M-Theory Brane as Giant Graviton and the Fractional Quantum Hall Effect

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Abstract: A small number of M-theory branes as giant gravitons in the M-theory sector of LLM geometry is studied as a probe. The abelian way shows that the low energy effective action for M-theory brane is exactly the 2d electron subject to a vertical magnetic field. We also briefly discuss the microscopic description of M2-brane giant graviton in this geometry, in the language of a combination of D0-branes as fuzzy 2-spheres. Then we go to the well-established Noncommutative Chern-Simons theory description. After quantization, well behaved Fractional Quantum Hall Effect is demonstrated. This goes beyond the original LLM description and should be some indication of novel geometry.

Keywords: AdS/CFT Correspondence, M-Theory, M(atrix) Theories, Chern-Simons Theories.
1. Introduction

Counting fundamental degrees of freedom of theoretical physics system remains a long term problem for theorists. In the brilliant work started by J. Maldacena which is known as AdS/CFT \(1\), a connection between the Supersymmetric Yang-Mills theory which lives on branes and the supergravity (which can be promoted to type IIB string/M-theory) on the asymptotic \(AdS_m \times S^n\) is established. Recently, the extension \(2\) (known as LLM) brings the whole \(1\) BPS sector to our scope, in which the original AdS/CFT correspondence can be viewed as a special case. Motivated by the gauge theory side work \(3\), it is shown that all the nonsingular \(\frac{1}{2}\) BPS configurations of the supergravity side that preserve certain symmetry can be described by a black-white 2d moduli space, which can be viewed as the 2d electron gas distribution of the Quantum Hall Effect. Once the distribution is given, the geometry which preserves half the supersymmetry is then uniquely determined by a Laplace/Toda equation.

As is known to all, the most mysterious aspect of the 2d electron gas when exposed to strong vertical magnetic field is the Fractional Quantum Hall Effect (FQHE) \(4\), which can be naively regarded as the composite particle carries only fraction charge while averaged to each parton. In \(5\), the origin of FQHE states in the LLM geometry is explored in the IIB sector, where a small number of giant gravitons are treated as probe. As mentioned in \(21\), the FQHE states in the AdS/CFT/QH formulism exceed the \(\frac{1}{2}\) BPS description,
which deserves further study.

Giant graviton [6, 7] plays an essential role in our treatment. The original ansatz adopted by LLM preserve the symmetry of $SO(4) \times SO(4)$ in the IIB sector and $SO(3) \times SO(6)$ in the M-Theory sector, which are exactly the symmetries preserved by both the internal and outer space giant gravitons in their separated sectors. This reflects the consideration that it is the condensation and interaction of giant gravitons that gives the novel LLM geometry. Giant graviton functions as element for constructing geometry, so it makes sense to separate a small number of giant gravitons as a probe and take others as background. This separation shed light on the possibility that is not covered by original LLM, especially, the Fractional QHE.

In the present paper, we will generalize [5] to the M-theory brane case. According to [8, 9], M-theory is the strong coupling limit of type IIA superstring theory, which takes the 11d supergravity as its low energy effective theory. M-theory has two kinds of branes, the M2-brane and the M5-brane, they couple electrically and magnetically to the 3-form gauge potential in the 11d supergravity. An exact microscopic description of M-theory is still unknown, but for the M2-brane case, there is a promising suggestion known as matrix model which suggests that the M2-brane is obtained by a combination of large (infinite) number of D0-branes [2, 10]. Although the original BFSS Matrix Model [1] is formulated in the flat spacetime, some further work show that the certain weakly coupled background case can also be covered [10, 11, 12, 13]. In [13] a set of D0-branes are combined into a fuzzy 2-sphere as a the M2-brane giant graviton in the $AdS_7 \times S^4$ and $AdS_4 \times S^7$ geometry, giving a microscopic description of M2-brane, which is also a nontrivial extension of BFSS Matrix Model Conjecture. This work shed light on our case, we will use similar technique in our treatment.

The paper is organized as follows. We present a brief review of giant graviton in AdS/CFT and the LLM geometry in section 2, emphasizing the M-theory sector. In section 3 we will show that the low energy effective action of several coincident M-theory branes are indeed governed by Quantum Hall Effect. First we use the macroscopic (or abelian) way of directly computing the induced metric and the pullback of the gauge potential for a single brane. Then some brief comments will also be made about the microscopic (or nonabelian) side, i.e., towards the DLCQ matrix model in a weakly coupled background, in which we extend it to multi-brane case. In the 4th section we will make use of the alternative (but somewhat standard) description of QHE, i.e. the noncommutative Chern-Simons theory (NCCS). We focus on the finiteness of giant gravitons and revise the theory for consistency. After quantization we will finally show the inverse filling factor quantization, i.e., the emergence of the FQHE states. Then a brief discussion of the interaction between the giant gravitons is presented. In the final section we make some explanation and discussion of the meaning of the solution and then conclude.
2. Review of Giant Graviton and LLM Geometry

2.1 M-Theory Brane as Giant Graviton

The $AdS_m \times S^n$ metric we adopt is [7]:

\[
\begin{align*}
  ds^2 &= ds^2_{AdS_m} + ds^2_{S^n} \\
  ds^2_{AdS_m} &= -\left(1 + \frac{r^2}{R_{AdS}^2}\right) dt^2 + \frac{dr^2}{1 + \frac{r^2}{R_{AdS}^2}} + r^2 d\Omega_{m-2}^2 \\
  ds^2_{S^n} &= R_S^2(d\theta^2 + \cos^2 \theta d\phi^2 + \sin^2 \theta d\tilde{\Omega}_{n-2}^2)
\end{align*}
\]  

The coordinates on two spheres are separately denoted by $\varphi_1, \ldots, \varphi_{m-2}$ and $\tilde{\varphi}_1, \ldots, \tilde{\varphi}_{n-2}$. In the M-theory, two cases are separately $AdS_4 \times S^7$ and $AdS_7 \times S^4$, with the curvature radii separately satisfying $2R_{AdS} = R_S$ and $R_{AdS} = 2R_S$.

Generally speaking, a giant graviton is a $p$-brane ($p \geq 1$) that have the same quantum number with the ordinary graviton but extended in several spatial (either internal or outer) dimensions. For simplicity we will mainly show the internal case, then a $p = n - 2$ brane can wrap a $S^{n-2}$ in the internal $S^n$ space, while in the uncompactified spacetime we see a worldline [6, 7]. Using the diffeomorphism, the brane can be chosen to follow the static gauge:

\[
\begin{align*}
  \xi_0 &= \tau = t \\
  \xi_1 &= \tilde{\varphi}_1 \\
  \ldots \\
  \xi_{n-2} &= \tilde{\varphi}_{n-2}
\end{align*}
\]

\[
 r = 0 \quad \theta = \text{Constant} \quad \phi = \phi(t)
\]

For outer noncompact space case we simply replace the condition by $\theta = 0, r = \text{Constant}$ and the spatial $\xi$s are the $\varphi$s in the $AdS$ space. Similar giant graviton solution can be constructed [7].

