New covering codes of radius $R$, codimension $tR$ and $tR + \frac{R}{2}$, and saturating sets in projective spaces

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Abstract. The length function $\ell_q(r, R)$ is the smallest length of a $q$-ary linear code of codimension $r$ and covering radius $R$. In this work we obtain new constructive upper bounds on $\ell_q(r, R)$ for all $R \geq 4$ and $r = tR$ with integer $t \geq 2$, and also for all even $R \geq 2$ and $r = tR + \frac{R}{2}$ with integer $t \geq 1$. The new bounds are provided by new infinite families of covering codes with fixed $R$ and growing codimension. The new bounds improve upon the known ones.

We propose a general regular construction (called “Line-Ovals”) of a minimal $\rho$-saturating $((\rho + 1)q + 1)$-set in the projective space $\PG(2\rho + 1, q)$ for all $\rho \geq 0$. Such a set corresponds to an $[Rq + 1, Rq + 1 - 2R]_qR$ locally optimal code of covering radius $R = \rho + 1$. In these codes, we investigate combinatorial properties regarding to spherical capsules (including the property to be a surface-covering code) and give corresponding constructions for code codimension lifting. Using the new codes as starting points in these constructions we obtained the desired infinite code families with growing $r = tR$.

In addition, we obtain new 1-saturating sets in the projective plane $\PG(2, q^2)$ and, founding on them, construct infinite code families with fixed even radius $R \geq 2$ and growing codimension $r = tR + \frac{R}{2}$, $t \geq 1$.

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1See the definitions at the end of Section 1.1.
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1 Introduction

1.1 Covering codes. The length function

Let $F_q$ be the Galois field with $q$ elements, $F_q^* = F_q \setminus \{0\}$. Let $F_q^n$ be the $n$-dimensional vector space over $F_q$. Denote by $[n, n-r]_q$ a $q$-ary linear code of length $n$ and codimension (redundancy) $r$, that is a subspace of $F_q^n$ of dimension $n-r$. The sphere of radius $R$ with center $c$ in $F_q^n$ is the set $\{v : v \in F_q^n, d(v, c) \leq R\}$ where $d(v, c)$ is the Hamming distance between the vectors $v$ and $c$.

**Definition 1.1.** (i) The covering radius of a linear $[n, n-r]_q$ code is the least integer $R$ such that the space $F_q^n$ is covered by the spheres of radius $R$ centered at the codewords.

(ii) A linear $[n, n-r]_q$ code has covering radius $R$ if every column of $F_q^n$ is equal to a linear combination of at most $R$ columns of a parity check matrix of the code, and $R$ is the smallest value with this property.

Definitions 1.1(i) and 1.1(ii) are equivalent. Let an $[n, n-r]_q R$ code be an $[n, n-r]_q$ code of covering radius $R$. An $[n, n-r]_q R$ code of minimum distance $d$ is denoted by $[n, n-r, d]_q R$ code. For an introduction to coverings of vector Hamming spaces over finite fields, see [5, 7].

The covering density $\mu$ of an $[n, n-r]_q R$-code is defined as the ratio of the total volume of all $q^{n-r}$ spheres of radius $R$ centered at the codewords to the volume $q^n$ of the space $F_q^n$. By Definition 1.1(i), we have $\mu \geq 1$. In the other words,

$$\mu = \left( q^{n-r} \sum_{i=0}^{R} (q-1)^i \binom{n}{i} \right) \frac{1}{q^n} = \frac{1}{q^r} \sum_{i=0}^{R} (q-1)^i \binom{n}{i} \geq 1. \quad (1.1)$$

The covering quality of a code is better if its covering density is smaller. For fixed $q, r, R$, the covering density of an $[n, n-r]_q R$ code decreases with decreasing $n$.

Codes investigated from the point of view of the covering quality are usually called covering codes [7]; see an online bibliography [28], works [5,9,14,17,20,26,27], and the references therein.

**Definition 1.2.** [5,7] The length function $\ell_q(r, R)$ is the smallest length of a $q$-ary linear code of codimension $r$ and covering radius $R$.
From (1.1), see also Definition 1.1(ii), one obtains an implicit lower bound on $\ell_q(r, R)$:

\[
\sum_{i=0}^{R} (q-1)^i \binom{\ell_q(r, R)}{i} \geq q^r.
\]  

(1.2)

In particular, for $R = 1$ we have $\ell_q(r, 1) \geq \frac{q^r-1}{q-1}$. This means that the perfect $[q^r-1, q^{r-1} - r, 3]_q$ Hamming code achieves the bound and has the covering density equal to one. The same is true for the perfect Golay codes $[23, 12, 7]_2$ and $[11, 6, 5]_2$. In the general case, note that the main term of the sum in (1.2) is $(q-1)^R \binom{\ell_q(r, R)}{R}$. If $n$ is considerable larger than $R$ (this is the natural situation in covering codes investigations) and if $q$ is large enough, we have

\[
\sum_{i=0}^{R} (q-1)^i \binom{\ell_q(r, R)}{i} \approx (q-1)^R \frac{(\ell_q(r, R))^R}{R!} \geq q^r,
\]

\[
\ell_q(r, R) \geq \sqrt[2]{R!} \cdot q^{(r-R)/R},
\]

and, in a more general form,

\[
\ell_q(r, R) \geq cq^{(r-R)/R},
\]  

(1.3)

where $c$ is independent of $q$ but it is possible that $c$ depends on $r$ and $R$.

Let $t, s, R^*$ be integers. Let $q'$ be a prime power. In [11, 13, 14, 17], see also the references therein, for the situations

(i) $r = tR$, arbitrary $q$,

(ii) $R = sR^*$, $r = tR + s$, $q = (q')^{R^*}$,

(iii) $r \neq tR$, $q = (q')^R$,

[n, n-r]_qR covering codes are obtained with lengths of the form

\[
n = c_1(r, R)q^{(r-R)/R} + \sum_{i \geq 2} c_i(r, R)q^{(r-R)/R-\mu_i}, \quad c_1(r, R) > 1, \quad \mu_i > 0,
\]  

(1.5)

where all $c_i(r, R)$ are constants independent of $q$. Also, for $i \geq 2$, one usually has $c_i(r, R) \geq 0$, but it is possible that $c_i(r, R) < 0$, see for example Propositions 1.6 and 1.7. For growing $q$, code length $n$ of (1.5) is close (by order) to the bound (1.3) since all $\mu_i > 0$.

In this work, we consider the case (i) of (1.4) for $R \geq 4$ and the situation (ii) for even $R$ with $R^* = 2$. We briefly describe the known results and then improve upon many of them by constructing new codes.

For new codes with $r = tR$ we note and use interesting and useful combinatorial properties connected with the locally optimality, $R$, $\ell$-capsules and $R$, $\ell$-objects.
Definition 1.3. [12] A linear covering code is called \textit{locally optimal} if one cannot remove any column from its parity check matrix without increase in covering radius. A locally optimal code can be called also \textit{non-shortening} in the sense mentioned.

Let $0 \leq \ell \leq R$. A \textit{spherical} $R, \ell$-capsule with center $c$ in $\mathbb{F}_q^n$ is the set $\{v : v \in \mathbb{F}_q^n, 0 \leq \ell \leq d(v, c) \leq R\}$ where $d(v, c)$ is the Hamming distance between the vectors $v$ and $c$, see [9, Rem. 5], [10, Rem. 2.1], [14, Sect. 2].

Definition 1.4. [9], [10, Sect. 2], [14, Sect. 2] Let $0 \leq \ell \leq R$. A linear $[n, n-r]_q R$ code of covering radius $R$ is called an $R, \ell$-\textit{object} and is denoted as an $[n, n-r]_q R, \ell$ code if any of following holds.

(i) The space $\mathbb{F}_q^n$ is covered by the $R, \ell$-capsules centered at the codewords.

(ii) Every column of the space $\mathbb{F}_q^r$ (including the zero column) is equal to a linear combination with \textit{nonzero coefficients} of at least $\ell$ and at most $R$ distinct columns of a parity-check matrix of the code.

(iii) Every coset of the code (including the code itself) contains a weight $w$ word of the space $\mathbb{F}_q^n$ such that $\ell \leq w \leq R$.

Definitions [1.4(i), 1.4(ii), and 1.4(iii)] are equivalent. In [9][10][14] widened definitions of $R, \ell$-objects are considered. But for this work, Definition 1.4 is sufficient.

Note that the $R, R$-capsule is the surface of the sphere of radius $R$.

Definition 1.5. An $[n, n-r]_q R, R$ code is called \textit{surface-covering code} of radius $R$.

The value of $\ell$ is important for code codimension lifting constructions, see Section 4.

1.2 The known results

Codes with radius $R = 2, 3$ and codimension $r = tR$ are widely investigated for arbitrary $q$, see [11], [14, Sects. 4, 5], [17] Ths. 9, 12. At the same time, codes with $R \geq 4$ and $r = tR$ are investigated insufficiently; moreover, the known results on these codes are obtained by use of codes with $R = 2, 3$ in the so-called direct sum construction [14, Sect. 4.2]. The following results on codes with $R \geq 4$ and $r = tR$ are described in the literature.

Proposition 1.6. [13, Sect. 2], [14, Ths. 6.1, 6.2] The following constructive upper bounds on the length function (provided by infinite families of codes with growing codimension) hold:

$$\ell_q(r, R) \leq R q^{(r-R)/R} + \left[\frac{R}{3}\right] q^{(r-2R)/R} + \delta_q(r, R), \quad R \geq 4, \quad r = tR, \quad t \geq 2,$$

where values of $\delta_q(r, R)$ with $w = 2R \pmod{3}$ are as follows:

$$\delta_q(r, R) = 0, \quad q \geq 4, \quad r = 2R \quad [14, \text{Th. 6.1}];$$
\[ \delta_q(r, R) = 0, \quad q = 16, \ q \geq 23, \ r = 3R \quad [14, \text{eq. (6.1)}], \ [17]; \]
\[ \delta_q(r, R) = 2w(q^{(r-3R)/R} + 1), \ q = 4, 5, 9, \ r = 4R \quad [14, \text{eq. (6.1)}], \ [11]; \]
\[ \delta_q(r, R) = w(q^{(r-3R)/R} + 1), \ q \geq 7, \ q \neq 9, \ r = 4R, 6R \quad [14, \text{eq. (6.1)}], \ [17]; \]
\[ \delta_q(r, R) = wq^{(r-3R)/R}, \ q = 5, 9, \ r \geq 5R, \ r \neq 6R \quad [14, \text{Th. 6.2}]; \]
\[ \delta_q(r, R) = 0, \quad q \geq 7, \ q \neq 9, \ r \geq 5R, \ r \neq 6R \quad [14, \text{Th. 6.2}]. \]

The following results on codes with even covering radius \( R \geq 2 \) and codimension \( r = tR + \frac{R}{2} \) are described in the literature.

**Proposition 1.7.** [11, Ex. 6, eq. (33)], [13, Sects. 4.4, 7] Let \( q' \) be a prime power. Let the covering radius \( R \geq 2 \) be even. Let the code codimension be \( r = tR + \frac{R}{2} \) with integer \( t \). The following constructive upper bounds on the length function \( \ell_q(r, R) \) (provided by infinite families of codes with growing codimension) hold:

\[
\ell_q(r, R) \leq R \left( 3 - \frac{1}{\sqrt{q}} \right) q^{(r-R)/R} + \frac{R}{2} \left[ q^{(r-2R)/R-0.5} \right], \ q = (q')^2 \geq 16, \ t \geq 1; \tag{1.7}
\]
\[
\ell_q(r, R) \leq R \left( 1 + \frac{1}{\sqrt{q}} + \frac{1}{\sqrt{q'}} \right) q^{(r-R)/R} + \frac{R}{2} \left[ q^{(r-2R)/R-0.5} \right], \ q = (q')^4, \ t \geq 1; \tag{1.8}
\]
\[
\ell_q(r, R) \leq R \left( 1 + \frac{1}{\sqrt{q}} + \frac{1}{\sqrt{q'}} + \frac{1}{\sqrt{q''}} \right) q^{(r-R)/R} + R \left[ q^{(r-2R)/R-0.5} \right], \ q = (q')^6, \tag{1.9}
\]

\( q' \leq 73 \) prime, \( t \geq 1, \ t \neq 4, 6. \)

**Problem 1.** Improve the known bounds on the length function \( \ell_q(r, R) \) collected in

(i) Proposition 1.6 where \( R \geq 4, \ r = tR, \ t \geq 2, \)

(ii) Proposition 1.7 where \( R \geq 2 \) is even, \( r = tR + \frac{R}{2}, \ t \geq 1. \)

### 1.3 Saturation sets in projective spaces

Effective methods to obtain upper bounds on \( \ell_q(r, R) \) are connected with *saturation sets in projective spaces*.

