Higher K-energy functionals and higher Futaki invariants

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1. Introduction

In 1986, Mabuchi \[1\] introduced the K-energy functional, which integrates the Futaki invariant \[8\] and whose critical points are metrics of constant scalar curvature. It was used by Bando and Mabuchi \[2\] to prove a uniqueness theorem for Kähler-Einstein metrics. The K-energy is strongly related to notions of stability in Geometric Invariant Theory and has been used by Tian \[13, 14, 16\] and Phong and Sturm \[12\], to give some results on the conjecture of Yau \[17, 18\] on the relationship between stability and the existence of Kähler-Einstein metrics. Tian \[15, 16\] has cast both the K-energy and the Futaki invariant in a more general setting using Bott-Chern forms and Donaldson functionals.

Higher K-energy functionals were defined by Bando and Mabuchi \[3\] and generalize the K-energy map to higher Chern classes of the manifold. They integrate higher Futaki invariants (see \[3, 1, 9, 10\]). This note presents these functionals with an emphasis on Bott-Chern forms. Two new formulas for the higher K-energy are given and the second K-energy is shown to be related to Donaldson’s Lagrangian applied to metrics on the tangent bundle.

Let \( M \) be a compact complex manifold of complex dimension \( n \) with Kähler metric \( g \). Let \( \omega = \sqrt{-1} g_{ij} dz^i \wedge d\bar{z}^j \) be the corresponding Kähler form. Define

\[
P(M, \omega) = \{ \phi \in C^\infty(M) \mid g_{ij} + \partial_i \phi \bar{\partial}_j \phi > 0 \}.
\]

For \( \phi \) in \( P(M, \omega) \), let \( \{\phi_t\}_{0 \leq t < 1} \) be a smooth path in \( P(M, \omega) \) with \( \phi_0 = 0 \), \( \phi_1 = \phi \). Write \( \omega_t \) for \( \omega + \sqrt{-1} \partial \bar{\partial} \phi_t \). Now define the \( k \)th K-energy functional by

\[
M_{k, \omega}(\phi) = -(n - k + 1) \int_0^1 \int_M \phi_t (c_k(\phi_t) - \mu_k \omega_t^k) \wedge \omega_t^{n-k} dt,
\]
where $c_k(\phi_t)$ (also written $c_k(\omega_t)$) is the representative of the $k$th Chern class of $M$ given by the metric $\omega_t$, and $\mu_k$ is the number given by

$$\mu_k = \frac{1}{V} \int_M c_k(\omega) \wedge \omega^{n-k}, \quad \text{where} \quad V = \int_M \omega^n.$$

The case $k = 1$ corresponds to the original K-energy as defined in \cite{11}. Theorems 1 and 2 below were proved by Bando and Mabuchi \cite{3} (also see \cite{9} for a proof of Theorem 1). We present an alternative derivation using Bott-Chern forms, leading to new formulas for the higher K-energy (Lemma 5.1 and Corollary 5.2) as well as Theorem 3 below, which gives the second K-energy in terms of Donaldson’s Lagrangian.

**Theorem 1.** $M_{k,\omega}(\phi)$ is independent of the choice of path $\{\phi_t\}$.

To define the higher Futaki invariants, let $c_k(\omega)$ and $Hc_k(\omega)$ be respectively the $k$th Chern form and its harmonic part. There is a real smooth $(k-1,k-1)$ form $f_k$ such that

$$c_k(\omega) - Hc_k(\omega) = \sqrt{-1} \partial \bar{\partial} f_k.$$

Define the $k$th Futaki invariant, which acts on a holomorphic vector field $X$, by

$$\mathcal{F}_{k,\omega}(X) = \int_M \mathcal{L}_X f_k \wedge \omega^{n-k+1},$$

where $\mathcal{L}_X$ is the Lie derivative. The case $k = 1$ corresponds to the original Futaki invariant if $c_1(M) = \mu_1 \omega$, so that $Hc_1(\omega) = \mu_1 \omega$. It is shown in \cite{11} (see also \cite{4}) that $\mathcal{F}_{k,\omega}$ is independent of choice of metric in the Kähler class.

