SPACE-TIME BLOCK CODES FROM NONASSOCIATIVE DIVISION ALGEBRAS

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Abstract. Associative division algebras are a rich source of fully diverse space-time block codes (STBCs). In this paper the systematic construction of fully diverse STBCs from nonassociative algebras is discussed. As examples, families of fully diverse $2 \times 2$, $2 \times 4$ multiblock and $4 \times 4$ STBCs are designed, employing nonassociative quaternion division algebras.

1. Introduction

Space-time coding is used for reliable high rate transmission over wireless digital channels with multiple antennas at both the transmitter and receiver ends. From a mathematical point of view, a space-time block code (STBC) consists of a family of matrices with complex entries (the codebook) that satisfies a number of properties which determine how well the code performs.

The first aim is to find fully diverse codebooks, where the difference of any two code words has full rank. Once a fully diverse codebook is found it is then further optimized to satisfy additional design criteria (see Section 6).

Using central simple associative division algebras to build space-time block codes allows for a systematic code design (see for instance [22], [13], [29], [15], [7], [8], [9] and the excellent survey [28]).

Most of the existing codes are built from cyclic division algebras over $F = \mathbb{Q}(i)$ or $F = \mathbb{Q}(\zeta_3)$ with $\zeta_3 = e^{2\pi i/3}$ a third root of unity. These fields are used for the transmission of QAM or HEX constellations, respectively.

There are two ways to embed an associative division algebra into a matrix algebra in order to obtain a codebook: the left regular representation of the algebra and the representation over some maximal subfield. For instance, a real $4 \times 4$ orthogonal design is obtained by the left regular representation of the real quaternions $\mathbb{H} = (-1, -1)_R$ (see for instance [30] p. 1458), whereas the Alamouti code [1] uses the representation of $\mathbb{H}$ over its maximal subfield $\mathbb{C}$.

In [29] p. 2608, the authors note that “the Alamouti code is the only rate-one STBC which is full rank over any finite subset of $\mathbb{C}$, which is due to the fact that the set of quaternions $\mathbb{H}$ is the only division algebra which has the entire complexes..."
as its maximal subfield.” There are however other, nonassociative, division algebras over \( \mathbb{R} \) of dimension 4 which contain \( \mathbb{C} \) as a subfield and which yield STBCs that are full rank over any finite subset of \( \mathbb{C} \): the nonassociative quaternion division algebras over \( \mathbb{R} \) which were classified in [4] and which we will employ here.

In this paper we show how nonassociative division algebras can be used to systematically construct fully diverse linear STBCs. If a nonassociative algebra \( A \) behaves well enough, one can also obtain fully diverse families of matrices using subfields of \( A \). We use 4-dimensional nonassociative quaternion division algebras to construct new examples of fully diverse \( 2 \times 2 \) and \( 4 \times 4 \) space-time block codes and of \( 2 \times 4 \) multiblock space-time codes (cf. [17]). We also investigate when these codes satisfy the non-vanishing determinant property.

The paper is organized as follows: in Sections 2 and 3 we present nonassociative algebras and the Cayley-Dickson doubling process, respectively. In Section 4 we define nonassociative quaternion division algebras (constructed via a generalized Cayley-Dickson doubling process). In Section 5 we explain the general framework for obtaining fully diverse STBCs from nonassociative division algebras. In Section 6 we list the design criteria used in the construction of STBCs. In Section 7 we look in more detail at the construction of fully diverse \( 2 \times 2 \) STBCs from nonassociative quaternion algebras. We also discuss the non-vanishing determinant and information lossless properties. This is followed by many examples. In Sections 8 and 9 we discuss the construction of fully diverse \( 2 \times 4 \) multiblock codes and \( 4 \times 4 \) codes from nonassociative quaternion algebras, respectively. In Appendix A we collect results from algebraic number theory that are needed in this paper.

2. Nonassociative algebras

Let \( F \) be a field and let \( A \) be a finite-dimensional \( F \)-vector space. We call \( A \) an algebra over \( F \) if there exists an \( F \)-bilinear map \( A \times A \to A \), \( (x, y) \mapsto xy \) (also denoted simply by juxtaposition \( xy \)), called multiplication, on \( A \). This definition does not imply that the algebra is associative; we only have \( c(xy) = (cx)y = x(cy) \) for all \( c \in F \), \( x, y \in A \). Hence we also call such an algebra a nonassociative algebra, in the sense that it is not necessarily associative. A (nonassociative) algebra \( A \) is called unital if there is an element in \( A \) (which can be shown to be uniquely determined), denoted by 1, such that \( 1x = x1 = x \) for all \( x \in A \). We will only consider unital nonassociative algebras.

For a nonassociative \( F \)-algebra \( A \), associativity in \( A \) is measured by the associator

\[
[x, y, z] = (xy)z - x(yz).
\]

The nucleus of \( A \) is defined as

\[
\mathcal{N}(A) = \{ x \in A \mid [x, A, A] = [A, x, A] = [A, A, x] = 0 \}.
\]

The nucleus is an associative subalgebra of \( A \) (it may be zero), and \( x(yz) = (xy)z \) whenever one of the elements \( x, y, z \) is in \( \mathcal{N}(A) \). In other words, the nucleus of the algebra \( A \) contains all the elements of \( A \) which associate with every other two elements in \( A \). Suppose \( K \subset A \) is a subfield of \( A \). An \( F \)-algebra \( A \) is \( K \)-associative if \( K \) is contained in the nucleus \( \mathcal{N}(A) \). The left nucleus of \( A \) is defined as

\[
\mathcal{N}_l(A) = \{ x \in A \mid [x, A, A] = 0 \},
\]

the middle nucleus of \( A \) is defined as

\[
\mathcal{N}_m(A) = \{ x \in A \mid [A, x, A] = 0 \}.
\]
and the right nucleus of $A$ is defined as
\[ \mathcal{N}_r(A) = \{ x \in A \mid [A, A, x] = 0 \}. \]

Their intersection is the nucleus $\mathcal{N}(A)$.

A nonassociative algebra $A$ is called a division algebra if for any $a \in A$, $a \neq 0$, the left multiplication with $a$, $\lambda_a(x) = ax$, and the right multiplication with $a$, $\rho_a(x) = xa$, are bijective. The algebra $A$ is a division algebra if and only if $A$ has no zero divisors [25, pp. 15, 16]. Note that if the $F$-algebra $A$ is associative and finite-dimensional as an $F$-vector space, this definition of division algebra coincides with the usual one for associative algebras.

3. THE CAYLEY-DICKSON DOUBLING PROCESS

The Cayley-Dickson doubling process is a well-known way to construct a new algebra with involution from a given algebra with involution. It can be motivated by the observation that the complex numbers can be viewed as pairs of real numbers with componentwise addition and a suitably defined multiplication:

**Example 3.1.** We define a multiplication on $\mathbb{R} \times \mathbb{R}$ via
\[ (u, v)(u', v') := (uu' - v'v, uv' + u'v), \]
for $u, v, u', v' \in \mathbb{R}$. The unit element for this multiplication is $(1, 0)$. Let $i = (0, 1)$. Then $i^2 = (-1, 0)$. We can now write the pair $(u, v)$ as $(u, v) = (u, 0) + (0, 1)(v, 0)$ and identify it with the element $u + iv \in \mathbb{R} \oplus i\mathbb{R}$. In this way we obtain the complex numbers
\[ \mathbb{C} = \mathbb{R} \oplus i\mathbb{R}. \]

For $x = u + iv, y = u' + iv'$ with $u, v, u', v' \in \mathbb{R}$, we have $xy = (uu' - v'v) + i(v'u + uv')$.

Let $\overline{\cdot}$ denote complex conjugation, given by $\overline{x} = u - iv$ for $x = u + iv$. Then we can also write $(u, v) = (u, -v)$.

The above process can be repeated with $\mathbb{C}$ instead of $\mathbb{R}$: define a multiplication on $\mathbb{C} \times \mathbb{C}$ via
\[ (u, v)(u', v') := (uu' - v'v, uv' + u'v), \]
for $u, v, u', v' \in \mathbb{C}$. The unit element for this multiplication is $(1, 0)$. Let $j = (0, 1) \in \mathbb{C} \times \mathbb{C}$. Then $j^2 = (-1, 0)$. We identify the element $(u, v) \in \mathbb{C} \times \mathbb{C}$ with $u + jv \in \mathbb{C} \oplus j\mathbb{C}$. In this way we obtain the Hamilton quaternions
\[ \mathbb{H} = \mathbb{C} \oplus j\mathbb{C}. \]

We define quaternion conjugation (again denoted $\overline{\cdot}$) via
\[ \overline{(u, v)} = (\overline{u}, -v). \]

Another iteration of this process, this time starting with $\mathbb{H}$, results in the (Cayley-Graves) octonion algebra $\mathbb{O}$.

**Remark 3.2.** In 1958 it was shown that finite-dimensional real division algebras can only have dimension 1, 2, 4 or 8 (cf. [10]). In addition to the well-known algebras $\mathbb{R}, \mathbb{C}, \mathbb{H}$ and $\mathbb{O}$, there exist other finite-dimensional real division algebras. The algebras $\mathbb{R}, \mathbb{C}, \mathbb{H}$ and $\mathbb{O}$ are just the alternative ones (see [25, p. 48]). Indeed, a complete classification of these algebras is still far away. Only certain subclasses are understood thus far. Moreover, the restriction on the dimension only holds for real and real closed fields (see [11]). Over number fields there exist also higher dimensional division algebras as well as division algebras which do not appear over the real numbers.
The previous construction of the Hamilton quaternions as a double of the complex numbers serves as a motivating example for obtaining generalized (associative) quaternion algebras as doubles, as we will do now.

