REGULARITY OF SOLUTIONS TO TIME-HARMONIC MAXWELL’S SYSTEM WITH VARIOUS LOWER THAN LIPSCHITZ COEFFICIENTS

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ABSTRACT. In this paper, we study the regularity of the solutions of Maxwell’s equations in a bounded domain. We consider several different types of low regularity assumptions to the coefficients which are all less than Lipschitz. We first develop a new approach by giving $H^1$ estimate when the coefficients are $L^\infty$ bounded; and then we derive $W^{1,p}$ estimates for every $p > 2$ when one of the leading coefficients is simply continuous; Finally, we obtain $C^{1,\alpha}$ estimates for the solution of the homogeneous Maxwell’s equations almost everywhere when the coefficients are $W^{1,p}$, $p > 3$ and close to the identity matrix in the sense of $L^\infty$ norm. The last two estimates are new, and the techniques and methods developed here can also be applied to other problems with similar difficulties.

Key Words: Maxwell’s equations; regularity theory; anisotropic;

1. Introduction

Let $\Omega \subset \mathbb{R}^3$ be an open bounded domain with $C^{1,1}$ boundary. The time-harmonic electromagnetic field $(E, H) \in \mathcal{H}_{\text{loc}} (\text{curl}; \Omega)$, where for any domain $\Omega$ we define

$$\mathcal{H}_{\text{loc}} (\text{curl}; \Omega) := \{ u \in L^2_{\text{loc}} (\Omega) \text{ such that } \nabla \times u \in L^2_{\text{loc}} (\Omega) \},$$

satisfies the Maxwell’s equations

$$\begin{cases}
\nabla \times E - i \omega \mu_0 \mu(x) H = J_m & \text{in } \Omega, \\
\nabla \times H + i \omega \varepsilon_0 \varepsilon(x) E = J_e & \text{in } \Omega, \\
E \wedge \nu = G \wedge \nu & \text{on } \partial \Omega,
\end{cases}$$

(1.1)

where $\varepsilon$ and $\mu$ are real matrix-valued functions in $L^\infty (\mathbb{R}^3)^{3\times 3}$, the terms $J_m, J_e \in L^2 (\Omega)$ are current sources and the boundary condition $G$ is in $\mathcal{H}_{\text{loc}} (\text{curl}; \Omega)$. We assume that $\varepsilon^{-1}(x)$ and $\mu^{-1}(x)$ are real, uniformly positive definite and bounded, that is, there exist $0 < \lambda_1 \leq \lambda_2 < \infty$ and $0 < \lambda_3 \leq \lambda_4 < \infty$ such that for all $\xi \in \mathbb{R}^3$ and almost every $x \in \mathbb{R}^3$,

$$\begin{align*}
\lambda_1 |\xi|^2 &\leq \varepsilon^{-1}(x) \xi \cdot \xi \leq \lambda_2 |\xi|^2, \\
\lambda_3 |\xi|^2 &\leq \mu^{-1}(x) \xi \cdot \xi \leq \lambda_4 |\xi|^2.
\end{align*}$$

(1.2)

The goal of this paper is to study the regularity of the electromagnetic fields with low regularity assumptions on the material parameters. The material parameters are $(\varepsilon(x), \mu(x))$ which are the matrix valued coefficients of Maxwell’s equations, and the importance of the low regularity assumptions on the coefficients lies in the consideration of the corresponding medium...
with complicated structure, such as liquid crystals. This research topic is originally motivated by the study of the electromagnetic inverse problems for liquid crystals, see [5] in which we study the molecular structure of liquid crystals by means of the optical measurement which is represented by the solutions of Maxwell’s equations. The question of whether the data measured is well understood in the sense of the object observed is of paramount importance in this context. There are some work have been done recently in this consideration; Capdebosq and Tsering-xiao [2] studied the reconstruction of the coefficients of anisotropic Maxwell’s equations using optical measurement of laser beam. Okaji [14] and Colton [3] studied the uniqueness of the solutions of Maxwell’s equations when the coefficients are smooth and Nguyen and Wang [13] and Ball, Capdebosq and Tsering-xiao [2] studied the uniqueness when the coefficients hold low regularity assumptions.

The regularity of the solutions of Maxwell’s equations has been studied by many people, see [11,12,15–17] for more detail. In the case of the high regularity assumptions on the coefficients, Leis [15] established well-posedness in $H^1(\Omega)$ when the coefficients are smooth matrices where $H^1(\Omega)$ is the Sobolev space and later Costabel [19] and Fernandes [9] showed the $C^1$ regularity for $C^{1,1}$ domain. However, Alberti and Capdebosq [1] is the only work so far that studied the regularity of the solutions with low regularity assumptions on the coefficients. In this paper, we extend the regularity results given in [1] in the case of real coefficients. We study the regularity and improve the results obtained in [1] encompasses three aspects: different approach, similar regularity results under weaker regularity assumptions on the coefficients, and better regularity results for the solutions under the same regularity assumptions. Specifically, we work on the system of Maxwell’s equations instead of studying the deduced scalar elliptic equations of second order as done in [1]. We first develop the approach through giving the $H^1$ estimates for the solutions when the coefficients are bounded; Based on the $H^1$ estimates, we derive $W^{1,p}$ estimates of $H$ for every $p > 2$ when the coefficients $\varepsilon$ is simply continuous and $\mu \in W^{1,p}$ for some $p > 3$; This result is not addressed in the literature, since one of the coefficients is simply continuous. Finally, we obtain that the solution $H$ of the homogeneous Maxwell’s equations is $C^{1+\alpha}$ almost everywhere when $\varepsilon \in C^\beta$, $\mu \in W^{1,p}$, $p > 3$, and the difference between $\mu$ and the identity matrix is small in the sense of $L^\infty$ norm.

Based on the regularity results of $H$ and taking into account the symmetric structure of the equations, we can easily derive the interior estimates of the solution $E$. However, the boundary conditions of $E$ and $H$ are different, so it is necessary to use a different approach to derive the boundary estimates for $E$. By developing a different approach for the boundary estimates of $E$, we obtain the exact same regularity for $E$ and $H$ if we give the same assumptions on the coefficients $\varepsilon$ and $\mu$ respectively.

Section 3 gives the proofs of the results given in Section 2. Each of the proofs is made of two main parts, of which the first part consists of the proofs corresponding to the homogeneous case, and the second part is the proof of each theorem given in Section 2.
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2. Main results

We investigate the regularity of the solutions of the equation \( (1.1) \) with inhomogeneous boundary conditions. Without loss of the generality, we will first study the regularity of the solution \( H \), since the regularity of \( E \) can be studied similarly. By \( (1.1) \), \( H \) satisfies the following equation

\begin{equation}
(2.1) \quad \nabla \wedge (\varepsilon^{-1}(x) \nabla \wedge H) = k_1^2 \mu(x) H - k_2 J_m + \nabla \wedge (\varepsilon^{-1}(x) J_e),
\end{equation}

and

\begin{equation}
(2.2) \quad \text{div}(\mu(x) H) = k_3 \text{div} J_m,
\end{equation}

where the constants are given by \( k_1^2 = \omega^2 \varepsilon_0 \mu_0, k_2 = i \omega \varepsilon_0, \) and \( k_3 = i/(\omega \mu_0). \) By \( (1.1) \), \( H \) satisfies the following boundary condition

\begin{equation}
(2.3) \quad (\varepsilon^{-1} \nabla \wedge H) \wedge \nu = (\varepsilon^{-1} J_e) \wedge \nu - k_2 G \wedge \nu \quad \text{on} \; \partial \Omega.
\end{equation}

Then by applying \( (3.52) \) in \( [1], \) i.e.,

\begin{equation}
(2.4) \quad \text{div}_{\partial \Omega}(E \wedge \nu) = (\nabla \wedge E) \cdot \nu,
\end{equation}

then

\begin{equation}
(2.5) \quad (\mu H) \cdot \nu = k_3 J_m \cdot \nu - k_3 (\nabla \wedge G) \cdot \nu \quad \text{on} \; \partial \Omega.
\end{equation}

**Theorem 1** (\( H^1 \) regularity). Let \( \mu(x) \in W^{1,3+\delta}(\Omega)^{3 \times 3} \), where \( \delta > 0. \) Suppose \( \varepsilon^{-1}(x) \in L^\infty(\Omega)^{3 \times 3} \) and satisfies \( (2.2) \). If \( H \) is a weak solution of equations \( (2.1) \) and \( (2.2) \), with the boundary conditions \( (2.3) \) and \( (2.5) \), then \( H \in H^1(\Omega) \) and

\[
\|H\|_{H^1(\Omega)} \leq C (\|\nabla \wedge G\|_{L^2(\Omega)} + \|J_m\|_{W^{1,2}(\Omega)} + \|H\|_{L^2(\Omega)} + \|J_e\|_{L^2(\Omega)} + \|(J_m - \nabla \wedge G) \cdot \nu\|_{H^1(\partial \Omega)}).
\]

**Remark 2.** The result given in Theorem 1 is same with the result addressed in Theorem 1 in Alberti and Capdeboscq [1], however we derive the result by developing a new approach. We obtain the subsequent new regularity results given in the theorems below based on the approach developed in this theorem.

Based on the \( H^1 \)-regularity, by applying the Stampacchia interpolation theorem, we derive the \( W^{1,p} \) estimate of \( H \) for every \( p \geq 2 \) when the coefficient \( \varepsilon^{-1}(x) \) is simply continuous.

**Theorem 3** (\( W^{1,p} \) regularity). Let \( \mu(x) \in W^{1,3+\delta}(\Omega)^{3 \times 3} \), where \( \delta > 0. \) Suppose \( \varepsilon^{-1}(x) \in C^0(\Omega)^{3 \times 3} \) is a positive definite matrix, \( J_m, J_e \in L^2(\Omega) \) and \( G \in H_{\text{loc}}(\text{curl}; \Omega). \) If \( H \) is a weak solution of equations \( (2.1) \) and \( (2.2) \), with the boundary conditions \( (2.3) \) and \( (2.5) \), then \( H \in W^{1,p}(\Omega) \) for every \( p \geq 2 \) and

\[
\|H\|_{W^{1,p}(\Omega)} \leq C (\|\nabla \wedge G\|_{L^p(\Omega)} + \|J_m\|_{W^{1,p}(\Omega)} + \|H\|_{L^2(\Omega)} + \|J_e\|_{L^p(\Omega)} + \|(J_m - \nabla \wedge G) \cdot \nu\|_{W^{-1,\frac{1}{p},q}(\partial \Omega)}),
\]

where \( \frac{1}{p} + \frac{1}{q} = 1. \)

**Remark 4.** To produce the same regularity result, we assume that \( \varepsilon^{-1}(x) \) is simply continuous and \( \mu(x) \in W^{1,3+\delta}(\Omega)^{3 \times 3} \) while Theorem 3 in [1] assumes that both \( \mu(x) \) and \( \varepsilon(x) \) are in \( W^{1,3+\delta}(\Omega)^{3 \times 3} \).
Now we extend the results and give $C^{1, \alpha}$-regularity of $H$ almost everywhere provided that $\varepsilon^{-1}(x)$ is Hölder continuous.

**Theorem 5** ($C^{1, \alpha}$ regularity). Let $\mu(x) \in W^{1, 3+\delta}(\Omega)^{3 \times 3}$, where $\delta > 0$. Suppose there exists $\delta_0 > 0$ small such that $\|\mu(x) - I\|_{C^0(\Omega)^{3 \times 3}} < \delta_0$ where $I$ is the $3 \times 3$ identity matrix, and $k_1$ is not a Maxwell eigenvalue. Suppose $\varepsilon^{-1}(x) \in C^\beta(\Omega)^{3 \times 3}$, $0 < \beta < 1$, and $H$ is a weak solution of equations (2.1) and (2.2) with $J_m = J_e = 0$, then there exists an open set $\Omega_h \subset \Omega$ and $\alpha \in (0, 1)$, such that $\text{meas}(\Omega \setminus \Omega_h) = 0$, and for any $x_0 \in \Omega_h$, there exists $r > 0$ such that

$$\|H\|_{C^{1, \alpha}(B_r(x_0))} \leq C(1 + \|H\|_{L^2(\Omega)}),$$

where $B_r(x_0)$ is the ball centering at $x_0$ with radius $r > 0$.

Here we say $k_1$ is a Maxwell eigenvalue if and only if there exist an integer $j$ and a point $x_0 \in \Omega$, such that $k_1 = \varepsilon_j$, where $\varepsilon_j$ is the $j$th Maxwell eigenvalue guaranteed by Theorem 4.18 of the book [15] (which is written as $k_j$ in [15]), with the coefficients $\varepsilon \equiv \varepsilon(x_0)$ and $\mu \equiv \mu(x_0)$.

**Remark 6.** The results obtained in Theorem 3 and Theorem 5 are new regularity results for the solution, and the low regularity assumption on the coefficients are important for many application problems, i.e. electromagnetic inverse problems for liquid crystals in which the permeability $\mu$ is identity matrix and permittivity $\varepsilon$ is necessarily to have a low regularity property, see [2] for more detail.

**Remark 7.** We study the $C^{1, \alpha}$ regularity property of the solutions of the homogeneous equation in Theorem 3 and one of the simple counterexamples for the inhomogeneous equation can be constructed when $H = (1 + \sigma V(x_1), 0, 0), \mu = \frac{1}{1 + \sigma V(x_1)} I, \varepsilon = I, J_e = (0, 0, 0), J_m = -i\omega(1, 0, 0)$, where $V(x_1)$ is the Volterra's function, $\sigma$ is any small positive constant, and $I$ is the $3 \times 3$ identity matrix.

Similar to the regularity properties of $H$ given above, we also derive the regularity properties for the electrical field $E$ when the coefficients are given the similar assumptions.

