Group stability and Property (T)

Oren Becker and Alexander Lubotzky

ABSTRACT. In recent years, there has been a considerable amount of interest in the stability of a finitely-generated group \( \Gamma \) with respect to a sequence of groups \( \{ G_n \}_{n=1}^\infty \), equipped with bi-invariant metrics \( \{ d_n \}_{n=1}^\infty \). In particular, in [5] and [13], it was shown that many groups with Property (T) are stable with respect to \( G_n = U(n) \), equipped with various metrics (e.g. the \( p \)-Schatten metrics for \( 1 < p < \infty \)). Here we show that the situation is very different for \( G_n = U(n) \) (resp. \( G_n = \text{Sym}(n) \)), equipped with the normalized Hilbert-Schmidt metric \( d_n^{\text{HS}} \) (resp. the normalized Hamming metric \( d_n^{\text{Hamming}} \)). Namely, an infinite hyperlinear (resp. sofic) group with Property (T) is not stable with respect to \( (U(n), d_n^{\text{HS}}) \) (resp. \( (\text{Sym}(n), d_n^{\text{Hamming}}) \)). We suggest a more flexible notion of stability that may repair this deficiency.

1. Introduction

Let \( G = \{(G_n, d_n)\}_{n=1}^\infty \) be a family of groups \( G_n \) endowed with bi-invariant metrics \( d_n \), i.e., \( d_n (ag_1b, ag_2b) = d_n (g_1, g_2) \) for all \( g_1, g_2, a, b \in G_n \). Here are some examples:

1. \( \mathcal{P} = \{(\text{Sym}(n), d_n^{\text{Hamming}})\}_{n=1}^\infty \), where \( d_n^{\text{Hamming}} \) is the normalized Hamming metric on \( \text{Sym}(n) \): for \( \sigma, \tau \in \text{Sym}(n) \),
   \[
   d_n^{\text{Hamming}}(\sigma, \tau) = \frac{1}{n} \cdot |\{ x \in [n] | \sigma(x) \neq \tau(x) \}| ,
   \]
   where \([n] = \{1, \ldots, n\}\).

2. \( \mathcal{HS} = \{(U(n), d_n^{\text{HS}})\}_{n=1}^\infty \), where \( d_n^{\text{HS}} \) is the normalized Hilbert-Schmidt metric: for \( A, B \in U(n) \),
   \[
   d_n^{\text{HS}}(A, B) = \| A - B \|_{\text{HS}} , \quad \text{where} \quad \|T\|_{\text{HS}} = \text{Tr} \left( \frac{1}{n} \cdot T^{*}T \right)^{1/2} .
   \]

3. \( \mathcal{G}^{(p)} = \{(U(n), d_n^{(p)})\}_{n=1}^\infty \), for any fixed \( 1 \leq p < \infty \), where \( d_n^{(p)} \) is the Schatten \( p \)-norm (see [4], Section IV.2): for \( A, B \in U(n) \),
   \[
   d_n^{(p)}(A, B) = \| A - B \|_p , \quad \text{where} \quad \|T\|_p = \left( \text{Tr} \left( (T^{*}T)^{p/2} \right) \right)^{1/p} .
   \]
The case \( p = 2 \) is of special interest: this is the standard \( L^2 \)-norm, a.k.a. the Frobenius norm, denoted \( \| \cdot \|_F \). Note that 
\[
\| d_n(2) \|_F = \sqrt{n} \cdot d_{n^2}^{\operatorname{HS}}.
\]
The proofs in Section 3 also make use of the Frobenius norm \( \| \cdot \|_F \) for non-square matrices, which is defined by the same formula: 
\[
\| T \|_F = \operatorname{Tr}(T^*T)^{1/2}.
\]

\[ G^{(\infty)} = \left\{ \left( \left( U(n), d_n^{(\infty)} \right) \right) \right\}_{n=1}^{\infty}, \]
where the metric \( d_n^{(\infty)} \) is defined for \( A, B \in U(n) \) by 
\[
\| A - B \|_\infty, \text{ where } \| \cdot \|_\infty\text{ is the operator norm.}
\]

Let \( \mathbb{F} \) be a free group on a finite set \( S \). Let \( \Gamma \) be a quotient of \( \mathbb{F} \), and denote the quotient map by \( \pi : \mathbb{F} \to \Gamma \). From now on, for a group \( G \), a function \( f : S \to G \) and an element \( w \in \mathbb{F} \), we write \( f(w) \) for the element of \( G \) resulting from applying the substitution \( s \mapsto f(s) \) to the word \( w \).

**Definition 1.1.**

i) A \( \mathcal{G} \)-stability-challenge for \( \Gamma \) is a sequence \( (f_n)_{k=1}^{\infty} \) of functions 
\[
f_n : S \to G_{n_k}, \quad n_k \to \infty,
\]
such that for every \( w \in \operatorname{Ker}(\pi) \),
\[
d_n(f_n(w), 1_{G_{n_k}}) \xrightarrow{k \to \infty} 0.
\]

ii) Let \( (f_n)_{k=1}^{\infty} \) be a \( \mathcal{G} \)-stability-challenge for \( \Gamma \). A solution for \( (f_n)_{k=1}^{\infty} \) is a sequence of functions \( (g_n)_{k=1}^{\infty} \), \( g_n : S \to G_{n_k} \), such that for every \( w \in \operatorname{Ker}(\pi) \), \( g_n(w) = 1_{G_{n_k}} \) (i.e., \( g_n \) defines a homomorphism \( \Gamma \to G_{n_k} \)), and
\[
\sum_{s \in S} d_n(f_n(s), g_n(s)) \xrightarrow{k \to \infty} 0.
\]

iii) The group \( \Gamma \) is \( \mathcal{G} \)-stable if every \( \mathcal{G} \)-stability-challenge for \( \Gamma \) has a solution.

