Social hierarchy promotes the cooperation prevalence

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Social hierarchy is important that can not be ignored in human socioeconomic activities and in the animal world. Here we incorporate this factor into the evolutionary game to see what impact it could have on the cooperation outcome. The probabilistic strategy adoption between two players is then not only determined by their payoffs, but also by their hierarchy difference — players in the high rank are more likely to reproduce their strategies than the peers in the low rank. Through simulating the evolution of Prisoners’ dilemma game with three different hierarchical distributions: uniform, exponential and power-law, we find that the levels of cooperation are enhanced in all cases, and the enhancement is optimal in the uniform case. The enhancement is due to the fact that the presence of hierarchy facilitates the formation of cooperation clusters with high-rank players acting as the nucleation cores. We also study a two-hierarchy model, where similar cooperation promotion is revealed and some theoretical analysis are provided. Our finding implies that social hierarchy may be not that harmful to cooperation than previously thought.

INTRODUCTION

From Confucius to Kant, cooperation is a central concern in our society not only for its humanity value but also its crucial role in the development of economy, technology, and science etc [1]. Cooperation is also ubiquitous in the natural world, like those prosocial species, such as ants and bees, whose survivals rely on their altruistic behaviors [2]. But according to the evolution theory of Darwinism, individuals are inherently selfish therefore they are only willing to change their strategies or behaviors to maximize their profits. Because defection generally brings more profits as least in the short term, no one is willing to cooperate and the world will be dominated by defectors. How the cooperation emerges and is maintained in selfish population is then a vitally important question that has attracted the attention of researchers in different fields in the past several decades.

In this regard, the evolutionary game theory provides a proper framework to explore the potential mechanisms behind cooperation [3, 4], and some prototypical game models are often adopted for this aim, such as the prisoner’s dilemma [5], the snowdrift game [6, 7] and the public good game [8] etc. The endeavor has successfully revealed quite a few mechanisms, such as direct reciprocity [9], indirect reciprocity [10–12], kin selection [13], group selection, spatial reciprocity [14, 15], voluntary participation [16–18] and punishment [19, 20]. More recent advance comes from the methodologies within machine learning to compare what outcome difference could be if the games are played by AI algorithms [21–23]. Note that, similar to the spirit of identical particle assumption in statistical mechanics, individuals in all these models are supposed to be indistinguishable and thus of equal position in the strategy updating.

However, a consensus is that our society is more often composed by heterogeneous individuals where they differ in many aspects such as their occupations, social statuses, cultural backgrounds etc [24] that potentially affect their decision- makings. The statement is also true in nature, like the reported examples homing pigeons [25] and white storks [26], where a small fraction of the group take the lead of decision-making regarding their movement direction and the social hierarchy potentially affects their swarming transitions [27]. Recently, researchers start to consider the impact of non-identical individuality and find that the population heterogeneity generally leads to better cooperation outcomes [28–37]. In [28], Perc et al study the impact of social diversity on the cooperation in the form of diversified payoff structures, and find that the levels of cooperation are improved in all cases. Social diversity is also able to promote cooperation in spatial multigames where two different games are played simultaneously [29]. In [30], Zhang et al unveil that the popularity-driven selection facilitates the formation of cooperator clusters, thus the promotion of cooperation. A coevolutionary model shows that a moderate popularity works best for boosting cooperation [31]. The diversity of teaching or learning ability is also found to lift the level of cooperation generally [32–35]. In addition, positive correlation between strategy persistence and teaching ability also provides an approach to promote cooperation [37]. In [36], hierarchically structured lattices are also studied, where players on different layers exhibit different levels of cooperation.

A more closely related question is to address how does the social hierarchy influence cooperation. Some researchers find that the social hierarchy is detrimental for the evolution of cooperative behaviors [38], but an experiment work shows that collaborative hierarchy maintains cooperation in asymmetric games [39], leading to a contradictory conclusion. Given the ubiquitous presence of hierarchy in all aspects of society and nature, it’s crucial to clarify the impact of social hierarchy on the cooperation prevalence.

