Contraction Theory for Dynamical Systems on Hilbert Spaces

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Abstract

Contraction theory for dynamical systems on Euclidean spaces is well-established. For contractive (resp. semi-contractive) systems, the distance (resp. semi-distance) between any two trajectories decreases exponentially fast. For partially contractive systems, each trajectory converges exponentially fast to an invariant subspace.

In this note, we develop contraction theory on Hilbert spaces. First, we provide a novel integral condition for contractivity, and for time-invariant systems, we establish the existence of a unique globally exponentially stable equilibrium. Second, we introduce the notions of partial and semi-contraction and we provide various sufficient conditions for time-varying and time-invariant systems. Finally, we apply the theory on a classic reaction-diffusion system.

Keywords: contraction theory, Hilbert spaces, Banach spaces, differential equations, stability

1 Introduction

Problem statement and motivation  Contraction theory establishes the exponential incremental stability of ordinary differential equations. Its mature development can be traced back to the work by Coppel [8], where linear systems were studied, and to the textbook treatment by Vidyasagar [37]. Later, a reformulation was proposed in the seminal work by Slotine [21]. We refer to [4] for an introduction and a survey of applications on contraction theory, and to [34] for extensions to Riemannian manifolds. Generalizations of the classical contraction theory have been proposed in the literature. The notion of partial contraction, first introduced in [31], studies the exponential convergence of trajectories to invariant subspaces [31, 12]. Recently, [18] introduces the concept of semi-contraction, which establishes the exponential incremental semi-stability of trajectories. Contraction theory has also been used for control design [24, 33], and extended to non-differentiable and discontinuous vector fields [22, 13, 6] and dynamics on Finsler manifolds [14].

To the best of our knowledge, a general contraction theory on Hilbert and Banach spaces is missing. The importance of working with systems defined on such general space is, for example, the wide scope of possible applications of systems based on partial differential equations, delayed differential equations, functional differential equations, and integro-differential equations (e.g., see [25, Chapter 9]). The purpose of this note is to concisely present such a theory, with the hope that it will be relevant in both theoretical and applied work. We also provide an application example to illustrate the theory. This work builds a bridge between the abstract theory of differential equations developed in mathematics [9][20] and the widely-established contraction theory in the field of systems and control.

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Literature review To the best of our knowledge, a first approach to contraction theory on general Banach spaces can be traced back to the 1972 book by Ladas & Lakshmikantham [20], in its Lemma 5.4.1 and 5.4.2. However, these results do not parallel much of the richer development of contraction theory in Euclidean spaces (see our Contributions below). Interestingly, the results in [20] seem to be unknown in the literature on contraction theory, which developed decades later. Applications of contraction theory have been proposed to specific classes of partial differential equations [4, 2, 3] and more recently to functional differential equations [29]. Besides these notable exceptions, the study of contraction theory on infinite dimensional systems has not received the same development as the Euclidean case, e.g., no concept of semi- or partially contractive systems on Hilbert spaces exists in the literature either.

In the controls community, the recent works [35, 19] have considered dynamical systems on Banach and Hilbert spaces and their applications to PDEs. Other recent interests in dynamical systems on these abstract spaces include controller design [32], event-triggered control [38], observability studies [15], optimal control [36], and stability characterizations [26].

Contributions First, we review two little-known results from [20] that establish a generalization of Coppel’s inequality to Banach spaces and that provide some sufficient conditions for contraction using operator measures when the vector fields are continuously differentiable. Then, we prove that every time-invariant contractive system has a unique globally exponentially stable equilibrium point. We also provide a sufficient condition using operator measures for when the norm of a time-invariant system has its vector field exponentially decreasing on trajectories of the system. In the case of systems on Hilbert spaces, we introduce a simpler sufficient condition for contraction without the differentiability requirement on the vector field: the integral contractivity condition. Moreover, under the differentiability requirement, we prove: (i) that the condition using operator measures presented in [20] can be relaxed and still imply contraction (in particular, it is no longer needed to check a bound for Fréchet derivative of the vector field); (ii) the integral contractivity condition is implied by the one using operator measures.

Second, associated with a surjective linear operator \( T \), we introduce the concepts of \( T \)-seminorms and \( T \)-operator semi-measures which can be considered as generalization of recently introduced concepts in the study of the classical Euclidean setting [18]. Then, we introduce the concepts of partial and semi-contraction for systems on Hilbert spaces. Using the concepts of seminorms and semi-measures, we provide sufficient conditions for partial contraction and semi-contraction. We present a series of novel results. Firstly, we introduce the integral partial contractivity condition, a sufficient condition for partial contraction. Secondly, we introduce the integral semi-contractivity condition, a sufficient condition for semi-contraction. For continuous differentiable vector fields, we prove this condition is implied by another sufficient condition for semi-contraction using operator semi-measures. When there exists an invariant subspace for the system, our conditions for semi-contraction imply partial contraction. We remark that, to the best of our knowledge, our characterization of partial and semi-contraction using integral conditions are new even in the classic Euclidean setting (with the usual inner product); e.g. as studied in the work [31] and in our previous work [18].

Finally, we present an example of a reaction-diffusion system and use partial contraction to prove the same result as [5]; moreover, we establish semi-contraction when the reaction term is linear in the state variable.

Paper organization Section 2 has preliminaries and notation. Sections 3 and 4 contain the main results on Banach and Hilbert spaces. Section 5 presents the application example and Section 6 is
2 Preliminaries and notation

2.1 Notation, definitions and useful results

A Banach space is a complete normed vector space \((\mathcal{X}, \| \cdot \|)\), where \(\mathcal{X}\) is a vector space and \(\| \cdot \|\) a norm over \(\mathcal{X}\). A Hilbert space is a pair \((\mathcal{X}, \langle \cdot, \cdot \rangle)\), where \(\mathcal{X}\) is a vector space and \(\langle \cdot, \cdot \rangle\) is an inner product over \(\mathcal{X}\), such that its induced norm \(\| \cdot \| := \sqrt{\langle \cdot, \cdot \rangle}\) makes the space a Banach space. In what follows we assume \(\mathcal{X}\) is a vector space over the field of real numbers.

