Axisymmetric deformations of neutron stars and gravitational-wave astronomy

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Einstein’s theory of general relativity predicts that the only stationary configuration of an isolated black hole is the Kerr spacetime, which has a unique multipolar structure and a spherical shape when non-spinning. This is in striking contrast to the case of other self-gravitating objects, which instead can in principle have arbitrary deformations even in the static case. Here we develop a general perturbative framework to construct stationary stars with small axisymmetric deformations, and study explicitly compact stars with an intrinsic quadrupole moment. The latter can be sustained, for instance, by crust stresses or strong magnetic fields. While our framework is general, we focus on quadrupolar deformations of neutron stars induced by an anisotropic crust, which continuously connect to spherical neutron stars in the isotropic limit. Deformed neutron stars might provide a more accurate description for stellar remnants formed in supernovae and in binary mergers, and can be used to improve constraints on the neutron-star equation of state through gravitational-wave detections and through the observation of low-mass X-ray binaries. We argue that, if the (dimensionless) intrinsic quadrupole moment is of a few percent or higher, the effect of the deformation is stronger than that of tidal interactions in coalescing neutron-star binaries, and might also significantly affect the electromagnetic signal from accreting neutron stars.

I. INTRODUCTION

Neutron stars (NSs) harbor the highest densities, the strongest magnetic fields, the highest binding energy per nucleon, and the strongest spacetime curvatures in the universe. Provided their interior and spacetime can be accurately modeled using nuclear physics and general relativity, NSs are unique probes of all fundamental interactions and ideal laboratories to test foundational physics and high-energy astrophysics [1].

However, at variance with black holes, NSs are not simple objects. Within Einstein’s theory of general relativity, the black-hole uniqueness theorems imply that the ultimate stationary outcome of the gravitational collapse must be a Kerr black hole [2, 3]. The latter has an infinite number of multipole moments [4] which are anyway uniquely determined in terms of its mass and angular momentum [5]. When non-spinning, any isolated black hole in the universe must be spherically symmetric and described by the Schwarzschild spacetime.

This remarkable simplicity does not hold true for other self-gravitating objects, in particular for NSs. There is no compelling reason preventing NSs formed by core-collapse supernovae [6, 7] or by a compact-binary coalescence [8, 9] to be arbitrarily deformed away from spherical symmetry, even when non-spinning. In fact, one might even argue the opposite, namely that spherical symmetry is a mere idealization and that all astrophysical formation processes are intrinsically asymmetric, e.g., due to magnetic fields, environmental effects, crust shears, elasticity, collimated neutrino fluxes, gravitational-wave (GW) emission, kicks, deformations of the progenitor proto-NS, etc. It is therefore natural to expect that a newly-born NS might be deformed to some degree and that it can reach a stationary, axisymmetric configuration through GW emission [10]. Non-axisymmetric deformations have been intensively studied as a source of quasi-monochromatic GWs from isolated spinning NSs [11]. Strong constraints exist on departures from axisymmetry, typically measured by the ellipticity of a NS [11, 12]. However, axisymmetric deformations are much less constrained. Although they do not destabilize the star through GW emission [13], they might give rise to important effects in isolated and binary NSs. The scope of this work is to discuss this scenario. Unless otherwise stated, we use $G = c = 1$ units henceforth.

II. AXISYMMETRIC DEFORMATIONS OF NSS: THIN-SHELL CRUST

We have developed a general-relativistic, perturbative framework to construct equilibrium configurations of self-gravitating bodies with small (but otherwise generic) axisymmetric deformations away from spherical symmetry. Our approach is based on a framework recently developed for vacuum spacetimes [16] and extends the latter to the case of matter fields, in particular perfect fluids, which provide an accurate description of the interior of cold NSs. We consider a spherical-harmonic decomposition of the spacetime and of the stress-energy tensor, and solve for Einstein’s field equations in the interior of the star perturbatively in the deformations. The numerical solution in the stellar interior is then matched to the

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\textsuperscript{1} GW emission from axisymmetric NSs can occur only if the stellar angular momentum is misaligned with the axis of symmetry, leading to precession. We focus here on stationary configurations, for which the angular momentum (if present) coincides with the axis of symmetry.
analytical solution known in the exterior [16]. Spherical NSs deformed by an intrinsic small angular momentum (constructed in the seminal papers by Hartle and Thorne [17, 18]) are a particular case of our general framework, which can be straightforwardly extended to arbitrary intrinsic multipole moments, to any perturbative order, and to different matter content. Details of the technique are given in Appendix A. In the following we shall focus on the most interesting case of quadrupolar deformations, being \( Q = Q/M^2 \) the dimensionless quadrupole moment of a star with mass \( M \).

In the absence of source terms (e.g. tidal fields, angular momentum, magnetic fields, shears, etc), the metric that describes a self-gravitating, perfect fluid with a quadrupolar deformation is discontinuous across the surface of the object. This shows that a (single) perfect-fluid star does not support deviations away from spherical symmetry in the static case. However, here we show how an anisotropic crust can support relatively large quadrupolar deformations. The standard equation of state (EoS) of a NS includes also the relatively low-density region of the crust [1, 19], but typically assuming a perfect fluid for the latter, and therefore neglecting possible anisotropies. In reality the crust of a NS is crystallized, i.e. atomic nuclei form a lattice [19, 21] whose elasticity can support asymmetric distributions [15]. The physical processes of the crust are complex and model dependent. In an attempt to build a more general and less model-dependent configuration, here we model the NS crust with a thin shell made of an anisotropic fluid and then match its properties with those of a more realistic crust model. In the deformed case the discontinuity of the metric across the stellar radius, together with the junction conditions at the surface [22, 23], dictate the properties of the thin shell.

We can solve numerically the equations for the deformation in the interior of the star and match them with the analytical solution in the exterior using the appropriate junction conditions (cf. Appendix A). We have thus constructed explicitly a novel two-parameter family of NS equilibrium configurations characterized – for each value of the central density \( \rho_c \) – by an intrinsic quadrupole moment \( Q \). It is straightforward to add angular momentum \( J \) perturbatively in our framework, so this family of solutions can be extended to a three-parameter model. For each value of \( (\rho_c, J, Q) \), we can compute the other properties of the star, including the mass \( M \) and radius \( R \). We stress that these solutions can be asymmetric even when non-rotating; for this reason we refer to them as deformed NSs.