Let $F$ be a field. Let $K$ be a separable quadratic field extension of $F$ with non-trivial Galois automorphism $\sigma: K \rightarrow K$. Let $b \in F^\times := F \setminus \{0\}$. Then the 4-dimensional $F$-vector space $K \times K$ can be made into a new unital associative (but not commutative) algebra over $F$ via the multiplication

$$(u, v)(u', v') := (uu' + bv'\sigma(v), \sigma(u)v' + u'v)$$

for $u, u', v, v' \in K$. The unit element is given by $(1, 0)$. The automorphism $\sigma$ induces an involution $\overline{\cdot}$ on $K \times K$ as follows:

$$\overline{(u, v)} := (\sigma(u), -v).$$

Let $j = (0, 1)$. Then $j^2 = (b, 0)$. We identify $(u, v) \in K \times K$ with $u + jv$ in $K \oplus jK$. The algebra $K \oplus jK$ is called the Cayley-Dickson double of $K$ (with scalar $b$) and denoted by $\text{Cay}(K,b)$ (cf. [2]).

The Cayley-Dickson double of $K$ yields a quaternion algebra over $F$. If $F$ has characteristic not 2 and $K = F(\sqrt{a}) = F(i)$, $\sigma: \sqrt{a} \mapsto -\sqrt{a}$, we have

$$\text{Cay}(K,b) \cong (a,b)_F.$$ 

The standard basis $\{1, i, j, k\}$ of the quaternion algebra $(a,b)_F$ satisfies $i^2 = a, j^2 = b, k = ij$ and $ij = -ji$.

The Cayley-Dickson doubling process depends on the scalar $b$ only up to an invertible square, i.e.

$$\text{Cay}(K,b) \cong \text{Cay}(K,bd^2)$$

for every $d \in F^\times$. The algebra $\text{Cay}(K,b)$ is a division algebra if and only if $b \notin N_{K/F}(K^\times)$, where $N_{K/F}$ is the norm of the field extension $K/F$.

The quadratic norm $N_A: A \rightarrow F$ of the algebra $A = \text{Cay}(K,b)$ is given by

$$N_A(u + jv) = N_{K/F}(u) - bN_{K/F}(v)$$

for $u, v \in K$. If $F$ has characteristic not 2, a straightforward computation shows that

$$N_A(x) = x\overline{x} = xx = q_0^2 - aq_1^2 - bq_2^2 + abq_3^2$$

for $x = q_0 + iq_1 + jq_2 + jq_3, q_i \in F$.

**Example 3.3.** Let $\mathbb{H} = (-1,-1)_R$ denote Hamilton’s quaternion algebra. This is just the algebra $\text{Cay}(\mathbb{C},-1)$, as already established above.

**Example 3.4.** The algebra $(5, i)_{\mathbb{Q}(i)} = \text{Cay}(K,i)$ with $K = \mathbb{Q}(i, \sqrt{5})$ is the quaternion algebra used in the construction of the Golden code [21]. This algebra is isomorphic to the cyclic algebra $(K/F, \sigma, i)$ where $\sigma: \sqrt{\frac{5}{2}} \mapsto -\sqrt{\frac{5}{2}}$. Since $K = \mathbb{Q}(i, \sqrt{5}) \cong \mathbb{Q}(i, \theta)$, where $\theta = \frac{1+i\sqrt{5}}{2}$ is the golden number, we also have that $\text{Cay}(\mathbb{Q}(i, \theta), i) \cong (5,i)_{\mathbb{Q}(i)}$.

**Remark 3.5.** The Cayley-Dickson doubling process can be iterated: the quaternion algebras double to octonion algebras, which in turn double to sedenion algebras. Continuing the doubling process results in successive generalized Cayley-Dickson algebras.
4. Nonassociative quaternion division algebras

Let $F$ be a field of characteristic not 2. Let $K$ be a quadratic field extension of $F$ with non-trivial Galois automorphism $\sigma$ and let $b \in K \setminus F$. We define an algebra structure on the $F$-vector space $K \times K$ via the multiplication

$$(u, v)(u', v') := \left( uu' + bv'\sigma(v), \sigma(u)v' + u'v \right)$$

for $u, u', v, v' \in K$. The multiplication is thus defined just as for quaternion algebras with the exception that we require the scalar $b$ to lie outside of $F$. We denote the algebra again by $\text{Cay}(K, b)$. Its unit element is $(1, 0)$.

Since $b \in K \setminus F$, the multiplication of $\text{Cay}(K, b)$ is not associative anymore. It is not even third power-associative, meaning that in general $(x^2)x \neq x(x^2)$. The algebra $\text{Cay}(K, b)$ with $b \in K$ and not in $F$ is called a nonassociative quaternion algebra over $F$.

**Remark 4.1.** Let $K = F(\sqrt{a}) = F(i)$ be a quadratic field extension and let $b \in K \setminus F$. Let $A = \text{Cay}(K, b)$ be a nonassociative quaternion division algebra. Put $j = (0, 1) \in \text{Cay}(K, b)$.

Then $A$ has $F$-basis $\{1, i, j, ji\}$ such that $i^2 = a, j^2 = b$ and $xj = j\sigma(x)$ for all $x \in K$ (so in particular $ij = -ji$).

**Theorem 4.2** ([34] or [24]). The nonassociative quaternion algebra $\text{Cay}(K, b)$ has nucleus $K$ and is a division algebra over $F$.

Thus products involving a factor from $K$ are still associative. Furthermore, nonassociative quaternion algebras are always division algebras, which is not the case for the usual associative quaternion algebras.

**Remark 4.3.** Let $F = \mathbb{R}$ and $b, b' \in \mathbb{C}$. Let $b = p + iq$. Then $\text{Cay}(\mathbb{C}, b) \cong \text{Cay}(\mathbb{C}, b')$ if and only if $b' = t(p \pm iq)$ for some positive $t \in \mathbb{R}$ [3 Thm. 14].

Over $\mathbb{Q}$, we can easily find non-isomorphic nonassociative quaternion division algebras: it was observed in [34] that two nonassociative quaternion algebras $\text{Cay}(K, b)$ and $\text{Cay}(L, c)$ can only be isomorphic if $L \cong K$. Moreover,

$$\text{Cay}(K, b) \cong \text{Cay}(K, c) \text{ iff } g(b) = d\sigma(d)c$$

for some automorphism $g \in \text{Aut}(K)$ and some non-zero $d \in K$.

Nonassociative quaternion algebras provided early examples of real nonassociative division algebras which were neither power-associative nor quadratic. They were investigated for the first time by Dickson [12] in 1935 and by Albert [3] in 1948. In 1987, Waterhouse [34] completely classified these algebras over a field of characteristic not 2.

The only division algebras which appear in the classification of 4-dimensional $K$-associative algebras, cf. [4] or [34], are the generalized quaternion division algebras and the nonassociative quaternion division algebras over $F$.

5. STBCs from nonassociative division algebras: the general setup

The general setup for constructing a fully diverse STBC from an associative division algebra $A$ is simple: associate to each nonzero element $x \in A$ a square matrix $X$ over a fixed subfield of $A$ (normally the base field, via the left regular representation, or a maximal extension of the base field). The difference of any two such matrices $X - X'$ (with $X \neq X'$) will then always be invertible. This procedure
can be adapted to work in the nonassociative case as well, as will be explained in this section.

5.1. The left regular representation. Let $A$ be a nonassociative division algebra over $F$ of dimension $n$ as an $F$-vector space. Let $a$ be any element in $A$. The left multiplication $\lambda_a : A \to A$ determined by $a$ is defined by $x \mapsto ax$ for all $x \in A$. The operator $\lambda_a$ is linear and the set $\{ \lambda_a \mid a \in A \}$ is a subspace of the associative algebra $\text{End}_F(A)$, the algebra of $F$-linear transformations on $A$. Consider the left regular representation

$$\lambda : A \to \text{End}_F(A), a \mapsto \lambda_a.$$  

If $\lambda_a = \lambda_b$ then $ax = bx$ for all $x \in A$, hence $(a - b)x = 0$ for all $x$, which yields $a = b$ ($A$ is a finite-dimensional division algebra and as such does not have zero divisors) and we have an injection.

After a choice of $F$-basis for $A$, we can embed $\text{End}_F(A)$ into the algebra $\text{Mat}_n(F)$ of $n \times n$-matrices with entries from $F$, where $n = \dim_F(A)$. In this way we get an embedding $\lambda : A \hookrightarrow \text{Mat}_n(F)$ of vector spaces.

Contrary to the situation for associative division algebras, this only embeds the vector space $A$ into the vector space $\text{Mat}_n(F)$; the algebra structure of $A$ is disregarded here. Nonetheless, all non-zero elements of $A$ are invertible, hence all $\lambda_a$ with $a \neq 0$ are bijective and so all non-zero matrices in $\lambda(A)$ have non-zero determinant. Now $X \pm Y \in \lambda(A)$ for all $X, Y \in \lambda(A)$. Thus $\lambda(A)$ constitutes a linear codebook which in addition is fully diverse, since the rank of the difference of two distinct codewords is maximal.

5.2. Representation over a maximal subfield. For coding purposes, an associative division algebra $A$ is often considered as a vector space over some subfield $K$ of the algebra $A$. Usually $K$ is maximal with respect to inclusion. Given a nonassociative $F$-algebra $A$ with a maximal subfield $K$, this is not always possible because of the nonexistence of the associative law. So what are the minimum requirements on a nonassociative algebra in order to have such a representation?

Let $K$ be a subfield of the $F$-algebra $A$. We need $A$ to be a right $K$-vector space, i.e. we need

$$x(cd) = (xc)d \text{ for all } x \in A, c, d \in K.$$  

This is satisfied for instance if $K \subset \mathcal{N}_r(A)$ or if $K \subset \mathcal{N}_m(A)$.

We also need that left multiplication $\lambda_a$ is a linear endomorphism of the right $K$-vector space $A$, i.e. that $(aa)x = a(ax)$ for all $a \in K$, $a, x \in A$, which is equivalent to $K \subset \mathcal{N}_l(A)$. Then

$$\lambda_{aa}(x) = (aa)x = a(ax) = a\lambda_a(x)$$  

for all $a, x \in A$, $a \in K$ and $\lambda_a \in \text{End}_K(A)$, so $\lambda : A \hookrightarrow \text{End}_K(A), a \mapsto \lambda_a$.