Let \(B(\mathcal{X})\) be the space of bounded linear operators with domain and codomain \(\mathcal{X}\). Let 0 be the null element of \(\mathcal{X}\), or the number zero, depending on the context. Let \(I\) be the identity operator. Given \(A \in B(\mathcal{X})\), \(\|A\|_{op} = \sup_{x \neq 0} \frac{\|Ax\|}{\|x\|}\) is its associated operator norm. Given an open set \(\Omega \subseteq \mathcal{X}\), we say a function \(H : \mathcal{X} \rightarrow \mathcal{X}\) is continuously Fréchet differentiable in \(\Omega\) when \(H\) is Fréchet differentiable at each \(x_0 \in \Omega\) (i.e., \(DH(x_0) \in B(\mathcal{X})\)) such that \(\lim_{\|x\| \rightarrow 0} \frac{\|H(x_0 + h) - H(x_0) - DH(x_0)h\|}{\|h\|} = 0\) exists [1]) and \(DH : \Omega \rightarrow B(\mathcal{X})\) is continuous. Finally, we say a subspace \(\mathcal{V} \subset \mathcal{X}\) is invariant for \(A \in \mathcal{X}\) if for any \(x \in \mathcal{V}\) then \(Ax \in \mathcal{V}\).

Let \(I_n\) be the \(n \times n\) identity matrix and \(0_n \in \mathbb{R}^n\) be the all-zeros column vector with \(n\) entries.

The concept of matrix measures or logarithmic norm, e.g., see [4], can be generalized as the following; e.g., see [20, Definition 5.4.2],

**Definition 2.1** (Operator measure). Let \(A \in B(\mathcal{X})\) and define the operator measure of \(A\) as:

\[
\mu(A) = \lim_{h \rightarrow 0^+} \frac{\|I + hA\|_{op} - 1}{h}.
\]

2.2 Dynamical systems on Banach spaces

Given the Banach space \((\mathcal{X}, \| \cdot \|)\) and the vector field \(F : \mathbb{R} \times \mathcal{X} \rightarrow \mathcal{X}\), consider the differential equation:

\[
\dot{x} = F(t, x)
\]  

(1)

with \(\dot{x} := \frac{dx}{dt}\). Following closely the setting in [25, Chapter 9], a continuous function \(\phi : [t_0, t_0 + c) \rightarrow \mathcal{X}\), \(c > 0\), is a solution of (1) if it is differentiable with respect to \(t\) for \(t \in [t_0, t_0 + c)\) and if \(\phi\) satisfies the equation \(\dot{\phi} = F(t, \phi(t))\) for all \(t \in [t_0, t_0 + c)\). When the system (1) is associated the initial condition \(x(t_0) = x_0\), we have an initial value problem or Cauchy problem. In this paper we assume that (i) \(t \mapsto F(t, x)\) is continuous on \(t \in \mathbb{R}_{\geq 0}\) for any \(x \in \mathcal{X}\), and that (ii) for any \(x_0 \in \mathcal{X}\), there exists at least one solution \(\phi(t, t_0, x_0)\) to the initial value problem with \(x(t_0) = x_0 = \phi(t_0, t_0, x_0)\) for all \(t \geq t_0\), \(t_0 \in \mathbb{R}_{\geq 0}\).

We say that a set \(\mathcal{U}\) is (positively) invariant for the system (1) if \(\phi(t', t_0, x_0) \in \mathcal{U}\) at some time \(t' \geq t_0\) implies \(\phi(t, t_0, x_0) \in \mathcal{U}\) for any \(t \geq t'\).

The dynamical system (1) is time-invariant whenever the vector field \(F\) is time-invariant, i.e., \(F\) does not explicitly depend on \(t\). If the system (1) is time-invariant, it has an equilibrium point \(x^*\) if \(F(x^*) = 0\).

The system (1) has exponential incremental stability or is contracting with respect to norm \(\| \cdot \|\) if, for any \(x_0, y_0 \in \mathcal{X}\), the trajectories \(\phi(t, t_0, x_0)\) and \(\phi(t, t_0, y_0)\) for any \(t \geq t_0\) satisfy \(\|\phi(t, t_0, x_0) - \phi(t, t_0, y_0)\| \leq e^{-c(t-t_0)}\|x_0 - y_0\|\), for some \(c > 0\). In the Euclidean case, a central tool for studying contractivity is the matrix measure [4], i.e., the operator measure taking matrices as arguments.
3 Contraction on Banach and Hilbert spaces

The following Lemma 3.1 was proved in [20, Lemma 5.4.1], and the next Theorem 3.1 is an application of [20, Lemma 5.4.2]. These are the only two existing results from the scarce literature on contraction on Banach spaces that we use.

Lemma 3.1 (Coppel’s inequality for Banach spaces [20, Lemma 5.4.1]). Consider the linear time-varying dynamical system

\[ \dot{x}(t) = A(t)x(t) \]

on the Banach space \((\mathcal{X}, \| \cdot \|)\), with \(A(t) \in B(\mathcal{X})\) and \(t \mapsto A(t)\) being continuous for every \(t \in \mathbb{R}_{\geq 0}\). Suppose that \(\phi(t, t_0, x_0)\) is a solution of the Cauchy problem, then

\[ \|x_0\| \exp \left( \int_{t_0}^{t} -\mu(-A(\tau))d\tau \right) \leq \|\phi(t, t_0, x_0)\| \leq \|x_0\| \exp \left( \int_{t_0}^{t} \mu(A(\tau))d\tau \right). \tag{2} \]

Theorem 3.1 (Contraction with operator measures on Banach Spaces. [20, Lemma 5.4.2]). Consider the dynamical system (1) on the Banach space \((\mathcal{X}, \| \cdot \|)\) with \(F(t, \cdot)\) continuously Fréchet differentiable for each \(t\) and such that \(\|DF(t, u)\|_{op} \leq a(t)\) for any \(u \in \mathcal{X}\), and for some continuous function \(a(t) \geq 0, t \geq 0\). Assume that \(\mu(DF(t, x)) \leq -c(t)\) for any \(x \in \mathcal{X}\), and some continuous function \(c(t) > 0, t \geq 0\). Then the system (1) is contractive, i.e.,

\[ \left\|\phi(t, t_0, x_0) - \phi(t, t_0, x'_0)\right\| \leq e^{-\int_{t_0}^{t} c(s)ds}\|x_0 - x'_0\| \tag{3} \]

for all \(t \geq t_0\) and any \(x_0, x'_0 \in \mathcal{X}\).

The proof can be found in the Appendix. The function \(c(t) > 0, t \geq 0\), in Theorem 3.1 is the contraction rate. When the contraction rate is time-invariant, then a contractive system has incremental exponential stability. Beginning now, all the following results presented in this paper are novel. We present additional properties for contractive systems when the vector field is time-invariant.

