PRESENTING HIGHER STACKS AS SIMPLICIAL SCHEMES

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ABSTRACT. We show that an $n$-geometric stack may be regarded as a special kind of simplicial scheme, namely a Duskin $n$-hypergroupoid in affine schemes, where surjectivity is defined in terms of covering maps, yielding Artin $n$-stacks, Deligne–Mumford $n$-stacks and $n$-schemes as the notion of covering varies. This formulation adapts to most HAG contexts, so in particular works for derived $n$-stacks (replacing rings with simplicial rings). We exploit this to describe quasi-coherent sheaves and complexes on these stacks, and to draw comparisons with Kontsevich’s dg-schemes. As an application, we show how the cotangent complex controls infinitesimal deformations of higher and derived stacks.

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INTRODUCTION

Although the usual approach to defining $n$-stacks ([TV2] and [Lur1]) undoubtedly yields the correct geometric objects, it has numerous drawbacks. The inductive construction does not lend itself easily to calculations, while the level of abstract homotopy theory involved can make $n$-stacks seem inaccessible to many. In this paper, we introduce a far more elementary concept, namely a Duskin-Glenn $n$-hypergroupoid in affine schemes, and show how it is equivalent to the concept of $n$-geometric stacks introduced in [TV2]. According to [Toë2], this is essentially the formulation of higher stacks originally envisaged by Grothendieck in [Gro2]. It is also closely related to the definition of Lie $n$-groupoids.

In [Dus1], Duskin defined an $n$-dimensional hypergroupoid to be a simplicial set $X \in \mathbb{S}$ for which the horn fillers all exist, and are moreover unique in levels greater than $n$. Explicitly, the $k$th horn $\Lambda^m_k \subset \Delta^m$ is defined by deleting the interior and the $k$th face, and he required that the map $\text{Hom}_S(\Delta^m, X) \to \text{Hom}_S(\Lambda^m_k, X)$ should be surjective for all $k, m$, and an isomorphism for $m > n$. When $n = 0$, this gives sets (with constant simplicial structure), and for $n = 1$, the simplicial sets arising in this way are precisely nerves of groupoids. This description is not as complicated as it first seems, since it suffices to truncate $X$ at the $(n + 2)$th level (Lemma 2.14). The combinatorial properties of these hypergroupoids were extensively studied by Glenn in [Gle].

It is well-known that Artin 1-stacks can be resolved by simplicial schemes, and this idea was exploited in [Ols1], [Ols2], [Aok1] and [Aok2] to study their quasi-coherent sheaves and deformations. It is therefore natural to expect that simplicial resolutions should exist for $n$-geometric Artin stacks. In Theorem 4.9, we show that every (quasi-compact, quasi-separated, . . .) $n$-geometric Artin stack $X$ (as defined in [TV2]) can be resolved by a simplicial affine scheme $X$, which is an Artin $n$-hypergroupoid in the sense that the morphism $\text{Hom}_S(\Delta^m, X) \to \text{Hom}_S(\Lambda^m_k, X)$ of affine schemes is a smooth surjection for all $k, m$, and an isomorphism for $m > n$. For $n$-geometric Deligne–Mumford stacks or $n$-geometric schemes there are similar statements, replacing smooth morphisms with étale morphisms or local isomorphisms. In particular, this means that any functor satisfying the conditions of Lurie’s Representability Theorem ([Lur1] Theorem 7.1.6) gives rise to such a simplicial scheme.

If $X$ is a quasi-compact semi-separated scheme, this resolution just corresponds to constructing a Čech complex. We may therefore think of Artin $n$-hypergroupoids as being analogues of atlases on manifolds. While they share with atlases an amenability to calculation, they also share the disadvantage of not being canonical. However, there is a notion of trivial relative Artin $n$-hypergroupoids $X' \to X$, which is analogous to refinement of an atlas, and we may regard two Artin $n$-hypergroupoids as being equivalent if they admit a common refinement of this type (Corollary 5.6). In this way, we can recover the entire $\infty$-category of $n$-geometric Artin stacks (Theorem 5.16).

Replacing rings with simplicial rings or dg-rings allows us to define derived Artin $n$-hypergroupoids, and these turn out to be equivalent to the $D^-$ geometric $n$-stacks of [TV2] (Theorem 7.21). Čech complexes allow us to make close comparisons between derived Artin stacks and Kontsevich’s dg-schemes (Remark 8.40.1). Indeed, this approach
adapts to all homotopical algebraic geometry contexts, so the rich theory of [TV2] can all be carried over to the setting of Duskin-Glenn hypergroupoids (see Remark 7.2 for details).

Quasi-coherent sheaves on an $n$-geometric stack then just correspond to Cartesian quasi-coherent sheaves on the associated Artin $n$-hypergroupoid (Corollary 6.7), and there is a similar statement for homotopy-Cartesian complexes on a derived $n$-geometric stack (Proposition 6.13). This facilitates a relatively simple description of the cotangent complex, and our main application of this theory is to describe infinitesimal deformations of $n$-geometric stacks in terms of the cotangent complex (Theorem 10.8).

The structure of the paper is as follows. Sections 1 and 2 are mostly a recapitulation of background material, with a few results proved in §2 for which the author does not know of any other reference. Section 3 introduces the main concepts to be used in the paper, the Artin $n$-hypergroupoids described above, and establishes their basic properties.

The technical heart of the paper is in Sections 4 and 5, where we establish an equivalence between the simplicial categories of $n$-geometric stacks and of affine $n$-hypergroupoids. The crucial result is Theorem 4.9, which uses an intricate induction taking $2^n - 1$ steps to construct an Artin $n$-hypergroupoid resolving a given $n$-geometric stack. Readers unfamiliar with $n$-geometric stacks can skip most of these sections and §1, instead just using Corollaries 5.6 and 5.13 to define the stack associated to an Artin $n$-hypergroupoid $X$ as a functor

$$X^♯ : \text{Aff}^{\text{op}} \to \mathcal{S},$$

and also to define the simplicial category of geometric stacks. Anyone wishing to make comparisons with [Lur1] should also read Remark 1.8 for differences in terminology.

Section 6 is dedicated to studying quasi-coherent sheaves. The equivalence between Cartesian quasi-coherent sheaves on an Artin $n$-hypergroupoid and quasi-coherent sheaves on the associated $n$-geometric Artin stack amounts to little more than cohomological descent. Inverse images of sheaves are easily understood in terms of this comparison. However, direct images and derived direct images prove far more complicated, and an explicit description is given in §6.4.

In Section 7, the main results of §§4–5 are adapted to the $n$-geometric $D^−$-stacks of [TV2] (or equivalently, the derived stacks of [Lur1]). The main results are Theorems 7.7 and 7.19 establishing the existence of hypergroupoid resolutions of geometric $D^−$-stacks, and Theorems 7.10 and 7.20 describing morphisms between two such stacks.

In Section 8, various alternative formulations of derived Artin hypergroupoids are developed. One of these builds on the Quillen equivalence in characteristic zero between simplicial algebras and dg algebras, permitting comparisons with the dg-schemes of [CFK]. The other formulations are based on the observation that in the homotopy category of simplicial rings, every object $R$ can be expressed as a filtered inverse limit of homotopy nilpotent extensions of the discrete ring $π_0R$. This allows us to replace any cosimplicial affine scheme $X$ with Zariski or étale neighbourhoods $X^l$ or $X^h$ (or sometimes even the formal neighbourhood $\hat{X}$) of $π_0X$ in $X$.

Section 9 adapts many of the results of Section 6 to derived stacks. It then applies them to construct the relative cotangent complex of a morphism in §9.2, compatibly with existing characterisations (Propositions 9.19 and 9.28). In §10, we then show how the cotangent complex governs deformations. We first consider deformations of morphisms of derived Artin stacks in Theorem 10.5, generalising the deformations of 1-morphisms of Artin 1-stacks considered in [Aok2]. We then consider deformations of derived Artin stacks in Theorem 10.8, generalising the deformations of Artin 1-stacks considered in [Aok1].

I would like to thank Barbara Fantechi for raising many important and interesting questions about the relation between $n$-stacks and simplicial schemes. I would also like to thank Bertrand Toën for his helpful comments, and Mathieu Anel for explaining properties of unique factorisation systems.
Notation and conventions. We will denote the category of simplicial sets by $\mathbf{S}$. A simplicial category is a category enriched in $\mathbf{S}$. In other words, for all objects $X, Y \in \mathcal{C}$, there is a simplicial set $\text{Hom}_\mathcal{C}(X, Y)$, and these Hom-spaces are equipped with the usual composition laws for morphisms. The category underlying $\mathcal{C}$ has morphisms $\text{Hom}_\mathcal{C}(X, Y) := \text{Hom}_\mathbf{S}(X, Y)$.

A simplicial structure on a category $\mathcal{C}$ is sometimes given by defining $\otimes : \mathbf{S} \times \mathcal{C} \to \mathcal{C}$ or $\wedge : \mathbf{S}^{\text{op}} \times \mathcal{C} \to \mathcal{C}$. In these cases, $\text{Hom}_\mathcal{C}$ is determined by the formulae

$$\text{Hom}_\mathcal{C}(X \otimes K, Y) = \text{Hom}_\mathbf{S}(K, \text{Hom}_\mathcal{C}(X, Y)) = \text{Hom}_\mathcal{C}(X, Y^K).$$

Definition 0.1. Given a simplicial abelian group $A_n$, we denote the associated normalised chain complex by $N^sA$. Recall that this is given by $N^s(A)_n := \bigcap_{i>0} \ker(\partial_i : A_n \to A_{n-1})$, with differential $\partial_0$. Then $H_n(N^sA) \cong \pi_n(A)$.

Definition 0.2. Given a cosimplicial abelian group $A^\bullet$, we denote the associated conormalised cochain complex by $N_cA$. Recall that this is given by $N_c(A)^n := \bigcap_i \ker(\sigma^i : A^n \to A^{n-1})$, with differential $\sum (-1)^i \partial^i$.

Definition 0.3. Given an affine scheme $X$, we will write $O(X) := \Gamma(X, \mathcal{O}_X)$.

1. Background on $n$-geometric stacks

We now recall various definitions and results from [TV2].

1.1. Artin and Deligne–Mumford $n$-stacks. Let $\text{Aff}$ be the category of affine schemes, and $s\text{Pr}(\text{Aff})$ the category of simplicial presheaves on $\text{Aff}$. Given $X \in \text{Aff}$, there is an associated presheaf $U \mapsto X(U)$.

We now fix a class $\mathcal{P}$ of morphisms of affine schemes, satisfying the conditions of [TV2] Assumption 1.3.2.11:

1. Morphisms in $\mathcal{P}$ are stable by compositions, isomorphisms and pull-backs.
2. Let $f : X \to Y$ be a morphism in $\text{Aff}$. We suppose that there exists a covering family $\{U_i \to X\}$ such that each $U_i \to X$ and each composite morphism $U_i \to Y$ lies in $\mathcal{P}$. Then $f$ belongs to $\mathcal{P}$.
3. For any two objects $X$ and $Y$ in $\text{Aff}$, the two natural morphisms $X \to X \sqcup Y, Y \to X \sqcup Y$ are in $\mathcal{P}$.
4. Let $f : X \to Y$ be a morphism in $\text{Aff}$. We suppose that there exists a covering family $\{U_i \to Y\}$ such that each pull back morphism $X \times_Y U_i \to U_i$ is in $\mathcal{P}$. Then $f$ belongs to $\mathcal{P}$.

Here, a covering family is defined relative to some topology $\tau$, usually taken to be fpqc, satisfying the conditions of [TV2] Assumption 1.3.2.2.

Examples of possible classes for $\mathcal{P}$ are smooth morphisms (which will give Artin stacks), étale morphisms (giving Deligne–Mumford stacks), local isomorphisms (giving schemes), fpqc and fppf morphisms.

Definition 1.1. We say that a morphism $f : X \to Y$ of schemes is a covering (resp. a $\mathcal{P}$-covering) if $X = \coprod U_i$ for a covering family (resp. a covering family of $\mathcal{P}$-morphisms) $\{U_i \to Y\}$.

A morphism $f : X \to Y$ of presheaves on $\text{Aff}^{\text{op}}$ is representable by a covering (resp. representable by a $\mathcal{P}$-covering) if for all maps $U \to Y$ with $U$ an affine scheme, $X \times_Y U \to U$ is a covering (resp. a $\mathcal{P}$-covering).

Definition 1.2. $n$-geometric stacks are defined in [TV2] Definition 1.3.3.1 as follows:

1. A stack is 0-geometric if it is a disjoint union of affine schemes.
2. For $n \geq 0$, a morphism of stacks $F \to G$ is $n$-representable if for any affine scheme $X$ and any morphism $X \to G$, the homotopy pull-back $F \times^h_G X$ is $n$-geometric.
(3) A morphism of stacks \( f : F \to G \) is in \( 0 - P \) if it is 0-representable, and if for any affine scheme \( X \) and any morphism \( X \to G \), the induced morphism \( F \times^h_G X \to X \) is a disjoint union of \( P \)-morphisms between affine schemes.

(4) Now let \( n > 0 \) and let \( F \) be any stack. An \( n \)-atlas for \( F \) is a morphism \( U \to F \) such that \( U \) is 0-geometric, and \( U \to F \) is a covering in \((n - 1) - P\).

(5) For \( n > 0 \), a stack \( F \) is \( n \)-geometric if the diagonal morphism \( F \to F \times F \) is \((n - 1)\)-representable and the stack \( F \) admits an \( n \)-atlas.

(6) For \( n > 0 \), a morphism of stacks \( F \to G \) is in \( n - P \) (or has the property \( n - P \), or is a \( n - P \)-morphism) if it is \( n \)-representable and if for any affine scheme \( X \) and any morphism \( X \to G \), there exists an \( n \)-atlas \( U \to F \times^h_G X \), such that \( U \to X \) is in \((n - 1) - P\).

Moreover, [TV2] Definition 1.3.3.6 says that a morphism of stacks is in \( P \) (or a \( P \)-morphism) if it is in \( n - P \) for some integer \( n \).

The following is [TV2] Proposition 1.3.3.5.

**Lemma 1.3.** Let \( f : F \to G \) be an \( n \)-representable morphism. If \( f \) is in \( P \) then it is in \( n - P \).

The following is [TV2] Corollary 1.3.4.5.

**Proposition 1.4.** Let \( f : F \to G \) be a cover of stacks and \( n \geq 1 \). If \( F \) is \( n \)-geometric and \( f \) is \((n - 1)\)-representable and in \( P \), then \( G \) is \( n \)-geometric.

The following results do not appear explicitly in [TV2], but prove extremely useful.

This shows how we can alternatively relax the other defining condition of an \( n \)-geometric stack:

**Lemma 1.5.** If \( \mathcal{X} \) is a stack equipped with a surjective \( P \)-morphism \( f : U \to \mathcal{X} \) from a 0-geometric stack, and such that the diagonal \( \mathcal{X} \to \mathcal{X} \times \mathcal{X} \) is \((n - 1)\)-representable, then \( \mathcal{X} \) is \( n \)-geometric.

**Proof.** We need to show that \( f : U \to \mathcal{X} \) is \((n - 1)\)-representable. Since \( f \) is surjective, it suffices to show that \( U \times^h_U U \) is \((n - 1)\)-geometric. This follows because the map \( U \times^h_U U \to U \times U \) is the pullback of the diagonal along \( (f, f) : U \times U \to \mathcal{X} \times \mathcal{X} \).

**Lemma 1.6.** Given a morphism \( f : \mathcal{X} \to \mathcal{Y} \) of stacks, the diagonal of the morphism

\[
\mathcal{X} \to \mathcal{X}^{\partial \Delta^{r-1}} \times^h_{\mathcal{Y}} \mathcal{Y},
\]

is isomorphic to

\[
\mathcal{X} \to \mathcal{X}^{\partial \Delta^r} \times^h_{\mathcal{Y}} \mathcal{Y}.
\]

**Proof.** The key observation is that we may express \( \partial \Delta^r \) as the pushout

\[
\partial \Delta^r = \Lambda^r_0 \cup_{\partial \Delta^{r-1}} \Delta^{r-1}.
\]

Since this is a pushout of cofibrations, we have

\[
\mathcal{X}^{\partial \Delta^r} \simeq \mathcal{X}^{\Lambda^r_0} \times^h_{\mathcal{X}^{\partial \Delta^{r-1}}} \mathcal{X}^{\Delta^{r-1}} \simeq \mathcal{X} \times^h_{\mathcal{X}^{\partial \Delta^{r-1}}} \mathcal{X},
\]

the second equivalence following because \( \Lambda^r_0 \) and \( \Delta^{r-1} \) are contractible.

This implies that the diagonal

\[
\mathcal{X} \to (\mathcal{X} \times^h_{\mathcal{X}^{\partial \Delta^{r-1}}} \mathcal{X}) \times^h_{\mathcal{Y}} \mathcal{Y}
\]

of \( \mathcal{X} \to \mathcal{X}^{\partial \Delta^{r-1}} \times^h_{\mathcal{Y}} \mathcal{Y} \) is just

\[
\mathcal{X} \to \mathcal{X}^{\partial \Delta^{r}} \times^h_{\mathcal{Y}} \mathcal{Y}.
\]
Lemma 1.7. Given an \( n \)-representable morphism \( X \to \mathcal{Y} \) of stacks, the map
\[
X \to X^\partial \Delta^r \times_{\mathcal{Y}^\partial \Delta^r} \mathcal{Y}
\]
is \((n-r)\)-representable for all \( 0 \leq r \leq n \), a \( 0 \)-representable closed immersion for \( r = n+1 \), and an equivalence for \( r > n+1 \).

Proof. We prove this by induction on \( r \). When \( r = 0 \), \( \partial \Delta^0 = \emptyset \), so the statement is trivially true. Now assume the statement for \( r - 1 \), with \( r \leq n \). Since \( X \to X^\partial \Delta^{r-1} \times_{\mathcal{Y}^\partial \Delta^{r-1}} \mathcal{Y} \) is \((n-r+1)\)-representable, its diagonal must be \((n-r)\)-representable, giving the required result by Lemma 1.6. The statement for \( r = n+1 \) follows because the diagonal of an affine morphism is a closed immersion. For \( r > n+1 \), observe that the diagonal of a closed immersion is an isomorphism.

Remark 1.8. When we take \( P \) to be the class of smooth morphisms, we can also make comparisons with [Lur1], but there are slight differences in terminology between [TV2] and [Lur1]. In the former, only disjoint unions of affine schemes are \( 0 \)-representable, so arbitrary schemes are 2-geometric stacks, and Artin stacks are 1-geometric stacks if and only if they have affine diagonal. In the latter, algebraic spaces are \( 0 \)-stacks. An \( n \)-stack in the sense of [Lur1] is called \( n \)-truncated in [TV2], and it follows easily that every \( n \)-geometric stack in [TV2] is \( n \)-truncated.

Conversely, for any \( n \)-truncated stack \( X \), the map \( X \to X^\partial \Delta^{n+2} \) is an isomorphism (adapting Lemma 1.7), hence \( 0 \)-representable. Thus \( X \) must be \((n+2)\)-geometric.

We can summarise this by saying that for a geometric stack \( X \) to be \( n \)-truncated means that \( X \to X^\partial \Delta^{n+2} \) is an equivalence, or equivalently that \( X \to X^\partial \Delta^{n} \) is representable by algebraic spaces. For \( X \) to be \( n \)-geometric means that \( X \to X^\partial \Delta^{n} \) is representable by disjoint unions of affine schemes.

As an example of this difference, consider the scheme \( X \) obtained by gluing two copies of \( \mathbb{A}^2 \), with intersection \( \mathbb{A}^2 - \{0\} \). This is certainly an algebraic space, so is \( 0 \)-truncated. However, the diagonal \( X \to X \times X \) is not an affine morphism, so \( X \) is not even 1-geometric, but merely 2-geometric.

Nevertheless, any Artin stack with affine diagonal (in particular any separated algebraic space) is 1-geometric.

1.2. Derived \( n \)-stacks. We now recall the geometric \( D^- \)-stacks of [TV2] Chapter 2.2, although there are similar constructions for any homotopic algebraic geometry context, in the sense of [TV2] Definition 1.3.2.13.

Let \( c\text{-}\text{Aff} \) be the category of cosimplicial affine schemes, which is opposite to the category of simplicial algebras. Given a simplicial algebra \( A \), write \( \text{Spec} \ A \) for the corresponding object of \( c\text{-}\text{Aff} \). Let \( s\text{Pr}(c\text{-}\text{Aff}) \) denote the category of simplicial presheaves on \( c\text{-}\text{Aff} \).

There is a simplicial model structure on \( c\text{-}\text{Aff} \), in which a map \( \text{Spec} \ B \to \text{Spec} \ A \) is a weak equivalence if \( \pi_n(A) \cong \pi_n(B) \) for all \( n \), and a cofibration if \( A \to B \) is surjective.

Definition 1.9. A map \( \text{Spec} \ B \to \text{Spec} \ A \) in \( c\text{-}\text{Aff} \) is said to be smooth (resp. etale, resp. a local isomorphism) if \( \pi_0(A) \to \pi_0(B) \) is smooth (resp. etale, resp. a local isomorphism), and \( \pi_n(B) \cong \pi_n(A) \otimes_{\pi_0(A)} \pi_0(B) \) for all \( n \).

Definition 1.10. Define the functor \( \pi^0 : c\text{-}\text{Aff} \to \text{Aff} \) by \( \text{Spec} \ A \mapsto \text{Spec} \pi_0 A \). A morphism \( f \) in \( c\text{-}\text{Aff} \) is said to be surjective if \( \pi^0(f) \) is surjective.

Definition 1.11. Given a fibrant object \( X \in c\text{-}\text{Aff} \), define \( \mathfrak{X} \in s\text{Pr}(c\text{-}\text{Aff}) \) by
\[
\mathfrak{X}(U) := \text{Hom}_{c\text{-}\text{Aff}}(U, X) \in \mathcal{S},
\]
for \( U \in c\text{-}\text{Aff} \). It follows from [TV2] 1.3.2.5 that \( \mathfrak{X} \) is a fibrant as an object of the category \( c\text{-}\text{Aff}^{-\text{op}} \) of stacks (defined as in [TV2] §1.3.2), and that \( \mathfrak{X} \) is a model for the sheafification
X^2 of X. Thus an object of sPr(cAff) is a 0-geometric stack if and only if it is weakly equivalent to a disjoint union of such stacks \( X^2 \).

2. \( n \)-DIMENSIONAL HYPERGROUPOIDS

Let \( S \) denote the category of simplicial sets.

Definition 2.1. Let \( \Delta^n \in S \) be the standard \( n \)-simplex, and \( \partial \Delta^n \in S \) its boundary. Given \( 0 \leq k \leq n \), define the \( k \)th horn \( \Lambda^k_n \) of \( \Delta^n \) to be the simplicial set obtained from \( \Delta^n \) by removing the interior and the \( k \)th face. See [GJ] §1.1 for explicit descriptions.

Definition 2.2. Given a simplicial set \( K \) and a simplicial object \( X_\bullet \) in a complete category \( C \), define the \( K \)-matching object in \( C \) by

\[
M_K X := \text{Hom}_S(K, X).
\]

Thus \( X_n = M_{\Delta^n} X \).

Remark 2.3. The matching object \( M_{\partial \Delta^n} X \) is usually denoted \( M_n X \). In [Gle], it is called the \( n \)th simplicial kernel, and denoted \( \Delta^i(n)(X) \), while \( M_{\Lambda^k_n} X \) is there denoted \( \Lambda^k(n)(X) \).

Recall the following definition from [Dus1].

Definition 2.4. An \( n \)-dimensional hypergroupoid (often also called a weak \( n \)-groupoid) is an object \( X \in S \) for which the partial matching maps

\[
X_m \rightarrow M_{\Lambda^n_k} X
\]

are surjective for all \( k, m \) (i.e. \( X \) is fibrant), and isomorphisms for all \( m > n \).

Remarks 2.5. (1) A 0-dimensional hypergroupoid is just a set \( X = X_0 \). A 1-dimensional hypergroupoid is the nerve of a groupoid \( G \), which can be recovered by taking objects \( X_0 \), morphisms \( X_1 \), source and target \( \partial_0, \partial_1 : X_1 \rightarrow X_0 \), identity \( \sigma_0 : X_0 \rightarrow X_1 \) and multiplication

\[
X_1 \times_{\partial_0, X_0, \partial_1} X_1 \xrightarrow{(\partial_2, \partial_0)^{-1}} X_2 \xrightarrow{\partial_1} X_1.
\]

Equivalently, \( G \) is the fundamental groupoid \( \pi_f X \) of \( X \).

(2) For an \( n \)-dimensional hypergroupoid \( X \), the factorisation \( \Lambda^n_k \rightarrow \partial \Delta^n \rightarrow \Delta^n \)

ensures that for all \( m > n \), \( X_m \rightarrow M_{\partial \Delta^n} \) has a section, so is surjective. This implies that \( X_m = 0 \) for all \( m > n \).

(3) Given an \( n \)-dimensional hypergroupoid \( X \) is equivalent to giving the sets \( X_i \) for \( i \leq n \), together with the operations between them, and an operation on certain \((n+1)\)-tuples in \( X_n \), satisfying the hyper-associativity and hyper-unit laws of [Gle] §3.2. A related result will be given in Lemma 2.14. Note that the definition in [Gle] differs slightly from ours, as it does not have any restriction on the partial matching maps for \( m \leq n \), which is equivalent to saying that \( X \) need not be fibrant.

(4) Under the Dold-Kan correspondence between \( \mathbb{N}_0 \)-graded chain complexes and abelian groups, \( n \)-dimensional hypergroupoid in abelian groups correspond to chain complexes concentrated in degrees \( \leq n \). This is essentially because the normalisation functor is given by \( N_n A = \text{ker}(A_n \rightarrow M^\circ_n A) \).

We also have a relative version of this definition.

Definition 2.6. Given \( Y \in S \), define a relative \( n \)-dimensional hypergroupoid over \( Y \) to be a morphism \( f : X \rightarrow Y \) in \( S \), such that the relative partial matching maps

\[
X_m \rightarrow M_{\Lambda^n_k}(X) \times_{M_{\Lambda^n_k} Y} Y_m
\]

are surjective for all \( k, m \) (i.e. \( f \) is a Kan fibration), and isomorphisms for all \( m > n \). In the terminology of [Gle], this says that \( f \) is a Kan fibration which is an exact fibration in all dimensions \( > n \).
Examples 2.7. (1) A morphism \( f : X \to Y \) between 1-dimensional hypergroupoids \( X, Y \) is a relative 1-dimensional hypergroupoid if and only if it satisfies the path-lifting property that for all objects \( x \in X_0 \), and all morphisms \( m \) in \( \pi_f Y \) with source \( fx \), there exists a unique morphism \( \tilde{m} \) in \( \pi_f X \) with source \( x \) and \( f(\tilde{m}) = m \).

(2) A relative 0-dimensional hypergroupoid \( f : X \to Y \) is just a space which is Cartesian over \( Y \), in the sense that the maps

\[
X_n \xrightarrow{(\partial_i, f)} X_{n-1} \times_{Y_{n-1}, \partial_i} Y_n
\]

are all isomorphisms.

Lemma 2.8. Given a fibration \( f : X \to Y \) and a morphism \( g : Y \to Z \) such that \( gf \) is a relative \( n \)-dimensional hypergroupoid, the morphism \( f \) must be a relative \( n \)-dimensional hypergroupoid.

Proof. Existence of the lifts follows from \( f \) being a fibration, and uniqueness follows from uniqueness for \( gf \).

Definition 2.9. Given \( Y \in \mathbb{S} \), define a trivial relative \( n \)-dimensional hypergroupoid over \( Y \) to be a morphism \( f : X \to Y \) in \( \mathbb{S} \), such that the relative matching maps

\[
X_m \to M_{\partial \Delta^m}(X) \times_{M_{\partial \Delta^m}Y} Y_m
\]

are surjective for all \( m \) (i.e. \( f \) is a trivial Kan fibration), and isomorphisms for all \( m \geq n \).

Definition 2.10. Let \( \Delta_n \subset \Delta \) be the full subcategory on objects \( i \), for \( i \leq n \). The category of \( n \)-truncated simplicial sets is \( \text{Set}^{\Delta_{n\text{opp}}} \). The forgetful functor from \( \mathbb{S} \) to \( \text{Set}^{\Delta_{n\text{opp}}} \) has a left adjoint denoted \( \text{sk}_n \), and a right adjoint denoted \( \text{cosk}_n \). Thus \( \text{Hom}(K, \text{cosk}_nX) = \text{Hom}(\text{sk}_nK, X) \) (suppressing notation of the forgetful functor).

Note that this implies the following Lemma.

Lemma 2.11. A morphism \( f : X \to Y \) is a trivial relative \( n \)-dimensional hypergroupoid if and only if \( X = Y \times_{\text{cosk}_{n-1}Y} \text{cosk}_{n-1}X \), and the \((n-1)\)-truncated morphism \( X_{<n} \to Y_{<n} \) satisfies the conditions of Definition 2.9 (up to level \( n-1 \)).

Lemma 2.12. A morphism \( f \) is a trivial relative \( n \)-dimensional hypergroupoid if and only if it is a relative \( n \)-dimensional hypergroupoid and a weak equivalence.

Proof. The ”only if” part is immediate. For the converse, note that \( f \) is a trivial fibration, so has the right lifting property (RLP) with respect to all cofibrations, and consider the maps \( \Lambda_0^i \to \partial \Delta^i \to \Delta^i \). Since \( f \) has exact RLP with respect to \( \Lambda_0^i \to \Delta^i \) for \( i > n \), and RLP with respect to \( \partial \Delta^i \to \Delta^i \) for all \( i \geq n \), it follows that \( f \) has exact RLP with respect to \( \partial \Delta^i \to \Delta^i \) for all \( i > n \). It therefore remains only to prove uniqueness of the lift for \( i = n \).

Consider the pushout \( \partial \Delta^{n+1} = \Lambda_0^{n+1} \cup_{\partial \Delta^n} \Delta^n \). Since \( \partial \Delta^n \to \Lambda_0^{n+1} \) is a cofibration, exact RLP with respect to \( \partial \Delta^n \to \Delta^n \) will follow from exact RLP with respect to \( \Lambda_0^{n+1} \to \partial \Delta^{n+1} \). This now follows from exact RLP with respect to \( \Lambda_0^{n+1} \to \Delta^{n+1} \), since both \( \Lambda_0^{n+1} \to \partial \Delta^{n+1} \) and \( \partial \Delta^{n+1} \to \Delta^{n+1} \) are cofibrations.

Definition 2.13. Say that a morphism of \( m \)-truncated simplicial sets is a relative \( n \)-dimensional hypergroupoid if it satisfies the conditions of Definition 2.6 (up to level \( m \)).

Lemma 2.14. A morphism \( f : X \to Y \) is a relative \( n \)-dimensional hypergroupoid if and only if \( X = Y \times_{\text{cosk}_{n+1}Y} \text{cosk}_{n+1}X \), and the \((n+2)\)-truncated morphism \( X_{\leq n+2} \to Y_{\leq n+2} \) is a relative \( n \)-dimensional hypergroupoid.
Proof. For the first part, note that for $i \geq n + 1$, the factorisation $\Lambda^i_0 \to \partial \Delta^i \to \Delta^i$ implies that the map

$$X_i \to Y_i \times_{M_i Y} M_i X$$

has a retraction $\rho$, so is injective. Since $\Lambda^i_0 \to \partial \Delta^i$ is the pushout of $\partial \Delta^{i-1} \to \Delta^i$ along $\partial^i: \partial \Delta^{i-1} \to \Lambda^i_0$, for $i \geq n + 2$ the map

$$M_i X \to M_i Y \times_{M^i_0 Y} M^i_0 X$$

also has a retraction, so

$$\rho: Y_i \times_{M_i Y} M_i X \to Y_i \times_{M^i_0 Y} M^i_0 X \simeq Y_i \times Y_i X_i = X_i$$

has a retraction, and so must be an isomorphism.

