Sphaleron solutions of the Skyrme model from Yang-Mills holonomy

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Abstract

We discuss how an approximation to the axially symmetric sphalerons in the Skyrme model can be constructed from the holonomy of a non-BPS Yang-Mills calorons. These configurations, both in the Skyrme model and in the Euclidean Yang-Mills theory, are characterized by two integers $n$ and $m$, where $\pm n$ are the winding numbers of the constituents and the second integer $m$ defines type of the solution, it has zero topological charge for even $m$ and for odd values of $m$ the corresponding chain has topological charge $n$. It is found numerically that the holonomy of the chains of interpolating calorons–anticalorons provides a reasonably good approximation to the corresponding Skyrmion–antiSkyrmion chains when the topological charge of the Skyrmion constituents is two times more than the Chern-Pontryagin index of the caloron.

1 Introduction

There is a very interesting interplay between different topological solitons in $d = 3 + 1$ and $d = 4$ dimensions. It was pointed out by M. Atiyah and N. Manton that the holonomy of Yang-Mills instantons in $d = 4$ provides a very good approximations to Skyrmion solutions of the Skyrme model in $d = 3 + 1$ [1], the energy of the lower charge $B = 1$ Skyrmion is reproduced within a percent accuracy. The remarkable exact topological results is that if the instanton holonomy is considered as a Skyrme field, the baryon number $B$ becomes exactly equal to the Chern-Pontryagin charge $N$. Furthermore, a suitable parametrization of the instanton solution yields also the symmetry of the Skyrme field of the higher charges [2] [3] [4] [5]. This analogy is especially noticeable because, unlike instantons, the Skyrmion
solutions are not Bogomolnyi-Prasad-Sommerfeld (BPS) states, the energy of the $B = 1$ Skyrmion exceeds the topological energy bound by about 23%.

This method can be extended to set a similar link between Skyrmions and calorons which are periodic instantons at finite temperature [6, 7]. Recently the caloron generated field on $S^1 \times \mathbb{R}^3$ was used to construct axially symmetric Skyrmion chains [8], in this approach the periodicity of a chain of Skyrmions matches the period of the gauge field.

Interestingly, if the time-periodic array of instantons corresponds to the non-trivial holonomy, the caloron configurations may possess constituent structure by itself [9]. As the size of the caloron is getting larger than the period $T$, the caloron is splitting into constituents, which represent monopole-antimonopole pair configuration. On the other hand, the BPS monopole solution of the Yang-Mills-Higgs theory in $d = 3 + 1$ is equivalent to an infinite chain of instantons directed along the Euclidean time axis [10]. Thus, it is possible to construct a family of calorons which interpolates between the monopoles at one end and the instantons at the other end [11].

Another similarity between the Skyrme model, Yang-Mills theory in $d = 4$ and Yang-Mills-Higgs theory in $d = 3 + 1$ is that all these models support axially symmetric sphaleron solutions which represent chains of interpolating Skyrmion–antiSkyrmions [12, 13], non-self-dual instantons [14, 15] and non-BPS monopole–antimonopole chains consisting of a charge $n$ and a charge $-n$ constituents [16], respectively. Since these sphaleron configurations, both in the Yang-Mills model, Yang-Mills-Higgs model and in the Skyrme model, are deformations of topological sectors with a given charge, the statement about the topological equivalence of the instanton induced holonomy and the corresponding Skyrme field cannot be applied straightforwardly.

The purpose of this letter is to compute the holonomy of the axially-symmetric caloron–anti-caloron configurations and compare it to the Skyrmion–antiSkyrmion chains. We show that it provides rather good approximation to the sphaleron solutions of the Skyrme model if the baryon number of the constituents in the latter sector is taken to be twice as much as the topological charge of the Yang-Mills calorons. This result is not very surprising because it is known that the the Skyrmion–antiSkyrmion chains may exist only for values $B \geq 2$ [12, 13]. It was also pointed out that the transition between the chains and ring-like configurations for the Skyrmion–antiSkyrmion pair it taken place as the charge $B$ increases above $B = 4$ [12], in the case of monopole-antimonopole pair similar transition occurs above $N = 2$ [16], this indicates that the gauge interaction between the constituents in the monopole-antimonopole pair is much stronger than the dipole-dipole interaction in the Skyrme–antiSkyrme pair [12, 13].

