On the Hodge Conjecture for products of certain surfaces

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October 15, 2021

Abstract

In this paper we prove the Hodge conjecture for arbitrary products of surfaces, $S_1 \times \cdots \times S_n$ such that $q(S_i) = 2, p_g(S_i) = 1$. We also prove the Hodge conjecture for arbitrary self-products of a K3 surface $X$ such that the field $E = \text{End}_{h^2}T(X)$ is CM.

0 Notation and preliminaries

Unless otherwise stated, we use the terms curve and surface to denote smooth projective curves and surfaces, resp. The term $p_g(S) = h^{2,0}(S)$ is called the geometric genus of $S$, and $q(S) = h^{1,0}(S) = \dim \text{Alb}(S)$ is known as the irregularity of $S$. For any complex projective manifold $X$, $H^k(X)$ will denote the group $H^k(X, \mathbb{Q})$ regarded as a (rational) Hodge structure of (pure) weight $k$. All Hodge structures appearing in this paper are rational and pure $[2]$; as usual, a Hodge cycle (of codimension $p$) or Hodge class of a Hodge structure $V$ is an element $v \in V^{p,p}_C \cap V$. We denote the subspace of Hodge cycles of $V$ by $\mathcal{H}(V)$, and also $\mathcal{H}^p(X) = \mathcal{H}(H^{2p}(X))$ for $X$ a smooth projective variety; consequently, $\mathcal{H}(X) = \bigoplus_{i=0}^{\dim(X)} \mathcal{H}^i(X)$ will denote the Hodge ring, or ring of Hodge classes of $X$.

We define the (rational) transcendental lattice $T(S)$ of a surface $S$ by the following orthogonal decomposition

$$H^2(S) = T(S) \oplus NS(S)_{\mathbb{Q}}$$
with respect to the cup-product. The cup-product induces, after a change of sign, a polarisation of the Hodge structure $T(S)$.

For $V$ and $W$ two (pure) Hodge structures of the same weight, we denote $\text{Hom}_{\text{hg}}(V,W)$ to be the space of linear maps from $V$ to $W$ respecting the Hodge structures. For an introduction see [2], [5].

For a Hodge structure $V$ as above we define the Hodge group $Hg(V)$ to be the minimal $\mathbb{Q}$-defined algebraic subgroup of $GL(V)$ such that $h(U(1)) \subset Hg(V)\mathbb{R}$; here $h$ is the representation corresponding to the Hodge bigraduation as in [2]. The following is basic in this paper:

**Proposition 0.1** [2] [3] Let $V$ be a polarisable Hodge structure. Then $Hg(V)$ is reductive. As a result, the category of polarisable Hodge structures is semisimple abelian.

For an comprehensive survey on the Hodge conjecture for abelian varieties, as well as a detailed introduction on the Hodge group $Hg(A)$, we refer the reader to [9] Appendix B.

**Acknowledgments:** I thank Prof. Bert van Geemen for a very useful and generous tutorial on the subject. I wish to express my gratitude to Prof. B.J. Totaro for valuable suggestions on the writing of this paper. I am grateful to the EU Research Training Network 'Arithmetic Algebraic Geometry' for their financial support during my PhD. The warm hospitality of the Isaac Newton Institute is gratefully acknowledged. I finally wish to thank Prof. G. E. Welters for introducing me to this subject.

## 1 Introduction

The purpose of this article is to prove the Hodge conjecture in two different situations of product of surfaces. The first one is the product $S_1 \times \cdots \times S_n$, where $q(S_i) = 2$ and $p_g(S_i) = 1$. (It turns out that these surfaces are birationally equivalent to abelian or elliptic isotrivial surfaces). This result generalises the Main Theorem in the author’s PhD thesis [18].

The other case we consider is the following: take a K3 surface $X$; then the transcendental lattice $T(X)$ is irreducible, and its endomorphism algebra is a number field $E = \text{End}_{hg}T(X)$, which can be either CM or totally real ([25] 1.5). We will prove that the Hodge conjecture for arbitrary powers of $X$ follows from the Hodge conjecture for $X \times X$. In the case when $E$ is CM, we use results of Mukai [14], together with an elementary lemma, to prove the Hodge conjecture for $X \times X$, and establish the result for $X^n$ for all $n$ by using invariant theory (see for instance [19] for similar arguments).