Now given a holomorphic vector field $X$, let $\Phi_t$ be the integral curve of $X_R = X + \mathbf{X}$. Then $\Phi_t : M \rightarrow M$ is holomorphic, and there exists a smooth path $\{\phi_t\}$ in $P(M,\omega)$ satisfying

$$\Phi_t^* \omega = \omega + \sqrt{-1} \partial \bar{\partial} \phi_t, \quad \text{and} \quad \int_M \phi_t \omega^n = 0.$$

Then the higher K-energy integrates the Futaki invariant in the following sense.

**Theorem 2.** Suppose that $c_k(M) = \mu_k [\omega^k]$. Then with $\{\phi_t\}$ as above,

$$\frac{d}{dt} M_{k,\omega}(\phi_t) = 2 \text{Re}(\mathcal{F}_{k,\omega}(X)).$$
Donaldson \cite{Donaldson1, Donaldson2} defined a Lagrangian which was used in his proof that Mumford-Takemoto stable vector bundles over projective algebraic varieties admit Hermitian-Einstein metrics. It is defined in terms of the Bott-Chern forms corresponding to $\text{Tr}(A)$ and $\text{Tr}(AB)$. It is unsurprising then that this Lagrangian is related to the first and second K-energy functionals, which correspond to the first and second Chern classes.

Let $E$ be a holomorphic vector bundle of rank $r$ over $M$, and let $H$ and $H_0$ be Hermitian metrics on $E$. Let $\{H_t\}_{0 \leq t \leq 1}$ be a smooth path of metrics between $H_0$ and $H_1 = H$. Let $F_t$ be the curvature of $H_t$. Then Donaldson’s Lagrangian is given by

$$L(H, H_0) = \int_M \int_0^1 n\sqrt{-1}\text{Tr}(\dot{H}_t H_t^{-1} F_t) \wedge \omega^{n-1} dt - \lambda \int_M \int_0^1 \text{Tr}(\dot{H}_t H_t^{-1}) \omega^n dt,$$

where

$$\lambda = \frac{2\pi n}{r} \frac{1}{V} \int_M c_1(E) \wedge \omega^{n-1}.$$

Now consider the case when $E = T'M$, the holomorphic tangent bundle. Let $H_0$ be the Kähler metric $g$, and, for $\phi$ in $P(M, \omega)$, let $H$ be $g_\phi$. Write $\omega_\phi$ for $\omega + \sqrt{-1}\partial \bar{\partial} \phi$ and $L(\phi)$ for $L(g_\phi, g)$. We have the following formula for the second K-energy functional.

**Theorem 3.** For $\phi$ in $P(M, \omega)$,

$$M_{2, \omega}(\phi) = -\frac{1}{(2\pi)^2 n} L(\phi)$$

$$- \frac{1}{4\pi n} \int_M \log \left( \frac{\omega_\phi^n}{\omega^n} \right) (2\mu_1 \omega - n c_1(\phi) - n c_1(0)) \wedge \omega^{n-1}$$

$$- \int_M \phi \left( c_2(\phi) \wedge \sum_{i=0}^{n-2} \omega^i \wedge \omega_\phi^{n-2-i} - \left( \frac{n-1}{n+2} \right) \mu_2 \sum_{i=0}^{n} \omega^i \wedge \omega_\phi^{n-i} \right).$$

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2. Chern classes and Bott-Chern forms

Before proving the above theorems, we will briefly discuss Chern classes and their Bott-Chern forms. For more details on Bott-Chern forms, see [4], [6] and [16]. Let $E$ be a holomorphic vector bundle of rank $r$ over the compact Kähler manifold $M$. If $H = H_{\alpha\beta}$ is a Hermitian metric on $E$, then its curvature is an endomorphism-valued $(1,1)$ form given by

$$(F_H)_{\alpha}^\beta = (F_H)_{\alpha}^\beta \overline{\sigma} \, dz^i \wedge d\overline{z}^j = \overline{\partial}((\partial H_{\alpha\beta}) H^\overline{\beta}).$$