Let PG\((N,q)\) be the \( N \)-dimensional projective space over the field \( \mathbb{F}_q \); see [21, 23] for an introduction to the projective spaces over finite fields, see also [19, 22, 26, 27] for connections between coding theory and Galois geometries.

**Definition 1.8.** (i) A point set \( \mathcal{S} \subseteq \text{PG}(N,q) \) is *\( \rho \)-saturating* if for any point \( A \) of \( \text{PG}(N,q) \setminus \mathcal{S} \) there exist \( \rho + 1 \) points in \( \mathcal{S} \) generating a subspace of \( \text{PG}(N,q) \) containing \( A \), and \( \rho \) is the smallest value with such property.

(ii) A point set \( \mathcal{S} \subseteq \text{PG}(N,q) \) is *\( \rho \)-saturating* if every point \( A \in \text{PG}(N,q) \) (in the homogeneous coordinates) can be written as a linear combination of at most \( \rho + 1 \) points of \( \mathcal{S} \), and \( \rho \) is the smallest value with such property (cf. Definition 1.1(ii)).
Definitions 1.8(i) and 1.8(ii) are equivalent.

Saturating sets are considered, for instance, in [5, 6, 10, 12, 17, 19, 20, 24, 26, 27, 30]. In the literature, saturating sets are also called “saturated sets”, “spanning sets”, “dense sets”.

Let \( s_q(N, \rho) \) be the smallest size of a \( \rho \)-saturating set in \( \text{PG}(N, q) \).

If \( q \)-ary positions of a column of an \( r \times n \) parity check matrix of an \( [n, n-r]_q \)\( R \) code are treated as homogeneous coordinates of a point in \( \text{PG}(r-1, q) \) then this parity check matrix defines an \( (R-1) \)-saturating set of size \( n \) in \( \text{PG}(r-1, q) \) [6,10,13,14,16,19,20,24,26,27]. So, there is a one-to-one correspondence between \( [n, n-r]_q \)\( R \) codes and \((R-1)\)-saturating \( n \)-sets in \( \text{PG}(r-1, q) \). Therefore,

\[
\ell_q(r, R) = s_q(r-1, R-1).
\]

Recall that the results of Proposition 1.6 are based on direct sum of codes of radius \( R = 2, 3 \). The following geometrical constructions make an important contribution to the structures of the best codes with \( R = 2, 3 \):

- “oval plus line” [6 p. 104], [10 Th. 5.1]; the construction obtains an 1-saturating \((2q+1)\)-set in \( \text{PG}(3, q) \) that corresponds to an \( [2q+1, (2q+1) - 4]_q 2 \) code with \( r = 2R \);
- “two ovals plus line” [16 Sect. 4]; the construction obtains a 2-saturating \((3q+1)\)-set in \( \text{PG}(5, q) \) that corresponds to a \([3q+1, (3q+1) - 6]_q 3 \) code with \( r = 2R \).

**Problem 2.** [14 Sect. 6.1] For all \( \rho \geq 3 \) obtain a general construction of a \( \rho \)-saturating \(((\rho + 1)q + 1)\)-set in \( \text{PG}(2\rho + 1, q) \) that corresponds to an \([Rq + 1, Rq + 1 - 2R]_q R \) code with \( R = \rho + 1 \). In other words, prove (constructively) that \( s_q(2\rho + 1, \rho) \leq (\rho + 1)q + 1 \) and thereby prove that \( \ell_q(2R, R) \leq Rq + 1 \).

Note that for \( n < (\rho + 1)q + 1 = Rq + 1 \), no examples of \( \rho \)-saturating \( n \)-sets in \( \text{PG}(2\rho + 1, q) \) (resp. \([n, n-2R]_q R \) codes with \( R = \rho + 1 \)) seem to be known. Moreover, in [14 Prop. 4.2], it is proved that \( \ell_q(4, 2) = s_q(3, 1) = 2 \cdot 4 + 1 \). This strengthens the interest to Problem 2 and gives rise to the following.

**Problem 3.** [14 Sects. 4, 5] Determining whether \( \ell_q(2R, R) = Rq+1 \), equivalently whether \( s_q(2\rho + 1, \rho) = (\rho + 1)q + 1 \).

**Definition 1.9.** A \( \rho \)-saturating set in \( \text{PG}(N, q) \) is minimal if it does not contain a smaller \( \rho \)-saturating set in \( \text{PG}(N, q) \).

If the positions of a column of a parity check matrix of an \([n, n-r]_q R \) locally optimal code are considered as homogeneous coordinates of a point in \( \text{PG}(r-1, q) \) then this parity check matrix defines a minimal \((R-1)\)-saturating \( n \)-set in \( \text{PG}(r-1, q) \) [12]. So, there is a one-to-one correspondence between \([n, n-r]_q R \) locally optimal codes and minimal \((R-1)\)-saturating \( n \)-sets in \( \text{PG}(r-1, q) \).
If for the solution of Problem 2 we obtain minimal \(((\rho + 1)q + 1)\)-sets in \(\text{PG}(2\rho + 1, q)\)
(resp. locally optimal \([Rq + 1, Rq + 1 - 2R]_q R\) codes), this advances the solution of Problem 3.

Note that the codes providing the bounds of Proposition 1.7 are based on 1-saturating sets in the projective plane of square order. Improvements of these bounds could be connected with new 1-saturating sets of relatively small sizes.

**Problem 4.** In \(\text{PG}(2, q^2)\), construct new 1-saturating sets with sizes smaller than the known ones.

### 1.4 The goals and the structure of the paper

The goals of this paper:
- solve Problem 2 and with the help of the new \([Rq + 1, Rq + 1 - 2R]_q R\) codes solve Problem 1(i) regarding codes of covering radius \(R \geq 4\) and codimension \(tR\);
- solve Problem 3 and with the help of the new 1-saturating sets solve Problem 1(ii) regarding codes with even covering radius \(R \geq 2\) and codimension \(tR + \frac{R}{2}\).

The paper is organized as follows. In Section 2 we collect the main results of the paper. In Section 3 we propose a construction “line plus \(\rho\) ovals” for \(\rho\)-saturating sets in \(\text{PG}(2\rho + 1, q)\) and codes of codimension 2\(R\). This solves Problem 2. In Section 4 we describe two constructions from the family of the so-called “\(q^m\)-concatenating constructions” for code codimension lifting. The constructions are convenient for \([n, n-r]_q R, \ell\) codes with \(\ell \in \{R-1, R\}\). In Section 5 we prove that the codes obtained in Section 3 have \(\ell = R\) for odd \(q\) and \(\ell = R-1\) for even \(q\). (So, for odd \(q\) we have surface-covering codes.) Then we use these codes as starting ones for the constructions of Section 4. As the result, we obtained new infinite code families with fixed radius \(R \geq 4\) and growing codimension \(tR\). This solves Problem 1(i) for the most part. In Section 6, using recent results on double blocking set, we obtain new 1-saturating sets in \(\text{PG}(2, q^2)\) that solve in part Problem 1(ii). Then basing on these sets, we obtain new infinite code families for all fixed even radii \(R \geq 2\) and growing codimension \(tR + \frac{R}{2}\). This solves in part Problem 1(ii).

### 2 The main results

The main results of this paper are as follows:
- Problem 2 is solved, see Section 3. For all \(\rho \geq 0\) we propose a general regular construction (“Line-Ovals”) of a minimal \(\rho\)-saturating \(((\rho + 1)q + 1)\)-set in \(\text{PG}(2\rho + 1, q)\). This set corresponds to an \([Rq + 1, Rq + 1 - 2R]_q R\) locally optimal code with \(R = \rho + 1\). Thereby we have proved that \(s_q(2\rho + 1, \rho) \leq (\rho + 1)q + 1\) and, equivalently, \(\ell_q(2R, R) \leq Rq + 1\). The minimality of the obtained \(\rho\)-saturating set allows to hope that Problem 3 can be solved.
• Problem 1(i) is solved for the most part, see Sections 4 and 5. We described two constructions for code codimension lifting. Using the \([Rq+1, Rq+1-2R]_q\) codes as a start for these constructions, we obtained infinite code families with fixed radius \(R \geq 4\) and growing codimension \(tR\). These families improve the known results collected in Proposition 1.6 apart from \(t=3\). New bounds on the length function obtained in this paper are given in Theorem 2.1 based on Theorems 3.8, 3.10, 5.3, 5.4.

**Theorem 2.1.** Let \(t\) be a growing integer. For the length function \(\ell_q(r, R)\) and for the smallest size \(s_q(r-1, R-1)\) of a \((R-1)\)-saturating set in the projective space \(\text{PG}(r-1, q)\) the following constructive upper bounds (provided by infinite families of codes) hold:

\[
\ell_q(r, R) = s_q(r-1, R-1) \leq Rq^{(r-R)/R} + q^{(r-2R)/R} + \Delta_q(r, R), \quad r = tR,
\]

where for \(m_1 = \lceil \log_q(R+1) \rceil + 1\) we have

(i) \(\Delta_q(r, R) = 0\) if \(t = 2\), \(q = 4\) and \(q \geq 7\), \(R \geq 4\);

(ii) \(\Delta_q(r, R) = 0\) if \(t = 2\), \(q = 5\), \(R = 4, 5\);

(iii) \(\Delta_q(r, R) = 0\) if \(t \geq \lceil \log_q R \rceil + 3\), \(q \geq 7\) odd, \(R \geq 4\);

(iv) \(\Delta_q(r, R) = \sum_{j=2}^{m_1+2} q^{(r-jR)/R}\) if \(m_1 + 2 < t < 3m_1 + 2\), \(q \geq 8\) even, \(R \geq 4\);

(v) \(\Delta_q(r, R) = \sum_{j=2}^{m_1+2} q^{(r-jR)/R}\) if \(t = m_1 + 2\) and \(t \geq 3m_1 + 2\), \(q \geq 8\) even, \(R \geq 4\).

The new bounds of Theorem 2.1 are better than the known ones of Proposition 1.6. In particular, in Proposition 1.6 the coefficient for \(q^{(r-2R)/R}\) is \(\lceil \frac{R}{3} \rceil\), whereas in Theorem 2.1 it is equal to 1 or 2, see (i)–(iii) and (iv)–(v), respectively. Note that in the cases (iv)–(v), the coefficient is equal to 2 since the term with \(j = 2\) of the sum in \(\Delta_q(r, R)\) is \(q^{(r-2R)/R}\).

• Problem 4 is solved in part, see Section 6.

Throughout the paper we use the following notation:

\[
\phi(q)\text{ is the order of the largest proper subfield of } \mathbb{F}_q; \quad (2.1)
\]

\[
f_q(r, R) = \begin{cases} 
0 & \text{if } r \neq \frac{9R}{2}, \frac{13R}{2} \\
q^{(r-3R)/R-0.5} + q^{(r-4R)/R-0.5} & \text{if } r = \frac{9R}{2}, \frac{13R}{2} 
\end{cases} \quad (2.2)
\]

In Theorem 6.3(v),(vi), using recent results on double blocking set, it is shown that in \(\text{PG}(2, q)\) there are 1-saturating sets of the following sizes:

\[
2\sqrt{q} + 2 \frac{\sqrt{q} - 1}{\phi(\sqrt{q}) - 1}, \quad q = p^{2h}, \quad h \geq 2, \quad p \geq 3 \text{ prime};
\]
\[ 2\sqrt{q} + 2\frac{\sqrt{q}}{p} + 2, \quad q = p^{2h}, \ h \geq 2, \ p \geq 7 \text{ prime}. \]

The new 1-saturating sets have smaller sizes than the known ones, see Remark [6.3]

- Problem (1) is solved in part, see Section [6]. Using the new 1-saturating sets in $PG(2, q)$, we obtained infinite families of codes with covering radius $R = 2$, see Theorem [6.9] and, basing on them, we constructed infinite code families with fixed even radius $R \geq 2$ and growing codimension $tR + \frac{R}{2}$, see Theorem [6.11] that gives rise to Theorem 2.2.