While the above definition of a \( \mathcal{G} \)-stable group made use of a given presentation of \( \Gamma \) as a quotient of a free group, it is in fact a group property, independent of the specific presentation.

In recent years, there has been a considerable amount of interest in “group stability” (see [11, 18, 2]). One of the main motivations is the study of \( \mathcal{G} \)-approximations of \( \Gamma \):

**Definition 1.2.** For \( \mathcal{G} \) and \( \Gamma \) as above, we say that \( \Gamma \) is \( \mathcal{G} \)-approximated if there is a sequence of integers \( \{n_k\}_{k=1}^{\infty} \), \( n_k \xrightarrow{k \to \infty} \infty \), and a sequence \( \{\varphi_{n_k}\}_{k=1}^{\infty} \) of functions \( \varphi_{n_k} : \Gamma \to G_{n_k} \), such that
\[
\forall g, h \in \Gamma \quad \lim_{k \to \infty} d_n(\varphi_{n_k}(gh), \varphi_{n_k}(g) \varphi_{n_k}(h)) = 0
\]
and
\[
\forall 1_\Gamma \neq g \in \Gamma \quad \limsup_k d_n(\varphi_{n_k}(g), 1_{G_{n_k}}) > 0
\]
In classical terminology, $\mathcal{P}$-approximated groups (for $\mathcal{P}$ as in Example 1 above) are called *sofic* groups, and HS-approximated groups are called *hyperlinear* groups. It is a well-known open problem, due to Gromov (resp. Connes), whether every group is sofic (resp. hyperlinear). Note that all sofic groups are hyperlinear.

In [5], it was shown for the first time that there are finitely presented groups $\Gamma$ which are not $(U(n), d_n^{(2)})$-approximated (i.e., Frobenius-approximated), and this result was extended in [13] to all $1 < p < \infty$. The groups $\Gamma$ in those papers are finite central extensions of suitable lattices $\bar{\Gamma}$ in simple Lie groups of rank $r \geq 3$ over local non-archimedean fields. The key point there is that these groups $\Gamma$ and $\bar{\Gamma}$ are $L^2$-stable (and even $L^p$-stable). This is proved as a corollary to the vanishing result $H^i(\Gamma, V) = 0$ for every $i = 1, \ldots, r - 1$ and for all actions of $\Gamma$ on Hilbert spaces $V$ (and the same for many Banach spaces). The case $i = 2$ gives the stability of $\Gamma$. Vanishing for $i = 1$ is equivalent to $\Gamma$ having Property (T), so all the groups treated there have Kazhdan’s Property (T).

The goal of the present paper is to show that these examples are neither $\mathcal{P}$-stable nor HS-stable. In fact, we prove a much more general result, which is of independent interest:

**Theorem 1.3.**

i) If $\Gamma$ is sofic and has Property (T), then it is not $\mathcal{P}$-stable, unless it is finite.

ii) If $\Gamma$ is hyperlinear and has Property (T), then it is not HS-stable, unless it is finite.

Theorem 1.3 is a corollary of the following:

**Theorem 1.4.** Assume that $\Gamma$ has Kazhdan’s Property (T), and is either $\mathcal{P}$-stable or HS-stable. Then, $\Gamma$ has only finitely many finite-index subgroups.

Theorem 1.4 is proved in Section 3. We can already show how it implies Theorem 1.3:

**Proof of Theorem 1.3.** Assume that $\Gamma$ is sofic and $\mathcal{P}$-stable. A well-known observation (see Theorem 2 in [7]) says that, in this case, $\Gamma$ is residually-finite. If further, $\Gamma$ has Property (T), then by Theorem 1.4, it has only finitely many finite-index subgroups, and so it is finite.

Assume, instead, that $\Gamma$ is hyperlinear and HS-stable. It is well-known that in this case too, $\Gamma$ is residually-finite. Indeed, arguing as in [7], we see that $\Gamma$ is residually-linear, and so it is residually-finite since finitely-generated linear groups are residually-finite. So, as before, if, further, $\Gamma$ has Property (T), it must be finite.

In Section 4, we give some remarks regarding variations of Property (T) (e.g., Property ($\tau$) and relative Property (T)) and the stability of
semidirect products and free products with amalgamation, and suggest problems for further research.

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3. The proof of Theorem 1.4

Before we begin, we record a simple observation regarding P-stability. Fix formal elements $\{v_i\}_{i=1}^{\infty}$, and for every $n \in \mathbb{N}$, let $B_n = (v_1, \ldots, v_n)$ serve as an ordered basis for a complex vector space $\mathcal{H}_n$. A permutation $\sigma \in \text{Sym}(n) \cong \text{Sym}(B_n)$ extends uniquely to an element of $U(\mathcal{H}_n) \cong U(n)$, giving an embedding $\iota : \text{Sym}(n) \to U(n)$. For permutations $\sigma, \tau \in \text{Sym}(n)$,

$$d_{\text{HS}}(\iota(\sigma), \iota(\tau)) = \sqrt{2 \cdot d_{\text{Hamming}}(\sigma, \tau)}.$$

Therefore, for $G_0 = \{\iota(\text{Sym}(n)) \mid n \in \mathbb{N}\}$, equipped the the Hilbert-Schmidt metric $d_{\text{HS}}$, the group $\Gamma$ is P-stable if and only if it is $G_0$-stable.

For a linear operator $T : \mathcal{H} \to \mathcal{H}$ on a finite-dimensional vector space $\mathcal{H}$ and a basis $\mathcal{B}$ for $\mathcal{H}$, we write $[T]_{\mathcal{B}}$ for the matrix representing $T$ with respect to $\mathcal{B}$.

