A bilinear Airy-estimate with application to gKdV-3

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Abstract

The Fourier restriction norm method is used to show local wellposedness
for the Cauchy-Problem

\[ u_t + u_{xxx} + (u^4)_x = 0, \quad u(0) = u_0 \in H^s_x(\mathbb{R}), \quad s > -\frac{1}{6} \]

for the generalized Korteweg-deVries equation of order three, for short gKdV-3. For real valued data \( u_0 \in L^2_x(\mathbb{R}) \) global wellposedness follows by the conservation of the \( L^2 \)-norm. The main new tool is a bilinear estimate for solutions of the Airy-equation.

The purpose of this note is to establish local wellposedness of the Cauchy-Problem

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for the generalized Korteweg-deVries equation of order three, for short gKdV-3. So far, local wellposedness of this problem is known for data \( u_0 \in H^s_x(\mathbb{R}), \quad s \geq \frac{1}{12} \). This was shown by Kenig, Ponce and Vega in 1993, see Theorem 2.6 in [KPV93]. Here we extend this result to data \( u_0 \in H^s_x(\mathbb{R}), \quad s > -\frac{1}{6} \). A standard scaling argument suggests that this is optimal (up to the endpoint). For real valued data \( u_0 \in L^2_x(\mathbb{R}) \) we obtain global wellposedness by the conservation of the \( L^2 \)-norm.

By the Fourier restriction norm method introduced in [B93] and further developed in [KPV96] and [GTV97] matters reduce to the proof of the estimate

\[
\| \partial_x \prod_{i=1}^4 u_i \|_{X_{s,b'}} \leq c \prod_{i=1}^4 \| u_i \|_{X_{s,b}}
\]

for suitable values of \( s, b \) and \( b' \). Here the space \( X_{s,b} \) is the completion of the Schwartz class \( \mathcal{S}(\mathbb{R}^2) \) with respect to the norm

\[
\| u \|_{X_{s,b}} = \left( \int d\xi d\tau < \tau - \xi^3 >^{2b} < \xi >^{2s} |\mathcal{F} u(\xi, \tau)|^2 \right)^{\frac{1}{2}},
\]

where \( \mathcal{F} \) denotes the Fourier transform in both variables. The main new tool for the proof of (1) is the following bilinear Airy-estimate:
Lemma 1 Let $I_s$ denote the Riesz potential of order $-s$ and let $I_s(f, g)$ be defined by its Fourier transform (in the space variable):

$$
\mathcal{F}_x I^s(f, g)(\xi) := \int_{\xi_1 + \xi_2 = \xi} d\xi_1 |\xi_1 - \xi_2|^s \mathcal{F}_x f(\xi_1) \mathcal{F}_x g(\xi_2).
$$

Then we have

$$
||I^s f^s(e^{-it \partial^2} u_1, e^{-it \partial^2} u_2)||_{L^2_t} \leq c ||u_1||_{L^2_t} ||u_2||_{L^2_t}.
$$

Proof: We will write for short $\hat{u}$ instead of $\mathcal{F}_x u$ and $\int_s d\xi_1$ for $\int_{\xi_1 + \xi_2 = \xi} d\xi_1$. Then, using Fourier-Plancherel in the space variable we obtain:

$$
\begin{align*}
||I^s f^s(e^{-it \partial^2} u_1, e^{-it \partial^2} u_2)||_{L^2_t}^2 & = c \int d\xi |\xi| \left| \int d\xi_1 |\xi_1 - \xi_2|^s e^{i(\xi_1 + \xi_2)} \hat{u}_1(\xi_1) \hat{u}_2(\xi_2) \right|^2 \\
& = c \int d\xi |\xi| \left| \int d\xi_1 d\eta e^{i(\xi_1 + \xi_2 - \eta_1 - \eta_2)} (|\xi_1 - \xi_2||\eta_1 - \eta_2|)^{\frac{1}{2}} \prod_{i=1}^2 \hat{u}_i(\xi_i) \overline{\hat{u}_i(\eta_i)} \right|^2 \\
& = c \int d\xi |\xi| \int d\xi_1 d\eta \delta(\eta_1^3 + \eta_2^3 - \xi_1^3 - \xi_2^3) (|\xi_1 - \xi_2||\eta_1 - \eta_2|)^{\frac{1}{2}} \prod_{i=1}^2 \hat{u}_i(\xi_i) \overline{\hat{u}_i(\eta_i)}.
\end{align*}
$$

