THE COHERENCE THEOREM FOR ANN-CATEGORIES

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February 1, 2008

abstract

This paper presents the proof of the coherence theorem for Ann-categories whose set of axioms and original basic properties were given in [9]. Let

\[ A = (\mathcal{A}, \mathfrak{A}, c, (0, g, d), a, (1, l, r), \mathcal{L}, \mathfrak{R}) \]

be an Ann-category. The coherence theorem states that in the category \( A \), any morphism built from the above isomorphisms and the identification by composition and the two operations \( \otimes \), \( \oplus \) only depends on its source and its target.

The first coherence theorems were built for monoidal and symmetric monoidal categories by Mac Lane [7]. After that, as shown in the References, there are many results relating to the coherence problem for certain classes of categories.

For Ann-categories, applying Hoang Xuan Sinh’s ideas used for Gr-categories in [2], the proof of the coherence theorem is constructed by faithfully “embedding” each arbitrary Ann-category into a quite strict Ann-category. Here, a quite strict Ann-category is an Ann-category whose all constraints are strict, except for the commutativity and left distributivity ones.

This paper is the work continuing from [9]. If there is no explanation, the terminologies and notations in this paper mean as in [9].

1 Canonical isomorphisms

In this section, we define some canonical isomorphisms induced by isomorphisms \( c, \mathcal{L}, \mathfrak{L}^A \) and the identification, laws \( \otimes, \oplus \) on the quite strict Ann-category \( A \).

Let \( I \) be a fully ordered limited set. If \( I \neq \emptyset \) and \( \alpha \) is the maximal of \( I \), we will denote \( I' = I \setminus \{\alpha\} \); and the notation \( |I| \) refers to the cardinal of \( I \).

**Definition 1.1.** [1] The canonical sum \( \sum_{i \in I} A_i \) where \( A_i \in \text{Ob} \mathcal{A}, i \in I \) is defined inductively as follows

1. \( \sum_{i \in I} A_i = 0 \) if \( I = \emptyset \) and \( \sum_{i \in I} A_i = A_\alpha \) if \( I = \{\alpha\} \).
2. \( \sum_{i \in I} A_i = (\sum_{i' \in I'} A_i \oplus A_\alpha) \) if \( |I| > 1 \).

**Definition 1.2.** We define the isomorphism

\[ \nu_{\sum A_i, \sum B_i} : (\sum_{i \in I} A_i) \oplus (\sum_{i \in I} B_i) \longrightarrow \sum_{i \in I} (A_i \oplus B_i), \]

(which is abbreviated by \( \nu_I \)) by induction on \( |I| \) as follows

1. \( \nu_I = \text{id} \) if \( |I| \leq 1 \).
2. if \( |I| > 2 \), \( \nu_I \) is defined by the following commutative diagram

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1. This paper has been published (in Vietnamese) in Vietnam Journal of Mathematics Vol. XVI, No 1, 1988.
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\[
(\sum_i A_i) \oplus (\sum_j B_j) \xrightarrow{id} (\sum_i A_i) \oplus A_\alpha \oplus (\sum_j B_j) \oplus B_\alpha
\]

where \( \nu = id \oplus c \oplus id \). We can see that the isomorphism \( \nu_1 \) is built only from isomorphisms \( c, id \) by law \( \oplus \). Moreover, the isomorphism \( \nu_1 \) is natural.

**Definition 1.3.** We define following isomorphisms

\[
u_{I,J} : \sum_i \sum_j (A_i \otimes B_j) \longrightarrow \sum_j \sum_i (A_i \otimes B_j)
\]

by induction on \(|I|\) as follows

1. \( \nu_{I,J} = id \) if \(|I| \leq 1\) or \( J = \emptyset \).
2. \( \nu_{I,J} = \nu_J (\nu_{I,\emptyset} \oplus id) \) if \(|I| > 1\).

So, isomorphisms \( \nu_{I,J} \) are also built from the isomorphisms \( c, id \) by law \( \oplus \) and these morphisms are functorial.

**Definition 1.4.** We define following isomorphisms

\[
F_{I,J} : (\sum_i A_i) \otimes (\sum_j B_j) \longrightarrow \sum_i \sum_j (A_i \otimes B_j) = \sum_{i \times J} (A_i \otimes B_j)
\]

where \( I \times J \) is ordered alphabetically as follows

1. \( F_{I,J} = id : 0 \otimes (\sum_j B_j) \rightarrow 0 \) if \( I = \emptyset \).