Thus, let $K$ be a subfield of $A$, maximal with respect to inclusion and assume that $K \subset \mathcal{N}_r(A) \cap \mathcal{N}_l(A)$ or $K \subset \mathcal{N}_l(A) \cap \mathcal{N}_m(A)$. Consider $A$ as a right $K$-vector space. After a choice of $K$-basis for $A$, we can embed $\text{End}_K(A)$ into the vector space $\text{Mat}_r(K)$ where $r = \dim_K(A)$. In this way we get an embedding

$$\lambda : A \hookrightarrow \text{Mat}_r(K)$$  

of vector spaces. Obviously, we have $X \pm Y \in \lambda(A)$ for all $X, Y \in \lambda(A)$. Thus $\lambda(A)$ constitutes a linear codebook.
Remark 5.1. If we want to consider $A$ as a left $K$-vector space, we require $K \subset N_\ell(A)$ and $K \subset N_m(A)$ and adjust the above construction accordingly.

More generally one can do the following: let $D$ be a subalgebra of $A$, assume that $D \subset N_\ell(A) \cap N_r(A)$ or that $D \subset N_m(A) \cap N_\ell(A)$ and suppose $A$ can be viewed as a free right $D$-module of rank $r$. After a choice of a $D$-basis for $A$, we can embed the right $D$-module $\text{End}_D(A)$ into the vector space $\text{Mat}_r(D)$. In this way we get an embedding $\lambda : A \to \text{Mat}_r(D)$ of $D$-modules. Obviously, we have $X \pm Y \in \lambda(A)$ for all $X, Y \in \lambda(A)$. Thus $\lambda(A)$ is a linear codebook.

Remark 5.2. It is not known whether there exist 8-dimensional real division algebras with some (left, middle or right) nucleus isomorphic to $\mathbb{H}$. The fact that there are no 8-dimensional real division algebras with two associative nuclei (left, middle or right) isomorphic to $\mathbb{H}$ suggests a negative answer [16, Proposition 3]. This need not be the case over other base fields, however.

In the remainder of this paper, all fields are assumed to be algebraic number fields unless stated otherwise.

6. STBC Design Criteria

Let $C \subset \text{Mat}_n(C)$ be a space-time block code. In order for $C$ to perform well, it should satisfy property (1) below (as remarked before) and as many of the other properties as possible.

1. It is fully diverse: $\det(X - X') \neq 0$ for all matrices $X \neq X', X, X' \in C$.
2. It has full rate, which means that the $n^2$ degrees of freedom are used to transmit $n^2$ information symbols.
3. It has non-vanishing determinant: the minimum determinant of the code,
   \[ \delta(C) = \inf_{X' \neq X'' \in C} |\det(X' - X'')|^2, \]
   is bounded below by a constant even if the codebook $C$ is infinite.
4. It has cubic shaping: each layer of a codeword is of the form $Rv$, where $R$ is a unitary matrix and $v$ is a vector containing the information symbols. As a consequence it is information lossless.
5. It induces uniform average energy per antenna: the $i$th antenna will transmit the $i$th row of the codeword; on average, the norms of the rows should be equal in order to have a balanced repartition of the energy at the transmitter.

These properties, originally considered for codes based on associative division algebras, also make sense in the nonassociative case. Codes that satisfy all the properties above are called perfect codes. We refer to [22] for more details. The Golden Code [6] is the best performing $2 \times 2$ perfect STBC, cf. [21].

7. $2 \times 2$ Codebooks from nonassociative quaternion division algebras

STBCs based on associative quaternion algebras seem to have been considered explicitly for the first time in [5]. See also [32] for more details. In this section we look at the construction of STBCs based on nonassociative quaternion algebras.

Roughly speaking, constructing a nonassociative quaternion division algebra boils down to choosing the nonzero scalar $b$ in the quadratic field extension $K = F(\sqrt{a})$ of the base field $F$, and not in $F$ itself. In contrast, $b$ is chosen in $F$ in the construction of a classical generalized quaternion algebra $(a, b)_F$ over $F$. This
we get an embedding \( \lambda \): \( K \) to inclusion. For \( x \in K \), consider the \( K \)-linear map \( \lambda : A \mapsto \text{End}_K(A) \), \( x \mapsto \lambda_x \).

7.1. **Fully diverse codebook construction.** Let \( K \) be a quadratic field extension of \( F \) with non-trivial Galois automorphism \( \sigma \). Let \( A = \text{Cay}(K, b) \) be a nonassociative quaternion division algebra over \( F \) (so \( b \in K \setminus F \)) with \( F \)-basis \( \{1, i, j, ji\} \) (see Remark 4.1).

The algebra \( A \) is \( K \)-associative by Theorem 4.2 hence we can consider \( A \) as a right vector space over the subfield \( K \) of \( A \). The field \( K \) is maximal with respect to inclusion. For \( x \in A \), the left multiplication \( \lambda_x : A \to A \), \( a \mapsto xa \), is a \( K \)-linear endomorphism of the right \( K \)-vector space \( A \). Therefore \( \lambda_x \in \text{End}_K(A) \) and we get an injective \( K \)-linear map

\[
\lambda : A \mapsto \text{End}_K(A), \quad x \mapsto \lambda_x.
\]

Consider the \( K \)-basis \( \{1, j\} \) of \( A \). Then \( \text{End}_K(A) \cong \text{Mat}_2(K) \) as vector spaces and we get an embedding \( \lambda : A \mapsto \text{Mat}_2(K) \) of vector spaces, which sends \( x \in A \) to the matrix of \( \lambda_x \) with respect to the basis \( \{1, j\} \).

**Lemma 7.1.**

\[
\lambda(A) \cong \left\{ \begin{bmatrix} x_0 & b\sigma(x_1) \\ x_1 & \sigma(x_0) \end{bmatrix} \right| x_0, x_1 \in K \}
\]

**Proof.** Let \( x = x_0 + jx_1 \in A \) with \( x_0, x_1 \in K \). Then

\[
\lambda_x(1) = x_0 + jx_1,
\]

\[
\lambda_x(j) = x_0j + jx_1j = j\sigma(x_0) + j^2\sigma(x_1) = b\sigma(x_1) + j\sigma(x_0)
\]

by the rules in Remark 4.1. \( \square \)

**Lemma 7.2.** For any

\[
0 \neq X = \begin{bmatrix} x_0 & b\sigma(x_1) \\ x_1 & \sigma(x_0) \end{bmatrix}
\]

with \( x_i \in K \) for \( i = 1, 2 \), we have

\[
\det(X) = N_{K/F}(x_0) - bN_{K/F}(x_1) \neq 0.
\]

**Proof.** For \( b = u + \sqrt{av} \in K \), \( u, v \in F \), \( v \neq 0 \), we compute

\[
\det(X) = x_0\sigma(x_0) - b\sigma(x_1)x_1 = (N_{K/F}(x_0) - uN_{K/F}(x_1)) - \sqrt{av}N_{K/F}(x_1).
\]

Since \( (x_0, x_1) \neq (0, 0) \) and since \( N_{K/F}(x) = 0 \) iff \( x = 0 \), we get \( \det(X) \neq 0 \). \( \square \)

Since \( X \pm Y \in \lambda(A) \) for all \( X, Y \in \lambda(A) \), the difference of any two distinct elements in \( \lambda(A) \) will have non-zero determinant. Therefore the (infinite linear) codebook built on \( A, C := \lambda(A) \), is fully diverse.

7.2. **Non-vanishing determinant.** We closely follow the approach in [8, §17]. The minimum determinant of \( C \) determines the coding gain and is defined as

\[
\delta(C) = \inf_{X' \neq X'' \in C} |\det(X' - X'')|^2.
\]

The discussion in [8, p. 73] can easily be adapted to the more general set-up of nonassociative algebras. Since the codebook \( C \) is linear (it is based on an algebra) we have

\[
\delta(C) = \inf_{0 \neq X \in C} |\det(X)|^2.
\]
Let us compute the minimum determinant of the codebook
\[ C = \left\{ \begin{array}{c|c}
    c + d\sqrt{a} & b(e - f\sqrt{a}) \\
    e + f\sqrt{a} & c - d\sqrt{a}
\end{array} \right\} \text{ for } c, d, e, f \in F, \]
obtained from the nonassociative quaternion division algebra \( A = \text{Cay}(K, b) \) with \( K = F(\sqrt{a}) \) and \( b \in K \setminus F \) in \( \leq 7.1 \). We obtain
\[ \delta(C) = \inf_{c, d, e, f \in F} |N_{K/F}(c + d\sqrt{a}) - bN_{K/F}(e + f\sqrt{a})|^2 \]
with the infimum taken over all \((c, d, e, f) \neq (0, 0, 0, 0)\), or equivalently
\[ \delta(C) = \inf_{c, d, e, f \in F} |c^2 - ad^2 - be^2 + abf^2|^2. \]
Thus
\[ \delta(C) \in K \cap \mathbb{R}^+. \]
Since \( A \) is a division algebra, \( \delta(C) \neq 0 \). If the code \( C \) is finite, i.e. if the information symbols \( c, d, e, f \) belong to a finite constellation in \( F \), then \( \delta(C) \) is bounded below by a constant. If the constellation size increases however, \( \delta(C) \) can get arbitrarily close to zero (e.g. let \((c, d, e, f) = (1, 0, 0, 0); \) as \( n \) increases, \( \delta(C) \) will approach zero). This will also be the case for infinite codes.

Codes whose minimum determinant is bounded below by a constant which is independent of the size of the constellation from which the information symbols are chosen are said to satisfy the non-vanishing determinant (NVD) property, cf. Section 6.

For associative division algebras over a number field \( F \) and with maximal subfield \( K \) infinite codes that satisfy the NVD property can often be obtained by restricting the entries in the codebook to the ring of integers \( \mathcal{O}_K \). If \( F = \mathbb{Q} \) or \( F \) is quadratic imaginary, then the resulting code will still be infinite, and its minimum determinant is guaranteed to be bounded away from zero, cf. [6, Cor. 17.8].

Let us look at what happens for a code \( \mathcal{C} \), based on a nonassociative quaternion division algebra.