For definiteness we consider $AdS_7 \times S^4$ case, which relates to the calculation we will perform in section 3. An M2-brane wrap a $S^2$ in the internal $S^4$. The induced metric gives the Nambu-Goto like action

\[
S_{NG} = -T_{M2} \int_V d^3\xi \sqrt{-\det \partial_i X^\mu \partial_j X^\nu G_{\mu\nu}}
\]

\[
= -4\pi T_{M2} \int d\tau R^2 \sin^2 \theta \sqrt{1 - R^2 \cos^2 \theta \dot{\phi}^2}
\]

where the $V$ denotes the worldvolume of the M2-brane.

The $N$ coincident M5-branes which generate the AdS geometry have background flux on the $S^4$. In the adopted coordinate system the potential is given by

\[
G^{(3)}_{\phi_12} = R^3 \sin^3 \theta \sqrt{\det g_{\tilde{\Omega}_2}}
\]

where $g_{\tilde{\Omega}_2}$ is the metric on the internal $S^2$. The pullback of the gauge potential gives the action

\[
S_{gauge} = T_{M2} \int_V G^{(3)} = 4\pi T_{M2} \int d\tau R^3 \sin^3 \theta \dot{\phi}
\]
So the overall lagrangian is given as

$$\mathcal{L} = 4\pi T_{M_2} R^2 \sin^2 \theta (-\sqrt{1 - R^2 \cos^2 \theta \dot{\phi}^2} + R \sin \theta \dot{\phi})$$ (2.9)

From the action we can obtain some classical steady configuration for giant graviton. The steps are the standard procedure of classical mechanics, namely the Hamilton canonical transformation. The Lagrangian doesn’t contain $\phi$, which means the canonical momentum $p_\phi$ is conserved. So the minimal energy of fixed $p_\phi$ is determined only by the value of $\theta$

$$H = \frac{1}{R} \sqrt{p_\phi^2 + \tan^2 \theta (p_\phi - 4\pi T_{M_2} R^3 \sin \theta)^2}$$ (2.10)

The minimal energy is given by

$$\frac{\partial H}{\partial \theta} \propto \sin \theta \cos^3 \theta (p_\phi - 4\pi T_{M_2} R^3 \sin \theta) [p_\phi - 4\pi T_{M_2} R^3 (2 \sin \theta - \sin^3 \theta)]$$ (2.11)

One obvious solution is $\sin \theta = \frac{p_\phi}{4\pi T_{M_2} R^3}$. This means a nonvanishing volume that the M2-brane takes in the $S^4$ is a steady configuration. Given that all the quantum number of the membrane is the same with a graviton, it is called a giant graviton, which also indicates that this configuration is a version of graviton that has spatial extension.

There is a further comment that we want to make. The ansatz we adopted contains only an angle $\phi$ in the internal space $S^n$ as a function of the brane worldvolume variables, and the solution we explicitly showed implies that the angular speed $\dot{\phi}$ is a constant. The giant graviton rotates in a plane around some fixed point. This is a strong evidence that while considering the giant graviton as electrons and the rotation is due to the transverse magnetic field, it is the Hall effect that the giant graviton experiences [6]. We will soon see that this effect extend to the LLM case.

### 2.2 The M-Theory Sector of LLM Geometry

The original M-theory sector of AdS/CFT correspondence includes both the $AdS_4 \times S^7$ case and the $AdS_7 \times S^4$ case, which are separately the throat geometry of large number of M2-branes and M5-branes. As we have seen, the giant graviton includes the same brane spectrum for both cases. Namely, both the M2-brane and the M5-brane can be implemented as giant graviton in either the $AdS_4 \times S^7$ case or the $AdS_7 \times S^4$ case, whereas they differ in whether the giant graviton is in the internal spheric space or the outer $AdS$ space. So the LLM geometry describing them can be uniformly obtained by requiring the $SO(3) \times SO(6)$ symmetry. Taking the $\frac{1}{2}$ BPS condition into consideration, the geometry is given in [2]

$$ds^2 = -4e^{2\lambda}(1 + y^2 e^{-6\lambda})(dt + V_i dx_i)^2 + \frac{e^{-4\lambda}}{1 + y^2 e^{-6\lambda}} [dy^2 + e^D(dx_1^2 + dx_2^2)]$$

$$+ 4e^{2\lambda} d\tilde{\Omega}_2^2 + y^2 e^{-4\lambda} d\tilde{\Omega}_2^2$$ (2.12)

$$G^{(4)} = F^{(2)} \wedge d^2 \tilde{\Omega}_2$$ (2.13)

$$e^{-6\lambda} = \frac{\partial_y D}{y(1 - y \partial_y D)}$$ (2.14)
where $i, j = 1, 2, *_3$ is the Hodge star of the 3d metric $dy^2 + e^D dx_1^2$ and $*_3$ is the 3d flat space Hodge star. The function $D$ which determines the solution obeys the 3d Toda equation

$$\left(\partial^2_1 + \partial^2_2\right) D + \partial^2_y e^D = 0$$

(2.19)

Note that the coordinate $y$ is related to the radii of the two spheres $S^2, S^5$ by $y = R_2 R_3^2$, where $R_2 = ye^{-2\lambda}$ and $R_3 = 2e^{\lambda}$. So if the coordinate $y$ goes to zero, one of the two radii must go to zero and the function $D$ is constrained in a subtle way in order to eliminate the singularity. The condition is given in $[2]$

$$y \to 0 \iff \begin{cases} \partial_y D = 0 & D = \text{finite} \ S^2 \text{ shrink} \\ D \sim \log y & S^5 \text{ shrink} \end{cases}$$

(2.20)

Although generically, the nonlinear Toda equation cannot be analytically solved, two exact solutions are shown in $[2]$. They are based on the above nonsingular analysis and separately correspond to the original AdS/CFT geometry. For the $AdS_4 \times S^7$ case

$$e^D = 4L^{-6} \sqrt{1 + \frac{r^2}{4}} \sin^2 \theta \quad x = \left(1 + \frac{r^2}{4}\right)^{\frac{1}{2}} \cos \theta \quad 2y = L^{-3} r \sin^2 \theta$$

(2.21)

For the $AdS_7 \times S^4$ case

$$e^D = \frac{r^2 L^{-6}}{r^2 + 4} \quad x = \left(1 + \frac{r^2}{4}\right) \cos \theta \quad 4y = L^{-3} r^2 \sin \theta$$

