Limiting sequential decompositions and applications in finance

Gero Junike∗, Hauke Stier†, Marcus Christiansen‡

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Abstract

The sequential updating (SU) decomposition is a well-known technique to obtain a profit and loss (P&L) attribution, e.g. of a bond portfolio, by dividing the time horizon into n subintervals and only vary one risk factor, e.g. FX, IR, CS or calendar time, in each subinterval. We show that the SU decomposition converges for large n if the P&L attribution can be expressed by a smooth function of the risk factors. We consider the average SU decomposition, which does not depend on the order or labeling of the risk factors. Sufficient conditions are given to reduce the computational complexity significantly when calculating the average SU decomposition.

Keywords: profit and loss attribution; sequential decompositions; change analysis; risk decomposition

1 Introduction

Profit and loss (P&L) attribution has a long history in risk management. P&L attribution is the process of analyzing the change between two valuation dates and explaining the development of the P&L by the movement of the sources (risk factors) between the two dates, see Candland & Latz (2014). That is, the change of the P&L over time is decomposed with respect to the different risk factors and each risk factor is assigned a contribution to the P&L.

One common way to carry out a decomposition is to divide the time horizon into subintervals and sequentially update only one risk factor in each subinterval while “freezing” all other risk factors. This idea dates back at least to Oaxaca

∗Corresponding author. E-Mail: gero.junike@uol.de, Carl von Ossietzky Universität, Institut für Mathematik, 26129 Oldenburg, Germany.
†E-Mail: hauke.stier@uol.de, Carl von Ossietzky Universität, Institut für Mathematik, 26129 Oldenburg, Germany.
‡E-Mail: marcus.christiansen@uol.de, Carl von Ossietzky Universität, Institut für Mathematik, 26129 Oldenburg, Germany.
(1973) and Blinder (1973), who developed a sequential updating (SU) decomposition technique to decompose the wage gap between people of different groups, i.e. black/white and male/female, into an observable influencing factor and an unobservable influencing factor, i.e. discrimination against a certain group. Their SU decomposition is static. The SU decomposition is called waterfall in Candland & Latz (2014). Shorrocks (2013) and Biewen (2012) introduced a multifactor, discrete sequential decomposition. Similar to Frei (2020), Jettes & Christiansen (2022) and Christiansen (2022) applied the SU technique recursively to multiple time periods and analysed the case that the length of the subintervals for the SU decomposition converges to zero and arrived at an infinitesimal sequential updating (ISU) decomposition principle.

The SU/ISU decompositions depend on the order in which the risk factors are updated. I.e. if there are $d$ risk factors, there are $d!$ different update orders and therefore $d!$ possible ways to compute the SU/ISU decompositions.

There are a number of desirable properties that a decomposition should have, see Shorrocks (2013). It should be symmetric, i.e. the contributions of the risk factors should be independent of the way in which the factors are labeled or ordered. A decomposition should also be exact or additive, which means that the sum of all contributions is equal to the total change of the P&L.

Jettes & Christiansen (2022) showed in an insurance context that the ISU decomposition is not only additive but also symmetric if there are no interaction effects, i.e. no covariation between the risk factors. Christiansen (2022) proved that the ISU decomposition is symmetric if it is stable with respect to small perturbations in the empirical observation of the risk factors. In Appendix A.3 we show that the ISU decomposition of a simple product of two correlated Brownian motions is not stable. So stability is a rather strong assumption.

Shorrocks (2013) and Jettes & Christiansen (2022) introduced the ASU and IASU decompositions, which are simply the arithmetic average of the $d!$ possible SU/ISU decompositions. The ASU and IASU decompositions are additive and symmetric by construction. The (static) ASU decomposition is also called Shapley decomposition, see Shorrocks (2013). In game theory, the Shapley decomposition is better known as Shapley value. It is the unique decomposition satisfying certain desirable properties, see Young (1985). The Shapley decomposition has been introduced axiomatically to risk allocation, see for example Denault (2001) and Powers (2007). An axiomatic introduction to P&L attribution can be found in Moehle et al. (2021, Section 3.1), where also possible numerical approximations of the ASU decomposition are discussed.

Beside the sequential updating concepts, the so-called one-at-a-time (OAT) decomposition and its infinitesimal counterpart IOAT are also popular decomposition approaches, see Biewen (2012), Candland & Latz (2014), Frei (2020) and Jettes & Christiansen (2022). The OAT decomposition is called bump and reset in Candland & Latz (2014). OAT and IOAT decompositions are symmetric but in general not additive. Frei (2020) proposes the OAT decomposition for a risk attribution and shows that the OAT decomposition converges if there are no interaction effects and the risk factors have continuous paths. In this article, we generalize Frei (2020) and allow general semimartingales and non-zero...
interaction effects.

In many applications the P&L or price can be expressed as a smooth function of the underlying (risk) factors. For example, the price of a foreign stock is simply the product of the foreign exchange rate and the stock price in its domestic currency. The price \( p_{CB} \) of a zero-coupon foreign corporate bond can be expressed by

\[
p_{CB}(t) = \eta(t)e^{-(r(t)+c(t))(T-t)},
\]

where \( \eta \) is the foreign exchange rate, \( r \) is the interest rate, \( c \) is the credit spread, \( t \) is the calendar time and \( T \) the maturity of the bond. In this article, we aim to decompose \( f(X(t))_{t \in [0, \infty)} \) with respect to \( X \) where \( X \) is a \( d \)-dimensional semimartingale describing the evolution of \( d \) risk factors, and \( f \) is a twice continuously differentiable function.

This paper makes the following contributions: In Section 3, we give a rigorous definition of the SU decomposition for arbitrary update orders in the risk factors. In Section 4, we prove that the ISU, IOAT and IASU decompositions of twice continuously differentiable functions of the risk factors always exists. We also provide sufficient conditions under which the computational complexity for calculating the ASU decomposition can be significantly reduced. In Section 5, we discuss several examples of P&L attributions, including an example on the decomposition of a Value-at-Risk process. Section 6 concludes.

2 Notation

We use a similar setting as in Schilling et al. (2020) and Christiansen (2022). Let \( X \) be a \( d \)-dimensional semimartingale and let \( \mathbb{F} = (\mathcal{F}_t)_{t \geq 0} \) be the natural filtration of \( X \). Let \((\Omega, \mathcal{F}, \mathbb{P})\) be a filtered probability space satisfying the usual conditions. \( \mathcal{F}_0 \) is trivial. Let \( X \) be the set of all \( \mathbb{F} \)-semimartingales. \( X \in \mathcal{X}^d \) is called a risk basis, risk factors, sources of risk or information basis. Let \( C_2 \) be the set of twice differentiable functions from \( \mathbb{R}^d \) to \( \mathbb{R} \). For \( f \in C_2 \), we write \( f_i \) and \( f_{ij} \) for \( i,j = 1,\ldots,d \) for the partial derivatives \( \partial_i f \) and \( \partial_i \partial_j f \). Let \( F : \mathcal{X}^d \to \mathcal{X} \) be a map. For example, the map \( F \) could be defined by \( F(Y) = (f(Y(t)))_{t \geq 0} \). \( Y \in \mathcal{X}^d \) for some \( f \in C_2 \). The aim of this paper is to decompose \( F(X) \) with respect to \( X \).

Let \( a \wedge b \) denote the minimum of real numbers \( a \) and \( b \). For a scalar \( s \geq 0 \) and a vector \( \tau \in \mathbb{R}_+^d \), we define stopped semimartingales

\[
X^s = (X^s_1, \ldots, X^s_d) \quad \text{and} \quad X^\tau = (X_1^{\tau_1}, \ldots, X_d^{\tau_d}).
\]

We denote the \( i^{th} \)-row of a real \( d \times d \) matrix \( M \) by

\[
M(i,:) = (M_{i,1}, \ldots, M_{i,d}).
\]

For a sequence \( (s_l)_{l \in \mathbb{N}} \subset \mathbb{R} \) and \( \beta \in \{0,1\}^d \), we denote by \( s_{l+\beta} \) the vector

\[
s_{l+\beta} = (s_{l+\beta_1}, \ldots, s_{l+\beta_d}) \in \mathbb{R}^d.
\]
We define for $\beta \in \{0, 1\}^d$ and $s \in (0, t]$
\[X(s \ast \beta) = \lim_{\varepsilon \searrow 0} (X_1(s - \varepsilon(1 - \beta_1)), ..., X_d(s - \varepsilon(1 - \beta_d))).\]
In words, $X(s \ast \beta)$ is the left-limit of $X$ at time $s$ at those positions where $\beta$ is equal to zero.

3 SU decomposition principle

A decomposition of $F(X)$ is a $d$-dimensional process $D = (D_1, ..., D_d)$. A decomposition is called additive or exact if
\[F(X) = D_1 + ... + D_d.\]

We interpret $D_i$ as the contribution of $X_i$ to $F(X)$ for $i = 1, ..., d$. A decomposition principle does not need to be additive, see for example the one-at-time decomposition in Biewen (2012) and Jetses & Christiansen (2022), which has remaining interaction effects. But additivity is certainly a desirable property in many applications, see Shorrocks (2013) and Christiansen (2022).

Before formally defining the sequential updating (SU) decomposition, we provide some motivation based on telescoping series for the case $d = 2$. That is, we decompose $(f(X_1(t), X_2(t)))_{t \geq 0}$ with respect to $X_1$ and $X_2$ for some $f \in C_2$.

