Towards the generalized gravitational entropy for spacetimes with non-Lorentz invariant duals

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ABSTRACT: Based on the Lewkowycz-Maldacena prescription and the fine structure analysis of holographic entanglement proposed in [1], we explicitly calculate the holographic entanglement entropy for warped CFT that duals to AdS\(_3\) with a Dirichlet-Neumann type of boundary conditions. We find that certain type of null geodesics emanating from the entangling surface \(\partial \mathcal{A}\) relate the field theory UV cutoff and the gravity IR cutoff. Inspired by the construction, we furthermore propose an intrinsic prescription to calculate the generalized gravitational entropy for general spacetimes with non-Lorentz invariant duals. Compared with the RT formula, there are two main differences. Firstly, instead of the homology constraint, we require the consistence between the boundary and bulk causal structures to determine the corresponding bulk extremal surface \(\mathcal{E}\). Secondly we use null geodesics (or hypersurfaces) emanating from \(\partial \mathcal{A}\) and normal to \(\mathcal{E}\) to regulate \(\mathcal{E}\) in the bulk. We apply this prescription to flat space in three dimensions and get the entanglement entropies straightforwardly.
1 Introduction

2 New observations from the Rindler method
   2.1 A brief introduction to Rindler method
   2.2 New observations from the Rindler method

3 Rindler method applied on AdS$_3$/WCFT correspondence
   3.1 AdS$_3$ with CSS boundary conditions
   3.2 Rindler method for AdS$_3$ with CSS boundary conditions

4 Modular flows and modular planes

5 Generalized gravitational entropy for AdS$_3$ with CSS boundary conditions
   5.1 The replica story on the boundary
   5.2 The replica story in the bulk
      5.2.1 Selecting the bulk curve by matching the bulk and boundary causal structures
      5.2.2 Bulk replica story
   5.3 Relating the UV and IR cutoffs with the modular planes (null geodesics)
   5.4 The intrinsic construction of the geometric picture

6 Towards the generalized gravitational entropy for spacetimes with non-Lorentzian duals

7 Generalised gravitational entropy for 3-dimensional flat space
   7.1 Null-orbifold
   7.2 Global Minkowski
   7.3 Flat Space Cosmological solutions

A Entanglement contour for WCFT
   A.1 Entanglement contour from the fine structure
   A.2 Testing the entanglement contour proposal

B The saddle that connect the two null curves $\gamma_{\pm}$
1 Introduction

Entanglement entropy, which describes the correlation structure of a quantum system, has played a central role in the study of modern theoretical physics. In the context of AdS/CFT correspondence [2–4], the Ryu-Takayanagi (RT) [5, 6] formula relates the entanglement entropy to a geometric quantity on the gravity side. More explicitly, for a static subregion \( \mathcal{A} \) in the boundary CFT and a minimal surface \( \mathcal{E}_A \) in the dual AdS bulk that anchored on the boundary \( \partial \mathcal{A} \) of \( \mathcal{A} \), the RT formula states that the entanglement entropy of \( \mathcal{A} \) is measured by the area of \( \mathcal{E}_A \) in Planck units,

\[
S_{EE} = \frac{\text{Area}(\mathcal{E}_A)}{4G}.
\]  

Soon the Hubeny-Rangamani-Takayanagi (HRT) [7] formula was proposed as the covariant version of the RT formula. Accordingly the minimal surface is generalized to the extremal surface in the HRT formula. The holographic picture of entanglement entropy has a huge impact on our understanding of holography itself as well as the emergent of spacetime.

One way to understand the RT formula is the Rindler method, which is first proposed in [8] and later generalized in [9, 10]. The key point of the Rindler method is to construct a Rindler transformation, which is a symmetry of the theory, that maps the causal development of a subregion to a thermal “Rindler space”. Thus the problem of calculating the entanglement entropy of a subregion is replaced by the problem of calculating the thermal entropy of the Rindler space. According to holography, the thermal entropy of the Rindler space equals to the thermal entropy of its bulk dual, which is usually a black string (or hyperbolic black hole). The horizon of the hyperbolic black hole is exactly what maps to the RT surface under the Rindler transformations in the bulk.

The other way is to extend the replica trick into the bulk and calculate the entanglement entropy using the partition function calculated by the path integral on the gravity side. This prescription is explicitly done by Lewkowycz and Maldacena [11] (see [12] for the covariant generalization). The entanglement entropy of a quantum system is defined as the von Neumann entropy \( S_A = -\text{Tr} \rho_A \log \rho_A \) of the reduced density matrix \( \rho_A \). Consider a quantum field theory on \( \mathcal{B} \), the replica trick first calculates the Renyi entropy \( S_A^{(n)} = \log \text{Tr} \rho_A^n \) for \( n = \mathbb{Z}_+ \), then analytically continues \( n \) away from integers. We get the entanglement entropy \( S_A \) when \( n \to 1 \). To calculate \( \rho_A^n \), we can cut \( \mathcal{B} \) open along \( \mathcal{A} \), glue \( n \) copies of them cyclically to a new manifold \( \mathcal{B}_n \), then we do path integral on \( \mathcal{B}_n \). The entanglement entropy is calculated by

\[
S_{EE}(\mathcal{A}) = -n \partial_n (\log Z_n - n \log Z_1) \big|_{n=1},
\]  

where \( Z_n \) is the partition function of the quantum field theory on \( \mathcal{B}_n \). Assuming holography and the unbroken replica symmetry in the bulk, the LM (Lewkowycz-Maldacena) prescription manages to construct the bulk dual of \( \mathcal{B}_n \), which is a replicated bulk geometry \( \mathcal{M}_n \) with its boundary being \( \mathcal{B}_n \). Then the partition function \( Z_n \) can be calculated by path integral on \( \mathcal{M}_n \) on the gravity side. The two main results of [11, 12] are:
the holographic entanglement entropy is calculated by the area of the codimension two surface $\mathcal{E}$ in Plank units, which is the set of all the fixed points of the bulk replica symmetry,

- the codimension two surface $\mathcal{E}$ is an extremal surface\(^1\).

As was indicated in [11, 12], these results are quite general even for holographies beyond AdS/CFT.

However the above results are not equivalent to the RT (or HRT) formula without the homology constraint and the repriscription to regulate the entanglement entropy via the UV/IR cutoff relation [15] in AdS/CFT. The homology constraint requires the extremal surface $\mathcal{E}$ to be anchored on $\partial \mathcal{A}$ and homologous to $\mathcal{A}$ [16–18], thus selects the right extremal surface that matches $\mathcal{A}$. The prescription for regulation tells us how to regulate the extremal surface $\mathcal{E}$ in the bulk when we regulate $\mathcal{A}$ on the boundary. Although the homology constraint and prescription for regulation in the RT formula seems quite natural, it has never been thoroughly studied in holographies beyond AdS/CFT.

In the context of AdS/CFT, since both of the boundary field theory and the bulk gravity are relativistic, the causal structures near the entangling surface $\partial \mathcal{A}$ and the RT surface $\mathcal{E}$ match with each other, so the consistence between bulk and boundary causal structures naturally lead to the homology constraint\(^2\). In this sense, the homology constraint should hold for holographies with relativistic duals. However, for holographies beyond AdS/CFT\(^3\), especially those with non-Lorentz invariant field theory duals (for example non-relativistic theories, ultra-relativistic theories or Lifshitz-type theories), it is reasonable to question the validity of the homology constraint. Also, the UV/IR cutoff relations, as well as their application to regulate the bulk extremal surface, in more general holographic models have not been discussed before. These are crucial to the validity of the RT formula in more general holographic models.

Recently a series of work [9, 10, 13] calculated the holographic entanglement entropy for spacetimes that are not asymptotically AdS and found the corresponding geometric quantities. Remarkably, these results challenge the validity of the RT formula in holographies beyond AdS/CFT. In the context of (warped) AdS/warped CFT correspondence [28, 36] and 3-dimensional flat holography [33–35], the geometric quantities $\mathcal{E}_\mathcal{A}$, which correspond to the entanglement entropy of an single interval $\mathcal{A}$ in warped CFT (WCFT) [28] and BMS\(_3\) invariant field theories (BMSFTs), are found [9, 10] respectively with the Rindler method.

\(^1\) The extremal condition is the result of imposing the equations of motion and replica symmetry on all the fields in the action. In [13], as the gauge fields are nondynamical and do not appear in the symplectic structure, thus should not be imposed with the replica symmetry (or periodic) condition. As a result, in that case the geometric quantity $\mathcal{E}$ that measures the entanglement entropy is not an extremal surface. See [14] for a simpler discussion on the extremal condition.

\(^2\) A proof for the homology constraint at topological level in AdS/CFT is given in [19].

\(^3\) Although the AdS/CFT is the mostly well studied holography theory, the holographic principle is assumed to be hold for general spacetimes. So far the holography beyond AdS/CFT that has been proposed include the d5/CFT correspondence [20], the Lifshitz spacetime/non-relativistic field theory duality [21–24], the Kerr/CFT correspondence [25], the WAdS/CFT [26, 27] or WAdS/WCFT [28, 29] correspondence, and flat holography in four dimensions [30–32] and three dimensions [33–35].
In both cases, the holographic calculations consist with the field theory (or other holographic) results [37–40]. The corresponding geometric quantity $\mathcal{E}_A$ are spacelike geodesics in the bulk, thus consistent with the results in [11, 12]. However, unlike the RT surfaces, the endpoints of $\mathcal{E}_A$ are not anchored on $\partial A_\pm$, thus do not satisfy the homology constraint.

For example, in 3-dimensional flat space, the endpoints of $\mathcal{E}_A$ are in the bulk and connected to $\partial A_\pm$ by two null geodesics $\gamma_\pm$ normal to $\mathcal{E}_A$ [10]. Some recent work related to this geometric picture can be found in [41–43]. This new geometric picture of entanglement entropy with the extra null geodesics $\gamma_\pm$ is reformulated in [41] in the spirit of the HRT covariant formulation [7]. More interestingly, following a similar prescription [44, 45] for calculating holographic conformal blocks in the probe limit in AdS/CFT, the authors of [41] calculated the Poincaré blocks (or global BMS blocks) holographically by extremizing the length of a network of geodesics connected to the operators at the boundary through certain null geodesics. Based on these null geodesics an extrapolate dictionary is proposed in [41] (see also [46]) for 3-d flat holography.

For (warped) AdS$_3$ which duals to a WCFT, the geometric picture for holographic entanglement entropy is also a spacelike geodesic $\mathcal{E}_{\text{reg}}$ with endpoints in the bulk [9]. We will show that (which is not addressed in [9]) the endpoints of $\mathcal{E}_{\text{reg}}$ are also connected to the endpoints of $\mathcal{A}$ at the cutoff boundary by two null geodesics $\gamma_\pm$, which are normal to $\mathcal{E}_{\text{reg}}$ (see Fig.1). One can clearly see that the geodesic $\mathcal{E}$ does not satisfy the homology constraint.

![Figure 1](image)

Figure 1. The blue solid line is the $\mathcal{E}_{\text{reg}}$ which is regulated from the spacelike geodesic $\mathcal{E}$. The red and green lines are the null geodesics $\gamma_\pm$ that connect the endpoints of $\mathcal{E}_{\text{reg}}$ and $\mathcal{A}$.

While for the Lifshitz spacetime, it has been shown in [47] that the normal null hyper-surfaces emanating from the RT (or HRT) surface do not reach the boundary and thus do not enclose a bulk region. Which implies the inconsistence between the bulk and boundary causal structures for RT surfaces in Lifshitz spacetime.

The above results also imply that the prescription to regulate the entanglement entropy via the UV/IR cutoff relations is different from the case of AdS/CFT. Instead of being cut off at an infinitely large radius, the IR cutoff of the curve $\mathcal{E}$ in these cases are at finite
radius and in some way controlled by the null geodesics $\gamma_{\pm}$ emanating from the boundary entangling surface $\partial A$. In this paper we try to understand this new geometric picture for holographic entanglement entropy following the LM prescription [11]. We focus on the case of AdS$_3$/WCFT correspondence. In this holography model the gravity side is AdS$_3$ with the Compere-Song-Strominger (CSS) [36] boundary conditions while the field theory side is a WCFT. We study the replica story both on the boundary and in the bulk and try to understand the role of the null geodesics in the replica story.

We will not re-derive the two main results of [11, 12] listed above and admit they are true in general holographic models. However these results are not enough to determine the geometric picture for holographic entanglement entropy without solving the remaining two problems:

- how to determine the extremal surface $E$ in the bulk that matches $A$ if we do not use the homology constraint?
- how to regulate $E$ in the bulk accordingly when we regulate $A$ on the boundary?

The above two problems are the main tasks of this paper. For the first problem, we need to explicitly study the causal structure of the boundary field theory and find its match in the bulk. To solve the second problem, we use the prescription of [1] to study the fine structure in holographic entanglement. More explicitly we use a new geometric quantity named the modular plane, to slice the entanglement wedge. Under this construction we get a fine correspondence between the points on $A$ and the points on $E$. The point where we cut $A$ off and the point where we cut $E$ off is related by this fine correspondence.

