Recent advances in exact solutions of pairing models

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Abstract. Following a brief overview on the subject of Exactly Solvable Pairing Models, we describe two recent developments in the field that could have a future impact in nuclear structure theory. One concerns a recent extension to include the continuum and the second concerns development of the hyperbolic pairing model as an exactly solvable approximation to Gogny pairing.

1. Introduction
In recent years, there has been great interest in the exact solution of the BCS pairing hamiltonian. Originally discovered by Richardson in the early 60s, this subject was rediscovered some 40 years later in the context of ultrasmall superconducting grains [1], to appropriately describe the crossover from superconductivity to a normal metal as a function of the grain size. Since then, there has been a flurry of activity in the field, from the extension of the Richardson exact solution to several families of exactly-solvable models, now called the Richardson-Gaudin (RG) models [2], to the application of these models in many different areas of quantum many-body physics. The current status of the subject, with particular emphasis on its role in nuclear structure physics, was recently reviewed [3] as a chapter of the book “Fifty Years of Nuclear BCS”. Here we focus on two recent developments in the field, developments that could perhaps prove useful in the ongoing program to develop a unified microscopic theory of finite nuclei.

The outline of the presentation is as follows. In Section 2, we briefly review Richardson’s solution of the BCS pairing hamiltonian and then in Section 3 discuss its extension by Id Betan [4, 5] to properly include the continuum. In Section 4, we briefly describe the two families of Richardson-Gaudin models associated with the SU(2) algebra and then in Section 5 focus on a recent application [6] of the hyperbolic model as an exactly solvable approximation to Gogny pairing. Finally, Section 6 provides a brief summary of the key points of the presentation.

2. Richardson’s solution of the BCS pairing hamiltonian
We focus on a pairing hamiltonian with constant strength \( G \) acting in a space of doubly-degenerate time-reversed states \( (k, \bar{k}) \),

\[
H_P = \sum_k \varepsilon_k c_k^\dagger c_k - G \sum_{k,k'} c_k^\dagger c_{\bar{k}}^\dagger c_{\bar{k}} c_k ,
\]  
(1)
where $\varepsilon_k$ are the single-particle energies for the doubly-degenerate orbits $k, \bar{k}$.

Cooper [7] considered the addition of a pair of fermions with an attractive pairing interaction on top of an inert Fermi sea (FS) under the influence of this hamiltonian, showing that the eigenstate is

$$|\Psi_{\text{Cooper}}\rangle = \sum_{k>k_F} \frac{1}{2\varepsilon_k - E} c_k^\dagger c_{\bar{k}}^\dagger |\text{FS}\rangle ,$$

with $E$ the energy eigenvalue.

Richardson [8, 9] proposed an ansatz for the exact solution of the hamiltonian (1) based closely on Cooper’s original idea. For a system with $2M + \nu$ particles, with $\nu$ of them unpaired, his ansatz involved the state

$$|\Psi\rangle = B_1^\dagger B_2^\dagger \cdots B_M^\dagger |\nu\rangle ,$$

with the collective pair operators $B_\alpha^\dagger$ that build this state having the form found by Cooper for the one-pair problem,

$$B_\alpha^\dagger = \sum_{l=1}^{L} \frac{1}{2\varepsilon_k - E_\alpha} c_{k}^\dagger c_{\bar{k}}^\dagger .$$

Here $L$ is the number of doubly-degenerate single-particle levels and

$$|\nu\rangle \equiv |\nu_1, \nu_2 \cdots, \nu_L\rangle$$

is a state of $\nu$ unpaired fermions ($\nu = \sum_k \nu_k$, with $\nu_k = 1$ or 0), defined by $c_k c_{\bar{k}} |\nu\rangle = 0$, and $n_k |\nu\rangle = \nu_k |\nu\rangle$.

In the one-pair problem, the quantity $E_1$ entering (4) is the eigenvalue of the pairing hamiltonian, as shown by Cooper. In the $M$-pair problem, Richardson proposed to use the $M$ quantities $E_\alpha$ (called the pair energies) as parameters chosen to fulfill (if possible) the eigenvalue equation $H_P |\Psi\rangle = E |\Psi\rangle$. He showed that it is indeed an exact eigenstate of the pairing hamiltonian if these pair energies satisfy a set of $M$ non-linear coupled equations

$$1 - \frac{G}{2} \sum_{k=1}^{L} \frac{2 - 2\nu_k}{2\varepsilon_k - E_\alpha} - 2G \sum_{\beta(\neq \alpha)=1}^{M} \frac{1}{E_\beta - E_\alpha} = 0 ,$$

which are now called the Richardson equations. The second term represents the interaction between particles in a given pair, whereas the third term represents the interaction between pairs. The eigenvalues of $H$ associated with a given set of pair energies emerging from (6) are given by

$$E = \sum_{k=1}^{L} \varepsilon_k \nu_k + \sum_{\alpha=1}^{M} E_\alpha ,$$

namely as a sum of the pair energies plus a contribution from the unpaired particles.

