The Hooke-Newton transmutation system of magnetic force

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Abstract

Recently, it was reported that the Hooke-Newton transmutation of magnetic force can be generated by a conformal mapping in a uniform magnetic field. We perform the classical analysis of the transmutation system in this paper. First, the action variables are calculated and the energy is expressed in terms of the variables. The quantum spectrum of the system is manifested by the condition of the angular variables for the closed trajectories. Second, the trajectory equations are presented for the charge in the transmuted Coulomb field whose characteristics, attractive or repulsive, are determined by the signatures of both the angular momentum and the charge. It is shown that, given the released energy $E$ and angular momentum $p_j$ of the charged particle, the types of trajectories can be classified by the critical energy associated with the magnetic field strength $E_C = |\omega/\hbar^2|/4$. The feature makes the system only trap the particles with energy $E < E_C$ and the definite signatures of angular momentum and charge.

1. Introduction

It was pointed out by Issac Newton that there exists duality between Hooke’s linear force law and the inverse square force law (Cor. III, Prop. VII, [1]). Newton revealed this duality by showing a transmutation between these two kinds of forces when given a planet orbit, and, amazingly, proved it only with the skills of elementary geometry. In modern terminology, the transmutation is established by a quadric conformal mapping (see, e.g., appendix I in [2, 3], and section 6 in [4]). Recently, it was shown that there exists a magnetic force version of the Hooke-Newton (H-N) transmutation for a charged quantum particle in a uniform magnetic field [5]. This is more or less unexpected since the magnetic force is quite different from the central force field, e.g., doing no work, and unlike the fields of the Newtonian gravitational attraction and of electrostatic interaction generated by a massive object and a charge, which are the sources of the inverse square force fields, there is no magnetic monopole discovered so far. Although the magnetic version of the H-N transmutation is due to a conformal image, it is possible to realize the system by reinterpreting the conformal structure as a physical effect by transformation design means [6–8]. In reference [5], the quantum mechanics of the magnetic transmutation system have been discussed. The purpose of this article is to carry out the classical analysis for the system.

The paper is arranged as follows: in section 2, the physical realization of the mechanical conformal image is formulated. In section 3, the H-N transmutation of a charge in the uniform magnetic field is investigated by way of the action and angular variables. The energy of the closed trajectory is expressed in terms of the action variables. The quantum spectrum of the system is then identified after the Bohr’s quantization rules have been adopted for the action variables. It follows from the action representation of energy that the angular variables are calculated, from which the relation between the allowed quantum bound states and the conditions of the angular variables for the closed trajectories are established. Section 4 is used to obtain the trajectory equations of a charge in the transmutation system. It is shown that the types of trajectories can be controlled by the strength of the magnetic field. Finally, the conclusion is provided in section 5.
2. Form-invariant Hamilton-Jacobi equation and mechanical conformal mapping

The purpose of this section is to show that a form-invariant Hamilton-Jacobi (H-J) equation can be generated by a kind of conformal mapping, and one can endow the conformal factor in the equation with the meaning of a potential field such that it is possible to carry out the transformed system by physical means. The evolution of a charge moving in a 2D space is governed by the solution of the H-J equation

\[ \frac{1}{2M} \sum_i g_{ij} \frac{\partial S}{\partial q^i} \frac{\partial S}{\partial q^j} = E, \]  

where \( S = S(q) \) is the reduced action (p. 149, [9]). It may be regarded as a function of the coordinates. The partial derivatives of the action with respect to the coordinates are equal to the corresponding generalized momenta:

\[ \frac{\partial S}{\partial q^i} = p_i. \]

Consider the form-invariant H-J equation due to a transformation

\[ \frac{1}{2M} \sum_i \tilde{g}^{ij} \frac{\partial S}{\partial \tilde{q}^i} \frac{\partial S}{\partial \tilde{q}^j} = E, \]

where the new coordinate variables \( \tilde{q}^i = q^i(q^j) \) are only the function of \( q^j \), and the metric coefficients \( \tilde{g}^{ij} \) are defined by \( \tilde{g}^{ij} = g^{ij}(\tilde{q}) \) obtained by replacing the variable \( q^j \) with \( \tilde{q}^i \) for the inverse of the covariant metric components \( g_{ij}(q) \) calculated by the standard definition

\[ g_{ij}(q) = \frac{\partial x^i}{\partial q^j} \frac{\partial x^j}{\partial q^i} = \frac{\partial x^i}{\partial q^j} \frac{\partial x^j}{\partial q^i} + \frac{\partial y^i}{\partial q^j} \frac{\partial y^j}{\partial q^i}. \]

Equation (3) can be expressed as

\[ \frac{1}{2M} \sum_i \tilde{g}^{ij} \frac{\partial q^j}{\partial q^i} \frac{\partial S}{\partial q^i} \frac{\partial S}{\partial q^j} = E. \]

Since \( q^j \) is only the function of \( q^i \),

\[ \frac{\partial q^j}{\partial q^i} = \frac{dq^j}{dq^i} \delta^i_j, \]

Equation (5) reduces to

\[ \frac{1}{2M} \sum_i \tilde{g}^{ij} \left( \frac{dq^j}{dq^i} \right)^2 \frac{\partial S}{\partial q^i} \frac{\partial S}{\partial q^j} = E. \]

Here, let us introduce the metric coefficients for the new coordinate system \( \{ \tilde{q}^j \} \)

\[ G_{ij}(\tilde{q}) = \frac{\partial x^i}{\partial \tilde{q}^j} \frac{\partial x^j}{\partial \tilde{q}^i} + \frac{\partial y^i}{\partial \tilde{q}^j} \frac{\partial y^j}{\partial \tilde{q}^i}. \]

