MORSE THEORY FOR THE HOFER LENGTH FUNCTIONAL

YASHA SAVEYEV

Abstract. Following [16], we develop here a connection between Morse theory for the (positive) Hofer length functional $L: \Omega\text{Ham}(M,\omega) \to \mathbb{R}$, with Gromov-Witten/Floer theory, for monotone symplectic manifolds $(M,\omega)$. This gives some immediate restrictions on the topology of the group of Hamiltonian symplectomorphisms (possibly relative to the Hofer length functional), and a criterion for non-existence of certain higher index geodesics for the Hofer length functional. The argument is based on a certain automatic transversality phenomenon which uses Hofer geometry to conclude transversality and may be useful in other contexts. Strangely the monotone assumption seems essential for this argument, as abstract perturbations necessary for the virtual moduli cycle, decouple us from underlying Hofer geometry, causing automatic transversality to break.

1. Introduction

Topology of the group of Hamiltonian diffeomorphisms $\text{Ham}(M,\omega)$ of a symplectic manifold $(M,\omega)$, is a very rich object of study connected to all techniques of modern symplectic topology. The first major investigations already appear in Gromov’s [2] for some four dimensional symplectic manifolds using Gromov-Witten theory. Much of the subsequent study of the subject was based on this. In the higher dimensional setting rather little is known outside special cases with high symmetry, e.g. [13]. The problem in general is that it is very hard to even construct good candidates for “cycles” in $\text{Ham}(M,\omega)$, which may be non-trivial. For $\pi_1$ there is one very natural candidate: Hamiltonian circle actions. It turns out [12] that if the circle action is in appropriate sense semi-free, it always represents a non-trivial class in $\pi_1$. There are some generalizations of this to certain geodesics of the Hofer length functional for example [11]. In these cases one crucial necessary condition on such a geodesic is that it must be index 0. From this point of view it is in a sense clear how to try generalize the above: consider more general higher index geodesics for the (positive) Hofer length functional, and their unstable manifolds, i.e. try to do some kind of Morse theory. One immediate problem is that in general we can only make sense of “unstable manifolds” locally, (see however [16] for examples of when global unstable manifolds do exist) and so must work with relative classes. Also to guarantee local smoothness of the Hofer length functional, we must restrict the class of geodesics to what we call “Ustilovsky geodesics”, which first appear in [17], and whose theory is further developed in [7] in more generality. Nevertheless the local Morse theory can be set up. This gives us candidates for “cycles”, when are they non-trivial? We show that this always happens under certain Floer theoretic assumptions on the Ustilovsky geodesic, which leads us to the notion of “robust Ustilovsky geodesic”, (technically we still have to perturb the geodesic). This is a strange phenomenon. The robust condition is some global Floer theoretic
condition, but is local from the point of view of Ham(M, ω), yet it is enough to deduce the global fact that the above cycles are non-trivial.

The main argument is based on a certain unusual automatic transversality phenomenon, which actually uses Hofer geometry to conclude transversality. This already appeared in [16] in less generality, but was somewhat obscured and had some inaccuracies.

1.1. The group of Hamiltonian symplectomorphisms and Hofer metric.

Given a smooth function \( H : M \times [0,1] \to \mathbb{R} \), there is an associated time dependent Hamiltonian vector field \( X_t \), \( 0 \leq t \leq 1 \), defined by

\[
\omega(X_t, \cdot) = -dH_t(\cdot).
\]

The vector field \( X_t \) generates a path \( \gamma : [0,1] \to \text{Diff}(M) \), starting at \( \text{id} \). Given such a path \( \gamma \), its end point \( \gamma(1) \) is called a Hamiltonian symplectomorphism. The space of Hamiltonian symplectomorphisms forms a group, denoted by \( \text{Ham}(M,\omega) \).

In particular the path \( \gamma \) above lies in \( \text{Ham}(M,\omega) \). It is well-known that any smooth path \( \gamma \) in \( \text{Ham}(M,\omega) \) with \( \gamma(0) = \text{id} \) arises in this way (is generated by \( H : M \times [0,1] \to \mathbb{R} \)). Given a general smooth path \( \gamma \), the Hofer length, \( L(\gamma) \) is defined by

\[
L(\gamma) := \int_0^1 \max_M H_t^\gamma - \min_M H_t^\gamma dt,
\]

where \( H^\gamma \) is a generating function for the path \( t \mapsto \gamma(t) \), \( 0 \leq t \leq 1 \). The Hofer distance \( \rho(\phi,\psi) \) is defined by taking the infimum of the Hofer length of paths from \( \phi \) to \( \psi \). We only mention it, to emphasize that it is a deep and interesting theorem that the resulting metric is non-degenerate, (cf. [3, 6]). This gives \( \text{Ham}(M,\omega) \) the structure of a Finsler manifold. A related functional, \( L^+ \), that more readily connects to Gromov-Witten theory is given by

\[
L^+(\gamma) := \int_0^1 \max_M H_t^\gamma dt,
\]

for \( H_t^\gamma \) the generating function normalized by \( \int_M H_t^\gamma \omega^n = 0 \), for every \( t \).

We now consider \( L^+ \) as a functional on the space of paths in \( \text{Ham}(M,\omega) \) starting at \( \text{id} \) and ending at some fixed end point \( \phi \), denote this by \( P_\phi \). It is shown by Ustilovsky [17] that \( \gamma \) is a smooth critical point of

\[
L^+ : P_\phi \to \mathbb{R},
\]

if there is a unique point \( x_{\text{max}} \in M \) maximizing the generating function \( H_t^\gamma \) at each moment \( t \), and such that \( H_t^{\gamma_{x_{\text{max}}}} \) is Morse at \( x_{\text{max}} \), at each moment \( t \).

Definition 1.1. We call such a \( \gamma \) an Ustilovsky geodesic.

Definition 1.2. Given a chain complex \((A_*, d)\) with some distinguished basis, and the inner product determined by this basis, (with respect to which it is orthonormal), we say that a chain \( c \) is semi homologically essential if \( c \) is orthonormal to \( d(c) \) for any \( e \).

This is of course automatic if \( A_* \) is perfect, which is often the case in Floer theoretic applications we consider. Although we don’t need this, it is relatively easy to see that if \( c \) is semi-homologically essential in a chain complex of vector spaces \((A_*, d)\), and is closed (\( dc = 0 \)) then it is homologically essential, meaning there is no quasi-isomorphic sub-complex of \( A_* \), which is orthogonal to \( c \). If \( c \) is
homologically essential then again a bit of elementary algebra implies that it is semi-homologically essential. (We need field coefficients for this.)

**Definition 1.3.** We will say that an Ustilovsky geodesic \( \gamma \in \mathcal{P}_{\phi} \) is robust, if \( \bar{\phi} \) is Floer non-degenerate and the constant, period one orbit \( o_{\text{max}} \) at \( x_{\text{max}} \) for the flow \( \gamma \) is semi-homologically essential in \( CF(\gamma) \). Here \( \bar{\phi} \in \text{Ham}(M, \omega) \) is the lift of \( \phi \) to the universal cover determined by \( \gamma \).

**Theorem 1.4.** \(^{[15]} \) Let \( \gamma \in \mathcal{P}_{\phi} \) be an Ustilovsky geodesic, then the Morse index of \( \gamma \) with respect to \( L^+ \) is

\[
|\text{CZ}(o_{\text{max}}) - \text{CZ}([M])|,
\]

where \( \text{CZ} \) is the Conley-Zehnder index and \( \text{CZ}([M]) \) denotes the Conley-Zehnder degree of the fundamental class, under the PSS isomorphism.