   (since \( \mathcal{R} = id \) we have \( \mathcal{R}^A = id \) for all \( A \in Ob A)\)
2. \( F_{I,J} = \hat{L}^X : X \otimes 0 \rightarrow 0 \) where \( X = \sum_i A_i \) if \( J = \emptyset \).
3. \( F_{I,J} = \sum I \alpha, \) if \( I \neq \emptyset \) and \( J \neq \emptyset \), where

\[
f_A : A \otimes (\sum_j B_j) \rightarrow \sum_j A \otimes B_j
\]

is defined as follows: If \(|J| = 1\), \( f_A = id \); whereas \( f_A \) is defined by induction on \(|J|\) by the following commutative diagram

\[
\begin{array}{ccc}
A \otimes (\sum_j B_j) & \xrightarrow{f_A} & (\sum_j (A \otimes B_j)) \\
\downarrow L^A & & \| \\
A \otimes (\sum_j B_j) \oplus (A \otimes B_\beta) & \xrightarrow{f_A \otimes id} & (\sum_j A \otimes B_j) \oplus (A \otimes B_\beta)
\end{array}
\]

where \( \beta \) is the maximal element of \( J \) and \( J' = J \setminus \{\beta\} \).

**Definition 1.5.** We define following isomorphisms

\[
K_{I,J} : (\sum_i A_i) \otimes (\sum_j B_j) \longrightarrow \sum_j \sum_i (A_i \otimes B_j)
\]

as follows

1. \( K_{I,J} = id \) if \( I = \emptyset \).
2. \( K_{I,J} = \hat{L}^X \), where \( X = \sum_i A_i \) if \( J = \emptyset \).
3. \( K_{I,J} = f_X \), where \( X = \sum_i A_i \) in other cases.
Then, we have the following proposition immediately

**Proposition 1.6.** With canonical sums \( \sum_I A_i, \sum_J A_j \), we have the relation

\[
K_{I,J} = u_{I,J} \cdot F_{I,J}.
\]

Applying this proposition we can prove

**Proposition 1.7.** Assume \( J_1, J_2 \) be non-empty subsets of \( J \) such that \( J = J_1 \coprod J_2 \) and \( j_1 < j_2 \) if \( j_1 \in J_1, j_2 \in J_2 \). Then for sums \( A = \sum_I A_i, \sum_J A_j, \sum_{J_1} A_j, \sum_{J_2} A_j \) we have following relations

\[
F_{I,J} = \nu_{I} \cdot (F_{I,J_1} \oplus F_{I,J_2}) \cdot \bar{L}^A
\]

\[
F_{J,I} = F_{J_1,I} \oplus F_{J_2,I}.
\]

We will give the proof of this proposition in detail to illustrate the proof using commutative diagrams. Hereafter, for convenience, we write \( AB \) instead of \( A \otimes B \) for all \( A, B \in \text{Obs}(A) \).

**Proposition 1.8.** In the Ann-category \( A \), the following diagrams

\[
\begin{array}{ccc}
(\sum_I A_i) \otimes (\sum_J B_j) \otimes (\sum_T C_t) & \xrightarrow{F_{I,J,T} \otimes id} & (\sum_I A_i \otimes B_j) \otimes (\sum_T C_t) \\
\downarrow{id \otimes F_{I,T}} & & \downarrow{id}
\end{array}
\]

\[
(\sum_I A_i) \otimes (\sum_{J \times T} B_j \otimes C_t) & \xrightarrow{F_{I,J \times T}} & \sum_{I \times J \times T} A_i \otimes B_j \otimes C_t
\]

commute.

**Proof.** 1. In case \( I = \emptyset \), we have the proposition proved since the diagram (1.1) becomes the following one

\[
\begin{array}{ccc}
0(\sum_J B_j)(\sum_T C_t) & \xrightarrow{id \otimes id} & 0(\sum_T C_t) \\
\downarrow{id \otimes F_{J,T}} & & \downarrow{id}
\end{array}
\]

whose commutativity follows from \( \tilde{R}^X = id \) and the property of the zero object (see Prop.3.2 [9]). In case \( J = \emptyset \) or \( T = \emptyset \) the proposition is proved similarly. Hence, we now can suppose that \( I, J, T \) are all not empty.