**Proposition 7.3.** Let \( F \) be a number field and let \( K = F(\sqrt{a}) \) for some nonzero square-free \( a \in \mathcal{O}_F \). Let \( b \in K \setminus F \). Let \( \mathcal{C} = \lambda(\text{Cay}(K, b)) \) and let \( \mathcal{C}_{\mathcal{O}_K} \) denote the code obtained from \( \mathcal{C} \) by restricting the elements of \( K \) to elements of \( \mathcal{O}_K \). Then there exists a constant \( c > 0 \) such that
\[ \delta(\mathcal{C}_{\mathcal{O}_K}) \in \frac{1}{c} \mathcal{O}_K \cap \mathbb{R}^+. \]

If \( K \) is quadratic imaginary, then there exists an integer \( d > 0 \) such that
\[ \delta(\mathcal{C}_{\mathcal{O}_K}) \geq \frac{1}{d} \]
(and so \( \mathcal{C}_{\mathcal{O}_K} \) satisfies the NVD property), otherwise \( \delta(\mathcal{C}_{\mathcal{O}_K}) \) can become arbitrarily small.

**Proof.** Write \( b \) as a fraction \( b = \frac{b_n}{b_d} \) with \( b_n, b_d \in \mathcal{O}_K \) (not necessarily unique) and \( b_d \neq 0 \). A codeword of \( \mathcal{C}_{\mathcal{O}_K} \) is of the form
\[ \begin{bmatrix} u & b\sigma(v) \\ v & \sigma(u) \end{bmatrix} \]
with \( u, v \in \mathcal{O}_K \). Thus we have
\[ \delta(\mathcal{C}_{\mathcal{O}_K}) = \inf_{u, v \in \mathcal{O}_K} |N_{K/F}(u) - bN_{K/F}(v)|^2 \]
\[
\text{with the infimum taken over all } (u, v) \neq (0, 0). \text{ Taking } c = |b_d|^2 \text{ establishes the first part of the proposition.}
\]

Assume that \( K \) is quadratic imaginary (i.e. \( F = \mathbb{Q} \) and \( a < 0 \)). Then it follows from Proposition A.1 that \( \mathcal{O}_K \cap \mathbb{R}^+ = \mathbb{N} \) and that \( |b_d|^2 = N_{K/\mathbb{Q}}(b_d) \) is a positive integer. Thus, among all possible pairs \((b_n, b_d) \in \mathcal{O}_K^2\) (with \( b_d \neq 0 \)) that satisfy \( b = \frac{b_n}{b_d} \), we can choose a pair \((b_n, b_d)\) in such a way that \( |b_d|^2 \) is minimal. Let \( d = |b_d|^2 \).

If \( K \) is not quadratic imaginary it follows from the Dirichlet Unit Theorem (Proposition A.3) that \( \mathcal{O}_K \) contains units \( u \) such that \( |u|^2 \) is arbitrarily large or small, so that \( \delta(\mathcal{O}_K) \) can become arbitrarily small. \( \square \)

Remark 7.4. It follows from the proposition that codes based on nonassociative quaternion algebras only satisfy the NVD property if we assume that \( K \) is a quadratic imaginary number field.

Assume that \( K \) is quadratic imaginary. If we assume in addition that \( \mathcal{O}_K \) is a unique factorization domain (or, equivalently, a principal ideal domain; cf. Appendix A), we can write \( b \) as an irreducible fraction \( b = b_n/b_d \) and \( b_d \) will be unique up to multiplication by a unit. By the Dirichlet Unit Theorem the only units in \( \mathcal{O}_K \) are roots of unity, so that \( |b_d|^2 \) is unique. For information symbols \( u, v \) taken from a QAM constellation we use the field \( K = \mathbb{Q}(i) \) which is quadratic imaginary and whose ring of integers \( \mathcal{O}_K = \mathbb{Z}[i] \) is a unique factorization domain.

7.3. Information lossless encoding. A code \( \mathcal{C} \) is information lossless if it is obtained from information symbols in such a way that the energy needed to transmit them is the same as the energy needed to transmit the information symbols without encoding. By [23, Prop. 3.5] it suffices to construct the layers of each codeword from the information symbols vector by applying a unitary matrix. This procedure is called cubic shaping (cf. [22, p. 3886]), as it corresponds to an isometry transformation of the cubic lattice \( \mathbb{Z}[i]^n \) in the case of information symbols taken from a QAM constellation. The term good shaping is also used.

The energy needed to transmit a complex number \( z \) is determined by \( |z|^2 \). The energy needed to transmit a codeword \( X = [x_{i,j}] \in \mathcal{C} \) is determined by its squared Frobenius norm \( \| X \|^2 = \sum_{i,j} |x_{i,j}|^2 \).

Information lossless encoding of information symbols \( c, d, e, f \in F \) into a codeword

\[
X = \begin{bmatrix}
 x_0 & b\sigma(x_1) \\
 x_1 & \sigma(x_0)
\end{bmatrix}
\]

of \( \mathcal{C} \) can be done as follows. Let \( \{u_0, u_1\} \) be an \( F \)-basis of \( K \). Let

\[
G = \begin{bmatrix}
 u_0 \\
 \sigma(u_0)
\end{bmatrix} \quad \begin{bmatrix}
 u_1 \\
 \sigma(u_1)
\end{bmatrix}
\]

be the matrix of the embeddings of the basis. Let \( x_0 = cu_0 + du_1 \) and \( x_1 = eu_0 + fu_1 \) be elements of \( K \) and consider the vectors \( x_0 = (c, d)^T, x_1 = (e, f)^T \), each containing two information symbols. Then

\[
Gx_0 = (x_0, \sigma(x_0))^T, \quad Gx_1 = (x_1, \sigma(x_1))^T.
\]
Let $\Gamma_1 = I_2$ be the identity matrix and
$$
\Gamma_2 = \begin{bmatrix}
0 & b \\
1 & 0
\end{bmatrix}.
$$
Then we can write $X$ as the sum of its layers,
$$
X = \Gamma_1 \text{diag}(Gx_0) + \Gamma_2 \text{diag}(Gx_1) = \begin{bmatrix}
x_0 & 0 \\
0 & \sigma(x_0)
\end{bmatrix} + \begin{bmatrix}
0 & b\sigma(x_1) \\
x_1 & 0
\end{bmatrix}.
$$
In order for the encoding to be information lossless, we want $\Gamma_2$ and $G$ to be unitary.

Note that the matrix $\Gamma_2$ is unitary if and only if $|b|^2 = 1$.

In order to find a good unitary matrix $G$ (if one exists) and also to satisfy the NVD property, one usually restricts $x_0$ and $x_1$ to the ring of integers $\mathcal{O}_K$ or an ideal $I$ of $\mathcal{O}_K$ with “good” properties. See [23, §4.3-4.4] for more details.

In the rest of this section we construct fully diverse $2 \times 2$ codebooks $\lambda(A)$, based on nonassociative quaternion algebras $A$. From these we construct codebooks $C$ that satisfy the NVD and/or cubic shaping properties in certain cases. The codebooks are all infinite. When restricting entries of the codewords to the ring of integers $\mathcal{O}_K$, we indicate this by writing $C_{\mathcal{O}_K}$.

### 7.4. Nonassociative Alamouti codes.

Let $F = \mathbb{R}$, $K = \mathbb{C}$ and let $\sigma = \overline{}$ be complex conjugation. Recall [1] that the Alamouti Code is obtained from the quaternion division algebra $\mathbb{H} = (-1, -1)_{\mathbb{R}}$ over $\mathbb{R}$ and yields codewords of the form
$$
\begin{bmatrix}
c + id & -e + if \\
e + if & c - id
\end{bmatrix},
$$
with $c + id, e + if$ the information symbols ($c, d, e, f \in \mathbb{R}$). Used with QAM symbols it achieves the diversity-multiplexing gain trade-off (DMT) of a MISO channel with 2 transmit antennas and 1 receive antenna.

**Example 7.5.** (i) Let $A = \text{Cay}(\mathbb{C}, i)$. Then
$$
\lambda(A) = \left\{ \begin{bmatrix} c + id & f + ie \\ e + if & c - id \end{bmatrix} \middle| c + id, e + if \in \mathbb{C} \right\}.
$$
The codebook obtained from $\text{Cay}(\mathbb{C}, i)$ closely resembles the Alamouti Code.

Consider also $B = \text{Cay}(\mathbb{C}, -i)$. Then
$$
\lambda(B) = \left\{ \begin{bmatrix} c + id & -(f + ie) \\ e + if & c - id \end{bmatrix} \middle| c + id, e + if \in \mathbb{C} \right\}
$$
is another Alamouti-like code. Note that the algebras $A$ and $B$ are isomorphic by Remark 4.3.

Let us now consider QAM symbols only.

(ii) Let $A = \text{Cay}(\mathbb{Q}(i), i)$ over $\mathbb{Q}$ and restrict the entries in $\lambda(A)$ to $\mathbb{Z}[i]$. Then
$$
\lambda(A)_{\mathbb{Z}[i]} = \left\{ \begin{bmatrix} c + id & f + ie \\ e + if & c - id \end{bmatrix} \middle| c + id, e + if \in \mathbb{Z}[i] \right\}.
$$
From $\lambda(A)_{\mathbb{Z}[i]}$ we obtain a code $C_A$ with good shaping as follows: $\{1, i\}$ is a $\mathbb{Z}$-basis of $\mathbb{Z}[i]$ and
$$
\frac{1}{\sqrt{2}} G = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & i \\ 1 & -i \end{bmatrix}
$$
is a unitary matrix. So a codeword of $C_A$ is given by

$$\frac{1}{\sqrt{2}} \begin{bmatrix} c + id & f + ie \\ e + if & c - id \end{bmatrix},$$

c, d, e, f ∈ \mathbb{Z}.