Here, we introduce and investigate an evolutionary game model of a hierarchical population, where each player is designated a social rank according to some distributions. For players in the high social rank, their strategies are more likely
increases, the level of cooperation continuously improves. Other parameters: \( L \) and \( K \). The model returns to the original PD game model when \( \alpha = 0 \). As \( \alpha \) increases, the level of cooperation continuously improves. Other parameters: \( L = 1024 \), \( K = 0.1 \).

to reproduce in their neighborhood than players in the low rank, even if their payoffs are close. Specifically, we study the evolution of prisoner’s dilemma game in three different hierarchy distributions, and we find that the levels of cooperation are significantly improved in all cases. The promotion is due to the hierarchy-induced spatial structures that effectively protect the survival of cooperators. We also develop a simplified model to give some analytic treatments.

**MODEL**

In our model, the system is composed of \( N \) individuals that are located on an \( L \times L \) square lattice with a periodic boundary condition. Each player is initially set either as a cooperator or defector with equal probability. They play pairwise game defined by three scenarios. Mutual cooperation brings each the reward \( R \) yet the temptation \( T \) for the defector. Strict prisoner’s dilemma (PD) requires \( T > R > P > S \), but here we adopt the common practice with parameterization \( R = 1 \), \( P = S = 0 \), and \( T = b \), which is known as the weak PD game. To incorporate the hierarchy, each player \( i \) is designated a random \( h_i \in [0, 1] \) at beginning, drawn from some distributions. Players with higher value of \( h \) are supposed to be in higher rank in the social hierarchy.

In an elementary step of the standard Monte Carlo (MC) simulation, the procedure is as follows. First, an individual \( i \) is randomly chosen and acquires its payoff \( \Pi_i \) by playing the game with all its neighbors \( \Omega_i \) defined by the underlying networks. Next, one of \( i \)'s neighbors \( j \) is selected randomly and also acquires its payoff \( \Pi_j \) by playing the game within its neighborhood \( \Omega_j \). Lastly, player \( i \) adopts the strategy of \( j \) with an imitation probability according to the Fermi rule [5]

\[
W(s_j \rightarrow s_i) = \frac{1}{1 + \exp((\Pi_i - \Pi_j(1 + \alpha \Delta h))/K)},
\]

where \( \Delta h = h_j - h_i \in (-1, 1) \) denotes the hierarchical difference between \( i \) and \( j \). \( \alpha \in [0, 1] \) is the hierarchal coefficient determining the degree to which the strategy adoption in the imitation process is influenced by the hierarchy. Obviously, the case of \( \alpha = 0 \) is recovered to the traditional evolutionary games, where the strategy updating is purely determined by their payoffs, irrespective of other factors. Instead, when \( \alpha > 0 \), the presence of hierarchical difference facilitates the strategy reproduction of those players in the high social rank. \( K \) quantifies the uncertainty in decision-making during the imitation, and is fixed at 0.1 throughout the whole study. A full MC step consists of \( N \) such elementary steps, which means that every player is going to update its strategy once on average.

To be specific, we consider the following three distributions: uniform \((\propto \text{const.})\), exponential \((\propto e^{-\beta h})\), and power-law types \((\propto h^{-1})\), which are drawn in practice as follows:

\[
\begin{align*}
    h &= \chi, \quad \chi \in [0, 1), \\
    h &= \frac{1}{2} \ln\left(\frac{1}{1 - \chi}\right), \quad \chi \in [0, 1 - \frac{1}{e^2}), \\
    h &= \left(\frac{1}{\epsilon}\right)^{\chi^{-1}}, \quad \chi \in [0, 1).
\end{align*}
\]

Here \( \chi \) are uniformly drawn in the given range, and \( h \in (\epsilon, 1) \) with \( \epsilon \to 0 \) for the last implementation. For most numerical experiments, 50 thousand MC steps are run to guarantee that the equilibrium is reached, and then we average the data for another 10 thousand MC steps.