**Theorem 2.2.** Assume that $p$ is prime, $q = p^{2n}$, $\eta \geq 2$, covering radius $R \geq 2$ is even, and code codimension is $r = tR + \frac{R}{2}$ with growing integer $t \geq 1$. Let $\phi(\sqrt{q})$ and $f_q(r, R)$ be as in (2.1), (2.2). The following constructive upper bounds on the length function (provided by infinite families of codes with growing codimension) hold:

(i) \[ \ell_q(r, R) \leq R \left( 1 + \frac{\sqrt{q} - 1}{\sqrt{q}(\phi(\sqrt{q}) - 1)} \right) q^{(r-R)/R} + R \left[ q^{(r-2R)/R-0.5} + \frac{R}{2} f_q(r, R) \right], \ p \geq 3; \]

(ii) \[ \ell_q(r, R) \leq R \left( 1 + \frac{1}{p} + \frac{1}{\sqrt{q}} \right) q^{(r-R)/R} + R \left[ q^{(r-2R)/R-0.5} + \frac{R}{2} f_q(r, R) \right], \ p \geq 7. \]

If $\sqrt{q} = p^n$ with $\eta \geq 3$ odd, the new bounds of Theorem 2.2 are better than the known ones of Proposition [1.7]. For example, if $q = p^6$, $\eta = 3$, then the bound of Theorem 2.2(ii) is by $Rq^{(r-R)/R-1/3}$ smaller than the known one of (1.9). Also, the new bound holds for all $p \geq 7$ whereas in (1.9) $p \leq 73$. Moreover, if $\eta \geq 5$ odd, the known bounds (1.7) have the main term $\frac{3}{2} Rq^{(r-R)/R}$ whereas for the new bounds it is $Rq^{(r-R)/R}$.

3 Construction “Line-Ovals” for $\rho$-saturating sets in $PG(2\rho + 1, q)$ and codes of codimension $2R$

**Notation.** Throughout the paper we denote by $x_i$, $i = 0, 1, \ldots, N$, the homogeneous coordinates of points of $PG(N, q)$. In the other words, a point $(x_0 x_1 \ldots x_N) \in PG(N, q)$. The leftmost nonzero coordinate is equal to 1. In general, by default, $x_i \in \mathbb{F}_q$. If $x_i \in \mathbb{F}_q^*$, we denote it as $\widehat{x}_i$. If $(x_i \ldots x_{i+m}) \neq (0 \ldots 0)$, we denote it as $\overline{x}_i \ldots \overline{x}_{i+m}$. Also, we can write explicit values 0,1 for some coordinates or denote coordinates by letters values of which is explained later.

### 3.1 The construction

Let $\mathbb{F}_q = \{a_1 = 0, a_2, \ldots, a_q\}$ be the Galois field of order $q$. Let $\mathbb{F}_q^* = \mathbb{F}_q \setminus \{0\} = \{a_2, \ldots, a_q\}$. Denote $\Sigma_\rho = PG(2\rho + 1, q)$. Let $\Sigma_u$ be the $(2u+1)$-subspace of $\Sigma_\rho$ such that

\[ \Sigma_u = \{(x_0 x_1 \ldots x_{2u+1} 0 \ldots 0) : x_i \in \mathbb{F}_q\}, \ u = 0, 1, \ldots, \rho. \]
In $\Sigma_u$, let $\pi_u$ be the plane such that
\[
\pi_u = \{(0\ldots0 x_{2u-1} x_{2u} x_{2u+1} 0\ldots0) : x_i \in \mathbb{F}_q\} \subset \Sigma_u, \ u = 1, 2, \ldots, \rho.
\]
In $\pi_u$, let $A_u^0$ and $A_u^\infty$ be the points of the form
\[
A_u^0 = (0\ldots0 100 0\ldots0) \in \pi_u, \ A_u^\infty = (0\ldots0 1a a^2 0\ldots0) \in \pi_u, \ u = 1, 2, \ldots, \rho.
\]
In $\pi_u$, let $C_u$ and $C_u^\ast$ be the conic and the truncated one, respectively, of the form
\[
C_u = C_u^\ast \cup \{A_u^0, A_u^\infty\}, \ C_u^\ast = \{(0\ldots0 1aa^2 0\ldots0) : a \in \mathbb{F}_q^*\}, \ u = 1, 2, \ldots, \rho.
\]
Let $T_u$ be the nucleus of $C_u$, if $q$ is even, or the intersection of the tangents to $C_u$ in $A_u^0$ and $A_u^\infty$, if $q$ is odd, so that
\[
T_u = (0\ldots0 010 0\ldots0) \in \pi_u, \ u = 1, 2, \ldots, \rho.
\]
Finally, in $\Sigma_0$, let $A_0^0$ and $A_0^\infty$ be the points of the form $A_0^0 = (10 0\ldots0)$, $A_0^\infty = (01 0\ldots0)$. Also, let $L_0$ and $L_0^\ast$ be the line and the truncated one, respectively, such that
\[
L_0 = L_0^\ast \cup \{A_0^0, A_0^\infty\} \subset \Sigma_0, \ L_0^\ast = \{(1a 0\ldots0) : a \in \mathbb{F}_q^*\} \subset \Sigma_0.
\]

**Construction S. (“Line-Ovals”)** Let $\rho \geq 0$. Let $S_\rho$ be a point $((\rho + 1)q + 1)$-subset of $\Sigma_\rho$. Let $P_j$ be the $j$-th point of $S_\rho$, $j = 1, 2, \ldots, (\rho + 1)q + 1$. We construct $S_\rho$ as follows:

\[
S_\rho = \{A_0^0\} \cup L_0^\ast \cup \bigcup_{u=1}^{\rho} \left( C_u^\ast \cup \{T_u\} \right) \cup \{A_\rho^\infty\} = \{P_1, P_2, \ldots, P_{(\rho+1)q+1}\} \tag{3.1}
\]
The points $P_j$ of $S$ have the form

$$P_1 = (100\ldots0) = A_0^0, \quad P_j = (1a_j0\ldots0), \quad a_j \in F_q^*, \quad j = 2, 3, \ldots, q; \quad (3.2)$$

$$P_{uq+j-1} = (0\ldots01a_ja_j^20\ldots0), \quad a_j \in F_q^*, \quad u = 1, 2, \ldots, \rho, \quad j = 2, 3, \ldots, q;$$

$$P_{(u+1)q} = (0\ldots00100\ldots0) = T_u, \quad u = 1, 2, \ldots, \rho; \quad P_{(\rho+1)q+1} = A_\rho^\infty.$$

**Example 3.1.** By (3.1), $S_0 = \{A_0^0\} \cup L_0^* \cup \{A_\infty^0\}$, $S_1 = \{A_0^0\} \cup L_0^* \cup C_1^* \cup \{T_1, A_\infty^1\}$, $S_2 = \{A_0^0\} \cup L_0^* \cup C_1^* \cup \{T_1\} \cup C_2^* \cup \{T_2, A_\infty^2\}$. By (3.1), (3.2), we have

$$S_0 = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & a_2 \ldots a_4 & 1 & 0 \\ - & - & - & - \\ A_0^0 & L_0^* & A_\infty^0 \end{pmatrix}, \quad S_1 = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & a_2 \ldots a_q & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ - & - & - & - & - & - & - & - \\ A_0^0 & L_0^* & C_1^* & T_1 & A_\infty^1 \end{pmatrix},$$

$$S_2 = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & a_2 \ldots a_4 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & a_2 \ldots a_q & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & a_2 \ldots a_q & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & a_2 \ldots a_q & 0 & 1 \\ - & - & - & - & - & - & - & - \\ A_0^0 & L_0^* & C_1^* & T_1 & C_2^* & T_2 & A_\infty^2 \end{pmatrix}.$$

### 3.2 Saturation of Construction $S$ for $0 \leq \rho \leq 2$

We say that a point $A \in \text{PG}(N, q)$ is $\rho$-covered by a set $S$ if $A$ is a linear combination of less than or equal to $\rho + 1$ points of $S$. A subset $G \subset \text{PG}(N, q)$ is $\rho$-covered by $S$ if all points of $G$ are $\rho$-covered by $S$.

**Definition 3.2.** Let $S$ be a $\rho$-saturating set in $\text{PG}(N, q)$. A point $A \in S$ is $\rho$-essential if $S \setminus \{A\}$ is no longer a $\rho$-saturating set. A point $A \in S$ is $\rho$-essential for a set $\overline{M}_\rho(A) \subset \text{PG}(N, q)$ if all points of $\overline{M}_\rho(A)$ are not $\rho$-covered by $S \setminus \{A\}$. We denote by $M_\rho(A)$ a set such that $\overline{M}_\rho(A) \subseteq M_\rho(A) \subset \text{PG}(N, q)$.

Note that by Definition 1.8, a 0-saturating set in $\text{PG}(N, q)$ is the whole space.

The following proposition is obvious.
Proposition 3.3. Let $q \geq 3$. Let $\Sigma_0 = \text{PG}(1,q)$. Let the set $\mathcal{S}_0 \subset \Sigma_0$ be as in (3.1), (3.2) see also Example 3.1. Then it holds that

(i) The $(q + 1)$-set $\mathcal{S}_0$ is a minimal 0-saturating set in $\Sigma_0$.

(ii) The point $A_0^\infty$ of $\mathcal{S}_0$ is 0-essential for the set $\tilde{\mathcal{M}}_0(A_0^\infty)$ such that

$$\tilde{\mathcal{M}}_0(A_0^\infty) = \mathcal{M}_0(A_0^\infty) = \{A_0^\infty\} = \{(01)\}. \quad (3.3)$$

(iii) The $q$-set $\mathcal{S}_0 \setminus \{A_0^\infty\}$ is 1-saturating in $\Sigma_0$.

Lemma 3.4. (i) Let $q = 4$ or $q \geq 7$. Then all points of $\pi_u \setminus \{A_u^0, A_u^\infty\}$ are 1-covered by $C_u^* \cup \{T_u\}$, $u = 1, \ldots, \rho$.

(ii) Let $q \geq 4$. Then all points of $\pi_\rho \setminus \{A_\rho^0\}$ are 1-covered by $C_\rho^* \cup \{T_\rho, A_\rho^\infty\}$.

Proof. (i) If $q$ is even, every point of a plane outside of a hyperoval $C_u \cup \{T_u\}$ lies on $(q + 2)/2$ its bisecants. If $q$ is odd, every point of a plane outside of a conic $C_u$ lies on at least $(q - 1)/2$ its bisecants. At most two of aforementioned bisecants will be removed if one removes $A_u^0, A_u^\infty$ from $C_u$. Thus, for $q = 4$ and $q \geq 7$, every point of $\pi_u \setminus \{A_u^0, A_u^\infty\}$ lies on at least one bisecant of $C_u^* \cup \{T_u\}$.

(ii) The proof is similar to the case (i) taking into account that here we remove only one point $A_\rho^0$ from $C_\rho$.\Box

Lemma 3.5. Let $q \geq 4$, $\rho \geq 2$. Then it holds that

(i) The point $A_u^\infty = A_{u+1}^0$, $u = 1, \ldots, \rho - 1$, is 2-covered by $C_u^*$ as well as by $C_{u+1}^*$.

(ii) The plane $\pi_u$, $u = 1, \ldots, \rho$, is 2-covered by $C_u^*$.

Proof. Any three points of a conic generate the plane in which it lies. As $q \geq 4$, we have $\#C_u^* \geq 3$.\Box

Proposition 3.6. Let $q = 4$ or $q \geq 7$. Let $\Sigma_1 = \text{PG}(3,q)$. Let the set $\mathcal{S}_1 \subset \Sigma_1$ be as in (3.1), (3.2), see also Example 3.1. Let $\mathcal{M}_0(A_0^\infty)$ be as in (3.3). Then it holds that

(i) The $(2q + 1)$-set $\mathcal{S}_1$ is a minimal 1-saturating set in $\Sigma_1$.

