Relativistic non-equilibrium thermodynamics revisited

L.S. García-Colín\textsuperscript{a,b} and A. Sandoval-Villalbazo\textsuperscript{c}
\textsuperscript{a} Departamento de Física, Universidad Autónoma Metropolitana
México D.F., 09340 México
\textsuperscript{b} El Colegio Nacional, Centro Histórico 06020 México D.F., México
E-Mail: lgcs@xanum.uam.mx
\textsuperscript{c} Departamento de Física y Matemáticas, Universidad Iberoamericana
Lomas de Santa Fe 01210 México D.F., México
E-Mail: alfredo.sandoval@uiam.mx

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Abstract

Relativistic irreversible thermodynamics is reformulated following the conventional approach proposed by Meixner in the non-relativistic case. Clear separation between mechanical and non-mechanical energy fluxes is made. The resulting equations for the entropy production and the local internal energy have the same structure as the non-relativistic ones. Assuming linear constitutive laws, it is shown that consistency is obtained both with the laws of thermodynamics and causality.

1 Introduction

Classical or non-equilibrium thermodynamics (LNT) as formulated originally by J. Meixner\textsuperscript{1,2} and given its final and more accessible version in the classical monograph by S.R. de Groot and P. Mazur\textsuperscript{3} first published in 1961 is, at present, the most complete and self consistent framework available to deal with non-equilibrium phenomena. It is firmly rooted in both statistical mechanics\textsuperscript{4} and in the kinetic theory of gases based on the Boltzmann equation\textsuperscript{2,3,5}. It is also well known that the theory has its limitations, specially when the so-called constitutive equations fail to describe correctly phenomena which occur in the presence of large gradients or when memory effects are non-negligible. The latter
issue is well known from the work of Kohlrausch and Weber in the first half of the nineteenth century, dealing with responses of systems like glass fibers and similar systems [7]. These observations are readily available today in rheological polymeric fluids, all types of glasses forming liquids in the neighborhood of the glass transition temperature, and others. The extension of LNT to deal with such phenomena has given rise to what is generically known as Extended Nonequilibrium Thermodynamics (ENT). Of about the so far seven different versions of this theory [8] [9] [10], none of them is completely convincing.

The main object of this paper is addressed to a natural question namely the relativistic version of LNT. Some authors have jumped to conclusions as to why something like the relativistic version of one of the existing theories belonging to ENT has to be used, specially in some astrophysical and cosmological phenomena without really understanding clearly the tenets of the former one. Although the reason for this attitude is easy to understand, what is astonishing is that the relativistic version of Meixner’s original theory has been only attempted in one occasion before [11] and, unfortunately, in a very clumsy representation. This is why in this paper we wish to go back to this question with the idea of convincing the reader that relativistic LNT is far more powerful than what has been hinted in previous works and that it may be regarded as consistent, both with the tenets of the theory of relativity and the first two laws of thermodynamics.

To clarify ideas let’s go back to their origin. In 1940 C. Eckart published three papers entitled *The thermodynamics of irreversible processes* [12], the first two dealing, in much the same fashion done by Meixner, with a simple fluid, fluid mixtures and the third one addressing the problem of a relativistic simple fluid. This happened just a few years before Meixner published his own full version of LNT which was first communicated in English by de Groot in 1952 [2] and finally brought to its present version twenty years later by de Groot and Mazur [3]. The point is that in his third paper, Eckart proceeded in the standard way except for the fact that he faced the problem of relating the first law of thermodynamics with the standard energy-momentum tensor of relativity, and, in his own words, could not find a heat density to combine it with the three vector associated with the flow of heat to form a four-vector. This is precisely the root of one of the difficulties found today with the papers dealing with this subject. In one way or another, several authors have followed Eckart, who proposed to construct a tensor which would include both the internal energy and the flow of heat. As had been pointed out already by other authors, this is against the tenets of the general theory of relativity, the stress energy tensor includes only all forms of mechanical energy [13] [14]. Heat cannot be incorporated into its structure. It is surprising that most of the papers written today on this questions keep at all cost this point of view. Since Eckart’s theory leads to results that violate causality and involves undesirable unstable modes, it has been patched up in several ways using the ideas introduced by Israel and coworkers [15] [16] [17] and even resorting to some of the versions of EIT [18] [19].