Let
\[\{0 = s_0 < s_1 < ... < s_{n-1} < t = s_n < s_{n+1} < ...\}\]
be an unbounded partition of $[0, \infty)$. Then by telescoping series
\[
\begin{align*}
f(X(t)) - f(X(0)) &= f(X_1(s_1), X_2(s_0)) - f(X_1(s_0), X_2(s_0)) + \\
&\quad f(X_1(s_1), X_2(s_1)) - f(X_1(s_1), X_2(s_0)) + \\
&\quad \cdots \\
&\quad f(X_1(s_n), X_2(s_{n-1})) - f(X_1(s_{n-1}), X_2(s_{n-1})) + \\
&\quad f(X_1(s_n), X_2(s_n)) - f(X_1(s_n), X_2(s_{n-1})).
\end{align*}
\]
That is, the period $[0, t]$ is divided into smaller sub-periods $(s_j, s_{j+1}]$ and in each sub-period, the risk factor $X_1$ is updated first, while the other risk factor $X_2$ stays constant. Then $X_2$ is updated while $X_1$ is hold fixed. We write Eq. (1) in shorter notation. Let $F(X)(t) := f(X(t))$, $t \geq 0$. Then it holds that
\[
\begin{align*}
F(X)(t) - F(X)(0) &= \sum_{l=0}^{\infty} [F(X_1^{s_l+1}, X_2^{s_l}) (t) - F(X_1^{s_l}, X_2^{s_l}) (t)] \\
&\quad + \sum_{l=0}^{\infty} [F(X_1^{s_l+1}, X_2^{s_l+1}) (t) - F(X_1^{s_l+1}, X_2^{s_l}) (t)].
\end{align*}
\]
The first sum on the right-hand-side of Eq. (2) is the sum of all those differences from the right-hand-side of Eq. (1) where $X_1$ varies, and we interpret it as
the contribution of $X_1$ to $f(X(t)) - f(X(0))$. The second sum of Eq. (2) is interpreted as the contribution of $X_2$.

One could also go the other way around and update $X_2$ first in each sub-period and then $X_1$. In the case $d = 2$, there are hence two possible SU decompositions. Now we formally define the two the SU decompositions.

**Definition 3.1.** (SU decomposition, case $d = 2$). Let $\gamma = \{0 = s_0 < s_1 < \ldots \}$ be an unbounded partition of $[0, \infty)$. The SU (sequential updating) approach with respect to $\gamma$ defines two decompositions $D^{SU} = (D^{SU}_1, D^{SU}_2)$ and $D^{SU'} = (D^{SU'}_1, D^{SU'}_2)$ of $F(X)$ for $t \geq 0$ by

$$D^{SU}_1(t) = \sum_{l=0}^{\infty} [F(X_{s_{l+1}}^{s_l+1}, X_{s_{l+1}}^{s_l}) (t) - F(X_{s_l+1}^{s_l}, X_{s_{l+1}}^{s_l}) (t)],$$

$$D^{SU}_2(t) = \sum_{l=0}^{\infty} [F(X_{s_{l+1}}^{s_l+1}, X_{s_{l+1}}^{s_l+1}) (t) - F(X_{s_{l+1}}^{s_l+1}, X_{s_{l+1}}^{s_l}) (t)]$$

and

$$D^{SU'}_1(t) = \sum_{l=0}^{\infty} [F(X_{s_{l+1}}^{s_l+1}, X_{s_{l+1}}^{s_l+1}) (t) - F(X_{s_{l+1}}^{s_l+1}, X_{s_{l+1}}^{s_l}) (t)],$$

$$D^{SU'}_2(t) = \sum_{l=0}^{\infty} [F(X_{s_{l+1}}^{s_l}, X_{s_{l+1}}^{s_l}) (t) - F(X_{s_{l+1}}^{s_l}, X_{s_{l+1}}^{s_l}) (t)].$$

The processes $D^{SU}_i$ or $D^{SU'}_i$ describe the contribution of $X_i$ for $i = 1, 2$.

At next, we define the SU decomposition principle for arbitrary dimensions. Let $d \in \mathbb{N}$. Define a matrix $A \in \mathbb{R}^{d \times d}$ by

$$A = \begin{pmatrix}
1 & 0 & \ldots & 0 \\
1 & 1 & \ldots & 0 \\
\vdots & \ddots & \ddots & \vdots \\
1 & 1 & \ldots & 1
\end{pmatrix}.$$  

This notation is inspired by Biewen (2012, Section 3.3). Let $\pi \in \sigma_d$, where $\sigma_d$ is the set of all $d!$ permutations of $\{1, \ldots, d\}$. Define matrix $B^\pi$ by

$$B^\pi_{i,j} = A_{\pi(i),\pi(j)}, \quad i, j = 1, \ldots, d.$$  

The matrix $B^\pi$ permutes the columns and rows of $A$ by $\pi$. Let

$$C^\pi = B^\pi - I,$$

where $I$ is the identity matrix. Then $C^\pi$ and $B^\pi$ are identical except at the main diagonal, where $B^\pi$ has a chain of ones and $C^\pi$ has a chain of zeros.
Example 3.2. For $d = 3$ and $\pi \in \sigma_d$ with $\pi(1) = 1$, $\pi(2) = 3$, $\pi(3) = 2$, it holds that
\[
A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}, \quad B^\pi = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix} \quad \text{and} \quad C^\pi = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}.
\]

Definition 3.3. (SU decomposition, $d \in \mathbb{N}$). Let $\gamma = \{0 = s_0 < s_1 < \ldots \}$ be an unbounded partition of $[0, \infty)$. The SU (sequential updating) approach with respect to $\gamma$ defines a set of $d!$ decompositions of $F(X)$ by
\[
D_{i}^{SU, \pi}(t) = \sum_{l=0}^{\infty} \{ F(X^{s_l + B^\pi(i, \ldots)}) (t) - F(X^{s_l + C^\pi(i, \ldots)}) (t) \}, \quad t \geq 0, \tag{3}
\]
where $i = 1, \ldots, d$ and $\pi \in \sigma_d$.

Remark 3.4. If $\pi = id$, then $B^\pi = A$ and the SU decomposition reduces to the formula stated in Jetses & Christiansen (2022) and Christiansen (2022). It is easy to see that Definition 3.3 generalizes Definition 3.1. By telescoping series, the SU decomposition principle is indeed additive if
\[
F(X^s) (t) = F(X)(t \wedge s), \quad s, t \geq 0,
\]
which holds for instance if $F(X)(t) = f(X(t))$, $t \geq 0$.

4 ISU decomposition principle

Jetses & Christiansen (2022) define the ISU decomposition by using finite partitions that differ for each interval $[0, t]$, $t \geq 0$. It is more convenient to follow the definition of Christiansen (2022) who uses infinite partitions that work for the whole time line. We recall the definitions. Let $\gamma^n = \{0 = s_0^n < s_1^n < \ldots \}$, $n \in \mathbb{N}$, be an increasing sequence, i.e. $\gamma^n \subset \gamma^{n+1}$ for all $n \in \mathbb{N}$, of unbounded partitions, i.e. $\lim_{k \to \infty} s_k^n = \infty$, of the interval $[0, \infty)$ with vanishing step lengths, i.e. $\lim_{n \to \infty} \sup_k |s_{k+1}^n - s_k^n| = 0$.

Definition 4.1. For $n \in \mathbb{N}$, let $D^{SU, \pi, n}$ be the SU decomposition of $F(X)$ with respect to $\pi \in \sigma_d$ and $\gamma^n$. If the limits
\[
D_{i}^{ISU, \pi}(t) = \lim_{n \to \infty} D_{i}^{SU, \pi, n}(t), \quad i \in \{1, \ldots, d\}, \quad t \geq 0,
\]
exist in probability, then $D^{ISU, \pi}$ is called ISU (infinitesimal sequential updating) decomposition of $F(X)$ with respect to order $\pi$ and partitions $(\gamma^n)_{n \in \mathbb{N}}$.

Provided existence, the ISU decomposition is unique up to modifications. Next, we give some sufficient conditions that ensure existence of ISU decompositions. For a one-dimensional semimartingale $Z$, we write
\[
\Delta Z(t) = Z(t) - Z(t-), \quad t \geq 0,
\]
\[ [X_i, Y_j]^{c} = [X_i, Y_j] - \sum_{0 < s \leq t} \Delta X_i(s) \Delta Y_j(s), \quad i, j = 1, \ldots, d, \ X, Y \in \mathcal{X}. \quad (4) \]

**Theorem 4.2.** Let \( f \in C_2 \). Then the ISU decomposition of \((f(X_i))_{i \geq 0}\) exists for each order \( \pi \in \sigma_d \) and each increasing sequence of unbounded partitions of \([0, \infty)\) with vanishing step lengths. For \( i = 1, \ldots, d \) and \( t \geq 0 \) it almost surely equals
\[
D^{ISU, \pi}_i(t) = \int_0^t f_i(X(s-)) dX_i(s) + \frac{1}{2} \int_0^t f_{ii}(X(s-)) d[X_i, X_i]^c(s) + \sum_{j=1}^d \int_0^t f_{ij}(X(s-)) d[X_i, X_j]^c(s) + \sum_{0 < s \leq t} \{ f(X(s \ast B^\pi(i, :))) - f(X(s \ast C^\pi(i, :))) - f_i(X(s-)) \Delta X_i(s) \}. \quad (5)
\]