Each independent solution of the set of Richardson equations defines a set of $M$ pair energies that completely defines a particular eigenstate (3, 4). All eigenstates of the pairing hamiltonian can be obtained in this way, both for systems with an odd or an even number of particles. The ground eigenstate is the energetically lowest solution in the $\nu = 0$ or $\nu = 1$ sector, depending on whether the system has an even or an odd number of particles, respectively.

There are few points worth noting here. First, if one of the pair energies $E_\alpha$ is complex, then its complex-conjugate $E_\alpha^*$ is also a solution, as required for $|\Psi\rangle$ to preserve the time-reversal invariance of the hamiltonian. Second, from the structure of the Richardson pair (4), we see that a pair energy that is close to the energy of an particular unperturbed pair $2\varepsilon_k$ is dominated
by this configuration, so that the associated pair is uncorrelated. In contrast, if a pair energy lies sufficiently far from any $2\varepsilon_k$ in the complex plane, the resulting pair will be highly correlated. Lastly, the set of coupled non-linear Richardson equations can be solved numerically, and efficient algorithms for doing so have been developed (see e.g. [10]).

3. Extension to the continuum

Richardson’s solution focused on pairing over a set of discrete states. In nuclear physics, a realistic single-particle spectrum includes both discrete states and continuum states. The latter are especially important when they are sufficiently close to the Fermi energy such that pair scattering into them can be important or when the Fermi level is itself in the continuum. This is the case both for weakly-bound systems and for unbound systems. A description of such systems requires explicit treatment of pair scattering into the continuum.

The first effort to include pair scattering to the continuum within a Richardson approach was reported by Hasegawa and Kaneko [11]. In that work, however, only the effect of resonances was considered. As a consequence, their calculations produced complex energies even for bound states of the system.

As is well known, a proper treatment of the continuum should treat not only resonances but also the background states obtained from contours in the complex plane that enclose all the resonances included. The first work that treated the full continuum was reported recently by Id Betan in two papers [4, 5]. Here we briefly review the key points of this work and summarize the key results.

In the presence of the continuum, Richardson’s equation (6) becomes

$$1 - \frac{G}{2} \sum_{k=1}^{L} \frac{d_k}{2\varepsilon_k - E_\alpha} - \frac{G}{2} \int_0^\infty \frac{g(\varepsilon)}{\varepsilon - E_\alpha} d(\varepsilon) - 2G \sum_{\beta(\neq \alpha)=1}^{M} \frac{1}{E_\beta - E_\alpha} = 0 \ , t$$

where $d_k$ is the occupation of level $k$, including blocking effects, and $g(\varepsilon)$ is the Continuum Single Particle Level Density (CSPLD).

In a proper treatment of the continuum, the CSPLD should include contributions both from resonances (res) and from background (bkgrd) states. In contrast to the earlier work of ref. [11], Id Betan included both contributions, viz.

$$g(\varepsilon) = g_{\text{res}}(\varepsilon) + g_{\text{bkgrd}}(\varepsilon)$$

The resonant contributions can be modeled in the usual Breit-Wigner form. The non-resonant (background) can be treated by rotating the integration contour of the resonant part to the imaginary axis and then using the Cauchy Theorem. More details can be found in ref. [5].

The most recent paper [5] used this formalism to treat a nuclear chain that includes both bound and unbound systems, namely the even-$A$ Carbon isotopes up to $^{28}C$. When the system is bound, the pair energies that contribute to the ground state emerged in complex conjugate pairs, thus preserving the real nature of the ground state energy. This can be seen clearly from Figure 3 of ref. [5], which gives the pair energies for $^{22}C$, the last of the bound even-$C$ isotopes.

