Quantizing the homogeneous linear perturbations about Taub using the Jacobi method of second variation and the method of invariants

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Abstract
Applying the Jacobi method of second variation to the Bianchi IX system in Misner variables \((\alpha, \beta, \gamma)\), we specialize to the Taub space background \((\beta = 0)\) and obtain the governing equations for linearized homogeneous perturbations \((\alpha', \beta', \gamma')\) thereabout. Employing a canonical transformation, we isolate two decoupled gauge-invariant linearized variables \((\beta'\text{ and } Q'=\rho_\alpha\alpha+\rho_\beta\beta)\), together with their conjugate momenta and linearized Hamiltonians. These two linearized Hamiltonians are of time-dependent harmonic oscillator form, and we quantize them to get time-dependent Schrödinger equations. For the case of \(Q'\), we are able to solve for the discrete solutions and the exact quantum squeezed states.

Keywords: canonical quantization, perturbative approach, method of invariants

1. Introduction

The Taub space is a special case of Bianchi IX; unlike Bianchi IX, whose exact solutions are known only asymptotically, the Taub metric is an exact solution to Einstein’s equations [1, 2]. In our previous work, we demonstrated the application of the modified semi-classical method to the canonically quantized vacuum Bianchi IX (Mixmaster) model, solving the relevant Wheeler–DeWitt equation asymptotically by integrating a set of linear transport equations along the flow of the Moncrief-Ryan (or ‘wormhole’) solution to the corresponding Euclidean-signature Hamilton–Jacobi equation [3]. We found that the excited state solutions, peaked away from the minisuperspace origin, are labeled by a pair of positive integers that can be plausibly interpreted as graviton excitation numbers for the two independent anisotropy degrees of freedom \((\beta_+, \beta_-)\).
In this paper, we seek to give a more intuitive explanation for the excited states found in our previous work by taking a different approach to the approximate solutions of the Bianchi IX model. To this end, we use the Jacobi method of second variation to study the homogeneous linearized perturbations about the Taub background solutions (For the method applied to the Schwarzschild metric, see [4].) Because the Taub background solutions are known exactly, we can write down the explicit solutions of the linearized Hamilton equations for the linear perturbations that stay within the Taub model. These explicit solutions, together with the method of invariants [5], allow us to solve for the quantum squeezed states for the quantized linearized Hamiltonians.

The paper is organized as follows: we start with ADM action of the background Bianchi IX expressed in Misner variables, paying special attention to the background Hamiltonian constraint. Next, we apply the Jacobi method of second variation, obtaining the linearized constraint and the linearized Hamiltonian for the perturbation variables. A canonical transformation decouples the linearized Hamiltonian into gauge-invariant and gauge-dependent parts. Of the gauge-invariant part of the linearized Hamiltonian, we identify the part that stays within the Taub family of solutions, writing down the explicit time-dependences of the new gauge-invariant perturbation variable from the exact solutions for the background Taub space. Finally, we use the method of invariants to calculate the quantum squeezed states from the exact classical solutions.

2. Taub models using Misner variables

We can express the dimensionless line element of Bianchi IX models using Misner variables \( \{\alpha, \beta_+, \beta_\perp\} \). The scale parameter \( \alpha(t) \) gives the size of the three-dimensional hyper surface relative to its initial size \( l_0 \); the anisotropy parameters \( \beta_+(t) \) and \( \beta_-(t) \) describe the anisotropy of the hyper surface. Together, they define the Bianchi IX minisuperspace. We set \( \beta_+(t) = 0 \) to describe the Taub family of models. We also divide the line element by \( r^2 = \frac{l^2}{6\alpha} \), and define a dimensionless time parameter \( \tau = \frac{t}{r} \);

\[
\frac{ds^2}{r^2} = -N^2\, dt^2 + e^{2\alpha} \left( e^{2\beta_+} 0 0 \right) \sigma' \sigma',
\]

where \( \sigma' \)'s are the invariant differential one-forms on the \( S^3 \) manifold satisfying \( d\sigma' = \frac{1}{2} e_{ijk} \, \sigma^i \wedge \sigma^j \wedge \sigma^k \). Following the method outlined in [3], we write the ADM action (where \( p_\alpha \) and \( p_+ \) are conjugate momenta):

\[
S_{\text{ADM}} = \int \left( p_\alpha \dot{\alpha} + p_+ \dot{\beta}_+ - NH_{\perp} \right) \, d\tau.
\]