The structure of this paper is in the following. In section 2 we present some interesting observations from the Rindler method which partially inspire the intrinsic prescription we will propose. Then we focus on the case of AdS$_3$/warped CFT correspondence. In section 3 we apply the Rindler method to this case. In section 4 we will explicitly study the bulk and boundary modular flows and define the modular planes. In section 5, we calculate the generalized gravitational entropy for AdS$_3$ with CSS boundary conditions with the help of modular planes. The goal of this section is to understand how do the null geodesics (or modular planes) relate the boundary and bulk cutoffs. Base on the above construction, in section 6 we propose an intrinsic prescription of calculate the generalized gravitational entropy for spacetimes with non-Lorentzian duals. At last, we apply our prescription to flat space in 3 dimensions in section 7. The Appendix A and B are written for special readers, who are interested in the extremal condition for $E_{\text{reg}}$ and the entanglement contour for WCFT.

2 New observations from the Rindler method

2.1 A brief introduction to Rindler method

In the field theory, the key step of the Rindler method is to construct a Rindler transformation, a symmetry transformation which maps the calculation of entanglement entropy to thermal entropy. The general strategy to construct Rindler transformations and its bulk
extension by using the symmetries of a QFT and holographic dictionary, is summarized in
the section 2 of [10]. Here we just give the main points of the Rindler method.
Consider a QFT on manifold $\mathcal{B}$ with a symmetry group $G$. The global symmetries,
whose generators are denoted by $h_j$, is a subset of the symmetry group. The Rindler
transformations $R$, which map a subregion $\mathcal{D}$ of $\mathcal{B}$ to a Rindler space $\tilde{\mathcal{B}}$ with infinitely far
away boundary, can be constructed in the following way:

1. The transformation $\tilde{x} = f(x)$ should be a symmetry transformation of the QFT.
2. The vectors $\partial_{\tilde{x}_i}$ in the Rindler space should be a linear combination of the global
generators in the original space
   \[
   \partial_{\tilde{x}_i} = \sum_j b_{ij} h_j, \tag{2.1}
   \]
   where $b_{ij}$ are arbitrary constants.
3. The bulk extension of the Rindler transformations is obtained by replacing the global
generators $h_j$ in (2.1) with their bulk duals, which are just the isometries of the bulk
space. Furthermore we require the metric of the Rindler bulk to satisfy the same
boundary conditions.

To construct the Rindler transformation, the strategy is to impose the requirement
of property 1 and property 2 simultaneously. The requirement that $R$ should be a local
symmetry will give extra confinements on the coefficients $b_{ij}$. The remaining independent
coefficients will control the size, position of $\mathcal{D}$ and the thermal circle of $\tilde{\mathcal{B}}$. Note that the
shape of $\mathcal{D}$ is determined by the symmetries and independent of the choice of the coefficients $b_{ij}$. The coordinate transformation $\tilde{x} = f(x)$ should be invariant under some imaginary
identification of the new coordinates $\tilde{x}_i \sim \tilde{x}_i + i\beta_i$. Such an identification will be referred
to as a “thermal” identification hereafter.

Our strategy to construct Rindler transformations only involves global generators, thus
has a natural way to extend to the bulk. According to the holographic dictionary, the global
generators $h_i$ of the asymptotic symmetry group are dual to the isometries of the bulk
spacetime. Then by replacing the $h_i$ generators with the generators of the bulk isometries
and requiring the Rindler bulk space to satisfy the same boundary conditions, we can get
the Rindler transformations in the bulk. The bulk extension of $\tilde{\mathcal{B}}$ (or the Rindler bulk)
should have a horizon whose Bekenstein-Hawking entropy gives the thermal entropy of the
field theory on $\mathcal{B}$. Using the inverse bulk Rindler transformations we can map this horizon
back to the bulk extension of the original field theory on $\mathcal{B}$. The image of this mapping will
give the geometric quantity $E$ that related to the holographic entanglement entropy. One
can consult [9, 10] for explicit examples of the Rindler method.

### 2.2 New observations from the Rindler method

In [10], with the inverse bulk Rindler transformations, the authors made several in-
teresting observations. In summary we have three ways to determine the position of the
extremal surface $E$:
• Firstly, the horizon of the Rindler bulk space is mapped to two codimension one null hypersurfaces $\mathcal{N}_\pm$ in the original bulk space. The curve $\mathcal{E}$ that related to entanglement entropy is the curve where $\mathcal{N}_-$ intersect with $\mathcal{N}_+$,

$$\mathcal{E} = \mathcal{N}_- \cap \mathcal{N}_+ .$$  \hfill (2.2)

• Secondly, the Hamiltonian in the Rindler bulk space and boundary $\tilde{\mathcal{B}}$ will be mapped to the modular Hamiltonian in the original bulk and boundary respectively. A modular flow $k_t (k_{t}^{bulk})$ is generated by the modular Hamiltonian. Then the curve $\mathcal{E}$ is determined by

$$k_{t}^{bulk}(\mathcal{E}) = 0 ,$$  \hfill (2.3)

which means $\mathcal{E}$ is the fixed points of the modular flow.

• Thirdly, the two null hypersurfaces $\mathcal{N}_\pm$ are formed by the normal null geodesic congruences of $\mathcal{E}$. The intersection between $\mathcal{N}_\pm$ and the boundary $\mathcal{B}$ gives a decomposition on the boundary, which is consistent with the causal structure of the dual field theory. This implies, on the other way around, $\mathcal{E}$ can be determined by the requirement that, the bulk causal decomposition associated to $\mathcal{E}$ should reproduce the causal structure of the boundary field theory associated to the corresponding boundary subregion $\mathcal{A}$.

Although these observations are made in 3d flat holography, the logic behind them should work for general holographic theories. The first way is just a mapping from the Rindler bulk to the original spacetime and follows the logic of Rindler method. The second way is equivalent to the statement that $\mathcal{E}$ is the fixed points of the bulk replica symmetry which is a key statement of the LM prescription [11]. In the third way, the requirement that the bulk and boundary causal structure should be consistent is obviously a requirement of holography\(^4\). Later we will explicitly show that the above observations are also true in the context of AdS\(_3\)/WCFT.

Note that, the first and second ways are hard to apply because they rely on the explicit information of the Rindler transformations and locally defined modular Hamiltonian, which may not even exist. While the third way is applicable for general spacetimes without the need to construct the Rindler transformations and find the explicit form for the locally defined modular Hamiltonians. With both sides of the holography model given, for a general extremal codimension two surface $\mathcal{E}$ we can always do the bulk causal decomposition using the normal null hypersurfaces $\mathcal{N}_\pm$ of $\mathcal{E}$. When $\mathcal{N}_\pm$ intersect with $\mathcal{B}$, we match the decomposition with the boundary causal structure associated to a subregion $\mathcal{A}$, thus select the right $\mathcal{E}$ that correspond to $\mathcal{A}$. In holographic models with non-Lorentzian duals, this requirement for the consistence of bulk and boundary causal structures can play the role of the homology constraint in the RT formula. Along this line, it is possible to give an intrinsic proposal for the geometric picture of holographic entanglement entropy for these holographic theories.

\(^4\)In the context of AdS/CFT this requirement has been discussed in detail in [18].
3 Rindler method applied on AdS$_3$/WCFT correspondence

3.1 AdS$_3$ with CSS boundary conditions

Let us give a quick review on the AdS$_3$/WCFT correspondence. In the Fefferman-Graham gauge, solutions to 3-dimensional Einstein gravity with a negative cosmological constant are in the following,

$$\frac{ds^2}{\ell^2} = \frac{d\eta^2}{\eta^2} + \eta^2 \left( g^{(0)}_{ab} + \frac{1}{\eta} g^{(2)}_{ab} + \frac{1}{\eta^4} g^{(4)}_{ab} \right) dx^a dx^b, \quad (3.1)$$

where $\eta$ is the radial direction, and $x^a$, $a = 1, 2$ parametrize the boundary. Under the Dirichlet boundary (Brown-Henneaux) conditions $\delta g^{(0)}_{ab} = 0$, the asymptotic symmetry are generated by two copies of Virasoro algebra [48], which indicates the dual field theory is a CFT$_2$.

In [36], a Dirichlet-Neumann type of boundary conditions

$$\delta g^{(0)}_{+-} = 0, \quad \delta g^{(0)}_{++} = 0, \quad \delta g^{(2)}_{--} = 0, \quad (3.2)$$

is considered for AdS$_3$ in Einstein gravity, which we refer as the CSS boundary conditions. Under the CSS boundary conditions the metric on the boundary is no longer fixed and is allowed to fluctuate. Without a fixed background metric the usual way to determine the causal structure of the boundary field theory with null geodesics (or hypersurfaces) associated to $\partial A$ is meaningless. In these cases, one can still define the causal development $D_A$ as the subregion in $\mathcal{B}$, which is mapped to the whole Rindler space $\tilde{\mathcal{B}}$ under Rindler transformations (see section 2.1). This definition of causal development is more general, and will reduce to the definition using null lines when the boundary has a fixed background metric.

Consider a BTZ metric

$$ds^2 = \ell^2 \left( T_u^2 du^2 + 2r du dv + T_v^2 dv^2 + \frac{dr^2}{4 (r^2 - T_u^2 T_v^2)} \right). \quad (3.3)$$

When we impose the CSS boundary conditions, the asymptotic symmetry group is featured by a Virasoro-Kac-Moody algebra [36],

$$[\hat{L}_n, \hat{L}_m] = (n - m) \hat{L}_{n+m} + \frac{\tilde{c}}{12} (n^3 - n) \delta_{n+m},$$

$$[\hat{L}_n, \hat{P}_m] = - m \hat{P}_{m+n} + m \hat{P}_0 \delta_{n+m},$$

$$[\hat{P}_n, \hat{P}_m] = \frac{\tilde{k}}{2} n \delta_{n+m}, \quad \tilde{k} = - \frac{G T_v^2}{T_u}, \quad (3.4)$$

with the central charge and Kac-Moody level given by

$$\tilde{c} = \frac{3\ell}{2G}, \quad \tilde{k} = - \frac{G T_v^2}{T_u}. \quad (3.5)$$

Here we have set $T_v$ fixed to satisfy the Neumann part of the boundary conditions. Then $u$ is the direction that keeps the $SL(2, \mathbb{R})$ global symmetry. Although the local isometry...
in the bulk is still \( SL(2, R) \times SL(2, R) \), only the \( SL(2, R) \times U(1) \) part consists with the boundary conditions.

The above asymptotic symmetry analysis indicates that the dual field theory is a WCFT featured by the algebra (3.4). The WCFT has the following local symmetries [28]

\[
    u = f(u') , \quad v = v' + g(u') .
\]  

(3.6)

A Similar story of asymptotic symmetry analysis happens for warped AdS\(_3\) [49], and the WAdS/WCFT correspondence [28] is conjectured.

The (warped) AdS\(_3\) WCFT correspondence has passed several key tests by matching the thermal entropy [28], entanglement entropy [9, 37], correlation functions [29] and one-loop determinants [45]. See [50–52] for a few examples of WCFT models.

Note that the Kac-Moody level in (3.4) is charge dependent. This is different from the canonical warped CFT algebra [28] which has a constant Kac-Moody level. These two algebras can be related by a state dependent coordinate transformation. One can also obtain the canonical algebra using the state-dependent asymptotic Killing vectors [53]. The mapping between the entanglement entropies of the theories featured by these two algebras is explicitly discussed in [9]. In this paper, by WCFT we mean the theory featured by the algebra (3.4), and will not do the further mapping to the canonical ones.

### 3.2 Rindler method for AdS\(_3\) with CSS boundary conditions

The global symmetries of the asymptotic symmetry group of AdS\(_3\) with CSS boundary conditions is \( SL(2, R) \times U(1) \), which consist of the following generators

\[
    \begin{align*}
        J_- &= \partial_u , \quad J_0 = u\partial_u - r\partial_r , \quad J_+ = u^2\partial_u - \frac{1}{2r}\partial_v - 2ru\partial_r , \quad \bar{J} = \partial_v ,
    \end{align*}
\]

(3.7)

Now we consider the AdS\(_3\) with \( T_u = 0, T_v = 1 \), and the AdS radius being \( \ell = 1 \),

\[
    ds^2 = 2rdudv + (dv)^2 + \frac{(dr)^2}{4r^2} .
\]

(3.8)

On the boundary we choose the interval to be

\[
    \mathcal{A} : \{(\frac{l_u}{2}, \frac{l_v}{2}) \rightarrow (\frac{l_u}{2}, \frac{l_v}{2})\} ,
\]

(3.9)

One can consult [9] for the cases with general temperatures.

