Kinetic theory of one-dimensional inhomogeneous long-range interacting
$N$-body systems at order $1/N^2$ without collective effects

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Long-range interacting systems irreversibly relax as a result of their finite number of particles, $N$. At order $1/N$, this process is described by the inhomogeneous Balescu–Lenard equation. Yet, this equation exactly vanishes in one-dimensional inhomogeneous systems with a monotonic frequency profile and sustaining only 1:1 resonances. In the limit where collective effects can be neglected, we derive a closed and explicit $1/N^2$ collision operator for such systems. We detail its properties highlighting in particular how it satisfies an $H$-theorem for Boltzmann entropy. We also compare its predictions with direct $N$-body simulations. Finally, we exhibit a generic class of long-range interaction potentials for which this $1/N^2$ collision operator exactly vanishes.

I. INTRODUCTION

Because they are composed of a finite number of particles, long-range interacting $N$-body systems unavoidably relax towards their thermodynamical equilibrium [1–4], should it exist [5, 6]. Such a dynamics is sourced by Poisson shot noise: for a fixed total mass, the larger the number of particles, $N$, the slower the diffusion. Kinetic theory aims at describing this long-term relaxation. Here, we are interested in inhomogeneous systems, i.e. systems with a non-trivial mean-field orbital structure [2]. When limited to $1/N$ effects, i.e. two-body correlations, the system’s evolution is generically described by the inhomogeneous Balescu–Lenard (BL) equation [7, 8]. When collective effects are neglected, i.e. when one neglects the system’s ability to amplify its own self-generated fluctuations [9], this intricate kinetic equation becomes the inhomogeneous Landau equation [10]. Since both equations are valid at order $1/N$, they describe evolutions on timescales of order $N T_d$, with $T_d$ the dynamical time.

For $1D$ inhomogeneous systems with a monotonic frequency profile and sustaining only 1:1 resonances, the BL collision term vanishes exactly [see, e.g., 11–17]. This is a kinetic blocking and we refer to [18] for a detailed review of the literature on that regard. Kinetically blocked systems can only evolve under the effect of three-body correlations. Their relaxation occurs therefore on a timescale of order $N^2 T_d$, or even larger. In this paper, we derive an appropriate kinetic equation for this (very) slow process.

Steps in that direction were successively performed by: (i) [19] which made a first attempt at deriving a $1/N^2$ equation for the one-dimensional (homogeneous) Hamiltonian Mean Field (HMF) model [20] starting from the BBGKY hierarchy and neglecting collective effects; (ii) this was further clarified by [21] which emphasised the main properties of this collision operator and compared it with numerical simulations; (iii) finally, these results were generalised in [18] to homogeneous systems with an arbitrary interaction potential.

The kinetic equation derived in [18] was restricted to homogeneous $1D$ systems. Here, we go beyond this limitation and focus on inhomogeneous systems, while still neglecting collective effects. In the limit where collective amplification can be neglected, we present a closed, explicit and well-posed kinetic equation that describes these systems’ relaxation on $N^2 T_d$ timescales. In addition to reviewing the key properties of this collision operator, we also quantitatively compare its predictions with direct $N$-body simulations. Remarkably, we present a class of interaction potentials for which this $1/N^2$ collision term exactly vanishes whatever the system’s distribution function: we call this a second-order kinetic blocking.

The paper is organised as follows. In §II, we spell out the inhomogeneous $1/N^2$ kinetic equation, as given by Eq. (4). In §III, we present the main properties of this equation, while in §IV, we investigate its steady states. In §V, we quantitatively compare the prediction of this theory with direct numerical simulations of particles interacting on the unit sphere. Finally, we conclude in §VI.

In all these sections, technical details are either deferred to Appendices or to appropriate references.

II. KINETIC EQUATION

We consider a population of $N$ particles of individual mass $\mu = M_{\text{tot}}/N$ with $M_{\text{tot}}$ the system’s total mass. The $1D$ canonical (specific) phase coordinates are denoted by $\mathbf{w} = (\theta, J)$ with $\theta$ the $2\pi$-periodic angle and $J$ the action [2]. The system’s total specific Hamiltonian reads

$$H = \sum_{i=1}^{N} U_{\text{ext}}(w_i) + \sum_{i<j}^{N} \mu U(w_i, w_j),$$

with $U_{\text{ext}}(w)$ a given external potential and $U(w, w')$ the pairwise interaction potential between the particles. We assume that the pairwise interaction satisfies the symmetries $U(w, w') = U(|\theta - \theta'|, \{J, J'\})$. When Fourier expanded w.r.t. the angles, it becomes

$$U(w, w') = \sum_{k=-\infty}^{\infty} U_k(J, J') e^{ik(\theta - \theta')},$$

where the coefficients, $U_k(J, J') \in \mathbb{R}$, satisfy the symmetries (i) $U_{-k}(J, J') = U_k(J, J')$; (ii) $U_k(J', J) = U_k(J, J')$. 


We describe the system with its distribution function (DF), \( F = F(w) \), normalised to \( \int dw F = N \). The system is in a quasi-stationary equilibrium, so that both the mean DF, \( \bar{F} = F(J) \), and the mean Hamiltonian, \( H_0(w) = U_{\text{ext}}(w) + \int dw' U(w, w') F(w') = H_0(J) \), depend only on the action. The present system is said to be inhomogeneous because (i) the coupling coefficients, \( U_k(J, J') \), depend on the particles’ actions; (ii) to every action is associated an orbital frequency, \( \Omega(J) = dH_0/J \). Characterising relaxation amounts to characterising \( \partial F/J_t/\partial t \).