Variation of the lapse function \( N \) leads to the Hamiltonian constraint

\[
H_{\perp} = -\rho \, e^{-3\alpha} \left[ \gamma - \frac{1}{\rho} \, e^{4\alpha} \left( e^{-8\beta_+} - 4 \, e^{-2\beta_\perp} \right) \right] = 0.
\]

We introduce the following abbreviations used throughout this paper:

\[
\rho = \frac{G}{24\pi r^3}, \quad \lambda = \frac{\pi r}{2G}.
\]
\[ \gamma_\pm = \pm \rho_0^2 \mp p_\pm^2. \]  

### 3. Jacobi method of second variation

The Jacobi method of second variation is a completely general variational method for the study of any perturbations (to linear order) about any family of background solutions (for a similar approach applied to the Schwarzschild metric, see [4]). In this paper, we apply the Jacobi method of second variation to write the linearized Hamiltonian for homogeneous perturbations about the background Taub solution. This is equivalent to a linear approximation about the Taub solution (denoted by the ‘0’ subscript):

\[ \alpha = \alpha_0 + \alpha', \]
\[ \beta_+ = \beta_{+0} + \beta_+ ', \]
\[ \rho_+ = \rho_{+0} + \rho_+ ', \]
\[ p_+ = p_{+0} + p_+ ', \]
\[ \mathcal{N} = \mathcal{N}_0 + \mathcal{N}'. \]

When applied to our ADM Hamiltonian formulation, the approximation yields a linearized constraint (we drop the ‘0’ subscript for clarity) and a linearized Hamiltonian for the variations. In effect, we obtain another variational principle whose Hamilton equations govern the homogeneous perturbations about the exact Taub solution.

Another way to think about the linear approximation is to consider each ‘background’ variable as a function of an extra parameter (say \( e \)), and define the ‘primed’ notation as:

\[ q' = \frac{\partial q(\tau, e)}{\partial e} \bigg|_{e=0}. \]

This is exactly what is done later in equations (50) and (52). Similarly, we can define the ‘double primed’ notation as:

\[ q'' = \frac{\partial^2 q(\tau, e)}{\partial e^2} \bigg|_{e=0}. \]

Write the resulting linearized action:

\[ S_{ADM}^* = 2 \int \left( \rho' \dot{\alpha}' + p'_+ \dot{\beta}_+ ' - H^* \right) \, d\tau, \]

where the linearized constraint and the linearized Hamiltonian are:

\[ H^* = \mathcal{N}' \mathcal{H}_\perp^* + \frac{\lambda}{2} \mathcal{N} \mathcal{H}_\perp^*, \]
\[ H_\perp^* = -\rho \, e^{-2\lambda} \left[ \gamma_- - 4 \frac{\lambda}{\rho} \, e^{4\lambda} f_A \left( \alpha' - \frac{2\rho}{f_A} \beta_+ ' \right) \right]. \]
\[ H^*_\pm = 2\rho e^{-3\alpha}
\left[-p'_\alpha^2 + p'_\beta^2 + 3\gamma'_\alpha \alpha'
+ 4\gamma_\alpha \left(-\frac{2\kappa}{f_\alpha} (\alpha')(\beta'_\alpha) + \frac{2\kappa}{f_\alpha} (\beta'_\alpha)^2\right)\right]. \] (17)

Variation of the linearized lapse function \( N' \) leads to the linearized constraint \( H'_\pm = 0 \). In the above equations and throughout this paper, the following abbreviations are used:

\[ f_A = e^{-8\phi_{\alpha}} - 4 e^{-2\phi_{\beta}}, \] (18)
\[ f_B = e^{-8\phi_{\beta}} - e^{-2\phi_{\beta}}, \] (19)
\[ f_C = 4 e^{-8\phi_{\alpha}} - e^{-2\phi_{\beta}}. \] (20)

Note that the linearized action does not involve the terms \( (N'', \alpha'', \beta''_{\alpha}, \beta''_{\beta}) \) or their conjugate momenta, as these are multiplied by terms that vanish identically when the background constraint and equations of motions are taken into account. For example, the term \( N'' \) would be multiplied by the background Hamiltonian \( H_{\perp} \), which is zero by equation (3). Note also that the linearized constraint equation (16) involves \( (\alpha', \beta'_{\alpha}) \) and their conjugate momenta only.

