Degree-equipartite graphs

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Abstract
A graph $G$ of order $2n$ is called degree-equipartite if for every $n$-element set $A \subseteq V(G)$, the degree sequences of the induced subgraphs $G[A]$ and $G[V(G) \setminus A]$ are the same. In this paper, we characterize all degree-equipartite graphs. This answers Problem 1 in the paper by Grünbaum et al [B. Grünbaum, T. Kaiser, D. Král, and M. Rosenfeld, Equipartite graphs, Israel J. Math. 168 (2008), 431-444].

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1 Introduction

In this paper, all graphs are assumed to be simple, i.e., without any loops and multiple edges. A graph $G$ of order $2n$ is called weakly equipartite if for every partition of $V(G)$ into two sets $A$ and $B$ of $n$ vertices each, the subgraphs of $G$ induced by $A$ and $B$ are isomorphic. If there is an automorphism of $G$ mapping $A$ onto $B$ then $G$ is called equipartite [4]. Plainly, every equipartite graph is weakly equipartite.

Theorem 1. [4, Theorem 13] A graph $G$ of order $2n$ is weakly equipartite if and only if it is one of the following graphs:

$$2nK_1, \ nK_2, \ 2C_4, \ K_{n,n} \setminus nK_2, \ and \ 2K_n$$

or one of their complements

$$K_{2n}, \ K_{2n} \setminus nK_2, \ K_8 \setminus 2C_4, \ 2K_n + nK_2 \ and \ K_{n,n}.$$  

It is interesting that the graphs in the theorem above are also equipartite. Therefore, we have:
Corollary 2. [4, Corollary 14] A graph $G$ of order $2n$ is equipartite if and only if it is weakly equipartite.

According to [4, Section 5], a relaxation of the notion of equipartite graphs seems to be of some interest:
A graph $G$ of order $2n$ is called degree-equipartite if for every $n$-element set $A \subseteq V(G)$, the degree sequences of the induced subgraphs $G[A]$ and $G[V(G)\setminus A]$ are the same. Grünbaum et al [4, Problem 1] asked: Which graphs $G$ are degree-equipartite? In particular, is there a degree-equipartite graph which is not equipartite?

By the definition of a degree-equipartite graph, one can immediately conclude that:

**Proposition 3.** The complement of a degree-equipartite graph is also degree-equipartite.

In this paper, we show that:

**Theorem 4.** A graph of even order is degree-equipartite if and only if it is weakly equipartite.

We prove this theorem through some modifications of the machinery of P. Kelly and D. Merriell in [5].

The notion of a weakly equipartite graph was initially introduced by Kelly and Merriell in [5], but they have a different terminology. Indeed, their phrase for a weakly equipartite graph, is a graph which “has all bisections” and they reach to the same characterization of such graphs as in Theorem 1 ([5, Theorems 4,5]). But if we carefully look at all statements proved there, that are for characterizing graphs which have all bisections, we find that the proofs can be modified for characterizing degree-equipartite graphs. In fact, Kelly and Merriell do not use the full power of the isomorphism between $G[A]$ and $G[V(G)\setminus A]$, in most situations. We realized that instead of using isomorphisms, they are working with the degree sequences of $G[A]$ and $G[V(G)\setminus A]$, except when they are characterizing disconnected weakly equipartite graphs; in which they are using the isomorphism between $G[A]$ and $G[V(G)\setminus A]$ ([5, Theorem 3]). We provide a proof for characterizing disconnected degree-equipartite graphs (Theorem 6 below) and modify other proofs of [5] to be applicable to connected degree-equipartite graphs.

## 2 Proofs and Techniques

First, we state the following theorem from [5] which is intriguing in its own right.

**Theorem 5.** [5, Theorem 1] An even order graph $G$ is regular if and only if for every partition of $V(G)$ into two equal-sized sets $A$ and $B$, the induced subgraphs $G[A]$ and $G[B]$ have the same number of edges.

By the theorem above, we conclude that every degree-equipartite graph of order $2n$ is regular with degree, say $k$. When $k = 0$, we have the empty graph, and when $k = 1$ we get $nK_2$; both degree-equipartite. So, let us study the case $k > 1$. First, we characterize all disconnected degree-equipartite graphs.

