Flow diagram of the longitudinal and Hall conductivities in ac regime in the disordered graphene quantum Hall system

Takahiro Morimoto and Hideo Aoki
Department of Physics, University of Tokyo, Hongo, Tokyo 113-0033, Japan
E-mail: morimoto@cms.phys.s.u-tokyo.ac.jp

Abstract. We numerically study the behavior of $\sigma_{xy}(\omega)$ and $\sigma_{xx}(\omega)$ for graphene QHE system in the ac (frequency $\omega$) domain. We interpret these conductivities with the dynamical scaling analysis. We also discuss the temperature flow of $\sigma_{xy}(\omega) - \sigma_{xx}(\omega)$ diagram for graphene QHE system in the ac region.

1. Introduction
In graphene massless Dirac fermion is realized as the low-energy physics, and there are bursting interests after the fabrication of graphene and the demonstration of the Dirac integer quantum Hall effect (QHE).[1, 2] While static properties have mainly been studied for the graphene QHE so far, dynamics of electrons in the integer QHE regime is an interesting, hitherto not fully explored problem in general and for graphene in particular. Theoretically, the question is how the static Hall conductivity, which may be regarded as a topological quantity[3], evolves into the optical Hall conductivity, where the relevant energy scale is the cyclotron energy (optically $\sim$ THz). We have recently shown that the plateau structure in $\sigma_{xy}(\omega)$ is retained in the ac (THz) regime in both the ordinary two-dimensional electron gas (2DEG) and in graphene, although the plateau height deviates from the quantized values in ac.[4, 5] The numerical result indicates that the plateau structure remains remarkably robust in the ac regime even when there is a significant disorder. We can attribute this to an effect of localization, which dominates the physics of electrons around the centers of Landau levels in disordered QHE systems. The ac plateau has actually been experimentally observed through THz Faraday rotation measurement for 2DEG system [6]. The dynamical scaling behavior of the plateau-to-plateau transition width in $\sigma_{xy}$ has also been studied theoretically, which shows a well-defined dynamical scaling behavior and a crossover from a sample-size dominated regime to a scattering-length dominated regime.[7]

In this paper we discuss the scaling properties of graphene QHE system by combining the optical Hall conductivity $\sigma_{xy}(\omega)$ with the longitudinal optical conductivity $\sigma_{xx}(\omega)$ to explore the temperature flow of $\sigma_{xy}(\omega) - \sigma_{xx}(\omega)$ diagram”. A focus of interest is whether we have a fixed-point behavior in graphene. The $\sigma_{xy} - \sigma_{xx}$ flow diagram has been known to beautifully capture the (static) scaling properties of QHE system, as discussed by Pruisken in terms of the nonlinear sigma model[8, 9]. Points are there exist (i) stable fixed points at $(\sigma_{xy}, \sigma_{xx}) = (N, 0)$ ($N$: Landau index), along with (ii) unstable fixed points characterizing delocalization at $(\sigma_{xy}, \sigma_{xx}) = (N + \frac{1}{2}, \sigma_{xx}^*)$. While this picture was originally proposed for the conventional
(2DEG) QHE system in the static regime [10], here we extend it to graphene QHE system for finite frequency responses.

2. Formalism
For graphene QHE system, we employ the two-dimensional effective Dirac model,

\[ H = v_F \sigma \cdot \pi + V(\mathbf{r}), \]