The junction conditions imply that the crust is made of an anisotropic fluid, whose surface energy density can be written in general as

\[
\sigma(\theta) = \sigma_0 + \sigma_2 Q P_2(\cos \theta),
\]

where the free parameter \( \sigma_0 \approx M_{\text{crust}}/(4\pi R^2) \) is the surface density of the crust in the spherical configuration, \( \sigma_2 \) is the (normalized) amplitude of the axisymmetric perturbation to the surface density, and \( P_2(\cos \theta) \) is the Legendre polynomial \( (\ell = 0, 1, 2, \ldots) \). The anisotropy of the fluid on the thin shell can be measured by the difference

\[
\Delta \gamma \equiv \gamma_\theta - \gamma_\phi = \frac{Q}{8\pi R^2 \sqrt{1 - 2M/R}} \frac{3 \sin^2 \theta}{[\xi]},
\]
between the surface pressure $\gamma$ along the two angular directions, where $[\xi]$ is the jump of the fluid displacement at the radius. Note that both the monopolar and quadrupolar components of the pressure are anisotropic.

In Fig. 1 we show $Q\sigma_2$ and the surface pressure anisotropy $\Delta \gamma$ at the equator as a function of the NS compactness for some relevant EoS. We normalize both quantities by the quadrupole value $|Q| = 0.1$, and by typical reference values, $\sigma_0 \sim 10^{18}$ g cm$^{-2}$ and $\gamma_0 \sim 10^{17}$ g cm$^{-2}$, respectively, roughly corresponding to a crust containing 1% of the total NS mass [19]. In particular, the values of the anisotropic pressure shown in Fig. 1 are compatible with the maximum allowed by elasticity of the crust, $\Delta \gamma / \gamma_0 \approx (0.005 - 0.04) Z^{1/3}$, where $Z \geq 26$ is the atomic number of the ions in the crust (mostly iron in the outer crust [19]) and the prefactor depends on the type and direction of elastic deformations of the lattice [15].

Due to the broken spherical symmetry of the system, there are values of the parameter space for which the density might be negative, i.e., when $Q\sigma_2 P_2(\theta) < -\sigma_0$. However, the surface density is always positive for a NS with $|Q| \lesssim 0.1$ and compactness $M/R \lesssim 0.15$. For the same range of quadrupole values, prolate stars have positive density for realistic ranges of compactness $(0.1 \lesssim M/R \lesssim 0.2)$, whereas more massive and compact $(M/R \gtrsim 0.16)$ oblate stars require smaller quadrupoles to maintain positive density at the poles.

More generically, it is noteworthy to study all the energy conditions for the thin-shell fluid. The null energy condition reads $\sigma + \gamma_i \geq 0$ (here $i = \theta, \phi$), the weak energy condition additionally requires $\sigma \geq 0$; in addition to the latter condition, the strong energy condition also requires $\sigma + \sum_i \gamma_i \geq 0$. Finally, the dominant energy condition requires only $\sigma > |\gamma_i|$. A detailed numerical exploration of the parameter space shows that all energy conditions are satisfied for any realistic ranges of compactness whenever $Q \lesssim M_{\text{crust}}/M$. For the reference value $M_{\text{crust}}/M \sim 0.01$ adopted here, the condition $|Q| \lesssim 0.01$ ensures that all energy conditions are satisfied for any compactness, whereas if $|Q| \lesssim 0.1$ all energy conditions are satisfied for NSs with compactness $M/R \lesssim 0.15$.

III. PHENOMENOLOGICAL IMPLICATIONS

The above discussion suggests that deformed NSs would have reasonable properties for values of the quadrupole as high as $Q = \mathcal{O}(0.1)$, with more conservative values being $Q = \mathcal{O}(0.01)$. This value should be compared with the spin-induced quadrupole moment of a slowly-spinning NS, which scales quadratically with the spin, $Q = -\gamma \chi^2$, where $\chi = J/M^2$ is the dimensionless spin parameter, and $\gamma \approx 4/7$ for compact stars with $M \approx 1.4M_\odot$, the precise number depending on the EoS [17, 18, 24, 25]. Since the spin $\chi$ of a NS is typically small [26] (roughly $\chi \approx 0.1$ for the fastest millisecond pulsars and likely much smaller for old NSs in coalescing binaries detectable by LIGO/Virgo), the spin-induced quadrupole moment is at most $|Q| \approx 7 \times 10^{-2}$ and typically smaller. This suggests that any putative quadrupole moment of a NS might actually be natal rather than spin-induced. If this is the case, a deformed NS might provide a better model for the external spacetime of the body.

Any quantity $X$ of a spinning, deformed NS contains independent $J$-induced and $Q$-induced corrections; schematically, up to second order in the deformation $|Q| $ \leq |Q_0| \approx (X_0 + X_{01} Q + X_{20} \chi^2 + X_{02} Q^2 + X_{11} \chi Q) $, (3)

where $X_0$ is the value of the corresponding spherically-symmetric star and $X_{ij}$ are corrections that only depend on the EoS and on the central density of the star. When $Q = 0$, we recover the well-known case of a slowly-spinning NS [17, 18].

The external metric can be used to study how the spacetime of a deformed NS is affected by its intrinsic quadrupole. All geodesic quantities – including the innermost stable circular orbit (ISCO) and the epicyclic frequencies – acquire corrections (see Eq. 3) which affect properties such as the innermost location of an accretion disk and, in turn, the corresponding electromagnetic flux from accreting low-mass X-ray binaries, whose signal originates very deep in the gravitational field of the accreting object, at distances down to a few gravitational radii [27].

To the linear order, the azimuthal frequency ($\nu_\phi$) and the vertical epicyclic frequency ($\nu_\theta$) at the ISCO read

$$\nu_\phi^{\text{ISCO}} \approx 1.57 \left(1 + 0.75\chi + 0.23\bar{Q}\right) \left(\frac{1.4M_\odot}{M}\right) \text{kHz}, (4)$$

$$\nu_\theta^{\text{ISCO}} \approx 1.57 \left(1 + 0.61\chi + 0.17\bar{Q}\right) \left(\frac{1.4M_\odot}{M}\right) \text{kHz}, (5)$$

independently of the NS EoS. When $Q \approx 0.1$, these frequencies can differ by a few percent relative to the spherical case, leading to deviations in the emitted flux of the same order. The quadrupolar correction is larger than the spin-induced linear term whenever $\bar{Q} \gtrsim 0.18 (\chi/M)$. NSs in compact binaries are assumed to be spherically symmetric at large orbital distance $d$, whereas they are deformed during the coalescence due to tidal interactions [30]. The tidally-induced quadrupole moment is proportional to the tidal field $\sim M/d^3$; the leading-order tidal correction to the GW phase enters at the fifth post-Newtonian order [31], i.e. $\phi_{\text{tidal}} \sim v^5$, and it is proportional to the tidal deformability $\Lambda$, which characterizes the size of the tidally induced quadrupole deformations of the two stars [9, 30].