Now we use $\delta(g(x)) = \sum_n \frac{1}{g'(x_n)} \delta(x - x_n), \text{ where the sum is taken over all simple}$

zeros of $g$, in our case:

$$
g(x) = 3\xi(x^2 + \xi(1-x) - \xi^2)
$$

with the zeros $x_1 = \xi_1$ and $x_2 = \xi - \xi_1$, hence $g'(x_1) = 3\xi(2\xi_1 - \xi)$ respectively $g'(x_2) = 3\xi(2\xi - \xi_1)$. So the last expression is equal to

$$
\begin{align*}
& c \int d\xi |\xi| \int d\xi_1 d\eta \frac{1}{|\xi_1 - \xi_2 - \xi|} \delta(\eta_1 - \xi_1)(|\xi_1 - \xi_2||\eta_1 - \eta_2|)^{\frac{1}{2}} \prod_{i=1}^2 \hat{u}_i(\xi_i) \overline{\hat{u}_i(\eta_i)} \\
+ & c \int d\xi |\xi| \int d\xi_1 d\eta \frac{1}{|\xi_1 - \xi_2 - \xi|} \delta(\eta_1 - \xi - \xi_1)(|\xi_1 - \xi_2||\eta_1 - \eta_2|)^{\frac{1}{2}} \prod_{i=1}^2 \hat{u}_i(\xi_i) \overline{\hat{u}_i(\eta_i)} \\
& = c \int d\xi \int d\xi_1 \prod_{i=1}^2 |\hat{u}_i(\xi_i)|^2 + c \int d\xi \int d\xi_1 d\xi_2 \hat{u}_1(\xi_1) \overline{\hat{u}_1(\xi_2)} \hat{u}_2(\xi_2) \overline{\hat{u}_2(\xi_1)} \\
& \leq c \prod_{i=1}^2 ||u_i||_{L^2_t}^2 + ||\hat{u}_1\hat{u}_2||_{L^2_t}^2 \leq c \prod_{i=1}^2 ||u_i||_{L^2_t}^2.
\end{align*}
$$

Arguing as in the proof of Lemma 2.3 in [GTV97] we get the following

Corollary 1 Let $b \succ 0$. Then the following estimate holds true:

$$
||I^s f^s(u, v)||_{L^2_t} \leq c ||u||_{X_{s,b}} ||v||_{X_{s,b}}
$$

In the next Lemma, some well known Strichartz type estimates for the Airy equation are gathered in terms of $X_{s,b}$-norms:
Lemma 2 For \( b > \frac{1}{2} \) the following estimates are valid:

i) \( \|u\|_{L^p_t(H^{s,q}_x)} \leq c\|u\|_{X_{0,b}}, \) whenever \( 0 \leq s = \frac{1}{p} \leq \frac{1}{2} \) and \( \frac{1}{q} = \frac{1}{2} - \frac{2}{p}, \)

ii) \( \|u\|_{L^p_t(L^q_x)} \leq c\|u\|_{X_{0,b}}, \) whenever \( 0 < \frac{1}{q} = \frac{1}{2} - \frac{2}{p} \leq \frac{1}{2}. \)

Quotation/proof: Theorem 2.1 in [KPV91] gives in the case of the Airy-equation

\[ \|e^{-it\delta^3}u_0\|_{L^p_t(H^{s,q}_x)} \leq c\|u_0\|_{L^2_t}, \]

provided \( 0 \leq s = \frac{1}{p} \leq \frac{1}{4} \) and \( \frac{1}{q} = \frac{1}{2} - \frac{2}{p}. \) Now Lemma 2.3 in [GTV97] is applied to obtain

\[ \|u\|_{L^p_t(H^{s,q}_x)} \leq c\|u\|_{X_{0,b}}, \quad b > \frac{1}{2} \]

for the same values of \( s, p \) and \( q. \) From this ii) follows by Sobolev’s embedding theorem (in the space variable). Especially we have

\[ \|u\|_{L^p_t} \leq c\|u\|_{X_{0,b}}, \quad b > \frac{1}{2}, \]

which, interpolated with the trivial case, gives

\[ \|u\|_{L^p_t} \leq c\|u\|_{X_{0,b}}, \quad b > \frac{1}{3}. \]