Let $\Phi$ be an invariant symmetric $k$-linear function on $\mathfrak{gl}(r, \mathbb{C})$. Then we have a Chern-Weil form

$$\Phi(H) = \Phi(F_H) = \Phi(F_H, \ldots, F_H) \in \Omega^{k,k}_M,$$

which represents a characteristic class.

Let $H$ and $H_0$ be two Hermitian metrics on $E$, and let $\{H_t\}$ be a smooth path of metrics between them. Then there exists a Bott-Chern form $BC_{\Phi}(H, H_0)$ in $\Omega^{k-1,k-1}_M/(\text{Im} \partial + \text{Im} \overline{\partial})$ given by

$$BC_{\Phi}(H, H_0) = -k\sqrt{-1} \int_0^1 \Phi(\dot{H}_t H_t^{-1}, F_{H_t}, \ldots, F_{H_t}) dt.$$ 

It is shown in [4] that this definition is independent of the choice of path. Moreover we have

$$-\sqrt{-1}\partial \overline{\partial} BC_{\Phi}(H, H_0) = \Phi(H) - \Phi(H_0),$$

which can be checked by differentiating with respect to some parameter.

We will restrict to the case of Chern classes of holomorphic vector bundles. The Chern-Weil form for the metric $H$ representing the $k$th Chern class of $E$ is given by

$$c_k(H) = \frac{1}{k!} \left(\frac{\sqrt{-1}}{2\pi}\right)^k \sum_{\pi \in \Sigma_k} \text{sgn}(\pi) F_{\alpha_{s(1)}}^{\alpha_{1}} \wedge F_{\alpha_{s(2)}}^{\alpha_{2}} \wedge \ldots \wedge F_{\alpha_{s(k)}}^{\alpha_{k}},$$

where $F = F_H$ and the usual summation convention is being used.

The Bott-Chern form for the $k$th Chern class is then given by

$$BC_k(H, H_0) = -k\sqrt{-1} \int_0^1 \frac{1}{k!} \left(\frac{\sqrt{-1}}{2\pi}\right)^k \sum_{\pi \in \Sigma_k} \text{sgn}(\pi) (\dot{H}_t \overline{H}_t^{-1})_{\alpha_{s(1)}}^{\alpha_{1}} F_{\alpha_{s(2)}}^{\alpha_{2}} \wedge \ldots \wedge F_{\alpha_{s(k)}}^{\alpha_{k}} dt,$$
where $F = F_{H_t}$. It satisfies

$$c_k(H) - c_k(H_0) = -\sqrt{-1} \partial \bar{\partial} BC_k(H, H_0).$$

Now consider the case where the vector bundle is $T'M$, the holomorphic tangent bundle of $M$. Then $g_{\bar{\partial} \phi}$, the Kähler metric on $M$, is a Hermitian metric on $T'M$. Let $\phi$ be in $P(M, \omega)$. Write $g_\phi$ for $g_{\bar{\partial} \phi}$ and $c_k(\phi)$ for $c_k(g_\phi)$. Then, in normal coordinates for $g_\phi$, we have

$$c_k(\phi) = \frac{1}{k!} \left( \frac{\sqrt{-1}}{2\pi} \right)^k \sum_{\pi \in \Sigma_k} \text{sgn}(\pi) F_{\alpha_1 \pi_1}^{\alpha_k} \wedge \ldots \wedge F_{\alpha_k \pi_k}^{\alpha_1},$$

writing $F^{\alpha_\beta}_{\alpha_\beta}$ for $R^{\alpha_\beta}_{\alpha_\beta i j} dz^i \wedge dz^j$, where $R^{\alpha_\beta}_{\alpha_\beta i j}$ is the curvature tensor for $g_\phi$.