In the limit where only 1\( \!/N \) effects are accounted for, the dynamics of \( F(J, t) \) is described by the inhomogeneous BL equation [7, 8]. It reads

\[
\frac{\partial F(J)}{\partial t} = 2\pi^2 \mu \frac{\partial}{\partial J} \left[ \int dJ_1 |\psi_{\text{tot}}^d(J, J_1)|^2 \delta_D(\Omega(J) - \Omega(J_1)) \right] \times \left( \frac{\partial}{\partial J} - \frac{\partial}{\partial J_1} \right) F(J) F(J_1),
\]

where we dropped the time dependence of the DFs for clarity. Here, the total coupling coefficients follow from \( |\psi_{\text{tot}}^d(J, J_1)|^2 = \sum_k |k||\psi_{kk}^d(J, J_1, \Omega(J))|^2 \), where the detailed expression of the dressed coupling coefficients, \( \psi_{kk}^d(J, J_1, \Omega) \), can be found in §G of [22]. Importantly, we emphasise that the symmetry of Eq. (2) imposes \( \psi_{kk}^d \propto \delta_{kk} \), i.e. only 1:1 resonances are permitted.

For a system with a monotonic frequency profile, \( J \rightarrow \Omega(J) \), the diffusion flux from Eq. (3) exactly vanishes. Indeed, the resonance condition, \( \delta_D(\Omega(J) - \Omega(J_1)) \), imposes that only local two-body resonances of the form \( J_1 = J \) are permitted. This leads to \( \partial F/\partial t = 0 \) in Eq. (3). As a consequence, one-dimensional inhomogeneous systems with a monotonic frequency profile and sustaining only 1:1 resonances generically have a vanishing BL flux. This is a kinetic blocking [see, e.g., 16, 17], i.e. these systems cannot relax via two-body correlations of order 1\( \!/N \). Relaxation is then significantly delayed as it can only occur through the weaker 1\( \!/N^2 \) three-body correlations. This is our focus here.

In order to derive an appropriate 1\( \!/N^2 \) kinetic equation for the present system, we generalise the result from [18], which focused on homogeneous systems with an arbitrary interaction potential. Yet, accounting for inhomogeneity makes it so that calculations become rapidly difficult to handle given the large number of terms appearing. All these aspects are dealt with within a Mathematica code that is distributed in the Supplemental Material [23].

Building upon [18], the key steps of the derivation are highlighted in §A. In short, we proceed by: (i) writing the usual BBGKY coupled evolution equations for the one-, two- and three-body DFs [24]; (ii) rewriting these equations as evolution equations for the one-body DF, \( F(J, t) \), and the two- and three-body correlation functions using the cluster expansion [25]; (iii) truncating these equations at order 1\( \!/N^2 \) and splitting the two-body correlation function in its 1\( \!/N \) and 1\( \!/N^2 \) components [19]; (iv) neglecting collective effects by assuming that the system is dynamically hot enough so as to not strongly amplify its own self-generated perturbations; (v) solving explicitly a sequence of four differential equations. Once these steps implemented, it remains to perform a large number of integration by parts, symmetrisations, and relabellings to reach a simple expression for the final 1\( \!/N^2 \) collision operator. This is, by far, the most challenging part of the calculation where the use of a computer algebra system appears mandatory. All the details are given in [23].

Ultimately, the kinetic equation reads

\[
\frac{\partial F(J)}{\partial t} = 2\pi^3 \mu^2 \frac{\partial}{\partial J} \sum_{k_1, k_2} \frac{1}{p} \int \frac{dJ_1}{(\Omega(J) - \Omega(J_1))^4} \int dJ_2 U_{k_1, k_2}(J, k) \delta_D(k \cdot \Omega) \left( k \cdot \frac{\partial}{\partial J} \right) F_3(J),
\]

where the sum over \( k_1, k_2 \) is restricted to the integers such that \( k_1 \), \( k_2 \) and \((k_1+k_2)\) are all nonzero. In that expression, we also shortened the notations by introducing the vectors \( J = (J_1, J_2) \), \( \Omega = (\Omega(J_1), \Omega(J_2)) \), \( \Omega(J) = (\Omega(J_1), \Omega(J_2)) \), \( \delta_D(k \cdot \Omega) \) is satisfied. Finally, Eq. (4) also differs from Eq. (3) because it does not involve collective effects.

In the limit of an homogeneous system, one may replace (i) the action \( J \) by the velocity \( v \); (ii) the orbital frequency, \( \Omega(J) \), by \( v \); (iii) the action-dependent coefficients, \( U_k(J, J') \), by the velocity-independent ones, \( U_k \). Following such replacements, one exactly recovers the homogeneous 1\( \!/N^2 \) kinetic equation already derived in [18].

Equation (4) is the main result of this work. It describes the (very) long-term relaxation of dynamically hot one-dimensional inhomogeneous systems, as driven by 1\( \!/N^2 \) effects. Equation (4) is rather general since it applies to any arbitrary long-range interaction potential that follows Eq. (2). In the coming section, we explore some of the main properties of this kinetic equation.

1 It is the same assumption that allows one to derive the (simpler) Landau equation from the BL one, at order 1\( \!/N \) [10].
III. PROPERTIES

A. Conservation laws

Equation (4) satisfies a couple of important conservation laws. Up to prefactors, those are

\[ M(t) = \int \frac{dJ}{\partial J} F(J, t) \quad \text{(total mass);} \]

\[ P(t) = \int \frac{dJ}{\partial J} F(J, t) \quad \text{(total momentum);} \]

\[ E(t) = \int \frac{dJ}{\partial J} H_0(J) F(J, t) \quad \text{(total energy).} \]

We refer to §C1 for the proof of these conservations.

