Thermal superspace is characterized by Grassmann variables which are time-dependent and antiperiodic in imaginary time, with a period given by the inverse temperature. The thermal superspace approach allows to define thermal superfields obeying consistent boundary conditions and to formulate a “super-KMS” condition for superfield propagators. Upon constructing thermal covariantizations of the supersymmetry generators, we define thermal covariant derivatives and provide a definition of thermal chiral and antichiral superfields. Thermal covariantizations of the generators of the super-Poincaré algebra are also constructed, and the thermal supersymmetry algebra is computed; it has the same structure as at \( T = 0 \). We then investigate realizations of this thermal supersymmetry algebra on systems of thermal fields. In doing so, we observe thermal supersymmetry breaking in terms of the lifting of the mass degeneracy, and of the non-invariance of the thermal action.

I. INTRODUCTION

These proceedings present an account of recent work in which realizations of supersymmetry at finite temperature have been investigated in terms of thermal superfields. These are defined in a thermally constrained superspace, baptized “thermal superspace”. The thermal superspace approach provides a new framework for the study of thermal supersymmetry breaking. Previous investigations of supersymmetry at finite temperature can be found in \[2, 3\].

Supersymmetry and thermal effects are incompatible as is. On the one hand, supersymmetry treats bosons and fermions on an equal footing, as members of the same supermultiplet. On the other hand, thermal bosons and fermions are strongly distinguished by their thermal behaviour. The thermal superspace approach allows to reconcile these conflicting notions. Thermal superspace is spanned by time-dependent and antiperiodic Grassmann parameters, and makes it possible to write consistent boundary conditions, as well as KMS conditions, at the level of thermal superfields. These conditions not only can be proven directly in thermal superspace, they also imply the correct, bosonic or fermionic, b.c.’s, resp. KMS conditions, for the superfield’s components. So thermal superspace is the correct superspace to be used at finite temperature, and the information on thermal supersymmetry breaking is encoded in the temperature-dependent constraints its Grassmann variables obey. Upon viewing the passage from \( T = 0 \) superspace to thermal superspace as a change of coordinates, we then easily construct the thermal covariant derivatives and show that they provide a consistent definition of the thermal superfields.

At \( T = 0 \), superspace provides a natural representation for the supersymmetry algebra. Taking the point of view that the same holds at \( T > 0 \), we construct thermal covariantizations of the supersymmetry generators and compute their algebra. The thermal supersymmetry algebra so obtained has the same form and number of supersymmetries as at \( T = 0 \). It is only when trying to realize this thermal algebra on thermal fields that one is faced with thermal supersymmetry breaking. Indeed, the boundary conditions as well as the KMS relations – which distinguish bosons from fermions at finite temperature – are of space-time global character. The supersymmetry algebra, being a local structure, is insensitive to such global conditions. At the level of thermal fields however, supersymmetry breaking is signalled by the lifting of the \( T = 0 \) mass degeneracy, as well as in terms of the non-invariance of the thermal action under thermal supersymmetry transformations.

Thermal superspace can be motivated through the following, heuristic argument. Consider first supersymmetry at zero temperature. Due to \( \{Q_\alpha, \overline{Q}_\dot{\alpha}\} = 2\sigma_{\alpha\dot{\alpha}}^\mu P_\mu \), the supersymmetry charges can be viewed as “square roots” of translations. Expressing the supercharges as generalized translations acting through derivatives only is however not possible in space-time \( x^\mu \), the parameter space of translations, and requires that one enlarges that parameter space to contain, in addition to \( x^\mu \), a new set of Grassmannian coordinates – denoted \( \theta^\alpha \) and \( \overline{\theta}^{\dot{\alpha}} \) – which are translated under the action of the supercharges. A point \( X \) in superspace has therefore coordinates \( X = (x^\mu, \theta^\alpha, \overline{\theta}^{\dot{\alpha}}) \), and since at zero temperature the parameters of supersymmetry transformations are space-time constant, the zero-temperature superspace coordinates \( \theta^\alpha \) and \( \overline{\theta}^{\dot{\alpha}} \) are
space-time constants as well.

However, one cannot make use of constant parameters in supersymmetry transformations rules at finite temperature: the supersymmetry parameters must be time-dependent and antiperiodic in imaginary time on the interval \([0, \beta]\), where \(\beta = 1/T\) denotes the inverse temperature (see also \[3\]). Adapting straightforwardly the zero-temperature argument above, it appears natural to require that the variables which are translated by the effect of the thermal supercharges bear the same characteristics as the thermal supersymmetry parameters. From this we conclude that thermal superspace must be spanned, in addition to usual space-time, by Grassmann parameters which are time-dependent and antiperiodic in imaginary time on the interval \([0, \beta]\). A point in thermal superspace has therefore coordinates

\[
\hat{X} = (x^\alpha, \tilde{\theta}(t), \tilde{\vartheta}(t)),
\]

where a “hat” is used to denote thermal quantities, and \(\tilde{\theta}(t), \tilde{\vartheta}(t)\) are subject to the antiperiodicity conditions

\[
\tilde{\theta}(t + i\beta) = -\tilde{\theta}(t), \quad \tilde{\vartheta}(t + i\beta) = -\tilde{\vartheta}(t).
\] (1)

These conditions induce a temperature-dependent constraint on the time-dependent superspace Grassmann coordinates \(\hat{\theta}(t)\) and \(\hat{\vartheta}(t)\).

The heuristic argument presented here is supported by an independent formal argument based on the KMS conditions, which we develop in Section II.

II. FROM KMS TO SUPER-KMS CONDITIONS

Consider first a free real scalar field \(\varphi\) carrying no conserved charges. The hamiltonian \(H\) being the time evolution operator, the field \(\varphi\) at \(x = (t, \mathbf{x})\) (in the Heisenberg picture and with \(\hbar = c = 1\)) can be obtained as

\[
\varphi(x) = \varphi(t, \mathbf{x}) = e^{iHT}\varphi(0, \mathbf{x})e^{-iHT},
\] (2)

with a time coordinate \(x^0 = t\) which is allowed to be complex. The thermal bosonic propagator \(D_C\) writes

\[
D_C(x_1, x_2) = \langle T_C\varphi(x_1)\varphi(x_2)\rangle_\beta = \theta_C(t_1 - t_2)D_C^\beta(x_1, x_2) + \vartheta_C(t_2 - t_1)D_C^\vartheta(x_1, x_2),
\]

where \(\langle ...\rangle_\beta\) denotes the (canonical) thermal average, \(T_C\) denotes path ordering, \(C\) is the path Heaviside function (see [3] for details), and the two-point functions \(D_C^\beta, D_C^\vartheta\) are defined as

\[
D_C^\beta(x_1, x_2) = \langle \varphi(x_1)\varphi(x_2)\rangle_\beta, \quad D_C^\vartheta(x_1, x_2) = \langle \varphi(x_2)\varphi(x_1)\rangle_\beta.
\]

The Boltzmann weight \(e^{-\beta H}\) can be interpreted as an evolution operator in imaginary time. Indeed, rewriting \([3]\) for a translation in imaginary time by \(t = i\beta\), one gets

\[
e^{-\beta H}\varphi(t, \mathbf{x})e^{\beta H} = \varphi(t + i\beta, \mathbf{x}).
\] (3)

Now, starting from \(D_C^\beta\), using the cyclicity of the thermal trace (upon inserting \(e^{\beta H}e^{-\beta H} = 1\)) and the evolution \([3]\), one deduces the bosonic KMS (Kubo-Martin-Schwinger) condition \([3]\). This condition relates \(D_C^\beta, D_C^\vartheta\) through a translation in imaginary time:

\[
D_C^\beta(t_1; x_1, t_2; x_2) = D_C^\beta(t_1 + i\beta; x_1, t_2; x_2). \tag{4}
\]

A similar derivation holds for fermionic fields. Defining the fermionic two-point functions \(S_C^\alpha_{ab}, S_C^{\bar{\alpha}}_{ab}\) [with \(a, b = 1, ..., 4\) for Dirac (four-component) spinors] as

\[
S_C^\alpha_{ab}(x_1, x_2) = \langle \psi^a_\alpha(x_1)\bar{\psi}^b_\beta(x_2)\rangle_\beta, \quad S_C^{\bar{\alpha}}_{ab}(x_1, x_2) = -\langle \bar{\psi}^b_\alpha(x_2)\psi^a_\beta(x_1)\rangle_\beta,
\]

and following the same procedure as in the bosonic case, one derives the fermionic KMS condition

\[
S_C^\alpha_{ab}(t_1; x_1, t_2; x_2) = -S_C^{\bar{\alpha}}_{ab}(t_1 + i\beta; x_1, t_2; x_2), \tag{5}
\]

which differs from the bosonic one by a relative sign.

