Fault-tolerant quantum gates with defects in topological stabiliser codes

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Braiding defects in topological stabiliser codes has been widely studied as a promising approach to fault-tolerant quantum computing. Here, we explore the potential and limitations of such schemes in codes of all spatial dimensions. We show that a universal gate set for quantum computing cannot be realised by supplementing locality-preserving logical operators with defect braiding, even in more than two dimensions. However, we also demonstrate that higher dimensional defect-braiding schemes have the potential to play an important role in realising fault-tolerant quantum computing. Specifically, we present an approach that allows for the full Clifford group to be implemented by braiding in any code that admits twists that allow for condensation of a fermion. We demonstrate two examples of this in higher dimensional codes, specifically in self-dual surface codes and the three dimensional Levin-Wen Fermion Model. We also present an example of how our no-go theorems can be circumvented to provide a universal scheme in three-dimensional surface codes without magic state distillation. Specifically, this scheme employs adaptive implementation of logical operators conditional on logical measurement outcomes to lift a combination of locality-preserving and braiding logical operators to universality.

I. INTRODUCTION

Quantum computers can solve certain problems more efficiently than their classical counterparts. However, the fragility of quantum coherence against noise means that quantum computers will likely require error correction to function on a large scale. In particular, a quantum computer will require that a universal set of logical operators—gates—can be implemented fault-tolerantly, meaning that typical errors remain controlled and correctable throughout the computation. Developing schemes allowing for such a universal, fault-tolerant gate set is thus of great importance, but is also challenging. In particular, the most natural approach to fault-tolerant operators, transversality, cannot allow for a universal set in any quantum error correcting code [1].

More specifically, special attention has been given to topological stabiliser codes [2–12]. These are a widely-studied and extremely promising class of codes, since they provide protection against general local errors, and their realisation by local Pauli Hamiltonian models provides a route to experimental feasibility. However, even stronger constraints apply to fault-tolerant logical operators in topological stabiliser codes. In particular, locality-preserving logical operators, which are a much more general class of operators than transversal operators, have been proven to be insufficient for universality in any topological stabiliser code [24]. A number of ways to construct a universal gate set in topological stabiliser codes have been proposed, including magic state distillation [13], stabiliser state injection [9], dimensional jumping [6] and just-in-time gauge fixing [14,15]. However, each of these techniques requires a high overhead in qubits, or a substantial increase in the complexity of the scheme, to move from its initial set of locality-preserving logical operators to a universal gate set. Thus, there is significant value in investigating if alternative methods of achieving universality may be possible that may be realised more naturally within a topological stabiliser code.

In this paper, we explore the power and limitations of a natural class of fault-tolerant logical operators in topological stabiliser codes that is more general than locality-preserving logical operators. Specifically, we allow for operators that may be implemented by braiding topological defects. Such operators have been extensively studied in two dimensional codes and a wide range of schemes have been proposed using holes [16–22] and twist defects [23–33]. It is known that there exist codes for which this set of logical operators implementable by braiding defects is larger than that of locality-preserving logical operators admitted by the code [21–25]. One may thus hope that braiding defects might allow the limitations on the power of locality-preserving logical operators to be overcome. In particular, defects allow abelian models to exhibit non-abelian braiding statistics. Since there are known examples of non-abelian braiding models that allow for universal fault-tolerant quantum computing [34,35], we may wonder if there exist topological stabiliser codes with defects that similarly allow for universality. This possibility seems all the more promising since there are known cases of topological models that do not admit a universal set of logical operators fault-tolerantly, but which do allow for universality when genons, a type of twist defect, are introduced and braided [30].

However, we show that the set of operators implementable by braiding defects in a topological stabiliser code cannot be universal. This remains true even when we combine these operators with the set of locality-preserving logical operators. We present the theoretical background underpinning these results in Sec. [11] In particular, we review topological stabiliser codes and their properties. We then discuss defects in such codes and how quantum information may be encoded and manipulated using such defects. In Sec. [11] we present and
discuss the results constraining the power of these logical operators. Specifically, we generalise Theorem 2 of our companion paper [36], which rules out universality by combining locality-preserving logical operators with defect braiding in a class of defect encodings to a wider range of schemes and to abelian quantum double models. For completeness, we also review Theorem 1 of the companion paper – that under appropriate assumptions defect braiding cannot be used to implement non-Clifford logical operators [35].

Having established the limitations in the power of braiding defects, we then consider schemes that saturate or even circumvent these limits. Specifically, in Sec. IV we explore schemes allowing for the full Clifford group to be implemented. We begin this exploration with a review of the well-known example of twists in the two-dimensional surface code [24, 25]. By adapting this scheme, we then present two novel schemes for the Clifford group by braiding in higher dimensional topological stabiliser codes. Specifically, these schemes are realisable in a class of even-dimensional surface codes and the three dimensional Levin-Wen fermion model respectively [4, 37]. These are the first proposed examples of defect-braiding schemes in higher dimensional codes. Even in light of our constraints on the power of such schemes, they are still valuable to consider for a number of reasons. In particular, such higher dimensional schemes can exhibit properties, such as single-shot error correction and self-correction, that cannot be realised in two dimensional schemes. They also provide examples of braiding phenomena that are unique to higher-dimensional codes, and may provide inspiration for schemes in more exotic higher dimensional models that are not constrained by our results. Finally, they provide illustrations of the types of schemes that are constrained by our no-go theorems, beyond the two dimensional schemes that had been presented previously.

Finally, in Sec. V we present a scheme that uses braiding of defects in a topological stabiliser code and does allow for a universal gate set. Specifically, we consider a scheme based on encoding qubits in punctures in three copies of a three-dimensional surface code. By supplementing locality-preserving and braiding logical operators with operators that are conditional on logical measurements, we circumvent the constraining results of Sec. III. In particular, this is possible because such conditioning uses non-local classical processing, which is known to allow for previous no-go theorems such as the Bravyi-König bound to be overcome [35]. Our scheme is similar in inspiration to that proposed in [9]. However, it has the advantage that braiding defects allows for a universal gate set to be implemented on arbitrarily many logical qubits within a single code block, without the need for lattice surgery. This illustrates how, even considering the limitations on the power of braiding defects in topological stabiliser codes, valuable schemes can still be realised that use such braiding.

II. TOPOLOGICAL STABILISER CODES AND DEFECTS

We begin this section by reviewing the relevant structure and properties of topological stabiliser codes, focussing on the structure of excitations (localised errors) of such codes and their relationship to locality-preserving logical operators (an important type of fault-tolerant logical operator in such codes). We then turn to the theory of defects in topological stabiliser codes. Specifically, we review the definition of topological defects, and then survey several examples. We then describe how defects can be used to encode information, the properties of the encoding and then finally how braiding defects allows for fault-tolerant logical operators.

A. Topological Stabiliser Codes

A topological stabiliser code is defined on a lattice of physical qubits in \( D \geq 2 \) spatial dimensions. To construct a topological stabiliser code, we begin by specifying a stabiliser group of Pauli operators, \( \mathcal{S} \). We choose this group so that the following conditions hold. Firstly, it must be abelian and not contain \((-I, I)\). This ensures that it is a valid stabiliser group. Secondly, it must admit some generating set, \( S = \{S_i\} \subseteq \mathcal{S} \), such that the all the \( S_i \) operators are local with respect to the arrangement of qubits in the lattice. This ensures that the interactions that must be created during encoding are local and that error correction can be performed using only local measurements. Thirdly, any operator that commutes with all elements of \( S \) but is not in \( S \) must be non-local and have support that is topologically non-trivial (i.e. cannot be deformed to a point). The codespace is then defined to be the \(+1\)-eigenspace of \( S \), i.e. the set of states stabilised by all elements of \( S \).

One of the nice properties of a topological stabiliser code is that we can construct a local Hamiltonian that possesses this codespace as its degenerate ground space. Specifically, a Hamiltonian consisting of commuting Pauli terms may be associated with any stabiliser code by choosing a set of stabiliser generators, \( S \), which we can then use to define the Hamiltonian \( H = -\sum_{S_i \in S} S_i \). The ground space of this Hamiltonian is then the common \(+1\)-eigenspace of all elements of \( S \), which is equivalent to the codespace (i.e. space stabilised by \( S \)). For a topological stabiliser code, the generating set \( S \) may be chosen to contain only local operators, which is not possible for more general stabiliser codes.

The locality of Hamiltonian terms in this model ensures that local operators can only give rise to local excitations. This makes the model robust against local noise, since the local excitations it gives rise to can be effectively detected and removed to return it to the correct ground state. This means that any local error on a topological stabiliser code is correctable. This robustness makes the model topologically ordered – it has degenerate ground
states that can only be interchanged by excitations propagating over a topologically non-trivial path. This very general protection against local noise provided by topologically ordered systems makes them well-suited for use as quantum memories.

1. Locality-Preserving Logical Operators

To perform quantum gates on the information encoded in a topological stabiliser code, it is necessary to implement logical operators that act nontrivially on the degenerate ground space. To ensure the protection from errors provided by the code is maintained, these logical operators must be fault-tolerant. By this we mean that if implemented on a code where a local error is present, this error must remain local after the operator. Since local errors in topological stabiliser codes are necessarily correctable, this ensures that correctable errors cannot become uncorrectable in a single step of computation. A locality-preserving logical operator is defined to be a unitary operator that acts as a logical operator while preserving the locality of errors present in the code. More precisely, $U$ is locality-preserving if it grows the support of any local operator by at most some constant amount under conjugation. Locality-preserving logical operators in a $D$-dimensional topological stabiliser code can be of any dimension $k$ such that $1 \leq k \leq D$. The spatial dimension of a locality-preserving logical operator is defined to be the smallest $k$ such that there exists a $k$-dimensional manifold that supports an implementation of the operator. That is, it is the smallest number $k$ for which there is an implementation of the operator that is effectively $k$-dimensional.

Locality-preserving logical operators may be considered the most direct type of fault-tolerant logical operators that may be implemented in a topological code. They have been extensively studied in previous work [2], [4] and a full classification for a large class of topological stabiliser codes is known [4]. We direct the reader to [4] for full details on the properties and limitations of locality-preserving logical operators in such codes.

