SYMPLYCTIC TRISECTIONS AND THE ADJUNCTION INEQUALITY

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ABSTRACT. In this paper, we establish a version of the adjunction inequality for closed symplectic 4-manifolds. As in a previous paper on the Thom conjecture, we use contact geometry and trisections of 4-manifolds to reduce this inequality to the slice-Bennequin inequality for knots in the 4-ball. As this latter result can be proved using Khovanov homology, we completely avoid gauge theoretic techniques. This inequality can be used to give gauge-theory-free proofs of several landmark results in 4-manifold topology, such as detecting exotic smooth structures, the symplectic Thom conjecture, and excluding connected sum decompositions of certain symplectic 4-manifolds.

1. Introduction

A fundamental problem in low-dimensional topology is to determine which smooth 4-manifolds admit a symplectic structure. After some basic constraints from algebraic topology, the key information comes from gauge theory, such as Seiberg-Witten invariants, Gromov invariants and Heegaard Floer homology. In a different direction, the asymptotically holomorphic techniques of Donaldson and Auroux characterized symplectic 4-manifolds in terms of Lefschetz pencils [Don99] and branched coverings [Aur00], respectively. These characterizations showed that 4-dimensional symplectic topology is encoded purely algebraically, in terms of mapping class groups of surfaces and braid groups, although in practice the algebra can be obscenely difficult. Recently, an open question for several years was to find a characterization of symplectic 4-manifolds in terms of trisections of 4-manifolds.

In [Lam18], the author proved the Thom conjecture in \(\mathbb{C}P^2\) using trisections, contact geometry and Khovanov homology. Attempting to understand and generalize these techniques has led to a surprising new and natural characterization of symplectic 4-manifolds in terms of trisections. The notion of a symplectic trisection was introduced in [LM18] and it was later shown that every symplectic 4-manifold admits such a trisection [LMS20]. From this perspective, the symplectic 4-manifold can be decomposed into three \(J\)-convex polytopes, glued together along their Levi-flat boundaries. Reversing the direction, the 4-dimensional symplectic topology is encoded by 3-dimensional geometry along the spine of the trisection. Specifically, up to diffeomorphism, the symplectic structure is determined by a triple \((\beta_1, \beta_2, \beta_3)\) of nonvanishing, harmonic 1-forms that define taut-like foliations on 3-dimensional handlebodies. Pairwise, these foliations can be perturbed and ‘grafted’ together into a contact structure, then filled in with a Weinstein 1-handlebody. This provides a new bridge between 3-dimensional geometric topology and 4-dimensional symplectic topology.

We further develop this perspective and recover many of the landmark results from the classical era of gauge theory on 4-manifolds. This approach uses 3-dimensional geometric topology – including contact geometry, taut foliations, and knot theory – instead of deep analytic methods. In particular, we obtain a very clean formulation of the adjunction inequality that sidesteps the disjointed approach that came from Seiberg-Witten. Applying this inequality, we can deduce the symplectic Thom conjecture.
and detect exotic smooth structures. This appears to be the first time combinatorial methods have been used to detect exotic smooth structures on closed 4-manifolds.

1.1. Adjunction Inequality. Let \((X, \omega)\) be a symplectic 4-manifold. A main goal of this paper is a geometric and combinatorial proof of the following inequalities.

**Theorem 1.1** (Adjunction Inequality). Let \((X, \omega)\) be a closed, symplectic 4-manifold with \([\omega]\) integral. If \(K \subset X\) is a smoothly embedded, essential surface with \([K]\) nontorsion, then

\[
\chi(K) \leq \langle c_1(\omega), [K] \rangle - K \cdot K
\]

We deduce the adjunction inequality from the following, slightly weaker inequality. Recall that the symplectic area of a surface \(K\) is the integral

\[
\text{SympArea}(K) = \int_K \omega
\]

**Theorem 1.2.** Let \((X, \omega)\) be a closed, symplectic 4-manifold with \([\omega]\) integral. If \(K \subset X\) is a smoothly embedded surface with positive symplectic area, then

\[
\chi(K) \leq \langle c_1(\omega), [K] \rangle - K \cdot K
\]

We can also deduce a stronger version of the adjunction inequality for some classes.

**Theorem 1.3.** Let \((X, \omega)\) be a closed, symplectic 4-manifold with \([\omega]\) integral. If \(K \subset X\) is a smoothly embedded, essential surface with zero symplectic area and \([K]\) nontorsion, then

\[
\chi(K) \leq -|\langle c_1(\omega), [K] \rangle| - K \cdot K
\]

**Proof.** Note the if \(\omega\) is symplectic with first Chern class \(c_1(\omega)\), then \(-\omega\) is also symplectic with first Chern class \(c_1(-\omega) = -c_1(\omega)\). Since the nondegeneracy of a closed 2-form is an open property, we can find a closed 2-form \(\eta\) such that \(\omega + \epsilon \eta\) and \(-\omega + \epsilon \eta\) are symplectic forms for small \(\epsilon > 0\) and the \(\eta\)-area of \(K\) is positive. Since the first Chern class is a deformation invariant, we have that

\[
c_1(\omega + \epsilon \eta) = c_1(\omega) \quad c_1(-\omega + \epsilon \eta) = c_1(-\omega) = -c_1(\omega)
\]

By choosing \(\epsilon \eta\) to be rational and then taking a large multiple, we obtain integral symplectic forms. We can then apply Theorem 1.2 with both \(c_1(\omega)\) and \(-c_1(\omega)\). \(\square\)

**Remark 1.4.** In contrast to the many adjunction inequality results in Seiberg-Witten theory, there are no restrictions on \(K^2\), \(\chi(K)\) or \(b_2^+\) in Theorems 1.1 – 1.3.

1.2. Symplectic Thom. Now, let \(K\) be an embedded symplectic surface in \((X, \omega)\). By a sufficiently small perturbation of \(\omega\), we can assume it is rational and that \(K\) is still symplectic. After scaling \(\omega\), we can assume it is integral and apply Theorem 1.1. The genus of an embedded symplectic surface is determined by the adjunction formula. In particular,

\[
\chi(K) = \langle c_1(\omega), [K] \rangle - [K]^2
\]

Since symplectic area is a homological invariant, we obtain a gauge-theory-free proof of the symplectic Thom conjecture.
**Theorem 1.5** (Symplectic Thom). Let $K$ be an embedded symplectic surface in $(X^4,\omega)$. Then $K$ minimizes genus in its homology class.

The original proof of Theorem 1.5 came from relations in Seiberg-Witten theory [OS00]. Ozsváth and Szabó gave a second proof using Heegaard Floer homology and Donaldson’s construction of Lefschetz pencils on symplectic manifolds [OS04]. The result here is similar to the latter, as Auroux’s branched covering result relies on the same techniques as Donaldson.

1.3. Detecting exotic smooth structures. The adjunction inequality (Theorem 1.1) can be used to detect exotic smooth structures on 4-manifolds without gauge theory. It is possible to use combinatorial methods to find exotic smooth structures on $\mathbb{R}^4$. A well-known argument uses a knot that is topologically slice but not smoothly slice. For example, the Conway knot has Alexander polynomial 1, so is topologically slice [Fre82], but is not smoothly slice and this can be detected using Khovanov homology [Pic20]. However, until now, methods from gauge theory were required to find exotic smooth structures on closed 4-manifolds. There has been some recent interest in extending Khovanov homology to give invariants of 4-manifolds and surfaces in 4-manifolds and to hopefully detect exotic smooth structures [MWW19, MSMW19, MN20]. The following results are known, but demonstrate that it is possible to use combinatorial methods to study smooth structures on closed 4-manifolds.

Let $V_d$ denote the complex hypersurface in $\mathbb{CP}^3$ of degree $d$. For $d$ odd, each $V_d$ is homeomorphic to a 4-manifold $S_d$ that is a connected sum of some number of copies of $\mathbb{CP}^2$ and $\overline{\mathbb{CP}^2}$, according to its Euler characteristic and signature. For $d$ even, we have $V_2 = S^2 \times S^2$ and $V_4 = K3$ and when $d \geq 6$ the surface $V_d$ is homeomorphic to a 4-manifold $S_d$ that is a connected sum of some number of copies of these two surfaces.

**Theorem 1.6.** For $d \geq 5$, the manifolds $V_d$ and $S_d$ are homeomorphic but not diffeomorphic.

**Proof.** The first Chern class $c_1(V_d)$ is Poincare dual to $(4 - d)H$ where $H$ is a hyperplane divisor. This hyperplane divisor is itself Poincare dual to a positive multiple of the induced symplectic form given by the embedding $V_d \subset (\mathbb{CP}^3,\omega_{FS})$. When $d \geq 5$, a (real) surface $K \subset V_d$ has nonnegative symplectic area if and only if $(c_1,K)$ is nonpositive. Consequently, when $d$ is odd, we can rule out $(-1)$-spheres and when $d$ is even, we can rule out $(+0)$-spheres using the adjunction inequality. \qed

More elaborate, ad-hoc arguments should suffice to detect exoticness in other closed examples. For example, in the construction of the Baldridge-Kirk exotic $\mathbb{CP}^2 \# 3\overline{\mathbb{CP}^2} \cong S^2 \times S^2 \# 2\overline{\mathbb{CP}^2}$ [BK08], it is easy to identify a dual pair of symplectic genus 2 surfaces (as opposed to dual symplectic spheres) and a disjoint pair of symplectic $(-1)$ tori (as opposed to symplectic $(-1)$ spheres).

1.4. Connected sums. Results of Witten [Wit94] and Taubes [Tau94] imply that a symplectic 4-manifold $(X,\omega)$ cannot split as the connected sum of two 4-manifolds $X_1, X_2$, each with $b_2^+$ positive. In certain examples we can deduce similar results. First, we recover a result of Donaldson that is relevant to the 11/8 conjecture [Don90].

**Theorem 1.7.** $K3$ does not have an $S^2 \times S^2$ summand.

**Proof.** $K3$ has a Kähler structure with vanishing first Chern class, so it follows immediately from Theorem 1.1 that $K3$ has no essential 2-spheres with trivial normal bundle. \qed
Secondly, we can rule out the existence of symplectic structures on some connected sums [Tau94].

**Theorem 1.8.** The manifold $X = \#3\mathbb{CP}^2$ has no symplectic structure.

**Proof.** Suppose that $X$ admits a symplectic structure $\omega$. Let $H_1, H_2, H_3$ denote the three projective lines generating $H_2(X; \mathbb{Z})$. It follows from the equations $c_1^2 = 2\chi + 2\sigma$ and $c_1 = w_2 \mod 2$ that, up to reordering the generators and flipping the orientations, we have that $\langle c_1, H_1 \rangle = \langle c_1, H_2 \rangle = 3$ and $\langle c_1, H_3 \rangle = 1$. The adjunction inequality implies any embedded surface representing $H_3$ must have nonpositive Euler characteristic, but this class can be represented by a sphere, which is a contradiction. \hfill \Box

1.5. **$J$-convex polytopes and grafted contact structures.** A standard approach in low-dimensional topology is to decompose a symplectic 4-manifold along a hypersurface of contact type, into a filling and a cap. Alternatively, one can start with a contact structure and look to find and classify symplectic fillings and caps, then glue them together. Due to convexity, however, there is sharp asymmetry between fillings and caps. In a trisection, we split a symplectic 4-manifold into three pieces where there is some cyclic symmetry and each piece is convex. Moreover, it can be useful to cut a 4-manifold into many pieces, not just three [IK19]. To formalize what is happening geometrically, we introduce the notions of a $J$-convex polytope and a grafted contact structure.

Intuitively, a $J$-convex polytope is a complexified version of a (geometrically) convex polytope in real affine space: it has ‘flat’ sides that meet at convex angles. One motivating and prototypical example of a $J$-convex polytope is the unit bidisk in $\mathbb{C}^2$

$$\Delta := \mathbb{D} \times \mathbb{D} = \{(z_1, z_2) : |z_1|, |z_2| \leq 1\} \subset \mathbb{C}^2$$

The boundary $\partial \Delta$ consists of two solid tori that are each foliated by holomorphic disks. Note that we can smooth the corner at $|z_1| = |z_2| = 1$ to be a 4-ball with smooth boundary. The field of complex tangencies along the boundary is the standard tight contact structure. More generally, a $J$-convex polytope has piecewise Levi-flat boundary, with each component foliated by $J$-holomorphic curves, meeting at convex angles. The boundary can be smoothed to be honestly $J$-convex with a contact structure given by the field of $J$-lines. We can describe this contact structure abstractly, by slightly perturbing the foliations to be contact structures and then grafting them together at the corners.