This makes (7) turn into

\[ \frac{1}{2M} \sum_i G_{ij} \tilde{g}^{ij} \frac{\partial S}{\partial q^i} \frac{\partial S}{\partial q^j} = E \]

for \( g^{ij} \equiv 1/g_{ij} \). From this, one gets to the conditions of conformal mapping that generate the invariant equation (3)

\[ G_{11} \tilde{g}^{11} = \sqrt{G} \sqrt{g}, \quad \text{and} \quad G_{22} \tilde{g}^{22} = \sqrt{G} \sqrt{g}. \]

where the determinants \( G = |G_{ij}| \) and \( g = |g_{ij}| \). The simultaneous equations determine the definite form of the transformation. Now equation (10) becomes

\[ \frac{1}{2M} \sqrt{\frac{G}{g}} \sum_i g^{ij} \frac{\partial S}{\partial q^i} \frac{\partial S}{\partial q^j} = E. \]
The invariant squared distance between two neighboring points of the transformed space is depicted by

\[ dl^2 = \sqrt{\frac{g}{G}} \sum_i g_{ij} dq^i dq^j. \]  \hspace{1cm} (13)

One interesting outcome which could emerge from the H-J equation is that the motion of the particle in the conformal space can be obtained with a physical effect by moving the factor \( \sqrt{g/G} \) to the right hand side of the equation, which yields

\[ \frac{1}{2M} \sum_i g^{ij} \frac{\partial S}{\partial q^i} \frac{\partial S}{\partial q^j} = (E - U(q)), \]  \hspace{1cm} (14)

with the effective potential

\[ U(q) = \left( 1 - \sqrt{\frac{g}{G}} \right) E, \]  \hspace{1cm} (15)

where \( E \) is the released energy of the particle. This reinterpretation turns the geometric effect of the transformation into a real mechanical effect. It allows us to reveal and construct novel systems through conformal mapping. Equation (14) is effective for higher dimensional space as long as the conformal conditions \( G_{ij} \bar{g}^{ij} = \sqrt{G/g} \) are solvable. In the coming sections, the formulation is applied to investigate the H-N transmutation in the uniform magnetic field.

3. Action variables and the energy of the Hooke-Newton transmutation system in the uniform magnetic field

The action variables of the transmutation system are calculated in this section so as to obtain the system energy. To generate the transmutation, let us evaluate the transformation functions with respect to the cylindrical coordinate system \((\rho, \varphi)\). According to the definitions of \( \bar{g}_{ij} \) and \( G_{ij} \) in (9), we have

\[ (\bar{g}_{ij}) = \text{diag}(1, \rho^2), \quad (G_{ij}) = \text{diag} \left( \frac{1}{(\rho^2)^2}, \frac{\rho^2}{(\varphi)^2} \right), \]  \hspace{1cm} (16)

where \( \rho' = d\rho/d\rho \), and \( \varphi' = d\varphi/d\varphi \). Solving the equations in (11) gives the transformation functions

\[ \bar{p} = \frac{\rho'}{\lambda^{-\nu}}, \quad \bar{\varphi} = \nu \varphi, \]  \hspace{1cm} (17)

where the parameter \( \nu \) is a real number, and \( \lambda \) has the units of length used to balance the units. For our consideration of the Hooke-Newton transmutation, the parameter \( \nu = 2 \), we have

\[ p = \frac{\rho^2}{\lambda}, \quad \varphi = 2\varphi. \]  \hspace{1cm} (18)

Substituting the transformation into the components in (16) and a short calculation give the conformal factor

\[ \sqrt{\frac{g}{G}} = \frac{\bar{p} \rho' \varphi'}{\rho} = \left( \frac{2\rho}{\lambda} \right)^2. \]  \hspace{1cm} (19)

So we obtain the invariant distance of transformed space

\[ dl^2 = \left( \frac{2\rho}{\lambda} \right)^2 (d\rho^2 + \rho^2 d\varphi^2) \]  \hspace{1cm} (20)

from (13), and the transformed H-J equation

\[ \frac{1}{2M} \left[ p_{\rho}^2 + \left( \frac{p_{\varphi}}{\rho} \right)^2 \right] = (E - U), \]  \hspace{1cm} (21)

with

\[ U = \left( 1 - 4\frac{p_{\rho}^2}{\lambda^2} \right) E, \]  \hspace{1cm} (22)

from (14). It turns out that the geometric effect is now achieved physically by the potential \( U \). It is not difficult to generate the potential field since it corresponds to a repulsive linear force which can be produced by the electric field \( E = F/q = -\nabla U/q \). Now let us turn our attention to consider the influence of a uniform magnetic field on the conformal image of a charge’s motion. For the magnetic field directed along the \( Z \)-axis, \( B = B\hat{z} \), the H-J equation is obtained by replacing \( p_{\rho}, \rho \) with the minimal coupling \( (p_{\rho,}/\rho) - qA_{\rho} \) since the magnetic field is calculated through \( B = \nabla \times A \), and one can choose the vector potential \( A = (B\rho/2)\hat{e}_\varphi \). The replacement
guarantees the conservation of the total momentum of the charge and field. The motion of the charge \( q \) is now governed by the H-J equation

\[
\frac{1}{2M} \left[ p^2 + \left( \frac{p}{\rho} - qA_{\phi} \right)^2 \right] = (E - U),
\]

with \( A_{\phi} = B\rho / 2 \). The angular variable \( \phi \) is obviously a cyclic coordinate. So the generalized momentum

\[
p_{\phi} = \frac{\partial S}{\partial \phi} = \text{constant}.
\]

This is the angular momentum of the system. The corresponding action variable to the momentum is

\[
I_{\phi} = \frac{1}{2\pi} \oint p_{\phi} d\phi = p_{\phi}.
\]

