The Halo Mass Function from Excursion Set Theory with a Non-Gaussian Trispectrum

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Abstract: A sizeable level of non-Gaussianity in the primordial cosmological perturbations may be induced by a large trispectrum, i.e. by a large connected four-point correlation function. We compute the effect of a primordial non-Gaussian trispectrum on the halo mass function, within excursion set theory. We use the formalism that we have developed in a previous series of papers and which allows us to take into account the fact that, in the presence of non-Gaussianity, the stochastic evolution of the smoothed density field, as a function of the smoothing scale, is non-markovian. In the large mass limit, the leading-order term that we find agrees with the leading-order term of the results found in the literature using a more heuristic Press-Schechter (PS)-type approach. Our approach however also allows us to evaluate consistently the subleading terms, which depend not only on the four-point cumulant but also on derivatives of the four-point correlator, and which cannot be obtained within non-Gaussian extensions of PS theory. We perform explicitly the computation up to next-to-leading order.
1. Introduction

Over the last decade a great deal of evidence has been accumulated from the Cosmic Microwave Background (CMB) anisotropy and Large Scale Structure (LSS) spectra that the observed structures originated from seed fluctuations generated during a primordial stage of inflation. While standard single-field models of slow-roll inflation predict that these fluctuations are very close to gaussian (see [1, 2]), non-standard scenarios allow for a larger level of non-Gaussianity (NG) (see [3] and references therein). A signal is gaussian if the information it carries is completely encoded in the two-point correlation function, all higher connected correlators being zero. Deviations from Gaussianity are therefore encoded, e.g., in the connected three- and four-point correlation functions which are dubbed the bispectrum and the trispectrum, respectively. A phenomenological way of parametrizing the level of NG is to expand the fully non-linear primordial Bardeen gravitational potential $\Phi$ in powers of the linear gravitational potential $\Phi_L$

$$\Phi = \Phi_L + f_{NL} (\Phi_L^2 - \langle \Phi_L^2 \rangle) + g_{NL} \Phi_{NL}^3. \quad (1.1)$$

The dimensionless quantities $f_{NL}$ and $g_{NL}$ set the magnitude of the three- and four-point correlation functions, respectively [3]. If the process generating the primordial non-Gaussianity is local in space, the parameter $f_{NL}$ and $g_{NL}$ in Fourier space are independent of the momenta entering the corresponding correlation functions; if instead the process which generates the primordial cosmological perturbations is non-local in space, like in models of inflation with non-canonical kinetic terms, $f_{NL}$ and $g_{NL}$ acquire a dependence on the momenta. It is clear that detecting a significant amount of non-Gaussianity and its shape either from the CMB or from the LSS offers the possibility of opening a window into the dynamics of the universe during the very first stages of its evolution. Non-Gaussianities are particularly relevant in the high-mass end of the power spectrum of perturbations, i.e. on the scale of galaxy clusters, since the effect of NG fluctuations becomes especially visible.
on the tail of the probability distribution. As a result, both the abundance and the clustering properties of very massive halos are sensitive probes of primordial non-Gaussianities [4, 5, 6, 7, 8, 9, 10, 11], and could be detected or significantly constrained by the various planned large-scale galaxy surveys, both ground based (such as DES, PanSTARRS and LSST) and on satellite (such as EUCLID and ADEPT) see, e.g. [12] and [13]. Furthermore, the primordial non-Gaussianity alters the clustering of dark matter halos inducing a scale-dependent bias on large scales [12, 14, 15, 16] while even for small primordial non-Gaussianity the evolution of perturbations on super-Hubble scales yields extra contributions on smaller scales [17, 18]. The strongest current limits on the strength of local non-Gaussianity set the $f_{NL}$ parameter to be in the range $-4 < f_{NL} < 80$ at 95% confidence level [19].

While the literature on NG has vastly focussed on the impact on observables induced by a non-vanishing bispectrum, only recently attention has been devoted to the impact of a non-vanishing trispectrum of cosmological perturbations [20, 21, 22, 23, 24]. This has been computed in several models, such as multifield slow-roll inflation model [25, 26, 27, 28], the curvaton model [29], theories with non-canonical kinetic terms both in single field [30, 31, 32, 33] and in the multifield case [34], and in the case in which the cosmological perturbations are induced by vector perturbations of some non-abelian gauge field [35]. While the most natural value of the $g_{NL}$ parameter is $O(f_{NL}^2)$, there are cases in which $|g_{NL}| \gg 1$ even if $f_{NL}$ is tiny [36]. The effects of a cubic correction to the primordial gravitational potential onto the mass function and bias of DM haloes have been recently analyzed in [37] where the theoretical predictions have been compared to the results extracted from a series of large $N$–body simulations. The limit $-3.5 \cdot 10^5 < g_{NL} < +8.2 \cdot 10^5$ has been obtained at 95% confidence level in the case in which the NG is of the local type.

The goal of this paper is to present the computation of the DM halo mass function from the excursion set theory in the presence of a trispectrum, thus extending our previous computation of the DM halo mass function when the NG is induced by a bispectrum [38]. The halo mass function can be written as

$$\frac{dn(M)}{dM} = f(\sigma) \frac{\bar{\rho}}{M^2} \frac{d\ln \sigma^{-1}(M)}{d\ln M},$$

where $n(M)$ is the number density of dark matter halos of mass $M$, $\sigma(M)$ is the variance of the linear density field smoothed on a scale $R$ corresponding to a mass $M$, and $\bar{\rho}$ is the average density of the universe. Analytical derivations of the halo mass function are typically based on Press-Schechter (PS) theory [39] and its extension [40, 41] known as excursion set theory (see [42] for a recent review). In excursion set theory the density perturbation evolves stochastically with the smoothing scale, and the problem of computing the probability of halo formation is mapped into the so-called first-passage time problem in the presence of a barrier.

