G-STRUCTURES DEFINED ON PSEUDO-RIEMANNIAN MANIFOLDS

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Concepts and techniques from the theory of G-structures of higher order are applied to the study of certain structures (volume forms, conformal structures, linear connections and projective structures) defined on a pseudo-Riemannian manifold. Several relationships between the structures involved have been investigated. The operations allowed on G-structures, such as intersection, inclusion, reduction, extension and prolongation, were used for it.

Keywords: Higher order G-structures; pseudo-Riemannian conformal structure; projective differential geometry; general relativity.

1. Introduction and motivation

A differential structure of a manifold $M$ is a $C^\infty$ maximal atlas and, indeed, the charts of the atlas make up the primordial structure. The idea of a geometrical structure can be realized by the concept of G-structure when choosing the allowable meaningful classes of charts.

General relativity is a physical theory, which is heavily based on differential geometry. The fundamental mathematical tools used by this theory to explain and to handle gravity are the geometrical structures. The space-time is described by a 4-dimensional manifold with a Lorentzian metric field, and the theory put the matter on space-time, being mainly represented by curves in the manifold or by the overall stress-energy tensor.

The law of inertia in the space-time is translated into a projective structure on the manifold, which is provided by the geodesics of the metric in keeping with the equivalence principle. Furthermore, the space-time in general relativity is a dynamical entity because the metric field is subject to the Einstein field equations, which almost equate Ricci curvature with stress-energy of matter.
Other main structures are the *volume form*, that is used to get action functionals by integration over the manifold, and the *Lorentzian conformal structure*, that gives an account of light speed invariance. Different approaches to gravity try to *separate* the geometry into independent compounds to promote the understanding about physical interpretation of geometric variables.

The theory of $G$-structures of higher order is possibly the more natural framework for studying the interrelations involved among the relevant structures. In a pseudo-Riemannian manifold there are defined unambiguously the following structures: *volume form*, *conformal structure*, *pseudo-Riemannian metric*, *symmetric linear connection* and *projective structure*. Volume, conformal and metric structures are $G$-structures of first order, but each of them lead to a *prolonged* second order structure. Symmetric linear connection and projective structure are inherently $G$-structures of second order. We will try to clarify this unified description.

2. The bundle of $r$-frames

A differentiable manifold $M$ is a set of points with the property that we can cover it with the charts of a $C^\infty$ $n$-dimensional maximal atlas $\mathcal{A}$. The *bundle of $r$-frames* $\mathcal{F}^r M$ is a quotient set over $\mathcal{A}$. Every class-point, an $r$-frame, collect the charts with equal origin of coordinates which produce identical $r$-th order Taylor series expansion of functions (see Refs. 1, 2).

An $r$-frame is an $r$-jet at 0 of inverses of charts of $M$; two charts are in the same $r$-jet if they have the same partial derivatives up to $r$-th order at the same origin of coordinates. Every $\mathcal{F}^r M$ is naturally equipped with a *principal bundle* structure with respect to the group $G^r_n$ of $r$-jets at 0 of diffeomorphisms of $\mathbb{R}^n$, $f_0 \phi$, with $\phi(0) = 0$.

The group of the *bundle of 1-frames* is $\text{GL}(n, \mathbb{R}) \cong G^1_n$. Its natural representation on $\mathbb{R}^n$ gives an *associated bundle* coinciding with the tangent bundle $TM$. In the end, we identify $\mathcal{F}^1 M$ with the *linear frame bundle* $LM$. Other representations of $G^1_n$ on subspaces of the tensorial algebra over $\mathbb{R}^n$ give associated bundles whose sections are the well-known tensor fields.

The *bundle of 2-frames* $\mathcal{F}^2 M$ is somehow more complicated. Every 2-frame is characterized by a *torsion-free transversal $n$-subspace* $H_l \subset T_l LM$. It happens that the chart’s first partial derivatives fix $l \in LM$ and the second partial derivatives give the ‘inclination’ of that $n$-subspace. The group $G^2_n$ is isomorphic to $G^1_n \times S^2_n$, a semidirect product, with $S^2_n$ the additive group of symmetric bilinear maps of $\mathbb{R}^n \times \mathbb{R}^n$ into $\mathbb{R}^n$; the multiplication rule is $(a, s)(b, t) := (ab, b^{-1}s(b, b) + t)$, for $a, b \in G^1_n$, $s, t \in S^2_n$. 
Let \( \mathfrak{g} \) denote the Lie algebra of \( G \subset G_1^1 \), then the named first prolongation of \( \mathfrak{g} \) is defined by \( \mathfrak{g}_1 := S^2 \cap L(\mathbb{R}^n, \mathfrak{g}) \). We obtain that \( G \times \mathfrak{g}_1 \) is a subgroup of \( G_1^1 \times S^2 \cong G^2_n \). This will be used in Sec. 4.