**Proof.** We use standard arguments from the proof of Ito’s formula, see for example Protter (1990, Thm. 32, Part II), but many new arguments are needed too. Let \( t > 0 \). Fix some \( i \in 1, \ldots, d \) and some permutation \( \pi \). Let \( \gamma^n = \{ 0 = s^n_0 < s^n_1 \ldots \} \) be an increasing sequence of unbounded partitions of \([0, \infty)\) with vanishing step length. To keep the notation simple, we use the following abbreviations: for a matrix \( M \in \{0, 1\}^{d \times d} \) let
\[
X^{M, l} = X^{s^n_{l+M(i, :)}}, \quad \delta X^{M, l} = X^{M, l} - X^{0, l},
\]
\( 0 \) and \( 1 \) denote the \( d \times d \) matrices containing only zeros, and only ones, respectively. Instead of \( B^\pi \) and \( C^\pi \) we simply write \( B \) and \( C \). By Protter (1990, Thm. 21 and 30, Part II) it holds for \( i, j = 1, \ldots, d \) that
\[
\sum_{l \in \mathbb{N}_0} f_i(X^{0, l}) \delta X^{1, l}_i \xrightarrow{p} \int_0^t f_i(X(s)) d(X_i)(s), \quad n \to \infty, \quad (6)
\]
\[
\sum_{l \in \mathbb{N}_0} f_{ij}(X^{0, l}) \delta X^{1, l}_i \delta X^{1, l}_j \xrightarrow{p} \int_0^t f_{ij}(X(s)) d[X_i, X_j](s), \quad n \to \infty \quad (7)
\]
and
\[
\sum_{l \in \mathbb{N}_0} \delta X^{1, l}_i \delta X^{1, l}_j \xrightarrow{p} [X_i, X_j](t), \quad n \to \infty, \quad (8)
\]
where \( \xrightarrow{p} \) denotes convergence in probability. We use the fact that a sequence of random variables converges in probability if and only if every subsequence has a further subsequence which converges almost surely. Let \( (n_k)_{k \in \mathbb{N}} \) a subsequence
of the sequence \( (n)_{n \in \mathbb{N}} \). There is another subsequence \( (a_n)_{n \in \mathbb{N}} \) of \( (n_k)_{k \in \mathbb{N}} \) such that the sums in Eq. (6-8) converge almost surely. Let \( \Omega_0 \subset \Omega \) be the maximal subsequence such that for all \( \omega \in \Omega_0 \) the paths \( u \mapsto X(\omega)(u) \) are càdlàg, Eq. (29) holds and the sums in Eq. (6-8) converges surely on \( \Omega_0 \) along the subsequence \( (a_n)_{n \in \mathbb{N}} \). Then \( P(\Omega_0) = 1 \). We will show that the SU decomposition with respect to \( (\gamma^{a_n})_{n \in \mathbb{N}} \) converges for all \( \omega \in \Omega_0 \) surely as \( n \to \infty \). That implies that the SU decomposition converges in probability.

In the next steps of the proof, we assume that \( \omega \in \Omega_0 \) is fixed. Let \( \mathcal{M}_\omega \subset \mathbb{R}^d \) be the closure of the set \( \{ X(\omega)(u), u \in [0, t] \} \). The path \( u \mapsto |X(\omega)(u)| \) is càdlàg and takes a maximum value on the compact set \( [0, t] \). This implies that \( \mathcal{M}_\omega \) is bounded and hence compact. The convex hull \( \text{conv}(\mathcal{M}_\omega) \) of the compact set \( \mathcal{M}_\omega \) is also compact by Carathéodory’s theorem, see Grünbaum (2013, Sec. 2.3). By the definition of \( B \) and \( C \), it holds that

\[
B_{i,j} = \begin{cases} 1 & , \pi(j) \leq \pi(i) \\ 0 & , \text{otherwise} \end{cases} \quad \text{and} \quad C_{i,j} = \begin{cases} 1 & , \pi(j) < \pi(i) \\ 0 & , \text{otherwise} \end{cases}
\] (9)

which implies \( B_{i,i} = 1 \) and \( C_{i,i} = 0 \) and hence \( \delta X_i^{C,l} = 0 \) and

\[
\delta X_i^{B,l} = \begin{cases} 0 & , \pi(j) > \pi(i) \\ \delta X_i^{1,l} & , \pi(j) \leq \pi(i) \\ \delta X_i^{C,l} & , j \neq i. \end{cases}
\] (10)

Let \( \alpha > 0 \) and let \( \mathcal{A}_\alpha = \mathcal{A}_\alpha(\omega) \subset [0, t] \) be defined as in Lemma A.2, i.e. \( \mathcal{A}_\alpha \) contains all time points in \( [0, t] \) where at least one component of \( u \mapsto X(u) \) has jumps greater than \( \alpha \). The SU decomposition \( D_{i,\pi,a}^{SU} \) with respect to \( \gamma^{a_n} \) can be written as

\[
D_{i,\pi,a}^{SU} = \sum_{l \in \mathbb{N}_0} \{ f(X^{B,l} - f(X^{C,l}) \} = \sum_{l \in \mathcal{A}_\alpha} \{ f(X^{B,l} - f(X^{C,l}) \} \\
+ \sum_{l \in \mathcal{A}_\alpha^c} \{ f(X^{B,l} - f(X^{C,l}) \},
\] (11)

where \( \mathcal{A}_\alpha^c = \mathbb{N}_0 \setminus \mathcal{A}_\alpha \) and

\[
\mathcal{A}_\alpha = \{ l \in \mathbb{N}_0 : \mathcal{A}_\alpha \cap (s_{i-l}^{\alpha n}, s_{i-l+1}^{\alpha n}] \neq \emptyset \}. \]

We now analyze the first sum on the right-hand-side of Eq. (11). The first sum of Eq. (11) converges for \( n \to \infty \) to

\[
\sum_{s \in \mathcal{A}_\alpha} \{ f(X(s \ast B(i,:))) - f(X(s \ast C(i,:))) \}.
\] (12)
Later, we need the following observation: it holds that

\[ f(X(s \ast B(i,:))) = f(X(s-)) + \sum_{h=1}^{d} f_h(X(s-)) \Delta X_h(s) \]

\[ + \frac{1}{2} \sum_{h,j=1}^{d} f_{hj}(X(s-)) \Delta X_h(s) \Delta X_j(s) + R^B(s), \]

where \( R^B \) is the remainder of the Taylor expansion, see Lemma A.3. The term \( f(X(s \ast C(i,:))) \) has a similar representation replacing \( B \) by \( C \) and \( \pi(h) \leq \pi(i) \) by \( \pi(h) < \pi(i) \). Hence for some \( \Psi \) that only involves quadratic terms \( \Delta X_h \Delta X_j \) it holds that

\[ f(X(s \ast B(i,:))) - f(X(s \ast C(i,:))) = f_i(X(s-)) \Delta X_i(s) + \Psi(s). \quad (13) \]

We now analyze the second sum of the right-hand-side of Eq. (11). We develop \( f \) around \( X_0,l \) by Taylor expansion, see Lemma A.3 and use Eq. (10). It holds that

\[ f(X^{B,l}) - f(X^{C,l}) \]

\[ = \sum_{h=1}^{d} f_h(X^{0,l}) \delta X^B_h + \frac{1}{2} \sum_{h,j=1}^{d} f_{hj}(X^{0,l}) \delta X^B_h \delta X^B_j + R^{B,l} \]

\[ - \sum_{h=1}^{d} f_h(X^{0,l}) \delta X^C_h - \frac{1}{2} \sum_{h,j=1}^{d} f_{hj}(X^{0,l}) \delta X^C_h \delta X^C_j - R^{C,l} \]

\[ = f_i(X^{0,l}) \delta X^1_i + \frac{1}{2} f_{ii}(X^{0,l}) \left( \delta X^1_i \right)^2 \]

\[ + \sum_{j=1}^{d} f_{ij}(X^{0,l}) \delta X^1_i \delta X^1_j + R^{B,l} - R^{C,l}. \quad (14) \]

The last equation follows by looking at the four cases 1) \( h, j = i \), 2) \( h, j \neq i \), 3) \( h = i \) and \( j \neq i \) and 4) \( h \neq i \) and \( j = i \). The remainder is defined by

\[ R^{B,l} = \frac{1}{2} \sum_{h,j=1}^{d} \left( f_{hj}(\xi^{B,l}) - f_{hj}(X^{0,l}) \right) \delta X^B_h \delta X^B_j \]

and for some \( \theta \in [0,1] \)

\[ \xi^{B,l} = \theta X^{B,l} + (1 - \theta)X^{0,l}. \]

\( R^{C,l} \) is similarly defined. Using \( \sum_{\ell \in \mathbb{H}_a} = \sum_{\ell \in \mathbb{N}_0} - \sum_{\ell \in \mathbb{H}_a} \) and Eq. (14), the
second sum of the right-hand-side of Eq. (11) is can be written as
\[
\sum_{l \in A} \{ f(X^{B,l}) - f(X^{C,l}) \}
\]
\[
= \sum_{l \in N_0} f_i(X^{0,l}) \delta X_i^{1,l} + \frac{1}{2} \sum_{l \in N_0} f_{ii}(X^{0,l}) \left( \delta X_i^{1,l} \right)^2
\]
\[
+ \sum_{l \in N_0} \sum_{j=1}^d f_{ij}(X^{0,l}) \delta X_i^{1,l} \delta X_j^{1,l}
\]
\[
- \sum_{l \in A} \left\{ f_i(X^{0,l}) \delta X_i^{1,l} + \frac{1}{2} f_{ii}(X^{0,l}) \left( \delta X_i^{1,l} \right)^2
\right. \\
\left. + \sum_{\pi(j) < \pi(i)}^d f_{ij}(X^{0,l}) \delta X_i^{1,l} \delta X_j^{1,l} \right\}
\]
\[
+ \sum_{l \in A} R_{B,l} - \sum_{l \in A} R_{C,l}.
\] (15)

The first, second and third sum of the right-hand-side of Eq. (15) converge by the definition of the sequence \((a_n)_{n \in N}\) for \(n \to \infty\). The forth sum of Eq. (15) converges for \(n \to \infty\) to
\[
- \sum_{s \in A} \left\{ f_i(X(s^-)) \Delta X_i(s) + \frac{1}{2} f_{ii}(X(s^-)) (\Delta X_i(s))^2
\right. \\
\left. + \sum_{\pi(j) < \pi(i)}^d f_{ij}(X(s^-)) \Delta X_i(s) \Delta X_j(s) \right\}.
\] (16)

