Theoretical modelling of arch-shaped carbon nanotube resonators exhibiting Euler–Bernoulli snap-through bi-stability

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Abstract In this work, we present a detailed static and dynamic analysis of a recently reported electrically actuated buckled carbon nanotube (CNT) resonator, based on the Euler–Bernoulli beam theory. The system behavior is analyzed using the Galerkin reduced-order model. We present a detailed two-dimensional analysis, which predicts static behavior similar to buckled micro-beams, despite the difference in the force dependency on the beam deflection. However, we show that buckled CNTs are the first type of buckled beams to exhibit out-of-plane static deflection, resulting in a unique three-dimensional snap-through transition, never before predicted. In addition, we show the criteria under which these devices can also exhibit latching phenomena, meaning that they can maintain their buckled configuration when no force is applied, making these devices appealing for mechanical memory applications.

Keywords CNT resonators · Snap-through buckling · Bi-stability · Latching · Mechanical memory · Galerkin ROM · Euler–Bernoulli beam theory · Suspended CNTs · NEMS · MEMS · Buckled beams

1 Introduction

Almost everywhere in our daily life we encounter micro-electro-mechanical systems (MEMS) technology by a large number of devices with a wide range of applications [1–3]. These devices are frequently found in electrical systems as filters, phase shifters, resonators, and radio-frequency (RF) switches [4,5]. MEMS structures can also be found in heat transfer and MHD applications [6–8]. In biological and chemical systems, they serve as ultra-sensitive gas and mass detectors, reaching the scale of small molecules or single cells [9]. In mechanical systems, they play an important role in accelerometers in air bag mechanism in vehicles, pressure sensors, actuators, gyroscopes, and much more [10–12].

Common MEMS designs are based on initially curved micro-beams, allowing the mechanical bi-stability of the device. While many actuations can result in bi-stability and latching phenomena, such as thermal environment [13], fluid–solid interaction [14], and magnetic actuation [15], the easiest and most controlled mechanism is by electrical actuation of the arched beam [16–18]. These devices possess diverse static behavior (snap-through (ST), pull-in, latching) as well as intriguing linear and nonlinear dynamics (mode-coupling, veering, harmonics, internal resonance) [19–23]. These phenomena are characterized by large static and dynamic amplitudes due to small variations in its geometric or physical parameters and have been widely...
studied in the context of both potential applications as well as fundamental understanding of mode coupling and nonlinear phenomena.

As fabrication capabilities and techniques are improved, smaller mechanical objects were fabricated, entering the regime of nano-electro-mechanical systems (NEMS) [24]. These miniaturized devices are able to achieve even better performance with respect to their MEMS counterparts [25]. Indeed, fabrication of bistable buckled nano-beams has also been demonstrated [26].

Recently, we reported the fabrication of a bistable NEMS device based on a buckled suspended carbon nanotube (CNT), exhibiting snap-through buckling bistability [27,28]. We showed that in addition to their small dimensions and record frequency electrostatic tuning, these devices exhibit a new type of mode coupling, which results from a three-dimensional static deflection, which was not possible in traditional micro- and nano-beams, mainly due to the large differences in the resonance vibrational modes of the out-of-plane and in-plane motion.

Mechanical nonlinear motion of doubly clamped CNT devices has been widely investigated, both experimentally and theoretically, and theoretical modeling has traditionally been based on EB beam theory [29,30], and their nonlinearity has been proposed for improved sensing [31,32]. However, in all of these studies, the CNT has always been either straight or slacked towards the gate, and hence, EB buckling instability has never been investigated in the context of CNT resonators.

It has been previously shown that for nano-beams, the scale effect is important, and nonlocal elasticity strain gradient and surface elasticity theories must be considered ([33]). As doubly clamped CNT is essentially a nano-beam, several studies address the importance of these effects on the mechanical behavior of CNT resonators [34–37]. According to these studies, when \( L \gg r \) (\( L \) is the CNT length and \( r \) is the CNT radius), the influence of these scaling effects is negligible [38,39]. This is not the sole criterion. Additionally, the strain \( \frac{d^2W}{dx^2} \) and the relative curvature \( L \frac{d^2W}{dx^2} \) should be much smaller than 1 [37].

In this work, we present a full theoretical study of the new mechanics enabled by these buckled nanotubes, based on coupled Euler–Bernoulli (EB) beam equations for the in-plane as well as the out-of-plane static and dynamic motions. We assume \( L/r > 500 \), \( \frac{\partial W}{\partial x} \ll 1 \), and \( L \frac{\partial^3 W}{\partial x^3} \ll 1 \), and hence, the local Euler–Bernoulli beam model is sufficient, and surface effects can be neglected. We use the Galerkin reduced-order model (ROM) to solve the EB equations, validated by a finite element method (FEM) approximation. This is the first snap-through buckling analysis of a system in which the force is inversely proportional to the distance between the CNT and the local gate (resulting from the capacitance model between a wire and a plane), and more significant is the unique three-dimensional static deflection, which cannot occur in traditional buckled beams. We show that after developing the full 3D model, we can predict a rotational continuous motion downward, a 3D EB ST buckling transition, or latching, depending on the initial configuration of the device. This model was used for the theoretical fitting in [27] and produces an excellent agreement with the experimental data.

2 Problem formulation—2D model

We begin by modeling the system as a standard doubly clamped beam, as illustrated in Fig. 1. Neglecting damping and limiting the CNT motion to the \( xz \)-plane, it can be described by the Euler–Bernoulli beam equation:

\[
EI \frac{\partial^4 \hat{w}}{\partial \hat{x}^4} + \rho A \frac{\partial^2 \hat{w}}{\partial \hat{t}^2} - T \frac{\partial^2 \hat{w}}{\partial \hat{x}^2} = \hat{k}
\]

(1)

where \( \hat{x} \) is the axis along the beam, \( \hat{t} \) is time, \( \hat{w}(\hat{x}, \hat{t}) \) is the deflection along the beam in the \( z \)-axis, \( L \) is the CNT length, \( E \) is the Young’s modulus, \( I \) is the moment of inertia, \( \rho \) is the mass density, \( A \) is the CNT cross-sectional area and \( \hat{k} \) is the electrostatic force exerted by the local gate along the \( z \) axis per unit length. \( T \) is the tension given by [40]:

![Fig. 1 Schematic illustration of an initially upward buckled CNT device. \( w(x) \) is the CNT \( z \) displacement](image)
Table 1  Physical parameters of a typical device simulated in this work

| Symbol | Physical interpretation | Value for simulations |
|--------|-------------------------|-----------------------|
| \( L \) | CNT length | \( 1 \ \mu \text{m} \) |
| \( r \) | CNT cross-section radius | \( 1 \ \text{nm} \) |
| \( g_0 \) | CNT height above the local gate | \( 150 \ \text{nm} \) |
| \( E \) | CNT Young’s modulus | \( 3.5 \ \text{TPascal} \) [46] |
| \( \rho \) | CNT mass density | \( 1300 \ \text{kg/m}^3 \) [47] |

\[
T = T_0 - \frac{EA}{2L} \int_0^L \left( \frac{\partial w}{\partial \hat{x}} \right)^2 d\hat{x}
\]  
(2)

in which \( L \) is the CNT length, \( A \) is the CNT cross-sectional area, \( T_0 \) is the initial axial tension when no external force is applied, and the second term represents the contribution of the CNT’s displacement to the induced built-in tension. All simulations in this paper are performed for a device with the physical parameters detailed in Table 1, unless specified otherwise.