(ii) The point $A_1^\infty$ of $\mathcal{S}_1$ is 1-essential for the set $\tilde{\mathcal{M}}_1(A_1^\infty)$ such that

$$\tilde{\mathcal{M}}_1(A_1^\infty) = \mathcal{M}_1(A_1^\infty) = \{(x_0 \ldots x_3) : (x_0x_1) \notin \mathcal{M}_0(A_0^\infty), (x_2x_3) = (x_3)\}. \quad (3.4)$$

(iii) The 2-set $\mathcal{S}_1 \setminus \{A_1^\infty\}$ is 2-saturating in $\Sigma_1$.

Proof. (i) By Proposition 3.3 (iii) and Lemma 3.4, $\Sigma_0$ consists of points $(x_0x_100)$ and $\pi_1$ (points $(0x_1x_2x_3)$) are 1-covered by $\{A_0^0\} \cup L_0^* \cup C_1 \cup \{T_1, A_1^\infty\}$. So, we should consider points of the form

$$B = (\bar{x}_0x_1\bar{x}_2x_3) = (1x_1\bar{x}_2x_3) \in \Sigma_1 \setminus (\Sigma_0 \cup \pi_1). \quad (3.5)$$

We show that $B$ in (3.5) is a linear combination of at most 2 points of $\mathcal{S}_1$.\Box
1) Let \((x_0 x_1) \in \mathcal{M}_0(A_0^\infty)\).

By the hypothesis, \((x_0 x_1) = (01)\). By (3.5), we have no such points \(B\).

2) Let \((x_0 x_1) \notin \mathcal{M}_0(A_0^\infty)\).

By the hypothesis, \((x_0 x_100)\) is 0-covered by \(S_0 \setminus \{A_0^\infty\}\), i.e. \((x_0 x_100) = (1 x_100) \in \{A_0^0\} \cup \mathcal{L}_0^\ast\).

For \(B\) of (3.5), we have

\[
B = (x_0 x_10 \hat{x}_3) = (x_0 x_100) + \hat{x}_3 (0001) = (x_0 x_100) + \hat{x}_3 A_1^\infty; \tag{3.6}
\]

\[
B = (x_0 x_1 \hat{x}_2 0) = (x_0 x_100) + \hat{x}_2 (0010) = (x_0 x_100) + \hat{x}_2 T_1;
\]

\[
B = (x_0 x_1 \hat{x}_2 \hat{x}_3) = (x_0 z00) + \frac{\hat{x}_2}{\hat{x}_3} (01yy^2), \ z = x_1 - \frac{\hat{x}_2}{\hat{x}_3}, \ y = \frac{\hat{x}_3}{\hat{x}_2}.
\]

Note that \((x_0 z00) = (1z00)\) is 0-covered by \(S_0 \setminus \{A_0^\infty\}\) for any \(z\).

From (3.6), we see that all points of \(S_1\) are 1-essential.

(ii) The assertion follows from (3.6).

(iii) We have, cf. (3.6), \((x_1 (0 \hat{x}_3) = (1z00) + (010\hat{x}_3)\), where \(z = x_1 - 1\) and \((010\hat{x}_3) \in \pi_1 \setminus \{A_0^0, A_1^\infty\}\) is 1-covered by \(C_1^* \cup \{T_1\}\), see Lemma 3.4. \(\square\)

**Proposition 3.7.** Let \(q = 4\) or \(q \geq 7\). Let \(\Sigma_2 = PG(5, q)\). Let the set \(S_2 \subset \Sigma_2\) be as in (3.1), (3.2), see also Example 3.1. Let \(\mathcal{M}_1(A_1^\infty)\) be as in (3.4). Then it holds that

(i) The \((3q + 1)\)-set \(S_2\) is a minimal 2-saturating set in \(\Sigma_2\).

(ii) The point \(A_2^\infty\) of \(S_2\) is 2-essential for the set \(\overline{\mathcal{M}_2(A_2^\infty)}\) such that

\[
\overline{\mathcal{M}_2(A_2^\infty)} \subset \mathcal{M}_2(A_2^\infty) = \{ (x_0 \ldots x_3) : (x_0 \ldots x_3) \notin \mathcal{M}_1(A_1^\infty), (x_4 x_5) = (0 \hat{x}_5) \}. \tag{3.7}
\]

(iii) The \(3q\)-set \(S_2 \setminus \{A_2^\infty\}\) is 3-saturating in \(\Sigma_2\).

**Proof.**

(i) By Propositions 3.3, 3.6 and Lemmas 3.4, 3.5, we have the following: \(\Sigma_0\) (points \((x_0 x_10000)\)) is 1-covered by \(\{A_0^0\} \cup \mathcal{L}_0^\ast\); \(\pi_1\) (points \((0 x_1 x_2 x_30)\)) and \(\pi_2\) (points \((000 x_3 x_4 x_5)\)) are 2-covered by \(C_1^*\) and \(C_2^*\), respectively; \(\pi_2 \setminus \{A_2^0\}\) is 1-covered by \(C_2^* \cup \{T_2, A_2^\infty\}\); \(\Sigma_1\) (points \((x_0 x_1 x_2 x_3000)\)) is 2-covered by \(S_1 \setminus \{A_1^\infty\}\). Recall that \(\Sigma_0 \cup \pi_1 \subset \Sigma_1\). So, we should consider points of the form

\[
B = (x_0 x_1 x_2 x_3 x_4 x_5) \in \Sigma_2 \setminus (\Sigma_1 \cup \pi_2). \tag{3.8}
\]

We show that \(B\) in (3.8) is a linear combination of at most 3 points of \(S_2\).

1) Let \((x_0 \ldots x_3) \in \mathcal{M}_1(A_1^\infty)\).

By the hypothesis and by (3.4), (3.8), we have

\[
(x_0 x_1) \notin \mathcal{M}_0(A_0^\infty), \ B = (x_0 x_10 \hat{x}_3 x_4 x_5) = (x_0 x_10000) + (000 \hat{x}_3 x_4 x_5),
\]

where \((x_0 x_10000)\) is 0-covered by \(S_0 \setminus \{A_0^0\}\) and \((000 \hat{x}_3 x_4 x_5) \in \pi_2 \setminus \{A_2^0, A_2^\infty\}\) is 1-covered by \(C_2^* \cup \{T_2\}\), see Lemma 3.4.

2) Let \((x_0 \ldots x_3) \notin \mathcal{M}_1(A_1^\infty)\).

By the hypothesis, \((x_0 \ldots x_30)\) is 1-covered by \(S_1 \setminus \{A_1^\infty\}\). We can write

\[
B = (x_0 \ldots x_30 \hat{x}_5) = (x_0 \ldots x_300) + \hat{x}_5 (000001) = (x_0 \ldots x_300) + \hat{x}_5 A_2^\infty; \tag{3.9}
\]

13
B = (x_0 \ldots x_3 \hat{x}_4) = (x_0 \ldots x_300) + \hat{x}_4(00010) = (x_0 \ldots x_300) + \hat{x}_4T_2; \quad (3.10)
B = (x_0 \ldots x_3 \hat{x}_4 \hat{x}_5) = (x_0x_1x_2z00) + \frac{\hat{x}_4^2}{\hat{x}_5^2}(0001yy^2), \ z = x_3 - \frac{\hat{x}_4^2}{\hat{x}_5^2}, \ y = \frac{\hat{x}_5}{\hat{x}_4}. \quad (3.11)

In (3.9), (3.10), B is a linear combination of at most \((1+1) + 1 = 3\) points. If \((x_0x_1x_2z) \notin \mathcal{M}_1(A_1^\infty)\), then the representation (3.11) is the needed linear combination. If \((x_0x_1x_2z) \in \mathcal{M}_1(A_1^\infty)\) whereas \((x_0 \ldots x_3) \notin \mathcal{M}_1(A_1^\infty)\), then the only possible situation is \((x_0x_1) \notin \mathcal{M}_0(A_0^\infty)\) with \((x_2x_3) = (00)\), see (3.4). In this case,

\[
B = (x_0x_100\hat{x}_4\hat{x}_5) = (1x_10000) + (0000\hat{x}_4\hat{x}_5), \quad (3.12)
\]

where \((1x_10000)\) is 0-covered by \(\{A_0^0\} \cup L_0^*\) and \((0000\hat{x}_4\hat{x}_5) \in \pi_2 \setminus \{A_2^0, A_2^\infty\}\) is 1-covered by \(C_2^\infty \cup \{T_2\}\), see Lemma 3.4. Thus, \(B\) in (3.12) is a linear combination of at most \((0+1) + (1+1) = 3\) points.

From (3.9)–(3.12) we see that all points of \(S_2 \setminus S_1\) are 2-essential. Also, we take into account that \(S_1\) is a minimal 1-saturating set.

(ii) The assertion follows from (3.9). For some (but not for all) points in (3.9) we could avoid use of \(A_2^\infty\); this explains the sign "\(\subset\)" in (3.7). For example, let \(B = (001\hat{x}_30\hat{x}_5) \notin \mathcal{M}_1(A_1^\infty)\). Then \(B = (001000) + \hat{x}_3 \left(0001\frac{\hat{x}_3}{x_3}\right)\), where \((001000) = T_1\) and \((0001\frac{\hat{x}_3}{x_3}) \in \pi_2 \setminus \{A_2^0, A_2^\infty\}\) is 1-covered by \(C_2^\infty \cup \{T_2\}\), see Lemma 3.4. However, if \(B = (00100\hat{x}_5) \notin \mathcal{M}_1(A_1^\infty)\), we are not able to avoid use of \(A_2^\infty\).

(iii) We have, cf. (3.9), \(B = (x_0 \ldots x_30\hat{x}_5) = (x_0x_1x_2z00) + (00010\hat{x}_5)\), where \(z = x_3 - 1\) and \((00010\hat{x}_5) \in \pi_2 \setminus \{A_2^0, A_2^\infty\}\) is 1-covered by \(C_2^\infty \cup \{T_2\}\), see Lemma 3.4. This representation of \(B\) is the needed linear combination of at most \((1+1) + (1+1) = 4\) columns if \((x_0x_1x_2z) \notin \mathcal{M}_1(A_1^\infty)\) whence \((x_0x_1x_2z00)\) is 1-covered by \(S_1 \setminus \{A_2^0\}\).

But if \((x_0x_1x_2z) \in \mathcal{M}_1(A_1^\infty)\), then by (3.4), \((x_0x_1) \notin \mathcal{M}_0(A_0^\infty)\) and we have, similarly to (3.12), \(B = (1x_1000\hat{x}_5) = (1x_10000) + \hat{x}_5(000001)\), where \((1x_1000)\) is 0-covered by \(\{A_0^0\} \cup L_0^*\) and \((000001) = A_2^\infty \in \pi_2\) is 2-covered by \(C_2^\infty\), see Lemma 3.5.

3.3 Saturation of Construction S for any \(\rho\)

Theorem 3.8. Let \(q = 4\) or \(q \geq 7\). Let \(\Upsilon \geq 1\). Let \(\Sigma_\rho = \text{PG}(2p+1, q)\). Let \(S_\rho\) be a point \(((\rho + 1)q + 1)\)-subset of \(\Sigma_\rho\) as in Construction S of (3.1), (3.2). Then it holds that

(i) The point \((\rho + 1)q + 1\)-set \(S_\rho\) is a minimal \(\rho\)-saturating set in \(\Sigma_\rho\), \(\rho = 0, 1, \ldots, \Upsilon\).

(ii) The point \(A_\rho^\infty\) of \(S_\rho\) is \(\rho\)-essential for the set \(\tilde{M}_\rho(A_\rho^\infty)\) such that

\[
\tilde{M}_0(A_0^\infty) = M_0(A_0^\infty) = \{(01)\},
\tilde{M}_1(A_1^\infty) = M_1(A_1^\infty) = \{(x_0 \ldots x_3): (x_0x_1) \notin M_0(A_0^\infty), (x_2x_3) = (0\hat{x}_3)\},
\tilde{M}_\rho(A_\rho^\infty) \subset M_\rho(A_\rho^\infty) = \{(x_0 \ldots x_{2\rho-1}): (x_0 \ldots x_{2\rho-1}) \notin M_{\rho-1}(A_{\rho-1}^\infty), (x_{2\rho}x_{2\rho+1}) = (0\hat{x}_{2\rho+1})\}, \quad \rho = 2, 3, \ldots, \Upsilon. \quad (3.13)
\]
(iii) The \((\rho + 1)q\)-set \(S_\rho \setminus \{A^\infty_\rho\}\) is \((\rho + 1)\)-saturating in \(\Sigma_\rho\), \(\rho = 0, 1, \ldots, \Upsilon\).