Imposing the three requirements in section 2.1, we can construct the most general Rindler transformations (see Appendix A in [9]). The coefficients \( b_{ij} \) controls the position, size of \( D \) and the thermal circle of \( \bar{\mathcal{B}} \). Here for simplicity, we settle down the position of \( D \) and the thermal circle of \( \bar{\mathcal{B}} \), which do not affect the entanglement entropy. Then we get the Rindler transformations from the AdS\(_3\) (3.8) to a Rindler \( \widetilde{\text{AdS}}_3 \) with \( \widetilde{T}_u = \widetilde{T}_v = 1 \)

\[
    ds^2 = d\tilde{u}^2 + 2\tilde{r}d\tilde{u}d\tilde{v} + d\tilde{v}^2 + \frac{d\tilde{r}^2}{4(\tilde{r}^2 - 1)} ,
\]

(3.10)
The Rindler transformations are given by

\[
\begin{align*}
\tilde{u} &= \frac{1}{4} \log \left( \frac{R^2 - 1}{R^2 + 1} \right), \\
\tilde{v} &= \frac{1}{4} \log \left( \frac{(R_+ - 1)(R_- + 1)}{(R_+ + 1)(R_- - 1)} \right) + v, \\
\tilde{r} &= \frac{(R_+ - 1)(R_- - 1)}{R_+ + R_-} + 1,
\end{align*}
\] (3.11)

where

\[R_\pm = r(l_u \mp 2u).\] (3.12)

Asymptotically, the transformations is given by

\[
\begin{align*}
\tilde{u} &= \text{Arctanh} \left( \frac{2u}{l_u} \right), \\
\tilde{v} &= v,
\end{align*}
\] (3.13)

which, as expected, is a warped conformal mapping (3.6). We see that the \((\tilde{u}, \tilde{v})\) coordinates only covers a subregion

\[\mathcal{D} : -\frac{l_u}{2} < u < \frac{l_u}{2}.\] (3.14)

with the shape of a strip. This means we find a “Rindler observer” who can only see the information inside a strip subregion. We define this strip \(\mathcal{D}\) as the causal development of the interval \(\mathcal{A}\) (see also [37]) in WCFT.

The bulk Rindler transformations (3.11) map the horizon of the Rindler \(\widetilde{\text{AdS}}_3\) at \(\tilde{r} = 1\) to two null surfaces \(\mathcal{N}_\pm : R_\pm = 1\), or equivalently

\[\mathcal{N}_+ : \quad r = \frac{1}{l_u - 2u}, \quad \mathcal{N}_- : \quad r = \frac{1}{l_u + 2u}.\] (3.15)

It is easy to see that, \(\mathcal{N}_\pm\) intersect with the asymptotic boundary on \(u = \pm \frac{l_u}{2}\), thus enclose a strip \(-\frac{l_u}{2} < u < \frac{l_u}{2}\) on the boundary, which is just the strip region \(\mathcal{D}\) (3.14). This reproduces the causal structure of the boundary WCFT. Also we find \(\mathcal{N}_\pm\) intersect at a curve in the bulk

\[\mathcal{E} = \mathcal{N}_- \cup \mathcal{N}_+ : \quad \left\{ u = 0, r = \frac{1}{l_u} \right\}.\] (3.16)

which is just the curve found in [9] that related to the holographic entanglement entropy. This is exactly the first observation we made from Rindler method.

According to the logic of Rindler method, the thermal entropy for \(\widetilde{\text{AdS}}_3\) gives the entanglement entropy of \(\mathcal{A}\). Since the thermal entropy is infinite, we need to regulate the interval \(\mathcal{A}\) by a cutoff \(\epsilon_u\) along the \(u\) direction, such that

\[\mathcal{A}_{\text{reg}} : \quad \left\{ \left( -\frac{l_u}{2} + \epsilon_u, -\frac{l_v}{2} \right) \to \left( \frac{l_u}{2} - \epsilon_u, \frac{l_v}{2} \right) \right\}.\] (3.17)
As a consequence the extension of the horizon in Rindler $\tilde{\text{AdS}}_3$, as well as the curve $E$ in the original $\text{AdS}_3$, are also regulated. We find the regulated $E$ is given by $[9]$

\[ E_{\text{reg}} : \left\{ (u,v,r) \mid u = 0, \ r = \frac{1}{l_u}, \ v = \ell \left( l_v + \log \frac{l_u}{\epsilon_u} \right) \left( \eta - \frac{1}{2} \right), \ \eta \in [0,1] \right\}. \] (3.18)

We see that $E$ is cut off in the bulk at a finite radius $r = \frac{1}{l_u}$, rather than the asymptotic boundary. The holographic entanglement entropy is then given by

\[ S_{E\text{E}} = \frac{\text{Length}(E_{\text{reg}})}{4G} = \frac{1}{4G} \left( l_v + \log \frac{l_u}{\epsilon_u} \right). \] (3.19)

Note that there is no need to introduce the cutoff $\epsilon_v$ along the $v$ direction since it can be taken to be zero without introducing extra divergence for the entanglement entropy.

Before going on, let us comment on the case of W$\text{AdS}_3$/WCFT. As was pointed out in $[9]$, the Rindler method applied on warped $\text{AdS}_3$ is indeed the same as the above story on $\text{AdS}_3$. Because the warping factor does not appear in neither the Rindler transformations nor the thermodynamic quantities. For simplicity we only focus on the case of $\text{AdS}_3$/WCFT.

4 Modular flows and modular planes

The Rindler method can also help us find the explicit formula for the modular flow. The generator of the normal Hamiltonian in Rindler space or Rindler bulk, which maps to the modular Hamiltonian in the original space, is the generator along the thermal circle. In other words $k_t = \tilde{\beta}^i \partial_{\tilde{x}^i}$. Since $\partial_{\tilde{x}^i}$ can be written as a linear combination of the global generators, $k_t$ should have the same property. Using the holography dictionary, we can easily get the bulk dual of $k_t$, which we call $k_t^{\text{bulk}}$. In order to map it to the original space, we need to solve the following differential equations

\[
\begin{align*}
\partial_u &= (\partial_u \tilde{u}) \partial_{\tilde{u}} + (\partial_u \tilde{v}) \partial_{\tilde{v}} + (\partial_u \tilde{r}) \partial_{\tilde{r}}, \\
\partial_v &= (\partial_v \tilde{u}) \partial_{\tilde{u}} + (\partial_v \tilde{v}) \partial_{\tilde{v}} + (\partial_v \tilde{r}) \partial_{\tilde{r}}, \\
\partial_r &= (\partial_r \tilde{u}) \partial_{\tilde{u}} + (\partial_r \tilde{v}) \partial_{\tilde{v}} + (\partial_r \tilde{r}) \partial_{\tilde{r}}.
\end{align*}
\] (4.1)

So we can get $\partial_{\tilde{u}}, \partial_{\tilde{v}}, \partial_{\tilde{r}}$, and furthermore $k_t^{\text{bulk}}$, in terms of $\partial_u, \partial_v, \partial_r$.

We plug (3.11) into (4.1). Solving the equations we get the bulk and boundary modular flow

\[
\begin{align*}
k_t^{\text{bulk}} &= -\tilde{\beta}^u \partial_u + \tilde{\beta}^u \partial_v = \pi (\partial_v - \partial_u) \\
&= \frac{\pi (r^{-2} + 4u^2 - l_u^2)}{2 l_u} \partial_u + \left( \pi - \frac{\pi}{l_u r} \right) \partial_v - \frac{4 \pi r u}{l_u} \partial_r, \quad (4.2) \\
k_t &= \frac{\pi (4u^2 - l_u^2)}{2 l_u} \partial_u + \pi \partial_v. \quad (4.3)
\end{align*}
\]

It is easy to check that the curve $E$ (3.16) we find by Rindler method can also be determined by

\[ k_t^{\text{bulk}}(E) = 0. \] (4.4)
Which means $\mathcal{E}$ is the fixed points of $k_t^{bulk}$, thus the fixed points to bulk replica symmetry. This is exactly the second observation we made from the Rindler method. Obviously the endpoints of $\mathcal{A}$ are neither the fixed points of $k_t$ nor $k_t^{bulk}$. This, from the modular flow point of view, means $\mathcal{E}$ should not be anchored on $\partial \mathcal{A}$, thus break the homology constraint.

We can get the explicit picture of the flow from (4.2) and (4.3). Solving the equation

$$\left( \frac{du(s)}{ds}, \frac{dv(s)}{ds} \right) = k_t,$$

(4.5)

we get the lines along modular flow on the boundary. They are given by

$$u(s) = -\frac{l_u}{2} \tanh (\pi s) \quad v(s) = \pi s + v_0$$

(4.6)

where we have set $u(0) = 0$ and $v_0$ is a integration constant that characterizes different modular flow lines. We can use $\mathcal{L}_{v_0}$ to denote all the boundary flow lines, hence $v_0$ is the $v$ coordinate of the point where $\mathcal{L}_{v_0}$ intersect with the line $u = 0$. It is easy to see that $\mathcal{L}_{v_0}$ is anti-symmetric with respect to its middle point $(0, v_0)$. More straight forwardly we can write

$$\mathcal{L}_{v_0} : \quad v = v_0 - \text{arctanh} \frac{2u}{l_u}.$$  

(4.7)

The picture of modular flow on the boundary is shown in Fig.2.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure2.png}
\caption{The orange lines with arrows depict the trajectory of the modular flow in WCFT. Note that the modular flow can never pass through $\partial \mathcal{D}$, which is depicted by the two purple lines.}
\end{figure}

Similarly, by solving the equation

$$\left( \frac{du(s)}{ds}, \frac{dv(s)}{ds}, \frac{dr(s)}{ds} \right) = k_t^{bulk},$$

(4.8)
we get the functions of the bulk modular flow lines. Then we change the parameter from \( s \) to \( r \), and find the modular flow lines are described by the following two branches of solutions

\[
\text{branch A : } \left\{ \begin{array}{l}
u(u) = -\frac{1}{2} u^{\sqrt{r - r_0}} \\
u(v) = \bar{v}_0 + \frac{1}{2} \log \left( \frac{1 + \sqrt{r - r_0}}{1 - \sqrt{r - r_0}} \right) 
\end{array} \right.
\]

\[
\text{branch B : } \left\{ \begin{array}{l}
u(u) = \frac{1}{2} u^{\sqrt{r - r_0}} \\
u(v) = \bar{v}_0 - \frac{1}{2} \log \left( \frac{1 + \sqrt{r - r_0}}{1 - \sqrt{r - r_0}} \right) 
\end{array} \right.
\]

The constants \( r_0 \) and \( \bar{v}_0 \) are the integration constants characterizing different bulk modular flow lines. The bulk modular flow lines have the following properties:

1. Each bulk modular flow line consist of A branch part and B branch part which are described by (4.9) and (4.10) respectively. These two parts are smoothly connected at the point \((u, v, r) = (0, \bar{v}_0, r_0)\), which is a turning point along the \( r \) direction.

2. The bulk modular flow lines and the turning points are in one-to-one correspondence. So we can denote them as \( \mathcal{L}_{\bar{v}_0} \).

3. The two branches (4.9) and (4.10) both give \( v(r) = \bar{v}_0 + \frac{1}{2} \log \left( \frac{1 + \sqrt{r - r_0}}{1 - \sqrt{r - r_0}} \right) \). So asymptotically we have

\[
v = \bar{v}_0 - \tanh^{-1} \left( \frac{2u}{I_u} \right) + O \left( \frac{1}{r} \right).
\]

\[
(4.11)
\]

Compare (4.11) with (4.7), we find the projection of \( \mathcal{L}_{\bar{v}_0} \) on the boundary is just the boundary modular flow line \( \mathcal{L}_{v_0} \) with

\[
\bar{v}_0 = v_0.
\]

(4.12)

With the explicit picture of bulk and boundary modular flows, following [1] we then define the geometric quantity which we call the modular plane. When \( r \to \infty \), all \( \mathcal{L}_{\bar{v}_0} \) will anchor on the two lines \( u = \pm \frac{1}{2} \) (or \( \partial D \)) at the boundary. However when we push the boundary into the bulk a little, or take a large but finite \( r = r_I \), the class of \( \mathcal{L}_{\bar{v}_0} \) with \( \bar{v}_0 \) fixed by (4.12), will intersect with the boundary on \( \mathcal{L}_{v_0} \). This class of bulk modular flow lines form a codimension one surface in the bulk, which we call the modular plane \( \mathcal{P}(v_0) \). In other words,

- the modular plane \( \mathcal{P}(v_0) \) is the orbit of the boundary modular flow line \( \mathcal{L}_{v_0} \) under the bulk modular flow.

The modular planes are in one-to-one correspondence with the boundary modular flow lines. See Fig.3 for an explicit diagram for modular plane. Later we will show that the modular planes play a crucial role on how we regulate \( \mathcal{E} \) in the bulk.

Another class of bulk modular flow lines are those with \( r_0 = \frac{1}{I_u} \). The A branch and B branch intersect at the fixed points \( \mathcal{E} \) of \( k^b \). It is easy to check that these modular flow
lines are null, and furthermore the A and B branch respectively form the two normal null hypersurfaces $N_{\pm}$ (3.15) emanating from $\mathcal{E}$. We denote this class of bulk modular flow lines as $\mathcal{L}_{v_0}$

$$\mathcal{L}_{v_0} : \begin{cases} u = \frac{1}{2} \left( \frac{1}{r} - l_u \right), \\ v = \bar{v}_0 + \frac{1}{2} \log(l_u r), \quad \text{branch A}, \\ u = -\frac{1}{2} \left( \frac{1}{r} - l_u \right), \\ v = \bar{v}_0 - \frac{1}{2} \log(l_u r), \quad \text{branch B}. \end{cases}$$  (4.13)

Figure 3. The left figure gives an explicit diagram for a modular plane $P(v_0)$. The blue line is $\mathcal{L}_{v_0}$ and the dashed line depict $u = 0$ on the boundary, which intersect with $\mathcal{L}_{v_0}$ at $(0, v_0, \infty)$. The orange lines are the class of $\mathcal{L}_{v_0}$ with $\bar{v}_0 = v_0$ and $r \geq \frac{1}{l_u}$. The red line is the one with $r = \frac{1}{l_u}$ which lies on the null hypersurfaces $N_{\pm}$. Its turning point is $\mathcal{E}(v_0) = (0, v_0, \frac{1}{l_u})$. The other two black points on the right and left hand side are the points where $\mathcal{L}_{v_0}$ intersect with the red line at $v_0 = \pm \infty$. The right figure is just the projection of the left figure to a flat plane. The orange arrows describe the direction of the flow.

