Eta-invariants and Anomalies in $U(1)$-Chern-Simons theory

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Abstract. This paper studies $U(1)$-Chern-Simons theory and its relation to a construction of Chris Beasley and Edward Witten (BW05). The natural geometric setup here is that of a three-manifold with a Seifert structure. Based on a suggestion of Edward Witten we are led to study the stationary phase approximation of the path integral for $U(1)$-Chern-Simons theory after one of the three components of the gauge field is decoupled. This gives an alternative formulation of the partition function for $U(1)$-Chern-Simons theory that is conjecturally equivalent to the usual $U(1)$-Chern-Simons theory (as in Man98). The goal of this paper is to establish this conjectural equivalence rigorously through appropriate regularization techniques. This approach leads to some rather surprising results and opens the door to studying hypoelliptic operators and their associated eta-invariants in a new light.

1. Introduction

In BW05 the authors study the Chern-Simons partition function (see BW05, (3.1)),
(1.1)

$$Z(k) = \frac{1}{\text{Vol}(G)} \left( \frac{k}{4\pi^2} \right)^{\Delta(G)} \int \mathcal{D}A \exp \left[ \frac{k}{4\pi} \int_X \text{Tr} \left( A \wedge dA + \frac{2}{3} A \wedge A \wedge A \right) \right],$$

where,

- $A \in \mathcal{A}_P = \{ A \in (\Omega^1(P) \otimes g)^G \mid A(\xi^\sharp) = \xi, \forall \xi \in g \}$ is a connection on a principal $G$-bundle $\pi : P \to X$ over a closed three-manifold $X$,
- $\mathfrak{g} = \text{Lie}G$ and $\xi^\sharp \in \Gamma(TX)$ is the vector field on $P$ generated by the infinitesimal action of $\xi$ on $P$,
- $k \in \mathbb{Z}$ (thought of as an element of $H^4(BG, \mathbb{Z})$ that parameterizes the possible Chern-Simons invariants),
- $\mathcal{G} := \{ \psi \in (\text{Diff}(P, P))^G \mid \pi \circ \psi = \pi \}$ is the gauge group,
- $\Delta(G)$ is formally defined as the dimension of the gauge group $G$.

1991 Mathematics Subject Classification. Primary 54C40.

Key words and phrases. Contact geometry, quantum field theory.

The first author was supported in part by a grant from NSERC.

We would like to thank John Bland, Eckhard Meinrenken, Raphaël Ponge, Edward Witten and especially Frédéric Rochon and Michel Rumin for helpful advice related to this work.

1In fact, BW05 consider only $G$ compact, connected and simple, and for concreteness one may assume $G = SU(2)$.

2Note that the definition of the Chern-Simons partition function in Eq. (1.1) is completely heuristic. The measure $\mathcal{D}A$ has not been defined, but only assumed to "exist heuristically," and
In general, the partition function of Eq. [1.1] does not admit a general mathematical interpretation in terms of the cohomology of some classical moduli space of connections, in contrast to Yang-Mills theory for example (cf. [Wit92]). The main result of [BW05], however, is that if \( X \) is assumed to carry the additional geometric structure of a Seifert manifold, then the partition function of Eq. [1.1] does admit a more conventional interpretation in terms of the cohomology of some classical moduli space of connections. Using the additional Seifert structure on \( X \), [BW05] decouple one of the components of a gauge field \( A \), and introduce a new partition function (cf. [BW05] ; Eq. 3.7),

\[
\tilde{Z}(k) = K \cdot \int \mathcal{D}A \mathcal{D}\Phi \exp \left[ i \frac{k}{4\pi} \left( CS(A) - \int_X 2\kappa \wedge \text{Tr}(\Phi F_A) + \int_X \kappa \wedge d\kappa \text{Tr}(\Phi^2) \right) \right],
\]

where

- \( K := \frac{1}{\text{Vol}(\mathcal{G})} \frac{1}{\text{Vol}(\mathcal{S})} \left( \frac{k}{4\pi} \right)^{\Delta(\mathcal{G})} \)
- \( \kappa \in \Omega^1(X,\mathbb{R}) \) is a contact form associated to the Seifert fibration of \( X \) (cf. [BW05] ; §3.2),
- \( \Phi \in \Omega^1(X,\mathfrak{g}) \) is a Lie algebra-valued zero form on \( X \),
- \( \mathcal{D}\Phi \) is a measure on the space of fields \( \Phi \),
- \( \mathcal{S} \) is the space of local shift symmetries that “acts” on the space of connections \( \mathcal{A}_p \) and the space of fields \( \Phi \) (cf. [BW05] ; §3.1),
- \( F_A \in \Omega^2(X,\mathfrak{g}) \) is the curvature of \( A \), and
- \( CS(A) := \int_X \text{Tr} (A \wedge dA + \frac{2}{3} A \wedge A \wedge A) \) is the Chern-Simons action.

Then give a heuristic argument showing that the partition function computed using the alternative description of Eq. [1.2] should be the same as the Chern-Simons partition function of Eq. [1.1]. In essence, they show

\[
Z(k) = \tilde{Z}(k),
\]

by gauge fixing \( \Phi = 0 \) using the shift symmetry. [BW05] then observe that the dependence in the integral can be eliminated by simply performing the Gaussian integral over \( \Phi \) in Eq. [1.2] directly. They obtain the alternative formulation:

\[
Z(k) = \tilde{Z}(k) = K' \cdot \int \mathcal{D}A \exp \left[ i \frac{k}{4\pi} \left( CS(A) - \int_X \frac{1}{\kappa \wedge d\kappa} \text{Tr} \left[ (\kappa \wedge F_A)^2 \right] \right) \right],
\]

the volume and dimension of the gauge group, \( \text{Vol}(\mathcal{G}) \) and \( \Delta(\mathcal{G}) \), respectively, are at best formally defined.

3 The measure \( \mathcal{D}\Phi \) is defined independently of any metric on \( X \) and is formally defined by the positive definite quadratic form

\[
(\Phi, \Phi) := -\int_X \kappa \wedge d\kappa \text{Tr}(\Phi^2),
\]

which is invariant under the choice of representative for the contact structure \((X, H)\) on \( X \), i.e. under the scaling \( \kappa \mapsto f\kappa \), \( \Phi \mapsto f^{-1}\Phi \), for some non-zero function \( f \in \Omega^0(X,\mathbb{R}) \).

4 \( \mathcal{S} \) may be identified with \( \Omega^1(X,\mathfrak{g}) \), where the “action” on \( \mathcal{A}_p \) is defined as \( \delta_\sigma(A) := \sigma\kappa \), and on the space of fields \( \Phi \) is defined as \( \delta_\sigma(\Phi) := \sigma \), for \( \sigma \in \Omega^1(X,\mathfrak{g}) \). \( \delta_\sigma \) denotes the action associated to \( \sigma \).

5 Note that the partition functions of Eq.’s [1.1] and [1.2] are defined implicitly with respect the pullback of some trivializing section of the principal G-bundle \( P \). Of course, every principal G-bundle over a three-manifold for \( G \) compact, connected and simple is trivializable. It is basic fact that the partition functions of Eq.’s [1.1] and [1.2] are independent of the choice of such trivializations.
where \( K' := \frac{1}{\text{Vol}(U)} \frac{1}{\text{Vol}(S)} (\frac{-ik}{4\pi})^{\Delta G/2} \).

The objective in this article is to study the partition function for \( U(1) \)-Chern-Simons theory using the analogue of Eq. 1.4 in this case. Thus, we are also assuming here that \( X \) is a Seifert manifold with a “compatible” contact structure, \((X, \kappa)\) (cf. [BW05]; §3.2). Note that any compact, oriented three-manifold possesses a contact structure and one aim of future work is to extend our results to all closed three-manifolds using this fact. For now, we restrict ourselves to the case of closed three-manifolds that possess contact compatible Seifert structures (see Definition 11 for example). We restrict to the gauge group \( U(1) \) so that the action is quadratic and hence the stationary phase approximation is exact. A salient point is that the group \( U(1) \) is not simple, and therefore may have non-trivial principal bundles associated with it. This makes the \( U(1) \)-theory very different from the \( SU(2) \)-theory in that one must now incorporate a sum over bundle classes in a definition of the \( U(1) \)-partition function. As an analogue of Eq. 1.1, our basic definition of the partition function for \( U(1) \)-Chern-Simons theory is now

\[
Z_{U(1)}(X, k) = \sum_{p \in \text{Tors} H^2(X; \mathbb{Z})} Z_{U(1)}(X, p, k)
\]

where

\[
Z_{U(1)}(X, p, k) = \frac{1}{\text{Vol}(\mathcal{G}_p)} \int_{\mathcal{A}_p} \mathcal{D}A \mathcal{E}^{\tau ik S_{X,P}(A)}
\]

recalling that the torsion subgroup \( \text{Tors} H^2(X; \mathbb{Z}) < H^2(X; \mathbb{Z}) \) enumerates the \( U(1) \)-bundle classes that have flat connections. Note that the bundle \( P \to X \) in Eq. 1.6 is taken to be any representative of a bundle class with first Chern class \( c_1(P) = p \in \text{Tors} H^2(X; \mathbb{Z}) \). Also note that some care must be taken to define the Chern-Simons action, \( S_{X,P}(A) \), in the case that \( G = U(1) \). We outline this construction in Appendix A.

The main results of this article may be summarized as follows. First, our main objective is the rigorous confirmation of the heuristic result of Eq. 1.3 in the case where the gauge group is \( U(1) \). This statement is certainly non-trivial and involves some fairly deep facts about the “contact operator” as studied by Michel Rumin (cf. [Rum94]). Recall that this is the second order operator “\( D \)” that fits into the complex,

\[
C^\infty(X) \xrightarrow{d_H} \Omega^1(H) \xrightarrow{D} \Omega^2(V) \xrightarrow{d_H} \Omega^3(X),
\]

and is defined by:

\[
D \alpha = \kappa \wedge [\mathcal{L}_\xi + d_H \ast_H d_H] \alpha, \quad \alpha \in \Omega^1(H).
\]

This operator is elaborated upon in [4] below. A somewhat surprising observation is that this operator shows up quite naturally in \( U(1) \)-Chern-Simons theory (see Prop. 17 below), and this leads us to make several conjectures motivated by the

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6Note that we have abused notation slightly by writing \( \frac{1}{\text{Vol}(U)} = \frac{1}{\text{Vol}(S)} (\frac{-ik}{4\pi})^{\Delta G/2} \). We have done this with the understanding that since \( \kappa \wedge dk \) is non-vanishing (since \( \kappa \) is a contact form), then \( \kappa \wedge F_A = \phi \kappa \wedge dk \) for some function \( \phi \in \Omega^1(X, g) \), and we identify \( \frac{1}{\text{Vol}(U)} = \frac{1}{\text{Vol}(S)} (\frac{-ik}{4\pi})^{\Delta G/2} \).

