Recycling the nonlocality of a qubit pair using only projective measurements

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Unsharp measurements are widely seen as the key resource for recycling the nonlocality of an entangled state shared between several sequential observers. Contrasting this, we investigate the recycling of nonlocality using only projective measurements and classical shared randomness. We show that one share of a maximally entangled qubit pair can be recycled, with such resources, for at least three sequential violations of the CHSH inequality. For two sequential violations, using the same state, we analytically obtain the optimal trade-off in the CHSH parameters. Furthermore, we show that non-maximally entangled qubit states enable larger sequential violations than the maximally entangled state. Our results establish projective measurements as a simple and useful resource for recycling quantum nonlocality.

Introduction.— In the simplest, Clauser-Horne-Shimony-Holt (CHSH) [1] Bell experiment, two parties perform local measurements on a shared pair of entangled qubits. Regardless of which state they share, the largest violation of the CHSH inequality is obtained using standard measurements, corresponding to basis projections. Because each qubit is only measured once, the parties do not need to consider the fact that such projections render the post-measurement state separable.

Beginning with the work of Silva et al. [2], there has in recent years been much interest in whether the post-measurement state in a Bell experiment can be re-used for showcasing nonlocality between several observers who perform sequential measurements. This has been explored in both theory [3–14] and experiment [15–19], and it has inspired similar investigations of other quantum correlation tasks (see e.g. [20–27]). In such scenarios, a two-qubit state is shared between measuring parties A and B\textsubscript{1}. The post-measurement state of B\textsubscript{1} is relayed to another, independent, measuring party B\textsubscript{2}. The relay process is continued until the qubit is measured by the final sequential party B\textsubscript{n} (see Figure 1). The aim is for every pair A – B\textsubscript{k} (for k = 1, . . . , n) to violate a Bell inequality. To this end, each sequential party (except B\textsubscript{n}) must perform a compromise measurement: it must produce strong enough correlations with A to elude local models, but still preserve enough of the entanglement between the qubits to enable the next party to do the same.

In such recycling protocols, the standard procedure is to let the measurement apparatus interact weakly with the incoming qubit [2]. It corresponds to an unsharp measurement, which can be realised by Lüders type quantum instruments [28]. By tuning B\textsubscript{1}’s sharpness, one obtains a trade-off between the magnitude of the Bell parameter observed between A – B\textsubscript{k} and the amount of nonlocality left for the remaining pairs A – B\textsubscript{j}, for j > k, to consume. Unsharp measurements performed on one share of a maximally entangled state can enable arbitrary many sequential violations of the CHSH inequality [10].

Here, we depart from unsharp measurements and consider the exclusive use of projective measurements for recycling violations of the CHSH inequality between independent parties measuring one share of a two-qubit state. At first sight, this may appear as a futile endeavour: a CHSH test, in which each party has two measurements, offers only three different classes of projective qubit measurement strategies. (i) Both measurements are basis projections. This renders the state separable, preventing a second violation. (ii) Both measurements are trivial, each corresponding to identity measurements, preventing a first violation. (iii) One measurement is a basis projection and the other is trivial: since one input is effectively discarded, it prevents a first violation. Our approach is to overcome the apparent inabilities of projective measurement strategies by leveraging shared classical randomness as a free resource in Bell inequality tests. By stochastically combining different, individually unsuccessful, projective measurement strategies, we show that at least three sequential violations of the CHSH inequality are possible. For two sequential violations, we analytically characterise the optimal trade-off between the two CHSH parameters, as obtained from a maximally entangled state. Then, we focus on pure, partially entangled, two-qubit states, and show that some pairs of CHSH parameters, that cannot be obtained with the maximally entangled state, are realisable deterministically with such weaker entanglement. Moreover, we also find that by actively exploiting shared randomness, suitably chosen partially entangled states allow for larger double violations of the CHSH inequality. We conclude

\begin{figure}
\centering
\includegraphics[width=\textwidth]{fig1}
\caption{A source distributes a two-qubit state $\psi$ to measuring parties A and B\textsubscript{1}. The qubit of B\textsubscript{1} is sequentially relayed, measured, relayed etc, until measured by a final party B\textsubscript{n}. Before the experiment begins, all parties may agree on sharing strings of classically correlated data $\Lambda$.}
\end{figure}
with some open problems on the use of projective measurements to recycle quantum nonlocality.

**Sequential scenario.**—The sequential Bell scenario is illustrated in Figure 1. A two-qubit state ψ is shared between parties A and B. They each privately select one of two inputs, denoted x ∈ {0,1} and y₁ ∈ {0,1} respectively, and perform a corresponding quantum measurement. The outcomes are denoted a ∈ {0,1} and b₁ ∈ {0,1}. Party B₁ then relays the post-measurement qubit to party B₂ who similarly selects y₂ ∈ {0,1}, performs an associated measurement, outputs b₂ ∈ {0,1} and relays the post-measurement qubit. This process continues until the final sequential party, Bₙ, selects yₙ ∈ {0,1} and outputs bₙ ∈ {0,1}. Before the experiment begins, the parties may also agree to share correlated strings of classical data, λ, subject to some probability distribution \{p(λ)\}. All parties select their inputs without bias and the sequential parties act independently: Bₖ relays no classical information about (yₖ, bₖ) to Bₖ₊₁.

Each pair A - Bₖ tests the CHSH inequality,

\[ Sₖ ≡ \sum_λ p(λ) S^{(λ)}_k ≤ 2, \tag{1} \]

where \( S^{(λ)}_k ≡ \sum_{x,yₖ} (−1)^{yₖ} \langle A^{(λ)}_x, B^{(λ)}_{yₖ} \rangle \psi^{(λ)} \). Here, \( \{A^{(λ)}_x, B^{(λ)}_{yₖ}\} \) denote the observables of the respective parties conditioned on λ and the expectation value is evaluated with respect to the average recycled state \( ψ^{(λ)}_k \) (with \( ψ^{(λ)}_1 = ψ \)). This state is recursively given by

\[ ψ^{(λ)}_{k+1} = \frac{1}{2} \sum_{yₖ,bₖ} \left( 1 \otimes \sqrt{K^{(λ)}_{bₖ|yₖ}} \right) ψ^{(λ)}_k \left( 1 \otimes \sqrt{K^{(λ)}_{bₖ|yₖ}} \right) \dagger, \tag{2} \]

where \( \{K^{(λ)}_{bₖ|yₖ}\} \) are the Kraus operators representing the quantum instrument used by Bₖ to realise the measurement \( B^{(λ)}_{bₖ|yₖ} = \left( K^{(λ)}_{bₖ|yₖ} \right) \dagger K^{(λ)}_{bₖ|yₖ} \) when advised by λ.

