Window Processing of Binary Polarization Kernels

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Abstract

A decoding algorithm for polar (sub)codes with binary $2^t \times 2^t$ polarization kernels is described. The proposed approach exploits the linear relationship of the considered kernels and the Arikan matrix. This relationship enables one to compute the kernel input symbol log-likelihood ratio (LLR) by computing path scores of several paths in Arikan successive cancellation (SC) decoding. Further complexity reduction is achieved by identification and reusing of common subexpressions arising in this computation.

The proposed algorithm is applied to kernels of size 16 and 32 with improved polarization properties. It enables polar (sub)codes with the considered kernels to provide better performance and lower decoding complexity compared with polar (sub)codes with Arikan kernel.

Index Terms

Polar codes, polarization kernels, fast decoding.

I. INTRODUCTION

Polar codes are a novel class of error-correcting codes, which achieve the symmetric capacity of a binary-input discrete memoryless channel $W$, have low complexity construction, encoding
and decoding algorithms [11]. However, the performance of polar codes of practical length is quite poor. The reasons for this are poor minimum distance of polar codes, and existence of imperfectly polarized bit subchannels, which are used for transmission of data. This causes the successive cancellation algorithm to be highly suboptimal. These problems can be alleviated by employing list successive cancellation algorithm (SCL) [2], as well as improved code constructions, such as polar subcodes, polar codes with CRC [3], [4], [5].

Polarization is a general phenomenon, and is not restricted to the Arikan matrix [6]. One can replace it by a larger matrix, called polarization kernel, which has better polarization properties. Polar codes with large kernels were shown to provide asymptotically optimal scaling exponent [7]. Many kernels with various properties were proposed [6], [8], [9], [10]. Until recently, polar codes with large kernels were believed to be impractical due to very high decoding complexity.

In this paper we propose the decoding algorithm for polar codes with $2^t \times 2^t$ kernels. The proposed approach exploits the relationship between the considered kernels and the Arikan matrix. Essentially, the input symbol LLRs of the considered kernels are obtained from the path scores of several in Arikan SC decoding. These path scores are obtained using LLRs computed via the Arikan recursive expressions. Further significant complexity reduction is achieved by identification and reusing of some common subexpressions, which arise in these expressions.

We discuss the application of the proposed decoding algorithm to polar codes with polarization kernels of size 16 and 32, which have improved rate of polarization and scaling exponent. The considered kernels of size 16 have rate of polarization 0.51828, scaling exponents 3.346 and 3.45. The proposed $32 \times 32$ kernel has rate of polarization 0.521936 and scaling exponent 3.41706.

The proposed decoding algorithm can be used with SCL decoding. Simulation results show, that polar codes with large kernels provide significant performance gain compared to polar codes with Arikan kernels and the same list size. Furthermore, the proposed approach results in lower decoding complexity compared to polar codes with Arikan kernel with the same performance.

The paper is organized as follows. The background on polar codes and polarization kernels
is presented in Section II. The window processing (WP) algorithm is introduced in Section III. Section IV presents a common subexpression identification and reusing method, which provides further complexity reduction of WP. Some further improvements of WP are discussed in Section V. The processing algorithms for the considered kernels are described in Section VI. Simulation results are presented in Section VII.

II. BACKGROUND

A. Notations

The following notations are used throughout the paper. For a positive integer \( n \), denote by \([n]\) the set of \( n \) integers \( \{0, 1, \ldots, n - 1\} \). \( u^b_a \) denotes subvector \((u_a, u_{a+1}, \ldots, u_b)\) of some vector \( u \). For vectors \( a \) and \( b \) we denote their concatenation by \( a.b \). \( M[i] \) is an \( i \)-th row of matrix \( M \). By \( M[i,j] \) we denote \( j \)-th element of \( M[i] \).

B. Polarizing transformation

Consider a binary-input memoryless channel with transition probabilities \( W\{y|c\}, c \in \mathbb{F}_2, y \in \mathcal{Y} \), where \( \mathcal{Y} \) is output alphabet. An \((n = l^m, k)\) polar code is a linear block code generated by \( k \) rows of matrix \( G_m = M^{(m)} K \otimes^m \), where \( \otimes^m \) is \( m \)-fold Kronecker product of matrix with itself, and \( M^{(m)} \) is a digit-reversal permutation matrix, corresponding to mapping \( \sum_{i=0}^{m-1} t_i l^i \to \sum_{i=0}^{m-1} t_{m-1-i} l^i, t_i \in [l] \). The encoding scheme is given by \( c_0^{n-1} = u_0^{n-1} G_m \), where \( u_i, i \in \mathcal{F} \) are set to some pre-defined values, e.g. zero (frozen symbols), \( |\mathcal{F}| = n - k \), and the remaining values \( u_i \) are set to the payload data.

It is possible to show that a binary input memoryless channel \( W \) together with matrix \( G_m \) gives rise to bit subchannels \( W^{(i)}_{m,K}(y_{0}^{n-1}, u_{0}^{i-1}|u_i) \) with capacities approaching 0 or 1, and fraction of noiseless subchannels approaching \( I(W) \) [6]. Selecting \( \mathcal{F} \) as the set of indices of low-capacity subchannels enables almost error-free communication.
It is convenient to define probabilities
\[
W_{m,K}^{(i)}(u_0^n|y_0^{n-1}) = \frac{W_{m,K}^{(i-1)}(y_0^{n-1}, u_0^{i-1}|u_i)}{2W(y_0^{n-1})} = \sum_{u_{i+1}^{n-1}} W_{m,K}^{(n-1)}(u_0^{n-1}|y_0^{n-1}) = \sum_{u_{i+1}^{n-1}} \prod_{i=0}^{n-1} W((u_0^{n-1}G_m)_i|y_i).
\]

(1)

Let us further define \( W_{m}^{(j)}(u_0^n|y_0^{n-1}) = W_{m,K}^{(j)}(u_0^n|y_0^{n-1}) \), where kernel \( K \) will be clear from the context.

Due to the recursive structure of \( G_m \), one has
\[
W_{m}^{(s+t)}(u_0^{s+t}|y_0^{n-1}) = \sum_{u_{s+t}^{l(s+1)-1}} \prod_{j=0}^{l-1} W_{m-1}^{(s)}(\theta_K[u_{l(s+1)-1}^{l(r+1)-1}, j]|y_j^{n,r-1}),
\]

(2)

where \( \theta_K[u_{l(r+1)-1}, j]_r = (u_{l(r+1)-1}^{l(r+1)}K)_j, r \in [s+1] \). Computing these probabilities reduces to soft-output decoding of non-systematically encoded codes generated by last \( l - t - 1 \) rows of \( K \). This problem was considered in [11].

At the receiver side, the successive cancellation (SC) decoding algorithm makes estimates
\[
\hat{u}_i = \begin{cases} 
\arg \max_{u_i \in \mathbb{F}_2} W_{m}^{(i)}(\hat{u}_0^{i-1}.u_i^{n-1}), & i \notin \mathcal{F}, \\
\text{the frozen value of } u_i & i \in \mathcal{F}.
\end{cases}
\]

(3)

C. Kernel processing

The decoding algorithms for polar codes require one to compute probabilities \( W_{m}^{(i)}(u_0^n|y_0^{n-1}) \) for a given polarization transform \( G_m \). Since the values \( W_{m}^{(i)}(u_0^n|y_0^{n-1}) \) are computed recursively according to (2), we assume for simplicity that \( m = 1 \). The corresponding task is referred to as kernel processing. The probabilities for one layer of the polarization transform are given by
\[
W_{1}^{(\phi)}(u_0^{l-1}|y_0^{l-1}) = \sum_{u_{\phi+1}^{l-1} \in \mathbb{F}_2^{\phi-1}} W_{1}^{(l-1)}(u_0^{l-1}|y_0^{l-1}).
\]

(4)

\footnote{Sometimes it is also referred to as kernel marginalization [12].}
The value of $\phi$ is referred to as processing phase, while vector $u_0^\phi$ is referred to as a path.

We introduce approximate probabilities

$$W_1^{(\phi)}(u_0^{l-1}|y_0^{l-1}) = \max_{u_{l-1}^{\phi-1} \in \mathbb{F}_2^{l-\phi-1}} W_1^{(l-1)}(u_0^{l-1}|y_0^{l-1}) \tag{5}$$

Note that the same probabilities were introduced in [13], [14], and shown to provide substantial reduction of the complexity of sequential decoding of polar codes.

Decoding can be implemented using the log-likelihood ratios (LLRs) of the approximate probabilities (5)

$$S_{1,\phi} = \ln \frac{W_1^{(\phi)}(u_0^{l-1}|y_0^{l-1})}{W_1^{(\phi)}(u_0^{l-1}1|y_0^{l-1})} = R(0) - R(1), \tag{6}$$

where $R(a) = \max_{u_{l-1}^{\phi+1}} \ln W_1^{(l-1)}(u_0^{\phi-1}.a.u_{\phi+1}^{l-1}|y_0^{l-1})$.

The above expression means that $S_{1,\phi}$ can be computed by performing ML decoding of the coset of a code generated by last $l-\phi-1$ rows of the kernel $K$, assuming that all $u_j, \phi < j < l$, are equiprobable. The particular coset representative is given by the product of $u_0^{\phi-1}$ and matrix consisting of top $\phi$ rows of kernel $K$.