**Theorem 8** (Regularity for $E$). Let $\varepsilon(x) \in W^{1, 3+\delta}(\Omega)^{3 \times 3}$, where $\delta > 0$. Suppose the condition (1.2) holds, and if $E$ is a weak solution of the equations (1.1), then $E \in H^1(\Omega)$ and

$$\|E\|_{H^1(\Omega)} \leq C(\|E\|_{L^2(\Omega)} + \|\nabla \cdot G\|_{L^2(\Omega)} + \|J_e\|_{W^{1, 2}(\Omega)} + \|J_m\|_{L^2(\Omega)}).$$

Moreover, if $\mu^{-1}(x) \in C^0(\Omega)^{3 \times 3}$ and satisfies the condition (1.2), then the following inequality holds

$$\|E\|_{W^{1, p}(\Omega)} \leq C(\|E\|_{L^2(\Omega)} + \|\nabla \cdot G\|_{L^p(\Omega)} + \|J_e\|_{W^{1, 2}(\Omega)} + \|J_m\|_{L^p(\Omega)}),$$

where $p \geq 2$ and the constant $C$ does not depend on the solution $E$; Suppose $\mu^{-1}(x) \in C^\beta(\Omega)^{3 \times 3}$, and $k_1$ is not a Maxwell eigenvalue; If there exits $\delta_0 > 0$ small such that $\|\varepsilon(x) - I\|_{C^0(\Omega)^{3 \times 3}} < \delta_0$, and exist an open set $\Omega_h \subset \Omega$ and $\alpha \in (0, 1)$, such that $\text{meas}(\Omega \setminus \Omega_h) = 0$, then for any $x_0 \in \Omega_h$ there exists
r > 0 such that the solution $E$ of the homogeneous Maxwell’s equations ($J_m = J_e = 0$) satisfies that

$$||E||_{L^{1,\alpha}(B_r(x_0))} \leq C(1 + ||E||_{L^2(\Omega)})$$

where $B_r(x_0)$ is the ball centering at $x_0$ with radius $r > 0$.

**Remark 9.** Clearly, if we exchange the role of $\varepsilon(x)$ and $\mu(x)$, then the arguments given in the theorems above can be easily applied to derive the regularity of $E$. However $E$ and $H$ hold different boundary conditions in Theorem 4 and Theorem 5, hence we use different methods to prove the boundary estimate of $E$.

We combine all the previous results together such that if $\mu(x)$ and $\varepsilon(x)$ are regular enough, then we can have the following general results.

**Theorem 10** (Regularity for $(E, H)$). Let $\mu(x), \varepsilon(x) \in W^{1,3+\delta}(\Omega)^{3\times3}$, where $\delta > 0$. Let $\varepsilon(x)$ and $\mu(x)$ satisfy (1.2). Then the solution $(E, H)$ satisfies the estimate that

$$||H||_{W^{1,p}(\Omega)} + ||E||_{W^{1,p}(\Omega)} \leq C(||\nabla \wedge G||_{L^p(\Omega)} + ||J_m||_{W^{1,p}(\Omega)} + ||J_e||_{W^{1,p}(\Omega)} + ||E||_{L^2(\Omega)} + ||H||_{L^2(\Omega)} + ||(J_m - \nabla \wedge G) \cdot \nu||_{W^{1,\frac{3}{2}}(\partial\Omega)}),$$

where $\frac{1}{p} + \frac{1}{q} = 1$. Moreover, we assume $k_1$ is not Maxwell eigenvalue, and there exists $\delta_0 > 0$ small such that $||\mu(x) - I||_{L^\infty(\Omega)^{3\times3}} + ||\varepsilon(x) - I||_{L^\infty(\Omega)^{3\times3}} < \delta_0$. Then there exists an open set $\Omega_h \subset \Omega$ and $\alpha \in (0, 1)$, such that $\text{mes} (\Omega \setminus \Omega_h) = 0$, and for any $x_0 \in \Omega_h$, there exists $r > 0$ such that, the solutions $E$ and $H$ of the homogeneous Maxwell’s equations ($J_m = J_e = 0$) satisfy the following estimates

$$||E||_{C^{1,\alpha}(B_r(x_0))} + ||H||_{C^{1,\alpha}(B_r(x_0))} \leq C(1 + ||E||_{L^2(\Omega)} + ||H||_{L^2(\Omega)}),$$

where $B_r(x_0)$ is the ball centering at $x_0$ with radius $r > 0$.

The proofs of the theorems are given in section 3.

### 3. Proof of the theorems

3.1. **Proof of Theorem 1**. To prove Theorem 1 we first develop the same regularity results for the homogeneous equations with homogeneous boundary conditions. In another word, we first make the structure of the equations simpler to develop a better approach which can be generalized to the non-homogeneous Maxwell’s equations.

**Proposition 11.** Let $\mu(x) \in W^{1,3+\delta}(\Omega)^{3\times3}$, where $\delta > 0$. Suppose $\varepsilon^{-1}(x) \in L^\infty(\Omega)^{3\times3}$ and that the condition (1.2) holds. If $H$ is a weak solution of the following equations

$$(3.1)\quad \nabla \wedge (\varepsilon^{-1}(x) \nabla \wedge H) = k^2 \mu(x) H, \quad \text{in} \ \Omega$$

with the boundary condition $(\varepsilon^{-1} \nabla \wedge H) \cdot \nu = 0$ on $\partial\Omega$, then $H \in H^1(\Omega)$ and

$$||H||_{H^1(\Omega)} \leq C ||H||_{L^2(\Omega)}.$$
Proof of Proposition 11. We first show the estimate of $\nabla \wedge H$ which can be proved by the simple energy estimate as follows.

**Step 1.** Estimates of $\nabla \wedge H$. Multiply both sides of the equation (3.1) by $\overline{H}$, and integrate it by parts, we get that

$$\int_{\Omega} \varepsilon^{-1}(x)(\nabla H \cdot (\nabla \wedge H)) dx + \int_{\partial \Omega} ((\varepsilon^{-1} \nabla \wedge H) \wedge \nu) \cdot H d\sigma = k^2 \int_{\Omega} (\mu H) \cdot \overline{H} dx.$$

Notice that $(\varepsilon^{-1} \nabla \wedge H) \wedge \nu = E \wedge \nu = 0$ on $\partial \Omega$, and by applying the condition (1.2), we obtain that

$$(3.2) \int_{\Omega} |\nabla \wedge H|^2 dx \leq C \int_{\Omega} |H|^2 dx.$$

**Step 2.** Helmholtz decomposition. From Lemma 28 which listed in the Appendix (see for more detail of the lemma in Amrouche, Seloula [6] and Amrouche, Bernardi, Dauge [7]), we know that $H$ can be decomposed as $H = \nabla \varphi + \nabla \wedge A$. Notice that $\nabla \wedge (\nabla \wedge A) = \nabla \wedge H$, and $\text{div}(\nabla \wedge A) = 0$, so

$$\nabla \wedge A \in W^{1,2}(\Omega),$$

provided if $\nabla \wedge H \in L^2(\Omega)$. More precisely, we have the estimates

$$(3.3) \|\nabla \wedge A\|_{W^{1,2}(\Omega)} \leq C \left( \|\nabla \wedge H\|_{L^2(\Omega)} + \|H\|_{L^2(\Omega)} \right).$$

Next from the equation (3.1), we get that

$$k^2 \text{div}(\mu \nabla \varphi + \mu \nabla \wedge A) = - \text{div}(\nabla \wedge E) = 0,$$

hence $\varphi$ satisfies the following elliptic equation

$$(3.4) \text{div}(\mu \nabla \varphi) = - \text{div}(\mu \nabla \wedge A) \quad \text{in } \Omega.$$

Since (3.1) is the homogeneous form of the Maxwell equation, hence we use the homogeneous form of (2.5) on the boundary,

$$0 = (\mu H) \cdot \nu = (\mu \nabla \varphi + \mu \nabla \wedge A) \cdot \nu \quad \text{on } \partial \Omega.$$

Then $\varphi$ satisfies

$$(3.5) (\mu \nabla \varphi) \cdot \nu = -(\mu \nabla \wedge A) \cdot \nu \quad \text{on } \partial \Omega.$$

Multiply $\varphi$ on both sides of the equation (3.3) and integrate it by part,

$$\int_{\partial \Omega} \overline{\varphi}(\mu \nabla \varphi) \cdot \nu d\sigma - \int_{\Omega} \mu \nabla \varphi \cdot \nabla \varphi dx = - \int_{\partial \Omega} \varphi(\mu \nabla \wedge A) \cdot \nu d\sigma + \int_{\Omega} \mu \nabla \wedge A \cdot \nabla \varphi dx.$$

By employing the boundary condition (3.5), and the Hölder inequality, we have that

$$(3.6) \|\nabla \varphi\|_{L^2(\Omega)} \leq C \|\mu \nabla \wedge A\|_{L^2(\Omega)} \leq C \|\nabla \wedge H\|_{L^2(\Omega)}.$$

**Step 3.** Interior estimates. Let $\eta$ be a smooth positive cut-off function such that $\eta \equiv 1$ in $B_r$ and $\eta = 0$ on $\partial B_R$, where $B_r = B_r(x_0)$ and $B_R = B_R(x_0)$ with $r < R$ and $B_R(x_0) \subset \Omega$. Then take derivatives $\partial_k$ on both
sides of the equation (3.4) and multiply $\eta^2 \partial_k \varphi$, where $k = 1, 2, 3$. After an integration by part, it gives that

$$\lambda \int_{B_R} \eta^2 |\partial_k \nabla \varphi|^2 dx \leq \int_{B_R} \eta^2 \mu \partial_k \nabla \varphi \cdot \partial_k \nabla \varphi dx$$

$$= - \int_{B_R} 2 \eta \partial_k \varphi \mu \nabla \eta \cdot \partial_k \nabla \varphi dx - \int_{B_R} \eta^2 \partial_k \mu \nabla \varphi \cdot \partial_k \nabla \varphi dx - 2 \int_{B_R} \eta \partial_k \varphi \partial_k \mu \nabla \eta \cdot \nabla \varphi dx$$

$$+ 2 \int_{B_R} \eta \partial_k \varphi \partial_k (\mu \nabla \varphi) \cdot \nabla \eta dx + \int_{B_R} \eta^2 \partial_k (\mu \nabla \varphi) \cdot \partial_k \nabla \varphi dx.$$ 

Let us set that

$$I_1 = - \int_{B_R} 2 \eta \partial_k \varphi \mu \nabla \eta \cdot \partial_k \nabla \varphi dx, \quad I_2 = - \int_{B_R} \eta^2 \partial_k \mu \nabla \varphi \cdot \partial_k \nabla \varphi dx,$$

$$I_3 = - 2 \int_{B_R} \eta \partial_k \varphi \partial_k \mu \nabla \eta \cdot \nabla \varphi dx, \quad I_4 = 2 \int_{B_R} \eta \partial_k \varphi \partial_k (\mu \nabla \varphi) \cdot \nabla \eta dx,$$

$$I_5 = \int_{B_R} \eta^2 \partial_k (\mu \nabla \varphi) \cdot \partial_k \nabla \varphi dx,$$

then we can get that

$$I_1 \leq \epsilon \int_{B_R} \eta^2 |\partial_k \nabla \varphi|^2 dx + C_\epsilon \|\mu\|_{L^\infty(B_R)} \int_{\Omega} |\nabla \eta|^2 |\partial_k \varphi|^2 dx,$$

$$I_2 \leq C \left( \int_{B_R} \eta^2 |\partial_k \nabla \varphi|^2 dx \right)^{\frac{3}{2}} \left( \int_{B_R} |\nabla \varphi|^6 dx \right)^{\frac{1}{2}} \left( \int_{B_R} \eta^3 |\partial_k \mu|^3 dx \right)^{\frac{1}{2}},$$

$$I_3 \leq C \left( \int_{B_R} \eta^3 |\nabla \eta|^3 |\partial_k \mu|^3 dx \right)^{\frac{1}{2}} \left( \int_{B_R} |\nabla \varphi|^3 dx \right)^{\frac{1}{2}},$$

$$I_4 \leq \epsilon \int_{B_R} \eta^2 |\partial_k (\mu \nabla \varphi)\varphi|^2 dx + C_\epsilon \int_{B_R} |\nabla \eta|^2 |\partial_k \varphi|^2 dx,$$

$$I_5 \leq \epsilon \int_{B_R} \eta^2 |\partial_k \nabla \varphi|^2 dx + C_\epsilon \int_{B_R} \eta^2 |\partial_k (\mu \nabla \varphi)\varphi|^2 dx,$$

where $\epsilon$ is a small constant. Notice that

$$||\partial_k (\mu \nabla \varphi)\varphi||_{L^2(\Omega)} \leq ||\mu||_{L^\infty(B_R)} ||\partial_k (\nabla \varphi)\varphi||_{L^2(B_R)} + ||\mu||_{W^{1,3}(B_R)} ||\nabla \varphi||_{L^6(B_R)}$$

$$\leq C(||\mu||_{W^{1,3}(B_R)} + ||\mu||_{L^\infty(B_R)}) ||\nabla \varphi||_{L^2(B_R)}.$$

Then when $\epsilon$ is small enough, we have $I_1 \leq C||\nabla \varphi||_{L^2(\Omega)}^2$ and then by applying the embedding theorem we can get that

$$I_2 \leq C \sum_{k=1}^3 \left( \int_{B_R} \eta^2 |\partial_k \nabla \varphi|^2 dx \right)^{\frac{3}{2}} \left( \int_{B_R} |\nabla \varphi|^2 dx \right)^{\frac{1}{2}} \left( \int_{B_R} |\partial_k \mu|^3 dx \right)^{\frac{3}{2}},$$

$$I_3 \leq C \sum_{k=1}^3 \left( \int_{B_R} |\partial_k \mu|^3 dx \right)^{\frac{3}{2}} \left( \int_{B_R} |\nabla \varphi|^2 dx \right)^{\frac{1}{2}} + C \left( \int_{B_R} |\nabla \varphi|^2 dx \right)^{\frac{3}{2}},$$

$$I_4 + I_5 \leq C(||\mu||_{W^{1,3}(B_R)}^3 + ||\mu||_{L^\infty(B_R)}^3) ||\nabla \varphi||_{L^2(B_R)}^3 + C ||\nabla \varphi||_{L^2(B_R)}^3.$$
Since \( \sum_{k} \int_{B_{R}} \eta^{2} |\partial_{k} \nabla \varphi|^{2} dx \leq \sum_{i=1}^{5} I_{i} \), and it leads to

\[
\sum_{k} \int_{B_{R}} \eta^{2} |\partial_{k} \nabla \varphi|^{2} dx \\
\leq C \sum_{k=1}^{5} (\int_{B_{R}} \eta^{2} |\partial_{k} \nabla \varphi|^{2} dx)^{1/2} (\int_{B_{R}} |\nabla^{2} \varphi|^{2} dx)^{1/2} (\int_{B_{R}} |\partial_{k} \mu|^{2 + \delta} dx)^{1/2 + \delta} R^{\delta/2} \\
+ C \sum_{k} (\int_{B_{R}} |\partial_{k} \mu|^{3} dx)^{1/3} (\epsilon \int_{B_{R}} |\nabla \varphi|^{2} dx)^{1/3} + C (\int_{B_{R}} |\nabla \varphi|^{2} dx)^{1/2} \\
+ C (||\mu||_{W^{1,3}(B_{R})}^{2} + ||\mu||_{L^{\infty}(B_{R})}^{2}) ||\nabla \wedge A||_{W^{1,2}(B_{R})} + C ||\nabla \varphi||_{L^{2}(B_{R})}^{2},
\]

Again let \( \epsilon \) and \( R \) be small enough, by using the embedding theorem \( ||\mu||_{W^{1,3}(B_{R})} + ||\mu||_{L^{\infty}(B_{R})} \leq C ||\mu||_{W^{1,3+i}(B_{R})} \) for any \( \delta > 0 \),

\[
(\int_{B_{R}} |\nabla^{2} \varphi|^{2} dx)^{1/2} \leq C (||\nabla \wedge A||_{W^{1,2}(B_{R})} + ||\nabla \varphi||_{L^{2}(B_{R})}) \leq C (||\nabla \wedge H||_{L^{2}(B_{R})} + ||H||_{L^{2}(B_{R})}),
\]

where the constant \( C \) depends on the norm \( ||\mu||_{W^{1,3+i}} \). Combining the equations \( (3.7) \) with \( (3.2) \), \( (3.3) \) and \( (3.4) \), then we obtain the \( H^{1} \) interior estimate of \( H \) as claimed in the proposition.