B. Dimensionless rescaling

It is enlightening to estimate the typical relaxation time predicted by Eq. (4) using a dimensionless rewriting. We denote the system’s typical frequency with \( \Omega_0 \) and set the dynamical time to \( T_d = 1/\Omega_0 \). We define the typical action \( J_0 \) via the DF’s action dispersion, and introduce the dimensionless DF \( \mathcal{T} = J_0 F/M_{\text{tot}} \). Finally, we assume that the interaction potential satisfies \( U \propto G \). Injecting these various elements in Eq. (4), we find \( \partial \mathcal{T}/\partial t = 1/(N^2 Q^4) \) with \( \mathcal{T} = t/T_d \) and all the remaining terms dimensionless. Here, we introduced the system’s dimensionless stability parameter [see, e.g., 26]

\[ Q = \frac{J_0 \Omega_0}{G M_{\text{tot}}}. \]

The larger \( Q \), the hotter the system, i.e. the weaker the collective effects\(^2\). The system’s relaxation time, \( T_r \), when driven by Eq. (4), therefore scales like

\[ T_r \simeq Q^4 N^2 T_d. \]

As expected, the hotter the system, the slower the long-term relaxation. Given that Eq. (4) has been derived by neglecting collective effects, it can only be applied to systems with \( Q \gg 1 \).

C. Well-posedness

Because it involves a high-order resonance denominator, it is not obvious that Eq. (4) is well-posed, i.e. that there are no divergences when \( J_1 \rightarrow J \). Following the same approach as [18], we show in §C2 that one can rewrite Eq. (4) under an alternative form for which the principal value can be computed.

IV. STEADY STATES

A. H-theorem

Up to prefactors, the system’s entropy is defined as

\[ S(t) = -\int dJ s[F(J, t)], \]

with \( s[F] = F \ln(F) \) Boltzmann’s entropy. As detailed in §D1, one can show that Eq. (4) drives an evolution of the entropy according to

\[ \frac{dS}{dt} = \frac{2\pi^3 \mu^2}{3} \sum_{k_1, k_2} \frac{dJ}{dJ} U_{k_1, k_2}(J) \frac{P((\Omega[J] - \Omega[J_1])^4)}{(k_1 + k_2)^2} \]

\[ \times \frac{\delta_D[\mathbf{k} \cdot \Omega]}{F_3(J)} \left( k_1 + k_2 \right)^2 \left( \frac{F(J)}{F_3(J)} - k_1 F(J_1) - k_2 F(J_2) \right)^2. \]

All the terms in this integral are positive. Hence, Eq. (4) satisfies an H-theorem, i.e.

\[ \frac{dS}{dt} \geq 0. \]

Equation (4) therefore drives an irreversible relaxation.

B. Boltzmann distribution

For the present case, the thermodynamical equilibria, i.e. the Boltzmann DFs, are generically of the form

\[ F_B(J) \propto e^{-\beta H_0(J) + \gamma J}, \]

with \( \beta, \gamma \) two Lagrange multipliers associated with the conservation of \( E(t) \) and \( P(t) \) in Eq. (5). When injected in Eq. (4), the DF from Eq. (11) gives

\[ \frac{\partial F_B(J)}{\partial t} \propto \delta_D[\mathbf{k} \cdot \Omega] \left\{ -\beta \left[ \mathbf{k} \cdot \Omega \right] + \gamma \left[ (k_1 + k_2) - k_1 - k_2 \right] \right\} = 0. \]

As expected, Boltzmann DFs are equilibria of Eq. (4).

C. Constraint from the H-theorem

Following the calculation of \( dS/dt \) in Eq. (9), let us now investigate what are the most generic steady states of Eq. (4). For simplicity, we assume that there exists \( (k_1, k_2) \) such that \( U_{k_1, k_2}(J) \neq 0 \) when \( \mathbf{k} \cdot \Omega = 0 \), i.e. at resonance. An obvious way of ensuring \( dS/dt = 0 \) is for the last term in Eq. (9) to systematically vanish. Since \( J \mapsto \Omega(J) \) is monotonic, we can define the function \( G(\Omega) = F'(J[\Omega])/F(J[\Omega]) \) and we find the constraint

\[ \forall \Omega_1, \Omega_2 : G \left( \frac{k_1 \Omega_1 + k_2 \Omega_2}{k_1 + k_2} \right) = \frac{k_1 G(\Omega_1) + k_2 G(\Omega_2)}{k_1 + k_2}, \]

\(^2\) The typical scaling of \( Q \) can readily be found from the system’s inhomogeneous response matrix [see §5.3 in 2].
namely a weighted average. For this constraint to be satisfied for all \( \Omega_1, \Omega_2 \), the function \( \Omega \rightarrow G(\Omega) \) must necessarily be affine, i.e. one has
\[
G(\Omega) = -\beta \Omega + \gamma. \tag{14}
\]
Integrating once w.r.t. \( J \), Eq. (14) recovers the Boltzmann DF from Eq. (11). Provided that there exists one \((k_1, k_2)\) for which \( U_{k_1 k_2}(J) \neq 0 \) at resonance, the only equilibrium DFs of Eq. (4) are the Boltzmann DFs.

D. Second-order kinetic blocking

It is interesting to determine whether or not one can design an interaction potential and a frequency profile so that, for all \((k_1, k_2)\) one has \( U_{k_1 k_2}(J) = 0 \) at resonance. Following Eq. (9), this would then impose \( \text{d}S/\text{d}t = 0 \) whatever the considered DF. For simplicity, in this section, we assume the simple frequency profile \( \Omega(J) \propto J \).

