On Embedding of Multidimensional Morse-Smale Diffeomorphisms into Topological Flows

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Abstract

J. Palis found necessary conditions for a Morse-Smale diffeomorphism on a closed \( n \)-dimensional manifold \( M^n \) to embed into a topological flow and proved that these conditions are also sufficient for \( n = 2 \). For the case \( n = 3 \) a possibility of wild embedding of closures of separatrices of saddles is an additional obstacle for Morse-Smale cascades to embed into topological flows. In this paper we show that there are no such obstructions for Morse-Smale diffeomorphisms without heteroclinic intersection given on the sphere \( S^n \), \( n \geq 4 \), and Palis’s conditions again are sufficient for such diffeomorphisms.

1 Introduction and statements of results

Let \( M^n \) be a smooth connected closed \( n \)-manifold. Recall that a \( C^m \)-flow \( (m \geq 0) \) on the manifold \( M^n \) is a continuously depending on \( t \in \mathbb{R} \) family of \( C^m \)-diffeomorphisms \( X^t : M^n \to M^n \) that satisfies the following conditions:

1) \( X^0(x) = x \) for any point \( x \in M^n \);
2) \( X^t(X^s(x)) = X^{t+s}(x) \) for any \( s, t \in \mathbb{R}, x \in M^n \).

A \( C^0 \)-flow is also called a topological flow. One says that a homeomorphism (diffeomorphism) \( f : M^n \to M^n \) embeds into a \( C^m \)-flow on \( M^n \) if \( f \) is the time one map of this flow.

Obviously, if a homeomorphism embeds in a flow then it is isotopic to identity. For a homeomorphism of the line and a connected subset of the line this condition also is necessary (see [5, 6]). If an orientation preserving homeomorphism \( f \) of the circle satisfies either one of the three conditions: 1) \( f \) has a fixed point, 2) \( f \) has a dense orbit, or 3) \( f \) is periodic then it embeds in a flow (see [7]). Sufficient conditions of embedding in topological flow for a homeomorphisms of a compact two-dimensional disk and of the plane one can find in review [35]. An analytical, \( \varepsilon \)-close to the identity diffeomorphism \( f : M^n \to M^n \) can be approximated with accuracy \( e^{-\varepsilon} \) by a diffeomorphism which embeds in an analytical flow, see [34].

Due to [27] the set of \( C^r \)-diffeomorphisms \( (r \geq 1) \) which embed in \( C^1 \)-flows is a subset of the first category in \( Diff^r(M^n) \). As Morse-Smale diffeomorphisms are structurally stable (see [28]) then for any manifold \( M^n \) there exists an open set (in \( Diff^1(M^n) \)) of Morse-Smale diffeomorphisms embeddable in topological flows. This set contains neighborhoods of time one maps of Morse-Smale flows without periodic trajectories (according to [30] such flows exist on an arbitrary smooth manifold).
Recall that a diffeomorphism \( f : M^n \to M^n \) is called a Morse-Smale diffeomorphism if it satisfies the following conditions:

- the non-wandering set \( \Omega_f \) is finite and consists of hyperbolic periodic points;
- for any two points \( p, q \in \Omega_f \) the intersection of the stable manifold \( W^s_p \) of the point \( p \) and the unstable manifold \( W^u_q \) of the point \( q \) is transversal.

In [26] J. Palis established the following necessary conditions of the embedding of a Morse-Smale diffeomorphism \( f : M^n \to M^n \) into a topological flow (we call them Palis conditions):

1. the non-wandering set \( \Omega_f \) coincides with the set of its fixed points;
2. the restriction of the diffeomorphism \( f \) to each invariant manifold of a fixed point \( p \in \Omega_f \) preserves the orientation of the manifold;
3. if for two distinct saddle points \( p, q \in \Omega_f \) the intersection \( W^s_p \cap W^u_q \) is not empty then it contains no compact connected components.

According to [26] these conditions are not only necessary but also sufficient for the case \( n = 2 \). For the case \( n = 3 \) a possibility of wild embedding of closures of separatrices of saddles is another obstruction for Morse-Smale cascades to embed in topological flows (phase portraits of such diffeomorphisms are shown on the Figure 1). In [12] examples of such cascades are described and a criteria for embedding of Morse-Smale 3-diffeomorphisms in topological flows is provided. In the present paper we establish that the Palis conditions are sufficient for Morse-Smale diffeomorphisms on \( S^n, n \geq 4 \), such that for any distinct saddle points \( p, q \in \Omega_f \) the intersection \( W^s_p \cap W^u_q \) is empty.

**Theorem 1.** Suppose that a Morse-Smale diffeomorphism \( f : S^n \to S^n, n \geq 4 \) satisfies the following conditions:

1. the non-wandering set \( \Omega_f \) of the diffeomorphism \( f \) coincides with the set of its fixed points;

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\[1\] Definitions of stable and unstable manifolds and of transversality are given in the section 4; see also the book [15] for references.
\textit{ii) the restriction of $f$ to each invariant manifold of a fixed point \( p \in \Omega_f \) preserves the orientation of the manifold;}

\textit{iii) the invariant manifolds of distinct saddle points of $f$ do not intersect.}

Then $f$ embeds into a topological flow.

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## 2 Comments to Theorem 1

Due to [26] the conditions $\text{i)}$ and $\text{ii)}$ are necessary for embedding a Morse-Smale diffeomorphism into a flow. Our condition that the ambient manifold is the sphere $S^n$ and the absence of heteroclinic intersections (condition $\text{iii)}$) are not necessary but violation of each of them allows to construct examples of Morse-Smale diffeomorphisms which do not embed in topological flows. Below we describe such examples.

In [23] V. Medvedev and E. Zhuzhoma constructed a Morse-Smale diffeomorphism $f_0 : M^4 \to M^4$ satisfying conditions $\text{i)} - \text{iii)}$ on a projective-like manifold $M^4$ (different from $S^4$) whose non-wandering set consists of exactly three fixed points: a source, a sink and a saddle. Invariant manifolds of the saddle are two-dimensional and the closure of each of them is a wild sphere (see [23], Theorem 4, item 2). Assume that $f_0$ embeds in a topological flow $X_{t0}$. Then $X_{t0}$ is a topological flow whose the non-wandering set consists of three equilibrium points with locally hyperbolic behavior. According to [36, Theorem 3] the closures of the invariant manifolds of the saddles are locally flat spheres. That is a contradiction because the closures of the invariant manifolds of the saddle singularities of $X_{t0}$ and $f_0$ coincide. Thus, $f_0$ does not embed into a flow.

In [24] T. Medvedev and O. Pochinka constructed an example of Morse-Smale diffeomorphism $f_1 : S^4 \to S^4$ satisfying to the conditions $\text{i)} - \text{ii)}$ of the Theorem 1. The non-wandering set of the diffeomorphism $f_1$ consists of two sources, two sinks and two saddles

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure.png}
\caption{The disk $D_p \subset W_{ps}^s$}
\end{figure}
p, q such that \( \text{dim } W^s_p = \text{dim } W^u_q = 3 \). The intersection \( W^s_p \cap W^u_q \) is not empty and its closure in \( W^s_p \) is a wildly embedded open disk \( D_p \) (see Fig. 2). If \( S^2 \subset W^s_p \) is a 2-sphere which bounds an open ball containing the point \( p \) then the intersection \( S^2 \cap D_p \) contains at least three connected components. Assume that \( f_1 \) embeds into a topological flow \( X'_1 \). Then due to \([12]\) the restriction \( X'_1 \) of \( X'_1 \) to \( W^s_p \) is topologically conjugated by means of a homeomorphism \( h : W^s_p \setminus p \to S^2 \times \mathbb{R} \) to a shift flow \( \chi^l(s, r) = (s, r + t), (s, r) \in S^2 \times \mathbb{R} \). Let \( \Sigma = h^{-1}(S^2 \times \{0\}) \). Then every trajectory of the flow \( X'_1 \) intersects the sphere \( \Sigma \) at a unique point. Since the disk \( D_p \) is invariant with respect to the flow \( X'_1 \), the intersection \( D_p \cap \Sigma \) consists of a unique connected component and that is a contradiction. Thus, \( f_1 \) does not embed into a flow.

3 The scheme of the proof of Theorem 1

The proof of Theorem 1 is based on the technique developed for classification of Morse-Smale diffeomorphisms on orientable manifolds in a series of papers \([2, 3, 4, 9, 17, 18, 11, 13]\). The idea of the proof consists of the following.

In section 1, we introduce a notion of Morse-Smale homeomorphism on a topological \( n \)-manifold and define the subclass \( G(S^n) \) of such homeomorphisms satisfying to conditions similar to (i) – (iii) of Theorem 1.

Let \( f \in G(S^n) \). In \([13\), Theorem 1.3] it is shown that the dimension of the invariant manifolds of the fixed points of \( f \) can be only one of 0, 1, \( n-1 \) or \( n \). Denote by \( \Omega^f \) the set of all fixed points of \( f \) whose unstable manifolds have dimension \( i \in \{0, 1, n-1, n\} \), and by \( m_f \) the number of all saddle points of \( f \).

Represent the sphere \( S^n \) as the union of pairwise disjoint sets

\[
A_f = \left( \bigcup_{\sigma \in \Omega_f^1} W^s_\sigma \right) \cup \Omega_f^0, \quad R_f = \left( \bigcup_{\sigma \in \Omega_f^{n-1}} W^s_\sigma \right) \cup \Omega_f^n, \quad V_f = S^n \setminus (A_f \cup R_f).
\]

Similar to \([16\) one can prove that the sets \( A_f, R_f, V_f \) are connected, the set \( A_f \) is an attractor, \( R_f \) is a repeller \( ^2 \) and \( V_f \) consists of wandering orbits of \( f \) moving from \( R_f \) to \( A_f \).

Denote by \( \hat{V}_f = V_f / f \) the orbit space of the action of \( f \) on \( V_f \) and by \( p_f : V_f \to \hat{V}_f \) the natural projection. Let

\[
\hat{L}_f^s = \bigcup_{\sigma \in \Omega_f^1} p_f(W^s_\sigma \setminus \sigma), \quad \hat{L}_f^u = \bigcup_{\sigma \in \Omega_f^{n-1}} p_f(W^u_\sigma \setminus \sigma).
\]

**Definition 3.1.** The collection \( S_f = (\hat{V}_f, \hat{L}_f^s, \hat{L}_f^u) \) is called the scheme of the homeomorphism \( f \in G(S^n) \).

**Definition 3.2.** Schemes \( S_f \) and \( S_{f'} \) of homeomorphisms \( f, f' \in G(S^n) \) are called equivalent if there exists a homeomorphism \( \hat{\varphi} : \hat{V}_f \to \hat{V}_{f'} \) such that \( \hat{\varphi}(\hat{L}_f^s) = \hat{L}_{f'}^s \) and \( \hat{\varphi}(\hat{L}_f^u) = \hat{L}_{f'}^u \).

The next statement follows from paper \([13\), Theorem 1.2] (in fact, Theorem 1.2 was proven for Morse-Smale diffeomorphisms but the smoothness plays no role in the proof).

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\(^2\) A set \( A \) is called an attractor of a homeomorphism \( f : M^n \to M^n \) if there exists a closed neighborhood \( U \subset M^n \) of the set \( A \) such that \( f(U) \subset \text{int } U \) and \( A = \bigcap_{n \geq 0} f(U) \). A set \( R \) is called a repeller of a homeomorphism \( f \) if it is an attractor for the homeomorphism \( f^{-1} \).
Statement 3.1. Homeomorphisms $f, f' \in G(S^n)$ are topologically equivalent if and only if their schemes $S_f, S_{f'}$ are equivalent.

