CYCLIC COHOMOLOGY AND CHERN CONNES PAIRING OF SOME CROSSED PRODUCT ALGEBRAS

SAFDAR QUDDUS

Abstract. We compute the cyclic and Hochschild cohomology groups for the algebras $A^{alg}_\theta \rtimes \mathbb{Z}_3$, $A^{alg}_\theta \rtimes \mathbb{Z}_4$ and $A^{alg}_\theta \rtimes \mathbb{Z}_6$. We also compute the partial Chern-Connes index table for each of these algebras.

0. Introduction

The homological properties of noncommutative algebras have interested several mathematicians in recent years. In the articles [C] and [Y], the (co)homology groups of smooth algebras having a $C^*$-algebra structure are studied. The classical noncommutative algebras have been studied by Alev and Lambre [AL], Baudry [B], Fryer [F], Berest et al. [BRT] and Quddus ([Q1] and [Q2]).

For given $\theta \notin \mathbb{Q}$, we associate the algebraic noncommutative torus as the algebra $A^{alg}_\theta$ defined to be

$$A^{alg}_\theta := \left\{ a = \sum_{(n,m) \in \mathbb{Z}^2} a_{n,m} U^n_1 U^m_2 | a_{n,m} = 0 \text{ for all but finitely many } (n,m) \right\},$$

where $U_1$ and $U_2$ are unitary generators satisfying $U_2 U_1 = \lambda U_1 U_2$, $\lambda = e^{2\pi i \theta}$.

The group $SL(2, \mathbb{Z})$ has the following action on $A^{alg}_\theta$. An element

$$g = \begin{bmatrix} g_{1,1} & g_{1,2} \\ g_{2,1} & g_{2,2} \end{bmatrix} \in SL(2, \mathbb{Z})$$

acts on the generators $U_1$ and $U_2$ as described below:

$$g \cdot U_1 = e^{(\pi i g_{1,1}, g_{2,1})\theta} U_1^{g_{1,1}} U_2^{-g_{2,1}} \text{ and } g \cdot U_2 = e^{(\pi i g_{1,2}, g_{2,2})\theta} U_1^{g_{1,2}} U_2^{g_{2,2}}.$$

The algebra $A^{alg}_\theta$ and associated algebras has been studied in several articles. While the authors of [B] and [O] computed some of the Hochschild homology groups of its $\mathbb{Z}_2, \mathbb{Z}_3, \mathbb{Z}_4$ and $\mathbb{Z}_6$ crossed products, these groups were completely known in [Q1]. In the paper [BRT] the authors calculated the Picard group and the Morita equivalence classes of $A^{alg}_\theta$.

In this article we compute the Hochschild and cyclic cohomology groups of the crossed product algebras $A^{alg}_\theta \rtimes \mathbb{Z}_3$, $A^{alg}_\theta \rtimes \mathbb{Z}_4$ and $A^{alg}_\theta \rtimes \mathbb{Z}_6$ and thereafter we present the Chern-Connes index table by pairing the known projections of each of these algebras with the cyclic cocycles calculated in this article. This article hence completes the Hochschild and cyclic cohomological description of the crossed product algebras obtained by the action of the discrete subgroups of $SL(2, \mathbb{Z})$ on the algebraic noncommutative torus algebra. In this article we confine ourselves to the notations used in [Q1] and [Q2].
1. Statements

The following are the statements of the theorems proved in this article.

**THEOREM 1.1.** The Hochschild cohomology of the crossed product algebras are as follows:

\[
H^0(A^\text{alg}_\theta \rtimes \Gamma, (A^\text{alg}_\theta \times \Gamma)^*) \cong \begin{cases} 
\mathbb{C}^7 & \text{for } \Gamma = \mathbb{Z}_3 \\
\mathbb{C}^8 & \text{for } \Gamma = \mathbb{Z}_4 \\
\mathbb{C}^9 & \text{for } \Gamma = \mathbb{Z}_6.
\end{cases}
\]

\[H^1(A^\text{alg}_\theta \rtimes \Gamma, (A^\text{alg}_\theta \times \Gamma)^*) \cong 0 \text{ for the finite subgroups } \Gamma = \mathbb{Z}_3, \mathbb{Z}_4 \text{ and } \mathbb{Z}_6.\]

\[H^2(A^\text{alg}_\theta \rtimes \Gamma, (A^\text{alg}_\theta \times \Gamma)^*) \cong \mathbb{C} \text{ for the finite subgroups } \Gamma = \mathbb{Z}_3, \mathbb{Z}_4 \text{ and } \mathbb{Z}_6.\]

\[H^k(A^\text{alg}_\theta \rtimes \Gamma, (A^\text{alg}_\theta \times \Gamma)^*) \cong 0 \text{ for all } k > 3 \text{ and the finite subgroups } \Gamma = \mathbb{Z}_3, \mathbb{Z}_4 \text{ and } \mathbb{Z}_6.\]

**THEOREM 1.2.** The periodic cyclic cohomology groups are as follows:

\[H^\text{even}(A^\text{alg}_\theta \rtimes \Gamma) \cong \begin{cases} 
\mathbb{C}^8 & \text{for } \Gamma = \mathbb{Z}_3 \\
\mathbb{C}^9 & \text{for } \Gamma = \mathbb{Z}_4 \\
\mathbb{C}^{10} & \text{for } \Gamma = \mathbb{Z}_6.
\end{cases}\]

\[H^\text{odd}(A^\text{alg}_\theta \rtimes \Gamma) \cong 0 \text{ for the finite subgroups } \Gamma = \mathbb{Z}_3, \mathbb{Z}_4 \text{ and } \mathbb{Z}_6.\]

**THEOREM 1.3.** Let \(\zeta = e^{\frac{2\pi i}{6}}\), the following are the Chern-Connes index tables for the respective crossed product algebras.

(a) For \(A^\text{alg}_6 \rtimes \mathbb{Z}_3\):

\[
\begin{array}{|c|c|c|c|c|c|c|c|}
\hline
S & S\xi_{0,0} & S\xi_{0,0}^2 & S\xi_{0,1} & S\xi_{0,1} & S\xi_{0,-1} & S\varphi \\
\hline
1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
p^0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & 0 & 0 & 0 \\
\hline
p^1 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & 0 & 0 & 0 \\
\hline
q^0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & 0 & \sqrt{\frac{3}{\zeta^2}} & 0 \\
\hline
q^1 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & 0 & 0 & 0 \\
\hline
r^0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & 0 & \sqrt{\frac{3}{\zeta^2}} & 0 \\
\hline
\end{array}
\]

(b) For \(A^\text{alg}_9 \rtimes \mathbb{Z}_4\):

\[
\begin{array}{|c|c|c|c|c|c|c|c|}
\hline
S & S\xi_{1,0} & S\xi_{0,0} & S\xi_{0,1} & S\xi_{0,1} & S\xi_{0,-1} & S\varphi \\
\hline
1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
p^0 & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & 0 & 0 & 0 & 0 \\
\hline
p^1 & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & 0 & 0 & 0 & 0 \\
\hline
q^0 & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & 0 & 0 & \sqrt{\frac{3}{\zeta^2}} & 0 \\
\hline
q^1 & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & 0 & 0 & 0 & 0 \\
\hline
q^2 & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & 0 & 0 & \sqrt{\frac{3}{\zeta^2}} & 0 \\
\hline
r^0 & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & 0 & 0 & 0 & 0 \\
\hline
\end{array}
\]
(c) For $A_θ^{alg} \rtimes Z_6$:

|   | $Sτ$ | $SD_{0,0}$ | $S(D_{1,0} + λ\sqrt{A}D_{1,0} + \sqrt{λ}D_{1,1})$ | $S(ε_{0,1}^ω + ε_{0,-1}^ω)$ | $Sε_{0,0}^ω$ | $S(ε_{0,1}^ω + ε_{0,-1}^ω)$ | $Sε_{0,0}^2$ | $SG_{0,0}^{−ω}$ | $SG_{0,0}^{−ω}$ | $φ$ |
|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $p_0^β$ | $\frac{1}{3}$ | $\frac{1}{3}$ | 0 | 0 | $\frac{1}{3}$ | 0 | $\frac{1}{3}$ | $\frac{1}{3}$ | $\frac{1}{3}$ | 0 |
| $p_1^β$ | $\frac{1}{3}$ | $\frac{1}{3}$ | 0 | 0 | $\frac{1}{3}$ | 0 | $\frac{1}{3}$ | $\frac{1}{3}$ | $\frac{1}{3}$ | 0 |
| $p_2^β$ | $\frac{1}{3}$ | $−\frac{1}{3}$ | 0 | 0 | $−\frac{1}{3}$ | 0 | $\frac{1}{3}$ | $\frac{1}{3}$ | $−\frac{1}{3}$ | 0 |
| $p_3^β$ | $\frac{1}{3}$ | $\frac{1}{3}$ | 0 | 0 | $\frac{1}{3}$ | 0 | $\frac{1}{3}$ | $−\frac{1}{3}$ | $\frac{1}{3}$ | 0 |
| $p_4^β$ | $\frac{1}{3}$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $q_0^β$ | $\frac{1}{3}$ | 0 | 0 | $\frac{1}{3}$ | 0 | $\frac{1}{3}$ | 0 | 0 | 0 | 0 |
| $q_1^β$ | $\frac{1}{3}$ | 0 | 0 | $−\frac{1}{3}$ | 0 | $−\frac{1}{3}$ | 0 | 0 | 0 | 0 |
| $r^β$ | $\frac{1}{3}$ | 0 | $−\frac{λ\sqrt{λ}}{2}$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
2. Hochschild Cohomology

We note that the dual of the algebraic noncommutative torus $A^\text{alg}_\theta$ is

$$A^\text{alg*}_\theta = \left\{ a \mid a = \sum_{(n,m) \in \mathbb{Z}^2} a_{n,m} U_1^n U_2^m \right\}.$$  

For $g \in \Gamma$, $gA^\text{alg}_\theta$ is the $g$-twisted $A^\text{alg}_\theta$ bimodule, it consists of elements of $A^\text{alg*}_\theta$ with the following twisted $A^\text{alg}_\theta$ bimodule structure. For $a \in gA^\text{alg*}_\theta$ and $\alpha \in A^\text{alg}_\theta$,

$$\alpha \cdot a = (g \cdot \alpha) a \text{ and } a \cdot \alpha = a \alpha.$$  

We outline the procedure to calculate the Hochschild cohomology groups, using the paracyclic decomposition technique [GJ, Proposition 4.6] we have the following decomposition

$$H^* (A^\text{alg}_\theta \rtimes \Gamma, (A^\text{alg}_\theta \rtimes \Gamma)^*) = \bigoplus_{g \in \Gamma} H^* (A^\text{alg}_\theta, gA^\text{alg*}_\theta \Gamma).$$  

Using the well-known bimodule resolution

$$0 \rightarrow (A^\text{alg}_\theta)^e \rightarrow (A^\text{alg}_\theta)^e \oplus (A^\text{alg}_\theta)^e \rightarrow (A^\text{alg}_\theta)^e$$  

for $A^\text{alg}_\theta$ with maps $1 \mapsto (U_2 \otimes 1 - \lambda U_2 \otimes 1, 1 \otimes U_1 - 1 \otimes U_1)$ and $(1, 0) \mapsto (U_1 \otimes 1 - 1 \otimes U_1)$ and $(0, 1) \mapsto (U_2 \otimes 1 - 1 \otimes U_2)$. Hence for any bimodule $M$ of $A^\text{alg}_\theta$, $H^* (A^\text{alg}_\theta, M)$ is computed from the following complex

$$M \rightarrow M \oplus M \rightarrow M \rightarrow 0$$  

in which the maps are $m \mapsto (U_1 m - m U_1, U_2 m - m U_2)$ and $(m_1, m_2) \mapsto (U_2 m_1 - \lambda m_1 U_2 - \lambda U_1 m_2 - m_2 U_1)$. To calculate the group $H^* (A^\text{alg}_\theta, gA^\text{alg*}_\theta)$ we use the modified Connes resolution for the algebra $A^\text{alg}_\theta$. We refer [Q1, page 326] for detailed discussion on the modified Connes resolution for the algebra $A^\text{alg}_\theta$. To locate the $\Gamma$ invariant -cocycles in $H^* (A^\text{alg}_\theta, gA^\text{alg*}_\theta)$ we push a cocycle into the bar Hochschild cohomology complex and after the $\Gamma$ action on it, we pull it back onto the Connes complex using the chain homotopy maps calculated explicitly in [Q1, page 329] and [C, page 134]. A comparison between the two -cocycles on the Connes complex will determine invariance.

2.1. The Hochschild cohomology groups $H^0 (A^\text{alg}_\theta \rtimes \Gamma, (A^\text{alg}_\theta \rtimes \Gamma)^*)$.

The case $\Gamma = \mathbb{Z}_3$.

The group $\mathbb{Z}_3$ is embedded in $SL(2, \mathbb{Z})$ through its generator $\omega = \left[ \begin{array}{cc} 0 & 1 \\ -1 & -1 \end{array} \right] \in SL(2, \mathbb{Z})$. The generator acts on $A^\text{alg}_\theta$ in the following way

$$U_1 \mapsto U_1^{-1}, U_2 \mapsto \frac{U_1 U_2^{-1}}{\sqrt{\lambda}}.$$  

The case $g = 1$ has been considered in the article [Q2], wherein the author calculated the Hochschild cohomology group $H^0 (A^\text{alg}_\theta, A^\text{alg*}_\theta)$. The said group is a one-dimensional group generated by the cocycle $\tau$ [Q2, Section 2]. To check the $\mathbb{Z}_3$ invariance of $\tau$ we use the method described above. We push the 0-cocycle $\tau$ to the bar complex and pull it back to the Connes complex after the action of the group $\mathbb{Z}_3$. We notice that in this case the cocycle $\tau$ is invariant under the $\mathbb{Z}_3$ action and hence $H^0 (A^\text{alg}_\theta, A^\text{alg*}_\theta)^{\mathbb{Z}_3} \cong \mathbb{C}$. It is evident that $H^0 (A^\text{alg}_\theta, A^\text{alg*}_\theta)^{\mathbb{Z}_3} \cong H^0 (A^\text{alg}_\theta, A^\text{alg*}_\theta)^{\mathbb{Z}_6} \cong \mathbb{C}$.
The cases $g = \omega$ and $g = \omega^2$ are similar and we shall consider one of them for computational purpose (say $g = \omega$). We notice that following is the Hochschild cohomology complex for $g = \omega$

$$\omega A^\text{alg*}_g \xrightarrow{\omega \alpha_1} \omega A^\text{alg*}_g \oplus \omega A^\text{alg*}_g \xrightarrow{\omega \alpha_2} \omega A^\text{alg*}_g \to 0,$$

wherein with the bimodule structure of $\omega A^\text{alg*}_g$, the maps are as follows:

$$\omega \alpha_1(\varphi) = (U_2^{-1} \varphi - \varphi U_1, \frac{U_1 U_2^{-1}}{\sqrt{\lambda}} \varphi - \varphi U_2); \omega \alpha_2(\varphi_1, \varphi_2) = \frac{U_1 U_2^{-1}}{\sqrt{\lambda}} \varphi_1 - \lambda \varphi_1 U_2 - \lambda U_2^{-1} \varphi_2 + \varphi_2 U_1.$$

Hence the $\omega$ twisted Hochschild cohomology group $H^0(A^\text{alg}_g, \omega A^\text{alg*}_g)$ is the group $\text{ker}(\omega \alpha_1)$. The group $\text{ker}(\omega \alpha_1)$ is the set of all elements $\varphi \in A^\text{alg*}_g$ such that the following relations are satisfied:

$$U_2^{-1} \varphi = \varphi U_1 \text{ and } \frac{U_1 U_2^{-1}}{\sqrt{\lambda}} \varphi = \varphi U_2.$$

Hence, we deduce that for $\varphi$ an element of $\text{ker}(\omega \alpha_1)$, its coefficients must satisfy the following:

$$\varphi_{n,m+1} = \lambda^{m+n} \varphi_{n-1,m} \text{ and } \varphi_{n-1,m+1} = \lambda^{n-\frac{1}{2}} \varphi_{n,m-1}.$$

**Lemma 2.1.** The $\omega$ twisted zeroth Hochschild cohomology group $H^0(A^\text{alg}_g, \omega A^\text{alg*}_g)$ is generated by the coefficients $\varphi_{0,0}$ (generates the cocycle $E^\omega_{0,0}$), $\varphi_{0,1}$ (generates the cocycle $E^\omega_{0,1}$) and $\varphi_{0,-1}$ (generates the cocycle $E^\omega_{0,-1}$). And satisfies the following relations:

For $m - n \equiv 0 \pmod{3}$, $\varphi_{n,m} = \lambda^{\frac{n^2 + n^2 + 4mn}{6}} \varphi_{0,0}$ and for $m - n \equiv \pm 1 \pmod{3}$, $\varphi_{n,m} = \lambda^{\frac{n^2 + n^2 + 4mn - 1}{6}} \varphi_{0,\pm 1}$.