**Step 4.** Boundary estimates. Notice that \( \varphi \) is a solution of the conformal derivatives problem of a scalar elliptic equation of second order, so one can use the standard argument to derive the boundary estimates of the solutions. More precisely, for any point \( x_{0} \in \partial \Omega \), we can introduce an orthogonal transformation of the coordinates,

\[
y = \Phi(x), \quad x = \Psi(y).
\]

In the new coordinates \( y = (y_{1}, y_{2}, y_{3}) \), and we define \( B_{R+}(y_{0}) := B_{R}(y_{0}) \cap \{y_{3} > 0\} \), where \( R > 0 \) and \( y_{0} = \Phi(x_{0}) \). Let \( \tilde{\varphi}(y) = \varphi(\Psi(y)) \), and for the simplicity, we write \( \varphi \) instead of \( \tilde{\varphi} \). Then \( \varphi \) satisfies the equation

\[
\nabla^{2} \varphi = -\nabla \wedge A,
\]

where \( \nabla = (\partial_{x_{1}}, \partial_{x_{2}}, \partial_{x_{3}}) \), \( \tilde{\mu}_{kl} = \sum_{i,j=1}^{3} \frac{\partial y_{i}}{\partial x_{j}} \frac{\partial y_{j}}{\partial x_{i}} \mu_{ij} \), \( \tilde{\mu}_{kj} = \sum_{i=1}^{3} \frac{\partial y_{i}}{\partial x_{j}} \mu_{ij} \), \( \tilde{b}_{m} = \sum_{i,j,k,l=1}^{3} \frac{\partial^{2} y_{m}}{\partial x_{i} \partial x_{j}} \frac{\partial y_{k}}{\partial x_{i}} \frac{\partial y_{l}}{\partial x_{j}} \mu_{ij} \), and \( \tilde{b}_{j} = \sum_{i,k,l=1}^{3} \frac{\partial^{2} y_{m}}{\partial x_{i} \partial x_{j}} \frac{\partial y_{k}}{\partial x_{i}} \mu_{ij} \). Moreover, \( \varphi \) satisfies the following boundary condition on \( \{y_{3} = 0\} \)

\[
(\tilde{\mu} \nabla \varphi) \cdot e_{3} = - (\tilde{\mu} \nabla \wedge A) \cdot e_{3},
\]

where \( e_{3} \) is the unit direction of \( y_{3} \)-axis. Since the boundary \( \partial \Omega \) is of \( C^{1,1} \), the coefficient \( \tilde{\mu} \) and \( \tilde{\mu} \) have the same regularity as \( \mu \), while \( \tilde{b} \) and \( \tilde{b} \) are \( L^{\infty} \) functions.

Again let \( \eta \) be a smooth positive cut-off function such that \( \eta \equiv 1 \) in \( B_{r}(y_{0}) \) and \( \eta = 0 \) on \( \partial B_{R}(y_{0}) \). Then let us take the tangential derivatives \( \partial_{k} \), where \( \partial_{k} = \partial_{y_{k}} \) with \( k = 1 \) or \( 2 \) on both sides of the equation \( (3.8) \) and multiply.
\[ \eta^2 \partial_k \varphi. \] After an integration by part, it gives that
\[
- \int_{B_{R+}(y_0)} \partial_k (\bar{\mu} \nabla \varphi) \cdot \bar{\nabla} (\eta^2 \partial_k \varphi) dy + \int_{B_{R+}(y_0)} \bar{b} \cdot \nabla \varphi \partial_k (\eta^2 \partial_k \varphi) dy \\
+ \int_{(y_3=0) \cap B_{R+}(y_0)} \eta^2 \partial_k \varphi \partial_k (\bar{\mu} \nabla \varphi) \cdot e_3 dy \\
= \int_{B_{R+}(y_0)} \partial_k (\bar{\mu} \nabla \wedge A) \cdot \bar{\nabla} (\eta^2 \partial_k \varphi) dy - \int_{B_{R+}(y_0)} \bar{b} \cdot \nabla_x \wedge A \partial_k (\eta^2 \partial_k \varphi) dy \\
- \int_{(y_3=0) \cap B_{R+}(y_0)} \eta^2 \partial_k \varphi \partial_k (\bar{\mu} \nabla \wedge A) \cdot e_3 dy.
\]

Applying the boundary condition (3.9), we have that
\[
\partial_k (\bar{\mu} \nabla \varphi) \cdot e_3 = - \partial_k (\bar{\mu} \nabla \wedge A) \cdot e_3,
\]
on \( y_3 = 0 \), for \( k = 1 \) or \( 2 \). Therefore, the boundary terms vanish and we have
\[
- \int_{B_{R+}(y_0)} \partial_k (\bar{\mu} \nabla \varphi) \cdot \bar{\nabla} (\eta^2 \partial_k \varphi) dy + \int_{B_{R+}(y_0)} \bar{b} \cdot \nabla \varphi \partial_k (\eta^2 \partial_k \varphi) dy \\
= \int_{B_{R+}(y_0)} \partial_k (\bar{\mu} \nabla \wedge A) \cdot \bar{\nabla} (\eta^2 \partial_k \varphi) dy - \int_{B_{R+}(y_0)} \bar{b} \cdot \nabla_x \wedge A \partial_k (\eta^2 \partial_k \varphi) dy.
\]

Then following Step 3, we can derive the estimates of \( \partial_k \nabla \varphi \) for \( k = 1 \) or \( 2 \). By applying the equation (3.9), it is easy to see that \( \partial_k^2 \varphi \) is bounded by \( \partial_k \nabla \varphi \) for \( k = 1 \) or \( 2 \). Then we have the boundary estimates in the \( y \)-coordinates which satisfies that
\[
\int_{B_{R+}(y_0)} |\nabla^2 \varphi|^2 dx \leq C (\|\nabla \wedge A\|_{W^{1,2}(B_{R+}(y_0))} + \|\nabla \varphi\|_{L^2(B_{R+}(y_0))}) \\
\leq C (\|\nabla \wedge H\|_{L^2(B_{R+}(y_0))} + \|H\|_{L^2(B_{R+}(y_0))}).
\]

**Step 5.** Global estimates. Finally, let \( \eta_i \) be cut-off functions which satisfies \( \sum_i \eta_i \equiv 1 \) and such that the set of all the subregions
\[ \Omega_i := \{ x \in \Omega \; ; \; \eta_i(x) > 0 \} \]
together is a finite cover of \( \Omega \) with the property that \( \text{diam}\{\Omega_i\} \leq R \). Based on the interior and boundary estimate, we can prove the \( H^1 \)-regularity of \( H \) as follows
\[
\|H\|_{H^1(\Omega)} \leq \|\nabla (\nabla \varphi)\|_{L^2(\Omega)} + \|\nabla (\nabla \wedge A)\|_{L^2(\Omega)} + \|H\|_{L^2(\Omega)} \\
\leq C (\sum_i \|\eta_i \nabla (\nabla \varphi)\|_{L^2(\Omega)} + \|\nabla \wedge H\|_{L^2(\Omega)} + \|H\|_{L^2(\Omega)}) \\
\leq C (\|\nabla \wedge H\|_{L^2(\Omega)} + \|H\|_{L^2(\Omega)}) \\
\leq C \|H\|_{L^2(\Omega)}.
\]

Based on the new approach, we are now going to prove theorem 1 for the non-homogeneous case.
Proof of theorem \[ \text{Let} \]
\[ (3.10) \]
\[ \tilde{E} = E - G, \quad \tilde{H} = H. \]

Obviously, \( \tilde{E} \) and \( \tilde{H} \) satisfy the system
\[ \begin{cases} 
\nabla \wedge \tilde{E} - i \omega \mu \mu(x) \tilde{H} = \tilde{J}_m \\
\nabla \wedge \tilde{H} + i \omega \varepsilon(x) \tilde{E} = \tilde{J}_e \\
\tilde{E} \wedge \nu = 0 
\end{cases} \quad \text{on } \partial \Omega. \]

where
\[ \tilde{J}_m := J_m - \nabla \wedge G, \quad \tilde{J}_e := J_e - i \omega \varepsilon(x) G \quad \text{in } \Omega. \]

Then \( \tilde{H} \) satisfies the equations
\[ (3.12) \quad \nabla \wedge (\varepsilon^{-1}(x) \nabla \wedge \tilde{H}) = k_1^2 \mu(x) \tilde{H} - k_2 \tilde{J}_m + \nabla \wedge (\varepsilon^{-1}(x) \tilde{J}_e), \]
and
\[ (3.13) \quad \text{div}(\mu(x) \tilde{H}) = k_3 \text{div}\tilde{J}_m = k_3 \text{div}J_m. \]

On the boundary \( \partial \Omega \), \( \tilde{H} \) satisfies that
\[ (3.14) \quad (\varepsilon^{-1} \nabla \wedge \tilde{H}) \wedge \nu = -i \omega \varepsilon \wedge \tilde{H} \wedge \nu = \varepsilon^{-1} \tilde{J}_e \wedge \nu. \]

**Step 1.** Interior Estimates

Multiply \( \tilde{H} \) on both sides of equation \( (3.12) \) and integrate it by part, then
\[ (3.15) \quad \int_\Omega (\varepsilon^{-1}(x) \nabla \wedge \tilde{H}) \cdot (\nabla \wedge \tilde{H}) \, dx + \int_{\partial \Omega} ((\varepsilon^{-1} \nabla \wedge \tilde{H}) \wedge \nu) \cdot \tilde{H} \, d\sigma = k_1^2 \int_\Omega (\mu \tilde{H}) \cdot \tilde{H} \, dx - k_2 \int_\Omega \tilde{J}_m \cdot \tilde{H} \, dx + \int_\Omega (\varepsilon^{-1} \tilde{J}_e) \cdot (\nabla \wedge \tilde{H}) \, dx + \int_{\partial \Omega} (\varepsilon^{-1} \tilde{J}_e \wedge \nu) \cdot \tilde{H} \, d\sigma. \]

By applying the identity \( (3.14) \) to the equation \( (3.15) \), it is easy to see that the following inequality holds,
\[ (3.17) \quad \| \nabla \wedge \tilde{H} \|_{L^2(\Omega)} \leq C (\| \tilde{H} \|_{L^2(\Omega)} + \| \tilde{J}_e \|_{L^2(\Omega)} + \| \tilde{J}_m \|_{L^2(\Omega)}). \]

Next, let \( \tilde{H} = \nabla \varphi + \nu \wedge A \). Obviously, we have that
\[ \| \nabla \wedge A \|_{L^1(\Omega)} \leq \| \nabla \wedge \tilde{H} \|_{L^2(\Omega)} \leq C (\| \tilde{H} \|_{L^2(\Omega)} + \| \tilde{J}_e \|_{L^2(\Omega)} + \| \tilde{J}_m \|_{L^2(\Omega)}), \]
and \( \varphi \) satisfies the following scalar elliptic equation of second order
\[ (3.18) \quad \text{div}(\mu \nabla \varphi) = -\text{div}(\mu \nabla \wedge A) + k_3 \text{div}J_m. \]