Thus the matching maps $X_i \to Y_i \times_{M_i Y} M_i X$ are isomorphisms for all $i \geq n + 1$, so

$$X = Y \times_{\cosk_{n+1} Y} \cosk_{n+1} X$$

Finally, note that $sk_{n+1} \Lambda^i_k \to sk_{n+1} \Delta^i$ is an isomorphism for all $i \geq n + 3$, so automatically has the exact LLP with respect to $X = Y \times_{\cosk_{n+1} Y} \cosk_{n+1} X$. This only leaves the lifting conditions from Definition 2.6 up to level $n + 2$. \(\square\)

Remarks 2.15. (1) When $n = 1$ and $Y$ is a point, we can compare this with the data required to define a groupoid. Levels 0 and 1 determine the objects, morphisms and identities, but the face maps from level 2 are needed to define the multiplication, and hence to construct the groupoid. Only once the face maps from level 3 have been defined can we ensure that the multiplication is associativ.

(2) When $n = 0$, a relative 0-dimensional hypergroupoid $X \to Y$ is a Cartesian morphism. Thus level 0 gives the fibres, level 1 the descent datum (thereby determining the fibration), and level 2 gives the cocycle condition, ensuring that the descent datum is effective.

(3) In Lemma 2.11, only the $(n-1)$-truncation was required for to describe a trivial relative $n$-dimensional hypergroupoid. In the 1-dimensional case, this says that a contractible groupoid is just a set, with a unique morphism between any pair of points.

Observe that an $n$-dimensional hypergroupoid $X$ is determined by $X_m$ for $m \leq n + 1$, while a trivial $n$-dimensional hypergroupoid $X$ is determined by $X_m$ for $m < n$. The next Lemma accounts for this apparent discrepancy.

Lemma 2.16. If $X \to Y$ is a relative $n$-dimensional hypergroupoid (resp. a trivial relative $n$-dimensional hypergroupoid), and $K \to L$ is a trivial cofibration (resp. a cofibration) giving an isomorphism $sk_{n-1} K \to sk_{n-1} L$ on $n$-skeleta, then the maps

$$M_L X \to M_L Y \times_{M_K Y} M_K X$$

are isomorphisms.

Proof. This is essentially the same as the proof that the maps $\Lambda^m_k \to \Delta^m$ (resp. $\partial \Delta^m \to \Delta^m$) generate the trivial cofibrations in $\mathcal{S}$. The same proof adapts to show that every map $K \to L$ satisfying the conditions above is a retract of a transfinite composition of pushouts of maps $\Lambda^m_k \to \Delta^m$ for $m > n$. \(\square\)

Definition 2.17. Given a simplicial set $K$ and a simplicial object $X_\bullet$ in a category $\mathcal{C}$, define the object $X^K \in \mathcal{C}^{\text{opp}}$ by

$$(X^K)_n := \text{Hom}_\mathcal{S}(K \times \Delta^n, X).$$

Thus $M_K X = (X^K)_0$. 

Lemma 2.18. If $f : X \to Y$ is a relative $n$-dimensional hypergroupoid, and $g : K \to L$ a cofibration in $S$, then the map
\[ F : X^L \to Y^L \times_{Y^K} X^L \]
is a relative $n$-dimensional hypergroupoid, which is trivial if either $f$ or $g$ is trivial. Moreover, if $\text{sk}_{n-1}g : \text{sk}_{n-1}K \to \text{sk}_{n-1}L$ is an isomorphism, then $F$ is a relative 0-dimensional hypergroupoid.

Proof. This amounts to describing when a diagram
\[
\begin{array}{ccc}
(A \times L) \cup_{(A \times K)} (B \times K) & \xrightarrow{f} & X \\
\downarrow & & \downarrow f \\
B \times L & \xrightarrow{g} & Y
\end{array}
\]
admits the lifting shown, and determining when the lift is unique. For all cofibrations $h : A \to B$, the map $h' : (A \times L) \cup_{(A \times K)} (B \times K) \to B \times L$ is a cofibration, so the lift must exist if $f$ is trivial. If either $g$ or $h$ is trivial, then $h'$ is trivial, so the lift exists.

Now,
\[ \text{sk}_{n-1}((A \times L) \cup_{(A \times K)} (B \times K)) \to \text{sk}_{n-1}(B \times L) \]
is an isomorphism whenever either of the maps $\text{sk}_{n-1}A \to \text{sk}_{n-1}B$ or $\text{sk}_{n-1}K \to \text{sk}_{n-1}L$ is an isomorphism, so Lemma 2.16 then implies that the lift is unique in this case. \hfill \Box

Corollary 2.19. If $f : X \to Y$ is a relative $n$-dimensional hypergroupoid, then
\[ X^{\Delta^n} \to Y^{\Delta^n} \times_{Y^{\partial \Delta^n}} X^{\Delta^n} \]
is a relative 0-dimensional hypergroupoid.

Definition 2.20. Define $\text{Dec}_+ : S \to S$ by $\text{Dec}_+(X)_n = X_{n+1}$, with $\partial_i^{\text{Dec}_+ X} = \partial_i^X$ and $\sigma_i^{\text{Dec}_+ X} = \sigma_i^X$. This is a comonad, with counit $\partial_{n+1}^X : \text{Dec}_+(X)_n \to X_n$, and comultiplication $\sigma_{n+1}^X : \text{Dec}_+(X)_n \to \text{Dec}_+^2(X)_n$.

Remark 2.21. $\text{Dec}_+$ is the functor denoted by DEC in [Gle].

Lemma 2.22. The maps $X_0 \xrightarrow{\sigma_0^X} \text{Dec}_+(X) \xrightarrow{\partial_0^X} X_0$ form a deformation retract, for all $X \in S$.

Proof. See [Dus2] §2.6. \hfill \Box

The following is immediate:

Lemma 2.23. $\text{Dec}_+$ has a left adjoint $F$, defined on simplices by $F(\Delta^n) = \Delta^{n+1}$, $F(\Delta^{n-1} \xrightarrow{\partial^n} \Delta^n) = (\Delta^n \xrightarrow{\partial^n} \Delta^{n+1})$, $F(\Delta^{n+1} \xrightarrow{\partial^n} \Delta^n) = (\Delta^{n+2} \xrightarrow{\partial^n} \Delta^{n+1})$.

Corollary 2.24. If $f : X \to Y$ is a relative $n$-dimensional hypergroupoid, then $\text{Dec}_+(X) \to \text{Dec}_+(Y) \times_Y X$ is a relative $(n-1)$-dimensional hypergroupoid.

Proof. It suffices to describe the maps $\alpha_k^m : F(\Lambda^m_k) \cup_{\Lambda^m_k} \Delta^m \to F(\Delta^m)$, where the natural transformation $\text{id} \to F$ is adjoint to the co-unit $\text{Dec}_+ \to \text{id}$. Now, $F(\Lambda^m_k)$ is the union of all faces of $\Delta^{m+1}$ except the $k$th and the $(m+1)$th, so $F(\Lambda^m_k) \cup_{\Lambda^m_k} \Delta^m = \Lambda^{m+1}_k$, and $\alpha_k^m$ is just the natural inclusion $\Lambda^{m+1}_k \hookrightarrow \Delta^{m+1}$.

The result now follows immediately since $\alpha_k^m$ lifts with respect to $F$ for all $m$, and lifts uniquely for $m > n - 1$. \hfill \Box
Remark 2.25. Given an \( n \)-dimensional hypergroupoid \( X \), this gives us a way to regard the homotopy fibre product \( X_0 \times^h_X X_0 \) as an \((n - 1)\)-dimensional hypergroupoid, since a model for it is given by the \((n - 1)\)-dimensional hypergroupoid
\[
\text{Dec}_+(X) \times_X X_0,
\]
as \( \text{Dec}_+(X) \to X \) is a fibration. This object is usually called the path-homotopy complex ([Dus2] §2.6).

Lemma 2.26. If \( f : X \to Y \) is a relative \( n \)-dimensional hypergroupoid, then \( \text{Dec}_+(Y) \times^Y_0 X_0 \) is a trivial relative \( n \)-dimensional hypergroupoid.

Proof. The retraction \( \text{Dec}_+ X \to X_0 \) corresponds to a section \( \pi_0 K \to F(\pi_0 K) \), functorial in \( K \). It thus suffices to describe the maps \( F(\partial \Delta^m) \cup \pi_0 \partial \Delta^m \pi_0(\Delta^m) \to F(\Delta^m) \). Calculation gives this map as \( \Lambda^{m+1}_k \to \Delta^{m+1} \). Therefore the \( m \)th relative matching map of \( \text{Dec}_+(X) \to \text{Dec}_+(Y) \times^Y_0 X_0 \) is surjective for \( m < n \), and an isomorphism for \( m \geq n \), as required. \( \square \)

3. \( n \)-DIMENSIONAL HYPERGROUPOIDS IN AFFINE SCHEMES

We now fix a class \( P \) of morphisms of affine schemes, satisfying the conditions of [TV2] Assumption 1.3.2.11 (as recalled in §1.1).

Definition 3.1. Let \( \coprod \text{Aff} \) be the category of disjoint unions of affine schemes. Let \( s(\coprod \text{Aff}) \) be the category of simplicial diagrams of disjoint unions of affine schemes.

Definition 3.2. Define an \((n, P)\)-hypergroupoid to be an object \( X_\bullet \in s(\coprod \text{Aff}) \), such that the partial matching maps
\[
X_m \to M^m(X)
\]
are \( P \)-coverings for all \( k, m \), and isomorphisms for all \( m > n \) and all \( k \).

When \( P \) consists of smooth (resp. étale) morphisms, we will call these Artin (resp. Deligne–Mumford) \( n \)-hypergroupoids.

Definition 3.3. Given a simplicial presheaf \( Y \) on \( \text{Aff}^\text{opp} \) (such as any simplicial algebraic space), define a relative \((n, P)\)-hypergroupoid over \( Y \) to be a morphism \( X_\bullet \to Y_\bullet \) of simplicial presheaves, satisfying the following:

1. for all \( m \), the morphisms \( f : X_m \to Y_m \) of simplicial presheaves are disjoint unions of affine morphisms over \( Y \);
2. the partial matching maps
\[
X_m \to M^m(X) \times_{M^m Y} Y_m
\]
are representable by \( P \)-coverings for all \( k, m \), and are isomorphisms for all \( m > n \) and all \( k \).

Definition 3.4. Given a simplicial presheaf \( Y \) on \( \text{Aff}^\text{opp} \) (such as any simplicial algebraic space), define a trivial relative \((n, P)\)-hypergroupoid over \( Y \) to be a morphism \( X \to Y \) of simplicial presheaves, satisfying the following:

1. for all \( m \), the morphism \( f : X_m \to Y_m \) of simplicial presheaves is a disjoint union of affine morphisms over \( Y \);
2. the matching maps
\[
X_m \to M_{\partial \Delta^m}(X) \times_{M_{\partial \Delta^m} Y} Y_m
\]
are representable by \( P \)-coverings for all \( m \), and are isomorphisms for all \( m \geq n \).

Remark 3.5. Note that trivial relative \((n, P)\)-hypergroupoids correspond to the truncated hypercovers considered in [Bek].
Definition 3.6. We say that a morphism $X \to Y$ of simplicial presheaves is strongly quasi-compact if each map $f_n : X_n \to Y_n$ is representable and quasi-compact. In particular, an $(n, P)$-hypergroupoid $X$ is quasi-compact if and only if each $X_n$ is an affine scheme.

Lemma 3.7. Any composition of (strongly quasi-compact) relative $(n, P)$-hypergroupoids is a (strongly quasi-compact) relative $(n, P)$-hypergroupoid.

Proof. This follows immediately by verifying the axioms. □

Lemma 3.8. If $f : X \to Y$ is a relative $(n, P)$-hypergroupoid, then $\text{Dec}_+(X) \to \text{Dec}_+(Y) \times_Y X$ is a relative $(n - 1, P)$-hypergroupoid, and $\text{Dec}_+(X) \to \text{Dec}_+(Y) \times_Y X_0$ is a trivial relative $(n, P)$-hypergroupoid.

Proof. The proofs of Corollary 2.24 and Lemma 2.26 carry over to this context. □

Lemma 3.9. If $f : X \to Y$ is a relative $(n, P)$-hypergroupoid, and $g : K \to L$ a cofibration of finite simplicial sets, then the map

$$F : X_L \to Y_L \times_Y K \times X_L$$

is a relative $(n, P)$-hypergroupoid, which is trivial if either $f$ or $g$ is trivial. Moreover, if $\text{sk}_{n-1} g : \text{sk}_{n-1} K \to \text{sk}_{n-1} L$ is an isomorphism, then $F$ is a relative $(0, P)$-hypergroupoid.

Proof. The proof of Lemma 2.18 carries over. □

Definition 3.10. Given a simplicial presheaf $F : \text{Aff}^{\text{opp}} \to \mathbb{S}$, define the sheafification $F^\sharp : \text{Aff}^{\text{opp}} \to \mathbb{S}$ as $Rj \circ a(F)$ in the notation of [TV2] §1.3.2. Sheaves in this sense correspond to stacks in the sense of [TV2] Definition 1.3.2.1, and if $F : \text{Aff}^{\text{opp}} \to \text{Set}$, then $F^\sharp$ is sheafification in the classical sense.

Lemma 3.11. If $f : X \to Y$, $g : Z \to Y$ are morphisms of simplicial presheaves on $\text{Aff}^{\text{opp}}$, such that every partial matching map

$$X_m \to M_{\Lambda^n_k} X$$

is representable by a covering, then we may describe the homotopy fibre product of the sheafifications by

$$X^\sharp \times_{Y^\sharp} Z^\sharp \simeq (X \times_Y Z)^\sharp$$

Proof. By [TV1] Theorem 4.6.1, a map $F \to G$ induces a weak equivalence $F^\sharp \to G^\sharp$ of the associated sheaves if and only if the maps $\pi_n F \to \pi_n G$ are local isomorphisms, i.e. the maps $(\pi_n F)^\sharp \to (\pi_n G)^\sharp$ are isomorphism.

Now, $(X \times^h_Y Z)^\sharp \simeq X^\sharp \times_{Y^\sharp} Z^\sharp$, so it suffices to show that the map

$$X \times_Y Z \to X \times_Y Z$$

is locally a weak equivalence.

Take an arbitrary affine scheme $U$, and an arbitrary point $x \to U$. We need to show that

$$\lim_{x \to U' \to U} \pi_n(X \times_Y Z)(U') \to \lim_{x \to U' \to U} \pi_n(X \times^h_Y Z)(U')$$

is an isomorphism, the limit being taken over all pointed $\tau$-morphisms $U' \to U$ (where $\tau$-morphisms are just flat maps when $\tau$ is the fpqc topology).

Now, homotopy groups commute with filtered colimits, so we just need to show that

$$\lim_{x \to U' \to U} (X \times_Y Z)(U') \to \lim_{x \to U' \to U} X(U') \times^h_{Y(U')} Z(U')$$

is a weak equivalence.

Now, observe that

$$\lim_{x \to U' \to U} X(U') \to \lim_{x \to U' \to U} Y(U')$$
is a fibration, since the partial matching maps are representable by $\tau$-coverings. Thus
\[
(\varprojlim_{x \to U' \to U} X(U')) \times (\varprojlim_{y \to Y(U')} (\varprojlim_{x \to U' \to U} Z(U'))) \cong (\varprojlim_{x \to U' \to U} X(U')) \times (\varprojlim_{y \to Y(U')} Z(U'))
\]
which gives the required result, since filtered colimits commute with homotopy fibre products.

Lemma 3.12. If $f : X \to Y$ is a trivial relative $(n, P)$-hypergroupoid, then $f^\sharp : X^\sharp \to Y^\sharp$ is an equivalence.

Proof. This follows because $f$ is locally a trivial fibration, so $f^\sharp$ is a trivial fibration, and hence a weak equivalence.

Proposition 3.13. If $f : X \to Y$ is a relative $(n, P)$-hypergroupoid, then the associated morphism
\[f^\sharp : X^\sharp \to Y^\sharp\]
of sheaves is an $n$-representable morphism in the sense of [TV2] Definition 1.3.3.1, which is strongly quasi-compact whenever $f$ is. In particular, if $Y$ is an affine scheme, then $X$ is an $n$-geometric stack.

If $f_0 : X_0 \to Y_0$ is also in $P$, then $f^\sharp$ is an $(n, P)$-morphism.

Proof. First, we reduce this to the case when $Y$ is an affine scheme. We need to show that $U \times_{Y^\sharp} X^\sharp$ is an $n$-geometric stack for any morphism $U \to Y^\sharp$ with $U$ an affine scheme. Since this property is local on $Y^\sharp$, we may replace $U$ by any affine covering $U' \to U$. Since the morphism $Y_0 \to Y^\sharp$ is locally surjective, there is such a covering $U' \to U$ for which $U' \to Y^\sharp$ factors through $U'' \to Y_0$. Now, Lemma 3.11 implies that
\[U' \times_{Y^\sharp} X^\sharp \cong (U' \times_Y X)^\sharp,\]
and we observe that $U' \times_Y X \to U'$ is a relative $(n, P)$-hypergroupoid, strongly quasi-compact whenever $f$ is, with $(U' \times_Y X)_{0} \to U'$ in $P$ whenever $X_0 \to Y_0$ is in $P$.

We now proceed by induction on $n$. For $n = 0$, the statements are immediate.

Now assume that the inductive hypothesis (for relative hypergroupoids) holds for $n - 1$, and take a relative $(n, P)$-hypergroupoid $X \to Y$, for $Y$ affine. Since $X$ is then an $(n, P)$-hypergroupoid, Lemma 3.8 implies that $\text{Dec}_+(X) \to X$ is a relative $(n - 1, P)$-hypergroupoid, with $\text{Dec}_+(X)_{0} \to X_0$ in $P$, so by induction $\text{Dec}_+(X)^\sharp \to X^\sharp$ is an $(n - 1, P)$-morphism. Moreover, Lemma 2.22 implies that $X_0 \cong \text{Dec}_+(X)^\sharp$, which means that we have a covering $(n - 1, P)$-morphism $X_0 \to X^\sharp$. Since $X_0$ is 0-geometric, Proposition 1.4 now implies that $X^\sharp$ is $n$-geometric. As $Y$ is assumed affine, this implies that $f^\sharp : X^\sharp \to Y$ is $n$-representable.

If $f$ is strongly quasi-compact, then both $X_0$ and the covering morphism are strongly quasi-compact, so $f^\sharp$ is also strongly quasi-compact. If $f_0 : X_0 \to Y$ is in $P$, then the $n$-atlas $X_0$ is a $P$-morphism over $Y$, so $X^\sharp \to Y$ is in $n$-$P$. This completes the induction.

Remark 3.14. If we took $P$ to be the class of smooth morphisms, and used algebraic spaces instead of disjoint unions of affine schemes in Definition 3.3, then Proposition 3.13 would adapt to give $n$-truncated (hence $(n + 2)$-representable) morphisms, similarly to Remark 1.8.

3.1. Quasi-compactness and colimits. In this section, we will show how to regard arbitrary relative $(n, P)$-hypergroupoids as filtered colimits of strongly quasi-compact relative $(n, P)$-hypergroupoids.

Definition 3.15. Recall that for a category $\mathcal{C}$, $\text{ind}(\mathcal{C})$ is the category of filtered direct systems $\{X_\alpha\}_{\alpha \in I}$ in $\mathcal{C}$, with morphisms defined by
\[\text{Hom}(\{X_\alpha\}, \{Y_\beta\}) = \varinjlim_{\alpha} \varprojlim_{\beta} \text{Hom}(X_\alpha, Y_\beta).\]
As in [AM] App. 3.2, every morphism in ind\(C\) is isomorphic to a level map, i.e. a morphism \(f : X \to Y\) for which \(X\) and \(Y\) have the same indexing category \(I\), with \(f : \{X_\alpha\}_{\alpha \in I} \to \{Y_\alpha\}_{\alpha \in I}\) consisting of compatible maps \(f_\alpha : X_\alpha \to Y_\alpha\).

**Definition 3.16.** A morphism in \(\text{Aff}\) is said to be a clopen immersion if it is simultaneously a closed immersion and an open immersion.

**Definition 3.17.** Given a simplicial diagram \(X_\bullet\) in a cocomplete category \(C\), recall from [GJ] §VII.1 that the \(n\)th latching object \(L_n X\) is defined to be \((\text{sk}_{n-1}X)_n\). Explicitly, this is the coequaliser
\[
\coprod_{i=0}^{n-1} \coprod_{j=0}^{i-1} X_{n-2} \xrightarrow{\alpha \beta} \coprod_{i=0}^{n-1} X_{n-1} \xrightarrow{\beta} L_n X,
\]
where for \(x \in X_{n-1}^{(i,j)}\), we define \(\alpha(x) = \sigma_j x \in X_n^{(i)}\), and \(\beta(x) = \sigma_{i-1} \in X_n^{(i)}\).

Note that there is a map \(L_n X \to X\) given by \(\sigma_i\) on \(X_n^{(i)}\).

The following lemma shows that we may regard \(s(\coprod \text{Aff})\) as a full subcategory of \(\text{ind}(\text{sAff})\).

**Lemma 3.18.** Given an object \(Z \in s(\coprod \text{Aff})\), there exists a filtered direct system \(W = \{W_\alpha\} \in \text{ind}(\text{sAff})\), with the transition maps \(W_\alpha \to W_{\alpha'}\) being clopen immersions, and
\[
Z \cong \lim_{\alpha} W_\alpha.
\]
Moreover, \(W\) is unique up to unique isomorphism.

**Proof.** Assume that we have constructed a filtered direct system \(\text{sk}_{n-1} W = \{(\text{sk}_{n-1} W_\alpha)\}_{\alpha \in I_{n-1}}\) of \((n-1)\)-truncated simplicial affine schemes, together with an isomorphism \(\lim_{\alpha} \text{sk}_{n-1} W_\alpha \to Z_{\leq n}\).

Let \((Z_n)_{(\alpha)} := Z_n \times_{M_n(Z)} M_n((\text{sk}_{n-1} W_\alpha))\); this is a disjoint union of affine schemes, so can be expressed as the colimit
\[
(Z_n)_{(\alpha)} = \lim_{\beta \in J(\alpha)} (W_n)_{\beta},
\]
of its finite sub-unions.

Since \(W_\alpha\) is affine, the latching object \(L_n W_\alpha\) exists (since arbitrary colimits exist in the category of affine schemes). For sufficiently large \(\beta\), the latching map \(L_n W_\alpha \to M_n W_\alpha \times_{M_n Z} Z_n\) lifts to \((W_n)_{\beta}\), since \(L_n W_\alpha\) is quasi-compact. We may therefore redefine \(J(\alpha)\) by requiring that this lift exists for all \(\beta \in J(\alpha)\).

Now let \(I_n\) be the filtered category of pairs \(\{(\alpha, \beta) : \alpha \in I_{n-1}, \beta \in J(\alpha)\}\); we have constructed
\[
\text{sk}_n W = \{(\text{sk}_n W)_{(\gamma)}\}_{\gamma \in I_n}
\]
satisfying the inductive hypothesis. That this is unique follows from the observation that \(\lim_{\to}\) is a full and faithful functor from \(\text{ind}(\text{sAff})\) to the category of simplicial schemes. □

**Proposition 3.19.** Given a relative (trivial) \((n, P)\)-hypergroupoid \(f : Z \to Y\) in \(s(\coprod \text{Aff})\), there exists a filtered direct system \(\{f_\alpha : Z_\alpha \to Y_\alpha\}_{\alpha \in I}\) of strongly quasi-compact relative (trivial) \((n, P)\)-hypergroupoids in \(\text{sAff}\), for which the transition maps \(Z_\alpha \to Z_{\alpha'}\) and \(Y_\alpha \to Y_{\alpha'}\) being clopen immersions, with \(f = \lim f_\alpha\). If \(f\) is levelwise quasi-compact, we may assume that the maps \(Z_\alpha \to Z_{\alpha'} \times_{Y_{\alpha'}} Y_\alpha\) are isomorphisms.

**Proof.** Lemma 3.18 allows us to write \(Y = \lim Y_\gamma\), for \(Y_\gamma \in \text{sAff}\). Replacing \(Z\) by \(Z \times_Y Y_\gamma\), we may therefore assume that \(Y \in \text{sAff}\).

We now construct \(W\) as in the proof of Lemma 3.18, but restrict to \(\beta \in J(\alpha)\) for which the relative partial matching maps (resp. relative matching maps)
\[
(W_\gamma)_{\beta} \to M_K W_\alpha \times_{M_K Y} Y_n
\]
are surjective or isomorphisms, as appropriate, noting that smoothness is automatically inherited from \( Z \). This is always possible, since the right-hand side is quasi-compact. \( \square \)

4. Resolutions

In this section, we establish the converse to Proposition 3.13. By way of motivation, we begin with the construction of a \((2, \mathcal{P})\)-hypergroupoid \( X \) resolving a 2-geometric stack \( \mathfrak{X} \).

Take a 2-atlas \( Z_0 \to \mathfrak{X} \), and a 1-atlas \( Z_1 \to Z_0 \times^h \mathfrak{X} Z_0 \). For some \( \mathcal{P} \)-morphism \( X_0 \to Z_0 \), there must exist a lift \( X_0 \to Z_1 \) of the diagonal map \( X_0 \to Z_0 \times^h \mathfrak{X} Z_0 \). Setting \( X_1 := Z_1 \times_{(Z_0 \times Z_0)} (X_0 \times X_0) \) gives a 1-truncated simplicial scheme \( X_{\leq 1} \) over \( \mathfrak{X} \), so we can take \( X \) to be the coskeleton

\[
X := \text{cosk}_1(X_{\leq 1}/\mathfrak{X}).
\]

Note that we cannot adapt the approach of [SD] to construct a split simplicial resolution, since that would entail taking \( Z_0 \) in level 0 and \( Z_1 \sqcup Z_0 \) in level 1. This fails because the diagonal map \( Z_0 \to Z_0 \times^h \mathfrak{X} Z_0 \) will seldom be a \( \mathcal{P} \)-morphism.

**Definition 4.1.** Given a set \( S \), define \( S^{\Delta^r} \in \mathcal{S} \) to be given by \( (S^{\Delta^r})_i = S^{\Delta^r_i} \).

**Lemma 4.2.** For \( K \in \mathcal{S} \), there is a natural isomorphism

\[
\text{Hom}_{\mathcal{S}}(K, S^{\Delta^r}) \cong S^{K^r}.
\]

Thus, for a morphism \( f : S \to T \) of sets, the \( i \)th relative matching map \( (S^{\Delta^r})_i \to M_i(S^{\Delta^r}) \times_{M_i(T^{\Delta^r})} (T^{\Delta^r})_i \) is

1. an isomorphism when \( i > r \);
2. a pullback of \( f \) when \( i = r \);
3. a pullback of several copies of \( f \) when \( i < r \).

**Proof.** The first statement follows because the functors \( K \mapsto \text{Hom}_{\mathcal{S}}(K, S^{\Delta^r}) \) and \( K \mapsto S^{K^r} \) are both colimit-preserving contravariant functors on \( \mathcal{S} \), and they agree on the objects \( \Delta^n \).

Thus we have canonical isomorphisms

\[
(S^{\Delta^r})_i \cong S^{\Delta^r_i} \cong S^{\partial \Delta^r_i} \times S^{\partial \Delta^r_i - \partial \Delta^r_i} \cong M_i(S^{\Delta^r}) \times S^{\partial \Delta^r_i - \partial \Delta^r_i},
\]

giving the required isomorphisms and pullbacks. \( \square \)

**Definition 4.3.** Let \( \Delta^+ \) be the subcategory of the ordinal number category \( \Delta \) containing only injective morphisms, and let \( \Delta_{n+} = \Delta^+ \cap \Delta_n \).

**Definition 4.4.** Given an object \( Z_\bullet \) in the homotopy category \( \text{Ho}((s \text{Pr}(\text{Aff}))^{\Delta^{\text{opp}}}) \) of \( \Delta^{\text{opp}} \)-diagrams (i.e. simplicial objects) in the model category \( s \text{Pr}(\text{Aff}) \) (with weak equivalences defined levelwise), define homotopy matching objects \( M^h_r Z \in \text{Ho}(s \text{Pr}(\text{Aff})) \) by

\[
M^h_r Z = \text{holim}_{(\Delta^+_r)^{\text{opp}}} Z^\Delta, \quad \text{lim}_{(\Delta^+_r)^{\text{opp}}} Z^\Delta,
\]

where \( (Z^\Delta)_i := Z^{\Delta^r_i} \), and we write \( \mathfrak{X}^K := \text{RHom}_{\mathcal{S}}(K, \mathfrak{X}) \) for \( \mathfrak{X} \in \text{Ho}(s \text{Pr}(\text{Aff})) \).

Note that if each \( Z_i \) is a scheme, then \( M^h_r Z \) is the usual simplicial matching object \( M_r Z \). If the simplicial structure on \( Z_\bullet \) is constant (so \( Z_i = Z \in \text{Ho}(s \text{Pr}(\text{Aff})) \) for all \( i \)), then \( M^h_r Z = Z^{\partial \Delta^n} \).

**Definition 4.5.** Given \( Z_\bullet \in \text{Ho}((s \text{Pr}(\text{Aff}))^{\Delta^{\text{opp}}}) \), let \( a_r(Z) \in \mathbb{N}_0 \) be the smallest number for which the homotopy matching morphism

\[
Z_r \to M^h_r Z
\]

is \( a_r(Z) \)-representable, whenever this is defined.
If the numbers $a_i(Z)$ are all defined and are eventually 0, define

$$\nu(Z) := \sum_{i \geq 0} 2^i (a_i(Z))_{\neq 0}.$$  

**Proposition 4.6.** Given $Z \in \text{Ho}((s \text{Pr(Aff)})^\Delta_{\text{opp}})$ for which $\nu(Z)$ is defined and $\nu(Z) < 2^n$, there exists a morphism $\tilde{Z} \to Z$ in $\text{Ho}((s \text{Pr(Aff)})^\Delta_{\text{opp}})$ with the following properties:

1. the relative homotopy matching maps
   $$\tilde{Z}_r \to (M^h_r \tilde{Z}) \times_{(M^h_r Z)} Z_r$$

   are covering $P$-morphisms for all $r$, and equivalences for all $r \geq n$;

2. each $\tilde{Z}_r$ is a 0-geometric stack.

If in addition there is a morphism $Z \to B$, with each $B_r$ 0-geometric, such that the maps $Z_r \to B_r$ are all strongly quasi-compact, then we may choose $\tilde{Z}$ such that the relative homotopy matching maps of $\tilde{Z} \to Z$ are strongly quasi-compact. The maps $\tilde{Z}_r \to B_r$ will also then be strongly quasi-compact.

**Proof.** We prove this by induction on $\nu(Z)$. If $\nu(Z) = 0$, then the matching maps are all 0-representable. Since 0-representability of the objects $Z_i$ for $i < r$ implies 0-representability of $M_r Z$, we deduce that each $Z_r$ is 0-representable, so we may set $\tilde{Z} = Z$.

Now assume that the result holds for all $Y$ with $\nu(Y) < \nu(Z)$. Let $r$ be the smallest number for which $a_r(Z) \neq 0$; since $\nu(Z) < 2^n$, we know that $r < n$. The stacks $Z_i$ are all 0-representable for $i < r$, so $M_r Z$ is 0-representable. As $Z_r \to M_r Z$ is 0-representable, we know that $Z_r$ is an $a_r(Z)$-geometric stack.