We are interested in quantum protocols based on projective qubit measurements. Thus, \( B^{(λ)}_{bₖ|yₖ} = \frac{1}{2} \left( B^{(λ)}_{bₖ|yₖ} \right) \dagger \). The Kraus operators then take the form \( K^{(λ)}_{bₖ|yₖ} = U^{(λ)}_{bₖ|yₖ} B^{(λ)}_{bₖ|yₖ} \) where \( U^{(λ)}_{bₖ|yₖ} \) are arbitrary unitary operators. This means that a party first measures projectively and then performs a unitary based on the input and output. Notice that for qubits, all projective measurements are either basis measurements, i.e. they correspond to measuring in the direction of some Bloch vector \((|Φ⟩, |−Φ⟩)\), or trivial identity projections, for which the outcome is independent of the state \((|0⟩, |0⟩)\). When performed on one share of an entangled qubit pair, the former renders the post-measurement state separable while the latter leaves it unchanged.

For the simplest scenario, namely \( n = 2 \), which is our main focus, we simplify the notations by calling the parties \{A, B, C\}, their inputs \( \{x,y,z\} \), their outcomes \( \{a,b,c\} \), their observables \( \{A^{(λ)}_x, B^{(λ)}_{yₖ}, C^{(λ)}_{zₖ}\} \) and B’s unitaries \( U^{(λ)}_{bₖ|yₖ} \).

**Simple double violation.**—The conceptually central question is whether projective measurements can at all be used to recycle violations of the CHSH inequality. We now show, through a simple example, that such is the case. Let the state be maximally entangled, \( |ψ⟩ = |Φ⁺⟩ \), where \( |Φ⁺⟩ \equiv \frac{1}{\sqrt{2}} \left( |00⟩ ± |11⟩ \right) \), and let the parties share one bit of randomness \( λ ∈ \{1,2\} \).

When \( λ = 1 \), the pair A - B runs the standard quantum strategy for attaining the Tsirelson bound [29]. Thus, \( B \) measures \( σ_X \) and \( σ_Z \), and A measures diagonally, yielding \( S^{(1)}_1 = 2\sqrt{2} \). Here \( \langle σ_X, σ_Y, σ_Z \rangle \) denote the standard Pauli matrices. We ignore the unitaries \( U^{(1)}_{bₖ|yₖ} = I \). Let C also measure \( σ_X \) and \( σ_Z \). This gives \( S^{(1)}_2 = \sqrt{2} \).

When \( λ = 2 \), we set B’s first observable to \( B₀ = I \) and choose the second observable as a basis projection. Since the local state is \( \frac{I}{2} \), we can w.l.o.g. choose \( B₁ = σ_Z \). Again ignoring the unitaries \( U^{(2)}_{bₖ|yₖ} = I \), the state of A - C becomes \( \frac{3}{4} φ^+ + \frac{1}{4} φ^- \). Via the Horodecki criterion [30], one finds that the state enables at most the CHSH value \( S^{(2)}_2 = \sqrt{5} \). The measurements that achieve this are \( \frac{1}{\sqrt{2}} σ_X ± \frac{1}{\sqrt{2}} σ_Z \) for A, and \( σ_X \) and \( σ_Z \) for C. These choices also imply \( S^{(2)}_1 = \frac{4}{\sqrt{5}} \).

Now, we use the shared randomness to mix the two projective measurement strategies together. Notice also that λ only is used to correlate A - B, since C performs the same measurements in both strategies. The final pair of CHSH parameters \( S^{(1)} \) becomes

\[ S₁ = qS^{(1)}_1 + (1 - q)S^{(2)}_1, \tag{3} \]
\[ S₂ = qS^{(2)}_2 + (1 - q)S^{(1)}_2, \tag{4} \]

where \( q = p(λ = 1) \) is the degree of mixture between the two strategies. If, for instance, we insist on both CHSH parameters being equal, we can immediately determine \( q \) and obtain the double violation \( S₁ = S₂ = \frac{6\sqrt{10}}{5\sqrt{2} + \sqrt{5}} ≈ 2.04 \). However, as we see next, a more sophisticated protocol can achieve larger double violations.

**Optimal trade-off for maximally entangled state.**—We focus on the simplest scenario, namely that featuring \{A, B, C\}, when the shared state is maximally entangled, and set out to determine the set of possible pair of CHSH parameters, \( (S₁, S₂) \) reachable under projective measurements. To this end, we must determine the optimal trade-off, i.e. the largest possible value of \( S₂ \) given a value of \( S₁ \).

Since A’s qubit is only measured once, the optimal measurements are rank-1 projective. These can w.l.o.g. be written as the observables \( A^{(λ)}_x = \cos θ^{(λ)} σ_X + (-1)^x \sin θ^{(λ)} σ_Z \). Then, using that \( O \otimes I |φ^+⟩ = I \otimes O^T |φ^+⟩ \), we can for given λ write the first CHSH parameter as

\[ S^{(λ)}_1 = \cos θ^{(λ)} Tr \left( σ_X B^{(λ)}_0 + I \otimes O^T |φ^+⟩ \right) \left( σ_Z B^{(λ)}_1 \right) \tag{5} \]

Using the state update rule (2) for projective measurements, and the completeness condition \( C_{1|z} = I - C_{0|z} \), we write the
1.0 2.5
1.0
1.5
2.0
2.5
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1.85 1.90 1.95
2.00
2.16
2.18
2.20
2.22
2.24
2.0 2.1 2.2 2.3
2.0
2.1
2.2
S1
S2
DET
SR

FIG. 2: Optimal trade-off between $S_2$ and $S_1$ for a maximally entangled qubit pair under projective measurements and shared randomness. The trade-off consists of four parts represented by red, orange, green and blue solid curves respectively. The first and third are stochastic (boundary points marked) while the second and fourth are deterministic. The blue and orange dashed curves are the optimal trade-offs for deterministic projective strategies of type (i) and (iii) respectively. The lower (upper) inset illustrates, in solid black, a deterministic (stochastic) model based on partially entangled states that outperforms the maximally entangled state. The dashed black lines are the classical and quantum bounds of the CHSH inequality.

