A modified Lax-Phillips scattering theory for quantum mechanics

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Abstract

The Lax-Phillips scattering theory is an appealing abstract framework for the analysis of scattering resonances. Quantum mechanical adaptations of the theory have been proposed. However, since these quantum adaptations essentially retain the original structure of the theory, assuming the existence of incoming and outgoing subspaces for the evolution and requiring the spectrum of the generator of evolution to be unbounded from below, their range of applications is rather limited. In this paper it is shown that if we replace the assumption regarding the existence of incoming and outgoing subspaces by the assumption of the existence of Lyapunov operators for the quantum evolution (the existence of which has been proved for certain classes of quantum mechanical scattering problems) then it is possible to construct a structure analogous to the Lax-Phillips structure for scattering problems for which the spectrum of the generator of evolution is bounded from below.

1 Introduction

The Lax-Phillips scattering theory [LP] had been originally developed for the analysis of scattering problems involving the solution of hyperbolic wave equations in domains exterior to compactly supported obstacles. As a theory formulated for such purposes, the Lax-Phillips theory, in its original form, is most suitable for dealing with resonances in the scattering of electromagnetic or acoustic waves off compact obstacles. The theory is based on a Hilbert space description of the propagating waves and the time evolution of these waves is given by a unitary evolution group.

Several aspects of the Lax-Phillips scattering theory distinguish it as an appealing abstract formalism for implementation even in situations outside of the strict range of problems for which it has been originally devised. The description of resonances in the framework of the Lax-Phillips theory possesses properties which may be considered as defining properties of an appropriate description of these objects. One such property is a dynamical characterization of resonances via their time evolution given in terms of a continuous, one parameter, strongly contractive semigroup known as the Lax-Phillips semigroup. Specifically, resonances are identified as eigenvalues of the generator of the Lax-Phillips semigroup. This corresponds to another desirable feature of the theory, namely, the fact that each resonance pole is associated with a resonance state (or more

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generally a subspace) in a Hilbert space. In fact, the Lax-Phillips semigroup is obtained by a projection of the unitary evolution of the full system onto the subspace spanned by resonance states.

Consider a Hilbert space $\mathcal{H}^{lp}$ and a continuous, one parameter, evolution group of unitary operators $\{U(t)\}_{t \in \mathbb{R}}$ on $\mathcal{H}^{lp}$. The starting point for the Lax-Phillips scattering theory is the assumption that there exist in $\mathcal{H}^{lp}$ two distinguished subspaces $\mathcal{D}_-$ and $\mathcal{D}_+$ with the properties

$$
\mathcal{D}_- \perp \mathcal{D}_+, \quad U(t)\mathcal{D}_- \subseteq \mathcal{D}_-, \quad \forall t \leq 0
$$

$$
U(t)\mathcal{D}_+ \subseteq \mathcal{D}_+, \quad \forall t \geq 0
$$

$$
\bigcap_{t \in \mathbb{R}} U(t)\mathcal{D}_\pm = \{0\}
$$

$$
\bigvee_{t \in \mathbb{R}} U(t)\mathcal{D}_\pm = \mathcal{H}^{lp}
$$

We call a Hilbert space $\mathcal{H}^{lp}$ on which the assumptions in Eq. (1) hold a Lax-Phillips Hilbert space. The subspaces $\mathcal{D}_-$ and $\mathcal{D}_+$ are called, respectively, the incoming subspace and outgoing subspace for the evolution $\{U(t)\}_{t \in \mathbb{R}}$. The subspace $\mathcal{D}_-$ corresponds to incoming waves which do not interact with the target prior to $t = 0$ and the subspace $\mathcal{D}_+$ corresponds to outgoing waves which do not interact with the target after $t = 0$. These properties are reflected in the stability properties of $\mathcal{D}_-$ and $\mathcal{D}_+$ in Eq. (1) above.

Let $P_-$ be the orthogonal projection in $\mathcal{H}^{lp}$ onto the orthogonal complement of $\mathcal{D}_-$ and $P_+$ be the orthogonal projection in $\mathcal{H}^{lp}$ onto the orthogonal complement of $\mathcal{D}_+$. The main object of study in the Lax-Phillips theory is the family $\{Z_{lp}(t)\}_{t \geq 0}$ of operators on $\mathcal{H}^{lp}$ defined by

$$
Z_{lp}(t) := P_+ U(t) P_-, \quad t \geq 0
$$

Lax and Phillips prove the following theorem:

**Theorem 1** The operators $Z_{lp}(t), t \geq 0$, annihilate $\mathcal{D}_-$ and $\mathcal{D}_+$, map the orthogonal complement subspace $\mathcal{H}^{lp}_{\text{res}} := \mathcal{H}^{lp} \ominus (\mathcal{D}_- \oplus \mathcal{D}_+)$ into itself and form a strongly continuous semigroup (i.e., $Z_{lp}(t_1)Z_{lp}(t_2) = Z_{lp}(t_1 + t_2), t_1, t_2 \geq 0$) of contraction operators on $\mathcal{H}^{lp}_{\text{res}}$. Furthermore, we have $s - \lim_{t \to \infty} Z_{lp}(t) = 0$.

The family of operators $\{Z_{lp}(t)\}_{t \geq 0}$ is known as the Lax-Phillips semigroup.

Let $L^2(\mathbb{R}, \mathcal{K})$ denote the space of Lebesgue square integrable functions defined on the real line $\mathbb{R}$ and taking their values in a separable Hilbert space $\mathcal{K}$. Ja. G. Sinai [10] proved that if the assumptions in Eq. (1) hold for the outgoing subspace $\mathcal{D}_+$ then the following theorem holds:

**Theorem 2** (Ja. G. Sinai) If $\mathcal{D}_+$ is an outgoing subspace with respect to the unitary group $\{U(t)\}_{t \in \mathbb{R}}$ defined on a Hilbert space $\mathcal{H}^{lp}$ then $\mathcal{H}^{lp}$ can be represented isometrically as the Hilbert space of functions $L^2(\mathbb{R}, \mathcal{K})$ for some Hilbert space $\mathcal{K}$ (called the auxiliary Hilbert space) in such a way that $U(t)$ goes to translation to the right by $t$ units and $\mathcal{D}_+$ is mapped onto $L^2(\mathbb{R}_+, \mathcal{K})$. This representation is unique up to an isomorphism of $\mathcal{K}$.

A representation of this kind is called an outgoing translation representation. An analogous representation theorem holds for an incoming subspace $\mathcal{D}_-$, i.e., if $\mathcal{D}_-$ is an incoming subspace with respect to the group $\{U(t)\}_{t \in \mathbb{R}}$ then there is a representation in which $\mathcal{H}^{lp}$ is mapped isometrically onto $L^2(\mathbb{R}_-, \mathcal{K})$, $U(t)$ goes to translation to the right by $t$ units and $\mathcal{D}_-$ is mapped onto $L^2(\mathbb{R}_-, \mathcal{K})$. This representation is called an incoming translation representation.

Let $W_{lp}^+ : \mathcal{H}^{lp} \mapsto L^2(\mathbb{R}, \mathcal{K})$ and $W_{lp}^- : \mathcal{H}^{lp} \mapsto L^2(\mathbb{R}, \mathcal{K})$ be the mappings of $\mathcal{H}^{lp}$ onto the outgoing and incoming translation representations respectively. The map $S_{lp} : L^2(\mathbb{R}, \mathcal{K}) \mapsto L^2(\mathbb{R}, \mathcal{K})$ defined by

$$
S_{lp} := W_{lp}^+ (W_{lp}^-)^{-1}
$$

2
is called the Lax-Phillips scattering operator. It was proved by Lax and Phillips that $S_{LP}$ is equivalent to the standard definition of the scattering operator. For most purposes it is more convenient not to work with the incoming and outgoing representations but rather with their Fourier transforms called, respectively, the incoming spectral representation and outgoing spectral representation. According to the Paley-Wiener theorem [PW] in the incoming spectral representation $D_-$ is represented by $\mathcal{H}^2_+ (\mathbb{R}, K)$ where $\mathcal{H}^2_+ (\mathbb{R}, K)$ is the space of boundary values on $\mathbb{R}$ of functions in the Hardy space $\mathcal{H}^2 (\mathbb{C}^+, K)$ of vector valued functions (with values in $K$) defined on the upper half-plane $\mathbb{C}^+$. By the same theorem in the outgoing spectral representation $D_+$ is represented by $\mathcal{H}^2_+ (\mathbb{R}, K)$ where $\mathcal{H}^2 (\mathbb{R}, K)$ is the space of boundary values on $\mathbb{R}$ of functions in the Hardy space $\mathcal{H}^2 (\mathbb{C}^-, K)$ of vector valued functions (with values in $K$) defined on the lower half-plane $\mathbb{C}^-$. The transformation to the spectral representations implies a transformation of the scattering operator $S_{LP}$ into the scattering operator in the spectral representation $\hat{S}_{LP}$ defined by

$$\hat{S}_{LP} := FS_{LP}F^{-1}$$

where $F$ is the Fourier transform operator. The operator $\hat{S}_{LP}$ is then realized in the spectral representation as a multiplicative, operator valued function $\hat{S}_{LP}(\cdot) : \mathbb{R} \rightarrow \mathcal{B}(K)$ (where $\mathcal{B}(K)$ is the space of all bounded operators on $K$) having the properties:

(a) $\hat{S}_{LP}(\cdot)$ is the boundary value on $\mathbb{R}$ of an operator valued function $\hat{S}_{LP}(\cdot) : \mathbb{C}^+ \rightarrow \mathcal{B}(K)$ analytic on $\mathbb{C}^+$,

(b) $\|\hat{S}_{LP}(z)\| \leq 1$, $\forall z \in \mathbb{C}^+$,

(c) $\hat{S}_{LP}(E)$, $E \in \mathbb{R}$ is, pointwise, a unitary operator on $K$.

The operator valued function $\hat{S}_{LP}(\cdot)$ is called the Lax-Phillips $S$-matrix. This function is characterized by its action on the subspace $\mathcal{H}^2_+ (\mathbb{R}, K)$ as being an inner (operator valued) function $SzNE [RR, Ho]$. The analytic continuation of $\hat{S}_{LP}(\cdot)$ from the upper half-plane to the lower half-plane is given by

$$\hat{S}_{LP}(z) := [\hat{S}_{LP}(\bar{\mu})]^{-1}, \quad \text{Im} \ z < 0$$

It can then be shown that the analytic continuation of the Lax-Phillips $S$-matrix to the whole complex plane is a meromorphic operator valued function. One of the main results of the Lax-Phillips scattering theory is:

**Theorem 3** Let $B$ denote the generator of the semigroup $\{ Z_{LP}(t) \}_{t \geq 0}$. If $\text{Im} \mu < 0$, then $\mu$ belongs to the point spectrum of $B$ if and only if $\hat{S}_{LP}(\mu)$ has a non-trivial null space. □

Theorem 3 implies that a pole of the Lax-Phillips $S$-matrix at a point $\mu$ in the lower half plane is associated with an eigenvalue $\mu$ of the generator of the Lax-Phillips semigroup. In other words, resonance poles of the Lax-Phillips $S$-matrix correspond to eigenvalues of the (generator of the) Lax-Phillips semigroup with well defined eigenvectors belonging to the resonance subspace $\mathcal{H}^2 \cap (D_+ \oplus D_-)$. The attractive properties of the Lax-Phillips scattering theory, mentioned above, have led to some efforts to adapt the formalism into the framework of quantum mechanics. Early work in this direction can be found, for example, in Refs. [Pav1, Pav2, FP, HP, EH] (see also Ref. [KMPY] for a more recent application of the Lax-Phillips structure to quantum problems). A general formalism was developed in Ref. [SH] and subsequently applied to several physical models in Refs. [SH1, SH2, BariH]. However, in general one cannot apply, without modification, the basic structure of the Lax-Phillips scattering theory in the context of standard quantum mechanical scattering problems since incoming and outgoing subspaces $D_{\pm}$ having the properties listed in Eq. (11) cannot be found for large classes of such problems. This can be seen, for example, by noting the fact that in the Lax-Phillips theory the continuous spectrum of the generator of evolution is
necessarily unbounded from below as well as from above, a requirement which is not met by most quantum mechanical Hamiltonians. Hence, the range of applications of quantum mechanical adaptations of the Lax-Phillips theory which essentially retain the original mathematical structure of the theory is rather limited.

A step forward in the efforts to approximate the structure of the Lax-Phillips theory within the context of quantum mechanics has been made in Ref. [S1] with the introduction into the framework of quantum mechanics of forward and backward Lyapunov operators, based on properties of Hardy spaces, and subsequent investigation of their properties and their applications in Refs. [S2, SSMH1, SSMH2]. If \( H \) is the Hilbert space corresponding to a given system and \( H \) is a self-adjoint generator of evolution of the system we define the trajectory \( \Phi_\varphi \) corresponding to a state \( \varphi \in H \) to be the set of states

\[
\Phi_\varphi := \{ \varphi(t) \}_{t \in \mathbb{R}} = \{ U(t) \varphi \}_{t \in \mathbb{R}},
\]

where \( U(t) = \exp(-iHt) \). Note that this definition extends the definition of trajectory in Ref. [S1] to include negative as well as positive times. Accordingly, the definition of a forward Lyapunov operator in Ref. [S1] is also extended as follows:

**Definition 1 (forward Lyapunov operator)** Let \( M \) be a bounded self-adjoint operator on \( H \). Let \( \Phi_\varphi \) be the trajectory corresponding to an arbitrarily chosen normalized state \( \varphi \in H \). Let \( M(\Phi_\varphi) := \{ (\psi, M\psi) \mid \psi \in \Phi_\varphi \} \) be the collection of all expectation values of \( M \) for states in \( \Phi_\varphi \). Then \( M \) is a forward Lyapunov operator if the mapping \( \tau_{M,\varphi} : \mathbb{R} \to M(\Phi_\varphi) \) defined by

\[
\tau_{M,\varphi}(t) = (\varphi(t), M\varphi(t))
\]

is one to one and monotonically decreasing. □

**Remark 1:** We assume throughout the present paper that all generators of evolution are time independent and, therefore, we have symmetry of the evolution with respect to time translations.

**Remark 2:** If in the definition above we require that \( \tau_{M,\varphi} \) be monotonically increasing instead of monotonically decreasing we also obtain a valid definition of a forward Lyapunov operator. The requirement that \( \tau_{M,\varphi} \) is monotonically decreasing is made purely for the sake of convenience.

