A study of the consistency between noncommutative quantum mechanics and Galilean isotropy.

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Abstract

A demonstration is given that the simplest model of quantum mechanics formulated on a plane non-commutative geometry endowed with a Galilean symmetry group in which the position and linear momentum-variable commutators are first order in the dynamical variables (and thus constitute a true Lie algebra) is incompatible with the hypothesis of spacial isotropy.

“Civilization advances by extending the number of important operations which we can perform without thinking about them.”
Alfred North Whitehead.

1 Introduction

The proposal to generalise Heisenberg commutation relations in order to include mutually noncommuting position variables is not a new avenue at all. But in recent years, the argument has received a renewed momentum in connection with the search for a valid generalization of quantum field theory. It is well-known that this theory gives rise to infinite quantities for whose removal we need to put to practice renormalisation procedures. One of these, known as the Pauli-Villars dimensional regularisation, consists in
decreeing a cutoff in the range of values for linear momentum contributing to the cross section. Thus, all ultraviolet contributions are removed, the finite part is extracted, and only at the end one makes the mentioned cutoff in the momentum space go to zero. Such techniques are not totally free from ambiguity and quantum field theory suffers from some problems at least of a methodologic nature as is, e. g., the problem of the vacuum energy (related to the value of the cosmologic constant in Einstein’s equations) or the non-renormalizability of the quantum theory of gravitation.

Among the alternative ideas that have been coming through, (as is the one of assuming a string structure beyond Plank’s length scale) ranks the theory of non-commutative geometries, which occupies my attention in this work.

The fundamental idea of this theory, expressed in a way rather more prosaic than usual in the specialised literature, stems from the generalisation of Heisenberg commutation relations

\[
[X_i, P_j] = i\hbar \delta_{ij}, \quad [X_i, X_j] = [P_i, P_j] = 0
\]  

(1)

to a more general one of the form

\[
[X_i, X_j] \neq 0, \quad [P_i, P_j] \neq 0
\]  

(2)

The most famous consequence of (1) is the impossibility to prepare states such that, once the pertinent measurements have been performed, lead to dispersion relations for \(X_i\) and \(P_j\) simultaneously better than those given by

\[
\Delta_\psi X_i \Delta_\psi P_j \geq \frac{1}{2} \hbar \delta_{ij}
\]  

(3)

as results from the more general version for two noncommuting quantum operators \(A\) and \(B\):

\[
\Delta_\psi A \Delta_\psi B \geq \frac{1}{2} |\langle \psi | [A, B] | \psi \rangle|
\]  

(4)

This more general formulation, allows to generalise the uncertainty principle in such a way that it precludes the possibility to construct states with the position defined in a point-like manner. The fact that, following (1) for the instance \(A = X_i\) and \(B = X_j\) with \(i \neq j\), it is impossible to define
both coordinates with unlimited precision, leads to ruling out all states having a localisation probability density point-like arranged. Now, it has been pointed out in numerous occasions the possibility that the infinities that plague quantum theories come from an excessive idealisation consisting in the handling of such “point-like particle states”. Should the hope for such a spacially extended model be realised, the doors may be open to the existence of a theory finite to every order (or, at least, renormalisable,) of which the present point-particle quantum field theory, would be but a low-energy version or an effective field theory.

2 Noncommutativity and isotropy

In the present work, I will investigate the consistency of a certain model of non-commutative structure for the dynamical variables in a Galilean, first-quantised quantum mechanical scenario. More in particular, I will assume the commutation relations in such context are of the form

\[
[X, P_j] = i\delta_{ij} + iu_{ijk}X_k + iv_{ijk}P_k
\]

\[
[X, X_j] = i\alpha_{ij} + if_{ijk}X_k + ig_{ijk}P_k
\]

\[
[P_i, P_j] = i\beta_{ij} + il_{ijk}X_k + im_{ijk}P_k
\]