5 Generalized gravitational entropy for AdS$_3$ with CSS boundary conditions

In this section we try to understand how the LM prescription [11, 12] works in AdS$_3$ with the CSS boundary conditions. In the rest of this section, we will use the terminologies in [12]. For simplicity we will not repeat the Schwinger-Keldysh construction or time-folded path integral as in [12], but calculate the trace of reduced density matrix by performing the path integral over the whole spacetime. The generalization to the time-folded path integral can be obtained following the lines in [12]. Before we proceed, we would like to give a brief summary on the similarities and differences between our construction and the construction in [11, 12].

The main stories are the same. We also try to extend the replica story of the boundary field theory into the bulk, and assume the replica symmetry is unbroken in the bulk. The main requirement is that the bulk replica story should reproduce the replica story of the dual field theory on the boundary. The way we calculate entanglement entropy are the same,
by using the equivalence of the partition functions on both sides. As we have mentioned
the main results of [11, 12] applies to holographies beyond AdS/CFT. More explicitly, the
holographic entanglement entropy is measured by the area of an extremal codimension two
surface $\mathcal{E}$ in Planck units, which is fixed under the bulk extended replica symmetry in the
bulk. However they are not enough to calculate the entanglement entropy for the cases
beyond AdS/CFT. To make the prescription complete, we also have significant differences.

- The main difference is that, the consistence between the causal structures of the
  boundary WCFT and the AdS$_3$ bulk require the extremal surface $\mathcal{E}$ not to touch the
  boundary. This means we should not impose the homology constraint. In this sense
  the extremal surface $\mathcal{E}$ is not a RT type.

- As well as the entanglement entropy, the length of the curve $\mathcal{E}$ is infinitely long. When
  we introduce a cutoff for the interval thus regulate the entanglement entropy, there
  should be a corresponding regulation for the length of $\mathcal{E}$. In other words we should
cut $\mathcal{E}$ off thus regulate it to an finite interval $\mathcal{E}_A$ in the bulk. As $\mathcal{E}$ does not go to
the boundary, it cannot be regulated by taking a cutoff on $r = r_I$ as the RT surfaces.
How to regulate $\mathcal{E}$ accordingly is the main problem of this paper. We will use the
modular planes to relate the bulk and boundary cutoffs.

We will give the replica story for WCFT in subsection 5.1, then we construct the dual
bulk replica story in subsection 5.2. Then in subsection 5.3 we follow the prescription
in [1] to study the fine structure of the bulk story using the modular planes. With the
fine structure known, we will show the UV and IR cutoffs can be naturally related by the
modular planes. This explains the role of the null geodesics in the new geometric picture for
entanglement entropy. At last, based on the fine structure analysis, we propose an intrinsic
prescription of construct the geometric picture of entanglement entropy in subsection 5.4.

5.1 The replica story on the boundary

The field theory dual of AdS$_3$ (3.8) with CSS boundary conditions is a WCFT with
the following thermal circle

$$(u, v) \sim (u, v - \pi i). \quad (5.1)$$

Similar to the strategy in [12] we can decompose the spacetime into several regions with
Rindler coordinates $(\tilde{u}_i, \tilde{v}_i)$. Since there is no fixed metric background, instead of using
the null lines we need to use the Rindler transformations to do the causal decomposition. We
can simultaneously refer to all these spacetime regions in question, by allowing the Rindler
coordinates to be complex and assigning them discrete imaginary parts. The subscript $i$
of the Rindler coordinates denote different spacetime regions. Moving along the modular
flow and jump from one region to another, we hop from one imaginary part to another. If
eventually we jump back to our starting point, the imaginary parts we have gone through
form a circle, which can be physically understood as the thermal circle measured by the
Rindler observer. Putting all the regions together, whose imaginary parts form the total
thermal circle, we should recover a pure state.
Cutting open the WCFT along $\mathcal{A}$ then gluing all the copies cyclically are not the whole replica story. We also need to know how the thermal circle changes under this cyclical gluing. The key to understand this is the assignment of the imaginary parts to all the spacetime regions. Since WCFT is not a relativistic field theory, its causal structure is quite abnormal compared with the relativistic ones, hence deserves discussion in detail.

For convenience, here we re-write the boundary Rindler transformations (3.13)

$$\tanh(\tilde{u}) = \frac{2u}{l_u}, \quad \tanh(\tilde{v}) = \tanh v.$$  \hspace{1cm} (5.2)

We write the second equation in such a way that the thermal circles in both of the original and Rindler space are exhibited in the coordinate transformation, ie. (5.1) and

$$(\tilde{u}, \tilde{v}) \sim (\tilde{u} + \pi i, \tilde{v} - \pi i).$$ \hspace{1cm} (5.3)

Then it is convenient to confine

$$0 \leq \text{Im}(\tilde{u}) \leq i\pi, \quad -i\pi \leq \text{Im}(\tilde{v}) \leq 0.$$ \hspace{1cm} (5.4)

As we have seen that, the Rindler transformations divide the original spacetime $\mathcal{B}$ into three spacetime regions by two “horizons”. The three regions include the strip $-\frac{l_u}{2} < u < \frac{l_u}{2}$ region, the left region $u < -\frac{l_u}{2}$ and the right region $u > \frac{l_u}{2}$. We require the assignment of imaginary parts should consist with the Rindler transformations, in other words the Rindler transformations should map the Rindler space $(\tilde{u}_i, \tilde{v}_i)$ with different imaginary parts to different spacetime regions on $\mathcal{B}$. Furthermore we require the imaginary part of each region should be unique and different from the other regions.

Note that, the assignment is not uniquely determined by the above requirements. For example, for the strip region we can chose either the assignment $\text{Im}[(\tilde{u}, \tilde{v})] = (0, 0)$ or $\text{Im}[(\tilde{u}, \tilde{v})] = (0, -i\pi)$. Both of the choices consist with the Rindler transformations. However all the choices satisfying our requirements can be related by a rotation or reversion of the thermal circle, thus do not change the physical story. Here we choose the following assignment for the three regions on $\mathcal{B}$

$$\text{Im}[(\tilde{u}, \tilde{v})] = \begin{cases} (0, 0), & \frac{l_u}{2} < u < \frac{l_u}{2}, \\ \left(\frac{\pi}{2}, 0\right), & u < -\frac{l_u}{2}, \\ \left(\frac{\pi}{2}, -\pi i\right), & u > \frac{l_u}{2}. \end{cases}$$ \hspace{1cm} (5.5)

According to uniqueness requirement for the imaginary part, there is no place on $\mathcal{B}$ for the assignment $\text{Im}[(\tilde{u}, \tilde{v})] = (0, -\pi i)$. This can be understood in the following way. The state on $\mathcal{B}$ is already mixed, which indicates the complete thermal circle involves some region outside $\mathcal{B}$ that purifies $\mathcal{B}$. We denote this region as $\mathcal{B}^c$ and consider $\mathcal{B} \cup \mathcal{B}^c$ as the spacetime that recovers the pure state. The proper place where we can assign the last imaginary part is $\mathcal{B}^c$

$$\mathcal{B}^c : \quad \text{Im}[(\tilde{u}, \tilde{v})] = (0, -\pi i).$$ \hspace{1cm} (5.6)

We will see that the above assignment is quite natural in the gravity side story.
It will be more convenient to introduce a coordinate that parametrize the modular flow. We chose the new coordinate to be the Rindler time
\[ \tau = \tilde{u} - \tilde{v}, \] (5.7)
thus the thermal circle is given by \( \tau \sim \tau + 2\pi i \). As in [12], we define \( \tau_m \) to be
\[ \tau_m = \tau + (m - 1)i\pi/2. \] (5.8)

According to (5.5) and (5.7), the three regions on \( \mathcal{B} \) can be denoted by \( \tau_1, \tau_2 \) and \( \tau_4 \) (see the left figure in Fig.4), while the \( \mathcal{B}^c \) can be denoted as \( \tau_3 \). For each time we cross the “horizon”, we add a \( i\pi/2 \) to \( \tau \).

Then we construct the replica story on the field theory side. We cut \( \mathcal{A} \) open then the strip region is further divided into \( \tau_1, \tau_5 \) (see the right figure in Fig.4). Starting from some point in the \( \tau_1 \) region then move along the modular flow and cross the horizons, we will get to the \( \tau_5 \) region. Then we consider \( n \) copies of \( \mathcal{B} \) and set boundary conditions for the fields in WCFT on the upper cut \( \mathcal{A}_+ \) and lower cut \( \mathcal{A}_- \) respectively
\[ \phi(\mathcal{A}_+) = \phi_+, \quad \phi(\mathcal{A}_-) = \phi_. \] (5.9)

The path integral with these boundary conditions on \( \mathcal{A}_\pm \) gives the reduced density matrix \( (\rho_{\mathcal{A}})^{++} \). To calculate \( \text{Tr}(\rho_{\mathcal{A}}^n) \) we need to glue \( n \) copies of \( \mathcal{B} \) cyclically in the following way
\[ \phi_I(\mathcal{A}_-) = \phi_{(I+1)}(\mathcal{A}_+), \quad \phi_n(\mathcal{A}_-) = \phi_1(\mathcal{A}_+), \quad I = 1, \cdots, n - 1. \] (5.10)

After the gluing we get a \( n \)-sheet manifold \( \mathcal{B}_n \) with replica symmetry. Then we perform path integral over \( \mathcal{B}_n \). For \( n = 1 \), the \( \tau_5 \) region is glued back to the \( \tau_1 \) region at \( \mathcal{A} \), in other words the thermal circle on \( \mathcal{B} \) is \( \tau \sim \tau + 2\pi i \). Similarly we find the thermal circle on \( \mathcal{B}_n \) becomes \( \tau \sim \tau + 2\pi ni \).
The fixed point of the thermal circle, or the replica symmetry, should be shrinking point of the thermal circle. More explicitly it should be the joint point of all the regions. However, unlike the case of CFT$_2$ (or other relativistic theories), there is no such point in WCFT. This is not surprising as we have already seen that the modular flow $k_t$ (4.3) is nonvanishing everywhere on $\mathcal{B}$. In other words we have replica symmetry on $\mathcal{B}_n$, but there is no fixed point for this symmetry on $\mathcal{B}_n$.

5.2 The replica story in the bulk

In this subsection we try to construct the bulk extension of the boundary replica story. As in [11, 12], we need to make the basic assumptions to extend the boundary replica story into the bulk. Firstly we assume the AdS$_3$/WCFT correspondence between the bulk and boundary theories, thus the partition functions on both sides are equivalent. The other assumption is that the replica symmetry can be extended into the bulk.

5.2.1 Selecting the bulk curve by matching the bulk and boundary causal structures

According to [11] the curve $\mathcal{E}$ that is fixed under the bulk replica symmetry is a geodesic in the bulk. We do the bulk causal decomposition using the two normal null hypersurfaces $\mathcal{N}_\pm$ emanating from $\mathcal{E}$. Then $\mathcal{N}_\pm$ decompose the bulk into four regions, which we can also denote with $\tau_m$. Here $\tau_m$ parametrizes the bulk modular flow. Also we expect $\mathcal{N}_\pm$ to intersect with the boundary and simultaneously give a decomposition for the boundary. This boundary decomposition is expected to be consistence with the causal structure of the dual field theory. With an AdS bulk and a RT surface, the boundary decomposition in this sense is consistent with the causal structure of a CFT. This has already been explicitly discussed in general in [17] (see also [1] from the Rindler method point of view). However, in our case the dual field theory is a WCFT, whose causal structure (see Fig. 4) is quite different from CFT. We need to chose geodesics that does not touch the boundary.