### 3.1. Decoupling the linearized Hamiltonian

The linearized Hamiltonian \( H^*_\pm \), together with the linearized constraint \( H'_\pm = 0 \), govern the equations of motion for the perturbations

\[ H''\big|_{H'_\pm = 0} = \frac{1}{2} NH^*_\pm = H^*_\pm, \] (21)

\[ H^*_\pm = \rho N e^{-3\alpha}
\left[-(p'_\alpha)^2 + (p'_\beta)^2 + 3\gamma'_\alpha \alpha'
+ 4\gamma_\alpha \left(-\frac{2\kappa}{f_\alpha} (\alpha')(\beta'_\alpha) + \frac{2\kappa}{f_\alpha} (\beta'_\alpha)^2\right)\right]. \] (22)

\( H^*_\pm \) can be decoupled into a gauge-invariant part and a gauge-dependent part through a canonical transformation from \( q'_{ij} = \{\alpha', \beta'_{\alpha}, \beta'_{\beta}\} \) to \( Q'_{ij} = \{Q'_\alpha, Q'_\beta\} \) such that one of the two new conjugate momenta \( (P'_\pm) \) is proportional to \( H'_\pm \): 

\[ Q'_i = \Lambda_{ij} q'_j, \] (23)

\[ P'^i = p'^i \Lambda^{-1} - \frac{\partial F}{\partial Q'_i}. \] (24)

\[ \Lambda = \begin{pmatrix} p'_\alpha & p'_\beta \\ -p'_\alpha & p'_\beta \end{pmatrix}, \] (25)

\[ K^*_\Lambda = H'_\Lambda(P', Q') + \frac{\partial F}{\partial q'} \Lambda_{ij} \left(\Lambda^{-1}\right)^k_i Q'_k. \] (26)
To summarize, we used a canonical transformation on \( H_\alpha^p \) to write \( K^{\alpha}_N(Q^\alpha, P^\alpha) \) such that one of the new conjugate momenta \( P^\alpha_\perp \) is directly proportional to the linearized constraint \( H^\perp_\alpha \). Let us write out \( K^\alpha_N \) in full. Note that all the parts multiplied by \( \rho \gamma \) go to zero when we impose the linearized constraint \( H^\perp_\alpha = 0 \).

\[
K^\alpha_N = \rho N \ e^{-3\alpha} \left[ \gamma N P^\alpha_\parallel^2 - \frac{72}{\tau_\alpha} \ e^{-10\beta^\alpha_\parallel Q^\alpha_\parallel^2} \right] + \rho N \ e^{-3\alpha} \left[ \frac{2}{\tau_\alpha} \left( 5 \rho^\alpha_\parallel - 3 \rho^\alpha_\parallel - 8 \frac{2\rho^\alpha_\parallel}{\tau_\alpha} \rho^\alpha_\perp \rho^\alpha_\perp \right) \rho^\alpha_\perp Q^\alpha_\perp^2 \right. + \frac{2}{\tau_\alpha} \left( 7 \rho^\alpha_\perp \rho^\alpha_\perp - 3 \rho^\alpha_\perp \rho^\alpha_\perp + \frac{4 \rho^\alpha_\parallel}{\tau_\alpha} \rho^\alpha_\parallel \rho^\alpha_\parallel \right) Q^\alpha_\perp - \left. \frac{2 \rho^\alpha_\parallel}{\tau_\alpha} + \frac{2 \rho^\alpha_\parallel}{\tau_\alpha} \rho^\alpha_\parallel - \frac{2 \rho^\alpha_\parallel}{\tau_\alpha} \right].
\]

(28)