7Recall the definition of the torsion of an abelian group is the collection of those elements which have finite order.
rigorous confirmation of the heuristic result of Eq. 1.3. Our main result is the following:

**Proposition 1.** Let \((X, \phi, \xi, \kappa, g)\) be a closed, quasi-regular K-contact three-manifold. If,

\[
\bar{Z}_{U(1)}(X, p, k) = k^n_X e^{\pi i k S_{X, p}(A_0)} e^{\frac{\pi i}{4} (\eta(-\star D) + \frac{1}{12} \sum_{I=1}^{d} \text{CS}(A_I))} \int_{\mathcal{M}_p} (T_{RS})^{1/2}
\]

where \(R \in C^\infty(X)\) = the Tanaka-Webster scalar curvature of \(X\), and \([\text{Man98}]\),

\[
Z_{U(1)}(X, p, k) = k^m_X e^{\pi i k S_{X, p}(A_0)} e^{\pi i \left(\eta(-\star d) + \frac{1}{12} \text{CS}(A_1)\right)} \int_{\mathcal{M}_p} (T_{RS})^{1/2}
\]

then,

\[
Z_{U(1)}(X, k) = \bar{Z}_{U(1)}(X, k)
\]

as topological invariants.

Following [Man98], we rigorously define \(\bar{Z}_{U(1)}(X, k)\) in §6 using the fact that the stationary phase approximation for our path integral should be exact. This necessitates the introduction of the regularized determinant of \(D\) in Eq. 7.2 which in turn naturally involves the hypoelliptic Laplacian of Eq. 7.3. The rigorous quantity that we obtain for the integrand of Eq. 6.2 in §6 is derived in Prop. 21. Using an observation from §5 that identifies the volume of the isotropy subgroup of the gauge group \(\mathcal{G}_P\), we identify the integrand of Eq. 6.2 with the contact analytic torsion \(T_{dC}\) defined in Def. 23. After formally identifying the signature of the contact operator \(D\) with the \(\eta\)-invariant of \(D\) in §8, we obtain our fully rigorous definition of \(\bar{Z}_{U(1)}(X, k)\) in Eq. 9.15 below, which is repeated in Eq. 1.9 above.

On the other hand, [Man98] provides a rigorous definition of the partition function \(Z_{U(1)}(X, k)\) that does not involve an a priori choice of a contact structure on \(X\). The formula for this is recalled in Eq. 9.16 below, and is the term \(Z_{U(1)}(X, p, k)\) in Eq. 1.10 of Prop. 1 above.

Our first main step in the proof of Prop. 1 is confirmation of the fact that the Ray-Singer analytic torsion (cf. [RS73]) of \(X\), \(T^d_{RS}\), is identically equal to the contact analytic torsion \(T^d_{dC}\).\(^8\) We observe that this result follows directly from ([RS08]; Theorem 4.2).

We also observe in Remark 22 that the quantities \(m_X\) and \(n_X\) that occur in Prop. 1 are also equal. This leaves us with the main final step in the confirmation of Prop. 1 which involves a study of the \(\eta\)-invariants, \(\eta(-\star d)\), \(\eta(-\star D)\), that naturally show up in \(Z_{U(1)}(X, k)\), \(\bar{Z}_{U(1)}(X, k)\), respectively. This analysis is carried out in §10 where we observe that the work of Biquard, Herzlich, and Rumin ([BHR07]) is our most pertinent reference. Our main observation here is that the quantum anomalies that occur in the computation of \(Z_{U(1)}(X, k)\) and \(\bar{Z}_{U(1)}(X, k)\) should, in an appropriate sense, be completely equivalent. In our case, these quantum anomalies are made manifest precisely in the failure of the \(\eta\)-invariants to represent topological invariants. As observed by Witten (cf. [Wit89]), this is deeply connected with the fact that in order to actually compute the partition function, one needs to make a

\(^8\)We consider the square roots thereof, viewed as densities on the moduli space of flat connections \(\mathcal{M}_X\).
choice that is tantamount to either a valid gauge choice for representatives of gauge classes of connections, or in some other way by breaking the symmetry of our problem. Such a choice for us is equivalent to a choice of metric, which is encoded in the choice of a quasi-regular K-contact structure on our manifold $X$. Witten observes in [Wit89] that the quantum anomaly that is introduced by our choice of metric may be canceled precisely by adding an appropriate “counterterm” to the $\eta$-invariant, $\eta(- \star d)$. This recovers topological invariance and effectively cancels the anomaly. This counterterm is found by appealing to the Atiyah-Patodi-Singer theorem, and is in fact identified as the gravitational Chern-Simons term

\begin{equation}
\text{CS}(A^\theta) := \frac{1}{4\pi} \int_X \text{Tr}(A^\theta \wedge dA^\theta + \frac{2}{3} A^\theta \wedge A^\theta \wedge A^\theta),
\end{equation}

where $A^\theta$ is the Levi-Civita connection on the spin bundle of $X$ for the metric,

\begin{equation}
g = \kappa \otimes \kappa + d\kappa(\cdot, J\cdot),
\end{equation}

on our quasi-regular K-contact three manifold, $(X, \phi, \xi, \kappa, g)$. In particular, we use the fact that,

\begin{equation}
\frac{\eta(- \star d)}{4} + \frac{1}{12} \frac{\text{CS}(A^\theta)}{2\pi},
\end{equation}

is a topological invariant of $X$, after choosing the canonical framing. As is discussed in §10 this leads us to conjecture that there exists an appropriate counterterm for the $\eta$-invariant associated to the contact operator $D$ that yields the same topological invariant as in Eq. 1.13. More precisely, we conjecture that there exists a counterterm, $C_T$, such that

\begin{equation}
e^{\pi i \left[ \frac{\eta(- \star d)}{4} + \frac{1}{12} \frac{\text{CS}(A^\theta)}{2\pi} \right]} = e^{\pi i \left[ \eta(- \star H D^1) + C_T \right]},
\end{equation}

as topological invariants. We establish the following in Proposition 31.

**Proposition 2.** $(X, \phi, \xi, \kappa, g)$ closed, quasi-regular K-contact three-manifold. The counterterm, $C_T$, such that $e^{\pi i \left[ \eta(- \star H D^1) + C_T \right]}$ is a topological invariant that is identically equal to the topological invariant $e^{\pi i \left[ \frac{\eta(- \star d)}{4} + \frac{1}{12} \frac{\text{CS}(A^\theta)}{2\pi} \right]}$ is

\begin{equation}
C_T = \frac{1}{512} \int_X R^2 \kappa \wedge d\kappa,
\end{equation}

where $R \in C^\infty(X)$ is the Tanaka-Webster scalar curvature of $X$.

This proposition is proven in §10 by appealing to the following result, which is established using a “Kaluza-Klein” dimensional reduction technique for the gravitational Chern-Simons term. This result is modeled after the paper [GLJP03], and is listed as Proposition 30.

**Proposition 3.** $(\mathcal{M}cL10)$ $(X, \phi, \xi, \kappa, g)$ closed, quasi-regular K-contact three-manifold,

\begin{equation}
U(1) \xrightarrow{\cdot} X \xrightarrow{\downarrow} \Sigma
\end{equation}

\footnote{In this case, topological invariance is recovered only up to a choice of two-framing for $X$. Of course, there is a canonical choice of such framing ([At90]), and we assume this choice throughout.}
Let $g := \epsilon^{-1} \kappa \otimes \kappa + \pi^* h$. After choosing a framing for $TX \oplus TX$, corresponding to a choice of vielbeins, then,

$$\text{CS}(A^g) = \left(\frac{\epsilon^{-1}}{2}\right) \int_{\Sigma} r \omega + \left(\frac{\epsilon^{-2}}{2}\right) \int_{\Sigma} f^2 \omega$$

where $r \in C_{\text{orb}}^\infty(\Sigma)$ is the (orbifold) scalar curvature of $(\Sigma, h)$, $\omega \in \Omega^2_{\text{orb}}(\Sigma)$ is the (orbifold) Hodge form of $(\Sigma, h)$, and $f := \ast h \omega$. In particular, the adiabatic limit of $\text{CS}(A^g)$ vanishes:

$$\lim_{\epsilon \to \infty} \text{CS}(A^g) = 0.$$

Finally, as a consequence of these investigations, we are able to compute in Proposition 32 the $U(1)$-Chern-Simons partition function fairly explicitly.

**Proposition 4.** $(X, \phi, \xi, \kappa, g)$ closed, quasi-regular K-contact three-manifold. Then,

$$\eta(- \ast d) + \frac{1}{3} \frac{\text{CS}(A^g)}{2\pi} = \eta(- \ast D) + \frac{1}{512} \int_X R^2 \kappa \wedge d\kappa$$

$$= 1 + \frac{d}{3} + 4 \sum_{j=1}^{N} s(\alpha_j, \beta_j),$$

where $d = c_1(X) = n + \sum_{j=1}^{N} \frac{\beta_j}{\alpha_j} \in \mathbb{Q}$ and

$$s(\alpha, \beta) := \frac{1}{4\alpha} \sum_{k=1}^{\alpha-1} \cot \left(\frac{\pi k}{\alpha}\right) \cot \left(\frac{\pi k^2}{\alpha}\right) \in \mathbb{Q}$$

is the classical Rademacher-Dedekind sum, where $[n; (\alpha_1, \beta_1), \ldots, (\alpha_N, \beta_N)]$ (for $\gcd(\alpha_j, \beta_j) = 1$) are the Seifert invariants of $X$. In particular, we have computed the $U(1)$-Chern-Simons partition function as:

$$Z_{U(1)}(X, p, k) = k^n X e^{\pi ik S_{X, P}(A_0)} e^{\frac{\pi}{4}(1 + \frac{4}{3} + 4 \sum_{j=1}^{N} s(\alpha_j, \beta_j))} \int_{M_P} \left(T^d_{RS}\right)^{1/2},$$

where $S_{X, P}(A_0)$ is the Chern-Simons invariant associated to $P$ for $A_0$ a flat connection on $P$. The derivation of Eq. (2.1) can be found in Appendix A. It is obtained by expanding the $U(1)$ analogue of Eq. (1.4) around a critical point $A_0$ of the action. Note that the critical points of this action, up to the action of the shift symmetry, are precisely the flat connections ([BW05] ; Eq. 5.3). In our notation, $A \in T_{A_0} A_P$.

2. Preliminary Results

Our starting point is the analogue of Eq. (1.4) for the $U(1)$-Chern-Simons partition function:

$$Z_{U(1)}(X, p, k) = e^{\pi i S_{X, P}(A_0)} \int_{A_P} DA \exp \left[ \frac{ik}{4\pi} \left( \int_{X} A \wedge dA - \int_{X} (\kappa \wedge dA)^2 \right) \right]$$

where $S_{X, P}(A_0)$ is the Chern-Simons invariant associated to $P$ for $A_0$ a flat connection on $P$. The derivation of Eq. (2.1) can be found in Appendix A. It is obtained by expanding the $U(1)$ analogue of Eq. (1.4) around a critical point $A_0$ of the action. Note that the critical points of this action, up to the action of the shift symmetry, are precisely the flat connections ([BW05] ; Eq. 5.3). In our notation, $A \in T_{A_0} A_P$. Let us define the notation

$$S(A) := \int_{X} A \wedge dA - \int_{X} (\kappa \wedge dA)^2$$

Let $S(A) := \int_{X} A \wedge dA - \int_{X} (\kappa \wedge dA)^2$
for the new action that appears in the partition function. Also, define

\[ S(A) := \int_X \frac{(\kappa \wedge dA)^2}{\kappa \wedge d\kappa} \]

so that we may write

\[ S(A) = CS(A) - \tilde{S}(A) \]

The primary virtue of Eq. 2.1 above is that it is exactly equal to the original Chern-Simons partition function of Eq. 1.6 and yet it is expressed in such a way that the action \( S(A) \) is invariant under the shift symmetry. This means that \( S(A + \sigma \kappa) = S(A) \) for all tangent vectors \( A \in T_{A_0}(\mathcal{A}_P) \simeq \Omega^1(X) \) and \( \sigma \in \Omega^0(X) \). We may naturally view \( A \in \Omega^1(H) \), the sub-bundle of \( \Omega^1(X) \) restricted to the contact distribution \( H \subset TX \). Equivalently, if \( \xi \) denotes the Reeb vector field of \( \kappa \), then \( \Omega^1(H) = \{ \omega \in \Omega^1(X) \mid \iota_\xi \omega = 0 \} \). The remaining contributions to the partition function come from the orbits of \( S \) in \( \mathcal{A}_P \), which turn out to give a contributing factor of \( \text{Vol}(S) \) (cf. [BW05] : Eq. 3.32). We thus reduce our integral to an integral over \( \bar{\mathcal{A}}_P := \mathcal{A}_P / S \) and obtain:

\[
Z_{U(1)}(X, p, k) = \frac{e^{\pi i k S_{X, p}(A_0)}}{\text{Vol}(\mathcal{G}_P)} \int_{\bar{\mathcal{A}}_P} \bar{D}A \exp \left[ \frac{i k}{4\pi} \left( \int_X A \wedge dA - \int_X \frac{(\kappa \wedge dA)^2}{\kappa \wedge d\kappa} \right) \right]
\]

where \( \bar{D}A \) denotes an appropriate quotient measure on \( \bar{\mathcal{A}}_P \), and we can now assume that \( A \in \Omega^1(H) \simeq T_{A_0}\bar{\mathcal{A}}_P \).