**Lemma 3.1.** Let $\mathcal{H}$ be a finite-dimensional complex Hilbert space with orthonormal ordered basis $\mathcal{B} = (v_1, \ldots, v_n)$. Let $B_0 = (v_1, \ldots, v_{n-1})$ and $\mathcal{H}_0 = \text{span}_C B_0$. Let $T : \mathcal{H} \to \mathcal{H}$ be a linear operator. Write $P_0 : \mathcal{H} \to \mathcal{H}_0$ for the orthogonal projection onto $\mathcal{H}_0$, and $T_0 = P_0 \circ T \mid_{\mathcal{H}_0} : \mathcal{H}_0 \to \mathcal{H}_0$. Then,

i) If $T$ permutes $\mathcal{B}$, then there is a linear operator $A_0 : \mathcal{H}_0 \to \mathcal{H}_0$, which permutes $B_0$, such that

$$\|T_0 - A_0\|_F \leq \sqrt{2}.$$

ii) If $T$ is unitary, then there is a unitary linear operator $A_0 : \mathcal{H}_0 \to \mathcal{H}_0$, such that

$$\|T_0 - A_0\|_F \leq 1.$$

In both cases, the inclusion map $f : \mathcal{H}_0 \to \mathcal{H}$ satisfies

$$\|T^{-1} \circ f \circ A_0 - f\|_F \leq \|T_0 - A_0\|_F + 1.$$

**Proof.** (i) Assume that $T$ permutes $\mathcal{B}$. Denote $T^{-1}(v_n) = v_{i_0}$, $1 \leq i_0 \leq n$. Define a linear operator $A_0 : \mathcal{H}_0 \to \mathcal{H}_0$ on the elements of
Let $A_0(v_i) = \begin{cases} T(v_i) & i \neq i_0 \\ T(v_n) = T(T(v_i)) & i = i_0 \end{cases}$. Then, only the $i_0$-th column of $[T_0 - A_0]_{B_0}$ may be nonzero, and its norm is 0 if $i_0 = n$, or $\sqrt{2}$ otherwise. In any case,

$$\|T_0 - A_0\|_F \leq \sqrt{2}$$

(ii) Assume that $T$ is unitary. Take a polar decomposition $T_0 = A_0 \cdot \sqrt{T_0^*T_0}$ of $T_0$, where $A_0 \in U(H_0)$ (see Theorem 3.1.9(c) in [9]). Note that, generally, $A_0$ is only guaranteed to exist, but is not unique. Then,

$$\|T_0 - A_0\|_F = \|A_0^{-1} \cdot (T_0 - A_0)\|_F = \|\sqrt{T_0^*T_0} - I\|_F$$

Let $u \in M_{1 \times (n-1)}(\mathbb{C})$ be the bottom row of $[T]_B$, with the rightmost entry removed. Since $T$ is unitary, we have $\|u\| \leq 1$. Partition $[T]_B$ as a block matrix, where $[T_0]_{B_0}$ is the top-left block, and $u$ is the bottom-left block. Since $T^*T = I$, we get $[T_0^*T_0]_{B_0} + u^*u = I_{n-1}$, i.e., $[T_0^*T_0]_{B_0} = I_{n-1} - u^*u$. The eigenvalues of $u^*u$ are 0 (with multiplicity $n - 2$), and $\langle u, u \rangle = \|u\|^2$ (with multiplicity 1, corresponding to the right eigenvector $u^*$). So, $\sqrt{T_0^*T_0}$ is a unitarily diagonalizable operator whose eigenvalues are 1, with multiplicity $n - 2$, and $\sqrt{1 - \|u\|^2}$, with multiplicity 1. So,

$$\|\sqrt{T_0^*T_0} - I\|_F = \sqrt{1 - \|u\|^2} - 1 \leq 1$$

which, together with (3.1), implies the desired result.

As for the last claim,

$$\|T^{-1} \circ f \circ A_0 - f\|_F = \|f \circ A_0 - T \circ f\|_F$$

$$= \|f \circ A_0 - T\|_{H_0} \|_F$$

$$\leq \|f \circ (A_0 - T_0)\|_F + \|f \circ T_0 - T\|_{H_0} \|_F$$

$$= \|T_0 - A_0\|_F + \|(f \circ P_0 \circ T - T)\|_{H_0} \|_{B_0} \|_F$$

$$= \|T_0 - A_0\|_F + \|u\|_F$$

$$\leq \|T_0 - A_0\|_F + 1$$

where $u$ is, again, the bottom-left row of $[T]_B$, with the rightmost entry removed.

\[\square\]

**Lemma 3.2.** Let $\mathcal{H}$ be a finite-dimensional complex Hilbert space. Let $U_1, \ldots, U_l \in U(\mathcal{H})$ and $E_1, \ldots, E_l \in \text{End}_\mathbb{C}(\mathcal{H})$. Let $c \geq 0$, and assume that $\|E_i\|_F \leq c$ for all $1 \leq i \leq l$. Then,

$$\left\| \prod_{i=1}^l (U_i + E_i) - \prod_{i=1}^l U_i \right\|_F \leq (c + 1)^l$$


PROOF. Let \( \emptyset \neq A_0 \subset [l] \) (where \([l] = \{1, \ldots, l\}\)). For each \( 1 \leq i \leq l \), denote \( M_i = \begin{cases} E_i & i \in A_0 \\ U_i & i \notin A_0 \end{cases} \). Let \( 1 \leq k \leq l \), and consider the product \( \prod_{i=1}^k M_i \). On one hand, if \( k \notin A_0 \), then

\[
\| \prod_{i=1}^k M_i \|_F = \| \left( \prod_{i=1}^{k-1} M_i \right) \cdot U_k \|_F = \| \left( \prod_{i=1}^{k-1} M_i \right) \|_F
\]

since the Frobenius norm \( \| \cdot \|_F \) is invariant under multiplication by unitary operators. On the other hand, if \( k \in A_0 \), then

\[
\| \prod_{i=1}^k M_i \|_F = \| \left( \prod_{i=1}^{k-1} M_i \right) \cdot E_k \|_F \leq \prod_{i=1}^{k-1} M_i \|_F \cdot \| E_k \|_F
\]

since \( \| \cdot \|_F \) is submultiplicative. So, we conclude by induction that

\[
\| \prod_{i=1}^k M_i \|_F \leq \prod_{i \in A_0} \| E_i \|_F \leq c^{|A_0|}.
\]

Together with the triangle inequality, this implies that

\[
\| \prod_{i=1}^l (U_i + E_i) - \prod_{i=1}^l U_i \|_F \leq \sum_{\emptyset \neq A \subset [l]} c^{|A|}
\]

\[
= \sum_{i=1}^l \binom{l}{i} \cdot c^i
\]

\[
\leq (c + 1)^l.
\]

\( \square \)

For a word \( w \in \mathbb{F} \), write \( |w| \) for length of \( w \), i.e., the length of \( w \) when written as a reduced word over \( S^* \). Recall that we write \( \pi \) for the fixed quotient map \( \pi : \mathbb{F} \to \Gamma \).