Thus, we have isolated the gauge-invariant part of the linearized Hamiltonian through a canonical transformation \( H^\alpha_N(\alpha^\alpha_\parallel, \beta^\alpha_\parallel, \rho^\alpha_\parallel, \rho^\alpha_\perp) \rightarrow K^\alpha_N(Q^\alpha, P^\alpha, P^\alpha_\perp) \), producing a time-dependent harmonic oscillator linearized Hamiltonian for \( Q^\alpha = \rho^\alpha_\parallel \alpha^\alpha_\parallel + \rho^\alpha_\parallel \beta^\alpha_\parallel \): 

\[
K^\alpha_N \mid_{P^\alpha_\perp=0} = \rho N \ e^{-3\alpha} \left[ \gamma N P^\alpha_\parallel^2 - \frac{72}{\tau_\alpha} \ e^{-10\beta^\alpha_\parallel Q^\alpha_\parallel^2} \right].
\]

(29)

Comparing with the standard time-dependent harmonic oscillator equation, we arrive at 

\[
K^\alpha_N = \frac{P^\alpha_\parallel^2}{2M} + \frac{1}{2} M \omega^2 Q^\alpha_\parallel^2,
\]

(30)

\[
M = \frac{G}{\pi \ e^{4N\gamma^\alpha}} ,
\]

(31)

\[
\omega^2 = -\frac{6N^2 \ e^{-10\beta^\alpha}}{\rho \ e^{2\alpha}}.
\]

(32)

Since the overall sign of a Hamiltonian can be changed without affecting the underlying equations of motion, we can assume without loss of generality that \( M \) and \( \omega^2 \) are real and positive for the range of values of \( \{ k, \tau, \tau_\parallel \} \) that we study. Later, when we set \( k = 1 \) and \( \tau_\parallel = 0 \), it can be shown that \( M \) and \( \omega^2 \) are real and positive for all values of \( \tau \). Moreover, even when \( \omega^2 \) changes sign, the results in this section is equally valid if \( \omega \) is imaginary; all that is necessary is that \( \omega^2 \) be real [5]. We now proceed to solve this gauge-invariant linearized Hamiltonian for \( Q^\alpha_\parallel \) explicitly. To do so, we will first need to use an explicit solution to the background Taub family.

3.2. Exact background Taub

The solution for the Taub metric as given in Taub’s 1951 paper [1] uses a notation and gauge that are similar to those used by Landau and Lifshitz [6] as well as others [2, 7] in describing
the diagonalized Bianchi IX metric. Write the dimensionless line element (after dividing by $r^2$):

$$ds^2 = -(abc)^2 dr^2 + \begin{pmatrix} a^2 & 0 & 0 \\ 0 & b^2 & 0 \\ 0 & 0 & c^2 \end{pmatrix} \sigma^i \sigma^j.$$  \hspace{2cm} (33)

The solutions are as follows:

$$a^2 = \frac{2k}{\cosh[2k(\tau - \tau_1)]},$$ \hspace{2cm} (34)

$$b^2 = c^2 = \frac{k \cosh[2k(\tau - \tau_1)]}{2\cosh[2k(\tau - \tau_2)]}^2,$$ \hspace{2cm} (35)

$$N_i = \sqrt{\frac{k^3 \cosh[2k(\tau - \tau_1)]}{2\cosh[2k(\tau - \tau_2)]^2}}.$$ \hspace{2cm} (36)

Here $k$, $\tau_1$, and $\tau_2$ are arbitrary, real parameters; $\tau$ could be shifted to absorb $\tau_2$, leaving two independent dimensionless parameters $k$ and $\tau_1$ (We set $\tau_2 = 0$ from here on.). In this gauge ($N_i = e^{3\omega}$), the Misner variables $\alpha$ and $\beta$ have the following $\tau$-dependence:

$$e^{6\alpha} = \frac{k^3 \cosh[2k(\tau - \tau_1)]}{2\cosh[k\tau]^4},$$ \hspace{2cm} (37)

$$e^{3\beta} = \frac{\cosh[2k(\tau - \tau_1)]}{2 \cosh k\tau},$$ \hspace{2cm} (38)

$$e^{2\alpha-\beta} = \frac{k}{\cosh k\tau}.$$ \hspace{2cm} (39)

