An Information Geometric Analysis of Entangled Continuous Variable Quantum Systems

D.-H. Kim  
Institute for the Early Universe, Ewha Womans University, Seoul 120-750, South Korea

S. A. Ali  
Department of Physics, State University of New York at Albany, 1400 Washington Avenue, Albany, NY 12222, USA
Department of Arts and Sciences, Albany College of Pharmacy and Health Sciences, 106 New Scotland Avenue, Albany, NY 12208, USA

C. Cafaro and S. Mancini  
School of Science and Technology, Physics Division, University of Camerino, I-62032 Camerino, Italy

Abstract. In this work, using information geometric (IG) techniques, we investigate the effects of micro-correlations on the evolution of maximal probability paths on statistical manifolds induced by systems whose microscopic degrees of freedom are Gaussian distributed. Analytical estimates of the information geometric entropy (IGE) as well as the IG analogue of the Lyapunov exponents are presented. It is shown that the entanglement duration is related to the scattering potential and incident particle energies. Finally, the degree of entanglement generated by an $s$-wave scattering event between minimum uncertainty wave-packets is computed in terms of the purity of the system.

1. Introduction
One of the most important, interesting and often debated features of (composite) quantum mechanical systems is their ability to become entangled [1, 2]. By quantum entanglement we mean quantum correlations among the distinct subsystems of the entire composite quantum system. For such correlated quantum systems, it is not possible to specify the quantum state of any subsystem independently of the remaining subsystems [1]. Quantum entanglement is an indispensable resource for quantum information processes. Continuous variable quantum systems (CVQS) are also an interesting topic in quantum information theory [3]. By CVQS we refer to quantum mechanical systems on which one can - in principle - perform measurements of certain observables whose eigenvalue spectrum is continuous. The most realistic approach to the generation of entangled continuous variable systems is via dynamical interaction, of which local scattering events (collisions) are a natural, ubiquitous type [4]. The primary focus of the present article is to explore the potential utility of the information geometric approach to chaos (IGAC) for analyzing quantum mechanical systems. The IGAC [5, 6, 7, 8, 9, 10] is a theoretical framework developed to study chaos in informational geodesic flows on statistical...
manifolds associated with probabilistic descriptions of physical, biological or chemical systems. For a system of two spin-0, non-relativistic particles in one dimension with no internal degrees of freedom, a complete set of commuting observables is furnished by the momentum operators of each particle [11]. The continuous variables in our case are therefore taken to be the momentum of each particle. In the present work we consider two CVQS with Gaussian continuous degrees of freedom that are prepared independently, interact via a scattering processes mediated by an interaction (scattering) potential and separate again. We investigate the entanglement of the two-particle wave-function of the system generated by such a scattering event.

The layout of this article is as follows. In Section II, we briefly review the phenomenon of quantum entanglement for two colliding wave packets. In Section III, we describe the general steps through which the statistical manifold $M$ in presence of micro-correlations is constructed. In Section IV, we apply IG techniques in conjunction with standard $s$-wave scattering theory to study our specific continuous two-variable micro-correlated Gaussian model. In Section V, the IGAC is used to investigate the chaoticity arising in the information constrained dynamics of our quantum entangled system. Our final remarks are presented in Section VI.

2. Entanglement and Wave-packet Scattering Processes

For the purpose of modeling a head-on collision we consider two identical spin-0 particles in momentum space, each represented by minimum uncertainty Gaussian wave-packets. Before collision, particles 1 and 2 are initially located far from each other - a linear distance $R_o$ - each having the initial average momentum $\langle p_1 \rangle_o = p_0 = \hbar k_o$ and $\langle p_2 \rangle_o = -p_0 = -\hbar k_o$, respectively, with equal momentum dispersion $\sigma_o = \hbar \sigma_{ko}$. The normalized, separable (i.e., non-entangled) two-particle Gaussian wave function representing the situation before collision is then given by [12]

$$\psi_{before} (k_1, k_2) = \left( \frac{1}{2\pi \sigma_{ko}^2} \right)^{1/2} \exp \left[ -\frac{(k_1 - k_o)^2}{4\sigma_{ko}^2} - \frac{(k_2 + k_o)^2}{4\sigma_{ko}^2} \right] \times e^{i[-\frac{1}{2}(k_1-k_o)R_o + \frac{1}{2}(k_2+k_o)R_o]}, \quad (1)$$