## 2 Surfaces $S$ with $p_g = 1, q = 2$

This section is devoted to understanding the geometry of surfaces with $p_g = 1, q = 2$. 2
Proposition 2.1 Let $S$ be a minimal surface with $p_g = 1, q = 2$. If $S$ is not abelian, then $S$ is of the form

$$S = (C' \times E')/G$$

where $C'$ is a curve, $E'$ is an elliptic curve and $G$ acts faithfully on both components.

Proof: One has $\chi(O_S) = 0 = 1 - q + p_g$. From Enriques’ classification we see that $S$ is non-ruled, and $K^2 \geq 0$. Also $e(S) \geq 0$ (see [1] Th. X.4), and by Noether’s formula we get $0 = e(S)$, i.e. $b_2(S) = 6$, and so therefore $K^2 = 0$, which yields $S$ elliptic. Finally, by [23] or [1] Exs. VI.22(4), VIII.22, we see that $S = (C' \times E')/G$ is a finite étale quotient such that $g(E') = 1$, and the proof is thus complete. ■

All the statements concerning motives are, unless otherwise stated, considered in the category of Chow motives modulo homological equivalence. We refer the reader to [21] for the basic notations and language.

Proposition 2.2 (Murre) [21] Let $X$ be a surface. Then there exists a decomposition $h(X) = \bigoplus_{i=0}^{4} h^{i}(X)$; i.e. a Chow-Künneth decomposition exists in the case of surfaces (in fact, modulo rational equivalence).

Remark 2.3 From the above and the standard conjectures for abelian varieties [8], it follows that the Hodge classes on $X \times X$ inducing the projectors $H^{*}(X) \to H^{i}(X) \subset H^{*}(X)$ on a variety $X$ which is a product of surfaces and abelian varieties are all algebraic, and thus $X$ admits a decomposition $h(X) \simeq \bigoplus_{i=0}^{2\dim(X)} h^{i}(X)$ modulo homological equivalence. This result will be used throughout.

Proposition 2.4 Let $S$ be a minimal, not abelian surface such that $p_g = 1, q = 2$. Notations being as in Proposition 2.1 the following cases hold:

(a) either $g(E'/G) = 1$ and the Albanese map $\alpha$ induces an isomorphism $h(S) \simeq h(\text{Alb } S)$, or

(b) $g(E'/G) = 1$ and the Albanese map $\alpha$ sends $S$ onto a curve $B$. It turns out that $B = C'/G$ and the Albanese fibration

$$S = (C' \times E')/G \to \alpha(S) = B = C'/G$$

is the canonical projection.

Proof: The following argument holds in both cases [4] [23]: $H^1((C' \times E')/G) = H^1(C'/G) \oplus H^1(E'/G)$. Since $q(S) = \frac{1}{2}b_1(S)$, we have

$$q(S) = g(C'/G) + g(E'/G)$$

and so the following cases are possible.

(a) $g(E'/G) = 1, g(C'/G) = 1$. In this case $G$ acts on $E'$ by translations, and $A = C'/G \times E'/G$ is an abelian surface; the natural map

$$\phi : S = (C' \times E')/G \to C'/G \times E'/G = A$$

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yields an isomorphism on $H^1$ by the above (and so on $H^3$); therefore $\text{Alb} \ S \sim A$, whence $h^1(S) \simeq h^1(A)$. On $H^2$, the following holds:

$$H^2(S) = H^2(C' \times E')^G = H^2(C'/G \times E'/G),$$

for $G$ acts freely on $C' \times E'$ and trivially on $H^\bullet(E')$; this proves that $\phi|H^2$ is an isomorphism, thus establishing the result.

(b) Let $B = C'/G$. In this case we have $g(E'/G) = 0$ and $g(B) = 2$. The natural map

$$p : S = (C' \times E')/G \to C'/G = B$$

satisfies $q(S) = g(B)$ by Formula (2), and therefore coincides with the Albanese fibration [1]; see also [24] Ch. 9. ■

2.1 The case $g(E'/G) = 0$

Let $S$ satisfy case (b) of Proposition 2.4 and let $H \subset G$ be the subgroup of translations on $E'$. Since $H - \{1\}$ coincides with the set of fixed-point-free transformations of $E'$ in $G$, we have a split exact sequence (we now fix a section $\sigma$)

$$1 \to H \to G \to \mathbb{Z}_n \to 1,$$

where $\mathbb{Z}_n \hookrightarrow \text{Aut}_P(E')$ for $P$ fixed point of a generator $\phi$ of $\sigma(\mathbb{Z}_n)$. Clearly $n \in \{2, 3, 4, 6\}$.