In §D2, we show that potentials of the form
\[
U(w, w') \propto |J - J'|^\alpha \sum_{k=1}^{\infty} \frac{1}{|k|^\alpha} \cos(k(\theta - \theta')) \tag{15}
\]
with \( \alpha \) an arbitrary power law index and \( d \geq 1 \) an arbitrary integer, are generically blocked for the \( 1/N^2 \) dynamics driven by Eq. (4). More precisely, such potentials ensure that the coupling function \( U_{k_1 k_2}(J) \) (§B) satisfies
\[
\forall k_1, k_2, J, J_1: U_{k_1 k_2}(J, J_1, J_2^{\text{res}}) = 0, \tag{16}
\]
with \( J_2^{\text{res}} = J_2^{\text{res}}[k, J, J_1] \), the resonant action complying with the constraint \( k: \Omega = 0 \). For the frequency profile \( \Omega(J) \propto J \), it simply reads \( J_2^{\text{res}} = [(k_1 + k_2)J - k_1 J_1]/k_2 \). In §D2, we show that (i) Eq. (15) allows one to recover the local interaction potentials already unveiled in [18]; (ii) for particular values of \( \alpha \), the harmonic sum from Eq. (15) can be explicitly performed.

Following Eq. (15), any (stable) DF, \( F = F(J) \), when embedded within the mean potential \( H_0 = 1/2J^2 \) and pairwise interaction from Eq. (15), satisfies \( \partial F / \partial t = 0 \) when plugged into Eq. (4). We call such a situation a second-order kinetic blocking, i.e. both the \( 1/N \) BL flux and the \( 1/N^2 \) flux from Eq. (4) exactly vanish. In the limit where collective effects can effectively be neglected, we expect that such systems will relax on the much longer timescales \( O(N^3 T_d) \). Given the difficulty of (i) deriving a kinetic equation at order \( 1/N^3 \); and (ii) numerically integrating such dynamics on extremely long timescales, we postpone their investigation to future works.

V. APPLICATION

In order to test Eq. (4), we perform numerical simulations of classical Heisenberg spins embedded within an external potential [see, e.g., 15, 16]. We refer to §E for a detailed presentation of the setup.

In Fig. 1, we illustrate the initial diffusion flux, \( \partial F / \partial t = \partial F / \partial J \). This figure shows a good quantitative agreement between the predicted and measured fluxes. The remaining slight mismatch likely stems from (i) non-vanishing contributions from the source term in \( G_2^{(1)} \times G_2^{(1)} \); (ii) remaining contributions associated with collective effects.

VI. CONCLUSIONS

We presented a \( 1/N^2 \) closed and explicit kinetic equation for long-range interacting one-dimensional inhomogeneous systems. The collision operator from Eq. (4) generalises the classical inhomogeneous Landau kinetic equation to regimes where the \( 1/N \) relaxation exactly vanishes by symmetry. Equation (4) conserves the total mass, momentum and energy, and satisfies an H-theorem. We exhibited a class of long-range interaction potentials for which this \( 1/N^2 \) collision term exactly vanishes. Finally, we showed how Eq. (4) quantitatively matches with measurements from direct \( N \)-body simulations.

Naturally, the present work is only one step toward ever more detailed characterisation of the long-term relaxation of finite-\( N \) systems. As such it would be worthwhile to (i) generalise Eq. (4) to also account for collective effects – a significantly challenging endeavour; (ii) account for the source term in \( G_2^{(1)} \times G_2^{(1)} \) (§A3); (iii) investigate, both analytically and numerically, systems driven by the interaction potential from Eq. (15), for which we expect a second-order kinetic blocking, i.e. a relaxation on \( N^3 T_d \) timescales.

Acknowledgments

This work is partially supported by the grant Segal ANR-19-CE31-0017 of the French Agence Nationale de
la Recherche and by the Idex Sorbonne Université. We thank S. Rouberol for the smooth running of the Infinity cluster, where the simulations were performed. We thank M. Roule, M. Petersen and C. Pichon for many stimulating discussions.

Appendix A: Derivation

Most of the derivation of Eq. (4) follows the same lines as in [18] (see §A therein). Here, we only highlight the key changes stemming from inhomogeneity.

1. BBGKY hierarchy

The dynamics of the system is exactly described by the BBGKY equations for the \( n \)-body DFs, \( F_n(w_1, ..., w_n, t) \) [see §A1 in 18]. It reads

\[
\frac{\partial F_n}{\partial t} + [F_n, H_n]_n + \int dw_{n+1} [F_{n+1}, \delta H_{n+1}]_n = 0, \tag{A1}
\]

with the Poisson bracket

\[
[f, h]_n = \sum_{i=1}^{n} \left( \frac{\partial f}{\partial \theta_i} \frac{\partial h}{\partial J_i} - \frac{\partial f}{\partial J_i} \frac{\partial h}{\partial \theta_i} \right). \tag{A2}
\]

In Eq. (A1), the \( n \)-body Hamiltonian, \( H_n \), follows from Eq. (1) with the replacement \( N \rightarrow n \). In Eq. (A1), the specific interaction energy, \( \delta H_{n+1} \), simply reads

\[
\delta H_{n+1}(w_1, ..., w_{n+1}) = \sum_{i=1}^{N} U(w_i, w_{n+1}). \tag{A3}
\]

Importantly, Eq. (A1) provides us with the evolution equations for the one-, two-, and three-body DFs \( F_1, F_2 \) and \( F_3 \). This is the starting point of the derivation.

2. Cluster expansion

To perform perturbative expansions w.r.t. the small parameter \( 1/N \), the \( n \)-body DFs are developed using the cluster expansion [25]. Doing so, one introduces the two- and three-body correlation functions, \( G_2(w_1, w_2) \) and \( G_3(w_1, w_2, w_3) \). As an example, \( G_2 \) is defined via

\[
F_2(w_1, w_2) = F(w_1) F(w_2) + G_2(w_1, w_2), \tag{A4}
\]

where, from now on, we write the one-body DF as \( F = F_1 \). Once this expansion performed, the quantities at our disposal w.r.t. \( N \) like \( F \sim 1 \), \( G_2 \sim 1/N \), \( G_3 \sim 1/N^2 \).

These expansions are then injected into Eq. (A1) to obtain evolution equations for \( \partial F/\partial t \), \( \partial G_2/\partial t \) and \( \partial G_3/\partial t \). All these calculations are explicitly performed in [23].

