Rational homotopy types of mapping spaces via cohomology algebras

Sang Xie, Jian Liu, Jianming Xiao, and Xiugui Liu

Abstract. Let $X$ be a connected finite CW-complex and $Y$ be a connected rational space with minimal Sullivan model of the form $(\Lambda(P \oplus Q), dP = 0, dQ \subset \Lambda P)$, where $P$ and $Q$ are graded spaces of finite type. In this paper, it is shown that the rational homotopy type of $\text{map}(X, Y)$ is determined by the cohomology algebra $H^\ast(X; \mathbb{Q})$ and the rational homotopy type of $Y$.

Mathematics Subject Classification. 55P62.

Keywords. Rational homotopy type, Mapping spaces, $L_\infty$-algebras, H-spaces.

1. Introduction. For two topological spaces $X$ and $Y$, let $\text{map}(X, Y)$ be the mapping space of all (free, continuous) maps of $X$ into $Y$. A fundamental problem is to describe the homotopy type of the mapping space $\text{map}(X, Y)$ in terms of homotopy types of $X$ and $Y$. Following the work of Haefliger [7], Møller and Raussen [12] considered the rational homotopy classification problem for the components of some mapping spaces $\text{map}(X, Y)$. In particular, Møller-Raussen gave an elegant formula for the rational homotopy type of the components of $\text{map}(X, S^n_\mathbb{Q})$ in terms of the cohomology algebra $H^\ast(X; \mathbb{Q})$ and the Sullivan model of $S^n_\mathbb{Q}$. In fact, Møller-Raussen proved that if $X$ and $X'$ are connected finite CW-complexes and the rational space $Y$ has the Sullivan model of the form $(\Lambda(x, y), dy = x^m)$ ($|y|$ denotes the degree of $y$), then $\text{map}(X, Y) \simeq \text{map}(X', Y)$.

Recently, the rational homotopy type of mapping spaces seems to be well described in terms of the theory of $L_\infty$-algebras [1,3,4,9]. By the homotopy type
transfer theorem for $L_\infty$-algebras, Buijs and Gutiérrez [3] gave an $L_\infty$-model for the mapping space $\text{map}(X, Y)$ with the cohomology of $X$ and the $L_\infty$-model of $Y$. Inspired by their work, we extend Møller-Raussen’s result by the following theorem.

**Theorem 1.1.** Let $X$ be a connected nilpotent finite CW-complex, and let $Y$ be a connected rational space with the minimal Sullivan model of the form

$$\left(\Lambda(P \oplus Q), dP = 0, dQ \subset \Lambda P\right),$$

where $P$ and $Q$ are graded spaces of finite type. Then the rational homotopy type of $\text{map}(X, Y)$ is determined by the cohomology algebra $H^*(X; \mathbb{Q})$ and the Sullivan algebra $(\Lambda(P \oplus Q), d)$.

**Remark 1.2.** In [2, Theorem 1.4], it was shown that it is always possible to obtain a model of $\text{map}(X, Y)$ of the form

$$\left(\Lambda(V \otimes H), d\right),$$

where $H$ stands for $H_*(X; \mathbb{Q})$, $(\Lambda V, d')$ is a Sullivan model of $Y$. The differential $d$ depends only on the differential in $(\Lambda V, d')$ and the coalgebra structure on $H$, that is, the product on $H^*(X; \mathbb{Q})$ (see [2, Section 7]).

**Remark 1.3.** Note that $S^n_{\mathbb{Q}}$ and $CP^n_{\mathbb{Q}}$ all satisfy the hypotheses of $Y$ in Theorem 1.1. More generally, the pure Sullivan algebras and the Sullivan model of homogeneous spaces and $F_0$-spaces also satisfy the hypotheses of $Y$ in Theorem 1.1 [5].

G. Lupton and S. Smith proved that for a CW-complex $Y$, the space $\text{map}(Y, Y; \text{id})$ is a group-like space [11].

Let $X, Y$ be as in Theorem 1.1. As applications of Theorem 1.1, we obtain the following two results (see Section 3).

**Corollary 1.4.** If $H^*(X; \mathbb{Q}) \cong H^*(Y; \mathbb{Q})$, then there exists a component of the mapping space $\text{map}(X, Y)$ which is a rational $H$-space.

