Rotations of Nuclei with Reflection Asymmetry Correlations

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Abstract

We propose a collective Hamiltonian which incorporates interactions capable to generate rotations in nuclei with simultaneous presence of octupole and quadrupole deformations. It is demonstrated that the model formalism could be applied to reproduce the staggering effects observed in nuclear octupole bands. On this basis we propose that the interactions involved would provide a relevant handle in the study of collective phenomena in nuclei and other quantum mechanical systems with reflection asymmetry correlations.

The properties of nuclear systems with octupole deformations are of current interest due to the increasing number of evidences for the presence of octupole instability in different regions of nuclear table. Various parametrizations of the octupole degrees of freedom have opened a useful tool for understanding the role of the reflection asymmetry correlations and for analysis of the collective properties of such kind of systems. As an important step in this direction it is necessary to elucidate the question: which are the collective nuclear interactions that correspond to the different octupole shapes and how do they determine the structure of the respective energy spectra? The physically meaningful answer could be obtained by taking into account the simultaneous presence of other collective degrees of freedom, such as the quadrupole ones.

In the present work we address the above problem by examining the interactions that generate collective rotations in a system with octupole deformations. Based on the octahedron point symmetry parametrization of the octupole shape, we propose a general collective Hamiltonian which incorporates the interactions responsible for the rotations associated with the different octupole deformations. It will be shown that after taking into account the quadrupole degrees of freedom and the appropriate higher order quadrupole-octupole interaction the model formalism would be able to reproduce schematically some interesting effects of the fine rotational structure of nuclear octupole bands. The study is strongly motivated by the need of theoretical explanation of the recently observed staggering patterns in octupole bands of light actinides as well as by the possibility to gain

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an insight into the fine structure of negative parity rotational bands based on octupole vibrations.

Our model formalism is based on the understanding that the collective properties of a physical system in which octupole correlations take place should be influenced by the following most general octupole field $V_3 = \sum_{\mu=-3}^{3} \alpha_{3\mu}^i Y_{3\mu}$, (in the intrinsic, body-fixed frame) which can be written in the form [4]:

$$V_3 = \epsilon_0 A_2 + \sum_{i=1}^{3} \epsilon_1(i) F_1(i) + \sum_{i=1}^{3} \epsilon_2(i) F_2(i), \tag{1}$$

where the quantities

$$A_2 = -\frac{i}{\sqrt{2}} (Y_{32} - Y_{3-2}) = \frac{1}{r^3} \sqrt{\frac{105}{4\pi}} x y z, \tag{2}$$

$$F_1(1) = Y_{30} = \frac{1}{r^3} \sqrt{\frac{7}{4\pi}} (z^2 - \frac{3}{2} x^2 - \frac{3}{2} y^2), \tag{3}$$

$$F_1(2) = -\frac{1}{4} \sqrt{5} (Y_{33} - Y_{3-3}) + \frac{1}{4} \sqrt{3} (Y_{31} - Y_{3-1})$$

$$\quad = \frac{1}{r^3} \sqrt{\frac{7}{4\pi}} [(x^2 - \frac{3}{2} y^2 - \frac{3}{2} z^2)], \tag{4}$$

$$F_1(3) = -i \frac{1}{4} \sqrt{5} (Y_{33} + Y_{3-3}) - i \frac{1}{4} \sqrt{3} (Y_{31} + Y_{3-1})$$

$$\quad = \frac{1}{r^3} \sqrt{\frac{7}{4\pi}} y (y^2 - \frac{3}{2} z^2 - \frac{3}{2} x^2), \tag{5}$$

$$F_2(1) = \frac{1}{\sqrt{2}} (Y_{32} + Y_{3-2}) = \frac{1}{r^3} \sqrt{\frac{105}{16\pi}} z (x^2 - y^2), \tag{6}$$

$$F_2(2) = \frac{1}{4} \sqrt{3} (Y_{33} - Y_{3-3}) + \frac{1}{4} \sqrt{5} (Y_{31} - Y_{3-1})$$

$$\quad = \frac{1}{r^3} \sqrt{\frac{105}{16\pi}} y (y^2 - z^2), \tag{7}$$

$$F_2(3) = -i \frac{1}{4} \sqrt{3} (Y_{33} + Y_{3-3}) + i \frac{1}{4} \sqrt{5} (Y_{31} + Y_{3-1})$$