**FIG. 1.** (Color online) Cooperator fraction \( f_c \) as a function of temptation \( b \) for a couple of hierarchy coefficients \( \alpha \) given three different hierarchical distributions: (a) uniform, (b) exponential, and (c) power-law. The model returns to the original PD game model when \( \alpha = 0 \). As \( \alpha \) increases, the level of cooperation continuously improves. Other parameters: \( L = 1024 \), \( K = 0.1 \).
RESULTS

Numerical simulations

To begin with, we first present the impact of hierarchy on the cooperation prevalence in the uniform distribution case. Fig. 1(a) gives the cooperation phase transitions as the function of the temptation \( b \) for a couple of hierarchical coefficients \( \alpha \). It shows that as the impact of hierarchy becomes stronger, the cooperative likelihood is increased. Specifically, the threshold for cooperation outbreak \( b_{c1} \) (below which cooperators start to appear) is continuously increased as \( \alpha \) becomes larger and this shift is the most significant for \( \alpha = 1 \). Meanwhile, the threshold for defector eradication \( b_{c2} \) (below which defectors go extinct) is also increased, whereby full cooperation is possible in those strong hierarchy cases for the given parameter region. For the temptation \( b_{c2} < b < b_{c1} \), the coexistence states of cooperators and defectors are expected.

For comparison, Fig. 1(b) and 1(c) illustrate the phase transitions of two nonuniform distributions, which show qualitatively the same cooperation enhancement. Compared to the case of uniform distribution, the enhancement is lesser in the population with exponential and power-law hierarchical distributions, especially the cooperation promotion is least in the latter scenario, where the defector eradication threshold \( b_{c2} \) is not even present in the shown parameter range. But still the cooperation outbreak thresholds \( b_{c1} \) are shifted to the right and all prevalences are greater than the case without hierarchical impact (\( \alpha = 0 \)).

To more systematically investigate the impact of hierarchy, Fig. 2 provides the phase diagrams in the \( b - \alpha \) parameter space for the three distributions. It confirms that the cooperation is promoted in all cases as the hierarchy coefficient \( \alpha \) becomes larger. In particular, both thresholds \( b_{c1,2} \) are monotonously shifted and they are almost linear functions of the hierarchical coefficient \( \alpha \). These results suggest that the enhancement of cooperation in the hierarchical population is universal, and the uniform distribution of social ranking comparatively yields the optimal promotion of cooperation.

Mechanism analysis

To explore the mechanism underlying the cooperation promotion due to the hierarchy impact, we first provide the time series of \( f_c \) with the uniform distribution for three different \( \alpha \), see Fig. 3. For the given temptation \( b = 1.05 \), the fraction of \( f_c \) gradually decreases as time goes by and becomes extinct when the hierarchical impact is absent (\( \alpha = 0 \)). By contrast, this fraction turns up in the midway after a couple of MC steps when \( \alpha \) increases and stabilizes at some nonzero values; full cooperation is possible when \( \alpha \) further increases for the given \( b \). The reason for the decay of cooperation at the early stage is well-known that well-mixed initial population is more beneficial to defectors that jeopardise the reproduction of cooperation, and the possible survival of cooperators later

FIG. 2. (Color online) Color map encoding the cooperator fraction \( f_c \) on the \( b - \alpha \) parameter plane respectively for three different types of hierarchical distributions: (a) uniform, (b) exponential, and (c) power-law. Other parameters: \( L = 256, K = 0.1 \).