There is one last issue of upmost importance to deal with, what authors in this field refer to as the “order” in the theories. The most common statement is to disqualify a theory if the entropy current contains only terms of first order
in the deviations from equilibrium. This is kind of meaningless. First of all one has to remember that the entropy balance equation is solely and uniquely a consequence of the local equilibrium hypothesis and the balance equations. The former one is by no means whatsoever a well established criteria to determine whether or not a system exhibits behavior which is close or far from equilibrium. Needless to mention here examples, but just to clarify matters we can recall the Burnett equations of hydrodynamics which as shown recently are the best option describing the structure of shock waves at high Mach numbers and are perfectly consistent with the local equilibrium assumption. It is the generalization of the linear (in the gradients) constitutive equations which account for this improvement. The entropy current in completely invariant to this feature and is still given its standard form:

$$\vec{J}_S = \frac{\vec{J}_Q}{T} + \rho s \vec{u}$$

(1)

where $\vec{J}_Q$ is the heat flow vector, $T$ and $\vec{u}$ the local values of the temperature and velocity, respectively, and $\rho$ the local entropy per unit of volume. Clearly, since now $\vec{J}_Q$ obeys a constitutive equation much more complicated than that arising from Fourier’s equation, the explicit form for $\vec{J}_S$ is such that it contains terms of order higher than the first in the gradients, and issue ignored by most authors. Thus, the question of ”high order deviations from equilibrium” is completely foreign to the generic form of $\vec{J}_S$.

All the features pointed out above are the topic of this paper. Indeed, if one follows the canonical rules behind the standard theory of LNT using the basic principles of the theory of relativity one obtains a set of equations of motion for the chosen local state variables whose order on the gradients arise because of the constitutive equations, which we underline, are foreign to the theory and that one chooses to relate the fluxes with the forces. The linear laws give rise the Navier-Stokes-Fourier equations of hydrodynamics in the case of a simple fluid which, by the way, are already non-linear in the gradients. Higher order in the gradients may be studied by using what we could call the general Burnett constitutive equations leading to what some authors refer to as ”second order in the deviations from equilibrium”, but a full discussion of these equations will be undertaken later. Lastly, we also bring to the fore another ignored feature inherent to a relativistic thermodynamic theory, namely, necessary and sufficient conditions that the linear constitutive equations must fulfill so that the theory obeys causality and consistency with the second law of thermodynamics.

To accomplish our task we have structured the paper as follows. In section two the basic assumptions behind the relativistic version of classical non-equilibrium thermodynamics is discussed in a very simple representation. In section 3 we derive the hydrodynamic or transport heat equation for a simple shear free fluid. Shearing stresses may be trivially included but we ignore them for pedagogical reasons. It is at this stage where consistency with causality and the second law enter into the formalism. A discussion on the results and some concluding remarks are left to section 4.
2 Relativistic linear non-equilibrium thermodynamics

As we mentioned in the introduction, our first task in this paper is to use
the structure of the conservation equations together with the local equilibrium
assumption to derive an equation for the entropy balance using a representation
in which the independent local variables are taken to be the number of particles
per unit of volume \( n(x^i, t) \), the four velocity \( u^\nu (\nu = 1, ..., 4) \) and the internal
energy per particle \( \varepsilon(x^i, t) \). The calculation will be done within the framework of
the general theory of relativity assuming that the fluid is isotropic and free from
shearing stresses, that is shear viscosity is neglected. Therefore, the mass-stress
tensor has the form:

\[
T_{\nu}^{\mu} = \rho u^{\mu} u_{\nu} + \tilde{p} h_{\nu}^{\mu}
\]

where \( \rho \) is the mass density and \( \tilde{p} = p + p' \), \( p \) being the local hydrostatic pressure
and \( p' \) the diagonal part of the stress tensor, namely the one responsible for
bulk stresses. Also, \( h_{\mu}^{\nu} \) by many authors referred to as the spatial projector for
reasons to be discussed later, is defined as

\[
h_{\mu}^{\nu} = \delta_{\mu}^{\nu} - \frac{1}{c^2} u_{\mu} u_{\nu}
\]

so that \( u^{\mu} u_{\mu} = c^2 \), \( u_{\mu} h_{\nu}^{\mu} = 0 \) and \( u^{\nu} h_{\nu}^{\mu} = -\theta \), where \( \theta \equiv u_{\nu}^{\mu} \).

