Viscous control of minimum uncertainty state

T. Koide
Instituto de Física, Universidade Federal do Rio de Janeiro, C.P. 68528, 21941-972, Rio de Janeiro, RJ, Brazil

A minimum uncertainty state for position and momentum is obtained in quantum viscous hydrodynamics which is defined through the Navier-Stokes-Korteweg (NSK) equation. This state is the generalization of the coherent state and its uncertainty is given by a function of the coefficient of viscosity. The uncertainty can be smaller than the standard minimum value in quantum mechanics, $\hbar/2$, when the coefficient of viscosity is smaller than a critical value which is similar in magnitude to the Kovtun-Son-Starinets (KSS) bound.

I. INTRODUCTION

The uncertainty relation is an important feature in quantum physics and its comprehension requires unceasing improvement [1–5]. A similar relation was recently proposed in classical viscous hydrodynamics [6, 7]. There, the minimum uncertainty for position and momentum of fluid elements is given by a function of the coefficient of viscosity. This suggests that the uncertainty of a quantum state can be controlled by viscosity.

Viscosity is one of dissipative effects but there is no established quantum theory of dissipation. The dissipative effect is introduced, for example, by the coarse-graining of environment variables using the influence functional method or the projection operator method. The derived dissipative equation however can describe an unphysical state with negative probability [8]. See also Refs. [9–16] for other approaches of quantum dissipation. At the same time, the behavior of quantum many-body systems is approximately given by viscous fluids in heavy-ion collision physics [17]. The viscous effect is indeed considered to be indispensable because it is believed that the coefficient of viscosity cannot be smaller than the Kovtun-Son-Starinets (KSS) bound [18].

As was discussed by Madelung, the Schrödinger equation can be expressed in the form of the Euler equation with the so-called quantum potential [19]. Thus it is natural to consider that quantum viscous hydrodynamics is described by the Navier-Stokes-Fourier (NSF) equation with the quantum potential term. In the D-dimensional Cartesian coordinate system, this equation is given by

\begin{equation}
(\partial_t + \mathbf{v} \cdot \nabla) \mathbf{v} = -\frac{1}{M} \partial_t P + 2\kappa \partial_i \left( \frac{\nabla^2 \sqrt{\rho}}{\sqrt{\rho}} \right) - \frac{1}{M \rho} \partial_i \left( P - \left( \mu + \frac{\eta}{D} \right) (\nabla \cdot \mathbf{v}) \right) + \frac{1}{M \rho} \sum_{j=1}^{D} \partial_j (\eta E_{ij}),
\end{equation}

(1)

where $\mathbf{v}$, $P$, $\eta$ and $\mu$ are the velocity field, the external potential, the pressure, the coefficient of viscosity and the second coefficient of viscosity, respectively. The traceless symmetric stress tensor is defined by

\[ E_{ij} = \frac{1}{2} \left( \partial_i v^j + \partial_j v^i \right) - \frac{1}{D} (\nabla \cdot \mathbf{v}) \delta_{ij}. \]

Normally, hydrodynamics is described using the mass distribution. For the sake of comparison with quantum mechanics, however, we use the distribution of constituent particles of the quantum viscous fluid $\rho$, which is normalized by the number of constituent particles $N$. Then the mass distribution is given by $M\rho$ with $M$ being the mass of constituent particles. The last term on the first line is the gradient of the quantum potential. Ignoring the contributions on the second line and choosing $\kappa = \hbar^2/(4M^2)$, Eq. (1) agrees with Madelung’s Euler equation. It is interesting to note that Eq. (1) is a special case of the Navier-Stokes-Korteweg (NSK) equation which was proposed to describe liquid-vapor fluids near phase transitions. Then the term associated with the quantum potential represents the capillary action [20–21].

The uncertainty relation for the quantum viscous fluid is derived by applying the method developed in Refs. [6, 7] to Eq. (1). However, the existence of the corresponding minimum uncertainty state has not yet been known. The purpose of this paper is to derive the minimum uncertainty state. The uncertainty of the derived state is a function of the coefficient of viscosity and can be smaller than the standard minimum value in quantum mechanics $\hbar/2$ for a sufficiently weak viscosity.