FIG. 3. (Color online) Time evolution of cooperation fraction \( f_c \) in the population with uniform hierarchical distribution. Three hierarchical strengths are shown \( \alpha = 0, 0.5, 1 \). When \( \alpha = 0 \), the case is recovered into the traditional case without hierarchical impact, where cooperation is absent. But as the hierarchical impact is involved, cooperation is possible and even becomes dominating. Parameters: \( L = 1024, K = 0.1 \), and \( b = 1.05 \).
FIG. 4. (Color online) Characteristic spatial patterns for $\alpha = 0, 0.5$, and 1. Red and blue respectively represent cooperators and defectors, and the snapshots are taken at $t = 1, 4, 40, 4000$ from left to right columns. When without hierarchy (a-d, $\alpha = 0$), cooperators gradually die out; for the middle level of hierarchical impact (e-h, $\alpha = 0.5$), cooperators survive and their clusters merge into even larger size; for even stronger impact (i-l, $\alpha = 1$), cooperation is dominating and the population tends to be full cooperation in the long term. Parameters: $L = 128$, $K = 0.1$, and $b = 1.05$.

on is due to the formation of cooperator clusters whereby they support each other and resist against the invasion of defectors at clusters’ boundaries. This suggests that the presence of hierarchy could better enhances the formation of cooperation clusters than the traditional case without the hierarchical impact.

To see this, Fig. 4 shows the corresponding characteristic snapshots to obtain some intuitions. Starting from the same random initial condition, we observe the densities of cooperations are all reduced in the short term (the second column), but afterwards some cooperation clusters are starting to form (the third column). In the upper panels ($\alpha = 0$), however, these clusters shrink and disappear in the end. By contrast in the other two cases ($\alpha = 0.5$ and 1), they grow up after shrinking, and some merge with each other to form even bigger clusters, thus these cooperators are able to survive or even dominating.

To understand how the hierarchy facilitates the formation of cooperation clusters, it’s crucial to investigate the evolution dynamics at the interaction boundaries between the cooperator and defector clusters. By the boundary, it is defined as the time-varying set $B(t)$ of any site with at least one different state in its four nearest neighbors. First, let’s monitor the time evolution of the fraction of boundaries in the population defined as $n^s(t)/N$, shown in Fig. 5 for three typical hierarchical cases ($\alpha = 0.5, 0.75, 1$). The rapid decrease in the first few steps in all cases is due to reduction of cooperators starting from the well-mixed condition, followed by a typical increase due to the formation and growth of cooperation clusters. While the increase saturates in the case of $\alpha = 0.5$, there is a nontrivial peak in the cases of $\alpha = 0.75$ and 1, especially for boundary cooperators. This reason lies in the fact that in these cases, many small cooperator clusters further merge into each other gradually, resulting in bigger ones which then reduce the boundaries as well as cooperators or defectors there.

To study the individual difference caused by the hierarchical rank, we classify all individuals into five subgroups on the basis of the hierarchical labeling as $L_g$ with $g = 1, 2, ..., 5$ if $h_i \in [0, 0.2), [0.2, 0.4), ..., [0.8, 1)$ respectively. Specifically, we monitor the evolution of relative composition with respect to the hierarchy for both cooperators and defectors at boundaries, and the relative composition fractions are defined as

$$f^s_{L_g} = \frac{n^s_{L_g}(t)}{n^s(t)}, s \in \{C, D\}. \quad (3)$$

Here, $n^s_{L_g}(t) = \sum_{i \in B_{L_g}(t)} \delta(s_i(t) - s)$ is the number of players within the state $s$ at the interface belonging to subgroup $L_g$. 
at time $t$, and $n^s(t) = \sum_{g=1}^{5} n^s_{L_g}(t)$ accordingly.

