A NOTE ON MAXIMAL SUBGROUPS OF FREE IDEMPOTENT GENERATED SEMIGROUPS OVER BANDS

IGOR DOLINKA

Department of Mathematics and Informatics, University of Novi Sad,
Trg Dositeja Obradovića 4, 21101 Novi Sad, Serbia
E-mail: dockie@dmi.uns.ac.rs

Abstract

We prove that all maximal subgroups of the free idempotent generated semigroup over a band \( B \) are free for all \( B \) belonging to a band variety \( V \) if and only if \( V \) consists either of left seminormal bands, or of right seminormal bands.

Let \( S \) be a semigroup, and let \( E = E(S) \) be the set of its idempotents; in fact, \( E \), along with the multiplication inherited from \( S \), is a partial algebra. It turns out to be fruitful to restrict further the domain of the partial multiplication defined on \( E \) by considering only the pairs \( e, f \in E \) for which either \( ef \in \{e, f\} \) or \( fe \in \{e, f\} \) (i.e. \( \{ef, fe\} \cap \{e, f\} \neq \emptyset \)). Note that if \( ef \in \{e, f\} \) then \( fe \) is an idempotent, and the same is true if we interchange the roles of \( e \) and \( f \). Such unordered pairs \( \{e, f\} \) are called basic pairs and their products \( ef \) and \( fe \) are basic products.

The free idempotent generated semigroup over \( E \) is defined by the following presentation:

\[
\text{IG}(E) = \langle E \mid e \cdot f = ef \text{ such that } \{e, f\} \text{ is a basic pair} \rangle.
\]

Here \( ef \) denotes the product of \( e \) and \( f \) in \( S \) (which is again an idempotent of \( S \)), while \( \cdot \) stands for the concatenation operation in the free semigroup \( E^+ \) (also to be interpreted as the multiplication in its quotient \( \text{IG}(E) \)). An important feature

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of \( IG(E) \) is that there is a natural homomorphism from \( IG(E) \) onto the subsemigroup of \( S \) generated by \( E \), and the restriction of \( \phi \) to the set of idempotents of \( IG(E) \) is a basic-product-preserving bijection onto \( E \), see e.g. [5, 9, 13].

An important background to these definitions is the notion of the biordered set [7] of idempotents of a semigroup and its abstract counterpart. The biordered set of idempotents of \( S \) is just a partial algebra on \( E(S) \) obtained by restricting the multiplication from \( S \) to basic pairs of idempotents. In this way we have that if \( B \) is a band (an idempotent semigroup), then, even though there is an everywhere defined multiplication on \( E(B) = B \), its biordered set [3] is in general still a partial algebra. Another way of treating biordered sets is to consider them as relational structures \((E(S), \leq^{(l)}, \leq^{(r)})\), where the set of idempotents \( E(S) \) is equipped by two quasi-order relations defined by

\[
e \leq^{(l)} f \text{ if and only if } ef = e,
\]

\[
e \leq^{(r)} f \text{ if and only if } fe = e.
\]

One of the main achievements of [4, 5, 9] is the result that the class of biordered sets considered as relational structures is axiomatisable: there is in fact a finite system of formulae satisfied by biordered sets such that any set endowed with two quasi-orders satisfying the axioms in question is a biordered set of idempotents of some semigroup. In this sense we can speak about the free idempotent generated semigroup over a biordered set \( E \). A fundamental fact which justifies the term ‘free’ is that \( IG(E) \) is the free object in the category of all semigroups \( S \) whose biordered set of idempotents is isomorphic to \( E \): if \( \psi : E \to E(S) \) is any isomorphism of biordered sets, then it uniquely extends (via the canonical injection of \( E \) into \( IG(E) \)) to a homomorphism \( \psi' : IG(E) \to S \) whose image is the subsemigroup of \( S \) generated by \( E(S) \). This is also true if \( \psi \) is a (surjective) homomorphism of biordered sets (taken as relational structures), so that the freeness property of \( IG(E) \) carries over to even wider categories of semigroups.