(iii) Similarly, $B = \text{Cay}(\mathbb{Q}(i), -i)$ yields

$$\lambda(B)\mathbb{Z}[i] = \left\{ \begin{bmatrix} c + id & -(f + ie) \\ e + if & c - id \end{bmatrix} \mid c + id, e + if ∈ \mathbb{Z}[i] \right\},$$

resulting in a shaped code $C_B$ with codewords

$$\frac{1}{\sqrt{2}} \begin{bmatrix} c + id & -(f + ie) \\ e + if & c - id \end{bmatrix},$$

c, d, e, f ∈ \mathbb{Z}.

In examples (ii) and (iii), $F = \mathbb{Q}$, $K = \mathbb{Q}(i)$ is a quadratic imaginary number field, $b = ±i$ and $\mathcal{O}_K = \mathbb{Z}[i]$ is a principal ideal domain. Hence, before shaping, the minimum determinant of each code is bounded below by 1 by Proposition 7.3. Thus the minimum determinant of both shaped codes is lower bounded by the constant $1/2$.

To summarize: the codes in (ii) and (iii) are fully diverse, satisfy the NVD property, have good shaping and clearly also satisfy the uniform average transmitted energy per antenna property. Used for a $2 \times 2$ MIMO channel they are only half-rate though, since 4 transmitted signals are used to transmit 2 QAM information symbols.

7.5. Nonassociative Golden Codes. Let $F = \mathbb{Q}(i)$ and let $K = \mathbb{Q}(i)(\sqrt{5})$. Then $\mathcal{O}_K = \mathbb{Z}[i][\frac{1+i}{\sqrt{5}}]$. The Golden Code [6] uses the maximal $\mathbb{Z}[i]$-order in the quaternion division algebra $(5, i)_{\mathbb{Q}(i)} = \text{Cay}(\mathbb{Q}(i)(\sqrt{5}), i)$ which can be described by the Cayley-Dickson doubling $\text{Cay}\left(\mathbb{Z}[i][\frac{1+i}{\sqrt{5}}], i\right)$, defined in the obvious way. (Note that associative quaternion algebras are precisely the cyclic algebras of dimension 4.) After shaping by $\frac{1}{\sqrt{5}}$, a codeword of $C$ is thus of the form

$$X = \frac{1}{\sqrt{5}} \begin{bmatrix} c + d\theta & e + f\theta \\ i(e + f\sigma(\theta)) & c + d\sigma(\theta) \end{bmatrix},$$

with $\theta = \frac{1+i}{\sqrt{5}}$ the golden number, $\sigma : \mathbb{Q}(i)(\sqrt{5}) \to \mathbb{Q}(i)(\sqrt{5})$, $\sigma(i) = i$, $\sigma(\sqrt{5}) = -\sqrt{5}$ and $c, d, e, f ∈ \mathbb{Z}[i]$. To obtain an energy-efficient code, the entries in the codewords are then restricted to elements in the principal ideal $I$ in $\mathcal{O}_K$ of norm 5 in $\mathbb{Q}$ which is generated by $\alpha = 1 + i - i\theta$. So, finally the Golden Code is given by the codewords

$$X = \frac{1}{\sqrt{5}} \begin{bmatrix} \alpha(c + d\theta) & \alpha(e + f\theta) \\ i\sigma(\alpha)(e + f\sigma(\theta)) & \sigma(\alpha)\sigma(c + d\sigma(\theta)) \end{bmatrix}$$

with $c, d, e, f ∈ \mathbb{Z}[i]$ and has minimum determinant $1/5$. We refer to [6] for the details.

Example 7.6. Consider the nonassociative quaternion division algebra

$$A = \text{Cay}(\mathbb{Q}(i)(\sqrt{5}), \frac{i + \sqrt{5}}{i - \sqrt{5}})$$

over $\mathbb{Q}(i)$, where $|\frac{i + \sqrt{5}}{i - \sqrt{5}}|^2 = 1$ guarantees that $\Gamma_2$ is unitary.
The codebook based on $A$ is

$$\lambda(A) = \left\{ \begin{bmatrix} c + d\theta & e + f\theta \\ \frac{e + f\theta}{c + d\theta} & \frac{e + f\theta}{c + d\theta} \end{bmatrix} \mid c, d, e, f \in \mathbb{Q}(i) \right\}. $$

(Compared to the general code construction in Lemma 7.1 we are transposing the matrices here in order to better compare them with the Golden Code matrices above. This does not influence the behaviour of the code.) The code has full diversity and uniform average transmitted energy per antenna. To obtain an energy-efficient code, we restrict the entries in the codeword again to elements in the principal ideal $I$ in $\mathcal{O}_K$ generated by $\alpha = 1 + i - i\theta$. Then a nonassociative Golden Code is given by the codewords

$$X = \frac{1}{\sqrt{5}} \begin{bmatrix} 1 & \theta \\ 1 & \sigma(\theta) \end{bmatrix}$$

with $c, d, e, f \in \mathbb{Z}[i]$. The choice of the ideal $I$ is optimal here for the exact same reasons as the ones given in [6, p. 1433] and yields good shaping. The code is also full rate, fully diverse and has uniform average transmitted energy per antenna.

With the same arguments codes can be constructed using any of the infinitely many scalars $b \in \mathbb{Q}(i)(\sqrt{5}) \setminus \mathbb{Q}(i)$ with $|b|^2 = 1$. All these codes, however, have vanishing determinant by Proposition 7.3. We give another example:

**Example 7.7.** Let $b = \frac{2i + \sqrt{5}}{3}$ and consider the nonassociative quaternion algebra

$$\text{Cay}\left(\mathbb{Q}(i)(\sqrt{5}), \frac{2i + \sqrt{5}}{3}\right)$$

over $\mathbb{Q}(i)$. Again $\left| \frac{2i + \sqrt{5}}{3} \right|^2 = 1$ and we obtain another code which has full diversity and uniform average transmitted energy per antenna. To obtain an energy-efficient code, we restrict the entries in the codeword again to elements in the principal ideal $I$ in $\mathcal{O}_K$ generated by $\alpha = 1 + i - i\theta$. Then another nonassociative Golden Code with good shaping is given by the codewords

$$X = \frac{1}{\sqrt{5}} \begin{bmatrix} 1 & \theta \\ 1 & \sigma(\theta) \end{bmatrix}$$

with $c, d, e, f \in \mathbb{Z}[i]$.

7.6. **Optimality of the Golden Code.** Oggier [21] shows that the Golden Code is optimal inside the class of cyclic algebra based $2 \times 2$ codes built over fields $K = \mathbb{Q}(i)(\sqrt{7})$ in the following sense: the minimum determinant of such codes is inversely proportional to $|d_{K/\mathbb{Q}(i)}|$, where $d_{K/\mathbb{Q}(i)}$ denotes the relative discriminant of $K/\mathbb{Q}(i)$. For the Golden Code $|d_{K/\mathbb{Q}(i)}| = 5$. While it is possible to consider fields $K = \mathbb{Q}(i)(\sqrt{d})$ with $|d_{K/\mathbb{Q}(i)}| < 5$, Oggier shows that the resulting codes are no longer fully diverse [21, III].

This problem does not occur in the nonassociative case by Theorem 1.2. It is possible to construct fully diverse nonassociative codes over fields $K = \mathbb{Q}(i)(\sqrt{d})$ with $|d_{K/\mathbb{Q}(i)}| < 5$, but by Proposition 7.3 these codes do not satisfy the NVD...
property. In the examples below we will consider the cases \(|d_{K/Q(i)}| = 4\) and \(|d_{K/Q(i)}| = 3\). The case \(|d_{K/Q(i)}| = 2\) does not exist, cf. Proposition \(7.3\).

**Example 7.8.** Let \(K = Q(i)(\sqrt{2})\). Then \(|d_{K/Q(i)}| = 4\) and \(\sigma(\sqrt{2}) = -\sqrt{2}\). Moreover, \(K = Q(i)(\zeta_8)\) where \(\zeta_8 = \frac{1+i}{\sqrt{2}}\) is an 8th root of unity and \(\sigma(\zeta_8) = -\zeta_8\). We have that \(\{1, \zeta_8\}\) is a \(\mathbb{Z}[i]\)-basis for the ring of integers \(\mathcal{O}_K = \mathbb{Z}[i][\zeta_8]\).

Consider the nonassociative quaternion division algebra \(A = \text{Cay}(Q(i)(\zeta_8), \zeta_8)\) over \(Q(i)\). The choice of \(b = \zeta_8\) guarantees that \(\Gamma_2\) is unitary, since \(|\zeta_8|^2 = 1\). We obtain the codebook

\[
\lambda(A) = \left\{ \begin{bmatrix} x_0 \\ x_1 \end{bmatrix} \begin{bmatrix} \zeta_8 \sigma(x_1) \\ \sigma(x_0) \end{bmatrix} \bigg| \begin{array}[]{c} x_0, x_1 \in Q(i)(\zeta_8) \end{array} \right\}
\]

Now \(\{1, \zeta_8\}\) is a \(Q(i)\)-basis of \(Q(i)(\zeta_8)\) and

\[
\frac{1}{\sqrt{2}} G = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & \zeta_8 \\ 1 & \sigma(\zeta_8) \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & \zeta_8 \\ 1 & -\zeta_8 \end{bmatrix}
\]

is a unitary matrix. So after multiplying the matrices in the codebook by \(\frac{1}{\sqrt{2}}\) and restricting the information symbols to \(\mathbb{Z}[i]\), we obtain a code that has good shaping:

\[
\mathcal{C} = \left\{ \frac{1}{\sqrt{2}} \begin{bmatrix} u_0 + \zeta_8 u_0 \\ u_1 + \zeta_8 u_1 \end{bmatrix} \begin{bmatrix} \zeta_8(u_1 - \zeta_8 u_1) \\ u_0 - \zeta_8 u_0 \end{bmatrix} \bigg| \begin{array}[]{c} u_0, u_1, w_0, w_1 \in \mathbb{Z}[i] \end{array} \right\}
\]

The code \(\mathcal{C}\) is full rate, has full diversity and good shaping. The factor \(\zeta_8\) in the first row of the codeword guarantees uniform average transmitted energy per antenna since \(|\zeta_8|^2 = 1\). This code does not satisfy the NVD property however by Proposition \(7.3\).