Theorem 3.2 (Properties of time-invariant systems). Consider the dynamical system (1) on the Banach space \((\mathcal{X}, \| \cdot \|)\) with \(F\) time-invariant. Pick \(c > 0\).

(i) If the system is contractive with contraction rate \(c\), then there exists a unique globally exponentially stable equilibrium point \(x^*\) such that

\[ \|\phi(t, t_0, x_0) - x^*\| \leq e^{-c(t-t_0)}\|x_0 - x^*\|, \]

for all \(t \geq t_0\) and any \(x_0 \in \mathcal{X}\).

(ii) If \(F\) is continuously Fréchet differentiable and \(\mu(DF(x)) \leq -c\) for every \(x \in \mathcal{X}\), then

\[ \|F(\phi(t, t_0, x_0))\| \leq e^{-c(t-t_0)}\|F(x_0)\|, \text{ for all } t \geq t_0 \text{ and any } x_0 \in \mathcal{X}. \]

Proof. We prove statement (i). Recalling that the system is contractive, we have \(\|\phi(t, t_0, x_0) - \phi(t, t_0, x'_0)\| \leq e^{-c(t-t_0)}\|x_0 - x'_0\|\) for any \(t \geq t_0, x_0, x'_0 \in \mathcal{X}\). Fix any \(t > t_0\). Since \(e^{-c(t-t_0)} < 1\), we can use the Banach fixed point theorem to conclude that there exists a unique fixed point \(x^*\) such that \(\phi(t, t_0, x^*) = x^*\), which implies that \(x^*\) is either an equilibrium point or is a point which is revisited by the trajectory at time \(t\). By contradiction, if we assume the latter, then any point

1The result [20, Lemma 5.4.1] does not prove the left inequality in equation (2), but this follows immediately from the same proof.
y* at time $t_0 \leq t' \leq t$ will be revisited at time $t + (t' - t_0)$ (since there is uniqueness of solutions from (3) and $F$ is time invariant) and thus $y^*$ is also a fixed point of $\phi(t, t_0, \cdot)$, which violates the uniqueness of $x^*$ as a fixed point. Then, $x^*$ must be the unique equilibrium of $F$. We just proved statement (i).

Finally, to prove statement (ii), observe that using the chain rule on Banach spaces [1, Theorem 2.4.3],
\[
\frac{d}{dt}F(\phi(t, 0, x_0)) = D\phi(t, t_0, x_0)\frac{d}{dt}\phi(t, t_0, x_0) = D\phi(t, t_0, x_0)F(\phi(t, t_0, x_0)),
\]
i.e., $F(\phi(t, t_0, x_0)) \in X$ satisfies a linear time-varying differential equation on Banach spaces. Now, using Lemma 3.1,
\[
\|F(\phi(t, t_0, x_0))\| \leq \|F(x_0)\| e^{\int_{t_0}^t \mu(D\phi(t, t_0, x_0))} \leq e^{-c(t-t_0)} \|F(x_0)\|,
\]
where we used $\mu(D\phi(x)) \leq -c$, for every $x \in X$.

Assume now that $X$ is also a Hilbert space (over the field of real numbers) equipped with some inner product $\langle \cdot, \cdot \rangle$. Then, a weaker and simpler sufficient condition for contractivity than the one in Theorem 3.1 can be obtained.

**Theorem 3.3** (Contraction on Hilbert spaces). Consider the dynamical system (1) on the Hilbert space $(X, \langle \cdot, \cdot \rangle)$.

(i) If the following integral contractivity condition holds
\[
\langle x-y, F(t, x) - F(t, y) \rangle \leq -c\|x-y\|^2
\]
for some continuous function $c(t) > 0$, $t \geq 0$, and any $x, y \in X$, then system (1) is contractive.

(ii) If $F$ is continuously Fréchet differentiable, and $\mu(DF(t, x)) \leq -c(t)$ for any $x \in X$, and some continuous function $c(t) > 0$, $t \geq 0$, then condition (5) holds.

**Proof.** Consider (5) and define $e := x - y$. Then, $\dot{e} = F(t, x) - F(t, y)$ and we obtain $\langle e, \dot{e} \rangle \leq -c\|e\|^2 \Rightarrow \frac{d\|e\|^2}{dt} = \frac{d(e, e)}{dt} = (\langle e, \dot{e} \rangle + \langle \dot{e}, e \rangle) \leq -2c\|e\|^2$, where we used the fact that the inner product is a bilinear function. Solving this differential inequality using the Grönwall’s Lemma, we obtain $\|e(t)\| \leq e^{-\int_{t_0}^t c(s)\, ds}\|e(t_0)\|$ for any $t \geq t_0$, establishing that the system is contractive and proving statement (i).

Now, we prove statement (ii) of the theorem. First, let $A \in B(X)$, then
\[
\mu(A) = \lim_{h \to 0^+} \sup_{x, y \in X} \frac{\|x, (I+hA)y\|}{\|x\|\|y\|} - 1 \geq \lim_{h \to 0^+} \frac{\|x, (I+hA)x\|}{\|x\|\|x\|} - 1 = \langle x, Ax \rangle = \langle x, x \rangle
\]
for any $x \in X$ and $x \neq 0$; the first equality follows from [28, p. 187]. However, note that $\mu(A) \langle x, x \rangle \geq \langle x, Ax \rangle$ does hold for any $x \in X$. Now, consider $F$ to be continuously Fréchet differentiable, and consider any $x, y \in X$. From the fundamental theorem of calculus for Fréchet derivatives [1, Proposition 2.4.7],
\[
F(t, x) - F(t, y) = \left(\int_0^1 DF(t, s\lambda(x, y))d\lambda\right)(x - y)
\]
for a fixed $t$, with $s\lambda(x, y) := x + \lambda(x - y)$, and where the integral is the Riemann integral on Banach spaces [10, Chapter 7][20, Section 1.3] ($B(X)$ is a Banach space with the operator norm).
Then, \( \langle x - y, F(t, x) - F(t, y) \rangle = \langle x - y, \int_0^1 DF(s\lambda(x, y))d\lambda(x - y) \rangle \), and using (6),

\[
\langle x - y, F(t, x) - F(t, y) \rangle \leq \mu \left( \int_0^1 DF(t, s\lambda(x, y))d\lambda \right) \| x - y \|^2 
\leq \int_0^1 \mu(DF(t, s\lambda(x, y)))d\lambda \| x - y \|^2 \leq -(c)\| x - y \|^2.
\]

We now justify the second inequality above. Let \( S \) be the nth partial sum of a Riemann integral \( \mathcal{I} \), and set \( q_n(h) = \| I + hS \|_{\text{op}}^{-1} \). Observe that (i) \( \lim_{h \to 0^+} q_n(h) = \mu(S_n) \) for each \( n \); (ii) \( \lim_{n \to \infty} q_n(h) = \| I + h\mathcal{I} \|_{\text{op}}^{-1} \) uniformly over \( h \). Then, the Moore-Osgood Theorem implies \( \lim_{n \to \infty} \mu(S_n) = \mu(\mathcal{I}) \). This and the sub-additive property of operator measures [20, Problem 5.4.1] prove the second inequality above.