We now examine separately the three types of projective measurement strategies, (i-iii), that may be employed by $B$.

**Case (i):** ($\lambda = 1$). Both measurements of $B_1$ are rank-1 projective. Let $A$ measure diagonally, by choosing $\theta = \frac{\pi}{4}$. Choose $B$’s observables as $B_0 = \cos \phi \sigma_X + \sin \phi \sigma_Z$ and $B_1 = \sin \phi \sigma_X + \cos \phi \sigma_Z$. Then, we let the unitaries only depend on the input, i.e., $U_{by} = U_y$ and choose $U_0 = \mathbb{1}$ and $U_1 = e^{i(\phi - \frac{\pi}{4})}\sigma_Y$. Finally, we choose $C_0 = C_1 = \cos \phi \sigma_X + \sin \phi \sigma_Z$. We obtain $S_1^{(1)} = 2\sqrt{2}\cos \phi$ and $S_2^{(1)} = \sqrt{2}(\cos \phi + \sin \phi)$. The trade-off becomes

$$S_2^{(1)} = \frac{1}{2} \left( S_1^{(1)} + \sqrt{8 - \left( S_1^{(1)} \right)^2} \right),$$

which we use in the range $2 \leq S_1^{(1)} \leq 2\sqrt{2}$. In the remaining range, $0 \leq S_1^{(1)} \leq 2$, one can straightforwardly choose observables to achieve the trivial, classical, trade-off $S_2^{(1)} = 2$. In Appendix A1, we prove that there exists no strategy based on rank-one projective measurements that outperforms the above.

**Case (ii):** ($\lambda = 2$). Both measurements of $B$ are identity projections. This trivialises the trade-off. We write $B_0 = B_1 = \mathbb{1}$. Rank-1 projective measurements for $A$ implies $S_1^{(2)} = 0$. Optimally choosing $U_{by} = \mathbb{1}$, the post-measurement state is left maximally entangled and hence we optimally choose $C_0 = \sigma_X$ and $C_1 = \sigma_Z$ and let $A$ measure diagonally to reach the Tsirelson bound $S_2^{(2)} = 2\sqrt{2}$.

**Case (iii):** ($\lambda = 3$). One measurement of $B$ is rank-1 projective and the other is an identity projection. We take setting $y = 1$ as the former and outcome $b = 0$ of setting $y = 0$ to correspond to the projector $\mathbb{1}$. This comes at no loss of generality because the CHSH parameter is invariant under the coordinated permutations $\{y \to \bar{y}, x \to \bar{x} : \bar{a} \to \bar{a}\}$ respectively, where the bar-sign denotes bitflip. Thus, we have $B_0 = \mathbb{1}$. If $B_1 = \sigma_Z$, $U_{by} = \mathbb{1}$, $C_0 = \sigma_X$ and $C_1 = \sigma_Z$. It yields $S_1^{(3)} = 2\sin \theta$ and $S_2^{(3)} = \cos \theta + 2\sin \theta$. The trade-off becomes

$$S_2^{(3)} = S_1^{(3)} + \frac{1}{2} \sqrt{4 - \left( S_1^{(3)} \right)^2},$$

which we use in the range $\frac{3}{\sqrt{2}} \leq S_1^{(3)} \leq 2$. Note that $S_1^{(3)} > 2$ is not possible due to the trivialisation of setting $y = 0$. The curve (8) has its maximum, $S_2^{(3)} = \sqrt{5}$, at $S_1^{(3)} = \frac{4}{\sqrt{5}}$. Therefore, in the range $0 \leq S_1^{(3)} \leq \frac{4}{\sqrt{5}}$ one can easily find the trivial trade-off $S_2^{(3)} = \sqrt{5}$. In Appendix A3, we prove that the given strategy is optimal for the given ranks of $B$.

**Mixing strategies:** We can now, via Eq. (1), use the distribution of the shared randomness $\{p(\lambda)\}_{\lambda=1}^5$ to stochastically combine the three above cases to obtain the boundary of the region of attainable pairs $(S_1, S_2)$. This analysis is detailed in Appendix A4. The boundary ends up being divided into four regions: a mixture between case (ii)&(iii), deterministic case (iii), a mixture between case (i)&(iii) and deterministic case (i). Specifically, it is given by

$$S_2 = \begin{cases} 
(1 - \frac{7}{2})S_1 + 2\sqrt{2} & \text{if } 0 \leq S_1 \leq \frac{\sqrt{2}}{2}, \\
S_1 + \frac{1}{2}\sqrt{4 - (S_1)^2} & \text{if } \sqrt{\frac{7}{2}} \leq S_1 \leq 3 \sqrt{\frac{2}{7}}, \\
\sqrt{\frac{10}{9}} - \frac{S_1}{2} & \text{if } 3 \sqrt{\frac{2}{7}} \leq S_1 \leq 4 \sqrt{\frac{2}{5}}, \\
\frac{S_1}{2} + \frac{1}{2}\sqrt{8 - (S_1)^2} & \text{if } 4 \sqrt{\frac{2}{5}} \leq S_1 \leq 2\sqrt{2}.
\end{cases}$$

(9)

This optimal trade-off is piecewise illustrated with solid lines in Figure 2. Notice that the region of double violation corresponds to the third line of (9) and that the largest achievable identical double violation is $S_1 = S_2 = \frac{2\sqrt{10}}{3} \approx 2.108$. As expected, this is weaker than in the case of unsharp measurements, where one can have $S_1 = S_2 = \frac{8\sqrt{2}}{3} \approx 2.263$ [2].