After collision, the wave function for the two-particle system in the long time limit takes the form [12]:

$$\psi_{after} (k_1, k_2, t) = (N)^{-1/2} \left[ \psi_{before} (k_1, k_2) e^{-i(k_1^2 + k_2^2)t/(2\mu)} + \varepsilon \psi_{scat} (k_1, k_2, t) \right], \quad (2)$$

where $N$ and $\varepsilon$ are normalization constants such that $\psi_{after}$ and $\psi_{scat}$ are both normalized to unity, and $\mu$ denotes the mass of each particle. We treat $|\varepsilon| \ll 1$ as a small number. Following [12], one can specify

$$\varepsilon \psi_{scat} (k_1, k_2, t) = \left( \frac{1}{2\pi \sigma_{ko}^2} \right)^{1/2} \exp \left[ -\frac{K^2 + 4(k - k_o)^2}{4\sigma_{ko}^2} \right] \times \frac{if(k)\sigma_{ko}^2}{2k_o - i\sigma_{ko}^2} e^{-\hbar K^2t/(2\mu)} e^{i[(k-k_o)R_o - \hbar k^2t/(2\mu)]}, \quad (3)$$

where we have adopted the center of mass and relative coordinates such that the conjugate momenta, $K \equiv k_1 + k_2$ and $k \equiv \frac{1}{2}((k_1 - k_2)$ are used along with the total mass, $M = 2m$ and the reduced mass, $\mu = m/2$, and $f(k) \equiv \frac{e^{i2\pi(k-\frac{1}{2})}}{2\pi k}$ is the $s$-wave scattering amplitude due to
the s-wave scattering phase shift, $\theta(k)$. In describing a head-on collision between two identical particles we may choose (for simplicity) the total momentum of the system to be vanishing, i.e. we choose the momenta of each particle to be equal in magnitude but opposite in sign such that $K = 0$. Then using (2) and (3), and assuming that our wave-packet is well-localized around $k = k_0$, we find

$$|\psi(k_1, k_2, t)|^2 \sim \exp \left( -\frac{k^2}{\sigma^2_{k_0}} \right) \left[ 1 + 2 \Re (g_0) \exp \left( -\frac{k^2}{2\sigma^2_{k_0}} \right) \right] + |g_0|^2 \exp \left( -\frac{k^2}{\sigma^2_{k_0}} \right),$$

(4)

where $\bar{k} \equiv \frac{1}{2} [(k_1 - k_0) - (k_2 + k_0)]$ and $g_0 \equiv \frac{i f(k_0) \sigma^2_{k_0}}{2k_0 - i \sigma^2_{k_0} \bar{K}}$.

It is worth noting that by choosing $K = 0$ in computing $|\psi(k_1, k_2, t)|^2$ we render $|\psi(k_1, k_2, t)|^2$ time-independent. This implies that with the appropriate choice of reference frame, in which $K = 0$ (experiencing an exact head-on collision), the probability associated with the quantum scattering process can be interpreted as static.

2.1. On the Two-variable Micro-correlated Gaussian Statistical Model

We want to make reliable macroscopic predictions when only partial knowledge about the micro-structure of a system is available. The complexity of such predictions is quantified in terms of the IGE and the Lyapunov exponents. Consider micro-correlated Gaussian statistical models with probability distribution $P(x, y|\mu_x, \mu_y, \sigma)$ [13],

$$P(x, y|\mu_x, \mu_y, \sigma) = \frac{\exp \left\{ -\frac{1}{2(1-r^2)} \left[ \frac{(x-\mu_x)^2}{\sigma_x^2} - 2r \frac{(x-\mu_x)(y-\mu_y)}{\sigma_x^2 \sigma_y^2} + \frac{(y-\mu_y)^2}{\sigma_y^2} \right] \right\}}{2\pi \sigma^2 \sqrt{1-r^2}},$$

(5)

which is a bivariate normal distribution, where $\mu_x \equiv \langle x \rangle$, $\mu_y \equiv \langle y \rangle$ and $\sigma = \sqrt{\langle (x - \langle x \rangle)^2 \rangle}$, $\sqrt{\langle (y - \langle y \rangle)^2 \rangle} > 0$, and the micro-correlation coefficient $r = r(x, y) \equiv \langle xy \rangle - \langle x \rangle \langle y \rangle / \sigma_x \sigma_y$ assumes values in the domain $(-1, 1)$.