3. First order \( G \)-structures

An \( r \)-th order \( G \)-structure of \( M \) is a reduction (see [3, p. 53]) of \( \mathcal{F}^r M \) to a subgroup \( G \subset G_1^n \). First order \( G \)-structures are just called \( G \)-structures. Let us see some of them.

Let’s define a volume on \( M \) as a first order \( G \)-structure, with \( G = \text{SL}^\pm_n := \{ a \in G_1^n : |\det(a)| = 1 \} \). If \( M \) is orientable, a volume on \( M \) has two components: two \( \text{SL}(n, \mathbb{R}) \)-structures for two equal, except sign, volume \( n \)-forms. For a general \( M \), a volume corresponds to an odd type \( n \)-form.

From bundle theory, \( \text{SL}_n^\pm \)-structures are sections of \( \mathcal{V}M \), the associated bundle to \( LM \) and the action of \( G_1^n \) on \( \text{SL}_n^\pm \), and they correspond to \( G_1^n \)-equivariant functions \( f \) of \( LM \) to \( G_1^n/\text{SL}_n^\pm \). The isomorphisms \( G_1^n/\text{SL}_n^\pm \cong H_n := \{ k I_n : k > 0 \} \cong \mathbb{R}^+ \) allow to write \( f : LM \rightarrow \mathbb{R}^+ \); the equivariance condition is \( f(la) = |\det a|^{\frac{1}{n}} f(l) \), for \( l \in LM, a \in G_1^n \).

**Theorem 3.1.** We have the bijections:

\[
\text{Volumes on } M \leftrightarrow \text{Sec } \mathcal{V}M \leftrightarrow C^\infty_{\text{equiv}}(LM, \mathbb{R}^+)
\]

Analogous bijective diagram can be obtained for every reduction of a principal bundle. The Lie algebra of \( \text{SL}_n^\pm \) is \( \mathfrak{s}(n, \mathbb{R}) \) and its first prolongation is \( \mathfrak{s}(n, \mathbb{R})_1 := \{ s \in S^2_n : \sum_k s_{ik} = 0 \} \); it’s a Lie algebra of infinite type.

We define a pseudo-Riemannian metric as an \( O_{q,n,q} \)-structure, with \( O_{q,n,q} := \{ a \in G_1^n : a^\dagger \eta a = \eta = \begin{pmatrix} -I_n & 0 \\ 0 & I_{n,q} \end{pmatrix} \} \). As in Th.3.1, we obtain bijections between the metrics and the sections of the associated bundle with typical fiber \( G_1^n/O_{q,n,q} \), and also with the equivariant functions of \( LM \) in \( G_1^n/O_{q,n,q} \). The first prolongation of \( \mathfrak{o}_{q,n,q} \) is \( \mathfrak{o}_{q,n,q} 1 = 0 \); a consequence of this fact is the uniqueness of the Levi-Civita connection.

A (pseudo-Riemannian) conformal structure is a \( \text{CO}_{q,n,q} \)-structure, with \( \text{CO}_{q,n,q} := O_{q,n,q} \times H_n \) (direct product); this definition is equivalent to consider a class of metrics related by a positive factor, and in the Lorentzian case, \( q = 1 \), a conformal structure is characterized by the field of null cones. The first prolongation of \( \mathfrak{co}_{q,n,q} \) is \( \mathfrak{co}_{q,n,q} 1 = \{ s \in S^2_n : s_{jk} = \delta_{ij} \mu_k + \delta_{jk} \mu_i - \sum \eta^{is} \eta_{jk} \mu_s, \mu = (\mu_i) \in \mathbb{R}^{n_*} \} \cong \mathbb{R}^{n_*} \). The named second prolongation \( \mathfrak{co}_{q,n,q} 2 \) is equal to 0 (i.e., \( \mathfrak{co}_{q,n,q} \) is of finite type 2); this deals with the existence and uniqueness of the normal Cartan connection but we do not deal with this here (see [2, \S\S VI.4.2, VII.3]).
Volumes on $M$ and conformal structures are extensions (see [4, p. 202]) of pseudo-Riemannian metrics because of the inclusion of $O_{q,n-q}$ in $SL_n^\pm$ and $CO_{q,n,q}$. Reciprocally:

**Theorem 3.2.** A pseudo-Riemannian metric field on $M$ is given by a pseudo-Riemannian conformal structure and a volume on $M$.