(2.22)

where $r$ is a radial coordinate on $AdS$ and $\theta$ is an angle on $S$, they function like the ones in (2.2), (2.3) but are not exactly the same things. We transform the Cartesian coordinates $x_1, x_2$ into polar coordinates and denote them by $x, \tilde{\phi}$

$$ds^2_2 = dx_1^2 + dx_2^2 = dx^2 + x^2 d\hat{\phi}^2$$

(2.23)

Only the $x$ appears in the above two formulae, which implies the rotation symmetry. $L^{-1}$ is the curvature radius of $AdS_4(S^4)$ in the former(latter) case. The variables $r, \theta$ of the function $D$ can be transformed into $x, y$ by the relation shown above. After a few calculation, they give back precisely to the $AdS \times S$ geometry as shown in (2.1), (2.2), (2.3). Some of the steps go as follows

$$AdS_4 \times S^7 \quad e^\lambda = \frac{\sin \theta}{L} \quad V_\phi = -\frac{4 \cos^2 \theta}{r^2 + 4 \sin^2 \theta} \quad \tilde{\phi} = \phi - t$$

$$ds^2 = \frac{1}{L^2} \left[-(r^2 + 4) dt^2 + \frac{dr^2}{r^2 + 4} + \frac{r^2}{4} d\hat{\phi}^2\right] + \frac{4}{L^2} [d\theta^2 + \cos^2 \theta d\phi^2 + \sin^2 \theta d\Omega^2_3]$$

(2.24)
\[ AdS_7 \times S^4 \quad e^{\lambda} = \frac{r}{2L} \quad V_\phi = \frac{2 \cos^2 \theta}{r^2 + 4 \sin^2 \theta} \quad \tilde{\phi} = 2t - \phi \]

\[ ds^2 = \frac{4}{L^2} \left[ -(r^2 + 4) dt^2 + \frac{dr^2}{r^2 + 4} + \frac{r^2}{4} d\Omega_5^2 \right] + \frac{1}{L^2} \left[ d\theta^2 + \cos^2 \theta d\phi^2 + \sin^2 \theta d\tilde{\Omega}_2^2 \right] \tag{2.25} \]

The \( AdS \) part of the geometry is not in the standard form given in (2.2). However, perform a diffeomorphism rescaling of \( r \rightarrow 2rL \), \( t \rightarrow \frac{1}{2} tL \) for the former case and \( r \rightarrow rL \), \( t \rightarrow \frac{1}{4} tL \) for the latter case, they readily become the standard \( AdS \times S \) geometry shown in (2.1). The relation between the curvature radii of the \( AdS \) and \( S \) parts is correctly reproduced.

3. From Multi Giant Graviton to QHE Action

3.1 The Macroscopic Description

Next we will consider the case of a number of M-theory branes in the LLM geometry. The most direct way is, as used by [6, 7], the induce metric and the pullback of the gauge potential. As a fundamental object, the induce metric is not explained as a kind of Born-Infeld action that governs the low energy dynamics of D-branes, but their appearance looks exactly the same. It is a direct further generalization of the Nambu-Goto action of the least worldsheet area

\[ S_{NG} = -T_M \int_V d^3 \xi \sqrt{-\text{det} P[G_{ab}]} \tag{3.1} \]

\[ S_{NG} = -T_M \int_V d^6 \xi \sqrt{-\text{det} P[G_{ab}]} \tag{3.2} \]

where \( P[G_{ab}] = \partial_a X^\mu \partial_b X^\nu G_{\mu\nu} \) is the pullback of the metric on the worldvolume. \( a, b \) label the longitudinal directions along the brane worldvolume, and the left transverse ones are denoted by \( i, j \). Going to the static gauge we have \( P[G_{ab}] = G_{ab} + G_{i(a} \partial_{b)} X^i X^j + \partial_a X^i \partial_b X^j G_{ij} \).

Since the M2-brane electrically couples to the M-theory 3 form potential and the M5-brane magnetically, the pullback of the gauge potential should also be included in the action. This gives the Chern-Simons like part

\[ S_{CS} = T_{M2} \int_V C^{(3)} \tag{3.3} \]

\[ S_{CS} = T_{M5} \int_V C^{(6)} \tag{3.4} \]

where \( C^{(3)} \) is the gauge potential coupled to M2-brane and \( C^{(6)} \) coupled to M5-brane. They are related by \( * (dC^{(6)}) = dC^{(3)} = G^{(4)} \).

We will use the LLM frame rather than the \( AdS \times S \) geometry to show the dynamics of the giant gravitons. The LLM geometry, taking the \( AdS \times S \) as a special case, will recover the whole \( \frac{1}{2} \) BPS sector which preserve certain bosonic symmetries. For simplicity we will only consider the M2-brane case, where we can directly use the 4-form field strength given in [2] (2.13) to determine the gauge potential. Note that physically we ask the background LLM geometry to be nonsingular, so what we are using is only taking the \( S^5 \)
shrinking condition (corresponding to the M2-brane case), i.e. factor 1 will be ignored when accompanied by \( y^2 e^{-6\lambda} \), because this term is in fact \( \frac{1}{4} \left( \frac{R_2}{R_5} \right)^2 \). Since we are only interested in the behavior of \( y = 0 \) plane, the \( e^D \) will be replaced by its asymptotic value \( y \). Then the Nambu-Goto part is

\[
S_{NG} = -4\pi T_{M2} \int dt [2y^3 e^{-6\lambda} (1 + V_i \dot{x}_i) - \frac{1}{4}(\dot{x}_1^2 + \dot{x}_2^2)] \tag{3.5}
\]

The pullback of the 3 form part is

\[
S_{CS} = 2\pi T_{M2} \int dt [4y^3 e^{-6\lambda} (1 + V_i \dot{x}_i) + x_1 \dot{x}_2 - x_2 \dot{x}_1] \tag{3.6}
\]

So the combined Lagrangian is

\[
\mathcal{L} = \frac{1}{2}(\dot{x}_1^2 + \dot{x}_2^2) + (x_1 \dot{x}_2 - x_2 \dot{x}_1) \tag{3.7}
\]

If we want to restore the dependence of the 11d M-theory Planck length \( \ell_P \), dimensional analysis gives the Lagrangian

\[
\mathcal{L} = \frac{1}{2\ell_P}(\dot{x}_1^2 + \dot{x}_2^2) + \frac{1}{\ell^2 P}(x_1 \dot{x}_2 - x_2 \dot{x}_1) \tag{3.8}
\]

It is exactly the one that describes the 2d charged particles which is exposed to vertical magnetic field and subject to the Lorentz force, namely, the one which governs the Quantum Hall Effect. We see that the \( x_1, x_2 \) coordinates in the original LLM geometry in our formulation is exactly 2d plane that the Quantum Hall Effect lives.