As described in Example 3.2, a second motivating example of a $J$-convex polytope is a Lefschetz fibration $\pi : X \to D^2$ with exact symplectic fibers. It is well-known that the Lefschetz fibration induces an open book decomposition of $\partial X$, which supports a contact structure for which $X$ is a symplectic filling. In this case, we have a decomposition $\partial X = \tilde{M} \cup \nu(B)$ where $\tilde{M}$ fibers over $S^1$ and $\nu(B)$ is a neighborhood of the binding that is fibered by meridional disks. Spinal open books were introduced by Lisi, Van Horn-Morris and Wendl to describe the contact structure on the boundary of Lefschetz fibrations over more general base surfaces [LHMW18, BVHM15]. Grafted contact structures generalize their construction in a different direction.

1.6. **Outline of the Adjunction Inequality proof.** The basic outline of the proof is as follows. Auroux and Katzarkov constructed a quasiholomorphic branched covering $f : (X, \omega) \to \mathbb{CP}^2$ over a nodal, cuspidal symplectic surface $R$ [Aur00, AK00]. Following [Lam19, LMS20], we obtain a Weinstein trisection of $(X, \omega)$ by isotoping the branch locus $R$ into transverse bridge position and pulling back the standard (Weinstein) trisection of $\mathbb{CP}^2$ by $f$. From this point, the proof essentially follows the same path as in [Lam18]. We can interpret $\chi(K), |K|^2, \langle c_1(X), K \rangle$ in terms of contact geometry and

\footnote{In horticulture, a fruiting tree like an apple or cherry can be grafted onto the rootstock of another species to increase yield.}
a trisection diagram of $\mathcal{K}$ and use the Slice-Bennequin inequality to deduce the adjunction inequality. The major new subtlety is accounting for the nonabelian fundamental group of the pieces of a Weinstein trisection of a general $(X, \omega)$. This requires fairly explicit control of the symplectic geometry along the spine of the trisection (see Section 4).

The material on $J$-convex polytopes and grafted contact structures (Section 3) is formally independent of the proof of the Adjunction Inequality. However, it motivated and was motivated by the latter and provides some geometric intuition regard ‘why’, morally speaking, the proof works. In particular, the symplectic trisection decomposes $(X, \omega)$ into three $J$-convex polytopes $Z_1, Z_2, Z_3$. Given a surface $\mathcal{K} \subset (X, \omega)$, we attempt to isotope $\mathcal{K}$ with respect to this decomposition such that it intersects $\partial Z_\lambda$ along a link transverse to the grafted contact structure on a smoothing of $\partial Z_\lambda$. The slice-Bennequin inequality can then be used to give a lower bound on how much topology of $\mathcal{K}$ ends up in the sector $Z_\lambda$.

**Figure 1.** The trisection decomposes $(X, \omega)$ into three $J$-convex polytopes $Z_1, Z_2, Z_3$, whose (smoothed) boundaries admit contact structures $\xi_1, \xi_2, \xi_3$.

**Remark 1.9.** When $(X, \omega)$ is a projective surface, the branched covering $f : X \rightarrow \mathbb{CP}^2$ arises from classical complex geometry by projecting $X \subset \mathbb{CP}^N$ onto a generic plane $\mathbb{CP}^2 \subset \mathbb{CP}^N$. In many of these cases, the branched coverings and associated trisections can be described explicitly [LM18].

1.7. **Further directions.**

1.7.1. **Minimal genus.** To first-order, the results of this paper are the strongest possible generalization of the trisection methods used to prove the Thom conjecture in $\mathbb{CP}^2$ [Lam18]. In particular, despite early hope, there are no ‘basic classes’ with associated adjunction inequalities to be constructed via trisections other than $c_1$ of a symplectic structure. Nonetheless, the method of proof of Theorem 1.1 differs greatly from previous versions, as it uses a local bound on the slice genus in $B^4$ as opposed to a global solution to the Seiberg-Witten equations. Consequently, there is hope that extensions of these techniques may yield genus information in smooth 4-manifolds with trivial Seiberg-Witten invariants, such as connected sums.
1.7.2. **Existence of symplectic surfaces.** Implicit in the proof of Theorem 1.2 is that, up to isotopy, every essential surface $K \subset (X,\omega)$ with positive symplectic area can be decomposed $K = F_1 \cup F_2$ where $F_1$ is a ribbon surface in a Weinstein $\natural_k S^1 \times B^3 \subset (X,\omega)$ and $F_2$ is a *symplectic hat*, in the terminology of [EG20]. If the inequality in Theorem 1.2 is sharp, then $\partial F_1$ is a transverse link that maximizes the Slice-Bennequin inequality. From the other direction, *quasipositive links* in $S^3$ are precisely those links that bound a holomorphic curve in $B^4$ and consequently, maximize the Slice-Bennequin inequality.

**Question 1.10.** Let $L$ be a link that maximizes the slice-Bennequin inequality. Is $L$ quasipositive?

By replacing $F_1$ with a holomorphic curve, a positive answer to this question would imply that the adjunction inequality (Theorem 1.2) is sharp on a homology class if and only if it can be represented by a symplectic surface. Some work on the Slice-Bennequin defect appears in [EVHM15, HS16, GLW17, HIK18].

1.7.3. **Symplectic cut-and-paste.** The existence result of [LMS20] implies that every closed symplectic 4-manifold can be decomposed into three $J$-convex polytopes. This decomposition appears similar to the $k$-fold symplectic sum introduced by Symington [Sym98], which was used by Fintushel and Stern to construct small exotic symplectic 4-manifolds [FS11]. It would be extremely interesting to further investigate cut-and-paste constructions with $J$-convex polytopes as these might be useful in constructing exotic 4-manifolds.

1.7.4. **Tightness criteria for grafted contact structures.** We have only defined the notion of $J$-convex polytopes and grafted contact structures here. However, they deserve a thorough investigation and accounting of their own, along the lines of [LHMW18, BVHM15]. Moreover, the existence of a symplectic structure on a trisected 4-manifold is directly tied to the grafted contact structures being tight. Thus, it would be extremely interesting to give criteria to determine when such a contact structure is tight.

**Problem 1.11.** Give criteria to determine when a grafted contact structure is tight/weakly fillable/strongly fillable/Stein fillable.

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2. Weinstein trisections

We will review some background material on Weinstein trisections of 4-manifolds and knotted surfaces, but refer the reader to [GK16, MZ17, MZ18, Lam18, LMS20] for a more detailed treatment.

2.1. Trisections of 4-manifolds. Let \( X \) be a closed, smooth, oriented 4-manifold. A trisection \( T \) of \( X \) is a decomposition \( X = Z_1 \cup Z_2 \cup Z_3 \) such that

1. Each \( Z_\lambda \) is diffeomorphic to \( \frac{1}{k_\lambda} (S^1 \times B^3) \) for some \( k_\lambda \geq 0 \),
2. Each double intersection \( H_\lambda = Z_{\lambda-1} \cap Z_\lambda \) is a 3-dimensional, genus \( g \) handlebody and
3. the triple intersection \( \Sigma = Z_1 \cap Z_2 \cap Z_3 \) is a closed, genus \( g \) surface with \( \Sigma = \partial H_\lambda \) for all \( \lambda = 1, 2, 3 \)

We will let \( Y_\lambda \) denote \( \partial Z_\lambda \). The decomposition \( Y_\lambda = H_\lambda \cup H_{\lambda+1} \) is a Heegaard splitting. Every closed, oriented, smooth 4-manifold admits a trisection decomposition [GK16].

Example 2.1 (Standard trisection of \( \mathbb{CP}^2 \)). The projective plane admits a standard genus 1 trisection as follows. In homogeneous coordinates \([z_1 : z_2 : z_3]\) on \( \mathbb{CP}^2 \), take

\[
Z_\lambda = \{ [z_1 : z_2 : z_3] | |z_\lambda|, |z_{\lambda+1}| \leq |z_{\lambda+2}| \}
\]

\[
H_\lambda = \{ [z_1 : z_2 : z_3] | |z_\lambda| \leq |z_{\lambda+1}| = |z_{\lambda+2}| \}
\]

For example, the first sector is precisely

\[
Z_1 = \{ [z_1 : z_2 : 1] | |z_1|, |z_2| \leq 1 \} \cong \mathbb{D} \times \mathbb{D}
\]

and its boundary is

\[
\partial Z_1 = S^1 \times \mathbb{D} \cup \mathbb{D} \times S^1 = H_1 \cup H_2
\]

The central surface \( \Sigma = \{ [z_1 : z_2 : 1] | |z_1| = |z_2| = 1 \} \) is a torus.

2.2. Bridge trisections of surfaces. A tangle \( \tau \) in a 3-manifold with boundary \( H \) is trivial if the arcs of \( \tau \) can be simultaneously isotoped to lie in \( \partial H \). A disk tangle \( D \) in a smooth 4-manifold with boundary \( Z \) is trivial if the disks of \( D \) can be simultaneously isotoped to lie in \( \partial Z \).

Let \( X \) be a 4-manifold with trisection \( T \) and \( \mathcal{K} \subset X \) a smoothly embedded surface. We say that \( \mathcal{K} \) is in bridge trisection position with respect to \( T \) if for each \( \lambda = 1, 2, 3 \)

1. \( \tau_\lambda = H_\lambda \cap \mathcal{K} \) is a trivial tangle,
2. \( \mathcal{D}_\lambda = Z_\lambda \cap \mathcal{K} \) is a trivial disk tangle.

Note that conditions (1) and (2) implies that \( H_\lambda = \partial Z_\lambda \cap Y_\lambda \) is an unlink in bridge position with respect to the induced Heegaard splitting on \( Y_\lambda \). Every smoothly embedded surface \( \mathcal{K} \) in trisection 4-manifold can be isotoped into bridge trisection position with respect to \( T \) [MZ18].

Let \( b = \frac{1}{2} \# (\mathcal{K} \cap \Sigma) \) be the bridge index and \( c_\lambda \) denote the number of components of the trivial disk tangle \( D_\lambda \). Then the Euler characteristic of a surface \( \mathcal{K} \) in bridge position satisfies the formula

\[
\chi(\mathcal{K}) = c_1 + c_2 + c_3 - b
\]

2.3. Weinstein trisections. Let \( X \) be a closed, smooth 4-manifold. A symplectic structure on \( X \) is a closed, nondegenerate 2-form \( \omega \). The nondegeneracy condition means that \( \omega \wedge \omega \) is a volume form on \( X \). A Liouville vector field \( \rho \) for \( \omega \) is a vector field satisfying

\[
\mathcal{L}_\rho \omega = \omega
\]
Using Cartan’s formula, we have that
\[ \omega = \mathcal{L}_{\rho} \omega = \iota_{\rho} (d\omega) + d(\iota_{\rho} \omega) = d(\iota_{\rho} \omega). \]
Thus, the 1-form \( \alpha = \iota_{\rho} \omega \) is a primitive for \( \omega \).

Let \( Y^3 \subset X \) be a smooth, oriented hypersurface. We say that \( Y \) is a \textit{hypersurface of contact type} if there exists a Liouville vector field positively transverse to \( Y \). In this case, the 1-form \( \alpha \) restricts to a (positive) contact form on \( Y \). The contact condition is that
\[ \alpha \wedge d\alpha = \omega(\rho, -) \wedge \omega(-, -) > 0 \]
is a volume form on \( Y \). This follows from the fact that \( \omega \wedge \omega \) is a volume form on \( X \) and via the positive-normal-first convention. Note that if a Liouville vector field \( \rho \) is \textit{negatively} transverse to \( Y \) then the 1-form \( \alpha \) determines a \textit{negative} contact structure.

A \textit{Weinstein domain} \( (X, \omega, \rho, \phi) \) is a compact symplectic manifold \( (X, \omega) \) with boundary, together with a globally defined Liouville vector field \( \rho \) that points outward along the boundary of \( X \), and a Morse function \( \phi : X \to \mathbb{R} \) such that \( f \) is locally constant along the boundary and such that \( \rho \) is gradient-like for \( \phi \). The Liouville vector field induces a contact form \( \alpha \) on the boundary \( \partial Y = \partial X \) of a Weinstein domain. We say that \( (X, \omega) \) is a \textit{strong symplectic filling} of the contact structure \( (Y, \ker(\alpha)) \).