\footnotetext{The post-Newtonian approach is a weak-field/slow-velocity expansion of Einstein’s equations, where the expansion parameter is the orbital velocity $v \ll 1$. A $n$-th post-Newtonian correction to the GW phase corresponds to a term that is suppressed by $v^{2n}$ relative to the leading-order, $\sim v^{-2}$, contribution.}
relative phase

We use reference values for correction reads at second post-Newtonian order \([31, 32]\). The phase correction due to the two inspiraling bodies affects the GW phase already and tidal deformability \(\Lambda = 190\) (normalized) intrinsic quadrupole moment as small \(\bar{\Lambda_2} \approx 0.01\) or larger, the effect of the intrinsic quadrupole moment induced by the magnetic field \(B\) can be parametrized as

\[
\bar{Q} = -\beta \frac{\pi B^2 R^8}{\mu_0 I M^3} \approx -0.01 \beta \frac{B_{16}^2 R_{12}^8}{I_{45} M_{1.4}^4},
\]

where \(B_{16} = \frac{B}{10^{16} \text{ Gauss}}\), \(R_{12} = \frac{R}{12 \text{ km}}\), \(M_{1.4} = \frac{M}{1.4 M_{\odot}}\), \(I_{45} = \frac{I}{10^{45} \text{ g cm}^2}\), \(I\) is the moment of inertia, \(\mu_0\) is the vacuum permeability, and \(\beta \sim \mathcal{O}(1)\) is the magnetic distortion factor which measures to what extent a star can be deformed by the magnetic field. Note the very strong dependence on the radius of the star \(R\) and the quadratic dependence on the magnetic field \(B\). By computing \(M\), \(R\), and \(I\) for a family of NSs with some tabulated EoS, one can check that the magnetic-induced \(\bar{Q}\) is of the order of what given in Eq. (7) for all configurations, and therefore very small unless the magnetic field is extreme. In the case of deformation sourced by a magnetic field, the exterior of the star is not vacuum, so strictly speaking our solution in the exterior is not valid. However, the magnetic field is mainly dipolar and decays as \(r^{-3}\). Sufficiently far from the star, the vacuum solution is a good approximation and the metric can be matched to the analytical one written in terms of the multipole moments \([16]\).

Observational bounds on parametrized corrections to the post-Newtonian coefficient at second order from binary-NS coalescence GW170817 \((\delta \varphi_2 \lesssim 3.5\) at 90% confidence level \([54]\) can be directly translated – using Eq. (6) – into a bound on the intrinsic quadrupole moments of the binary components, yielding \(\bar{Q} \lesssim 0.14\) (assuming equal masses). This justifies our perturbative treatment but does not exclude that the binary components of GW170817 had some significant intrinsic deformation. This also suggests that a putative measurement \(\delta \varphi_2 \neq 0\) in the future might be due to an intrinsic deformation of the NSs rather than to a fundamental departure from general relativity.

\[ \delta \varphi_2 \leq 3.5 \] at 90% confidence level

On the other hand, an intrinsic quadrupole moment of the two inspiraling bodies affects the GW phase already at second post-Newtonian order \([31, 32]\). The phase correction reads

\[
\phi_{\text{quadrupole}} = \frac{75}{64} \frac{(m_1^2 \bar{Q}_1 + m_2^2 \bar{Q}_2)}{m_1 m_2} \approx \frac{75}{32} \frac{\bar{Q}}{v}, \tag{6}
\]

where in the last step we assumed the same masses \((m_1 = m_2 = M)\) and the same intrinsic quadrupole moment \((\bar{Q}_1 = \bar{Q}_2 = \bar{Q})\) for the two stars. As long as \(\bar{Q} \approx 0.01\) or larger, the effect of the intrinsic quadrupole will dominate over the tidal term, especially at low frequencies (Fig. 2). To the best of our knowledge, this effect has never been considered before, but can dramatically affect the parameter estimation of NS binaries and significantly modify the constraints on the NS EoS \([9, 28, 29]\).

IV. DISCUSSION

These findings could have important implications for high-energy astrophysics, GW astronomy, and nuclear physics. Deformed NSs might provide a more accurate description for the interior of a realistic compact star and, in turn, for all the strong-gravity phenomena which are analyzed to infer the NS EoS.

It is difficult to estimate the degree of deformation expected for NSs formed in realistic situations, e.g. in a core-collapse supernova or in a merger, since the latter are often simulated with simplified initial data, with some assumed degree of symmetry, and the evolution is typically stopped before the remnant has reached a truly stationary configuration. More importantly, simulations typically use perfect fluids, which cannot sustain deformations away from spherical symmetry in the static case. On the other hand, deformations might be the rule rather than the exception \([15]\), as relatively large quadrupole moments can be sustained even by modest anisotropic stresses.

We have explicitly focused on the case of an anisotropic crust, but non-spherical deformations might also be sourced by other processes, for example by a magnetic field. Relativistic models of axisymmetric, highly-magnetized NSs have been studied in the past \([33]\). The quadrupole moment induced by the magnetic field \(B\) can be parametrized as

\[
\bar{Q} = -\beta \frac{\pi B^2 R^8}{\mu_0 I M^3} \approx -0.01 \beta \frac{B_{16}^2 R_{12}^8}{I_{45} M_{1.4}^4}, \tag{7}
\]

where \(B_{16} = \frac{B}{10^{16} \text{ Gauss}}\), \(R_{12} = \frac{R}{12 \text{ km}}\), \(M_{1.4} = \frac{M}{1.4 M_{\odot}}\), \(I_{45} = \frac{I}{10^{45} \text{ g cm}^2}\), \(I\) is the moment of inertia, \(\mu_0\) is the vacuum permeability, and \(\beta \sim \mathcal{O}(1)\) is the magnetic distortion factor which measures to what extent a star can be deformed by the magnetic field. Note the very strong dependence on the radius of the star \(R\) and the quadratic dependence on the magnetic field \(B\). By computing \(M\), \(R\), and \(I\) for a family of NSs with some tabulated EoS, one can check that the magnetic-induced \(\bar{Q}\) is of the order of what given in Eq. (7) for all configurations, and therefore very small unless the magnetic field is extreme. In the case of deformation sourced by a magnetic field, the exterior of the star is not vacuum, so strictly speaking our solution in the exterior is not valid. However, the magnetic field is mainly dipolar and decays as \(r^{-3}\). Sufficiently far from the star, the vacuum solution is a good approximation and the metric can be matched to the analytical one written in terms of the multipole moments \([16]\).