If the Conley-Zehnder index is normalized as in \(^{[14]} \text{CZ}(o_{\text{max}}) - \text{CZ}([M]) \leq 0 \). (With \( \text{CZ}([M]) = -n \)) This is the normalization that will be assumed for the grading of the Floer chain complex. (Although it will be implicit.)

1.2. Statement of main theorems.

**Definition 1.5.** Let \( \gamma \in \mathcal{P}_{\phi} \) be an index \( k \) Ustilovsky geodesic and let \( B^k \) denote the standard \( k \)-ball in \( \mathbb{R}^n \), centered at the origin 0. Let \( \mathcal{P}_{\phi,E^{\gamma}} \) denote the \( E^{\gamma} \) sub-level set of \( \mathcal{P}_{\phi} \), with respect to \( L^+ \), with \( 0 < E^{\gamma} < L^+(\gamma) \). A local unstable manifold at \( \gamma \) is a pair \( (f_{\gamma}, E^{\gamma}) \), with \( f_{\gamma} : B^k \rightarrow \mathcal{P}_{\phi} \), s.t. \( f_{\gamma}(0) = \gamma \), \( f_{\gamma}^*L^+ \) is a function Morse at the unique maximum \( 0 \in B^k \), and s.t. \( f_{\gamma}(\partial B) \subset \mathcal{P}_{\phi,E^{\gamma}} \).

Note that local unstable manifolds for a given Ustilovsky geodesic always exist as can be immediately deduced from \(^{[17]} \text{2.2A} \), for geodesics coming from circle actions elegant explicit local unstable manifolds are constructed in \(^{[4]} \).

**Theorem 1.6.** Suppose that \((M, \omega)\) is a monotone symplectic manifold and \( \gamma \in \mathcal{P}_{\phi} \) is an index \( k \) robust Ustilovsky geodesic. Then there is a \( \gamma' \in \mathcal{P}_{\phi'} \), which is a robust Ustilovsky geodesic arbitrarily \( C^\infty \)-close to \( \gamma \), a number \( E^{\gamma'} < L^+(\gamma') \) arbitrarily close to \( L^+(\gamma') \) and \( (f_{\gamma'}, E^{\gamma'}) \) a local unstable manifold at \( \gamma' \), s.t.

\[
0 \neq [f_{\gamma'}] \in \pi_k(\mathcal{P}_{\phi'}, \mathcal{P}_{\phi',E^{\gamma'}}) \otimes \mathbb{Q}.
\]

Below we list some examples of robust Ustilovsky geodesics.

**Example 1.7.** An Ustilovsky geodesic \( \gamma \in \mathcal{P}_{\phi} \) is robust, for example if \( CF(\gamma) \) is perfect. An explicit example for \( M = S^2 \) with arbitrary index could be obtained as follows. Suppose \( H \) generates a \( k + 1/2 \) fold rotation of \( S^2 \). In this case \( H \) is Floer non-degenerate and \( CF(\gamma) \) is perfect simply by virtue of not having odd degree generators. (It is generated by \( o_{\text{max}}, o_{\text{min}} \) as a module over the Novikov ring.) The index of \( \gamma \) is 2\( k \).

**Example 1.8.** We may generalize the above example as follows. Suppose we have a symplectic manifold \( M \) and \( H \) a Morse Hamiltonian generating a semi-free circle action in time 1. Here semi-free means that the isotropy group of every point in \( M \) is either trivial or the whole group. Then the time 1 flow map for \( \tau \cdot H \), with \( \tau \in \mathbb{N} \) is Floer non-degenerate, and has no non-constant period 1 orbits. The CZ index of all the constant period 1 orbits, i.e. critical points of \( H \) in this case must be even,
which can be readily checked, as the linearized flow at the critical points is a path in \( U(n) \). (Linearizing the circle action itself we get an \( S^1 \) subgroup of \( \text{Symp}(\mathbb{R}^{2n}) \) which must be unitary as \( U(n) \) is the maximal compact subgroup of \( \text{Symp}(\mathbb{R}^{2n}) \)). In particular the Floer complex is perfect, cf. [5], [11].

We may consequently get a more explicit version of 1.6 as follows.

**Theorem 1.9.** Suppose \( \gamma \) is an index \( k \) Ustilovsky geodesic, which in addition is a path determined by a Hamiltonian circle action, (is a restriction thereof). If \( (f_\gamma, E_\gamma) \) is a local unstable manifold for \( \gamma \) then

\[
0 \neq [f_\gamma] \in \pi_k(\mathcal{P}_\phi, \mathcal{P}_{\phi', E'}) .
\]

This follows by [1.2.4], [1.2.1] and 1.8. It is worth pointing out for comparison that the length/energy functional on the path space (with fixed end points) of a smooth Riemannian manifold \( X, g \), may have lots of critical points (i.e. geodesics) which do not satisfy the analogue of the property above. It is tricky to describe higher index examples without getting completely side-tracked. We can for example produce a Riemannian manifold, and a submanifold \( U \) of the path space with \( U \) shaped like a heart shaped sphere for the energy functional, with non-degenerate critical points (geodesics) of index 2,1,2,0. The index 1 geodesic in this case clearly does not have the analogue of the property of Theorem 1.6 above. In fact it is possible to show that all the geodesics of \( X, g \) satisfy the property above and are all non-degenerate then the energy functional is perfect, i.e. the \( i \)th Betti number of the path space is the number of index \( i \) geodesics. In the index 0 case here is a very elementary concrete example. Deform the \( z = 0 \) isometric embedding of \( \mathbb{R}^2 \) into \( \mathbb{R}^3 \) so that the image acquires a single mountain (and is flat elsewhere). Pull-back the metric. Take points \( p, q \in \mathbb{R}^2 \) to be on the opposite side of the mountain. There are a pair of geodesics going around the base of the mountain. We may shape the mountain, so that none of their sufficiently small perturbations are length minimizing.

Theorem 1.6 of course also give restrictions on absolute homotopy groups. Considering the classical long exact sequence for relative homotopy groups (tensored with the flat \( \mathbb{Z} \) module \( \mathbb{Q} \)) we get the following:

**Corollary 1.10.** Under assumptions of the theorem above, either \( \pi_{k-1}(\mathcal{P}_{\phi', E'}, \mathbb{Q}) \neq 0 \) or \( \pi_k(\mathcal{P}_\phi, \mathbb{Q}) \simeq \pi_{k+1}(\text{Ham}(M, \omega), \mathbb{Q}) \neq 0 \).

More explicitly, if the boundary of the local unstable manifold \( (f_{\gamma'}, E_{\gamma'}) \) can be contracted inside \( \mathcal{P}_{\phi', E'} \) we get a sphere representing a non-trivial class in \( \pi_k(\mathcal{P}_\phi, \mathbb{Q}) \simeq \pi_{k-1}(\text{Ham}(M, \omega), \mathbb{Q}) \).