2.1 Electrostatic force

The force acting on the CNT per unit length, exerted by the local gate (LG), is given by:

\[
\dot{\vec{k}} = -\frac{1}{2} \frac{\partial C_g}{\partial \hat{z}} \left( V_0 + V_{gDC} + V_{gAC} \right)^2
\]

where \( V_{gDC} \) and \( V_{gAC} \) are the DC voltage and AC actuation amplitude applied to the local gate, respectively. Since the harmonic excitation is small \( (V_{gAC} \ll V_{gDC}) \), we neglect the quadratic term, such that:

\[
(V_0 + V_{gDC} + V_{gAC})^2 \\
\approx (V_0 + V_{gDC})^2 + 2(V_0 + V_{gDC})V_{gAC}
\]

In general, we assume that there may exist a small electrostatic force between the LG and the CNT even at \( V_{gDC} = 0 \) [41,42], due to the work function difference between the CNT and metal electrodes, resulting in transfer of charges to or from the CNT upon contact. This effect is represented by the \( V_0 \) term, and its significance will be clarified in the following section.

The capacitance between the CNT and the local gate (per unit length) is taken as the capacitance of a wire parallel to plane, and assuming \( \frac{(g_0 + w)^2}{r^2} \gg 1 \), it can be approximated as [43]:

\[
C_g(z) = \frac{2\pi \varepsilon_0}{\log \left( \frac{2(g_0 - \hat{w})}{r} \right)}
\]

where \( \varepsilon_0 \) is the vacuum permittivity, \( r \) is the CNT radius, and \( g_0 \) is the LG-CNT distance when the tube is straight and \( w = 0 \). The derivative is therefore given by:

\[
\frac{\partial C_g}{\partial \hat{z}} = -\frac{2\pi \varepsilon_0}{(g_0 - \hat{w})\log \left( \frac{2(g_0 - \hat{w})}{r} \right)^2}
\]

(6)

In typical devices, \( \frac{\hat{w}}{g_0} \ll \frac{1}{2} \), and since the tube radius is approximately 1nm, \( \log \left( \frac{2(g_0 - \hat{w})}{r} \right) \approx \log \left( \frac{2g_0}{r} \right) - \frac{\hat{w}}{g_0} \approx \log \left( \frac{2g_0}{r} \right) \). Therefore, we choose to neglect the CNT displacement inside the \( \log \) argument. Justification for this approximation is presented in the supplementary information.

We shall emphasize a significant difference between this analysis and previous studies on buckled beams [44,45,49,50]. In those studies, the capacitance derivative between two plates is inversely proportional to the displacement-squared \((F \propto \frac{1}{w^2}, [45]) \), whereas in our case, it is inversely dependent on the displacement \((F \propto \frac{1}{w}) \).

To conclude, our initial 2D model is based on the following EB beam equation:

\[
EI \frac{\partial^4 \hat{w}}{\partial \hat{x}^4} + \rho A \frac{\partial^2 \hat{w}}{\partial t^2} - T \frac{\partial^2 \hat{w}}{\partial \hat{x}^2} = \frac{\pi \varepsilon_0}{(g_0 - \hat{w})\log \left( \frac{2g_0}{r} \right)^2} \left( (V_0 + V_{gDC})^2 + 2(V_0 + V_{gDC})V_{gAC} \right)
\]

(7)

2.2 Non-dimensional beam equation

It is customary to work in dimensionless units. All non-dimensional parameters used in the formulation are detailed in Table 2. After the transformation, (7) and (2) reduce to:

\[
w^{('')'} + Pw^{('')} - \alpha w^{('')} \int_0^1 w^2 dx + \ddot{w} = (\kappa_0 + \kappa_s + \kappa_d) \frac{1}{1 - w}
\]

(8)

where prime (’) and dot (·) represent the derivative with respect to \( x \) and time, respectively. Doubly clamped boundary conditions are imposed: \( w(0, t) = w(1, t) = w'(0, t) = w'(1, t) = 0 \). We describe the non-dimensional deflection as a superposition of the initial
buckling \((w_0)\), the static deflection \((w_s)\) due to the DC voltage, and the dynamic oscillation \((w_d)\) due to the AC actuation (Fig. 1):

\[ w(x, t) = w_0(x) + w_s(x) + w_d(x, t) \tag{9} \]

Substituting (9) into (8), the CNT motion is described by a set of three nonlinear integro-differential equations: initial state (10), static deflection due to the DC force, \(\kappa_s\) (11), and the dynamic motion due to the AC actuation \(\kappa_d\) (12):

\[
\left( w_{0}^{\prime\prime\prime\prime} + P w_{0}^{\prime\prime} - \alpha w_{0}^{\prime}\int_{0}^{1} w_{0}^{\prime2}\,dx \right) \cdot (1 - w_{0}) = \kappa_0
\]

\[
\left[ w_{s}^{\prime\prime\prime\prime} + \left( P - \alpha \int_{0}^{1} w_{0}^{\prime2}\,dx \right) w_{s}^{\prime\prime} \right. \\
- \alpha \left( w_{0}^{\prime\prime} + w_{s}^{\prime\prime} \right) \int_{0}^{1} \left( 2w_{0}'w_{s}' + w_{s}^{\prime2} \right)\,dx \left(1 - w_{0} - w_{s} \right) \right.
\]

\[
- \left( w_{0}^{\prime\prime\prime\prime} + P w_{0}^{\prime\prime} - \alpha w_{0}^{\prime}\int_{0}^{1} w_{0}^{\prime2}\,dx \right) w_{s} = \kappa_s
\]

\[
\left[ w_{d}^{\prime\prime\prime\prime} + \left( P - \alpha \int_{0}^{1} \left( w_{0}^{\prime} + w_{s}^{\prime} \right)^2\,dx \right) w_{d}^{\prime\prime} + \ddot{w}_{d} \right. \\
- \alpha \left( w_{0}^{\prime\prime} + w_{s}^{\prime\prime} \right) \int_{0}^{1} \left( 2(w_{0}^{\prime} + w_{s}^{\prime})w_{d}^{\prime} + w_{d}^{\prime2} \right)\,dx \left(1 - w_{0} - w_{s} - w_{d} \right) - \left[ w_{d}^{\prime\prime\prime\prime} + w_{d}^{\prime\prime\prime\prime} \right. \\
+ \left. \left( P - \alpha \int_{0}^{1} \left( w_{0}^{\prime} + w_{s}^{\prime} \right)^2\,dx \right) \left( w_{0}^{\prime\prime} + w_{s}^{\prime\prime} \right) \right]\] \\
\left. (1 - w_{0} - w_{s} - w_{d}) \right) w_{d} = \kappa_d
\tag{12}
\]

2.3 Galerkin reduced-order model

We approximate the beam deflection as:

\[
w(x, t) = \sum_{i=1}^{N} \left( q_{0i} + q_{si} + q_{di}(t) \right) \varphi_i(x) \tag{13}\]

where \(\varphi_i(x)\) is the \(i\)th eigenmode of an unactuated doubly clamped buckled beam, given by

\[
\varphi_i(x) = \beta_i \left[ \cos(\lambda_i x) - \frac{1}{\sin(\lambda_i x)} \sin(\lambda_i x) - \frac{\lambda_i}{\sin(\lambda_i x)} x + 1 \right]
\tag{14}\]

where \(\beta_i\) are chosen such that \(\max_{x\in[0,1]}(\varphi_i(x)) = 1\) and \(\lambda_i\) are the eigenvalues, found as the solution to \(2(1 - \cos(\lambda_i)) - \lambda_i \sin(\lambda_i) = 0\). We shall note that \(\varphi_i\) with odd indices are symmetric functions of \(x \in [0, 1]\), whereas \(\varphi_i\) with even indices are asymmetric functions. The different \(q_{0i}, q_{si}, q_{di}\) represent the initial, static, and dynamic normalized amplitudes of the \(i\)th eigenmode, respectively.