where the numbers \(\alpha_{ij}\) and \(\beta_{ij}\) constitute real skew-symmetric matrices and the structure constants \(f_{ijk}, g_{ijk}, l_{ijk}\) and \(m_{ijk}\) are also real and skew-symmetric in their first 2 indices. That the dynamical-variable commutators (operators) should lead just to first-order terms in these is not, to be sure, a logical necessity (they could be, e.g., polynomials in \(X\) and \(P\), or perhaps more general functions). However, it seems the most natural hypothesis to demand a closure condition or, in other words, for these dynamical variables to form a proper Lie algebra. In order for this to be complied with, it is necessary that such commutations produce linear combinations of the operators we started from. As a matter of fact, the most general theory of noncommutative geometries developed by Connes et al., either assumes a more general structure or does not make an issue of it at all. But notice that the possibility of the occurrence of, e.g., quadratic terms in \(X\) or \(P\) would be far more serious a
problem than it seems at first sight. Indeed, the fact that such operators
are not bounded in the test-function space, renders every successive multi-
PLICATION of any of them problematic, as it forces us to expect very drastic
changes in the domain. Thus, the sheer writing of (1) already meets problems
in ordinary quantum mechanics, for the respective domains of the identity
operator on the one hand and the mutually isomorphic ones $X$ and $P$ do not
coincide. That is why, the rigorous way to write (1) is actually

$$[X_i, P_j] \subseteq i\hbar\delta_{ij} \quad (8)$$

in the understanding that every assignation from domain to image in the l.h.s.
are included among the assignations given by the r.h.s., but the converse not
being necessarily true. The successive multiplications of position and linear
momentum variables would then gradually “erode” the domain in such a way
that the operators corresponding to the dynamical functions $f(X, P)$ would
not be well-defined anymore. Weyl’s version of (8) is another way of saying
the same:

$$\exp(i\mathbf{a} \cdot \mathbf{X}) \exp(i\mathbf{b} \cdot \mathbf{P}) = e^{-i\mathbf{a} \cdot \mathbf{b}} \exp(i\mathbf{b} \cdot \mathbf{P}) \exp(i\mathbf{a} \cdot \mathbf{X}) \quad (9)$$
as results from being expressed in terms of operators free from domain-related
problems.

The central idea of the present work consists in exploring the successive
commutations of these operators under restrictions (5)-(7) using the Leibniz
rule for the commutator:

$$[A, BC] = [A, B] C + B [A, C] \quad (10)$$

that is: “the commutator is a derivative”. We have then, on the one hand,
by directly using (5)-(7):

$$[X_i, [X_j, P_k]] = iu_{jkl} [X_i, X_l] + iv_{jkl} [X_i, P_l] =$$

$$-(u_{jkl}\alpha_{il} + v_{jki}) - (u_{jkl}f_{ilm} + v_{jkl}u_{ilm}) X_m - (u_{jkl}g_{ilm} + v_{jkl}v_{ilm}) P_m$$

and on the other hand, using (10), and also (5)-(7) again

$$[X_i, [X_j, P_k]] = [X_i, X_j P_k] - [X_i, P_k X_j] =$$
\[ [X_i, X_j] P_k + X_j [X_i, P_k] - [X_i, P_k] X_j - P_k [X_i, X_j] = \]

\[ \sum_{i,j} X_{ij} P_k + ig_{ij} [P_i, P_k] + iu_{ikl} [X_j, X_l] + iv_{ikl} [X_j, P_l] = \]

\[-f_{ijk} - v_{ijk} - g_{ijk} \delta_{lk} - u_{ikl} \alpha_{jl} - (f_{ijl} u_{lk} + g_{ijl} l + u_{ikl} f_{jlm} + v_{ikl} u_{jlm}) X_m \]

\[-(f_{ijl} v_{lk} + g_{ijl} m_{lk} + u_{ikl} g_{jlm} + v_{ikl} v_{jlm}) P_m \]

which, after identification of similar terms with respect to the former expression \((I, X_i \text{ and } P_i \text{ should constitute a basis of our algebra})\), allow us to obtain:

\[ f_{ijk} + v_{ijk} + g_{ijk} \delta_{lk} + u_{ikl} \alpha_{jl} = u_{jkl} \alpha_{il} + v_{jki} \]  \(11\)

\[ f_{ijl} u_{lk} + g_{ijl} l + u_{ikl} f_{jlm} + v_{ikl} u_{jlm} = u_{jkl} f_{ilm} + v_{jkl} u_{ilm} \]  \(12\)

\[ f_{ijl} v_{lk} + g_{ijl} m_{lk} + u_{ikl} g_{jlm} + v_{ikl} v_{jlm} = u_{jkl} g_{ilm} + v_{jkl} v_{ilm} \]  \(13\)