If \( M \) is a forward Lyapunov operator then we are able to find the time ordering of states in the trajectory \( \Phi_\varphi \) according to the ordering of expectation values in \( M(\Phi_\varphi) \). Hence, the existence of a Lyapunov operator introduces temporal ordering into the Hilbert space \( H \) of a problem for which such an operator can be constructed. The definition of a backward Lyapunov operator is similar to that of a forward Lyapunov operator, but with respect to the reversed direction of time. The significance of the existence of forward and backward Lyapunov operators in the construction of a formalism analogous to the Lax-Phillips theory within quantum mechanics can be understood if we consider again the original Lax-Phillips formalism, and, in particular, the properties of the projection operators \( P_+ \) and \( P_- \). In fact, from the representation of \( P_+ \) in the outgoing translation representation as an orthogonal projection on the subspace \( L^2(\mathbb{R}_+;K) \) (or, indeed, directly from the definition of \( P_+ \) and the properties of \( D_+ \) in Eq. [1]) it is evident that \( P_+ \) is a forward Lyapunov operator for the evolution in the Lax-Phillips theory. For every \( \psi \in H^{LP} \) we have

\[
(\psi(t_2), P_+ \psi(t_2)) \leq (\psi(t_1), P_+ \psi(t_1)), \quad t_1 \leq t_2, \quad \lim_{t \to \infty} (\psi(t), P_+ \psi(t)) = 0.
\]

Likewise, from the representation of \( P_- \) in the incoming translation representation as an orthogonal projection on the subspace \( L^2(\mathbb{R}_-;K) \) it is evident that \( P_- \) is a backward
Lyapunov operator for the Lax-Phillips evolution satisfying
\[
(\psi(t_2), P_-\psi(t_2)) \leq (\psi(t_1), P_-\psi(t_1)), \quad t_2 \leq t_1, \quad \lim_{t \to -\infty} (\psi(t), P_-\psi(t)) = 0.
\]
Note that, if we define \( Z(t) := P_+U(t) \) for \( t \geq 0 \) then, by the stability properties of \( D_+ \), we have for \( t_1, t_2 \geq 0 \)
\[
Z(t_1)Z(t_2) = P_+U(t_1)P_+U(t_2) = P_+U(t_1)(P_+ + P_+)U(t_2) = P_+U(t_1 + t_2) = Z(t_1 + t_2),
\]
where \( P_+^+ = I - P_+ \). Hence, the family of operators \( \{Z(t)\}_{t \geq 0} \) is a continuous, one parameter, contractive semigroup on \( \mathcal{H}_{\text{LP}} \). It is easy to show, in addition, that \( s - \lim_{t \to -\infty} Z(t) = 0. \) Moreover, we have
\[
P_+U(t) = P_+U(t)(P_+ + P_+) = P_+U(t)P_+ = Z(t)P_+, \quad t \geq 0,
\]
so that for non-negative times \( P_+ \) intertwines the unitary evolution \( U(t) \) with the semigroup evolution \( Z(t) \). Finally, observe that by the intertwining relation in Eq. (3) we have
\[
Z_{\text{LP}}(t) = P_+U(t)P_- = Z(t)P_+P_-, \quad t \geq 0
\]
We turn now to consider Lyapunov operators in quantum mechanics. Following the basic existence results proved in Ref. \[S1\] it has been shown in Refs. \[S2, SSMH1, SSMH2\] that if a quantum mechanical scattering problem satisfies the assumptions that:
(a) The absolutely continuous spectrum of the unperturbed and perturbed Hamiltonians is \( \sigma_{ac}(H_0) = \sigma_{ac}(H) = \mathbb{R}^+ \),
(b) The multiplicity of the a.c. spectrum of \( H \) is uniform,
(c) The incoming and outgoing Møller wave operators \( \Omega_\pm(H_0, H) \) exist and are complete; then, if \( \mathcal{H}_{ac} \) is the subspace of \( \mathcal{H} \) corresponding to the a.c. spectrum of \( H \), there exists a self-adjoint, contractive, injective and non-negative forward Lyapunov operator \( M_+ : \mathcal{H}_{ac} \to \mathcal{H}_{ac} \) for the quantum evolution, i.e., for any \( \psi \in \mathcal{H}_{ac} \) we have
\[
(\psi(t_2), M_+\psi(t_2)) \leq (\psi(t_1), M_+\psi(t_1)), \quad t_1 \leq t_2, \quad \lim_{t \to -\infty} (\psi(t), M_+\psi(t)) = 0
\]
where \( \psi(t) = U(t)\psi = \exp(-iHt)\psi \). In addition, it is shown in Refs. \[S2, SSMH1, SSMH2\] that \( \text{Ran } M_+ \) is dense in \( \mathcal{H}_{ac} \). Similarly, under the same assumptions, there exists a self-adjoint, contractive, injective and non-negative backward Lyapunov operator \( M_- : \mathcal{H}_{ac} \to \mathcal{H}_{ac} \) for the quantum evolution
\[
(\psi(t_2), M_-\psi(t_2)) \leq (\psi(t_1), M_-\psi(t_1)), \quad t_2 \leq t_1, \quad \lim_{t \to -\infty} (\psi(t), M_-\psi(t)) = 0,
\]
with \( \text{Ran } M_- \) dense in \( \mathcal{H}_{ac} \).

Set \( \Lambda_+ := M_+^{1/2} \) and \( \Lambda_- := M_-^{1/2} \). Upon comparison to the definition of the Lax-Phillips semigroup in Eq. (2) we are led to define for a quantum mechanical scattering problem a family of operators \( \{Z_{\text{app}}(t)\}_{t \geq 0} : \mathcal{H}_{ac} \to \mathcal{H}_{ac} \), to which we refer as the \emph{approximate Lax-Phillips semigroup}, via the definition
\[
Z_{\text{app}}(t) := \Lambda_+U(t)\Lambda_-, \quad t \geq 0.
\]
Note that if we apply similar definitions of \( \Lambda_\pm \), as square roots of the Lyapunov operators, in the Lax-Phillips case we obtain \( \Lambda_+ = P_+^{1/2} = P_+ \) and \( \Lambda_- = P_-^{1/2} = P_- \) so that in this case we have \( Z_{\text{app}}(t) = Z_{\text{LP}}(t), \forall t \geq 0 \).

It is shown in Refs. \[S2, SSMH1\] that there exists a continuous, strongly contractive, one parameter semigroup \( \{Z_+(t)\}_{t \geq 0} \) such that for each \( \psi \in \mathcal{H}_{ac} \) we have
\[
\|Z_+(t_2)\psi\| \leq \|Z_+(t_1)\psi\|, \quad t_2 \geq t_1 \geq 0, \quad s - \lim_{t \to -\infty} Z_+(t) = 0,
\]
and the following intertwining relation holds

\[ \Lambda_+ U(t) = Z_+(t) \Lambda_+ , \quad U(t) = e^{-iHt}, \quad t \geq 0 \]  \hspace{1cm} (6)

(a similar semigroup and intertwining relation can be found for \( \Lambda_- \) in the backward direction of time). Using this intertwining relation we obtain

\[ Z_{\text{app}}(t) := \Lambda_+ U(t) \Lambda_- = Z_+(t) \Lambda_+ \Lambda_-, \quad t \geq 0. \]  \hspace{1cm} (7)

Eq. (6) is to be compared with Eq. (3) and Eq. (7) is to be compared with Eq. (4). Note, however, that \( \{Z_{\text{app}}(t)\}_{t \in \mathbb{R}^+} \) is not an exact semigroup.

For a quantum mechanical scattering problem satisfying assumptions (a)-(c) the scattering operator \( S_{\text{QM}} = \Omega_-^{-1} \Omega_+ \), where \( \Omega_+ \) and \( \Omega_- \) are, respectively, the incoming and outgoing Møller wave operators, has a representation as a mapping from the incoming energy representation to the outgoing energy representation in terms of the scattering matrix \( \hat{S}_{\text{QM}}(\cdot) : \mathbb{R}^+ \mapsto \mathcal{U}(\mathcal{K}) \), where \( \mathcal{U}(\mathcal{K}) \) is the set of unitary operators on the multiplicity Hilbert space \( \mathcal{K} \) (note that \( \hat{S}_{\text{QM}}(\cdot) \) is, in fact, a representation of the scattering operator \( S_{\text{QM}} \) in the spectral representation of the unperturbed Hamiltonian \( H_0 \)). The scattering matrix \( \hat{S}_{\text{QM}}(\cdot) \) in the quantum case is analogous to the Lax-Phillips scattering matrix \( \hat{S}_{\text{LP}}(\cdot) \) in the Lax-Phillips case, which is also a mapping between incoming and outgoing spectral representations of the generator of evolution. Adding the correspondence of these two objects to the list of analogous constructions for the quantum mechanical scattering theory and the Lax-Phillips scattering theory discussed above we may produce, for a quantum mechanical scattering problem satisfying assumptions (a)-(c), the following list of correspondences between objects in the Lax-Phillips scattering theory and corresponding objects in the case of quantum mechanical scattering

\[
\begin{array}{ll}
\text{LP scattering theory} & \text{QM scattering theory} \\
U(t) = e^{-iKt} & U(t) = e^{-iHt} \\
P_\pm & \Lambda_\pm \\
Z_{\text{LP}}(t) = P_+ U(t) P_- , \ t \geq 0 & Z_{\text{app}}(t) = \Lambda_+ U(t) \Lambda_-, \ t \geq 0 \\
\hat{S}_{\text{LP}}(E) , \ E \in \mathbb{R} & \hat{S}_{\text{QM}}(E) , \ E \in \mathbb{R}^+ 
\end{array}
\]  \hspace{1cm} (8)

Our goal in the present paper is to construct, in the context of quantum mechanical scattering, a formalism analogous to the Lax-Phillips scattering theory. Thus far we have considered in the quantum mechanical case a set of objects analogous to the central objects of the Lax-Phillips theory. However, beyond analogy in the construction of certain objects what we seek for is a result analogous to Theorem 3 the central theorem of the Lax-Phillips scattering theory associating resonance poles of the Lax-Phillips \( S \)-matrix to eigenvalues and eigenfunctions of the Lax-Phillips semigroup. Note that we cannot expect to obtain in the quantum mechanical case an exact parallel of Theorem 3 since, as mentioned above, \( \{Z_{\text{app}}(t)\}_{t \in \mathbb{R}^+} \) is not an exact semigroup. The task of proving a theorem analogous to Theorem 3 in the case of quantum mechanical scattering processes, at least in an appropriately defined approximate sense, is taken up in Section 3 and Section 4 where it is proved that to a resonance pole of the quantum mechanical \( S \)-matrix \( \hat{S}_{\text{QM}}(\cdot) \) in the second sheet of the complex energy Riemann surface, at a point \( \mu \) with \( \text{Im} \mu < 0 \), there corresponds a state \( \psi^{res}_\mu \) which is an approximate eigenfunction of each element of \( \{Z_{\text{app}}(t)\}_{t \in \mathbb{R}^+} \) in the sense that

\[
Z_{\text{app}}(t) \psi^{res}_\mu = e^{-i\mu t} \psi^{res}_\mu + \text{small corrections , } \ t \geq 0.
\]

The state \( \psi^{res}_\mu \) is considered to be a resonance state corresponding to the resonance pole of \( \hat{S}_{\text{QM}}(\cdot) \) at \( z = \mu \). By establishing a result analogous to Theorem 3 in the context
of quantum mechanical resonance scattering (in an appropriately defined approximate sense) we complete the construction of a framework analogous to the Lax-Phillips theory for quantum mechanical scattering. We note that \( \psi_R^{res} \) is an exact eigenstate of each element of the semigroup \( \{ Z_\mu(t) \}_{t \geq 0} \) in the same way that a resonance state in the Lax-Phillips theory is an eigenstate of each element of the semigroup \( \{ \tilde{Z}(t) \}_{t \geq 0} \).

The arrangement of the rest of the present paper is as follows: Section 2 provides a detailed discussion of Lyapunov operators for the case of a scattering problem satisfying assumptions (i)-(ii) below. As mentioned above, since incoming and outgoing subspaces \( \mathcal{D}_\pm \) cannot be found, in general, for standard quantum mechanical scattering problems, and hence the construction of a formalism analogous to the Lax-Phillips scattering theory cannot be based on the existence of such subspaces, the basic objects involved in the construction of a structure approximating the Lax-Phillips structure in the quantum case are the Lyapunov operators analogous to the Lyapunov operators \( \tilde{P}_\pm \) of the Lax-Phillips theory. The existence of the Lyapunov operators in the quantum case leads to the definition of objects (such as the approximate Lax-Phillips semigroup) and representations (called the forward and backward transition representations) analogous to the objects and representations of the Lax-Phillips theory. These are also discussed in Section 2. Section 3 is centered on the discussion, in the quantum mechanical context, of a result analogous to Theorem 3, the main result of the Lax-Phillips theory associating with each resonance pole of the Lax-Phillips S-matrix a resonance state in the Lax-Phillips Hilbert space \( \mathcal{H}^{LP} \). An analogous (approximate) result in the quantum mechanical case is given by Theorem 5 in Section 3. The proof of Theorem 5 is contained in Section 4. Conclusions are given in Section 5.

2 Lyapunov operators and transition representations in Lax-Phillips theory and in quantum mechanics

Let \( \mathcal{K} \) be a separable Hilbert space and let \( L^2(\mathbb{R}; \mathcal{K}) \) be the Hilbert space of Lebesgue square integrable \( \mathcal{K} \)-valued functions defined on \( \mathbb{R} \). Let \( \hat{E} \) be the operator of multiplication by the independent variable on \( L^2(\mathbb{R}; \mathcal{K}) \). Let \( \{ u(t) \}_{t \in \mathbb{R}} \) be the continuous, one parameter, unitary evolution group on \( L^2(\mathbb{R}; \mathcal{K}) \) generated by \( \hat{E} \), i.e.,

\[
[u(t), f](E) = [e^{-\hat{E}t} f](E) = e^{-it} f(E), \quad f \in L^2(\mathbb{R}; \mathcal{K}), \quad E \in \mathbb{R}.
\]

(9)

Let \( \mathcal{H}^2(\mathbb{C}^+; \mathcal{K}) \) and \( \mathcal{H}^2(\mathbb{C}^-; \mathcal{K}) \) be, respectively, the Hardy space of \( \mathcal{K} \)-valued functions analytic in \( \mathbb{C}^+ \) and \( \mathbb{C}^- \). As mentioned in the introduction, the Hilbert space \( \mathcal{H}^2_+ (\mathbb{R}; \mathcal{K}) \) consisting of nontangential boundary values on the real axis of functions in \( \mathcal{H}^2(\mathbb{C}^+; \mathcal{K}) \) is isomorphic to \( \mathcal{H}^2(\mathbb{C}^+; \mathcal{K}) \). Similarly, the Hilbert space \( \mathcal{H}^2_- (\mathbb{R}; \mathcal{K}) \) of non-tangential boundary value functions of functions in \( \mathcal{H}^2(\mathbb{C}^-; \mathcal{K}) \) is isomorphic to \( \mathcal{H}^2(\mathbb{C}^-; \mathcal{K}) \). The spaces \( \mathcal{H}^2_+ (\mathbb{R}; \mathcal{K}) \) are orthogonal subspaces of \( L^2(\mathbb{R}; \mathcal{K}) \) and we have

\[
L^2(\mathbb{R}; \mathcal{K}) = \mathcal{H}^2_+ (\mathbb{R}; \mathcal{K}) \oplus \mathcal{H}^2_- (\mathbb{R}; \mathcal{K}).
\]

We denote the orthogonal projections in \( L^2(\mathbb{R}; \mathcal{K}) \) on \( \mathcal{H}^2_+ (\mathbb{R}; \mathcal{K}) \) and \( \mathcal{H}^2_- (\mathbb{R}; \mathcal{K}) \), respectively, by \( \hat{P}_+ \) and \( \hat{P}_- \).