### D. Processing of Arikan kernel

Straightforward evaluation of (6) for arbitrary kernel has complexity $O(2^l)$. However, there is a simple explicit recursive procedure for computing these values for the case of the Arikan matrix $F_t = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}^\otimes t$.

Let $l = 2^t$. Consider encoding scheme

$$e_0^{l-1} = v_0^{l-1} F_t. \tag{7}$$
Similarly to (5), define approximate probabilities
\[
\tilde{W}_t^{(i)}(v_0^i | y_0^{i-1}) = \max_{v_{l+1}^{l-1} \in F_{l-1}^{l-1}} W_t^{(l-1)}(v_0^{l-1} | y_0^{l-1})
\]
and modified log-likelihood ratios
\[
S_t^{(i)}(v_0^{i-1}, y_0^{l-1}) = \log \frac{\tilde{W}_t^{(i)}(v_0^{i-1.0} | y_0^{l-1})}{\tilde{W}_t^{(i)}(v_0^{i-1.1} | y_0^{l-1})}.
\]

It can be seen that (15)
\[
S^{(2i)}(v_0^{2i-1}, y_0^{N-1}) = Q(a, b) = \text{sign}(a) \text{ sign}(b) \min(|a|, |b|) \quad (8)
\]
\[
S^{(2i+1)}(v_0^{2i}, y_0^{N-1}) = P(a, b, v_{2i}) = (-1)^{v_{2i}} a + b, \quad (9)
\]
\[
a = S_t^{(i)}(v_0^{2i-1}, v_0^{2i-1}, y_0^{N-1}), \quad (10)
\]
\[
b = S_t^{(i)}(v_0^{2i-1}, y_0^{N-1}). \quad (11)
\]
where \(N = 2^l\). The initial values for this recursion are given by \(S_0^{(0)}(y_i) = \log \frac{W(0|y_i)}{W(1|y_i)}\). These expressions can be readily recognized as the min-sum approximation of the list SC algorithm (16). However, these are also the exact values, which reflect the probability of the most likely continuation of a given path \(v_0^{i-1}\) in the code tree.

Observe that one does not need to compute all values given by recursion (8)-(9) at each phase. It is possible to reuse intermediate LLRs \(S^{(j)}(v_0^{i-1}, y_0^{N-1})\), obtained at previous phases. Namely, at phase \(i > 0\) one needs to compute only values \(S^{(i/2^k-j)}(v_0, y_0^{N-1})\), \(0 \leq k \leq s\), where \(s\) is the largest integer such that \(2^s\) divides \(i\). See (14) or (17) for details.

The log-likelihood of a path \(v_0^i\) can be obtained as (15)
\[
R_y(v_0^i) = R(v_0^i | y_0^{i-1}) = \log \tilde{W}_t^{(i)}(v_0^i | y_0^{i-1}) = R_y(v_0^{i-1}) + \tau \left(S_t^{(i)}(v_0^{i-1}, y_0^{i-1}), v_i\right), \quad (12)
\]
where $R_y(\epsilon)$ can be set to 0, $\epsilon$ is an empty sequence, and

$$\tau(S, v) = \begin{cases} 
0, & \text{sign}(S) = (-1)^v \\
-|S|, & \text{otherwise}.
\end{cases}$$

E. Fundamental parameters of polar codes

1) Rate of polarization: The rate of polarization shows how fast bit subchannels of $K^{\otimes m}$ approach either almost noiseless or noisy channel with $n = l^m$.

Suppose we constructed $(n, k)$ polar code $C$ with kernel $K$. Let $P_e(n)$ be a block error probability of $C$ under transmission over $W$ and decoding by SC algorithm. It was proven [6], that if $n/k < I(W)$ and $\beta < E(K)$, then $P_e(n) \leq 2^{-n^\beta}$.

It turns out that the rate of polarization is independent of channel $W$. The method of its computation is proposed in [6]. It is proven that by increasing kernel dimension $l$ to infinity, it is possible to obtain $l \times l$ kernels with rate of polarization arbitrarily close to 1 [6]. Explicit constructions of kernels with high rate of polarization are provided in [9], [18].

2) Scaling exponent: Another crucial property of polarization kernels is the scaling exponent. Let us fix a B-DMC $W$ of capacity $I(W)$ and a desired block error probability $P_e$. Given $W$ and $P_e$, suppose we wish to communicate at rate $I(W) - \Delta$ using a family of $(n, k)$ polar codes with kernel $K$. It has been shown that this value of $n$ scales as $O(\Delta^{-\mu(K)})$, where the constant $\mu(K)$ is known as the scaling exponent [8].

The scaling exponent depends on channel. Unfortunately, the algorithm of its computing is only known for the case on binary erasure channel [19], [8]. It is possible to show [20], [21] that there exist $l \times l$ kernels $K_i$, such that $\lim_{l \to \infty} \mu(K_i) = 2$, i.e. the corresponding polar codes provide optimal scaling behaviour. Constructions of kernels with good scaling exponent are provided in [22], [23].
III. WINDOW PROCESSING

In this section we present a detailed overview of the window-based kernel processing method, which is referred to as window processing (WP) algorithm. The basic idea of this method is to calculate input symbols probabilities of desired kernel $K$ via probabilities of Arikan matrix $F_t$, which are easy to compute. This idea was originally proposed in [24] and reinvented in [10].

Basically, WP was proposed for computation of bit subchannel probabilities. In this section we also present an implementation of WP in LLR domain of approximated probabilities (6).

A. Input vectors of polarizing transforms

WP algorithm calculates $\phi$-th bit subchannel probabilities $W_1^{(\phi)}(u_0^l|y_0^{l-1})$ for $l \times l, l = 2^t$, kernel $K$ via probabilities $W_t^{(i)}(v_i^l|y_0^{l-1}), i \geq \phi$ for Arikan matrix $F_t$. This requires establishing the relations between the input vectors $u$ and $v$ of polarizing transforms $K$ and $F_t$ respectively.

Indeed, since $K$ and $F_t$ are invertible, we can write $TK = F_t$, where the matrix $T$ is referred to as transition matrix. Hence, any vector can be represented as $c_0^{l-1} = v_0^{l-1}F_t = u_0^{l-1}K$. It implies that $u_0^{l-1} = v_0^{l-1}T$, or

$$u_\phi = \sum_{s=0}^{l-1} v_s T[s, \phi] = \sum_{T[s, \phi] = 1}^{\tau_\phi} v_s,$$  

(13)

where $\tau_\phi$ is the position of the last non-zero symbol in the $\phi$-th column of $T$.

The formula (13) can be rewritten to obtain

$$v_{\tau_\phi} = u_\phi + \sum_{T[s, \phi] = 1}^{\tau_\phi - 1} v_s$$  

(14)

However, some $\tau_\phi$ might be the same. It means that some $v_i, i \in [l]$ are not directly defined by (14). For further discussion we rewrite (14) to obtain the equations for all $v_i, i \in [l]$.  

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Indeed, vectors $u_{l-1}^i$ and $v_{l-1}^i$ satisfy the equation

$$\Theta'(u_{l-1}, \ldots, u_1, u_0, v_0, v_1, \ldots, v_{l-1})^T = 0,$$

where $\Theta' = (T I)$, and $l \times l$ matrix $T$ is obtained by transposing $T^{-1}$ and reversing the order of columns in the obtained matrix. By applying elementary row operations, matrix $\Theta'$ can be transformed into a minimum-span form $\Theta$, such that the first and last non-zero elements of the $\phi$-th row are located in columns $\phi$ and $z_\phi$, respectively, where all $z_\phi$ are distinct. This enables one to obtain symbols of vector $u$ as

$$u_\phi = \sum_{s=0}^{\phi-1} u_s \Theta_{l-1-\phi, l-1-s} + \sum_{t=0}^{\omega_\phi} v_t \Theta_{l-1-\phi, l+t},$$

(15)

where $\omega_\phi = z_{l-1-\phi} - l$. Using the expression (15) we can obtain the equations for vector $v_{l-1}^0$ as follows:

$$v_{\omega_\phi} = \sum_{s=0}^{\phi} u_s \Theta_{l-1-\phi, l-1-s} + \sum_{t=0}^{\omega_\phi-1} v_t \Theta_{l-1-\phi, l+t},$$

(16)

In this paper we apply window processing approach to kernels of size $2^t \times 2^t$ only. We also consider kernels with all $\tau_\phi$, $\phi \in [l]$ distinct. In this case we do not need to compute the matrix $\Theta$ to obtain (15). For such kernels we have $\omega_\phi = \tau_\phi$, $\phi \in [l]$, and can immediately use (13).