**Proof.** Using the relation $\varphi_{n,m+1} = \lambda^{n+m} \varphi_{n-1,m}$ we infer that the lattice points $(n, m)$ and $(n - 1, m - 1)$ belong to the same coboundary hence represent the same cocycle element. Furthermore the second relation $\varphi_{n-1,m+1} = \lambda^{n-\frac{1}{2}} \varphi_{n,m-1}$ relates $\varphi_{0,k}$ with $\varphi_{-1,k+2}$. Thus, we conclude that the group $H^0(A^\text{alg}_g, \omega A^\text{alg*}_g)$ is generated by the coefficients $\varphi_{0,0}$, $\varphi_{0,1}$ and $\varphi_{0,-1}$. This can be pictorially understood through the following diagram.
Using the two relations, we easily see that for the case \( m - n \equiv 0 \pmod{3} \), \( \phi_{0,3k} = \lambda^{\frac{3k^2}{2}} \phi_{0,0} \). Further we notice that

\[
\phi_{r,3k+1} = \lambda(3k^2 + 3k + 1) + \cdots (3k+2r-1) \phi_{0,3k} = \lambda(3k^2 + r^2) \lambda^{\frac{3k^2}{2}} \phi_{0,0} = \lambda \left( 2r^2 + 6kr + 3k^2 + 2r + 3 \right) \phi_{0,0}.
\]

Hence, we have for \( m - n \equiv 0 \pmod{3} \), \( \varphi_{n,m} = \lambda^{\frac{m^2 + n^2 + 4mn}{6}} \varphi_{0,0} \). Now we consider the case \( m - n \equiv 1 \pmod{3} \). Using the relations \( \varphi_{n-1,m+1} = \lambda^{\frac{1}{2}} \varphi_{n,m-1} \) and \( \varphi_{n,m+1} = \lambda^{m+n} \varphi_{n-1,m} \) successively we get that

\[
\varphi_{0,3k+1} = \lambda^{\frac{3k^2}{2} + k} \varphi_{0,1}.
\]

As in the previous case we notice that

\[
\varphi_{r,3k+r+1} = \lambda(3k^2 + 3k + 1) + \cdots (3k+2r) \phi_{0,3k+1} = \lambda(3k^2 + r^2 + 1) \phi_{0,3k+1} = \lambda \left( 2r^2 + 6kr + 3k^2 + 2r + 2k \right) \phi_{0,1}.
\]

Hence, we have that for \( m - n \equiv 1 \pmod{3} \), \( \varphi_{n,m} = \lambda^{\frac{m^2 + n^2 + 4mn - 1}{6}} \varphi_{0,1} \). A similar computation for the final case \( m - n \equiv -1 \pmod{3} \) completes the proof. \( \square \)

**Lemma 2.2.** \( H^0(\mathcal{A}_g^{alg}, \omega_{\mathcal{A}_g^{alg}})^{\mathbb{Z}_3} \simeq \mathbb{C}^3 \).

**Proof.** The entry \( \varphi_{n,m}U_1^n U_2^m \) under the action of \( \omega \) transforms to \( \varphi_{n,m}(U_2^{-n})(\frac{U_1}{\sqrt{\lambda}})^m \).

Using the fact that \( (U_1 U_2^{-1})^m = \lambda^{\frac{m(m-1)}{2}} U_1^m U_2^{-m} \). We have
\[ \varphi_{n,m}(U_2^{-n})(U_1U_2^{-1})^m = \varphi_{n,m} \frac{m}{2} U_2^{-n}(U_1U_2^{-1})^m = \varphi_{n,m} \frac{m(m-1)}{2} \lambda^\frac{m}{2} U_2^{-n}U_1^mU_2^{-m} = \varphi_{n,m} \frac{m^2}{2} \lambda^{-nm}U_1^mU_2^{-n-m}. \]

We note that \((-n - m - m) - (m - n) = -3m \equiv 0 \pmod{3}\), hence the coefficient of \(\omega \cdot \varphi_{n,m}U_1^mU_2^m\) is generated by the same generator (\(\varphi_{0,0}\) or \(\varphi_{0,\pm 1}\)) which generates \(\varphi_{n,m}\).

Let us consider the case \(m - n \equiv 0 \pmod{3}\), in this case we have

\[
\varphi_{n,m} \lambda^{-\frac{m^2}{2} \lambda^{-nm}U_1^mU_2^{-n-m}} = \lambda^{n^2+m^2+4nm} \lambda^{-\frac{m^2}{2} \lambda^{-nm} \varphi_{0,0}U_1^mU_2^{-n-m}} = \\
\lambda^{n^2-2m^2-2nm} \varphi_{0,0}U_1^mU_2^{-n-m} = \lambda^{m^2+(-n-m)^2+4m(-n-m)} \varphi_{0,0}U_1^mU_2^{-n-m} = \\
\varphi_{m,-n-m}U_1^mU_2^{-n-m}
\]

Hence we see that for \(m - n \equiv 0 \pmod{3}\), \(\omega \cdot \varphi_{n,m}U_1^mU_2^m = \varphi_{m,-n-m}U_1^mU_2^{-n-m}\) and therefore is invariant under the action of \(\omega\), the generator of the group \(\mathbb{Z}_3\). For the cases \(m - n \equiv \pm 1 \pmod{3}\), the proof is similar and the key to the invariance is the polynomial \(n^2 + m^2 + 4mn\), which under the transformation \(n \mapsto m\) and \(m \mapsto -n - m\) satisfies the relation

\[
n^2 + m^2 + 4mn - 3m^2 - 6nm = m^2 + (-n - m)^2 + 4m(-n - m).
\]

\(\square\)

**THEOREM 2.3.** \(H^0(\mathcal{A}_{a^g} \rtimes \mathbb{Z}_3, (\mathcal{A}_{a^g} \rtimes \mathbb{Z}_3)^*) \cong \mathbb{C}^7.\)

**Proof.** Using the paracyclic decomposition of the group \(H^0(\mathcal{A}_{a^g} \rtimes \mathbb{Z}_3, (\mathcal{A}_{a^g} \rtimes \mathbb{Z}_3)^*)\) we have the following relation

\[
H^0(\mathcal{A}_{a^g} \rtimes \mathbb{Z}_3, (\mathcal{A}_{a^g} \rtimes \mathbb{Z}_3)^*) = H^0(\mathcal{A}_{a^g}, \omega^g \mathcal{A}_{a^g})^\mathbb{Z}_3 \oplus H^0(\mathcal{A}_{a^g}, \omega^2 \mathcal{A}_{a^g})^\mathbb{Z}_3 \oplus H^0(\mathcal{A}_{a^g}, \mathcal{A}_{a^g}^{alg*})^\mathbb{Z}_3 = \\
\mathbb{C}^3 \oplus \mathbb{C}^3 \oplus \mathbb{C} = \mathbb{C}^7.
\]

Hence proved. \(\square\)

The case \(\Gamma = \mathbb{Z}_4\).

The group \(\mathbb{Z}_4\) is embedded in \(SL(2,\mathbb{Z})\) through its generator \(i = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \in SL(2,\mathbb{Z}).\)

The generator acts on \(\mathcal{A}_{a^g}\) in the following way

\[ U_1 \mapsto U_2^{-1}, U_2 \mapsto U_1. \]

Below is the paracyclic decomposition of the cohomology group \(H^0(\mathcal{A}_{a^g} \rtimes \mathbb{Z}_4, (\mathcal{A}_{a^g} \rtimes \mathbb{Z}_4)^*)\);

\[
H^0(\mathcal{A}_{a^g} \rtimes \mathbb{Z}_4, (\mathcal{A}_{a^g} \rtimes \mathbb{Z}_4)^*) = \bigoplus_{g \in \mathbb{Z}_4} H^0(\mathcal{A}_{a^g}^{alg}, g \mathcal{A}_{a^g}^{alg*})^{\mathbb{Z}_4}.
\]

We recall that the zeroth Hochschild cohomology group \(H^0(\mathcal{A}_{a^g}^{alg}, \mathcal{A}_{a^g}^{alg*})\) is one dimensional and the generator \(\tau\) is invariant under \(\mathbb{Z}_4\) action. The \(-1\) twisted zeroth cohomology group \(H^*(\mathcal{A}_{a^g}^{alg}, -1 \mathcal{A}_{a^g}^{alg*})\) has been calculated in \([Q2]\). For \(g = \pm i\), we notice that the \(i\) twisted Hochschild cohomology group \(H^*(\mathcal{A}_{a^g}^{alg}, i \mathcal{A}_{a^g}^{alg})\) and the \(-i\) twisted Hochschild cohomology group \(H^*(\mathcal{A}_{a^g}^{alg}, -i \mathcal{A}_{a^g}^{alg})\) are isomorphic. Hence we consider the case \(g = i\) for our computational purpose.

For \(g = i\) the following is the Hochschild cohomology complex for the Connes resolution.
\[ i\Lambda^\text{alg}_\theta \to i\Lambda^\text{alg}_\theta \oplus i\Lambda^\text{alg}_\theta \to i\Lambda^\text{alg}_\theta \to 0, \]

where the cochain maps are as follows:

\[ i\alpha_1(\varphi) = (U_2^{-1}\varphi - \varphi U_1, U_1\varphi - \varphi U_2); \]
\[ i\alpha_2(\varphi_1, \varphi_2) = U_1\varphi_1 - \lambda\varphi_1 U_2 + \varphi_2 U_1 - \lambda U_2^{-1}\varphi_2. \]

Hence the twisted Hochschild cohomology group \( H^0(i\Lambda^\text{alg}_\theta, i\Lambda^\text{alg}_\theta) \) is the group \( \ker(i\alpha_1) \).

The group \( \ker(i\alpha_1) \) is the set of all entries \( \varphi \in i\Lambda^\text{alg}_\theta \) such that the following relations hold:

\[ U_2^{-1}\varphi = \varphi U_1 \text{ and } U_1\varphi = \varphi U_2. \]

Hence, we deduce that \( \varphi_{n,m+1} = \lambda^{m+n}\varphi_{n-1,m} \) and \( \varphi_{n-1,m} = \varphi_{n,m-1} \).

**Lemma 2.4.** The twisted zeroth Hochschild cohomology group \( H^0(i\Lambda^\text{alg}_\theta, i\Lambda^\text{alg}_\theta) \) is generated by the coefficients \( \varphi_{0,0} \) (generates the cocycle \( F^i_{0,0} \)) and \( \varphi_{0,1} \) (generates the cocycle \( F^i_{0,1} \)).

And satisfies the following relations:

For \( m + n \equiv 0 \pmod{2} \), \( \varphi_{n,m} = \lambda^{m^2 + n^2 + 2mn} \varphi_{0,0} \) and

for \( m + n \equiv 1 \pmod{2} \), \( \varphi_{n,m} = \lambda^{m^2 + n^2 + 2mn - 1} \varphi_{0,1} \).

**Proof.** The relation \( \varphi_{n,m+1} = \lambda^{n+m}\varphi_{n-1,m} \) relates all the lattice points \((n,m)\) with \((n-1,m-1)\). Furthermore, using the relation \( \varphi_{n-1,m} = \varphi_{n,m-1} \) we see that the lattice points \((n,m)\) with \((n-1,m+1)\) are generated by the same coefficient. It can be easily concluded that \( \varphi_{0,0} \) generates all the coefficients \( \varphi_{r,s} \) such that \( r + s \) is even and the coefficient \( \varphi_{0,1} \) generates all the coefficients \( \varphi_{r,s} \) such that \( r + s \) is odd. This can be pictorially realised by the following diagram.

Using the two relations, we easily see that \( \varphi_{0,2k} = \lambda^{k^2}\varphi_{0,0} \).
\[ \varphi_{r,2k+r} = \lambda^{(2k+1)+(2k+3)+\cdots+(2k+(2r-1))} \varphi_{0,2k} = \lambda^{(2r+1)} \lambda^{2k} \varphi_{0,0} = \lambda^{(r+k)^2} \varphi_{0,0}. \]

Hence we have proved the theorem for case \( m + n \equiv 0 \pmod{2} \). A similar argument for the case \( m + n \equiv 1 \pmod{2} \) completes the proof of the lemma. \( \square \)

**Lemma 2.5.** \( H^0(A^\text{alg}_1, i_A^\text{alg}_2)Z_4 \cong \mathbb{C}^2 \).

**Proof.** Equivalently, we need to show that the two cocycles \( i_{0,0}^j \) and \( i_{0,1}^j \) are \( Z_4 \) invariant. The entry \( \varphi_{nm} U_1^m U_2^n \) under the action of \( i \) transforms to \( \varphi_{nm} (U_2^{-n} U_1^m) \). Using the fact that \( U_2^{-n} U_1^m = \lambda^{-nm} U_1^m U_2^{-n} \). We notice that if \( m + n \equiv 0 \pmod{2} \) then \( m - n \equiv 0 \pmod{2} \) and similar is the case for \( m + n \equiv 1 \pmod{2} \). Hence it makes sense to compare the coefficient of \( i \cdot \varphi_{nm} U_1^m U_2^n \) with \( \varphi_{nm} \).

Using the result of previous lemma, for \( m + n \equiv 0 \pmod{2} \), we have

\[ \varphi_{nm} U_1^m U_2^n = \lambda^{-nm} \varphi_{nm} U_1^m U_2^n = \lambda^{-nm} \lambda^{2n^2 + 2nm} \varphi_{00} U_1^m U_2^n = \lambda^{\frac{n^2 + 2nm}{4}} \varphi_{00} U_1^m U_2^n = \varphi_{m-n} U_1^m U_2^n. \]

Hence we see that for \( m + n \equiv 0 \pmod{2} \), \( i \cdot \varphi_{nm} U_1^m U_2^n = \varphi_{m-n} U_1^m U_2^n \) and therefore \( i \) leaves the cocycle \( i_{0,0}^j \) invariant under its action. For the case \( m + n \equiv 1 \pmod{2} \), the proof is similar and the key to the invariance is the polynomial \( n^2 + m^2 + 2nm \), which is under the transformation \( n \mapsto m \) and \( m \mapsto -n \) satisfies the relation

\[ n^2 + m^2 + 2nm - 4nm = m^2 + (-n)^2 + 2m(-n). \]

\( \square \)

**Lemma 2.6.** \( H^0(A^\text{alg}_1, i_A^\text{alg}_2)Z_4 \cong \mathbb{C}^3. \)

**Proof.** The cocycles of \( H^0(A^\text{alg}_1, i_A^\text{alg}_2) \) are described in [Q2]. They are generated by the coefficients \( \varphi_{00}, \varphi_{10}, \varphi_{01} \) and \( \varphi_{11} \). A typical cocycle of \( H^0(A^\text{alg}_1, i_A^\text{alg}_2) \) is of the form

\[ \Phi = aD_{00} + bD_{01} + cD_{10} + dD_{11} \text{ where, } a, b, c, d \in \mathbb{C}. \]

For \( 0 \leq i, j \leq 1 \) the cocycles \( D_{i,j} \) are generated by the coefficients \( \varphi_{i,j} \). Consider the cocycle \( D_{00}, \) its entries are of the form \( \varphi_{2n,2m} \). These are generated by \( \varphi_{00} \) and satisfies the relation \( \varphi_{2n,2m} = \lambda^{2nm} \varphi_{00} \). By the following calculations we conclude that the cocycle \( D_{00} \) is invariant under the action of \( Z_4 \)

\[ i \cdot \varphi_{2n,2m} U_1^m U_2^n = \varphi_{2n,2m} U_1^m U_2^n = \varphi_{2n,2m} \lambda^{-4nm} U_1^m U_2^n = \lambda^{2nm} \lambda^{-4nm} \varphi_{00} U_1^{2m} U_2^n = \lambda^{-2nm} \varphi_{00} U_1^{2m} U_2^n = \varphi_{-2n,2m} U_1^{2m} U_2^n. \]

Similarly we consider the cocycle \( D_{11}, \) its entries satisfy the relation \( \varphi_{2n+1,2n+1} = \lambda^{2nm+n+m} \varphi_{11} \). We have the following relations

\[ i \cdot \varphi_{2n+1,2n+1} U_1^{2n+1} U_2^{2n+1} = \varphi_{2n+1,2n+1} U_1^{2n+1} U_2^{2n+1} = \varphi_{2n+1,2n+1} \lambda^{2nm+n+m} \lambda^{-2n-1} U_1^{2n+1} U_2^{2n+1} = \varphi_{1,1} \lambda^{2n-2nm-n} U_1^{2n+1} U_2^{2n+1} = \varphi_{1,1} \lambda^{2n-2nm-n} U_1^{2n+1} U_2^{2n+1} = \varphi_{2n+1,2n+1} \lambda^{2(-n-1)m+(-n-1)} U_1^{2n+1} U_2^{2n+1} = \varphi_{2n+1,2n+1} \lambda^{2(-n-1)m+(-n-1)} U_1^{2n+1} U_2^{2n+1}. \]
Hence the cocycle $D_{1,1}$ belongs to the group $H^0(\mathcal{A}_\theta^{alg}, -1\mathcal{A}_\theta^{alg*})\mathbb{Z}_4$.

