AN ALGEBRAIC PROOF OF DETERMINANT FORMULAS OF GROTHENDIECK
POLYNOMIALS

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Abstract. We give an algebraic proof of the determinant formulas for factorial Grothendieck polynomials obtained by Hudson–Ikeda–Matsumura–Naruse in [4] and by Hudson–Matsumura in [5].

1. Definition and Theorem

In [11] and [10], Lascoux and Schützenberger introduced (double) Grothendieck polynomials indexed by permutations as representatives of K-theory classes of structure sheaves of Schubert varieties in a full flag variety. In [9] and [8], Fomin and Kirillov described them combinatorially in the framework of Yang-Baxter equations. Let $x = (x_1, \ldots, x_d), \ b = (b_1, b_2, \ldots)$ be sets of indeterminants. A Grassmannian permutation with descent at $d$ corresponds to a partition $\lambda$ of length at most $d$, i.e. a sequence of non-negative integers $\lambda = (\lambda_1, \ldots, \lambda_d)$ such that $\lambda_i \geq \lambda_{i+1}$ for each $i = 1, \ldots, d-1$. For such permutation, Buch [1] gave a combinatorial expression of the corresponding Grothendieck polynomial $G_{\lambda}(x)$ as a generating series of set-valued tableaux, a generalization of semi-standard Young tableaux by allowing a filling of a box in the Young diagram to be a set of integers. In [14], McNamara gave an expression of factorial (double) Grothendieck polynomials $G_{\lambda}(x|b)$ also in terms of set-valued tableaux.

In this paper, we prove the following Jacobi–Trudi type determinant formula for $G_{\lambda}(x|b)$. For each non-negative integer $k$ and an integer $m$, let $G_{\lambda}^{(k)}(x|b)$ be a function of $x$ and $b$ given by

$$G_{\lambda}^{(k)}(u) := \sum_{m \in \mathbb{Z}} G_{m}^{(k)}(x|b)u^m := \frac{1}{1 - \beta u - 1} \prod_{i=1}^{d} \frac{1 + \beta x_i}{1 - x_i u} \prod_{j=1}^{k} (1 + (u + \beta)b_j),$$

where $\beta$ is a formal variable of degree $-1$ and $\frac{1}{1 - \beta u - 1}$ is expanded as $\sum_{s \geq 0} (-1)^s \beta^s u^{-s}$. We use the generalized binomial coefficients $\binom{n}{i}$ given by $(1 + x)^n = \sum_{i \geq 0} \binom{n}{i} x^i$ for $n \in \mathbb{Z}$ with the convention that $\binom{n}{i} = 0$ for all integers $i > 0$.

Theorem 1.1. For each partition $\lambda$ of length at most $d$, we have

$$G_{\lambda}(x|b) = \det \left( \sum_{s \geq 0} \binom{i-d}{s} \beta^s G_{\lambda_i+j-i+s}^{(d-i)}(x|b) \right)_{1 \leq i, j \leq d} \quad (1.1)$$

$$= \det \left( \sum_{s \geq 0} \binom{i-j}{s} \beta^s G_{\lambda_i+j-i+s}^{(d-i)}(x|b) \right)_{1 \leq i, j \leq d} \quad (1.2)$$

In particular, we have $G_{(k,0,\ldots,0)}(x|b) = G_{k}^{(k+d-1)}(x|b)$.

The formulas (1.1) and (1.2) were originally obtained in the context of degeneracy loci formulas for flag bundles by Hudson–Matsumura in [5] and Hudson–Ikeda–Matsumura–Naruse in [4] respectively. The proof in this paper is purely algebraic, generalizing Macdonald’s argument in [13 (3.6)] for Jacobi–Trudi formula.
of Schur polynomials. We use the following “bi-alternant” formula of $G_\lambda(x|b)$ described by Ikeda–Naruse in [6]:

\begin{equation}
G_\lambda(x|b) = \frac{\det ([x_j|b]^{a_i+d-i}(1+\beta x_j)^{i-1})_{1\leq i,j \leq d}}{\prod_{1 \leq i < j \leq d} (x_i - x_j)}.
\end{equation}

Here we denote $x + y := x + y + \beta xy$ and $[y|b]^k := (y \oplus b_1) \cdots (y \oplus b_k)$ for any variable $x, y$. Note that the Grothendieck polynomial $G_\lambda(x)$ described in [1] coincides with $G_\lambda(x|b)$ by setting $\beta = -1$ and $b_i = 0$.