where \( v_F \) is Fermi velocity, \( \sigma = (\sigma_x, \sigma_y) \) the Pauli matrix, and \( \pi = p + e\mathbf{A} \) with \( p = (p_x, p_y) \) being the momentum and \( \mathbf{A} \) the vector potential. Disorder is introduced by a random potential, \( V(\mathbf{r}) = \sum_{i,j} u_{i,j} \exp(-|\mathbf{r} - \mathbf{R}_{i,j}|^2/2d^2)/(2\pi d^2) \), composed of Gaussian scattering centers of range \( d \) and \( u_{i,j} \) takes a value in \((-u, u)\) randomly. Impurity sites \( \mathbf{R}_{i,j} \) are periodically placed on \( \mathbf{R} = (L/n)(i, j) = (2\pi^2/L)(i, j), \) with \( L^2/(2\pi^2) = n \) for a computational facility. The measure of disorder is the Landau level broadening[11], \( \Gamma = (uL/\pi)|2\pi(f^2 + 2d^2)|^{-1/2} \), where \( \ell = \sqrt{h/eB} \) is the magnetic length. We can compare this with the cyclotron energy given by \( \omega_c = \sqrt{2v_F/\ell} = v_F\sqrt{2eB/h} \). Here we take \( d = 0.7\ell \). Diagonalization of the Hamiltonian is done for the subspace spanned by the five lowest Landau level’s (LL’s) for \( L \times L \) systems with \( L/\ell \) varied over 20 – 40. With the obtained wave functions and energy eigenvalues \( \epsilon_a \) the optical Hall conductivity is evaluated from the Kubo formula,

\[ \sigma_{\alpha\beta}(\omega) = \frac{e^2\hbar}{i\pi} \int d\varepsilon \frac{f(\varepsilon)}{\hbar\omega} \left[ \text{Tr} \left( j_{\alpha a} \text{Im} G(\varepsilon) j_{\beta b} (G^+(\varepsilon + \hbar\omega) - G^+(\varepsilon)) \right) \right. \\
\left. - \text{Tr} \left( j_{\alpha a} (G^-(-\varepsilon) - G^-(-\varepsilon - \hbar\omega)) j_{\beta b} \text{Im} G(\varepsilon) \right) \right], \]

where \( \alpha, \beta = x, y \), \( f(\varepsilon) \) the Fermi distribution, and \( G^\pm \equiv G(\varepsilon \pm i0) \) Green’s function. At \( T = 0 \) which we assume here, the optical Hall conductivity is reduced to

\[ \sigma_{xy}(\varepsilon_F, \omega) = \frac{i\hbar e^2}{L^2} \sum_{\varepsilon_a, \varepsilon_b} \frac{f(\varepsilon_b) - f(\varepsilon_a)}{\epsilon_b - \epsilon_a} \frac{j_{ab}^x j_{ba}^y}{\epsilon_b - \epsilon_a - \hbar\omega - i\eta}, \]

where \( j_{ab}^x \) is the current matrix element[4], which has a special selection rule for Dirac model (\(|N| \leftrightarrow |N| \pm 1 \) with \( N \) the Landau index) distinct from that (\( N \leftrightarrow N \pm 1 \)) for 2DEG.

On the other hand, the longitudinal optical conductivity is given by

\[ \text{Re}\sigma_{xx}(\omega) = \frac{e^2}{\pi} \int d\varepsilon \frac{f(\varepsilon + \hbar\omega) - f(\varepsilon)}{\hbar\omega} \text{Tr} \left[ j_x \text{Im} G(\varepsilon) j_x \text{Im} G(\varepsilon + \hbar\omega) \right] \\
= \frac{e^2}{\pi} \sum_{\varepsilon_a < \varepsilon < \varepsilon_b} \frac{f(\varepsilon_b) - f(\varepsilon_a)}{\varepsilon_b - \varepsilon_a} \frac{|j_{ab}^x|^2}{(\varepsilon_b - \varepsilon_a - \hbar\omega)^2/\eta + \eta}, \]

The quantity \( \eta \) controls the low-energy cutoff, and it affects the \( \omega \sim 0 \) behavior of \( \sigma_{xx}(\omega) \). Physically, the cutoff \( \eta \) should be chosen close to the Thouless energy, which is typically of the order of the level spacing \( \sim 1/L^2 \). The conductance is averaged over a few thousands samples with different disorder realizations. The averaged conductivity is hereafter denoted by the same symbol \( \sigma_{xy}(\varepsilon_F, \omega) \). For the scaling analysis the calculation is repeated for varied sample size \( L \), Fermi energy \( \varepsilon_F \) and frequency \( \omega \).