For convenience of the treatment in the next subsection, we will explore some aspect of the QHE side. Where our interest mainly focus in, the Lowest Landau Level (LLL) captures the long distance (or low energy) behavior of a QHE system. In the above case it is equivalent to \( \ell_P \to 0 \) limit, in other words, to omit the kinetic part of the Lagrangian. Then only left with

\[
\mathcal{L} = x_1 \dot{x}_2 - x_2 \dot{x}_1 = x^2 \dot{\phi} \tag{3.9}
\]

in the RHS we use the coordinate transformation (2.23). Then we can see in the LLL limit it coincides with the original ansatz of giant graviton. Only the derivative of the coordinate \( \dot{\phi} \) appears in the Lagrangian and not the coordinate itself, which means the symmetry along the \( \phi \) direction. Effectively the \( \phi \) is an isometric direction in this limit.

One should not be surprised about the result. Going back to the original giant graviton solution in \( AdS \times S \), we can see the rotation movement in \( \phi \) direction. The above Lagrangian is just an extension of this circumnutation. Based on this consideration, we would argue that in the M5-brane case the corresponding Lagrangian will be exactly the same, according to the fact that it also contains the rotating giant graviton as a special case.

### 3.2 Towards the DLCQ Matrix Model in Weakly Curved Background

In this subsection we will focus on the M2-brane case where the microscopic description is somehow formulated. The above description is purely macroscopic, which ignore the
internal degree of freedom of the M2-brane. The most crucial point that the transverse direction $X_1, X_2$ should be matrices, is also completely obscured.

As we have noted in the introduction, it is believed that the M2-brane takes the BFSS matrix model as a microscopic description. In flat 11d spacetime and large $N$ limit the description is exact [1], while later Susskind [11] have extended it. M-theory compactified on a light like direction with $N$ units of KK momenta (discrete light cone quantization or DLCQ) is described by the dynamics of $N$ D0-branes, i.e. 0+1 dimensional $U(N)$ Super Yang-Mills. In [11] the interaction between two matrix entities are computed at the linearized order. With the addition of linearized supergravity action, such interaction between two matrix model entities is translated into the determination of the linearized supergravity currents, and it is done in [11]. Then viewed as a matrix entity, an 11d light like graviton in some general background can be treated in such formulism of the matrix model literature. Namely given the 11d weakly curved spacetime and the flux, the additional action of a graviton beyond the flat space matrix model action is determined by the linearized supercurrent formula. In [12, 13] the theory is directly formulated in general weakly curved background, and it is shown that additional linearized supergravity action is equivalent to the low energy Born-Infeld and Chern-Simons action of D0-branes. Finally in [13] it is focused in the M-theory AdS/CFT geometry, a microscopic description of M2-brane giant graviton is given. Giant graviton in $AdS \times S$ geometry is described by a combination of D0-branes as a fuzzy 2-sphere. This is related to the dielectric effect studied by Myers [14] which implies a lower dimensional D(p-2n)-brane can couple to a higher dimensional RR charge which originally should be taken by Dp-brane. We should check carefully whether the techniques can be implemented in our treatment.

First of all, we must note that the DLCQ procedure requires an isometric compact direction in which the 11d gravitation wave propagates, then we can change our coordinate into light cone system and discretize the Kuluza-Kleyn mode spectrum. On the D0-brane side this direction is eliminated, in order to guarantee that the D0-brane live in nine transverse direction. In the original $AdS \times S$ case the giant graviton ansatz automatically satisfies this requirement, it is the isometric direction $\phi$. At first glance, in the LLM geometry (2.12) we do not have such a direction that is obviously the propagation direction of the KK modes. However, it is not difficult to see that in the LLL limit of the effective action (3.9) the $\phi$ direction is such a direction. That's because in the LLL limit and the coordinate system after the transform (2.23), the system is equivalently described by the ansatz $X = \text{Constant}, \tilde{\phi} = \hat{\phi}(t)$, the dynamics are the same. Physically in that limit all the movement of 11d gravitation wave in other directions such as radial $x$ are all small fluctuation that can be ignored. So the DLCQ still can be carried out.

On the D0-brane side the effective metric must be introduced, which eventually eliminate one isometric direction so that the D0-brane feels ten spacetime dimension. Meanwhile the gauge potential in the M-theory case should be inner producted by some transverse direction to give the dielectric effect for them to couple to D0-branes. It is checked several times [12, 13] that the action in the weakly curved background is recovered by the nonabelian dielectric D0-brane Born-Infeld action and Chern-Simons action, which is considered as nontrivial test of the D0-brane matrix model description for 11d M-Theory or its
type IIA compactification. The checking is valid for arbitrary background geometry and flux, so all the following microscopic treatment and the matrix substitution are guaranteed. For our purpose we will not repeat the checking procedure but use the following action directly

\[ S_{BI} = - T_0 \int d\tau \text{Str} \left\{ k^{-1} \sqrt{-P[E_{00} + E_{0i}(Q^{-1} - \delta)ij E_{j0}] \text{det}(Q^i)} \right\} \] (3.10)

\[ S_{CS} = T_0 \int d\tau \text{Str} \left\{ iP[(iY_i Y_j)C^{(3)}] \right\} \] (3.11)

where \( i = 1, \ldots, 9 \) label the transverse direction and in all directions \( E_{\mu\nu} = G_{\mu\nu} + k^{-1}(i_k C^{(3)})_{\mu\nu} \) where \( G_{\mu\nu} = G_{\mu\nu} - k^2 k_\mu k_\nu \) is the effective metric. \( k_\mu \) is the Killing vector of the isotropic spacelike direction in which the 11d gravitation wave propagate and \( k = |k_\mu| \). In our case some coordinate transformation is needed when we fix this direction, namely we can perform \((2,23)\) and the \( \tilde{\phi} \) is such a direction. The definition of effective metric \( G_{\mu\nu} \) naturally subtracts the contribution from the \( k \) direction so that the D0-brane feels nine transverse direction. We also take \( Q^i_j = \delta^i_j + ik[Y^i, Y^j]E_{kj} \). Finally \( iY \) is the inner product of gauge potential with the transverse Killing vector and \((iY_i Y_j)C_{\mu_1 \mu_2 \ldots \mu_p} = Y^i Y^j C_{ij \mu_1 \mu_2 \ldots \mu_p} \). \( Y \) labels arbitrary transverse direction and generally becomes matrix valued.

Corresponding to the giant graviton in \([13]\), we also need some fuzzy 2-sphere ansatz

\[ Y^i = \frac{ye^{-2\lambda}}{\sqrt{N^2 - 1}} J^i \] (3.12)

where \( i = 1, 2, 3 \) and \( J^i \) form an \( \tilde{N} \times \tilde{N} \) representation of \( SU(2) \). Then \((Y^1)^2 + (Y^2)^2 + (Y^3)^2 = ye^{-4\lambda} \) implies the fuzzy 2-sphere has the fixed radius. All the other directions remain Abelian, i.e. proportional to \( \mathbb{1} \), the commutator between them vanishes. This is because we only have a single M2-brane as a whole, the transverse direction should not be matrix valued. Note that corresponding to the LLM limit, we have eliminated the propagation direction of the graviton wave, i.e. the only coordinate which is function of time, so merely the steady configuration can be studied in the above formulism and also the Hamiltonian. We can not get explicitly the dynamical low energy effective action. It can be parallelly shown that up to order \( O(\frac{1}{N}) \) the above microscopic description should be coincident with the macroscopic way, i.e. treatment carried out in the above subsection.