Assuming that \(w_0(x) = \sum_{i=1}^{N} q_{0i}\varphi_i(x)\) is a solution to (10), we can substitute it into (11) as well as \(w_s(x) = \sum_{i=1}^{N} q_{si}\varphi_i(x)\). Using the standard Galerkin decomposition \([44, 48]\), of multiplying the equation by \(\varphi_i(x)\) and integrating over \(x\) within \([0, 1]\) integral, (11) is transferred to a system of coupled nonlinear algebraic equations, as detailed in the following sections.

For compact writing, let us define:

\[
\int_{0}^{1} \varphi_i dx = e_i, \quad \int_{0}^{1} \varphi_i^2 dx = a_{ii}, \quad \int_{0}^{1} \varphi_i \varphi_j dx = b_{ij} \quad \int_{0}^{1} \varphi_i^{\prime\prime\prime\prime} \varphi_k dx = d_{ijk}, \quad \int_{0}^{1} \varphi_i \varphi_j^{\prime\prime\prime\prime} \varphi_k dx = u_{ijk}, \quad \int_{0}^{1} \varphi_i \varphi_j^{\prime\prime\prime\prime} \varphi_k dx = p_{ijk}, \quad \int_{0}^{1} \varphi_i \varphi_j \varphi_k dx = x_{ijk}.
\]
2.4 Single-mode decomposition

In order to understand the significance of each mode to the overall behavior, let us begin taking only the first Galerkin mode. In this case, all equations can be solved analytically. The initial configuration is described by:

\[ q_{01} \left( d_{11} - (P - \alpha b_{11} q_{01}^2)b_{11} \right) (1 - q_{01}) = \kappa_0 e_1 \]

(15)

For \( \kappa_0 = 0 \), we can extract a simple relation between the initial axial strain and the initial midpoint deflection: \( q_{01}^2 = (P b_{11} - d_{11})/(\alpha b_{11}^2) \), where we choose the negative root for upward buckling and the positive root for downward buckling. In order to receive some initial buckling (i.e., a real root), we must demand \( P > d_{11}/b_{11} \), which translates to the standard EB buckling criterion \( |T_0| > 4\pi^2 E l^2/L^2 \). However, if we allow \( \kappa_0 \neq 0 \), then \( P \) and \( q_{01} \) can be chosen as two independent parameters, yielding more diverse possibilities for the static and dynamic behavior of the device.

Assuming independent \( P \) and \( q_{01} \), and taking only one mode \( (N = 1) \), one receives a cubic equation for \( q_{s1} \) for every static load \( \kappa_s \):

\[ q_{s1} \left[ d_{11} - (P - \alpha (Q_0 + Q_s)) b_{11} \right] (1 - \lambda q_{01} - \lambda q_{s1}) + \alpha q_{01} b_{11} Q_s (1 - \lambda q_{01}) = \kappa_s e_1 \]

(16)

where \( Q_0 = b_{11} q_{01}^2 \), \( Q_s = 2b_{11} q_{01} q_{s1} + q_{s1}^2 b_{11} \), and \( \lambda = 1 \) (its role will be clarified in the results analysis).

As a sanity check, one can notice that for \( \kappa_s = 0 \) we get the trivial solution, which corresponds to our definition of the static displacement: \( q_{s1} = 0 \).

Moving to the dynamic equation, we perform the single-mode Galerkin decomposition for (12), yielding:

\[ \omega^2 = \kappa_d e_1 \]

(17)

where \( Q_d = 2(q_{01} + q_{s1}) q_{d1} b_{11} + q_{d1}^2 b_{11} \). Assuming small vibrations, we can neglect the nonlinear dynamic terms and solve for the resonance frequencies.

2.5 Single-mode decomposition—results and discussion

2.5.1 Static analysis—single mode with constant force

The static equation (16) defines the dependence of the static deflection of the beam, \( q_{s1} \), on the applied static force, \( \kappa_s \) (Fig. 2). Since the force can be regarded as the derivative of the potential energy \( (\kappa_s = dU/dq_{s1}) \), physical solutions (i.e., stable solutions) exist only for non-negative values of the derivative of (16), meaning \( d^2 U/dq_{s1}^2 \geq 0 \). Snap-through transition occurs at the extremum points of this curve, where \( d \kappa_s/dq_{s1} = 0 \iff d^2 U/dq_{s1}^2 = 0 \). Hence, if one wishes to find the critical loads at which the ST transition occurs, one should compare the derivative of (16) to zero.

In practice, we have used a difference, equivalent, method to examine the stability of (16). We solved the linearized (17)—real roots are actual resonance frequencies, corresponding to a stable static configuration, able to sustain a periodic response when perturbed. Imaginary roots correspond with unstable static configurations, and hence, the resonance frequency does not exist.

If we assume small initial and static deflection \( (\dot{w} \ll g_0) \), we can approximate the static electrostatic force as a constant: \( (\kappa_s \approx (\pi\varepsilon_0 g_0 V^2_{\text{DC}})/(2g_0^2r)) \). Substituting into (1), transforming the equation to...

\[ \text{(Color figure online)} \]
dimensionless units, and performing the Galerkin decomposition process for a single mode (N=1), we obtain the same static equation as in (16), but with \( \lambda = 0 \):

\[
q_{s1}(d_{11} - (P - \alpha(Q_0 + Q_s))b_{11}) + \alpha q_{01} b_{11} Q_s = \kappa_s e_1
\]

Substituting \( Q_0 \) and \( Q_s \), taking the derivative of (18) with respect to \( q_{s1} \), and comparing to zero yields

\[
q_{s1}^{\text{extremum}} = \left(-6q_{01} \pm \sqrt{-12(d_{11} - Pb_{11})/\alpha/b_{11}^2}/6 \right)
\]

From here, we find the non-dimensional criterion for buckling \( P_c = d_{11}/b_{11} \), as in previous studies [49,50]. In the case where \( P < P_c \), the roots are imaginary and hence the transition from upward to downward curvature is continuous, but for \( P > P_c \), there exist two real extremum points in which the snap-through and release transitions occur (marked by the dashed arrows in Fig. 2). If we were to assume that \( \kappa_0 = 0 \) and substitute the condition, it imposes on \( q_{01} \) into (16), the continuous transition is not possible. Furthermore, bistability in this case (\( \kappa_0 = 0 \)) results only in latching [23], meaning a snap-back (release) transition at \( \kappa_s < 0 \).