It follows from (23) that the momentum \( p_{\rho} \) can be evaluated by

\[
p_{\rho} = \sqrt{2M(E - U) - \frac{1}{\rho^2} (p_{\rho} - q\rho A_{\phi})^2}
\]

\[
= \frac{1}{\rho} \sqrt{-p_{\rho}^2 + M\omega_c^2 \rho^2 - \left[ \frac{M\omega_c^2}{2} - \frac{8ME}{\lambda^2} \right] \rho^4},
\]

where the cyclotron frequency \( \omega_c = qB/M \) has been introduced. Using the new coordinate variable \( \bar{\rho} = \rho^2 / \lambda \), the radial action variable is calculated from the integration

\[
I_{\rho} = \frac{1}{2\pi} \oint p_{\rho} d\rho = \frac{1}{2\pi} \int \frac{-p_{\rho}(2\rho^2)}{\bar{\rho}} + \frac{M(\lambda/2)\omega_c(p_{\rho}/2)}{\bar{\rho}} + 2M \left( E - \frac{M(\lambda/2)^2(\omega_c/2)^2}{2} \right) d\bar{\rho}.
\]

We note that this is an action variable for an inverse square force generated by the Coulomb field \((\lambda\omega_c p_{\rho}/8)/\bar{\rho}\).

Therefore, the mapping in (18) does indeed generate the H-N transmutation of the magnetic force in the uniform magnetic field. It is remarkable to note that the force which is attractive or repulsive is determined by both the signatures of charge \( q \) in \( \omega_c \) and the angular momentum \( p_{\phi} \). Given a kind of charged particle in the Coulomb field, a right or left turn will determine the particles with closed or unclosed trajectories. To perform the integration of \( I_{\rho} \), let us put

\[
a = \frac{(p_{\rho}/\lambda)^2}{\left( \frac{M\omega_c^2}{2} - \frac{8ME}{\lambda^2} \right)}; \quad b = \frac{M\omega_c p_{\rho}/(2\lambda)}{\left( \frac{M\omega_c^2}{2} - \frac{8ME}{\lambda^2} \right)};
\]

then

\[
I_{\rho} = \frac{\lambda}{4\pi} \sqrt{\left( \frac{M\omega_c^2}{2} - \frac{8ME}{\lambda^2} \right)} \int_{\rho_{\min}}^{\rho_{\max}} \frac{1}{\bar{\rho}} \sqrt{-a + 2b\bar{\rho} - \bar{\rho}^2} d\bar{\rho},
\]

where

\[
\rho_{\max} = b + \sqrt{b^2 - a}; \quad \rho_{\min} = b - \sqrt{b^2 - a}.
\]

Since \( \rho \geq 0 \), it implies \( b \geq 0 \). So the signature of \( q \) has to be the same as that of the angular momentum \( p_{\phi} \), and energy satisfies

\[
E < \frac{M}{32}(\lambda\omega_c)^2.
\]

The conditions of the signature and the inequality actually stand for the attractive force and closed trajectory conditions as we shall see later. With the integral formula (p. 102, [10]),

\[
\int \frac{1}{\xi} \sqrt{-a + 2b\xi - \xi^2} d\xi
\]

\[
= (-a + 2b\xi - \xi^2)^{1/2} - \sqrt{a} \sin^{-1} \left( \frac{-a + b\xi}{\sqrt{b^2 - a}} \right) - b \sin^{-1} \left( \frac{b - \xi}{\sqrt{b^2 - a}} \right).
\]
The radial action variable is found to be

\[ I_\rho = \frac{I_\rho}{2} \left[ \frac{M\omega_c/2}{\sqrt{\left(M\omega_c/2\right)^2 - \frac{8ME}{\lambda^2}}} - 1 \right]. \tag{33} \]

Solve the equation for the energy. It is expressed in terms of action variables by

\[ E = \frac{M(\omega_c/2)^2(\lambda/2)^2}{2} \left[ 1 - \frac{(I_\rho/2)^2}{(I_\rho/2 + I_p)^2} \right]. \tag{34} \]

Action variables are adiabatic invariant. Let us choose the Bohr’s quantization conditions for the actions

\[ I_\rho = (n_\rho + 1/2)\hbar, \quad n_\rho = 0, 1, 2, \ldots, \tag{35} \]

and

\[ I_p = |m|\hbar, \quad m = \pm 1, \pm 2, \ldots. \tag{36} \]

Here we remember that \( I_\rho = p_\rho = 0 \). Otherwise, the attractive force no longer exists. The energy turns into the discrete spectrum

\[ E = \frac{M(\omega_c/2)^2(\lambda/2)^2}{2} \left[ 1 - \frac{(m/2)^2}{(n_\rho + |m|/2 + 1/2)^2} \right]. \tag{37} \]

This is exactly the quantum spectrum of the transmutation system (See equation (46) in [5]). Equations (34) and (37) show that the existence of \( \omega_c \), and thus the magnetic field, is the crucial point of the closed trajectories and bound states. With the representation in (34), the angular variables can be calculated, yielding

\[ \omega_\rho = \frac{\partial E}{\partial I_\rho} = \frac{2C}{I_\rho}(I_\rho/2)^2(I_\rho/2 + I_p)^3, \tag{38} \]

where \( C \) is the prefactor in formula (34), and

\[ \omega_p = \frac{\partial E}{\partial I_p} = -\frac{C}{I_p} \frac{I_pI_\rho/2}{(I_\rho/2 + I_p)^3}. \tag{39} \]

Their ratio is

\[ \frac{\omega_\rho}{\omega_p} = -\frac{I_\rho}{I_p}. \tag{40} \]

When the quantum rules in (35) and (36) are inserted, the ratio becomes

\[ \frac{\omega_\rho}{\omega_p} = -\frac{|m|}{n_\rho + 1/2} = -\frac{2|m|}{2(n_\rho + 1/2)}. \tag{41} \]

It is a rational number, and the condition of the closed trajectories. This exhibits the corresponding relation between the bound states in quantum mechanics and the closed trajectories in classical mechanics.