The computation of the effect of a primordial trispectrum on the mass function has been performed in [9, 43, 37], and is based on NG extensions of Press-Schechter theory [39]. Performing the computation using the two different approximations proposed in [9] and [43], respectively, to evaluate the impact of a non-vanishing bispectrum on the DM
halo mass functions, one finds that, to leading order in the large mass limit, the predicted halo mass functions are the same for the two methods, but differ in the subleading terms, i.e. in the intermediate mass regime. Apart from understanding what is the correct result for the subleading terms, there is yet a fundamental reason why we wish to apply the excursion set theory to compute the DM mass function with a non-vanishing trispectrum. The derivation of the PS mass function in [41] requires that the density field $\delta$ evolves with the smoothing scale $R$ (or more precisely with the variance $S(R)$ of the smoothed density field) in a markovian way. Only under this assumption one can derive the correct factor of two that Press and Schechter were forced to introduce by hand. As we have discussed at length in [44], this markovian assumption is broken by the use of a filter function different from a sharp filter in momentum space and, of course, it is further violated by the inclusion of non-Gaussian corrections. The non-markovianity induced by the NG introduces memory effects which have to be appropriately accounted for in the excursion set. As we will see, the mass function indeed gets “memory” corrections in the intermediate mass regime, which depend on derivatives of the correlators, and therefore cannot be computed with the NG extensions of PS theory studied in [9, 43].

This paper is organized as follows. In Section 2, we recall the basic points of the formalism developed in [44] for gaussian fluctuations, and extended in [38] to the NG case. In Section 3 we compute the NG corrections with the excursion set method induced by a large trispectrum, and we present our results for the halo mass function. Finally, in Section 4 we present our conclusions.

2. The basic principles of the computation

2.1 Excursion set theory

In excursion set theory one considers the density field $\delta$ smoothed over a radius $R$ with a tophat filter in coordinate space, and studies its stochastic evolution as a function of the smoothing scale $R$. As it was found in the classical paper [41], when the density $\delta(R)$ is smoothed with a sharp filter in momentum space, and the density fluctuations have gaussian statistics, the smoothed density field satisfies the equation

$$\frac{\partial \delta(S)}{\partial S} = \eta(S),$$

(2.1)

where $S = \sigma^2(R)$ is the variance of the linear density field smoothed on the scale $R$ and computed with a sharp filter in momentum space, while $\eta(S)$ is a stochastic variable that satisfies

$$\langle \eta(S_1)\eta(S_2) \rangle = \delta_D(S_1 - S_2),$$

(2.2)

where $\delta_D$ denotes the Dirac delta function. Equations (2.1) and (2.2) are the same as a Langevin equation with a Dirac-delta noise $\eta(S)$, with the variance $S$ formally playing the role of time. Let us denote by $\Pi(\delta, S)d\delta$ the probability density that the variable $\delta(S)$ reaches a value between $\delta$ and $\delta + d\delta$ by time $S$. A textbook result in statistical physics is that, if a variable $\delta(S)$ satisfies a Langevin equation with a Dirac-delta noise,
the probability density $\Pi(\delta, S)$ satisfies the Fokker-Planck (FP) equation
\[
\frac{\partial \Pi}{\partial S} = \frac{1}{2} \frac{\partial^2 \Pi}{\partial \delta^2}.
\] (2.3)
The solution of this equation over the whole real axis $-\infty < \delta < \infty$, with the boundary condition that it vanishes at $\delta = \pm \infty$, is
\[
\Pi^0(\delta, S) = \frac{1}{\sqrt{2\pi S}} e^{-\delta^2/(2S)}.
\] (2.4)
and is nothing but the distribution function of PS theory. Since large $R$, i.e. large halo masses, correspond to small values of the variance $S$, in ref. [41] it was realized that we are actually interested in the stochastic evolution of $\delta$ against $S$ only until the “trajectory” crosses for the first time the threshold $\delta_c$ for collapse. All the subsequent stochastic evolution of $\delta$ as a function of $S$, which in general results in trajectories going multiple times above and below the threshold, is irrelevant, since it corresponds to smaller-scale structures that will be erased and engulfed by the collapse and virialization of the halo corresponding to the largest value of $R$, i.e. the smallest value of $S$, for which the threshold has been crossed. In other words, trajectories should be eliminated from further consideration once they have reached the threshold for the first time. In ref. [41] this is implemented by imposing the boundary condition
\[
\Pi(\delta, S)|_{\delta = \delta_c} = 0.
\] (2.5)
The solution of the FP equation with this boundary condition is
\[
\Pi(\delta, S) = \frac{1}{\sqrt{2\pi S}} \left[ e^{-\delta^2/(2S)} - e^{-(2\delta_c - \delta)^2/(2S)} \right],
\] (2.6)
and gives the distribution function of excursion set theory. The first term is the PS result, while the second term in eq. (2.6) is an “image” gaussian centered in $\delta = 2\delta_c$. Integrating this $\Pi(\delta, S)$ over $d\delta$ from $-\infty$ to $\delta_c$ gives the probability that a trajectory, at “time” $S$, has always been below the threshold. Increasing $S$ this integral decreases because more and more trajectories cross the threshold for the first time, so the probability of first crossing the threshold between “time” $S$ and $S + dS$ is given by $\mathcal{F}(S)dS$, with
\[
\mathcal{F}(S) = -\frac{\partial}{\partial S} \int_{-\infty}^{\delta_c} d\delta \Pi(\delta; S).
\] (2.7)
With standard manipulations (see e.g. [42] or [44]) one then finds that the function $f(\sigma)$ which appears in eq. (1.2) is given by
\[
f(\sigma) = 2\sigma^2 \mathcal{F}(\sigma^2),
\] (2.8)
where we wrote $S = \sigma^2$. Using eq. (2.6) one finds
\[
f(\sigma) = \left( \frac{2}{\pi} \right)^{1/2} \frac{\delta_c}{\sigma} e^{-\delta_c^2/(2\sigma^2)},
\] (2.9)
Observe that, when computing the first-crossing rate, the contribution of the gaussian centered in $\delta = 0$ and of the image gaussian in eq. (2.6) add up, giving the factor of two that was missed in the original PS theory.
2.2 Refinements of excursion set theory

