OBSERVABLES AND STATES IN p-MECHANICS

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Abstract. This is an up-to-date survey of the p-mechanical construction, which is a consistent physical theory suitable for a simultaneous description of classical and quantum mechanics. Observables in p-mechanics are defined to be convolution operators on the Heisenberg group \( H^n \). Under irreducible representations of \( H^n \) the p-observables generate corresponding observables in classical and quantum mechanics. p-States are defined as positive linear functionals on p-observables. It is shown that both states and observables can be realised as certain sets of functions/distributions on the Heisenberg group. The dynamical equations for both p-observables and p-states are provided. The construction is illustrated by the forced and unforced harmonic oscillators. Connections with the contextual interpretation of quantum mechanics are discussed.

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1. Introduction

The Copenhagen interpretation deals with quantum uncertainties through the concepts of physical systems and observers. However it is not correct to say that in the orthodox interpretation system and observer are “separated” since they are both declared to be meaningful only through their interaction during the measurement process. Effectively in most cases observer is replaced by the state of a physical system and a measurement gives the probability for an observer to find the system in a particular state. Thus the triad of system-state-measurement forms the starting point in any mathematical model of quantum mechanics [39]. Even the schemes opposing the Copenhagen interpretation, e.g Bohmian mechanics [8, 7] and contextual interpretation [18, 19], could not escape this structure implemented in one or another way. For example, the Bohm approach [8, 7] may be interpreted in these terms as follows:

| Copenhagen interpretation | Bohmian approach |
|---------------------------|-----------------|
| a physical system governed by the Schrödinger equation | an independent observer |
| an independent observer | a hidden wave described by the master equation |
| a measurement of the system by the observer with probabilistic outcome | an accidental choice of |
| | a trajectory by the particle along the wave |

Originally quantum mechanics was implemented through the Hamiltonian formalism borrowed from classical mechanics. It became important from the idealogical point of view to find a system-state-measurement triple in the classical framework. The description of classical mechanics in these words could be found in the introductory chapters of almost all textbooks on quantum mechanics [31] since the only “king road” into the quantum world is still going through the classical “pathway”. On the other hand textbooks devoted only to classical theory [2, 13] usually ignore an analysis of this observable-state-measurement triad as having little relevance in classical picture.

It is commonly accepted that observables form an algebra both in quantum and classical mechanics: non-commutative in the former case and commutative in the later. Then states are defined as positive linear functionals on those algebras: in quantum mechanics pure states are labelled by certain vectors in a Hilbert spaces and classic pure states correspond to points of the phase space. Measurements are represented by evaluation of observables on particular states.
In this paper we are going to present a description of observables and states within the framework of \( p \)-mechanics. \( p \)-Mechanics unifies Hamilton formulations of quantum and classical mechanics on the grounds of the representation theory of the Heisenberg group. Therefore our approach coincides with the traditional route in many principal positions. On the other hand \( p \)-mechanics (being based only on the natural properties of Heisenberg group) is in better agreement with physical requirements and sheds an additional light on the known results and constructions in both mechanics.

For example, the widespread agreement about observables being elements of an algebra contradicts the basic physical principle that all measurements (and thus observables) are evaluated in certain physical units. The multiplication of two physical quantities with different units is natural, for example, velocity multiplied by time gives length. However the addition of two physical quantities in different units is unacceptable, for example, you cannot add something measured in kilograms with something measured in centimetres. In \( p \)-mechanics we don’t allow addition to be freely permitted by replacing “algebra” with “symmetric space”. For a more in depth description of this dimensional analysis see \( [26, §§ 1.1, 1.2] \).

\( p \)-Mechanical observables \( [24, 26] \) are identified with convolution operators on the Heisenberg group and labelled by kernels of these convolutions. By the representations of the Heisenberg group convolutions are transformed into well known images of quantum (operators on a Hilbert space) and classical (functions on the phase space) observables. Therefore it is natural to define \( [9] \) \( p \)-states in line with quantum and classical cases as positive linear functionals on the space of \( p \)-observables. Elaboration of this approach is the main purpose of the present paper.

The paper outline is as follows. In the next Section we present the representation theory of the Heisenberg group based on the orbit method of Kirillov \( [23] \) and utilising Fock–Segal–Bargmann spaces \( [12, 16] \). We emphasise the existence and applicability of the family of one-dimensional representations: they play for classical mechanics exactly the same rôle as the infinite dimensional representations—for quantum. In section 3 we introduce the concept of both states and observables in \( p \)-mechanics and describe relations with their quantum and classical counterparts. These links are provided by the representations of the Heisenberg group and wavelet transforms. In subsections 4.1 and 4.3 we study \( p \)-mechanical brackets and the associated dynamic equation together with its classical and quantum representations. While in subsections 4.2 and 4.3 we describe the time evolution of \( p \)-Mechanical states and prove that it agrees with the time evolution of observables in doing so we exhibit the pictures of \( p \)-dynamics. In section 4.5 we introduce a system of coherent states for \( p \)-mechanics. Finally in section 5 we demonstrate the theory through the examples of the forced and unforced harmonic oscillator.

2. The Heisenberg Group and Its Representations

We start from the representation theory of the Heisenberg group \( \mathbb{H}^n \) based on the orbit method of Kirillov. Analysis of the unitary dual of \( \mathbb{H}^n \) in Subsection 2.2 suggests that the family of one-dimensional representations of \( \mathbb{H}^n \) forms the phase space of a classical system. Infinite dimensional representations in a Fock type space are described in Subsection 2.4.
As was mentioned in the Introduction, p-mechanics does not associate observables with an algebra since this contradicts the physical reality. In order to emphasise that feature of p-mechanics we supply the analysis of dimensions along our construction.

Let $M$ be a unit of mass, $L$—of length, $T$—of time. Then coordinates of a point in phase space, $(q, p)$ are measured in units $L$ and $ML/T$ (momentum) respectively. Derivatives $\frac{\partial}{\partial q}$, $\frac{\partial}{\partial p}$, $\frac{\partial}{\partial t}$ of an observables with respect to coordinates, momentum, and time are measured by $1/L$, $T/(ML)$, and $1/T$ respectively. Corresponding differentials $dq$, $dp$, and $dt$ are measured in the according units: $L$, $ML/T$, and $T$ in order to make the inner product of vectors and 1-forms a dimensionless pure number. Throughout this paper only physical quantities of the same dimension can be added or subtracted together.

### 2.1. Representations $\mathbb{H}^n$ and the Method of Orbits.

Let $(s, x, y)$, where $x, y \in \mathbb{R}^n$ and $s \in \mathbb{R}$, be an element of the Heisenberg group $\mathbb{H}^n$. We assign to $x$ and $y$ components of $(s, x, y)$ physical units $1/L$ and $T/(LM)$ respectively. We chose these units so that $qx$ and $py$ are dimensionless products.

The group law on $\mathbb{H}^n$ is given as follows:

$$ (s, x, y) * (s', x', y') = (s + s' + \frac{1}{2} \omega(x, y; x', y'), x + x', y + y'), \quad (2.1) $$

where the non-commutativity is solely due to $\omega$—the **symplectic form** [2, § 37] on the Euclidean space $\mathbb{R}^{2n}$:

$$ \omega(x, y; x', y') = xy' - x'y. \quad (2.2) $$

Consequently the parameters $s$ should be measured in $T/(L^2 M)$—the product of units of $x$ and $y$. The Lie algebra $\mathfrak{h}^n$ of $\mathbb{H}^n$ is spanned by a basis $S, X_j, Y_j$, $j = 1, \ldots, n$ which may be represented by either left- or right-invariant vector fields on $\mathbb{H}^n$:

$$ S^{(l)} = \pm \frac{\partial}{\partial s}, \quad X_j^{(r)} = \pm \frac{\partial}{\partial x_j} - \frac{y_j}{2} \frac{\partial}{\partial s}, \quad Y_j^{(l)} = \pm \frac{\partial}{\partial y_j} + \frac{x_j}{2} \frac{\partial}{\partial s} \quad (2.3) $$

with the Heisenberg commutator relations

$$ [X_i^{(l)}, Y_j^{(r)}] = \delta_{i,j} S^{(l)} \quad (2.4) $$

all other commutators (including any between left and right vector fields) vanish.

Units to measure $S^{(l)}$, $X_j^{(r)}$, and $Y_j^{(l)}$ are inverse to $s, x, y$—i.e. $L^2 M/T$, $L$, and $LM/T$ respectively—which are obviously compatible with $\mathbb{H}^n$.

The exponential map $\exp : \mathfrak{h}^n \to \mathbb{H}^n$ is provided by the formula:

$$ \exp : sS + \sum_{j=1}^n (x_j X_j + y_j Y_j) \mapsto (s, x_1, \ldots, x_n, y_1, \ldots, y_n). $$

which respects multiplication [21] and the Heisenberg commutator relations [24].

The composition of the exponential map with representations [23] of $\mathfrak{h}^n$ by the left(right)-invariant vector fields produces the right (left) regular representation $\lambda_r(l)$ of $\mathbb{H}^n$ by right (left) shifts. Linearised [21, § 7.1] to $L_2(\mathbb{H}^n)$ they are:

$$ \lambda_r(g) : f(h) \mapsto f(hg), \quad \lambda_l(g) : f(h) \mapsto f(g^{-1}h), \quad \text{where } f(h) \in L_2(\mathbb{H}^n). \quad (2.5) $$

As any group $\mathbb{H}^n$ acts on itself by the conjugation automorphisms $A(g)h = g^{-1}hg$, which fix the unit $e \in \mathbb{H}^n$. The differential $\text{Ad} : \mathfrak{h}^n \to \mathfrak{h}^n$ of $A$ at $e$ is a linear map which can be differentiated again to the representation $\text{ad}$ of the Lie
algebra \( \mathfrak{h}^n \) by the commutator: \( \text{ad}(A) : B \mapsto [B, A] \). The adjoint space \( \mathfrak{h}^*_n \) of the Lie algebra \( \mathfrak{h}^n \) can be realised by the left invariant first order differential forms on \( \mathbb{H}^n \). By the duality between \( \mathfrak{h}^n \) and \( \mathfrak{h}^*_n \) the map \( \text{Ad} \) generates the \textit{co-adjoint representation} \( \mathfrak{h}^*_n \rightarrow \mathfrak{h}^*_n \):

\[
\text{Ad}^*(s, x, y) : (h, q, p) \mapsto (h, q + hy, p - hx), \quad \text{where} \quad (s, x, y) \in \mathbb{H}^n
\]

and \( (h, q, p) \in \mathfrak{h}^*_n \) in bi-orthonormal coordinates to the exponential ones on \( \mathfrak{h}^n \). These coordinates \( h, q, p \) should have units of action \( M\ell^2/T, \text{position} L, \text{and momenta} \ell M/T \) respectively.

There are two types of orbits in \( \mathfrak{h}^n \) for \( \text{Ad}^* \): the Euclidean spaces \( \mathbb{R}^{2n} \) and single points:

\[
\mathcal{O}_h = \{(h, q, p) : \text{for a fixed} \ h \neq 0 \text{ and all} \ (q, p) \in \mathbb{R}^{2n}\},
\]

\[
\mathcal{O}_{(q,p)} = \{(0, q, p) : \text{for a fixed} \ (q, p) \in \mathbb{R}^{2n}\}.
\]

The \textit{orbit method} of Kirillov \cite{kirillov197615}, \cite{kirillov1976} starts from the observation that the above orbits parametrise all irreducible unitary representations of \( \mathbb{H}^n \). All representations are \textit{induced} \cite{kirillov197615} \cite{kirillov1976} by a character \( \chi_h(s, 0, 0) = e^{2\pi ihs} \) of the centre of \( \mathbb{H}^n \) generated by \( (h, 0, 0) \in \mathfrak{h}^*_n \) and shifts \( 2.0 \) from the “left hand side” (i.e. by \( g^{-1} \) on orbits). Using \cite{kirillov197613} \cite{kirillov197615} \cite{kirillov1976} \cite{casini2018} we get a neat formula, which (unlike some others in the literature, e.g. \cite{kirillov1976} Chap. 1, (2.23)) respects the rule that you cannot add any two physical quantities of different units:

\[
\rho_h(s, x, y) : f_h(q, p) \mapsto e^{-2\pi i(hs+qx+py)}f_h\left(q - \frac{h}{2}y, p + \frac{h}{2}x\right),
\]

Exactly the same formula is obtained if we apply the Fourier transform \( \hat{\cdot} \) : \( L_2(\mathbb{H}^n) \rightarrow L_2(\mathfrak{h}^*_n) \) given by:

\[
(\mathcal{F}(\phi))(Y) = \hat{\phi}(Y) = \int_{\mathbb{H}^n} \phi(\exp X)e^{-2\pi i(X,Y)}dX \quad \text{where} \quad X \in \mathfrak{h}^n, \ Y \in \mathfrak{h}^*_n
\]

to the left regular action \( \mathbb{H}^n \), that is

\[
\lambda_i(g)\mathcal{F} = \mathcal{F}\rho_h(g).
\]

See \cite{kirillov1976} \cite{kirillov1976} \cite{kirillov1976} \cite{kirillov1976} for relations between the Fourier transform \( \mathcal{F} \) and the orbit method.

The derived representation \( d\rho_h \) of the Lie algebra \( \mathfrak{h}^n \) defined on the vector fields \( \mathfrak{h} \) is:

\[
d\rho_h(S) = -2\pi ihI, \quad d\rho_h(X_j) = \frac{h}{2}\partial_{p_j} - 2\pi iq_jI, \quad d\rho_h(Y_j) = -\frac{h}{2}\partial_{q_j} - 2\pi ip_jI,
\]

which clearly represent the commutation rules \( \mathfrak{h} \). The representation \( \rho_h \) is reducible on the whole of \( L_2(O_h) \) as can be seen from the existence of the set of “right-invariant”, i.e. commuting with \( \mathfrak{h} \), differential operators:

\[
d\rho_h^*(S) = 2\pi ihI, \quad \frac{h}{2}\partial_{p_j} - 2\pi iq_jI, \quad \frac{h}{2}\partial_{q_j} - 2\pi ip_jI.
\]

These vectors fields represent the commutation rules \( \mathfrak{h} \) as well.