This statement is proved in Ref. 5 by the fact that $G^1_n = SL_n^\pm \cdot CO_{q,n,-q}$ and $O_{q,n-q} = SL_n^\pm \cap CO_{q,n,-q}$ imply that volume and conformal $G$-structures intersect in $O_{q,n-q}$-structures.

4. Second order $G$-structures

A symmetric linear connection (SLC) on $M$ is a distribution on $LM$ of torsion-free transversal $n$-subspaces, which is invariant by the action of $G^1_n$. Identifying $G^1_n \simeq G^1_n \times 0 \subset G^2_n$, we can define an SLC on $M$ as a second order $G^1_n$-structure. From bundle theory, as in Th.3.1, every SLC $\nabla$ is a section of the associated bundle to $F^2M$ and the action of $G^2_n$ on $G^2_n/G^1_n \simeq S^2_n$, and corresponds to an equivariant function $f^{\nabla}: F^2M \to S^2_n$, verifying $f^{\nabla}(z(a,s)) = a^{-1}f^{\nabla}(z)(a,a) + s$, for $z \in F^2M$, $a \in G^1_n$, $s \in S^2_n$.

Let $P$ be a first order $G$-structure on $M$; a symmetric connection on $P$ is a distribution on $P$ of torsion-free transversal $n$-subspaces, which is invariant by the action of $G$, thereby producing a second order $G$-structure, whose $G^1_n$-extension is an SLC on $M$. Reciprocally, a second order $G$-structure determines a first order $G$-structure and a symmetric connection on it.

Noteworthy examples of this are: i) A pseudo-Riemannian metric and its Levi-Civita connection are given by a second order $O_{q,n-q}$-structure. ii) An equiaffine structure on $M$ is a SLC with a parallel volume; it is given by a second order $SL_n^\pm$-structure. iii) A Weyl structure is a conformal structure with a SLC compatible; it is given by a second order $CO_{q,n,q}$-structure.

The following result is an important theorem, arisen from the Weyl’s ‘Raumproblem’, studied by Cartan and others. The theorem is proved in Ref. 6, with a correction revealed in Ref. 7.

**Theorem 4.1.** Let $G$ be a subgroup of $G^1_n$, with $n \geq 3$. Any first order $G$-structure admits a symmetric connection if and only if $g$ is one of these: $\mathfrak{sl}(n,\mathbb{R})$, $o_{q,n-q}$, $co_{q,n,q}$, $\mathfrak{gl}_{n,W}$ (algebra of endomorphisms with an invariant 1-dimensional subspace $W$), $\mathfrak{gl}_{n,W,c}$ (certain subalgebra of the last one, for each $c \in \mathbb{R}$) or, for $n = 4$, $\mathfrak{osp}(2,\mathbb{R})$.

A $G$-structure $P$ can admit many torsion-free transversal $n$-subspaces in $T_lP$, for every $l \in P$. We have the following result (see [2, p.150-155]):
Theorem 4.2. If a first order $G$-structure $P$ admits a symmetric connection, the set $P^2$ of 2-frames corresponding with torsion-free transversal $n$-subspaces included in $TP$ is a second order $G \times \mathfrak{g}_1$-structure.

We name $P^2$ the (first holonomic) prolongation of $P$. Necessarily, a second order $G \times \mathfrak{g}_1$-structure is the prolongation of a $G$-structure.

A (differential) projective structure is a set of SLCs which have the same geodesics up to reparametrizations; we can define it as a second order $G\times_{\mathfrak{p}}\mathfrak{p}$-structure with $\mathfrak{p} := \{ s \in S^2_n : s^{jk} = \delta^j_i \mu_k + \mu_j \delta^i_k, \mu = (\mu_i) \in \mathbb{R}^n \} \simeq \mathbb{R}^n$. Considering geometrical structures on second order, with the same techniques than in the Th.3.2 for the first order (see Ref. 5), we obtain:

Theorem 4.3. A projective structure and a volume on $M$ give an SLC belonging to the former and making the volume parallel.

Hence, a volume select a class of affine parametrizations for the paths of a projective structure. Contrarily, a projective structure and the prolongation of a conformal structure not always intersect; if they intersect we get a Weyl structure.

5. Concluding remarks

The study of integrability conditions of higher order, like curvatures, with respect to the interrelations of these $G$-structures is probably the natural next step following this work.

The geometrical structures described herein can be considered components of the space-time geometry. In this context, the named causal set theory make a conceptual separation between volume and conformal structures, and Stachel proposes an approach, similar to the metric-affine variational principle, using conformal and projective structures as independent variables. In this line of thought, and from the above results, I suggest considering the volume on space-time as a set of independent dynamical variables to make a variational analysis.

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