While excursion set theory is quite elegant, and gives a first analytic understanding of the halo mass function, it suffers of two important set of problems. First, it is based on the spherical (or ellipsoidal) collapse model, which is a significant oversimplification of the actual complex dynamics of halo formation. We have discussed these limitations in detail in [45], where we also proposed that some of the physical complications inherent to a realistic description of halo formation can be included in the excursion set theory framework, at least at an effective level, by taking into account that the critical value for collapse is not a fixed constant \( \delta_c \), as in the spherical collapse model, nor a fixed function of the variance \( \sigma \) of the smoothed density field, as in the ellipsoidal collapse model, but rather is itself a stochastic variable, whose scatter reflects a number of complicated aspects of the underlying dynamics. The simplest implementation of this idea consists in solving the first-passage time problem in the presence of a barrier that performs a random walk, with diffusion coefficient \( D_B \), around an average value given by the constant barrier of the spherical collapse model (more generally, one should consider a barrier that fluctuates over an average value given by the ellipsoidal collapse model). In this simple case, we found in [45] that the exponential factor in the Press-Schechter mass function changes from \( \exp\left(-\frac{\delta_c^2}{2\sigma^2}\right) \) to \( \exp\left(-a\frac{\delta_c^2}{2\sigma^2}\right) \), where \( a = 1/(1 + D_B) \). In this approach all our ignorance on the details of halo formation is buried into \( D_B \). The numerical value of \( D_B \), and therefore the corresponding value of \( a \), depends among other things also on the details of the algorithm used for identifying halos (e.g. on the link-length in a Friends-of-Friends algorithm). Observe that the replacement of \( \exp\left(-\frac{\delta_c^2}{2\sigma^2}\right) \) with \( \exp\left(-a\frac{\delta_c^2}{2\sigma^2}\right) \) (with \( a \) taken however as a fitting parameter) is just the replacement that was made in refs. [46, 47], in order to fit the results of N-body simulations.

The second set of problems of excursion set theory is of a more technical nature, and is due to the fact that the Langevin equation with Dirac-delta noise, which is at the basis of the whole construction of ref. [41], can only be derived if one works with a sharp filter in momentum space, and if the fluctuations are gaussian. However, as it is well known, and as we have discussed at length in ref. [44], with such a filter it is not possible to associate a halo mass to the smoothing scale \( R \). A unambiguous relation between \( M \) and \( R \) is rather obtained with a sharp filter in coordinate space, in which case one simply has \( M = \frac{4}{3} \pi R^3 \rho \), where \( \rho \) is the density. When one uses a sharp filter in coordinate space, the evolution of the density with the smoothing scale however becomes non-markovian, and therefore the problem becomes technically much more difficult. In particular, the distribution function \( \Pi(\delta, S) \) no longer satisfies a local differential equation such as the FP equation. The issue is particularly relevant when one wants to include non-Gaussianities in the formalism, since again the inclusion of non-Gaussianities renders the dynamics non-markovian.

In refs. [44, 38] we have developed a formalism that allows us to generalize excursion set theory to the case of a non-markovian dynamics, either generated by the filter function or by primordial non-Gaussianities. The basic idea is the following. Rather than trying to derive a simple, local, differential equation for \( \Pi(\delta, S) \) (which, as we have shown in ref. [44], is
impossible; in the non-markovian case $\Pi(\delta, S)$ rather satisfies a very complicated equation which is non-local with respect to “time” $S)$, we construct the probability distribution $\Pi(\delta, S)$ directly by summing over all paths that never exceeded the threshold $\delta_c$, i.e. by writing $\Pi(\delta, S)$ as a path integral with boundaries. To obtain such a representation, we consider an ensemble of trajectories all starting at $S_0 = 0$ from an initial position $\delta(0) = \delta_0$ and we follow them for a “time” $S$. We discretize the interval $[0, S]$ in steps $\Delta S = \epsilon$, so $S_k = k\epsilon$ with $k = 1, \ldots, n$, and $S_n \equiv S$. A trajectory is then defined by the collection of values $\{\delta_1, \ldots, \delta_n\}$, such that $\delta(S_k) = \delta_k$. The probability density in the space of trajectories is

$$W(\delta_0; \delta_1, \ldots, \delta_n; S_n) \equiv \langle \delta_D(\delta(S_1) - \delta_1) \cdots \delta_D(\delta(S_n) - \delta_n) \rangle,$$  \hspace{1cm} (2.10)

where $\delta_D$ denotes the Dirac delta. Then the probability of arriving in $\delta_n$ in a “time” $S_n$, starting from an initial value $\delta_0$, without ever going above the threshold, is$^1$

$$\Pi_\epsilon(\delta_0; \delta_n; S_n) \equiv \int_{-\infty}^{\delta_c} d\delta_1 \cdots \int_{-\infty}^{\delta_c} d\delta_{n-1} W(\delta_0; \delta_1, \ldots, \delta_{n-1}, \delta_n; S_n).$$  \hspace{1cm} (2.11)

The label $\epsilon$ in $\Pi_\epsilon$ reminds us that this quantity is defined with a finite spacing $\epsilon$, and we are finally interested in the continuum limit $\epsilon \to 0$. As we discussed in $[14, 38]$, $W(\delta_0; \delta_1, \ldots, \delta_{n-1}, \delta_n; S_n)$ can be expressed in terms of the connected correlators of the theory,

$$W(\delta_0; \delta_1, \ldots, \delta_n; S_n) = \int \mathcal{D}\lambda$$

$$\exp \left\{ i \sum_{i=1}^{n} \lambda_i \delta_i + \sum_{p=2}^{\infty} \frac{(-i)^p}{p!} \sum_{i_1=1}^{n} \cdots \sum_{i_p=1}^{n} \lambda_{i_1} \cdots \lambda_{i_p} \langle \delta_{i_1} \cdots \delta_{i_p} \rangle_c \right\},$$

where

$$\int \mathcal{D}\lambda \equiv \int_{-\infty}^{\infty} \frac{d\lambda_1}{2\pi} \cdots \frac{d\lambda_n}{2\pi},$$  \hspace{1cm} (2.13)