4. \textbf{$Q^\tau$ Hamiltonian}

Now let us write the linearized Hamiltonian with all the explicit $\tau$-dependences:

$$K^\tau_A = \frac{k^2}{12\rho} \left[ \tanh^2 \left[ \frac{2k(\tau - \tau_1)}{2} \right] - \tanh^2 k\tau \right] P_+^2$$

$$- 18\rho \left( \frac{\cosh k\tau \cosh[2k(\tau - \tau_1)]}{\cosh^2 k\tau - \cosh^2[2k(\tau - \tau_1)]} \right)^2 Q_+^2.$$ \hspace{2cm} (40)

Comparing with the standard time-dependent harmonic oscillator equation, we arrive at

$$K^\tau_A = \frac{P_+^2}{2M(\tau)} + \frac{1}{2} M(\tau) \omega^2(\tau) Q_+^2,$$ \hspace{2cm} (41)

$$M = \frac{-6\rho}{k^2 \left[ \tanh^2 \left[ \frac{2k(\tau - \tau_1)}{2} \right] - \tanh^2 k\tau \right]}.$$ \hspace{2cm} (42)
\( \omega^2 = -6k^2 \left[ \tanh^2 \left( 2k (\tau - \tau_1) \right) - \tanh^2 k\tau \right] \)

\[
\times \left( \frac{\cosh k\tau \cosh[2k(\tau - \tau_1)]}{\cosh^2 k\tau - \cosh^2[2k(\tau - \tau_1)]} \right)^2.
\]

(43)

Following [8], we transform this to \( H_1(\tau) \)

\[
H_1 = \frac{p^2}{2m} + \frac{1}{2} m\Omega^2(\tau)q^2.
\]

where

\[
\Omega^2 = \omega^2 + 1 \left( \frac{M}{M} \right)^2 - \frac{1}{2} \left( \frac{M}{M} \right)
\]

(45)

and the new canonical variables are related to the old by:

\[
q' = \left( \frac{M(\tau)}{m} \right)^{1/2} Q_+.
\]

(46)

\[
q' = \left( \frac{M(\tau)}{m} \right)^{1/2} \frac{1}{2\rho} \left[ \frac{\partial \beta^*}{\partial \tau} + \frac{\partial \alpha}{\partial \tau} - \frac{\partial \alpha}{\partial \tau_1} - \frac{\partial \beta^*}{\partial \tau_1} \right].
\]

(47)

\[
p' = \left( \frac{m}{M(\tau)} \right)^{1/2} \left( P_+ + \frac{1}{2} (mM(\tau))^{1/2} \frac{M}{M} \right).
\]

(48)

Now, from our explicit equations for \( \alpha \) and \( \beta^* \) as given in equations (37) and (38), we can write down two independent explicit \( \tau \)-dependences of the linear perturbation \( q' \) in equation (47) by simply differentiating \( \alpha \) and \( \beta^* \) with respect to the two parameters \( \tau_1 \) and \( k \) (recall our definition earlier in equation (12)).

\[
q' \left[ \frac{\partial}{\partial \tau_1} \right] = \left( \frac{M}{m} \right)^{1/2} \frac{1}{2\rho} \left[ \frac{\partial \beta^*}{\partial \tau} \frac{\partial \alpha}{\partial \tau_1} - \frac{\partial \alpha}{\partial \tau} \frac{\partial \beta^*}{\partial \tau_1} \right]
\]

(49)

\[
q' \left[ \frac{\partial}{\partial k} \right] = \frac{ik}{\sqrt{6}\rho} \frac{\tanh k\tau \tanh[2k(\tau - \tau_1)]}{\left( \tanh^2[2k(\tau - \tau_1)] - \tanh^2 k\tau \right)^{1/2}}
\]

(50)

\[
q' \left[ \frac{\partial}{\partial k} \right] = \left( \frac{M}{m} \right)^{1/2} \frac{1}{2\rho} \left[ \frac{\partial \beta^*}{\partial \tau} \frac{\partial \alpha}{\partial k} - \frac{\partial \alpha}{\partial \tau} \frac{\partial \beta^*}{\partial k} \right]
\]