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FIG. 2. GW phase. Phase contribution, \(\phi_{\text{quadrupole}}\), for the coalescence of two NSs with intrinsic quadrupole moment relative to the leading-order Newtonian term in the LIGO band. The red band corresponds to the range \(\bar{Q} \in (0.01, 0.2)\). The effect of the intrinsic deformation is larger than that of the tidal deformability (blue band) below 100–200 Hz even for (normalized) intrinsic quadrupole moment as small \(\bar{\Lambda_2} \approx 0.01\), and it always dominates at lower frequencies or for higher deformations. We use reference values \(m_1 = m_2 = 1.4M_{\odot}\), and tidal deformability \(\Lambda = 190^{+390}_{-120}\) \([28, 29]\). The quadrupole moment induced by the magnetic field \(B\) can be parametrized as

\[
\bar{Q} = -\beta \frac{\pi B^2 R^8}{\mu_0 I M^3} \approx -0.01 \beta \frac{B_{16}^2 R_{12}^8}{I_{45} M_{1.4}^4}, \tag{7}
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where \(B_{16} = \frac{B}{10^{16} \text{ Gauss}}\), \(R_{12} = \frac{R}{12 \text{ km}}\), \(M_{1.4} = \frac{M}{1.4 M_{\odot}}\), \(I_{45} = \frac{I}{10^{45} \text{ g cm}^2}\), \(I\) is the moment of inertia, \(\mu_0\) is the vacuum permeability, and \(\beta \sim \mathcal{O}(1)\) is the magnetic distortion factor which measures to what extent a star can be deformed by the magnetic field. Note the very strong dependence on the radius of the star \(R\) and the quadratic dependence on the magnetic field \(B\). By computing \(M\), \(R\), and \(I\) for a family of NSs with some tabulated EoS, one can check that the magnetic-induced \(\bar{Q}\) is of the order of what given in Eq. (7) for all configurations, and therefore very small unless the magnetic field is extreme. In the case of deformation sourced by a magnetic field, the exterior of the star is not vacuum, so strictly speaking our solution in the exterior is not valid. However, the magnetic field is mainly dipolar and decays as \(r^{-3}\). Sufficiently far from the star, the vacuum solution is a good approximation and the metric can be matched to the analytical one written in terms of the multipole moments \([16]\).

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An outstanding issue concerns the stability of deformed NSs at equilibrium. Performing a linear stability analysis is particularly challenging due to the broken symmetry of the equilibrium configuration. Nevertheless, such an analysis can be performed using advanced perturbation-theory techniques developed for slowly-spinning compact objects \cite{35,37} (see Appendix A for details). A preliminary analysis shows that deformed NSs are stable against radial perturbations for masses below the maximum mass, exactly as spherically-symmetric NSs \cite{10} and other thin-shell objects like gravastars \cite{38}. Within our perturbative framework, possible instabilities might only come from (axial) nonaxisymmetric fluid perturbations, which are the only ones containing a zero mode in the spherical case. This mode can turn unstable due to the correction proportional to $Q$, similarly to the case of the r-mode instability of slowly-spinning NSs \cite{39–41}. If present, an r-mode-like instability might remove part of the intrinsic quadrupole moment until the mode is saturated by GW emission or other mechanisms.

To second order in the deformations, the coupling between $J$ and $Q$ moments induces novel multipole moments through the standard angular-momentum addition rules \cite{16}. In particular, it induces a mass hexadecapole and a current octupole, as well as a shift of the quadrupole moment (see Appendix A). These corrections are quadratic in the deformation and therefore subleading. For example, for $\bar{Q}_0 \approx 0.1$ in the non-spinning case the NS mass acquires a correction approximately at the percent level, which is comparable to the mass of the crust \cite{19}. On the other hand, higher-order corrections might be important to model highly-deformed stars more accurately.

To summarize, using a simple thin-shell model, we showed that anisotropic crust stresses can support significant quadrupolar deformations while satisfying all energy conditions. Based on this model, we argue that if NSs display some departure from spherical symmetry of the order of at least a few tenths of a percent, their natal quadrupole moment would introduce corrections that are more important than the spin and tidal deformations. We advocate the urgency of quantifying NS intrinsic deformations and of including them in the parameter estimation of electromagnetic and GW signals from isolated and binary NSs.

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when the only nonvanishing deformations are due to the mass quadrupole moment (and, possibly, to the spin). We provide the general framework for independent multipolar deformations, which reduces to the case considered in the main text. We consider a deformed metric of the form

where $g^{(0)}_{\mu\nu} = \text{diag} \{-e^{\nu(r)}, 1/(1 - 2M(r)/r), e^2, r^2 \sin^2 \theta\}$ is the background metric which describes the spherically-symmetric star and can be obtained by solving for the Tolman-Oppenheimer-Volkoff equations. The specific solution considered in the main text, only the mass quadrupole $Q$ (and, possibly, the angular moment $J$) are present at the leading order; in this case the physical expansion parameters are the dimensionless quantities $\epsilon Q/M^2$ and $\epsilon J/M^2$, which are independent from one another.

Stationary and axisymmetric deformations can be expanded in a complete basis of Legendre polynomials:

where $P_\ell = P_\ell (\cos \theta)$ and $P_\ell' = \frac{dP_\ell (\cos \theta)}{d\cos \theta}$. The parameter $\ell$ is related to the multipole moment sourced at each given order $n$ of the perturbative scheme. We separate the perturbations in two sets, according to how they transform

Appendix A: Details on the computation

1. Equilibrium configurations

Our technique is an extension of the perturbative, general-relativistic framework recently developed to study generic axisymmetric departures from spherical symmetry in vacuum spacetimes [10]. We provide the general framework for an arbitrary number of independent multipolar deformations, which reduces to the case considered in the main text when the only nonvanishing deformations are due to the mass quadrupole moment (and, possibly, to the spin).

