Geometrical characteristics of phase and group velocity surfaces in anisotropic media

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ABSTRACT
The phase and group velocity surfaces are essential for wave propagation in anisotropic media. These surfaces have certain features that, especially, for shear waves result in complications for modelling and inversion of recorded wavefields. To analyse wave propagation in an anisotropic model, it is important to identify these features in both the phase and group domains. We propose few characteristics for this analysis: the energy flux angle, decomposed in the polar and azimuth angle correction angles and enhancement factor, which is able to characterize both singularity points and triplication zones. The very simple equation that controls the triplications is derived in the phase domain. The proposed characteristics are illustrated for elastic and acoustic anisotropic models of different symmetry classes.

Key words: Anisotropy, Modelling, Velocity.

INTRODUCTION
The wave propagation in anisotropic elastic media is very complex subject of research. In particular, the very common features of S waves in low symmetry anisotropic models are singularity points and triplications of group velocity surfaces. These features make the S wave and converted wave data processing practically impossible in the vicinity of triplication points and triplication cusps. Ray theory by Červený (2001) among others has problems in these points, and the wave amplitudes are not defined. At finite frequencies, these points result in complex waveforms that can be an issue for elastic imaging and Full Wave Inversion (FWI), specifically with linearized gradient (Oh and Alkhalifah, 2016). The details of the connection between the complexity of the group velocity surface and practical applications can be found in Grechka (2017).

There are many papers and books devoted to this problem (Fedorov, 1968; Musgrave, 1970; Achenbach, 1973; Auld, 1973; Fryer and Frazer, 1987; Arnold, 1989; Every and Kim, 1994; Helbig, 1994; Wolfe, 1998; Červený, 2001; Sharma, 2002; Chapman, 2004). The phase and group velocity surfaces with complex topological properties (Alshits and Lothe, 1979; O’Neill, 2006) are essential features of wave propagation in anisotropic media (Shuvalov, 1998). While the phase velocity surface is relatively simple and given by equation of sixth order, the group velocity surface can be very complex (Wang, 1995; Zhou and Greenhalgh, 2004) and given by equation up to 150th order (Fedorov, 1968; Musgrave, 1970; Helbig, 1994; Grechka, 2017). The classification of critical points, caustics and wave fronts can be found in Arnold et al. (1987). In particular, for shear waves, the group velocity surface can by folded due to triplications shaped with caustics or cusps (cuspidal lines and edges) (Musgrave, 1970; Payton, 1983; Every, 1986; Wolfe, 1995, 1998; Červený, 2002; Thomsen and Dellingler, 2003; Vavryčuk, 2003a,b; Roganov and Stovas, 2010; Stovas, 2016; Xu and Stovas, 2018) and might have different type of singularities (wedge and conical) (Crampin and Yedlin, 1981; Crampin, 1991; Schoenberg and Helbig, 1997; Vavryčuk, 1999, 2001, 2003b; Grechka, 2015, 2017; Ivanov and Stovas, 2017; Ivanov, 2019; Xu and Stovas, 2019a) or acoustic axes (Khatkevich, 1963, 1977; Musgrave, 1985; Darinskij, 1994; Shuvalov, 1998; Boulanger and Hayes, 1998; Wolfe, 1998; Alshits and Lothe, 2004, 2016; Norris, 2020).
where coefficients in (3) are invariants of symmetric positive definite matrix $\Gamma$, that is, $P = \text{tr}(\Gamma)$, $Q = \text{tr} (\text{cof}(\Gamma))$ and $R = \text{det} (\Gamma)$ (Helbig, 1994; Červený, 2001), where $\text{tr} (\cdot)$, $\text{cof} (\cdot)$, $\text{det} (\cdot)$ stand for trace, cofactor and determinant of Christoffel matrix. Equation (3) has three real positive roots which are shown in Appendix A.

The velocity of the wave packet formed by the superposition of plane waves is called the group (ray) velocity $V$. In a homogeneous non-dispersive medium (Musgrave, 1970; Červený, 2001),

$$V = \nabla v,$$  

(4)

with the group velocity projections given by

$$V_i = c_{ijkl} \partial g / \partial x_k,$$  

(5)

where $p_i$ is the slowness projection, $g_i$ are the corresponding normalized eigenvector projections, and $V = \left(\frac{\partial}{\partial \theta_1}, \frac{\partial}{\partial \theta_2}, \frac{\partial}{\partial \theta_3}\right)$.

The equation for the group velocity can be found in Červený (2001). The group velocity vector $V$ (from equation (4)) is defined by

$$V = vn + (I - nn^t) \frac{\partial v}{\partial n}.$$  

(6)

The projections $n_1$, $n_2$ and $n_3$ are given by $n_1 = \sin \theta \cos \phi$, $n_2 = \sin \theta \sin \phi$, $n_3 = \cos \theta$, with $\phi$ and $\theta$ being tilt and azimuth phase angles, respectively. We can arrive to equations for group velocity projections (Fedorov, 1968; Ben-Menahem and Sena, 1990),

$$V_1 = v \sin \theta \cos \phi + \frac{\partial v}{\partial \theta} \cos \theta \cos \phi - \frac{\partial v}{\partial \phi} \sin \theta,$$

$$V_2 = v \sin \theta \sin \phi + \frac{\partial v}{\partial \theta} \cos \theta \sin \phi + \frac{\partial v}{\partial \phi} \cos \theta,$$

$$V_3 = v \cos \theta - \frac{\partial v}{\partial \theta} \sin \theta.$$  

(7)

From equations (7), we can compute the group velocity magnitude

$$V^2 = V_1^2 + V_2^2 + V_3^2 = v^2 \left(1 + \left(\frac{1}{v} \frac{\partial v}{\partial \theta}\right)^2 + \frac{1}{\sin^2 \theta \left(\frac{1}{v} \frac{\partial v}{\partial \phi}\right)^2}\right).$$  

(8)

From equations (7), we can also compute the group velocity direction vector, $N$ with projections $N_i = V_i / V$, $N_1 = \sin \theta \cos \Phi$, $N_2 = \sin \theta \sin \Phi$, $N_3 = \cos \Theta$, where $\Theta$ and $\Phi$ are the tilt and azimuth group angles.

