Computation of $\beta'(g_c)$ at $O(1/N^2)$ in the $O(N)$ Gross Neveu model in arbitrary dimensions.

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Abstract. By using the corrections to the asymptotic scaling forms of the fields of the $O(N)$ Gross Neveu model to solve the dressed skeleton Schwinger Dyson equations, we deduce the critical exponent corresponding to the $\beta$-function of the model at $O(1/N^2)$. 
1 Introduction.

One of the persistent problems of quantum field theories is a lack of total knowledge of the renormalization group functions, such as the $\beta$-function, which are important for having a precise picture of the quantum structure of a theory. For certain models the functions can be computed to three or four orders as a power series in a perturbative coupling constant which is assumed to be small. However, in explicit calculations one has to work appreciably harder to gain new information which is partly due to the increased number of Feynman graphs one has to analyse within some renormalization scheme such as $\overline{\text{MS}}$, even in theories with the simplest of interactions. It is therefore important to develop different techniques to give an alternative picture of the perturbative series. One method which achieves this is the large $N$ expansion for those theories where one has an $N$-tuplet of fundamental fields. Then $1/N$ is a small quantity for $N$ large and this can be used as an alternative expansion parameter. Whilst a conventional leading order analysis is relatively straightforward to carry out for most theories it turns out that it is not useful for going to subsequent orders in $1/N$. To obviate these difficulties methods were developed for the $O(N)$ $\sigma$ model which were successful in solving that model at $O(1/N^2)$, [1, 2]. In particular the method uses a different approach to the conventional renormalization of the large $N$ expansion in that one solves the field theory precisely at its $d$-dimensional critical point, ie the non-trivial zero of the $\beta$-function, by solving for the critical exponents. Performing the analysis at the fixed point of the renormalization group means that there are several simplifying features. First, the theory is finite. Second, since $\beta(g)$ is zero the theory has a conformal symmetry and the fields are massless. This has two consequences, one of which is that the critical exponents of various Green’s functions can be determined order by order in $1/N$ in arbitrary dimensions, [1, 2]. Second, the masslessness of the fields simplifies the Feynman integrals which occur and allows one to compute the graphs in arbitrary dimensions, which are otherwise intractable in the conventional (massive) large $N$ renormalization. By solving for the exponents in this fashion one can then relate the results through an analysis of the renormalization group equation at criticality to the critical renormalization group functions. (See, for example, [3].) Hence, one gains, albeit by a seemingly indirect method, information on the perturbation series of the theory to all orders in the coupling at the order in $1/N$ one is interested in via the $\epsilon$-expansion of the exponent. Clearly, this has important implications for gaining a new insight into the renormalization.
group functions at large orders of the coupling as well as allowing one to check the series with explicit calculations at low orders. Furthermore, since one calculates in arbitrary dimensions a three dimensional result will always be determined simultaneously.

Since the earlier work of [1, 2] the method has been extended to models with fermions, [4, 5], supersymmetry [6, 7] and theories with gauge fields [8-11]. In this paper we present the detailed evaluation of the $\beta$-function exponent for the four-fermi theory or the $O(N)$ Gross Neveu model, [12]. The motivation for such a calculation is that first of all knowledge of $2\lambda = -\beta'(g_c)$ at $O(1/N^2)$ will mean that the field theory will be solved completely to this order. In [4] the exponent $\eta$, which is the fermion anomalous dimension, was calculated at $O(1/N^2)$ and more recently the vertex or $\bar{\psi}\psi$ anomalous dimension was also computed to the same order, [5]. Together with $\lambda$ one can deduce the remaining thermodynamic exponents for this model through the hyperscaling laws discussed in [13]. Secondly, the computation of $\lambda$-type exponents for theories with fermions as their fundamental field is not as straightforward as the case where one deals with purely bosonic fields. As was noted in [10] there is a subtle reordering of the graphs in the formalism and it is important to have a complete understanding of this feature if one is to apply similar methods to deduce results in physical gauge theories. Also to a lesser extent the methods which we had to develop to solve the current problem, which essentially is the evaluation of massless four loop Feynman diagrams will prove to be extremely useful in other contexts. Finally, we are interested in going well beyond the leading order in the three dimensional Gross Neveu model to compare estimates of various exponents from our analytic work with Monte Carlo simulations currently being carried out, [14]. The leading order results are not precise enough to be able to compare with the relatively low values of $N$ which are being simulated and the $O(1/N^2)$ results therefore must be computed.

The paper is organised as follows. In section 2, we introduce our notation and review the leading order formalism used to compute the exponent $\lambda$ at $O(1/N)$ which will serve as the foundation for the $O(1/N^2)$ corrections. This formal extension is discussed in section 3 where we derive finite consistency equations and explain the need to compute some three and four loop Feynman graphs. The explicit evaluation of these is discussed in sections 4 and 5 whilst the $O(1/N^2)$ corrections to a 2-loop integral which appears at $O(1/N)$ are derived in section 6. We conclude our calculation in section 7 by giving an arbitrary dimensional expression for $\lambda$ at $O(1/N^2)$ and discuss the numerical predictions deduced from it for the three dimensional model.
2 Preliminaries.

The theory we consider involves self interacting fermions $\psi^i$, $1 \leq i \leq N$, where $1/N$ will be the expansion parameter for $N$ large. One can formulate the lagrangian either by using the explicit four point interaction or by introducing a bosonic auxiliary field, $\sigma$, which is the version we use. The quantum theory of both are equivalent. Thus we take, [12],

$$L = \frac{1}{2} \bar{\psi} \frac{\partial}{\partial \psi} + \frac{1}{2} \sigma \bar{\psi} \psi - \frac{\sigma^2}{2g^2}$$  \hspace{1cm} (2.1)

where $g$ is the perturbative coupling constant which is dimensionless in two dimensions. The aim will be to calculate the $O(1/N^2)$ corrections to the $\beta$-function and we note that the three loop structure of this has already been calculated perturbatively in dimensional regularization using the $\overline{\text{MS}}$ scheme, [13, 16, 17, 18], as

$$\beta(g) = (d-2)g - (N-2)g^2 + (N-2)g^3 + \frac{1}{4}(N-2)(N-7)g^4$$  \hspace{1cm} (2.2)

where the coupling constant in (2.2) is related to that of (2.1) by a factor $2\pi$ which we omit here since it will play a totally passive role in the rest of the discussion. It is important to note that in carrying out perturbative calculations with dimensional regularization that (2.2) is what one determines as the $\beta$-function in $d$-dimensions prior to setting $d = 2$ to obtain the renormalization group functions in the original dimension. There, of course, the theory is asymptotically free. However, the $d$-dimensional $\beta$-function (2.2) can be viewed from a different point of view for the large $N$ critical point analysis of the present work. For instance, when $d > 2$ which is the case we will deal with for the rest of the paper, there exists a non-trivial zero of the $\beta$-function at a value $g_c$ given by

$$g_c \sim \frac{\epsilon}{(N-2)}$$  \hspace{1cm} (2.3)

at leading order in large $N$ where the corrections are $O(\epsilon^2)$ and $O(1/(N-2)^2)$ and $d = 2 + \epsilon$. This corresponds to a phase transition which is apparent in the explicit three dimensional work of [19, 20]. Indeed similar approaches were examined in the work of [21] for the $O(N)$ $\sigma$ model. When one is in the neighbourhood of a phase transition, it is well known that physical quantities possess certain power law behaviour. For physical systems the power or critical exponent fundamental to the power law totally characterizes the
properties of the system. In continuum field theory, in the neighbourhood of a phase transition, the Green’s functions also exhibit a power law structure where the critical exponent, by the universality principle, has certain properties. For instance, it depends only on the spacetime dimension and any internal parameters of the underlying field theory. More importantly, though, for our purposes, one can solve the renormalization group equation at criticality and relate several exponents to the fundamental functions of the renormalization group equation, which like (2.2) are ordinarily calculated order by order in perturbation theory. In the alternative critical point approach one can compute the exponents at several orders in $1/N$ which then gives independent information on that critical renormalization group function. Since the location $g_c$ is known at leading order as a function of $\epsilon$ and $1/N$, one can undo the relations between exponent and critical renormalization group function to deduce the coefficients appearing in the perturbative series. Clearly this is a powerful alternative method of computing, say, $\beta$-functions.