Table I presents $16 \times 16$ kernels $K'_{16}$ and $K_{16}$ in hexadecimal notation. These kernels have improved polarization properties and are suitable for window processing. $K'_{16}$ and $K_{16}$ were
TABLE II: Input symbols $u_\phi$ for kernels $K_{16}'$, $K_{16}$ as functions of input symbols $v$ for $F_4$

| $\phi$ | $v_0$ | $v_1$ | $v_2$ | $v_3$ | $v_8$ | $v_{6} \oplus v_9$ | $v_{5} \oplus v_6 \oplus v_{10}$ | $v_{12}$ | $v_6$ | $v_7$ | $v_11$ | $v_{11}$ | $v_{13}$ | $v_{14}$ | $v_{15}$ |
|-------|-------|-------|-------|-------|-------|------------------|------------------|-------|-------|-------|-------|-------|-------|-------|-------|
| 0     | $\{\}$ | $\{\}$ | $\{\}$ | $\{\}$ | $\{3, 5, 6, 7\}$ | $\{3, 5, 6, 7\}$ | $\{3, 5, 6, 7\}$ | $\{\}$ | $\{\}$ | $\{\}$ | $\{\}$ | $\{\}$ | $\{\}$ | $\{\}$ | $\{\}$ |
| 1     | $\{\}$ | $\{\}$ | $\{\}$ | $\{\}$ | $\{\}$ | $\{\}$ | $\{\}$ | $\{\}$ | $\{\}$ | $\{\}$ | $\{\}$ | $\{\}$ | $\{\}$ | $\{\}$ | $\{\}$ |
| 2     | $\{\}$ | $\{\}$ | $\{\}$ | $\{\}$ | $\{\}$ | $\{\}$ | $\{\}$ | $\{\}$ | $\{\}$ | $\{\}$ | $\{\}$ | $\{\}$ | $\{\}$ | $\{\}$ | $\{\}$ |
| 3     | $\{\}$ | $\{\}$ | $\{\}$ | $\{\}$ | $\{\}$ | $\{\}$ | $\{\}$ | $\{\}$ | $\{\}$ | $\{\}$ | $\{\}$ | $\{\}$ | $\{\}$ | $\{\}$ | $\{\}$ |
| 4     | $\{\}$ | $\{\}$ | $\{\}$ | $\{\}$ | $\{\}$ | $\{\}$ | $\{\}$ | $\{\}$ | $\{\}$ | $\{\}$ | $\{\}$ | $\{\}$ | $\{\}$ | $\{\}$ | $\{\}$ |
| 5     | $\{\}$ | $\{\}$ | $\{\}$ | $\{\}$ | $\{\}$ | $\{\}$ | $\{\}$ | $\{\}$ | $\{\}$ | $\{\}$ | $\{\}$ | $\{\}$ | $\{\}$ | $\{\}$ | $\{\}$ |
| 6     | $\{\}$ | $\{\}$ | $\{\}$ | $\{\}$ | $\{\}$ | $\{\}$ | $\{\}$ | $\{\}$ | $\{\}$ | $\{\}$ | $\{\}$ | $\{\}$ | $\{\}$ | $\{\}$ | $\{\}$ |
| 7     | $\{\}$ | $\{\}$ | $\{\}$ | $\{\}$ | $\{\}$ | $\{\}$ | $\{\}$ | $\{\}$ | $\{\}$ | $\{\}$ | $\{\}$ | $\{\}$ | $\{\}$ | $\{\}$ | $\{\}$ |
| 8     | $\{\}$ | $\{\}$ | $\{\}$ | $\{\}$ | $\{\}$ | $\{\}$ | $\{\}$ | $\{\}$ | $\{\}$ | $\{\}$ | $\{\}$ | $\{\}$ | $\{\}$ | $\{\}$ | $\{\}$ |
| 9     | $\{\}$ | $\{\}$ | $\{\}$ | $\{\}$ | $\{\}$ | $\{\}$ | $\{\}$ | $\{\}$ | $\{\}$ | $\{\}$ | $\{\}$ | $\{\}$ | $\{\}$ | $\{\}$ | $\{\}$ |
| 10    | $\{\}$ | $\{\}$ | $\{\}$ | $\{\}$ | $\{\}$ | $\{\}$ | $\{\}$ | $\{\}$ | $\{\}$ | $\{\}$ | $\{\}$ | $\{\}$ | $\{\}$ | $\{\}$ | $\{\}$ |
| 11    | $\{\}$ | $\{\}$ | $\{\}$ | $\{\}$ | $\{\}$ | $\{\}$ | $\{\}$ | $\{\}$ | $\{\}$ | $\{\}$ | $\{\}$ | $\{\}$ | $\{\}$ | $\{\}$ | $\{\}$ |
| 12    | $\{\}$ | $\{\}$ | $\{\}$ | $\{\}$ | $\{\}$ | $\{\}$ | $\{\}$ | $\{\}$ | $\{\}$ | $\{\}$ | $\{\}$ | $\{\}$ | $\{\}$ | $\{\}$ | $\{\}$ |
| 13    | $\{\}$ | $\{\}$ | $\{\}$ | $\{\}$ | $\{\}$ | $\{\}$ | $\{\}$ | $\{\}$ | $\{\}$ | $\{\}$ | $\{\}$ | $\{\}$ | $\{\}$ | $\{\}$ | $\{\}$ |
| 14    | $\{\}$ | $\{\}$ | $\{\}$ | $\{\}$ | $\{\}$ | $\{\}$ | $\{\}$ | $\{\}$ | $\{\}$ | $\{\}$ | $\{\}$ | $\{\}$ | $\{\}$ | $\{\}$ | $\{\}$ |
| 15    | $\{\}$ | $\{\}$ | $\{\}$ | $\{\}$ | $\{\}$ | $\{\}$ | $\{\}$ | $\{\}$ | $\{\}$ | $\{\}$ | $\{\}$ | $\{\}$ | $\{\}$ | $\{\}$ | $\{\}$ |

constructed by algorithm [22]. The expressions (13) for $K_{16}'$ and $K_{16}$ are given in Table II.

B. Decoding window

In the previous section we discussed the relations between input vectors $u$ and $v$ of polarizing transforms $K$ and $F_t$. In this section we use these relations to compute probabilities of $K$ via probabilities of $F_t$.

Let $h_\phi = \max_{0 \leq \phi' \leq \phi} \tau_{\phi'}$ be a processing front. Observe that it is possible to reconstruct $u_{h_\phi}^\phi$ from $v_{h_\phi}^{h_\phi}$ via expression (15). Recall that computation of probabilities $W_t^{(h_\phi)}(v_0^{h_\phi} | y_0^{l-1})$ (11) requires exact values of all elements of $v_{h_\phi}^{h_\phi}$. However, if $h_\phi > \phi$, then some values of $v_{h_\phi}^{h_\phi}$ are independent from $u_{\phi-1}^\phi$ and, therefore, unknown. It means that one should consider vectors $v_{h_\phi}^{h_\phi}$ with all possible values of unknown bits and compute the corresponding probabilities $W_t^{(h_\phi)}(v_0^{h_\phi} | y_0^{l-1})$.

Let $D_\phi = [h_\phi + 1] \setminus \{\omega_0, \omega_1, \ldots, \omega_\phi\}$ be a decoding window, i.e. the set of indices of independent (from $u_\phi^\phi$) components of $v_0^{h_\phi}$. Note that $|D_\phi| = |[h_\phi + 1]| - |\{\omega_0, \omega_1, \ldots, \omega_\phi\}| = h_\phi - \phi$, since all $\omega_\phi$ are distinct and $\{\omega_0, \omega_1, \ldots, \omega_\phi\} \subseteq [h_\phi + 1]$. 
Theorem 1. Let $K$ be an $l \times l, l = 2^t$ polarization kernel. Then, the probability of input vector $u_0^\phi$ of polarizing transform $K$ is given by

$$W_1^{(\phi)}(u_0^\phi|y_0^{l-1}) = \sum_{v_0^h \in Z_\phi^{(u_\phi)}} W_t^{(h_\phi)}(v_0^h|y_0^{l-1}),$$

(17)

where $Z_\phi^{(b)}$ is the set of vectors $v_0^h$, where $v_s \in \mathbb{F}_2, s \in D_\phi$, the values of $v_t$, $t \in [h_\phi + 1]\setminus D_\phi$, are obtained according to expression (16) under condition that $u_\phi = b$.

Proof: By definition (4), we have

$$W_1^{(\phi)}(u_0^\phi|y_0^{l-1}) = \sum_{u_{0+1}^{l-1} \in \mathbb{F}_2^{l-h_\phi}} W_1^{(l-1)}(u_0^{l-1}|y_0^{l-1}) = \sum_{u_{0+1}^{l-1} \in \mathbb{F}_2^{l-h_\phi}} \prod_{i=0}^{l-1} W((u_0^{l-1}K)_i|y_i)$$

$$= \sum_{v_0^{h_\phi} \in Z_\phi^{(u_\phi)}} \prod_{i=0}^{l-1} W((v_0^{l-1}T)K)_i|y_i = \sum_{v_0^{h_\phi} \in Z_\phi^{(u_\phi)}} \sum_{v_{0+1}^{l-1} \in \mathbb{F}_2^{l-h_\phi}} \prod_{i=0}^{l-1} W((v_0^{l-1}F_i)_i|y_i)$$

$$= \sum_{v_0^{h_\phi} \in Z_\phi^{(u_\phi)}} \sum_{v_{0+1}^{l-1} \in \mathbb{F}_2^{l-h_\phi}} W_t^{(l-1)}(v_0^{l-1}|y_0^{l-1}) = \sum_{v_0^{h_\phi} \in Z_\phi^{(u_\phi)}} W_t^{(h_\phi)}(v_0^h|y_0^{l-1}).$$

Equality (1) follows from (15) and definition of $Z_\phi$. Note that $|Z_\phi^{(u_\phi)}| + |\mathbb{F}_2^{l-h_\phi}| = |\mathbb{F}_2^{l-h_\phi}|$.

Equality (2) is given by definition $TK = F_t$ from Section III-A. Equality is derived from (4).

The expression (17) is the window processing algorithm in probabilistic domain. It means that computation of the probability $W_1^{(\phi)}(u_0^\phi|y_0^{l-1})$ requires considering $2^{|D_\phi|}$ paths $v_0^{h_\phi} \in Z_\phi^{(u_\phi)}$ in context of Arikan SC decoding and computing $2^{|D_\phi|}$ probabilities $W_t^{(h_\phi)}(v_0^h|y_0^{l-1})$.