To obtain an irreducible representation defined by \( \mathfrak{h} \) we need to restrict it to a subspace of \( L_2(O_h) \) where the operators \( \mathfrak{h} \) act as scalars, e.g. use a \textit{polarisation} from \textit{geometric quantisation} \cite{polarisation}. Consider for \( h > 0 \) the vector field \( -X_j + iq_jY_j \) from the complexification of \( \mathfrak{h}^n \), where a constant \( c_1 \) has the dimension \( T/M, \) the
numerical value of $c_i$ in the given units can be assumed to be 1. We introduce 
operators $D^j_h$, $1 \leq j \leq n$ representing vectors $-X_j + ic_i Y_j$:

$$D^j_h = d\rho_h (-X_j + ic_i Y_j) = \frac{\hbar}{2}(\partial_{p_j} + c_i \partial_{q_j}) + 2\pi (c_i p_j + i q_j) I = \hbar \partial_{z_j} + 2\pi z_j I \quad (2.14)$$

where $z_j = c_i p_j + i q_j$. For $h < 0$ we define $D^j_h = d\rho_h^r(-c_i Y_j + i X_j)$. Operators (2.14) are used to give the following classical result in terms of orbits:

**Theorem 2.1** (Stone–von Neumann, cf. [12] Chap. 1, § 5, [21] § 18.4). All unitary irreducible representations of $\mathbb{H}^n$ are parametrized up to equivalence by two classes of orbits (2.17) and (2.8) of the co-adjoint representation (2.2) in $\mathfrak{h}_n^*$:

1. The infinite dimensional representations by transformation $\rho_h$ (2.1) for $h \neq 0$ in Fock [12, 15] space $F_2(\mathcal{O}_h) \subset L_2(\mathcal{O}_h)$ of null solutions to the operators $D^j_h$ (2.14):

$$F_2(\mathcal{O}_h) = \{ f_h(q,p) \in L_2(\mathcal{O}_h) \mid D^j_h f_h = 0, \ 1 \leq j \leq n \} \quad (2.15)$$

2. The one-dimensional representations as multiplication by a constant on $\mathbb{C} = L_2(\mathcal{O}_{(q,p)})$ which drops out from (2.14) for $h = 0$:

$$\rho_{(q,p)}(s, x, y) : c \mapsto e^{-2\pi i (q x + p y)} c, \quad (2.16)$$

with the corresponding derived representation

$$d\rho_{(q,p)}(S) = 0, \quad d\rho_{(q,p)}(X_j) = -2\pi i q_j, \quad d\rho_{(q,p)}(Y_j) = -2\pi i p_j. \quad (2.17)$$

2.2. **Structure and Topology of the Unitary Dual of $\mathbb{H}^n$**. The structure of the unitary dual object to $\mathbb{H}^n$—the collection of all different classes of unitary irreducible representations—as it appears from the method of orbits is illustrated by Figure 1 cf. [22] Chap. 7, Fig. 6 and 7. The adjoint space $\mathfrak{h}_2^*$ is sliced into “horizontal” hyperplanes. A plane with a parameter $h \neq 0$ forms a single orbit (2.17) and corresponds to a particular class of unitary irreducible representations (2.16). The plane with parameter $h = 0$ is a family of one-point orbits $(0, q, p)$ (2.8), which produce one-dimensional representations (2.16). The topology on the dual object is the factor topology inherited from the adjoint space $\mathfrak{h}_n^*$ under the above identification, see [23] § 2.2.

**Example 2.2**. A set of representations $\rho_h$ (2.1) with $h \to 0$ is dense in the whole family of one-dimensional representations (2.16), as can be seen either from the Figure 1 or the analytic expressions (2.9) and (2.10) for those representations.

Non-commutative representations $\rho_h$, $h \neq 0$ (2.9) have been connected with quantum mechanics from the very beginning [12], this explains, for example, the name of the Heisenberg group. In contrast the commutative representations (2.10) are always neglected and only mentioned for completeness in mathematical formulations of the Stone–von Neumann theorem. The development of p-mechanics starts [23] from the observation that the union of all representations $\rho_{(q,p)}$, $(q, p) \in \mathbb{R}^{2n}$ naturally acts as the classical phase space. The appropriateness of the single union

$$\mathcal{O}_0 = \bigcup_{(q,p) \in \mathbb{R}^{2n}} \mathcal{O}_{(q,p)} \quad (2.18)$$

rather than unrelated sets of disconnected orbits manifests itself in several ways:
The adjoint space $\mathfrak{h}_n^*$ of the algebra $\mathfrak{h}^n$    The unitary dual of $\mathbb{H}^n$

Figure 1. Structure of the unitary dual object to $\mathbb{H}^n$ appearing from the method of orbits. The space $\mathfrak{h}_n^*$ is sliced into “horizontal” hyperplanes. Planes with $h \neq 0$ form single orbits and correspond to different classes of unitary irreducible representation. The plane $h = 0$ is a family of one-point orbits $(0, q, p)$, which produce one-dimensional representations. The topology on the dual object is the factor topology inherited from the $\mathfrak{h}_n^*$. [23 § 2.2].

(1) The topological position of $O_0$ as the limiting case (cf. Example 2.2) of quantum mechanics for $h \to 0$ realises the correspondence principle between quantum and classical mechanics.

(2) Symplectic automorphisms of the Heisenberg group (see Subsection 4.4) produce the metaplectic representation in quantum mechanics and transitively acts by linear symplectomorphisms on the whole set $O_0 \setminus \{0\}$.

(3) We got the Poisson brackets (4.7) on $O_0$ from the same source (4.2) which leads to the correct Heisenberg equation in quantum mechanics.

Our form (2.9) of representations of $\mathbb{H}^n$ given in Theorem 2.1 has at least two following advantages which are rarely combined together:

(1) There is the explicit physical meaning of all entries in (2.9) as will be seen below. In contrast the formula (2.23) in [38, Chap. 1] contains terms $\sqrt{h}$ (in our notations) which could be hardly justified from a physical point of view.

(2) The one-dimensional representations (2.16) explicitly correspond to the case $h = 0$ in (2.9). The Schrödinger representation (the most used in quantum mechanics!) is handicapped in this sense: the transition for $h \to 0$ from $\rho_h$ in the Schrödinger form to $\rho_{(q,p)}$ requires a long discussion [22 Ex. 7.11].

We finish the discussion of the unitary dual of $\mathbb{H}^n$ by a remark about negative values of $h$. Due to its position in the Heisenberg equation the negative value of $\hbar$ will revert the flow of time. Thus representations $\rho_h$ with $h < 0$ seem to be suitable for a description of anti-particles with the explicit (cf. Figure 1) mirror symmetry between matter and anti-matter through classical mechanics. In this paper however we will consider only the case of $h > 0$.

2.3. Fock Spaces $F_2(O_h)$. Our Fock type spaces (2.15) are not very different [26 Ex. 4.3] from the standard Segal–Bargmann spaces.
Definition 2.3. [12, 15] The Segal–Bargmann space (with a parameter \( h > 0 \)) consists of functions on \( \mathbb{C}^n \) which are holomorphic in \( z \), i.e. \( \partial_z f(z) = 0 \), and square integrable with respect to the measure \( e^{-2|z|^2/h} dz \) on \( \mathbb{C}^n \):

\[
\int_{\mathbb{C}^n} |f(z)|^2 e^{-2|z|^2/h} dz < \infty.
\]

Noticing the \( \partial_z \) component in the operator \( D_h^j \) [24] we obviously obtain

Proposition 2.4. [26] A function \( f_h(q, p) \) is in \( F_2(\mathcal{O}_h) \) [2, 15] for \( h > 0 \) if and only if the function \( f_h(z)e^{|z|^2/h}, z = p + iq \) is in the classical Segal–Bargmann space.

The space \( F_2(\mathcal{O}_h) \) can also be described in the language of coherent states, (also known as wavelets, matrix elements of representation, Berezin transform, etc., see [1]). Since the representation \( \rho_h \) is irreducible any vector \( f_0 \) in \( F_2(\mathcal{O}_h) \) is cyclic, i.e. vectors \( \rho_h(g)f_0 \) for all \( g \in G \) span the space \( F_2(\mathcal{O}_h) \). However even if all vectors are equally good in principle, some of them are more equal for particular purposes. Our best option is to take the vector in \( F_2(\mathcal{O}_h) \) corresponding to the vacuum state of the harmonic oscillator with classical Hamiltonian \( \frac{1}{2}(m\omega^2q^2 + \frac{1}{2}p^2) \) where \( \omega \) is the constant frequency (measured in units \( \frac{1}{h} \)) and \( m \) is the constant mass:

\[
f_0(q,p) = \exp\left(-\frac{2\pi}{h}((\omega m)q^2 + (\omega m)^{-1}p^2)\right), \tag{2.19}
\]

which corresponds to the minimum level of energy. Note also that \( f_0(q,p) \) is destroyed by the annihilation operators (sf. [2, 12] and 2.14):

\[
A_h^j = d\rho_h(X_j + i\alpha Y_j) = \frac{h}{2}(\partial_{p_j} - i\alpha\partial_{q_j}) + 2\pi(c\partial_{p_j} - i\alpha)I. \tag{2.20}
\]

We introduce a dimensionless inner product on \( F_2(\mathcal{O}_h) \) by the formula:

\[
\langle f_1,f_2 \rangle = \left( \frac{4}{h} \right)^n \int_{\mathbb{R}^{2n}} f_1(q,p)\bar{f_2}(q,p) dq dp \tag{2.21}
\]

With respect to this product the vacuum vector \( f_0 \) [2, 14] is normalised: \( \| f_0 \| = 1 \). For a dimensionless vector \( f \in F_2(\mathcal{O}_h) \) the formula defines a state

\[
\langle Af,f \rangle = \left( \frac{4}{h} \right)^n \int_{\mathbb{R}^{2n}} Af(q,p)\bar{f}(q,p) dq dp \tag{2.22}
\]

which for any observable \( A \) will give an expectation in the units of \( A \), since the inner product is dimensionless. The term \( h^{-n} \) in (2.21) not only normalises the vacuum and fixes the dimensionality of the inner product; it is also related to the Plancherel measure [12, (1.61)], [38, Chap. 1, Th. 2.6] on the unitary dual of \( \mathbb{H}^n \).

Elements \( (s,0,0) \) of the centre of \( \mathbb{H}^n \) trivially act in the representation \( \rho_h \) [2, 14] as multiplication by scalars, e.g. any function is a common eigenvector of all operators \( \rho_h(s,0,0) \). Thus the essential part of the operator \( \rho_h(s, y) \) is determined solely by \( (x, y) \in \mathbb{R}^{2n} \). The coherent states for \( F_2(\mathcal{O}_h) \), \( f_{(x,y)}(q, p) \), are “left shifts” of the vacuum vector \( f_0(q, p) \) by operators [24]:

\[
f_{(x,y)}(q, p) = \rho_h(0, x, y)f_0(q, p) \tag{2.23}
\]

\[
= \exp\left(-2\pi i(qx + py) \right) - \frac{2\pi}{h} \left( \omega m \left(q - \frac{h}{2}y \right)^2 + \omega m)^{-1} \left(p + \frac{h}{2}x \right)^2 \right) \right).
\]
Now any function from the space $F_2(O_\hbar)$ can be represented \cite{25, Ex. 4.3} as a linear superposition of coherent states:

$$f(q, p) = [\hat{M}_h f](q, p) = \hbar^n \int_{\mathbb{R}^{2n}} \hat{f}(x, y) f_{(x, y)}(q, p) \, dx \, dy$$ (2.24)

where $\hat{f}(x, y)$ is the wavelet transform \cite{1, 25} of $f(q, p)$:

$$\hat{f}(x, y) = [\mathcal{W}_h f](x, y) = \langle f, f_{(x, y)} \rangle_{F_2(O_\hbar)}$$ (2.25)

The formula (2.24) can be regarded \cite{25} as the inverse wavelet transform $\hat{M}$ of $\hat{f}(x, y)$.

This set of coherent states, $f_{(x, y)}$, are useful as an overcomplete system of vectors in $F_2(O_\hbar)$ and in exhibiting relations between $p$-mechanics and Berezin quantisation (subsection 3.2). Unfortunately the "classical limits" for $\hbar \to 0$ of all these coherent states are functions supported in the neighbourhood of $(0, 0)$. Instead we want them to be supported around different classical states $(q, p)$. This defect is resolved in section 4.5 when we have a clearer definition of what $p$-mechanical states are.

3. $p$-Mechanics: Statics

We define $p$-mechanical observables to be convolutions on the Heisenberg group. The next subsection describes their multiplication and commutator as well as their quantum and classical representations. The Berezin quantisation in the form of a wavelet transform is considered in subsection 3.2. This is developed in subsection 3.3 into a construction of $p$-observables out of either quantum or classical ones. $p$-Mechanical states are introduced in subsection 3.4, as functionals on the set of observables, which come in two forms: kernels and elements of a Hilbert space.

3.1. Observables in $p$-Mechanics, Convolutions and Commutators. In line with the standard quantum theory we give the following definition:

**Definition 3.1.** \cite{21, 26} Observables in $p$-mechanics ($p$-observables) are presented by operators on $L_2(\mathbb{H}^n)$.

It is important for subsection 3.4 to note that as the observables are operators on a Hilbert space they form a $C^*$-algebra \cite{24}. Actually we will need here\footnote{More general operators are in use for a string-like version of $p$-mechanics, see \cite{26, Sect 5.2.3].} only operators generated by convolutions on $L_2(\mathbb{H}^n)$. Let $dg$ be a left invariant measure \cite{21, § 7.1} on $\mathbb{H}^n$, which coincides with the standard Lebesgue measure on $\mathbb{R}^{2n+1}$ in the exponential coordinates $(s, x, y)$. Then a function $B_1$ from the linear space $L_1(\mathbb{H}^n, dg)$ acts on $B_2 \in L_2(\mathbb{H}^n, dg)$ by a convolution as follows:

$$(B_1 * B_2)(g) = c_h^{n+1} \int_{\mathbb{H}^n} B_1(g_1) B_2(g^{-1}_1 g) \, dg_1$$ (3.1)

where the constant $c_h$ has the value 1 in the units of action. Then $c_h^{n+1}$ has units inverse to $dg$. Thus the convolution $B_1 * B_2$ is measured in units which are the...
product of the units for $B_1$ and $B_2$. We can alternatively write the convolution of two functions on the Heisenberg group as

$$ (B_1 * B_2)(g) = c_h^{n+1} \int_{\mathbb{H}^n} B_1(h) \lambda_0(h) dh B_2(g) $$  \hspace{1cm} (3.2)

where $\lambda_0$ is as defined in equation (2.6). This form of convolution is shown to be useful in subsection 3.4.