3. Truncation at order \( 1/N^2 \)

The next step is to truncate these three evolution equations at order \( 1/N^2 \). We perform the operations:

(I) We introduce the small parameter \( \epsilon = 1/N \) and decompose the two-body correlation function as

\[
G_2 = \epsilon G_2^{(1)} + \epsilon^2 G_2^{(2)}. \tag{A5}
\]

Similarly, the other parameters at our disposal are rescaled as \( \mu \rightarrow \epsilon \mu \), \( G_3 \rightarrow \epsilon^2 G_3 \).

(II) We truncate the BBGKY evolution equations up to order \( \epsilon^2 \). We split the evolution equation for \( \partial G_2/\partial t \) into two, respectively for \( \partial G_2^{(1)}/\partial t \) and \( \partial G_2^{(2)}/\partial t \).

(III) We leverage our assumption of a quasi-stationarity, i.e. \( F = F(J) \) and \( H_0 = H_0(J) \), hence introducing the orbital frequencies, \( \Omega(J) \).

(IV) We neglect the contributions from collective effects, i.e. we perform replacements of the form

\[
\int dw_3 G_2^{(1)}(w_2, w_3) \partial_\theta U(w_1, w_3) \rightarrow 0, \tag{A6}
\]

and similarly for \( G_2^{(2)} \) and \( G_3 \).

(V) In the hot limit, we neglect the source term in \( G_2^{(1)} \times G_2^{(1)} \) in \( \partial G_3/\partial t \), as its contribution is a factor \( 1/Q \) smaller than the source term in \( G_2^{(2)} \).

4. Solving the equations

Following all these steps, we have at our disposal a set of four coupled partial differential equations. These equations can be solved sequentially, starting with \( G_2^{(1)} \), then \( G_3 \), \( G_2^{(2)} \) and finally \( F \). This is done as follows:

(I) We perform Fourier expansions w.r.t. the angles \( \theta_i \).

(II) We rely on the assumption of timescale separation, hence fixing \( F(J, t) = \text{cst.} \) when solving for fluctuations.

(III) We impose the initial conditions \( G_2^{(1)}(t=0) = 0 \) and similarly for \( G_2^{(2)} \) and \( G_3 \).

Ultimately, we obtain an explicit time-dependent expression for \( G_2^{(2)}(t) \). Leveraging once again timescale separation, we consider the limit \( t \rightarrow +\infty \) of \( G_2^{(2)}(t) \) in the evolution equation for \( \partial F/\partial t \). A typical time integral is then replaced asymptotically by

\[
\lim_{t \rightarrow +\infty} \int_0^t dt_1 e^{i(t-t_1)\omega_R} = \pi \delta_D(\omega_R) + i P \left( \frac{1}{\omega_R} \right), \tag{A7}
\]

where \( \omega_R \in \mathbb{R} \) is a linear combination of \( \Omega \).

5. Simplifying the expressions

At this stage, we are left with a kinetic equation involving thousands of terms. The computer algebra system allows for efficient manipulations of these expressions. The key steps are as follows:
(I) We implement systematic relabellings of the actions $(J_1, J_2)$ and resonance numbers $(k_1, k_2)$, so that the resonant frequencies are all of the form $\omega_R = k \cdot \Omega$.

(II) We integrate by parts w.r.t. the actions so that no derivatives act on $\delta_D$, and only first-order derivatives of the DF and coupling coefficients are present.

(III) We use the scaling relations of $\delta_D$ and $P$, e.g., $\delta_D(\alpha x) = \delta_D(x)/|\alpha|$, to take out resonance numbers.

(IV) The frequency profile being monotonic, we use

$$\int dJ_2 f(J_1, J_2) \delta_D(\Omega[J_1] - \Omega[J_2]) = \frac{f(J_1, J_1)}{|F(J_1)|}.$$  \hspace{1cm} (A8)

(V) We use the resonance condition $\delta_D[k \cdot \Omega]$ to make the replacements $(\Omega - \Omega_2) \rightarrow -(k_1/k_2)(\Omega - \Omega_1)$ and $(\Omega_1 - \Omega_2) \rightarrow -(k_1 + k_2)/k_2)(\Omega - \Omega_1)$, with the shortened notation $(\Omega, \Omega_1, \Omega_2) = \Omega$. Principal values are therefore only expressed as functions of $(\Omega - \Omega_1)$.

All these manipulations are automated using tailored rules in Mathematica, that can all be found in [23]. Ultimately, one obtains the closed result from Eq. (4).

**Appendix B: Coupling coefficients**

The coupling coefficients, $U_{k_1, k_2}(J)$, appearing in Eq. (4) are generically given by

$$U_{k_1, k_2}(J) = [(\Omega[J] - \Omega[J_1])]U_{k_1, k_2}^{(1)}(J) + k_2 U_{k_1, k_2}^{(2)}(J).$$  \hspace{1cm} (B1)

The coupling functions appearing in this expression read

$$U_{k_1, k_2}^{(1)}(J) = k_2(k_1 + k_2)\left\{U_{k_1+k_2}(J, J_2) \partial J_1 U_{k_1}(J_1, J_2) - U_{k_2}(J, J_2) \partial J_1 U_{k_1}(J_1, J)\right\}$$

$$+ k_1(k_1 + k_2)\left\{U_{k_1}(J, J_1) \partial J_2 U_{k_2}(J, J_2) - U_{k_1+k_2}(J, J_2) \partial J_2 U_{k_2}(J_2, J_1)\right\}$$

$$- k_1k_2\left\{U_{k_1}(J_1, J_2) \partial J_1 U_{k_1+k_2}(J, J_2) - U_{k_1}(J_1, J_2) \partial J_2 U_{k_1+k_2}(J, J_2)\right\},$$  \hspace{1cm} (B2)

and

$$U_{k_1, k_2}^{(2)}(J) = (k_1 + k_2)\frac{d \Omega}{d J_1} U_{k_1}(J, J_1) U_{k_2}(J, J_2)$$

$$- k_1 \frac{d \Omega}{d J_1} U_{k_1+k_2}(J, J_1) U_{k_2}(J_1, J_2)$$

$$- k_2 \frac{d \Omega}{d J_2} U_{k_1}(J_1, J_2) U_{k_1+k_2}(J, J_2).$$  \hspace{1cm} (B3)

We refer to [23] for the associated derivation.