**Theorem 6.** If $G$ is a disconnected $k$-regular degree-equipartite graph of order $2n$ with $k > 1$, then it consists of two components; these are either two complete graphs of order $n$ or else are two 4-cycles.
Proof. Let $G_1, \ldots, G_h$ denote the components of $G$ with the corresponding orders $r_1 \geq r_2 \geq \ldots \geq r_h \geq k + 1$. Let $i$, $1 \leq i \leq h$, be the first index such that $s := \sum_{j=1}^{i} r_j \geq n$. If $s > n$, put $\alpha = r_i - s + n$. Clearly, $0 < \alpha < r_i$. Choose arbitrary $R \subseteq V(G_i)$ with $|R| = \alpha$. Now consider the $n$-element subset $A = V(G_i) \cup \ldots \cup V(G_{i-1}) \cup R$ of $V(G)$ and its complement $B$. If $R$ contains an isolated vertex $u$, since $G$ is degree-equipartite, there exists a corresponding isolated vertex $b$ in $B$, which must belong to $B \cap V(G_i)$. By switching $u$ and $b$, the number of edges in $G[A]$ and $G[B]$ increases, and hence by continuing this process as long as isolated vertices exist in $G[A]$, we eventually obtain the $n$-element subset $A'$ with no isolated vertex in $G[A']$. Now choose a vertex $u \in R' = A' \cap V(G_i)$ that has a neighbour in $V(G_i) \setminus R'$ and switch it with a vertex $v \in V(G_i)$. The vertex $v$ in $A'' = (A' \setminus \{u\}) \cup \{v\}$ is isolated in $G[A'']$ while there is no isolated vertex in $G[V(G) \setminus A'']$. This contradiction shows that $s = n$, i.e., $|V(G_1) \cup \ldots \cup V(G_i)| = |V(G_{i+1}) \cup \ldots \cup V(G_h)| = n$. If $h > 2$, then $i < h - 1$. Now consider two adjacent vertices $x$ and $y$ in $G_1$, and two vertices $z \in V(G_{h-1})$ and $t \in V(G_h)$. The subgraphs induced by $(V(G_1) \cup \ldots \cup V(G_i) \setminus \{x, y\}) \cup \{z, t\}$ and $(V(G_{i+1}) \cup \ldots \cup V(G_h) \setminus \{z, t\}) \cup \{x, y\}$ have not the same degree sequences. So $h = 2$ and $r_1 = r_2 = n$. If $G_1$ is not complete, then there exist two nonadjacent vertices $p$ and $q$ in $G_1$. On the other hand, let $p'$ and $q'$ be two adjacent vertices in $G_2$. Then the subgraphs induced by $(V(G_1) \setminus \{p, q\}) \cup \{p', q'\}$ and $(V(G_2) \setminus \{p', q'\}) \cup \{p, q\}$ have not the same degree sequences, unless $G = 2C_4$. If $G \neq 2C_4$, then $G_1$ and similarly $G_2$, are complete graphs.

To characterize connected degree-equipartite graphs, we follow the approach of Kelly and Merrill in [5] with somewhat different proofs. For a vertex $v$ of a graph $G$, denote by $N(v)$ the set of neighbours of $v$ in $G$. Also, we put $\hat{N}(v) = N(v) \cup \{v\}$ and

$$F(v) = \{x \in V(G) : \hat{N}(x) \cap \hat{N}(v) = \emptyset\}.$$ 

To begin, we adopt Lemmas 1 and 2 of [5] for degree-equipartite graphs:

**Lemma 7.** If $G$ is a connected $k$-regular degree-equipartite graph of order $2n$ with $1 < k \leq n - 1$, then $N(u) \neq N(v)$ for every two distinct vertices $u$ and $v$ of $G$.

**Proof.** Assume that $N(u) = N(v)$ for some distinct vertices $u$ and $v$ of $G$. Consider an $n$-subset $A$ of $V(G)$ with $u \in A$, and $\hat{N}(u) \subseteq B$, where $B = V(G) \setminus A$. Since $G$ is connected, we can choose $B$ such that $G[B]$ is connected. But in this case $G[A]$ has an isolated vertex $u$, while $G[B]$ has not. □

**Lemma 8.** If $G$ is a connected $k$-regular degree-equipartite graph of order $2n$ with $1 < k \leq n - 1$, then $|F(u)| = n - k$ for every vertex $u$ of $G$.