Let $\mathcal{M}(K)$ denote the max$_{i \in [l]} |D_i|$ for kernel $K$. In general, one has $\mathcal{M}(K) = O(l)$ for an arbitrary $K$. There are some methods to construct polarization kernels, which admit window processing with low complexity [22], [25].
Example 1. Consider computing $W^{(6)}_1 (u_6^0 | y_0^{15})$ for $K'_{16}$. By definition (4) one should compute

$$W^{(6)}_1 (u_6^0 | y_0^{15}) = \sum_{u_7^{15} \in \mathbb{F}_2^9} W^{(15)}_1 (u_7^{15} | y_0^{15}).$$

(18)

The processing front $h_6 = 10$ indicates the largest index of dependent element of $v$ (from $u_6^0$). It implies that for fixed $v_0^{10}$ we can compute probabilities $W^{(10)}_4 (v_0^{10} | y_0^{15})$ via simple expressions [2]. However, in $v_0^{10}$ we have 4 independent elements: $v_3$ and $v_5$. Indices of these elements forms the decoding window $D_6 = \{3, 5, 6, 7\}$.

The expressions for each $v_i$ are given by (14). This enables us to construct the set of paths $v_0^{10} \in \mathcal{Z}_6$. In this example we have $\mathcal{Z}_6^{(u_6)} = \{v_0^{10} | v_i = \mathbb{F}_2, i \in D_6, v_j = u_j, j \in [3], v_4 = u_3, v_8 = u_4, v_9 = u_6 \oplus v_5, v_{10} = v_5 \oplus v_6 \oplus u_6\}$. Note that $|\mathcal{Z}_6| = 16$. Thus, according to Theorem 1,

$$W^{(6)}_1 (u_6^0 | y_0^{15}) = \sum_{v_0^{10} \in \mathcal{Z}_6^{(u_6)}} W^{(10)}_4 (v_0^{10} | y_0^{15}).$$

The decoding windows of $K'_{16}, K_{16}$ are presented in Table II together with the cost of window processing of each phase. The detailed description of processing algorithm for $K_{16}$ kernel is presented in Section VI-B.

C. Log-likelihood domain

Similarly to (5) we can rewrite (17) for the case of the approximate probabilities

$$\tilde{W}^{(6)}_1 (u_0^0 | y_0^{l-1}) = \max_{v_0^{l-1} \in \mathcal{Z}_\phi^{(0)}} \tilde{W}^{(l-1)}_1 (v_0^{l-1} | y_0^{l-1}) = \max_{v_0^{l-1} \in \mathcal{Z}_\phi^{(0)}} \max_{v_0^{l-1} \in \mathcal{Z}_\phi^{(1)}} W^{(l-1)}_1 (v_0^{l-1} | y_0^{l-1}).$$

(19)

Using (12), we can go to LLR domain

$$S_{1, \phi} = \max_{v_0^{l-1} \in \mathcal{Z}_\phi^{(0)}} R_y (v_0^{l-1}) - \max_{v_0^{l-1} \in \mathcal{Z}_\phi^{(1)}} R_y (v_0^{l-1}).$$

(20)

Calculation of LLRs $S_{1, \phi}$ via (20) is referred to as the window processing algorithm in LLR domain. Let $\mathcal{Z}_\phi = \mathcal{Z}_\phi^{(0)} \cup \mathcal{Z}_\phi^{(1)}$. The number of path scores to be computed in (20) is equal to
\[ |\mathcal{Z}_\phi| = 2^{|\mathcal{D}_\phi|+1}. \] It is also possible to reuse intermediate LLRs for each path (see Section II-D).

Although the number of paths scores \( R_y(v_0^{\phi}), v_0^{\phi} \in \mathcal{Z}_\phi \) is \( 2^{|\mathcal{D}_\phi|+1} \), they can be computed with \( 2^{|\mathcal{D}_\phi|} \) operations only. Namely, during the calculation of \( R_y(v_0^{\phi}), v_0^{\phi} \in \mathcal{Z}_\phi \), two terms \( \tau(S_t^{(\phi)}(v_0^{\phi-1}|y_0^{l-1}), b), b \in \mathbb{F}_2 \) are arising for each path \( v_0^{\phi-1} \in \mathcal{Z}_\phi \). Due to the definition of \( \tau \) function, there is such value of \( a \in \mathbb{F}_2 \), that \( \text{sign} S_t^{(\phi)}(v_0^{\phi-1}|y_0^{l-1}) = (-1)^a \), and, therefore, \( \tau(S_t^{(\phi)}(v_0^{\phi-1}|y_0^{l-1}), a) = 0 \). It remains to compute only \( \tau(S_t^{(\phi)}(v_0^{\phi-1}|y_0^{l-1}), a \oplus 1) \). The number of such non-zero terms is given by \( 2^{|\mathcal{D}_\phi|} \).

The equations (20) and (8)–(12) can be immediately used to perform kernel processing. However, in the next sections we show how to substantially reduce the complexity of WP.

### IV. Common Subexpressions Identification

In this section we propose a simplification of WP algorithm. Conventional WP considers several paths in Arikan SC decoding. For each path the algorithm recursively computes its LLR, (which is used to obtain the path score). It turns out that the intermediate LLRs induced by this recursion can be the same for different paths. Thus, one can compute only unique intermediate LLRs, and, therefore, substantially reduce the processing complexity.

#### A. Motivation

Suppose we want to compute \( \phi \)-th input symbol LLR \( S_{1,\phi} \) of the \( l \times l, l = 2^t \), polarization kernel \( K \) by WP method. It computes several scores \( R_y(v_0^{\phi}) \) of paths \( v_0^{\phi} \), considered in the context of Arikan SC decoding. The number of such paths is determined by the size of the \( \phi \)-th decoding window \( \mathcal{D}_\phi \). The values \( \phi \) and \( h_\phi \) are referred to as external phase and internal phase, respectively.

According to (12), to obtain a single path score \( R_y(v_0^{\phi}) \) one need to compute the LLR \( S_t^{(\phi)}(v_0^{\phi-1}|y_0^{l-1}) \) and the path score \( R_y(v_0^{\phi-1}) \) from the previous phase \( h_\phi - 1 \). The value \( S_t^{(\phi)} \) is computed recursively via (8)–(11). On the each layer \( \lambda \in [t+1] \) of this recursion,
\[ M = 2^{t-\lambda} \] intermediate LLRs \( S^{(j)}_\lambda = S^{(j)}_\lambda (\tilde{v}^{j-1}_0, \tilde{y}^{N-1}_0) \), \( N = 2^\lambda \) are arising. The value \( \tilde{v}^{j-1}_0 \) denotes a partial sum of some elements from \( v_0^{h_0-1} \). The structure of these sums is determined by (10) and (11) as well as a particular subvector \( \tilde{y}^{N-1}_0 \) of \( y_0^{l-1} \).

It is possible to write \( \tilde{v}^{j-1}_0 \) and \( \tilde{y}^{N-1}_0 \) explicitly for all \( S^{(j)}_\lambda \). We introduce an array \( C^{(j)}_\lambda [\beta] \), \( \beta \in [M] \), of partial sums \( \tilde{v}^{j-1}_0 \) at layer \( \lambda \)

\[
C^{(j)}_\lambda [\beta] = \begin{cases} 
(c_0^{(j)}, c_1^{(j)}, \ldots, c_{j-1}^{(j)}), & j > 0, \\
\epsilon, & \text{otherwise.}
\end{cases}
\]

where \( c^{(i)} = v^{M+i+M-1}_M F_{t-\lambda}, i \in [j] \). We also introduce an array \( S^{(j)}_\lambda [\beta], \beta \in [M] \), of LLRs \( S^{(j)}_\lambda (\tilde{v}^{j-1}_0, \tilde{y}^{N-1}_0) \) defined as

\[
S^{(j)}_\lambda [\beta] = S^{(j)}_\lambda (C^{(j)}_\lambda [\beta], (y_\beta, y_{\beta+M}, \ldots, y_{\beta+M(N-1)})).
\]

Similarly to (14), the LLRs can be computed as

\[
S^{(j)}_\lambda [\beta] = \begin{cases} 
Q(S^{(j)}_{\lambda-1}[\beta], S^{(j)}_{\lambda-1}[\beta + N]), & j \text{ even,} \\
P(S^{(j)}_{\lambda-1}[\beta], S^{(j)}_{\lambda-1}[\beta + N], (C^{(j)}_\lambda [\beta])_{j-1}), & j \text{ odd,}
\end{cases}
\]

where \( \hat{j} = [j/2] \).

WP method requires calculation of \( 2|S_\phi| \) LLRs \( S_t^{(h_\phi)} \) given by \( v_0^{h_\phi-1} \in Z_\phi \). In a straightforward implementation, all intermediate LLRs \( S^{(j)}_\lambda [\beta] \) should be separately computed for each \( S_t^{(h_\phi)} \). It turns out, that \( S^{(j)}_\lambda [\beta] \) induced by recursive calculation of \( S_t^{(h_\phi)} \) for different \( v_0^{h_\phi-1} \in Z_\phi \) may be the same. For example, the number of distinct vectors \( \tilde{v}^{j-1}_0 \) and length-\( N \) subvectors \( \tilde{y}^{N-1}_0 \) of \( y_0^{l-1} \) arising in (8)–(9) is at most \( 2^j \) and \( 2^{l-\lambda} \), respectively. Thus, the number of distinct pairs \( (\tilde{v}^{j}_0, \tilde{y}^{N-1}_0) \leq 2^{j+t-\lambda} \), which can be even less than \( |S_t^{(h_\phi)}| \).

