A novel mechanism for probing the Planck scale

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The Planck length, energy and time scales, first mentioned by Planck himself in [3][4], continues to play special roles in physics. While these are believed to be the scales where Quantum Gravity (QG) effects will most certainly appear, given the immense gap between the electroweak scale (≃ 1 TeV) and Planck scale (≃ 10^{16} TeV), it is conceivable that some of these effects may show up in this intermediate region, even if indirectly. It is also believed that the Planck scale signifies an absolute minimum measurable length scale in Nature, beyond which the notion of a continuum spacetime ceases to exist. Arguments in favour of a Minimum Length Scale (MLS) can also be found in early works of Heisenberg [5], Yang [6], Deser [7] and Mead [8, 9]. They have been refined appropriately 'filter out' the relativistic effects and extract the free Hamiltonian. Width of wavepackets are often measured in Atomic-Molecular-Optical (AMO) experiments for various purposes (e.g. [11, 12]). In this work, we re-examine this effect, but in light of a Hamiltonian which is still free, but modified from the canonical Hamiltonian due to the Generalized Uncertainty Principle (GUP), which encapsulates a MLS and is implied by it. Such a generic modification of the Heisenberg Uncertainty Principle (HUP) has been argued from many theories of QG, including String Theory, Loop Quantum Gravity, Doubly Special Relativity, black hole physics etc, and its implications were examined [13][23].

Although the Planck scale, MLS and the QG scale are often assumed to be of the same order of magnitude, per se, there is no reason or evidence behind this assumption. We will therefore relax this, and assume that new physical effects, including QG effects may potentially show up in the vast arena of 15 orders of magnitude intervening between the electroweak and the Planck scales. Therefore, in the absence of a direct probe beyond the LHC scale energy (≃ 10 TeV), it is imperative that one looks for potential experimental signatures and new physics that may be present in the aforementioned energy range.

In this paper, we examine promising experimental paths which might be able to detect GUP modifications with a high accuracy. In particular, the “doubling time difference (DTD)” (difference in times taken for a wavepacket to double in size, with and without GUP) was computed in [11][2]. It was also shown there however, that the DTD only becomes experimentally measurable, once the velocity of the travelling wave-packets are quite large (≃ 10^5 − 10^6 m/s). This is because the GUP effects are momentum (and hence velocity) dependent and gets enhanced with increasing velocity of the wavepackets. While this is encouraging, one encounters the following issue: for these velocities, the relativistic corrections are of the order of (v/c)^2 ≃ 10^{-6} − 10^{-4}, and it has to be determined whether these corrections will be comparable or exceed the GUP corrections for the energy and momenta range under consideration. It is precisely this important point that we will examine in this paper and show that the GUP effects are still potentially measurable! In an attempt to systematically study both the relativistic and GUP effects, we in fact find that the two get mixed in a non-trivial way. However, it is still possible to appropriately ‘filter out’ the relativistic effects and extract the
GUP corrections, which are again just within the realm of current and future experimental accuracies.

We start by considering the Hamiltonian for a free particle of mass $m$ in $(1+1)$-dimensions, including the leading order relativistic correction term

$$H = \frac{p^2}{2m} - \frac{p^4}{8m^3c^2}$$

(1)

Now, as per GUP, the fundamental commutator between position and momentum is modified to [21]

$$[x, p] = i\hbar [1 - 2\alpha_0 + 4\alpha^2 p^2] .$$

(2)

The above defines a minimum measurable length and a maximum measurable momentum, in terms of the GUP parameter $\alpha$ [1]

$$\langle \Delta x \rangle_{\text{min}} = \frac{3\alpha_0}{2} \ell_{Pl}; \quad \langle \Delta p \rangle_{\text{max}} = \frac{M_{Pl}c}{2\alpha_0}$$

(3)

where we have defined $\alpha = \alpha_0/M_{Pl}c$, $\alpha_0$ being dimensionless. $M_{Pl}$ is the Planck mass, $M_{Pl}c^2 \approx 10^{16}$ TeV the Planck energy and $\ell_{Pl} \approx 10^{-35}$ m is the Planck length. We do not assume any specific value of $\alpha_0$, rather we hope that experiments will shed light on the allowed values of $\alpha_0$. Since no evidence of a MLS has not been found in experiments at the LHC, one is forced to put an upper bound on $\alpha_0$. Together with a lower bound on it corresponding to the Planck scale, one arrives at the following allowed range: $1 \leq \alpha_0 \leq 10^{16}$.