In this short note we consider \( IG(B) \), the free idempotent generated semigroup over (the biordered set of) a band \( B \); more precisely, we are interested in the question whether the maximal subgroups of these semigroups are free. It was conjectured in [8] that each maximal subgroup of any semigroup of the form \( IG(E) \) is a free group. Recently, this was disproved [11] (see also [2]), where a certain 72-element semigroup was found whose biordered set \( E \) of idempotents yields a maximal subgroup in \( IG(E) \) isomorphic to \( \mathbb{Z} \oplus \mathbb{Z} \), the rank 2 free abelian group. Here we will see that a particular 20-element regular band suffices for the same purpose. In fact, as proved by Gray and Ruškuc in [6], every group can be isomorphic to a maximal subgroup of some \( IG(E) \), while the assumption that the semigroup \( S \) with \( E = E(S) \) is finite yields a sole restriction that the groups...
in question are finitely presented. This puts forward many new questions, one of which is the characterisation of bands $B$ for which all subgroups of $\text{IG}(B)$ are free.

More specifically, as a first approximation to the latter question, we may ask for a description of all varieties $V$ of bands with the property that for each $B \in V$ the maximal subgroups of $\text{IG}(B)$ are free. To facilitate the discussion, we depict in Fig. 1 the bottom part of the lattice $\mathcal{L}(B)$ of all band varieties, along with their standard labels (see also [15, Diagram II.3.1]).

![Figure 1: The bottom part of the lattice of all varieties of bands](image)

The main result of this note is the following.

**Theorem 1.** Let $V$ be a variety of bands. Then $\text{IG}(B)$ has all its maximal subgroups free for all $B \in V$ if and only if $V$ is contained either in $\text{LSNB}$ or in $\text{RSNB}$.

This theorem is a direct consequence of the following two propositions.

**Proposition 2.** For any left (right) seminormal band $B$, all maximal subgroups of $\text{IG}(B)$ are free.

**Proposition 3.** There exists a regular band $B$ such that $\text{IG}(B)$ has a maximal subgroup isomorphic to $\mathbb{Z} \oplus \mathbb{Z}$.

The first of these propositions is a generalisation of the well known result of Pastijn [13, Theorem 6.5] (cf. also [10, 112]) that all maximal subgroups of
\(\text{IG}(B)\) are free for any normal band \(B\). The other one supplies a simpler example with the same non-free maximal subgroup than the one considered in [11, Section 5]. The method used is the one from [6], which is based on the Reidemeister-Schreier type rewriting process for obtaining presentations of maximal subgroups of semigroups developed in [11]. So, before turning to the proofs of the above two propositions, we briefly present this general method yielding presentations for maximal subgroups of \(\text{IG}(E)\), \(E = E(S)\), for an arbitrary semigroup \(S\), and then we explain its particular case when \(S\) is a band. Along the way, we assume some familiarity with the most basic notions of semigroup theory, such as Green’s relations and the structure of bands, see, for example. [7, 15].

Let \(S\) be a semigroup and let \(D\) be a \(D\)-class of \(S\) containing an idempotent \(e_0 \in E(S)\). We are going to label the \(R\)-classes contained in \(D\) by \(R_i\), \(i \in I\), while \(L_j\), \(j \in J\), is the list of all \(L\)-classes of \(D\). The \(H\)-class \(R_i \cap L_j\) will be denoted by \(H_{ij}\). Define \(K = \{(i, j) : H_{ij} \text{ is a group}\}\); as is well known, \((i, j) \in K\) if and only if \(H_{ij}\) contains an idempotent, which we denote by \(e_{ij}\). There is no loss of generality if we assume that both \(I\) and \(J\) contain an index 1, so that \(e_0 = e_{11}\).

For a word \(w \in E^*\), let \(\overline{w}\) denote the image of \(w\) under the canonical monoid homomorphism of \(E^*\) into \(S^1\): in other words, when \(w\) is non-empty, \(\overline{w}\) is just the element of \(S\) obtained by multiplying in \(S\) the idempotents the concatenation of which is \(w\). We say that a system of words \(r_j, r_j' \in E^*, j \in J\), is a Schreier system of representatives for \(D\) if for each \(j \in J\):

- the right multiplications by \(\overline{r_j}\) and \(\overline{r_j'}\) are mutually inverse \(R\)-class preserving bijections \(L_1 \rightarrow L_j\) and \(L_j \rightarrow L_1\), respectively (so, in particular, right multiplication by \(r_1\) is the identity mapping on \(L_1\));

- each prefix of \(r_j\) coincides with \(r_{j'}\) for some \(j' \in J\) (in particular, the empty word is just \(r_1\)).