We now observe the action of $i$ on the cocycle $D_{0,1}$, an entry $\varphi_{2n,2m+1}U_1^{2n}U_2^{2m+1}$ when acted by $i$ has the following transformations

$$i \cdot \varphi_{2n,2m+1}U_1^{2n}U_2^{2m+1} = \varphi_{2n+1,2m+1}U_1^{2n-1}U_2^{2m+1} = \varphi_{2n,2m+1}\lambda^{-2n}(2m+1)U_1^{2m+1}U_2^{2n} = \varphi_{0,1}\lambda^{-2n}mU_1^{2m+1}U_2^{2n} = \varphi_{0,1}\varphi_{2m+1,-2n}U_1^{2m+1}U_2^{2n}.$$ 

Similarly, when $i$ acts on the cocycle $D_{1,0}$, the entries of the form $\varphi_{2n+1,2m}U_1^{2n+1}U_2^{2m}$ have the following transformations

$$i \cdot \varphi_{2n+1,2m}U_1^{2n+1}U_2^{2m} = \varphi_{2n+1,2m}U_1^{2n}U_2^{2m-1} = \varphi_{2n+1,2m}\lambda^{-2n}(2m+1)U_1^{2m}U_2^{2m-1} = \varphi_{1,0}\lambda^{-2n}mU_1^{2m}U_2^{2m-1} = \varphi_{1,0}\varphi_{-2m,2n}U_1^{2m}U_2^{2n-1}.$$ 

Let us consider the action of $i$ on a typical cocycle $\Phi$. Then we have

$$i \cdot \Phi = a(i \cdot D_{0,0}) + b(i \cdot D_{0,1}) = c(i \cdot D_{1,0}) + d(i \cdot D_{1,1}) = aD_{0,0} + bD_{1,0} + cD_{0,1} + dD_{1,1}.$$ 

Hence, the $\mathbb{Z}_4$ invariant subspace of $H^0(\mathcal{A}_\theta^{alg}, -1\mathcal{A}_\theta^{alg*})$ is a three dimensional subspace. □

**THEOREM 2.7.** $H^0(\mathcal{A}_\theta^{alg} \rtimes \mathbb{Z}_4, (\mathcal{A}_\theta^{alg} \rtimes \mathbb{Z}_4)^*) \cong \mathbb{C}^8$.

**Proof.** Using the paracyclic decomposition of the group $H^0(\mathcal{A}_\theta^{alg} \rtimes \mathbb{Z}_4, (\mathcal{A}_\theta^{alg} \rtimes \mathbb{Z}_4)^*)$ we have the following relation

$$H^0(\mathcal{A}_\theta^{alg} \rtimes \mathbb{Z}_4, (\mathcal{A}_\theta^{alg} \rtimes \mathbb{Z}_4)^*) = H^0(\mathcal{A}_\theta, i\mathcal{A}_\theta^{alg*})\mathbb{Z}_4 \oplus H^0(\mathcal{A}_\theta, -i\mathcal{A}_\theta^{alg*})\mathbb{Z}_4 \oplus H^0(\mathcal{A}_\theta, \mathcal{A}_\theta^{alg*})\mathbb{Z}_4 = \mathbb{C}^2 \oplus \mathbb{C}^2 \oplus \mathbb{C}^3 \oplus \mathbb{C} = \mathbb{C}^8$$

□

The case $\Gamma = \mathbb{Z}_6$.

This group is generated in $SL(2, \mathbb{Z})$ through its generator $-\omega = \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix} \in SL(2, \mathbb{Z})$. The generator acts on $\mathcal{A}_\theta^{alg}$ in the following way

$$U_1 \mapsto U_2, U_2 \mapsto \frac{U_1^{-1}U_2}{\sqrt{\lambda}}.$$ 

We know that $H^0(\mathcal{A}_\theta^{alg}, \omega\mathcal{A}_\theta^{alg})$ also we notice that the $-\omega$ twisted Hochschild cohomology group $H^*(\mathcal{A}_\theta^{alg}, -\omega\mathcal{A}_\theta^{alg})$ and the $-\omega^2$ twisted Hochschild cohomology group $H^*(\mathcal{A}_\theta^{alg}, -\omega^2\mathcal{A}_\theta^{alg})$ are isomorphic. Hence we consider the case $g = -\omega$ for our computational purpose.

For $g = -\omega$ the following is the Hochschild cohomology complex for the Connes resolution. 

$$-\omega\mathcal{A}_\theta^{alg*} \xrightarrow{-\omega^1} -\omega\mathcal{A}_\theta^{alg*} \oplus -\omega\mathcal{A}_\theta^{alg*} \xrightarrow{-\omega^2} -\omega\mathcal{A}_\theta^{alg*} \rightarrow 0,$$

where the cochain maps are as follows:

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\[ -\omega \alpha_1(\varphi) = (U_2\varphi - \varphi U_1, U_{1}^{-1}U_2\varphi - \varphi U_2); \]

\[ -\omega \alpha_2(\varphi_1, \varphi_2) = \frac{U_1^{-1}U_2\varphi_1}{\sqrt{\lambda}} - \lambda \varphi_1 U_2 - \lambda U_2 \varphi_2 + \varphi_2 U_1. \]

Hence the \(-\omega\) twisted Hochschild cohomology group \(H^0(A_{\theta}^{alg}, -\omega A_{\theta}^{alg})\) is the group \(\ker(-\omega \alpha_1)\). The group \(\ker(-\omega \alpha_1)\) is the set of all entries \(\varphi \in A_{\theta}^{alg}\) such that the following relations hold:

\[ U_2 \varphi = \varphi U_1 \text{ and } \frac{U_1^{-1}U_2}{\sqrt{\lambda}} \varphi = \varphi U_2. \]

Hence, we deduce that \(\varphi_{n-1,m} = \lambda^n \varphi_{n,m-1}\) and \(\varphi_{n,m} = \lambda^{n+\frac{1}{2}} \varphi_{n+1,m-1}\).

**Lemma 2.8.** If \(\varphi\) is a cocycle of \(H^0(A_{\theta}^{alg}, -\omega A_{\theta}^{alg})\) then, \(\varphi\) is generated by the coefficient \(\varphi_{0,0}\). And for all \((n, m) \in \mathbb{Z}^2\), \(\varphi_{n,m} = \lambda^{-\frac{m^2+n^2}{2}} \varphi_{0,0}\).

**Proof.** We observe that all the coefficients \(\varphi_{r, r}\) are generated by \(\varphi_{0, r}\). Furthermore the relation \(\varphi_{n-1,m} = \lambda^n \varphi_{n,m-1}\) gives us that \(\varphi_{r, r}\) are generated by \(\varphi_{0, 0}\). Hence the \(-\omega\) twisted zero Hochschild cohomology group \(H^0(A_{\theta}^{alg}, -\omega A_{\theta}^{alg})\) is a one dimensional group generated by the coefficient \(\varphi_{0,0}\).

The second relation gives us that

\[ \varphi_{n, \bullet} = \lambda^{-\frac{n(n-1)}{2}} \varphi_{0, \bullet} = \lambda^{-\frac{n^2}{2}} \varphi_{0, \bullet}. \]

Similarly, using both the relations we deduce that \(\varphi_{\bullet, m} = \lambda^{-(m-1)+\frac{1}{2}} \varphi_{\bullet, m-1}\). Hence we have

\[ \varphi_{0, m} = \lambda^{-\frac{m(m-1)}{2}} \varphi_{0, 0} = \lambda^{\frac{m^2}{2}} \varphi_{0, 0}. \]

Therefore, \(\varphi_{n, m} = \lambda^{-\frac{n^2}{2}} \varphi_{0, m} = \lambda^{-\frac{n^2}{2}} \lambda^{-\frac{m^2}{2}} \varphi_{0, 0} = \lambda^{-\frac{n^2+m^2}{2}} \varphi_{0, 0}\).

**Lemma 2.9.** \(H^0(A_{\theta}^{alg}, -\omega A_{\theta}^{alg}) \mathbb{Z}_6 \cong \mathbb{C}\).

**Proof.** We need to show that the lone cocycle of the group \(H^0(A_{\theta}^{alg}, -\omega A_{\theta}^{alg})\) is invariant under the action of \(-\omega\). The cocycle \(G_{0,0}^{-\omega}\) generated by \(\varphi_{0, 0}\) as described in the lemma above has a typical entry of the form \(\varphi_{n, m} U_1^n U_2^m\). When \(-\omega\) acts on this entry we have the following

\[ -\omega \cdot \varphi_{n, m} U_1^n U_2^m = \varphi_{n, m} U_2^m (U_1^{-1}U_2)^m = \varphi_{n, m} \lambda^{-\frac{m^2+n^2}{2}} U_2^m(U_1^{-1}U_2)^m = \varphi_{n, m} \lambda^{-\frac{mn}{2}} \lambda^{-\frac{m(m-1)}{2}} U_2^n U_1^{-m} U_2^m = \varphi_{n, m} \lambda^{-\frac{m^2+n^2}{2}} U_2^m U_1^{-m} U_2^m = \varphi_{0, 0} \lambda^{-\frac{m^2+n^2}{2}} U_1^{-m} U_2^m = \varphi_{0, 0} \lambda^{-\frac{m^2+n^2}{2}} U_1^{-m} U_2^m. \]

Hence we see that the cocycle \(G_{0,0}^{-\omega}\) is invariant under the action of \(\mathbb{Z}_6\).

**Lemma 2.10.** \(H^0(A_{\theta}^{alg}, \omega A_{\theta}^{alg}) \mathbb{Z}_6 \cong \mathbb{C^2}\).

**Proof.** The actions of \(-\omega\) on the twisted zero Hochschild cocycles \(E_{0,0}^\omega, E_{0,1}^\omega\) and \(E_{0,-1}^\omega\) are described below.

We consider \(E_{0,0}^\omega\), a typical entry \(\varphi_{n, m} U_1^n U_2^m\) when acted by \(-\omega\) transforms to
Hence, we conclude that
\[
\varphi_{n,m}U_2^m\left(\frac{U_1^{-1}U_2}{\sqrt{\lambda}}\right)^m = \varphi_{n,m}\lambda^{-\frac{m}{2}}\frac{U_2^n(U_1^{-1}U_2)^m}{U_2^nU_1^{-m}U_2^m} = \varphi_{n,m}\lambda^{-\frac{m}{2}}\frac{mn(n+1)}{6}\lambda^{-nmU_1^{-m}U_2^m} = \\
\varphi_{0,0}\lambda^{n^2+m^2+4nm} \lambda^{-\frac{m}{2}}\lambda^{-nmU_1^{-m}U_2^m} = \\
\varphi_{0,0}\lambda^{n^2+m^2+4nm} \lambda^{-\frac{m}{2}}\lambda^{-nmU_1^{-m}U_2^m}.
\]

Similarly, it is trivial to figure that
\[
D \equiv (m + n - m - n) \equiv 0 \pmod{3}
\]
Hence, \(-\omega \cdot \varphi_{n,m}U_2^m\) is an entry in \(E_0^0\), so, \(-\omega \cdot \varphi_{n,m}U_2^0 = \varphi_{n,m}^0\).

Similarly for the cocycle \(E_0^1\) we have
\[
\varphi_{n,m}U_2^m\left(\frac{U_1^{-1}U_2}{\sqrt{\lambda}}\right)^m = \varphi_{n,m}\lambda^{-\frac{m}{2}}\frac{U_2^n(U_1^{-1}U_2)^m}{U_2^nU_1^{-m}U_2^m} = \\
\varphi_{n,m}\lambda^{-\frac{m}{2}}\lambda^{-nmU_1^{-m}U_2^m} = \\
\varphi_{0,1}\lambda^{n^2+m^2+4nm} \lambda^{-\frac{m}{2}}\lambda^{-nmU_1^{-m}U_2^m} = \\
\varphi_{-m,n+m}U_1^{-m}U_2^m.
\]

Unlike the previous case, for the cocycle \(E_0^1\) we have
\[
(n + m) - (m - n) = m + 2n = m - n + 3n \equiv m - n \pmod{3}
\]
Hence, \(-\omega \cdot \varphi_{n,m}U_2^0 = \varphi_{n,m}^0\).

Similarly, it is trivial to figure that \(-\omega \cdot \varphi_{n,m}U_2^1 = \varphi_{n,m}^1\). Hence, the two dimensional \(-\omega\) invariant subspace of \(H^0(\mathcal{A}_\theta^{alg}, \omega \mathcal{A}_\theta^{alg*})\) is generated by \(E_0^0\) and \(E_0^1\).

**Lemma 2.11.** \(H^0(\mathcal{A}_\theta^{alg}, \mathcal{A}_\theta^{alg*})\mathbb{Z}_6 \cong \mathbb{C}^2\).

**Proof.** The four \(-1\) twisted zero Hochschild cohomology cocycles are \(D_{0,0}, D_{0,1}, D_{1,0}\) and \(D_{1,1}\). Consider the action of \(-\omega\) on \(D_{0,0}\),
\[
-1 \cdot \varphi_{2n,2m}U_1^{-2m}U_2^{2m} = \varphi_{2n,2m}U_2^{2m}\left(\frac{U_1^{-1}U_2}{\sqrt{\lambda}}\right)^{2m} = \\
\varphi_{2n,2m}\lambda^{-m}\lambda^{-\frac{2n(2m-1)}{2}}U_2^{2m}U_1^{-2m}U_2^{2m} = \\
\varphi_{0,0}\lambda^{2n}\lambda^{-2m}\lambda^{-4nm}\lambda^{-2m}U_1^{-2m}U_2^{2m} = \\
\varphi_{-2n,2m+2m}U_1^{-2m}U_2^{2m}.
\]

Hence, we conclude that \(D_{0,0}\) is invariant under the action of \(-\omega\).

Similar computations for \(D_{0,1}, D_{1,0}\) and \(D_{1,1}\) yield the following
\[
-\omega \cdot D_{0,1} = \sqrt{\lambda}D_{1,1} ; \quad -\omega \cdot D_{0,1} = D_{0,1} ; \quad -\omega \cdot D_{1,1} = \lambda D_{1,0}.
\]

Hence the \(-\omega\) invariant sub-space of \(H^0(\mathcal{A}_\theta^{alg}, \mathcal{A}_\theta^{alg*})\) is 2 dimensional and is generated by \(D_{0,0}\) and \(D_{1,0} + \lambda\sqrt{\lambda}D_{1,0} + \sqrt{\lambda}D_{1,1}\).

**Theorem 2.12.** \(H^0(\mathcal{A}_\theta^{alg} \otimes \mathbb{Z}_6, (\mathcal{A}_\theta^{alg} \otimes \mathbb{Z}_6)^*) \cong \mathbb{C}^0\).
Proof. Using the paracyclic decomposition of the group \(H^0(\mathcal{A}_\theta^{alg} \times \mathbb{Z}_6, (\mathcal{A}_\theta^{alg} \times \mathbb{Z}_6)^*)\) we have the following relation

\[
H^0(\mathcal{A}_\theta^{alg} \times \mathbb{Z}_6, (\mathcal{A}_\theta^{alg} \times \mathbb{Z}_6)^*) = H^0(\mathcal{A}_\theta, -\omega \mathcal{A}_\theta^{alg*})^\mathbb{Z}_6 \oplus H^0(\mathcal{A}_\theta, -\omega^2 \mathcal{A}_\theta^{alg*})^\mathbb{Z}_6 \oplus H^0((\mathcal{A}_\theta, -\omega \mathcal{A}_\theta^{alg*}))^\mathbb{Z}_6 \oplus H^0((\mathcal{A}_\theta, -\mathcal{A}_\theta^{alg*}))^\mathbb{Z}_6 \oplus C \oplus C \oplus C^2 \oplus C \oplus C \oplus C = C^9.
\]

\[\square\]

2.2. The Hochschild cohomology groups \(H^2(\mathcal{A}_\theta^{alg} \times \Gamma, (\mathcal{A}_\theta^{alg} \times \Gamma)^*))\).