**Appendix C: Properties**

1. **Conservation laws**

We can generically rewrite Eq. (4) as

$$\frac{\partial F(J)}{\partial t} = \frac{\partial F(J)}{\partial J},$$  \hspace{1cm} (C1)

with $F(J)$ the diffusion flux. The time derivatives of Eq. (5) then read

$$\frac{dM}{dt} = \int dJ \frac{\partial F}{\partial J},$$

$$\frac{dP}{dt} = -\int dJ F(J),$$

$$\frac{dE}{dt} = -\int dJ \Omega(J) F(J).$$  \hspace{1cm} (C2)

The conservation of the total mass, $M(t)$, follows from the absence of any boundary contributions.

For the conservation of $P(t)$ and $E(t)$, we investigate

$$\int dJ F(J) = \sum_{k_1, k_2} (k_1 + k_2) \int dJ A_{k_1, k_2}(J),$$  \hspace{1cm} (C3)

with $A_{k_1, k_2}(J)$ given by Eq. (4). With the relabellings $J \leftrightarrow J_2$ and $(k_1, k_2) \rightarrow (-k_1 - k_2, k_2)$, Eq. (C3) becomes [23]

$$\int dJ F(J) = - \sum_{k_1, k_2} k_1 \int dJ A_{k_1, k_2}(J).$$  \hspace{1cm} (C4)

Similarly, with the relabellings $J \leftrightarrow J_2$ and $(k_1, k_2) \rightarrow (-k_1, k_1 + k_2)$, Eq. (C3) becomes [23]

$$\int dJ F(J) = - \sum_{k_1, k_2} k_2 \int dJ A_{k_1, k_2}(J).$$  \hspace{1cm} (C5)

We can now go back to the computation of the conserved quantities in Eq. (C2). By adding $\frac{1}{3}$ of Eqs. (C3)–(C5), we finally obtain

$$\frac{dP}{dt} = - \frac{1}{3} \sum_{k_1, k_2} \int dJ A_{k_1, k_2}(J) \left\{(k_1 + k_2) - k_1 - k_2\right\} = 0,$$

$$\frac{dE}{dt} = - \frac{1}{3} \sum_{k_1, k_2} \int dJ A_{k_1, k_2}(J) \left\{k \cdot \Omega\right\} = 0,$$  \hspace{1cm} (C6)

where the final equality stems from the presence of the resonance condition, $\delta_D[k \cdot \Omega]$, in Eq. (4).

2. **Well-posedness**

Following the same approach as in [18], we define the set of fundamental resonances as

$$\{k, k' \mid 0 < k, k'\}.$$  \hspace{1cm} (C7)

For a given fundamental resonance, $(k, k')$, we define the associated set of resonance pairs, $(k_1, k_2)$, with

$$\mathcal{R}(k, k') = \{(k, k'), (k + k', -k), (k, k' - k'), (k', k), (k + k', -k'), (k', k - k')\}.$$  \hspace{1cm} (C8)

