell \in \mathbb{N}$ such that $p = \frac{1}{6\kappa} \log \left( \frac{1}{\ell} \right)$ has $p \leq \frac{1}{2}$. There exists a $\kappa$-spread hypergraph $H$ such that $P(\exists S \in H \text{ s.t. } S \subseteq X_p) < 1 - \epsilon$.

In conclusion, this paper has successfully generalized the asymptotically optimal bound from the $\kappa$-spread setting to the $q$-small setting, using a different proof technique from those used to achieve this bound in the $\kappa$-spread setting [8, 11, 4, 6].

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