**Pure partially entangled states.**—Now we extend the discussion beyond the maximally entangled state and consider pure partially entangled states, i.e. states of the form $|\psi_\varphi\rangle = \cos \varphi |00\rangle + \sin \varphi |11\rangle$, for some angle $\varphi \in [0, \frac{\pi}{4}]$. Does there exist pairs of CHSH parameters $(S_1, S_2)$ that cannot be attained with the maximally entangled state, i.e. they lie outside the boundary described in (9), but are reachable with a projective quantum strategy based on a suitable choice of $\varphi$? We answer this in the affirmative.
To this end, we first show how one can go beyond the boundary (9) without using shared randomness. In fact, as noted in Appendix B, trivial examples are possible based on type (ii) strategies. Here, we show the more interesting case, namely that deterministic type (iii) strategies are sufficient. Party A measures $A_x = (-1)^{x} \cos \theta \sigma_x + \sin \theta \sigma_z$ while party B measures $B_0 = \mathbb{1}$ and $B_1 = \sigma_x$, ignoring any subsequent unitary ($U_{B_0} = \mathbb{1}$). Party C measures $C_0 = \sigma_y$ and $C_1 = \sigma_y$. A direct calculation gives $S_1 = 2 \sin (\theta + 2 \varphi)$ and $S_2 = \sin \theta + 2 \cos \theta \sin (2 \varphi)$. The former implies $\theta = \pi - 2 \varphi - \arcsin (\frac{2\varphi}{\xi})$, which for the latter is $S_2 = \sin (2 \varphi) \sqrt{1 - \left(\frac{S_1}{2}\right)^2 (1 - 2 \cos (2 \varphi)) + \frac{S_1}{2} (2 \sin^2 (2 \varphi) + \cos (2 \varphi))}. \quad (10)$

The optimal choice of $\varphi$ is found to be

$$\varphi = \arccos \left(\frac{1}{\sqrt{4 \sqrt{g(S_1)} + \sqrt{33 - g(S_1) + \frac{8(S_1)^2}{\sqrt{g(S_1)}}}}}, \quad (11)\right)$$

where $g(x) = 11 + h(x) + (121 - 242^x)/h(x)$ and $h(x) = \sqrt{8x^2 - 396x^2 + 8x^2 \sqrt{x^4 + 117x^2 - 484 + 1331^1/3}}$. This strategy is illustrated in the lower inset in Figure 2 where we see that it outperforms the maximally entangled state in the range $1.84 \leq S_1 \leq 1.99$. As $S_1$ increases, the entanglement becomes weaker, reaching about $\varphi \approx 0.686$ at the upper demarcation ($S_1 \approx 1.99$). We have also numerically investigated deterministic, type (iii), strategies in this range but only found a tiny improvement in $S_2$ as compared to the above analytical construction: the improvement increases with $S_1$ and was at most found to be roughly of size $2 \times 10^{-3}$. Since a strategy of this type cannot produce a double violation, we proceed with investigating whether shared randomness can be used to achieve stronger double violations than those based on the maximally entangled state. For this, we use a numerical search, considering mixtures of type (i) and type (iii) strategies. The results are presented in the upper inset in Figure 2. We see that partially entangled states indeed enable stronger double violations.

Finally, in Appendix B, we numerically explore the boundary of the set of pairs $(S_1, S_2)$ reachable with projective measurements at a fixed value of $\varphi$. There, we also show that by mixing type (i) and type (ii) strategies, one can obtain a double violation of the CHSH inequality for every entangled $|\psi_\varphi\rangle$. This case we now go beyond the case of two sequential parties and show that the maximally entangled state $|\psi_+\rangle$ enables also a triple violation of the CHSH inequality. To show this, let all parties share a trit of randomness $\lambda \in \{1, 2, 3\}$. For each $\lambda$, we tailor a quantum strategy such that the pair $A - B_{\lambda}$ violates the CHSH inequality while the other two pairs only nearly fail to achieve a violation.

For $\lambda = 1$, let $A$ measure $A_x = \sigma_x + (-1)^{\lambda} \sigma_z$, and choose $B_0^1 = \cos \sigma_x \phi + \sin \phi \sigma_z$ and $B_1^1 = \sin \phi \sigma_x + \cos \phi \sigma_z$ followed by unitaries $U_{B_{\lambda}0} = U_{B_{\lambda}1} = \mathbb{1}$, with $U_{B_{\lambda}0} = \mathbb{1}$ and $U_{B_{\lambda}1} = e^{i(\phi - \frac{\pi}{4}) \sigma_y}$. The remaining two sequential parties perform no unitaries ($U = \mathbb{1}$) and identical measurements: $B_{\lambda}^0 = B_{\lambda}^1 = \cos \phi \sigma_x + \sin \phi \sigma_z$ independently of $\lambda$. This leads to $S_1^{(1)} = 2 \sqrt{2} \cos \phi$ and $S_1^{(2)} = S_1^{(3)} = \sqrt{2} (\cos \phi + \sin \phi)$. For $\lambda = 2$, no party performs a unitary ($U = \mathbb{1}$). Choose $A_x = \cos \phi \sigma_x + (-1)^{\lambda} \sin \phi \sigma_z$, $B_0^1 = \mathbb{1}$ and $B_1^1 = \sigma_x$. $B_2$ and $B_3$ perform identical measurements $B_2^0 = B_2^1 = \sigma_x$ and $B_3^0 = B_3^1 = \sigma_z$. This leads to $S_1^{(2)} = 2 \sin \phi$, $S_1^{(2)} = \cos \phi + 2 \sin \phi$ and $S_1^{(3)} = \cos \phi + 4 \sin \phi$.

Now we obtain the final three CHSH parameters by mixing the three strategies. Following (1), we have for $k = 1, 2, 3$, $S_k = q_1 S_k^{(1)} + q_2 S_k^{(2)} + q_3 S_k^{(3)}$. One can find many choices of angles $(\phi, \psi, \phi, \psi)$ and probabilities $q_\lambda = p(\lambda)$ such that $S_k > 2$ for all three $k$. For example, insisting that $S_1 = S_2 = S_3 = S$ fixes $(q_1)$ and one obtains $S = S(\phi, \psi, \phi, \psi$). Choosing $(\phi, \psi, \phi, \psi) = (\frac{1}{129}, 0.088, 0.088, 0.088)$, one obtains the triple violation $S \approx 2.00227$.

