Abstract

Search has played a fundamental role in computer game research since the very beginning. And while online search has been commonly used in perfect information games such as Chess and Go, online search methods for imperfect information games have only been introduced relatively recently. This paper addresses the question of what is sound search in an imperfect information setting of two-player zero-sum games? We argue that the fixed-strategy definitions of exploitability and epsilon-Nash equilibria are ill suited to measure the worst-case performance of an online search algorithm. We thus formalize epsilon-soundness, a concept that connects the worst-case performance of an online algorithm to the performance of an epsilon-Nash equilibrium. As epsilon-soundness can be difficult to compute in general, we also introduce a consistency framework – a hierarchy that connects the behavior of an online algorithm to a Nash equilibrium. Our multiple levels of consistency describe in what sense an online algorithm plays “just like a fixed Nash equilibrium”. These notions further illustrate the difference in perfect and imperfect information settings, as the same consistency guarantees have different worst-case online performance in perfect and imperfect information games. Our definition of soundness and the consistency hierarchy finally provide appropriate tools to analyze online algorithms in imperfect information games. We thus inspect some of the previous online algorithms in a new light, bringing new insights into their worst case performance guarantees.

1 Introduction

From the very dawn of computer game research, search was a fundamental component of many algorithms. Turing’s chess algorithm from 1950 was able to think two moves ahead [Cop04], and Shannon’s work on chess from 1950 includes an extensive section on how an evaluation function can be used within search [Sha50]. Samuel’s checkers from 1959 already combine search and learning of a value function, which is approximated through self-play method and bootstrapping [Sam59]. The combination of search and learning has been a crucial component in the remarkable milestones where computers outperformed their human counterparts in long standing challenging games: DeepBlue in Chess [CHH02], AlphaGo in Go [Sil+17], DeepStack and Libratus in Poker [Mor+17; BS18].

As online search methods for approximating Nash equilibria in imperfect information games appeared only in the last few years [LLB15; BS17; Mor+17; BS18; BS19], we investigate
what it takes for an online algorithm to be sound in imperfect information settings. While it has been known that search with imperfect information is more challenging than with perfect information [FB98, LLB15], the problem is maybe more complex than previously thought. Online algorithms “live” in a fundamentally different setting, and they need to be evaluated appropriately.

Previously, the common approach to evaluate online algorithms was to compute a corresponding offline strategy by “querying” the online algorithm at each state (“tabularization” of the strategy) [LLB15, SKL19]. One would then report the exploitability of the resulting offline strategy. We show that not only is this not generally possible, naive tabularization can also lead to incorrect conclusions about the worst-case performance of the online algorithm. As a consequence we show that some algorithms previously considered to be sound are not.

We first give a simple example of how an online algorithm can lose to an adversary in a repeated game setting, while being seemingly unexploitable based on a naive tabularization. We build on top of this example to introduce a framework for evaluating the performance of an online algorithm. Within this framework, we introduce the definition of a sound and $\epsilon$-sound algorithm. Like the exploitability of a strategy in the offline setting, the soundness of an algorithm is a measure of if its performance against a worst case adversary. Importantly, this notion collapses to the previous notion of exploitability when the algorithm follows a fixed strategy profile.

We then introduce a consistency framework, a hierarchy that allows us to formally state in what sense an online algorithm plays “consistently” with an $\epsilon$-equilibrium. This allows to state multiple bounds on the soundness of the algorithm, based on the $\epsilon$-equilibrium and the type of consistency. The stronger the consistency in our hierarchy, the stronger the bounds. This further illustrates the discrepancy of search in perfect and imperfect information settings, as these bounds sometimes differ for perfect and imperfect information games.

Our definitions of soundness and the consistency hierarchy finally provide appropriate tools to analyze online algorithms in imperfect information games. We thus inspect some of the previous online algorithms in a new light, bringing new insights into their worst case performance guarantees. Namely, we focus on the Online Outcome Sampling (OOS) [LLB15] algorithm. Consider the following statement from the OOS publication: “We show that OOS is consistent, i.e. it is guaranteed to converge to an equilibrium strategy as search time increases. To the best of our knowledge, this is not the case for any existing online game playing algorithm...” The problem is that OOS provides only the weakest of our introduced consistencies – local consistency. As the local consistency gives no guarantee for imperfect information games (in contrast to perfect information games), OOS (and potentially other locally consistent algorithms) can be highly exploited by an adversary. Our experimental section then verifies this issue for OOS in two small imperfect information games.

2 Background

We present our results using the recent formalism of factored-observations games [Kov+19]. The entirety of the paper trivially applies to the extensive form formalism [OR94] as well (as we are only relying on the notion of states and rewards). We believe this choice of formalism makes it easier to incorporate our definitions in the future online algorithms, as sound search in imperfect information critically relies on the notion of common/public information [BJB14]. Indeed, all the recently introduced search algorithms in imperfect information games rely on these notions [Mor+17, BS18, SKL19].

**Definition 1.** A factored-observations game is a tuple $G = (N, W, w^0, A, T, R, O)$, where:

- $N = \{1, 2\}$ is a player set. We use symbol $n$ for a player and -$n$ for its opponent.
- $W$ is a set of **world states** and $w^0 \in W$ is a designated initial world state.
- $A = A_1 \times \cdots \times A_N$ is a space of **joint actions**. The subsets $A_n(w) \subseteq A_n$ and $A(w) = A_1(w) \times \cdots \times A_N(w) \subseteq A$ specify the (joint) actions legal at $w \in W$. For $a \in A$, we write $a = (a_1, \ldots, a_N)$. $A_n(w)$ for $n \in N$ are either all non-empty or all empty. A world state with no legal actions is **terminal**.


• After taking a (legal) joint action \( a \) at \( w \), the transition function \( T \) determines the next world state \( w' \), drawn from the probability distribution \( T(w,a) \in \Delta(W) \).

• \( R = (R_1, \ldots , R_N) \) and \( R_n(w,a) \) is the reward player \( n \) receives when a joint action \( a \) is taken at \( w \).

• \( O = (O_{\text{priv}(1)} \ldots , O_{\text{priv}(N)}, O_{\text{pub}}) \) is the observation function, where \( O_{\text{(c)}} : W \times A \times W \rightarrow O_{\text{(c)}} \) specifies the private observation that \( n \) receives, resp. the public observation that every player receives, upon transitioning from world state \( w \) to \( w' \) via some joint action \( a \).

