HOLONOMY GROUPS OF PSEUDO-RIEMANNIAN MANIFOLDS

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Abstract. We prove the uniqueness of the decomposition of holonomy groups of pseudo-Riemannian manifolds into its indecomposable normal subgroups, based on the discussion on de Rham decomposition of pseudo-Riemannian manifolds.

1. Introduction

Holonomy groups play an important role in the study of geometric structures of manifolds equipped with a non-degenerate metric, that is, pseudo-Riemannian manifolds. It links geometric and algebraic properties, and then provides a tool of algebra to geometric questions. In fact, we can describe parallel sections in geometric vector bundles associated to the manifold, such as the tangent bundle, tensor bundles, or the spin bundle, as invariant objects by the action of the holonomy group and by algebraic means. Hence, the classification of holonomy groups gives a framework in which geometric structures on pseudo-Riemannian manifolds can be studied. Moreover the classification has applications to physics, in particular to string theory.

Many interesting developments in differential geometry were initiated or driven by the study and the knowledge of holonomy groups, such as the study of so-called special geometries in Riemannian geometry. These developments were based on the classification result of Riemannian holonomy groups, which was achieved by the de Rham decomposition theorem [5] and the Berger list of irreducible pseudo-Riemannian holonomy groups [1].

For pseudo-Riemannian manifolds, this question was widely open and untackled for a long time. The main difficulty in the case of pseudo-Riemannian manifolds is that we can't reduce the algebraic aspect of the classification problem to irreducible representations. Based on the Wu theorem [15], the classification of holonomy groups of pseudo-Riemannian manifolds is reduced to the case of indecomposable groups (any such group does not preserve any proper nondegenerate subspace of the tangent space). It leads to many progress on the classification of indecomposable holonomy groups of Lorentzian manifolds [3, 6, 7, 9, 12], pseudo-Riemannian manifolds of index

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2 \cite{11} and pseudo-Riemannian manifolds with neutral signature \cite{4}. The paper \cite{10} gives a good survey in this field.

However Wu \cite{15} only proves the uniqueness of the decomposition when the maximal trivial subspace of the holonomy group is nondegenerate, and there is an example in \cite{15} to show that the decomposition isn’t necessary to be unique. The goal of this paper is to discuss when the decomposition is unique up to an order based on the reformulated de Rham decomposition of pseudo-Riemannian manifolds. All manifolds in this paper are assumed complete, connected and simply connected unless otherwise specified.

Section 2 lists some facts and results on the holonomy groups of pseudo-Riemannian manifolds whose maximal trivial subspaces of the holonomy groups are nondegenerate. Theorem 2.6 shows that the condition in Theorem 2.5, i.e. the unique decomposition theorem in \cite{15}, can be relaxed.

In Section 3 we prove the unique decomposition theorem of holonomy groups of pseudo-Riemannian manifolds satisfying Condition Φ. Firstly we give the definition of pseudo-Riemannian manifolds satisfying Condition Φ, which include pseudo-Riemannian manifolds whose maximal trivial subspaces of the holonomy groups are nondegenerate as a subclass. Then we discuss the reformulated de Rham decomposition theorem, i.e Theorem 3.2. Theorem 3.9 gives a sufficient condition for a pseudo-Riemannian manifold to satisfy Condition Φ. Based on Theorem 3.2 we prove the unique decomposition theorem, i.e. Theorem 3.11.

Section 4 discusses the decomposition of holonomy groups of pseudo-Riemannian manifolds dissatisfying Condition Φ. Example 4.1 shows that the decomposition isn’t unique for some pseudo-Riemannian manifold which is a direct product of two indecomposable factors, where the tangent space of every factor admits a nontrivial decomposition into holonomy invariant subspaces. Furthermore Theorem 4.3 proves the unique decomposition of every pseudo-Riemannian manifold which is a direct product of a pseudo-Riemannian manifold satisfying Condition Φ and an indecomposable pseudo-Riemannian manifold whose tangent space admits a nontrivial decomposition into holonomy invariant subspaces.

2. PSEUDO-RIEMANNIAN MANIFOLDS Whose MAXIMAL TRIVIAL SUBSPACE OF THE HOLO NYOMY GROUP IS NONDEGENERATE

This section is to recall the work on the decomposition of holonomy groups for pseudo-Riemannian manifolds. Firstly, we give some notations. Let $V$ be a pseudo-Riemannian space and let $G$ be a subgroup of the orthogonal group
\[ \{ g \in GL(V) \mid \langle gx, gy \rangle = \langle x, y \rangle \text{ for any } x, y \in V \}. \]
Let $S(V, G)$ denote the vector space extended by $v - gv$ for any $v \in V$ and $g \in G$, and define the maximal trivial subspace of the group $G$ in $V$ by
\[ V^G = \{ v \in V \mid gv = v \text{ for any } g \in G \}. \]
Lemma 2.1. Let notation be as above. Then \( V^G = (S(V,G))^\perp \).

Proof. In fact, \( v \in (S(V,G))^\perp \) if and only if \( \langle v, w - gw \rangle = 0 \) for any \( w \in V \) and \( g \in G \), if and only if \( \langle v, w \rangle - \langle v, gw \rangle = 0 \) for any \( w \in V \) and \( g \in G \), if and only if \( \langle v - gv, w \rangle = 0 \) for any \( w \in V \) and \( g \in G \), if and only if \( \langle v - gv \rangle = 0 \) for any \( g \in G \), i.e., \( v \in V^G \). \( \square \)

Definition 2.2. A pseudo-Riemannian manifold \( M \) is a smooth manifold \( M \) with a nondegenerate inner product \( \langle , \rangle \) on the fibers of its tangent bundle \( TM \). Let the expression \((n_+, n_-)\), where \( n_+ + n_- = \dim M \), denote the signature of \( \langle , \rangle \). The manifold \( M \) is Riemannian in the case where \( \langle , \rangle \) has signature \((\dim M, 0)\), i.e. is positive definite.

By Lemma 2.1 and the above definition, we have:

Proposition 2.3. Let \( M \) be a pseudo-Riemannian manifold and \( H \) be its holonomy group at the point \( m \). Then \( (M_m)^H = (S(M_m,H))^\perp \).

The main theorem in [15] describes the decomposition theorem of holonomy groups.

Theorem 2.4 ([15]). Let \( M \) be a pseudo-Riemannian manifold and \( H \) be its holonomy group at the point \( m \). Then \( H \) is the direct product of a finite number of its normal subgroups which are indecomposable. This decomposition is unique up to the order if the maximal trivial subspace is nondegenerate.

