A Tempered Gronwall Inequality and Its Application in Tempered Fractional Order System

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Abstract. Gronwall inequalities are widely used in analytical science, especially in ordinary or fractional differential systems. In this paper, a tempered Gronwall inequality is proposed and applied to tempered fractional system. By the tempered Gronwall inequality, dependences of the solution of tempered fractional systems on initial value, fractional order and tempered parameter are obtained. Finally, a new tempered fractional chaotic system is proposed to show the validity of the dependences.

Keywords: Gronwall inequality, tempered, fractional order, system.

1. Introduction

Integral inequalities are strong tools for studying the qualitative and quantitative properties of differential equations. In order to meet the needs of various applications, there has been an increasingly growing of interest in different integral inequalities. Gronwall inequalities are widely used to estimate the properties of the solution of integer-order and fractional differential equation. Gronwall inequality states that for functions satisfy a certain differential equation or integral equation, there are corresponding inequalities for that equation. Many different types of Gronwall inequalities have been proposed, such as classic Gronwall inequality [1–4], discrete Gronwall inequality [5], generalized fractional Gronwall inequality [6], multidimensional nonlinear Gronwall inequality [7], weakly singular discrete Gronwall inequalities [8]. It has been widely used to prove the uniqueness of solutions to ordinary or fractional systems.

Fractional integrals and derivatives are generalization of the classic integer-order calculus to deal with nonlocal problems. It is one of the most rapidly growing research fields due to the valuable results found in the fractional models for real world problems [9]. Many different definitions of fractional calculus have been proposed to deal with different problems. In the course of modeling the problem under investigation, it is very important to choose the most suitable operator. Recently, researchers have paid more interest to tempered fractional calculus. It is not only a generalization of classic fractional and integer-order calculus, but also can better describe some physical phenomena, such as the finite life span of particles [10–12]. Many papers have discussed these types of fractional operators [13-15]. However, the solution of tempered fractional system depend on the fractional order, tempered parameter and the initial condition for a tempered fractional differential equations has not been discussed until now.
A new version of generalized Gronwall inequality is proposed in this paper, which can be used for tempered fractional system. By the tempered Gronwall inequality, the dependences of the initial value, fractional order and the tempered parameter on the solution of the tempered fractional system are discussed. A new fractional chaotic system with infinitely many hidden chaotic attractors is proposed. By the chaos system, numerical tests are presented to show the validity of the theoretical analysis.

2. Preliminaries and tempered Gronwall inequality
Some basic definitions and tempered Gronwall inequality are given as follows.

**Definition 1** ([11,12]). For $\alpha > 0, \lambda \geq 0$, the definition of tempered fractional integral is given as

$$I_{b,t}^{\alpha,\lambda} v(t) = \frac{1}{\Gamma(\alpha)} \int_{b}^{t} e^{-\lambda(t-s)} (t-s)^{\alpha-1} v(s) ds,$$

where $\Gamma$ presents the well-known Euler gamma function.

**Definition 2** ([11,12]). For $n-1 \leq \alpha < n, \lambda \geq 0$, the definition of tempered fractional Riemann-Liouville derivative is given as

$$RL_{b,t}^{\alpha,\lambda} v(t) = \frac{e^{-\lambda t}}{\Gamma(n-\alpha)} \frac{d^{n}}{dt^{n}} \int_{b}^{t} e^{\lambda s} v(s) (t-s)^{\alpha-n+1} ds.$$

Base on the Theorem 1 in the Reference [6], we could easily establish the following tempered Gronwall inequality.

**Theorem 1.** Suppose that $\beta > 0, \lambda \geq 0, x_{0} \leq x \leq X$ (some $X \leq +\infty$), $a(x) \geq 0$ and $v(x) \geq 0$ are locally integral functions, $f(x)$ is a continuous and nondecreasing function, $0 \leq f(x) \leq M$ ($M$ is a constant), and $v(x)$ satisfies

$$v(x) \leq a(x) + f(x) \int_{a_{0}}^{x} (x-s) e^{-\lambda(x-s)} v(s) ds. \quad (1)$$

Then

$$v(x) \leq a(x) + \int_{x_{0}}^{x} e^{-\lambda(x-s)} a(s) \sum_{n=1}^{\infty} \frac{(x-s)^{n\beta-1} (f(x)\Gamma(\beta))^{n}}{\Gamma(n\beta)} ds.$$

3. Application to tempered fractional system
Next, we will use the tempered Gronwall inequality to investigate solution of tempered Riemann-Liouville fractional system depends on the fractional order, the tempered parameter, and the initial condition.

