A Topological Approach to Quantum Electrodynamics

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Abstract
The physical reasons in favour of a two dimensional topological model of quantum electrodynamics are discussed. It is shown that in accord with this model there is a new uncertainty relation for photon which is compatible with QED.
We discuss a two dimensional topological approach to quantum electrodynamics which can be helpful to understand dynamical aspects of two dimensional topological quantum effects such as flux quantization, cyclotron motion, Aharonov-Bohm effect and QHE. Note that all these effects can be considered as two dimensional quantum electrodynamical phase effects caused by a magnetic field which is perpendicular to the two dimensional surface of motion of electrons.

Our motivation in considering such a theory is based on the physical fact that the quantum field of electromagnetic interaction, the photon, possess only two degrees of freedom which refers to a two dimensional geometrical background. Thus even in the four dimensional Maxwell theory one reduces the original four degrees of freedom of the electromagnetic field to the two physical degrees of freedom, e. g. by the use of radiation gauge. Further it is based on the phenomenological fact that the mentioned two dimensional quantum phase, i. e. \[ \oint_{(\text{contour})} A_m dx^m = N \hbar; m, n = 1, 2; N \in \mathbb{Z} \] where \( e \) and \( A_m \) are the electric charge and the electromagnetic potential, is verified by flux quantization and Aharonov-Bohm effect. It is based also on theoretical consideration that even the four dimensional Maxwell action:

\[
\int F \wedge *F, F = F_{mn} dx^m \wedge dx^n \]

must be described by the substructure of electromagnetic two-forms \( F \) and \( *F \).

Note also that in view of global character of quantum theory which is manifested through the globality of quantum state (\( \sim \) wave function), the global aspects of the underlying manifolds or bundle manifolds are essential for the structure of quantum theories which are defined on this manifold. In this sense topological invariants, e. g. cohomology, homology or harmonics on the mentioned manifolds determine global aspects of the mentioned quantum theories [1]. Accordingly we show that in view of the main role played by the second cohomology \( H^2(M_{4D}) \cong \text{Harm}^2(M_{4D}) \) of space-time four manifold in electrodynamics, such two-forms and their "dual" surfaces are essential even in four dimensional quantum electrodynamics.

Hence with respect to topological quantum effects, two dimensional quantum theories defined by such two forms on such surfaces can replace four dimensional quantum theories.

We will show first that the structure of action and so the equations of motion of four dimensional Maxwell theory can be considered as conditions which restrict the electromagnetic two-forms to be defined on a two dimensional submanifold of the original four dimensional space-time. Recall that Maxwell equations in QED are considered also as conditions on the quantum state [2].
There are various local or differential and global or topological hints about the main role played by the two dimensional substructures of the four dimensional classical structure in four dimensional classical and quantum electrodynamics, i.e. about the restriction of the relevant structures in both theories to two-forms and two dimensional submanifolds. The first one is the absence of three- and simple four-forms in the Lagrangian of electrodynamics in view of its restriction to two-forms, although the underlying manifold is a four dimensional one. This fact alone can be considered as the irrelevance of higher than simple two-forms in electromagnetism, since if there were simple electromagnetic four or three-forms, they should be involved in such a general four dimensional theory. On the other hand to define a two form a two dimensional manifold is sufficient, so that an electromagnetic field strength can be defined on a two dimensional submanifold of the \((3 + 1)\)-dimensional space-time. A reasonable two dimensional boundaryless non-boundary submanifold: \(M_{2D} \in H_2(M_{4D})\) given as a part of three dimensional space seems to be suitable for our objection.

The second hint is that, accordingly, the vacuum equations of motions of both electrodynamics, i.e. \(dF = 0\) and \(d^\dagger F = 0\) restrict the involved two-forms which are assumed to be defined on a four manifold \(M_{4D}\) to be harmonic forms: \(F \in Harm^2(M_{4D})\). Thus the relevant electrodynamical two-form \(F \in Harm^2(M_{4D})\) have no contribution from higher than two dimensional structures of the underlying four manifold, although the general possible electrodynamical two form on a compact orientable four manifold without boundary is given by the Hodge decomposition \(F = dA \oplus d^\dagger \Omega^3(A, F) \oplus Harm^2\) [3].