The composition of two convolution operators $K_1$ and $K_2$ with kernels $B_1$ and $B_2$ has the kernel defined by the same formula (3.1). This produces inner derivations $D_B$ of $L_1(\mathbb{H}^n)$ by the commutator:

$$ D_B : f \mapsto [B, f] = B * f - f * B $$

$$ = c_h^{n+1} \int_{\mathbb{H}^n} B(g_1) (f(g_1^{-1} g) - f(g g_1^{-1}) ) dg_1. \hspace{1cm} (3.3) $$

Since we only consider observables which are convolutions on $\mathbb{H}^n$ we can extend a unitary representation $\rho_h$ of $\mathbb{H}^n$ to a $*$-representation of $L_1(\mathbb{H}^n, dg)$ by the formula:

$$ [\rho_h(B)](q, p) = c_h^{n+1} \int_{\mathbb{H}^n} B(g) \rho_h(g) f(q, p) dg $$

$$ = c_h^n \int_{\mathbb{R}^{2n}} \left( c_h \int_{\mathbb{R}} B(s, x, y) e^{-2\pi i hs} ds \right) e^{-2\pi i (qx + py)} f \left( \frac{q-h}{2}, y + \frac{h}{2} x \right) dx dy. \hspace{1cm} (3.4) $$

The last formula in the Schrödinger representation defines for $h \neq 0$ a pseudodifferential operator on $L^2(\mathbb{R}^n)$, which are known to be quantum observables in the Weyl quantisation. For representations $\rho_{(q,p)}$ (2.10) the expression analogous to (3.4) defines an operator of multiplication on $O_0$ by the Fourier transform of $B(s, x, y)$:

$$ \rho_{(q,p)}(B) = \hat{B}(0, q, p) = c_h^{n+1} \int_{\mathbb{H}^n} B(s, x, y) e^{-2\pi i (qs + py)} ds dx dy, \hspace{1cm} (3.5) $$

where the direct $\hat{}$ and inverse $\check{}$ Fourier transforms are defined by the formulae:

$$ \hat{f}(v) = \int_{\mathbb{R}^n} f(u) e^{-2\pi i vu} du \quad \text{and} \quad f(u) = (\hat{f})^{-\check{}}(u) = \int_{\mathbb{R}^n} \hat{f}(v) e^{2\pi i vu} dv. $$

For reasons discussed in subsections 2.2 and 1.1 we regard the functions $\hat{\rho_{(q,p)}}$ on $O_0$ as classical observables. Again both the representations $\rho_h(B)$ and $\rho_{(q,p)}(B)$ are measured in the same units as the function $B$.

From (3.4) it follows that $\rho_h(B)$ for a fixed $h \neq 0$ depends only from $\hat{B}_s(h, x, y)$—the partial Fourier transform $s \rightarrow h$ of $B(s, x, y)$. Then the representation of the composition of two convolutions depends only from

$$ (B' * B)_s = c_h \int_{\mathbb{R}} e^{-2\pi i hs} c_h^{n+1} \int_{\mathbb{H}^n} B'(s', x', y') \times B(s - s' + \frac{1}{2}(xy' - yx'), x - x', y - y') ds' dx' dy' ds $$

$$ = c_h^{n+1} \int_{\mathbb{R}^{2n}} e^{i h(xy' - yx')} \hat{B}'_s(h, x', y') \hat{B}_s(h, x - x', y - y') dx' dy'. \hspace{1cm} (3.6) $$

Note that if we apply the Fourier transform $(x, y) \rightarrow (g, p)$ to the last expression in (3.6) then we get the star product of $\hat{B}'$ and $\hat{B}$ known in deformation quantisation,
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Consequently the representation $\rho_h((B', B)]$ of commutator depends only on:

$$
[B', B]^s = \epsilon_h^n \int_{\mathbb{R}^{2n}} \left( e^{i\pi \hbar (xy' - yx')} - e^{-i\pi \hbar (xy' - yx')} \right)
\times \hat{B}'_s(-h, x', y') \hat{B}_s(h, x - x', y - y')
= 2i \epsilon_h^n \int_{\mathbb{R}^{2n}} \sin (\pi \hbar (xy' - yx')) \hat{B}'_s(h, x', y') \hat{B}_s(h, x - x', y - y')
\, dx' \, dy'.
$$

The integral turns out to be equivalent to the Moyal brackets for the (full) Fourier transforms of $B'$ and $B$. It is commonly accepted that the method of orbits is the mathematical side of geometric quantisation. Our derivation of the Moyal brackets in terms of orbits shows that deformation and geometric quantisations are closely connected and both are not very far from the original quantisation of Heisenberg and Schrödinger. Yet one more close relative can be identified as the Berezin quantisation, see the next subsection.

Remark 3.2. The expression vanishes for $h = 0$ as can be expected from the commutativity of representations. Thus, it does not produce anything interesting on $O_0$, that supports the common negligence to this set.

Summing up, $p$-mechanical observables, i.e. convolutions on $L^2(\mathbb{H}^n)$ are transformed

1. by representations $\rho_h$ into quantum observables with the Moyal bracket between them;

2. by representations $\rho(q, p)$ into classical observables.

We haven’t got a meaningful bracket on the set of classical observables yet, this will be done in Section 4.1.

3.2. Berezin Quantisation and Wavelet Transform. There is the following construction, known as the Berezin quantisation, which allows us to assign a function to an operator (observable) and an operator to a function. The scheme is based on the construction of coherent states, which can be derived from different sources. We prefer the group-theoretic origin of Perelomov coherent states in this section we use the coherent states in $F_2(O_h)$ defined in equation. Later in this paper we construct a more general system of coherent states independent of a Hilbert space. Following we introduce a covariant symbol $a(g)$ of an operator $A$ on $F_2(O_h)$ by the simple expression:

$$
a(g) = \langle Af_g, f_g \rangle,
$$

i.e. we get a map from the linear space of operators on $F_2(O_h)$ to a linear space of functions on $\mathbb{H}^n$. A map in the opposite direction assigns to a function $\tilde{a}(g)$ on $\mathbb{H}^n$ a linear operator $A$ on $F_2(O_h)$ by the formula

$$
A = \epsilon_h^{n+1} \int_{\mathbb{R}^{2n}} \tilde{a}(g) P_g \, dg,
$$

where $P_g$ is the projection $P_g f = \langle f, f_g \rangle f_g$.

The function $\tilde{a}(g)$ is called the contravariant symbol of the operator $A$.

The co- and contravariant symbols of operators are defined through the coherent states, in fact both types of symbols are realisations of the direct
and inverse \textit{[25] wavelet transforms. Let us define a representation \( \rho_{bh} \) of the group \( \mathbb{H}^n \times \mathbb{H}^n \) in the space \( \mathcal{B}(F_2(O_h)) \) of operators on \( F_2(O_h) \) by the formula:

\[
\rho_{bh}(g_1,g_2) : A \mapsto \rho_{h}(g_1^{-1})A \rho_{h}(g_2), \quad \text{where } g_1,g_2 \in \mathbb{H}^n.
\]  

(3.10)

According to the scheme from \textit{[25] for any state \( l_0 \) on \( \mathcal{B}(F_2(O_h)) \) we get a wavelet transform \( \mathcal{W}_l : \mathcal{B}(F_2(O_h)) \rightarrow \mathcal{C}(\mathbb{H}^n \times \mathbb{H}^n) : \)

\[
\mathcal{W}_l : A \mapsto \bar{a}(g_1,g_2) = \langle \rho_{bh}(g_1,g_2)A, l_0 \rangle.
\]  

(3.11)

The important particular case is given by \( l_0 \) defined through the vacuum vector \( f_0 \) \textit{[2.19] by the formula \( \langle A, l_0 \rangle = \langle Af_0, f_0 \rangle \). Then the wavelet transform \( \textit{[3.11] \textit{produces the covariant presymbol } \bar{a}(g_1,g_2) \textit{of operator } A \). Its restriction } a(g) = \bar{a}(g,g) \textit{to the diagonal } D \textit{of } \mathbb{H}^n \times \mathbb{H}^n \textit{is exactly } \textit{[25] the Berezin covariant symbol } \textit{[3.8] of } A \textit{. Such a restriction to the diagonal is done without a loss of information due to holomorphic properties of } \bar{a}(g_1,g_2) \textit{[4].}

Another important example of the state \( l_0 \) is given by the trace:

\[
\langle A, l_0 \rangle = \text{tr } A = h^n \int_{\mathbb{R}^{2n}} \langle Af(x,y), f(x,y) \rangle_{F_2(O_h)} \, dx \, dy,
\]  

(3.12)

where coherent states \( f(x,y) \) are again defined in \textit{[2.28]}. The operators \( \rho_{bh}(g,g) \) from the diagonal \( D \) of \( \mathbb{H}^n \times \mathbb{H}^n \) trivially act on the wavelet transform \( \textit{[3.11] \textit{generated by the trace } \textit{[3.11] \textit{since the trace is invariant under } \rho_{bh}(g,g). According to the general scheme we can consider the reduced wavelet transform } \textit{[25] \textit{on the homogeneous space } \mathbb{H}^n \times \mathbb{H}^n/D \textit{instead of the entire group } \mathbb{H}^n \times \mathbb{H}^n. The space } \mathbb{H}^n \times \mathbb{H}^n/D \textit{is isomorphic to } \mathbb{H}^n \textit{with an embedding } \mathbb{H}^n \rightarrow \mathbb{H}^n \times \mathbb{H}^n \textit{given by } g \mapsto (g,0). Furthermore the centre } Z \textit{of } \mathbb{H}^n \textit{acts trivially in the representation } \rho_{bh} \textit{as usual. Thus the only essential part of } \mathbb{H}^n \times \mathbb{H}^n/D \textit{in the wavelet transform is the homogeneous space } \Omega = \mathbb{H}^n/Z. A Borel section } s : \Omega \rightarrow \mathbb{H}^n \times \mathbb{H}^n \textit{in the principal bundle } G \rightarrow \Omega \textit{can be defined as } s(x,y) \mapsto ((0,x,y);(0,0,0)). We got the reduced realisation } \mathcal{W}_r \textit{of the wavelet transform } \textit{[3.11] \textit{in the form:}

\[
\mathcal{W}_r : A \mapsto \bar{a}_r(x,y) = \langle \rho_{bh}(s(x,y))A, l_0 \rangle = \text{tr } (\rho_{h}((0,x,y)^{-1})A) = h^n \int_{\mathbb{R}^{2n}} \langle \rho_{h}((0,x,y)^{-1})Af(x',y'), f(x',y') \rangle_{F_2(O_h)} \, dx' \, dy' = h^n \int_{\mathbb{R}^{2n}} \langle Af(x',y'), f(x,y)s(x',y') \rangle_{F_2(O_h)} \, dx' \, dy' .
\]  

(3.13)

(3.14)

The formula \textit{[3.13] \textit{is the principal ingredient of the inversion formula for the Heisenberg group } \textit{[12 Chap. 1, (1.60)]}, \textit{[35 Chap. 1, Th. 2.7]}, which reconstructs the kernel of convolution } B(g) \textit{out of operators } \rho_{h}(B). \textit{Therefore if we define the mother wavelet to be the identity operator } I \textit{the inverse wavelet transform } (cf. \textit{[25,21] \textit{will be}

\[
\mathcal{M}_r \bar{a}_r = h^n \int_{\mathbb{R}^{2n}} a(x,y)\rho_{bh}(s((0,x,y)^{-1}))I \, dx \, dy = h^n \int_{\mathbb{R}^{2n}} a(x,y)\rho_{h}(0,x,y) \, dx \, dy.
\]  

(3.15)

The inversion formula for \( \mathbb{H}^n \) insures that
Proposition 3.3. [26] The composition $\mathcal{M}_r \circ \mathcal{W}_r$ is the identity map on the representations $\rho_h(B)$ of convolution operators on $\mathcal{O}_h$.

Example 3.4. The wavelet transform $\mathcal{W}_r$ applied to the quantum coordinate $Q = dp_h(X)$, momentum $P = dp_h(Y)$ (see (2.12)), and the energy function of the harmonic oscillator $(m\omega^2Q^2 + \frac{1}{m}P^2)/2$ produces the distributions on $\mathbb{R}^{2n}$:

\[
Q \mapsto \frac{1}{2\pi} \delta^{(1)}(x)\delta(y),
\]
\[
P \mapsto \frac{1}{2\pi} \delta(x)\delta^{(1)}(y),
\]
\[
\frac{1}{2} \left( m\omega^2Q^2 + \frac{1}{m}P^2 \right) \mapsto -\frac{1}{8\pi^2} \left( m\omega^2\delta^{(2)}(x)\delta(y) + \frac{1}{m}\delta(x)\delta^{(2)}(y) \right),
\]

where $\delta^{(1)}$ and $\delta^{(2)}$ are the first and second derivatives of the Dirac delta function $\delta$ respectively. We will use them later in Example 3.7.

3.3. From Classical and Quantum Observables to $p$-Mechanics. It is commonly accepted that we can not deal with quantum mechanics directly and thus classical dynamics serve as an unavoidable intermediate step. The passage from classical observables to quantum ones—known as a quantisation—is a huge field with many concurring approaches (geometric, deformation, Weyl, Berezin, etc. quantisations) each having its own merits and demerits. Similarly one has to construct $p$-mechanical observables starting from classical or quantum ones by some procedure (should it be named “$p$-mechanisation”?), which we are about to describe now.

The transition from a $p$-mechanical observable to a classical one is given by the formula (3.16), which in turn is a realisation of the inverse wavelet transform (2.12):

\[
\rho_{(q,p)}B = \hat{B}(0,q,p) = c_h^{n+1} \int_{\mathbb{R}^n} B(s,x,y) e^{-2\pi i (qx+py)} \, ds \, dx \, dy. \tag{3.16}
\]

Just like in the case of quantisation the classical image $\rho_{(q,p)}(B)$ contains only partial information about a $p$-observable $B$ unless we make some additional assumptions. Let us start from a classical observable $c(q,p)$ and try to construct the corresponding $p$-observable. As follows from general considerations (see [25] and Section 2.3) we can partially invert formula (3.16) by the wavelet transform (2.12):

\[
\hat{c}(x,y) = \mathcal{W}_0c(x,y) = \langle cv_{(0,0)}, v_{(x,y)} \rangle = c_h^{n} \int_{\mathbb{R}^{2n}} c(q,p) e^{2\pi i (qx+py)} \, dq \, dp, \tag{3.17}
\]

where $v_{(x,y)} = \rho_{(q,p)}v_{(0,0)} = e^{-2\pi i (qx+py)}$.