We consider a deformed metric of the form

where $g^{(0)}_{\mu\nu} = \text{diag} \{-e^{\nu(r)}, 1/(1 - 2M(r)/r), e^2, r^2 \sin^2 \theta\}$ is the background metric which describes the spherically-symmetric star and can be obtained by solving for the Tolman-Oppenheimer-Volkoff equations. The specific solution considered in the main text, only the mass quadrupole $Q$ (and, possibly, the angular moment $J$) are present at the leading order; in this case the physical expansion parameters are the dimensionless quantities $\epsilon Q/M^2$ and $\epsilon J/M^2$, which are independent from one another.

Stationary and axisymmetric deformations can be expanded in a complete basis of Legendre polynomials:

with $P_\ell = P_\ell (\cos \theta)$ and $P_\ell' = \frac{dP_\ell (\cos \theta)}{d\cos \theta}$. The parameter $\ell$ is related to the multipole moment sourced at each given order $n$ of the perturbative scheme. We separate the perturbations in two sets, according to how they transform

$$h^{(n)}_{\mu\nu} = \sum_{\ell} \left( \begin{array}{cccc} g^{(0)}_{00} H_3^{\ell} P_\ell & 0 & 0 & h_0^{\ell} P_\ell' \\ 0 & g^{(0)}_{rr} H_2^{\ell} P_\ell & 0 & 0 \\ 0 & 0 & r^2 K^{\ell} P_\ell & 0 \\ h_0^{\ell} P_\ell' & 0 & 0 & r^2 \sin^2 \theta K^{\ell} P_\ell \end{array} \right),$$

$$\epsilon Q/M^2 \text{ and } \epsilon J/M^2,$$
under parity. The odd (or axial) sector contains only the radial function \( h_0^{\ell}(r) \), which is associated to the current multipole moments, \( S_\ell \). The even (or polar) sector contains the radial functions \( H_0^{\ell}(r) \), \( H_2^{\ell}(r) \), and \( K_n^{\ell}(r) \), which are associated to the mass multipole moments, \( M_\ell \). Note that in the text we defined the mass quadrupole moment as \( Q \); a more standard and general nomenclature is \( Q \equiv M_2 \) [4, 5].

We consider Einstein’s equations coupled to a perfect fluid that describes the interior of the star. The fluid’s stress-energy tensor reads

\[
T^{\mu\nu} = (P + \rho)u^{\mu}u^{\nu} + Pg^{\mu\nu}, \tag{A3}
\]

where

\[
P = P^{(0)}(r) + \sum_{n=1}^{\infty} \sum_{\ell} c^n \delta P^{n\ell}(r) P_\ell \cos(\theta), \tag{A4}
\]

\[
\rho = \rho^{(0)}(r) + \sum_{n=1}^{\infty} \sum_{\ell} c^n \delta \rho^{n\ell}(r) P_\ell \cos(\theta), \tag{A5}
\]

are the pressure and the energy density, respectively, the background values of which are denoted by \( P^{(0)}(r) \) and \( \rho^{(0)}(r) \). The functions \( \delta P^{n\ell}(r) \) and \( \delta \rho^{n\ell}(r) \) are polar quantities. The four-velocity of a fluid element reads

\[
u^\mu = \frac{1}{\sqrt{-g_{tt} - 2\epsilon \Omega g_{t\phi} - g_{\phi\phi}\epsilon^2 \Omega^2}} \{1, 0, 0, \epsilon \Omega\}, \tag{A6}
\]

where \( \Omega \) is the fluid angular velocity and the normalization constant ensures that \( u^2 = -1 \).

By inserting the above decomposition for the metric and the fluid variables into Einstein’s equations, \( G_{\mu\nu} = 8\pi T_{\mu\nu} \), and separating the angular dependence by using the orthogonality of the Legendre polynomials [16, 42], we obtain a set of ordinary differential equations for the deformation functions \( h^{(n)}_{\mu\nu}(r) \), \( \delta P^{(n)}(r) \), and \( \delta \rho^{(n)}(r) \). The system is closed by assuming a barotropic EoS, \( P = P(\rho) \).

In the exterior, the solution to the system can be found analytically at any given order and it depends on an arbitrary number of (mass and current) multipole moments [16]. The explicit values of the multipole moments can be obtained by matching the external solution to the internal one at the radius \( R \) of the star, defined by \( P^{(0)}(r = R) = 0 \) at the leading order. The internal solution is obtained numerically using a Runge-Kutta 4-th order scheme with adaptive meshes [43].

The couplings between multipoles follow the standard addition rules for angular momenta in quantum mechanics, so that if two modes with \( \ell_1 \) and \( \ell_2 > \ell_1 \) are present at a given order in \( \epsilon \), to the next order they will source multipole moments with \( \ell \) such that \( \ell_2 - \ell_1 \leq \ell \leq \ell_2 + \ell_1 \), provided some terms are not forbidden by parity and equatorial-symmetry selection rules [16]. In the specific case of the solution discussed in the main text, only the \( \ell = 1 \) axial deformation and the \( \ell = 2 \) polar deformation are present to \( \mathcal{O}(\epsilon) \). Therefore, all induced multipole moments of this family of solutions can be written as a combination of terms sourced by the spin and by the quadrupole; to the leading order:

\[
\begin{align*}
\bar{M}_\ell &= \sum_{p=0}^{\ell/2} \alpha_p \chi^{\ell-2p} \bar{Q}^p, \quad \text{even } \ell \geq 4, \\
\bar{S}_\ell &= \sum_{p=0}^{(\ell-1)/2} \beta_p \chi^{\ell-2p} \bar{Q}^p, \quad \text{odd } \ell \geq 3, 
\end{align*}
\]

whereas \( M_\ell = 0 \) and \( S_\ell = 0 \) for odd \( \ell \) and even \( \ell \), respectively. In the above equation, \( \bar{M}_\ell = M_\ell/M^{\ell+1} \) and \( \bar{S}_\ell = S_\ell/M^{\ell+1} \) are the normalized multipole moments of degree \( \ell \). The prefactors \( \alpha_p \) and \( \beta_p \) depend on the central density and on the EoS and have to be computed numerically.