Since, if one considers the term \(d^\dagger \Omega^3(A, F)\) as the contribution of the higher than two dimensional structure of the four manifold to the structure of two forms, then the absence of this term shows that the relevant electromagnetic two-forms are those which can be defined only on a two dimensional submanifold \(M_{2D}\) of the four manifold. Hence \(F = dA \oplus Harm^2\) is an element of \(H^2(M_{4D})\) which is isomorphic to \(Harm^2(M_{4D})\). Note also that the dimension of the \(H^2(M_{2D}) \cong H^2(M_{2D})\) is closely related with the invariant aspects of two dimensional manifolds.

The third hint is that in QED only slowly varying field strengths: \(dF \ll F\) and \(\partial_t F \ll F\) produce finite terms which enables one to renormalize QED [4], i.e. again only fields with \(dF = 0\) and in view of equations of motion also with \(d^\dagger F = 0\) are QED relevant. It means that only the solutions of Maxwell equations \(dF = 0, d^\dagger F = 0\) produce physical, i.e. finite, results in QED. This is in accord with the path
integral quantization idea where only the real classical path, which is the solution of classical equations of motion, contributes to the phase of quantum state.

As the last local hint let us mention that the electromagnetic field strength in the Landau gauge, which is the usual one for quantization in presence of magnetic field, is restricted to the two dimensional field which is defined on the two dimensional spatial submanifold: $F(M_{2D})$. Thus even the phenomenological description of magnetic quantization is obliged to use the two dimensional two-forms $F(M_{2D})$ instead of the four dimensional ones: $F(M_{4D})$.

In other words the four dimensional classical as well as quantum electrodynamics are not only based on the two dimensional substructure of two-forms, but both theories restrict these two-forms to be harmonic two-forms which should be defined on the $M_{2D}$ submanifolds of the four manifold. Moreover recall that in QED only average of field strengths over finite space-time regions have a well defined meaning. Hence these averages can be considered as to be averaged over the rest $(1 + 1)$ dimensional part and to be defined only on $M_{2D}$ submanifold of the $(3 + 1)$ dimensional space-time. Thus one can consider the averaged field strength to be constant with respect to the rest and to depend only on variables on the $M_{2D}$ submanifold.

Also the topological invariant properties of Maxwell action $\int F \wedge * F$, which are essential with respect to the topological (global) character of quantum phases, refer to two dimensional invariants, since all relevant invariants here are constructed from the electromagnetic two form $F$.

The electromagnetic elements of the four manifold cohomology invariants are the action function $f_{(em)}(A(x)) = \int F(A) \wedge * F(A); \ f_{(em)}(A(x)) \in H^0(M_{4D})$, the closed non-exact electromagnetic two-form $F \in H^2(M_{4D})$ and the Maxwell-form $\Omega^4(M_{4D}) := F \wedge * F; \ \Omega^4_{(em)}(M_{4D}) \in H^4(M_{4D})$ are constructed from the electromagnetic two-form $F = dA \oplus Harm^2_{(em)}; \ F \in H^2_{(em)}(M_{4D})$. In other words $H^2_{(em)}(M_{4D}) \times H^2_{(em)}(M_{4D}) \to H^4_{(em)} \cong H^0_{(em)}(M_{4D})$. Note that in view of Hodge theorem on compact orientable Riemannian manifolds: $H^r(M) \cong Harm^r(M)$ and $dimH^r(M) = dimHarm^r(M)$ all above isomorphisms are given also between harmonics, so that the solutions of Maxwell equations $\in Harm^2(M_{4D})$ determine in this way global aspects of $M_{4D}$ like the Euler characteristic $\chi(M_{4D}) = \Sigma(-1)^r dimHarm^r(M_{4D})$. Thus in the simply connected case of interest where $H^1 \cong H^3 = 0$ only $dimHarm^2$ will determine the Euler characteristic of our $M_{4D}$. Therefore even the topological aspects of four dimensional QED: $H^4 \cong Harm^4 \cong Harm^0 \cong H^0$. 