The first part is based on the discussion of de Rham decomposition for pseudo-Riemannian manifolds [13, 14] and the second part is given in the following theorem.

Theorem 2.5 ([15], The Full de Rham Decomposition Theorem). Let \( M \) be a pseudo-Riemannian manifold and \( H \) be its holonomy group at the point \( m \). Suppose that the maximal trivial subspace \( M^0_m = (M_m)^H \) of \( H \) in \( M_m \) is nondegenerate.

(1) \( M_m \) admits an orthogonal decomposition into \( H \)-invariant subspaces \( M_m = M^0_m \oplus M^1_m \oplus \cdots \oplus M^p_m \) which is unique up to the order, where \( M^i_m \) is indecomposable for any \( 1 \leq i \leq p \).

(2) \( M \) is isometric to a direct product \( M^0 \times M^1 \times \cdots \times M^p \) which is unique up to the order, where \( M^j \) is the maximal integral manifold of the distribution obtained by parallel translating \( M^j_m \) over \( M \) for any \( 1 \leq i \leq p \). Moreover, \( M^0 \) is flat, and \( M^i \) is indecomposable for any \( 1 \leq i \leq p \).

(3) \( H \) is the direct product of its normal subgroups \( H^1 \times \cdots \times H^p \) which is unique up to the order, where \( H^i \) is the holonomy group of \( M^i \) for any \( 1 \leq i \leq p \). Each \( H^i \) is indecomposable and \( H^i \) acts trivially on \( M^k_m \) if \( k \neq i \).

The uniqueness of Theorem 2.5 is based on the orthogonal decomposition. In fact, we have:
Theorem 2.6. Let $M$ be a pseudo-Riemannian manifold and $H$ be its holonomy group at the point $m$. Suppose that the maximal trivial subspace $M_1^m = (M_m)^H$ of $H$ in $M_m$ is nondegenerate. If $M_m$ admits a decomposition into nondegenerate $H$-invariant subspaces

$$M_m = M_0^m \oplus M_1^m \oplus \cdots \oplus M_p^m,$$

where $\langle x, y \rangle = 0$ for any $x \in M_0^m$ and $y \in M_1^m \oplus \cdots \oplus M_p^m$, and $M_{in}^m$ is indecomposable for any $1 \leq i \leq p$. Then the decomposition is orthogonal.

Proof. It is clear when $p = 1$. Assume that $p \geq 2$. Let $N_m = M_1^m \oplus \cdots \oplus M_p^m$ and $N_1^m = M_2^m \oplus \cdots \oplus M_p^m$. Then $N_m = M_1^m \oplus (M_1^m)^\perp$ since both $N_m$ and $M_1^m$ are nondegenerate. Here $(M_1^m)^\perp$ means the orthogonal complement of $M_1^m$ in $N_m$. Let $H = H_1 \times H_2$ be the decomposition of $H$ corresponding to the decomposition $N_m = M_1^m \oplus (M_1^m)^\perp$. Consider the projection $f$ from $N_1^m$ to $M_1^m$. For any $h \in H_1$ and $x \in N_1^m$,

$$hf(x) - f(x) = hx - x \in M_1^m \cap N_1^m = 0.$$

It follows that $f(x) \in (M_1^m)^H = 0$. Namely $\langle x, y \rangle = 0$ for any $x \in M_1^m$ and $y \in N_1^m$. The theorem follows from similar discussion. \hfill $\square$

3. Pseudo-Riemannian manifolds satisfying Condition $\Phi$

This section is to prove the unique decomposition theorem of holonomy groups of pseudo-Riemannian manifolds satisfying Condition $\Phi$.

Definition 3.1. Let $M$ be a pseudo-Riemannian manifold and $H$ be its holonomy group at the point $m$. For every indecomposable nondegenerate $H$-invariant subspace $V$ of $M_m$ satisfying $(V)^H \neq 0$, if $V$ doesn’t admit a nontrivial decomposition into $H$-invariant subspaces, then $M$ is said to satisfy Condition $\Phi$.

It is clear that every pseudo-Riemannian manifold whose maximal trivial subspace of the holonomy group is nondegenerate satisfies Condition $\Phi$, but the inverse doesn’t hold any more.

3.1. De Rham decomposition of pseudo-Riemannian manifolds satisfying Condition $\Phi$. This subsection is to prove the reformulated de Rham decomposition theorem:

Theorem 3.2. Let $M$ be a pseudo-Riemannian manifold and $H$ be its holonomy group at the point $m$. Then $M_m$ admits an orthogonal decomposition into $H$-invariant subspaces:

$$M_m = M_0^m \oplus M_1^m \oplus \cdots \oplus M_{p_1}^m \oplus M_{p_1+1}^m \oplus \cdots \oplus M_{p_1+p_2}^m,$$

where $M_0^m$ is a maximal nondegenerate subspace of the maximal trivial subspace of $H$ in $M_m$, $M_{in}^m$ is indecomposable for any $1 \leq i \leq p_1 + p_2$, $(M_i^m)^H = 0$ for any $1 \leq i \leq p_1$, and $(M_i^m)^H$ is nonzero and isotropic for any $p_1 + 1 \leq i \leq p_1 + p_2$. 
Assume that $M$ satisfies Condition $\Phi$. Let
\[ M_m = N_m^0 \oplus N_m^1 \oplus \cdots \oplus N_m^{q_1} \oplus N_m^{q_1+1} \oplus \cdots \oplus N_m^{q_1+q_2}, \]
be another orthogonal decomposition of $M_m$ into $H$-invariant subspaces, where $N_m^0$ is a maximal nondegenerate subspace of the maximal trivial subspace of $H$ in $M_m$, $N_m^0$ is indecomposable for any $1 \leq j \leq q_1 + q_2$, $(N_m^0)^H = 0$ for any $1 \leq j \leq q_1$, and $(N_m^j)^H$ is nonzero and isotropic for any $q_1 + 1 \leq j \leq q_1 + q_2$. Then we have:

1. $p_1 = q_1$ and $p_2 = q_2$;
2. Changing the subscripts if necessary, $M_{m_i} = N_{m_i}^i$ for any $1 \leq i \leq p_1$,
   \[ \dim M_{m_i} = \dim N_{m_i}^i \text{ for any } p_1 + 1 \leq i \leq p_1 + p_2 \text{ and } S(M_{m_i}, H) = S(N_{m_i}^i, H); \]
3. There exist $\pi_i : 0 \leq i \leq p_1 + p_2$ from $M_{m_i}$ to $N_{m_i}^i$ such that $\pi_i$ is
   \[ 1 - 1 \text{ and } (\pi_i(x), \pi_i(x)) = (x, x) \text{ for any } i \in M_{m_i}. \]

So $\pi = (\pi_0, \cdots, \pi_{p_1+p_2})$ is distance-preserving. That is, the decomposition is unique up to a distance-preserving map. Here $\pi_0$ is the projection and $\pi_j = \text{id}$ for any $1 \leq j \leq p_1$.