In this section we study the second Hochschild cohomology groups of the \(\mathbb{Z}_3, \mathbb{Z}_4\) and \(\mathbb{Z}_6\) noncommutative algebraic torus orbifold. From [Q2] we have \(H^2(\mathcal{A}_\theta^{alg} , -1, \mathcal{A}_\theta^{alg*})Z_2 = 0\). We conclude that

\[
H^2(\mathcal{A}_\theta^{alg} , -1, \mathcal{A}_\theta^{alg*})^\mathbb{Z}_4 = H^2(\mathcal{A}_\theta^{alg} , -1, \mathcal{A}_\theta^{alg*})^\mathbb{Z}_6 = 0.
\]

Similarly we also conclude that

\[
H^2(\mathcal{A}_\theta^{alg} , \omega^2 \mathcal{A}_\theta^{alg*})^\mathbb{Z}_6 = H^2(\mathcal{A}_\theta^{alg} , \omega \mathcal{A}_\theta^{alg*})^\mathbb{Z}_6 = 0.
\]

The case \(\Gamma = \mathbb{Z}_3\).

We notice that the \(\omega\) twisted Hochschild cohomology group \(H^2(\mathcal{A}_\theta^{alg} , \omega \mathcal{A}_\theta^{alg*}) = \omega \mathcal{A}_\theta^{alg*}/\text{Im}(\omega \alpha_2)\) is generated by the classes \(\varphi_{0,0}\) (generated by \(U_1^0 U_2^0\)), \(\varphi_{1,0}\) (generated by \(U_1^{-1} U_2^0\)) and \(\varphi_{0,1}\) (generated by \(U_1^0 U_2^{-1}\)).

We argue the case as in [Q2], for \(\varphi \in \omega \mathcal{A}_\theta^{alg*}\) and let \(\bar{\varphi}\) be the corresponding element of \(\text{Hom}_{\mathcal{A}_\theta^{alg}}(J_2, \omega \mathcal{A}_\theta^{alg*})\). Where \(J_4\) is the chain complex \(J_4 Z_4\) defined in [Q1, Page 340, Section 8] and \(B_\theta^{alg} = \mathcal{A}_\theta^{alg} \otimes (\mathcal{A}_\theta^{alg})^f\). We have the following relation

\[
\bar{\varphi}(a \otimes b \otimes e_1 \otimes e_2)(x) = \varphi(\omega \cdot b) xa,
\]

for all \(a, b, x \in \mathcal{A}_\theta^{alg}\). Let \(\psi = \kappa_2^* \bar{\varphi} = \bar{\varphi} \circ \kappa_2\). We also have

\[
\psi(x, x_1, x_2) = \bar{\varphi}(\kappa_2(I \otimes x_1 \otimes x_2))(x),
\]

for all \(x, x_1, x_2 \in \mathcal{A}_\theta^{alg}\). The group \(\omega\) acts on \(\mathcal{A}_\theta^{alg}\) in the bar complex as

\[
\omega \cdot \chi(x, x_1, x_2) = \chi(\omega \cdot x, \omega \cdot x_1, \omega \cdot x_2).
\]

Further we pull the map \(\omega \psi := \omega \cdot \psi\) back on to the Connes complex via the map \(h_2^*\). Let \(w = h_2^*(\omega \psi)\) denote the pullback of \(\omega \psi\) on the Connes complex. We have

\[
w(x) = \omega \psi(x, U_2, U_1) - \lambda_\psi(x, U_1, U_2) = \psi(\omega \cdot x, U_1, U_2) - \psi(\omega \cdot x, U_2, U_1) = \bar{\varphi}(\kappa_2(I \otimes U_1^{-1} U_2^{-1}))(x) - \lambda \bar{\varphi}(\kappa_2(I \otimes U_2^{-1} U_1^{-1}))(x).
\]

Following the calculations from [C] and [Q1, Section 6], we have

\[
k_2(I \otimes U_1 U_2^{-1} U_1^{-1}) - \lambda k_2(I \otimes U_2^{-1} U_1 U_2^{-1}) = (U_2^{-1} \otimes U_2^{-2}).
\]

Applying this we conclude that

\[
\frac{1}{\sqrt{\lambda}}(\bar{\varphi}((k_2(I \otimes U_1 U_2^{-1} \otimes U_2^{-1}))(\omega \cdot x) - \lambda \bar{\varphi}(k_2(I \otimes U_2^{-1} \otimes U_1 U_2^{-1}))(\omega \cdot x)) = \frac{1}{\sqrt{\lambda}}(U_2^{-1} \otimes U_2^{-2})(\omega \cdot x) = \sqrt{\lambda} \varphi(U_1^{-2} U_2^2 \cdot (\omega \cdot x) \cdot U_2^{-1}).
\]
To assert the $Z_3$ invariance of a 2-cocycle $\varphi$, we need to show that for $x \in A^\text{alg}_\theta$, $\varphi(x)$ and $\sqrt{\lambda}\varphi(U_1^{-2}U_2^2 \cdot (\omega \cdot x) \cdot U_2^{-1})$ are the same.

We consider the 2-cocycle $\varphi_{0,0}$. On one hand we have $\varphi_{0,0}(x) = x_{0,0}$ while on the other hand

$$\varphi_{0,0}(U_1^{-2}U_2^2 \cdot (\omega \cdot (x_{-1,2}U_1^{-1}U_2^2)) \cdot U_2^{-1}) = \frac{1}{\sqrt{\lambda}} \varphi(x_{-1,2}U_1^{-2}U_2^2 \cdot (U_2U_2^{-2}) \cdot U_2^{-1}) =$$

$$\frac{1}{\sqrt{\lambda}} \varphi(x_{-1,2}U_1^{-2}U_2^2U_2^{-3}) = \frac{1}{\sqrt{\lambda}} \lambda^6 \varphi(x_{-1,2}U_1^0U_2^0) = \lambda^5 \sqrt{\lambda} x_{-1,2}.$$

The cocycle $\varphi_{0,0}$ and $\lambda^3 \sqrt{\lambda} \varphi_{-1,2}$ represent the same cocycle as they are separated by a coboundary element. Hence we conclude that the 2-cocycle $\varphi_{0,0}$ is not invariant under the action of $\omega$. Therefore $\varphi_{0,0} \notin H^2(A^\text{alg}_\theta, \omega A^\text{alg*}_\theta)^{Z_3}$. Repeating the above computations for the cocycles $\varphi_{1,0}$ and $\varphi_{0,1}$ we finally conclude that $H^2(A^\text{alg}_\theta, \omega A^\text{alg*}_\theta)^{Z_3} = 0$.

Now we consider the group $H^2(A^\text{alg}_\theta, A^\text{alg*}_\theta)$ generated by $U_2U_1$[Q2, Lemma 3.2]. To check the invariance of the untwisted 2-cocycle (say $\varphi_{-1,-1}$), we need to compare $\varphi_{-1,-1}(x)$ and $\frac{1}{\sqrt{\lambda}} \varphi((U_2^{-2}(\omega \cdot x)U_2^{-1}))$ for all $x \in A^\text{alg}_\theta$. If they are the same, the cocycle is $Z_3$ invariant. else not.

On one hand we have $\varphi_{-1,-1}(x) = x_{-1,-1}$ and on the other hand

$$\frac{1}{\sqrt{\lambda}} \varphi_{-1,-1}(U_2^{-2}(\omega \cdot x)U_2^{-1}) = \frac{1}{\sqrt{\lambda}} \varphi_{-1,-1}(U_2^{-2}(\omega \cdot (x_{-1,-1}U_1^{-1}U_2^{-1}))U_2^{-1}) =$$

$$\frac{1}{\sqrt{\lambda}} \varphi_{-1,-1}(U_2^{-2}x_{-1,-1}U_2(U_1U_2^{-1}U_2U_1^{-1}U_2^{-1}) = \varphi_{-1,-1}(x_{-1,-1}U_1^{-1}U_2^{-1}) = x_{-1,-1}.$$  

Hence the cocycle is invariant under the action of the group $Z_3$ and $H^2(A^\text{alg}_\theta, A^\text{alg*}_\theta)^{Z_3} \cong \mathbb{C}$.

**THEOREM 2.13.** $H^2(A^\text{alg}_\theta \rtimes Z_3, (A^\text{alg}_\theta \rtimes Z_3)^*) \cong \mathbb{C}$.

**Proof.** Using the paracyclic decomposition of the group $H^2(A^\text{alg}_\theta \rtimes Z_3, (A^\text{alg}_\theta \rtimes Z_3)^*)$ we have the following relation

$$H^2(A^\text{alg}_\theta \rtimes Z_3, (A^\text{alg}_\theta \rtimes Z_3)^*) = H^2(A^\text{alg}_\theta, \omega A^\text{alg*}_\theta)^{Z_3} \oplus H^2(A^\text{alg}_\theta, A^\text{alg*}_\theta)^{Z_3} \oplus H^2(A^\text{alg}_\theta, A^\text{alg*}_\theta)^{Z_3} = 0 \oplus 0 \oplus \mathbb{C} = \mathbb{C}$$

\[\square\]

The case $\Gamma = Z_4$.

We notice from [Q2] that $H^2(A^\text{alg}_{\theta, \cdot A^\text{alg*}_\theta})^{Z_2} = H^2(A^\text{alg}_{\theta, \cdot A^\text{alg*}_\theta})^{Z_4} = 0$ and the $i$ twisted Hochschild cohomology group $H^2(A^\text{alg}_{\theta, \cdot i A^\text{alg*}_\theta}) = iA^\text{alg*}_\theta / Im(i\alpha_2)$ is generated by the classes $\varphi_{0,0}$ (generated by $U_1^{0}U_2^{0}$) and $\varphi_{1,0}$ (generated by $U_1^{-1}U_2^{0}$).

As in the previous case and using the results of [C] and [Q1, Section 6] we are finally down to comparing $\varphi(x)$ with the following

$$\varphi(U_2^{-1} \otimes U_2^{-1})(i \cdot x).$$

On one hand we have $\varphi_{0,0}(x) = x_{0,0}$ while on the other hand
\[
\varphi(U_1^{-1}(i \cdot x)U_2^{-1}) = \varphi(U_1^{-1}(i \cdot (x_{-1,1}U_1^{-1}U_2))U_2^{-1}) = \varphi(x_{-1,1}U_1^{-1}(i \cdot (U_1^{-1}U_2))U_2^{-1}) = \\
\varphi(x_{-1,1}U_1^{-1}U_2U_1^{-1}U_2^{-1}) = \lambda x_{-1,1}.
\]

But since \(\varphi_{0,0}\) and \(\varphi_{-1,1}\) represent the same cocycle being separated by a coboundary element. We conclude that the 2-cocycle \(\varphi_{0,0}\) is not invariant under the action of \(i\). Hence \(\varphi_{0,0} \notin H^2(\mathcal{A}_\theta^{alg}, \mathcal{A}_\theta^{alg})\mathbb{Z}_4\). Repeating the above computations for the cocycle \(\varphi_{1,0}\) we observe that \(H^2(\mathcal{A}_\theta^{alg}, i\mathcal{A}_\theta^{alg})\mathbb{Z}_4 = 0\).

We consider the group \(H^2(\mathcal{A}_\theta^{alg}, \mathcal{A}_\theta^{alg})\) generated by \(U_2U_1\). To check the invariance of this untwisted 2-cocycle \(\varphi_{-1,-1}\), we need to compare \(\varphi_{-1,-1}(x)\) and \(\varphi_{-1,-1}(U_2^{-1}(i \cdot x)U_2^{-1})\) for all \(x \in \mathcal{A}_\theta^{alg}\). If they are the same, the cocycle is \(\mathbb{Z}_4\) invariant, else not.

On one hand we have \(\varphi_{-1,-1}(x) = x_{-1,-1}\) and on the other hand
\[
\varphi_{-1,-1}(U_2^{-1}(i \cdot x)U_2^{-1}) = \varphi_{-1,-1}(U_2^{-1}(i \cdot (x_{-1,1}U_1^{-1}U_2))U_2^{-1}) = \\
\varphi_{-1,-1}(x_{-1,1}U_1^{-1}U_2U_1^{-1}U_2^{-1}) = x_{-1,-1}.
\]

Hence the cocycle is invariant under the action of the group \(\mathbb{Z}_3\) and \(H^2(\mathcal{A}_\theta^{alg}, \mathcal{A}_\theta^{alg})\mathbb{Z}_4 \cong \mathbb{C}\).

**Theorem 2.14.** \(H^2(\mathcal{A}_\theta^{alg} \times \mathbb{Z}_4, (\mathcal{A}_\theta^{alg} \times \mathbb{Z}_4)^{*}) \cong \mathbb{C}\).

**Proof.** Using the paracyclic decomposition of the group \(H^2(\mathcal{A}_\theta^{alg} \times \mathbb{Z}_4, (\mathcal{A}_\theta^{alg} \times \mathbb{Z}_4)^{*})\) we have the following relation
\[
H^2(\mathcal{A}_\theta^{alg} \times \mathbb{Z}_4, (\mathcal{A}_\theta^{alg} \times \mathbb{Z}_4)^{*}) = H^2(\mathcal{A}_\theta^{alg}, \mathcal{A}_\theta^{alg} \times \mathbb{Z}_4) \oplus H^2(\mathcal{A}_\theta^{alg}, -i\mathcal{A}_\theta^{alg} \times \mathbb{Z}_4) \oplus \\
H^2(\mathcal{A}_\theta^{alg}, \mathbb{Z}_4) \times H^2(\mathcal{A}_\theta^{alg}, \mathcal{A}_\theta^{alg} \times \mathbb{Z}_4) = 0 \oplus 0 \oplus 0 \oplus \mathbb{C} = \mathbb{C}
\]

This completes the proof. \(\square\)

The case \(\Gamma = \mathbb{Z}_6\).

We notice from preceding theorems that
\[
H^2(\mathcal{A}_\theta, \omega \mathcal{A}_\theta^{alg})\mathbb{Z}_6 = H^2(\mathcal{A}_\theta, \omega \mathcal{A}_\theta^{alg})\mathbb{Z}_6 = H^2(\mathcal{A}_\theta, \omega \mathcal{A}_\theta^{alg})\mathbb{Z}_6 = H^2(\mathcal{A}_\theta, \omega \mathcal{A}_\theta^{alg})\mathbb{Z}_6 = \\
H^2(\mathcal{A}_\theta, -i \mathcal{A}_\theta^{alg})\mathbb{Z}_6 = H^2(\mathcal{A}_\theta, -i \mathcal{A}_\theta^{alg})\mathbb{Z}_6 = H^2(\mathcal{A}_\theta, -i \mathcal{A}_\theta^{alg})\mathbb{Z}_6 = 0.
\]

The \(-\omega\) twisted Hochschild cohomology group \(H^2(\mathcal{A}_\theta^{alg}, -\omega \mathcal{A}_\theta^{alg}) = -\omega \mathcal{A}_\theta^{alg}/\text{Im}(\omega \alpha_2)\) is generated by the classes \(\varphi_{0,0}\).