The following proposition is a reduction to the case $G = \mathbb{Z}_n, H = \{1\}$.

**Proposition 2.5** Let $C = C'/H$, $E = E'/H$. If the natural action $\mu$ of $\mathbb{Z}_n = G/H$ on $C \times E$ is étale, then the natural map

$$\beta : S = (C' \times E')/G \to S' = (C \times E)/\mathbb{Z}_n$$

yields an isomorphism of motives $h(S) \simeq h(S')$.

**Proof:** The proof is similar to that of Proposition 2.4(b).

2.2 $\mu$ is free

We suppose $g(E'/G) = 0$, notations being as above. We are going to prove that this case meets the hypotheses of Proposition 2.5.

**Remark 2.6** Consider the Hodge structure

$$V = [H^1(C') \otimes H^1(E')]^G = [H^1(C) \otimes H^1(E)]^{\mathbb{Z}_n}.$$

Then

$$H^2(S) = V \oplus \mathbb{Q}(-1)^{\oplus 2}$$

and $V$ has Hodge numbers $\dim V^{2,0} = 1$, $\dim V^{0,2}$, $\dim V^{1,1} = 2$. 

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Consider the action of $G/H$ on $JC$; let $\phi$ be a generator of $G/H$ such that $\phi^n|H^1,0(E) = \omega$ where $\omega = e^{2\pi i/n}$; let $Q_n(x)$ denote the cyclotomic polynomial of order $n$.

**Theorem 2.7** Let $P := \ker Q_n(\phi)^0 \subset JC$. Then $\dim P = 1$ for $n = 2$ and $\dim P = 2$ for $n = 3, 4, 6$. The quotient map $C \to C/Z_n = B$ is étale in all cases.

**Proof of Theorem 2.7** Consider $V$ as above. It is clear that

\[ V = (H^1(P) \otimes H^1(E))^\mathbb{Z}_n. \]

In the case $n = 2$, $\phi$ acts on both vector spaces as $-Id$, so $V = H^1(P) \otimes H^1(E)$, whence $\dim P = 1$ by inspection. For $n = 3, 4, 6$, let $\chi$ be the character of $\mathbb{Z}_n$ such that $H^{1,0}(E) = \chi$; then $H^{1,0}(P) = a\chi \oplus b\chi$. Inspecting Hodge numbers as above and using Remark 2.6, we find $a = b = 1$, which in turn yields $\dim P = 2$.

From the above we conclude that the action $\mu$ of $\mathbb{Z}_n$ on $C$ has no fixed points. This follows from \[20\] Lemma 1.5; alternatively one can derive this result from several Riemann-Hurwitz type inequalities.

**Corollary 2.8** The motive of a surface $S = (C' \times E')/G$ with $G \neq H$ is isomorphic to that of a surface $(C \times E)/\mathbb{Z}_n$ with $H = \{1\}$. In other words, the conclusion of Proposition 2.5 holds true always.

### 2.3 $h^2(S) \simeq h^2(A)$

We now consider $S$ as above, i.e. with cyclic $G = \mathbb{Z}_n$, such that $B = C/G$ is a genus 2 étale quotient, and find an abelian surface $A$ such that an isomorphism of Hodge structures $H^2(S) \cong H^2(A)$ holds. The first step is to decompose $P$.

**Lemma 2.9** The abelian surface $P$ above splits as $P \simeq E_1 \times E_1$.

**Proof:** Indeed, suppose that $P$ is simple. Then $Hg(P \times E) = Hg(P) \times Hg(E)$ (due to F. Hazama; see e.g. \[9\] B.7.6.2; see also \[11\]), whence the Hodge structure $W = H^1(P) \otimes H^1(E)$ is irreducible (with $\dim W^{2,0} = 2$). Hence $W$ cannot contain $V$, which contradicts our hypothesis. Therefore $P$ must split; using the $\mathbb{Z}_n$-decomposition of $H^1(P)$ from the Proof of Theorem 2.7 and an elementary argument we obtain $P \simeq E_1 \times E_1$ for $E_1$ an elliptic curve, thereby completing the proof.