2. Excitations

Excitations of topological stabiliser code have exotic properties, due to the topological ordering of the model. In two spatial dimensions, the localised excitations of a topological stabiliser code are anyons. The surface code provides the simplest example of such an anyon model, admitting four types of anyons: the vacuum ($1$), electric charges ($e$), magnetic fluxes ($m$) and a composite excitation produced by fusing a charge and a flux ($em$) [30]. Each type of anyon may be distinguished by its fusion and braiding rules, and collectively these rules define the anyon model.

In $D \geq 3$ spatial dimensions, excitations need not be point-like. Specifically, they may be of any spatial dimension $j$ such that $0 \leq j \leq D - 2$. We refer to such generalised versions of anyons as eigenstate excitations [4]. For example, the three dimensional surface code has the same types of eigenstate excitations as the anyons of the two dimensional code, but with magnetic fluxes as one-dimensional loop-like objects while electric charges are still point-like quasiparticles. While point-like excitations can only be condense in the bulk of a code in pairs, higher dimensional excitations can condense individually provided they form closed loops or surfaces (or more generally, hypersurfaces).

Another peculiarity in $D \geq 3$ spatial dimensions is that the elementary topological objects may not be energy eigenstates at all. Three and higher dimensional codes may also admit non-eigenstate excitations of at least one dimension [4, 40–45]. The existence of such excitations has been inferred independently from multiple different considerations. Mathematically, the existence of Cheshire charge – a delocalised charge across a loop-like excitation that cannot be decomposed into local charges – is implied by analysis of three dimensional topological stabiliser codes as corresponding to braided fusion 2-categories [10]. Excitations carrying Cheshire charge are non-eigenstate excitations. Independently of this, study of the action of non-Clifford locality-preserving logical operators on Pauli errors in topological stabiliser codes has led to the identification of linking charges [11]. These linking charges are also equivalent to non-eigenstate excitations (and, indeed, carry Cheshire charge). The route to understanding non-eigenstate excitations that is most relevant to our work, however, is due to Yoshida [42–44]. From this perspective, non-eigenstate excitations are best understood as boundaries of non-Pauli locality-preserving logical operators. It is this approach that we pursue in the following subsection to further elucidate the nature of such excitations.

3. Excitations and locality-preserving logical operators

In topological stabiliser codes, there is a relationship between the localised (eigenstate and non-eigenstate) excitations and the locality-preserving logical operators that the code supports. Specifically, excitations in topological stabiliser codes can be considered boundaries of locality-preserving logical operators [4] [42]. Indeed, if the restriction of a $k$-dimensional locality-preserving logical operator is applied to a region of a $k$-dimensional subspace, then an excitation arises at the boundary of the region. Since any topologically non-trivial path of an excitation can be used to implement some locality-preserving logical operator [4] [42], all excitations must in fact be of this type.

This relationship is well understood for the case of eigenstate excitations. For example, in the two dimensional surface code, a logical Pauli operator is imple-
mented by a string of Pauli operators between an appropriate pair of boundaries or defects, or across a topologically non-trivial loop. Restricting a string of this kind so that it has endpoints in the bulk of the code gives rise to a pair of anyons at these endpoints. More generally, restricting a $k$-dimensional logical Pauli operator to have a boundary in the bulk of the code, rather than on code boundaries or defects, gives rise to a $(k-1)$-dimensional excitation at this boundary of the operator. This excitation will be an eigenstate excitation, since the Pauli operators that give rise to it necessarily commute or anticommute with each stabiliser meaning that the state that results from their application will be an energy eigenstate. Since logical Pauli operators are necessarily locality-preserving, this is an instance of the general phenomenon described above.

Non-eigenstate excitations arise from applying the same analysis to non-Pauli locality-preserving logical operators. Specifically, non-eigenstate excitations arise at the boundary of non-Pauli locality-preserving logical operators that are restricted to terminate in the bulk of a code. Since non-Pauli locality-preserving logical operators are necessarily of dimension $k \geq 2$ \cite{42}, their boundaries cannot be point-like, and so can only support excitations in codes of more than two dimensions.

As an example, consider the three-dimensional colour code, which admits a two-dimensional locality-preserving logical Clifford phase operator, $S$ \cite{3,12}. The restriction of $S$ to a compact region gives rise to a loop-like one-dimensional boundary. At this boundary, a non-eigenstate excitation, labelled $s$, is realised. This excitation is not an energy eigenstate, since the $S$ operator is a superposition of Pauli operators that neither commutes nor anticommutes with the $X$-type stabilisers at the boundary of the region. This excitation is thus a superposition of eigenstate excitations (specifically, the vacuum and point-like $e$ excitations), but cannot be projected onto an eigenstate excitation by local operators — such projection requires operators to act on its entire interior. For this reason, it must be considered a distinct excitation that cannot be realised by simply assembling a configuration of eigenstate excitations, just as its corresponding locality-preserving logical operator is a superposition of logical Pauli operators but cannot be realised by a product of Pauli operators. As with eigenstate excitations, this $s$ excitation can be characterised by its braiding statistics \cite{42}.

More general non-eigenstate excitations have similar properties. As noted, general $(k-1)$-dimensional non-eigenstate excitation arises at the boundary of a $k$-dimensional non-Pauli locality-preserving logical operator. This excitation cannot be an energy eigenstate, since non-Pauli operators necessarily have some Pauli operator with which they neither commute nor anticommutate. However, as with $s$, such excitations cannot be decomposed into eigenstate excitations. In addition, non-eigenstate excitations can be characterised by their braiding statistics with eigenstate excitations, that correspond to the (nested) commutation relations between their corresponding logical operators and logical Pauli operators \cite{42}.

B. Defects in Topological Stabiliser Codes

To define defects, assume that we initially have a topological stabiliser code that is translationally invariant. A defect is defined to be a $k$-dimensional region introduced to this code where this translational invariance is broken, with $0 \leq k < D$. This can be viewed as a region of the code where the Hamiltonian terms are altered. We note that, unlike the original stabilisers, these altered Hamiltonian terms are not required to be Pauli operators. However, we do require that they are local. We refer to a defect at which excitations can condense as a topological defect, as this condensation allows the defect to carry topological charge. To illuminate this definition, we survey several examples of topological defects.

1. Holes

A hole or puncture is a compact region of the code where the stabilisers are removed from the Hamiltonian (so that this region is equivalent to the vacuum). The boundary of such a hole is thus similar to a code boundary, and acts as a topological defect. For example, the surface code admits two types of boundaries; rough boundaries which can condense electric charge ($e$) excitations and smooth boundaries which can condense magnetic flux ($m$) excitations \cite{46}. It correspondingly admits two types of holes (those with rough and smooth boundaries respectively) that condense $e$ and $m$ type excitations respectively. The boundaries of holes are closed loops (or, more generally, hypersurfaces) of some finite size, independent of the size of the code. This means that holes can be moved around through the bulk of the code, allowing for braiding.

To illustrate how such holes can be used to encode quantum information, consider the example of the two dimensional surface code. As discussed above, holes in a two dimensional surface codes can be constructed to have rough boundaries that allow $e$ excitations, but not $m$, to condense. Consider introducing a pair of holes of this type into a surface code. Assume that there is no total net charge across the pair. We may then associate a logical qubit with the charge in the region of either of the holes; i.e. the parity of $e$ excitations in this region. Provided the holes are separated by a distance, $d$, this parity can only be changed by an operator of weight at least $d$. Similarly, provided each hole has a circumference of at least $d$, the parity can only be distinguished by an operator of weight at least $d$. We can thus define a logical qubit with computational basis states $|0\rangle$ and $|1\rangle$ corresponding to even and odd parity of $e$ excitations respectively in the region of either hole.
The Pauli operators for this logical qubit are then naturally transversal. Specifically, a logical $X$ operator may be implemented by transferring an $e$ excitation between the pair of holes. This can be implemented by a string of $Z$ operators between the holes. A logical $Z$ operator may be implemented by braiding an $m$ excitation around either hole, since this results in a phase of $-1$ if and only if the parity of $e$ in this region is odd. This can be implemented by a string of $X$ operators around a hole. These logical operators are illustrated in Fig. 1.

Schemes for realising a universal set of logical operators on logical qubits encoded in holes have been developed for both the two dimensional surface [20] and colour codes [21]. These schemes use braiding to achieve an entangling CNOT operator between logical qubits. However, they must then be supplemented in two ways. Firstly, to realise single qubit logical operators, braiding must be supplemented by locality-preserving logical operators. This can be done to realise a Hadamard operator in the surface code [20] and the full single qubit Clifford group in the colour code [21]. Secondly, locality-preserving and braiding logical operators must be supplemented by some additional type of operator to realise universality. In both the surface and colour code this may be done by using magic state distillation to realise the (non-Clifford) $T$ gate [20, 21].

2. Domain Walls and Twists

A second type of defect arises from considering boundaries between two copies of the same code. Such boundaries are referred to as (transparent, gapped) domain walls. Unlike code boundaries, they do not absorb excitations, but instead may transform them from one type of excitation into another. They may be introduced by applying a restriction of a locality-preserving logical operator to a region of the code. Domain walls in a $D$-dimensional code may be $k$-dimensional for any $k$ such that $1 \leq k < D$ [4] [12].

As a simple example of a domain wall, consider the two dimensional colour code. This has stabilisers that are generated by products of $X$ operators and products of $Z$ operators around each plaquette of the lattice [17]. It thus clearly has a symmetry under qubit-wise Hadamard operators that interchange $X$ and $Z$ operators. If this symmetry is applied to some compact region of the code, then the result is a gapped boundary around the boundary of the region – a domain wall. This boundary is a defect since some but not all of the qubits in the support of stabilisers that lie along it are acted on by the symmetry. This means that these stabilisers will have a different structure after the symmetry has been applied from stabilisers in the rest of the code, and so the symmetry breaks the translational invariance of the code. Excitations that cross this domain wall are transformed in accordance with the symmetry applied to the interior region.

As described so far, however, domain walls do not allow for the condensation of excitations. Such condensation is made possible by terminating domain walls in boundaries of their own; topological defects referred to as twists. For details on how to construct such twists, we refer readers to the procedure presented in [33].

Since twists allow excitations to condense, they are topological defects. For example, in the case of the Hadamard symmetry in the two dimensional colour code, a pair of $e$ excitations may be condensed from the vacuum, and one of these excitations brought across the domain wall, converting it to an $m$. This $e$ and $m$ may then be brought together by allowing them to meet beyond the twist, producing the composite excitation $em$. More generally, domain walls may be classified by their action on excitations of the code. To see this, denote by $b^*$ an excitation that annihilates to the vacuum with $b$. (If $b$ is a quasiparticle, $b^*$ is the antiparticle of $b$.) A domain wall that takes $a \rightarrow b$ will then allow composite excitation $a^*b$ to condense. This is as shown in Fig. 2.