Proof. We prove by induction on \(\Upsilon\).

For \(\Upsilon = 3\) the theorem is proved in Propositions 3.3, 3.6, 3.7.

Assumption: let the assertions (i)–(iii) hold for some \(\Upsilon \geq 3\).

We show that under Assumption, the assertions hold for \(\Gamma = \Upsilon + 1\). (i) By Propositions 3.3, 3.6, 3.7, Lemmas 3.4, 3.5 and Assumption, we have the following: \(\Sigma_0\) (points \((x_0x_10\ldots0)\)) is 1-covered by \(\{A^0_0 \cup L^*_0\}; \pi_1 \setminus \{A^\infty_1\}, \pi_u \setminus \{A^0_u, A^\infty_u\}\), \(u = 2, 3, \ldots, \Upsilon\), are 1-covered by \(\{A^0_u \cup L^*_0 \cup \bigcup\limits_{u=1}^\Upsilon (C^*_u \cup \{T_u\}); \pi_\Upsilon \setminus \{A^0_\Upsilon\}\) is 1-covered by \(C^*_\Upsilon \cup \{T_\Upsilon, A^\infty_\Upsilon\}\); \(\pi_1\) (points \((0x_1x_20\ldots0)\)), \(\pi_2\) (points \((000x_3x_40\ldots0)\)), \ldots, \(\pi_\Upsilon\) (points \((0\ldots0x_{2\Gamma-1}x_{2\Gamma}x_{2\Gamma+1})\)) are 2-covered by \(C^*_1, C^*_2, \ldots, C^*_\Upsilon\), respectively; \(\Sigma_\Upsilon\) is \(\Gamma\)-covered by \(S_\Upsilon \setminus \{A^\infty_\Upsilon\}\). Recall that \(\Sigma_0 \cup \bigcup\limits_{u=1}^\Upsilon \pi_u \subset \Sigma_\Upsilon\). So, we should consider points of the form

\[
B = (x_0 \ldots x_{2\Gamma-2}x_{2\Gamma-1}x_{2\Gamma}x_{2\Gamma+1}) \in \Sigma_\Upsilon \setminus (\Sigma_\Upsilon \cup \pi_\Upsilon). \tag{3.14}
\]

We show that \(B\) in (3.14) is a linear combination of at most \(\Gamma + 1\) points of \(S_\Upsilon\).

1) Let \((x_0 \ldots x_{2\Gamma-1}) \in M_\Upsilon(A^\infty_\Upsilon)\).

By the hypothesis and by (3.13), \((x_0 \ldots x_{2\Gamma-1}) \notin M_{\Upsilon-1}(A^\infty_{\Upsilon-1})\). Therefore, \((x_0 \ldots x_{2\Gamma-1}0000)\) is \((\Upsilon - 1)\)-covered by \(S_{\Upsilon-1} \setminus \{A^\infty_{\Upsilon-1}\}\). Now by (3.14), we have

\[
B = (x_0 \ldots x_{2\Upsilon-1}0\tilde{x}_{2\Upsilon-1}x_{2\Upsilon}x_{2\Upsilon+1}) = (x_0 \ldots x_{2\Upsilon-1}0000) + (0 \ldots 0\tilde{x}_{2\Upsilon-1}x_{2\Upsilon}x_{2\Upsilon+1}), \tag{3.15}
\]

where \((0 \ldots 0\tilde{x}_{2\Upsilon-1}x_{2\Upsilon}x_{2\Upsilon+1}) \in \Sigma_\Upsilon \setminus \{A^0_\Upsilon, A^\infty_\Upsilon\}\) is 1-covered by \(C^*_\Upsilon\), see Lemma 3.4. Thus, \(B\) in (3.15) is a linear combination of at most \((\Upsilon - 1 + 1) + (1 + 1) = \Gamma + 1\) points.

2) Let \((x_0 \ldots x_{2\Gamma-1}) \notin M_\Upsilon(A^\infty_\Upsilon)\).

By the hypothesis, \((x_0 \ldots x_{2\Gamma-1}0)\) is \(\Upsilon\)-covered by \(S_\Upsilon \setminus \{A^\infty_\Upsilon\}\). We can write

\[
B = (x_0 \ldots x_{2\Gamma-1}0\tilde{x}_{2\Gamma+1}) = (x_0 \ldots x_{2\Gamma-1}000) + \tilde{x}_{2\Gamma+1}A^\infty_{\Upsilon}; \tag{3.16}
\]

\[
B = (x_0 \ldots x_{2\Gamma-1}\tilde{x}_{2\Gamma}0) = (x_0 \ldots x_{2\Gamma-1}000) + \tilde{x}_{2\Gamma}T_\Upsilon; \tag{3.17}
\]

\[
B = (x_0 \ldots x_{2\Gamma-1}\tilde{x}_{2\Gamma}\tilde{x}_{2\Gamma+1}) = (x_0 \ldots x_{2\Gamma-2}000) + \frac{\tilde{x}_{2\Gamma}^2}{\tilde{x}_{2\Gamma+1}}(0 \ldots 0yy^2), \tag{3.18}
\]

\[
z = x_{2\Gamma-1} - \frac{\tilde{x}_{2\Gamma}^2}{\tilde{x}_{2\Gamma+1}}, \quad y = \frac{\tilde{x}_{2\Gamma+1}}{\tilde{x}_{2\Gamma}}.
\]

In (3.16), (3.17), \(B\) is a linear combination of at most \((\Upsilon + 1) + 1 = \Gamma + 1\) points. If \((x_0 \ldots x_{2\Gamma-2}0) \notin M_\Upsilon(A^\infty_\Upsilon)\), then the representation (3.18) is the needed linear combination. If \((x_0 \ldots x_{2\Gamma-2}0) \in M_\Upsilon(A^\infty_\Upsilon)\) while \((x_0 \ldots x_{2\Gamma-1}0) \notin M_\Upsilon(A^\infty_\Upsilon)\), then the only possible situation is \((x_0 \ldots x_{2\Gamma-1}) \notin M_{\Upsilon-1}(A^\infty_{\Upsilon-1})\) with \((x_{2\Gamma-2}x_{2\Gamma-1}) = (00)\), see (3.13). In this case,

\[
B = (x_0 \ldots x_{2\Upsilon-1}000\tilde{x}_{2\Upsilon}\tilde{x}_{2\Upsilon+1}) = (x_0 \ldots x_{2\Upsilon-1}0000) + (0 \ldots 0\tilde{x}_{2\Upsilon}\tilde{x}_{2\Upsilon+1}), \tag{3.19}
\]
where \((x_0\ldots x_{2^\Gamma-1}0000)\) is \((\Upsilon - 1)\)-covered by \(S_{\Upsilon - 1} \setminus \{A^\infty_{\Upsilon - 1}\}\) and \((0\ldots 0\hat{x}_{2^\Gamma+1})\in\pi_{\Upsilon}\setminus\{A^0_{\Upsilon}, A^\infty_{\Upsilon}\}\) is 1-covered by \(C^*_{\Upsilon} \cup \{T_{\Upsilon}\}\), see Lemma 3.4. Thus, \(B\) in (3.19) is a linear combination of at most \((\Upsilon + 1) + (1 + 1) = \Gamma + 2\) points if (\((x_0\ldots x_{2^\Gamma-2}z00)\in\pi_{\Upsilon}\setminus\{A^0_{\Upsilon}, A^\infty_{\Upsilon}\}\) is 1-covered by \(C^*_{\Upsilon}\), see Lemma 3.4. This representation of \(B\) is the needed linear combination of at most \((\Upsilon + 1) + (1 + 1) = \Gamma + 2\) points if \((x_0\ldots x_{2^\Gamma-2}z)\notin M_{\Upsilon}(A^\infty_{\Upsilon})\) whence \((x_0\ldots x_{2^\Gamma-2}z00)\) is \(\Upsilon\)-covered by \(S_{\Upsilon} \setminus A^\infty_{\Upsilon}\).

But if \((x_0\ldots x_{2^\Gamma-2}z)\in M_{\Upsilon}(A^\infty_{\Upsilon})\), then by (3.13), \((x_0\ldots x_{2^\Upsilon-1}000)\notin M_{\Upsilon-1}(A^\infty_{\Upsilon-1})\), and we have, cf. (3.19), \((x_0\ldots x_{2^\Upsilon-1}0000)\in\pi_{\Upsilon}\setminus\{A^0_{\Upsilon}, A^\infty_{\Upsilon}\}\) is 2-covered by \(C^*_{\Upsilon}\), see Lemma 3.5.

By computer search for \(q = 5\) we have proved the following proposition.

**Proposition 3.9.** Let \(q = 5\). Let \(0 \leq \rho \leq 4\). Let \(\Sigma_\rho = PG(2\rho + 1, 5)\). Let the \((5\rho + 1)\)-set \(S_\rho \subset \Sigma_\rho\) be as in (3.1), (3.2). Then \(S_\rho\) is a minimal \(\rho\)-saturating set in \(\Sigma_\rho\).

### 3.4 Codes of covering radius \(R\) and codimension \(2R\)

In the coding theory language, the results of this section give the following theorem.

**Theorem 3.10.** Let \(\hat{V}_\rho\) be the code such that the columns of its parity check matrix are the points (in the homogeneous coordinates) of the \(\rho\)-saturating \(((\rho + 1)q + 1)\)-set \(S_\rho\) of Construction S (3.1), (3.2).

(i) Let \(q = 4\) or \(q \geq 7\). Then for all \(R \geq 1\), the code \(\hat{V}_\rho\) is a \([Rq + 1, Rq + 1 - 2R, 3]_qR\) locally optimal code of covering radius \(R = \rho + 1\).

(ii) Let \(q = 5\). Then for \(1 \leq R \leq 5\), the code \(\hat{V}_\rho\) is a \([5R + 1, 5R + 1 - 2R, 3]_5R\) locally optimal code of covering radius \(R = \rho + 1\).

**Proof.** We use Theorem 3.8 and Proposition 3.9. The code \(\hat{V}_\rho\) is locally optimal as the corresponding \(\rho\)-saturating set \(S_\rho\) is minimal. Minimum distance \(d = 3\) is due to \(L^*_\rho\). 

**Conjecture 3.11.** (i) Let \(q = 5\). Let \(\Sigma_\rho = PG(2\rho + 1, 5)\). Let the \((5\rho + 1)\)-set \(S_\rho \subset \Sigma_\rho\) be as in (3.1), (3.2). Then for all \(\rho \geq 0\) it holds that \(S_\rho\) is a minimal \(\rho\)-saturating set in \(\Sigma_\rho\).

(ii) Let \(q = 5\). Let \(\hat{V}_\rho\) be as in Theorem 3.10. Then for all \(R \geq 1\), the code \(\hat{V}_\rho\) is a \([5R + 1, 5R + 1 - 2R, 3]_5R\) locally optimal code with radius \(R = \rho + 1\).
The $q^m$-concatenating constructions for code codimension lifting

The $q^m$-concatenating constructions are proposed in [9] and are developed in [10–12, 14, 17, 18], see also [5, 7, Sec. 5.4] and the references in these works. By using a starting code as a “seed”, a $q^m$-concatenating construction yields an infinite family of new codes with a fixed covering radius, growing codimension and with almost the same covering density.

We give versions of the $q^m$-concatenating constructions convenient for our goals. Several other versions of such constructions can be found in [9–12, 14, 17, 18] and the references therein. In Construction QM1 below, we use a surface-covering code as a starting one, whereas for Construction QM2 we need to start with an $[n, n - r]_q R, \ell$ code, $\ell = R - 1$. Resulting codes of both the constructions are surface-covering.