**Remark 3.4** (About contraction on Hilbert spaces).

(i) The integral contractivity condition does not require the vector field \( F \) to be Fréchet differentiable.

(ii) When \( F \) is continuously Fréchet differentiable, \( DF(t, \cdot) \) is no longer required to be bounded as in Theorem 3.1 (which follows from [20, Lemma 5.4.2]). Then, statement (ii) of Theorem 3.3 provides a more relaxed condition for contraction using operator measures when the dynamical system is on Hilbert spaces.

(iii) The integral contractivity condition generalizes a known sufficient condition of contractivity (e.g., [7, Lemma 2.1]) that has been established in the Euclidean space and is related to the so-called QUAD condition for dynamical systems [11, 30].

**Remark 3.5** (Uniqueness of solutions). For any system satisfying the assumptions of Theorem 3.1 or Theorem 3.3, the existence of a solution implies its uniqueness.

### 4 Semi- and partial contraction on Hilbert spaces

In this section, let \( (\mathcal{X}, \langle \cdot, \cdot \rangle_\mathcal{X}) \) and \( (\mathcal{Y}, \langle \cdot, \cdot \rangle_\mathcal{Y}) \) be Hilbert spaces and let \( \mathcal{T} : \mathcal{X} \to \mathcal{Y} \) be linear, surjective and bounded.\(^2\) A classic example is \( \mathcal{X} = \mathbb{R}^n \), \( \mathcal{Y} = \mathbb{R}^m \) with \( m \leq n \), and \( \mathcal{T} \in \mathbb{R}^{m \times n} \) being a full rank matrix.

Define the bilinear function \( \langle \cdot, \cdot \rangle_\mathcal{T} : \mathcal{X} \times \mathcal{X} \to \mathbb{R} \) by \( \langle x_1, x_2 \rangle_\mathcal{T} = \langle Tx_1, Tx_2 \rangle_\mathcal{Y} \), and define the seminorm \( \| x_1 \|_\mathcal{T} := \sqrt{\langle x_1, x_1 \rangle_\mathcal{T}} \). Let \( \mathcal{T}^+ \) be the Moore-Penrose (generalized) inverse of \( \mathcal{T} \), which is a well-defined operator since \( \mathcal{T} \) is surjective (and trivially has closed range) [39, Corollary 11.1.1].

**Definition 4.1** (Partial and semi-contraction). The system (1) is

(i) partially contractive with respect to \( \| \cdot \|_\mathcal{T} \) if there exists a continuous function \( c(t) > 0, t \geq 0 \), such that, for any \( x_0 \in \mathcal{X} \) and \( t \geq t_0 \),

\[
\| \phi(t, t_0, x_0) \|_\mathcal{T} \leq e^{-\int_{t_0}^t c(s)ds} \| x_0 \|_\mathcal{T};
\]

(ii) semi-contractive with respect to \( \| \cdot \|_\mathcal{T} \) if there exists a continuous function \( c(t) > 0, t \geq 0 \), such that, for any \( x_0, y_0 \in \mathcal{X} \) and \( t \geq t_0 \),

\[
\| \phi(t, t_0, x_0) - \phi(t, t_0, y_0) \|_\mathcal{T} \leq e^{-\int_{t_0}^t c(s)ds} \| x_0 - y_0 \|_\mathcal{T}.
\]

\(^2\)In this case, the operator norm of \( \mathcal{T} \) is \( \| \mathcal{T} \|_{\text{op}} = \sup_{x \neq 0} \frac{\| \mathcal{T}x \|_\mathcal{Y}}{\| x \|_\mathcal{X}} \).
We remark that the concept of partial and semi-contraction have not been formalized before on Hilbert spaces. Indeed, in the Euclidean space (with the usual inner product) with time-invariant contraction rates, our formalization becomes the classic cases studied in [18] and [31] respectively, where $T$ becomes an $n \times m$, $n < m$, full-row rank matrix.

We introduce the following useful concepts.

**Definition 4.2 ($T$-seminorms and $T$-operator semi-measures).** Consider a linear, surjective, bounded operator $T : X \rightarrow Y$ and let $A \in B(X)$. The associated $T$-seminorm of $A$ as $\|A\|_{T,\text{op}} = \sup_{x \in \text{Ker}(T)^\perp, x \in X, x \neq 0} \frac{\|Ax\|_Y}{\|x\|_Y}$

and the $T$-operator semi-measure of $A$ as $\mu_T(A) = \lim_{h \rightarrow 0^+} \frac{\|I + hA\|_{T,\text{op}} - 1}{h}$.

The definition of $T$-operator semi-measure is well-posed, since the existence of directional derivatives follows from the subaddivity property of the seminorm $\|\cdot\|_{T,\text{op}}$ (with the argument in $B(X)$) as shown in [10, Example 7.7.].

**Theorem 4.1 (Partial contraction on Hilbert spaces).** Let $(X, \langle \cdot, \cdot \rangle_X)$ and $(Y, \langle \cdot, \cdot \rangle_Y)$ be Hilbert spaces and $T : X \rightarrow Y$ be linear, surjective and bounded. Consider the dynamical system (1) on $(X, \langle \cdot, \cdot \rangle_X)$ with $F(t, \cdot)$ continuously Fréchet differentiable for each $t$. Assume that

(i) there exists a continuous function $c(t) > 0$ such that $\mu_T(DF(t, x)) = \mu(TDF(t, x)T^\dagger) \leq -c(t)$ for every $(t, x) \in \mathbb{R}_{\geq 0} \times X$ (with the operator measure $\mu$ associated to $\|\cdot\|_Y$),

(ii) the subspace $\text{Ker}(T)$ is positively invariant.