**Discussion.**—Contrasting the common wisdom, we have here shown that projective measurements are a resource for recycling quantum nonlocality. We have shown this for the most relevant case, namely that in which observers are independent, perform unbiased measurements and share a two-qubit entangled state. Given that such measurements also have been found useful for recycling quantum communication advantages [25], it appears plausible that many other sequential quantum information protocols, e.g. for entanglement witnessing [20], steering [21] and contextuality [23], also can be based on projective measurements.

The relevance of projective measurements is not only interesting from a conceptual point of view. It also comes with practical advantages. L"uders quantum instruments, that realise objective measurements. The relevance of projective measurements is not only inter-...
We found three violations with the maximally entangled state but did not find a fourth. (2) We found that partially entangled states can produce pairs of CHSH parameters that go beyond those reachable with the maximally entangled state. What is the optimal trade-off between the two CHSH parameters for a given partially entangled state $|\psi^+\rangle$? In particular, what is the optimal trade-off when evaluated over all two-qubit entangled states? (3) There is considerable evidence [12, 13] that double violations of the CHSH inequality, under general measurements, are not possible when both qubits in the entangled pair are recycled. Could the introduction of shared randomness be interesting for this problem?

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Appendix A: Optimal trade-off for the maximally entangled state under projective measurements

Consider that $A$ and $B$ share the maximally entangled state $|\psi\rangle = |\phi^+\rangle = \frac{1}{\sqrt{2}} (|00\rangle + |11\rangle)$. The optimal measurements of $A$ are rank-1 projective (these are extremal), because the qubit is not recycled afterwards. This amounts to standard observables $A_0 = \hat{a}_0 \cdot \sigma$ and $A_1 = \hat{a}_1 \cdot \sigma$ with $|\hat{a}_x| = 1$. W.l.g, we can take $\hat{a}_0 = (\cos \theta, 0, \sin \theta)$ and $\hat{a}_1 = (\cos \theta, 0, -\sin \theta)$ because any global rotation of $\hat{a}_0$ and $\hat{a}_1$ can be absorbed into a global rotation of $B$’s measurements via the relation $O \otimes |\phi^+\rangle = |\phi^+\rangle \otimes O^T |\phi^+\rangle$. Thus, the unnormalised states remotely prepared by $A$ on $B$’s side correspond to the eigenvectors of the observables, namely $\rho_{a|x} = \frac{1}{4} (\mathbb{1} + (-1)^y \hat{a}_x \cdot \sigma)$. Here, $p(a|x) = Tr(\rho_{a|x}) = \frac{1}{2}$ which follows from the fact that $A$’s measurement operators are trace-one and the local state is maximally mixed. We can then define $\rho_x = \rho_{0|z} - \rho_{1|z} = \frac{\hat{a}_x \cdot \sigma}{2}$. The CHSH parameter between $A$ and $B$ now reads

$$S_1 = \sum_{x,y=0,1} (-1)^{x+y} Tr(\rho_x B_y),$$

where $B_y$ is $B$’s observable.

Every quantum instrument that realises a measurement $\{B_{0|y}, B_{1|y}\}$ can be represented as a square-root instrument followed by an arbitrary CPTP map [10]. The square-root instrument performs on the input state the transformation $\rho \rightarrow \sqrt{B_{0|y}} \theta \sqrt{B_{0|y}}$. Since extremal CPTP maps taking a qubit to a qubit are unitary, we can write the Kraus operators of the quantum instrument as $K_{0|y} = U_{by} \sqrt{B_{0|y}}$. Hence, state shared between $A$ and $C$ after $B$’s measurement reads

$$\psi_2 = \frac{1}{2} \sum_{y,b} \left( \mathbb{1} \otimes U_{by} \sqrt{B_{0|y}} \right) \psi \left( \mathbb{1} \otimes \sqrt{B_{0|y}} U_{by} \right).$$

Similarly, the CHSH parameter between $A$ and $C$ becomes

$$S_2 = \frac{1}{2} \sum_{x,z=0,1} (-1)^{x+z} \sum_{y,b} Tr \left( B_{by} \sqrt{B_{by} \rho_x \sqrt{B_{by} U_{by}^\dagger C_{0|z}}} \right)$$

$$= \cos \theta \sum_{y,b} Tr \left( \sqrt{B_{by} \sigma_X} \sqrt{B_{by} U_{by}^\dagger C_{0|1} U_{by}} \right) + \sin \theta \sum_{y,b} Tr \left( \sqrt{B_{by} \sigma_Z} \sqrt{B_{by} U_{by}^\dagger C_{0|1} U_{by}} \right),$$

where we for simplicity have used normalisation to make the substitution $C_z = 2C_{0|z} - \mathbb{1}$ and used the cyclicity of the trace. We can without loss of generality restrict $C$’s measurements to be rank-1 projective.

We now proceed to examine the trade-off between $S_1$ and $S_2$ when $B$’s measurements are restricted to being projective, i.e. $B_{by} B_{by}^\dagger = \delta_{b,b} B_{by} B_{by}^\dagger$, causing the relevant Kraus operators take the form $K_{by} = U_{by} B_{by}$. Thus, the second CHSH parameter simplifies into

$$S_2 = \cos \theta \sum_{y,b} Tr \left( B_{by} \sigma_X B_{by} U_{by}^\dagger C_{0|0} U_{by} \right) + \sin \theta \sum_{y,b} Tr \left( B_{by} \sigma_Z B_{by} U_{by}^\dagger C_{0|1} U_{by} \right).$$

Classifying the projectors by their rank, there are only three qualitatively different cases.

(i) Both $B$’s measurements are projections onto two orthogonal vectors (standard qubit measurement).

(ii) Both $B$’s measurements are projections onto the identity and the zero projector respectively (trivial measurement).

(iii) One of $B$’s measurements is a projection onto two orthogonal vectors and the other is trivial.

In what follows, we examine these three cases one by one.

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1. Case (i): projection onto a basis

We now characterise the optimal trade-off between $S_1$ and $S_2$ when $B$ is restricted to performing rank-one projective measurements. Note that such strategies enable $S_1 > 2$ but not $S_2 > 2$ because the instrument of $B$ is entanglement breaking.