Next, for calculational convenience, we define an auxiliary momentum variable $p_0$, which is ‘canonical’ in the sense that $[x, p_0] = i\hbar$, and therefore as an operator, one can write $p_0 = -i\hbar d/dx$. This is related to the physical (i.e. measurable) momentum $p$ via the relation $p = p_0(1 - \alpha p_0 + 2\alpha^2 p_0^2)$. Substituting in Eq. (1), one obtains the following effective Hamiltonian for a relativistic system, incorporating GUP

$$H = H_{NR} + H_{rel} + H_{LGUP} + H_{QGUP} + H_{rel}^{\text{GUP}}$$

(4)

where, (i) $H_{NR} = \frac{p_0^2}{2m}$, (ii) $H_{rel} = -\frac{m}{8\alpha_0^3 p_0^2}$, (iii) $H_{LGUP} = -\frac{m}{2}p_0^3$, (iv) $H_{QGUP} = \frac{5}{2m^2}p_0^4$, and (v) $H_{rel}^{\text{GUP}} = \frac{\alpha}{2m^2 c^2}p_0^5$.

In the above, (i) is the standard non-relativistic Hamiltonian, (ii) the leading order relativistic correction, (iii) the linear GUP correction (proportional to $\alpha$), (iv) the quadratic GUP correction (proportional to $\alpha^2$) and (v) the hybrid or mixed term, which includes both the relativistic and linear GUP correction.

Next, we move on to the study of evolution of free wavepackets under the above Hamiltonian. It is textbook knowledge that a free wave-packet tends to broaden itself due to the Heisenberg’s uncertainty principle. Use of the Ehrenfest theorem is one of the direct ways of estimating this broadening. Here our interest is to consider the modified broadening rate of the free wave-packet with the full Hamiltonian [4]. As is well-known, the Ehrenfest’s theorem gives the time derivative of the expectation values of the position ($x$) and its canonically conjugate momentum ($p_0$) operators as follows:

$$\frac{d}{dt}\langle x \rangle = \frac{\partial}{\partial x} \langle H \rangle = \frac{\partial}{\partial x} \langle \alpha \rangle$$

and

$$\frac{d}{dt}\langle p_0 \rangle = \frac{\partial}{\partial p_0} \langle H \rangle = -\frac{\partial}{\partial p_0} \langle \alpha \rangle .$$

These can be extended to the expectation of any operator of course, and in particular to $p_0^n$, which appear in [4] for various integer values of $n$. For the above, one obtains $\frac{d}{dt}\langle p_0^n \rangle = \frac{1}{n!} \langle [p_0, H] \rangle = 0$, implying that $\langle p_0^n \rangle = \text{constant}$ in time.

Next, to estimate the DTD, we first write the first and second time-derivatives of the square of the width (or variance) of the quantum mechanical wave-packet, which is defined as $\xi = \Delta x^2 = \langle x^2 \rangle - \langle x \rangle^2$;

$$\dot{\xi} = \frac{d\xi}{dt} = \frac{d}{dt} \langle x^2 \rangle - 2\langle x \rangle \frac{d\langle x \rangle}{dt}$$

(5)

$$\ddot{\xi} = \frac{d^2\xi}{dt^2} = \frac{d^2}{dt^2} \langle x^2 \rangle - 2\left(\frac{d\langle x \rangle}{dt}\right)^2 - 2x \frac{d^2\langle x \rangle}{dt^2} .$$

(6)

The above can be simplified using the Ehrenfest theorem and the Hamiltonian given in [4].