4
which are essential in topological quantum effects are given by the two dimensional topological aspects, since these topological invariants of four manifold are given in terms of topological invariants of the two dimensional submanifold. Thus, if with regard to the Maxwell-form $H^2_{(em)}(M_4D) \times H^2_{(em)}(M_4D) \rightarrow H^4_{(em)}(M_4D)$, also the "dual" map $H_2(M_4D) \times H_2(M_4D) \rightarrow H_0(M_4D) \cong H_4(M_4D)$ is given, then the homological invariants of our four manifold are also given by the invariants of its two dimensional submanifold: $M_{2D} \in H_2(M_4D)$.

Moreover recall that the invariant of $F$, i.e. $\int_{\text{surface}} F$, is obtained with respect to a two dimensional surface and also that in view of the absence of three-forms on two dimensional manifolds $M_{2D}$ any two-form on these manifolds is a closed two form, i.e. in our case $dF(M_{2D}) = 0$. Note also that the two dimensional equations of motion $dF^\dagger(M_{2D}) = 0$, which result from the mentioned invariant action $\int_{\text{surface}} F$, are together with the two dimensionality condition: $dF(M_{2D}) = 0$ equivalent to the equations of motion in the four dimensional case: $dF^\dagger(M_{4D}) = 0$, $dF(M_{4D}) = 0$. So that in two dimensional case the relevant $F$ is given by $F \in Harm^2(M_{2D})$ and in the four dimensional case the relevant $F$ is given by $F \in Harm^2(M_{4D})$. Nevertheless, as it is discussed above, the construction of the underlying four manifold is so that only the spatial two dimensional submanifold seems to be relevant for definition of the physical electromagnetic two-form. Thus in two dimensional case there is an isomorphism $H^0(M_{2D}) \cong H^2(M_{2D})$ which replaces the four dimensional isomorphism $H^2(M_{4D}) \times H^2(M_{4D}) \rightarrow H^4(M_{4D}) \cong H^0(M_{4D})$. Furthermore recall that the second cohomology of a four manifold $H^2(M_{4D})$ is "destroyed" by removing the $M_{2D} \in H_2(M_{4D})$ surfaces from the four manifold $M_{4D}$. Therefore the Maxwell action $H^2_{(em)}(M_{4D}) \times H^2_{(em)}(M_{4D}) \rightarrow H^4_{(em)}(M_{4D}) \cong H^0_{(em)}(M_{4D})$ depends entirely on the two dimensional submanifold $M_{2D} \in H_2(M_{4D})$. Accordingly also the constructing two-from $F \in H^2_{(em)}(M_{4D})$ depends only on the surface $M_{2D} \in H_2(M_{4D})$ and "effectively" it should be defined on such a surface.

Note also that the usual four dimensional coupling term $\int A_\mu j^\mu dt$ with $j^\mu = ne\hat{\epsilon}^\mu$ where $n$ is the electronic density, is equal to $Q \int A_\mu dx^\mu$: $Q = ne$ which reduces in the two dimensional case to $Q \int A_m dx^m$.

Therefore the whole four dimensional action $\int_{M_4D} F \wedge *F + \int A_\mu \cdot j^\mu dt$ reduces in the two dimensional single electron case to $e \int_{M_2D} A_m dx^m = \int_{\text{surface}} F$, since in this case also $\int_{4D} F \wedge *F$ reduces to $\int_{M_{2D}} F$. Recall that in two dimensions $*\Omega^2 = \Omega^0$, then in view of $\Omega^0 \cdot \Omega^2 \sim \Omega^2$ one has in this case $F(M_{2D}) \wedge *F(M_{2D}) \sim$
Therefore, in order to adopt all these facts in the theory, we conjecture a two dimensional invariant of two-form $F$, i.e., \( \int_{\text{surface}} F \), for the electromagnetic action which avoids problems with extra conditions for renormalization and with constraints of a four dimensional QED.