3. Contact structures

At this point, we further restrict the structure on our 3-manifold and assume that the Seifert structure is compatible with a contact metric structure \((\phi, \xi, \kappa, g)\) on \( X \). In particular, we restrict to the case of a quasi-regular K-contact manifold. Let us review some standard facts about these structures in the case of dimension three.

Remark 5. Our three manifolds \( X \) are assumed to be closed throughout this paper.

Definition 6. A \( K \)-contact manifold is a manifold \( X \) with a contact metric structure \((\phi, \xi, \kappa, g)\) such that the Reeb field \( \xi \) is Killing for the associated metric \( g \), \( \mathcal{L}_\xi g = 0 \).

where,

- \( \kappa \in \Omega^1(X) \) contact form, \( \xi = \text{Reeb vector field} \).
- \( H := \ker \kappa \subset TX \) denotes the horizontal or contact distribution on \((X, \kappa)\).
- \( \phi \in \text{End}(TX), \phi(Y) = JY \) for \( Y \in \Gamma(H) \), \( \phi(\xi) = 0 \) where \( J \in \text{End}(H) \) complex structure on the contact distribution \( H \subset TX \).
- \( g = \kappa \otimes \kappa + d\kappa(\cdot, \phi) \)

Remark 7. Note that we will assume that our contact structure is “co-oriented,” meaning that the contact form \( \kappa \in \Omega^1(X) \) is a global form. Generally, one can take
the contact structure to be defined only locally by the condition $H := \ker \kappa$, where $\kappa \in \Omega^1(U)$ for open subsets $U \in X$ contained in an open cover of $X$.

**Definition 8.** The characteristic foliation $\mathcal{F}_\xi$ of a contact manifold $(X, \kappa)$ is said to be quasi-regular if there is a positive integer $j$ such that each point has a foliated coordinate chart $(U, x)$ such that each leaf of $\mathcal{F}_\xi$ passes through $U$ at most $j$ times. If $j = 1$ then the foliation is said to be regular.

Definitions 9.9 and 9.10 together define a quasi-regular $K$-contact manifold, $(X, \phi, \xi, \kappa, g)$. Such three-manifolds are necessarily “Seifert” manifolds that fiber over a two-dimensional orbifold $\hat{\Sigma}$ with additional structure. Recall:

**Definition 9.** A Seifert manifold is a three manifold $X$ that admits a locally free $U(1)$-action. Thus, Seifert manifolds are simply $U(1)$-bundles over an orbifold $\hat{\Sigma}$,

$$\xymatrix{U(1) \ar@{^{(}->}[rr] & & X \ar[d] \ar@{_{(}->}[ll]^\Sigma}$$

We have the following classification result: $X$ is a quasi-regular K-contact three manifold $\iff$

- ([BG08], Theorem 7.5.1, (i)) $X$ is a $U(1)$-Seifert manifold over a Hodge orbifold surface, $\hat{\Sigma}$.
- ([BG08], Theorem 7.5.1, (iii)) $X$ is a $U(1)$-Seifert manifold over a normal projective algebraic variety of real dimension two.

**Example 3.1.** All 3-dimensional Lens spaces, $L(p, q)$ and the Hopf fibration $S^1 \hookrightarrow S^3 \to \mathbb{C}P^1$ possess quasi-regular $K$-contact structures. Note that any trivial $U(1)$-bundle over a Riemann surface $\Sigma_g$, $X = U(1) \times \Sigma_g$, possesses no $K$-contact structure ([Ito97]), however, and our results do not apply in this case.

**Remark 10.** Note that in fact our results apply to the class of all closed Sasakian three-manifolds. This follows from the observation that every Sasakian three manifold is $K$-contact (cf. [Bla76]; Corollary 6.5), and every $K$-contact manifold possesses a quasi-regular $K$-contact structure (cf. [BG08]; Theorem 7.1.10).

A useful observation for us is that for a quasi-regular $K$-contact three-manifold, the metric tensor $g$ must take the following form (cf. [BG08]; Theorem 6.3.6):

$$g = \kappa \otimes \kappa + \pi^* h$$

where $\pi : X \to \Sigma$ is our quotient map, and $h$ represents any (orbifold)Kähler metric on $\hat{\Sigma}$ which is normalized so that the corresponding (orbifold)Kähler form, $\hat{\omega} \in \Omega^2_{orb}(\Sigma, \mathbb{R})$, pulls back to $d\kappa$.

Note that the assumption that the Seifert structure on $X$ comes from a quasi-regular $K$-contact structure $(\phi, \xi, \kappa, g)$ is equivalent to assuming that $X$ is a CR-Seifert manifold (cf. [BG08]; Prop. 6.4.8). Recall the following

**Definition 11.** A CR-Seifert manifold is a three-dimensional compact manifold endowed with both a strictly pseudoconvex CR structure $(H, J)$ and a Seifert structure, that are compatible in the sense that the circle action $\psi : U(1) \to \text{Diff}(X)$ preserves the CR structure and is generated by a Reeb field $\xi$. In particular,
given a choice of contact form $\kappa$, the Reeb field is Killing for the associated metric $g = \kappa \otimes \kappa + d\kappa$.

The assumption that $X$ is CR-Seifert (hence quasi-regular K-contact) is sufficient to ensure that the assumption in [BW05; Eq. 3.27], which states that the $U(1)$-action on $X$, $\psi: U(1) \to \text{Diff}(X)$, acts by isometries, is satisfied.

We now employ the natural Hodge star operator $\ast$, induced by the metric $g$ on $X$, that acts on $\Omega^\bullet(X)$ taking $k$ forms to $3-k$ forms. As a result of this normalization convention, we have $\ast 1 = \kappa \wedge d\kappa$ and $\ast \kappa = d\kappa$. Now let

$$
\ast_H = -\iota_\xi \circ \ast
$$

as in equation (3.30) of [BW05]. This operator then satisfies

$$
\ast_H \kappa = 0
$$

$$
\ast_H (\kappa \wedge d\kappa) = 0
$$

$$
\ast_H 1 = -d\kappa
$$

$$
(\ast_H)^2 = -1
$$

as is shown in (BW05; pg. 20). We also define a horizontal exterior derivative $d_H$ as the usual exterior derivative $d$ restricted to the space of horizontal forms $\Omega^\bullet(H)$.

Our key observation is that the action $S(A)$ may now be expressed in terms of these horizontal quantities. Let us start with the term $\tilde{S}(A)$. Firstly, the term $\kappa \wedge dA$ in $\tilde{S}(A)$ is equivalent to $\kappa \wedge d_H A$ since the vertical part of $dA$ is annihilated by $\kappa$ in the wedge product. The term $\frac{\kappa \wedge dA}{\kappa \wedge d\kappa}$ is equivalent to $\ast (\kappa \wedge d_H A)$ by the properties of $\ast$ above. By the definition of $\ast_H$, $\ast (\kappa \wedge d_H A) = \ast_H d_H A$. We then have,

$$
\tilde{S}(A) = \int_X \frac{(\kappa \wedge dA)^2}{\kappa \wedge d\kappa}
$$

$$
= \int_X \ast_H (d_H A) \wedge \kappa \wedge d_H A
$$

$$
= \int_X \kappa \wedge [d_H A \wedge \ast_H (d_H A)]
$$

We claim that $\tilde{S}(A)$ is now expressed in terms of an inner product on $\Omega^2 H$. More generally, we define an inner product on $\Omega^l H$ for $0 \leq l \leq 2$:

**Definition 12.** Define the pairing $\langle \cdot, \cdot \rangle^l_\kappa : \Omega^l H \times \Omega^l H \to \mathbb{R}$ as

$$
\langle \alpha, \beta \rangle^l_\kappa := (-1)^l \int_X \kappa \wedge [\alpha \wedge \ast_H \beta]
$$

for any $\alpha, \beta \in \Omega^l H$, $0 \leq l \leq 2$.

**Proposition 13.** The pairing $\langle \cdot, \cdot \rangle^l_\kappa$ is an inner product on $\Omega^l H$.

**Proof.** It can be easily checked that this pairing is just the restriction of the usual $L^2$-inner product, $\langle \cdot, \cdot \rangle : \Omega^l X \times \Omega^l X \to \mathbb{R}$,

$$
\langle \alpha, \beta \rangle := \int_X \alpha \wedge \ast \beta
$$
restricted to horizontal forms. i.e. for any $\beta \in \Omega^lH$, $0 \leq l \leq 2$, we have $\ast \beta = \kappa \wedge \ast_H \beta$. We then have $\alpha \wedge \ast \beta = (-1)^l \kappa \wedge [\alpha \wedge \ast_H \beta]$ for any $\alpha, \beta \in \Omega^lH$, $0 \leq l \leq 2$. Thus, $\langle \cdot, \cdot \rangle^l_\kappa = \langle \cdot, \cdot \rangle$ on $\Omega^lH$ and therefore defines an inner product.

By our definition, we may now write $\bar{S}(A) = \langle d_H A, d_H A \rangle^2_\kappa$. We make the following

**Definition 14.** Define the formal adjoint of $d_H$, denoted $d^*_H$, via:

$$\langle d^*_H \gamma, \phi \rangle^{l-1}_\kappa = \langle \gamma, d_H \phi \rangle^l_\kappa$$

for $\gamma \in \Omega^l(H)$, $\phi \in \Omega^{l-1}(H)$ where $l = 1, 2$ and $d^*_H \gamma = 0$ for $\gamma \in \Omega^0(H)$.

**Proposition 15.** $d^*_H = (-1)^l \ast_H d_H \ast_H : \Omega^l(H) \to \Omega^{l-1}(H)$, $0 \leq l \leq 2$, where $\Omega^{-1}(H) := 0$.

**Proof.** This just follows from the definition of $d^*$ relative to the ordinary inner product $\langle \cdot, \cdot \rangle$, and the facts that $\langle \cdot, \cdot \rangle^{l-1}_\kappa$ is just this ordinary inner product restricted to horizontal forms and $d^* = (-1)^l \ast d\ast$. 