Before we give the right geodesic $\mathcal{E}$, we would like to find out all the geodesics in the bulk. The spacelike geodesics in the AdS$_3$ (3.3) with $T_u = 0$ satisfy the following equations of motion

\[ \frac{c_1}{\ell^2} = r \dot{v}, \]  \tag{5.11}

\[ \frac{c_2}{\ell^2} = r \dot{u} + T_v^2 \dot{v}, \]  \tag{5.12}

\[ \frac{1}{\ell^2} = T_v^2 \dot{v}^2 + 2r \dot{u} \dot{v} + \frac{\dot{r}^2}{4v^2}, \]  \tag{5.13}

where we $c_1$ and $c_2$ are two integration constants, satisfying $c_1 c_2 > 0$, and dot represent differential with respect to the affine parameter $s$. From (5.11) and (5.12) we get

\[ \dot{u} = \frac{c_2 r - c_1 T_v^2}{\ell^2 r^2}, \quad \dot{v} = \frac{c_1}{r \ell^2}. \]  \tag{5.14}

Substituting the above equations into (5.13) we get a radial equation

\[ \dot{r} = \pm \frac{2 \sqrt{r (\ell^2 r - 2c_1 c_2) + T_v^2 c_1^2}}{\ell}, \]  \tag{5.15}
We get three types of spacelike geodesics by adjusting the value of $|c_2|$. Firstly when
\[
|c_2| > \ell T_v ,
\]
we find $\dot{r} = 0$ at
\[
r_{\pm} = \frac{c_1 c_2 \pm \sqrt{c_1^2 (c_2^2 - \ell^2 T_v^2)}}{\ell^2} > 0.
\]
This type of geodesic emanating from $r = \infty$ will turn around at $r_{\pm}$ then go back to the boundary $r = \infty$, thus belongs to the RT curves.

The Second type of geodesic satisfy $|c_2| < \ell T_v$, which has no turning points. They emanate from the $r = \infty$ boundary and goes through the horizon, and at last touch the other boundary at $r = -\infty$.

Both of the above two types do not satisfy our requirement. However the third type of geodesics that never touch the boundary arises when
\[
|c_2| = \ell T_v .
\]
In this case we find that, at $r = r_h = \frac{c_1 T_v}{\ell}$,
\[
\dot{r} = \dot{u} = 0 , \quad \dot{v} = \frac{1}{c_2} = \pm \frac{1}{\ell T_v}.
\]
The corresponding geodesics $E$ lines along the $v$ direction at a fixed radius $r = r_h$, thus never touch the boundary. Using a parameter $s$ we can write down the function of $E$
\[
E : \{ r = \frac{c_1 T_v}{\ell} , \ u = u_0 , \ v = \frac{s}{\ell T_v} + v_0 \},
\]
where $u_0$, $v_0$ are arbitrary constants. In the case (3.8) we study, we set $T_v = 1$. Note that, for our coordinates (3.3), the global AdS correspond to $T_u = T_v = \frac{1}{2}$ and Poincaré AdS correspond to $T_u = T_v = 0$. In these two cases there are no spacelike geodesics that do not touch the boundary.

It is interesting that, the geodesics (5.20) coincide with the curve (3.16) we found by Rindler method if we define $c_1 = \frac{1}{\ell u}$. This is a strong evidence that the geodesics (5.20) will be the curves that satisfy our requirements. Let us give a more concrete argument. As we have discussed in section 4, the bulk modular flows $\tilde{L}_{i_0}$ (4.13) that intersect at $E$ are null. It is not hard to check that $\tilde{L}_{i_0}$ are also null geodesics in the bulk and normal to $E$. These together indicate that the normal null hypersurfaces associated to $E$ (5.20) are just the $N_{\pm}$ given by (3.15). As we have already shown that $N_{\pm}$ intersect with the boundary at $\partial D$, i.e. the two “horizons” $u = \pm \ell T_v$, thus the boundary decomposition consists with the causal structure for the boundary WCFT (see Fig.5). The bulk subregion enclosed by $B$ and $N_{\pm}$ is the analogue of the entanglement wedge $W_A$ in the case of AdS/CFT. In this paper we also denote this region as $W_A$.

Note that the $\tau_3$ region in the bulk does not overlap with the boundary $B$. This is consistent with our statements that there is no $\tau_3$ region on $B$. 

– 19 –
Figure 5. The causal decomposition of the bulk spacetime $\mathcal{M}$ and the boundary $\mathcal{B}$. The two surfaces that intersect at $\mathcal{E}$ are the two normal null hypersurfaces $\mathcal{N}_\pm$ that emanating from $\mathcal{E}$.

5.2.2 Bulk replica story

Now we try to construct the bulk replica story. Similarly we use bulk Rindler coordinates $\tau_m$ to denote all the bulk regions. Also we allow $\tau_m$ to be complex and refer to all the bulk regions in question by using $\tau_m = \tau + \frac{(m-1)}{2}i\pi$. The assignment\(^5\) of the imaginary parts are explicitly shown in Fig.5. We expect the boundary branched cover structure inherent in the replica construction to be inherited by the holographic map in the bulk. In other words, the bulk geometry should be a replicated geometry glued cyclically from $n$ copy of the bulk spacetime.

Before we go ahead, we briefly review the bulk story in AdS/CFT in [12]. In this case the RT surface $\mathcal{E}$ is homologous to $\mathcal{A}$. We denote the spacelike codimension one surface enclosed by $\mathcal{E}$ and $\mathcal{A}$ as $\mathcal{R}_A$, which satisfy $\partial \mathcal{R}_A = \mathcal{A} \cup \mathcal{E}$. Firstly, for each copy of bulk $\mathcal{M}_I$, we cut them open along $\mathcal{R}_{I\mathcal{A}}$ to $\mathcal{R}_{I\mathcal{A}+}$ and $\mathcal{R}_{I\mathcal{A}-}$. Then we get the replicated geometry by gluing the open cuts cyclically

$$\mathcal{R}_{A-}^I = \mathcal{R}_{A}^{(I+1)+}, \quad \mathcal{R}_{A-}^n = \mathcal{R}_{A+}^1.$$  \hspace{1cm} (5.21)

In the case of AdS/WCFT, in order to conduct the replica trick we also need to cut the bulk open along some codimension one surface, which we also denote as $\mathcal{R}_A$. Since $\mathcal{E}$ is the fixed points of the replica symmetry, we require $\mathcal{E} \subset \partial \mathcal{R}_A$. On the boundary, to reproduce the boundary replica story we require $\mathcal{A} \subset \partial \mathcal{R}_A$. However, in this case $\mathcal{E}$ is not homologous to $\mathcal{A}$, so $\partial \mathcal{R}_A$ contains other parts. In additional, $\mathcal{R}_A$ have two more boundaries $\gamma_{\pm}$ that connect the two endpoints $\partial \mathcal{A}_{\pm}$ of the boundary interval and the endpoints of $\mathcal{E}$ at $v = \pm\infty$. Later we will argue that $\gamma_{\pm}$ should be the null geodesics on $\mathcal{N}_{\pm}$. In summary we have

$$\partial \mathcal{R}_A = \mathcal{A} \cup \mathcal{E} \cup \gamma_{+} \cup \gamma_{-}.$$ \hspace{1cm} (5.22)

With the boundary $\partial \mathcal{R}_A$ settled down, the surface $\mathcal{R}_A$ has the freedom to vibrate as long as we keep $\mathcal{R}_A$ spacelike everywhere except at the two boundary null lines $\gamma_{\pm}$. Then we

\(^5\)The assignment in the bulk can also be obtained from the bulk Rindler transformations, see also [54, 55] for discussions related to the assignment.
cut $\mathcal{R}_A$ open to $\mathcal{R}_A^+$ and $\mathcal{R}_A^-$ in each copy of the bulk and then glue them cyclicly to the replicated geometry $\mathcal{M}_n$.

Figure 6. The red surface is the $\mathcal{R}_A$ where we cut the bulk open.

Note that when we cut $\mathcal{R}_A$ open we divide the $\tau_1$ region into two regions denoted by $\tau_1$ and $\tau_5$ (see Fig. 6). Starting from some point in the $\tau_1$ region, we move along a bulk modular flow line, after crossing the horizons $\mathcal{N}_\pm$ for four times, we will arrive at the $\tau_5$ region. Then we pass through $\mathcal{R}_A$ and enter the next copy of bulk spacetime. The cyclic gluing of $n$ copies of the bulk make us pass through the horizons for $4n$ times to get back to the starting point. This induces the thermal circle $\tau \sim \tau + 2\pi n i$ in $\mathcal{M}_n$, which shrinks at $\mathcal{E}$. On the boundary this story reproduces the boundary replica story.

5.3 Relating the UV and IR cutoffs with the modular planes (null geodesics)

Then we focus on our second task: how to regulate the length of $\mathcal{E}$ when we regulate the boundary interval. In the case of AdS/CFT, the relation [15] between the UV cutoff on the field theory side and the IR cut off on the gravity side is used to regulate the holographic entanglement entropy in the RT formula. However the prescription for using the UV/IR relation to regulate entanglement entropy is not thoroughly justified. Using Rindler method, this application is implied by the Rindler transformations [8] (see also [9][10]). The deeper understanding of this application is recently given by [1] using the modular planes as a slicing of the entanglement wedge. This fine structure analysis of holographic entanglement gives a one-to-one correspondence between the points on the boundary interval $\mathcal{A}$ and the points on the RT surface $\mathcal{E}$. When the point on $\mathcal{A}$ approaches an end point, its partner on $\mathcal{E}$ will approach the boundary following precisely the UV/IR relation [15] in AdS/CFT.

In the case of AdS$_3$/WCFT, according to [9] we need to cut off $\mathcal{E}$ along the $v$ direction rather than the $r$ direction, so the prescription of the RT formula does not work here. On the other hand with the bulk and boundary modular flows clear, we can also define the
modular planes as in [1] and perform the fine structure analysis to determine the cutoff point in the bulk. As in the case of AdS$_3$/CFT$_2$ [1], we will show that the modular planes can give the expected relationship between the boundary UV cutoff and the bulk IR cutoff along the $v$ direction.

The modular plane $\mathcal{P}(v_0)$ is defined as the orbit of a boundary modular flow line under the bulk modular flow, which is a codimension one surface in the bulk. Its construction in this case is discussed in details in section 4. The modular planes are in one-to-one correspondence with the boundary modular flow lines $\mathcal{L}_{v_0}$. By definition we have

$$\mathcal{P}(v_0) \cap \mathcal{B} = \mathcal{L}_{v_0}. \quad (5.23)$$

As the boundary can be viewed as a slicing of modular flow lines $\mathcal{L}_{v_0}$, the entanglement wedge $\mathcal{W}_A$ can also be viewed as a slicing of the modular planes $\mathcal{P}(v_0)$. The normal null geodesics on $\mathcal{N}_\pm$ are also bulk modular flow lines, which indicates the modular planes will intersect with $\mathcal{N}_\pm$ on these normal null geodesics, ie.

$$\mathcal{P}(v_0) \cap \mathcal{N}_\pm = \mathcal{N}_\pm(v_0) = \mathcal{L}_{v_0}. \quad (5.24)$$

Also we have shown that $\mathcal{L}_{v_0}$ (or $\mathcal{P}(v_0)$) intersect with the bulk curve curve $\mathcal{E}$ (3.16) at

$$\mathcal{E}(v_0) : (u, v, r) = (0, v_0, \frac{1}{l_u}). \quad (5.25)$$

Also each modular plane will intersect with $\mathcal{R}_A$ on a line

$$\mathcal{R}_A(v_0) = \mathcal{P}(v_0) \cap \mathcal{R}_A. \quad (5.26)$$

See Fig.3 for a typical modular plane.

Translation along a bulk modular flow line is a translation on the real part of $\tau$ while keeping the imaginary part fixed. When we apply the replica trick, the bulk and boundary are cyclically glued, the orbit of the modular flow changes, as well as the distribution of the imaginary part $\text{Im}[\tau]$. Let us consider the cyclic gluing of a single point $\mathcal{A}(v'_0)$ and see how it changes the modular flow picture both in the bulk and boundary. Here $v'_0$ denote the $v$ coordinate of the point $\mathcal{A}(v'_0)$. We denote the boundary modular flow line that passes this point as $\mathcal{L}_{v_0}$. On the boundary, $\mathcal{L}_{v_0}$ will enter the next copy of $\mathcal{B}$ when it passes through $\mathcal{A}(v'_0)$. By definition all the bulk modular flow lines that emanating from $\mathcal{L}_{v_0}$ will return back to $\mathcal{L}_{v_0}$. As $\mathcal{L}_{v_0}$ enters the next copy of $\mathcal{B}$, the bulk modular flow emanating from $\mathcal{L}_{v_0}$ also need to enter the next copy of bulk to get back to the same $\mathcal{L}_{v_0}$. For $n$ copy of the bulk and modular planes, the natural bulk extension of the cyclic gluing of the point $\mathcal{A}(v'_0)$ is the cyclic gluing of the modular plane $\mathcal{P}(v_0)$ on $\mathcal{R}_A(v_0)$, ie.

$$\psi_1(\mathcal{R}_A-(v_0)) = \psi_{(I+1)}(\mathcal{R}_A+(v_0)), \quad \psi_n(\mathcal{R}_A-(v_0)) = \psi_1(\mathcal{R}_A+(v_0)), \quad (5.27)$$

where $\psi$ denote all the bulk metric and matter fields.

We denote the cyclically glued modular plane as $\mathcal{P}_n(v_0)$. Following the modular flows we can keep track of the imaginary part of $\tau$ everywhere on $\mathcal{P}_n(v_0)$ and find the thermal circle on $\mathcal{P}_n(v_0)$ becomes

$$\tau \sim \tau + 2\pi ni. \quad (5.28)$$
Accordingly the induced metric on $\mathcal{P}_n(v_0)$ becomes

$$ds^2 = n^2 d\rho^2 - \rho^2 d\tau^2 + \cdots,$$

(5.29)

where $\rho$ denotes the distance from the fixed point $\mathcal{E}(v_0)$, and the dots means the high order terms. See Fig. 7 for replica story on the modular plane with $n = 2$. The whole bulk replica story can be considered as a slicing of replica stories on all the modular planes.