We shall be interested ahead in deriving KMS conditions for superfields. Superfields are usually formulated using two-component Weyl spinors \(\psi_\alpha\) and \(\bar{\psi}^\alpha\), which are related to Dirac spinors through \(\psi_\alpha = \left(\begin{array}{cc} \bar{\psi}^\alpha \\ \psi_\alpha \end{array}\right)\). The KMS condition for Dirac spinors \([3]\) thus translates into a set of KMS conditions for two-component spinors \(\psi_\alpha, \bar{\psi}^\alpha\). Defining the thermal two-point functions \(S_C^\alpha_{a\beta}, S_C^{\bar{\alpha}}_{a\beta}\) for two-component spinors as, e.g., \([3]\),

\[
S_C^\alpha_{a\beta}(x_1, x_2) = \langle \psi_\alpha(x_1)\bar{\psi}^\beta(x_2)\rangle_\beta, \quad S_C^{\bar{\alpha}}_{a\beta}(x_1, x_2) = -\langle \bar{\psi}^\beta(x_2)\psi_\alpha(x_1)\rangle_\beta,
\]

we derive from \([3]\) the fermionic KMS condition for two-component Majorana spinors:

\[
S_C^{\alpha\beta}_{a\bar{\beta}}(t_1; x_1, t_2; x_2) = -S_C^{\bar{\alpha}\beta}_{a\bar{\beta}}(t_1 + i\beta; x_1, t_2; x_2). \tag{6}
\]

The KMS conditions derived above provide an essential, mandatory characterization of thermal effects at the level of thermal Green’s functions, and induce a clear distinction between bosons and fermions. So, thermal physics is in conflict with supersymmetry, which treats

\[1\] This is the only relation we shall need for practical purposes. Of course, similar relations can be derived for \(S_C^{a\beta_{ab}}, S_C^{{a\beta}}_{ab}\) and \(S_C^{a\bar{\beta}_{ab}}\).
bosons and fermions on an equal footing, as members of closed supermultiplets. One convenient way of describing supermultiplets is to use the language of superfields. Superfields are superspace expansions which contain as components the bosonic and fermionic degrees of freedom of supermultiplets. And imposing the KMS conditions for the superfield components at finite temperature – if feasible – must result in temperature-dependent constraints on the superfield expansion parameters. So we first grant some freedom to the superspace Grassmann parameters by allowing them to depend on imaginary time. Then we show that the superfield formalism can be reconciled with thermal physics provided superspace is constrained by requiring that the Grassmann variables be antiperiodic in their imaginary time variable, with a period given by the inverse temperature.

Let us first settle some notions in the zero-temperature case. \( T = 0 \) chiral superfields, noted \( \phi \), and \( T = 0 \) antichiral superfields, denoted by \( \overline{\phi} \), are defined by the conditions

\[
\overline{D}_\alpha \phi = 0, \quad D_\alpha \overline{\phi} = 0, \tag{7}
\]

where the \( T = 0 \) covariant derivatives \( D_\alpha \), \( \overline{D}_\alpha \) write

\[
D_\alpha = \frac{\partial}{\partial \theta^\alpha} - i \sigma^\mu_{\alpha \beta} \overline{\sigma}^\beta \partial_\mu, \tag{8}
\]

\[
\overline{D}_\alpha = \frac{\partial}{\partial \bar{\theta}^\alpha} - i \bar{\sigma}^\mu_{\alpha \beta} \sigma^\beta \partial_\mu. \tag{9}
\]

For chiral and antichiral superfields, it is technically more convenient to use the so-called chiral and antichiral coordinates, instead of the usual superspace coordinates \((x^\mu, \theta, \bar{\theta})\). Chiral, resp. antichiral, coordinates are given by \((y^\mu, \theta^\alpha, \bar{\theta}^\alpha)\) and \((\overline{y}^\mu, \bar{\theta}^\alpha, \theta^\alpha)\), with \( y \) and \( \overline{y} \) defined by the combinations

\[
y^\mu = x^\mu - i \theta^\alpha \sigma^\mu_{\alpha \beta} \bar{\sigma}^\beta, \quad \bar{y}^\mu = x^\mu + i \bar{\theta}^\alpha \sigma^\mu_{\alpha \beta} \sigma^\beta. \tag{10}
\]

In these variables, the expansions of \( T = 0 \) chiral and antichiral superfields are simply

\[
\phi(y, \theta) = z(y) + \sqrt{2} \theta \psi(y) - \theta \theta f(y), \tag{11}
\]

\[
\bar{\phi}(\overline{y}, \bar{\theta}) = \overline{z}(\overline{y}) + \sqrt{2} \bar{\theta} \bar{\psi}(\overline{y}) - \bar{\theta} \bar{\theta} \bar{f}(\overline{y}). \tag{12}
\]

The components of the superfields \( \phi \) and \( \overline{\phi} \) form a chiral supermultiplet which contains two complex scalar fields \( z \) and \( f \) and a Majorana spinor with Weyl components \( \psi \) and \( \bar{\psi} \).

Consider now the \( T = 0 \) chiral-antichiral superfield propagator \( G(y_1, \overline{y}_2, \theta_1, \bar{\theta}_2) = \langle 0| T \phi(y_1, \theta_1) \overline{\phi}(\overline{y}_2, \bar{\theta}_2)|0 \rangle \). Its superspace expansion can equivalently be cast in the variables \((x, \theta, \bar{\theta})\) or the chiral ones, as

\[
G(y_1, \overline{y}_2, \theta_1, \bar{\theta}_2) = D(y_1 - \overline{y}_2) - 2 \theta_1 \overline{\theta}_2 \beta S^\alpha_\beta(y_1 - \overline{y}_2) + \theta_1 \bar{\theta}_2 \bar{F}(y_1 - \overline{y}_2) \tag{13}
\]

\[
= D(x_1 - x_2) - 2 \theta_1 \overline{\theta}_2 \beta S^\alpha_\beta(x_1 - x_2) + \theta_1 \bar{\theta}_2 \bar{F}(x_1 - x_2) + \text{derivative terms},
\]

and includes the \( T = 0 \) Green’s functions for the superfield’s components:

\[
D(x_1 - x_2) \equiv \langle 0| T \phi(x_1) \overline{\phi}(x_2)|0 \rangle, \quad
\bar{F}(x_1 - x_2) \equiv \langle 0| T \bar{\phi}(x_1) \phi(x_2)|0 \rangle, \quad S^\beta_\alpha(x_1 - x_2) \equiv \langle 0| T \psi_\alpha(x_1) \bar{\psi}^\beta(x_2)|0 \rangle.
\]

We now put this system of propagating component fields in a thermal bath at some finite temperature \( T \). As the heat bath affects propagation, the \( T = 0 \) propagators above must be replaced by their thermal counterparts:

\[
D_C(x_1 - x_2) \equiv \langle 0| T_C \phi(x_1) \overline{\phi}(x_2)|0 \rangle, \quad
\bar{F}_C(x_1 - x_2) \equiv \langle 0| T_C \bar{\phi}(x_1) \phi(x_2)|0 \rangle, \quad S_C^\beta_\alpha(x_1 - x_2) \equiv \langle 0| T_C \psi_\alpha(x_1) \bar{\psi}^\beta(x_2)|0 \rangle, \tag{14}
\]

\[
D_C^\beta_\alpha(t_1; x_1, t_2; x_2) = D_C^\beta_\alpha(t_1 + i \beta; x_1, t_2; x_2), \tag{15}
\]

\[
\bar{F}_C^\beta_\alpha(t_1; x_1, t_2; x_2) = \bar{F}_C^\beta_\alpha(t_1 + i \beta; x_1, t_2; x_2), \tag{16}
\]

while the thermal propagator of the fermionic component has to satisfy the fermionic constraint \( \bar{C} \),

\[
S_C^\beta_\alpha(t_1; x_1, t_2; x_2) = - S_C^\alpha_\beta(t_1 + i \beta; x_1, t_2; x_2). \tag{16}
\]

But the presence of a heat bath not only enforces the KMS conditions for the superfields components. It also obliges us to adapt the notion of superfield to the thermal context. Indeed the usual \( T = 0 \) superfield formulation is by construction supersymmetric, and is therefore incompatible with thermal effects, as we have argued above.