(51)
\[ \frac{1}{2k\sqrt{6}\rho} \tanh k\tau - 2 \tanh \left[ 2k \left( \tau - \tau_1 \right) \right] \]

\[ + \frac{i}{2k\sqrt{6}\rho} \frac{\tau_1}{ \frac{1}{2} \left( \tanh^2 \left[ 2k \left( \tau - \tau_1 \right) \right] - \tanh^2 k\tau \right)^{\frac{1}{2}}} \]  

\[ \quad \text{(52)} \]

By a linear combination of these two independent \( q' \) functions (let us call them \( q_{i\tau} \) and \( q_k \) respectively), we define two independent functions \( \alpha_1 \) and \( \alpha_2 \):

\[ \alpha_1 = \frac{\sqrt{6}\rho}{ik} q_{i\tau}', \quad \text{(53)} \]

\[ \alpha_1 = \frac{\tanh k\tau \tanh \left[ 2k \left( \tau - \tau_1 \right) \right]}{ \left( \tanh^2 \left[ 2k \left( \tau - \tau_1 \right) \right] - \tanh^2 k\tau \right)^{\frac{1}{2}}} \quad \text{(54)} \]

\[ \alpha_2 = \frac{k\sqrt{6}\rho}{i} \left( q_{i\tau}' - \frac{\tau_1}{k} q_k \right) \quad \text{(55)} \]

\[ \alpha_2 = \frac{1}{2} \frac{\tanh k\tau - 2 \tanh \left[ 2k \left( \tau - \tau_1 \right) \right]}{ \left( \tanh^2 \left[ 2k \left( \tau - \tau_1 \right) \right] - \tanh^2 k\tau \right)^{\frac{1}{2}}} \quad \text{(56)} \]

By construction, then, \( \alpha_i \) are solutions to the second order Euler–Lagrange equation that results from the time-dependent harmonic oscillator Hamiltonian equation (44):

\[ \dot{\alpha}_i + \Omega^2 \alpha_i = 0. \quad \text{(57)} \]

We also note that they satisfy the following initial conditions:

\[ \alpha_i(0) = 0, \quad \text{(58)} \]

\[ \dot{\alpha}_i(0) = -k \text{ sign}(k\tau_1), \quad \text{(59)} \]

\[ \alpha_2(0) = \text{ sign}(k\tau_1), \quad \text{(60)} \]

\[ \dot{\alpha}_2(0) = \frac{k}{2 \left| \tanh 2k\tau_1 \right|}. \quad \text{(61)} \]

### 4.1. Discrete solutions and squeezed states

Now, we are in a position to write down the discrete solutions and squeezed state solutions to the time-dependent Schrödinger’s equation:

\[ \hat{H}_d \Psi(q', \tau) = -\frac{\hbar^2}{2m} \frac{\partial^2 \Psi}{\partial q'^2} + \frac{1}{2} m\Omega^2 q^2 \Psi = i\hbar \frac{\partial \Psi}{\partial \tau}. \quad \text{(62)} \]
For discrete solutions, we quote the results of Dantas et al [8], but using a re-scaled variable \( a \rightarrow \sqrt{\frac{\hbar}{2m}} a \) from the ones used in their work. Their discrete solutions to the time-dependent Schrödinger’s equation are:

\[
\Psi_n(q', \tau) = \frac{1}{(\sqrt{\pi} \sqrt{n!} \sqrt{2} a')^{\frac{1}{2}}} \exp \left\{ -\frac{1}{2} \left( \frac{q'}{\sqrt{2} a'} \right)^2 \right\} \\
\times \exp \left\{ -i \left( n + \frac{1}{2} \right) \int_0^\tau \frac{d\tau'}{2ma'^2} \right\} \\
\times \exp \left\{ \frac{i m a a}{\hbar} \left( \frac{q'}{\sqrt{2} a'} \right)^2 \right\} H_n \left( \frac{q'}{\sqrt{2} a'} \right),
\]

where \( H_n \) are Hermite polynomials and \( a(\tau) \) is a solution to the following nonlinear equation:

\[
\ddot{a} + \Omega^2 a = -\frac{\hbar^2}{4ma'^2}.
\]

As expected of a (time-dependent) harmonic oscillator Hamiltonian, the discrete states are Gaussian envelopes modulated by a Hermite polynomial, labeled by a positive integer \( n \) that corresponds to the number of nodes. Thus, we have explicitly shown the existence of a spectrum of discrete states of corresponding to the homogeneous linearized perturbations about the Taub solution labeled by one positive integer.