PHASE AND GROUP VELOCITIES

The phase velocity (velocity of a plane wave) in anisotropic medium can be obtained from solving the Christoffel equation,

$$\text{det}(\Gamma - v^2 I) = 0,$$  

(1)

where $I$ is the unit matrix, $\Gamma$ is the Christoffel matrix with the elements (Helbig, 1994)

$$\Gamma_{ij} = c_{ijkl} n_j n_k,$$  

(2)

where $c_{ijkl}$ is the reduced (density normalized) stiffness tensor, and $n = (n_1, n_2, n_3)^T$ is the phase directional vector.

The characteristic equation (1) is the cubic equation for phase velocity squared,

$$v^6 - P v^4 + Q v^2 - R = 0,$$  

(3)

where

$$P = \text{tr}(\Gamma), \quad Q = \text{tr}(\text{cof}(\Gamma))$$

and

$$R = \text{det}(\Gamma).$$

The last feature is associated with degeneracy of the Christoffel matrix (Aleshits et al., 1985; Goldin, 2013). Triplications and caustics can also be observed in the vicinity of singularity points (Grechka and Obolentsueva, 1993; Shuvalov and Every, 1996, 1997; Vavryčuk, 1999, 2003b,c). A numerical method to compute the phase velocities for a given group velocity direction is developed in Vavryčuk (2006) and Grechka (2013). The longitudinal directions (in which there exists a propagating plane wave with a polarization vector being parallel to the slowness vector) are computed in Helbig (1993) for arbitrary anisotropy. There is an extensive literature describing different mathematical problems when computing these surfaces. For example, polynomial equations defining the special point of the slowness surface can be transformed with the help of the Gröbner basis (Becker and Weispfenning, 1993). We are not pretending to give a review for all the issues related to anisotropic surfaces but just briefly mention a few related problems.

In this paper, we introduce a few geometrical characteristics of transformations from the phase velocity surface to the group one and vice versa. One of these characteristics, the enhancement factor, results in a very simple equation defining the triplications for anisotropic media of arbitrary symmetry. The proposed characteristics are evaluated for anisotropy models of different symmetry: transversely isotropic with a vertical symmetry axis, elastic, acoustic and elliptic orthorhombic model, monoclinic model with a horizontal symmetry plane and triclinic model.
The expression for phase velocity is defined through the slowness notations (the duality between the phase and group domains is exploited),

$$\frac{v}{v^2} = NV^{-1} + (1 - NN^T) \frac{\partial(V^{-1})}{\partial N}. \quad (9)$$

The phase velocity projections can be explicitly given by

$$\frac{v_1}{v^2} = \frac{1}{V_1} \sin \Xi \cos \phi - \frac{1}{V_2} \cos \Xi \cos \phi + \frac{1}{V_2} \sin \Xi \sin \phi, \quad (10)$$

$$\frac{v_2}{v^2} = \frac{1}{V_1} \sin \Xi \sin \phi - \frac{1}{V_2} \frac{\partial V}{\partial \Xi} \cos \Xi \sin \phi - \frac{1}{V_2} \frac{\partial V}{\partial \phi} \sin \Xi \sin \phi, \quad (11)$$

$$\frac{v_3}{v^2} = \frac{1}{V_1} \cos \Xi + \frac{1}{V_2} \frac{\partial V}{\partial \Xi} \sin \Xi,$$

where

$$\frac{1}{v^2} = \frac{1}{V^2} \left(1 + \left(\frac{1}{V} \frac{\partial V}{\partial \Xi}\right)^2 + \frac{1}{\sin^2 \Xi} \left(\frac{1}{V} \frac{\partial V}{\partial \phi}\right)^2\right). \quad (12)$$

From comparison of equations (8) and (11), we have the relation between velocity derivatives in the phase and group domains

$$\frac{1}{v} \frac{\partial v}{\partial \theta} + \frac{1}{\sin^2 v} \left(\frac{1}{v} \frac{\partial v}{\partial \phi}\right)^2 = \left(\frac{1}{V} \frac{\partial V}{\partial \Xi}\right)^2 + \frac{1}{\sin^2 \Xi} \left(\frac{1}{V} \frac{\partial V}{\partial \phi}\right)^2. \quad (13)$$

GEOMETRICAL CHARACTERISTICS

The geometrical characteristics of the phase velocity surface $v = v(\theta, \phi)$ (solution of equation (3)) are defined by the derivatives of velocity upon the polar and azimuth phase angles.

By using first-order derivatives, we can compute different geometrical characteristics.

The simplest one defines the angle between the phase and group velocity vectors and called the energy flux (or power flow) angle $\psi$,

$$v = V \cos \psi, \quad (14)$$

where

$$\cos \psi = n^T N. \quad (15)$$

From equation (8), it follows that

$$\tan \psi = \left(\frac{1}{v} \frac{\partial v}{\partial \theta}\right)^2 + \frac{1}{\sin^2 \theta} \left(\frac{1}{v} \frac{\partial v}{\partial \phi}\right)^2. \quad (16)$$

This angle can be interpreted as a measure of local anisotropy in a sense that in case of weak-anisotropy approximation, the energy flux angle is assumed to be zero. In the case of a singularity point, the phase velocity derivatives go to infinity (Appendix B), and the energy flux angle $\psi = \pi/2$.

Following from equation (12), the energy flux angle can also be defined in a group domain,

$$\tan \psi = \sqrt{\left(\frac{1}{V} \frac{\partial V}{\partial \Xi}\right)^2 + \frac{1}{\sin^2 \Xi} \left(\frac{1}{V} \frac{\partial V}{\partial \phi}\right)^2}. \quad (17)$$

Other geometrical characteristics defined by the first-order derivatives are the polar and azimuth angle corrections, $\Delta \theta$ and $\Delta \phi$. They define the difference between the phase and group angles. For the longitudinal direction according to Helbig (1993), both velocity derivatives are zero, and, thus, the energy flux angle being computed for P wave is zero.