We introduce the Fisher-Rao information metric [20]

$$g_{ab}(\Theta) = \int dXP(X|\Theta) \partial_a \ln P(X|\Theta) \partial_b \ln P(X|\Theta)$$

$$= 4 \int dX \partial_a \sqrt{P(X|\Theta)} \partial_b \sqrt{P(X|\Theta)},$$

(6)

where $a, b = 1, 2, 3$ and $X = (x, y)$, $\Theta = (\mu_x, \mu_y, \sigma)$, $\partial_a = \frac{\partial}{\partial \theta^a}$, which assigns an information geometry to the space of macrostates. Upon substituting (5) in (6), the Fisher-Rao information metric $g_{ab}(\mu_x, \mu_y, \sigma; r)$ becomes

$$g_{ab}(\mu_x, \mu_y, \sigma; r) = \frac{1}{\sigma^2} \begin{pmatrix}
-\frac{1}{r^2-1} & \frac{r}{r^2-1} & 0 \\
\frac{r}{r^2-1} & \frac{1}{r^2-1} & 0 \\
0 & 0 & 4
\end{pmatrix}.$$ 

(7)
The metric for the non-correlated case, \( g_{ab} (\mu_x, \mu_y, \sigma; 0) \) can be obtained from (7) by setting \( r = 0 \). With \( r \ll 1 \), we may split the micro-correlated geometry into two pieces, i.e. 
\[
\text{g}_{ab} (\mu_x, \mu_y, \sigma; r) = \text{g}_{ab} (\mu_x, \mu_y, \sigma; 0) + \text{h}_{ab} (\mu_x, \mu_y, \sigma; r).
\]
We comment on this metric decomposition in our Final Remarks.

In what follows we limit our analysis to the study of non-negative micro-correlations, that is we will consider \( r \in [0, 1) \).

2.2. Information Dynamics on \( \mathcal{M}^{3D}_{\text{corr.}} \) (three-dim statistical manifold in presence of micro-correlations)

The information dynamics can be derived from a standard principle of least action of Jacobi type [14]. The geodesic equations for the macro-variables of our Gaussian model are given by

\[
d^2\Theta^a d\tau^2 + \Gamma^a_{bc} \frac{d\Theta^b}{d\tau} \frac{d\Theta^c}{d\tau} = 0,
\]

where \( a, b, c = 1, 2, 3 \) and we denote \( \Theta^1 = \mu_1 = \mu_x, \Theta^2 = \mu_2 = \mu_y, \Theta^3 = \sigma \). The connection coefficients \( \Gamma^a_{bc} \) appearing in (8) are defined as \( \Gamma^a_{bc} = \frac{1}{2} g^{ad} (\partial_b g_{dc} + \partial_c g_{bd} - \partial_d g_{bc}) \) [15], and computed by means of 7. After integration of (8), the geodesic trajectories become: for the correlated Gaussian system,

\[
\begin{align*}
\mu_1(\tau; r) &= -\sqrt{1-r} (p_o^2 + 2\sigma_o^2) \tanh (A_o \tau), \\
\mu_2(\tau; r) &= \sqrt{1-r} (p_o^2 + 2\sigma_o^2) \tanh (A_o \tau), \\
\sigma(\tau; r) &= \sqrt{\frac{1}{2} p_o^2 + \frac{1}{2} \sigma_o^2 \cosh (A_o \tau)},
\end{align*}
\]

where \( p_o \equiv \mu_1 (-\tau_0; 0), \sigma_o \equiv \sigma (-\tau_0; 0) \) and the quantity \( A_o \) is defined as

\[
A_o \equiv \frac{1}{\tau_o} \sinh^{-1} \left( \frac{p_o}{\sqrt{2}\sigma_o} \right)
\]

\[
\frac{p_o}{\sigma_o} \ll 1 \quad \frac{1}{\tau_o} \ln \left( \frac{\sqrt{2} p_o}{\sigma_o} \right) + \frac{1}{2} \left( \frac{\sigma_o}{p_o} \right)^2 - 3 \left( \frac{\sigma_o}{p_o} \right)^4 + O \left( \left( \frac{\sigma_o}{p_o} \right)^6 \right) > 0.
\]