Adding the sums in Eq. (16) and Eq. (12) we obtain
\[
\sum_{s \in A} \left\{ f(X(s \ast B(i,:))) - f(X(s \ast C(i,:))) - f_i(X(s^-)) \Delta X_i(s) \right\}
\] (17)
\[
- \sum_{s \in A} \frac{1}{2} f_{ii}(X(s^-)) (\Delta X_i(s))^2
\] (18)
\[
- \sum_{s \in A} \sum_{\pi(j) < \pi(i)}^d f_{ij}(X(s^-)) \Delta X_i(s) \Delta X_j(s).
\] (19)

Note that \(f\) and all its derivatives take a maximum on the set \(\text{conv}(\mathcal{M}_a)\), which is compact. By Eq. (13), each of the three sums only depend on quadratic terms \(\Delta X_i \Delta X_j\) and by Lemma A.1, the sums (17), (18) and (19) are absolutely
convergent for $\alpha \to 0$ and converges for $\alpha \to 0$ to
\[
\sum_{0<s\leq t} \left\{ f (X(s \ast B(i,:))) - f (X(s \ast C(i,:))) - f_i (X(s^-)) \Delta X_i(s) \right\}
- \sum_{0<s\leq t} \frac{1}{2} f_{ii} (X(s^-)) (\Delta X_i(s))^2 \\
- \sum_{0<s\leq t} \sum_{j=1}^d \sum_{\pi(j) < \pi(i)} f_{ij} (X(s^-)) \Delta X_i(s) \Delta X_j(s).
\]

*At last, we treat the remainder.* Applying the same arguments as in the proof of the classical Ito formula, one can show that the remainder converges to zero: note that $\xi^{B,l}_{t} \in \text{conv}(\mathcal{M}_\omega)$ and the second derivatives of $f$ are uniformly continuous on $\text{conv}(\mathcal{M}_\omega)$. Hence, for any $\varepsilon > 0$ there is a $\eta > 0$ such that for all $h,j$ we have
\[
|f_{hj}(x) - f_{hj}(y)| < \varepsilon, \quad x,y \in \text{conv}(\mathcal{M}_\omega), \quad |x-y| < \eta,
\]
where $|.|$ denotes the Euclidean norm. That implies there is a non-decreasing function $\varphi : [0, \infty) \to \mathbb{R}$ such that $\varphi(c) \to 0$ for $c \to 0$ and
\[
R^{B,l} \leq \varphi \left( |\xi^{B,l} - X^{0,l}| \right) \sum_{h,j=1}^d \left\{ (\delta(X_h + X_j)^{B,l})^2 + (\delta X_h^{B,l})^2 + (\delta X_j^{B,l})^2 \right\},
\]
where we applied Eq. (30) as well. Let $\varepsilon > 0$. Let $\alpha > 0$ such that
\[
\sum_{s\in[0,t] \setminus A_\alpha} (\Delta X_h(s))^2 < \varepsilon^2
\]
for all $h = 1, \ldots, d$, which exists by Lemma A.2. Eq. (20) implies that the largest absolute jump for $s \in [0, t] \setminus A_\alpha$ is less than $\varepsilon$. Therefore
\[
\limsup_{n \to \infty} \sum_{h,j=1}^d R^{B,l}_{h,j} \leq \limsup_{n \to \infty} \varphi \left( |\xi^{B,l} - X^{0,l}| \right) \\
\left( \sum_{h,j=1}^d \limsup_{n \to \infty} \sum_{l \in \mathbb{N}_0} \left\{ (\delta(X_h + X_j)^{B,l})^2 + (\delta X_h^{B,l})^2 + (\delta X_j^{B,l})^2 \right\} \right) \\
\leq \limsup_{\gamma \searrow 0} \varphi \left( \sqrt{d} \gamma \right) \sum_{h,j=1}^d \left\{ |X_h + X_j| + |X_h - X_j| + |X_h - X_j| \right\}
\]
is arbitrarily small for $\varepsilon$ close enough to zero.

The next example shows that the assumption $f \in C_2$ in Theorem 4.2 is important for the existence of the ISU decomposition.
Example 4.3. Let $Z$ be a stochastic process with independent increments and $Z_0 = 0$. Jumps of $Z$ shall only occur at fixed times $J = \{2 - l^{-1}, l \in \mathbb{N}\}$, and for each $l \in \mathbb{N}$ the process jumps by $\pm l^{-1}$ with equal probability for upward and downward movements. The process $Z$ stays constant between jumps. Then $Z$ is a semimartingale, see Černý & Ruf (2021). Let

$$ f(x_1, x_2) = |x_1 - x_2|, $$

so $f \notin \mathcal{C}_2$. Let $(s^l_t)_{t \in \mathbb{N}}$ be a sequence of unbounded partitions of $[0, \infty)$ with vanishing step lengths such that the set $\{s^l_t, t \in \mathbb{N}\}$ contains the first $n$ smallest elements of $J$ but its intersection with $(2 - n^{-1}, 2]$ being empty. Assume $X = (Z, Z)$. Then it holds for $t \geq 2$ that

$$ \sum_{l=0}^{\infty} \left\{ f(X_{1}^{s^l_{n+1}}(t), X_{2}^{s^l_{n}}(t)) - f(X_{1}^{s^l_{n}}(t), X_{2}^{s^l_{n}}(t)) \right\} = \sum_{l=1}^{n} t^{-1}, \quad (21) $$

which is divergent for $n \to \infty$, so the ISU decomposition does not exist here.

4.1 One-at-a-time and average decompositions

The next corollary and remark state that the ISU decomposition is symmetric, i.e. independent of the way the risk factors are labeled, if the covariations between different risks factors are zero. This has been observed as well by Jetses & Christiansen (2022) in an insurance context.

Corollary 4.4. Let $f \in \mathcal{C}_2$. If $X_1, \ldots, X_d$ have no simultaneous jumps, the ISU decomposition of $(f(X(t)))_{t \geq 0}$ with respect to $\pi \in \sigma_d$ is given for $i = 1, \ldots, d$ and for $t \geq 0$ almost surely by

$$ D_i^{ISU, \pi}(t) = \int_{0}^{t} f_i(X(s-))dX_i(s) + \frac{1}{2} \int_{0}^{t} f_{ii}(X(s-))d[X_i, X_i]^c(s) $$

$$ + \sum_{j=1}^{d} \int_{0}^{t} f_{ij}(X(s-))d[X_i, X_j]^c(s) $$

$$ + \sum_{0 \leq s \leq t \atop \Delta X_i(s) \neq 0} \left\{ f(X(s)) - f(X(s-)) - f_i(X(s-))\Delta X_i(s) \right\}. \quad (22) $$

Proof. It holds that $B_{ij}^\pi = C_{ij}^\pi, i \neq j$, see Eq. (9). Let $0 < s \leq t$. In case of $\Delta X_i(s) = 0$ it holds that $X_i(s) = X_i(s-)$ and hence

$$ f(X(s \ast B^\pi(i, :))) - f(X(s \ast C^\pi(i, :))) $$

$$ = f(X(s \ast C^\pi(i, :))) - f(X(s \ast C^\pi(i, :))) = 0. $$

In case of $\Delta X_i(s) \neq 0$ it holds that $X_j(s) = X_j(s-)$ for all $j \neq i$ and hence

$$ f(X(s \ast B^\pi(i, :))) - f(X(s \ast C^\pi(i, :))) = f(X(s)) - f(X(s-)). \quad \square $$
Remark 4.5. If $[X_i, X_j] = 0$ for $i \neq j$, it follows $\Delta X_i \Delta X_j = \Delta [X_i, X_j] = 0$, i.e. $X_i$ and $X_j$ do not have simultaneous jumps. The third term on the right-hand-side of Eq. (22) disappears and the ISU decomposition is invariant with respect to $\pi \in \sigma_d$, i.e. the ISU decomposition does not depend on the order or labeling of the risk factors.

Another common decomposition principle is the OAT (one-at-a-time) decomposition, see Jetses & Christiansen (2022), Frei (2020), Biewen (2012) and Candland & Latz (2014). In contrast to the SU decomposition, the OAT decomposition is symmetric but it is not additive. In Corollary 4.7 we show that the OAT decomposition also exists in the limit. But any interaction effects between the risk factors are not assigned to any particular risk factor, see Corollary 4.10.

Definition 4.6. (OAT decomposition). Let $\gamma = \{0 = s_0 < s_1 < \ldots \}$ be an unbounded partition of $[0, \infty)$. The OAT (one-at-a-time) decomposition with respect to $\gamma$ defines a decomposition of $F(X)$ for $i = 1, \ldots, d$ and for $t \geq 0$ by

$$D_{i}^{OAT}(t) = \sum_{l=0}^{\infty} \int_{s_l}^{s_{l+1}} \{ F(X_{s_l}^{i}, \ldots, X_{i-1}^{s_l}, X_{i+1}^{s_l}, \ldots, X_{d}^{s_l}) - F(X_s)(t) \} ds.$$  

If the limit of the OAT decomposition exists in probability for an increasing sequence of unbounded partitions with vanishing step length, the limit is called IOAT (infinitesimal one-at-a-time) decomposition. The next corollary generalizes Frei (2020, Proposition 1).