Therefore, we propose to compute \( S^{(j)}_\lambda [\beta] = S^{(j)}_\lambda (\tilde{v}^{j}_0, \tilde{y}^{N-1}_0) \) only for distinct pairs \( (\tilde{v}^{j}_0, \tilde{y}^{N-1}_0) \) arising at each layer of the recursion (8)–(11). We denote by \( X_\lambda = X^{(h_\phi/2^{t-\lambda})}_\lambda, \lambda \in [t+1], \) the
Algorithm 1: GetCSEPairs(h, \phi)

1. \( \hat{Z}_{\phi,h} = \{ v_i^h | v_i \in \{0, 1\}, i \in D_\phi, u_j = \hat{u}_j, j \in [\phi] \} \);
2. \( y_{l-1}^0 \leftarrow (y_0, y_1, \ldots, y_{l-1}) \) in symbolic form;
3. \( X_i \leftarrow \{ (v_i^{h-1}, y_{i-1}^0) | v_i^{h-1} \in \hat{Z}_{\phi,h} \} \);
4. \( I \leftarrow h; \)
5. \textbf{for } \lambda \leftarrow t \textbf{ downto } 1 \textbf{ do}
6. \hspace{1em} \( X_{\lambda-1} \leftarrow \emptyset; N \leftarrow 2^h; \)
7. \hspace{1em} \( I' \leftarrow I - (I \mod 2); \)
8. \hspace{1em} \textbf{for each } (\tilde{v}_0^{l-1}, \tilde{y}_0^{N-1}) \textbf{ in } X_\lambda \textbf{ do}
9. \hspace{1.5em} \( X_{\lambda-1} \leftarrow X_{\lambda-1} \cup \{ (v_0^{l-1} \oplus v_0^{l-1}, y_0^{N-1}) \}; \)
10. \hspace{1.5em} \( X_{\lambda-1} \leftarrow X_{\lambda-1} \cup \{ (v_0^{l-1}, y_0^{N-1}) \}; \)
11. \hspace{1.5em} \( I \leftarrow \lfloor I/2 \rfloor; \)
12. \textbf{return } \{ X_\lambda | \lambda \in [t + 1] \}

set of these distinct pairs \((\tilde{v}_0^j, \tilde{y}_0^{N-1})\) and refer it to as a set of common subexpressions (CSE).

In case when \( h_\phi - 1 \) is greater than \( h_{\phi-1} \), one needs to compute several input symbol LLRs \( S_t^{(h)} \) of paths \( v_0^{h-1} \in \hat{Z}_\phi \) using CSE for each internal phase \( h \), where \( h_{\phi-1} < h < h_\phi \).

B. The CSE identification algorithm

The sets \( X_\lambda \) of CSE can be identified offline for a given polarization kernel \( K \) of length \( l = 2^t \), external phase \( \phi \) and internal phase \( h \), \( h_{\phi-1} < h < h_\phi+1 \). To do that, one should run the procedure \( \text{GetCSEPairs}(h, \phi) \) illustrated in Algorithm 1. This procedure iteratively enumerates pairs \((\tilde{v}_0^{l-1}, \tilde{y}_0^{N-1})\) according to recursion in \((8)-(11)\), and stores the unique ones only. It starts from symbolic pairs \((v_0^{h-1} | y_0^{l-1})\), where \( v_0^{h-1} \in \hat{Z}_{\phi,h}, \hat{Z}_{\phi,h} = \{ v_i^{h-1} | v_i \in \{0, 1\}, i \in D_\phi, u_j = \hat{u}_j, j \in [\phi] \} \). Recall that \( \hat{u}_i \) is an already estimated symbol of \( u \) by SC decoding. Observe that \( X_0 = \{ S_0^{(0)}(\epsilon, y_i), i \in [l] \} \) for any \( h \).

Example 2. Consider the identification of CSE for kernel \( K_{16} \) at external phase 5 and internal phase \( h_5 = 8 \). The decoding window \( D_5 = \{5, 6, 7\} \). Thus, under condition \( u_0^4 = \hat{u}_0^4 \) we have \( \hat{Z}_{5,8} = \{ v_i^7 | v_i^7, v_i = \{0, 1\}, i \in D_5 \} \), which is initialized at line 7.
All steps of computation of $X_\lambda$, $0 \leq \lambda < 4$, by $\text{GetCSEPairs}(8, 5)$ are illustrated in Fig. 1. The set $X_4$ is initialized as $\left\{ (v_0^7 | y_0^{15}) | v_0^7 \in \hat{Z}_5^{(8)} \right\}$ at line 3. The elements of $X_3$ are obtained from elements of $X_4$ at lines 7-10 of Algorithm 1. Each pair $(v_0^7 | y_0^{15}) \in X_4$ is connected with two distinct pairs $(\hat{v}_0^5 | y_0^{15})$ from $X_3$. One of them is given by (10) and connected by black solid line, while another one is given by (11) and connected with a red dashed line. This process is performed at lines 9-10 of Algorithm 1. The same process is performed to obtain sets $X_2, X_1, X_0$.

In a straightforward implementation of WP for each $S_3^{(8)}$, one should compute 2 values of $S_3^{(4)}$, 4 values of $S_2^{(2)}$, and 8 values of $S_1^{(1)}$. This results in $8 \cdot (1 + 2 + 4 + 8) = 120$ operations. According to Fig. 1, identification of CSE allows one to compute $S_5^{(8)}$ only with $|X_4| + |X_3| + |X_2| + |X_1| = 8 \cdot 3 + 16 = 40$ operations.

C. CSE LLR computation

Finally, to obtain the LLRs $S_t^{(h)}(v_0^{h-1}, y_0^{l-1})$ given by $v_0^{h-1} \in Z_\phi$ at the external phase $\phi$ of the kernel processing, one should firstly perform $\text{GetCSEPairs}(h, \phi)$ to obtain CSE $X$, and
Algorithm 2: ComputeLLRs($X, h, \phi$)

1. $S^{(0)}_0(\epsilon, y_i) \leftarrow \log \frac{W(\epsilon|y_i)}{W(\bar{\epsilon}|y_i)}, i \in [l]$;
2. for $\lambda \leftarrow 1$ to $t$ do
3. \hspace{1em} $I \leftarrow \lfloor h/2^{t-\lambda} \rfloor$;
4. \hspace{1em} $I' \leftarrow I - (I \mod 2)$;
5. \hspace{1em} $N \leftarrow 2^{t-\lambda}$;
6. for each $(\bar{v}_i^{-1}, \bar{y}_N^{-1})$ in $X_\lambda$ do
7. \hspace{2em} $a \leftarrow S^{(I')/2}(v_i^{I-1} \oplus v_0^{I-1}, y_0^{N-1})$;
8. \hspace{2em} $b \leftarrow S^{(I')/2}(v_0^{I-1}, y_0^{N-1})$;
9. \hspace{2em} if $I$ is even then
10. \hspace{3em} $S^{(I)}_\lambda(\bar{v}_i^{-1}, \bar{y}_N^{-1}) \leftarrow \text{sign}(a) \text{sign}(b) \min(|a|, |b|)$;
11. \hspace{2em} else
12. \hspace{3em} evaluate $\bar{v}_I$ using $\hat{u}^{\phi-1}_0$ as $\bar{c}$;
13. \hspace{3em} $S^{(I)}_\lambda(\bar{v}_0^{-1}, \bar{y}_0^{N-1}) \leftarrow (-1)^{\epsilon} a + b$;
14. return an array of $S^{(h)}_t(v_i^{h-1}, y_0^{l-1})$, for $v_i^{h-1} \in \mathbb{Z}_{\phi,h}$.

run ComputeLLRs($X, h, \phi$) procedure, presented in Algorithm 2. This procedure uses CSE pairs $(\bar{v}_i^{-1}, \bar{y}_N^{-1}) \in X_\lambda$ and iteratively computes all intermediate LLRs $S^{(j)}_\lambda(\bar{v}_i^{-1}, \bar{y}_N^{-1})$ corresponding to these pairs.

We would like to emphasise that CSE pairs $(\bar{v}_i^{-1}, \bar{y}_N^{-1}) \in X_\lambda$ are used in symbolic form, while the corresponding LLRs $S^{(j)}_\lambda(\bar{v}_i^{-1}, \bar{y}_N^{-1})$ are obtained as evaluated numbers in Algorithm 2. The pairs $(\bar{v}_i^{-1}, \bar{y}_N^{-1})$ are used as indices of $S^{(j)}_\lambda$. We assume that one can access to any value $S^{(j)}_\lambda$ in constant time. In a software implementation one can replace pairs $(\bar{v}_i^{-1}, \bar{y}_N^{-1})$ by integer indices and consider $S^{(j)}_\lambda$ as arrays.

D. Reusing of CSE at different phases

Arikan SC decoding allows one to reuse intermediate LLRs to compute LLR for next phases as we mentioned in Section II-D. Similar approach can be used in WP with CSE.