Thus, we may now write $\bar{S}(A) = \langle A, d^*_H d_H A \rangle^l_\kappa$ and identify this piece of the action with the second order operator $d^*_H d_H$ on horizontal forms.

Now we turn our attention to the Chern-Simons part of the action $CS(A) = \int_X A \wedge dA$. We would like to reformulate this in terms of horizontal quantities as well. This is straightforward to do; simply observe that $dA = \kappa \wedge \mathcal{L}_\xi A + d_H A$. Thus, we have:

\begin{align}
(3.9) \quad CS(A) &= \int_X A \wedge dA \\
(3.10) &= \int_X A \wedge [\kappa \wedge \mathcal{L}_\xi A + d_H A] \\
(3.11) &= \int_X A \wedge [\kappa \wedge \mathcal{L}_\xi A] + \int_X A \wedge d_H A \\
(3.12) &= \int_X A \wedge [\kappa \wedge \mathcal{L}_\xi A]
\end{align}

where the last line follows from the fact that $A \wedge d_H A = 0$ since both forms are horizontal. Putting this all together, we may now express the total action $S(A)$ in terms of horizontal quantities as follows:

$$S(A) = CS(A) - \bar{S}(A) = \int_X A \wedge [\kappa \wedge \mathcal{L}_\xi A] + \int_X A \wedge [\kappa \wedge d_H \ast_H d_H A] = \int_X A \wedge [\kappa \wedge (\mathcal{L}_\xi + d_H \ast_H d_H) A]$$

4. **The contact operator $D$**

A surprising observation is that $\kappa \wedge (\mathcal{L}_\xi + d_H \ast_H d_H) \kappa$ turns out to be well known. It is the second order operator "$D$" that fits into the complex,

$$C^\infty(X) \xrightarrow{d_H} \Omega^1(H) \xrightarrow{D} \Omega^2(V) \xrightarrow{d_H} \Omega^3(X)$$

where,

$$\Omega^\bullet(V) := \{ \kappa \wedge \alpha \mid \alpha \in \Omega^\bullet(H) \} = \kappa \wedge \Omega^\bullet(H)$$
and for \( f \in C^\infty(X) \), \( d_H f \in \Omega^1(H) \) stands for the restriction of \( df \) to \( H \) as usual, while

\[
(4.3) \quad d_H : \Omega^2(V) \to \Omega^3(X)
\]

is just de Rham’s differential restricted to \( \Omega^2(V) \) in \( \Omega^3(X) \). \( D \) is defined as follows: since \( d \) induces an isomorphism

\[
(4.4) \quad d_0 : \Omega^1(V) \to \Omega^2(H), \quad \text{with} \quad d_0(f\kappa) = fd\kappa|_{\Lambda^2(H)}
\]

then any \( \alpha \in \Omega^1(H) \) admits a unique extension \( \ell(\alpha) \) in \( \Omega^1(X) \) such that \( d_\ell(\alpha) \) belongs to \( \Omega^2(V) \); i.e. given any initial extension \( \bar{\alpha} \) of \( \alpha \), one has

\[
(4.5) \quad \ell(\alpha) = \bar{\alpha} - d_0^{-1}(d\bar{\alpha})|_{\Lambda^2(H)}
\]

We then define

\[
(4.6) \quad D\alpha := d\ell(\alpha)
\]

We then have \([\textit{BHR07}; \text{Eq. 39}]\),

\[
(4.7) \quad D\alpha = \kappa \wedge [\mathcal{L}_{\xi} + d_H \ast_H d_H]|\alpha
\]

for any \( \alpha \in \Omega^1(H) \). Thus,

\[
(4.8) \quad S(A) = \int_X A \wedge [\kappa \wedge (\mathcal{L}_{\xi} + d_H \ast_H d_H)A]
\]

\[
(4.9) \quad = \int_X A \wedge DA
\]

\[
(4.10) \quad = \langle A, - \ast DA \rangle
\]

where \( \langle \cdot, \cdot \rangle \) is the usual \( L^2 \) inner product on \( \Omega^1(X) \).

Alternatively, we make the following

**Definition 16.** Let \( D^1 : \Omega^1(H) \to \Omega^1(X) \) denote the operator

\[
(4.11) \quad D^1 := \mathcal{L}_{\xi} + d_H \ast_H d_H
\]

and observe that we can also write \( S(A) = \langle A, - \ast_H D^1 A \rangle_\kappa \), identifying \( S(A) \) with the operator \( - \ast_H D^1 \) on \( \Omega^1(H) \). Thus, we have proven the following

**Proposition 17.** The new action, \( S(A) \), as defined in Eq. \([2.2]\) for the “shifted” partition function of Eq. \([2.1]\) can be expressed as a quadratic form on the space of horizontal forms \( \Omega^1(H) \) as follows:

\[
(4.12) \quad S(A) = \langle A, - \ast DA \rangle
\]

or equivalently as,

\[
(4.13) \quad S(A) = \langle A, - \ast_H D^1 A \rangle_\kappa
\]

where \( D \) and \( D^1 \) are the second order operators defined in Eq.’s \([4.7]\) and \([4.11]\) respectively. \( \langle \cdot, \cdot \rangle \) is the usual \( L^2 \) inner product on \( \Omega^1(X) \), and \( \langle \cdot, \cdot \rangle_\kappa \) is defined in Eq. \([3.7]\).
5. Gauge group and the isotropy subgroup

In order to extract anything mathematically meaningful out of this construction we will need to divide out the action of the gauge group $G_P$ on $A_P$. At this point we observe that the gauge group $G_P \simeq \text{Maps}(X \to U(1))$ naturally descends to a "horizontal" action on $\bar{A}_P$, which infinitesimally can be written as:

$$\theta \in \text{Lie}(G_P) : A \mapsto A + d_H \theta$$

Following [Sch79b], we let $H_A$ denote the isotropy subgroup of $G_P$ at a point $A \in \bar{A}_P$. Note that $H_A$ can be canonically identified for every $A \in \bar{A}_P$, and so we simply write $H$ for the isotropy group. The condition for an element of the gauge group $h(x) = e^{i\theta(x)}$ to be in the isotropy group is that $d_H \theta = 0$, given definition 5.1 above. By [Rum94 : Prop. 12], we see that the condition $d_H \theta = 0$ implies that $\theta$ is harmonic, and so $L_\theta \theta = 0$. Therefore we have $d \theta = 0$ since $d = d_H + \kappa \wedge L_\xi$. Thus, the group $H$ can be identified with the group of constant maps from $X$ into $U(1)$; hence, is isomorphic to $U(1)$. We let $\text{Vol}(H)$ denote the volume of the isotropy subgroup, computed with respect to the metric induced from $G_P$, so that

$$\text{Vol}(H) = \left[ \int_X \kappa \wedge d\kappa \right]^{1/2} = \left[ n + \sum_{j=1}^{N} \frac{\beta_j}{\alpha_j} \right]^{1/2}$$

where $[n; (\alpha_1, \beta_1, \ldots, (\alpha_N, \beta_N)]$ are the Seifert invariants of our Seifert manifold $X$. The last equality in Eq. 5.2 above follows from Eq. 3.22 of [BW05].

6. The partition function

We now have

$$Z_{U(1)}(X, p, k) = \frac{e^{\pi i k S_{X, P}(A_0)}}{\text{Vol}(G_P)} \int_{\bar{A}_P} DA e^{\frac{i k}{4} \kappa S(A)}$$

$$= \frac{\text{Vol}(G_P) e^{\pi i k S_{X, P}(A_0)}}{\text{Vol}(H) \text{Vol}(G_P)} \int_{\bar{A}_P/G_P} e^{\frac{i k}{4} \kappa S(A)} \left[ \text{det}'(d_H^* d_H) \right]^{1/2} \mu$$

$$= \frac{e^{\pi i k S_{X, P}(A_0)}}{\text{Vol}(H)} \int_{\bar{A}_P/G_P} e^{\frac{i k}{4} \kappa S(A)} \left[ \text{det}'(d_H^* d_H) \right]^{1/2} \mu$$

(6.1)

where $\mu$ is the induced measure on the quotient space $\bar{A}_P/G_P$ and $\text{det}'$ denotes a regularized determinant to be defined later. Since $S(A) = \langle A, - *_H D^1 A \rangle$ is quadratic in $A$, we may apply the method of stationary phase ([Sch79a, GS77]) to evaluate the oscillatory integral (6.1) exactly. We obtain,

$$Z_{U(1)}(X, p, k) = \frac{e^{\pi i k S_{X, P}(A_0)}}{\text{Vol}(H)} \int_{\bar{M}_P} e^{\frac{i k}{4} \text{sgn}(-*_H D^1)} \left[ \text{det}'(d_H^* d_H) \right]^{1/2} \left[ \text{det}'(-k *_H D^1) \right]^{1/2} \nu$$

(6.2)

where $\bar{M}_P$ denotes the moduli space of flat connections modulo the gauge group and $\nu$ denotes the induced measure on this space. Note that we have included a factor of $k$ in our regularized determinant since this factor occurs in the exponent multiplying $S(A)$. 

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7. Zeta function determinants

We will use the following to define the regularized determinant of $-k \star_H D^1$

**Proposition 18.** [Sch79b] Let $\mathcal{H}_0, \mathcal{H}_1$ be Hilbert spaces, and $S : \mathcal{H}_1 \to \mathcal{H}_1$ and $T : \mathcal{H}_0 \to \mathcal{H}_1$ such that $S^2$ and $TT^*$ have well defined zeta functions with discrete spectra and meromorphic extensions to $\mathbb{C}$ that are regular at 0 (with at most simple poles on some discrete subset). If $ST = 0$, and $S^2$ is self-adjoint, then

$$\det'(S^2 + TT^*) = \det'(S^2) \det'(TT^*)$$

(7.1)

**Proof.** This equality follows from the facts that $S^2TT^* = 0$ and $TT^*S^2 = 0$ (i.e. these operators commute), which both follow from $ST = 0$ and the fact that $S^2$ and $TT^*$ are both self-adjoint. \qed

Following the notation of Eq.’s (3)-(6) in section 2 of [Sch79b], we set the operators $S = -k \star_H D^1$ and $T = kd_H d_{H}^!$ on $\Omega^1(H)$ and observe that $ST = 0$ since (4.1) is a complex. With Prop. [18] as motivation, we make the formal definition

$$\det'(-k \star_H D^1) := C(k, J) : \frac{[\det'(S^2 + TT^*)]^{1/2}}{[\det'(TT^*)]^{1/2}}$$

(7.2)

where $S^2 + TT^* = k^2((D^1)^* D^1 + (d_H d_{H}!)^2)$, $TT^* = k^2(d_H d_{H}!)^2$ and

$$C(k, J) := k \left( \int_X R^2 \kappa \wedge d\kappa \right)$$

(7.3)

is a function of $R \in C^\infty(X)$, the Tanaka-Webster scalar curvature of $X$, which in turn depends only on a choice of a compatible complex structure $J \in \text{End}(H)$. That is, given a choice of contact form $\kappa \in \Omega^1(X)$, the choice of complex structure $J \in \text{End}(H)$ determines uniquely an associated metric. We have defined $\det'(-k \star_H D^1)$ in this way to eliminate the metric dependence that would otherwise occur in the $k$-dependence of this determinant. The motivation for the definition of the factor $C(k, J)$ comes explicitly from Prop. [20] below.