PROPOSITION 3.3. Let \( (\mathcal{H}, \alpha) \) be a finite-dimensional unitary representation of \( \Gamma \). Let \( \mathcal{H}_0 \subset \mathcal{H} \) be a subspace of co-dimension 1. Let \( \mathcal{B}_0 \subset \mathcal{B} \) be orthonormal bases for \( \mathcal{H}_0, \mathcal{H} \), respectively. Then,

i) There is a function \( \rho : S \to U(\mathcal{H}_0) \), such that \( \| \rho(w) - I \|_F \leq 4|w| \)

for every \( w \in \text{Ker}(\pi) \), and the inclusion map \( f : \mathcal{H}_0 \to \mathcal{H} \) satisfies

\[
\| \alpha(s^{-1}) \circ f \circ \rho(s) - f \|_F \leq \sqrt{2} + 1
\]

for each \( s \in S \).

ii) If, furthermore, each \( \alpha(s) \) permutes \( \mathcal{B} \), then \( \rho \) above can be chosen such that each \( \rho(s) \) permutes \( \mathcal{B}_0 \).

PROOF. Define \( \alpha_0 : S \to \text{End}_\mathbb{C} \mathcal{H} \) by \( \alpha_0(s) = P_0 \circ \alpha(s) |_{\mathcal{H}_0} \), where \( P_0 : \mathcal{H} \to \mathcal{H}_0 \) is the orthogonal projection. By Lemma 3.1 applied to
\[ \alpha(s) \] for each \( s \in S \) separately, there is a function \( \rho : S \to U(H_0) \), such that
\[ \| \alpha_0(s) - \rho(s) \|_F \leq \sqrt{2} \] (3.2)
and if each \( \alpha(s) \) permutes \( B \), then \( \rho \) can be chosen so that each \( \rho(s) \) permutes \( B_0 \). In any case, Lemma 3.1 guarantees that
\[ \| \alpha(s) \circ f \circ \rho(s) - f \|_F \leq \sqrt{2} + 1 \]
for each \( s \in S \). Let \( s \in S \). Since \( \alpha(s) \) is unitary,
\[ \| \rho(s) \odot 1_{H_0} - \alpha(s) \|_F^2 \leq \| \rho(s) - \alpha_0(s) \|_F^2 + 3, \]
and so, from (3.2), we see that
\[ \| \rho(s) \odot 1_{H_0} - \alpha(s) \|_F \leq \left( \| \rho(s) - \alpha_0(s) \|_F^2 + 3 \right)^{1/2} \]
\[ = \left( \sqrt{2} + 3 \right)^{1/2} \]
\[ \leq 3. \]

We would like to bound \( \| \rho(\cdot) \odot 1_{H_0} - \alpha(\cdot) \|_F \), evaluated at a word \( w \in \mathbb{F} \), and so we also need to bound \( \| \rho(s)^{-1} \odot 1_{H_0} - \alpha(s)^{-1} \|_F \). But, in general \( \| A^{-1} - B^{-1} \|_F = \| A - B \|_F \) for \( A, B \in U(H) \), and so
\[ \| \rho(s)^{-1} \odot 1_{H_0} - \alpha(s) \|_F \leq 3 \]

Let \( w \in \mathbb{F} \). By Lemma 3.2, the above implies that
\[ \| \rho(w) \odot 1_{H_0} - \alpha(w) \|_F \leq (3 + 1)^{|w|} = 4^{|w|}. \]

Assume further that \( w \in \text{Ker}(\pi) \). Then \( \alpha(w) = I \), and so,
\[ \| \rho(w) - I \|_F = \| \rho(w) \odot 1_{H_0} - I \|_F \]
\[ \leq \| \rho(w) \odot 1_{H_0} - \alpha(w) \|_F + \| \alpha(w) - I \|_F \]
\[ \leq 4^{|w|} \]

Henceforth, given representations \( H_1 \) and \( H_2 \) of \( \Gamma \), we treat \( \text{Hom}_C(H_1, H_2) \) as a \( \Gamma \)-representation with the action given by \( g \cdot f = g \circ f \circ g^{-1} \) for \( g \in \Gamma \) and \( f \in \text{Hom}_C(H_1, H_2) \).

**Proposition 3.4.** Let \( H_0 \subseteq H \) be finite-dimensional complex Hilbert spaces, and write \( f : H_0 \to H \) for the inclusion map. Let \( \alpha : \Gamma \to U(H) \) and \( \beta : \Gamma \to U(H_0) \) be unitary representations. Then,

i) If \( (H, \alpha) \) is irreducible, then \( \| f - h \|_F = \| f \|_F \) for every morphism of \( \Gamma \)-representations \( h : H_0 \to H \).

ii) If \( B_0 \subset B \) are orthonormal bases for \( H_0, H \), respectively, each \( \beta(s) \) permutes \( B_0 \), each \( \alpha(s) \) permutes \( B \), and the action \( \Gamma \times^\alpha B \) of \( \Gamma \) on \( B \) through \( \alpha \) is transitive, then \( \| f - h \|_F \geq \frac{1}{\sqrt{2}} \| f \|_F \) for every morphism of \( \Gamma \)-representations \( h : H_0 \to H \).
PROOF. (i) Since $\dim_C \mathcal{H}_o < \dim_C \mathcal{H}$ and $(\mathcal{H}, \alpha)$ is irreducible, Schur’s Lemma implies the only morphism of representations $\mathcal{H}_o \to \mathcal{H}$ is the zero morphism, and so the result follows.