The two dimensional electromagnetic action of interest is given by the classical flux function

\[
S_{(cl)} = \Phi_{(cl)} = e \oint_{\text{contour}} A_m dx^m = e \int_{\text{surface}} F_{mn} dx^m \wedge dx^n, \tag{1}
\]

where \( e, A_m \) and \( F_{mn} \) are, respectively, the electric charge of the electron, the electromagnetic potential and the magnetic field strength interacting with the electron and \( m, n = 1, 2 \). Here the domain of electromagnetic potential and magnetic field strength is a non-simply connected region containing of two regions which is similar to the case of Aharonov-Bohm effect: One is the flux surface where a constant magnetic field \( B_{(\text{surface})} = B_{(\text{constant})} := \epsilon_{mn} F_{(\text{surface})}^{mn} \) is present and \( A_{(\text{surface})}^m = B_{(\text{constant})} x_n \epsilon^{mn} \). The second is the contour region which surrounds this flux surface where the magnetic field is absent \( B_{(\text{contour})} = 0 \) and \( A_{(\text{contour})}^m = \partial^m \Phi \), i.e. \( dA_{(\text{contour})} = B_{(\text{contour})} = 0 \). Thus, the integral \( e \oint_{\text{contour}} A_m dx^m \) is defined on the contour region, whereas the equivalent integral \( e \int_{\text{surface}} F_{mn} dx^m \wedge dx^n \) is defined on the surface region.

We will prove that the canonical quantization of \( S_{(cl)} \) is given by the commutator postulate:

\[
\epsilon [\hat{A}_m, \hat{x}_m] = -i \hbar \quad \text{which is related with a new uncertainty relation } \epsilon \Delta A_m \cdot \Delta x_m \geq \hbar \text{ for photon.}
\]

Hereby functions \( x_m \) are the position coordinates of an electron interacting with the magnetic field \( F_{mn} \) of photon \( A_m \). In other words in this approach photon is considered, in accord with the equivalence between quantum fields and quantum particles in quantum field theory, as a quantum particle with usual uncertainty properties in measurements (see below).

With respect to the commutator postulate note that, in view of \( A_{(\text{surface})}^m = B_{(\text{constant})} x_n \epsilon^{mn} \) and \( A_{(\text{contour})}^m := \partial^m \Phi \), in both relevant regions \( A_m \) is not a function of \( x_m \) and so there is no a priori reason for the commutativity of \( \hat{A}_m \) and \( \hat{x}_m \) operators. Further recall that although the \( A_m \) potential is non-observable in view of its gauge dependence, however the quantized integral \( e \oint_{\text{contour}} A_m dx^m \) is in view of Aharonov-Bohm effect or flux quantization an observable phase. Thus the difference of two gauge potential \( \Delta A_m \sim (\delta A_m = \hat{A}_m - A_m) \) is, in view of gauge transformations: \( A'_m = A_m + \partial_m \Lambda \), \( \hat{A}'_m = \hat{A}_m + \partial_m \Lambda \), a gauge invariant quantity.
Not that the quantization postulates $S_Q = e \oint (\text{contour}) A_m dx^m = N\hbar$, $e[\hat{A}_m, \hat{x}_m] = -i\hbar$ and $e\Delta A_m \cdot \Delta x_m \geq \hbar$ can be compared with the canonical quantization postulates $\oint P_m dq^m = N\hbar$, $[\hat{P}_m, \hat{q}_m] = -i\hbar$ and $\Delta P_m \cdot \Delta q_m \geq \hbar$. Thus the quantization $S_Q = e \int (\text{surface}) F_{mn} dx^m \wedge dx^n = N\hbar$ is just the integrality condition for the first Chern class $c_1 = F$ and in this sense it is a well defined geometric quantization [10].