Let us introduce now the isotropy hypothesis. This requires that, provided \(t_{ijk}\) is any among \(f_{ijk}, g_{ijk}, l_{ijk}, m_{ijk}, u_{ijk}\) or \(v_{ijk}\), the condition: \(t_{ijk} = t \varepsilon_{ijk}\) be satisfied, where \(t\) is a scalar factor which is different \((f, g, l, m, u \text{ or } v)\) in each case. But, that being true, (11)-(13) are transformed into:

\[ f_{ijk} + v_{ijk} + g_{ijk} \delta_{lk} + u_{ikl} \alpha_{jl} = u_{jkl} \alpha_{il} + 2v_{jki} \]  \(14\)

\[ (fu + gl) \varepsilon_{ijk} = u(f + v) \varepsilon_{jkl} = u(f + v) \varepsilon_{jkl} \]  \(15\)

\[ (fv + gm) \varepsilon_{ijk} + (ug + v^2) \varepsilon_{jkl} = (ug + v^2) \varepsilon_{jkl} \]  \(16\)

that is:

\[ f_{ijk} + g_{ijk} \delta_{lk} + u_{ikl} \alpha_{jl} = u_{jkl} \alpha_{il} + 2v_{jki} \]  \(17\)

\[ (fu + gl)(\delta_{ik} \delta_{jm} - \delta_{im} \delta_{jk}) + u(f + v)(\delta_{im} \delta_{kj} - \delta_{ij} \delta_{km}) = \]

\[ u(f + v)(\delta_{jm} \delta_{ki} - \delta_{ji} \delta_{km}) \]  \(18\)

\[ (fv + gm)(\delta_{ik} \delta_{jm} - \delta_{im} \delta_{jk}) + (ug + v^2)(\delta_{im} \delta_{kj} - \delta_{ij} \delta_{km}) = \]
Let us consider now the subgroup of the Galilei group $SO(3)$, as being a necessary invariance group for the commutation relations (3)-(7). Such condition is satisfied by all canonical transformations of the form

$$X'_i = a_{ij}X_j$$

$$P'_i = a_{ij}P_j$$

with $a_{il}a_{jl} = \delta_{ij}$, leading us from (3)-(7) to the new ones

$$[X'_i, P'_j] =$$

$$[a_{il}X_l, a_{jm}P_m] = a_{il}a_{jm}[X_l, P_m] = a_{il}a_{jm}(i\delta_{lm} + iug\varepsilon_{lmk}X_k + iv\varepsilon_{lmk}P_k) =$$

$$ia_{il}a_{jl} + iua_{il}a_{jm}a_{nk}\varepsilon_{lmk}X'_n + iv\varepsilon_{lmk}P'_n =$$

$$i\delta_{ij} + iug\varepsilon_{ijn}X'_n + iv\varepsilon_{ijn}P'_n$$

(22)

$$[X'_i, X'_j] =$$

$$[a_{il}X_l, a_{jm}X_m] = a_{il}a_{jm}[X_l, X_m] = a_{il}a_{jm}(i\alpha_{lm} + if\varepsilon_{lmk}X_k + ig\varepsilon_{lmk}P_k) =$$

$$ia_{il}a_{jm}\alpha_{lm} + ifa_{il}a_{jm}a_{nk}\varepsilon_{lmk}X'_n + ig\varepsilon_{lmk}P'_n =$$

$$ia_{il}a_{jm}\alpha_{lm} + if\varepsilon_{ijn}X'_n + ig\varepsilon_{ijn}P'_n$$

(23)

$$[P'_i, P'_j] =$$

$$[a_{il}P_l, a_{jm}P_m] = a_{il}a_{jm}[P_l, P_m] = a_{il}a_{jm}(i\beta_{lm} + il\varepsilon_{lmk}X_k + im\varepsilon_{lmk}P_k) =$$

$$ia_{il}a_{jm}\beta_{lm} + ila_{il}a_{jm}a_{nk}\varepsilon_{lmk}X'_n + im\varepsilon_{lmk}P'_n =$$

$$ia_{il}a_{jm}\beta_{lm} + il\varepsilon_{ijn}X'_n + im\varepsilon_{ijn}P'_n$$

(24)