Here is one concrete corollary:

**Corollary 1.11.** Given a local unstable manifold \( (f_{\gamma}, E_{\gamma}) \) at a robust index \( k > 2 \) Ustilovsky geodesic \( \gamma \) in the path space of \( \text{Ham}(S^2) \), \( \mathcal{P}_\phi \), and \( \gamma' \) as in 1.7, the class of the boundary map \( [\partial f_{\gamma'}] : B^{k-1} \to \mathcal{P}_{\phi', E_{\gamma'}} \) is non-zero in \( \pi_{k-1}(\mathcal{P}_{\phi', E_{\gamma'}}, \mathbb{Q}) \). The same holds for \( M = \mathbb{C}P^2 \) if \( k > 4 \).

This follows upon noting that for \( k > 2 \), \( \pi_k(\mathcal{P}_{\phi'}, \mathbb{Q}) = \pi_{k+1}(\text{Ham}(S^2, \mathbb{Q}) = 0 \), and for in case of \( M = \mathbb{C}P^2 \) for \( k > 4 \), \( \pi_k(\mathcal{P}_{\phi'}, \mathbb{Q}) = \pi_{k+1}(\text{Ham}(\mathbb{C}P^2, \mathbb{Q}) = \pi_{k-1}(\text{PSU}(3), \mathbb{Q}) = 0 \). (Note that \( \text{PSU}(3) \) does have non-vanishing rational homotopy group in degree 5.)

Note that there are robust Ustilovsky geodesics in \( \text{Ham}(S^2) \) of arbitrary even index, and \( \phi, \phi' \) can be taken to be arbitrarily close to \( \text{id} \) see Example 1.7.
particular although $\Omega \text{Ham}(S^2) \simeq \Omega SO(3)$ has vanishing rational homotopy groups in degree greater than 2, there are $E$ sub-level sets for the (positive) Hofer length functional, with $E$ arbitrarily large, which do not.

Here is another geometric/dynamical application of 1.6, formally reminiscent of the non-existence result [7] for length minimizing Hofer geodesics in $\text{Ham}(S^2)$.

**Corollary 1.12.** Given a monotone $(M,\omega)$, and $\phi \in \text{Ham}(M,\omega)$ s.t. the space of paths $\delta$ close to minimizing the positive Hofer length (a.k.a $\delta$-minimizing paths) from $\text{id}$ to $\phi$ has vanishing rational homotopy groups, for some $\delta > 0$, there is no index $k > 0$ robust Ustilovsky geodesic from identity to $\phi$, $\delta$-close to minimizing the Hofer length.

We expect that the condition on $\phi$ holds for any $\phi$ sufficiently close to identity. For $\text{Ham}(S^2)$ this would follow for example if it was known that $\text{Ham}(S^2)$ had some contractible epsilon ball with respect to the (positive) Hofer metric. (Having vanishing rational homotopy groups would suffice for the above.) Although at the moment it is not clear how to verify this, in a joint work in progress with Misha Khanevsky we intend to show that there is $\epsilon_k > 0$ and a Hofer $\epsilon_k$-ball $B_{\epsilon_k}$ in the space of Lagrangians in $S^2$ Hamiltonian isotopic to the equator, with the number of intersections with the equator at most $k$, such that $B_{\epsilon_k}$ is contractible. Some initial results in the spirit of this discussion are obtained in [8], in particular we show there that for $\phi \in \text{Ham}(S^2)$ sufficiently close to identity, there is a $\delta$ so that inclusion map from the space of $\delta$-minimizing paths into the space of all paths vanishes on rational homotopy groups.

As another corollary we have:

**Theorem 1.13.** Let $\gamma \in P_\phi$ be a robust, index 0, Ustilovsky geodesic, then $\gamma$ globally minimizes $L^+$ in its homotopy class, relative to end points.

This result is new, although there are related existing results, for example [11], [5]. These are essentially based on variations of sufficient conditions on $o_{\max}$ for being homologically essential, (in the second case this is generalized to take into account action intervals).

**Theorem 1.14.** Under the conditions of theorem 1.9 above, the local unstable manifold

$$f_\gamma : (D^k, \partial D^k) \to (P_\phi, P_{\phi,E^\gamma})$$

realizes the minimum in the definition of the semi-norm:

$$||f|| = \inf_{f' \in [f]} \max_{b \in D^k} L^+(f'(b)).$$

This is a special case of Theorem 1.22.

1.3. **Outline of the argument.** Our symplectic manifold $(M,\omega)$ is assumed everywhere to be monotone. We first construct, using a kind of parametric Floer continuation map, a group-homomorphism (for $k > 0$, otherwise just a set map):

$$\Psi_\gamma : \pi_k(P_\phi, P_{\phi,E^\gamma}) \otimes \mathbb{Q} \to \mathbb{Q},$$

for $E^\gamma$ sufficiently close to $L^+(\gamma)$. Here and from now on $P_\phi$ denotes the component of the path space in the homotopy class $[\gamma]$. As indicated the homomorphism
depends on a particular Ustilovsky geodesic \( \gamma \). The setup for Gromov-Witten theory needed in the definition of \( \Psi_\gamma \) is somewhat unusual, and we need to take time to describe the class of almost complex structures needed for this. Let \( C \) denote the space of Hamiltonian connections on \( M \times \mathbb{C} \) satisfying the first pair of conditions in the Definition 1.15 below.

For \( A \in C \) or \( \Omega \) a coupling 2-form on \( M \times \mathbb{C} \), inducing a connection in \( C \) define

\[
\text{area}(A) = \inf \{ \int_{C} \alpha [\bar{\Omega}_A + \pi^*(\alpha) \text{ is nearly symplectic} \},
\]

\[
\text{area}(\Omega) = \inf \{ \int_{C} \alpha [\bar{\Omega} + \pi^*(\alpha) \text{ is nearly symplectic} \},
\]

where \( \bar{\Omega}_A \) is the coupling form inducing \( A \), (\cite{9} Theorem 6.21) \( \alpha \) a 2-form on \( \mathbb{C} \) and nearly symplectic means that

\[
(1.4) \quad (\bar{\Omega}_A + \pi^*(\alpha))(\bar{v}, jv) \geq 0,
\]

for \( \bar{v}, jv \) horizontal lifts with respect to \( A \), of \( v, jv \in T_z \mathbb{C} \), for all \( z \). It is not hard to see that the infimum is attained on a uniquely defined 2-form \( \alpha_A \):

\[
(1.5) \quad \alpha_A(v, w) = \max_M R_A(v, w),
\]

where \( R_A \) is the Lie algebra valued curvature 2-form of \( A \), and we are using the isomorphism \( \text{lieHam}(M, \omega) \cong C^\infty_{\text{norm}}(M) \), see Section 3.2. By assumptions this form has compact support.