Now let us see how to replace \( \dot{H}^{s,q}_x \) by \( H^{s,q}_x \) in ii) in the endpoint case, i. e. \( s = \frac{1}{p} = \frac{1}{4}, \) \( q = \infty: \) Using the projections \( p = \mathcal{F}^{\infty}X_{(\xi|\leq 1)}\mathcal{F} \) and \( P = \text{Id} - p \) we have

\[ \|u\|_{L^p_t(H^{s,q}_x)} \leq \|Pu\|_{L^p_t(H^{s,q}_x)} + \|pu\|_{L^p_t(H^{s,q}_x)} =: I + II. \]

For \( I \) we use (2) to obtain

\[ I \leq c\|Pu\|_{L^p_t[H^{s,q}_{2+\frac{1}{2}}]} \leq c\|u\|_{X_{0,b}}, \]

while for \( II \) by Sobolev’s embedding theorem we get

\[ II \leq c\|pu\|_{L^p_t[H^{s,q}_{2+\frac{1}{2}}]} \leq c\|pu\|_{X_{2+\frac{1}{2}, b}} \leq c\|u\|_{X_{0,b}}. \]

This gives i) in the endpoint case, from which the general case follows by interpolation with Sobolev’s embedding theorem (in the time variable).

Remark: The endpoint case in ii) is also valid - see e. g. Lemma 3.29 in [KPV93] - but we shall not make use of this here.

Now we are prepared to prove the crucial nonlinear estimate:

Theorem 1 For \( 0 \geq s > -\frac{1}{9}, -\frac{1}{2} < b' < s - \frac{1}{9} \) and \( b > \frac{1}{2} \) the estimate (1) is valid.

Proof: Writing \( f_i(\xi, \tau) = <\tau - \xi^3 >^b' <\xi >^s F u_i(\xi, \tau), \) \( 1 \leq i \leq 4, \) we have

\[ \|\partial_\xi \prod_{i=1}^4 u_i\|_{X_{s,b'}} \leq c\|\tau - \xi^3 >^b' <\xi >^s |f| \|d\nu \prod_{i=1}^4 <\tau - \xi^3 >^b' <\xi >^s f_i(\xi, \tau)\|_{L^2_{\xi, \tau}}, \]

where \( d\nu = d\xi_1 . . d\xi_4 d\tau_1 . . d\tau_4 \) and \( \sum_{i=1}^4 (\xi_i, \tau_i) = (\xi, \tau). \) Now the domain of integration is devideed into the regions \( A, B \) and \( C = (A \cup B)^c, \) where in \( A \) we assume \( |\xi_{\max}| \leq c. \) (Here \( \xi_{\max} \) is defined by \( |\xi_{\max}| = \max_{i=1}^4 |\xi_i|, \) similarly \( \xi_{\min}. \) ) Then for the region \( A \) we have the upper bound

\[ c\|f| \|d\nu \prod_{i=1}^4 <\tau - \xi^3 >^b' f_i(\xi, \tau)\|_{L^2_{\xi, \tau}} = c\|\prod_{i=1}^4 J^s u_i\|_{L^2_{s, s^s}} \leq c\prod_{i=1}^4 \|J^s u_i\|_{L^2_{s, s^s}} \leq c\prod_{i=1}^4 \|u_i\|_{X_{0, b}}, \]

where in the last step Lemma [2], part ii), with \( p = q = 8 \) was applied.

Besides \( |\xi_{\max}| \geq c (\Rightarrow <\xi_{\max} > \leq c|\xi_{\max}|) \) we shall assume for the region \( B \) that
Then the region $B$ can be splitted again into a finite number of subregions, so that for any of these subregions there exists a permutation $\pi$ of $\{1, 2, 3, 4\}$ with $\xi_i > 0$.

Then we have

$$|\xi| < \xi >^s \prod_{i=1}^{4} < \xi_i >^{-s} \leq c|\xi(1) + \xi(2)|^\frac{4}{2} |\xi(1) - \xi(2)|^\frac{4}{2} < \xi(3) >^{-\frac{4}{2}} < \xi(4) >^{-\frac{4}{2}}.$$ 

Assume $\pi = id$ for the sake of simplicity now. Then we get the upper bound

$$\|<\tau - \xi^3>^b f(\xi_1 + \xi_2)\|_{L^q_0(L^2_x)} \leq c \|f(\xi_1, \xi_2)\|_{L^q_{\tau,\xi}},$$

To estimate the latter expression, we fix some Sobolev- and Hölder-exponents:

1. $\frac{1}{q_0} = \frac{1}{b} - b'$ so that $L^{q_0}(L^2_x) \subset X_{0,b'},$
2. $\frac{1}{p} = \frac{1}{q_0} - \frac{1}{b} = -b'$,
3. $\frac{1}{q} = \frac{1}{b} - \frac{2}{p} = \frac{1}{b} + b'$ so that by Lemma $\|J^\frac{2}{p} f\|_{L^q_0(L^2_x)} \leq c \|f\|_{X_{0,b'}}$,
4. $\epsilon = \frac{1}{p} + \frac{2}{q} > \frac{1}{q}$ (since $s > \frac{1}{3} + b'$) so that $H^{\epsilon,\alpha}_2 \subset L^\infty.$

Now we have

$$\|<\xi>^b f(\xi_1 + \xi_2)\|_{L^q_0(L^2_x)} \leq c \|f(\xi_1, \xi_2)\|_{L^q_{\tau,\xi}},$$

Now by Lemma $\|J^\frac{2}{p} f\|_{L^q_0(L^2_x)} \leq c \|f\|_{X_{0,b'}}$.

The third factor can be treated in precisely the same way. So for the contributions of the region $B$ we have obtained the desired bound.

Finally we consider the remaining region $C$: Here the $|\xi_i|$, $1 \leq i \leq 4$, are all very close together and $\geq c < \xi_i >$. Moreover, at least three of the variables $\xi_i$ have the same sign. Thus for the quantity $c.q.$ controlled by the expressions $<\tau - \xi^3>$, $<\tau_i - \xi^3_i >, 1 \leq i \leq 4$, we have in this region:

$$c.q. := |\xi^3 - \sum_{i=1}^{4} \xi^3| \geq c \sum_{i=1}^{4} < \xi_i >^3 \geq c < \xi >^3$$

and hence, since $s > \frac{1}{b} + b'$ is assumed,

$$|\xi| < \xi >^s \prod_{i=1}^{4} < \xi_i >^{-s} \leq c <\tau - \xi^3 >^{-b'} + \sum_{i=1}^{4} < \tau_i - \xi^3_i >^{-b'} \chi C_i,$$
where in the subregion $C_i$, $1 \leq i \leq 4$, the expression $< \tau_i - \xi^3_i >$ is dominant. The first contribution can be estimated by

$$c \| \int d\nu \prod_{i=1}^4 <\tau_i - \xi^3_i>^{-b} f_i(\xi_i, \tau_i) \|_{L^2_{\xi,\tau}}$$

$$= c \| \prod_{i=1}^4 J^s u_i \|_{L^2_{x,t}} \leq c \prod_{i=1}^4 \| J^s u_i \|_{L^8_{x,t}} \leq c \prod_{i=1}^4 \| u_i \|_{X_{s,b}},$$

where we have used Lemma 2, part ii). For the contribution of the subregion $C_1$ we take into account that $< \tau_1 - \xi^3_1 > = \max \{ < \tau - \xi^3 >, < \tau_i - \xi^3_i >, 1 \leq i \leq 4 \}$, which gives

$$< \tau - \xi^3 >^{b+b'} |\xi| < \xi >^s \prod_{i=1}^4 < \xi_i >^{-s} \leq c < \tau_1 - \xi^3_1 >^b.$$

So, for this contribution we get the upper bound

$$c \| < \tau - \xi^3 >^{-b} \int d\nu < \tau_1 - \xi^3_1 >^b \prod_{i=1}^4 <\tau_i - \xi^3_i>^{-b} f_i(\xi_i, \tau_i) \|_{L^2_{\xi,\tau}}$$

$$\leq c \| \mathcal{F}^{-1} f_1 \prod_{i=2}^4 J^s u_i \|_{X_{0,-b}} \leq c \| \mathcal{F}^{-1} f_1 \prod_{i=2}^4 J^s u_i \|_{L^2_{x,t}}$$

$$\leq c \| \mathcal{F}^{-1} f_1 \|_{L^2_{x,t}} \prod_{i=2}^4 \| J^s u_i \|_{L^8_{x,t}} \leq c \prod_{i=1}^4 \| u_i \|_{X_{s,b}}.$$

Here we have used the dual version of the $L^8$-Strichartz estimate, Hölder and the estimate itself. For the remaining subregions $C_i$ the same argument applies. \qed

**Corollary 2** For $s \geq 0$, $-\frac{1}{2} < b' < -\frac{1}{3}$ and $b > \frac{1}{2}$ the estimate (4) holds true.

**Proof:** For $s = 0$ this is contained in the above theorem, while for $s > 0$ one only has to use $< \xi > \leq c \prod_{i=1}^4 < \xi_i >$. \qed

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