**Example 7.9.** Let \(K = Q(i)(\sqrt{3})\). Then \(|d_{K/Q(i)}| = 3\) and \(\sigma(\sqrt{3}) = -\sqrt{3}\). Moreover, \(K = Q(i)(\zeta_3)\) where \(\zeta_3 = e^{2\pi i/3} = -\frac{1+i}{\sqrt{3}}\) is a third root of unity. We have \(\sigma(\zeta_3) = -\frac{1+i}{\sqrt{3}} = \overline{\zeta_3}\). We know that \(\{1, \zeta_3\}\) is a \(\mathbb{Z}[i]\)-basis for the ring of integers \(\mathcal{O}_K = \mathbb{Z}[i][\zeta_3]\).

Consider the nonassociative quaternion division algebra \(A = \text{Cay}(Q(i)(\zeta_3), \zeta_3)\) over \(Q(i)\). We obtain the codebook

\[
\lambda(A) = \left\{ \begin{bmatrix} u_0 + \zeta_3 u_0 \\ u_1 + \zeta_3 u_1 \end{bmatrix} \begin{bmatrix} \zeta_3(u_1 + \overline{\zeta_3} w_1) \\ u_0 + \zeta_3 w_0 \end{bmatrix} \bigg| \begin{array}[]{c} u_0, u_1, w_0, w_1 \in Q(i) \end{array} \right\}
\]

This time the matrix \(G\) (up to scaling) is not unitary. Thus the energy required to send the linear combination of the information symbols on each layer is higher than the energy needed to send the information symbols themselves and we would still have to optimize for energy efficiency. In addition the discriminant of this code is not bounded away from zero by Proposition \(7.3\).

8. 2 \times 4 MultiBlock Space-Time Codes from Nonassociative Quaternion Algebras

Let \(F\) be a number field, let \(a \in F^\times\) and let \(K = F(\sqrt{a})\) be a quadratic field extension of \(F\) with non-trivial Galois automorphism \(\sigma\) and norm \(N_{K/F}(x) = \sigma(\sigma(x))\). Let \(b \in K \setminus F\), so that \(A = \text{Cay}(K, b)\) is a nonassociative quaternion division
algebra. A $2 \times 4$ multiblock space-time code based on $A$ is a set of matrices of the form $Y = [X\sigma(X)]$ where $X \in \lambda(A)$ (see [17] for a more general construction in the associative case).

In this set-up, we want the code to satisfy a generalized $Y$-property.

We let $d = x_1 b (\sigma(x_1) \sigma(x_0))$, with $x_0, x_1 \in K$. They have full rank since $X$ comes from the division algebra $A$. In this set-up, we want the code to satisfy a generalized NVD property (see [18]) which can be achieved by bounding the generalized minimum determinant

$$\delta_g(C) = \inf_{0 \neq X \in \lambda(A)} |\det(X)\det(\sigma(X))| \text{ away from zero.}$$

**Proposition 8.1.** Let $F$ be a number field and let $K = F(\sqrt{a})$ for some nonzero square-free $a \in \mathcal{O}_F$. Let $b \in K \setminus F$. Let $A = \text{Cay}(K, b)$ and let

$$C_{\mathcal{O}_K} = \{ [X\sigma(X)] \mid X \in \lambda(A)\mathcal{O}_K \}$$

denote the $2 \times 4$ multiblock code obtained from $C$ by restricting the elements of $K$ to elements of $\mathcal{O}_K$. Then there exists a constant $c > 0$ such that

$$\delta_g(C_{\mathcal{O}_K}) \leq \frac{1}{c} \mathcal{O}_F \cap \mathbb{R}^+.$$

If $F = \mathbb{Q}$ or $F$ is quadratic imaginary, then there exists an integer $d > 0$ such that

$$\delta_g(C_{\mathcal{O}_K}) \geq \frac{1}{\sqrt{d}}$$

(and so $C_{\mathcal{O}_K}$ satisfies the generalized NVD property), otherwise $\delta_g(C_{\mathcal{O}_K})$ can become arbitrarily small.

**Proof.** Write $b$ as a fraction $b = \frac{b_n}{b_d}$ with $b_n, b_d \in \mathcal{O}_K$ (not necessarily unique) and $b_d \neq 0$. We have

$$\delta_g(C_{\mathcal{O}_K}) = \inf_{0 \neq X \in \lambda(A)\mathcal{O}_K} |\det(X)\det(\sigma(X))|$$

$$= \inf_{0 \neq X \in \lambda(A)\mathcal{O}_K} |\det(X)\sigma(\det(X))|$$

$$= \inf_{0 \neq X \in \lambda(A)\mathcal{O}_K} |N_{K/F}(\det(X))|$$

$$= \inf_{x_0, x_1 \in \mathcal{O}_K (x_0, x_1) \neq (0, 0)} |N_{K/F}(N_{K/F}(x_0) - bN_{K/F}(x_1))|$$

$$= \inf_{x_0, x_1 \in \mathcal{O}_K (x_0, x_1) \neq (0, 0)} \frac{1}{|N_{K/F}(b_d)|} |N_{K/F}(b_d N_{K/F}(x_0) - b_n N_{K/F}(x_1))|$$

$$\in \frac{1}{|N_{K/F}(b_d)|} \mathcal{O}_F \cap \mathbb{R}^+$$

since $N_{K/F}(\mathcal{O}_K) \subset \mathcal{O}_F$. Taking $c = |N_{K/F}(b_d)|$ establishes the first part of the proposition.

Assume that $F = \mathbb{Q}$. Then $|N_{K/F}(b_d)|$ is a positive integer. Thus, among all possible pairs $(b_n, b_d) \in \mathcal{O}_K^2$ (with $b_d \neq 0$) that satisfy $b = \frac{b_n}{b_d}$, we can choose a pair $(b_n, b_d)$ in such a way that $|N_{K/F}(b_d)|$ is minimal. Furthermore, $\mathcal{O}_F \cap \mathbb{R}^+ = \mathbb{N}$. We let $d = |N_{K/F}(b_d)|^2$ in this case.
Next assume that $F$ is quadratic imaginary (i.e. $F = \mathbb{Q}(\sqrt{m})$ and $m < 0$). Then it follows from Proposition 3.1 that $O_F \cap \mathbb{R}^+ = \mathbb{N}$ and that $|N_{K/F}(b_d)|^2 = N_{F/Q}(N_{K/F}(b_d))$ is a positive integer. Thus, among all possible pairs $(b_n, b_d) \in O_K^2$ (with $b_d \neq 0$) that satisfy $b = \frac{b_n}{b_d}$, we can choose a pair $(b_n, b_d)$ in such a way that $|N_{K/F}(b_d)|^2$ is minimal. Let $d = |N_{K/F}(b_d)|^2$.

If $F$ is not $\mathbb{Q}$ or not quadratic imaginary it follows from the Dirichlet Unit Theorem (Proposition A.8) that $O_F$ contains units $u$ such that $|u|^2$ is arbitrarily large or small, so that $\delta_g(C_{O_K})$ can become arbitrarily small. \hfill $\Box$

**Remark 8.2.** If $F = \mathbb{Q}$ or $F$ is quadratic imaginary, the generalized minimum determinant of $C_{O_K}$ is lower bounded by a positive constant and the generalized NVD property is satisfied. As a consequence the code will achieve the diversity-multiplexing gain trade-off, as explained in [17] p. 5232.

If we assume in addition that $O_K$ is a unique factorization domain (or, equivalently, a principal ideal domain; cf. Appendix A), we can write $b$ as an irreducible fraction $b = b_n/b_d$ and $b_d$ will be unique up to multiplication by a unit. By the Dirichlet Unit Theorem the only units in $O_K$ are roots of unity, so that $|N_{K/F}(b_d)|^2$ is unique.

For QAM constellations we take $F = \mathbb{Q}(i)$, so that $K = \mathbb{Q}(i)(\sqrt{m})$ for some square-free non-zero integer $m$. Proposition A.6 lists the fields $K$ whose ring of integers $O_K$ is a unique factorization domain.

In the setting of multiblock space-time codes it is again natural to ask that $|b|^2 = 1$, cf. [17] p. 5232.

**Example 8.3.** Let $F = \mathbb{Q}(i)$ and $K = \mathbb{Q}(i)(\sqrt{5})$. Let $\theta = \frac{1 + \sqrt{5}}{2}$ be the golden number, $\sigma : \mathbb{Q}(i)(\sqrt{5}) \to \mathbb{Q}(i)(\sqrt{5})$, $\sigma(i) = i$, $\sigma(\sqrt{5}) = -\sqrt{5}$, $b = \frac{1 + \sqrt{5}}{2} \frac{1 - \sqrt{5}}{2}$, and $A = \text{Cay}(\mathbb{Q}(i)(\sqrt{5}), b)$ over $\mathbb{Q}(i)$ as in Example 7.6. In order to obtain an energy-efficient code, we restrict the entries in the codewords to elements in the principal ideal $I$ in $O_K$, generated by $\alpha = 1 + i - i\theta$. Then

$$X = \frac{1}{\sqrt{5}} \begin{bmatrix} \alpha(c + d\theta) & b\sigma(\alpha)(e + f\sigma(\theta)) \\ \alpha(e + f\theta) & \sigma(\alpha)(c + d\sigma(\theta)) \end{bmatrix}$$

with $c, d, e, f \in \mathbb{Z}[i]$ and the code $C_{O_K}$ consists of block matrices of the form

$$Y = \frac{1}{\sqrt{5}} \begin{bmatrix} \alpha(c + d\theta) & b\sigma(\alpha)(e + f\sigma(\theta)) & 1 \\ \alpha(e + f\theta) & \sigma(\alpha)(c + d\sigma(\theta)) & \sigma(b)\alpha(e + f\theta) \end{bmatrix}.$$

Note that

$$\delta_g(C_{O_K}) \geq \frac{1}{|N_{K/F}(\sqrt{5}(i - \sqrt{5}))|} = \frac{1}{30},$$

guaranteeing that the code satisfies the generalized NVD property.