Schemes for quantum computing with twists have been studied for a range of two dimensional codes. These codes include the surface code [24, 25], the colour code [20], subsystem colour codes [24] and the $Z_2$ abelian quantum double model (which generalises the surface code to qutrits) [28]. All of these schemes allow only Clif-
ford gates by braiding, and require additional methods, such as magic state distillation [13] or topological charge measurement [28], to realise universality. We note that genons, a particular type of twist defect, have also attracted study for their braiding properties across a range of models, and allow for universality by braiding in more exotic codes than topological stabiliser codes [30, 31].

C. Encoding Quantum Information in Defects

We now build a general formalism for encoding quantum information in defects of any type in any spatial dimension. Specifically, the excitations that can condense at a defect form an abelian group under fusion. We focus on the case where this group is $\mathbb{Z}_2$, but our results naturally generalise to other groups by considering generalised Pauli and Clifford groups for qudits.

Such defects can be used to encode logical qubits. Specifically, we first assume that there is no net charge across the defect setup. We may then associate computational basis states for a qubit with the relative charge associated with the condensing excitation across one or more defects. This can be seen most clearly using two defects. Then, the $|0\rangle$ and $|1\rangle$ states may be associated with even and odd parity of the excitation respectively on either of the defects. This encoding is used for all the two-dimensional schemes reviewed in the previous subsection. We note that in more than two dimensions, we may have the excitation used for encoding be of dimension greater than or equal to one. Such excitations arise at the boundary of locality-preserving logical operators of more than one dimension. Since such a boundary can be a single connected object, these excitations can thus condense individually from the vacuum. Thus, to ensure that the parity of the excitation remains the same between the pair of defects, we must thread an additional defect through the pair used for encoding that prevents the excitation condensing from the vacuum. An example of this in three dimensions is shown in Fig. 3. More than one logical qubit may then be encoded by adding further defects and associating the computational basis states of new qubits with the relative charge between other pairs of defects.

In more than two dimensions, it can also be possible to encode a logical qubit in a single defect. Specifically, consider a defect allowing for the condensation of an excitation that is not point-like, for example a loop-like excitation. Such an excitation can condense individually from the vacuum and then grow to be absorbed by this single defect. Thus, we may associate the computational basis states of a logical qubit with the presence or absence of this excitation on the single defect. Since this excitation can condense from the vacuum, the net charge of the defect must remain neutral in either state, with only the distribution of this charge across the defect distinguishing the states. An example of this type of encoding is shown in Fig. 4.

**FIG. 3.** An example of a logical qubit encoded in a pair of defects. The two tori are holes in the three dimensional surface code at which magnetic fluxes may condense. The additional loop threaded through is a puncture (which may be viewed as a narrow tube) at which magnetic fluxes cannot condense. The computational basis states of the qubit correspond to the parity of fluxes at either of the toroidal holes, which is well-defined since the threaded defect prevents these fluxes from condensing from the vacuum. The $\bar{Z}$ operator is a surface corresponding to propagating a flux between the holes. The logical $\bar{Z}$ is realised by braiding an electric charge around either hole.

**FIG. 4.** An example of a logical qubit encoded in a single defect. The defect is a toroidal hole in a 3D toric code that can condense one dimensional $m$ excitations. The logical $\bar{X}$ operator is implemented by a process by which an $m$ condenses from the vacuum and then grows to be absorbed by the defect. This is equivalent to applying a membrane of $X$ operators that meets the torus in a topologically non-trivial loop. The logical $\bar{Z}$ is implemented by braiding an $e$ excitation around the defect. This is equivalent to a loop of $Z$ operators around the torus.
By combining encodings of the kinds described across arbitrarily many defects in a code, we may account for all natural encodings of protected quantum information in defects. Indeed, the potential for defects to encode information arises from the additional ground state degeneracies they can introduce to a code. This degeneracy is associated with the fact that it is possible for such defects to have different charges while remaining in the ground space. These different charges are associated with excitations that can condense at the defects. Thus, the ground state degeneracy introduced by defects can be associated with the distributions of excitations across the defect setup that are possible while still maintaining no net charge across the entire code. This leads to the types of encodings we have presented.

We note that all of these encodings necessarily admit locality-preserving logical $\bar{X}$ operators on all encoded qubits. Indeed, for a qubit in which the $|\bar{0}\rangle$ and $|\bar{1}\rangle$ states are associated with the parity of excitation $a$, these states may be interchanged by moving $a$ to condense on defects as necessary. This movement of an excitation can be realised as a locality-preserving logical operator. This operator is then a locality-preserving logical $\bar{X}$ operator. In the case where excitation $a$ is an eigenstate excitation, it is also clear that $\bar{Z}$ can be implemented as a locality-preserving logical operator. Specifically, it can be realised by choosing another eigenstate excitation, $b$, that braids with $a$ to give a phase of $-1$, and then braiding $b$ around one of the defects used to encode $a$. Examples of this are illustrated in Fig. 3 and Fig. 4. In the case where $a$ is not an eigenstate excitation, however, the existence of such an excitation $b$ is not guaranteed. This means that the implementation of $\bar{Z}$ in this case may be more complicated. For this reason, we in general assume only that $\bar{X}$ operators are locality-preserving and not that $\bar{Z}$ operators are.

D. Braiding

One reason logical qubits encoded in defects are interesting is that they allow for logical operators to be implemented by braiding defects. Braiding defects consists of any process by which the positions of defects in the code are altered smoothly. To ensure fault-tolerance, we require that this is done in a way that maintains the distance of the encoding, i.e. so that defects remain sufficiently large and well-separated throughout. This process is generally proposed to be done by code deformation in which the Hamiltonian is adiabatically transformed through successive expressions corresponding to gradually changed positions of defects [48]. However, our analysis and results are independent of the particular method used for braiding. For example, Zhu et al. have recently proposed a low time-overhead approach to braiding non-abelian anyons [49, 50]. If this approach could be adapted to the braiding of defects considered here, our analysis would still apply to operators implemented in this way.

Examples of braiding processes are illustrated in Fig. 5, Fig. 6 and Fig. 7. By contrast, note that processes that involve discontinuous deformations, such as lattice surgery [51], are not included as braiding. We note that, as evident in Fig. 7, braiding processes in models of more than two dimensions may be more complicated and varied than in two dimensional codes. Indeed, in general such processes can involve defects of any spatial dimension, and can involve more than two defects at a time [52]. This greater variety of braiding processes may lead to the hope that universal quantum computing may be achievable in higher dimensional codes by using the power of such braiding processes. We show the limitations of this approach in the next section.
III. LIMITATIONS OF BRAIDING DEFECTS

In this section, we present the limitations of performing fault-tolerant logical operators by braiding defects in topological stabiliser codes. We begin by proving a very general result – that the set of logical operators implementable by any combination of braiding and locality-preserving logical operators cannot be universal. Thus, adding the ability to braid defects does not allow the challenges to universal fault-tolerance presented by the results of Eastin-Knill [1] and Bravyi-König [2] to be overcome. We then consider how this results may be naturally generalised from topological stabiliser codes to apply to the broader class of abelian quantum double models. Finally, we comment on how, with further assumptions, an even stronger constraint that requires all braiding logical operators to be contained in the Clifford group can be established. This result first appeared in the companion paper [36], and is presented there in more detail.

A. Universality No-Go Result

The main result we present is that a universal gate set cannot be achieved by using only locality-preserving logical operators and braiding logical operators in a topological stabiliser code of any spatial dimension. This builds on the result of Bravyi and König [2] that the group of locality-preserving logical operators admitted by a topological stabiliser code cannot be universal. Specifically, we show that the addition of logical operators implementable by braiding defects is not sufficient to allow for universality.

We first prove two lemmas, and then the theorem itself.

Lemma 1. Let \( \bar{B} \) be a logical operator implementable by braiding defects, and \( \bar{U} \) be a locality-preserving logical operator. Then \( \bar{U} \bar{B} \bar{U}^\dagger \) is also a locality-preserving logical operator.

Proof. Consider a \( k \)-dimensional locality-preserving logical operator \( \bar{U} \) and braiding logical operator \( \bar{B} \). Note that the operator \( \bar{B} \bar{U} \bar{B}^\dagger \) corresponds to the resultant realisation of \( \bar{U} \) after \( \bar{B} \) has been applied. The effect of \( \bar{B} \) may be decomposed into two parts. Firstly, there are finite translations and rotations in the support of defects that are implemented when no defect is crossing \( \bar{U} \). These can introduce translations, rotations and stretchings of the support of \( \bar{U} \). Secondly, there is the action of defects (such as domain walls) crossed by \( \bar{U} \). Since defects can only have local interactions, however, this action must be locality-preserving on \( \bar{U} \). Thus, both of these types of transformations take locality-preserving logical operators to locality-preserving logical operators and so \( \bar{B} \bar{U} \bar{B}^\dagger \) is a \( k \)-dimensional locality-preserving logical operator.

Lemma 2. Any finite set of logical qubits encoded in defects in a topological stabiliser code necessarily admits a non-empty, but finite set of locality-preserving logical operators.

Proof. As explained in Sec. II C, the logical \( \bar{X} \) operator acting on each logical qubit is necessarily a locality-preserving logical operator. Thus, the set of locality-preserving logical operators acting on any set of encoded qubits is non-empty.

To show finiteness, consider first topological stabiliser codes without defects. In that case, the group of localised (eigenstate and non-eigenstate) excitations must be finite. To see this, note that a topological stabiliser code on a surface with periodic boundary conditions has at least as many locality-preserving logical operators as it has localised excitations. Indeed, each process by which a localised excitations traverses a topologically non-trivial loop around the surface gives rise to a distinct locality-preserving logical operator. Thus, since defect-free topological stabiliser codes admit only a finite set of locality-preserving logical operators [2], the group of localised excitations must also be finite.

Now, note that the group of localised excitations is independent of the boundary conditions and defects of the code. Thus, in the presence of defects, we must still have a finite group of localised excitations. Any finite set of logical qubits must be encoded in only finitely many defects. Thus, there are only a finite set of topologically distinct paths an excitation can take to implement a logical operator. Thus, since locality-preserving logical operators correspond to topologically distinct paths of excitations, the set of locality-preserving logical operators acting on any finite set of qubits encoded in defects must be finite.