However the function $\hat{c}(x,y)$ is not defined on the entire of $\mathbb{H}^n$. The natural domain of $\hat{c}(x,y)$ according to the construction of the reduced wavelet transform (2.25) is the homogeneous space $\Omega = G/Z$, where $G = \mathbb{H}^n$ and $Z$ is its normal subgroup of central elements $(s,0,0)$. Let $s : \Omega \to G$ be a Borel section in the principal bundle $G \to \Omega$, which is used in the construction of induced representations, see [21] §13.1. For the Heisenberg group $\mathbb{H}^n$ it can be simply defined as $s : (x,y) \in \Omega \mapsto (0,x,y) \in \mathbb{H}^n$. One can naturally transfer functions from $\Omega$ to the image $s(\Omega)$ of the map $s$ in $G$. However the range $s(\Omega)$ of $s$ has often (particularly for $\mathbb{H}^n$) a zero Haar measure in $G$. Probably two simplest possible ways out are:
(1) To increase the “weight” of function $\tilde{c}(s, x, y)$ vanishing outside of the range $s(\Omega)$ of $s$ by a suitable Dirac delta function on the subgroup $Z$. For the Heisenberg group this can be done, for example, by the map:

$$E : \tilde{c}(x, y) \mapsto \tilde{c}(s, x, y) = \delta(s)c(x, y),$$  \hspace{1cm} (3.18)

where $\tilde{c}(x, y)$ is given by the inverse wavelet (Fourier) transform \[5.17\]. As we will see in Proposition \[3.6\] this is related to the Weyl quantisation and the Moyal brackets.

(2) To extend the function $\tilde{c}(x, y)$ to the entire group $G$ by a tensor product with a suitable function on $Z$, for example $e^{-s^2}$:

$$\tilde{c}(x, y) \mapsto \tilde{c}(s, x, y) = e^{-s^2} \tilde{c}(x, y).$$

In order to get the \textit{correspondence principle} between classical and quantum mechanics (cf. Example \[2.2\]) the function on $Z$ has to satisfy some additional requirements. For $\mathbb{H}^n$ it should vanish for $s \to \pm \infty$, which is fulfilled for both $e^{-s^2}$ and $\delta(s)$ from the previous item. In this way we get infinitely many essentially different quantisations with non-equivalent \textit{deformed} Moyal brackets between observables.

There are other more complicated possibilities not mentioned here, which can be of some use if some additional information or assumptions are used to extend functions from $\Omega$ to $G$. We will focus here only on the first “minimalistic” approach from the two listed above.

\textbf{Example 3.5.} The composition of the wavelet transform $\mathcal{W}_0$ \[3.17\] and the map $E$ \[3.18\] applied to the classical coordinate, momentum, and the energy function of the harmonic oscillator produces the distributions on $\mathbb{H}^n$:

$$q \mapsto \frac{1}{2\pi i} \delta(s)\delta^{(1)}(x)\delta(y),$$  \hspace{1cm} (3.19)

$$p \mapsto \frac{1}{2\pi i} \delta(s)\delta(x)\delta^{(1)}(y),$$  \hspace{1cm} (3.20)

$$\frac{1}{2} \left( m\omega^2 q^2 + \frac{1}{m} p^2 \right) \mapsto -\frac{1}{8\pi i} \left( m\omega^2 \delta(s)\delta^{(2)}(x)\delta(y) + \frac{1}{m} \delta(s)\delta(x)\delta^{(2)}(y) \right).$$  \hspace{1cm} (3.21)

We will use the notation $X = \frac{1}{2\pi i} \delta(s)\delta^{(1)}(x)\delta(y)$ and $Y = \frac{1}{2\pi i} \delta(s)\delta(x)\delta^{(1)}(y)$, these distributions are used later in Section \[5\].

If we apply the representation $\rho_h$ \[3.4\] to the function $\tilde{c}(s, x, y)$ \[3.18\] we will get the operator on $F_2(\mathcal{O}_h)$:

$$\mathcal{Q}_h(c) = c^{n+1}_h \int_{\mathbb{R}^n} \tilde{c}(s, x, y)\rho_h(s, x, y) ds \, dx \, dy$$

$$= c^n_h \int_{\mathbb{R}^{2n}} \tilde{c}(x, y)e^{x \cdot dp_h(X) + y \cdot dp_h(Y)} \, dx \, dy,$$  \hspace{1cm} (3.22)

where the last expression is exactly the \textit{Weyl quantisation} (the \textit{Weyl correspondence} \[12\] § 2.1) if the Schrödinger realisation with $dp_h(X) = q$ and $dp_h(Y) = ih\partial_q$ on $L_2(\mathbb{R}^n)$ is chosen for $\rho_h$. Thus we demonstrate that

\textbf{Proposition 3.6.} \[2.6\] \textit{The Weyl quantisation} $\mathcal{Q}_h$ \[3.22\] \textit{is the composition of the wavelet transform} \[3.17\], \textit{the extension} $E$ \[3.18\], \textit{and the representation} $\rho_h$ \[2.7\] :

$$\mathcal{Q}_h = \rho_h \circ E \circ \mathcal{W}_0.$$  \hspace{1cm} (3.23)
Figure 2. The relations between:
- \(Q_h\) — the Weyl quantisation from classical mechanics to quantum;
- \(C_{h \to 0}\) — the classical limit \(h \to 0\) of quantum mechanics;
- \(\rho_h\) and \(\rho_{(q,p)}\) — unitary representations of Heisenberg group \(\mathbb{H}^n\);
- \(\mathcal{W}_r\) and \(\mathcal{W}_0\) — wavelet transforms defined in (3.13) and (3.17);
- \(\mathcal{E}\) — extension of functions from \(\Omega = \mathbb{H}^n / \mathbb{Z}\) to the whole group \(\mathbb{H}^n\).

Note the relations \(Q_h = \rho_h \circ \mathcal{E} \circ \mathcal{W}_0\) and \(C_{h \to 0} = \rho_{(q,p)} \circ \mathcal{E} \circ \mathcal{W}_r\).

A similar construction can be carried out if we have a quantum observable \(A\) and wish to recover the related \(p\)-mechanical object. The wavelet transform \(\mathcal{W}_r\) maps \(A\) into the function \(a(x, y)\) defined on \(\Omega\) and we again face the problem of extending \(a(x, y)\) to the entire group \(\mathbb{H}^n\). It will be solved as in the classical case by a tensor product with the delta function \(\delta(s)\). We get the following formula:

\[
A \mapsto a(s, x, y) = \mathcal{E} \circ \mathcal{W}_r(A) = h^n \delta(s) \int_{\mathbb{R}^{2n}} \langle Av(x', y'), v(x, y) - (x', y') \rangle F_2(O_h) \, dx' \, dy'.
\]

We can apply to this function \(a(s, x, y)\) the representation \(\rho_{(q,p)}\) and obtain a classical observable \(\rho_{(q,p)}(a)\). For a reasonable quantum observable \(\rho_{(q,p)}(A)\) its classical image \(\rho_{(q,p)} \circ \mathcal{E} \circ \mathcal{W}_r(A)\) will coincide with its classical limit \(C_{h \to 0} A\):

\[
C_{h \to 0} = \rho_{(q,p)} \circ \mathcal{E} \circ \mathcal{W}_r, \quad (3.24)
\]

which is expressed here through integral transformations and does not explicitly use the limit \(h \to 0\). Figure 2 illustrates various transformations between quantum, classical, and \(p\)-observables. Besides the mentioned decompositions (3.23) and (3.24) there are presentations of identity maps on classical and quantum spaces correspondingly:

\[
\mathcal{I}_c = \rho_{(q,p)} \circ \mathcal{E} \circ \mathcal{W}_0, \quad \mathcal{I}_h = \rho_h \circ \mathcal{E} \circ \mathcal{W}_h.
\]

Example 3.7. The wavelet transform \(\mathcal{W}_r\) applied to the quantum coordinate \(Q\), momentum \(P\), and the energy function of the harmonic oscillator \((m\omega^2 Q^2 + \frac{1}{m} P^2) / 2\) was calculated in Example 3.4. A composition with the above map \(\mathcal{E}\) yields the
distributed:

\[ Q \mapsto \frac{1}{2\pi i} \delta(s)\delta^{(1)}(x)\delta(y), \]

\[ P \mapsto \frac{1}{2\pi i} \delta(s)\delta^{(1)}(x)\delta(y), \]

\[ \frac{1}{2} \left( m\omega^2 Q^2 + \frac{1}{m} P^2 \right) \mapsto -\frac{1}{2\pi i} \left( m\omega^2 \delta(s)\delta^{(2)}(x)\delta(y) + \frac{1}{m} \delta(s)\delta(x)\delta^{(2)}(y) \right), \]

which are exactly the same as in the Example 3.5.

3.4. \( p \)-Mechanical States. In this subsection we introduce states to \( p \)-mechanics — these are positive linear functionals on the \( C^* \)-algebra \( \mathbb{H}^n \) of \( p \)-mechanical observables (cf. subsection 3.1). According to the GNS construction for a general \( C^* \)-algebra \( \mathbb{H}^n \) § 1.6]

- an arbitrary state could be decomposed as a linear combination of the pure states; and
- the pure states correspond to irreducible representations.

Since irreducible representations of \( L_1(\mathbb{H}^n) \) are given by Theorem 2.11 and are associated in \( p \)-mechanics with quantum and classical pictures then the pure states in \( p \)-mechanics also corresponds to quantum and classical states. We give here several equivalent descriptions of these states.

For each \( h \neq 0 \) (the quantum case) we give two equivalent forms of states: the first form we give is as elements of a Hilbert space, the second is as integration with an appropriate kernel since the second one is not essentially different from the former.

Definition 3.8. [9] The Hilbert space \( \mathcal{H}_h, \ h \in \mathbb{R} \setminus \{0\} \), is the subset of functions on \( \mathbb{H}^n \) defined by

\[ \mathcal{H}_h = \left\{ e^{2\pi i h s} f(x, y) : E_j^h f = 0 \quad 1 \leq j \leq n \right\} \] (3.25)

where the operator \( E_j^h = \frac{h}{2}(y_j + ic_jx_j)I + 2\pi(c_i \frac{\partial}{\partial y_j} + i \frac{\partial}{\partial x_j}) \) (this is the Fourier transform of \( D_j^h \)). The inner product on \( \mathcal{H}_h \) is defined as

\[ \langle v_1, v_2 \rangle_{\mathcal{H}_h} = \left( \frac{4}{h} \right)^n \int_{\mathbb{R}^{2n}} v_1(s, x, y)\overline{v_2}(s, x, y) \, dx \, dy. \] (3.26)

Note in equation 3.26 there is no integration over the \( s \) variable since for any two functions \( v_1 = e^{2\pi i h s} f_1(x, y) \) and \( v_2 = e^{2\pi i h s} f_2(x, y) \) in \( \mathcal{H}_h \)

\[ \langle v_1, v_2 \rangle = \int_{\mathbb{R}^{2n}} e^{2\pi i h s} e^{-2\pi i h s} f_1(x, y)\overline{f_2}(x, y) \, dx \, dy = \int_{\mathbb{R}^{2n}} f_1(x, y)\overline{f_2}(x, y) \, dx \, dy \]

and hence there is no \( s \)-dependence. It is important to note that all the \( \mathcal{H}_h \) are shift-invariant and thus invariant under convolutions. Since the Fourier transform intertwines multiplication and differentiation we have

\[ \mathcal{H}_h = \left\{ e^{2\pi i h s} F_{x, y}(f(q, p)) : f \in F^2(\mathcal{O}_h) \right\}. \] (3.27)

\( \mathcal{H}_h \) is mapped into another Hilbert space \( \mathcal{I}_h \) by the Fourier transform. This Hilbert space \( \mathcal{I}_h \) is

\[ \mathcal{I}_h = \left\{ \delta(h'-h)f(q, p) : f \in F^2(\mathcal{O}_h) \right\}, \]
where \( \delta \) is the Dirac delta distribution. The inner product for \( j_1(h', q, p) = \delta(h' - h)f_1(q, p) \) and \( j_2(h', q, p) = \delta(h' - h)f_2(q, p) \) in \( I_h \) is

\[
\langle j_1, j_2 \rangle_{I_h} = \left( \frac{4}{\hbar} \right)^n \int_{\mathbb{R}^{2n+1}} j_1(h', q, p) j_2(h', q, p) \, dh' \, dq \, dp = \langle f_1, f_2 \rangle_{F_2(O_h)}.
\]

We define a set of states for each \( h \neq 0 \) using \( \mathcal{H}_h \) (later in this subsection we will define a set of states for \( h = 0 \) which are defined using a kernel and a set of states for \( h = 0 \) by a kernel).

**Definition 3.9.** A \( h \)-state corresponding to a vector \( v \in \mathcal{H}_h \) is defined on a \( p \)-mechanical observable \( B \) by

\[
\langle B * v, v \rangle_{\mathcal{H}_h}.
\]

For any vector \( f \in F_2(O_h) \) equation (2.42) gives us a corresponding state. We now introduce a map \( S_h \) which maps vectors in \( F_2(O_h) \) to vectors in \( \mathcal{H}_h \)

\[
S_h(f(q, p)) = e^{2\pi i h f}(x, y).
\] (3.28)

The following Theorem proves that the states corresponding to vectors \( f \) and \( S_h f \) give the same expectation values for observables \( B \) and \( \rho_h(B) \) respectively.

**Theorem 3.10.** For any observable \( B \) and any \( v_1, v_2 \in \mathcal{H}_h, h \in \mathbb{R} \setminus \{0\} \), of the form \( v_1(s, x, y) = S_h f_1, v_2(s, x, y) = S_h f_2 \) we have the relationship

\[
\langle B * v_1, v_2 \rangle_{\mathcal{H}_h} = \langle \rho_h(B) f_1, f_2 \rangle_{F_2(O_h)}.
\] (3.29)

**Proof.** From the Plancherel identity for \( \mathbb{R}^{2n} \) we have

\[
\langle B * v_1, v_2 \rangle_{\mathcal{H}_h} = \langle \hat{B} * \hat{v}_1, \hat{v}_2 \rangle_{I_h}
\] (3.30)

where again \( \hat{\cdot} \) is the Fourier transform on the Heisenberg group as described in equation (2.10). Using (2.11) equation (3.30) can be written as

\[
\langle B * v_1, v_2 \rangle_{\mathcal{H}_h} = \langle \hat{B} * \hat{v}_1, \hat{v}_2 \rangle_{I_h}
\] (3.31)

Using (2.11) equation (3.31) becomes

\[
\langle B * v_1, v_2 \rangle_{\mathcal{H}_h} = \langle \hat{B} * \hat{v}_1, \hat{v}_2 \rangle_{I_h}
\] (3.32)

\[
= \left( \frac{4}{\hbar} \right)^n \int \rho_h(B) \delta(h' - h) f_1(q, p) \delta(h' - h) f_2(q, p) \, dq \, dp \, dh'
\]

\[
= \left( \frac{4}{\hbar} \right)^n \int \rho_h(B) f_1(q, p) f_2(q, p) \, dq \, dp.
\]

Hence the result has been proved. \( \square \)

Taking \( v_1 = v_2 \) in (3.29) shows that the states corresponding to \( f \) and \( S_h f \) give the same expectation values for \( \rho_h(B) \) and \( B \) respectively. If we take \( B \) to be a time development operator we can get probability amplitudes between states \( v_1 \neq v_2 \). The map \( S_h \) can be realised as a map from the set of functionals on the quantum observables to the set of functionals on the set of \( p \)-mechanical observables. This map is the adjoint of \( \rho_h \) when realised as a map from \( p \)-observables to quantum observables.