For this paper, we extend the earlier work of which was based on the pioneering techniques developed for the bosonic $\sigma$ model on $S^N$, and. The physical ideas behind the method are relatively simple. In the neighbourhood of $g_c$ the model is conformally symmetric and therefore the Green’s functions scale. To analyse the critical theory one postulates the most general structure the Green’s functions can take which is consistent with Lorentz and conformal symmetry. The critical exponents of these scaling forms involves two pieces. One is related in the case of a propagator to the canonical dimension of the field as defined by the fact that the classical action with lagrangian (2.1) is a dimensionless object. Since quantum fluctuations will always alter the canonical dimension a non-zero anomalous dimension is appended to the canonical dimensional and it carries the information relevant for the renormalization group functions.

To be more concrete and to fix notation, for (2.1) the scaling forms of the propagators in coordinate space as $x \to 0$ are,

$$
\psi(x) \sim \frac{A}{(x^2)^{\alpha}}, \quad \sigma(x) \sim \frac{B}{(x^2)^{\beta}}
$$

(2.4)

where

$$
\alpha = \mu + \frac{1}{2} \eta, \quad \beta = 1 - \eta - \chi
$$

(2.5)

and $\eta$ is the fermion anomalous dimension, $\chi$ is the vertex anomalous dimension and $d = 2\mu$ is the spacetime dimension. Both $\eta$ and $\chi$ have been
calculated at $O(1/N^2)$ within the self consistency approach, [4, 5], and are

$$\eta_1 = \frac{2(\mu - 1)^2 \Gamma(2\mu - 1)}{\Gamma(2 - \mu) \Gamma(\mu + 1) \Gamma^2(\mu)}$$  \hspace{1cm} (2.6)

$$\eta_2 = \frac{\eta_1^2}{2(\mu - 1)^2} \left[ \frac{(\mu - 1)^2}{\mu} + 3\mu + 4(\mu - 1) + 2(\mu - 1)(2\mu - 1)\Psi(\mu) \right]$$  \hspace{1cm} (2.7)

$$\chi_1 = \frac{\eta_1}{(\mu - 1)}$$  \hspace{1cm} (2.8)

$$\chi_2 = \frac{\mu \eta_1^2}{(\mu - 1)^2} \left[ 3\mu(\mu - 1)\Theta(\mu) + (2\mu - 1)\Psi(\mu) - \frac{(2\mu - 1)(\mu^2 - \mu - 1)}{(\mu - 1)} \right]$$  \hspace{1cm} (2.9)

where $\Psi(\mu) = \psi(2\mu - 1) - \psi(1) + \psi(2 - \mu) - \psi(\mu)$, $\Theta(\mu) = \psi'(\mu) - \psi'(1)$ and $\psi(\mu)$ is the logarithmic derivative of the $\Gamma$-function. The expression (2.6) was first derived in [22] and later in [23-25] where $\chi_1$ was also given in [23]. Each expression (2.6)-(2.9) agrees with the respective three loop perturbative results of the corresponding renormalization group functions in $d = 2 + \epsilon$ dimensions which were given in [16]. The quantities $A$ and $B$ in (2.4) are the amplitudes of $\psi$ and $\sigma$ respectively and are independent of $x$.

The method to deduce (2.6) and (2.7) is to take the ansätze (2.4) and (2.5) and substitute them into the skeleton Dyson equations with dressed propagators of the 2-point function, [1, 4], which are valid for all values of the coupling including the critical coupling. One subsequently obtains a set of self consistent equations which represent the critical Dyson equations. Their solution fixes $\eta_i$ which is the only unknown at the $i$th order where $\eta = \sum_{i=1}^{\infty} \eta_i/N^i$. Further the vertex anomalous dimension $\chi$ is determined by considering the scaling behaviour of the $\sigma\bar{\psi}\psi$ vertex also in the critical region using a method developed in [5] which extended the earlier work of [26] to $O(1/N^2)$.

To determine the corrections to the $\beta$-function one follows the analogous procedure used in [26] and developed for models with fermion fields in [4, 11]. If we set $2\lambda = -\beta'(g_c)$ then the critical slope of the $\beta$-function can be computed by considering the corrections to the asymptotic scaling, [4], ie

$$\psi(x) \sim \frac{A}{(x^2)^{\lambda}} \left[ 1 + A'(x^2)^{\lambda} \right]$$
\[ \sigma(x) \sim \frac{B}{(x^2)^\beta} \left[ 1 + B'(x^2)^\lambda \right] \]  \hspace{1cm} (2.10)

where \( A' \) and \( B' \) are new amplitudes and \( \lambda = \mu - 1 + \sum_{i=1}^\infty \lambda_i/N^i \) from (2.2). The idea then is to compute \( \lambda \) at \( O(1/N^2) \) in arbitrary dimensions. Once obtained we can use the relation between \( \lambda \) and \( \beta'(g_c) \) to deduce \( \beta(g) \) as a power series in \( g \) at the same approximation in large \( N \), since knowledge of \( \lambda_1 \) allows us to determine the value of \( g_c \) to undo the relation.

We close the section by reviewing the method of [4, 11] to deduce \( \lambda_1 \). As indicated we use the skeleton Dyson equations which are illustrated in fig. 1. To deduce \( \eta_1 \), the equations were truncated by including only the one loop graphs of fig. 1. However, it turns out, as we will recall below, that for \( \lambda_1 \) one has to consider the additional two loop graph of the \( \sigma \) equation. The quantities \( \psi^{-1} \) and \( \sigma^{-1} \) are the respective two point functions and their asymptotic scaling forms have been deduced from (2.4) by inverting in momentum space using the Fourier transform

\[ \frac{1}{(x^2)\alpha} = \frac{a(\alpha)}{2^{2\pi \mu}} \int_k \frac{e^{ikx}}{(k^2)^{\mu-\alpha}} \]  \hspace{1cm} (2.11)

where \( a(\alpha) = \Gamma(\mu - \alpha)/\Gamma(\alpha) \). Thus as \( x \to 0 \), [4],

\[ \psi^{-1}(x) \sim \frac{r(\alpha - 1)^f}{A(x^2)^{2\mu-\alpha+1}} \left[ 1 - A's(\alpha - 1)(x^2)^\lambda \right] \]  \hspace{1cm} (2.12)

\[ \sigma^{-1}(x) \sim \frac{p(\beta)}{B(x^2)^{2\mu-\beta}} \left[ 1 - B'q(\beta)(x^2)^\lambda \right] \]  \hspace{1cm} (2.13)

where

\[ p(\beta) = \frac{a(\beta - \mu)}{\pi^{2\mu}a(\beta)} , \quad r(\alpha) = \frac{\alpha p(\alpha)}{(\mu - \alpha)} \]  \hspace{1cm} (2.14)

\[ q(\beta) = \frac{a(\beta - \mu + \lambda)a(\beta - \lambda)}{a(\beta - \mu)a(\beta)} , \quad s(\alpha) = \frac{\alpha(\alpha - \mu)q(\alpha)}{(\alpha - \mu + \lambda)(\alpha - \lambda)} \]