Theorem 1. The set of logical operators implementable by any combination of locality-preserving logical operators and braided logical operators in a topological stabiliser code cannot be universal.

Proof. By Lemma 1, both braiding and locality-preserving logical operators permute the set of locality-preserving logical operators under conjugation. By Lemma 2, the set of locality-preserving logical operators acting on any finite set of logical qubits encoded in defects is finite and non-empty. Thus, the set of locality-preserving logical operators is a finite, non-empty subset of the set of logical operators that is invariant under combinations of locality-preserving and braiding logical operators. Thus, the set of combinations of locality-preserving and braiding logical operators cannot be dense in the set of logical operators and thus cannot be universal.

Theorem 1 shows that including logical operators implemented by braiding defects is not sufficient to lift the set of locality-preserving logical operators to universality in a defect setup in any topological stabiliser code.
B. Generalisation to Abelian Quantum Double Models

In this paper we primarily consider topological stabiliser codes. However, we may also consider whether Theorem 1 may be generalised to include other classes of topological quantum error correcting codes. For example, Escobar-Velásquez et al. have studied braiding holes in two dimensional Dijkgraaf-Witten theories, which include topological stabiliser codes as well as more general abelian and non-abelian quantum double models [55]. They showed that the set of gates implementable by braiding holes in such codes cannot be universal. In light of our results, it is natural to ask whether this result could be generalised to more general defects and to models of higher spatial dimensions.

We here argue that our results can at least be generalised to all defects implemented in codes in the class of abelian quantum double models of all spatial dimensions. We confine our attention to this class since the excitations and locality-preserving logical operators they admit have been studied and classified [4] and shown to be entirely analogous to that of topological stabiliser codes. This means that the theory of Sec. [1] and results used to prove Theorem 1 hold entirely and straightforwardly analogously for abelian quantum double models. The class of codes is also interesting, since the power of braiding its defects has been explored in previous work. In particular, the technique of topological charge measurement has been shown to provide universality in the double of $\mathbb{Z}_3$ when used to supplement braiding logical operators [28].

Abelian quantum double models are defined by specifying some abelian group, $G$ [39]. Generalised logical Pauli operators are defined such that $X$ and $Z$ operators each generate groups isomorphic to $G$. The generalised Clifford group for a given $G$ can be defined to be the normaliser of the generalised Pauli group. With these definitions, we may then state the following result. The proof of this results is entirely analogous to that presented for Theorem 1.

**Theorem 2.** The set of logical operators implementable by any combination of locality-preserving logical operators and braided logical operators in an abelian quantum double model cannot be universal.

C. Restriction to the Clifford Group

Having presented a very general result, we now consider a more specific setup. In particular, we consider a pair of topological defects that allows for some group of excitations to condense. For clarity we assume that this group has the form $\mathbb{Z}_2^k$ generated by excitations $a_i$ for $1 \leq i \leq k$. This allows us to restrict to the case of logical qubits, but what we discuss generalises naturally to all abelian groups. We can encode logical qubits by associating the computational basis states of a logical qubit $i$ with the parity of excitation $a_i$ in the region of either of the defects. We assume here that $a_i$ is an eigenstate excitation. We may introduce further logical qubits by adding more pairs of defects and encoding further logical qubits analogously. We note that this setup is a natural generalisation of the braiding schemes discussed in the previous section in two dimensional topological stabiliser codes. In particular, the schemes are all based on encoding logical qubits in this way either in pairs of domain walls terminating in twists or in pairs of holes.

Generalising this approach to higher dimensional codes may naturally be expected to allow for a larger set of braiding logical operators. In particular, it is known that braiding twists in the two dimensional surface code allows for the full Clifford group to be implemented. This is the same as the set of locality-preserving logical operators that are possible in a two dimensional topological stabiliser code, according to the Bravyi-König bound [2]. One may expect this coincidence of the allowed set of locality-preserving and braiding logical operators to extend to higher dimensional codes. Specifically, one may expect generalising to a $d$-dimensional topological stabiliser code would allow for braiding logical operators from the $d^\text{th}$ level of the Clifford hierarchy. This may be expected in particular by using twists that terminate domain walls corresponding to locality-preserving logical operators in the $d^\text{th}$ level of the Clifford hierarchy to encode logical qubits, and following a similar approach to that used for the two dimensional surface code.

However, perhaps surprisingly, this expectation is not met. Instead, it can be shown that encoding logical qubits in this way only allows for braiding logical operators in the Clifford group, regardless of the spatial dimension of the code. This further illustrates that braiding logical operators in topological stabiliser codes do not realise the possibilities for fault-tolerant quantum computing that may have been imagined. This result is presented formally, proven and discussed in more detail in our companion paper [36]. Examples are given in the following section, where we consider schemes that encode logical qubits in the way described here, and can be seen to only allow for the Clifford group by braiding.

We also note that the generalisation to abelian quantum double models presented in the previous subsection may also naturally be applied to these results. In particular, the Pauli and Clifford groups admit natural generalisations to the qudits of abelian quantum double models. Using these generalisations, the results of [36] straightforwardly extend to abelian quantum double models under the same assumptions as for topological stabiliser codes.

IV. SCHEMES FOR IMPLEMENTING THE CLIFFORD GROUP BY BRAIDING

In this section, we explore examples of schemes that allow for the Clifford group to be implemented by braiding. In particular, we generalise the scheme for imple-
menting the Clifford group by braiding twists in the two-dimensional surface code to higher dimensional codes. In doing so we demonstrate a very general result: that any code with domain walls that can condense a \textit{generalised fermion} can be used to realise the full Clifford group by braiding. By a generalised fermion, we mean an eigenstate excitation of any spatial dimension, such that a exchanging a pair of such excitations gives a phase of \( -1 \). The Clifford group can be used to achieve universality when supplemented by magic state distillation. Thus, the generalisation allows for an approach to realising universality in a large class of topological codes.

We begin by reviewing braiding in the two dimensional surface code, following \cite{24}, as the motivating example for this result. We then provide and describe two examples of twist setups using domain walls in codes of more than two dimensions that condense generalised fermions. Specifically, we first generalise the example of the two dimensional surface code to \textit{self-dual surface codes}. These are surface codes for which the logical Pauli \( X \) and \( Z \) operators are of the same dimension or, equivalently, where electric and magnetic excitations are of the same dimension. This includes the two-dimensional surface code, as well as instances of all even spatial dimensions. While these generalised schemes can only be realised in more than three dimensions, they are worthy of consideration because they offer the potential for self-correction and display braiding phenomena that do not appear in braiding in two dimensional codes. We also introduce the example of braiding in the three dimensional Levin-Wen fermion model \cite{37,56}. This is an interesting example for several reasons, including that it offers an example of a three dimensional code that allows for the Clifford group to be implemented fault-tolerantly.

We show explicitly how the single qubit Clifford group may be implemented in each defect setup. This can be extended to the Clifford group on any number of encoded qubits by introducing ancilla holes, analogously to \cite{24}, however we omit the details of this from our illustrative examples for the sake of simplicity. We note that the schemes we consider all use the type of encoding presented in Sec. \[1TC\] and so we have already seen that they cannot allow for non-Clifford logical operators.

\subsection{Two Dimensional Surface Code}

The two-dimensional surface code admits a domain wall that interchanges \( e \) and \( m \) type excitations, corresponding to the locality-preserving logical Hadamard operator \( S \) \cite{4,23}. This wall allows for the condensation of a composite \( em \) type excitation around a twist at one of its endpoints. A pair of such walls with twists at their ends can allow for a single logical qubit to be encoded. Specifically, we encode the states \( |0\rangle \) and \( |1\rangle \) as corresponding to an even and odd parity of \( em \) excitations in one of the domain walls. The logical Pauli \( X \) operator is then realised by a process that allows an \( em \) excitation to be transferred between the two walls. The logical Pauli \( Z \) operator is realised by a process that results in a phase of \( -1 \) if the parity of \( em \) excitations is odd. This can correspond to enclosing either of the domain walls in a loop traced out by an \( e \) or \( m \) excitation. An example of each of these Pauli logical operators is shown in Fig. \[8\]. We note that this encoding in twists is different from standard surface code encodings in boundaries or holes, which do not allow for the full Clifford group by locality-preserving logical operators or braiding.

We may then implement logical Clifford operators by interchanging twists to interchange logical Pauli operators \cite{24}. Specifically, the \( X \) and \( Z \) logical operators may be interchanged by a braiding process with the effect of swapping a pair of twists diagonally (for example the top-right and bottom-left in Fig. \[8\]). This braid implements the logical Hadamard operator, \( H \). Similarly, swapping a pair of twists vertically (for example the two left twists) interchanges the logical \( X \) and \( Y \) operators (up to a phase of \( -1 \)) and so realises the logical Clifford phase gate, \( S \). The \( S \) gate can also be understood by noting that it introduces a phase of \( \pm i \) if and only if the parity \( em \) on the domain wall is odd, since the excitation is a fermion being rotated by \( \pi \). These logical operators are shown in Fig. \[8\]. Together these logical operators generate the full single qubit Clifford group.

\subsection{Self-Dual Surface Codes}

We now present our first new example of a setup that allows for logical operators by braiding in more than two
dimensions. It is a generalisation of the scheme in the two dimensional surface code we have just presented. Specifically, we show how a surface code with equal dimensions of its logical $X$ and $Z$ operators admits the full Clifford group by braiding.

We begin by focussing on the example of the four dimensional surface code, since this is the simplest case beyond two dimensions. The four dimensional surface code has one dimensional loop-like $e$ and $m$ type excitations. It admits a three dimensional domain wall that interchanges these excitations \[4, 57\]. We seek to generalise the setup used for the two dimensional surface code. We do so by constructing two pairs of concentric (two dimensional) tori as higher-dimensional twists, with the three dimensional volume between each pair as domain walls. A three dimensional cross-section of this is sketched in Fig. 9. Such domain walls allow for loop-like $em$ excitations to condense, which are generalised fermions. We note that to ensure that these $em$ excitations cannot condense from the vacuum away from these twists, we require an additional defect to be threaded through the twist setup, as shown in Fig. 9. This is not required in the two-dimensional case, since its point-like $em$ excitations cannot condense individually from the vacuum. However, such threaded defects are generally required when higher-dimensional excitations are used to encode, since such excitations can condense individually from the vacuum as closed loops (or more generally, hypersurfaces).