Let $\text{MC}(H^*(X; \mathbb{Q}) \otimes L)$ denote the Maurer-Cartan set of $H^*(X; \mathbb{Q}) \otimes L$, where $L$ is the minimal $L_\infty$-model of $Y$. Given an isomorphism $\psi: H^*(X; \mathbb{Q}) \to H^*(X; \mathbb{Q})$ between cohomology algebras, we have

**Corollary 1.5.** If $h \otimes l \in \text{MC}(H^*(X, \mathbb{Q}) \otimes L)$, then $\psi(h) \otimes l \in \text{MC}(H^*(X, \mathbb{Q}) \otimes L)$ and $\text{map}(X, Y; f) \simeq_\mathbb{Q} \text{map}(X, Y; f')$, where $h \otimes l$ and $\psi(h) \otimes l$ represent the maps $f: X \to Y$ and $f': X \to Y$, respectively.

The paper is organized as follows. In Sect. 2, a brief exposition of the $L_\infty$-model for mapping spaces and the homotopy transfer theorem for $L_\infty$-algebras is given. Section 3 contains the proof of Theorem 1.1 and its applications.

**2. $L_\infty$-algebras and the homotopy transfer theorem.** In this section, we recall some basic facts about $L_\infty$-algebras.
2.1. $L_{\infty}$-algebras.

**Definition 2.1.** An $L_{\infty}$-algebra $(L, \{\ell_k\})$ is a graded vector space $L$ together with linear maps

$$\ell_k: L^\otimes k \to L, \quad x_1 \otimes \cdots \otimes x_k \mapsto [x_1, \cdots, x_k],$$

of degree $k - 2$ for $k \geq 1$, satisfying the following two conditions:

(i) Anti-symmetry,

$$[\cdots, x, y, \cdots] = -(-1)^{|x||y|}[\cdots, y, x, \cdots],$$

(ii) The generalized Jacobi identity,

$$\sum_{i+j=n+1} \varepsilon\varepsilon(-1)^{(i-1)}[[x_{\sigma(1)}, \cdots, x_{\sigma(i)}], x_{\sigma(i+1)}, \cdots, x_{\sigma(n)}] = 0,$$

where $S(i, n-i)$ denotes the set of $(i, n-i)$ shuffles.

An $L_{\infty}$-algebra $(L, \{\ell_k\})$ is called minimal if $\ell_1 = 0$. For two $L_{\infty}$-algebras $(L, \{\ell_k\}_{k \geq 1})$ and $(L', \{\ell'_k\}_{k \geq 1})$, an $L_{\infty}$-morphism $f: L \to L'$ is a family of skew-symmetric linear maps $\{f^n: L^\otimes n \to L'\}$ of degree $n - 1$ ($n \geq 1$) which satisfy an infinite sequence of equations involving the brackets $\ell_k$ and $\ell'_k$ (see for instance [8]). It is called an $L_{\infty}$ quasi-isomorphism if $f^{(1)}: (L, \ell_1) \to (L', \ell'_1)$ is a quasi-isomorphism of complexes.

A model for a space (not necessarily connected) is a commutative differential graded algebra whose simplicial realization has the same homotopy type as the singular simplicial approximation of the rationalization of $X$ [13]. Similarly, an $L_{\infty}$-model for a space $X$ is an $L_{\infty}$-algebra $L$ such that $\mathcal{C}^*(L)$ is a commutative differential graded algebra model for $X$, where $\mathcal{C}^*(L)$ is the Chevalley–Eisenberg construction for the $L_{\infty}$-algebra $L$ [1].

**Theorem 2.2** ([1, Theorem 1.4]). If $(A, d_A)$ is a finite dimensional commutative differential graded algebra model for $X$ and $(L, \{\ell_k\}_{k \geq 1})$ is an $L_{\infty}$-model for $Y$, then the following $L_{\infty}$-algebra is a model of $\text{map}(X, Y)$:

$$(A \otimes L, \{\ell_k\}_{k \geq 1}),$$

where the brackets are defined by

$$\ell_1(x \otimes a) = d_A(x) \otimes a \pm x \otimes \ell'_1(a),$$

$$[x_1 \otimes a_1, \cdots, x_r \otimes a_r] = \pm x_1 \cdots x_r \otimes [a_1, \cdots, a_r], \quad r > 1,$$

where $x, x_1, \ldots, x_r, a, a_1, \ldots, a_r \in L$.