Recall that in the Lax-Phillips theory \( \hat{P}_+ \) is the orthogonal projection on the orthogonal complement of the outgoing subspace \( \mathcal{D}_+ \). In the outgoing translation representation the Lax-Phillips Hilbert space \( \mathcal{H}^{LP} \) is mapped isometrically onto the function space \( L^2(\mathbb{R}; \mathcal{K}) \) where \( \mathcal{K} \) is the auxiliary Hilbert space and \( \mathcal{D}_+ \) is mapped onto the subspace \( L^2(\mathbb{R}_+; \mathcal{K}) \). The evolution \( U(t) \) is represented by translation to the right by \( t \) units. The outgoing spectral representation is a spectral representation of the generator \( \hat{K} \) of the unitary evolution group \( \{ U(t) \}_{t \in \mathbb{R}} = \{ \exp(-i\hat{K}t) \}_{t \in \mathbb{R}} \) of the Lax-Phillips theory and is obtained
by Fourier transform of the outgoing translation representation. In this representation \( \mathcal{H}^{LP} \) is represented by \( L^2(\mathbb{R}; \mathcal{K}) \), the evolution group \( \{ U(t) \}_{t \in \mathbb{R}} \) is represented by the group \( \{ u(t) \}_{t \in \mathbb{R}} \) in Eq. \( \Phi \) and, by the Paley-Wiener theorem, \( \mathcal{D}_+ \) is represented by the Hardy space \( \mathcal{H}^2(\mathbb{R}; \mathcal{K}) \). Therefore \( P_+ \) is represented in this representation by the projection \( \hat{P}_+ \) on \( \mathcal{H}^2_+(\mathbb{R}; \mathcal{K}) \). Hence if \( \hat{W}^{LP}_+ : \mathcal{H}^{LP} \to L^2(\mathbb{R}; \mathcal{K}) \) is the mapping of \( \mathcal{H}^{LP} \) onto the outgoing spectral representation we have

\[
P_+ = (\hat{W}^{LP}_+)^{-1} \hat{P}_+ \hat{W}^{LP}_+.
\] (10)

In a similar manner, in the incoming translation representation the Lax-Phillips Hilbert space \( \mathcal{H}^{LP} \) is mapped isometrically onto the function space \( L^2(\mathbb{R}; \mathcal{K}) \) and the incoming subspace \( \mathcal{D}_- \) is mapped onto the subspace \( L^2(\mathbb{R}^-; \mathcal{K}) \). The evolution \( U(t) \) is again represented by translation to the right by \( t \) units. The incoming spectral representation, obtained by Fourier transform of the incoming translation representation, is a spectral representation of the generator \( K \) of the evolution group \( \{ U(t) \}_{t \in \mathbb{R}} \). In this representation \( \mathcal{H}^{LP} \) is represented by the function space \( L^2(\mathbb{R}; \mathcal{K}) \), the evolution group \( \{ U(t) \}_{t \in \mathbb{R}} \) is represented by the group \( \{ u(t) \}_{t \in \mathbb{R}} \) of Eq. \( \Phi \) and, by the Paley-Wiener theorem, \( \mathcal{D}_- \) is represented by the Hardy space \( \mathcal{H}^2_-(\mathbb{R}; \mathcal{K}) \). Therefore \( P_- \), the projection on the orthogonal complement of \( \mathcal{D}_- \), is represented in the incoming spectral representation by the projection \( \hat{P}_- \) on \( \mathcal{H}^2_-(\mathbb{R}; \mathcal{K}) \) and hence, if \( \hat{W}^{LP}_- : \mathcal{H}^{LP} \to L^2(\mathbb{R}; \mathcal{K}) \) is the mapping of \( \mathcal{H}^{LP} \) onto the incoming spectral representation, we have

\[
P_- = (\hat{W}^{LP}_-)^{-1} \hat{P}_- \hat{W}^{LP}_-.
\] (11)

Observe that Eqs. (10) and (11) provides us with an explicit procedure for the construction of the forward and backward Lyapunov operators \( P_\pm \) in the Lax-Phillips theory.

Next, we turn to consider the construction of Lyapunov operators in the quantum mechanical case. Let \( L^2(\mathbb{R}_\pm; \mathcal{K}) \) be the subspaces of \( L^2(\mathbb{R}; \mathcal{K}) \) consisting of functions supported on \( \mathbb{R}_\pm \). Then we have another orthogonal decomposition of \( L^2(\mathbb{R}; \mathcal{K}) \)

\[
L^2(\mathbb{R}; \mathcal{K}) = L^2(\mathbb{R}_+; \mathcal{K}) \oplus L^2(\mathbb{R}_-; \mathcal{K}).
\]

We denote the orthogonal projections on the subspaces \( L^2(\mathbb{R}_+; \mathcal{K}) \) and \( L^2(\mathbb{R}_-; \mathcal{K}) \), respectively, by \( P_{\mathbb{R}_+} \) and \( P_{\mathbb{R}_-} \). Let \( \hat{E}_+ \) be the operator of multiplication by the independent variable on \( L^2(\mathbb{R}_+; \mathcal{K}) \). Let \( \{ u_+(t) \}_{t \in \mathbb{R}} \) be the continuous, one parameter, unitary evolution group generated by \( \hat{E}_+ \), i.e.,

\[
[u_+(t)f](E) = e^{-i\hat{E}_+t}f(E) = e^{-iEt}f(E), \quad f \in L^2(\mathbb{R}_+; \mathcal{K}), \quad E \in \mathbb{R}_+.
\]

The following two theorems, first proved in Ref. \[S1\], form the basis for the present discussion of Lyapunov operators and their applications in the context of quantum mechanical scattering:

**Theorem 4** Let \( M_F : L^2(\mathbb{R}^+_+; \mathcal{K}) \to L^2(\mathbb{R}^+_+; \mathcal{K}) \) be the operator defined by

\[
M_F := (P_{\mathbb{R}_+} \hat{P}_+ P_{\mathbb{R}_+})|_{L^2(\mathbb{R}^+_+; \mathcal{K})}.
\] (12)

Then \( M_F \) is a positive, contractive, injective operator on \( L^2(\mathbb{R}^+_+; \mathcal{K}) \), such that \( \text{Ran} M_F \) is dense in \( L^2(\mathbb{R}^+_+; \mathcal{K}) \) and \( M_F \) is a Lyapunov operator in the forward direction, i.e., for every \( \psi \in L^2(\mathbb{R}^+_+; \mathcal{K}) \) we have

\[
(\psi(t_2), M_F \psi(t_2)) \leq (\psi(t_1), M_F \psi(t_1)), \quad t_2 \geq t_1 \geq 0, \quad \psi(t) = u_+(t)\psi,
\]

and, moreover,

\[
\lim_{t \to \infty} (\psi(t), M_F \psi(t)) = 0.
\]

\[\square\]
Theorem 5 Let $\Lambda_F := M_F^{1/2}$. Then $\Lambda_F : L^2(\mathbb{R}^+; \mathcal{K}) \mapsto L^2(\mathbb{R}^+; \mathcal{K})$ is a positive, contractive, injective operator such that $\text{Ran} \, \Lambda_F$ is dense in $L^2(\mathbb{R}^+; \mathcal{K})$. Furthermore, there exists a continuous, strongly contractive, one parameter semigroup $\{Z_F(t)\}_{t \in \mathbb{R}^+}$ with $Z_F(t) : L^2(\mathbb{R}^+; \mathcal{K}) \mapsto L^2(\mathbb{R}^+; \mathcal{K})$, such that for every $\psi \in L^2(\mathbb{R}^+; \mathcal{K})$ we have
\[
\|Z_F(t_2)\psi\| \leq \|Z_F(t_1)\psi\|, \quad t_2 \geq t_1 \geq 0
\]
and
\[
s - \lim_{t \to \infty} Z_F(t) = 0
\]
and the following intertwining relation holds:
\[
\Lambda_F u_+(t) = Z_F(t) \Lambda_F, \quad t \geq 0.
\] □

In a manner similar to the construction of a forward Lyapunov operator $M_F$ one is able to construct a backward Lyapunov operator $M_B$. The theorems analogous to Theorem 5 and Theorem 6 in this case are

Theorem 6 Let $M_B : L^2(\mathbb{R}^+; \mathcal{K}) \mapsto L^2(\mathbb{R}^+; \mathcal{K})$ be the operator defined by
\[
M_B := (P_{\mathbb{R}^+} \hat{P}_{\mathbb{R}^+}) |_{L^2(\mathbb{R}^+; \mathcal{K})},
\]
(14)
Then $M_B$ is a positive, contractive, injective operator on $L^2(\mathbb{R}^+; \mathcal{K})$, such that $\text{Ran} \, M_B$ is dense in $L^2(\mathbb{R}^+; \mathcal{K})$ and $M_B$ is a Lyapunov operator in the backward direction, i.e., for every $\psi \in L^2(\mathbb{R}^+; \mathcal{K})$ we have
\[
(\psi(t_2), M_B \psi(t_2)) \leq (\psi(t_1), M_B \psi(t_1)), \quad t_2 \leq t_1 \leq 0, \quad \psi(t) = u_+(t)\psi,
\]
and, moreover,
\[
\lim_{t \to -\infty} \psi(t), M_B \psi(t)) = 0.
\] □

Theorem 7 Let $\Lambda_B := M_B^{1/2}$. Then $\Lambda_B : L^2(\mathbb{R}^+; \mathcal{K}) \mapsto L^2(\mathbb{R}^+; \mathcal{K})$ is a positive, contractive, injective operator such that $\text{Ran} \, \Lambda_B$ is dense in $L^2(\mathbb{R}^+; \mathcal{K})$. Furthermore, there exists a continuous, strongly contractive, one-parameter semigroup $\{Z_B(t)\}_{t \in \mathbb{R}^-}$ with $Z_B(t) : L^2(\mathbb{R}^+; \mathcal{K}) \mapsto L^2(\mathbb{R}^+; \mathcal{K})$, such that for every $\psi \in L^2(\mathbb{R}^+; \mathcal{K})$ we have
\[
\|Z_B(t_2)\psi\| \leq \|Z_B(t_1)\psi\|, \quad t_2 \leq t_1 \leq 0
\]
and
\[
s - \lim_{t \to -\infty} Z_B(t) = 0
\]
and the following intertwining relation holds:
\[
\Lambda_B u_+(t) = Z_B(t) \Lambda_B, \quad t \leq 0.
\] □

Note that Theorem 4 and Theorem 6 refer, respectively, to positive and negative times. However, due to the time translation invariance of the evolution, the restriction that $t_2$, $t_1$ are non-negative in Theorem 4 and the $t_2$, $t_1$ are non-positive in Theorem 6 can be removed (keeping the time ordering between $t_2$ and $t_1$ in both cases) and the Lyapunov property extends to all values of time. It is evident from Theorems 4 and 6 that both the forward and backward Lyapunov operators are defined on a rather abstract level in terms of certain functions spaces and that no relation to any concrete class of physical problems has been made yet. We amend this by introducing Lyapunov operators specifically in the context of quantum mechanical scattering problems.

In the following we consider quantum mechanical scattering problems satisfying the following two assumptions:
(i) Let $\mathcal{H}$ be a separable Hilbert space corresponding to a given quantum mechanical scattering problem. Assume that a self-adjoint "free" unperturbed Hamiltonian $H_0$ and a self-adjoint perturbed Hamiltonian $H$ are defined on $\mathcal{H}$ and form a complete scattering system, i.e., we assume that the Møller wave operators $\Omega_{\pm} \equiv \Omega_{\pm}(H_0, H)$ exist and are complete.

(ii) We assume that $\sigma_{ac}(H) = \sigma_{ac}(H_0) = \mathbb{R}^+$. Moreover, we assume that the multiplicity of the absolutely continuous spectrum is uniform over $\mathbb{R}^+$.

Under assumptions (i)-(ii) above, there exist two mappings $\hat{W}_{\pm}^{\text{QM}} : \mathcal{H}_{ac} \mapsto L^2(\mathbb{R}^+; \mathcal{K})$ that map the subspace $\mathcal{H}_{ac} \subseteq \mathcal{H}$ isometrically onto the function space $L^2(\mathbb{R}^+; \mathcal{K})$ for some Hilbert space $\mathcal{K}$ whose dimension corresponds to the multiplicity of $\sigma_{ac}(H)$ and the Schrödinger evolution $U(t) = \exp(-iHt)$ is represented by the group $\{u_+(t)\}_{t \in \mathbb{R}}$. The representation of the scattering problem in the function space $L^2(\mathbb{R}^+; \mathcal{K})$ obtained by applying $\hat{W}_{\pm}^{\text{QM}}$ is known as the outgoing energy representation and is a spectral representation for $H$ in which the action of $H$ is represented by multiplication by the independent variable. In a similar manner, the representation obtain by applying the mapping $\hat{W}_{-}^{\text{QM}}$ is another spectral representation for $H$, known as the incoming energy representation of the problem. We note that all of the objects $M_F$, $M_B$, $\Lambda_F$, $\Lambda_B$, $Z_F(t)$ and $Z_B(t)$ in Theorems 4.17 are defined on the level of such spectral representations of $H$ and their construction is made irrespective of the specific spectral representation in which one is working. However, when applied to scattering problems, we need to distinguish between the corresponding objects defined within the incoming energy representation and the outgoing energy representation.