A logical qubit may be encoded in the parity of $em$ loops in one of these domain walls. The logical $X$ operator is implemented by a process where a pair of $m$ loops is condensed from the vacuum, one of these loops passes through the pair of domain walls to change to $e$ and then back to $m$, then grows to be larger than the outer twists before returning back to annihilate with the other condensed $m$. Note that this process does indeed have the effect of passing an $em$ between the walls. It also creates a torus that encloses the two outer twists, analogously to how the two dimensional setup had a logical $X$ operator that encloses the top two twists, as shown in Fig. 8. The logical $Z$ is also a torus. It is implemented by a process by which a pair of $m$ loops condense passing through the hole in the left domain wall but not the right one. One of these $m$ loops then rotates through the dimension perpendicular to the three dimensional subspace in which $\tilde{X}$ is embedded. This $\tilde{Z}$ operator then encloses the two left twists and so is analogous to the $\tilde{Z}$ operator in the two dimensional setup which also enclosed a pair of twists from the same domain wall. A cross-section with these Pauli logical operators are illustrated in Fig. 9.

Clifford logical operators by braiding can now be described by identifying twists of this setup with those of the two dimensional setup. Specifically, inner twists here correspond to bottom twists in the two dimensional setup and left twists in this setup correspond to left twists in the two dimensional setup. Pictured in this way, it is straightforward to see that swapping the left twists will implement the logical phase operator, $S$, since it interchanges the two twists enclosed by $\tilde{Z}$. Interchanging the inner-left with outer-right twist will implement the logical Hadamard operator, $H$, since it swaps the twists that are enclosed by $\tilde{X}$ but not $\tilde{Z}$ and vice versa. Thus, the setup does indeed allow for the full single qubit Clifford group to be implemented by braiding.

This picture can be generalised to a $2k$-dimensional surface code with $(k - 1)$-dimensional $e$ and $m$ excitations. Specifically, we choose two concentric pairs of hyper-surface of the form $S^{k-1} \times S^{k-1}$ as twists, where $S^k$ is a $k$-dimensional sphere and the regions between each concentric pair to be domain walls. The $\tilde{X}$ and $\tilde{Z}$ operators will then be surfaces of the form $S^{k-1} \times S^{k-1}$ enclosing appropriate twists. The $S$ and $H$ operators can be implemented by appropriate interchanges of the twists.

While the scheme laid out here cannot be realised in
less than four spatial dimensions, it is of some theoretical interest. One avenue of interest is that all surface codes considered here are self-correcting. We may expect that property to carry over to their braiding schemes, since all logical operators are of at least two spatial dimensions. Properties of this braiding scheme may thus offer guidance on how self-correction may be achieved in more general braiding schemes. Moreover, this example may provide more guidance on how to generalise two dimensional braiding schemes to higher dimensions.

C. Three Dimensional Levin-Wen Fermion Model

The three dimensional Levin-Wen fermion model is a three dimensional topological stabiliser code that admits point-like fermionic excitations \([37, 56]\), which we label \(e\). This means that it is not equivalent to the three dimensional surface code, although the structure of its excitations is similar; both codes admit a point-like \(e\) and a loop-like \(m\). Defined on a cube with boundaries (similarly to the planar surface code) it encodes two or three logical qubits, depending on whether the length of the cube is an even or odd number of qubits. This code with boundaries admits only (two-dimensional) \(CZ\) operators between each pair of logical qubits as locality-preserving logical operators \([4]\). The \(CZ\) logical operator has a corresponding (one-dimensional) domain wall that allows for the fermionic \(e\) excitations to condense. Thus, we may expect that it should admit the full Clifford group by braiding. We show that this is the case. We note, however, that unlike three-dimensional surface codes, this model does not admit a locality-preserving logical \(CCZ\) operator \([4]\), and so this scheme does not allow for a universal gate set.

We construct a setup to allow this braiding analogous to the two dimensional surface code. Specifically, we have two domain walls, each ending in a pair of point-like twists. The computational basis states for a logical qubit can then be encoded in the parity of \(e\) excitations in either one of these domain walls. The logical \(Z\) operator is then a surface enclosing one of the domain walls traced out by an \(m\) loop growing from the vacuum passing around the domain wall and then shrinking back to the vacuum. The logical \(X\) operator is implemented by an \(m\) loop growing from the vacuum, then passing so that one point passes through a domain wall so that an \(e\) is appended, then passing through the other wall so that the \(e\) may annihilate with another \(e\) that is attached, and finally shrinking back to the vacuum. These Pauli logical operators are illustrated in Fig. 10. We may then immediately see that the single qubit Clifford group may be implemented analogously to in the two dimensional surface code. Specifically, interchanging the two left twists will implement the phase operator, \(S\), since it introduces a phase of \(\pm i\) if and only if the parity of \(e\) in the left domain wall is odd. Interchanging the bottom-left and top-right twists implements the Hadamard operator, \(H\), since it swaps the logical \(X\) and \(Z\) operators.

This scheme is interesting for a number of reasons. Firstly, it is an instance of how the Clifford group may be implemented without state injection or dimensional jumping in a three dimensional topological stabiliser code. Specifically, the Levin-Wen fermion model allows for the Clifford group to be implemented by braiding point-like defects. This is interesting as it is known that no translationally-invariant (i.e. without defects) and scale-symmetric three dimensional topological stabiliser code can admit the full Clifford group by locality-preserving logical operators \([4]\) and so offers an example of where braiding defects can allow for fault-tolerant operators not admitted otherwise. The scheme is also notable as an instance of a code where truly string-like Pauli logical operators are relatively rare (since they can only be realised by strings between the pair of twists). Nonetheless, we may expect that the code is not self-correcting, since membrane operators implementing logical Paulis must only be of length \(d\) in one dimension and can be arbitrarily small in the other dimension. The interesting properties of this three dimensional braiding setup may inspire and guide future research into braiding schemes in more exotic models.

Finally, the scheme offers new insight into the types of objects and braids that may yield interesting behaviour in more than two dimensions. For example, it is an example of non-trivial braiding of point-like twists in a three dimensional code. This is interesting since point-like anyons in a three dimensional space are known to have trivial (bosonic or fermionic) statistics. It can, however, be understood as reflecting that the point-like twists are only endpoints to the one-dimensional domain wall, and that it is the rotation of this object that has a non-trivial...
action on the excitations present in it. Nonetheless, it highlights an interesting limitation to the conventional picture of twists as analogous to non-abelian anyons.

D. Discussion of General Scheme

The schemes presented clarify a sufficient requirement for a code to allow the full Clifford group by braiding. Specifically, this requirement is that the code admits domain walls with twists that can condense a generalised fermion (of any spatial dimension). For completeness, we now briefly summarise how a defect setup admitting the Clifford group can be constructed if this requirement is satisfied.

We begin by noting that the significance of a $D$-dimensional code admitting twists that allow a generalised fermion to condense is that it implies that the corresponding domain wall interchanges two excitations that braid with one another to give a phase of $-1$. Indeed, a twist that allows generalised fermion $a$ to condense must act on some eigenstate excitation $b$ by the mapping $b \rightarrow ab^*$ as discussed in Sec. II B 2. Since $b$ and $a$, must have the same phase under self-exchange (since they are interchanged by a domain wall), but the excitation $a$ produced by composing them is a generalised fermion, they must give a phase of $-1$ when braided around one another. This implies that the operators used to propagate them must anticommute [4] and that the sum of the dimensions of excitations $a$ and $b$ must be $D - 2$. This in turn implies that the dimension of the twist at which $a$ condenses must be twice the dimension of $a$ [1]. These facts justify the validity of the following construction.

Assume a code admits some $k$-dimensional domain wall with twists that allows for some $j$-dimensional generalised fermion, $a$, to condense. Such a domain wall must act on an eigenstate excitation $b$ by the mapping $b \rightarrow ab^*$ to allow for excitation $a$ to condense. Construct a pair of such domain walls such that each has a pair of concentric twists topologically equivalent to $S^{j-1} \times S^{j-1}$ at their boundaries. Add a further puncture threaded through the holes of the domain walls if $j > 1$ at which $a$ cannot condense. This prevents excitations $a$ that condense on a topologically non-trivial loop of one of the domain walls from being absorbed by the vacuum. The logical $X$ operator can then be specified by a process that propagates excitation $a$ between the pair of domain walls. This is equivalent up to stabilisers to a pair of excitations $b$ and $b^*$ condensing from the vacuum, excitation $b$ being propagated through the pair of domain walls, and then returning to its initial position and size without crossing through the walls again. This is the form in which the logical $X$ operator was presented for the examples illustrated above. The logical $Z$ operator can be realised by an operator that propagates excitation $b$ to trace out a hypersurface that encloses one of the domain walls. This choice of logical $X$ and logical $Z$ are valid since they anticommute, by the arguments of the previous paragraph. Swapping two defects from the same domain wall will now implement a logical $\hat{S}$ operator, and swapping the inner defect from one domain wall with the outer defect from the other will implement a logical $\hat{H}$ operator. These operators generate the full single qubit Clifford group. This can be extended to the full Clifford group on a larger set of qubits by adding additional holes and using these as ancillary defects to allow for entangling gates, analogously to in [24].

V. A UNIVERSAL SCHEME

We now illustrate one scheme by which the no-go theorem presented in Sec. III A can be circumvented to allow for universal quantum computing with defects in a topological stabiliser code. Specifically, we present a scheme for realising a universal gate set on logical qubits encoded in defects in a stack of three 3D surface codes without magic state distillation. Our scheme is similar to that of Vasm and Browne [9], but has the advantage that it can be performed entirely by manipulating defects in the bulk of the code and so allows for arbitrarily many logical qubits to be encoded in a single memory. In particular, we show that braiding of these defects can achieve the necessary entangling gates between these logical qubits for universality, without the need for lattice surgery between different code blocks. This illustrates how, despite the limitations discussed in Sec. III, braiding defects in higher dimensional codes can still be a useful tool in the pursuit of universality.

