Subsonic and supersonic nucleus-acoustic solitary waves in thermally degenerate plasmas with heavy nucleus species

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A fully ionized multi-nucleus plasma system (containing thermally degenerate electron species, non-degenerate warm light nucleus species, and low dense stationary heavy nucleus species) is considered. The basic features of thermal and degenerate pressure driven small and arbitrary amplitude subsonic and supersonic nucleus-acoustic solitary waves in such a plasma system are studied by the pseudo-potential approach. The effects of stationary heavy nucleus, non-relativistically and ultra-relativistically electron degeneracies, and light nucleus temperature on small and arbitrary amplitude subsonic and supersonic nucleus-acoustic solitary waves are also examined. It is found that (i) the presence of stationary heavy nucleus species with Boltzmann distributed electron species supports the existence of subsonic nucleus-acoustic solitary waves, and that the effects of electron degeneracies and light nucleus temperature reduce the possibility for the formation of these subsonic nucleus-acoustic solitary waves; (ii) the amplitude (width) of the subsonic nucleus-acoustic solitary waves increases (decreases) with the rise of the number density of heavy nucleus species; (iii) the amplitude of the supersonic nucleus-acoustic solitary waves in the situation of no-relativistically degenerate electron species is much smaller than that of ultra-relativistically degenerate electron species, but is much larger than that of isothermal electron species; (iv) their width in the situation of non-relativistically degenerate electron species is much wider than that of ultra-relativistically degenerate electron species; (v) their amplitude (width) decreases (increases) with the rise of the light nucleus temperature. The applications of the results in astrophysical, space, and laboratory plasma situations are briefly discussed.

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I. INTRODUCTION

The ion-acoustic (IA) waves \[1–3\] are thermal pressure driven longitudinal electro-acoustic waves in which the ion mass density (electron thermal pressure) provides the inertia (restoring force). They propagate as compression and rarefaction (and vice-versa) of the inertial ion fluid (a macroscopically neutral substance containing many interacting charged and neutral particles, which exhibit collective behavior due to the long-range Coulomb forces). The linear dispersion relation for the IA waves propagating in a pure electron-ion plasma [containing Boltzmann distributed electron species (BDES) and cold inertial ion fluid] is given by

\[
\omega = \frac{kC_i}{\sqrt{1 + k^2\lambda_{De}^2}},
\]

(1)

where \(\omega = 2\pi f\) and \(k = 2\pi/\lambda\) in which \(f (\lambda)\) is the IA wave frequency (wavelength); \(C_i = (k_BT_e/m_i)^{1/2}\) is the IA speed in which \(k_B\) is the Boltzmann constant, \(T_e\) is the electron temperature, and \(m_i\) is the ion mass; \(\lambda_{De} = C_i/\omega_{pi}\) is the IA wave length scale in which \(\omega_{pi} = (4\pi N_{i0}e^2/m_i\gamma_e)^{1/2}\) is the ion plasma frequency, \(N_{i0}\) is the ion number density at equilibrium, and \(e\) is the magnitude of an electron charge. We note that for a pure electron-ion plasma \(N_{e0} = N_{i0}\), where \(N_{e0}\) is the electron number density at equilibrium. The dispersion relation \(1\) indicates that for a long-wavelength limit (viz. \(k\lambda_{De} \ll 1\)), we have \(\omega \simeq kC_i\), and for a short wavelength limit (viz. \(k\lambda_{De} \gg 1\)), we have \(\omega \simeq \omega_{pi}\). So, the upper limit of \(\omega\) for the IA waves is \(\omega_{pi}\).

The IA waves do not exit for cold electron limit (\(T_e = 0\)). However, in degenerate plasma (which is significantly different from usual electron-ion plasma because of its extra-ordinarily high density \[4–11\], and low temperature \[12–16\]), Mamun \[17\] predicted the existence of degenerated pressure (generated due to Heisenberg’s uncertainty principle with infinitesimally small uncertainty in position and infinitely large uncertainty in momentum of degenerate electron species) driven nucleus-acoustic (NA) waves in absolutely cold degenerate plasmas. The linear dispersion relation for such degenerate pressure driven (DPD) NA waves in such a cold degenerate electron-nucleus plasma is given by

\[
\omega = \frac{\sqrt{\gamma_e}kC_i}{\sqrt{1 + \gamma_e k^2\lambda_{De}^2}},
\]

(2)

where \(C_i = (Z\epsilon_0\epsilon_{i0}/m_i)^{1/2}\) is the DPD NA speed in which \(\epsilon_{i0} = KN_{i0}\gamma_e^{-1}\) is the cold electron degenerate energy associated with the degenerate electron pressure \[17\]. \(Z_i\) is the charge state of the nucleus species, \(m_i\) is the mass of the nucleus species, \(K = 3\pi h^2/5m_e\) (where \(m_e\) is the mass of an electron and \(h\) is the reduced Plank constant) for \(\gamma_e = 5/3\) (non-relativistically degenerate electron species \[9–10\]), \(K = 3hc/4\) (where \(c\) is speed of light).
of light in vacuum) for \( \gamma_e = 4/3 \) (ultra-relativistically
degenerate electron species \([9,10]\), and \( K \) is unknown for
\( \gamma_e = 1 \) (which, thus, cannot be considered in \([2]\),
\( N_{e0} \) is the degenerate electron number density at equilibrium;
\( \lambda_q = C_l/\omega_{pl} \) is the DPD NA wave length scale;
\( \omega_{pl} = (4\pi N_{i0} Z_l^2 e^2/m_i) \) is the nucleus plasma
frequency. We note that for a degenerate electron-nucleus
plasma \( N_{e0} = Z_l N_{i0} \), where \( N_{i0} \) is the non-degenerate
nucleus number density at equilibrium. The dispersion
relation \([2]\) indicates that for the appropriate (long wavelength)
limit, viz. \( k\lambda_D \ll 1 \), we have \( \omega \simeq kC_l \), which
indicates that in DPD NA waves, the nucleus mass density
(electron degenerate pressure) provides the inertia
(restoring force). We note that the DPD NA waves disappear
in absence of the electron degenerate pressure, which is
independent of temperature of any plasma species.