3. Higher K-energy functionals

In this section we give an alternative proof of Theorem 1. Let $\{\phi_t\}_{0 \leq t \leq 1}$ be a smooth path in $P(M, \omega)$ with $\phi_0 = \phi_1$, and let $\lambda$ be any constant. Calculate

$$\int_0^1 \int_M \dot{\phi}_t (c_k(\phi_t) - \lambda \omega^k_t) \wedge \omega^{n-k}_t dt$$

$$= \int_0^1 \int_M \dot{\phi}_t (c_k(\phi_t) - c_k(\phi_0) + c_k(\phi_0) - \lambda \omega^k_t) \wedge \omega^{n-k}_t dt$$

$$= \int_0^1 \int_M \dot{\phi}_t (-\sqrt{-1} \partial \bar{\partial} BC_k(\phi_t, \phi_0) + c_k(\phi_0) - \lambda \omega^k_t) \wedge \omega^{n-k}_t dt$$

$$= \int_0^1 \int_M (-\sqrt{-1} \partial \bar{\partial} \dot{\phi}_t) \wedge BC_k(\phi_t, \phi_0) \wedge \omega^{n-k}_t dt$$

$$+ \int_0^1 \int_M \dot{\phi}_t (c_k(\phi_0) - \lambda \omega^k_t) \wedge \omega^{n-k}_t dt$$

$$= -\frac{1}{n - k + 1} \int_0^1 \int_M BC_k(\phi_t, \phi_0) \wedge \frac{\partial}{\partial t} (\omega^{n-k+1}_t) dt$$

$$+ \int_0^1 \int_M \dot{\phi}_t (c_k(\phi_0) - \lambda \omega^k_t) \wedge \omega^{n-k}_t dt$$

$$= \frac{1}{n - k + 1} \int_0^1 \int_M \frac{\partial}{\partial t} BC_k(\phi_t, \phi_0) \wedge \omega^{n-k+1}_t dt$$

$$+ \int_0^1 \int_M \dot{\phi}_t (c_k(\phi_0) - \lambda \omega^k_t) \wedge \omega^{n-k}_t dt.$$

The next two lemmas will show that this expression is zero. The proof of the theorem will then follow immediately.
Lemma 3.1 \textit{With} \{\phi_t\} \textit{as above},

$$
\int_0^1 \int_M \dot{\phi}_t (c_k(\phi_0) - \lambda \omega^k_t) \wedge \omega^{n-k}_t dt = 0.
$$

\textbf{Proof} \textit{The argument is almost identical to one given in [13], but we will include it for the reader’s convenience. Define}

$$
\phi_{t,s} = s \phi_t, \quad \text{and} \quad \omega_{t,s} = \omega + \sqrt{-1} \partial \bar{\partial} \phi_{t,s},
$$

$$
D(s) = \int_0^1 \int_M \frac{\partial \phi_{t,s}}{\partial t} (c_k(\phi_0) - \lambda \omega^k_t,\omega^{n-k}_t) dt.
$$

Notice that \(D(0) = 0\). We will show that \(D'(s) = 0\).

\[
D'(s) = \int_0^1 \int_M \frac{\partial^2 \phi_{t,s}}{\partial t \partial s} (c_k(\phi_0) - \lambda \omega^k_t,\omega^{n-k}_t) dt \\
+ \int_0^1 \int_M \frac{\partial \phi_{t,s}}{\partial t} ((n-k)c_k(\phi_0)\omega^{n-k}_t - \lambda n \omega^{n-1}_t) \wedge \sqrt{-1} \partial \bar{\partial} (\phi_{t,s}) dt \\
= \int_0^1 \int_M \frac{\partial \phi_{t,s}}{\partial t} ((n-k)c_k(\phi_0)\omega^{n-k}_t - \lambda n \omega^{n-1}_t) \wedge \sqrt{-1} \partial \bar{\partial} (\phi_{t,s}) dt \\
+ \int_0^1 \int_M \frac{\partial \phi_{t,s}}{\partial s} ((n-k)c_k(\phi_0)\omega^{n-k}_t - \lambda n \omega^{n-1}_t) \wedge \sqrt{-1} \partial \bar{\partial} (\phi_{t,s}) dt \\
= \int_0^1 \int_M \frac{\partial}{\partial t} \left( \frac{\partial \phi_{t,s}}{\partial s} (c_k(\phi_0) - \lambda \omega^k_t,\omega^{n-k}_t) \right) dt \\
= 0.
\]