2.2. The homotopy transfer theorem. Now, we recall how to transfer $L_{\infty}$-structure from homotopy retract. Let $(L, \ell_k)$ be an $L_{\infty}$-algebra. Consider the following diagram

$$\begin{array}{ccc}
\mathbb{K} & \xrightarrow{\iota} & (L, d) \\ & \searrow & \downarrow q \\
& & (H, 0)
\end{array}$$

(2.1)

in which $H = H(L, \ell_1)$, $i$ is a quasi-isomorphism, $qi = \text{id}_H$, and $K$ is a chain homotopy between $\text{id}_L$ and $iq$, i.e., $\text{id}_L - iq = \partial K + K\partial$. We encode this data as $(L, i, q, K)$ and call it a homotopy retract of $L$. In this setting, the classical homotopy transfer theorem reads:
There exists an $L_\infty$-algebra structure \( \{ \ell'_k \} \) on \( H \), unique up to isomorphism, and $L_\infty$ quasi-isomorphisms \( (L, \partial) \xrightarrow{Q} (H, \{ \ell'_k \}) \) such that \( I^{(1)} = i \) and \( Q^{(1)} = q \). Moreover, the transferred higher brackets can be explicitly described by the following
\[
\ell'_k = \sum_{T \in \mathcal{T}_k} \frac{\ell_T}{|\text{Aut}(T)|},
\]
(2.2)
where \( \mathcal{T}_k \) is the set of isomorphism classes of directed binary rooted trees with \( k \) leaves, \( \text{Aut}(T) \) is the automorphism group of the tree \( T \), and \( \ell_T \) is described below.

We describe here the item \( \ell_T \) in the formula (2.2). For \( T \in \mathcal{T}_k \), we define the linear map \( \ell_T \) as follows. The leaves of the tree \( T \) are labeled by the map \( i \), each internal edge is labeled by the chain homotopy \( K \) and the root edge is labeled by \( q \). By moving down from the leaves to the root, we define the linear map \( \ell_T \). For example, the tree
\[
\begin{array}{c}
\text{i} \\
\ell_2 \\
K \\
\ell_2 \\
\text{i} \\
q
\end{array}
\]
gives the map
\[
\ell_T = q \circ \ell_2 \circ (K \otimes \text{id}) \circ (\ell_2 \otimes \text{id}) \circ (i \otimes i \otimes i).
\]

3. Proof of Theorem 1.1 and some applications. We begin with the following

Lemma 3.1. Let \((AV, d)\) be the commutative differential graded algebra of finite type of the form (1.1), i.e.,
\[
(AV, d) = (A(P \oplus Q), dP = 0, dQ \subset \Lambda P).
\]
The Chevalley–Eilenberg construction gives an $L_\infty$-algebra \((L, \{ \ell_k \}_{k \geq 1})\) such that \( C^* \circ (L) = (AV, d) \). Then the $L_\infty$-structure \( \{ \ell_k \}_{k \geq 1} \) satisfies
\[
[\cdots, [\cdots, \cdots]] = 0.
\]

Proof. Recall that \( V \) and \( \text{sL} \) (the suspension of \( L \)) are dual graded vector spaces and the $L_\infty$-structure \( \{ \ell_k \}_{k \geq 1} \) is determined by the differential \( d = d_1 + d_2 + \cdots \), where derivations \( d_k \) raise the wordlength by \( k - 1 \). More precisely, we have
\[
\langle d_k v; sx_1 \wedge \cdots \wedge sx_k \rangle = \pm \langle v; s\ell_k(x_1, \ldots, x_k) \rangle
= \sum_{\sigma \in S_k} \pm \langle v_{\sigma(1)}; s x_1 \rangle \cdots \langle v_{\sigma(k)}; s x_k \rangle,
\]
where \(d_k v = v_1 \cdots v_k, \langle \cdot ; \cdot \rangle \) is the Sullivan pairing, \(v, v_1, \ldots, v_k \in V\), and \(x_1, \ldots, x_k \in L \cong s^{-1}V^\#\).