Correction angles in phase domain

The polar group angle $\Xi$ and the azimuth group angle $\Phi$ can be computed from equations (7) as follows:

$$\tan \Xi = \frac{V_2}{V_1} = \frac{\tan(\theta + \Delta \theta)}{\cos \Delta \phi}, \quad (18)$$

$$\tan \Phi = \frac{V_2}{V_1} = \tan(\phi + \Delta \phi), \quad (19)$$

where correction angles $\Delta \theta$ and $\Delta \phi$ are defined by

$$\tan \Delta \theta = \frac{1}{v} \frac{\partial v}{\partial \theta}, \quad \tan \Delta \phi = \frac{1}{\sin \theta + \cos \theta \frac{1}{v} \frac{\partial v}{\partial \phi}}. \quad (20)$$

The correction angles can be interpreted as the magnitude of anisotropy being decomposed into the polar and azimuth angles.

In vertical symmetry planes of an anisotropic medium with orthorhombic symmetry (ORT), $\partial v/\partial \phi = 0$, therefore, $\Delta \phi = 0$, equation (17) reduces to the a transversely isotropic medium with a vertical symmetry axis (VTI), and $\Delta \theta = \psi$. In the case of singularity point (see Appendix B), the polar correction angle tends to $-\pi/2$, since $\tan \Delta \theta \rightarrow -\infty$. The azimuth correction angle is given by

$$\tan \Delta \phi = \frac{1}{\sin \theta} \frac{\frac{1}{v} \frac{\partial v}{\partial \phi}}{\frac{1}{v} \frac{\partial v}{\partial \theta}}. \quad (21)$$

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Correction angles in group domain

The correction angles in the group domain can be introduced in a similar way,

\[
\tan \theta = \frac{\tan(\Xi - \Delta \Xi)}{\cos \Delta \Phi}, \quad \tan \phi = \tan(\Phi - \Delta \Phi), \tag{21}
\]

with

\[
\tan \Delta \Xi = \frac{1}{V} \frac{\partial V}{\partial \Xi}, \quad \tan \Delta \Phi = \frac{1}{\sin \Xi} \frac{1}{\sin V} \frac{1}{\cos \Xi} \frac{1}{\cos V} \frac{1}{\cos \Phi} \cos \Delta \Phi \tan \Xi. \tag{22}
\]

From equations (17), (18), (21) and (22), we can compute the difference between group and phase azimuth angles defined in group and phase domains,

\[
\Phi - \phi = \Delta \Phi = \Delta \phi
\]

\[
equiv \frac{1}{\sin \Xi} \frac{1}{\sin V} \frac{1}{\cos \Xi} \frac{1}{\cos V} \frac{1}{\cos \Phi} \cos \Delta \Phi \tan \Xi
\]

and the difference between group and phase polar angles defined in group and phase domains,

\[
\Xi - \theta = a \tan \left[ \frac{1}{\sin \Xi} \frac{1}{\sin V} \frac{1}{\cos \Xi} \frac{1}{\cos V} \frac{1}{\cos \Phi} \frac{1}{\cos \Delta \Phi} \tan \Xi \right]
\]

\[
\Xi = \theta - a \tan \left[ \frac{1}{\sin \Xi} \frac{1}{\sin V} \frac{1}{\cos \Xi} \frac{1}{\cos V} \frac{1}{\cos \Phi} \frac{1}{\cos \Delta \Phi} \tan \Xi \right]
\]

\[
equiv \frac{1}{\sin \Xi} \frac{1}{\sin V} \frac{1}{\cos \Xi} \frac{1}{\cos V} \frac{1}{\cos \Phi} \frac{1}{\cos \Delta \Phi} \tan \Xi
\]

Enhancement factor

A more complex geometrical characteristic is the enhancement factor (Maris, 1971; Jaeken and Cottenier, 2016) \( A \), which can be introduced as follows:

\[
A = \frac{d\omega}{d\Omega}, \tag{26}
\]

where \( d\omega \) and \( d\Omega \) are the solid angles spanned by a set of phase velocity vectors \( \mathbf{n} \) and group velocity vectors \( \mathbf{N} \), respectively. Characteristic \( A \) from equation (26) is proportional to the ratio of the Gaussian curvatures of the phase and group velocity surfaces at corresponding points and allows to control the abnormal behaviour of these surfaces (Dubrovin et al., 1984). The enhancement factor can also be associated with the ray intensity parameter proposed by Červený (1972), geometrical spreading (Kendall and Thomson, 1989) and wave propagation metric tensor (Klimeš, 2002).

The angle \( d\omega \) has a very simple expression (Fig. 1):

\[
d\omega = \left| \frac{\partial \mathbf{n}}{\partial \theta} \times \frac{\partial \mathbf{n}}{\partial \phi} \right| d\theta d\phi = \sin \theta d\theta d\phi, \tag{27}
\]

while the angle \( d\Omega \) is given by

\[
d\Omega = \left| \frac{\partial \mathbf{N}}{\partial \theta} \times \frac{\partial \mathbf{N}}{\partial \phi} \right| d\theta d\phi, \tag{28}
\]

where \( \times \) denotes the vector product.

Substituting equations (27) and (28) into equation (26) results in

\[
A = \frac{\sin \theta}{\left| \frac{\partial \mathbf{n}}{\partial \theta} \times \frac{\partial \mathbf{n}}{\partial \phi} \right|} = \sin \theta \left| \frac{\partial \mathbf{N}}{\partial \theta} \times \frac{\partial \mathbf{N}}{\partial \phi} \right|^{-1}. \tag{29}
\]

The enhancement factor defined in equation (29) can be computed in the phase domain by taking the derivatives of
the group velocity projections over phase angles (Appendix C). After tedious algebraic manipulations, the final expression for the enhancement factor is given by

\[ A = \frac{V^3 \sin^4 \theta}{V^3 |D|} = \frac{V^3}{v^3} \sin^4 \theta \left| R_1 R_3 \sin^2 \theta - R_3^2 \right|, \]

where the terms \( R_1, R_1, \) and \( R_3 \) are defined by the second-order derivatives of the phase velocity:

\[ R_1 = 1 + \frac{\partial^2 v}{v \partial \theta^2}, \]
\[ R_2 = \sin^2 \theta + \sin \theta \cos \theta \frac{\partial v}{v \partial \theta} + \frac{\partial^2 v}{v \partial \phi^2}, \]
\[ R_3 = \frac{\partial v}{v \partial \phi} \cos \theta - \frac{\partial^2 v}{v \partial \theta \partial \phi} \sin \theta. \]

The equation for \( D \) controls the abnormal behaviour of group velocity vectors. If \( D \to 0 \) (\( A \to \infty, \ d\Omega \to 0 \)), this point results in a cusp shaping the triplication zone; if \( D \to \infty \) (\( A \to 0, \ d\Omega \to \infty \)), this point it is a singularity one.