a. Explicit external solution to leading order and epicyclic frequencies

The solution used in this work can be computed to any given order using the set of rules [16] discussed above. Nonetheless, to linear order in the perturbation framework the solution is very simple since it is not affected by the
nonlinearities of the field equations. In the exterior of the object the solution can be written as

\[ g_{tt} = -\left(1 - \frac{2M}{r}\right) - \frac{5Q(2M(2M^3 + 4M^2r - 9Mr^2 + 3r^3) + 3r^2(r - 2M)^2 \log \left(1 - \frac{2M}{r}\right))}{8M^5 r^2} P_2, \]  
(A7)

\[ g_{rr} = \left(1 - \frac{2M}{r}\right)^{-1} - \frac{5Q(2M(2M^3 + 4M^2r - 9Mr^2 + 3r^3) + 3r^2(r - 2M)^2 \log \left(1 - \frac{2M}{r}\right))}{8M^5(r - 2M)^2} P_2, \]  
(A8)

\[ g_{\theta\theta} = r^2 - \frac{5Qr(-4M^3 + (3r^3 - 6Mr^2) \log \left(1 - \frac{2M}{r}\right) + 6Mr^2 + 6Mr^2)}{8M^5} P_2, \]  
(A9)

\[ g_{\phi\phi} = g_{\theta\theta} \sin^2 \theta, \]  
(A10)

\[ g_{t\phi} = -\frac{2I \sin^2 \theta}{r}. \]  
(A11)

The epicyclic frequencies can be computed for a generic stationary, axisymmetric spacetime [44]. Setting the spin to zero, to the leading order in \(Q\), the azimuthal frequency and the vertical and radial epicyclic frequencies read

\[ \nu_\phi = \frac{1}{2\pi M x^{3/2}} \left(1 + \Delta_\phi \bar{Q}\right), \]  
(A12)

\[ \nu_\theta = \frac{1}{2\pi M x^{3/2}} \left(1 + \Delta_\theta \bar{Q}\right), \]  
(A13)

\[ \nu_r = \frac{\sqrt{x - 6}}{2\pi M x^2} \left(1 + \Delta_r \bar{Q}\right), \]  
(A14)

respectively, where \(x \equiv r/M\) and

\[ \Delta_\phi = -15(x - 2)x(x^3 - 2) \log \left(\frac{x - 2}{x}\right) + 10x(x(2 - 3(x - 1)x) + 8) - 60, \]  
\[ \frac{32(x - 2)x}{x}, \]  
\[ \Delta_\theta = \frac{5}{32} \left(6x(2x - 7) - \frac{12}{(x - 2)x} + 3(x - 1)(x - 2)^2 \log \left(\frac{x - 2}{x}\right) + 34\right), \]  
\[ \Delta_r = \frac{10(48 + x(30 + x(26 + x(3(25 - 4x)x - 127)))) - 15(x - 2)^2 x^2(x(4x - 13) - 2) \log \left(\frac{x - 2}{x}\right)}{32(x - 6)(x - 2)x}. \]

For completeness, we also provide the leading-order quadrupolar corrections to the ISCO radius in closed form, including also the well-known spin term at the linear order:

\[ r_{\text{ISCO}} = 6M \left[1 - \frac{2\sqrt{2}}{3\sqrt{3}} \chi + \left(\frac{9325}{96} - 480 \coth^{-1}(5)\right) \bar{Q}\right], \]  
(A15)

and the corresponding analytical expressions for \(\nu_\phi\) and \(\nu_\theta\) at the ISCO:

\[ \nu_\phi^{\text{ISCO}} = \frac{1}{12\pi \sqrt{6}M} \left[1 + \frac{11}{6\sqrt{6}} \chi + \frac{5}{32} \left(5892 \coth^{-1}(5) - 1193\right) \bar{Q}\right], \]  
(A16)

\[ \nu_\theta^{\text{ISCO}} = \frac{1}{12\pi \sqrt{6}M} \left[1 + \frac{\sqrt{3}}{2\sqrt{2}} \chi + \left(555 \coth^{-1}(5) - \frac{3595}{32}\right) \bar{Q}\right]. \]  
(A17)

An approximate version of the above equations has been presented in the main text.

\[ b. \text{ Thin-shell crust and junction conditions} \]

We can solve numerically the equations for the deformation in the interior of the star and match them with the above analytical solution in the exterior using the appropriate junction conditions [23, 23, 38, 45].

The first junction condition imposes continuity of the induced metric \(\eta_{ab}\) across the thin shell,

\[ [\eta_{ab}] = 0, \]  
(A18)

where \([X] := X_{\text{out}} - X_{\text{in}}\) denotes a jump of a quantity across the radius of the thin shell. To zeroth order in the deformations the first junction condition implies that

\[ e^{\nu(R)} = 1 - 2M/R, \]  
(A19)
which is the usual continuity of time-time component of the metric, and can be imposed with a rescaling of the time coordinate in the interior of the star. To linear order this condition relates the jump of the metric functions with the quadrupole of the NS,

\[
[[H^0_0]] = - \left[ \left[ \frac{f'}{f} \xi_2 \right] \right],
\]

\[
[[K^1_0]] = - \frac{2}{R} \left[ [\xi_2] \right],
\]

where \( f_{\text{out}} = 1 - 2M/r, f_{\text{in}} = e^\nu \) and \( \xi := \xi_2 P_2(\theta) \) is the displacement of the thin shell.

The second junction condition relates the jump of the extrinsic curvature \( K_{ab} \) with the stress-energy tensor,

\[
S_{ab} = \frac{1}{8\pi} ([[K_{ab}]] - \eta_{ab}[[K]]) ,
\]

with \( K = \eta_{ab} K^{ab} \). The surface energy density \( \sigma \) of the thin shell can be obtained as the eigenvalue of the stress energy tensor, namely

\[
S^a_b u^b = -\sigma u^a .
\]

The surface energy density can be written in general as

\[
\sigma = \sigma_0 + \sigma_2 \tilde{Q} P_2(\theta) ,
\]

where \( \sigma_0 \) is the surface density of the thin shell in the spherically-symmetric configuration, whereas \( \sigma_2 \) is the amplitude of the axisymmetric perturbation to the surface density. By solving Eq. (A23) we find the expressions for the surface density components in terms of the jump of metric functions. The zeroth order spherical case is particularly simple,

\[
\sigma_0 = -\frac{1}{4\pi R} \left[ \left[ (g^{(0)}_{rr})^{-1/2} \right] \right].
\]

Obviously, in the spherical-symmetric case the continuity of the mass function implies the absence of any shell of matter. On the contrary, if we assume the existence of a thin shell with mass \( M_{\text{crust}} = \delta M := M - M(R) \) this implies a \( [[g^{(0)}_{rr}]] \neq 0 \) and the presence of a surface energy density given by Eq. (A25). For small thin-shell masses Eq. (A25) becomes

\[
\sigma_0 = \frac{1}{4\pi R^2} \frac{\delta M}{\sqrt{1 - 2M/R}} + O(\delta M^2) ,
\]

which reduces to the Newtonian case \( \sigma_0 = \delta M/(4\pi R^2) \) when \( M/R \ll 1 \).