We will show that, indeed for the canonical conjugate variables of phase space of the two dimensional electromagnetic system which is represented by $S_{(cl)}$, the commutator of related operators is non-trivial. The key point is the correct choice of phase space, i.e. the choice of true canonical conjugate variables for the two dimensional electromagnetic system under consideration.

The point of departure is the two dimensional electromagnetic action functional:

$$S_{(cl)} = \Phi_{(cl)} = e \int (\text{surface}) F_{mn} dx^m \wedge dx^n = e \int (\text{surface}) dA_n \wedge dx^n = e \oint (\text{contour}) A_n dx^n, \quad (1)$$

with $dA_n := \partial_m A_n dx^m \epsilon_{mn}$.

The action is defined on the electromagnetic $U(1)$ bundle over the two dimensional manifold which consists of a two dimensional non-interacting electronic system in magnetic field in the "single electron picture".

First we show that $S_{(cl)} = e \int (\text{surface}) F_{mn} dx^m \wedge dx^n$ is a well defined action functional from which one can derive the equations of motion for $A_m$, so that it can be quantized canonically in order to describe the quantized dynamics of the two dimensional electromagnetic system.

In view of the fact that $A_m$ depends on $x^n$ by $A_{m(\text{surface})} = B_{\text{constant}} x_n \epsilon_{mn}$, the variation of action $\delta S_{(cl)}$ needs to be considered only with respect to the variation of $\delta x^n$, since the variation $\delta A_m$ is proportional to $\delta x_n$. Hence, one has to consider $dx^l = \frac{\partial x^l}{\partial x^m} dx^m$. The Euler-Lagrange equations $\frac{\partial L}{\partial x^m} = \partial_n \frac{\partial L}{\partial \partial_n x^m}$ which result from the variation of this action with respect to the variation $\delta x^n$ are:

$$\partial_n \partial_n A_m - \partial_n \partial_m A_n = 0 \quad (2)$$
These are the usual equations of motion for $A_m$ potential in vacuum. Nevertheless, in view of the fact that $A_m^{(surface)} = B_{(constant)} x^n \epsilon_{mn}$, the second term is identically zero and one is left with the Laplace equation in two dimensions $\partial^n \partial_n A_m = 0$. Note that also Maxwell equations in vacuum together with Lorentz condition $d^I A = 0$ result in the four dimensional Laplace equation $d d^I A + d^I dA = 0$.

Therefore, the action $S_{(cl)} = e \oint_{(surface)} F_{mn} dx^m \wedge dx^n = e \oint_{(contour)} A_m dx^m$ is a well defined action functional for our two dimensional system, which can be canonically quantized in order to describe the quantum behaviour of photon.

To quantize the phase space of a classical system which is represented by an action functional $S_{(cl)}$, one should determine first the canonical conjugate variables of phase space and then one should postulate the quantum commutator for operators which are related to these variables. Now to determine the phase space variables of the system represented by the action functional $S_{(cl)}$ one can use the Legendre transformation formula $P_m := \frac{\partial L}{\partial \dot{q}_m}$ which is defined for the phase space of canonical action: $\oint_{phase \ space} P_m \dot{q}_m dt$. Thus the phase space of our system which is represented by the action $S_{(cl)} = e \oint_{(contour)} A_m dx^m = e \oint_{(contour)} A_m \dot{x}_m dt$ has the canonical conjugate variables $\{A_m, x^m\}$ and it can be quantized directly in comparison with the phase space of canonical action as mentioned above. Nevertheless to be precise we perform the quantization of this system in accord with the general formalism of geometric quantization [10]:

Then, the globally Hamiltonian vector fields of our system with the symplectic two-form:

$\omega = dA_n \wedge dx^n = F_{mn} dx^m \wedge dx^n$ are given by the following differential operators [10], [12]:

$$X_{A_m} = \frac{\partial}{\partial x^m} \, , \, \, X_{x^m} = -\frac{\partial}{\partial A_m}$$