Then take derivatives \( \partial_k \) on both sides of the equation \( (3.18) \) and multiply \( \eta^2 \partial_k \varphi \). After an integration by part, it gives that
\[ \lambda_1 \int_\Omega \eta^2 |\partial_k \nabla \varphi|^2 \, dx \leq \int_\Omega \eta^2 \mu \partial_k \nabla \varphi \cdot \partial_k \nabla \varphi \, dx \]
\[ = -\int_\Omega 2 \eta \partial_k \mu \nabla \eta \cdot \partial_k \nabla \varphi \, dx - \int_\Omega \eta^2 \partial_k \mu \nabla \varphi \cdot \partial_k \nabla \varphi \, dx - 2 \int_\Omega \eta \partial_k \varphi \partial_k \mu \nabla \eta \cdot \nabla \varphi \, dx \]
\[ + 2 \int_\Omega \eta \partial_k \varphi \partial_k (\mu \nabla \wedge A) \cdot \nabla \eta \, dx + \int_\Omega \eta^2 \partial_k (\mu \nabla \wedge A) \cdot \partial_k \nabla \varphi \, dx \]
\[ + 2 \int_\Omega \eta \partial_k \varphi \partial_k (k_3 J_m) \cdot \nabla \eta \, dx + \int_\Omega \eta^2 \partial_k (k_3 J_m) \cdot \partial_k \nabla \varphi \, dx. \]
Similar to the approach that we used in the argument for the homogeneous equations, we use the following notation,

\[ I_1 = - \int_{B_R} 2\eta \bar{\theta} \bar{\theta} \mu \nabla \eta \cdot \partial_k \nabla \varphi dx, \quad I_2 = - \int_{B_R} \eta^2 \partial_k \mu \nabla \varphi \cdot \partial_k \nabla \varphi dx, \]
\[ I_3 = -2 \int_{B_R} \eta \bar{\theta} \bar{\theta} \mu \nabla \eta \cdot \nabla \varphi dx, \quad I_4 = 2 \int_{B_R} \eta \bar{\theta} \bar{\theta} \mu (\mu \nabla \wedge A) \cdot \nabla \varphi dx, \]
\[ I_5 = \int_{\Omega} \eta^2 \partial_k (\mu \nabla \wedge A) \cdot \partial_k \nabla \varphi dx, \quad I_6 = 2 \int_{\Omega} \eta \bar{\theta} \bar{\theta} \partial_k (k_3 J_m) \cdot \nabla \varphi dx, \]
\[ I_7 = \int_{\Omega} \eta^2 \partial_k (k_3 J_m) \cdot \partial_k \nabla \varphi dx, \]

then we can get that

\[ I_1 \leq \epsilon \int_{B_R} \eta^2 |\partial_k \nabla \varphi|^2 dx + C \epsilon \eta^2 \int_{\Omega} |\nabla \varphi|^2 |\partial_k \varphi|^2 dx, \]
\[ I_2 \leq C \left( \int_{B_R} \eta^2 |\partial_k \nabla \varphi|^2 dx \right)^{\frac{1}{2}} \left( \int_{\Omega} |\nabla \varphi|^3 dx \right)^{\frac{1}{2}}, \]
\[ I_3 \leq C \left( \int_{B_R} \eta^3 |\nabla \eta| |\partial_k \varphi|^3 dx \right)^{\frac{1}{2}} \left( \int_{\Omega} |\nabla \varphi|^3 dx \right)^{\frac{1}{2}}, \]
\[ I_4 \leq \epsilon \int_{B_R} \eta^2 |\partial_k (\mu \nabla \wedge A)|^2 dx + C \epsilon \int_{\Omega} |\nabla \varphi|^2 |\partial_k \varphi|^2 dx, \]
\[ I_5 \leq \epsilon \int_{B_R} \eta^2 |\partial_k \nabla \varphi|^2 dx + C \epsilon \int_{B_R} \eta^2 |\partial_k (\mu \nabla \wedge A)|^2 dx, \]
\[ I_6 \leq \epsilon \int_{B_R} \eta^2 |\partial_k (J_m)|^2 dx + C \epsilon \int_{B_R} \eta^2 |\partial_k \varphi|^2 dx, \]
\[ I_7 \leq \epsilon \int_{B_R} \eta^2 |\partial_k \nabla \varphi|^2 dx + C \epsilon \int_{B_R} \eta^2 |\partial_k (J_m)|^2 dx. \]

Therefore, when the \( \epsilon \) is small enough, we have the interior estimates for any \( x_0 \in \Omega \),

\[ \left( \int_{B_r(x_0)} |\nabla^2 \varphi|^2 dx \right)^{\frac{1}{2}} \leq C (\int_{B_r} |\nabla \wedge \tilde{H}|^2 dx + \int_{\Omega} |\nabla \varphi|^2 |\partial_k \varphi|^2 dx, \]
\[ \leq C (\int_{B_r} |\nabla \wedge \tilde{H}|^2 dx + \int_{\Omega} |\nabla \varphi|^2 |\partial_k \varphi|^2 dx). \]

**Step 2. Boundary Estimates.** For the boundary estimates, by (3.17), it is easy to see that for any \( x_0 \in \partial \Omega \),

\[ (3.19) \quad \int_{B_{r+\epsilon} \cap \Omega} |\nabla \wedge \tilde{H}|^2 dx \leq C(R) \int_{\Omega} (|\tilde{H}|^2 + |\tilde{J}|^2 + |\tilde{J}_m|^2) dx. \]

Moreover, in the new \( y \)-coordinates as introduced in Step 4 of the proof of Proposition 11, \( \varphi := \tilde{\varphi}(y) \) satisfies the equation

\[ (3.20) \quad \text{div}(\mu \nabla \varphi) - \tilde{b} \cdot \nabla \varphi = -\text{div}(\mu \nabla \wedge A) + \tilde{b} \cdot \nabla \wedge A + k_3 \text{div} x J_m, \]

with the boundary condition on \( \{ y_3 = 0 \} \)

\[ (3.21) \quad (\mu \nabla \varphi) \cdot e_3 = -(\mu \nabla \wedge A) \cdot e_3 + k_3 \tilde{J}_m \cdot e_3. \]
Let η be a smooth positive cut-off function such that η ≡ 1 in B_r(y_0) and η = 0 on ∂B_R(y_0). Then take the tangential derivatives ∂_k on equation (3.21), where ∂_k = ∂_{y_k} and k = 1 or 2 on both sides of the equation above and multiply η^2 ∂_k φ. After an integration by part, it gives that

\[
- \int_{B_{R+}(y_0)} \partial_k (\tilde{\mu} \nabla \varphi) \cdot \nabla (\eta^2 \partial_k \varphi) \, dy + \int_{B_{R+}(y_0)} \tilde{b} \cdot \nabla \varphi \partial_k (\eta^2 \partial_k \varphi) \, dy + \int_{\{y_3=0\} \cap B_{R+}(y_0)} \eta^2 \partial_k \varphi \partial_k (\tilde{\mu} \nabla \varphi) \cdot e_3 \, d\sigma \\
= \int_{B_{R+}(y_0)} \partial_k (\tilde{\mu} \nabla_x \wedge A) \cdot \nabla (\eta^2 \partial_k \varphi) \, dy - \int_{B_{R+}(y_0)} \tilde{b} \cdot \nabla_x \wedge A \partial_k (\eta^2 \partial_k \varphi) \, dy - \int_{\{y_3=0\} \cap B_{R+}(y_0)} \eta^2 \partial_k \varphi \partial_k (\tilde{\mu} \nabla_x \wedge A) \cdot e_3 \, d\sigma.
\]

Here we use the fact that all the integrals on the boundary ∂B_{R+}(y_0) ∩ \{y_0 > 0\} is 0, due to the cut-off function η.

Then by the boundary condition (5.21) and the trace theorem,

\[
\int_{\{y_3=0\} \cap B_{R+}(y_0)} \eta^2 \partial_k \varphi \partial_k (\tilde{\mu} \nabla \varphi) \cdot e_3 \, d\sigma + \int_{\{y_3=0\} \cap B_{R+}(y_0)} \eta^2 \partial \varphi \partial_k (\tilde{\mu} \nabla_x \wedge A) \cdot e_3 \, d\sigma = k_3 \int_{\{y_3=0\} \cap B_{R+}(y_0)} \eta^2 \partial_k \varphi \partial_k \tilde{J}_m \cdot e_3 \, d\sigma \\
\leq C ||\eta^2 \partial_k \varphi||_{H^{\frac{1}{2}}(\{y_3=0\} \cap B_{R+}(y_0))} ||\partial_k (\tilde{J}_m \cdot \nu)||_{H^{-\frac{1}{2}}(\{y_3=0\} \cap B_{R+}(y_0))} \\
\leq C ||\varphi||_{L^2(B_{R+}(y_0))} ||\tilde{J}_m \cdot \nu||_{H^{\frac{1}{2}}(\{y_3=0\} \cap B_{R+}(y_0))}.
\]

Therefore, by the Hölder inequality, if the radius R is small enough, then

\[
\left( \int_{B_{R+}(y_0)} |\nabla^2 \varphi|^2 \, dx \right)^{\frac{1}{2}} \\
\leq C \left( ||\nabla \wedge A||_{W^{1,2}(B_{R+}(y_0))} + ||\nabla \varphi||_{L^2(B_{R+}(y_0))} + ||\tilde{J}_m||_{W^{1,2}(B_{R+})} + ||\tilde{J}_m \cdot \nu||_{H^{\frac{1}{2}}(\{y_3=0\} \cap B_{R+}(y_0))} \right) \\
\leq C \left( ||\tilde{H}||_{L^2(B_{R+})} + ||\tilde{J}_e||_{L^2(B_{R+})} + ||\tilde{J}_m||_{L^2(B_{R+})} + ||\tilde{J}_m||_{W^{1,2}(B_{R+})} + ||\tilde{J}_m \cdot \nu||_{H^{\frac{1}{2}}(\{y_3=0\} \cap B_{R+}(y_0))} \right) \\
\leq C \left( ||\nabla \wedge G||_{L^2(B_{R+})} + ||\tilde{J}_m||_{W^{1,2}(B_{R+})} + \left( ||\tilde{H}||_{L^2(B_{R+})} + ||\tilde{J}_e||_{L^2(\Omega)} + ||\tilde{J}_m||_{L^2(\Omega)} + ||\tilde{J}_m||_{W^{1,2}(\Omega)} + ||\tilde{J}_m \cdot \nu||_{H^{\frac{1}{2}}(\partial \Omega)} \right) \right).
\]

Hence by summarizing all the interior and boundary estimates together, we have that

\[
||\tilde{H}||_{H^1(\Omega)} = ||\tilde{H}||_{H^1(\Omega)} \\
\leq ||\nabla \wedge A||_{L^2(\Omega)} + ||\varphi||_{H^2(\Omega)} \\
\leq C \left( ||\tilde{H}||_{L^2(\Omega)} + ||\tilde{J}_e||_{L^2(\Omega)} + ||\tilde{J}_m||_{L^2(\Omega)} + ||\tilde{J}_m||_{W^{1,2}(\Omega)} + ||\tilde{J}_m \cdot \nu||_{H^{\frac{1}{2}}(\partial \Omega)} \right) \\
\leq C \left( ||\nabla \wedge G||_{L^2(\Omega)} + ||\tilde{J}_m||_{W^{1,2}(\Omega)} + \left( ||\tilde{H}||_{L^2(\Omega)} + ||\tilde{J}_e||_{L^2(\Omega)} + ||\tilde{J}_m||_{L^2(\Omega)} + ||\tilde{J}_m \cdot \nu||_{H^{\frac{1}{2}}(\partial \Omega)} \right) \right).
\]

It completes the proof of this theorem. \(\square\)
3.2. Proof of Theorem 3. As before, we develop the exact regularity results for the homogeneous equation before giving the proof of the theorem [3]. Notice that in the bounded domain $\Omega$, the Rellich-Kondratychov Theorem provides that, $\|H\|_{L^p(\Omega)} \leq C\|\nabla H\|_{L^q(\Omega)} + C\|H\|_{L^2(\Omega)}$ for $p \geq 2$, so we only need to estimate $\|\nabla H\|_{L^p(\Omega)}$. We first give the following Lemma to estimate the solutions of elliptic equations with constant coefficients. For any $x_0 \in \Omega$, let $0 < R \leq \text{dist}(x_0, \partial \Omega)$, and let $B_R = B_R(x_0) := \{x; |x - x_0| < R\}$.

**Lemma 12.** If $f_i^{m} \in L^p(B_R)$, suppose $u \in H_0^1(B_R)$ satisfies the following equation

$$
\sum_{m,\beta,i,j=1}^{3} \int_{B_R} A_{ij}^{m\beta} \nabla m u^i \nabla \beta \varphi^j \, dx = \sum_{i,m=1}^{3} \int_{B_R} f_i^{m} \nabla m \varphi \, dx, \quad \forall \varphi \in H_0^1(B_R),
$$

where $\nabla = (\nabla_1, \nabla_2, \nabla_3)$, $A_{ij}^{m\beta}$ are constants and satisfy that $|A_{ij}^{m\beta}| \leq M$. Further there is a positive constant $\lambda > 0$ such that $\sum_{i,j,m,\beta=1}^{3} A_{ij}^{m\beta} \eta^i \xi^j \mid \mid |\xi|^2$ for any $\eta^i, \xi^j \in \mathbb{R}$, then for any $2 \leq p < \infty$,

$$
\mid \mid \nabla u \mid \mid _{L^p(B_R)} \leq C\mid \mid f \mid \mid _{L^p(B_R)}.
$$

where $C$ depends on $p$.

**Proof.** By choosing $\varphi = \pi$, it is easy to see that $\mid \mid \nabla u \mid \mid _{L^2(B_R)} \leq C\mid \mid f \mid \mid _{L^2(B_R)}$. Again it is not hard to obtain the estimate $\mid \mid \nabla u \mid \mid _{\text{BMO}(B_R)} \leq \mid \mid \nabla u \mid \mid _{L^2(B_R)} \leq C\mid \mid f \mid \mid _{L^2(B_R)} \leq C\mid \mid f \mid \mid _{L^\infty(B_R)}$ for the elliptic equations with constant coefficients where the BMO space and $L^{2,n}$ space are introduced in the Appendix.

Now the Lemma is proved by the Stampacchia interpolation theorem. See the details of the Stampacchia interpolation theorem in the Appendix. \(\square\)

**Proposition 13.** Let $\mu(x) \in W^{1, 3+\delta}(\Omega)^{3 \times 3}$, where $\delta > 0$. Suppose $H \in H^1(\Omega)$ is a weak solution of the equations [3.1], with the boundary condition $(\varepsilon^{-1} \nabla \wedge H) \wedge \nu = 0$ on $\partial \Omega$, where $\varepsilon^{-1}(x) \in C^0(\Omega)^{3 \times 3}$ and satisfies the condition [1.2], then the following inequality holds

$$(3.22) \quad \|H\|_{W^{1,p}(\Omega)} \leq C\|H\|_{L^2(\Omega)}, \quad p \geq 2,$$

where the constant $C$ does not depend on the solutions $H$.

**Proof of Proposition 13.** We divide the proof into three steps.

**Step 1.** Interior regularity. For any $x_0 \in \Omega$, let $B_R := B_R(x_0) \subset \Omega$. Suppose $\eta$ is a cut-off function so that $0 \leq \eta \leq 1$ in $B_R$, $\eta = 1$ in $B_r$ and $\eta = 0$ on $\partial B_R$, where $0 < r < R$. Moreover, $|\nabla \eta| \leq \frac{C}{R-r}$.

For any $\psi(x) \in H_0^1(B_R(x_0))$, we have that
where \( p \) and \( q \) are defined by \( p = \frac{2}{2 - \frac{n}{p}} \) and \( q = \frac{2}{2 - \frac{n}{q}} \) respectively.