By using the projection tensor \( q_{ab} := \eta_{ab} + u_a u_b \), we can define the projected stress-energy tensor of the thin shell,

\[
\gamma_{ab} = S^{cd} q_{ac} q_{bd} .
\]

Again, the spherical case is particularly simple and it reduces to the stress-energy tensor of a perfect fluid,

\[
\gamma_{ab} = \gamma_0 q_{ab} ,
\]

where \( \gamma_0 \) is the surface pressure of the shell. Comparing Eqs. (A27) with (A28) yields

\[
\gamma_0 = \frac{1}{8\pi R} \left[ \left[ \frac{1 + R \ell \ell'}{\sqrt{g^{(0)}_{rr}}} \right] \right] \sim \frac{M}{8\pi R^3} \left( 1 - 2M/R \right)^{3/2} + O(\delta M^2) .
\]

To allow for quadrupolar deformations we will consider an anisotropic fluid for the matter of the thin shell, whose stress-energy tensor reads

\[
\gamma_{ab} = \gamma^\theta k_a k_b + \gamma^\phi \chi_a \chi_b ,
\]

where \( k_a \) and \( \chi_a \) are two normal vectors tangent to the thin shell, orthogonal to the fluid velocity \( u_a \) and among themselves, i.e. \( \chi_a k^a = k_a u^a = \chi_a u^a = 0, k_a k^a = 1 = \chi_a \chi^a \). The latter five conditions do not specify the two vectors completely. There is one remaining degree of freedom which is related to the orientation of the two vectors
with respect to the coordinate axis. One can fix this freedom by choosing to align $k_a$ with the $\theta$-axis and $\chi_a$ with the $\phi$-axis. With this choice $\gamma^\theta$ and $\gamma^\phi$ are the components of the surface pressure measured along the $\theta$ and $\phi$ directions, respectively. To calculate the pressure components it is useful to decompose each of them as $\gamma^i = \gamma^i_0 + \gamma^i_2 P_2(\theta)$, and solve Eqs. (A27) and (A30) for the four independent pressure functions.

For example, the quadrupolar component of the surface energy density reads

$$\sigma_2 = \frac{1}{8\pi R} \left( \left[ \sqrt{1 - 2M/R} H_2 \right] - \left[ \frac{4 + 6M/R}{R \sqrt{1 - 2M/R}} \xi_2 \right] - \left[ R \sqrt{1 - 2M/R} K' \right] \right).$$  (A31)

FIG. 3. **Deformed NSs.** Illustrative embedding diagrams of a deformed NS. The arrow is the angular momentum vector and (if present) coincides with the axis of symmetry. The colors are weighted to represent the current multipole moment. In this example the body on the left is oblate due to a negative intrinsic quadrupole moment which adds to the spin-induced term (the latter contributes to make the star oblate), whereas the body on the right is prolate because the effect of a positive intrinsic quadrupole moment is stronger than the spin contribution.

c. **Examples of deformed NS configurations**

To summarize, the metric and fluid variables obtained numerically in the interior of the object are then matched to the external solution through the above junction conditions. The numerical solutions form a 3-parameter family, which depends on the central density ($\rho_c$), the angular momentum $J$ (aligned with the symmetry axis, see Fig. 3) and the mass quadrupole moment $Q \equiv M^2$.

As a reference, in Table I we provide some relevant quantities of the equilibrium configurations obtained with this procedure and assuming the AP4 EoS. A representative example of the fluid and metric variables in the interior of a deformed NS is shown in Fig. 4.

2. **Linear stability analysis**

Performing a linear-stability analysis of deformed NSs is a challenging task, owing to the lack of symmetry of the background equilibrium solutions. To deal with this problem, we adopted a multi-parameter perturbative scheme, extending the techniques recently developed to study perturbations of slowly-rotating compact objects [35–37]. The computation can be summarized in the following steps [35]: (i) consider a complete set of small perturbations to the metric and the fluid variables, expanded in a basis of spherical harmonics $Y_{lm}(\theta, \phi)$; (ii) solve for the dynamical equations perturbatively to leading order in the perturbation and to any given order in the background deformation; (iii) Fourier-transform the perturbation functions and reduce the dynamical equations to a system of ordinary (radial) differential equations; (iv) finally, solve for the system as an eigenvalue problem and obtain the characteristic modes of vibration. The eigenfrequencies $\omega$ are generically complex numbers and the sign of the imaginary part allows us to discriminate between stable and unstable configurations, owing to the $e^{-i\omega t}$ time dependence of the perturbations.

This standard procedure is made more involved by the broken symmetry of the background and by the presence of a thin shell. In an axially symmetric background, perturbations with different values of the azimuthal number $m$ are
different multipolar indices. For heuristic purposes, let us first consider the dynamics of a test scalar field on the geometry of a deformed NS. By expanding \( \psi = \sum_{\ell m} R_{\ell m}(r)r^{-1}Y_{\ell m}(\theta, \phi)e^{-i\omega t} \), the Klein-Gordon equation, \( \Box \psi = 0 \), reduces to

\[
\sum_{\ell=0}^{\infty} \left( A_\ell(r)Y_{\ell m} + D_\ell(r)P_2(\cos \theta)Y_{\ell m}(\theta, \phi) \right) = 0 ,
\]