2.5.2 Static analysis—single-mode general solution

Realistic device parameters force us to consider the nonlinearity of the electrostatic force. Unfortunately, an analytical solution is not possible. Therefore, to investigate the system, we introduce the non-dimensional parameter \( \lambda \) to (16), where \( \lambda = 0 \) reduces the equation to the case of constant force (18) and \( \lambda = 1 \) is the general equation (16):

\[
q_{s1}(d_{11} - (P - \alpha(Q_0 + Q_s))b_{11})(1 - 2\lambda q_{01} - \lambda q_{s1}) + \alpha q_{01} b_{11} Q_s(1 - \lambda q_{01}) = \kappa_s e_1
\]

Examining the effect of \( \lambda \) reveals that the qualitative behavior does not change, only that a smaller force is required for ST and release (Fig. 3).

2.5.3 Dynamic analysis—single-modes resonance frequencies

As displayed in Fig. 4a, if \( P < P_c \) the resonance frequency dependence on the gate voltage is continuous (orange curve), whereas for \( P > P_c \) the resonance frequency decreases to zero after which it exhibits a “jump” (blue curve). The dark and light grey arrows present an upward or downward gate voltage sweeps, respectively. Purple line represents the ST hysteresis window, i.e., the difference between static loads at which the ST (dark grey) and release (light grey) jumps occur. b–c CNT shape for several static loads in the vicinity of the transition from upward to downward buckling, corresponding to the curves in (a), presenting both continuous transition for \( P < P_c \) (b) and ST “jump” (c). (Color figure online)
this result is very encouraging. However, the “jumps” measured in our experiments occur at nonzero resonance frequency, and we did not detect zero resonance frequency in any of the dozens of fabricated devices.

Previous studies [49,50] suggest that symmetry breaking can account for such observations. In order to introduce such symmetry breaking into the system, we must take two or more modes into our Galerkin-based approximation.

2.6 Two-mode decomposition

In previous studies, even when a second mode was introduced, the initial shape still consisted of only a single mode, \( q_{01} \) [49]. This makes sense for a buckled micro-beam since the width is typically much bigger than the thickness. However, for a naturally grown CNT, the width and thickness are the same, and the chances of the CNT to grow in a perfect symmetric arch shape are quite slim. In addition, we show that if the deflection is very small such that the electrostatic nonlinearity can be neglected, taking \( q_{02} = 0 \) in (18) eliminates the geometric nonlinearity and results in a linear dependence of \( \kappa \) on \( q_{s1} \) (see Supplementary Information for details). Therefore, we must introduce a second mode to the initial state as well.

After the Galerkin analysis for two modes, a solution to (11) can be approximated by solving the following set of coupled equations:

\[
\begin{align*}
q_{s1} d_{11} - (P - \alpha Q_{0}) q_{s1} b_{11} & + \alpha (q_{01} + q_{s1}) b_{11} \cdot Q_{1} \cdot (1 - q_{01} - q_{s1}) \\
& - [q_{s2} u_{221} + (P - \alpha Q_{0}) q_{s2} p_{221}] \\
& - \alpha (q_{02} + q_{s2}) p_{221} \cdot Q_{1} \cdot (q_{02} + q_{s2}) \\
& - q_{s1}[q_{01} d_{11} - (P - \alpha Q_{0}) q_{01} b_{11}] \\
& - q_{s2}[q_{02} u_{221} + (P - \alpha Q_{0}) q_{02} p_{221}] - \epsilon_{1} \kappa_{x} = 0
\end{align*}
\]

\[
\begin{align*}
q_{s2} d_{22} - (P - \alpha Q_{0}) b_{22} q_{s2} & + \alpha (q_{02} + q_{s2}) b_{22} \cdot Q_{1} \\
& - [q_{s2} u_{122} + (P - \alpha Q_{0}) q_{s2} p_{122}] \\
& - \alpha (q_{02} + q_{s2}) p_{122} \cdot Q_{1} \cdot (q_{02} + q_{s2}) \\
& - [q_{s1} u_{212} + (P - \alpha Q_{0}) q_{s1} p_{212}] \\
& - \alpha (q_{01} + q_{s1}) p_{212} \cdot Q_{1} \cdot (q_{02} + q_{s2}) \\
& - q_{s1}[q_{02} u_{122} + (P - \alpha Q_{0}) q_{02} p_{122}] \\
& - q_{s2}[q_{01} u_{212} + (P - \alpha Q_{0}) q_{01} p_{212}] = 0
\end{align*}
\]

where \( Q_{0} = \int_{0}^{1} (w_{0}')^{2} dx = q_{01}^{2} b_{11} + q_{02}^{2} b_{22} \) and \( Q_{s} = \int_{0}^{1} [2 w_{0}' w_{s}' + (w_{s}')^{2}] dx = b_{11} q_{s1} (2 q_{01} + q_{s1}) + b_{22} q_{s2} (2 q_{02} + q_{s2}). \)

Moving to the dynamics, we perform the Galerkin decomposition process for (12), which transforms into:

\[
q_{d1} d_{11} - (q_{01} + q_{s1} + q_{d1}) q_{d1} u_{111} \\
- (q_{02} + q_{s2} + q_{d2}) q_{d2} u_{221} \\
- (q_{01} + q_{s1}) q_{d1} u_{111} + (q_{02} + q_{s2}) q_{d2} u_{221} - (P - \alpha Q_{0d}) \\
\cdot \left( q_{d1} b_{11} + (q_{01} + q_{s1}) q_{d1} p_{111} + (q_{02} + q_{s2}) q_{d2} p_{221} + \right) \\
\left( q_{01} + q_{s1} + q_{d1} q_{d1} p_{111} + (q_{02} + q_{s2} + q_{d2}) q_{d2} p_{221} \right) \\
+ \alpha Q_{d} \cdot \left[ (q_{01} + q_{s1} + q_{d1}) b_{11} \right. \\
\left. + (q_{01} + q_{s1}) q_{d1} u_{111} + (q_{02} + q_{s2} + q_{d2}) q_{d2} u_{221} \right] \\
- \omega^{2} \cdot [q_{d1} a_{111} - (q_{01} + q_{s1}) q_{d1} x_{111}] \\
- (q_{02} + q_{s2} + q_{d2}) q_{d2} x_{221}] - \epsilon_{1} \kappa_{x} = 0
\]

\[
q_{d2} d_{22} - (q_{02} + q_{s2}) q_{d2} u_{122} - (q_{01} + q_{s1}) q_{d2} u_{212} \\
- (q_{01} + q_{s1} + q_{d1}) q_{d2} u_{122} - (q_{02} + q_{s2} + q_{d2}) q_{d2} u_{212} \\
- (P - \alpha Q_{0d}) \\
\cdot \left( q_{d2} b_{22} + (q_{02} + q_{s2}) q_{d2} p_{122} + (q_{01} + q_{s1}) q_{d2} p_{212} + \right) \\
\left( q_{01} + q_{s1} + q_{d1} q_{d1} p_{122} + (q_{02} + q_{s2} + q_{d2}) q_{d2} p_{212} \right) \\
+ \alpha Q_{d} \cdot \left[ (q_{02} + q_{s2} + q_{d2}) b_{22} \\
+ (q_{01} + q_{s1} + q_{d1}) q_{d2} + q_{s2} + q_{d2}) q_{d2} (p_{122} + p_{212}) \right] \\
- \omega^{2} \cdot [q_{d2} x_{222} - (q_{01} + q_{s1} + q_{d1}) q_{d2} x_{122}] \\
- (q_{02} + q_{s2} + q_{d2}) q_{d2} x_{212} = 0
\]