Then the system (1) is partially contractive with respect to $\|\cdot\|_T$.

**Proof.** We first observe that

$$\|A\|_{T,\text{op}} = \sup_{x \in \text{Ker}(T)^\perp, x \in X, x \neq 0} \frac{\|TAT^\dagger x\|_Y}{\|x\|_Y} = \sup_{y \neq 0, y \in Y} \frac{\|TAT^\dagger y\|_Y}{\|y\|_Y} = \|TAT^\dagger\|_{Y,\text{op}} \quad (10)$$

where the first equality follows from the fact that $T^\dagger T$ is a projection operator on $\text{Ker}(T)^\perp$ [17, Theorem 3.5.8]. Then, using the fact that $T T^\dagger = I$, which follows from $T$ being surjective [39, Definition 11.1.3], it follows that

$$\mu_T(A) = \lim_{h \rightarrow 0^+} \frac{\|T(I + hA)T^\dagger\|_{Y,\text{op}} - 1}{h} = \lim_{h \rightarrow 0^+} \frac{\|I + hTAT^\dagger\|_{Y,\text{op}} - 1}{h} = \mu(TAT^\dagger). \quad (11)$$

Now, set $y = T x$, with $x$ being the state of the system, and by the chain rule, $y$ is differentiable with respect to time and $\dot{y} = T \dot{x} = T F(t, x)$. Now, since $T$ is a bounded linear operator, $\text{Ker}(T)$ is a closed linear subspace of $X$, and so, we have the following decomposition $X = \text{Ker}(T) \oplus \text{Ker}(T)^\perp$ [23,
Theorem 1, Section 3.4. Set $U := I - T^\dagger T$. Then, for any trajectory $t \mapsto x(t)$, we have $x(t) = T^\dagger T x(t) + U x(t) = T^\dagger y(t) + U x(t)$, with $T^\dagger y(t) \in \text{Ker}(T)^\perp$ and $U x(t) \in \text{Ker}(T)$. Then,

$$\dot{y} = TF(t, T^\dagger y + U x(t))$$

(12)
is a time-varying dynamical system on the Hilbert space $(\mathcal{Y}, \langle \cdot, \cdot \rangle_{\mathcal{Y}})$, and so the Fréchet derivative, using the chain rule, of the right-hand side of (12) (with respect to $y$) is $TDF(t, T^\dagger y + U x(t))T^\dagger$. Then, from (11), it easily follows from Theorem 3.3 that: if $\mu(TDF(t, x)T^\dagger) \leq -c(t)$ as in the theorem statement and assumption (i), then the dynamical system (12) is contracting with respect to the norm $\| \cdot \|_{\mathcal{Y}}$. Now, we make the following observation: let us consider a solution such that initial condition $x_o \in \text{Ker}(T)$ at time $t_0$ implies $\phi(t, t_0, x_o) \in \text{Ker}(T)$ and so $T \phi(t, t_0, x_o) = 0$ for any $t \geq t_0$ (by assumption (ii)). Differentiating, we obtain $\dot{T}F(t, \phi(t, t_0, x_o)) = 0$, which let us conclude that if $u \in \text{Ker}(T)$, then $\dot{T}F(t, u) = 0$. In conclusion, there are two solutions known for the system (12): $y = 0$ (because if $y = 0$, then $\dot{y} = TDF(t, U x) = 0$ follows from $U x \in \text{Ker}(T)$ as we just showed) and $t \mapsto y(t) = T x(t)$, and these two solutions should exponentially converge to each other due to contraction. Then, equation (8) follows from $\|y\|_{\mathcal{Y}} = \|Tx\|_{\mathcal{Y}} = \|x\|_{\mathcal{T}}$.

**Theorem 4.2** (Integral partial contractivity condition). Consider the dynamical system (1) on the Hilbert space $(\mathcal{X}, \langle \cdot, \cdot \rangle_{\mathcal{X}})$ and the linear, surjective, bounded operator $T : \mathcal{X} \to \mathcal{Y}$. If the following integral partial contractivity condition holds

$$\langle x, F(t, x) \rangle_{\mathcal{T}} \leq -c(t) \|x\|_{\mathcal{T}}^2$$

(13)

for some continuous function $c(t) > 0$, $t \geq 0$, and any $x \in \mathcal{X}$, then the system is partially contractive with respect to $\| \cdot \|_{\mathcal{T}}$, and, as a consequence, $\text{Ker}(T)$ is a positively invariant subspace.

**Proof.** The proof is very similar to the first part of Theorem 4.3 for proving its respective integral condition, and thus is omitted.

We now introduce the counterpart of Theorem 3.3 for semi-contractive systems.

**Theorem 4.3** (Integral semi-contractivity condition). Consider the dynamical system (1) on the Hilbert space $(\mathcal{X}, \langle \cdot, \cdot \rangle_{\mathcal{X}})$ and the linear, surjective, bounded operator $T : \mathcal{X} \to \mathcal{Y}$.

(i) If the following integral semi-contractivity condition holds

$$\langle x - y, F(t, x) - F(t, y) \rangle_{\mathcal{T}} \leq -c(t) \|x - y\|_{\mathcal{T}}^2$$

(14)

for some continuous function $c(t) > 0$, $t \geq 0$, and any $x, y \in \mathcal{X}$, then the system (1) is semi-contractive with respect to $\| \cdot \|_{\mathcal{T}}$.

(ii) If $F$ is continuously Fréchet differentiable, $\mu(TDF(t, x)T^\dagger) \leq -c(t)$, for some continuous function $c(t) > 0$, $t \geq 0$, and $\text{Ker}(T)$ is an invariant subspace for $DF(t, x)$, for every $x \in \mathcal{X}$ and $t \geq 0$, then condition (14) holds.

**Proof.** First, consider the inequality (14) and define $e := x - y$ and follow the same procedure as in the proof of Theorem 3.3 to show that $\|c(t)\|_{\mathcal{T}} \leq e^{-\int_{t_0}^t c(s)ds} \|c(t_0)\|_{\mathcal{T}}$ for any $t \geq t_0$, thus establishing the system is semi-contractive and proving statement (i).