A legal world history (or trajectory) is a finite sequence \( h = (w^0, a^0, w^1, a^1, \ldots , w^t) \), where \( w^k \in W \), \( a^k \in A(w^k) \), and \( w^{k+1} \in W \) is in the support of \( T(w^k, a^k) \). We denote the set of all legal histories by \( H \), and the set of all sub-sequences of \( h \) that are legal histories as \( H(h) \subseteq H \).

Since the last world state in each \( h \in H \) is uniquely defined, the notation for \( W \) can be overloaded to work with \( H \) (e.g., \( A(h) := A(\text{the last } w \text{ in } h) \), \( h \text{ being terminal, } \ldots \)). We use \( Z \) to denote the set of all terminal histories, i.e. histories where the last world state is terminal.

The cumulative reward of \( n \) at \( h \) is \( \text{sum}\{R_n(w^k, a^k) | k \leq \ell(h)\} := \sum_{k=0}^{\ell(h)} R_n(w^k, a^k) \). When \( h \) is a terminal history, cumulative rewards can also be called utilities, and denoted as \( u_n(h) \). We assume games are zero-sum, so \( u_n(h) = -u_n(h) \) \( \forall h \in H \). The maximum difference of utilities is \( \Delta = \max_{z \in Z} u_1(z) - \min_{z \in Z} u_1(z) \).

Player \( n \)'s information state or private history at \( h = (w^0, a^0, w^1, a^1, \ldots , w^t) \) is the action-observation sequence \( s_n(h) := (O_{\text{pub}}^0, a_n^0, O_{\text{pub}}^1, a_n^1, \ldots , O_{\text{pub}}^t) \), where \( O_{\text{pub}}^n = O_n(w^{k-1}, a^{k-1}, w^k) \). The space \( S_n \) of all such sequences can be viewed as the private tree of \( n \).

A strategy profile is a tuple \( \sigma = (\sigma_1, \ldots , \sigma_N) \), where each (behavioral) strategy \( \sigma_n : s_n \in S_n \mapsto \sigma_n(s_n) \in \Delta(A_n(s_n)) \) specifies the probability distribution from which player \( n \) draws their next action (conditional on having information \( s_n \)). We denote the set of all strategies of player \( n \) as \( \Sigma_n \) and the set of all strategy profiles as \( \Sigma \).

The reach probability of a history \( h \in H \) under \( \sigma \) is defined as \( \pi^\sigma(h) = \pi_1^\sigma(h) \pi_2^\sigma(h) \pi_3^\sigma(h) \cdots \pi_n^\sigma(h) \), where each \( \pi_n^\sigma(h) \) is a product of probabilities of the actions taken by player \( n \) between the root and \( h \), and \( \pi_n^\sigma(h) \) is the product of stochastic transitions. The expected utility for player \( n \) of a strategy profile \( \sigma \) is \( u_n(\sigma) = \sum_{z \in Z} \pi^\sigma(h) u_n(h) \).

We define a best response of player \( n \) to the other players strategies \( \sigma_{\setminus n} \) as the strategy \( br(\sigma_n) = \max_{\sigma_{\setminus n}} \arg \max_{\sigma_{\setminus n}} \epsilon(u_n(\sigma_n, \sigma_{\setminus n}) - u_n(\sigma^*, \sigma_{\setminus n})) \). The profile \( \sigma \) is an \( \epsilon \)-Nash equilibrium if \( \forall n \in N \) : \( u_n(\sigma) \geq \max_{\sigma_{\setminus n}} \arg \max_{\sigma_{\setminus n}} u_n(\sigma_n, \sigma_{\setminus n}) - \epsilon \). We denote the set of all \( \epsilon \)-equilibrium strategies of player \( n \) as \( \mathcal{NE}_n^\epsilon \). The strategy exploitability is \( \exp \rho_n(\sigma_n) := [u_n(\sigma_n) - \min_{\sigma_{\setminus n}} u_n(\sigma_n, \sigma_{\setminus n})] \) where \( \sigma^* \) is an equilibrium strategy. The game value \( u^* = u_1(\sigma^*) \) is the utility player 1 can achieve under a Nash equilibrium strategy profile.

3 Online Search

The environment we are concerned with is that of a repeated game, consisting of a sequence of individual matches. As a match progresses, the algorithm produces a strategy for a visited information state on-line: that is, once it actually observes the state. This common framework of repeated games is particularly suitable for analysis of online algorithms, as the online algorithm can be conditioned on the past experience (e.g. by trying to adapt to the opponent or by re-using parts of the previous computation). We are then interested in the accumulated reward of the agent during the span of the repeated game. Of particular interest will be the expected reward against a worst-case adversary.

3.1 Coordinated Matching Pennies

We now introduce a small imperfect information game that will be used throughout the article – “Coordinated Matching Pennies” (CMP). It is a variation on the well-known Matching
Pennies game OR94, where we additionally introduce a publicly observable chance event just after the first player acts. See Figure 1 for details.

Let $p$ and $q$ denote the probability of playing Heads in information states $s_1$ and $s_2$ respectively. An equilibrium strategy for the second player (Blue) is then any strategy that satisfies $p + q = 1$. He thus has to coordinate the actions between his two information states. The first player has a unique uniform equilibrium. Similar coordination happens also in Kuhn Poker, where equilibria of the first player are also parametrized with a single parameter, while the other player has a unique equilibrium as well [Kuh50].

3.2 Naive Tabularization

We now show that if one naively tries to convert an online algorithm into a fixed strategy, the resulting exploitability is not always representative of the worst-case performance of the online algorithm. Consider the following algorithm PlayCache for the repeated game of CMP. PlayCache keeps an internal state, a cache -- a mapping of information state to probability distribution over the actions, and it gradually fills the cache during the game.

Concretely, PlayCache plays for the second player as follows:

- Initialize algorithm’s state $\theta_0$ to an empty cache.
- Given an information state $s$ visited during a game, there are three possible cases:
  i) The cache is empty: play Heads and store \{s, Heads\} into the cache.
  ii) The cache is non-empty and contains $s$: play the cached strategy for $s$.
  iii) The cache is non-empty and does not contain $s$: play Tails and store \{s, Tails\}.

Consider what happens if one tries to naively tabularize the PlayCache by querying the algorithm for all the information states. If we query the algorithm for states $s_1$, $s_2$, we get the resulting offline strategy $s_1 : Heads$, $s_2 : Tails$. Querying the algorithm for states in reverse order, i.e. $s_2$, $s_1$ results in $s_1 : Tails$, $s_2 : Heads$. And while both of these offline strategies have zero exploitability, one can still exploit the algorithm during the repeated game. This follows from the fact that the very first time the PlayCache gets to act, it always plays Heads. The first player can thus simply play Heads during the first match and is guaranteed to win the match. As we will show later, PlayCache falls within a class of algorithms that can be exploited, but where the average reward is guaranteed to converge to the game value as we repeatedly keep playing the game.