Lemma 3.2 \textit{With} \{\phi_t\} \textit{as above},

$$
\int_M \frac{\partial}{\partial t} (BC_k(\phi_t, \phi_0)) \wedge \omega^{n-k+1}_t = 0.
$$

\textbf{Proof} \textit{Writing} \(g\) \textit{for} \(g_{\phi_t}\) \textit{we have}

$$
\dot{g} g \bar{g}^k = (\partial_i \bar{\partial}_j \phi_t) g \bar{g}^k.
$$

Hence, using the formula for the Bott-Chern form, and working in normal coordinates,

$$
\int_M \frac{\partial}{\partial t} BC_k(\phi_t, \phi_0) \wedge \omega^{n-k+1}_t
$$
Integrating by parts, using the Bianchi identity, and defining \( \tau \) to be the transposition in \( \Sigma_k \) which interchanges 1 and 2, we see that one term in the integrand of (1) will be

\[- \sum_{\pi \in \Sigma_k} \text{sgn}(\pi) \nabla_{\pi_1} \phi_t \nabla_{\alpha_{x(1)}} F_{\alpha_{x(2)}} \pi_2 \wedge F_{\alpha_{x(3)}} \pi_3 \wedge \ldots \wedge F_{\alpha_{x(k)}} \pi_k \wedge \omega^{n-k+1}_t.\]

Every other term is zero by the same argument, and the lemma is proved.

The proof of Theorem 1 is now complete.

**Remark 3.3** \( M_{k,\omega} \) satisfies a cocycle condition. Namely, if \( \omega' \) is another Kähler metric with \( \omega' = \omega + \sqrt{-1} \partial \bar{\partial} \psi \), then

\[ M_{k,\omega}(\phi) - M_{k,\omega'}(\phi - \psi) = M_{k,\omega}(\psi). \]

**Remark 3.4** The critical points of \( M_{k,\omega} \) are metrics \( \omega \) with \( \Lambda^k c_k(\omega) \) constant. An example of such a metric is the Fubini-Study metric on \( \mathbb{C}P^n \).

### 4. Higher Futaki invariants

In this section we give a proof of Theorem 2. Let \( X \) be a holomorphic vector field. Recall that the Lie derivative of \( X \) on forms is given by

\[ \mathcal{L}_X = i_X \circ d + d \circ i_X. \]
The interior product $i_X \omega$ is a $(0,1)$ form which is $\overline{\partial}$-closed since $\omega$ is Kähler. Hence there exists a smooth function $\theta_X$ and a harmonic $(0,1)$ form $\alpha$ such that

$$i_X \omega = \alpha - \overline{\partial}\theta_X.$$  

We have the following lemma.

**Lemma 4.1**

$$F_{k,\omega}(X) = -(n - k + 1) \sqrt{-1} \int_M \theta_X(c_k(\omega) - Hc_k(\omega)) \wedge \omega^{n-k},$$

where $\theta_X$ is as given above.

