Supershell Effect and Stability of Classical Periodic Orbits in Reflection-Asymmetric Superdeformed Oscillator

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Abstract

A semiclassical analysis is made of the origin of an undulating pattern in the smoothed level density for a reflection-asymmetric superdeformed oscillator potential. It is suggested that, when the octupole-type deformation increases, an interference effect between two families of periodic orbit with the ratio of periods approximately 2:1 becomes stronger and thus a pronounced “supershell” structure appears.

The quantum-energy spectrum in the axially-symmetric oscillator potential with frequency ratio $\omega_\perp/\omega_z=2$ (called “superdeformed” oscillator) is known to have “supershell” structure (i.e., modulation with periodicity $2\hbar\omega_{sh}$ in the oscillating level density, $\omega_{sh}$ being the basic frequency of the superdeformed oscillator). We have indicated in our previous paper\(^1\) that the supershell effect is significantly enhanced when octupole ($Y_{30}$) deformation is added to the 2:1 deformed harmonic-oscillator potential, and suggested that this enhancement might be responsible for the odd-even effect (with respect to the shell quantum number $N_{sh}$) in stability of superdeformed states against octupole deformed shape, discussed by Bengtsson et al.,\(^2\) Höller and Åberg,\(^3\) and Nazarewicz and Dobaczewski.\(^4\)
The single-particle Hamiltonian we used in our analysis is

\[ h = \frac{p^2}{2M} + \sum_{i=x,y,z} \frac{M\omega^2_i x_i^2}{2} - \lambda_3 K M\omega^2_0 \left( r^2 Y_3 K(\Omega) \right)'' , \tag{1} \]

where \( \omega_x = \omega_y = 2\omega_z \equiv 2\omega_{sh} \) and \( \omega^3_0 = \omega_x\omega_y\omega_z \). The double primes denote that the variables in parenthesis are defined in terms of the doubly-stretched coordinates\(^5\) \( x''_i = (\omega_i/\omega_0) x_i \).

In the following, we limit to the case \( K = 0 \). When the variables are scale-transformed to dimensionless ones, the Hamiltonian (1) is rewritten as

\[ h = \frac{p^2}{2} + \left( \frac{r^2}{2} - \lambda_30 r^2 Y_{30}(\theta) \right)'' , \tag{2} \]

where \( x'' = 2x, y'' = 2y \) and \( z'' = z \).

As is well known, the quantum level density \( g(E) = \sum_n \delta(E - E_n) \) is expressed in semiclassical theory as\(^6\)

\[ g(E) \approx \bar{g}(E) + \sum_{\gamma} A_\gamma(E) \cos \left( \frac{S_\gamma(E)}{\hbar} - \text{(phases)}_{\gamma} \right) , \tag{3} \]

where \( \bar{g} \) is an average level density corresponding to the Thomas-Fermi approximation, and the second term in the r.h.s. represents the oscillatory contributions from periodic orbits, \( S_\gamma \) is the action integral \( \oint_{\gamma} p \cdot dq \), and the amplitude factor \( A_\gamma \) is mainly related to the stability of the orbit \( \gamma \). When one is interested in an undulating pattern in \( g(E) \) smoothed to a finite resolution \( \delta E \) (i.e., shell structure), it is sufficient to only consider short periodic orbits with the periods \( T_\gamma < 2\pi \hbar/\delta E \). The supershell structure is expected to arise from interference effects between orbits with different periods \( T_\gamma \). The short periodic orbits are calculated by Monodromy Method\(^7\) and shown in Fig. 1.

[Fig. 1]

For the Hamiltonian system under consideration whose phase space is constructed with both regular and chaotic regions, evaluation of \( A_\gamma \) in eq. (3) is not always easy because
the stationary-phase approximation breaks down near resonances which take places rather frequently in the regular regions. Fortunately, however, by virtue of the scaling property, $h(\alpha p, \alpha q) = \alpha^2 h(p, q)$, we can use the Fourier-transformation techniques and extract informations about classical periodic orbits from quantum energy spectrum. The scaling rules for variables in eq. (3) are

$$\bar{g}(E) = E^2 \bar{g}(1),$$
$$S_\gamma(E) = E T_\gamma,$$
$$A_\gamma(E) = E^{k\gamma} A_\gamma(1) \quad \begin{cases} k_\gamma = 0 & \text{for isolated orbits}, \\ k_\gamma = \frac{1}{2} & \text{otherwise}. \end{cases}$$