$\delta_i = \delta(S_i)$, and $\langle \delta_{i_1} \cdots \delta_{i_p} \rangle_c$ denotes the connected $n$-point correlator. So

$$\Pi_\epsilon(\delta_0; \delta_n; S_n) \equiv \int_{-\infty}^{\delta_c} d\delta_1 \cdots d\delta_{n-1} \int \mathcal{D}\lambda$$

$$\exp \left\{ i \sum_{i=1}^{n} \lambda_i \delta_i + \sum_{p=2}^{\infty} \frac{(-i)^p}{p!} \sum_{i_1=1}^{n} \cdots \sum_{i_p=1}^{n} \lambda_{i_1} \cdots \lambda_{i_p} \langle \delta_{i_1} \cdots \delta_{i_p} \rangle_c \right\}.$$  \hspace{1cm} (2.14)

When $\delta(S)$ satisfies eqs. (2.1) and (2.2) (which is the case for sharp filter in momentum space) the two-point function can be easily computed, and is given by

$$\langle \delta(S_i)\delta(S_j) \rangle = \min(S_i, S_j).$$  \hspace{1cm} (2.15)

$^1$In eqs. (2.3) and (2.4) we implicitly assumed $\delta_0 = 0$. In the following however it will be necessary to keep track also of the initial position $\delta_0$. 

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If furthermore we consider gaussian fluctuations, all $n$-point connected correlators with $n \geq 3$ vanish, and the probability density $W$ can be computed explicitly,

$$W^\text{gm}(\delta_0; \delta_1, \ldots, \delta_n; S_n) = \frac{1}{(2\pi \epsilon)^n/2} \exp \left\{ -\frac{1}{2\epsilon} \sum_{i=0}^{n-1} (\delta_{i+1} - \delta_i)^2 \right\},$$

(2.16)

where the superscript “\text{"gm"}” (gaussian-markovian) reminds that this value of $W$ is computed for gaussian fluctuations, whose dynamics with respect to the smoothing scale is markovian. Using this result, in [44] we have shown that, in the continuum limit, the computed for gaussian fluctuations, whose dynamics with respect to the smoothing scale is markovian. Using this result, in [44] we have shown that, in the continuum limit, the distribution function $\Pi_{\epsilon=0}(\delta; S)$, computed with a sharp filter in momentum space, satisfies a Fokker-Planck equation with the boundary condition $\Pi_{\epsilon=0}(\delta_c, S) = 0$, and we have therefore recovered, from our path integral approach, the distribution function of excursion set theory, eq. (2.6).

When the consider a different filter, eq. (2.15) is replaced by

$$\langle \delta(S_i)\delta(S_j) \rangle = \min(S_i, S_j) + \Delta(S_i, S_j),$$

(2.17)

where $\Delta(S_i, S_j)$ describes the deviations from a markovian dynamics. For instance, for a sharp filter in coordinate space, which is the most interesting case, the function $\Delta(S_i, S_j)$ is very well approximated by

$$\Delta(S_i, S_j) \simeq \kappa \frac{S_j(S_j - S_i)}{S_j},$$

(2.18)

(for $S_i \leq S_j$, and is symmetric under exchange of $S_i$ and $S_j$), with $\kappa \simeq 0.45$. The non-markovian corrections can then be computed expanding perturbatively in $\kappa$. The computation, which is quite non-trivial from a technical point of view, has been discussed in great detail in [44]. Let us summarize here the crucial points. First of all, expanding to first order in $\Delta_{ij}$ and using $\lambda_i e^{i \sum_k \lambda_k \delta_k} = -i \partial_i e^{i \sum_k \lambda_k \delta_k}$, where $\partial_i = \partial/\partial \delta_i$, the first-order correction to $\Pi_\epsilon$ is

$$\Pi_\epsilon^\Delta 1(\delta_0; \delta_n; S_n) \equiv \int_{-\infty}^{\delta_c} d\delta_1 \ldots d\delta_{n-1} \frac{1}{2} \sum_{i,j=1}^{n} \Delta_{ij} \partial_i \partial_j$$

$$\times \int \mathcal{D}\lambda \exp \left\{ i \sum_{i=1}^{n} \lambda_i \delta_i - \frac{1}{2} \sum_{i,j=1}^{n} \min(S_i, S_j) \lambda_i \lambda_j \right\}$$

$$= \frac{1}{2} \sum_{i,j=1}^{n} \Delta_{ij} \int_{-\infty}^{\delta_c} d\delta_1 \ldots d\delta_{n-1} \partial_i \partial_j W^\text{gm}(\delta_0; \delta_1, \ldots, \delta_n; S_n),$$

(2.19)

where we use the notation $\Delta_{ij} = \Delta(S_i, S_j)$. One then observes that the derivatives $\partial_i$ run over $i = 1, \ldots, n$, while we integrate only over $d\delta_1 \ldots d\delta_{n-1}$. Therefore, derivatives $\partial_i$ with $i = n$ can be simply carried outside the integrals. Derivatives $\partial_i$ with $i = 1, \ldots, n - 1$ are dealt as follows. Consider for instance the terms with $i < n$ and $j = n$ (together with $j < n$ and $i = n$, which gives a factor of two). These are given by

$$\sum_{i=1}^{n-1} \Delta_{in} \partial_n \int_{-\infty}^{\delta_c} d\delta_1 \ldots d\delta_{n-1} \partial_i W^\text{gm}(\delta_0; \delta_1, \ldots, \delta_n; S_n),$$