Now, we prove statement (ii) of the theorem. Consider $F$ to be continuously Fréchet differentiable, and consider any $x, y \in \mathcal{X}$. Then, using the fundamental theorem of calculus for Fréchet derivatives [1, Proposition 2.4.7], we obtain, for a fixed $t$, $F(t, x) - F(t, y) = B(t, x, y)(x - y)$ with $B(t, x, y) := \int_0^1 DF(t, y + \lambda(x - y))d\lambda$. 

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Then,
\[ \langle T(x - y), T(F(t, x) - F(t, y)) \rangle_Y = \langle T(x - y), TB(t, x, y)(I - T^T T + T^T T)(x - y) \rangle_Y \]
\[ = \langle T(x - y), TB(t, x, y)T^T T(x - y) \rangle_Y \]
\[ \leq \mu(TB(t, x, y)) \langle T(x - y), T(x - y) \rangle_Y \]
\[ = \mu(TB(t, x, y)) \|x - y\|^2_Y. \quad (15) \]

We now justify the second equality in (15). First, the invariance assumption implies that \( DF(t, u)v \in \text{Ker}(T) \) for any \( v \in \text{Ker}(T) \) and \( u \in X \), and so:
\[ TB(t, x, y)v = T \int_0^1 DF(t, y + \lambda(x - y))d\lambda v = \int_0^1 TDF(t, y + \lambda(x - y))d\lambda v = 0. \]
Then, we use this to obtain the second equality: \((I - T^T T)(x - y) \in \text{Ker}(T)\), and so \( B(t, x, y)(I - T^T T)(x - y) \in \text{Ker}(T) \) and so \( TB(t, x, y)(I - T^T T)(x - y) = 0 \).

Now, observe that
\[ \mu(TB(t, x, y)T^\dagger) = \mu(\int_0^1 TDF(t, y + \lambda(x - y)T^\dagger d\lambda) \leq \int_0^1 \mu(TDF(t, y + \lambda(x - y)T^\dagger))d\lambda \leq -c(t), \]
where the first inequality is justified in the same way as in the last part of the proof of Theorem 3.3. Then, using this relationship in (15), we obtain \( \langle T(x - y), T(F(t, x) - F(t, y)) \rangle_Y \leq -c(t)\|x - y\|^2_Y \), which is condition (14). \( \square \)

**Remark 4.4** (About partial and semi-contraction).

(i) The integral semi- and partial contractivity conditions do not require \( F \) to be continuously Fréchet differentiable.

(ii) If \( \text{Ker}(T) \) is positively invariant for the system, then the integral condition in Theorem 4.3 implies partial contractivity.

(iii) For continuous differentiable vector fields on Euclidean spaces, the semi-contraction condition in Theorem 4.3 was first introduced in [18].

5 Application to reaction-diffusion systems

Reaction-diffusion PDEs have a long history of study due to their importance in chemistry and biology [27]. Of particular interest are conditions under which the system does not present the phenomenon of pattern formation, which occurs from diffusion-driven instabilities [5]. Particular instances of these systems have been studied using analysis related to contraction [2, 3, 4].

Consider a bounded and convex domain \( \Omega \subset \mathbb{R}^m \) with smooth boundary \( \partial \Omega \). For any function \( h : \mathbb{R}^m \rightarrow \mathbb{R}^n \), define the vector Laplacian operator \( \nabla^2 \) by \( \nabla^2 h = (\nabla^2 h_1, \ldots, \nabla^2 h_n)^\top \) and \( (\nabla^2 h(x))_i = \sum_{j=1}^n \frac{\partial^2 h(x)}{\partial x_j^2} \). Let \( \mathcal{L}^2(\Omega) \) be the space of squared-integrable functions \( h : \Omega \subset \mathbb{R}^m \rightarrow \mathbb{R}^n \) with \( \int_\Omega h_i^2dx < \infty, i \in \{1, \ldots, n\} \), endowed with inner product \( \langle u, v \rangle = \int_\Omega u^\top vdx \) for any \( u, v \in \mathcal{L}^2(\Omega) \) and induced norm \( \|u\| = \sqrt{\langle u, u \rangle} \). It is known that \( (\mathcal{L}^2(\Omega), \langle \cdot, \cdot \rangle) \) is a Hilbert space.

Given a continuously differentiable reaction function \( f : \mathbb{R}^n \rightarrow \mathbb{R}^n \) and a nonnegative matrix of diffusion rates \( \Gamma \in \mathbb{R}^{m \times n} \), the reaction-diffusion system with Neumann boundary conditions is
\[ \frac{\partial u}{\partial t} = f(u) + \Gamma \nabla^2 u \]
\[ \nabla u_i(t, x) \cdot \hat{n}(x) = 0 \quad \text{for all } x \in \partial \Omega, i \in \{1, \ldots, n\}, \quad (16) \]
for \( u \in \mathcal{L}^2(\Omega) \) and \((t, x) \in \mathbb{R}_{\geq 0} \times \Omega \), with \( \hat{n}(x) \) being the vector normal to \( x \in \partial \Omega \). We refer to [5] and references therein for the system’s well-posedness and existence of classical solutions \( u = u(x, t) \).
such that \( u(t, \cdot) \) is twice continuously differentiable for each fixed \( t \in \mathbb{R}_{>0} \), and that \( t \mapsto u(t, \cdot) \) is a twice continuously differentiable function on \( \Omega \). We assume that classical solutions exist.

A Neumann eigenvalue \( \lambda \in \mathbb{R} \) for the Laplacian operator \( \nabla^2 \) on \( \Omega \) is defined by
\[
- \nabla^2 u = \lambda u
\]
\[
\nabla u_i(x) \cdot \vec{n}(x) = 0 \text{ for all } x \in \partial \Omega, i \in \{1, \ldots, n\}.
\]
The set of Neumann eigenvalues of the Laplacian operator consists of countably many nonnegative values with no finite accumulation point [16, Section 7.1]: \( 0 = \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_k \leq \ldots \), i.e., \( \lim_{k \to \infty} \lambda_k = \infty \). For our \( \Omega \), the eigenspace associated with the lowest eigenvalue \( \lambda_1 = 0 \) is
\[
S = \{ h \in \mathcal{L}^2(\Omega) \mid h(x) = c \text{ for all } x \in \Omega \text{ and some constant vector } c \}.
\]
The volume of \( \Omega \) is \( |\Omega| = \int_{\Omega} dx \) and the spatial average of \( h \in \mathcal{L}^2(\Omega) \) over \( \Omega \) is \( \mathcal{I} = \frac{1}{|\Omega|} \int_{\Omega} h(x) dx \in \mathbb{R}^n \). Define the operator \( \Pi_S : \mathcal{L}^2(\Omega) \to S^\perp \) by
\[
\Pi_S(u) = u - \mathcal{I}.
\]
One can easily check that \( \Pi_S \) is the orthogonal projection onto \( S^\perp \). Since \( \Pi_S \) is surjective, i.e., \( \text{Im}(\Pi_S) = S^\perp \), it follows from the definition of the Moore-Penrose pseudoinverse and its uniqueness that \( \Pi_S^2 = \Pi_S \). Thus, for every solution \( u \in \mathcal{L}^2(\Omega) \), it follows from the definition of the Moore-Penrose pseudoinverse and its uniqueness that \( \Pi_S^2 = \Pi_S \) is given by \( \Pi_S(u) = u \).