This paper is organized as follows. In Sec. II, quantum viscous hydrodynamics is formulated in the framework of the stochastic variational method [6, 7, 22–28]. In Sec. III, the uncertainty relation is derived by applying the method in Refs. [6, 7]. The minimum uncertainty state in quantum viscous hydrodynamics is derived in Sec. IV. Section V is devoted to concluding remarks.

II. STOCHASTIC VARIATIONAL METHOD

To define the uncertainty relation in viscous systems, we formulate Eq. (1) in the stochastic variational method (SVM) [6, 7, 22–28]. As is well known, the behavior of a fluid can be described by the ensemble of fluid elements. We thus consider the variation of the trajectory of a fluid
element in SVM. A fluid element is an abstract volume element and constituent particles inside of it are assumed to be thermally equilibrated. As is shown later, however, the derivation of the uncertainty relation is irrelevant to the assumption of the local thermal equilibrium. For the sake of simplicity, we thus identify a fluid element with a constituent particle in the following discussion. See Ref. [7] for more details on the uncertainty relation for fluid elements.

In SVM, the viscous and quantum potential terms are induced through the fluctuations of constituent particles. Then the trajectory of a constituent particle is supposed to be given by the forward stochastic differential equation (SDE),

$$d\tilde{\mathbf{r}}(t) = \mathbf{u}_+(\tilde{\mathbf{r}}(t),t)dt + \sqrt{2\nu}d\tilde{\mathbf{W}}(t) \quad (dt > 0).$$

The second term on the right-hand side represents the noise of Brownian motion. We used (·) to denote stochastic variables and $d\tilde{\mathbf{A}}(t) = \tilde{\mathbf{A}}(t+dt) - \tilde{\mathbf{A}}(t)$ for an arbitrary $\tilde{\mathbf{A}}(t)$. The standard Wiener process is described by $\tilde{\mathbf{W}}(t)$ which satisfies

$$E[d\tilde{\mathbf{W}}(t)] = 0, \quad E[d\tilde{\mathbf{W}}(t)d\tilde{\mathbf{W}}(t')] = |dt| \delta_{tt'} \delta_{ij},$$

where $E[\cdot]$ denotes the ensemble average for the Wiener process. Note that $\mathbf{u}_+(\tilde{\mathbf{r}}(t),t)$ is stochastic because of $\tilde{\mathbf{r}}(t)$, but $\mathbf{u}_+(\mathbf{x},t)$ is a smooth function. The field $\mathbf{u}_+(\mathbf{x},t)$ is associated with the velocity of constituent particles. SVM determines its form by applying the variational principle. The noise intensity $\nu$ controls the stochasticity of the trajectory.

The standard definition of velocity is not applicable because the left and right-hand limits of the inclination do not agree in the stochastic trajectory. To distinguish this difference, we consider the backward time evolution of the trajectory described by the backward SDE,

$$d\tilde{\mathbf{r}}(t) = \mathbf{u}_-(\tilde{\mathbf{r}}(t),t)dt + \sqrt{2\nu}d\tilde{\mathbf{W}}(t) \quad (dt < 0),$$

where $\tilde{\mathbf{W}}(t)$ satisfies the same correlation properties as Eq. (2) using $|dt| = -dt$. The field $\mathbf{u}_-(\mathbf{x},t)$ is associated with the velocity backward in time. Because of this ambiguity of velocity, Nelson introduced two different time derivatives [29]: one is the mean forward derivative $D_+$ and the other the mean backward derivative $D_-$, which are defined by

$$D_\pm \tilde{\mathbf{r}}(t) = \lim_{dt \to 0^\pm} E \left[ \frac{\tilde{\mathbf{r}}(t+dt) - \tilde{\mathbf{r}}(t)}{dt} \right] |_{\tilde{\mathbf{r}}(t)} = \mathbf{u}_\pm(\tilde{\mathbf{r}}(t),t).$$

Here the expectation value is the conditional average for fixing $\tilde{\mathbf{r}}(t)$ and we used that $\tilde{\mathbf{r}}(t)$ is Markovian. When these are operated to a function $f(\tilde{\mathbf{r}})(t)$, we find

$$D_\pm f(\tilde{\mathbf{r}}(t),t) |_{\tilde{\mathbf{r}}(t)=\mathbf{x}} = \left( \delta_t + \mathbf{u}_\pm(\mathbf{x},t) \cdot \nabla \pm \nu \nu^2 \right) f(\mathbf{x},t),$$

where $f(\mathbf{x},t)$ is an arbitrary smooth function and we used Itô’s lemma [7]. That is, $D_+$ and $D_-$ correspond to material derivatives along the stochastic trajectories described by the forward and backward SDE’s, respectively.