2. In case \( |I| = 1 \). Firstly, consider the case in which \( |J| = 1 \). We prove the proposition by induction on \( |T| \). Then

If \( |T| = 1 \), the proof is obvious.
If \( |T| = 2 \), the diagram commutes thanks to the axiom (1.1) of Ann-categories (see [9]).
If \( |T| > 2 \), consider the diagram (1.2).
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In this diagram, the region (I) commutes thanks to the axiom (1.1) in [9]; regions (II), (IV), (V) commute thanks to definitions of isomorphisms \( f_{AB}, f_A, f_B \); (VI) commute thanks to the inductive supposition; the parameter commutes since \( \tilde{L}^A \) is a functorial isomorphism. Therefore, the region (III) commutes. This completes the proof.

After that, still with the condition \(|I| = 1\), we can prove the proposition with \(|J| > 1\) by induction on \(|J|\).

3. Now if \(|I| > 1\), consider the diagram (1.3). In this diagram, the region (I) commute since \( R = id \) is a functorial isomorphism; the region (II) commutes thanks to the inductive supposition for the first component of sums, for the second component since the case \(|I| = 1\) has just been aproved above; the region (III) commutes thanks to the property of the isomorphism \( F_{I,J} \) (see the Prop.1.7); regions (IV), (V) commute thanks to definitions of \( F_{I,J} \) and \( F_{I,J \times T} \). So the parameter commutes. This completes the proof. \qed
2 The coherence theorem for Ann-categories

Let $\mathcal{A}$ be a quite strict Ann-category. Assume $X_s, s \in \Omega$ be a non-empty, limited family of objects of $\mathcal{A}$ and $Y$ be an expression of the family $X_s$ with operations $\otimes$ and $\oplus$. With distributivity constraints $\mathcal{L}, \mathcal{R} = \text{id}$, induced isomorphisms $\hat{L}^A$ and isomorphisms $\mathfrak{A} = \text{id}$, $g, d = \text{id}$, $a = \text{id}$, $l, r = \text{id}$, we can write $Y$ as a sum of monomials of objects of $X_s$ by using the following isomorphism

$$h : Y \rightarrow \sum_i A_i$$

where $A_i \neq 0$ for all $i$ if $I \neq \emptyset$ and $h$ is built from the identification, and isomorphisms $\mathcal{L}, \hat{L}^A$. A such a pair $(h, \sum_i A_i)$ is called an expansion form of $Y$. We now define a canonical expansion form of $Y$ by induction on its length, where the length of expansion form $Y$ is the total number of times of appearances of objects $A_i$ in $Y$. It is easily to see that any $Y$ whose length is more than 1 can be written in the form of $U \otimes V$ or $U \oplus V$. That implies

**Definition 2.1.** The canonical expansion form

$$h : Y \rightarrow \sum_i A_i$$

of $Y$ is defined as follows

1. If $Y = \sum_i X_i$, $h = \text{id} : \sum_i X_i \rightarrow \sum_i X_i$, where $I$ is a subset of $\Omega$; whereas $I'$ is the set of indexes $i$ such that $X_i \neq 0$.
2. If $Y = U \otimes V$, the isomorphism $h$ is the composition

$$Y = U \otimes V \xrightarrow{u \otimes v} (\sum J B_j) \otimes (\sum T C_t) \xrightarrow{F_{J,T}} \sum J \times T B_j C_t$$

where $(u, \sum J B_j), (v, \sum T C_t)$ are, respectively, canonical expansion forms of $U, V$ and defined by the induction supposition.
3. If $Y = U_1 \oplus U_2$, $h$ is the composition

$$Y = U_1 \oplus U_2 \xrightarrow{u_1 \oplus u_2} (\sum I_1 B_i) \otimes (\sum I_2 B_i) \xrightarrow{id} \sum I' B_i$$

where $u_1, u_2$ are defined by the inductive supposition; $I = I_1 \sqcup I_2$; $i_1 < i_2$ if $i_1 \in I_1, i_2 \in I_2$ and $I'$ is the set of indexes $i \in I$ such that $B_i \neq 0$.

**Proposition 2.2.** If $Y_1, Y_2$ are expressions of objects $X_s, s \in \Omega$ and $\varphi : Y_1 \rightarrow Y_2$ is the morphism built from morphisms $c, \mathcal{L}, \hat{L}^A$, the identification and laws $\otimes, \oplus$ together with the composition, then they can be embedded into the following commutative diagram

$$\begin{array}{ccc}
Y_1 \xrightarrow{h_1} & \sum_i A_i & \\
\varphi \downarrow & & \downarrow u \\
Y_2 \xrightarrow{h_2} & \sum_i A_{\sigma(i)} & \\
\end{array}$$

where $(h_1, \sum_i A_i), (h_2, \sum_i A_{\sigma(i)})$ are, respectively, canonical expansion forms of $Y_1, Y_2$; $\sigma$ is a permutation of the set $I$ and $u$ is an isomorphism built from the morphism $c$, the identification, the law $\oplus$ together with the composition.