Let $f : S \to Z_r$ be an $a_r(Z)$-atlas for $Z_r$, observe that Lemma 4.1 gives a canonical morphism $Z \to Z_{r, S}^\Delta$, and let

$$Z' := Z \times_{Z_{r, S}^\Delta} S^\Delta.$$  

Since $f$ is a covering $(a_r(Z) - 1, P)$-morphism, Lemma 4.1 now implies that the $i$th relative homotopy matching map of $Z' \to Z$ is a covering $P$-morphism and is

1. an equivalence when $i > r$;
2. a pullback of $f$ when $i = r$.

As the homotopy matching maps of $Z'$ are simply the composition of these relative homotopy matching maps with a pullback of the matching maps for $Z$, the $r$th matching map is given by pulling back the composition

$$S \to Z_r \to M^h_r Z$$

along $M^h_r Z' \to M^h_r Z$. Since $S$ and $M^h_r Z$ are both 0-representable, it follows that this map is 0-representable, so $a_r(Z') = 0$. Likewise, $a_i(Z') \leq a_i(Z)$ for all $i > r$. Moreover, $a_i(Z')$ is defined for all $i$, since the relative matching homotopy maps are $P$-morphisms, so $k$-geometric for some $k$.

Thus $\nu(Z') < \nu(Z)$ (as $2^r > \sum_{i < r} 2^i$), so satisfies the inductive hypothesis, giving $\tilde{Z} \to Z'$ satisfying the conditions of the proposition. Now observe that the composition $\tilde{Z} \to Z' \to Z$ also satisfies these conditions, which completes the proof of the inductive step.

For the final part, note that the homotopy matching maps of a morphism $U \to V$ are strongly quasi-compact if and only if the maps $U_r \to V_r$ are all strongly quasi-compact. Thus we may choose each atlas $f : S \to Z_r$ in such a way that $S$ is strongly quasi-compact over the 0-geometric stack $M_r Z \times_{M_r B} B_r$, thereby making $f$ strongly quasi-compact. This ensures that the relative homotopy matching maps of $\tilde{Z} \to Z$ are strongly quasi-compact, so the matching maps of $\tilde{Z} \to B$ are also quasi-compact, which means that each $\tilde{Z}_r \to B_r$ is quasi-compact, as required.  \qed
Remark 4.7. Here is an explanation of why the induction has \(2^n - 1\) steps. If we let \(f(r)\) be the number of steps required to set \(a_i\) to 0 for all \(i < r\), then one further step sets \(a_r\) to 0, at the expense of the \(a_i\) becoming non-zero for \(i < r\). We therefore require \(f(r)\) further steps to get \(a_i = 0\) for all \(i < r + 1\). Thus we have the recurrence relation \(f(r + 1) = 2f(r) + 1\), with initial condition \(f(0) = 0\), so \(f(n) = 2^n - 1\).

Proposition 4.8. The simplicial diagram \(\bar{Z}\) of 0-geometric stacks constructed in Proposition 4.6 is represented by a simplicial diagram of disjoint unions of affine schemes, unique up to unique isomorphism. In the strongly quasi-compact case, \(\bar{Z}\) is represented by a simplicial affine scheme.

Proof. The functor \(\text{Aff} \to s\Pr(\text{Aff})\) is full and faithful. Since the individual objects \(\bar{Z}_r\) lift (essentially uniquely) to objects \(\bigsqcup\text{Aff}\), the \(\Delta^{\text{op}}\)-diagram \(\bar{Z}\) must also lift uniquely. If each \(\bar{Z}_r\) is strongly quasi-compact, their associated schemes are quasi-compact, hence affine, so give a simplicial affine scheme.

Alternatively, we could pass to the homotopy category at each level, with \(\bar{Z}\) inducing an object of \(\text{Ho}(s\Pr(\text{Aff}))^{\Delta^{\text{op}}}\). Since \(\text{Aff} \to \text{Ho}(s\Pr(\text{Aff}))\) is full and faithful, this determines an object of \(s(\bigsqcup\text{Aff})\).

\[\Box\]

Theorem 4.9. Given an \(n\)-geometric stack \(\mathcal{X}\), there exists an \((n, \mathbf{P})\)-hypergroupoid \(X\) whose sheafification \(X^\sharp\) is equivalent to \(\mathcal{X}\). Moreover, we may choose \(X\) to be strongly quasi-compact whenever \(\mathcal{X}\) is.

Proof. We apply Proposition 4.8 to the simplicial stack \(Z\) given by the constant object \(\mathcal{X}\). By Lemma 1.7, the homotopy matching maps

\[
\mathcal{X} \simeq Z_r \to M^h Z \simeq X^\partial \Delta^r
\]

are \((n - r)\)-geometric for \(r \leq n\), and 0-geometric for \(r > n\), so \(a_r(Z) \leq n - r\) for \(r \leq n\), and \(a_r(Z) = 0\) for \(r \geq n\). Thus \(\nu(Z)\) is defined and \(\nu(Z) \leq 2^n - 1\).

Proposition 4.8 now gives \(X \in s(\bigsqcup\text{Aff})\), such that the relative homotopy matching maps

\[
X_r \to (M_r X) \times^{\mathcal{H}}_{\mathcal{X}^\Delta^r} \mathcal{X}^\Delta^r
\]

are covering \(\mathbf{P}\)-morphisms for all \(r\), and equivalences for \(r \geq n\). If \(\mathcal{X}\) is strongly quasi-compact, then we may assume that each \(X_r\) is an affine scheme.

Since the relative homotopy matching maps are coverings, the map \(X \to \mathcal{X}\) of simplicial presheaves is locally a trivial fibration, so \(X^\sharp \to \mathcal{X}\) is a weak equivalence.

By considering compositions of pushouts, we see that for any cofibration \(K \hookrightarrow L\) of finite simplicial sets, the map

\[
M_L X \to M_K X \times^h_{\mathcal{X}^L} \mathcal{X}^L
\]

is a covering \(\mathbf{P}\)-morphism, and moreover an equivalence if \(\text{sk}_{n-1} K \cong \text{sk}_{n-1} L\). In particular,

\[
X_r \to (M^K_r X) \times^h_{\mathcal{X}^{\Delta^n}} \mathcal{X}^{\Delta^n}
\]

is a covering \(\mathbf{P}\)-morphism for all \(k, r\), and an equivalence for \(r > n\). Since the simplicial sets \(\Delta^r, \Delta^n\) are contractible, \(\mathcal{X} \simeq \mathcal{X}^{\Delta^r} \simeq \mathcal{X}^{\Delta^n}\), so

\[
M^K_r \times^h_{\mathcal{X}^{\Delta^r}} \mathcal{X}^{\Delta^n} = M^K_r X,
\]

and \(X\) is indeed an \((n, \mathbf{P})\)-hypergroupoid, as required. \(\Box\)

There is also the following relative version.

Theorem 4.10. Given \(Y \in s(\bigsqcup\text{Aff})\), and an \(n\)-representable morphism \(f : \mathcal{X} \to Y^\sharp\) to the sheafification of \(Y\), there exists a relative \((n, \mathbf{P})\)-hypergroupoid \(X \to Y\) whose sheafification \(X^\sharp\) is equivalent to \(\mathcal{X}\) over \(Y^\sharp\). Moreover, if \(f\) is a \(\mathbf{P}\)-morphism, then \(X_0 \to Y_0\) is in \(\mathbf{P}\). If \(Y\) and \(f\) are strongly quasi-compact, then we may choose \(X \to Y\) to be strongly quasi-compact.
Proof. We first let \( \mathcal{Y} := Y^\sharp \), and form the simplicial stack \( Z \) given by \( Z_r := X \times^h_\mathcal{Y} Y_r \). The \( r \)-th relative homotopy matching map of \( Z \to Y \) is just the pullback of that for \( X \to \mathcal{Y} \), so by Lemma 1.7, it is \((n-r)\)-representable for \( r \leq n \), and \( 0 \)-representable for \( r > n \). Since the matching maps of \( Y \) are \( 0 \)-representable (being morphisms of disjoint unions of affine schemes), it follows that \( a_r(Z) \leq n-r \) for \( r \leq n \), and \( a_r(Z) = 0 \) for \( r \geq n \). Thus \( \nu(Z) \) is defined and \( \nu(Z) \leq 2^n - 1 \).

Proposition 4.8 now gives \( X \in s(\prod \text{Aff}) \), such that the relative homotopy matching maps

\[
X_r \to (M_rX) \times^h_{M_rZ} Z_r
\]

are covering \( P \)-morphisms for all \( r \), and equivalences for \( r \geq n \). If \( Y \) and \( f \) are strongly quasi-compact, then we may take each \( X_r \) to be quasi-compact.

As in Theorem 4.9, this implies that \( X^\sharp \to (\text{diag} Z)^\sharp \) is a weak equivalence, where \( \text{diag} \) is the diagonal functor from bisimplicial sets to simplicial sets. Now,

\[
(\text{diag} Z)^\sharp \cong X \times^h_\mathcal{Y} Y^\sharp \cong X,
\]

so we have a weak equivalence \( X^\sharp \to X \).

For any cofibration \( K \hookrightarrow L \) of finite simplicial sets, the map

\[
M_LX \to M_KX \times^h_{M_KZ} M_LZ = (M_KX \times M_KY M_LY) \times^h_{\mathcal{Y}^h Z} \mathcal{Y}^L X^L
\]

is a covering \( P \)-morphism, and moreover an equivalence if \( \text{sk}_{n-1}K \cong \text{sk}_{n-1}L \). If \( K \) and \( L \) are contractible, this map just becomes

\[
M_LX \to M_KX \times M_KY M_LY.
\]

Arguing as in Theorem 4.9, we see that \( X \to Y \) is indeed an \((n,P)\)-hypergroupoid, and is strongly quasi-compact whenever \( Y \) and \( f \) are. Finally, if \( f \) is a \( P \)-morphism, then

\[
X^\sharp \times^h_{Y^\sharp} Y_0 \to Y_0
\]

is also a \( P \)-morphism. Since \( X_0 \to X^\sharp \times^h_{Y^\sharp} Y_0 \) is an \( n \)-atlas, it is a \( P \)-morphism, so \( X_0 \to Y_0 \) is in \( P \). \qed

5. Morphisms

Now that we have described objects in the category of \( n \)-geometric stacks, we need to describe the morphisms. Just looking at the familiar case of \( X \) an affine scheme and \( Y = BG \), for \( G \) a smooth affine group scheme, we have

\[
\text{Hom}_{\text{Aff}}(X,BG) = \text{Hom}_{\text{Aff}}(X,(BG)_0) = \text{Hom}_{\text{Aff}}(X,\text{Spec} Z) = \ast,
\]

the one-point set, whereas in the category of Artin stacks, \( \text{Hom}(X,(BG)^\sharp) \) is given by isomorphism classes in the groupoid of \( G \)-torsors on \( X \). Thus the sheafification functor from \((n,P)\)-hypergroupoids to \( n \)-geometric stacks is not full.

Lemma 5.1. Given a relative \((n,P)\)-hypergroupoid \( Y \to S \), the \( i \)-th relative homotopy matching map of

\[
Y \to S \times^h_{S^\sharp} Y^\sharp
\]

is a \( P \)-covering for all \( i \), and an equivalence for \( i \geq n \). If \( i \leq n \), it is \((n-i)\)-representable.

Proof. Let \( \mathcal{S} = S^\sharp, \mathcal{Y} = Y^\sharp \). The matching map is

\[
\mu_i : Y_i \to (M_iY \times^h_{\mathcal{Y}^h \Delta^i} \mathcal{Y}) \times^h_{M_iS \times^h \mathcal{S}^h \Delta^i} \mathcal{S}_i
\]

which is the sheafification of

\[
Y_i \to (M_iY \times_{Y^h \Delta^i} Y^\Delta^i) \times_{M_iS \times_{S^h \Delta^i} \mathcal{S}^h \Delta^i} \mathcal{S}_i.
\]
which is given in simplicial level 0 by
\[ Y_i \to (M_i Y \times_M Y Y_i) \times_{M_i S \times M_i S_i} S_i = Y_i \times_S S_i = Y_i. \]

This is certainly a covering \( P \)-morphism, so \( \mu_i \) is also a covering \( P \)-morphism by Proposition 3.13.

If \( i \geq n \), then Lemma 1.7 implies that \( \mu_i \) is 0-representable, and if \( i \leq n \), then it is \((n - i)\)-representable.

**Proposition 5.2.** Take a morphism \( X \to S \) in \( s(\coprod \text{Aff}) \) (for instance a relative \((m, P)\)-hypergroupoid), a relative \((n, P)\)-hypergroupoid \( Y \to S \), and a morphism
\[ f : X^2 \to Y^2 \]
in the homotopy category of simplicial presheaves over \( S^2 \).

Then there exists a trivial relative \((n, P)\)-hypergroupoid \( \pi : \tilde{X} \to X \), and a morphism \( \tilde{f} : \tilde{X} \to Y \) of simplicial schemes over \( S \), such that \( f \circ \pi^2 = \tilde{f}^2 \). Moreover, the map \( (\pi, \tilde{f}) : \tilde{X} \to X \times_S Y \) is a relative \((n, P)\)-hypergroupoid.

If \( Y \) is strongly quasi-compact over \( S \), then we may also take \( \pi : \tilde{X} \to X \) to be strongly quasi-compact.

**Proof.** Write \( \mathcal{S} := S^2 \), \( \mathcal{Y} := Y^2 \) and \( \mathcal{X} := X^2 \). Define the simplicial stack \( Z \) by
\[ Z_r := X_r \times^h_{(\mathcal{Y} \times^h_{\mathcal{S}} \mathcal{S}_r)} Y_r, \]
and observe that the relative homotopy matching maps of \( Z \to X \) are obtained by pulling back the relative homotopy matching maps of \( Y \to \mathcal{Y} \times^h_{\mathcal{S}} S \) along \( X \to \mathcal{Y} \).

Now by Lemma 5.1, the \( i \)-th such map is a covering \( P \)-morphism, which is an equivalence for \( i \geq n \), and \((n - i)\)-representable for \( i \leq n \). Since the matching maps of \( X \) are 0-geometric, we may apply Proposition 4.8 to \( Z \), obtaining \( \tilde{X} \in s(\coprod \text{Aff}) \), with the \( i \)-th relative homotopy matching map of \( \tilde{X} \to Z \) being a covering \( P \)-morphism for all \( i \), and an equivalence for \( i \geq n \). If \( Y \to S \) is strongly quasi-compact, then \( Z \) is strongly quasi-compact over \( X \), so we may also take \( \pi : \tilde{X} \to X \) to be strongly quasi-compact.

We therefore conclude that the \( i \)-th matching map of \( \pi : \tilde{X} \to X \) is a covering \( P \)-morphism for all \( i \), and an equivalence for \( i \geq n \), so \( \pi \) is a trivial relative \((n, P)\)-hypergroupoid. Projection \( Z \to Y \) on the second factor gives the map \( \tilde{f} : \tilde{X} \to Y \).

Finally, observe that \( Z \to X \times_S Y \) is a pullback of the \((n - 1)\)-geometric map \( \mathcal{Y} \to \mathcal{Y} \times^h_{\mathcal{S}} \mathcal{Y} \), so the relative partial matching maps of \( \tilde{X} \to X \times_S Y \) are covering \( P \)-morphisms, and are isomorphisms in levels above \( n \), as required.

**Proposition 5.3.** Take a morphism \( X \to S \) in \( s(\coprod \text{Aff}) \), a relative \((n, P)\)-hypergroupoid \( Y \to S \), and two morphisms
\[ f, g : X \to Y \]
over \( S \), such that \( f^2 = g^2 \) in the homotopy category of simplicial presheaves over \( S^2 \).

Then there exists a trivial relative \((n, P)\)-hypergroupoid \( \pi : \tilde{X} \to X \), and a morphism \( h : \tilde{X} \to Y^{\Delta^1} \times_{S^{\Delta^1}} S \) of simplicial schemes over \( S \), such that \( ev_0 \circ h = f \) and \( ev_1 \circ h = g \).

If \( Y \) is strongly quasi-compact over \( S \), then we may also take \( \pi : \tilde{X} \to X \) to be strongly quasi-compact.

**Proof.** Let \( \mathcal{Y} = Y^2 \), \( \mathcal{S} = S^2 \), and define \( W \in \text{Ho}(s \text{Pr}(\text{Aff})^{\Delta \text{op}}) \) by
\[ W_r := (Y_r \times_S Y_r) \times^h_{(\mathcal{Y} \times^h_{\mathcal{S}} \mathcal{S}_r)} \mathcal{Y} \cong (Y_r \times^h_{\mathcal{Y}} Y_r) \times_{(S \times^h_{\mathcal{S}} S_r)} S_r. \]

Let \( PY = Y^{\Delta^1} \times_{S^{\Delta^1}} S \); there is a natural map \( PY \to W \), since \( W \cong (Y \times_S Y) \times (Y \times_S Y)^{\Delta^1} \) \( (PY)^{\Delta^1} \). Now, the map \( PY \to Y \times_S Y \) is a relative \((n, P)\)-hypergroupoid (by Lemma 3.9), so we may apply Lemma 5.1 to see that the \( i \)-th matching map of \( PY \to W \) is a \( P \)-morphism, and is an equivalence if \( i \geq n \).
By hypothesis, there is a natural map $X \to W$ whose projection on $Y \times S$ is given by $(f, g)$, so we may set $Z = X \times^W_{\bot} PY$. Since the matching maps of $X$ are 0-representable, it follows that $Z$ satisfies the conditions of Proposition 4.8, so we have a $\tilde{X} \in s(\mathbb{I} \Aff)$, with the $i$th relative homotopy matching map of $\tilde{X} \to Z$ being a covering $P$-morphism for all $i$, and an equivalence for $i \geq n$. Since the map $Z \to X$ is a pullback of $PY \to W$, the same conditions hold for the composition $\pi : \tilde{X} \to X$. In other words, $\pi$ is a trivial relative $(n, P)$-hypergroupoid.

Now, we just let $h : \tilde{X} \to PY$ be the composite of the maps $\tilde{X} \to Z \to PY$, and observe that this has the required properties.

If $Y \to S$ is strongly quasi-compact, then $PY \to W$, and hence $Z \to X$, are also strongly quasi-compact. We may therefore take $\pi$ to be strongly quasi-compact. □

**Definition 5.4.** Given simplicial schemes $X, Y$ over $S \in s(\mathbb{I} \Aff)$, let $\text{Hom}_{s(\mathbb{I} \Aff)S}(X, Y)_n := \text{Hom}_{s(\mathbb{I} \Aff)S}(X, Y_{\Delta^n} \times_{S \Delta^n} S)$. $\text{Hom}_{s(\mathbb{I} \Aff)S}(X, Y)_n := \text{Hom}_{s(\mathbb{I} \Aff)S}(X, Y_{\Delta^n} \times_{S \Delta^n} S)$. $\text{Hom}_{s(\mathbb{I} \Aff)S}(X, Y)_n := \text{Hom}_{s(\mathbb{I} \Aff)S}(X, Y_{\Delta^n} \times_{S \Delta^n} S)$.

**Definition 5.5.** Define the category $TP_n(X)$ to be the full subcategory of $s(\mathbb{I} \Aff) \downarrow X$ consisting of trivial $(n, P)$-hypergroupoids over $X$.

If morphisms in $P$ are finitely presented, observe that $TP_n(X)$ is equivalent to a small category (i.e. one whose objects form a set), so it makes sense to take limits indexed by $TP_n(X)$. This covers all our motivating cases except for fpqc stacks, but the following formulae will still make sense, as sheafification for the fpqc site has to land in a larger universe.

**Corollary 5.6.** Take a morphism $X \to S$ in $s(\mathbb{I} \Aff)$ and a relative $(n, P)$-hypergroupoid $Y \to S$. Then homomorphisms in the homotopy category of simplicial presheaves over $S^2$ are given by

$$\text{Hom}_{\text{Ho}(s \text{Pr}(\mathbb{I} \Aff)S^2)}(X^2, Y^2) \cong \pi_0(\lim_{\tilde{X} \in TP_n(X)} \text{Hom}_{s(\mathbb{I} \Aff)S}(\tilde{X}, Y))$$

If $X \to S$ is strongly quasi-compact, then we may restrict to strongly quasi-compact $\tilde{X} \to X$.

In particular, the homotopy category of $n$-geometric stacks over $S^2$ is obtained by localising the category of relative $(n, P)$-hypergroupoids over $S$ at the class of trivial relative $(n, P)$-hypergroupoids.

**Proof.** By Lemma 3.12, trivial relative $(n, P)$-hypergroupoids map to equivalences, so there is a natural map

$$\phi : \pi_0(\lim_{\tilde{X} \in TP_n(X)} \text{Hom}_{s(\mathbb{I} \Aff)S}(\tilde{X}, Y)) \to \text{Hom}_{\text{Ho}(s \text{Pr}(\mathbb{I} \Aff)S^2)}(X^2, Y^2),$$

which is surjective on by Proposition 5.2. By Proposition 5.3, $\phi$ is also injective, hence an isomorphism.

Finally, observe that sheafification defines a functor from the localised category of $(n, P)$-hypergroupoids to the category of $n$-geometric stacks. Proposition 4.10 implies that this functor is essentially surjective, and the calculation above shows that it is full and faithful. This is because the maps $Y_{\Delta^1} \times_{S \Delta^1} S \to Y$ are trivial relative $(n, P)$-hypergroupoids, by Lemma 3.9.

**Remarks 5.7.** (1) Since $Y \to S$ is an $n$-hypergroupoid (adapting Lemma 2.14), $Y = \cosk_{n+1}(Y/S)$, so $\text{Hom}_S(U, Y) = \text{Hom}_S(U_{\leq n+1}, Y_{\leq n+1})$.

(2) Observe that in this categorisation of the category of $n$-geometric stacks, the topology $\tau$ does not feature. Thus we obtain the same category for any other topology $\tau'$ with respect to which the class of $P$-morphisms is local, and with the same notion of surjectivity for a $P$-morphism.
(3) The properties here strongly resemble a simplicial model structure, but as \( s(\mathbb{I} \text{Aff}) \) does not contain arbitrary coequalisers, it cannot be a closed model category. However, if we are willing to restrict attention to strongly quasi-compact \( n \)-geometric stacks, then it seems likely that there is a simplicial model structure on simplicial affine schemes, with (trivial) fibrations cogenerated by finitely presented (trivial) relative \((n, \mathcal{P})\)-hypergroupoids. Then \( n \)-geometric stacks would just correspond to fibrant objects satisfying some finiteness conditions, and cofibrant approximation would send \( X \) to \( \varprojlim_{X \in 1} \tilde{X} \).

(4) These results suggest that the notion of \((n, \mathcal{P})\)-hypergroupoids could be generalised to far more general contexts. For instance, Zhu’s Lie \( n \)-groupoids ([Zhu]) are obtained by replacing affine schemes with differential manifolds, and taking \( \mathcal{P} \)-coverings to be surjective submersions. Replacing rings with \( C^\infty \)-rings would give an alternative model for differential stacks. Since \( C^\infty \)-rings cannot (as far as the author is aware) be realised as commutative monoids in any abelian category, this is a scenario for which there is no suitable HAG context in the sense of [TV2].

**Example 5.8.** If we take \( X \) to be an affine scheme, then a trivial relative Artin 1-hypergroupoid \( \tilde{X} \to X \) is just given by \( \tilde{X} = \cosk_0(X'/X) \), for a smooth surjection \( X' \to X \), with \( X' := \tilde{X}_0 \).

If \( Y = BG \), for \( G \) a smooth affine group scheme, then an element of \( \text{Hom}(\tilde{X}, BG) \) is a \( G \)-torsor \( P \) on \( X \) equipped with a trivialisation \( \theta : P \times_X X' \cong G \times X' \), while an element of \( \text{Hom}(\tilde{X}, (BG)^\Delta^1) \) consists of two such pairs \((P_1, \theta_1), (P_2, \theta_2)\), and an isomorphism \( g \in G(X') \) between them.

Thus \( \pi_0(\varprojlim \text{Hom}(\tilde{X}, BG)) \) consists of isomorphism classes of \( G \)-torsors on \( X \), as expected.

**Lemma 5.9.** In the scenario of Corollary 5.6, the image of \( TP_n(X) \) under the functor \( \pi_0\text{Hom}_{s(\mathbb{I} \text{Aff})\mathcal{S}}(-, Y) \) is a filtered direct category. 

Thus the image of \( \text{Hom}_{s(\mathbb{I} \text{Aff})\mathcal{S}}(-, Y) \to \text{Ho}(\mathcal{S}) \) is a filtered direct category.

**Proof.** This is similar to [Fri] Proposition 3.1. We need to show that for any pair \( f, g : U \to V \) of arrows in \( TP_n(X) \), there exists \( h : T \to U \), with \( fh \) and \( gh \) mapping to the same morphism. Let \( T = U \times_{V \times V} V^\Delta^1 \times_X X \); this reduces the question to the case \( U = V^\Delta^1 \times_X X \).

Given a map \( V \to Y \) over \( S \), there is an associated map \( V^\Delta^1 \times_X \mathcal{S} \to Y^\Delta^1 \times_{\mathcal{S}^\Delta^1} \mathcal{S} \), giving the required homotopy between \( fh \) and \( gh \).

The final statement is just the observation that \( \text{Hom}_{s(\mathbb{I} \text{Aff})\mathcal{S}}(U, Y) \) is determined by the property that for all finite simplicial sets \( K \),

\[
\text{Hom}_{\text{Ho}(\mathcal{S})}(K, \text{Hom}_{s(\mathbb{I} \text{Aff})\mathcal{S}}(U, Y)) = \pi_0(\text{Hom}_{s(\mathbb{I} \text{Aff})\mathcal{S}}(U, Y)^K) = \pi_0\text{Hom}_{s(\mathbb{I} \text{Aff})\mathcal{S}}(U, Y^K \times_{\mathcal{S}^K} \mathcal{S}).
\]

This implies that we can also recover derived \( \text{Hom} \)-spaces \( \mathcal{R}\text{Hom}(X^2, Y^2) \subset \text{Ho}(\mathcal{S}) \) from \( X \) and \( Y \):

**Corollary 5.10.**

\[
\mathcal{R}\text{Hom}_{\mathcal{S} \text{Pr}(\text{Aff})\mathcal{S}^2}(X^2, Y^2) \simeq \varprojlim_{X \in TP_n(X)^{\text{op}}} \text{Hom}_{\mathcal{S}\text{Sch}\mathcal{S}}(\tilde{X}, Y),
\]

where \( \mathcal{S}\text{Sch} \) is the category of simplicial schemes.
Proof. For all finite simplicial sets \( K \),
\[
\text{Hom}_{\text{Ho}(\mathcal{S})}(K, \text{RHom}_{\text{Pr(Aff)}}(X^2, Y^2)) = \pi_0\text{Hom}_{\text{Ho}(\text{Pr(Aff))}}(X^2, (Y^2)_K \times (S^2)_K)
\]
\[
= \pi_0\lim_{TP_n(X)^{\text{op}}} \underline{\text{Hom}}_{\mathcal{S}h\mathcal{S}}(\tilde{X}, Y^K \times S^K)
\]
\[
= \pi_0\lim_{TP_n(X)^{\text{op}}} \underline{\text{Hom}}_{\mathcal{S}h\mathcal{S}}((\tilde{X}, Y)_K).
\]
Since the image category \( \mathcal{J} \) of \( TP_n(X) \) is a filtered category, we have
\[
\lim_{TP_n(X)^{\text{op}}} \underline{\text{Hom}}_{\mathcal{S}h\mathcal{S}}(\tilde{X}, Y)_K = \lim_{\mathcal{J}} \underline{\text{Hom}}_{\mathcal{S}h\mathcal{S}}(\tilde{X}, Y)_K
\]
\[
= (\lim_{TP_n(X)^{\text{op}}} \underline{\text{Hom}}_{\mathcal{S}h\mathcal{S}}(\tilde{X}, Y))_K,
\]
since filtered colimits commute with finite limits. \( \square \)

5.1. \( \acute{E} \text{tale hypercoverings} \). When \( P \) is the class of smooth morphisms and \( X \) a disjoint union of affine schemes, Corollary 5.6 still seems inadequate, since we would expect to be able to define the sheafification by considering just \( \acute{E} \text{tale} \) hypercoverings, rather than smooth hypercoverings. We will now show how to establish this in a more general setting.

Let \( E \subset P \) be a class of morphisms of affine schemes also satisfying the conditions of [TV2] Assumption 1.3.2.11 (as listed in §3), together with the following conditions:

E1 for all covering morphisms \( X \to Y \) in \( P \), there exists a morphism \( W \to X \), for which the composition \( W \to Y \) is a covering morphism in \( E \);

E2 for all diagrams \( Z \to X \to Y \), with \( g \) in \( P \) and \( gf \) a closed immersion, there exists a factorisation \( Z \to W \to X \) of \( f \), for which \( g\beta \) is in \( E \).

The usual case to consider will be when \( P \) is the class of smooth morphisms, and \( E \) the class of \( \acute{E} \text{tale} \) morphisms.

Lemma 5.11. If \( P \) is the class of smooth morphisms, then the class of \( \acute{E} \text{tale} \) morphisms satisfies the conditions for \( E \) above.

Proof. Condition (E1) follows from [Gro1] 17.16.3, so it only remains to prove (E2). Since \( Z \to Y \) is a closed immersion, we may form the henselisation \( Z \to Y^h \to Y \), as in [Gre2]; then \( (Z, Y^h) \) is a henselian pair (as in [Laf]), and \( Y^h \to Y \) is a filtered inverse limit of \( \acute{E} \text{tale} \) morphisms.

Now, [Gru] Theorem I.8 shows that the map \( Z \to X \) extends to \( Y^h \to X \) over \( Y \), since \( g \) is smooth. Finally, since \( g \) is finitely presented, and \( f' \) is pro-\( \acute{E} \text{tale} \), the map \( Y^h \to X \) factors through some quotient \( W \) with \( W \to Y \) \( \acute{E} \text{tale} \). This has all the required properties. \( \square \)

Proposition 5.12. Given a diagram \( Z \to X \to Y \) in \( s(\prod \text{Aff}) \), with \( g \) a trivial relative \( (n, P) \)-hypergroupoid and \( gf \) a levelwise closed immersion, there exists a factorisation \( Z \to W \to X \) of \( f \), for which \( g\beta \) is a trivial relative \( (n, E) \)-hypergroupoid. If \( g \) is strongly quasi-compact, then we may take \( g\beta \) to be so.

Proof. By Proposition 3.19, we may restrict to diagrams in \( s\text{Aff} \). The construction of \( W \) is inductive. For \( r < n \), assume that we have constructed an \( (r-1) \)-truncated simplicial affine scheme \( sk_{r-1}W \), with \( sk_{r-1}Z \to sk_{r-1}W \to sk_{r-1}X \) satisfying the required properties up to level \( r \).

Since we are working with affine schemes, all colimits exist, including latching objects, and we now seek \( W_r \) fitting into the diagram
\[
L_r W \cup_{L_rZ} Z_r \to W_r \to M_r W \times_{M_rX} X_r,
\]
with the relative matching map \( W_r \to M_r W \times_{M_rY} Y_r \) a covering \( E \)-morphism.
Since \( X_r \to M_rX \times_{M_rY} Y_r \) is a covering \( P \)-morphism, pulling back along \( M_rW \to M_rX \) gives a covering \( P \)-morphism \( M_rW \times_{M_rX} X_r \to M_rW \times_{M_rY} Y_r \). Condition (E1) now provides a map \( S \to M_rW \times_{M_rX} X_r \), with the composition \( S \to M_rW \times_{M_rY} Y_r \) a covering \( E \)-morphism.