Recall that in Arikan SC decoding at phase $i > 0$ one need to recompute only intermediate LLRs $S^{(i/2^{s-k})}_{t-k}$, $0 \leq k \leq s$, where $s = \psi(i)$ is the largest integer such that $2^s$ divides $i$. 
Observe that $S_{t-s}^{(i/2^j)}$ is calculated via $P$ function (9) using partial sums $C_{t-s}^{(i/2^j)}[\beta]$, while remaining $S_{t-k}^{((i/2^n-k)}$, $0 \leq k < s$ are computed via $Q$ function.

The above principle is applicable to the case of window processing with CSE as well. Consider processing of $\phi$-th phase of kernel $K$. Suppose we computed all CSE at internal phase $h_{\phi}$. Let $s' = \psi(h + 1)$, and $|D_{\phi}| \geq |D_{\phi+1}|$. In this case $X_{\lambda}^{(h_{\phi}+1)} \subseteq X_{\lambda}^{(h_{\phi})}$, $\lambda \in [s']$. Thus, at phase $h_{\phi} + 1$ we need to recompute only CSE LLRs $S_{\lambda}^{(h_{\phi}+1)}$ at layers $s' \leq \lambda \leq t$. In other words, one can start the Algorithm 2 from layer $s'$. In contrast, in case of $|D_{\phi}| < |D_{\phi+1}|$ we have $X_{\lambda}^{(h_{\phi})} \subseteq X_{\lambda}^{(h_{\phi}+1)}$, $\lambda \in [t + 1]$, and need to update all $X_{\lambda}$. However, even in this case we can reuse $|X_{\lambda}^{(h_{\phi}+1)}|/|X_{\lambda}^{(h_{\phi})}|$ LLRs at layers $\lambda \in [s']$.

**Example 3.** At external phase $\phi = 5$ of kernel $K'_{16}$ we have $u_5 = v_6 \oplus v_9$, $h_5 = 9$, and $D_5 = D_4$. In this case $X_{\lambda}^{(5)} = X_{\lambda}^{(4)}$, $\lambda \in [4]$. Thus, one can start Algorithm 2 from layer 4. The values $S_5^{(9)}$ are computed at lines 12–13 with $v_8$ evaluated as $\hat{u}_4$.

As a result, the arithmetic complexity (number of summation and comparison operations) of computing $S_{\phi}^{(h)}$ is given by $\sum_{\lambda=s'}^{t} |X_{\lambda}|$, while straightforward implementation of the WP algorithm requires $|D_{\phi}| \cdot \sum_{\lambda=s'}^{t} 2^{t-\lambda}$ operations. It can be seen that $\sum_{\lambda=s'}^{t} |X_{\lambda}| \leq |D_{\phi}| \cdot \sum_{\lambda=s'}^{t} 2^{t-\lambda}$ by definition of the CSE.

V. FURTHER IMPROVEMENTS OF WINDOW PROCESSING

In this section we show how to further simplify the window processing by exploiting some properties of path scores maximization.

A. Path score computation

We consider the WP of $l \times l$ polarization kernel $K$. Suppose for some external phase $\phi$ we have $h_{\phi} > h_{\phi-1} + 1$, or, in other words, $|D_{\phi}| > |D_{\phi} + 1|$. As we discussed in Section IV-A, in this case we needs to compute several input symbol LLRs $S_{t}^{(h)}$, where $h_{\phi-1} < h < h_{\phi}$. 
Essentially, in this case WP requires to obtain path scores

\[ R_y(v_0^h \phi) = R_y(v_0^{h-1}) + \tau \left( S_t^{(h\phi)}(v_0, h\phi) \right) = R_y(v_0^{h-1}) + \sum_{h=h_{d-1}+1}^{h_{d}} \tau \left( S_t^{(h\phi)}, v_h \right). \] (24)

It turns out that the last term of the above expression can be computed in another way. Let

\[ E(v_0^{n-1}, s_0^{n-1}) = \sum_{i=0}^{n-1} \tau(s_i, c_i) \]

be the ellipsoidal weight (also known as correlation discrepancy) of vector \( v_0^{n-1} \in \mathbb{R}_2^n \) with respect to \( s_0^{n-1} \) [26], [27].

**Theorem 2.** [17] The ellipsoidal weight of the vector \( v_0^{2^t-1} F_t \) with respect to the input LLRs \( s \) is equal to score of the path \( v_0^{2^t-1} \) in the SC decoder, i.e. \( E(v_0^{2^t-1} F_t, s) = \sum_{i=0}^{2^t-1} \tau(S_t(i)(v_0^{i-1}, y_0^{2^t-1}), v_i), \) where \( s = (S_0(0)(y_0), \ldots, S_0(0)(y_{2^{m-1}})) \).

**Theorem 2** implies the following:

**Corollary 1.** Consider Arikan SC decoding of \((n = 2^t, k)\) polar code at phase \( \phi = i + 2^q - 1 \), where \( q \) is the largest integer such that \( 2^q \) divides \( i \). Then,

\[ \sum_{\beta=1}^{i+2^q-1} \tau(S_t(\beta)(v_0^{\beta-1}, y_0^{n-1}), v_j) = \sum_{\beta=0}^{2^q-1} \tau(S_{\lambda-q}^{(\lfloor \beta/2^q \rfloor)}[\beta], c_\beta), \]

where \( c_0^{2^q-1} = u_1^{i+2^q-1} F_q. \)

Corollary [1] allows one to avoid computation of some \( S_t^{(l)} \) in [24] and use \( S_{t-q}^{(\lfloor l/2^q \rfloor)}[\beta] \) instead.

In some cases one can obtain \( R(v_0^h | y_0^{n-1}) \) in lower number of operations.

Consider computing the LLR for \( u_4 \) of \( K_t \) kernel. According to (12), one has \( R_y(v_0^8) = R_y(v_0^7) + \tau \left( S_t^{(8)}(v_0^7, y_0^{15}), v_8 \right) \). By definition we have \( R_y(v_0^7) = \sum_{j=0}^{7} \tau(S_4^{(j)}(v_0^{j-1}, y_0^{15}), v_j) \). It can be verified, that the computation of this path score for all \( v_0^7 \in \mathbb{Z}_4 \) with CSE requires 36 operations. Using Corollary [1], one can rewrite this expression in a following way:

\[ R_y(v_0^7) = R_y(v_0^3) + \sum_{\beta=0}^{3} \tau(S_2^{(1)}[\beta], (v_1^3 F_2)^\beta). \] (25)
Note that $S_2^{(1)}[\beta] = S_2^{(1)}((v_0^3 F_2)_\beta|((y_\beta, y_{\beta+4}, y_{\beta+8}, y_{\beta+12})))$. Unlike (24), one does not need to compute $S_4^{(5)}$, $S_4^{(6)}$ and $S_4^{(7)}$.

Observe that $\bar{Z}_b = \{v_4^7 F_2 | v_4 = b, v_5^7 \in \mathbb{F}_2^3\}$ is a coset of Reed-Muller code $RM(1, 2) \oplus b(F_2[0])$. Furthermore, $R_y(v_4^7) = \frac{1}{2} \left( \sum_{\beta=0}^{3} (-1)^c S_2^{(1)}[\beta] - \sum_{\beta=0}^{3} |S_2^{(1)}[\beta]| \right)$, \hspace{1cm} (26)

where $c_0^3 = v_4^7 F_2$. Assume for simplicity that $v_4 = 0$. Then the first term in (26) can be obtained for each $v_4^7 \in \bar{Z}_0$ via the fast Hadamard transform (FHT) [28] in 8 operations, and the second one does not need to be computed, since it cancels in (20). We have the index $3 \in \mathcal{D}_4$, thus, to obtain $R(v_0^7 | y_0^{15})$ for all $v_0^7 \in \mathcal{Z}_4$ one need to evaluate FHT with respect to LLRs $S_2^{(1)}[\beta]$, computed for $v_3 \in \{0, 1\}$. These LLRs was already obtained at phase 3, since $\mathcal{D}_3 = \{3\}$. In sum, Corollary 1 and FHT allowed us to reduce path score computation from 36 to 16 operations.

B. Recursive maximum computation

Consider the case when the decoding window is consequently decreasing at next phases of window processing. In this situation the complexity of (20) can be significantly reduced.

Let $h_\phi = h_{\phi+1} = \cdots = h_{\phi+q}$. It implies that $|\mathcal{D}_{\phi+i}| = |\mathcal{D}_\phi| - i, i \in [q + 1]$ and, therefore $\mathcal{Z}_{\phi+i} \subset \mathcal{Z}_{\phi+i-1} \subset \cdots \subset \mathcal{Z}_\phi$. Thus, each path score $R_y(v_0^{h_{\phi+i}})$ for $v_0^{h_{\phi+i}} \in \mathcal{Z}_{\phi+i}$ is computed at the phase $\phi$. Recall that WP (20) for phase $\phi + i$ requires the maximization of $R_y(v_0^{h_{\phi+i}})$ over $v_0^{h_{\phi+i}} \in \mathcal{Z}_{\phi+i}^{(b)}$, $b \in \mathbb{F}_2$. Since $\mathcal{Z}_{\phi+i} \subset \mathcal{Z}_\phi$, we propose to perform this maximization during the computation of $S_{1,\phi}$.