In order to prove Theorem 3.2 we need to discuss by several steps. Firstly assume that the maximal trivial subspace of $H$ in $M_m$ is isotropic. Then $M_m$ admits an orthogonal decomposition into $H$-invariant subspaces
\[ M_m = M_m^1 \oplus \cdots \oplus M_m^p, \]
where $M_m^i$ is indecomposable for any $1 \leq i \leq p$. Let $H = H^1 \times \cdots \times H^p$ be the decomposition of $H$ associated with the decomposition $M_m = M_m^1 \oplus \cdots \oplus M_m^p$. Let $M_m = N_m^1 \oplus \cdots \oplus N_m^q$ be another decomposition and $H = G^1 \times \cdots \times G^q$ be the decomposition of $H$ respectively.

**Lemma 3.3.** $S(M_m^i, H^1) \neq 0$.

**Proof.** If not, $M_m^i = (M_m^i)^{H^1}$ by Lemma 2.1. Then $M_m^i \subset (M_m)^H$, which contradicts the fact that the maximal trivial subspace of $H$ is isotropic. \qed

Then there exists $x \in M_m^i$ and $h \in H^1$ such that $x - hx \neq 0$. Let $x = x_1 + \cdots + x_q$ and $h = g_1 g_2 \cdots g_q$ be the expression of $x$ and $H$ associated with $M_m = N_m^1 \oplus \cdots \oplus N_m^q$ and $H = G^1 \times \cdots \times G^q$ respectively. Then
\[ 0 \neq x - hx = \sum_{i=1}^q (x_i - hx_i) = \sum_{i=1}^q (x_i - g_i x_i). \]
Without loss of generality, assume that $x_1 - g_1 x_1 \neq 0$. That is,
\[ M_m^1 \cap N_m^1 \neq 0 \]
since $x_1 - g_1 x_1 = x - g_1 x \in M_m^1 \cap N_m^1$.

Furthermore assume that $M$ satisfies Condition $\Phi$. That is, if a nondegenerate subspace $U$ of $M_m$ is indecomposable and $U = U_1 + U_2$ where both $U_1$ and $U_2$ are $H$-invariant, then $U_1 = 0$ or $U_2 = 0$. Then we have:
Lemma 3.4. \( M^1_m \cap (N^2_m \oplus \cdots \oplus N^q_m) = 0 \) and \( N^1_m \cap (M^2_m \oplus \cdots \oplus M^p_m) = 0 \).

Proof. If \( M^1_m = (M^1_m \cap N^1_m) \oplus (M^1_m \cap (N^2_m \oplus \cdots \oplus N^q_m)) \), then \( M^1_m \cap (N^2_m \oplus \cdots \oplus N^q_m) = 0 \) by the assumption. Or there exists \( x \in M^1_m \) such that \( x \not\in (M^1_m \cap N^1_m) \oplus (M^1_m \cap (N^2_m \oplus \cdots \oplus N^q_m)) \). Let

\[
x = x_1 + x_2
\]

be the expression of \( x \) associated with the decomposition \( M_m = N^1_m \oplus (N^2_m \oplus \cdots \oplus N^q_m) \). Then let

\[
x_1 = x_1 + x_2^2 \quad \text{and} \quad x_2 = x_2^1 + x_2^2
\]

be the expression of \( x_i, i = 1, 2 \) associated with the decomposition \( M_m = M^1_m \oplus (M^2_m \oplus \cdots \oplus M^p_m) \). So \( x = x_1^1 + x_2^1 + x_2^2 + x_2^1 = 0 \).

Clearly \( x_1^1 - hx_1^1 \in M^1_m \) and \( x_2^1 - hx_2^1 \in M^1_m \) for any \( h \in H \). For any \( h \in H \), let \( h = h_1 h_2 \) be the expression of \( h \) associated with the decomposition \( H = H^1 \times (H^2 \times \cdots \times H^p) \). Then

\[
x_1^1 - hx_1^1 = x_1^1 - h_1 x_1^1 = x_1^1 - x_2^1 - h_1 (x_1^1 - x_2^1) = x_1^1 - h_1 x_1 \in N^1_m,
\]

\[
x_2^1 - hx_2^1 = x_2^1 - h_1 x_2^1 = x_2^1 - x_2^2 - h_1 (x_2^1 - x_2^2) = x_2^1 - h_1 x_2 \in N^1_m.
\]

It follows that for any \( h \in H \),

\[
hx_1^1 \in x_1^1 + M^1_m \cap N^1_m \quad \text{and} \quad hx_2^1 \in x_2^1 + M^1_m \cap (N^2_m \oplus \cdots \oplus N^q_m).
\]

That is, \( b_1 = \mathbb{R}x_1^1 + M^1_m \cap N^1_m \) and \( b_2 = \mathbb{R}x_2^1 + M^1_m \cap (N^2_m \oplus \cdots \oplus N^q_m) \) are \( H \)-invariant. Since \( x \not\in (M^1_m \cap N^1_m) \oplus (M^1_m \cap (N^2_m \oplus \cdots \oplus N^q_m)) \) and \( x = x_1^1 + x_2^2 \), we have

\[
(M^1_m \cap N^1_m) \oplus (M^1_m \cap (N^2_m \oplus \cdots \oplus N^q_m)) \subsetneq b_1 \oplus b_2.
\]

If \( M^1_m = b_1 \oplus b_2 \), \( b_2 = 0 \) by the assumption. In particular, \( M^1_m \cap (N^2_m \oplus \cdots \oplus N^q_m) = 0 \). If \( M^1_m \neq b_1 \oplus b_2 \), since \( \dim M^1_m \leq \infty \), repeating the above discussion, there exist \( H \)-invariant subspaces \( b_1^k \) and \( b_2^k \) satisfying

\[
M^1_m = b_1^k \oplus b_2^k.
\]

By the assumption, \( b_2^k = 0 \). In particular, \( M^1_m \cap (N^2_m \oplus \cdots \oplus N^q_m) = 0 \). Similarly, we have \( N^1_m \cap (M^1_m \oplus \cdots \oplus M^p_m) = 0 \). □

Lemma 3.5. The projection \( \pi_1 \) from \( M^1_m \) to \( N^1_m \) is \( 1 - 1 \) and distance-preserving.