The dispersion relations \([1]\) and \([2]\) along with their
interpretations indicate that the DPD NA waves \([17]\) are
different from the IA waves \([11,3]\), and are completely
new in view of restoring force, and are found to exist
only in degenerate plasmas. There are large number of
investigations on the properties of linear and non-linear
ion-acoustic, nucleus-acoustic, electron-acoustic,
and positron-acoustic waves in degenerate plasma sys-
tems under different situations carried out during last
one decade or more \([18–47]\). The investigations (men-
tioned above and made so far) on IA or NA solitary
waves (SWs) in degenerate plasmas (DPs), which are
of our present interest, are based absolutely cold degen-
erate plasma approximation and reductive perturbation
method \([48]\), and thus, are not valid for \( T_e \neq 0 \) and
\( T_l \neq 0 \) (where \( T_l \) is the light nucleus temperature),
and are also not valid for arbitrary amplitude SWs. So, \([1]\)
is not valid when the degenerate pressure of the electron
species is comparable to or greater than its thermal pres-
sure, and that \([2]\) is not valid when the electron thermal
pressure is comparable to or greater than its degenerate
pressure. We, therefore, propose a more general degener-
ate plasma model by considering a thermally degener-
ate plasma (TDP) system containing thermally
degenerate electron species (TDES), non-degenerate cold/warm
light nucleus specie (since the degeneracy in nucleus species is at least \((m_e/m_i)\) times lower than that in elec-
tron species \([10,17,35,37]\), and stationary heavy nu-
cleus species (SHNS). The advantage of this TDP model
is that the TDES is valid for the arbitrary value of \( \gamma_e \), i.e.
valid for \( \gamma_e = 1 \) (BDES), \( \gamma_e = 5/3 \) (non-relativistically
TDES), \( \gamma_e = 4/3 \) (ultra-relativistically TDES), etc. The
linear dispersion relation for the NA waves in such a TDP
system with the cold NDLNS \((T_l = 0)\) is given by

\[
\omega = \sqrt{\frac{\gamma_e}{1 + \mu}} \frac{kC_q}{\sqrt{1 + \frac{\gamma_e}{1 + \frac{\mu}{1 + \mu}} k^2 \lambda_q^2}}
\]

where \( \mu = Z_h N_{h0}/Z_l N_{i0} \) with \( Z_h \) (\( N_{h0} \)) is the
charge state (number density) of the SHNS; \( C_q =
(Z_l \varepsilon_{ed} / m_i) \) is the energy associated with the electron degenerate pressure,
\( \varepsilon_{ed} = k_B T_e N_{e0}^{-1} \) is that associated with the electron thermal
pressure, and \( \lambda_q = C_q/\omega_{pl} \). We note that for
BDES \((\gamma_e = 1)\), \( \varepsilon_{ed} = k_B T_e, C_q = C_l \), and \( \lambda_q = \lambda_{De} \).
So, in absence of SHNS \((\mu = 0)\), \([3]\) reduces to \([1]\).

On the other hand, for the cold degenerate electron
species \((T_e = 0)\), \( \varepsilon_{ed} = k_B T_e, C_q = C_l \), and \( \lambda_q = \lambda_{De} \).
So, in absence of the SHNS \((\mu = 0)\), \([3]\) reduces to \([2]\).

The dispersion relation \([3]\) indicates that for the appro-
priate (long wavelength) limit, viz. \( k\lambda_q \ll 1 \), we have \( \omega \simeq \sqrt{\gamma_e kC_q \sqrt{T + \mu}} \), which indicates that in the NA
waves, the light nucleus mass density (sum of electron
degenerate and thermal pressures) provides the inertia
(restoring force), and that the phase speed of the NA
waves decreases (increases) with the rise of the value
of \( \mu \) (\( \gamma_e \)). We note that the upper limit of \( \omega \) for the NA
waves defined by \([2]\) and \([3]\) is \( \omega_{pl} \).

To the best knowledge of the author, no investigation
has been made on NA SWs in any TDP system. There-
fore, in the present work, this new TDP model is con-
sidered to investigate the arbitrary amplitude subsonic
and supersonic SWs associated the linear waves defined
by \([3]\). The pseudo-potential approach \([49,50]\), which
is valid for the arbitrary amplitude SWs, is used. The
new TDP model proposed in the present work is so gen-
eral that it is valid not only for hot white dwarfs \([51–
55]\), but also for many space \([56–59]\) and laboratory \([60–
62]\) plasma situations, where non-degenerate electron-ion
plasma with heavy positively charged particles (as impu-
ritiy or dust) occur.

The manuscript is organized as follows. The governing
equations in dimensional and normalized forms are given
in Sec. II. The conditions for the formation of subsonic
and supersonic NA SWs associated with non-degenerate
light nucleus species (NDLNS) in both cold and warm
adiabatic situations are described in Sec. III. Their basic
features are also illustrated in the same section (Sec. III).
The discussion in short is provided in Sec. IV.