By applying Lemma 12 to equation (3.24), for any \( p \geq 2 \), we have that

\[
\int_{B_R} \varepsilon^{-1}(x_0) \nabla (\eta H) \cdot \nabla \psi(x) dx
\]

\[
= \int_{B_R} [\varepsilon^{-1}(x) - \varepsilon^{-1}(x)] \nabla (\eta H) \cdot \nabla \psi(x) dx + \int_{B_R} \varepsilon^{-1}(x) \nabla (\eta H) \cdot \nabla \psi(x) dx
\]

\[
= \int_{B_R} [\varepsilon^{-1}(x) - \varepsilon^{-1}(x)] \nabla (\eta H) \cdot \nabla \psi(x) dx + \int_{B_R} \varepsilon^{-1}(x) \nabla (\eta H) \cdot \nabla \psi(x) dx
\]

\[
+ \int_{B_R} \eta \varepsilon^{-1}(x) \nabla H \cdot \nabla \psi(x) dx
\]

\[
= \int_{B_R} [\varepsilon^{-1}(x) - \varepsilon^{-1}(x)] \nabla (\eta H) \cdot \nabla \psi(x) dx + \int_{B_R} \varepsilon^{-1}(x) \nabla (\eta H) \cdot \nabla \psi(x) dx
\]

\[
- \int_{B_R} \varepsilon^{-1}(x) \nabla H \cdot \nabla \psi(x) dx + \int_{B_R} \varepsilon^{-1}(x) \nabla (\eta H) \cdot \nabla \psi(x) dx
\]

\[
= \int_{B_R} [\varepsilon^{-1}(x) - \varepsilon^{-1}(x)] \nabla (\eta H) \cdot \nabla \psi(x) dx + \int_{B_R} \mathcal{G}_i \psi_i^i dx + \int_{B_R} \mathcal{F}_i^m \nabla_m \psi_i^i dx,
\]

where, by equation (3.1), we assume that

\[
\int_{B_R} \mathcal{G}_i \psi_i^i dx = \int_{B_R} k^2 \mu(x) H \cdot \eta \psi dx - \int_{B_R} \varepsilon^{-1}(x) \nabla H \cdot \nabla \eta \psi(x) dx,
\]

\[
\int_{B_R} \mathcal{F}_i^m \nabla_m \psi_i^i dx = \int_{B_R} \varepsilon^{-1}(x) \nabla \eta \cdot \nabla \psi(x) dx.
\]

Let \( \omega \in \mathcal{H}_0^1(B_R) \) satisfy

\[
(3.24) \quad - \int_{B_R} \delta^{m_3} \delta_{ij} \nabla \bar{\omega}_i \nabla_m \phi^j dx = \int_{B_R} \delta^{m_3} \mathcal{G}_i \nabla_m \phi^j dx, \quad \forall \phi \in \mathcal{C}^{1,1}(B_R),
\]

then we can write that

\[
(3.25) \quad \int_{B_R} \varepsilon^{-1}(x_0) \nabla (\eta H) \cdot \nabla \psi(x) dx = \int_{B_R} [\varepsilon^{-1}(x_0) - \varepsilon^{-1}(x)] \nabla (\eta H) \cdot \nabla \psi(x) dx
\]

\[
+ \int_{B_R} \tilde{\mathcal{F}}_i^m \nabla_m \psi_i^i dx,
\]

where \( \tilde{\mathcal{F}}_i^m = \mathcal{F}_i^m + \omega_i^m \).

By applying Lemma 12 to equation (3.24), for any \( p \geq 2 \), we have that

\[
(3.26) \quad ||\omega||_{L^{p^*}(B_R)} \leq C ||\nabla \omega||_{L^p(B_R)} \leq ||\mathcal{G}||_{L^q(B_R)},
\]

where \( p^* \) is the Sobolev conjugate of \( p \), so \( \frac{1}{p^*} = \frac{1}{p} - \frac{1}{n} \). By applying the Hölder inequality to equation (3.26), we obtain that

\[
(3.27) \quad ||\nabla (\eta H)||_{L^q(B_R)} \leq C ||[\varepsilon^{-1}(x_0) - \varepsilon^{-1}(x)] \nabla (\eta H)||_{L^q(B_R)} + C ||\tilde{\mathcal{F}}||_{L^q(B_R)},
\]

where \( q = p^* \) and we take \( p = 2 \) here, hence \( q = 2^* = 6 > 2 \) since \( 2^* = \frac{2n}{n-2} \) and \( n = 3 \) in this paper.
Let $R$ be small enough so that $C||\varepsilon^{-1}(x_0) - \varepsilon^{-1}(x)||_{L^\infty(B_R)} < 1$, then

$$(3.28)$$

$$||\nabla \wedge (\eta H)||_{L^q(B_R)} \leq C||\tilde{F}||_{L^q(B_R)} \leq C(||\omega||_{L^q(B_R)} + ||F||_{L^q(B_R)})$$

$\leq C(||G||_{L^2(B_R)} + ||F||_{L^q(B_R)}) \leq \frac{C}{R - r}||\nabla H||_{L^q(B_R)} + ||H||_{L^q(B_R)} \leq \frac{C}{R - r}||H||_{H^1(B_R)}.$

Let $H = \nabla \wedge A + \nabla \varphi$, and then we have that

$$||\nabla \wedge A||_{W^{1,q}(B_{r_1})} \leq C(||\nabla \wedge (\eta H)||_{L^q(B_R)} + ||H||_{L^q(B_R)}) \leq \frac{C}{R - r}||H||_{H^1(B_R)}.$$

Let $\tilde{\psi}$ be the test function in $B_r$, such that $\tilde{\psi} \equiv 1$ in $B_{r_1}$, where $0 < r_1 < r$ and $\tilde{\psi} = 0$ on $\partial B_r$, then for any $k = 1, \cdots, 3$, the equation $(3.4)$ leads to

$$\int_{B_r(x_0)} \mu(x) \nabla (\eta \partial_k \varphi) \cdot \nabla \tilde{\psi} dx = \int_{B_r(x_0)} (\mu(x) - \mu(x)) \nabla (\eta \partial_k \varphi) \cdot \nabla \tilde{\psi} + \int_{B_r(x_0)} G \tilde{\psi} dx + \sum_{i=1}^3 \int_{B_r(x_0)} F_i \partial_i \tilde{\psi} dx,$$

where

$$G = -\mu \partial_k (\nabla \wedge A) \cdot \nabla \eta - \partial_k (\mu) \nabla \wedge A \cdot \nabla \eta - \partial_k \mu \nabla \varphi \cdot \nabla \eta - \mu \nabla (\partial_k \varphi) \cdot \nabla \eta$$

and

$$\sum_{i=1}^3 F_i \partial_i \tilde{\psi} = -\mu \eta \partial_k (\nabla \wedge A) \cdot \nabla \tilde{\psi} - \eta \partial_k \mu \nabla \wedge A \cdot \nabla \tilde{\psi} - \eta \partial_k (\nabla \varphi) \cdot \nabla \tilde{\psi} + \mu \partial_k \varphi \nabla \eta \cdot \nabla \tilde{\psi}.$$ 

Similarly, for any $k = 1, 2, 3$, when $r$ is sufficiently small, we have

$$||\nabla (\eta \partial_k \varphi)||_{L^q(B_r)} \leq C(||G||_{L^2(B_r)} + ||F||_{L^q(B_r)})$$

$\leq C(||\partial_k (\nabla \wedge A)||_{L^q(B_r)} + ||\partial_k (\nabla \varphi)||_{L^2(B_r)} + ||\nabla \varphi||_{L^q(B_r)}) \leq \frac{C}{(R - r)(r - r_1)}||H||_{H^1(B_R)}.$

and so we obtain that

$$(3.29)$$

$$||\nabla H||_{L^q(B_r)} \leq \frac{C}{R - r}||H||_{H^1(B_R)}.$$ 

The following argument improves the inequality $(3.29)$ and gives that $\nabla H \in L^p(B_{R_2})$, for every $p \geq 2$. Suppose we replace $2$ and $2^\ast$ by $l$ and $l^\ast$ subsequently in the above argument, and redefine $l^\ast := \infty$ if $l^\ast < 0$, then we can get that

$$(3.30)$$

$$||\nabla H||_{L^{l^\ast}(B_{R_2})} \leq \frac{C}{R - r}||H||_{W^{1,1}(B_{R_2})}.$$ 

Now let us introduce a new Ball with the radius between $R_2$ and $R$ as follows

$$\frac{R}{2} < R_1 < R.$$ 

For any $2 \leq q \leq 6 = 2^\ast$,

$$||\nabla H||_{L^q(B_{R_2})} \leq C||\nabla H||_{L^6(B_{R_2})} \leq \frac{C}{R}||H||_{H^1(B_R)}.$$
And for any $p \geq 6$, by the estimate (3.30),
\begin{equation}
\|\nabla H\|_{L^p(B_R^2)} \leq C \|\nabla H\|_{L^\infty(B_R^2)} \leq \frac{C \|\nabla H\|_{L^6(B_{R_1}^2)}}{R_1 - \frac{R}{2}} \leq \frac{C \|H\|_{H^1}}{(R - R_1)(R_1 - \frac{R}{2})}.
\end{equation}
Hence
\begin{equation}
\|\nabla H\|_{L^p(B_R^2)} \leq C \|H\|_{H^1(B_R)}.
\end{equation}

**Step 2.** Boundary regularity. Let $x_0 \in \partial \Omega$, and by taking the assumption $\partial \Omega$ is $C^{1,1}$, we can always find a ball $B$ centred at $x_0$ and an orthogonal coordinates transformation $\Phi \in C^{1,1}(B; R^3)$ such that $(y_1, y_2, y_3) = (\Phi_1(x), \Phi_2(x), \Phi_3(x))$, and that $\Phi(B \cap \Omega) = \{y_3 > 0\} \cap B_R(0)$ in the new coordinates. Let upper and lower indices denote contravariant and covariant components respectively, and let the components of tensors be marked by "$\tilde{}"$ if they are expressed in the $y_i$-coordinate system, whereas unmarked components refer to the cartesian coordinates. Then the following transformation rules are valid, see [16] for more detail.

Let
\begin{align*}
\mathcal{E}_i &= \sum_k E_k \frac{\partial x_k}{\partial \Phi_i}, \quad \mathcal{H}_i = \sum_k H_k \frac{\partial x_k}{\partial \Phi_i}, \quad \tilde{\varepsilon}^{ij} = \sum_{k,l} \varepsilon_{kl} \frac{\partial \Phi_i}{\partial x_k} \frac{\partial \Phi_j}{\partial x_l}, \quad \tilde{\mu}^{ij} = \sum_{k,l} \mu_{kl} \frac{\partial \Phi_i}{\partial x_k} \frac{\partial \Phi_j}{\partial x_l}
\end{align*}
with $i, j, k, l \in \{1, 2, 3\}$.

Let us set that $\mathcal{F} := \nabla \wedge E$, and $\mathcal{G} := \nabla \wedge H$ then we have that
\begin{align}
\sum_k \mathcal{F}_k \frac{\partial x_k}{\partial \Phi_i} &= D \cdot (\tilde{\nabla} \wedge \mathcal{E})_i, \quad \mathcal{F} := \nabla \wedge E, \\
\sum_k \mathcal{G}_k \frac{\partial x_k}{\partial \Phi_i} &= D \cdot (\tilde{\nabla} \wedge \mathcal{H})_i, \quad \mathcal{G} := \nabla \wedge H.
\end{align}

where $D = \det(\frac{\partial \Phi_i(1)}{\partial x_j})$ for $i, j = 1, 2, 3$ and $\tilde{\nabla} = (\partial y_1, \partial y_2, \partial y_3)$.

Then we can see that $\mathcal{E}$ and $\mathcal{H}$ satisfy the following equations
\begin{align}
\tilde{\nabla} \wedge \mathcal{E} &= iw_0 \mu \mathcal{H} \\
\tilde{\nabla} \wedge \mathcal{H} &= -iw_0 \varepsilon \mathcal{E}
\end{align}
with the boundary conditions on $\{y_3 = 0\}$ that $\mathcal{E}_1 = \mathcal{E}_2 = \mathcal{H}_3 = 0$. Moreover, notice that $\nabla \Phi \in W^{1,\infty}$ which assures that $\varepsilon$ and $\mu$ satisfy all the assumptions which $\varepsilon$ and $\mu$ satisfy.

Then using the half balls in place of the balls, we can follow the exact procedure of Step 1 to derive the estimates near the boundary. One can treat the boundary integration terms as exactly same with the terms we have analyzed in Step 1 (for $\nabla \wedge \mathcal{H}$) and Step 4 (for $\nabla \cdot \mathcal{H}$) in the proof of Proposition 11. Hence we omit the details here for the shortness, and we also refer [8] and [10] for more detail.

**Step 3.** Global regularity. After applying the interior estimates (step 1 in the proof), boundary estimates (step 2 in the proof) and the standard finite covering technique (step 5 in the proof of Proposition 11) as well as the result of Proposition 11, one can easily complete the proof of this theorem. □
Next we give the proof of Theorem\[3\] in which we obtain the $W^{1,p}$ estimate for every $p \geq 2$.

Proof of the theorem\[3\]. Using the interpolation theorem, then one only needs to prove that
\[
||\nabla H||_{L^p(\Omega)} \leq C( ||H||_{L^2(\Omega)} + ||J_e||_{L^p(\Omega)} + ||\nabla \wedge G||_{L^p(\Omega)} + ||J_m||_{L^p(\Omega)} + ||\text{div} J_m||_{L^p(\Omega)}).
\]
Let us define the terms $\tilde{H}$, $\tilde{E}$, $\tilde{J}_e$ and $\tilde{J}_m$ as in the proof of Theorem\[4\]. Then we only need to prove that
\[
||\nabla \tilde{H}||_{L^p(\Omega)} \leq C( ||\tilde{H}||_{L^2(\Omega)} + ||\tilde{J}_e||_{L^p(\Omega)} + ||\tilde{J}_m||_{L^p(\Omega)} + ||\text{div} \tilde{J}_m||_{L^p(\Omega)}).
\]
We follow the similar approach as given in the proof of Proposition\[13\]. Because of the source terms, $G_i$ and $F_i^\alpha$ are now replaced by
\[
\int_{B_R} \hat{G}_i \varphi^i dx = \int_{B_R} k_1^2 \mu \cdot \eta \varphi dx - \int_{B_R} \varepsilon^{-1}(x)(\nabla \wedge \tilde{H} - \tilde{J}_e) \cdot D\eta \wedge \varphi(x) dx - \int_{B_R} k_2 \tilde{J}_m \cdot \eta \varphi dx
\]
\[
\int_{B_R} \hat{F}_i^\alpha \nu \varphi^i dx = \int_{B_R} \varepsilon^{-1}(x)(D\eta \wedge \tilde{H} - \tilde{J}_e) \cdot \nabla \wedge \varphi(x) dx.
\]
Following the same argument as the proof to the estimate \[3.28\], we can have the estimate of $\nabla \wedge (\eta \tilde{H})$. At the same time, $\text{div}(\eta \tilde{H})$ is estimated through equation \[3.13\] by applying the technics developed in Step 4 of the proof of Proposition\[13\] and the inhomogeneous terms are treated using the exact same method used in the proof of Theorem\[4\]. Then we can easily obtain the $W^{1,p}$-estimates. Here we need to use the following different estimate to bound the integration on the boundary
\[
\int_{\partial \Omega \cap B_R(y_0)} \eta^2 \partial_k \varphi \partial_k \tilde{J}_m \cdot e_3 d\sigma
\]
\[
\leq ||\eta^2 \partial_k \varphi||_{W^{1,\frac{p}{p+1}}(\partial \Omega)} ||\partial_k (\tilde{J}_m \cdot \nu)||_{W^{-\frac{1}{2}, \frac{p}{p+1}}(\partial \Omega)}
\]
\[
\leq C( ||\nabla \varphi||_{W^{1,p}(\Omega)} + ||\nabla \varphi||_{L^p(\Omega \cap B_R(y_0))} ||\tilde{J}_m \cdot \nu||_{W^{1,\frac{p}{p+1}}(\partial \Omega)}
\]
where $\frac{1}{p} + \frac{1}{q} = 1$.