**Figure 7.** This figure is taken from [1] and shows the replica story for a cyclically glued modular plane $\mathcal{P}_n(v_0)$ with $n = 2$. In the first copy the boundary modular flow line $\mathcal{L}_{v_0}$ parametrized by $\tau_1$ (blue line) passes through $\mathcal{A}(v_0)$ and get into the next copy. The bulk flow (blue arrow) should also go through $\mathcal{R}_\mathcal{A}(v_0)$ (the green line) to the next copy then go back to $\mathcal{L}_{v_0}$. A similar story happens to the flow lines parametrized by $\tau_5$ (the red line and arrow). The dashed arrows means the cyclic gluing of $\mathcal{R}_\mathcal{A}(v_0)$.

In summary, the cyclic gluing of a point $\mathcal{A}(v'_0)$ on the boundary interval induces a replica story on the corresponding modular plane. Following the calculations in [11, 12], this turns on nonzero contribution to the entanglement entropy on the fixed point $\mathcal{E}(v_0)$ where the modular plane intersect with $\mathcal{E}$. In other words, this gives an one-to-one correspondence between the points $\mathcal{A}(v'_0)$ on $\mathcal{A}$ and the points $\mathcal{E}(v_0)$ on $\mathcal{E}$ (see Fig. 8). More explicitly, if we consider $\mathcal{A}$ to be a straight line

$$u = \frac{l_u}{l_v} v, \quad -\frac{l_v}{2} \leq v \leq \frac{l_v}{2},$$

(5.30)

the two points that correspond to each other are related by

$$v'_0 + \text{arctanh} \frac{2v'_0}{l_v} = v_0.$$  

(5.31)

When $\mathcal{A}(v'_0)$ approaches $\partial_{+}\mathcal{A}$, ie. $v'_0 = \pm \frac{l_v}{2}$, the partner points become $\mathcal{E}(v'^+_0)|_{v'^+_0 = \pm \infty}$. Note that the endpoints $\partial_{+}\mathcal{A}$ also lie on the null hypersurfaces $\mathcal{N}_{\pm}$, they can be connected to their partners $\mathcal{E}(v'^\pm_0)$ by the two null geodesics $\gamma_{\pm} = \hat{\gamma}'_{v_0}$. This indicates that all the lines that connects these two pair of points will contain timelike part except the two null lines $\gamma_{\pm}$. Since we do not expect time-like parts on the surface $\mathcal{R}_\mathcal{A}$, the only choice for $\mathcal{R}_\mathcal{A}(v'^\pm_0)$ are $\gamma_{\pm}$. This is how we determine $\partial \mathcal{R}_\mathcal{A}$ to be (5.22).
Figure 8. This figure shows the correspondence between the point \( A(v'_0) \) on \( A \) and its partner \( E(v_0) \) on \( E \) with \( v_0 \) and \( v'_0 \) satisfying (5.31). Also they are the points where the modular plane \( \mathcal{P}(v_0) \) intersect with \( A \) and \( E \). The green line is \( \mathcal{R}_A(v_0) \).

In the same sense, an arbitrary sub-interval \( A_2 \) of \( A \) correspond to a sub-interval \( E_2 \) on \( E \) (see Fig.9). We denote the \( v \) coordinate of the two endpoints of \( A_2 \) as \( v'_1 \) and \( v'_2 \) and denote the \( v \) coordinates of the two endpoints of \( E_2 \) as \( v_1 \) and \( v_2 \), which are related by (5.31). According to our prescription, the contribution from \( E_2 \) to the total entanglement entropy \( S_A \) is turned on by the cyclic gluing of \( A_2 \) on the boundary. Thus it is natural to propose that the length of \( E_2 \) captures the contribution to \( S_A \) from the sub-interval \( A_2 \).

Let us go a step further and interpret the regulated entanglement entropy as the part in the total entanglement entropy contributed from \( A_2 = A_{reg} \), with \( A_{reg} \) given by (3.17). In other words we set

\[
  v'_1 = \frac{l_v}{2} + \epsilon_u \frac{l_v}{l_u}, \quad v'_2 = \frac{l_v}{2} - \epsilon_u \frac{l_v}{l_u}.
\]  

Then the end two points \( E(v_1) \) and \( E(v_2) \) of \( E_2 = E_{reg} \) should be the points where the

\footnote{The additional terms proportional to \( \epsilon_u \) in (5.32) appear because we regulate the \( u \) direction with \( \epsilon_u \) and keep the endpoints of \( A_{reg} \) on the straight line.}
geodesic $\mathcal{E}$ is cut off. Applying (5.31), we find

$$v_1 = -\frac{1}{2} \left( l_v + \log \frac{l_u}{\epsilon_u} \right) + \mathcal{O}(\epsilon_u), \quad v_2 = \frac{1}{2} \left( l_v + \log \frac{l_u}{\epsilon_u} \right) + \mathcal{O}(\epsilon_u),$$

which are exactly the cutoff points we found by Rindler method. For an overall picture of our prescription, see Fig.10.

The entanglement wedge as a slicing of the modular planes. We denote the thick blue line in the bulk as $\mathcal{E}_{\text{reg}}$. We only depicted the modular planes that go through $A_{\text{reg}}$. The modular planes $\mathcal{P}(v_1)$ and $\mathcal{P}(v_2)$ intersect with $A$ and $\mathcal{E}$ on the points where they are cut off.

The regulated entanglement entropy is then given by

$$S_A = \frac{\text{Length}(\mathcal{E}_{\text{reg}})}{4G} = \frac{c}{6} \left( l_u + \log \frac{l_u}{\epsilon_u} \right),$$

where we have used the relation $c = \frac{3\ell}{2G}$. As expected the result coincide with the result (3.19) we get by Rindler method.

As was pointed out by [1], the fine correspondence between the points on $A$ and the points on $\mathcal{E}$ defines a entanglement contour function that describes the distribution of the entanglement on $A$. People who are interested in the entanglement contour should consult the Appendix A, where we calculate the contour function based on this fine correspondence (5.31), and further more we use the result to test the proposal [1] for entanglement contour function for general theories.

5.4 The intrinsic construction of the geometric picture

The above geometric picture absolutely contains more information than the total entanglement entropy, however it depends on the explicit picture of the bulk and boundary modular flows. However this information is very complicated to get and may not even have local description in more general cases. Then we come to the important question: is there a prescription to construct the geometric picture intrinsically without the construction of
Rindler transformations as well as the information from the modular flows? Inspired by the above construction we try to propose such a prescription for the case we study.

The prescription also involves a cutoff at large radius \( r_I \), which is related to the cutoff \( \epsilon_u \) in the WCFT by

\[
  r_I = \frac{1}{\epsilon_u}.
\]

In this case the radius cutoff is imposed on the two null geodesics \( \gamma_{\pm} \) rather than the spacelike geodesic \( \mathcal{E} \). However the way we impose this cutoff is a little tricky. The \( \gamma_{\pm} \) emanating from the boundary endpoints \( \partial_{\pm} \mathcal{A} \) at the real boundary will intersect with \( \mathcal{E} \) at \( v_0^\pm = \pm \infty \). Cutting off these two \( \gamma_{\pm} \) at \( r_I \) does not regulate the entanglement entropy.

The right way to do the regulation is in the following. We first need to push the WCFT to the cutoff boundary at \( r = r_I \). During we push the boundary, we should keep \( \partial_{\pm} \mathcal{A} \) on \( \mathcal{N}_{\pm} \) thus adapt to the bulk decomposition. Furthermore we should keep the \( v \) coordinate of \( \partial_{\pm} \mathcal{A} \) fixed since there is no cutoff along the \( v \) direction. Then the two null geodesics that emanating from the \( \partial_{\pm} \mathcal{A} \) at \( r = r_I \) will intersect with \( \mathcal{E} \) on the right cutoff points. See Fig. 11.

Following the above prescription, the endpoints \( \partial_{\pm} \mathcal{A} \) are pushed to the following positions

\[
  \partial_{\pm} \mathcal{A} : \left( \pm \frac{l_v}{2} \mp \frac{1}{2} \frac{l_u}{r_I}, \pm \frac{l_v}{2}, r_I \right). \tag{5.36}
\]

Note that all the null geodesics \( \mathcal{L}_{v_0} \) lie on \( \mathcal{N}_{\pm} \) are given in (4.13) and \( v_0 \) denotes the \( v \) coordinate of the points where \( \mathcal{L}_{v_0} \) intersect with \( \mathcal{E} \). It is easy to find that the two null geodesics \( \gamma_{\pm} \) emanating from \( \partial_{\pm} \mathcal{A} \) (5.36) are just given by \( \mathcal{L}_{v_0} \) with \( v_0 = \pm \left( \frac{l_v}{2} + \frac{1}{2} \log(l_u r_I) \right) \), ie.

\[
  \gamma_- : \quad u = \frac{1}{2} \left( \frac{1}{r} - l_u \right), \quad v = - \left( \frac{l_v}{2} + \frac{1}{2} \log(l_u r_I) \right) + \frac{1}{2} \log(l_u r), \tag{5.37}
\]
\[
  \gamma_+ : \quad u = - \frac{1}{2} \left( \frac{1}{r} - l_u \right), \quad v = \frac{l_v}{2} + \frac{1}{2} \log(l_u r_I) - \frac{1}{2} \log(l_u r). \tag{5.38}
\]

See Fig. 12 for a complete description for our construction.

The length of the regulated curve \( \mathcal{E}_{reg} \) is just

\[
  \text{Length}(\mathcal{E}_{reg}) = l_v + \log(l_u r_I), \tag{5.39}
\]

which reproduces the result (5.34) with the UV/IR relation (5.35). In the Appendix B, we show that, similar to the flat case [10], the \( \mathcal{E}_{reg} \) is the saddle among all the geodesics that connect the two null geodesics \( \gamma_{\pm} \).

6 Towards the generalized gravitational entropy for spacetimes with non-Lorentzian duals

Based on the above discussions, we are ready to generalize our intrinsic prescription to calculate the generalized gravitational entropy for general spacetimes with non-Lorentzian
Here the boundary $\mathcal{B}$ as well as the interval $\mathcal{A}$ is pushed to the cutoff boundary $r = r_I$. We assume $\mathcal{E}(v_0^\pm)$, with $v_0^- = -\infty$, is the point that connected to $\partial_- \mathcal{A}$ at the real boundary through the null geodesics $\gamma^-$, while $\mathcal{E}(v_1)$ is where we cut $\mathcal{E}$ off. When $\partial_- \mathcal{A}$ moves on $\mathcal{N}_-$ and gets into the bulk, the modular plane, as well as the null geodesic $\gamma^-$, that goes through it will change accordingly. In such a way the curve $\mathcal{E}$ is regulated through the null geodesics.

In this figure the interval $\mathcal{A}$ is on the cutoff boundary $r = r_I$ and its endpoints $\partial_\pm \mathcal{A}$ are on $\mathcal{N}_\pm$. The two red lines are the two null geodesics (5.37) and (5.38) which intersect with $\mathcal{E}$ at the endpoints of $\mathcal{E}_{\text{reg}}$. The solid blue line is just the $\mathcal{E}_{\text{reg}}$.

duals. As expected the prescription only evolves extremal surfaces $\mathcal{E}$ and their associated normal null hypersurfaces $\mathcal{N}_\pm$, which are available when the bulk metric is given.

In case we have a bulk spacetime $\mathcal{M}$ and its asymptotic boundary $\mathcal{B}$ at $r \to \infty$, suppose a holography is conjectured between the bulk gravity theory and the field theory on $\mathcal{B}$. The prescription is in the following

1. Firstly we should figure out the causal structure of the boundary field theory, using either the boundary null geodesics (or hypersurfaces) when the metric background is fixed, or the Rindler method. In other words, for a given subregion $\mathcal{A}$ we should find out the corresponding causal development $\mathcal{D}_\mathcal{A}$. For non-Lorentzian field theories, the causal developments usually have the shape of a strip (or a solid cylinder) rather than a diamond.
2. For a general spacelike extremal surface we study its normal null hypersurfaces $\mathcal{N}_\pm$.
Then the $\mathcal{E}$ that matches the boundary subregion $\mathcal{A}$ is determined by the requirement\(^7\) that the intersection of $\mathcal{N}_\pm$ and $\mathcal{B}$ should cover $\partial \mathcal{D}_\mathcal{A}$, i.e.
\[
\mathcal{N}_\pm \cap \mathcal{B} \supset \partial \mathcal{D}_\mathcal{A}.
\]
(6.1)
The above requirement is just the requirement for the consistence between the bulk and boundary causal structures.

3. Then we push the dual field theory to the cutoff boundary at some large radius $r = r_I$\(^8\). During we push the boundary, it is crucial how the entangling surface $\partial \mathcal{A}$ moves. The main requirement is that $\partial \mathcal{A}$ should adapt to the consistence of the bulk and boundary causal structures on $r = r_I$\(^9\). For the cases with non-Lorentzian duals, we should keep $\partial \mathcal{A}$ on $\mathcal{N}_\pm$ and keep the coordinates, whose UV cutoff can be taken to be zero, fixed.