Once the system becomes unbound, however, this ceases to be the case. Now the pair energies that contribute to the ground state do not occur in complex conjugate pairs. More specifically, as can be seen from figure 11 of Ref. [5], where the pair energies for $^{28}C$ are displayed, there are a series of pair energies for which the real part is negative and these occur in nearly complex conjugate pairs. These are indeed the pair energies that are analogous to those for bound $^{22}C$. The remaining pair energies all have positive real parts and are typically far from being in complex conjugate pairs. From these results, we see how complex energies and thus widths arise for the states of an unbound system within a Richardson approach that includes the true continuum.
4. Generalization to the Richardson-Gaudin class of integrable models

Here we discuss how to generalize the exactly-solvable BCS pairing model to a wider variety of exactly-solvable models, the so-called Richardson-Gaudin models [12], all of which are based on the SU(2) algebra.

We begin by introducing the generators of SU(2),

\[
K_0^j = \frac{1}{2} \left( \sum_{m} a_{jm}^\dagger a_{jm} - \Omega_j \right), \quad K_+^j = \sum_{m} a_{jm}^\dagger a_{jm}, \quad K_-^j = (K_+^j)^\dagger.
\] (10)

Here \( a_{jm}^\dagger \) creates a fermion in single-particle state \( jm \), \( jm \) is the time reverse of \( jm \), and \( \Omega_j = j + \frac{1}{2} \) is the pair degeneracy of orbit \( j \).

These operators fulfill the usual SU(2) commutation algebra \( [K_0^j, K_{\pm}^j] = \pm 2 \delta_{jj'} K_{\pm}^j \).

We next consider the most general set of \( L \) Hermitian and number-conserving operators that can be built up from the generators of SU(2) for each level \( j \) with linear and quadratic terms only,

\[
R_i = K_0^i + 2g \sum_{j \neq i} \left[ \frac{X_{ij}}{2} \left( K_+^j K_-^j + K_-^j K_+^j \right) + Y_{ij} K_0^j K_0^i \right].
\] (11)

It turns out that there are essentially two sets of conditions on the matrices \( X \) and \( Y \) under which the set of \( R \) operators commute and are complete. The two families of solutions associated with these conditions are referred to as the rational and hyperbolic families, respectively.

i. The rational family

\[
X_{ij} = Y_{ij} = \frac{1}{\eta_i - \eta_j}
\] (12)

ii. The hyperbolic family

\[
X_{ij} = 2 \frac{\sqrt{\eta_i \eta_j}}{\eta_i - \eta_j}, \quad Y_{ij} = \frac{\eta_i + \eta_j}{\eta_i - \eta_j}
\] (13)

In both families, the parameters \( \eta_i \) are a set of \( L \) free real numbers.

A third family, called the trigonometric, was shown to be equivalent to the hyperbolic family in [12], as you can pass from the hyperbolic to the trigonometric family by simply replacing real etas by imaginary etas.

The traditional pairing model is an example of the rational family, as it can be obtained as a linear combination of the integrals of motion, \( H_P = \sum_j \varepsilon_j R_j(\varepsilon_j) \), with \( \eta_j = \varepsilon_j \).

But it is not the only one.

Any hamiltonian that can be expressed as a linear combination of the \( R_i \) operators, whether for the rational family or for the hyperbolic family, can be solved exactly using the same methods as were used by Richardson for the pure pairing model. In all cases, one is led to a system of coupled non-linear equations, whose solutions can be used to generate all eigensolutions of that integrable hamiltonian.

5. Application of the hyperbolic model in nuclear physics

We now discuss the one application in nuclear physics reported for the hyperbolic class of SU(2) RG models [6]. What was shown in this work is that hyperbolic model gives rise to a separable pairing hamiltonian with two free parameters, which after appropriate choice gives a good approximation to Gogny pairing, but which can be can be solved exactly.

The hamiltonian for the hyperbolic model we will discuss can be obtained as a simple linear combination of the hyperbolic integrals of motion, \( H = \lambda \sum_i \eta_i R_i \). By defining \( \lambda = \ldots \)
\[ [1 + 2\gamma(1 - M) + \gamma L_C]^{-1} \text{ and carrying out some algebraic manipulations, we end up with a hamiltonian} \]

\[ H = \sum_i \eta_i K_i^0 - G \sum_{i,j} \sqrt{\eta_i \eta_j} K_i^+ K_j^- , \]

where \( G = 2\lambda \gamma \) is a free parameter.