Now, we know the only isotropic form of a 2-index tensor with 3-valued indexes is $\alpha_{ij} = 0$ (resp. $\beta_{ij} = 0$). Then
\( \alpha_{ij} = \beta_{ij} = 0 \) \hspace{1cm} (25)

In this way, equation (17) is reduced to

\[ f \varepsilon_{ijk} = 2v \varepsilon_{ijk} \] \hspace{1cm} (26)

from which

\[ f = 2v \] \hspace{1cm} (27)

With these, equations (18) and (19) are somewhat simplified

\[ (2uv + gl)\left( \delta_{ik}\delta_{jm} - \delta_{im}\delta_{jk} \right) + 3uv\left( \delta_{jm}\delta_{ki} - \delta_{ij}\delta_{km} \right) = 3uv\left( \delta_{jm}\delta_{ki} - \delta_{ij}\delta_{km} \right) \] \hspace{1cm} (28)

\[ (2v^2 + gm)\left( \delta_{ik}\delta_{jm} - \delta_{im}\delta_{jk} \right) + \left( ug + v^2 \right)\left( \delta_{jm}\delta_{ki} - \delta_{ij}\delta_{km} \right) = \left( ug + v^2 \right)\left( \delta_{jm}\delta_{ki} - \delta_{ij}\delta_{km} \right) \] \hspace{1cm} (29)

The argument so far exposed is completely analogous by permutating the parameters, due to the fact that equations (5)-(7) are symmetrical with respect to the “duality” transformation

\[ X_i \rightarrow P_i, \quad u \rightarrow v^{t(1,2)} = -v, \quad \alpha \rightarrow \beta, \quad f \rightarrow m, \quad g \rightarrow -l \]
\[ P_i \rightarrow -X_i, \quad v \rightarrow u^{t(1,2)} = -u, \quad \beta \rightarrow \alpha, \quad m \rightarrow -f, \quad l \rightarrow g \] \hspace{1cm} (30)

where \( u^{t(1,2)} \) and \( v^{t(1,2)} \) represent the three-index tensors \( u_{ijk} \) and \( v_{ijk} \) transposed with respect to their first two indices. But, as we have already seen both tensors have to be proportional to the \( \varepsilon \) tensor, all the equations are exactly the same as before with the shuffling indicated in table (30). All this leads us to

\[ m = -2u \] \hspace{1cm} (31)

with which (29) is further simplified:

\[ (2v^2 - 2ug)\left( \delta_{ik}\delta_{jm} - \delta_{im}\delta_{jk} \right) + \left( ug + v^2 \right)\left( \delta_{jm}\delta_{ki} - \delta_{ij}\delta_{km} \right) = \left( ug + v^2 \right)\left( \delta_{jm}\delta_{ki} - \delta_{ij}\delta_{km} \right) \] \hspace{1cm} (32)
By contracting in (28) and (32) \( i \) with \( k \) and \( j \) with \( m \):

\[
\begin{align*}
gl &= uv \\
v^2 &= 3ug
\end{align*}
\]

(33)  
(34)

Other contractions produce either redundant conditions or identities (but no contraction produces contradiction!).

Let us try now with other forms of commutators:

Type \([P, [X, P]]\) (recall that already we have \( \alpha = \beta = 0 \)):

\[
[P, [X, P]] = [P, i\delta_{jk} + iu\varepsilon_{jkl}X_l + iv\varepsilon_{jkl}P_l] = iu\varepsilon_{jkl} [P, X_l] + iv\varepsilon_{jkl} [P, P_l] =
\]

\[
-iv\varepsilon_{jkl}(i\delta_{li} + iu\varepsilon_{lik}X_k + iv\varepsilon_{lik}P_k) + iv\varepsilon_{jkl}(il\varepsilon_{ikl}X_k + im\varepsilon_{ikl}P_k) =
\]

\[
u\varepsilon_{jki} + (u^2\varepsilon_{jkl}\varepsilon_{lik} - v\varepsilon_{jkl}\varepsilon_{ilk})X_k + (uv\varepsilon_{jkl}\varepsilon_{lik} - vm\varepsilon_{jkl}\varepsilon_{ilk})P_k
\]