(12)

The geodesic trajectories on \( \mathcal{M}^{3D}_{\text{non-corr.}} \) for the non-correlated Gaussian system may be obtained from (9), (10) and (11) by setting \( r = 0 \).

3. Application to Quantum Physics

We employ IG techniques in conjunction with standard partial wave quantum scattering theory to investigate quantum entanglement produced by a head-on collision between two Gaussian wave-packets in momentum space.

3.1. Momentum-space Gaussian Statistical Models

By letting \( x \to p_1 = \hbar k_1, y \to p_2 = \hbar k_2 \) and \( \mu_x \to \langle p_1 \rangle_o = p_o = h k_o, \mu_y \to \langle p_2 \rangle_o = -p_o = -h k_o \) in (5), we can identify the two particle probability density before collision, given by (1), with the non-correlated probability distribution obtained from (5) with \( r = 0 \). In a similar manner, the probability density \( |\psi(k_1, k_2, t)|^2 \) in (4) is approximated with the Gaussian probability distribution (5) with \( r \neq 0 \). Comparison of (4) and (5) suggests that when both \( |\varrho_o| \) and \( r \)
are very small: $P_{Q\text{M}} \simeq P_{\text{Corr-Gaussian}}$, where

$$
P_{Q\text{M}} = \frac{1}{\sqrt{\pi} \sigma_{k_0}} \left[ 1 + \frac{2 \Re (\theta_0)}{\sqrt{\pi} \sigma_{k_0}} \right] \exp \left( -\frac{\tilde{k}^2}{2 \sigma_{k_0}^2} \right)
$$

$$
\times \left[ 1 + 2 \Re (\theta_0) \exp \left( -\frac{\tilde{k}^2}{2 \sigma_{k_0}^2} \right) + |\theta_0|^2 \exp \left( -\frac{\tilde{k}^2}{\sigma_{k_0}^2} \right) \right],
$$

(13)

and

$$
P_{\text{Corr-Gaussian}} = \frac{1}{\sqrt{\pi} \sigma_{k_0} \sqrt{1 - r}} \exp \left[ -\frac{\tilde{k}^2}{\sigma_{k_0}^2} (1 - r) \right],
$$

(14)

respectively. Here we have reduced the expression via $K \equiv (k_1 - k_0) + (k_2 + k_0) = K = 0$ and $\tilde{k} \equiv \frac{1}{2} ((k_1 - k_0) - (k_2 + k_0))$, and the normalization factor for each distribution has been weighted.

We are thus able to model the system of pair of particles, which describes the pre-collision and post-collision scenarios, by the normalized, bipartite Gaussian statistical system. In this system we join two different charts of Gaussian statistical manifolds, one without correlation (before collision) and the other with correlation (after collision). To examine how the correlation affects the geodesic curves, we compute the relative conjugate-momentum curves, $\langle p(\tau; 0) \rangle \equiv \frac{1}{2} [\mu_2 (\tau; 0) - \mu_1 (\tau; 0)]$ and $\langle p (\tau; r) \rangle \equiv \frac{1}{2} [\mu_2 (\tau; r) - \mu_1 (\tau; r)]$ and find that at any arbitrary time $\tau \geq 0$

$$
\langle p (\tau; 0) \rangle \geq \langle p (\tau; r) \rangle = \sqrt{1 - r} \langle p (\tau; 0) \rangle.
$$

(15)