Corollary 4.7. Let $f \in C_2$. Then the IOAT decomposition of $(f(X_t))_{t \geq 0}$ exists for each increasing sequence of unbounded partitions of $[0, \infty)$ with vanishing step lengths. For $i = 1, \ldots, d$ and for $t \geq 0$ it almost surely equals

$$D_{i}^{IOAT}(t) = \int_{0}^{t} f_i(X(s-)) dX_i(s) + \frac{1}{2} \int_{0}^{t} f_{ii}(X(s-)) d[X_i, X_i](s)$$

$$+ \sum_{0 < s \leq t} \{ f(X_{s-}, \ldots, X_{i-1}(s-), X_i(s), X_{i+1}(s-), \ldots, X_{d}(s-)) - f(X(s-)) - f_i(X(s-)) \Delta X_i(s) \}. \tag{23}$$

Proof. Let $i \in \{1, \ldots, d\}$ and take $\pi \in \sigma_d$ such that $\pi(i) = 1$. Then

$$B^\pi(i, :) = (0, \ldots, 0, 1, 0, \ldots, 0)$$

and $C^\pi(i, :) = (0, \ldots, 0)$. Then the SU decomposition $D_{i}^{SU, \pi}$ is identical to the OAT decomposition $D_{i}^{OAT}$ and Eq. (23) follows immediately from Theorem 4.2.

If some of the covariations between different risk factors are non-zero, we propose to work with the ASU and IASU decompositions, which are simply the arithmetic average of the $d!$ possible SU/ ISU decompositions and are symmetric and additive by construction, see Shorrocks (2013) and Jetses & Christiansen (2022).
Definition 4.8. Let $D^{SU, \pi}$ and $D^{ISU, \pi}$ be the SU, respectively the ISU decomposition, with respect to $\pi \in \sigma_d$. For $i = 1, \ldots, d$ and for $t \geq 0$ the ASU and IASU decomposition are defined by

$$D^{ASU}_i(t) = \frac{1}{d!} \sum_{\pi \in \sigma_d} D^{SU, \pi}_i(t) \quad \text{and} \quad D^{IASU}_i(t) = \frac{1}{d!} \sum_{\pi \in \sigma_d} D^{ISU, \pi}_i(t).$$

The next corollary allows to reduce the computational complexity calculating the ASU decomposition as an approximation of the IASU decomposition if there are no simultaneous jumps, compare with Corollary 4.12.

Corollary 4.9. Let $f \in C^2$. If $X_1, \ldots, X_d$ have no simultaneous jumps, the IASU decomposition of $(f(X(t))_{t \geq 0}$ is given for $i = 1, \ldots, d$ and for $t \geq 0$ almost surely by

$$D^{IASU}_i(t) = \int_0^t f_i(X(s))dX_i(s) + \frac{1}{2} \sum_{j=1}^d \int_0^t f_{ij}(X(s))d[X_i, X_j](s) + \sum_{0 \leq s \leq t, \Delta X_i(s) \neq 0} \{ f(X(s)) - f(X(s-)) - f_i(X(s-))\Delta X_i(s) \}. \quad (24)$$

Proof. If $d = 1$, Eq. (24) is trivially true. Assume $d \geq 2$. Fix $i \in \{1, \ldots, d\}$. Note that

$$\sum_{\pi \in \sigma_d} 1_{\{\pi(j) < \pi(i)\}} = \begin{cases} \frac{d!}{2}, & j \neq i \\ 0, & j = i. \end{cases}$$

Let $a_{ij}$ be real numbers. It follows that

$$\frac{1}{d!} \sum_{\pi \in \sigma_d} \sum_{j=1}^d a_{ij} = \sum_{j=1}^d \left\{ a_{ij} \frac{1}{d!} \sum_{\pi \in \sigma_d} 1_{\{\pi(j) < \pi(i)\}} \right\}$$

$$= \frac{1}{2} a_{i1} + \ldots + \frac{1}{2} a_{i(i-1)} + \frac{1}{2} a_{i(i+1)} + \ldots + \frac{1}{2} a_{id}$$

$$= \frac{1}{2} \sum_{j \neq i} a_{ij}. \quad (25)$$

Let $D^{ISU, \pi}$ be the ISU decomposition with respect to $\pi \in \sigma_d$. Then Eq. (24) follows since $D^{IASU}_i = \frac{1}{d!} \sum_{\pi \in \sigma_d} D^{ISU, \pi}_i$ and by Corollary 4.4 and Eq. (25). \qed

The three decomposition principles ISU, IOAT and IASU have a particularly appealing form if $X$ is continuous, as the next corollary shows.

Corollary 4.10. Let $X$ have almost surely continuous paths, and define

$$I_{ij} = \int_0^t f_{ij}(X(s))d[X_i, X_j](s), \quad i, j = 1, \ldots, d.$$
The ISU, IOAT and IASU decompositions of \((f(X_t))_{t \geq 0}\) are given for \(i = 1, \ldots, d\) and for \(t \geq 0\) by

\[
D_{ISU,\pi}^i(t) = \int_0^t f_i(X(s)) \, dX_i(s) + \frac{1}{2} I_{ii} + \sum_{j=1}^d I_{ij} \quad (26)
\]

\[
D_{IOAT}^i(t) = \int_0^t f_i(X(s)) \, dX_i(s) + \frac{1}{2} I_{ii} \quad (27)
\]

\[
D_{IASU}^i(t) = \int_0^t f_i(X(s)) \, dX_i(s) + \frac{1}{2} I_{ii} + \frac{1}{2} \sum_{j \neq i} I_{ij}. \quad (28)
\]

**Proof.** Eq. (26) follows from Theorem 4.2. Eq. (27) follows from Corollary 4.7. Eq. (28) follows from Corollary 4.9. \(\square\)

**Remark 4.11.** We see that the three decompositions differ only in the attribution of the interaction effects \(I_{ij}\). For the ISU decomposition, the interaction effects are assigned to the different risk factors depending on the order \(\pi\) of the risk basis. In the case of the IOAT decomposition, the interaction effects are not assigned to any particular risk factor but have to be reported separately. The IASU on the other hand, assigns half of the interaction effect to each risk factor \(X_i\) and \(X_j\), i.e. splits the interaction effects “fifty-fifty”. If the components of \(X\) have simultaneous jumps, we see from Theorem 4.2 and Corollary 4.7 that the ISU principle treats simultaneous jumps depending on \(\pi\), the IOAT principle ignores simultaneous jumps and the IASU principle takes the average over simultaneous jumps.

### 4.2 Numerical approximation of the IASU decomposition

How can the IASU decomposition be approximated numerically? One solution could be to directly approximate the integrals appearing in Theorem 4.2 in the \(d!\) ISU decompositions. This has the disadvantage that the approximation is not perfectly additive anymore. Justified by Theorem 4.2, we propose to approximate the IASU decomposition by the ASU decomposition.

To obtain the approximating ASU decomposition, we need to compute \(d!\) SU decompositions. That becomes unfeasible for large \(d\). In the next corollaries we provide sufficient conditions to reduce the numerical complexity. If \(X_1, \ldots, X_d\) have no simultaneous jumps, we only need two SU decompositions instead of \(d!\) to obtain a perfectly additive approximation of the IASU decomposition as the next corollary shows.

**Corollary 4.12.** Let \(f \in C_2\). Assume \(X_1, \ldots, X_d\) have no simultaneous jumps. For \(i = 1, \ldots, d\) and for \(t \geq 0\) the IASU decomposition of \((f(X(t)))_{t \geq 0}\) can then be obtained from the average of two ISU decompositions, i.e.

\[
D_{IASU}^i(t) = \frac{1}{2} (D_{ISU,\pi}^i(t) + D_{ISU,\pi'}^i(t)),
\]
where $\pi \in \sigma_d$ is the identity, i.e. $\pi(i) = i$ and $\pi' \in \sigma_d$ is the "reverse" identity, i.e. $\pi'(i) = d + 1 - i$ for $i = 1, \ldots, d$.

**Proof.** Corollary 4.4 and Corollary 4.9 imply that

\[
\frac{1}{2} (D_i^{ISU,\pi}(t) + D_i^{ISU,\pi'}(t))
= \int_0^t f_i(X(s)) dX_i(s) + \frac{1}{2} \int_0^t f_{ii}(X(s)) d[X_i, X_i](s)
+ \frac{1}{2} \sum_{j=1, j<i}^d \int_0^t f_{ij}(X(s)) d[X_i, X_j](s)
+ \frac{1}{2} \sum_{j=1, j>i}^d \int_0^t f_{ij}(X(s)) d[X_i, X_j](s)
+ \sum_{0 < s \leq t \Delta X_i(s) \neq 0} \left\{ f(X(s)) - f(X(s-)) - f_i(X(s-)) \Delta X_i(s) \right\}
= D_i^{IASU}(t).
\]

In the next corollary we show that the numerical complexity can be reduced significantly for general $X$ if $f$ has more structure, e.g. if $f$ can be described as a sum of smooth functions that do not each depend on all of the risks factors.

This might for instance be the case with a (large) portfolio, whereby each individual asset only depends on a few risk factors. For instance, a foreign investment in $d^2$ stocks has $d$ risk factors: the stocks and the foreign exchange rates. The next corollary shows that in this example, we need only $2d$ SU decompositions to obtain an approximation of the IASU decomposition.

**Corollary 4.13.** Let $d_1, \ldots, d_m \in \mathbb{N}$ such that $d_1 + \ldots + d_m = d$. Let $f \in \mathcal{C}_2$ and assume

\[ f = g^1 + \ldots + g^m, \]

where $g^h \in \mathcal{C}_2$ depends only on $d_h$ risk factors. The IASU decomposition of $(f(X(t)))_{t \geq 0}$ can be obtained from $\sum_{h=1}^m d_h!$ ISU decompositions.

**Proof.** By Theorem 4.2, the IASU decomposition is linear in the argument $f$. The assertion follows immediately, because the IASU decomposition of $g^h$ can be obtained by $d_h!$ ISU decompositions by Theorem 4.2.