Two Fokker-Planck equations for $\rho$ are obtained using the two SDE’s independently but these are equivalent. To satisfy this condition, $\mathbf{u}_\pm(\mathbf{x},t)$ should satisfy the consistency condition,

$$\mathbf{u}_+(\mathbf{x},t) = \mathbf{u}_-(\mathbf{x},t) + 2\nu\nabla \ln \rho(\mathbf{x},t).$$

See Ref. [41] for details. The consistency condition corresponds to the commutation relation in quantum mechanics as discussed later. It is also noteworthy that a similar condition plays an important role in the derivation of the modified NSF equation called bivelocity hydrodynamics [30, 31, 32].

We apply SVM to the classical Lagrangian,

$$L_{cla}(\mathbf{r},d\mathbf{r}/dt) = \frac{M}{2} \left( \frac{d\mathbf{r}}{dt} \right)^2 - V - \frac{\varepsilon}{\rho},$$

where $\varepsilon$ is an internal energy density given by a function of the particle distribution and the entropy density. Applying the classical variation, this Lagrangian gives the Euler equation [4]. As mentioned before, the viscous and quantum potential terms are induced through the fluctuating trajectory in SVM and hence quantum viscous hydrodynamics [1] is obtained by applying SVM to this Lagrangian. To find the corresponding stochastic Lagrangian, we have to replace $d\mathbf{r}/dt$ with $D_+$ and $D_-$ in Eq. (4). Due to this ambiguity in the replacement, we introduce two real parameters $\alpha_A$ and $\alpha_B$. Then the stochastic Lagrangian is defined by

$$L_{sto}(\mathbf{r},D_+\tilde{\mathbf{r}},D_-\tilde{\mathbf{r}}) = \frac{M}{2}(D_+\tilde{\mathbf{r}},D_-\tilde{\mathbf{r}})M \left( \frac{D_+\tilde{\mathbf{r}}}{D_-\tilde{\mathbf{r}}} \right) - V - \frac{\varepsilon}{\rho},$$

with

$$M = \left( \frac{1}{2} + \alpha_A \right) \left( \frac{1}{2} + \alpha_B \right) - \frac{2\varepsilon}{\rho}, \quad \left( \frac{1}{2} - \frac{\alpha_A}{2} \right) \left( \frac{1}{2} - \alpha_B \right).$$

In the vanishing limit of $\nu$, $D_\pm$ coincide with $d/dt$ and then the stochastic Lagrangian [7] is reduced to the corresponding classical one [6] independently of $\alpha_A$ and $\alpha_B$. The parameters $\alpha_A$ and $\alpha_B$ are absorbed into the definitions of $\kappa$ and $\eta$ as shown later in Eq. (11).

In the classical variation, a trajectory is entirely determined for a given velocity. This is however not the case with SVM due to the noise terms in the two SDE’s. Thus only the averaged behavior of the stochastic Lagrangian is optimized by variation. The action is then

$$I_{sto}[\tilde{\mathbf{r}}] = \int_{t_i}^{t_f} dt E[L_{sto}(\mathbf{r},D_+\tilde{\mathbf{r}},D_-\tilde{\mathbf{r}})],$$

with an initial time ($t_i$) and a final time ($t_f$). Here, the initial distribution of constituent particles is omitted but it does not affect the result of the stochastic variation. See, for example, Eq. (116) in Ref. [7].