This implies that the correlation causes the momentum to reduce for any $\tau \geq 0$ (relative to the non-correlated situation). This situation is analogous to the change in momentum caused by a repulsive scattering potential. It is then reasonable to assume there exists some connection between the scattering potential and the correlation. That is, with the correlation, the wave-packets experience the effect of a repulsive potential; the magnitude of the wave vectors (momenta) decreases relative to the corresponding non-correlated value. Given the potential $V(x) = V$ for $0 \leq x \leq L$ and $V(x) = 0$ for $x > L$, one may write [21]

$$
k_r \cot (k_r L) = k_o L + \theta_o,
$$

(16)

$$
k_r = \frac{\sqrt{2 \mu (E - V)}}{\hbar}, \quad 0 < x < L, \quad k_o = \frac{\sqrt{2 \mu E}}{\hbar}, \quad x > L.
$$

(17)

The quantities $V$ and $L$ denote the height ($V > 0$; repulsive potential) and range of the potential, respectively, and $\theta_o \equiv \theta (k_o)$ the $s$-wave scattering phase shift, and $k_r$ and $k_o$ the wave vectors with and without the correlation, respectively, and $\mu$ and $E$ being the reduced mass and kinetic energy of the two-particle system in the relative coordinates, respectively. From (15) one finds that the correlation renders

$$
k_o \rightarrow k_r \equiv \sqrt{1 - r k_o}.
$$

(18)

Then using (17) and (18), we determine the scattering potential,

$$
V = r E = r \frac{p_o^2}{2 \mu},
$$

(19)

where $p_o = \hbar k_o$. With the potential determined, one can determine the scattering phase shift for low energy $s$-wave scattering $k_o L \ll 1$ by combining (16), (17) and (19), the result being

$$
\theta_o \approx -\frac{r k_o^3 L^3}{3} = -\frac{r p_o^3 L^3}{3 \hbar^3} = -\frac{2 \mu V p_o L^3}{3 \hbar^3},
$$

(20)

which is in agreement with [16].
3.2. Correlation vs. Entanglement: Connection Established via Scattering and Purity

The purity of a system of two particles engaged in a head-on collision was calculated in [12] by deriving the two-particle wave function modified by s-wave scattering amplitudes. They utilized the purity function $P$ as a measure of entanglement. For pure two-particle states, the smaller the value of $P$ the higher the entanglement. That is, the loss of purity provides an indicator of the degree of entanglement. Hence, a disentangled product state corresponds to $P = 1$. For the system being considered, it was found in [12] that the purity of final state is evaluated approximately as

$$P \approx 1 - 2|\varepsilon|^2; \ |\varepsilon| \approx \sqrt{2}\,|f(p_o)|\,\frac{\sigma_o}{\hbar\chi}$$

(21)

for the wave-packets whose widths are very narrow in momentum space: the subscript, “o” denotes the initial state, and $f(p_o) \equiv \frac{\hbar(\varepsilon^{p_o} - 1)}{2p_o}$ is the s-wave scattering amplitude with $\theta_o$ representing the s-wave scattering phase shift, and $\chi \equiv \sqrt{1 + \left(\frac{\sigma^2 p}{2lp_o}\right)^2}$ corresponding to the spreading factor of the spatial width of the wave-packets. Using (21), and employing the scattering cross section $\Sigma(p_o) = 4\pi|f(p_o)|^2$, and requiring that if the particles are not correlated (i.e. if $r = 0$) after collision then no entanglement should be present (i.e. $P = 1$ implying $\theta_o = f(p_o) = \Sigma(p_o) = 0$), we may express the purity in an alternative manner as

$$P = 1 - 4\left(\frac{\sigma_o}{\hbar\chi}\right)^2 |f(p_o)|^2 = 1 - \frac{\left(\frac{\sigma_o}{\hbar\chi}\right)^2 \Sigma(p_o)}{\pi} = 1 - \frac{16\mu^2\sigma^2_p V^2 L^6}{9\hbar^6 \left[1 + \left(\frac{\sigma^2 p}{2lp_o}\right)^2\right]}.$$  

(22)