\( Q_{0d} = \int_{0}^{1} (w_{0}')^{2} dx = (q_{01} + q_{s1})^{2} b_{11} + (q_{02} + q_{s2})^{2} b_{22} \) and \( Q_{d} = \int_{0}^{1} [2 w_{0}' w_{s}' + 2 w_{s}' w_{s}'] dx = 2 b_{11} q_{d1} q_{s1} + 2 b_{22} q_{d2} q_{s2} + \frac{q_{d1}^{2} b_{11} + q_{d2}^{2} b_{22}}{2}. \) We wish to solve for the resonance frequencies, so we neglect all nonlinear dynamic terms.

2.7 Two mode decomposition—results and discussion

2.7.1 Static two-mode analysis

In the single-mode case, the only parameter responsible for the formation of snap-through buckling was the initial axial strain \( P \). In the two-mode approximation, however, the initial \( q_{02} \) also significantly influences the static response of the system, and whether a snap-through or a continuous transition will occur. Figure 5 presents the solutions of (20) for the force dependence on the symmetric static mode \( (q_{s1}) \), solved for
Fig. 5 Static response solution for the two-mode approximation. The dependence of the static response on the initial second mode $q_{02}$. The different colored lines represent different initial $q_{02}$ values, where the thick parts are stable solutions and the thin lines represent an unstable solution. The grey line is the solution for a single-mode approximation, in which $P > P_c$. $q_{02} \to 0$ approaches the single-mode solution, but the ST and release transitions (light blue dots) occur before reaching the extremum points of the grey curve (yellow dots). Larger $q_{02}$ results in a continuous transition. (Color figure online)

varying asymmetric initial configurations (i.e., varying $q_{02}$ values). As with the single mode case, solution stability was assessed by the curvature of the energy. It can be observed that when $q_{02}$ is small, the stable solution (thick line) is not continuous, and when $q_{02}$ increases, the stable solution becomes continuous (green line). Note that even when $q_{02} \to 0$, while the mathematical solution (blue lines) approaches the single mode solution (grey line), the physically stable part (thick blue line) is different, and the ST transition occurs at a new extremum point of the $\kappa_s$ versus $q_{s1}$ curve (at the edges of the thick lines, marked by the light blue dots), which is far from the single-mode solution extrema (yellow dots, grey line).

Let us explain. The first equation in (20) defines a two-dimensional surface of the force, $\kappa_s(q_{s1}, q_{s2})$ (colored surface in Fig. 6). The left column presents a three-dimensional view of the force for increasing $q_{02}$ values. Together with the second equation, $\kappa_s(q_{s1}, q_{s2}(q_{s1}))$ is reduced to a curve in this three-dimensional space, depicted by the black (stable) and red (unstable) lines in Fig. 6. The middle column presents a “front view” of the same graph as the left column, almost perpendicular to the $\kappa_s - q_{s2}$ plane. The snap-through transition occurs only when reaching a local extremum point along the $\kappa_s(q_{s1}, q_{s2}(q_{s1}))$ curve, where $d\kappa_s / dq_{s1} = \delta \kappa_s / \delta q_{s1} + \delta \kappa_s / \delta q_{s2} \cdot \delta q_{s2} / \delta q_{s1} = 0$ (exactly at the edges of the black curve in Fig. 6 mid-column or the edges of the thick colored lines in Fig. 5). When $q_{02}$ increases, while the landscape of $\kappa_s$ is only slightly modified, the solution curve moves away from the saddle point along the $q_{2}$ axis, bypassing the instability. In the mechanical system, what actually happens to the CNT is referred to as symmetry breaking. The CNT shape can now deform before the transition from upward to downward buckling, both for the ST jump (Fig. 7b) and for the continuous transition (Fig. 7a). Since $q_{01} \gg q_{02}$, the initial beam shape is almost symmetric. When approaching the transition point, however, the symmetric mode nearly vanishes and the beam shape transforms to be nearly asymmetric. This corresponds well with the shift along the $q_{s2}$ axis, apparent in the right column of Fig. 6 (see also Fig. S1).
The dependence of the static response on the initial second mode $q_{02}$. As $q_{02}$ increases (top to bottom), the force curve is shifted from the saddle point and the bifurcation is avoided. Left column is a three-dimensional view. Middle column is the $\kappa_s - q_{s1}$ cross-section (same as in Fig. 5), and right column is the $\kappa_s - q_{s2}$ cross-section. The black and red lines indicate the $\kappa_s(q_{s1}, q_{s2}(q_{s1}))$ curve in this three-dimensional space that is the solution to (20), where black parts represent stable solution and red parts represent unstable solutions. (Color figure online)

2.7.2 Dynamic analysis—two-modes resonance frequencies

To obtain the resonance frequencies, Eq. (21) is linearized with respect to $q_{d1}, q_{d2}$. We substitute an oscillatory solution and disregard the external dynamic force. We are then left with an eigenvalue problem, from which we calculate the frequencies, presented in Fig. 8. It depicts the first resonance mode dependence on the static force for the case of a small asymmetric initial component (blue), which results in a snap-through transition vs. the case of a large asymmetric initial component (orange), which results in a continuous transition. For small $q_{02}$ (blue curve), the resonance frequency decreases due to compression (dark grey arrows) until it reaches zero, and then, the snap-through transition occurs, which translates to a “jump” in the resonance frequency (marked by the vertical dark grey arrow). Increasing the force further (dark grey arrows) causes the resonance frequency to rise due to stretch-
See the image for text content.
external force, causing it to rotate. For a CNT with a downward slack ($q_{01} > 0$), the static out-of-plane motion is less likely, since the torque attracts the CNT towards the in-plane motion. However, for a CNT with $q_{01} < 0$, the torque pushes the tube away from the in-plane motion, and the whole movement from upward curvature towards the gate electrode can include out-of-plane motion as well and cannot be ignored. Therefore, we are forced to describe the CNT motion as a superposition of in-plane ($w(x, t)$) and out-of-plane ($v(x, t)$) components:

\[
\begin{align*}
\dot{w}(x, t) &= w_0(x) + w_s(x) + w_d(x, t) \\ 
\dot{v}(x, t) &= v_0(x) + v_s(x) + v_d(x, t)
\end{align*}
\]

(24)  
(25)

In this case, we receive two coupled non-dimensional Euler–Bernoulli beam equations for each component, where we still restrict the force exerted by the gate to be solely along the z direction (in-plane):

\[
\dddot{w} + P\ddot{w}'' - \alpha \dddot{w}'' \int_0^1 \left( (w')^2 + (v')^2 \right) dx + \ddot{w} = 0
\]

(26)

\[
\dddot{v} + P\ddot{v}'' - \alpha \dddot{v}'' \int_0^1 \left( (w')^2 + (v')^2 \right) dx + \ddot{v} = 0
\]

