Asymptotic expansions of the inverse of the Beta distribution

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Abstract

In this work in progress, we study the asymptotic behaviour of the $p$-quantile of the Beta distribution, i.e. the quantity $q$ defined implicitly by $\int_0^q t^{a-1}(1-t)^{b-1}dt = pB(a,b)$, as a function of the first parameter $a$. In particular, we derive asymptotic expansions of $q$ and its logarithm at $0$ and $\infty$. Moreover, we provide some relations between Bell and Nørlund Polynomials, a generalisation of Bernoulli numbers. Finally, we provide Maple and Sage algorithms for computing the terms of the asymptotic expansions.

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1 Introduction

1.1 Background

Granted a probability distribution on $\mathbb{R}$, its median is defined as the value $m \in \mathbb{R}$ that leaves exactly half of the “mass” of the distribution on its left and half on its right. Instead of requiring that $m$ splits the mass exactly in two equal parts, one may choose a $p \in [0,1]$ and define the more general notion of the $p$-quantile value of the probability distribution:\[4\]

Definition 1 Let $F$ be a cumulative distribution function on some subset $I \subset \mathbb{R}$. Let $p \in [0,1]$. A $p$-quantile of $F$ is a point $q \in I$ such that $F(q) = p$. If $p = 1/2$, a $1/2$-quantile is called median.

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For an arbitrary probability distribution on $\mathbb{R}$, not always do $p$-quantiles exist, neither do they have to be unique, but for a distribution with density wrt to Lebesgue measure $p$-quantile values always exist, as then the distribution function is continuous and increasing, and if furthermore the density is a.e. non-zero, they are also unique, as the distribution function shall be strictly increasing.

One point of interest has been the study of the $p$-quantiles, including medians, of a parametrised family of probability distributions as a function of the parameter, given a fixed value of $p$. Such a function is well defined if the distribution has density wrt to the Lebesgue measure which is a.e. non-zero. Questions that may arise in this context have to do with analyticity, monotonicity, geometric properties and approximations, in particular asymptotic expansions, of the implicit function $q(a)$ defined by an equation of the form $F_a(q(a)) = p$, where $F_a$ is a family of cumulative distribution functions. Because of the implicit definition, the study of its properties can be challenging. An example is the median of the gamma distribution, which has been studied in several occasions, for example in [7], [5], and many connections have been found, for example with the Ramanujan’s rational approximation of $e^x$, see [2], [8] and [1], while in [6] it was also proved that it is a convex function.

In this paper, considering $p$ fixed in $(0,1)$, we focus on studying the $p$-quantile of the beta distribution, i.e. the distribution on $[0,1]$ with the density function $t \mapsto t^{a-1}(1 - t)^{b-1}$, as a function of the parameter $a$ considering $b$ fixed. The $p$-quantile of the beta distribution has been considered by Temme in [12], who studied the asymptotic behaviour of the $p$-quantile (or in his notation, the inverse of the normalised beta incomplete function) under restrictions over relations between the two parameters of the beta distribution. Also, see [11] for some inequalities on the median. This preprint is to be a continuation of our work in [4], which deals with convexity/concavity properties.

The $p$-quantile of the beta distribution, as a function of the first parameter, is defined as:

**Definition 2** Fix $p \in (0,1)$ and $b \in (0,\infty)$. The function $q : (0,\infty) \to (0,1)$ defined implicitly by

$$
\int_0^{q(a)} t^{a-1}(1 - t)^{b-1} dt = p \int_0^1 t^{a-1}(1 - t)^{b-1} dt
$$

is called the $p$-quantile of the beta distribution with parameters $a$ and $b$.