The mappings $\hat{W}_{+}^{\text{QM}}$ and $\hat{W}_{-}^{\text{QM}}$ correspond, respectively, to incoming and outgoing solutions of the the Lippmann-Schwinger equation. If $\{\phi_{E, \xi}\}_{E \in \mathbb{R}^+ \cup \Xi}$ are outgoing solutions of the Lippmann-Schwinger equation, where $\xi$ corresponds to degeneracy indices of the energy $E$, and if $\{\phi_{E, \xi}^\pm\}_{E \in \mathbb{R}^+ \cup \Xi}$ are incoming solutions of the Lippmann-Schwinger equation, and $\psi \in \mathcal{H}_{ac}$ is any scattering state, then

$$
(\hat{W}_{+}^{\text{QM}} \psi)(E, \xi) = (\phi_{E, \xi}^+, \psi) \\
(\hat{W}_{-}^{\text{QM}} \psi)(E, \xi) = (\phi_{E, \xi}^-, \psi).
$$

With the help of the two mapping $\hat{W}_{\pm}^{\text{QM}}$ which are, in fact, associated with the two Møller wave operators $\Omega_{\pm}$, we define the forward Lyapunov operator for the quantum scattering problem to be

$$
M_+ := (\hat{W}_{+}^{\text{QM}})^{-1} M_F \hat{W}_{+}^{\text{QM}}. \tag{16}
$$

By Theorem 4.17 the operator $M_+$ is a positive, contractive, injective operator on $\mathcal{H}_{ac}$, such that $\text{Ran } M_+$ is dense in $\mathcal{H}_{ac}$ and $M_+$ is a forward Lyapunov operator with respect to the quantum evolution on $\mathcal{H}_{ac}$. Similarly, the backward Lyapunov operator for the quantum scattering problem is defined to be

$$
M_- := (\hat{W}_{-}^{\text{QM}})^{-1} M_B \hat{W}_{-}^{\text{QM}}. \tag{17}
$$

According to Theorem 5.65 the operator $M_-$ is a positive, contractive, injective operator on $\mathcal{H}_{ac}$, such that $\text{Ran } M_-$ is dense in $\mathcal{H}_{ac}$ and $M_-$ is a backward Lyapunov operator with respect to the quantum evolution on $\mathcal{H}_{ac}$. (The Lyapunov operators $M_+$ and $M_-$ are identical, respectively, to the outgoing forward Lyapunov operator $M_{F,+}$ and the incoming backward Lyapunov operator $M_{B,-}$ of Ref. [22]). Eqns. (16) and (17) are to be compared to Eqns. (10) and (11).

We presently show that the Lax-Phillips Lyapunov operators $P_{\pm}$ and the quantum mechanical Lyapunov operators $M_{\pm}$ are but two particular instances of a more general
construction. Consider a scattering problem, defined on a Hilbert space \( \mathcal{H} \), for which \( H \) is the generator of the evolution group \( \{ U(t) \}_{t \in \mathbb{R}} \) of the system under consideration, i.e., \( U(t) = \exp(-iHt) \). Denote by \( \sigma_{ac}(H) \) the absolutely continuous part of the spectrum of \( H \) and let \( \mathcal{H}_{ac} \subseteq \mathcal{H} \) denote the subspace corresponding to \( \sigma_{ac}(H) \). Assume, furthermore, that the multiplicity of the absolutely continuous spectrum is uniform over \( \sigma_{ac}(H) \) and that the Møller wave operators exist and are complete. Then there exist two unitary mappings \( \hat{W}_\pm : \mathcal{H}_{ac} \to L^2(\mathbb{R}; \mathcal{K}) \), corresponding to the outgoing and incoming wave operators of the problem, which map \( \mathcal{H}_{ac} \) onto a function space \( L^2(\sigma_{ac}(H), \mathcal{K}) \subseteq L^2(\mathbb{R}, \mathcal{K}) \), where \( \mathcal{K} \) is a Hilbert space whose dimension corresponds to the multiplicity of \( \sigma_{ac}(H) \) and such that \( \hat{W}_\pm \) maps the action of the generator of evolution \( H \) into multiplication by the independent variable in \( L^2(\sigma_{ac}(H), \mathcal{K}) \). The two representations of the scattering problem thus obtained are the incoming and outgoing energy representations for the problem. Let \( P_{\sigma_{ac}(H)} : L^2(\mathbb{R}, \mathcal{K}) \to L^2(\mathbb{R}, \mathcal{K}) \) be the orthogonal projection in \( L^2(\mathbb{R}, \mathcal{K}) \) on the subspace \( L^2(\sigma_{ac}(H), \mathcal{K}) \). Hence, if \( P_{\sigma_{ac}(H)} : \mathcal{H} \to \mathcal{H} \) is the orthogonal projection on \( \mathcal{H} \) on \( \mathcal{H}_{ac} \), we have

\[
\hat{W}_\pm P_{\sigma_{ac}(H)} \mathcal{H} = \hat{W}_\pm \mathcal{H}_{ac} = L^2(\sigma_{ac}(H), \mathcal{K}) = P_{\sigma_{ac}(H)} L^2(\mathbb{R}, \mathcal{K}).
\]

Define two operators \( M_\pm(H) : \mathcal{H}_{ac} \to \mathcal{H}_{ac} \) by

\[
M_\pm(H) := \hat{W}_\pm^{-1} P_{\sigma_{ac}(H)} \hat{U} \hat{P}_\pm P_{\sigma_{ac}(H)} \hat{W}_\pm,
\]

where \( \hat{P}_\pm \) are, respectively, the projections in \( L^2(\mathbb{R}, \mathcal{K}) \) on the Hardy subspaces \( \mathcal{H}_{ac}^\pm(\mathbb{R}, \mathcal{K}) \). It is readily verified that \( M_\pm(H) \) are positive, contractive operators on \( \mathcal{H}_{ac} \). Now, if \( \sigma_{ac}(H) = \mathbb{R}^+ \) we find that \( P_{\sigma_{ac}(H)} = P_{\hat{W}_+} \) and hence in this case we obtain

\[
M_+(H) = \hat{W}_+^{-1} P_{\sigma_{ac}(H)} \hat{P}_+ \hat{P}_\pm P_{\sigma_{ac}(H)} \hat{W}_+ = \hat{W}_+^{-1} P_{\hat{W}_+} \hat{P}_+ \hat{W}_+ = \hat{W}_+^{-1} M_{\hat{W}_+} \hat{W}_+ = M_+,
\]

and similarly

\[
M_-(H) = \hat{W}_-^{-1} P_{\sigma_{ac}(H)} \hat{P}_- \hat{P}_\pm P_{\sigma_{ac}(H)} \hat{W}_- = \hat{W}_-^{-1} P_{\hat{W}_-} \hat{P}_- \hat{W}_- = \hat{W}_-^{-1} M_{\hat{W}_-} \hat{W}_- = M_-.
\]

If, on the other hand, we consider the Lax-Phillips scattering theory the generator \( K \) of the unitary evolution group \( \{ U(t) \}_{t \in \mathbb{R}} = \{ e^{-iKt} \} \) defined on the Lax-Phillips Hilbert space \( \mathcal{H}_{1/2} \), satisfies \( \sigma_{ac}(K) = \mathbb{R} \) and we have \( P_{\sigma_{ac}(K)} = I_{L^2(R, K)} \). Therefore, in this case we have

\[
M_+(K) = \hat{W}_+^{-1} P_{\sigma_{ac}(K)} \hat{P}_+ P_{\sigma_{ac}(K)} \hat{W}_+ = \hat{W}_+^{-1} I_{L^2(\mathbb{R}, \mathcal{K})} \hat{P}_+ I_{L^2(\mathbb{R}, \mathcal{K})} \hat{W}_+ = \hat{W}_+^{-1} \hat{P}_+ \hat{W}_+ = P_+,
\]

and

\[
M_-(K) = \hat{W}_-^{-1} P_{\sigma_{ac}(K)} \hat{P}_- P_{\sigma_{ac}(K)} \hat{W}_- = \hat{W}_-^{-1} I_{L^2(\mathbb{R}, \mathcal{K})} \hat{P}_- I_{L^2(\mathbb{R}, \mathcal{K})} \hat{W}_- = \hat{W}_-^{-1} \hat{P}_- \hat{W}_- = P_-.
\]

Since the operators \( M_\pm(H) \) defined in Eq. (18) are positive, contractive operators on \( \mathcal{H}_{ac} \) then their square roots

\[
\Lambda_\pm(H) := M_\pm^{1/2}(H)
\]

are well defined as operators on \( \mathcal{H}_{ac} \). Define a family of operators \( \{ Z_{app}(t) \}_{t \geq 0} : \mathcal{H}_{ac} \to \mathcal{H}_{ac} \) by

\[
Z_{app}(t) := \Lambda_+(H)U(t)\Lambda_-(H), \quad t \geq 0.
\]

Applying the definitions of \( \Lambda_\pm(H) \) in the Lax-Phillips scattering theory we obtain

\[
\Lambda_\pm(K) := M_\pm^{1/2}(K) = P_\pm^{1/2} = P_\pm.
\]

Eq. (19) then yields

\[
Z_{app}(t) = \Lambda_+(K)U(t)\Lambda_-(K) = P_+U(t)P_- = Z_{LP}(t), \quad t \geq 0,
\]

11
so that \( \{Z_{\text{app}}(t)\}_{t \geq 0} \) is in this case exactly the Lax-Phillips semigroup \( \{Z_{LP}(t)\}_{t \geq 0} \). Applying the definitions in the case of a quantum mechanical scattering problem satisfying assumptions (i)-(ii) above we get

\[
\Lambda_\pm(H) := M_\pm^{1/2}(H) = M_\pm^{1/2} = \Lambda_\pm,
\]

where \( \Lambda_\pm := M_\pm^{1/2} \), and hence

\[
Z_{\text{app}}(t) = \Lambda_+(H)U(t)\Lambda_-(H) = \Lambda_+ U(t)\Lambda_-, \quad t \geq 0.
\]

The family of operators \( \{Z_{\text{app}}(t)\}_{t \geq 0} \) is identified in this case as the approximate Lax-Phillips semigroup of Eq. (5).

We make a formal definition of the approximate Lax-Phillips semigroup for a scattering problem satisfying conditions (i)-(ii):

**Definition 2 (Approximate Lax-Phillips semigroup)** Consider scattering problem satisfying assumptions (i)-(ii) above. Let \( \Lambda_+ = M_+^{1/2} \) and \( \Lambda_- = M_-^{1/2} \) where \( M_+ \) is the forward Lyapunov operator and \( M_- \) is the backward Lyapunov operator for the problem. Then the approximate Lax-Phillips semigroup is defined to be the family of operators

\[
\Lambda_+ U(t)\Lambda_-, \quad t \geq 0, \quad U(t) = e^{-iHt}.
\]  

Note that Theorem 5 implies that there exists a continuous, strong ly contractive, one parameter semigroup \( \{Z_+(t)\}_{t \in \mathbb{R}^+} \) with \( Z_+(t) : \mathcal{H}_{ac} \mapsto \mathcal{H}_{ac} \), such that for every \( \psi \in \mathcal{H}_{ac} \) we have

\[
\|Z_+(t_2)\psi\| \leq \|Z_+(t_1)\psi\|, \quad t_2 \geq t_1 \geq 0
\]

and the following intertwining relation holds

\[
\Lambda_+ U(t) = Z_+(t)\Lambda_+, \quad t \geq 0. \tag{21}
\]

In fact, we have \( Z_+(t) = (\hat{W}^{-QM}_+)^{-1}Z_F(t)\hat{W}^{QM}_+ \), where \( Z_F(t) \) are elements of the semigroup \( \{Z_F(t)\}_{t \geq 0} \) in Theorem 5. Similarly, Theorem 5 implies that there exists a continuous, strongly contractive, one parameter semigroup \( \{Z_-(t)\}_{t \in \mathbb{R}^-} \) with \( Z_-(t) : \mathcal{H}_{ac} \mapsto \mathcal{H}_{ac} \), such that for every \( \psi \in \mathcal{H}_{ac} \) we have

\[
\|Z_-(t_2)\psi\| \leq \|Z_-(t_1)\psi\|, \quad t_2 \leq t_1 \leq 0
\]

and the following intertwining relation holds

\[
\Lambda_- U(t) = Z_-(t)\Lambda_-, \quad t \leq 0. \tag{22}
\]

It is precisely due to the central importance of the intertwining relations in Eqs. (21) and (22) that the definition of the approximate Lax-Phillips semigroup in Eqs. (19) and (20) is made using the square roots \( \Lambda_\pm \) of the Lyapunov operators \( M_\pm \) and not the Lyapunov operators themselves (as mentioned in the introduction, it is evident that in the Lax-Phillips case a definition using the Lyapunov operators or their square roots would yield the same family of objects).
We complete the set of relations between constructions of the Lax-Phillips theory and the corresponding constructions for quantum mechanical scattering by considering the incoming and outgoing representations. Observe that the outgoing (spectral or translation) representations of the Lax-Phillips theory are distinguished by the representation of the outgoing subspace $D_+$. Hence, for example, if $W^{lp}_+$ is the mapping of $H^{lp}$ onto the outgoing spectral representation then $W^{lp}_+D_+ = H^2(R; K)$ and $W^{lp}(K \ominus D_-) = H^2(R; K)$. Thus the outgoing representations are centered on the separation of the outgoing part of an evolving state $\psi(t) = U(t)\psi$ from the other components of that state which is achieved by the application of the projection $P_+$. This implies a decomposition of an evolving state $\psi(t) = \exp(-iKt)\psi$, corresponding to an initial state $\psi \in H^{lp}$, according to

$$\psi(t) = P_+\psi(t) + P_-\psi(t) = \psi_+^f(t) + \psi_+^b(t),$$  \hfill (23)

where $\psi_+^f(t) := P_+\psi(t)$ and $\psi_+^b(t) := P_+\psi(t)$. It is readily verified, using the outgoing translation representation, that $\psi_+^b(t)$ is a backward asymptotic component of $\psi(t)$, i.e., $\psi_+^b(t)$ vanishes in the forward time asymptote as $t \to \infty$ and is asymptotic to $\psi(t)$ in the backward time asymptote as $t \to -\infty$. Similarly, $\psi_+^f(t)$ is a forward asymptotic component of $\psi(t)$, i.e., $\psi_+^f(t)$ vanishes in the backward time asymptote as $t \to -\infty$ and is asymptotic to $\psi(t)$ in the forward time asymptote as $t \to \infty$. The evolution of $\psi(t)$ is then represented as a transition from $\psi_+^f(t)$ to $\psi_+^b(t)$. We call the representation of the evolution obtained by the decomposition in Eq. (23) a forward transition representation and emphasize again its direct relation to the outgoing (translation or spectral) representations in the Lax-Phillips theory. Note that the name given to this representation of the evolution registers both the fact that the representation involves a transition between different components of the evolving state and the fact that the decomposition in Eq. (23) is obtained using the forward Lyapunov operator $P_+$.