**Proof** Integrating by parts,

$$F_{k,\omega}(X) = \int_M \mathcal{L}_X f_k \wedge \omega^{n-k+1}$$

$$= - \int_M f_k \wedge \mathcal{L}_X \omega^{n-k+1}$$

$$= - \int_M f_k \wedge \partial i_X \omega^{n-k+1}$$

$$= -(n - k + 1) \int_M f_k \wedge \partial (\alpha - \overline{\partial}\theta_X) \wedge \omega^{n-k}$$

$$= (n - k + 1) \int_M f_k \wedge \partial \overline{\partial}\theta_X \wedge \omega^{n-k}$$

$$= -(n - k + 1) \sqrt{-1} \int_M \theta_X(c_k(\omega) - Hc_k(\omega)) \wedge \omega^{n-k}.$$  

To finish the proof of the theorem, let $\Phi_t$ be the integral curve of $X_R$ as before, and let $\phi_t$ be given by

$$\Phi_t^* \omega = \omega_t = \omega + \sqrt{-1} \overline{\partial}\phi_t, \quad \int M \phi_t \omega^n = 0.$$  

Then

$$\mathcal{L}_{X_R} \omega_t = \sqrt{-1} \overline{\partial}\phi_t.$$  

Hence

$$\partial i_X \omega_t + \overline{\partial} i_X \omega_t = \frac{1}{2} \sqrt{-1} (\partial \overline{\partial}\phi_t - \overline{\partial}\partial\phi_t)$$

and so

$$i_X \omega_t - \frac{1}{2} \sqrt{-1} \overline{\partial}\phi_t = \alpha_t,$$  

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where $\alpha_t$ is a $(0,1)$ form satisfying $\text{Re}(\partial \alpha_t) = 0$. It follows from above that

$$2\text{Re}(\mathcal{F}_{k,\omega_t}(X)) = -(n-k+1) \int_M \phi_t(c_k(\omega_t) - H c_k(\omega_t)) \wedge \omega_t^{n-k}$$

$$= -(n-k+1) \int_M \phi_t(c_k(\omega_t) - \mu_k \omega_t^k) \wedge \omega_t^{n-k}$$

$$= \frac{d}{dt} M_{k,\omega}(\phi_t).$$

Finally we must show that $\mathcal{F}_{k,\omega_t}(X) = \mathcal{F}_{k,\omega}(X)$. This is immediate from [1], or alternatively can be seen as follows. Since

$$\mathcal{L}_{X_t} \omega_t = \Phi_t^* \mathcal{L}_{X_t} \omega,$$

we have

$$\partial \Phi_t = \partial \Phi_t^* \Phi_0,$$

and so $\dot{\phi}_t = \Phi_t^* \dot{\phi}_0$. Hence

$$\int_M \phi_t(c_k(\omega_t) - \mu_k \omega_t^k) \wedge \omega_t^{n-k} = \int_M \Phi_t^* \dot{\phi}_0(\Phi_t^* c_k(\omega) - \mu_k \Phi_t^* \omega^k) \wedge \Phi_t^* \omega^{n-k}$$

$$= \int_M \dot{\phi}_0(c_k(\omega) - \mu_k \omega^k) \wedge \omega^{n-k}.$$  

5. Higher K-energy functionals and Donaldson’s Lagrangian

In this section we give a proof of Theorem 3 and give two alternative formulas for the higher K-energy functionals. It is pointed out in [15] (see also [5] and [13]) that the K-energy can be written without a path integral as

$$M_{1,\omega}(\phi) = \int_M \frac{1}{2\pi} \log \left( \frac{\omega_\phi^n}{\omega^n} \right) \omega_\phi^n$$

$$- \int_M \phi \left( \frac{\sqrt{-1}}{2\pi} \text{Ric}(\omega) \wedge \sum_{i=0}^{n-1} \omega_i \wedge \omega_\phi^{n-1-i} - \frac{\mu_1}{n+1} \sum_{i=0}^{n} \int_M \phi \omega_i \wedge \omega_\phi^{n-i} \right).$$

