Clifton’s exact solution of $f(R) = R^{1+\delta}$ gravity describing a dynamical spherical metric which is asymptotically Friedmann–Lemaître–Robertson–Walker is studied. It is shown that it harbours a strong spacetime singularity and that this singularity is naked at late times.

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1. Introduction

Type Ia supernovae have provided us with the knowledge that the universe is currently in a phase of accelerated expansion [1]. This acceleration has been modeled in various ways; the most common models are probably dark energy ones based on general relativity (hereafter GR, see [2] for a list of references). However, the exotic and ad hoc dark energy leaves many cosmologists dissatisfied and attempts have been made to model the cosmic acceleration without dark energy. $f(R)$ theories of gravity akin to the quadratic theories required by the renormalization of GR have been introduced in the metric [3], Palatini [4], and metric-affine [5] formulations and have received much attention in recent years (see [6] for a review and [7] for short introductions).

Along with cosmological and other considerations (e.g., stability, weak-field limit, ghost content), it is important to understand spherically symmetric solutions in these theories, a task which has proved to be non-trivial (see [8] and references therein). For definiteness, we consider here metric $f(R)$ gravity described by the action

$$S = \frac{1}{2\kappa} \int \, d^{4}x \sqrt{-g} \, f(R) + S^{\text{matter}},$$

where $f(R)$ is a nonlinear function of its argument and $S^{\text{matter}}$ is the matter part of the action. $R$ denotes the Ricci scalar of the metric $g_{\mu\nu}$ with determinant $g$, $\kappa = 8\pi G$ where $G$ is Newton’s constant, and we adopt the notations of [9].
It is well known that the Jebsen–Birkhoff theorem does not hold in these theories, which adds to the richness and variety of spherically symmetric solutions. Of particular interest are black holes in generalized gravity, which have been studied especially in relation to their thermodynamics\(^1\) (e.g. [11]). Since \(f(R)\) theories are designed to produce a time-varying effective cosmological constant, the spherically symmetric and black hole solutions of interest likely represent central objects embedded in cosmological backgrounds. Not much is known about this kind of object even in the context of GR, although a few examples are available [12–19] in Einstein’s theory. Even less is known about \(f(R)\) black holes and spherically symmetric solutions, which deserve to be understood better. Here we consider a specific solution proposed in \(f(R) = R^{1+\delta}\) gravity in [20]. The observational constraints set the limits \(\delta = (-1.1 \pm 1.2) \times 10^{-5}\) on the parameter \(\delta\) [20, 21], while local stability requires \(f''(R) \geq 0\) [22, 23], i.e., \(\delta > 0\) hence we will use positive values of this parameter.

The solution proposed in [20] is dynamical and presumably represents some kind of dynamical central object embedded in a spatially flat FLRW background in vacuum \(f(R) = R^{1+\delta}\) gravity. This solution is made possible by the fact that the fourth-order field equations of vacuum metric \(f(R)\) gravity

\[
f'(R)R_{ab} - \frac{f(R)}{2} g_{ab} = \nabla_a \nabla_b f'(R) - g_{ab} \square f'(R)
\]

(2)

can be rewritten in the form of effective Einstein equations with geometric terms acting as a form of effective matter as

\[
R_{ab} - \frac{1}{2} g_{ab} R = \frac{1}{f'(R)} \left[ \nabla_a \nabla_b f' - g_{ab} \square f' + g_{ab} \frac{(f - Rf'')}{2} \right].
\]

(3)

In this picture the effective matter spoils the Jebsen–Birkhoff theorem and fuels the cosmic acceleration. Alternatively, an equivalent representation of \(f(R)\) gravity as a Brans–Dicke theory with a scalar field potential exhibits a massive spin zero degree of freedom that causes these effects [6]. Since exact spherically symmetric dynamical solutions of \(f(R)\) gravity in asymptotically FLRW backgrounds are harder to find than in GR (where only a few are known anyway) and are therefore valuable, we study Clifton’s solution in the following.