The basis for this scheme is the use of stabiliser state injection to allow for a logical Hadamard operator to be implemented on qubits encoded in defects in a three-dimensional surface code. This injection circuit is not constrained by our no-go results since it requires adaptive implementation of a logical operator to be applied conditionally on the outcome of a logical measurement. This necessitates supplementing locality-preserving and braiding logical operators with non-local classical processing [35] (see Appendix 1 for further discussion). In particular, such Hadamard operators could otherwise not be realised by braiding or as a locality-preserving logical operator. Together with logical $CCZ$ operators, this gives universality. In addition to the scheme of Vasm and Browne that applies this stabiliser state injection technique to the 3D surface code [9], the technique has also previously been applied to allow for universality in the 3D colour code [5] and a class of 2D Bacon-Shor codes [58]. To illustrate the key ideas of our scheme, we begin by discussing the case of three logical qubits, before then proceeding to the general case.

A. Logical Qubits and Transversal Gates

We begin with a code that is locally equivalent to a stack of three 3D surface codes [4], labelled codes one,
FIG. 11. Defect setup for the universal scheme we present. All five defects are punctures in a stack of three 3D surface codes. $h_1$ and $h_2$ are rough with respect to surface code one and smooth with respect to surface codes two and three. $h_\alpha$ and $h_\beta$ are rough with respect to surface code two and smooth with respect to surface codes one and three. $h_3$ is threaded through all four other holes, and is rough with respect to surface codes one and two and smooth with respect to surface code three.

two and three. Three logical qubits can be encoded in five defects in this code, as illustrated in Fig. 11. Specifically, this setup consists of four toroidal defects arranged with a fifth toroidal defect threaded through the hole of each other defect. Appropriately choosing boundaries of these defects to be rough and smooth with respect to the three surface codes allows for three logical qubits to be encoded, with logical Pauli operators as illustrated in Fig. 12, Fig. 13 and Fig. 14. Further details of these defects are provided in Appendix 2. These logical qubits admit transversal $\text{CZ}_{ij}$ operators, implemented by applying $\text{CZ}_{ij}$ transversally across the support of $\bar{X}_k$ (for $\{i, j, k\} = \{1, 2, 3\}$). Transversally applying $\text{CCZ}_{123}$ across the whole defect setup implements the logical operator $\text{CCZ}_{123}$. We note that these transversal operators are all analogous to the transversal operators implementable across three 3D surface codes with boundaries used for encoding instead of defects.

B. Logical Hadamard Operator by State Injection

To realise a universal gate set on three logical qubits, we now need only to supplement the transversal gates with a technique for implementing a logical Hadamard operator on each logical qubit. This can be done using stabiliser state injection, using the circuit shown in Fig. 15. However, there are a number of issues that must be addressed to allow for this circuit to be used to realise logical Hadamard operators on each logical qubit. These are fully addressed in Appendix 3 - here we simply summarise these issues and their solutions.

Firstly, the circuit requires the use of an ancilla that can be prepared in the logical $|\bar{+}\rangle$ state. Fig. 12 and Fig. 13 show how two ancilla qubits, $a$ and $b$, can be encoded. Either of these ancillae can be prepared in state $|\bar{+}\rangle$ by measuring their logical $X$ operators and applying Pauli corrections if necessary. Logical $\text{CZ}_{3a}$ and $\text{CZ}_{3b}$ operators can be implemented transversally by implementing $\text{CZ}_{31}$ on the support of $\bar{X}_b$ and $\text{CZ}_{23}$ on the support of $\bar{X}_a$ respectively. This can be used to allow for a logical $\bar{H}_3$ operator to be applied using these ancillae, as described in Appendix 3e.
\( |\bar{\psi}\rangle \xrightarrow{M_X} |\bar{\psi}\rangle \)

\( |\bar{+}\rangle \xrightarrow{X} |\bar{+}\rangle \xrightarrow{\hat{H}} |\bar{\psi}\rangle \)

Applying \( \hat{H}_1 \) and \( \hat{H}_2 \) is more challenging since transversal CCZ operators cannot be realised transversally between logical qubits one and two and the ancillae. This problem is overcome by braiding. Specifically, braiding defect \( h_1 \) around \( h_\alpha \) implements the logical operator \( \text{CNOT}_{a1}\text{CNOT}_{2b} \). This can be seen to be analogous to braiding holes in the two dimensional surface code which implements entangling CNOT operators between encoded logical qubits. This entangling operator allows for logical information to be transferred between the logical qubits 1 and 2 and the encoded ancilla qubits, which then allows for the circuit necessary for logical \( \hat{H}_1 \) and \( \hat{H}_2 \) to be implemented using transversal CZ operators. This is described in more detail in Appendix B. These logical \( \hat{H} \) operators combined with the transversal CCZ operator suffice for a universal gate set.

\section{Universal Gate Set on \( N \) Logical Qubits}

Having established how a universal gate set may be implemented on three logical qubits, we now consider a larger number of logical qubits. In particular, we show that \( N \) logical qubits admitting a universal gate set may be encoded in \( N + 2 \) defects (for odd \( N \geq 3 \)). Specifically, we maintain a single threaded defect through the holes of \( \frac{N+1}{2} \) pairs of puncture defects. In one pair of puncture defects we encode a pair of ancillae as in the three logical qubit case, while we encode a pair of logical qubits in each other pair of defects and one logical qubit using the threaded defect analogously to qubit three in the previous case.

The universal gate set on this set of logical qubits again emerges from combining transversal CCZ logical operators with Hadamard logical operators implemented by state injection. The full details of how this can be done are presented in Appendix B. We here consider only at a high level the two major issues that emerge in the general case that did not in the three qubit case.

Firstly, universality requires that we can implement CCZ between any three logical qubits; even those associated with different defect pairs. This is similar to the problem of implementing a CCZ operator between logical qubits in different code blocks in the scheme of Vasmer and Browne [9]. They address this using lattice surgery, which allows for logical qubits to be swapped between different code blocks. An advantage of our scheme based on defects is that braiding may be used to swap logical qubits between different defect pairs, using CNOT operators. This allows for logical qubits to be swapped around to bring them into a configuration that admits a CCZ operator, without requiring separate code blocks of lattice surgery.

Secondly, the presence of many logical qubits in the same code block adds a challenge in implementing transversal gates. In particular, the transversal CCZ operator described in Sec. V A has support on all qubits across the whole defect setup. However, applying such an operator in the presence of many logical qubits would implement all possible CCZ operators simultaneously. Specifically, it would entangle the logical qubits in each defect pair with the logical qubit encoded using the threaded defect, without being able to select out the logical qubits of a particular pair. This can be overcome by isolating the pair of logical qubits associated out from the other logical qubits temporarily to allow for a CCZ operator to be implemented on this pair and the logical qubit associated with the threaded defect, without involving other logical qubits. This can be done by braiding defects to implement CNOT operators between the qubits to be isolated and the encoded ancilla. Once logical qubits are stored in a defect pair along with an ancilla, the ancilla may be used to control whether or not the logical qubit is affected by the implementation of the transversal CCZ operator. This allows for CCZ operators to be implemented on just three logical qubits, and not across all logical qubits, as required.

\section{Discussion of Scheme}

We emphasise that this universal scheme has the advantage over similar alternative schemes, such as that of Vasmer and Browne [9], that it allows for universality
on an arbitrary number of qubits within a single quantum memory. This means that we do not require a large number of code blocks, including ancilla blocks, to be entangled using lattice surgery. This is made possible by the use of braiding to entangle logical qubits associated with different defect pairs. Such braiding is made possible only by our consideration of encoding logical qubits in defects. In particular, it can only play the role it does in our universal scheme as we have considered braiding in higher dimensional codes. Moving beyond two dimensions is what allows for non-Clifford locality-preserving logical operators, which can be supplemented by the entangling gates by braiding we have presented.

It also has the feature that it does not require additional encoded ancilla qubits as the number of logical qubits increases. This is made possible by our technique of shifting logical information onto ancilla qubits when necessary to allow for desired gates to be implemented only on specific logical qubits. This technique could be more broadly useful for other schemes that have many logical qubits encoded in the same memory, that often face the problem of transversal gates not being specific to desired logical qubits.

VI. CONCLUSION

In this paper, we have explored both the limitations and potential for fault-tolerant quantum computing by braiding defects in topological stabiliser codes. In particular, we have shown that universal quantum computing cannot be achieved by braiding alone, nor by supplementing braiding logical operators by locality-preserving logical operators. This extends the result of Bravyi and König that shows that locality-preserving logical operators are insufficient for universality to a far larger class of fault-tolerant operators [2]. Indeed, we have also noted that this constraint applies more generally to abelian quantum double models, extending the results of [4].

Our result has implications for realising universal fault-tolerant quantum computing. In particular, it means that schemes based on braiding defects in topological stabiliser codes or abelian quantum double models must be supplemented by other techniques to permit universality. The most widely-employed approach to this is magic state distillation, which can lift schemes allowing for the full Clifford group to universality [16, 17]. We have furthered the potential for this approach by demonstrating that the Clifford group may be realised by braiding in a large class of topological stabiliser codes – those with twists that allow for condensation of a generalised fermion. We have also presented two examples of higher dimensional codes in the class, both of which admit Clifford braiding schemes with interesting features that make them worthy of consideration. These features may prove valuable in inspiring universal schemes based on higher dimensional braiding in models beyond topological stabiliser codes and abelian quantum double models. In particular, it would be worthwhile to consider whether higher dimensional analogues of non-abelian anyon models may exhibit interesting braiding behaviour similar to that we have explored for defects in higher dimensional surface codes and the three dimensional Levin-Wen fermion model.

Beyond magic state distillation, other approaches to universality can also be considered, including those that make more explicit use of topological protection. We have presented such an approach by adapting the scheme of Vasmer and Browne based on stabiliser state injection to develop a universal gate set on logical qubits encoded in defects [9]. This scheme uses a combination of locality-preserving and braiding logical operators, but supplements these operators with non-local classical processing to circumvent our no-go results. It has the advantage that it allows for arbitrarily many logical qubits to be encoded in a single code block, removing the need for lattice surgery. This scheme could be used as a model for adapting existing approaches to universality on qubits encoded in boundaries to qubits encoded in defects. In particular, future work could seek to adapt other techniques used to achieve universality such as dimensional jumping [6] and just-in-time gauge fixing [14, 15] to defect encodings. Future work could also explore and seek to develop approaches to universality that are specifically designed for defect schemes, such as topological charge measurement [28]. The potential of all these different approaches should also be assessed by investigating the overhead of each technique to determine if they offer an advantage over the conventional alternative of using the two dimensional surface code with magic state distillation.