Before finishing the discussion of the section, let us determine the constant \( \lambda \). The best choice is

\[ \lambda = \sqrt{\frac{8\hbar^2}{M|\omega_c||p_\rho|}}, \tag{42} \]

the characteristic length of the system\(^1\). The expression is particularly useful when we would like to compare the interaction strengths of the magnetic H-N system and the electric Coulomb system, as we shall see in the notes of the conclusion. Furthermore, the choice also makes the wave function normalized to unity [5]. To obtain the expression, it is noted that the characteristic length of the hydrogen atom system with the electric Coulomb interaction \( q_e^2/\rho \) is the Bohr radius \( a_B = h^2/Mq_e^2 \) with \( q_e \) being the charge of an electron. We now have the interaction of the transmutation Coulomb field \( (\lambda|\omega_c|p_\rho)|/\beta \) and the characteristic length of the system

\(^1\) In [5], the quantum number \(|m|/2\) in \( \lambda \) was factored out and put aside every \( \rho/\rho \) in the wave function, see equation (49) therein. A better choice is absorbing the factor \(|m|/2\) into the dimensionless variable \( \rho/a = \rho^2/a^2 \) becoming \( \rho/\lambda = \rho^2/\lambda^2 \) with \( \lambda = \sqrt{8\hbar^2/M|\omega_c||m|\hbar} \) as presented in this paper since the Coulomb interaction of the transmutation system depends on the quantum number \( m \).
\[ a_T = \frac{\hbar^2}{M(\lambda_\omega p_\gamma / \rho)} \tag{43} \]

Here \( \hbar \) is just a constant quantity, with no relevance to the quantization. The constant is then found by the ratio

\[ \bar{\rho} \approx \frac{\rho^2 / \lambda}{8\hbar^2 / (\lambda M \omega p_\gamma / \rho)} = \frac{\rho^2}{\sqrt[8]{8\hbar^2 / (\lambda M \omega p_\gamma / \rho)}} \tag{44} \]

which indicates that \( \lambda = a_T \sqrt[8]{8\hbar^2 / (\lambda M \omega p_\gamma / \rho)} \). This completes the proof of (42). Let us estimate the order of \( \lambda \) when the magnetic field \( B = 1 \) Tesla. According to quantum mechanics, the angular momentum is always quantized by \( \lambda_\omega = \pm 1, \pm 2, \ldots \), so that

\[ \lambda = \frac{1}{\sqrt{|m|}} \sqrt[8]{8\hbar / M \omega} \tag{45} \]

The substitution of the cyclotron frequency \( \omega_c = qB / M \approx 1.6 \times 10^{13} \text{ Hz} \) for an electron under the field, and \( \hbar = 6.58 \times 10^{-22} \text{ Mev-sec} \) gives

\[ \lambda \approx \frac{1}{\sqrt{|m|}} 8.12 \times 10^{-2} \mu\text{m}. \tag{46} \]

A weaker magnetic field will increase the length.

4. Trajectory equations for the Hooke-Newton transmutation system of magnetic force

The trajectory is defined by the equation in the H-J formulation of mechanics (p. 149, [9])

\[ \varphi + \frac{\partial S(\rho)}{\partial \rho} = \text{constant}. \tag{47} \]

Equation (23) shows that the differential form of the H-J equation is

\[ \frac{1}{2M} \left[ \left( \frac{\partial S}{\partial \rho} \right)^2 + \left( \frac{p_\varphi}{\rho} - qA_\varphi \right)^2 \right] = (E - U). \tag{48} \]

Since \( S = S(\rho) \) is a function of \( \rho \), and \( dS/d\rho = (2\rho / \lambda) dS/d\rho \) solving the above equation for the factor \( dS/d\rho \) gives

\[ \frac{dS}{d\rho} = \sqrt{2ME - \left( \frac{\lambda M \omega_c}{\rho} \right)^2 + \frac{\rho/2}{\rho} \left( \frac{p_\varphi}{\rho} / \bar{\rho} \right)^2 - \left( \frac{p_\varphi}{\rho} / \bar{\rho} \right)^2}. \tag{49} \]

The reduced action is then given by

\[ S = \int \sqrt{2ME - \left( \frac{\lambda M \omega_c}{\rho} \right)^2 + \frac{\rho/2}{\rho} \left( \frac{p_\varphi}{\rho} / \bar{\rho} \right)^2 - \left( \frac{p_\varphi}{\rho} / \bar{\rho} \right)^2} d\rho. \tag{50} \]

Using the representation, we get

\[ \frac{\partial S}{\partial p_\varphi} = \frac{1}{2} \int \frac{p_\varphi / \bar{\rho}}{\bar{\rho} \sqrt{\Psi(\bar{\rho})}} d\bar{\rho} + \frac{1}{2} \int \frac{(\lambda M \omega_c / \rho)^2}{\bar{\rho} \sqrt{\Psi(\bar{\rho})}} d\bar{\rho} \tag{51} \]

by introducing the abbreviation

\[ \Psi(\bar{\rho}) = \left[ 2ME - \left( \frac{\lambda M \omega_c}{\rho} \right)^2 + \frac{\rho/2}{\rho} \left( \frac{p_\varphi}{\rho} / \bar{\rho} \right)^2 - \left( \frac{p_\varphi}{\rho} / \bar{\rho} \right)^2 \right] \tag{52} \]

for short. To go further, let us define

\[ \bar{E} \equiv \left[ E - \frac{M}{2} \left( \frac{\lambda \omega_c}{\rho} \right)^2 \right] = \left[ E - \frac{\hbar^2}{4} \frac{\lambda \omega_c}{p_\varphi} \right], \tag{53} \]

where we have used (42) to get to the second equality. The action (50) may be for the attractive or repulsive force depending on the coupling constant \( (\lambda \omega_c p_\gamma / 8) > 0 \) or \( < 0 \). We shall first consider the attractive case.