To represent the graphs of fig. 1 one merely substitutes (2.10), (2.12) and (2.13) for the lines of each of the graphs to obtain

\[ 0 = r(\alpha - 1)[1 - A's(\alpha - 1)(x^2)^\lambda] + z[1 + (A' + B')(x^2)^\lambda] \]  \hspace{1cm} (2.15)

\[ 0 = \frac{p(\beta)}{(x^2)^{2\mu-\beta}} [1 - B'q(\beta)(x^2)^\lambda] + \frac{Nz}{(x^2)^{2\alpha-\Gamma}} [1 + 2A'(x^2)^\lambda] \]  \hspace{1cm} (2.16)
where we have not cancelled the powers of \( x^2 \) in (2.10) and the quantities \( \Pi_1, \Pi_{1A} \) and \( \Pi_{1B} \) are the values of the two loop integral in the respective cases when there are no \( (x^2)^\lambda \) contributions, when \( (x^2)^\lambda \) is included on a \( \psi \) line and when it is included on the \( \sigma \) field. We have also set \( z = A^2B \). As in [2, 4] the terms of (2.15) and (2.16) involving powers of \( (x^2)^\lambda \) decouple from those which do not to leave two sets of consistency equations. One set yields \( \eta_1 \) whilst the second determine \( \lambda_1 \). To achieve this one forms a \( 2 \times 2 \) matrix which has \( A' \) and \( B' \) as the basis vectors and sets its determinant to zero to have a consistent solution. It is the subtlety of taking this determinant which necessitates the inclusion of \( \Pi_{1B} \), [4], in (2.16) as discussed in [4]. Basically when one substitutes the leading order values for \( \alpha \) and \( \beta \) into the basic functions (2.14) one finds

\[
s(\alpha - 1) = O(N) \quad , \quad r(\alpha - 1) = O \left( \frac{1}{N} \right) \quad , \quad q(\beta) = O \left( \frac{1}{N} \right) \quad (2.17)
\]

Thus analysing the leading order \( N \) dependence of each of the elements of the matrix one finds that the contribution from the terms in \( \sigma^{-1} \) involving \( B' \) are of the same order as the (finite) two loop graph \( \Pi_{1B} \). Thus it cannot be neglected and we note that explicit evaluation gave

\[
\Pi_{1B} = \frac{2\pi^{2\mu}}{(\mu - 1)^2 \Gamma^2(\mu)} \quad (2.18)
\]

Thus substituting into the equation

\[
\det \left( \begin{array}{cc}
- r(\alpha - 1)s(\alpha - 1) & z \\
2z & - \frac{\bar{\rho}(\beta)q(\beta)}{N} - \frac{\pi^2}{2} \Pi_{1B}
\end{array} \right) = 0 \quad (2.19)
\]

one deduces

\[
\lambda_1 = - (2\mu - 1)\eta_1 \quad (2.20)
\]

as was recorded in [4]. This completes our review of the previous work in this area and lays the foundation for the subsequent higher order calculations.

### 3 Master equation.

In this section, we derive the formal master equation whose solution will yield \( \lambda_2 \). As already indicated in the previous section this involves truncating the Dyson equations at the next order and including the appropriate corrections. For the moment we concentrate on the equation for \( \psi \) as it has
a simpler structure compared to (2.16). The additional $O(1/N^2)$ correction we consider is illustrated in fig. 2 and we denote it by $\Sigma$. Including it in (2.15) we have

$$0 = \frac{r(\alpha-1)}{(x^2)^{2\mu-\alpha+1}}[1 - A's(\alpha - 1)(x^2)^\lambda] + \frac{zm^2}{(x^2)^{\alpha+\beta-\Delta}}[1 + (A' + B')(x^2)^\lambda]$$

$$+ \frac{z^2}{(x^2)^{3\alpha+2\beta-2\mu-1-2\Delta}}[\Sigma + (A'\Sigma_A + B'\Sigma_B)(x^2)^\lambda]$$

(3.1)

where the subscript on the corrections $\Sigma_A$ and $\Sigma_B$ correspond to the insertion of $(x^2)^\lambda$ on either the $\psi$ or $\sigma$ lines of the graph of fig. 2 and $\Sigma$, $\Sigma_A$ and $\Sigma_B$ are the values of the respective integrals. There are two graphs making up $\Sigma_B$ due to the presence of two $\sigma$ lines and each give the same contribution. For $\Sigma_A$, there are three graphs, one of which gives a different value from the other two where the insertion is on a line adjacent to the external vertex. In (3.1) we have included the additional quantities $\Delta$ and $m$. The graph $\Sigma$ arises in [4] in the determination of $\eta_2$ and it is in fact infinite which can be seen by the explicit computation using the uniqueness method developed first in [27] and later in [2, 28]. Consequently, one has to introduce a regularization by shifting the exponent of the $\sigma$ field by an infinitesimal quantity $\Delta$, i.e $\beta \rightarrow \beta - \Delta$. To remove the infinities from $\Sigma$, $\Sigma_A$ and $\Sigma_B$ one uses the counterterm available from the leading order one loop graph. Thus formally setting

$$\Sigma = \frac{K}{\Delta} + \Sigma', \quad \Sigma_{A,B} = \frac{K_{A,B}}{\Delta} + \Sigma'_{A,B}$$

(3.2)

in (3.1) and expanding

$$m = 1 + \frac{m_1}{\Delta N} + O\left(\frac{1}{N^2}\right)$$

(3.3)

the divergent terms of (3.1) are set to zero minimally to obtain a finite consistency equation i.e

$$m_1 = - \frac{z_1K_A}{2} = - \frac{z_1K_B}{2}$$

(3.4)

which implies $K_A = K_B$ and this will provide a check on the explicit calculation described later. In order to proceed to the critical region one must exclude the $\ln x^2$ style terms which remain which is achieved by exploiting the freedom in the definition of the vertex anomalous dimension by setting

$$\chi_1 = - \frac{z_1K_A}{2} = - \frac{z_1K_B}{2}$$

(3.5)
Agreement with (2.8) will be another check. This will leave a finite set of equations which are valid as $x \to 0$ which again decouples into one which is relevant for $\eta_2$ and the other for $\lambda_2$, i.e.

$$0 = r(\alpha - 1) + z + z^2\Sigma'$$

(3.6)

from which $z_2$ can be derived and

$$0 = A'[z - r(\alpha - 1)s(\alpha - 1) + z^2\Sigma'_A] + B'[z + z^2\Sigma'_B]$$

(3.7)

However, by analysing the $N$-dependence of each term of the $A'$ coefficient of (3.7) the correction $\Sigma'_A$ is $O(1/N^2)$ with respect to $r(\alpha - 1)s(\alpha - 1)$ and therefore does not need to be computed explicitly since it will contribute to $\lambda_3$ and not $\lambda_2$.