Proof. Let \( \pi_1 : M^1_m \to N^1_m \) be the projection from \( M^1_m \) to \( N^1_m \). Since \( \ker \pi \subseteq M^1_m \cap (N^2_m \oplus \cdots \oplus N^q_m) = 0 \), we have that \( \pi \) is injective. Thus \( \dim M^1_m \leq \dim N^1_m \). Reversing the role of \( M^1_m \) and \( N^1_m \) and by \( N^1_m \cap (M^1_m \oplus \cdots \oplus M^p_m) = 0 \), \( \dim N^1_m \leq \dim M^1_m \). Namely

\[
\dim M^1_m = \dim N^1_m
\]

and \( \pi \) is a \( 1 - 1 \) correspondence. For any \( x \in M^1_m \), let \( x = x_1 + x_2 \) be the expression of \( x \) associated with the decomposition \( M_m = N^1_m \oplus (N^2_m \oplus \cdots \oplus N^q_m) \).
orthogonal decomposition of $M^H$ in the holonomy group at the point $m$. Assume that $M^H$ is indecomposable for any $i$. By Theorem 3.6, let $S$ be a pseudo-Riemannian manifold and $H$ be its holonomy group at the point $m$. Assume that the maximal trivial subspace of $M^H$ is determined. Then $\langle x, x \rangle = \langle x_1 + x_2, x_1 + x_2 \rangle = \langle x_1, x_1 \rangle = \langle \pi_1(x), \pi_1(x) \rangle$. □

For any $x \in M^H$, $x = x_1 + x_2$ where $x_1 \in N_1^H$ and $x_2 \in (M^H)^H$ by Lemma 3.5. It follows that $S(M^H, H) = S(N_1^H, G^H) = S(N_1^H, H)$. Similar $S(N_1^H, G^H) = S(N_1^H, H) \subset S(M^H, H) = S(M^H, H^1)$. That is,

$$S(M^H, H) = S(M^H, H^1) = S(N_1^H, G^H) = S(N_1^H, H).$$

Repeating the above discussion for $j = 2, 3, \ldots, p$, we have:

**Theorem 3.6.** Let $M$ be a pseudo-Riemannian manifold and $H$ be its holonomy group at the point $m$. Assume that the maximal trivial subspace of $H$ in $M^H$ is isotropic. Then $M^H$ admits an orthogonal decomposition into $H$-invariant subspaces:

$$M^H = M^1_m \oplus \cdots \oplus M^p_m,$$

where $M^i_m$ is indecomposable for any $1 \leq i \leq p$.

Assume that $M$ satisfies Condition $\Phi$. Let $M^H = N_1^m \oplus \cdots \oplus N_q^m$ be an orthogonal decomposition of $M^H$ into $H$-invariant subspaces, where $N^i_m$ is indecomposable for any $1 \leq i \leq q$. Then we have:

1. $p = q$;
2. Changing the subscripts if necessary, $\dim M^i_m = \dim N^i_m$ for any $1 \leq i \leq p$ and $S(M^i_m, H) = S(N^i_m, H)$;
3. Each projection $\pi_i : M^i_m \to N^i_m$ is $1 - 1$ and $\langle \pi_i(x), \pi_i(x) \rangle = \langle x, x \rangle$ for any $x \in M^i_m$ and $1 \leq i \leq p$. So $\pi = (\pi_1, \ldots, \pi_p)$ is distance-preserving. That is, the decomposition is unique up to a distance-preserving map.

By Theorem 3.6, we can get the reformulated de Rham decomposition when the maximal trivial nondegenerate subspace is determined.

**Theorem 3.7.** Let $M$ be a pseudo-Riemannian manifold, $H$ be its holonomy group at the point $m$, and $M^0_m$ be a maximal nondegenerate subspace of the maximal trivial subspace of $H$ in $M^H$. Assume that $M$ satisfies Condition $\Phi$ and

$$M^H_m = M^0_m \oplus M^1_m \oplus \cdots \oplus M^{p_1}_m \oplus M^{p_1+1}_m \oplus \cdots \oplus M^{p_1+p_2}_m = M^0_m \oplus N^1_m \oplus \cdots \oplus N^{q_1}_m \oplus N^{q_1+1}_m \oplus \cdots \oplus N^{q_1+q_2}_m$$

are orthogonal decompositions of $M^H_m$ into $H$-invariant subspaces, where both $M^i_m$ and $N^i_m$ are indecomposable for any $1 \leq i \leq p_1 + p_2$ and $1 \leq j \leq q_1 + q_2$, $(M^i_m)^H = (N^j_m)^H = 0$ for any $1 \leq i \leq p_1$ and $1 \leq j \leq q_1$, and both $(M^i_m)^H$
and \((N^i_m)^H\) are nonzero and isotropic for any \(p_1 + 1 \leq i \leq p_1 + p_2\) and \(q_1 + 1 \leq j \leq q_1 + q_2\). Then we have:

1. \(p_1 = q_1\) and \(p_2 = q_2\);
2. Changing the subscripts if necessary, \(M^i_m = N^i_m\) for any \(1 \leq i \leq p_1\), \(\dim M^i_m = \dim N^i_m\) for any \(p_1 + 1 \leq i \leq p_1 + p_2\) and \(S(M^i_m, H) = S(N^i_m, H)\);
3. Each projection \(\pi_i : M^i_m \to N^i_m\) is 1-1 and \(\langle \pi_i(x), \pi_i(x) \rangle = \langle x, x \rangle\) for any \(x \in M^i_m\) where \(p_1 + 1 \leq i \leq p_1 + p_2\). Therefore \(\pi = (\text{id}, \ldots, \text{id}, \pi_{p_1+1}, \ldots, \pi_{p_1+p_2})\) is distance-preserving. That is, the decomposition is unique up to a distance-preserving map.