A straightforward computation shows that
\[
\langle \ldots, q, \ldots \rangle = 0, \quad q \in s^{-1}Q^\#,
\]
and
\[
[p_1, \ldots, p_i] \in \mathbb{Q}q_1 \oplus \cdots \oplus \mathbb{Q}q_j, \quad p_1, \ldots, p_i, q_1, \ldots, q_j \in s^{-1}P^\#.
\]
It follows that
\[
[x_1, \ldots, x_2, \ldots, x_k] = 0, \quad x_1, x_2 \in s^{-1}V^#.
\]

Suppose that \(L\) is the minimal \(L_\infty\)-model of \(Y\) and \((A, d)\) is a finite dimensional commutative differential graded algebra model of \(X\). By Theorem 2.2, the \(L_\infty\)-algebra \((A \otimes L, \{\ell_k\}_{k \geq 1})\) is the \(L_\infty\)-model of \(\text{map}(X, Y)\). Let \(H = H(A, d)\), then we have the following decomposition
\[
A = B \bigoplus dB \bigoplus H \quad (3.1)
\]
with basis \(\{b_i\}, \{\theta b_i\}, \) and \(\{h_j\}\), respectively. Note that \(d = 0\) in \(H\) and \(d: B \to dB\) is an isomorphism. It is easy to check that the decomposition (3.1) induces a homotopy retract:
\[
K' \xrightarrow{(A \otimes L, \ell_1)} (H \otimes L, \ell'_1),
\]
where \(K' = K \otimes id, q' = q \otimes id, i' = i \otimes id\). By Theorem 2.3, we obtain an \(L_\infty\)-structure \(\{\ell'_k\}_{k \geq 1}\) on \(H \otimes L\) and the following quasi-isomorphism between \(L_\infty\)-algebras
\[
(A \otimes L, \{\ell_k\}_{k \geq 1}) \xrightarrow{\simeq} (H \otimes L, \{\ell'_k\}_{k \geq 1}).
\]
Then, we have

**Lemma 3.2 ([3]).** The \(L_\infty\)-algebra \(H \otimes L\) is an \(L_\infty\)-model of \(\text{map}(X, Y)\).

Now, we prove our main result in this paper.

**Proof of Theorem 1.1.** To prove Theorem 1.1, it is sufficient to show that
\[
\ell'_k(h_1 \otimes x_1, \ldots, h_k \otimes x_k) = h_1 \cdot_H h_2 \cdot_H \cdots \cdot_H h_k \otimes [x_1, \ldots, x_k], \quad k \geq 2,
\]
where \(h_1 \otimes x_1, \ldots, h_k \otimes x_k \in H \otimes L\), and \(\cdot_H\) denotes the product on \(H\).

We proceed by induction on \(k\). The case \(k = 1\) follows immediately from the definition. For \(k = 2\), the explicit formula for \(\ell'_2\) is provided by the tree
\[
\begin{array}{c}
i \otimes id \\ \ell_2 \\ i \otimes id \\ q \otimes id.
\end{array}
\]
Note that \( q \circ i = id_H \), we have
\[
\ell'(h_1 \otimes x_1, h_2 \otimes x_2) = (q \otimes id) \circ \ell_2 \circ (i \otimes id, i \otimes id)(h_1 \otimes x_1, h_2 \otimes x_2) \\
= (q \otimes id) \circ \ell_2(h_1 \otimes x_1, h_2 \otimes x_2) \\
= q(h_1 \cdot_A h_2) \otimes [x_1, x_2] \\
= h_1 \cdot_H h_2 \otimes [x_1, x_2],
\]
where \( h_1, h_2 \in H, x_1, x_2 \in L \), and \( \cdot_A \) denotes the product of \( A \).