Equation \( D = 0, \)

\[ (1 + \frac{\partial^2 v}{v \partial \theta^2}) \left( \sin^2 \theta + \sin \theta \cos \theta \frac{\partial v}{v \partial \theta} + \frac{\partial^2 v}{v \partial \phi^2} \right) \sin^2 \theta - \left( \frac{\partial v}{v \partial \phi} \cos \theta - \frac{\partial^2 v}{v \partial \theta \partial \phi} \sin \theta \right)^2 = 0 \]

is controlling the triplications in an arbitrary anisotropic medium. For the azimuthally independent medium, \( \partial v/\partial \phi = 0 \), equation (32) reduces to a well-known condition for triplication in a VTI medium (Dellinger, 1991; Thomsen and Dellinger, 2003; Roganov and Stovas, 2010),

\[ 1 + \frac{\partial^2 v}{v \partial \theta^2} = 0. \]
In the vicinity of a singularity point,

\[ D = \frac{\sin^2 \theta}{\nu^2} \left( \frac{\partial^2 \psi}{\partial \theta^2} \frac{\partial^2 \nu}{\partial \phi^2} - \left( \frac{\partial^2 \psi}{\partial \theta \partial \phi} \right)^2 \right), \tag{34} \]

being proportional to the determinant of the matrix of second-order derivatives of the phase velocity upon phase angles. The enhancement factor in vicinity of the singularity point is given by

\[ A = \frac{\left( \frac{\partial \nu}{\partial \phi} \right)^2 \sin^2 \theta + \left( \frac{\partial \nu}{\partial \theta} \right)^2}{\nu \sin \theta \left( \frac{\partial^2 \psi}{\partial \theta^2} \frac{\partial^2 \nu}{\partial \phi^2} - \left( \frac{\partial^2 \psi}{\partial \theta \partial \phi} \right)^2 \right)^{3/2}}. \tag{35} \]

Special cases

For the isotropic medium, all velocity derivatives are zero; therefore, \( \psi = 0 \), \( \Delta \theta = \Delta \phi = 0 \) and \( A = 1 \).

Transversely isotropic medium with a vertical symmetry axis

In case of the transversely isotropic medium with a vertical symmetry axis (VTI), \( \partial \nu / \partial \phi = 0 \), we have \( \Delta \Phi = \Delta \phi = 0 \), and the energy flux angle (16) is reduced to

\[ \psi = \Delta \theta = a \tan \left( \frac{1}{\nu} \frac{\partial \nu}{\partial \theta} \right) = a \tan \left( \frac{\partial V}{V} \frac{\partial \theta}{\partial \theta} \right). \tag{36} \]

The enhancement factor (30) is simplified to

\[ A = \frac{V^3 \sin \theta}{\nu^3 \left( \frac{1}{\nu} \frac{\partial^2 \psi}{\partial \phi^2} + 1 \right) \left( \sin \theta + \frac{1}{\nu} \frac{\partial \nu}{\partial \theta} \cos \theta \right)}. \tag{37} \]

where the group velocity (8) is defined by

\[ V^2 = \nu^2 \left( 1 + \left( \frac{1}{\nu} \frac{\partial \nu}{\partial \theta} \right)^2 \right). \tag{38} \]

In this case, \( \sin \theta + \frac{1}{\nu} \frac{\partial \nu}{\partial \theta} \cos \theta \geq 0 \), and the term \( \frac{1}{\nu^2} \frac{\partial^2 \psi}{\partial \phi^2} + 1 \) controls the SV wave triplications. There are no singularity points since the SH wave is decoupled from equation (3).
Phase and group velocity surfaces in anisotropic media

Figure 6 Azimuth correction angle computed for the elastic ORT model (standard) with parameters defined in equation (D1). P wave is shown to the left, S1 wave in the middle and S2 wave to the right. The positions of singularity points are indicated by black points.

Figure 7 Energy flux angle computed for the elastic ORT model (standard) with parameters defined in equation (D1). P wave is shown to the left, S1 wave in the middle and S2 wave to the right. The positions of singularity points are indicated by black points.

Symmetry planes in orthorhombic medium

For vertical symmetry planes in the ORT medium (the first-order azimuth angle derivative is zero, \( \partial v/\partial \phi = 0 \)), the angles are the same as for the VTI medium (equation 36), but the second-order derivative over the azimuth angle is not zero. Equation (B6) takes the form

\[
v_{\phi\phi} = -\left(\frac{a_{\phi\phi}v^4 + b_{\phi\phi}v^2 + c_{\phi\phi}}{2v(3v^2 + 2av^2 + b)}\right) \neq 0.
\]  

Therefore, the term \( R_3 = 0 \), and the enhancement factor is different VTI equation (37),

\[
A = v^3 \left( \frac{V^2 \sin^2 \theta}{\frac{1}{v} \frac{\partial^2 v}{\partial \phi^2} + 1} \right) \left( \sin^2 \theta + \frac{1}{v} \frac{\partial^2 v}{\partial \phi^2} + \frac{1}{v} \frac{\partial v}{\partial \theta} \sin \theta \cos \theta \right) \left( \sin^2 \theta + \frac{1}{v} \frac{\partial^2 v}{\partial \phi^2} + \frac{1}{v} \frac{\partial v}{\partial \theta} \sin \theta \cos \theta \right).
\]  

The expression in the denominator of (40) is equivalent to the one obtained in Stovas et al. (2019) for SV–SH wave

In this case, the term \( R_1 = \frac{1}{v} \frac{\partial^2 v}{\partial \phi^2} + 1 \) controls in-plane triplication, and the term \( R_2 = \sin^2 \theta + \frac{1}{v} \frac{\partial^2 v}{\partial \phi^2} + \frac{1}{v} \frac{\partial v}{\partial \theta} \sin \theta \cos \theta \) controls out-of-plane triplication. The singularity points are achieved when the derivatives of the phase velocity are not defined.