**Proof.** First to show that there exists some \(1 \leq i \leq p_1\) such that \(N^i_m = M^i_m\) for any \(1 \leq j \leq q_1\). The proof is similar with that for the full de Rham decomposition theorem of Wu [15]. Let \(H = H^1 \times \cdots \times H^{p_1+p_2}\) be the decomposition of \(H\) associated with the decomposition \(M_m = M^0_m \oplus M^1_m \oplus \cdots \oplus M^{p_1}_m \oplus M^{p_1+1}_m \oplus \cdots \oplus M^{p_1+p_2}_m\). Take \(0 \neq x \in N^1_m\), and let

\[
x = x_1 + \cdots + x_{p_1+p_2}
\]

be the expression of \(x\) associated with \(M^0_m \oplus M^1_m \oplus \cdots \oplus M^{p_1}_m \oplus M^{p_1+1}_m \oplus \cdots \oplus M^{p_1+p_2}_m\). Since \((N^1_m)^H = 0\), we have that there exists \(h = h_1 \cdots h_{p_1+p_2}\) such that \(x - hx \neq 0\). Then

\[
0 \neq x - hx = \sum_{i=1}^{p_1+p_2} (x_i - h_i x_i) = \sum_{i=1}^{p_1+p_2} (x_i - h_i x_i).
\]

Then there exists \(k\) such that \(x_k - h_k x_k \neq 0\). It follows that

\[
N^1_m \cap M^k_m \neq 0
\]

since \(x_k - h_k x_k = x - h_k x \in N^1_m \cap M^k_m\). Furthermore \(N^1_m \cap M^k_m\) is nondegenerate. In fact, assume that there exists \(x \in N^1_m \cap M^k_m\) such that \(\langle x, y \rangle = 0\) for any \(y \in N^1_m \cap M^k_m\). For any \(z \in N^1_m\) and \(h \in H\),

\[
\langle x, z - h z \rangle = \langle x, \sum_{i=1}^{p_1+p_2} (z_i - h_i z_i) \rangle = \langle x, \sum_{i=1}^{p_1+p_2} (z_i - h_i z_i) \rangle = \langle x, z_k - h_k z_k \rangle = 0
\]

since \(z_k - h_k z_k = z - h_k z \in N^1_m \cap M^k_m\), where \(z = z_1 + \cdots + z_{p_1+p_2}\) and \(h = h_1 \cdots h_{p_1+p_2}\) are the decompositions of \(z\) and \(h\) associated with the decomposition \(M^0_m \oplus M^1_m \oplus \cdots \oplus M^{p_1}_m \oplus M^{p_1+1}_m \oplus \cdots \oplus M^{p_1+p_2}_m\) and \(H^1 \times \cdots \times H^{p_1+p_2}\) respectively. Equivalently, \(\langle x - h x, z \rangle = 0\) for any \(x \in N^1_m\) and \(h \in H\). Then \(h x = x\) for any \(h \in H\) since the restriction of \(\langle \cdot, \cdot \rangle\) to \(N^1_m\) is nondegenerate. Thus \(x = 0\) by the assumption. That is, \(N^1_m \cap M^k_m \neq 0\) is nondegenerate. Since both \(N^1_m\) and \(M^k_m\) are indecomposable, we have \(N^1_m = M^k_m = N^1_m \cap M^k_m\).

By the above discussion, we have \(q_1 \leq p_1\). Similarly, \(p_1 \leq q_1\). Namely \(p_1 = q_1\). Thus the theorem follows from Theorem 3.6. \(\square\)
Theorem 3.8. Let $M$ be a pseudo-Riemannian manifold and $H$ be its holonomy group at the point $m$. Assume that the maximal trivial subspace of $H$ in $M_m$ isn’t isotropic. Then $M_m$ admits an orthogonal decomposition into $H$-invariant subspaces which is unique up to a distance-preserving map:

$$M_m = M_m^0 \oplus M_m^1,$$

where $M_m^0 \subset (M_m)^H$ is nondegenerate and $(M_m^1)^H$ is isotropic.

Proof. Let $M_m = N_m^0 \oplus N_m^1$ be another such decomposition. Then we have

$$S(M_m, H) = S(M_m^1, H) = S(N_m^1, H).$$

Since $(M_m^1)^H$ is isotropic, by Lemma 2.1, we have $(M_m^1)^H \subset ((M_m^1)^H)^\perp = S(M_m^1, H)$. It follows that

$$(M_m^1)^H = S(M_m^1, H) \cap (M_m)^H = S(N_m^1, H) \cap (M_m)^H = (N_m^1)^H.$$

Namely, $\dim M_m^1 = \dim N_m^1$, and then $\dim M_m^0 = N_m^0$.

Let $\{x_1, \cdots, x_r, x_{r+1}, \cdots, x_q, x_{q+1}, \cdots, x_{q+r}\}$ be a basis of $M_m^1$, where

$$(M_m^1)^H = L(e_1, \cdots, e_r), S(M_m^1, H) = L(e_1, \cdots, e_q),$$

and the matrix of the metric associated with the basis is

$$
\begin{pmatrix}
0 & 0 & I_r \\
0 & A_{q-r} & 0 \\
I_r & 0 & 0
\end{pmatrix},
$$

where $I_r$ is the identity matrix of $r \times r$ and $A_{q-r}$ is a diagonal matrix with the element $\epsilon_i$, i.e., the sign. Let $\pi_0$ and $\pi_1$ denote the projections

$$\pi_0 : M_m^0 \to N_m^0 \text{ and } \pi_1 : M_m^1 \to N_m^1.$$

Then $\pi_0$ and $\pi_1$ are $1 - 1$, $\langle \pi_0(x), \pi_0(x) \rangle = \langle x, x \rangle$ for any $x \in M_m^0$, $\pi_1(x) = x$ for any $x \in S(M_m^1, H)$, and $\langle \pi_1(x_1), \pi_1(x_j) \rangle = \langle x_i, x_j \rangle$ for any $1 \leq i \leq q + r$ and $1 \leq j \leq q$.