As in [5] for the case of the median of the gamma distribution, to study the $p$-quantile we consider and study an auxiliary function related to its logarithm

$$
\varphi(a) := -a \log q(a)
$$

and it will become clear that studying the logarithm gives more information on the behaviour of the $p$-quantile. One may also consider $\varphi$ itself as the $(1 - p)$-
quantile of some distribution. Indeed, using change of variables in \((1.1)\)

\[
\int_0^{x(a)} e^{-s}(1 - e^{-s/a})^{b-1}ds = (1 - p) \int_0^\infty e^{-s}(1 - e^{-s/a})^{b-1}ds
\]

Later, Bernoulli numbers and a generalisation of them known as Nørlund polynomials will become useful. The Bernoulli numbers \(B_n\) are classically defined through their generating function

\[
\frac{x}{e^x - 1} = \sum_{n=0}^{\infty} B_n \frac{x^n}{n!}
\]

(1.4)

They can be generalised to the Bernoulli polynomials \(B_n(t)\), defined similarly through the generating function

\[
\frac{x e^{tx}}{e^x - 1} = \sum_{n=0}^{\infty} B_n(t) \frac{x^n}{n!}
\]

(1.5)

Another generalisation of Bernoulli numbers are the Nørlund polynomials \(B_n^{(s)}\) defined through the generating function

\[
\left(\frac{x}{e^x - 1}\right)^s = \sum_{n=0}^{\infty} B_n^{(s)} \frac{x^n}{n!}
\]

(1.6)

They are polynomials in \(s\). If \(s \in \mathbb{N}\), then \(B_n^{(s)}\) is the \(s\)-fold convolution of Bernoulli numbers. An account on Nørlund polynomials can be found in [9, 24.16] and references within. Bernoulli and Nørlund appear often when we consider asymptotic expansions of the gamma and related functions (see e.g. [13]).

1.2 Main results

We state the following propositions regarding first order asymptotics. They are proved in [4]. In the rest, \(\gamma_b\) denotes the \((1 - p)\)-quantile of the gamma distribution with parameter \(b\).

**Proposition 1** [4, Proposition 1.2] The \(p\)-quantile of the beta distribution \(q(a)\) is a real analytic, increasing function of \(a\). It has limits

\[
\lim_{a \to 0} q(a) = 0
\]

and

\[
\lim_{a \to \infty} q(a) = 1
\]
Proposition 2 \[4, \text{Proposition 1.3}\] The function $\varphi(a) = -a \log q(a)$ is real analytic and increasing for $b > 1$, constant for $b = 1$ and decreasing for $b < 1$. It has limits

$$\lim_{a \to 0} \varphi(a) = \log p \quad (1.7)$$

and

$$\lim_{a \to \infty} \varphi(a) = \gamma_b \quad (1.8)$$

To study the asymptotic behaviour of $q$ and $\varphi$ in more depth, we shall try to find the asymptotic expansions of $\varphi$ at 0 and $\infty$. Studying asymptotic expansions of implicit functions can be highly non-trivial, as the method and the obstacles arising depend much on the form of the defining implicit relation. For the $p$-quantile of the beta distribution, we consider the cases of asymptotic expansions of $\varphi$ centered at 0 and at $\infty$. In both cases, we shall combine differentiation and Faà di Bruno’s formula (2.10), and the existence of the expansion has to be proved inductively.

For the case of 0, we shall compute the limits of the derivatives. For the case of $\infty$, for the same purpose, we shall introduce the differential operator $D$ defined by

$$Df(x) = x^2 \partial f(x)$$

where $\partial$ denotes the common differentiation operator. The calculus of $D$ is studied in subsection 3.1.

This operator has the importance that it can give, under certain conditions, the asymptotic expansion of a suitably smooth function at infinity, which is summarized in the following lemma, which is proved in subsection 3.1:

**Lemma 1.1** Let $f \in C^n(0, \infty)$ for some $n \in \mathbb{N}$. Then, the following hold:

i. If $\lim_{x \to \infty} D^m f(x)$ exists in $\mathbb{R}$ for all $m \leq n$, we have the asymptotic expansion

$$f(x) \sim \sum_{k=0}^{n-1} \frac{c_k}{x^k} + O\left(\frac{1}{x^n}\right)$$

where

$$c_k = \frac{(-1)^k}{k!} \lim_{a \to \infty} D^k f(a), \quad m < n$$

ii. Assume, conversely, that $f$ has asymptotic expansion of order $n$, i.e. $f(x) \sim \sum_{k=0}^{n} \frac{c_k}{x^k} + O\left(\frac{1}{x^n}\right)$, as well as that its derivatives $f^{(m)}$ admit asymptotic expansions of orders $m+n$, for $m \leq n$. Then, we have

$$c_k = \frac{(-1)^k}{k!} \lim_{a \to \infty} D^k f(a)$$

We note that, if conditions in i. hold, we may apply the lemma to $D^k f$ and get asymptotic expansions of higher derivatives, hence the expansion in i. can be
differentiated. Also, if in the previous lemma \( f \in C^\infty(0, \infty) \) and its conditions hold for all \( n \), then we may get the whole asymptotic expansion of \( f \).