(2.20)
To compute this expression we integrate \( \partial_i \) by parts,

\[
\int_{-\infty}^{\delta_c} d\delta_1 \ldots d\delta_{n-1} \partial_i W^{gm}(\delta_0; \delta_1, \ldots, \delta_n; S_n)
\]

\[
= \int_{-\infty}^{\delta_c} d\delta_1 \ldots \hat{d}\delta_i \ldots d\delta_{n-1} W(\delta_0; \delta_1, \ldots, \delta_i = \delta_c, \ldots, \delta_{n-1}, \delta_n; S_n),
\]

where the notation \( \hat{d}\delta_i \) means that we must omit \( d\delta_i \) from the list of integration variables. We next observe that \( W^{gm} \) satisfies

\[
W^{gm}(\delta_0; \delta_1, \ldots, \delta_i = \delta_c, \ldots, \delta_n; S_n)
\]

\[
= W^{gm}(\delta_0; \delta_1, \ldots, \delta_{i-1}, \delta_c; S_i) W^{gm}(\delta_c; \delta_{i+1}, \ldots, \delta_n; S_n - S_i),
\]

as can be verified directly from its explicit expression (2.16). Then

\[
\int_{-\infty}^{\delta_c} d\delta_1 \ldots d\delta_{i-1} \int_{-\infty}^{\delta_c} d\delta_{i+1} \ldots d\delta_{n-1}
\]

\[
\times W^{gm}(\delta_0; \delta_1, \ldots, \delta_{i-1}, \delta_c; S_i) W^{gm}(\delta_c; \delta_{i+1}, \ldots, \delta_n; S_n - S_i)
\]

\[
= \Pi^c_{\epsilon}(\delta_0; \delta_c; S_i) \Pi^c_{\epsilon}(\delta_c; \delta_n; S_n - S_i),
\]

(2.23)

and to compute the expression given in eq. (2.20) we must compute

\[
\sum_{i=1}^{n-1} \Delta_{in} \Pi^{gm}_{\epsilon}(\delta_0; \delta_c; S_i) \Pi^{gm}_{\epsilon}(\delta_c; \delta_n; S_n - S_i).
\]

(2.24)

To proceed further, we need to know \( \Pi^c_{\epsilon}(\delta_0; \delta_c; S_i) \). By definition, for \( \epsilon = 0 \) this quantity vanishes, since its second argument is equal to the the threshold value \( \delta_c \), compare with eq. (2.25). However, in the continuum limit the sum over \( i \) becomes \( 1/\epsilon \) times an integral over an intermediate time variable \( S_i \),

\[
\sum_{i=1}^{n-1} \rightarrow \frac{1}{\epsilon} \int_{0}^{S_n} dS_i,
\]

(2.25)

so we need to know how \( \Pi^c_{\epsilon}(\delta_0; \delta_c; S_i) \) approaches zero when \( \epsilon \rightarrow 0 \). In [44] we proved that it vanishes as \( \sqrt{\epsilon} \), and that

\[
\Pi^c_{\epsilon}(\delta_0; \delta_c; S) = \sqrt{\epsilon} \frac{1}{\sqrt{\pi}} \frac{\delta_c - \delta_0}{S^{3/2}} e^{-(\delta_c-\delta_0)^2/(2S)} + O(\epsilon).
\]

(2.26)

Similarly, for \( \delta_n < \delta_c \),

\[
\Pi^c_{\epsilon}(\delta_c; \delta_n; S) = \sqrt{\epsilon} \frac{1}{\sqrt{\pi}} \frac{\delta_c - \delta_n}{S^{3/2}} e^{-(\delta_c-\delta_n)^2/(2S)} + O(\epsilon).
\]

(2.27)

Therefore the two factor \( \sqrt{\epsilon} \) from eqs. (2.26) and (2.27) produce just an overall factor of \( \epsilon \) that compensate the factor \( 1/\epsilon \) in eq. (2.23), and we are left with a finite integral over \( dS_i \). Terms with two or more derivative, e.g. \( \partial_i \partial_j \), or \( \partial_i, \partial_j \partial_k \) acting on \( W_{ij} \), with all indices \( i, j, k \) smaller than \( n \), can be computed similarly, and have been discussed in detail in [44].
3. Contribution of the trispectrum to the halo mass function

The effect of non-Gaussianities can be computed similarly, expanding perturbatively eq. (2.14) in terms of the higher-order correlators. In [38] we have examined the three-point correlator, i.e. the bispectrum. Here we compute the effect of the trispectrum.

If in eq. (2.14) we only retain the four-point correlator, and we use the tophat filter in coordinate space, we have

\[
\Pi^c_4(\delta_0; \delta_n; S_n) = \int_{-\infty}^{\delta_c} d\delta_1 \ldots d\delta_{n-1} \int D\lambda \\
\times \exp \left\{ i\lambda_i \delta_i - \frac{1}{2} \left[ \text{min}(S_i, S_j) + \Delta_{ij} \right] \lambda_i \lambda_j + \frac{(-i)^4}{24} \langle \delta_i \delta_j \delta_k \delta_l \rangle \lambda_i \lambda_j \lambda_k \lambda_l \right\}.
\]

(3.1)

Expanding to first order, \( \Delta_{ij} \) and \( \langle \delta_i \delta_j \delta_k \delta_l \rangle \) do not mix, so we must compute

\[
\Pi^c_4(\delta_0; \delta_n; S_n) \equiv \frac{1}{24} \sum_{i,j,k,l=1}^{n} \langle \delta_i \delta_j \delta_k \delta_l \rangle \int_{-\infty}^{\delta_c} d\delta_1 \ldots d\delta_{n-1} \partial_i \partial_j \partial_k \partial_l W^{gm},
\]

(3.2)

where the superscript (4) in \( \Pi^c_4 \) refers to the fact that this is the contribution linear in the four-point correlator.