(ii) For $b_0 \in \mathcal{B}_o$ and $b \in \mathcal{B}$, let $E_{b_0, b} : \mathcal{H}_o \to \mathcal{H}$ be the linear map sending $b_0 \mapsto b$, and sending every other element of $\mathcal{B}_o$ to zero. Then, \( \{ E_{b_0, b} \}_{(b_0, b) \in \mathcal{B}_o \times \mathcal{B}} \) is a basis for $\text{Hom}_C (\mathcal{H}_o, \mathcal{H})$. The inner product for which $\{ E_{b_0, b} \}_{(b_0, b) \in \mathcal{B}_o \times \mathcal{B}}$ is an orthonormal basis makes the $\Gamma$-representation $\text{Hom}_C (\mathcal{H}_o, \mathcal{H})$ unitary. The group $\Gamma$ acts on $\mathcal{B}_o \times \mathcal{B}$ entrywise. A map $T \in \text{Hom}_C (\mathcal{H}_o, \mathcal{H})$, represented as $T = \sum_{(b_0, b) \in \mathcal{B}_o \times \mathcal{B}} \lambda_{b_0, b} \cdot E_{b_0, b}$, is a morphism of representations if and only if the mapping $(b_0, b) \mapsto \lambda_{b_0, b}$ is constant on each $\Gamma$-orbit of $\mathcal{B}_o \times \mathcal{B}$. We claim that the cardinality of each such orbit is at least $2 \cdot |\mathcal{B}_o|$. Indeed, let $(b_0, b) \in \mathcal{B}_o \times \mathcal{B}$. Then,

\[
\text{Stab}_\Gamma ((b_0, b)) = \text{Stab}_\Gamma (b_0) \cap \text{Stab}_\Gamma (b) \leq \text{Stab}_\Gamma (b_0) . 
\] (3.3)

The action $\Gamma \curvearrowright \mathcal{B}$ is transitive, and so $|\mathcal{B}| = [\Gamma : \text{Stab}_\Gamma (b_0)]$. Thus,

\[
[\Gamma : \text{Stab}_\Gamma (b_0)] \leq |\mathcal{B}_o| < |\mathcal{B}| = [\Gamma : \text{Stab}_\Gamma (b_0)] .
\]

In particular, $\text{Stab}_\Gamma (b_0)$ is not a subgroup of $\text{Stab}_\Gamma (b)$, and so the inclusion in (3.3) is strict. Hence, for all $(b_0, b) \in \mathcal{B}_o \times \mathcal{B}$

\[
[\Gamma \cdot (b_0, b)] = [\Gamma : \text{Stab}_\Gamma ((b_0, b))]
\]

\[
= [\Gamma : \text{Stab}_\Gamma (b_0) \cdot \text{Stab}_\Gamma ((b_0, b))]
\]

\[
\geq 2 \cdot [\Gamma : \text{Stab}_\Gamma (b_0)]
\]

\[
= 2 \cdot |\mathcal{B}_o| ,
\]

as claimed. For each $\Gamma$-orbit $o$ of $\mathcal{B}_o \times \mathcal{B}$, let $c (o)$ be the number of elements $(b_0, b) \in o$ for which $f (b_0) = b$, i.e., $c (o) = |o \cap \{(b_0, b_0) \mid b_0 \in \mathcal{B}_o\}|$. Then, $\sum_o c (o) = |\mathcal{B}_o|$. Let $h : \mathcal{H}_o \to \mathcal{H}$ be the orthogonal projection of the given inclusion map $f$ into $\text{Hom}_C (\mathcal{H}_o, \mathcal{H})$. Then, $h$ is the morphism of representations which is closest to $f$ under $\| \cdot \|_F$, and $h$ is obtained by averaging out $f$ in each $\Gamma$-orbit separately. Write $f = \sum_{(b_0, b) \in \mathcal{B}_o \times \mathcal{B}} \lambda_{b_0, b} \cdot E_{b_0, b}$ and $h = \sum_{(b_0, b) \in \mathcal{B}_o \times \mathcal{B}} \mu_{b_0, b} \cdot E_{b_0, b}$. Then, for each $\Gamma$-orbit $o$ of $\mathcal{B}_o \times \mathcal{B}$, the map $\lambda : o \to \mathbb{C}$, defined by $\lambda (b_0, b) = \lambda_{b_0, b}$, has, in its image, $c (o)$ 1-s and $(|o| - c (o))$ 0-s, while the map $\mu : o \to \mathbb{C}$, defined by $\mu (b_0, b) = \mu_{b_0, b}$, is constant, mapping all elements to $c (o) / |o|$. So,

\[
\| f - h \|^2_F = \sum_o \left( c (o) \cdot \left( 1 - \left( \frac{c (o)}{|o|} \right) \right) + \left( |o| - c (o) \right) \cdot \left( 0 - \left( \frac{c (o)}{|o|} \right) \right) \right)
\]

\[
= \sum_o c (o) - \sum_o \frac{c (o)^2}{|o|}
\]

\[
\geq |\mathcal{B}_o| - \frac{1}{2 |\mathcal{B}_o|} \sum_o c (o)^2
\]
But,

\[
\frac{1}{2 |B_0|} \sum_o c(o)^2 \leq \frac{1}{2 |B_0|} \left( \sum_o c(o) \right)^2 = \frac{1}{2} \cdot |B_0|
\]

Thus,

\[
\| f - h \|_F^2 \geq \frac{1}{2} \cdot |B_0| = \frac{1}{2} \cdot \| f \|_F^2,
\]

and so, taking square roots finishes the proof.