**Example 8.4.** Replacing $b$ by $\frac{2 + \sqrt{5}}{3}$ (cf. Example 7.7) in the previous example results in a code $C_{O_K}$ such that

$$\delta_g(C_{O_K}) \geq \frac{1}{|N_{K/F}(3\sqrt{5})|} = \frac{1}{45},$$

guaranteeing that the code satisfies the generalized NVD property.
Example 8.5. Let \( F = \mathbb{Q}(i) \) and \( K = \mathbb{Q}(i)(\sqrt{2}) = \mathbb{Q}(i)(\zeta_8) \) where \( \zeta_8 = \frac{1+i}{\sqrt{2}} \) is an 8th root of unity as in Example 7.8. Let \( A = \text{Cay}(\mathbb{Q}(\zeta_8), \zeta_8) \). Then

\[
X = \frac{1}{\sqrt{2}} \begin{bmatrix}
    u_0 + \zeta_8 w_0 & \zeta_8 (u_1 - \zeta_8 w_1) \\
    u_1 + \zeta_8 w_1 & u_0 - \zeta_8 w_0
\end{bmatrix}
\]

with \( u_0, u_1, w_0, w_1 \in \mathbb{Z}[i] \) and the code \( C_{O_K} \) consists of block matrices of the form

\[
Y = \frac{1}{\sqrt{2}} \begin{bmatrix}
    u_0 + \zeta_8 w_0 & \zeta_8 (u_1 - \zeta_8 w_1) & u_0 - \zeta_8 w_0 & -\zeta_8 (u_1 + \zeta_8 w_1) \\
    u_1 + \zeta_8 w_1 & u_0 - \zeta_8 w_0 & u_1 - \zeta_8 w_1 & u_0 + \zeta_8 w_0
\end{bmatrix}
\]

We have

\[
\delta_g(C_{O_K}) \geq \frac{1}{|N_{K/F}((\sqrt{2})^2)|} = \frac{1}{4},
\]

guaranteeing that the code satisfies the generalized NVD property.

9. 4 \times 4 Codebooks from nonassociative quaternion division algebras

Let \( \mathbb{H} = (-1, -1)_{\mathbb{R}} \) be Hamilton’s quaternion division algebra. Its left regular representation with respect to the basis \( \{1, i, j, -ij\} \) consists of matrices of the form

\[
\begin{bmatrix}
    x_0 & -x_1 & -x_2 & -x_3 \\
    x_1 & x_0 & x_3 & -x_2 \\
    x_2 & -x_3 & x_0 & x_1 \\
    x_3 & x_2 & -x_1 & x_0
\end{bmatrix}
\]

with \( x_\ell \in \mathbb{R}, \ell = 0, \ldots, 3 \) (cf. [29] Example 8). This is exactly the four-dimensional real orthogonal design from [30] Section III-A.

Let us look at the left regular representation of a nonassociative quaternion algebra \( A \), this time over its base field rather than over a maximal subfield.

9.1. Fully diverse codebook construction. Let \( F \) be a number field and let \( K = F(\sqrt{a}) = F(i) \) with \( i^2 = a \in F_{\times} \) be a quadratic field extension with non-trivial Galois automorphism \( \sigma : \sqrt{a} \mapsto -\sqrt{a} \). Let \( A = \text{Cay}(K, b) \) be a nonassociative quaternion division algebra over \( F \) with \( b = p + qi \in K \setminus F \), so \( p, q \in F \) with \( q \neq 0 \). For the basis \( \{1, i, j, -ij\} \) of \( A \) over \( F \) the matrix representation of left multiplication with \( x = x_0 + x_1 i + x_2 j - x_3 ij \) yields the fully diverse \( 4 \times 4 \) space-time block code

\[
C = \lambda(A) = \left\{ \begin{bmatrix}
    x_0 & a x_1 & p x_2 - a q x_3 & a q x_2 - a p x_3 \\
    x_1 & x_0 & q x_2 - p x_3 & p x_2 - a q x_3 \\
    x_2 & a x_3 & x_0 & -a x_1 \\
    x_3 & x_2 & -x_1 & x_0
\end{bmatrix} \mid x_0, x_1, x_2, x_3 \in F \right\}.
\]

Example 9.1. Let \( i^2 = -1 \). The \( \mathbb{R} \)-algebra \( A = \text{Cay}(\mathbb{C}, i) \) yields the fully diverse \( 4 \times 4 \) space-time block code

\[
\begin{bmatrix}
    x_0 & -x_1 & x_3 & -x_2 \\
    x_1 & x_0 & x_2 & x_3 \\
    x_2 & -x_3 & x_0 & x_1 \\
    x_3 & x_2 & -x_1 & x_0
\end{bmatrix} \quad x_0, x_1, x_2, x_3 \in \mathbb{R}.
\]

Its matrices are not orthogonal, but their first two column vectors and, respectively, their last two, are orthogonal to each other.
9.2. Non-vanishing determinant. Let $X \in \mathcal{C}$. Then
\[
\det(X) = [(x_0^2 - ax_1^2) - p(x_2^2 - ax_3^2)]^2 - a q (x_2^2 - ax_3^2)^2 \in F.
\]
Since the codebook is based on a division algebra, its minimum determinant equals
\[
\delta(\mathcal{C}) = \inf_{0 \neq X \in \mathcal{C}} |\det(X)|^2
\]
and is non-zero. If the information symbols $x_0, x_1, x_2, x_3$ belong to a finite constellation in $F$, then $\delta(\mathcal{C})$ is bounded below by a constant which depends on the constellation size. If the constellation size increases, $\delta(\mathcal{C})$ can get arbitrarily close to zero. By restricting the entries in $\mathcal{C}$ to the ring of integers $\mathcal{O}_F$ we obtain for certain number fields $F$ infinite codes that satisfy the NVD property:

**Proposition 9.2.** Let $F$ be a number field and let $K = F(\sqrt{a})$ for some nonzero square-free $a \in \mathcal{O}_F$. Let $b = p + q \sqrt{a} \in K \setminus F$ with $p, q \in F$ (so that $q \neq 0$). Let $\mathcal{C} = \lambda(Cay(K, b))$ and let $\mathcal{C}_{\mathcal{O}_F}$ denote the code obtained from $\mathcal{C}$ by restricting the elements of $F$ to elements of $\mathcal{O}_F$. Then there exists a constant $c > 0$ such that
\[
\delta(\mathcal{C}_{\mathcal{O}_F}) \in \frac{1}{c} \mathcal{O}_F \cap \mathbb{R}^+.
\]
If $F = \mathbb{Q}$ or $F$ is quadratic imaginary, then there exists an integer $d > 0$ such that
\[
\delta(\mathcal{C}_{\mathcal{O}_F}) \geq \frac{1}{d}
\]
(and so $\mathcal{C}_{\mathcal{O}_F}$ satisfies the NVD property), otherwise $\delta(\mathcal{C}_{\mathcal{O}_F})$ can become arbitrarily small.

**Proof.** Write $p = \frac{p_n}{p_d}$, $q = \frac{q_n}{q_d}$ with $p_n, p_d, q_n, q_d \in \mathcal{O}_F$ (not necessarily unique) and $p_d \neq 0$, $q_d \neq 0$. An easy calculation confirms that
\[
\delta(\mathcal{C}_{\mathcal{O}_F}) \in \frac{1}{|q_d|^4} \mathcal{O}_F \cap \mathbb{R}^+ \text{ if } p = 0, \quad \delta(\mathcal{C}_{\mathcal{O}_F}) \in \frac{1}{|p_d q_d|^4} \mathcal{O}_F \cap \mathbb{R}^+ \text{ if } p \neq 0.
\]
Letting $c = |q_d|^4$ if $p = 0$ and $c = |p_d q_d|^4$ if $p \neq 0$ establishes the first part of the proposition.

Assume for the sake of argument that $p = 0$. The case $p \neq 0$ can be settled in a similar manner.

Assume that $F = \mathbb{Q}$. Then $|q_d|^4$ is a positive integer. Thus, among all possible pairs $(q_n, q_d) \in \mathcal{O}_F^2 = \mathbb{Z}^2$ (with $q_d \neq 0$) that satisfy $q = \frac{q_n}{q_d}$, we can choose a pair $(q_n, q_d)$ in such a way that $|q_d|^4$ is minimal. Furthermore, $\mathcal{O}_F \cap \mathbb{R}^+ = \mathbb{N}$. We let $d = |q_d|^4$ in this case.

Next assume that $F$ is quadratic imaginary (i.e. $F = \mathbb{Q}(\sqrt{m})$ and $m < 0$). Then it follows from Proposition A.1 that $\mathcal{O}_F \cap \mathbb{R}^+ = \mathbb{N}$ and that $|q_d|^4 = N_{F/\mathbb{Q}}(q_d)^2$ is a positive integer. Thus, among all possible pairs $(q_n, q_d) \in \mathcal{O}_F^2$ (with $q_d \neq 0$) that satisfy $q = \frac{q_n}{q_d}$, we can choose a pair $(q_n, q_d)$ in such a way that $|q_d|^4$ is minimal. Let $d = |q_d|^4$. If $F$ is not $\mathbb{Q}$ or not quadratic imaginary it follows from the Dirichlet Unit Theorem (Proposition A.3) that $\mathcal{O}_F$ contains units $\mu$ such that $|\mu|^2$ is arbitrarily large or small, so that $\delta(\mathcal{C}_{\mathcal{O}_F})$ can become arbitrarily small. \qed

**Remark 9.3.** If $F = \mathbb{Q}$ or $F$ is quadratic imaginary, the minimum determinant of $\mathcal{C}_{\mathcal{O}_F}$ is lower bounded by a positive constant and the NVD property is satisfied.

If we assume in addition that $\mathcal{O}_F$ is a unique factorization domain (or, equivalently, a principal ideal domain; cf. Appendix A), we can write $p$ and $q$ as irreducible
fractions $p = \frac{u}{p_d}$, $q = \frac{v}{q_d}$ and $p_d$, $q_d$ will be unique up to multiplication by a unit. By the Dirichlet Unit Theorem the only units in $\mathcal{O}_F$ are roots of unity, so that $|q_d|^4$, resp. $|p_dq_d|^4$, is unique.