Fig. 6 further shows the relative hierarchical compositions $f^C_{L_g}, f^D_{L_g}$ respectively for the five subgroups. For boundary cooperators, it shows that at the very early stage only the relative fraction of the highest level of cooperators $L_5$ exhibits obvious increase. This observation is understandable because for well-mixed population at this stage defectors are at relatively advantage position over cooperators, but the high rank for those cooperators compensates their disadvantageous competitiveness, therefore they survive better than those in lower subgroups. This explains increasing trend for those higher rank subgroups of cooperators. Interestingly, this trend is reversed as time goes by that cooperators of lower subgroups are dominating at boundaries and the fraction differences become larger as the hierarchical impact becomes stronger. This means that in the long term the cooperation clusters are more likely surrounded by low-rank cooperators; and accordingly those high-rank cooperators are more probably located within the center position of cooperation clusters. This is reasonable because those high-rank cooperators who survive better in the early phase naturally act as the nucleation core for the growth of cooperation clusters at the late stage.

The time evolution of composition fractions for defectors, however, shows some different dynamical features, see the lower row in Fig. 6. The long term evolution shows qualitatively the same property that low-rank defectors dominate at the interface. But this feature evolves at the very beginning of evolution unlike the cooperator case. These means that high-rank defectors are not likely to appear at the interaction interface from the beginning. This difference lies in the fact that high-rank cooperators act as the core of cooperation clusters but those high-rank defectors are more often embedded in a connected defection sea.

To characterize the interaction interface in more details, we also survey the fractions of cooperators in their neighborhood centered around players at boundaries. According to both the state and hierarchy of the center players, we compute the evolution of cooperators fraction in the neighborhood respectively for cooperator and defector being the center player,

$$f^s_{L_g}(C) = \frac{\sum_{i \in B_{L_g}(t)} \delta(s_i - s) \delta(s_j - C)}{\sum_{i \in B_{L_g}(t)} \delta(s_i - s)}, s \in \{C, D\} \tag{4}$$

where the population is also divided into five subgroups. The result is shown in Fig. 7. In all cases, the cooperation fractions first decrease then followed by an increase and saturate in the end, qualitatively the same as the typical time series shown in Fig. 3. The most significant observation is that for cooperators at the boundary higher rank of their social hierarchy convinces more of their neighbors to be cooperative as well, and vice versa, while for defectors the opposite is true that higher ranks lead to much less cooperators in their neighborhood.

Altogether, these observations constitute the following picture: due to the presence of social hierarchy, those high-rank cooperators survive from exploitation starting from random conditions, and they act as the nucleation cores whereby cooperation clusters grow by attracting more and more low rank individuals around; at interaction boundaries high-rank cooperators facilitate the growth of cooperation clusters while high-rank defectors do the opposite. Without social hierarchy, the nucleation process is absent in the cases when the temptation $b$ is large, thus the cooperation cannot be expected.

Finally, to gain some analytical insight into the hierarchy impact, we adopt a further simplified model, where only two hierarchies are assumed within the population (i.e. $h_i \in \{0, 1\}$), and the replicator rule is used. The probability of strategy adoption is as following:

$$W(s_j \to s_i) = \max\{\Pi_i(1 + \alpha \triangle h) - \Pi_i \frac{4b(1 + \alpha)}{4b(1 + \alpha)}, 0\}, \quad \Pi_i \Pi_j \tag{5}$$

where $\Pi_i$ and $\Pi_j$ are the payoffs of $i$ and $j$ as above. In this two-hierarchy model, $\triangle h = h_j - h_i$ only has three values: $0, \pm 1$. When the effective payoff $\Pi_i(1 + \alpha \triangle h) > \Pi_i$, the player $i$ adopts player $j$’s strategy with a nonzero probability, otherwise there is no change in $s_i$. The presence of $4b(1 + \alpha)$ is for the probability normalization. The reason for the adoption
FIG. 6. (Color online) The time series of relative composition fraction at boundaries $f_{L_j}^C(t)$ for three hierarchical strengths $\alpha = 0.5, 0.75, 1$. (a-c) and (d-f) are respectively for cooperators and defectors. In each subplot, the population is divided into five subgroups according to their hierarchy. $L_1, L_2, ..., L_5$ correspond to the lowest to the highest ranks. Other parameters: $L = 1024, K = 0.1$, and $b = 1.05$.