In the non-factorial case, Jacobi–Trudi type formulas different from the ones in Theorem 1.1 have been also obtained by Lenart in [12] and by Kirillov in [8] (see also the papers [10] by Lascoux–Naruse and [14] by Yeliussizov). It is worth mentioning that in [11] Buch obtained the Littlewood–Richardson rule for the structure constants of Grothendieck polynomials $G_\lambda(x)$, and hence the Schubert structure constants of the $K$-theory of Grassmannians (see also the paper [7] by Ikeda-Shimazaki for another proof). For the equivariant $K$-theory of Grassmannians (or equivalently for the factorial Grothendieck polynomials $G_\lambda(x|b)$), the structure constants were recently determined by Pechenik and Yong in [15] by introducing a new combinatorial object called genomic tableaux.

## 2. PROOF OF (1.1)

By (1.3), it suffices to show the identity

\[
\frac{\det ([x_j|b]^{a_i+d-i}(1+\beta x_j)^{i-1})_{1\leq i,j \leq d}}{\prod_{1 \leq i < j \leq d} (x_i - x_j)} = \det \left( \sum_{s \geq 0} \binom{i-d}{s} \beta^s G_{a_i+j-i+s}^{(a_i+d-i)}(x|b) \right)_{1 \leq i,j \leq d},
\]

for each $(a_1, \ldots, a_d) \in \mathbb{Z}^d$ such that $a_i + d - i \geq 0$. For each integer $j$ such that $1 \leq j \leq d$, we let

\[
E^{(j)}(u) := \sum_{p=0}^{d-1} e_p^{(j)}(x) u^p := \prod_{1 \leq i \neq j} (1 + x_i u).
\]

We denote $\bar{y} := \frac{-y}{1 + \beta y}$. Since $1 + (u + \beta)y = \frac{1 - \bar{y} u}{1 + \beta \bar{y}}$, we have

\[
G^{(k)}(u) = \frac{1}{1 + \beta u^{-1}} \prod_{i=1}^{d} \frac{1 + \beta x_i}{1 - x_i u} \prod_{\ell=1}^{k} \frac{1 - \bar{b}_\ell u}{1 + \beta \bar{b}_\ell}.
\]

Consider the identity

\begin{equation}
G^{(k)}(u) E^{(j)}(-u) = \frac{1}{1 + \beta u^{-1}} \prod_{i=1}^{d} \frac{1 + \beta x_i}{1 - x_i u} \prod_{\ell=1}^{k} \frac{1 - \bar{b}_\ell u}{1 + \beta \bar{b}_\ell}.
\end{equation}

\[
\sum_{p=0}^{d-1} G^{(k)}_{m-p}(x|b)(-1)^p e_p^{(j)}(x) = x_j^{m-k} \prod_{i=1}^{k} (x_j - \bar{b}_i) \prod_{1 \leq i \neq j} (1 + \beta x_i),
\]

Since $\frac{y - \bar{b}}{1 + \beta b} = y \oplus b$, we have

\begin{equation}
\sum_{p=0}^{d-1} G^{(k)}_{m-p}(x|b)(-1)^p e_p^{(j)}(x) = x_j^{m-k}[x_j|b]^k \prod_{1 \leq i \neq j} (1 + \beta x_i), \quad (m \geq k).
\end{equation}

Consider the matrices

\[
H := \left( \sum_{s \geq 0} \binom{i-d}{s} \beta^s G_{a_i+j-i+s}^{(a_i+d-i)}(x|b) \right)_{1 \leq i,j \leq d}, \quad \text{and} \quad M := \left( (-1)^{d-i} e_{d-i}^{(j)}(x) \right)_{1 \leq i,j \leq d}.
\]
By using (3.2), we find that the \((i, j)\)-entry of \(HM\) is
\[
(HM)_{ij} = [x_j | b]^{a_i+d-i} (1 + \beta x_j)^{i'-1} \prod_{1 \leq \ell \leq d} (1 + \beta x_\ell).
\]
By taking the determinant of \(HM\), the factor \(\prod_{1 \leq j \leq d} (1 + \beta x_j)^{-d} \prod_{1 \leq \ell \leq d} (1 + \beta x_\ell)^d\) which turns to be 1 comes out, and therefore we obtain
\[
det H \det M = \det \left( [x_j | b]^{a_i+d-i} (1 + \beta x_j)^{i'-1} \right)_{1 \leq i, j \leq d}.
\]
By dividing by \(\det M\), we obtain the desired identity since \(\det M = \prod_{1 \leq i < j \leq d} (x_i - x_j)\) (see [13, p.42]). □