$$\quad = \frac{1}{r^3} \sqrt{\frac{105}{16\pi}} y (z^2 - x^2), \tag{8}$$

(with $r^2 = x^2 + y^2 + z^2$) belong to the irreducible representations (irreps) of the octahedron group (O). $A_2$ is one-dimensional, while $F_1$ and $F_2$ are three-dimensional irreps. The seven real parameters $\epsilon_0$ and $\epsilon_r(i)$ ($r = 1, 2; i = 1, 2, 3$) determine the amplitudes of the octupole deformation. Their relation to the $\alpha_{3\mu}^i$ is given in [4].

Our proposition is that the general collective Hamiltonian which incorporates the shape characteristics of the octupole field [4] can be constructed on the basis of the above octahedron irreps. For this purpose we introduce operator forms of the quantities $A_2$, $F_1(i)$ and $F_2(i)$ ($i = 1, 2, 3$) in which the cubic terms of the Cartesian variables $x$, $y$ and $z$ in Eqs (2)–(8) are replaced by appropriately symmetrized combinations of cubic terms.
of the respective angular momentum operators \( \hat{I}_x, \hat{I}_y, \hat{I}_z \) (with \( \hat{r}^2 = \hat{I}_x^2 + \hat{I}_y^2 + \hat{I}_z^2 \)). The following Hamiltonian is then obtained:

\[
\hat{H}_{\text{oct}} = \hat{H}_{A_2} + \sum_{r=1}^{2} \sum_{i=1}^{3} \hat{H}_{F_i}(r),
\]  

(9)

with

\[
\hat{H}_{A_2} = a_2 \frac{1}{4} \left[ (\hat{I}_x \hat{I}_y + \hat{I}_y \hat{I}_x) \hat{I}_z + \hat{I}_z (\hat{I}_x \hat{I}_y + \hat{I}_y \hat{I}_x) \right],
\]

(10)

\[
\hat{H}_{F1(1)} = \frac{1}{2} f_{11} \hat{I}_z (5\hat{I}_z^2 - 3\hat{I}^2),
\]

(11)

\[
\hat{H}_{F1(2)} = \frac{1}{2} f_{12} (5\hat{I}_x^3 - 3\hat{I}_x \hat{I}_z^2),
\]

(12)

\[
\hat{H}_{F1(3)} = \frac{1}{2} f_{13} (5\hat{I}_y^3 - 3\hat{I}_y \hat{I}_z^2),
\]

(13)

\[
\hat{H}_{F2(1)} = f_{21} \frac{1}{2} [\hat{I}_z (\hat{I}_z^2 - \hat{I}_y^2) + (\hat{I}_x - \hat{I}_y \hat{I}_z)],
\]

(14)

\[
\hat{H}_{F2(2)} = f_{22} (\hat{I}_x \hat{I}_z^2 - \hat{I}_x \hat{I}_y^2 - \hat{I}_y \hat{I}_z^2),
\]

(15)

\[
\hat{H}_{F2(3)} = f_{23} (\hat{I}_y \hat{I}_z^2 + \hat{I}_z \hat{I}_y^2 - \hat{I}_x \hat{I}_z^2).
\]

(16)

The Hamiltonian parameters \( a_2 \) and \( f_{ri} (r = 1, 2; i = 1, 2, 3) \) are formally related to the parameters in (1) as follows \( a_2 = \epsilon_0 \sqrt{105/(4\pi)} \), \( f_{1i} = \epsilon_1(i) \sqrt{7/(4\pi)} \), \( f_{2i} = \epsilon_2(i) \sqrt{105/(16\pi)} \), \( i = 1, 2, 3 \).

During the procedure described above, the \( r^3 \) factors appearing in the denominators of Eqs (2)–(8) are replaced by \( \hat{r}^3 \) factors. In the final result, Eqs (10)–(16), we normalize with respect to \( \hat{r}^3 \), i.e. we multiply the results by \( \hat{r}^3 \), an operation which is equivalent to the transition to a unit sphere, a natural thing to do since we are interested in surface shapes.

We remark that the terms of the Hamiltonian obtained (as a function of the angular momentum operators \( \hat{I}_x, \hat{I}_y, \hat{I}_z \)) correspond to the same octupole shapes which appear in Eqs (2)–(8) and belong to the same irreps of the octahedron group. In other words, through the above procedure we determine the octahedron point symmetry properties of the system in angular momentum space.