4. On $\mathcal{N}_+ \cup \mathcal{N}_-$, there are null geodesics $\gamma_\pm$ (or codimension two null hypersurfaces) emanating from $\partial \mathcal{A}$ at the cutoff boundary $r = r_I$. The place where $\gamma_\pm$ intersect with $\mathcal{E}$ are where we cut off the extremal surface $\mathcal{E}$ in the bulk. Then we get the regulated extremal surface $\mathcal{E}_\text{reg}$ and the holographic entanglement entropy is just given by
\[
S_\mathcal{A} = \frac{\text{Area} (\mathcal{E}_\text{reg})}{4G}.
\]
(6.2)
The first two steps show how to use the consistence of the causal structures to determine the $\mathcal{E}$ corresponding to the boundary subregion $\mathcal{A}$ in question. While the last two steps tell us how to regulate $\mathcal{E}$ using the null geodesics on $\mathcal{N}_+ \cup \mathcal{N}_-$, which are just the modular flows. The above construction remind us of the construction of the light-sheet by Bousso [56]. In [7], using the light-sheets the authors propose a prescription to construct the covariant geometric picture for entanglement entropy in the context of AdS/CFT. They require the light-sheet associated to $\mathcal{E}$ should intersect with the boundary on the boundary light-sheet associated to $\partial \mathcal{A}$. This is equivalent to our requirement of the consistence between the bulk and boundary causal structures (6.1).

For the cases with locally defined modular Hamiltonian, the fine structure analysis with the modular planes gives a strong support for the validity of our prescription. However in more general cases our prescription is still applicable. Though in general the modular Hamiltonian is non-local, effectively it can be locally defined in some special bulk and boundary regions: the region near the $\mathcal{N}_\pm$ and the region near $\partial \mathcal{D}$. In the same sense the modular planes can

\(^7\)For spacetimes with relativistic duals, this requirement naturally lead to the homology constraint.

\(^8\)The cutoff radius $r_I$ should be properly related to the UV cutoff of the field theory, and the relation should be discussed case by case.

\(^9\)For example, in the relativistic holography cases, the only way to satisfy this requirement is to push $\partial \mathcal{A}$ along $\mathcal{N}_+ \cup \mathcal{N}_- = \mathcal{E}$ thus $\mathcal{E}$ is cut off at $r = r_I$. 

also be locally defined in these regions. Since the endpoints $\partial A$, the extremal surface $E$ and the null geodesics (or hypersurfaces) $\gamma_{\pm}$ are all inside these regions, our prescription is still applicable. Also we conjecture that our prescription can still give the right holographic entanglement entropy.

In the following we give an argument for this statement. We can divide the $A$ and $E$ into two parts

$$A = A_{\text{cut}} \cup A_{\text{reg}}, \quad E = E_{\text{cut}} \cup E_{\text{reg}}. \quad (6.3)$$

Here $A_{\text{cut}}$ are the part of $A$ we cut off during the regulation. Also the total entanglement entropy can be divided into two part which are contributed from $A_{\text{cut}}$ and $A_{\text{reg}}$ respectively

$$S_A = S_{A_{\text{cut}}} \cup S_{A_{\text{reg}}} = \frac{\text{Area}(E)}{4G}. \quad (6.4)$$

Though the fine correspondence for the points on $A_{\text{reg}}$ no longer exists when the modular Hamiltonian is non-local, it still holds between the points on $A_{\text{cut}}$ and the points on $E_{\text{cut}}$. This is because $A_{\text{cut}}$ is in the near $\partial D$ region, where the modular flow effectively has a local description. So we have

$$S_{A_{\text{cut}}} = \frac{\text{Area}(E_{\text{cut}})}{4G}. \quad (6.5)$$

Combine with (6.4), we get

$$S_{A_{\text{reg}}} = \frac{\text{Area}(E_{\text{reg}})}{4G}. \quad (6.6)$$

7 Generalised gravitational entropy for 3-dimensional flat space

In this section we apply our prescription to the case of 3-dimensional flat holography [33–35]. In this holography, the 3-dimensional asymptotic flat spacetime is conjectured to be dual to a field theory invariant under the BMS$_3$ group (BMSFT). The BMS$_3$ group is the asymptotic symmetry group of flat space enhanced from the Poincaré group, and the BMSFT can be considered as the ultra-relativistic limit of a CFT$_2$. The holographic calculation, as well as the geometric picture, of the entanglement entropy for BMSFTs are given in [10] with the Rindler method. We will show that the above prescription can easily reproduce the results in [10] without the Rindler transformations and the modular flows. It will be very interesting to apply our prescription higher dimensional flat spacetimes.

In particular we consider the following classical solutions of Einstein gravity with vanishing cosmological constant in Bondi gauge

$$ds^2 = Mdu^2 - 2dudr + Jdud\phi + r^2d\phi^2. \quad (7.1)$$

The above solutions are usually classified into three types:

- $M = -1, J = 0$: Global Minkowski, which duals to the zero temperature BMSFT on the cylinder with $\phi \sim \phi + 2\pi$. 

• $M = J = 0$: Null-orbifold, with $\phi$ decompactified this duals to the zero temperature BMSFT on the plane.

• $M > 0$: Flat Space Cosmological solutions (FSC), which duals to BMSFT at finite temperature.

The asymptotic boundary (null infinity) $B$ settles at $r = r_I \to \infty$ with a fixed background metric

$$ds^2 = 0 du^2 + d\phi^2,$$

(7.2)

which is degenerate. On $B$ the null direction is characterized by $u$. The subregion we study is a single interval

$$A : (-l_u/2, -l_\phi/2) \to (l_u/2, l_\phi/2).$$

(7.3)

Since $u$ is the null direction, the domain of causality $D_A$ is just a strip along the $u$ direction

$$D_A : \{-\frac{l_\phi}{2} \leq \phi \leq \frac{l_\phi}{2}\}.$$

(7.4)

Asymptotically we should have

$$\partial D_A : \phi = \pm \frac{l_\phi}{2} + O\left(\frac{1}{r_I}\right), \quad r = r_I.$$

(7.5)

It will be quite subtle to apply our prescription in the Bondi gauge, so we apply it in the Cartesian coordinates. Another advantage of using the Cartesian coordinates is that we do not need to solve the Einstein equations, because the geodesics and their null normal hypersurfaces are just straight lines and null planes. In the following we will map the picture to the Cartesian coordinates, thus our prescription can be much simpler.

7.1 Null-orbifold

We choose the coordinate transformation between the Null-orbifold and the Cartesian coordinates to be

$$t = \frac{l_\phi}{4} r + \frac{2}{l_\phi} (u + \frac{r \phi^2}{2}),$$

(7.6)

$$x = \frac{l_u}{l_\phi} + r \phi,$$

(7.7)

$$y = \frac{l_\phi}{4} r - \frac{2}{l_\phi} (u + \frac{r \phi^2}{2}).$$

(7.8)

Here we have adjusted the transformation in advance by some proper Poincaré transformation and the constants are chosen to be related to the parameters $l_u$ and $l_\phi$ which characterize the boundary interval $A$, hence the curve $E$ can be characterized by a single coordinate $y$ and settled at $t = x = 0$. Of course one can begin with free parameters and
settle them down one by one through the matching condition (6.1). One can check that, up to a Poincaré transformation for the Cartesian coordinates, we have the following equation,

\[ ds^2 = -2dudr + v^2d\phi^2 = -dt^2 + dx^2 + dy^2. \] (7.9)

Then in Cartesian coordinates \( \partial \mathcal{D}_A \) (7.5) and the two endpoints \( \partial_{\pm} \mathcal{A} \) of \( \mathcal{A} \) are given by

\[ \partial \mathcal{D}_A : \{t, x, y\} = \left\{ \frac{l_{\phi}}{2} r_I + \mathcal{O} \left( r_I^0 \right), \pm \frac{l_{\phi}}{2} r_I + \mathcal{O} \left( r_I^0 \right), -\frac{2u}{l_{\phi}} \right\}, \] (7.10)

\[ \partial_{\pm} \mathcal{A} : \{t, x, y\} = \left\{ \frac{l_{\phi}}{2} r_I \pm \frac{l_u}{l_{\phi}}, \pm \frac{l_{\phi}}{2} r_I + \frac{l_u}{l_{\phi}}, \mp \frac{l_u}{l_{\phi}} \right\}. \] (7.11)

Firstly we determine the \( \mathcal{E} \) that satisfies the requirement (6.1) for the consistence of the bulk and boundary causal structures. It is easy to see that the spacelike geodesic \( \mathcal{E} \) and the associated \( \mathcal{N}_{\pm} \) that asymptotically satisfy (6.1) are just given by

\[ \mathcal{E} : \{x = 0, \ t = 0\}, \] (7.12)

\[ \mathcal{N}_{\pm} : \{x = \pm t\}, \] (7.13)

as the \( \mathcal{N}_{\pm} \) (7.13) will asymptotically go through \( \partial \mathcal{D}_A \) (7.10).

Then we regulate \( \mathcal{E} \) using null geodesics on \( \mathcal{N}_{\pm} \). The two null geodesics \( \gamma_{\pm} \) on \( \mathcal{N}_+ \cup \mathcal{N}_- \) that emanating from \( \partial_{\pm} \mathcal{A} \) are given by

\[ \gamma_{\pm} : \left\{ x = \pm t, \ y = \mp l_u \frac{l_{\phi}}{l_{\phi}} \right\}. \] (7.14)

Note that in this case there is no need to introduce UV cutoffs in the \( u \) and \( \phi \) direction since they can be taken to zero without introducing divergence to the entanglement entropy\(^{10}\). So when we push the boundary, we keep the \( u \) and \( \phi \) coordinate of \( \partial_{\pm} \mathcal{A} \) fixed. This means \( \partial_{\pm} \mathcal{A} \) moves along \( \gamma_{\pm} \) and the \( \mathcal{E}_{\text{reg}} \) will be independent of the choice of \( r_I \), We will see this later. The whole picture of the construction is shown in Fig. 13.

According to our prescription, the points where \( \gamma_{\pm} \) intersect with \( \mathcal{E} \) are the place we cut \( \mathcal{E} \) off. Then we get

\[ \mathcal{E}_{\text{reg}} : \left\{ x = t = 0, \ -\frac{l_u}{l_{\phi}} \leq y \leq \frac{l_u}{l_{\phi}} \right\}. \] (7.15)

Accordingly we have

\[ S_\mathcal{A} = \frac{\text{Area}(\mathcal{E}_{\text{reg}})}{4G} = \frac{1}{2G} \frac{l_u}{l_{\phi}}, \] (7.16)

which reproduces the results in [10, 39–41].\(^{10}\) However when we consider gravity with gravitational anomaly, for example the topological massive gravity [57, 58], a divergent contribution will arise due to the anomaly [10].
Figure 13. The two red lines are the null rays $\gamma_{\pm}$ emanating from $\partial A$ and normal to $E$. The straight line $E$ is cut off where it intersect with $\gamma_{\pm}$. The solid blue segment is just our $E_{reg}$.

7.2 Global Minkowski

The global Minkowski space

$$ds^2 = -2dudr + r^2d\phi^2,$$

(7.17)

can be transformed to the Cartesian coordinates by the following transformation

$$t = (r + u) \csc \frac{\phi}{2} + \frac{r \cos \phi}{2} \left( \tan \frac{\phi}{4} - \cot \frac{\phi}{4} \right),$$

$$x = r \sin \phi + \frac{1}{2}l_u \csc \frac{\phi}{2},$$

$$y = r \cos \phi \csc \frac{\phi}{2} - (r + u) \cot \frac{\phi}{2}.$$  

(7.18)

Then in Cartesian coordinates we have

$$\partial D_A : \{t, x, y\} = \left\{ \sin \frac{\phi}{2} r_I + O \left( r^0_I \right), \pm \sin \frac{\phi}{2} r_I + O \left( r^0_I \right), -u \cot \frac{\phi}{2} \right\}$$

(7.19)

$$\partial_{\pm} A : \{t, x, y\} = \left\{ \csc \frac{\phi}{2} \left( l_u \pm 2r_I \sin^2 \frac{\phi}{2} \right), \pm r_I \sin \frac{\phi}{2}, \mp l_u \cot \frac{\phi}{2} \right\}$$

(7.20)

Similarly we get the spacelike geodesic $E$ and the associated $N_{\pm}$ that asymptotically satisfy the requirement (6.1)

$$E : \{x = 0, \ t = 0\},$$

(7.21)

$$N_{\pm} : \{x = \pm t\}.$$

(7.22)

The two null geodesics $\gamma_{\pm}$ on $N_+ \cup N_-$ that emanating from $\partial_{\pm} A$ are just given by

$$\gamma_{\pm} : \left\{ x = \pm t, \ y = \mp l_u \cot \frac{\phi}{2} \right\}.$$  

(7.23)
Quite straightforwardly we get
\[ S_A = \frac{\text{Area} \left( \mathcal{E}_{\text{reg}} \right)}{4G} = \frac{l_u}{2G} \cot \frac{l_\phi}{2}. \] (7.24)

### 7.3 Flat Space Cosmological solutions

The coordinate transformation from FSC to the Minkowski space can be given by
\[ r = \sqrt{M(t'^2 - x'^2) + r_c^2}, \] (7.25)
\[ \phi = -\frac{1}{\sqrt{M}} \log \frac{\sqrt{M}(t' - x')}{r + \frac{J}{2\sqrt{M}}}, \] (7.26)
\[ u = \frac{1}{M} \left( r - \sqrt{M}y' - \frac{J}{2} \phi \right). \] (7.27)

The above transformations show that the FSC can be considered as a quotient of the Minkowski space, since the region outside the cosmological horizon \( r_c = \frac{J}{2\sqrt{M}} \) only covers a quarter of the Minkowski space \( t' \geq |x'| \).