(27)

As before, these can be divided into initial state, static equations and dynamic equations:

\[
\begin{align*}
\left[ w_0'''' + P w_0''' - \alpha w_0'' Q_0 \right] \cdot (1 - w_0) &= \kappa_0 \\
\left[ v_0'''' + P v_0''' - \alpha v_0'' Q_0 \right] &= 0
\end{align*}
\]

\[
\begin{align*}
\left[ w_s'''' + (P - \alpha Q_0) w_s''' - \alpha (w_s'' + w_s') Q_s \right] \\
\left[ v_s'''' + (P - \alpha Q_0) v_s''' - \alpha (v_s'' + v_s') Q_s \right]
\end{align*}
\]

\[
\begin{align*}
(1 - w_0 - w_s) \\
(1 - w_0 - w_d)
\end{align*}
\]

where the in-plane and out-of-plane coupling results from the built-in tension along the CNT, represented by the following integral terms:

\[
Q_0 \triangleq \int_0^1 \left( w_0''^2 + v_0''^2 \right) dx
\]

\[
Q_s \triangleq \int_0^1 \left( 2 w_0' w_s'' + w_s^2 + 2 v_0' v_s'' + v_s^2 \right) dx
\]

\[
Q_d \triangleq \int_0^1 \left( 2 w_0'' + w_s'' \right) w_d'' + w_d^2
\]

\[
+ 2 (v_0'' + v_s'') v_d'' + v_d^2 \right) dx
\]

(29)

3.1 Galerkin decomposition

For the static deflection, we substitute

\[
\begin{align*}
w_0(x) &= q_{01} \phi_1(x) + q_{02} \phi_2(x) \\
w_s(x) &= q_{11} \phi_1(x) + q_{12} \phi_2(x) \\
v_0(x) &= v_{01} \phi_1(x) + v_{02} \phi_2(x) \\
v_s(x) &= v_{11} \phi_1(x) + v_{12} \phi_2(x)
\end{align*}
\]

(30)

into equations (28) and receive a set of four coupled algebraic static equations:

\[
[q_{s1} d_{11} - (P - \alpha Q_0) q_{s1} b_{11} + \alpha (q_{01} + q_{s1}) b_{11} \cdot Q_s] \cdot (1 - q_{01} - q_{s1})
\]

\[
- [q_{s2} u_{211} + (P - \alpha Q_0) q_{s2} p_{211} - \alpha (q_{02} + q_{s2}) p_{211} \cdot Q_s] (q_{02} + q_{s2})
\]

\[
- q_{s1} [q_{01} d_{11} - (P - \alpha Q_0) q_{01} b_{11} - q_{s1} [q_{02} u_{211} + (P - \alpha Q_0) q_{02} p_{211} - \alpha (q_{02} + q_{s2}) p_{211} \cdot Q_s] (q_{02} + q_{s2})]
\]

\[
- q_{s2} [q_{01} d_{211} - (P - \alpha Q_0) q_{02} p_{212} - \alpha (q_{01} + q_{s1}) p_{212} \cdot Q_s] (q_{01} + q_{s1})
\]

\[
- q_{s1} [q_{02} u_{212} + (P - \alpha Q_0) q_{02} p_{212} - \alpha (q_{02} + q_{s2}) p_{212} \cdot Q_s] (q_{02} + q_{s2})
\]

\[
- q_{s2} [q_{01} d_{212} + (P - \alpha Q_0) q_{01} p_{212}] = 0
\]

\[
[\alpha (v_{01} + v_{s1}) b_{11} \cdot Q_s] (q_{01} + q_{s1})
\]

\[
+ [\alpha (v_{02} + v_{s2}) b_{22} \cdot Q_s] (q_{02} + q_{s2})
\]

\[
+ \alpha (v_{01} + v_{s1}) b_{11} \cdot Q_s = 0
\]

\[
[\alpha (v_{02} + v_{s2}) b_{22} \cdot Q_s] (q_{02} + q_{s2}) = 0
\]

(31)
remain the same as (20), and the coupling to the out-of-plane motion is manifested through the built-in tension integrals, $Q_0$ and $Q_s$.

For the dynamic analysis, we define

$$w_d(x) = q_d1\varphi_1(x) + q_d2\varphi_2(x)$$
$$v_d(x) = v_d1\varphi_1(x) + v_d2\varphi_2(x)$$

The dynamic equations in (28) are transformed to the following set of algebraic coupled equations:

$$q_d1d11 - (q_01 + q_11)q_d1u_{111} - (q_02 + q_22)q_d2u_{221}$$
$$- (q_01 + q_1 + q_11)q_d1\varphi_111 - (q_02 + q_2 + q_22)q_d2\varphi_221$$
$$- (P - \alpha Q_d)$$
$$\left( q_d1b_{11} + (q_01 + q_11)q_d1p_{111} + (q_02 + q_22)q_d2p_{221} + \right)$$
$$\left( q_01 + q_1 + q_11)q_d1\varphi_111 + (q_02 + q_2 + q_22)q_d2\varphi_221 \right)$$
$$+ \alpha Q_d[q_d1d11 - (q_01 + q_11)$$
$$+ q_11q_d1x_{111} - (q_02 + q_22)q_d2x_{221}] - \omega^2$$

$$q_d2d22 - (q_02 + q_22)q_d2u_{222} - (q_01 + q_11)q_d2u_{212}$$
$$- (q_01 + q_1 + q_11)q_d2u_{222} - (q_02 + q_2 + q_22)q_d2u_{212}$$
$$- (P - \alpha Q_d)$$
$$\left( q_d2b_{22} + (q_02 + q_22)q_d2p_{222} + (q_01 + q_11)q_d2p_{212} + \right)$$
$$\left( q_01 + q_1 + q_11)q_d2\varphi_222 + (q_02 + q_2 + q_22)q_d2\varphi_221 \right)$$
$$+ \alpha Q_d[q_d2d22 - (q_01 + q_11 + q_11)q_d2x_{222}$$
$$- (q_02 + q_22)q_d2x_{212}] = 0$$

$$v_d1d11 - (P - \alpha Q_d)v_d1b_{111}$$
$$+ \alpha Q_d[v_01 + v_1 + v_11]b_{111} - \omega^2a_{11}v_1 = 0$$

$$v_d2d22 - (P - \alpha Q_d)v_d2b_{222}$$
$$+ \alpha Q_d[v_02 + v_2 + v_22]b_{222} - \omega^2a_{22}v_2 = 0$$

where $Q_{0d} = ((q_01 + q_11)^2 + (v_01 + v_11)^2)b_{111} + ((q_02 + q_22)^2 + (v_02 + v_22)^2)b_{222}$ and $Q_d = 2b_{111}(q_01 + q_11)q_d1v_1 + (v_11 + v_11)q_d1v_1 + 2b_{222}(q_02 + q_22)q_d2v_2 + (v_02 + v_22)v_2d) + b_{11}(q^2_{d1} + v^2_{d1}) + b_{22}(q^2_{d2} + v^2_{d2}).$

### 3.2 Results and discussion

#### 3.2.1 Three-dimensional static deflection

Equations (31) can be solved numerically. Figure 9 presents the effect of the initial out-of-plane component $v_{01}$ on the static response of the beam; $q_{02} = -1e - 5$ was chosen such that if $v_{01} = 0$, 2D ST should occur. $v_{02}$ is negligible for all curves. For higher values of $v_{01}$, but even for very small values, we see that the snap-through transition is eliminated, and the CNT moves in a continuous torsional motion downwards (Fig. 10). The effect of $v_{02}$ is very similar to that of $q_{02}$ in Fig. 5. Figure 10a presents a three-dimensional rotational transition of the static CNT shape from upward to downward curvature as the voltage applied to the LG is increased, for the case where $v_{01} = -1e - 5$. Bottom-left panel (Fig. 10b) is a “side-view” (projection on the $zy$-plane), showing how the out-of-plane rotational motion evolves to avoid ST buckling, and the bottom-right panel (Fig. 10c) is a “front-view” (projection on the $xz$-plane), showcasing a continuous transition; no “jump” is observed.