Our analysis shows that the operator \( \hat{H}_{F1(1)} \), Eq. (14), which corresponds to \( Y_{30} \) (with axial deformation) is the only one octupole operator possessing diagonal matrix elements in the states with collective angular momentum \( I \). Below it will be shown that it is of major importance for determining the fine structure of collective bands with octupole correlations. Actually, it is well known that the \( Y_{30} \) (axial) deformation is the leading mode in the systems with reflection asymmetric shape (See for review [3]).

Further, it is known that the use of the pure octupole field (1) is not sufficient to incorporate the collective shape properties of the system. More specifically a unique parametrization of the pure octupole field in an intrinsic frame has not been obtained yet in a consistent way [3]. In this respect the consideration of octupole degrees of freedom together with the quadrupole deformations is important. A general treatment of a combined quadrupole-octupole field is proposed in the framework of a general collective model for coupled multipole surface modes [4, 5].
Based on the above consideration we suggest that the most general collective Hamiltonian of a system with octupole correlations should contain also the standard (axial) quadrupole rotation part
\[ \hat{H}_{\text{rot}} = A \hat{I}^2 + A' \hat{I}_z^2 , \]  
(17)
where \( A \) and \( A' \) are the inertial parameters. In addition the following higher order diagonal quadrupole-octupole interaction term (corresponding to the product \( Y_{20} \cdot Y_{30} \)) could be introduced:
\[ \hat{H}_{\text{qoc}} = f_{\text{qoc}} \frac{1}{I^2} (15 \hat{I}_z^5 - 14 \hat{I}_z^3 \hat{I}^2 + 3 \hat{I}_z \hat{I}^4) . \]  
(18)
This operator is normalized with respect to the multiplication factor \( I^3 \). (More precisely we use the product \( I^3 Y_{20} \cdot Y_{30} \) so as to keep all non-quadrupole Hamiltonian terms of the same order.)

Then the Hamiltonian of the system can be written as
\[ \hat{H} = \hat{H}_{\text{bh}} + \hat{H}_{\text{rot}} + \hat{H}_{\text{oct}} + \hat{H}_{\text{qoc}} . \]  
(19)

Here
\[ \hat{H}_{\text{bh}} = \hat{H}_0 + f_k \hat{I}_z , \]  
(20)
is a pure phenomenological part introduced to reproduce the bandhead energy in the form
\[ E_{\text{bh}} = E_0 + f_k K , \]  
(21)
were \( E_0 \) and \( f_k \) are free parameters. The \( K \)-dependence of \( E_{\text{bh}} \), which can be reasonably referred to the intrinsic motion, provides the correct value of the bandhead angular momentum projection \( K \) in the variation procedure described below.

We remark that the Hamiltonian (13) is not a rotational invariant in general. It does not commute with the total angular momentum operators and any state with given angular momentum is energy split with respect to the quantum number \( K \). Therefore, the physical relevance of this Hamiltonian depends on the possibility to determine in an unique way the angular momentum projection \( K \). The basic assumption of our consideration is that \( K \) is not frozen within the states of the collective rotational band. We suggest that for any given angular momentum it should be determined so as to minimize the respective collective energy. The resulting energy spectrum represents the yrast sequence of energy levels for our model Hamiltonian. We remark that similar procedure is used in Refs. \([9, 10]\) in reference to the \( \Delta I = 2 \) staggering effect in superdeformed nuclei.

As a first step in testing our Hamiltonian we consider its diagonal part
\[ \hat{H}^d = \hat{H}_{\text{bh}} + \hat{H}_{\text{rot}} + \hat{H}_{\text{oct}}^d + \hat{H}_{\text{qoc}} . \]  
(22)
were the operator \( \hat{H}_{\text{oct}}^d \equiv \hat{H}_{F_{1(1)}} \) represents the diagonal part of the pure octupole Hamiltonian \( \hat{H}_{\text{oct}} \), Eq. (9).