Following a similar line of argument we may use the backward Lyapunov operator, i.e., the projection $P_-$, to obtain a decomposition of an evolving state $\psi(t)$ in the form

$$\psi(t) = P_-\psi(t) + P_+\psi(t) = \psi_-^f(t) + \psi_-^b(t),$$  \hfill (24)

where $\psi_-^f(t) := P_-\psi(t)$ and $\psi_-^b(t) := P_-\psi(t) = (I - P_-)\psi(t)$. Here $\psi_-^f(t)$ is a forward asymptotic component of $\psi(t)$ and $\psi_-^b(t)$ is a backward asymptotic component of $\psi(t)$ and we obtain another transition representation of the evolution of $\psi(t)$ which we call the backward transition representation. In a manner similar to the case of the forward transition representation, the backward transition representation is directly associated with the Lax-Phillips incoming (spectral or translation) representations. It is evident from the structure of the Lax-Phillips theory that such transition representations are useful for the description of transient phenomena in scattering processes, such as resonances.

Turning to the quantum mechanical case we may define forward and backward transition representations analogous to those defined in the Lax-Phillips theory using the following two propositions, proved in Ref. [2]:

**Proposition 1** For $\psi(t) = u_+(t)\psi$, $\psi \in L^2(R_+; K)$, $t \in R$, apply the following decomposition

$$\psi(t) = \psi_F^b(t) + \psi_F^f(t)$$  \hfill (25)

where

$$\psi_F^b(t) := \Lambda_F \psi(t), \quad \psi_F^f(t) := (I - \Lambda_F)\psi(t).$$

Then

$$\lim_{t \to -\infty} \|\psi(t) - \psi_F^b(t)\| = 0, \quad \lim_{t \to -\infty} \|\psi_F^f(t)\| = 0,$$

$$\lim_{t \to -\infty} \|\psi_F(t)\| = 0, \quad \lim_{t \to -\infty} \|\psi(t) - \psi_F(t)\| = 0.$$
Proposition 2 For $\psi(t) = u_+(t)\psi$, $\psi \in L^2(\mathbb{R}_+; K)$, $t \in \mathbb{R}$, apply the following decomposition

$$\psi(t) = \psi_B^b(t) + \psi_B^f(t)$$

(26)

where

$$\psi_B^b(t) := (I - \Lambda_B)\psi(t), \quad \psi_B^f(t) := \Lambda_B\psi(t).$$

Then

$$\lim_{t \to -\infty} \|\psi(t) - \psi_B^b(t)\| = 0, \quad \lim_{t \to -\infty} \|\psi_B^b(t)\| = 0,$$

$$\lim_{t \to -\infty} \|\psi_B^f(t)\| = 0, \quad \lim_{t \to -\infty} \|\psi(t) - \psi_B^f(t)\| = 0.$$

Proposition 2 states that $\psi(t)$ can be decomposed into a sum of two components, $\psi_B^b(t)$ and $\psi_B^f(t)$ such that $\psi_B^b(t)$ is a backward asymptotic component and $\psi_B^f(t)$ is a forward asymptotic component of $\psi(t)$. Via the decomposition in Eq. (26) the evolution of $\psi(t)$ is represented as a transition from the backward asymptotic component to the forward asymptotic component and we obtain a transition representation of the evolution which, by the fact that the decomposition is defined using the (square root of the) forward Lyapunov operator, is a forward transition representation. By Proposition 2 we have a different decomposition of $\psi(t)$ into a backward asymptotic component $\psi_B^b(t)$ and a forward asymptotic component $\psi_B^f(t)$. The resulting transition representation of the evolution of $\psi(t)$ in this case is a backward transition representation, i.e., the decomposition of the evolving state $\psi(t)$ into the two components in Eq. (26) is achieved via the use of the backward Lyapunov operator.

Consider a scattering problem satisfying assumptions (i)-(ii) above. Defining the forward Lyapunov operator $M_+$ for the scattering problem as in Eq. (16) and noting that $\Lambda_+ = (\hat{W}^{QM})^{-1}M_+^{1/2}\hat{W}^{QM}$ we immediately obtain, using proposition 1 a forward transition representation for the quantum evolution. The formal definition of this representation is:

Definition 3 (forward transition representation) Let $\Lambda_+ := M_+^{1/2}$ be the square root of $M_+$. For any $\psi \in \mathcal{H}_{ac}$ the forward transition representation of the evolution of $\psi$ is defined to be the decomposition

$$\psi(t) = \psi_+^b(t) + \psi_+^f(t),$$

where $\psi_+^b(t) := \Lambda_+\psi(t)$, $\psi_+^f(t) := (I - \Lambda_+)\psi(t)$, $\psi(t) = U(t)\psi$ and $U(t) = \exp(-iHt)$ is the Schrödinger evolution in $\mathcal{H}$.

The asymptotic behavior over time of the two components $\psi_+^b(t)$, $\psi_+^f(t)$ of $\psi(t)$ follows directly from Proposition 1. The backward transition representation is defined in a similar manner following the definition of the backward Lyapunov operator $M_-$ for the scattering problem in Eq. (17) and the fact that $\Lambda_- = (\hat{W}^{QM})^{-1}M_-^{1/2}\hat{W}^{QM}$:

Definition 4 (backward transition representation) Let $\Lambda_- := M_-^{1/2}$ be the square root of $M_-$. For any $\psi \in \mathcal{H}_{ac}$ the backward transition representation of the evolution of $\psi$ is defined to be the decomposition

$$\psi(t) = \psi_-^b(t) + \psi_-^f(t),$$

where $\psi_-^b(t) := (I - \Lambda_-)\psi(t)$, $\psi_-^f(t) := \Lambda_-\psi(t)$, $\psi(t) = U(t)\psi$ and $U(t) = \exp(-iHt)$ is the Schrödinger evolution in $\mathcal{H}$.
Of course, the asymptotic behavior over time of the two components \( \psi_b(t) \), \( \psi_f(t) \) of the backward transition representation follows directly from Proposition 2.

By defining the two transition representations in Definitions 3 and 4 we complete the construction within the framework of quantum mechanics of objects and representations analogous to the central objects and representations of the Lax-Phillips theory. We may then extend Eq. (8) as follows:

\[
\begin{align*}
\text{LP scattering theory} & \quad U(t) = e^{-iKt} \\
\text{QM scattering theory} & \quad U(t) = e^{-iHt} \\
\psi(t) = P_+ \psi(t) + P_\perp \psi(t) & \quad \psi(t) = \Lambda_+ \psi(t) + (I - \Lambda_+) \psi(t) \\
\psi(t) = P_\perp \psi(t) + P_- \psi(t) & \quad \psi(t) = (I - \Lambda_-) \psi(t) + \Lambda_- \psi(t) \\
Z_{\text{LP}}(t) = P_+ U(t) P_- & \quad t \geq 0 \\
\hat{S}_{\text{LP}}(E), E \in \mathbb{R} & \quad \hat{S}_{\text{QM}}(E), E \in \mathbb{R}^+ 
\end{align*}
\]

(27)

We remark that the forward transition representation, corresponding to the decomposition on the right hand side of the third line in Eq. (27) above, has already been used successfully in Ref. [S2] for the description of quantum mechanical resonance scattering processes and results in a clear separation of the outgoing probability waves from the incoming waves and the formation of a resonance (the forward transition representation is called in Ref. [S2] the outgoing forward transition representation).

3 Resonance poles, resonance states and the approximate Lax-Phillips semigroup in quantum mechanical scattering

Upon completion of the set of relations in Eq. (27) we are left with an important task, i.e., to establish in the context of quantum mechanical scattering a theorem analogous to Theorem 3 relating resonance poles of the Lax-Phillips scattering matrix to eigenvalues and eigenvectors of the Lax-Phillips semigroup. Hence, an appropriate definition of resonance states and investigation of their relation to the approximate Lax-Phillips semigroup, defined in the previous section, is a central ingredient in the development of the formalism introduced in the present work. Of course, we do not expect to define resonance states as exact eigenvectors of elements \( Z_{\text{app}}(t) \) of the approximate Lax-Phillips semigroup since, as its name suggests, it is not an exact semigroup. However, we may try to find resonance states, associated with resonance poles of the quantum mechanical scattering matrix, which are eigenvectors of \( Z_{\text{app}}(t) \) in some approximate sense and make an effort to quantify the quality of such an approximation.

The problem of the definition of appropriate resonance states corresponding to resonance poles of the scattering matrix in quantum mechanical scattering has been addressed in the context of the recent development of the formalism of semigroup decomposition of resonance evolution [S3, S4, SHV] (of course, there are several other formalisms for dealing with the problem of scattering resonances in quantum mechanics, notably complex scaling [AC, BC, Sim1, Sim2, Hum, SZ, HS] and rigged Hilbert spaces [BaSch, Baum, BG, HoSi, PGS]. Here we consider the framework most suitable, in terms of its mathematical constructions, for the development of the formalism introduced in the present paper). The semigroup decomposition formalism utilizes basic mathematical constructions of the Lax-Phillips and the Sz.-Nagy-Foias theory for the formulation of a time dependent theory for the description of the evolution of scattering resonances in quantum mechanics. Significant progress has been achieved in the development of the structure of
the general formalism under certain simplifying assumptions which we continue to apply in the present paper. Thus, we add to assumptions (i)-(ii) above the assumptions

(iii) The absolutely continuous spectrum of the free Hamiltonian $H_0$ and the full Hamiltonian $H$ is simple, i.e., the multiplicity of the absolutely continuous spectrum is one.

(iv) Denote by $C^+$, respectively by $C^-$ the upper and lower half-planes of the complex plane $C$. We assume that the $S$-matrix in the energy representation, denoted by $S_{QM}(\cdot)$, has an extension into a function $\hat{S}_{QM}(\cdot)$, holomorphic in some region $\Sigma^+ \subset C^+$ above the positive real axis $R^+$ and having an analytic continuation across $R^+$ into a region $\Sigma^- \subset C^-$ such that the resulting analytically continued function, again denoted by $\hat{S}_{QM}(\cdot)$, is meromorphic in an open, simply connected region $\Sigma = \Sigma^+ \cup \Sigma^- \cup (\Sigma \cap R^+)$. We assume that $\hat{S}_{QM}(\cdot)$ has a single, simple, resonance pole at a point $z = \mu \in \Sigma^-$ and no other singularity in $\Sigma$ (the closure of $\Sigma$).

We emphasize that the semigroup decomposition formalism may be applied under much less stringent conditions than those assumed here. However, conditions (iii)-(iv) make the discussion below more transparent and, in fact, facilitate the development of the formalism in the present paper.

It is shown in Ref. \[S2\] that if we apply the operator $\Lambda_\mu$ to any state $\psi \in H_{ac}$ then the resonance pole of the $S$-matrix $\hat{S}_{QM}(\cdot)$ at $z = \mu$ (see assumption (iv) above) induces a decomposition of the state $\psi_{\Lambda_\mu} := \Lambda_\mu \psi$ of the form

$$\psi_{\Lambda_\mu} = \Lambda_\mu \psi = b(\psi; \mu) + (\psi_{\mu}^{app}, \psi)\|\psi_{\mu}^{res}\|^2 \psi_{\mu}^{res}. \quad (28)$$

The state $\psi_{\mu}^{app} \in H_{ac}$ is referred to as an approximate resonance state and the state $\psi_{\mu}^{res} \in H_{ac}$ is referred to as the resonance state corresponding to the resonance pole at $z = \mu$. Note that in Ref. \[S2\] the resonance state $\psi_{\mu}^{res}$ is denoted by $\psi_{\mu}^{ir}$. The states $\psi_{\mu}^{app}$ and $\psi_{\mu}^{res}$ are related. In fact, we have

$$\psi_{\mu}^{app} = \Lambda_\mu \psi_{\mu}^{res}.$$

Thus we may write equation (28) in the form

$$\psi_{\Lambda_\mu} = \Lambda_\mu \psi = b(\psi; \mu) + (\psi_{\mu}^{app}, \psi)\|\psi_{\mu}^{res}\|^2 \psi_{\mu}^{res} = b(\psi; \mu) + (\Lambda_\mu \psi_{\mu}^{res}, \psi)\|\psi_{\mu}^{res}\|^2 \psi_{\mu}^{res} = b(\psi; \mu) + (\psi_{\mu}^{res} \Lambda_\mu, \psi_{\mu}^{res})\|\psi_{\mu}^{res}\|^2 \psi_{\mu}^{res} = b(\psi; \mu) + P_{res} \psi_{\Lambda_\mu}, \quad (29)$$

where $P_{res}$ is the projection on the subspace $H_{res} := P_{res} H_{ac} \subset H_{ac}$ spanned by the resonance state $\psi_{\mu}^{res}$. We emphasize again that the decomposition in Eq. (28) or Eq. (29) is not arbitrary but naturally induced by the existence of the resonance pole of the scattering matrix $\hat{S}_{QM}(\cdot)$. It is shown furthermore in Ref. \[S2\] that

$$Z_+(t) \psi_{\mu}^{res} = e^{-i\mu t} \psi_{\mu}^{res}, \quad t \geq 0. \quad (30)$$

where $Z_+(t)$ are elements of the semigroup $\{ Z_+(t) \}_{t \geq 0}$, appearing on the right hand side of Eq. (21). Now define a linear subspace $(H_{ac})_{\Lambda_\mu} := \Lambda_\mu H_{ac}$. Since $\Lambda_\mu$ is injective and since Ran $\Lambda_\mu$ is dense in $H_{ac}$ we have that $(H_{ac})_{\Lambda_\mu}$ is a dense linear subspace of $H_{ac}$ and for any state $\varphi \in (H_{ac})_{\Lambda_\mu}$ the state $\bar{\varphi} := \Lambda_\mu^{-1} \varphi$ is well defined in $H_{ac}$. Taking arbitrary states $\varphi \in (H_{ac})_{\Lambda_\mu}$, $\psi \in H_{ac}$ we may use Eqs. (28), (21) and (30) to obtain

$$(\varphi, U(t) \psi) = (\Lambda_\mu \bar{\varphi}, U(t) \psi) = (\bar{\varphi}, \Lambda_\mu U(t) \psi) = (\bar{\varphi}, Z_+(t) \Lambda_\mu \psi) = (\bar{\varphi}, Z_+(t) \psi_{\Lambda_\mu}) = (\bar{\varphi}, Z_+(t)[b(\psi; \mu) + P_{res} \psi_{\Lambda_\mu}]) = B(\varphi, \psi, \mu, t) + (\bar{\varphi}, Z_{QM}(t) \psi_{\Lambda_\mu}), \quad t \geq 0. \quad (31)$$
where \( B(\varphi, \psi, \mu, t) \) and \( Z_{QM}(t) \) are defined in Eqs. (21) and (22), respectively. Since \( P_{res} \) is a projection on a subspace of eigenvalues of \( Z_{+}(t) \), the family of operators \( \{Z_{QM}(t)^{1/2}\}_{t \geq 0} \) annihilates \( \mathcal{H}_{ac} \) and forms a continuous, one parameter, contractive semigroup on the subspace \( \mathcal{H}_{res} \). The right hand side of Eq. (31) is the semigroup decomposition of the matrix element \((\varphi, U(t)\psi)\) of the Schrödinger evolution for \( t \geq 0 \).