The operator

$$\Delta := (D^1)^* D^1 + (d_H d_{H}!)^2$$

(7.4)

is actually equal to the middle degree Laplacian defined in Eq. (10) of [RS08] and has some nice analytic properties. In particular, it is maximally hypoelliptic and invertible in the Heisenberg symbolic calculus (See [RS08]; §3.1). We define the regularized determinant of $\Delta$ via its zeta function ([RS08]; Pg. 10)

$$\zeta(\Delta)(s) := \sum_{\lambda \in \text{spec}^+ (\Delta)} \lambda^{-s}$$

(7.5)

Note that our definition agrees with [RS08] up to a constant term $\dim H^1(X, D)$, which is finite by hypoellipticity ([RS08]; Pg. 11). Also, $\zeta(\Delta)(s)$ admits a meromorphic extension to $\mathbb{C}$ that is regular at $s = 0$ ([Pon07]; §4). Thus, we define the regularized determinant of $\Delta$ as

$$\det'(\Delta) := e^{-\zeta'(\Delta)(0)}$$

(7.6)

Let $\Delta_0 := (d_H^* d_H)^2$ on $\Omega^0(X)$, $\Delta_1 := \Delta$ on $\Omega^1(H)$ and define $\zeta_i(s) := \zeta(\Delta_i)(s)$. We claim the following
Proposition 19. For any real number $0 < c \in \mathbb{R}$,
\begin{equation}
(7.7) \quad \det'(c\Delta_i) := e^{\zeta_i(0)} \det'(\Delta_i)
\end{equation}
for $i = 0, 1$.

**Proof.** To prove this claim, recall that $\zeta_i(s) = \zeta(\Delta_i)(s)$ for $i = 0, 1$, scale as follows:
\begin{equation}
(7.8) \quad \zeta(c\Delta_i)(s) = e^{-s}\zeta(\Delta_i)(s).
\end{equation}
From here we simply calculate the scaling of the regularized determinants using the definition
\begin{equation}
(7.9) \quad \det'(\Delta_i) := e^{-\zeta'(\Delta_i)(0)}
\end{equation}
and the claim is proven. \qed

The following will be useful.

Proposition 20. For $\Delta_0 := (d_H^*d_H)^2$ on $\Omega^0(X)$, $\Delta_1 := \Delta$ on $\Omega^1(H)$ defined as above and $\zeta_i(s) := \zeta(\Delta_i)(s)$, we have
\begin{equation}
(7.10) \quad \zeta_0(0) - \zeta_1(0) = \left( -\frac{1}{512} \int_X R^2 \kappa \wedge d\kappa \right) + \dim \text{Ker}\Delta_1 - \dim \text{Ker}\Delta_0
\end{equation}
\begin{equation}
(7.11) \quad = \left( -\frac{1}{512} \int_X R^2 \kappa \wedge d\kappa \right) + \dim H^1(X, d_H) - \dim H^0(X, d_H).
\end{equation}
where $R \in C^\infty(X)$ is the Tanaka-Webster scalar curvature of $X$ and $\kappa \in \Omega^1(X)$ is our chosen contact form as usual.

**Proof.** Let
\begin{align*}
\hat{\zeta}_0(s) & := \dim \text{Ker}\Delta_0 + \zeta_0(s) \\
\hat{\zeta}_1(s) & := \dim \text{Ker}\Delta_1 + \zeta_1(s)
\end{align*}
denote the zeta functions as defined in [RS08]. From ([RS08 ; Cor. 3.8]), one has that
\begin{equation}
\hat{\zeta}_1(0) = 2\hat{\zeta}_0(0)
\end{equation}
for all 3-dimensional contact manifolds. By ([BHR07 ; Theorem 8.8]), one knows that on CR-Seifert manifolds that
\begin{equation}
\hat{\zeta}_0(0) = \hat{\zeta}(\Delta_0)(0) = \hat{\zeta}(\Delta_0^2)(0) = \frac{1}{512} \int_X R^2 \kappa \wedge d\kappa
\end{equation}
Thus,
\begin{equation}
\hat{\zeta}_1(0) = \frac{1}{256} \int_X R^2 \kappa \wedge d\kappa
\end{equation}
By our definition of the zeta functions, which differ from that of [RS08] by constant dimensional terms, we therefore have
\begin{align*}
\zeta_0(0) &= \frac{1}{512} \int_X R^2 \kappa \wedge d\kappa - \dim \text{Ker}\Delta_0 \\
\zeta_1(0) &= \frac{1}{256} \int_X R^2 \kappa \wedge d\kappa - \dim \text{Ker}\Delta_1
\end{align*}
Hence,

\[
\zeta_0(0) - \zeta_1(0) = \left[ \frac{1}{512} \int_X R^2 \kappa \wedge d\kappa - \dim \ker \Delta_0 \right] - \left[ \frac{1}{256} \int_X R^2 \kappa \wedge d\kappa - \dim \ker \Delta_1 \right]
\]

\[
= \left( -\frac{1}{512} \int_X R^2 \kappa \wedge d\kappa \right) + \dim \ker \Delta_1 - \dim \ker \Delta_0
\]

\[
= \left( -\frac{1}{512} \int_X R^2 \kappa \wedge d\kappa \right) + \dim H^1(X, d_H) - \dim H^0(X, d_H).
\]

and the result is proven.

We now have the following

**Proposition 21.** The term inside of the integral of Eq. [6.2] has the following expression in terms of the hypoelliptic Laplacians, \(\Delta_0\) and \(\Delta_1\), as defined in Prop. 20:

\[
(7.12) \quad \frac{[\det'(d_H^* d_H)]^{1/2}}{[\det'(-k \ast H \, D^1)]^{1/2}} = k^{n_X} \frac{[\det'((\Delta_0))]^{1/2}}{[\det'(\Delta_1)]^{1/4}}
\]

where

\[
(7.13) \quad n_X := \frac{1}{2} (\dim H^1(X, d_H) - \dim H^0(X, d_H)).
\]

**Proof.**

\[
(7.14) \quad \frac{[\det'(d_H^* d_H)]^{1/2}}{[\det'(-k \ast H \, D^1)]^{1/2}} = C(k, J)^{-1} \cdot \frac{[\det'(d_H^* d_H)]^{1/4}}{[\det'(k^2 \Delta)]^{1/4}} \cdot \frac{[\det'(d_H^* d_H)]^{1/4}}{[\det'(k^2 \Delta)]^{1/4}}
\]

\[
(7.15) \quad = C(k, J)^{-1} \cdot \frac{k^{\zeta_0(0)/2} [\det'((\Delta_0))]^{1/4}}{[\det'(\Delta_1)]^{1/4}} \cdot \frac{[\det'(\Delta_1))]^{1/4}}{[\det'(\Delta_1)]^{1/4}}
\]

\[
= C(k, J)^{-1} \cdot C(k, J) \cdot k^{n_X} \frac{[\det'(\Delta_0)]^{1/2}}{[\det'(\Delta_1)]^{1/4}}, \text{Prop. 20}
\]

\[
= k^{n_X} \frac{[\det'(\Delta_0)]^{1/2}}{[\det'(\Delta_1)]^{1/4}}
\]

where the second last line comes from Eq. [7.11]. Also note that \(d_H^* d_H\) and \(d_H d_H^*\) have the same eigenvalues (by standard arguments), which allows us to proceed to Eq. [7.15] from Eq. [7.14].

**Remark 22.** Note that by [RS08 : Prop. 2.2], the definition of \(n_X\) (see Eq. 7.13) here is exactly equal to the quantity \(m_X := \frac{1}{2} (\dim H^1(X, d) - \dim H^0(X, d))\) of [Man98 : Eq. 5.18]. This shows that our partition function has the same \(k\)-dependence as that in [Man98].
8. The eta invariant

Next we regularize the signature $sgn(-\ast_H D^1)$ via the eta-invariant and set

$sgn(-\ast_H D^1) = \eta(-\ast_H D^1)(0) =: \eta(-\ast_H D^1)(s)$

where

$$\eta(-\ast_H D^1)(s) := \sum_{\lambda \in \text{spec}^*(\ast_H D^1)} (sgn\lambda)|\lambda|^{-s}$$

Finally, we may now write the result for our partition function

$$Z_{U(1)}(X, p, k) = k^{{n_X}} e^{\frac{i\pi k s}{2}} e^{\frac{i\pi}{4} \eta(-\ast_H D^1)} \int_{\mathcal{M}_P} \frac{1}{Vol(H)} \left[\frac{\text{det}'(\Delta_0)}{\text{det}'(\Delta_1)}\right]^{1/2} \nu$$

where $n_X := \frac{1}{2}(\dim H^1(X, d_H) - \dim H^0(X, d_H))$. Note that $\nu$ is a measure on $\mathcal{M}_P$ (the moduli space of flat connections modulo the gauge group) relative to the horizontal structure on the tangent space of $\mathcal{M}_P$.

9. Torsion

Now we will study the quantity $\nu$ inside of the integral in Eq. 8.2, and in particular how it is related to the analytic contact torsion $T_C$. First, recall that (RS08, Eq. 16)

$$T_C := \exp \left( \frac{1}{4} \sum_{q=0}^{3} (-1)^q w(q) \zeta'(\Delta_q)(0) \right)$$

where

$$w(q) = \begin{cases} q & \text{if } q \leq 1, \\ q + 1 & \text{if } q > 1. \end{cases}$$

in the case where $\dim(X) = 3$. Note that we have chosen a sign convention that leads to the inverse of the definition of $T_C$ in RS08. Recall (RS08, Eq. 10),

$$\Delta_q = \begin{cases} (d_H^*d_H + d_Hd_H^*)^2 & \text{if } q = 0, 3, \\ D^*D + (d_Hd_H^*)^2 & \text{if } q = 1, \\ DD^* + (d_Hd_H^*)^2 & \text{if } q = 2. \end{cases}$$

We would, however, like to work with torsion when viewed as a density on the determinant line

$$|\text{det}H^*(X, d_H)^*| = |\text{det}H^0(X, d_H)| \otimes |\text{det}H^1(X, d_H)^*| \otimes |\text{det}H^2(X, d_H)| \otimes |\text{det}H^3(X, d_H)^*|$$

We follow RS73 and Man98 and make the analogous definition.

**Definition 23.** Define the analytic torsion as a density as follows

$$T_C^D := T_C \cdot \delta_{|\text{det}H^*(X, d_H)|}$$

where $T_C$ is as defined in Eq. 9.1 and

$$\delta_{|\text{det}H^*(X, d_H)|} := \otimes_{q=0}^{\dim X} |\nu_q^2 \wedge \cdots \wedge \nu_q^2|(-1)^q$$
where \( \{ \nu_i^0, \ldots, \nu_i^q \} \) is an orthonormal basis for the space of harmonic contact forms \( H^q(X, dH) \) with the inner product defined in Eq. \( (9.7) \). Note that \( H^q(X, dH) \) is canonically identified with the cohomology space \( H^q(X, dH) \), and \( b_q := \dim(H^q(X, dH)) \) is the \( q \)-th contact Betti number.