Regarding the functions \( \varphi \) and \( q \), we have the following two pairs of theorems and Corollaries on their asymptotic expansions, which are proved in sections 2 and 3 respectively. In the following, \( \Psi(n, z) := 2^{n+1} \log \Gamma(z) \) denotes the polygamma function. Also, \( (m)_n \) denotes the Pochhammer symbol of \( m \), i.e. \( (m)_n = m(m + 1)\ldots(m + n - 1) \). If \( m \in \mathbb{N} \), then we have \( (m)_n = \frac{(m+n-1)!}{(m-1)!} \), and \( (-m)_n = (-1)^n \frac{m!}{(m-n)!} \) if \( n \leq m \), and \( (-m)_n = 0 \) if \( n > m \). These identities will be widely used in this paper.

**Theorem 1** The function \( \varphi \) admits the asymptotic expansion

\[
\varphi(a) \sim \sum_{n=0}^{\infty} c_n a^n \quad \text{at} \quad 0,
\]

with \( c_0 = \log p \) and

\[
c_n = \frac{\Psi(n-1, b) - \Psi(n-1, 1)}{n!} = \frac{(-1)^{n+1}}{n!} \int_0^\infty u^{n-1} \left( \frac{e^{-u} - e^{-bu}}{1 - e^{-u}} \right) du, \quad n \geq 1
\]

For \( b \in \mathbb{N} \), we have in particular

\[
c_n = (-1)^{n+1} (n-1)! \sum_{k=1}^{n-1} \frac{1}{k^n} \tag{1.9}
\]

**Corollary 1** An approximation for \( \varphi \) for values of \( a \) close to 0 is

\[
\varphi(a) \sim \log \frac{\Gamma(a + b)}{\Gamma(a + 1)\Gamma(b)} - \log p
\]

and for \( q \)

\[
\frac{q(a)}{p^{1/a}} \sim \left( \frac{\Gamma(a + b)}{\Gamma(a + 1)\Gamma(b)} \right)^{1/a}
\]

each having a remainder term vanishing faster than \( a^n \) at 0, \( \forall n \in \mathbb{N} \). Hence, we have the asymptotic expansion

\[
\frac{q(a)}{p^{1/a}} \sim e^{-\gamma - \Psi(0, b)} \left( \sum_{n=0}^{\infty} \frac{B_n(c_1, c_2, \ldots, c_n)}{n!} a^n \right) \tag{1.10}
\]

where \( \gamma \) is the Euler constant, \( c_n = \frac{\Psi(n-1, b) - \Psi(n-1, 1)}{n!} \) and \( B_n \) denotes the \( n \)th complete Bell polynomial (see Remark 2.1).

**Theorem 2** The function \( \varphi \) admits the asymptotic expansion

\[
\varphi(a) \sim \sum_{n=0}^{\infty} \varphi_n (-1)^n \frac{1}{n! a^n}, \quad a \to \infty
\]
at $\infty$, with $\varphi_n$ satisfying the system of recursive relations

\[
\varphi_n = -\sum_{j=1}^{n-1} \binom{n-1}{j} \varphi_{n-j} \delta(0, j, 0) - \sum_{k=0}^{n-2} \sum_{j=0}^{k} \binom{k}{j} \varphi_{k-j+1} \delta(0, j, n-k-1)
+ B_n^{(1-b)} \sum_{k=0}^{n-1} (b + n - k) \gamma_b^n \tag{1.11}
\]

\[
\delta(k, m, n) = \delta(k, m - 1, n + 1) + \sum_{j=0}^{m-1} \binom{m-1}{j} \varphi_{m-j} \delta(k + 1, j, n) \tag{1.12}
\]

and the initial conditions

\[
\varphi_0 = \gamma_b \tag{1.13}
\]

\[
\delta(k, 0, n) = B_n^{(1-b)} \sum_{j=0}^{k} \binom{k}{j} (-1)^{k-j} (b + n - j) \gamma_b^{n-j} \tag{1.14}
\]