However, this is only a single giant graviton which is combined by a number of D0-branes. In the LLM geometry we are interested in a number of \( N \) giant gravitons, so we should revise our ansatz into the block diagonal form

\[ Y^i = \text{diag} \left( \frac{R_1}{\sqrt{N_1^2 - 1}} J^i_1, \frac{R_2}{\sqrt{N_2^2 - 1}} J^i_2, \ldots, \frac{R_N}{\sqrt{N_N^2 - 1}} J^i_N \right) \] (3.13)

The matrices have \( \sum_{n=1}^{N} N_n \) columns and rows. Our aim is to treat each block along the diagonal as a size \( \tilde{N}_n \times \tilde{N}_n \) unit of the above giant graviton, where \( J^i_n \) is the \( \tilde{N}_n \times \tilde{N}_n \) \( SU(2) \) representation. In all we will have \( \sum_{n=1}^{N} N_n \) D0-branes to combine into \( N \) M2-brane giant gravitons. Correspondingly we relax our requirement and do not ask the other directions
to be identity matrices, especially, in the $x$ direction of the $(2.23)$.

Now we are not interested in the microscopic side of each graviton, so we can ignore the $\tilde{N}_n \times \tilde{N}_n$ identity matrices contents and simply replace each of them for number 1. Correspondingly in other effective transverse directions we should also do this simplification, then each transverse coordinate shrinks to $N \times N$ matrix to describe the $N$ M2-brane giant gravitons. This is equivalent to treat each giant graviton as a particle, and ignore all the inner structure and dynamics of the each of them. If the radial direction $X$ is still diagonal, it is merely some superposition of noninteracting giant gravitons, just the trivial ground state configuration. In general it can be arbitrary matrix when the full dynamics is taken into consideration.

Now turn to the low energy effective action. Using the correspondence between the microscopic and macroscopic description, the Lagrangian corresponding to $(3.8)$ is very simply revised. The transverse coordinates $x_1, x_2$ should be replaced by matrices, and correspondingly a trace should be added. We see from the LLL limit $(1.3)$ that it is in fact only one effective direction, namely radial $X$ in $(2.23)$, so there should be no commutator between the two $X_i$

$$\mathcal{L} = \frac{1}{2\ell_P} \text{Tr}(\dot{X}_1^2 + \dot{X}_2^2) + \frac{1}{\ell_P^2} \text{Tr}(X_1\dot{X}_2 - X_2\dot{X}_1)$$  (3.14)

This is our starting point of the next formulation of the QHE side.

Finally let’s make some comment on the M5-brane case. Generally the microscopic description of the M5-brane is unclear, so we can’t perform such kind of microscopic analysis. But we can see that in the M2-brane case, the only visible change from the macroscopic description is the substitution of the coordinate $x_i$ by their matrix counterpart $X_i$, which is an indication of the essential nonabelian property of a stack of M2-branes. At least according to some dimensional reduction result in string theory (say, the M5-brane double-dimensional reduction is the IIA string D4-brane $\mathcal{I}$), the nonabelian property of transverse direction should be inherited by the M5-brane from the D-brane side. So corresponding to the macroscopic description, in the M5-brane case we expect that the Lagrangian should also be revised as $(3.14)$. Then the following formulation is also valid for the M5-brane case.

4. The Noncommutative Chern-Simons Description

Recall that we want to perform a deep exploration on the Quantum Hall Effect side of the correspondence, in order to shed light on the uncovered aspect of the corresponding geometry. A well known approach is the Noncommutative Chern-Simons theory description $\mathcal{I}$, which will be our main guidance.

First of all, we note that the matrix nature of the coordinates implies that the system is fermionic. Since the Lagrangian is invariant under $U(N)$ gauge symmetry, one of the $Xs$, say $X_1$, can be diagonalized by a gauge transformation. In this eigenvalue basis, with notation $(X_1)_{mn} = \delta_{mn}x_{1n}$, $(X_2)_{mn} = y_{mn}$, $y_{nn} = x_{2n}$, a typical classical Lagrangian of the
matrix model reads
\[ \mathcal{L} = \frac{1}{2\ell_P} \sum_{i,n} \dot{x}_{1n}^2 + \frac{1}{2\ell_P} \sum_{m\neq n} \dot{y}_{mn} y_{mn} + \frac{1}{\ell_P} \sum_n (x_{1n}\dot{x}_{2n} - x_{2n}\dot{x}_{1n}) \] (4.1)

where \( i = 1, 2 \). The Hamiltonian description can be established in the eigenvalue basis. However, quantum mechanically there is change of measure in path integral from the matrix-element basis to the eigenvalue basis, the Van der Monde determinant of the eigenvalue \( \Delta(x_1) = \prod_{n<m} (x_{1n} - x_{1m}) \) is inserted [18]. So the Hamiltonian in the quantum theory is given by
\[ H = \frac{1}{\Delta(x_1)} \tilde{H} \Delta(x_1) \] (4.2)

where the \( \tilde{H} \) on the RHS is the direct Legendre transformation of the Lagrangian (4.1). Obviously the eigenfunction \( \Psi \) of the quantum Hamiltonian is related to the original eigenfunction by \( \Psi(x) = \Delta(x_1)^{-1}\tilde{\Psi}(x) \). The minus sign which is produced by exchanging the two elements in the Van der Monde determinant just indicates the fermionic nature of the system. So not only the dynamics of a single giant graviton but the statistics of the interaction system enable us to identify the whole probe brane system as a realization of the Quantum Hall Effect.

From now on we will change the coefficient by \( \frac{1}{\ell_P} \rightarrow m \) as the effective giant graviton mass and \( \frac{1}{\ell_P^2} \rightarrow B \) as the magnetic field, in order to borrow some concepts from the study of QHE. Note that we have set the ‘electric charge’ \( e \) to unit. We are going to the well-established Noncommutative Chern-Simons Description, which means that we only focus on the long distance behavior. As mentioned above, a prior of doing so is to ignore the kinetic part of the Lagrangian. Because all the phenomena we are interested in lie in the LLL, in which the kinetic energy of the electron system is degenerated into the lowest level [4, 21], the kinetic part can be dropped. On the M-theory side, this is equivalent to taking the \( \ell_P \rightarrow 0 \) limit, i.e., ignoring all the high order corrections, which consists with the low energy limit. At the moment, we must point out that there is a subtlety for doing this, and postpone treating it in a little later.