Let $x_s = x_{s_0} + x_{s_1}$ be the expression of $x_s$ associated with the decomposition $M_m = N_m^0 + N_m^1$ for any $q + 1 \leq s \leq q + r$. For any $q + 1 \leq t \leq q + r$, we have

$$0 = \langle x_s, x_t \rangle = \langle x_{s_0}, x_{t_0} \rangle + \langle x_{s_1}, x_{t_1} \rangle.$$

Let $x_{s_1}' = x_{s_1} + \frac{1}{2}\langle x_{s_0}, x_{s_0} \rangle x_{s-q} + \sum_{l=s+1}^{q+r} \langle x_{l_0}, x_{l_0} \rangle x_{l-q}$. It is easy to check

$$\langle x_{s_1}', x_{s_1}' \rangle = \langle x_{s_1}, x_{s_1} \rangle + \langle x_{s_0}, x_{s_0} \rangle = 0, \quad q + 1 \leq s \leq q + r;$$

$$\langle x_{s_1}', x_{t_1}' \rangle = \langle x_{s_1}, x_{t_1} \rangle + \langle x_{s_0}, t_{s_0} \rangle = 0, \quad q + 1 \leq s < t \leq q + r.$$

Define $\pi_1' : M_m^1 \to N_m^1$ by

$$\pi_1'(x_j) = x_j, 1 \leq j \leq q; \pi_1'(x_j) = x_j', q + 1 \leq j \leq q + r.$$

Then $\pi_1'$ is $1 - 1$ from $M_m^1$ onto $N_m^1$ and $\langle \pi_1'(x), \pi_1'(x) \rangle = \langle x, x \rangle$ for any $x \in M_m^1$. Then $\pi = (\pi_0, \pi_1')$ is a distance-preserving map of $M_m$. 

\qed
The proof of Theorem 3.2. It is enough to prove the second part. Similar to the proof of Theorem 3.7 we have that \( p_1 = q_1 \) and \( M^i_m = N^i_m \) for any \( 1 \leq i \leq p_1 \) by changing the subscripts if necessary. Similar to Lemma 3.3, we have \( S(M^{p_1+1}_m, H) \neq 0 \). Then there exists \( 0 \leq k \leq q_1 + q_2 \) such that \( M^{p_1+1}_m \cap N^k_m \neq 0 \). Obviously \( k \neq 0 \). If \( 1 \leq k \leq p_1 \), we have \( M^{p_1+1}_m = N^k_m \cap N^k_m \) by the proof of Theorem 3.7. It is a contradiction. So \( q_1 + 1 \leq k \leq q_1 + q_2 \). Without loss of generality, assume that \( k = q_1 + 1 = p_1 + 1 \). Similar to the proof of Lemma 3.4, we have

\[
M^{p_1+1}_m \cap (N^0_m \oplus \cdots \oplus N^{p_1+1}_m) = 0,
\]

\[
N^{p_1+1}_m \cap (M^0_m \oplus \cdots \oplus M^{p_1+1}_m) = 0.
\]

Similar to the proof of Lemma 3.5, the projection \( \pi'_{p_1+1} \) from \( M^{p_1+1}_m \) to \( N^{p_1+1}_m \) is 1 - 1. But the projection maybe isn't distance-preserving since the projection from \( M^{p_1+1}_m \) to \( N^0_m \) maybe isn't zero. Anyway, we have \( S(M^{p_1+1}_m, H) = S(N^{p_1+1}_m, H) \). Using the proof of Theorem 3.8 we can modify the projection \( \pi'_{p_1+1} \) to \( \pi_{p_1+1} \) such that \( \pi_{p_1+1} \) is 1 - 1 and distance-preserving. Thus the theorem follows by induction.

The following is a sufficient condition for a pseudo-Riemannian manifold to satisfy Condition \( \Phi \).

Theorem 3.9. Let \( M \) be a pseudo-Riemannian manifold and \( H \) be its holonomy group at the point \( m \). If there is an orthogonal decomposition of \( M_m \) into \( H \)-invariant subspaces:

\[
M_m = M^0_m \oplus M^1_m \oplus \cdots \oplus M^p_m
\]

where \( M^0_m \) is a maximal nondegenerate subspace in \( (M_m)^H \), for any \( 1 \leq i \leq p \), \( M^i_m \) is indecomposable and \( M^i_m \) doesn’t admit a nontrivial decomposition into \( H \)-invariant subspaces when \( (M^i_m)^H \neq 0 \), then \( M \) satisfies Condition \( \Phi \).

Proof. Assume that \( M \) dissatisfies Condition \( \Phi \). That is, there exists an indecomposable \( H \)-invariant subspace \( N^1_m \) of \( M_m \) such that \( (N^1_m)^H \neq 0 \) and \( N^1 = V_1 \oplus V_2 \) where \( V_1 \) and \( V_2 \) are nontrivial \( H \)-invariant subspaces. Let

\[
M_m = N^0_m \oplus N^1_m \oplus \cdots \oplus N^q_m
\]

be an orthogonal decomposition into \( H \)-invariant subspaces, where \( N^i_m \) is indecomposable for any \( 1 \leq i \leq q \). We can assume that \( (N^i_m)^H \neq 0 \) for any \( 1 \leq i \leq q \) and \( (M^i_m)^H \neq 0 \) for any \( 1 \leq j \leq p \) by the proof of Theorem 3.7. Clearly, \( S(N^1_m, H) \neq 0 \). Then there exists \( 1 \leq k \leq p \) such that \( M^k_m \cap N^1_m \neq 0 \). Without loss of generality, assume that \( k = 1 \). Similar to the proof of Lemma 3.4 we have

\[
M^1_m \cap (N^0_m \oplus N^2_m \oplus \cdots \oplus N^p_m) = 0,
\]
since $M^1_m$ doesn’t admit a nontrivial decomposition into $H$-invariant subspaces. Similar to the proof of Lemma 3.5, the projection $\pi_t$ from $M^1_m$ to $N^1_m$ is an injection, and for any $x \in M^1_m$, $x = x_1 + x_2$ where $x_1 \in N^1_m$ and $x_2 \in (M_m)^H$. It follows that

$$S(M^1_m, H) \subset S(N^1_m, H).$$

Let $\{x_1, \cdots, x_s, \cdots, x_t, \cdots, x_{t+s}\}$ be a basis of $M^1_m$ such that $(M^1_m)^H = L(e_1, \cdots, e_s)$, $S(M^1_m, H) = L(e_1, \cdots, e_t)$ and the matrix of the metric associated with the basis is

$$\begin{pmatrix} 0 & 0 & I_s \\ 0 & A & 0 \\ I_s & 0 & 0 \end{pmatrix},$$

where $I_s$ is the identity matrix of $s \times s$ and $A$ is a diagonal matrix with the element $\pm 1$. Thus $\{x_1, \cdots, x_s, \cdots, x_t, \pi(x_{t+1}), \cdots, \pi(x_{t+s})\}$ is a basis of $\pi_1(M^1_m)$, and the matrix of the metric associated with the basis is

$$\begin{pmatrix} 0 & 0 & I_s \\ 0 & A & 0 \\ I_s & 0 & B \end{pmatrix}.$$  

Obviously, it is nondegenerate. Furthermore, for any $h \in H$ and $t + 1 \leq r \leq t + s$,  