**Definition 1.15.** For a \( 0 < \delta < (L^+(\gamma) - E^+)/2 \) and a given \( f : (B^k, \partial B^k) \to (\mathcal{P}_0, \mathcal{P}_0, E^+) \), a family of Hamiltonian connections \( A_b \) on \( M \times \mathbb{C} \) is said to be \( \delta \)-admissible with respect to \( f \), if:

\begin{itemize}
  \item Using the modified polar coordinates \((r, \theta), 0 \leq r < \infty, 0 \leq \theta \leq 1, \text{ for } r \geq 2\) each \( A_b \) is flat and invariant under the dilation action of \( \mathbb{R} \) on \( \mathbb{C} \).
  \item The holonomy path \( \pi(A_b) \) of \( A_b \) over the circles \( \{ r \} \times S^1 \) is \( f(b) \), for \( r \geq 2 \).
  \item \( |\text{area}(A_b) - L^+(f(b))| \leq \delta \).
\end{itemize}

We will say that \( A \) is \( \delta \)-admissible with respect to \( p \in \mathcal{P}_0 \), if it satisfies the conditions above with respect to \( p \). Fix a family \( \{ j_{r, \theta} \} \), on the vertical tangent bundle \( T_{\text{vert}}(M \times \mathbb{C}) \), invariant under the dilation action of \( \mathbb{R} \) for \( r > 2 \), so that each \( j_{r, \theta} \) is compatible with \( \omega \) : \( \omega(\cdot, j \cdot) > 0 \), for \( \cdot \neq 0 \). Then a Hamiltonian connection \( A \) on \( M \times \mathbb{C} \) induces an almost complex structure \( J_A \) on \( M \times \mathbb{C} \) having the properties:

\begin{itemize}
  \item Each \( J_A \) coincides on the vertical tangent distribution of \( M \times \mathbb{C} \) with \( \{ j_{r, \theta} \} \).
  \item The projection map \( \pi : M \times \mathbb{C} \to \mathbb{C} \) is \( J_A \) holomorphic.
  \item \( J_A \) preserves the \( A \)-horizontal distribution on \( M \times \mathbb{C} \).
\end{itemize}

(We don’t specify \( \{ j_{r, \theta} \} \) in the notation for \( J_A \), the dependence will be implicit). For a family \( \{ A_b \} \) \( \delta \)-admissible with respect to \( f \), \( b \in B^k \), we define

\[
(1.6) \quad j_{b, r, \theta} = \psi_{b, r, \theta}^* ;
\]

\[
(1.7) \quad \psi_b : M \times \mathbb{C} \to M \times \mathbb{C},
\]

\[
\psi_b(x, r, \theta) = (\gamma_\theta \circ f_{b, \theta}^{-1} x, r, \theta) \text{ for } r \geq 2,
\]

\[
\psi_b(x, r, \theta) = (x, r, \theta) \text{ for } r \leq 1,
\]

with \( \psi_b \) for \( 1 < r < 2 \) being an interpolation determined by the contraction of the loop \( \gamma_\theta \circ f_{b, \theta}^{-1} \), \( \theta \in S^1 \) of Hamiltonian diffeomorphism, which is obtained by
concatenating \( f(p_b) \), where \( p_b \) is a smooth geodesic path constant near end points from \( b \) to 0 in \( B^k \) (for the flat metric), with a fixed smooth path \( p_0 \) also constant near end points, from \( f(0) \) to \( \gamma \) in \( \mathcal{P}_\phi \).

Let
\[
\overline{\mathcal{M}}(\{J_b^A\}) \equiv \overline{\mathcal{M}}(\{J_b^A\}, \sigma_{\text{max}}, \sigma_{\text{max}})
\]
denote the compactified space of pairs \((u, b)\), for \( u \) a class \([\sigma_{\text{max}}]\), \( J_b \)-holomorphic section of \( M \times \mathbb{C} \) with \( u|_{\{r\} \times S^1} \) asymptotic as \( r \to \infty \) to the flat section \( o_{\text{max}, b} \in CF(p(A_b)) \). Where the generator, \( o_{\text{max}, b} \) corresponds to \( o_{\text{max}} \in CF(\gamma) \) under the canonical identification of generators of \( CF(p(A_b)) \) with those of \( CF(p(A_0)) = CF(\gamma) \), via the action of the loop \( \gamma_\theta \circ f_{-1, \theta}^{-1}, \theta \in S^1 \) of Hamiltonian diffeomorphisms. This identification is in fact an isomorphism of chain complexes
\[
CF(p(A_b), \{j_b, r, \theta \}_{r, \theta}) \to CF(\gamma, j),
\]
see Section 3.1. The class \([\sigma_{\text{max}}]\) is the class of the section \( z \mapsto x_{\text{max}} \), for \( x_{\text{max}} \) the maximizer of \( H_\gamma^\flat \) as before. An element \((u, b)\) is said to be in class \([\sigma_{\text{max}}]\) if the “section” \( \psi_b \circ o \) asymptotic to \( o_{\text{max}} \), is in the class \([\sigma_{\text{max}}]\), where “class” is now unambiguous.

For this class the virtual dimension of \( \overline{\mathcal{M}}(\{J_b^A\}) \) will be 0. The compactification is the classical compactification in Floer theory. Elements in the boundary of \( \overline{\mathcal{M}}(\{J_b^A\}) \), may have vertical bubbles lying in the fibers \( M \) of the projection \( M \times \mathbb{C} \to \mathbb{C} \), and or may have breaks as in usual Floer theory. These breaks happen in the \( r > 2 \), flat, dilation invariant part of \( M \times \mathbb{C} \). Projecting the “section” to \( M \) in the \( r > 2 \) region we get the usual picture for breaking of Floer trajectories. We will not say much more on this as this part of classical Floer theory.

Since \( M \) is monotone and since the expected dimension of the moduli space is 0, we may regularize so that \( \overline{\mathcal{M}}(\{J_b^A\}) \) consists only of smooth curves. However, we will have to deal with breaking (but not bubbling) when studying deformations of the data \( \{A_b\} \).

The map \( \Psi_\gamma \) is defined as the Gromov-Witten invariant
\[
\int_{\overline{\mathcal{M}}(\{J_b^\text{reg}, A_b\})} 1.
\]
The fact that regularization is possible via perturbation of the family \( \{A_b\} \) is not immediate but readily follows by [10, Theorem 8.3.1]. This is going to be of paramount importance for the main argument.

**Remark 1.16.** We need monotonicity as opposed to semi-positivity, as we have to deal with families of almost complex structures on \( M \times \mathbb{C} \). The analogous condition of being semi-positive that would be necessary is that for a generic (in parametric sense) \( k \)-family of almost complex structures on \( M \times \mathbb{C} \) there are no vertical holomorphic spheres in \( M \times \mathbb{C} \) with negative Chern number. Clearly this condition becomes more restrictive as \( k \) increases, on the other hand monotonicity insures this for all \( k \) at once.

**Remark 1.17.** The monotonicity assumption is not due to avoidance of the virtual moduli cycle, it appears to be rather necessary for the argument to go through at all.

**Lemma 1.18.** \( \Psi_\gamma([f]) \) is independent of the choice of the family \( \{A_b\} \) admissible with respect to \( f' \), and of \( f' \in [f] \).
To prove this we first need to show that for a deformation \( \{A_{b,t}\}, \, 0 \leq t \leq 1 \), there are no elements 
\[(u, b, t) \in \mathcal{M}(\{J_{b,t}^A\}), \]
for \( b \) near \( \partial B^k \). For this we need the special nature of the class \([\sigma_{\text{max}}]\).

**Proposition 1.19.** For \( f \) as before, there is an assignment \( f \mapsto A_{b,f} \), where \( \{A_{b,f}\} \) is admissible with respect to \( f \). Denote by \( \{J_{b,f}^A\} \) the induced family of almost complex structures, then for a local unstable manifold \( f_\gamma \) at \( \gamma \), the space \( \mathcal{M}(\{J_{b,f}^A\}) \) consists of only one point: \( (\sigma_{\text{max}}, 0) \).