As in both the previous cases, to check the \(\mathbb{Z}_6\) invariance of an \(-\omega\) twisted Hochschild cocycle \(\varphi\), we compare \(\varphi(x)\) with the following
\[
\frac{1}{\sqrt{\lambda}} \tilde{\varphi}(U_1^{-1} \otimes U_1^{-1}U_2)(-\omega \cdot x).
\]

If \(\varphi(x)\) equals \(\frac{1}{\sqrt{\lambda}} \tilde{\varphi}(U_1^{-1} \otimes U_1^{-1}U_2)(-\omega \cdot x)\), we say that the the cocycle \(\varphi\) is \(\mathbb{Z}_6\) invariant, else not. Now let us consider the 2-cocycle \(\varphi_{0,0}\). On one hand we have \(\varphi_{0,0}(x) = x_{0,0}\) while on the other hand
\[
\frac{1}{\sqrt{\lambda}} \tilde{\varphi}_{0,0}(U_1^{-1} \otimes U_1^{-1}U_2)(-\omega \cdot x) = \frac{1}{\lambda} \varphi_{0,0}(U_2^{-1}U_1^{-1}U_2(-\omega \cdot x)U_1^{-1}) = \\
\frac{1}{\lambda} \varphi_{0,0}(U_1^{-1}(-\omega \cdot x)U_1^{-1})) = \frac{1}{\lambda} x_{2,-2}.
\]
Since in this case the cocycle $\varphi_{0,0}$ and $\lambda^2 \varphi_{2,-2}$ are the same cocycle. We conclude that the 2-cocycle $\varphi_{0,0}$ is not invariant under the action of $-\omega$. Hence we observe that

$$H^2(A_{alg}^{\theta}, -\omega A_{alg}^{\theta}) \cong \mathbb{Z}_6 = 0.$$ 

Now we check the invariance the group $H^2(A_{alg}^{\theta}, A_{alg}^{\theta})$ generated by $U_2 U_1$. For this we need to compare $\varphi_{-1,-1}(x)$ and $\sqrt{\lambda} \varphi((U_1^{-1}(-\omega \cdot x) U_1^{-1} U_2))$ for all $x \in A_{alg}^{\theta}$. If they are the same, the cocycle is $\mathbb{Z}_6$ invariant. else not.

On one hand we have $\varphi_{-1,-1}(x) = x_{-1,-1}$ and on the other hand

$$\frac{1}{\sqrt{\lambda}} \varphi_{-1,-1}((U_1^{-1} U_2 (-\omega \cdot x) U_1^{-1})) = \frac{1}{\sqrt{\lambda}} \varphi_{-1,-1}((U_1^{-1} (-\omega \cdot (x_{-1,-1} U_1^{-1} U_2^{-1})) U_1^{-1} U_2)) = \frac{1}{\sqrt{\lambda}} \varphi_{-1,-1}(x_{-1,-1} U_1^{-1} U_2^{-1} (\frac{U_1^{-1} U_2}{\sqrt{\lambda}})^{-1} U_1^{-1}) = \varphi_{-1,-1}(x_{-1,-1} U_1^{-1} U_2^{-1}) = x_{-1,-1}.$$

Hence the cocycle is invariant under the action of the group $\mathbb{Z}_6$ and $H^2(A_{alg}^{\theta}, A_{alg}^{\theta}) \cong \mathbb{C}$.

**Theorem 2.15.** $H^2(A_{alg}^{\theta} \times \mathbb{Z}_6, (A_{alg}^{\theta} \times \mathbb{Z}_6)^*) \cong \mathbb{C}$.

**Proof.** Using the paracyclic decomposition of the group $H^2(A_{alg}^{\theta} \times \mathbb{Z}_6, (A_{alg}^{\theta} \times \mathbb{Z}_6)^*)$ we have the following relation

$$H^2(A_{alg}^{\theta} \times \mathbb{Z}_6, (A_{alg}^{\theta} \times \mathbb{Z}_6)^*) = H^2(A_{alg}^{\theta}, -\omega A_{alg}^{\theta} \mathbb{Z}_6) \oplus H^2(A_{alg}^{\theta}, -\omega^2 A_{alg}^{\theta} \mathbb{Z}_6) \oplus H^2(A_{alg}^{\theta}, - \mathbb{Z}_6) \oplus H^2(A_{alg}^{\theta}, -1 A_{alg}^{\theta} \mathbb{Z}_6) \oplus H^2(A_{alg}^{\theta}, -\omega A_{alg}^{\theta} \mathbb{Z}_6) \oplus H^2(A_{alg}^{\theta}, -\omega^2 A_{alg}^{\theta} \mathbb{Z}_6) \oplus 0 \oplus 0 \oplus 0 \oplus 0 \oplus 0 \oplus \mathbb{C} = \mathbb{C}$$

2.3. The Hochschild cohomology groups $H^1(A_{alg}^{\theta} \times \Gamma, (A_{alg}^{\theta} \times \Gamma)^*)$.

The case $\Gamma = \mathbb{Z}_3$.

In this case we have the following paracyclic decomposition of the first Hochschild cohomology group for $A_{alg}^{\theta} \times \mathbb{Z}_3$.

$$H^1(A_{alg}^{\theta} \times \mathbb{Z}_3, (A_{alg}^{\theta} \times \mathbb{Z}_3)^*) = H^1(A_{alg}^{\theta}, A_{alg}^{\theta} \mathbb{Z}_3) \oplus H^1(A_{alg}^{\theta}, \omega A_{alg}^{\theta} \mathbb{Z}_3) \oplus H^1(A_{alg}^{\theta}, -1 A_{alg}^{\theta} \mathbb{Z}_3) \oplus H^1(A_{alg}^{\theta}, -\omega A_{alg}^{\theta} \mathbb{Z}_3).$$

We observe that $H^1(A_{alg}^{\theta}, A_{alg}^{\theta} \mathbb{Z}_3) = \mathbb{C}$, generated by $\varphi_{1,-1,0}$ and $\varphi_{2,-1,0}^2$ [C]. We need to identify its $\mathbb{Z}_3$ invariant subgroup. For $a, b \in \mathbb{C}$, we consider the cocycle $\chi := (a \varphi_{1,-1,0}, b \varphi_{2,-1,0}^2) \in A_{alg}^{\theta} \mathbb{Z}_3 \oplus A_{alg}^{\theta} \mathbb{Z}_3$ and push forward to the bar complex, thereafter we pull it back to the bar complex after the $\mathbb{Z}_3$ action of it. Proceeding in a similar way as in [Q2] we finally observe that $h_1(\omega \cdot (k_{1,1}\chi))(x) = (-a\lambda \sqrt{x}_{-1,-1} - b \lambda^{-1} x_{-2,-1})$. Hence we conclude that $H^1(A_{alg}^{\theta}, A_{alg}^{\theta} \mathbb{Z}_3) = 0$. Since we have $H^1(A_{alg}^{\theta}, A_{alg}^{\theta} \mathbb{Z}_2) = H^1(A_{alg}^{\theta}, A_{alg}^{\theta} \mathbb{Z}_3) = 0$, we conclude that

$$H^1(A_{alg}^{\theta}, A_{alg}^{\theta} \mathbb{Z}_4) = H^1(A_{alg}^{\theta}, A_{alg}^{\theta} \mathbb{Z}_6) = 0.$$ 

Now we turn our attention to understand the group $H^1(A_{alg}^{\theta}, \omega A_{alg}^{\theta} \mathbb{Z}_3)$. We deploy the same method which we had used to calculate $H^1(A_{alg}^{\theta}, -1 A_{alg}^{\theta} \mathbb{Z}_3)$ in [Q2]. Below, a typical portion of the infinite graph corresponding to a $\omega \alpha_2$ kernel equation is plotted on the $\mathbb{Z}^2$ lattice plane.
The big dots corresponds to the $\varphi^1$ elements while the smaller dots are $\varphi^2$'s. For a given cocycle $\sigma$ let $\text{Dia}(\sigma) \subset \mathbb{Z}^2 \oplus \mathbb{Z}^2 \oplus \mathbb{Z}^2$ be the associated diagram [Q1, Page 334]. Let $\pi_i$ denote the $i^{th}$ projection map on the lattice $\mathbb{Z}^2$. With out loss of generality assume that in the projected diagram $\pi_1(\text{Dia}(\sigma))_{0,0}$ the element $\varphi^2_{0,0} = 0$ and $\varphi^1_{0,0} \neq 0$. For each $c \in \mathbb{Z}$ we shall construct below $\gamma_c \in A^\text{alg}_g$ such that $\pi_1(\text{Dia}(\omega \alpha_1(\gamma_c)))_{n,m} = \pi_1(\text{Dia}(\sigma))_{n,m}$ for all $n, m \in \mathbb{Z}$ such that $m - n = 3c$. Thereafter, we shall prove that for $\gamma := \sum_{c \in \mathbb{Z}} \gamma_c$
\[\pi_1(\text{Dia}(\omega \alpha_1(\gamma)))_{r,s} - \pi_1(\text{Dia}(\sigma))_{r,s} = (0, \bullet)\] for all $r, s \in \mathbb{Z}$.

Hence by repetition of the arguments for the projected diagrams $\pi_2(\text{Dia}(\sigma))_{0,0}$ and $\pi_3(\text{Dia}(\sigma))_{0,0}$, we shall conclude that every cocycle is the trivial cocycle.

**LEMMA 2.16.** For $\sigma \in \ker \omega \alpha_2$. There exists $\gamma \in A^\text{alg}_g$ such that $\pi_1(\text{Dia}(\omega \alpha_1(\gamma)))_{n,m} = \pi_1(\text{Dia}(\sigma))_{n,m}$.

**Proof.** Without loss of generality we assume that $\pi_1(\text{Dia}(\sigma))_{0,0} = (\bullet, 0)$. Let $\gamma_0$ be defined as
\[(\gamma_0)_{n,m} = \begin{cases} -\varphi^1_{0,0} & \text{for } (n, m) = (-1, 0) \\ 0 & \text{for } (n, m) = (0, 1) \text{ and for } (n, m) \text{ such that } m - n \neq 1. \end{cases}\]

Thereafter we define $(\gamma_0)_{n,m}$ for $n < 0$ and $m - n = 1$ in the following iterated way.
\[(\gamma_0)_{n,m} = \lambda^{-m}\varphi^{1'}_{n+1,m}.\]

Where $\varphi^{1'}_{n+1,m}$ is first entry of $\pi_1(\text{Dia}(\sigma - \omega \alpha_1(\gamma_0)))_{n+1,m}$. We see that
\[\pi_1(\text{Dia}(\sigma - \omega \alpha_1(\gamma_0)))_{n,m} = (0, 0) \text{ for all } (n, m) \text{ such that } m = n \leq 0.\]

Similarly we can define $(\gamma_0)_{n,m}$ for $n > 0$ and $m - n = 1$ in the following way:
\[(\gamma_0)_{n,m} = \lambda^n\varphi^{1'}_{n,m-1}.\]

With this we have defined $\gamma_0$ at all lattice points on the plane. And we can easily see that:
\[\pi_1(\text{Dia}(\sigma - \omega \alpha_1(\gamma_0)))_{n,m} = (0, 0) \text{ for all } (n, m) \text{ such that } m = n.\]

Putting a similar argument for any given $c \in \mathbb{Z}$, we get $\gamma_c$ such that:
\[\pi_1(\text{Dia}(\sigma - \omega \alpha_1(\gamma_c)))_{n,m} = (0, 0) \text{ for all } (n, m) \text{ such that } m - n = 3c.\]

Furthermore the element $\gamma := \sum_{c \in \mathbb{Z}} \gamma_c$ satisfies the property that:
\[\pi_1(\text{Dia}(\sigma - \omega \alpha_1(\gamma)))_{n,m} = (0, \bullet) \text{ for all } (n, m).\]

But, all $\varphi^1$ elements zero in the diagram $\pi_1(\text{Dia}(\sigma - \omega \alpha_1(\gamma)))$. It can be easily shown that the $\varphi^2$ elements constitute a zero cocycle. \qed
THEOREM 2.17. $H^1(\mathcal{A}_g^{alg} \times \mathbb{Z}_3, (\mathcal{A}_g^{alg} \times \mathbb{Z}_3)^*) = 0$.

Proof. Using the paracyclic decomposition of the group $H^1(\mathcal{A}_g^{alg} \times \mathbb{Z}_3, (\mathcal{A}_g^{alg} \times \mathbb{Z}_3)^*)$ we have the following relation

$$H^1(\mathcal{A}_g^{alg} \times \mathbb{Z}_3, (\mathcal{A}_g^{alg} \times \mathbb{Z}_3)^*) = H^1(\mathcal{A}_g, \omega_1 \mathcal{A}_g^{alg} \mathbb{Z}_3) \oplus H^1(\mathcal{A}_g, \omega_2 \mathcal{A}_g^{alg} \mathbb{Z}_3) \oplus H^1(\mathcal{A}_g, \mathcal{A}_g^{alg} \mathbb{Z}_3) = 0 \oplus 0 \oplus 0 = 0.$$ 

□

The case $\Gamma = \mathbb{Z}_4$.

Like the previous case, we associate to a given twisting cocycle a diagram and thereafter prove that the cocycle is trivial. A typical portion of the infinite graph resembles the one below.

The squares corresponds to the $\varphi^1$ elements while the filled dots are $\varphi^2$'s. The diagram of a given cocycle $\sigma$, $\text{Dia}(\sigma) \subset \mathbb{Z}^2 \oplus \mathbb{Z}^2$. With out loss of generality assume that in the projected diagram $\pi_1(\text{Dia}(\sigma))_{0,0}$ the element $\varphi^2_{0,0}, \varphi^1_{0,0} \neq 0$

LEMMA 2.18. For $\sigma \in \ker \alpha_2$. There exists $\gamma \in \mathcal{A}_g^{alg}$ such that $\pi_1(\text{Dia}(\iota \alpha_1(\gamma)))_{n,m} = \pi_1(\text{Dia}(\sigma))_{n,m}$.

Proof. Let $\gamma_0$ be defined as

$$(\gamma_0)_{n,m} = \begin{cases} -\varphi^1_{0,0} & \text{for } (n,m) = (-1,0) \\ 0 & \text{for } (n,m) = (0,1) \text{ and for } (n,m) \text{ such that } m - n \neq 1. \end{cases}$$

Thereafter we define $(\gamma_0)_{n,m}$ for $n < 0$ and $m - n = 1$ in the following iterated way.

$$(\gamma_0)_{n,m} = -\lambda^{-m} \varphi^1_{n+1,m}.$$ 

Where $\varphi^1_{n+1,m}$ is first entry of $\pi_1(\text{Dia}(\sigma - \omega \alpha_1(\gamma)))_{n+1,m}$. We see that $\pi_1(\text{Dia}(\sigma - \omega \alpha_1(\gamma)))_{n,m} = (0,0)$ for all $(n,m)$ such that $m = n \leq 0$.

Similarly we can define $(\gamma_0)_{n,m}$ for $n > 0$ and $m - n = 1$ in the following way:

$$(\gamma_0)_{n,m} = \lambda^n \varphi^1_{n,m-1}.$$
With this we have defined $\gamma_0$ at all lattice points on the plane. And similar computations gives us that for $c \in \mathbb{Z}$:

$$\pi_1(Dia(\sigma - i_1(\gamma_c)))_{n,m} = (0,0) \text{ for all } (n,m) \text{ such that } m - n = 2c.$$ 

Finally the element $\gamma := \sum_{c \in \mathbb{Z}} \gamma_c$ satisfies the property that:

$$\pi_1(Dia(\sigma - i_1(\gamma)))_{n,m} = (0, \bullet) \text{ for all } (n,m).$$

Since, all $\varphi^1$ elements zero in the diagram $\pi_1(Dia(\sigma - \omega_1(\gamma)))$, it can be easily shown that the $\varphi^2$ elements constitute a zero cocycle.

**THEOREM 2.19.** $H^1(A_{\theta}^{alg} \times \mathbb{Z}_4, (A_{\theta}^{alg} \times \mathbb{Z}_4)^*) = 0$.

**Proof.** Using the previous lemma and the paracyclic decomposition of the group $H^1(A_{\theta}^{alg} \times \mathbb{Z}_4, (A_{\theta}^{alg} \times \mathbb{Z}_4)^*)$ we have the following relation

$$H^1(A_{\theta}^{alg} \times \mathbb{Z}_4, (A_{\theta}^{alg} \times \mathbb{Z}_4)^*) = H^1(A_{\theta}, i_{A_{\theta}^{alg}*} \mathbb{Z}_4) \oplus H^1(A_{\theta}, -i_{A_{\theta}^{alg}*} \mathbb{Z}_4) \oplus H^1(A_{\theta}, A_{\theta}^{alg*} \mathbb{Z}_4) = 0 \oplus 0 \oplus 0 = 0.$$

The case $\Gamma = \mathbb{Z}_6$.

We associate to a given $-\omega$ twisted cocycle $\sigma$, a diagram $Dia(\sigma)$ and thereafter prove that the cocycle is trivial. The argument here is similar to the previous two cases.

The squares corresponds to the $\varphi^1$ elements while the filled dots are $\varphi^2$’s. For a given cocycle $\sigma$, its diagram $Dia(\sigma) \subset \mathbb{Z}^2$.

**LEMMA 2.20.** For $\sigma \in \ker -\omega \alpha_2$. There exists $\gamma \in A_{\theta}^{alg*}$ such that $Dia(-\omega_1(\gamma))_{n,m} = Dia(\sigma)_{n,m}$.