Because the details are similar, we omit them for brevity. \qed

3.3. Proof of Theorem\[5\]. Before giving the proof, we first address the following three lemmas. The first lemma develops the local energy estimate of the solutions with constant coefficients.

Lemma 14. If $V$ satisfies
\[
\nabla \wedge (\varepsilon^{-1}(x_0) \nabla \wedge V) = k_1^2 \mu(x_0) V \quad \text{and} \quad \text{div}(\mu(x_0) V) = 0 \quad \text{in } B_R(x_0),
\]
then
\[
\int_{B_r(x_0)} |\nabla V|^2 dx \leq C \left( \frac{r}{R} \right)^n \int_{B_R(x_0)} |\nabla V|^2 dx, \quad \text{for all } 0 \leq r \leq R
\]
where $n = 3$ is the dimension.

Proof. Since $\partial_k V$ satisfies the same equation \[3.36\], we only need to prove
\[
\int_{B_r(x_0)} |V|^2 dx \leq C \left( \frac{r}{R} \right)^3 \int_{B_R(x_0)} |V|^2 dx, \quad \text{for all } 0 \leq r \leq R
\]
For the simplicity, we assume that \(\mu(x_0)\) is an identity matrix, for the non-identity matrix case we apply the argument in Step 3 of the proof of Proposition \(\ref{prop:11}\) to estimate \(\text{div}\bar{V}\).

Let us use the substitution \(y = \frac{x-x_0}{R}\) in the previous equation and write

\[\bar{V} = \hat{V}(y) = V(x),\]

then \(B_R(x_0)\) becomes \(B_1(0)\). Let \(\eta \in C_0^\infty(B_1(0))\) and \(\eta \equiv 1\) in \(B_\tau(0)\), \(|\nabla\eta| \leq \frac{C}{1-\tau}\) and \(0 \leq \eta \leq 1\). Let us multiply the equation \(\ref{eq:3.38}\) by \(\eta^2\bar{V}\) and integrate it by part,

\[
\int_{B_1(0)} \varepsilon^{-1}(x_0) \nabla \cdot \bar{V} \cdot \nabla \cdot (\eta^2 \bar{V}) dy = \int_{B_1(0)} k_1^2 \eta^2 \bar{V} \cdot \nabla \eta dy,
\]

\[
\int_{B_1(0)} \eta^2 \nabla \cdot \bar{V}^2 dy \leq C \int_{B_1(0)} \eta^2 |\bar{V}|^2 dy + C \int_{B_1(0)} |\nabla\eta|^2 |\bar{V}|^2 dy.
\]

It gives that

\[
\int_{B_\tau(0)} |\nabla \cdot \bar{V}|^2 dy \leq \int_{B_1(0)} \eta^2 |\nabla \cdot \bar{V}|^2 dy \leq C(\tau) \int_{B_1(0)} |\bar{V}|^2 dy.
\]

Furthermore, because \(\text{div}(\bar{V}) = 0\) in \(B_1(0)\), we have that

\[
\int_{B_\tau(0)} |\nabla \bar{V}|^2 dy \leq C(\tau) \int_{B_1(0)} |\bar{V}|^2 dy.
\]

Therefore, by the induction, we get that

\[
\int_{B_{\tau/2}(0)} |\nabla^{(2)} \bar{V}|^2 dy \leq C \int_{B_{\tau/4}(0)} |\nabla \bar{V}|^2 dy \leq C \int_{B_1(0)} |\bar{V}|^2 dy.
\]

Hence

\[
||\bar{V}||_{L_2(B_{\tau/2}(0))}^2 \leq C \sum_{k=0}^{2} \int_{B_{\tau/4}(0)} |\nabla^{(k)} \bar{V}|^2 dy \leq C \int_{B_1(0)} |\bar{V}|^2 dy.
\]

If \(0 < \frac{\tau}{R} < \frac{1}{2}\), then \(\ref{eq:3.39}\) leads to

\[
\int_{B_{\tau/2}(0)} |\bar{V}|^2 dy \leq C \int_{B_1(0)} |\bar{V}|^2 dy \cdot |B_{\tau/2}(0)| = C(\frac{\tau}{R})^3 \int_{B_1(0)} |\bar{V}|^2 dy.
\]

It follows that if \(0 < r < \frac{R}{2}\), then

\[
\int_{B_r(x_0)} |V|^2 dx = R^3 \int_{B_{\tau/2}(0)} |\bar{V}|^2 dy \leq C(\frac{\tau}{R})^3 R^3 \int_{B_1(0)} |\bar{V}|^2 dy = C(\frac{\tau}{R})^3 \int_{B_R(x_0)} |V|^2 dx,
\]

where the constant \(C\) does not depend on \(r\) and \(R\).

On the other hand, if \(\frac{R}{2} \leq r < R\), then we easily obtain that

\[
\int_{B_r(x_0)} |V|^2 dx \leq 2^3(\frac{r}{R})^3 \int_{B_{R}(x_0)} |V|^2 dx.
\]
Remark 15. We actually proved that if $V$ satisfies (3.35), then

$$\int_{B_r(x_0)} |V|^2 dx \leq C(\frac{r}{R})^3 \int_{B_R(x_0)} |V|^2 dx, \quad \text{for all} \quad 0 \leq r \leq R. \tag{3.43}$$

Based on Lemma 14, we further have the following lemma for the solutions of the equation with constant coefficients.

Lemma 16. If $V$ satisfies (3.35) in $B_R(x_0)$, then

$$\int_{B_r(x_0)} |V - \bar{V}_{x_0,r}|^2 dx \leq C\left(\frac{r}{R}\right)^{n+2} \int_{B_R(x_0)} |V - \bar{V}_{x_0,r}|^2 dx, \tag{3.44}$$

where $\bar{V}_{x_0,r} := \frac{1}{|B_r(x_0)|} \int_{B_r(x_0)} V(x) dx$ and $|B_r(x_0)|$ is the volume of the ball $B_r(x_0)$, and $n = 3$ is the dimension.

Proof. Notice that $\partial_t V$ satisfies the same equation as $V$. If $r < \frac{R}{2}$, then by the Poincare inequality and Lemma 14, we get that

$$\int_{B_r(x_0)} |V - \bar{V}_{x_0,r}|^2 dx \leq Cr^2 \int_{B_r(x_0)} |\nabla V|^2 dx \leq C\left(\frac{r}{R}\right)^n \int_{B_R(x_0)} |\nabla V|^2 dx. \tag{3.45}$$

Now let us introduce a cut-off function $\eta$ and multiply on both sides of the given equation by $\eta^2(V - \bar{V}_{x,R})$. After an integration by parts, we get that

$$C_r^2\left(\frac{r}{R}\right)^n \int_{B_R(x_0)} |\nabla V|^2 dx \leq C_r^2\left(\frac{r}{R}\right)^n \frac{1}{R^2} \int_{B_R(x_0)} |V - \bar{V}_{x,R}|^2 dx. \tag{3.46}$$

and it is obvious in the case of $r > \frac{R}{2}$. \hfill \Box

The next lemma is classical and the previous methods used could be referred to as Caccioppoli estimates. This lemma is used to absorb the error terms described by the small parameter $\delta$.

Lemma 17. If $\Phi(\rho)$ is monotone increasing and satisfies that

$$\Phi(\rho) \leq A\left[\frac{\rho}{R}\right]^\tau_1 + \delta \Phi(R) + BR^{\tau_2},$$

for every $\rho$, such that $0 < \rho < R \leq R_0$, and $\tau_2 < \tau_1$, then let $\delta$ be small enough such that there exists $\tau_2 < \nu < \tau_1$, and gives that

$$\Phi(\rho) \leq C\left[\frac{\rho}{R}\right]^\nu \Phi(R) + B\rho^{\tau_2}. \tag{3.47}$$

Proof. For all $\theta \in (0,1)$, $\Phi(\theta R) \leq A\theta^\tau_1 [1 + \theta^{-\tau_1} \delta] \Phi(R) + BR^{\tau_2}$. Let $\tau_2 < \nu < \tau_1$. Suppose that $2A\theta^{\tau_1} = \theta^{\nu}$, and let $\delta$ be small enough such that $\theta^{\nu - \tau_1} \leq 1$, then it gives that

$$\Phi(\theta R) \leq 2A\theta^{\tau_1} \Phi(R) + BR^{\tau_2} \leq \theta^{\nu} \Phi(R) + BR^{\tau_2}. \tag{3.48}$$

Let us take $k$ for any $0 < r < R$, such that $\theta^{1+k} R < r < \theta^k R$, then it follows that

$$\Phi(r) \leq \Phi(\theta^k R) \leq \theta^{k\nu} \Phi(R) + B\theta^{(k-1)\tau_1} R^{\tau_2} \leq \frac{1}{\theta^{\tau_2} - \theta^{\nu}} \leq C\left(\frac{r}{R}\right)^{\tau_2} \left(\frac{r}{R}\right)^{\nu - \tau_2} \Phi(R) + BR^{\tau_2},$$

and so it leads to the conclusion. \hfill \Box

Now we are ready to prove Theorem 5.
Proof of theorem 4. Step 1 Consider \((V, \tilde{E})\) be the solution of the system:

\[
\begin{aligned}
\nabla \wedge \tilde{E} - i \omega \mu_0 \mu(x_0) V &= 0 \quad \text{in } B_R(x_0),
\nabla \wedge V + i \omega \varepsilon_0 \varepsilon(x_0) \tilde{E} &= 0 \quad \text{in } B_R(x_0),
\tilde{E} \wedge \nu &= E \wedge \nu \quad \text{on } \partial B_R(x_0).
\end{aligned}
\]

(3.47)

Note that \(J_e = J_m = 0\), then \(V\) satisfies the equation (3.35) in \(B_R(x_0)\), and the following boundary condition

\[
(\varepsilon^{-1}(x_0) \nabla \wedge V) \wedge \nu = -i \omega \varepsilon_0 \tilde{E} \wedge \nu = -i \omega \varepsilon_0 E \wedge \nu = (\varepsilon^{-1}(x) \nabla \wedge H) \wedge \nu \quad \text{on } \partial B_R(x_0)
\]

The existence of \(V\) is guaranteed by Corollary 4.19 of the book [15], since \(k_1\) is not a Maxwell eigenvalue.

Let \(W = H - V\). Let \(B_R := B_R(x_0)\). Let \(B_r := B_r(x_0)\). Then

\[
\int_{B_r} |\nabla H|^2 dx \leq 2 \int_{B_r} |\nabla W|^2 dx + 2 \int_{B_r} |\nabla V|^2 dx.
\]

(3.48)

Because the coefficients of the equations (3.35) are constants, by Lemma 14, we can derive the inequality as follows

\[
\int_{B_r} |\nabla H|^2 dx \leq C \left( \frac{r}{R} \right)^3 \int_{B_R} |\nabla V|^2 dx + C \int_{B_R} |\text{div}W|^2 dx + C \int_{B_R} |\nabla \wedge W|^2 dx.
\]

(3.49)

Step 2 Note that by the boundary condition of \(V\) and Proposition 13, \(H\) and \(V\) are in \(W^{1,p}(B_R)\). Then \(W \in W^{1,p}(B_R)\), with the boundary condition \(W \wedge \nu = 0\) on \(\partial B_R\), then \(W\) satisfies the following equations in \(B_R\):

\[
\nabla \wedge [\varepsilon^{-1}(x_0) \nabla \wedge W] = k_1^2 \mu(x_0) W + k_2^2 (\mu(x) - \mu(x_0)) H + \nabla \wedge [(\varepsilon^{-1}(x_0) - \varepsilon^{-1}(x)) \nabla \wedge H],
\]

(3.50)

and

\[
\text{div}(\mu(x_0) W) = \text{div}(\mu(x_0) - \mu(x)) H),
\]

(3.51)

with the boundary condition

\[
(\varepsilon^{-1}(x_0) \nabla \wedge W) \wedge \nu = ((\varepsilon^{-1}(x_0) - \varepsilon^{-1}(x)) \nabla \wedge H) \wedge \nu \quad \text{on } \partial B_R.
\]

(3.52)
Multiply equation (3.50) by $W$ on the both sides and integrate it by parts, then
\[
\lambda \int_{B_R} |\nabla \wedge W|^2 dx 
\]
\[
\leq \int_{B_R} (\varepsilon^{-1}(x_0) \nabla \wedge W) \nabla \wedge W dx
\]
\[
= \int_{B_R} W(\nabla \wedge (\varepsilon^{-1}(x_0) \nabla \wedge W)) dx + \int_{\partial B_R} W(\varepsilon^{-1}(x_0) \nabla \wedge W) \wedge \nu ds
\]
\[
= \int_{B_R} k_1^2 \mu(x_0) W W dx + \int_{B_R} k_1^2 (\mu(x) - \mu(x_0)) H W dx
\]
\[
+ \int_{\partial B_R} ((\varepsilon^{-1}(x_0) - \varepsilon^{-1}(x)) \nabla \wedge H) \nabla \wedge W dx
\]
\[
- \int_{\partial B_R} ((\varepsilon^{-1}(x_0) - \varepsilon^{-1}(x)) \nabla \wedge H) \nu W ds
\]
\[
+ \int_{\partial B_R} W(\varepsilon^{-1}(x_0) \nabla \wedge W) \wedge \nu ds
\]
By the condition (3.52), we know that if $\mu \in C^\alpha(\Omega)$ and $\varepsilon \in C^\alpha$, then
\[
(3.53) \quad \int_{B_R} |\nabla \wedge W|^2 dx
\]
\[
\leq C \int_{B_R} |W|^2 dx + CR^{2\alpha} \int_{B_R} |H|^2 dx + C R^{2\alpha} \int_{B_R} |\nabla \wedge H|^2 dx
\]
\[
\leq CR^{2\alpha} \int_{B_R} (|H|^2 + |\nabla \wedge H|^2) dx + CR^2 ||W||^2_{H^1}
\]