**Example 4.14.** Annually, the EIOPA (European Insurance and Occupational Pensions Authority) performs a market and credit risk comparison study. All insurance companies participating have to model the risk of different synthetic instruments, which are used to build a set of realistic investment portfolios.
The benchmark portfolios consist of in total 71 instruments, which cover all relevant asset classes, i.e. risk-free interest rates, sovereign bonds, corporate bonds, equity indices and property, see Table A2 in Flaig & Junike (2022) for an overview. About 47 risk factors are necessary for evaluating all instruments. Each individual instrument has up to three risk-factors (exposure to foreign currencies is supposed to be hedged). For example, a France sovereign bond with five years left to maturity is an instrument that has three risk factors: EUR interest rate (5y), France credit spread (5y) and time decay. A linear combination of all instruments builds a portfolio with price \( f = g_1 + \ldots + g_{71} \), where \( g_k, k = 1, \ldots, 71 \), is a (weighted) instrument and depends on three or less risk factors. Corollary 4.13 implies that no more than \( 71 \cdot 3! = 426 \) SU decompositions need to be computed to obtain the ASU decomposition of that portfolio. A naive approach to obtain the ASU decomposition would require \( 47! \approx 2.59 \cdot 10^{59} \) SU decompositions.

5 Applications

In this section, we provide some applications. In Section 5.1 we consider an European investor who invests one EUR in the US stock market. Her return is driven by the S&P 500 and the USD-EUR exchange rate in a product structure. What is the contribution of the stock movements to the overall P&L, what is the contribution of the evolution of the currency rate to the overall P&L? We also decompose the one-year value at risk of an investment in Section 5.1.

We also look at financial instruments with a finite maturity, i.e. we provide a formula for the P&L attribution of a call option in Section 5.2 on some stock and decompose the P&L of a foreign corporate zero-coupon bond into the four contributions of foreign exchange rate, interest rate, credit spread and calendar time.

5.1 P&L and VaR decomposition of a foreign stock in domestic currency

We would like to decompose the price \( P = X_1X_2 \) of a foreign stock, where \( X_1 \) is the foreign exchange rate and \( X_2 \) the stock price in the foreign currency. The instantaneous P&L of the foreign stock in home currency is given by

\[
dP(t) = X_1(t-)dX_2(t) + X_2(t-)dX_1(t) + d[X_1, X_2](t),
\]

i.e. it can be decomposed into variation of the foreign exchange rate, variation of the stock price and interaction effects. The latter is distributed equally between \( D_1^{IASU} \) and \( D_2^{IASU} \) as the next corollary shows.

Corollary 5.1. Let \( d = 2 \) and \( F(X) = X_1X_2 \). Then for \( t \geq 0 \) the IASU
decomposition is equal to
\[ D_{IASU}^{(1)}(t) = \int_0^t X_2(s-)dX_1(s) + \frac{1}{2}[X_1, X_2](t), \]
\[ D_{IASU}^{(2)}(t) = \int_0^t X_1(s-)dX_2(s) + \frac{1}{2}[X_1, X_2](t). \]

**Proof.** By Theorem 4.2 we obtain
\[ D_{IASU}^{(1)}(t) = \int_0^t X_2(s-)dX_1(s) + \frac{1}{2}[X_1, X_2](t) \]
\[ + \frac{1}{2} \sum_{0<s\leq t} \{X_1(s)X_2(s) - X_1(s-)X_2(s-) \]
\[ + X_1(s)X_2(s) - X_1(s-)X_2(s) - 2X_2(s-) (X_1(s) - X_1(s-)) \}
\[ = \int_0^t X_2(s-)dX_1(s) + \frac{1}{2}[X_1, X_2](t). \]

By Ito’s lemma and the additivity of the IASU decomposition we also have
\[ D_{IASU}^{(2)}(t) = \int_0^t X_1(s-)dX_2(s) + \frac{1}{2}[X_1, X_2](t). \]

Suppose that \( X_1 \) describes an investment of one EUR in the S&P 500 index and that \( X_2 \) the USDEUR exchange rate. In Figure 1 we can see that the exchange rate has risen by 18%, while the S&P 500 has fallen by 17%. This has the effect that both risk factors cancel each other out at \( t = 1 \), such that the P&L in EUR barely changes over the time horizon of one year. Accordingly, the contributions of the S&P 500 and the exchange rate are of similar size.
Next we look at the value at risk over some time interval $T > 0$ of the product of two geometric Brownian motions describing for instance the value of a foreign stock in domestic currency. We first define the value at risk formally. The following definition is in the spirit of Example 11.7. in Föllmer & Schied (2016). $L_0$ denotes the space of finite valued random variables. By $\mathcal{F}_\infty$ we denote the smallest $\sigma$-Algebra which contains all $\mathcal{F}_t$, $t \geq 0$.

**Definition 5.2.** Let $T > 0$. Let $G \subset \mathcal{F}_\infty$ be a $\sigma$-Algebra. Let $X$ be a $\mathcal{F}_\infty$ measurable random variable. The conditional Value at Risk of $X$ given $G$ at level $\lambda \in (0, 1]$ is defined by

$$\text{CVaR}_\lambda(X|G) = \text{ess inf}\{M \in L_0(G) : P(X \leq M|G) \geq \lambda\}.$$  

The essential infimum is defined in Föllmer & Schied (2016, Thm A.37) and recalled in Section A.2. Given $G$, the probability that $X$ is greater than $\text{CVaR}_\lambda(X|G)$ is less or equal than $1 - \lambda$ a.s.

Let $\Xi_{a,b}$ denote the cumulative distribution function of a lognormal distribution with parameters $a \in \mathbb{R}$ and $b > 0$, i.e. the distribution function of the random variable $e^{a + bZ}$, where $Z$ is standard normal.

**Proposition 5.3.** Let $X_1, X_2$ be correlated geometric Brownian motions, i.e.
for two independent Brownian motions $W$ and $B$ and $\rho \in [0,1]$. Let

$$
\mu := \left( a_1 + a_2 - \frac{1}{2} \left( b_1^2 + b_2^2 \right) \right) \quad \text{and} \quad \sigma^2 := \left( b_1^2 + b_2^2 + 2\rho b_1 b_2 \right).
$$

For a fixed time horizon $T$ and $0 \leq t < T$, it holds that

$$
\text{CVaR}_\lambda(X_1(T + t)X_2(T + t)|F_t) = X_1(t)X_2(t) \Xi^{-1}_{\mu_T,\sigma_T}(\lambda).
$$

Proof. Let $0 \leq t < T$. It holds that

$$
R(T, t) := \frac{X_1(T + t)X_2(T + t)}{X_1(t)X_2(t)} \exp \left( \mu_T + \sigma_T Z \right)
$$

where $Z$ is a standard normal random variable and $R(T, t)$ is independent of $F_t$. For the conditional value at risk we obtain

$$
\text{CVaR}_\lambda(F(X)(T + t)|F_t)
= \text{ess inf}\left\{ M \in L_0(F_t) : P(X_1(T + t)X_2(T + t) \leq M|F_t) \geq \lambda \right\}
= X_1(t)X_2(t) \text{ess inf}\left\{ M \in L_0(F_t) : P\left( \frac{X_1(T + t)X_2(T + t)}{X_1(t)X_2(t)} \leq M|F_t \right) \geq \lambda \right\}
= X_1(t)X_2(t) \text{ess inf}\left\{ M \in L_0(F_t) : \Xi^{-1}_{\mu_T,\sigma_T}(M) \geq \lambda \right\}
= X_1(t)X_2(t) \Xi^{-1}_{\mu_T,\sigma_T}(\lambda).
$$

Hence, the conditional value at risk is equal to the product of $X_1$ and $X_2$ weighted by $w := \Xi^{-1}_{\mu_T,\sigma_T}(\lambda)$. For the IASU decomposition of the conditional value at risk

$$
t \mapsto \text{CVaR}_\lambda(X_1(T + t)X_2(T + t)|F_t),
$$

we obtain

$$
D_1^{\text{IASU}}(t) = w \left( \int_0^t X_2(u)dX_1(u) + \frac{1}{2}[X_1, X_2](t) \right),
$$

$$
D_2^{\text{IASU}}(t) = w \left( \int_0^t X_1(u)dX_2(u) + \frac{1}{2}[X_1, X_2](t) \right).
$$

The processes $D_1^{\text{IASU}}(t)$ and $D_2^{\text{IASU}}(t)$ capture the changes of the conditional value at risk of $X_1(T + t)X_2(T + t)$ due to movements in $X_1$ and $X_2$, respectively. These contributions are also obtained by Frei (2020, Proposition 1) if there are no interaction effects, i.e. $[X_1, X_2] = 0$.  