Where did this discrepancy between the exploitability of the tabularized strategy and the exploitability of the online algorithm come from? It is simply because the tabularized strategy does not properly describe the game dynamics of PlayCache. In fact, there is no fixed strategy that does so! We will now formalize an appropriate framework to describe the rewards and dynamics of online algorithms, which will allow us to define notions for the expected reward and the worst-case performance in the online setting.

3.3 Online Settings

The repeated game $p$ consists of a finite sequence of $k$ individual matches $p = (z_1, z_2, \ldots, z_k)$, where each match $z_i \in \mathcal{Z}$ is a sequence of world states and actions $z_i = (w_i^0, a_i^0, w_i^1, a_i^1, \ldots, a_i^{l_i-1}, w_i^{l_i})$, ending in a terminal world state $w_i^{l_i}$. For each visited world state in the match, there is a corresponding information state, i.e. a player’s private perspective of the game (for perfect information games, the notion of information state and world state collapse as the player gets to observe the world perfectly). An online algorithm $\Omega$ then simply maps an information state observed during a match to a strategy, while possibly using its internal algorithm state (Def. 2).
Given two players $\Omega_1, \Omega_2$, we use $p_k^{\Omega_1, \Omega_2}$ to denote the distribution over all the possible repeated games $p$ of length $k$ when these two players face each other. The average reward of $p$ is $R_n(p) = 1/k \sum_{i=1}^{k} u_n(z_i)$ and we denote $E_{p \sim p_k^{\Omega_1, \Omega_2}} [R_n(p)]$ to be the expected average reward when the players play $k$ matches. From now on, if player $n$ is not specified, we assume without loss of generality it is player 1. The proofs of the theorems can be found in Appendix C.

**Definition 2.** Online algorithm $\Omega$ is a function $S_n \times \Theta \to \Delta(\mathcal{A}_n(s_n)) \times \Theta$, that maps an information state $s_k \in S_n$ to the strategy $\sigma_n(s_n) \in \Delta(\mathcal{A}_n(s_n))$, while possibly making use of algorithm’s state $\theta_0 \in \Theta$ and updating it. We denote the algorithm’s initial state as $\theta_0$. A special case of an online algorithm is stateless search, where the output of the function is independent of the algorithm’s state (thus independent of the previous matches). If the output depends on the algorithm’s state, we say the search is stateful.

As the game progresses, the online algorithm produces strategies for the visited information states and updates its algorithm state. This allows it to potentially output different strategies for the same information state visited in different matches. We thus use $\Omega^p(s_n)$ to denote the resulting strategy in the information state $s_n$ after the algorithm has already played the matches $p = z_1, \ldots, z_k$. Note that players can not visit the same information state twice in a single match.

**Remark 3.** The interpretation of the algorithm’s state is up to the specific implementation of the algorithm. If we need to encode a stochastic algorithm, we can do it formally as taking the initial state to be a realization of a random variable. The initial state should be extended to encode how the algorithm should act (seemingly) randomly in any possible game-play situation beforehand.

### 3.4 Soundness of Online Algorithm

We are now ready to formalize the desirable properties of an online algorithm in our settings. Exploitability / $\epsilon$-equilibrium considers the expected utility of a fixed strategy against a worst-case adversary in a single match. We thus define a similar concept for the settings of an online algorithm in a repeated game: $\epsilon$-soundness. Intuitively, an online algorithm is $\epsilon$-sound if and only if it is guaranteed the same reward as if it followed a fixed $\epsilon$-equilibrium.

**Definition 4.** For an $\epsilon$-sound online algorithm $\Omega$, the expected average reward against any opponent is at least as good as if it followed an $\epsilon$-Nash equilibrium fixed strategy $\sigma$ for any number of matches $k$:

$$\forall k \forall \Omega_2 : E_{p \sim p_k^{\Omega_1, \Omega_2}} [R(p)] \geq E_{p \sim p_k^{\Omega_1, \Omega_2}} [R(p)].$$

Note that this definition guarantees that an online algorithm that simply follows a fixed $\epsilon$-equilibrium is $\epsilon$-sound. And while the online algorithm can certainly play as a fixed strategy, online algorithms are far from limited to doing so. (e.g. PlayCache from Section 3). The $\epsilon$-soundness for $k = 1$ of PlayCache is 1, showing that this algorithm is highly exploitable. Additionally, an online algorithm may be sound ($\epsilon = 0$), but there might not be any offline equilibrium that produces the same distribution of matches (example in Appendix A).

### 3.5 Response Game

To compute the $\epsilon$-soundness as in Def. 4, we construct a repeated game [OR94] in the FOG formalism, where we replace the decisions of the online algorithm with stochastic (chance) transitions. As we allow the online algorithm to be stateful and thus produce strategies depending on the game trajectory, the response game must also reflect this possibility. The resulting game $G^k_\Omega$ is thus exponential in size as it reflects all possible trajectories of $k$ matches. We call this single-player game a $k$-step response game.

The $k$-step response game allows us to compute the best response value of a worst-case adversary in $k$-match game-play. We will use overloaded notation $brv(G^k_\Omega)$ to denote this value.

**Theorem 5.** If $\forall k \ brv(G^k_\Omega) \leq k \epsilon$, then algorithm $\Omega$ is $\epsilon$-sound.
3.6 Tabularized Strategy

When an online algorithm produces the same strategy for an information state regardless of the previous matches, there is no need for the k-response game. Fixed strategy notion sufficiently describes the behavior of the online algorithm and thus the exploitability of the fixed strategy matches the soundness. To compute this fixed strategy, one simply queries the online algorithm for all the information states in the game. See Appendix B for a formal definition of the tabularized strategy that also handles the case of stochastic algorithms.

4 Relating ϵ-Soundness and ϵ-Nash

Unfortunately, our notion of ϵ-soundness is often infeasible to reason about. In this section, we introduce the concept of consistency that allows one to formally state that the online algorithm plays “consistently” with an ϵ-equilibrium. Our consistency notion allows us to directly bound the ϵ-soundness of an online algorithm. We introduce three hierarchical levels of consistency, with varying restrictions and corresponding bounds.

4.1 Local Consistency

Local consistency simply guarantees that every time we query the online algorithm, there is an ϵ-equilibrium that is consistent with the produced behavioral strategy. All algorithms are locally consistent with some ϵ-equilibria.