This appears to be both a natural and useful expression for the K-energy (see [12]). Notice that

$$c_1(\omega) = \frac{\sqrt{-1}}{2\pi} \text{Ric}(\omega) \quad \text{and} \quad \frac{1}{2\pi} \log \left( \frac{\omega_\phi^n}{\omega^n} \right) = \text{BC}_1(\phi,0),$$

where to be precise, the above is only a representative of $\text{BC}_1(\phi,0)$. The following lemma generalizes the above formula to the higher K-energy functionals.
Lemma 5.1

\[ M_{k,\omega}(\phi) = \int_M BC_k(\phi, 0) \land \omega^n_{\phi} - \int_M \phi \left( c_k(0) \land \sum_{i=0}^{n-k} \omega^i \land \omega^n_{\phi} - \frac{(n-k+1)}{n+1} \mu_k \sum_{i=0}^n \omega^i \land \omega^n_i \right). \]

**Proof** For a path \( \{ \phi_t \} \) in \( P(M, \omega) \), calculate

\[ \int_M \frac{\partial}{\partial t} (BC_k(\phi_t, 0) \land \omega^n_{\phi}^{n-k+1}) \]

\[ = \int_M \frac{\partial}{\partial t} (BC_k(\phi_t, 0)) \land \omega^n_{\phi}^{n-k+1} + (n-k+1) \int_M BC_k(\phi_t, 0) \land \omega^n_{\phi}^{n-k} \land \sqrt{-1} \partial \bar{\theta} \dot{\phi}_t \]

\[ = -(n-k+1) \int_M \dot{\phi}_t(c_k(\phi_t) - c_k(0)) \land \omega^n_{\phi}^{n-k}, \quad (2) \]

using Lemma 3.2 for the first term and integrating by parts for the second.

Also

\[ \int_M \frac{\partial}{\partial t} (\phi_t c_k(0) \land \sum_{i=0}^{n-k} \omega^i \land \omega^n_{\phi}^{n-k-i}) \]

\[ = \int_M \dot{\phi}_t c_k(0) \land \sum_{i=0}^{n-k} \omega^i \land \omega^n_{\phi}^{n-k-i} \]

\[ + \int_M \phi_t c_k(0) \land \sum_{i=0}^{n-k} (n-k-i) \omega^i \land \omega^n_{\phi}^{n-k-i} \land \sqrt{-1} \partial \bar{\theta} \dot{\phi}_t \]

\[ = \int_M \dot{\phi}_t c_k(0) \land \sum_{i=0}^{n-k} \omega^i \land \omega^n_{\phi}^{n-k-i} \]

\[ + \int_M \dot{\phi}_t c_k(0) \land \sum_{i=0}^{n-k} (n-k-i) \omega^i \land \omega^n_{\phi}^{n-k-i} \land (\omega_t - \omega) \]

\[ = \int_M \dot{\phi}_t c_k(0) \land \left( \sum_{i=0}^{n-k} (n-k-i+1) \omega^i \land \omega^n_{\phi}^{n-k-i} \right) \]

\[ - (n-k-i) \omega_t^{i+1} \land \omega^n_{\phi}^{n-k-i-1} \]

\[ = (n-k+1) \int_M \dot{\phi}_t c_k(0) \land \omega^n_{\phi}^{n-k}. \quad (3) \]
Similarly,
\[
\int_M \frac{\partial}{\partial t} \left( \frac{(n-k+1)\mu_k}{n+1} \phi t \sum_{i=0}^n \omega_i \wedge \omega^{n-i}_t \right) = (n-k+1)\mu_k \int_M \dot{\phi} \omega^n_t. \tag{4}
\]

The lemma follows from (2), (3) and (4).

We will need a slightly different formula for \( M_{k,\omega} \).