There is something important that we can not ignore, namely the fermionic statistics. Viewing the 2d electron gas as a dissipationless fluid, the fermionic statistics is shown by the property that the droplet can not be compressed. This is guaranteed in the treatment of [16] by introducing a potential \( U \), which can be viewed as the short-distance statistics effect and have an equilibrium configuration when the droplet is not compressed. In order to achieve this we can introduce another set of ‘comoving’ coordinates \( y_i \), of which the graduation in the real space is everywhere (gauge) adjustable, according to the real space separation of the nearby electrons. In other words, the electron is statically distributed on this ‘comoving’ coordinates with even density. The real space position is a map from the comoving coordinates, and it depends on the time. After normalizing the comoving coordinates density as \( \rho_0 \), the real space electron density is given by the Jacobi \( \rho = \rho_0 \frac{\partial y}{\partial x} = \rho_0 \frac{\partial (y_1, y_2)}{\partial (x_1, x_2)} \). The incompressibility is reflected by requiring that the Jacobi is unity when the potential reaches its minimum and the configuration is equilibrium. This
is also called the Gauss Law Constraint. The action is given by
\[ \mathcal{L} = \int dy^2 \rho_0 \left[ \frac{B}{2} \epsilon_{ij} \dot{x}_i \dot{x}_j + U \left( \rho_0 \frac{\partial(y_1, y_2)}{\partial(x_1, x_2)} \right) \right] \] (4.3)
where we temporarily ignore the matrix nature of the \( X_i \)s.

The theory has the area preserving diffeomorphism (APD) symmetry. If we consider some transformation of the comoving coordinates \( y'_I = y_I + f_i(y) \) and requiring the Jacobi remains the same, it can be factorized as \( \frac{\partial y'}{\partial x} = \frac{\partial y}{\partial x} \frac{\partial y'}{\partial y} \) and the solution is readily obtained
\[ f_i = \epsilon_{ij} \frac{\partial \Lambda(y)}{\partial y_j} \] (4.4)

So the induced transformation of the \( x_a \) will be
\[ \delta x_a = \frac{\partial x_a}{\partial y_i} f_i(y) = \epsilon_{ij} \frac{\partial x_a}{\partial y_i} \frac{\partial \Lambda(y)}{\partial y_j} \] (4.5)

Obviously there is a trivial solution to the equilibrium configuration, i.e., we can choose the comoving coordinates such that \( x_i = y_i \). While we are not interested in this trivial solution, we can perform a deformation to obtain new interesting solutions, using the above APD symmetry technique. For instance
\[ x_i = y_i + \epsilon_{ij} \frac{1}{2\pi \rho_0} \]

The most direct way to achieve this APD is by introducing an auxiliary field \( A_0 \), and requiring the equation of motion of \( A_0 \) gives the desired Gauss Law Constraint. The action is constructed in [16]
\[ \mathcal{L} = \int dy^2 \left\{ \frac{B \rho_0}{2} \epsilon_{ij} \left[ (\dot{x}_i - \frac{1}{2\pi \rho_0} \{ x_i, A_0 \}) x_j + \frac{\epsilon_{ij}}{2\pi \rho_0} A_0 \right] - \frac{1}{2} \rho_0 m \omega^2 x_i^2 \right\} \] (4.6)
where classically the commutator \( \{ , \} \) is defined as the Poisson Bracket with derivatives to \( y_i \). The variation of the auxiliary field \( A_0 \) gives the constraint
\[ \{ x_1, x_2 \} = 1 \] (4.7)
This is exactly the inverse version of the equilibrium condition, from which the APD symmetry can also be deduced. There is no constraint on the value of \( A_0 \). For simplicity we can even choose the gauge \( A_0 = 0 \), then the new Lagrangian goes back to the old one (4.3).

We also add a term \(-\frac{1}{2} \int dy^2 \rho_0 m \omega^2 x_i^2 \) in the Lagrangian, where \( \omega = \frac{B}{m} \) is the cyclotron frequency. Recall that the frequency also equals to the reciprocal of the 11d Planck length \( \ell_P^{-1} \) from our original QHE description (3.14), so we can see that the LLL condition \( \omega \to \infty \) indeed match the low energy limit of the M-theory side \( \ell_P \to 0 \) very well. It is the subtlety we pointed out above. Let us see the reason of doing this. What we are interested in is the ‘droplet’ solution that all the giant gravitons concentrate. The fermionic exclusion principal as well as the fixed cyclotron frequency gives us a picture that classically the droplet is a
rigid body made up of incompressible fermions that experience an overall circumnutation. What is more, in fact we do not treat the rotation movement explicitly in the following. In this sense going to the LLL can be interpreted as the frame of reference transformation, i.e. we are going to the rotating frame and the giant gravitons seem static to combine into a droplet. From classical mechanics we know that when transformed into a noninertial frame an inertia force must be introduced, which is equivalent to the potential we added. From the droplet constraint we notice that increasing the radius means more energy, in this way the sign of the term can be determined.

Right now we treat the real space coordinates as ordinary number, and this is equivalent to treating the system as uniform continuous fluid. But indeed they are some finite number of giant gravitons, which have the discrete property. So it is high time that we replace the simplified $x_i$ by their $N \times N$ matrices counterparts $X_i$, meanwhile the Poisson bracket gives way to matrix commutator, integral replaced by matrix trace and the coefficient should be revised

$$\mathcal{L} = \frac{B}{2} \text{Tr}\{\epsilon_{ij}(\dot{X}_i - i[X_i, A_0])X_j + 2\theta A_0 - \omega X_i^2}\}
$$

(4.8)

where $\theta = \frac{1}{\sqrt{\rho_0}}$. The continuous comoving coordinate $y$s are replaced by the comoving lattices which are mapped into the real space matrix element. Recall that the dynamics we obtained in (3.14) is already in terms of matrix, so it is the matrix version that just corresponds to the original action.

The relationship between this theory and the standard noncommutative Chern-Simons theory is elaborated in [16, 19]. Ignoring the centrifugal potential, the standard Noncommutative Chern-Simons form is obtained

$$\mathcal{L}_{NC} = \frac{1}{4\pi\nu} \epsilon_{\mu\nu\rho} \left( \hat{A}_\mu \ast \partial_\nu \hat{A}_\rho + \frac{2i}{3} \hat{A}_\mu \ast \hat{A}_\nu \ast \hat{A}_\rho \right)$$

(4.9)

But we will not explore further in such pure noncommutative Chern-Simons theory direction. In fact, knowing (4.8) is enough for our formulation.