Define the rank-one projectors $P_{z,y} = U_{z,y}^* C_{z,y} U_{z,y}$. We can place an upper bound on $S_2$ by assuming that $P_{z,y}$ can be aligned with the eigenvector of $B_{y,0} \sigma_x B_{y,0}$ (when $z = 0$) and the eigenvector of $B_{y,1} \sigma_z B_{y,1}$ (when $z = 1$), both associated to the largest eigenvalue. This gives the upper bound

$$S_2 \leq \cos \theta \sum_{y,b} \lambda_{\max} \left( B_{y,b} \sigma_x B_{y,b} \right) + \sin \theta \sum_{y,b} \lambda_{\max} \left( B_{y,b} \sigma_z B_{y,b} \right). \quad (A5)$$

Notice now that operators of the form $P(\vec{u} \cdot \vec{\sigma})$, where $P$ is a rank-one projector, are rank-one. Hence, their spectra is of the form $(0, \lambda)$. We may write $\lambda_{\max} \left( P(\vec{u} \cdot \vec{\sigma}) \right) = \max \{ 0, \text{Tr} \left( (\vec{u} \cdot \vec{\sigma}) P \right) \}$. Since the second argument is just the expectation value of measuring the state $P$ with the observable $\vec{u} \cdot \vec{\sigma}$, it follows that for each $y$ and each of the two terms in (A5), we get a contribution from only one value of $b$. Moreover, since the states remotely prepared by $A$ for $B$ are in the XZ-plane, it is optimal to assign Bloch vectors for $B$ in the same plane: $\vec{b}_y = (\cos \phi_y, 0, \sin \phi_y)$. Then, one finds

$$S_2 \leq \cos \theta (|\cos \phi_0| + |\cos \phi_1|) + \sin \theta (|\sin \phi_0| + |\sin \phi_1|). \quad (A6)$$

We may also write the CHSH parameter between $A$ and $B$ as

$$S_1 = (\vec{a}_0 + \vec{a}_1) \cdot \vec{b}_0 + (\vec{a}_0 - \vec{a}_1) \cdot \vec{b}_1 = 2 \cos \theta \cos \phi_0 + 2 \sin \theta \sin \phi_1. \quad (A7)$$

Thus, the pair $(S_1, S_2)$ is characterised by the variables $(\theta, \phi_0, \phi_1)$.

Firstly, we note that by choosing $\phi_0 = \phi_1 = -\theta$, we obtain $S_1 = 2 \cos (2\theta)$ and $S_2 = 2$. By taking $\theta \in [0, \frac{\pi}{4}]$, we cover the classical boundary, namely $S_2(S_1) = 2$ for $0 \leq S_1 \leq 2$. This is optimal because a rank-1 projective strategy leaves the state between $A - C$ separable and hence it can at best saturate the CHSH inequality.

Secondly, we consider the non-trivial range $2 \leq S_1 \leq 2\sqrt{2}$. We claim that for every pair $(S_1, S'_2)$ obtained at $(\theta, \phi_0, \phi_1)$, there exists another pair $(S_1, S_2)$, with $S_2 \geq S'_2$, obtained at $(\theta = \frac{\pi}{4}, \phi_0 = \phi, \phi_1 = \frac{\pi}{4} - \phi)$ for some $\phi \in [0, \frac{\pi}{4}]$. To show this, we must first reproduce $S_1$ with the new strategy. This implies that we must find a $\phi$ such that

$$2 \cos \theta \cos \phi_0 + 2 \sin \theta \sin \phi_1 = 2\sqrt{2} \cos \phi. \quad (A8)$$

The left-hand-side is maximal for $\phi_0 = 0$, $\phi_1 = \frac{\pi}{4}$ and $\theta = \frac{\pi}{4}$, for which it becomes $2\sqrt{2}$. Thus, for any value of the left-hand-side we can always choose $\phi$ such that equality holds. Next, we show that with this choice of $\phi$, we have $S_2 \geq S'_2$. This condition becomes

$$\cos \theta (|\cos \phi_0| + |\cos \phi_1|) + \sin \theta (|\sin \phi_0| + |\sin \phi_1|) \leq \sqrt{2} (\cos \phi + \sin \phi). \quad (A9)$$

In the range of our interest, we can w.l.g. take $\phi_0, \phi_1 \in [0, \frac{\pi}{4}]$ and drop the absolute values. Squaring both sides and using (A8) to substitute for $\phi$, we arrive at

$$\cos^2 \theta (\cos^2 \phi_0 + \sin^2 \phi_1) + \sin^2 \theta (\sin^2 \phi_0 + \sin^2 \phi_1) + \sin(2\theta) \sin \phi_0 + \phi_1 \leq 2. \quad (A10)$$

Differentiating the left-hand-side w.r.t. $\phi_0$ and $\phi_1$ respectively, we find that they have two joint roots, at $\theta = \frac{\pi}{4}$, $\phi_0 = \phi_1 = \frac{\pi}{4}$, and at $\theta = \phi_0 = \phi_1$. In both cases, the derivative w.r.t. $\theta$ vanishes. Inserting this into the left-hand-side of (A10), we find that the maximum is $2$, thus proving the inequality to hold.

Hence, we need only to consider strategies of the form $\theta = \frac{\pi}{4}$, $\phi_0 = \phi$ and $\phi_1 = \frac{\pi}{4} - \phi$. This gives

$$S_1 = 2\sqrt{2} \cos \phi \quad (A11)$$

$$S_2 \leq 2 \sqrt{2} (\cos \phi + \sin \phi). \quad (A12)$$

Substituting the former into the latter, we see that the trade-off between the two is given by

$$S_2(S_1) \leq \frac{S_1}{2} + \frac{1}{2} \sqrt{8 - (S_1)^2}. \quad (A13)$$

This is optimal (tight) because it equals the trade-off obtained from an explicit quantum strategy in the main text. Moreover, notice that the maximum of this function occurs at $S_1 = 2$, where we have $S_2 = 2$. Thus, at the lower demarcation of its range of validity, $2 \leq S_1 \leq 2\sqrt{2}$, it meets the trivial (classical) trade-off encountered for $0 \leq S_1 \leq 2$. 