It is convenient to define the array

$$M_i[u_\phi^{\phi+i}] = \max_{v_0^{h_{\phi+i}} \in \mathcal{Z}_{\phi+i}^{(b)}, u_{\phi+i-1}^{\phi+i-1} = u_\phi^{\phi+i-1}} R_y(v_0^{h_{\phi+i}}),$$  \hspace{1cm} (27)
where $i \in [q + 1]$. This array can be also defined recursively as

$$M_t[\hat{u}_\phi^{q+i}] = \max(M_{t+1}[\hat{u}_{\phi+i}^q], M_{t+1}[\bar{u}_{\phi+i}^q]).$$  \hspace{1cm} (28)

The base of this recursion is given by $M_q[\hat{u}_{\phi+q}]$, which should be computed by (27). As a result, after recursive computation of $M_0[b], b \in \mathbb{F}_2$, the $\phi + i$ LLR $S_{1,\phi+i}$ can be obtained in one subtraction as $M_0[\hat{u}_{\phi+i}^{q-1}] - M_0[\bar{u}_{\phi+i}^{q-1}]$.

C. Reusing of maximums from the previous phase

Expression (20) requires computation of two maximums over path scores $R_y(v_{h_0}^{h_{\phi+1}})$ for $v_{0}^{h_{\phi}} \in \mathcal{Z}_{\phi}^{(b)}, b \in \mathbb{F}_2$. We made an observation that allowed us to reduce the complexity of this step.

Namely, consider the case $h_{\phi+1} = h_0 + 1$, which implies that $\mathcal{D}_{\phi+1} = \mathcal{D}_0$. According to (12), the path score $R_y(v_{0}^{h_{\phi+1}}) = R_y(v_{0}^{h_{\phi}}) + \tau(S_{t}^{(h_{\phi}+1)}(v_{0}^{h_{\phi}}, y_{0}^{l-1}), v_{h_0+1})$. Recall that $\tau(S, c)$ function is zero for one of $c \in \mathbb{F}_2$, therefore, the path score $R_y(v_{0}^{h_{\phi+1}}) = R_y(v_{0}^{h_{\phi}})$ for some $v_{h_0+1}$.

Suppose the decoder made estimate of symbol $\hat{u}_\phi$. Let $v_{0}^{h_{\phi}} = \arg \max_{v_{0}^{h_{\phi}} \in \mathcal{Z}_{\phi}^{(b)}} R_y(v_{0}^{h_{\phi}})$. Let $b \in \mathbb{F}_2$ be a such value that $\tau(S_{t}^{(h_{\phi}+1)}(v_{0}^{h_{\phi}}, y_{0}^{l-1}), b) = 0$. Using (16) for $v_{h_0+1}$ with $u_{\phi+1} = b$ we can obtain such value $a \in \mathbb{F}_2$ so that vector $v_{0}^{h_{\phi}}a \in \mathcal{Z}_{\phi}^{(b)}$. Therefore, $R_y(v_{0}^{h_{\phi}}a) = R_y(v_{0}^{h_{\phi}})$ and $v_{0}^{h_{\phi}}a = \arg \max_{v_{0}^{h_{\phi}+1} \in \mathcal{Z}_{\phi+1}^{(b)}} R_y(v_{0}^{h_{\phi}+1})$. It means that by tracking the value of $b$ we can directly obtain maximum for $u_{\phi+1} = b$ in (20). It remains to compute only $\max_{v_{0}^{h_{\phi}+1} \in \mathcal{Z}_{\phi+1}^{(b)}} R_y(v_{0}^{h_{\phi}+1})$.

It is also possible to employ this approach in case of reduced decoding window at next phases.

VI. EFFICIENT PROCESSING OF SOME POLARIZATION KERNELS

In this section we briefly describe the window processing of some polarization kernels with good polarization properties and small size of decoding window. We carefully compute the arithmetic complexity and point out all simplifications used.

The processing algorithm for $K_{16}^I$ is described in [29].
Algorithm 3: Improved window processing algorithm for phase $\phi$ of $2^t \times 2^t$ kernel $K$

Step 1: If $h_\phi > h_{\phi-1} + 1$, then use Corollary 1 to obtain $R_y(v_0^{h_\phi-1})$, $v_0^{h_\phi-1} \in Z_\phi$.
Step 2: Compute $S_t^{(h_\phi)}(v_0^{h_\phi-1}|y_0^{l-1})$, $v_0^{h_\phi-1} \in Z_\phi$ using CSE via Algorithm 2.
   Reuse intermediate LLRs from previous phases if it is possible (see Section IV-D).
Step 3: Compute $R_y(v_0^{h_\phi})$ for $v_0^{h_\phi} \in Z_\phi$ with $2|D_\phi| + 1 - 2$ operations. Some modifications are possible:
   a) If $h_\phi = h_{\phi-1} + 1$, then reuse computations from previous phase to obtain one of $R_b$ and compute only $R_b \oplus 1$ with $2|D_\phi| - 1$ operations (see Section V-C).
   b) If $h_\phi = h_{\phi+1} = \cdots = h_{\phi+q}$, then compute $R_b$ as $M_0[b]$ using (28) with $2|D_\phi| + 1 - 2$ operations.
Step 4: Compute $S_{h_\phi} = \max_{v_0 \in Z_\phi} R_y(v_0^{h_\phi})$ for $b \in [2]$ with $2|D_\phi| + 1 - 2$ operations. Some modifications are possible:

A. Overall processing algorithm

We consider the processing of $2^t \times 2^t$ polarization kernel $K$. Algorithm 3 illustrates all stages of WP of phase $\phi$ with improvements proposed at previous phases. Observe that the transition matrix $T$, equation (16), and CSE pairs (see Section IV-B) should be computed offline.

In some cases, Algorithm 3 can be skipped for some phase $\phi$. If $D_\phi = \emptyset$, then WP is equal to Arikan SC decoding, i.e., $S_{h_\phi} = S_t^{(h_\phi)}(v_0^{h_\phi-1}|y_0^{l-1})$. In case of $h_{\phi-p} = \cdots = h_{\phi-q}$ we assume that line 0b of Algorithm 3 was performed at phase $h_{\phi-p}$, therefore, $S_{h_\phi} = M_p[\hat{u}_{\phi-p}^{h_\phi-1}] - M_p[\hat{u}_{\phi-p}^{h_\phi-1}]$ (see Section V-B).

B. $K_{16}$ kernel

At phases 0–4, 11–15, $D_\phi = \emptyset$, thus, WP is equal to Arikan SC decoding. The processing complexity for these phases is given in Table III. Observe that all intermediate LLRs for phases $\geq 11$ can be retrieved from CSE LLRs computed at previous phases.

We also have $h_1 = h_2 = h_3$, thus, $D_8 = D_7 \setminus \{5\}$, $D_9 = D_8 \setminus \{6\}$ and $D_{10} = D_9 \setminus \{7\}$. It implies that recursive maximum computation (see Section V-B) can be used for phases 8–10.

Table III demonstrates the complexity of all steps of Algorithm 3 for phases 5–7, including...
TABLE III: Complexity of Algorithm 3 for $K_{16}$ kernel

| $\phi$ | $h_\phi$ | $|D_\phi|$ | Complexity of Algorithm 3 steps | Total complexity |
|-------|--------|-----------|-------------------------------|-----------------|
| 5     | 8      | 3         | $X_1$ $X_2$ $X_3$ $X_4$ $X_5$ | 10              |
| 6     | 9      | 3         | -                             | 7 (4a)          |
| 7     | 10     | 3         | -                             | 14 (4b)         |

TABLE IV: $K_{32}$ kernel with improved polarization

| $i$   | $K_{32}[i]$ | $i$   | $K_{32}[i]$ | $i$   | $K_{32}[i]$ |
|-------|-------------|-------|-------------|-------|-------------|
| 0     | 80000000    | 1     | C0000000    | 2     | A0000000    |
| 4     | 88000000    | 5     | 80800000    | 6     | C0C00000    |
| 8     | AC600000    | 9     | 6AC00000    | 10    | FF000000    |
| 12    | 80008000    | 13    | C000C000    | 14    | 4888C000    |
| 16    | A000A000    | 17    | F000F000    | 18    | 5AAAF000    |
| 20    | 88008800    | 21    | 80808080    | 22    | C0C0C0C0    |
| 24    | AC60AC60    | 25    | 6AC06AC0    | 26    | FF00FF00    |
| 28    | 88888888    | 29    | CCCCCCCC    | 30    | AAAAAAAA    |

sizes CSE. Recall that CSE LLRs can be reused, what also shown in this table. We also indicate the type of simplification implemented at step 4.

At step 1 of phase 5 the path scores $R_y(v_7^0)$ for $v_7^0 \in Z_5$ can be computed with 8 operations (see Section V-A). Moreover, using properties of FHT, one can find $\arg\max_{v_7^0 \in Z_5} R_y(v_7^0)$ with 3 operations and obtain one of $R_b$. The remaining $R_{b+1}$ is computed with 7 operations.

The total processing complexity of $K_{16}$ is 181 operations.

C. $K_{32}$ kernel

We present a new $32 \times 32$ polarization kernel $K_{32}$, written in Table IV in hexadecimal notation. This kernel has rate of polarization $E(K_{32}) = 0.521936$ and scaling exponent $\mu(K_{32}) = 3.417$.