4.1. Attractive force

Equation (50) shows an inverse square force generated by the Coulomb field \( \lambda \omega_c p_\gamma / 8 / \bar{\rho} \). Attraction is caused by the coupling \( (\lambda \omega_c p_\gamma / 8) > 0 \). The calculation of \( \partial S/\partial p_\varphi \) would be divided into three regions: 1. \( \bar{E} < 0 \), 2.
\( E = 0 \), and \( 3. \bar{E} > 0 \), according to the strength of the magnetic field when the released energy and angular momentum of the charge are given.

### 4.1.1. Trajectory equation for \( \bar{E} < 0 \)

In this case the strength of the magnetic field is strong enough such that

\[
\frac{\hbar^2}{4} \left| \frac{\omega_z}{p_z} \right| > \bar{E}.
\]

(54)

The first integral in (51) can be manipulated to yield

\[
\int \frac{\left( p_x/\bar{E} \right)}{\bar{E}^2 \sqrt{\rho(\bar{E})}} d\bar{E} = \int -\bar{E} \rho \frac{p_x}{\bar{E}^2} d\bar{E},
\]

\[
= \int \frac{p_x}{\bar{E}^2} \sqrt{\frac{-2M|\bar{E}| + (2M\alpha)/\bar{E} - \bar{E}^2/\bar{E}^2}{1 - (\bar{E}^2/\bar{E}^2 - 2M|\bar{E}| + (2M\alpha)/\bar{E}^2)}} - \bar{E}^2/\bar{E}^2 d\bar{E},
\]

(55)

where the abbreviations,

\[
\tilde{p}_x = \frac{p_x}{2}, \text{ and } \alpha = \frac{\lambda \omega_z}{8},
\]

(56)

have been used. Put

\[
x = \frac{(\bar{E} - M\alpha/\bar{E})}{\sqrt{-2M|\bar{E}| + (2M\alpha)/\bar{E}^2}}.
\]

(57)

One gets the analytic result of the integral

\[
\int dx \frac{-1}{\sqrt{1 - x^2}} = \cos^{-1} x.
\]

(58)

The second integral in (51)

\[
\int \frac{1}{\rho \sqrt{\rho(\bar{E})}} d\rho = \int \frac{1}{\sqrt{-2M|\bar{E}|\bar{E}^2 + (2M\alpha)\bar{E} - \bar{E}^2}} d\bar{E},
\]

\[
= \frac{1}{\sqrt{2M|\bar{E}|}} \int \frac{1}{\sqrt{1 - \left[ \alpha / (2|\bar{E}|) \right]^2 - \tilde{p}_x^2 / (2M|\bar{E}|)}} d\bar{E}.
\]

(59)

Put

\[
y = \frac{\bar{E} - \alpha / (2|\bar{E}|)}{\sqrt{\left[ \alpha / (2|\bar{E}|) \right]^2 - \tilde{p}_x^2 / (2M|\bar{E}|)}}.
\]

(60)

The integral becomes

\[
\frac{1}{\sqrt{2M|\bar{E}|}} \int \frac{dy}{\sqrt{1 - y^2}} = \frac{-1}{\sqrt{2M|\bar{E}|}} \cos^{-1} y.
\]

(61)

The trajectory equation for \( \bar{E} < 0 \) is then given by

\[
\varphi - \frac{1}{2} \cos^{-1} x = -\frac{\lambda \omega_z}{2} \frac{1}{\sqrt{2M|\bar{E}|}} \cos^{-1} y = \text{constant}.
\]

(62)

It is convenient to introduce the parameters

\[
e = \sqrt{1 - \frac{2|\bar{E}|\tilde{p}_x^2}{M\alpha^2}}, \text{ and } L = \frac{\tilde{p}_x^2}{M\alpha}.
\]

(63)

The range of \( e \) can only be

\[
0 \leq e < 1
\]

(64)

for the case \( \bar{E} < 0 \). It may be referred to as the quasi-eccentricity. Without loss of generality, the constant in (62) is chosen as zero, and the trajectories must satisfy the equation...
where the minus (plus) sign is for the negative (positive) charged particle. To reflect the cyclic characteristic of $\varphi$ for a general value of $\epsilon$, let us take the cosine operation with respect to both sides of (65), yielding

$$\cos 2\varphi = x_1 \cos \left( \frac{1}{\sqrt{1 - \epsilon^2}} x_2 \right) \mp \sqrt{1 - x_1^2} \sin \left( \frac{1}{\sqrt{1 - \epsilon^2}} \cos^{-1} x_2 \right),$$

(66)

where

$$x_1 = \left( \frac{L/p - 1}{e} \right), \quad x_2 = \left( \frac{(1 - \epsilon^2)p/L - 1}{e} \right).$$

(67)

The top sign is for the positive charge. It is possible to obtain an alternative representation of the trajectory equation without resorting to the inverse function. In order to prove the statement, we need the known equalities (p. 58, \[10\])

$$\cos (\beta \cos^{-1} (x)) = \frac{1}{2} [(x + \sqrt{x^2 - 1})^\beta + (x - \sqrt{x^2 - 1})^\beta],$$

(68)

$$\cos^{-1} x + \cos^{-1} y = \begin{cases} \cos^{-1} (xy - \sqrt{1 - x^2} \sqrt{1 - y^2}), & \text{for } x + y \geq 0, \\ 2\pi - \cos^{-1} (xy - \sqrt{1 - x^2} \sqrt{1 - y^2}), & \text{for } x + y < 0, \end{cases}$$

(69)

and

$$\cos^{-1} x - \cos^{-1} y = \begin{cases} -\cos^{-1} (xy + \sqrt{1 - x^2} \sqrt{1 - y^2}), & \text{for } x \geq y, \\ \cos^{-1} (xy + \sqrt{1 - x^2} \sqrt{1 - y^2}), & \text{for } x < y. \end{cases}$$

(70)