To calculate the contributions for all the terms in [4], we consider each term in addition to the free nonrelativistic term ($p_0^2/2m$) separately, and write $H = \frac{p_0^2}{2m} + Dp_0^n$ with $n > 2$ and $D$ a constant, and compute the corresponding correction, using the Ehrenfest theorem and $[x, p_0] = i\hbar$. Finally, we plug-in the appropriate value of $n$ and $D$ for each correction term in [4] and add them together to find the total correction. A straightforward calculation of the [5] and [6] then yields,

$$\dot{\xi} = \frac{1}{m} \left(\langle x p_0 + p_0 x \rangle - 2\langle p_0 \rangle \langle x \rangle \right) + nD \langle x p_0^{n-1} + p_0^{n-1} x \rangle - 2\langle p_0^{n-1} \rangle \langle x \rangle$$

(7)

$$\ddot{\xi} = \frac{2}{m^2} \Delta p_0^2 + \frac{4nD}{m} \langle (p_0^n - \langle p_0 \rangle (p_0^{n-1}) \rangle + 2nD^2 \Delta p_0^{(n-1)^2} .$$

(8)

In the above, $\Delta p_0^2 = \langle p_0^2 \rangle - \langle p_0 \rangle^2$ is the variance of the canonical momentum and $\Delta p_0^{(n-1)^2} = \langle p_0^{(n-1)^2} \rangle - \langle p_0^{n-1} \rangle^2$, that of the $(n - 1)$-th power of the canonical momentum. We can now identify $n$ and $D$ for all higher order corrections to the NR Hamiltonian and put them in the above expression of $\xi$ to obtain

$$\ddot{\xi}_{\text{full}} = \frac{2}{m^2} \Delta p_0^2 + C_{\text{rel}} + C_{\text{LGUP}} + C_{\text{QGUP}} + C_{\text{rel}}^{\text{GUP}} , \quad (9)$$
where

\[ C_{\text{rel}} = -\frac{2}{m^4 c^2} \left( \langle p_0^4 \rangle - \langle p_0 \rangle \langle p_0^3 \rangle \right) + \frac{\Delta p_0^4}{2m^6 c^4} \]  

\[ C_{\text{LGUP}} = -\frac{12\alpha}{m^2} \left( \langle p_0^3 \rangle - \langle p_0 \rangle \langle p_0^1 \rangle \right) + \frac{18\alpha^2}{m^2} \Delta p_0^2 \]  

\[ C_{\text{QGUP}} = \frac{40\alpha^2}{m^2} \left( \langle p_0^3 \rangle - \langle p_0 \rangle \langle p_0^1 \rangle \right) + \frac{200\alpha^4}{m^2} \Delta p_0^2 \]  

\[ C_{\text{rel}}^{\text{GUP}} = -\frac{2\alpha}{m^3 c^2} \left( \langle p_0^4 \rangle - \langle p_0 \rangle \langle p_0^3 \rangle \right) + \frac{25\alpha^2}{2m^6 c^4} \Delta p_0^4 \]  

\[ \Delta x(t) = \sqrt{\xi_{\text{in}} + \xi_{\text{in}} t^2 + \frac{1}{2} \left( C_{\text{rel}} + C_{\text{GUP}} + C_{\text{QGUP}} + C_{\text{rel}}^{\text{GUP}} \right) t^2}, \]

The master equation [1] has the following solution giving the rate of broadening of the free wavepacket under the combined influence of the relativistic and GUP corrections

\[ \Delta x(t) = \sqrt{\xi_{\text{in}} + \xi_{\text{in}} t^2 + \frac{1}{2} \left( C_{\text{rel}} + C_{\text{LGUP}} + C_{\text{QGUP}} + C_{\text{rel}}^{\text{GUP}} \right) t^2}, \]

where, the subscript “in” corresponds to the initial value of the various quantities, such as the initial width (\(\sqrt{\xi_{\text{in}}}\)), the initial rate of expansion \(\dot{\xi}_{\text{in}}\) and the initial variance of the canonical momentum (\(\Delta p_{0 \text{in}}^2\)), and new corrections due to the relativistic and GUP effects appearing in [9].