The variation of the stochastic trajectory is defined by $\tilde{\mathbf{r}}(t) \rightarrow \tilde{\mathbf{r}}(t) = \tilde{\mathbf{r}}(t) + \delta \mathbf{f}(\tilde{\mathbf{r}}(t),t)$, where an infinitesimal
smooth function $\delta f(x, t)$ satisfies $\delta f(x, t_f) = 0$. We further define the fluid velocity field by

$$v(x, t) = \frac{u_+(x, t) + u_-(x, t)}{2}. $$

Then the stochastic variation of Eq. (8) leads to

$$\left[ \frac{D_- p_+ + D_+ p_-}{2} \right] \frac{\nabla V}{\rho} = -\nabla \left( \mu \nabla \cdot v \right) \right]_{\bar{f}(t) = \infty} = 0.$$  

(9)

Here, $\mu$ is obtained through the variation of the entropy density. See Sec. 5.1 in Ref. [24] for details. To derive $\bar{f}$, $\varepsilon$ is assumed to satisfy the local thermal equilibrium, but this assumption does not affect the definitions of the two momenta, which are introduced through the Legendre transformation of the stochastic Lagrangian,

$$p_\pm(x, t) = 2 \frac{\partial L_{sto}}{\partial (D_\pm f)} \bigg|_{\bar{f} = \infty},$$

(10)

where the factors 2 in the definitions of $p_\pm(x, t)$ are introduced for a convention to reproduce the classical result in the vanishing limit of $\nu$ [6]. Note that the operations $\overline{\cdot \cdot \cdot}$ and $\overline{\cdot \cdot \cdot}$ in Eqs. (4). Then Eq. (9) is shown to be equivalent to Eq. (1) with the identification,

$$\kappa = 2 \alpha_B \nu^2, \quad \eta = 2 \alpha_A (1 + 2 \alpha_B) \nu M \rho.$$ 

(11)

### III. UNCERTAINTY RELATIONS

The emergence of the two momenta is attributed to the non-differentiability of the stochastic trajectory. As seen in Eq. (9), $p_\pm(x, t)$ contribute to our equation of motion on an equal footing. Therefore it is natural to define the standard deviation of momentum by the average of the two contributions, $p_+(x, t)$ and $p_-(x, t)$.

We define the standard deviations of position and momentum. The former is given by

$$\sigma_{x_i}^{(2)} = \langle \delta x^2 \rangle,$$

where $\delta f = f(x, t) - \langle f \rangle$ and we introduced the following expectation value,

$$\langle f \rangle = \frac{1}{N} \int \rho(x, t) f(x, t) \, dx,$$

with $N$ being the number of constituent particles. As discussed above, the latter is given by the average,

$$\sigma_{p_i}^{(2)} = \frac{\langle \delta p_i^2 \rangle + \langle \delta p_i^R \rangle^2}{2} = \left( \frac{\delta p_i^+ + \delta p_i^-}{2} \right)^2 + \left( \frac{\delta p_i^+ - \delta p_i^-}{2} \right)^2,$$

where

$$\left( p_+(x, t) - p_-(x, t) \right) = 2 M G \left( -\nu \nabla \ln \rho(x, t) \right) v(x, t).$$

The symmetric matrix $G$ is defined by

$$G = \begin{pmatrix} \frac{\kappa}{\nu} & -\frac{\kappa}{\mu} \\ -\frac{\kappa}{\mu} & 1 \end{pmatrix},$$

with the kinematic viscosity,

$$\kappa = \frac{\eta}{2 M \rho}.$$ 

The consistency condition [6] is used in Eq. (12). The above definitions of the standard deviations reproduce the corresponding quantum-mechanical quantities as shown later in Eq. (15).

Using these definitions and the Cauchy-Schwarz inequality $\langle A^2 \rangle \langle B^2 \rangle \geq \langle [AB] \rangle^2$, the product of $\sigma_{x_i}^{(2)}$ and $\sigma_{p_i}^{(2)}$ satisfies the inequality,

$$\sigma_{x_i}^{(2)} \sigma_{p_i}^{(2)} \geq M^2 \left( \frac{\nu^2 \lambda_i^2}{\lambda_+ + \lambda_- - \lambda_+ \lambda_-} \delta_{ij} + (\lambda_+ + \lambda_- - \lambda_+ \lambda_-) \right)^2 \times \left( \frac{\nu^2 (\lambda_+ + \lambda_-)(1 - \lambda_+)(1 - \lambda_-)}{\lambda_+ + \lambda_- - \lambda_+ \lambda_-} \right)^2 \right) \right)$$

$$= \frac{M^2 (\xi^2 - \kappa)^2}{\nu^2 + \xi^2} \delta_{ij} + \frac{M^2 (\xi^2 - \kappa)^2}{\nu^2 + \xi^2} \delta_{ij},$$

(13)

where $\lambda_\pm = \{1 + \kappa/\nu^2 \pm \sqrt{(1 - \kappa/\nu^2)^2 + 4 \xi^2/\nu^2}\}$ are the eigenvalues of $G$. This was derived in Ref. [6] for the first time. The right-hand side becomes minimum when $\delta x^2 \delta v^2 \leq \delta_{ij} (\nu^2 + \kappa)/\nu^2 + \xi^2$.