The correlation coefficient $r$ can now be expressed in terms of the physical quantities such as the scattering potential, the scattering cross section and purity as follows

$$r = \frac{2\mu V}{p_o^2} = \frac{3\hbar^2 \Sigma(p_o)}{2\sqrt{\pi} p_o^3 L^3} = \frac{3\hbar^3 \sqrt{1 - P}}{2p_o^2 \sigma_o L^3}.$$  

(23)

From (23) we observe that the micro-correlation coefficient $r$ is directly associated with the quantum scattering process, and thus with the quantum entanglement. For example, the cross term $\langle p_1 p_2 \rangle$ in $r = r(p_1, p_2) \overset{\text{def}}{=} \frac{\langle p_1 p_2 \rangle - \langle p_1 \rangle\langle p_2 \rangle}{\sigma^2_o}$ may represent the average interference between transmitted/reflected modes in the momentum degrees of freedom of particle 1 and 2, respectively.

From (15) it is observed that for the micro-correlated Gaussian system considered here, more time is required to attain the same momentum value compared with the non-correlated Gaussian system. For example, to attain the same value of initial momentum $p_o$, the non-correlated system and the micro-correlated system would require time intervals $\tau_o$ and $\tau_*$, respectively, where

$$p_o = \sqrt{p_o^2 + 2\sigma^2_o} \tanh (A_o \tau_o),$$

$$p_o = \sqrt{(1 - r) (p_o^2 + 2\sigma^2_o)} \tanh (A_o \tau_*).$$  

(24)

(25)

Combining (24) and (25) and using (12), we find

$$\Delta \propto \ln \left\{1 - \left[1 - (1 - r)^{-1/2} - 1\right] \cdot \eta_\Delta\right\},$$

(26)

where $\Delta \equiv \tau_* - \tau_o$ represents the prolongation and

$$\eta_\Delta \equiv \frac{1}{2} e^{2A_o \tau_o} = \left(\frac{p_o}{\sigma_o}\right)^2 \exp \left[\left(\frac{\sigma_o}{p_o}\right)^2 - \frac{3}{4} \left(\frac{\sigma_o}{p_o}\right)^4 + O\left(\left(\frac{\sigma_o}{p_o}\right)^6\right)\right]$$

for $\frac{\sigma_o}{p_o} \ll 1$. See the Final Remarks for further discussion of the prolongation.
4. Complexity
In this Section, we investigate the complexity of the dynamical system being considered.

4.1. Information Geometric Indicators of Chaoticity

4.1.1. Jacobi Fields and Lyapunov Exponents
The relative geodesic spread on $\mathcal{M}_{\text{corr}}^{3D}$, formally characterized by the Jacobi-Levi-Civita (JLC) equation [22, 23],

$$\frac{D^2 J^a}{D\tau^2} + R_{abcd}^a \frac{\partial \Theta^b}{\partial \tau} J^c \frac{\partial \Theta^d}{\partial \tau} = 0,$$  \hspace{1cm} (27)

where $a, b, c, d = 1, 2, 3$ and $\frac{D^2}{D\tau^2}$ denotes the second order covariant derivatives and the Riemann curvature tensor $R_{abcd}$ is defined as $R_{abcd} = \partial_c \Gamma^b_{ad} - \partial_d \Gamma^b_{ac} + \Gamma^b_{fc} \Gamma^f_{ad} - \Gamma^b_{fd} \Gamma^f_{ac}$, and the Jacobi vector field components $J^a$ are given by $J^a = \delta_i \Theta^a \equiv \frac{\partial \Theta^a(\tau, \varsigma)}{\partial \varsigma^i} \delta_i^b$ with $\tau$ and $\varsigma = (\varsigma^1, \varsigma^2, \varsigma^3)$ denoting affine parameters. Through some involved analysis by means of (7) and (27), we reduce the JLC equation to

$$\frac{D^2 J_M}{D\tau^2} - A_o^2 J_M = 0.$$

where the Jacobi vector field intensity is defined as $J_M = \left( g_{ab} J^a J^b \right)^{\frac{1}{2}} = \left( J^a J^a \right)^{\frac{1}{2}}$ and $A_o$ is given by (12). Solutions of equation (28) assume the form $J_M(\tau) = \text{const} \times \sinh (A_o \tau)$. In our approach the quantity $\lambda_M$,

$$\lambda_M \equiv \lim_{\tau \to \infty} \frac{1}{\tau} \ln \left[ \frac{\| J_M(\tau) \|^2 + \left\| \frac{dJ_M(\tau)}{d\tau} \right\|^2}{\| J_M(0) \|^2 + \left\| \frac{dJ_M(0)}{d\tau} \right\|^2} \right] = 2A_o > 0$$  \hspace{1cm} (29)

would play the role of the conventional Lyapunov exponents. It is evident from (29) that the Lyapunov exponents can be determined solely from the initial conditions.