3. Behavior of \( \sigma_{xx}(\omega) \) and \( \sigma_{xy}(\omega) \)
First we discuss the behavior \( \sigma_{xx}(\omega) \) and \( \sigma_{xy}(\omega) \) separately. Fig.1(a) depicts \( \sigma_{xx}(\omega) \) against the Fermi energy \( \varepsilon_F \) and the frequency \( \omega \) in a small-frequency region, where \( \sigma_{xx}(\omega) \)-peak at each
Landau level persists over some range of frequency. These peaks, coming from metallic states, correspond to the plateau-to-plateau transition and are subject to the dynamical scaling. As $L$ is increased, the width of $\sigma_{xx}(\omega = 0)$ peak shrinks, while its height stays constant (Fig.1(b)). Concomitantly the transition width of $\sigma_{xy}(\omega = 0)$ sharpens with $L$ (Fig.1(c)).

We have previously done a dynamical scaling analysis of the optical Hall conductivity $\sigma_{xy}(\omega, L)$, where we found a well-defined dynamical scaling behavior with two, qualitatively different regimes, i.e., the dc and ac regimes[7]. It is interesting to extend this picture to incorporate a scaling analysis for the longitudinal $\sigma_{xx}(\omega, L)$.

4. Scaling and $\sigma_{xy}(\omega) - \sigma_{xx}(\omega)$ diagram

Now we are in a position to examine the dynamical scaling analysis for the flow diagram combining those for $\sigma_{xx}(\epsilon_F, \omega)$ and $\sigma_{xy}(\epsilon_F, \omega)$. Let us assume that the optical conductivity depends on Fermi energy $\epsilon_F$ and frequency $\omega$ only through the ratio $L/\xi$ and $L_\omega/\xi$. Here we have the localization length, $\xi \sim 1/|\epsilon_F - \epsilon_c|^{\nu}$, where $\epsilon_c$ is the critical energy which coincides with the center of the LL, $\nu$ is the localization critical exponent. The dynamical length scale behaves as $L_\omega \sim \nu^{1/\nu}$, where $z$ is the dynamical critical exponent, which is found to be $z = 2$.[7] Then the dynamical scaling ansatz for the longitudinal and Hall conductivities reads

$$\sigma_{xx}(\epsilon_F, \omega, L) = \frac{e^2}{h} F_{xx}(\delta \epsilon_F L^{1/\nu}, \omega L^z),$$

$$\sigma_{xy}(\epsilon_F, \omega, L) = \frac{e^2}{h} F_{xy}(\delta \epsilon_F L^{1/\nu}, \omega L^z),$$

where $F_{xx}, F_{xy}$ are universal scaling functions, and $\delta \epsilon_F \equiv \epsilon_F - \epsilon_c$. This ansatz implies that the flow of $(\sigma_{xy}(\omega), \sigma_{xx}(\omega))$ in the ac region depends on the frequency $\omega$ only through the rescaled frequency $\omega L^z$, if we have assumed the dynamical exponent $z = 2$.

In the dc limit the temperature defines an effective thermal length, which is assumed to obey a power law $L_T \sim 1/T^{1/\nu'}$, where $\nu'$ is the critical exponent associated with the thermal length.