The possibility of embedding of $f \in G(S^n)$ into a topological flow follows from triviality of the scheme in the following sense.

Let $a^t$ be the flow on the set $\mathbb{S}^{n-1} \times \mathbb{R}$ defined by $a^t(x, s) = (x, s + t), x \in \mathbb{S}^{n-1}, s \in \mathbb{R}$ and let $a$ be the time-one map of $a^t$. Let $\mathbb{Q}^n = \mathbb{S}^{n-1} \times \mathbb{S}^1$. Then the orbit space of the action $a$ on $\mathbb{S}^{n-1} \times \mathbb{R}$ is $\mathbb{Q}^n$. Denote by $p_{qn} : \mathbb{S}^{n-1} \times \mathbb{R} \to \mathbb{Q}^n$ the natural projection. Let $m \in \mathbb{N}$ and $c_1, ..., c_m \subset \mathbb{S}^{n-1}$ be a collection of smooth pairwise disjoint $(n - 2)$-spheres. Let $Q^n = \bigcup_{i=1}^{m} Q^n_i$ and $\tilde{L}_m = p_{qn}(L_m)$.

Definition 3.3. The scheme $S_f = (\hat{V}_f, \hat{L}_f^x, \hat{L}_f^y)$ of a homeomorphism $f \in G(S^n)$ is called trivial if there exists a homeomorphism $\hat{\psi} : \hat{V}_f \to \mathbb{Q}^n$ such that $\hat{\psi}(\hat{L}_f^x \cup \hat{L}_f^y) = \tilde{L}_m$.

In the section 5 we prove the following key lemma.

Lemma 3.1. If $f \in G(S^n)$ then its scheme $S_f$ is trivial.

In the section 6 we construct a topological flow $X_f^t$ whose time one map belongs to the class $G(S^n)$ and has the scheme equivalent to $S_f$. According to Statement 3.1 there exists a homeomorphism $h : S^n \to S^n$ such that $f = hX_f^t h^{-1}$. Then the homeomorphism $f$ embeds into the topological flow $Y_f^t = hX_f^t h^{-1}$.

4 Morse-Smale homeomorphisms

This section contains some definitions and statements which was introduced and proved in [13].

4.1 Basic definitions

Remind that a linear automorphism $L : \mathbb{R}^n \to \mathbb{R}^n$ is called hyperbolic if its matrix has no eigenvalues with absolute value equal one. In this case a space $\mathbb{R}^n$ have a unique decomposition into the direct sum of $L$-invariant subsets $E^s, E^u$ such that $||L|_{E^s}|| < 1$ and $||L^{-1}|_{E^u}|| < 1$ in some norm $|| \cdot ||$ (see, for example, Propositions 2.9, 2.10 of Chapter 2 in [24]).

According to Proposition 5.4 of the book [24] any hyperbolic automorphism $L$ is topologically conjugated with a linear map of the following form:

$$a_{\lambda, \mu, \nu}(x_1, x_2, ..., x_{\lambda}, x_{\lambda+1}, x_{\lambda+2}, ..., x_n) = (2\mu x_1, 2x_2, ..., 2x_\lambda, \frac{1}{2} \nu x_{\lambda+1}, \frac{1}{2} x_{\lambda+2}, ..., \frac{1}{2} x_n), \ (1)$$

where $\lambda = \dim E^u \in \{0, 1, ..., n\}$, $\mu = -1$ ($\mu = 1$) if the restriction $L|_{E^u}$ reverses (preserves) an orientation of $E^u$, and $\nu = -1$ ($\nu = 1$) if the restriction $L|_{E^s}$ reverses (preserves) an orientation of $E^s$.

Put $E^s_\lambda = \{(x_1, ..., x_n) \in \mathbb{R}^n \mid x_1 = x_2 = \cdots = x_\lambda = 0\}, \ E^u_\lambda = \{(x_1, ..., x_n) \in \mathbb{R}^n \mid x_{\lambda+1} = x_{\lambda+2} = \cdots = x_n = 0\}$ and denote by $P^s_x (P^u_y)$ a hyperplane that parallel to the hyperplane $E^u_\lambda (E^s_\lambda)$ and contain a point $x \in E^u_\lambda (y \in E^s_\lambda)$. Unions $P^s_x = \{P^s_x \}_{x \in E^s_\lambda}, \ P^u_y = \{P^u_y \}_{y \in E^s_\lambda}$ form the $a_{\lambda, \mu, \nu}$-invariant foliation.
Suppose that $M^n$ is an $n$-dimensional topological manifold, $f : M^n \to M^n$ is a homeomorphism and $p$ is a fixed point of the homeomorphism $f$. We will call the point $p$ a topologically hyperbolic point of index $\lambda_p$, if there exists its neighborhood $U_p \subset M^n$, numbers $\lambda_p \in \{0, 1, \ldots, n\}, \mu_p, \nu_p \in \{+1, -1\}$, and a homeomorphism $h_p : U_p \to \mathbb{R}^n$ such that $h_p f|_{U_p} = a_{\lambda_p, \mu_p, \nu_p} h_p|_{U_p}$ when the left and right sides are defined. Call the sets $W^s_{p, \text{loc}} = h_p^{-1}(E^s), W^u_{p, \text{loc}} = h_p^{-1}(E^u)$ the local invariant manifolds of the point $p$, and the sets $W^s_p = \bigcup_{i \in \mathbb{Z}} f^i(W^s_{p, \text{loc}}), W^u_p = \bigcup_{i \in \mathbb{Z}} f^i(W^u_{p, \text{loc}})$ the stable and unstable invariant manifolds of the point $p$.

It follows form the definition that $W^s_p = \{x \in M^n : \lim_{i \to +\infty} f^i(x) = p\}, W^u_p = \{x \in M^n : \lim_{i \to -\infty} f^i(x) = p\}$ and $W^s_p \cap W^u_q = \emptyset (W^s_p \cap W^u_q = \emptyset)$ for any distinct hyperbolic points $p, q$. Moreover, there exists an injective continuous immersion $J : \mathbb{R}^{\lambda_p} \to M^n$ such that $W^u_p = J(\mathbb{R}^{\lambda_p})$.

A hyperbolic fixed point is called the source (the sinks) if its indice equals $n$ ($0$), a hyperbolic fixed point $p$ of index $0 < \lambda_p < n$ is called the saddle point.

A periodic point $p$ of period $m_p$ of a homeomorphism $f$ is called a topologically hyperbolic saddle point (source, saddle) periodic point if it is the topologically hyperbolic (source, saddle) fixed point for the homeomorphism $f^{n_p}$. The stable and unstable manifolds of the periodic point $p$ considered as the fixed point of the homeomorphism $f^{n_p}$ are called the stable and unstable manifolds of the point $p$. Every connected component of the set $W^s_p \setminus p (W^u_p \setminus p)$ is called the stable (the unstable) separatrix and is denoted by $l^s_p (l^u_p)$.

The linearizing homeomorphism $h_p : U_p \to \mathbb{R}^n$ induces a pair of transversal foliations $F^s_p = h_p^{-1}(P^s_{\lambda_p}), F^u_p = h_p^{-1}(P^u_{\lambda_p})$ on the set $U_p$. Every leaf of the foliation $F^s_p (F^u_p)$ is an open disk of dimension $\lambda_p (n - \lambda_p)$. For any point $x \in U_p$ denote by $F^s_{p,x}, F^u_{p,x}$ the leaf of the foliation $F^s_p, F^u_p$, respectively, containing the point $x$.

The invariant manifolds $W^s_p$ and $W^u_p$ of saddle periodic points $p, q$ of a homeomorphism $f$ intersect consistently transversally if one of the following conditions holds:

1. $W^s_p \cap W^u_p = \emptyset$;
2. $W^s_p \cap W^u_q \neq \emptyset$ and $F^s_{q,x} \subset W^s_p, F^u_{p,y} \subset W^u_q$ for any points $x \in W^s_p \cap U_q, y \in W^u_q \cap U_p$.

**Definition 4.1.** A homeomorphism $f : M^n \to M^n$ is called the Morse-Smale homeomorphism if it satisfies the next conditions:

1. its non-wandering set $\Omega_f$ finite and any point $p \in \Omega_f$ is topologically hyperbolic;
2. invariant manifolds of any two saddle points $p, q \in \Omega_f$ intersect consistently transversally.

### 4.2 Properties of Morse-Smale homeomorphisms

**Statement 4.1.** Let $f : M^n \to M^n$ be a Morse-Smale homeomorphism. Then:

1. $W^s_p \cap W^u_p = p$ for any saddle point $p \in \Omega_f$;
2. for any saddle points $p, q, r \in \Omega_f$ the conditions $(W^s_p \setminus p) \cap (W^u_q \setminus q) \neq \emptyset, (W^s_q \setminus q) \cap (W^u_r \setminus r) \neq \emptyset$;

\footnote{A map $J : \mathbb{R}^m \to M^n$ is called immersion if for any point $x \in \mathbb{R}^m$ there exists a neighborhood $U_x \in \mathbb{R}^m$ such that the restriction $J|_{U_x}$ of the map $J$ on the set $U_x$ is a homeomorphism.}
3. there are no sequence of distinct saddle points \( p_1, p_2, \ldots, p_k \in \Omega_f, k > 1, \) such that 
\((W_{p_i}^s \setminus p_i) \cap (W_{p_{i+1}}^u \setminus p_{i+1}) \neq \emptyset \) for \( i \in \{1, \ldots, k-1\} \) and 
\((W_{p_k}^s \setminus p_k) \cap (W_{p_1}^u \setminus p_1) \neq \emptyset \).

**Statement 4.2.** Let \( f : M^n \to M^n \) be a Morse-Smale homeomorphism. Then:

1) \( M^n = \bigcup_{p \in \Omega_f} W_p^u; \)

2) for any point \( p \in \Omega_f \) the manifold \( W_p^u \) is a topological submanifold of the manifold \( M^n; \)

3) for any point \( p \in \Omega_f \) and any connected component \( l_p^u \) of the set \( W_p^u \setminus p \) the following equality holds: \( \text{cl} l_p^u \setminus (l_p^u \cup p) = \bigcup_{q \in \Omega_f ; W_q^s \setminus q \neq \emptyset} W_q^u. \)

**Corollary 4.1.** If \( f : M^n \to M^n \) is a Morse-Smale homeomorphism and \( p \in \Omega_f \) is a saddle point such that \( l_p^u \cap W_q^s = \emptyset \) for any saddle point \( q \neq p, \) then there exists a unique sink \( \omega \in \Omega_f \) such that \( \text{cl} l_p^u = l_p^u \cup p \cup \omega \) and \( \text{cl} l_p^u \) is either a compact arc in case \( \lambda_p = 1 \) or a sphere of dimension \( \lambda_p \) in case \( \lambda_p > 1. \)

For an arbitrary point \( q \in \Omega_f \) and \( \delta \in \{u, s\} \) put \( V_q^\delta = W_q^\delta \setminus q \) and denote by \( \hat{V}_q^\delta = V_q^\delta / f \) the orbit space of the action of the homeomorphism \( f \) on the set \( V_q^\delta. \) The following statement is proved in the book [9] (Proposition 2.1.5).