We now turn to the $\sigma$ equation. In the same way we had to consider the higher order two loop graph of fig. 1 to deduce $\lambda_1$, we now have to include the analogous set of graphs for the next order to determine $\lambda_2$. As we are using graphs with dressed propagators it turns out there are only five graphs which arise. These are illustrated in figs 3 and 4 and we have given each a label. The subscript $B$ indicates that we need only consider the graphs where there is an $(x^2)^\lambda$ insertion on the $\sigma$ line. Again the insertions on the $\psi$ lines will be relevant for $\lambda_3$. We have grouped the graphs which are divergent and therefore require regularization by $\Delta$. The origin of the infinity is the same as the vertex infinity which occurs in $\Sigma$. Indeed in fig. 3 each graph corresponds to the usual vertex correction of the two loop graph $\Pi_1$. For the $\lambda$ consistency equation of the $\sigma$ Dyson equation there is an insertion of $(x^2)^\lambda$ in one of the $\sigma$ lines of the graphs. For the case $\Pi_{3B}$, for example, the insertion in one line removes the infinity from that vertex since the presence of the exponent $\lambda$ on the $\sigma$ lines moves the overall exponent of that line from the value which gives infinity. Thus the graph has a simple pole in $\Delta$ which is removed by the same vertex counterterm as (3.4). For $\Pi_{2B}$ one of the insertions on a $\sigma$ line makes the graph finite and we call it $\Pi_{2B2}$ and no regularization is required. The other case, $\Pi_{2B1}$, is divergent but is again rendered finite by (3.4) in the consistency equation. By contrast the graphs of fig. 4 do not involve any divergent vertex subgraphs and when any one of the $\sigma$ lines has an $(x^2)^\lambda$ insertion each graph is completely finite. For notational convenience we define $\Pi_{5B1}$ to be the graph with an insertion on the top $\sigma$ line and $\Pi_{5B2}$ to have an insertion on the central $\sigma$ line. Similarly, we denote the graph of $\Pi_{6B}$ with the bottom $\sigma$ line corrected by $\Pi_{6B1}$ and the other case by $\Pi_{6B2}$. Therefore, there are five distinct finite massless Feynman graphs to evaluate.
We have dealt at length with the higher order graphs which need to be included in the corrections to (2.16). However, there are also $O(1/N^2)$ contributions coming from $\Pi_{1B}$. For, $\lambda_1$, one considers this graph with $\alpha = \mu$, $\beta = 1$ and $\lambda = \mu - 1$. These are only the leading order values of the exponents and since we are dealing with fields with non-zero anomalous dimensions which are $O(1/N)$ these give contributions to $\lambda_2$ when $\Pi_{1B}$ is expanded in powers of $1/N$. Therefore, these ought not to be neglected in deriving the master equation for $\lambda_2$.

Rather than reproduce the analogous renormalization of the $\sigma$ consistency equation, which proceeds along the same straightforward lines as discussed earlier, we obtain the following finite equation which includes the corrections to (2.16)

$$0 = 2zA' - B' \left[ \frac{p(\beta)q(\beta)}{N} + \frac{z^2}{2}\Pi_{1B} + \frac{z^3}{2}\Pi_{B2} \right]$$ (3.8)

where

$$\Pi_{B2} = 2\Pi_{2B1} + 2\Pi_{2B2} + 2\Pi_{3B} + 2\Pi_{4B}$$

$$- z_1[2\Pi_{5B1} + \Pi_{5B2} + 2\Pi_{6B1} + 4\Pi_{6B2}]$$ (3.9)

and the prime is understood on the integrals which are divergent. We have preempted the explicit calculation of later sections by using the fact that $\Pi'_{1A} = 0$ in writing down (3.8) and ignoring the corrections where there is an insertion on the $\psi$ lines of the graphs of figs. 3 and 4 since they are relevant for $\lambda_3$.

This completes the derivation of the formal consistency equations which yield $\lambda_2$. One again sets the determinant of the $2 \times 2$ matrix formed by $A'$ and $B'$ as the basis vectors in (3.7) and (3.8) to zero, and all that remains is the evaluation of $\Pi_{1B}$ and $\Pi_{B2}$.

4 Computation of divergent graphs.

In this section we discuss the computation of the divergent graphs $\Pi_{1A}$, $\Sigma_{1A}$, $\Pi_{2B1}$ and $\Pi_{3B}$. First, though we recall the basic tool we use for computing massless Feynman graphs, which is the uniqueness construction first used in [27] and developed for large $N$ work in [2] and other applications in [28]. The basic rule for a bosonic vertex which we require is illustrated in fig. 5 whilst that for a $\sigma\bar{\psi}\psi$ type vertex is given in fig. 6, [4]. In each case the arbitrary
exponents $\alpha_i$ and $\beta_i$ are constrained to be their uniqueness value, $\sum_i \alpha_i = 2\mu$ and $\sum_i \beta_i = 2\mu + 1$, whence the integral over the internal coordinate space vertex can be completed and is given by the product of propagators on the right side represented by a triangle. The quantity $\nu(\alpha_1, \alpha_2, \alpha_3)$ is defined to be $\pi^\mu \prod_{i=1}^3 a(\alpha_i)$.

It is easy to observe that for the $\sigma \bar{\psi} \psi$ vertex of (2.1) we have $2\alpha + \beta = 2\mu + 1$ at leading order so that in principle the integration rule of fig. 6 can be used. However, if we recall that there is a non-zero regularization $\Delta$ this upsets the uniqueness condition. To proceed with the determination of the divergent graph one instead uses the method of subtractions of [2]. Since we need only the simple pole with respect to $\Delta$ and the finite part of each divergent graph one subtracts from the particular integral another integral which has the same divergence structure but which can be calculated for non-zero $\Delta$. The difference of these two integrals is $\Delta$-finite and therefore can be computed directly by uniqueness. To determine the two loop graphs $\Pi_{1A}$ and $\Sigma_{1A}$ which occur for $\lambda_2$ we refer the interested reader to the elementary definition of the subtracted integrals given in [2] since the treatment of the three loop graphs is new and will be detailed here.