Let us get back to our $H^2(S)$. We had by Formula (3) and Lemma 2.9

\[ H^2(S) = \mathbb{Q}(-1)^2 \oplus (H^1(P) \otimes H^1(E))^\mathbb{Z}_n \subset \mathbb{Q}(-1)^2 \oplus [H^1(E_1) \otimes H^1(E)]^2. \]

Again, since the transcendental part of $H^1(E_1) \otimes H^1(E)$ has one-dimensional $(2,0)$-part (and is thus irreducible \[9\] \[23\]), by Formula (4) $H^2(S)$ and $H^2(E_1 \times E)$ differ only by powers of the Tate Hodge structure, which implies $H^2(S) \cong H^2(E_1 \times E)$ by counting dimensions. We have thus proven the following Proposition.

**Proposition 2.10** Under the hypotheses of Proposition 2.4(b), the abelian surface $A = E_1 \times E$ is such that $H^2(S) \cong H^2(A)$ (as Hodge structures).
We now proceed to construct an algebraic cycle inducing the described isomorphism. The scheme is the following. Choose a \( \phi \)-equivariant projection \( u : JC \to P \), and consider the correspondence \( \beta = (u_\ast \circ (alb_C)_\ast, id_E) \circ \pi^\ast \) from \( S \) to \( P \times E \), where \( \pi : C \times E \to (C \times E)/\mathbb{Z}_m = S \) is the natural projection. This correspondence from \( S \) to \( P \times E \) realises the inclusion in Formula (4). The final step in this construction will be to cook up a correspondence from \( P \times E \sim E_1 \times E_1 \times E \) sending the image of \( V \) onto \( H^1(E_1) \otimes H^1(E) \) in \( E_1 \times E \), which after composing can be easily extended to the sought-after isomorphism.

**Lemma 2.11** Let \( E_1, E_2 \) be two elliptic curves. For every Hodge substructure \( V \) of \( H^1(E_1) \otimes H^1(E_2) \) isomorphic to \( H^1(E_1) \otimes H^1(E_2) \) there exists an algebraic correspondence \( \alpha \) from \( E_1 \times E_1 \times E_2 \) to \( E_1 \times E_2 \) such that \( \alpha_*V = H^1(E_1) \otimes H^1(E_2) \).

**Proof of Lemma 2.11** It suffices to prove that every Hodge correspondence between \( H^1(E_1) \otimes H^1(E_2) \) and \( H^1(E_1 \times E_1) \otimes H^1(E_2) \) is algebraic. This follows from the Hodge conjecture for products of elliptic curves, due to Imai [9] [11] (see also Proposition 2.18 below.)

We are now ready to prove the following result:

**Theorem 2.12** With the assumptions of this Section, the motives \( h^2(S) \) and \( h^2(E_1 \times E) \) are isomorphic (modulo homological equivalence).

**Proof:** To conclude the proof, consider the correspondence \( \beta \) above, which takes \( V \) to its image inside \( H^1(P) \otimes H^1(E) \) of \( P \times E \sim E_1 \times E_1 \times E \). Choose a projection

\[
\alpha : H^1(E_1 \times E_1) \otimes H^1(E) \to H^1(E_1) \otimes H^1(E)
\]

such that \( \alpha|_V \) is a (Hodge) isomorphism. \( \alpha \) is algebraic by Lemma 2.11, and so the composition \( \alpha \circ \beta \), also algebraic, yields the desired isomorphism. \( \blacksquare \)

**Remark 2.13** An explicit isomorphism could be obtained by fiddling with \( \phi^\ast \) as an element of \( M_2(\text{End}(E_1) \otimes \text{End}(E)) \), without the use of Lemma 2.11. We leave this to the reader.

### 2.4 The Hodge Conjecture for \( S_1 \times \cdots \times S_m \), \( p_g(S_i) = 1 \), \( q(S_i) = 2 \)

We are going to prove the following theorem:

**Theorem 2.14** Let \( S_i \) be surfaces such that \( p_g(S_i) = 1 \), \( q(S_i) = 2 \) (\( S_i \) need not be minimal). Then the Hodge conjecture holds for \( S_1 \times \cdots \times S_m \).