II. MODEL EQUATIONS

We consider a TDP system containing TDES, NDLNS,
and SHNS. Thus, at equilibrium we have \( N_{e0} = Z_l N_{i0} +
Z_h N_{h0} \). We also consider the propagation of the NA
waves in such a TDP system. The dynamics of nonlinear
NA waves in such a TDP system is described by

\[
\frac{\partial N_i}{\partial T} + \frac{\partial}{\partial X} (N_i U_j) = 0,
\]

\[
\frac{\partial P_{\parallel q}}{\partial T} + U_j \frac{\partial P_{\parallel q}}{\partial X} + \gamma_j \frac{\partial U_j}{\partial X} = 0,
\]

\[
\frac{\partial}{\partial X} (P_{ed} + P_{ct}) - N_e \frac{\partial \Phi}{\partial X} = 0,
\]

\[
\frac{\partial U_l}{\partial T} = - \frac{Z_l}{m_l} \frac{\partial \Phi}{\partial X} - \frac{1}{N_l m_l} \frac{\partial}{\partial X} (P_{ld} + P_{dl}),
\]

\[
\frac{\partial^2 \Phi}{\partial X^2} = 4\pi e (N_e - Z_l N_l - Z_l N_{h0}),
\]
where $\dot{D}_T = \partial / \partial T + U_j \partial / \partial X$; $\Phi$ is the electrostatic NA wave potential; $N_j (U_j)$ is number density (fluid speed) of the plasma species $j$ (with $j = e$ for the TDES, $j = l$ for the NDLNS); $P_{jq}$ in (5) and (7) is the outward pressure for the species $j$ of the type $q$ (with $q = d$ for the degenerate pressure, and $q = t$ for the thermal pressure); $\gamma_j$ is the adiabatic index for the plasma species $j$; $X (T)$ is the space (time) variable. The nonlinear equations (4), describing the nonlinear propagation of the NA waves in the TDP system under consideration can be interpreted as follows:

- **Equation (4)** is the continuity equation for the plasma species $j$, where the effects of the source and sink terms have been neglected.

- **Equation (5)** is the energy equation for the plasma species $j$ for arbitrary $\gamma_j$. The use of this equation is meaningful if and only if the temperature $T_j$ is not constant since for constant $T_j$, i.e. for $T_j = T_{j0}$ (where $T_{j0}$ is the temperature of the plasma species $j$ at equilibrium) and $\gamma_j = 1$ (BDES), (4) and (5) are identical.

- **Equation (6)** is the momentum balance equation for TDES, and is due to the fact that the sum of degenerate and thermal pressures ($P_{ed} + P_{et}$) of the electron species counterbalances the electrostatic pressure ($N_e e \Phi$) associated with the NA waves. It is valid for $\omega/k << (E_{eq}/m_e)^{1/2}$, where $E_{eq}$ is the energy associated with the electron thermal $(q = t)$ or degenerate $(q = d)$ pressure at equilibrium.

- **Equation (7)** is the momentum balance equation for the NDLNS. The last term of (7) is due to the effect of the sum of degenerate and thermal pressures ($P_{ed} + P_{et}$) of the NDLNS.

- **Equation (8)** is Poisson’s equation for the NA wave potential, which has closed the set of our basic equations (4)–(7). The consideration of heavy nucleus species being stationary is valid since heavy nucleus plasma frequency is much less than the NA wave frequency because of the heavy mass and low number density of the heavy nucleus species.

It is important to mention that the gravitational force acting in TDES and NDLNS, which is inherently very small compared to the other forces under consideration, is neglected for the study of the NA waves.

To find the expressions for $P_{jq}$, we start with (4) and (5) and assume that all dependent variables in them depend on a single variable $\zeta = X - MT$ (with $M$ being the phase speed of the DA waves), and that $\partial / \partial T \to 0$. These assumptions allow us to express (4) and (5) as

$$M \frac{dN_j}{d\zeta} - \frac{d}{d\zeta}(N_j U_j) = 0, \quad (9)$$

$$M \frac{dP_{jq}}{d\zeta} - U_j \frac{dP_{jq}}{d\zeta} - \gamma_j P_j \frac{dU_j}{d\zeta} = 0. \quad (10)$$

At equilibrium $N_j = N_{j0}$ and $U_j = 0$. Thus, (9) can be expressed as

$$U_j = M \left( 1 - \frac{N_{j0}}{N_j} \right). \quad (11)$$

Now, substituting (11) into (10), and dividing the resulting equation by $N_{j0}$, we obtain

$$\frac{1}{N_{j0}^2} \frac{dP_{jq}}{d\zeta} - \gamma_j \frac{dN_j}{N_j (\gamma_j + 1)} = 0,$$

which, after rearrangement, can be expressed as

$$\frac{d}{d\zeta} \left( \frac{P_{jq}}{N_j^{\gamma_j}} \right) = 0. \quad (13)$$

The integration of (13) yields

$$P_{jq} = K_{jq} N_j^{\gamma_j}, \quad (14)$$

where $K_{jq} = \delta_{jq} N_{j0}^{\gamma_j - 1}$ is the integration constant in which $\delta_{jq}$ is equilibrium energy associated with the outward pressure for the species $j$ of type $q$.