**Theorem 1.20.** Let \( \gamma \) be an Ustilovsky geodesic, s.t. the associated real linear Cauchy-Riemann operator (rest stands for restricted: the full operator has \( T_0B \) as a summand for domain):
\[ D_{\sigma_{\text{max}}}^{\text{rest}} : \Omega^0(\sigma_{\text{max}}^* T^\text{vert}(M \times \mathbb{C})) \rightarrow \Omega^{0,1}(\sigma_{\text{max}}^* T^\text{vert}(M \times \mathbb{C})), \]
has no kernel. Let \( f_\gamma : (B^k, \partial B^k) \rightarrow (\mathcal{P}_\phi, \mathcal{P}_{\phi,E^\gamma}) \) be a local unstable manifold at \( \gamma \), then there is an admissible family \( \{A_b\}, \, A_0 = A_{0,f_\gamma} \), such that \( \mathcal{M}(\{J_{0,b}^A\}) \) consists only of \( (\sigma_{\text{max}}, 0) \) and this element is regular. And so if \( \gamma \) is in addition robust then
\[ 0 \neq [f_\gamma] \in \pi_k(\mathcal{P}_\phi, \mathcal{P}_{\phi,E^\gamma}). \]

The first half of the statement is the “automatic transversality”, although the term is used in a somewhat looser sense than usual. The point is that the full real linear CR operator with domain \( \Omega^0(\sigma_{\text{max}}^* T^\text{vert}(M \times \mathbb{C})) \oplus T_0B \) may still have kernel on the \( T_0B \) component (and hence cokernel as the index is 0), but any such kernel is “removable” in the sense that there is a regularizing Fredholm perturbation of the Cauchy-Riemann section, which does not change the 0 locus. The above theorem is the main ingredient for Theorem 1.6.

**Proposition 1.21.** For \( \gamma \) as in 1.15 by the proof of 1.6 the condition on the CR operator is always satisfied for an almost complex structure \( j \) on \( M \) integrable and invariant under the action of \( \gamma \) in a neighborhood of \( x_{\text{max}} \), and admitting a Kahler chart to \( \mathbb{C}^n \) at \( x_{\text{max}} \).

**Proof.** Pulling back the \( \gamma \) action to \( \mathbb{R}^{2n} \), by the Kahler chart at \( x_{\text{max}} \), we get an action of \( S^1 \) on a neighborhood \( U \) of 0 in \( \mathbb{C}^n \) which preserves the standard complex structure and symplectic form, hence is an action by complex isometries of \( U \) fixing 0 \( \in U \). Since such an isometry is linear, this determines a homomorphism \( S^1 \rightarrow U(\iota) \). The proof of 1.6 in this case gives that the normal bundle of \( \sigma_{\text{max}} \) is naturally holomorphic and a neighborhood of the 0-section is biholomorphic to a neighborhood of \( \sigma_{\text{max}} \) in \( M \times \mathbb{C} \) with respect to the almost complex structure \( J_{\mathcal{A}_\gamma} \).

Completing the proof 1.6 we get the desired claim.

**Theorem 1.22.** Under the conditions of theorem 1.20 above, the local unstable manifold
\[ f_\gamma : (B^k, \partial B^k) \rightarrow (\mathcal{P}_\phi, \mathcal{P}_{\phi,E^\gamma}) \]
realizes the minimum in the definition of the semi-norm:
\[ ||f|| = \inf_{f' \in [f]} \max_{b \in B^k} L^+(f'(b)). \]
The proof of Theorem 1.6 proceeds by constructing $\gamma'$ from $\gamma$ satisfying the condition on the CR operator.

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3. Some preliminaries and notation

3.1. Floer chain complex. Classically, the generators of the Hamiltonian Floer chain complex associated to $H : M \times S^1 \to \mathbb{R}$, are pairs $(o, o)$, with $o$ a time 1 periodic orbit of the Hamiltonian flow generated by $H$, and $o$ a homotopy class of a disk bounding the orbit. The function $H$ determines a Hamiltonian connection $A_H$ on the bundle $M \times S^1 \to S^1$. The horizontal spaces for $A_H$ are the $\tilde{\Omega}_H$ orthogonal spaces to the vertical tangent spaces of $M \times S^1$, where

\[ \tilde{\Omega}_H = \omega - d(Hd\theta). \]

To remind the reader our convention for the Hamiltonian flow is:

\[ \omega(X_t, \cdot) = -dH_t(\cdot). \]

The horizontal (a.k.a flat) sections for $A_H$ correspond to periodic orbits of $H$, in the obvious way. The homotopy classes $\tilde{o}$ induce homotopy classes of bounding disks in $M \times D^2$ of the corresponding flat sections. The connection $A_H$, induces an obvious $\mathbb{R}$-translation invariant connection on $M \times \mathbb{R} \times S^1$, trivial in the $\mathbb{R}$ direction. For an $\mathbb{R}$ translation invariant family $\{j_\theta\}$ of almost complex structures on the vertical tangent bundle of $M \times \mathbb{R}$, we have an induced almost complex structure $J_H$ on $M \times \mathbb{R} \times S^1$ as explained in the introduction. The differential in the classical Hamiltonian Floer chain complex is obtained via count of $J_H$-holomorphic sections $u$ of $M \times \mathbb{R} \times S^1 \to \mathbb{R} \times S^1$, whose projections to $M \times S^1$ are asymptotic in backward, forward time to generators $(o_-, \tilde{o}_-)$, respectively $(o_+, \tilde{o}_+)$ of $CF(H)$, such that $\tilde{o}_- + [u] - \tilde{o}_+ = 0 \in \pi_2(M \times S^2)$, with the obvious interpretation of this equation, and such that

\[ CZ(o_+, \tilde{o}_+) - CZ(o_-, \tilde{o}_-) = 1. \]

We shall omit further details. The corresponding chain complex will be denoted by $CF(p, \{j_\theta\})$, if $p$ is the path from $id$ to $\phi \in \text{Ham}(M, \omega)$ generated by $H$. If $\{j_\theta\}$ is independent of $\theta$ we just write $CF(p, j)$, or just $CF(p)$ if $\{j_\theta\}$ is implicit.

3.2. Coupling forms. This material appears in [9, Chapter 6], in slightly more generality of locally Hamiltonian fibrations, so we review it here only briefly. A Hamiltonian fibration is a smooth fiber bundle

\[ M \hookrightarrow P \to X, \]
with structure group $\text{Ham}(M, \omega)$. A coupling form for a Hamiltonian fibration $M \to P \to X$, is a closed 2-form $\Omega$ whose restriction to fibers coincides with $\omega$ (with respect to a Hamiltonian trivialization), and which has the property:

$$\int_M \Omega^{n+1} = 0 \in \Omega^2(X).$$

Such a 2-form determines a Hamiltonian connection $A_\Omega$, by declaring horizontal spaces to be $\tilde{\Omega}$ orthogonal to the vertical tangent spaces. A Hamiltonian connection $A$ in turn determines a coupling form $\tilde{\Omega}_A$ as follows. First we ask that $\tilde{\Omega}_A$ generates the connection $A$ as above. This determines $\tilde{\Omega}_A$, up to values on $\mathcal{A}$ horizontal lifts $\tilde{v}, \tilde{w} \in T_P \mathcal{M}$ of $v, w \in T_x X$. We specify these values by the formula

$$\tilde{\Omega}_A(\tilde{v}, \tilde{w}) = R_A(v, w)(p),$$

where $R_A|_x$ is the curvature 2-form with values in $C_\text{norm}^\infty(P^{-1}(x))$. For a connection $A_H$ induced by $H : M \times S^1 \to \mathbb{R}$, the associated coupling form $\tilde{\Omega}_{A_H}$ is given by:

\begin{equation}
\tilde{\Omega}_{A_H} = \tilde{\Omega}_H = \omega - d(Hd\theta).
\end{equation}

In particular for a section $u$ of $M \times \mathbb{C}$ asymptotic to a flat section $o$, the integral of $\tilde{\Omega}_{A_H}$ over $u$ is the action of $o$ as a periodic orbit of $H$, (with bounding disk determined by $u$).