**Definition 2.2.** A \textit{Weinstein trisection} of a symplectic 4-manifold \( (X, \omega) \) consists of a trisection \( T \), which induces the decomposition \( X = Z_1 \cup Z_2 \cup Z_3 \), and a Weinstein structure \( (Z_\lambda, \omega|_{Z_\lambda}, \rho_\lambda, \phi_\lambda) \) on each sector.

**Remark 2.3.** The sectors \( \{Z_\lambda\} \) are manifolds with boundary and \textit{corners}. We can formalize the transversality condition along \( \partial Z_\lambda \) as follows. For each point \( x \in \partial Z_\lambda \), there exists an open neighborhood \( U \) of \( x \in Z_\lambda \) and a collection of functions \( \{f_1, \ldots, f_n\} \) such that \( \partial Z_\lambda \cap U \) is the level set \( \{\max(f_1, \ldots, f_n) = 0\} \). Then \( \rho \) points outward along \( \partial Z_\lambda \) if and only if \( df_i(\rho) > 0 \) for \( i = 1, \ldots, n \).

**Example 2.4.** The standard trisection of \( \mathbb{CP}^2 \) can be made into a Weinstein trisection with respect to the Fubini-Study symplectic structure. This is defined as
\[ \omega_{FS} = \frac{1}{2} \partial \bar{\partial} \log(|z_1|^2 + |z_2|^2 + |z_3|^2) \]
The function \( \phi_\lambda = \log(|z_1|^2 + |z_2|^2 + |z_3|^2) \) is the Kähler potential. It is well-defined in homogeneous coordinates up to adding a constant and therefore its derivatives are well-defined. In the coordinate chart \( z_{\lambda-1} = 1 \), we can equivalently describe the Kähler form as
\[ \omega_{FS} = d\bar{d} \log(1 + |z_\lambda|^2 + |z_{\lambda+1}|^2) \]
Moreover, the form
\[ \alpha_\lambda = d\bar{d} \log(1 + |z_\lambda|^2 + |z_{\lambda+1}|^2) \]
is a primitive for \( \omega_{FS} \) and is \( \omega_{FS} \)-dual to a Liouville vector field \( \rho_\lambda \). In polar coordinates \( z_j = r_je^{i\theta_j} \), the Liouville form is
\[ \alpha_\lambda = \frac{1}{1 + r_\lambda^2 + r_{\lambda+1}^2} (2r_\lambda dr_\lambda(J-) + 2r_{\lambda+1} dr_{\lambda+1}(J-)) = \frac{1}{1 + r_\lambda^2 + r_{\lambda+1}^2} (2r_\lambda^2 d\theta_\lambda + 2r_{\lambda+1}^2 d\theta_{\lambda+1}) \]
A tedious calculation shows that the \( \rho_\lambda \) is a gradient-like vector field for the Kähler potential \( \phi_\lambda \). Another calculation shows that the Liouville form \( \alpha_\lambda \) is a positive contact form on \( \partial Z_\lambda \), implying that \( \rho_\lambda \) points outward from \( Z_\lambda \) (c.f. [Lam18, Section 3.1]). Consequently, the data \( (Z_\lambda, \omega_{FS}|_{Z_\lambda}, \rho_\lambda, \phi_\lambda) \) is a Weinstein structure on \( Z_\lambda \).

**Theorem 2.5 ([LMS20]).** Every closed symplectic 4-manifold admits a Weinstein trisection.
2.4. Foliations on handlebodies. Suppose we are given a Weinstein trisection. Let \( H_\lambda = Z_\lambda \cap Z_{\lambda - 1} \). Along \( H_\lambda \), there is a Liouville vector field \( \rho_\lambda \) pointing out of \( Z_\lambda \) and into \( Z_{\lambda - 1} \) and a Liouville vector field \( \rho_{\lambda - 1} \) pointing into \( Z_\lambda \) and out of \( Z_{\lambda - 1} \). Orienting \( H_\lambda \) as subset of the boundary of \( Z_\lambda \), the form \( \alpha_\lambda = \iota_{\rho_\lambda} \omega \) is a positive contact form on \( H_\lambda \) and \( \alpha_{\lambda - 1} = \iota_{\rho_{\lambda - 1}} \omega \) is a negative contact form.

We say that a \( k \)-form \( \beta \) on \( M \) is intrinsically harmonic if there exists a Riemannian metric \( g \) on \( M \) such that \( \beta \) is \( g \)-harmonic (See Section 3.2).

**Lemma 2.6.** The form \( \beta_\lambda = \alpha_\lambda - \alpha_{\lambda - 1} \) is closed, nonvanishing and intrinsically harmonic.

**Proof.** Since \( \alpha_\lambda \) is a positive contact form and \( \alpha_{\lambda - 1} \) is a negative contact form, we have that
\[
\alpha_\lambda \wedge d\alpha_\lambda > 0 \quad \alpha_{\lambda - 1} \wedge d\alpha_{\lambda - 1} < 0
\]
But \( d\alpha_\lambda = d\alpha_{\lambda - 1} = \omega \). This clearly implies that \( \beta_\lambda \) is closed. Moreover, we have that
\[
\beta_\lambda \wedge \omega = \alpha_\lambda \wedge d\alpha_\lambda - \alpha_{\lambda - 1} \wedge d\alpha_{\lambda - 1} > 0
\]
and consequently \( \beta_\lambda \) cannot vanish. Moreover, we can choose a Riemannian metric such that \( *\beta_\lambda = \omega \), which implies that \( \beta_\lambda \) is closed and coclosed and therefore harmonic with respect to this metric. \( \square \)

Consequently, we can interpret \( \beta_\lambda \) as defining a ‘taut’ foliation \( \mathcal{F}_\lambda = \ker(\beta_\lambda) \) of \( H_\lambda \).

**Example 2.7.** Recall from Example 2.4 that in the coordinate chart \([z_1 : z_2 : 1]\) on \( \mathbb{C}P^2 \), the Liouville form is
\[
\alpha_1 = \frac{1}{1 + r_1^2 + r_2^2} \left( 2r_1^2 d\theta_1 + 2r_2 d\theta_2 \right)
\]
Moreover, in the coordinate chart \([w_1 : 1 : w_3]\) on \( \mathbb{C}P^2 \), the Liouville form is
\[
\alpha_3 := d^C \phi_3 = \frac{1}{1 + s_1^2 + s_2^2} \left( 2s_1^2 d\psi_1 + 2s_3^2 d\psi_3 \right)
\]
where \( w_j = s_j e^{i\psi_j} \). Now, we make the change of coordinates \( w_1 = \frac{\rho_1}{r_2} \) and \( w_3 = \frac{1}{r_2} \), which implies that
\[
s_1 = \frac{r_1}{r_2} \quad \psi_1 = \theta_1 - \theta_2 \quad s_3 = \frac{1}{r_2} \quad \psi_3 = -\theta_2
\]
Consequently, the Liouville form \( \alpha_3 \) becomes
\[
\alpha_3 = \frac{1}{1 + \left( \frac{r_1}{r_2} \right)^2 + \left( \frac{1}{r_2} \right)^2} \left( 2 \left( \frac{r_1}{r_2} \right)^2 (d\theta_1 - d\theta_2) + 2 \left( \frac{1}{r_2} \right)^2 (-d\theta_2) \right)
\]
As a result, we have that
\[
\beta_1 = \alpha_1 - \alpha_3 = 2d\theta_2
\]
Recall that \( H_1 = \{[z_1 : z_2 : 1] : |z_1| \leq 1 = |z_2| \} \cong S^1 \times \mathbb{D} \). The leaves of the foliation \( \mathcal{F}_1 \) determined by \( \beta_1 \) are holomorphic disks of the form
\[
\mathbb{D}_{\theta_2} = \{\theta_2\} \times \mathbb{D}
\]
In particular, the induced taut foliation is precisely the foliation of the solid torus by disks.

When the genus of \( H_\lambda \) is greater than 2, the induced foliation is singular on the boundary \( \partial H_\lambda = \Sigma \) and thus is not strictly ‘taut’. Nonetheless, these foliations obey many properties of taut foliations, such as: the existence of a loop transverse to every leaf; the leaves minimize the Thurston norm; the leaves are \( \pi_1 \)-injective.
2.5. **Compatible almost-complex structures.** Recall that an almost-complex structure on $X$ is a bundle automorphism $J : TX \to TX$ satisfying $J^2 = -I$. If $X$ admits a symplectic structure $\omega$, we say that $J$ is **compatible** with $\omega$ if

$$\omega(Jx, Jy) = \omega(x, y)$$

for all tangent vectors $x, y$. Every symplectic manifold admits a compatible almost-complex structure. In particular, given a Riemannian metric $g$ on $X$, we can define a compatible almost-complex structure $J$ by declaring

$$g(Jx, y) = \omega(x, y)$$

for all tangent vectors $x, y$.

Let $F$ a smoothly immersed, real surface in $(X, \omega, J)$. We say that a point $x \in F$ is a **complex point** if the tangent plane $T_x F$ is a $J$-complex line. Furthermore, the surface $F$ is **$J$-holomorphic** if every point is a complex point.

**Definition 2.8.** An almost-complex structure $J$ is **compatible** with a Weinstein trisection $T$ of $(X, \omega)$ if $J$ is compatible with $\omega$ and the leaves of the foliation of $H_\lambda$ defined by $\beta_\lambda$ are $J$-holomorphic.

**Example 2.9.** The standard complex structure on $\mathbb{CP}^2$ is compatible with the standard Weinstein trisection of $(\mathbb{CP}^2, \omega_{FS})$.

2.6. **Transverse bridge position.** Let $(X, \omega)$ be a symplectic manifold, let $T$ be a Weinstein trisection of $(X, \omega)$ with associated 1-forms $\beta_1, \beta_2, \beta_3$ and let $J$ be an almost-complex structure compatible with $T$.

**Definition 2.10.** Let $K$ be a surface in bridge position with respect to $T$. We say that $K$ has **complex bridge points** if $K$ is $J$-holomorphic in an open neighborhood of each bridge point.

Let $\beta$ be a closed, nonvanishing 1-form on a compact 3-manifold with boundary $M$. We say that a properly embedded, oriented arc $\tau$ in $M$ is $\beta$-positive if $\beta(\tau') > 0$ at every point along $\tau$. In other words, the arc $\tau$ is everywhere moving transversely to the foliation defined by $\ker(\beta)$.

**Definition 2.11.** A (singular) symplectic surface $K$ in $(X, \omega)$ is in **transverse bridge position** if

1. the surface $K$ has complex bridge points, and
2. for each $\lambda = 1, 2, 3$, the multiarc $\tau_\lambda$ is $\beta_\lambda$-positive.

3. Grafted contact structures and $J$-convex polytopes

In this section, we define $J$-convex polytope and a grafted contact structure and prove some basic results. This section is formally independent from the proof of Theorem 1.1, although gives some geometric motivation and insight.

3.1. **$J$-convex polytopes.** Let $(X, J)$ be an almost-complex manifold. For a function $\phi : X \to \mathbb{R}$, we define the forms

$$d^C \phi := d\phi(J-), \quad \omega_\phi := -dd^C \phi$$

If $J$ is integrable, then $\omega_\phi$ is $J$-invariant and can equally be written as $\omega_\phi = 2i\partial \bar{\partial} \phi$. A function $\phi : X \to \mathbb{R}$ is $J$-convex if $\omega_\phi$ is positive on all $J$-complex lines. This is nonstandard terminology, but we will say that $\phi$ is $J$-flat if $\omega_\phi$ is identically 0 (i.e. if $d^C \phi$ is a closed 1-form). In particular, if $J$ is integrable and $f : X \to \mathbb{C}$ is holomorphic, then the real and imaginary parts of $f$ are $J$-flat. Now let $Y \subset (X, J)$ be a smooth hypersurface. The **field of $J$-complex tangencies** along $Y$ is the hyperplane field $\xi \subset TY$ defined pointwise as

$$\xi_p = T_p Y \cap J(T_p Y)$$
If \( Y = \phi^{-1}(0) \) for some function \( \phi : X \to \mathbb{R} \), then the field of \( J \)-complex tangencies satisfies
\[
\xi = \ker(d^c \phi)
\]
The Levi form of \( Y \) is the 2-form
\[
\omega_Y := \omega|_{\xi}
\]
where \( \phi \) is any function defining \( Y \). The form is well-defined up to multiplication by a positive function. The hypersurface \( Y \) is Levi-flat if \( \omega_Y \equiv 0 \). In this case, the Frobenius integrability condition implies that \( \xi \) integrates to a real codimension 1 foliation by \( J \)-holomorphic leaves. The hypersurface \( Y \) is \( J \)-convex if \( \omega_Y \) is positive on all \( J \)-complex lines. If \( Y \) has dimension 3, then in this case the plane field \( \xi \) is a positive contact structure on \( Y \). Note that the level sets of a \( J \)-convex (resp. \( J \)-flat) function are \( J \)-convex (resp. Levi-flat).