On the other hand:

\[
[P, [X, P]] = [P, X_j P_k] - [P, P_k X_j] =
\]

\[
[P, X_j] P_k + X_j [P, P_k] - [P, P_k] X_j - P_k [P, X_j] =
\]

\[
-(i\delta_{ji} + iu\varepsilon_{jil}X_l + iv\varepsilon_{jil}P_l)P_k + X_j(il\varepsilon_{ikl}X_l + im\varepsilon_{ikl}P_l) - \\
(il\varepsilon_{ikl}X_l + im\varepsilon_{ikl}P_l)X_j + P_k(i\delta_{ji} + iu\varepsilon_{jil}X_l + iv\varepsilon_{jil}P_l) =
\]

\[
iu\varepsilon_{jil} [P_k, X_l] + iv\varepsilon_{jil} [P_k, P_l] + il\varepsilon_{ikl} [X_j, X_l] + im\varepsilon_{ikl} [X_j, P_l] =
\]

\[
u\varepsilon_{jik} + (u^2\varepsilon_{jil}\varepsilon_{klm} - v\varepsilon_{jil}\varepsilon_{klm} - fl\varepsilon_{ikl}\varepsilon_{jlm} - mu\varepsilon_{ikl}\varepsilon_{jlm})X_m + \\
(uv\varepsilon_{jil}\varepsilon_{klm} - vm\varepsilon_{jil}\varepsilon_{klm} - gl\varepsilon_{ikl}\varepsilon_{jlm} - mv\varepsilon_{ikl}\varepsilon_{jlm})P_m
\]
By equaling similar coefficients as before:

\[ u \varepsilon_{jik} = u \varepsilon_{jki} \]

from which the vanishing of \( u \) is inferred. The argument is valid for \( v \) as well, and then

\[ u = v = 0 \]  \hspace{1cm} (35)

and, substituting in (33)

\[ gl = 0 \]  \hspace{1cm} (36)

from which one (or both) \( g \) and \( l \) has to vanish also.

But, as we have \( u = v = 0 \), from (31) and (27), we have also

\[ m = f = 0 \]  \hspace{1cm} (37)

But lets us make a brief recapitulation before closing the argument motivating the present article. So far, the only commutation relations compatible with spacial isotropy have been reduced to the form

\[ [X_i, P_j] = i \delta_{ij} \]  \hspace{1cm} (38)

\[ [X_i, X_j] = ig_{ijk} P_k \]  \hspace{1cm} (39)

\[ [P_i, P_j] = il_{ijk} X_k \]  \hspace{1cm} (40)

where one of both constants, \( g \) or \( l \) is zero. The commutation relations in the more general isotropy-compliant instance are then, either on the one hand

\[ [X_i, P_j] = i \delta_{ij}, \; [X_i, X_j] = ig \varepsilon_{ijk} P_k, \; [P_i, P_j] = 0 \]  \hspace{1cm} (41)

or else

\[ [X_i, P_j] = i \delta_{ij}, \; [X_i, X_j] = 0, \; [P_i, P_j] = il \varepsilon_{ijk} X_k \]  \hspace{1cm} (42)

Both forms are invariant under the group of transformations (20), (21).

Both provide, besides, a unique constant with dimensions of length: \( L_1 = g^{1/3} \)
or else $L_2 = l^{-1/3}$, thus both are attractive as to the determination of a theory with good perspectives to deal with the problem of gravitation.

Allow me to put forward the following observation as an argument that looks quite upbeat in favour of the noncommutative-geometry idea before facing the whole thing with a somewhat more critical mind. The question is to try to decide oneself between both possibilities: the form given by (41) is particularly interesting due to a rather physical argument. The uncertainty principle (4) establishes for such election that

$$\Delta_\psi X \Delta_\psi Y \geq \frac{g}{2} |\langle P_z \rangle_\psi|$$

This constraint has a beautiful and intuitive interpretation: if we prepare a particle beam in a collision state such that

$$\langle P_z \rangle_\psi \gg \hbar g^{-1/3}$$

then it will be in general impossible that it be arbitrarily small in the transverse direction to the direction of collision, that is, with $\Delta_\psi X \simeq \Delta_\psi Y \simeq 0$, due to the fact that, for (43), we must have $\Delta_\psi X \Delta_\psi Y \gg \hbar g^{2/3}/2$. It is very reasonable to think of this relation as playing the role of an effective cutoff for the ultraviolet range, as it precludes the contribution of pointlike states in the transverse direction in the dispersion relations. And it seems plausible to use this as a working hypothesis for lack of a better phenomenological grasp of the full-fledged noncommutative theory.