For \( k = 3 \), the explicit formula for \( \ell_3' \) is provided by the following trees

\[ T_1, T_2, \ldots \]

By Lemma 3.1, we have
\[
[[x_1, x_2], x_3] = 0,
\]
where \( x_1, x_2, x_3 \in L \). Note that the tree \( T_1 \) has more than one vertex, we have
\[
\ell_{T_1}'(h_1 \otimes x_1, h_2 \otimes x_2, h_3 \otimes x_3) \\
= (q \otimes id) \circ \ell_2 \circ (K \otimes id, id) \circ (l_2, id) \circ (i \otimes id, i \otimes id, i \otimes id)(h_1 \otimes x_1, h_2 \otimes x_2, h_3 \otimes x_3) \\
= (q \otimes id) \circ \ell_2 \circ (K \otimes id, id)(\pm h_1 h_2 \otimes [x_1, x_2], h_3 \otimes x_3) \\
= (q \otimes id) \circ (\pm K(h_1 h_2) h_3 \otimes [x_1, x_2], x_3] \\
= 0,
\]
where \( h_1, h_2, h_3 \in H, x_1, x_2, x_3 \in L \). This implies that the explicit formula for \( \ell_3' \) only depends on the tree \( T_2 \).

For \( k \geq 4 \), the formula for \( \ell_k' \) is determined by the set of rooted trees with \( k \) leaves:
\[
\mathcal{T}_k = \left\{ \cdots, \cdots, \cdots \right\}.
\]
Recall that the \( k \)-corolla of \( \mathcal{T}_k \) is the last element in the above set. Note that the \( k \)-corolla is the only one which has one vertex. Consider \( T_i \in \mathcal{T}_k \) which has at least two vertices. By a similar proof as in the case \( k = 3 \), we have
\[
\ell'_{T_i} = 0.
\]
It follows that \( \ell_k' \) only depends on the \( k \)-corolla and
\[
\ell_k'(h_1 \otimes x_1, \ldots, h_k \otimes x_k) = h_1 \cdot_H h_2 \cdot_H \cdots \cdot_H h_k \otimes [x_1, \ldots, x_k], \quad k \geq 3.
\]
Thus the theorem is proved. \( \Box \)
Example 3.3. Consider the rational space $X$ with the same cohomology algebra as $S^2 \vee S^2 \vee S^5$ (note that there are two rational homotopy types whose cohomology algebras are both $H^*(S^2 \vee S^2 \vee S^5)$). Let $Y$ be the rational space with Sullivan model $(\Lambda(x,y,z),d)$ with $|x| = 3$, $|y| = 5$, $|z| = 7$, and $dx = dy = 0$, $dz = xy$. By Theorem 1.1, $\text{map}(X,Y) \simeq \text{map}(S^2 \vee S^2 \vee S^5, Y)$. Let $H = H^*(S^2 \vee S^2 \vee S^5; \mathbb{Q}) = \mathbb{Q}1 \oplus \mathbb{Q}e_2 \oplus \mathbb{Q}e'_2 \oplus \mathbb{Q}e_5$, and let $L = \mathbb{Q}x_2 \oplus \mathbb{Q}y_4 \oplus \mathbb{Q}z_6$ with $[x_2, y_4] = z_6$ be the $L_\infty$-model of $Y$, where the subscripts denote degrees. Then $H \otimes L$ is the $L_\infty$-model of $\text{map}(X,Y)$. A basis for $H \otimes L$ is given by

- degree 0: $e_2 \otimes x$, $e'_2 \otimes x$;
- degree 1: $e_5 \otimes z$;
- degree 2: $1 \otimes x$, $e_2 \otimes y$, $e'_2 \otimes y$;
- degree 4: $1 \otimes y$, $e_2 \otimes z$, $e'_2 \otimes z$;
- degree 6: $1 \otimes z$.

The non-trivial $L_\infty$-structure is described by

$$[e_2 \otimes x, 1 \otimes y] = e_2 \otimes z, \quad [e'_2 \otimes x, 1 \otimes y] = e'_2 \otimes z, \quad [1 \otimes x, e_2 \otimes y] = e_2 \otimes z,$$

$$[1 \otimes x, e'_2 \otimes y] = e'_2 \otimes z, \quad [1 \otimes x, 1 \otimes y] = 1 \otimes z.$$

A direct computation shows that $\text{map}(X,Y)$ has two components. One component has the same rational homotopy type as $K(\mathbb{Q},2) \times Z$,

where $Z$ is the rational space with minimal model

$$(\Lambda(a_1, a'_1, a_3, a'_3, a''_3, a_5, a'_5, a''_5, a_7), d)$$

with $da'_5 = a_1 a_5 + a_3 a'_3$, $da''_5 = a'_1 a_5 + a_3 a''_3$, and $da_7 = a_3 a_5$. Another component has the same rational homotopy type as

$$S^3 \times S^3 \times S^7 \times Z',$$

where $Z'$ is the rational space with minimal model

$$(\Lambda(a_1, a'_1, a_5, a'_5, a''_5, a_7), da'_5 = a_1 a_5, da''_5 = a'_1 a_5).$$

H-space structures of mapping spaces. An interesting question is to determine whether a mapping space is of the rational homotopy type of an H-space.