**Definition 3.1.** A \( J \)-convex polytope \( P \) in an almost-complex manifold \((V, J)\) is a codimension 0 subset such that, near each point \( x \in \partial P \), it locally it can be defined as the sublevel set
\[
\{ \max(\phi_1, \ldots, \phi_k) \leq 0 \}
\]
where \( \phi_1, \ldots, \phi_k \) are \( J \)-flat functions.

The corners of a \( J \)-convex polytope consist of the set of points \( p \in Y \) where at least 2 functions are required to define \( Y \) in every open neighborhood of \( p \). The faces of \( Y \) are the connected components in \( Y \) of the complement of the corners.

**Example 3.2.** Let \( X^4 \to D^2 \) be a Lefschetz fibration whose regular fiber is a compact surface with boundary \( \Sigma \). Then \( \partial X \) consists of two pieces:

1. the vertical boundary is a fibration \( Y \to S^1 \) whose fibers are fibers of the Lefschetz fibration.
2. the horizontal boundary is \( \partial \Sigma \times D^2 \).

In particular, the vertical and horizontal components of the boundary are each foliated by surfaces. The boundary components meet along the corner at \( \partial D^2 \times \partial \Sigma \), which is the union of copies of \( T^2 \), one for each component of \( \partial \Sigma \). Moreover, it is clear that the leaves of these foliations meet transversely along \( T^2 \), since their boundary components span one of the two circular directions on \( T^2 \). Smoothing the boundary results in the contact structure supported by the open book on \( \partial X \) induced by the Lefschetz fibration.

### 3.2. Intrinsically harmonic 1-forms.

**Definition 3.3.** A 1-form \( \beta \) on \( M \) is intrinsically harmonic if there exists a Riemannian metric \( g \) for \( M \) such that \( \beta \) is harmonic with respect to \( g \). In particular, this implies that \( \beta \) is a closed 1-form and \( *_g \beta \) is a closed \((n-1)\)-form.

Calabi gave an intrinsic characterization of intrinsically harmonic 1-forms \([Cal69]\). To state his criterion, we make the following definitions. We say that \( \beta \) is generic if, when viewed as a section of \( T^*M \), it is transverse to the 0-section. If \( \beta \) is generic and \( p \) is a zero of \( \beta \), we can choose a neighborhood \( U \) of \( p \) and a Morse function \( f \) on \( U \) that nondegenerate critical point at \( p \) and such that \( \beta|_U = df \).

The index of a zero of \( \beta \) is the index of this critical point. Finally, a path \( \gamma : [0, 1] \to M \) is \( \beta \)-positive if \( \beta(\gamma'(t)) > 0 \) for all \( t \in [0, 1] \).

**Theorem 3.4** (Calabi \([Cal69]\)). Let \( M \) be a closed, oriented manifold. A generic, closed 1-form \( \beta \) is intrinsically harmonic if and only if

1. \( \beta \) does not have any zeroes of index 0 or \( n \), and
2. for any two points \( p, q \) that are not zeroes, there is a \( \beta \)-positive path from \( p \) to \( q \).
The Morse theory of harmonic 1-forms was further investigated by Farber, Katz and Levine [FKL98], Honda [Hon99] and Volkov [Vol08].

Intrinsically harmonic 1-forms are related to taut foliations. Recall that a foliation $F$ of $M$ is taut if for every point $p \in M$ there exists an immersed loop $\gamma_p : S^1 \to M$ passing through $p$ and everywhere transverse to $F$. It follows easily from Calabi’s condition that nonvanishing, intrinsically harmonic 1-forms define taut foliations.

**Corollary 3.5.** Let $\beta$ be a nonvanishing, closed 1-form on a closed 3-manifold $Y$ and let $F$ be the foliation defined by $\ker(\beta)$. If $\beta$ is intrinsically harmonic then $F$ is a taut foliation of $Y$.

### 3.3. Grafted contact structures

Let $M$ be a closed, orientable 3-manifold. Fix a decomposition $M = M_1 \cup \cdots \cup M_k$ such that

1. each $M_i$ is a compact 3-manifold with boundary,
2. each double intersection $M_i \cap M_j$ is a closed, orientable surface with boundary $\Sigma_{i,j}$ (possibly empty),
3. each triple intersection is empty.

**Remark 3.6.** For simplicity in this paper, we restrict to 3-manifolds where the triple intersections are empty. It is possible to define grafted contact structures in higher dimensions as well as for more complicated decompositions of $M$.

Next, we require the following geometric data on each piece of the decomposition:

1. a nonvanishing, intrinsically harmonic 1-form $\beta_i$ on $M_i$ such that the restriction of $\beta_i$ to $\partial M_i$ is also intrinsically harmonic,
2. an exact 2-form $d\mu_i$ on $M_i$ such that $\beta_i \wedge d\mu_i > 0$ everywhere on $M_i$.

**Definition 3.7.** We say that the collections $\mathcal{B} = (\beta_1, \ldots, \beta_n)$ and $\mathcal{W} = (d\mu_1, \ldots, d\mu_n)$ of differential forms are compatible with a decomposition $M = M_1 \cup \cdots \cup M_k$ if for each pair $1 \leq i, j \leq n$

1. we have that $(\beta_i \wedge \beta_j)|_{\Sigma_{i,j}} \geq 0$, where $\Sigma_{i,j}$ is oriented as the boundary of $M_i$,
2. at each singularity $s \in \beta_i|_{\Sigma_{i,j}}^{-1}(0) = \beta_j|_{\Sigma_{i,j}}^{-1}(0)$, the 2-forms $d\mu_i$ and $d\mu_j$ induce the same orientation on $T_s \Sigma_{i,j}$.

**Theorem 3.8.** Suppose that $\mathcal{B}, \mathcal{W}$ be compatible with a decomposition $M = M_1 \cup \cdots \cup M_k$. There exists a contact structure $\xi_{\mathcal{B}}$ on $M$ such that on each piece $M_i$, the contact structure $\xi_{\mathcal{B}}$ is $C^\infty$-close to the foliation $\ker(\beta_i)$ outside a $C^0$-small neighborhood of $\partial M_i$.

We will construct this contact structure in stages.

**Step 1.** First, on each piece $M_j$ of the decomposition, we can perturb the foliation into a contact structure. Specifically, the form $\alpha_j = \beta_j + \epsilon \mu_j$ where $\omega_j = d\mu_j$, is contact for small $\epsilon > 0$.

Next, we will define how to graft these foliations into a contact structure. For simplicity, we will assume a decomposition $M = M_1 \cup M_2$ into two pieces. The general case follows easily as we only need to describe the contact structure along the interface $\Sigma = M_1 \cap M_2$.

Let $\beta_1, \beta_2$ denote the restrictions of $\beta_1, \beta_2$ to $\Sigma$. Without loss of generality, we can assume that

1. $\beta_1 \wedge \beta_2 \geq 0$. 

Claim: $\tilde{\beta}_1, \tilde{\beta}_2$ have the same singularities.

To see this, let $\Lambda_i = \ker(\tilde{\beta}_i)$ be the oriented line field. Around every point $x$, we can choose coordinates and trivialize $T\Sigma$. With respect to this trivialization, the line fields have an index corresponding to how many times, up to homotopy, the line field rotates relative to the trivialization. Equation 1 implies that around every point, the indices of the line fields agree. Moreover, we can always trivialize $T\Sigma$ such that nonsingular points have index 0 while singular points have nonzero index.

**Step 2** Fix a step function $\phi(t)$, increasing in the interval $(-1,1)$, that equals 0 for $t \leq -1$ and 1 for $t \geq 1$.

Define

$$\alpha_0 := \phi \beta_1 + (1 - \phi) \beta_2$$

**Lemma 3.9.** The form $\alpha_0$ is contact for $t \in (-1,1)$, except along arcs of the form $\{s\} \times (-1,1)$.

**Proof.** We have

$$d\alpha_0 = \phi' dt \wedge \beta_1 - \phi' dt \wedge \beta_2$$

and

$$\alpha_0 \wedge d\alpha_0 = (\phi\phi' + (1 - \phi)\phi') dt \wedge \beta_1 \wedge \beta_2 \geq 0$$

Equation 1 implies that $dt \wedge \beta_1 \wedge \beta_2$ is a volume form. Thus the form $\alpha_0$ is contact for $t \in (-1,1)$, except along singular arcs. This is because $\alpha_0, d\alpha_0$ vanish along those arcs. □

**Step 3** Let $s \in \Sigma$ be a singularity of $\beta_1, \beta_2$. With respect to the orientation determined by Equation 1, we can partition the singularities of $\tilde{\beta}_1, \tilde{\beta}_2$ according to sign. Specifically, a singularity is positive if the orientation on $T_x\Sigma$ induced by $\omega_1, \omega_2$ agrees with the orientation on $\Sigma$ and is negative otherwise. Choose coordinates in $\Sigma$ near $s$ and let $t$ denote a unit normal coordinate to $\Sigma$. Define the contact form

$$a_s := \psi_s \cdot (dt + xdy - ydx)$$

Here, $\psi_s$ is a bump function supported in some neighborhood of $s$ and not supported near any other singularity. If it is a positive singularity, then $\psi_s = 1$ near $s$; otherwise $\psi_s = -1$ near $s$.

Now, define the *inosculation contact form*

$$\alpha_I := \alpha_0 + \delta \sum_s q\alpha_s$$

where $\delta > 0$ and $q(t)$ is a bump function supported in $(-1,1)$. We can extend $\alpha_I$ by $\beta_i$ on $M_i$ to get a 1-form on the whole of $M$.

**Lemma 3.10.** For $\delta > 0$ sufficiently small, the inosculation form $\alpha_I$ is contact.

**Proof.** We have

$$d\alpha_I = d\alpha_0 + \delta \sum_s d(q\alpha_s)$$

and

$$\alpha_I \wedge d\alpha_I = \alpha_0 \wedge d\alpha_0 + \delta \sum_s \alpha_0 \wedge d(q\alpha_s) + \delta \sum_s q\alpha_s \wedge d\alpha_0 + \delta^2 \sum_s q^2\alpha_s \wedge d\alpha_s$$

This is a contact form for small $\delta > 0$. To see this, note that along an arc $\{s\} \times (-1,1)$, we have

$$\alpha_I \wedge d\alpha_I = \delta^2 q^2\alpha_s \wedge d\alpha_2.$$

This is a positive volume form. Thus, the perturbation term is positive in some neighborhood of $s$ and for sufficiently small $\delta$ we can assume that $\alpha_I$ is contact. □
Step 4 Finally, we define the grafted contact form
\[ \alpha_g = \epsilon \psi_1 \mu_1 + \alpha_I + \epsilon \psi_2 \mu_2 \]
where
1. \( \mu_i \) is a primitive for \( \omega_i \),
2. \( \epsilon, \epsilon_0 > 0 \) are small positive constants.
3. \( \psi_1 \) is a step function that equals 1 for \( t \leq -1 + \epsilon_0 \) and 0 for \( t \geq -1 + 2\epsilon_0 \).
4. \( \psi_2 \) is a step function that equals 0 for \( t \leq 1 - 2\epsilon_0 \) and 1 for \( t \geq 1 - \epsilon_0 \).

Moreover, we can extend \( \psi_1, \psi_2 \) by 0 or 1 to be defined on the whole of \( M \).

**Proposition 3.11.** For \( \epsilon, \epsilon_0 > 0 \) sufficiently small, the grafted form \( \alpha_g \) is contact on the whole of \( M \).

**Proof.** We have
\[ d\alpha_g = d\alpha_I + \epsilon d(\psi_1 \mu_1 + \psi_2 \mu_2) \]
and
\[ \alpha_g \wedge d\alpha_g = \alpha_I \wedge d\alpha_I + \epsilon (\psi_1 \mu_1 + \psi_2 \mu_2) \wedge d\alpha_I + \epsilon_0 (\alpha_I \wedge d(\psi_1 \mu_1 + \psi_2 \mu_2) + \epsilon^2 (\psi_1 \mu_1 + \psi_2 \mu_2)) \wedge d(\psi_1 \mu_1 + \psi_2 \mu_2) \]
For \( \epsilon \) sufficiently small, this is clearly contact on the interval \([-1 + \epsilon_0, 1 - \epsilon_0]\) since \( \alpha_I \wedge d\alpha_I \) is contact. For \( t \in (-1, -1 + \epsilon_0) \), we have
\[ \alpha_g \wedge d\alpha_g = \alpha_I \wedge d\alpha_I + \epsilon (\alpha_I \wedge d\alpha_I + \alpha_I \wedge d\alpha_I) + \epsilon^2 \alpha_I \wedge d\alpha_I \]
The perturbation term is positive at \( t = -1 \) and therefore, for \( \epsilon \) sufficiently small, so is \( \alpha_g \wedge d\alpha_g \).