We shall note a significant difference between the in-plane and out-of-plane static deflection. Figure 11 depicts two scenarios extracted from the model. For initial configurations in which $v_{01,02} \ll q_{01}$ the evolution of the out-of-plane static modes ($v_{1,2}$) is plotted in Fig. 11a and b. Figure 11a presents their amplitudes with respect to the mid-point $z$-deflection ($q_{01} + q_{11}$), and Fig. 11b depicts their relative ratio ($v_{1}/v_{2}$). One can observe that indeed at the ST transition the two...
components reach their maximum values with a 10-percent increase of the symmetric mode \(v_{s1}\) compared to the anti-symmetric one \(v_{s2}\). However, for higher values of the out-of-plane initial configurations \(v_{01}, v_{02}\), out-of-plane centering of the CNT shape is observed (Fig. 11c and d). Unlike the in-plane symmetry breaking in which near the transition, the in-plane symmetric mode, \(q_{s1}\), vanishes and the asymmetric mode, \(q_{s2}\), is at its maximum (Fig. 7 and Fig. S1), the out-of-plane motion act in an opposite manner. Near the transition, when the out-of-plane deflection is at its maximum, the asymmetric component, \(v_{s2}\), is oppressed compared to the symmetric component, \(v_{s1}\) (Fig. 11c and d). Moreover, since the tube length increases due to stretching by the LG, the maximum out-of-plane deflection occurs at positive z \(q_{01} + q_{s1} > 0\).

In order to achieve ST transition, we must increase the initial tension, \(P\) and the initial \(v_{01}, v_{02}\) accordingly. Figure 12a presents a novel three-dimensional snap-through transition. As before, out-of-plane rotational motion evolves with the electrostatic force (note the “side-view” in b), but at the critical point, the out-of-plane stretching is insufficient and the CNT cannot
compress downward any further, exhibiting a 3D EB instability, resulting in a “jump” to a downward curvature configuration, noticeable in the “front view” in (c).

The evolution of the ST buckling in this case depends on five initial parameters: $P, q_{01}, q_{02}, v_{01}$ and $v_{02}$. Since we cannot visualize a five-dimensional phase space, we shall investigate a three-dimensional phase space while keeping two initial parameters fixed. First, we fix $q_{01}$ and $q_{02}$ while varying the out-of-plane initial configurations (Fig. 13a), and then, we fix $v_{01}$ and $v_{02}$ and vary the in-plane initial configurations (Fig. 13b). Solving (31) for various initial configurations (as depicted by the dots in Fig. 13a and b) affirms that the initial configuration of the CNT determines the static response of the system. If the total tension of the CNT at zero gate voltage, $(P - \alpha Q_0)$, is lower than a critical value $P_c^*$ (depicted by the lowest surface separating the blue and green dots, Fig. 13a and b), then the CNT advances upward to a downward curvature continuously through a rotational motion (as shown in Fig. 10). When the initial total tension is higher than the critical value $P_c^*$, the system exhibits EB ST buckling transition despite the out-of-plane movement (Fig. 12). Unlike the single-mode ST criterium $(P > P_c = d_{11}/b_{11})$, which is easily calculated analytically and is independent of any specific parameters of the device, an expression for $P_c^*$ is not trivial, and its value (although non-dimensional) depends on the physical parameters of the device. Hence, further research needs to be conducted in order to establish the exact criteria for 3D ST buckling.

The black dots in Fig. 13a and b present a third regime (between the green and grey surfaces), in which latching bi-stability is observed [23]. In this regime, when force is applied, the CNT compresses downward ($q_{s1}$ increases), until the bifurcation point, at which the CNT “snaps” downward (marked by the top dashed arrow in Fig. 14). When the electrostatic force is removed (i.e., going back along the blue curve until the force along the z-axis equals zero), the CNT remains in downward buckling configuration. In order to achieve the release (“snap-back”) transition, a negative force must be applied, meaning that a second gate electrode must be used. This configuration can be utilized for realizing a non-volatile mechanical memory element, as was previously suggested with lateral buckling configurations [26]. In order to detect the predicted latching experimentally, suitable devices with a second gate electrode must be designed. As in the case for the ST criteria, the latching critical value is also dependent on the physical parameters of the device, requiring further research to arrive at the general criteria for latching. We shall note that the initial tension, $(P - \alpha Q_0)$, has a third critical value (depicted by the grey surface in Fig. 13a and b), such that higher values result in an unstable physical solution.
Theoretical modeling of arch-shaped carbon nanotube

Fig. 14 Static response for initial parameters satisfying the latching criteria. Note that the release occurs at negative force, meaning that at zero gate voltage the system remains in the downward-buckled state \( q_{01} + q_{s1} > 0 \) and can hence serve as a non-volatile memory.

3.2.2 Resonance frequencies of the three-dimensional system

In-plane initial configuration

Let us begin with the assumption that there is no out-of-plane initial deflection at all, meaning \( v_0(x) = 0 \). This results in zero out-of-plane static deflection, \( v_s = 0 \) (grey curve in Fig. 9). The in-plane and out-of-plane coupling is manifested through the tension integrals \( Q_{0d}, Q_d \). For \( v_0 = v_s = 0 \), only the quadratic dynamic terms preserve this coupling, represented by \( Q_d^* = b_{11}(q_{d1}^2 + v_{d1}^2) + b_{22}(q_{d2}^2 + v_{d2}^2) \). In the linear response analysis for the resonance modes, nonlinear dynamic terms are neglected (small vibrations), and the in-plane and out-of-plane coupling is removed. Thus, the dynamics in (33) can be solved as two separate systems: (1) two coupled in-plane equations and (2) two uncoupled out-of-plane equations.

Fig. 15a presents the resonance frequencies gate dependence for the lowest two in-plane and lowest two out-of-plane resonance modes. All the resonance frequencies decrease when the static force is applied due to compression. The lowest out-of-plane mode (purple) reaches zero frequency, after which there is a frequency “jump” (Fig. 15b). Applying additional force causes an increase in frequency (also known as “hardening”) due to stretching of the CNT in its downward configuration. However, an asymmetric vibration will stretch both parts (the upward and downward curvatures) of the tube, and therefore, the anti-symmetric mode stretches the CNT, which translates into an increase of the second in-plane resonance mode.