Since the basic conservation equation reads as

\[
T_{\nu;\mu}^{\mu} = 0
\]

straightforward algebra leads to the result that

\[
(\rho - \frac{\tilde{p}}{c^2}) \dot{u}_{\nu} + u_{\nu}(\dot{\rho} + p \theta - \frac{p}{c^2} \theta) = -\dot{\tilde{p}}_{\nu} h_{\nu}^{\mu}
\]

where \( \dot{u}_{\nu} = u^{\mu} u_{\nu;\mu} \). Multiplication of both sides of Eq. (5) with \( u^{\nu} \) yields the
mechanical energy balance equation namely,

\[
\dot{\rho} c^2 + \rho c^2 \theta - \dot{\tilde{p}} \theta = 0
\]

On the other hand, if we assume that the number of particles is conserved and
defining the particle flux as

\[
N^{\mu} = n u^{\mu}
\]

such conservation requirement is met by the condition that

\[
N^{\mu}_{;\mu} = \dot{n} + n \theta = 0
\]

Eqs. (6) and (8) will be particularly useful in the construction of balance equa-
tions for internal energy per particle \( \varepsilon \) as well as for the entropy per unit of
volume \( ns \). To do so we recall that in Meixner’s formulation of classical irre-
versible thermodynamics [1]-[2] one assumes that the local total energy density
is conserved. This implies that the total energy flux contains both the flux of mechanical energy plus the non-mechanical one, namely the heat flux. This statement is equivalent to assuming the validity of the first law of thermodynamics \[3\]. Thus, we write such flux as:

\[ J_\mu = u^\nu T_\nu + n\varepsilon u^\mu + J^\mu_{[Q]} \]  

where clearly, \( u^\nu T_\nu = \rho u^\mu c^2 \) is the mechanical energy flux, \( n\varepsilon u^\mu \) is the internal energy flux, and \( J^\mu_{[Q]} \), a four vector, is the heat flux.

Total energy conservation now requires that \( J_\mu_{[T];\mu} = 0 \), whence from Eqs. \[6\], \[8\] and \[9\], it follows that

\[ n\dot{\varepsilon} = -\dot{\rho} - J^\mu_{[Q];\mu} \]  

is the sought result for the internal energy balance equation. Eqs \[9\] and \[10\] require some additional remarks since they are not always acknowledged nor understood.

The controversy regarding the correct method to deal with relativistic properties of a simple fluid originated from the pioneering paper on the subject written by C. Eckart in 1940 mentioned before \[12\]. The main issue is that in this paper the stress-energy tensor was constructed allowing the inclusion of heat flow and internal energy. This seems to be against the tenets of general relativity, as emphasized in the introduction. The energy-momentum tensor is the most general expression involving mechanical energy and, as emphasized in Tolman’s book \[13\], contains no room for the first law of thermodynamics. In fact, Eckart’s version of Eq. \[10\] contains a term of the form \( \frac{1}{c^2} q^\alpha u_\alpha \), \( q^\alpha \) representing the heat flow. This term, difficult to interpret, is identified with a heat flow of accelerated matter and is foreign to the structure of classical irreversible thermodynamics as we shall see below.

Continuing with our argument, we now derive the entropy balance equation using the local equilibrium assumption \[2\] namely, the entropy per particle is a time independent functional of \( n \) and \( \varepsilon \),

\[ s = s(n, \varepsilon) \]  

so that the Gibbs relation reads

\[ n\dot{s} = \frac{n}{T}\dot{\varepsilon} - \frac{p}{nT}\dot{n} \]  

where the differential coefficients \( (\frac{\partial s}{\partial n})_\varepsilon \) and \( (\frac{\partial s}{\partial \varepsilon})_n \) have been evaluated using once more the local equilibrium assumption. Direct substitution of Eqs. \[10\] and \[12\] into Eq. \[13\] yields immediately that

\[ n\dot{s} + (\frac{J^\mu_{[Q]}}{T});\mu = -\frac{J^\mu_{[Q]}T_\mu}{T^2} - \frac{\rho'\theta}{T} \]  

after a slight rearrangement of the resulting equation. Here \( T \) is the local equilibrium temperature. The reader has to recognize that this is a very encouraging
result. The entropy balance equation in general relativity using the \( n, \varepsilon \) representation is identical in structure to the canonical classical equation. The entropy flow is simply the divergence of the ratio \( \frac{J_{\mu}^\nu}{T} \) and the entropy production, here represented by \( \sigma \) is given by

\[
\sigma = -\frac{J_{[\mu}^\nu T_{\nu]}^\mu}{T^2} - \frac{p' \theta}{T} \tag{14}
\]