**Example 9.4.** The $\mathbb{Q}$-algebra $A = \text{Cay}(\mathbb{Q}(i), i)$ yields the fully diverse $4 \times 4$ space-time block code

$$C_\mathbb{Z} = \left\{ \begin{bmatrix} x_0 & -x_1 & x_3 & -x_2 \\ x_1 & x_0 & x_2 & x_3 \\ x_2 & -x_3 & x_0 & x_1 \\ x_3 & x_2 & -x_1 & x_0 \end{bmatrix} \middle| x_0, x_1, x_2, x_3 \in \mathbb{Z} \right\}.$$  

We obtain

$$\delta(C_\mathbb{Z}) = \inf_{x_0, x_1, x_2, x_3 \in \mathbb{Z}} |(x_0^2 + x_1^2)^2 + (x_2^2 + x_3^2)^2|^2 = 1.$$

**Example 9.5.** The $\mathbb{Q}$-algebra $A = \text{Cay}(\mathbb{Q}(i), -i)$ yields the fully diverse $4 \times 4$ space-time block code

$$C_\mathbb{Z} = \left\{ \begin{bmatrix} x_0 & -x_1 & -x_3 & x_2 \\ x_1 & x_0 & -x_2 & -x_3 \\ x_2 & x_3 & x_0 & x_1 \\ x_3 & x_2 & -x_1 & x_0 \end{bmatrix} \middle| x_0, x_1, x_2, x_3 \in \mathbb{Z} \right\},$$

again with minimum determinant 1.

In the previous two examples the first two column vectors of any codeword and, respectively, the last two, are orthogonal to each other.

**Example 9.6.** Consider the $\mathbb{Q}(i)$-algebra $A = \text{Cay}(\mathbb{Q}(i)(\zeta_8), \zeta_8)$ where $\zeta_8 = \frac{1 + \sqrt{2}}{2}$ is an 8th root of unity. Note that $a = \zeta_8^2 = i$ and $b = \zeta_8$ (so $p = 0$ and $q = 1$). We obtain the fully diverse codebook

$$C_{\mathbb{Z}[i]} = \left\{ \begin{bmatrix} x_0 & ix_1 & -ix_3 & ix_2 \\ x_1 & x_0 & x_2 & -ix_3 \\ x_2 & ix_3 & x_0 & -ix_1 \\ x_3 & x_2 & -x_1 & x_0 \end{bmatrix} \middle| x_0, x_1, x_2, x_3 \in \mathbb{Z}[i] \right\}$$

whose minimum determinant is

$$\delta(C_{\mathbb{Z}[i]}) = \inf_{x_0, x_1, x_2, x_3 \in \mathbb{Z}[i]} |(x_0^2 + ix_1^2)^2 - i(x_2^2 - ix_3^2)^2|^2 = 1.$$  

### 9.3. Information lossless encoding.

For a (transposed) matrix

$$X = \begin{bmatrix} x_0 & x_1 & x_2 & x_3 \\ ax_1 & x_0 & ax_3 & x_2 \\ px_2 - ax_3 & qx_2 - px_3 & x_0 & -x_1 \\ aqx_2 - apx_3 & px_2 - aqx_3 & -ax_1 & x_0 \end{bmatrix}$$

in $\mathcal{C}$ we use the following encoding: let $I_4$ be the identity matrix and let

$$\Gamma_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ a & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & -a & 0 & 0 \end{bmatrix}, \quad \Gamma_2 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ p & q & 0 & 0 \\ aq & p & 0 & 0 \end{bmatrix}, \quad \Gamma_3 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & a & 0 \\ -aq & -p & 0 & 0 \\ -ap & -aq & 0 & 0 \end{bmatrix}.$$

The codeword $X$ is encoded as

$$X = I_4 \text{diag}(x_0) + \Gamma_1 \text{diag}(x_1) + \Gamma_2 \text{diag}(x_2) + \Gamma_3 \text{diag}(x_3),$$

where, for $\ell = 0, \ldots, 3$,

$$\text{diag}(x_\ell) = \begin{bmatrix} x_\ell & 0 & 0 & 0 \\ 0 & x_\ell & 0 & 0 \\ 0 & 0 & x_\ell & 0 \\ 0 & 0 & 0 & x_\ell \end{bmatrix}.$$

The matrix $\Gamma_3$ is unitary if and only if $|a|^2 = 1$, $aq + p\overline{q} = 0$ and $|p|^2 + |q|^2 = 1$. The matrix $\Gamma_2$ is unitary if and only if $|p|^2 + |q|^2 = 1$, $|a|^2|q|^2 + |p|^2 = 1$ and $aq + p\overline{q} = 0$. The matrix $\Gamma_1$ is unitary if and only if $|a|^2 = 1.$

Thus, $C$ is information lossless if $|a|^2 = |p|^2 + |q|^2 = 1$ and $p\overline{q} + aq = 0$.

It is not difficult to verify that the codes in Examples 9.4, 9.5 and 9.6 are all information lossless.

**Appendix A. Facts from Number Theory**

In this appendix we collect some results from algebraic number theory for the convenience of the reader.

Let $K$ be a number field. The ring of integers $\mathcal{O}_K$ of $K$ is a Dedekind domain [20, I(3.1)].

Let $d_K$ denote the discriminant of $K$.

**Proposition A.1** ([20, p.15]). Let $m \neq 0$ be a square-free integer and let $K = \mathbb{Q}(\sqrt{m})$. Then

$$d_K = \begin{cases} 4m & \text{if } m \equiv 2, 3 \pmod{4} \\ m & \text{if } m \equiv 1 \pmod{4} \end{cases}.$$ 

An integral basis of $K$ is given by $\{1, \sqrt{m}\}$ in the first case, by $\{1, \frac{1}{2}(1 + \sqrt{m})\}$ in the second case and by $\{1, \frac{1}{2}(m + \sqrt{m})\}$ in both cases. Thus

$$\mathcal{O}_K = \begin{cases} \mathbb{Z}[\sqrt{m}] & \text{if } m \equiv 2, 3 \pmod{4} \\ \mathbb{Z}[\frac{1 + \sqrt{m}}{2}] & \text{if } m \equiv 1 \pmod{4} \end{cases}.$$ 

Let $h_K$ denote the class number of $K$, then $h_K = 1$ if and only if $\mathcal{O}_K$ is a principal ideal domain [20 I,§6] if and only if $\mathcal{O}_K$ is a unique factorization domain (since $\mathcal{O}_K$ is a Dedekind domain [19 Prop. 3.18]).

**Proposition A.2** ([19 p. 48]). Let $m$ be a positive square-free integer and let $K = \mathbb{Q}(\sqrt{-m})$. Then $h_K = 1$ if and only if $m \in \{1, 2, 3, 7, 11, 19, 43, 67, 163\}.$

For an extension of number fields $K/F$, let $d_{K/F}$ denote the relative discriminant of $K$ over $F$.

**Proposition A.3** ([14 p. 443]). Let $L \supset K \supset F$ be a chain of number fields, then

$$d_{L/F} = N_{K/F}(d_{L/K})d_{K/F}^n,$$

where $n = [K : F].$

Let $i = \sqrt{-1}$. We collect some useful facts about quadratic extensions of $\mathbb{Q}(i)$. 

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20 SUSANNE PUMPLÜN AND THOMAS UNGER
Proposition A.4 ([26 Satz 2.1]). Let \( m \neq 0 \) be a square-free integer and let \( K = \mathbb{Q}(i)(\sqrt{m}) \). Then

\[
d_K = \begin{cases} 
16m^2 & \text{if } m \equiv 1, 3 \mod 4 \\
64m^2 & \text{if } m \equiv 2 \mod 4 
\end{cases}
\]

Proposition A.5. Let \( m \neq 0 \) be a square-free integer and let \( K = \mathbb{Q}(i)(\sqrt{m}) \). Then there exists a relative integral basis of \( K \) over \( \mathbb{Q}(i) \). Furthermore,

\[
|d_{K/\mathbb{Q}(i)}| = \begin{cases} 
|\sqrt{d_K}| & \text{if } m \equiv 1, 3 \mod 4 \\
2|\sqrt{d_K}| & \text{if } m \equiv 2 \mod 4 
\end{cases}
\]

Proof. The existence of a relative integral basis follows from [26 Satz 4.2a]. Consider the chain \( K \supset \mathbb{Q}(i) \supset \mathbb{Q} \). From Proposition A.4 it follows that \( d_{\mathbb{Q}(i)} = -4 \). Thus, \( d_K = |d_{K/\mathbb{Q}(i)}|^2(-4)^2 \) by Proposition A.3 which shows that \( |d_{K/\mathbb{Q}(i)}| = \frac{1}{4}\sqrt{d_K} \). We conclude with Proposition A.4.

Proposition A.6 ([33 pp. 915–916]). Let \( m \) be a positive square-free integer and let \( K = \mathbb{Q}(i)(\sqrt{m}) \). Then \( h_K = 1 \) if and only if \( m \in \{2, 3, 5, 7, 11, 13, 19, 37, 43, 67, 163\} \).

Remark A.7. Note that \( \mathbb{Q}(i)(\sqrt{m}) = \mathbb{Q}(i)(\sqrt{-m}) \). Also note that \( \mathbb{Q}(i)(\sqrt{2}) = \mathbb{Q}(\zeta_8) \) and \( \mathbb{Q}(i)(\sqrt{3}) = \mathbb{Q}(\zeta_{12}) \), where \( \zeta_n \) denotes a primitive \( n \)-th root of unity.

Proposition A.8 (Dirichlet’s Unit Theorem [20 I(7.4)]). Let \( K \) be a number field with ring of integers \( \mathcal{O}_K \). Let \( r \) be the number of real embeddings of \( K \) and \( s \) the number of pairs of complex conjugate embeddings of \( K \). Let \( \mu(K) \) denote the finite cyclic group of roots of unity that lie in \( K \). The group of units of \( \mathcal{O}_K \) is the direct product of \( \mu(K) \) and a free abelian group of rank \( r + s - 1 \).

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