The recursive relations (1.11) and (1.12) in the foregoing lemma work inductively. We know $\varphi_0$ and once we have computed $\varphi_0, \ldots, \varphi_{n-1}$, in order to compute $\varphi_n$ we use (1.11), where the maximum of the second argument of $\delta$ that is at most $n - 1$, and we can compute these terms using (1.12) and the initial conditions, as $\varphi_k$ appears there in orders at most equal to the second argument of $\delta$, and that we already have computed. This algorithm can give us the first terms of the asymptotic expansion:

\[
\varphi(a) = \gamma b - \frac{\gamma b (b - 1)}{2a} + \frac{\gamma b (-1 + b) (7b + \gamma b - 5)}{24a^2} - \frac{\gamma b (-1 + b)^2 (3b + \gamma b - 1)}{16a^3} + O \left( \frac{1}{a^4} \right) \tag{1.15}
\]

Also, for $q$ we then get:

**Corollary 2** For $a \to \infty$, an asymptotic expansion for $q$ is

\[
q(a) \sim \sum_{n=0}^{\infty} \frac{B_n(-\varphi_0, 2\varphi_1, -3\varphi_2, \ldots, (-1)^n n \varphi_{n-1})}{n!} \frac{1}{a^n} \tag{1.16}
\]

where $\varphi_n$ is the sequence defined in Theorem 2.

In section 4 we state some relations between Nørlund, Bernoulli and Bell polynomials that we came upon and we could not find in the literature. These relations come out by considering the coefficients of Bernoulli generating functions as taylor coefficients, i.e. as limits of derivatives, and using Faà di Bruno’s formula, and its relation to Bell polynomials, to compute these derivatives. Finally, in the appendix we implement the recursive relations of Theorem 2 as Maple and Sage algorithms and give coefficients of asymptotic expansions for some specific values.
2 Asymptotics at 0

For computing the asymptotic expansion of \( \varphi \) at 0, our method consists of iterated differentiation of relations that implicitly contain the \( p \)-quantile and use Faà di Bruno’s formula. Then, taking limits for \( a \to 0 \) and computing the limits of all the terms, we compute the limits of the derivatives which then yields the asymptotic expansion, as, if \( f \in C^\infty(0, \varepsilon) \), for some \( \varepsilon > 0 \), and \( \lim_{x \to 0} f^{(n)}(x) \) exists in \( \mathbb{R} \) for all \( n \), denoting this limit by \( f^{(n)}(0) \) we have \( f \sim \sum_{k=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n \).

The converse is not necessarily valid: if \( f \) admits asymptotic expansion at 0 it is not necessary that the limits of the derivatives exist, as there may be oscillations. The limits of the derivatives of \( \varphi \) will be computed then inductively.

First, we use integration by parts in (1.1) getting
\[
e^{-\varphi(a)}(1-q(a))^{b-1} + (b-1) \int_0^{q(a)} t^a(1-t)^{b-2}dt = ap\frac{\Gamma(b)\Gamma(a)}{\Gamma(a+b)} = p\Gamma(b)W(a)
\]
where \( W(a) = \Gamma(a+1)/\Gamma(a+b) \). This function \( W \) is studied in \([3, C4]\), and in a generalised form in \([10]\), where several properties, such as complete monotonicity, are proved. We consider the logarithmic derivative of \( W \) and we note that, as \( \log W(a) = \log \Gamma(a+1) - \log \Gamma(a+b) \), by the integral representation of the digamma function \( \Psi(0, z) \) (see \([3\, Theorem 1.6.1]\)), we get that
\[
(\log W(a))' = \Psi(0, a+b) - \Psi(0, a+1) = -\int_0^\infty e^{-au} \left( \frac{e^{-bu} - e^{-tu}}{1-e^{-u}} \right) du \quad (2.2)
\]