Moreover, the quantum differential operators on the quantized phase space of this system should be proportional to these vector fields by a complex factor, i.e. usually by $(-i\hbar)$, and so they should be given by:

$$\hat{A}_m = -i\hbar \frac{\partial}{\partial x^m} \, , \, \, \hat{x}_m = i\hbar \frac{\partial}{\partial A_m}$$
On the other hand, the real quantized phase space of a quantum system should be polarized in the sense that the $\Psi$ wave function of system should be a function of only half of the variables of the original phase space \[10\]. This means that it is either in the $\Psi(A_m, t)$- or in the $\Psi(x^m, t)$ representation. Then the quantum operators are given, respectively, by the set \[\{\hat{A}_m = A_m, \hat{x}_m = -i\hbar \frac{\partial}{\partial A_m}\}\] or by the set \[\{\hat{A}_m = -i\hbar X_{A_m} = -i\hbar \frac{\partial}{\partial x^m}, \hat{x}^m = x^m\}\]. Thus in both representations the commutator between the quantum operators is given by:

\[c[\hat{A}_m, \hat{x}_n]\Psi = -i\hbar \delta_{mn}\Psi, \quad (5)\]

which is gauge invariant \[13\]. Equivalently in accord with quantum mechanics there is a true uncertainty relation:

\[\epsilon \Delta A_m \cdot \Delta x_m \geq \hbar \quad (6)\]

Here $\Delta x_m$ is the position uncertainty of the electron observed by the light which is proportional to the wave length of light \[14\].

This approach considers the photon as a quantum particle with its typical uncertainties; Since in the same way that a measurement of momentum or position of an electron needs its interaction with a photon, the measurement of electric field strength of a photon needs its interaction with an electron or with a charged test body (see also Ref. \[14\]). Thus for a time dependent electromagnetic potential, e. g. $A_m = E_m \cdot t$, there is also an uncertainty relation for the electric field strength which is given by $\epsilon \Delta E_m \cdot \Delta t \cdot \Delta x_m \geq \hbar$. Note also that a similar quantum relation exists also in the four dimensional QED, although it is introduced phenomenologically in addition to the usual canonical uncertainty relation of QED \[14\]. Rather this additional quantum relation, i. e. $Q \Delta E_x \cdot T \cdot \Delta x \geq \hbar$; $Q = N'e$, $N' \in \mathbb{Z}$, is essential for the consistency of the usual uncertainty relations of QED \[13\]. Considering $A_x = E_x \cdot T$, it is obvious that the additional quantum relation in QED is the same as the canonically obtained uncertainty relation in the two dimensional model for $N'$ non-interacting electrons which has the action
Moreover considering the QED uncertainty relation \( \Delta G_x \cdot \Delta x \geq \hbar \) where \( G_i = \int d^3x \epsilon_{ijk}E_jB_k; i,j,k = 1,2,3 \) is the momentum of light which observes the position \( x \) of electron \([14]\); a comparison with additional QED quantum relation \( Q \Delta E_x \cdot T \cdot \Delta x \geq \hbar \) shows that: \( \Delta G_x = Q \Delta E_x \cdot T = Q \Delta A_x \). Thus, in two dimensions, the momentum of four dimensional QED \( G_m \) can be identified with the momentum of the two dimensional model of quantum electrodynamics \( QA_m \). This shows the compatibility of four and two dimensional quantum models with respect to the momentum structure. To underline this property note that using Gauss’s law for closed surfaces and \( A_i = \epsilon_{ijk}x_j B_k \) for solutions of Maxwell equations, i. e. for constant \( B_k \), the momentum in four dimensional case \( G_i = \int d^3x \epsilon_{ijk}E_jB_k \), is given by \( G_i = QA_i \). Hence the canonical identification of momentum of two dimensional photon with \( A_m \), as it is performed above, is in agreement with the momentum concept in four dimensional QED. Although the four dimensional theory do not present any canonical conjugate position variable for this momentum. At any case the momentum of photon \( A_m \) is correlated with the momentum of electron \( P_m \). Thus the \textit{flat} connection \( A_m \) on the contour region is defined, in accord with \( F_{mn}^{\text{contour}} = 0 \), by \((-i\hbar \partial_m - eA_m)\Psi = 0 \) where \( \Psi \) is the wave function of electron. Hence, in view of \( e\Delta A_x = \Delta P_x \) in accord with \( P_m \Psi = eA_m \Psi \), the momentums as well as uncertainty relations for interacting electron and photon becomes correlated as expected, e. g. in Ref. \([14]\).