This inequality reproduces the well-known result in quantum mechanics by choosing

$$\left( \alpha_A, \alpha_B, \nu \right) = \left( 0, \frac{1}{2}, \frac{\hbar}{2M} \right).$$

(14)

Then Eq. (14) or equivalently Eq. (15) coincides with Medelung’s Euler equation, and our uncertainty relation [13] gives the Robertson-Schrödinger inequality,

$$\sigma_{x_i}^{(2)} \sigma_{p_i}^{(2)} \geq \frac{\hbar^2}{4} \delta_{ij} + \left\{ \left( \langle p_i^{op} \rangle \right)^2 - \left( \langle p_i^{op} \rangle \right)^2 \right\}.$$ 

In this derivation, we used that the quantum-mechanical expectation values are expressed as

$$\langle x_i^{op} \rangle = \left\langle \left( x_i^{op} \right)^2 \right\rangle = \sigma_{x_i}^{(2)} = \langle \langle x_i^{op} \rangle \rangle,$$

$$\langle p_i^{op} \rangle = \left\langle \left( p_i^{op} \right)^2 \right\rangle = \sigma_{p_i}^{(2)} = \langle \langle p_i^{op} \rangle \rangle,$$

(15)

where $x_{op}$ and $p_{op}$ are the position and momentum operators, respectively, and $\langle \rangle$ denotes the expectation value with a wave function. See Refs. [6, 17] for details. Note that the paradox for the angular uncertainty relation is resolved in the present approach [28]. For a quantum-mechanical uncertainty relation in different stochastic approaches, see Refs. [33, 34]. The advantage of the present approach compared to the standard operator formalism is discussed in Sec. IV.
IV. MINIMUM UNCERTAINTY STATE IN QUANTUM VISCOUS HYDRODYNAMICS

To derive the minimum uncertainty state of the inequality \(13\), we consider the one-dimensional system and assume

\[
\rho(x) = \sqrt{\frac{A}{\pi}} e^{-A(x-x_0)^2}, \quad v(x) = v_0 + Bx, \quad (16)
\]

where \(x_0, v_0, A\) and \(B\) are real constants. The standard deviations, \(\sigma_x^{(2)}\) and \(\sigma_p^{(2)}\), are easily calculated using this state and then we find

\[
\sigma_x^{(2)} \sigma_p^{(2)} = M^2 \frac{(\kappa - \xi^2)^2}{\nu^2 + \xi^2} + M^2 \left( 1 + \frac{\xi^2}{\nu^2} \right) \left( \frac{B}{2A} - \frac{\xi(\nu^2 + \kappa)}{\nu^2 + \xi^2} \right)^2 \quad (17)
\]

The right-hand side becomes minimum by choosing

\[
\frac{B}{2A} = \left[ \delta x \delta v \right] = \frac{\xi(\nu^2 + \kappa)}{\nu^2 + \xi^2}. \quad (18)
\]

The right-hand side of Eq. \((17)\) is nothing but that of Eq. \((13)\), and hence our viscous minimum uncertainty state is defined by Eqs. \((16)\) and \((18)\).

This state is the generalization of the (standard) coherent state. As is shown in, for example, Ref. \([19]\), \(\rho(x)\) and \(v(x)\) for a given wave function \(\Psi(x)\) are defined by

\[
\rho(x) = |\Psi(x)|^2, \quad v(x) = -i \frac{\hbar}{M} \text{Im} \left[ \partial_x \ln \Psi(x) \right]. \quad (19)
\]