The following diagonal matrix element is then obtained:
\[ E_K(I) = E_0 + f_k K + AI(I + 1) + A'K^2 + f_{11} \left( \frac{5}{2} K^3 - \frac{3}{2} K I(I + 1) \right) \]
\[ + f_{\text{qoc}} \frac{1}{I^2} \left( 15 K^5 - 14 K^3 I(I + 1) + 3 K I^2 (I + 1)^2 \right) . \]  
(23)
Following the above assumption for the third angular momentum projection, we determine the yrast sequence $E(I)$ after minimizing Eq. (23) as a function of integer $K$ in the range $-I \leq K \leq I$. The obtained energy spectrum depends on six model parameters: $E_0$ essentially responsible for the bandhead energy; $f_k$ which provides minimal energy for $K = K_{bh} = I_{bh}$; $A$ and $A'$ are the quadrupole inertial parameters which should generally correspond to the known quadrupole shapes (axes ratios) of nuclei; $f_{11}$ and $f_{qoc}$ are the parameters of the diagonal octupole (11) and quadrupole-octupole (18) interactions respectively. We consider the latter two parameters as free parameters.

We applied several exemplary sets of the above parameters and obtained the corresponding schematic energy spectra. One of them is given in Table 1. It is seen that the “yrast” values of the quantum number $K$ gradually increase with the increase of the angular momentum $I$. We remark that they correspond to the local minima of Eq. (23) as a function of $K$. This is illustrated on Fig. 1. We see that these minimums are well determined and their depth increases with the increase of the angular momentum. Such a behavior of the spectrum corresponds to a wobbling motion and could also be interpreted as a multiband-crossing phenomenon. The obtained yrast sequence can be considered as the envelope of the curves with different values of the quantum number $K$ as it is illustrated in Fig. 2.

In addition we see that the $K$-values of the odd and the even sequence of levels are grouped by couples which imply the presence of odd–even staggering effect. Indeed, the presence of such an effect is demonstrated in Fig. 3(a)–(e), where the quantity

$$Stg(I) = 6\Delta E(I) - 4\Delta E(I - 1) - 4\Delta E(I + 1) + \Delta E(I + 2) + \Delta E(I - 2),$$

with $\Delta E(I) = E(I+1) - E(I)$, is plotted as a function of angular momentum $I$ for several different sets of model parameters. (The quantity $Stg(I)$ is the discrete approximation of the fourth derivative of the function $\Delta E(I)$, i.e. the fifth derivative of the energy $E(I)$. Its physical relevance has been discussed extensively in Refs [6, 11].)

Fig. 3(a) illustrates a long $\Delta I = 1$ staggering pattern with several irregularities, which looks similar to the “beats” observed in the octupole bands of some light actinides such as $^{220}$Ra, $^{224}$Ra and $^{226}$Ra [6]. Also, it is rather similar to the staggering patterns observed in rotational spectra of diatomic molecules [12]. In Fig. 3(b) the increased values of $f_{11}$ and $f_{qoc}$ provide a wide angular momentum region (up to $I \sim 40$) with a regular staggering pattern. The further increase of $f_{qoc}$ results in a staggering pattern with different amplitudes, shown in Fig. 3(c). These two figures resemble the staggering behavior of some rotational (negative parity) bands based on octupole vibrations [13]. The further increase of $f_{11}$ and $f_{qoc}$ leads to a staggering pattern with many “beats”, as shown in Fig 3(d). Notice that in Fig. 3(d) the first three “beats” are completed by $I \approx 40$, while in Fig. 3(a) the first three “beats” are completed by $I \approx 70$. An example with almost constant staggering amplitude is shown in Fig. 3(e). It resembles the form of the odd–even staggering predicted in the SU(3) limit of various algebraic models (see Ref. [6] for details and relevant references). It also resembles the odd–even staggering seen in some octupole bands of light actinides, such as $^{222}$Rn [1].

Now we can discuss the general Hamiltonian structure (19) including the various non-diagonal terms (10), (12)–(16). Here, the major problem is the circumstance that $K$ is generally not a good quantum number. However we are able to provide our analysis for small values of the respective parameters which conserve $K$ “asymptotically” good. This requirement assumes a weak $K$-bandmixing interaction which guarantees that for
any explicit energy minimum appearing in the diagonal case the corresponding perturbed Hamiltonian eigenvalue will be uniquely determined. Thus we are able to obtain respective \( K \)-mixed yrast energy sequence. Our numerical analysis of the Hamiltonian eigenvector systems shows that the parameters of the non-diagonal terms should be by an order smaller in value than the parameter \( f_{11} \). In addition, we established that the following couples of non-diagonal terms give the same contribution in the energy spectrum: \( \hat{H}_{F_1(2)} \) and \( \hat{H}_{F_1(3)} \); \( \hat{H}_{F_2(1)} \) and \( \hat{H}_{F_2(2)} \) and \( \hat{H}_{F_2(3)} \).