Finally, for \( t \leq -1 \), we have \( \alpha_g = \epsilon \alpha_I \), which is contact.

A similar argument shows it is contact near \( t = 1 \).

**Proposition 3.12.** The grafted contact structure \( \xi_g = \ker(\alpha_g) \) is well-defined. In particular, up to isotopy the construction does not depend on the choices made. Moreover, a family \( \mathcal{B}_1, \mathcal{M}_1 \) of compatible forms induces an isotopy of grafted contact structures.

**Proof.** Shrinking any of the small constants \( \delta, \epsilon, \epsilon_0 \) gives a 1-parameter family of contact forms. Moreover, given 1-parameter families of the functions \( \phi, \psi, \psi_1, \psi_2 \), the closed 1-forms \( \beta_1, \beta_2 \) or the primitives \( \mu_1, \mu_2 \), we can choose the constants sufficiently small to obtain a 1-parameter family of grafted contact forms. Gray stability then implies that we have an isotopy of grafted contact structures.

4. An explicit Weinstein trisection of \( (X, \omega) \)

Using asymptotically holomorphic techniques, Auroux constructed branched coverings \( f : (X, \omega) \rightarrow (\mathbb{C}P^2, \omega_{FS}) \) whenever \( \omega \) is integral [Aur00]. Strengthening this result, Auroux and Katzarkov showed that the ramification locus \( R \subset \mathbb{C}P^2 \) is a so-called quasiholomorphic curve [AK00]. For our purposes, the important implication is that \( R \) can be encoded algebraically in terms of a factorization of the full twist of the braid group. When the ramification curve \( R \) is in so-called transverse bridge position, the standard trisection of \( \mathbb{C}P^2 \) pulls back to a Weinstein trisection of \( (X, \omega) \) [LMS20]. This isotopy can be done by a symplectic isotopy of \( R \), as in [LMS20]. However we need different geometric control of the Weinstein trisection of \( X \) to prove the adjunction inequality. By Lemma 4.2 it is sufficient to instead follow [Lam19] and take a smooth isotopy of \( R \) into bridge position.
4.1. Torus diagram of branch locus. Surfaces in $\mathbb{CP}^2$ can be encoded by torus diagrams on the central surface $\Sigma$ of the standard trisection. If $K$ is in general position with respect to the trisection, it intersects $\Sigma$ in a finite number of points and each solid torus $H_\lambda$ along a tangle $\tau_\lambda$. Generically, this tangle misses the core $B_\lambda$ of $H_\lambda$ and there is a projection $H_\lambda \setminus B_\lambda \cong \Sigma \times (0,1) \to \Sigma$. The tangles $\tau_1, \tau_2, \tau_3$ project onto collections of arcs that we label $A, B, C$, respectively. See [Lam18, Section 2.2].

Let $\Sigma = T^2$ be the central surface of the trisection of $\mathbb{CP}^2$. We have coordinates $(x, y)$ for $x, y \in [0,1]$ such that the foliations are

$$\beta_1 = dy \quad \beta_2 = -dx \quad \beta_3 = dx - dy$$

We can extend these foliations across the handlebodies. Let $\alpha_1 = (x, 0)$ and $\alpha_2 = (0, x)$ and $\alpha_3 = (-x, -x)$ for $x \in [0,1]$; these curves bound disks in $H_1$ and $H_2$ and $H_3$, respectively, that are leaves of the foliations $F_1$ and $F_2$ and $F_3$, respectively. Let $p = (0, 0) = \alpha_1 \cap \alpha_2$ be a fixed basepoint.

A key result of [AK00] is that the branch locus $R$ is encoded algebraically by a braid factorization of the full twist. Specifically, if $R$ is a degree $d$ surface in $\mathbb{CP}^2$, it is determined by a factorization

$$\Delta_d^2 = (g_1 \sigma_1^{-1} g_1^{-1}) \cdots (g_n \sigma_i^{-1} g_n^{-1})$$

where $\sigma_1$ is the first half twist in the Artin braid group and $i_j \in \{-2, 1, 2, 3\}$. Reconstructing $R$ from this factorization is described in [AK00, Lam19].

A torus diagram for $R$ in transverse bridge position in constructed in [Lam19, Proposition 4.6]. It is determined by stacking local models for each conjugate half-twist $g_j \sigma_1^{i_j} g_j^{-1}$, as in Figure 2a. Crossings in the conjugating braid $g_i, g_i^{-1}$ can be removed by a mini-stabilization (Figure 2b). The only modification we require here is to dictate that each $B$ arc in the local model of a half-twist intersects the curve $\alpha_2$ transversely once. In each sector, we obtain a transverse link $R_\lambda = R \cap Y_\lambda$. Each component of $R_\lambda$ is either an (a) unknot, (b) Hopf link or (c) right-handed trefoil. The surface $R$ is obtained by capping off each component of $R_\lambda$ with either (a) trivial disk, (b) pair of transversely intersecting trivial disks, or (c) a cone on a right-handed trefoil.

4.2. Symplectic geometry of the cover. The $n$-fold branched covering $f : X \to \mathbb{CP}^2$ over the ramification locus $R$ is determined by a map

$$\phi : \pi_1(\mathbb{CP}^2 \setminus R, p) \to S_n$$

that sends every meridian to an elementary transposition. In particular, the smooth topology of the cover does not depend on the ambient isotopy of $R$. Given a symplectic ramification curve, we can perturb $f^{-1}(\omega_{FS})$ to a nondegenerate closed 2-form $\omega$ on the cover [Aur00].

Lemma 4.1. The first Chern class of $(X, \omega)$ is

$$c_1(X, \omega) = f^*(c_1(\mathbb{CP}^2)) - PD(f^{-1}(R))$$

Proof. Recall that $f$ is locally either a diffeomorphism, fold or cusp. A simple computation shows that a section of the determinant bundle of $\mathbb{CP}^2$, pulled back to $X$, also vanishes to first-order at the fold and cusp points. $\square$

Lemma 4.2. Let $R, R'$ be nodal, cuspidal symplectic surfaces that are smoothly isotopic and let $(X, \omega)$ and $(X, \omega')$ be the induced symplectic branched covers. Then $[\omega] = [\omega']$ and $c_1(\omega) = c_1(\omega')$.

Proof. The isotopy is equivalent to a family $f_t : X \to \mathbb{CP}^2$ of branched coverings. Pulling back $c_1(\mathbb{CP}^2)$ and $\omega_{FS}$ by $f_t$, we get families of cohomologous closed 2-forms. By Lemma 4.1, the first Chern class is further determined by the preimage of the branch locus, which only changes by a smooth isotopy, and therefore stays in the same homology class. $\square$
Let $\tilde{\Sigma}$ denote the preimage of $\Sigma$; let $\tilde{H}_\lambda$ denote the preimage of $H_\lambda$; and let $\tilde{\beta}_\lambda$ denote the induced foliation on $\tilde{H}_\lambda$ by the Weinstein trisection.

First, we check that the foliation structure agrees with the branched covering in the expected way.

**Lemma 4.3.** The induced foliation $\tilde{\beta}_\lambda$ agrees with $f^*(\beta_\lambda)$.

**Proof.** The Weinstein structure is constructed in [LMS20, Theorem 6.1]. Outside a neighborhood of the branch locus, the map $f$ is a diffeomorphism. In particular

$$\omega = f^*(\omega_{FS})$$

$$\tilde{\rho}_\lambda = df^{-1}(\rho_\lambda)$$

$$\tilde{\alpha}_\lambda = f^*(\alpha_\lambda)$$

consequently

$$\tilde{\beta}_\lambda = \tilde{\alpha}_\lambda - \tilde{\alpha}_{\lambda-1} = f^*(\alpha_\lambda - \alpha_{\lambda-1}) = f^*(\beta_\lambda)$$

Along the branch locus, we have that $f^*(\omega_{FS})$ becomes degenerate and the resulting symplectic form is $f^*(\omega_{FS}) + \sum d\phi_j$ where $\phi_j$ is a collection of compactly supported 1-forms. The Liouville forms are then

$$\tilde{\alpha}_\lambda = f^*(\alpha_\lambda) + \epsilon \sum \phi_j$$

But clearly this does not affect $\tilde{\beta}_\lambda$, as the perturbation terms cancel out. \qed

**Lemma 4.4.** The lifted trisection admits a compatible almost-complex structure $\tilde{J}$.

**Proof.** Away from the branch locus, $f$ is a local diffeomorphism and therefore we can pull back $\omega, J, \beta_\lambda$ by $f$ and preserve compatibility. Secondly, since $\tilde{\beta}_\lambda = f^*(\beta_\lambda)$ and the leaves are $J$-holomorphic away from the branch locus, we can locally ensure that the leaves are $J$-holomorphic along the arcs of the branch locus in $\tilde{H}_\lambda$. Finally, we have assumed that $R$ is locally $J$-holomorphic in an integral complex structure at the bridge points. We can therefore define $\tilde{J}$ so that $f$ to be honestly holomorphic (i.e. a fold) here. \qed

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{torus_diagram.png}
\caption{Component pieces of a torus diagram for the branch locus $R$}
\end{figure}
Proposition 4.5. For each \( \lambda = 1, 2, 3 \), there exists a sequence \( \{ \hat{Y}_{\lambda,N}, \hat{\xi}_{\lambda,N} \} \) where \( \hat{Y}_{\lambda,N} \) is a hypersurface in \( \hat{Z}_\lambda \) and \( \hat{\xi}_{\lambda,N} \) is the field of complex tangencies such that

1. for \( N \) sufficiently large, \( \hat{Y}_{\lambda,N} \) is \( C^0 \)-close to \( \hat{Y}_\lambda \)
2. let \( U \) be a fixed open neighborhood of \( \hat{\Sigma} \). For \( N \) sufficiently large, the hypersurface \( \hat{Y}_{\lambda,N} \) is \( C^\infty \)-close to \( \hat{Y}_\lambda \) outside \( U \),
3. \( \hat{\xi}_{\lambda,N} \) is a tight contact structure
4. if \( K \) is a geometrically transverse surface with \( K \cap R \) disjoint from the spine of the trisection of \( X \), then for \( N \) sufficiently large, the intersection \( \hat{K}_\lambda = K \cap \hat{Y}_{\lambda,N} \) is a transverse link.

In addition, there exists a section \( \nu \subset \hat{\xi}_{\lambda,N} \) such that

1. in a fixed open neighborhood \( \hat{\Sigma} \times D^2 \) of the central surface, the vector field \( \nu_\lambda \) is tangent to the \( \hat{\Sigma} \) leaves,
2. the vector field vanishes positively along \( f^{-1}(B_\lambda - B^{\lambda+1}) \) and negatively along \( f^{-1}(R_\lambda) \)
3. along \( \hat{H}_\lambda \) and outside the fixed neighborhood \( U \), the vector fields \( \nu_\lambda \) and \( -\nu_{\lambda-1} \) are \( C^\infty \)-close.

Proof. The first half is a straightforward generalization of [Lam18, Lemma 3.5, Proposition 3.8] under branched covering. A vector field \( \partial_h \) is described in the proof of [Lam18, Proposition 3.3] and pushed forward to \( \hat{\xi}_{\lambda,N} \) in the proof of [Lam18, Lemma 3.9]. It points radially in the solid torus \( H_\lambda \), vanishing positively along \( B_\lambda \) and negatively along \( B^{\lambda+1} \). Define \( \nu_\lambda \) to be the pullback of this vector field under the branched covering. Near \( R_\lambda \), we can choose coordinates \( (x, y, t) \) where \( \beta_\lambda = dx \) and \( R_\lambda = \{ y = t = 0 \} \) and the branched covering is given by the map \( (x, y, t) \mapsto (x, y^2 - t^2, 2yt) \). Pulling back \( \partial_t \), we see that \( \nu_\lambda \) locally looks like \( 2yt\partial_y - 2t\partial_t \). In particular, it vanishes negatively along \( f^{-1}(R_\lambda) \). \qed
By construction, we have a torus diagram for the branch locus \( R \) such that \( \tau_1 \) projects onto \( T^2 \) as a collection \( A \) of disjoint arcs. This diagram also pulls back to a diagram on \( S^1 \times \mathbb{R} \) for \( \tau_1 \).