We note that (i) all $(k_1, k_2)$ in $\mathcal{R}(k, k')$ satisfy $k_1 \geq 0$; (ii) even for $k = k'$, this set still contains six elements.
Following these definitions, we rewrite Eq. (4) as [23]
\[
\frac{\partial F(J)}{\partial t} = 2\pi^3 \mu^2 \frac{\partial}{\partial J} \left[ \sum_{k,k'>0} \mathcal{F}_{kk'}(J) \right],
\]
(C9)
where \( \mathcal{F}_{kk'}(J) \) stands for the flux generated by the fundamental resonance \((k,k')\). It reads
\[
\mathcal{F}_{kk'}(J) = \mathcal{P} \int \frac{dJ_1}{(\Omega|J|-\Omega|J_1|)^4} \sum_{(k_1,k_2) \in \mathcal{R}(k,k')} C_{k_1,k_2}(J,J_1).
\]
(C10)
Here, \( C_{k_1,k_2}(J,J_1) \) follows from Eq. (4) and reads
\[
C_{k_1,k_2}(J,J_1) = \int dJ_2 \frac{U_{k_1,k_2}(J)}{k_1^2(k_1+k_2)} \delta_\Omega |\mathbf{k}| \Omega \left( \mathbf{k} \cdot \frac{\partial}{\partial J} \right) F_3(J).
\]
(C11)
Combining Eqs. (C10) and (C11), we must ultimately perform an integration w.r.t. \( dJ_1,dJ_2 \). As in [18], we can propose an alternative writing for \( \mathcal{C}_{k_1,k_2}(J,J_1) \) by performing the relabelling \( J_1 \leftrightarrow J_2 \). We obtain [23]
\[
\mathcal{C}_{k_1,k_2}(J,J_1) = \int dJ_2 \frac{U_{k_1,k_2}(J_2)}{k_2^2(k_1+k_2)} \delta_\Omega |\mathbf{k}| \Omega \left( \mathbf{k} \cdot \frac{\partial}{\partial J} \right) F_3(J).
\]
(C12)
As in Eq. (4), we introduced \( \mathbf{k} = (k_1+k_2,-k_2,-k_1) \) along with the coupling function
\[
\Omega_{k_1,k_2}(J) = \left| \Omega(J) - \Omega(J_1) \right| \Omega_{k_1,k_2}(J) - k_1 \Omega_{k_1,k_2}(J_2)^2 \right|^4.
\]
(C13)
In that expression, \( \Omega_{k_1,k_2}(J) \) (resp. \( \Omega_{k_1,k_2}(J) \)) are directly obtained from Eq. (B2) (resp. Eq. B3) by performing the relabelling \( J_1 \leftrightarrow J_2 \).
To obtain a well-posed expression for the flux, \( \mathcal{F}_{kk'}(J) \), it remains to appropriately combine the two possible writings, \( \mathcal{C}_{k_1,k_2}(J,J_1) \) and \( \mathcal{C}_{k_1,k_2}(J,J_2) \). Once again, the computer algebra system proves undoubtly useful [23].
Starting from Eqs. (C10), we can rewrite the writing
\[
\mathcal{F}_{kk'}(J) = \mathcal{P} \int dJ_1 \frac{U_{k_1,k_2}(J)}{(\Omega|J|-\Omega|J_1|)^4} \sum_{(k_1,k_2) \in \mathcal{R}(k,k')} C_{k_1,k_2}(J,J_1).
\]
(C14)
\[
\times \left\{ \left( C_{kk'} + C_{k'k} \right) + C_{k+k',-k} + C_{k+k',-k} + \left( \gamma_{kk'} C_{k,k'-k} + (1-\gamma_{kk'}) C_{k,k'-k} \right) + \left( \gamma_{kk'} C_{k,k'-k} + (1-\gamma_{kk'}) C_{k,k'-k} \right) \right\},
\]
with weight
\[
\gamma_{kk'} = \frac{(k-k')^2}{k^2+k'^2}.
\]
(C15)
Such a choice is always well-defined since \( k,k'>0 \) (see Eq. C7). This symmetrisation is the same as in [18].
We can then rewrite Eq. (C14) as
\[
\mathcal{F}_{kk'}(\Omega) = \mathcal{P} \int d\Omega_1 \frac{K_{kk'}(\Omega,\Omega_1)}{(\Omega-\Omega_1)^4},
\]
(C16)
where we used the fact that \( J \rightarrow \Omega(J) \) is monotonic, introducing therefore \( \Omega=\Omega(J) \) and \( \Omega_1=\Omega[J_1] \). After calculation [23], for \( \delta \Omega \rightarrow 0 \), one finds
\[
K(\Omega,\Omega+\delta\Omega) \simeq O\left((\delta\Omega)^3\right),
\]
(C17)
making the principal value in Eq. (C16) well-posed.

**Appendix D: Steady states**

1. **H-theorem**

Starting from Eq. (8), the time-derivative of Boltzmann’s entropy is given by
\[
\frac{dS}{dt} = \int dJ \frac{F'(J)}{F(J)} F(J).
\]
(D1)
Using the same symmetrisations as in Eq. (C6), we find
\[
\frac{dS}{dt} = \frac{3}{4} \sum \int dJ A_{k_1,k_2}(J)
\]
(D2)
\[
\times \left\{ (k_1+k_2) F'(J_1) - k_1 F'(J_1) - k_2 F'(J_2) \right\}.
\]
It then only remains to inject the expression of \( A_{k_1,k_2}(J) \) (Eq. C3) to obtain \( dS/dt \) as given in Eq. (9).

2. **Second-order kinetic blocking**

Let us consider interaction potentials of the form
\[
U_{k}(J,J') \propto \frac{\delta_{k^id} \delta_{|J-J'|^\alpha}}{|k|^d},
\]
(D3)
with \( \alpha \) an arbitrary power law index and \( d \geq 1 \) an arbitrary integer. We also introduced the function \( \delta_{k^id} = 1 \) if \( k \equiv 0 \mod d \) and 0 otherwise.
As detailed in [23], for such a potential, when embedded within the frequency profile \( \Omega(J) \propto J \), one finds
\[
U_{k_1,k_2}(J,J_1,J_2^{\text{ref}}) \propto \left\{ (k_1+k_2) \delta_{k_1} \delta_{k_2} \delta_{k_1} \delta_{k_2} \right\}
\]
(D4)
\[
-k_1 \delta_{k_1+k_2} \delta_{k_2} \delta_{k_1} \delta_{k_2} - k_2 \delta_{k_1+k_2} \delta_{k_1} \delta_{k_2}^2 = 0,
\]
where we used \( \delta_{k^id} \delta_{k^id} = \delta_{k^id} \delta_{k^id} \). We have therefore devised a generic second-order kinetic blocking.

For \( \alpha = 0 \), Eq. (D3) becomes the simple interaction potential \( U_k(J,J') \propto \delta_{|J-J'|} \). This makes the pairwise interaction independent of the particles’ actions. This class of
potential was already unveiled in [18], when investigating homogeneous 1D systems at order $1/N^2$. In particular, [18] showed that this interaction potential becomes

$$ U(w, w') \propto 1 - \frac{d-1}{d} \sum_{k=0}^{d-1} \delta_{p}[(\theta - \theta') - k\frac{\pi}{d}]. \quad (D5) $$

As discussed in [18], such potentials amount to exactly local interactions; they do not drive any relaxation.

For $d=1$ and $\alpha > 0$ an even integer, one can explicitly compute $U(w, w')$ from $U_k(J, J')$ as given by Eq. (D3). Up to prefactors, one obtains [see 9.622.3 in 27]

$$ U(w, w') \propto (J - J')^{2n} B_{2n}[\frac{1}{2\pi} w_{2\pi}(\theta - \theta')], \quad (D6) $$

with $n \geq 1$ an integer. In that expression, $B_{2n}(x)$ is a Bernoulli polynomial, e.g., $B_2(x) = x^2 - x + \frac{1}{6}$. We also introduced the “wrapping” function

$$ w_{2\pi}(\theta) \equiv \theta [\mod 2\pi]; \quad 0 \leq w_{2\pi}(\theta) < 2\pi. \quad (D7) $$

Investigating systems driven by interaction potentials as in Eq. (D6) will be the subject of future works.