**Statement 4.3.** The space \( \hat{V}_q^u \) is homeomorphic to \( S^{\lambda_q - 1} \times S^1 \) and the space \( \hat{V}_q^s \) is homeomorphic to \( S^{n-\lambda_q - 1} \times S^1. \)

Remark that \( S^0 \times S^1 \) means a union of two disjoint closed curves.

**Proposition 4.1.** Suppose \( f : M^n \to M^n \) is a Morse-Smale homeomorphism, \( n \geq 4, \) and \( \sigma \in \Omega_f \) is a saddle point of index \( (n-1) \) such that \( l_\sigma^u \cap W_q^s = \emptyset \) for any saddle point \( q \neq p. \) Then the sphere \( \text{cl} l_\sigma^u \) is bicollared.

**Proof:** Let \( \omega \in \Omega_0 \) be a sink point such that \( l_\sigma^u \subset W_\omega^s. \) Due to Corollary 4.1 and the item 2 of Statement 4.2 the set \( \text{cl} l_\sigma^u = l_\sigma^u \cup \omega \) is an \( (n-1) \)-sphere which is locally flat embedded in \( M^n \) at all its points apart possibly one point \( \omega. \) According to [3, 20] an \( (n-1) \)-sphere in a manifold \( M^n \) of dimension \( n \geq 4 \) is either locally flat or have more than countable set of points of wildness. Therefore the sphere \( \text{cl} l_\sigma^u \) is locally flat at point \( \omega. \) According to [11] a locally flat sphere is bicollared. \( \diamond \)

By \( G(S^n) \) we denoted a class of Morse-Smale homeomorphism on the sphere \( S^n \) such that any \( f \in G(S^n) \) satisfy the following conditions:

i) \( \Omega_f \) consists of fixed points;

ii) \( W_p^s \cap W_q^u = \emptyset \) for any distinct saddle points \( p, q \in \Omega_f; \)

iii) the restriction of a homeomorphism \( f \) on every invariant manifolds of an arbitrary fixed point \( p \in \Omega_f \) preserves its orientation.

**Proposition 4.2.** If \( f \in G(S^n) \), then any saddle fixed point has index 1 and \( (n-1). \)

\[4\text{Here } \text{cl} l_p^u \text{ means the closure of the set } l_p^u.\]
Proof: Suppose that, on the contrary, there exists a point \( \sigma \in \Omega_f \) of index \( j \in (1, n-1) \). According to Corollary 4.1 the closures \( \text{cl} \, W^u_\sigma, \text{cl} \, W^s_\sigma \) of the stable and unstable manifolds of the point \( \sigma \) are spheres of dimensions \( j \) and \( n-j \) correspondingly. Due to item 1 of Statements 4.1 the spheres \( S^j = \text{cl} \, W^u_\sigma, S^{n-j} = \text{cl} \, W^s_\sigma \) intersect at a single point \( \sigma \). Therefore their intersection index equals either \( 1 \) or \( -1 \) (depending on the choice of orientations of the spheres \( S^j, S^{n-j} \) and \( S^n \)). Since homology groups \( H_j(S^n), H_{n-j}(S^n) \) are trivial it follows that there is a sphere \( \tilde{S}^j \) homological to the sphere \( S^j \) and having the empty intersection with the sphere \( S^{n-j} \). Then the intersection number of the spheres \( S^j, S^{n-j} \) must be equal to zero as the intersection number is the homology invariant (see, for example, [32], §69). This contradiction proves the statement.

4.3 Canonical manifolds connected with saddle fixed points of a homeomorphism \( f \in G(S^n) \)

It follows from Statement 4.2 that for each saddle point of a homeomorphism \( f \in G(S^n) \) there exists a neighborhood where \( f \) is topologically conjugated either with the map \( a_1: \mathbb{R}^n \to \mathbb{R}^n \) defined by \( a_1(x_1, x_2, \ldots, x_n) = (2x_1, \frac{1}{2}x_2, \ldots, \frac{1}{2}x_n) \) or with the map \( a_1^{-1} \). In this section we describe canonical manifolds defined by the action of the map \( a_1 \) and prove Proposition 4.3 allowing to define similar canonical manifolds for the homeomorphism \( f \in G(S^n) \).

Put \( U_\tau = \{(x_1, \ldots, x_n) \in \mathbb{R}^n | x_1^2(x_2^2 + \ldots + x_n^2) \leq \tau^2, \tau \in (0, 1), U = U_1; U_0 = \{(x_1, \ldots, x_n) \in \mathbb{R}^n | x_1 = 0\}, N^s = U \setminus O x_1, N^u = U \setminus U_0, \tilde{N}^s = N^s / a_1, \tilde{N}^u = N^u / a_1 \). Denote by \( p_s: N^s \to \tilde{N}^s \), \( p_u: N^u \to \tilde{N}^u \) the natural projections and put \( \tilde{V}^s = p_s(U_0) \).

Figure 3: Fundamental domains \( \tilde{N}^s, \tilde{N}^u \) of the action of the homeomorphism \( a_1 \) on the sets \( N^s, N^u \)

The following statement is proved in [11] (Propositions 2.2, 2.3).

**Statement 4.4.** The space \( \tilde{N}^s \) is homeomorphic to the direct product \( S^{n-2} \times S^1 \times [-1, 1] \), the space \( \tilde{N}^u \) consists of two connected components each of which is homeomorphic to the direct product \( \mathbb{R}^{n-1} \times S^1 \).
Recall that an *annulus* of dimension $n$ is a manifold homeomorphic to $S^{n-1} \times [0,1]$.

On the Figure [3] we present the neighborhoods $N^s$, $N^a$ and the fundamental domains $\tilde{N}^s = \{ (x_1, \ldots, x_n) \in \mathbb{N}_s^{\sum} | x_1^2 + \cdots + x_n^2 \leq 1 \}$, $\tilde{N}^a = \{ (x_1, \ldots, x_n) \in \mathbb{N}^{\sum} | x_1 \in [1,2] \}$ of the action of the diffeomorphism $a$. Put $C = \{ (x_1, \ldots, x_n) \in \mathbb{R}^n | x_1^2 + \cdots + x_n^2 \leq 1 \}$. The set $\tilde{N}^s$ is the union of the hyperplanes $L_1 = \{ (x_1, \ldots, x_n) \in \mathbb{N} | x_1 = 1 \}$ and $\tilde{N}^a$ is the union of the pairs of annuli $K_1 = L_1 \cap C, t \in [-1,1]$ and the space $\tilde{N}^a$ can be obtained from $\tilde{N}^s$ by gluing the connected components of the boundary of each annulus by means of the diffeomorphism $a$.

The set $\tilde{N}^a$ is the union of two connected components each of which is homeomorphic to the direct product $B^{n-1} \times [0,1]$. The space $\tilde{N}^s$ is obtained from $\tilde{N}^a$ by gluing the disk $B_1 = \{ (x_1, \ldots, x_n) \in \mathbb{N} | x_1 = 1 \}$ to the disk $B_2 = \{ (x_1, \ldots, x_n) \in \mathbb{N} | x_1 = 2 \}$ and the disk $B_{-1} = \{ (x_1, \ldots, x_n) \in \mathbb{N} | x_1 = -1 \}$ to the disk $B_{-2} = \{ (x_1, \ldots, x_n) \in \mathbb{N} | x_1 = -2 \}$ by means of the diffeomorphism $a$.

**Proposition 4.3.** Suppose $f \in G(S^n)$; then there exists a set of pair-wise disjoint neighborhoods $\{ N_\sigma \}_{\sigma \in \Omega_f} \cup \Omega_f^0$ such that for any neighborhood $N_\sigma$ there exists a homeomorphism $\chi_\sigma : N_\sigma \to \mathbb{R}^n$ such that $\chi_\sigma f|_{N_\sigma} = a_1 \chi_\sigma|_{N_\sigma}$ whenever $\lambda_\sigma = 1$ and $\chi_\sigma f|_{N_\sigma} = a_{-1} \chi_\sigma|_{N_\sigma}$ whenever $\lambda_\sigma = n - 1$.

**Proof:** Put $V_\delta^\sigma = \bigcup_{q \in \Omega_f^\sigma} V_q^{\sigma} \subset \mathbb{N}^{\sum}$, $\hat{V}_\delta^\sigma = \bigcup_{q \in \Omega_f^\sigma} \hat{V}_q^{\sigma} \subset \mathbb{N}^{\sum}$, $\delta \in \{ 0,1, n-1,n \}$, $\sigma$ denote by $p_\delta^{\sigma} : V_\delta^\sigma \to \hat{V}_\delta^\sigma$ the natural projection such that $p_\delta^{\sigma}|_{V_q^{\sigma}} = p_\delta^{\sigma}|_{\hat{V}_q^{\sigma}}$ for any point $q \in \Omega_f^\sigma$.

Put $\Sigma_f = \bigcup_{\sigma \in \Omega_f^\sigma} \Omega_f^\sigma \cup \Omega_f^{n-1}$. Then $\Sigma_f = \bigcup_{\sigma \in \Omega_f^\sigma} \Omega_f^\sigma \cup \Omega_f^{n-1}$.

Let $U_\sigma \subset \tilde{N}^a$ be a neighborhood of the point $\sigma$ such that a homeomorphism $g_\sigma : U_\sigma \to \mathbb{R}^n$ satisfying the condition $g_\sigma f|_{U_\sigma} = a_\sigma g_\sigma|_{U_\sigma}$ is defined.

Put $u_\sigma = \{ (x_1, \ldots, x_n) \in U_\sigma | x_1^2 + \cdots + x_n^2 \leq 1, |x_1| = 2 \tau, D_\tau^\sigma = \{ (x_1, \ldots, x_n) \in U_\sigma | \tau < |x_1| \leq 2 \tau \}, \bar{u}_\tau = g_\sigma^{-1}(u_\tau), \bar{D}_\tau^\sigma = g_\sigma^{-1}(D_\tau^\sigma)$, $\delta \in \{ s,u \}$, and $N_\tau = \bigcup_{\delta \in \{ s,u \}} f_{\delta}^{-1}(\bar{u}_\tau)$.

Let us show that there is a number $\tau_1 > 0$ such that for any $i \in \mathbb{N}$ the intersection $f_{\delta}^{-1}(\bar{u}_{\tau_1}) \cap \bar{u}_{\tau_1}$ is empty. Suppose $\sigma \in \Omega_f^{n-1}$ (the argument for the case $\sigma \in \Omega_f^1$ is similar).

By the Statement 4.2, the set $\bigcup_{i \in \mathbb{N}} f_{\delta}^{-1}(\bar{u}_\tau)$ lies in the stable manifold of a unique sink point $\omega$. Since the homeomorphism $f$ is locally conjugated with the linear compression $a_0$ in a neighborhood of the point $\omega$, we have that there exists a ball $B^n \subset W_a \setminus U_\sigma$ such that $\omega \subset B^n$ and $f(B^n) \subset int B^n$. Since $\tilde{D}_\tau^\sigma$ is compact, there is $i_0 > 0$ such that $f_{\delta}(\tilde{D}_\tau^\sigma) \cap U_\sigma \subset B^n$ for all $i > i_0$. Hence the set of numbers $i_0$ such that $f_{\delta}(\tilde{D}_\tau^\sigma) \cap \bar{u}_\tau \neq \emptyset$ is finite. Then one can choose $\tau_1 \in (0, \tau)$ such that $\bar{u}_{\tau_1} \cap f_{\delta}(\tilde{D}_\tau^\sigma) = \emptyset$ and therefore $\bar{u}_{\tau_1} \cap f_{\delta}(\tilde{D}_{\tau_1}^\sigma) = \emptyset$ for any $i \in \mathbb{N}$. Similarly one can show that there exists a number $\tau_2 \in (0, \tau_1] \subset [0, \tau]$ such that for any $i \in \mathbb{N}$ the intersection of $f_{\delta}(-\tilde{D}_{\tau_2}^\sigma) \cap \bar{u}_{\tau_2}$ is empty.