2. Case (ii): identity projections

The trade-off between $S_1$ and $S_2$ when $B$ performs trivial projective measurements, corresponding to deterministically choosing $b$ based on $y$ without regard to the quantum state, is trivial. Such measurements are represented as either $(\mathbb{1}, 0)$ (always output $b = 0$) or $(0, \mathbb{1})$ (always output $b = 1$). From (A1) it follows that $S_1 = 0$. Notice that this can be increased to $S_1 = 2$ if $A$ also performs trivial measurements, but this is not interesting in our context because it implies $S_2 \leq 2$. Thus, $B$’s instrument is essentially reduced only to a unitary: the post-measurement state becomes $\phi^+ = \frac{1}{2} \sum_y (\mathbb{1} \otimes V_y) \phi^+ (\mathbb{1} \otimes V_y^†)$ where $V_y$ is the element in $\{U_0, U_1, U_2\}$ associated to the single output (unit probability event) of $B$’s measurements. Clearly, one optimal choice is $V_y = \mathbb{1}$, for which $\tilde{\phi}^+ = \phi^+$. Then, $C$ can achieve the Tsirelson bound $S_2 = 2\sqrt{2}$ by performing measurements $\sigma_X$ and $\sigma_Z$, while $A$ chooses $\theta = \frac{\pi}{4}$. In contrast to case (i) and case (iii), this trivial trade-off is just a single point: $(S_1, S_2) = (0, 2\sqrt{2})$.

3. Case (iii): one identity projection and one basis projection

Recall from the main text that we can w.l.o.g. choose to associate $B$’s identity measurement to the setting $y = 0$ and the single relevant outcome to $b = 0$. Thus, we have $B_0 = \mathbb{1}$. The second measurement is a standard basis projection, corresponding to measuring in the direction of the unit Bloch vector $\vec{b}$. The observable is $B_1 = \vec{b} \cdot \vec{\sigma}$. Due to the freedom of a global shift in the unitaries $U_{by}$, we may always fix the first one as a reference $U_{y0} = \mathbb{1}$. Moreover, since $B_{y0} = 0$, the choice of $U_{10}$ does not influence the post-measurement state (A2). Furthermore, for $y = 1$, recall from case (i) that the eigenvector of $B_{y0} (\vec{u} \cdot \vec{\sigma}) B_{y0}^†$, for any unit vector $\vec{u}$, associated to its largest eigenvalue, is identical for $b = 0$ and $b = 1$ (one eigenvalue is positive, the other is zero). In Eq (A4), the optimal unitaries $U_{y1}$ aim to align both $C_{y0}$ and $C_{y1}$ with said eigenvector. Since this vector does not depend on $b$, we optimally choose $U_{y1} = U_{11} \equiv U_1$. The post-measurement state becomes

$$\frac{1}{2} \theta^+ + \frac{1}{4} (\mathbb{1} \otimes U_1) \left( \left| \vec{b}, \vec{b} \right\rangle \left\langle \vec{b}, \vec{b} \right| + \left| -\vec{b}, -\vec{b} \right\rangle \left\langle -\vec{b}, -\vec{b} \right| \right) (\mathbb{1} \otimes U_1^†).$$

(A14)

Since the states prepared remotely by $A$ for $B$ are in the XZ-plane, it is optimal to choose $\vec{b} = (\cos \phi, \sin \phi)$. This gives $S_1 = 2 \sin \theta \sin \phi$. The only remaining unitary is then optimally taken as a rotation in the same plane, $U_1 = e^{i \mu \sigma_Y}$. Writing $C$’s measurement Bloch vectors as $\vec{c}_2 = (\cos \phi_2, 0, \sin \phi_2)$, we have

$$S_2 = \cos \theta \left( \cos (2 \mu - \phi_0) + \cos \phi \cos (2 \mu + \phi - \phi_0) \right) + \sin \theta \left( \sin \phi \cos (2 \mu + \phi - \phi_1) - \sin (2 \mu - \phi_1) \right).$$

(A15)

Considering the derivative w.r.t. $\mu$, $\phi_0$, $\phi_1$, $\phi$ respectively, one finds that they are simultaneously zero when $\phi_0 = 2 \mu$, $\phi_1 = \frac{\pi}{2} + 2 \mu$ and $\phi = \frac{\pi}{2}$. This corresponds to the optimum, at which we obtain

$$S_2 = \cos \theta + 2 \sin \theta.$$  

(A16)

Note that $\phi = \frac{\pi}{2}$ also optimises $S_1$, yielding $S_1 = 2 \sin \theta$. Thus, the optimal trade-off is given by

$$S_2 = S_1 + \sqrt{1 - \frac{(S_1)^2}{4}}.$$  

(A17)

This reaches its maximal value at $S_1 = \frac{4}{\sqrt{5}}$, where we have $S_2 = \sqrt{5} > 2$. In the range $0 \leq S_1 \leq \frac{4}{\sqrt{5}}$, the above solution is no longer optimal. However, in this range the optimal trade-off is trivially given by $S_2(S_1) = \sqrt{5}$ because the largest value of $S_2$, when no consideration is given to $S_1$ other than fixing the ranks of $B$’s measurements, is $\sqrt{5}$. This is seen from the fact that choosing $\theta = \frac{\pi}{4}$, $\phi = -\frac{\pi}{4}$, $\phi_0 = -\arctan \frac{1}{3}$ and $\phi_1 = \pi - \arctan 3$ gives $S_2 = \sqrt{5}$ and simultaneously constitutes a root of all relevant derivatives of $S_2$.

4. Mixing via shared randomness

We deduce the optimal trade-off between $S_1$ and $S_2$ by stochastically combining the projective strategies using shared randomness. Recall that case (i) enables a first violation but not a second violation, whereas cases (ii) and (iii) do not enable a first violation but do enable a second violation. Notice that in all three deterministic cases, the trade-offs are concave. Thus, it is only useful to leverage shared randomness in order to mix between deterministic strategies associated to different rank combinations (i-iii). We consider the mixing case by case.
Mixing (ii) and (iii): Define \( f(x) = x + \frac{1}{2} \sqrt{4 - x^2} \), corresponding to the function form of the optimal trade-off (A17) for strategies of class (iii). We have \( \frac{df}{dx} = 1 - \frac{x}{2\sqrt{4-x^2}} \). The tangent of \( f \) is then obtained from solving
\[
(1 - \frac{x}{2\sqrt{4-x^2}}) x + k = f(x) \Rightarrow k = \frac{2}{\sqrt{4-x^2}}.
\] (A18)

We are seeking the tangent that intersects the point \((0, 2\sqrt{2})\) associated to case (ii), which means putting \( k = 2\sqrt{2} \). This gives \( x = \sqrt{\frac{7}{4}} \). Thus, one piece of the boundary in the space of \((S_1, S_2)\) becomes
\[
S_2 = \left(1 - \frac{\sqrt{7}}{2}\right) S_1 + 2\sqrt{2},
\] (A19)
which is valid for \( 0 \leq S_1 \leq \sqrt{\frac{7}{2}} \).