First, we consider the case $\Pi_{3B}$. The subtraction we used is given in fig. 7 where the right vertex subgraph is divergent and the subtraction is given by removing the internal $\psi$ line to join to the external right vertex. This graph can be computed for non-zero $\Delta$ and for $\alpha$, $\beta$ and $\lambda$ given by their leading order values. After integrating two chains one is left with the two loop integral

$$\langle \bar{\mu}, \bar{\nu}, \mu - \Delta, \bar{\mu}, 2 - \mu \rangle$$

(4.1)

where the general definition is given in fig. 8. To compute (4.1) we first make the transformations $\mathcal{N}$ and $\mathcal{V}$ in the notation of [2] which leaves the integral $\langle 1, \bar{\mu}, \bar{\mu}, 1 - \Delta, \mu - 1 \rangle$ which has been computed already in [7]. Thus overall we have

$$\langle \bar{\mu}, \bar{\nu}, \mu - \Delta, \bar{\mu}, 2 - \mu \rangle = \frac{2\pi^{2\mu}}{(\mu - 1)^2 \Gamma^2(\mu)} \times \left[ 1 - \frac{\Delta}{2} \left( 3(\mu - 1)\Theta + \frac{1}{(\mu - 1)} \right) \right]$$

(4.2)

To determine the remaining finite part one uses fig. 6 and as an intermediate step a temporary regularization $\delta$ has to be introduced to perform integrations in both graphs in different orders, as in [2, 4, 6]. Useful in obtaining
the correct answer is the result
\[
(\hat{\mu}, \hat{\mu}, \hat{\mu}, 2 - \mu - \Delta) = \frac{2\pi^{2\mu}}{(\mu - 1)^2 \Gamma^2(\mu)} \times \left[ 1 - \frac{3(\mu - 1)\Delta}{2} \left( \Theta + \frac{1}{(\mu - 1)^2} \right) \right]
\]
computed in a similar fashion to (4.2). After a little algebra the sum of the finite piece and subtracted integral yield
\[
\Pi_{3B} = -\frac{2\pi^{4\mu}}{(\mu - 1)^3 \Gamma^4(\mu)\Delta} \left[ 1 - \frac{\Delta(\mu - 1)}{2} \left( 3\Theta + \frac{1}{(\mu - 1)^2} \right) \right]
\]
(4.4)
To determine \(\Pi_{2B1}\) we have illustrated one of two possible subtractions in fig. 7. The procedure is the same as that for \(\Pi_{3B}\) and also makes use of (4.2) and (4.3). We obtain
\[
\Pi_{2B1} = -\frac{2\pi^{4\mu}}{(\mu - 1)^3 \Gamma^4(\mu)\Delta} \left[ 1 - \frac{\Delta(\mu - 1)}{2} \left( 3\Theta + \frac{2}{(\mu - 1)^2} \right) \right]
\]
(4.5)
Finally, we note
\[
\Sigma_{1B} = -\frac{2\pi^{2\mu}}{(\mu - 1) \Gamma^2(\mu)\Delta}, \quad \Pi_{1A} = \frac{8\pi^{2\mu}}{(\mu - 1)^2 \Gamma^2(\mu)\Delta}
\]
(4.6)
where the finite part is zero in both cases and (4.6) are consistent with our choice of \(\chi_1\) in the renormalization of the previous section.

5 Computation of finite integrals.

The remaining higher order graphs are \(\Delta\)-finite and therefore do not need to be regularized. Moreover, we need only compute them for the leading order values of \(\alpha, \beta\) and \(\lambda\). In other models, the higher order graphs were computed by uniqueness and we used this technique extensively for the calculation though we had to employ some novel methods which deserve discussion. As the integrals we need to determine involve massless propagators one can introduce conformal changes of variables on the internal vertices on integration. For example, one conformal transformation is
\[
x_\mu \rightarrow \frac{x_\mu}{x^2}
\]
(5.1)
from which it follows that
\[ x^2 \rightarrow \frac{1}{x^2} \quad (5.2) \]
which was used in [2] to compute two loop graphs. For an integral with fermions the analogous situation for the propagators is
\[ \not{x} \rightarrow \frac{\not{x}}{x^2} \quad (5.3) \]
from which we deduce that for a fermion propagating from \( x \) to \( y \) where both are internal vertices
\[ (\not{x} - \not{y}) \rightarrow -\frac{\not{x}(\not{x} - \not{y})\not{y}}{x^2y^2} \quad (5.4) \]
This latter transformation provides the starting point for computing each integral as it allows us to carry out several integrations over the internal vertices immediately.

For example, we consider the three loop graph of fig. 4. First, integrating one of the unique vertices and then making a conformal transformation on the subsequent integral, which requires (5.1)-(5.4), one ends up with the first graph of fig. 9. The fermion trace is taken over the endpoints of the propagator with exponent \( \mu \) joining to the vertex with two bosonic propagators and the propagator with exponent 1. The bosonic triangle of this graph is unique and can be replaced by a unique vertex. After integrating several chains one is left with the two loop graph of fig. 9. The techniques to compute two loop graphs are elementary. One performs the fermion trace which yields a series of chains of integrals and another integral proportional to the basic integral \( ChT(1,1) \) defined in [2] as \( ChT(\alpha_1, \alpha_2) = \langle \mu - 1, \alpha_1, \alpha_2, \mu - 1, \mu - 1 \rangle \) and evaluated as

\[
ChT(\alpha_1, \alpha_2) = \frac{\pi^{2\mu} a(2\mu - 2)}{\Gamma(\mu - 1)} \left[ a(\alpha_1)a(2 - \alpha_1) \right] \frac{a(\alpha_2)a(2 - \alpha_2)}{(1 - \alpha_2)(\alpha_1 + \alpha_2 - 2)} + \frac{a(\alpha_2 - 1)a(3 - \alpha_1 - \alpha_2)}{(\alpha_1 - 1)(\alpha_2 - 1)} \right] \quad (5.5)
\]
Thus,
\[
\Pi_{4B} = \frac{\pi^{4\mu}}{(\mu - 1)^2 \Gamma^4(\mu)} \left[ 3\Theta + \frac{1}{(\mu - 1)^2} \right] \quad (5.6)
\]
For the four loop graphs of fig. 4, the conformal transformations (5.1)-
(5.4) are again the starting point. For instance, in $\Pi_{6B1}$ the location of
the exponent $(2 - \mu)$ and the topological structure of the graph means that
elementary integrations quickly result in an integral of the form of fig. 10,
where there is a fermion trace over the two propagators with both exponents
1 and another over the remaining four propagators. In fact the unique vertex
present in fig. 10 means that one only has to compute one two loop integral.
This is again achieved by taking the fermion trace and one finds

$$\Pi_{6B1} = \frac{\pi^{6\mu}a(2\mu - 2)}{(\mu - 1)^5T^3(\mu)} \left[ \frac{1}{(\mu - 1)} - \frac{5}{2(\mu - 1)^2} - \frac{(2\mu - 1)\Psi}{(\mu - 1)} \right] + \frac{\pi^{6\mu}a^2(2\mu - 2)}{(\mu - 1)^8}$$

(5.7)

The remaining three integrals, however, turned out to require a significant
amount of effort. We consider $\Pi_{5B2}$ first. After several integrations one is
left with the first graph of fig. 11 where again there are two fermion traces,
one of which is over the two propagators with exponents 1. Taking this trace
explicitly yields three graphs. Two of these are equivalent after several chain
integrations and are proportional to the basic integral $\text{tr} G(2 - \mu, 1)$ which
was defined and evaluated in [7]. The remaining integral is equal to the
second graph of fig. 11, after performing the transformations $\leftarrow$ on the
right external vertex. Again taking the fermion trace yields two graphs
which are equivalent and elementary to compute as they are proportional
to $ChT(2 - \mu, 1)$ and a purely bosonic integral which is the third graph of
fig. 11. It is completely finite. However, to handle infinities which cancel in
our manipulations of it, we have introduced a temporary regulator $\delta$ in the
graph, which is a standard technique in the evaluation of such complicated
integrals. It is easy to see that each of the top and bottom vertices is
one step from uniqueness and this suggests one uses integration by parts
on the internal vertex which includes the line with exponent $(3 - \mu)$. The
rule for this has been given several times in previous work such as [2, 28].
Consequently, one obtains the difference, after one integration, of two two
loop graphs

$$\langle 3 - \mu - \delta, \mu - 1, \mu - 1 + \delta, 1, \mu - 1 - \delta \rangle - \langle 3 - \mu, \mu - 1, \mu - 1 + \delta, 1, \mu - 1 - \delta \rangle$$