---

A fundamental domain of the action of a group $G$ on a set $X$ is a closed set $D_G \subset X$ containing a subset $\tilde{D}_G$ with the following properties: 1) $cl \tilde{D}_G = D_G$; 2) $g(\tilde{D}_G) \subset \tilde{D}_G = \emptyset$ for any $g \in G$ distinct from the neutral element; 3) $\bigcup_{g \in G} g(\tilde{D}_G) = X$. 

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Suppose $\lambda_\sigma = 1$, put $N_\sigma = \bigcup_{i \in \mathbb{Z}} f^i(\tilde{u}_{r_2})$, and define a homeomorphism $\chi^*_\sigma : N_\sigma \to U_{r_2}$ by the following: $\chi^*_\sigma(x) = g_\sigma(x)$ whenever $x \in \tilde{u}_{r_2}$, and $\chi^*_\sigma(x) = a_N^{-k}(g_\sigma(f^k(x)))$ whenever $x \in N_\sigma \setminus \tilde{u}_{r_2}$, where $k \in \mathbb{Z}$ is such that $f^k(x) \in \tilde{u}_{r_2}$. The homeomorphism $\chi^*_\sigma$ conjugates the homeomorphism $f|_{N_\sigma}$ with the linear diffeomorphism $a_1|_{\tilde{u}_{r_2}}$. Since the homeomorphism $a_1|_{\tilde{u}_{r_2}}$ is topologically conjugated with $a_1|_U$ by means of the diffeomorphism $g(x_1, ..., x_n) = (\frac{x_1}{\sqrt{r_2}}, ..., \frac{x_n}{\sqrt{r_2}})$, we see that the superposition $\chi_\sigma = g\chi^*_\sigma : N_\sigma \to U$ topologically conjugates $f|_{N_\sigma}$ with $a_1|_U$. A homeomorphism $\chi_\sigma$ for the case $\lambda_\sigma = n - 1$ can be constructed in the same way.

\[\diamond\]

Put $N_\sigma^u = N_\sigma \setminus W^s_\sigma$, $N_{r,\sigma} = \chi^{-1}_\sigma(U_r)$, $N_\sigma^s = N_\sigma \setminus W^s_\sigma$, $\hat{N}_\sigma^s = N_\sigma^s/f$, $\hat{N}_\sigma^u = N_\sigma^u/f$.

## 5 Triviality of the scheme of the homeomorphism $f \in G(S^n)$

This section is devoted to the proof of Lemma 3.1. In subsections 5.1-5.3 we establish some axillary results.

### 5.1 Introduction results on the embedding of closed curves and their tubular neighborhoods in a manifold $M^n$

Further we denote by $M^n$ a topological manifold possibly with non-empty boundary.

Recall that a manifold $N^k \subset M^n$ of dimension $k$ without boundary is **locally flat in a point** $x \in N^k$ if there exists a neighborhood $U(x) \subset M^n$ of the point $x$ and a homeomorphism $\varphi : U(x) \to \mathbb{R}^k$ such that $\varphi(N^k \cap U(x)) = \mathbb{R}^k$, where $\mathbb{R}^k = \{ (x_1, ..., x_n) \in \mathbb{R}^n | x_{k+1} = x_{k+2} = ... = x_n = 0 \}$.

A manifold $N^k$ is **locally flat in $M^n$** or the **submanifold** of the manifold $M^n$ if it is locally flat at each its point.

If the condition of local flatness fails in a point $x \in N^k$ then the manifold $N^k$ is called **wild** and the point $x$ is called the **point of wildness**.

A topological space $X$ is called $m$-**connected** (for $m > 0$) if it is non-empty, path-connected and its first $m$ homotopy groups $\pi_i(X)$, $i \in \{1, ..., m\}$ are trivial. The requirements of being non-empty and path-connected can be interpreted as $(-1)$-connected and 0-connected correspondingly.

A topological space $P$ generated by points of a simplicial complex $K$ with the topology induced from $\mathbb{R}^n$ is called the **polyhedron**. The complex $K$ is called the **partition** or the **triangulation** of the polyhedron $P$.

A map $h : P \to Q$ of polyhedra is called **piece-vise linear** if there exists partitions $K,L$ of polyhedra $P,Q$ correspondingly such that $h$ move each simplex of the complex $K$ into a simplex of the complex $L$ (see for example [29]).

A polyhedron $P$ is called the **piece-vise linear manifold** of dimension $n$ with boundary if it is a topological manifold with boundary and for any point $x \in \text{int} P$ ($y \in \partial P$) there is a neighborhood $U_x$ ($U_y$) and a piece-vise linear homeomorphism $h_x : U_x \to \mathbb{R}^n$ ($h_y : U_y \to \mathbb{R}^n_+ = \{ (x_1, ..., x_n) \subset \mathbb{R}^n | x_1 \geq 0 \}$).

The following important statement follows from Theorem 4 of [19].
Statement 5.1. Suppose that $N^k, M^n$ are compact piece-vise linear manifolds of dimension $k, n$ correspondingly, $N^k$ is the manifold without boundary, $M^n$ possibly has a non-empty boundary, $\tilde{e}, e : N^k \to \text{int } M^n$ are homotopic piece-vise linear embeddings, and the following conditions hold:

1. $n - k \geq 3$;
2. $N^k$ is $(2k - n + 1)$-connected;
3. $M^n$ is $(2k - n + 2)$-connected.

Then there exists a family of piece-vise linear homeomorphisms $h_t : M^n \to M^n, t \in [0, 1]$, such that $h_0 = \text{id}, h_t \tilde{e} = e, h_t|_{\partial M^n} = \text{id}$ for any $t \in [0, 1]$.

We will say that a topological submanifold $N^k \subset M^n$ of the manifold $M^n$ is an essential if a homomorphism $e_{\gamma_k} : \pi_1(N^k) \to \pi_1(M^n)$ induced by an embedding $e_{\gamma_k} : N^k \to M^n$ is the isomorphism. We will call an essential manifold $\beta$ homeomorphic to the circle $S^1$ the essential knot.

Let $\beta \in M^n$ be an essential knot and $h : B^{n-1} \times S^1 \to M^n$ be a topological embedding such that $h(\{O\} \times S^1) = \beta$. Call the image $N_\beta = h(B^{n-1} \times S^1)$ the tubular neighborhood of the knot $\beta$.

Proposition 5.1. Suppose that $\mathbb{P}^{n-1}$ is either $\mathbb{S}^{n-1}$ or $\mathbb{B}^{n-1}$, $\beta_1, \ldots, \beta_k \subset \text{int } \mathbb{P}^{n-1} \times S^1$ are essential knots and $x_1, \ldots, x_k \subset \text{int } \mathbb{P}^{n-1}$ are arbitrary points. Then there is a homeomorphism $h : \mathbb{P}^{n-1} \times S^1 \to \mathbb{P}^{n-1} \times S^1$ such that $h(\bigcup \beta_i) = \bigcup \{x_i\} \times S^1$ and $h|_{\beta_i \times S^1} = \text{id}$.

**Proof:** Put $b_i = \{x_i\} \times S^1, i \in \{1, \ldots, k\}$. Choose pair-vise disjoint neighborhoods $U_1, \ldots, U_k$ of knots $\beta_1, \ldots, \beta_k$ in $\text{int } \mathbb{P}^{n-1} \times S^1$. It follows from Theorem 1.1 of the paper [10] that there exists a homeomorphism $g : \mathbb{P}^{n-1} \times S^1 \to \mathbb{P}^{n-1} \times S^1$ that is identity outside the set $\bigcup U_i$ and such that for any $i \in \{1, \ldots, k\}$ the set $g(\beta_i)$ is a subpolyhedron.

By assumption, piece-vise linear embeddings $\tilde{e} : S^1 \times \mathbb{Z}_k \to \mathbb{P}^{n-1} \times S^1, e : S^1 \times \mathbb{Z}_k \to \mathbb{P}^{n-1} \times S^1$ such that $\tilde{e}(S^1 \times \mathbb{Z}_k) = \bigcup g(\beta_i), e(S^1 \times \mathbb{Z}_k) = \bigcup b_i$ are homotopic. By Statement 5.1 there exists a family of piece-vise linear homeomorphisms $h_t : \mathbb{P}^{n-1} \times S^1 \to \mathbb{P}^{n-1} \times S^1, t \in [0, 1]$, such that $h_0 = \text{id}, h_t \tilde{e} = e, h_t|_{\beta \times S^1} = \text{id}$ for any $t \in [0, 1]$. Then $h_1$ is the desired homeomorphism.

The following Statement 5.2 is proved in the paper [11] (see Lemma 2.1).

**Statement 5.2.** Let $h : \mathbb{B}^{n-1} \times S^1 \to \text{int } \mathbb{B}^{n-1} \times S^1$ be a topological embedding such that $h(\{O\} \times S^1) = \{O\} \times S^1$. Then a manifold $\mathbb{B}^{n-1} \times S^1 \setminus \text{int } h(\mathbb{B}^{n-1} \times S^1)$ is homeomorphic to the direct product $S^n \setminus \{0, 1\}$.

**Proposition 5.2.** Suppose that $Y$ is a topological manifold with boundary, $X$ is a closed component of its boundary, $Y_1$ is a manifold homeomorphic to $X \times [0, 1]$, and $Y \cap Y_1 = X$. Then a manifold $Y \cup Y_1$ is homeomorphic to $Y$. Moreover, if the manifold $Y$ is homeomorphic to the direct product $X \times [0, 1]$ then there exists a homeomorphism $h : X \times [0, 1] \to Y \cup Y_1$ such that $h(X \times \{\frac{1}{2}\}) = X$.
Proof: By \( \Box \) (Theorem 2), there exists a topological embedding \( h_0 : X \times [0, 1] \to Y \) such that \( h_0(X \times \{1\}) = X \). Put \( Y_0 = h_0(X \times [0, 1]) \). Let \( h_1 : X \times [0, 1] \to Y_1 \) be a homeomorphism such that \( h_1(X \times \{0\}) = X = h_0(X \times \{1\}) \).