20
5.2 P&L decomposition of a call option

Assume a Black-Scholes market with a stock and a call option on the stock.

\[ S(t) = S_0 e^{rt - \frac{1}{2} \sigma^2 t + \sigma W_t}, \quad t \geq 0, \]

is a geometric Brownian motion describing the stock price for some \( \sigma > 0 \). The price of a call option with strike \( K > 0 \), maturity \( T > 0 \) and interest rate \( r \in \mathbb{R} \) can be described by

\[ f(S(t), t), \quad t \geq 0 \]

where \( t \) is the calendar time,

\[ f(x, t) = \Phi(d_1(x, t))x - \Phi(d_2(x, t))K e^{-r(T-t)} \in \mathbb{C}^2, \]

\( \Phi \) is the cumulative distribution function of the standard normal distribution, and

\[ d_1(x, t) = \frac{1}{\sigma \sqrt{T-t}} \left[ \log \left( \frac{x}{K} \right) + (r + \frac{\sigma^2}{2})(T-t) \right], \]

\[ d_2(x, t) = d_1(x, t) - \sigma \sqrt{T-t}. \]

The idea is now to decompose the P&L of a call option in the Black-Scholes model with respect to calendar time and stock price movement. For \( t \geq 0 \) the IASU decomposition yields

\[ D_{1}^{\text{IASU}}(t) = \int_0^t \Delta(S(u), u) dS(u) + \frac{1}{2} \int_0^t \Gamma(S(u), u) d[S, S](u), \]

\[ D_{2}^{\text{IASU}}(t) = \int_0^t \Theta(S(u), u) du, \]

where \( \Delta, \Gamma \) and \( \Theta \) are the Greeks Delta, Gamma and Theta of the call option. As in Eq. (2) in Carr & Wu (2020), we can attribute the instantaneous P&L of the option investment to the variation of calendar time and stock price by

\[ df(S(t), t) = \Delta dS(t) + \frac{1}{2} \Gamma d[S, S](t) + \Theta dt, \]

i.e. \( D_{1}^{\text{IASU}} \) captures price movements and second order effects, and \( D_{2}^{\text{IASU}} \) captures calendar time.

5.3 P&L decomposition of a foreign zero coupon bond

Assume the price \( P \) of a foreign, zero coupon bond with maturity \( T > 0 \) can be described by \( P(t) = f(Z(t), R(t), C(t), t) \), where

\[ f(x_1, x_2, x_3, t) = x_1 e^{-(x_2 + x_3)(T-t)}, \quad x_1, x_2, x_3 \in \mathbb{R}, \quad t \in [0, T], \]
and $R$ is the (continuous compounded) interest rate (IR), $C$ is the credit spread (CS), and $Z$ is some foreign exchange rate (FX), possibly correlated with $R$ and $C$. The IASU decomposition with respect to the risk factors $Z$, $R$, $C$ and calendar time $\tau$ yields

$$D_{IASU}^{FX} = -\int_{0}^{\tau} e^{-(R(u)+C(u))(T-u)}dZ(u) + \frac{1}{2}(I_{ZR} + I_{ZC}),$$

$$D_{IASU}^{R} = -\int_{0}^{\tau} (T-u)P(u)dR(u) + \frac{1}{2}\int_{0}^{\tau} (T-u)^2 P(u)d[R,R](u)$$

$$+ \frac{1}{2}(I_{ZR} + I_{CR}),$$

$$D_{IASU}^{C} = -\int_{0}^{\tau} (T-u)P(u)dC(u) + \frac{1}{2}\int_{0}^{\tau} (T-u)^2 P(u)d[C,C](u)$$

$$+ \frac{1}{2}(I_{CR} + I_{ZC}),$$

$$D_{IASU}^{\tau} = \int_{0}^{\tau} (R(u) + C(u))P(u)du,$$

where

$$I_{ZR} = -\int_{0}^{\tau} (T-u)e^{-(R(u)+C(u))(T-u)}d[R,Z](u),$$

$$I_{ZC} = -\int_{0}^{\tau} (T-u)e^{-(R(u)+C(u))(T-u)}d[C,Z](u),$$

$$I_{CR} = \int_{0}^{\tau} (T-u)^2 P(t)d[R,C](u)$$

are the interaction effects between interest rate and foreign exchange rate. I.e. $D_{IASU}^{Z}$ captures foreign exchange rate movements, $D_{IASU}^{R}$ captures interest rate movements, $D_{IASU}^{C}$ represents credit spread evolution and $D_{IASU}^{\tau}$ captures calendar time. $D_{IASU}^{Z}$, $D_{IASU}^{R}$, $D_{IASU}^{C}$ also include second order and interaction effects.

Let us model the interest rate $R$ by a Vasicek model, which can be described by the following stochastic differential equation, see Brigo & Mercurio (2001, Sec. 3.2),

$$dR(t) = \kappa(\theta - R(t))dt + \sigma W(t),$$

where $\kappa = 0.5$, $\theta = 0.02$, $\sigma = 0.01$, $R(0) = 0.01$ and $W$ is a Brownian motion. The foreign exchange rate $Z$ is modeled by a geometric Brownian motion

$$Z(t) = e^{-\frac{\nu}{2}t + \nu B(t)}, \quad t \geq 0,$$

where $\nu = 0.01$ and $B$ is a Brownian motion correlated with $W$ by $\rho = -0.5$. The process $C$ is modeled by a pure jump process, which starts at zero and jumps from 0 to 0.01 at time $t = 0.5$, independent of $R$ and $Z$. For the time horizon we choose $T = 1$. 

22
In Figure 2 we can see contributions of the risk factors to the P&L of the bond over time. Figure 3 provides a waterfall chart for the two time periods $[0, 0.4]$ and $[0, 1]$ for the value of the bond.

![Figure 2: Decomposition of a foreign zero coupon bond with respect to interest rate IR, credit spread CS, foreign exchange rate FX and calendar time $\tau$. For a better visualization the exchange rate is shifted to zero at $t = 0$ and we omitted the risk factor calendar time (which is the identity).]
Figure 3: Waterfall chart of the value changes in the domestic currency of a foreign zero coupon bond \( P \) with respect to interest rate \( IR \), credit spread \( CS \), foreign exchange rate \( FX \) and calendar time \( \tau \) for the periods \([0, 0.4]\) (left) and \([0, 1]\) (right). A black (gray) area corresponds to a positive (negative) contributions to the P&L.

6 Conclusion

It is common practice to use sequential updating (SU) decompositions for obtaining profit and loss attributions of portfolios. In this article, we provided a definition of the SU decomposition for multiple time periods and arbitrary orders of the risk factors. We analyzed the case that the P&L of the portfolio can be described by a twice differentiable function of the risk factors. We allow non-zero interaction effects and describe the risk factors by general semimartingales. We proved the existence of the ISU decomposition. To obtain a symmetric decomposition, i.e. a decomposition that does not depend on the order or labeling of the risk factors, we studied the IASU decomposition, giving closed form that has complexity \( d! \) and providing conditions under which the complexity can be reduced.

We also discussed several examples and showed that in many applications the P&L can indeed be described by a twice differentiable function of the risk factors.
A Appendix

A.1 Auxiliary results

Lemma A.1. Let $X$ be a $d$-dimensional semimartingale. Then for all $t \geq 0$

\[
\sum_{0 < s \leq t} |\Delta X_h(s)\Delta X_j(s)| < \infty \quad \text{a.s.,} \quad h, j = 1, \ldots, d.
\]  

(29)

Proof. For any numbers $a, b \in \mathbb{R}$ it holds that $2ab = (a + b)^2 - a^2 - b^2$, hence

\[
2|ab| \leq (a + b)^2 + a^2 + b^2.
\]  

(30)

That implies,

\[
2 \sum_{0 < s \leq t} |\Delta X_h(s)\Delta X_j(s)| \leq \sum_{0 < s \leq t} (\Delta X_h(s) + \Delta X_j(s))^2 + \sum_{0 < s \leq t} (\Delta X_h(s))^2 + \sum_{0 < s \leq t} (\Delta X_j(s))^2
\]

\[
\leq [X_j + X_h, X_j + X_h](t) + [X_j, X_j](t) + [X_h, X_h](t)
\]

\[
< \infty.
\]

Lemma A.2. Let $\omega \in \Omega$ such that the path $u \mapsto X(\omega)(u)$ is càdlàg and Eq. (29) holds. Let $t > 0$ and

\[
\mathcal{A}_\alpha := \mathcal{A}_\alpha(\omega) := \left\{ s \in (0, t], \max_{j=1,\ldots,d} |\Delta X_j(\omega)(s)| > \alpha \right\}, \quad \alpha > 0.
\]

$\mathcal{A}_\alpha$ contains all time points in $[0, t]$, where at least one component of $u \mapsto X(\omega)(u)$ has jumps greater than $\alpha$. Then $\mathcal{A}_\alpha$ contains only finitely many points. Let

\[
a_k := \max_{j, h=1,\ldots,d} \sum_{s \in \mathcal{A}_{1/k}} |\Delta X_j(\omega)(s)\Delta X_h(\omega)(s)|, \quad k \in \mathbb{N}.
\]

Then the sequence $(a_k)_{k \in \mathbb{N}}$ converges. Further, for any $\varepsilon > 0$ there is a $\alpha > 0$ such that

\[
\sum_{s \in (0, t] \setminus \mathcal{A}_\alpha} |\Delta X_j(\omega)(s)\Delta X_h(\omega)(s)| < \varepsilon.
\]

Proof. By Lemma A.1, the number of jumps of $u \mapsto X(\omega)(u)$ which are greater or equal $\alpha > 0$ must be finite. Again by Lemma A.1 it holds that

\[
a_k \leq \max_{j, h=1,\ldots,d} \sum_{0 < s \leq t} |\Delta X_j(\omega)(s)\Delta X_h(\omega)(s)| < \infty \quad \text{a.s.,} \quad k \in \mathbb{N}.
\]
The sequence \((a_k)_{k \in \mathbb{N}}\) is monotone increasing and bounded and hence convergent. Therefore, there is \(\alpha > 0\) such that for \(\varepsilon > 0\) it holds that
\[
\varepsilon > \left| \sum_{0 < s \leq t} |\Delta X_j(\omega)(s)\Delta X_h(\omega)(s)| - \sum_{s \in A_\alpha} |\Delta X_j(\omega)(s)\Delta X_h(\omega)(s)| \right|
= \sum_{s \in [0,t] \setminus A_\alpha} |\Delta X_j(\omega)(s)\Delta X_h(\omega)(s)|.
\]

**Lemma A.3.** (Taylor’s theorem). Let \(U \subset \mathbb{R}^d\) open. Let \(x, a \in U\) such that \(\lambda x + (1 - \lambda)a \in U\) for all \(\lambda \in [0,1]\). Let \(f : U \to \mathbb{R}\) be twice continuously differentiable. Then it holds that
\[
f(x) = f(a) + \sum_{h=1}^{d} f_h(a) (x_h - a_h) + \frac{1}{2} \sum_{h,j=1}^{d} f_{hj}(a) (x_h - a_h)(x_j - a_j) + R,
\]
where the remainder \(R\) can be expressed for some \(\theta \in [0,1]\) and \(\xi = \theta x + (1 - \theta)a\) by
\[
R = \frac{1}{2} \sum_{h,j=1}^{d} (f_{hj}(\xi) - f_{hj}(a)) (x_h - a_h)(x_j - a_j).
\]

*Proof.* Forster (2017, Satz 2, Sec. 7).