We now compute the expansion rates by considering a normalized Gaussian wave-packet of the form

\[ \psi(x) = \frac{1}{(2\pi \xi)^{1/2}} \exp \left( \frac{i}{\hbar} p_0 x - \frac{(x - \bar{x})^2}{4\xi} \right), \]

which represents a minimum wave-packet with \(\langle x \rangle = \bar{x}, \langle p_0 \rangle = p_0, \Delta x^2 = \langle x^2 \rangle - \langle x \rangle^2 = \xi, \) and \(\Delta p_0 = \frac{\hbar}{\sqrt{\xi}}\). Its Fourier transformation in momentum space is

\[ \phi(p_0) = \frac{1}{\sqrt{2\pi \hbar}} \int_{-\infty}^{+\infty} \psi(x) e^{-ip_0 x / \hbar} \]  

\[ = \left( \frac{2\xi}{\pi \hbar^2} \right)^{1/4} \exp \left( \frac{-i x_0}{\hbar} (p - p_0) - \frac{(p - p_0)^2 \xi}{\hbar^2} \right) \]

Since our results contain moments of \(p\) up to the eighth order, and using the standard quantum mechanical definition \(\langle p_0^n \rangle = \int_{-\infty}^{+\infty} \phi^*(p_0) p_0^n \phi(p_0)\) we calculated following coefficients for the gaussian wavepacket,

\[ C_{\text{rel}} = \frac{3\hbar^2}{128c^4 m^6 \xi_{\text{in}}^3} \left( -16c^2 m^2 \xi_{\text{in}} \left( 4p_0^2 \xi_{\text{in}} + \hbar^2 \right) + 48p_0^2 \xi_{\text{in}} \hbar^2 + 48p_0^2 \xi_{\text{in}}^2 + 5\hbar^4 \right) \]  

\[ C_{\text{LGUP}} = \frac{3\alpha \hbar^2}{4m^2 \xi_{\text{in}}^2} \left( 3\alpha \hbar^2 + 24\alpha p_0^2 \xi_{\text{in}} - 8p_0 \xi_{\text{in}} \right) \]  

\[ C_{\text{QGUP}} = \frac{15\alpha^2 \hbar^2}{8m^2 \xi_{\text{in}}^2} \left( 25\alpha^2 \hbar^4 + 16p_0^2 \left( 15\alpha^2 \xi_{\text{in}} \hbar^2 + \xi_{\text{in}}^2 \right) + 240p_0^4 \xi_{\text{in}}^2 + 4\xi_{\text{in}}^2 \right) \]  

\[ C_{\text{rel}}^{\text{GUP}} = \frac{\alpha \hbar^2}{16c^4 m^6 \xi_{\text{in}}^3} \left[ 8c^2 m^3 p_0 \xi_{\text{in}}^2 \left( 4p_0^2 \xi_{\text{in}} + 3\hbar^2 \right) + 25\alpha \left( 84p_0^4 \xi_{\text{in}} \hbar^2 + 48p_0^2 \xi_{\text{in}}^2 + 32p_0^4 + 3\hbar^6 \right) \right]. \]

As can be seen from [10-13], the above modifications contain moments up to the eighth order in momentum space. It is indeed a nontrivial result, as it shows that not only the standard deviation, but also higher order moments such as the skewness (3rd), kurtosis (4th), hyperkurtois (5th), hypertailedness (6th) etc. all dictate the broadening rate, albeit with decreasing importance. Equipped with the above, we ask our primary question of interest - can the above GUP modifications be observed in an experiment, similar to earlier analyses where it was shown that a large parameter space of GUP can be probed by measuring the DTD for large molecular wavepackets such as \(C_{60}, \ C_{176}\) [11, 12]? The corresponding results incorporating relativistic effects are given by [15] - [18]. It can be easily checked that the results of [11, 12] are recovered in the \(c \to \infty\) limit. As in the above references, we address this question numerically.

The DTD is defined as

\[ \Delta t_{\text{double}} = t_{\text{GUP}}^{\text{double}} - t_{\text{HUP}}^{\text{double}}, \]
where the first and second terms on the right hand side signify the times required for a free wavepacket to double its width following (14), and by the same equation in the \( \alpha \to 0 \) limit (i.e. no GUP). Note that even in the latter limit, the relativistic effect, in terms of \( C_{\text{rel}} \) is always present in (14).