Since the relative matching map of a surjection of cosimplicial abelian groups is abelian, the same is true for cosimplicial rings. Thus the latching maps of a levelwise closed immersion of simplicial affine schemes are all closed immersions. In particular, \( L_rW \cup_{L_rZ} Z_r \to M_rW \times_{M_rY} Y_r \) is a closed immersion, so condition (E2) provides a factorisation \( L_rW \cup_{L_rZ} Z_r \to T \to M_rW \times_{M_rY} Y_r \), with the composition \( T \to M_rW \times_{M_rY} Y_r \) an \( E \)-morphism.

Setting \( W_r = S \cup T \) completes the inductive step. Having constructed \( sk_{n-1}W \), we set \( W = \cosk_{n-1}(sk_{n-1}W) \times_{\cosk_{n-1}Y} Y \).

**Corollary 5.13.** Take a morphism \( X \to S \) in \( s([\square] \text{Aff}) \) (for instance a relative \((m, P)\)-hypergroupoid), a relative \((n, P)\)-hypergroupoid \( Y \to S \), and a morphism

\[
f : X^\sharp \to Y^\sharp
\]

in the homotopy category of simplicial presheaves over \( S^\sharp \).

Then there exists a trivial relative \((n, E)\)-hypergroupoid \( \pi : \tilde{X} \to X \), and a morphism \( \tilde{f} : \tilde{X} \to Y \) of simplicial schemes over \( S \), such that \( f \circ \pi^\sharp = \tilde{f}^\sharp \). Moreover, the map \((\pi, \tilde{f}) : \tilde{X} \to X \times_S Y\) is a relative \((n, P)\)-hypergroupoid.

Thus we may define the direct limits in Proposition 5.6 and Corollary 5.10 over trivial relative \((n, E)\)-hypergroupoids only.

If \( Y \) is strongly quasi-compact over \( S \), then we may also take \( \pi : \tilde{X} \to X \) to be strongly quasi-compact.

**Proof.** Combine Propositions 5.12 and 5.2. \( \square \)

**Remark 5.14.** Further to Remark 5.7.(2), this implies that we obtain the same description of Artin \( n \)-stacks regardless of whether descent is with respect to flat hypercovers or just étale hypercovers. This accounts for the main discrepancy in definition between [TV2] and [Lur1].

5.2. The \( \infty \)-category of \( n \)-geometric Artin stacks.

**Definition 5.15.** Define the category \( TE_n(X) \) (resp. \( TE_n(X)^{sq} \)) to be the full subcategory of \( s([\square] \text{Aff}) \downarrow X \) (resp. \( s\text{Aff} \downarrow X \)) consisting of trivial Deligne–Mumford \( n \)-hypergroupoids over \( X \).

Summarising the main results so far in the case when \( P \) is the class of smooth surjections gives the following.

**Theorem 5.16.** The \( \infty \)-category of \( n \)-geometric Artin stacks is weakly equivalent to the \( \infty \)-category \( G_n \) whose objects are Artin \( n \)-hypergroupoids, with morphisms given by

\[
\text{Hom}_{G_n}(X, Y) = \lim_{\tilde{X} \in TE_n(X)^{\text{opp}}} \text{Hom}_{s\text{Sch}S}^{\text{opp}}(\tilde{X}, Y),
\]

where \( s\text{Sch} \) is the category of simplicial schemes.

The \( \infty \)-category of strongly quasi-compact \( n \)-geometric Artin stacks is weakly equivalent to the \( \infty \)-category \( G_n^{sq} \) whose objects are strongly quasi-compact Artin \( n \)-hypergroupoids, with morphisms given by

\[
\text{Hom}_{G_n^{sq}}(X, Y) = \lim_{\tilde{X} \in (TE_n(X)^{sq})^{\text{opp}}} \text{Hom}_{s\text{Sch}S}^{\text{opp}}(\tilde{X}, Y).
\]
Proof. The functor from $G_n$ to $n$-geometric Artin stacks is given by $X \mapsto X^2$, where
\[ X^2(A) := \text{Hom}_{G_n}(\text{Spec } A, X) \]
for rings $A$. Essential surjectivity of this functor is given by Theorem 4.9, while full faithfulness is Corollary 5.10, as modified in Corollary 5.13. \qed

6. Quasi-coherent sheaves

The sheaves with which we are primarily concerned are quasi-coherent modules. Theorem 4.9 would allow us to apply the results of [Ols1] concerning sheaves on simplicial algebraic spaces, thus extending their consequences from Artin 1-stacks to higher stacks. However, the use of affine schemes rather than algebraic spaces yields many simplifications. It is easier to define quasi-coherent modules as presheaves than as sheaves. This becomes especially important when we work with quasi-coherent complexes, since the definition of a hypersheaf valued in an $\infty$-category (as in [Lur1] §4.1) or equivalently a stack valued in a model category ([TV1] Definition 4.6.5) is more complicated, involving hyperdescent rather than descent.

In this section, we assume that all morphisms in $P$ are flat, and that all $P$-coverings are faithfully flat. These assumptions hold for all the standard contexts.

6.1. Quasi-coherent modules.

Definition 6.1. Given a simplicial site $X_\bullet$, define a presheaf $P$ on $X_\bullet$ to consist of presheaves $P|_{X_n}$ on $X_n$, equipped with operations $\partial^i : \partial^{-1}_i(P|_{X_{n-1}}) \to P|_{X_n}$ and $\sigma^i : \sigma^{-1}_i(P|_{X_{n+1}}) \to P|_{X_n}$, satisfying the usual cosimplicial identities.

Say that a presheaf $P$ is Cartesian if the maps $\partial^i, \sigma^i$ are all isomorphisms.

Remarks 6.2. Note that it suffices to verify that the maps $\partial^i$ are isomorphisms, since $\sigma^i$ has right inverse $\sigma_{is} \partial^i$.

Definition 6.1 is essentially the same definition as [Del] 5.1.6. Thus we may also describe presheaves on $X_\bullet$ as presheaves on the site of pairs $(n, U)$, for $n \in \Delta^{op}$ and $U \to X_n$.

Giving a Cartesian presheaf $P$ is equivalent to just giving the presheaf $\mathcal{G} = P|_{X_0}$ together with the descent datum $\omega := (\partial^1_0)^{-1} \omega \partial^0_0 : \partial^{-1}_0 \mathcal{G} \to \partial^{-1}_1 \mathcal{G}$, satisfying the conditions $\sigma^0 \omega = \text{id}, (\partial^2_1 \omega) \circ (\partial^0_0 \omega) = \partial^1_1 \omega$.

Definition 6.3. Given a geometric stack $X$, define $\text{Aff}_P(X)$ to be the $\infty$-category whose objects are $P$-morphisms (in the sense of Definition 1.2) $U \to X$ in the $\infty$-category of stacks, for $U$ affine, and whose morphisms are all morphisms $V \to U$ over $X$. This has a Grothendieck topology whose basis is given by covering families of $P$-morphisms.

Definition 6.4. Define the presheaf $\mathcal{O}_X$ on $\text{Aff}_P(X)$ by $\mathcal{O}_X(\text{Spec } A) := A$. An $\mathcal{O}_X$-module $\mathcal{M}$ of presheaves on $\text{Aff}_P(X)$ is said to be quasi-coherent if the transition maps $\mathcal{M}(U) \otimes_{\mathcal{O}_X(U)} \mathcal{O}_X(V) \to \mathcal{M}(V)$ are isomorphisms for all morphisms $V \to U$ in $\text{Aff}_P(X)$.

Proposition 6.5. Given an $(n, P)$-hypergroupoid $X_\bullet$, the category of quasi-coherent modules over $X_\bullet$ is equivalent to the category of Cartesian quasi-coherent $\mathcal{O}_X$-modules on the simplicial site $n \mapsto \text{Aff}_P(X_n)$.

Proof. Write $X := X^2$. Given a quasi-coherent module $\mathcal{M}$ on $X$, we define a presheaf $a^{-1}\mathcal{M}$ on $X_\bullet$ by $(a^{-1}\mathcal{M})|_{X_n} := a_n^{-1}\mathcal{M}$, for $a_n : X_n \to X^2$ the canonical map (which is a $P$-morphism). It follows immediately that $(a^{-1}\mathcal{M})|_{X_n}$ is quasi-coherent, and that $a^{-1}\mathcal{M}$ is Cartesian.

Conversely, given a Cartesian quasi-coherent module $\mathcal{N}$ on $X_\bullet$, define $a_*\mathcal{N}$ as follows. If $f : U \to X$ is in $\text{Aff}_P(X)$, then it is an $r - P$-morphism for some $r$, so Theorem 4.9 provides a relative $(r, P)$-hypergroupoid $\tilde{f} : \tilde{U}_\bullet \to X$, with $f_0 : \tilde{U}_0 \to X_0$ a $P$-morphism,
and }\pi: \bar{U}^2 \to U\text{ a trivial relative } (n, P)\text{-hypergroupoid. Thus the morphisms }\tilde{f}_n : \bar{U}_n \to X_n\text{ are all } P\text{-morphisms, so we may define }\tilde{f}^{-1}\mathcal{N} \text{ on } \bar{U}_\bullet.

Now, define the quasi-coherent } \mathcal{O}_{\bar{U}}\text{-module }\pi_*\tilde{f}^{-1}\mathcal{N} \text{ on } U \text{ as the equaliser

\[ \pi_*\tilde{f}^{-1}\mathcal{N} \xrightarrow{\pi_0(\tilde{f}^{-1}\mathcal{N})(\tilde{U}_0)} \frac{\partial}{\partial} \mathcal{O}_{\bar{U}_\bullet}(\tilde{f}^{-1}\mathcal{N})|_{\tilde{U}_1}. \]

Since }\tilde{f}^{-1}\mathcal{N}\text{ is a Cartesian quasi-coherent } \mathcal{O}_{\bar{U}}\text{-module and }\pi\text{ is a flat hypercover, faithfully flat descent implies that }\tilde{f}^{-1}\mathcal{N} \cong \pi^{-1}\pi_*\tilde{f}^{-1}\mathcal{N}. \text{ Define } (a_*\mathcal{N})(U) := (\pi_*\tilde{f}^{-1}\mathcal{N})(U).

Since any two choices of hypercover } U \text{ admit a common hypercover by a third, this construction of } a_*\mathcal{N} \text{ is well-defined.}

To see that these functors are mutually inverse, first consider } (a^{-1}a_*\mathcal{N})|_{X_n} = a_n^{-1}a_*\mathcal{N}.\text{ The construction of } a_*\mathcal{N} \text{ gives a hypercover } \pi : \bar{X}_n \to X_n \text{ and a levelwise } P\text{-morphism }\tilde{a}_n : \bar{X}_n \to X. \text{ Then

\[ \pi^{-1}a_n^{-1}a_*\mathcal{N} = \bar{a}_n^{-1}\mathcal{N} = \pi^{-1}(\mathcal{N})|_{X_n}, \]

so } (a^{-1}a_*\mathcal{N}) \cong \mathcal{N}. \text{ Conversely, given } \mathcal{M} \text{ on } X \text{ and } f : U \to X \text{ in Aff}_P(X), \text{ there are } \pi : \bar{U} \to U, \text{ and } \bar{f} : \bar{U} \to X \text{ as above, with

\[ \pi^{-1}\bar{f}^{-1}(a_*a^{-1}\mathcal{M}) \cong \tilde{f}^{-1}a^{-1}\mathcal{M} \cong \pi^{-1}f^{-1}\mathcal{M}, \]

so } f^{-1}(a_*a^{-1}\mathcal{M}) \cong f^{-1}\mathcal{M}, \text{ as required.} \]

\[ \square \]

Remark 6.6. Similarly, for any class } C \text{ of sheaves on affine schemes with respect to which } P\text{-coverings are universal descent morphisms, it makes sense to define } C\text{-sheaves on } X \text{ as Cartesian } C\text{-sheaves on } X_\bullet. \text{ This is because Corollary 5.6 implies that } X_\bullet \text{ is unique up to a } P\text{-hypercover, and } [SD] \text{ then implies that the categories of } C\text{-sheaves are equivalent. For } \acute{e}tale \text{ sheaves, this means that we may apply many results from } [AM] \text{ and } [Fri] \text{ concerning } \acute{e}tale \text{ homotopy types. For Artin hypergroupoids, it means that we can define finite } \acute{e}tale \text{ sheaves, but not arbitrary } \acute{e}tale \text{ sheaves.}

Corollary 6.7. If } X \text{ is a strongly quasi-compact } (n, P)\text{-hypergroupoid, then the category of quasi-coherent modules on } X^2 \text{ is equivalent to the category of Cartesian (cosimplicial modules of the cosimplicial algebra } n \mapsto O(X_n).\]

Proof. This just combines Proposition 6.5 with the observation that since } X_n \text{ is affine, the categories of } O(X_n)\text{-modules and quasi-coherent } X_n\text{-modules are equivalent (for } O(X_n) \text{ as in Definition 0.3).} \]

\[ \square \]

Corollary 6.8. If } X \text{ is an arbitrary } (n, P)\text{-hypergroupoid, take a filtered direct system } \{ X_\alpha \}_{\alpha \in I} \text{ of strongly quasi-compact } (n, P)\text{-hypergroupoids, with } X = \varinjlim X_\alpha, \text{ as in Proposition 3.19. Then the category of quasi-coherent modules on } X^2 \text{ is equivalent to the category of Cartesian modules of the } \Delta \times \mathbb{I}^{opp}\text{-representation in algebras given by } (n, \alpha) \mapsto O(\langle X_\alpha \rangle_n).\]

6.2. Quasi-coherent complexes.

Definition 6.9. For a geometric stack } X, \text{ define a quasi-coherent complex on } X \text{ to be a presheaf } \mathcal{M}_\bullet \text{ of chain complexes of } \mathcal{O}_X\text{-modules on } \text{Aff}_P(X), \text{ such that the homology presheaves } H_n(\mathcal{M}_\bullet) \text{ are quasi-coherent } \mathcal{O}_X\text{-modules. A morphism } f : \mathcal{M}_\bullet \to \mathcal{N}_\bullet \text{ of quasi-coherent complexes is said to be a weak equivalence if } H_\ast(f) : H_\ast(\mathcal{M}_\bullet) \to H_\ast(\mathcal{N}_\bullet) \text{ is an isomorphism of presheaves. The homotopy category of quasi-coherent complexes is obtained by formally inverting weak equivalences, while the } \infty\text{-category is given by Dwyer-Kan localisation of weak equivalences (as in } [DK]).\]

Definition 6.10. Given a simplicial site } X_\bullet, \text{ say that a presheaf } \mathcal{F}_\bullet \text{ of chain complexes on } X_\bullet \text{ is homotopy-Cartesian if the homology presheaves } H_n(\mathcal{F}) \text{ are Cartesian.}
Remarks 6.11. Note that Definition 6.10 tallies with the definition of quasi-coherent complexes in [Toë] 3.7, which anticipated the simplicial characterisation of $n$-geometric stacks given in Theorem 4.9.

However, this differs crucially from the construction of $D_{\text{cart}}(\mathcal{O}_X)$ in [Ols] 4.6. The latter takes complexes $\mathcal{F}$ of sheaves of $\mathcal{O}_X$-modules whose homology sheaves $\mathcal{H}_n(\mathcal{F})$ are Cartesian. Thus the sheafification $\mathcal{F}^\mathbb{Z}$ of any homotopy-Cartesian $\mathcal{O}_X$-module $\mathcal{F}$ lies in $D_{\text{cart}}(\mathcal{O}_X)$, but $\mathcal{F}^\mathbb{Z}$ might not be homotopy-Cartesian. One consequence is that whereas projective resolutions exist in the category of quasi-coherent complexes, they do not exist in $D_{\text{cart}}(\mathcal{O}_X)$. This means that derived pullbacks $\mathcal{L}f^*$ exist automatically for quasi-coherent complexes (as will be exploited in Proposition 9.19), but are constructed only with considerable difficulty in [Ols].

Definition 6.9 corresponds to the definition of a homotopy-Cartesian module in [TV2] Definition 1.2.12.1. These are called quasi-coherent complexes in [Lur] 4.1, which anticipated the simplicial characterisation of quasi-coherent complexes in [Ols] 3.10, which takes complexes of sheaves of $\mathcal{O}_X$-modules whose homology sheaves are quasi-coherent. The differences arise because quasi-coherent complexes are hypersheaves (as explained in §6.3) rather than sheaves.

For a quasi-compact semi-separated scheme $X$, [Hüt] shows that the homotopy category of quasi-coherent complexes $X$ is equivalent to the homotopy category of complexes of quasi-coherent sheaves on $X$. By [BN], this in turn is equivalent to $D_{\text{cart}}(\mathcal{O}_X)$ under the same hypotheses.

**Definition 6.12.** Given a simplicial hypercover $\tilde{U}_\bullet \rightarrow U$, and a presheaf $\mathcal{P}$ on $U$, define the cosimplicial complex $\tilde{\mathcal{C}}^\bullet(U/U, \mathcal{P})$ by $\tilde{\mathcal{C}}^n(U/U, \mathcal{P}) = \mathcal{P}(\tilde{U}_n)$. If $\mathcal{P}$ is abelian, define $\tilde{H}^\bullet(U/U, \mathcal{P}) := H^\bullet(\tilde{\mathcal{C}}^\bullet(U/U, \mathcal{P}))$.

**Proposition 6.13.** Given an $(n, \mathcal{P})$-hypergroupoid $X$, there is a weak equivalence between the $\infty$-category of quasi-coherent complexes on $X^\mathbb{Z}$ and the $\infty$-category of homotopy-Cartesian chain complexes of quasi-coherent $\mathcal{O}_X$-modules on the simplicial site $n \mapsto \text{Aff}_\mathcal{P}(X_n)$. There is thus an equivalence between the corresponding homotopy categories.

**Proof.** We adapt the proof of Proposition 6.5. Take a filtered direct system $\{X_\alpha\}_{\alpha \in \Omega}$ of strongly quasi-compact $(n, \mathcal{P})$-hypergroupoids with $X = \lim_{\rightarrow} X_\alpha$, as in Proposition 3.19. Write $\mathcal{X} = X^\mathbb{Z}$, and let $R$ be the $\Delta \times \mathbb{I}$-representation in algebras given by $(n, \alpha) \mapsto O((X_\alpha)_n)$.

Given a quasi-coherent complex $\mathcal{M}$ on $\mathcal{X}$, the chain complex $a^{-1}\mathcal{M}$ of $\mathcal{O}_X$-modules will be homotopy-Cartesian. However, $(a^{-1}\mathcal{M})|_{X_n}$ is only a quasi-coherent complex, not a complex of quasi-coherent modules.

However, for a quasi-coherent complex $\mathcal{F}$ on an affine scheme $U$, the map $\Gamma(U, \mathcal{F}) \otimes_{\Gamma(U, \mathcal{O}_U)} \mathcal{O}_U \rightarrow \mathcal{F}$ is a weak equivalence of presheaves. Thus $a^{-1}\mathcal{M}$ is equivalent to the homotopy-Cartesian chain complex of quasi-coherent $\mathcal{O}_X$-modules associated to the $R$-module $(n, \alpha) \mapsto \Gamma((X_\alpha)_n, \mathcal{M}|_{X_\alpha}_n)$.

Conversely, given a homotopy-Cartesian chain complex $\mathcal{F}$ of $\mathcal{O}_X$-modules on $\text{Aff}_\mathcal{P}(X)$, we now define $\mathcal{R}a_* \mathcal{F}$. Given $f : U \rightarrow \mathcal{X}$ is in $\text{Aff}_\mathcal{P}(\mathcal{X})$, we form $\tilde{f} : \tilde{U}_\bullet \rightarrow \mathcal{X}$ as in Proposition 6.5, and set $(\mathcal{R}a_* \mathcal{F})(U) := \text{Tot}^{\Pi} \tilde{\mathcal{C}}^\bullet(U/U, \tilde{f}^{-1}\mathcal{F})$, where $\text{Tot}^{\Pi}(V^\bullet)_n = \prod_i V^i_{n+i}$.

Faithfully flat descent implies that this complex is well-defined (i.e. independent of $\tilde{U}_\bullet$) and functorial in the homotopy category (and hence in the $\infty$-category), and that the functors $a^{-1}$ and $\mathcal{R}a_*$ are weak inverses, in the sense that the natural transformations $a^{-1} \mathcal{R}a_* \mathcal{F} \rightarrow \mathcal{F}$ and $\mathcal{R}a_* a^{-1} \mathcal{M} \rightarrow \mathcal{M}$ are weak equivalences.\qed
**Corollary 6.14.** If $X$ is a strongly quasi-compact $(n, P)$-hypergroupoid, then the $\infty$-category of quasi-coherent complexes on $X^\Delta$ is weakly equivalent to the $\infty$-category of homotopy-Cartesian complexes of (cosimplicial) modules of the cosimplicial algebra $n \mapsto O(X_n)$.

**Proof.** This follows from the proof of Proposition 6.13. □

**Corollary 6.15.** If $X$ is an arbitrary $(n, P)$-hypergroupoid, take a filtered direct system $\{X_\alpha\}_{\alpha \in \mathbb{I}}$ of strongly quasi-compact $(n, P)$-hypergroupoids, with $X = \lim_\to X_\alpha$. Then the $\infty$-category of quasi-coherent complexes on $X^\Delta$ is weakly equivalent to the $\infty$-category of homotopy-Cartesian complexes of modules of the $\Delta \times \mathbb{I}^{\text{op}}$-representation in algebras given by $(n, \alpha) \mapsto O((X_\alpha)_n)$.

### 6.3. Sheaves and hypercovers.

**Proposition 6.16.** Given an $(n, P)$-hypergroupoid $X$, there is an equivalence of categories between sheaves on $\text{Aff}_P(X^\Delta)$, and Cartesian sheaves on the simplicial site $n \mapsto \text{Aff}_P(X_n)$.

**Proof.** Let $\mathfrak{X} := X^\Delta$. Since $X_* \to \mathfrak{X}$ is a simplicial hypercover of $\mathfrak{X}$ in $\text{Aff}_P(\mathfrak{X})$, the category of sheaves on $\text{Aff}_P(\mathfrak{X})$ is equivalent to the category of Cartesian sheaves on the simplicial site $n \mapsto \text{Aff}_P(X_n)$. Explicitly, the equivalence is given by the functors $a^{-1}, a_*$ defined as in Proposition 6.5. □

**Definition 6.17.** Given a cosimplicial object $X^\bullet$ in a model category $\mathcal{C}$, define $\text{Tot}_C X \in \text{Ho}(\mathcal{C})$ by

$$\text{Tot}_C X := \text{holim}_{n \in \Delta} X^n.$$  

**Examples 6.18.** If $\mathcal{C}$ is a category with trivial model structure (all morphisms are fibrations and cofibrations, isomorphisms are the only weak equivalences), then $\text{Tot}_C X$ is the equaliser in $\mathcal{C}$ of the maps $\partial^0, \partial^1 : X^0 \to X^1$.

$\text{Tot}_C$ is the functor $\text{Tot}$ of a cosimplicial space defined in [GJ] §VIII.1. Explicitly,

$$\text{Tot}_C X^\bullet = \{ x \in \prod_n (C^n)^{\Delta^n} : \partial_X^i x_n = (\partial^i_{\Delta})^* x_{n+1}, \sigma_X^i x_n = (\sigma^i_{\Delta})^* x_{n-1} \},$$

for $X$ Reedy fibrant. Homotopy groups of the total space are related to a spectral sequence given in [GJ] Proposition VIII.1.15.

In the category $s\text{Ab}$ of simplicial abelian groups, all cosimplicial objects are Reedy fibrant, and $\text{Tot}_{s\text{Ab}} A = \text{Tot}_C A$. Likewise for simplicial modules and simplicial rings. In all these cases, we may compute homotopy groups by $\pi_n(\text{Tot}_{s\text{Ab}} A) = H_n(\text{Tot}_{\text{Ch}^{\geq 0}} N^s A)$, for $N^s A$ the normalised cochain complex associated to $A$, and $\text{Tot}_{\text{Ch}^{\geq 0}}$ defined below. The unnormalised associated chain complex also gives the same result.

For the category $\text{Ch}$ of unbounded chain complexes, $\text{Tot}_{\text{Ch}} \simeq \text{Tot}^\Pi$, or the quasi-isomorphic functor $\text{Tot}_{\text{Ch}}^N$, where $N_C$ is conormalisation (see Definition 0.2), and $(\text{Tot}^\Pi V)_n = \prod_{i=0}^n V^{i+n}$, Similarly, $\text{Tot}_{\text{CoCh}^{\geq 0}} \simeq \text{Tot}^\Pi$ for the category $\text{CoCh}^{\geq 0}$ of non-negatively graded cochain complexes. However, for non-negatively graded chain complexes $\text{Ch}^{\geq 0}$, we have $\text{Tot}_{\text{Ch}^{\geq 0}} \simeq \tau_{\geq 0} \text{Tot}^\Pi$, where $\tau$ is good truncation.

For the categories $d_0\text{Alg}$ and $d_0^{\geq 0}\text{Alg}$ of unbounded and non-positively graded differential graded (chain) algebras, $\text{Tot}_{d_0\text{Alg}}$ and $\text{Tot}_{d_0^{\geq 0}\text{Alg}}$ are modelled by the functor $\text{Th}$ of Thom-Sullivan (or Thom-Whitney) cochains. In the category $d_0^{\geq 0}\text{Alg}$ of differential non-negatively graded (chain) algebras, $\text{Tot}_{d_0^{\geq 0}\text{Alg}} \simeq \tau_{\geq 0}\text{Th}$.

**Definition 6.19.** Given a model category $\mathcal{C}$ and a site $X$, a $\mathcal{C}$-valued presheaf $\mathcal{F}$ on $X$ is said to be a hypersheaf if for all hypercovers $U_* \to U$, the natural map

$$\mathcal{F}(U) \to \text{Tot}_C \mathcal{F}(U/U, \mathcal{F})$$

is a weak equivalence, for $\mathcal{C}$ as in Definition 6.12.
Remarks 6.20. If $C$ has trivial model structure, note that hypersheaves in $C$ are precisely $C$-sheaves. A sheaf $F$ of modules is a hypersheaf when regarded as a presheaf of non-negatively graded chain complexes, but not a hypersheaf when regarded as a presheaf of unbounded chain complexes unless $H^i(U, F) = 0$ for all $i > 0$ and all $U$. Beware that the sheafification of a hypersheaf will not, in general, be a hypersheaf.

Remark 6.21. A comment on terminology. Hypersheaves are usually known as $\infty$-stacks or $\infty$-sheaves, but are referred to as stacks in [TV1], and as sheaves in [Lur1]. They are also sometimes known as homotopy sheaves, but we avoid this terminology for fear of confusion with homotopy groups of a simplicial sheaf. The stacks featuring in §1 were hypersheaves.

Proposition 6.22. For all of the abelian model categories $C$ listed in Examples 6.18, every quasi-coherent complex $\mathcal{M}$ on a geometric stack $X$ is a hypersheaf in $C$.

Proof. Given an object $U \to X$ in $\text{Aff}_P(X)$ and a $P$-hypercover $\tilde{U} \to U$, we need to show that

$$\mathcal{M}(\mathcal{U}) \to \text{Tot}_C \tilde{C}(\tilde{U}/U, \mathcal{M})$$

is a weak equivalence. Now, there is spectral sequence

$$H^i(\tilde{U}/U, H_j\mathcal{M}) \Rightarrow H_{j-i}\text{Tot}^H(\tilde{C}(\tilde{U}/U, \mathcal{M})), $$

which converges weakly, by [Wei] §5.6, since $\tilde{C}(\tilde{U}/U, \mathcal{M}_j) = 0$ for $i < 0$ (and hence for $i < 0, j < 0$). Since $U$ is affine and $H_j\mathcal{M}$ quasi-coherent, $H^i(\tilde{U}/U, H_j\mathcal{M}) = 0$ for all $i > 0$, and the spectral sequence converges, giving

$$H_n\text{Tot}^H(\tilde{C}(\tilde{U}/U, \mathcal{M})) \cong H^0(\tilde{U}/U, H_n\mathcal{M}) = (H_n\mathcal{M})(U).$$

Thus the maps

$$(H_n\mathcal{M})(U) \to \tau_{\geq 0}\text{Tot}^H(\tilde{C}(\tilde{U}/U, \mathcal{M})) \to \text{Tot}^H(\tilde{C}(\tilde{U}/U, \mathcal{M}))$$

are weak equivalences, as required. \qed

Proposition 6.23. Given an $(n, P)$-hypergroupoid $X$, there is a weak equivalence of $\infty$-categories between hypersheaves on $\text{Aff}_P(X^\sharp)$, and homotopy-Cartesian hypersheaves on the simplicial site $n \mapsto \text{Aff}_P(X_n)$.

Proof. The proof is as for Proposition 6.16. The quasi-inverse to $a^{-1}$ is now given by the functor $R a_*$, which is given on $U$ by $\text{Tot}_C \tilde{C}(\tilde{U}/U, -)$, for $\tilde{U}$ as constructed in Proposition 6.16. \qed

6.4. Direct images and cohomology. Given any strongly quasi-compact morphism $f : X \to Y$ of $(n, P)$-hypergroupoids, there is a natural exact functor $f_*$ from quasi-coherent sheaves on $\text{Aff}_P(X)$ to sheaves on $\text{Aff}_P(Y)$. However, this does not preserve Cartesian sheaves, so we will now give a construction corresponding to the direct image functor for stacks.

In general, for any (not necessarily Cartesian) sheaf $\mathcal{F}$ on $\text{Aff}_P(X_\bullet)$, we may form the direct image $a_*\mathcal{F}$ on $X^\sharp$ as in the proof of Proposition 6.16, and the derived direct image $R a_*\mathcal{F}$ as in the proof of Proposition 6.23, for $a : X \to X^\sharp$ the canonical map associated to sheafification of a presheaf. Thus the functors $a^{-1}a_*$ and $a^{-1}R a_*$ can be regarded as Cartesianification and homotopy-Cartesianification.

Definition 6.24. For an $(l, P)$-hypergroupoid $X_\bullet$ and a model category $C$, let $\text{Shf}(X_\bullet, C)$ be the category of $C$-valued hypersheaves on the simplicial site $\text{Aff}_P(X_\bullet)$, and let $\text{Shf}(X_\bullet, C)_{\text{cart}}$ be the full subcategory consisting of homotopy-Cartesian sheaves. Let $R c_{\text{cart}} : \text{Ho}(\text{Shf}(X_\bullet, C)) \to \text{Ho}(\text{Shf}(X_\bullet, C)_{\text{cart}})$ be the functor $a^{-1}R a_*$, as described in Lemma 6.23. Let $c_{\text{cart}}^{-1} : \text{Ho}(\text{Shf}(X_\bullet, C)_{\text{cart}}) \to \text{Ho}(\text{Shf}(X_\bullet, C))$ be the inclusion functor, noting that $R c_{\text{cart}} c_{\text{cart}}^{-1}$ is an equivalence. If the model structure on $C$ is trivial, we simply denote $R c_{\text{cart}}$ by $c_{\text{cart}}$. 
Definition 6.25. Given a simplicial scheme $X_\bullet$, let $(\text{Dec}_+)\circ X_\bullet$ be the bisimplicial scheme given in level $n$ by $(\text{Dec}_+)^{n+1}X_\bullet$, for $\text{Dec}_+$ as in Definition 2.20. The simplicial structure is defined as in [Wei] 8.6.4, using the comonadic structure of $\text{Dec}_+$. This admits an augmentation $\partial : (\text{Dec}_+)_0 \rightarrow X_\bullet$, given by the co-unit.

This also has an augmentation $\alpha$ in the orthogonal direction, given by $(\partial_\alpha)^{n+1} : (\text{Dec}_+)^{n+1}X_\bullet \rightarrow ((\text{Dec}_+)^{n}X_\bullet)_0 = X_n$, corresponding to the retraction of Lemma 2.22.