Let
\[
\nu^{(q)} := \nu_1^q \land \cdots \land \nu_{b_q}^q
\]
and write the analytic torsion of a compact connected Seifert 3-manifold \( X \) as
\[
(9.4) \quad T^d_C = T_C \times |\nu_1^{(0)}| \otimes |\nu_1^{(1)}|^{-1} \otimes |\nu_2^{(2)}| \otimes |\nu_3^{(3)}|^{-1}.
\]
In terms of regularized determinants, we have
\[
(9.5) \quad T_C = \left[ (\text{det}'(\Delta_0))^0 \cdot (\text{det}'(\Delta_1))^1 \cdot (\text{det}'(\Delta_2))^{-3} \cdot (\text{det}'(\Delta_3))^4 \right]^{1/4}
\]
where \( \Delta_q, \ 0 \leq q \leq 3 \), denotes the Laplacians on the contact complex as defined in ([RS08] : Eq. 10) and recalled in Eq. \( (9.3) \) above. This notation agrees with our notation for \( \Delta_0, \Delta_1 \) as in Eq. \( (7.7) \). The Hodge \( \ast \)-operator induces the equivalences \( \Delta_q \simeq \Delta_{3-q} \) (see [RS08]; Theorem 3.4) and allows us to write
\[
(9.6) \quad T_C = \left[ (\text{det}'(\Delta_0))^0 \cdot (\text{det}'(\Delta_1))^1 \cdot (\text{det}'(\Delta_2))^{-3} \cdot (\text{det}'(\Delta_3))^4 \right]^{1/4}
\]
\[
(9.7) \quad \frac{\text{det}'(\Delta_0)}{(\text{det}'(\Delta_1))^{1/2}}
\]
Also, from the isomorphisms \( H^q(X, \mathbb{R}) \simeq H^q(X, dH) \) of Prop. 2.2 of [RS08], we have Poincaré duality \( H^q(X, dH) \simeq H^{3-q}(X, dH)^\ast \), and therefore
\[
(9.8) \quad T^d_C = T_C \times |\nu_1^{0}|^{\otimes 2} \otimes (|\nu_1^{1}|^{-1})^{\otimes 2}
\]
Moreover, by [Rum94] (Prop. 12), \( H^q(X, dH) = H^q(X, \mathbb{R}) \), and thus any orthonormal basis \( \nu^{(0)} \) of \( H^0(X, dH) \simeq \mathbb{R} \) is a constant such that
\[
(9.9) \quad |\nu_1^{(0)}| = \left[ \int_X \kappa \land d\kappa \right]^{-1/2}
\]
Also, recall that the tangent space \( T_A \mathcal{M}_P \simeq H^1(X, dH) \simeq H^1(X, \mathbb{R}) \), at any point \( A \in \mathcal{M}_P \). The measure \( \nu \) on \( \mathcal{M}_P \) that occurs in Eq. \( (8.2) \) is defined relative to the metric on \( H^1(X, dH) \simeq H^1(X, dH) \), which can be identified with the usual \( L^2 \)-metric on forms. Thus the measure \( \nu \) may be identified with the inverse of the density \( |\nu_1^{(1)}| \) by dualizing the orthogonal basis \( \{ \nu_1^0, \ldots, \nu_b^0 \} \) for \( H^1(X, dH) \); i.e.
\[
(9.10) \quad \nu = |\nu_1^{(1)}|^{-1} = |\nu_1^0 \land \cdots \land \nu_b^0|^{-1}
\]
Putting together Equations \( (8.7) \) \( (9.9) \) \( (9.10) \) into Equation \( (9.8) \) we have
\[
(9.11) \quad T^d_C = T_C \times |\nu_1^{0}|^{\otimes 2} \otimes (|\nu_1^{1}|^{-1})^{\otimes 2}
\]
\[
(9.12) \quad = \frac{\text{det}'(\Delta_0)}{(\text{det}'(\Delta_1))^{1/2}} \cdot \left[ \int_X \kappa \land d\kappa \right]^{-1} \nu^{\otimes 2}
\]
\[
(9.13) \quad = \text{Vol}(H)^{-2} \frac{\text{det}'(\Delta_0)}{(\text{det}'(\Delta_1))^{1/2}} \cdot \nu^{\otimes 2}
\]
We have thus proven the following,
Proposition 24. The contact analytic torsion, when viewed as a density $T_d^C$ as in definition [23] can be identified as follows:

\[(T_d^C)^{1/2} = \frac{1}{\text{Vol}(H)} \left[ \text{det}'(\Delta_0) \right]^{1/2} \nu \]

Our partition function is now

\[(9.15) \overline{Z}_{U(1)}(X, p, k) = k^{n_X} e^{\pi i k S_X, p(A_0)} e^{\frac{2\pi}{12} \eta(- \star H D^1)} \int_{M_P} (T_d^C)^{1/2} \]

This partition function should be completely equivalent to the partition function defined in ([Man98]: Eq. 7.27):

\[(9.16) Z_{U(1)}(X, p, k) = k^{m_X} e^{\pi i k S_X, p(A_0)} e^{\frac{2\pi}{12} \eta(- \star d)} \int_{M_P} (T_{RS}^d)^{1/2}. \]

Our goal in the remainder is to show that this is indeed the case. Our first observation is that $(T_d^C)^{1/2}$ is equal to the Ray-Singer torsion $(T_{RS}^d)^{1/2}$ that occurs in ([Man98]: Eq. 7.27). This follows directly from ([RS09]: Theorem 4.2); note that their sign convention makes $T_C$ the inverse of our definition.

10. Regularizing the eta-invariants

Since we have seen that our $k$-dependence matches that in [Man98] (i.e. $m_X = n_X$; cf. Remark 22), the only thing left to do is to reconcile the eta invariants, $\eta(- \star_H D^1)$ and $\eta(- \star d)$. As observed in [Wit89], the correct quantity to compare our eta invariant to would be

\[(10.1) \quad \frac{\eta(- \star d)}{4} + \frac{1}{2} \frac{\text{CS}(A^g)}{2\pi}, \]

where,

\[(10.2) \quad \text{CS}(A^g) = \frac{1}{4\pi} \int_X Tr(A^g \wedge dA^g + \frac{2}{3} A^g \wedge A^g \wedge A^g) \]

is the gravitational Chern-Simons term, with $A^g$ the Levi-Civita connection on the spin bundle of $X$ for a given metric $g$ on $X$. See Appendix [E] for a short exposition on the regularization of $\eta(- \star d)$ in Eq. (10.1). It was noticed in [Wit89] that in the quasi-classical limit, quantum anomalies can occur that can break topological invariance. Invariance may be restored in this case only after adding a counterterm to the eta invariant. Our job then is to perform a similar analysis for the eta invariant $\eta(- \star_H D^1)$, which depends on a choice of metric. Of course, our choice of metric is natural in this setting and is adapted to the contact structure. One possible approach is to consider variations over the space of such natural metrics and calculate the corresponding variation of the eta invariant, giving us a local formula for the counterterm that needs to be added. Such a program has already been initiated in [BHR07].

Our starting point is the conjectured equivalence that results from the identification of Eq.’s [8.13] and [9.16]

\[(10.3) \quad e^{\pi i \left[ \frac{\eta(- \star d)}{4} + \frac{1}{2} \frac{\text{CS}(A^g)}{2\pi} \right]} \left[ \frac{\eta(- \star_H D^1) + C_T}{4} \right] \]

where $C_T$ is some appropriate counterterm that yields an invariant comparable to the left hand of this equation. As noted in Appendix [E] the left hand side of this
equation depends on a choice of 2-framing on $X$, and since we have a rule (cf. Eq. [B.9]) for how the partition function transforms when the framing is twisted, we basically have a topological invariant. Alternatively, as also noted in Appendix [B] one can use the main result of [Ati90] and fix the canonical 2-framing on $TX \oplus TX$. We therefore expect the same type of phenomenon for the right hand side of this equation, having at most a $\mathbb{Z}$-dependence on the regularization of our eta invariant, along with a rule that tells us how the partition function changes when our discrete invariants are “twisted,” once again yielding a topological invariant.

Let us first make the statement of the conjecture of Eq. [10.3] more precise. We should have the following

**Conjecture 25.** $(X, \phi, \xi, \kappa, g)$ a closed quasi-regular K-contact three-manifold. Then there exists a counterterm, $C_T$, such that

$$e^{\frac{i}{\pi} \left[ \eta(- \star_H D^1) + C_T \right]}$$

is a topological invariant that is identically equal to the topological invariant

$$e^{\frac{i}{\pi} \left[ \frac{1}{4} \star_H - d + \frac{1}{12} \text{CS}(A^\theta) \right]}$$

where $\text{CS}(A^\theta)$ and all relevant operators are defined with respect to the metric $g$ on $X$ and we use the canonical 2-framing [Ati90].

Our regularization procedure for $\eta(- \star_H D^1)$ will be quite different than that used for $\eta(- \star d)$. Since we are restricted to a class of metrics that are compatible with our contact structure, we are really only concerned with finding appropriate counterterms for $\eta(- \star_H D^1)$ that will eliminate our dependence on the choice of contact form $\kappa$ and complex structure $J \in \text{End}(H)$. In the case of interest, we observe that our regularization may be obtained in one stroke by introducing the renormalized $\eta$-invariant, $\eta_0(X, \kappa)$, of $X$ that is discussed in ([BHR07] ; §3). Before giving the definition of $\eta_0(X, \kappa)$, we require the following

**Lemma 26.** ([BHR07] ; Lemma 3.1) Let $(X, J, \kappa)$ be a strictly pseudoconvex pseudohermitian 3-manifold. The $\eta$-invariants of the family of metrics $g_\epsilon := \epsilon^{-1} \kappa \otimes \kappa + d\kappa(\cdot, J \cdot)$ have a decomposition in homogeneous terms:

$$\eta(g_\epsilon) = \sum_{i=-2}^{2} \eta_i(X, \kappa) \epsilon^i.$$

The terms $\eta_i$ for $i \neq 0$ are integrals of local pseudohermitian invariants of $(X, \kappa)$, and the $\eta_i$ for $i > 0$ vanish when the Tanaka-Webster torsion, $\tau$, vanishes.

We then make the following

**Definition 27.** Let $(X, \kappa)$ be a compact strictly pseudoconvex pseudohermitian 3-dimensional manifold. The renormalized $\eta$-invariant $\eta_0(X, \kappa)$ of $(X, \kappa)$ is the constant term in the expansion of Eq. [10.4] for the $\eta$-invariants of the family of metrics $g_\epsilon := \epsilon^{-1} \kappa \otimes \kappa + d\kappa(\cdot, J \cdot)$.