(5.8)

Since the expression is multiplied by $a(\mu - \delta)$ one needs the $O(\delta)$ term of
(5.8). This is achieved by Taylor expanding each integral of (5.7) in powers
of $\delta$ but since the location of $\delta$ is common in several exponents in each term, expanding (5.8) gives

$$\delta \left[ \frac{\partial}{\partial \epsilon} ChT(3 - \mu - \epsilon, 1) \right]_{\epsilon=0}$$  \hfill (5.9)

which can now be easily evaluated. Collecting terms and setting $\delta$ to zero the third graph of fig. 11 is equivalent to

$$\frac{\pi^3 a(2\mu - 2)}{(\mu - 2)^2 T^2(\mu - 1)} \left[ a(3 - \mu)a(\mu - 1) \left( \Phi + \Psi^2 - \frac{1}{2(\mu - 1)^2} \right) \right.$$

$$\left. + \frac{2}{(2\mu - 3) - 2\Psi \left( \frac{1}{2\mu - 3} + \frac{1}{\mu - 2} - \frac{1}{2(\mu - 1)} \right) + \frac{2}{2(\mu - 3)(\mu - 2)} \right]$$

$$- \frac{1}{(2\mu - 3)(\mu - 1) - (\mu - 2)(\mu - 1)} + \frac{2\alpha^2(1)}{(\mu - 2)^2} \right]$$ \hfill (5.10)

where $\Phi(\mu) = \psi'(2\mu - 1) - \psi'(2 - \mu) - \psi'(\mu) + \psi'(1)$. This completes the steps required to compute $\Pi_{5B2}$. The final result is

$$\Pi_{5B2} = -\frac{\pi^6 a(2\mu - 2)}{(\mu - 1)^5 T^3(\mu)} \left[ \frac{(2\mu - 3)}{(\mu - 2)} \left( \Phi + \Psi^2 - \frac{1}{2(\mu - 1)^2} \right) \right.$$

$$\left. - \frac{(3\mu - 4)\Psi}{(\mu - 1)(\mu - 2)^2} + \frac{1}{(\mu - 2)^2} \right] + \frac{2\pi^6 a^2(2\mu - 2)}{(\mu - 1)^6 (\mu - 2)^2}$$ \hfill (5.11)

For $\Pi_{5B1}$ and $\Pi_{6B2}$ a common integral lurks within each and deserves separate treatment. It is illustrated in fig. 12 and after the transformation $\to$ one obtains the second integral of fig. 12 where we have again introduced a temporary regulator $\delta$ in advance of using integration by parts on the left top internal vertex. This yields a set of four integrals, two of which are finite and proportional to the two loop graphs $ChT(1, 3 - \mu)$ and $ChT(1, 1)$ and two which are divergent but they arise in such a way that the $1/\delta$ infinity cancels ie

$$\pi^\mu a(\mu - \delta)a(2\mu - 3)a^2(1)$$

$$\times \left[ a(1 + \delta)ChT(1, \mu - 1 - \delta) - \frac{a(1)a(\mu - 1 + \delta)}{a(\mu - 1)}ChT(1 - \delta, \mu - 1) \right]$$ \hfill (5.12)

The finite part of (5.12) can easily be deduced by Taylor expanding each two loop integral so that overall the sum of contributions to the integral of
fig. 12 means that it is
\[
\frac{(2\mu - 3)\pi^3a^3(1)a^2(2\mu - 2)}{2(\mu - 2)} \left[ 6\Theta + \frac{13}{2(\mu - 1)^2} - \Phi - \Psi^2 \right] - \frac{2}{(2\mu - 3)^2} + \frac{1}{(2\mu - 3)(\mu - 1)} + \frac{\Psi}{(2\mu - 3)(\mu - 1)} \right] \tag{5.13}
\]
This is the hardest part of \(\Pi_{5B1}\) and \(\Pi_{6B2}\) to compute. The remaining pieces of each can easily be reduced to two loop integrals which can be determined by methods we have already discussed. The final result for each is
\[
\Pi_{5B1} = \frac{(2\mu - 3)\pi^6a(2\mu - 2)}{2(\mu - 1)^5(\mu - 2)\Gamma(\mu)} \left[ 6\Theta - \Phi - \Psi^2 + \frac{5}{2(\mu - 1)^2} - \frac{8}{(2\mu - 3)} \right] + \frac{1}{(2\mu - 3)(\mu - 1)} + \frac{2(\mu - 2)\Psi}{(\mu - 1)} + \frac{(\mu - 2)}{(\mu - 1)^2} \right] \tag{5.14}
\]
and
\[
\Pi_{6B2} = -\frac{(2\mu - 3)\pi^6a(2\mu - 2)}{(\mu - 1)^5(\mu - 2)} \left[ \frac{\Phi}{2} + \frac{\Psi^2}{2} - \frac{3(\mu - 1)}{(\mu - 3)} \left( \Theta + \frac{1}{(\mu - 1)^2} \right) - \frac{\Psi}{2(\mu - 3)(\mu - 1)} + \frac{(\mu - 2)}{2(2\mu - 3)(\mu - 1)} + \frac{2}{(\mu - 1)^2} \right] \tag{5.15}
\]

6 Calculation of \(\Pi_{1B}\).

There remains only one integral to evaluate. As we have already recalled one has to include the integral \(\Pi_{1B}\) of fig. 1 in order to obtain the exponent \(\lambda_1\) correctly at leading order. In \([4, 11]\) it was determined at leading order in \(1/N\). However, its \(O(1/N)\) correction needs to be included for \(\lambda_2\) with the anomalous dimensions of \(\alpha, \beta\) and \(\lambda\) now non-zero and we therefore define the \(1/N\) expansion of the integral as
\[
\Pi_{1B} = \Pi_{1B1} + \Pi_{1B2} + O\left(\frac{1}{N^2}\right) \tag{6.1}
\]
and \(\Pi_{1Bi} = O(1)\). The formalism to determine \(\Pi_{1B2}\) has been discussed extensively in \([11]\). Basically, by using recursion relations it is possible to
rewrite the two bosonic integrals which occur in $\Pi_{1B}$ after taking the fermion trace as a sum of graphs which are finite at $\alpha = \mu + \frac{1}{2} \eta$, $\beta = 1 - \eta - \chi$ and $\lambda = \mu - 1 + O(1/N)$. As most of the integrals which then occur have coefficients which are $O(1/N)$ one can write down the contributions of these integrals when the fields have zero anomalous dimensions. For one integral, though, $\langle \alpha - 1, \alpha - 2, \alpha - 1, \alpha - 1, \xi + 2 \rangle$, this is not the case where $\xi = 3 - \mu - \eta - \chi - \lambda'$ and we have set $\lambda = \mu - 1 + \lambda'$. We now detail its evaluation. Manipulation using recursion relations, [11], and using the transformations $\uparrow$ and $\downarrow$ in the notation of [2] gives