We now substitute $P_{ed}$ and $P_{et}$ [which can be obtained from (14)] into (6), and express $n_e (= N_e/N_{ed})$ in terms of $\phi (= e \Phi/E_{ed})$, where $E_{ed} = E_{ed} + E_{et}$ as

$$n_e = \left( 1 + \frac{\gamma_e - 1}{\gamma_e} \phi \right)^{\frac{\gamma_e}{\gamma_e - 1}}. \quad (15)$$

It is important to note that (15) is valid for arbitrary value of $\gamma_e$, and is, thus, valid for non-relativistically ($\gamma_e = 5/3$) as well as ultra-relativistically ($\gamma_e = 4/3$) TDES. We note that for cold DES, $E_{et} = 0$ and $E_{ed} = E_{ed} = K_{ed} N_{j0}^{\gamma_e - 1}$, which mean that $\phi = e \Phi/E_{ed}$. On the other hand, for BDES, $E_{ed} = 0$ and $E_{et} = E_{et} = k_B T_e$, which indicate that $\phi = e \Phi/k_B T_e$ and $\sigma_l = T_l / Z_i T_e$. It is also important to mention that we cannot put $\gamma_e = 1$ [corresponding to BDES, $n_e = \exp(\phi)$ with $\phi = e \Phi/k_B T_e$] directly into (15). To consider this limit, we expand $n_e$ [defined by (15)] as

$$n_e = \left( \frac{1}{\gamma_e} \right) \phi + \left( \frac{\gamma_e}{2!} \right) \phi^2 + \left( \frac{\gamma_e^2 \gamma_3}{3!} \right) \phi^3 + \cdots, \quad (16)$$

where $\gamma_2 = 2 - \gamma_e$ and $\gamma_3 = 3 - 2 \gamma_e$. The substitution of $\gamma_e = 1$ into (16) leads to

$$n_e = 1 + \phi + \frac{\phi^2}{2!} + \frac{\phi^3}{3!} + \cdots = \exp(\phi). \quad (17)$$

Thus, (15) [by rewriting it in the form of (16)] is also valid for $\gamma_e = 1$ corresponding to BDES, $n_e = \exp(\phi)$ with $\phi = e \Phi/k_B T_e$ and $\sigma_l = T_l / Z_i T_e$.

On the other hand, substituting $P_{et}$ [which can be obtained from (14)] into (7), the basic equations (4) and
It is valid for cold NDLNS (\(\gamma = 1\)) which is valid for arbitrary value of work since the degeneracy in light nucleus species is at least \((m_e/m)\) times lower than that in electron species [10, 17, 30, 37]. The cold NDLNS \((\sigma_l = 0)\) gives (28) to

\[
n_l = \frac{1}{\sqrt{1 - 2\phi/\Lambda^2}}.
\]

This means that we can consider \(\sigma_l = 0\), but cannot consider \(\gamma_l = 1\) with \(\sigma_l \neq 0\) to study the effect of the NDLNS temperature on the basic features of arbitrary amplitude SWs by the pseudo-potential approach.

Now, for warm adiabatic NDLNS \((\sigma_l \neq 0)\) and \(\gamma_l = 3\) we can express \(n_l\) from (28) as

\[
3\gamma_l n_l^4 - (M^2 + 3\sigma_l - 2\phi)n_l^2 + M^2 = 0.
\]

The solution of (30), which is a quadratic equation for \(n_l^2\), for \(n_l\) is given by

\[
n_l = \left[\frac{1}{\delta \sigma_l} \left(\Phi_0 - \sqrt{\Phi_0^2 - 12\sigma_l M^2}\right)\right]^{\frac{1}{2}},
\]

where \(\Phi_0 = M^2 + 3\sigma_l - 2\phi\).

Now, multiplying both side of (25) by \(d\phi/d\xi\) and integrating with respect to \(\xi\), we obtain

\[
\frac{1}{2} (d\phi/d\xi)^2 + V(\phi) = 0,
\]

where

\[
V(\phi) = -\int [(1 + \mu)n_e - n_l - \mu]d\phi,
\]

in which \(n_e\) is given by (15), which is valid \(\gamma = 5/3\) (non-relativistically TDES) and \(\gamma = 4/3\) (ultra-relativistically TDES) or (16), which is valid for \(\gamma = 1\) (BDES), and \(n_l\) is given by (29), which valid for \(\sigma_l = 0\) or (31), which is valid for \(\gamma_l = 3\) (adiabatic NDLNS). We note that (32) is an energy integral for a pseudo-particle of unit mass with \(\xi\) as pseudo-time, \(\phi\) as pseudo-position, and \(V(\phi)\) as pseudo-potential. Therefore, from the analysis of \(V(\phi)\), one can not only find the conditions for the formation of the NA SWs, but also can study their basic features. We now consider the following two situations of NDLNS number density \(n_l\), and investigate the basic features of the NA SWs.