**Proof.** Let $\gamma_0$ be defined as

$$(\gamma_0)_{n,m} = \begin{cases} -\varphi^1_{0,0} & \text{for } (n,m) = (-1,0) \\ 0 & \text{for } (n,m) \text{ such that } m \neq 0. \end{cases}$$

Thereafter we define $(\gamma_0)_{n,m}$ for $n < 0$ and $m = 0$ in the following iterated way.

$$(\gamma_0)_{n,m} = -\lambda^{-m} \varphi^{1'}_{n+1,m}.$$ 

Where $\varphi^{1'}_{n+1,m}$ is first entry of $Dia(\sigma - \omega_1(\gamma))_{n+1,m}$. We see that

$$Dia(\sigma - \omega_1(\gamma))_{n,m} = (0, 0) \text{ for all } (n,m) \text{ such that } m = 0.$$
Similarly we can define \((\gamma_0)_{n,m}\) for \(n > 0\) and \(m = 0\) in the following way:
\[
(\gamma_0)_{n,m} = \lambda^{-n}\varphi^1_{n,m-1}.
\]
With this we have defined \(\gamma_0\) at all lattice points on the plane satisfying:
\[
\text{Dia}(\sigma - \omega\alpha_1(\gamma_0))_{n,m} = (0,0) \quad \text{for all} \quad (n,m) \quad \text{such that} \quad m = 0.
\]
Similarly for \(c \in \mathbb{Z}\), we get construct \(\gamma_c\) and the element \(\gamma := \sum_{c \in \mathbb{Z}} \gamma_c\) satisfies the following property:
\[
\text{Dia}(\sigma - \omega\alpha_1(\gamma))_{n,m} = (0,0,\bullet) \quad \text{for all} \quad (n,m).
\]
But, all \(\varphi^1\) elements zero in the diagram \(\pi_1(\text{Dia}(\sigma - \omega\alpha_1(\gamma)))\). It can be easily shown that the \(\varphi^2\) elements constitute a zero cocycle.

**THEOREM 2.21.** \(H^1(A^\text{alg}_\theta \times \mathbb{Z}_6, (A^\text{alg}_\theta \times \mathbb{Z}_6)^*) = 0.\)

**Proof.** Using the paracyclic decomposition of the group \(H^1(A^\text{alg}_\theta \times \mathbb{Z}_6, (A^\text{alg}_\theta \times \mathbb{Z}_6)^*)\) we have the following relation
\[
H^1(A^\text{alg}_\theta \times \mathbb{Z}_6, (A^\text{alg}_\theta \times \mathbb{Z}_6)^*) = H^1(A^\text{alg}_\theta, -\omega A^\text{alg}_\theta)^Z_\mathbb{Z}_6 \oplus H^1(A^\text{alg}_\theta, -\omega^2 A^\text{alg}_\theta)^Z_\mathbb{Z}_6 \oplus H^1(A^\text{alg}_\theta, \omega A^\text{alg}_\theta)^Z_\mathbb{Z}_6
\]
\[
H^1(A^\text{alg}_\theta, \omega^2 A^\text{alg}_\theta)^Z_\mathbb{Z}_6 \oplus H^1(A^\text{alg}_\theta, -1 A^\text{alg}_\theta)^Z_\mathbb{Z}_6 \oplus H^1(A^\text{alg}_\theta, A^\text{alg}_\theta)^Z_\mathbb{Z}_6 = 0 = 0 + 0 + 0 + 0 + 0 + 0 = 0.
\]

**Proof of Theorem 1.1.** The Theorems 2.3 2.7 and 2.12 describe the zeroth Hochschild cohomology groups while the Theorems 2.17 2.19 and 2.21 resolves the dimension of the first Hochschild cohomology of these three orbifolds. The Theorems 2.13 2.14 and 2.15 state that the dimension of the second Hochschild cohomology in all the three cases is 1.

3. **Periodic Cyclic Cohomology**

3.1. **Cyclic cohomology groups.**

**LEMMA 3.1.** For the algebraic noncommutative toroidal orbifold \(A^\text{alg}_\theta \times \mathbb{Z}_3\), we have,
\[
HC^0(A^\text{alg}_\theta \times \mathbb{Z}_3) \cong \mathbb{C}^7; \quad HC^1(A^\text{alg}_\theta \times \mathbb{Z}_3) \cong 0; \quad HC^2(A^\text{alg}_\theta \times \mathbb{Z}_3) \cong \mathbb{C}^8.
\]

**Proof.** We consider the \(B, S, I\) [L, Section 2.4] long exact sequence for Hochschild and cyclic cohomology.
\[
\cdots \to H^1(A^\text{alg}_\theta, \omega A^\text{alg}_\theta)^Z_\mathbb{Z}_3 \xrightarrow{B} HC^0(A^\text{alg}_\theta, \omega A^\text{alg}_\theta)^Z_\mathbb{Z}_3 \xrightarrow{S} HC^2(A^\text{alg}_\theta, \omega A^\text{alg}_\theta)^Z_\mathbb{Z}_3 \xrightarrow{I} H^2(A^\text{alg}_\theta, \omega A^\text{alg}_\theta)^Z_\mathbb{Z}_3 \xrightarrow{B} HC^1(A^\text{alg}_\theta, \omega A^\text{alg}_\theta)^Z_\mathbb{Z}_3 \xrightarrow{S} \cdots
\]

In the above sequence \(H^\bullet(A^\text{alg}_\theta, \omega A^\text{alg}_\theta)^Z_\mathbb{Z}_3 = 0 = HC^\bullet(A^\text{alg}_\theta, \omega A^\text{alg}_\theta)^Z_\mathbb{Z}_3\) for \(\bullet = 1\), hence
\[
HC^2(A^\text{alg}_\theta, \omega A^\text{alg}_\theta)^Z_\mathbb{Z}_3 \cong HC^0(A^\text{alg}_\theta, \omega A^\text{alg}_\theta)^Z_\mathbb{Z}_3 \oplus H^2(A^\text{alg}_\theta, \omega A^\text{alg}_\theta)^Z_\mathbb{Z}_3 = HC^0(A^\text{alg}_\theta, \omega A^\text{alg}_\theta)^Z_\mathbb{Z}_3 \cong \mathbb{C}^3.
\]

Similarly we have
\[
HC^2(A^\text{alg}_\theta, A^\text{alg}_\theta)^Z_\mathbb{Z}_3 \cong HC^0(A^\text{alg}_\theta, A^\text{alg}_\theta)^Z_\mathbb{Z}_3 \oplus H^2(A^\text{alg}_\theta, A^\text{alg}_\theta)^Z_\mathbb{Z}_3 = HC^0(A^\text{alg}_\theta, A^\text{alg}_\theta)^Z_\mathbb{Z}_3 \cong \mathbb{C} + \mathbb{C} = \mathbb{C}^2.
\]

Hence for the second cyclic cohomology we finally conclude that :
\[
HC^2(A^\text{alg}_\theta \times \mathbb{Z}_3) \cong HC^2(A^\text{alg}_\theta, A^\text{alg}_\theta)^Z_\mathbb{Z}_3 \oplus HC^2(A^\text{alg}_\theta, \omega A^\text{alg}_\theta)^Z_\mathbb{Z}_3 \oplus HC^2(A^\text{alg}_\theta, \omega^2 A^\text{alg}_\theta)^Z_\mathbb{Z}_3 \cong \mathbb{C}^8.
\]
It is straightforward to conclude that \( HC^0(\mathcal{A}_{\theta}^{alg} \times \mathbb{Z}_3) \cong \mathbb{C}^7 \) while \( HC^1(\mathcal{A}_{\theta}^{alg} \times \mathbb{Z}_3) \) is trivial.

**LEMMA 3.2.** For the \( \mathbb{Z}_4 \) orbifold, we have:

\[
HC^0(\mathcal{A}_{\theta}^{alg} \times \mathbb{Z}_4) \cong \mathbb{C}^8; \quad HC^1(\mathcal{A}_{\theta}^{alg} \times \mathbb{Z}_4) \cong 0; \quad HC^2(\mathcal{A}_{\theta}^{alg} \times \mathbb{Z}_4) \cong \mathbb{C}^9.
\]

**Proof.** Again we consider the \( B, S, I \) long exact sequence for Hochschild and cyclic cohomology.

\[
\cdots \to H^1(\mathcal{A}_{\theta}^{alg}, i_{\mathcal{A}_{\theta}^{alg})*Z_4} \xrightarrow{B} HC^0(\mathcal{A}_{\theta}^{alg}, i_{\mathcal{A}_{\theta}^{alg})*Z_4} \xrightarrow{S} HC^2(\mathcal{A}_{\theta}^{alg}, i_{\mathcal{A}_{\theta}^{alg})*Z_4} \xrightarrow{I} H^2(\mathcal{A}_{\theta}^{alg}, i_{\mathcal{A}_{\theta}^{alg})*Z_4} \xrightarrow{B} HC^1(\mathcal{A}_{\theta}^{alg}, i_{\mathcal{A}_{\theta}^{alg})*Z_4} \xrightarrow{S} \cdots
\]

In the above sequence \( H^\bullet(\mathcal{A}_{\theta}^{alg}, i_{\mathcal{A}_{\theta}^{alg})*Z_4} = 0 = HC^\bullet(\mathcal{A}_{\theta}^{alg}, i_{\mathcal{A}_{\theta}^{alg})*Z_4} \) for \( \bullet = 1 \), hence

\[
HC^2(\mathcal{A}_{\theta}^{alg}, i_{\mathcal{A}_{\theta}^{alg})*Z_4} \cong HC^0(\mathcal{A}_{\theta}^{alg}, i_{\mathcal{A}_{\theta}^{alg})*Z_4} \oplus H^2(\mathcal{A}_{\theta}^{alg}, i_{\mathcal{A}_{\theta}^{alg})*Z_4) = HC^0(\mathcal{A}_{\theta}^{alg}, i_{\mathcal{A}_{\theta}^{alg})*Z_4} \cong \mathbb{C}^2.
\]

Similarly we have \( HC^2(\mathcal{A}_{\theta}^{alg}, \mathcal{A}_{\theta}^{alg})*Z_4 \cong \mathbb{C}^2 \) and \( HC^2(\mathcal{A}_{\theta}^{alg}, -\mathcal{A}_{\theta}^{alg})*Z_4 \cong \mathbb{C}^3 \). Hence for the second cyclic cohomology we finally conclude that:

\[
HC^2(\mathcal{A}_{\theta}^{alg} \times \mathbb{Z}_4) \cong HC^2(\mathcal{A}_{\theta}^{alg}, \mathcal{A}_{\theta}^{alg})*Z_4 \oplus HC^2(\mathcal{A}_{\theta}^{alg}, -\mathcal{A}_{\theta}^{alg})*Z_4 \oplus HC^2(\mathcal{A}_{\theta}^{alg}, -i_{\mathcal{A}_{\theta}^{alg})*Z_4} \cong \mathbb{C}^9.
\]

The dimension of \( HC^0(\mathcal{A}_{\theta}^{alg} \times \mathbb{Z}_4) \) and \( HC^1(\mathcal{A}_{\theta}^{alg} \times \mathbb{Z}_4) \) can be easily computed.

**LEMMA 3.3.**

\[
HC^0(\mathcal{A}_{\theta}^{alg} \times \mathbb{Z}_6) \cong \mathbb{C}^9; \quad HC^1(\mathcal{A}_{\theta}^{alg} \times \mathbb{Z}_6) \cong 0; \quad HC^2(\mathcal{A}_{\theta}^{alg} \times \mathbb{Z}_6) \cong \mathbb{C}^{10}.
\]

**Proof.** Again we consider the \( B, S, I \) long exact sequence for Hochschild and cyclic cohomology.

\[
\cdots \to H^1(\mathcal{A}_{\theta}^{alg}, -\omega_{\mathcal{A}_{\theta}^{alg})*Z_6} \xrightarrow{B} HC^0(\mathcal{A}_{\theta}^{alg}, -\omega_{\mathcal{A}_{\theta}^{alg})*Z_6} \xrightarrow{S} HC^2(\mathcal{A}_{\theta}^{alg}, -\omega_{\mathcal{A}_{\theta}^{alg})*Z_6} \xrightarrow{I} H^2(\mathcal{A}_{\theta}^{alg}, -\omega_{\mathcal{A}_{\theta}^{alg})*Z_6} \xrightarrow{B} HC^1(\mathcal{A}_{\theta}^{alg}, -\omega_{\mathcal{A}_{\theta}^{alg})*Z_6} \xrightarrow{S} \cdots
\]

In the above sequence \( H^\bullet(\mathcal{A}_{\theta}^{alg}, -\omega_{\mathcal{A}_{\theta}^{alg})*Z_6} = 0 = HC^\bullet(\mathcal{A}_{\theta}^{alg}, -\omega_{\mathcal{A}_{\theta}^{alg})*Z_6} \) for \( \bullet = 1 \), hence

\[
HC^2(\mathcal{A}_{\theta}^{alg}, -\omega_{\mathcal{A}_{\theta}^{alg})*Z_6} \cong HC^0(\mathcal{A}_{\theta}^{alg}, -\omega_{\mathcal{A}_{\theta}^{alg})*Z_6} \oplus H^2(\mathcal{A}_{\theta}^{alg}, -\omega_{\mathcal{A}_{\theta}^{alg})*Z_6) = HC^0(\mathcal{A}_{\theta}^{alg}, -\omega_{\mathcal{A}_{\theta}^{alg})*Z_6) \cong \mathbb{C}.
\]

We also have \( HC^2(\mathcal{A}_{\theta}^{alg}, \mathcal{A}_{\theta}^{alg})*Z_6 \cong \mathbb{C}^2 \), \( HC^2(\mathcal{A}_{\theta}^{alg}, -\mathcal{A}_{\theta}^{alg})*Z_6 \cong \mathbb{C}^2 \) and \( HC^2(\mathcal{A}_{\theta}^{alg}, i_{\mathcal{A}_{\theta}^{alg})*Z_6} \cong \mathbb{C}^2 \). Hence we finally conclude that:

\[
HC^2(\mathcal{A}_{\theta}^{alg} \times \mathbb{Z}_6) \cong HC^2(\mathcal{A}_{\theta}^{alg}, \mathcal{A}_{\theta}^{alg})*Z_6 \oplus HC^2(\mathcal{A}_{\theta}^{alg}, -\omega_{\mathcal{A}_{\theta}^{alg})*Z_6) \oplus HC^2(\mathcal{A}_{\theta}^{alg}, -\omega_{\mathcal{A}_{\theta}^{alg})*Z_6) \oplus HC^2(\mathcal{A}_{\theta}^{alg}, \omega_{\mathcal{A}_{\theta}^{alg})*Z_6) \oplus HC^2(\mathcal{A}_{\theta}^{alg}, -\omega_{\mathcal{A}_{\theta}^{alg})*Z_6) \oplus HC^2(\mathcal{A}_{\theta}^{alg}, -\omega_{\mathcal{A}_{\theta}^{alg})*Z_6) \cong \mathbb{C}^{10}.
\]

The dimension of \( HC^0(\mathcal{A}_{\theta}^{alg} \times \mathbb{Z}_4) \) and \( HC^1(\mathcal{A}_{\theta}^{alg} \times \mathbb{Z}_4) \) can be easily computed.

**Proof of Theorem 1.2.** From the modified Connes complex we have \( H^\bullet(\mathcal{A}_{\theta}^{alg} \times \Gamma, (\mathcal{A}_{\theta}^{alg} \times \Gamma)^*) = 0 \) for \( \Gamma = \mathbb{Z}_3, \mathbb{Z}_4 \) and \( \mathbb{Z}_6 \) and \( \bullet \geq 1 \). We also have the isomorphism \( HC^\bullet(\mathcal{A}_{\theta}^{alg} \times \Gamma, (\mathcal{A}_{\theta}^{alg} \times \Gamma)^*) \cong HC^{\bullet+2}(\mathcal{A}_{\theta}^{alg} \times \mathbb{Z}_2, (\mathcal{A}_{\theta}^{alg} \times \mathbb{Z}_2)^*) \) for \( \bullet > 1 \). Now, using the results of Lemma 3.1 we arrive at the desired results:

\[
HF^{even}(\mathcal{A}_{\theta}^{alg} \times \mathbb{Z}_3) \cong \mathbb{C}^8 \quad \text{and} \quad HF^{odd}(\mathcal{A}_{\theta}^{alg} \times \mathbb{Z}_3) = 0.
\]
Similarly the Lemmas 3.2 and 3.3 yields the following:
\[ HP^{\text{even}}(A_{0}^{\text{alg}} \times \mathbb{Z}_4) \cong \mathbb{C}^9 \text{ and } HP^{\text{odd}}(A_{0}^{\text{alg}} \times \mathbb{Z}_4) = 0. \]
and
\[ HP^{\text{even}}(A_{0}^{\text{alg}} \times \mathbb{Z}_6) \cong \mathbb{C}^{10} \text{ and } HP^{\text{odd}}(A_{0}^{\text{alg}} \times \mathbb{Z}_6) = 0. \]
This completes the proof. 

\[ 4. \text{ Chern-Connes Index} \]

The projections of the noncommutative smooth torus orbifolds, \( A_{\theta} \times \mathbb{Z}_3 \), \( A_{\theta} \times \mathbb{Z}_4 \) and \( A_{\theta} \times \mathbb{Z}_6 \) were calculated in [ELPH]. In each of these cases all except one projection is algebraic and hence an element of the respective \( K_0 \) group. In this section we shall pair these projections with the above calculated periodic even cohomology cocycles and hence obtain a table with the Chern-Connes index for the noncommutative algebraic orbifolds. 