In the Introduction, we have motivated the fact that, at finite temperature, the superspace Grassmann variables should be dependent on imaginary time and antiperiodic. Consequently, we promote \( \theta \) and \( \overline{\theta} \) to become thermal superspace coordinates \( \theta \rightarrow \tilde{\theta} = \tilde{\theta}(t) \), \( \overline{\theta} \rightarrow \overline{\tilde{\theta}} = \overline{\tilde{\theta}}(t) \) obeying the antiperiodicity properties \( \tilde{\theta} \).

In taking superspace Grassmann coordinates \( \tilde{\theta}(t), \overline{\tilde{\theta}}(t) \), we also introduce a formal time-dependence in the second terms of the variables \( y \) and \( \overline{y} \) [10], which we henceforth denote by:

\[
\tilde{y}^\mu(t) = x^\mu - i \tilde{\theta}(t) \sigma^\mu \tilde{\theta}(t), \quad 
\tilde{\overline{y}}^\mu(t) = x^\mu + i \tilde{\theta}(t) \sigma^\mu \tilde{\theta}(t). \tag{17}
\]

2 Similar expansions can be written for the “chiral-chiral” superfield propagator \( \langle 0| T \phi(y_1, \theta_1) \phi(y_2, \theta_2)|0 \rangle \) and for the conjugate, “antichiral-antichiral” \( \langle 0| T \overline{\phi}(\overline{y}_1, \overline{\theta}_1) \overline{\phi}(\overline{y}_2, \overline{\theta}_2)|0 \rangle \).
In thermal superspace, we define chiral and antichiral superfields at finite temperature, denoted by the “hat” notation $\hat{\phi}$, resp. $\hat{\phi}$, similarly to (11), (12), but with the thermal superspace Grassmann coordinates $\hat{\theta}(t), \hat{\bar{\theta}}(t)$ as the expansion parameters. This yields

\begin{align*}
\hat{\phi}[\hat{y}, \hat{\bar{\theta}}] &= z[\hat{y}] + \sqrt{2} \hat{\theta} \psi[\hat{y}] - \hat{\theta} \theta f[\hat{y}], \\
\hat{\bar{\theta}}[\bar{\hat{y}}, \hat{\bar{\theta}}] &= \bar{z}[\bar{\hat{y}}] + \sqrt{2} \hat{\theta} \bar{\psi}[\bar{\hat{y}}] - \hat{\theta} \theta \bar{f}[\bar{\hat{y}}].
\end{align*}

The consistency of these thermal expansions is discussed in Section 11 ahead, where we also give the $(x, \hat{\theta}, \hat{\bar{\theta}})$-expansion for $\phi$, eq. (20). In the same spirit, we next define the thermal chiral-antichiral superfield propagator

\begin{equation*}
G_C[\hat{y}_1, \hat{y}_2, \hat{\bar{\theta}}_1, \hat{\bar{\theta}}_2] = \langle T_C \hat{\phi}[\hat{y}_1, \hat{\bar{\theta}}_1] \hat{\bar{\theta}}[\bar{\hat{y}}_2, \hat{\bar{\theta}}_2] \rangle_{\beta},
\end{equation*}

and expand it in thermal superspace, in analogy to (13), as

\begin{equation*}
G_C[\hat{y}_1(t_1), \hat{y}_2(t_2), \hat{\bar{\theta}}_1(t_1), \hat{\bar{\theta}}_2(t_2)] = D_C[\hat{y}_1(t_1) - \hat{y}_2(t_2)] - 2 \hat{\theta}_1(t_1) \hat{\bar{\theta}}_2(t_2) S_C \alpha \beta[\hat{y}_1(t_1) - \hat{y}_2(t_2)]
+ \hat{\theta}_1(t_1) \hat{\bar{\theta}}(t_1) \hat{\bar{\theta}}_2(t_2) \hat{\bar{\theta}}_2(t_2) F_C[\hat{y}_1(t_1) - \hat{y}_2(t_2)].
\end{equation*}

The thermal superfield two-point functions $G_C$, resp. $G_C^\sigma$, can be defined in relation to $G_C$ through

\begin{equation*}
G_C[\hat{y}_1(t_1), \hat{y}_2(t_2), \hat{\bar{\theta}}_1(t_1), \hat{\bar{\theta}}_2(t_2)] = \theta_C(t_1 - t_2) G_C^\sigma[\hat{y}_1(t_1), \hat{y}_2(t_2), \hat{\bar{\theta}}_1(t_1), \hat{\bar{\theta}}_2(t_2)]
+ \theta_C(t_2 - t_1) G_C \sigma[\hat{y}_1(t_1), \hat{y}_2(t_2), \hat{\bar{\theta}}_1(t_1), \hat{\bar{\theta}}_2(t_2)],
\end{equation*}

with

\begin{align*}
G_C^\sigma[\hat{y}_1, \hat{y}_2, \hat{\bar{\theta}}_1, \hat{\bar{\theta}}_2] &= \langle \phi[\hat{y}_1, \hat{\bar{\theta}}_1] \hat{\bar{\theta}}[\bar{\hat{y}}_2, \hat{\bar{\theta}}_2] \rangle_{\beta}, \tag{20} \\
G_C^\sigma[\hat{y}_1, \hat{y}_2, \hat{\bar{\theta}}_1, \hat{\bar{\theta}}_2] &= \langle \phi[\hat{y}_2, \hat{\bar{\theta}}_2] \hat{\bar{\theta}}[\bar{\hat{y}}_1, \hat{\bar{\theta}}_1] \rangle_{\beta}. \tag{21}
\end{align*}

The KMS condition can now be formulated at the level of thermal superfield propagators. The superfield KMS (or super-KMS) condition is:

\begin{equation*}
G_C^\sigma[\hat{y}_1(t_1), \hat{y}_2(t_2), \hat{\bar{\theta}}_1(t_1), \hat{\bar{\theta}}_2(t_2)] = G_C^\sigma[\hat{y}_1(t_1 + i\beta), \hat{y}_2(t_2), \hat{\bar{\theta}}_1(t_1 + i\beta), \hat{\bar{\theta}}_2(t_2)], \tag{22}
\end{equation*}

with the time-translated variable $\hat{y}_1(t_1 + i\beta)$ given by

\begin{equation*}
\hat{y}_1(t_1 + i\beta) = \hat{y}_1(t_1) + (i\beta; 0), \tag{23}
\end{equation*}

upon making use of the antiperiodicity (1).

\[ \text{3Here and in the sequel, we simplify the notation by occasionally using } \hat{y}_1 \text{ and } \hat{\bar{\theta}}_1 \text{ instead of } \hat{y}_1(t_1) \text{ and } \hat{\bar{\theta}}_1(t_1) \text{ in non ambiguous situations.} \]

Clearly, the superfield KMS condition (22) is of bosonic type, since chiral and antichiral superfields are bosonic objects. This condition can be proven at the superfield level in a way similar to the case of the scalar field. Let us start by formulating the evolution in imaginary time for a, e.g., chiral, thermal superfield. Applying this evolution, eq. (23), to the components in $\hat{\phi}$, and using (23), we get

\begin{align*}
&= z[\hat{y}_1(t_1 + i\beta; \hat{\bar{\theta}}(t_1) + i\beta; \hat{\bar{\theta}}(t_1))] \\
&= \hat{\phi}[\hat{y}_1(t_1 + i\beta; \hat{\bar{\theta}}(t_1))].
\end{align*}

(24)

Note that the time arguments of $\hat{\theta}(t)$ and $\hat{\bar{\theta}}(t)$ are not shifted. The thermal Grassmann variables which are coordinates – do not undergo dynamical evolution in imaginary time under the Hamiltonian.