It is well-known that such a Schreier system always exists. In the following, we assume that one particular Schreier system has been fixed.

In addition, we will assume that a mapping \(i \mapsto j(i)\) has been specified such that \((i, j(i)) \in K\): such \(j(i)\) must exist for each \(i \in I\), since \(D\) is a regular \(D\)-class (as it contains an idempotent), and so each \(R\)-class \(R_i\) must contain an idempotent. The index \(j(i) \in J\) is called the anchor of \(R_i\).

Finally, call a square a quadruple of idempotents \((e, f, g, h)\) in \(D\) such that

\[
\begin{array}{ccc}
  e & R & f \\
  \mathcal{L} & \mathcal{L} \\
  g & R & h.
\end{array}
\]
Then there are \( i, k \in I \) and \( j, \ell \in J \) such that \( e \in H_{ij}, f \in H_{it}, g \in H_{kj} \) and \( h \in H_{k\ell} \). For an idempotent \( \varepsilon \in S \) we say that it singularises the square \((e, f, g, h)\) if any of the following two cases takes place:

(a) \( \varepsilon = e \) and \( \varepsilon g = g \), while \( e = f \varepsilon \); or

(b) \( e = \varepsilon g \), along with \( \varepsilon e = e \) and \( f \varepsilon = f \).

Note that case (a) implies \( \varepsilon f = f \), \( \varepsilon h = h \), \( \varepsilon e = e \) and \( g = g \varepsilon = h \varepsilon \), while conditions \( \varepsilon e = e \), \( f = \varepsilon f = \varepsilon h \), \( g \varepsilon = g \) and \( h \varepsilon = h \) follow from (b). The square \((e, f, g, h)\) is singular if it is singularised by some idempotent of \( S \). Let \( \Sigma \) be the set of all quadruples \((i, k; j, \ell) \in I \times I \times J \times J \) (to be called singular rectangles) such that \((e_{ij}, e_{i\ell}, e_{k\ell}, e_{k\ell})\) is a singular square in \( D \).

The required general result of \([6]\) can be now paraphrased as follows.

**Theorem 4** (Theorem 5 of \([6]\)). Let \( S \) be a semigroup with a non-empty set of idempotents \( E = E(S) \). With the notation as above, the maximal subgroup of the free idempotent generated semigroup \( 1G(E) \) containing \( e_{11} \in E \) is presented by \( \langle \Gamma | \mathcal{R} \rangle \), where \( \Gamma = \{f_{ij} : (i, j) \in K\} \), while \( \mathcal{R} \) consists of three types of relations:

(i) \( f_{i,j(i)} = 1 \) for all \( i \in I \);

(ii) \( f_{ij} = f_{it} \) for all \( i \in I \) and \( j, \ell \in J \) such that \( r_{j} \cdot e_{i\ell} = r_{\ell} \);

(iii) \( f_{ij}^{-1} f_{it} = f_{k\ell}^{-1} f_{k\ell} \) for all \( (i, k; j, \ell) \in \Sigma \).

For our purpose, we would like to focus on the particular case when \( S \) is a band. Then, clearly, \( K = I \times J \) and \( D = \{e_{ij} : i \in I, j \in J\} \). Since \( D = J \) in any band, the set of all \( D \)-classes of \( B \) is partially ordered; it instantly turns out that, by definition, if \( \varepsilon \) singularises a square \((e, f, g, h)\) in \( D \), then \( D_{\varepsilon} \supseteq D \).