We recall the definition of Kazhdan’s Property (T) (see Section 1.1 of \[3\]). Let \( Q \subset \Gamma \) and \( \kappa > 0 \). Recall that for a unitary representation \((\mathcal{H}, \rho)\) of \( \Gamma \) and a nonzero vector \( v \in \mathcal{H} \), we say that \( v \) is \((Q, \kappa)\)-invariant if
\[
\sup_{x \in Q} \| \rho(x) \cdot v - v \| < \kappa \cdot \| v \|.
\]
We say that \((Q, \kappa)\) is a Kazhdan pair for \( \Gamma \) if every unitary representation \((\mathcal{H}, \rho)\) of \( \Gamma \) satisfies:

\[
\text{if } \mathcal{H} \text{ contains a } (Q, \kappa)\text{-invariant vector, then it also contains a } \Gamma\text{-invariant non-zero vector.}
\]

\[3\text{.4}\]

We say that the group \( \Gamma \) has Kazhdan’s Property (T) if it has a Kazhdan pair \((Q, \kappa)\) for which \( Q \) is finite (and \( \kappa > 0 \)). Every discrete group with Property (T) is finitely-generated \([3,10]\). If \( \Gamma \) has Property (T), then for every finite generating set \( Q \) of \( \Gamma \), there is \( \kappa > 0 \) for which \((Q, \kappa)\) is a Kazhdan pair for \( \Gamma \), and we call such \( \kappa \) a Kazhdan constant for \((\Gamma, Q)\).

**Lemma 3.5.** Assume that \( \Gamma \) has Property (T) with Kazhdan constant \( \kappa > 0 \) for \((\Gamma, S)\). Let \((\mathcal{H}_1, \alpha)\) and \((\mathcal{H}_2, \beta)\) be finite-dimensional unitary representations of \( \Gamma \). Let \( \epsilon > 0 \), and let \( f : \mathcal{H}_1 \to \mathcal{H}_2 \) be a nonzero linear map, such that for each \( s \in S \),
\[
\| \alpha(s^{-1}) \circ f \circ \beta(s) - f \| < \epsilon \cdot \| f \|
\]

Then, there is a morphism \( h : \mathcal{H}_1 \to \mathcal{H}_2 \) of \( \Gamma \)-representations, such that
\[
\| f - h \| < \frac{\epsilon}{\kappa} \cdot \| f \|
\]

**Proof.** The map \( f \) is an \((S, \epsilon)\)-invariant vector in the representation \( \text{Hom}_\mathbb{C}(\mathcal{H}_1, \mathcal{H}_2) \) of \( \Gamma \). So, there is a \( \Gamma \)-invariant linear map \( h \in \text{Hom}_\mathbb{C}(\mathcal{H}_1, \mathcal{H}_2) \) such that
\[
\| f - h \| < \frac{\epsilon}{\kappa} \cdot \| f \|
\]
(see Remark 1.1.10 of \[3\]). The invariance of \( h \) is equivalent to \( h \) being a morphism of \( \Gamma \)-representations.

We are now ready to prove the main theorem:

**Proof of Theorem** \[1,4\] Before we begin, note that for each \( n \in \mathbb{N} \), \( \Gamma \) has only finitely many finite-index subgroups of index \( n \) because \( \Gamma \) is finitely-generated. Since, in addition, \( \Gamma \) has Property (T), it has only finitely many irreducible unitary representations of any given dimension \( n \in \mathbb{N} \) (up to isomorphism). For the last assertion, see Theorem 2.6 of
where the last inequality follows from (3.7) and the fact that

Thus, by Lemma 3.5, there are morphisms of representations

together with the compactness of $U(n)$ and Schur’s Lemma.

Let $\kappa > 0$ be a Kazhdan constant for $\Gamma$ with respect to $S$. First, assume that $\Gamma$ is P-stable. Assume, for the sake of contradiction, that $\Gamma$ has infinitely many finite-index subgroups, and let $\{\Lambda_n\}_{n=1}^{\infty}$ be a sequence of such subgroups for which $[\Gamma : \Lambda_n] \to \infty$. Fix $n \in \mathbb{N}$. Denote $\mathcal{B}_n = \Gamma/\Lambda_n = \{x_1, \ldots, x_k\}$, where $k = [\Gamma : \Lambda_n]$. Write $\alpha_n : \Gamma \to U(\mathbb{C}[\mathcal{B}_n])$ for the permutation representation produced by the action of $\Gamma$ on $\mathcal{B}_n$ by multiplication from the left. Write $\mathcal{B}_n^0 = \{x_1, \ldots, x_{k-1}\}$, and let $f_n : \mathbb{C}[\mathcal{B}_n^0] \to \mathbb{C}[\mathcal{B}_n]$ be the inclusion map. By Proposition (3.3)(ii), there is a function $\rho_n : S \to U(\mathbb{C}[\mathcal{B}_n^0])$, such that:

$$\forall s \in S \quad \rho_n(s) \text{ permutes } \mathcal{B}_n^0 \quad (3.5)$$

$$\forall w \in \ker(\pi) \quad \|\rho_n(w) - I\|_{\text{F}} \leq 4^{\|w\|} \quad (3.6)$$

$$\forall s \in S \quad \|\alpha_n(s^{-1}) \circ f_n \circ \rho_n(s) - f_n\|_{\text{F}} \leq \sqrt{2} + 1 \quad (3.7)$$

Inequality (3.6) above is equivalent to

$$\forall w \in \ker(\pi) \quad \|\rho_n(w) - I\|_{\text{HS}} \leq \frac{4^{\|w\|}}{[\mathcal{B}_n^0]^{1/2}}. \quad (3.8)$$