Using the fact that, for \( t \geq 0 \), we have \( Z_{QM}(t) = Z_{+}(t)P_{res} = e^{-itP_{res}} \) we can obtain from Eq. (31)

\[
(\varphi, U(t)\psi) = B(\varphi, \psi, \mu, t) + (\varphi, Z_{QM}(t)\psi_{\Lambda_{+}}) = B(\varphi, \psi, \mu, t) + (\varphi, P_{res}\psi_{\Lambda_{+}}) e^{-itP_{res}}, \quad t \geq 0,
\]

and if we insert on the right hand side the explicit form of the projection \( P_{res} \) we obtain

\[
(\varphi, U(t)\psi) = B(\varphi, \psi, \mu, t) + \left|\psi_{\mu}^{res}\right|^{-2} (\varphi, \psi_{\mu}^{res}) (\psi_{\mu}^{res}, \psi_{\Lambda_{+}}) e^{-itP_{res}} = B(\varphi, \psi, \mu, t) + \left|\psi_{\mu}^{res}\right|^{-2} (\varphi, \psi_{\mu}^{res}) (\psi_{\mu}^{app}, \psi) e^{-itP_{res}}, \quad t \geq 0.
\]

This is the form of the semigroup decomposition appearing in Ref. [S2]. The term \( B(\varphi, \psi, \mu, t) \) is a background term and the second term on the right hand side is the resonance term. The resonance state \( \psi_{\mu}^{res} \), begin an eigenstate of the generator of the semigroup \( \{Z_{+}(t)\}_{t \geq 0} \), determines the time evolution of the resonance term. Note that if the state \( \psi \) is chosen to be orthogonal to \( \psi_{\mu}^{app} \) then the resonance term in Eq. (32) vanishes. Hence the state \( \psi_{\mu}^{app} \) is directly associated with the appearance of the resonance contribution on the right hand side of Eq. (32). The reference to \( \psi_{\mu}^{app} \) as an approximate resonance state follows from the fact that it can be shown that there is no choice of \( \varphi \) and \( \psi \) in Eq. (32) for which we obtain a pure resonance behavior, i.e., there is no choice of \( \varphi \) and \( \psi \) for which the background term disappears. In fact, it can be shown that \( \psi_{\mu}^{res} \in \mathcal{H}_{ac} \setminus \{0\} \), i.e., \( \psi_{\mu}^{res} \) is not in the range of \( \Lambda_{+} \), the term \( b(\psi, \mu) = (I - P_{res})\psi_{\Lambda_{+}} \) on the right hand side of Eq. (29) cannot disappear and the background term \( B(\varphi, \psi, \mu, t) \) cannot be identically zero for any choice of \( \varphi \) and \( \psi \) (see Ref. [S4]).

It is of particular interest to apply Eq. (32) to \( \psi_{\mu}^{app} \). If we set in Eq. (32) \( \varphi = \psi = \psi_{\mu}^{app} \) and use the fact that \( \psi_{\mu}^{res} = \Lambda_{+}^{-1} \psi_{\mu}^{app} \) we get

\[
\frac{\langle \psi_{\mu}^{app}, U(t)\psi_{\mu}^{app} \rangle}{\left|\psi_{\mu}^{app}\right|^2} = B_{\mu}(t) + e^{-itP_{res}},
\]

where \( B_{\mu}(t) := \left|\psi_{\mu}^{app}\right|^2 B(\psi_{\mu}^{app}, \psi_{\mu}^{app}, \mu, t) \). It is shown in Ref. [SHV] that in this case it is possible to obtain an estimate on the background term of the form

\[
|B_{\mu}(t)| \leq \left( \frac{\left|\psi_{\mu}^{res}\right|^4}{\left|\psi_{\mu}^{app}\right|^2} - 1 \right)^{1/2}, \quad t \geq 0.
\]

(note that since \( \psi_{\mu}^{app} = \Lambda_{+} \psi_{\mu}^{res} \) and since \( \Lambda_{+} \) is contractive we always have \( \left|\psi_{\mu}^{app}\right| \leq \left|\psi_{\mu}^{res}\right| \)). For sharp (non-threshold) resonances, i.e., for resonance poles close to the real axis in the complex energy plane, the right hand side of the above inequality is small and hence the background term is small (see Ref. [S4]).

Next, consider the approximate Lax-Phillips semigroup. By the intertwining relation in Eq. (21) we have

\[
Z_{app}(t) = \Lambda_{+} U(t) \Lambda_{-} = Z_{+}(t) \Lambda_{+} \Lambda_{-} = Z_{+}(t) P_{res} + Z_{+}(t)(\Lambda_{+} \Lambda_{-} - P_{res}) =
= Z_{QM}(t) + Z_{+}(t)(\Lambda_{+} \Lambda_{-} - P_{res}).
\]

Applying the approximate Lax-Phillips semigroup to the resonance state \( \psi_{\mu}^{res} \) we obtain

\[
Z_{app}(t)\psi_{\mu}^{res} = Z_{QM}(t)\psi_{\mu}^{res} + Z_{+}(t)(\Lambda_{+} \Lambda_{-} - P_{res})\psi_{\mu}^{res} =
= e^{-itP_{res}}\psi_{\mu}^{res} + Z_{+}(t)(\Lambda_{+} \Lambda_{-} - P_{res})\psi_{\mu}^{res} = e^{-itP_{res}}\psi_{\mu}^{res} + Z_{+}(t)(\Lambda_{+} \Lambda_{-} \psi_{\mu}^{res} - \psi_{\mu}^{res}),
\]

(33)
so that
\[ \| Z_{\text{app}}(t) \tilde{\psi}_{\mu}^{\text{res}} - e^{-i\mu t} \tilde{\psi}_{\mu}^{\text{res}} \| \leq \| \tilde{\psi}_{\mu}^{\text{res}} - \Lambda_+ \Lambda_- \tilde{\psi}_{\mu}^{\text{res}} \|, \]
where \( \tilde{\psi}_{\mu}^{\text{res}} = \| \psi_{\mu}^{\text{res}} \|^{-1} \psi_{\mu}^{\text{res}} \) is the normalized resonance state. We can now state our main result for this section

**Theorem 8** Assume a scattering problem for which conditions (i)-(iv) hold and let \( \psi_{\mu}^{\text{res}} \), \( \tilde{\psi}_{\mu}^{\text{res}} \) and \( \psi_{\mu}^{\text{app}} \) be as above. Let \( \tilde{W}_\mu^{\text{QM}} : H_\mu \to L^2(\mathbb{R}^+) \) be the mapping to the outgoing energy representation for the problem and let \( \psi_{\mu,+}^{\text{app}} \in L^2(\mathbb{R}^+) \) be given by \( \psi_{\mu,+}^{\text{app}}(E) = [\tilde{W}_\mu^{\text{QM}} \psi_{\mu}^{\text{app}}](E) \). Then we have the estimate

\[ \| \tilde{\psi}_{\mu}^{\text{res}} - \Lambda_+ \Lambda_- \tilde{\psi}_{\mu}^{\text{res}} \| \leq C \left( 1 - \frac{\| \psi_{\mu}^{\text{app}} \|^2}{\| \tilde{\psi}_{\mu}^{\text{res}} \|^2} \right)^{1/2} + \left( \int_0^\infty dE \left| 1 - \frac{E - \mu}{E - \mu_{\text{QM}}(E)} \right|^2 \frac{\| \psi_{\mu,+}^{\text{app}}(E) \|^2}{\| \tilde{\psi}_{\mu}^{\text{res}} \|^2} \right)^{1/2}. \]

\[ (35) \]

Let us consider the two terms on the right hand side of Eq. \( (35) \). In accord with the remark below Eq. \( (33) \), if the resonance pole is at \( z = \mu \) in the complex energy plane and if the resonance is narrow, i.e., if \( |\text{Im} \mu| \ll 1 \), then we have

\[ 1 - \frac{\| \psi_{\mu}^{\text{app}} \|^2}{\| \tilde{\psi}_{\mu}^{\text{res}} \|^2} \ll 1. \]

Note that the first term on the right hand side of Eq. \( (35) \) is small exactly when the estimate on the size of the background term in the evolution of the survival amplitude of \( \psi_{\mu}^{\text{app}} \) in Eq. \( (33) \) is also small. We see that the first term on the right hand side of Eq. \( (35) \) measures the proximity of the resonance pole to the real axis, associated with the sharpness of the resonance at \( z = \mu \).

Turning to the second term on the right hand side of Eq. \( (35) \), we recall that our assumption is that the S-matrix \( \hat{S}_{\text{QM}}(\cdot) \) has a simple pole at \( z = \mu \). This leads us naturally to express \( \hat{S}_{\text{QM}}(E) \) in the form \( \hat{S}_{\text{QM}}(E) = \frac{E - \mu}{E - \mu_{\text{QM}}(E)} \hat{S}_1(E) \). If the S-matrix \( \hat{S}_{\text{QM}}(E) \) were purely rational, i.e., \( \hat{S}_{\text{QM}}(E) = \frac{E - \mu}{E - \mu_{\text{QM}}(E)} \) (as is the case in the pure Lax-Phillips theory), we would have \( 1 - \frac{E - \mu_{\text{QM}}(E)}{E - \mu} \hat{S}_{\text{QM}}(E) = 0 \) and then the second term on the right hand side of Eq. \( (35) \) would vanish. We conclude that the factor \( |1 - \frac{E - \mu}{E - \mu_{\text{QM}}(E)} \hat{S}_{\text{QM}}(E)| \) gives a measure of the deviation of the phase shift corresponding to the actual resonance at \( z = \mu \) from the phase shift of an ideal, Breit-Wigner shaped, resonance associated with a purely rational S-matrix. In the second term on the right hand side of Eq. \( (35) \), the factor \( |1 - \frac{E - \mu}{E - \mu_{\text{QM}}(E)} \hat{S}_{\text{QM}}(E)| \) is multiplied by the probability density function \( \| \psi_{\mu}^{\text{app}} \|^2 \| \psi_{\mu,+}^{\text{app}}(E) \|^2 \) which has a peak at the energy of the resonance and if the resonance is narrow then this peak is rather sharp. Hence the multiplication with the energy probability density of \( \psi_{\mu}^{\text{app}} \) implies that the deviations of the factor \( |1 - \frac{E - \mu}{E - \mu_{\text{QM}}(E)} \hat{S}_{\text{QM}}(E)| \) from zero are evaluated in the vicinity of the resonance energy. We conclude that the second term on the right hand side of Eq. \( (35) \) measures the deviation of the phase shifts of the resonance at \( z = \mu \) from the phase shifts of an ideal resonance.

To summarize, to the resonance pole of the scattering matrix \( \hat{S}_{\text{QM}}(\cdot) \) at the point \( z = \mu \) we assign a resonance state \( \tilde{\psi}_{\mu}^{\text{res}} \). If the right hand side of the inequality in Eq. \( (35) \) is small then \( \tilde{\psi}_{\mu}^{\text{res}} \) is a good approximation to an eigenstate of the elements of the approximate Lax-Phillips semigroup \( \{ Z_{\text{app}}(t) \}_{t \geq 0} \) with eigenvalue \( e^{-i\mu t} \). Thus, in the context of quantum mechanical resonance scattering and under the conditions that both
terms on the right hand side of Eq. (35) are small, i.e., the resonance is sharp and exhibits
phase shift which is close to that of an ideal resonance, Eq. (34) and Theorem 3 taken
together, provide a result analogous to Theorem 3 of the Lax-Phillips theory.

4 Proof of Theorem 8

Let \( \hat{W}_+^{QM} : H_{ac} \rightarrow L^2(\mathbb{R}^+, K) \) be the mapping to the outgoing energy representation
 corresponding to the scattering problem considered in Theorem 8. As mentioned in Section 2
explicit expression for the mapping \( W_+^{QM} : H_{ac} \rightarrow L^2(\mathbb{R}^+, K) \) is obtained by finding a
complete set of outgoing solutions of the Lippmann-Schwinger equation. In this section
we shall utilize the Dirac notation and denote an outgoing solution of the Lippmann-Schwinger
equation corresponding to energy \( E \in \mathbb{R}^+ \) by \( |E^-\rangle \) (recall that we assume that
the a.c. spectrum is simple, i.e., the multiplicity of the a.c. spectrum is one, so that there are
no degeneracy indices for the spectrum). The complete set of outgoing solutions is
then \( \{|E^-\rangle\}_{E \in \mathbb{R}^+} \). Similarly, If \( \hat{W}_-^{QM} : H_{ac} \rightarrow L^2(\mathbb{R}^+, K) \) is the mapping to the incoming
energy representation then an explicit expression for this mapping is obtained by finding
a complete set of incoming solutions of the Lippmann-Schwinger equation. Again we use the
Dirac notation and denote an incoming solution of the Lippmann-Schwinger equation
 corresponding to energy \( E \in \mathbb{R}^+ \) by \( |E^+\rangle \). The complete set of incoming solutions is then
\( \{|E^+\rangle\}_{E \in \mathbb{R}^+} \).