We now go on to show that each of these states can also be realised by an appropriate kernel.
Theorem 3.11. \[9\] If \( l(s, x, y) \) is defined to be the kernel
\[
l(s, x, y) = \left( \frac{4}{h} \right)^n \int_{\mathbb{R}^{2n}} v((s, x, y)^{-1}(s', x', y')) v((s', x', y')) \, dx \, dy'.
\] (3.32)
then
\[
\langle B* v, v \rangle_{\mathcal{H}_h} = \int_{\mathbb{H}^n} B(s, x, y) l(s, x, y) \, ds \, dx \, dy.
\]

Proof. It is easily seen that
\[
\langle B* v, v \rangle = \left( \frac{4}{h} \right)^n \int_{\mathbb{H}^n} \int_{\mathbb{R}^{2n}} B((s, x, y)) v((s, x, y)^{-1}(s', x', y'))
\times v((s', x', y')) \, ds \, dx \, dy \, dx' \, dy'
\] (3.33)
\[
= \left( \frac{4}{h} \right)^n \int_{\mathbb{H}^n} B((s, x, y)) \times \left( \int_{\mathbb{R}^{2n}} v((s, x, y)^{-1}(s', x', y')) v((s', x', y')) \, dx' \, dy' \right) \, ds \, dx \, dy
\]
Note that there is no integration over \( s' \) by the definition of the \( \mathcal{H}_h \) inner product. \( \square \)

The quantum states defined through their kernels instead of vectors of a Hilbert space are particularly suitable for contextual probability interpretation \[18, 19, 28\] of quantum mechanics, see the discussion in Section \[6\]. Thus we collect them together under the following definition.

Definition 3.12. \[9\] We denote the set of kernels corresponding to the elements in \( \mathcal{H}_h \) as \( \mathcal{L}_h \).

Now we introduce \((q, p)\)-states in \( p \)-mechanics, which correspond to classical states, they are again functionals on the \( C^* \)-algebra of \( p \)-mechanical observables. Pure states in classical mechanics evaluate observables at particular points of phase space, they can be realised as kernels \( \delta(q - q', p - p') \) for fixed \( q, p \) in phase space, that is
\[
\int_{\mathbb{R}^{2n}} F(q, p) \delta(q - q', p - p') \, dq' \, dp' = F(q, p). \tag{3.34}
\]
We now give the \( p \)-mechanical equivalent of pure classical states.

Definition 3.13. \[9\] A \((q, p)\)-pure state is defined to be the set of functionals, \( k_{(0,q,p)} \), for fixed \( (q,p) \in \mathbb{R}^{2n} \) which act on observables by
\[
k_{(0,q,p)}(B(s, x, y)) = \int_{\mathbb{H}^n} B(s, x, y) e^{-2\pi i (qx + py)} \, dx \, dy. \tag{3.35}
\]
Each classical pure state \( k_{(0,q,p)} \) is defined entirely by its kernel \( l_{(0,q,p)} \)
\[
l_{(0,q,p)} = e^{-2\pi i (qx + py)}. \tag{3.36}
\]
By equation \[3.16\] we have
\[
\int_{\mathbb{H}^n} B(s, x, y) e^{-2\pi i (qx + py)} \, ds \, dx \, dy = F(q, p) \tag{3.37}
\]
where \( F \) is the classical observable corresponding to \( B \) (using the relation exhibited in subsection \[5.5\]), hence when we apply state \( k_{(0,q,p)} \) to a \( p \)-mechanical observable.
we get the value of its classical counterpart at the point \((q, p)\) of phase space. We introduce the map \(S_0\) which maps classical pure state kernels to \(p\)-mechanical classical pure state kernels

\[
S_0(\xi(q, p)) = \hat{\xi}(x, y).
\]

This equation is almost identical to the relation in equation (3.28). The kernels \(l_{0}(0, q, p) = e^{-2\pi i(q'q + p'p)}\), are the Fourier transforms of the delta functions \(\delta(q - q', p - p')\), hence pure \((q, p)\) states are just the image of pure classical states.

Mixed states, as used in statistical mechanics [15], are linear combinations of pure states. In \(p\)-mechanics \((q, p)\) mixed states are defined in the same way.

**Definition 3.14.** [9] Define \(L_0\), to be the space of all linear combinations of \((q, p)\) pure state kernels \(l_{0}(0, q, p)\), that is the set of all kernels corresponding to \((q, p)\) mixed states.

The map \(S_0\) exhibits the same relations on mixed states as pure states due to the linearity of the Fourier transform. Note that if we consider the map \(S_0\) as mapping from functional to functional, that is going from the dual space of classical observables on \(O_0\) to the dual space of the set of \(p\)-mechanical observables then it is the adjoint of \(\rho_{(q,p)}\).

**Remark 3.15.** The above description of classical states corresponds to quantum states defined through their kernels (3.32). It is possible to define classical states through vector in the Hilbert space \(L_2(O_0)\) as well. Indeed a classical observable \(B(q, p)\) acts on \(L_2(O_0)\) by multiplication. Then a vector \(v(q, p) \in L_2(O_0)\) defines the state by the natural formula \(B(q, p) \mapsto \langle B(q, p)v(q, p), v(q, p) \rangle\). This permits quantum superpositions of states while the dynamics of observables is governed by the classical Hamilton equation (4.11).

In accordance with the general theory of \(C^*\)-algebras mentioned in the beginning of this subsection we could now describe a general \(p\)-mechanical state:

**Proposition 3.16.** An arbitrary \(p\)-mechanical state is a superposition of quantum \(h\)-states given by Definition 3.9 and classical \((q, p)\)-states described in Definition 3.13.

Consequently a comprehensive study of \(p\)-mechanical states, notably their dynamics, could be done through this decomposition. The various relations between \(p, h,\) and \((q,p)\)-states could be derived from Figure 2. Indeed since all types of states form the dual spaces to the corresponding spaces of observables, the reverse of arrows on Figure 2 provides the maps between states through the adjoint operators to \(\rho_h, \rho_0, W_r, W_0, E, Q_h, C_h, 0\). The adjoint operator to the wavelet transform \(W_r\) was identified with the inverse wavelet transform in [25].

We conclude this section by the following result describing relations between eigenvectors in \(F_2\) and their images under \(S_h\).

**Theorem 3.17.** [9] For a \(p\)-observable \(B \in L_1(\mathbb{R}^n)\) and \(f_1 \in F_2(O_h)\), \(\rho_h(B)f_1 = \lambda f_1\), if and only if for \(v_1(s, x, y) = S_h f_1 = e^{2\pi ihs} f_1(x, y) \in \mathcal{H}_h\)

\[
\langle B \ast v_1, v_2 \rangle = \lambda \langle v_1, v_2 \rangle
\]

holds for all \(v_2 \in \mathcal{H}_h\).
4. \( p \)-Mechanics: Dynamics

We introduce the \( p \)-mechanical brackets which fulfil all essential physical requirements and have a non-trivial classical representation coinciding with the Poisson brackets. A consistent \( p \)-mechanical dynamic equation for observables is given in subsection 4.1. In subsection 4.2 we give two equivalent dynamic equations for \( p \)-mechanical states. Symplectic automorphisms of the Heisenberg groups produce symplectic symmetries of \( p \)-mechanical, quantum, and classical dynamics in subsection 4.4.

4.1. \( p \)-Mechanical Brackets and Dynamic Equation on \( \mathbb{H}^n \).

Having observables as convolutions on \( \mathbb{H}^n \) we need a dynamic equation for their evolution. To this end we seek a time derivative generated by the observable associated with energy. The first candidate is the derivation coming from commutator (3.3). However the straight commutator has at least two failures. The first failure is that it can’t produce any dynamics on \( \mathcal{O}_0 \) (2.18), see Remark 3.2. The second failure is due to a mismatch in units: the \( p \)-mechanical energy, \( B_H \), is measured in units \( ML^2/T^2 \) whereas the time derivative should be measured in \( 1/T \), that is the mismatch is in units of action \( ML^2/T \).

Fortunately, there is a possibility to fix both the above defects of the straight commutator at the same time. Let us define a multiple \( A \) of a right inverse operator to the vector field \( S \) (2.3) on \( \mathbb{H}^n \) by its actions on exponents—characters of the centre \( Z \in \mathbb{H}^n \):

\[
SA = 4\pi^2I, \quad \text{where} \quad A e^{2\pi i hs} = \begin{cases} 2\pi e^{2\pi i hs} & \text{if } h \neq 0, \\ 4\pi^2 s, & \text{if } h = 0. \end{cases} \tag{4.1}
\]

An alternative definition of \( A \) as a convolution with a distribution is given in [27].

We can extend \( A \) by linearity to the entire space \( L_1(\mathbb{H}^n) \). As a multiple of a right inverse to \( S \) the operator \( A \) is measured in \( T/(ML^2) \)—exactly that we need to correct the mismatch of units in the straight commutator. Thus we introduce [27] a modified convolution operation \( \ast \) on \( L_1(\mathbb{H}^n) \):

\[
B' \ast B = (B' \ast B)A \tag{4.2}
\]

and the associated modified commutator (\( p \)-mechanical brackets):

\[
\{[B', B]\} = [B', B]A = B' \ast B - B \ast B'. \tag{4.3}
\]

Obviously (4.3) is a bilinear antisymmetric form on the convolution kernels. It was also demonstrated in [27] that the \( p \)-mechanical brackets satisfy the Leibniz and Jacoby identities. They are all important for consistent dynamics [10] along with the dimensionality condition given in the beginning of this subsection.

From (3.4) one gets \( \rho_h(AB) = \frac{1}{2\pi} \rho_h(B) \) for \( h \neq 0 \). Consequently the modification of the commutator for \( h \neq 0 \) is only slightly different from the original one:

\[
\rho_h \{[B', B]\} = \frac{1}{ih} \rho_h(B'), \quad \text{where} \quad h = \frac{h}{2\pi} \neq 0. \tag{4.4}
\]
The integral representation of the modified commutator kernel becomes (cf. (2.18)):

\[
\langle [B', B] \rangle = e^{i\hbar} \int_{\mathbb{R}^{2n}} \frac{4\pi}{\hbar} \sin(\pi h(x'y' - y'x')) \hat{B}'(h, x', y') \hat{B}(h, x - x', y - y') \, dx' dy',
\]

where we may understand the expression under the integral as

\[
\frac{4\pi}{\hbar} \sin(\pi h(x'y' - y'x')) = 4\pi^2 \sum_{k=1}^{\infty} (-1)^{k+1} (\pi h)^{2(k-1)} \frac{(xy' - y'x')^{2k-1}}{2k-1}!
\]

This makes the operation (4.5) for \( h = 0 \) significantly distinct from the vanishing integral (3.7). Indeed it is natural to assign the value 4\pi^2(xy' - y'x') to (4.6) for \( h = 0 \). Then the integral in (4.5) becomes the Poisson brackets for the Fourier transforms of \( B' \) and \( B \) defined on \( \mathcal{O}_0 \) (2.18):

\[
\rho_{(q,p)} \langle [B', B] \rangle = \frac{\partial \hat{B}'(0, q, p)}{\partial q} \frac{\partial \hat{B}(0, q, p)}{\partial p} - \frac{\partial \hat{B}'(0, q, p)}{\partial p} \frac{\partial \hat{B}(0, q, p)}{\partial q}.
\]

The same formula is obtained [27, Prop. 3.5] if we directly calculate \( \rho_{(q,p)} \langle [B', B] \rangle \) rather than resolve the indeterminacy for \( h = 0 \). This means there is continuity in our construction at \( h = 0 \) which represents the correspondence principle between quantum and classical mechanics.

We have now arrived at the conclusion the Poisson brackets and the inverse of the Planck constant \( 1/\hbar \) have the same dimensionality because they are the image of the same object (anti-derivative (1.11)) under different representations (2.0) and (2.16) of the Heisenberg group.

Note that functions \( X = \delta(s)\delta^{(1)}(x)\delta(y) \) and \( Y = \delta(s)\delta(x)\delta^{(1)}(y) \) (see (3.19) and (3.20)) on \( \mathbb{H}^n \) are measured in units \( L \) and \( ML/T \) (inverse to \( x \) and \( y \)) correspondingly as respective derivatives of the dimensionless function \( \delta(s)\delta(x)\delta(y) \). Then the \( p \)-mechanical brackets \( \langle [X, \cdot] \rangle \) and \( \langle [Y, \cdot] \rangle \) with those functions have dimensionality of \( T/(ML) \) and \( 1/L \) correspondingly. Their representation \( \rho_* \langle [X, \cdot] \rangle \) and \( \rho_* \langle [Y, \cdot] \rangle \) (for both type of representations \( \rho_h \) and \( \rho_{(q,p)} \)) are measured by \( L \) and \( ML^2/T \) and are just derivatives:

\[
\rho_* \langle [X, \cdot] \rangle = \frac{\partial}{\partial p}, \quad \rho_* \langle [Y, \cdot] \rangle = \frac{\partial}{\partial q}.
\]

Thus \( \rho_* \langle [X, \cdot] \rangle \) and \( \rho_* \langle [Y, \cdot] \rangle \) are generators of shifts on both types of orbits \( \mathcal{O}_h \) and \( \mathcal{O}_0 \) independent from \( h \).

Since the modified commutator (1.3) with a \( p \)-mechanical energy has the dimensionality \( 1/T \)—the same as the time derivative—we introduce the dynamic equation for an observable \( B(s, x, y) \) on \( \mathbb{H}^n \) based on that modified commutator as follows

\[
\frac{dB}{dt} = \langle [B, B_H] \rangle.
\]

Remark 4.1. It is a general tendency to make a Poisson bracket or quantum commutator out of any two observables and say that they form a Lie algebra. However there is a physical meaning to do that if at least one of the two observables is an energy, coordinate or momentum: in these cases the bracket produces the time derivative (4.9) or corresponding shift generators (1.3) of the other observable.

A simple consequence of the previous consideration is that the \( p \)-dynamic equation (1.10) is reduced.
(1) by the representation $\rho_h$, $h \neq 0$ \((2.9)\) on $F'_2(O_h)$ \((2.1)\) to Moyal’s form of Heisenberg equation \((4.1)\) \((8)\) based on the formulae \((4.4)\) and \((4.5)\):

$$\frac{d\rho_h(B)}{dt} = \frac{1}{ih}[\rho_h(B), H_h], \quad \text{where the operator } H_h = \rho_h(B_H); \quad (4.10)$$

(2) by the representations $\rho_{(q,p)}$ \((2.16)\) on $O_0$ \((2.18)\) to Poisson’s equation \([2, \S\, 39]\) based on the formula \((4.7)\):

$$\frac{d\hat{B}}{dt} = \{\hat{B}, H\} \quad \text{where the function } H(q,p) = \rho_{(q,p)}(B_H) = \hat{B}_H(0,q,p). \quad (4.11)$$

The same connections are true for the solutions of the three equations \((4.9) - (4.11)\), this equation is demonstrated in section 5.