2. Clifton’s spherically symmetric dynamical solution

Clifton’s spherically symmetric dynamical solution in vacuum \(f(R) = R^{1+\delta}\) gravity [20] is given by

\[
d s^2 = -A_2(r) \, dt^2 + a^2(t) B_2(r) \left( dr^2 + r^2 d\Omega^2 \right),
\]

(4)

where \(d\Omega^2 = d\theta^2 + \sin^2 \theta \, d\phi^2\) is the line element on the unit 2-sphere,

\[
A_2(r) = \left( \frac{1 - C_2/r}{1 + C_2/r} \right)^{2/\delta},
\]

(5)

\[
B_2(r) = \left( 1 + \frac{C_2}{r} \right)^4 A_2(r)^{q+2\delta-1},
\]

(6)

\[
a(t) = t^{\frac{1+2\delta}{2\delta^2}},
\]

(7)

\[
q^2 = 1 - 2\delta + 4\delta^2,
\]

(8)

\(^1\) The thermodynamics of local Rindler horizons in \(f(R)\) gravity, which is used to derive the classical field equations as an equation of state [10] is modeled after the thermodynamics of dynamical \(f(R)\) black holes.

\(^2\)
in isotropic coordinates and using the notation of [20] for the metric components. There are two distinct classes of solutions when \( \delta \) is fixed: one with \( C_2 qr > 0 \) and the other with \( C_2 qr < 0 \). The line element (4) reduces to the Friedmann–Lemaître–Robertson–Walker one in the limit \( C_2 \to 0 \). In the limit \( \delta \to 0 \) in which the theory reduces to GR, the metric (4) reduces to the Schwarzschild solution in isotropic coordinates provided that \( C_2 qr > 0 \), so both positive and negative values of \( r \) are possible according to the sign of the parameter \( C_2 \).

In the following, we take \( r > 0, C_2 > 0 \) and the positive root in the expression \( q = \pm \sqrt{1 - 2 \delta + 4 \delta^2} \) deriving from equation (8), so that \( q \simeq 1 - \delta \) in the limit \( |\delta| \ll 1 \). Moreover, as said before, we assume that \( \delta > 0 \) for stability. The solution (4)–(8) is conformal to the Fonarev solution [24] which is conformally static [25], and therefore is also conformally static. This is a property shared with the Sultana–Dyer solution [12] and with some representatives of the class of generalized McVittie solutions [19].

We now want to write the metric (4) in the Nolan gauge, in which it is straightforward to identify the apparent horizons (if they exist). To this end, we make use of the Schwarzschild-like radial coordinate

\[
\tilde{r} \equiv r \left( 1 + \frac{C_2}{r} \right)^2,
\]

in terms of which \( dr = \left( 1 - \frac{C_2}{r} \right)^{-1} d\tilde{r} \) and we eventually transform to the areal radius

\[
\rho = \frac{a(t) \sqrt{B_2(t) r}}{(1 + \frac{C_2}{r})^2} = a(t) \tilde{r} A_2(\tilde{r})^{\frac{q+2}{2}}.
\]

The line element (4) then becomes

\[
ds^2 = -A_2 dr^2 + a^2 A_2^{2q-1} d\tilde{r}^2 + \rho^2 d\Omega^2.
\]

Using the fact that

\[
d\tilde{r} \equiv \frac{d\rho}{a A_2 \sqrt{B_2(r)} \tilde{r}} \equiv \frac{d\rho - A_2^{q+2} \dot{\tilde{r}} dt}{a A_2^{-q} C(r)},
\]

where an overdot denotes differentiation with respect to \( t \) and

\[
C(r) = 1 + \frac{2(q + 2 \delta - 1)}{q} \frac{C_2}{r} A_2^{-q} = 1 + \frac{2(q + 2 \delta - 1)}{q} \frac{C_2 a}{\rho} A_2^{q+2-\frac{1}{2}}
\]

the metric assumes the Painlevé–Gullstrand-like form

\[
ds^2 = -A_2 \left[ 1 - \frac{A_2^{2q-1} - \dot{\tilde{r}}^2 r^2}{C_2^2} \right] dt^2 - \frac{2A_2^{q+2} \dot{\tilde{r}}}{C_2^2} \dot{\tilde{r}} dr + \frac{d\rho^2}{A_2^2 C^2} + \rho^2 d\Omega^2.
\]

In order to eliminate the cross-term in \( dt \, d\rho \) we introduce the new time coordinate \( \bar{t} \) defined by

\[
d\bar{t} = \frac{1}{F(t, \rho)} [dt + \beta(t, \rho) d\rho],
\]