In principle this expression can be computed with the same technique discussed above, separating the various contributions to the sum according to whether an index is equal or smaller than \( n \). In this way, however, the computations faces some technical difficulties.\(^2\)

Fortunately, the problem simplifies considerably in the limit of large halo masses, which is just the physically interesting limit. Large masses mean small values of \( \sigma \), i.e. for \( \sigma/\delta_c \ll 1 \), and the leading contribution to the halo mass function is given by the term in eq. (3.4) with \( p = q = r = s = 0 \). At next-to-leading order we must also include the contribution of the

\[G_4^{(p,q,r,s)}(S_n) = \left[ \frac{d^p}{dS_i^p} \frac{d^q}{dS_j^q} \frac{d^r}{dS_k^r} \frac{d^s}{dS_l^s} \langle \delta(S_i) \delta(S_j) \delta(S_k) \delta(S_l) \rangle \right]_{S_i=S_j=S_k=S_l=S_n}.\]

(3.3)

Then

\[
\langle \delta(S_i) \delta(S_j) \delta(S_k) \delta(S_l) \rangle = \sum_{p,q,r,s=0}^{\infty} \frac{(-1)^{p+q+r+s}}{p!q!r!s!} \times (S_n - S_i)^p (S_n - S_j)^q (S_n - S_k)^r (S_n - S_l)^s G_4^{(p,q,r,s)}(S_n).
\]

(3.4)

Terms with more and more derivatives give contributions to the function \( f(\sigma) \), defined in eq. (1.2), that are subleading in the limit of small \( \sigma \), i.e. for \( \sigma/\delta_c \ll 1 \), and the leading contribution to the halo mass function is given by the term in eq. (3.4) with \( p = q = r = s = 0 \). At next-to-leading order we must also include the contribution of the

\(^2\)In particular, when we have terms with three or more derivatives, we need to generalize eq. (2.27), including terms up to \( O(e^{3/2}) \), which is quite non-trivial.
terms in eq. (3.4) with \( p + q + r + s = 1 \), i.e. the four terms \((p = 1, q = 0, r = 0, s = 0), (p = 0, q = 1, r = 0, s = 0), (p = 0, q = 0, r = 1, s = 0)\) and \((p = 0, q = 0, r = 0, s = 1)\); at next-to-next-to-leading order we must include the contribution of the terms in eq. (3.4) with \( p + q + r = 2 \), and so on. For the purpose of organizing the expansion in leading term, subleading terms, etc., we can reasonably expect that, for small \( S_n \)

\[
G_4^{(p,q,r,s)}(S_n) \sim S_n^{-(p+q+r+s)} \langle \delta^4(S_n) \rangle ,
\]

(3.5)
i.e. each derivative \( \partial/\partial S_i \), when evaluated in \( S_i = S_n \), gives a factor of order \( 1/S_n \). This ordering will be assumed when we present our final result for the halo mass function below. However, our formalism allows us to compute each contribution separately, so our results below can be easily generalized in order to cope with a different hierarchy between the various \( G_4^{(p,q,r,s)}(S_n) \).

The leading term in \( \Pi^{(4)} \) is

\[
\Pi^{(4,L)}_\epsilon(\delta_0; \delta_n; S_n) = \frac{\langle \delta^4 \rangle}{24} \sum_{i,j,k,l=1}^{n} \int_{-\infty}^{\delta_c} \ldots d\delta_n \ldots d\delta_i d\partial_i d\partial_j d\partial_k d\partial_l W^{gm} ,
\]

(3.6)

where the superscript “L” stands for “leading”. Since in the end we are interested in the integral over \( d\delta_n \) of \( \Pi^{(4,L)}_\epsilon(\delta_0; \delta_n; S_n) \), see eq. (2.7), we can write directly

\[
\int_{-\infty}^{\delta_c} d\delta_n \Pi^{(4,L)}_\epsilon(0; \delta_n; S_n) = \frac{\langle \delta^4(S_n) \rangle}{24} \sum_{i,j,k,l=1}^{n} \int_{-\infty}^{\delta_c} \ldots d\delta_n d\partial_i d\partial_j d\partial_k d\partial_l W^{gm} .
\]

(3.7)

This expression can be computed very easily by making use of identities that we proved in [4, 38]. Namely, we consider the derivative of \( \Pi^{gm} \) with respect to the threshold \( \delta_c \) (which, when we use the notation \( \Pi^{gm}(\delta_0; \delta_n; S_n) \), is not written explicitly in the list of variable on which \( \Pi^{gm} \) depends, but of course enters as upper integration limit in eq. (2.11)). Then one can show that

\[
\sum_{i=1}^{n} \int_{-\infty}^{\delta_c} d\delta_n \ldots d\partial_i W^{gm} = \frac{\partial}{\partial \delta_c} \int_{-\infty}^{\delta_c} d\delta_n \Pi^{gm} ,
\]

(3.8)

\[
\sum_{i,j=1}^{n} \int_{-\infty}^{\delta_c} d\delta_n \ldots d\partial_i d\partial_j W^{gm} = \frac{\partial^2}{\partial \delta_c^2} \int_{-\infty}^{\delta_c} d\delta_n \Pi^{gm} ,
\]

(3.9)

and similarly for all higher-order derivatives, so in particular

\[
\sum_{i,j,k,l=1}^{n} \int_{-\infty}^{\delta_c} d\delta_n \ldots d\partial_i d\partial_j d\partial_k d\partial_l W^{gm} = \frac{\partial^4}{\partial \delta_c^4} \int_{-\infty}^{\delta_c} d\delta_n \Pi^{gm} .
\]

(3.10)

Therefore, in the continuum limit, the right-hand side of eq. (3.7) is computed very simply, just by inserting in eq. (3.10) the value of \( \Pi^{gm}_\epsilon \) for \( \epsilon \to 0 \),

\[
\Pi^{gm}_{\epsilon=0}(\delta_0 = 0; \delta_n; S_n) = \frac{1}{\sqrt{2\pi S_n}} \left[ e^{-\delta^2_c/(2S_n)} - e^{-(2\delta_c - \delta_n)^2/(2S_n)} \right] ,
\]