The tempered Riemann-Liouville fractional system is considered as follows:

$$RL_{0,x}^{\alpha,\lambda} y(x) = g(x, y(x)), \quad n-1 < \alpha < n, \lambda \geq 0, \quad \text{where} \quad k = 0, 1, 2, \cdots, n-1, \quad 0 \leq x < X \leq +\infty \quad \text{and} \quad f : [0, X] \times \mathbb{R} \to \mathbb{R}.$$

Problem (2)-(3) can be converted to an equivalent Volterra integral equation [25],

$$y(x) = \sum_{k=0}^{n-1} \eta_{k} e^{-\lambda x} x^{\alpha-k-1} + \frac{1}{\Gamma(\alpha)} \int_{0}^{x} e^{-\lambda(x-\tau)} (x-\tau)^{\alpha-1} g(\tau, y(\tau)) d\tau. \quad (4)$$

**Theorem 2.** Assume continuous function $g(x, y)$ satisfies Lipschitz condition with $y$, that is

$$|g(x, y) - g(x, z)| \leq L |y - z|,$$

where $L$ is a constant. For $0 \leq x \leq h < X$, suppose that $y$ and $z$ are the solutions of the tempered fractional system (2)-(3) and
\[ RL D_{0,x}^{\alpha - \sigma \tau} z(x) = g(x, z(x)), \quad n - 1 < \alpha - \sigma \alpha < \alpha < n, \tau \geq 0, \quad (6) \]

\[ RL D_{0,x}^{\alpha - \sigma - k - \tau} (e^{\sigma x} y(x))|_{x=0} = \overline{\eta}_k, \quad k = 0, 1, 2, \ldots, n - 1, \quad (7) \]

respectively. Then for \( 0 < x \leq h \),

\[ |z(x) - y(x)| \leq M(x) \]

\[ + \int_0^x e^{-\lambda(t-x)} M(s) \sum_{n=1}^a \frac{L^2 \Gamma^n((\alpha - \sigma)(t-s)) e^{a(\alpha - \sigma)} ds}{\Gamma^n(\alpha)(\alpha(\alpha - \sigma))} \]

where \( \|g\| = \max_{0 \leq x \leq h} |g(x, y)|. \)

**Proof** From (4), the solutions of the tempered fractional systems (2)-(3) and (6)-(7) are given by

\[ y(x) = \sum_{k=0}^{n-1} \eta_k e^{-\lambda x} x^{a-k} + \frac{1}{\Gamma(\alpha)} \int_0^x e^{-\lambda(x-t)} (x-t)^{a-1} g(t, y(t)) d\tau \quad (8) \]

and

\[ z(x) = \sum_{k=0}^{n-1} \frac{\overline{\eta}_k e^{-\lambda x} x^{a-k}}{\Gamma(\alpha - \sigma - k)} + \frac{1}{\Gamma(\alpha)} \int_0^x e^{-\lambda(x-t)} (x-t)^{a-1} g(t, z(t)) d\tau, \quad (9) \]

respectively. From (8) and (9), one has

\[ |z(x) - y(x)| \leq \sum_{k=0}^{n-1} \left| \frac{\overline{\eta}_k e^{-\lambda x} x^{a-k}}{\Gamma(\alpha - \sigma - k)} - \frac{\eta_k e^{-\lambda x}}{\Gamma(\alpha - k)} \right| x^{a-k-1} \]

\[ + \left| \frac{1}{\Gamma(\alpha - \sigma)} - \frac{1}{\Gamma(\alpha)} \right| \int_0^x e^{-\lambda(x-t)} (x-t)^{a-1} g(t, z(t)) d\tau \]

\[ + \frac{1}{\Gamma(\alpha)} \int_0^x (e^{-\lambda(x-t)} - e^{-\lambda(x-t)}) (x-t)^{a-1} g(t, z(t)) d\tau \]

\[ + \frac{1}{\Gamma(\alpha)} \int_0^x e^{-\lambda(x-t)} [(x-t)^{a-1} - (x-t)^{a-1}] g(t, z(t)) d\tau \]