Given a matrix \( A \in \mathbb{R}^{n \times n} \), the matrix measure associated to the standard Euclidean 2-norm, \( \mu_2(A) \), has the following property [4]: \( \mu_2(A) \leq c \) if and only if \( A + A^\top \geq cI_n \), i.e., \( cI_n - \frac{A + A^\top}{2} \) is positive semi-definite.

**Theorem 5.1** (Partial and semi-contraction of reaction-diffusion systems). Consider the reaction-diffusion system (16) with the standard assumptions on \( f, \Gamma \), and over a bounded and convex set domain \( \Omega \subset \mathbb{R}^m \). Suppose that there exists a positive definite matrix \( P \in \mathbb{R}^{n \times n} \) such that \( \mu_2(P^1/2) \geq 0 \) and \( \mu_2(P(Df(x) - \lambda_2 \Gamma)) \leq -c \) for all \( x \in \Omega \) and some constant \( c > 0 \). Define \( u \mapsto \| u \|_{\Pi_S, P^{1/2}} := \| \Pi_S(P^{1/2}u) \| \) with the set \( S \) as in (17), and let \( \lambda_{\max}(P) \) be the largest eigenvalue of \( P \). Then,

(i) system (16) is partially contractive with respect to \( \| \cdot \|_{\Pi_S, P^{1/2}} \), that is, for every solution \( u : \mathbb{R}_{\geq 0} \times \Omega \to \mathbb{R}^n \),
\[
\| u(t, \cdot) \|_{\Pi_S, P^{1/2}} \leq e^{-\frac{c}{\lambda_{\max}(P)}} \| u(0, \cdot) \|_{\Pi_S, P^{1/2}},
\]

(ii) \( \text{Ker}(\Pi_S) = S \) is an invariant subspace and all trajectories exponentially converge to it; and

(iii) if additionally \( f(u) = Au \), \( A \in \mathbb{R}^{n \times n} \), then (16) is semi-contractive with respect to \( \| \cdot \|_{\Pi_S, P^{1/2}} \), that is, for every solution \( u, v : \mathbb{R}_{\geq 0} \times \Omega \to \mathbb{R}^n \),
\[
\| u(t, \cdot) - v(t, \cdot) \|_{\Pi_S, P^{1/2}} \leq e^{-\frac{c}{\lambda_{\max}(P)}} \| u(0, \cdot) - v(0, \cdot) \|_{\Pi_S, P^{1/2}}.
\]

The proof can be found in the Appendix.

**Remark 5.2.** Statements (i) and (ii) of Theorem 5.1 are essentially the same result as [5, Theorem 1]; however, these statements and statement (iii) are now consequences of a general contraction theory.

In Figure 1 we simulate the following instance of (16): \( u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, \quad \Gamma = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}, \quad f(u) = 0.05 \begin{bmatrix} 1/2 \\ 1/3 \end{bmatrix}, \quad \Omega = [0, 1]. \) Set \( \alpha(x) := -0.25 + 0.9 \sin(15x) \) and \( \beta(x) := 1.75x - 0.75, \) \( x \in \Omega. \) In the upper plot of Figure 1, we plot the quantity \( \| u(t, \cdot) - v(t, \cdot) \|_{\Pi_S, P^{1/2}} \), \( t \in [0, 0.8], \) for signals \( u \) and \( v \) under the initial conditions \( u_1(0, \cdot) = 10\alpha, \) \( u_2(0, \cdot) = 10\beta, \) \( v_1(0, \cdot) = 5\beta, \) and \( v_2(0, \cdot) = 2.5\alpha. \) As predicted by the statement (iii) of Theorem 5.1, this quantity has exponential decay due to semi-contraction. In the lower three plots, we plot \( u_1(t, \cdot) \) for different values of \( t. \)
6 Conclusion

This paper presents a general contraction theory for dynamical systems on Hilbert spaces. We provide sufficient conditions for contraction, semi-contraction and partial contraction based on operator measures or operator semi-measures, and on the differentiability of the vector field. Moreover, when the system is time-invariant, we present weaker conditions that do not require differentiability. Finally, we present an example of reaction-diffusion systems.

Our work brings the machinery of contraction theory, so far mainly applied to ODEs, to other possible application domains related to a variety of systems that can be expressed as dynamical systems on functional spaces.

Acknowledgment

The authors thank Prof. Sam Coogan for insightful discussions about contraction theory.
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First, observe that, for any \( u, h \in X \), \( \|DF(t, u)h\| \leq a(t)\|h\| \). Moreover, observe that the differential equation \( \dot{r} = a(t)r \) satisfies that if it has a solution \( r(t) \) such that \( r(t_0) = 0 \), then \( r(t) = 0 \) for any \( t \geq t_0 \). These two conditions satisfy the hypotheses of [20, Theorem 5.3.3], and thus we can use [20, Lemma 5.4.2] from which equation (3) follows. \( \square \)

**Proof of Theorem 5.1.** Note that the reaction-diffusion system we are analyzing can be written as \( \frac{\partial u}{\partial t} = F(u) \) where \( F : \mathcal{L}^2(\Omega) \to \mathcal{L}^2(\Omega) \) is defined by \( F(u) := f(u) + \Gamma \nabla^2 u \). Let \( \langle \cdot, \cdot \rangle_{\Pi_S,P^{1/2}} := \langle \Pi_S(P^{1/2}), \Pi_S(P^{1/2}) \rangle \).

We start by proving statement (i). Consider any \( u \in \mathcal{L}^2(\Omega) \), and set \( \bar{u} = u - \tilde{u} \), so that \( \tilde{u} \in \mathcal{S}^\perp \).