(3.11)

\[ - 10 - \]
and therefore, in the continuum limit,
\[ \int_{-\infty}^{\delta_c} d\delta_n \Pi_{\epsilon=0}^{(4L)}(0; \delta_n; S_n) = \frac{\langle \delta_4 \rangle}{12 \sqrt{2\pi} S_5^5/2} \delta_c \left( 3 - \frac{\delta_c^2}{S_n} \right) e^{-\delta_c^2/(2S_n)}. \]  
(3.12)

We now insert this result into eqs. (2.7) and (2.8) and we express the result in terms of the normalized kurtosis
\[ S_4(\sigma) \equiv \frac{1}{6\sigma^4} \langle \delta^4(S) \rangle. \]  
(3.13)

Putting the contribution of \( \Pi^{(4L)} \) together with the gaussian contribution, and writing \( S = \sigma^2 \), we find
\[ f(\sigma) = \left( \frac{2}{\pi} \right) \frac{\delta_c}{\sigma} e^{-\delta_c^2/(2\sigma^2)} \left[ 1 + \frac{\sigma^2 S_4(\sigma)}{24} \left( \frac{\delta_c^4}{\sigma^4} - 4 \frac{\delta_c^2}{\sigma^2} - 3 \right) + \frac{1}{24 \sigma^2} \frac{dS_4}{d\ln \sigma} \left( \frac{\delta_c^2}{\sigma^2} - 3 \right) \right]. \]  
(3.14)

Let us emphasize that the variance \( \sigma^2 \) is to be computed applying the linear transfer function to the primordial gravitational potential \( \Pi_0 \) containing the extra \( g_{NL} \)-piece. The result given in eq. (3.14) agrees with the one obtained in [38] by performing the Edgeworth expansion of a non-Gaussian generalization of Press-Schechter theory. However, just as we have discussed in [38] for the case of the bispectrum, eq. (3.14) cannot be taken as the full result beyond leading order. If we want to compute consistently to NL order, we need to include the terms with \( p + q + r + s = 1 \) in eq. (3.4), which is given by
\[ \int_{-\infty}^{\delta_c} d\delta_n \Pi_{\epsilon=0}^{(4NL)}(\delta_0; \delta_n; S_n) = -\frac{4}{24} G_{4(1,0,0,0)}^{(1,0,0,0)}(S_n) \sum_{i=1}^{n} (S_n - S_i) \times \sum_{j,k,l=1}^{n} \int_{-\infty}^{\delta_c} d\delta_1 \ldots d\delta_{n-1} d\delta_n \partial_j \partial_k \partial_l W_{gm}, \]  
(3.15)

where the superscript “NL” in \( \Pi_{\epsilon=0}^{(4NL)} \) stands for next-to-leading, and we used the fact that the four terms \( (p = 1, q = 0, r = 0, s = 0), \ldots, (p = 0, q = 0, r = 0, s = 1) \) give the same contribution. We now use the same trick as before to eliminate \( \sum_{j,k,l=1}^{n} \partial_j \partial_k \partial_l \) in favor of \( \partial^3/\partial\delta_c^3 \),
\[ \int_{-\infty}^{\delta_c} d\delta_n \Pi_{\epsilon=0}^{(4NL)}(\delta_0; \delta_n; S_n) = -\frac{4}{24} G_{4(1,0,0,0)}^{(1,0,0,0)}(S_n) \sum_{i=1}^{n} (S_n - S_i) \frac{\partial^3}{\partial\delta_c^3} \int_{-\infty}^{\delta_c} d\delta_1 \ldots d\delta_{n-1} d\delta_n W_{gm}. \]  
(3.16)

The remaining path integral can be computed using the technique discussed in eqs. (2.21)–(2.27), and we get
\[ \int_{-\infty}^{\delta_c} d\delta_n \Pi_{\epsilon=0}^{(4NL)}(\delta_0; \delta_n; S_n) = -\frac{4}{24\pi} G_{4(1,0,0,0)}^{(1,0,0,0)}(S_n) \int_{0}^{S_n} dS_i \frac{1}{S_i^{3/2} (S_n - S_i)^{1/2}} \times \frac{\partial^3}{\partial\delta_c^3} \left[ \delta_c e^{-\delta_c^2/(2S_i)} \int_{-\infty}^{\delta_i} d\delta_n (\delta_c - \delta_n) \exp \left\{ -\frac{(\delta_c - \delta_n)^2}{2(S_n - S_i)} \right\} \right]. \]  
(3.17)

Observe that this expression involves an integral over all values of an intermediate “time” variable \( S_i \) which ranges over \( 0 \leq S_i \leq S_n \). As we have discussed in detail in [44, 38] these
terms are “memory” terms that depend on the whole past history of the trajectory, and reflect the non-markovian nature of the stochastic process.

The integral over \( d\delta_n \) is easily performed writing

\[
(\delta_c - \delta_n) \exp \left\{ -\frac{(\delta_c - \delta_n)^2}{2(S_n - S_i)} \right\} = (S_n - S_i) \partial_n \exp \left\{ -\frac{(\delta_c - \delta_n)^2}{2(S_n - S_i)} \right\},
\]

so it just gives \((S_n - S_i)\). Carrying out the third derivative with respect to \( \delta_c \) and the remaining elementary integral over \( dS_i \) we get

\[
\int_{-\infty}^{\delta_c} d\delta_n \Pi_c^{(4, NL)}(\delta_0; \delta_n; S_n) = \frac{\delta_c}{3\sqrt{2\pi}} \frac{G_{4}^{(1,0,0,0)}(S_n)}{S_n^{3/2}} e^{-\delta_c^2/(2S_n)}.
\]