Define homeomorphisms \( g : X \times [0, 1] \to X \times [0, 1], \) \( h_1 : X \times [0, 1] \to Y_1, \) \( h : X \times [0, 1] \to Y_0 \cup Y_1 \) by \( g(x, t) = (h_1^{-1}(h_0(x, 1)), t), \) \( h_1 = h_1g, \)

\[
h(x, t) = \begin{cases}
h_0(x, 2t), & t \in [0, \frac{1}{2}]; \\
h_1(x, 2t - 1), & t \in (\frac{1}{2}; 1],
\end{cases}
\]

and define a homeomorphism \( H : Y \cup Y_1 \to Y \) by

\[
H(x) = \begin{cases}
h_0(h^{-1}(x)), & x \in Y_0 \cup Y_1; \\
x, & x \in Y \setminus Y_0.
\end{cases}
\]

To prove the second item of the statement it is enough to put \( Y = Y_0 \). Then the homeomorphism \( h : X \times [0, 1] \to Y \cup Y_1 \) defined above is the desired one. \( \Box \)

**Proposition 5.3.** Suppose that \( \mathbb{P}^{n-1} \) is either the ball \( \mathbb{B}^{n-1} \) or the sphere \( \mathbb{S}^{n-1}, \) \( \beta_1, ..., \beta_k \subset \) \( \text{int} \mathbb{P}^{n-1} \times \mathbb{S}^1 \) are essential knots, \( N_{\beta_1}, ..., N_{\beta_k} \subset \mathbb{P}^{n-1} \times \mathbb{S}^1 \) are their pair-wise disjoint neighborhoods, \( D_1^{n-1}, ..., D_k^{n-1} \subset \mathbb{P}^{n-1} \) are pair-wise disjoint disks, and \( x_1, ..., x_k \) are inner points of the disks \( D_1^{n-1}, ..., D_k^{n-1} \) correspondingly. Then there exist a homeomorphism \( h : \mathbb{P}^{n-1} \times \mathbb{S}^1 \to \mathbb{P}^{n-1} \times \mathbb{S}^1 \) such that \( h(\beta_i) = \{x_i\} \times \mathbb{S}^1, h(N_{\beta_i}) = D_i^{n-1} \times \mathbb{S}^1, i \in \{1, ..., k\} \) and \( h|_{\partial \mathbb{P}^{n-1} \times \mathbb{S}^1} = \text{id}. \)

**Proof:** By Proposition 5.1, there exists a homeomorphism \( h_0 : \mathbb{P}^{n-1} \times \mathbb{S}^1 \to \mathbb{P}^{n-1} \times \mathbb{S}^1 \) such that \( h_0(\beta_i) = \{x_i\} \times \mathbb{S}^1, h_0(N_{\beta_i}) = \text{id}. \) By \( \Box \), there exist topological embeddings \( e_i : \mathbb{S}^{n-2} \times \mathbb{S}^1 \times [0, 1] \to \text{int} \mathbb{P}^{n-1} \times \mathbb{S}^1 \) such that \( e_i(\mathbb{S}^{n-2} \times \mathbb{S}^1 \times \{1\}) = \partial \tilde{N}_{\beta_i}, e_i(\mathbb{S}^{n-2} \times \mathbb{S}^1 \times [0, 1]) \cap e_j(\mathbb{S}^{n-2} \times \mathbb{S}^1 \times [0, 1]) = \emptyset \) for \( i \neq j, i, j \in \{1, ..., k\} \). Put \( U_i = e_i(\mathbb{S}^{n-2} \times \mathbb{S}^1 \times [0, 1]) \cup \tilde{N}_i. \)

Suppose that \( D_0^{n-1}, ..., D_{0,k}^{n-1}, D_{1,1}^{n-1}, ..., D_{1,k}^{n-1} \subset \mathbb{P}^{n-1} \) are disks such that \( x_i \subset \text{int} D_{j,i}^{n-1}, D_{j,i}^{n-1} \subset \text{int} D_0^{n-1}, j \in \{0, 1\}, D_{0,0}^{n-1} \subset \text{int} D_{1,0}^{n-1}, D_{1,0}^{n-1} \subset \text{int} \tilde{N}_i. \)

By Proposition 5.2, every set \( \tilde{N}_i \setminus (\text{int} D_{1,i}^{n-1} \times \mathbb{S}^1), (\text{int} D_{0,i}^{n-1} \times \mathbb{S}^1) \subset \mathbb{S}^{n-2} \times \mathbb{S}^1 \) is homeomorphic to the direct product \( \mathbb{S}^{n-2} \times \mathbb{S}^1 \times [0, 1]. \) By Proposition 5.2, there exists a homeomorphism \( g_i : \mathbb{S}^{n-2} \times \mathbb{S}^1 \times [0, 1] \to U_i \setminus \text{int} D_{0,i}^{n-1} \times \mathbb{S}^1 \) such that \( g_i(\mathbb{S}^{n-2} \times \mathbb{S}^1 \times \{t_1\}) = \partial \tilde{N}_i, g_i(\mathbb{S}^{n-2} \times \mathbb{S}^1 \times \{t_2\}) = \partial D_{1,i}^{n-1} \times \mathbb{S}^1 \) for some \( t_1, t_2 \subset (0, 1). \) Let \( \xi : [0, 1] \to [0, 1] \) be a homeomorphism that is identity on the ends of the interval \([0, 1]\) and such that \( \xi(t_1) = t_2. \) Define a homeomorphism \( \tilde{g}_i : \mathbb{S}^{n-2} \times \mathbb{S}^1 \times [0, 1] \to \mathbb{S}^{n-2} \times \mathbb{S}^1 \times [0, 1] \) by \( \tilde{g}_i(x, t) = (x, \xi(t)). \)

Define a homeomorphism \( h_1 : \mathbb{P}^{n-1} \times \mathbb{S}^1 \to \mathbb{P}^{n-1} \times \mathbb{S}^1 \) by

\[
h_i(x) = \begin{cases}
g_i(\tilde{g}_i(\xi^{-1}(x))), & x \in U_i \setminus \text{int} D_{0,i}^{n-1} \times \mathbb{S}^1; \\
x, & x \in (\mathbb{P}^{n-1} \times \mathbb{S}^1 \setminus U_i).
\end{cases}
\]

The superposition \( \eta = h_k \cdot \cdots \cdot h_1 \cdot h_0 \) maps every knot \( \beta_i \) into the knot \( \{x_i\} \times \mathbb{S}^1, \) the neighborhood \( N_{\beta_i} \) into the set \( D_{i,i}^{n-1} \times \mathbb{S}^1, \) and keeps the set \( \partial \mathbb{P}^{n-1} \times \mathbb{S}^1 \) fixed. Construct a homeomorphism \( \Theta : \mathbb{P}^{n-1} \times \mathbb{S}^1 \to \mathbb{P}^{n-1} \times \mathbb{S}^1 \) that be identity on the set \( \partial \mathbb{P}^{n-1} \times \mathbb{S}^1 \) and on the knots \( \{x_1\} \times \mathbb{S}^1, ..., \{x_k\} \times \mathbb{S}^1 \) and move the set \( D_{i,i}^{n-1} \times \mathbb{S}^1 \) into the set \( D_{i,i}^{n-1} \times \mathbb{S}^1 \) for
Corollary 5.1. If \( N \subset S^{n-1} \times S^1 \) is a tubular neighborhood of an essential knot than the manifold \( (S^{n-1} \times S^1) \setminus \text{int} \ N \) is homeomorphic to the direct product \( \mathbb{B}^{n-1} \times S^1 \).

5.2 A surgery of the manifold \( S^{n-1} \times S^1 \) along an essential submanifold homeomorphic to \( S^{n-2} \times S^1 \)

Recall that we put \( Q^n = S^{n-1} \times S^1 \). Suppose that \( N \subset Q^n \) is an essential submanifold homeomorphic to \( \mathbb{B}^{n-1} \times S^1 \), \( T = \partial N \), and \( e_T : S^{n-2} \times S^1 \times [-1; 1] \to Q^n \) is a topological embedding such that \( e_T(S^{n-2} \times S^1 \times \{0\}) = T \). Put \( K = e_T(S^{n-2} \times S^1 \times [-1; 1]) \) and denote by \( N_+, N_- \) connected components of the set \( Q^n \setminus \text{int} \ K \). It follows from Propositions 5.3, 5.2 that the manifolds \( N_+, N_- \) are homeomorphic to \( \mathbb{B}^{n-1} \times S^1 \). Let \( N'_\delta, N'_j \) manifolds homeomorphic to \( \mathbb{B}^{n-1} \times S^1 \). Denote by \( \psi_\delta : \partial N_\delta \to \partial N'_\delta \) an arbitrary homeomorphism reversing the natural orientation, by \( \pi_\delta : (N_\delta \cup N'_j) \to Q_\delta \) the natural projection, \( \delta \in \{+, -\} \).

We will say that the manifolds \( Q_+, Q_- \) are obtained from \( Q^n \) by the surgery along the submanifold \( T \).

Note that \( S^{n-2} \times S^1 \) is the boundary of \( \mathbb{B}^{n-1} \times S^1 \). By [22] (Theorem 2), the following statement holds.

Statement 5.3. Let \( \psi : S^{n-2} \times S^1 \to S^{n-2} \times S^1 \) be an arbitrary homeomorphism. Then there exists a homeomorphism \( \Psi : \mathbb{B}^{n-1} \times S^1 \to \mathbb{B}^{n-1} \times S^1 \) such that \( \Psi|_{S^{n-2} \times S^1} = \psi|_{S^{n-2} \times S^1} \).

Proposition 5.4. The manifolds \( Q_+, Q_- \) are homeomorphic to \( Q^n \).

Proof: Let \( D^{n-1} \subset S^{n-1} \) be an arbitrary disk, \( N_\delta = D^{n-1} \times S^1 \) and \( h_\delta : \pi_\delta(N_\delta) \to N_\delta \) be an arbitrary homeomorphism. Put \( \hat{\psi}_\delta = h_\delta \pi_\delta \hat{\psi}_\delta \pi_\delta^{-1} h_\delta^{-1}|\partial N_\delta \). Due to Proposition 5.3 a homeomorphism \( \hat{\psi}_\delta \) can extend up to a homeomorphism \( h'_\delta : \pi_\delta(N'_j) \to Q^n \setminus \text{int} \ N_\delta \). Then a map \( H_\delta : Q_\delta \to Q^n \) defined by \( H_\delta(x) = h_\delta(x) \) whenever \( x \in \pi_\delta(N_\delta) \) and \( H_\delta(x) = h'_\delta(x) \) whenever \( x \in \pi_\delta(N'_j) \) is the desired homeomorphism.

5.3 A surgery of manifolds homeomorphic to \( S^{n-1} \times S^1 \) along essential knots

Let \( Q^n_1, \ldots, Q^n_{k+1} \) be manifolds homeomorphic to \( Q^n \). Denote by \( \beta_1, \ldots, \beta_{2k} \subset \bigcup_{i=1}^{k+1} Q^n_i \) essential knots such that for any \( j \in \{1, \ldots, k\} \) knots \( \beta_{2j-1}, \beta_{2j} \) belongs to distinct manifolds

\[\text{every } i \in \{1, \ldots, k\}. \text{ It follows from the Annulus Theorem} \text{ that the set } D^{n-1}_i \setminus \text{int } D^{n-1}_i \text{ is homeomorphic to the annulus } S^{n-2} \times [0, 1]. \text{ Then apply the construction similar to one described above to define a homeomorphism } \theta : \mathbb{P}^{n-1} \to \mathbb{P}^{n-1} \text{ such that } \theta(x_i) = x_i, \theta(D^{n-1}_i) = D^{n-1}_i, \theta|_{\partial^{n-1}} = id. \text{ Put } \Theta(x, t) = (\theta^{-1}(x), t), x \in \mathbb{P}^{n-1}, t \in S^1. \text{ Then } h = \Theta \eta \text{ is the desired homeomorphism.} \]
from the union \( \bigcup_{i=1}^{k+1} Q_i^n \) and every manifold \( Q_i^n \) contains at least one knot from the set \( \beta_1, ..., \beta_{2k} \). Let \( N_{\beta_1}, ..., N_{\beta_{2k}} \) be tubular neighborhoods of the knots \( \beta_1, ..., \beta_{2k} \) correspondingly.