Construction QM1. Let columns $h_j$ belong to $\mathbb{F}_q^{r_0}$ and let $H_0 = [h_1, h_2, \ldots, h_{n_0}]$ be a parity check matrix of an $[n_0, n_0 - r_0]_q R, R$ starting surface-covering code $V_0$ with $R \geq 2$. Let $m \geq 1$ be an integer such that $q^m \geq n_0 - 1$. To each column $h_j$ we associate an element $\beta_j \in \mathbb{F}_q \cup \{\ast\}$ so that $\beta_i \neq \beta_j$ if $i \neq j$. Let a new code $V$ be the $[n, n - (r_0 + Rm)]_q R, \ell$ code with $n = q^m n_0$ and parity check matrix of the form

$$H_V = [B_1, B_2, \ldots, B_{n_0}],$$

$$B_j = \begin{bmatrix} h_j & h_j & \cdots & h_j \\ \xi_1 & \xi_2 & \cdots & \xi_q^m \\ \beta_1 \xi_1 & \beta_2 \xi_2 & \cdots & \beta_j \xi_q^m \\ \vdots & \vdots & \cdots & \vdots \\ \beta_j^{-R-1} \xi_1 & \beta_j^{-R-1} \xi_2 & \cdots & \beta_j^{-R-1} \xi_q^m \end{bmatrix} \quad \text{if } \beta_j \in \mathbb{F}_q^m, \quad B_j = \begin{bmatrix} h_j & h_j & \cdots & h_j \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & 0 \\ \xi_1 & \xi_2 & \cdots & \xi_q^m \end{bmatrix} \quad \text{if } \beta_j = \ast,$$

where $B_j$ is an $(r_0 + Rm) \times q^m$ matrix, 0 is the zero element of $\mathbb{F}_q^m$, $\xi_u$ is an element of $\mathbb{F}_q^m$, $\{\xi_1, \xi_2, \ldots, \xi_q^m\} = \mathbb{F}_q^m$. An element of $\mathbb{F}_q^m$ written in $B_j$ denotes an $m$-dimensional $q$-ary column vector that is a $q$-ary representation of this element.

We denote $b_j(\xi_u) = (h_j, \xi_u, \beta_j \xi_u, \beta_j^2 \xi_u, \ldots, \beta_j^{R-1} \xi_u)$ the $u$-th column of $B_j$ with $\beta_j \in \mathbb{F}_q^m$. If $\beta_j = \ast$, we have $b_j(\xi_u) = (h_j, 0, \ldots, 0, \xi_u)$.

Theorem 4.1. In Construction QM1, the new code $V$ with the parity check matrix \(4.1)\), \(4.2)\) is an $[n, n - (r_0 + Rm), 3]_q R, R$ surface-covering code with covering radius $R$ and length $n = q^m n_0$. Moreover, if the starting code $V_0$ is locally optimal (non-shortening), then the new code $V$ is locally optimal too.

Proof. The length of the code $V$ directly follows from the construction.

The minimum distance $d$ is equal to 3 since for any pair of columns $b_j(\xi_{u_1}), b_j(\xi_{u_2})$ of $B_j$, a 3-rd one can be found such that the column triple corresponds to a codeword of
weight 3. Take \(a, b, c \in \mathbb{F}_q^*\) with \(a + b + c = 0\). Put \(\xi_{u_3} = (-a\xi_{u_1} - b\xi_{u_2})/c\). Then for all \(j\) we have
\[
ab\ell_j(\xi_{u_1}) + bb\ell_j(\xi_{u_2}) + c\ell_j(\xi_{u_3}) = 0,
\]
where \(\mathbf{0}\) is the zero \((r_0 + Rm)\)-positional column.

We show that covering radius \(R_V\) of \(V\) is equal to \(R\).

Consider an arbitrary column \(t = (fs) \in \mathbb{F}_{r_0+Rm}\) with \(f \in \mathbb{F}_{r_0}^q, s \in \mathbb{F}_{Rm}^q, s = (s_1, s_2, \ldots, s_{Rm})\), \(s_i \in \mathbb{F}_q\). We partition \(s\) by \(m\)-vectors so that \(s = (S_0, S_1, \ldots, S_{R-1})\), \(S_v = (s_{vm+1}, s_{vm+2}, \ldots, s_{vm+m})\), \(v = 0, 1, \ldots, R - 1\). We treat \(S_v\) as an element of \(\mathbb{F}_{qm}\).

Since \(V_0\) is an \([n_0, n_0 - r_0]qR, R\) code, there exists a linear combination of the form
\[
f = \sum_{k=1}^{R} c_k h_{jk}, \ c_k \in \mathbb{F}_q^* \text{ for all } k,
\]
see Definition [1.4] Now we can represent \(t\) as a linear combination (with nonzero coefficients) of \(R\) distinct columns of \(H_V\). We have, see (4.2),
\[
t = \sum_{k=1}^{R} c_k b_{jk}(x_k), \ c_k \in \mathbb{F}_q^* \text{ and } x_k \in \mathbb{F}_q^m \text{ for all } k,
\]
where values of \(x_k\) are obtained from the linear system with nonzero determinant. If for \(j_k\) in (4.4) we have \(\beta_{jk} \in \mathbb{F}_{qm}\) for all \(k\), then the system has the form
\[
\sum_{k=1}^{R} c_k \beta_{jk} x_k = S_v, \ v = 0, 1, \ldots, R - 1.
\]
(4.6)
As usual, we put \(0^0 = 1\). If in (4.4) we have, for example, \(\beta_{jR} = \ast\), then the system is as follows:
\[
\sum_{k=1}^{R-1} c_k \beta_{jk} x_k = S_v, \ v = 0, 1, \ldots, R - 2; \ \sum_{k=1}^{R-1} c_k \beta_{jk}^{R-1} x_k + c_R x_R = S_{R-1}.
\]
(4.7)
If \(V_0\) is a locally optimal code, then every column \(h_j\) of \(H_0\) takes part in a representation of the form (4.4). If we remove \(b_{jk}(\xi_u)\) from \(B_{jk}\) then there is \((s_1, s_2, \ldots, s_{Rm})\) such that the system (4.6) or (4.7) gives \(x_k = \xi_u\). As a result, for some \(t\) the representation (4.5) becomes impossible. So, all columns of \(H_V\) are essential and the code \(V\) is locally optimal. □

**Construction QM2.** Let \(\theta_{m,q} = q^{m+1} - 1\). Let columns \(h_j\) belong to \(\mathbb{F}_{q^0}\) and let \(H_0 = [h_1 h_2 \ldots h_{n_0}]\) be a parity check matrix of an \([n_0, n_0 - r_0]qR, \ell_0\) \(\ast\) \(\ast\) starting code \(V_0\) with \(\ell_0 = R - 1, R \geq 2\). Let \(m \geq 1\) be an integer such that \(q^m \geq n_0\). To each column \(h_j\)
we associate an element $\beta_j \in \mathbb{F}_{q^m}$ so that $\beta_i \neq \beta_j$ if $i \neq j$. Let a new code $V$ be the $(n, n - (r_0 + Rm)]_q R V, \ell _V$ code with $n = q^m n_0 + \theta_{m,q}$ and parity check matrix of the form

\[
H_V = [C \ B_1 \ B_2 \ldots \ B_{n_0}],
\]

(4.8)

where $B_j$ is an $(r_0 + Rm) \times q^m$ matrix as in (4.2), $C$ is an $(r_0 + Rm) \times \theta_{m,q}$ matrix,

\[
C = \begin{bmatrix} 0_{r_0+(R-1)m} \\ W_m \end{bmatrix},
\]

(4.9)

$0_{r_0+(R-1)m}$ is the zero $(r_0 + (R - 1)m) \times \theta_{m,q}$ matrix, $W_m$ is a parity check $m \times \theta_{m,q}$ matrix of the $[\theta_{m,q}, \theta_{m,q} - m, 3]_q 1$ Hamming code.

**Theorem 4.2.** In Construction QM2, the new code $V$ with the parity check matrix (4.8), (4.9), (4.2) is an $[n, n - (r_0 + Rm), 3]_q R, R$ surface-covering code with covering radius $R$ and length $n = q^m n_0 + \frac{q^{m+1} - 1}{q-1}$. Moreover, if the starting code $V_0$ is locally optimal (non-shortening), then the new code $V$ is locally optimal too.

**Proof.** The length of the code $V$ directly follows from the construction.

The minimum distance is equal to 3 as the Hamming code is a code with $d = 3$. Also we can use (4.3) from the proof of Theorem 4.1.

We show that covering radius $R_V$ of $V$ is equal to $R$.

Consider an arbitrary column $t = (fs) \in \mathbb{F}_{q^{r_0+Rm}}$ with $f \in \mathbb{F}_{q^{r_0}}, \ s \in \mathbb{F}_{q^{Rm}}, \ s = (s_1, s_2, \ldots, s_{Rm}), \ s_i \in \mathbb{F}_q$. We partition $s$ by $m$-vectors so that $s = (S_0, S_1, \ldots, S_{R-1}), \ S_v = (s_{vm+1}, s_{vm+2}, \ldots, s_{vm+m}), \ v = 0, 1, \ldots, R - 1$. We treat $S_v$ as an element of $\mathbb{F}_{q^m}$.

Since $V_0$ is an $[n_0, n_0 - r_0]_q R, \ell_0$ code with $\ell_0 = R - 1$, there exists a linear combination of $\varphi(f)$ distinct columns of $H_0$ of the form

\[
f = \sum_{k=1}^{\varphi(f)} c_k h_{jk}, \ c_k \in \mathbb{F}_q^* \text{ for all } k, \varphi(f) \in \{ R - 1, R \},
\]

see Definition 1.4. If $\varphi(f) = R$ we act similarly to the proof of Theorem 4.1.

Let $\varphi(f) = R - 1$. We represent $t$ as a linear combination (with nonzero coefficients) of at most $R$ distinct columns of $H_V$. We have, see (4.2), (4.9),

\[
t = \eta c + \sum_{k=1}^{R-1} c_k b_{jk} (x_k), \ c_k \in \mathbb{F}_q^* \text{ and } x_k \in \mathbb{F}_{q^m} \text{ for all } k, \eta \in \mathbb{F}_q,
\]

(4.10)

where $c$ is a column of $C$ and $\eta = 0$ means that the summand $\eta c$ is absent. Also, in (4.10), values of $x_k$ are obtained from the linear system

\[
\sum_{k=1}^{R-1} c_k \beta_{jk}^v x_k = S_v, \ v = 0, 1, \ldots, R - 2,
\]

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with nonzero determinant. Finally, in (4.10), \( \mathbf{c} = (0_\mathbf{w}) \) where \( 0_\mathbf{w} \) is the zero \((r_0 + (R - 1)m)\)-positional column and \( \mathbf{w} \) is a column of \( \mathbf{W}_m \) that satisfies the equality

\[
\eta\mathbf{w} + \sum_{k=1}^{R-1} c_k \beta_{j_k}^{R-1} x_k = S_{R-1}.
\]  

(4.11)

In (4.11), if \( \sum_{k=1}^{R-1} c_k \beta_{j_k}^{R-1} x_k = S_{R-1} \) we have \( \eta = 0 \). If \( \sum_{k=1}^{R-1} c_k \beta_{j_k}^{R-1} x_k \neq S_{R-1} \), the needed column \( \eta\mathbf{w} \) always exists as the Hamming code has covering radius 1.

Now we show that \( \mathbf{V} \) is an \([n, n-(r_0 + Rm), 3]_q R, R \) code, i.e. \( \ell_{\mathbf{V}} = R \). The critical situation is when in (4.10) and (4.11) \( \eta = 0 \), i.e. the summand \( \eta\mathbf{c} \) is absent. We use the approach of the proof of Theorem 4.1 regarding (4.3). In (4.3) we put \( j = j_1, \xi_{u_1} = x_1, a = -c_1 \) with \( j_1, x_1, c_1 \) taken from (4.10). Then

\[
\mathbf{t} = -c_1 b_{j_1}(x_1) + b b_{j_1}(\xi_{u_2}) + c b_{j_1}(\xi_{u_3}) + \sum_{k=1}^{R-1} c_k b_{j_k}(x_k)
\]

\[
= b b_{j_1}(\xi_{u_2}) + c b_{j_1}(\xi_{u_3}) + \sum_{k=2}^{R-1} c_k b_{j_k}(x_k).
\]

Thus, we always can represent \( \mathbf{t} \in \mathbb{F}_{r_0+Rm}^q \) as a linear combination with nonzero coefficients of exactly \( R \) columns of \( \mathbf{H}_\mathbf{V} \).