Lemma 6.26. Given an $(l, \mathcal{P})$-hypergroupoid $X_\bullet$ and a $\mathcal{C}$-valued sheaf $\mathcal{F}$ on the simplicial site $\text{Aff}_\mathcal{P}(X_\bullet)$, we may describe $R_{\text{cart}}\mathcal{F}$ as the cosimplicial complex $R_{\alpha, \partial_\alpha^{-1}}\mathcal{F}$.

Explicitly, for $U \rightarrow X_m$,

$$(R_{\text{cart}}\mathcal{F})(U) = \lim_{n \in \Delta} \Gamma(U \times X_m, ((\text{Dec}_+)^{m+1}X)_n, (\partial_\alpha^{-1})^{m+1}\mathcal{F}|_{X_n})$$

$$= \text{Tot}_C \mathcal{C}(U \times X_m, (\text{Dec}_+)^{m+1}X, (\partial_\alpha^{-1})^{m+1}\mathcal{F})$$

Proof. This is just the observation that $(\text{Dec}_+)^{m+1}X$ is a $\mathcal{P}$-hypercovering of $X_m$ with a natural map to $X$, corresponding to the map $X_m \rightarrow X^2$.

Definition 6.27. For any morphism $f : X \rightarrow Y$ of $(n, \mathcal{P})$-hypergroupoids, we may now define $R_{f_*}^{\text{cart}} : \text{Shf}(X_\bullet, \mathcal{C})_{\text{cart}} \rightarrow \text{Shf}(Y_\bullet, \mathcal{C})_{\text{cart}}$ by

$$R_{f_*}^{\text{cart}} := R_{\text{cart}} \circ R_{f_*} \circ \text{cart}^{-1},$$

where $R_{f_*} : \text{Shf}(X_\bullet, \mathcal{C}) \rightarrow \text{Shf}(Y_\bullet, \mathcal{C})$ is given on $X_n$ by $f_{n_*}$. This has the property that $(R_{f_*}^{\text{cart}})^2 = R(f^*_*)$.

Note that if we restrict attention to quasi-coherent $\mathcal{O}_X$-modules and $f$ is strongly quasi-compact, then $R_{f_*} = f_*$, so

$$R_{f_*}^{\text{cart}} := R_{\text{cart}} \circ f_* \circ \text{cart}^{-1} : dg\text{Mod}(X)_\text{cart} \rightarrow dg\text{Mod}(Y)_\text{cart}.$$

The following is a generalisation of the vanishing of higher direct images of an affine morphism of schemes.

Lemma 6.28. If $f : X \rightarrow Y$ is a strongly quasi-compact relative $(0, \mathcal{P})$-hypergroupoid over an $(n, \mathcal{P})$-hypergroupoid $Y$, then

$$R_{f_*}^{\text{cart}} \simeq f_* : dg\text{Mod}(X)_\text{cart} \rightarrow dg\text{Mod}(Y)_\text{cart}.$$

Proof. Since $f$ is a relative $(0, \mathcal{P})$-hypergroupoid, it is Cartesian, so $f_* M$ is Cartesian for all $M \in dg\text{Mod}(X)_\text{cart}$. Now

$$R_{f_*}^{\text{cart}} M = R_{\text{cart}} f_* \text{cart}^{-1} M = R_{\text{cart}} \text{cart}^{-1} f_* M \simeq f_* M,$$

as required.

Remark 6.29. Note that the computation of cohomology groups is simpler, since we just need to compute cohomology on the simplicial site $X_\bullet$. This gives the familiar spectral sequence $H^n(X_n, \mathcal{F}|_{X_n}) \Rightarrow H^{n+1}(X_\bullet, \mathcal{F})$. For a quasi-coherent sheaf $\mathcal{F}$, cohomology is just cohomology of the cosimplicial complex given in degree $n$ by $\Gamma(X_n, \mathcal{F}|_{X_n})$.

7. Derived Hypergroupoids

In the constructions of the previous sections, there was nothing special about the category $\text{Aff}$ of affine schemes. In fact, the main results hold for any homotopic algebraic geometry context, in the sense of [TV2] Definition 1.3.2.13. The only exceptions were Proposition 4.8 and its corollaries, which utilised special properties of $\text{Aff}$. This means that the correct notion of an $(n, \mathcal{P})$-hypergroupoid in an arbitrary context is as a simplicial diagram of 0-representable stacks, rather than a simplicial diagram of disjoint unions of objects in $\text{Aff}_C$. 

The main remaining HAG context of interest comprises the geometric $D^\cdot$-stacks of [TV2] Chapter 2.2. We will now describe how Sections 3–6 carry over to this context, including analogues of Proposition 4.8.

7.1. **Strongly quasi-compact derived stacks.** See §1.2 for definitions and notation concerning derived stacks.

**Lemma 7.1.** Given an object $\bar{Z}_* \in \text{Ho}((s\text{Pr}(\text{cAff}))^{\Delta_{n}^{\text{opp}}})$ for which the homotopy matching maps $\bar{Z}_r \to (M^h_r \bar{Z})$ are 0-representable and strongly quasi-compact for all $r$, there exists a fibrant object $X_* \in$ in the Reedy category in $(\text{cAff})^{\Delta_{n}^{\text{opp}}}$, unique up to weak equivalence, with $X_* \cong \bar{Z}_*$ in $\text{Ho}((s\text{Pr}(\text{cAff}))^{\Delta_{n}^{\text{opp}}})$.

**Proof.** We will construct the $n$-truncated objects $X_{\leq n}$ by induction.

Assume that $\bar{Z}$ is fibrant, and that we have constructed a Reedy fibrant $n$-truncated simplicial object $X_{\leq n} \in (\text{cAff})^{\Delta_{n}^{\text{opp}}}$, together with a weak equivalence $(X_{\leq n})^\sharp \to \bar{Z}_{\leq n}$ in $(s\text{Pr}(\text{cAff}))^{\Delta_{n}^{\text{opp}}}$. If we let $L_{n+1}X$ and $M_{n+1}X$ denote the $(n+1)$th latching and matching objects, then it suffices to construct a factorisation of $L_{n+1}X \to M_{n+1}X$ through $X_{n+1}$, for suitable $X_{n+1}$.

Since $\bar{Z}_{n+1} \to M_{n+1}\bar{Z}$ is 0-representable and strongly quasi-compact, there exists a fibrant $X_{n+1} \in \text{cAff} \downarrow M_{n+1}X$, and a weak equivalence

$$X_{n+1} \to M_{n+1}X \times_{M_{n+1}\bar{Z}} \bar{Z}_{n+1}.$$ 

Since this is a weak equivalence of fibrant objects over $M_{n+1}X$, the map $L_{n+1}X \to M_{n+1}X \times_{M_{n+1}\bar{Z}} \bar{Z}_{n+1}$ lifts (up to homotopy relative to $M_{n+1}X$) to a map $L_{n+1}X \to X_{n+1}$, as $L_{n+1}X$ is automatically cofibrant.

Now, $\text{Hom}(L_{n+1}X, X_{n+1}) = X_{n+1}(L_{n+1}X)_0 = \text{Hom}(L_{n+1}X, X_{n+1})$, so we have defined the necessary latching map, giving a fibrant $(n+1)$-truncated simplicial object $X_{\leq n+1}$ satisfying the inductive hypothesis. \qed

**Remark 7.2.** Note that this proof carries over to any homotopic algebraic geometry context ([TV2] Definition 1.3.2.13) in which all objects of the model category $\text{Aff}^c$ are cofibrant, since we may choose the cofibrant resolution functor $\Gamma^c$ so that $\Gamma^0Y = Y$, giving

$$\text{Map}_{\text{Aff}^c}(Y, T)_0 = \text{Hom}_{\text{Aff}^c}(\Gamma^0Y, T) = \text{Hom}_{\text{Aff}^c}(Y, T).$$

If $\text{Aff}^c$ has objects that are not cofibrant, the lemma still holds. However, in this case the proof has to be modified so that $X_{\leq n}$ is constructed to be Reedy fibrant and cofibrant, thereby ensuring that the latching objects are cofibrant. We then have to choose $X_{n+1}$ so that the $n$th latching map is a cofibration (by taking a cofibration-trivial fibration factorisation $L_{n+1}X \to X'_{n+1} \to X_{n+1}$, then replacing $X_{n+1}$ with $X'_{n+1}$).

Let $\text{scAff}$ be the category $(\text{cAff})^{\Delta_{\infty}^{\text{opp}}}$, equipped with its Reedy model structure. Recall that a morphism $X_* \to Y_*$ in $\text{scAff}$ is said to be a Reedy fibration if the matching maps

$$X_m \to M_\partial\Delta_m(X) \times_{M_\partial\Delta_m(Y)} Y_m$$

are fibrations in $\text{cAff}$, for all $m$.

**Definition 7.3.** Given $Y_* \in \text{scAff}$, define a homotopy strongly quasi-compact relative derived Artin $n$-hypergroupoid over $Y_*$ to be a morphism $X_* \to Y_*$ in $\text{scAff}$, satisfying the following:

1. For each $k, m$, the homotopy partial matching map

$$X_m \to M_{\Delta k}^m(X) \times_{M_{\Delta k}^m(Y)} Y_m$$

is a smooth surjection (in the sense of Definitions 1.9 and 1.10);
(2) for all $m > n$ and all $k$, the homotopy partial matching maps

$$X_m \to M_{\partial\Delta^m}(X) \times_{M_{\partial\Delta^m}(Y)} Y_m$$

are weak equivalences.

A strongly quasi-compact homotopy relative derived Artin $n$-hypergroupoid is said to be a strongly quasi-compact relative derived Artin $n$-hypergroupoid if it is also a Reedy fibration. It is said to be smooth if $X_0 \to Y_0$ is smooth.

A strongly quasi-compact homotopy trivial relative derived Artin $n$-hypergroupoid $X_\bullet \to Y_\bullet$ is a morphism in scAff satisfying the following:

1. for each $m$, the homotopy partial matching map

$$X_m \to M_{\partial\Delta^m}(X) \times_{M_{\partial\Delta^m}(Y)} Y_m$$

is a smooth surjection (in the sense of Definitions 1.9 and 1.10);

2. for all $m \geq n$, the homotopy partial matching maps

$$X_m \to M_{\partial\Delta^m}(X) \times_{M_{\partial\Delta^m}(Y)} Y_m$$

are weak equivalences.

A strongly quasi-compact homotopy trivial relative derived Artin $n$-hypergroupoid is said to be a strongly quasi-compact relative derived Artin $n$-hypergroupoid if it is also a Reedy fibration.

Note that for Reedy fibrations, the homotopy partial matching maps are just ordinary matching maps (with fibre products, not homotopy fibre products).

**Remark 7.4.** Given a strongly quasi-compact homotopy relative derived Artin $n$-hypergroupoid $X \to Y$, the Reedy model structure gives us a Reedy fibration $f : \hat{X} \to Y$, with $X \to \hat{X}$ a levelwise weak equivalence. Since the conditions are invariant under levelwise weak equivalences, $f$ will be a strongly quasi-compact relative derived Artin $n$-hypergroupoid.

For example, if $Y \in c\text{Aff}$, then any morphism $X \to Y$ in $c\text{Aff}$ is a homotopy relative derived Artin 0-hypergroupoid. Taking a weakly equivalent fibration $X' \to Y$ in cAff, we may then set $\hat{X} = X' \times_{\Delta^n} Y$, as defined in Definition 1.11. Explicitly, if $X' = \text{Spec } B$ and $Y = \text{Spec } A$, then $\hat{X}_n = \text{Spec } (B \otimes A^n \otimes A \otimes A^n A)$.

**Remark 7.5.** If $X_\bullet \to Y_\bullet$ is a Reedy fibration, the matching maps are fibrations in $c\text{Aff}$. Thus Definition 7.3 is closely related to the concept of quasi-smoothness for simplicial cosimplicial formal schemes introduced in [Pri2], where a morphism was said to be quasi-smooth (resp. trivially smooth) if the matching maps were fibrations and the partial matching maps (resp. the matching maps) were formally smooth fibrations.

**Proposition 7.6.** If $f : X \to Y$ is a strongly quasi-compact homotopy relative derived Artin $n$-hypergroupoid, then the associated morphism

$$f^\sharp : X^\sharp \to Y^\sharp$$

of sheaves is a strongly quasi-compact $D^-$ $n$-representable morphism in the sense of [TV2] Definition 1.3.3.1.

In particular, if $Y$ is a cosimplicial affine scheme, then $X$ is a $D^-$ $n$-geometric stack. If $f_0 : X_0 \to Y_0$ is also smooth, then $f^\sharp$ is a smooth $n$-morphism.

**Proof.** The proof of Proposition 3.13 carries over to this context. \qed

Fix a simplicial cosimplicial affine scheme $S$. 
Theorem 7.7. Given a strongly quasi-compact $m$-representable morphism $f : X \to S^2$ to the sheafification of $S$, there exists a strongly quasi-compact relative derived Artin $m$-hypergroupoid $X \to S$ whose sheafification $X^\sharp$ is equivalent to $X$ over $S^2$. Moreover, if $f$ is a smooth morphism, then $X_0 \to S_0$ is smooth.

Proof. The proof of Theorem 4.10 carries over, using Lemma 7.1 and Proposition 4.6 instead of Proposition 4.8.

Definition 7.8. The Reedy model category $sc\text{Aff}$ has a natural simplicial structure. If we regard an object $X \in sc\text{Aff}$ as a contravariant functor from $c\text{Aff}$ to $S$, then for $K \in S$, the object $X^K \in sc\text{Aff}$ is given by

$$X^K(U) := X(U)^K,$$

for $U \in c\text{Aff}$. Note that this simplicial structure is independent of the natural simplicial structure on $c\text{Aff}$.

We then define $\underline{\text{Hom}}_{sc\text{Aff}}(X, Y) \in S$ by

$$\underline{\text{Hom}}_{sc\text{Aff}}(X, Y)_n := \text{Hom}_{sc\text{Aff}}(X, Y^{\Delta^n}) .$$

The constructions so far hold for any class $P$ of morphisms satisfying [TV2] Assumption 1.3.2.11.

Definition 7.9. We define relative derived Deligne–Mumford hypergroupoids in the same way as derived Artin hypergroupoids, replacing smooth morphisms with étale morphisms. Likewise, we can define relative derived Nisnevich or Zariski hypergroupoids, with surjectivity for closed (rather than geometric) points, and étale morphisms or local isomorphisms instead of smooth morphisms.

Theorem 7.10. Take a morphism $X \to S$ in $sc\text{Aff}$, and a strongly quasi-compact relative derived Artin $n$-hypergroupoid $Y \to S$. Then homomorphisms in the $\infty$-category and homotopy category of simplicial presheaves over $S^\sharp$ are given by

$$\text{RHom}_{s \text{Pr}(c\text{Aff}) S^\sharp}(X^2, Y^2) \simeq \lim_{\to} \text{Hom}_{sc\text{Aff} S}(\tilde{X}, Y),$$
$$\text{Hom}_{\text{Ho}(s \text{Pr}(c\text{Aff}) S^\sharp)}(X^2, Y^2) \cong \pi_0(\lim_{\to} \text{Hom}_{sc\text{Aff} S}(\tilde{X}, Y)).$$

Here, the limit is taken over the category of strongly quasi-compact trivial relative derived Deligne–Mumford $n$-hypergroupoids.

In particular, the homotopy category (resp. $\infty$-category) of strongly quasi-compact trivial relative derived Deligne–Mumford $n$-hypergroupoids

$$\text{Ho}_{sc\text{Aff} S}(U, Y) \rightarrow \text{RHom}_{sc\text{Aff} S}(U, Y),$$
$$\text{Hom}_{sc\text{Aff} S}(U, Y) \rightarrow \text{Ho}_{sc\text{Aff} S}(U, Y)$$

is surjective for all $U$.

Remark 7.11. Similarly to Remark 5.7.(3), the properties here strongly resemble a simplicial model structure on $sc\text{Aff}$. In this model structure, (trivial) fibrations would be cogenerated by (trivial) relative $(n, P)$-hypergroupoids. This is very similar to the geometric model structure in [Pri2] §?? on simplicial cosimplicial formal schemes.

7.2. Disjoint unions. We now show how to address the case of an $n$-geometric stack not being strongly quasi-compact.
Definition 7.12. A morphism $X \to Y$ in $\text{cAff}$ is said to be a clopen immersion if for some $U \in \text{cAff}$, $Y$ is weakly equivalent to $X \sqcup U$.

A morphism $f : X \to Y$ in $\text{scAff}$ is said to be a levelwise clopen immersion if each map $f_n : X_n \to Y_n$ is a clopen immersion in $\text{cAff}$.

Definition 7.13. Say that a level map $f : X \to Y$ in $\text{ind}(\text{scAff})$ is a Reedy weak equivalence (resp. a Reedy fibration) if each $f_{\alpha} : X_{\alpha} \to Y_{\alpha}$ is a Reedy equivalence (resp. a Reedy fibration) in $\text{scAff}$. Say that an arbitrary map $f$ is a Reedy weak equivalence (resp. a Reedy fibration) if it is isomorphic to a level map which is a Reedy equivalence (resp. a Reedy fibration).

By [FI] Theorem 5.15, this defines a model structure, with cofibrations described in [FI] Definition 5.3.

Definition 7.14. Define $s(\coprod \text{cAff})$ to be the full subcategory of $\text{ind}(\text{scAff})$ consisting of objects $\{X_{\alpha}\}_{\alpha \in I}$ whose transition maps $X_{\alpha} \to X_{\alpha'}$ are levelwise clopen immersions.

Lemma 7.15. Given an object $\tilde{Z} \in \text{Ho}(s \text{Pr}(\text{cAff}))^{\Delta^{op}}$ for which the homotopy matching maps $\tilde{Z}_r \to (M_r^{\mathbb{Z}} \tilde{Z})$ are 0-representable for all $r$, there exists a filtered direct system $W = \{W_{\alpha}\} \in \text{ind}(\text{Ho}(s \text{Pr}(\text{cAff}))^{\Delta^{op}})$, with all the homotopy matching maps of each $W_{\alpha}$ being quasi-compact and 0-representable, the transition maps $W_{\alpha} \to W_{\alpha'}$ being clopen immersions, and

$$\tilde{Z} \simeq \lim\alpha W_{\alpha}.$$  

Moreover, $W$ is unique up to unique isomorphism.

Thus there exists an object $X_{\bullet}$ in $s(\coprod \text{cAff})$, Reedy fibrant in $\text{ind}(\text{scAff})$ and unique up to weak equivalence, with $X_{\bullet} \cong \tilde{Z}_{\bullet}$ in $\text{Ho}(s \text{Pr}(\text{cAff}))^{\Delta^{op}}$.

Proof. The construction of $W$ is the same as in Lemma 3.18, taking quasi-compact 0-geometric stacks instead of affine schemes.

Finally, Lemma 7.1 allows us to replace $W$ with a direct system of fibrant objects in $\text{scAff}$, as required.

Definition 7.16. Observe that the functor $\pi^0 : \text{cAff} \to \text{Aff}$ of Definition 1.10 extends to a functor from $\coprod \text{cAff}$ to $\coprod \text{Aff}$. A morphism $f$ in $\coprod \text{cAff}$ is said to be surjective if $\pi^0(f)$ is surjective.

Definition 7.17. Say that a map in $s(\coprod \text{cAff})$ is a (homotopy) relative derived Artin $n$-hypergroupoid if it is isomorphic to a filtered direct system of strongly quasi-compact (homotopy) relative derived Artin $n$-hypergroupoids.

A map in $s(\coprod \text{cAff})$ is a (homotopy) trivial relative derived Artin $n$-hypergroupoid if it is isomorphic to a filtered direct system of strongly quasi-compact (homotopy) trivial relative derived Artin $n$-hypergroupoids.

A level map $\{f_{\alpha} : X_{\alpha} \to Y_{\alpha}\}_{\alpha \in I}$ in $s(\coprod \text{cAff})$ is strongly quasi-compact if the maps $X_{\alpha} \to Y_{\alpha} \times_{Y_{\beta}} X_{\beta}$ are weak equivalences for all arrows $\alpha \to \beta$ in $I$.

Lemma 7.15 and Proposition 4.6 have the following consequences. Fix $S \in s(\coprod \text{cAff})$.

Proposition 7.18. If $f : X \to S$ is a homotopy relative derived Artin $m$-hypergroupoid, then the associated morphism

$$f^\sharp : X^\sharp \to S^\sharp$$

is a $D^-$ $m$-representable morphism.

Proof. The proof of Proposition 7.6 carries over to this context. □

Theorem 7.19. Given a $D^-$ $m$-representable morphism $f : X \to S^\sharp$ to the sheafification of $S$, there exists a relative derived Artin $m$-hypergroupoid $X \to S$ whose sheafification $X^\sharp$ is equivalent to $X$ over $S^\sharp$. Moreover, if $f$ is a smooth morphism, then $X_0 \to S_0$ is smooth.
Proof. The proof of Theorem 7.7 adapts, substituting Proposition 7.15 for Proposition 7.1, and applying Proposition 3.19 to \( \pi^0 X \) for the surjectivity conditions.

\[ \text{Theorem 7.20. Take a morphism } X \to S \text{ in } s(\coprod\text{Aff}), \text{ and a relative derived Artin } n\text{-hypergroupoid } Y \to S. \text{ Then homomorphisms in the homotopy category of simplicial presheaves over } S^2 \text{ are given by} \]
\[
\text{Hom}_{\text{Ho}(s(\text{Pr(cAff)});S^2)}(X^2, Y^2) \cong \pi_0(\lim_{\to} \text{Hom}_{\text{ind}(\text{scAff});S}(\tilde{X}, Y)).
\]

Here, the limit is taken over the category of trivial relative derived Artin \( n \)-hypergroupoids \( \tilde{X} \to X \). In fact, we can restrict to trivial relative derived Deligne–Mumford \( n \)-hypergroupoids.

In particular, the homotopy category of \( n \)-geometric derived Artin stacks over \( S^2 \) is obtained by localising the category of derived Artin \( n \)-hypergroupoids over \( S \) at the class of trivial relative derived Artin \( n \)-hypergroupoids.

Proof. The proof of Theorem 7.10 adapts, with the same substitutions as Theorem 7.7.

\[ \text{\textbf{7.3. The } \infty\text{-category of } n\text{-geometric derived Artin stacks.} \text{ Summarising the main results of this section gives the following.} \]

\[ \text{\textbf{Theorem 7.21. The } \infty\text{-category of strongly quasi-compact } n\text{-geometric derived Artin stacks is weakly equivalent to the } \infty\text{-category } \mathcal{G}_n^{\text{strict}}, \text{ whose objects are strongly quasi-compact homotopy derived Artin } n\text{-hypergroupoids, with morphisms given by} \]
\[
\text{Hom}_{\mathcal{G}_n^{\text{strict}}}(X, Y) = \lim_{\to} \text{Hom}_{\text{ind}(\text{scAff})}(\tilde{X}, \hat{Y}),
\]

where \( Y \to \hat{Y} \) is a Reedy fibrant replacement, and the limit is taken over the category of strongly quasi-compact trivial relative derived Deligne–Mumford \( n \)-hypergroupoids \( \tilde{X} \to X \).

The \( \infty\text{-category of } n\text{-geometric derived Artin stacks is weakly equivalent to the } \infty\text{-category } \mathcal{G}_n^{\text{d}}, \text{ whose objects are homotopy derived Artin } n\text{-hypergroupoids, with morphisms given by} \]
\[
\text{Hom}_{\mathcal{G}_n^{\text{d}}}(X, Y) = \lim_{\to} \text{Hom}_{\text{ind}(\text{scAff})}(\tilde{X}, \hat{Y}),
\]

where \( Y \to \hat{Y} \) is a Reedy fibrant replacement, and the limit is taken over the category of trivial relative derived Deligne–Mumford \( n \)-hypergroupoids \( \tilde{X} \to X \).

Proof. The functor from \( \mathcal{G}_n^{\text{d}} \) to \( n \)-geometric Artin stacks is given by \( X \mapsto X^2 \), where

\[ X^2(A) := \text{Hom}_{\mathcal{G}_n^{\text{d}}}(\text{Spec } A, X) \]

for simplicial rings \( A \).

Essential surjectivity of this functor is given by Theorem 7.19 (or Theorem 7.7 in the quasi-compact case). Full faithfulness is given in Theorem 7.20 (Theorem 7.10 in the quasi-compact case).

\[ \text{\textbf{8. Alternative formulations}} \]

In this section, we will restrict to the strongly quasi-compact case in order to simplify the exposition. However, the statements carry over to the general case by taking suitable filtered colimits, as in Definition 7.14.

Intuitively, a derived Artin \( n \)-stack \( X \) should be a small neighbourhood of the associated Artin \( n \)-stack \( \pi^0 X \) given in Definition 1.10. We will see how any homotopy derived Artin \( n \)-hypergroupoid \( X \) is equivalent to the Zariski neighbourhood \( X' \) of \( \pi^0 X \) in \( X \), and even the étale neighbourhood \( X^h \) of \( \pi^0 X \) in \( X \). Under fairly strong Noetherian hypotheses, \( X \) is even equivalent to the formal neighbourhood \( \tilde{X} \).
8.1. Approximation and completion. We use use the terms formally étale and formally smooth in the sense of [TV2], meaning that a morphism $f : A \to B$ of simplicial rings is formally étale (resp. formally smooth) if the cotangent complex $L_{B/A} \simeq 0$ (resp. $L_{B/A}^*$ is equivalent to a retract of a direct sum of copies of $B$). Note that when $A$ and $B$ are discrete rings, these notions are stronger than those used classically (e.g. in [Mil]).

**Definition 8.1.** Recall from [TV2] Definition 2.2.2.1 that a morphism $f : A \to B$ in sRing is said to be strong if the maps $\pi_n(A) \otimes_{\pi_0(A)} \pi_0(B) \to \pi_n(B)$ are isomorphisms for all $n$.

**Lemma 8.2.** A morphism $f : A \to B$ in sRing is formally étale if and only if the relative cotangent complex $L_{B/A} \otimes_B \pi_0 B$ is contractible as a simplicial $\pi_0(B)$-module, i.e. $\pi_*(L_{B/A} \otimes_B \pi_0 B) = 0$.

**Proof.** The argument from the proof of [TV2] Lemma 2.2.2.12 shows that contractibility of $L_{B/A} \otimes_B \pi_0 B$ implies contractibility of the simplicial $B$-module $L_{B/A}$, which in turn means that $f$ is formally étale. This implies that $f$ is a weak equivalence.

**Lemma 8.3.** A morphism $f : A \to B$ in sRing is a weak equivalence if and only if $f$ is formally étale and $\pi_0(f)$ is an isomorphism.

**Proof.** [TV2] Proposition 2.2.2.7 implies that a formally étale map $f$ is étale if and only if $\pi_0 f$ is finitely presented. Since $\pi_0 f$ is an isomorphism, this holds, so $f$ is étale. Finally, [TV2] Theorem 2.2.2.10 states that étale morphisms $f$ in sRing are precisely strong morphisms for which $\pi_0(f)$ is étale. This implies that $f$ is a weak equivalence.

**Lemma 8.4.** If $f : A \to B$ is a morphism in sRing, for which each map $f_n : A_n \to B_n$ is formally étale (in the sense of [TV2]), then $f$ is formally étale (i.e. the cotangent complex $L_{B/A}$ is contractible). In fact, we only need $L_{B/A} \otimes_B \pi_0 B$ to be contractible for all $n$.

**Proof.** Let $\tilde{B}_n$ be the cotriple resolution of $B_n$ as an $A_n$-algebra. Thus $\tilde{B}_n$ is a bisimplicial ring, and $A \to \operatorname{diag} \tilde{B}_n$ is a cofibrant approximation to $B$ in $A \downarrow \text{sRing}$.

Therefore

$$L_{B/A} \otimes_B \pi_0 B \simeq \Omega(\operatorname{diag} \tilde{B}_n/A_n) \otimes_{\operatorname{diag} \tilde{B}_n} \pi_0 B,$$

and this is the diagonal of the bisimplicial diagram

$$M_{nm} := \Omega(\tilde{B}_{nm}/A_n) \otimes_{\tilde{B}_{nm}} \pi_0 B.$$

Since $f_n$ is formally étale,

$$L_n := \pi_* M_{nm} = \pi_* (L_{B/n} \otimes_B \pi_0 B) = 0,$$

for all $n$. Since $\pi_*(\operatorname{diag} M_{nm}) \cong H_*(\operatorname{Tot} M_{nm})$ (Eilenberg-Zilber), we have a convergent spectral sequence

$$0 = \pi_0(0) = \pi_*(L_n) \implies \pi_*(\operatorname{diag} M_{nm}),$$

so $f$ is formally étale by Lemma 8.2. \hfill \Box

**Definition 8.5.** Given a cosimplicial affine scheme $X$, define $\hat{X} \in \text{cAff}$ by setting $\hat{X}^n$ to be the formal neighbourhood of $\pi^0 X$ in $X^n$, so $O(\hat{X}^n) = \varprojlim_m O(X^n)/(I_n)^m$, where $I_n = \ker(O(X^n) \to \pi_0 O(X))$.

**Proposition 8.6.** For $X \in \text{cAff}$ leftheavy Noetherian, the morphism $f : \hat{X} \to X$ is a weak equivalence.

**Proof.** Write $X^n = \text{Spec} A_n$ and $\hat{X}^n = \text{Spec} \hat{A}_n$. We first describe the cotangent complex of $f_n : A_n \to \hat{A}_n$. Since $A_n$ is Noetherian, [Mat] Theorem 8.8 implies that $f_n$ is flat. Thus flat base change shows that

$$L_{\hat{A}_n/A_n} \otimes A_n \pi_0 A \simeq L_{(\hat{A}_n \otimes A_n \pi_0 A)/\pi_0 A},$$

where $L_{\hat{A}_n/A_n} \otimes A_n \pi_0 A$ is the cotangent complex of $f_n$. Since $\hat{A}_n$ is a flat $A_n$-algebra, its cotangent complex $L_{\hat{A}_n/A_n}$ is equivalent to $\pi_0 A$ as a simplicial $A_n$-module. Since $\pi_0 A$ is a direct sum of copies of $A_n$, the cotangent complex $L_{\hat{A}_n/A_n}$ is a retract of a direct sum of copies of $A_n$. Therefore, $f_n$ is formally étale. Since $f_n$ is flat, $f$ is a weak equivalence.
and note that then
$$\mathbb{L}\hat{A}_n/A_n \otimes_{\hat{A}_n} \pi_0 A = \mathbb{L}(\hat{A}_n/A_n \otimes \pi_0 A) \otimes_{\hat{A}_n/A_n} \pi_0 A.$$ [Mat] Theorem 8.7 implies that $\hat{A}_n \otimes_{A_n} \pi_0 A = \pi_0 A$, so we have shown that
$$\mathbb{L}\hat{A}_n/A_n \otimes_{\hat{A}_n} \pi_0 A \simeq 0,$$
and hence that $f$ is formally étale, by Lemma 8.4.

In order to satisfy the conditions of Lemma 8.3, it only remains to prove that $\pi_0 \hat{A} = \pi_0 A$. This follows by letting $I = \ker(A \to \pi_0 A)$, and observing that the surjections $I^\otimes n \to I^n$ imply that $\pi_0(I^n) = 0$ for all $n > 0$, so $\pi_0(\lim A/I^n) = \pi_0 A$. \qed

Note that [Mat] Theorem 8.12 implies that $\hat{X}$ is then also levelwise Noetherian.

**Corollary 8.7.** Any levelwise Noetherian homotopy derived Artin $n$-hypergroupoid $X$ is weakly equivalent to the homotopy derived Artin $n$-hypergroupoid $\hat{X}$, defined by setting $(\hat{X})_n := (X_n)$.  