**Step 3** Note that by equation (3.51), we can obtain that
\[
(3.54) \quad \text{div}W = \text{div}((I - \mu(x_0))W) + \text{div}((\mu(x_0) - \mu(x))H)
\]
If $|I - \mu(x_0)| < \delta_0$, $\mu \in C^\alpha$, and $\mu \in W^{1,3+\delta}$, then
\[
\int_{B_R} |\text{div}W|^2 dx \leq C \delta_0 \int_{B_R} |\nabla W|^2 dx + CR^{2\alpha} \int_{B_R} |\nabla H|^2 dx + C \int_{B_R} |\nabla \mu|^2 |H|^2 dx
\]
Then
\[
\int_{B_R} |\text{div}W|^2 dx \leq C \delta_0 \int_{B_R} |\nabla \wedge W|^2 dx + CR^{2\alpha} \int_{B_R} |\nabla H|^2 dx + CR^{\frac{2\delta}{\delta + 3}} \int_{B_R} |\nabla H|^2 dx
\]
Let $\alpha^* = \min\{\alpha, \frac{4}{9+3\delta}\}$, then by (3.53) (we write $\alpha$ instead of $\alpha^*$ in the following equations for the convenience)
\[
(3.55) \quad \int_{B_R} |\nabla W|^2 dx \leq CR^{2\alpha} ||H||^2_{H^1} + CR^2 ||W||^2_{H^1}
\]

**Step 4** By (3.49), we now know that
\[
\int_{B_R} |\nabla H|^2 dx \leq C\left[\left(\frac{T}{R}\right)^3 + R^{2\alpha}\right] \int_{B_R} (|\nabla H|^2 + |H|^2) dx + CR^2 ||W||^2_{H^1}
\]
Similarly,
\[
\int_{B_r} |H|^2 dx \leq C \left( \frac{r}{R} \right)^3 \int_{B_R} |V|^2 dx + C \int_{B_R} |W|^2 dx
\]
\[
\leq C \left[ \left( \frac{r}{R} \right)^3 + R^{2\alpha} \right] \int_{B_R} \left( |\nabla H|^2 + |H|^2 \right) dx + CR^2 \|W\|^2_{H^1}.
\]
Set
\[
\Phi(r) = \int_{B_r} \left( |\nabla H|^2 + |H|^2 \right) dx,
\]
and set \( \delta := R^{2\alpha} \) where \( R \) is sufficiently small such that \( \delta \) is small. By Lemma \[17\] if \( \|\nabla W\|^2_{L^2(B_R)} < CR^{1+\alpha} \) for any \( 0 < R \leq R_0 \), then we obtain that
\[
\Phi(r) \leq C \left[ \left( \frac{r}{R} \right)^\nu \Phi(R) + Cr^{3+\alpha} \right]
\]
for any \( 1 + 2\alpha < \nu < 3 \). It means
\[
\int_{B_r} \left( |\nabla H|^2 + |H|^2 \right) dx \leq C \left( \frac{r}{R} \right)^\nu \int_{B_R} \left( |\nabla H|^2 + |H|^2 \right) dx + Cr^{3+\alpha}.
\]
Moreover, if \( \|\nabla H\|^2_{L^2(B_R)} < CR^\nu \) for any \( 0 < R \leq R_0 \), then by (3.55), we have that
\[
\int_{B_R} |\nabla W|^2 dx \leq CR^{2\alpha+\nu} \|H\|^2_{H^1(B_{R_0})} + CR^{3+2\alpha}, \quad \text{for any } 0 < r \leq R \leq R_0
\]

**Step 5** Note that
\[
\int_{B_r} |\nabla H - (\nabla H_{x_0,r})|^2 dx \leq \int_{B_r} |\nabla V - (\nabla H_{x_0,r})|^2 dx + \int_{B_r} |\nabla W|^2 dx
\]
\[
\leq \int_{B_r} |\nabla V - (\nabla H_{x_0,r})|^2 dx + C \int_{B_r} |\nabla W|^2 dx,
\]
where \( \nabla H_{z,r} := \frac{1}{|B_r(z)|} \int_{B_r(z)} \nabla H(z) dz \). Moreover,
\[
\int_{B_R} |(\nabla H_{x_0,R}) - (\nabla V_{x_0,R})|^2 dx = |B_R| \left( \frac{1}{|B_R|} \right) \int_{B_R} \nabla W dx \right|^2 \leq \int_{B_R} |\nabla W|^2 dx.
\]
After an application of the Lemma \[16\] with \( n = 3 \), we get that for any \( 0 < r \leq R \leq R_0 \),
\[
\int_{B_r} |\nabla V - (\nabla V_{x_0,r})|^2 dx + C \int_{B_r} |\nabla W|^2 dx
\]
\[
\leq C \left( \frac{r}{R} \right)^5 \int_{B_r} |\nabla V - (\nabla V_{x_0,R})|^2 dx + C \int_{B_r} |\nabla W|^2 dx
\]
\[
\leq C \left( \frac{r}{R} \right)^5 \int_{B_R} |\nabla V - (\nabla V_{x_0,r})|^2 dx + C \int_{B_R} |\nabla W|^2 dx
\]
\[
\leq C \left( \frac{r}{R} \right)^5 \int_{B_r} |\nabla H - (\nabla H_{x_0,r})|^2 dx + C \int_{B_r} |\nabla W|^2 dx
\]
\[
\leq C \left( \frac{r}{R} \right)^5 \int_{B_R} |\nabla H - (\nabla H_{x_0,R})|^2 dx + CR^{2\alpha+\nu} \|H\|^2_{H^1(B_{R_0})} + CR^{3+2\alpha}
\]
where the constant $C$ does not depend on $r$ and $R$. Let

$$\Phi(r) = \int_{B_r(x_0)} |\nabla H - (\nabla H_{x,r})|^2 dx,$$

and choose $\nu$ such that $2\alpha + \nu > 3$. Let

$$\sigma := \min(2\alpha + \nu, 3 + 2\alpha, 5) \in (3, 5),$$

then by Lemma 17,

$$\int_{B_r} |\nabla H - (\nabla H_{x,R})|^2 dx \leq C(rR)^\sigma \int_{B_R} |\nabla H - (\nabla H_{x,R})|^2 dx + Cr^\sigma.$$

Hence

$$||\nabla H||_{L^2(B_r)} \leq C.$$

By Lemma 22 which is listed in the Appendix, we obtain that

$$||\nabla H||_{C^{0,\alpha}(B_r)} \leq C,$$

where $\alpha = \frac{\sigma - n}{2} > 0$. Therefore we get that

$$H \in C^{1+\alpha}(B_r).$$

**Step 6.** Till now, we have proved that for any given $x_0 \in \Omega$, if there exists $R_0$ such that

$$(3.59) \quad ||\nabla W||_{L^2(B_R(x_0))} < CR^{1+\alpha} \quad \text{and} \quad ||\nabla H||_{L^2(B_R)} < CR^\nu,$$

for any $0 < R \leq R_0$, then we know the solution $H \in C^{1+\alpha}(B_R(x_0))$. Let $\Omega_h$ be the set of all the points in $\Omega$ such that (3.59) holds. Note that $||\nabla W||_{L^2(B_R(x_0))}$ and $||\nabla H||_{L^2(B_R)}$ are continuous with respect to $x_0$, so $\Omega_h$ is open. Next by the Lebesgue differentiation theorem, we know that for almost every point in $\Omega$, (3.59) holds, since $|B_R(x_0)| = \frac{4\pi}{3} R^3$ and $(\nabla H, \nabla W) \in (L^2(\Omega))^2$. Therefore, we have that

$$\text{meas}(\Omega \setminus \Omega_h) = 0.$$

Then we can finish the proof of the theorem 5.

\[\square\]

### 3.4. Proof of theorem 8 and 10

Using the symmetric structure of the system, we can show the similar interior estimate to the solutions $E$. However, one needs to take care of the different boundary condition satisfied by $E$ and $H$, namely the solutions $E$ satisfies $E \wedge \nu = G \wedge \nu$ on the boundary and $H$ satisfies $(\varepsilon^{-1} \nabla \wedge H) \wedge \nu = (\varepsilon^{-1} J_e) \wedge \nu - k_2 G \wedge \nu$ on the boundary. Because the difference between the proof in this subsection and the proof of the estimates of $H$ in the previous subsection comes from the different boundary conditions, Hence, in this subsection, we study the boundary estimates of $E$ in the homogeneous equations case. Then the estimates for the inhomogeneous case can be done in a similar way. Precisely, we give the detail of the argument to derive the boundary estimates of the homogeneous equations in the proof of the propositions (18), and the argument corresponding to the inhomogeneous source terms can be achieved from the proof of theorem 1–5 by applying the symmetry structure of the system. Since the argument is very long and tedious, so we omit the detail for the shortness.
The regularity results we developed for the solution $E$ of the homogeneous Maxwell’s equations are given as below.

**Proposition 18** (Regularity for $E$—Homogeneous). Let $\varepsilon(x) \in W^{1,3+\delta}(\Omega)^{3\times 3}$, where $\delta > 0$. Suppose $\mu^{-1}(x) \in L^{\infty}(\Omega)^{3\times 3}$ and that the condition (1.2) holds. If $E$ is a weak solution of the following equations,

\begin{align}
\nabla \wedge (\mu^{-1}(x)\nabla \wedge E) = k^2\varepsilon(x)E, \quad \text{div}(\varepsilon E) = 0 \quad \text{in} \ \Omega
\end{align}

with the boundary condition $E \wedge \nu = 0$ on $\partial \Omega$, then $E \in H^1(\Omega)$ and

\begin{align}
\|E\|_{H^1(\Omega)} \leq C\|E\|_{L^2(\Omega)}.
\end{align}

Moreover, if $\mu^{-1}(x) \in C^0(\Omega)^{3\times 3}$ and satisfies the condition (1.2), then the following inequality holds

\begin{align}
\|E\|_{W^{1,p}(\Omega)} \leq C\|E\|_{L^2(\Omega)}, \quad p \geq 2,
\end{align}

where the constant $C$ does not depend on the solutions $E$; and finally assume $k$ is not a Maxwell eigenvalue, and if $\mu^{-1}(x) \in C^0(\Omega)^{3\times 3}$, and $\|\varepsilon(x) - I\|_{L^{\infty}(\Omega)^{3\times 3}} < \delta_0$, then there exists an open set $\Omega_h \subset \Omega$ and $\alpha \in (0, 1)$, such that $\text{meas}(\Omega \setminus \Omega_h) = 0$, and for any $x_0 \in \Omega_h$, there exists $r > 0$ and $C > 0$ such that

\begin{align}
\|E\|_{C^{1,\alpha}(B_r(x_0))} \leq C(1 + \|E\|_{L^2(\Omega)}).
\end{align}

**Remark 19.** The result (3.61) and (3.62) can be extended to the inhomogeneous case; Similar to the regularity of $H$, the result (3.63) only works with the homogeneous equation $J_e = J_m = 0$ case.

Using all the results obtained before, we can address the following estimate to both $E$ and $H$.

**Proof of Proposition 18.** The only difference in this case is the property of the terms on the boundary $\partial \Omega$, since $E$ and $H$ satisfy the different boundary conditions. Let us multiply both sides of the equation (3.60) by $\overline{E}$, and integrate it by parts, we get that

\[
\int_{\Omega} \mu^{-1}(x)(\nabla \wedge E) \cdot (\nabla \wedge E) \, dx + \int_{\partial \Omega} (\mu^{-1} \nabla \wedge E) \cdot (E \wedge \nu) \, d\sigma = k^2 \int_{\Omega} |E|^2 \, dx.
\]

Notice that $E \wedge \nu = 0$ on $\partial \Omega$, and so

\[
\int_{\Omega} |\nabla \wedge E|^2 \, dx \leq C \int_{\Omega} |E|^2 \, dx.
\]

Based on the estimate of $\nabla \wedge E$, we can show the interior estimate of $|\nabla E|$ exactly as done in the Step 3 of the proof of Proposition 11 where we only need to exchange the role of $\varepsilon$ and $\mu$.

Now we are going to derive the boundary estimates. Let $x_0 \in \partial \Omega$ and $B_r$ be the ball with the center $x_0$ and radius $r$. Let $\eta$ be a smooth positive cut-off function such that $\eta \equiv 1$ in $B_r$ and $\eta = 0$ on $\partial B_r$. Then multiply $\eta^2 \overline{E}$ on the both sides of the equation (3.60), after an integration by part,

\begin{align}
\int_{B_r} \mu^{-1}(x)(\nabla \wedge E) \cdot (\nabla \wedge (\eta^2 E)) \, dx + \int_{\partial B_r} (\mu^{-1} \nabla \wedge E) \cdot (\eta^2 E \wedge \nu) \, d\sigma &= k^2 \int_{B_r} |\eta E|^2 \, dx.
\end{align}
Notice that $\eta^2 E \wedge \nu = 0$ on $\partial B_R$, so by employing the Hölder inequality, it is easy to have that
\begin{equation}
(3.65) \quad \int_{B_r \cap \Omega} |\nabla \wedge E|^2 \, dx \leq C(R) \int_{B_r \cap \Omega} |E|^2 \, dx,
\end{equation}
where the constant $C(R)$ depends on $R$ but does not depend on the solutions $E$ and $H$.