where \( F(t, \rho) \) is an integrating factor that satisfies the equation

\[
\frac{\partial}{\partial \rho} \left( \frac{1}{F} \right) = \frac{\partial}{\partial t} \left( \frac{\beta}{F} \right)
\]
to ensure that $d\bar{t}$ is an exact differential. The line element then becomes
\[
\mathrm{d}s^2 = -A_2 \left[ 1 - \frac{A_2^{2(\delta-1)}}{C^2} \dot{\bar{r}}^2 \right] F^2 \mathrm{d}\bar{r}^2 + 2F \left\{ A_2 \beta \left[ 1 - \frac{A_2^{2(\delta-1)}}{C^2} \dot{\bar{r}}^2 \right] - \frac{A_2^{-1} \dot{\bar{r}}}{C^2} \right\} \mathrm{d}\bar{t} \mathrm{d}\rho + \left\{ -A_2 \left[ 1 - \frac{A_2^{2(\delta-1)}}{C^2} \dot{\bar{r}}^2 \right] \beta^2 + \frac{2A_2^{-1} \dot{\bar{r}}}{C^2} + \frac{1}{A_2^2 C^2} \right\} \mathrm{d}\rho^2 + \rho^2 \mathrm{d}\Omega^2.
\]
By setting
\[
\beta = \frac{A_2^{-1}}{C^2} \frac{\dot{\bar{r}}}{1 - \frac{A_2^{-1}}{C^2} \dot{\bar{r}}^2}
\]
the $\mathrm{d}t \mathrm{d}\rho$ cross-term disappears and we are left with the Nolan gauge metric
\[
\mathrm{d}s^2 = -A_2 DF^2 \mathrm{d}\bar{r}^2 + \frac{1}{A_2^2 C^2} \left[ 1 + A_2^{-q-1} \frac{H^2 \rho^2}{C^2 D} \right] \mathrm{d}\rho^2 + \rho^2 \mathrm{d}\Omega^2,
\]
where $H \equiv \dot{a}/a$ and
\[
D \equiv 1 - \frac{A_2^{2(\delta-1)}}{C^2} \dot{\bar{r}}^2 = 1 - \frac{A_2^{-q-1}}{C^2} H^2 \rho^2.
\]
Using the second of these equations, the line element (19) simplifies to
\[
\mathrm{d}s^2 = -A_2 DF^2 \mathrm{d}\bar{r}^2 + \frac{\mathrm{d}\rho^2}{A_2^2 C^2 D} + \rho^2 \mathrm{d}\Omega^2.
\]
The apparent horizons, if they exist, are located at $g^{\rho\rho} = 0$. This equation is satisfied if $A_2^2 C^2 D = 0$, which yields
\[
A_2^2 \left( C^2 - H^2 R^2 A_2^{-q-1} \right) = 0,
\]
hence $g^{\rho\rho}$ vanishes if $A_2 = 0$ or $H^2 R^2 = C^2 A_2^{-q-1}$. $A_2$ vanishes at $r = C_2$, which describes the Schwarzschild horizon in the limit $\delta \to 0$ in which the theory reduces to GR, and corresponds to a spacetime singularity. In fact, the Ricci scalar is
\[
R = \frac{6(H + 2H^2)}{A_2(r)}
\]
and diverges as $r \to C_2$ (it reduces to the familiar value $6(H + 2H^2)$ in the $C_2 \to 0$ limit). Furthermore, this singularity is a strong one in the sense of Tipler’s classification [26]: the areal radius $\rho = a \bar{r} A_2^{2/2-1}$ vanishes when $r = C_2$ for $\delta > 0$. This is in contrast with the case $\delta = 0$ (the Schwarzschild metric of GR) in which $\rho = \bar{r} = 4C_2$ at $r = C_2$.

Let us focus on the second possibility $H^2 \rho^2 = C^2 A_2^{-q-1}$, or
\[
H \rho = \pm \left[ 1 + \frac{2(q + 2\delta - 1)}{q} C_2 A_2^{-2-\frac{1}{q}} \right] A_2^{2+\delta},
\]
where the positive sign is to be chosen in an expanding universe. In the limit of small $\delta$ this equation reduces to $H \rho = \left[ 1 + \frac{2(q + 2\delta - 1)}{q} \right] A_2^{1-\delta}$.

To gain some insight, consider the following two limits. In the limit $C_2 \to 0$ in which the central object disappears and the solution is a FLRW space, $r = \bar{r}$ becomes a comoving
Figure 1. The apparent horizons of the Clifton solution for parameter values $C_2 = 1$ and $\delta = 0.1$. The apparent horizon radius $\rho$ is plotted versus the time $t$. Two inner horizons appear after the big bang and then merge and disappear, covering the $\rho = 0$ singularity only for a finite period of time. A third, cosmological horizon corresponding to the upper branch of the curve keeps expanding.

radius and $\rho$ becomes a proper radius, while equation (24) reduces to $H\rho = 1$ with solution $\rho_c = 1/H$, the radius of the cosmological horizon. In the limit $\delta \to 0$ in which the theory reduces to GR, equation (24) reduces to $A_2 = 0$ or $r = C_2$ with $H \equiv 0$.