For the squeezed state solutions, we follow the approach given by Nassar [9]. For more information on the method of explicitly time-dependent invariants, see [5, 8, 10]. Nassar’s full quantum squeezed state is:

\[
\Psi(q', \tau) = \frac{1}{(\sqrt{\pi} \sqrt{\frac{1}{2} \Omega} \sqrt{2} a')^{\frac{1}{2}}} \exp \left\{ -\frac{1}{2} \left( \frac{q'-X}{\sqrt{\sqrt{2} a'}} \right)^2 \right\} \\
\times \exp \left\{ \frac{i}{\hbar} \int_0^\tau d\tau' \left\{ \frac{1}{2} mX'^2 - \frac{1}{2} m\Omega^2 X^2 - \frac{\hbar^2}{4ma'^2} \right\} \right\} \\
\times \exp \left\{ \frac{i}{\hbar} \left( m a a' \left( \frac{q'}{\sqrt{2} a'} \right)^2 + mX (q' - X) \right) \right\},
\]

Figure 1. Plot of \(|\Psi(q', \tau)|\) for \( X = a_1 |_{k=1,n=0} \) and \( a = \sqrt{a_1^2 + a_2^2} |_{k=1,n=0} \).
where \( a(\tau) \) is a solution to equation (64) as before and corresponds to the width of the wave packet; and \( X(\tau) \) (which corresponds to the position of the wave packet) is a solution to the following homogeneous equation:

\[
\ddot{X} + \Omega^2 X = 0. \tag{66}
\]

Davydov [11] showed that one can always write down a solution of equation (64) given two independent solutions of equation (66), by viewing the two independent solutions as cartesian \((x, y)\)-coordinates of a particle, and then realizing that equation (64) is satisfied by the amplitude component \( a = \sqrt{x^2 + y^2} \) when we transform into polar coordinates.

Figure 1 shows a plot of \(|\Psi|\) for the case \( X = \alpha_1 \mid k = 1, n = 0 \) and \( a = \sqrt{\alpha_1^2 + \alpha_2^2} \mid k = 1, n = 0 \). The position of the peak starts from a large positive \( q' \) at a finite \( \tau \) in the past \((\tau < 0)\) and swings by the origin at \( \tau = 0 \), moving off to large positive \( q' \) again at \( \tau > 0 \). During this time, the width of the Gaussian peak (given by \( a \)) narrows to a sharp peak at \( \tau = 0 \) and broadens out again. This behavior was found to be quite robust with respect to changes in the parameter choices and also the choice of \( X \). For example, for \( X = \alpha_2 \), we see that the position of the peak starts from a large positive \( q' \) at a finite \( \tau \) in the past, and goes through the origin, moving off to large negative \( q' \) again at \( \tau > 0 \). The width of the peak narrows to a sharp peak \( t \tau = 0 \) and broadens out again.

Here we note that the squeezed states of Nassar equation (65) at \( X \to 0 \) correspond exactly to the discrete solution states of Dantas et al. equation (63) at \( n \to 0 \). The squeezed states also yield the uncertainty relation, where \( \Delta q^2 = \langle q^2 \rangle - \langle q \rangle^2 \) and \( \Delta p^2 = \langle p^2 \rangle - \langle p \rangle^2 \):

\[
\Delta q \Delta p = \frac{1}{2} \hbar \left( 1 + \frac{4m^2}{\hbar^2 \alpha^2 a^2} \right)^{\frac{1}{2}}. \tag{67}
\]

In the time-independent limit \((\dot{a} \to 0)\), we see that the uncertainty relation reduces to the familiar minimum \(2\hbar\). Moreover, unlike in the time-independent case, where the coherent states do not spread out in time, the squeezed states are not of minimum uncertainty, and they spread out in time according to \((\Delta q)^2 \propto a^2 [10]\).