**Lemma 4.7.** The subgroup \( M_r \) is generated by a collection of meridians, one for each embedded arc in the torus diagram in the region \( S^1 \times [0, r] \) or \( S^1 \times [r, 0] \).

**Proof.** Immediate application of the Wirtinger presentation, where the tangle is inside the solid torus and we view it from the outside. \( \square \)

**Lemma 4.8.** If there are no bridge points in the cylinder \( S^1 \times [s, r] \), then \( M_s = M_r \).

**Proof.** All of the arcs are embedded and end at bridge points. Thus the number of arcs can only increase at a bridge point. \( \square \)

**Proposition 4.9.** There exist a subgroup \( R \subset \pi_1(\Sigma \setminus b, p) \) such that

1. \( M_r \subset M_s \ast R \) if \( 0 < s < r \) or \( r < s < 0 \)
2. \( M \subset M_0 \ast R \)
3. \( R \) is invariant under conjugation by the longitude \( l \)
4. every loop in \( R \) contracts in \( H_3 \setminus \tau_3 \)
5. every loop in \( R \) maps to a flat loop in \( H_2 \setminus \tau_2 \)

**Proof.** We will prove part (1) by induction; then (2) clearly follows from (1). The properties of \( R \) will come out of the construction.

For simplicity, we will assume that \( 0 < s < r \); the negative case is essentially identical. The group \( M_r \) changes at a discrete collection of values of \( r \), corresponding to two cases: (a) a crossing in the torus diagram, or (b) a local model of a half-twist.

**Case (a): crossings.** Suppose that the crossing happens at level \( r \). In the local picture \( 2b \), we have three \( \tau_1 \)-arcs and two bridge points. We can choose loops \( a, b \) in \( \Sigma \) that descend to the Wirtinger meridians of the two arcs ending at the basepoints, so that \( M_{r+\epsilon} \) is obtained from \( M_{r-\epsilon} \) by adding the generator \( b \). Note that \( a, b \) map to the same meridians of \( \tau_2 \) in \( H_2 \) and \( \tau_3 \) in \( H_3 \); thus \( ab^{-1} \) contracts in \( H_2 \setminus \tau_2 \) and in \( H_3 \setminus \tau_3 \). Include \( \rho = ab^{-1} \) in \( R \); clearly it satisfies properties (3)-(5) and \( b = a\rho^{-1} \).

**Case (b): half-twist.** In the local model, we have four bridge points. Choose four loops \( a, b, c, d \) in \( \Sigma \) based at \( p \) as in Figure 3b; we can assume they agree outside this local picture. Pushing these loops into \( H_\lambda \), they become meridians of \( \tau_\lambda \). In \( H_1 \), they become the Wirtinger meridians of the four incident \( \tau_1 \) arcs.

First, we claim that each of the four loops maps to a flat meridian in \( H_2 \). Note that \( a = c \) and \( b = d \) in \( H_2 \). Then each \( B \)-arc in the projection intersects \( \alpha_2 \) transversely in a point. We can then represent these by a path that travels from \( p \) along \( \alpha_2 \), then makes a small meridian loop, and returns to \( p \) along \( \alpha_2 \).

Secondly, we claim that the following loops, depending on the value of \( k \), contract in \( H_3 \setminus \tau_3 \). See Figure 3a. These are precisely the loops corresponding to the arcs of \( \tau_3 \).

| \( k \) | \( \rho_1 \) | \( \rho_2 \) |
|------|------|------|
| 1    | \( ad \) | \( cb \) |
| 2    | \( a^{-1}dab \) | \( cb^{-1}ab \) |
| -2   | \( ab^{-1}a^{-1}cab \) | \( b^{-1}dbaba^{-1} \) |
| 3    | \( a^{-1}dab^{-1}ab \) | \( cb^{-1}a^{-1}bab \) |
In each case, we have that
\[ c = \rho_1 c_0 \quad d = \rho_2 d_0 \]
where \( c_0, d_0 \) lie in \( M_s \) and \( \rho_1, \rho_2 \) are loops that bound in \( H_3 \setminus \tau_3 \) and are flat loops in \( H_2 \).

We now let \( R \) consist of all \( l \)-conjugates of all the relations at each half-twist. Thus, it satisfies property (3) by definition. Since each relation vanishes in \( H_3 \setminus \tau_3 \), so does every conjugate and therefore every element in \( R \). Finally we can homotope the longitude to lie in the leaf of \( H_2 \) bounded by \( \alpha_2 \). This implies that all \( l \)-conjugates of flat loops in \( H_2 \) are also flat.

\[ \square \]

**Figure 3.** Local fundamental group calculation

**Proposition 4.10.** Every element \( \gamma \in \pi_1(H_1 \setminus \tau_1, p) \) can be factored as
\[ \gamma = l^p \gamma_0 \]
where \( p \) is an integer and \( \gamma_0 \in M_0 \ast R \).

**Proof.** Clearly, \( \gamma \) can be factored into a product of the longitude and the meridians of \( \tau_1 \). For any meridian \( x \), we have that
\[ xl = l(l^{-1}xl) = lT \]
where \( \bar{x} = (t^{-1}x,t) \) is a different meridian. Consequently, we can move all longitudes to the front of the factorization. We are then left with a product of meridians \( \gamma_0 \), which lies in \( M_0 \ast R \) by Proposition 4.9.

Let \( \pi_1(\widetilde{H}_1,p) \) denote the set of homotopy classes of paths in \( \widetilde{H}_1 \) connecting any two points \( p_i, p_j \) in the preimage of \( p \). It consists precisely of all lifts of all paths in \( \pi_1(H_1 \setminus \tau_1, p) \). Let \( \widetilde{M}_0 \) and \( \widetilde{R} \) denote the set of all lifts of \( M_0 \) and \( R \).

**Proposition 4.11.** Every element \( \gamma \in \pi_1(\widetilde{H}_1,p_1) \) can be factored in \( \pi_1(\widetilde{H}_1,p) \) into the form

\[
\gamma = l_1^m \gamma_0
\]

where \( \gamma_0 \) is a sequence of paths in \( \widetilde{M}_0 \) and \( \widetilde{R} \) and \( l_1 \) is the lift of \( l \) based at \( p_1 \).

**Proof.** Every loop in the branched cover projects to a loop in \( H_1 \setminus \tau_1 \) based at \( p \). We can use Proposition 4.10 to factor it. Then each longitude \( l \) lifts to \( l_1 \), the lift based at \( p_1 \), and each element of \( \widetilde{M}_0 \) and \( \widetilde{R} \) lifts to a path in \( \widetilde{M}_0 \) or \( \widetilde{R} \).

**Lemma 4.12.** There exists lifts \( l_{1,B}, l_{1,C} \in \pi_1(\Sigma,p_1) \) of \( l_1 \in \pi_1(\widetilde{H}_1,p_1) \) such that \( l_{1,B} \) contracts in \( \widetilde{H}_2 \) and \( l_{1,C} \) contracts in \( \widetilde{H}_3 \).

**Proof.** In \( \Sigma = T^2 \), we can find a loops \( l_B \) and \( l_C \) through \( p \) homotopic to \( \alpha_2 \) and \( \alpha_3 \), respectively, that miss the embedded arcs \( B \) and \( C \), respectively, of the torus diagram. By construction, they contract in the appropriate handlebody and lift to loops through \( p_1 \) with the same property.

5. Homological information

5.1. Symplectic area of surfaces in \( X \). The triple \((\bar{\beta}_1, \bar{\beta}_2, \bar{\beta}_3)\) encodes cohomological information about the symplectic form \( \omega \). See also [LC20, Section 3].

**Lemma 5.1.** Let \( K \subset X \) be a surface in bridge position. Then

\[
\int_K \omega = \sum \int_{\tau_\lambda} \bar{\beta}_\lambda
\]

where \( \tau_\lambda \) is oriented as a subset of the surface \( D_\lambda = K \cap Z_\lambda \).

**Proof.** By Stokes’s Theorem

\[
\int_K \omega = \sum_\lambda \int_{D_\lambda} \omega = \sum_\lambda \int_{\tau_\lambda - \tau_{\lambda+1}} \bar{\alpha}_\lambda = \sum_\lambda \int_{\tau_\lambda} (\bar{\alpha}_\lambda - \bar{\alpha}_{\lambda-1}) = \sum_\lambda \int_{\tau_\lambda} \bar{\beta}_\lambda
\]

**5.2. First Chern Class.** Recall that the Weinstein trisection of \((X,\omega)\) is obtained by a symplectic branched covering map \( f : X \to \mathbb{CP}^2 \) over a nodal, cuspidal surface \( \mathcal{R} \).

**Lemma 5.2.** The first Chern class of \( (\mathbb{CP}^2,\omega_{FS}) \) is represented by a surface that intersects each \( H_\lambda \) along a core \( B_\lambda \).

**Proof.** Each \( B_\lambda \) is the intersection of a projective line with \( H_\lambda \). An anticanonical divisor has degree three, so taking three projective line – one each intersecting \( H_1, H_2, H_3 \) – is dual to \( c_1 \).

Let \( \Lambda = f^{-1}(B_1, B_2, B_3) \cup f^{-1}(R) \) denote the oriented 1-complex in the spine of the trisection of \( X \).
Lemma 5.3. We have that \( \langle c_1(\omega), K \rangle = \sum_\lambda lk(K_\lambda, \Lambda_\lambda - \Lambda_{\lambda+1}) \)

Proof. By Lemma 4.1, we can represent \( PD(c_1) \) by a surface \( C \) that intersects along \( \Lambda \). The signed count of intersections of \( K \) with \( C \) in the sector \( Z_\lambda \) is precisely this linking number and the total algebraic intersection number is obtained by summing over all three sectors. \( \square \)

5.3. Self-linking formula. A key result in establishing the adjunction inequality is relating the self-linking numbers of the transverse links \( K_1, K_2, K_3 \) to the bridge index and homological information about \( K \).

Proposition 5.4. Suppose \( K \) is in transverse bridge position. Then
\[
sl(K_1) + sl(K_2) + sl(K_3) = K \cdot K - \langle c_1(\omega), [K] \rangle - b
\]

This formula follows from the following facts. Let \( w_\lambda(K_\lambda) \) denote the framing on \( K_\lambda \) induced by the vector field \( v_\lambda \).

Proposition 5.5. Suppose that \( K_\lambda \) is a transverse link. The self-linking number of \( K_\lambda \) is given by the formula
\[
sl(K_\lambda) = w_\lambda(K_\lambda) - lk(K_\lambda, \Lambda_\lambda - \Lambda_{\lambda+1})
\]

Proof. Choose a Seifert surface \( F \) for \( K_\lambda \). There exists a nonvanishing section \( s \) of \( \xi \), restricted to \( S \). The self-linking number is given by \( lk(K_\lambda, K'_\lambda) \) where \( K'_\lambda \) is a pushoff using the framing \( s \). The framing determined by \( v_\lambda \) differs from the framing determined by \( s \) by a signed count of the zeros of \( v_\lambda \) as a section of \( \xi|_F \). Positive elliptics and negative hyperbolics count \( -1 \) and negative elliptics and positive hyperbolics count \( +1 \). This count is precisely given by the signed intersection of \( F \) with \( \Lambda_\lambda - \Lambda_{\lambda+1} \). \( \square \)

Proposition 5.6. Let \( K \) be a geometrically transverse, immersed surface in general position. Then
\[
K^2 = w_1(K_1) + w_2(K_2) + w_3(K_3) + b
\]

Proof. To compute \( K^2 \), we choose a pushoff \( K' \) and compute the algebraic intersection number \( K \cdot K' \). The \( \epsilon \)-index of a bridge point is defined in [Lam18, Section 2.5] and by [Lam18, Lemma 3.6] is always \( +1 \) at a complex bridge point. Near each bridge point, we can choose the pushoff \( K' \) to project on \( \tilde{\Sigma} \) as in Figure 4 and then extend the pushoff along \( \tau_3 \) using the vector field \( v_\lambda \). By Proposition 4.5, the framings of \( \tau_\lambda \) by \( v_\lambda \) and \( v_{\lambda-1} \) agree. This produces a framed pushoff \( K'_\lambda \) of \( K_\lambda \). By construction, we have that
\[
lk(K_1, K'_1) = w_1(K_1) \quad lk(K_2, K'_2) = w_2(K_2)
\]
However, the pushoff \( K'_3 \) differs from the surface-framed pushoff by a positive full twist along each arc of \( \tau_3 \) (see also [Lam18, Section 2.5]). Thus we have that
\[
lk(K_3, K'_3) = w_3(K_3) + b
\]
Consequently
\[
K \cdot K' = \sum_\lambda lk(K_\lambda, K'_\lambda) = w_1(K_1) + w_2(K_2) + w_3(K_3) + b
\]
\( \square \)
6. Fixing clasps

6.1. Homotopically transverse surfaces. Let $\mathcal{K} \subset X$ be a surface in bridge position with positive symplectic area.