**8.2. Local thickenings.**

**Definition 8.8.** Given a morphism $f : X \to Y$ of affine schemes, define $(X/Y)^{\text{loc}}$ to be the localisation of $Y$ at $X$. By [Ane] Proposition 3.3, this is given by
$$(X/Y)^{\text{loc}} = \text{Spec} \lim_{f(X) \subset U \subset Y \text{ open}} \Gamma(U, \mathcal{O}_Y) = \text{Spec} \Gamma(X, f_{\mathcal{P}r}^{-1} \mathcal{O}_Y^{\text{Zar}}),$$
where $f_{\mathcal{P}r}^{-1}$ is the presheaf inverse image functor.

**Lemma 8.9.** If $X_0 \to X$ is a morphism of affine schemes, and $X^l := (X_0/X)^{\text{loc}}$, then $X_0 \times_X X^l = X_0$.

**Proof.** Given any affine scheme $Y$ under $X_0$, write $Y^l$ for the localisation $(X_0/Y)^{\text{loc}}$. Properties of localisation from [Ane] give
$$(X_0 \times_X X^l)^l = X_0^l \times_X X^l = X_0,$$
so the morphism $X_0 \to X_0 \times_X X^l$ is conservative (i.e. an inverse limit of open immersions), and hence a monomorphism. Since $X^l \to X$ is also conservative, the projection $X_0 \times_X X^l \to X_0$ is also a monomorphism, so must in fact be an isomorphism. \qed

**Definition 8.10.** Given a cosimplicial affine scheme $X$, define $X^l \in \text{cAff}$ by $(X^l)^n := (\pi^0 X/X^n)^{\text{loc}}$.

**Proposition 8.11.** For $X \in \text{cAff}$, the morphism $f : X^l \to X$ is a weak equivalence.

**Proof.** Given any affine scheme $Y$ under $\pi^0 X$, write $Y^l$ for the localisation $(\pi^0 X/Y)^{\text{loc}}$.

By construction, the maps $f^n : (X^l)^n \to X^n$ have trivial cotangent complexes (being filtered limits of open immersions), so Lemma 8.4 implies that $f$ is formally étale. By Lemma 8.3, it only remains to prove that $\pi^0 f$ is an isomorphism.

Observe that the functor $Y \mapsto Y^l$ from affine schemes under $\pi^0 X$ to localisations preserves limits, since it has a right adjoint given by inclusion. Thus $\pi^0(X^l)$, being the equaliser of the maps $\partial^0, \partial^1 : (X^l)^0 \to (X^l)^1$, is just the localisation of the equaliser $\pi^0 X$ of $\partial^0, \partial^1 : X^0 \to X^1$. Thus
$$\pi^0(X^l) = (\pi^0 X)^l = \pi^0 X.$$ \qed

**Corollary 8.12.** Any homotopy derived Artin $n$-hypergroupoid $X$ is weakly equivalent to the homotopy derived Artin $n$-hypergroupoid $X^l$, defined by setting $(X^l)_n := (X_n)^l$. 

**Remark 8.13.** In fact, $s\text{Ring}_l$ has a simplicial model structure, in which a morphism is a weak equivalence or fibration whenever the underlying map in $s\text{Ring}$ is so.

**Theorem 8.14.** Given an $n$-geometric scheme $\mathfrak{Z}$ over a ring $R$, the $\infty$-category of $n$-geometric derived schemes $\mathfrak{X}$ over $R$ with $\pi^0\mathfrak{X} \simeq \mathfrak{Z}$ is weakly equivalent to the $\infty$-category of presheaves $\mathcal{A}^\bullet$ of simplicial $R$-algebras on the category $\text{Aff}_{\text{Zar}}(\mathfrak{Z})$ of affine open subschemes of $\mathfrak{Z}$, satisfying the following:

1. $\pi_0(\mathcal{A}^\bullet) = \mathcal{O}^\bullet_{\mathfrak{Z}}$;
2. for all $i$, the presheaf $\pi_i(\mathcal{A}^\bullet)$ is a quasi-coherent $\mathcal{O}^\bullet_{\mathfrak{Z}}$-module.

In particular, the corresponding homotopy categories are equivalent.

**Proof.** The key idea behind this proof is that given a local morphism $B \to A$, we can define a presheaf $(B/\mathcal{O}_Y)^{\text{loc}}$ on $Y := (\text{Spec } A)_{\text{Zar}}$, by $(B/\mathcal{O}_Y)^{\text{loc}}(\text{Spec } C) = (B/C)^{\text{loc}}$. By Proposition 3.19 and §7.2, it suffices to assume that $\mathfrak{Z}$ (and hence $\mathfrak{X}$) is strongly quasi-compact.

We begin by observing that Proposition 6.13 adapts to show that $\mathcal{A}^\bullet$ is a hypersheaf, so pairs $(\mathfrak{Z}, \mathcal{A}^\bullet)$ as above correspond to pairs $(Z, \mathcal{B}^\bullet)$, where $Z^\bullet$ is a Zariski $n$-hypergroupoid (with $a : Z \to \mathfrak{Z}$ such that $Z^0 \simeq \mathfrak{Z}$), and $\mathcal{B} = a^{-1}\mathcal{A}^\bullet$ is a homotopy-Cartesian presheaf of simplicial algebras on the simplicial site $\text{Aff}_{\text{Zar}}(Z^\bullet)$, with $\pi_0\mathcal{B}^\bullet \cong \mathcal{O}_Z$, and $\pi_n\mathcal{B}^\bullet$ a (Cartesian) $\mathcal{O}_Z$-module.

We then define the simplicial cosimplicial scheme $X^\bullet$ by letting $X_n$ be the cosimplicial scheme $\text{Spec } \Gamma(Z_n, \mathcal{B}^\bullet)$. It follows immediately from the properties of $\mathcal{B}^\bullet$ that $X^\bullet$ is a homotopy derived Zariski $n$-hypergroupoid, and that $\pi^0X = Z$. Therefore $\mathfrak{X}(\mathfrak{Z}, \mathcal{A}^\bullet) := X^\bullet$ has all the required properties.

If we replace $Z$ by a trivial relative Zariski $n$-hypergroupoid $Z' \to Z$ in the above construction, we get a homotopy trivial derived relative Zariski $n$-hypergroupoid $X' \to X$, so $(X')^\bullet \simeq X^\bullet$, and the functor $\mathfrak{X}(\_,-)$ is well-defined.

For the inverse construction, take a strongly quasi-compact derived $n$-geometric scheme $\mathfrak{X}$ with $\pi^0\mathfrak{X} \simeq \mathfrak{Z}$, and apply Theorem 7.7 to form a derived Zariski $n$-hypergroupoid $X$ with $X^\bullet \simeq \mathfrak{X}$. Let $Z := \pi^0X$, and for $\iota : Z \to X$, consider the presheaf $\mathcal{B}^\bullet := \iota_{\text{Pr}1}^\bullet \mathcal{O}_X$ on the simplicial site $\text{Aff}_{\text{Zar}}(Z^\bullet)$ — this is a Zariski presheaf of simplicial algebras. Since $\iota_{\text{Pr}1}$ commutes with colimits, $\pi_0(\mathcal{B}^\bullet) = \mathcal{O}_Z$. Properties of localisation imply that $(\mathcal{B}^\bullet(U)) = (O(X^n)/O(U))^{\text{loc}}$, so

$$\pi_i(\mathcal{B}_n^\bullet(U)) \cong \pi_0(\mathcal{B}_n^\bullet(U)) \otimes_{\pi_0(\mathcal{A}_n^\bullet(Z))} \pi_i(\mathcal{B}_n^\bullet(Z)) = \mathcal{O}_Z(U) \otimes_{\mathcal{O}_Z(Z)} \pi_i(\mathcal{B}_n^\bullet(Z)),$$

which means that $\pi_i(\mathcal{B}^\bullet)$ is homotopy-Cartesian and quasi-coherent.

Now, similarly to Proposition 6.13, this defines a presheaf $\mathcal{A}^\bullet(\mathfrak{X}) := R\mathcal{A}_a(\mathcal{B}^\bullet)$, for $a : Z^\bullet \to \mathfrak{Z}$. If we replace $X$ by a homotopy trivial relative Zariski $n$-hypergroupoid $X' \to X$ in the above construction, we get a weakly equivalent presheaf, so $\mathcal{A}^\bullet(\mathfrak{X})$ is well-defined.

Moreover, $a^{-1}\mathcal{A}^\bullet(\mathfrak{X}) \simeq \mathcal{B}^\bullet$, and $\text{Spec } \Gamma(Z_n, \mathcal{B}^\bullet) = X'^\bullet_n$, so

$$\mathfrak{X}(\mathfrak{Z}, \mathcal{A}^\bullet(\mathfrak{X})) \simeq (X'^\bullet)_n \simeq \mathfrak{X},$$

by Proposition 8.11. Conversely, to compute $\mathcal{A}^\bullet(\mathfrak{X}(\mathcal{A}^\bullet(\_)))$, we may choose the resolution $X^\bullet$ of $\mathfrak{X}$ used in the construction of $\mathfrak{X}(\mathcal{A}^\bullet)$, and then we have

$$\mathcal{A}^\bullet(\mathfrak{X}(\mathcal{A}^\bullet(\_))) = R\mathcal{A}_a(\Gamma(Z, a^{-1}\mathcal{A}^\bullet)/\mathcal{O}_Z)^{\text{loc}};$$

which is weakly equivalent to $R\mathcal{A}_a(a^{-1}\mathcal{A}^\bullet)$ by Proposition 8.11. This in turn is weakly equivalent to $\mathcal{A}^\bullet$, as $\mathcal{A}^\bullet$ is a hypersheaf.

**Remark 8.15.** In fact, since $\mathcal{A}^\bullet$ is a hypersheaf, we can replace the category $\text{Aff}_{\text{Zar}}(\mathfrak{Z})$ with any subcategory $\mathcal{U}$ generating the Zariski topology on $\mathfrak{Z}$. This means that the objects of $\mathcal{U}$ must cover $\mathfrak{Z}$, and that for any $U, V \in \mathcal{U}$, the $n$-geometric scheme $U \times^h V$ can be covered by objects of $\mathcal{U}$.\[\square\]
8.3. Henselian thickenings.

**Definition 8.16.** Given a morphism \( f : X \to Y \) of affine schemes, define \((X/Y)\)\text{\textsuperscript{hen}}\) to be the Henselisation of \( Y \) at \( X \). It follows from [Ane] Proposition 3.15 that this is given by

\[
(X/Y)\text{\textsuperscript{hen}} = \text{Spec} \left( \varprojlim_{X \to Y} \Gamma(U, \mathcal{O}_U) \right) = \text{Spec} \Gamma(X, f_{\mathcal{P}1}^{-1} \mathcal{O}_{Y, \text{ét}}),
\]

where \( f_{\mathcal{P}1}^{-1} \) is the presheaf inverse image functor.

**Remark 8.17.** Note that if \( f \) is also a closed immersion, then \((X,Y)\) is a Henselian pair (as in [Laf] or [Ray] §11).

**Lemma 8.18.** If \( X_0 \to X \) is a closed immersion of affine schemes, and \( X^h := (X_0/X)\)\text{\textsuperscript{hen}}, then \( X_0 \times_X X^h = X_0 \).

**Proof.** Given any affine scheme \( Y \) under \( X_0 \), write \( Y^h \) for the localisation \((X_0/Y)\)\text{\textsuperscript{hen}}. Properties of henselisation from [Ane] give

\[
(X_0 \times_X X^h) = X_0^h \times_X X^h = X_0,
\]

so the morphism \( X_0 \to X_0 \times_X X^h \) is pro-étale. It is automatically a closed immersion, so we must have \( X_0 \times_X X^h = X_0 \cup T \), for some \( T \). However, by [Gre1] Theorem 4.1, components of \( X^h \times_X X_0 \to X^h \) correspond to components of \( X_0 \times_X X_0 \to X_0 \). The latter is an isomorphism, so \( X_0 \to X_0 \times_X X^h \) must be an isomorphism on the set of components, hence an isomorphism.

**Definition 8.19.** Given a cosimplicial affine scheme \( X \), define \( X^h \in \text{cAff} \) by \((X^h)_n := (\pi^n X/X^n)\)\text{\textsuperscript{hen}}.

**Proposition 8.20.** For \( X \in \text{cAff} \), the morphism \( f : X^h \to X \) is a weak equivalence.

**Proof.** The proof of Proposition 8.11 carries over.

**Corollary 8.21.** For any simplicial cosimplicial affine scheme \( X \), there is a (levelwise) weak equivalence \( X^h \to X \) in \( \text{scAff} \), where \( X^h \) is defined by setting \((X^h)_i := (X_i)^h \). In particular, this applies when \( X \) is any homotopy derived Artin \( n \)\text{-hypergroupoid (in which case \( X^h \) is also).

**Remarks 8.22.** The maps \( \pi^0(X) \to X^h \) are levelwise closed immersions, so \( \pi^0(X) \to X^h \) is a Henselian pair. By [Gru] Theorem I.8, Henselian pairs are lifted by all smooth morphisms of affine schemes. This means that many techniques involving infinitesimal extensions can be adapted for Henselian pairs. Henselian pairs are also the largest class of maps with this lifting property, since lifting with respect to all standard étale morphisms forces a map to be Henselian, while lifting with respect to \( \mathbb{A}^1 \) forces a map to be a closed immersion. There is, in fact, a simplicial model structure on \( \text{sRing}^h \), for which a morphism is a weak equivalence or fibration whenever the underlying map in \( \text{sRing} \) is so. For this model structure, smooth rings are cofibrant.

**Theorem 8.23.** Given an \( n \)\text{-geometric Deligne–Mumford stack \( \mathcal{Z} \) over a ring \( R \), the \( \infty \)\text{-category of \( n \)\text{-geometric derived Deligne–Mumford stacks \( X \) over \( \mathcal{Z} \) with \( \pi^0 X \simeq \mathcal{Z} \) is weakly equivalent to the \( \infty \)\text{-category of presheaves \( \mathcal{A}_\bullet \) of simplicial \( R \)\text{-algebras on the category \( \text{Aff}_{\text{ét}}(\mathcal{Z}) \) of affine schemes étale over \( \mathcal{Z} \), satisfying the following:

(1) \( \pi_0(\mathcal{A}_\bullet) = \mathcal{O}_\mathcal{Z} \);

(2) for all \( i \), the presheaf \( \pi_i(\mathcal{A}_\bullet) \) is a quasi-coherent \( \mathcal{O}_\mathcal{Z} \)\text{-module.}

In particular, the corresponding homotopy categories are equivalent.

**Proof.** The proof of Theorem 8.14 carries over to this context, replacing Proposition 8.11 with Proposition 8.20.
Remark 8.24. As in Remark 8.15, we can replace the category $\text{Aff}_{\text{et}}(\mathcal{Z})$ with any subcategory $U$ generating the étale topology on $\mathcal{Z}$.

Remark 8.25. Let $\mathcal{X} = X^2$; since $X^h \to X$ is a weak equivalence and $\pi^0 X \to X^h$ is a Henselian pair on each level, the étale sites of $\mathcal{X}$ and $\pi^0 \mathcal{X}$ are weakly equivalent.

[Another way to look at this phenomenon is that taking the Postnikov tower of $O(X)$ (as in [GJ] Definition VI.3.4) allows us to describe $X$ as the colimit of a sequence $\pi^0 X \hookrightarrow \pi^{\leq 1} X \hookrightarrow \pi^{\leq 2} X \hookrightarrow \ldots$ of derived Artin hypergroupoids, with the $i$th closed immersion defined by an ideal sheaf weakly equivalent to $\pi_i O(X)[-i]$. Thus $\mathcal{X}$ can be expressed as the colimit of the sequence $\pi^0 \mathcal{X} \to \pi^{\leq 1} \mathcal{X} \to \pi^{\leq 2} \mathcal{X} \to \ldots$ of homotopy square-zero extensions, so the étale sites of $\mathcal{X}$ and $\pi^0 \mathcal{X}$ are isomorphic.]

Thus the corresponding $\infty$-topoi are equivalent, so giving the étale sheaf $\mathcal{O}_\mathcal{X}$ on $\mathcal{X}$ is equivalent to giving the étale sheaf $\iota^{-1} \mathcal{O}_\mathcal{X}$ on $\pi^0 \mathcal{X}$. The same is not true for smooth morphisms, where the $\infty$-topoi differ (although the homotopy categories are the same), which suggests that we cannot expect an exact analogue for derived Artin stacks. However, Corollary 10.11 will provide a strong substitute.

8.4. DG stacks and dg-schemes. Using the Eilenberg-Zilber shuffle product, the normalisation functor $N^s$ from Definition 0.1 extends to a functor

$$N^s : \text{sRing} \to \text{dg} \text{Ring}$$

from simplicial rings to (graded-commutative) algebras in non-negatively graded chain complexes. Where no confusion is likely, we will denote this functor by $N$. It has a left adjoint $N^*$, and there is a model structure on $\text{dg} \text{Ring}$, defined by setting weak equivalences to be quasi-isomorphisms, and fibrations to be maps which are surjective in strictly positive degrees.

In non-zero characteristic, the normalisation functor is poorly behaved. However, the functor

$$N : \text{sAlg}_\mathbb{Q} \to \text{dg}_+ \text{Alg}_\mathbb{Q}$$

is a right Quillen equivalence by [Qui], where $\text{Alg}_\mathbb{Q}$ is the category $\mathbb{Q} \downarrow \text{Ring}$ of $\mathbb{Q}$-algebras, and $\text{dg}_+ \text{Alg}_\mathbb{Q} = \mathbb{Q} \downarrow \text{dg}_+ \text{Ring}$. In particular, this implies that it gives a weak equivalence of the associated $\infty$-categories, and hence that

$$N : \text{Ho}(\text{sAlg}_A) \to \text{Ho}(\text{dg}_+ \text{Alg}_A)$$

is a weak equivalence for any $\mathbb{Q}$-algebra $A$.

Remark 7.2 now applies, showing that Theorems 7.7 and 7.10 adapt to the HAG context given by smooth morphisms and $\text{dg}_+ \text{Alg}_A$. Note that this is not the same as the HAG context of $D$-stacks in [TV2], since our chain complexes are non-negatively graded. However, since $N$ is a Quillen equivalence, this HAG context is equivalent to that of $D^-$-stacks. Explicitly:

Definition 8.26. A morphism $f : A \to B$ in $\text{dg} \text{Ring}$ is said to be smooth if $H_0(f) : H_0 A \to H_0 B$ is smooth, and the maps $H_n(A) \otimes_{H_0(A)} H_0(B) \to H_n(B)$ are isomorphisms for all $n$.

Definition 8.27. Let $\text{DG}^+ \text{Aff}_\mathbb{Q}$ be the opposite category to $\text{dg}_+ \text{Alg}_\mathbb{Q}$. Define the denormalisation functor $D : \text{DG}^+ \text{Aff}_\mathbb{Q} \to \text{cAff}_\mathbb{Q}$ to be opposite to $N^*$, so $DX(A) = X(NA)$, for $X \in \text{DG}^+ \text{Aff}_\mathbb{Q}, A \in \text{sAlg}_\mathbb{Q}$. This has left adjoint $D^*$, given by $D^* \text{Spec} A = \text{Spec} NA$.

Definition 8.28. Given $Y_\bullet \in \text{s}(\text{DG}^+ \text{Aff})$, define a (homotopy) strongly quasi-compact (trivial) DG Artin $n$-hypergroupoid over $Y_\bullet$ to be a morphism $X_\bullet \to Y_\bullet$ in $\text{sDG}^+ \text{Aff}$, satisfying the relevant conditions of Definition 7.3.

The following is an immediate consequence of the Quillen equivalence between simplicial and dg algebras:
Proposition 8.29. If \( f : X \to S \) is a relative DG Artin n-hypergroupoid, then \( Df : DX \to DS \) is a relative derived Artin n-hypergroupoid, which is trivial whenever \( f \) is. If \( S = D^*Z \) for some \( Z \in \text{scAff} \), then the map
\[ D^*DX \times_{D^*DS} S \to X \]
is a weak equivalence.

If \( g : Y \to T \) is a homotopy relative derived Artin n-hypergroupoid, then \( D^*g : D^*Y \to D^*T \) is a homotopy strongly quasi-compact relative DG Artin n-hypergroupoid, which is trivial whenever \( g \) is. If \( D^*Y \) is a Reedy fibrant approximation to \( D^*Y \) over \( D^*T \), then the map
\[ Y \to DD^*Y \times_{DD^*T} T \]
is a weak equivalence.

Note that the functor \( D \) only behaves well when applied to Reedy fibrations, so the proposition does not apply if \( g \) is only a homotopy relative DG Artin n-hypergroupoid.

However, we can define a homotopy inverse to \( N \), similar to the Thom-Sullivan functor, which does preserve weak equivalences.

Definition 8.30. Let \( \Omega_n \) be the cochain algebra
\[ \mathbb{Q}[t_0, t_1, \ldots, t_n, dt_0, dt_1, \ldots, dt_n]/\left( \sum t_i - 1, \sum dt_i \right) \]
of rational differential forms on the \( n \)-simplex \( \Delta^n \). These fit together to form a simplicial diagram \( \Omega_* \) of DG-algebras.

Definition 8.31. Given a simplicial module \( S_* \) and a cosimplicial module \( C^* \), define \( S \otimes_{\Delta^*} C \) by
\[ S \otimes_{\Delta^*} C := \{ x \in \prod_n S_n \otimes C^n : (1 \otimes \partial^i)x_n = (\partial_i \otimes 1)x_{n+1}, (1 \otimes \sigma^i)x = (\sigma_i \otimes 1)x_{n-1} \}. \]

Similarly, given a chain complex \( S_* \) and a cochain complex \( C^* \), define \( S \otimes_{\Delta^*} C \) by
\[ S \otimes_{\Delta^*} C := \{ x \in \prod_n S_n \otimes C^n : (1 \otimes d)x_n = (d \otimes 1)x_{n+1} \}. \]

Definition 8.32. Define the functor \( T : dg_+\text{Alg}_\mathbb{Q} \to \text{sAlg}_\mathbb{Q} \) by \( T(B)_n := \Omega_n \otimes_{\Delta^*} B \).

Define \( T : DG^+\text{Aff}_\mathbb{Q} \to \text{cAff}_\mathbb{Q} \) by \( T(\text{Spec} A) := \text{Spec} TA \).

Proposition 8.33. For a \( \mathbb{Q} \)-algebra \( A \), the functor \( T \) is quasi-inverse to normalisation \( N : \text{Ho}(\text{sAlg}_A) \to \text{Ho}(dg_+\text{Alg}_A) \).

Proof. Let \( \mathbb{Q}^\Delta \) be the simplicial cosimplicial \( \mathbb{Q} \)-algebra given by \( (\mathbb{Q}^\Delta)_m^n = \mathbb{Q}^{\Delta^n_m} \). As in [HS], there are cochain quasi-isomorphisms \( \int : \Omega_m \to N_c(\mathbb{Q}^\Delta)_m \) for all \( m \), where \( N_c \) denotes cosimplicial normalisation. Explicitly,
\[ (\int \omega)(f) = \int_{|\Delta^n|} f^*\omega, \]
for \( \omega \in \Omega^n_m, f \in \Delta^n_m \). Denormalisation gives a quasi-isomorphism \( \int : N^{-1}_c\Omega_m \to (\mathbb{Q}^\Delta)_m \) of cosimplicial complexes, and analysis of the shuffle product shows that this is a quasi-isomorphism of cosimplicial algebras.

Now, for \( A \in \text{sAlg}_A \),
\[ T(N^sA) = (N^sA) \otimes_{\Delta^*} \mathbb{Q} \cong A \otimes_{\Delta^*} (N_c\Omega), \]
and \( \int \) gives a quasi-isomorphism
\[ TN^sA \to A \otimes_{\Delta^*} (\mathbb{Q}^\Delta) = A \]
of simplicial algebras, so \( T \) is a quasi-inverse to \( N^s \) on the homotopy categories. \( \square \)
Corollary 8.34. If $f : X \to S$ is a homotopy relative DG Artin $n$-hypergroupoid, then $Tf : TX \to TS$ is a homotopy relative derived Artin $n$-hypergroupoid, which is trivial whenever $f$ is. If $S = D^+Z$ for some $Z \in \text{scAff}$, then the map

$$D^+TX \times_{D^+TS} S \to X$$

is a weak equivalence.

We now introduce an analogue of the results of §§8.1–8.3.

Definition 8.35. Given $A \in d\mathcal{g}_+\mathcal{A}_{\mathbb{Q}}$, define localisation and henselisation of $A$ by

$$A^l := (A_0/H_0A)_{\text{loc}} \otimes_{A_0} A, \quad A^h := (A_0/H_0A)_{\text{hen}} \otimes_{A_0} A.$$  
Define completion by letting $I := \ker(A_0 \to H_0A)$ and setting

$$\hat{A} := \lim_{\leftarrow n} A/I^nA.$$

Lemma 8.36. The maps $A \to A^l$ and $A \to A^h$ are weak equivalences. If $A_0$ is Noetherian and each $A_n$ is a finite $A_0$-module, then $A \to \hat{A}$ is also a weak equivalence.

Proof. We begin by noting that the maps $A_0 \to A_0^l$ and $A_0 \to A_0^h$ are flat, so

$$H_\ast(A^l) \cong H_\ast(A) \otimes_{A_0} A_0^l, \quad H_\ast(A^h) \cong H_\ast(A) \otimes_{A_0} A_0^h.$$  
Now, Lemmas 8.9 and 8.18 imply that $H_0(A^l) \otimes_{A_0} A_0^l \cong H_0(A)$ and $H_0(A^h) \otimes_{A_0} A_0^h \cong H_0(A)$. Since $H_\ast(A)$ is automatically an $H_0(A)$-module, this completes the proof in the local and Henselian cases.

If $A_0$ is Noetherian, then [Mat] Theorem 8.8 implies that $A_0 \to \hat{A}_0$ is flat. If $A_n$ is a finite $A_0$-module, then [Mat] Theorem 8.7 implies that $\hat{A}_n = \hat{A}_0 \otimes_{A_0} A_n$. Thus

$$H_\ast(\hat{A}) \cong H_\ast(A) \otimes_{A_0} \hat{A}_0,$$

and applying [Mat] Theorem 8.7 to the $A_0$-module $H_0A$ gives that $H_0(A) \otimes_{A_0} \hat{A}_0 \cong H_0(A)$, so $H_\ast(A) \cong H_\ast(\hat{A})$, as required. \qed

Corollary 8.37. For any simplicial diagram $X$ in $D^+\text{Aff}$, there are (levelwise) weak equivalences $X^l \to X$ and $X^h \to X$ in $\text{scAff}$, where $X^l$ is defined by setting $(X_i)^l := (X_i)^l$ and similarly for $X^h$. In particular, this applies when $X$ is any homotopy DG Artin $n$-hypergroupoid (in which case $X^l$ and $X^h$ are also). If $X^0$ is levelwise Noetherian, with each $\mathcal{O}(X^n)$ levelwise coherent on $X^0$, the same is true of the levelwise completion $\hat{X} \to X$.

We are now in a position to describe the $D^-$-stack associated to a dg-scheme.

Definition 8.38. Recall from [CFK] Definition 2.2.1 that a dg-scheme $X = (X^0, \mathcal{O}_X)$ consists of a scheme $X^0$, together with a sheaf $\mathcal{O}_X$ of quasi-coherent non-negatively graded (chain) dg-algebras over $\mathcal{O}_{X^0}$, with $\mathcal{O}_{X^0,0} = \mathcal{O}_{X^0}$.

Definition 8.39. The degree 0 truncation $\pi^0X$ of a dg-scheme $X$ is defined to be the closed subscheme

$$\pi^0X := \text{Spec} \mathcal{H}_0(\mathcal{O}_X) \subset X^0.$$  
This is denoted by $\pi_0X$ in [CFK].

A morphism $f : X \to Y$ of dg-schemes is said to be a quasi-isomorphism if it induces an isomorphism $(\pi^0X, \mathcal{H}_0(\mathcal{O}_X)) \to (\pi^0Y, \mathcal{H}_0(\mathcal{O}_Y))$.

Note that Lemma 8.36 implies that for a dg-scheme $X = (X^0, \mathcal{O}_X)$, the maps

$$(\pi^0X/X)^{\text{hen}}, a^*b^* \mathcal{O}_X) \xrightarrow{a} (\pi^0X/X)^{\text{loc}}, b^* \mathcal{O}_X) \xrightarrow{b} X$$

are quasi-isomorphisms. If $X^0$ is locally Noetherian and each $\mathcal{O}_{X,n}$ coherent, then the map

$$((\pi^0X/X)^{\text{hen}}, c^*a^*b^* \mathcal{O}_X) \xrightarrow{c} ((\pi^0X/X)^{\text{loc}}, a^*b^* \mathcal{O}_X)$$

is also a weak equivalence.
is also a quasi-isomorphism, where \((\pi^0 X / X)\) is the formal completion of \(X\) along \(\pi^0 X\).

Given a dg-scheme \(X = (X^0, \mathcal{O}_X)\), with \(X^0\) semi-separated, we may take an affine cover \(U = \coprod_i U_i\) of \(X^0\), and define the simplicial scheme \(Y^0\) by \(Y^0 := \cosk_0(U/X^0)\). Hence
\[
Y^0_n = \coprod_{i_0, \ldots, i_n} U \times_{X^0} U \times_{X^0} \ldots \times_{X^0} U,
\]
so \(Y^0_n\) is a disjoint union of affine schemes (as \(X^0\) is semi-separated, i.e. \(X^0 \to X^0 \times X^0\) is affine). Thus \(Y^0\) is a \((1, \text{open})\)-hypergroupoid and \(a : Y^0 \to X^0\) a resolution, and we may define a homotopy DG \((1, \text{open})\)-hypergroupoid \(Y\) by
\[
\Gpd(X) := \text{Spec}(a^{-1}\mathcal{O}_X).
\]

A fortiori, this is a DG Artin 1-hypergroupoid.

Remarks 8.40. (1) Note that \(Y^0\) is a variant of the unnormalised Čech resolution for \(X^0\). The standard normalised Čech resolution is given by \(\coprod_{i_0 \leq \ldots \leq i_n} U_{i_0} \cap \ldots \cap U_{i_n}\) in level \(n\), so the standard unnormalised Čech resolution \((X^0)'\) is given by \((X^0)'_n = \coprod_{i_0 \leq \ldots \leq i_n} U_{i_0} \cap \ldots \cap U_{i_n}\). There is a canonical morphism \((X^0)' \to Y\), but the partial matching map
\[
(X^0)'_2 \to M_{A^2}(X^0)'
\]
is not surjective, so \((X^0)'\) is not a relative Zariski \(n\)-hypergroupoid for any \(n\).

(2) If \(X^0\) is quasi-compact, then we may take \(U\) to be affine, and so \(Y\) will be a strongly quasi-compact DG hypergroupoid. If \(X^0\) is not semi-separated, we could instead apply Theorem 4.9 to obtain a \((2, \text{open})\)-hypergroupoid \(Y^0\), and then a homotopy DG \((2, \text{open})\)-hypergroupoid \(Y\) by the formula above.

(3) If \(X\) is a dg-manifold in the sense of [CFK], then \(Y^h\) will be Reedy fibrant over the model category \(sDG^+\text{Aff}^h\) (defined analogously to Remarks 8.22). [CFK] Theorem 2.6.1 shows that if \(X^0\) is quasi-projective, then there exists a dg-manifold \(X'\) quasi-isomorphic to \(X\), which can be thought of as a form of fibrant approximation. In fact, the construction is such that \((X')'\) is Reedy fibrant over the model category \(sDG^+\text{Aff}^l\), since \(\mathcal{O}_{X'}\) is locally free for the Zariski topology.