We define the function \( \psi \) by
\[
\psi(a) := -\log \frac{\Gamma(a+1)}{\Gamma(a+b)} - \log p\Gamma(b) = -\log W(a) - \log p\Gamma(b) \quad (2.3)
\]
which implies that \( e^{-\psi(a)} = p\Gamma(b)W(a) \) and (2.3) can be rewritten as
\[
e^{-\varphi(a)}(1-q(a))^{b-1} + (b-1) \int_0^{q(a)} t^a(1-t)^{b-2}dt = e^{-\psi(a)} \quad (2.4)
\]
Hence, as \( \psi \in C^\infty(-1, \infty) \), denoting the limit of the \( k \)th derivative of \( \psi \) at 0 by \( \psi^{(k)}(0) \), by (2.2) we have
\[
\psi^{(k)}(0) = \Psi(k-1, b) - \Psi(k-1, 1) = (-1)^{k-1} \int_0^{\infty} u^{k-1} \left( \frac{e^{-u} - e^{-bu}}{1-e^{-u}} \right) du \quad (2.5)
\]
Let denote by \( \varphi^{(n)}(0) \) the right limit of \( \varphi^{(n)} \) at 0, supposing it exists. We already have, combining \([1.7]\) and \([2.3]\), that \( \varphi(0) = \psi(0) = -\log p \). Our goal is to prove that for the limits of all the derivatives of \( \varphi \) and \( \psi \) at 0 are the same, i.e. we have \( \varphi^{(k)}(0) = \psi^{(k)}(0) \). Differentiating (2.3) we get
\[
-\psi'(a)e^{-\psi(a)} + \varphi'(a)e^{-\varphi(a)}(1-q(a))^{b-1} = (b-1) \int_0^{q(a)} t^a(1-t)^{b-2} \log t dt
\]
We define the functions
\[ \rho(a) := (1 - q(a))^{b-1} \] (2.6)
\[ \sigma(a) := \int_0^a t^a (1 - t)^{b-1} \log t \, dt \] (2.7)
and hence the last equation can be rewritten as
\[ -\psi'(a)e^{-\psi(a)} + \varphi'(a)e^{-\varphi(a)} \rho(a) = (b - 1)\sigma(a) \] (2.8)
We will use this equality to find the limits of the derivatives of \( \varphi \). This will be done inductively, differentiating (2.8) at each step. Our strategy is, at the kth step, where we will want to compute the limit of the kth derivative, that we use the results from the previous steps about the asymptotic behaviour of \( \varphi \) up to the kth derivative to find the asymptotic behaviour of the derivatives of \( q \) up to \( k \), and then use this result to find the behaviour of the derivatives of \( g \) and \( h \) up to \( k \), so that we finally compute the limit of the kth derivative of \( \varphi \). The first part will be done in the next lemmas, and the inductive proof will be given in the end of the section.

We state the following well known differentiation formulas that we will be constantly using, see (1.4.12) and (1.4.13) in [9]: The product formula for derivation,
\[ \left( \prod_{i=1}^k f_i(x) \right)^{(n)} = \sum_{\{j \in \mathbb{N} | \sum_{i=1}^k j_i = n\}} \frac{n!}{m_1!m_2!...m_n!} f_1(j_1) f_2(j_2) ... f_k(j_k) \] (2.9)
and the Faà di Bruno formula, for the derivatives of composite functions,
\[ (f \circ g)^{(n)}(x) = \sum_{\{m \in \mathbb{N}^n | j_i = m_i, \sum_{i=1}^n j_i = n\}} \frac{n!}{m_1!m_2!...m_n!} f_1(j_1) f_2(j_2) ... f_k(j_k) \] (2.10)
The latter, in case \( f(x) = \log(x) \), can take the simpler form
\[ (\log g(x))^{(n)} = \sum_{\{m \in \mathbb{N}^n | \sum_{i=1}^n j_i = m_i\}} C_m \prod_{j=1}^n \left( \frac{g^{(j)}(x)}{g(x)} \right)^{m_j} \] (2.11)
where
\[ C_m = (-1)^{1+\sum_{j=1}^n m_j} \frac{n! \left( \sum_{j=1}^n m_j - 1 \right)!}{m_1!m_2!...m_n!} \prod_{j=1}^n \frac{1}{j^{m_j}} \]
and for \( f(x) = e^x \),
\[ (e^{g(x)})^{(n)} = e^g(x) \sum_{\{m \in \mathbb{N}^n | \sum_{i=1}^n j_i = m_i\}} \frac{n!}{m_1!m_2!...m_n!} \prod_{j=1}^n \left( \frac{g^{(j)}(x)}{j!} \right)^{m_j} \] (2.12)
Remark 2.1 Faà di Bruno formula (2.10) is related to the polynomials that are known as Bell polynomials. The (complete) Bell polynomials are defined by the relation
\[ B_n(x_1, x_2, \ldots, x_n) = \sum_{\{k_j \geq 0 \mid \sum_j jk_j = n\}} \frac{n!}{k_1!k_2! \cdots k_n!} \prod_j \left( \frac{x_j}{j!} \right)^{k_j} \] (2.13)