5. \( \beta' \) Hamiltonian

We can repeat the Jacobi method of second variation from the full Bianchi IX ADM action, with \( \beta \) not set to zero. The resulting linearized Hamiltonian would include the following part involving the \( \beta' \) variable:

\[
H^\prime|_{\beta'} = 0 = \frac{1}{2} NH^\prime_{\beta'} = H_{\beta'}^\prime + H_{B}^\prime,
\]

\[
H_{B}^\prime = \rho \left( \rho' \right)^2 + 24 \lambda e^{2\phi} \left( 2 e^{4\phi} - e^{-2\rho} \right) \left( \rho' \right)^2. \tag{69}
\]

Because \( H_B^{\prime} \) involves \( \beta' \) and \( \rho' \) only, it is gauge-invariant: the equations of motion for these quantities do not depend on our choice of the linearized lapse function \( N' \). Here, we cannot use ‘differentiation with respect to parameters’ to generate explicit forms of the linearized variables, as the perturbations do not stay within the Taub family \((\beta = 0)\), and hence we cannot make use of the exact known Taub solutions. However, all the results from above still apply to this Hamiltonian, and this means that there is another spectrum of discrete states corresponding to homogenous linearized perturbations, that are orthogonal to the first, and are labeled by another independent integer \( n \). In total, then, we find that the homogeneous
linearized perturbations about the Taub model give rise to a spectrum of discrete solutions that can be labeled by two independent integers \((m, n)\).

6. Discussion

In our previous work, we applied the modified semi-classical method to the canonically quantized vacuum Bianchi IX (Mixmaster) model, solving the relevant Wheeler–DeWitt equation asymptotically by integrating a set of linear transport equations along the flow of the Moncrief-Ryan (or ‘wormhole’) solution to the corresponding Euclidean-signature Hamilton–Jacobi equation [3]. We discovered that the excited state solutions, peaked away from the minisuperspace origin, are labeled by a pair of positive integers that can be plausibly interpreted as graviton excitation numbers for the two independent anisotropy degrees of freedom \((\beta_+, \beta_-)\). Is there some way to help us interpret these states in a more intuitive way?

In this paper, we have taken a different approach to the approximate solutions of the Bianchi IX model, using the Jacobi method of second variation to study the homogeneous linearized perturbations about the Taub background solution.

By way of visualization afforded by the explicit quantum squeezed states of the linearized Hamiltonian for \(Q = p_\alpha' a_\alpha' + \beta_\alpha' \beta_\alpha'\), we have sought to give a more intuitive explanation for the excited states found in our previous work. Moreover, there exists another set of discrete quantum states to the other gauge-invariant linearized Hamiltonian \(H''\) that describe homogeneous perturbations about the Taub models in an orthogonal direction. Although the second set of discrete quantum states have not been found explicitly, it is clear that, taken together with the explicit discrete solutions in equation (63), we have two sets of discrete solutions each labeled by a positive integer. Thus, we have a tentative connection between the discrete spectrum of excited solutions to our Bianchi IX Wheeler–DeWitt equation in [3] that are labeled by two independent positive integers \((m, n)\), and the two sets of discrete states discussed in this paper. Currently, the full comparison between the two is being left for future study, since it seems to be more subtle than initially anticipated.

In conclusion, we have applied the method of invariants to a linearized perturbation model about the exact Taub background. Our methods used here apply to a wider set of models used in quantum cosmology, as the Jacobi method of second variation can be extended to even inhomogeneous perturbations about any background model formulated in the variational principle. Once the linearized perturbations are expressed in their respective linearized Hamiltonians that are of time-dependent harmonic oscillator form, this paper describes how a series of canonical transformations can be used to separate out the gauge-invariant Hamiltonians from the gauge-dependent ones, and also to use the results in the literature to write down the explicit forms of discrete and squeezed state solutions where the exact background solutions are known.

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