Definition 6. Algorithm Ω is locally consistent with ϵ-equilibria if

\[ \forall k \forall p = (z_1, z_2, \ldots, z_k) \forall h \in \mathcal{H}(z_k) \exists \sigma \in \mathcal{NE}_n : \Omega^{(z_1, \ldots, z_k-1)}(s(h)) = \sigma(s(h)). \]  

(2)

While this suggests that the algorithm plays like some equilibrium, it is not so, and the resulting strategy can still be highly exploitable. This is because one cannot combine local behavioral strategies from different ϵ-equilibria and hope to preserve their exploitability. Consider the CMP game with two strategies \( \sigma_1 = \{(s_1, p = 1), (s_2, q = 0)\} \) and \( \sigma_2 = \{(s_1, p = 0.5), (s_2, q = 0.5)\} \). While both strategies are equilibria, if one plays in the states \( s_1 \) and \( s_2 \) based on the first and second equilibrium respectively, it corresponds to an exploitable strategy \( \{(s_1, p = 1), (s_2, q = 0.5)\} \).

Note that this can happen even in perfect information games (example in Appendix A). Interestingly, local consistency is sufficient if the algorithm is consistent with a subgame perfect equilibrium.

Theorem 7. In perfect information games, an algorithm that is locally consistent with a subgame perfect equilibrium is sound.

A particularly interesting example of an algorithm that is only locally consistent is Online Outcome Sampling \([LLB15]\) (OOS). See Section for detailed discussion and experimental evaluation, where we show that this algorithm can produce highly exploitable strategies in imperfect information games.

4.2 Global Consistency

Local consistency guaranteed consistency only for individual states. The problem we have then seen is that the combination of these visited local strategies might produce highly exploitable overall strategy. A natural extension is then to guarantee consistency for all the visited states in combination: global consistency.

Definition 8. Algorithm Ω is globally consistent with ϵ-equilibria if

\[ \forall k \forall p = (z_1, z_2, \ldots, z_k) \exists \sigma \in \mathcal{NE}_n \forall i \in \{1, \ldots, k\} \forall h \in \mathcal{H}(z_i) : \Omega^{(z_1, \ldots, z_i-1)}(s(h)) = \sigma(s(h)). \]  

(3)

The PlayCache algorithm is globally consistent, but unfortunately we have seen that it is exploitable during the first match \( (k = 1) \).

Theorem 9. An algorithm that is globally consistent with ϵ-equilibria might not be ϵ-sound.
But what if the algorithm keeps on playing the repeated game? While the global consistency with equilibria does not guarantee soundness, it guarantees that the expected average reward converges to the game value in the limit.

**Theorem 10.** For an algorithm $\Omega$ that is globally consistent with $\epsilon$-equilibria,

$$\forall k \forall \Omega_2 : \mathbb{E}_{p \sim P_{\Omega_2}} [R(p)] \geq u^* - \epsilon - \frac{|S_1| \Delta}{k},$$

(4)

### 4.3 Strong Global Consistency

Essentially, the problem with global consistency is that it guarantees the existence of consistent equilibrium for any game-play after the game-play is generated. Strong global consistency additionally guarantees that the game-play itself is generated consistently with an equilibrium; and as in global consistency, the partial strategies for this game-play also correspond to an $\epsilon$-equilibrium. In other words, the online algorithm simply exactly follows a predefined equilibrium.

**Definition 11.** Online algorithm $\Omega$ is strongly globally consistent with $\epsilon$-equilibrium $\sigma$ if

$$\forall k \forall p = (z_1, z_2, \ldots, z_k) \forall h \in \mathcal{H}(z_k) : \Omega(z_1, \ldots, z_{k-1})(s(h)) = \sigma(s(h)).$$

(5)

Strong global consistency guarantees that the algorithm can be tabularized, and the exploitability of the tabularized strategy matches $\epsilon$-soundness of the online algorithm. Canonical examples of strongly globally consistent search algorithms are DeepStack/Libratus. In general, algorithms that use a notion of safe (continual) resolving are strongly globally consistent as it essentially re-solves some epsilon equilibrium (albeit unknown one) that it follows.

**Theorem 12.** If a globally consistent algorithm is stateless then it is also strongly globally consistent.

### 5 Experiments

A particularly interesting example of an algorithm that is only locally consistent is OOS [LLB15]. We use it to demonstrate the theoretical ideas in this paper with empirical experiments. We show that local consistency does in fact fail to result in $\epsilon$-soundness in the online setting. The problem we demonstrate is also not specific to OOS, but in general to any adaptation of an offline algorithm to the online setting where the algorithm attempts to improve its strategy during online play.

At high level, OOS runs the offline MCCFR algorithm in the full game (while also gradually building the tree), parametrized to increase the sampling probability of the current information state. The algorithm then plays based on the resulting strategy for that particular state. The problem is that these individual MCCFR runs can converge to different $\epsilon$-equilibria as the MCCFR is parametrized differently in each information state. In other words, the OOS algorithm exactly suffers from the fact that it is only locally consistent.

We use two games in our experiments: Coordinated Matching Pennies from Section 3 and Kuhn Poker [Kuh50]. We present the Coordinated Matching Pennies results here. See Appendix D for the complete experimental details and a similar experiment for Kuhn Poker.

The code for all the experiments can be found at url will be published for camera-ready version.

Within a single match of Coordinated Matching Pennies, the second player will act either in $s_1$ or $s_2$. OOS will therefore bias MCCFR samples to whichever information state that actually occurs in the match. These two situations are distinct and result in two different strategies for the whole game (including the non-visited state), similarly to the example in Section 4.1. To emulate what OOS does, we parametrize MCCFR runs to bias samples into $s_1$ and $s_2$ respectively, and initialize the regrets in $s_1, s_2$ so that the MCCFR is likely to produce diverse sets of strategies. As MCCFR is stochastic, we average the strategies over $3 \cdot 10^4$ random seeds.

In Figure 2 we plot exploitability for the average strategies, and unbiased MCCFR for reference. The two biased variants of MCCFR actually converge at a similar rate to unbiased
MCCFR, confirming that OOS is locally consistent: it quickly converges to an $\epsilon$-equilibrium for $s_1$ and $s_2$ individually. However, the tabularized strategy — the strategy OOS follows online — is many orders of magnitude more exploitable even with hundreds of thousands of online iterations. The problem is that adapting its strategy online at $s_1$ and $s_2$ causes it to not be globally consistent with any $\epsilon$-equilibria.