Recall however that whereas in the two dimensional approach the uncertainty relations \( e\Delta A_m \cdot \Delta x_m \geq \hbar \) result directly from the canonical quantization of the action, in the four dimensional QED one needs various assumptions to introduce the ”inaccuracy” relation: \( Q \Delta E_x \cdot \Delta x \cdot T \geq \hbar \) which is not deducable from the quantum structure of the four dimensional QED \([14]\). This seems to be an advantage of the two dimensional approach to quantum electrodynamics.

Moreover, in accord with \( A_m^{\text{surface}} = B_{(constant)} x^m \epsilon_{mn} \) or with \( \Delta A_m = B \cdot \Delta x^m \epsilon_{mn} \) there should be also an equivalent uncertainty relation which is given by: \( eB \Delta x_m \cdot \Delta x_n \geq \hbar |\epsilon_{mn}| \), i. e. for \( m \neq n \). This uncertainty relation is related to the quantum commutator postulate \( eB[\hat{x}_m , \hat{x}_n] = -i\hbar \epsilon_{mn} \) which is equivalent to the quantization postulate: \( S_{(cl)} = eB_{(constant)} \epsilon_{mn} \int_{(\text{surface})} dx^m \wedge dx^n = Nh \). Recall that the commutator \( eB[\hat{x}_m , \hat{x}_n] = -i\hbar \epsilon_{mn} \) is known, phenomenologically, as the commutator of relative electron coordinates operators in the cyclotron motion \([16]\) and it seems to be related with the so
called "Peierls substitution" for electrons in strong magnetic fields \[17\]. Furthermore in accord with the
uncertainty relations the quantized electromagnetic gauge potential possess a maximal uncertainty of
\[
\Delta A_m = (\Delta A_m)_{\text{maximum}} = \frac{\hbar}{eB} \text{ for the case } \Delta x_m = (\Delta x_m)_{\text{minimum}} = l_B.
\]
Using \( \Delta A_m = eB \cdot l_B \) one obtains from uncertainty relation the independent phenomenological definition of magnetic length:
\[
l_B^2 = \frac{\hbar}{eB},
\]
which proves the consistency of this approach. Therefore this two dimensional quantum theory
which is represented by the equivalent canonical quantum postulates: i. e. by
\[
e \oint (\text{contour}) A_m dx^m = e \int (\text{surface}) F_{mn} dx^m \land dx^n = \Phi(Q) = Nh, N \in \mathbb{Z}; e[\hat{A}_m, \hat{x}_n] = -i\hbar \delta_{mn} \text{ or } eB[\hat{x}_m, \hat{x}_n] = -i\hbar \epsilon_{mn}
\]
can be considered as an appropriate theory to describe two dimensional quantum effects in presence of magnetic
fields.

In conclusion let us remark that in accord with this canonical quantized model there should be a quantum
of length equal to the magnetic length \( l_B \) in two dimensional quantum electrodynamical systems,
which was introduced phenomenologically in the magnetic quantization. Accordingly in such quantum
systems all relevant lengths and areas are quantized in units of \( l_B \) and \( l_B^2 \), respectively, since
\[
(\Delta x_m)_{\text{minimum}} = l_B \text{ and } (\Delta x_m \cdot \Delta x_n)_{\text{minimum}} = l_B^2.
\]
Furthermore the introduced quantum commutator postulates \( e[\hat{A}_m, \hat{x}_n] = -i\hbar \delta_{mn} \) or \( eB[\hat{x}_m, \hat{x}_n] = -i\hbar \epsilon_{mn} \) are equivalent, in accord with
\[
A_m^{(\text{surface})} = B(\text{constant}) x^m \epsilon_{mn},
\]
to the commutator \( e[\hat{A}_m, \hat{A}_n] = -i\hbar B \) which is equivalent to the quantum commutator postulate for the 2 + 1 dimensional Chern-Simons theory in the \( A_\tau = 0 \) gauge \[18\].
This circumstance relates our gauge free model with the gauged Chern-Simons models of topological field
theories which are used also to describe QHE \[19\]. Moreover the present model seems to be related, by
quantum postulate \( eB[\hat{x}_m, \hat{x}_n] = -i\hbar \epsilon_{mn} \), to the Chern-Simons quantum mechanics \[20\].