### A. Cold NDLNS \((\sigma_l = 0)\)

The cold NDLNS \((\sigma_l = 0)\) is valid for \(\omega/k \gg (k_BT_0/m)\), which is usual for the NA waves and plasma system under consideration. We, thus, first substitute (15) and (29) into (33), and express (33) as

\[
V(\phi) = C_0 - \delta \left(1 + \frac{\phi}{\gamma_e}\right) - M^2 \sqrt{1 - \frac{2\phi}{\Lambda^2} + \mu\phi},
\]

where \(C_0 = 1 + \mu + M^2\) is the integration constant, and \(\gamma_e = \gamma_e/(\gamma_e - 1)\).
It is usual for almost all non-degenerate and degenerate plasma situations that $E_{eT} > E_{eT}$, and $|\phi| < 1$ since $E_{eT} = E_{ed} + E_{et}$, $\phi = \Phi/E_{eT}$, and $E_{et} \approx k_BT_{et}$. Thus, $V(\phi)$ [defined by (34)] can be expanded as

$$V(\phi) = C_2\phi^2 + C_3\phi^3 + \cdots,$$

(35)

where

$$C_2 = \frac{1}{2} \left[ \frac{1}{M^2} - \frac{1}{\gamma_c}(1 + \mu) \right],$$

(36)

$$C_3 = \frac{1}{3} \left[ \frac{3}{M^4} - \frac{1}{\gamma_c}(2 - \gamma_c)(1 + \mu) \right].$$

(37)

It is clear from (35) that the constant $C_0$, and the coefficient of $\phi$ in the expansion of $V(\phi)$ vanish because of the choice of the integration constant, and because of the equilibrium charge neutrality condition, respectively. Thus, the NA SWs exist if and only if $|d^2V/d\phi^2|_{\phi=0} < 0$ so that the fixed point at the origin is unstable, and $|d^3V/d\phi^3|_{\phi=0} > (\phi=0)$ for the existence of the NA SWs with $\phi > 0$ ($\phi < 0$) [50]. These imply that the NA SWs exist if $C_2 < 0$, i.e. if $M > M_c$, where $M_c$ is the critical Mach number (the minimum value of the Mach number above which the NA SWs exist), and is given by

$$M_c = \sqrt{\frac{\gamma_c}{1 + \mu}}.$$  

(38)

On the other hand, the NA SWs exist with $\phi > 0$ ($\phi < 0$) if $C_3(M = M_c) > 0$ ($< 0$), where $C_3(M = M_c)$ is

$$C_3(M = M_c) = \left( \frac{1 + \mu}{3! \gamma_c} \right) (1 + \gamma_c + 3\mu),$$

(39)

which implies that $C_3(M = M_c) > 0$ since $\mu \geq 0$ and $\gamma_c \geq 1$, and that the NA SWs only with $\phi > 0$ exist for any possible values of $\mu$ and $\gamma_c$ (so, from now ‘SWs’ will be used to mean SWs with $\phi > 0$). We have examined the variation of $M_c$ with $\mu$ for $\gamma_c = 1$ (BDES), $\gamma_c = 5/3$ (non-relativistically DES), and $\gamma_c = 4/3$ (ultra-relativistically DES) as shown, respectively, in solid, dotted, and dashed curves of figure 1. This figure clearly indicates that for BDES, the subsonic NA SWs exist for those values of $\mu$, which are in between solid and dotted curves, and that above the dotted curve, there exist the supersonic NA SWs. On the other hand, for realistic values of $\mu$ (e.g. $0 < \mu < 0.3$), the non-relativistically and ultra-relativistically DES are in against for the formation of subsonic NA SWs, but are in favor of the formation of supersonic NA SWs.

We now study the basic features of small amplitude NA SWs by considering the approximation $V(\phi) = C_2\phi^2 + C_3\phi^3$. This approximation along with the condition $V(\phi_m) = 0$ (where $\phi_m \neq 0$) is the amplitude of the solitary waves) reduces the SW solution of (32) to

$$\phi = \left( -\frac{C_2}{C_3} \right) \mathrm{sech}^2 \left( \sqrt{\frac{C_2}{2}} \xi \right),$$

(40)

which has been derived in Appendix A.

We have graphically represented [40] to observe the basic features of small amplitude subsonic ($M = 0.99$) NA SWs for $\gamma_c = 1$ (BDES), $\gamma_c = 5/3$ (non-relativistically DES), and $\gamma_c = 4/3$ (ultra-relativistically DES) for different values of $\mu$ (viz. $\mu = 0.1$, $\mu = 0.15$, and $\mu = 0.2$). The results are displayed in figures 2-4. We have also reexamined these basic features of these subsonic and supersonic NA SWs by the direct analysis of pseudo-potential $V(\phi)$ defined by (34) for the same set of plasma parameters. The results are displayed in figures 5-7.

It is obvious from figures 2-7 that (i) the presence of SHNS supports the existence of small amplitude subsonic NA SWs; (ii) the amplitude (width) of the subsonic NA SWs increases (decreases) with the increase
in number density (represented by \( \mu \)) of the SHNS; (iii) the effects of non-relativistically \( (\gamma_e = 5/3) \) and ultra-relativistically \( (\gamma_e = 4/3) \) DES are in against the formation of the subsonic NA SWs, and thus, give rise to the formation of the supersonic NA SWs; (iv) the amplitude of the supersonic NA SWs in no-relativistically DES \( (\gamma_e = 5/3) \) is much smaller than that in ultra-relativistically DES \( (\gamma_e = 4/3) \), but is much larger than that in BDES \( (\gamma_e = 1) \); (iv) the width of the supersonic NA SWs in no-relativistically DES \( (\gamma_e = 5/3) \) is much wider than that in ultra-relativistically DES \( (\gamma_e = 4/3) \); (v) the small amplitude approximation provides almost the same results as the direct analysis of the pseudopotential \( V(\phi) \) [defined by (34)] does.