6 Related literature

There are several known pathologies that occur in imperfect information games that are not present in the perfect information case. The pathologies that happen in the offline setting present also a problem in the online setting. In [FB98] they identified two problems: i) strategy-fusion and ii) non-locality.

These two problems can easily arise for algorithms designed to solve only perfect-information games, such as mini-max or reinforcement learning algorithms. However, both of these problems can be captured by the computation of exploitability. The proposed $\epsilon$-soundness is similar in its spirit to non-locality: composition of partial strategies (that correspond to parts of distinct equilibria) produced by an online algorithm may not be an overall equilibrium strategy. However $\epsilon$-soundness cannot always be computed with simple exploitability. This problem has not been studied thoroughly in online settings, to our best understanding of the current literature.

Tabularization has been used in [SKL19] to compute an offline strategy and its exploitability. In [LLB15] they consider computing this tabularization (they refer to it as “brute-force” approach), but it is a very expensive procedure. Instead they use an “aggregate method”, which “stitches” strategy from a small number of matches and defines the strategy as uniform in non-visited information states. They do not state whether such approximation of tabularization is indeed correct. In [Mor+17; BS18], they use some form of continual re-solving, which is strongly globally consistent. This guarantees soundness of the algorithms.

7 Conclusion

We first introduced Coordinated Matching Pennies (CMP). This game illustrates the consistency issues that can arise for online algorithms in imperfect information games. Motivated by the observation that fixed strategy exploitability is not an appropriate measure of an algorithm’s performance in online settings, we introduced a formal framework for studying online (search) algorithms. This allowed us to define $\epsilon$-soundness that just like $\epsilon$-exploitability, measures the performance against the worst-case adversary. Soundness generalizes exploitability as it collapses to it when an online algorithm follows a fixed strategy. We then introduced a hierarchical consistency framework that formalizes in what sense an online algorithm can be consistent with a fixed strategy. Namely, we introduced three levels of consistency i) local ii) global iii) strongly global, connecting the behavior of an online algorithm to that of a fixed strategy with increasing tightness. This allowed us to state various bounds on soundness based on the exploitability of a consistent fixed strategy. Interestingly, the implications are in some cases different for perfect and imperfect information games.

Within this framework, we see that local consistency in imperfect information games does not guarantee a worst case performance. Based on this result, we argued that OOS, previously
considered sound, can be in fact exploited. This illustrates that these subtle problems with online algorithms can easily be missed. Our experimental section included experiments in CMP and Kuhn and showed OOS a large discrepancy between the worst case performance and the bound previously thought to hold.

Extending our results for the case of non-zero sum games and multiple players might be an interesting path for future work.
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Broader Impact  This is a theoretical paper, broader impact discussion is not applicable.

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A  Consistency Examples

Example 13. An online algorithm may be sound (\( \epsilon = 0 \)), but there might not be any offline equilibrium that produces the same distribution of matches.

Suppose we have a game where each player acts once, chooses from actions \{A, B, C\} and receives zero utility (i.e. a normal-form game with 3x3 zero payoff matrix). All strategies are equilibria. If we play \( k = 3 \) matches and the players play pure strategies \( A, B \) and \( C \) in each match, we get a distribution of matches \( p = (z_1, z_2, z_3) \) that cannot be achieved with fixed offline equilibrium. In this case, the distribution is \( ((w^0, (A, A)), (w^0, (B, B)), (w^0, (C, C))) \) with probability one, and all other terminal histories with probability zero.

Example 14. An algorithm that is locally consistent with equilibria can be exploited in a perfect information game.

Suppose we have a single player game as in figure on the right. Both blue \( L, Y \) and red \( R, X \) pure strategies are equilibria. However, if the top node is locally consistent with the blue strategy, and the bottom node with the red strategy, the resulting strategy the algorithm actually plays is \( L, X \), which is sub-optimal.

B  Tabularization

We can consider two ways how the online setting can be realized, with respect to how players’ state changes between \( k \) matches in the repeated game: i) no-memory, where the players take turns in a match, and their memory is reset when each match is over (players are allowed to retain memory within the individual matches), or ii) with-memory, where the players are allowed to retain memory between the matches.

As exploitability of tabularized strategy is guaranteed to reflect \( \epsilon \)-soundness only for strongly globally consistent algorithms, we assume their use only. The with-memory case then collapses to the no-memory case: strongly globally consistent algorithm simply plays as some predefined (offline) equilibrium. The following text then simply defines how to compute the offline equilibrium by querying the algorithm in all states.

Definition 15 (Partial strategy). For a terminal history \( z = (w^0, a^0, w^1, a^1, \ldots, w^l, a^l, w^{l+1}) \) player \( n \) has a corresponding sequence of information states \( s = (s^0, s^1, \ldots, s^l, s^{l+1}) \)\(^3\).

We say a partial strategy \( \sigma_n^{\Omega, \theta_0}(z) \) for player \( n \) who uses search \( \Omega \) and starts with state \( \theta_0 \), is an expected behavioral strategy defined only for the visited information states:

\[
\sigma_n^{\Omega, \theta_0}(z) = \{ (s^t, \mu_t) \mid (\mu_t, \theta_{t+1}) = \mathbb{E}_{\theta_t} [\Omega(s^t, \theta_t)|s^t] \forall t \in \{0, \ldots, l\} \}.
\]

Note that when we compute the strategy \( \mu_t \) from \( \mathbb{E}_{\theta_t} [\Omega(s^t, \theta_t)|s^t] \), we must compute it as a weighted average to respect the structure of the private tree. The weights are reach probabilities of the information state \( s^t \): cumulative product of the player’s strategy over the sequence of information states \( s^0, \ldots, s^{t-1} \), leading to target information state \( s^t \). See \[Zin+08\] Eq. 4\(^4\) for more details.

Definition 16. A composition of partial strategies for terminals \( Z \) is a tabularized strategy

\[
\sigma_n^{\Omega, \theta_0}(Z) = \bigcup_{z_i \in Z} \sigma_n^{\Omega, \theta_0}(z_i).
\]

C  Proofs of Theorems

Theorem 5. If \( \forall k \ |brv(G^k_{\Omega})| \leq \epsilon k \), then algorithm \( \Omega \) is \( \epsilon \)-sound.

Proof. If we used \( \epsilon \)-equilibrium strategy \( \sigma \) in each response game \( G^k_n \), then the \( |brv(G^k_{\Omega})| = \epsilon \) \( \forall k \) because adversary can gain at most \( \epsilon \) in each match. Since \( \epsilon \)-sound algorithm should play at least as well as some offline \( \epsilon \)-equilibrium, it must have \( |brv(G^k_{\Omega})| \leq \epsilon \) \( \forall k \).