4.1.2. Information Geometric Entropy
It is known [18, 19] that a suitable indicator of temporal complexity within the IGAC framework is provided by the information geometric entropy (IGE) $S_{\mathcal{M}_{\text{corr}}^{3D}}(\tau)$,

$$S_{\mathcal{M}_{\text{corr}}^{3D}}(\tau) \equiv \lim_{\tau \to \infty} \ln \tilde{\text{vol}} \left[ D_{\Theta}^{\text{(geodesic)}}(\tau) \right],$$  \hspace{1cm} (30)

where the quantity $\tilde{\text{vol}} \left[ D_{\Theta}^{\text{(geodesic)}}(\tau) \right]$ is termed the information geometric complexity (IGC). The IGC is defined as the temporal average of the dynamical statistical volume, and for the system under investigation, via (7), (9), (10) and (11) it is given by

$$\tilde{\text{vol}} \left[ D_{\Theta}^{\text{(geodesic)}}(\tau) \right] \equiv \lim_{\tau \to \infty} \frac{1}{\tau} \int_0^\tau \left( \int_{\mu_1(\tau')} \int_{\mu_2(\tau')} \int_{\sigma(0)} \sqrt{g} d\mu_1 d\mu_2 d\sigma \right) \hspace{1cm} (31)

\hspace{2cm} = \frac{4}{A_o} \left[ -\frac{3}{2} A_o + \frac{1}{4} \frac{\sinh (2A_o \tau)}{\tau} + \frac{\tanh (A_o \tau)}{\tau} \right].$$

By means of (30) and (31), the IGE for the correlated Gaussian statistical model is given by

$$S_{\text{corr.}}(\tau; r) \equiv \lambda_M \tau + \frac{1}{2} \ln \left( \frac{1 - r}{1 + r} \right).$$  \hspace{1cm} (32)
5. Final Remarks

In this article, micro-correlated and non-correlated Gaussian statistical models were used to model scattering-induced quantum entanglement. The manifolds $\mathcal{M}^{3D}_{\text{corr.}}$ and $\mathcal{M}^{3D}_{\text{non-corr.}}$ were used to model the quantum entanglement induced by head-on scattering of two spin-0 particles, each represented by minimum uncertainty wave-packets. When $r = 0$, the quantities $\theta_0$, $\Sigma$, and $V$ are each zero while $\mathcal{P} = 1$ (indicating that the system is not entangled). When $r \neq 0$, the wave-packets experience the effect of a repulsive potential and the magnitude of the wave vectors (momenta) decreases relative to their corresponding non-correlated value. It was found that the micro-correlation coefficient $r$ depends inversely on $p_0$. The role played by $r$ in the quantities $\mathcal{P}$, $\Sigma$, $\theta_0$, and $V$ suggests that information about quantum scattering and therefore about quantum entanglement is encoded in the statistical micro-correlation, specifically in the covariance term appearing in the definition of $r$.

Furthermore, from the notion $g_{ab}(\mu_x, \mu_y, \sigma; r) = g_{ab}(\mu_x, \mu_y, \sigma; 0) + h_{ab}(\mu_x, \mu_y, \sigma; r)$ for $r \ll 1$, we observe that the quantum entanglement manifests as a geometric perturbation of the statistical space in analogy to the interpretation of a static gravitational field as a perturbation of flat space. After scattering the two particles maintain a correlation among their microscopic momentum degrees of freedom regardless of the extent of their separation in statistical space. This fact, together with the time-independence of the statistical geometry (i.e. the information metric is Riemannian since its signature is positive definite) leads to a notion of statistical non-locality. The perturbation of statistical geometry is associated with the scattering phase shift in the statistical momentum space.