In the next section we discuss the axially symmetric ansatz which we apply to parametrize the action of the Euclidean Yang-Mills theory, and the boundary conditions imposed to get regular solution. The numerical results are presented in Section 3 where we evaluated the holonomy of the caloron–anti calorion chains and compare it to the numerical solutions of the Skyrme model. We give our conclusions and remarks in the final section.
2 Sphaleron calorons in Euclidean \(SU(2)\) Yang-Mills

As a starting point we consider the \(SU(2)\) Yang-Mills action in \(R^3 \times S^1\) with one periodic dimension \(x_0 \in [0, T]\)

\[
S = \frac{1}{2} \int d^4x \text{Tr} (F_{\mu\nu}F_{\mu\nu}) = \frac{1}{4} \int d^4x \left( F_{\mu\nu} + \tilde{F}_{\mu\nu} \right)^2 = \frac{1}{2} \int d^4x \text{Tr} (F_{\mu\nu}\tilde{F}_{\mu\nu})
\]  

(1)

Here \(F_{\mu\nu} = \partial_{\mu}A_{\nu} - \partial_{\nu}A_{\mu} + i[A_{\mu}, A_{\nu}]\) are the components of the \(su(2)\)-valued field strength and the topological charge is defined as

\[
N = \frac{1}{32\pi^2} \varepsilon_{\mu\nu\rho\sigma} \int d^4x \text{Tr} (F_{\mu\nu}F_{\rho\sigma})
\]  

(2)

A lower bound on the action then is \(S \geq 8\pi^2 N\), it is saturated by self-dual configurations.

As discussed in [15], non-self dual axially symmetric regular caloron solutions in this theory can be constructed using the Ansatz for the gauge field

\[
A_k dx^k = \left( \frac{K_1}{r} dr + (1 - K_2) d\theta \right) \frac{\tau_r^{(n)}}{2e} - n \sin \theta \left( \frac{K_3}{2e} \tau_r^{(n,m)} + (1 - K_4) \frac{\tau_\theta^{(n,m)}}{2e} \right) d\phi;
\]

\[
A_0 = A_0^a \tau^a = \left( \frac{K_5}{2} \tau_r^{(n,m)} + K_6 \tau_\theta^{(n,m)} \right)
\]

which was previously applied in the Yang-Mills-Higgs theory to obtain various monopole-antimonopole chains and vortex-like configurations [16]. The Ansatz is written in the basis of \(su(2)\) matrices \(\tau_r^{(n,m)}, \tau_\theta^{(n,m)}\) and \(\tau^{(n)}\) which are defined as the dot product of the Cartesian vector of Pauli matrices \(\vec{\tau}\) and the spacial unit vectors

\[
\epsilon_r^{(n,m)} = \left( \sin(m\theta) \cos(n\phi), \sin(m\theta) \sin(n\phi), \cos(m\theta) \right),
\]

\[
\epsilon_\theta^{(n,m)} = \left( \cos(m\theta) \cos(n\phi), \cos(m\theta) \sin(n\phi), -\sin(m\theta) \right),
\]

\[
\epsilon_\phi^{(n)} = \left( -\sin(n\phi), \cos(n\phi), 0 \right),
\]

(3)

respectively. The functions \(K_i, i = 1, \ldots, 6\) depend on the coordinates \(r\) and \(\theta\). The solutions are labeled by two integers \(n, m\) and the topological charge of configurations is \(N = \frac{n}{2} [1 - (-1)^m]\). Evidently, for even \(m\) the configuration has zero topological charge while for odd \(m\) is has a charge \(n\). Thus, the solutions represent periodic arrays of the calorons and the anticalorons of topological charge \(\pm n\), which are located on the axis of symmetry in alternating order [15]. For configuration with \(m \geq 2\) the energy bound cannot be attained.

The finiteness of the Euclidean action [11] requires \(\text{Tr}(F_{\mu\nu}F_{\mu\nu}) \to O(r^{-4})\) as \(r \to \infty\). In the regular gauge the value of the component of the gauge potential \(A_0\) approaches a constant as \(r \to \infty\), i.e.,

\[
A_0 \to \frac{i\beta}{2} \tau_r^{(n,m)}
\]

(4)
where $\beta \in [0 : 2\pi/T]$ and $T$ is the period in the imaginary time direction. Using the classical scale invariance we can fix $\beta = 1$.

Note that these configurations should be more akin to the monopole–antimonopole chains [16], rather than the instanton-anti–instantons sphalerons discussed in [14], indeed there is a caloron-anticaloron chain consisting of a charge $n = \pm 1$ constituents whereas the corresponding instanton-anti-instanton chain may exist only for values $n \geq 2$. These configurations remain above the self-duality bound.

## 3 Skyrmion-antiSkyrmion chains from calorons

We discuss now, how an approximation to the sphalerons in the Skyrme model can be obtained from the holonomy of a suitable non-self dual calorons. The Atiyah-Manton approach of generating approximate Skyrmion configurations on $\mathbb{R}^3$ is to evaluate the holonomy of the corresponding Yang-Mills instantons along the lines parallel to the Euclidean time axis. The construction is topologically natural, since the holonomy of an instanton with topological charge $N$ is a Skyrmion with baryon charge $B = N [1]$.