From (3.5) and (3.8), we see that $(\rho_n)_{n=1}^{\infty}$ is a P-stability-challenge for $\Gamma$. Since $\Gamma$ is P-stable, there is a solution $(\tilde{\rho}_n)_{n=1}^{\infty}$ for $(\rho_n)_{n=1}^{\infty}$. We may extend each $\tilde{\rho}_n : S \to U(\mathbb{C}[\mathcal{B}_n^0])$ to a representation $\tilde{\rho}_n : \Gamma \to U(\mathbb{C}[\mathcal{B}_n^0])$. Now, for each $s \in S$,

$$\frac{1}{\|f_n\|_F} \cdot \|\alpha_n(s^{-1}) \circ f_n \circ \tilde{\rho}_n(s) - f_n\|_F$$

$$\leq [\mathcal{B}_n^0]^{-1/2} \cdot \|\alpha_n(s^{-1}) \circ f_n \circ \rho_n(s) - f_n\|_F$$

$$+ [\mathcal{B}_n^0]^{-1/2} \cdot \|\alpha_n(s^{-1}) \circ f_n \circ (\tilde{\rho}_n(s) - \rho_n(s))\|_F$$

$$\leq [\mathcal{B}_n^0]^{-1/2} \cdot (\sqrt{2} + 1) + \|\tilde{\rho}_n(s) - \rho_n(s)\|_{\text{HS}},$$

where the last inequality follows from (3.7) and the fact that $\alpha_n(s^{-1})$ and $f_n$ are unitary operators. So, since $(\tilde{\rho}_n)_{n=1}^{\infty}$ is a solution for $(\rho_n)_{n=1}^{\infty}$, we deduce that

$$\frac{1}{\|f_n\|_F} \cdot \|\alpha_n(s^{-1}) \circ f_n \circ \tilde{\rho}_n(s) - f_n\|_F \xrightarrow{n \to \infty} 0.$$ 

Thus, by Lemma 3.5, there are morphisms of representations $(h_n)_{n=1}^{\infty}$, $h_n : \mathbb{C}[\mathcal{B}_n^0] \to \mathbb{C}[\mathcal{B}_n^0]$, such that

$$\frac{1}{\|f_n\|_F} \cdot \|f_n - h_n\|_F \to 0.$$
in contradiction with Proposition 3.4(ii). This finishes the proof under the assumption that \( T \) is \( P \)-stable.

Now, assume instead that \( T \) is HS-stable. Arguing as above, using Proposition 3.3(i) and Proposition 3.4(i) instead of Proposition 3.3(ii) and Proposition 3.4(ii), respectively, we deduce that \( T \) has only finitely many irreducible finite-dimensional representations. Assume, for the sake of contradiction, that \( T \) has infinitely many subgroups of finite-index, and let \( \{ \Lambda_n \}_{n=1}^{\infty} \) be a sequence of such subgroups, for which \( T : \Lambda_n \to \infty \). Write \( \Lambda_0 = T \). We may assume, without loss of generality, that the subgroups \( \{ \Lambda_n \}_{n=1}^{\infty} \) are normal in \( T \), and that \( \Lambda_n \not\subseteq \Lambda_{n-1} \) for all \( n \in \mathbb{N} \). Fix \( n \in \mathbb{N} \). Take \( \gamma_n \in \Lambda_{n-1} \setminus \Lambda_n \). The regular representation \( C[\Gamma/\Lambda_n] \) of the finite group \( \Gamma/\Lambda_n \) is faithful, and it decomposes as a direct sum of irreducible representations of \( T \). So, for at least one of these irreducible representations, call it \( V_n \), \( \gamma_n \) does not act on \( V_n \) as the identity. But \( \gamma_n \in \Lambda_{n-1} \), and so it acts as the identity on \( V_i \) for each \( 1 \leq i < n \). Therefore, we produced a sequence \( \{ V_n \}_{n=1}^{\infty} \) of pairwise non-isomorphic finite-dimensional irreducible representations of \( T \), a contradiction. \( \square \)

4. Remarks and suggestions for further research

4.1. Stability of hyperbolic groups. It is clear that free groups are both \( P \)-stable and HS-stable. On the other hand, lattices in the rank one simple Lie groups \( \text{Sp}(n,1) (n \geq 2) \) have Property (T) (see [11] or [3]), and so they are neither \( P \)-stable nor HS-stable by Theorem 1.3. However both free groups and the cocompact lattices among the aforementioned lattices are hyperbolic [8]. So, hyperbolicity by itself does not suffice to determine whether a group is stable. An interesting question is whether surface groups of genus \( g \geq 2 \) are \( P \)-stable or HS-stable.

4.2. Property (\( \tau \)) and Property (T;FD). The arguments presented in Section 3 do not require the full strength of Property (T) in the sense that they only go through finite-dimensional unitary representations of \( T \). Focusing on \( P \)-stability (rather than HS-stability), even more is true: only finite-dimensional unitary representations that factor through finite quotients of \( T \) are relevant. Recall that the finitely-generated group \( T \) has Property (\( \tau \)) if it has a pair \((Q,\kappa)\), \( |Q| < \infty \), and \( \kappa > 0 \), such that Condition 3.3 from the definition of Property (T) holds for all finite-dimensional representations of \( T \) that factor through finite quotients, and it has Property (T;FD) (see [15]) if the same holds for all finite-dimensional representations of \( T \). We get the following more general result:

**Theorem 4.1.**

i) If \( T \) has Property (\( \tau \)) and is \( P \)-stable, then \( T \) has only finitely many finite-index subgroups. Hence, a sofic group with Property (\( \tau \)) is not \( P \)-stable, unless it is finite.
ii) If $\Gamma$ has Property $(T; FD)$ and is HS-stable, then $\Gamma$ has only finitely many finite-index subgroups. Hence, a hyperlinear group with Property $(T; FD)$ is not HS-stable, unless it is finite.