In Fig. 3(f) a staggering pattern with a presence of \( K \)-band-mixing is illustrated. In fact we added the following three non-diagonal terms \( \hat{H}_{F_1(2)} \), \( \hat{H}_{F_2(1)} \) and \( \hat{H}_{F_2(2)} \) to the already considered diagonal Hamiltonian \( \hat{H}_{F_1(1)} \), with the parameters of the latter being kept the same as in Fig. 3(b) (and in Table 1). We see that the mixing leads to a decrease in the staggering amplitude with the increase of angular momentum so that the staggering pattern is reduced completely in the higher spin region. This pattern resembles the experimental situation in \( ^{218}\text{Rn} \) and \( ^{228}\text{Th} \) [6] (odd–even staggering with amplitude decreasing as a function of \( I \)).

So, the staggering patterns illustrated so far (Fig. 3) cover almost all known \( \Delta I = 1 \) staggering patterns in nuclei and molecules. The amplitudes obtained for the examined sets of parameters vary up to 300 keV. Some reasonable theoretical patterns with \( Stg(I) \sim 500\text{keV} \) can be easily obtained. On this basis we suppose that the model parameters can be adjusted appropriately so as to reproduce the staggering effects in nuclear octupole bands as well as in some rotational negative parity bands built on octupole vibrations. Also, an application of the present formalism to the spectra of diatomic molecules could be reasonable.

Here the following comments on the structure of the collective interactions used and the related symmetries would be relevant:

1) The equal contribution of the three couples of non-diagonal terms (mentioned above) indicates that only four octupole Hamiltonian terms are enough to determine the energy spectrum. This result reflects the circumstance that in the intrinsic frame three octupole degrees of freedom, from the seven ones, are related to the orientation angles. For example we could suggest that the following terms [applied in Fig. 3(f)] give an independent contribution in the total Hamiltonian: \( \hat{H}_{F_1(1)} \), \( \hat{H}_{F_1(2)} \), \( \hat{H}_{F_2(1)} \) and \( \hat{H}_{F_2(2)} \). We remark that our analysis (related to the collective rotations of the system) gives a natural way to determine the four collective octupole interaction terms.

2) From symmetry point of view we remark that the diagonal term \( \hat{H}_{F_1(1)} \) which corresponds to \( Y_{30} \) possesses an axial symmetry while the non-diagonal terms \( \hat{H}_{F_1(2)} \), \( \hat{H}_{F_2(1)} \) and \( \hat{H}_{F_2(2)} \) (of previous item 1) are constructed by using the combinations \( (Y_{31} - Y_{3 -1}) \) with \( C_{2v} \) symmetry, \( (Y_{32} + Y_{3 -2}) \) with \( T_d \) symmetry and \( (Y_{33} - Y_{3 -3}) \) with \( D_{3h} \) symmetry. So, our analysis shows that the axial symmetric term should play the major role in the structure of the collective rotational Hamiltonian while the non-axial parts could be considered as small \( K \)-band-mixing interactions. From microscopic point of view, a detailed analysis of the above spherical harmonic combinations and the respective symmetries has been provided on the basis of the one particle spectra of the octupole-coupled two-level model [4].

3) The observed influence of the non-diagonal Hamiltonian terms on the fine structure of our “schematic” spectra suggests an important physical conclusion: the non-diagonal \( K \)-mixing interactions suppress the staggering pattern. In such a way we find that the axial symmetric term \( \hat{H}_{F_1(1)} \) is the only one pure octupole degree of freedom which provides
“beat” staggering behavior of the quantity $\langle 24 \rangle$ (See Fig. 3(e)). (The quadrupole–octupole term $\hat{H}_{qoc}$ gives an additional contribution and provides wider angular momentum regions with regular staggering.) So, our analysis suggests that the $\Delta I = 1$ staggering effect observed in systems with octupole deformations could be referred to the dominant role of the axial symmetric “pear-like” shape.