3. Proof of (3.2)

By (1.3), it suffices to show the identity
\[
\frac{\det \left( [x_j | b]^{a_i+d-i} (1 + \beta x_j)^{i'-1} \right)_{1 \leq i, j \leq d}}{\prod_{1 \leq i < j \leq d} (x_i - x_j)} = \det \left( \sum_{s \geq 0} \left( \begin{array}{c} i-j \s \end{array} \right) \beta^s G_{a_i+j-i+s}^{(a_i+d-i)} (x | b) \right)_{1 \leq i, j \leq d}
\]
for each \((a_1, \ldots, a_d) \in \mathbb{Z}^d\) such that \(a_i + d - i \geq 0\). Fix \(j\) such that \(1 \leq j \leq d\). Let
\[
E^{(j)}(u) := \sum_{p=0}^{d-1} e_p^{(j)} (-\bar{x}) u^p := \prod_{1 \leq \ell < j} (1 - \bar{x}_\ell u).
\]
Since \(1 + (u + \beta y) = \frac{1 - \bar{y} u}{1 + \beta y}\), we have the identity
\[
G^{(k)}(u) E^{(j)}(-u - \bar{\beta}) = \frac{1}{1 + \beta u - 1} \frac{1 + \beta x_j}{1 + \beta \bar{\beta} x_j} \prod_{1 \leq \ell \leq k} \frac{1 - \bar{y} u}{1 + \beta \bar{\beta} \bar{\epsilon}_\ell}.
\]
By comparing the coefficient of \(u^m, m \geq k\) in (3.1) we obtain
\[
\sum_{p=0}^{d-1} \sum_{s=0}^{p} \left( \begin{array}{c} p \s \end{array} \right) \beta^s G^{(k)}_{m-p+s} (x | b)(-1)^p e_p^{(j)}(-\bar{x}) = x_j^{m-k} \prod_{1 \leq \ell \leq k} \frac{x_j - \bar{\beta} u}{1 + \beta \bar{\beta} \bar{\epsilon}_\ell} = x_j^{m-k} [x_j | b]^k
\]
where the last equality follows from the identity \(x - \bar{y} u = x \oplus y\) for any variable \(x, y\). Consider the matrices
\[
H' := \left( \sum_{s \geq 0} \left( \begin{array}{c} i-j \s \end{array} \right) \beta^s G_{a_i+j-i+s}^{(a_i+d-i)} (x | b) \right)_{1 \leq i, j \leq d}
\]
and \(\bar{M} := \left( (-1)^{d-i} e_p^{(j)}_{d-i}(-\bar{x}) \right)_{1 \leq i, j \leq d}\).

We write the \((i, j)\)-entry of the product \(H' \bar{M}\) as
\[
(H' \bar{M})_{ij} = \sum_{p=0}^{d-1} \sum_{s \geq 0} \left( \begin{array}{c} i-d+p \s \end{array} \right) \beta^s G_{a_i+d-i+p}^{(a_i+d-i)} (x | b)(-1)^p e_p^{(j)}(-\bar{x}).
\]
By writing \(\left( \begin{array}{c} i-d+p \s \end{array} \right) = \sum_{\ell \geq 0} \left( \begin{array}{c} i-d \ell \s \end{array} \right) \left( \begin{array}{c} p \ell \s \end{array} \right)\) using a well-known identity of binomial coefficients and then applying (3.2), we obtain
\[
(H' \bar{M})_{ij} = [x_j | b]^{a_i+d-i} (1 + \beta x_j)^{i'-1} (1 + \beta x_j)^{1-d}.
\]
By taking the determinant of \(H' \bar{M}\), we have
\[
det H' \det \bar{M} = \left( \prod_{1 \leq \ell \leq d} (1 + \beta x_\ell)^{1-d} \right) \det \left( [x_j | b]^{a_i+d-i} (1 + \beta x_j)^{i'-1} \right)_{1 \leq i, j \leq d}.
\]
Since we have (see [13, p.42])

\[
\det \mathbf{M} = \prod_{1 \leq i < j \leq d} (\bar{x}_j - \bar{x}_i) = \prod_{1 \leq i < j \leq d} \frac{x_i - x_j}{(1 + \beta x_i)(1 + \beta x_j)} = \prod_{1 \leq i \leq d} (1 + \beta x_i)^{1-d} \prod_{1 \leq i < j \leq d} (x_i - x_j),
\]

we obtain the desired identity. \(\square\)

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