At the same time, the coordinate representation of the coherent state is given by

\[
(x|\alpha) = \left( \frac{C}{\pi} \right)^{1/4} e^{-\frac{1}{2}(\sqrt{\kappa}x - \alpha_R)^2} e^{i\alpha_I(\sqrt{\kappa}x - \alpha_R/2)},
\]

where \(\alpha_R, \alpha_I\) and \(C\) are real constants and \(\alpha = (\alpha_R + i\alpha_I)/\sqrt{2}\) is the eigenvalue of the lowering operator in a quantum harmonic oscillator \([35]\). Substituting this into Eq. \((19)\), we find that \(\rho(x)\) and \(v(x)\) for the coherent state are reproduced from Eq. \((16)\) by choosing

\[
A = C, \quad x_0 = \frac{\alpha_R}{\sqrt{\kappa}}, \quad B = 0, \quad v_0 = \frac{\hbar}{\pi \sqrt{\kappa} \alpha_I}.
\]

Here \(B = 0\) means the absence of the kinematic viscosity in Eq. \((18)\).

We see that the viscous minimum uncertainty state is sensitive to the inhomogeneity of the mass distribution. Indeed, for the homogeneous distribution where \(A\) is very small, \(B\) defined by Eq. \((18)\) can be negligible and the viscous minimum uncertainty state is hardly distinguishable from the coherent state. Moreover, the viscous minimum uncertainty state has a position-dependent velocity characterized by Eqs. \((16)\) and \((18)\) and thus expands to homogenize the mass distribution. Therefore the persistence time of the viscous minimum uncertainty state is short. To see the influence of the viscous uncertainty, we should observe short time evolutions in small inhomogeneous systems as is realized in heavy-ion collisions. See also the discussion in Sec. \(V\).

So far we did not specify the parameter \(\kappa\). We now choose \(\kappa = \nu^2/h^2/(4M^2)\) so that Eq. \((1)\) coincides with Madelung’s Euler equation in the vanishing limits of the coefficients of viscosities, \(\eta\) and \(\mu\). Then our minimum uncertainty obtained from Eqs. \((16)\) and \((18)\) is simplified as

\[
\sqrt{\sigma_x^{(2)}} \sqrt{\sigma_p^{(2)}} = M \sqrt{\frac{(\kappa - \xi^2)^2}{\nu^2 + \xi^2}} = \frac{1}{2} \frac{|\hbar^2 - 4M^2\xi^2|}{\sqrt{\nu^2 + 4M^2\xi^2}} \quad (20)
\]

The right-hand side is controlled by \(\xi\), and can be smaller than the standard minimum value in quantum mechanics, \(\hbar/2\), for a sufficiently weak kinematic viscosity satisfying

\[
0 < \xi < \frac{\sqrt{3} \hbar}{2M} \quad (21)
\]

The behavior of the minimum uncertainty \((20)\) is shown as a function of \(M\xi/h\) in Fig. 1. For the sake of comparison, the dashed line represents the quantum mechanical minimum value. The effects induced by the quantum potential term and the viscous term cancels each other out and hence the product \((\sigma_x^{(2)})^{1/2}(\sigma_p^{(2)})^{1/2}/\hbar\) can be smaller than \(1/2\) for \(M\xi/h < \sqrt{3}/2\). The viscous minimum uncertainty vanishes when \(M\xi/h = 1/2\) which correspond to \(\kappa - \xi^2 = 0\) but this choice is forbidden, because \(\kappa - \xi^2 \propto \text{det}(M)\) but \(\text{det}(M) \neq 0\) to define...
the momenta through the Legendre transformation \( (10) \) \[ [6, 7] \). For a larger \( \xi \), the effect of the viscous term becomes dominant and then the number of the collisions among particles (fluid elements) increases. Because the non-differentiability of trajectories is enhanced by the collisions, the viscous minimum uncertainty behaves as an increasing function of \( \xi \).

**V. CONCLUDING REMARKS**

We showed that there exists the minimum uncertainty state of the uncertainty relation in quantum viscous hydrodynamics. The corresponding uncertainty is a function of the coefficient of viscosity and can be smaller than the standard minimum value in quantum mechanics, \( h/2 \), for a sufficiently weak kinematic viscosity.