Now we make use of Theorem 1.1 to show the following

Corollary 3.4. Let $X$ and $Y$ be as in Theorem 1.1. If $H^*(X; \mathbb{Q}) \cong H^*(Y; \mathbb{Q})$ as algebras, then there exists a component of the mapping space $\text{map}(X,Y)$ which is a rational H-space.

Proof. By Theorem 1.1, $\text{map}(X,Y) \simeq \text{map}(Y,Y)$. Then there exists a component $\text{map}(X,Y; f)$ of $\text{map}(X,Y)$ such that $\text{map}(X,Y; f) \simeq \text{map}(Y,Y; \text{id})$. Note that $\text{map}(Y,Y; 1)$ is a group-like space [11, Theorem 3.6]. It follows that $\text{map}(X,Y; f)$ is a rational H-space. $\square$
Example 3.5. Let $Y$ be a rational space with Sullivan model $(\Lambda(x, y, z), d)$ with $|x| = 3$, $|y| = 5$, $|z| = 7$, and $dx = dy = 0$, $dz = xy$. Let $X$ be a rational space with the commutative differential graded algebra model $H$ which is isomorphic to $H^*(Y)$. Note that $H = \mathbb{Q}1 \oplus \mathbb{Q}x \oplus \mathbb{Q}yz \oplus \mathbb{Q}xyz$ with the only non-trivial product $x \cdot yz = xyz = yxz$. The space $X$ is not rational homotopy equivalent to $Y$ since $Y$ is not formal. By Corollary 3.4, there exists a component map $(X, Y; f)$ which is a rational $H$-space. A straightforward computation shows that

$$\text{map}(X, Y; f) \simeq K(Q, 2) \times K(Q, 2) \times K(Q, 3) \times K(Q, 7).$$

Classifying the components of mapping spaces. A fundamental problem is to classify the components of map$(X, Y)$ up to homotopy type. Let $X$ and $Y$ be as in Theorem 1.1. By Theorem 1.1, we have an $L_\infty$-model $H \otimes L$ for the map$(X, Y)$, where $H = H^*(X; Q)$, $L$ is the minimal $L_{\infty}$-model of $Y$. Recall ([1, Theorem 1.5]) that

$$[X, Y] \cong \mathcal{MC}(H \otimes L),$$

where the moduli space $\mathcal{MC}(H \otimes L) = MC(H \otimes L)/ \sim$, the quotient set of equivalence classes of the Maurer-Cartan element. Let $z \in MC(H \otimes L)$, we have a new $L_{\infty}$-algebra $H \otimes L^z$ [6, Proposition 4.4]. The truncated and twisted $L_{\infty}$-algebra $(H \otimes L^z)_{\geq 0}$ is an $L_{\infty}$-model for the component map$(X, Y; f)$, where the Maurer-Cartan element $z$ represents the map $f : X \to Y$.

Given an isomorphism $\psi : H^*(X) \to H^*(X)$ between cohomology algebras, we have

Corollary 3.6. If $h \otimes l \in MC(H \otimes L)$, then $\psi \otimes id(h \otimes l) = \psi(h) \otimes l \in MC(H \otimes L)$ and map$(X, Y; f) \simeq_{\mathbb{Q}} \text{map}(X, Y; f')$, where $h \otimes l$ and $\psi(h) \otimes l$ represent the maps $f : X \to Y$ and $f' : X \to Y$, respectively.