In order to understand Fig. 15, let us examine the eigenvalue problem as before:

\[
\begin{pmatrix}
H_{11} & H_{12} & 0 & 0 \\
H_{12} & H_{22} & 0 & 0 \\
0 & 0 & H_{33} & 0 \\
0 & 0 & 0 & H_{44}
\end{pmatrix}
\begin{pmatrix}
q_{d1} \\
q_{d2} \\
v_{1d} \\
v_{2d}
\end{pmatrix}
= \omega^2
\begin{pmatrix}
q_{d1} \\
q_{d2} \\
v_{1d} \\
v_{2d}
\end{pmatrix}
\]

where \( H_{33} = \frac{\partial^2 H}{\partial v_{d1}^2} \) and \( H_{44} = \frac{\partial^2 H}{\partial v_{d2}^2} \). (The other parameters are defined the same as in (22).) The eigenvalues are extracted from the requirement that \( \det(H) = 0 \) and one obtains the following relation:

\[
(H_{33} - \omega^2)(H_{44} - \omega^2)((H_{11} - \omega^2)(H_{22} - \omega^2) - H_{12}H_{21}) = 0
\]

We shall note that the second in-plane mode (blue) displays hardening (an increase in frequency) near the ST transition. We attribute this hardening to the fact that the transition is anti-symmetric (Fig. 12c). A symmetric vibration around the asymmetric static configuration will stretch one part of the CNT and compress the other, with an overall negligible effect on the CNT tension. However, an asymmetric vibration will stretch both parts (the upward and downward curvatures) of the tube, and therefore, the anti-symmetric mode stretches the CNT, which translates into an increase of the second in-plane resonance mode.

Fig. 15a presents the resonance frequencies gate dependence for the lowest two in-plane and lowest two out-of-plane resonance modes. Note that the out-of-plane modes are lower and that the lowest out-of-plane mode is the one to reach zero frequency and dictate the ST “jump”. Fig. 15b shows a zoom-in on the “jump” of the lowest out-of-plane mode when reaching zero, and Fig. 15c the consequent “jump” of the lowest in-plane mode at nonzero frequency.

Fig. 15 Three-dimensional dynamics. a Resonance frequencies gate dependence for the lowest two in-plane and lowest two out-of-plane resonance modes. Note that the out-of-plane modes are lower and that the lowest out-of-plane mode is the one to reach zero frequency and dictate the ST “jump”. b Zoom-in on the “jump” of the lowest out-of-plane mode when reaching zero, and c the consequent “jump” of the lowest in-plane mode at nonzero frequency.
At the critical ST transition, the Jacobian determinant must also equal zero (at the bifurcation point):

\[ H_{33}H_{44}(H_{11}H_{22} - H_{12}H_{21}) = 0 \]  

(36)

Substituting (36) into the expanded (35) eliminates the \( \omega^0 \) term, forcing a solution of \( \omega^2 = 0 \), which means that the zero frequency at the snap-through transition cannot be avoided. However, solving the dynamic equations reveals that the out-of-plane mode is always the lowest, and therefore, the mode to reach zero frequency is actually the out-of-plane mode (Fig. 15). This explains why zero resonance has never been evidenced in the experiments, since the only modes detected are the in-plane [27].

**Full 3D statics and dynamics**

Experimentally, fabricating a device with zero out-of-plane initial deflection is extremely unlikely, effectively impossible. Also, we shall emphasize the singularity of zero initial out-of-plane conditions, which results in static deflection that is only in-plane (two-dimensional). For even an extremely small out-of-plane initial component (0 < \( |v_0| \ll |q_{01}| \)), static out-of-plane motion will evolve and the out-of-plane and in-plane dynamic coupling cannot be neglected, even at the linear regime.

Therefore, we must introduce small out-of-plane components to the initial configuration (\( v_0 \neq 0 \)) as well. We solve the static equations for initial configurations, which satisfy the criteria for ST buckling, as discussed in Sect. 3.2.1. Then, in order to find the resonance frequencies, we assume small vibrations (linear regime) and neglect dynamic nonlinear terms in (33). We transfer the system of equations to a matrix form:

\[
\begin{pmatrix}
H_{11} - \omega^2 & H_{12} & H_{13} & H_{14} \\
H_{21} & H_{22} - \omega^2 & H_{23} & H_{24} \\
H_{31} & H_{32} & H_{33} - \omega^2 & H_{34} \\
H_{41} & H_{42} & H_{43} & H_{44} - \omega^2
\end{pmatrix}
\begin{pmatrix}
q_{d1} \\
q_{d2} \\
v_{1d} \\
v_{2d}
\end{pmatrix} = 0
\]

and solve for \( det(M) = 0 \) for every static load. Each \( H_{ij} \) element represents the second derivative of the Hamiltonian with respect to the relevant coordinates. Figure 16 presents the resulting resonance frequencies gate dependence of the suspended CNT. It can be easily seen, that just as in the case of no out-of-plane statics, the lowest resonance mode is the first out-of-plane mode. This is true both for the case of a rotational continuous transition (Fig. 16a) and for the case of 3D EB ST transition (Fig. 16b). The lowest out-of-plane mode (light blue) reaches zero frequency at the ST critical point and dictates the ST “jump” for all of the other modes, and thus, the lowest in-plane mode (blue) follows and “jumps” at the same load from nonzero frequency.

Solving the three-dimensional equations for the resonance modes gate dependence predicts the behavior of all types of devices discussed in [27] and can be used for fitting experimental data and learn about the physical motion of the device.

**4 Conclusion**

We analyze the theoretical behavior of the recently reported CNT resonators with initial upward curvature using the framework of continuous mechanics. We model the CNT as a doubly clamped beam subjected to electrostatic force from the local gate. We use the Euler–Bernoulli beam theory to describe both the in-plane and the out-of-plane static and dynamic motions, analyzed using the Galerkin reduced-order model. In the process of arriving at a satisfactory model,
we gather several surprising insights on the mechanical system behavior. We show that the addition of a second in-plane mode cannot achieve a physical solution without adding the asymmetric mode to the initial configuration. Then, we prove that a two-mode Galerkin decomposition results in a satisfactory approximation and that the electrostatic force can be rightfully approximated as inversely proportional to the displacement. Most importantly, we prove that the experimental data of snap-through buckling at finite frequency can only be explained by the addition of the out-of-plane degrees of motion. We show that the case of in-plane initial configuration is a very unique singularity, such that a realistic model must include an out-of-plane initial configuration as well. We show how such initial configurations result in a hybrid three-dimensional static transition from upward to downward curvature for all types of transitions (continuous, ST or latching). Finally, we analyze the criteria for ST buckling as well as latching, which depend on the initial tension and configuration of the CNT. We believe that the model formulated in this work is of fundamental significance to the understanding of buckled CNTs and essential for designing the next generation of buckled CNT devices for practical applications.

Acknowledgements This study was supported by the ISF (Grant No. 1854/19) and the Russell Berrie Nanotechnology Institute. The work made use of the Micro Nano Fabrication Unit at the Technion. S.R. acknowledges support by the Council for Higher Education and the Russel Berrie scholarships.

Data availability Data sharing was not applicable to this article as no datasets were generated or analyzed during the current study. Source codes for theoretical calculations are available from the corresponding author upon reasonable request.

Declarations

Conflict of interest The authors declare that they have no conflict of interest.

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