Now any such \( \varepsilon \in B \) induces a pair of transformations on \( I \) and \( J \), respectively, in the following sense. For each \( i \in I \) and \( j \in J \) there are \( i', k \in I \) and \( j', \ell \in J \) such that \( \varepsilon e_{ij} = e_{i'\ell} \) and \( e_{ij} \varepsilon = e_{k\ell}' \). One immediately sees that it must be \( \ell = j \) and \( k = i \), so that \( B \) acts on the left on \( I \) and on the right on \( J \). Thus it is convenient to write the transformation \( \sigma = \sigma_{(i)}(\ell) \) induced by \( \varepsilon \) on \( I \) to the left of its argument (so that \( \varepsilon e_{ij} = e_{\sigma(i)j} \)), while the analogous transformation \( \sigma' = \sigma_{(\ell)}(r) \) on \( J \) is written to the right (resulting in the rule \( e_{ij} \varepsilon = e_{i(j)\sigma'} \)).

**Corollary 5.** Let \( B \) be a band, let \( D \) be a \( D \)-class of \( B \), and let \( e_{11} \in D \). Then the maximal subgroup \( G_{e_{11}} \) of \( IG(B) \) containing \( e_{11} \) is presented by \( \langle \Gamma | \mathcal{R} \rangle \), where \( \Gamma = \{f_{ij} : i \in I, j \in J\} \) and \( \mathcal{R} \) consists of relations

\[
f_{i1} = f_{1j} = f_{11} = 1
\]  

(1)
for all \( i \in I \) and \( j \in J \), and
\[
f^{-1}_{ij}f_{it} = f^{-1}_{kj}f_{kt},
\]
where for some \( \varepsilon \in B \) such that \( D_\varepsilon \supseteq D \) the indices \( i, k \in I \), \( j, \ell \in J \) satisfy one of the following two conditions:
\[
\begin{align*}
\text{(a)} & \quad \sigma^{(l)}_\varepsilon(i) = i, \quad \sigma^{(l)}_\varepsilon(k) = k, \quad \text{and} \quad (j)\sigma^{(r)}_\varepsilon = (\ell)\sigma^{(r)}_\varepsilon = \ell, \\
\text{(b)} & \quad \sigma^{(l)}_\varepsilon(i) = \sigma^{(l)}_\varepsilon(k) = k, \quad (j)\sigma^{(r)}_\varepsilon = j \quad \text{and} \quad (\ell)\sigma^{(r)}_\varepsilon = \ell.
\end{align*}
\]

**Proof.** Since \( \mathcal{K} = I \times J \), we have a generator \( f_{ij} \) for each \( i \in I \) and \( j \in J \). Furthermore, the same reason allows us to choose \( j(i) = 1 \) as the anchor for each \( i \in I \). Such a choice will imply that the relations of type (i) from Theorem 4 take the form \( f_{11} = 1 \), \( i \in I \). In particular, we have \( f_{11} = 1 \). As for the Schreier system, we can choose \( r_1 \) to be the empty word, \( r_j = e_{1j} \) for all \( j \in J \setminus \{1\} \) and \( r'_j = e_{11} \) for all \( j \in J \). The system \( r_j, j \in J \), of words over \( E \) is obviously prefix-closed. Since \( e_{11}e_{ij} = e_{ij} \) and \( e_{ij}e_{11} = e_{11} \) holds for all \( i \in I \), \( j \in J \), the right multiplications by \( e_{ij} \) and \( e_{11} \) are indeed mutually inverse bijections between \( L_1 \) and \( L_j \) and between \( L_j \) and \( L_{11} \), respectively. Hence, the relations of type (ii) reduce to \( f_{11} = f_{1j} \), that is, \( f_{1j} = 1 \), for all \( j \in J \). Thus we have all the relations (1). Finally, the conditions (a) and (b) express precisely the singularisation of a square \((e_{ij}, e_{ik}, e_{kj}, e_{kl})\) in \( D \) by an element \( \varepsilon \in B \); therefore, the relations (2) correspond to relations of type (iii).