For a horizontal symmetry plane in the ORT medium (the first-order polar angle derivative is zero), the polar angles are the same. \( \Delta \theta = 0 \) and the azimuth angle equation are the same as for VTI case:

\[
\Delta \Phi = \Delta \phi = a \tan \left( \frac{1}{v} \frac{\partial v}{\partial \phi} \right) = a \tan \left( \frac{1}{V} \frac{\partial V}{\partial \Phi} \right).
\]

The second-order derivative over the polar angle,

\[
v_{\theta\theta} = -\left(\frac{a_{\theta\theta}v^4 + b_{\theta\theta}v^2 + c_{\theta\theta}}{2v(3v^2 + 2av^2 + b)}\right) \neq 0.
\]
Figure 8 Enhancement factor computed for the elastic ORT model (standard) with parameters defined in equation (D1). P wave is shown to the left, S1 wave in the middle and S2 wave to the right. The positions of singularity points are indicated by black points.

Table 1 Singularity points in the first quadrant computed for the elastic ORT model (standard) with parameters defined in equation (D1)

|   | θ, deg | φ, deg |
|---|--------|--------|
| 1 | 20.1   | 0      |
| 2 | 59.8   | 0      |
| 3 | 72.5   | 90     |
| 4 | 46.53  | 44.89  |

and the terms $R_2 = 1 + \frac{1}{v^2} \frac{\partial^2 v}{\partial \phi^2}$ and $R_3 = 0$. The enhancement factor is given by

$$A = \frac{V^3}{v^3} \left| \left(1 + \frac{1}{v^2} \frac{\partial^2 v}{\partial \phi^2}\right) \left(1 + \frac{1}{v^2} \frac{\partial^2 v}{\partial \theta^2}\right) \right|^3. \tag{43}$$

In this case, the term $1 + \frac{1}{v^2} \frac{\partial^2 v}{\partial \phi^2}$ controls in-plane triplication, and the term $1 + \frac{1}{v^2} \frac{\partial^2 v}{\partial \theta^2}$ controls out-of-plane triplication.

When velocity derivatives are not defined, we have the singularity points.

**Ellipsoidal case**

Explicit equations can be obtained for the ellipsoidal case. In this case, the phase and group velocity equations are given, respectively, by

$$v^2 = v_1^2 \sin^2 \theta \cos^2 \phi + v_2^2 \sin^2 \theta \sin^2 \phi + v_3^2 \cos^2 \theta \tag{44}$$

and

$$\frac{1}{V^2} = \frac{\sin^2 \Theta \cos^2 \Phi}{v_1^2} + \frac{\sin^2 \Theta \sin^2 \Phi}{v_2^2} + \frac{\cos^2 \Theta}{v_3^2}. \tag{45}$$

Figure 9 Azimuth correction angle computed for the monoclinic model with parameters defined in equation (D3). P wave is shown to the left, S1 wave in the middle and S2 wave to the right. The positions of singularity points are indicated by black points.
Figure 10 Energy flux angle computed for the monoclinic model with parameters defined in equation (D3). P wave is shown to the left, S1 wave in the middle and S2 wave to the right. The positions of singularity points are indicated by black points.

Figure 11 Enhancement factor computed for the monoclinic model with parameters defined in equation (D3). P wave is shown to the left, S1 wave in the middle and S2 wave to the right. The positions of singularity points are indicated by black points.

The correction angles $\Delta \phi$, $\Delta \theta$ and the energy flux $\psi$ are, respectively, defined as

$$\tan \Delta \phi = \frac{(v_2^3 - v_1^3) \sin \phi \cos \phi}{v_1^2 \cos^2 \phi + v_2^3 \sin^2 \phi}, \quad (46)$$

$$\tan \Delta \theta = \frac{\sqrt{v_4^1 \cos^2 \phi + v_4^2 \sin^2 \phi - v_1^3} \tan \theta}{v_1^3 + \tan^2 \theta \sqrt{v_4^1 \cos^2 \phi + v_4^2 \sin^2 \phi}}, \quad (47)$$

$$\tan \psi = \frac{(v_4^1 \sin^2 \theta \cos^2 \phi + v_4^2 \sin^2 \theta \sin^2 \phi + v_1^3 \cos^2 \phi)}{v_1^3 \sin^2 \theta \cos^2 \phi + v_1^3 \sin^2 \theta \sin^2 \phi + v_4^3 \cos^2 \phi \cos^2 \phi} \sin \theta, \quad (48)$$

The enhancement factor is given by

$$A = \left(\frac{v_1^3 \sin^2 \theta \cos^2 \phi + v_4^2 \sin^2 \theta \sin^2 \phi + v_1^3 \cos^2 \phi}{v_1^2 v_2^2 v_3^2}\right)^{3/2}. \quad (49)$$

Table 2 Singularity points in the first quadrant computed for a monoclinic model with parameters defined in equation (D3)

| $\theta$, deg | $\phi$, deg |
|-------------|-------------|
| 1           | 33.16       |
| 2           | 67.41       |

In this case, we do not have any triplications. In Figure 2, we show the geometrical characteristics computed from equations (46), (48) and (49) for (P wave–based) ellipsoidal velocity model with parameters given in equation (D5). One can see that the behaviour of all characteristics is smooth and simple.

Acoustic orthorhombic medium

This model was introduced in Alkhalifah (2003), Stovas (2015) and Masmoudi and Alkhalifah (2016). This special case is an acoustic ORT model (on-axes S wave phase
velocities are zero). In this case, the stiffness coefficient matrix takes the form (Abedi et al., 2019)

\[
\begin{aligned}
c_{ij} &= \begin{cases} 
v_j^2, & j = 1, 2, 3 \\
0, & j = 4, 5, 6 
\end{cases} \\
c_{jk} &= \frac{v_jv_k}{\sqrt{1+2\eta_l}}, & j, k, l = 1, 2, 3; \ j \neq k \neq l,
\end{aligned}
\]

(50)

where \(v_j\) are the on-axis P wave velocities, and \(\eta_1, \eta_2\) and \(\eta_3\) are the anellipticity parameters defined in the (x,y), (x,z) and (y,z) symmetry planes, respectively. In this case, the P wave is similar to elastic ORT case, while two S wave artefacts are completely different from their counterparts in the elastic case (Xu and Stovas, 2019b).