It was suggested to consider a periodic version of this construction to obtain an approximation to Skyrme crystal [5] and chains of Skyrmions [8], then the Skyrme field can be approximated via the path-ordered exponential integral

$$U(\mathbf{r}) = \mathcal{P} \exp \left( \int_0^T A_0(\mathbf{r}, x_0) dx_0 \right),$$

where the period $T$ is related with finite temperature $\Theta$ as $T = 1/k\Theta$, and $\mathcal{P}$ denotes the path ordering.
Figure 2: The $\sigma$ field for the $2n$ Skyrmion and charge $-2n$ antiSkyrmion solutions (right column) and its approximation from the holonomy of charge axially symmetric $n$ calorons and charge $-n$ anticalorons (m=2) (left column) are shown as function of the coordinates $z$ and $\rho = \sqrt{x^2 + y^2}$.
Figure 3: The $\sigma$ field for the $\pm 2n$ S-A-S solutions (right column) and its approximation from the holonomy of axially symmetric charge $n$ calorons and charge $-n$ anticalorons ($m=3$) (left column) are shown as function of the coordinates $z$ and $\rho = \sqrt{x^2 + y^2}$. 
To construct Skyrmion–antiSkyrmion pairs we shall consider deformations of the holonomy, now the points \( \tau = \pm T/4 \) of each periodic interval along the Euclidean time axis should be identified with the middle point of the double sphere which is obtained from \( S^3 \) compactification of \( \mathbb{R}^3 \times S^1 \) when the latter is twisted by \( \pi \) and squeezed to a point in the equatorial hyperplane, as illustrated in Fig. 1. This corresponds to the boundary conditions we are imposing on the Yang-Mills field of the caloron-anticaloron pair [16], as \( r \to \infty \) we have \( A_\mu \to i\partial_\mu U + i\pi^a U^a \) where \( U = \exp \{-i\theta \tau (x)\} \), so it is element of unity as \( \theta = 0 \) and it is \(-1\) as \( \theta = \pi \).

Having constructed the non-BPS caloron-anticaloron chains we now wish to make use of this to compute a Skyrme field. Since we are only concerned with axially symmetric fields, the \( SU(2) \) valued Skyrme field \( U(r) = \sigma \cdot I + i\pi^a \cdot \tau^a \) can be parameterized as [12, 13]

\[
\begin{align*}
\pi^1 &= \phi^1 \cos(n\varphi); \\
\pi^2 &= \phi^1 \sin(n\varphi); \\
\pi^3 &= \phi^2; \\
\sigma &= \phi^3
\end{align*}
\]

where the triplet of scalar fields \( \phi^a \) on unit sphere is a function only of radial variable \( r \) and polar angle \( \theta \). Similar to the axially symmetric caloron–anticaloron systems, the baryon number is \( B = \frac{n}{2}[1 - (-1)^m] \) where the second integer \( m \) corresponds to the number of the constituents of the configuration which can be identified with individual charge \( n \) Skyrmions and charge \(-n \) antiSkyrmions [13].

For an axially-symmetric caloron-anticaloron chain configuration [3] the holonomy [5] produces a Skyrme field

\[
\begin{align*}
\phi^1 &= \frac{1}{||A_0||} (K_5 \sin m\theta + K_6 \cos m\theta) \sin \frac{T||A_0||}{2}, \\
\phi^2 &= \frac{1}{||A_0||} (K_5 \cos m\theta - K_6 \sin m\theta) \sin \frac{T||A_0||}{2}, \\
\phi^3 &= \cos \frac{T||A_0||}{2},
\end{align*}
\]

where \( ||A_0|| = \sqrt{(K_5)^2 + (K_6)^2} \). Although this construction does not give exact solutions to the Skyrme model it gives fields which are good approximations to the corresponding Skyrmion-antiSkyrmion chains [12, 13] if we suppose that the charge \( 2n \) Skyrmion and charge \(-2n \) antiSkyrmion configurations corresponds to the charge \( n \) caloron and charge \(-n \) anticaloron system. Indeed, Figs 2,3 demonstrate that the holonomy of the axially symmetric charge \( n \) Yang-Mills calorons and charge \(-n \) anticalorons yields a good approximation to the charge \( \pm 2n \) Skyrmion-antiSkyrmion (S-A) pair and to the \( \pm 2n \) Skyrmion-antiSkyrmion-Skyrmion (S-A-S) configuration. Furthermore, such a good agreement was observed for all components of the Skyrme field [7]. This result is not very surprising because it is known that the the Skyrmion–antiSkyrmion chains may exist only for value of charge \( n \geq 2 \) [12, 13]. It was also pointed out that the transition between the chains and ring-like configurations for the Skyrme–antiSkyrme pair it taken place as the charge \( n \) increases above \( n = 4 \) [12], in the case of monopole-antimonopole pair similar transition occurs above \( n = 2 \) [16], it indicates that the gauge interaction between the constituents
in the monopole-antimonopole pair is much stronger than the dipole-dipole interaction in the Skyrmion–antiSkyrmion pair \cite{12,13}.