Let \( K_1, ..., K_k \) be manifolds homeomorphic to the direct product \( S^{n-2} \times S^1 \times [-1;1] \). For every \( j \in \{1, ..., k\} \) denote by \( T_j \subset K_j \) a manifold homeomorphic to \( S^{n-2} \times S^1 \) that cuts \( K_j \) into two connected components whose closures are homeomorphic to \( S^{n-2} \times S^1 \times \{0;1\} \), and by \( \psi_j : \partial N_{2j-1} \cup \partial N_{2j} \to \partial K_j \) an arbitrary reversing the natural orientation homeomorphism.

Glue manifolds \( \tilde{Q} = \bigcup_{i=1}^{k+1} Q_i^n \setminus \bigcup_{\nu=1}^{2k} int N_{\nu} \) and \( K = \bigcup_{j=1}^{k} K_j \) by means of the homeomorphisms \( \psi_1, ..., \psi_k \), denote by \( Q \) the obtained manifold and by \( \pi : \tilde{Q} \cup K \to Q \) the natural projection. We will say that the manifold \( Q \) is obtained from \( Q_1^n, ..., Q_k^n \) by the surgery along knots \( \beta_1, ..., \beta_{2k} \) and call every pair \( \beta_{2j-1}, \beta_{2j} \) the binding pair; \( j \in \{1, 2, ..., k\} \).

**Proposition 5.5.** The manifold \( Q \) is homeomorphic to \( Q^n \) and every manifold \( \pi(T_j) \) cuts \( Q \) into two connected components whose closures are homeomorphic to \( \mathbb{B}^{n-1} \times S^1 \).

**Proof:** Prove the proposition by induction on \( k \). Consider the case \( k = 1 \). Due to Propositions 5.3, 5.2, manifolds \( \tilde{N}_1 = Q_1^n \setminus int N_1, \tilde{N}_2 = Q_2^n \setminus int N_2, \tilde{N}_1 \cup_{\psi_1} K_1 \) are homeomorphic to the direct product \( \mathbb{B}^{n-1} \times S^1 \). By definition, the manifold \( T_1 \) cuts the manifold \( K_1 \) into two connected components whose closures are homeomorphic to \( Q^n \times [0, 1] \). It follows from Proposition 5.2 that \( T_1 \) cuts \( \tilde{N}_1 \cup_{\psi_1} K_1 \) into two connected components such that the closure of one of which, denote it by \( \tilde{N} \), is homeomorphic to \( \mathbb{B}^{n-1} \times S^1 \) and the closure of another is homeomorphic to \( Q^n \times [0, 1] \). Suppose that \( D_0^{n-1} \subset S^{n-1} \) is an arbitrary disk, \( N_0 = D_0^{n-1} \times S^1 \) and \( h_0 : \pi(\tilde{N}_1 \cup K_1) \to N_0 \) is an arbitrary homeomorphism. Put \( \tilde{\psi}_1 = h_0 \circ \psi_1 \circ \pi^{-1} \circ h_0^{-1} \). In virtue of Proposition 5.3 a homeomorphism \( \tilde{\psi} \) can be extended up to a homeomorphism \( h_1 : \pi(\tilde{N} \cup K_1) \to Q^n \setminus int N_0 \). Then the map \( h : Q \to Q^n \) defined by \( h(x) = h_0(x) \) for \( x \in \pi(\tilde{N}_1 \cup K_1) \) and \( h(x) = h_1(x) \) for \( x \in \pi(\tilde{N}_2) \) is the desired homeomorphism. The manifold \( \pi(T_1) \) cuts \( Q \) into two connected components such that the closure of one of them is \( \pi(N) \) which is homeomorphic to \( \mathbb{B}^{n-1} \times S^1 \). By Corollary 5.1 the closure of another connected component is also homeomorphic to \( \mathbb{B}^{n-1} \times S^1 \).

Suppose that the statement is true for all \( \lambda = k \) and show that it is true also for \( \lambda = k + 1 \). Since \( 2k \geq k + 1 \) we have that there exists at least one manifold among the manifolds \( Q^n_1, ..., Q^n_{\lambda+1} \); say \( Q^{n}_{\lambda+1} \), containing exactly one knot from the set \( \beta_1, ..., \beta_{2k} \) (if every of that manifolds would contain no less than two knots, then the total number of all knots be no less than \( 2k + 2 \)). Let \( \beta_2 \subset Q^n_{\lambda+1}, \beta_{2\lambda-1} \subset Q^n_i, i \in \{1, ..., \lambda\}, \) be a binding pair. By the induction hypothesis and Corollary 5.1 the manifold \( Q_{\lambda} \) obtained by the surgery of manifolds \( Q^n_i, ..., Q^n_{\lambda} \) along knots \( \beta_1, ..., \beta_{2\lambda-2} \) is homeomorphic to \( Q^n \); the projection of every manifold \( (T_j) \) cuts \( Q_{\lambda} \) into two connected components such that the closure of each of which is homeomorphic to \( \mathbb{B}^{n-1} \times S^1 \); and the projection of the knot \( \beta_{2\lambda-1} \) is the essential knot. Now apply the surgery to manifolds \( Q_{\lambda}, Q^n_{\lambda+1} \) along knots \( \pi(\beta_{2\lambda-1}), \beta_{2\lambda} \) and use the first step arguments to obtain the desired statement.

### 5.4 Proof of Lemma 3.1

**Step 1.** Prove of the fact that the manifold \( \tilde{V}_f \) is homeomorphic to \( Q^n \) and every connected
component $Q^{n-1}$ of the set $\hat{L}^n_f \cup \hat{L}^s_f$ cuts $\hat{V}_f$ into two connected components whose closures are homeomorphic to $\mathbb{B}^{n-1} \times S^1$.

Put $k_i = |\Omega^s_f|, i \in \{0, 1, n - 1, n\}$. Due to Statement 4.2 and the fact that the closure of every separatrix of dimension $(n - 1)$ cuts the ambient sphere $S^n$ into two connected components one gets $k_0 = k_1 + 1, k_n = k_{n-1} + 1$.

Denote by $\beta_1, ..., \beta_{2k_1}$ the essential knots in the set $\hat{V} = \bigcup_{\omega \in \Omega^s_f} \hat{V}_\omega^s$ which are projections (by means of $p_{\hat{V}}$) of all one-dimension unstable separatrices of the diffeomorphism $f$. Without loss of generality assume that knots $\beta_{2j-1}, \beta_{2j}$ are the projection of the separatrices of the same saddle point $\sigma_j \in \Omega^s_f, j \in \{1, ..., k_1\}$.

It follows from Statement 4.2 that every manifold $\hat{V}_\omega^s$ contains at least one knot from the set $\beta_1, ..., \beta_{2k_1}$. Since stable and unstable manifolds of different saddle points do not intersect we have that for any $j \in \{1, ..., k_1\}$ knots $\beta_{2j-1}, \beta_{2j}$ belong to distinct connected components of $\hat{V}$. Indeed, if one suppose that $\beta_{2j-1}, \beta_{2j} \subset \hat{V}_\omega^s$ for some $j, \omega$, then the set $cl\, W_\omega^u = W_\sigma^u \cup \omega$ is homeomorphic to the circle. Since $cl\, W_\omega^u$ divides the sphere $S^n$ into two parts and intersect the circle $cl\, W_\omega^u$, at the point $\sigma_j$ we have that there exists at least one point in $cl\, W_\sigma^u \cap cl\, W_\omega^s$ different from $\sigma_j$. This fact contradicts to the item 1 of Statement 4.1.

Let $N_{\sigma_j}, \chi_{\sigma_j} : N_{\sigma_j} \to \mathbb{U}$ be the neighborhood of the point $\sigma_j$ and the homeomorphism defined in Proposition 4.3. Further we use denotations of the sections 4.2, 4.3. Denote by $N_{2j-1}, N_{2j}$ the connected components of the set $N_{\sigma_j}$ containing knots $\beta_{2j-1}, \beta_{2j}$ correspondingly. Let $\psi : \partial \tilde{\mathbb{N}}^u \to \partial \tilde{\mathbb{N}}^s$ be a homeomorphism such that $\psi p_{\{\mathbb{U}\}} = p_{\{\mathbb{U}\}}$. Put $K_j = \tilde{N}_\sigma^s, T_j = \tilde{V}_\sigma^s$ and define homeomorphisms $\varphi_{u,j} : N_{2j-1} \cup N_{2j} \to \tilde{\mathbb{N}}^u, \varphi_{s,j} : K_j \to \tilde{\mathbb{N}}^s, \psi_j : \partial N_{2j-1} \cup \partial N_{2j} \to \partial K_j$ by

$$\varphi_{u,j} = p_u \chi_{\sigma_j} p_{\hat{V}_f}^{-1}|_{N_{2j-1} \cup N_{2j}},$$

$$\varphi_{s,j} = p_s \chi_{\sigma_j} p_{\hat{V}_f}^{-1}|_{K_j},$$

$$\psi_j = \varphi_{s,j}^{-1} \psi \varphi_{u,j} |_{\partial N_{2j-1} \cup \partial N_{2j}},$$

and denote by

$$\Psi : \bigcup_{j=1}^{k_1} (\partial N_{2j-1} \cup \partial N_{2j}) \to \bigcup_{j=1}^{k_1} K_j$$

the homeomorphism such that

$$\Psi |_{\partial N_{2j-1} \cup \partial N_{2j}} = \psi_j |_{\partial N_{2j-1} \cup \partial N_{2j}}.$$
\[
\hat{V}_f = \left( \hat{V}_f \setminus \left( \bigcup_{\sigma \in \Omega_j^f} \hat{N}_\sigma^a \right) \right) \cup_{\psi} \left( \bigcup_{\sigma \in \Omega_j^f} \hat{N}_\sigma^a \right) = \left( \hat{V}_f \setminus \left( \bigcup_{j=1}^{k_1} N_j \right) \right) \cup_{\psi} \left( \bigcup_{j=1}^{k_1} K_j \right).
\]

So, the manifold \( \hat{V}_f \) is obtained from \( \bigcup_{\omega \in \Omega_j^f} \hat{V}_\omega^s \) by the surgery along knots \( \beta_1, ..., \beta_{2k_1} \).

Due to Proposition 5.5, the manifold \( \hat{V}_f \) is homeomorphic to \( \mathbb{Q}^n \) and every connected component of the set \( \hat{L}_f^s \) cuts the set \( \hat{V}_f \) into two connected components such that the closure of each of which is homeomorphic to \( \mathbb{B}^{n-1} \times S^1 \).

From the other hand
\[
V_f = \left( \bigcup_{\alpha \in \Omega_j^f} V_{\alpha}^u \setminus \left( \bigcup_{\sigma \in \Omega_j^f} V_{\sigma}^a \right) \right) \bigcup_{\psi} \left( \bigcup_{\sigma \in \Omega_j^f} V_{\sigma}^a \right) = \left( V_f \setminus \left( \bigcup_{\sigma \in \Omega_j^f} N_{\sigma}^a \right) \right) \bigcup_{\psi} \left( \bigcup_{\sigma \in \Omega_j^f} N_{\sigma}^a \right).
\]

Similar to previous arguments one can conclude that the set \( \hat{V}_f \) is obtained from \( \bigcup_{\alpha \in \Omega_j^f} \hat{V}_{\alpha}^u \) by the surgery along the projections of all one-dimensional stable separatrices of the saddle points of the diffeomorphism \( f \). In virtue of Proposition 5.5, every connected component of the set \( \hat{L}_f^y \) cuts the set \( \hat{V}_f \) into two connected components such that the closure of each of which is homeomorphic to \( \mathbb{B}^{n-1} \times S^1 \).