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Appendix

In this appendix, we provide a complete presentation of the construction summarised in Sec. [V] Specifically, we begin by discussing the background to this approach to universality, and how it violates the assumptions of the no-go theorem of Sec. [III]A We then present a scheme using punctures in a code equivalent to a stack of three 3D surface codes that allows for this approach. We go on to show how that scheme admits a universal gate set on three logical qubits generated by Hadamard operators on each qubit and a CCZ between the three logical qubits. Finally, we explain how this scheme may be extended to allow for a universal gate set on $N$ logical qubits in a setup with a total of $N + 2$ punctures (for any odd $N \geq 3$).

1. Approach to Circumventing the No-Go Theorem

   a. Role of Non-Local Classical Processing

   The Eastin-Knill and Bravyi-Konig theorems rule out a universal set of locality-preserving logical operators in any topological stabiliser codes. Nonetheless, multiple schemes are known that allow for a universal set that appear to be locality-preserving. An explanation to this paradox has been provided by Bombin, who has highlighted the role of non-local classical processing as a resource that can lift locality-preserving quantum schemes to universality [35]. We explore this resource further now to work towards presenting our scheme.

   A requirement that we place on quantum computing schemes is that they must be fault-tolerant. Specifically, we require that a local error that occurs in a quantum memory must not be spread around the memory to become non-local when logical operators are performed on the memory. When considering classical computing, however, we have an error rate that is negligible compared to that for any foreseeable quantum computer. Thus, we do not need to apply this same constraint to classical computing – we may permit a local error in a classical memory to be spread to a non-local error – since the rate of all classical errors is very small. Thus, we may consistently constrain quantum operators to be locality-preserving even while allowing for operations that can spread errors in classical memories arbitrarily.

   We may view this ability to perform classical computing non-locally as a resource that manifests itself in the ability to perform quantum operators conditional on outcomes of measurements of the quantum memory. To see this, consider an operation that performs a logical $\hat{Z}$ measurement of an ancilla logical qubit, and then applies a (locality-preserving) logical operator $\hat{U}$ on a data logical qubit if and only if the measurement outcome is $\hat{Z} = 1$. Such an operation clearly cannot be implemented as a unitary operator on the quantum memory. Instead, we must break it up into parts as follows.

   1. $\hat{Z}$ is measured on the ancilla. This consists of many local measurements of qubits across some topologically non-trivial region of the code. Each local measurement is repeated multiple times to ensure fault-tolerance and a majority vote taken, so that local measurement errors cannot cause the wrong logical measurement outcome. Alternatively, single-shot error correction used to achieve the same result.

   2. Classical processing is used to determine the logical measurement outcome from all of the local measurements.

   3. Operator $\hat{U}$ is applied to the data logical qubit if and only if the measurement outcome is $\hat{Z} = 1$.

   This procedure is fault-tolerant with respect to errors in the quantum memory. Specifically, repeated measurements ensure that local measurement errors cannot propagate to cause logical errors and since $\hat{U}$ is locality-preserving it does not cause local errors in the quantum memory to become non-local. However, a classical error in a single bit can cause failure. Specifically, if an error occurs to the bit that is used to control whether or not $\hat{U}$ is implemented, then the result will be a logical $\hat{U}$ or $\hat{U}^\dagger$ error on the data logical qubit. That is, the single bit error in the classical memory would be spread to an uncorrectable (non-local) error in the quantum memory. This cannot be avoided through repetition, since
any such system would ultimately need to be resolved by a majority vote that would reduce it to a single bit. Thus, this procedure is not allowed within a computing scheme that is strictly locality-preserving, but is permitted if classical errors are assumed to be sufficiently rare that we do not require tolerance to them.

This situation will occur whenever a quantum operator is implemented conditionally on the solution to a decision problem solved on a classical computer. This is because this solution must ultimately be reduced to a bit that could then have an error occur to it before the conditional operator is applied. Indeed, as an example, it applies to quantum error correction, since decoder algorithms must determine whether or not a logical $X$ or $Z$ operator should be applied to correct a logical qubit.

The arguments also apply to all known schemes for achieving universal quantum computing in topological stabiliser codes. For example, dimensional jumping [9] depends on implementing logical Pauli operators conditionally on an error correction problem applied to the outcomes of measurements used in jumping from 3D to 2D to ensure that the memory remains free of logical errors after jumping. This is also required for the just-in-time gauge fixing used in recent work by Bombin [14] and Brown [15]. State injection (including magic state injection) also require a conditional operator to be applied conditional on a logical outcome [28]. We note that an error occurs to it before the conditional operator is applied. Indeed, as an example, it applies to quantum error correction, since decoder algorithms must determine whether or not a logical $X$ or $Z$ operator should be applied to correct a logical qubit.

Theorem 1, which shows that a universal gate set cannot be realised using only braiding and locality-preserving logical operators, can be circumvented by allowing for non-local classical processing. Specifically, the scheme we present uses state injection to achieve logical operators, provided that logical operators may be applied conditionally on logical measurement outcomes [28]. Our scheme applies this technique to a setup in which logical qubits are encoded in defects, using a combination of locality-preserving and braiding logical operators.

### 2. Defects and Encoding

We now describe the defects and encoding that allow us to realise a universal gate set. This setup can be realised in a code locally equivalent to three copies of a surface code. We refer to these as codes 1, 2 and 3.

We begin by considering a pair of toroidal punctures. We choose these punctures to be rough with respect to surface code 1 and smooth with respect to surface codes 2 and 3. This means that they allow for $e_1$, $e_2$ and $m_3$ to condense, but not $m_1$, $e_2$ or $e_3$. We note that $s_{ij}$ excitations can condense at a boundary if and only if the boundary is rough with respect to code $i$ or $j$. Thus, these punctures allow for $s_{12}$ and $s_{31}$ to condense, but not $s_{23}$. We label the two punctures $h_1$ and $h_2$.

We introduce a further toroidal puncture threaded through these two punctures. We choose this puncture to be rough with respect to codes 1 and 2 and smooth with respect to code 3. This means it allows $e_1$, $e_2$, $m_3$, $s_{12}$, $s_{31}$ and $s_{31}$ to condense, but not $m_1$, $m_2$ or $e_3$. This threaded defect thus prevents $m_2$ that condense at one of the first two punctures to be absorbed into the vacuum. We label this threaded defect $h_3$. The defect setup is presented in Fig. 11.

These three defects allow for three logical qubits to be encoded. We define these logical qubits implicitly by specifying their logical Pauli operators. These are shown in Fig. 12, Fig. 13 and Fig. 14. Each of the three logical qubits is implemented by physical operators acting on the corresponding surface code. Specifically, qubit one has logical $X_1$ operator realised by a torus of physical $X_1$ operators acting around defect $h_1$. It has logical $Z_1$ operator implemented by a line of physical $Z_1$ operators between $h_1$ and $h_2$. Qubit two has $X_2$ operator that is a cylinder of physical $X_2$ operators between $h_1$ and $h_2$. Qubit three has $X_3$ operator that is a membrane of physical $X_3$ operators terminating in a loop on defect $h_2$ and $Z_3$ operator that is a loop of $Z_3$ operators around $h_3$.

We can verify that these operators define a valid encoding of three qubits by checking commutation relations of these logical operators. Specifically, logical operators acting on different logical qubits commute since they consist of physical operators acting on different surface codes. Logical $X_i$ and $Z_i$ operators consist of physical $X$ and $Z$ operators respectively acting on the same surface code, and intersect at a point, and so anticommute.

### 3. Universal Gate Set for Three Qubits

We now describe how to realise universal quantum computing on three logical qubits encoded in these defects. Specifically, we first show how transversal $\text{CCZ}$ operators may be implemented pairwise between logical qubits and then how transversal $\text{CCZ}$ may be implemented between all three. We then demonstrate that
any of the logical qubits with the addition of two ancilla qubits. Specifically, we may realise a transversal $CZ_{ab}$ by preparing one of the ancilla qubits in state $\ket{\psi}$, as shown in Fig. 12 and Fig. 13. This allows us to entangle qubit 3 with the ancilla qubits. We also note that the $CZ_{12}$ operator identified previously acts as $CZ_{ab}$ now that we have ancilla qubits present. We can ensure that this operator acts only as $CZ_{12}$ however by preparing one of the ancilla qubits in state $\ket{0}$ so that $CZ_{ab}$ acts trivially.

The introduction of additional defects including ancilla qubits also allows for braiding logical qubits that entangle data qubits with ancilla qubits. Specifically, we note that braiding $h_2$ around $h_1$ acts as $CNOT_{a_1}CNOT_{b_2}$. To see this, note that it is analogous to the braiding of holes in the two dimensional surface code. Specifically, here, when a defect is braided around another, a cylindrical operator that terminates on one of the holes being braided will have a torus around the other braided defect appended to it. Thus, considering Fig. 12 and Fig. 13, the braiding operator maps $X_a \leftrightarrow X_b X_c$ and $X_a \leftrightarrow X_b X_1$. Also, a line that terminates on one of the holes being braided will have a loop around the other braided defect appended to it. Thus, again considering Fig. 12 and Fig. 13, the braiding operator maps $Z_a \leftrightarrow Z_b Z_c$ and $Z_a \leftrightarrow Z_b Z_2$. We note that we can also implement $CNOT_{a_1}$ and $CNOT_{b_2}$ individually by appropriately preparing ancilla qubits to make the other part of the operator act trivially.
c. Logical Measurements and Preparation

In addition to the transversal and braiding logical operators we have identified, we can also implement logical measurements in the $\bar{Z}$ and $\bar{X}$ eigenbases for any of the encoded qubits. This can be done by leveraging the transversal $\bar{X}$ and $\bar{Z}$ operators. Specifically, we can perform a (non-fault-tolerant) logical $Z_i$ by measuring $Z_i$ on each qubit on a path that supports a logical $\bar{Z}_i$ operator (or similarly for $\bar{X}_i$). This can be done fault-tolerantly by using $d$ (physical) ancilla qubits for each (physical) code qubit, where $d$ is the weight of the logical operator. This is shown in Fig. 17 for the case where $d = 3$. We note that these logical measurements allow us to fault-tolerantly prepare states $|0\rangle$ and $|+\rangle$ by performing measurements in the $\bar{Z}$ and $\bar{X}$ bases respectively and performing necessary Pauli corrections.

d. Gadgets

We now define two gadgets that we can use as tools to realise a Hadamard logical operator. The first of these gadgets is that described in Sec. V D and shown in Fig. 18. We call this gadget $H_{xy}$. Here, $x$ is initially a data qubit in logical state $|\psi\rangle$ that ends up encoding no information after the gadget is applied. Qubit $y$ is initially an ancilla qubit that ends up storing logical information and in the state $\bar{Z}|\psi\rangle$.