To calculate DTD using (14), we first notice that, since we are working with a Gaussian wavepacket, the term \( \dot{\xi}_i \) is zero. Next, one can replace initial value of \( \Delta p_0 \) in terms of the initial position uncertainty \( \Delta x \) in (14), using the following minimum uncertainty relation (again since \( [x, p_0] = i\hbar \))

\[
(\Delta x)_{\text{in}}(\Delta p_0)_{\text{in}} = \frac{\hbar}{2},
\]

and solve for the doubling time with GUP in which the width becomes \( 2\Delta x_0 \). To find the doubling time without GUP, we simply set the GUP parameter to zero in the above result. This enables us to calculate the doubling time difference (19) which becomes a function of the above result. This enables us to calculate the doubling time difference (19) which becomes a function of the initial width \( (\Delta x_0) \), mass \( (m) \), mean velocity \( (v) \) of the wavepacket, as well as, the Planck constant \( (\hbar) \), speed of light \( (c) \), and the value of the GUP parameter \( (\alpha) \).

We calculate the DTD numerically for three different molecular wavepackets: these are - (i) Buckyball \( C_{60} \), (ii) Buckyball \( C_{176} \) and (iii) Tetraphenylporphyrin or TPPF152 molecule \( (C_{168}H_{94}F_{152}O_8N_4S_4) \). Relevant physical parameters for these molecular wavepackets are given in the accompanying table. It is important to note that the aforementioned systems behave quantum mechanically and are stable against decoherence, at least for their assumed widths, as shown for example by means of double-slit experiments [21-26]. Therefore the GUP applies to them and would affect the broadening rates of these wavepackets. 1

Results of our numerical analysis are depicted in the left panel of Figure 1 which is a log-log plot relating the GUP parameter \( \alpha \) with the doubling time difference \( \Delta t_{\text{double}} \). The plot covers the entire region between the

| Molecules | Mass (kg) | Width (m) | Velocity (ms\(^{-1}\)) |
|-----------|----------|-----------|----------------------|
| \( C_{60} \) | \( 1.1967 \times 10^{-23} \) | \( 7 \times 10^{-10} \) | 10\(^{3} \) |
| \( C_{176} \) | \( 3.5070 \times 10^{-23} \) | \( 1.2 \times 10^{-9} \) | 10\(^{3} \) |
| TPPF152   | \( 8.8174 \times 10^{-24} \) | \( 6 \times 10^{-9} \) | 10\(^{3} \) |

TABLE I: Physical parameters of the wavepackets under consideration. While mass and width of the wavepackets are known from experiments, mean velocity is assumed by us which provides measurable effects.

1 Although it has been claimed that the GUP needs to be applied cautiously for a composite system (such as one with many constituent atoms) [27], we adopt the point of view that GUP would apply to the quantum system as a whole [22, 28, 29]. In the end, it is for experiments to decide on its correctness.
their evolution are insignificant. We computed the time taken for the corresponding wavepackets to double in size and the showed that QG/GUP effects entail a measurable difference in the doubling times, which may just be measurable with current precision of time-measurements, or those that are projected in the near future, as clearly demonstrated in the accompanying figures! Taking the required relativistic effects into account, we showed that the some of the earlier conclusions don’t just remain, they in fact get further solidified. Again, the unprecedented accuracy of time measurements should aid in this measurement, which of course would get progressively even better in the future. Note that the detection of a $\alpha_0 \gg 1$ would signify a length scale intermediate between the electroweak and the Planck scale. Even if the predicted effects are not observed, that would provide the best constraints on the GUP parameter $\alpha$ to date. For example, with an attosecond accuracy, we would provide up to 5 orders of magnitude tighter bound than previous best bound by measuring Lamb shift [21]. Furthermore, with the latest implementation of time measurement in the zeptosecond order, we would be able scan the whole GUP parameter space, and thus verifying or ruling out the linear GUP modification altogether! We hope to continue our study of similar effects in other quantum systems that can be prepared in the laboratory and report elsewhere.

Acknowledgement: This work is supported by the Natural Sciences and Engineering Research Council of Canada. Research of SKM is supported by CONACyT research grant CB/2017-18/A1S-33440, Mexico.

FIG. 1: Doubling time difference versus GUP parameter plot in logarithmic scale. The left panel of the figure corresponds to the results with the relativistic correction, as carried out in the present paper, and the right panel is without considering the relativistic correction, as carried out in the earlier work [1]. The blue shaded regions correspond to the GUP parameter space that could be scanned by measuring DTD with attosecond ($10^{-18}$ s) accuracy. The full GUP parameter space, up to the Planck scale could be scanned with zeptosecond accuracy ($10^{-21}$ s). For details see text.

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