Using equations (7) and (10), the left-hand side of equation (24) is expressed as

$$HR = \frac{\delta (1 + 2\delta)}{1 - \delta} \left( \frac{\frac{2^{2\delta+4}}{1-\delta} C_2}{x} \frac{(1-x)^{\frac{2\delta+5}{2\delta}}}{(1+x)^{\frac{2\delta+1}{2\delta}}} \right),$$

where $x \equiv C_2/r$, while the right-hand side of the same equation is

$$\frac{(1-x)^{\frac{2\delta}{1-\delta}}}{(1+x)^{\frac{2\delta}{1-\delta}}} \left[ 1 + \frac{2(q+2\delta-1)}{q} \frac{x}{(1-x)^2} \right].$$

Equation (24) then becomes

$$\frac{1}{t^{\frac{1-2\delta}{1-\delta}}} = \frac{(1-\delta) x (1+x)^{\frac{2\delta+5}{2\delta}}}{(1-x)^{\frac{2\delta+1}{2\delta}}} \left[ 1 + \frac{2(q+2\delta-1)}{q} \frac{x}{(1-x)^2} \right],$$

(note that $1-\frac{2\delta-2\delta^2}{1-\delta}$ is positive for $0 < \delta < \frac{\sqrt{2}-1}{2} \simeq 0.366$).

At late times $t$, the left-hand side of equation (27) vanishes, which implies that $x \simeq 0$. Hence, at late times, there is a unique root of the equation locating the apparent horizons. This unique late-time horizon is identified as a cosmological horizon, as can be deduced by the fact that $r \to \infty$ as $x = C_2/r \to 0$. The limit $x \to 0$ can also be obtained when the parameter $C_2 \to 0$, in which case $H\rho \to 1$ and $r \simeq \rho \approx H^{-1} = \frac{1-\delta}{20\delta^2} t$ is the radius of the cosmological horizon of the FLRW space without a central object. We conclude that there is only a cosmological apparent horizon and no black hole apparent horizon at late times, therefore the central singularity at $\rho = 0$ becomes naked at late times.
The time $t$ and the radius $\rho$ of the apparent horizons can be expressed parametrically as functions of $x$ by

$$t(x) = \left\{ \frac{(1 - \delta)}{\delta (1 + 2\delta)} C_2^2 \frac{x (1 + x)^{2^{2 + 2\delta - 1}/q}}{(1 - x)^{2^{2 + 2\delta - 1}/q}} \left[ 1 + \frac{2 (q + 2\delta - 1) x}{q(1 - x)^2} \right]^{\frac{q}{2^{2 + 2\delta - 1}}} \right\}^{1 - \delta^2},$$

$$\rho(x) = t(x)^{\frac{x^{-2\delta}}{1 - \delta}} C_2^2 \frac{x (1 - x)^{2^{2 + 2\delta - 1}/q} (1 + x)^{2 + 2\delta - 1}}{q}.\tag{29}$$

A plot of $\rho$ versus $t$ is presented in figure 1 for the parameter values $C_2 = 1$ and $\delta = 0.1$. As is clear from this figure, two inner horizons develop at a certain time after the big bang and cover the central singularity at $\rho = 0$. Later on, these apparent horizons approach each other, merge and disappear while a third, cosmological horizon keeps expanding. The $\rho = 0$ singularity becomes naked after this time.

3. Conclusions

In view of the fact that cosmology may be showing us the first-ever detected deviations from Einstein’s gravity and of the attention given to $f(R)$ gravity theories as possible models of the cosmic acceleration, it is of great interest to understand black holes and other spherically symmetric solutions of $f(R)$ gravity. Since the Jebsen–Birkhoff theorem does not hold in these theories, spherically symmetric solutions do not have to be static. These theories are designed to produce an effective dynamical cosmological constant to reproduce the current acceleration of the universe and, therefore, dynamical exact solutions with spherical symmetry describing a central object embedded in a cosmological background are particularly valuable. Unfortunately, this kind of solution is poorly understood even in the context of GR and deserves more attention in the future. A few examples are available [12–19] and further work is in progress on generalized McVittie solutions [27]. In particular, it seems difficult to find generic black hole solutions embedded in cosmological backgrounds. The goal of finding interior solutions for spherically symmetric $f(R)$ gravity (mainly with numerical methods) seems also a worthy one [8]. All these issues will be addressed in future publications.

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