(A32)

where \( A_\ell \) and \( D_\ell \) are complicated functions of \( R_{\ell m}(r) \) and of its derivative up to second order, and they also depend on the frequency \( \omega \) and on the multipolar index \( \ell \). The function \( A_\ell \) contains only quantities at zeroth order in the deformation, whereas the function \( D_\ell \) is proportional to the quadrupolar deformation. Using the properties of the spherical harmonics \([35]\), the above equation can be written as a system involving couplings between modes with harmonic index \( \ell \) and \( \ell \pm 2 \):

\[
\frac{2}{3} A_\ell + \left( C_{\ell+1}^2 + C_\ell^2 - \frac{1}{3} \right) D_\ell + C_\ell C_{\ell-1} D_{\ell-2} + C_{\ell+1} C_{\ell+2} D_{\ell+2} = 0 ,
\]

(A33)

where \( C_\ell = \sqrt{\ell^2 - m^2 / 4\ell^2 - 1} \). Formally, the above equation represents an infinite cascade of coupled ordinary differential equations since each \( \ell \) mode is coupled to \( \ell \pm 2 \). However, assuming only \( \ell \)-led perturbations to the leading order, the coupling to \( \ell \pm 2 \) modes is subleading (because \( D_{\ell \pm 2} = O(Q^2) \)) and Eq. (A33) reduces to

\[
A_\ell + \frac{3}{2} \left( C_{\ell+1}^2 + C_\ell^2 - \frac{1}{3} \right) D_\ell = 0 ,
\]

(A34)

which is now a single differential equation for the variable \( R_{\ell m} \) (for each \( \ell \)).

Remarkably, for radial perturbations (\( \ell = m = 0 \)) the second term in Eq. (A34) vanishes identically and therefore radial perturbations are described by an equation of the form \( A_0 = 0 \), which coincides with the spherically-symmetric case. For \( \ell > 0 \), this property is lost and the second term in Eq. (A34) yields some corrections. In this case the perturbation equation can be reduced to a Schrödinger-like equation of the form \( d^2 R_{\ell m}/dx^2 + (\omega^2 - V_{\ell m})R_{\ell m} = 0 \), where \( x \) is a new coordinates and the effective potential acquires a correction proportional to \( Q \). In the exterior, the form of the potential can be computed analytically. An analysis of the potential and corresponding eigenvalue problem shows that there are no unstable modes for any \( \ell \).

Let us now turn to the more interesting case of gravitational perturbations. This is much more involved due to the tensorial nature of the perturbations and the coupling to the fluid modes. Nonetheless, in the radial case we can still

| \( \rho_\ell [10^{15} g/cm^3] \) | \( \Omega/\Omega_K \) | \( Q/10^{-2} \) | \( M(R) \ [M_\odot] \) | \( R \ [km] \) | \( I \) | \( Q_{\sigma_3}/\sigma_0 \) | \( |\Delta \gamma/\gamma_0|_{\text{equator}} \) | \( \nu^{\text{ISCO}} \ [kHz] \) | \( \nu^{\text{ISCO}} \ [kHz] \) |
|----------------|-------------|-------------|----------|--------|------|--------|------------------------|-------------|-------------|
| 1.540 | 0 | 0 | 2.00 | 11.0 | 6.18 | 0 | 0 | 1.10 | 1.10 |
| 0.985 | 0 | 0 | 1.40 | 11.4 | 11.1 | 0 | 0 | 1.57 | 1.57 |
| 0.875 | 0 | 0 | 1.20 | 11.5 | 14.3 | 0 | 0 | 1.84 | 1.84 |
| 1.540 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 1.540 | 0 | 5 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |

### TABLE I. Parameters of a deformed NS for AP4 EoS.

\( \rho_\ell \) is the central energy density; \( \Omega/\Omega_K \) is the angular velocity normalized by the mass-shedding limit \( \Omega_K = \sqrt{M/R^3} \); \( Q \) is the dimensionless quadrupole moment, \( M \) and \( R \) are the stellar mass and radius, \( I \) is the normalized moment of inertia, \( Q_{\sigma_3}/\sigma_0 \) is the amplitude of the shell’s surface density due to the quadrupole moment \( Q \) normalized by a typical surface density, \( \sigma_0 \approx 10^{18} g/cm^2 \), corresponding to 1\% of the stellar mass, \( (\Delta \gamma/\gamma_0)_{\text{equator}} \) is the anisotropy of the shell at the equator normalized by the pressure of the spherical shell, and \( \nu^{\text{ISCO}} \) and \( \nu^{\text{ISCO}}_q \) are the azimuthal frequency and the vertical epicyclic frequency at the ISCO.
FIG. 4. **Internal solutions.** Radial profiles of the fluid and metric variables in the interior of a deformed NS evaluated along the equatorial direction ($\theta = \pi/2$, dashed lines) and polar direction ($\theta = 0$, solid lines). We show the profile of the pressure deformation (top panel) due to an intrinsic quadrupole deformation normalized by the central pressure of the object, $\tilde{p}_2 \equiv (P(r, \theta) - P_0(r))/P_0(0)$, and the metric deformation functions $h_{\mu\nu} \equiv g_{\mu\nu} - g_{\mu\nu}^0$ (bottom panel). The solution corresponds to a non-spinning prolate object with $M = 1.4M_\odot$ and $Q = 0.1$ to linear order in the deformation. To this order the deformations describing a non-rotating oblate object with the same magnitude of quadrupole ($Q = -0.1$) are identical to those shown in this plot but with the opposite sign.

obtain a system of equations which is formally equivalent to Eq. (A34), where now $A_\ell$ and $D_\ell$ are vector functions of all metric and fluid perturbations. Thus, we obtain the remarkable result that, to linear order in the quadrupolar deformation, radial perturbations of a deformed NS are governed by the same equation as in the spherical case. In particular, this implies that deformed NSs are stable under radial perturbations for configurations below the maximum mass [10]. The case of axial dipolar ($\ell = 1$) perturbations and of axial and polar perturbations with $\ell \geq 2$ is much more involved, due to the coupling among different sectors, and is left for future work. We note that, within our perturbative framework, the characteristic frequencies acquire small corrections proportional to $Q$. Therefore, only marginally stable modes in the spherical case can turn unstable. Instabilities, if they exist, can only come from the fluid axial sector for $\ell \geq 2$, which contains a zero mode ($\omega = 0$) in the spherical case. This mode can turn unstable due to the term proportional to $Q$, similarly to the case of the r-mode instability of slowly-spinning NSs [39–41], for which the unstable mode has $\omega = m\Omega$. If present, an r-mode-like instability might remove part of the intrinsic quadrupole moment until the mode is saturated.