The amplitude and the width of the NA SWs are also visualized from figures 5–7. The potential wells in figures 5–7 indicate the amplitude \( \phi_m \) (value of \( \phi \) at the point where the \( V(\phi) \) vs. \( \phi \) curve crosses the \( \phi \)-axis), and the width \( W \) [defined as \( W = \phi_m / \sqrt{|V_m|} \), where \( |V_m| \) is the maximum value of \( V(\phi) \) in the potential wells. Thus, the figures 5–7 indicate that the amplitude (with) of both subsonic and supersonic NA increases (decrease) with the rise of \( \mu \), and that their amplitude (width) of both subsonic and supersonic NA decrease (increase) with the rise of \( \gamma_e \), since in comparison with an increase in \( \phi_m \), a very slight increase/decrease in \( |V_m| \) causes a very significant decrease/increase in \( W \). The same results have already been obtained from the analysis of the SW solution (40), which is valid for the small, but finite amplitude subsonic and supersonic NA SWs.
γ must have an unstable fixed point at the origin and charge neutrality condition at equilibrium (viz. $V\Phi = 0$) to the formation of supersonic NA SWs) for $\gamma_c = 4/3$, $M = 1.24$, $\mu = 0.1$ (solid curve), $\mu = 0.15$ (dotted curve), and $\mu = 0.2$ (dashed curve).

B. Warm NDLNS ($\sigma_l > 0$)

We now consider warm adiabatic NDLNS of number density defined by (31). The latter is valid when $P_{tl} \ll P_{tr}$, which is valid not only for hot hot white dwarfs [51–55], but also for many space [56–59] and laboratory [60–62] plasma situations. Thus, substituting (15) and (31) into (33), and following the same procedure as adopted before, we can express the pseudo-potential $V(\phi)$ [in the energy integral defined by (32)] as

$$V(\phi) = C_0^2 + \mu \phi - (1 + \mu) \left[ 1 + \left( \frac{\gamma_e - 1}{\gamma_e} \right) \phi \right] \frac{\sigma_l}{\gamma_e - 1}$$

$$- \frac{\sqrt{2}}{3\sqrt{3}\sigma_l} \left( \sqrt{\Phi_0 - \Phi_1} \right) \left( \Phi_0 + \frac{1}{2} \Phi_1 \right),$$

where $C_0^2 = 1 + \mu + \sigma_l + M^2$ is the integration constant chosen in such a way that $V(\phi) = 0$ at $\phi = 0$, $\Phi_0 = M^2 + 3\sigma_l - 2\phi$, and $\Phi_1 = \sqrt{\Phi_0 - 12\sigma_l M^2}$. The charge neutrality condition at equilibrium (viz. $n_e = 1$ and $n_i = 1$) leads to $[dV/d\phi]_{\phi=0} = 0$. To have the NA SW solution of (32), its pseudo-potential $V(\phi)$ [defined (41)] must have an unstable fixed point [50] at the origin ($\phi = 0$), i.e. $[d^2V/d\phi^2]_{\phi=0} < 0$, and at the same time (i.e. satisfying this condition) if $[d^3V/d\phi^3]_{\phi=0} > 0$ ($\phi > 0$), the NA SWs with $\phi > 0$ ($\phi < 0$) exist [50].

To find the conditions for the existence of the NA SWs analytically, we expand $V(\phi)$ defined by (41) as

$$V(\phi) = C_2^2 \phi^2 + C_3^2 \phi^3 + \cdots,$$

where

$$C_2^2 = \frac{1}{2!} \left[ \frac{1}{M^2 - 3\sigma_l} - \frac{1}{\gamma_e} (1 + \mu) \right],$$

$$C_3^2 = \frac{1}{3!} \left[ \frac{3(M^2 + \sigma_l)}{(M^2 - 3\sigma_l)^3} - \frac{1}{\gamma_e^3} (2 - \gamma_e) (1 + \mu) \right].$$

The coefficient of $\phi^2$ (viz. $C_2^2$) indicates from $[d^2V/d\phi^2]_{\phi=0} < 0$ that the SW solution of (32) with (41) exists if and only if $C_2^2 < 0$. Thus, the NA SWs exist if $M > M_c^c$, where $M_c^c$ is given by

$$M_c^c = \sqrt{\frac{\gamma_e}{1 + \mu} + 3\sigma_l}.$$  \hspace{1cm} \text{(45)}$$

On the other hand, the NA SWs exist with $\phi > 0$ ($\phi < 0$) if $C_3(M = M_c^c) > 0$ ($< 0$), where $C_3(M = M_c^c)$ is

$$C_3(M = M_c^c) = \left( \frac{\delta}{3!\gamma_e^2} \right) \left[ \gamma_0 + 3\mu + \frac{12}{\gamma_e} \sigma_l \delta^2 \right],$$  \hspace{1cm} \text{(46)}$$

where $\delta = 1 + \mu$ and $\gamma_0 = 1 + \gamma_e$. Equation (46) implies that $C_3(M = M_c^c) > 0$ (since $\mu \geq 0$, $\gamma_e \geq 1$, and $\sigma_l \geq 0$) that the NA SWs only with $\phi > 0$ exist for all possible values of $\mu$, $\gamma_e$, and $\sigma_l$. It is clear from (45) that $M_c^c = M_c$ for $\sigma_l = 0$. We have graphically shown the variation of $M_c^c$ with $\sigma_l$ for $\mu = 0.2$. This is shown in figure 8 which shows that (i) $M_c^c$ increases with $\sigma_l$; (ii) the increase in $\sigma_l$ increases the minimum value of $\mu$ for which the subsonic NA SWs exist; (iii) the existence of subsonic NA SWs for $\mu = 0.3$ and a short range of $\sigma_l$ is shown in between solid and dot-dashed curves, and that above the dot-dashed curve, there exist the supersonic NA SWs; (iv) for realistic values of $\mu$ (e.g. $\mu < 0.3$), the non-relativistically and ultra-relativistically electron degeneracies as well as light nucleus temperature are in against the formation of subsonic NA SWs, but are in favor of the formation of supersonic NA SWs.