We constructed this kernel via algorithm, presented in [22]. Table IV demonstrates the transition matrix, decoding windows and processing complexity for each phase.
TABLE V: Input symbols \( u_\phi \) for kernel \( K_{32} \) as functions of input symbols \( v \) for \( F_5 \)

| \( \phi \) | \( u_\phi \) | \( D_\phi \) | Cost | \( \phi \) | \( u_\phi \) | \( D_\phi \) | Cost |
|---|---|---|---|---|---|---|---|
| 0 | \( v_0 \) | \{\} | 31 | 16 | \( v_{18} \) | \{14, 15\} | 16 |
| 1 | \( v_1 \) | \{\} | 1 | 17 | \( v_{14} \oplus v_{19} \) | \{14, 15\} | 15 |
| 2 | \( v_2 \) | \{\} | 3 | 18 | \( v_{14} \) | \{\} | 1 |
| 3 | \( v_3 \) | \{\} | 1 | 19 | \( v_{15} \) | \{\} | 1 |
| 4 | \( v_4 \) | \{\} | 7 | 20 | \( v_{20} \) | \{\} | 7 |
| 5 | \( v_5 \oplus v_6 \oplus v_9 \) | \{5, 6, 7\} | 67 | 21 | \( v_{24} \oplus v_{22} \oplus v_{21} \) | \{21, 22, 23\} | 67 |
| 6 | \( v_5 \oplus v_{10} \) | \{5, 6, 7\} | 47 | 23 | \( v_{21} \oplus v_{26} \) | \{21, 22, 23\} | 47 |
| 7 | \( v_5 \) | \{6, 7\} | 1 | 24 | \( v_{21} \) | \{22, 23\} | 1 |
| 8 | \( v_6 \) | \{7\} | 1 | 25 | \( v_{22} \) | \{23\} | 1 |
| 9 | \( v_7 \) | \{\} | 1 | 26 | \( v_{23} \) | \{\} | 1 |
| 10 | \( v_{11} \) | \{\} | 1 | 27 | \( v_{27} \) | \{\} | 1 |
| 11 | \( v_{16} \) | \{12, 13, 14, 15\} | 127 | 28 | \( v_{28} \) | \{\} | 7 |
| 12 | \( v_{12} \oplus v_{17} \) | \{12, 13, 14, 15\} | 63 | 29 | \( v_{29} \) | \{\} | 1 |
| 13 | \( v_{12} \) | \{13, 14, 15\} | 1 | 30 | \( v_{30} \) | \{\} | 3 |
| 14 | \( v_{13} \) | \{14, 15\} | 1 | 31 | \( v_{31} \) | \{\} | 1 |

TABLE VI: Complexity of Algorithm \( 3 \) for \( K_{32} \) kernel

| \( \phi \) | \( h_\phi \) | \( |D_\phi| \) | Complexity of Algorithm \( 3 \) steps | Total complexity |
|---|---|---|---|---|
| 12 | 16 | 4 | \( X_1, X_2, X_3, X_4, X_5 \) | 127 |
| 13 | 17 | 4 | \( X_1, X_2, X_3, X_4, X_5 \) | 63 |
| 16 | 18 | 2 | \( X_1, X_2, X_3, X_4, X_5 \) | 16 |
| 17 | 19 | 2 | \( X_1, X_2, X_3, X_4, X_5 \) | 15 |

At phases 0–4, 11, 20, 27–31, \( D_\phi = \emptyset \), the processing complexity is equal to Arikan SC decoding. Surprisingly, the phases 5-10 and 21-26 of \( K_{32} \) have absolutely the same structure and processing complexity as phases 5-10 of \( K_{16} \) (see Section VI-B). The only difference is in expressions for \( v_h \) and in specific CSE pairs. However, the sizes of CSE sets are the same.

Table VI demonstrates the complexity of all steps of Algorithm \( 3 \) for phases 12–13 and 16–17, including sizes of CSE. At phase 12, we have

\[
R_y(v_0^{15}) = R_y(v_0^{12}) + \sum_{\beta=0}^{3} \tau(S_3^{(3)}[\beta], c_\beta),
\]  

\( (29) \)
where $c_3^0 = v_{12}^{15} F_2$ (see Corollary 1). The structure $D_{12}$ implies that $c_3^0 \in \mathbb{F}_2^4$. The equation (29) requires 4 values $S_3^{(3)}[\beta]$. After that, using the properties of $\tau$ function, last part of (29) can be computed with 11 operations, while $R_y(v_{12}^{12})$ can be omitted. Moreover, since $c_3^0 = v_{12}^{15} F_2$, there is $c_3^0$ such that $\sum_{\beta=0}^3 \tau(S_3^{(3)}[\beta], c_\beta) = 0$, thus, $\arg \max_{v_{15}^0 \in \mathbb{Z}_{15}} R_y(v_{12}^{12})$ can be immediately obtained with 0 operations.

The total processing complexity for $K_{32}$ is 571 operations.

The above described techniques can be also used to implement SCL decoder for polar codes with the considered kernels, using a straightforward generalization of the algorithm and data structures presented in [2].

VII. NUMERICAL RESULTS

The performance of all codes was investigated for the case of AWGN channel with BPSK modulation. The sets of frozen symbols were obtained by method proposed in [30]. The SCL algorithm was implemented using the randomized order statistic algorithm for selection of the paths to be killed at each phase, which has complexity $O(L)$.

We constructed (4096, 2048) and (1024, 512) polar codes with kernels $K_{16}', K_{16}$ and $K_{32}$
respectively. Figure 2a illustrates the performance of (4096, 2048) plain polar codes, polar codes with CRC and polar subcodes (PSCs) [4]. The same polar code constructions were used to construct (1024, 512) polar codes, whose performance is demonstrated in Fig. 2b. It can be seen that the codes based on kernels $K'_16, K_{16}$ and $K_{32}$ with improved polarization rate $E(K'_16) = E(K_{16}) = 0.51828, E(K_{32}) = 0.521936$ provide significant performance gain compared with polar codes with Arikan kernel $F_1$. Observe also that randomized polar subcodes provide better performance compared to polar codes with CRC.

PSCs with kernels $K'_16, K_{16}$ under SCL with $L = 8$ have almost the same performance as PSCs with $F_1$ kernel under SCL with $L = 32$. Observe also that the codes based on kernels with lower scaling exponent exhibit better performance. In the case of polar subcodes with kernel $K_{32}$ its performance under SCL with $L = 8$ is slightly better than the performance of polar subcode with $F_1$ kernel under SCL with $L = 16$.

Figure 3a presents simulation results for (4096, 2048) PSCs with different kernels under SCL with different $L$ at $E_b/N_0 = 1.25$ dB. It can be seen that the kernels with rate of polarization 0.51828 require significantly lower list size $L$ to achieve the same performance as the code with $F_1$ kernel, which has rate of polarization 0.5. Moreover, this gap grows with $L$. This is due to improved rate of polarization, which results in smaller number of unfrozen imperfectly polarized bit subchannels. The size of the list needed to correct possible errors in these subchannels grows exponentially with their number (at least for the genie-aided decoder considered in [31]). On the other hand, lower scaling exponent gives better performance with the same list $L$, but the slope of the curve remains the same for both kernels $K'_16, K_{16}$.

Figure 3c presents simulation results for (1024, 512) PSC with different kernels under SCL with different $L$ at $E_b/N_0 = 1.5$ dB. Similarly to the case of $16 \times 16$ kernels, $32 \times 32$ kernel $K_{32}$ with improved polarization properties ($E(K_{32}) = 0.521936, \mu(K_{32}) = 3.417$) provides significant performance gain compared with polar subcode with $F_1$.

Figure 3b presents the performance of SCL decoding of (4096, 2048) PSCs with $K'_16$ and $K_{16}$
kernels in terms of the decoding complexity. Recall that proposed kernel processing algorithm uses only summations and comparisons. Observe that the PSC based on kernel $K_{16}$ can provide better performance with the same decoding complexity for $\text{FER} \leq 7 \cdot 10^{-3}$ ($L \geq 8$ for $K_{16}$ and $L \geq 22$ for $F_2$). This is due to higher slope of the corresponding curve in Figure 3a, which eventually enables one to compensate relatively high complexity of the LLR computation algorithm presented in Section VI-B.

Unfortunately, $K_{16}'$ kernel with lower scaling exponent has greater processing complexity than $K_{16}$, so that its curve intersects the one for the $F_1$ kernel only at $\text{FER}= 2 \cdot 10^{-3}$.

Figure 3d presents the results of SCL decoding of $(1024, 512)$ PSC with $K_{32}$ in terms of the decoding complexity. Similarly to $K_{16}$ kernel, the PSC with $K_{32}$ can provide better performance with the same decoding complexity. Namely, starting from $\text{FER} \leq 2 \cdot 10^{-2}$ PSC with $K_{32}$

![Graphs showing performance and decoding complexity for different kernels.](image-url)
(L ≥ 4) maintain approximately the same performance-complexity tradeoff as PSC with F₁ (L ≥ 9). Observe that K₃₂ kernel enables one to reduce required list size in more than two times. From FER ≤ 4 · 10⁻³ PSC with K₃₂ (L ≥ 20) provides better performance with the same decoding complexity compared with PSC with F₁ (L ≥ 48).

VIII. CONCLUSIONS

In this paper a reduced complexity decoding algorithm for polar codes with 2ᵗ × 2ᵗ polarization kernels was proposed. Application of this approach to kernels of size 16 and 32 was considered. The algorithm computes kernel input symbols LLRs via the ones for the Arikan kernel, and exploits the structure of recursive calculations induced by the kernel to identify and reuse the values of some common subexpressions. It was shown that in the case of SCL decoding with sufficiently large list size, the proposed approach results in lower decoding complexity compared to the case of polar (sub)codes with Arikan kernel with the same performance.

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