The last equality is obtained under the stationary-phase approximation. Using these relations, it is easy to see that the Fourier transform of eq. (3) multiplied by an appropriate weighting factor $E^{-k}$ will exhibit peaks at the periods $T_\gamma$ of classical periodic orbits and the heights of the peaks represent the strengths of their contributions. In Fig. 2, we show the power spectrum $P(s)$ for several values of $\lambda_{30}$, taking $k = \frac{1}{2}$ appropriate to non-isolated orbits;

$$P(s) = \left| \sum_n e^{i s E_n} \right|^2.$$  

![Fig. 2](image)

We see nice correspondence between peak locations of $P(s)$ and periods of classical periodic orbits. The most important observation is that relative intensity between peaks at $s \approx \pi$ and $s \approx 2\pi$ changes as the octupole deformation parameter $\lambda_{30}$ increases. This result indicates that the enhancement of the supershell effect (shown in Fig. 6 of Ref. 1 ) may be explained as due to the growth of the interference effect between classical periodic orbits with periods $T \approx \pi$ and those with $T \approx 2\pi$.

To understand the cause of the change in relative intensity mentioned above, let us investigate properties of the classical periodic orbits. Calculating periodic orbits by the
Monodromy Method,\textsuperscript{7)} we obtain stability matrices $M_\gamma$ for the periodic orbits $\gamma$. They are linearized Poincaré maps at the periodic orbits defined as

$$
\begin{pmatrix}
\delta p (T_\gamma) \\
\delta q (T_\gamma)
\end{pmatrix} = M_\gamma \begin{pmatrix}
\delta p (0) \\
\delta q (0)
\end{pmatrix} + \mathcal{O} (\delta^2),
$$

(6)

where $(\delta p, \delta q)$ represent deviations from the periodic orbits $\gamma$ in phase space. These six-dimensional matrices $M_\gamma$ are real and symplectic, so that eigenvalues of each $M_\gamma$ appear in pairs $\pm (e^\alpha, e^{-\alpha})$, $\alpha$ being real or pure imaginary. As the Hamiltonian (2) is axially-symmetric, classical orbits are usually non-isolated. For such orbits, each stability matrix has 4 unit eigenvalues. Values of $\text{Tr} M$ written in Fig. 1 are sums of the remaining 2 eigenvalues which determine stabilities of the periodic orbits; $\alpha=iv$ is pure imaginary and $|\text{Tr} M| = |2 \cos v| \leq 2$ when the orbit is stable, while $\alpha=u$ is real and $|\text{Tr} M| = |\pm 2 \cosh u| > 2$ when the orbit is unstable. Under the stationary-phase approximation, the amplitude factors $A_\gamma$ in eq. (3) are inversely proportional to $\sqrt{|\text{Tr} M_\gamma - 2|}$. Fig. 3 shows values of $\text{Tr} M$ for relevant orbits calculated as functions of the octupole-deformation parameter $\lambda_{30}$.

From this figure, we see that the orbits with $T \approx \pi$ are always stable and their values of $\text{Tr} M$ approach to 2 with increasing $\lambda_{30}$, while the orbit $B$ ($C,C'$) with $T \approx 2\pi$ become unstable (more unstable). Therefore, we can expect that the contributions from orbits with $T \approx \pi$ increase when $\lambda_{30}$ becomes large, while those from orbits with $T \approx 2\pi$ decrease. This result suggests that the enhancement of the supershell effect stems from the difference of the stability against octupole deformation between these two families of periodic orbit.

A more detailed analysis of the supershell structure in reflection-asymmetric superdeformed oscillator potentials will be given in a forthcoming full-length paper.\textsuperscript{9)} The author thanks Prof. Matsuyanagi for carefully reading the manuscript.
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Fig. 1. Short periodic orbits for the Hamiltonian (2) with $\lambda_{30} = 0.4$. Upper part: Planar orbits in the plane containing the symmetric axis $z$. Lower part: A circular orbit in the plane perpendicular to the symmetry axis (A’) and a three-dimensional orbit (C’). Their projections on the $(x, y)$ plane and on the $(z, y)$ plane are shown.
Fig. 2. Power spectra $P(s)$ defined by eq. (5) for $\lambda_{30} = 0.2 \sim 0.4$. The summation is taken up to $n=200$. Arrows indicate periods of the classical periodic orbits (see Fig. 1) and their repetitions.
Fig. 3. Traces of the stability matrices $M$ for the periodic orbits shown in Fig. 1 (see text for their definitions). For the isolated orbit $A'$, the stability matrix $M$ has 2 unit eigenvalues and the remaining 4 eigenvalues appear in two pairs $(e^{\alpha_a}, e^{-\alpha_a})$ and $(e^{\alpha_b}, e^{-\alpha_b})$. Thus, $A'_a$ and $A'_b$ denote the traces of these pairs, respectively.