References

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Electrodynamics", edited by: J. Schwinger, (Dover Publications, Inc. New York, 1958).
[3] As long as there is no quantum prescription to define or to measure the boundary of a manifold under quantum conditions, a general manifold should be considered as boundaryless.

[4] V. Weisskopf, Danske Vid., (1936): in "Selected Papers on Quantum Electrodynamics", edited by: J. Schwinger, (Dover Publications, Inc. New York, 1958).

[5] L. D. Landau, E. M. Lifschitz, III Vol. We mean here the Landau-gauge $A_m := \epsilon_{mn} B \cdot x^n$;

$$\epsilon_{mn} = -\epsilon_{nm} = -1$$

where $B$ is a constant magnetic field: $B = B_{mn} F^{mn}, dB = 0$.

[6] N. Bohr and L. Rosenfeld, Phys. Rev., 78, 794, (1950).

[7] R. Bott, Canad. Math. Bull. Vol. 28(2), 1985.

[8] Obviously the main constraint in a general four dimensional field theory is due to the fact that the introduced time component of the "field" is no dynamical variable, therefore such a constraint can be avoided in the absence of such a time component.

[9] We denote functions on the phase space by $x$, $A$, etc. and related quantum operators on the quantized phase space by $\hat{x}$, $\hat{A}$, etc. and set the velocity of light $C = 1$. Although the model is performed for a single electron, however it can be applied also to non-interacting two dimensional electronic system where the "single electron picture" is available.

[10] N. Woodhouse, "Geometric Quantization", (Clarendon Press, 1980, 1990) Oxford University.

[11] The resulting equations of motion from

$$S_{(cl)} = \oint_{(contour)} e A_m dx^m,$$

i. e. $\partial_m A^m = 0 \sim d^i A = 0$

together with the contour condition $dA = 0$, are equivalent to Laplace equation $\partial_n \partial^n A_m = (dd + d^i d) A = 0$.

[12] According to Ref. [10] the classical Hamiltonian vector fields related with the canonical conjugate variables $\{A_m, x^m\}$ are given in general by:

$$X_{A_m} = \frac{\partial A_m}{\partial A_n} \frac{\partial}{\partial x^n} - \frac{\partial A_m}{\partial x^n} \frac{\partial}{\partial A_n}, \quad X_{x^m} = \frac{\partial x^m}{\partial A_n} \frac{\partial}{\partial x^n} - \frac{\partial x^m}{\partial x^n} \frac{\partial}{\partial A_n}$$

Furthermore the inner product of any globally Hamiltonian vectorfield $X_f$ with symplectic two-form of the system, i. e. $\omega$, should result in: $<X_f, \omega> = -df$. 
[13] In accord with Refs. [10] and [12] the vector field of $d\lambda$ have vanishing inner product with the symplectic two-form of the system, i. e. $< X_{d\Lambda}, \omega > = -d^2 \Lambda \equiv 0$. The corresponding operator is then a constant operator proportional to the identity operator which commutes with all others.

[14] W. Heitler: The Quantum Theory of Radiation, Third Edition, (Dover Publications, Inc, New York 1984): (II-9 and II-7).

[15] Recall that the "charged test body" of Ref. [14] obeys the same uncertainty relations as the usual electron. Moreover the discussed two dimensional quantum electrodynamics is not changed, if the constant $e$ is replaced by $Q$.

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