We again study small amplitude NA SWs for which $V(\phi) = C_2^2 \phi^2 + C_3^2 \phi^3$ holds good. This approximation along with the condition $V(\phi_m) = 0$ allows us to write the SW solution of (32) as

$$\phi = -\left( \frac{C_2^2}{C_3^2} \right) \text{sech}^2 \left( \sqrt{\frac{C_2^2}{C_3^2}} \xi \right),$$  \hspace{1cm} \text{(47)}$$
which is also derived in Appendix A.

To observe the effect of light nucleus temperature ($\sigma_l$) on the basic features of small amplitude subsonic and supersonic NA SWs, we have graphically analyzed [47] for $\gamma_e = 1$ (BDES), $\gamma_e = 5/3$ (non-relativistically DES), and $\gamma_e = 4/3$ (ultra-relativistically DES). The results are displayed in figures 9−11. We have also reexamined these basic features of these subsonic and supersonic NA SWs by the direct analysis of the pseudo-potential $V(\phi)$ [defined by (41)] for the same set of plasma parameters. The results are displayed in figures 12−14.

It is obvious from figures 9−11 that (i) the effect of light nucleus temperature significantly reduces the possibility for the formation of the subsonic NA SWs, and in the case of more light nucleus temperature ($\sigma_l$), we need more number of the SHNS to have the existence of subsonic NA SWs; (ii) the combined effects of light nucleus temperature ($\sigma_l$), non-relativistically ($\gamma_e = 5/3$), and ultra-relativistically ($\gamma_e = 4/3$) DES are in against for the formation of subsonic NA SWs, and thus, give rise to the formation of the supersonic NA SWs; (iii) the amplitude of the supersonic NA SWs in non-relativistically DES ($\gamma_e = 5/3$) is much smaller than that in ultra-relativistically DES ($\gamma_e = 4/3$), but is much larger than that BDES ($\gamma_e = 1$); (iv) the width of the supersonic NA SWs in non-relativistically DES ($\gamma_e = 5/3$) is much wider than that in ultra-relativistically DES ($\gamma_e = 4/3$); (v) the amplitude (width) of both subsonic and supersonic NA SWs decreases (increases) with the rise of the
IV. DISCUSSION

We consider a fully ionized multi-nucleus plasma system containing thermally degenerate electron, cold/warm non-degenerate light nucleus, and low dense heavy nucleus species. The basic features of thermal and degenerate pressure driven arbitrary amplitude subsonic and supersonic nucleus-acoustic solitary waves in such a plasma system have been investigated by the pseudo-potential approach, which is valid for arbitrary amplitude nucleus-acoustic solitary waves. The results, which have been found from this investigation, can be pinpointed as follows:

- The presence of stationary heavy nucleus species ($\mu > 0$) in electron-nucleus plasma supports the existence of subsonic nucleus-acoustic solitary waves with $\phi > 0$. This is due to the fact that the phase speed ($\omega/k$) of the nucleus-acoustic waves decreases with the rise of the number density of the stationary heavy nucleus species.
- It has been observed that in the case of higher light nucleus temperature ($T_l$), we need more number of stationary heavy nucleus species to have the existence of the subsonic nucleus-acoustic solitary waves.
- The effects of non-relativistically ($\gamma_e = 5/3$) and ultra-relativistically ($\gamma_e = 4/3$) degenerate electron species, and light nucleus temperature ($\sigma_l$) reduce the possibility for the formation of subsonic nucleus-acoustic solitary waves, and thus, give rise to the formation of the supersonic nucleus-acoustic solitary waves with $\phi > 0$. This is because that the phase speed ($\omega/k$) increases with the increase in value of the index $\gamma_e$ and light nucleus temperature (represented by $\sigma_l$).
- The amplitude of the supersonic nucleus-acoustic solitary waves in non-relativistically degenerate electron species ($\gamma_e = 5/3$) is much smaller than that in ultra-relativistically degenerate electron species ($\gamma_e = 4/3$), but is much larger than that in Boltzmann distributed electron species ($\gamma_e = 1$).
- The amplitude (width) of the subsonic and supersonic nucleus-acoustic solitary waves increases (decreases) with the rise of the number density of the heavy nucleus species, represented by $\mu$. The width of the supersonic nucleus-acoustic solitary waves in non-relativistically degenerate electron species ($\gamma_e = 5/3$) is much wider than that in ultra-relativistically degenerate electron species ($\gamma_e = 4/3$).
- The amplitude (width) of both subsonic and supersonic nucleus-acoustic solitary waves decreases (increases) with the rise of the light nucleus temperature, represented by $\sigma_l$. This is due to the fact that the temperature of the light nucleus fluid enhances the random motion of light nucleus, which causes to decrease the amplitude of the NA solitary structures.
- The correctness of the results are verified by obtaining the same results from the analysis of analytical solitary wave solution of the energy integral [defined by (32)] with the pseudo-potential $V(\phi)$ provided in (34) and (41) and the direct analysis of the pseudo-potential $V(\phi)$ [provided in (34) and (41)].
There are many hot white dwarfs [51–55], where the electron thermal pressure can be comparable to or greater than its degenerate pressure, and where in addition to degenerate electron species, non-degenerate light and heavy nucleus species exist. On the other hand, non-degenerate electron species [defined by (17) as a special case of γ_e = 1], ions [identical to light nucleus species considered here, and defined by (31)], and positively charged particle (impurity/dust) species [identical to stationary heavy nucleus species considered here] are observed in both space [56–59] and laboratory [60–62] plasma systems.