\[ + \frac{1}{\Gamma(\alpha)} \int_0^x e^{-\lambda(x-t)} (x-t)^{a-1} [g(t, z(t)) - g(t, y(t))] d\tau \]

\[ \leq M(x) + \frac{L}{\Gamma(\alpha)} \int_0^x e^{-\lambda(x-t)} (x-t)^{a-1} |z(t) - y(t)| d\tau, \]

where

\[ M(x) = \sum_{k=0}^{n-1} \left| \frac{\overline{\eta}_k e^{-\lambda x} x^{a-k}}{\Gamma(\alpha - \sigma - k)} - \frac{\eta_k e^{-\lambda x}}{\Gamma(\alpha - k)} \right| x^{a-k-1} \]

\[ + \left| \frac{1}{\Gamma(\alpha - \sigma)} - \frac{1}{\Gamma(\alpha)} \right| \|g\| x^{a-\sigma} \]

\[ + \frac{2\alpha + |\alpha x^{\sigma} - (\alpha - \sigma)| x^\sigma}{(\alpha - \sigma)\Gamma(\alpha + 1)} \|g\|. \]

From Theorem 1, the Theorem 2 is proved.
4. Numerical test
In this part, an example is proposed to show the validity of theoretical results.

A new fractional system with infinitely many hidden chaotic attractors is given as follows

\[
\begin{align*}
\alpha & \frac{\mathrm{RL}}{D} D^{\alpha,\lambda}_{0,t} x(t) = k_1 \sin(z(t)), \\
\alpha & \frac{\mathrm{RL}}{D} D^{\alpha,\lambda}_{0,t} y(t) = k_2 \sin(z(t)) + ay(t), \\
\alpha & \frac{\mathrm{RL}}{D} D^{\alpha,\lambda}_{0,t} z(t) = x(t)[b + k_3 \sin(z(t)) + k_4 \cos(z(t)) + c y^2(t)] \\
& + \sin(z(t))[k_5 y(t) + k_6 \cos(z(t))].
\end{align*}
\]

where \( k \ (i = 1, 2, \ldots, 6) \) and \( a, b, c \) are constants.

Let

\[
(k_1, k_2, k_3, k_4, k_5, k_6, a, b, c) = (1.5, 6, 8.8, -1, 10, -0.09, -1, -0.4, -10/3),
\]

system (10) has infinitely equilibria \((0, 0, k \pi)(k \in \mathbb{Z})\). From Figure 1, we can see that the tempered fractional system (10) is also chaotic. Choosing fractional order \( \alpha = 0.998 \), tempered parameter \( \lambda = 0.001 \), and different initial values \((0, 0.1, 0.1, 0.1 + 2k \pi), k = 0, \pm 1, \pm 2, \pm 3\), Figure 2 can be clearly seen that the solution of tempered fractional system (10) dependents on initial value. Let tempered parameter \( \lambda = 0 \) and initial value \((0.1, 0.1, 0.1)\), Figure 3 shows that the solution of tempered fractional system (10) depends on the fractional order. Let fractional order \( \alpha = 0.998 \) and initial value \((0.1, 0.1, 0.1)\), from Figure 4, the tempered parameters also can impact the solution of system (10).

**Figure 1.** System (10) with initial value \((0.1, 0.1, 0.1)\), tempered parameter \( \alpha = 0.001 \) and fractional order \( \alpha = 0.998 \). (a) chaotic attractor. (b) projection on plane \( z = 0 \). (c) projection on plane \( x = 0 \). (d) projection on plane \( y = 0 \).
Figure 2. System (10) with different initial values $(0.1, 0.1, 0.1 + 2k\pi), k = 0, \pm 1, \pm 2, \pm 3$. (a) chaotic attractors. (b) projection on plane $y = 0$.

Figure 3. System (10) with different fractional orders $\alpha = 1, 0.99, 1.01$. (a) chaotic attractors. (b) projection on plane $y = 0$.

Figure 4. System (10) with different tempered parameters $\lambda = 0, 0.01, 0.02$. (a) chaotic attractors. (b) projection on plane $y = 0$. 
From the above analysis, we get that the solution of tempered fractional system depends on the initial value, the fractional order and the tempered parameter. That is, the initial value, the fractional order and the tempered parameter have direct effect on the behavior of tempered fractional systems.

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