Next, from the equation $\nabla \wedge H + i \omega \xi \xi(x) \, E = 0$, we have that
\[ \text{div}(\xi E) = 0, \]
with boundary condition $E \wedge \nu = 0$. As the argument in Step 2 of the proof of Proposition 13 we introduce the transformation of coordinates such that $\nu = c_3$ in the new coordinates $(y_1, y_2, y_3)$, and $E$ satisfies that $\text{div}(\xi E) = 0$, with boundary conditions $E \wedge c_3 = 0$ on the boundary $y_3 = 0$ in $B_R(x_0)$. The boundary condition is actually that
\[ \tilde{E}_1 = \tilde{E}_2 = 0. \]

We first work on the estimates of $\tilde{E}_1$, and the estimate of $\tilde{E}_2$ will be derived similarly. By the equation,
\[ 0 = \partial_{y_1} (\text{div}(\xi \tilde{E})) = \text{div}(\partial_{y_1} (\xi \tilde{E} + \xi \partial_{y_1} (\tilde{E}))) \]
Note that
\[ \partial_{y_1} \tilde{E}_2 = \partial_{y_2} \tilde{E}_1 + (\nabla \wedge \tilde{E})_3, \quad \text{and} \quad \partial_{y_1} \tilde{E}_3 = \partial_{y_3} \tilde{E}_1 - (\nabla \wedge \tilde{E})_2. \]

Let us define a vector $b$ as the following
\[ b := (0, (\nabla \wedge \tilde{E})_3, -(\nabla \wedge \tilde{E})_2)^T, \]
then $\tilde{E}_1$ satisfies the following elliptic equation of second order,
\[ \text{div}(\xi \nabla \tilde{E}_1) = -\text{div}(\partial_{y_1} (\xi \tilde{E} + \xi b)), \]
with the boundary condition $\tilde{E}_1 = 0$ on $\partial \Omega$.

Multiply $\eta^2 \tilde{E}_1$ on the both sides of the equation, and integrate by part in $B_R(y_0)$, where $y_0$ is the corresponding point of $x_0$ in the new coordinates, then
\begin{equation}
(3.66) \quad \int_{B_r \cap \Omega} |\nabla \tilde{E}_1|^2 \, dy \leq C(\lambda) \int_{B_r \cap \Omega} |\tilde{E}|^2 |\nabla \xi|^2 \, dy + \frac{C(\lambda)}{R} \int_{B_r \cap \Omega} (|\tilde{E}|^2 |\nabla \xi| + ||\xi||_{L^\infty}^2 |b|^2) \, dy,
\end{equation}
where the constant $C(\lambda)$ only depends on the elliptic ratio. Since
\begin{align*}
\int_{B_r \cap \Omega} |\tilde{E}|^2 |\nabla \xi|^2 \, dy &\leq ||\nabla \xi||_{L^{3+\delta}}^2 \left( \int_{B_R(\cap \Omega)} |\tilde{E}|^{\frac{6+2\delta}{1+\delta}} \, dy \right)^{\frac{1+\delta}{2+\delta}}, \\
\left( \int_{B_R(\cap \Omega)} |\tilde{E}|^{\frac{6+2\delta}{1+\delta}} \, dy \right)^{\frac{1+\delta}{2+\delta}} &\leq ||\tilde{E}||_{L^6(B_R(\cap \Omega))}^{\frac{2\delta}{1+\delta}} \leq CR^{\frac{2\delta}{1+\delta} ||\nabla \tilde{E}||_{L^2(B_R(\cap \Omega))}^2},
\end{align*}
Secondly,
\begin{align*}
\int_{B_r \cap \Omega} |\nabla \xi|^2 \, dy &\leq ||\xi||_{L^{3+\delta}}^2 \left( \int_{B_R(\cap \Omega)} |\tilde{E}|^{\frac{6+2\delta}{1+\delta}} \, dy \right)^{\frac{2+\delta}{3+\delta}}, \\
\left( \int_{B_R(\cap \Omega)} |\tilde{E}|^{\frac{6+2\delta}{1+\delta}} \, dy \right)^{\frac{2+\delta}{3+\delta}} &\leq ||\tilde{E}||_{L^6(B_R(\cap \Omega))}^{\frac{2\delta}{1+\delta}} \leq CR^{\frac{2\delta}{1+\delta} ||\nabla \tilde{E}||_{L^2(B_R(\cap \Omega))}^2}.
\end{align*}
with
\[
\left( \int_{B_R \cap \Omega} |\tilde{E}|^{\frac{6+2\delta}{3+\delta}} dy \right)^{\frac{3+\delta}{6+2\delta}} \leq ||\tilde{E}||^2_{L^2(B_R \cap \Omega)} |B_R \cap \Omega|^{\frac{3+2\delta}{3+\delta}} \leq CR^{\frac{3+2\delta}{3+\delta}} ||\nabla \tilde{E}||_{L^2(B_R \cap \Omega)}^2.
\]
Moreover,
\[
\int_{B_R \cap \Omega} |b|^2 dy \leq \int_{B_R \cap \Omega} |\nabla \wedge \tilde{E}|^2 dy.
\]
Combining all the inequalities above together, we have
\[
\begin{aligned}
\int_{B_r \cap \Omega} |\nabla \tilde{E}|^2 dy & \leq C(\lambda)(R^{\frac{24}{3+\delta}} ||\nabla \tilde{\varepsilon}||_{L^{3+\delta}}^2 + R^{\frac{3+2\delta}{3+\delta}} ||\nabla \tilde{\varepsilon}||_{L^{3+\delta}}) \int_{B_R \cap \Omega} |\nabla \tilde{E}|^2 dy \\
& + C(R)||\tilde{\varepsilon}||_{L^\infty}^2 \int_{B_R \cap \Omega} |\nabla \wedge \tilde{E}|^2 dy + C(R) \int_{B_R \cap \Omega} |\tilde{E}|^2 dy.
\end{aligned}
\]
Obviously \(\tilde{E}_2\) satisfies the same estimate.
Finally, we are going to estimate \(\tilde{E}_3\). Note that
\[
\partial_y \tilde{E}_3 = \partial_{y_3} \tilde{E}_1 - (\nabla \wedge \tilde{E})_2, \quad \text{and} \quad \partial_{y_2} \tilde{E}_3 = \partial_{y_3} \tilde{E}_2 + (\nabla \wedge \tilde{E})_1,
\]
and by the equation \(\text{div} (\tilde{\varepsilon} \tilde{E}) = 0\),
\[
\tilde{\varepsilon}_{3,3} \partial_{y_3} \tilde{E}_3 = -(\sum_{i,j=1}^3 \tilde{\varepsilon}_{ij} \partial_{y_i} \tilde{E} + \tilde{\varepsilon}_{3,3} \partial_{y_3} \tilde{E}_3) - \sum_{i,j=1}^3 \partial_{y_i} \tilde{\varepsilon}_{ij} \tilde{E}_j,
\]
where \(\tilde{\varepsilon}_{ij}\) is the \((i,j)\)th entry of matrix \(\tilde{\varepsilon}\), and \(\tilde{\varepsilon}_{3,3} \geq \lambda > 0\). So
\[
\int_{B_r \cap \Omega} |\nabla \tilde{E}_3|^2 dy \leq C(\lambda) \int_{B_r \cap \Omega} |\tilde{E}|^2 |\nabla \tilde{\varepsilon}|^2 dy + C(\lambda)||\tilde{\varepsilon}||_{L^\infty}^2 \int_{B_r \cap \Omega} (|\nabla \wedge \tilde{E}|^2 + |\nabla \tilde{E}_1|^2 + |\nabla \tilde{E}_2|^2) dy.
\]
Then
\[
\int_{B_r \cap \Omega} |\nabla \tilde{E}|^2 dy \leq C_1 (R^{\frac{24}{3+\delta}} + R^{\frac{3+2\delta}{3+\delta}}) \int_{B_R \cap \Omega} |\nabla \tilde{E}|^2 dy \\
+ C(R)||\tilde{\varepsilon}||_{L^\infty}^2 \int_{B_R \cap \Omega} |\nabla \wedge \tilde{E}|^2 dy + C(R) \int_{B_R \cap \Omega} |\tilde{E}|^2 dy,
\]
where the constant \(C_1\) only depends on \(||\tilde{\varepsilon}||_{L^{1+\delta}}\) and the elliptic ration.

Apply the estimates of \(\nabla \wedge \tilde{E}\) which we obtained before, and after the change of the coordinates back, we have
\[
(3.67) \quad \int_{B_r \cap \Omega} |\nabla E|^2 dx \leq C_1 (R^{\frac{24}{3+\delta}} + R^{\frac{3+2\delta}{3+\delta}}) \int_{B_R \cap \Omega} |\nabla \tilde{E}|^2 dx + C(R) \int_{B_R \cap \Omega} |E|^2 dx.
\]
Finally, let \(\eta_i\) be cut-off functions which satisfies \(\sum_i \eta_i \equiv 1\) and such that the set of all the subregions \(\Omega_i := \{x \in \Omega ; \eta_i(x) > 0\}\) together is a finite
cover of $\Omega_i$, with the property that $\text{diam}\{\Omega_i\} \leq R$. Then
\[
\|E\|_{H^1(\Omega)} \leq \sum_i \|\eta_i \nabla E\|_{L^2(\Omega)} + \|E\|_{L^2(\Omega)}
\leq C(R^{4+2\beta} + R^{4+2\gamma}) \sum_i \|\nabla E\|_{L^2(\Omega_i)} + C\|E\|_{L^2(\Omega)}
\leq C(R^{4+2\beta} + R^{4+2\gamma}) \sup_i \|\nabla E\|_{L^2(\Omega_i)} + C\|E\|_{L^2(\Omega)}.
\]

Therefore, let $R > 0$ sufficiently small, we complete the proof of the estimate (3.61).

Similarly, for the remaining two estimates, the interior estimates of $E$ and the boundary estimate of $\nabla \wedge E$ can be achieved by exchanging the role of $\varepsilon$ and $\mu$. The estimate of $\text{div} E$ near the boundary $\partial \Omega$ can be deduced by using the relation $\text{div}(\varepsilon E) = 0$ carefully as done in the proof of the estimate (3.61) (i.e., from (3.61) to (3.67)), then all the three estimates yield the remaining two estimates of $E$ in the proposition.

\textbf{Proof of Theorem 10.} By the embedding theorem, $\mu(x) \in C^\beta(\Omega)$ and $\varepsilon(x) \in C^\delta(\Omega)$. Then the theorem follows from Theorem 5 and Proposition 16. Moreover, the subset $\Omega_i$ is determined by considering the $H^1$-estimates of the two functions $W_E$ and $W_H$ together, where $W_E := E - V_E$ and $W_H := H - V_H$. $V_H$ is the $V$ constructed in the proof of Theorem 5, while $V_E$ is similar but with respect to $E$. Clearly, we have that $\Omega_h$ is open and $\text{meas}(\Omega \setminus \Omega_h) = 0$. \hfill \square

\section*{Appendix}

In the appendix, we will introduce several important functional spaces and properties which are used in this paper. Because most of them are well-known, we only list the lemmas and omit the proof for the shortness.

\begin{definition}[Morrey Space] We denote by $C^{\rho,\nu}(\Omega)$ the Morrey spaces, and for $1 \leq p < \infty$ and $\nu \in (0, n)$, its norm is defined as
\[
\|u\|_{C^{\rho,\nu}(\Omega)} = \left\{ \sup_{x \in \Omega, 0 < r < d} r^{-\nu} \int_{B(x, r)} |u(z)|^p dz \right\}^{\frac{1}{p}}
\]
where $\Omega(x, r) = \Omega \cap B(x, r)$, $B(x; r)$ is a ball of which centre is $x \in \Omega$. We call $u(x) \in C^{\rho,\nu}(\Omega)$ if $\|u\|_{C^{\rho,\nu}(\Omega)} < \infty$.
\end{definition}

\begin{definition}[Campanato space] We denote by $L^{\rho,\nu}(\Omega)$ the Campanato spaces, and for the same $\rho, \nu$ used in the definition 20, the norm of the Campanato spaces is given as
\[
\|u\|_{L^{\rho,\nu}(\Omega)} = \|u\|_{C^{\rho,\nu}(\Omega)} + \left\{ \sup_{x \in \Omega, 0 < r < d} r^{-\nu} \int_{B(x, r)} |u(z) - \bar{u}_{x,r}|^p dz \right\}^{\frac{1}{p}}
\]
where $\bar{u}_{x,r} = \frac{1}{|B(x, r)|} \int_{B(x, r)} u(z) dz$.

The Morrey space and Campanato space have the following two well-known and important properties.

\begin{lemma} $C^{0,\delta} \cong L^{0,\delta}$, where $\delta = \frac{\nu - n}{p}$ and $n < \nu < n + p$.
\end{lemma}

\begin{lemma} $L^{p,n}(\Omega) \cong L^\infty$.
\end{lemma}
Lemma 24. $L^{p,\nu}(\Omega) \cong L^{p,\nu}$, for $0 \leq \nu < n$.

Definition 25 (BMO space). We denote by $\text{BMO}(\Omega)$ the bounded mean oscillation function space, and its norm is defined as

$$||u||_{\text{BMO}(\Omega)} = ||u||_{L^1(\Omega)} + \sup_{Q \subset \Omega} \frac{1}{|Q|} \int_Q |u - \bar{u}_Q| dx$$

where $Q$ is a hypercube in $\Omega$ and $|Q|$ denotes its volume. We call $u \in \text{BMO}(\Omega)$ if $||u||_{\text{BMO}(\Omega)} < \infty$.

Lemma 26. For any open bounded domain $\Omega$, we have that $||u||_{\text{BMO}(\Omega)} \leq ||u||_{L^{1,\nu}(\Omega)}$.

By employing the BMO space, we have the following interpolation theorem.

Lemma 27 (Stampacchia interpolation theorem). Let $1 < q < +\infty$, and $T$ be a linear operator. If

$$||Tu||_{L^q(\Omega)} \leq C_1 ||u||_{L^q(\Omega)}, \quad \forall u \in L^q(\Omega),$$

$$||Tu||_{\text{BMO}(\Omega)} \leq C_2 ||u||_{L^\infty(\Omega)}, \quad \forall u \in L^\infty(\Omega).$$

Then for $p \in [q, +\infty)$, $||Tu||_{L^p(\Omega)} \leq C ||u||_{L^q(\Omega)}$.

The final one is the Helmholtz decomposition in the vector analysis.

Lemma 28 (Helmholtz Decomposition). Suppose $\Omega$ is a bounded, simply-connected, Lipschitz domain. Every square-integrable vector field $u \in (L^2(\Omega))^3$ has an orthogonal decomposition:

$$u = \nabla \varphi + \nabla \times A,$$

where $\varphi \in H^1(\Omega)$ is a scalar function, and $A \in H^1(\text{curl}, \Omega)$.

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