Proof. By Theorem 1.1, the $L_{\infty}$-structure $\{l_k\}_{k \geq 1}$ of $H \otimes L$ depends on the product of $H$ and the $L_{\infty}$-structure of $L$, i.e., $l_k(h \otimes l, \ldots, h \otimes l) = h \cdot_H h \cdot_H \cdots h \otimes [l, \ldots, l]_L$. Thus,

$$0 = \psi \otimes id\left(\sum_{k \geq 0} \frac{1}{k!} l_k(h \otimes l, \ldots, h \otimes l)\right)$$

$$= \sum_{k \geq 0} \frac{1}{k!} l_k(\psi(h) \otimes l, \ldots, \psi(h) \otimes l)$$

$$= \sum_{k \geq 0} \frac{1}{k!} \psi(h \cdot_H h \cdot_H \cdots h) \otimes [l, \ldots, l].$$

It follows that $\psi(h) \otimes l \in MC(H \otimes L)$.

Let the Maurer-Cartan elements $h \otimes l$ and $\psi(h) \otimes l$ represent the maps $f$ and $f'$, respectively. Note that, as graded vector spaces,

$$\psi \otimes id : (H \otimes L^{h \otimes l})_{\geq 0} \overset{\cong}{\to} (H \otimes L^{\psi(h) \otimes l})_{\geq 0}.$$ 

It is easy to check that this isomorphism is compatible with their $L_{\infty}$-structures, i.e., $(H \otimes L^{h \otimes l})_{\geq 0}$ and $(H \otimes L^{\psi(h) \otimes l})_{\geq 0}$ are isomorphic as $L_{\infty}$-algebras. Thus we have that
map(X, Y; f) ≃_Q map(X, Y; f'). □

Publisher's Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Springer Nature or its licensor holds exclusive rights to this article under a publishing agreement with the author(s) or other rightsholder(s); author self-archiving of the accepted manuscript version of this article is solely governed by the terms of such publishing agreement and applicable law.

References

[1] Berglund, A.: Rational homotopy theory of mapping spaces via Lie theory for L_\infty-algebras. Homology Homotopy Appl. 17(2), 343–369 (2015)
[2] Brown, E.H., Szczarba, R.H.: On the rational homotopy type of function spaces. Trans. Amer. Math. Soc. 349(12), 4931–4951 (1997)
[3] Buijs, U., Gutiérrez, J.J.: Homotopy transfer and rational models for mapping spaces. J. Homotopy Relat. Struct. 11, 309–332 (2016)
[4] Buijs, U., Murillo, A.: Algebraic models of non-connected spaces and homotopy theory of L_\infty algebras. Adv. Math. 236, 60–91 (2013)
[5] Félix, Y., Halperin, S., Thomas, J.-C.: Rational Homotopy Theory. Graduate Texts in Mathematics, vol. 205. Springer, Berlin (2000)
[6] Getzler, E.: Lie theory for nilpotent L_\infty-algebras. Ann. of Math. (2) 170(1), 271–301 (2009)
[7] Haefliger, A.: Rational homotopy of the space of sections of a nilpotent bundle. Trans. Amer. Math. Soc. 273, 609–620 (1982)
[8] Kontsevich, M.: Deformation quantization of Poisson manifolds. Lett. Math. Phys. 66(3), 157–216 (2003)
[9] Lazarev, A.: Maurer–Cartan moduli and models for function spaces. Adv. Math. 235, 296–320 (2013)
[10] Loday, J.-L., Vallette, B.: Algebraic Operads. Grundlehren der mathematischen Wissenschaften vol. 346. Springer, Heidelberg (2012)
[11] Lupton, G., Smith, S.-B.: Criteria for components of a function space to be homotopy equivalent. Math. Proc. Camb. Philos. Soc. 145(1), 95–106 (2008)
[12] Møller, J.M., Raussen, M.: Rational homotopy of spaces of maps into spheres and complex projective spaces. Trans. Amer. Math. Soc. 292(2), 721–732 (1985)
[13] Sullivan, D.: Infinitesimal computations in topology. Inst. Hautes Études Sci. Publ. Math. 47, 269–331 (1978)

Sang Xie
College of Mathematics and Physics
Chengdu University of Technology
Chengdu 610059
People’s Republic of China
e-mail: xiesangxl@qq.com
Acknowledgments

We are grateful to the anonymous referees for their valuable comments and suggestions, which helped us to improve the presentation of this paper.

We would like to express our appreciation to the editors for their patience and support throughout the publication process.

Finally, we would like to thank the Department of Mathematics for providing the necessary resources and support to carry out this research.

Received: 25 May 2022
Revised: 16 August 2022
Accepted: 1 September 2022.