$$\langle \alpha - 1, \alpha - 2, \alpha - 1, \alpha - 1, \xi + 2 \rangle = a^2(a - 1)(\xi + 1)(2\mu - 2\alpha - \xi)$$
$$\times \left[ 2(4\alpha + 2\xi - 3\mu - 1)(2\alpha + \xi - \mu - 1) \right.$$
$$\times \langle \alpha - 1, 2\alpha + \xi - \mu - 1, 2\alpha + \xi - \mu - 1, \alpha - 1, 2\mu - 2\alpha - \xi + 1 \rangle$$
$$+ \left( \frac{(3\mu - 4\alpha - \xi + 2)(4\alpha + \xi - 2\mu - 3) - 1}{(\xi + 1)(\mu - \xi - 2)} \right)$$
$$\times \langle \alpha - 1, 2\alpha + \xi - \mu - 1, 2\alpha + \xi - \mu - 1, \alpha - 1, 2\mu - 2\alpha - \xi + 1 \rangle \right]$$

(6.2)

The coefficient of each of the two integrals can easily be expanded in powers of $1/N$, whilst the integrals themselves need to be expanded to the same order. For the latter integral this is achieved by rewriting it as

$$\left[ \langle \mu - 1 + \frac{1}{2} \eta, \mu - 1 + \frac{1}{2} \eta, 1 - \frac{1}{2} \eta, 1 - \frac{1}{2} \eta, 2 - \chi - \lambda \rangle$$
$$- \langle \mu - 1 + \frac{1}{2} \eta, \mu - 1 + \chi + \lambda - \frac{1}{2} \eta, 1 - \frac{1}{2} \eta, 1 - \frac{1}{2} \eta, 2 - \chi - \lambda \rangle \right]$$
$$+ \langle \mu - 1 + \frac{1}{2} \eta, \mu - 1 + \chi + \lambda - \frac{1}{2} \eta, 1 - \frac{1}{2} \eta, 1 - \frac{1}{2} \eta, 2 - \chi - \lambda \rangle$$

(6.3)

which is an exact result where we have first of all made a conformal transformation based on the right external vertex, [2], followed by mapping the integral to momentum space. However, in the second term there is uniqueness at one of the internal vertices of integration which means it can be computed using fig. 5 exactly and then expanded to $O(1/N)$. For the first two terms of (6.3), since we have a difference it is easy to see that this will be $O(1/N)$. In other words we have chosen to subtract off an integral whose leading order value coincides with that of the integral we require, in much the same way as the method of subtractions is used for $\Delta$ divergent graphs.
However, here both graphs have the same structure as in fig. 8. In this linear combination the exponents of all but one propagator coincide. Therefore the first two integrals of (6.3) simply become

\[
\left. \frac{\eta_1 - \chi_1 - \lambda_1}{N} \left[ \frac{\partial}{\partial \epsilon} \langle \mu - 1, \mu - 1 + \epsilon, 1, 1, 2 \rangle \right] \right|_{\epsilon=0} \quad (6.4)
\]

at leading order. The two loop integral can be deduced through a recursion relation which gives a sum of \( ChT(\alpha_1, \alpha_2) \) type integrals. We record

\[
\langle \mu - 1, \mu - 1 + \epsilon, 1, 1, 2 \rangle = \frac{2\pi^{2\mu} a(1)a(2\mu - 2)(2\mu - 3)(\mu - 3)}{(\mu - 2)} \times \left[ 1 + \epsilon \left( \frac{1}{\mu - 2} - \frac{2}{\mu - 3} - 2 \right) \right.
\]

\[
\left. \left[ 3(\mu - 1)(\mu - 2)\epsilon \left( \Theta + \frac{1}{(\mu - 1)^2} \right) \right] \right] \quad (6.5)
\]

Thus

\[
\langle \alpha - 1, 2\alpha + \xi - \mu - 1, 2\alpha + \xi - \mu - 1, \alpha - 1, 2\mu - 2\alpha - \xi + 1 \rangle
\]

\[
= \frac{\pi^{2\mu} (\mu - 3)(2\mu - 3)a(1)a(2\mu - 2)}{(\mu - 2)} \left[ 1 - \eta \frac{(\mu - 3)}{N} \left( \frac{1}{(\mu - 2)} \right) \left( 1 - \Psi + \frac{1}{(2\mu - 3)} - \frac{1}{2(\mu - 1)} \right) \right.
\]

\[
\left. + \frac{(2\mu - 1)(\mu - 2)}{(\mu - 1)(\mu - 2)} \left( 1 + \frac{1}{2(\mu - 3)} \right) + \frac{(2\mu^2 - 4\mu + 1)}{(\mu - 3)} \left( 1 + \frac{1}{(\mu - 1)^2} \right) \right] \quad (6.6)
\]

Following a similar set of steps yields

\[
\langle \alpha - 1, 2\alpha + \xi - \mu - 1, 2\alpha + \xi - \mu, \alpha - 1, 2\mu - 2\alpha - \xi + 1 \rangle
\]

\[
= \frac{\pi^{2\mu} (\mu - 2)(2\mu - 3)a(1)a(2\mu - 2)}{2(\mu - 3)} \left[ 1 + \eta \frac{(2\mu - 1)(\mu - 2)}{(\mu - 1)} \right.
\]

\[
\left. \times \left( \Psi - \frac{1}{(2\mu - 3)} - \frac{1}{2(\mu - 2)} + \frac{1}{2(\mu - 1)} - \frac{3}{2} \right) \right.
\]

\[
\left. - \frac{(2\mu^2 - 4\mu + 1)(\mu - 5)}{2(\mu - 1)(\mu - 3)} - \frac{(\mu - 1)(\mu - 4)(2\mu - 5)}{2(\mu - 2)^2(\mu - 3)} \right] \quad (6.7)
\]

The remaining amount of effort in determining \( \Pi_{1B2} \) lies in simply adding up all the \( O(1/N) \) terms which we believe we have done correctly due to
the frequent cancellation of denominator factors such as \((2\mu - 5), (\mu - 4), (\mu - 3)\) and \((2\mu - 3)\). The cancellation of the latter is reassuring as its potential appearance in the final answer would have indicated unwelcome singular behaviour in three dimensions. Overall we obtained the relatively simple result

\[
\Pi_{1B} = \frac{2\pi^{2\mu}}{(\mu - 1)^2T^2(\mu)} \left[ 1 - \frac{\eta_1}{N} \left( \frac{2}{(\mu - 1)} - \frac{3\mu(2\mu - 3)}{2} \left( \Theta + \frac{1}{(\mu - 1)^2} \right) \right) \right]
\]

(6.8)

where we record that we have used the results

\[
\langle \mu - 3, \mu - 1, \mu - 1, \mu - 1, 5 - \mu \rangle = a(5 - \mu)(\mu - 1)\frac{[a(\alpha) + a(2 - \alpha)]}{(\mu - 1)^2} + a(2 - \mu)\frac{[a(\beta) + a(2 - \beta)]}{(\mu - 1)^2} + \frac{1}{(\mu - 1)^2} \left( \Theta + \frac{1}{(\mu - 1)^2} \right) \]

(6.9)

which were incorrectly given in [11], but were not required for that work.