4. The proofs

Proof of Lemma 1.18. Let $\mathcal{O} \to B^k$, be a fibration with fiber over $b$ the space of Hamiltonian connections $\mathcal{A}$ on $M \times \mathbb{C}$, $\delta$-admissible with respect to $f(b)$. By Proposition 1.17, the fiber is non-empty, and is contractible (it is a $\delta$-ball in an affine space), moreover it readily follows by Proposition 1.19 that $\mathcal{O}$ is a Serre fibration. Consequently the space of sections of $\mathcal{O}$ is connected by classical obstruction theory. But this is exactly the space of families $\{A_b\}$, $\delta$-admissible with respect to $f$.

Suppose we are given a one parameter family $\{A_{b,t}\}$, $0 \leq t \leq 1$, with each $A_{b,t}$ $\delta$-admissible with respect to $f_t(b)$, $[f_t] = [f]$. To show that the invariant $\Psi_\gamma([f])$ is well defined, we need to show that a regular one parameter family induces a one-dimensional compact cobordism

$$\overline{M}(\{J_{b,0,t}^{\text{reg}, A}\}),$$

between $\overline{M}(\{J_{b,0,t}^{\text{reg}, A}\}), \overline{M}(\{J_{b,1,t}^{\text{reg}, A}\})$. We should elaborate on what regular means. First the chain complexes $CF(p(A_{b,t}, \{J_{b,t,r,\theta}\})_{r,\theta})$, are meant to be regular for each $b, t$ but this follows by construction of $\{J_{b,t,r,\theta}\}$ (analogous to construction of $\{h_{b,t,\theta}\}$, see 1.16) assuming $j$ was taken to be regular. Next, denote by $\mathcal{B}$ the space of triples $(u, b, t)$, $u \in \mathcal{B}_b$, $t \in [0, 1]$ with $\mathcal{B}_b$ denoting the space of class $[\sigma_{\text{max}}]$-smooth sections of $M \times \mathbb{C}$, asymptotic to $o_{\text{max}, b}$. This is a Frechet bundle over $B^k$, the charts can be constructed using the diffeomorphisms $\psi_{b,t}$, (see 1.17). After appropriate Sobolev completions which we don’t specify (as this is classical), we get a Banach bundle

$$\mathcal{E} \to B,$$

whose fiber over $u_b = (u, b, t)$ is $\Omega^{0,1}(S^2, u^* T^\text{vert}(M \times \mathbb{C}))$, and the section we call $\mathcal{F}_f$,

$$\mathcal{F}_f(u_b) = \tilde{\partial} f_{b,t}(u).$$
The space $\mathcal{M}(\{J^A_{b,t,f}\})$ is identified with the 0-locus of this section. As $\mathcal{B}$ fibers over $B \times [0,1]$, the so called vertical differential at $u \in \mathcal{M}(J^A_{b,t,f})$, is a real linear Cauchy-Riemann operator of the form

$$D_u : \Omega^0(u^*T^{vert}(M \times \mathbb{C})) \oplus T(B \times [0,1]) \to \Omega^{0,1}(u^*T^{vert}(M \times \mathbb{C})),$$

and is Fredholm of index

$$CZ(o_{max}) - CZ([M]) + \dim B + 1 = 1,$$

by assumption that $\dim B = \text{index} \gamma = -(CZ(o_{max}) - CZ([M]))$. We say $u$ is regular if this operator is surjective. For a general $u \in \mathcal{M}(J^A_{b,t,f})$, we will say that it is regular if the analogue of the operator above is surjective for $u_{princ}$: the principal component of the “section” $u$, i.e. the component of the holomorphic building not entirely contained in the translation invariant part of $(M \times \mathbb{C}, \{A_b,t\})$, and which is not a vertical bubble. We say $\{J^A_{b,t,f}\}$ is regular if all the elements of the corresponding compactified moduli space are regular. This latter regularity can be obtained, by perturbing the family of connections $\{A_b,t\}$, because the monotonicity assumption rules out holomorphic bubbles in the fiber with negative Chern number. Under above regularity vertical bubbling cannot happen. This is because a vertical bubble in class $A$ drops the Fredholm index of the CR operator at the principal component by $2\{c_1TM,A\} \geq 2$, (monotonicity assumption) which would make the Fredholm index at any such principal component negative. To show that we have such a cobordism (between $\overline{\mathcal{M}}(\{J^r_{b,t,f}\})$, $\overline{\mathcal{M}}(\{J^p_{b,t,f}\})$) we need two things. First that there is an $\epsilon > 0$, s.t. there are no $(u,b,t) \in \overline{\mathcal{M}}(\{J^r_{b,t,f}\})$ with $b$ in the $\epsilon$-neighborhood of $\partial B^k$, denoted by $\partial B^k_\epsilon$. Take $\epsilon$ so that $f(\partial B^k_\epsilon) \subset P_{\phi,\varepsilon} + \delta - P_{\phi,\varepsilon}$. By remark following [4.1], we have $[\Omega_{\partial B^k_\epsilon}](\{u\}) = -L^+(\gamma)$. Then for $(u,b,t) \in \overline{\mathcal{M}}(\{J^r_{b,t,f}\})$, $b \in \partial B^k_\epsilon$ we have:

$$0 \leq \langle [\Omega_{\partial B^k_\epsilon} + \pi^*(\alpha_{A_b,t})], [u] \rangle = -L^+(\gamma) + \text{area}(A_{b,t}) < -L^+(\gamma) + E_{\varepsilon} + \delta + \delta < 0,$$

for $\alpha_{A_b,t}$, as in [4.1], but this is impossible.

Next we need to show that the signed count of boundary points of the manifold $\overline{\mathcal{M}}(\{J^r_{b,t,f}\})$, corresponding to broken holomorphic sections is 0. As $\{J^r_{b,t,f}\}$ is by assumption regular, a broken holomorphic section $(u,b,t)$ will have a pair of components (levels of the holomorphic building): a principal component $(u_1,b,t)$ with $u_1$ a $J^r_{b,t,f}$ holomorphic section of $M \times \mathbb{C}$, asymptotic to a generator $g$ of $CF(p(A_{b,t}))$ with Conley-Zehnder index $CZ(o_{max}) + 1$, and a component corresponding to a Floer gradient trajectory from $g$ to $o_{max}$. Regularity rules out all other non-compactness possibilities by dimension counting.

But in this case either the signed count of flow lines from $g$ to $o_{max}$ is zero, which would be what we want or $o_{max}$ is not semi homologically essential, which contradicts our hypothesis.