The characteristic equation (3) for stiffness coefficient

\[
P = v_j^2\sin^2\theta\cos^2\phi + v_j^2\sin^2\theta\sin^2\phi + v_j^2\cos^2\theta.
\]

\[
Q = \sin^2\theta \left( 2\eta_1 - v_j^2v_k^2\cos^2\theta\sin^2\phi + \frac{2\eta_2}{1+2\eta_1} \right)
\times v_j^2v_k^2\sin^2\theta\cos^2\phi + \frac{2\eta_3}{1+2\eta_2} v_j^2v_k^2\sin^2\theta\sin^2\phi\cos^2\phi,
\]

(51)

\[
R = v_j^2v_k^2v_l^2 \left( \frac{2\eta_1 + 2\eta_2 + 2\eta_3 - 8\eta_1\eta_2\eta_3}{(1+2\eta_1)(1+2\eta_2)(1+2\eta_3)} \right)
\times \sin^4\theta\cos^2\theta\sin^2\phi\cos^2\phi.
\]

In any symmetry plane, \(R = 0\), and characteristic equation reduces to

\[
v_j^2 \left( v^4 - P v^2 - Q \right) = 0,
\]

(52)

which indicates that one of S wave artefact does not exist in the symmetry plane, and another one is coupled with P wave.
Figure 14 Enhancement factor computed for the triclinic model with parameters defined in equation (D2). P wave is shown to the left, S1 wave in the middle and S2 wave to the right. The positions of singularity points are indicated by black points. Note singularity points for P–S1 and S1–S2 waves.

Table 3 Singularity points in the first quadrant computed for a triclinic model with parameters defined in equation (D2)

| θ, deg | φ, deg |
|--------|--------|
| 1      | 36.79  | 60.81  |
| 2      | 65.42  | 33.86  |
| 3      | 80.33  | 45.41  |
| 4      | 88.27  | 26.99  |

For example, the non-trivial solution of equation (52) in the horizontal symmetry plane is given by

\[
v^2 = \frac{1}{2} \left( v_1^2 \cos^2 \phi + v_2^2 \sin^2 \phi \right) \pm \sqrt{\left( v_1^2 \cos^2 \phi + v_2^2 \sin^2 \phi \right)^2 - \frac{8 \eta_3}{1 + 2 \eta_3} v_1^2 v_2^2 \sin^2 \phi \cos^2 \phi},
\]  

where the signs + and – correspond to P wave and S wave artefacts, respectively.

For the S wave artefact, in the horizontal symmetry plane, we have \( R_3 = 0, R_1 \geq 0 \) and \( R_2 < 0 \). Therefore, the S wave artefact in the symmetry plane is given by the out-of-plane triplication branch. The caustic point is defined by equation \( R_1 = 0 \) and is located at an azimuth angle of

\[
tan \phi_c = \frac{v_1}{v_2} \sqrt{\frac{(1 + 2 \eta_2)}{(1 + 2 \eta_1)}} \left( 1 + 2 \eta_1 \right) \left( 1 + 2 \eta_2 \right). 
\]  

The caustic condition is given by inequalities

\[
\frac{1}{1 + 2 \eta_1} < \frac{1 + 2 \eta_2}{1 + 2 \eta_1} < 1 + 2 \eta_3.
\]  

The singularity point is located at \( \theta = 0 \), and there are no other singularity points in symmetry planes since only one S wave artefact exists there.

Similar analysis can be performed for vertical symmetry planes.

The azimuth correction angle (equation (19)) computed for the acoustic ORT model (standard) with parameters defined in equation (D4) is shown in Figure 3, and the energy flux angle (equation (15)) is shown in Figure 4. The enhancement factor for acoustic ORT can be seen in Figure 5. The singularity point is clearly seen at phase angles \( \theta = 62.1^\circ \) and \( \phi = 16.8^\circ \) (the position can be computed analytically, see Xu and Stovas (2019b)) for both S1 and S2 waves. The cusps are identified at \( \phi = 0 \) and \( \theta = 90^\circ \) and \( \phi = 90^\circ \) and \( \theta = 90^\circ \) for the S1 wave. The S1 and S2 waves in the acoustic orthorhombic model are considered as artefacts. These artefacts are given by triplication sheets.

For the elastic ORT model, the azimuth correction angle, energy flux angle and enhancement factor are shown in Figures 6, 7 and 8, respectively. From the azimuth correction angle and energy flux angle plots (Fig. 6), we can see that the accuracy of the weak-anisotropy approximation is very low for this model. The maximum energy flux angle for P waves reaches 16 degrees, and the one for S1 waves reaches 25 degrees. The singularity points (Table 1) are clearly seen on all characteristics computed for S waves by abrupt increase in density of the contour lines. One can see that results are very different for elastic and acoustic models. The largest difference is observed for S1 and S2 waves (since the S waves in acoustic ORT models are considered as artefacts). For low symmetry models (monoclinic and triclinic), we still plot (for simplicity) only the first quadrant keeping in mind that monoclinic model...
is not symmetric in azimuth angle, and triclinic model has no
symmetry in the polar and azimuth angles. Azimuth correction
angle, energy flux angle and enhancement factor computed for
the monoclinic model are shown in Figures 9, 10 and 11,
respectively. The accuracy of weak-anisotropy approximation is
even worse for this model. The maximum energy flux angle is
30 degrees (P wave) and 45 degrees (S waves). Two singularity
points (Table 2) are clearly seen in all the S wave plots. The
azimuth angle correction factor (Fig. 9) also assists in identifying
the triplication zone clearly seen at around $\theta = 30^\circ$ (there
is a jump in the azimuth correction factor from $-90$ to $90$
degrees). Azimuth correction angle, energy flux angle and en-
hancement factor computed for the triclinic model are shown
in Figures 12, 13 and 14, respectively. The week-anisotropy
approximation is extremely inaccurate (maximum energy flux
angles are 45, 60 and 70 degrees for P, S1 and S2 waves, re-
respectively. For this model, we notice the singularity points for
P and S1 waves and the singularity points for S1 and S2 waves
(Table 3). The triplication zones are very complex in shape (es-
pecially for S2 waves).