In order to prove the superfield KMS relation (22) we start from the thermal superfield two-point function $G_C^\sigma$

\begin{equation*}
G_C^\sigma[\hat{y}(t_1), \hat{y}(t_2), \hat{\bar{\theta}}(t_1), \hat{\bar{\theta}}(t_2)] = \frac{1}{Z(\beta)} \text{Tr} \left\{ e^{-\beta H} \hat{\phi}[\hat{y}(t_1), \hat{\bar{\theta}}(t_1)] \hat{\bar{\theta}}[\hat{\bar{y}}(t_2), \hat{\bar{\theta}}(t_2)] \right\},
\end{equation*}

and introduce the thermal component expansions for the superfields [eqs. (18)–(19)]. We then rotate cyclically $\hat{\phi}$ to the front, insert the identity $e^{\beta H} e^{-\beta H} = 1$, and rotate $e^{-\beta H}$ to the front. The right side therefore rewrites as:

\begin{equation*}
\frac{1}{Z(\beta)} \text{Tr} \left\{ e^{-\beta H} \left( \hat{z}[\hat{y}_2] - \sqrt{2} \hat{\theta}_2 \hat{\bar{\theta}}[\hat{y}_2] - \hat{\bar{\theta}}_2 \hat{\bar{\theta}} F[\hat{y}_2] \right) \right. \times e^{-\beta H} \left( z[\hat{y}] + \sqrt{2} \hat{\theta}_1 \psi[\hat{y}] - \hat{\theta}_1 \hat{\bar{\theta}} f[\hat{y}] \right) \right\}.
\end{equation*}

The negative sign in front of the fermionic component of $\hat{\phi}$ follows from the anticommutativity of the Grassmann variables. We now insert the superfield time evolution (24) and recast the last expression as:

\begin{equation*}
\frac{1}{Z(\beta)} \text{Tr} \left\{ e^{-\beta H} \left( \hat{z}[\hat{y}_2] - \sqrt{2} \hat{\theta}_2 \hat{\bar{\theta}}[\hat{y}_2] \right) \times \left( z[\hat{y}_1(t_1 + i\beta)] + \sqrt{2} \hat{\theta}_1 \psi[\hat{y}_1(t_1 + i\beta)] - \hat{\theta}_1 \hat{\bar{\theta}} f[\hat{y}_1(t_1 + i\beta)] \right) \right\}.
\end{equation*}

Making use of the antiperiodicity (1) of the Grassmann variables, we set $\hat{\theta}_1(t_1) = -\hat{\theta}_1(t_1 + i\beta)$ and get
\[
\frac{1}{Z(\beta)} \text{Tr} \left\{ e^{-\beta H} \left( \tilde{\psi} [\tilde{\varphi}_2(t_2)] - \sqrt{2} \tilde{\varphi}_2(t_2) \tilde{\psi} [\tilde{\varphi}_2(t_2)] \right) \\
- \tilde{\varphi}_2(t_2) \tilde{\varphi}_2(t_2) F_C [\tilde{\varphi}_1(t_1), \tilde{\varphi}_2(t_2)] \right\}.
\]

Observing that fermionic fields do not propagate into bosonic fields, and vice-versa, we replace the two negative signs in front of the fermionic components by two positive signs. The second thermal superfield in the product then identifies to \( \tilde{\varphi} \) with all time arguments shifted by \( i\beta \). As a result, the above computation yields just the superfield KMS condition (22), which is hereby proved.

We verify now that the superfield KMS condition (22) yields the expected component relations (14)–(15) and (16). Expanding \( \tilde{\varphi} \) and \( \varphi \) along (15)–(17) in eqs. (24)–(21) yields

\[
G_C^\alpha [\tilde{\varphi}_1(t_1), \tilde{\varphi}_2(t_2)] = D_C^\alpha [\tilde{\varphi}_1(t_1), \tilde{\varphi}_2(t_2)] - 2\tilde{\theta}^{\alpha} [t_1] \tilde{\theta}^{\beta} [t_2] S_{C,\alpha}^\beta [\tilde{\varphi}_1(t_1), \tilde{\varphi}_2(t_2)] \\
+ \tilde{\theta}_1(t_1) \tilde{\theta}_1(t_1) \tilde{\theta}_2(t_2) \tilde{\theta}_2(t_2) F_C^\alpha [\tilde{\varphi}_1(t_1), \tilde{\varphi}_2(t_2)],
\]

and

\[
G_C^\alpha [\tilde{\varphi}_1(t_1), \tilde{\varphi}_2(t_2)] = D_C^\alpha [\tilde{\varphi}_1(t_1), \tilde{\varphi}_2(t_2)] - 2\tilde{\theta}^{\alpha} [t_1] \tilde{\theta}^{\beta} [t_2] S_{C,\alpha}^\beta [\tilde{\varphi}_1(t_1), \tilde{\varphi}_2(t_2)] \\
+ \tilde{\theta}_1(t_1) \tilde{\theta}_1(t_1) \tilde{\theta}_2(t_2) \tilde{\theta}_2(t_2) F_C^\alpha [\tilde{\varphi}_1(t_1), \tilde{\varphi}_2(t_2)].
\]

Replacing these developments in the superfield KMS condition (22) leads then to the following: (i) For the scalar component,

\[
D_C^\alpha [\tilde{\varphi}_1(t_1), \tilde{\varphi}_2(t_2)] = D_C^\alpha [\tilde{\varphi}_1(t_1), \tilde{\varphi}_2(t_2)],
\]

which reduces, when returning to the variables \( x = (t; \mathbf{x}) \) by Taylor expanding around \( x^\mu \), to the bosonic KMS relation (14). (ii) For the fermionic component,

\[
\tilde{\theta}^{\alpha} [t_1] \tilde{\theta}^{\beta} [t_2] S_{C,\alpha}^\beta [\tilde{\varphi}_1(t_1), \tilde{\varphi}_2(t_2)] \\
= \tilde{\theta}^{\alpha} [t_1 + i\beta] \tilde{\theta}^{\beta} [t_2] S_{C,\alpha}^\beta [\tilde{\varphi}_1(t_1 + i\beta), \tilde{\varphi}_2(t_2)].
\]

With the antiperiodicity condition (16), \( \tilde{\theta}^{\alpha} [t_1 + i\beta] = -\tilde{\theta}^{\alpha} [t_1] \), we obtain

\[
S_{C,\alpha}^\beta [\tilde{\varphi}_1(t_1), \tilde{\varphi}_2(t_2)] = -S_{C,\alpha}^\beta [\tilde{\varphi}_1(t_1 + i\beta), \tilde{\varphi}_2(t_2)],
\]

which yields, in the variables \( x = (t; \mathbf{x}) \), the fermionic KMS condition (15) with the correct relative sign. Finally (iii) for the auxiliary field, one gets

\[
\tilde{\theta}_1(t_1) \tilde{\theta}_1(t_1) \tilde{\theta}_2(t_2) \tilde{\theta}_2(t_2) F_C^\alpha [\tilde{\theta}_1(t_1), \tilde{\theta}_2(t_2)] \\
= \tilde{\theta}_1(t_1 + i\beta) \tilde{\theta}_1(t_1 + i\beta) \tilde{\theta}_2(t_2) \tilde{\theta}_2(t_2) F_C^\alpha [\tilde{\theta}_1(t_1 + i\beta), \tilde{\theta}_2(t_2)].
\]

With \( \tilde{\theta}_1(t_1 + i\beta) = -\tilde{\theta}_1(t_1) \), and in the variables \( x = (t; \mathbf{x}) \), this is the bosonic KMS condition (16).