□

Proof of Proposition 4.13. For a given $f : (B^k, \partial B^k) \to (P_{\phi}, P_{\phi,\varepsilon})$ and $b \in B^k$, the construction of $A_{b,f}$ is as follows. We first obtain a coupling form $\Omega_{b,f}$ on the trivial fibration $M \times \mathbb{C} \to \mathbb{C}$. 
This is a form with support in \( \{ r \geq 1 \} \subset \mathbb{C} \), such that under a fixed identification
\[
\{ r \geq 1 \} \subset \mathbb{C} \simeq \mathbb{R}^{\geq 1} \times S^1,
\]
it has the form
\[
(4.2) \quad \widehat{\Omega}_{b,f} = \omega - d(\eta H_b d\theta),
\]
where \( 0 \leq r < \infty, 0 \leq \theta \leq 1 \), (recall that we are using modified polar coordinates) \( H_b \) is the normalized generating function for \( f(b) \), and \( \eta : \mathbb{R} \rightarrow [0,1] \) is a smooth, monotonely increasing function, with support in \( [1, \infty] \) satisfying
\[
(4.3) \quad \eta(r) = \begin{cases} 
1 & \text{if } 2 - \kappa \leq r < \infty, \\
1 - r & \text{if } 1 + \kappa \leq r \leq 2 - 2\kappa,
\end{cases}
\]
for a small \( \kappa > 0 \). The Hamiltonian connection \( A_{b,f} \) is then defined by taking the horizontal spaces for \( A_{b,f} \) to be the \( \Omega_b \) orthogonal spaces to the vertical tangent spaces.

If \( f_\gamma \) is a local unstable manifold at \( \gamma \), then for
\[
(u, b) \in \overline{\mathcal{M}(\{ J^A_{b,f,\gamma} \})},
\]
for \( b \neq 0 \), \( L^+(f(b)) < L^+(\gamma) \) and so
\[
0 \leq \langle [\Omega_{b,f} + \pi^*\alpha_{A_{b,f}}, [u]] \rangle = -L^+(\gamma) + L^+(f(b)),
\]
with the calculation \( \pi^*\alpha_{A_{b,f}}([u]) = L^+(f(b)) \), being elementary from definitions, but this is impossible and so \( b = 0 \). We need to check that the section \( \sigma_{\max} \), is the only element of \( \overline{\mathcal{M}(J^A_{0,f,\gamma})} \). It is simple to check that it is the only smooth element, for given another smooth \( u \in \overline{\mathcal{M}(J^A_{0,f,\gamma})} \) we have
\[
(4.4) \quad 0 = \langle [\omega - \eta(r)dH_0 \wedge d\theta - H_0 d\eta \wedge d\theta + \max_{x \in M} H_0(x) d\eta \wedge d\theta], [u] \rangle.
\]
Note that \( u \) is necessarily horizontal, for otherwise right hand side is positive by \( 1.4 \). Hence the form \( \omega - \eta(r)dH_0 \wedge d\theta \) must vanish on \( u \), as the horizontal subspaces are spanned by vectors \( \frac{\partial}{\partial r} + \frac{\partial}{\partial \theta} + \eta(r)X_{H_0} \). Thus, \( 4.4 \) can only happen if \( u \) is \( \sigma_{\max} \). Upon a moment of reflection we see that the same argument works for a general \( u \in \overline{\mathcal{M}(J^A_{0,f,\gamma})} \).

\[ \square \]

**Proof of Theorem 1.20** By the assumption that \( f_\gamma \) is Morse at \( \gamma \), and by \( 1.19 \)
\[
\overline{\mathcal{M}(\{ J^A_{f_\gamma,\gamma} \})}
\]
is a zero dimensional manifold consisting of a single point \( \sigma_{\max} \). This is the expected dimension as the Fredholm index of
\[
D_{\sigma_{\max}} : \Omega^0(\sigma^*_{\max} T^{vert}(M \times \mathbb{C})) \oplus T_0B \rightarrow \Omega^0(\sigma^*_{\max} T^{vert}(M \times \mathbb{C})),
\]
is
\[
CZ(\sigma_{\max}) - CZ([M]) + k = CZ(\sigma_{\max}) - CZ([M]) + \text{index } \gamma = 0,
\]
by Theorem 1.4. The restricted operator
\[
D^*_{\sigma_{\max}} : \Omega^0(\sigma^*_{\max} T^{vert}(M \times \mathbb{C})) \rightarrow \Omega^0(\sigma^*_{\max} T^{vert}(M \times \mathbb{C})),
\]
has no kernel by assumption, and so the dimension of its cokernel is
\[
-(CZ(\sigma_{\max}) - CZ([M])) = k.
\]
The point of the following construction is to perturb the family \( \{ A_{b,f} \} \) so that this cokernel is covered by the \( T_0B \) component of the total vertical differential.

Let \( \mathcal{H} \) denote the space of coupling forms of the form \( \tilde{\Omega}_{A_p} + \Pi \), where \( A_p \) is the Hamiltonian connection on \( M \times \mathbb{C} \), induced by \( p \in \mathcal{P}_0 \) as in Proposition 1.19 and \( \Pi \) is of the form:

\[
\tilde{\Omega}_p + d(Gd\theta),
\]

\( G \) is a normalized function with support in \( \{ 1 + \kappa \leq r \leq 2 - 2\kappa \} \subset \mathbb{C} \), \( \kappa \) as in the proof of Proposition 1.19. Such that identifying \( \{ r \geq 1 \} \subset \mathbb{C} \) with \( \mathbb{R}^{2} \times S^1 \), \( G \) has the form:

\[
(4.5) \quad G : M \times \mathbb{R}^{2} \times S^1 \to \mathbb{R}, \quad G_r = G|_{(M \times \{ r \}) \times S^1} = \zeta(r) \cdot K,
\]

where \( K : M \times S^1 \to \mathbb{R} \), such that \( K(x_{\text{max}}) = 0 \) and \( \zeta : \mathbb{R}^{2} \to \mathbb{R} \) is a function with compact support. (Here \( x_{\text{max}} \) is the extremizer of the generating function of \( \gamma \) as before.) To emphasize we are not fixing \( p,K,\zeta \). Let \( \mathcal{C}_H \) denote the associated space of Hamiltonian connections.

**Lemma 4.1.** Let \( \{ A_b \} \) with \( A_b \in \mathcal{C}_H \), \( A_0 = A_\gamma \) be a family of Hamiltonian connections on \( M \times \mathbb{C} \). Then we have

\[
d\text{area}(A_b) (0) = 0,
\]

where

\[
\text{area}(A_b) (b) = \text{area}(A_b).
\]

**Proof.** Suppose we have a variation \( A_{b(\tau)} \), with \( b(\tau), \quad -1 \leq \tau \leq 1 \) a smooth path through \( 0 \in B^k \), \( b(0) = 0 \). Denote by \( x_{\text{max},\tau,r,\theta} \) the maximizer of \( *R_{A_{b(\tau)}} (r,\theta) \), with the latter denoting the Hodge star (evaluate curvature on an orthonormal pair) of the curvature 2-form of \( A_{b(\tau)} \), at \( r,\theta \) (identifying the lie algebra with \( C_\text{norm}(M) \)). Here the Hodge star is taken with respect to a metric \( g_C \) on \( \mathbb{C} \), for which the identification \( \{ r \geq 1 \} \subset \mathbb{C} \simeq \mathbb{R}^{2} \times S^1 \) is an isometry, for the classical metric \( g_{st} \) on the latter. (I.e. \( A_{b(\tau)} = *R_{A_{b(\tau)}} \otimes_R \omega_{st} \) for \( \omega_{st} \) the classical volume form on \( \mathbb{R} \times S^1 \), thinking of \( *R_{A_{b(\tau)}} \) as a lie algebra valued function.)