**Proof.** We prove this lemma in three steps. Let $u$ be any vertex of $G$. First, we show that $|F(u)| \geq n - k$. Consider an $n$-subset $A$ of $V(G)$ such that $\hat{N}(u) \subseteq A$. Set $B = V(G) \setminus A$. Since $G[A]$ and $G[B]$ have the same degree sequences, there exists a vertex $u_1 \in B$ such that $u_1$ has degree $k$ in $G[B]$. So $w_1 \in F(u)$. If $k = n - 1$, then clearly $|F(u)| \geq n - k = 1$. Now suppose $k < n - 1$. Let $v_1, v_2, \ldots, v_{n-k-1}$ denote the vertices in $A$ which are not in $\hat{N}(u)$. Switch $v_1$ and $w_1$, that is, let $A' = (A \setminus \{v_1\}) \cup \{w_1\}$ and $B' = (B \setminus \{w_1\}) \cup \{v_1\}$. Obviously, $w_1$ is an isolated vertex of $G[A']$ which belongs to $F(u)$. Again, we have a vertex $w_2$ of degree $k$ in $G[B']$, which we switch it with $v_2$. Continuing in this way, we ultimately get an $n$-subset $A_1$ of $V(G)$ such that

$$A_1 = \hat{N}(u) \cup \{w_1, w_2, \ldots, w_{n-k-1}\} \text{ and } \{w_1, w_2, \ldots, w_{n-k-1}\} \subseteq F(u).$$

Furthermore, we have still a vertex, say $w_{n-k}$, of degree $k$ in $G[B_1]$ which belongs to $F(u)$, too. Therefore, $|F(u)| \geq n - k$. 

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Second, we prove that \(|F(u)| < n - 1\). In contrary, suppose that \(|F(u)| \geq n - 1\). Consider an \(n\)-subset \(A\) of \(V(G)\) such that \(u \in A\) and \(|A \cap F(u)| = n - 1\). Then \(N(u) \subseteq B\), where \(B = V(G) \setminus A\). Since \(u \in G[A]\) is isolated, there exists an isolated vertex \(v\) in \(G[B]\). In view of \(k > 1\), we have also \(v \in F(u)\). Hence \(|F(u)| \geq n\). Now let \(A_1\) be an \(n\)-subset of \(V(G) \cap F(u)\) and \(B_1 = V(G) \setminus A_1\). We have \(N(u) \subseteq B_1\). If there is an isolated vertex \(w\) in \(G[B_1]\), then as before \(w \in F(u)\); switch it with some vertex \(x\) in \(A_1\) which has a neighbour in \(B_1\). Such a vertex \(x\) exists in \(A_1\) because \(G\) is connected. Continuing in this way, we finally arrive at two disjoint \(n\)-subsets \(A_2\) and \(B_2\) of \(V(G)\) such that \(A_2 \subseteq F(u)\), \(\hat{N}(u) \subseteq B_2\), and \(G[B_2]\) has no isolated vertex. Now switch \(u\) and a vertex \(z\) in \(A_2\) that has a neighbour in \(B_2\). Let \(A_3 = (A_2 \setminus \{z\}) \cup \{u\}\) and \(B_3 = (B_2 \setminus \{u\}) \cup \{z\}\). Then \(G[A_3]\) has the isolated vertex \(u\), while \(G[B_3]\) has not, because \(k > 1\). This contradiction shows that \(|F(u)| < n - 1\).

Third, consider an \(n\)-subset \(C\) of \(V(G)\) such that \(u \in C\), \(F(u) \subseteq C\), and \(N(u) \subseteq D\), where \(D = V(G) \setminus C\). Since every vertex of \(D\) which is not in \(N(u)\) has some neighbour in \(N(u)\), so the isolated vertex, say \(s\), of \(G[D]\) corresponding to \(u \in C\) is in \(N(u)\). Then \(N(s) \subseteq C\). Since \(N(s)\) and \(F(u)\) both are subsets of \(C\) and disjoint, we have \(|F(u)| \leq n - k\).

Now we apply the techniques of Theorem 5 of [5] to connected degree-equipartite graphs.

**Theorem 9.** If \(G\) is a connected \(k\)-regular degree-equipartite graph of order \(2n\) with \(1 < k \leq n - 1\), then \(G = K_{n,n} \setminus nK_2\).

**Proof.** Let \(u\) be an arbitrary vertex of \(G\). Consider the class \(C\) of all partitions \((A, B)\) of \(V(G)\) with \(|A| = |B| = n\), such that \(\{u\} \cup F(u) \subseteq A\) and \(N(u) \subseteq B\). The isolated vertex of \(G[B]\) corresponding to \(u \in A\) must necessarily belong to \(N(u)\); otherwise it would be in \(F(u)\) and hence in \(A\). Furthermore, for each partition in \(C\), \(N(u)\) has exactly one vertex isolated in \(G[B]\). For if there were two, say \(v_1\) and \(v_2\), then \(N(v_1)\) and \(N(v_2)\) would both consist of the \(k\) vertices of \(A\) not in \(F(u)\), contradicting Lemma 7.