By above, if we remove any column of \( \mathbf{H}_\mathbf{V} \), some representation of \( \mathbf{t} \) becomes impossible. So, all columns of \( \mathbf{H}_\mathbf{V} \) are essential and the code \( \mathbf{V} \) is locally optimal. \( \square \)

5 New infinite code families with fixed radius \( R \geq 4 \) and growing codimension \( tR \)

In the minimal \( \rho \)-saturating set of Construction S (3.1), (3.2), we consider a point \( P_j \) (in the homogeneous coordinates) as a column \( \mathbf{h}_j \) of the parity check matrix \( \mathbf{\tilde{H}}_\rho \) that defines the \([qR + 1, qR + 1 - 2R, 3]_q R, \ell \) locally optimal code \( \mathbf{\tilde{V}}_\rho \) of covering radius \( R = \rho + 1 \).

We consider some properties of \( \mathbf{\tilde{H}}_\rho \) useful to estimate \( \ell \). Let \( \mathbf{f} \in \mathbb{F}_q^r \). Let \( \mathcal{J}(\mathbf{f}) = \{\mathbf{h}_{j_1}, \ldots, \mathbf{h}_{j_\beta}\} \) and \( \mathcal{I}_w = \{\mathbf{h}_{i_1}, \ldots, \mathbf{h}_{i_w}\} \) be sets of distinct columns of \( \mathbf{\tilde{H}}_\rho \) such that

\[
\mathbf{f} = \sum_{k=1}^{\beta} c_k \mathbf{h}_{j_k}, \quad \mathbf{h}_{j_k} \in \mathcal{J}(\mathbf{f}) \text{ and } c_k \in \mathbb{F}_q^* \text{ for all } k; \quad (5.1)
\]

\[
\sum_{k=1}^w m_k \mathbf{h}_{i_k} = \mathbf{0}, \quad \mathbf{h}_{i_k} \in \mathcal{I}_w \text{ and } m_k \in \mathbb{F}_q^* \text{ for all } k; \quad (5.2)
\]

\( 0 \in \mathbb{F}_q^r \) is the zero column;
\[ f = \sum_{k=1}^{\beta} c_k h_{j_k} + \mu \sum_{k=1}^{w} m_k h_{l_k}, \mu \in \mathbb{F}_q^*. \] (5.3)

Note that \( I_w \) is a set of columns corresponding to a weight \( w \) codeword of \( \hat{V}_\rho \).

In the representation (5.3), the number of distinct columns of \( \hat{H}_\rho \), say \( \beta_{\text{new}} \), depends on the intersection \( I_w \cap J(f) \) and the values of nonzero coefficients \( c_k, m_k, \mu \). For example,

\[
\beta_{\text{new}} = \begin{cases} 
\beta + w & \text{if } I_w \cap J(f) = \emptyset \\
\beta + w - 1 & \text{if } |I_w \cap J(f)| = 1, h_{j_\beta} = h_{i_w}, \ c_\beta + \mu m_w \neq 0 \\
\beta + w - 2 & \text{if } I_w \cap J(f) = 2, h_{j_\beta} = h_{i_w}, \ c_\beta + \mu m_w = 0, h_{j_{\beta-1}} = h_{i_{w-1}}, c_{\beta-1} + \mu m_{w-1} \neq 0 \end{cases} \] (5.4)

To use (5.3), (5.4), note that submatrices of \( \hat{H}_\rho \) can be treated as parity check matrices of codes; we call them component codes and write in Table 1 where \( u = 1, \ldots, \rho \), “MDS” notes a minimum distance separable code and “AMDS” says on an Almost MDS code.

| rows of \( \hat{H}_\rho \) | columns of \( \hat{H}_\rho \) | geometrical object | code parameters | \( q \) | code name | code type |
|---------------------------|---------------------------|-------------------|----------------|-------|-----------|-----------|
| 1,2                       | \( h_1 \ldots h_q \)    | \( \{ A^0_U \} \cup \mathcal{L}_0^* \) | \( q, q - 2, 3 \) | 2     | \( \mathbb{L}_0 \) | MDS       |
| 2, 2u, 2u + 1, 2u + 2    | \( h_{qu+1} \ldots h_{qu+q-1} \) | \( C^*_u \)       | \( [q - 1, q - 4, 4]_q \) | 3     | \( \mathbb{C}_u \) | MDS       |
| 2, 2u + 1, 2u + 2        | \( h_{qu} \ldots h_{qu+q} \) | \( C^*_u \cup \{ T_u \} \) | \( |q - 3, 4| \) | even | \( \mathbb{C}_u^T \) | MDS       |
| 2, 2u + 1, 2u + 2        | \( h_{qt} \ldots h_{qt+q} \) | \( C^*_u \cup \{ T_u \} \) | \( [q - 3, 4]_q \) | 3     | \( \mathbb{C}_u \) | AMDS      |
| 2u, 2u + 1, 2u + 2       | \( h_{qu+1} \ldots h_{qu+q+1} \) | \( C^*_u \cup \{ A^0_{\rho} \} \) | \( q, q - 3, 4 \) | 3     | \( \mathbb{C}_\rho \) | MDS       |
| 2u, 2u + 1, 2u + 2       | \( h_{qp+q} \ldots h_{qp+q+1} \) | \( C^*_u \cup \{ A^0_{\rho}, T_u \} \) | \( |q + 1, q - 2, 4| \) | 3     | \( \mathbb{C}_{\rho}^T \) | MDS       |
| 2\rho, 2\rho + 1, 2\rho + 2 | \( h_{qp+1} \ldots h_{qp+q+1} \) | \( C^*_u \cup \{ A^0_{\rho}, T_u \} \) | \( |q + 1, q - 2, 3| \) | 3     | \( \mathbb{C}_{\rho}^T \) | AMDS      |
| \( \rho \)               | \( h_{qp+1} \ldots h_{qp+q+1} \) | \( C^*_u \cup \{ A^0_{\rho}, T_u \} \) | \( |q + 1, q - 2, 3| \) | 3     | \( \mathbb{C}_{\rho}^T \) | AMDS      |

Remark 5.1. The weight spectrum of MDS codes is known, see e.g. [29]. In particular, in \([n, n - r, d]_q \) MDS code any \( d \) columns of a parity check matrix correspond to a weight \( d \) codeword. If \( q \) odd, for AMDS component codes \( C^*_u \) and \( C_{\rho}^* \) we note that \( T_u \) lies on two tangents to \( \mathcal{C}_u \) (in \( A^0_U, A^\infty_U \)) and on \( \frac{d - 1}{2} \) bisecants of \( C^*_u \). Every of these bisecants gives rise to a weight 3 codeword. The \((q - 1)\)-set of points of \( C^*_u \) is partitioned by \( \frac{d - 1}{2} \) point pairs; every pair forms a bisecant through \( T_u \).

Note that from the proofs of Section 3 it can be seen that for the representation of a column \( f \in \mathbb{F}_q^* \) it is sufficient to use (for every \( u \)) at most 3 columns corresponding to \( C^*_u \). Similarly, one can use 2 columns corresponding to \( \{ A^0_U \} \cup \mathcal{L}_0^* \). Therefore, if \( q \geq 7 \) we have
in \( \{ A_0^q \} \cup L_0^q \) and in every \( C_\rho^q \) several points (columns) that can be used to form sets \( I_w \) useful to increase \( \beta \) and \( \beta^{\text{new}} \) in (5.2)–(5.4).

Assume that for a column \( f \in F_q^r \) we have the representation (5.1) with \( 1 \leq \beta < R \). Then using weight \( w \) codewords of the component codes we can increase \( \beta \) by \( w, w-1, w-2 \), see (5.4). The increase by \( w-1, w-2 \) is possible if some column of \( J(f) \) and \( I_w \) corresponds to the same component code. In particular, the situations with \( w = 3, w-2 = 1 \) can be provided if some column or a column pair of \( J(f) \) and \( I_w \) correspond to the same code \( L_0 \) (for all \( q \)) or to the same code \( C_u^T, C_\rho^{\infty T} \) (for \( q \) odd). There exist columns \( f \in F_q^r \) such that \( L_0 \) is not used for their representation. Therefore, in general, for even \( q \) (where MDS codes \( C_u^T, C_\rho^{\infty T} \) have minimum distance \( d = 4 \)) we are not able to do \( \beta^{\text{new}} = R \) when \( \beta = R-1 \), see (5.3), (5.4). In the other side, for odd \( q \), AMDS codes \( C_u^T, C_\rho^{\infty T} \) have \( d = 3 \) that allows us to increase \( \beta \) by \( w-2 = 1 \). Note also, see Remark 5.1 that for \( q \geq 7 \) the structure of minimum weight codewords in the component codes provides the situation that some columns of \( J(f) \) and \( I_w \) correspond to the same code.

By above, we have the following lemma.

**Lemma 5.2.** Let \( q \geq 7 \). Let \( R \geq 4 \). Let an \([n, n-r]_q R, \ell \) code be defined as in Definition 4.1. Let \( \tilde{V}_\rho \) be the \([Rq+1, Rq+1-2R, 3]_q R, \ell \) locally optimal code such that the columns of its parity check matrix correspond to points (in the homogeneous coordinates) of the minimal \( \rho \)-saturating set of Construction S (3.1), (3.2) with \( \rho = R-1 \). Then \( \ell = R \) if \( q \) is odd (i.e. we have a surface-covering code) and \( \ell = R-1 \) if \( q \) is even.

In Theorems 5.3 and 5.4 we consider \( R \geq 4 \) since for \( R = 1, 2, 3 \), several short covering codes with \( r = tR \) are given in detail in [11][13][14][16][17] and the references therein.

**Theorem 5.3.** Let \( q \geq 7 \) be odd. Let \( t \) be an integer. Then for all \( R \geq 4 \) there is an infinite family of \([n, n-r, 3]_q R, R \) locally optimal surface-covering codes with the parameters

\[
 n = Rq^{(r-R)/R} + q^{(r-2R)/R}, \quad r = tR, \quad t = 2 \quad \text{and} \quad t \geq \lceil \log_q R \rceil + 3.
\]

**Proof.** We take the \([Rq+1, Rq+1-2R, 3]_q R, R \) code \( \tilde{V}_\rho \), see Lemma 5.2 as the starting code \( V_0 \) of Construction QM.1. By Theorem 4.1 we obtain an \([n, n-r, 3]_q R, R \) code with \( n = (qR+1)q^m, \ r = 2R + mR \). Obviously, \( m + 1 = \frac{R}{R-1} \). The condition \( q^m \geq n_0 - 1 \) implies \( q^m \geq qR \) whence \( m \geq \lceil \log_q R \rceil + 1 \). Finally, we put \( t = m + 2 \).