It is easy to verify that the trajectory equation can be expressed in terms of

$$\cos 2\varphi = [x_1 z + \sqrt{1 - x_1^2} \sqrt{1 - x_2^2}],$$

(71)

where

$$z = \frac{1}{2} [(x_1 + \sqrt{x_2^2 - 1})^\beta + (x_1 - \sqrt{x_2^2 - 1})^\beta],$$

(72)

with $\beta = 1/\sqrt{1 - \epsilon^2}$. For the special case $\beta = 2$, the equation reduces to

$$\cos 2\varphi = [x_1 (2x_2^2 - 1) + \sqrt{1 - x_1^2} \sqrt{1 - (2x_2^2 - 1)^2}].$$

(73)

The top sign is for the positive charge. Figure 1 shows the patterns of the trajectories with the value $\beta = 2$, i.e. $e = \sqrt{3}/2$, where $L = 1$ was chosen. The left pattern is for the negative charge corresponding to the minus sign in (66) or the plus sign in (71). The patterns can be obtained either through (66) or by (71). Figure 2 shows the trajectories for several different values of $\epsilon$. They exhibit simpler designs when the charge is moving in the stronger magnetic fields, corresponding to the smaller values of $\epsilon$. Each of them only spans a smaller region of the trajectory plane. When the charge is moving in the weaker magnetic fields, the larger values of $\epsilon$, the trajectories
get complicated. This is reasonable since the trapping force of the uniform magnetic field \( F \sim \omega_c \rho_i \) is proportional to the strength of the magnetic field in \( \omega_c \). The quantitative proof of the statement that the strength of the magnetic field controls the value of \( e \) for a charge with given released energy and angular momentum can be proved as follows:

\[
\alpha^2 = \left( \frac{\lambda \omega_c \rho_i}{8} \right)^2 = \frac{\hbar^2}{4M} |\bar{\omega}_i| \quad (74)
\]

from (56) and (42). So the factor in \( e \)

\[
\frac{2|\bar{\mathbf{P}}|^2}{M \alpha^2} = \frac{8|\bar{\mathbf{E}}|}{\hbar^2} \left| \frac{\bar{\mathbf{P}}_e}{\omega_i} \right| = \left| \frac{E}{\hbar^2 \omega_i / 4p^e} - 1 \right| = \left( 1 - \frac{E}{\hbar^2 \omega_i / 4p^e} \right)
\]

(75)

from (53) and (54), where \( \bar{\mathbf{P}}_e = p_e / 2 \). The substitution of the factor for \( e \) in (63) then gives

\[
e = \sqrt{\frac{E}{\hbar^2 \omega_i / 4p^e}} < 1. \quad (76)
\]

Figure 3 shows the trajectories for \( e = 0.99 \) and 0.995. The trajectories are always finite, and closed. Some points in figure 3 where the trajectory is not smooth should be due to the fact that the second term with \( \cos^{-1}(x) / \sqrt{1 - x^2} \) in (66) oscillates very rapidly when \( e \to 1 \). The closed characteristic of the trajectory for \( e \to 1 \) can also be inspected by directly taking the limit in equation (66). Under the operation, the second inverse cosine function in (66) has the multivalued limit

\[
\cos^{-1}\left( \frac{(1 - e^2) \bar{p}/L - 1}{e} \right) \to \pm n\pi. \quad (77)
\]

This makes the asymptotic representation of the trajectory equation become

\[
\cos\left( 2\varphi \pm \frac{n\pi}{\sqrt{1 - e^2}} \right) = \left( \frac{L/\bar{p} - 1}{e} \right). \quad (78)
\]

It is easy to see that the trajectories of the equation are elliptic with two different tilting angles for a given \( n \).
4.1.2. Trajectory equation for $\vec{E} = 0$.

In this case the strength of the magnetic field satisfies the condition

$$\frac{\hbar^2}{4} \left| \frac{\omega_c}{p_c} \right| = E.$$  \hfill (79)

The condition is equivalent to $e = 1$. As calculated above, the first integral in (51) can be performed, and gives

$$\int \frac{\vec{p}_c}{\sqrt{\left(2M\alpha/\bar{p} - \vec{p}_c^2/\bar{p}^2\right)}} d\bar{p} = \cos^{-1} \left( \frac{\vec{p}_c/\bar{p} - M\alpha/\bar{p}_c}{M\alpha/\bar{p}_c} \right) = \cos^{-1} \left( \frac{L}{\bar{p} - 1} \right).$$ \hfill (80)

The second integral for $\vec{E} = 0$ becomes

$$\frac{\lambda M\omega_c}{4} \int \frac{1}{\sqrt{\left(2M\alpha\bar{p} - \vec{p}_c^2\right)}} d\bar{p} = \sqrt{\frac{2}{L}} \left( \bar{p} - \frac{L}{2} \right)^{1/2}.$$ \hfill (81)

*Figure 3.* Trajectories for $e = 0.99$, and $e = 0.995$. The patterns show that, given the strength of the magnetic field, the trajectories are always closed as long as the energy $E$ and angular momentum $p_c$ of the charged particle satisfies $E < |\omega_c/p_c|\beta^2/4$. 

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The second integral for $\vec{E} = 0$ becomes

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Accordingly, the equation of the trajectory is

\[ 2 \varphi = \cos^{-1} \left( \frac{L}{\rho} - 1 \right) \pm \sqrt{\frac{2}{L} \left( \frac{L}{2} - \frac{\rho}{2} \right)} \]

(82)
The plus is for the negative charged particle. The patterns in the first row of figure 4 exhibit the trajectory, where \( L = 1 \) was chosen. It is a kind of uniform spiral. The patterns in the second row are for the positive charged particle.