Eq. (14) clearly expresses the full meaning of Clausius uncompensated heat, it is the sum of the products between the "macroscopic" gradients of the intensive variables \( T \) and \( u^\alpha \) and the corresponding flows they generate. This result, together with Eq. (10) are the main results of the first part of this paper. Once more, these results differ from those obtained by Eckart not only in the methodology used by this author, but by the appearance of a term \( \frac{1}{T} q^{\alpha} u_\alpha \), which may hardly be interpreted as a product of a thermodynamic force and its corresponding flow. It is also worth noting that Eckart’s formalism cannot be naturally reconciled with the Onsager’s symmetry relations, while the canonical classical equation shows this desirable feature.

What we have therefore accomplished in this section may be safely considered as being the natural extension of classical irreversible thermodynamics to the context of the general theory of relativity. In many ways the results are an improvement over these obtained by the same authors about ten years ago [11] using the mass density \( \rho \) as an independent variable instead of \( n \). The interested reader may look at the analogous of Eqs. (12) and (14) as written in those papers.

We now go on to the next important question raised in the introduction namely the structure and meaning of the transport equations which arise from these theory. To fix our attention we shall concentrate on the heat transport equation leaving the momentum transport equation for future work.

### 3 The heat transport equation

Transport equations arise when the unknown quantities namely, the fluxes appearing in the conservation equations are expressed in terms of the independent variables \( n, u^\alpha \) and \( \varepsilon \) in our case. This is achieved through the so-called constitutive equations which, as is well known, are foreign to the theory. They must be extracted either from experiment or from a microscopic model, this latter possibility being rather difficult to achieve except for some very simple systems.

In our equations, the unknowns are \( p' \), the bulk momentum flow, and \( J_{\mu}^\nu \) the heat flux vector. Also, due to practical reasons, it is useful to eliminate \( \varepsilon \) in terms of the local temperature, a much more accessible variable, and \( n \). This is allowed by the local equilibrium hypothesis since one may always write that \( \varepsilon = \varepsilon(n, T) \), so that

\[
n\dot{\varepsilon} = n(\frac{\partial \varepsilon}{\partial n})_T \dot{n} + n(\frac{\partial \varepsilon}{\partial T})_n \dot{T} \tag{15}
\]
the thermodynamical coefficients in Eq. (15) are given by:

\[
\frac{\partial \epsilon}{\partial n}_T = -\frac{T \beta}{n^2 \kappa_T} + \frac{p}{n^2}
\]  

(16)

and

\[
\frac{\partial \epsilon}{\partial T}_n = C_n
\]  

(17)

In Eqs. (16) and (17) \( \beta \) is the thermal expansion coefficient, \( \kappa_T \) is the isothermal compressibility and \( C_n \) is the specific heat at constant numerical density. Eq. (15) transforms into

\[
\frac{\partial \epsilon}{\partial T}_n = C_n \dot{T} - \frac{p}{n} \dot{n}
\]  

(18)

which, when combined with Eqs. (8) (10) leads to an equation for \( \dot{T} \)

\[
-\dot{\rho} \theta - J_{\mu}^{\mu} = \left( \frac{T \beta}{\kappa_T} + p \right) \theta + C_n \dot{T}
\]  

(19)

where now \( p' \) and \( J_{\mu}^{\mu} \) have to be expressed in terms of \( u^\nu, T \) and \( n \) through constitutive equations. This brings us to a rather delicate question in all this formalism. In fact, if we were to proceed according to the postulates of Meixner’s theory there should be a coupling, linear, between fluxes and forces. One way to write these relations is

\[
p' = -\eta_B \theta
\]  

(20)

where \( \eta_B \) is the bulk viscosity coefficient and

\[
J_{[Q]}^{\mu} = -K g^{\mu \alpha} T_{\alpha}
\]  

(21)

where \( K \) is the heat conductivity. The reader may wonder about the time component of Eq. (21), since the absence of a projector operator implies a kind of energy density associated with the heat flux. It is interesting to notice that, indeed, even in non-relativistic irreversible thermodynamics a non-vanishing density energy associated to heat is needed to recover the first law of thermodynamics from the internal energy balance equation. This issue is discussed in Ref. [3]. Substitution of Eqs. (20) and (21) into Eq. (19) leads to the equation

\[
\eta_B \theta^2 + (K g^{\mu \alpha} T_{\alpha})_{;\mu} = \left( \frac{T \beta}{\kappa_T} + p \right) \theta + nC_n \dot{T}
\]  