\(^3\)We omit the index \( n \) for information state \( s_n \) for clarity.

\(^4\)
Theorem 7. In perfect information games, an algorithm that is locally consistent with a subgame perfect equilibrium is sound.

Proof. In perfect information games the notions of an information state and a history blend together, as there is a one-to-one correspondence between them. Expected utility of a history is the same for all subgame perfect equilibria. It corresponds to the best achievable value against worst-case adversary, given that the history occurred. This property implies that the worst-case expected utility of a history remains optimal, if the player plays only actions that are in the support of any subgame perfect equilibrium in the consequent states. A formal proof can be constructed by induction on the maximal distance from a terminal history.

Notice that this exactly happens if the player plays according to an algorithm locally consistent with subgame perfect equilibria. The expected worst-case utility for any history will be optimal, and the worst-case expected utility of a match will correspond to the worst-case expected utility of the history at the beginning of the game. Therefore the worst-case expected utility of each match is also optimal and the algorithm is sound.

Theorem 9. An algorithm that is globally consistent with $\epsilon$-equilibria might not be $\epsilon$-sound.

Proof. Counter-example can be shown with the toy algorithm PlayCache as in Section 3. PlayCache is not sound ($\epsilon = 0$), because it can be exploited in the first match.

We will now prepare the ground to prove Thm. 10.

When an algorithm that is globally consistent with an $\epsilon$-equilibrium is queried in some information states in a match in the repeated game, it will always keep playing the same behavioral strategy in these situations in subsequent matches. We call this as “filling in” strategy. Once the algorithm fills the strategy in all player’s information states, we are guaranteed to get match reward of $u^\epsilon = u^\star - \epsilon$ on average against a worst-case adversary.

Informally speaking, the bound in Thm. 10 can be easily seen to be true for a game like Coordinated Matching Pennies, or some generalization which will have a larger number of information states that need to be coordinated (think of “Coordinated Rock-Paper-Scissors”). At every match, we can incur a loss of at most $\Delta$ when reaching an unfilled history. This is a rather pessimistic lower bound on the value, but it lets us ignore the algorithm state: we are either playing at filled information states, or achieving the worst possible value. The problematic part is making sure the bound holds also when we (repeatedly) visit previously filled information states. For each of the possible future subgames, there are two cases. In both cases, the number of $\Delta$-sized losses in utility plus the number of unfilled information states does not increase, so we can use induction on the length of the game to prove the claim. Along branches where the opponent had an opportunity to exploit the algorithm by playing into an unfilled information state, the algorithm loses at most $\Delta$ utility compared to the equilibrium, but must fill in at least one information state to do so. Along branches where the agent played through filled information states, the algorithm is playing identically to the equilibrium strategy and thus achieves the same value.

To prove Thm. 10 we will need to establish a Lemma 17 a bound of difference of utilities a player can gain if he plays according to a partially filled $\epsilon$-equilibrium strategy compared to $u^\epsilon$ within an arbitrary match. The idea of the proof for Thm. 10 is then to bound this difference for any number of non-visited information states and any number of remaining matches within the response game using induction.

An online algorithm can fill in the strategy only into information states found on the trajectory to a terminal history, as it will be queried only in these situations. So after playing through a match $z = (w^0, a^0, w^1, a^1, \ldots, w^l, a^l, w^{l+1})$, the algorithm’s response at $s_n(w^i)$ will be fixed as $\sigma(s_n(w^i))$ for all visited worlds $w^i$ on the trajectory $z$.

To talk about possible filled strategies within a single match, we will partition $Z$ into two non-empty sets of terminal histories $Z_\bullet$ (pronounced “filled”) and $Z_\circ$ (pronounced “empty” or “unfilled”). The partition has a special property of “being possible to realize in online setting”: all information states on the trajectory to terminals $Z_\bullet$ are filled, and all terminals that can be reached just through these filled information states are also in $Z_\bullet$ (we are not
Lemma 17. For a probability of reaching a filled terminal \( P(\bullet) = \sum_{z_{\bullet} \in \mathcal{Z}_{\bullet}} \pi^\sigma(z_{\bullet}) \), an expected received utility for filled terminals \( u(\bullet) = \frac{\sum_{z_{\bullet} \in \mathcal{Z}_{\bullet}} \pi^\sigma(z_{\bullet}) u_1(z_{\bullet})}{\sum_{z_{\bullet} \in \mathcal{Z}_{\bullet}} \pi^\sigma(z_{\bullet})} \) and an utility of playing outside of filled histories \( u(\times) \), it holds that
\[
P(\bullet)(u(\bullet) - u^\epsilon) \geq -(1 - P(\bullet))(u(\times) - u^\epsilon + \Delta),
\]
assuming \( 0 < P(\bullet) < 1 \).

Proof. For any strategy profile \( \sigma = (\sigma_1, \sigma_2) \) with an \( \epsilon \)-equilibrium strategy \( \sigma_1^\epsilon \) and arbitrary opponent strategy \( \sigma_2 \) it holds that
\[
\sum_{z_{\bullet} \in \mathcal{Z}_{\bullet}} \pi^\sigma(z_{\bullet}) u_1(z_{\bullet}) + \sum_{z_0 \in \mathcal{Z}_0} \pi^\sigma(z_0) u_1(z_0) \geq u^\epsilon.
\]
The terms can be simplified and rewritten as factorization of product of probabilities and (weighted) utilities as
\[
P(\bullet)u(\bullet) = \sum_{z_{\bullet} \in \mathcal{Z}_{\bullet}} \pi^\sigma(z_{\bullet}) u_1(z_{\bullet}),
\]
and similarly for the "\( \circ \)" partition. It also holds that \( P(\bullet) + P(\circ) = 1 \), as the probability of reaching a terminal history within a match is equal to one.