Recall that \( \Lambda_+ = M_+^{1/2} = (\hat{W}_+^{QM})^{-1} \Lambda_F (\hat{W}_+^{QM}) \) and \( \Lambda_- = M_-^{1/2} = (\hat{W}_-^{QM})^{-1} \Lambda_B (\hat{W}_-^{QM}) \).
We then have

\[
\langle E^- | \Lambda_+ \Lambda_- | E'^- \rangle = \int_{0}^{\infty} dE_1 \langle E^- | \Lambda_+ | E_1^- \rangle \langle E_1^- | \Lambda_- | E'^- \rangle =
\]

\[
= \int_{0}^{\infty} dE_1 \langle E^- | (\hat{W}_+^{QM})^{-1} \Lambda_F \hat{W}_+^{QM} | E_1^- \rangle \langle E_1^- | (\hat{W}_-^{QM})^{-1} \Lambda_B \hat{W}_-^{QM} | E'^- \rangle =
\]

\[
= \int_{0}^{\infty} dE_1 \int_{0}^{\infty} dE_2 \int_{0}^{\infty} dE_3 \Lambda_F (E, E_1) \langle E_1^- | | E_2^+ \rangle \Lambda_B (E_2, E_3) \langle E_3^+ | E'^- \rangle.
\]

Now,

\[
\langle E_1^- | E_2^+ \rangle = S_{QM}(E_1) \delta(E_1 - E_2), \quad \langle E_3^+ | E'^- \rangle = S_{QM}(E_3) \delta(E_3 - E'),
\]

where \( S_{QM}(\cdot) \) is the S-matrix. Therefore,

\[
\langle E^- | \Lambda_+ \Lambda_- | E'^- \rangle = \int_{0}^{\infty} dE_1 \Lambda_F (E, E_1) S_{QM}(E_1) \Lambda_B (E_1, E') \hat{S}_{QM}(E').
\]

Applying this expression to \( \psi_{\mu}^{res} \) we get

\[
\langle E^- | \Lambda_+ \Lambda_- \psi_{\mu}^{res} \rangle = \int_{0}^{\infty} dE' \langle E^- | \Lambda_+ \Lambda_- | E'^- \rangle \langle E'^- | \psi_{\mu}^{res} \rangle =
\]

\[
= \int_{0}^{\infty} dE' \int_{0}^{\infty} dE_1 \Lambda_F (E, E_1) S_{QM}(E_1) \Lambda_B (E_1, E') \hat{S}_{QM}(E') \psi_{\mu,+}^{res}(E'), \quad (36)
\]

where \( \psi_{\mu,+}^{res}(E') := [\hat{W}_+^{QM} \psi_{\mu}^{res}](E) = \langle E' | \psi_{\mu}^{res} \rangle \). The transformation on the right hand
side of Eq. (36) is between different functional representations of \( H_{ac} \), but it is understood
that all of these transformations are acting in the function space $L^2(\mathbb{R}^+)$. In order to continue we write Eq. (36) in the form

$$
(E^-|\Lambda_1, \Lambda_-\psi_{\mu+}^{\text{res}}) = \int_0^\infty dE' \int_0^\infty dE_1 \Lambda_F(E, E_1)\hat{S}_\text{QM}(E_1)(\Lambda_B - I)(E_1, E')\hat{S}_\text{QM}^*(E')\psi_{\mu+}^{\text{res}}(E')
$$

$$
+ \int_0^\infty dE' \Lambda_F(E, E')\psi_{\mu+}^{\text{res}}(E') =
$$

$$
= \int_0^\infty dE_1 \Lambda_F(E, E_1)\hat{S}_\text{QM}(E_1)[(\Lambda_B - I)\hat{S}_\text{QM}^*(\psi_{\mu+}^{\text{res}})](E_1) + \int_0^\infty dE' \Lambda_F(E, E')\psi_{\mu+}^{\text{res}}(E') =
$$

$$
= [\Lambda_F\hat{S}_\text{QM}(\Lambda_B - I)\hat{S}_\text{QM}^*(\psi_{\mu+}^{\text{res}})](E) + [\Lambda_F\psi_{\mu+}^{\text{res}}](E).
$$

(37)

where $I \equiv I_{L^2(\mathbb{R}^+)}$ is the identity operator on $L^2(\mathbb{R}^+)$. We will show that under appropriate conditions the norm $\|(\Lambda_B - I)\hat{S}_\text{QM}^*(\psi_{\mu+}^{\text{res}})\|_{L^2(\mathbb{R}^+)}$ is small. Note first that by the positivity of $\Lambda_B$ we have the inequality

$$
(\varphi, (I + \Lambda_B)\varphi)_{L^2(\mathbb{R}^+)} \geq \|\varphi\|^2_{L^2(\mathbb{R}^+)}, \quad \forall \varphi \in L^2(\mathbb{R}^+),
$$

by which we obtain that

$$
\|\varphi\|^2_{L^2(\mathbb{R}^+)} = (\varphi, (I + \Lambda_B)^{-1}(I + \Lambda_B)\varphi)_{L^2(\mathbb{R}^+)} =
$$

$$
= ((I + \Lambda_B)^{-1/2}\varphi, (I + \Lambda_B)^{-1/2}(I + \Lambda_B)^{-1/2}(I + \Lambda_B)^{-1/2}\varphi)_{L^2(\mathbb{R}^+)} \geq ((I + \Lambda_B)^{-1/2}\varphi, (I + \Lambda_B)^{-1/2}\varphi)_{L^2(\mathbb{R}^+)} =
$$

$$
= (\varphi, (I + \Lambda_B)^{-1}\varphi)_{L^2(\mathbb{R}^+)} \geq (\varphi, \varphi)_{L^2(\mathbb{R}^+)}, \quad \forall \varphi \in L^2(\mathbb{R}^+).
$$

From this inequality we get that $\|(I + \Lambda_B)^{-1}\|_{L^2(\mathbb{R}^+)} \leq 1$. Noting that

$$
M_F + M_B = (P_{\mathbb{R}^+}, \hat{P}_{+}P_{\mathbb{R}^+})|_{L^2(\mathbb{R}^+)} + (P_{\mathbb{R}^+}, \hat{P}_{-}P_{\mathbb{R}^+})|_{L^2(\mathbb{R}^+)} = (P_{\mathbb{R}^+}, (\hat{P}_{+} + \hat{P}_{-})P_{\mathbb{R}^+})|_{L^2(\mathbb{R}^+)} = I,
$$

we have

$$
M_F = I - M_B = (I + \Lambda_B)(I - \Lambda_B) \implies I - \Lambda_B = (I + \Lambda_B)^{-1}M_F.
$$

Thus,

$$
\|(I - \Lambda_B)\hat{S}_{\text{QM}}^*(\psi_{\mu+}^{\text{res}})\|^2_{L^2(\mathbb{R}^+)} =
$$

$$
= \|(I - \Lambda_B)^{1/2}(I - \Lambda_B)^{1/2}\hat{S}_{\text{QM}}^*(\psi_{\mu+}^{\text{res}})\|^2_{L^2(\mathbb{R}^+)} \leq \|(I - \Lambda_B)^{1/2}\hat{S}_{\text{QM}}^*(\psi_{\mu+}^{\text{res}})\|^2_{L^2(\mathbb{R}^+)} =
$$

$$
= (\hat{S}_{\text{QM}}^*(\psi_{\mu+}^{\text{res}}, (I - \Lambda_B)\hat{S}_{\text{QM}}^*(\psi_{\mu+}^{\text{res}}))L^2(\mathbb{R}^+) = (\hat{S}_{\text{QM}}^*(\psi_{\mu+}^{\text{res}}, (I + \Lambda_B)^{-1}M_F\hat{S}_{\text{QM}}^*(\psi_{\mu+}^{\text{res}}))L^2(\mathbb{R}^+) =
$$

$$
= (\hat{S}_{\text{QM}}^*(\psi_{\mu+}^{\text{res}}, (I + \Lambda_B)^{-1}\Lambda_F^*\hat{S}_{\text{QM}}^*(\psi_{\mu+}^{\text{res}}))L^2(\mathbb{R}^+) =
$$

$$
= (\hat{S}_{\text{QM}}^*(\psi_{\mu+}^{\text{res}}, (I + \Lambda_B)^{-1}M_F\hat{S}_{\text{QM}}^*(\psi_{\mu+}^{\text{res}}))L^2(\mathbb{R}^+) = (\hat{S}_{\text{QM}}^*(\psi_{\mu+}^{\text{res}}, M_F\hat{S}_{\text{QM}}^*(\psi_{\mu+}^{\text{res}}))L^2(\mathbb{R}^+) =
$$

In the derivation of this inequality we have used the fact that $[(I - \Lambda_B), \Lambda_F] = [I - (I - M_F)^{1/2}, M_F^{1/2}] = 0$. To summarize, we have

$$
\|(I - \Lambda_B)\hat{S}_{\text{QM}}^*(\psi_{\mu+}^{\text{res}})\|^2_{L^2(\mathbb{R}^+)} \leq (\hat{S}_{\text{QM}}^*(\psi_{\mu+}^{\text{res}}, M_F\hat{S}_{\text{QM}}^*(\psi_{\mu+}^{\text{res}}))L^2(\mathbb{R}^+) = (38)
$$

Plugging the definition $M_F = P_{\mathbb{R}^+}, \hat{P}_{+}P_{\mathbb{R}^+}$ into the inequality in Eq. (38) we obtain

$$
\|(I - \Lambda_B)\hat{S}_{\text{QM}}^*(\psi_{\mu+}^{\text{res}})\|^2_{L^2(\mathbb{R}^+)} \leq (\hat{S}_{\text{QM}}^*(\psi_{\mu+}^{\text{res}}, M_F\hat{S}_{\text{QM}}^*(\psi_{\mu+}^{\text{res}})) =
$$

$$
= (\hat{S}_{\text{QM}}^*(\psi_{\mu+}^{\text{res}}, \hat{P}_{+}P_{\mathbb{R}^+}, \hat{S}_{\text{QM}}^*(\psi_{\mu+}^{\text{res}})) = (\hat{S}_{\text{QM}}^*(\psi_{\mu+}^{\text{res}}, \hat{P}_{+}P_{\mathbb{R}^+}, \hat{S}_{\text{QM}}^*(\psi_{\mu+}^{\text{res}})).
$$

(39)
It is shown in Ref. [SHV] that
\[ \psi_{\mu}^{\text{app}} = \Lambda_+ \psi_{\mu}^{\text{res}} = \int_0^\infty dE \frac{1}{E - \mu} |E^-|. \]
so that in the outgoing energy representation we have \( \psi_{\mu, +}^{\text{app}}(E) = (E^- | \psi_{\mu}^{\text{app}}) = \frac{1}{E - \mu}. \)
Define
\[ \psi_{\mu}^{\text{app}} = \int_0^\infty dE \frac{1}{E - \mu} |E^+| = \int_0^\infty dE |E^+| \psi_{\mu, -}^{\text{app}}(E), \]
where \( \psi_{\mu}^{\text{app}} := (E - \mu)^{-1}. \) Representing \( \psi_{\mu}^{\text{app}} \) in the outgoing energy representation we have
\[ \psi_{\mu}^{\text{app}}(E) := (E^- | \psi_{\mu}^{\text{app}}) = \int_0^\infty dE' \frac{1}{E' - \mu} (E^- | E'^+) = \hat{\mathcal{S}}_{\text{QM}}(E) \psi_{\mu}^{\text{app}}(E). \]

Then, using Eq. (39), we have
\[ \| (I - \lambda B) \hat{\mathcal{S}}_{\text{QM}} \psi_{\mu, +}^{\text{res}} \|_{L^2(\mathbb{R}_+)} \leq \| \hat{\mathcal{P}} + \hat{\mathcal{S}}_{\text{QM}} \psi_{\mu, +}^{\text{res}} \|_{L^2(\mathbb{R}_+)} \]
\[ \leq \| \hat{\mathcal{P}} + \hat{\mathcal{S}}_{\text{QM}} (\psi_{\mu, +}^{\text{res}} - \psi_{\mu}^{\text{app}}) \|_{L^2(\mathbb{R}_+)} + \| \hat{\mathcal{P}} + \hat{\mathcal{S}}_{\text{QM}} \psi_{\mu, +}^{\text{res}} \|_{L^2(\mathbb{R}_+)} \]
\[ \leq \| \psi_{\mu, +}^{\text{res}} \|_{L^2(\mathbb{R}_+)} + \| \psi_{\mu}^{\text{app}} \|_{L^2(\mathbb{R}_+)} + \| \hat{\mathcal{P}} + \hat{\mathcal{S}}_{\text{QM}} \psi_{\mu, +}^{\text{app}} \|_{L^2(\mathbb{R}_+)} \]
\[ = \| \psi_{\mu}^{\text{res}} + \Lambda_+ \psi_{\mu}^{\text{app}} - \Lambda_+ \psi_{\mu}^{\text{app}} \|_{L^2(\mathbb{R}_+)} = \| \psi_{\mu}^{\text{res}} - \Lambda_+ \psi_{\mu}^{\text{app}} \|. \]

We shall find an appropriate bound for each term on the right hand side of Eq. (40). First, we have
\[ \| \psi_{\mu, +}^{\text{res}} - \psi_{\mu}^{\text{app}} \|_{L^2(\mathbb{R}_+)} = \| \psi_{\mu}^{\text{res}} - \psi_{\mu}^{\text{app}} \| \leq \| (I + \lambda \Lambda) (\psi_{\mu}^{\text{res}} - \psi_{\mu}^{\text{app}}) \| = \| \psi_{\mu}^{\text{res}} + \Lambda_+ \psi_{\mu}^{\text{app}} - \Lambda_+ \psi_{\mu}^{\text{app}} \| = \| \psi_{\mu}^{\text{res}} + \Lambda_+ \psi_{\mu}^{\text{app}} - \Lambda_+ \psi_{\mu}^{\text{app}} \| = \| \psi_{\mu}^{\text{res}} - \Lambda_+ \psi_{\mu}^{\text{app}} \|. \]

Using Eq. (29) we get
\[ \Lambda_+ \psi_{\mu}^{\text{app}} = b(\psi_{\mu}^{\text{app}}; \mu) + (\Lambda_+ \psi_{\mu}^{\text{app}}) \| \psi_{\mu}^{\text{res}} \|^{-2} \psi_{\mu}^{\text{res}} = b(\psi_{\mu}^{\text{app}}; \mu) + \| \psi_{\mu}^{\text{app}} \|^2 \| \psi_{\mu}^{\text{res}} \|^2 \]
so that
\[ \| \psi_{\mu, +}^{\text{res}} - \psi_{\mu}^{\text{app}} \|_{L^2(\mathbb{R}_+)} \leq \| \psi_{\mu}^{\text{res}} - \Lambda_+ \psi_{\mu}^{\text{app}} \| = \| \psi_{\mu}^{\text{res}} - \Lambda_+ \psi_{\mu}^{\text{app}} \| = \left( 1 - \frac{\| \psi_{\mu}^{\text{app}} \|^2}{\| \psi_{\mu}^{\text{res}} \|^2} \right) \| \psi_{\mu}^{\text{res}} \| + \| b(\psi_{\mu}^{\text{app}}; \mu) \|. \]

By Eq. (29) and by the orthogonality of \( b(\psi_{\mu}^{\text{app}}; \mu) \) and \( \psi_{\mu}^{\text{res}} \) we obtain
\[ \| \psi_{\mu}^{\text{res}} \|^2 \geq \| \psi_{\mu}^{\text{app}} \|^2 \geq \| \Lambda_+ \psi_{\mu}^{\text{app}} \|^2 = \| b(\psi_{\mu}^{\text{app}}; \mu) \|^2 + \| \psi_{\mu}^{\text{app}} \|^2 \| \psi_{\mu}^{\text{res}} \|^2. \]