But let us look further beyond in the same direction we started from. Indeed, we know we have not used all the information at our disposal to maximally determine the structure constants. In order to resume, we apply again the same procedure used to other third-order commutators, but now with the commutation relations defining the algebra reduced to the form (41). Let us assume, e.g., that we have $l = 0$, $g \neq 0$. That is, we apply to the “promising so far” commutation relations (41) the iterated commutation using Leibniz’s rule in search of consistency (or inconsistency!). On the one hand

$$[X_i, [X_j, X_k]] = ig\varepsilon_{jkl} [X_i, P_l] = ig\varepsilon_{jkl}(i\delta_{il}) = -g\varepsilon_{jki}$$

But, on the other hand

$$[X_i, [X_j, X_k]] = [X_i, X_j X_k] - [X_i, X_k X_j] =$$
\[
[X_i, X_j] X_k + X_j [X_i, X_k] - [X_i, X_k] X_j - X_k [X_i, X_j] =
\]

\[
(ig\varepsilon_{ijl} P_l) X_k + X_j (ig\varepsilon_{ikl} P_l) - (ig\varepsilon_{ikl} P_l) X_j - X_k (ig\varepsilon_{ijl} P_l) =
\]

\[
ig\varepsilon_{ijl} [P_l, X_k] + ig\varepsilon_{ikl} [X_j, P_l] =
\]

\[
g\varepsilon_{ijk} - g\varepsilon_{ikj} = 2g\varepsilon_{ijk}
\]

By identifying, as before

\[
2g\varepsilon_{ijk} = -g\varepsilon_{jki} \Rightarrow g\varepsilon_{ijk} = 0 \Rightarrow g = 0
\]

The argument is completely analogous to prove that \( l = 0 \). Thus, the only commutation relations of the general form \((\ref{eq:commutation})\)-(\ref{eq:commutation2}) consistent with spatial isotropy in a first-quantised Galilean scenario, are Heisenberg’s; \((\ref{eq:heisenberg})\).

### 3 Final remarks

With this work I have tried to put emphasis on the dangers involved when dwelling in intrinsic formalisms without ever bringing the arguments down to the level of particular coordinations. Valuable though such intrinsic methods are, it is the particular coordinations that endorse any theory with physical content. In this sense, it is easy to browse hundreds of pages concerning noncommutative geometries without our eyes ever meeting a single concrete expression of commutation relations in a definite, even just nominally declared, coordinate system. It is sure that some arguments are better expressed in a particular coordinate system than in an intrinsic manner or, at least, by starting out from such a coordinate system in order to further consider the freedom the theory decrees by introducing a transformation group.

On the other hand, this work is restricted to the study of the position and momentum canonical operators on a configuration space of just one particle, that is, it is focused to drawing a model of quantum mechanics on a flat “noncommutative manifold”. In particular, it is not a study within the context of quantum field theory nor does it contemplate the complications of curvature and/or connection. It has to be interpreted, thus, as a mere preliminary study. Today we know that any theory aspiring to some generality...
must incorporate the feature of reducing itself to a field theory at low energies. In consequence, a more serious attempt would imply the generalisation of (3)-(7) to the Fock space and refer them to field operators. A possible attempt at generalising (41) could be the equal-time commutation relations (for a bosonic field)

\[
[\varphi_a(x,0), \pi_b(y,0)] = i\delta_{ab} \delta^{(3)}(x - y) \tag{45}
\]

\[
[\varphi_a(x,0), \varphi_b(y,0)] = ig_{abc} \pi_c(x - y) \tag{46}
\]

\[
[\pi_a(x,0), \pi_b(y,0)] = 0 \tag{47}
\]

that seems more promising than the “naive” version (41) as the isotropy argument cannot be exported to the inner space of field multiplets. The problem now seems to be that (45)-(47) have too many free parameters with length dimensions for a general invariance group. The reader can try these out and check that there is no easy way in which relations (45)-(47) can be be consistent and nonzero.

To end these comments, let as say that it is interesting (as an argumental whetting of experimental appetite) to think that empirical checks on the limits to the isotropy of space could serve as an indirect test for the validity of the noncommutative model.

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