Then,

\[
\langle u, F(u) \rangle_{\Pi_S,P^{1/2}} = \langle u, f(u) \rangle_{\Pi_S,P^{1/2}} + \langle u, \Gamma \nabla^2 u \rangle_{\Pi_S,P^{1/2}}
\tag{19}
\]

First, from the first term in the right-hand side of (19),

\[
\langle u, f(u) \rangle_{\Pi_S,P^{1/2}} = \langle \Pi_S(P^{1/2}(u)), \Pi_S(P^{1/2}(f(u))) \rangle
= \int_\Omega \tilde{u}^T P(f(u) - f(\bar{u})) dx \tag{20}
\]

where for the second equality we repeatedly used \( \int_\Omega \tilde{u}^T a dx = 0 \) for any constant vector \( a \in \mathbb{R}^n \) in \( \Omega \), and, since \( \Omega \) is convex, the mean-value theorem: \( f(u) - f(\bar{u}) = \int_0^1 Df(s(\gamma))\tilde{u} d\gamma \) with \( s(\gamma) = u + \gamma(\bar{u} - u) \) for the last inequality. Now, from the second term in the right-hand side of (19),

\[
\langle u, \Gamma \nabla^2 u \rangle_{\Pi_S,P^{1/2}} = \int_\Omega \tilde{u}^T \Gamma \nabla^2 \tilde{u} dx,
\tag{21}
\]

where we used \( \Pi_S(\nabla^2 u) = \nabla^2 u - \frac{1}{|\Omega|} \int_\Omega \nabla^2 u dx = \nabla^2 \bar{u} \), since \( \int_\Omega \nabla^2 u_i dx = \int_{\partial \Omega} \nabla u_i \cdot \bar{n} d\partial \Omega = 0 \) (where the surface integral after the first equality follows from the divergence theorem, and the last equality follows from the boundary condition in (16)). Note that, for every \( i \in \{1, \ldots, n\} \), we have \( \nabla^2 \tilde{u}_i = \nabla \cdot (\nabla \tilde{u}_i) \). Now, by the product rule, we have that for every \( i \in \{1, \ldots, n\} \), \( \nabla \cdot \)
\[
(\sum_{j=1}^{n} \nabla u_i(P\Gamma)_{ij} \nabla u_j) = \sum_{j=1}^{n} \nabla u_i(P\Gamma)_{ij} \nabla^2 u_j - \sum_{j=1}^{n} (\nabla u_i)_{ij} \nabla u_j. \]

Moreover, one can check that \( \langle \partial Q, (\sum_{j=1}^{n} \nabla u_i(P\Gamma)_{ij} \nabla u_j \cdot \hat{n}) dS = 0 \) where the last equality follows from the boundary condition in (16). Then, from the identity \( \tilde{u}^T P\Gamma \nabla^2 \tilde{u} = \sum_{i=1}^{n} \sum_{j=1}^{n} \nabla (P\Gamma)_{ij} \nabla \tilde{u}_j \) we get

\[
\int_{\Omega} \tilde{u}^T P\Gamma \nabla^2 \tilde{u} dx = - \int_{\Omega} \sum_{i=1}^{n} \sum_{j=1}^{n} (\nabla u_i)_{ij} \nabla \tilde{u}_j dx.
\]

Moreover, one can check that

\[
\sum_{i=1}^{n} \sum_{j=1}^{n} (\nabla u_i)_{ij} = \sum_{k=1}^{m} \frac{\partial u}{\partial x_k} \nabla \tilde{u}_j = \sum_{k=1}^{m} \frac{\partial \tilde{u}}{\partial x_k} \nabla u_j = \sum_{k=1}^{m} \frac{\partial Q\tilde{u}}{\partial x_k} = \sum_{i=1}^{n} (\nabla ((Q\hat{u})), \nabla ((Q\hat{u})) = \sum_{i=1}^{n} (\nabla u_i)_{ij} = \sum_{i=1}^{n} (\nabla \tilde{u}_j).
\]

Combining all of these results, we finally obtain \( \int_{\Omega} \tilde{u}^T P\Gamma \nabla^2 \tilde{u} dx = - \sum_{i=1}^{n} \| \nabla (Q\hat{u})_i \|^2 \). Now, since \( \int_{\Omega} Q\tilde{u} dx = Q \int_{\Omega} \tilde{u} dx = 0 \), we can use the Poincaré inequality on simply connected domains \([16, \text{Section 1.3}]) since our domain is convex, and obtain \( \| \nabla (Q\hat{u})_i \|^2 \geq \lambda_2 \| (Q\hat{u})_i \|^2 \). As a result, we get \( \int_{\Omega} \tilde{u}^T P\Gamma \nabla^2 \tilde{u} dx \leq -\lambda_2 \sum_{i=1}^{n} \| (Q\hat{u})_i \|^2 = \int_{\Omega} \tilde{u}^T (\lambda_2 P\Gamma) \tilde{u} dx \). Replacing this result in (21), and then replacing the resulting expression with the one in (20) back in (19):

\[
\langle u, F(u) \rangle_{\Pi_S, P^{1/2}} = \int_{0}^{1} \int_{\Omega} \tilde{u}^T P(Df(s(\lambda)) - \lambda_2 \Gamma) \tilde{u} dx d\gamma \leq -c \int_{0}^{1} \int_{\Omega} \tilde{u}^T \tilde{u} dx d\gamma \leq -\frac{c}{\lambda_{\max}(P)} \| P^{1/2} \tilde{u} \| = -\frac{c}{\lambda_{\max}(P)} \| \tilde{u} \|_{\Pi_S, P^{1/2}} = \frac{c}{\lambda_{\max}(P)} \| u \|_{\Pi_S, P^{1/2}}
\]

where the inequality comes from the assumption \( \mu_2(P(Df(x) - \lambda_2 \Gamma)) \leq -c \) for any \( x \in \Omega \). This expression has the form of the integral partial contractivity condition. Although the set of classical solutions endowed with \( \langle \cdot, \cdot \rangle \) is not a Hilbert space, we follow the proof of Theorem 4.2 using the Leibniz rule to differentiate the inner product and obtain partial contraction as in statement (i).

Statement (ii) follows by noting that \( \| u \|_{\Pi_S, P^{1/2}} = 0 \implies P^{1/2} u \in \ker(\Pi_S) \implies P^{1/2} u \in S \implies u \in S \). Finally, statement (iii) is proved in a similar way to statement (i), using the difference of any two solutions as a new state variable.

\[\square\]