Our assumption that $X$ is K-contact ensures that the Reeb flow preserves the metric. In this situation, it is known that the Tanaka-Webster torsion necessarily
vanishes (cf. [BHR07, §3]). In the case where the torsion of \((X, \kappa)\) vanishes, the terms \(\eta_i(X, \kappa)\) in Eq. (10.4) vanish for \(i > 0\), so that when \(\epsilon \to \infty\), one has
\[
\eta_0(X, \kappa) = \lim_{\epsilon \to \infty} \eta(g_\epsilon) := \eta_{ad}
\]
The limit \(\eta_{ad}\) is known as the adiabatic limit and has been studied in [BC89] and [Dai91], for example. The adiabatic limit is the case where the limit is taken as \(\epsilon\) goes to infinity,
\[
\eta_{ad} := \lim_{\epsilon \to \infty} \eta(g_\epsilon),
\]
while the the renormalized \(\eta\)-invariant, \(\eta_0(X, \kappa)\), is naturally interpreted as the constant term in the asymptotic expansion for \((\eta(g_\epsilon))\) in powers of \(\epsilon\), when \(\epsilon\) goes to 0. This reverse process of taking \(\epsilon\) to 0 is also known as the diabatic limit. When torsion vanishes (i.e. when the Reeb flow preserves the metric), Eq. (10.5) is the statement that the diabatic and adiabatic limits agree. One of the main challenges for our future work will be to extend beyond the case where torsion vanishes. This will naturally involve the study of the diabatic limit. For now, we are restricted to the case of vanishing torsion. In this case, the main result that we will use is the following

**Theorem 28.** ([BHR07, Theorem 1.4]) Let \(X\) be a compact CR-Seifert 3-manifold, with \(U(1)\)-action generated by the Reeb field of an \(U(1)\)-invariant contact form \(\kappa\). If \(R\) is the Tanaka-Webster curvature of \((X, \kappa)\) and \(D^1\) is the middle degree operator of the contact complex (cf. Eq. 4.1 and 4.11), then
\[
\eta_0(X, \kappa) = \eta(- \star_H D^1) + \frac{1}{512} \int_X R^2 \kappa \wedge d\kappa.
\]

Theorem 28 compels us to conjecture that \(C_T = \frac{1}{512} \int_X R^2 \kappa \wedge d\kappa\). Our motivation for this comes from the fact that \(\eta_0(X, \kappa)\) is a topological invariant in our case. We have the following,

**Theorem 29.** ([BHR07, cf. Remark 9.6 and Eq. 27]) If \(X\) is a CR-Seifert manifold, then \(\eta_0(X, \kappa)\) is a topological invariant and
\[
\eta_0(X, \kappa) = 1 + \frac{d}{3} + 4 \sum_{j=1}^{N} s(\alpha_j, \beta_j),
\]
where \(d \in \mathbb{Q}\) is the degree of \(X\) as a compact \(U(1)\)-orbifold bundle and
\[
s(\alpha, \beta) := \frac{1}{4\alpha} \sum_{k=1}^{\alpha-1} \cot \left( \frac{\pi k}{\alpha} \right) \cot \left( \frac{\pi k\beta}{\alpha} \right)
\]
is the classical Rademacher-Dedekind sum, where \([n; (\alpha_1, \beta_1), \ldots, (\alpha_N, \beta_N)]\) (for \((\alpha_i, \beta_i) = 1\) relatively prime) are the Seifert invariants of \(X\).

Thus, we are led to consider the natural topological invariant \(e^{\frac{\pi i}{4} \eta_0(X, \kappa)}\) and how it compares with the topological invariant \(e^{\pi i \left[ \frac{n(-\epsilon_d)}{4} + \frac{CS(A^\#)}{2\pi} \right]}\). We consider the limit
\[
\lim_{\epsilon \to \infty} e^{\pi i \left[ \frac{n(-\epsilon_d)}{4} + \frac{CS(A^\#)}{2\pi} \right]}
\]
where \( g_e = e^{-1} \kappa \otimes \kappa + \text{d} \kappa(\cdot, J\cdot) \) is the natural metric associated to \( \Sigma \). On the one hand, since this is a topological invariant, and is independent of the metric, we must have

\[
\lim_{\epsilon \to \infty} e^{\pi i \left[ \frac{(n(-\star d) - 4)}{4} + \frac{1}{2} \frac{\text{CS}(A^g)}{2\pi} \right]} = e^{\pi i \left[ \frac{(n(-\star d) - 4)}{4} + \frac{1}{2} \frac{\text{CS}(A^g)}{2\pi} \right]},
\]

where we take \( g_1 := g \) so that \( \ast g_1 := \ast \).

On the other hand, since \( \eta(g_e) = \eta(\epsilon \ast d) \) by definition, and we know that its limit exists as \( \epsilon \to \infty \) (in fact \( \eta_0(X, \kappa) = \lim_{\epsilon \to \infty} \eta(g_e) \)), we have

\[
\lim_{\epsilon \to \infty} e^{\pi i \left[ \frac{(n(-\star d) - 4)}{4} + \frac{1}{2} \frac{\text{CS}(A^g)}{2\pi} \right]} = e^{\pi i \left[ \frac{n(X, \kappa)}{4} + \lim_{\epsilon \to \infty} \frac{1}{2} \frac{\text{CS}(A^g)}{2\pi} \right]}.
\]

Thus, we have

\[
e^{\pi i \left[ \frac{n(-\star d) - 4}{4} + \frac{1}{2} \frac{\text{CS}(A^g)}{2\pi} \right]} = e^{\pi i \left[ \frac{n_0(X, \kappa)}{4} + \lim_{\epsilon \to \infty} \frac{1}{2} \frac{\text{CS}(A^g)}{2\pi} \right]}.
\]

We therefore see that if we can understand the limit \( \lim_{\epsilon \to \infty} \frac{1}{2} \frac{\text{CS}(A^g)}{2\pi} \), we will obtain crucial information for our problem. The following has been established using a “Kaluza-Klein” dimensional reduction technique modeled after the paper GLJP03.

**Proposition 30.** ([McL10]) \((X, \phi, \xi, \kappa, g)\) quasi-regular K-contact three-manifold,

\[
U(1) \xrightarrow{\phi} X \xrightarrow{\kappa} \Sigma
\]

Let \( g_e := e^{-1} \kappa \otimes \kappa + \pi \ast h \). After choosing a framing for \( TX \oplus TX \), corresponding to a choice of vielbeins, then,

\[
\text{CS}(A^g) = \left( \frac{e^{-1}}{2} \right) \int_{\Sigma} r \omega + \left( \frac{e^{-2}}{2} \right) \int_{\Sigma} f^2 \omega
\]

where \( r \in C^\infty_{\text{orb}}(\Sigma) \) is the (orbifold) scalar curvature of \((\Sigma, h)\), \( \omega \in \Omega^2_{\text{orb}}(\Sigma) \) is the (orbifold) Hodge form of \((\Sigma, h)\), and \( f := \ast h \omega \). In particular, the adiabatic limit of \( \text{CS}(A^g) \) vanishes:

\[
\lim_{\epsilon \to \infty} \text{CS}(A^g) = 0.
\]

**Proposition 30** combined with Eq. \(10.13\) and Theorem \(28\) gives us the following,

**Proposition 31.** \((X, \phi, \xi, \kappa, g)\) closed, quasi-regular K-contact three-manifold. The counterterm, \( C_T \), such that \( e^{\pi i \left[ \frac{n(\ast D) + 1}{2} + \frac{1}{2} \frac{\text{CS}(A^g)}{2\pi} \right]} \) is a topological invariant that is identically equal to the topological invariant \( e^{\pi i \left[ \frac{n(-\star d) + 4}{4} + \frac{1}{2} \frac{\text{CS}(A^g)}{2\pi} \right]} \) is

\[
C_T = \frac{1}{512} \int_X R^2 \kappa \wedge d\kappa.
\]

Given Proposition \(31\) and Theorem \(29\) we conclude the following as an immediate consequence,

**Proposition 32.** \((X, \phi, \xi, \kappa, g)\) closed, quasi-regular K-contact three-manifold. Then,

\[
\eta(- \ast d) + \frac{1}{3} \frac{\text{CS}(A^g)}{2\pi} = \eta(- \ast D) + \frac{1}{512} \int_X R^2 \kappa \wedge d\kappa
\]

\[
= 1 + \frac{d}{3} + 4 \sum_{j=1}^N s(\alpha_j, \beta_j),
\]
where \( d = c_1(X) = n + \sum_{j=1}^{N} \frac{\beta_j}{\alpha_j} \in \mathbb{Q} \) and
\[
s(\alpha, \beta) := \frac{1}{4\alpha} \sum_{k=1}^{\alpha-1} \cot\left(\frac{\pi k}{\alpha}\right) \cot\left(\frac{\pi k \beta}{\alpha}\right) \in \mathbb{Q}
\]
is the classical Rademacher-Dedekind sum, where \([n; (\alpha_1, \beta_1), \ldots, (\alpha_N, \beta_N)]\) (for \(\gcd(\alpha_j, \beta_j) = 1\)) are the Seifert invariants of \(X\). In particular, we have computed the \(U(1)\)-Chern-Simons partition function as:
\[
Z_{U(1)}(X,p,k) = k^n e^{\pi i k s_X, p(A_0)} e^{\frac{\pi i}{4} (1 + \frac{d}{4} + 4 \sum_{j=1}^{N} s(\alpha_j, \beta_j))} \int_{\mathcal{M}_P} (T^d)_{1/2},
\]
\[
= k^m e^{\pi i k s_X, p(A_0)} e^{\frac{\pi i}{4} (1 + \frac{d}{4} + 4 \sum_{j=1}^{N} s(\alpha_j, \beta_j))} \int_{\mathcal{M}_P} (T^d_{RS})_{1/2}.
\]

**Appendix A. Basic construction of \(U(1)\)-Chern-Simons theory**

Let \(X\) be a closed oriented 3-manifold. For any \(U(1)\)-connection \(A \in \mathcal{A}_P\), \cite{Man98} defined an induced \(SU(2)\)-connection \(\hat{A}\) on an associated principal \(SU(2)\)-bundle \(\hat{P} = P \times_{U(1)} SU(2)\). i.e.
\[
\hat{A}|_{\{p,g\}} = Ad_g - 1 (\rho_* p_1^* A|_p) + p_2^* \theta_g
\]
where \(\rho : U(1) \to SU(2)\) is the diagonal inclusion, \(p_1 : P \times SU(2) \to P\) and \(p_2 : P \times SU(2) \to SU(2)\). Since for any 3-manifold \(X\), \(\hat{P}\) is trivializable, let \(\hat{s} : X \to \hat{P}\) be a global section. The definition we use for the Chern-Simons action is as follows:

**Definition 33.** The Chern-Simons action functional of a \(U(1)\)-connection \(A \in \mathcal{A}_P\) is defined by:
\[
S_{X, p}(A) = \int_X \hat{s}^* \alpha(\hat{A}) \pmod{\mathbb{Z}}
\]
where \(\alpha(\hat{A}) \in \Omega^3(\hat{P}, \mathbb{R})\) is the Chern-Simons form of the induced \(SU(2)\)-connection \(\hat{A} \in \mathcal{A}_{\hat{P}}\),
\[
\alpha(\hat{A}) = Tr(\hat{A} \wedge F_{\hat{A}}) - \frac{1}{6} Tr(\hat{A} \wedge [\hat{A}, \hat{A}])
\]

We then define the partition function for \(U(1)\)-Chern-Simons theory as (as in \cite{Man98, MPR93}):
\[
Z_{U(1)}(X,k) = \sum_{p \in Tors\mathcal{H}^2(X; \mathbb{Z})} Z_{U(1)}(X,p,k)
\]
where,
\[
Z_{U(1)}(X,p,k) = \frac{1}{Vol(G_P)} \int_{\mathcal{A}_P} DA e^{\pi i k s_X, p(A)}
\]
and
\[
S_{X, p}(A) = \int_X \hat{s}^* \alpha(\hat{A})
\]
Then for any principal \(U(1)\)-bundle \(P\) we follow \cite{BW05} and define a new action
\[
S_{X, p}(A, \Phi) := S_{X, p}(A - \kappa \Phi)
\]
where we may view $\Phi \in \Omega^0(X)$ and,

\begin{align}
S_{X,P}(A, \Phi) &= \int_X \alpha(\hat{A} - \kappa \Phi) \\
&= \int_X \alpha(\hat{\dot{A}} - \kappa \hat{\dot{\Phi}}) \\
&= S_{X,P}(A) - \int_X [2\kappa \wedge \text{Tr}(\hat{\Phi} \wedge F_{\hat{A}}) - \kappa \wedge d\kappa \text{Tr}(\hat{\Phi}^2)]
\end{align}

where the second equality follows from the definition of $\hat{A}$ and $\hat{\Phi}$ (where $\hat{\Phi}|_{P,g} = \text{Ad}_{g^{-1}}(\rho^*\Phi|_g)$) on $\hat{P} = P \times U(1) SU(2)$. The third equality follows from Eq. 3.6 of [BW05]. We then define a new partition function

\begin{equation}
\bar{Z}_{U(1)}(X,p,k) := \frac{1}{\text{Vol}(S)} \frac{1}{\text{Vol}(G_P)} \int_{A(P)} DA D\Phi \ e^{i k S_{X,P}(A, \Phi)}
\end{equation}

where $D\Phi$ is defined by the invariant, positive definite quadratic form,

\begin{equation}
(\Phi, \Phi) = -\frac{1}{4\pi^2} \int_X \Phi^2 \kappa \wedge d\kappa
\end{equation}