Let \(v\) denote the unique isolated vertex of \(G[B]\) corresponding to \(u \in A\). So \(A = F(u) \cup N(v)\). Therefore, different partitions in \(C\) have different vertices isolated in \(G[B]\). Consequently, \(|C| \leq k\) because we have \(k\) vertices in \(N(u)\). On the other hand, \(A\) has \(k - 1\) unspecified vertices out of \(n - 1\) vertices not fixed in \(A\) or \(B\). So, there are \(\binom{n - 1}{k - 1}\) elements in \(C\). If \(k < n - 1\), then

\[
\binom{n - 1}{k - 1} \geq n - 1 > k.
\]

This contradiction shows that \(k = n - 1\), and hence there exist \(\binom{n - 1}{k - 1} = n - 1\) elements in \(C\). Thus, we have \(n - 1\) isolated vertices in \(G[N(u)]\), i.e., \(N(u)\) is a stable set in \(G\). Now consider \(A_1 = F(u) \cup N(u)\) which is a stable \(n\)-subset of \(V(G)\). Hence \(B_1 = V(G) \setminus A_1\) is also a stable set. It follows that \(G\) is a \((n - 1)\)-regular bipartite graph, and in fact \(G = K_{n,n} \setminus nK_2\).

Now, we are ready to characterize all degree-equipartite graphs, which results in Theorem 4.

**Theorem 10.** A graph \(G\) of order \(2n\) is degree-equipartite if and only if it is one of the following graphs:

\[2nK_1, \ nK_2, \ 2C_4, \ K_{n,n} \setminus nK_2, \ and \ 2K_n\]

or one of their complements

\[K_{2n}, \ K_{2n} \setminus nK_2, \ K_8 \setminus 2C_4, \ 2K_n + nK_2 \ and \ K_{n,n}\].

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Proof. By Theorem 1, all the graphs above are weakly equipartite, and so are degree-equipartite. For the converse, let \( G \) be a \( k \)-regular degree-equipartite graph of order \( 2n \). We have the following cases:

- If \( k = 0 \), then \( G = 2nK_1 \).
- If \( k = 1 \), then \( G = nK_2 \).
- If \( k > 1 \), and \( G \) is disconnected, then \( G = 2K_n \) or \( G = 2C_4 \), by Theorem 6.
- If \( 1 < k \leq n - 1 \), and \( G \) is connected, then \( G = K_{n,n} \setminus nK_2 \).
- If \( n - 1 < k < 2n - 2 \), then \( G \) is connected. If \( G^c \) is disconnected, then \( G^c \) is a \((2n - k - 1)\)-regular degree-equipartite graph by Proposition 3. Hence, since \( 2n - k - 1 > 1 \), we have \( G^c = 2K_n \), or \( G^c = 2C_4 \) by Theorem 6. Consequently, \( G = K_{n,n} \), or \( G = K_8 \setminus 2C_4 \). If \( G^c \) is connected, since we have \( 1 < 2n - k - 1 \leq n - 1 \), then \( G^c = K_{n,n} \setminus nK_2 \) by Theorem 9, and so \( G = 2K_n + nK_2 \).
- If \( k = 2n - 2 \), then \( G^c \) is a 1-regular graph, and hence \( G = K_{2n} \setminus nK_2 \).
- If \( k = 2n - 1 \), then \( G = K_{2n} \).

\( \square \)

3 Some Remarks

Igor Shparlinski (\[6\]) proposed the following problem:

**Problem 1.** Let \( G \) be a graph of order \( 2n \) such that for every \( n \)-element set \( A \subseteq V(G) \), the induced subgraphs \( G[A] \) and \( G[V(G) \setminus A] \) are isospectral. Let us call these graphs spectral-equipartite. Which graphs are spectral-equipartite?

Suppose \( G \) is a graph of order \( 2n \) and size \( m \), with eigenvalues \( \lambda_1, \ldots, \lambda_{2n} \). It is known that \( m = \frac{1}{2} \sum_{i=1}^{2n} \lambda_i^2 \) (see, e.g., \[3\]). So, by Theorem 5, the suggested graphs by Shparlinski are regular. A full characterization of spectral-equipartite graphs seems to be an interesting problem.

**Remark 2.** We independently introduced the notion of weakly equipartite graphs in \[1\]. We called such graphs well-bisective, and using techniques very similar but slightly weaker than those used in \[5\] and \[4\], could characterize all disconnected well-bisective graphs and also all bipartite well-bisective graphs. Later, we found that similar work are done in \[5\] and \[4\].

**Remark 3.** The definition of a degree-equipartite graph (and so, weakly equipartite graph) has no nontrivial generalization to more than two parts, by Exercise 2.2.25 of \[2\].

**Remark 4.** The proof techniques which are used in \[5\] may be useful to attack the Reconstruction Conjecture.

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