**Step 2. Proof of the fact that there is a set \( \hat{L}_{m_f} \subset \mathbb{Q}^n \) and a homeomorphism \( \hat{\varphi} : \hat{V}_f \to \mathbb{Q}^n \) such that \( \varphi(\hat{L}_f^y \cup \hat{L}_f^u) = \hat{L}_{m_f} \).**

Denote by \( \mathbb{Q}_1^{n-1}, ..., \mathbb{Q}_{n_1+k_{n-1}-1}^{n-1} \) all elements of the set \( \hat{L}_f^y \cup \hat{L}_f^u \) and suppose that \( \mathbb{Q}_1^{n-1} \) is an element such that all elements of the set \( \hat{L}_f^y \cup \hat{L}_f^u \setminus \mathbb{Q}_1^{n-1} \) are contained exactly in one of the connected components of the manifold \( \hat{V}_f \setminus \mathbb{Q}_1^{n-1} \). Denote by \( N_1 \) the closure of this connected component. By Step 1, \( N_1 \) is homeomorphic to \( \mathbb{B}^{n-1} \times S^1 \). By Proposition 5.3, there exists a disk \( D_1^{n-1} \subset S^{n-1} \) and a homeomorphism \( \psi_0 : \hat{V}_f \to \mathbb{Q}^n \) such that \( \psi_0(N_1) = D_1^{n-1} \times S^1 \).

If \( k_1 + k_{n-1} = 1 \) then the proof is complete and \( \hat{\varphi} = \psi_0 \), \( \hat{L}_{m_f} = \partial D_1^{n-1} \times S^1 \).

Let \( k_1 + k_{n-1} > 1 \). Denote the images of \( \mathbb{Q}_1^{n-1}, ..., \mathbb{Q}_{n_1+k_{n-1}-1}^{n-1} \) under the homeomorphism \( \psi_0 \) by the same symbols as their originals. For \( i \in \{2, ..., k_1 + k_{n-1}\} \) denote by \( N_i \) the connected component of the set \( \mathbb{Q}^n \setminus \mathbb{Q}_1^{n-1} \) contained in the set \( D_1^{n-1} \times S^1 \). Without loss of generality suppose that the numeration of the sets \( \mathbb{Q}_1^{n-1}, ..., \mathbb{Q}_{n_1+k_{n-1}-1}^{n-1} \) is chosen in such a way that there exist a number \( l_1 \in [2, k_1 + k_{n-1}] \) and pair-wise disjoint sets \( N_2, ..., N_{l_1} \) such that \( \bigcup_{i=2}^{l_1} N_i = k_1+k_{n-1} \). Choose in the interior of the disk \( D_1^{n-1} \) arbitrary pair-wise disjoint disks \( D_2^{n-2}, ..., D_{l_1}^{n-2} \). Due to Proposition 5.3, there exists a homeomorphism \( \psi_1 : \mathbb{Q}^n \to \mathbb{Q}^n \) such that \( \psi_1|_{\mathbb{Q}^n \setminus \text{int } D_1^{n-1} \times S^1} = id \), \( \psi_1(N_i) = D_i^{n-1} \times S^1 \), \( i \in \{2, ..., l_1\} \). If \( l_1 = k_1 + k_{n-1} \) then the proof is complete and \( \hat{\varphi} = \psi_1 \psi_0 \), \( \hat{L}_{m_f} = \bigcup_{i=1}^{l_1} \partial D_i^{n-1} \times S^1 \).

Let \( l_1 < k_1 + k_{n-1} \). Denote the images of \( \mathbb{Q}_1^{n-1}, ..., \mathbb{Q}_{k_1+k_{n-1}}^{n-1} \) and \( N_1, ..., N_{k_1+k_{n-1}} \) under the homeomorphism \( \psi_1 \) by the same symbols as their originals. Put \( \mathcal{N} = \bigcup_{i=l_1+1}^{k_1+k_{n-1}} N_i \).
If for fixed \( i \in \{2, \ldots, l_1\} \) the set \( N_i \) has non-empty intersection with the set \( N \), then denote by \( l_i, \tilde{k}_i, l_i \leq \tilde{k}_i \), the positive numbers such that \( N_{i,1}, \ldots, N_{i,\tilde{k}_i} \) are all elements from \( N_i \cap N \) and \( N_{i,1}, \ldots, N_{i,l_i} \) are pair-vise disjoint elements from \( N_i \cap N \) such that \( \bigcup_{j=1}^{l_i} N_{i,j} = \bigcup_{j=2}^{\tilde{k}_i} N_{i,j} \). Choose in the interior of the every disk \( D_{i,j}^{n-1} \) pair-vise disjoint disks \( D_{i,j}^{n-1}, \ldots, D_{\tilde{k}_i,j}^{n-1} \). It follows from Proposition 5.3 that there exists a homeomorphism \( \psi_i : \mathbb{Q}^n \to \mathbb{Q}^n \) such that \( \psi_i|_{\mathbb{Q}^n \setminus \text{int} N_i} = \text{id} \), \( \psi_i(N_{i,j}) = D_{i,j}^{n-1} \times S^1, j \in \{1, \ldots, l_i\}, i \in \{2, \ldots, l_1\} \). If \( N_i \cap N = \emptyset \), put \( \psi_i = \text{id} \).

If \( l_i = \tilde{k}_i \) for any \( i \in \{2, \ldots, l_1\} \) such that the numbers \( l_i, \tilde{k}_i \) are defined, then the proof is complete and \( \hat{\varphi} = \psi_{l_1} \psi_{l_1-1} \cdots \psi_{1}, \hat{\psi}_{m_f} = \bigcup_{i=1}^{l_1} \bigcup_{j=1}^{l_i} \partial D_{i,j}^{n-1} \times S^1 \). Otherwise, continue the process and after finite number of steps get the desired set \( \hat{\psi}_{m_f} \) and the desired homeomorphism \( \hat{\varphi} \) as a superposition of all constructed homeomorphisms.

6 Embedding of diffeomorphisms from the class \( G(M^n) \) into topological flows

6.1 Free and properly discontinuous action of a group of maps

In this section we collect an axillary facts on properties of the transformation group \( \{g^n, n \in \mathbb{Z}\} \) which is an infinite cyclic group acting freely and properly discontinously on a topological (in general, non-compact) manifold \( X \) and generated by a homeomorphism \( g : X \to X \).

Denote by \( X/g \) the orbit space of the action of the group \( \{g^n, n \in \mathbb{Z}\} \) and by \( p_{X/g} : X \to X/g \) the natural projection. In virtue of [33] (Theorem 3.5.7 and Proposition 3.6.7) the natural projection \( p_{X/g} : X \to X/\text{g} \) is a covering map and the space \( X/\text{g} \) is a manifold.

Denote by \( \eta_{X/g} : \pi_1(X/\text{g}) \to \mathbb{Z} \) a homeomorphism defined in the following way. Let \( \hat{c} \subset X/\text{g} \) be a loop non-homotopic to zero in \( X/\text{g} \) and \( [\hat{c}] \in \pi_1(X/\text{g}) \) be a homotopy class of \( \hat{c} \). Choose an arbitrary point \( \hat{x} \in \hat{c} \), denote by \( p_{X/g}^{-1}(\hat{x}) \) the complete inverse image of \( \hat{x} \), and fix a point \( \tilde{x} \in p_{X/g}^{-1}(\hat{x}) \). As \( p_{X/g} \) is the covering map then there is a unique path \( \tilde{c}(t) \) beginning at the point \( \tilde{x} \) \((\tilde{c}(0) = \tilde{x})\) and covering the loop \( c \) (such that \( p_{X/g}(\tilde{c}(t)) = \hat{c} \)). Then there exists the element \( n \in \mathbb{Z} \) such that \( \tilde{c}(1) = f^n(\tilde{x}) \). Put \( \eta_{X/g}([\hat{c}]) = n \). It follows from [21] (Lemma 18) that the homomorphism \( \eta_{X/g} \) is an epimorphism.

The next statement 6.1 can be found in [21] (Theorem 5.5) and [3] (Propositions 1.2.3 and 1.2.4).

**Statement 6.1.** Suppose that \( X, Y \) are connected topological manifolds and \( g : X \to X, h : Y \to Y \) are homeomorphisms such that groups \( \{g^n, n \in \mathbb{Z}\}, \{h^n, n \in \mathbb{Z}\} \) acts freely on \( X \) and \( Y \) respectively. Then there exists a homeomorphism \( \hat{\varphi} : X \to Y \) such that

\[
\hat{\varphi}(g^n \cdot x) = h^n \cdot \hat{\varphi}(x) \quad \text{for all } x \in X, n \in \mathbb{Z}, \quad \hat{\varphi}(g x) = h \cdot \hat{\varphi}(x) \quad \text{for all } x \in X, x \in \mathbb{R}^n, \quad \hat{\varphi}(1) = 1.
\]

\*A group \( \mathcal{G} \) acts on the manifold \( X \) if there is a map \( \zeta : \mathcal{G} \times X \to X \) with the following properties:

1) \( \zeta(e, x) = x \) for all \( x \in X \), where \( e \) is the identity element of the group \( \mathcal{G} \);

2) \( \zeta(g, \zeta(h, x)) = \zeta(gh, x) \) for all \( x \in X \) and \( g, h \in \mathcal{G} \).

A group \( \mathcal{G} \) acts freely on a manifold \( X \) if for any different \( g, h \in \mathcal{G} \) and for any point \( x \in X \) an inequality \( \zeta(g, x) \neq \zeta(h, x) \) holds.

A group \( \mathcal{G} \) acts properly discontinuously on the manifold \( X \) if for every compact subset \( K \subset X \) the set of elements \( g \in \mathcal{G} \) such that \( \zeta(g, K) \cap K \neq \emptyset \) is finite.
and properly discontinuously on $X$, $Y$ correspondingly. Then:

1) if $\varphi : X \to Y$ is a homeomorphism such that $h = \varphi g \varphi^{-1}$ and $\varphi_\pi : \pi_1(X/h) \to \pi_1(Y/h)$ is the induced homomorphism, then a map $\hat{\varphi} : X/g \to Y/h$ defined by $\hat{\varphi} = p_{Y/h} \circ p_{X/g}^{-1}$ is a homeomorphism and $\eta_{X/g} = \eta_{Y/h} \circ \varphi_\pi$;

2) if $\hat{\varphi} : X/g \to Y/h$ is a homeomorphism such that $\eta_{X/g} = \eta_{Y/h} \circ \varphi_\pi$ and $\hat{x} \in X/g$, $\tilde{x} \in p_{X/g}^{-1}(x)$, $y = \hat{\varphi}(x)$, $\tilde{y} \in p_{Y/h}^{-1}(y)$, then there exists a unique homeomorphism $\varphi : X \to Y$ such that $h = \varphi g \varphi^{-1}$ and $\varphi(\tilde{x}) = \tilde{y}$.

6.2 Proof of Theorem

Suppose that a Morse-Smale diffeomorphism $f : S^n \to S^n$ has no heteroclinic intersection and satisfy Palis conditions. To prove the theorem it is enough to construct a topological flow $X_f$ such that its time one map $X_f^1$ belongs to the class $G(S^n)$ and the scheme $S_{X_f^1}$ is equivalent to the scheme $S_f$ (see Section 3).