As observed in [BW05], our new partition function is identically equal to our original partition function defined for $U(1)$-Chern-Simons theory. On the one hand, we can fix $\Phi = 0$ above using the shift symmetry, $\delta \Phi = \sigma$, which will cancel the pre-factor $\text{Vol}(S)$ from the resulting group integral over $S$ and yield exactly our original partition function:

\begin{equation}
Z_{U(1)}(X,p,k) = \frac{1}{\text{Vol}(G_P)} \int_{A(P)} DA \ e^{i k S_{X,P}(A)}
\end{equation}

Thus, we obtain the heuristic result,

\begin{equation}
\bar{Z}_{U(1)}(X,p,k) = Z_{U(1)}(X,p,k).
\end{equation}

On the other hand, we obtain another description of $\bar{Z}_{U(1)}(X,p,k)$ by integrating $\Phi$ out. We will briefly review this computation here. Our starting point is the formula for the shifted partition function

\begin{equation}
\bar{Z}_{U(1)}(X,p,k) := \frac{1}{\text{Vol}(S)} \frac{1}{\text{Vol}(G_P)} \int_{A(P)} DA D\Phi \ e^{i k S_{X,P}(A, \Phi)}
\end{equation}

where

\begin{equation}
S_{X,P}(A, \Phi) = S_{X,P}(A) - \int_X [2\kappa \wedge \text{Tr}(\hat{\Phi} \wedge F_{\hat{A}}) - \kappa \wedge d\kappa \text{Tr}(\hat{\Phi}^2)]
\end{equation}

We formally complete the square with respect to $\Phi$ as follows:

\begin{align}
\int_X [\kappa \wedge d\kappa \text{Tr}(\hat{\Phi}^2) - 2\kappa \wedge \text{Tr}(\hat{\Phi} \wedge F_{\hat{A}})] &= \int_X \left[\text{Tr}(\hat{\Phi}^2) - \frac{2\kappa \wedge \text{Tr}(\hat{\Phi} \wedge F_{\hat{A}})}{\kappa \wedge d\kappa}\right] \kappa \wedge d\kappa \\
&= \int_X \text{Tr} \left(\hat{\Phi}^2 - \frac{2\kappa \wedge F_{\hat{A}}}{\kappa \wedge d\kappa} \hat{\Phi}\right) \kappa \wedge d\kappa \\
&= \int_X \text{Tr} \left(\hat{\Phi} - \frac{\kappa \wedge F_{\hat{A}}}{\kappa \wedge d\kappa}\right)^2 - \left[\frac{\kappa \wedge F_{\hat{A}}}{\kappa \wedge d\kappa}\right]^2 \kappa \wedge d\kappa
\end{align}
We then only need to compute the Gaussian
\[
\int D\Phi \exp \left[ \pi i k \int_X \text{Tr} \left( \Phi - \frac{\kappa \wedge F_A}{\kappa \wedge d\kappa} \right)^2 \kappa \wedge d\kappa \right]
\]
\[
= \int D\Phi \exp \left[ \pi i k \int_X \text{Tr} (\Phi^2) \kappa \wedge d\kappa \right]
\]
\[
= \int D\Phi \exp \left[ \frac{ik}{4\pi} \int_X \Phi^2 \kappa \wedge d\kappa \right]
\]
\[
= \int D\Phi \exp \left[ \frac{1}{2} (\Phi, A\Phi) \right]
\]
where we take \( A = 2\pi ik I \) acting on the space of fields \( \Phi \) and the inner product \((\Phi, \Phi)\) is defined as in Eq. A.11. We then formally get
\begin{align}
\int D\Phi \exp \left[ -\frac{1}{2} (\Phi, A\Phi) \right] &= \sqrt{\frac{(2\pi)^{\Delta G}}{\text{det} A}} \\
&= \left( -\frac{i}{k} \right)^{\Delta G/2}
\end{align}

where the quantity \( \Delta G \) is formally the dimension of the gauge group \( G \). Note that we have abused notation slightly throughout by writing \( \frac{1}{\kappa \wedge d\kappa} \). We have done this with the understanding that since \( \kappa \wedge F_A \neq 0 \wedge d\kappa \), then \( \kappa \wedge F_A = \phi \kappa \wedge d\kappa \) for some function \( \phi \in 2\pi i \Omega^0(X) \), and we identify \( \kappa \wedge F_A \kappa \wedge d\kappa := \phi \).

Our new description of the partition function is now,
\begin{align}
\tilde{Z}_{U(1)}(X, p, k) &= C \int_{A_P} DA \exp \left[ \pi i k \left( S_{X, P}(A) - \int_X \text{Tr} \left[ (\kappa \wedge F_A)^2 \right] \kappa \wedge d\kappa \right) \right] \\
&= C \int_{A_P} DA \exp \left[ \frac{ik}{4\pi} \left( \int_X A \wedge dA - \int_X \frac{(\kappa \wedge dA)^2}{\kappa \wedge d\kappa} \right) \right]
\end{align}

where
\[ C_1 = \frac{e^{\pi i k S_{X, P}(A)}}{Vol(S) Vol(G_P)} \left( -\frac{i}{k} \right)^{\Delta G/2} \]

We may further simplify Eq. A.18 by reducing \( A_P \) to its quotient under the shift symmetry \( A_P := A_P / S \), effectively canceling the factor of \( Vol(S) \) out front of the integral. We obtain:
\begin{align}
\tilde{Z}_{U(1)}(X, p, k) &= C_2 \int_{\tilde{A}_P} D\tilde{A} \exp \left[ \frac{ik}{4\pi} \left( \int_X A \wedge dA - \int_X \frac{(\kappa \wedge dA)^2}{\kappa \wedge d\kappa} \right) \right]
\end{align}

where \( C_2 = C_1 Vol(S) \).

Note that we are justified in excluding the factor \( \left( -\frac{i}{k} \right)^{\Delta G/2} \) from Eq. 2.1 since this factor would cancel in the stationary phase approximation in any case.
Appendix B. Framing dependence and the gravitational Chern-Simons term

As observed in ([Man98; Eq. 7.17], Eq. 1.6) can also be rigorously defined by setting

\begin{equation}
Z_{U(1)}(X, p, k) = e^{\pi i k S_X, p(A_P)} \frac{\text{Vol} U(1)}{\nu} \int_{\mathcal{M}_P} e^{\frac{\nu}{2} \text{sgn}(- \ast d)} \frac{[\det'(d^* d)]^{1/2}}{[\det'(-k \ast d)]^{1/2}}
\end{equation}

where \( \nu \) is the metric induced on the moduli space of flat connections on \( P, \mathcal{M}_P \). This last expression has rigorous mathematical meaning if the determinants and signatures of the operators are regularized. The signature of the operator \(- \ast d\) on \( \Omega^1(X; \mathbb{R})\) is regularized via the eta invariant, so that \( \text{sgn}(- \ast d) = \eta(- \ast d) + \frac{1}{24} \text{CS}(A^g) \), where

\begin{equation}
\eta(- \ast d) = \lim_{s \to 0} \sum_{\lambda_j \neq 0} \text{sign} \lambda_j |\lambda_j|^{-s}
\end{equation}

and \( \lambda_j \) are the eigenvalues of \(- \ast d\), and

\begin{equation}
\text{CS}(A^g) = \frac{1}{4\pi} \int_X \text{Tr}(A^g \wedge dA^g + \frac{2}{3} A^g \wedge A^g \wedge A^g)
\end{equation}

is the gravitational Chern-Simons term, with \( A^g \) the Levi-Civita connection on the spin bundle of \( X \). The determinants are regularized as in Remark 7.6 of [Man98]. It is straightforward to see that the the term inside of the integral

\begin{equation}
\frac{1}{\text{Vol} U(1)} \frac{[\det'(d^* d)]^{1/2}}{[\det'(-k \ast d)]^{1/2}}
\end{equation}

may be identified with the Reidemeister torsion of the 3-manifold \( X \), \( T^d_{RS} \) (cf. [Man98; Eq. 7.22]). We obtain,

\begin{equation}
Z_{U(1)}(X, p, k) = k^{m_X} e^{\pi i k S_X, p(A_P)} e^{\pi i \left( \frac{d^* d}{4} + \frac{\text{CS}(A^g)}{2\pi} \right)} \int_{\mathcal{M}_P} (T^d_{RS})^{1/2}
\end{equation}

where \( m_X = \frac{1}{2} (\dim H^1(X; \mathbb{R}) - \dim H^0(X; \mathbb{R})) \). The Atiyah-Patodi-Singer theorem says that the combination

\begin{equation}
\frac{\eta(- \ast d)}{4} + \frac{1}{12} \frac{\text{CS}(A^g)}{2\pi}
\end{equation}

is a topological invariant depending only on a 2-framing of \( X \). Recall ([Ati90]) that a 2-framing is choice of a homotopy equivalence class \( \pi \) of trivializations of \( TX \oplus TX \), twice the tangent bundle of \( X \) viewed as a Spin(6) bundle. The possible 2-framings correspond to \( \mathbb{Z} \). The identification with \( \mathbb{Z} \) is given by the signature defect defined by

\begin{equation}
\delta(X, \pi) = \text{sign}(M) - \frac{1}{6} p_1(2TX, \pi)
\end{equation}

where \( M \) is a 4-manifold with boundary \( X \) and \( p_1(2TX, \pi) \) is the relative Pontryagin number associated to the framing \( \pi \) of the bundle \( TX \oplus TX \). The canonical 2-framing \( \pi^c \) corresponds to \( \delta(X, \pi^c) = 0 \). Either we can choose the canonical framing, and work with this throughout, or we can observe that if the framing of \( X \) is twisted by \( s \) units, then \( \text{CS}(A^g) \) transforms by

\begin{equation}
\text{CS}(A^g) \to \text{CS}(A^g) + 2\pi s
\end{equation}
and so the partition function \( Z_{U(1)}(X, k) \) is transformed by

\[
Z_{U(1)}(X, k) \rightarrow Z_{U(1)}(X, k) \cdot \exp \left( \frac{2\pi i s}{24} \right)
\]

Then \( Z_{U(1)}(X, k) \) is a topological invariant of framed, oriented 3-manifolds, with a transformation law under change of framing. This is tantamount to a topological invariant of oriented 3-manifolds without a choice of framing.

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ETA-INVARINTS AND ANOMALIES IN U(1)-CHERN-SIMONS THEORY

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