As in the previous two cases, we can apply an additional Poincaré transformation to a new set of Cartesian coordinates \( \{t, x, y\} \) thus the \( E \) and \( N_{\pm} \) are just given by (7.12) and (7.13). Under this requirement \( \partial D_A \) (7.5) should asymptotically satisfy
\[ t = t' \cosh \eta - y' \sinh \eta, \] (7.28)
\[ x = x' + s_0, \] (7.29)
\[ y = y' \cosh \eta - t' \sinh \eta, \] (7.30)

where
\[ \eta = \text{arccosh} \left[ \coth \frac{l_\phi \sqrt{M}}{2} \right], \quad s_0 = \frac{(Jl_\phi + 2l_u M)}{4\sqrt{M}} \frac{\text{csch} \left( \frac{l_\phi \sqrt{M}}{2} \right)}{2}. \] (7.31)

Then in Cartesian coordinates we have
\[ \partial D_A : \{t, x, y\} = \left\{ \frac{\sinh \frac{l_u \sqrt{M}}{2}}{\sqrt{M}} r_I + \mathcal{O} \left( r_I^0 \right), \pm \frac{\sinh \frac{l_\phi \sqrt{M}}{2}}{\sqrt{M}} r_I + \mathcal{O} \left( r_I^0 \right), \right. \]
\[ \pm 2J \mp \sqrt{M} \left( Jl_\phi \pm 4Mu \right) \coth \frac{l_\phi \sqrt{M}}{2}, \]
\[ \frac{4M}{\sqrt{M}} \left( Jl_\phi \pm 2Mu \right) \}
\] (7.32)
\[ \partial_{\pm} A : y = \pm 2J \mp \sqrt{M} \coth \left( \frac{\sqrt{M}l_\phi}{2} \right) \left( Jl_\phi \pm 2Mu \right). \] (7.33)

For simplicity we only listed the \( y \) coordinates of \( \partial_{\pm} A \). The two null geodesics \( \gamma_{\pm} \) on \( N_+ \cup N_- \) that emanating from \( \partial_{\pm} A \) are given by
\[ \gamma_{\pm} : \{x = \pm t, \quad y = \pm 2J \mp \sqrt{M} \coth \left( \frac{\sqrt{M}l_\phi}{2} \right) \left( Jl_\phi \pm 2Mu \right) \}. \] (7.34)

Again we reproduce the right result
\[ S_A = \frac{\text{Area} \left( \mathcal{E}_{\text{reg}} \right)}{4G} = \frac{1}{4G} \left| \frac{(Jl_\phi + 2Mu)}{2\sqrt{M}} \coth \left( \frac{\sqrt{M}l_\phi}{2} \right) - \frac{J}{M} \right|. \] (7.35)
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A Entanglement contour for WCFT

A.1 Entanglement contour from the fine structure

The entanglement contour function is a density function of entanglement. In other words it describes the distribution of the contribution to the total entanglement entropy from each point of $\mathcal{A}$

$$S_\mathcal{A} = \int_{\mathcal{A}} s_\mathcal{A}(v) dv.$$  (A.1)

Here we parametrize $\mathcal{A}$ with the $v$ coordinate. The authors of [59] proposed a set of requirements for the contour functions\(^{11}\). Few analysis of the contour functions for bipartite entanglement have been explored in [59–63]. Also its fundamental definition is still not established.

In the previous section we propose that $\text{Length} (\mathcal{E}_2)$ captures the contribution from $\mathcal{A}_2$ to the entanglement entropy $S_\mathcal{A}$. In other words our fine structure analysis gives a holographic interpretation for the contour function. Consider $\mathcal{A}$ to be a straight line (5.30), according to the fine correspondence (5.31) we get the contour function $s_\mathcal{A}(v)$ for $S_\mathcal{A}$

$$s_\mathcal{A}(v) = \frac{1}{4G} \left( 1 + \frac{2l_v}{l_v^2 - 4v^2} \right).$$  (A.2)

Let us define

$$s_\mathcal{A}(\mathcal{A}_2) = \int_{\mathcal{A}_2} s_\mathcal{A}(v) dv = \frac{\text{Length} (\mathcal{E}_2)}{4G}.$$  (A.3)

According to the fine correspondence (5.31) or the contour function (A.2), we have

$$s_\mathcal{A}(\mathcal{A}_2) = \frac{1}{4G} \left( v'_2 - v'_1 + \text{arctanh} \frac{2v'_2}{l_v} - \text{arctanh} \frac{2v'_1}{l_v} \right),$$  (A.4)

where $v'_1$ and $v'_2$ are the $v$ coordinates of the two endpoints of $\mathcal{A}_2$.

Furthermore we consider two intervals $\mathcal{A}$ and $\mathcal{A}'$ that have the same causal development $\mathcal{D}_\mathcal{A} = \mathcal{D}_{\mathcal{A}'}$. The two arbitrary boundary modular flow lines $\mathcal{L}_{v_1}$ and $\mathcal{L}_{v_2}$ that divide $\mathcal{A}$ ($\mathcal{A}'$) into three part $\mathcal{A}_1, \mathcal{A}_2$ and $\mathcal{A}_3$ ($\mathcal{A}'_1, \mathcal{A}'_2$ and $\mathcal{A}'_3$). See Fig.14. According to our prescription, any $\mathcal{A}_2$ that goes through the same bunch of modular planes should correspond to the same $\mathcal{E}_2$ on $\mathcal{E}$. Then we should have the following causal property for $S_\mathcal{A}(\mathcal{A}_2)$

$$S_\mathcal{A}(\mathcal{A}_2) = S_{\mathcal{A}'}(\mathcal{A}'_2).$$  (A.5)

\(^{11}\)However the complete list of requirements that uniquely determines the contour is still not available.
A.2 Testing the entanglement contour proposal

It is proposed in [1] that in general theories the $s_A(A_2)$ can be written as a linear combination of the entanglement entropies of single subintervals inside $A$

$$s_A(A_2) = \frac{1}{2} (S_{A_1 \cup A_2} + S_{A_2 \cup A_3} - S_{A_1} - S_{A_3}).$$

Here we would like to test this proposal for WCFT. Using the (5.34) for all these subintervals on the straight interval $A$ (5.30) we have

$$s_A(A_2) = \frac{c}{6} (v_2' - v_1' + \frac{1}{2} \log \frac{(v_2' + \frac{l}{2})(v_1' - \frac{l}{2})}{(v_2' - \frac{l}{2})(v_1' + \frac{l}{2})}) = \frac{c}{6} (v_2 - v_1),$$

which coincide with the result (A.4) we get from the entanglement contour (A.2).

Then we test the causal property (A.5) for the proposal (A.6). We let the two endpoints of $A_2$ run along the boundary modular flow lines $\mathcal{L}_{v_1}$ and $\mathcal{L}_{v_2}$ that passing through them (see Fig. 14),

$$(u_1', v_1') = (u_1', v_1 - \text{arctanh} \frac{2u_1'}{l_u})$$

$$(u_2', v_2') = (u_2', v_2 - \text{arctanh} \frac{2u_2'}{l_u}),$$

The new subinterval $A_2'$ passes through the same class of modular planes as $A_2$, so satisfy (A.5). With all the endpoints of subintervals known, we apply (5.34) and find

$$S_{A_1 \cup A_2'} + S_{A_2' \cup A_3} - S_{A_1} - S_{A_3} = \frac{c}{3} (v_2 - v_1)$$

$$= S_{A_1 \cup A_2} + S_{A_2 \cup A_3} - S_{A_1} - S_{A_3}.$$

This indicates that the linear combination in (A.6) reproduce the right causal property for the contour function.
B The saddle that connect the two null curves $\gamma_{\pm}$

In this section of the appendix we prove that the regulated curve $\mathcal{E}_{\text{reg}}$ is the saddle among all the geodesics that connect $\gamma_-$ (5.37) and $\gamma_+$ (5.38). To prove this we need to calculate the proper distances between arbitrary two points in the bulk, then we fix the endpoints along $\gamma_+$ and $\gamma_-$ respectively and find out the saddle among all the geodesics. It is easier to start from calculating the proper length of arbitrary two points in Poincaré AdS, then we rewrite the distance in terms of the variables in the AdS space with nonzero temperatures via a coordinate transformation.

For simplicity we consider the Poincaré AdS$_3$ spacetime

$$ds^2 = \ell^2 \left( \frac{d\rho^2}{4\rho^2} + 2\rho dU dV \right). \quad (B.1)$$

with the geodesics given by

$$U = \frac{l_U}{2} \tanh \tau + c_U, \quad V = \frac{l_V}{2} \tanh \tau + c_V, \quad \rho = \frac{2 \cos^2 \tau}{l_U l_V}. \quad (B.2)$$

Here $c_U$ and $c_V$ are arbitrary constants, while $l_U$ and $l_V$ are the distances between the endpoints on the boundary along the $U$ and $V$ directions respectively, $\tau$ is the parameter that parametrize the geodesic. Along this line we have

$$ds^2 = \ell^2 d\tau^2 \quad (B.3)$$

so the proper length is just

$$L_{\text{AdS}} = \ell (\tau_1 - \tau_2) \quad (B.4)$$

Note that any two spacelike separated points, for example $(U_1, V_1, \rho_1)$ and $(U_2, V_2, \rho_2)$, in the bulk can be connected by a geodesic line described by (B.2), thus the distance between them is just (B.4). Using (B.2), this proper length (B.4) can be expressed in terms of the coordinates of the two endpoints

$$L_{\text{AdS}} (U_1, V_1, \rho_1, U_2, V_2, \rho_2) = \frac{1}{2} \log \left( \frac{\rho_2 (\rho_2 + X) + \rho_1 (\rho_2 Y (2\rho_2 + X) + X) + (\rho_1 + \rho_2 \rho_1 Y)^2}{2 \rho_1 \rho_2} \right) \quad (B.5)$$

where

$$Y = 2 (U_1 - U_2) (V_1 - V_2)$$

$$X = \sqrt{\rho_1^2 + 2 \rho_2 \rho_1 (\rho_1 Y - 1) + (\rho_2 + \rho_1 \rho_2 Y)^2} \quad (B.6)$$

Then we use the transformation from Poincaré AdS$_{(u,v,\rho)}$ to AdS$_{(u,v,r)}$ (3.3) with nonzero temperatures $T_u$ and $T_v$

$$U = e^{2 T_u u} \sqrt{1 - \frac{2 T_u T_v}{r + T_u T_v}},$$

$$V = e^{2 T_v v} \sqrt{1 - \frac{2 T_u T_v}{r + T_u T_v}},$$

$$\rho = \frac{(r + T_u T_v) e^{-2(T_u u + T_v v)}}{4 T_u T_v}, \quad (B.7)$$
to re-express the distance in terms of the coordinates of the two endpoints in \( \text{AdS}_{(u,v,r)} \)

\[
L_{\text{AdS}}(u_1, v_1, r_1, u_2, v_2, r_2).
\]  

(B.8)

Then we set \( T_u = 0 \), \( T_v = 1 \) (note that we should not set \( T_u = 0 \) at first, or the first equation in (B.7) will be trivial) and set the endpoints on the null geodesics (5.37)-(5.38). Finally we find the distance \( L_{\text{AdS}}(r_1, r_2) \) as a function of only \( r_1 \) and \( r_2 \). We will not write down the explicit expression for \( L_{\text{AdS}}(u_1, v_1, u_2, v_2, r_1, r_2) \) and \( L_{\text{AdS}}(r_1, r_2) \) since they are very complicated. Solving the saddle points equation

\[
\frac{\partial L_{\text{AdS}}(r_1, r_2)}{\partial r_1} = \frac{1 - r_2 l_u}{\sqrt{(l_u (r_2^2 e^{2l_u} - r_1 r_2) + r_1 + r_2)^2 - 4 r_2^2 e^{2l_u}}} = 0,
\]

\[
\frac{\partial L_{\text{AdS}}(r_1, r_2)}{\partial r_2} = \frac{1 - r_1 l_u}{\sqrt{(l_u (r_2^2 e^{2l_u} - r_1 r_2) + r_1 + r_2)^2 - 4 r_1^2 e^{2l_u}}} = 0,
\]

we get

\[
r_1 = r_2 = \frac{1}{l_u}.
\]

(B.9)

We see that the saddle is independent of \( r_I \) and \( l_v \). It is clear to see that the saddle geodesic is just our curve \( \mathcal{E}_{\text{reg}} \).

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