**Corollary 5.2**

\[
M_{k,\omega}(\phi) = \int_M BC_k(\phi, 0) \wedge \omega^{n-k+1} \\
- \int_M \phi \left( c_k(\phi) \wedge \sum_{i=0}^{n-k} \omega_i \wedge \omega^{n-k-i}_\phi - \frac{(n-k+1)}{n+1} \mu_k \sum_{i=0}^n \omega_i \wedge \omega^{n-i}_\phi \right).
\]

**Proof**

\[
\int_M BC_k(\phi, 0) \wedge \omega^{n-k+1}_\phi \\
= \int_M BC_k(\phi, 0) \wedge (\omega + \sqrt{-1} \partial \bar{\partial} \phi)^{n-k+1} \\
= \int_M BC_k(\phi, 0) \wedge \sum_{i=0}^{n-k+1} \binom{n-k+1}{i} \omega^i \wedge (\omega \Phi - \omega)^{n-k+1-i} \\
= \int_M BC_k(\phi, 0) \wedge \omega^{n-k+1} \\
+ \int_M BC_k(\phi, 0) \wedge \sqrt{-1} \partial \bar{\partial} \phi \wedge \sum_{i=0}^{n-k} \binom{n-k+1}{i} \omega^i \wedge (\omega \Phi - \omega)^{n-k-i} \\
= \int_M BC_k(\phi, 0) \wedge \omega^{n-k+1} - \int_M \phi(c_k(\phi) - c_k(0)) \wedge \sum_{i=0}^{n-k} \omega^i \wedge \omega^{n-k-i}_\phi,
\]

using, for the last line, the elementary identity
\[
\sum_{i=0}^m \binom{m+1}{i} x^i (y-x)^{m-i} = \sum_{i=0}^m x^i y^{m-i},
\]
which can be seen by observing that each side is equal to
\[
(y^{m+1} - x^{m+1})/(y-x).
\]

The corollary now follows immediately from Lemma 5.1.
We now prove Theorem 3. First note that we have

\[
BC_1(\phi, 0) = \frac{1}{2\pi} \int_0^1 \text{Tr}(\dot{g}_t g_t^{-1}) dt, \quad \text{and} \\
BC_2(\phi, 0) = \frac{\sqrt{-1}}{(2\pi)^2} \int_0^1 \left( \text{Tr}(\dot{g}_t g_t^{-1}) \text{Tr}(F_t) - \text{Tr}(\dot{g}_t g_t^{-1} F_t) \right) dt,
\]

where \( F_t \) is the curvature of \( g_t \). Then using the fact that \( \lambda = 2\pi \mu_1 \) we have

\[
L(\phi) = n(2\pi)^2 \int_M \int_0^1 \frac{\sqrt{-1}}{(2\pi)^2} \text{Tr}(\dot{g}_t g_t^{-1} F_t) \wedge \omega^{n-1} dt \\
- (2\pi)^2 \mu_1 \int_M \int_0^1 \frac{1}{2\pi} \text{Tr}(\dot{g}_t g_t^{-1}) \omega^n dt \\
= -n(2\pi)^2 \int_M BC_2(\phi, 0) \wedge \omega^{n-1} - (2\pi)^2 \mu_1 \int_M BC_1(\phi, 0) \omega^n \\
+ n\sqrt{-1} \int_M \int_0^1 \text{Tr}(\dot{g}_t g_t^{-1}) \text{Tr}(F_t) \wedge \omega^{n-1} dt. \tag{5}
\]

The last term is equal to

\[
n\sqrt{-1} \int_M \int_0^1 (\Delta_t \phi_t) \text{Ric}(\omega_t) \wedge \omega^{n-1} dt \\
= \frac{n\sqrt{-1}}{2} \int_M \log \left( \frac{\omega^n}{\omega^n} \right) (\text{Ric}(\omega_t) + \text{Ric}(\omega)) \wedge \omega^{n-1} \\
= n\pi \int_M \log \left( \frac{\omega^n}{\omega^n} \right) (c_1(\phi) + c_1(0)) \wedge \omega^{n-1}, \tag{6}
\]

where a straightforward integration by parts has been used for the second line. The proof of the theorem now follows from Corollary 5.2 and equations (5) and (6).

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