**Definition 6.1.** Let $\mathbf{x} = (x_1, \ldots, x_n) \in \tilde{\Sigma}$ be a collection of points and $U_e$ the union of small $D^2$-neighborhoods of each $x_i$. We say that $\mathcal{K}$ is based at $\mathbf{x}$ if all of the bridge points of $\mathcal{K}$ lie in $U_e$ and the tangles $\tau_2, \tau_3$ project to embedded arcs in $U_e$.

**Lemma 6.2.** There exists an isotopy of $\mathcal{K}$ so that it is based at the point $p_1 \in \tilde{\Sigma}$ (which is a lift of the basepoint $p \in T^2$).

**Proof.** Every bridge trisection can be isotoped so that one section is standard. Here we choose to standardize $\mathcal{K}_2 = \tau_2 \cup -\tau_3$. \hfill $\square$

Up to homotopy, there is a unique path in $U_e$ from each bridge point to $p_1$. Concatenating each arc $\tau_{\lambda,i}$ with the paths at its endpoints, we get a homotopy class $[\tau_{\lambda,i}] \in \pi_1(\tilde{\mathcal{H}}_{\lambda}, p_1)$. When $\lambda = 1$, by Proposition 4.11, we can factor this class as

$[\tau_{1,i}] = t_1^{k_i} \gamma_i$

where $t_1$ is the lift of the longitude through $p_1$ and $\gamma_i = \zeta_{i,1} \cdots \zeta_{i,m_i} \in \tilde{M}_0 * \tilde{R}$ is a product of paths that are flat in either $\tilde{H}_1$ or $\tilde{H}_2$.

**Definition 6.3.** Let $\mu$ be an immersed loop in $\tilde{\Sigma}$ based at bridge point $x$ of $\mathcal{K}$ and disjoint from all bridge points. The point-pushing isotopy of $\mathcal{K}$ along $\gamma$ consists of dragging the bridge point once around $\mu$ and extending this to an isotopy of $\mathcal{K}$.

Recall that the bridge points are oriented as $\partial \tau_\lambda$ and this orientation is independent of $\lambda$. We can thus speak of positive and negative bridge points. Let $\tau_{\lambda,x}$ denote the arc of $\tau_\lambda$ incident to $x$.

**Lemma 6.4.** If $x$ is a positive bridge point, then pushing $x$ along $\mu$ changes the homotopy class by

$[\tau_{\lambda,x}] \mapsto [\tau_{\lambda,x} \ast [\mu]]$

If $x$ is a negative bridge point, then pushing $x$ along $\mu$ changes the homotopy class by

$[\tau_{\lambda,x}] \mapsto [\mu]^{-1} \ast [\tau_{\lambda,x}]$

**Proposition 6.5.** Let $\mathcal{K}$ be a surface based at $p_1$. There exists an isotopy to a surface based at $p_1$ such that the homotopy class $[\tau_{1,i}]$ has no longitudes for all $i = 2, \ldots, b$ and the homotopy classes of the $[\tau_{2,i}], [\tau_{3,i}]$ arcs remain trivial for all $i = 1, \ldots, b$. 

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**Figure 4.** Pushoffs in the direction of $v_1$ at a positive (left) and negative (right) bridge point.
Proof. We can transfer longitudes between $\tilde{\tau}_1$ arcs as follows. Let $\tau_{2,j}$ be an arc oriented from $x$ to $y$. The effect of dragging $x$ and $y$ along the loop $\alpha_3$ changes the homotopy classes of the incoming arcs by

\[
[\tau_{1,x}] \mapsto [\alpha_3]^{-1} \ast [\tau_{1,x}] = l_1^{-1} [\tau_{1,x}] \quad \quad \quad [\tau_{1,y}] \mapsto [\tau_{1,y}] * [\alpha_3] = [\tau_{1,y}] * l_1
\]

Moreover, since $\tau_{2,x} = \tau_{2,y}$ and $|\tau_{2,x}| = 1$, the effect is

\[
[\tau_{2,x}] \mapsto [\alpha_3]^{-1} * l_1 * [\alpha_3] = 1
\]

In particular, this does not change the homotopy classes of $\tau_2, \tau_3$ but transfers longitudes between $\tau_1$-arcs. There is a corresponding effect by dragging the endpoints of a $\tau_3$-arc along $\alpha_2$.

Finally, since the surface $K$ is connected, any two $\tau_1$-arcs are connected by a path in $\tau_2 \cup \tau_3$. Furthermore, as in Proposition 6.6, we can commute the longitudes between the beginning and the end of the factorization of each $[\tau_{1,i}]$. Thus, we can transfer all longitudes to $\tau_{1,1}$. \hfill $\square$

**Proposition 6.6.** There exists an isotopy of $K$ to a surface based at $\overline{p} = f^{-1}(p)$ such that for all $\lambda = 1, 2, 3$ and $i = 1, \ldots, b$, either

1. $\lambda = 1$ and $[\tau_{1,i}] = l_1^{\pm 1}$, or
2. $[\tau_{\lambda,i}]$ is flat.

**Proof.** Recall that we can factor $[\tau_{\lambda,i}]$ as $l_1^{\mu_i} \zeta_{i,m_i}$, where each $\zeta$ is a path in $\overline{H}_1$ between two points in $\overline{p} = f^{-1}(p)$. By pushing $\tau_{\lambda,i}$ to the boundary near these points, we can perform a ministabilization that decomposes $\tau_{\lambda,i}$ into multiple arcs and adds trivial arcs to $\tau_2, \tau_3$. By a sequence of such stabilizations, we can assume that $[\tau_{\lambda,i}]$ has a factorization consisting of a single element, either a longitude $l_1^{\pm 1}$ or a path in either $\overline{M}_0$ or $\overline{R}$. In the first two cases, we are done. Now suppose $[\tau_{1,i}] = \zeta \in \overline{R}$. Then by Proposition 4.9, there exists a loop $\rho$ in $\Sigma$ such that (a) $\rho$ is homotopic in $\overline{H}_1$ to $\zeta$; (b) $\rho$ is flat in $\overline{H}_2$ and contractible in $\overline{H}_3$. By pushing the negative endpoint of $\tau_{1,i}$ along $\rho$, we can make $[\tau_{1,i}]$ trivial, keep $[\tau_{3,x}]$ trivial, and make $[\tau_{2,x}]$ nontrivial but flat. \hfill $\square$

**Lemma 6.7.** The surface $K$ has positive symplectic area if and only if the number of longitudes is positive.

**Proof.** Since $K$ is based at $\overline{p}$, we can assume all bridge points lie in an $\epsilon$-ball of some $p_i$. As $\epsilon \to 0$, the integral of $\overline{\beta}_1$ along a flat path goes to $0$ and the integral of $\overline{\beta}_1$ along $l_1$ goes to $1$. \hfill $\square$

**Lemma 6.8.** Suppose $K$ has positive symplectic area. Then by an isotopy, we can assume $\int_{\tau_{1,i}} \overline{\beta}_1$ is strictly positive along every arc.

**Proof.** The proof is essentially the same as [Lam18, Proposition 4.4]. Since $K$ is based at $\overline{p}$, we can start by assuming that for $\lambda = 2, 3$ and $i = 1, \ldots, b$

\[\int_{\tau_{\lambda,i}} \overline{\beta}_1 = O(\epsilon) > 0\]

Then by Proposition 6.6 and Lemma 6.7, we have that

\[\int_{\tau_{1,i}} \overline{\beta}_1 \simeq \int_{\Sigma} \omega > 0 \quad \quad \quad \int_{\tau_{1,i}} \overline{\beta}_1 = O(\epsilon) \text{ for } i = 2, \ldots, b\]

As in [Lam18, Proposition 4.4] and Proposition 6.5, we can shift $\overline{\beta}_1$ length between arcs of $\tau_1$, preserving the $\overline{\beta}_2$ lengths of $\tau_2$ and $\overline{\beta}_3$ lengths of $\tau_3$, until all are positive. \hfill $\square$
Definition 6.9. A surface $K \subset X$ is **homotopically transverse** with respect to the Weinstein trisection of $(X, \omega)$ if for $\lambda = 1, 2, 3$, each arc of $\tau_\lambda$ is homotopic to an arc that is positive with respect to $\tilde{\beta}_\lambda$.

Proposition 6.10. If $K$ has positive symplectic area, then it is isotopic to a homotopically transverse surface.

**Proof.** Since $K$ has positive symplectic area, we can assume by Lemma 6.8 that each arc of $\tau_\lambda$ has positive $\tilde{\beta}_\lambda$ length. By Proposition 6.6, each arc is homotopic either to $l_1$, which clearly has a $\tilde{\beta}_1$-positive representative, or a perturbation of a flat arc, which we can assume is homotopic to a positive arc as well. 

6.2. Clasps and Whitney arcs. We can avoid most of the work in [Lam18, Sections 4.3,4.4] by considering Whitney arcs instead of so-called simple clasps.

Let $\tau \subset H$ be a tangle in a 3-manifold and let $\tau_1$ be a regular homotopy of tangles rel boundary. We can decompose the homotopy into a finite collection of

1. ambient isotopies,
2. crossing changes, where the tangle passes through itself.

The crossing changes can be encoded by a collection of auxiliary arcs as follows. Suppose that $\tau_{1+\epsilon}$ is obtained from $\tau_{1-\epsilon}$ by a crossing change. We can choose an embedded arc $\zeta$, with endpoints on $\tau_{1+\epsilon}$, contained in a small neighborhood of the crossing. We refer to this arc as a Whitney arc and it directs a homotopy that reverses the crossing change.

![Figure 5. A crossing change in a homotopy of a tangle before (Left) and after (Right). The Whitney arc $\zeta$ encodes how to reverse the crossing change.](image)

6.3. Fixing clasps. From here on, we essentially follow the discussion in [Lam18, Section 6]. The important point is that in a general symplectic $(X, \omega)$, everything can be localized to a neighborhood of Whitney arcs.

Let $K$ be a surface with positive symplectic area. Let $K_\lambda = K \cap \tilde{Y}_{\lambda,N}$ for some $N \gg 0$. By Proposition 6.10, we can isotope $K$ to be homotopically transverse. We can extend a homotopy of $\tau_\lambda$ to a regular homotopy of $K$, where crossing changes in $\tau_\lambda$ correspond to finger moves of the surface. Consequently, there is a regular homotopy of $K$ to some immersed surface $L$ such that each $L_\lambda = L \cap \tilde{Y}_{\lambda,N}$ is a transverse link.

Set $(Y, \xi) = (\tilde{Y}_{1,N}, \tilde{\xi}_{1,N}) \amalg (\tilde{Y}_{2,N}, \tilde{\xi}_{2,N}) \amalg (\tilde{Y}_{3,N}, \tilde{\xi}_{3,N})$ and $K = K_1 \amalg K_2 \amalg K_3$ and $L = L_1 \amalg L_2 \amalg L_3$. The regular homotopy of $K$ to $L$ is encoded by a collection of $n$ Whitney arcs. Specifically, there are $n_\lambda$ Whitney arcs in $H_\lambda$ corresponding to the homotopy of $\tau_\lambda$. Each Whitney arc in $H_\lambda$ lifts to two arcs, one in $\tilde{Y}_{\lambda,N}$ with boundary on $L_\lambda$ and another in $\tilde{Y}_{\lambda-1,N}$ with boundary on $L_{\lambda-1}$; note that we can recover $K$ from $L$ by using these arcs to defined a homotopy. For each such pair of Whitney arcs, we can perform 0-surgery on $Y$ by perturbing the Whitney arcs, removing $B^3$ neighborhoods and
identifying the boundaries. This 0-surgery can be performed in the contact category as well. Let ($\hat{Y}, \hat{\xi}$) denote the resulting contact 3-manifold.