Mixing (i) and (iii): Consider now a mixture of the two non-trivial strategies. Thus, we look for the common tangent connecting the curves (A13) and (A17). For the former we write \( g(x) = \frac{x}{2} + \frac{1}{2} \sqrt{8 - x^2} \) and for the latter we write \( f(x) = x + \sqrt{1 - x^2} \).

To find the tangent, we solve the two equations
\[
f'(x_1) = g'(x_2) = \frac{f(x_1) - g(x_2)}{x_1 - x_2},
\] (A20)
where \( f' \) and \( g' \) are the derivatives of \( f \) and \( g \). The solution is
\[
x_1 = 3\sqrt{\frac{2}{5}}, \quad x_2 = 4\sqrt{\frac{2}{5}}.
\] (A21)
Returning to our original notations, this gives the tangent
\[
S_2 = \sqrt{10} - \frac{S_1}{2},
\] (A22)
which constitutes another piece of the boundary in the space of \((S_1, S_2)\), valid in the interval \( 3\sqrt{\frac{2}{5}} \leq S_1 \leq 4\sqrt{\frac{2}{5}} \).

Mixing (i) and (ii): Computing the tangent between the point \((0, 2\sqrt{2})\) and the curve (A13) is straightforward. However, it turns out to not correspond to the boundary: one can achieve a better trade-off by mixing other strategies, e.g. as in the above cases. Thus, this case is not interesting.

Intermediate regions: We are left to determine the optimal trade-off for the remaining two parts of the interval \( 0 \leq S_1 \leq 2\sqrt{2} \). These are
\[
\sqrt{\frac{7}{2}} \leq S_1 \leq 3\sqrt{\frac{2}{5}},
\] (A23)
\[
4\sqrt{\frac{2}{5}} \leq S_1 \leq 2\sqrt{2}.
\] (A24)
Since these are not covered by the first two cases (considered above), and the mixture of (i) and (ii) is suboptimal, we must attribute them to deterministic strategies. Specifically, in the first part, namely in the interval (A23), the deterministic strategy (iii) is optimal (see Eq. (A17)). In the second part, namely in the interval (A24), the deterministic the deterministic strategy (i) is optimal (see Eq. (A13)). Thus, the full boundary of the set of \((S_1, S_2)\) reachable under projective measurements and shared randomness is given by a four-part piecewise function.

Appendix B: Sequential violations with partially entangled states

Consider that the shared state is partially entangled, \(|\psi_\varphi\rangle = \cos \varphi |00\rangle + \sin \varphi |11\rangle\), for some \( \varphi \in [0, \pi] \). Party A’s and party C’s measurements can w.l.o.g. be restricted to the XZ-plane. Similarly we can restrict those measurements of \( B \) that are rank-1 to the XZ-disk as well. We have investigated the optimal value of \( S_2 \) for a given value of \( S_1 \) by numerically optimising the former over the measurements of \( A, B \) and \( C \), as well as the unitaries of \( B \) (which may be restricted to the XZ-plane) and the distribution \( \{p(\lambda)\}_{\lambda=1}^3 \) of a trit of shared randomness. The results for a few selected values of \( \varphi \) are illustrated in Figure 3.
This can be compared to the line segment $S_2(S_1) = \left(1 - \frac{\sqrt{7}}{2}\right)S_1 + 2\sqrt{2}$ of the boundary associated to the maximally entangled state. We have $\varphi = \frac{1}{2} \arccos \left( \frac{S_2}{2} \right)$ which gives $S_2 = \sqrt{8 - (S_1)^2}$ for the type (ii) strategy. It exceeds the line segment whenever $0 < S_1 < h$ where

$$h = \frac{8\sqrt{2}}{113} \left(7\sqrt{7} - 2\right) \approx 1.65.$$
that has \( S_1 < 2 \) and \( S_2 > 2 \). As \( \varphi \) decreases, this point approaches \((2, 2)\). Secondly, the above type (i) strategy \((B3)\) only depends on \( S_1 \) to second order in the vicinity of \( S_1 = 2 \). Putting \( S_1 = 2 + \epsilon \) for some small \( \epsilon > 0 \), the Tailor expansion of \((B3)\) gives

\[
S_2 = 2 - \frac{\csc^2(2\varphi)}{4} \epsilon^2 + O(\epsilon^3).
\] (B6)

Thus, for sufficiently small \( \epsilon \approx 0 \), the trade-off is essentially flat. Mixing these two strategies over shared randomness allows us to realise the secant that connects the point \((B4)\) with the curve \((B6)\). Thus, by choosing an infinitesimal \( \epsilon \) we assure the validity of the tailor expansion, and necessitate that the secant passes through the double violation region \((S_1, S_2) > 2\).

One can also the result through direct calculation, e.g. by evaluating the tangent between the point \((B4)\) and the curve \((B3)\), and then check the points at which this tangent intersects the line \( S_2 = 2 \) and \( S_1 = 2 \) respectively. The expression for these two points is cumbersome, but we plot it in Figure 4. Note that both curves exceed the local bound whenever \( \varphi \neq 0 \). For small \( \varphi \), namely \( \varphi \approx 0 \), we have the tailor expansions

\[
S_1(S_2 = 2) = 2 + 4 \left( \sqrt{2} - 1 \right) \varphi^2
\] (B7)

\[
S_2(S_1 = 2) = 2 + 2 \left( 2 - \sqrt{2} \right) \varphi^2,
\] (B8)

which demonstrate that both points are above the local bound.