Are Erdős- Rényi Random Graphs Topologically Random?

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March 3, 2017

Abstract

No.

1 Introduction

The random graph is a widely applicable construct of study and analysis. Paul Erdős and Alfréd Rényi first proposed a random graph in the late 1950’s \cite{1}, as a graph where all edges are chosen based on a uniform probability of appearing in the graph. Since then, the topic of graphs and uses of random graphs has exploded \cite{2}. It was later suggested that these random graphs are the “average” form of graphs \cite{3}, however they have generally failed at capturing the characteristics of topologies ubiquitous in nature \cite{4}. The reason for this failure has until now remained enigmatic. Smith & Escudero recently demonstrated that the topological characteristics of random graphs had a smaller variance than other models and real EEG brain networks at all densities \cite{5}. From this they conjectured that E-R random networks were not topologically random. Here, we show that Erdős-Rényi (E-R) random graphs on $n$ nodes and $m$ edges are random only in the sense of binary vector combinatorics and not in the sense of graph isomorphism. Thus, indeed, they cannot be said to be the topologically “average” graph. Future work will begin to uncover the extent to which these so-called random graphs deviate from the true topological average.

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2 Erdös- Rényi random graphs are not topologically random

Let $G(n,m)$ denote the E-R random graph ensemble of size $n$ with $m$ uniformly distributed edges. Here, all possible combinations of $m$ edges are equally likely to occur.

**Definition 1** (Node switching). A node switching switches the positions of two nodes, $i$ and $j$, in the adjacency matrix, $A$, of a graph which is done by simply switching the $i$th and $j$th rows and columns of $A$. We thus obtain a new adjacency matrix, $\bar{A}$, which is topologically equivalent (i.e. isomorphic) to $A$.

**Definition 2** (Redundant node switching). A redundant node switching of non-isolated nodes $i$ and $j$ is a node switching for which $\bar{A} = A$ and we say $i$ and $j$ are 'symmetric' within the graph. That is, for any node $k \in V$, $(i,k) \in E \iff (j,k) \in E$, so $i$ and $j$ are adjacent to all the same nodes in $G$.

**Lemma 1.** The following exists for all $n > 4$ and $1 < m < n(n-2)/2$:

(a) A graph, $G = (V,E)$, with $n$ vertices and $m$ edges with at least one redundant node switching

(b) A graph, $G' = (V',E')$, with $n$ vertices and $m$ edges with no redundant node switchings

**Proof.** (a) Let $i$ and $j$ be two distinct vertices in $G$. We construct an increasing density quasi-star graph in the following manner. Take node $i$ and one by one add edges adjacent to $i$ and all other nodes in the graph to make a star graph with central node $i$. At each step, for $m > 1$, we have at least one redundant node switching (in fact these accumulate since every node that gets attached to $i$ has redundant node switching with all other nodes except $i$). Now take node $j$ and repeat this process. Again, at each step a redundant node switching exists. At the first step the first new node with an adjacent edge to $j$ has a redundant node switching with $j$ (in fact all other nodes still have redundant node switchings too except when $n = 4$ and there is only one other node). At the second step, now the two nodes sharing edges with $j$ have redundant node switchings. Once $i$ and $j$ have degree $n-1$, i.e. all possible adjacent edges, then $\{i,j\}$ is a redundant node switching since $i$ and $j$ are connected to exactly the same nodes.
After this, in fact, one can then add edges in whichever manner one chooses- \( \{i, j\} \) will always be a redundant node switching.

(b) For \( m = 2 \) we simply place the edges such that for any 4 distinct nodes, \( i, j, k, l, (i, j), (k, l) \in \mathcal{E} \). For \( m = 3 \) the path of length 3 has no redundant node switchings. We then begin to construct an increasing density ring lattice, edge by edge, by arranging the nodes in a circle (with the path of length 3 such that the node attached to both other nodes is in the middle of the two). We go around the nodes in a clockwise direction, adding another edge to each node in turn made adjacent to the next (clockwise) not adjacent node. All such graphs for \( m < n(n-2)/2 \) (i.e. until we begin to get nodes of degree \( n-1 \)) contain no redundant node switchings since the nodes are necessarily connected ‘out of step’.

For directed graphs, we simply do the above processes for all outgoing edges and then repeat for all in-going edges and the result remains.

**Theorem 1.** \( G(n, m) \), for \( n > 4 \) and \( 1 < m < n(n-2)/2 \) is not a topologically random graph ensemble in the sense that resulting samples of graphs from \( G(n, m) \) are not uniformly distributed between the set of graph isomorphic classes.

**Proof.** We shall first consider undirected random graphs from which the directed case follows arbitrarily. We denote by \( \Phi \) the set of all binary vectors of length \( n(n-1)/2 \) with \( m \) non-zero entries. Since \( n(n-1)/2 \) is the total number of possible edges for a graph of size \( n \), \( \Phi \) is equivalent to the set of graphs from which \( G(n, m) \) chooses uniformly at random, see Fig. 1 (a).

Now we consider a node switching of some nodes \( i \) and \( j \). There are exactly \( \binom{n}{2} \) possible node switchings which can be performed for a graph with \( n \) nodes. Importantly, the number of permutations, \( \phi \in \Phi \), one can construct for a graph with redundant node switchings is clearly less than the number one can construct for a graph without redundant node switchings since a node switching provides the same permutation \( \phi \in \Phi \) where as a non-redundant node switching provides two separate permutation in \( \Phi \). Since constructing a graph with \( n \) nodes and \( m \) edges with redundant node switchings and without node switchings is always possible, from lemma 1, it follows that the E-R random graph is not topologically uniformly random-the probability of obtaining a graph without redundant node switchings will always be larger than the probability of obtaining a graph with redundant
node switchings. For a directed graph we take $\Phi$ as the entire non-diagonal array of entries in the adjacency matrix and the same result follows.

As an illustration of this result, we shall consider a simple case which has immediate relevance to fundamental network science concepts: $n = 4$, $m = 3$. Here, the E-R random graph construction chooses with equal probability between $\binom{6}{3} = 20$ permutations. Now there are three distinct possible isomorphisms for graphs with 4 nodes and 3 edges: the triangle (relevant to network segregation), the three-pronged star (relevant to network irregularity) and the path of length 3, Fig. 1(b). However, as Table 1 shows, the probability of obtaining a path in $G(4,3)$ is $12/20 = 3/5$, whereas the probability of obtaining the triangle is $1/5$ leaving also $1/5$ for the star.

![Figure 1: (a) Mapping edges from the adjacency matrix to a vector. (b) The three possible topologies with $n$ nodes and $m$ edges.](image)

References

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Table 1: All equally likely graphs from random edge selection for 4 nodes and 3 edges and the corresponding unequally distributed topologies

| Binary vector | Topology | Binary vector | Topology |
|---------------|----------|---------------|----------|
| 111000        | Star     | 011100        | Path     |
| 110100        | Triangle | 011010        | Path     |
| 110010        | Path     | 011001        | Triangle |
| 110001        | Path     | 010110        | Path     |
| 101100        | Path     | 010101        | Star     |
| 101010        | Triangle | 010011        | Path     |
| 101001        | Path     | 001110        | Path     |
| 100110        | Star     | 001101        | Path     |
| 100101        | Path     | 001011        | Star     |
| 100011        | Path     | 000111        | Triangle |

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