**Remark 2.15** Let \( S \) be a surface such that \( p_g = 1 \), \( q = 2 \). In the former sections we have actually proven that the motive of such a surface (minimal or not) is generated (in the Tannakian sense, see [3]) by motives of abelian surfaces and elliptic curves.

The following lemma follows easily from [3] (see also [9] Appendix B) and some linear algebra.
Lemma 2.16 Let $A$ be an abelian variety of dimension $\geq 2$. Then $Hg(H^2(A)) = Hg(A)/\mu_2$. In particular, for $A$ an abelian surface of simple CM type $(F, \Phi)$ one has $U_F(1) \simeq Hg(T(A)) = Hg(A)/\mu_2 = U_F(1)/\mu_2$.

Let $\theta$ be a primitive element of the real quadratic extension $F(x)$ is described by $\theta$. Indeed, the homomorphism of abstract groups $\rho : F^{\times} \to GL(H^{2,0}(A))$ is described by $x \mapsto \rho(x) = \sigma_1(x)\sigma_2(x)$ where $\sigma_i|F_0$ are different. A little Galois theory shows that if $\theta$ is described as above and $\theta_1 \neq \pm \theta$ is an algebraic conjugate then $\theta_1^2 = \tau(\alpha)$ and $E = \mathbb{Q}[\theta + \theta_1]$. One can see that the element $(\theta + \theta_1)^2$ is a primitive element of the real quadratic extension $\mathbb{Q}(\sqrt{N_{F_0}|Q(\alpha)}) \neq F_0$ since $Gal(N|Q) = D_{2,4}$. The Lemma is thus established.

Proof of Lemma 2.17 One need only observe that the subfield $E$ of $\mathbb{C}$ spanned by the action of $F^{\times}$ on $T(A)$ (which can be read on $H^{2,0}(A)$) is quartic CM and not isomorphic to $F$. Indeed, the homomorphism of abstract groups

\[
\rho : F^{\times} \to GL(H^{2,0}(A))
\]

is described by $x \mapsto \rho(x) = \sigma_1(x)\sigma_2(x)$ where $\sigma_i|F_0$ are different. A little Galois theory shows that if $\theta$ is described as above and $\theta_1 \neq \pm \theta$ is an algebraic conjugate then $\theta_1^2 = \tau(\alpha)$ and $E = \mathbb{Q}[\theta + \theta_1]$. One can see that the element $(\theta + \theta_1)^2$ is a primitive element of the real quadratic extension $\mathbb{Q}(\sqrt{N_{F_0}|Q(\alpha)}) \neq F_0$ since $Gal(N|Q) = D_{2,4}$. The Lemma is thus established.

We state the following proposition and prove only the cases not included in Moonen and Zarhin [11]:

Proposition 2.18 Let $A_i$ be abelian varieties of dimension 1 or 2. Then the Hodge conjecture holds for $A_1 \times \cdots \times A_r$ for $r$ an arbitrary natural number.
Proof of Proposition 2.18: By Goursat’s Lemma and the results of Hazama and Moonen-Zarhin one needs only prove the following statement. For $A_i$ such that $\dim A_i \leq 2$ and $\text{Hom}(A_1, A_2) = 0$, one has $Hg(A_1 \times A_2) = Hg(A_1) \times Hg(A_2)$. By Hazama, Moonen and Zarhin the only case left is the following.

Let $A_i$ be simple abelian varieties of CM type. Then $Hg(A_i) = U_{F_i}(1)$ and $Hg(A_1 \times A_2) \subset Hg(A_1) \times Hg(A_2)$ surjects onto both components, so either $Hg(A_1 \times A_2)$ is simple (and the projections are isogenies) or the former inclusion is an equality. Suppose that the projections are isogenies; in this case, $T(A_1) \otimes T(A_2)$ has a Hodge class (in fact, four such classes), and thus there is a Hodge isomorphism $T(A_1) \cong T(A_2)$. This implies that $E_1 = \text{End}_{hg}T(A_1)$ are isomorphic number fields; in the case where the Galois group of $N_1 = F_1^{gal}$ over $\mathbb{Q}$ is $D_{2,4}$, we have $E_1 \simeq E_2$ and it follows from Lemma 2.17 that $F_1 \simeq F_2$ as well. The Proposition follows in this case from Proposition 4.2. For the remaining cases, there is only one CM type for $F$ up to automorphisms of $F|\mathbb{Q}$ and the proof is similar.