We can restate \((7)\) as
\[
P(\bullet)(u(\bullet) - u^\epsilon) + (1 - P(\bullet))(u(\circ) - u^\epsilon) \geq 0.
\]
Suppose that for the partition "\( \circ \)" we didn’t use an equilibrium strategy for player 1, but arbitrary strategy profile \( \sigma' \) satisfying \( P^{\sigma'}(\bullet) + P^{\sigma'}(\circ) = 1 \). We will denote its utility
\[
u(\times) = \frac{\sum_{z_0 \in \mathcal{Z}_0} \pi^{\sigma'}(z_0) u_1(z_0)}{\sum_{z_0 \in \mathcal{Z}_0} \pi^{\sigma'}(z_0)}.
\]
The value of any two strategies cannot differ by more than the maximum difference of utilities in the game:
\[
u(\circ) \leq u(\times) + \Delta.
\]
Putting \((10)\) back to \((8)\), we get the lemma that lower bounds the difference of filled partition and \( u^\epsilon \) for an arbitrary match:
\[
P(\bullet)(u(\bullet) - u^\epsilon) \geq -(1 - P(\bullet))(u(\times) - u^\epsilon + \Delta).
\]

Theorem 10. For an algorithm \( \Omega \) that is globally consistent with \( \epsilon \)-equilibria,
\[
\forall k \forall \Omega_2 : \mathbb{E}_{p \sim P^k_{\bullet, \Omega_2}}[\mathcal{R}(p)] \geq u^* - \epsilon - \frac{|S_1| \Delta}{k},
\]

Proof. Let us rewrite the theorem slightly:
\[
\forall k \forall \Omega_2 : k \mathbb{E}_{p \sim P^k_{\bullet, \Omega_2}}[\mathcal{R}(p)] - ku^\epsilon \geq -|S_1| \Delta.
\]
Since \( \mathcal{R}(p) \) is average reward, multiplying by \( k \) we get cumulative utilities in the game-play \( p = (z_1, z_2, \ldots, z_k) \):
\[
\forall k \forall \Omega_2 : \mathbb{E}_{p \sim P^k_{\bullet, \Omega_2}} \left[ \sum_{i=1}^{k} u_1(z_i) \right] - ku^\epsilon \geq -|S_1| \Delta.
\]
So on the left side of the inequality we have a difference of cumulative (expected) utilities and of cumulative $u^\epsilon$. We use cumulative values because we are now in the setting of a $k$-repeated game.

Let $v$ be the number of non-visited information states of player 1 (resp. the number of unfilled information states) in a match, i.e. $0 \leq v \leq |S_1|$, and let $l$ be the number of next matches (including the current one), i.e. $1 \leq l \leq k$. We will use $a_{v,l}$ to denote the difference between expected cumulative rewards and cumulative $u^\epsilon$ from the current match (inclusively) until the end of the game, if we are playing against worst-case adversary. The left side of (12) corresponds to a value equal or greater than $a_{|S_1|,k}$ (so we need to prove that $a_{|S_1|,k} \geq -|S_1|\Delta$. It is sufficient to consider only the worst-case adversary, as the bound on $a_{v,l}$ will hold for any other opponent as well.

We will prove the theorem by induction on $a_{v,l}$ using $v$ and $l$ simultaneously. Let us characterize the base case. If we have visited all information states ($\forall v = 0$), we filled $\epsilon$-equilibrium strategy everywhere. So at each visit of such a match we receive a reward of $u^\epsilon$, and the difference between expected cumulative rewards and cumulative $u^\epsilon$ is zero:

$$a_{0,l} = 0 \quad \forall l. \quad (13)$$

The induction hypothesis is

$$a_{x,y} \geq -x\Delta \quad \forall x \leq v \quad \forall y < l. \quad (14)$$

There are two possibilities of what can happen in a match. We either “hit” the filled information states, receive some (expected) reward $u(v,l)$ and possibly continue into next match where we receive $a_{v,l-1}$ (if the current match is not the last one, i.e. $l > 1$). Or we “miss” the filled information states, meaning we visit arbitrary number of new information states previously not visited. This will also change $v$ to be smaller for all subsequent matches.

We state this with an abuse of notation as

\[
a_{v,l} = P(v,l)(u(v,l) - u^\epsilon + a_{v,l-1}) \\
+ P(v-1,l)(u(v-1,l) - u^\epsilon + a_{v-1,l-1}) \\
+ P(v-2,l)(u(v-2,l) - u^\epsilon + a_{v-2,l-1}) \\
+ \ldots \\
+ P(v-v,l)(u(v-v,l) - u^\epsilon + a_{v-v,l-1}),
\]

where the terms $P(v-i,l)$ and $u(v-i,l)$ are defined similarly to how we defined them for $P(\bullet)$ and $u(\bullet)$. They correspond to the probability and utilities received when we visit $i$ new (previously unfilled) information states with $l$ remaining matches (including current one). It holds that $P(v,l) + P(v-1,l) + P(v-2,l) + \ldots + P(v-v,l) = 1$ as the probability of reaching a terminal history within a match is equal to one.

By using the induction hypothesis (14) on terms $a_{x,l-1}$ $\forall x < v$ we get a lower bound $a_{x,l-1} \geq -x\Delta$ on all of $x$. By comparing these bounds we can deduce that $a_{v-1,l-1}$ lower bounds all of $a_{x,l-1}$ with

$$a_{v-1,l-1} \geq -(v-1)\Delta. \quad (16)$$

Using this bound in (15) we get

\[
a_{v,l} \geq P(v,l)(u(v,l) - u^\epsilon + a_{v,l-1}) \\
+ P(v-1,l)(u(v-1,l) - u^\epsilon - (v-1)\Delta) \\
+ P(v-2,l)(u(v-2,l) - u^\epsilon - (v-1)\Delta) \\
+ \ldots \\
+ P(v-v,l)(u(v-v,l) - u^\epsilon - (v-1)\Delta).
\]

We can factor it out as

\[
a_{v,l} \geq P(v,l)(u(v,l) - u^\epsilon + a_{v,l-1}) \\
+ (1 - P(v,l))(- (v-1)\Delta - u^\epsilon) \\
+ P(v-1,l)u(v-1,l) + P(v-2,l)u(v-2,l) + \ldots + P(v-v,l)u(v-v,l).
\]
We replace the utilities \( u(v-1), u(v-2), \ldots, u(v-v) \) by \( u(\times) \) from (9):
\[
a_{v,l} \geq P(v,l)(u(v,l) - u^\ell + a_{v,l-1}) + (1 - P(v,l)) (u(\times) - u^\ell - (v-1)\Delta).
\]  
(19)

By using the induction hypothesis \((14)\) we get
\[
a_{v,l} \geq P(v,l)(u(v,l) - u^\ell - v\Delta) + (1 - P(v,l)) (u(\times) - u^\ell - (v-1)\Delta).
\]  
(20)