### III. THERMAL COVARIANT DERIVATIVES

Deriving expressions for the covariant derivatives and supercharges on thermal superspace can be done simply by performing the change of variables from usual, zero temperature, superspace to thermal superspace, i.e.,

\[
(x^\mu, \theta, \bar{\theta}) \rightarrow (x'^\mu = x^\mu, \theta' = \hat{\theta}(t), \bar{\theta}' = \hat{\bar{\theta}}(t)),
\]

with \( t = x^0 \). Under this change of variables, the partial derivatives with respect to \( x, \theta \) and \( \bar{\theta} \) transform trivially,

\[
\frac{\partial}{\partial x} \rightarrow \frac{\partial}{\partial x'}, \quad \frac{\partial}{\partial \theta} \rightarrow \frac{\partial}{\partial \theta'}, \quad \frac{\partial}{\partial \bar{\theta}} \rightarrow \frac{\partial}{\partial \bar{\theta}^*},
\]

while the time derivative has to take the time-dependence of the thermal Grassmann variables into account:

\[
\frac{\partial}{\partial t} \rightarrow \frac{\partial}{\partial t'} + \frac{\partial}{\partial t} \frac{\partial}{\partial \theta} + \frac{\partial}{\partial \bar{\theta}^*} \frac{\partial}{\partial \bar{\theta}} \left( \frac{\partial t'}{\partial t} = 1 \right).
\]

Consequently, we define the partial time derivative at finite temperature as

\[
\tilde{\partial}_t \equiv \frac{\partial}{\partial t} - \Delta, \quad \Delta = \frac{\partial}{\partial t} \frac{\partial}{\partial \theta} + \frac{\partial}{\partial \bar{\theta}^*} \frac{\partial}{\partial \bar{\theta}}.
\]

We call this object the thermal time derivative; \( \Delta \) accounts for the thermal corrections. Accordingly, we also define a thermal space-time derivative as

\[
\tilde{\partial}_\mu = \left( \frac{\partial}{\partial t} - \Delta ; \frac{\partial}{\partial x} \right).
\]

To construct the thermal covariant derivatives, we replace in the expressions of the zero-temperature covariant derivatives (8)–(9) the \( T = 0 \) Grassmann variables and derivative operators by their thermal counterparts. This means that (i) we replace the zero-temperature, constant Grassmann parameters \( \theta, \bar{\theta} \) by the thermal, time-dependent and antiperiodic parameters \( \tilde{\theta}, \tilde{\bar{\theta}} \), and that (ii) the derivative operators \( \partial_\mu, \partial/\partial \theta \) and \( \partial/\partial \bar{\theta} \) are replaced by \( \tilde{\partial}_\mu, \partial/\partial \tilde{\theta} \) and \( \partial/\partial \tilde{\bar{\theta}} \). This yields thermal covariant derivatives \( \tilde{D}_\alpha \) and \( \tilde{\bar{D}}_\alpha \) in the form:
\[ \hat{D}_\alpha = \frac{\partial}{\partial \theta^\alpha} - i \sigma^\mu_{\alpha \dot{\alpha}} \tilde{\theta}^\dot{\alpha} \partial_\mu + i \sigma^0_{\alpha \alpha} \tilde{\theta}^\dot{\alpha} \Delta, \]
\[ \hat{\nabla}_\alpha = \frac{\partial}{\partial \theta} - i \hat{\theta}^\alpha \sigma^\mu_{\alpha \alpha} \partial_\mu + i \hat{\theta}^\alpha \sigma^0_{\alpha \alpha} \Delta. \]

In order to validate these expressions, we observe that they play, in thermal superspace, the same role as the usual covariant derivatives of supersymmetry in \( T = 0 \) superspace.

First, the thermal covariant derivatives obey the same anticommutation relations as at \( T = 0 \). This can readily be checked by direct computation of the anticommutators. One obtains, in perfect analogy to the \( T = 0 \) case,
\[ \{ \hat{D}_\alpha, \hat{D}_\beta \} = -2i \sigma^\mu_{\alpha \dot{\alpha}} \delta_{\mu \beta}, \quad \{ \hat{D}_\alpha, \hat{D}_\beta \} = (\hat{D}_\alpha, \hat{D}_\beta) = 0. \]

This is actually obvious upon noticing that the thermal space-time derivative \( \hat{\partial}_\mu \) gives zero when acting on the \( t \)-dependent variables \( \hat{\theta}, \hat{\theta}^\dot{\alpha} \), since
\[ \hat{\partial}_0 \hat{\theta}^\alpha = \frac{\partial}{\partial t} \hat{\theta}^\alpha - \frac{\partial}{\partial t} \hat{\theta}^\gamma \hat{\theta}^\dot{\gamma} = 0, \]
\[ \hat{\partial}_0 \hat{\theta}^{\dot{\alpha}} = \frac{\partial}{\partial t} \hat{\theta}^{\dot{\alpha}} - \frac{\partial}{\partial t} \hat{\theta}^\gamma \hat{\theta}^{\dot{\gamma}} = 0. \]

Therefore \( \hat{D}_\mu \) plays the same role for the thermal Grassmann variables \( \hat{\theta}, \hat{\theta}^\dot{\alpha} \) as that of the usual space-time derivative \( \partial_\mu \) for the \( t \)-independent, non thermal \( \theta_t \) and \( \theta^\dot{\alpha} \). In this sense, the thermal time (and consequently the thermal space-time) derivative is a covariantization, with respect to thermal superspace, of the zero-temperature time (space-time) derivative.

Second, the thermal covariant derivatives \( \hat{D}_\alpha \) and \( \hat{\nabla}_\alpha \) provide a definition of the thermal chiral and antichiral superfields \( \hat{\phi}, \hat{\bar{\phi}} \), eqs. [18] and [19], as the solution to the thermal generalization of the conditions (\( \mathbb{1} \)):
\[ \hat{D}_\alpha \hat{\phi} = 0, \quad \hat{\nabla}_\alpha \hat{\phi} = 0. \]

Our thermal superfield expansions (18) and (19) can easily be seen to be consistent also from the point of view of the fields’ boundary conditions. Thermal, e.g., chiral, superfields being bosonic objects, they must obey a superfield periodic boundary condition in the form:
\[ \hat{\phi}[g(t), \hat{\theta}(t)] = \hat{\phi}[g(t+i\beta), \hat{\theta}(t+i\beta)]. \]

In the variables \((x, \hat{\theta}, \hat{\theta}^\alpha), x = (t; \mathbf{x})\), this condition writes
\[ \hat{\phi}[t; \mathbf{x}, \hat{\theta}(t), \hat{\theta}^\alpha(t)] = \hat{\phi}[t+i\beta; \mathbf{x}, \hat{\theta}(t+i\beta), \hat{\theta}^\alpha(t+i\beta)]. \]

Expanding now both sides in thermal superspace along \( \hat{\phi}[x, \hat{\theta}, \hat{\theta}^\alpha] = z(x) + \sqrt{2} \tilde{\theta} \psi(x) - \hat{\theta} \hat{\theta} f(x) - i(\hat{\theta} \sigma^\mu \hat{\theta}) \partial_\mu z(x) + i \sqrt{2} \hat{\theta} \hat{\theta} \psi(x) \sigma^\mu \hat{\theta} - \frac{1}{4} \hat{\theta} \hat{\theta} \hat{\theta} \hat{\theta} \Box z(x) \), (28)
we immediately get from (28) (i) for the scalar field \( z \)
the periodic b.c.
\[ z(t; \mathbf{x}) = z(t+i\beta; \mathbf{x}), \]

(ii) for the fermionic field \( \psi \), upon replacing \( \hat{\theta}(t+i\beta) = -\hat{\theta}(t) \) [eq. (\( \mathbb{1} \))], the antiperiodic b.c.
\[ \psi(t; \mathbf{x}) = -\psi(t+i\beta; \mathbf{x}), \]

and (iii) for the scalar field \( f \), with \( \hat{\theta}(t+i\beta)\hat{\theta}(t+i\beta) = \hat{\theta}(t)\hat{\theta}(t) \) [eq. (\( \mathbb{1} \))], the periodic b.c.
\[ f(t; \mathbf{x}) = f(t+i\beta; \mathbf{x}), \]

as well as additional conditions for the fields’ derivatives.

IV. THERMAL COVARIANTIZATION OF THE SUPERSYMMETRY ALGEBRA

The main interest of superspace lies in the natural representation it provides for the super-Poincaré algebra in terms of derivatives with respect to superspace coordinates. The purpose of this section is to construct the supersymmetry generators acting on thermal superspace, and to compute their algebra. However, the existence of a supersymmetry algebra on thermal superspace should not be assimilated to a statement that supersymmetry does not break at finite \( T \). That such an algebra exists does not imply that a supersymmetric field theory can be constructed carrying the same symmetry algebra.