If we define \( \eta_i = 2(\varepsilon_i - \alpha) \), where \( \varepsilon_i \) is the single-particle energy of level \( i \) and \( \alpha \) plays the role of an interaction cutoff and if we make use of the pair representation of the \( SU(2) \) generators, we are led to a hamiltonian of the form

\[ H = \sum_i \varepsilon_i \left( c_i^\dagger c_i + c_i^\dagger c_i \right) - 2G \sum_{i,j} \sqrt{(\alpha - \varepsilon_i)(\alpha - \varepsilon_j)} c_i^\dagger c_j^\dagger c_j c_i . \]

By introducing the interaction cutoff \( \alpha \), we are led to a separable and exactly-solvable pairing hamiltonian with state-dependent pairing strengths in which the strengths do not grow unphysically as we increase the energy.

This separable pairing hamiltonian can be solved exactly using the RG approach. The resulting energies are given by

\[ E = 2\alpha M + \sum_i \varepsilon_i \nu_i + \sum_{\beta} E_{\beta} \]

where the pair energies \( E_{\beta} \) are solutions of the set of non-linear coupled Richardson equations

\[ \frac{1}{2} \sum_i \frac{1}{\eta_i - E_{\beta}} - \sum_{\beta' \neq \beta} \frac{1}{E_{\beta'} - E_{\beta}} = \frac{Q}{E_{\beta}} , \]

where \( Q = \frac{1}{\eta_i M} - \frac{L}{2} + M - 1 \). Each solution of Eq. (17) defines a unique eigenstate of the hyperbolic hamiltonian (15).

To see whether this model hamiltonian is useful, the authors of Ref. [6] applied it to two well-deformed nuclei, \(^{238}\text{U}\) and \(^{154}\text{Sm}\). Here we discuss their results and conclusions for \(^{238}\text{U}\).

To carry out analysis, they first took the single-particle energies \( \varepsilon_i \) from a Gogny HFB calculation for this nucleus. and then, using a BCS treatment of the hyperbolic hamiltonian (straightforward because of the separable character of the interaction), fitted the two parameters \( \alpha \) and \( G \) to the gaps and pairing tensors of the Gogny HFB calculation. An optimal fit was obtained with the choice of parameters \( \alpha = 25.25 \text{ MeV} \) and \( G = 2 \times 10^{-3} \text{ MeV} \). Comparison of the results of the Gogny HFB calculation and the BCS results for the hyperbolic hamiltonian with this choice of parameters is shown in Fig. 1. Note that the hyperbolic model reproduces Gogny’s fall off in \( \Delta_i \) with increasing energy, in contrast to constant-\( G \) pairing for which \( \Delta_i \) would remain constant. Overall, the hyperbolic model when treated in mean field gives a very good description of the corresponding mean-field physics of Gogny.

What is perhaps more important, however, is that this hamiltonian is exactly solvable, so that it can model in an approximate way the physics of Gogny beyond mean-field. This too was studied in Ref. [6]. The calculations were carried out in an energy window that extended 30 \text{ MeV} above and below the Fermi surface. For such a window of energies, the full space involves 148 single-particle levels for the 46 active proton pairs, for which the exact shell-model space contains \( 4.8 \times 10^{38} \) states. Diagonalization in such a space is clearly prohibitive. In contrast, the exact solution of the hyperbolic hamiltonian requires the quite tractable solution of 46 coupled non-linear equations. When these calculations were carried out, it was found that the exact solution of the hyperbolic hamiltonian gives roughly 2 \text{ MeV} more correlation energy than the BCS solution, suggesting the importance of going beyond mean field. As noted earlier, calculations were also carried out for \(^{154}\text{Sm}\) and can in principle be carried out systematically to treat pairing in heavy nuclei beyond mean field.
Figure 1. Gaps $\Delta_i$ and pairing tensor $u_i v_i$ for protons in $^{238}U$. Open circles are Gogny HFB results in $MeV$. Solid lines are BCS results of the hyperbolic hamiltonian in $MeV$.

6. Summary and concluding remarks
Following a brief review of Richardson’s exact solution of the pairing model and its generalization to a wider class of exactly-solvable RG models, we discussed two recent advances that have the potential of impacting current efforts to develop a unified microscopic theory of atomic nuclei. One was the recent extension by Id Betan of Richardson’s solution to the continuum, where we showed that when the continuum is treated appropriately the proper behavior is achieved on transitioning from bound to unbound systems. This could prove useful in the description of very weakly bound and unbound nuclear systems. The other was a first application of the hyperbolic model to nuclear systems, where it was shown that with appropriate choice of the parameters of the model it provides a very good and exactly-solvable approximation to Gogny pairing. This could prove useful in systematic HF + exact pairing calculations of heavy nuclei.

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