$$h(\pi_1(x_r)) - \pi_1(x_r) = h(x_r) - x_r \subset S(M^1_m, H).$$

It follows that $\pi_1(M^1_m)$ is $H$-invariant and nondegenerate. Then

$$\pi_1(M^1_m) = N^1_m.$$  

By the assumption, $M^1_m = \pi^{-1}(V_1) \oplus \pi^{-1}(V_2)$. For any $x \in \pi^{-1}(V_1)$ and $h \in H$, $hx = y_1 + y_2$, where $y_i \in \pi^{-1}(V_i)$. It follows that

$$h(\pi(x)) = \pi(y_1) + \pi(y_2).$$

Since $V_1$ is $H$-invariant, we have that $\pi(y_2) = 0$. That is, $y_2 = 0$. Namely, $\pi^{-1}(V_1)$ is $H$-invariant. Similarly $\pi^{-1}(V_2)$ is $H$-invariant. It is a contradiction. Then the theorem follows. \[ \square \]

**Remark 3.10.** Assume that an indecomposable nondegenerate subspace $M^i_m$ of $M_m$ preserves a nontrivial decomposition $M^i_m = V_1 \oplus V_2$. It is given in [2] that there exists also an $H$-invariant decomposition $M^i_m = U_1 \oplus U_2$ into the direct sum of two totally isotropic subspaces, in particular, $M^i_m$ must have a neutral signature. Thus if every factor with a non zero maximal trivial subspace in the decomposition hasn’t a neutral signature, then $M$ satisfies Condition $\Phi$ by Theorem 3.9.

3.2. **Holonomy groups of pseudo-Riemannian manifolds satisfying Condition $\Phi$.** Let $M$ be a pseudo-Riemannian manifold and $H$ be its holonomy group at the point $m$. Then $M_m$ admits an orthogonal decomposition into $H$-invariant subspaces:

$$M_m = M^0_m \oplus M^1_m \oplus \cdots \oplus M^p_m,$$
where $M^0_m$ is a maximal nondegenerate subspace of the maximal trivial subspace of $H$ in $M_m$ and $M^i_m$ is indecomposable for any $1 \leq i \leq p$. Let $H = H^1 \times \cdots \times H^p$ be the corresponding decomposition of $H$ associated with the decomposition $M_m = M^0_m \oplus M^1_m \oplus \cdots \oplus M^p_m$. Here $H^i$ is a normal subgroup of $H$ for any $1 \leq i \leq p$, each $H^i$ is indecomposable and $H^i$ acts trivially on $M^k_m$ if $k \neq i$.

Assume that $M$ satisfies Condition $\Phi$. Let

$$M_m = N^0_m \oplus N^1_m \oplus \cdots \oplus N^q_m$$

be another orthogonal decomposition of $M_m$ into $H$-invariant subspaces, where $N^0_m$ is a maximal nondegenerate subspace of the maximal trivial subspace of $H$ in $M_m$ and $N^j_m$ is indecomposable for any $1 \leq j \leq q$. By Theorem 3.2 we can assume that

$$M_m = N^0_m \oplus N^1_m \oplus \cdots \oplus N^p_m,$$

where $\dim M^i_m = \dim N^i_m$ and $S(M^i_m, H) = S(N^i_m, H)$ for any $1 \leq i \leq p$. Let $H = G^1 \times \cdots \times G^p$ be the decomposition of $H$ associated with the decomposition $M_m = N^0_m \oplus N^1_m \oplus \cdots \oplus N^p_m$. Here $G^i$ is a normal subgroup of $H$ for any $1 \leq i \leq p$, each $G^i$ is indecomposable and $G^i$ acts trivially on $N^k_m$ if $k \neq i$.

For any $h \in H^i$, let $h = g_1 \cdots g_p$ be the expression of $h$ associated with the decomposition $H = G^1 \times \cdots \times G^p$. For any $x \in N^j_m$, by the discussion in the previous section,

$$x = x_1 + x_2,$$

where $x_1 \in M^j_m$ and $x_2 \in (M_m)^H$. It follows that, when $i \neq j$,

$$hx = hx_1 + hx_2 = x_1 + x_2 = x = g_jx.$$

It follows that $g_j$ is the identity map when $j \neq i$. That is, $H^i \subset G^i$. Similarly, $G^i \subset H^i$. Namely

$$H^i = G^i.$$

Then we have the following theorem:

**Theorem 3.11.** Let $M$ be a pseudo-Riemannian manifold and $H$ be its holonomy group at the point $m$. Then $H$ is the direct product of a finite number of its normal subgroups which are indecomposable. If $M$ satisfies Condition $\Phi$, then the decomposition is unique up to the order.

4. PSEUDO-RIEMANNIAN MANIFOLDS DISSATISFYING CONDITION $\Phi$

In Section 3 we prove the unique decomposition of holonomy groups of pseudo-Riemannian manifolds satisfying Condition $\Phi$. But it doesn’t hold for every pseudo-Riemannian manifolds. The following is an example given in [15].
Example 4.1. There exists an indecomposable and non-irreducible Kahlerian symmetric space diffeomorphic to \( \mathbb{R}^4 \) whose holonomy group is the following one-parameter subgroup of \( SO(2,2) \):

\[
H = \begin{pmatrix}
1 & -t & 0 & t \\
t & 1 & -t & 0 \\
0 & -t & 1 & t \\
t & 0 & -t & 1
\end{pmatrix}
\]

relative to an orthonormal basis of type \((+,-,-,-)\). Consider \( M = \mathbb{R}^4 \times \mathbb{R}^4 \), where each factor is equipped with the pseudo-Riemannian structure mentioned above. Let \( \{e_1, e_2, e_3, e_4\} \) and \( \{f_1, f_2, f_3, f_4\} \) be an orthonormal basis of \( (\mathbb{R}^4_0,0) \) and \( (0,\mathbb{R}^4_0) \), where 0 denotes the origin. Corresponding to the decomposition \( M_{(0,0)} = \mathbb{R}^4_0 \oplus \mathbb{R}^4_0 \), we have a decomposition of the holonomy group of \( M \) at \( (0,0) \): \( H = H_1 \times H_2 \), where \( H_1 \) and \( H_2 \) are the holonomy group of the first and second factor respectively. Now let

\[
W_1 = \text{span}\{e_1 - (f_1 + f_3), e_2, e_3 + (f_1 + f_3), e_4\},
\]

\[
W_2 = \text{span}\{f_1 - (e_1 + e_3), f_2, f_3 + (e_1 + e_3), f_4\}.
\]

Then \( M_{(0,0)} = W_1 \oplus W_2 \) is another decomposition of \( M_{(0,0)} \) into mutually orthogonal nondegenerate subspaces which are \( H \)-invariant. Consequently, there is another decomposition of the holonomy group. Here are, then, two distinct decompositions of the holonomy group as a direct of its normal subgroups.