**Proof.** We can prove the proposition in case $\varphi$ is one of isomorphisms $c, \mathcal{L}, \hat{L}^A, \mathfrak{A} = \text{id}$, $g = d = \text{id}$, $a = \text{id}$, $l = r = \text{id}$. Next, we can prove it easily in case $\varphi$ is the sum $\otimes$ or the product $\oplus$ of two morphisms of above-mentioned ones. \[\square\]
Theorem 2.3. Let $Y_1, Y_2, ..., Y_n$ be expressions of the family $X_s, s \in \Omega$ of objects in the quite strict Ann-category $\mathcal{A}$. Let $\varphi_{i,i+1} : Y_i \to Y_{i+1}$ ($i=1, 2, ..., n$), $\varphi_{n,1} : Y_n \to Y_1$ be isomorphisms built from morphisms $c, \Sigma, \hat{L}^\mathcal{A}$, identification by laws $\otimes, \oplus$ and the composition. Then, the following diagram

\[
\begin{array}{cccccc}
Y_1 & \xrightarrow{\varphi_{1,2}} & Y_2 & \xrightarrow{\varphi_{2,3}} & Y_3 & \cdots & \xrightarrow{\varphi_{n,1}} & Y_n \\
\downarrow \varphi_{n,1} \\
\end{array}
\]

commutes.

Proof. Let $(h_i, \sum_J A_{\sigma(i)})$ denote the canonical expansion form of $Y_i$. Consider the following diagram

\[
\begin{array}{cccccccc}
Y_1 & \xrightarrow{\varphi_{1,2}} & Y_2 & \xrightarrow{\varphi_{2,3}} & \cdots & Y_{n-1} & \xrightarrow{\varphi_{n-1,n}} & Y_n \\
\downarrow h_1 & & \downarrow h_2 & & \cdots & \downarrow h_{n-1} & & \downarrow h_n \\
\sum_J A_{\sigma_1(j)} & \xrightarrow{u_{1,2}} & \sum_J A_{\sigma_2(j)} & \xrightarrow{\cdots} & \sum_J A_{\sigma_{n-1}(j)} & \xrightarrow{u_{n-1,n}} & \sum_J A_{\sigma_n(j)} \\
\downarrow u_{n,1} \\
\end{array}
\]

Then we have the diagram (2.2), where morphisms $u_{i,i+1}$ ($i = 1, 2, ..., n-1$), $u_{n,1}$ make regions $(1), ..., (n)$ and the parameter commute according to Prop.2.2. They are built from $e, id$ and the laws $\otimes$, so according to the coherence theorem for a symmetric monoidal category, the region (b) commutes. Therefore, the region (a) commutes. This completes the proof.

3 The general case

In this last section, we assume that $\mathcal{A}, \mathcal{A'}$ are Ann-categories with, respectively, constraints

\[
(\mathfrak{A}, c, (0, g, d), a, (1, l, r), \Sigma, \mathfrak{R}) \quad (\mathfrak{A}', c', (0', g', d'), a', (1', l', r'), \Sigma', \mathfrak{R}')
\]

and $(F, \bar{F}, \bar{F}) : \mathcal{A} \to \mathcal{A}'$ is a faithful Ann-functor such that the pair $(F, \bar{F})$ is compatible with the unitivity constraints $(1, l, r), (1', l', r')$. In addition, let $\hat{F} : F1 \to 1'$ denote the isomorphism induced by the above compatibility.

Let $(X_i), i \in I$ be a non-empty, limited family of objects of $\mathcal{A}$, and $Y = \mathcal{H}(X_i)$ be a certain expression of the family $(X_i), i \in I$. Then, the expression $Y' = \mathcal{H}(X_i')$ is called the canonical
image of $Y = \mathcal{H}(X_i)$ under $F$ if

$$
\begin{align*}
X_i' &= 0' & \text{when } X_i &= 0 \\
X_i' &= 1' & \text{when } X_i &= 1 \\
X_i' &= FX_i & \text{otherwise}
\end{align*}
$$