**Theorem 5.4.** Let \( q \geq 8 \) be even. Let \( t \) be an integer. Then for all \( R \geq 4 \) there are infinite families of \([n, n-r, 3]_q R, R \) locally optimal surface-covering codes with the parameters

\[
 (i) \ n = Rq^{(r-R)/R} + 2q^{(r-2R)/R} + \sum_{j=3}^{t} q^{(r-jR)/R}, \quad r = tR, \quad m_1 + 2 < t < 3m_1 + 2,
\]

\[
 m_1 = \lceil \log_q (R+1) \rceil + 1;
\]
\[(ii) \quad n = Rq^{(r-R)/R} + 2q^{(r-2R)/R} + \sum_{j=3}^{m_1+2} q^{(r-jR)/R}, \quad r = tR, \quad t = m_1 + 2 \text{ and } t \geq 3m_1 + 2.\]

**Proof.** (i) We take the \([qR+1,qR+1-2R,3]_q R, \ell \text{ code } \tilde{V}_\ell \text{ with } \ell = R-1, \text{ see Lemma } 5.2\] as the starting code \(V_0\) of Construction QM_2. By Theorem [4.12] we obtain an \([n,n-r,3]_q R, R \text{ code with } n = (qR+1)q^m + \frac{q^{m+1}-1}{q-1}, \quad r = 2R + mR.\] Obviously, \(m-\frac{j-2}{R} = \frac{r-jR}{R}.\] The condition \(q^m \geq n_0\) implies \(q^n \geq qR+1\) whence \(m \geq \lceil \log_q (qR+1) \rceil = \lceil \log_q (R+1) \rceil + 1.\] The restriction \(m < 3m_1\) is introduced as for \(m \geq 3m_1\) we have codes of (i) that are better than ones in (ii). For \(m = m_1\), codes of (i) and (ii) are the same. Finally, we put \(t = m+2.\)

(ii) In the relation (i), we put \(t = m_1 + 2\) and obtain an \([n_1,n_1-r,3]_q R, R \text{ code with } n_1 = (qR+1)q^{m_1} + \frac{q^{m_1+1}-1}{q-1}, \quad r_1 = 2R + m_1 R.\] We take this code as the starting code \(V_0\) of Construction QM_1. By Theorem [4.11] we obtain an \([n,n-r,3]_q R, R \text{ code with } r = 2R + m_1 R + m_2 R, \quad q^{m_2} \geq n_1, \quad n = n_1 q^{m_2} = (qR+1)q^{m_1} + m_2 + \sum_{i=0}^{m_1} q^{m_1+m_2-i}.\] Obviously, \(m_1 + m_2 - i = \frac{r-(i+2)R}{R}.\] Since \((R+1)q^{m_1+1} > n_1\), the condition \(q^{m_2} \geq n_1\) is satisfied when \(q^{m_2} \geq (R+1)q^{m_1+1}\) whence \(m_2 \geq \lceil \log_q (R+1) \rceil + m_1 + 1 = 2m_1.\) Then we denote \(2 + m_1 + m_2\) by \(t.\)

\section{New infinite code families with fixed even radius \(R \geq 2\) and growing codimension \(tR + \frac{R}{2}\)}

In the projective plane \(PG(2,q)\), a blocking (resp. double blocking) set \(S\) is a set of points such that every line of \(PG(2,q)\) contains at least one (resp. two) points of \(S.\)

There is an useful connection between double blocking sets and 1-saturating sets.

**Proposition 6.1.** [14 Cor. 3.3], [25] Let \(q\) be a square. Any double blocking set in the subplane \(PG(2,\sqrt{q}) \subset PG(2,q)\) is a 1-saturating set in the plane \(PG(2,q)\).

In future we use the following results, see also [11, 3, 14] Sect. 3.2.

**Proposition 6.2.** Let \(p\) be prime. Let \(\phi(q)\) be as in (2.1). The following bounds on the smallest size \(\tau_2(2,q)\) of a double blocking set in \(PG(2,q)\) hold:

\[
\tau_2(2,q) \leq 2(q + q^{2/3} + q^{1/3} + 1), \quad q = p^h, \quad p^h \equiv 2 \text{ mod } 7 \quad [3 \text{ Th. 5.5}];
\]

\[
\tau_2(2,q) \leq 2 \left( q + \frac{q-1}{\phi(q) - 1} \right), \quad q = p^h, \quad h \geq 2, \quad p \geq 3 \quad [11 \text{ Cor. 1.9}];
\]

\[
\tau_2(2,q) \leq 2 \left( q + \frac{q}{p} + 1 \right), \quad q = p^h, \quad h \geq 2, \quad p \geq 7 \quad [2 \text{ Th. 1.8, Cor. 4.10}].
\]

Now we give a list of 1-saturating sets in the projective plane of square order. The sets (iv)-(vi) are new.
Theorem 6.3. Let \( q \) be a square. Let \( p \) be prime. Let \( \phi(\sqrt q) \) be as in (2.1). Then in PG(2, q) there are 1-saturating sets of the following sizes:

(i) \[ 3\sqrt q - 1, \quad q = p^{2h} \geq 4, \ h \geq 1 \] [10, Th. 5.2];
(ii) \[ 2\sqrt q + 2\sqrt q + 2, \quad q = p^{4h} \geq 16, \ h \geq 1 \] [13, Th. 3.3], [14, Th. 3.4], [25];
(iii) \[ 2\sqrt q + 2\sqrt q + 2\sqrt q + 2, \quad q = p^6, \ p \leq 73 \] [13, Th. 3.4], [14, Th. 3.5];
(iv) \[ 2\sqrt q + 2\sqrt q + 2\sqrt q + 2, \quad q = p^{6h}, \ p^h \equiv 2 \mod 7; \]
(v) \[ 2\sqrt q + 2\sqrt q + 2\sqrt q + 2, \quad q = p^{2h}, \ h \geq 2, \ p \geq 3; \]
(vi) \[ 2\sqrt q + 2\sqrt q + 2, \quad q = p^{2h}, \ h \geq 2, \ p \geq 7. \]

Proof. For (i), a geometric construction is proposed in [10, Th. 5.2]. We describe it in Remark 6.5. The 1-saturating sets of (ii), (iii) are considered in [13], [14], [25]. For (iv)–(vi) we use Propositions 6.1 and 6.2.

Remark 6.4. In Theorem 6.3 if \( \sqrt q = p^\eta \) with \( \eta \geq 3 \) odd, then the new 1-saturating sets of (iv)–(vi) have smaller sizes than the known ones of (i)–(iii). For example, if \( q = p^6 \), \( \eta = 3 \), then the new size of (vi) is \( 2\sqrt q + 2\sqrt q + 2 \), cf. (iii). If \( \eta \geq 5 \) odd, the known sets have size \( 3\sqrt q - 1 \) whereas new sizes are \( 2\sqrt q + o(\sqrt q) \). For example, if \( q = p^{30} \), \( \eta = 15 \), then the new size of (iv), (v) is \( 2\sqrt q + 2\sqrt q + 2\sqrt q + 2 \), cf. (i). In general, if \( \eta \geq 3 \) is prime, then the case (vi) gives smaller sizes than other variants. If \( \eta \) is odd non-prime, then the variant (v) is the best.

The case (iv) gives the same size as (v), if \( 3|\eta \). Therefore, in future we consider new codes and bounds resulting from Theorem 6.3(v),(vi).

Note also that if \( q = p^2 \), i.e. \( \eta = 1 \), then the size (i) is the smallest in Theorem 6.3. It is why we pay attention to this case, see Remarks 6.5, 6.7 and Problem 5 below.

Remark 6.5. Let a point of PG(2, q) have the form \( (x_0, x_1, x_2) \) where \( x_i \in \mathbb{F}_q \), the leftmost nonzero coordinate is equal to 1. Let \( \beta \) be a primitive element of \( \mathbb{F}_q \).

In [10, Th. 5.2, eq. (30)], the following construction of a 1-saturating \((3\sqrt q - 1)\)-set \( S \) in PG(2, q), \( q \) square, is proposed:

\[
S = \{(1, 0, x_2) | x_2 \in \mathbb{F}_{\sqrt q} \} \cup \{(1, 0, c\beta) | c \in \mathbb{F}_{\sqrt q}^* \} \cup \{(0, 1, x_2) | x_2 \in \mathbb{F}_{\sqrt q} \}. \quad (6.1)
\]

We describe this construction in more detail than in [10] using, for the description, the Baer sublines similarly to [4, Prop. 3.2]. In [10], see (6.1), specific Baer sublines are noted. Here we explain the structure and role of these specific sublines. Two Baer subplanes \( B_1 \) and \( B_2 \) are considered. In the points of \( B_1 \), all coordinates \( x_i \in \mathbb{F}_{\sqrt q} \). Also, \( B_2 = B_1 \Phi \) where \( \Phi \) is the collineation such that \( (x_0, x_1, x_2) \Phi = (x_0, x_1\beta, x_2\beta) \). Let \( L_i \subset \text{PG}(2, q) \) be the “long” line of equation \( x_i = 0 \). Let \( L_{i,j} = L_i \cap B_j \) be the Baer subline of \( L_i \).
Remark 6.6. In [30, Ex. B] and [4, Prop. 3.2], constructions of a 1-saturating 3-set in PG(2, q), q square, are proposed. In [30], the set is minimal; it consists of three non-concurrent Baer sublines in a Baer subplane. In [4], the set is non-minimal; it is similar to one of the construction [11, Th. 5.2], see its description in Remark 6.5. However, in [4], the intersection point of the three Baer sublines is not removed from the 1-saturating set.

Remark 6.7. Let \( \phi \) be prime. To construct a 1-saturating \((3p - 1)\)-set in PG(2, p\(^2\)), another way than in [11] is possible. One can apply Proposition 6.1 to a double blocking set in PG(2, p). However, double blocking \((3p - 1)\)-sets in PG(2, p) are known only for \( q = 13, 19, 31, 37, 43 \), see [8] and the references therein. Moreover, in PG(2, p), no double blocking sets of size less than \( 3p - 1 \) are known.

In PG(2, p\(^2\), p prime, by [14, Tab. 2], we have the following sporadic examples of 1-saturating \( k \)-sets with \( k < 3p - 1 \): \( p^2 = 9, k = 6 \); \( p^2 = 25, k = 12 \); \( p^2 = 49, k = 18 \).

**Problem 5.** Develop a general construction of a 1-saturating \( k \)-set in PG(2, p\(^2\), p prime, such that \( k < 3p - 1 \).

In [11, Ex. 6], a lift-construction is given. It provides the following result.

**Proposition 6.8.** [11, Ex. 6], [14, Th. 4.4] Let \( an_{q, n_q - 3} \) \( q \)-code exist. Let \( n_q < q \) and \( q + 1 \geq 2n_q \). Let \( f_q(r, 2) \) be as in (2.2). Then there is an infinite family of \([n, n-r]q\)-codes with growing odd codimension \( r = 2t+1 \geq 5 \) and length \( n = n_qq^{(r-3)/2}+2q^{(r-5)/2}+f_q(r, 2) \).

**Theorem 6.9.** Assume that \( p \) is prime, \( q = p^{2h}, h \geq 2 \), and covering radius \( R = 2 \). Let \( \phi(\sqrt{q}) \) and \( f_q(r, 2) \) be as in (2.1), (2.2). Then there exist infinite families of \([n, n-r]q\)-codes with growing odd codimension \( r = 2t+1 \geq 4, t \geq 1 \), and length

\[
n = \left(2 + 2\frac{\sqrt{q} - 1}{\phi(\sqrt{q}) - 1}\right)q^{(r-2)/2} + 2q^{(r-5)/2} + f_q(r, 2), \quad p \geq 3;
\]
\[ n = \left( 2 + \frac{2}{p} + \frac{2}{\sqrt{q}} \right) q^{(r-2)/2} + 2q^{(r-5)/2} + f_q(r, 2), \quad p \geq 7. \]

**Proof.** Let \( n_q \) be the size of the 1-saturating sets of Theorem 6.3(iii),(iv). We treat every point (in homogeneous coordinates) of the set as a column of an \( 3 \times n_q \) parity check matrix of an \([n_q, n_q - 3]_q\) code. For these codes it can be shown that \( n_q < q \) and \( q + 1 \geq 2n_q \). Then we use Proposition 6.8.

The direct sum construction \[14\], Sect. 4.2 gives the following lemma.

**Lemma 6.10.** Let covering radius \( R \geq 2 \) be even. Let an \([n'', n'' - r'']_q\) code exist. Then there is an \([Rn'', Rn'' - Rr'']_q\) code.

**Theorem 6.11.** Assume that \( p \) is prime, \( q = p^{2h} \), \( h \geq 2 \), \( R \geq 2 \) even, and code codimension is \( r = tR + \frac{R}{2} \) with growing integer \( t \geq 1 \). Let \( \phi(\sqrt{q}) \) and \( f_q(r, R) \) be as in (2.1), (2.2). Then for all even \( R \geq 2 \) there are infinite families of \([n, n-r\]_q\) codes with fixed covering radius \( R \), growing codimension \( r = tR + \frac{R}{2} \), \( t \geq 1 \), and length

\[
\begin{align*}
\text{if } p & \geq 3; \\
\text{if } p & \geq 7.
\end{align*}
\]

**Proof.** We take codes of Theorem 6.9 as the codes \([n'', n'' - r'']_q\) of Lemma 6.10.

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