Therefore, the thermally degenerate plasma model under consideration is so general that it can be applied not only in astrophysical degenerate plasma systems [51–55], but also in many space [56–59] and laboratory [60–62] plasma systems. It may be added here that to examine the effects of the dynamics of heavy nucleus species and non-relativistic degeneracy in light nucleus species on the nucleus-acoustic subsonic and supersonic solitary waves (investigated in the present work) may also be a problem of great importance for some other degenerate plasma systems, but beyond the scope of the present work. However, it is expected that the present work is useful in understanding the physics of localized electrostatic disturbances in a number of astrophysical [51–55], space [56–59], and laboratory [60–62], plasma systems.

Appendix A: SW solution of $\frac{1}{2} \left( \frac{d\phi}{d\xi} \right)^2 + V(\phi) = 0$, where $V(\phi) = C_2\phi^2 + C_3\phi^3$

To obtain the solitary wave (SW) solution of this energy integral, two conditions must be satisfied. These are (i) $\left[ d^2\phi/d\xi^2 \right]_{\phi=0} < 0$ and (ii) $V(\phi_m) = 0$. The condition (i) means that the point of $V(\phi)$ vs. $\phi$ curve at the origin $(0,0)$ is unstable, which is satisfied if $C_3 < 0$. The condition (ii) is satisfied if $C_2 + C_3\phi_m = 0$, which gives rise to $C_2 = -C_3\phi_m$ or $\phi_m = -C_2/C_3$. Now, substituting $C_2 = -C_3\phi_m$ into the energy integral, we have

$$
\frac{1}{2} \left( \frac{d\phi}{d\xi} \right)^2 = C_3\phi^2(\phi_m - \phi)
\Rightarrow \left( \frac{d\phi}{d\xi} \right)^2 = 2C_3\phi^2(\phi_m - \phi)
\Rightarrow \frac{d\phi}{d\xi} = \sqrt{2C_3}\phi\sqrt{\phi_m - \phi}
\Rightarrow \frac{d\phi}{\phi\sqrt{\phi_m - \phi}} = \sqrt{2C_3}d\xi.
$$

(A1)

To integrate (A1), we let $\sqrt{\phi_m - \phi} = z$, which yields $\phi = (\phi_m - z^2)$ and $d\phi = -2zd\xi$. These along with $C_3\phi_m = -C_2$ reduce (A1) to

$$
\Rightarrow -\left[ \frac{1}{\sqrt{\phi_m + z}} + \frac{1}{\sqrt{\phi_m - z}} \right] dz = \sqrt{-2C_2}d\xi.
$$

(A2)

The integration of (A2) gives rise to

$$
\log(\sqrt{\phi_m - z}) - \log(\sqrt{\phi_m + z}) = \sqrt{-2C_2}\xi + K_0
\Rightarrow \log \left( \frac{\sqrt{\phi_m - z}}{\sqrt{\phi_m + z}} \right) = -\sqrt{2C_2}\xi + K_0
\Rightarrow \frac{\sqrt{\phi_m - z}}{\sqrt{\phi_m + z}} = K_1 \exp \left( \sqrt{-2C_2}\xi \right),
$$

(A3)

where $K_0$ is the integration constant, and $K_1 = \exp(K_0)$ is another constant to be determined. We have $\phi = \phi_m$ and $z = 0$ at $\xi = 0$ for the solitary wave solution. Thus, $K_1$ is determined from (A3) as $K_1 = 1$, and (A3) can be expressed as

$$
z = \phi_m \left[ 1 - \exp \left( \frac{\sqrt{2C_3}\phi_m\xi}{1 + \exp(\sqrt{2C_3}\phi_m\xi)} \right) \right].
$$

(A4)

Therefore, (A4) reduces $\phi = (\phi_m - z^2)$ to

$$
\phi = \phi_m \left[ 1 - \left( \frac{1 - \exp(\sqrt{2C_3}\phi_m\xi)}{1 + \exp(\sqrt{2C_3}\phi_m\xi)} \right)^2 \right]
= \phi_m \left[ \frac{2 \exp \left( \frac{\sqrt{C_3}\phi_m\xi}{2} \right)}{1 + \exp(2\sqrt{C_3}\phi_m\xi)} \right]^{1/2}
= \phi_m \left[ \frac{2}{\exp \left( \frac{\sqrt{C_3}\phi_m\xi}{2} \right) + \exp \left( -\frac{\sqrt{C_3}\phi_m\xi}{2} \right)} \right]^{1/2}
= \phi_m \text{sech}^2 \left( \sqrt{\frac{C_3}{2}}\phi_m\xi \right).
$$

(A5)

We note that the last few final step of (A5) are obtained by using the basic properties of hyperbolic functions. Thus, substituting $\phi_m = -C_2/C_3$ into (A5), we finally obtain

$$
\phi = \left( \frac{C_2}{C_3} \right) \text{sech}^2 \left( \sqrt{\frac{C_2}{2}}\xi \right).
$$

(A6)

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