There are two different commutativity in the theory. One originates quantum mechanically, the canonical conjugation of the coordinate matrix is $\Pi_i = \frac{B}{2} \epsilon_{ij} X_j$ so the canonical quantum condition gives

$$[X_1, X_2]_{QM} = i \frac{B}{2} \epsilon_{ij} X_j$$

(4.10)

The other is the APD symmetry constraint from the equation of motion of the auxiliary field $A_0$

$$[X_1, X_2]_{APD} = i \theta$$

(4.11)

In the following we will see that the ratio of the two commutative parameters is related to the filling factor of the FQHE.

4.1 Finite Noncommutative Chern-Simons and the Edge Excitation

However, the matrix version has some intrinsic inconsistence, which must be revised. Recall that the matrix order $N$ is the number of giant gravitons, which is large but finite. The problem is that it is impossible to satisfy the APD symmetry in the finite $N$ case. So the
action should be revised by adding the edge excitation \[17\]. We must admit that in the M-theory side, because our ignorance of the microscopic dynamics of the theory itself, a satisfactory microscopic picture is still lacking.

\[
\mathcal{L} = \frac{B}{2} \text{Tr}\{\epsilon_{ij}(\dot{X}_i - i[X_i, A_0])X_j + 2\theta A_0 - \omega X_i^2\} + \Psi^{\dagger}(i\dot{\Psi} - A_0\Psi)
\]  

(4.12)

where the introduced field \(\Psi\) lives only on the edge of the droplet which is formed by finite \(N\) discrete giant gravitons. The term of \(\Psi\) looks like the Dirac field the dynamics of which is the first order equation, and correspondingly, it lives in the fundamental representation of the \(SU(N)\) Lie algebra. The \(X_i\)s are in the adjoint representation. Namely, the symmetry transformation is

\[
X_i \rightarrow UX_iU^{-1} \quad \Psi \rightarrow U\Psi
\]

(4.13)

The extended Gauss Law Constraint is still the equation of motion of the auxiliary field \(A_0\). It is

\[
G = iB[X_1, X_2] - \Psi\Psi^{\dagger} + B\theta = 0
\]

(4.14)

Taking the trace of the above equation gives

\[
\Psi^{\dagger}\Psi = BN\theta
\]

(4.15)

Note that in the finite \(N\) case where the trace of commutator of finite dimensional matrices is zero, if we do not introduce the \(\Psi\), doing the same steps above immediately gives us the inconsistency. This explains the introduction of \(\Psi\). The equation of motion for \(\Psi\) in the \(A_0 = 0\) gauge is \(\dot{\Psi} = 0\), so we can take it to be

\[
\Psi = \sqrt{BN}\theta|\psi\rangle
\]

(4.16)

where \(|\psi\rangle\) is a constant vector of unit length. Then the traceless part of (4.14) reads

\[
[X_1, X_2] = i\theta(1 - N|\psi\rangle\langle\psi|)
\]

(4.17)

The equation of motion of the \(X_i\) field is \(\dot{X}_i = \omega\epsilon_{ij}X_j\). This is just a matrix oscillator and solved by

\[
X_1 + iX_2 = e^{i\omega t}A
\]

(4.18)

where \(A\) is any \(N \times N\) matrix satisfying

\[
[A, A^{\dagger}] = 2\theta(1 - N|\psi\rangle\langle\psi|)
\]

(4.19)

To find the ground state of the system, we must minimize the potential

\[
V = \frac{B\omega}{2} \text{Tr}(X_1^2 + X_2^2)
\]

(4.20)

with the constraint (4.17) or (4.19). This can be done with

\[
A = \sqrt{2\theta} \sum_{n=0}^{N-1} \sqrt{n}|n - 1\rangle\langle n| \quad |\psi\rangle = |N - 1\rangle
\]

(4.21)
This is essentially a quantum harmonic oscillator and hamiltonian projected to the lowest $N$ energy eigenstates. It is easy to check that the above satisfies (4.19). It represents a circular quantum Hall droplet of radius $\sqrt{2N\theta}$. The radius squared matrix coordinate $R^2$ is

$$R^2 = X_1^2 + X_2^2 = \frac{1}{2}(A^\dagger A + AA^\dagger)$$

$$= \sum_{n=0}^{N-2} \theta(2n+1)|n\rangle\langle n| + \theta(N-1)|N-1\rangle\langle N-1| \quad (4.22)$$

The highest eigenvalue of $R^2$ is $(2N-1)\theta$. So the giant graviton density is $\rho = \frac{N}{\pi R^2} = \frac{1}{2\pi\theta} = \rho_0$, gives back to the original density. Classically they rotate around the origin with the frequency $\omega = \frac{B}{m} = \ell_P^{-1}$.

This is the ground state of the system. For excitation state we are interested in the ‘quasiparticle’ and ‘quasihole’ states. For a quasihole of charge $-q$ sets at the origin the solution is given in [17] as

$$A = \sqrt{2\theta} \left( \sqrt{q}|N-1\rangle\langle 0| + \sum_{n=1}^{N-1} \sqrt{n+q}|n-1\rangle\langle n| \right) \quad q > 0 \quad (4.23)$$

The required condition (4.19) can also be checked explicitly. Meanwhile, the coefficient of the matrix $|n\rangle\langle n|$ goes like $2\theta(n+q)$, so the lowest nonzero mode is (approximately) $q$ and this is the picture of a quasihole. And now the outer radius of the droplet is shifted to $\sqrt{2(N+q)\theta}$. A number of $q$ giant gravitons in the center of the droplet are excited to the outer of the droplet, and explicitly the area of the droplet remains the same, which is just the APD requirement and the reflection of the Fermion nature.

Finally, the most general solution that can be viewed as giant gravitons excitation is

$$A = \sqrt{2\theta} \sum_{i=1}^{m} \left( \sqrt{q_i}|n_i\rangle\langle n_i-1| + \sum_{n=n_i+1}^{n_i-1} \sqrt{n+q_i}|n-1\rangle\langle n| \right) \quad (4.24)$$

where $|n_0\rangle = |0\rangle$ and $|n_m\rangle = |N-1\rangle$. One can see that it describes $m$ groups of giant gravitons excitation.