Finally, let us note that the holonomy also provides a reasonable approximation to the energy of the corresponding Skyrmion–antiSkyrmion system. This is illustrated in Fig. 4 there the curves joining the triangles represent the holonomy approximated energy as function of the double topological charge $2n$ and the curves joining the squares represent the exact energy of the corresponding Skyrmion–antiSkyrmion chains as function of the topological charge of the constituents $n$.

### 4 Conclusions

We have studied construction of the axially symmetric sphaleron solutions of the Skyrme model from caloron-anticaloron holonomy using the Atiyah-Manton approach. These configurations, both in the Skyrme model and in the Euclidean Yang-Mills theory, are characterized by two integers $n$ and $m$, where $\pm n$ are the winding numbers of the constituents and the second integer $m$ defines type of the solution, the total topological charge of any such finite action/energy configuration vanishes when $m$ is even, and equals $n$ when $m$ is odd. This construction provides another example of similarity which may be observed between monopoles, instantons, calorons and Skyrmions.

We found that the caloron-anticaloron holonomy provides a good approximation to the Skyrmion-antiSkyrmion chains when the baryon number of the constituents is taken...
to be two times more than the topological charge of the Yang-Mills calorons. Since in that case the usual statement about the topological equivalence of the instanton induced holonomy and the corresponding Skyrme field cannot be applied straightforwardly, it would be interesting to understand if there are some topological roots of this correspondence.

Finally, let us note that the caloron-anticaloron system may exist for \( n = 1 \), there is a difference between the spalerons in the Yang-Mills theory at finite temperature and the instantonanti-instanton pair which exists only for values \( n \geq 2 \) \[14\]. Since similar observation is made for the Skyrmion-antiSkyrmion pair \[12\], it might be interesting to analyse instanton-anti-instanton holonomy to generate Skyrme fields along the lines of our discussion above.

Acknowledgements
We would like to acknowledge numerous valuable discussions with Eugen Radu and Paul Sutcliffe. YS is very grateful to E. Norvaisas for kind hospitality at the Institute of Theoretical Physics and Astronomy, University of Vilnius. This work is partially supported by BMU-MID Research Fellowships (YS).

References

[1] M.F. Atiyah and N.S. Manton, Phys. Lett. B 222, 438 (1989); Commun. Math. Phys. 153, 391 (1993).
[2] R.A. Leese and N.S. Manton, Nucl. Phys. A 572, 575 (1994).
[3] M.A. Singer and P.M. Sutcliffe, Nonlinearity 12, 987 (1999).
[4] P.M. Sutcliffe, Proc. R. Soc. Lond. A 460, 2903 (2004).
[5] N.S. Manton and P.M. Sutcliffe, Phys. Lett. B 342, 196 (1995).
[6] M.A. Nowak and I. Zahed, Phys. Lett. B 230, 108 (1989).
[7] J. Dey and J.M. Eisenberg, Phys. Lett. B 334, 290 (1994).
[8] D. Harland and R. Ward, JHEP 0812 (2008) 093
[9] T.C. Kraan and P. van Baal, Phys. Lett. B 428 (1998) 268; T.C. Kraan and P. van Baal, Phys. Lett. B 435 (1998) 389; K.M. Lee and C.H. Lu, Phys. Rev. D 58 (1998) 025011.
[10] P. Rossi, Nucl. Phys. B 149, 170 (1979).
[11] R.S. Ward, Phys. Lett. B 582, 203 (2004).
[12] S. Krusch and P.M. Sutcliffe, J. Phys. A 37, 9037 (2004).
[13] Y. Shnir and D. H. Tchrakian, J. Phys. A 43, 025401 (2010).

[14] E. Radu and D. H. Tchrakian, Phys. Lett. B 636 (2006) 201.

[15] Y. Shnir, Europhys. Lett. 77, 21001 (2007).

[16] B. Kleihaus, J. Kunz, Y. Shnir, Phys. Lett. B 570, 237 (2003); B. Kleihaus, J. Kunz, Y. Shnir, Phys. Rev. D 68, 101701 (2003) B. Kleihaus, J. Kunz, Y. Shnir, Phys. Rev. D 70, 065010 (2004).