In addition, it is important to remark that the fine (staggering) behavior of our schematic energy spectra reflects the structure of the interactions considered through the K- sequences generated in the above minimization procedure. Thus our analysis suggests that in the high angular momentum region some high K band structures should be involved. From microscopic point of view the values $K = 0, 1, 2, 3$ have been included in calculations, showing that in the beginning of the rare earth region the values $K = 0, 1$ are important for the lowest $3^-$ state, while in the middle of the region the values $K = 1, 2$ are important and in the far end of the region the values $K = 2, 3$ are important [15]. The same authors deal with nuclei with $A \geq 222$ in Ref. [15]. One of the authors of Refs. [15, 16] in Ref. [17] finds that the restriction to $K \leq 3$ is not justifiable for large energies. This is in agreement with our findings of Table 1.

In conclusion, we remark that the collective interactions considered in this work suggest the presence of various fine rotational band structures in quantum mechanical systems with collective octupole correlations. In particular, they provide various forms of staggering patterns which appear as the results of a delicate interplay between the terms of pure octupole field and the terms of high order quadrupole–octupole interaction. The analysis carried out outlines the dominant role of the axial symmetric “pear-like” shape for the presence of a $\Delta I = 1$ staggering effect. The obtained multi K- band crossing structures could be referred to a wobbling collective motion of the system. We propose that the interactions involved would provide a relevant handle in the study of collective phenomena in nuclei and other quantum mechanical systems with complex shape correlations.

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Table 1: The “yrast” energy levels, $E(I)$ (in KeV), and the respective $K$- values obtained by Eq. (23) for the parameter set $E_0 = 500$keV, $f_k = -7.5$keV, $A = 12$keV, $A' = 6.6$keV, $f_{11} = 0.56$keV, $f_{qoc} = 0.085$keV.

| $I$ | $E(I)$  | $K$ | $I$ | $E(I)$  | $K$ | $E(I)$  | $K$ |
|-----|---------|-----|-----|---------|-----|---------|-----|
| 1   | 522.772 | 1   | 13  | 2335.81 | 5   | 5453.12 | 11  |
| 2   | 568.327 | 1   | 14  | 2576.57 | 6   | 5694.49 | 12  |
| 3   | 637.095 | 1   | 15  | 2827.57 | 6   | 5935.5  | 12  |
| 4   | 728.71  | 1   | 16  | 3082.36 | 7   | 6157.5  | 13  |
| 5   | 840.857 | 2   | 17  | 3344.94 | 7   | 6378.29 | 13  |
| 6   | 971.155 | 2   | 18  | 3608.18 | 8   | 6575.37 | 14  |
| 7   | 1123.22 | 2   | 19  | 3877.05 | 8   | 6770.62 | 14  |
| 8   | 1288.09 | 3   | 20  | 4143.16 | 9   | 6937.23 | 15  |
| 9   | 1472.71 | 3   | 21  | 4413.03 | 9   | 7101.62 | 15  |
| 10  | 1668.56 | 4   | 22  | 4676.45 | 10  | 7232.21 | 16  |
| 11  | 1880.56 | 4   | 23  | 4942.01 | 10  | 7360.44 | 16  |
| 12  | 2101.68 | 5   | 24  | 5197.18 | 11  | 7449.45 | 17  |

**Figure Captions**

**Figure 1.** The diagonal energy matrix element $E_K(I)$ (in MeV), Eq. (23), is plotted as a function of $K$ for $I = 1, 2, ..., 10$, for the parameter set $E_0 = 500$keV, $A = 12$keV, $A' = 6.6$keV, $f_{11} = 0.56$keV, $f_{qoc} = 0.085$keV.

**Figure 2.** The diagonal energy matrix element $E_K(I)$ (in MeV), Eq. (23), is plotted as a function of $I$ for $K = 10, 11, 12, 13$, for the parameter set of Figure 1.

**Figure 3.** $\Delta I = 1$ staggering patterns [Eq. (24)] obtained: (a) – (e) by the diagonal Hamiltonian (22) for several different sets of model parameters; (f) by adding three non-diagonal terms $\hat{H}_{F_1(2)}$ [Eq. (13)], $\hat{H}_{F_2(1)}$ [Eq. (14)] and $\hat{H}_{F_2(2)}$ [Eq. (15)] to the diagonal Hamiltonian (22).