Remark 4.2. In the above example, let

\[
V_1 = \text{span}\{e_1 + e_3, e_2 + e_4\},
\]

\[
V_2 = \text{span}\{e_1 - e_3 + t(e_2 + e_4), t(e_1 + e_3) + e_2 - e_4\}.
\]

Then the first factor of \( M_{(0,0)} \) is the direct sum of \( V_1 \) and \( V_2 \), where \( V_1 \) and \( V_2 \) are \( H \)-invariant and isotropic. Similarly it holds for the second factor of \( M_{(0,0)} \). That is, the decomposition of the holonomy group isn’t necessary to be unique if the pseudo-Riemannian manifold admits a decomposition into two indecomposable submanifolds such that the tangent space of every factor admits a nontrivial decomposition into \( H \)-invariant subspaces.

Furthermore we can prove:

Theorem 4.3. Let \( M \) be a pseudo-Riemannian manifold and \( H \) be its holonomy group at the point \( m \). Let

\[
M_m = M_m^0 \oplus M_m^1 \oplus \cdots \oplus M_m^p
\]

be an orthogonal decomposition of \( M_m \) into \( H \)-invariant subspaces, where \( M_m^0 \) is a maximal nondegenerate subspace in \( (M_m)^H \) and \( M_m^i \) is indecomposable for any \( i > 0 \). If there exists only one \( j > 0 \) such that \( (M_m^j)^H \neq 0 \) and \( M_m^j \) admits a nontrivial decomposition into \( H \)-invariant subspaces, then the corresponding decomposition of the holonomy group \( H \) into indecomposable normal subgroups is unique up to the order.
Proof. Let $M^p_m$ satisfy the assumption in the theorem, and let

$$M_m = N^0_m \oplus N^1_m \oplus \cdots \oplus N^q_m$$

be an orthogonal decomposition into $H$-invariant subspaces, where $N^i_m$ is indecomposable for any $1 \leq i \leq q$. We can assume that $(N^i_m)^H \neq 0$ for any $1 \leq i \leq q$ and $(M^1_m)^H \neq 0$ for any $1 \leq j \leq p$ by the proof of Theorem 3.7. Obviously $S(M^1_m, H) \neq 0$. Then there exists $1 \leq k \leq q$ such that $M^1_m \cap N^k_m \neq 0$. Without loss of generality, assume that $k = 1$. Similar to the proof of Lemma 3.3, we have

$$M^1_m \cap (N^0_m \oplus N^2_m \oplus \cdots \oplus N^p_m) = 0,$$

since $M^1_m$ doesn’t admit a nontrivial decomposition into $H$-invariant subspaces. Similar to the proof of Lemma 3.5, the projection $\pi_1$ from $M^1_m$ to $N^1_m$ is an injection, and for any $x \in M^1_m$, $x = x_1 + x_2$ where $x_1 \in N^1_m$ and $x_2 \in (M^1_m)^H$. It follows that

$$S(M^1_m, H) \subset S(N^1_m, H).$$

By the proof of Theorem 3.9, $\pi_1(M^1_m) = N^1_m$ and $N^1_m$ doesn’t admit a nontrivial decomposition. That is, $\pi_1$ is subjective. Reversing the role of $M^1_m$ and $N^1_m$,

$$S(M^1_m, H) = S(N^1_m, H).$$

Similar discuss to $i = 2, \cdots, p - 1$, then

1. $p = q$ and every $N^i_m$ doesn’t admit a nontrivial decomposition when $1 \leq i \leq p - 1$;
2. Changing the subscripts if necessary, $S(M^i_m, H) = S(N^i_m, H)$ for any $1 \leq i \leq p - 1$;
3. Each projection $\pi_i : M^i_m \to N^i_m$ is $1 - 1$, and for any $x \in M^i_m$, $x = x_1 + x_2$ where $x_1 \in N^i_m$ and $x_2 \in (M^i_m)^H \cap \oplus_{j \neq i} N^j_m$.

The following is to discuss $M^p_m$ and $N^p_m$. Take $x \in N^p_m \oplus \oplus_{i=0}^{p-1} M^i_m$. Let $x = x_0 + x_1 + \cdots + x_{p-1}$ be an expression of $x$ corresponding to $\oplus_{i=0}^{p-1} M^i_m$. Consider $hx - x$ for any $h \in H$, we have that $x_1 \in (M^i_m)^H$ for any $1 \leq i \leq p - 1$ by $S(M^i_m, H) = S(N^i_m, H)$ for any $1 \leq i \leq p - 1$. That is, $x \in (M^i_m)^H \cap N^p_m$. Then $x$ is isotropic, which implies $x_0 = 0$. Also for any $1 \leq i \leq p - 1$, $x_i \in (M^i_m)^H \subset S(M^i_m, H) = S(N^i_m, H)$. By the expression of $x$, $x_i = 0$ for any $1 \leq i \leq p - 1$. Then $x = 0$. Namely $N^p_m \cap \oplus_{i=0}^{p-1} M^i_m = 0$. Similarly $M^p_m \cap \oplus_{i=0}^{p-1} N^i_m = 0$. It follows that

1. $S(M^p_m, H) = S(N^p_m, H)$;
2. The projection $\pi_p : M^p_m \to N^p_m$ is $1 - 1$, and for any $x \in M^p_m$, $x = x_1 + x_2$ where $x_1 \in N^p_m$ and $x_2 \in (M^p_m)^H \cap \oplus_{j \neq p} N^j_m$.

Similar to the proof of Theorem 3.11, the theorem holds. \qed

The proof of Theorem 4.3 gives the de Rham decomposition for those pseudo-Riemannian manifolds. But we don’t list it here. Thanks to Theorems 3.11 and 4.3, we have the following theorem for Lorentzian manifolds:
Theorem 4.4. Let $M$ be a Lorentzian manifold and $H$ be its holonomy group at the point $m$. Then the decomposition of $H$ into its indecomposable normal subgroups is unique up to the order.

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