The zero-temperature supercharges are, in our conventions:
\[ Q_\alpha = -i \frac{\partial}{\partial \theta^\alpha} + \sigma^\mu_{\alpha \dot{\alpha}} \tilde{\theta}^\dot{\alpha} \partial_\mu, \]
\[ \bar{Q}_\alpha = i \frac{\partial}{\partial \theta^\alpha} - \theta^\alpha \sigma^\mu_{\alpha \dot{\alpha}} \partial_\mu. \]

The corresponding thermal objects are constructed using the same procedure as the one used above for the thermal covariant derivatives, that is, we replace \( \theta, \theta^\dot{\alpha} \) by \( \hat{\theta}, \hat{\theta}^\dot{\alpha} \), \( \partial_\mu \), \( \partial/\partial \theta \), \( \partial/\partial \theta^\alpha \) by \( \hat{\partial}_\mu \) [eq. (28)], \( \partial/\partial \hat{\theta} \) and \( \partial/\partial \hat{\theta}^\dot{\alpha} \). This yields the following expressions for the thermal supercharges:
\[ \hat{Q}_\alpha = -i \frac{\hat{\partial}}{\partial \theta^\alpha} + \sigma^\mu_{\alpha \dot{\alpha}} \tilde{\theta}^\dot{\alpha} \hat{\partial}_\mu, \]
\[ -\sigma^\mu_{\alpha \dot{\alpha}} \hat{\partial}_\mu \left( \frac{\hat{\partial} \hat{\theta}^\alpha \partial}{\partial \theta^\alpha} + \frac{\hat{\partial} \hat{\theta}^\dot{\alpha} \partial}{\partial \theta^\dot{\alpha}} \right), \]
\[ \hat{\bar{Q}}_\dot{\alpha} = i \frac{\hat{\partial}}{\partial \theta^\dot{\alpha}} - \hat{\theta}^\dot{\alpha} \sigma^\mu_{\alpha \dot{\alpha}} \hat{\partial}_\mu \]
\[ + \hat{\theta}^\dot{\alpha} \sigma^0_{\alpha \dot{\alpha}} \hat{\partial}_\mu \left( \frac{\hat{\partial} \hat{\theta}^\alpha \partial}{\partial \theta^\alpha} + \frac{\hat{\partial} \hat{\theta}^\dot{\alpha} \partial}{\partial \theta^\dot{\alpha}} \right). \]
or, in a compact form,

\[ \hat{Q}_\alpha = -i \frac{\partial}{\partial \theta^\alpha} + \sigma^\mu_{\alpha \bar{\alpha}} \hat{\theta}_\mu, \]

\[ \hat{\bar{Q}}_{\bar{\alpha}} = i \frac{\partial}{\partial \bar{\theta}^{\bar{\alpha}}} - \bar{\sigma}^\nu \sigma^\mu_{\alpha \bar{\alpha}} \hat{\bar{\theta}}_\nu. \]

It is straightforward to check that thermal supercharges obey the same anticommutation relations with thermal covariant derivatives as at \( T = 0 \):

\[ \{ \hat{Q}_\alpha, \hat{D}_\beta \} = \{ \hat{\bar{Q}}_{\bar{\alpha}}, \hat{\bar{D}}_{\bar{\beta}} \} = \{ \hat{Q}_\alpha, \hat{\bar{D}}_{\bar{\beta}} \} = (\hat{\bar{Q}}_{\bar{\alpha}}, \hat{D}_\beta) = 0. \]

Furthermore we have

\[ \{ \hat{Q}_\alpha, \hat{\bar{Q}}_{\bar{\alpha}} \} = 2i \sigma^\mu_{\alpha \bar{\alpha}} \hat{\theta}_\mu, \quad \{ \hat{Q}_\alpha, \hat{Q}_\beta \} = \{ \hat{\bar{Q}}_{\bar{\alpha}}, \hat{\bar{Q}}_{\bar{\beta}} \} = 0. \]

In order to compute the full thermal super-Poincaré algebra, we need in addition expressions for the thermal translations and thermal Lorentz generators. At finite temperature, the translation and Lorentz generators above are modified – similarly to the thermal covariant derivatives and the thermal supercharges – by replacing \( \theta, \bar{\theta} \) by \( \hat{\theta}, \hat{\bar{\theta}} \), and \( \partial/\partial \theta, \partial/\partial \bar{\theta} \) by \( \hat{\partial}_\mu \) [eq. (46)], \( \partial/\partial \hat{\theta} \) and \( \partial/\partial \hat{\bar{\theta}} \). Therefore we define the action of thermal translation and thermal Lorentz generators on a thermal scalar superfield through

\[ [\hat{P}^\mu, \hat{\phi}(x, \hat{\theta}, \hat{\bar{\theta}})] = -i \hat{\partial}^\mu \hat{\phi}(x, \hat{\theta}, \hat{\bar{\theta}}), \]

\[ [\hat{M}^{\mu \nu}, \hat{\phi}(x, \hat{\theta}, \hat{\bar{\theta}})] = \left[ i(x^\mu \hat{\partial}^\nu - x^\nu \hat{\partial}^\mu) + \frac{i}{2} (\sigma^{\mu \nu})_{\alpha \beta} \hat{\theta}^\alpha \hat{\bar{\theta}}^\beta \right] \hat{\phi}(x, \hat{\theta}, \hat{\bar{\theta}}). \]

As only the derivative in the time direction is modified at finite temperature, we now distinguish between the generators which are genuinely thermal [that is, which involve the operator \( \Delta \) in (43)] and those generators of which the only thermal character comes from the superspace coordinates being the thermal ones. We drop the “hat” for the latter operators, and hence decompose the thermal translations \( \hat{P}^\mu \) into thermal time translations \( \hat{P}^0 \) and space translations \( P^i \), while the thermal Lorentz generators \( \hat{M}^{\mu \nu} \) are separated into thermal Lorentz boosts \( \hat{M}^{0i} \) and space rotations \( M^{ij} \). A straightforward computation of the commutation rules yields the thermal Poincaré algebra – the bosonic sector of the thermal super-Poincaré algebra:

\[ [\hat{M}^{0i}, \hat{P}^0] = -i P^i, \]

\[ [\hat{M}^{0i}, P^j] = i \eta^{ij} \hat{P}^0, \]

\[ [M^{ij}, \hat{P}^0] = 0, \]

\[ [M^{ij}, P^k] = -i (\eta^{jk} P^i - \eta^{ij} P^k), \]

\[ [M^{ij}, M^{kl}] = -i (\eta^{ik} M^{jl} + \eta^{jl} M^{ik} - \eta^{il} M^{jk} - \eta^{jk} M^{il}), \]

\[ [\hat{M}^{0i}, M^{jk}] = -i (\eta^{ik} \hat{M}^{0j} - \eta^{jk} \hat{M}^{0i}), \]

\[ [\hat{M}^{0i}, \hat{M}^{0j}] = -i M^{ij}, \]

\[ [\hat{P}^0, \hat{P}^0] = [\hat{P}^0, P^i] = [P^i, \hat{P}^0] = 0, \]

while the fermionic sector is given by

\[ \{ \hat{Q}_\alpha, \hat{\bar{Q}}_{\bar{\beta}} \} = -2 \left( \sigma^0_{\alpha \beta} \hat{P}_0 - \sigma^1_{\alpha \beta} \hat{P}_1 \right), \]

\[ [\hat{M}^{0i}, \hat{Q}_\alpha] = \frac{i}{2} (\sigma^0)^{\alpha \beta} \hat{\theta}^\beta \hat{\bar{Q}}_{\bar{\beta}}, \]

\[ [\hat{M}^{0i}, \hat{\bar{Q}}_{\bar{\alpha}}] = \frac{i}{2} (\sigma^0)^{\alpha \beta} \hat{\bar{\theta}}^\alpha \hat{Q}_\beta, \]

\[ [\hat{M}^{ij}, \hat{Q}_\alpha] = \frac{i}{2} (\sigma^0)^{\alpha \beta} \hat{\theta}^\beta \hat{\bar{Q}}_{\bar{\beta}}, \]

\[ [\hat{M}^{ij}, \hat{\bar{Q}}_{\bar{\alpha}}] = \frac{i}{2} (\sigma^0)^{\alpha \beta} \hat{\bar{\theta}}^\alpha \hat{Q}_\beta, \]

and

\[ [\hat{P}^0, \hat{Q}_\alpha] = [\hat{P}^0, \hat{\bar{Q}}_{\bar{\alpha}}] = [P^i, \hat{Q}_\alpha] = [P^i, \hat{\bar{Q}}_{\bar{\alpha}}] = 0. \]