**Step 1.** It follows from Lemma 3.1 and Proposition 6.1 that there exists a homeomorphism $\psi_f : V_f \to \mathbb{S}^{n-1} \times \mathbb{R}$ such that:

1) $f|_{V_f} = \psi_f^{-1} \psi_f$, where $a$ is the time one map of the flow $a^t(x,s) = (x,s + t)$, $x \in S^{n-1}, s \in \mathbb{R}$;

2) for $(n-1)$-dimensional separatrix $l_\sigma$ of an arbitrary saddle point $\sigma \in \Omega_f$ there exists a sphere $S_\sigma^{n-2} \subset S^{n-1}$ such that $\psi_f(l_\sigma) = \bigcup_{t \in \mathbb{R}} a^t(S_\sigma^{n-2})$.

Recall that we denote by $L_f^s$ and $L_f^u$ the union of all $(n-1)$-dimensional stable and unstable separatrices of the diffeomorphism $f$ correspondingly. Put $L_f^s = \psi_f(L_f^s)$, $L_f^u = \psi_f(L_f^u)$. Then $L_f^\delta$ is the union of pair-vise disjoint cylinders $Q_\delta^k \cup \cdots \cup Q_\delta^k$, $\delta \in \{s,u\}$. Denote by $N(L_f^s) = N(Q_\delta^s) \cup \cdots \cup N(Q_\delta^s)$ the set of their pair-vise disjoint closed tubular neighborhoods such that $N(Q_\delta^s) = K_\delta^s \times \mathbb{R}$, where $K_\delta^s \subset S^{n-1}$ is an annulus of dimension $(n-1)$, $i = 1, \ldots, k^\delta$.

Define a flow $a_1^s$ on the set $U = \{x_1, \ldots, x_n \in \mathbb{R}^n| \ x_1^2(x_2^2 + \ldots + x_n^2) \leq 1 \}$ by $a_1^s(x_1, x_2, \ldots, x_n) = (2^i x_1, 2^{-i} x_2, \ldots, 2^{-i} x_n)$. It follows from Statements 4.4, 6.1 that there exists a homeomorphism $\chi_i^s : N(Q_\delta^s) \to \mathbb{N}_s$ such that $a_1^s|_{\mathbb{N}_s} = \chi_i^s a^s(\chi_i^s)|_{\mathbb{N}_s}$. Denote by $\chi^s : N(L_f^s) \to U \times \mathbb{Z}_{k^s}$ a homeomorphism such that $\chi^s|_{N(Q_\delta^s)} = \chi_i^s$ for any $i \in \{1, \ldots, k^s\}$. Put $Q^s = (S^{n-1} \times \mathbb{R}) \cup_{\psi_s} (U \times \mathbb{Z}_{k^s})$. A topological space $Q^s$ is a connected oriented $n$-manifold without boundary.

Denote by $\pi_s : (S^{n-1} \times \mathbb{R}) \cup (U \times \mathbb{Z}_{k^s}) \to Q^s$ a natural projection. Put $\pi_{s,1} = \pi_s|_{S^{n-1} \times \mathbb{R}}$, $\pi_{s,2} = \pi_s|_{U \times \mathbb{Z}_{k^s}}$. Define a flow $\hat{Y}_s^t$ on the manifold $Q^s$ by

$$\hat{Y}_s^t(x) = \begin{cases} \pi_{s,1}(a_1^s(\pi_{s,1}^{-1}(x))), & x \in \pi_{s,1}(S^{n-1} \times \mathbb{R}); \\ \pi_{s,2}(a_1^u(\pi_{s,2}^{-1}(x))), & x \in \pi_{s,2}(U \times \{i\}), i \in \mathbb{Z}_{k^s} \end{cases}.$$

By construction the non-Wandering set of the flow $\hat{Y}_s^t$ consists of $k^s$ equilibria such that the flow $\hat{Y}_s^t$ is locally topologically conjugated with the flow $a_1^s$ at the neighborhood of each equilibrium.

**Step 2.** Denote the images of the sets $L_1^u$, $N(L_1^u)$ by means of the projection $\pi_1$ by the same symbols as their originals. Due to Statements 4.4, 6.1 there exists a homeomorphism $\chi_i^u : N(Q_\delta^u) \to \mathbb{N}_u$ such that $\chi_i^u|_{\mathbb{N}_u} = \chi_i^u a^u(\chi_i^u)^{-1}$, $i = 1, \ldots, k^u$. Denote by $\chi^u : N(L_1^u) \to U \times \mathbb{Z}_{k^u}$ the homeomorphism such that $\chi^u|_{N(Q_\delta^u)} = \chi_i^u|_{N(Q_\delta^u)}$ for any $i = 1, \ldots, k^u$. Put $Q^u = Q^s \cup_{\psi_u} (U \times \mathbb{Z}_{k^u})$. A topological space $Q^u$ is a connected oriented $n$-manifold without boundary.
Denote by \( \pi_u : Q^s \cup (U \times \mathbb{Z}_{k^u}) \to Q^u \) the natural projection. Put \( \pi_{u,1} = \pi_u|Q^s \), \( \pi_{u,2} = \pi_u|U \times \mathbb{Z}_{k^u} \). Define a flow \( \tilde{Y}_u^t \) on the manifold \( Q^u \) by
\[
\tilde{Y}_u^t(x) = \begin{cases}
\pi_{u,1}(Y_u^t(\pi_u^{-1}(x))), & x \in \pi_{u,1}(Q^s); \\
\pi_{u,2}(a_{t}^{-1}(\pi_u^{-1}(x))), & x \in \pi_{u,2}(U \times \{i\}),
\end{cases} \quad i \in \mathbb{Z}_{k^u}.
\]

The non-wandering set \( \Omega_{\tilde{Y}_u^t} \) of the flow \( \tilde{Y}_u^t \) consists of \( k^u \) equilibria such that the flow \( \tilde{Y}_u^t \) is locally topological conjugated with the flow \( a_{t}^{-1} \) in each of their neighborhoods and \( k^u \) equilibria such that the flow \( \tilde{Y}_u^t \) is locally topologically conjugated with the flow \( a_{t}^{-1} \) in each of their neighborhoods.

**Step 3.** Put \( R^s = Q^u \setminus W_{\Omega_{\tilde{Y}_u^t}} \), denote by \( \rho_1^s, \ldots, \rho_n^s \) connected components of the set \( R^s \) and put \( \tilde{\rho}_i^s = \rho_i^s/\gamma_{\tilde{u}}^s \). A union of the orbit spaces \( \bigcup_{i=1}^{n^s} \tilde{\rho}_i^s \) is obtained from the manifold \( \tilde{\nu} \) by a sequence of the surgeries along essential submanifolds of codimension 1. In virtue of Proposition 5.4, for any \( i \in \{1, \ldots, n^s\} \) the manifold \( \tilde{\rho}_i^s \) is homeomorphic to \( \mathbb{S}^{n-1} \times \mathbb{S}^1 \), the manifold \( \rho_i^s \) is homeomorphic to \( \mathbb{S}^{n-1} \times \mathbb{R} \) and the flow \( \tilde{Y}_u^t|\tilde{\rho}_i^s \) is topologically conjugated with the flow \( a_t^{-1}|\rho_i^s \) by means of a homeomorphism \( \nu_i^s \). Denote by \( \nu^s : R^s \to (\mathbb{R}^n \setminus \{0\}) \times \mathbb{Z}_{n^s} \) the homeomorphism consisting of the homeomorphisms \( \nu_1^s, \ldots, \nu_n^s \). Put \( M^s = Q^u \cup \rho_i^s (\mathbb{R}^n \times \mathbb{Z}_{n^s}) \). Then \( M^s \) is a connected oriented \( n \)-manifold without boundary.

Put \( M^s = M^s \cup \rho_i^s (\mathbb{R}^n \times \mathbb{Z}_{n^s}) \) and denote by \( q_i : M^s \to M^s \) the natural projection. Put \( q_{u,1} = q_i|Q^s \), \( q_{u,2} = q_i|\mathbb{R}^n \times \mathbb{Z}_{n^s} \). Define a flow \( \tilde{X}_u^t \) on the manifold \( M^s \) by
\[
\tilde{X}_u^t(x) = \begin{cases}
q_{u,1}(Y_u^t(q_{u,1}^{-1}(x))), & x \in q_{u,1}(Q^s); \\
q_{u,2}(a_t^{-1}(q_{u,2}^{-1}(x))), & x \in q_{u,2}(\mathbb{R}^n \times \{i\}),
\end{cases} \quad i \in \mathbb{Z}_{n^s}.
\]

By construction the non-wandering set of the time one map of the flow \( \tilde{X}_u^t \) consists of \( k^s \) saddle topologically hyperbolic fixed points of index 1, \( k^s \) saddle topologically hyperbolic fixed points of index \( n-1 \) and \( n^s \) sink topologically hyperbolic fixed points.

**Step 4.** Put \( R^u = M^s \setminus W_{\tilde{X}_u^t} \) and denote by \( \rho_1^u, \ldots, \rho_n^u \) connected components of the set \( R^u \). Similar to Step 3 one can prove that every component \( \rho_i^u \) is homeomorphic to \( \mathbb{S}^{n-1} \times \mathbb{R} \) and the flow \( \tilde{X}_u^t|\rho_i^u \) is conjugated with the flow \( a_t^{-1}|\rho_i^u \setminus \{O\} \) by a homeomorphism \( \mu_i^u \). Denote by \( \mu^u : R^u \to (\mathbb{R}^n \setminus \{O\}) \times \mathbb{Z}_{n^u} \) a homeomorphism consisting of the homeomorphisms \( \mu_1^u, \ldots, \mu_n^u \). Put \( M^u = M^s \cup \mu_i^u (\mathbb{R}^n \times \mathbb{Z}_{n^u}) \). \( M^u \) is a connected closed oriented \( n \)-manifold.

Put \( M^u = M^s \cup (\mathbb{R}^n \times \mathbb{Z}_{n^u}) \), denote by \( q_u : M^u \to M^u \) the natural projection, and put \( q_{u,1} = q_u|M^s \), \( q_{u,2} = q_u|\mathbb{R}^n \times \mathbb{Z}_{n^u} \). Define a flow \( \tilde{X}_u^t \) on the manifold \( M^u \) by
\[
\tilde{X}_u^t(x) = \begin{cases}
q_{u,1}(X_u^t(q_{u,1}^{-1}(x))), & x \in q_{u,1}(M^s); \\
q_{u,2}(a_t^{-1}(q_{u,2}^{-1}(x))), & x \in q_{u,2}(\mathbb{R}^n \times \{i\}),
\end{cases} \quad i \in \mathbb{Z}_{n^u}.
\]

By construction the non-wandering set of the time one map of the flow \( \tilde{X}_u^t \) consists of \( k^u \) saddle topologically hyperbolic fixed points of index 1, \( k^u \) saddle topologically hyperbolic fixed points of index \( n-1 \), \( n^u \) sink and \( n^u \) source topologically hyperbolic fixed points.

**Step 5.** Put \( f = \tilde{X}_u^1 \). By construction \( f \) is a Morse-Smale homeomorphism on the manifold \( M^u \) and its restriction \( f|\nu_f \) is topologically conjugated with the diffeomorphism \( f|\nu_f \) by a homeomorphism mapping the \( (n-1) \)-dimensional separatrices of the diffeomorphism \( f \) to the \( (n-1) \)-dimensional separatrices of the diffeomorphism \( f \) and preserving their stability. Due to Statement 3.1 homeomorphisms \( f \) and \( f \) are topologically conjugated. Hence \( M^u = S^u \) and \( X^t = X_u^t \) is the desired flow.
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