Theorem 8.41. Given an \(n\)-geometric scheme (resp. \(n\)-geometric Deligne–Mumford stack) \(\mathcal{X}\) over a \(\mathbb{Q}\)-algebra \(R\), the \(\infty\)-category of \(n\)-geometric derived schemes (resp. \(n\)-geometric derived Deligne–Mumford stacks) \(\mathcal{X}\) over \(R\) with \(\pi^0 \mathcal{X} \simeq \mathcal{X}\) is weakly equivalent to the \(\infty\)-category of presheaves \(\mathcal{A}_p\) of non-negatively graded dg \(R\)-algebras on the category \(\text{Aff}_{zar}(3)\) (resp. \(\text{Aff}_{et}(3)\)), satisfying the following:

1. \(\pi_0(\mathcal{A}_p) = \mathcal{O}_3\);
2. for all \(i\), the presheaf \(H_i(\mathcal{A}_p)\) is a quasi-coherent \(\mathcal{O}_3\)-module.

In particular, the corresponding homotopy categories are equivalent.

Proof. The proofs of Theorems 8.14 and 8.23 carry over to this context, replacing \(\iota^{-1}_p \mathcal{O}_X\) with the presheaves \((O(X^0)/\mathcal{O}^Z)_{\text{loc}} \otimes_{O(X^0)} O(X)\bullet\) or \((O(X^0)/\mathcal{O}^Z)_{\text{hens}} \otimes_{O(X^0)} O(X)\bullet\).

\[\square\]

Corollary 8.42. If \(\mathcal{X}\) is an \(n\)-geometric Deligne–Mumford stack with \(\pi^0 \mathcal{X}\) a quasi-affine scheme, then there exists a dg-scheme \(X\) with \(\Gpd(X)^\ell \simeq \mathcal{X}\).
Proof. Let $Y = \text{Spec} \Gamma(\pi^0 \mathcal{X}, \mathcal{O}_{\pi^0 \mathcal{X}})$; since $\pi^0 \mathcal{X} \to Y$ is an open immersion, the complement $Z$ is closed. Take a presheaf $\mathcal{A}$ as in theorem 8.41, and let $W := \text{Spec} \Gamma(\pi^0 \mathcal{X}, \mathcal{O}_0)$, noting that $Y \to W$ is a closed immersion. Now set $X^0$ to be the quasi-affine scheme $W - Z$, and define $\mathcal{O}_{X^n}$ to be the quasi-coherent sheaf associated to the module $\Gamma(\pi^0 \mathcal{X}, \mathcal{A}_n)$ on the affine $W$.

9. Derived Quasi-coherent Sheaves and the Cotangent Complex

Given a simplicial ring $A$, the simplicial normalisation functor $N^s$ induces an equivalence between the categories $s\text{Mod}(A)$ of simplicial $A$-modules, and $dg\text{Mod}(N^sA)$ of $N^sA$-modules in non-negatively graded chain complexes, where $N^sA$ is given the graded-commutative product given by the Eilenberg-Zilber shuffle product. As observed in [TV2] §2.2.1, this extends to give a weak equivalence between the $\infty$-category of stable $A$-modules, and the $\infty$-category $dg\text{Mod}(N^sA)$ of $N^sA$-modules in $\mathbb{Z}$-graded chain complexes, and hence an equivalence between the corresponding homotopy categories.

Definition 9.1. If $X$ is a strongly quasi-compact homotopy derived Artin $l$-hypergroupoid, define the simplicial cosimplicial algebra $O(X)_\bullet$ by $O(X)_m^n = \Gamma(X^m_n \otimes \mathcal{O}_X)$. Define the category $dg\text{Mod}(X)$ to have objects $M$ consisting of complexes $M^n \in dg\text{Mod}(N^sO(X)_n)$ for all $n$, together with morphisms

$$\partial^i : M^n \otimes N^sO(X)_n \cdot \partial^i \to M^{n+1}$$

$$\sigma^i : M^n \otimes N^sO(X)_n \cdot \sigma^i \to M^{n-1},$$

satisfying the usual cosimplicial identities.

A morphism $f : M \to N$ in $dg\text{Mod}(X)$ is said to be a weak equivalence if the maps $f^n : M^n \to N^n$ are all weak equivalences of (simplicial) $O(X)_n$-modules.

Define $dg\text{Mod}(X)_{\text{cart}} \subset dg\text{Mod}(X)$ to be the full subcategory consisting of those $M$ for which the morphisms $\partial^i$ are all quasi-isomorphisms. Let $\text{Ho}(dg\text{Mod}(X))$ and $\text{Ho}(dg\text{Mod}(X)_{\text{cart}})$ be the categories obtained by localising at weak equivalences. Objects of $dg\text{Mod}(X)_{\text{cart}}$ are called derived quasi-coherent sheaves on $X$.

Definition 9.2. If $X$ is an arbitrary homotopy derived Artin $l$-hypergroupoid, write $X = \lim_{\alpha \in I} X_\alpha$ for $X_\alpha$ strongly quasi-compact, and define the category $dg\text{Mod}(X)_{\text{cart}} \subset \lim dg\text{Mod}(X_\alpha)_{\text{cart}}$ of derived quasi-coherent sheaves on $X$ to consist of objects $\{M_\alpha\}_{\alpha \in I}$ such that for $f : \alpha \to \beta$ in $I$, the maps

$$Lf^*M_\beta \to M_\alpha$$

are weak equivalences.

Proposition 9.3. The $\infty$-category of derived quasi-coherent sheaves on $X$ is equivalent to the $\infty$-category of homotopy-Cartesian modules on $X^2$ (as in [TV2] Definition 1.2.12.1) or equivalently, of quasi-coherent complexes on $X^2$ (in the sense of [Lur1] §5.2).

Proof. The proof of Corollary 6.15 carries over to this context.

9.1. Derived direct images.

Definition 9.4. Define $R_{\text{cart}} : dg\text{Mod}(X) \to dg\text{Mod}(X)$ using the formulae of Lemma 6.26.

Proposition 9.5. If $X$ is an arbitrary homotopy derived Artin $l$-hypergroupoid and $\mathcal{F} \in dg\text{Mod}(X)$, the complex $R_{\text{cart}}\mathcal{F}$ is homotopy-Cartesian.

Proof. We could reason as in §6.4. Here is an alternative proof. First take $X$ strongly quasi-compact.
Consider the maps $\partial_i(\Dec_+) : \Dec^{n+1}_+ X \to \Dec^n_+ X$ associated to the comonad $\Dec_+$. If we write $\varepsilon_Y : \Dec_+ Y \to Y$ for the counit of the adjunction, then $\partial_i(\Dec_+) = (\Dec_+)^i \varepsilon_{\Dec^{n-i}_+ Y}$. From Lemma 3.8, it follows that the morphisms

$$\partial_i(\Dec_+), \partial_0 : \Dec^{n+1}_+ X \to \Dec^n_+ X \times_{X_{n-1}} X_n$$

are trivial relative homotopy derived Artin l-hypergroupoids.

Since $\partial_i : X_n \to X_{n-1}$ is faithfully flat (being a smooth covering), flat base change implies that

$$\partial_i^\ast : \Dec^\ast((\Dec^+)_{n} X, (\partial^+_i)^{n} \mathcal{F}) \simeq \Dec^\ast((\Dec^+_n X \times_{X_{n-1}}, \partial_i X_n, \pr_1^\ast((\partial^+_i)^{n} \mathcal{F})),$$

Since $(\partial_i(\Dec_+), \partial_0) : \Dec^{n+1}_+ X \to \Dec^n_+ X \times_{X_{n-1}} X_n$ is a trivial relative homotopy derived Artin l-hypergroupoid, it is a flat hypercovering, so a cohomological descent morphism. Thus

$$\Dec^\ast((\Dec^+_n X \times_{X_{n-1}}, \partial_i X_n, \pr_1^\ast((\partial^+_i)^{n} \mathcal{F}) \simeq \Dec^\ast((\Dec^+)_{n+1} X, (\partial^+_i)^{n+1} \mathcal{F}),$$

and we have shown that

$$\partial_i^\ast : \Dec^\ast((\Dec^+)_{n} X, \partial_i^\ast \mathcal{F})|_{X_{n-1}} \to \Dec^\ast((\Dec^+)_{n} X, \partial_i^\ast \mathcal{F})|_{X_n}$$

is a quasi-isomorphism, as required.

If $X$ is not strongly quasi-compact, write $X = \{X_\alpha\}_{\alpha \in I}$, for $X_\alpha$ quasi-compact, and note that $\Dec^\ast$ applied levelwise in $I$ defines an object of $\dgMod(X)$, by flat base change. \qed

**Definition 9.6.** We may therefore write $\Dec^\ast : \dgMod(X) \to \dgMod(X)_{\cart}$. Denote the forgetful functor by $\cart^\ast : \dgMod(X)_{\cart} \to \dgMod(X)$.

**Proposition 9.7.** For $X$ as above and $\mathcal{F} \in \dgMod(X)$, there is a natural transformation $\varepsilon_{\cart} : \Dec^\ast \mathcal{F} \to \mathcal{F}$, which is a weak equivalence whenever $\mathcal{F}$ is homotopy-Cartesian.

**Proof.** Given a module $\mathcal{G} \in \dgMod(Y)$ for a simplicial cosimplicial affine scheme $Y$, we may define $\Dec^+_\mathcal{G} \in \Mod(\Dec_+ Y)$ by $(\Dec^+_\mathcal{G})|_{(\Dec_+ Y)_m} = \mathcal{G}|_{Y_{m+1}}$, with $\partial^i_{\Dec^+_\mathcal{G}} = \partial^i_G$ and $\sigma^i_{\Dec^+_\mathcal{G}} = \sigma^i_G$.

For any $\mathcal{F} \in \dgMod(X)$, this gives us a module $(\Dec^+_\mathcal{F})$ on the bisimplicial scheme $(\Dec_+)_\bullet X$, and there is a natural map

$$\partial^\top : \Dec^\top \mathcal{F} \to (\Dec^+_\mathcal{F}),$$

which is a quasi-isomorphism whenever $\mathcal{F}$ is Cartesian. Applying the functor $\Tot H^\ast \Dec^\ast$ gives a map

$$\partial^\top : \Dec^\ast \mathcal{F} \to \Tot H^\ast \Dec^\ast((\Dec_+)_\bullet X, (\Dec^+_\mathcal{F}).$$

Now, the deformation retraction $\partial_0 : \Dec_+ Y \to Y_0$ corresponds to a deformation retraction

$$\mathcal{G}|_{Y_0} \xrightarrow{\partial^0} \Dec^\ast((\Dec_+)_\bullet X, (\Dec^+_\mathcal{F})) \xrightarrow{\partial^0_{\Dec^+_\mathcal{F}}(\sigma^0)} \mathcal{G}|_{Y_0}$$

of cosimplicial complexes on $Y_0$. Applying this to $Y = (\Dec^+_n X)$ and $\mathcal{G} = (\Dec^+_n \mathcal{F})$ for all $n$ gives a deformation retraction

$$\mathcal{F} \xrightarrow{\partial^0} \Dec^\ast((\Dec_+)_\bullet X, (\Dec^+_\mathcal{F})) \xrightarrow{\partial^0} \mathcal{F}.$$

We now define $\varepsilon_{\cart}$ to be

$$\partial^0 : \Dec^\ast((\Dec_+)_\bullet X, (\Dec^+_\mathcal{F})) \xrightarrow{\partial^0} \mathcal{F}.$$

\qed

**Definition 9.8.** Let $\HOM$ be $\mathbb{Z}$-graded derived Hom for complexes. Explicitly, $\HOM(U, V)_n$ is the space of degree $n$ graded homomorphisms from $U$ to $V$ (ignoring the differentials), and the chain complex structure on $\HOM$ is given by $df = d_V \circ f \pm f \circ d_U$. Note that $N^\ast \HOM(U, V) = \tau_{\geq 0} \HOM(U, V)$, where $N^\ast$ is simplicial normalisation and $\tau$ is good truncation.
Corollary 9.9. For $X$ as above, the functor $R\text{cart}_* \colon \text{Ho}(dg\text{Mod}(X)) \to \text{Ho}(dg\text{Mod}(X)_{\text{cart}})$, which is right adjoint to $\text{cart}^*$, with $R\text{cart}_* \circ \text{cart}^* \simeq \text{id}$.

If $L, M \in dg\text{Mod}(X)$, with $L$ cofibrant and homotopy-Cartesian, then

$$\varepsilon_{\text{cart}^*} : \text{HOM}_{dg\text{Mod}(X)}(L, M) \to \text{HOM}_{dg\text{Mod}(X)}(L, R\text{cart}_* M)$$

is a weak equivalence.

Proof. Only the final statement is not immediate. We note that

$$H_1 \text{HOM}_{dg\text{Mod}(X)}(L, M) \cong \text{Hom}_{\text{Ho}(dg\text{Mod}(X))}(\text{cart}^* L[-1], M),$$

and that $L[-1]$ is also cofibrant and homotopy-Cartesian. Now,

$$\text{Hom}_{\text{Ho}(dg\text{Mod}(X))}(\text{cart}^* L[-1], M) \cong \text{Hom}_{\text{Ho}(dg\text{Mod}(X)_{\text{cart}})}(L[-1], R\text{cart}_* M) \cong \text{Hom}_{\text{Ho}(dg\text{Mod}(X))}(L[-1], R\text{cart}_* M),$$

as required. \qed

Definition 9.10. Given a strongly quasi-compact morphism $f : X \to Y$ of homotopy derived Artin $l$-hypergroupoids, we may now define

$$Rf_*^{\text{cart}} := R\text{cart}_* \circ f_\ast \circ \text{cart}^* : dg\text{Mod}(X)_{\text{cart}} \to dg\text{Mod}(Y)_{\text{cart}}.$$

9.2. Cotangent complexes. For simplicity, we will assume in this section that all derived Artin hypergroupoids are strongly quasi-compact. All constructions and statements can, however, be extended to arbitrary derived Artin hypergroupoids by taking filtered limits.

Fix a strongly quasi-compact derived Artin $m$-hypergroupoid $X \to S$.

Definition 9.11. We make $dg\text{Mod}(X)$ into a simplicial category by setting (for $K \in S$)

$$(M^K)^n := (M^n)^K,$$

as an $\mathbb{N}^sO(X)^n$-module in chain complexes. This has a left adjoint $M \mapsto M \otimes K$. Given a cofibration $K \hookrightarrow L$, we write $M \otimes (L/K) := (M \otimes L)/(M \otimes K)$.

Given $M \in dg\text{Mod}(X)$, define $\underline{M} \in (dg\text{Mod}(X))^\Delta$ to be the cosimplicial complex given in cosimplicial level $n$ by $M \otimes \Delta^n$.

Lemma 9.12. If $N^s\Omega(X/S) \in dg\text{Mod}(X)$ denotes the chain complex given on $X_n$ by $N^s\Omega_{X_n/S_n}$, then

$$N^s\Omega(X/S) \otimes K = \eta^* N^s\Omega(X^K/S^K),$$

for $\eta : X \to X^K$ corresponding to the constant map $K \to \bullet$.

Proof. This follows immediately from the definitions of $X^K$ and $M \otimes K$. \qed

Definition 9.13. Given $M \in dg_{+}\text{Mod}(X)$, define the derived deformation space by

$$\underline{\text{Der}}(X/S, M) := \underline{\text{Hom}}_{\text{scAff}_{X/S}}(\text{Spec}(\mathcal{O}_X \oplus N^{-1}_s M), X) \in S,$$

for the simplicial structure of Definition 7.8. Since $\text{Spec}(\mathcal{O}_X \oplus N^{-1}_s M)$ is an abelian cogroup object in $X \downarrow \text{scAff} \downarrow S$, $\underline{\text{Der}}(X/S, M)$ has the natural structure of a simplicial abelian group.

Lemma 9.14. For $M \in dg_{+}\text{Mod}(X)$, the simplicial group $\underline{\text{Der}}(X/S, M)$ is given in level $n$ by $\text{Hom}(\Omega(X/S)^n, M)$.

Definition 9.15. Define the cotangent complex $\mathbb{L}^{X/S} \in dg\text{Mod}(X)$ by $\mathbb{L}^{X/S} := \text{Tot} N_c \gamma^s \Omega(X/S)$, where $N_c$ denotes cosimplicial conormalisation (Definition 0.2), and $\text{Tot}$ is the total functor from cochain chain complexes to chain complexes. Note that Lemma 9.12 implies that $N_c^\gamma \Omega(X/S) \equiv \eta^* N\gamma^s \Omega(X^{\Delta^i}/X^{\Delta^j} \times_{S^{\Delta^i}} S^{\Delta^j})$. 

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Lemma 9.16. If we write \( i : \pi^0 X \to X \), then \( i^* \mathbb{L}^{X/S} \) is equivalent to the complex
\[
i^* N^s \Omega(X/S) \to \Omega(\pi^0 X/\pi^0 S) \otimes (\Delta^1/\Delta_0) \to \cdots \to \Omega(\pi^0 X/\pi^0 S) \otimes (\Delta^m/\Delta_0^m),
\]
in (chain) degrees \([-m, \infty)\). If moreover \( f : X \to S \) is smooth, then \( i^* \mathbb{L}^{X/S} \) is equivalent to the complex
\[
\Omega(\pi^0 X/\pi^0 S) \to \Omega(\pi^0 X/\pi^0 S) \otimes (\Delta^1/\Delta_0) \to \cdots \to \Omega(\pi^0 X/\pi^0 S) \otimes (\Delta^m/\Delta_0^m),
\]
which is locally projective and concentrated in degrees \([-m, 0]\).

Proof. The definition of cosimplicial normalisation shows that
\[
N^i c_N^i \Omega(X/S) \cong \eta^* N^s \Omega(X^\Delta^i/X^\Delta^i \times S^\Delta^i S^\Delta^i),
\]
and 3.9 implies that for \( i \geq m \), the maps \( \theta_i(f) : X^\Delta^i \to X^\Delta^i \times Y^\Delta^i \) are trivial relative 0-hypergroupoids, hence levelwise weak equivalences, so \( N^i c_N^i \Omega(X/S) \) is contractible for \( i > m \). For all \( i > 0 \), the maps \( \theta_i(f) \) are smooth relative \( m \)-hypergroupoids, hence levelwise smooth, so
\[
i^* N^i c_N^i \Omega(X/S) \cong N^i c_N^i \Omega(\pi^0 X/\pi^0 S) = N^i c_N^i \Omega(\pi^0 X/\pi^0 S),
\]
using the fact that \( \Omega(\pi^0 X/\pi^0 S) \) is a module rather than a complex.

Finally, if \( X \to S \) is smooth, then it is levelwise smooth, so \( i^* N^s \Omega(X/S) \cong \Omega(\pi^0 X/\pi^0 S) \). This is locally projective in the sense that each \( \Omega(\pi^0 X/\pi^0 S) \) is projective on \( X_n \). Note that in general it is not, however, projective in \( \text{dgMod}(X) \), since it is not Reedy cofibrant. \( \square \)

Remark 9.17. To understand the distinction between locally projective and projective objects \( P \) in \( \text{Mod}(X) \), note that projectivity implies vanishing of the higher \( \text{Ext} \) groups \( \text{Ext}^i_{\text{dgMod}(X)}(\mathcal{O}_X \oplus N_s^{-1} M)^2, X^2 \) for all \( i > 0 \).

Lemma 9.18.
\[
\text{Der}(X/S, M) \simeq \text{Hom}_{\text{Spec}(\mathcal{O}_X \oplus N_s^{-1} M)^2, X^2}).
\]

Proof. We just apply Proposition 7.10, noting that since \( Y := \text{Spec}(\mathcal{O}_X \oplus N_s^{-1} M) \) is an infinitesimal thickening of \( X \), its étale site is isomorphic to that of \( X \), so every homotopy derived trivial relative Deligne–Mumford \( m \)-hypergroupoid \( Y \to Y \) is levelwise weakly equivalent to one of the form \( Y \times_X \tilde{X} \), for a homotopy derived trivial relative Deligne–Mumford \( m \)-hypergroupoid \( \tilde{X} \to X \). \( \square \)

Proposition 9.19. \( \mathbb{L}^{X/S} \) gives rise to a cotangent complex in the sense of [TV2] Definition 1.4.1.7 (or equivalently [Lur1] §3.2) by associating to any morphism \( f : U \to X^2 \) with \( U \in \text{cAff} \), the complex \( f^* \mathbb{L}^{X/S} \in \text{dgMod}(U)_{\text{cart}} \). In other words,
\[
\text{Hom}_{\text{dgMod}(U)}(f^* \mathbb{L}^{X/S}, N) \simeq \text{Hom}_{\text{Spec}(\mathcal{O}_U \oplus N_s^{-1} N)^2, X^2}),
\]
functorially in \( N \in \text{dgMod}(U)_{\text{cart}} \) and in \( f \).

Proof. Passing to a suitable étale cover, we may assume that \( f \) factors as \( f : U \to X \). If we set \( M := f_* N \in \text{dgMod}(X) \) (which need not be Cartesian and is not to be confused with \( f_* \text{cart} N \)), then
\[
\text{Hom}_{\text{Spec}(\mathcal{O}_U \oplus N_s^{-1} N)^2, X^2}) \simeq \text{Der}(X/S, f_* N)
\]
by Lemma 9.18, so it suffices to show that
\[
\text{Hom}_{\text{dgMod}(X)}(\mathbb{L}^{X/S}, M) \simeq \text{Der}(X/S, M),
\]
for all \( M \in \text{dgMod}(X) \).
Given a cochain chain complex $L$, observe that $\text{HOM}(\text{Tot } L, M) \cong \text{Tot}^1 \text{HOM}(L, M)$, noting that $\text{HOM}(L, M)$ is a bichain complex. Thus

$$N^* \text{Hom}(L^X, S, M) \cong \tau_{\geq 0} \text{Tot}^1 \text{HOM}(N^*_c \Omega(X/S), M).$$

Now, $N^*_c \Omega(X/S) = \eta_c^* \eta_c^* \Omega(X/S)$. Since $X^\Delta_n \to X^\Delta_n \times S^\Delta_n$ is a trivial derived Artin $m$-hypergroupoid, we know that $N^*_c \Omega(X/S)$ is projective in the category $dg_+ \text{Mod}(X)$ (since the smoothness conditions imply that the latching maps are monomorphisms with projective cokernel).

For $M \in dg_+ \text{Mod}(X)$, the map $M[-i] \oplus M[1-i] \xrightarrow{id, d} M[-i]$ is a surjection in $dg_+ \text{Mod}(X)$ for $i > 0$; this implies that

$$\text{HOM}_{1-i}(N^*_c \Omega(X/S), M) \xrightarrow{d} \text{Z}_i \text{HOM}(N^*_c \Omega(X/S), M)$$

is surjective, so $\tau_{\geq 0} \text{HOM}(N^*_c \Omega(X/S), M) \cong \text{HOM}(N^*_c \Omega(X/S), M)$ for all $n > 0$. We have therefore shown that

$$N^* \text{Hom}(L^X, S, M) \cong \text{Tot} \tau_{\geq 0} \text{HOM}(N^*_c \Omega(X/S), M).$$

Now, the Reedy fibration conditions on $X \to S$ imply that for all $n$, the map

$$N^*_c \Omega(X/S)/dN^*_c \Omega(X/S) \to N^*_c \Omega(X/S)$$

is a monomorphism, with projective cokernel in the category $g \text{Mod}(X)$ of $N^*O(X)$-modules in graded abelian groups (ignoring the simplicial differential $d$). This implies that the sequence

$$\ldots \xrightarrow{d_n} \text{HOM}(N^*_c \Omega(X/S), M) \xrightarrow{d_n} \text{HOM}(N^*_c \Omega(X/S), M) \xrightarrow{d_n} \ldots$$

is exact. Since $\tau_{\geq 0} \text{HOM}_n = \text{HOM}_n$ for $n > 0$, and $(\tau_{\geq 0} \text{HOM})_0 = \text{Hom}$, this implies that

$$\text{Tot} \tau_{\geq 0} \text{HOM}(N^*_c \Omega(X/S), M) \cong \text{Hom}(N^*_c \Omega(X/S), M) \cong N^* \text{Der}(X/S, M),$$

by Lemma 9.14. \qed

**Remark 9.20.** Beware that the cotangent complex of [Lur1] is not the same as the better-known cotangent complex of [Lur2]. The difference is the that the former is based on simplicial rings, while the latter uses symmetric spectra. Thus they correspond locally to André–Quillen and topological André–Quillen homology, respectively. Roughly speaking, simplicial rings serve to apply homotopy theory to algebraic geometry, while symmetric spectra are used to do the opposite. As an example of the differences, some higher topological André–Quillen homology groups of $\mathbb{Z}[t]$ over $\mathbb{Z}$ are non-zero — in other words, $\mathbb{Z} \to \mathbb{Z}[t]$ is not formally smooth as a morphism of symmetric spectra.

**Definition 9.21.** Given a morphism $X \to \mathcal{S}$ of $m$-geometric derived Artin stacks, use Theorem 7.7 to form a relative derived Artin $m$-hypergroupoid $X \to S$ to a derived Artin $m$-hypergroupoid, with $X^\sharp \cong X, S^\sharp \cong \mathcal{S}$, and define the quasi-coherent complex $L^X/S$ on $X$ by $a^*L^X/S \cong L^X/S$, for $a : X \to \mathcal{X}$, using the equivalence of Proposition 9.9.3. The characterisation in Proposition 9.19 ensures that this is well defined.

**Corollary 9.22.** If $f : X \to S$ is a trivial derived Artin $m$-hypergroupoid, then $L^X/S \cong 0$.

**Proof.** Since $f$ takes small extensions to trivial fibrations, $\text{Der}(X/S, M)$ is trivially fibrant, so $\text{Der}(X/S, M) \cong 0$. Therefore $L^X/S \cong 0$. \qed

**Lemma 9.23.** $L^X/S$ is homotopy-Cartesian.

**Proof.** We need to show that for each map $\partial_l : X_{n+1} \to X_n$, the map

$$\partial^*: \partial^*(L^X/S|_{X_n}) \to L^X/S|_{X_{n+1}}$$

is a quasi-isomorphism of chain complexes of quasi-coherent sheaves on $X_{n+1}$.  


From the definition of $\mathbb{L}^{X/S}$, it follows that $\partial^i$ is a monomorphism, and that the cokernel is

$$C := \text{Tot} N_{c}(N^s \Omega(X/S) \otimes (\Delta^{n+1}/\Delta^n) \otimes \Delta^*);$$

we wish to show that $C$ is contractible.

Now, observe that $C$ is $L(X\Delta^{n+1}/X\Delta^n \times S\Delta^n S^{\Delta^{n+1}})^0$ on $X_n = (X\Delta^n)_0$. Since $\partial^i : \Delta^n \to \Delta^{n+1}$ is a trivial cofibration, $X\Delta^{n+1} \to X\Delta^n \times S\Delta^n S^{\Delta^{n+1}}$ is a trivial relative hypergroupoid by Lemma 3.9, so Corollary 9.22 implies that $C \simeq 0$, as required. □

**Corollary 9.24.** For $M \in dg\text{Mod}(X)$,

$$\text{HOM}_{dg\text{Mod}(X)}(\mathbb{L}^{X/S}, M) \simeq \text{HOM}_{dg\text{Mod}(X)}(\mathbb{L}^{X/S}, R\text{cart}*M).$$

**Proof.** Apply Corollary 9.9, noting that $\mathbb{L}^{X/S}$ is cofibrant, since it has the left lifting property with respect to all acyclic surjections. □

**Lemma 9.25.** If $f : X \to S$ is a derived Artin 0-hypergroupoid, then $\mathbb{L}^{X/S} \simeq N^s \Omega(X/S)$.

**Proof.** It suffices to show that each map $\partial^i : N^s \Omega(X/S) \otimes \Delta^n \to \Omega(X/S) \otimes \Delta^{n+1}$ is a weak equivalence in $dg_{+}\text{Mod}(X)$. This map is a monomorphism, with cokernel

$$\eta^s \Omega(X^{\Delta^{n+1}}/X^{\Delta^n} \times S\Delta^n S^{\Delta^{n+1}}).$$

By Lemma 3.9, $X^{\Delta^{n+1}} \to X^{\Delta^n} \times S\Delta^n S^{\Delta^{n+1}}$ is a trivial relative Artin 0-hypergroupoid, which is equivalent to saying that it is a levelwise weak equivalence, so $\Omega(X^{\Delta^{n+1}}/X^{\Delta^n} \times S\Delta^n S^{\Delta^{n+1}})$ is contractible, as required. □

**Corollary 9.26.** If $X \to S$ is a derived Artin $m$-hypergroupoid, then

$$\mathbb{L}^{X/S} \simeq N^s \Omega(X/S) \otimes (\Delta^m/\partial \Delta^m)[m].$$

In particular, this implies that $H_i\mathbb{L}^{X/S} = 0$ for $i < m$.

**Proof.** We use the characterisation of the cotangent complex from the proof of Proposition 9.19, namely that

$$\text{Hom}_{dg\text{Mod}(X)}(\mathbb{L}^{X/S}, M) \simeq \text{Der}(X/S, M),$$

for all $M \in dg_{+}\text{Mod}(X)$.

Now, simplicial considerations show that

$$\text{Der}(X^{\Delta^m}/X^{\partial \Delta^m} \times S\Delta^m S^{\Delta^m}, \eta_i M) = \ker(\text{Der}(X/S, M)^{\Delta^m} \to \text{Der}(X/S, M)^{\partial \Delta^m}),$$

which is a model for the iterated loop space $\Omega^m \text{Der}(X/S, M)$, so

$$\Omega^m \text{Hom}_{dg\text{Mod}(X)}(\mathbb{L}^{X/S}, M) \simeq \text{Hom}_{dg\text{Mod}(X)}(\eta^s \mathbb{L}^{X^{\Delta^m}}/X^{\partial \Delta^m} \times S\Delta^m S^{\Delta^m}, M).$$

Since $X^{\Delta^m} \to X^{\partial \Delta^m} \times S\Delta^m S^{\Delta^m}$ is a derived Artin 0-hypergroupoid (by Lemma 3.9), Lemma 9.25 then implies that

$$\Omega^m \text{Hom}_{dg\text{Mod}(X)}(\mathbb{L}^{X/S}, M) \simeq \text{Hom}_{dg\text{Mod}(X)}(N^s \Omega(X/S) \otimes (\Delta^m/\partial \Delta^m), M),$$

for all $M \in dg_{+}\text{Mod}(X)$.

Moreover,

$$N^s \Omega^m \text{Hom}_{dg\text{Mod}(X)}(\mathbb{L}^{X/S}, M) \simeq \tau_{\geq 0}(N^s \text{Hom}_{dg\text{Mod}(X)}(\mathbb{L}^{X/S}, M)[m]) = N^s \text{Hom}_{dg\text{Mod}(X)}(\mathbb{L}^{X/S}[-m], M).$$

Thus, for $M \in dg_{+}\text{Mod}(X)$, we have

$$\text{Hom}_{dg\text{Mod}(X)}(\mathbb{L}^{X/S}[-m], M) \simeq \text{Hom}_{dg\text{Mod}(X)}(N^s \Omega(X/S) \otimes (\Delta^m/\partial \Delta^m), M),$$

from which we deduce that

$$\mathbb{L}^{X/S}[-m] \simeq N^s \Omega(X/S) \otimes (\Delta^m/\partial \Delta^m),$$

as required. □
Definition 9.27. Given a homotopy derived Artin \( m \)-hypergroupoid \( X \to S \), define \( L^{X/S} \in dg\text{-}Mod(X)_{\text{cart}} \) by taking a Reedy fibrant approximation \( u : X \to \hat{X} \) over \( S \), and setting \( L^{X/S} := u^*L^{\hat{X}/S} \). In particular, we may apply this when \( X \to S \) is an Artin \( m \)-hypergroupoid (with trivial derived structure), and then exploit the description of \( L^{X/S} \) in Lemma 9.16.

9.2.1. Comparison with Olsson. To compare our definition of the cotangent complex with Olsson’s (in [Ols1]), we could simply note that Corollary 9.24 ensures that his must be the sheafification of ours. However, a more direct comparison is possible.