The thermal super-Poincaré algebra has hence the same structure as at \( T = 0 \), and contains the same number of supercharges, once one has appropriately covariantized the generators with respect to thermal superspace. The thermal time translation operator \( \hat{P}^0 = -i \hat{\partial}^0 \) (the thermal covariantization of \( P^0 \)) can be interpreted as a central charge of the subalgebra one obtains upon removing the thermal Lorentz boosts \( \hat{P}^i \).
Inserting this into (32) leads to
\[ \tilde{\delta} \phi(y, \theta) = \left( \epsilon^{\alpha} \frac{\partial}{\partial y^\alpha} - 2i(\bar{\theta} \sigma^\mu \bar{\epsilon}) \frac{\partial}{\partial y^\mu} \right) \phi(y, \theta). \]

Defining then \( \frac{\partial}{\partial y^\alpha} \varphi(y) \equiv \partial_\mu \varphi \), for \( \varphi = z \) or \( \psi \), we get:
\[ \tilde{\delta} \phi(y, \theta) = \epsilon^{\alpha} \left[ \sqrt{2} \psi_\alpha(y) - 2 \bar{\theta}_\alpha f(y) \right] \\
- 2i(\bar{\theta} \sigma^\mu \bar{\epsilon}) \left[ \partial_\mu z(y) + \sqrt{2} \theta^\alpha \partial_\mu \psi_\alpha(y) \right]. \]

Comparison with the component expansion of \( \tilde{\delta} \phi(y, \theta) \) immediately leads to:
\[ \tilde{\delta} z = \sqrt{2} \epsilon^\alpha \psi_\alpha, \]
\[ \tilde{\delta} \psi_\alpha = -\sqrt{2} \epsilon^\alpha f - i \sqrt{2}(\sigma^\mu \bar{\epsilon})_\alpha (\partial_\mu z), \]
\[ \tilde{\delta} f = -i \sqrt{2}(\sigma^\mu \bar{\epsilon})^a (\partial_\mu \psi_\alpha), \]  \( \text{(36)} \)

where the unique difference with the case of zero temperature is the appearance of the thermal (time-dependent and antiperiodic) spinorial parameter \( \epsilon \), \( \epsilon(t+i\beta) = -\epsilon(t) \), in place of the constant spinorial parameter \( \epsilon \) of \( T = 0 \) supersymmetry. The nature of \( \epsilon \) is however deeply related to finite temperature. The time-dependence and antiperiodicity of \( \epsilon \) are, in this thermal superspace formalism, the manifestation of the breaking of global supersymmetry at finite temperature.

VI. REALIZATIONS OF THERMAL SUPERSYMMETRY – WESS-ZUMINO MODEL

Bosonic and fermionic fields at finite temperature are characterized by periodic, resp. antiperiodic, boundary conditions [eqs. (31), (32) and (33)]. At the level of Green’s functions, thermal effects induce a differentiation between bosons and fermions through the corresponding KMS conditions (14)–(15), resp. (16). Both the fields’ boundary conditions and the KMS conditions carry information that is of global character, in the sense that it relates the values of the field at distant regions in spacetime. This is why the thermal superalgebra, which is a local structure, is insensitive to such global conditions and preserves its structure at finite temperature. In particular, the antiperiodicity conditions on \( \bar{\theta}, \bar{\bar{\theta}} \), eq. (4), have no influence on the algebra. It is only the local statement that the superspace Grassmann variables should be allowed to depend on time which makes it necessary to covariantize the algebra generators. However, in investigating realizations of the thermal supersymmetry algebra, we shall be dealing with thermal fields, at the level of which the thermal boson/fermion distinctions enter as global conditions. Therefore, we expect to see signs of supersymmetry breaking when realizing the thermal supersymmetry algebra on thermal bosonic fermionic fields.

A common way of introducing the fields’ global periodicity properties is to develop them thermally à la Matsubara. In the Matsubara expansion, bosons are expanded in thermal modes as
\[ z(t, x) = \frac{1}{\sqrt{\beta}} \sum_n z_n(x) e^{i\omega_n^B t}, \quad \omega_n^B = \frac{2n\pi}{\beta} \]

where \( \omega_n^B \) are the bosonic Matsubara frequencies, while fermions are developed as
\[ \psi(t, x) = \frac{1}{\sqrt{\beta}} \sum_n \psi_n(x) e^{i\omega_n^F t}, \quad \omega_n^F = \frac{(2n+1)\pi}{\beta} \]

with the fermionic Matsubara frequencies \( \omega_n^F \). Clearly, these developments contain the information on the periodicity or antiperiodicity in time, as one may immediately check upon shifting the time argument by \( i\beta \). The Matsubara expansion, after rotation to euclidean time, is a realization of the imaginary time formalism for finite temperature field theory. It is an expansion on \( S^1 \times \mathbb{R}^d \), the circle \( S^1 \) having length \( \beta = 1/T \). In a supergravity theory, it could be regarded as a particular Scherk-Schwarz compactification \( \mathbb{R}^d \) scheme of the time direction.

Since we have considered only non-interacting scalar and fermionic matter fields described by chiral and antichiral superfields, the natural zero-temperature limiting field theory – to be studied now at finite temperature – is the free \( T = 0 \) (off-shell) Wess-Zumino model
\[ S^{d=4} = \int d^4 x \left( \mathcal{L}_{\text{kin}}^{d=4} + \mathcal{L}_{\text{mass}}^{d=4} \right), \]  \( \text{(37)} \)

with kinetic and mass lagrangians given by
\[ \mathcal{L}_{\text{kin}}^{d=4} = \frac{1}{2} (\partial_\mu A)^2 + \frac{1}{2} (\partial_\mu B)^2 + \frac{1}{2} \bar{\psi} i \gamma^\mu \partial_\mu \psi + \frac{1}{2} (F^2 + G^2), \]
\[ \mathcal{L}_{\text{mass}}^{d=4} = -M_4 \left( \frac{1}{2} \bar{\psi} \bar{\psi} + AF + BG \right). \]

\( M_4 \) is the mass, \( \psi \) a four-component Majorana fermion and \( A, B, F, G \) are real scalar fields which relate to the complex scalar \( z \) and \( f \) through
\[ z(x) = \frac{1}{\sqrt{2}} [A(x) + iB(x)], \quad f(x) = \frac{1}{\sqrt{2}} [F(x) + iG(x)]. \]

The supersymmetry transformations at \( T = 0 \) write
\[ \delta A = \bar{\epsilon} \psi, \quad \delta B = i \bar{\epsilon} \gamma_5 \psi, \]
\[ \delta F = i \bar{\epsilon} \gamma^\mu \partial_\mu \psi, \quad \delta G = - \bar{\epsilon} \gamma_5 \gamma^\mu \partial_\mu \psi, \]
\[ \delta \bar{\psi} = -i \gamma^\mu \partial_\mu (A + iB\gamma_5) + F + iG\gamma_5 |\epsilon|, \]  \( \text{(38)} \)

and under these, the kinetic and mass contributions to the action \( S^{d=4} \) are separately invariant. Concretely, omitting in each case a space-time derivative which integrates to zero.
\[ \delta \int d^4x \mathcal{L}_{\text{kin}}^{d=4} = \int d^4x \bar{\psi} \gamma^\mu \gamma^\nu [\partial_\mu (A + iB\gamma_5)] \partial_\nu \epsilon = 0, \]
\[ \delta \int d^4x \mathcal{L}_{\text{mass}}^{d=4} = -iM_4 \int d^4x \bar{\psi} \gamma^\mu (A + iB\gamma_5) \partial_\mu \epsilon = 0, \]
which of course vanish at zero temperature where \( \epsilon \) is a constant spinor.