Rectangles \((i, k; j, \ell) \in I \times J \) of type (a) will be said to be left-right singular, while those of type (b) are up-down singular (with respect to \( \varepsilon \)). Another, more compact way of expressing condition (a) is \( i, k \in \text{Im} \sigma^{(l)}_\varepsilon, \ell \in \text{Im} \sigma^{(r)}_\varepsilon \) and \((j, \ell) \in \text{Ker} \sigma^{(r)}_\varepsilon \), while (b) is equivalent to \( k \in \text{Im} \sigma^{(l)}_\varepsilon, (i, k) \in \text{Ker} \sigma^{(l)}_\varepsilon \) and \( j, \ell \in \text{Im} \sigma^{(r)}_\varepsilon \).

We can now turn to proving our aforementioned result.

**Proof of Proposition 2.** Without any loss of generality, assume that \( B \in \text{RSNB} \) (the case when \( B \) belongs to \( \text{LSNB} \) is dual). Recall (e.g. from [15] Proposition II.3.8]) that the variety \( \text{RSNB} \) satisfies (and is indeed defined by) the identity \( tuv = tvtuv \). Therefore, if \( B = \bigcup_{\alpha \in Y} B_\alpha \) is the greatest semilattice decomposition of \( B \), \( a \in B \) and \( x, y \in D = B_\alpha \) for some \( \alpha \in Y \), then \( x = xyx \) and \( y = yxy \). Hence, we have \( ax = ax(yx) = ayxaxyx \) and \( ay = ay(xy) = axyayxy \), implying \( ax Ra ay \). In particular, for any \( \varepsilon \in B \) such that \( D_\varepsilon \supseteq D \), \( \varepsilon e_{ij} R \varepsilon e_{k\ell} \) holds in \( D \) for all \( i, k \in I \), \( j, \ell \in J \), so the transformation \( \sigma^{(l)}_\varepsilon \) is a constant function on \( I \).

We conclude that there are no proper (non-degenerate) rectangles \((i, k; j, \ell)\) that are left-right singular with respect to some \( \varepsilon \in B \). In other words, all proper
singular rectangles in $I \times J$—and thus all nontrivial relations of $G_{e_{11}}$—are of the up-down kind:

$$f_{ij}^{-1} f_{i\ell} = f_{k_0 j}^{-1} f_{k_0 \ell},$$

where $j, \ell$ are two fixed points of $\sigma_\varepsilon^{(r)}$, $i \in I$ is arbitrary, and (since in this context $\sigma_\varepsilon^{(l)}$ is constant) $\text{Im} \sigma_\varepsilon^{(l)} = \{k_0\}$, for some $\varepsilon \in B$. However, now it is straightforward to deduce the relation (2) for all $i, k \in I$ and fixed points $j, \ell$ of $\sigma_\varepsilon^{(r)}$. Thus we are led to define an equivalence $\theta_B$ of $\bigcup_{\varepsilon \in B, D, \varepsilon \geq D} \text{Im} \sigma_\varepsilon^{(r)} = J$ which is the transitive closure of the relation $\rho_B$ defined by $(j_1, j_2) \in \rho_B$ if and only if $j_1, j_2 \in \text{Im} \sigma_\varepsilon^{(r)}$ for some $\varepsilon \in B$. Now it is almost immediate to see that for all $i, k \in I$ and $j, \ell \in J$ such that $(j, \ell) \in \theta_B$ we have that

$$f_{ij}^{-1} f_{i\ell} = f_{kj}^{-1} f_{k\ell}$$

holds in $G_{e_{11}}$. This immediately implies $f_{k\ell} = 1$ for all $k \in I$ and $\ell \in 1/\theta_B$, as well as

$$f_{kj} = f_{k\ell}$$

for all $k \in I$, whenever $(j, \ell) \in \theta_B$. So, let $j_1 = 1, j_2, \ldots, j_m \in J$ be a cross-section of $J/\theta_B$. Then it is straightforward to eliminate all the relations from the presentation of $G_{e_{11}}$ while reducing its generating set to