CONCLUSIONS

We derived geometrical characteristics of the phase and group
velocity surfaces in anisotropic media. The energy flux angle is
decomposed into the polar and azimuth correction angles. The
new characteristic, enhancement factor is derived and used to
control different features of the surfaces like singularity points and
triplications. This factor provides an easy way to identify the
complications in the group velocity surface. From the en-
hancement factor, a simple equation controlling triplications is
derived. This equation is split into three factors responsible for
in-plane and out-of-plane triplications in the group velocity
surface.

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DATA AVAILABILITY STATEMENT

The data that support the findings of this study are available
from the corresponding author upon reasonable request.

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APPENDIX A: PHASE VELOCITIES

The solutions of the cubic equation for phase velocity squared (3),

\[ v^6 + av^4 + bv^2 + c = 0, \]  

has three roots corresponding to P, S1 and S2 wave modes. In case of three real roots, it is convenient to use the trigonometric solution for equation (A1) (Korn and Korn, 1968),

\[
v_P^2 = 2 \sqrt{-\frac{d}{3}} \cos \left( \frac{1}{3} \arccos \left( -\frac{q}{2} \sqrt{-\frac{27}{d^3}} \right) \right) - \frac{a}{3},
\]

\[
v_{S1}^2 = 2 \sqrt{-\frac{d}{3}} \cos \left( \frac{1}{3} \arccos \left( -\frac{q}{2} \sqrt{-\frac{27}{d^3}} + \frac{2\pi}{3} \right) \right) - \frac{a}{3},
\]

\[
v_{S2}^2 = 2 \sqrt{-\frac{d}{3}} \cos \left( \frac{1}{3} \arccos \left( -\frac{q}{2} \sqrt{-\frac{27}{d^3}} + \frac{4\pi}{3} \right) \right) - \frac{a}{3},
\]

where

\[
d = -\frac{a^2}{3} + b,
\]

\[
q = \frac{2a^3}{27} - \frac{ab}{3} + c.
\]

APPENDIX B: PHASE VELOCITY DERIVATIVES

Equation (3) is the cubic equation for phase velocity squared,

\[ v^6 + av^4 + bv^2 + c = 0, \]  

with coefficients defined from the Christoffel matrix, \( a = -\text{tr}(\Gamma), b = \text{tr}(\text{cot}(\Gamma)) \) and \( c = -\det(\Gamma) \). The notations are explained in the main text.

These coefficients depend on the polar and azimuth phase angles \( \theta \) and \( \phi \).

Differentiation of equation (B1) on the polar angle gives

\[ 6v^5 \nu_\theta + a_\theta v^4 + 4av^3 \nu_\theta + b_\theta v^2 + 2bv_\theta + c_\theta = 0. \]  

From equation (B2), we obtain

\[ \nu_\theta = -\frac{a_\theta v^4 + b_\theta v^2 + c_\theta}{2v (3v^4 + 2av^2 + b)}. \]
In a similar way, the velocity derivative on the azimuth angle can be computed as

\[ v_\phi = -\frac{a_\phi v^4 + b_\phi v^2 + c_\phi}{2v (3v^4 + 2av^2 + b)}. \]  
(B4)

The second-order derivatives can be obtained from the differentiation of equation (B2):

\[ v_{\theta\theta} = -\left( a_{\theta\theta} v^4 + b_{\theta\theta} v^2 + c_{\theta\theta} \right) + 4v v_\theta \left( 2a_{\theta v} v^2 + b_{\theta} \right) + 2v^2 (15v^4 + 6av^2 + b) \frac{1}{2v (3v^4 + 2av^2 + b)}. \]  
(B5)

In a similar way,

\[ v_{\phi\phi} = -\left( a_{\phi\phi} v^4 + b_{\phi\phi} v^2 + c_{\phi\phi} \right) + 4v v_\phi \left( 2a_{\phi v} v^2 + b_{\phi} \right) + 2v^2 (15v^4 + 6av^2 + b) \frac{1}{2v (3v^4 + 2av^2 + b)}. \]  
(B6)

The cross-derivative is given by

\[ v_{\theta\phi} = -\left( a_{\theta\phi} v^4 + b_{\theta\phi} v^2 + c_{\theta\phi} \right) + 2v v_\theta \left( 2a_{\theta v} v^2 + b_{\theta} \right) + 2v v_\phi \left( 2a_{\phi v} v^2 + b_{\phi} \right) + 2v_\theta v_\phi (15v^4 + 6av^2 + b) \frac{1}{2v (3v^4 + 2av^2 + b)}. \]  
(B7)

In a singularity point, all the velocity derivatives given in equations (B3)–(B7) go to infinity. This condition is reduced to

\[ 3v^4 + 2av^2 + b = 0, \]  
(B8)

with solutions

\[ v_{1,2}^2 = \frac{1}{3} \left( -a \pm \sqrt{a^2 - 3b} \right) = \frac{1}{3} \left( -a \pm \sqrt{-3d} \right), \]  
(B9)

where \( d \) is defined in equation (A3). Velocity \( v_2 \) satisfies the condition \( v_{32} \leq v_2 \leq v_{31} \) and can approximately be considered as average S wave velocity.

Eliminating phase velocity from equations (B1) and (B8) results in the following expression:

\[ F = -4a^3 + a^2 b^2 - 4b^3 + 18abc - 27c^2 = 0, \]  
(B10)

which is the well-known condition for multiple roots of the cubic equation.