We should however pay special attention to the existence of such a weak kinematic viscosity. As mentioned before, it is considered that the coefficient of viscosity cannot be smaller than the KSS bound \[18\],

\[
\eta \geq \frac{h}{s} = \frac{h}{4\pi k_B} \rho,
\]

where \( s \) is the entropy density. This bound is based on the anzatz of the AdS/CFT correspondence. When we suppose \( s \sim k_B \rho \), Eq. (22) reads the inequality for the kinematic viscosity,

\[
\xi \geq \frac{1}{8\pi} \frac{h}{M}.
\]

Comparing this with Eq. (21), we find that the maximum value of \( \xi \) in Eq. (21) is still larger than the KSS bound, but the difference is only slight. This may suggest that the lower bound of the coefficient of viscosity appears so that viscosity does not improve uncertainty beyond the standard minimum value \( h/2 \). Of course, the KSS bound is a conjecture and further studies are required to confirm the above discussion.

Quantum viscous hydrodynamics \[1\] is reduced to the NSF equation by setting \( \kappa = 0 \) and thus the state defined by Eqs. (16) and (18) describes also the minimum uncertainty state of a classical viscous fluid. For the minimum value of the uncertainty for water at room temperature, see the discussion in Ref. [7].

In the standard formulation of quantum mechanics, the non-commutativity of operators leads to the uncertainty relation, while the same property is reproduced from the non-differentiability of particle trajectories in the present approach. The operator formalism is established in various applications of quantum mechanics and thus one may wonder about the significance of the alternative interpretation for the uncertainty relation. The advantage of the present approach is its applicability to generalized coordinate systems. For example, the angle variable and the angular momentum form a pair of canonical variables in polar coordinates, but the corresponding operator representations are not established because there is no self-adjoint multiplicative operator which satisfies the periodicity and the canonical commutation relation at the same time. See Ref. [28] and references therein. Therefore, in the discussion of the angular uncertainty relation, the angle operator is introduced exclusively by altering one of those conditions. By contrast, the present approach is applicable to quantize generalized coordinate systems without introducing additional condition \[27\] and the uncertainty relation in generalized coordinates is obtained \[28\].

In the standard formulation of quantum mechanics, the finite minimum uncertainty is attributed to the non-commutativity of operators. The corresponding quantity in SVM is given by the consistency condition \[1\]. Using the parameters for quantum mechanics \[14\], the consistency condition \[1\] gives

\[
[x^i p^j_\pm - x^j p^i_\pm] = \hbar \delta_{ij}.
\]

This remind us of the canonical commutation relation and the above equation is reexpressed as

\[
[x^i p^j_\pm - x^j p^i_\pm] = (-i)\langle x_{op} p_{op} - p_{op} x_{op} \rangle \cdot (23)
\]

To see the role of the imaginary unit, we consider the time-reversal transformation. Using Eqs. (3) and (10) with the parameters \[14\], we find that the two momenta are transformed as

\[
P_\pm(x, t) \rightarrow -P_\mp(x, t).
\]

Therefore the left-hand side of Eq. (23) is time-reversal symmetric. On the other hand, for the right-hand side to be time-reversal symmetric, the imaginary factor \((-i)\) should be multiplied to the commutation relation. This partially explains the raison d’etre of the imaginary unit in the operator formulation of quantum mechanics.

The viscous uncertainty characterizes the motion of fluid elements. The fluid element is an abstract volume element and thus its direct observation will be difficult. However, the descriptions based on hydrodynamic models sometimes depend on the motions of fluid elements and thus the viscous uncertainty triggers the modification of the descriptions. Physics in relativistic heavy-ion collisions is one example \[17\]. The vacuum is excited by high-energy nucleus collisions and the behavior of the excited vacuum is approximately described by viscous hydrodynamics. The experimentally observed particles, called hadrons, are assumed to be produced by the thermal radiation from each fluid element of the viscous fluid. It is known that this hydrodynamic model explains experimental data very well. In this model, however, we assume that the fluid elements pass along the streamline of the viscous fluid, but such a view is too simple. Our result shows that the currents of the fluid elements fluctuate around streamlines of the viscous fluid and this fluctuation is characterized by \( \sigma_p^{(2)} \). Moreover the behavior of \( \sigma_p^{(2)} \) is restricted by that of \( \sigma_x^{(2)} \) which can reflect
the inhomogeneity of the matter distribution. Because of the lack of the above mentioned effect, the standard hydrodynamic model may underestimate the effect of the spatial inhomogeneity of the excited vacuum and hence the anisotropy of the hadron production. A more quantitative analysis is left as a future work.

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