4.2. $p$-Mechanical Dynamics for States. We now go on to show how $p$-mechanical states evolve with time. We first show how the elements of $L_h$, for all $h \in \mathbb{R}$ evolve with time and that this time evolution agrees with the time evolution of $p$-observables. In doing this we show that for the particular case of $L_0$ the time evolution is the same as classical states under the Liouville equation. Then we show how the elements of $H_h$ evolve with time and prove that they agree with the Schrödinger picture of motion in quantum mechanics. Before we can do any of this we need to give the definition of a Hermitian convolution.

**Definition 4.2.** \([9]\) We call a $p$-mechanical observable $B$ Hermitian if it corresponds to a Hermitian convolution, that is for any functions $f_1, f_2$ on the Heisenberg group

$$\int_{\mathbb{H}^n} (B \ast f_1)(g)f_2(g)dg = \int_{\mathbb{H}^n} f_1(g)(B \ast f_2)(g)dg.$$ 

If a $p$-observable $B$ is Hermitian then $B(g) = \overline{B(g^{-1})}$, this is the result of a trivial calculation. From now on we denote $\overline{B(g^{-1})}$ as $B^*$. For our purposes we just need to assume that the distribution or function, $B$, corresponding to the observable is real and $B(s,x,y) = B(-s,-x,-y)$.

**Definition 4.3.** \([9]\) If we have a system with energy $B_H$ then an arbitrary kernel $l \in L_h$, $h \in \mathbb{R}$, evolves under the equation

$$\frac{dl}{dt} = \{l, B_H\}. \quad (4.12)$$

We now show that the time evolution of these kernels coincides with the time evolution of $p$-mechanical observables.

**Theorem 4.4.** \([9]\) If $l$ is a kernel evolving under equation \((4.12)\) then for any observable $B$

$$\frac{d}{dt} \int_{\mathbb{H}^n} B l \, dg = \int_{\mathbb{H}^n} \{B, B_H\} \, l \, dg.$$
Proof. This result can be verified by the direct calculation,

\[
\frac{d}{dt} \int_{\mathbb{H}^n} B(s, x, y) l(s, x, y) \, ds \, dx \, dy \\
= \int_{\mathbb{H}^n} B(s, x, y) A(B_H \ast l - l \ast B_H)(s, x, y) \, ds \, dx \, dy \\
= -\int_{\mathbb{H}^n} AB(s, x, y)(B_H \ast l - l \ast B_H)(s, x, y) \, ds \, dx \, dy \\
= \int_{\mathbb{H}^n} A((B \ast B_H)(s, x, y)l(s, x, y) \\
- (B_H \ast B)(s, x, y)l(s, x, y)) \, ds \, dx \, dy \\
= \int_{\mathbb{H}^n} \{[B, B_H]\}(s, x, y)l(s, x, y) \, ds \, dx \, dy.
\]

At (4.13) we have used integration by parts while (4.14) follows since $B_H$ is Hermitian. □

If we take the representation $\rho_{(q,p)}$ of equation (4.12) we get the Liouville equation [15, Eq. 5.42] for a kernel $S^{-1}(l)$ moving in a system with energy $\rho_{(q,p)}(B_H)$. This only holds for elements in $L_0$ and can be verified by a similar calculation to [27, Propn. 3.5].

We now show how the vectors in $\mathcal{H}_h$ evolve with time. First we extend our definition of $A$ which was initially introduced in equation (4.1). $A$ can also be defined as an operator on each $\mathcal{H}_h$, $h \in \mathbb{R} \setminus \{0\}$, $A : \mathcal{H}_h \mapsto \mathcal{H}_h$ by

\[
A v = \frac{2\pi}{i\hbar} v.
\]

As the derivative operator the antiderivative $A$ is skew-symmetric, i.e. $A^* = -A$, on each $\mathcal{H}_h$, $h \in \mathbb{R} \setminus \{0\}$.

**Definition 4.5.** [9] If we have a system with energy $B_H$ then an arbitrary vector $v \in \mathcal{H}_h$ evolves under the equation

\[
\frac{dv}{dt} = AB_H \ast v = B_H \ast A v \\
(4.15)
\]

The operation of left convolution preserves each $\mathcal{H}_h$ so this time evolution is well defined. Equation (4.15) implies that if we have $B_H$ time-independent then for any $v \in \mathcal{H}_h$

\[
v(t; s, x, y) = e^{tAB_H} v(0; s, x, y)
\]

where $e^{AB_H}$ is the exponential of the operator of applying the left convolution of $B_H$ and then applying $A$. There is no mismatch in units here since $A$ has units $T/ML^2$ and $B_H$ has units $ML^2/T^2$, hence their product has units $1/T$.

**Theorem 4.6.** [9] If we have a system with energy $B_H$ (assumed to be Hermitian) then for any state $v \in \mathcal{H}_h$ and any observable $B$

\[
\frac{d}{dt} \langle B \ast v, v \rangle = \langle \{[B, B_H]\} \ast v, v \rangle.
\]
Proof. The result follows from the direct calculation:
\[
\frac{d}{dt} \langle B \ast v(t), v(t) \rangle = \langle B \ast \frac{d}{dt} v(t), v(t) \rangle + \langle B \ast v(t), \frac{d}{dt} v(t) \rangle
\]
\[
= \langle B \ast AB_H \ast v(t), v(t) \rangle + \langle B \ast v(t), AB_H \ast v(t) \rangle
\]
\[
= \langle B \ast AB_H \ast v(t), v(t) \rangle - \langle AB \ast v(t), B_H \ast v(t) \rangle
\]
\[
= \langle B \ast AB_H \ast v(t), v(t) \rangle - \langle ABH \ast B \ast v(t), v(t) \rangle
\]
\[
= \langle \{ [B, B_H^*] \} \ast v(t), v(t) \rangle.
\]
Equation (4.16) follows since \( A \) is skew-adjoint. At (4.17) we have used the fact that \( B_H \) is Hermitian.

This Theorem tells us that the time evolution of states in \( H_h \) coincides with the time evolution of observables as described in equation (4.9). We now give a Corollary to show that the time evolution of \( p \)-mechanical states in \( H_h, h \in \mathbb{R} \setminus \{0\} \) is the same as the time evolution of quantum states.

**Corollary 4.7.** If we have a system with energy \( B_H \) (assumed to be Hermitian) and an arbitrary state \( v = S_h f = e^{2 \pi i h s} f(x, y) \) (assuming \( h \neq 0 \)) then for any observable \( B(t; s, x, y) \)
\[
\frac{d}{dt} \langle B \ast v(t), v(t) \rangle_{H_h} = \frac{d}{dt} \langle \rho_h(B(t)) f(t), f(t) \rangle_{F_2(O_h)}.
\]
Where \( \frac{d}{dt} = \frac{d}{dt} \rho_h(B_H) f \) (this is just the usual Schrödinger equation).

**Proof.** From Theorem 4.6, we have
\[
\frac{d}{dt} \langle B \ast v, v \rangle = \langle \{ [B, B_H^*] \} \ast v, v \rangle
\]
\[
= \langle A(B \ast B_H - B_H \ast B) \ast v, v \rangle
\]
\[
= \langle (B \ast B_H - B_H \ast B) \ast B, v \rangle
\]
\[
= \frac{2 \pi}{\hbar} \langle (B \ast B_H - B_H \ast B) \ast v, v \rangle
\]
\[
= \frac{1}{\hbar} \langle (B \ast B_H - B_H \ast B) \ast v, v \rangle - \langle B \ast v, B_H \ast v \rangle
\]
The last step follows since \( B_H \) is Hermitian. Using equation (4.24), the above equation becomes,
\[
\frac{d}{dt} \langle B \ast v, v \rangle = \frac{1}{\hbar} \langle (\rho_h(B) \rho_h(B_H) f, f)_{F_2(O_h)} - \langle \rho_h(B) f, \rho_h(B_H) f \rangle_{F_2(O_h)} \rangle
\]
\[
= \frac{d}{dt} \langle \rho_h(B) f, f \rangle_{F_2(O_h)}
\]
which completes the proof. \( \square \)

Hence the time development in \( H_h \) for \( h \neq 0 \) gives the same time development as in \( F_2(O_h) \). If \( l(s, x, y) = (\frac{1}{\hbar})^n \int_{\mathbb{R}^n} v((s', x', y')) v((s', x', y')^{-1}(s, x, y)) dx' dy' \) then by Theorems 4.5 and 4.6 we have that
\[
\frac{d}{dt} \langle B \ast v, v \rangle_{H_h} = \frac{d}{dt} \int_{\mathbb{R}^n} B l \ dg. \tag{4.18}
\]

We conclude this subsection touching the question on mixing between quantum and classical states. A simple application of the representation theory yields the following “no-go” results equivalent to the main conclusion of the paper [36]:
**Theorem 4.8.** If the Hamiltonian of a \( p \)-mechanical system is given by a convolution operator then there is no mixing between quantum and classical states during the induced evolution.

Obviously, this result essentially relies on the assumption that the Hamiltonian is a convolution operator. Examples of mixing for quantum and classic states for more general Hamiltonians will be discussed somewhere else.

4.3. The \( p \)-Mechanical Interaction Picture. In the Schrödinger picture, time evolution is governed by the states and their equations \( \frac{d\psi}{dt} = ABH \ast \psi \quad \frac{d\psi}{dt} = \{BH, l\} \). In the Heisenberg picture, time evolution is governed by the observables and the equation \( \frac{dB}{dt} = \{B, BH\} \). In the interaction picture we divide the time dependence between the states and the observables. This is suitable for systems with a Hamiltonian of the form \( BH = BH_0 + BH_1 \) where \( BH_0 \) is time independent. The interaction picture has many uses in perturbation theory [30].

Let a \( p \)-mechanical system have the Hamiltonian \( BH = BH_0 + BH_1 \) where \( BH_0 \) is time independent. We first describe the interaction picture for elements of \( \mathcal{H}_h \). Define \( \exp(tABH_0) \) as the operator on \( \mathcal{H}_h \), which is the exponential of the operator of convolution by \( tABH_0 \). Now if \( B \) is an observable let

\[
\tilde{B} = \exp(tABH_0)B \exp(-tABH_0) \tag{4.19}
\]

If \( v \in \mathcal{H}_h \), define \( \tilde{v} = \exp(-tABH_0)v \), then we get

\[
\frac{d}{dt}\tilde{v} = \frac{d}{dt}(\exp(-tABH_0)v) \tag{4.20}
\]

\[
= -ABH_0 \ast \tilde{v} + \exp(-tABH_0)(A(BH_0 + BH_1) \ast v)
= ABH_0 \ast \tilde{v} + ABH_0 \ast \exp(-tABH_0)v + \exp(-tABH_0)ABH_1 \ast v
= (\exp(-tABH_0)ABH_1, \exp(tABH_0))(\tilde{v})
\]

Now we describe the interaction picture for a state defined by a kernel \( l \). Define

\[
\tilde{l} = e^{-t([BH_0, l])}l = \exp(-tABH_0)l \exp(+tABH_0)
\]

then

\[
\frac{d\tilde{l}}{dt} = ABH_0 \ast \tilde{l} + \exp(-tABH_0)\{BH_0 + BH_1, l\} \exp(tABH_0) - \tilde{l} \ast ABH_0
= \exp(-tABH_0)(l, \{BH_1, l\} \exp(tABH_0))
= \exp(-tABH_0)(A(BH_1) \ast \exp(tABH_0)\tilde{l} \exp(-tABH_0)
- \exp(tABH_0)l \exp(-tABH_0) \ast BH_1) \exp(tABH_0)
\]

\[
\frac{d\tilde{l}}{dt} = \{\exp(-tABH_0)BH_1, \exp(tABH_0), l\} \tag{4.21}
\]

This shows us how interaction states evolve with time, while the observables evolve by \( [l, BH_0] \). Note that if we take \( BH_0 = BH \) we have the Heisenberg picture, while if we take \( BH_0 = BH \) we have the Schrödinger picture. The interaction picture is very useful in studying the forced harmonic oscillator as will be shown in subsection 5.4.
Heisenberg group

\[ \rho_h \]

\[ \rho_{(q,p)} \]

Parameter \( h \neq 0 \)

Phase space \((h = 0)\)

Figure 3. Automorphisms of \( \mathbb{H}^n \) generated by the symplectic group \( \text{Sp}(n) \) do not mix representations \( \rho_h \) with different Planck constants \( h \) and act by the metaplectic representation inside each of them. In the contrast those automorphisms of \( \mathbb{H}^n \) act transitively on the set of one-dimensional representations \( \rho_{(q,p)} \) joining them into the tangent space of the classical phase space \( \mathbb{R}^{2n} \).

4.4. Symplectic Invariance from Automorphisms of \( \mathbb{H}^n \). Let \( A : \mathbb{R}^{2n} \to \mathbb{R}^{2n} \) be a linear symplectomorphism \([2, \S 4.1], [12, \S 4.1]\), i.e. a map defined by the \( 2n \times 2n \) matrix:

\[
A : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax + by \\ cx + dy \end{pmatrix}
\]

preserving the symplectic form \((4.22)\):

\[
\omega (A(x, y); A(x', y')) = \omega(x, y; x', y').
\] (4.22)

All such transformations form the symplectic group \( \text{Sp}(n) \). It follows from the identities \((4.22)\) and \((2.1)\) that the linear transformation \( \alpha : \mathbb{H}^n \to \mathbb{H}^n \) such that

\[
\alpha(s, x, y) = (s, A(x, y))
\]

is an automorphism of \( \mathbb{H}^n \). Let us also denote by \( \tilde{\alpha} = \tilde{\alpha}_A \) a unitary transformation of \( L_2(\mathbb{H}^n) \) in the form

\[
\tilde{\alpha}(f)(s, x, y) = \sqrt{\det A} f(s, A(x, y)),
\]

which is well defined \([12, \S 4.2]\) on the double cover \( \tilde{\text{Sp}}(n) \) of the group \( \text{Sp}(n) \). The correspondence \( A \mapsto \tilde{\alpha}_A \) is a linear unitary representation of the symplectic group in \( L_2(\mathbb{H}^n) \). One can also check the intertwining property

\[
\lambda_{l(r)}(g) \circ \tilde{\alpha} = \tilde{\alpha} \circ \lambda_{l(r)}(\alpha(g))
\] (4.23)

for the left (right) regular representations \([2, \S 4.5]\) of \( \mathbb{H}^n \).