(22)

Since \( \theta^2 \) is a quadratic form in \( u_\nu u^\nu \) and \( T \beta \) is usually a small number for ordinary gases, neglecting these last two terms we get that

\[
(K g^{\mu \alpha} T_{\alpha})_{;\mu} = nC_n \dot{T}
\]  

(23)

which is a hyperbolic type equation for \( T \). Causality is not violated and no temperature perturbations can propagate with a velocity larger than \( c \). Moreover, these constitutive equations lead to an entropy production.
\[
\sigma = \frac{kT^\mu T_\mu}{T^2} + \frac{\eta_0 T^2}{T}
\]  
which is a non-negative quadratic form since the transport coefficients are known to be positive. Therefore, \( \sigma > 0 \), in complete agreement with the second law of thermodynamics. There is, however, one problem. For isotropic homogeneous systems often encountered in cosmological applications, the spatial gradients vanish and Eq. (21) would imply that

\[
J^4_{[Q]} = -K \frac{\partial T}{c^2 \partial t}
\]  
This implies that the fourth component of the heat flow four-vector seems to "dissipate in time". The coefficient \( \frac{\Delta}{c^2} \) will in general be a very small number, but not zero.

Many authors do not accept this argument and keep the constitutive equations strictly spatially projected. For this purpose, it is proposed that Eq. (21) should read as:

\[
J^\mu_{[Q]} = -K h^{\mu\alpha} T_{,\alpha}
\]  
so, when \( h^{4\alpha} T_{,\alpha} \) is computed in the co-moving system it is trivial to see that its value is zero so that indeed Eq. (24) reduces to the ordinary Fourier’s equation. However, the second order in time derivative disappears and the counterpart of Eq. (23) becomes parabolic.

Thus, we have two alternatives. One is to accept some kind of dissipation along the time axis in the four dimensional space. This guarantees causality and consistency with the second law [24]. The other one is to project out from the constitutive equations the temporal components of the fluxes and reduce them to their classical expressions, losing causality and direct consistency with the second law. These features of both formulations are perhaps the main reason as to why relativistic non-equilibrium thermodynamics has been somewhat ignored.

4 Concluding remarks

As we have clearly shown in the previous sections, a relativistic generalization of classical non-equilibrium thermodynamics can be achieved without using arguments which are foreign to the Meixner scheme. The entropy balance equation arises solely from the conservation equations and the local equilibrium assumption. The introduction of linear constitutive equations yield a positive entropy production in agreement with the second law. Generalization of these constitutive equations, which are foreign to the theory, may be done consistently with the local equilibrium assumption, but the local positive definiteness of \( \sigma \) is lost. This already occurs in the non-relativistic limit, as has been extensively discussed in the literature that the Burnett and higher order corrections do not
yield a local positive entropy production, but only a global one. This is the main difference between our approach and the ones followed by Israel and collaborators and more recently by Pavon, Zimdahl and others [17-19]. They solve the problem using ideas of ENT incorporating fluxes themselves as state variables. Hence, the entropy of the system cannot be defined [7] [9] as well as the second law. So, the quadratic terms in the entropy flux proposed one way or the other by these authors, postulating Maxwell-Cattaneo type equations as constitutive relations leads to a theory in which neither Clausius entropy nor the second law in its conventional form have a place on their own.

The other remark comes from the results obtained some time ago by Hiscock and Lindblom [20] and fully availed by Israel [23] asserting that the relativistic version of the Navier-Stokes-Fourier hydrodynamics predict instabilities. It may be that this is not necessarily the case, since the linearized version of the transport equations used in those works introduce the heat flow in the stress-energy tensor. The well known Rayleigh-Brillouin (RB) spectrum, corresponding to damped perturbations can be calculated from the linearized equations and verified by experiment. It will be part of future work the computation of the relativistic counterpart of the RB spectrum with the version here presented of relativistic irreversible thermodynamics. The modifications by gravity to the RB spectrum in the Newtonian case has already been investigated [25] [26]. It is also interesting to consider other relativistic hydrodynamical systems such as the quark-gluon plasma [27].

In our opinion much more work is needed to fully access the physical content of the relativistic Meixner theory before it is virtually discarded incorporating arguments of other thermodynamic approaches whose physical content, even in its non-relativistic version, is still rather controversial. More evidence along these lines will be given in the next future.

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