$$\{f_{ij} : i \in I \setminus \{1\}, \ 2 \leq r \leq m\}.$$ 

In other words, $G_{e_{11}}$ is a free group of rank $(|I| - 1)(m - 1)$. \qed

Proof of Proposition 3. Let $B$ be the subband of the free regular band on four generators $a, b, c, d$ consisting of two $D$-classes: a $2 \times 2$ class $D_1$ consisting of elements $ab, aba, ba, bab$ and a $4 \times 4$ class $D_0$ consisting of elements of the form $u_1 v u_2$, where $u_1, u_2 \in \{ab, ba\}$ and $v \in \{cd, cdc, dc, dcd\}$. So, we can take $I = \{abcd, abdc, bacd, badc\}$, the set of all initial parts of words from $D_0$, and $J = \{cdba, dcba, cdab, dcab\}$, the set of all final parts of those words. A direct computation shows that

$$\sigma_{ab}^{(l)} = \sigma_{aba}^{(l)} = \begin{pmatrix} abcd & abdc & badc & bacd \\ abcd & abdc & abdc & abed \end{pmatrix},$$

$$\sigma_{ba}^{(l)} = \sigma_{bab}^{(l)} = \begin{pmatrix} abcd & abdc & badc & bacd \\ badc & badc & badc & badc \end{pmatrix},$$

$$\sigma_{ab}^{(r)} = \sigma_{bab}^{(r)} = \begin{pmatrix} cdca & cdab & dcab & dcba \\ cdab & cdab & dcab & dcab \end{pmatrix},$$

$$\sigma_{ba}^{(r)} = \sigma_{aba}^{(r)} = \begin{pmatrix} cdca & cdab & dcab & dcba \\ cdab & cdab & dcab & dcab \end{pmatrix}.$$
If we enumerate (for brevity of further calculations) \(abcd \rightarrow 1, abdc \rightarrow 2, badc \rightarrow 3, bacd \rightarrow 4\) and \(cdba \rightarrow 1, cdab \rightarrow 2, dcab \rightarrow 3, dcba \rightarrow 4\), we get

\[
\sigma^{(l)}_{ab} = \sigma^{(l)}_{aba} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 2 & 1 \end{pmatrix}, \quad \sigma^{(l)}_{ba} = \sigma^{(l)}_{bab} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 3 & 4 \end{pmatrix},
\]

\[
\sigma^{(r)}_{ab} = \sigma^{(r)}_{bab} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 2 & 3 & 3 \end{pmatrix}, \quad \sigma^{(r)}_{ba} = \sigma^{(r)}_{aba} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 4 & 4 \end{pmatrix}.
\]

Hence, the list of singular rectangles is exhausted by:

- \((1, 2; 1, 2)\), \((1, 2; 3, 4)\), \((3, 4; 1, 2)\), \((3, 4; 3, 4)\),
- \((1, 4; 2, 3)\), \((1, 4; 1, 4)\), \((2, 3; 2, 3)\), \((2, 3; 1, 4)\).

This results in \(f_{11} = f_{12} = f_{13} = f_{14} = f_{21} = f_{31} = f_{41} = f_{22} = f_{44} = 1\) and

\[
\begin{align*}
f_{23} &= f_{24}, & f_{24} &= f_{34}, & f_{33}^{-1} &= f_{33}^{-1} f_{34} \\
f_{32} &= f_{42}, & f_{42} &= f_{43}, & f_{23} &= f_{32}^{-1} f_{33}.
\end{align*}
\]

This completes the proof of Theorem 1.

**Remark 6.** The band \(B\) from the previous proof can be also realised as a regular subband of the free band \(FB_3\) on three generators \(a, b, c\) whose elements are from \(D'_1 = \{ab, aba, ba, bab\}\) and \(D'_0 = \{u \mathbf{v}: u, \mathbf{v} \in D'_1\}\).
We finish the note by several problems that might be subjects of future research in this direction.

**Problem 1.** Characterise all bands $B$ with the property that $\text{IG}(B)$ has a non-free maximal subgroup.

**Problem 2.** Characterise all groups that arise as maximal subgroups of $\text{IG}(B)$ for some band $B$. The same problem stands for regular bands $B$, and in fact for $B \in V$ for any particular band variety $V \supseteq \text{RB}$.

**Problem 3.** Given a band variety $V$ and an integer $n \geq 1$, describe the maximal subgroups of $\text{IG}(\tilde{F}_n V)$, where $\tilde{F}_n V$ denotes the $V$-free band on a set of $n$ free generators [16].

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