Because \( \alpha \) is an automorphism of \( \mathbb{H}^n \) the map \( \alpha^* : B(g) \mapsto B(\alpha(g)) \) is an automorphism of the convolution algebra \( L_1(\mathbb{H}^n) \) with the multiplication \(*\) \([1, \S 4.1]\), i.e. \( \alpha^*(B_1) * \alpha^*(B_2) = \alpha^*(B_1 * B_2) \). Moreover \( \alpha^* \) commutes with the antiderivative \( A \) \([4, \S 4.1]\), thus \( \tilde{\alpha} \) is an automorphism of \( L_1(\mathbb{H}^n) \) with the modified multiplication \( \star \) \([1, \S 4.2]\) as well, that is

\[
\alpha^*(B_1) \star \alpha^*(B_2) = \alpha^*(B_1 * B_2).
\] (4.24)
By the linearity we can extend the intertwining property (4.23) to the convolution operator $K$ as follows:

\[ \alpha^* K \circ \tilde{\alpha} = \tilde{\alpha} \circ K. \] (4.25)

Since $\alpha$ is automorphism of $\mathbb{H}^n$ it fixes the unit $e$ of $\mathbb{H}^n$ and its differential $d\alpha : \mathfrak{h}^n \to \mathfrak{h}^n$ at $e$ is given by the same matrix as $\alpha$ in the exponential coordinates. Obviously $d\alpha$ is an automorphism of the Lie algebra $\mathfrak{h}^n$. By the duality between $\mathfrak{h}^n$ and $\mathfrak{h}^*_n$ we obtain the adjoint map $d\alpha^* : \mathfrak{h}^*_n \to \mathfrak{h}^*_n$ defined by the expression

\[ d\alpha^* : (h, q, p) \mapsto (h, A^t(q, p)), \] (4.26)

where $A^t$ is the transpose of $A$. Obviously $d\alpha^*$ preserves any orbit $O_h$ (2.7) and maps the orbit $O_{(q,p)}$ (2.8) to $O_{A^t(q,p)}$.

Identity (4.26) indicates that both representations $\rho_h$ and $(\rho_h \circ \alpha)(s, x, y) = \rho_h(s, A(x, y))$ for $h \neq 0$ correspond to the same orbit $O_h$. Thus they should be equivalent, i.e. there is an intertwining operator $U_A : F_2(O_h) \to F_2(O_h)$ such that $U_A^{-1} \rho_h U_A = \rho_h \circ \alpha$. Then the correspondence $\sigma : A \mapsto U_A$ is a linear unitary representation of the double cover $\tilde{Sp}(n)$ of the symplectic group called the metaplectic representation [12, §4.2]. Thus we have

**Proposition 4.9.** [26] The $p$-mechanical brackets are invariant under the symplectic automorphisms of $\mathbb{H}^n$: $\{[\tilde{\alpha} B_1, \tilde{\alpha} B_2] = \tilde{\alpha} \{B_1, B_2\}$. Consequently the dynamic equation (4.9) has symplectic symmetries which are reduced

- (1) by $\rho_h$, $h \neq 0$ on $O_h$ (2.7) to the metaplectic representation in quantum mechanics;
- (2) by $\rho_{(q,p)}$ on $O_{(q,p)}$ (2.18) to the symplectic symmetries of classical mechanics [21 §38].

Combining intertwining properties of all three components (3.23) in the Weyl quantisation we get

**Corollary 4.10.** [26] The Weyl quantisation $Q_h$ (3.22) is the intertwining operator between classical and metaplectic representations.

The two equations for the time evolution of states (4.12), (4.15) are both invariant under the symplectic automorphisms of $\mathbb{H}^n$. The invariance of equation (4.15) is a consequence of (4.24) while the invariance of (4.12) follows from the invariance of the $p$-mechanical brackets.

### 4.5. Coherent States.

The coherent states defined in section 2.3 all had a function supported at $(0,0) \in \mathbb{R}^{2n}$ as their classical limit, rather than being supported around different classical states $(q,p)$. In this section we rectify this problem by introducing an overcomplete system of vectors in $\mathcal{H}_h$ through a representation of $\mathbb{H}^n$. The states which correspond to these vectors are an overcomplete system of coherent states for each $h \neq 0$. We then show that these vectors correspond to a system of kernels in $\mathcal{L}_h$, whose limit is the $(q,p)$ pure state kernels.

Initially we need to introduce a vacuum vector in $\mathcal{H}_h$. The vector in $F_2(O_h)$ corresponding to the ground state is (c.f. equation (2.19))

\[ f_0(q,p) = \exp \left(- \frac{2\pi}{h} (\omega m q^2 + (\omega m)^{-1} p^2) \right), \quad h > 0. \]
The image of this under $S_h$ is
\[ e^{2\pi i hs} \mathcal{F}(f_0) = e^{2\pi i hs} \int_{\mathbb{R}^2} e^{-\frac{2\pi}{\hbar}(m\omega q^2 + (\omega m)^{-1}p^2)} e^{-2\pi i(qx + py)} \, dq \, dp. \]

Using the basic formula
\[ \int_{\mathbb{R}} \exp(-ax^2 + bx + c) \, dx = \left( \frac{\pi}{a} \right)^{\frac{1}{2}} \exp \left( \frac{b^2}{4a} + c \right), \quad \text{where } a > 0 \quad (4.27) \]
we get
\[ S_h(f_0) = e^{2\pi i hs} \mathcal{F}(f_0) = \left( \frac{\hbar}{2} \right)^n \exp \left( 2\pi i hs - \frac{\pi h}{2} \left( \frac{x^2}{\omega m} + y^2 \omega m \right) \right), \]
which is the element of $\mathcal{H}_h$ corresponding to the ground state.

**Definition 4.11.** Define the vacuum vector in $\mathcal{H}_h$ as
\[ v_{(h,0,0)} = \left( \frac{\hbar}{2} \right)^n \exp \left( 2\pi i hs - \frac{\pi h}{2} \left( \frac{x^2}{\omega m} + y^2 \omega m \right) \right), \]
where $\omega$ and $m$ are constants representing frequency and mass respectively.

Now we calculate the kernel, $l_{(h,0,0)}$, for the ground state by the relationship (3.32) between kernels and vectors.

\[
l_{(h,0,0)}(s, x, y) = \left( \frac{4}{\hbar} \right)^n \int_{\mathbb{R}^2} v_{(h,0,0)}((-s, -x, -y)(s', x', y')) v_{(h,0,0)}(s', x', y') \, ds' \, dy'
\]

\[
\quad = \exp \left( -2\pi i hs - \frac{\pi h}{2} \left( \frac{x^2}{\omega m} + \omega my^2 \right) \right)
\times \exp \left( \frac{\pi h}{4} \left( \omega m \left( iy + \frac{x}{\omega m} \right)^2 + \frac{1}{\omega m} (\omega my - ix)^2 \right) \right)
\]

at (4.28) we have used formula (4.27). By a simple calculation it can be shown that
\[ \omega m \left( iy + \frac{x}{\omega m} \right)^2 + \frac{1}{\omega m} (\omega my - ix)^2 = 0 \]

hence
\[ l_{(h,0,0)} = \exp \left( -2\pi i hs - \frac{\pi h}{2} \left( \frac{x^2}{\omega m} + \omega my^2 \right) \right). \]

Recalling functions $X$ and $Y$ from equations (3.19) and (3.20)
\[ X = \frac{1}{2\pi i} \delta(s) \delta^{(1)}(x) \delta(y) \quad \text{and} \quad Y = \frac{1}{2\pi i} \delta(s) \delta(x) \delta^{(1)}(y). \]

Under left and right convolution $X$ and $Y$ generate left $2.12$ and right $2.13$ invariant vector fields respectively. That is, if $B$ is a function or distribution on $\mathbb{H}^n$ then
\[ X \ast B = \frac{1}{2\pi i} \left( \frac{\partial}{\partial x} - y \frac{\partial}{\partial s} \right) B, \quad B \ast X = \frac{1}{2\pi i} \left( \frac{\partial}{\partial x} + y \frac{\partial}{\partial s} \right) B; \]
\[ Y \ast B = \frac{1}{2\pi i} \left( \frac{\partial}{\partial y} + x \frac{\partial}{\partial s} \right) B, \quad B \ast Y = \frac{1}{2\pi i} \left( \frac{\partial}{\partial y} - x \frac{\partial}{\partial s} \right) B. \]
Consider the action of $\mathbb{H}_n$ on $\mathcal{H}_h$ by
\[
\zeta_{(r,q,p)}v(s, x, y) = e^{-2\pi i rs}e^{-2\pi i A(-pX + qY)v(s, x, y)},
\]
where $e^X$ is exponential of the operator of convolution by $X$. The elements $(r, 0, 0)$ act trivially in the representation, $\zeta$, thus the essential part of the operator $\zeta_{(r,q,p)}$ is determined by $(q, p)$. If we apply this representation with $r = 0$ to $v(h, 0, 0)$ we get a system of vectors $v(h, q, p)$,
\[
v_{(h,q,p)}(s, x, y) = \zeta_{(0,q,p)} \left( \left( \frac{\hbar}{2} \right)^n \exp \left( 2\pi i hs - \frac{\pi \hbar}{2} \left( \frac{x^2}{\omega m} + y^2 \omega m \right) \right) \right).
\]
By (4.18) the vectors $v_{(h,q,p)}$ are equivalent to the kernels $l_{(h,q,p)}$
\[
l_{(h,q,p)} = e^{2\pi i (-pX + qY)}l_{(h,0,0)}.
\]
Since for any function or distribution, $B$, on $\mathbb{H}_n$
\[
\{ [-pX + qY, B] = -(px + qy)B
\]
we have
\[
l_{(h,q,p)} = \exp \left( -2\pi i(qx + py) - 2\pi ihs - \frac{\pi \hbar}{2} \left( \frac{x^2}{\omega m} + \omega my^2 \right) \right).
\]

Definition 4.12. [9] For $h \in \mathbb{R} \setminus \{0\}$ and $(q, p) \in \mathbb{R}^{2n}$ define the system of coherent states $k_{(h,q,p)}$ by
\[
k_{(h,q,p)}(B) = \langle B * v_{(h,q,p)}, v_{(h,q,p)} \rangle = \int_{\mathbb{H}_n} B(g) l_{(h,q,p)}(g) dg.
\]

It is clear that the limit as $h \to 0$ of the kernels $l_{(h,q,p)}$ will just be the kernels $l_{(0,q,p)}$. This proves that the system of coherent states we have constructed have the $(q, p)$ pure states, $k_{(0,q,p)}$, from equation (3.35), as their limit as $h \to 0$, which is the content of the next Theorem.

Theorem 4.13. [9] If we have any $p$-observable $B$ which is of the form $\delta(s)\hat{F}(x, y)$ (that is, $B$ is the $p$-mechanisation of $F$ see equations (3.17) and (3.18)) then
\[
\lim_{h \to 0} k_{(h,q,p)}(B) = k_{(0,q,p)}(B) = F(q, p).
\]

We have used $p$-mechanics to rigorously prove, in a simpler way to previous attempts [14], the classical limit of coherent states.

Remark 4.14. If we apply the unitary transformation $\tilde{\alpha}_A$ (from subsection 4.4) for some $A \in Sp(n, \mathbb{R})$ to some kernel of a $(q, p)$ coherent state, $l_{(0,q,p)}$, we will get another $(q, p)$ coherent state, $l_{(0,A\hat{q}(q,p))}$,
\[
l_{(0,q,p)}(s, A(x, y)) = l_{(0,A\hat{q}(q,p))}(s, x, y).
\]

5. Examples

We now demonstrate the theory through applying it to two examples: the forced and unforced harmonic oscillator.
5.1. **Unforced Harmonic Oscillator.** For one account of the unforced harmonic oscillator see [27], the account we give here is slightly different.

Let the $p$-mechanical energy function of a harmonic oscillator be as obtained in Examples 3.5 and 3.7:

$$B_H(s, x, y) = -\frac{1}{8\pi^2} \left( m\omega^2\delta(s)\delta^{(2)}(x)\delta(y) + \frac{1}{m}\delta(s)\delta(x)\delta^{(2)}(y) \right), \quad (5.1)$$

Then the $p$-dynamic equation (4.9) on $H^n$ is

$$\frac{d}{dt}B(t; s, x, y) = \sum_{j=1}^{n} \left( \frac{1}{m}x_j \frac{\partial}{\partial y_j} - m\omega^2 y_j \frac{\partial}{\partial x_j} \right) B(t; s, x, y). \quad (5.2)$$

Solutions to the above equations are well known to be rotations in each of the $(x_j, y_j)$ planes given by:

$$B(t; s, x, y) = B_0 \left( s, x \cos(\omega t) - m\omega y \sin(\omega t), \frac{x}{m\omega} \sin(\omega t) + y \cos(\omega t) \right). \quad (5.3)$$

Since the dynamics on $L_2(\mathbb{H}^n)$ is given by a symplectic linear transformation of $\mathbb{H}^n$ its Fourier transform to $L_2(\mathfrak{h}^*_n)$ will be the adjoint symplectic linear transformations of orbits $O_h$ and $O_0$ in $\mathfrak{h}^*_n$, see Figure 4.

The representations $\rho_h$ transform the energy function $B_H$ (5.1) into the operator

$$H_h = -\frac{1}{8\pi^2}(m\omega^2 Q^2 + \frac{1}{m} P^2), \quad (5.4)$$

where $Q = d\rho_h(X)$ and $P = d\rho_h(Y)$ are defined in (2.12). The representation $\rho_{(q,p)}$ transforms $B_H$ into the classical Hamiltonian

$$H(q, p) = \frac{m\omega^2}{2} q^2 + \frac{1}{2m} p^2. \quad (5.5)$$
The \( p \)-dynamic equation (4.9) in form (5.2) is transformed by the representations \( \rho_h \) into the Heisenberg equation
\[
\frac{d}{dt} f(t; Q, P) = \frac{1}{i\hbar} [f, H_h],
\]
where
\[
\frac{1}{i\hbar} [f, H_h] = m\omega^2 p \frac{\partial f}{\partial q} - \frac{1}{m} q \frac{\partial f}{\partial p},
\]
defined by the operator \( H_h \) (5.3). The representation \( \rho_{(q,p)} \) produces the Hamilton equation
\[
\frac{d}{dt} f(t; q, p) = m\omega^2 p \frac{\partial f}{\partial q} - \frac{1}{m} q \frac{\partial f}{\partial p}
\]
defined by the Hamiltonian \( H(q, p) \) (5.5). Finally, to get the solution for equations (5.6) and (5.7) it is enough to apply representations \( \rho_h \) and \( \rho_{(q,p)} \) to the solution (5.3) of the \( p \)-dynamic equation (5.2). To conclude our description of the unforced harmonic oscillator we give an alternative form of the Hamiltonian which will be of use when considering the forced harmonic oscillator.