**A.2 Essential infimum**

The following theorem and definition are taken from Föllmer & Schied (2016).

**Theorem A.4.** Let \(\Phi\) be a set of random variables. There exists a random variable \(\varphi^*\) such that

i. \(\varphi^* \geq \varphi\) a.s. for all \(\varphi \in \Phi\)

ii. \(\varphi^* \leq \psi\) for all \(\psi\) such that \(\psi \geq \varphi\) a.s. for all \(\varphi \in \Phi\).

*Proof.* Föllmer & Schied (2016, Thm. A.37).

**Definition A.5.** \(\varphi^*\) in the Theorem is called the *essential supremum* of \(\Phi\). In Notation
\[
\text{ess sup} \Phi = \text{ess sup}_{\varphi \in \Phi}(\varphi) := \varphi^*.
\]

The *essential infimum* is defined by
\[
\text{ess inf} \Phi := -\text{ess sup}_{\varphi \in \Phi}(-\varphi).
\]
A.3 Stability

In this section, we use the notation as in Christiansen (2022). For \( i = 1, 2 \) let \( \tau_i : [0, \infty) \to [0, \infty) \) with \( \tau_i(t) \leq t \) for all \( t \geq 0 \). The function

\[
\tau(t) = (\tau_1(t), \tau_2(t))
\]

is called a delay. A delay is called phased if there exists an unbounded partition \( \{0 = s_0 < s_1 < ...\} \) of \([0, \infty)\) such that on each interval \((s_i, s_{i+1}]\) at most one component of \( \tau \) is non-constant. Let \( (\tau^n)_{n \in \mathbb{N}} \) be a refining sequence of delays that increase to identity (rsdii), i.e.

\[
\tau^n_i([0, t]) \subset \tau^{n+1}_i([0, t]), \quad n \in \mathbb{N}, \quad \text{and} \quad \bigcup_{n \in \mathbb{N}} \tau^n_i([0, t]) = [0, t], \quad i = 1, 2.
\]

Let \( \mathcal{T} \) be a set containing at least one phased rsdii. Let \( X = (X_1, X_2) \) be a semimartingale and define

\[
X \circ \tau := (X_1 \circ \tau_1, X_2 \circ \tau_2), \quad \tau \in \mathcal{T}.
\]

Let

\[
X = \{X \circ \tau : \tau \in \mathcal{T}\} \cup \{X\}.
\]

Let \( \varrho : X \to \mathbb{D}_0 \). A map \( \delta : X \to \mathbb{D}_0^2 \) is called decomposition scheme of \( \varrho \). \( \delta \) assigns each \( Y \in X \) a decomposition of \( \varrho(Y) \). The ISU decomposition scheme is abbreviated by \( \delta^{ISU} \). A decomposition scheme is called stable at \( X \) if

\[
\delta(X \circ \tau^n)(t-) \overset{P}{\to} \delta(X)(t-)
\]

for all rsdii \( (\tau^n)_{n \in \mathbb{N}} \subset \mathcal{T} \).

Proposition A.6. Assume \( X = (X_1, X_2) \) with \( X_1 = X_2 = B \) for a Brownian motion \( B \). Let \( \varrho(Y) = Y_1Y_2 \) be a simple product. Then there exists a set \( \mathcal{T} \) of continuous phased rsdii such that the ISU decomposition \( \delta^{ISU} \) of \( \varrho \) is not stable at \( X \).

Proof. Suppose that \( \mathcal{T} \) contains a continuous phased rsdii \( (\tau^n) = (\tau^n_1, \tau^n_2), n \in \mathbb{N} \) with \( \tau^n_1 \leq \tau^n_2, n \in \mathbb{N} \). For a partition \( (a^n_{l,i}, b^n_{l,i}], l \in \mathbb{N}_0, i = 1, 2 \) of \([0, \infty)\) such that \( (\tau^n_j)_{j \neq i} \) is constant on \( (a^n_{i,l}, b^n_{i,l}] \) let \( \tau^n_1(a^n_{i,2}) = \tau^n_2(a^n_{i,2}), n \in \mathbb{N}, l \in \mathbb{N}_0 \). In addition, let \( \mathcal{T} \) contain also \( (\tau^n_{n \in \mathbb{N}} = ((\tau^n_2, \tau^n_1))_{n \in \mathbb{N}} \).

Since \( \tau^n_2(a^n_{i,1}) = \tau^n_2(b^n_{i,1}) = \tau^n_1(b^n_{i,1}) \) and by the multidimensional Taylor theorem we have

\[
\delta_1^{ISU}(X \circ \tau^n)(t) = \sum_i (\varrho((X \circ \tau^n)^{b^n_{i,1}, \wedge t}) - \varrho((X \circ \tau^n)^{a^n_{i,1}, \wedge t}))
\]

\[
= \sum_i \varrho_1((X \circ \tau^n)^{a^n_{i,1}, \wedge t})(X_1(\tau^n_1(b^n_{i,1} \wedge t)) - X_1(\tau^n_1(a^n_{i,1} \wedge t))).
\]
By the definition of $X_1$, $X_2$ and $\rho$ we obtain
\[
\delta_1^{ISU}(X \circ \tau^n)(t) = \sum_i B(\tau^n_i(a^n t_i \wedge t))(B(\tau^n_i(b^n t_i \wedge t)) - B(\tau^n_i(a^n t_i \wedge t)))
\]
\[
= \sum_i B(\tau^n_i(b^n t_i \wedge t))(B(\tau^n_i(b^n t_i \wedge t)) - B(\tau^n_i(a^n t_i \wedge t)))
\]
\[
= \sum_i B(\tau^n_i(b^n t_i \wedge t)) \left( B(\tau^n_i(b^n t_i \wedge t)) - B(\tau^n_i(a^n t_i \wedge t)) \right)
\]
\[
= 2 \sum_i \frac{(B(t_i) + B(t_i-1))}{2} \left( B(\tau^n_i(b^n t_i \wedge t)) - B(\tau^n_i(a^n t_i \wedge t)) \right)
\]
\[
- \sum_i B(t_{i-1}) \left( B(\tau^n_i(b^n t_i \wedge t)) - B(\tau^n_i(a^n t_i \wedge t)) \right)
\]
for $t^n_i := \tau^n_i(b^n t_i) = \tau^n_i(a^n_{i+1,1}) = \tau^n_2(b^n_{i-1,2}) = \tau^n_2(a^n_{i,2})$. Let $\int_0^t B_s dB_s$ denote the Stratonovich integral and $\int_0^t B_s dB_s$ the Ito integral. Let \( \overset{p}{\rightarrow} \) and \( \overset{plim}{\rightarrow} \) denote the convergence in probability, then
\[
\delta_1^{ISU}(X \circ \tau^n)(t) \overset{p}{\rightarrow} 2 \int_0^t B_s dB_s - \int_0^t B_s dB_s
\]
\[
= \frac{1}{2} B_t^2 + \frac{1}{2} t, \quad n \rightarrow \infty.
\]
With the same arguments we have
\[
\delta_1^{ISU}(X \circ \tilde{\tau}^n)(t) = \sum_i \left( \varphi((X \circ \tilde{\tau}^n)b^n_{i,2} \wedge t) - \varphi((X \circ \tilde{\tau}^n)a^n_{i,2} \wedge t) \right)
\]
\[
= \sum_i \varphi((X \circ \tilde{\tau}^n)a^n_{i,2} \wedge t) \left( X_2(\tau^n_2(b^n_{i,2} \wedge t)) - X_2(\tau^n_2(a^n_{i,2} \wedge t)) \right)
\]
\[
= \sum_i B(\tau^n_2(a^n_{i,2} \wedge t)) \left( B(\tau^n_2(b^n_{i,2} \wedge t)) - B(\tau^n_2(a^n_{i,2} \wedge t)) \right)
\]
\[
= \sum_i B(\tau^n_2(a^n_{i,2} \wedge t)) \left( B(\tau^n_2(b^n_{i,2} \wedge t)) - B(\tau^n_2(a^n_{i,2} \wedge t)) \right)
\]
\[
= \sum_i B(t_i) \left( B(t_{i+1} \wedge t) - B(t_i \wedge t) \right)
\]
\[
\overset{p}{\rightarrow} \int_0^t B_s dB_s
\]
\[
= \frac{1}{2} B_t^2 - \frac{1}{2} t.
\]
for $n \rightarrow \infty$. Therefore we have
\[
\overset{plim}{\rightarrow} n \rightarrow \infty \delta_1^{ISU}(X \circ \tau^n)(t) \neq \overset{plim}{\rightarrow} n \rightarrow \infty \delta_1^{ISU}(X \circ \tilde{\tau}^n)(t), \quad i = 1, 2,
\]
for $t > 0$, and hence the ISU decomposition of $\varphi(X)$ cannot be stable at $X$. \(\square\)
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