As in [Lam18, Figure 16], for each Whitney arc, we can add a pair of untwisted, symmetric bands to $K$ and to $L$, respectively, that run across the 0-surgery 2-sphere. Let $\hat{K}$ and $\hat{L}$ denote the resulting links.

**Proposition 6.11.** Let $\hat{K}$ and $\hat{L}$ be the links obtained from $K$ and $L$, respectively, by attaching $2n$ symmetric, untwisted bands. Then

1. the links $\hat{K}$ and $\hat{L}$ are isotopic,
2. the link $\hat{K}$ bounds a ribbon surface $F$ with
   \[ \chi(F) = c_1 + c_2 + c_3 - 2n \]
3. the link $\hat{L}$ admits a transverse representative with
   \[ sl(\hat{L}) = \mathcal{K} \cdot \mathcal{K} - \langle c_1(X), [\mathcal{K}] \rangle - b + 2n \]

Statements (1) and (2) are proved exactly analogously to [Lam18, Proposition 6.3]. Statement (3) is analogous to [Lam18, Proposition 6.4] and the proof proceeds in a similar way. We know from Proposition 5.4 that

\[ sl(L) = \mathcal{K}^2 - \langle c_1(\omega), \mathcal{K} \rangle - b \]

It remains to show that each band can be attached to $L$ so that the result is transverse and so that each band increases the self-linking number by 1. From here, the proof proceeds exactly as the proof of [Lam18, Proposition 6.4], except that we will use Proposition 6.13 to determine the contact 0-surgery.

**Theorem 6.12** (Giroux flexibility [Gir91]). Let $(M, \xi)$ be a contact 3-manifold, let $\Sigma$ be an abstract closed surface, and let $i : \Sigma \to M$ be an embedding such that $i(\Sigma)$ is convex with dividing set $\Gamma_\Sigma$. Let $\mathcal{F}$ be any singular foliation on $\Sigma$ that is divided by $\Gamma = i^{-1}(\Gamma_\Sigma)$. Then for any neighborhood $U$ of $i(\Sigma)$, there exists an isotopy $\phi_t : \Sigma \to M$ for $t \in [0, 1]$ such that

1. $\phi_0 = i$
2. $\phi_t$ is fixed on $\Gamma$,
3. $\phi_t(\Sigma) \subset U$ for all $t \in [0, 1]$,
4. $\phi_t(\Sigma)$ is convex with dividing set $\Gamma_\Sigma$ for all $t \in [0, 1]$,
5. the characteristic foliation on $\phi_t(\Sigma)$ pulls back to $\mathcal{F}$.

In addition, if $\mathcal{F} = \mathcal{F}'$ on some compact set $V \subset \Sigma$, the map $\phi_t$ is constant on $V$ for all $t \in [0, 1]$.

The last part follows from carefully understanding of the proof.

**Proof.** We can find vertically-invariant neighborhood $\Sigma \times \mathbb{R}$. The strategy is to construct the isotopy here and then push forward to $M$. Out of the proof, we obtain a family of vertically-invariant contact forms $\alpha_t$ such that $\alpha_0$ defines $i^*(\xi)$ and $\alpha_t$ prints the desired foliation. Moreover, we can assume that $\alpha_t|_V$ is constant. Thus, when applying Moser’s method to obtain the isotopy, the resulting isotopy is fixed since $\alpha'_t = 0$ when restricted to $V$. 

**Lemma 6.13.** Let $\zeta$ be an (oriented) Whitney arc that is geometrically transverse near its endpoints, let $\zeta'$ be a horizontal pushoff, let $U_{\pm}$ be open neighborhoods of $\partial \zeta'$, and let $\nu(\zeta')$ be a tubular neighborhood such that the characteristic foliation on $\partial \nu(\zeta')$ has exactly one positive (resp. negative) elliptic singularity and no hyperbolic singularities in $U_+$ (resp. $U_-$). There exists a $C^0$-small isotopy of $\partial \nu(\zeta')$, that is the identity on $(U_+ \cup U_-) \cap \partial \nu(\zeta')$, such that the characteristic foliation consists of exactly two elliptic singularities.
Proof. First, by a $C^\infty$-small perturbation of $\partial \nu(\zeta)$, we can assume that it is a convex 2-sphere. By Giroux’s Criterion, the characteristic foliation is divided by a single closed curve that separates the positive and negative singularities of the characteristic foliation. We can build a second foliation $\mathcal{F}'$ that agrees with $\mathcal{F}$ in a neighborhood of the two distinguished elliptic singularities and consists of arcs connecting these two singularities and transverse to the dividing set. By Giroux Flexibility, there exists a $C^0$-small isotopy such that this is the induced foliation and this isotopy is the identity near the singularities, since $\mathcal{F} = \mathcal{F}'$ on a neighborhood of these singularities.

Proof of Proposition 6.11. Part (3) follows by the same proof as [Lam18, Proposition 6.4], as this analysis is purely local to the neighborhoods $U_\pm$ in Proposition 6.13.

6.4. Adjunction Inequality. The final step is to use the Slice-Bennequin inequality, which can be proved using Rasmussen’s $s$-invariant in Khovanov homology, to deduce the adjunction inequality.

Theorem 6.14 (Slice-Bennequin Inequality [Rud93, Ras10, Lam18, MMSW19]). Let $\bar{L}$ be a transverse link in the standard tight contact structure on $\#_k S^1 \times S^2$ bounding a slice surface $F$ in the 1-handlebody $\#_k S^1 \times B^3$. Then

$$sl(\bar{K}) \leq -\chi(F)$$

The adjunction inequality (Theorem 1.2) follows by combining Proposition 6.11 with the Slice-Bennequin inequality and the fact that $\chi(K) = c_1 + c_2 + c_3 - b$.

References

[AK00] Denis Auroux and Ludmil Katzarkov. Branched coverings of $\mathbb{C}P^2$ and invariants of symplectic 4-manifolds. Invent. Math., 142(3):631–673, 2000.

[Aur00] Denis Auroux. Symplectic 4-manifolds as branched coverings of $\mathbb{C}P^2$. Invent. Math., 139(3):551–602, 2000.

[BK08] Scott Baldridge and Paul Kirk. A symplectic manifold homeomorphic but not diffeomorphic to $\mathbb{C}P^2 \# 3\overline{\mathbb{C}P^2}$.Geom. Topol., 12(2):919–940, 2008.

[BVHM15] R. İnanç Baykur and Jeremy Van Horn-Morris. Families of contact 3-manifolds with arbitrarily large Stein fillings. J. Differential Geom., 101(3):423–465, 2015. With an appendix by Samuel Lisi and Chris Wendl.

[Cal69] Eugenio Calabi. An intrinsic characterization of harmonic one-forms. In Global Analysis (Papers in Honor of K. Kodaira), pages 101–117. Univ. Tokyo Press, Tokyo, 1969.

[Don90] S. K. Donaldson. Polynomial invariants for smooth four-manifolds. Topology, 29(3):257–315, 1990.

[Don99] S. K. Donaldson. Lefschetz pencils on symplectic manifolds. J. Differential Geom., 53(2):205–236, 1999.

[EG20] John B. Etnyre and Marco Golla. Symplectic hats. arXiv e-prints, page arXiv:2001.08978, January 2020.

[EVHM15] John B. Etnyre and Jeremy Van Horn-Morris. Monoids in the mapping class group. In Interactions between low-dimensional topology and mapping class groups, volume 19 of Geom. Topol. Monogr., pages 319–365. Geom. Topol. Publ., Coventry, 2015.

[FKL98] Michael Farber, Gabriel Katz, and Jerome Levine. Morse theory of harmonic forms. Topology, 37(3):469–483, 1998.

[Fre82] Michael Hartley Freedman. The topology of four-dimensional manifolds. J. Differential Geometry, 17(3):357–453, 1982.

[FS11] Ronald Fintushel and Ronald J. Stern. Pinwheels and nullhomologous surgery on 4-manifolds with $b^+ = 1$. Algebr. Geom. Topol., 11(3):1649–1699, 2011.

[Gir91] Emmanuel Giroux. Convexité en topologie de contact. Comment. Math. Helv., 66(4):637–677, 1991.

[GBK16] David Gay and Robion Kirby. Trisecting 4-manifolds. Geom. Topol., 20(6):3097–3132, 2016.

[GLW17] J. Elisenda Grigsby, Anthony M. Licata, and Stephan M. Wehrli. Annular Khovanov-Lee homology, braids, and cobordisms. Pure Appl. Math. Q., 13(3):389–436, 2017.

[HIK18] Jesse Hamer, Tetsuya Ito, and Keiko Kawamuro. Positivities of knots and links and the defect of Bennequin inequality. arXiv e-prints, page arXiv:1809.10836, September 2018.

[Hon99] Ko Honda. A note on Morse theory of harmonic 1-forms. Topology, 38(1):223–233, 1999.

[HS16] Diana Hubbard and Adam Saltz. An annular refinement of the transverse element in Khovanov homology. Algebr. Geom. Topol., 16(4):2305–2324, 2016.
[IK19] Gabriel Islambouli and Michael Klug. Representing smooth 4-manifolds as loops in the pants complex, 2019.

[Lam18] Peter Lambert-Cole. Bridge trisections in $\mathbb{CP}^2$ and the Thom conjecture. arXiv e-prints, page arXiv:1807.10131, July 2018.

[Lam19] Peter Lambert-Cole. Symplectic surfaces and bridge position. arXiv e-prints, page arXiv:1904.05137, April 2019.

[LC20] Peter Lambert-Cole. Lecture notes on trisections and cohomology, 2020.

[LHMW18] Samuel Lisi, Jeremy Van Horn-Morris, and Chris Wendl. On symplectic fillings of spinal open book decompositions i: Geometric constructions, 2018.

[LM18] Peter Lambert-Cole and Jeffrey Meier. Bridge trisections in rational surfaces. arXiv e-prints, page arXiv:1810.10450, October 2018.

[LMS20] Peter Lambert-Cole, Jeffrey Meier, and Laura Starkston. Symplectic 4-manifolds admit Weinstein trisections. arXiv e-prints, page arXiv:2004.01137, April 2020.

[MMSW19] Ciprian Manolescu, Marco Marengon, Sucharit Sarkar, and Michael Willis. A generalization of rasmussen’s invariant, with applications to surfaces in some four-manifolds, 2019.

[MN20] Ciprian Manolescu and Ikshu Neithalath. Skin lasagna modules for 2-handlebodies. arXiv e-prints, page arXiv:2009.08520, September 2020.

[MWW19] Scott Morrison, Kevin Walker, and Paul Wedrich. Invariants of 4-manifolds from Khovanov-Rozansky link homology. arXiv e-prints, page arXiv:1907.12194, July 2019.

[MZ17] Jeffrey Meier and Alexander Zupan. Bridge trisections of knotted surfaces in $S^4$. Trans. Amer. Math. Soc., 369(10):7343–7386, 2017.

[MZ18] Jeffrey Meier and Alexander Zupan. Bridge trisections of knotted surfaces in 4-manifolds. Proc. Natl. Acad. Sci. USA, 115(43):10880–10886, 2018.

[OS00] Peter Ozsváth and Zoltán Szabó. The symplectic Thom conjecture. Ann. of Math. (2), 151(1):93–124, 2000.

[OS04] Peter Ozsváth and Zoltán Szabó. Holomorphic triangle invariants and the topology of symplectic four-manifolds. Duke Math. J., 121(1):1–34, 2004.

[Pic20] Lisa Piccirillo. The Conway knot is not slice. Ann. of Math. (2), 191(2):581–591, 2020.

[Ras10] Jacob Rasmussen. Khovanov homology and the slice genus. Invent. Math., 182(2):419–447, 2010.

[Rud93] Lee Rudolph. Quasipositivity as an obstruction to sliceness. Bull. Amer. Math. Soc. (N.S.), 29(1):51–59, 1993.

[Sym98] Margaret Symington. A new symplectic surgery: the 3-fold sum. volume 88, pages 27–53. 1998. Symplectic, contact and low-dimensional topology (Athens, GA, 1996).

[Tau94] Clifford Henry Taubes. The Seiberg-Witten invariants and symplectic forms. Math. Res. Lett., 1(6):809–822, 1994.

[Vol08] Evgeny Volkov. Characterization of intrinsically harmonic forms. J. Topol., 1(3):643–650, 2008.

[Wit94] Edward Witten. Monopoles and four-manifolds. Math. Res. Lett., 1(6):769–796, 1994.

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