**Definition 5.1.** \[9\] We define the \( p \)-mechanical creation and annihilation operators respectively as convolution by the following distributions
\[
\begin{align*}
a^+ &= \frac{1}{2\pi i} \left( m\omega \delta(s) \delta^{(1)}(x) \delta(y) - i\delta(s) \delta(x) \delta^{(1)}(y) \right), \\
a^- &= \frac{1}{2\pi i} \left( m\omega \delta(s) \delta^{(1)}(x) \delta(y) + i\delta(s) \delta(x) \delta^{(1)}(y) \right).
\end{align*}
\]
The \( p \)-mechanical harmonic oscillator Hamiltonian has the equivalent form
\[
B_H = \frac{1}{2m} (a^+ * a^- + i\omega m^2 \delta^{(1)}(s) \delta(x) \delta(y)).
\]
We denote the \( p \)-mechanical normalised eigenfunctions of the harmonic oscillator by \( v_n \in \mathcal{H}_h \) (note here that \( v_0 = v_{(h,0,0)} \)); they have the form
\[
\begin{align*}
 v_n &= \left( \frac{1}{n!} \right)^{1/2} \left( Aa^+ \right)^n * v_{(h,0,0)} \\
 &= \left( \frac{1}{n!} \right)^{1/2} \left( \frac{\hbar}{2} \right)^n e^{2\pi i hs} e^{i\omega my} \exp \left( \frac{-\pi h}{2} \left( \frac{x^2}{\omega m} + \frac{y^2}{\omega m} \right) \right).
\end{align*}
\]
It can be shown by a trivial calculation that these creation and annihilation operators raise and lower the eigenfunctions of the harmonic oscillator respectively. It is important to note that these states are orthogonal under the \( \mathcal{H}_h \) inner product defined in equation (5.26).

5.2. The \( p \)-Mechanical Forced Oscillator: The Solution and Relation to Classical Mechanics. The classical forced oscillator has been studied in great depth for a long time — for a description of this see \[17\] and \[13\]. The quantum case has also been heavily researched — see for example \[33\], Sect 14.6], \[32\]. Of interest in the quantum case has been the use of coherent states, this is described in \[34\]. Here we extend these approaches to give a unified quantum and classical solution of the problem based on the \( p \)-mechanical approach. In \[9\] there is a more in depth description of this example and a description of the \( p \)-mechanical scattering matrix.

The classical Hamiltonian for a harmonic oscillator of frequency \( \omega \) and mass \( m \) being forced by a real function of a real variable \( z(t) \) (measured in units \( \frac{MT^2}{L^2} \)) is
\[
H(t, q, p) = \frac{1}{2} \left( m\omega^2 q^2 + \frac{1}{m} p^2 \right) - z(t)q.
\]
Then for any observable \( f \in C^\infty(\mathbb{R}^{2n}) \) the dynamic equation is
\[
\frac{df}{dt} = \{f, H\} = \frac{p}{m} \frac{\partial f}{\partial q} - \omega^2 m q \frac{\partial f}{\partial p} + z(t) \frac{\partial f}{\partial p}.
\] (5.10)

Through the procedure of \( p \)-mechanisation as described in subsection 3.3 we get the \( p \)-mechanical forced oscillator Hamiltonian to be
\[
B_H(t; s, x, y) = -\frac{1}{8\pi^2} \left( \omega^2 \delta(s) \delta^{(2)}(x) \delta(y) + \frac{1}{m} \delta(s) \delta^{(1)}(x) \delta(y) \right) - \frac{z(t)}{2\pi i} \delta(s) \delta^{(1)}(x) \delta(y).
\]

From equation (4.9) the dynamic equation for an arbitrary observable \( B \) is
\[
\frac{dB}{dt} = \frac{x}{m} \frac{\partial B}{\partial y} - \omega^2 m y \frac{\partial B}{\partial x} - z(t) y B.
\] (5.11)

By substituting the following expression into equation (5.11) we see that it is a solution of the \( p \)-dynamic equation
\[
B(t; s, x, y) = \exp \left( 2\pi i \left( \frac{1}{m\omega} \int_0^t z(\tau) \sin(\omega \tau) \, d\tau X(t) - \int_0^t z(\tau) \cos(\omega \tau) \, d\tau Y(t) \right) \right) \times B(0; s, X(t), Y(t)),
\] (5.12)

where
\[
X(t) = x \cos(\omega t) - m\omega y \sin(\omega t), \quad \text{and} \quad Y(t) = \frac{x}{m\omega} \sin(\omega t) + y \cos(\omega t).
\]

Let \( F(q, p) = \rho(q, p)(B(s, x, y)) \) (i.e. \( F \) is the classical observable corresponding to \( B \) under the relationship described in [26, Sect. 3.3]).
\[
F(t; q, p) = \int_{\mathbb{R}^{2n+1}} B(t; s, x, y) e^{2\pi i (qx + py)} \, ds \, dx \, dy
\]
\[
= F \left( 0; q \cos(\omega t) - \frac{p}{m\omega} \sin(\omega t) + \frac{1}{m\omega} \int_0^t z(\tau) \cos(\omega \tau) \, d\tau, \right.
\[
\left. qm\omega \sin(\omega t) + p \cos(\omega t) - \int_0^t z(\tau) \cos(\omega \tau) \, d\tau \right) \ (5.13)
\]

This flow satisfies the classical dynamic equation (5.10) for the forced oscillator — this is shown in [17].

5.3. A Periodic Force and Resonance. In classical mechanics the forced oscillator is of particular interest if we take the external force to be \( z(t) = Z_0 \cos(\Omega t) \) [17], that is the oscillator is being driven by a harmonic force of constant frequency \( \Omega \) and constant amplitude \( Z_0 \). By a simple calculation we have these results for \( \Omega \neq \omega \)
\[
\int_0^t \cos(\Omega \tau) \sin(\omega \tau) \, d\tau = \frac{2}{(\Omega^2 - \omega^2)} \left[ \Omega \cos(\Omega t) \cos(\omega t) + \omega \sin(\Omega t) \sin(\omega t) \right] \ (5.14)
\]
\[
\int_0^t \cos(\Omega \tau) \cos(\omega \tau) \, d\tau = \frac{2}{(\Omega^2 - \omega^2)} \left[ -\Omega \sin(\Omega t) \cos(\omega t) + \omega \cos(\Omega t) \sin(\omega t) \right] \ (5.15)
\]
When these are substituted into (5.12) we see that in p-mechanics using a periodic force the p-mechanical solution is the flow of the unforced oscillator multiplied by an exponential term which is also periodic. However this exponential term becomes infinitely large as Ω comes close to ω. If we substitute (5.14) and (5.15) into (5.13) we obtain a classical flow which is periodic but with a singularity as Ω tends toward ω. These two effects show a correspondence between classical and p-mechanics. The integrals have a different form when Ω = ω

\[
\int_0^t \cos(\omega \tau) \sin(\omega \tau) d\tau = \frac{1 - \cos(2\omega t)}{4\omega} \tag{5.16}
\]

\[
\int_0^t \cos(\omega \tau) \cos(\omega \tau) d\tau = \frac{t}{2} + \frac{1}{4\omega} \sin(2\omega t) \tag{5.17}
\]

Now when these new values are substituted into the p-mechanical solution (5.12) the exponential term will expand without bound as t becomes large. When (5.16) and (5.17) are substituted into (5.13) the classical flow will also expand without bound — this is the effect of resonance.

5.4. The Interaction Picture of the Forced Oscillator. We now use the interaction picture to get a better description of the p-mechanical forced oscillator. In [9] we use a different approach to the interaction picture using the \( H \) states, here we use the kernels. The p-mechanical forced oscillator Hamiltonian has the equivalent form

\[
B_H = \frac{1}{2m} \left( a^+ a^- + i\omega m^2 \delta^{(1)}(s) \delta(x) \delta(y) \right) - \frac{z(t)}{2} (a^- + a^+) \tag{5.18}
\]

(\( a^+ \) and \( a^- \) are the distributions defined in equations (5.8) and (5.9)). We now proceed to solve the forced oscillator in p-mechanics using the interaction picture

\[
\frac{d\tilde{l}}{dt} = \left\{ [C(s, x, y), \tilde{l}] \right\}
\]

where

\[
C(s, x, y) = \exp \left( -\frac{t}{2m} A(a^+ a^- + i\omega m^2 \delta^{(1)}(s) \delta(x) \delta(y)) \right) \times \left( \frac{z(t)}{2} (a^- + a^+) \right) \exp \left( \frac{t}{2m} A(a^+ a^- + i\omega m^2 \delta^{(1)}(s) \delta(x) \delta(y)) \right),
\]

where \( \tilde{l} = e^{-\tilde{t}[B_H, \tilde{l}]} \).

Lemma 5.2. We have the relations

\[
\left\{ [a^+ , a^-] \right\} = i\omega m \delta(s) \delta(x) \delta(y) \tag{5.19}
\]

\[
\left\{ [a^+ , a^+ + a^-] \right\} = i\omega m a^+ \tag{5.20}
\]

\[
\left\{ [a^- , a^- + a^+] \right\} = -i\omega m a^- \tag{5.21}
\]

Proof: Equation (5.19) follows from simple properties of commutation for convolutions of Dirac delta functions. Equations (5.20) and (5.21) follow from (5.19) and the fact that \( \left\{ [\cdot, \cdot] \right\} \) are a derivation. □
**Lemma 5.3.** If \( B_1, B_2 \) are functions or distributions on \( \mathbb{H}^n \) such that \([B_1, B_2] = \gamma B_2\) where \( \gamma \) is a constant then we have
\[
e^{-\alpha \gamma B_1} B_2 e^{\alpha \gamma B_1} = e^{-\gamma} B_2.
\]
Here \( e^{\alpha \gamma B_1} \) is the exponential of the operator of convolution by \( \alpha \gamma B_1 \).

The combination of Lemmas 5.2 and 5.3 simplifies equation (5.18) to
\[
\frac{d\tilde{l}}{dt} = -\left\{ \left[ \frac{z(t)}{2} (a e^{-i \omega t} + a^* e^{i \omega t}), \tilde{l} \right] \right\}
\]
\[
= \left\{ \left[ z(t) \cos(\omega t) X - z(t) \sin(\omega t) Y, \tilde{l} \right] \right\}
\]
where \( X \) and \( Y \) are from equations (5.19) and (5.20) respectively. This simplifies to
\[
\frac{d\tilde{l}}{dt} = 2 \pi i (z(t) \cos(\omega t) x + z(t) \sin(\omega t) y) \tilde{l}
\]
from which it follows that
\[
\tilde{l}(t_2, s, x, y) = \exp \left( 2 \pi i \left( \int_{t_1}^{t_2} z(\tau) \cos(\omega t) d\tau x + \int_{t_1}^{t_2} z(\tau) \sin(\omega t) d\tau y \right) \right) \tilde{l}(t_1, s, x, y).
\]

If \( \tilde{l}(t_1, s, x, y) = l(q,p)(s, x, y) \) then \( \tilde{l}(t_2, s, x, y) = \tilde{l}(q+\alpha, p+\beta)(s, x, y) \) where \( \alpha = \int_{t_1}^{t_2} z(\tau) \cos(\omega t) d\tau \) and \( \beta = \int_{t_1}^{t_2} z(\tau) \sin(\omega t) d\tau \). So if the system starts in a coherent state it will remain in a coherent state as time evolves. This result has been found in a much simpler manner than the method used in [35 Sect. 14.6].

6. In Conclusion: p-Mechanics and Contextuality

The presented construction of observables as (convolution) operators on \( L_2(\mathbb{H}^n) \) and states as positive linear functionals on them naturally unifies the quantum and classical pictures of mechanics. Moreover the p-mechanical description of states through their kernels \([18, 28]\) and the Liouville-type equation \([18, 28]\) for the dynamics of these kernels is suitable for the contextual interpretation \([18, 19, 28]\) of quantum mechanics.

Indeed the main point of the contextual approach \([18, 19]\) is that in a realistic model the total probability \( P(E_{12}) \) of two disjoint events \( P(E_1) \) and \( P(E_2) \) should not be calculated by a simplistic addition rule \( P(E_{12}) = P(E_1) + P(E_2) \). The typical example, when this formula fails, is the two slits experiment. However the textbook conclusion that “quantum particles do not have trajectories” is not legitimate in the contextual framework \([20, 28]\).

Contextuality requires that probabilities of events should depend from the context of experiments. For example, the probabilities of an electron to pass the first slit could be either \( P(E_1|S_1) \) or \( P(E_1|S_{12}) \) depending correspondingly from the context \( S_1 \) (only the first slit is open) or \( S_{12} \) (both slits are open). The similar notations \( P(E_2|S_1) \) or \( P(E_2|S_{12}) \) are used for the second slit and in general:
\[
P(E_1|S_1) \neq P(E_1|S_{12}), \quad \text{and} \quad P(E_2|S_2) \neq P(E_2|S_{12}),
\]
Then instead of the wrong probabilities addition rule
\[
P(E_{12}|S_{12}) = P(E_1|S_1) + P(E_2|S_2)
\]
the true contextual addition of probabilities is:
\[
P(E_{12}|S_{12}) = P(E_1|S_{12}) + P(E_2|S_{12}).
\]
Using some relations between contextual probabilities \( P(E_1|S_1), P(E_2|S_2), \) and \( P(E_1|S_{12}), P(E_2|S_{12}), \) which could be derived from a physical model, one can improve the wrong formula (6.1) to the “quantum addition” of probabilities:

\[
P(E_{12}|S_{12}) = P(E_1|S_1) + P(E_2|S_2) + \alpha \sqrt{P(E_1|S_1)P(E_2|S_2)},
\]

(6.3)

where \( \alpha \) is a real number. The presence of the square root in (6.3) could be motivated by the consideration of dimensions. If \( |\alpha| \leq 1 \) one can identify \( \alpha = \cos \phi \) for a quantum phase \( \phi \) and this would be the standard superposition of states in quantum theory. In remaining cases \( |\alpha| > 1 \) the formula (6.3) represents the hyperbolic version of quantum theory.

The contextual calculus of probabilities in quantum mechanics does not require a superposition of states as linear combinations of vectors in Hilbert space. Instead the outcome of combined experiments could be directly calculated from the contextual probabilities in a way similar to (6.2). \( p \)-Mechanical equation (4.12) for dynamics of states (i.e. corresponding contextual probabilities) describes the dynamics in a way similar to classical statistical mechanics. Therefore the combination of contextual probabilities and \( p \)-mechanical dynamics form a reliable model for quantum phenomenons. This combination of two approaches requires further study.

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