4.1.3. Trajectory equation for \( \bar{E} > 0 \).

The strength of the magnetic field satisfies the condition

\[ \frac{\hbar^2}{4} \left| \frac{\omega_c}{\bar{p}_c} \right| < E. \]

(83)
The equivalent condition to this is \( e > 1 \). The first integral in (51) in this case is

\[
\int \frac{(\bar{p}_c/2)}{\bar{p}^2 \sqrt{\psi(\bar{p})}} d\bar{p} \quad = \int \frac{\bar{p}_c}{\bar{p}^2 \sqrt{2M(\bar{E} + \alpha/\bar{p}) - \bar{p}_c^2/\bar{p}^2}} d\bar{p} \\
= \int dx \frac{-1}{\sqrt{1 - x^2}} = \cos^{-1} x,
\]

(84)
where
\[
x = \frac{(\hat{p}_\varphi / \rho - M \alpha / \hat{p}_\varphi)}{\sqrt{2mE + (M \alpha / \hat{p}_\varphi)^2}}.
\]  
(85)

We have the second integral
\[
\frac{\lambda M \omega_c}{4} \int \frac{1}{\hat{p}_\varphi \sqrt{\Psi(\rho)}} d\rho
\]
\[
= \frac{\lambda M \omega_c}{4} \int \frac{1}{\sqrt{(2mE) \rho^2 + (2M \alpha) \rho - \hat{p}_\varphi^2}} d\rho
\]
\[
= \frac{\lambda M \omega_c}{4} \frac{1}{\sqrt{2mE}} \cosh^{-1} y,
\]  
(86)

where
\[
y = \frac{\rho + \alpha/(2E)}{\sqrt{[\alpha/(2E)]^2 + \hat{p}_\varphi^2/(2mE)}}.
\]  
(87)

The solution of the inverse hyperbolic cosine has taken the branch \(\cosh^{-1} y > 0\) for \(y > 0\). Introducing the parameters
\[
e = \sqrt{1 + \frac{2E \hat{p}_\varphi^2}{M \omega_c^2}}, \quad \text{and} \quad L = \frac{\hat{p}_\varphi^2}{M \alpha},
\]  
(88)

the trajectory equation is found to be
\[
2 \varphi = \cos^{-1} \left( \frac{L/p - 1}{e} \right) \pm \frac{1}{\sqrt{e^2 - 1}} \cosh^{-1} \left( \frac{(e^2 - 1)p/L + 1}{e} \right).
\]  
(89)

As above, the plus is for the negative charge. The patterns in the first (second) row of figure 5 show two trajectories of the negative (positive) charged particle produced from the equation. They are a non-uniform spiral. The diversities of the trajectory are distinguished by the critical energy \(E_C = |\omega_c / p_\varphi| \hbar^2 / 4\). Let us estimate its order for an electron. Consider the angular momentum \(p_\varphi = |m| \hbar, m = \pm 1, \pm 2, \ldots\).

\[
E_C = \frac{\hbar^2}{4} \left| \frac{\omega_c}{p_\varphi} \right| = \frac{\hbar \omega_c}{4|m|}.
\]  
(90)

For the magnetic field \(B = 1\) Tesla, \(\hbar \omega_c \approx 10^{-4}\) eV. We have
\[
E_C \approx \frac{10^{-4}}{4|m|} \text{ eV}.
\]  
(90)

With \(\lambda \approx 8.12 \times 10^{-2} / \sqrt{|m|} \mu\text{m}\) from (46), the electric field which generates the corresponding effective potential is given by
\[
E = -\nabla U / q_e = \frac{8 E_C}{\lambda^2 q_e \rho \hat{e}_\rho} = \frac{8 \times 10^{-4}}{4|m|} \text{ eV} \times 1 \text{ m} = \frac{(8.12)^2 \times 10^{-4} / |m| (\mu\text{m})^2 \times (-1.6 \times 10^{-19} \text{ C})}{(8.12)^2 \times 10^{-19} / |m| (\mu\text{m})^2 \times (-1.6 \times 10^{-19} \text{ C})} \approx -3 \times 10^6 \hat{e}_\rho \text{ (N/C)},
\]  
(91)

where the variable is simply chosen as \(\rho = 1\) meter (m). A realized setup may be much smaller. The strength of the electric field is about 1/10 of an electron in a hydrogen atom. For the magnetic field \(B = 100\) Gauss, \(\omega_c \approx 1.6 \times 10^9 \text{ Hz}, \) and \(\lambda \approx 8.12 \times 10^{-1} / \sqrt{|m|} \mu\text{m},\) the critical energy would decrease to
\[
E_C \approx \frac{10^{-6}}{4|m|} \text{ eV}.
\]  
(92)

The corresponding electric field for the potential \(U\) is
\[
E \approx -3 \times 10^6 \hat{e}_\rho \text{ (N/C)}.
\]  
(93)

The strength is about 10 times the electric field in a photocopier and is easy to achieve. Equations (90) and (92) show that the electric and magnetic fields of the magnetic transmutation system can select and trap electrons with extremely low energy and a definite signature of angular momentum. This function is obviously effective for arbitrary charged particles. It thus offers a novel means for the choosing and trapping of the low energy charged particles in the uniform magnetic field.
4.2. Repulsive force

In accordance with equation (50), the force generated by the Coulomb field of the transmutation system would be repulsive when the charge \( q \) and momentum \( p_j \) have different signatures, i.e., \( \alpha = (\lambda \omega/l_p) / 8 < 0 \) in this case. With the definitions

\[
e = \sqrt{1 + \frac{2E\lambda^2}{M\omega^2}}, \quad \text{and} \quad L = \frac{\lambda^2}{M|\alpha|},
\]

it can be proved that only the case of \( \tilde{E} > 0 \), namely \( e > 1 \), has a physical solution. The other two cases, \( \tilde{E} = 0 \) and \( \tilde{E} < 0 \), only have solutions in the complex plane. A derivation similar to the previous subsection shows that the trajectory equation is given by

\[
2\varphi = \cos^{-1}\left(\frac{L/p + 1}{e}\right) \pm \frac{1}{\sqrt{e^2 - 1}} \cosh^{-1}\left(\frac{(e^2 - 1)p/L - 1}{e}\right).
\]

The top (bottom) pattern in figure 6 is a trajectory of the negative (positive) charged particle generated from the equation with the plus (minus) sign. It is a non-uniform spiral like the previous cases of \( \tilde{E} > 0 \) and \( \alpha > 0 \). However, it escapes more quickly from the region of the electromagnetic field.