Expanding the terms
\[
a_{v,l} \geq P(v,l)(-v\Delta)
+ (1 - P(v,l))(-v\Delta + (v-1)\Delta)
+ P(v,l)(u(v,l) - u^\ell)
+ (1 - P(v,l))(u(\times) - u^\ell)
\]
and using Lemma 17 with \( \bullet = v,l \) we have
\[
a_{v,l} \geq P(v,l)(-v\Delta) + (1 - P(v,l))(-v\Delta - (v-1)\Delta)
- (1 - P(v,l))(u(\times) - u^\ell + \Delta)
+ (1 - P(v,l))(u(\times) - u^\ell).
\]  
(22)

Simplifying, we get a bound on \( a_{v,l} \):
\[
a_{v,l} \geq -v\Delta.
\]  
(23)

Note that this bound holds also if \( P(v,l) = 1 \) or \( P(v,l) = 0 \):

- \( P(v,l) = 0 \): Then (20) becomes
\[
a_{v,l} \geq u(\times) - u^\ell - (v-1)\Delta.
\]
Using the same argument as in (10),
\[
a_{v,l} \geq -\Delta - (v-1)\Delta = -v\Delta.
\]

- \( P(v,l) = 1 \): We use (8), which becomes \( u(v,l) - u^\ell \geq 0 \). Then (20) becomes
\[
a_{v,l} \geq u(v,l) - u^\ell - v\Delta \geq -v\Delta.
\]

Since at the beginning of the game-play there are \( |S_1| \) unfilled information states, we arrive at the original theorem
\[
a_{|S_1|,k} \geq -|S_1|\Delta.
\]

\[\square\]

**Theorem 12.** If a globally consistent algorithm is stateless then it is also strongly globally consistent.

**Proof.** The definition of a stateless algorithm implies that for an information state \( s \) the algorithm always produces the same behavioral strategy \( \sigma(s) \) as the search algorithm is deterministic (all stochasticity is encoded within the algorithm state \( \theta \), see Remark 3).

This means that whatever \( \epsilon \)-equilibria the algorithm is globally consistent with is independent of the current game-play or match number. This allows us to swap the quantifiers:
\[
\forall k \forall p = (z_1, z_2, \ldots, z_k) \exists \sigma \in \mathcal{NE}_n \exists i \in \{1, \ldots, k\} \forall h \in \mathcal{H}(z_i) : \Omega(z_1,\ldots,z_{i-1})(s(h)) = \sigma(s(h))
\]  
(24)

\[
\exists \sigma \in \mathcal{NE}_n \forall k \forall p = (z_1, z_2, \ldots, z_k) \forall i \in \{1, \ldots, k\} \forall h \in \mathcal{H}(z_i) : \Omega(z_1,\ldots,z_{i-1})(s(h)) = \sigma(s(h)).
\]  
(25)

Using the same argument we can treat the different matches \( z_i \) as an iteration over \( k \), leading us to strong global consistency
\[
\exists \sigma \in \mathcal{NE}_n \forall k \forall p = (z_1, z_2, \ldots, z_k) \forall h \in \mathcal{H}(z_k) : \Omega(z_1,\ldots,z_{k-1})(s(h)) = \sigma(s(h)).
\]  
(26)

\[\square\]
Figure 3: Coordinated Matching Pennies. Left: While individual MCCFR strategies have low exploitability of $\sim 10^{-3}$, the tabularized OOS strategy has high exploitability of 0.17 after $10^6$ iterations. Right: Normalized histograms of probabilities of playing Heads in $s_1$ - $p$ and $s_2$ - $q$ after $10^6$ iterations. The histograms within columns are correlated, as they approximately satisfy equilibrium condition $p + q = 1$. Tabularized strategy, combination of $(p, s_1)$ and $(q, s_2)$ violates this constraint, resulting in high exploitability.

D Experiment details

As OOS runs MCCFR samples biased to particular information states, individual MCCFR runs can converge to different $\epsilon$-equilibria, as the MCCFR is parametrized differently in each information state. Additionally OOS runs in an online setting, where the algorithm is given a time budget for computing the strategy, and it may make different numbers of samples in each targeted information state.

We emulate this experimentally by slightly modifying initial regrets to produce distinct convergence trajectories. We show it is possible to highly exploit the online algorithm: in fact, it is possible to exploit the algorithm more than the worst of any individual biased strategies it produces, not just the expected strategies. This modification is sound: the initial regrets will “vanish” over longer sampling and the strategies will converge to an equilibrium in the limit. This is justified by the MCCFR regret bound [Lan+09, Theorem 5].

We use two games: Coordinated Matching Pennies (CMP) from Section 3 and Kuhn Poker [Kuh50]. We use the no-memory online setting. Nash equilibria in both games are parametrized with a single parameter $\alpha \in (0, 1)$ for one player, while the opponent has only a single unique equilibrium. In both games, equilibria require the strategies to be appropriately balanced, an effect of non-locality problem [FB98] present only in imperfect information games. When we compose the final strategy from partial online strategies, this balance can be lost, resulting in high exploitability of the composed strategy.

We modify the initial regrets with following procedure:

- Choose a distinct value of $\alpha$, one for each of the player’s top-most information states in the game. Compute an equilibrium strategy according to $\alpha$.
- Directly copy the behavioral strategy into regret accumulators, and multiply them by a constant $\mu$.

This simple procedure effectively kick-starts the algorithm to produce distinct trajectories based on $\alpha$.

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4 In CMP, $p = \alpha$ (playing Heads in $s_1$) and $q = 1 - \alpha$ (playing Heads in $s_2$). In Kuhn Poker, constructing equilibrium strategy based on $\alpha$ is more complicated and we refer the reader to [Kuh50] or [Hoe+05] for more details.
In Figure 3 and in Figure 4, we show that individual biased strategies converge to Nash equilibria, but the tabularized strategy has higher exploitability even than the worst individual strategy. In CMP, we bias the second player to play in information state $s_1$ ($\alpha = 0.5$) or $s_2$ ($\alpha = 1$) information states. In Poker, we bias the first player to play Jack ($\alpha = 0$), Queen ($\alpha = 1/2$) or King ($\alpha = 1$) card. For both experiments, exploration was set to 0.6, biasing to 0.1, and $\mu = 500$, a small regret that can be accumulated after less than 500 samples. Within our search framework, the state $\theta$ consists of regrets and average strategy accumulators for all information states, and from the state of the pseudo-random number generator, which has distinct initial seeds for each match. The expected strategies are estimated as an average over $3 \cdot 10^4$ seeds. We plot the worst strategy from these individual biased strategies over all the seeds for all iterations. We plot also MCCFR strategy for reference, to see the influence of biasing and regret initialization.