4.2 Quantization and the Fractional Quantum Hall Effect

We now come to the question of quantization of the above matrix model. After obtaining the matrix eigenstates, now we have no reason to neglect the matrix effect and the canonical quantization condition (4.10) becomes

$$[(X_1)_{mn}, (X_2)_{kl}] = \frac{i}{B} \delta_{ml} \delta_{nk} \quad (4.25)$$

or in terms of $A = X_1 + iX_2$

$$[A_{mn}, A^\dagger_{kl}] = \frac{1}{B} \delta_{mk} \delta_{nl} \quad (4.26)$$
Now we have the new edge excitation field $\Psi$ and its quantization should also be included. We quantize it as boson

$$[\Psi_m, \Psi_n] = \delta_{mn} \quad (4.27)$$

So the system is a priori just $N(N+1)$ uncoupled oscillators. What couples the oscillators and reduces the system to effectively $2N$ phase space variables (the planar coordinates of the giant gravitons) is the Gauss law constraint (4.14). Following the treatment of [17], we show note that $X, \Psi$ terms $(G_X, G_\Psi)$ in (4.14) is exactly the quantum generator of the $U(N)$ algebra. From group representation theory we know that if $R_{\alpha\beta}^a$ is the matrix element of the generators of $SU(N)$ in any representation, and $a_\alpha, a_\beta$ a set of mutually commuting oscillators, then the operator

$$G^a = a_\alpha^\dagger R_{\alpha\beta}^a a_\beta \quad (4.28)$$

satisfy the $SU(N)$ algebra. Recall that $X_i$ lives in the adjoint representation and $\Psi$ in the fundamental, the equivalent form of $G_X, G_\Psi$ in terms of oscillator basis is

$$G_X^a = -i a_\alpha^\dagger f^{abc} a_c \quad G_\Psi^a = \Psi_m^\dagger T_{mn}^a \Psi_n \quad (4.29)$$

where the $f^{abc}$ is the structure constant of the $SU(N)$ Lie algebra. So finally, in the form of acting on physical states, the constraint that expressed separately in traceless and trace part are

$$\text{traceless} \quad (G_X^a + G_\Psi^a)|_{phys} = 0 \quad (4.30)$$
$$\text{trace} \quad (\Psi_n^\dagger \Psi_n - BN\theta)|_{phys} = 0 \quad (4.31)$$

Focusing on the trace part, we notice that the former part, the occupation number operator $\Psi_n^\dagger \Psi_n$ takes integer eigenvalues, so the $BN\theta$ should also take integer value. What is more, the traceless equation shows that the physical state is in a singlet representation of the $SU(N)$. Since physical states are invariant under the sum of $G_X$ and $G_\Psi$, the representations of $G_X$ and $G_\Psi$ must be conjugate to each other, so their product contains the singlet. Therefore, the irreducible representation of $G_\Psi$ must also have a number of boxes in Young tableau which is a multiple of $N$. The oscillator realization (4.31) contains all the symmetric irreducible representation of $SU(N)$, whose Young tableau consists of a single row. The number of boxes equals the total number operator of the oscillators $\Psi_n^\dagger \Psi_n$. So we conclude that $BN\theta$ must be an integer multiple of $N$, that is

$$B\theta = \nu^{-1} = k \quad k = \text{integer} \quad (4.32)$$

This is nothing but the FQHE inverse filling factor quantization condition. The reciprocal $\nu = k^{-1}$ is the more used filling factor. Classically the integer can be any nonnegative integer, but quantum mechanically there is a shift $k \rightarrow k + 1$, which can be viewed as the effect of zero point energy [5]. So the inverse filling factor is strictly positive. Further from standard argument in QHE we know that the odd $k$ corresponds to the fermionic statistics and the even the bosonic. Given the fermionic statistics, we get that the $k$ should be odd numbers. So finally we obtain the fractional QHE behavior with the odd inverse filling factor.
4.3 The Effective Interaction of the Giant Graviton

To determine the interaction in the generalized Quantum Hall Effect, it is convenient to adopt the Hamiltonian approach. The Hamiltonian of the system is

\[ H = V = \frac{B\omega}{2} \text{Tr} X_1^2 = \frac{B\omega}{4} \text{Tr}(AA^\dagger + A^\dagger A) \] (4.33)

At the classical level, the Gauss law constraint can be solved in the eigenvalue basis of \( X_1 \)

\[ (X_1)_{mn} = x_n \delta_{mn} \quad (X_2)_{mn} = y_n \delta_{mn} + \frac{i\theta}{x_m - x_n} (1 - \delta_{mn}) \] (4.34)

Substituting the solution into the classical Hamiltonian, one obtains the Hamiltonian in terms of the variables \( x_n \)

\[ H = \sum_{n=1}^{N} \left( \frac{\omega}{2B} p_n^2 + \frac{B\omega}{2} x_n^2 \right) + \sum_{n \neq m} \frac{k(k-1)}{(x_m - x_n)^2} \] (4.35)

The latter term describes the interaction potential between different giant gravitons. It is nothing but the integrable one-dimensional Calogero model for non-relativistic particles on a line.

We see that going back to the IQHE \( k = 1 \) the interaction term vanishes. So the interaction is a novel property of the FQHE. We can also see the Hamiltonian imply that interaction between the giant gravitons is repulsive, for compressing the system means greater potential. This is common in the condense matter physics.

The ground-state wave function of the Calogero model

\[ \Psi_0(x_1, x_2, ...) = \prod_{m<n} (x_m - x_n)^k \exp\{-\frac{B}{2} \sum_n x_n^2\} \] (4.36)

is nothing but the 1d representation (in the Landau gauge) of the Laughlin wave function in the LLL with \( \nu = k^{-1} \) on a disk geometry. In this way the relationship between the \( k \) and the statistics can be clearly seen.

5. Conclusion and Speculation

We have explicitly shown that at the probe limit, the M-theory sector of the AdS/CFT/QH correspondence in the LLM geometry can adopt Fractional QHE. In fact, the low energy effective action of the probe giant graviton in the LLM geometry is exactly the QHE, this is enough for all the later derivation. Because the original correspondence only relates the Integer QHE sector (for example, see [21]), finding the exact counterpart in the supergravity side and the CFT side should be an interesting open question.

Although the dual geometry is an interesting open question, we want to make some speculation on it. In the IIB case, the nonsingular condition requires the function \( z \) to be \( \pm \frac{1}{2} \). After a simple shift \( \tilde{z} = z - \frac{1}{2} \) the 2d configuration is truly black-white, so the fractional particle density form the fractional QHE side is obviously identified as \( \tilde{z} \). However, in the
M-theory sector the nonsingular condition (2.20) is far from directly identified as a fermionic density-like function that adopts two values, which makes the fractional density explanation difficult. To attack the problem, corresponding to the IIB case, we can make an intuitive identification between the two nonsingular condition and the occupied/unoccupied states of the giant gravitons. For example, we can put the $S^5$ shrinking condition in the M2-brane case corresponding to the unoccupied state as the background. Then we speculate that the fractional charge density requirement from the fractional QHE side acting on the function $D$ just corresponds to a state different from and in some sense between the two $D$s that satisfy the separated nonsingular condition. A direct linear combination, say $D = \rho D_1 + (1-\rho)D_2$ where $D_1$ and $D_2$ satisfy separately the $S^2$ and $S^5$ shrinking condition of (2.20), will not work. That’s because the required Toda equation is not a linear equation. Needless to say, in our explanation the geometry of such kind of solution, if translated by the standard LLM dictionary [2], is singular. This is just because the obvious deviation from the nonsingular condition.

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