5. Temperature flow in $\sigma_{xy}(\omega) - \sigma_{xx}(\omega)$ diagram

Now we discuss the temperature dependence of the low-frequency $\sigma_{xx}(\omega)$ in Fig.2. First, Fig.2(a) shows a behavior of $\sigma_{xx}(\omega)$ for a fixed $\epsilon_F = 0$. The magnitude of $\sigma_{xx}(\omega)$ is fairly constant against frequency in this small $\omega$ region, while the overall value increases as the temperature is lowered, which reflects the metallic behavior of the delocalized states around the center of Landau level. Fig.2(b) depicts $\sigma_{xx}(\epsilon_F)$ with a fixed $\omega L^2 = 1.5$. The system is metallic ($\sigma_{xx}(\omega)$ increases with
Figure 2. Temperature dependence of $\sigma_{xx}(\omega)$ for various temperatures $k_B T/h\omega_c = (0.001, 0.002, 0.005)$ with $L = 30$. $\sigma_{xx}$ is plotted (a) against frequency $\omega L^2$ fixing $\varepsilon_F = 0$, (b) against Fermi energy $\varepsilon_F = 0$ fixing $\omega L^2 = 1.5$. (c) Temperature flow of $(\sigma_{xy}, \sigma_{xx})$ diagram with $\omega L^2 = 1.5$.

decreasing $T$) when $\varepsilon_F$ is close to LL center, while insulating $(\sigma_{xx}(\omega)$ decreases with decreasing $T$) away from LL center the system. Thus this picture holds for small but finite frequency region from Fig.2(b). Finally we plot both of the $\sigma_{xy}, \sigma_{xx}$ on the $(\sigma_{xy}, \sigma_{xx})$ plane for varied temperature with a fixed $\omega L^2 = 1.5$ in Fig.2(c). Right at the Landau level center the flow is metallic, i.e., the smaller temperature the larger the longitudinal conductivity. Then the flows are attracted to the quantum-Hall fixed point at $(\sigma_{xy}, \sigma_{xx}) = (\pm 1/2, 0)$, where the system is insulating and shows a quantum Hall plateau. Thus Fig.2(c) shows this picture, originally proposed for dc, still holds for ac regime, which is consistent with the observation of ac Hall plateau structure.

6. Summary
We have numerically obtained $\sigma_{xy}(\omega) - \sigma_{xx}(\omega)$ diagram for the graphene QHE system. The flow diagram found is robust against changes in the frequency at small but finite $\omega$ region, and gives a physical understanding for the existence of ac plateau structure observed with THz Faraday rotation measurements.

While we have focused on the Dirac field model, the original honeycomb lattice model is shown to have a fixed-point-like behavior (step-function-like rotation measurements). It is interesting to do a calculation going back to the honeycomb lattice model and examine the flow diagram.

References
[1] Novoselov K, Geim A, Morozov S, Jiang D, Katsnelson M, Grigorieva I, Dubonos S and Firsov A 2005 Nature 438 197–200
[2] Zhang Y, Tan Y W, Stormer H L and Kim P 2005 438 201
[3] Thouless D J, Kohmoto M, Nightingale M P and den Nijs M 1982 Phys. Rev. Lett. 49 405–408
[4] Morimoto T, Hatsuqai Y and Aoki H 2009 Phys. Rev. Lett. 103 116803
[5] Kawarabayashi T, Morimoto T, Hatsugai Y and Aoki H 2010 Phys. Rev. B 82 195426
[6] Ikebe Y, Morimoto T, Masutomi R, Okamoto T, Aoki H and Shimano R 2010 Phys. Rev. Lett. 104 256802
[7] Morimoto T, Avishai Y and Aoki H 2010 Phys. Rev. B 82 081404
[8] Pruisken A M M 1988 Phys. Rev. Lett. 61 1297–1300
[9] Pruisken A M M 1985 Phys. Rev. B 32 2636–2639
[10] Aoki H and Ando T 1986 Surface Science 170 249–255
[11] Ando T 1975 J. Phys. Soc. Jpn. 34 989
[12] Thouless D and Kirkpatrick S 1981 Journal of Physics C: Solid State Physics 14 235
[13] Nomura K, Koshino M and Ryu S 2007 Phys. Rev. Lett. 99 146806
[14] Nomura K, Ryu S, Koshino M, Mudry C and Furusaki A 2008 Phys. Rev. Lett. 100 246806