We close our discussion of the evaluation of our graphs by noting that the correct expressions for the integral \(F(\alpha, \beta)\) defined in [3] is

\[
F(\alpha, \beta) = -\frac{\pi^{2\mu}a(\mu - 1)a(2\mu - 2)}{(\mu - 1)^2}[a(\alpha)a(2 - \alpha) + a(\beta)a(2 - \beta)] + \frac{(\alpha + \beta + \mu - 3)(2 - \alpha - \beta)ChT(\alpha, \beta)}{(\mu - 1)^2} \]

(6.10)

which is valid for all \(\alpha\) and \(\beta\).

7 Discussion.

The previous three sections have been devoted to the evaluation of the integrals which appear in the formal master equation (3.7) and (3.8). It is now a straightforward matter of substituting for the various expressions in each
equation and evaluating the $O(1/N^2)$ correction to the determinant. As an intermediate step we note,

$$
\Pi_{2B} = \frac{\pi^{4\mu}}{(\mu - 1)^2 \Gamma^4(\mu)} \left[ \frac{8(2\mu - 3)}{(\mu - 2)} (\Phi + \Psi^2) - \frac{12(2\mu - 1)\Theta}{(\mu - 2)} \right] 
+ \Psi \left( \frac{16}{(\mu - 1)} + \frac{4(2\mu - 3)(\mu - 3)}{(\mu - 1)(\mu - 2)^2} \right) + \frac{16}{(\mu - 1)^2} - \frac{2}{(\mu - 2)} 
+ \frac{10}{(\mu - 1)} + \frac{2}{(\mu - 2)^2} - \frac{16}{\mu(\mu - 2)^2 \eta_1} \right] 
$$

(7.1)

and also record that

$$
z_2 = \frac{\mu \Gamma^2(\mu)\eta_1^2}{2\pi^2(\mu - 1)} \left[ \frac{\mu}{(\mu - 1)} + 2 + (2\mu - 1)\Psi(\mu) \right] 
$$

(7.2)

With these expressions together with the expansions of $p(\beta)$, $q(\beta)$, $r(\alpha - 1)$ and $s(\alpha - 1)$ to the next to leading order values, we find from the vanishing of the determinant of the matrix defined by (3.7) and (3.8) at $O(1/N^2)$

$$
\lambda_2 = \frac{2\mu \eta_1^2}{(\mu - 1)} \left[ \frac{2}{(\mu - 1)^2 \eta_1} - \frac{(2\mu - 3)\mu}{(\mu - 2)} (\Phi + \Psi^2) \right] 
+ \Psi \left( \frac{1}{(\mu - 2)^2} + \frac{1}{2(\mu - 2)} - \frac{2\mu^2 - 3}{2} - \frac{1}{2\mu} - \frac{3}{\mu - 1} \right) 
+ \frac{3\mu\Theta}{4} \left( 9 - 2\mu + \frac{6}{\mu - 2} \right) + 2\mu^2 - 5\mu - 3 + \frac{5}{4\mu} - \frac{1}{4\mu^2} 
- \frac{7}{2(\mu - 1)} + \frac{1}{(\mu - 1)^2} + \frac{1}{4(\mu - 2)} - \frac{1}{2(\mu - 2)^2} \right] 
$$

(7.3)

which is an arbitrary dimensional expression for the $O(1/N^2)$ corrections to the $\beta$-function of (2.2). It is worth recording that an independent check on the correctness of (7.3) is that it ought to agree with the expansion of the critical $\beta$-function slope computed explicitly from the three loop result of (2.2). We have checked that this is indeed the case. Further, we can deduce the value of the exponent in three dimensions as

$$
\lambda = \frac{1}{2} - \frac{16}{3\pi^2 N} + \frac{32(27\pi^2 + 632)}{27\pi^4 N^2} 
$$

(7.4)

As two independent exponents $\nu = 1/(2\lambda)$ and $\eta$ or $\eta + \chi$ are now known at $O(1/N^2)$ this implies that the remaining thermodynamic exponents of
the model can be deduced through the hyperscaling relations which were recently checked at leading order in [13].

With (7.4) we can now gain an improved estimate of the exponent $\nu$ in three dimensions and compare with recent lattice simulations where the same exponent is calculated for the case $N = 8$ in our notation, [14]. Thus

$$\nu = 1 + \frac{32}{3\pi^2 N} - \frac{64(27\pi^2 + 584)}{27\pi^4 N^2}$$

and employing a Padé-Borel technique widely used in improving estimates of exponents, [21, 29, 30], we have

$$\nu = N \int_0^\infty dt e^{-Nt} \left[ 1 - \frac{16t}{3\pi^2} + \frac{32(27\pi^2 + 608)t^2}{81\pi^4} \right]^{-1} \bigg|_{N=8} = 0.98 \ (7.6)$$

Recent simulations, [14], give $\nu = 0.98(7)$ and so (7.6) is in excellent agreement with that Monte Carlo result. Also the exponent $2 - \eta - \chi$, in our notation, has been calculated numerically as 1.26(3) in [14] and we record that from (2.6)-(2.9), at $N = 8$, we find $2 - \eta - \chi = 1.25$ again in good agreement.

We conclude by making several remarks. First, the Gross Neveu model has now been solved at $O(1/N^2)$ The techniques we have had to employ are different from those used to perform the analogous calculation in the $O(N)$ model, due to the appearance of several four loop graphs. More importantly, though, we have laid a substantial amount of the groundwork for performing the same calculation for QED. Whilst this is a more complicated theory the basic techniques to treat the integrals have been developed here.

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**Note added.** Whilst in the final stages of this work we received a preprint, [31], where $\lambda_2$ is stated and we record that it and (7.3) are in agreement. We believe the method of [31] is different from the one given here.
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Figure Captions.

Fig. 1. Skeleton Dyson equations to determine $\lambda_1$.

Fig. 2. Higher order graph for $\psi$ consistency equation.

Fig. 3. Divergent higher order corrections to $\sigma$ consistency equation.

Fig. 4. Finite higher order graphs for $\sigma$ consistency equation.

Fig. 5. Uniqueness rule for a bosonic vertex.

Fig. 6. Uniqueness rule for a fermionic vertex.

Fig. 7. Subtractions for $\Pi_{2B1}$ and $\Pi_{3B}$.

Fig. 8. Definition of $\langle \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5 \rangle$ and $\langle \tilde{\alpha}_1, \tilde{\alpha}_2, \tilde{\alpha}_3, \tilde{\alpha}_4, \tilde{\alpha}_5 \rangle$.

Fig. 9. Intermediate integrals in the evaluation of $\Pi_{4B}$.

Fig. 10. $\Pi_{6B1}$ after several integrations.

Fig. 11. Intermediate integrals in the evaluation of $\Pi_{5B2}$.

Fig. 12. Common integrals in the evaluation of $\Pi_{5B1}$ and $\Pi_{6B2}$.