From this notion we give the following definition

**Definition 3.1.** We define a canonical isomorphism

$$f : FY = F(\mathcal{H}(X_i)) \to F(\mathcal{H}(X_i'))$$

by induction on $Y$’s length as follows

1. If $Y$’s length is equal to 1, $Y = X_\alpha$ then

$$
\begin{align*}
f &= \hat{F} : F0 \to 0' & \text{in case } X_\alpha &= 0 \\
f &= \check{F} : F1 \to 1' & \text{in case } X_\alpha &= 1 \\
f &= \text{id} : FX_\alpha \to FX_\alpha & \text{in other cases}
\end{align*}
$$

2. If $Y$’s length is more than 1, $Y = U_1 \otimes U_2$ or $Y = U_1 \oplus U_2$. Then, the isomorphism $f$ is, respectively, the following compositions

$$
\begin{align*}
FY = F(U_1 \otimes U_2) & \xrightarrow{\hat{F}} FU_1 \otimes FU_2 \xrightarrow{f_1 \otimes f_2} \mathcal{H}_1(X_i') \otimes \mathcal{H}_2(X_i') \\
FY = F(U_1 \oplus U_2) & \xrightarrow{\check{F}} FU_1 \oplus FU_2 \xrightarrow{f_1 \oplus f_2} \mathcal{H}_1(X_i') \oplus \mathcal{H}_2(X_i')
\end{align*}
$$

where $f_1$, $f_2$ are canonical isomorphisms determined by inductive supposition.

**Proposition 3.2.** Suppose that $\varphi : Y_1 \to Y_2$ is a morphism built from isomorphisms $\mathfrak{A}$, $c$, $g$, $d$, $a$, $l$, $r$, $\mathfrak{L}$, $\mathfrak{R}$, $\hat{L}^A$, $\hat{R}^A$ in the Ann-category $\mathcal{A}$. Then, $\varphi$ can be embedded into the following commutative diagram

$$
\begin{array}{ccc}
FY_1 & \xrightarrow{f_1} & Y_1' \\
F(\varphi) \downarrow & & \downarrow \varphi' \\
FY_2 & \xrightarrow{f_2} & Y_2'
\end{array}
$$

where $f_i$ are canonical isomorphisms corresponding to canonical images $Y_i'$ of $Y_i$ ($i = 1, 2$), whereas $\varphi'$ is a morphism built from isomorphisms $\mathfrak{A}'$, $c'$, $g'$, $d'$, $a'$, $l'$, $r'$, $\mathfrak{L}'$, $\mathfrak{R}'$, $\hat{L}'^A$, $\hat{R}'^A$ in the Ann-category $\mathcal{A}'$.

**Proof.** The proof is completely similar to the one of Proposition 2.2.

Following is the main result of this paper.

**Theorem 3.3.** Let $Y_1, Y_2, ..., Y_n$ be expressions of the limited family of objects $(X_i)_{i \in I}$ of an Ann-category $\mathcal{A}$. Let $\varphi_{i+1} : Y_i \to Y_{i+1}$ $(i = 1, 2, ..., n-1)$, $\varphi_{n} : Y_n \to Y_1$ be isomorphisms built from the isomorphisms $\mathfrak{A}$, $c$, $g$, $d$, $a$, $l$, $r$, $\mathfrak{L}$, $\mathfrak{R}$, $\hat{L}^A$, $\hat{R}^A$, the identification and laws $\otimes$, $\oplus$. Then, the diagram (2.1) commutes.

**Proof.** From the theorem 2.4 [9], the Ann-category $\mathcal{A}$ can be faithfully embedded into a quite strict Ann-category $\mathcal{A}'$ by the faithful Ann-functor $(F, \hat{F}, \check{F})$. Moreover, $(F, \check{F})$ is compatible with the unitivity constraints. In order to prove the diagram (2.1) commutative, we consider its image under $F$.
where \( f_i \) are canonical isomorphisms, whereas \( \varphi_{i,i+1} \) \( (i = 1, 2, ..., n-1) \), \( \varphi_{n,1} \) are morphisms making regions from (1) to (n) and the parameter commute according to the Prop.3.2. These morphisms are built from isomorphisms \( c', L', \hat{L}', A \), \( id \) and by laws \( \otimes, \oplus \). Applying Theorem 2.3, the region (b) commutes. This implies that the region (a) commutes. This completes the proof.

**Remark.** The coherence theorem can be stated in another way as follows: Between two objects of the Ann-category \( \mathcal{A} \), there exists no more than one morphism built from morphisms \( \mathfrak{A}, c, g, d, a, l, r, L, \hat{L}, \hat{R}, A \), \( id \) and by laws \( \otimes, \oplus \).
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