We now define

\[
\mathcal{U}_4(\sigma) \equiv \frac{4G_{4}^{(1,0,0,0)}(S)}{S^2},
\]

where as usual \( S = \sigma^2 \). When the ordering given in eq. (3.5) holds, \( \mathcal{U}_4(\sigma) \) is of the same order as the normalized kurtosis \( S_4(\sigma) \). Computing the contribution to \( f(\sigma) \) from eq. (3.19) and we finally find

\[
f(\sigma) = \left( \frac{2}{\pi} \right)^{1/2} \frac{\delta_c}{\sigma} e^{-\delta_c^2/(2\sigma^2)} \left[ 1 + \frac{\sigma^2 S_4(\sigma)}{24} \left( \frac{\delta_c^4}{\sigma^4} - \frac{4\delta_c^2}{\sigma^2} - 3 \right) + \frac{1}{24} \sigma^2 \frac{dS_4}{d\ln \sigma} \left( \frac{\delta_c^2}{\sigma^2} - 3 \right) - \frac{\sigma^2 \mathcal{U}_4(\sigma)}{24} \left( \frac{\delta_c^2}{\sigma^2} + 1 \right) - \frac{1}{24} \sigma^2 \frac{d\mathcal{U}_4}{d\ln \sigma} \right].
\]

The above result only holds up to NL order. If one wants to use it up to NNL order, the terms in square bracket where \( \sigma^2 S_4(\sigma), \sigma^2 dS_4/d\ln \sigma, \sigma^2 \mathcal{U}_4(\sigma), \) and \( \sigma^2 d\mathcal{U}_4/d\ln \sigma \) are multiplied by factors of \( O(1) \) must be supplemented by the computation of the terms with \( p + q + r + s = 2 \) in eq. (3.4), which will give a contribution analogous to the term \( \mathcal{V}_3(\sigma) \) computed in [18]. However, with present numerical accuracy, NNL terms are not yet relevant for the comparison with \( N \)-body simulations with non-Gaussian initial conditions.

Our result may be refined in several ways. Until now we have worked with a barrier with a fixed height \( \delta_c \) and we neglected the corrections due to the filter. We can now include the modifications due to the fact that the height of the barrier may be thought to diffuse stochastically, as discussed in [15], and also the corrections due to the filter. To compute the non-Gaussian term proportional to the four-point correlator with the diffusing barrier we recall, from [15], that the first-passage time problem of a particle obeying a diffusion equation with diffusion coefficient \( D = 1 \), in the presence of a barrier that moves stochastically with diffusion coefficient \( D_B \), can be mapped into the first-passage time problem of a particle with effective diffusion coefficient \( (1 + D_B) \), and fixed barrier. This can be reabsorbed into a rescaling of the “time” variable \( S \to (1 + D_B)S = S/a \), and therefore \( \sigma \to \sigma/\sqrt{a} \). At the same time the four-point correlator must be rescaled according to \( \langle \delta_n^4 \rangle \to a^{-2} \langle \delta_n^4 \rangle \) since, dimensionally, \( \langle \delta_n^4 \rangle \) is the same as \( S^2 \), which means that \( S_4 \to aS_4 \). As a final ingredient, we must add the effect of the tophat filter function in coordinate space. For a tophat filter in coordinate space, we have found [14] that the
two-point correlator is given by eqs. (2.17) and (2.18). Including the non-markovianity induced by the tophat smoothing function in real space, using the computations already performed in [44, 38], we end up with

\[
f(\sigma) = (1 - \tilde{\kappa}) \left( \frac{2}{\pi} \right)^{1/2} \frac{a^{1/2} \delta_c}{\sigma} e^{-a\delta_c^2/(2\sigma^2)} \times \left[ 1 + \frac{\sigma^2 S_4(\sigma)}{24} \left( \frac{a^2 \delta_c^4}{\sigma^4} - \frac{4a\delta_c^2}{\sigma^2} - 3 \right) + \frac{1}{24} \sigma^2 \frac{dS_4}{d\ln \sigma} \left( \frac{a\delta_c^2}{\sigma^2} - 3 \right) \right. \\
- \frac{\sigma^2 U_4(\sigma)}{24} \left( \frac{a\delta_c^2}{\sigma^2} + 1 \right) - \frac{1}{24} \sigma^2 \frac{dU_4}{d\ln \sigma} \left. \right] + \frac{\tilde{\kappa}}{\sqrt{2\pi}} \frac{a^{1/2} \delta_c}{\sigma} \Gamma \left( 0, \frac{a\delta_c^2}{2\sigma^2} \right),
\]

where \( \tilde{\kappa} = \kappa/(1 + D_B) \). This is our final result. More generally, also the term proportional to the incomplete Gamma function could get non-Gaussian corrections, which in principle can be computed evaluating perturbatively a “mixed” term proportional to \( \Delta_{ij}\langle \delta_k \delta_l \delta_m \delta_n \rangle \partial_i \partial_j \partial_k \partial_l \partial_m \partial_n \). However we saw in [44] that in the large mass limit, where the non-Gaussianities are important, the term proportional to the incomplete Gamma function is subleading, so we will neglect the non-Gaussian corrections to this subleading term.

4. Conclusions

In this paper we have computed the DM halo mass function as predicted within the excursion set theory when a NG is present under the form of a trispectrum. We thus have extended our previous results presented in ref. [38], where a similar computation was performed in the presence of a NG bispectrum. Our computation accounts for the non-markovianity of the random walk of the smoothed density contrast, which inevitably arise when deviations from gaussianity are present. While our result coincides at the leading order \( O(\delta_c^2/\sigma^4) \) with that obtained in [38] through PS theory, it is different at the order \( O(\delta_c^2/\sigma^2) \). This is due to the memory effects induced by the non-markovian excursion set which are not present in the PS approach. Our final expression (3.22) takes into account as well the non-markovian effects due to the choice of the tophat filter in real space and the proper exponential decay at large DM masses.

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