Now Theorem 2.14 follows easily from Proposition 2.18 and Remark 2.15.

3 The case of powers of a K3 surface

Let $X$ be a K3 surface, and let $H^\bullet(X) \subset H^\bullet(X)$ be the ring of Hodge classes of $X$. Then $H^\bullet(X) = T(X) \oplus H^\bullet(X)$. $T(X)$ is an irreducible Hodge structure and if $E = \text{End}_{hg}T(X)$ we have an inclusion

$$E \hookrightarrow \text{End}_\mathbb{C}(H^{2,0}(X)) = \mathbb{C}$$

which renders $E$ a number field. It can be shown that $E$ is either totally real or CM.

The following proposition holds:

Proposition 3.1 The Hodge conjecture for $X^n$, for arbitrary $n$, holds if it holds for $X \times X$.

Proof: The ring of Hodge classes $H^\bullet(X^n)$ is, by the above, generated by the Hodge classes in the tensor powers of $T(X)$ up to order $n$ and by pullbacks of algebraic classes on $X$ via the canonical projections. Thus our result amounts to show that the ring of tensor invariants of the $Hg(X)$-module $T(X)$ is generated by those of degree 2 as an algebra; it is known (see [25]) that $Hg(X)_\mathbb{C}$ is isomorphic to a product of special orthogonal or general linear groups, which shows (see [19]) that the ring of tensor invariants of $Hg(X)$ is generated by the degree-2 invariants, thereby establishing the result.

We now prove the Hodge conjecture for self-products of a K3 surface $X$ in the case where $E$ is a CM field. We need the following elementary lemma.

Lemma 3.2 Let $E$ be a CM number field. Then $E$ is spanned as a vector space over $\mathbb{Q}$ by elements $\alpha_i \in E$ such that $\alpha_i \overline{\alpha}_i = 1$. 8
Proof of Lemma 3.2: Let $\chi_0: E^\times \to E^\times$ be given by $\chi_0(\alpha) = \alpha/\bar{\alpha}$. Suppose that the images of $\chi_0$ do not span $E$ over $\mathbb{Q}$; then there exists $\theta \in E$ such that

$$\text{Tr}_{E|\mathbb{Q}}(\theta \chi_0(\alpha)) = 0 \text{ for all } \alpha \in E^\times.$$ 

Now let $\chi_\sigma = \sigma \circ \chi_0$ for $\sigma: E \hookrightarrow \mathbb{C}$ an embedding; by Artin’s linear independence of characters, there are embeddings $\sigma \neq \tau$ such that $\chi_\sigma = \chi_\tau$, which amounts to saying that $\sigma(\alpha)/\tau(\alpha)$ is always real. It is not difficult to see that, since $E$ is non-real CM, this cannot hold if $\sigma$ and $\tau$ are different; indeed, evaluating at $\alpha$ and $1+\alpha$ for $\alpha \in E$ neither real nor purely imaginary, we see that $1+\sigma(\alpha)$ does not belong to $\mathbb{R}(1+\tau(\alpha))$, which leads to a contradiction, thereby establishing the Lemma.

We are now ready to prove our Theorem. See Morrison [12] for an earlier result in this direction.

Theorem 3.3 Let $X$ be a K3 surface such that $E = \text{End}_{h^0}(T(X))$ is a CM field. Then the Hodge conjecture holds for arbitrary powers of $X$.

Proof of Theorem 3.3 By the above Lemma 3.2, it suffices to prove algebraicity for $\alpha \in E$ such that $\alpha \cdot \bar{\alpha} = 1$, i.e. for the Hodge isometries of the polarised Hodge structure $(T(X), Q)$ [25]. This is a result established by Mukai, by refining former results on his theory of moduli:

Theorem 3.4 [14] Let $X_1$ and $X_2$ be K3 surfaces, and let $\psi: T(X_1) \to T(X_2)$ be a Hodge isometry. Then $\psi$ is induced by an algebraic cycle.

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