5. Conclusion

This article performs the classical analysis for the Hooke-Newton transmutation in the uniform magnetic field. The first part is devoted to discussing the physical realization of mechanical conformal mapping of the orthogonal systems based on the form-invariant Hamilton-Jacobi equation. Then, it is applied to calculate the action and angular variables of the transmutation system since they are the easiest way to manifest the coherence of energy between classical and quantum mechanics. The second part of the article is used to evaluate the different kinds of trajectory equations for the motion of a charge in the system. It is shown that the types of trajectories for the attractive field are classified by the magnetic field parameter \( |\omega/l_p| \kappa^2 / 4 \). Although the transmutation system is created by the quadratic conformal mapping, it is possible to actualize the system in physics by reinterpreting the conformal structure as an action of the force field. Several notes are worth making as follows: (i) The interaction strength of the transmuted Coulomb field is weak, but adjustable. To appreciate
this quantitatively, we assume that the angular momentum is to take the quantization values
\[ p_{\phi} = |m| \hbar, \quad m = \pm 1, \pm 2, \ldots, \]
according to quantum mechanics. For an electron moving in the magnetic field of \( B = 1 \) Tesla, \( \lambda \approx 8.12 \times 10^{-2}/\sqrt{|m|} \) \( \mu m \) from (46), and \( \hbar \omega_{c} \approx 10^{-4} \) eV. The transmuted fine-structure constant is given by

\[ \frac{\alpha_{M}}{\hbar c} = \frac{\lambda \omega_{c} p_{\phi}}{8 \hbar c} = \frac{|m| \lambda \hbar \omega_{c}}{8} = 0.4 \times 10^{-5} \sqrt{|m|}. \]  \( \text{(96)} \)

Here we have used a suffix ‘M’ to denote that the coupling is due to the magnetic field. The ratio between the transmuted and electric fine-structure constants is

\[ \frac{\alpha_{M}/\hbar c}{\alpha_{e}/\hbar c} \approx \frac{0.4 \times 10^{-5} \sqrt{|m|}}{1/137} = 0.55 \times 10^{-5} \sqrt{|m|}. \]  \( \text{(97)} \)

Therefore, the interaction strength is only the order of 1/1000 relative to the common electric Coulomb interaction when \( B = 1 \) Tesla and \( |m| \) is a small integer. However, it is not impossible to reverse the outcome as long as the charge is in the stronger magnetic field and has a large angular momentum \(|m| \hbar\). (ii) The signature of the angular momentum \( p_{\phi} \) for a particle with negative (positive) charge in the magnetic field \( B = B_{c} \) can only be prescribed as negative (positive) in the presented analysis. Otherwise, it would violate the physical fact that a charged particle is always trapped in a strong-enough uniform magnetic field. Nevertheless, quantum mechanics allows an alternative choice of the signature since it is well-known that the angular momentum of a particle is always quantized by \(|m| \hbar\) in the 2D central force field problem. Consequently, it is plausible to postulate that the presented transmutation system in quantum mechanics could be applied to select and trap charged particles with the corresponding signatures under the critical energy \( E_{c} = |\omega_{c} p_{\phi}| \hbar^{2}/4 \) since \( E_{c} \) is also the upper bound energy for a bound state according to equations (37) and (55). (iii) In general, there are many kinds of transmutations that can be generated from the magnetic force of the uniform magnetic field by conformal mapping. This is proved by taking the general conformal transformation from (17)

\[ \tilde{\rho} = \frac{\rho^{\nu}}{\chi^{1}}, \quad \text{and} \quad \tilde{\varphi} = \nu \varphi. \]  \( \text{(98)} \)

Substituting them into equation (13) gives rise to

\[ d\tilde{l}^{2} = \left( \frac{\nu \rho^{\nu-1}}{\chi^{1}} \right)^{2} (d\rho^{2} + \rho^{2} d\varphi^{2}). \]  \( \text{(99)} \)

The conformal factor thus results in a different kind of transmutation of force when a different value of \( \nu \) is chosen. Let us illustrate it with an example by choosing \( \nu = -1 \). The radial action variable becomes in this case...
This is obviously a central force field problem. The second term in the square root is just the centrifugal energy for a motion in the field. The third term stands for an attractive force proportional to $1/r^2$, while the fourth term is a repulsive (attractive) force when $\omega_\phi$ and $p_\phi$ have the same (different) signatures. Here we see again that the topological signature of angular momentum is also significant, not just the signature of the charge. Physically, the system can be created by applying the effective potential $U = (1 - \lambda^2/r^2)E$ and the uniform magnetic field to the environment of the charge.

All in all, the presented system has hybrid properties of the Coulomb and simple harmonic oscillator (SHO) systems. First, it has an unclosed trajectory like the Coulomb system, while the SHO only allows a closed orbit. Second, the system has a closed trajectory (corresponding to the discrete bound state at the quantum level) while the energy is restricted by the potential of the SHO type

$$E < \frac{M(\omega_\phi/2)^2(\lambda/2)^2}{2}.$$  

Finally, unlike the Coulomb system, the trapping force is due to the linear force of the constant magnetic field. This makes it easy to achieve in a laboratory. Transmutation or duality between two different systems established by a transformation has long been of interest. It narrates the internal connection of two systems, and the unity of the different branches. It is our hope that this presentation will be helpful in further revealing interesting systems and applications.

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