The prolongation of the system entanglement, denoted $\Delta$, was defined as the time required for the observed momentum difference between the correlated and non-correlated momenta to vanish. From (26), it was found that for $r$ values close to its upper bound, the prolongation $\Delta$ becomes infinitely large. On the other hand, with $r$ vanishing (i.e., no micro-correlation) $\Delta$ is identically zero. With $r$ fixed however, the prolongation $\Delta$ depends on $p_0$ and $\sigma_0$. Thus, the prolongation $\Delta$ may be taken to represent the duration of quantum entanglement for a given correlated system where the entanglement duration can be controlled by the initial conditions $p_0$ and $\sigma_0$ as well as $r$. Maximal prolongation occurs when $r$ is greatest and the ratio $\sigma_0/p_0$ is smallest. For small initial $r$ and $p_0$, $\Delta$ would be correspondingly small, suggesting that for such scenarios quantum entanglement is transient.

The Lyapunov exponents on both manifolds were found to be constant and positive definite, i.e. $\lambda_M = 2A_0 > 0$. In the presence of micro-correlations the IGE assumes a smaller initial value relative to the non-correlated case while the growth characteristics of both correlated and non-correlated IGEs were found to be similar. Furthermore, in the presence of micro-correlations, the IGE is attenuated in a correlation-dependent manner such that $S_{\text{corr.}}(\tau; r)$ decreases as the magnitude of the correlation increases. It is important to observe that this is not an asymptotic (long-time limit) feature of the IGE.

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References

[1] Einstein A, Podolsky B, Rosen N 1935 Phys. Rev. 47 777
[2] Schrödinger E 1935 Naturwiss. 48 807; 49 823; 50 844
[3] Law C K 2004 Phys. Rev. A 70 062311
[4] Bubhardt M and Freyberger M 2007 Phys. Rev. A 75 052101
[5] Cafaro C and Ali S A 2008 EJTP 5 139
[6] Cafaro C and Ali S A 2007 Physica D 234 70
[7] Cafaro C and Ali S A 2008 *Physica A* **387** 6876
[8] Ali S A, Cafaro C, Kim D H, Mancini S 2010 *Physica A* **389** 3117
[9] Cafaro C and Mancini S 2010 *Phys. Scr.* **82**, 035007
[10] Cafaro C, Giffin A, Ali S A and Kim D H 2010 *Appl. Math. Comput.* **217** 2944
[11] Harshman N L and Hutton G 2007 *Preprint* arXiv:quant-ph/0710.5776
[12] Wang J, Law C K and Chu M C 2006 *Phys. Rev. A* **73** 034302
[13] Rozanov Y A 1997 *Probability Theory: A Concise Course* (New York, NY: Dover Publications)
[14] Caticha A 2002 *Bayesian Inference and Maximum Entropy Methods in Science and Engineering* ed Fry R L (*AIP Conf. Proc.* **617** 302)
[15] De Felice F and Clarke J S 1990 *Relativity on Curved Manifolds* (Cambridge, UK: Cambridge University Press)
[16] Mishima K, Hayashi M and Lin S H 2004 *Phys. Lett. A* **333** 371
[17] Caticha A 2001 *Maximum Entropy and Bayesian Methods in Science and Engineering* ed. Mohammad-Djafari A (*AIP Conf. Proc.* **568** 72)
[18] Cafaro C 2008 *The Information Geometry of Chaos* (SUNY at Albany, NY: PhD Thesis)
[19] Cafaro C 2009 *Chaos, Solitons & Fractals* **41** 886
[20] Amari S and Nagaoka H 2000 *Methods of Information Geometry* (Oxford, UK: Oxford University Press)
[21] Hunt M *Lecture notes on Nuclear Physics II* (available online at http://www.physics.gla.ac.uk)
[22] Misner C W, Thorne K S and Wheeler J A 1973 *Gravitation* (San Francisco, CA: Freeman & Co.)
[23] Do Carmo M P 1992 *Riemannian Geometry* (Boston, MA: Birkhauser)