**Warning:** The weaker notion of Property $(\tau)$ with respect to a family of finite-index subgroups $\{N_i\}_{i=1}^\infty$ does not suffice to deduce the conclusion of Theorem 4.1(i), even if the family is separating (i.e. $\cap N_i = \{1\}$). For example, the group $\Gamma = \langle \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \rangle$ is free, so it is clearly P-stable, and has Property $(\tau)$ with respect to the family of congruence subgroups $\{\Gamma \cap \text{Ker} (\text{SL}_2(\mathbb{Z}) \to \text{SL}_2(\mathbb{Z}/m\mathbb{Z}))\}_{m=1}^\infty$ (it has the so called Selberg property \cite{14}).

Note that it is easy to see that a free product of stable groups is stable (for all versions of stability). An interesting corollary of Theorem 4.1(i) is that a free product of two P-stable groups, amalgamated along a finite-index subgroup, is not necessarily P-stable. Indeed, for $p$ an odd prime, look at

$$\Gamma(2) = \text{Ker} \left( \text{SL}_2 \left( \mathbb{Z} \left[ \frac{1}{p} \right] \right) \to \text{SL}_2 \left( \mathbb{Z} \left[ \frac{1}{p} \right] / 2\mathbb{Z} \left[ \frac{1}{p} \right] \right) \right).$$

This is an amalgamated product of two free groups along a finite-index subgroup (see \cite{17}, Chapter II, Section 1.4, Corollary 2), and, as with the example of $\text{SL}_2(\mathbb{Z})$ above, it has the Selberg property (\cite{14}). However, unlike $\text{SL}_2(\mathbb{Z})$, the group $\text{SL}_2 \left( \mathbb{Z} \left[ \frac{1}{p} \right] / 2\mathbb{Z} \left[ \frac{1}{p} \right] \right)$ satisfies the congruence subgroup property (\cite{16}), and so from the Selberg property we deduce that it has Property $(\tau)$, and so the same is true for our amalgamated product, hence the latter is not P-stable.

### 4.3. Relative Property $(T)$

Recall that the group $\Gamma$, generated by the finite set $S$, has relative Property $(T)$ with respect to a subgroup $N \leq \Gamma$ if there is $\kappa > 0$, such that every unitary representation $(\mathcal{H}, \rho)$ of $\Gamma$ that has an $(S, \kappa)$-invariant vector $v \in \mathcal{H}$, also has an $N$-invariant non-zero vector. If $\Gamma$ has relative Property $(T)$ relative to a subgroup $N \leq \Gamma$, rather than Property $(T)$, we may deduce a weak form of Lemma 3.5, where the constructed morphism $h$ is merely a morphism of $N$-representations. Using this variant of the lemma, and arguing as in the proof of Theorem 1.4 we deduce the following:

**Theorem 4.2.** Assume that $\Gamma$ is P-stable and has relative Property $(T)$ with respect to a subgroup $N \leq \Gamma$. Then, the collection

$$\{L \leq \Gamma \mid [\Gamma : L] < \infty \text{ and } \Gamma = NL\}$$

is finite.

We exhibit an application of Theorem 4.2. The group $\text{SL}_2(\mathbb{Z})$ acts on $\mathbb{Z}^2$ by matrix multiplication, giving rise to a semi-direct product
$\mathbb{Z}^2 \rtimes \text{SL}_2(\mathbb{Z})$. It is well-known that this semi-direct product has relative Property (T) with respect to the subgroup $\mathbb{Z}^2 \rtimes \{1\}$. So, the infinite collection $\left\{ (n\mathbb{Z}^2) \rtimes \text{SL}_2(\mathbb{Z}) \right\}_{n=1}^{\infty}$ of finite-index subgroups, exhibits the non-P-stability of $\mathbb{Z}^2 \rtimes \text{SL}_2(\mathbb{Z})$. More interestingly, letting $H$ be the finite-index subgroup of $\text{SL}_2(\mathbb{Z})$ generated by $\left( \begin{array}{cc} 1 & 2 \\ 0 & 1 \end{array} \right)$ and $\left( \begin{array}{cc} 1 & 0 \\ 2 & 1 \end{array} \right)$, we may deduce in the same manner that $\mathbb{Z}^2 \rtimes H$ is not P-stable as well. Note that since $\mathbb{Z}^2$ is abelian and $H$ is free, we know that both are stable [1]. We conclude:

**Corollary 4.3.** A semidirect product of finitely-generated (even finitely-presented) P-stable groups is not necessarily P-stable.

### 4.4. A variant of P-stability.

Finally, let us make a remark and a suggestion for further research. All of our proofs of non-P-stability start with a true action of $\Gamma$ on a set $X$ with $n$ points, which is then deformed a bit into an almost action on a set with $n-1$ points. For $\Gamma$ to be P-stable, this almost action must be close to an actual action on $n-1$ points. We proved that it is never the case if $\Gamma$ has Property (T) and the action $\Gamma \curlyarrowright X$ is transitive. However, the action on $n-1$ points is clearly close to a true action on a set with $n$ points since we started with such an action.

One may suggest a notion of “flexible P-stability”, which would require that every almost action can be corrected to an action by allowing to add a few more points to the set before correcting it. By “few more points”, one could mean $o(n)$ points, but allowing the addition of $O(n)$ points is interesting as well. Note that the observation of [7], claiming that a sofic P-stable group is residually finite, is valid even if we relax the definition of P-stability to such $O(n)$-flexible P-stability. This suggests a path towards finding a non-sofic group by finding a non-residually-finite group which is $O(n)$-flexibly P-stable. It is possible that the examples treated in this paper and the non-residually-finite groups in [5], while not P-stable, are still $O(n)$-flexibly P-stable. In this case, they would provide the desired example of non-sofic groups. In fact, combined with what was shown by [1], it would suffice to prove $O(n)$-P-flexible weak-stability, where "weak-stability" stands for stability only with respect to challenges that come from sofic approximations.

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O.B., Hebrew University, Israel
E-mail address: oren.becker@mail.huji.ac.il

A.L., Hebrew University, Israel
E-mail address: alex.lubotzky@mail.huji.ac.il