Appendix E: Numerical simulations

We consider the same system as in [28], i.e. a set of $N$ particles evolving on the unit sphere. This setup mimics “vector resonant relaxation” in galactic nuclei [29]. Given an axis $\hat{z}$, the spherical coordinates are denoted with $(\theta, \phi)$, so that $w = [\phi = \theta, \cos(\theta) = J]$ are canonical coordinates with the associated unit vector, $\hat{L} = \hat{L}(w)$.

Following [16], we fix the external potential, $U_{\text{ext}}$, and the pairwise interaction potential, $U$, to be

$$ U_{\text{ext}}(\hat{L}) = D (\hat{L} \cdot \hat{z})^2; \quad U(\hat{L}, \hat{L}') = G \hat{L} \cdot \hat{L}'. \quad (E1) $$

Following [29], the individual equations of motion read $d\hat{L}_i/dt = \partial H/\partial \hat{L}_i \times \hat{L}_i$. For the particular choice from Eq. (E1), we readily find

$$ \frac{\partial H}{\partial \hat{L}_i} = 2D (\hat{L}_i \cdot \hat{z}) \hat{z} + G S, \quad (E2) $$

with $S = \sum_{i=1}^N \mu \hat{L}_i$ the system’s magnetisation. Since $S$ is “shared” by all the particles, the rates of changes, $\{d\hat{L}_i/dt\}_i$, can be computed in $O(N)$ operations.

The invariants of the present system are

$$ \forall i: \|\hat{L}_i\| = 1; \quad S_z = S \cdot \hat{z}; \quad E_{\text{tot}} = H. \quad (E3) $$

Because of the gauge invariance associated with the constraints, $\|\hat{L}_i\| = 1$, $d\hat{L}_i/dt$ is uniquely defined, while the gradient $\partial H/\partial \hat{L}_i$ is not. As such, we define, unambiguously, the precession vector, $\Omega_i = \hat{L}_i \times d\hat{L}_i/dt$, so that

$$ \frac{d\hat{L}_i}{dt} = \Omega_i \times \hat{L}_i; \quad \Omega_i \cdot \hat{L}_i = 0. \quad (E4) $$

In order to exactly conserve all the $\|\hat{L}_i\|$, the system’s dynamics is integrated using the structure-preserving classical fourth-order Munthe–Kaas scheme [30]. We refer to §5.1 in [31] for a presentation of this explicit scheme.

For the results presented in Fig. 1, we considered $N = 1024$, $G = -1$, $D = 15$, $M_{\text{tot}} = 1$. The integration was performed with a fixed timestep $h = 5 \times 10^{-4}$, with a dump every $\Delta t = 100$ and integrated up to $t = 2 \times 10^5$. Running one realisation requires about 32h of computation on a single core. At the final times, the relative errors in $S_z$ and $E_{\text{tot}}$ were typically $10^{-7}$.

For the initial conditions, as in [28], we consider

$$ F(w) = C e^{-(J/\sigma)^4}, \quad (E5) $$

with $C$ a normalisation constant. We fixed the DF’s initial dispersion to $\sigma = 0.45$. This ensures that the system is linearly stable and hot enough [see fig. 6 in 28].

To obtain Fig. 1, we performed a total of $1024$ independent realisations. The action, $-1 \leq J \leq 1$, is binned in 50 equal size bins. For each realisation, each action bin and each time dump, we determine the number of particles left to that action. This is then averaged over the available realisations. Finally, we fit the associated time series with an affine time dependence: its slope is proportional to the local diffusion flux, $F(J, t=0)$. To estimate the measurement errors, we follow the same bootstrap approach as in §F of [21]. In Fig. 1, we report the 16% and 84% level lines over 100 bootstrap resamplings.
[16] J. Barré and S. Gupta, JSMTE 2014, 02017 (2014).
[17] C. R. Lourenço and T. M. Rocha Filho, Phys. Rev. E 92, 012117 (2015).
[18] J.-B. Fouvry, P.-H. Chavanis, and C. Pichon, Phys. Rev. E 102, 052110 (2020).
[19] T. M. Rocha Filho, A. E. Santana, M. A. Amato, and A. Figueiredo, Phys. Rev. E 90, 032133 (2014).
[20] M. Antoni and S. Ruffo, Phys. Rev. E 52, 2361 (1995).
[21] J.-B. Fouvry, B. Bar-Or, and P.-H. Chavanis, Phys. Rev. E 100, 052142 (2019).
[22] J.-B. Fouvry and B. Bar-Or, MNRAS 481, 4566 (2018).
[23] J.-B. Fouvry, Mathematica notebook (2022), URL https://github.com/jbfouvry/No-Inh-Land.
[24] D. Swanson, Plasma Kinetic Theory (CRC Press, 2008).
[25] R. Balescu, Statistical Dynamics: Matter out of Equilibrium (Imperial Coll., London, 1997).
[26] A. Toomre, ApJ 139, 1217 (1964).
[27] I. S. Gradshteyn and I. M. Ryzhik, Table of integrals, series, and products (Elsevier, 2007).
[28] J.-B. Fouvry, B. Bar-Or, and P.-H. Chavanis, Phys. Rev. E 99, 032101 (2019).
[29] B. Kocsis and S. Tremaine, MNRAS 448, 3265 (2015).
[30] H. Munthe-Kaas, Appl. Numer. Math. 29, 115 (1999).
[31] J.-B. Fouvry, W. Dehnen, S. Tremaine, and B. Bar-Or, ApJ 931, 8 (2022).