As \( *R_{A_{b(0)}} (r,\theta) = H_\theta^\gamma \), for \( \{ 1 + \kappa \leq r \leq 2 - 2\kappa \} \) is Morse at \( x_{\text{max}} \) the point \( x_{\text{max},\tau,r,\theta} \) is uniquely determined and varies smoothly with \( \tau \) for \( \tau \) small.

The derivative at \( \tau = 0 \) of \( \text{area}(\tau) = \text{area}(A_{b(\tau)}) \), is

\[
\text{area}'(0) = \frac{d}{d\tau} \bigg|_{\tau=0} L^+ (b(\tau)) + \int_G dH^\gamma_{x,\theta} \left( \frac{d}{d\tau} \bigg|_{\tau=0} x_{\text{max},\tau,r,\theta} \right) d\text{vol}_{g_C} + \int_G \frac{d}{d\tau} \bigg|_{\tau=0} K_{A_{b(\tau)},r,\theta} (x_{\text{max}}) d\text{vol}_{g_C}.
\]

For clarity we mention that the above expansion of the derivative comes from the chain rule for the composition

\[
\mathbb{R} \to \mathbb{R}^3 \to C^\infty (\mathbb{C},\mathbb{R}) \xrightarrow{f_k} \mathbb{R},
\]

with the obvious maps not indicated. The first term vanishes as \( b(0) = \gamma \) is critical for \( L^+ \) by assumption. The second term vanishes as \( x_{\text{max}} \) is critical for \( H_\theta^\gamma \) for each \( \theta \) by assumption. The last term vanishes by assumption that \( K(x_{\text{max}}) = 0 \) in (1.6). \( \square \)
Each element in \( H \) (or \( C_H \)) determines an almost complex structure on \( M \times \mathbb{C} \) as before, so we have the universal differential
\[
D^\text{univ}_{\sigma_{\max}, A_0} : \Omega^0(\sigma^*_b T^{\text{vert}}(M \times \mathbb{C})) \oplus T_{A_0} H \to \Omega^{0,1}(\sigma^*_b T^{\text{vert}}(M \times \mathbb{C})),
\]
which by the proof of \cite{10} Theorem 8.3.1 and \cite{10} Remark 3.2.3 is surjective, (we are dropping all mentions of Sobolev completions as this is standard.)

Let \( V \) denote a fixed complement to the image \( D_{\sigma_{\max}}, 0 \leq \dim V \leq k \). By the above we may find a subspace \( S \subset H, 0 \leq \dim S \leq k \) so that
\[
D^\text{univ}_{\sigma_{\max}, A_0}(0 \oplus S) = V.
\]
We now take \( \{ A_b \}, A_b \in H, A_0 = A_\gamma \), such that this is a family \( \delta \)-admissible with respect to \( f \), such that the natural differential
\[
T_0 B \to T_{A_\gamma} H
\]
is onto \( S \), and such that \( \{ A_b \} \) is sufficiently \( C^\infty \) close to \( \{ A_b, f \} \) so that the function area\(_\gamma\)(\( A_b \)) : \( B^k \to \mathbb{R} \) is still Morse at the unique maximum \( b = 0 \). This last condition is possible by Lemma \ref{1.18} and the following elementary observation.

**Lemma 4.2.** Suppose \( f : B^k \to \mathbb{R} \) is a function Morse at the unique maximizer \( 0 \in B^k \), and \( f' \) a smooth function sufficiently \( C^\infty \) close to \( f \), and such that \( 0 \) is a critical point of \( f' \). Then \( 0 \) is also a unique maximizer of \( f' \), and moreover \( f' \) is Morse at \( 0 \).

**Proof.** This follows by Morse Lemma the proof is omitted. \( \square \)

By the proof of Proposition \ref{1.19} \( (\sigma_{\max}, 0) \) is the only element of \( \overline{\mathcal{M}}(\{ J^A_b \}) \), and so this moduli space is regular and the Gromov-Witten invariant is \( \pm 1 \). \( \square \)

**Proof of Theorem 1.22.** Clearly \( |f_\gamma| = L^+(\gamma) \). On the other hand if there is an \( f' \in [f_\gamma] \), with \( |f'| < L^+(\gamma) \), then by the proof of Lemma \ref{1.18} the moduli space \( \overline{\mathcal{M}}(\{ J^A_{b, \gamma} \}) \) is empty but this contradicts \ref{1.20} \( \square \)

**Proof of theorem 1.6.** Given our robust Ustilovsky geodesic \( \gamma \), and \( H_{\theta}^r \) its generating Hamiltonian, let \( H' : M \times S^1 \to \mathbb{R} \) be a function whose pull-back to \( \mathbb{C}^n \) by a fixed Darboux chart of \( x_{\max} \in M \), coincides with the Hessian of \( H_{\theta}^r \) at \( x_{\max} \): \( Hess(H'_{\theta})(x_{\max}) \) as a function \( \mathbb{C}^n \to \mathbb{R} \), for each \( \theta \). Taking this Darboux neighborhood to be suitably small the resulting \( H' \) can be made arbitrarily \( C^\infty \) close to \( H_\theta^r \). Consequently the resulting path \( \gamma' \) is still a robust Ustilovsky geodesic, (it may have a different end point). We may also suppose without loss of generality that the Darboux chart \( \mathbb{C}^n \to M \), is holomorphic with respect to \( j \) on \( M \). Let \( A' \) denote the Hamiltonian connection on \( M \times \mathbb{C} \), induced by \( H' \). It follows that a normal neighborhood of \( \sigma_{\max} \subset (M \times \mathbb{C}, J^{A'}) \), is biholomorphic to a neighborhood of a 0 section of a \( \mathbb{C}^n \) vector bundle \( E \) over \( \mathbb{C} \), whose almost complex structure is induced by the unitary connection \( A' \), determined by
\[
Hess(H'_{\gamma})(x_{\max}) : \mathbb{C}^n \times S^1 \to \mathbb{R},
\]
(exactly as in Proposition \ref{1.19}). Such an almost complex structure must be integrable \cite[Section 5]{1}. Consequently, the operator
\[
D^\text{rest}_{\sigma_{\max}, A_0} : \Omega^0(\sigma^*_b T^{\text{vert}}(M \times \mathbb{C})) \to \Omega^{0,1}(\sigma^*_b T^{\text{vert}}(M \times \mathbb{C})),
\]
is the Dolbeault operator for the holomorphic structure on \( E \) above. If \( \xi \neq 0 \) is in the kernel of the operator above, then since a normal neighborhood of \( \sigma_{\max} \subset \)
(\(M \times \mathbb{C}, J^A\)) is biholomorphic to a neighborhood of the 0 section of \(E\), there would be an element of \(\mathcal{M}(\{J^A\})\), corresponding to \(\epsilon \cdot \xi\), \(\epsilon > 0\) small enough, and such an element would different from \(\sigma_{\text{max}}\), which is impossible. The claim then follows by Theorem 1.20.

Proof of Theorem 1.13. By Theorem 1.16 for any robust Ustilovsky geodesic \(\gamma\) there is an arbitrarily \(C^\infty\) close Ustilovsky geodesic \(\gamma'\), (with possibly different right end point) which is minimizing in its homotopy class relative end points. It immediately follows that \(\gamma\) itself must be minimizing in its homotopy class relative end points. \(\Box\)

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ICMAT Madrid
E-mail address: yasha.savelyev@gmail.com