Let \( \zeta = e^{\frac{2\pi i}{3}} \) and \( t \) satisfy the relations \( t^3 = 1 \), \( tU_1t^{-1} = \frac{U_1^{-1}U_2}{\sqrt{\lambda}} \) and \( tU_2t^{-1} = U_1^{-1} \). The known projections of \( A_{\theta}^{\text{alg}} \times \mathbb{Z}_3 \) are as follows:

(i) \([1]\]
(ii) \([p_{0}^{\theta}]\), where \( p_{0}^{\theta} = \frac{1}{3}(1 + t + t^2) \).
(iii) \([p_{1}^{\theta}]\), where \( p_{1}^{\theta} = \frac{1}{3}(1 + \zeta^2 t + (\omega t)^2) \).
(iv) \([q_{0}^{\theta}]\), where \( q_{0}^{\theta} = \frac{1}{3}(1 + e^{\frac{2\pi i(2+\theta)}{6}}(U_1t) + (e^{\frac{2\pi i(2+\theta)}{6}}(U_1t)^2)) \).
(v) \([q_{1}^{\theta}]\), where \( q_{1}^{\theta} = \frac{1}{3}(1 + e^{\frac{2\pi i(2+\theta)}{6}}(\zeta^2 U_1t) + (e^{\frac{2\pi i(2+\theta)}{6}}(\zeta^2 U_1t)^2)) \).
(vi) \([r_{0}^{\theta}]\), where \( r_{0}^{\theta} = \frac{1}{3}(1 + U_1^2 t + (U_1^2 t)^2) \).
(vii) \([r_{1}^{\theta}]\), where \( r_{1}^{\theta} = \frac{1}{3}(1 + \zeta^2 U_1^2 t + (\zeta^2 U_1^2 t)^2) \).

\[ \text{Proof of Theorem 1.3. (a) The following are the Chern-Connes indices for } A_{\theta}^{\text{alg}} \times \mathbb{Z}_3. \]

\[ \text{Pairing of } [S\tau] \]
1. \( \langle [1], [S\tau] \rangle = 1 \)
2. \( \langle [p_{0}^{\theta}], [S\tau] \rangle = \frac{1}{3} \)
3. \( \langle [p_{1}^{\theta}], [S\tau] \rangle = \frac{1}{3} \)
4. \( \langle [q_{0}^{\theta}], [S\tau] \rangle = \frac{1}{3} \)
5. \( \langle [q_{1}^{\theta}], [S\tau] \rangle = \frac{1}{3} \)
6. \( \langle [r_{0}^{\theta}], [S\tau] \rangle = \frac{1}{3} \)
7. \( \langle [r_{1}^{\theta}], [S\tau] \rangle = \frac{1}{3} \)

\[ \text{Pairing of } [S\mathcal{E}_{0,0}] \]
1. \(\langle 1, [S\xi_{0,0}] \rangle = 0\)
2. \(\langle p_0^0, [S\xi_{0,0}] \rangle = \frac{1}{3}\)
3. \(\langle p_1^1, [S\xi_{0,0}] \rangle = \frac{\zeta^2}{3}\)
4. \(\langle q_0^0, [S\xi_{0,0}] \rangle = 0\)
5. \(\langle q_1^1, [S\xi_{0,0}] \rangle = 0\)
6. \(\langle r_0^0, [S\xi_{0,0}] \rangle = 0\)
7. \(\langle r_1^1, [S\xi_{0,0}] \rangle = 0\).

Pairing of \([S\xi_{0,0}^2]\)
1. \(\langle 1, [S\xi_{0,0}^2] \rangle = 0\)
2. \(\langle p_0^0, [S\xi_{0,0}^2] \rangle = \frac{1}{3}\)
3. \(\langle p_1^1, [S\xi_{0,0}^2] \rangle = \frac{\zeta^2}{3}\)
4. \(\langle q_0^0, [S\xi_{0,0}^2] \rangle = 0\)
5. \(\langle q_1^1, [S\xi_{0,0}^2] \rangle = 0\)
6. \(\langle r_0^0, [S\xi_{0,0}^2] \rangle = 0\)
7. \(\langle r_1^1, [S\xi_{0,0}^2] \rangle = 0\).

Pairing of \([S\xi_{0,1}]\)
1. \(\langle 1, [S\xi_{0,1}] \rangle = 0\)
2. \(\langle p_0^0, [S\xi_{0,1}] \rangle = 0\)
3. \(\langle p_1^1, [S\xi_{0,1}] \rangle = 0\)
4. \(\langle q_0^0, [S\xi_{0,1}] \rangle = 0\)
5. \(\langle q_1^1, [S\xi_{0,1}] \rangle = 0\)
6. \(\langle r_0^0, [S\xi_{0,1}] \rangle = 0\)
7. \(\langle r_1^1, [S\xi_{0,1}] \rangle = 0\).

Pairing of \([S\xi_{0,1}^2]\)
1. \(\langle 1, [S\xi_{0,1}^2] \rangle = 0\)
2. \(\langle p_0^0, [S\xi_{0,1}^2] \rangle = 0\)
3. \(\langle p_1^1, [S\xi_{0,1}^2] \rangle = 0\)
4. \(\langle q_0^0, [S\xi_{0,1}^2] \rangle = \frac{\zeta^2}{3\sqrt{\lambda^2}}\)
5. \(\langle q_1^1, [S\xi_{0,1}^2] \rangle = \frac{1}{3\sqrt{\lambda^2}}\)
6. \(\langle r_0^0, [S\xi_{0,1}^2] \rangle = 0\)
7. \(\langle r_1^1, [S\xi_{0,1}^2] \rangle = 0\).

Pairing of \([S\xi_{0,-1}]\)
1. \(\langle 1, [S\xi_{0,-1}] \rangle = 0\)
2. \(\langle p_0^0, [S\xi_{0,-1}] \rangle = 0\)
3. $\langle [p_1^0], [S_0^ε_{0,-1}] \rangle = 0$
4. $\langle [q_0^0], [S_0^ε_{0,-1}] \rangle = \frac{\zeta \sqrt{\lambda}}{3}$
5. $\langle [q_1^0], [S_0^ε_{0,-1}] \rangle = -\frac{\sqrt{\lambda}}{3}$
6. $\langle [r_0^0], [S_0^ε_{0,-1}] \rangle = 0$
7. $\langle [r_1^0], [S_0^ε_{0,-1}] \rangle = 0$.

Pairing of $[S_0^ε_{0,-1}]$

1. $\langle [1], [S_0^ε_{0,-1}] \rangle = 0$
2. $\langle [p_0^0], [S_0^ε_{0,-1}] \rangle = 0$
3. $\langle [p_1^0], [S_0^ε_{0,-1}] \rangle = 0$
4. $\langle [q_0^0], [S_0^ε_{0,-1}] \rangle = 0$
5. $\langle [q_1^0], [S_0^ε_{0,-1}] \rangle = 0$
6. $\langle [r_0^0], [S_0^ε_{0,-1}] \rangle = 0$
7. $\langle [r_1^0], [S_0^ε_{0,-1}] \rangle = 0$.

Pairing of $[S_φ]$

1. $\langle [1], [S_φ] \rangle = 0$
2. $\langle [p_0^0], [S_φ] \rangle = 0$
3. $\langle [p_1^0], [S_φ] \rangle = 0$
4. $\langle [q_0^0], [S_φ] \rangle = 0$
5. $\langle [q_1^0], [S_φ] \rangle = 0$
6. $\langle [r_0^0], [S_φ] \rangle = 0$
7. $\langle [r_1^0], [S_φ] \rangle = 0$.

The case $Γ = Z_4$.

Let $t$ satisfy the relations $t^4 = 1$, $tU_1t^{-1} = U_2$ and $tU_2t^{-1} = U_1^{-1}$. The known projections of $A_{θ}^{alg} \rtimes Z_4$ are as follows:

(i) $[1]$  
(ii) $[p_0^0]$, where $p_0^0 = \frac{1}{4}(1 + t + t^2 + t^3)$.  
(iii) $[p_1^0]$, where $p_1^0 = \frac{1}{4}(1 + it - t^2 - it^3)$.  
(iv) $[p_2^0]$, where $p_2^0 = \frac{1}{4}(1 - t + t^2 - t^3)$.  
(v) $[q_0^0]$, where $q_0^0 = \frac{1}{4}(1 + i\sqrt{\lambda}U_1t - [\sqrt{\lambda}U_1t]^2 - i[\sqrt{\lambda}U_1t]^3)$.  
(vi) $[q_1^0]$, where $q_1^0 = \frac{1}{4}(1 - \sqrt{\lambda}U_1t + [\sqrt{\lambda}U_1t]^2 - [\sqrt{\lambda}U_1t]^3)$.  
(vii) $[q_2^0]$, where $q_2^0 = \frac{1}{4}(1 - i\sqrt{\lambda}U_1t - [\sqrt{\lambda}U_1t]^2 + i[\sqrt{\lambda}U_1t]^3)$.  
(viii) $[r_0^0]$, where $r_0^0 = \frac{1}{2}(1 - U_1t^2)$.  

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Proof of Theorem 1.3. (b) The following are the Chern-Connes indices for $A_{\theta}^{alg} \times \mathbb{Z}_4$.

Pairing of $[S\tau]$
1. $\langle [1], [S\tau] \rangle = 1$
2. $\langle [p^0_0], [S\tau] \rangle = \frac{1}{4}$
3. $\langle [p^1_0], [S\tau] \rangle = \frac{1}{4}$
4. $\langle [p^2_0], [S\tau] \rangle = \frac{1}{4}$
5. $\langle [q^0_0], [S\tau] \rangle = \frac{1}{4}$
6. $\langle [q^1_0], [S\tau] \rangle = \frac{1}{4}$
7. $\langle [q^2_0], [S\tau] \rangle = \frac{1}{4}$
8. $\langle [r^0_0], [S\tau] \rangle = \frac{1}{2}$

Pairing of $[SD_{1,1}]$
1. $\langle [1], [SD_{1,1}] \rangle = 0$
2. $\langle [p^0_0], [SD_{1,1}] \rangle = 0$
3. $\langle [p^1_0], [SD_{1,1}] \rangle = 0$
4. $\langle [p^2_0], [SD_{1,1}] \rangle = 0$
5. $\langle [q^0_0], [SD_{1,1}] \rangle = \frac{-1}{4\sqrt{\lambda}}$
6. $\langle [q^1_0], [SD_{1,1}] \rangle = \frac{1}{4\sqrt{\lambda}}$
7. $\langle [q^2_0], [SD_{1,1}] \rangle = \frac{-1}{4\sqrt{\lambda}}$
8. $\langle [r^0_0], [SD_{1,1}] \rangle = 0$

Pairing of $[SD_{0,0}]$
1. $\langle [1], [SD_{0,0}] \rangle = 0$
2. $\langle [p^0_0], [SD_{0,0}] \rangle = \frac{1}{4}$
3. $\langle [p^1_0], [SD_{0,0}] \rangle = \frac{-1}{4}$
4. $\langle [p^2_0], [SD_{0,0}] \rangle = \frac{1}{4}$
5. $\langle [q^0_0], [SD_{0,0}] \rangle = 0$
6. $\langle [q^1_0], [SD_{0,0}] \rangle = 0$
7. $\langle [q^2_0], [SD_{0,0}] \rangle = 0$
8. $\langle [r^0_0], [SD_{0,0}] \rangle = 0$

Pairing of $[S(D_{0,1} + D_{1,0})]$ 
1. $\langle [1], [S(D_{0,1} + D_{1,0})] \rangle = 0$
2. $\langle [p^0_0], [S(D_{0,1} + D_{1,0})] \rangle = 0$
3. $\langle [p^1_0], [S(D_{0,1} + D_{1,0})] \rangle = 0$
4. $\langle [p^2_0], [S(D_{0,1} + D_{1,0})] \rangle = 0$
5. $\langle [q_0^i], [S(D_{0,1} + D_{1,0})] \rangle = 0$.
6. $\langle [q_1^i], [S(D_{0,1} + D_{1,0})] \rangle = 0$.
7. $\langle [q_2^i], [S(D_{0,1} + D_{1,0})] \rangle = 0$.
8. $\langle [r_0^i], [S(D_{0,1} + D_{1,0})] \rangle = -\frac{1}{2}$.

Pairing of $[S\mathcal{F}^{1}_{0,0}]$

1. $\langle [1], [S\mathcal{F}^{1}_{0,0}] \rangle = 0$
2. $\langle [p_0^i], [S\mathcal{F}^{1}_{0,0}] \rangle = \frac{1}{4}$
3. $\langle [p_1^i], [S\mathcal{F}^{1}_{0,0}] \rangle = \frac{-i}{4}$
4. $\langle [p_2^i], [S\mathcal{F}^{1}_{0,0}] \rangle = \frac{-1}{4}$
5. $\langle [q_0^i], [S\mathcal{F}^{1}_{0,0}] \rangle = 0$.
6. $\langle [q_1^i], [S\mathcal{F}^{1}_{0,0}] \rangle = 0$.
7. $\langle [q_2^i], [S\mathcal{F}^{1}_{0,0}] \rangle = 0$.
8. $\langle [r_0^i], [S\mathcal{F}^{1}_{0,0}] \rangle = 0$.

Pairing of $[S\mathcal{F}^{1}_{0,1}]$

1. $\langle [1], [S\mathcal{F}^{1}_{0,1}] \rangle = 0$
2. $\langle [p_0^i], [S\mathcal{F}^{1}_{0,1}] \rangle = 0$
3. $\langle [p_1^i], [S\mathcal{F}^{1}_{0,1}] \rangle = 0$
4. $\langle [p_2^i], [S\mathcal{F}^{1}_{0,1}] \rangle = 0$
5. $\langle [q_0^i], [S\mathcal{F}^{1}_{0,1}] \rangle = \frac{i\sqrt{\lambda}}{4}$.
6. $\langle [q_1^i], [S\mathcal{F}^{1}_{0,1}] \rangle = \frac{-\sqrt{\lambda}}{4}$.
7. $\langle [q_2^i], [S\mathcal{F}^{1}_{0,1}] \rangle = \frac{-i\sqrt{\lambda}}{4}$.
8. $\langle [r_0^i], [S\mathcal{F}^{1}_{0,1}] \rangle = 0$.

Pairing of $[S\mathcal{F}^{-i}_{0,0}]$

1. $\langle [1], [S\mathcal{F}^{-i}_{0,0}] \rangle = 0$
2. $\langle [p_0^i], [S\mathcal{F}^{-i}_{0,0}] \rangle = \frac{1}{4}$
3. $\langle [p_1^i], [S\mathcal{F}^{-i}_{0,0}] \rangle = \frac{-i}{4}$
4. $\langle [p_2^i], [S\mathcal{F}^{-i}_{0,0}] \rangle = \frac{-1}{4}$
5. $\langle [q_0^i], [S\mathcal{F}^{-i}_{0,0}] \rangle = 0$.
6. $\langle [q_1^i], [S\mathcal{F}^{-i}_{0,0}] \rangle = 0$.
7. $\langle [q_2^i], [S\mathcal{F}^{-i}_{0,0}] \rangle = 0$.
8. $\langle [r_0^i], [S\mathcal{F}^{-i}_{0,0}] \rangle = 0$. 
Pairing of $[S\mathcal{F}^{-1}_{0,1}]$
1. $\langle [1], [S\mathcal{F}^{-1}_{0,1}] \rangle = 0$
2. $\langle [p^0], [S\mathcal{F}^{-1}_{0,1}] \rangle = 0$
3. $\langle [p^0], [S\mathcal{F}^{-1}_{0,1}] \rangle = 0$
4. $\langle [p^0], [S\mathcal{F}^{-1}_{0,1}] \rangle = 0$
5. $\langle [q^0], [S\mathcal{F}^{-1}_{0,1}] \rangle = -\frac{i}{4\sqrt{\lambda}}$
6. $\langle [q^0], [S\mathcal{F}^{-1}_{0,1}] \rangle = -\frac{1}{4\sqrt{\lambda}}$
7. $\langle [q^0], [S\mathcal{F}^{-1}_{0,1}] \rangle = -\frac{i}{4\sqrt{\lambda}}$
8. $\langle [r^0], [S\mathcal{F}^{-1}_{0,1}] \rangle = 0$.