**APPENDIX C: GROUP VELOCITY DERIVATIVES**

The group velocity projections are given in equation (7),

\[ V_1 = v \sin \theta \cos \phi + \frac{\partial v}{\partial \theta} \cos \theta \cos \phi - \frac{\partial v}{\partial \phi} \sin \phi \sin \theta, \]

\[ V_2 = v \sin \theta \sin \phi + \frac{\partial v}{\partial \theta} \cos \theta \sin \phi + \frac{\partial v}{\partial \phi} \cos \phi \sin \theta, \]

\[ V_3 = v \cos \theta - \frac{\partial v}{\partial \theta} \sin \theta. \]  
(C1)
The derivatives of these projections over the phase angles can be directly computed from equations (C1):

\[
\frac{\partial V_1}{\partial \theta} = v \cos \theta \cos \phi + \frac{\partial v}{\partial \phi} \cos \theta \sin \phi - \frac{\partial^2 v}{\partial \theta \partial \phi} \sin \phi \sin \theta + \frac{\partial^2 v}{\partial \phi^2} \cos \theta \cos \phi,
\]

\[
\frac{\partial V_1}{\partial \phi} = -v \sin \theta \sin \phi - \frac{\partial v}{\partial \theta} \cos \theta \sin \phi - \frac{\partial v}{\partial \phi} \cos^2 \theta \cos \phi \sin \theta + \frac{\partial^2 v}{\partial \theta \partial \phi} \cos \phi \sin \theta + \frac{\partial^2 v}{\partial \phi^2} \sin \phi \sin \theta,
\]

\[
\frac{\partial V_2}{\partial \theta} = v \cos \theta \sin \phi - \frac{\partial v}{\partial \phi} \cos \theta \cos \phi \sin^2 \theta + \frac{\partial^2 v}{\partial \theta \partial \phi} \cos \phi \sin \theta + \frac{\partial^2 v}{\partial \phi^2} \cos \theta \sin \phi,
\]

\[
\frac{\partial V_2}{\partial \phi} = v \sin \theta \cos \phi + \frac{\partial v}{\partial \theta} \cos \theta \cos \phi - \frac{\partial v}{\partial \phi} \cos^2 \theta \sin \phi \sin \theta + \frac{\partial^2 v}{\partial \theta \partial \phi} \cos \theta \sin \phi + \frac{\partial^2 v}{\partial \phi^2} \cos \phi \sin \phi \sin \phi.
\]

\[
\frac{\partial V_3}{\partial \theta} = -v \sin \theta - \frac{\partial^2 v}{\partial \theta^2} \sin \theta,
\]

\[
\frac{\partial V_3}{\partial \phi} = \frac{\partial v}{\partial \phi} \cos \theta - \frac{\partial^2 v}{\partial \theta \partial \phi} \sin \theta.
\]

After algebraic manipulations, we obtain equation (30)

\[
A = \frac{V_3^3 \sin^3 \theta}{\sin^2 \theta \left( 1 + \frac{1}{v^2} \frac{\partial v}{\partial \phi} \right) \left( \sin^2 \theta + \sin \theta \cos \theta \frac{\partial v}{\partial \theta} + \frac{1}{v^2} \frac{\partial^2 v}{\partial \phi^2} \right) - \left( \frac{1}{v^2} \frac{\partial v}{\partial \phi} \cos \theta + \frac{1}{v^2} \frac{\partial^2 v}{\partial \theta \partial \phi} \sin \theta \right)^2}.
\]

**APPENDIX D: ANISOTROPIC MODELS**

In order to illustrate the derived geometrical characteristics, we select the models with different symmetries.

The ORT model (model standard) is defined by the density normalized stiffness matrix (Schoenberg and Helbig, 1997):

\[
C_{\text{ORT}} = \begin{pmatrix}
9 & 3.6 & 2.25 \\
9.84 & 2.4 & \\
5.938 & 2 & 1.6 \\
2.182 & &
\end{pmatrix}
\]

The triclinic model is adopted from Vavryčuk (2005a):

\[
C_{\text{TRI}} = \begin{pmatrix}
137 & 52 & 57 & -13 & 32 & -20 \\
147 & 18 & -6 & 20 & -9 & \\
100 & 22 & -15 & 5 & \\
52 & 26 & -7 & & \\
75 & -40 & & & & \\
30 & & & & &
\end{pmatrix}
\]
and monoclinic (with a horizontal symmetry plane) is obtained from triclinic one by setting corresponding stiffness coefficients to zero,

\[
C_{\text{MONO}} = \begin{pmatrix}
137 & 52 & 57 & 0 & 0 & -20 \\
147 & 18 & 0 & 0 & -9 \\
100 & 0 & 0 & 5 \\
52 & 26 & 0 & 0 & 5 \\
75 & 0 & & & \\
30 & & & & \\
\end{pmatrix}.
\] (D3)

The acoustic orthorhombic model is obtained from the standard orthorhombic (defined in equation (D1), based on equations (43),

\[
C_{\text{ORT2}} = \begin{pmatrix}
\frac{v_1^2}{\sqrt{1 + 2\eta_3}} & \frac{v_1 v_3}{\sqrt{1 + 2\eta_3}} & \frac{v_1 v_3}{\sqrt{1 + 2\eta_3}} \\
\frac{v_2^2}{\sqrt{1 + 2\eta_2}} & \frac{v_2 v_3}{\sqrt{1 + 2\eta_2}} & \frac{v_2 v_3}{\sqrt{1 + 2\eta_2}} \\
\frac{v_3^2}{\sqrt{1 + 2\eta_1}} & \frac{v_1 v_3}{\sqrt{1 + 2\eta_1}} & \frac{v_1 v_3}{\sqrt{1 + 2\eta_1}} \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
\end{pmatrix}
\end{pmatrix} = \begin{pmatrix}
9 & 7.05 & 5.46 \\
9.84 & 6.41 \\
5.938 & & \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
\end{pmatrix}.
\] (D4)

The ellipsoidal model (for P wave) can be obtained from the acoustic orthorhombic model given by equation (D4) by setting all anelliptic parameters to zero,

\[
C_{\text{ORTe}} = \begin{pmatrix}
\frac{v_1^2}{\sqrt{1 + 2\eta_3}} & \frac{v_1 v_3}{\sqrt{1 + 2\eta_3}} & \frac{v_1 v_3}{\sqrt{1 + 2\eta_3}} \\
\frac{v_2^2}{\sqrt{1 + 2\eta_2}} & \frac{v_2 v_3}{\sqrt{1 + 2\eta_2}} & \frac{v_2 v_3}{\sqrt{1 + 2\eta_2}} \\
\frac{v_3^2}{\sqrt{1 + 2\eta_1}} & \frac{v_1 v_3}{\sqrt{1 + 2\eta_1}} & \frac{v_1 v_3}{\sqrt{1 + 2\eta_1}} \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
\end{pmatrix} = \begin{pmatrix}
9 & 9.41 & 7.31 \\
9.84 & 7.64 \\
5.938 & & \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
\end{pmatrix}.
\] (D5)