FIG. 7. (Color online) The time series of neighborhood composition fraction at boundaries $\tilde{f}_{L_j}^C(t)$ for three hierarchical strengths $\alpha = 0.5, 0.75, 1$. (a-c) and (d-f) are respectively for cooperators and defectors. Also, the population is divided into five subgroups. Other parameters: $L = 1024, K = 0.1$, and $b = 1.05$. 
The system evolves 510000 MC steps and the data is the average of the last 10000 points. Parameters: $L = 256$, $K = 0.1$. (b) Two microscopic schemes for the derivation of full cooperation conditions: (Left) a low rank defector is surrounded by cooperators with one of them being of a high rank; (Right) a cluster of defectors encounter a cluster of cooperators, where the focal defector/cooperator is of high/low rank. White and black color indicate the social rank being 0 and 1, respectively, and grey sites could be in either rank.

of the replicator rule is because it is more readily for analytical treatment than the Fermi rule.

Fig. 8(a) shows the results of numerical experiments with this simplified model, which exhibit qualitatively the same behaviors that the presence of hierarchy is able to enhance cooperation prevalence. In particular, the defector eradication threshold $b_{\text{eq}}$ is also shown to be a linear function of the hierarchical parameter $\alpha$. Though new complexities are revealed that an upper threshold of full cooperation arises. Though new complexities are revealed.

To have a stable full cooperation state, one can consider an extreme case where a single defector as a perturbation is surrounded by a group of cooperators, and find out under what condition this defector is going to die or at least there is a possibility for it to be invaded by cooperators of any hierarchy, and thus reaching absorbing state of full cooperation is possible. When this defector is of low rank ($h_D = 0$), a necessary condition is that if one of its cooperators neighbors is of high rank ($h_C = 1$) and the state transfer probability requires $W(C \rightarrow D) > 0$ (shown in the left panel of Fig. 8(b)). This scheme corresponds to the loosest scenario for defector extinction. The effective payoffs are $3(1 + \alpha)$ and $4b$ respectively for the focal cooperator and defector. This leads to the following inequality

$$\alpha > 4b/3 - 1. \quad (6)$$

A tough situation occurs when the defector is of high rank ($h_D = 1$). Since its effective payoff is higher than its any cooperator neighbor, it will convince some of its neighbors to be defectors after a few steps. To become full cooperation, this defector cluster has to be invadable. The most difficult scenario in this case is shown in the right panel of Fig. 8(b), and also $W(C \rightarrow D) > 0$ is required. Here the effective payoffs of the two focal players are respectively $3(1 - \alpha)$ and $b$ for C and D, and we have

$$\alpha < 1 - b/3. \quad (7)$$

The equations of the inequality (6) and (7) constitute the boundaries of the full cooperation region, which are well matched by numerical results, see the black lines in Fig. 8(a). But since the analytic boundaries are derived from the necessary conditions, the two boundaries only encircle the full cooperation region, cannot reproduce its exact boundary.

**SUMMARY**

In summary, we aim to address how the social hierarchy could affect the cooperation outcome in the population. In the numerical experiments, players in high social rank are more likely to reproduce their strategies than those in low rank even if they have similar payoffs. By using prisoner’s dilemma with three different hierarchical distributions, we find that the social hierarchy boosts cooperation in all cases, which is counterintuitive because previously social hierarchy was thought to be detrimental for cooperative behaviors [38]. The mechanism for cooperation promotion lies in the fact that the hierarchy in the population facilitates the formation of cooperation clusters that effectively protect the evasion of defection. A further simplified model provides some analytical insight into the processes. Our findings may provide an explanation of the ubiquitousness of social hierarchies and implies that introducing some degree of hierarchy into the population seems an optional strategy for institutional design to boost cooperation. Besides, our work calls for behavioral experiments for further confirm.

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