Performing the thermal expansion of (37) (see [1] for details), we get the \( d = 3 \) euclidean expression
\[ S^{d=3} = \int d^3x \sum_n \left\{ \frac{1}{2} \partial_i A_n^* \partial_i A_n + \frac{1}{2} \partial_i B_n^* \partial_i B_n \right. \]
\[ + \frac{1}{2} (M_{3,n}^2) (A_n^* A_n + B_n^* B_n) + \frac{1}{2} \left[ \lambda_n^1 (\sigma^1 \partial_i - \omega_n^F) \lambda_n \right. \]
\[ + M_4 \lambda_n^1 i \sigma^2 \lambda_n^* \big] + \text{h.c.} \bigg), \]
(40)
where the thermal mass of the \( n \)-th \( d = 3 \) bosonic mode \( M_{3,n}^B \) obeys
\[ (M_{3,n}^B)^2 = M_4^2 + (\omega_n^B)^2, \quad (\omega_n^B)^2 = \frac{4\pi^2 n^2}{\beta^2}. \]

For fermions, the mass matrix in (36) can be written:
\[ \frac{1}{2} \sum_n \left( \lambda_n^1 \lambda_n^* \right) \left( \frac{\omega_n^F}{M_4} \right) \left( \frac{\lambda_n^1}{\omega_n^F} \right) + \text{h.c.} \]
and possesses two opposite eigenvalues \( \pm M_{3,n}^F \) verifying
\[ (M_{3,n}^F)^2 = M_4^2 + (\omega_n^F)^2, \quad (\omega_n^F)^2 = \frac{\pi^2 (2n + 1)^2}{\beta^2}. \]

From these mass relations, it is clear that thermal effects lift the mass degeneracy characteristic of \( T = 0 \) supersymmetry. The lifting of the mass degeneracy by temperature effects is a clear signature of thermal supersymmetry breaking at the level of thermal fields.

We also expect to see thermal supersymmetry breaking when trying to realize the thermal supersymmetry algebra on systems of thermal fields. In order to investigate this, we first need to pin down the thermal supersymmetry transformations of the component fields, as we have expressed our Wess-Zumino model in components. In Section [3], we have shown that component transformations under thermal supersymmetry have the same form as at \( T = 0 \), but with the space-time constant supersymmetry parameter \( \epsilon \) replaced by the thermal, time-dependent and antiperiodic quantity \( \tilde{\epsilon} \). This allows us to identify immediately the thermal version of the transformations (35) as
\[ \tilde{\delta} A = \tilde{\epsilon} \bar{\psi}, \quad \tilde{\delta} B = i\tilde{\epsilon} \gamma_5 \psi, \]
\[ \tilde{\delta} F = i\tilde{\epsilon} \gamma^\mu (\partial_\mu \psi), \quad \tilde{\delta} G = -\tilde{\epsilon} \gamma_5 \gamma^\mu (\partial_\mu \psi), \]
\[ \tilde{\delta} \psi = -i\gamma^\mu (\partial_\mu (A + iB\gamma_5)) + F + i\gamma_5 \tilde{\epsilon}. \]

These expressions can be translated into transformations of the three-dimensional Matsubara modes, as is shown in [3]. When doing so, one must take into account the fact that, the supersymmetry parameters being now time-dependent, they must be developed thermally (with constant modes). One therefore obtains, at the level of the \( d = 3 \) thermal modes, thermal supersymmetry transformations with parameters which carry an index of the modes, and as a consequence mix the modes of bosons and fermions [1]. This should be contrasted with what one has at zero temperature, in which case the supersymmetry parameters are space-time constants that one does not develop thermally when dimensionally reducing to \( d = 3 \).

The thermal action has – in contrast to (39) – the following non-trivial variation under thermal supersymmetry [1] [a rotation to imaginary (euclidean) time is understood]:
\[ \tilde{\delta} \int d^4x \mathcal{L}_{\text{kin}}^{d=4} = \int d^4x \bar{\psi} \gamma^\mu \gamma_5 \partial_\mu (A + iB\gamma_5) \partial_5 \tilde{\epsilon}, \]
\[ \tilde{\delta} \int d^4x \mathcal{L}_{\text{mass}}^{d=4} = -iM_4 \int d^4x \bar{\psi} (A + iB\gamma_5) \partial_5 \tilde{\epsilon}, \]
(42)
where \( \partial_5 \tilde{\epsilon} \) does not vanish as \( \tilde{\epsilon} \) depends on time. Clearly, neither the kinetic action nor the mass action are invariant under the thermal supersymmetry transformations. Upon inserting the Matsubara mode expansions, these variations can be translated into temperature-dependent, \( d = 3 \) expressions [1]. The variation of the total action is then seen to be proportional to \( \omega_n^F \sim T \), as a consequence of thermal supersymmetry breaking. In the \( T \to 0 \) limit, one expects supersymmetry to be thermally unbroken. The variations \( \delta S_{\text{kin}}^{d=4} \) and \( \delta S_{\text{mass}}^{d=4} \) indeed vanish separately in that limit.

VII. CONCLUSIONS

Immersing a physical system in a heat bath results in the fields acquiring different properties according to their statistics. E.g., finite-temperature bosonic fields obey periodic boundary conditions, while fermionic fields satisfy antiperiodic b.c.’s. Such a distinction can be seen also at the level of the Green’s functions. Depending on their statistics, thermal propagators obey either a bosonic KMS condition, or a fermionic one. Therefore, thermal effects induce a clear and mandatory distinction between bosons and fermions. As a consequence, finite temperature environments are incompatible with \( T = 0 \) supersymmetry: the supersymmetry transformations are indeed unable to take into account the distinct thermal behaviours of bosons and fermions. The parameters of
supersymmetry transformations being antiperiodic at finite temperature, as advocated in [3], thermal supersymmetry transforms periodic bosons into antiperiodic fermions, and vice-versa. At the level of superfields, our approach makes it possible to reconcile the component fields’ distinct boundary conditions within a superfield boundary condition. For chiral superfields, this b.c. is of bosonic type and is formulated in thermal superspace. The latter is spanned by usual space-time and by time-dependent and antiperiodic Grassmann coordinates, with a period given by the inverse temperature. The superspace Grassmann coordinates are thus subject to a temperature-dependent constraint similar to that obeyed by the supersymmetry parameters.

Thermal superfield propagators are shown to obey a KMS condition formulated directly at the level of superfields. Its proof makes an essential use of the antiperiodicity of the Grassmann coordinates. This antiperiodicity is crucial as well in proving that the superfield KMS condition implies the correct, bosonic or fermionic, KMS condition for the superfield components.

In this sense, the thermal superspace approach allows to reconcile the notions of finite temperature physics and of supersymmetry, yielding a formalism for bosons and fermions in interaction with a heat bath in which thermal supersymmetry breaking is encoded. Such a formalism is particularly welcome for, e.g., cosmology. Thermal superspace is therefore the correct superspace for finite temperature situations. It is shown to provide a natural representation for the thermal supersymmetry algebra, which is obtained upon covariantizing thermally the supercharges as well as the Lorentz and translation generators. The thermal supersymmetry algebra has the same structure as at $T = 0$ and the same number of supersymmetries, while thermal supersymmetry breaking is encoded in the temperature-dependent conditions we impose on superspace.

It is only when trying to realize the thermal supersymmetry algebra on systems of thermal fields that one encounters explicit thermal supersymmetry breaking. Indeed, the conditions which distinguish bosons from fermions at finite temperature – the fields’ b.c.’s or the KMS conditions – carry information that is of global nature in space-time. The supersymmetry algebra, being a local structure, is insensitive to this global information. It only needs to be covariantized with respect to the local statement that the superspace Grassmann parameters depend on imaginary time. At the level of thermal actions, we encounter thermal supersymmetry breaking in two ways. First, upon developing thermally the $T = 0$ Wess-Zumino model, we observe that the mass degeneracy is lifted. Second, we compute the variation of the thermal action under thermal supersymmetry and observe that the action is non-invariant. The variation is given, in terms of thermal modes, by a temperature-dependent expression which vanishes in the $T \rightarrow 0$ limit where one expects supersymmetry to be thermally unbroken.

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