The other gadget that we make use of we label $I_{xy}$ and show in Fig. 19. This gadget takes logical information on data qubit $x$ and moves it to an ancilla qubit $y$. We note that we cannot implement unitary logical SWAP operators between data qubits 1 or 2 and ancilla qubits $a$ or $b$ since the CNOT operators that we can implement between them by braiding can only act in one direction (i.e. the control and target logical qubits cannot swapped). Nonetheless, by using this gadget we begin with qubit $x$ in state $|\psi\rangle$ and end with $y$ in this same state. Qubit $x$ ends up encoding no logical information.

e. Logical Hadamard Operators

We now describe how logical Hadamard operators can be implemented on each of the three logical qubits to complete a universal gate set on these qubits. We consider first logical qubit three, since it allows the easiest implementation, and then describe the more complicated approach taken for qubits one and two.

To implement $H_3$, we can implement the gadget $H_{xy}$ three times in a cycle then begins and ends on qubit three. In this way, the state, $|\psi\rangle$, on qubit three will have $H^3 = H$ applied to it and end the process back on qubit three. In particular, we can do this with ancilla qubits $a$ and $b$ by applying $H_{3b}H_{ab}H_{3a}$. This is possible since $\bar{CZ}_{3b}$ and $\bar{CZ}_{3a}$ are both implementable transversally. We can implement $\bar{CZ}_{ab}$ by first preparing qubit $b$ in state $|0\rangle$, applying the transversal operator $\bar{CZ}_{12}\bar{CZ}_{ab}$, then preparing $b$ in $|+\rangle$ and applying $\bar{CZ}_{12}\bar{CZ}_{ab}$ again. This has the effect of implementing $\bar{CZ}_{12}$ twice and so does not affect these qubits, but does implement $\bar{CZ}_{ab}$ on the ancilla qubits as necessary since the first implementation of $\bar{CZ}_{12}\bar{CZ}_{ab}$ acts trivially on these qubits. We can diagrammatically represent the process used to implement $H_3$ in the way shown in Fig. 20. We can also draw the circuit used explicitly as shown in Fig. 21.

For logical qubits 1 and 2 we are more limited in the entangling gates we can perform than for logical qubit 3. This means we must use a more complicated scheme to realise $H_1$ and $H_2$ than for
FIG. 20. Digramatic representation of how the sequence of gadgets $H_{3a}H_{ab}H_{3b}$ can be used to implement the logical operator $\hat{H}_1$.

\[
\begin{align*}
|\bar{Q}_1\rangle & \quad \quad |\bar{Q}_1\rangle \\
|\bar{Q}_2\rangle & \quad \quad (-1)^n |\bar{Q}_2\rangle \\
|\bar{Q}_3\rangle & \quad \quad X^n M x \quad Z^n \quad X^n M x \quad (-1)^x |\bar{Q}_3\rangle \\
|\bar{Q}_4\rangle & \quad \quad (-1)^x |\bar{Q}_4\rangle \\
|\bar{Q}_5\rangle & \quad \quad (\bar{x} M x) |\bar{Q}_5\rangle \\
\end{align*}
\]

FIG. 21. Explicit circuit that can be used to implement $\hat{H}_2$. Double lines show the measurement outcome of logical measurements. The parameters specifying these outcomes are then used to control future logical operators.

$\hat{H}_3$. In particular, we can implement $\hat{H}_1$ by applying $H_{a3}H_{31}H_{3b}H_{a3}H_{a1}$. Similarly, we can implement $\hat{H}_2$ by applying $H_{b3}H_{32}H_{3b}H_{a3}H_{a1}$. Verifying that these indeed implement the required logical Hadamard operators is best done by considering them diagrammatically, as presented in Fig. 22 and Fig. 23. The explicit circuits to implement this can be realised by combining together the circuits for each gadget.

We note two interesting features of the circuits used to implement the logical Hadamard operators on logical qubits 1 and 2. Firstly, we note that both $\hat{H}_1$ and $\hat{H}_2$ rely on braiding defects, since they use the gadget $I_{xy}$ which involves performing a CNOT operator by braiding. We also note that $\hat{H}_1$ and $\hat{H}_2$ both use a technique of shifting logical information around between encoded qubits as a tool to allow for the right gate to be implemented. In particular, the gadgets $H_{a3}$ and $H_{a1}$ in $\hat{H}_1$ (and similar for $\hat{H}_2$) are used only to move logical information from logical qubit 3 onto an ancilla qubit. This allows for logical qubit 3 to be used as an ancilla, while protecting the logical information of the qubit on qubit a. Such shifting of logical information around plays an important role here, and will also do so in the following section.

4. Universal Gate Set for n Qubits

Having demonstrated that a universal gate set on three logical qubits can be realised with this defect setup, we now show that this may be generalised to give universality on an arbitrarily large number of logical qubits. This procedure uses only locality-preserving logical operators and braiding supplemented by logical measurement and non-local classical processing. In particular, it does not require additional techniques such as lattice surgery to transfer information between different code blocks, or non-local operators.

We add more logical qubits to the scheme by adding more pairs of defects identical to $h_1$ and $h_2$, which also have the threaded defect threaded through them. Each new defect pair then adds two more logical qubits defined analogously to logical qubits one and two, but using this new pair of defects. In this way, we can construct an encoding of $N$ logical qubits in a setup with a total of $N + 2$ punctures (for odd $N \geq 3$). We note that we will label the qubit known as qubit three in the three-qubit case (encoded using the threaded defect) by qubit $N$, and label the pairs of logical qubits encoded in the $k$th pair of defects by qubit $2k - 1$ and $2k$. Indeed, braiding one of the punctures in the $k$th pair around hole $a$ with qubit $b$ prepared in state $|+\rangle$ allows the necessary entangling gate, and logical measurement and Pauli operators can be applied as before.

To justify that we have a universal gate set on this full set of logical qubits, we must now show that our scheme allows for fault-tolerant Hadamard logical operators on

FIG. 22. Digramatic representation of how the sequence of gadgets $H_{a3}H_{31}H_{3b}H_{3a}H_{a1}$ can be used to implement the logical operator $\hat{H}_1$.

\[
\begin{align*}
1 & \quad |\bar{\psi}\rangle \quad I_{1a} \quad H_{31} \quad \hat{H} |\bar{\psi}\rangle \\
3 & \quad |\bar{\varphi}\rangle \quad |\bar{\varphi}\rangle \quad |\bar{\psi}\rangle \quad |\bar{\varphi}\rangle \\
\bar{a} & \quad |\bar{\psi}\rangle \quad H_{3a} \quad \hat{H} |\bar{\varphi}\rangle \quad \hat{H} |\bar{\varphi}\rangle \\
\bar{b} & \quad \hat{H} |\bar{\psi}\rangle \quad H_{a3} \quad \hat{H} |\bar{\varphi}\rangle
\end{align*}
\]

FIG. 23. Digramatic representation of how the sequence of gadgets $H_{3a}H_{33}H_{3b}H_{33}H_{a3}I_{2a}$ can be used to implement the logical operator $\hat{H}_2$. 
each logical qubit, and CCZ operators on each subset of three logical qubits.

We consider CCZ first. Specifically, with $N + 2$ punctures, we now have that transversal CCZ acts as in equation 1

$$\text{CCZ}_{a,b,N} \prod_{k=1}^{N-1} \text{CCZ}_{(2k-1),2k,N}$$  \hspace{0.5cm} (1)

To implement $\text{CCZ}_{(2k-1),2k,N}$, we must thus isolate out the logical qubits we wish to act on. This can be done by using the gadget $I_{2k-1,a}$ to move logical information on qubit $K$ temporarily onto the ancilla qubit $a$. Note now that qubits $2K - 1$ and $b$ now do not carry logical information. Thus, we can prepare both these qubits in $|0\rangle$. This then means that $\text{CCZ}_{a,b,N} \text{CCZ}_{(2k-1),2k,N}$ acts trivially, and so applying CCZ transversally implements the logical operator in equation 2

$$\prod_{k=1}^{K-1} \text{CCZ}_{(2k-1),2k,N} \prod_{k=K+1}^{N-1} \text{CCZ}_{(2k-1),2k,N}$$  \hspace{0.5cm} (2)

We can then return the logical information being stored in qubit $a$ back to qubit $2K - 1$ by applying the gadget $I_{2k-1,a}$. With $b$ still in state $|0\rangle$ we can now implement CCZ transversally again. This then implements the operator in equation 3

$$\prod_{k=1}^{N-1} \text{CCZ}_{(2k-1),2k,N}$$  \hspace{0.5cm} (3)

Thus, the net effect of applying all of these operations is that the required operator $\text{CCZ}_{2K-1,2k,N}$ has been applied, since this is equivalent to the operator in equation 4

$$\prod_{k=1}^{K-1} \text{CCZ}_{(2k-1),2k,N} \prod_{k=K+1}^{N-1} \text{CCZ}_{(2k-1),2k,N} \prod_{k=1}^{N-1} \text{CCZ}_{2k-1,2k,N}$$  \hspace{0.5cm} (4)

The full circuit used to perform this operator, in the case $N = 5$ is shown in Fig. 24

To conclude, we now note that we may swap any logical qubit, $x$, with qubit $N$ by the operator in equation 7

$$\text{SWAP}_{x,N} = H_N \text{CCZ}_{x,N} H_N H_x \text{CCZ}_{x,N} H_x H_N \text{CCZ}_{x,N} H_N$$  \hspace{0.5cm} (7)

This means that we may swap any two logical qubits $x$ and $y$ by implementing $\text{SWAP}_{x,N} \text{SWAP}_{y,N} \text{SWAP}_{x,N}$. Thus, we may implement $\text{CCZ}_{xy,z}$ between any three logical qubits $x, y, z$. 

FIG. 24. Circuit that implements CCZ 235 in the case of the scheme with five logical qubits.