Pairing of $[S\varphi]$
1. $\langle [1], [S\varphi] \rangle = 0$
2. $\langle [p^0], [S\varphi] \rangle = 0$
3. $\langle [p^0], [S\varphi] \rangle = 0$
4. $\langle [p^0], [S\varphi] \rangle = 0$
5. $\langle [q^0], [S\varphi] \rangle = 0$
6. $\langle [q^0], [S\varphi] \rangle = 0$
7. $\langle [q^0], [S\varphi] \rangle = 0$
8. $\langle [r^0], [S\varphi] \rangle = 0$.

The case $\Gamma = \mathbb{Z}_6$.

Let $t$ satisfy the relations $t^6 = 1$, $tU_1t^{-1} = U_2$ and $tU_2t^{-1} = \frac{U_{U_2}}{-\sqrt{\lambda}}$. The known projections of $\mathcal{A}_\theta^\text{alg} \rtimes \mathbb{Z}_6$ are as follows:

(i) $[1]$
(ii) $[p^0]$, where $p^0 = \frac{1}{6}(1 + t + t^2 + t^3 + t^4 + t^5)$.
(iii) $[p^1]$, where $p^1 = \frac{1}{6}(1 + \zeta + \zeta^2 t^2 - t^3 + \zeta^4 t^4 + \zeta^5 t^5)$.
(iv) $[p^2]$, where $p^2 = \frac{1}{6}(1 + \zeta^2 t + \zeta^4 t^2 + t^3 - t^4 + \zeta^4 t^5)$.
(v) $[p^3]$, where $p^3 = \frac{1}{6}(1 - t + t^2 - t^3 + t^4 - t^5)$.
(vi) $[p^4]$, where $p^4 = \frac{1}{6}(1 + \zeta^4 t + \zeta^2 t^2 + t^3 + \zeta^4 t^4 + \zeta^2 t^5)$.
(vii) $[q^0]$, where $q^0 = \frac{1}{3}(1 + e^{(2\pi i (2+\theta)/6)}(U_1 t^2) + [e^{(2\pi i (2+\theta)/6)}(U_1 t^2)]^2)$.
(viii) $[q^1]$, where $q^1 = \frac{1}{3}(1 + \zeta^2 e^{(2\pi i (2+\theta)/6)}(U_1 t^2) + \zeta^4 [e^{(2\pi i (2+\theta)/6)}(U_1 t^2)]^2)$.
(ix) $[r^0]$, where $r^0 = \frac{1}{2}(1 - U_1 t^3)$.

Proof of Theorem 1.3. (c) The following are the Chern-Connes indices for $\mathcal{A}_\theta^\text{alg} \rtimes \mathbb{Z}_6$.

Pairing of $[S\tau]$
1. $\langle [1], [S\tau] \rangle = 1$
2. \( \langle [p_0^0], [S\tau] \rangle = \frac{1}{6} \)
3. \( \langle [p_1^0], [S\tau] \rangle = \frac{1}{6} \)
4. \( \langle [p_2^0], [S\tau] \rangle = \frac{1}{6} \)
5. \( \langle [q_0^0], [S\tau] \rangle = \frac{1}{6} \)
6. \( \langle [q_1^0], [S\tau] \rangle = \frac{1}{3} \)
7. \( \langle [q_2^0], [S\tau] \rangle = \frac{1}{3} \)
8. \( \langle [r_0^0], [S\tau] \rangle = \frac{1}{2} \)

**Pairing of \([SD_{0,0}]\)**

1. \( \langle [1], [SD_{0,0}] \rangle = 0 \)
2. \( \langle [p_0^0], [SD_{0,0}] \rangle = \frac{1}{6} \)
3. \( \langle [p_1^0], [SD_{0,0}] \rangle = -\frac{1}{6} \)
4. \( \langle [p_2^0], [SD_{0,0}] \rangle = \frac{1}{6} \)
5. \( \langle [p_3^0], [SD_{0,0}] \rangle = \frac{1}{6} \)
6. \( \langle [q_0^0], [SD_{0,0}] \rangle = 0 \)
7. \( \langle [q_1^0], [SD_{0,0}] \rangle = 0 \)
8. \( \langle [r_0^0], [SD_{0,0}] \rangle = 0 \)

**Pairing of \([SD_{1,0} + \lambda \sqrt{\lambda}D_{1,0} + \sqrt{\lambda}D_{1,1}]\)**

1. \( \langle [1], [SD_{1,0} + \lambda \sqrt{\lambda}D_{1,0} + \sqrt{\lambda}D_{1,1}] \rangle \rangle = 0 \)
2. \( \langle [p_0^0], [SD_{1,0} + \lambda \sqrt{\lambda}D_{1,0} + \sqrt{\lambda}D_{1,1}] \rangle \rangle = 0 \)
3. \( \langle [p_1^0], [SD_{1,0} + \lambda \sqrt{\lambda}D_{1,0} + \sqrt{\lambda}D_{1,1}] \rangle \rangle = 0 \)
4. \( \langle [p_2^0], [SD_{1,0} + \lambda \sqrt{\lambda}D_{1,0} + \sqrt{\lambda}D_{1,1}] \rangle \rangle = 0 \)
5. \( \langle [p_3^0], [SD_{1,0} + \lambda \sqrt{\lambda}D_{1,0} + \sqrt{\lambda}D_{1,1}] \rangle \rangle = 0 \)
6. \( \langle [q_0^0], [SD_{1,0} + \lambda \sqrt{\lambda}D_{1,0} + \sqrt{\lambda}D_{1,1}] \rangle \rangle = 0 \)
7. \( \langle [q_1^0], [SD_{1,0} + \lambda \sqrt{\lambda}D_{1,0} + \sqrt{\lambda}D_{1,1}] \rangle \rangle = 0 \)
8. \( \langle [r_0^0], [SD_{1,0} + \lambda \sqrt{\lambda}D_{1,0} + \sqrt{\lambda}D_{1,1}] \rangle \rangle = 0 \)
9. \( \langle [r_0^0], [SD_{1,0} + \lambda \sqrt{\lambda}D_{1,0} + \sqrt{\lambda}D_{1,1}] \rangle \rangle = -\frac{\lambda \sqrt{\lambda}}{2} \)

**Pairing of \([S(\xi_{0,1}^0 + \xi_{0,-1}^0)]\)**

1. \( \langle [1], [S(\xi_{0,1}^0 + \xi_{0,-1}^0)] \rangle \rangle = 0 \)
2. \( \langle [p_0^0], [S(\xi_{0,1}^0 + \xi_{0,-1}^0)] \rangle \rangle = 0 \)
3. \( \langle [p_1^0], [S(\xi_{0,1}^0 + \xi_{0,-1}^0)] \rangle \rangle = 0 \)
4. \( \langle [p_2^0], [S(\xi_{0,1}^0 + \xi_{0,-1}^0)] \rangle \rangle = 0 \)
5. \( \langle [p_3^0], [S(\xi_{0,1}^0 + \xi_{0,-1}^0)] \rangle \rangle = 0 \)
6. \( \langle [p_6^2], [S(\mathcal{E}_{0,1}^\omega + \mathcal{E}_{0,-1}^\omega)] \rangle = 0 \).
7. \( \langle [q_0^6], [S(\mathcal{E}_{0,1}^\omega + \mathcal{E}_{0,-1}^\omega)] \rangle = \frac{\zeta}{3} \).
8. \( \langle [q_1^6], [S(\mathcal{E}_{0,1}^\omega + \mathcal{E}_{0,-1}^\omega)] \rangle = -\frac{1}{3} \).
9. \( \langle [r_0^6], [S(\mathcal{E}_{0,1}^\omega + \mathcal{E}_{0,-1}^\omega)] \rangle = 0 \).

**Pairing of \([S\mathcal{E}_{0,0}^\omega] \)**
1. \( \langle [1], [S\mathcal{E}_{0,0}^\omega] \rangle = 0 \)
2. \( \langle [p_0^0], [S\mathcal{E}_{0,0}^\omega] \rangle = \frac{1}{6} \)
3. \( \langle [p_1^0], [S\mathcal{E}_{0,0}^\omega] \rangle = \frac{\zeta^2}{3} \)
4. \( \langle [p_2^0], [S\mathcal{E}_{0,0}^\omega] \rangle = -\frac{\zeta}{3} \)
5. \( \langle [p_3^0], [S\mathcal{E}_{0,0}^\omega] \rangle = \frac{1}{6} \)
6. \( \langle [q_0^6], [S\mathcal{E}_{0,0}^\omega] \rangle = \frac{\zeta^2}{6} \)
7. \( \langle [q_1^6], [S\mathcal{E}_{0,0}^\omega] \rangle = 0 \)
8. \( \langle [q_1^6], [S\mathcal{E}_{0,0}^\omega] \rangle = 0 \)
9. \( \langle [r^6], [S\mathcal{E}_{0,0}^\omega] \rangle = 0 \).

**Pairing of \([S(\mathcal{E}_{0,1}^\omega + \mathcal{E}_{0,-1}^\omega)] \)**
1. \( \langle [1], [S(\mathcal{E}_{0,1}^\omega + \mathcal{E}_{0,-1}^\omega)] \rangle = 0 \)
2. \( \langle [p_0^0], [S(\mathcal{E}_{0,1}^\omega + \mathcal{E}_{0,-1}^\omega)] \rangle = 0 \)
3. \( \langle [p_1^0], [S(\mathcal{E}_{0,1}^\omega + \mathcal{E}_{0,-1}^\omega)] \rangle = 0 \)
4. \( \langle [p_2^0], [S(\mathcal{E}_{0,1}^\omega + \mathcal{E}_{0,-1}^\omega)] \rangle = 0 \)
5. \( \langle [p_3^0], [S(\mathcal{E}_{0,1}^\omega + \mathcal{E}_{0,-1}^\omega)] \rangle = 0 \)
6. \( \langle [q_0^6], [S(\mathcal{E}_{0,1}^\omega + \mathcal{E}_{0,-1}^\omega)] \rangle = \frac{\zeta^2}{3\sqrt{\lambda}} \)
7. \( \langle [q_1^6], [S(\mathcal{E}_{0,1}^\omega + \mathcal{E}_{0,-1}^\omega)] \rangle = -\frac{\zeta}{3\sqrt{\lambda}} \)
8. \( \langle [r^6], [S(\mathcal{E}_{0,1}^\omega + \mathcal{E}_{0,-1}^\omega)] \rangle = 0 \).

**Pairing of \([S\mathcal{E}_{0,0}^\omega] \)**
1. \( \langle [1], [S\mathcal{E}_{0,0}^\omega] \rangle = 0 \)
2. \( \langle [p_0^0], [S\mathcal{E}_{0,0}^\omega] \rangle = \frac{1}{6} \)
3. \( \langle [p_1^0], [S\mathcal{E}_{0,0}^\omega] \rangle = -\frac{\zeta}{6} \)
4. \( \langle [p_2^0], [S\mathcal{E}_{0,0}^\omega] \rangle = -\frac{1}{6} \)
5. \( \langle [p_3^0], [S\mathcal{E}_{0,0}^\omega] \rangle = \frac{1}{6} \).
6. \( \langle [p_0^0], [S_0^0] \rangle = \frac{-\zeta}{6} \)
7. \( \langle [q_0^0], [S_0^0] \rangle = 0 \)
8. \( \langle [q_1^0], [S_0^0] \rangle = 0 \)
9. \( \langle [r^0], [S_0^0] \rangle = 0 \)

Pairing of \( [SG_{0,0}^\omega] \)
1. \( \langle [1], [SG_{0,0}^\omega] \rangle = 0 \)
2. \( \langle [p_0^0], [SG_{0,0}^\omega] \rangle = \frac{1}{6} \)
3. \( \langle [p_1^0], [SG_{0,0}^\omega] \rangle = \frac{\zeta}{6} \)
4. \( \langle [p_2^0], [SG_{0,0}^\omega] \rangle = \frac{\zeta^2}{6} \)
5. \( \langle [p_3^0], [SG_{0,0}^\omega] \rangle = -\frac{1}{6} \)
6. \( \langle [p_4^0], [SG_{0,0}^\omega] \rangle = -\frac{\zeta}{6} \)
7. \( \langle [q_0^0], [SG_{0,0}^\omega] \rangle = 0 \)
8. \( \langle [q_1^0], [SG_{0,0}^\omega] \rangle = 0 \)
9. \( \langle [r^0], [SG_{0,0}^\omega] \rangle = 0 \)

Pairing of \( [SG_{0,0}^{\omega^2}] \)
1. \( \langle [1], [SG_{0,0}^{\omega^2}] \rangle = 0 \)
2. \( \langle [p_0^0], [SG_{0,0}^{\omega^2}] \rangle = \frac{1}{6} \)
3. \( \langle [p_1^0], [SG_{0,0}^{\omega^2}] \rangle = \frac{-\zeta^2}{6} \)
4. \( \langle [p_2^0], [SG_{0,0}^{\omega^2}] \rangle = \frac{-\zeta}{6} \)
5. \( \langle [p_3^0], [SG_{0,0}^{\omega^2}] \rangle = \frac{1}{6} \)
6. \( \langle [p_4^0], [SG_{0,0}^{\omega^2}] \rangle = \frac{\zeta^2}{6} \)
7. \( \langle [q_0^0], [SG_{0,0}^{\omega^2}] \rangle = 0 \)
8. \( \langle [q_1^0], [SG_{0,0}^{\omega^2}] \rangle = 0 \)
9. \( \langle [r^0], [SG_{0,0}^{\omega^2}] \rangle = 0 \)

Pairing of \( [S_\varphi] \)
1. \( \langle [1], [S_\varphi] \rangle = 0 \)
2. \( \langle [p_0^0], [S_\varphi] \rangle = 0 \)
3. \( \langle [p_1^0], [S_\varphi] \rangle = 0 \)
4. \( \langle [p_2^0], [S_\varphi] \rangle = 0 \)
5. \( \langle [p_3^0], [S_\varphi] \rangle = 0 \)
6. \( \langle [p_4^0], [S_\varphi] \rangle = 0 \)
7. \( \langle [q_0^0], [S_\varphi] \rangle = 0 \)
8. \( \langle [q_1^0], [S_\varphi] \rangle = 0 \)
5. Conclusion and Conjectures

The homology and cohomology groups \([Q1], [Q2]\) and computed in this article gives us a complete understanding of the noncommutative algebraic noncommutative torus orbifold. Through our computations, we see the following dualities:

\[
H_\bullet(\mathbb{A}^{alg}_\theta \rtimes \Gamma, \mathbb{A}^{alg}_\theta \rtimes \Gamma) \cong H^\bullet(\mathbb{A}^{alg}_\theta \rtimes \Gamma, (\mathbb{A}^{alg}_\theta \rtimes \Gamma)^*) .
\]

\[
HP_\bullet(\mathbb{A}^{alg}_\theta \rtimes \Gamma) \cong HP^\bullet(\mathbb{A}^{alg}_\theta \rtimes \Gamma).
\]

We conjecture that this duality will hold for the noncommutative smooth orbifold \(\mathbb{A}_\theta \rtimes \Gamma\) under similar restriction on \(\theta\) as in \([C]\). Further we conjecture the following:

**Conjecture 5.1.** \(K_0(\mathbb{A}^{alg}_\theta \rtimes \Gamma) \cong \begin{cases} 
\mathbb{Z}^7 & \text{for } \Gamma = \mathbb{Z}_3 \\
\mathbb{Z}^8 & \text{for } \Gamma = \mathbb{Z}_4 \\
\mathbb{Z}^9 & \text{for } \Gamma = \mathbb{Z}_6.
\end{cases} \)

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Safdar Quddus,
School of Mathematical Sciences,
National Institute of Science Education and Research, Bhubaneswar, India.
Email: safdar@niser.ac.in.