If \( X \) is a quasi-compact Artin 1-stack, we may take a presentation \( X_0 \to X \), for \( X_0 \) affine. If \( X \) is semi-separated (i.e. the diagonal of \( X \) is affine), then \( X \) is 1-geometric, and \( X := \cosk_0(X_0/X) \) will be a simplicial affine scheme.

Given a strongly quasi-compact relative Artin 1-hypergroupoid \( f : X \to Y \), for \( Y \) a strongly quasi-compact Artin 1-hypergroupoid, let \( X := X^\sharp \) and \( \underline{Y} := Y^\sharp \) be the associated stacks, and \( u : X \to \hat{X} \) a Reedy fibrant approximation over \( Y \) in the category of simplicial cosimplicial affine schemes. Then [Ols1] 8.2.3 defines the cotangent complex to be the sheafification of the complex

\[
L^{X/Y}_{\underline{X}/\underline{Y}} := \text{Tot} (u^*N^\delta \Omega(\hat{X}/Y) \to \Omega(X/Y \times^{h_\underline{Y}} \underline{X}))
\]

in \( dg_+ \text{-}Mod(X) \).

Proposition 9.28. For \( X, Y, X, \underline{Y} \) as above, \( L^{X/Y}_{\underline{X}/\underline{Y}} \) is the sheafification of the cotangent complex \( L^{X/Y} \).

Proof. The characterisation in Definition 9.27 reduces to

\[
L^{X/Y} = \text{Tot} (u^*\Omega(\hat{X}/Y) \to \eta^*\Omega(X^{\Delta^1}/Y^{\Delta^1} \times_{\partial_0} X)),
\]

so it will suffice to show that \( \Omega(X/Y \times^{h_\underline{Y}} \underline{X}) \cong \eta^*\Omega(X^{\Delta^1}/Y^{\Delta^1} \times_{\partial_0} X) \).

Now, if we write \( \delta : X \to X \times^h X \), then

\[
\Omega(X/Y \times^{h_\underline{Y}} \underline{X}) = \delta^*(X \times^h X \xrightarrow{(pr_1,f)} X \times^{h_\underline{Y}} Y)).
\]

However, \( X^{\Delta^1} \cong X^{\Delta^1} \times^h X^{\Delta^1} \), so (since \( X = \cosk_0(X_0/X) \))

\[
X^{\Delta^1} = X^{\Delta^1} \times^h X^{\Delta^1} \underline{X} = (X \times X) \times^{h_\underline{X}} \underline{X} = X \times^{h_\underline{X}} X,
\]

and similarly for \( Y \). This gives the required isomorphism. \( \square \)

10. Deformations

10.1. Deforming morphisms of derived stacks.

Definition 10.1. Define \( \Delta_* \) to be the subcategory of the ordinal number category \( \Delta \) containing only those morphisms \( f \) with \( f(0) = 0 \). Given a category \( \mathcal{C} \), define the category \( c_+\mathcal{C} \) of semi-cosimplicial complexes over \( \mathcal{C} \) to consist of functors \( \Delta_* \to \mathcal{C} \). Thus a semi-cosimplicial object \( X^* \) consists of objects \( X^n \in \mathcal{C} \), with all of the operations \( \partial^i, \sigma^i \) of a cosimplicial complex except \( \partial^0 \), satisfying the usual relations.

Definition 10.2. Taking the dual construction to \( \text{Dec}^{opp}_+ \), we may define, for any semi-cosimplicial object \( Y \), a semi-cosimplicial object \( \text{Dec}^{opp} Y \) given by \( (\text{Dec}^{opp} Y)^n = Y^{n+1} \), with \( \partial^i_{\text{Dec}^{opp} Y} = \partial^{i+1}_Y \). The operations \( \partial^0_Y : Y \to \text{Dec}^{opp} Y \) define a semi-cosimplicial map, which has a retraction \( \sigma^0_Y \). \( \text{Dec}^{opp} \) is a comonad on semi-cosimplicial complexes, with coproduct given by \( \partial^1_Y : \text{Dec}^{opp} Y \to \text{Dec}^{opp} \text{Dec}^{opp} Y \), the co-unit being \( \sigma^0_Y \).
Proposition 10.3. Take a diagram $Z \xleftarrow{g} X \xrightarrow{f} S$ of simplicial cosimplicial affine schemes, with $f$ a Reedy fibration. If $Z \hookrightarrow \tilde{Z}$ is a levelwise closed immersion over $S$ defined by a square-zero ideal $\mathcal{I}$, then the obstruction to extending $g$ to a morphism $\tilde{g} : \tilde{Z} \to X$ over $S$ lies in

$$H_{-1} \text{Hom}_{d^g \text{Mod}(X)}(N^s \Omega^{X/S}, g_* N^s \mathcal{I}).$$

If the obstruction is zero, then the Hom-space of possible extensions is given by

$$N^s \text{Hom}_{d^g \text{ABLS}}(\tilde{Z}, X) \simeq \tau_{\geq 0} \text{Hom}_{d^g \text{Mod}(X)}(\mathbb{L}^{X/S}, N^s g_* \mathcal{I}).$$

Proof. Since $f$ is a Reedy fibration, the simplicial matching maps

$$X_n \to M_n X \times_{M_n S} S_n$$

are fibrations of cosimplicial affine schemes. For a morphism $V \to W$ of cosimplicial affine schemes, it is a fibration implies that the cosimplicial matching maps $V^j \to W^j \times_{M^j W} M^j V$ have the right lifting property (RLP) with respect to all closed immersions of affine schemes, so in particular are formally smooth.

Now, $\Delta \times \Delta^{\text{opp}}$ is a Reedy category (as in [Hov] §5.2), and we may regard $f$ as a morphism of $\Delta \times \Delta^{\text{opp}}$-diagrams in affine schemes. The results above imply that the $\Delta \times \Delta^{\text{opp}}$ matching maps

$$X^j_n \to (M_n X^j \times M^{j-1} X_n) \times_{(M_n S^j \times M_n M^{j-1} S M^{j-1} S_n)} S^j_n$$

have the RLP for all closed immersions.

Likewise, the simplicial latching maps

$$L_n \tilde{Z} \cup_{L_n Z} Z_n \to \tilde{Z}_n$$

are levelwise closed immersions. However, the cosimplicial latching maps of a levelwise closed immersion $g : V \to W$ of cosimplicial affine schemes will only be closed immersions if $g$ is a weak equivalence. Thus we instead consider the underlying diagram of hemicosimplicial affine schemes. For a simplicial ring $A_*$, the hemicosimplicial latching object $L^j \text{Spec}(A_*)$ is given by $\text{Spec} (M_{A_j} A)$, so the hemicosimplicial matching maps of a levelwise closed immersion are closed immersions.

Letting $\mathbb{I}$ be the Reedy category $\Delta_* \times \Delta^{\text{opp}}$, this implies that the latching maps

$$L(\mathbb{I})^j_n \tilde{Z} \cup_{L(\mathbb{I})^i Z} Z^j_n \to \tilde{Z}^j_n$$

are closed immersions, which allows us inductively to construct a lift

$$\lambda : \tilde{Z}^*_n \to X^*_n$$

of simplicial hemicosimplicial affine schemes.

Now, the obstruction to $\lambda$ being a cosimplicial morphism is the map

$$\partial_0 \lambda^i - \lambda^i \partial_0 : O(X)_n \to O(\tilde{Z})_{n-1},$$

whose image is contained in $\ker(O(\tilde{Z}) \to O(Z)) = \mathcal{I}$. Thus we have a map

$$O(X)_n \to \mathcal{I}_{n-1};$$

since $\mathcal{I}$ is a square-zero ideal, this map is an $O(S)$-linear $g^s \partial_0$-derivation, so corresponds to an $(O(X), g^s \partial_0)$-linear map

$$\delta : \Omega(X/S)_n \to \mathcal{I}_{n-1}.$$
where the sum is over shuffle permutations \((I, J)\) of \([0, n-1]\), \pm is the sign of the permutation, and \(\sigma_I = \sigma_{i_r} \ldots \sigma_{i_2} \sigma_{i_1}\) for \(I\) the sequence \(i_1 < i_2 < \ldots < i_r\).

Now, \(\sigma I m = 0\) if \(I\) contains 0, so the only non-zero terms in the sum have \(0 \in J\), in which case \(\partial_0 \sigma_{K\cup[0]} a = \sigma_{K-1}a\). We also have \(\delta(\sigma I m) = \sigma I-1 m\) in this case, so renumbering \(I\) and \(K\) gives

\[
\delta(m \nabla a) = \delta(m) \nabla g^s a;
\]

in other words, \(N^\ast \delta\) is an \((N^\ast \Omega(X), g^s)^\ast\) linear map.

The conditions on \(\partial_0\) ensure that \(N^\ast \delta\) is also a chain map, so we have

\[
N^\ast \delta \in Z_{-1} \text{HOM}_{N^\ast \Omega(X)}(N^\ast \Omega(X/S), N^\ast g_s \mathcal{I}).
\]

However, the choice of \(\lambda\) was not unique. Another choice \(\lambda'\) gives rise to \(\beta := \lambda^* - (\lambda')^*\). Similar reasoning shows that this is a \(g^s\)-derivation \(\beta : \Omega(X/S) \to g_s \mathcal{I}\), and corresponds to an \(N^\ast \Omega(X)^\ast\) linear map on normalised complexes. The operation \(\delta'\) is given by \(\delta - \partial_0 \beta + \beta \partial_0\), so \(N^\ast \delta' = N^\ast \delta - [d, N^\ast \beta]\). Thus the obstruction lies in

\[
H_{-1} \text{HOM}_{N^\ast \Omega(X)}(N^\ast \Omega(X/S), N^\ast g_s \mathcal{I}).
\]

Finally, \(\tilde{Z} \cup Z \cong \text{Spec}(\mathcal{O}_Z \oplus \mathcal{I} \epsilon)\), for \(\epsilon^2 = 0\), so the choice of \(\tilde{g}\) gives

\[
\text{Hom}_{\text{ZpcAff}}(\tilde{Z}, X) \cong \text{Hom}_{\text{ZpcAff}}(\tilde{Z} \cup Z \circ \text{Spec}(\mathcal{O}_Z \oplus \mathcal{I} \epsilon), X) \cong \text{Hom}_{\text{dgMod}(X)}(N^\ast \Omega(X/S), N^\ast g_s \mathcal{I}),
\]

and Lemma 9.18 completes the proof. 

**Lemma 10.4.** If \(f : X \to S\) is a relative derived Artin \(m\)-hypergroupoid, and \(M \in dg_+ \text{Mod}(X)\), then

\[
H_i \text{HOM}_{dg\text{Mod}(X)}(L^{X/S}, M) \cong H_i \text{HOM}_{dg\text{Mod}(X)}(\Omega(X/S), M)
\]

for \(i \leq 0\).

**Proof.** As in Lemma 9.16, the map \(L^{X/S} \to N^\ast \Omega(X/S)\) is surjective, and the kernel is the total complex of

\[
0 \to N^\ast \Omega(X) X^\Delta^i / X^{\Delta^i} \times S^\Lambda^i_{0} S^\Delta^i \to \ldots \to N^\ast \Omega(X) X^\Delta^i / X^{\Delta^i} \times S^\Lambda^i_{0} S^\Delta^i \to \ldots
\]

Since \(X^\Delta^i \to X^{\Lambda^i_0} \times S^\Lambda^i_{0} S^\Delta^i\) is a trivial relative derived Artin \(m\)-hypergroupoid, its simplicial matching maps are smooth. By [TV2] Definition 1.2.7.1, this means that the cotangent complexes associated to the matching maps are projective, so the chain complex \(N^\ast \Omega(X/S) \otimes (\Delta^i/\Lambda^i_0)\) is a projective object of \(dg_+ \text{Mod}(X)\) (but not of \(dg\text{Mod}(X)\)). Hence

\[
H_j \text{HOM}_{dg\text{Mod}(X)}(N^\ast \Omega(X/S) \otimes (\Delta^i/\Lambda^i_0), M) = 0
\]

for all \(i > 0, j < 0\) and \(M \in dg_+ \text{Mod}(X)\).

If we set \(\Lambda^i_0 := 0\), then we have a spectral sequence

\[
H_j \text{HOM}_{dg\text{Mod}(X)}(N^\ast \Omega(X/S) \otimes (\Delta^i/\Lambda^i_0), M) \implies H_{j+i} \text{HOM}_{dg\text{Mod}(X)}(L^{X/S}, M).
\]

When \(M \in dg_+ \text{Mod}(X)\), this combines with the calculation above to show that

\[
H_j \text{HOM}_{dg\text{Mod}(X)}(L^{X/S}, M) \cong H^j \text{HOM}_{dg\text{Mod}(X)}(N^\ast \Omega(X/S), M)
\]

for \(j < 0\), as required. 

**Theorem 10.5.** Take a diagram \(\mathcal{Z} \xrightarrow{g} X \xrightarrow{f} \mathcal{G}\) of derived \(m\)-geometric Artin stacks. If \(\mathcal{Z} \hookrightarrow \mathcal{Z}\) is a levelwise closed immersion over \(\mathcal{G}\) defined by a square-zero quasi-coherent complex \(\mathcal{I}\), then the obstruction to extending \(g\) to a morphism \(\tilde{g} : \tilde{Z} \to X\) over \(\mathcal{G}\) lies in

\[
\text{Ext}^{1}_{\mathcal{O}_X}(L^{X/\mathcal{G}}, Rg_* \mathcal{I}).
\]

If the obstruction is zero, then the Hom-space of possible extensions is given by

\[
\pi_1 \text{Hom}_{\text{ZpcAff}}(\tilde{Z}, X) \cong \text{Ext}_{\mathcal{O}_X}^{-1}(L^{X/\mathcal{G}}, Rg_* \mathcal{I}).
\]
Proof. Apply Theorem 7.7 to obtain a derived Artin $m$-hypergroupoid $S$ with $S^2 \simeq S$, and relative derived Artin $m$-hypergroupoids $Z \to X \to S$ with $X^2 \simeq X$ and $Z^2 \simeq Z$. It will follow from the proof of Theorem 10.8 that square-zero deformations of $Z$ correspond to square-zero deformations of $S$, so there exists an extension $Z \to \tilde{Z}$ of simplicial cosimplicial affine schemes with $\tilde{Z}^2 \simeq \tilde{Z}$. Thus the extension is defined by the square-zero ideal $a^* \mathcal{I}$, for $a : \tilde{Z} \to \tilde{Z}$.

We are now in the scenario of Proposition 10.3, and sheafification defines a functor from lifts $\tilde{Z} \to X$ to lifts $\tilde{S} \to X$. Applying Lemma 10.4 and Lemma 9.24, we see that these lifts are governed by $H_* \text{Hom}_{\text{dgMod}(X)}(\mathbb{L}^X/S, g_* \mathcal{I}) \cong H_* \text{Hom}_{\text{dgMod}(X)}(\mathbb{L}^X/S, R\text{cart}_* g_* \mathcal{I}) \cong \text{Ext}_S^{-*}(\mathbb{L}^X/S, Rg_* \mathcal{I})$.

Since this final description is independent of the choice $Z \to X \to S$ of resolutions, we deduce that another choice $Z' \to X' \to S'$ of resolutions would give an equivalent space of lifts. Explicitly, we could take a third sequence $Z'' \to X'' \to S''$, with trivial relative derived Artin hypergroupoids $\pi_Z : Z'' \to Z$, $Z'' \to Z'$ and similarly for $X'', S''$, compatible with the other morphisms. Since $\pi_Z^* \mathbb{L}^X/S \simeq \mathbb{L}^{X''}/S''$, and $R\pi_Z^* \pi_Z^* \mathcal{I} \simeq \mathcal{I}$, pulling back along $X'' \to X$ gives an equivalence of lifting spaces, and similarly for $X'' \to X'$.

10.2. Deformations of derived stacks.

**Proposition 10.6.** Take a Reedy fibration $f : X \to S$ of simplicial cosimplicial affine schemes. If $S \hookrightarrow \tilde{S}$ is a levelwise closed immersion defined by a square-zero ideal $\mathcal{I}$, then the obstruction to lifting $f$ to a levelwise flat morphism $\tilde{f} : \tilde{X} \to \tilde{S}$, with $\tilde{X} \times \tilde{S} = X$, lies in

$$H_{-2} \text{Hom}_{\text{dgMod}(X)}(N^s \Omega^X/S, f^* N^s \mathcal{I}).$$

If the obstruction is zero, then the isomorphism class of liftings is (non-canonically) isomorphic to

$$H_{-1} \text{Hom}_{\text{dgMod}(X)}(N^s \Omega^X/S, f^* N^s \mathcal{I}).$$

**Proof.** We adapt the proof of Proposition 10.3.

Letting $\mathbb{I}$ be the Reedy category $\Delta_s \times \Delta^{\text{opp}}$, we have seen that the matching maps

$$X_n^j \to M(\mathbb{I})_n^j X \times_{M(\mathbb{I})_n^j S} S_n^j$$

lift closed immersions (so are pro-smooth), and that the latching maps

$$L(\mathbb{I})^j_n X \to X_n^j$$

are closed immersions. Using the fact that deformations of pro-smooth morphisms of affine schemes are unobstructed and unique, we may inductively construct a lift

$$\tilde{f} : \tilde{X}^s \to \tilde{S}^s$$

in the category of simplicial hemi-cosimplicial affine schemes, with $\tilde{f} \mathbb{I}$-Reedy fibrant, and note that this lift is unique up to isomorphism. It therefore remains only to understand how the operation $\partial^{n+1}$ on $X^n$ can deform.

Using the formal smoothness of $\tilde{f}$, we may lift $\partial^0 X : X \to \text{Dec}_\text{opp}^+ X$ to a hemi-cosimplicial map

$$\lambda : \tilde{X} \to \text{Dec}_\text{opp}^+ \tilde{X},$$

satisfying $\tilde{f}\lambda = \partial^0 \tilde{f}$ and $\sigma^0 \lambda = \text{id}$. This will make $\tilde{X}$ into a cosimplicial scheme if and only if $\lambda^2 = \partial^1 \lambda$, since all the other conditions are automatic.

Thus the obstruction to $(\tilde{X}^s, \lambda)$ being a cosimplicial scheme is the map

$$(\lambda^2)^2 - \lambda^2 \partial_1 : O(\tilde{X})_n \to O(\tilde{X})_{n-2},$$
whose image is contained in $\ker(O(\tilde{X}) \to O(X)) = f^*\mathcal{I}$. Since $\mathcal{I}$ is a square-zero ideal, we know that $f^*\mathcal{I}$ must also map to 0. Thus we have a map
\[ O(X)_n \to (f^*\mathcal{I})_{n-2} \]
This map is an $O(S)$-linear $(\partial_0)^2$-derivation, so corresponds to an $(O(X), (\partial_0)^2)$-linear map
\[ \delta : \Omega(X/S)_n \to (f^*\mathcal{I})_{n-2}. \]
Moreover, $\partial_i \delta = \delta \partial_{i+2}$ for all $i > 0$, so $\delta$ descends to a map $N^s \delta$ on the normalised complexes, which will be 0 if and only if $\delta = 0$. Moreover, $\delta \sigma = \sigma_{i-2} \delta$ for $i \geq 2$, and 0 for $i = 0, 1$. A calculation similar to that in Proposition 10.3 shows that $N^s \delta$ is an $N^s O(X)$-linear map.

We need to show that $\delta(\lambda^2 - \partial_1 + \partial_2) = \lambda^2 \delta$ to ensure that $N^s \delta$ is a chain map.
\[ \delta(\lambda^2 - \partial_1 + \partial_2) = [(\lambda^2 - (\lambda^2)^2) \partial_1 + \lambda^2 \partial_2] - [\lambda^2 \partial_1 \lambda^2] = (\lambda^2)^3 - (\lambda^2)^2 \partial_1 = \lambda^2 \delta, \]
as required, so $N^s \delta \in Z_{-2} \text{HOM}_{N^s O(X)}(N^s \Omega(X/S), N^s f^*\mathcal{I})$.

However, the choice of $\lambda$ was not unique. Another choice $\lambda'$ gives rise to $\beta := \lambda^* - (\lambda')^*$. Similar reasoning shows that this is a $\partial^0$-derivation $\beta : \Omega(X/S) \to f^*\mathcal{I}$, and corresponds to an $N^s O(X)$-linear map on normalised complexes. The operation $\delta'$ is given by $\delta - \partial_0 \beta + \beta(\partial_1 - \partial_0)$, so $N^s \delta' = \delta - [d, N^s \beta]$. Thus the obstruction lies in
\[ H_{-2} \text{HOM}_{N^s O(X)}(N^s \Omega(X/S), N^s f^*\mathcal{I}). \]

To understand the isomorphism class of the deformations, fix a deformation $(\hat{X}, \lambda)$. Another choice of lift is given by $(\lambda')^* = \lambda^* - \beta$, and for this to be unobstructed, we need $[d, N^s \beta] = 0$, so $N^s \beta \in Z_1 \text{HOM}_{N^s O(X)}(N^s \Omega(X/S), N^s f^*\mathcal{I})$. Now, an isomorphism of $\hat{X}$ is equivalent to a map $\alpha \in \text{HOM}_{N^s O(X)}(N^s \Omega(X/S), N^s f^*\mathcal{I})^0$, and transformation by $\alpha$ sends $N^s \beta$ to $N^s \beta + [d, \alpha]$. Thus the isomorphism class is
\[ H_{-1} \text{HOM}_{N^s O(X)}(N^s \Omega(X/S), N^s f^*\mathcal{I}). \]

\[ \square \]

**Lemma 10.7.** If the morphism $f$ in Proposition 10.6 is a relative derived Artin $m$-hypergroupoid, the simplicial automorphism group $\text{Aut}_{\tilde{\mathcal{S}}}(\tilde{X})_X$ of a deformation $\tilde{X}$ is given by
\[ N^s \text{Aut}_{\tilde{\mathcal{S}}} (\tilde{X})_X \simeq \tau_{\geq 0} \text{HOM}_{d \text{Mod}(X)}(L^X/S, f^*N^s \mathcal{I}). \]

**Proof.** This follows immediately from Proposition 9.19. \[ \square \]

**Theorem 10.8.** Take a morphism $f : \mathfrak{X} \to \mathfrak{S}$ of derived $m$-geometric Artin stacks. If $\mathfrak{S} \hookrightarrow \tilde{\mathfrak{S}}$ is a 0-representable closed immersion defined by a square-zero quasi-coherent complex $\mathcal{I}$, then the obstruction to lifting $f$ to a morphism $\tilde{f} : \tilde{\mathfrak{X}} \to \tilde{\mathfrak{S}}$ with $\tilde{\mathfrak{X}} \times_{\tilde{\mathfrak{S}}} \mathfrak{S} \simeq \mathfrak{X}$ lies in
\[ \text{Ext}^2_{\tilde{\mathcal{I}}} (L^{X/S}, f^*\mathcal{I}). \]

If this obstruction is 0, then the equivalence class of deformations is
\[ \text{Ext}^1_{\tilde{\mathcal{I}}} (L^{X/S}, f^*\mathcal{I}). \]
and the simplicial automorphism group $\text{Aut}_{\tilde{\mathcal{S}}} (\tilde{X})_X$ of any deformation $\tilde{X}$ is given by
\[ \pi_1 \text{Aut}_{\tilde{\mathcal{S}}} (\tilde{X})_X \cong \text{Ext}^1_{\tilde{\mathcal{I}}} (L^{X/S}, f^*\mathcal{I}). \]

**Proof.** Apply Theorem 7.7 to obtain a derived Artin $m$-hypergroupoid $\tilde{\mathcal{S}}$ with $\tilde{\mathcal{S}}^2 \simeq \tilde{\mathcal{S}}$, and pull back $\mathcal{I}$ to give a levelwise closed immersion $S \hookrightarrow \tilde{S}$. Similarly, there exists a relative derived Artin $m$-hypergroupoid $X \to S$, with $X^2 \simeq \tilde{\mathfrak{X}}$. Proposition 10.6, Lemma 10.4 and Lemma 10.7 now imply that deformations of $X$ are governed by
\[ \text{Ext}^1_{\mathcal{I}} (L^{X/S}, f^*\mathcal{I}), \]
and sheafification defines a functor from the $\infty$-groupoid of deformations of $X$ to the $\infty$-groupoid of deformations of $\mathbb{X}$, so we just need to show that this is an equivalence.

Given a deformation $\tilde{\mathbb{X}}$ of $\mathbb{X}$, there exists a relative derived Artin $m$-hypergroupoid $\tilde{X}' \to \tilde{S}$, with $(\tilde{X}')^\beta \simeq \tilde{\mathbb{X}}$. Setting $X' := \tilde{X}' \times_{\tilde{S}} \tilde{S}$ gives $(X')^\beta \simeq X$. By Theorem 7.10, there exists a derived Artin $m$-hypergroupoid $X''$ equipped with morphisms $X'' \to X'$, $X'' \to X$ which are trivial relative derived Artin $m$-hypergroupoids.

It therefore suffices to show that the simplicial groupoid $\Gamma$ of deformations of a trivial relative derived Artin $m$-hypergroupoid $T \to Y$ is contractible. By Corollary 9.22, $L^{T/Y} \simeq 0$, so Proposition 10.6, Lemma 10.4 and Lemma 10.7 imply that $\Gamma$ has one equivalence class of objects, with contractible automorphism group.

\[ \square \]

**Remark 10.9.** Theorem 10.8 has an alternative proof, as a Corollary of Theorem 10.5, which we now sketch.

Given a quasi-coherent complex $\mathcal{F}$ on $\mathbb{G}$, let $\mathbb{G} \oplus \mathcal{F} := \text{Spec}(\mathcal{O}_\mathbb{G} \oplus (-)^{\leq 0} \mathcal{F})$, with $e^0 = 0$. Then there exists a morphism $t : \mathbb{G} \oplus \mathcal{F}[-1] \to \mathbb{G}$ such that $\tilde{\mathbb{G}}$ is the homotopy cofibre product $\mathbb{G} \cup_h t \mathbb{G}$. The category of deformations $\tilde{\mathbb{X}}$ over $\tilde{\mathbb{G}}$ is therefore equivalent to the simplicial category of equivalences $\alpha : t^* \mathbb{X} \to 0^* \mathbb{X}$ over $\mathbb{G} \oplus \mathcal{F}[-1]$, fixing the fibre $\mathbb{X}$ over $\mathbb{G}$. This is the same as the simplicial category of morphisms $t^* \mathbb{X} \to \mathbb{X}$ over $\mathbb{G}$, fixing $\mathbb{X}$. Since $\mathbb{X} \to t^* \mathbb{X}$ is a square-zero extension, Theorem 10.5 describes these morphisms in terms of the cotangent complex.

**Remark 10.10.** If $\mathbb{X}$ and $\mathbb{G}$ are (non-derived) geometric Artin stacks, then flatness of $\tilde{\mathbb{X}} \to \tilde{\mathbb{G}}$ is sufficient to ensure that $\tilde{\mathbb{X}} \times_{\tilde{\mathbb{G}}} \mathbb{G} \simeq \tilde{\mathbb{X}} \times \mathbb{G}$. In this case, Theorems 10.5 and 10.8 have alternative proofs, by considering the hemisimplicial schemes underlying the associated simplicial schemes. It is then necessary to study deformations of $X_0$ and of $\partial^1_0 X$, rather than of $\partial^1_0 \mathbb{X}$. This is the approach taken in [Pri1] §10.

**Corollary 10.11.** Given a derived $m$-geometric Artin stack $\mathbb{X}$, and an Artin $m$-hypergroupoid $Z$ with $Z^2 \simeq \pi^0 \mathbb{X}$, then there exists a derived Artin $m$-hypergroupoid $X$ with $X^2 \simeq \mathbb{X}$ and $\pi^0 X = Z$.

**Proof.** It suffices to find a homotopy derived Artin groupoid $X$ with these properties, since we may then take a Reedy fibrant approximation. Writing $\mathbb{X}$ as the colimit $\pi^0 \mathbb{X} \to \pi^1 \mathbb{X} \to \pi^2 \mathbb{X} \to \ldots$ of homotopy square-zero extensions as in Remark 8.25, we will inductively construct $a : X(i) \to \mathbb{X}$ with $X(i)^2 \simeq \mathbb{X}(i)$ and $\pi^0 X(i) = Z$.

Assume that we have constructed $X(i-1)$. We will now construct the schemes $X(i)_n$ inductively, so assume that we have done this compatibly for all $j < n$, giving latching and matching objects $L_n X(i), M_n X(i)$. We seek a diagram

\[
L_n X(i) \xrightarrow{\alpha_n} X(i)_n \xrightarrow{\beta_n} M_n X(i) \times_{X(i)}^h X(i)
\]

lifting

\[
L_n X(i-1) \xrightarrow{\alpha_{n-1}} X(i-1)_n \xrightarrow{\beta_{n-1}} M_n X(i-1) \times_{X(i-1)}^h X(i-1).
\]

Since the embedding $X(i-1) \to X(i)$ is defined by the square-zero sheaf $\pi_i O_X[-i]$, deformations of $\beta_{n-1}$ governed by the complex

\[
C_* := \text{HOM}_{\text{dgMod}(X(i-1)_n)}(L X(i-1)_n/M_n X(i-1) \times_{X(i-1)}^h X(i-1), a_n^* \pi_i O_X[-i])
\]

\[
\cong \text{HOM}_{\text{dgMod}(Z_n)}(L Z_n/M_n Z \times_{\pi^0 \mathbb{X}} \pi^0 \mathbb{X}, a_n^* \pi_i O_X[-i]).
\]

Since $a : Z \to \pi^0 \mathbb{X}$ is a resolution, the map $Z_n \to M_n Z \times_{\pi^0 \mathbb{X}} \pi^0 \mathbb{X}$ is a smooth $m$-representable morphism, so Lemma 9.16 implies that the cotangent complex $L_* Z_n/M_n Z \times_{\pi^0 \mathbb{X}} \pi^0 \mathbb{X}$ is equivalent to a complex of locally projective modules concentrated
in degrees $[-m,0]$. Since $Z_n$ is affine, locally projective modules on $Z_n$ are projective, so $H_j(C_\bullet) = 0$ for all $j < i$, and in particular for $j=0,-1,-2$.

Since $i > 0$, Theorem 10.8 then implies that $\beta_i : X(i)_n \to M_n X(i) \times \Delta^n X(i)$ exists, and is unique up to unique isomorphism in the homotopy category, so we may choose a Reedy fibrant representative. In particular, $\beta_i$ is smooth, and since $L_n X(i-1) \to L_n X(i)$ is a homotopy square-zero extension, Theorem 10.5 implies that deformations of $\alpha_{i-1}$ are governed by the complex

$$D_\bullet := \text{HOM}_{\text{dgMod}(X(i-1)_n)}(LX(i-1)_n/M_n X(i-1) \times \Delta^n X(i-1), R\alpha_{i-1} \pi_i \mathcal{O}_{L_n, X}[-i])$$

$$\cong \text{HOM}_{\text{dgMod}(Z_n)}(LZ_n/M_n Z \times \Delta^n, R\alpha_{i-1} \pi_1 \mathcal{O}_{L_n, X}[-i])$$

$$\cong \text{HOM}_{\text{dgMod}(Z_n)}(LZ_n/M_n Z \times \Delta^n, \alpha_{i-1} \pi_1 \mathcal{O}_{L_n, X}[-i])$$

the final isomorphism following by Lemma 6.28. As for $C_\bullet$, we deduce that $H_j(D_\bullet) = 0$ for all $j < i$. In particular, this holds for $j=0,-1$, so $\alpha_i$ exists, and is unique up to weak equivalence.

This completes the inductive step for constructing $X(i)$, and we set $X = \varinjlim X(i)$. \hfill $\square$

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