Hence
\[ \| b(\psi_{\mu}^{\text{app}}; \mu) \|^2 \leq \left( 1 - \frac{\| \psi_{\mu}^{\text{app}} \|^4}{\| \psi_{\mu}^{\text{res}} \|^4} \right) \| \psi_{\mu}^{\text{res}} \|^2 \leq 2 \left( 1 - \frac{\| \psi_{\mu}^{\text{app}} \|^2}{\| \psi_{\mu}^{\text{res}} \|^2} \right) \| \psi_{\mu}^{\text{res}} \|^2. \]
and finally
\[
\| \psi_{\mu,+}^{res} - \psi_{\mu,+}^{app} \|_{L^2(\mathbb{R}_+)} \leq \left( 1 - \frac{\| \psi_{\mu,+}^{app} \|^2}{\| \psi_{\mu,+}^{res} \|^2} \right) \| \psi_{\mu,+}^{res} \| + \sqrt{2} \left( 1 - \frac{\| \psi_{\mu,+}^{app} \|^2}{\| \psi_{\mu,+}^{res} \|^2} \right)^{1/2} \| \psi_{\mu,+}^{res} \|
\]
\[
\leq (1 + \sqrt{2}) \left( 1 - \frac{\| \psi_{\mu,+}^{app} \|^2}{\| \psi_{\mu,+}^{res} \|^2} \right)^{1/2} \| \psi_{\mu,+}^{res} \|. \tag{41}
\]

Next we obtain a convenient expression for the second term on the right hand side of Eq. (40). For this we write the S-matrix \( \hat{S}_{QM}(E) \) in the form
\[
\hat{S}_{QM}(E) = \frac{E - \mu}{E - \mu} \hat{S}_1(E),
\]
and, according to the assumptions of Theorem 8, \( \hat{S}_1(E) \) does not have a pole at \( E = \mu \). We then have,
\[
\hat{S}_{QM}(E) \psi_{\mu,+}^{app}(E) = \frac{E - \mu}{E - \mu} \hat{S}_1(E) \frac{1}{E - \mu} = \frac{1}{E - \mu} = \hat{S}_1(E) \psi_{\mu,+}^{app}(E),
\]
Hence we get that
\[
\| \psi_{\mu,+}^{app} - \hat{S}_{QM} \psi_{\mu,+}^{app} \|_{L^2(\mathbb{R}_+)} = \left( \int_0^\infty dE \left| (1 - \hat{S}_1(E))^2 \right| \psi_{\mu,+}^{app}(E)^2 \right)^{1/2} = \left( \int_0^\infty dE \left| 1 - \frac{E - \mu}{E - \mu} \hat{S}_{QM}(E) \right|^2 \psi_{\mu,+}^{app}(E)^2 \right)^{1/2}. \tag{42}
\]
Finally, we obtain a convenient expression for the third term on the right hand side of
Eq. (40). We have

\[
\| \hat{P}_+ \psi^{app}_{\pi^-} \|^2_{L^2(\mathbb{R}^+)} = (\psi^{app}_{\pi^-}, P_+ \psi^{app}_{\pi^-})_{L^2(\mathbb{R}^+)} = - \frac{1}{2\pi i} \int \int_0^\infty dE' \frac{1}{E' - \mu} \frac{1}{|E' - i0^+ - \bar{\mu}|} = \text{Re} \left[ - \frac{1}{2\pi i} \int \int_0^\infty dE' \frac{1}{E' - \mu} \frac{1}{|E' - i0^+ - \bar{\mu}|} \right] = \text{Re} \left[ - \frac{1}{2\pi i} \int \int_0^\infty dE' \frac{E - \bar{\mu}}{|E - \mu|^2} \frac{(E - E') - i\epsilon}{(E - E')^2 + \epsilon^2} \frac{E' - \mu}{|E' - \bar{\mu}|^2} \right] = - \frac{1}{2\pi i} \lim_{\epsilon \to 0^+} \text{Im} \left[ \int \int_0^\infty dE' \frac{E - \bar{\mu}}{|E - \mu|^2} \frac{(E - E') - i\epsilon}{(E - E')^2 + \epsilon^2} \frac{E' - \mu}{|E' - \bar{\mu}|^2} \right] = - \frac{1}{2\pi} \lim_{\epsilon \to 0^+} \int_0^\infty dE' \left( \frac{\text{Im} \bar{\mu}}{\epsilon} (E - E')^2 - \epsilon (E - \text{Re} \bar{\mu})(E' - \text{Re} \bar{\mu}) + (\text{Im} \bar{\mu})^2 \right) = - \frac{1}{2\pi} \left( \int_0^\infty \frac{1}{|E - \mu|^2} \right)^2 + \frac{1}{2} \int_0^\infty \frac{1}{|E - \mu|^2} = 1 - \frac{\| \psi^{app}_{\mu^+} \|^2_{L^2(\mathbb{R}^+)}^2}{\| \psi^{app}_{\mu^+} \|^2_{L^2(\mathbb{R}^+)}} = 1 - \frac{\| \psi_{\mu^+}^{res} \|^2}{\| \psi_{\mu^+} \|^2} \| \psi_{\mu^+}^{app} \|^2.
\]

The last two equalities here follow from the fact that

\[
\| \psi_{\mu}^{app} \|^2 = \| \psi_{\mu^+}^{app} \|^2 = \int_0^\infty \frac{1}{|E - \mu|^2} \| \psi_{\mu^+}^{res} \|^2 = \| \psi_{\mu^+}^{res} \|^2 = \int_{-\infty}^\infty \frac{1}{|E - \mu|^2} = \frac{\pi}{\text{Im} \bar{\mu}}
\]

(see Ref. [41]). Thus we get that

\[
\| \hat{P}_+ \psi^{app}_{\pi^-} \|^2_{L^2(\mathbb{R}^+)} = \frac{1}{\sqrt{2}} \left( 1 - \frac{\| \psi_{\mu}^{app} \|^2}{\| \psi_{\mu}^{res} \|^2} \right)^{1/2} \| \psi_{\mu}^{app} \|.
\] (43)

Collecting the estimates in Eqs. (31), (32), (33) and inserting in Eq. (40) we find that

\[
\| (I - \Lambda_B) \hat{S}^{\mu}_{\pi^-} \|_{L^2(\mathbb{R}^+)} \leq \| \psi_{\mu^+}^{res} - \psi_{\mu}^{app} \|_{L^2(\mathbb{R}^+)} + \| \psi_{\mu^+}^{app} - \hat{S}^{\mu}_{\pi^-} \|_{L^2(\mathbb{R}^+)} + \| \hat{P}_+ \psi^{app}_{\pi^-} \|_{L^2(\mathbb{R}^+)} \leq (1 + \sqrt{2}) \left( 1 - \frac{\| \psi_{\mu}^{app} \|^2}{\| \psi_{\mu}^{res} \|^2} \| \psi_{\mu}^{res} \| + \left( \int_0^\infty \frac{1}{|E - \mu|^2} \hat{S}^{\mu}_{\pi^-}(E) \right)^{1/2} \right)^{1/2} \| \psi_{\mu}^{app} \|.
\] (44)
In order to complete the proof we note that Eq. (37) lead to
\[
\langle E^- | \Lambda_+ \Lambda_- \psi^{res}_\mu \rangle - \langle E^- | \psi^{res}_\mu \rangle = \\
= [\Lambda_F \hat{S}_{QM}(\Lambda_B - I) \hat{S}_{QM}^* \psi^{res}_\mu](E) + [\Lambda_F \psi^{res}_\mu](E) - \psi^{res}_\mu(E) = \\
= [\Lambda_F \hat{S}_{QM}(\Lambda_B - I) \hat{S}_{QM}^* \psi^{res}_\mu](E) + \psi^{app}_\mu(E) - \psi^{res}_\mu(E),
\]
and hence, using Eq. (44) and Eq. (41), we obtain
\[
\| \psi^{res}_\mu - \Lambda_- \psi^{res}_\mu \| \leq \| \Lambda_F \hat{S}_{QM}(\Lambda_B - I) \hat{S}_{QM}^* \psi^{res}_\mu \|_{L^2(\mathbb{R})} + \| \psi^{app}_\mu - \psi^{res}_\mu \|_{L^2(\mathbb{R})}
\]
\[
\leq C \left( 1 - \frac{\| \psi^{app}_\mu \|^2}{\| \psi^{res}_\mu \|^2} \right)^{1/2} \| \psi^{res}_\mu \| + \left( \int_0^\infty \frac{dE}{\| \psi^{app}_\mu \|^2} \right)^{1/2}.
\]
This last inequality, together with the fact that \( \| \psi^{res}_\mu \| \geq \| \psi^{app}_\mu \| \), yields the desired result.

5 Conclusions

We have seen that in the context of quantum mechanical scattering it is possible to construct a structure analogous to the Lax-Phillips scattering theory. In fact, the main objects and representations of the Lax-Phillips theory are obtained as a particular example of a more universal construction. Let a scattering problem be defined on a Hilbert space \( \mathcal{H} \) with a unitary evolution group \( \{ U(t) \}_{t \in \mathbb{R}} \) having a self-adjoint generator \( H \), such that \( \sigma_{ac}(H) \neq \emptyset \), the multiplicity of the absolutely continuous spectrum is uniform and the mappings \( \hat{W}_\pm : \mathcal{H}_{ac} \mapsto L^2(\mathbb{R}; \mathcal{K}) \) onto the incoming and outgoing spectral (energy) representations for \( H \) exist. Denoting \( P_{\sigma_{ac}(H)} : L^2(\mathbb{R}; \mathcal{K}) \mapsto L^2(\mathbb{R}; \mathcal{K}) \) the projection in \( L^2(\mathbb{R}; \mathcal{K}) \) on the subspace \( L^2(\sigma_{ac}(H); \mathcal{K}) \), we construct the operators \( M_\pm(H) : \mathcal{H}_{ac} \mapsto \mathcal{H}_{ac} \) defined by
\[
M_\pm(H) = \hat{W}_\pm^{-1} P_{\sigma_{ac}(H)} \hat{P}_\pm P_{\sigma_{ac}(H)} \hat{W}_\pm,
\]
where \( \hat{P}_\pm \) are, respectively, projections on the upper and lower half-plane Hardy spaces \( \mathcal{H}^2(\mathbb{R}; \mathcal{K}) \). Then \( M_\pm(H) \) are positive, contractive operators on \( \mathcal{H}_{ac} \). We set
\[
\Lambda_\pm(H) := M_\pm^{1/2}(H),
\]
define
\[
Z_{app}(t) := \Lambda_+(H) U(t) \Lambda_-(H),
\]
and obtain a generalized form of Eq. (27) in terms of the following list of correspondences between objects and representations of the Lax-Phillips case and the general case:

| LP scattering theory | QM scattering theory |
|----------------------|----------------------|
| \( U(t) = e^{-iKt} \) | \( U(t) = e^{-iHt} \) |
| \( P_\pm \) | \( \Lambda_\pm(H) \) |
| \( \psi(t) = P_+ \psi(t) + P_- \psi(t) \) | \( \psi(t) = \Lambda_+(H) \psi(t) + (I - \Lambda_+(H)) \psi(t) \) |
| \( \psi(t) = P_- \psi(t) + P_+ \psi(t) \) | \( \psi(t) = (I - \Lambda_-(H)) \psi(t) + \Lambda_-(H) \psi(t) \) |
| \( Z_{LP}(t) = P_+ U(t) P_- \), \( t \geq 0 \) | \( Z_{app}(t) = \Lambda_+(H) U(t) \Lambda_-(H), \ t \geq 0 \) |
| \( \hat{S}_{LP}(E), \ E \in \mathbb{R} \) | \( \hat{S}_{QM}(E), \ E \in \mathbb{R}^+ \) |
The objects and representations on the left hand side of Eq. (45), i.e., those of the Lax-Phillips theory, are obtained from those of the right hand side of Eq. (45) in the particular case that \( P\sigma_{ac}(H) = I_{L^2(\mathbb{R},\mathbb{C})} \), i.e., in the case that \( \sigma_{ac}(H) = \mathbb{R} \). For a scattering system satisfying assumptions (i)-(ii) in Section 2 we have \( \sigma_{ac}(H) = \mathbb{R} \) and \( P\sigma_{ac}(H) = P_{\mathbb{R}} \) and the objects and representations on the right hand side of Eq. (45) in this case are those listed on the right hand side of Eq. (27) (see the discussion preceding Eq. (27) in Section 2). For a problem satisfying assumptions (i)-(ii) the results of Refs. [S2, SSMH1, SSMH2] characterize \( M_{\pm} \) as Lyapunov operators for the evolution, analogous to the Lyapunov operators \( P_{\pm} \) of the Lax-Phillips theory, and the decompositions on the right hand sides of the third and fourth lines in Eq. (27) are, respectively, forward and backward transition representations for the evolution. If in addition to (i)-(ii) we assume that the scattering system satisfies (iii)-(iv) in Section 3, then the results of Refs. [S3, S4, SHV] and Theorem 8 of the present paper imply that to a resonance pole of the scattering matrix \( S_{QM}(\cdot) \) at a point \( z = \mu, \, \text{Im} \, \mu < 0 \), there is associated a resonance state \( \psi_{\mu}^{res} \in \mathcal{H}_{ac} \) (or an eigensubspace in the more general case) which is an approximate eigenstate of the elements of the approximate Lax-Phillips semigroup \( \{Z_{opp}(t)\}_{t \geq 0} \). This last result is the analogue of Theorem 3 a central result of the Lax-Phillips scattering theory associating with each pole of the Lax-Phillips scattering matrix \( S_{LP}(\cdot) \) a resonance state (or eigensubspace) in the Lax-Phillips Hilbert space \( \mathcal{H}^{LP} \). The quality of the approximation to an exact semigroup behavior is quantified by the inequality in Eq. (35). For an ideal resonance the two terms on the right hand side of Eq. (35) vanish identically and the resonance state \( \psi_{\mu}^{res} \) becomes an eigenstate of an exact semigroup.

The most obvious way in which the results of the present paper may be extended is by showing that the list of correspondences in Eq. (45) is valid not only in terms of the formal construction of objects and representations but by proving that the objects and representations on the right hand side of Eq. (45) have all the necessary properties beyond the case of a scattering system satisfying assumptions (i)-(iv) discussed in the present paper. This, together with an extension of Theorem 3 to an analogous result in the general case, would constitute a generalization of the Lax-Phillips theory into a formalism applicable to a much broader range of problems than those satisfying the strict assumptions of the original theory listed in Eq. (1).

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