We can express the special case (2.12) of Faà di Bruno’s formula for the exponential in terms of these Bell polynomials
\[ e^{g(x)} \right)^{(n)} = e^{g(x)} B_n(g'(x), g''(x), \ldots, g^{(n)}(x)) \]

Lemma 2.1 Let \( k, l \in \mathbb{N} \). Then,
\[ \lim_{a \to 0} \frac{q(a) \log^k q(a)}{a^l} = 0 \] (2.14)

Proof We have
\[ \log \left( \frac{q(a)}{a^m} \right) = \log q(a) - m \log a \to -\infty \]
for \( a \to 0 \), as, by (1.7),
\[ a(\log q(a) - m \log a) = a \log q(a) - ma \log a \to \log p \]
This implies that
\[ \lim_{a \to 0} \frac{q(a)}{a^m} = 0 \]
Also (1.7) gives
\[ \lim_{a \to 0} a^k \log^k q(a) = \log^k p \]
Hence
\[ \lim_{a \to 0} \frac{q(a) \log^k q(a)}{a^l} = \lim_{a \to 0} \frac{q(a) \log^k q(a)}{a^{l-k}} \frac{a^k}{a^k} = 0 \]

Lemma 2.2 Let \( N \in \mathbb{N}^* \) and assume that \( \lim_{a \to 0} \varphi^{(k)}(a) \) exists in \( \mathbb{R} \), \( \forall k \leq N \). Then, \( \forall k \leq N \),
\[ \lim_{a \to 0} \frac{a^{2k} \varphi^{(k)}(a)}{q(a)} \] exists in \( \mathbb{R} \) (2.15)
In particular, we have that
\[ \lim_{a \to 0} \frac{\varphi^{(k)}(a)}{a^m} = 0, \quad m \geq 0 \] (2.16)
Proof. For $k = 1$, as $\varphi'(a) = -\log q(a) - aq'(a)/q(a)$, we have that

$$a^2q'(a)/q(a) = -a\varphi'(a) - a\log q(a) \to -\log p$$

so (2.15) holds. Assume that $1 \leq n < N$ and that (2.15) holds $\forall k \leq n$. We will prove that (2.15) holds for $k = n + 1$. Indeed, using (2.11), we get, for some coefficients $c_k$ and $d_k$,

$$-\varphi^{(n+1)}(a) = a(\log q(a))^{(n+1)} + (n+1)(\log q(a))^{(n)}$$

$$= a \sum_{\{k|j=1 \ldots j_k=n+1\}} c_k \prod_{j=1}^{n+1} \left( \frac{q^{(j)}(a)}{q(a)} \right)^{k_j}$$

$$+ (n+1) \sum_{\{k|j=1 \ldots j_k=n\}} d_k \prod_{j=1}^{n} \left( \frac{q^{(j)}(a)}{q(a)} \right)^{k_j}$$

But one can write

$$\sum_{\{k|j=1 \ldots j_k=n+1\}} c_k \prod_{j=1}^{n+1} \left( \frac{q^{(j)}(a)}{q(a)} \right)^{k_j} = \frac{q^{(n+1)}(a)}{q(a)} + \sum_{\{k|j=1 \ldots j_k=n+1\}} c_k \prod_{j=1}^{n} \left( \frac{a^{2j}q^{(j)}(a)}{q(a)} \right)^{k_j}$$

hence, rearranging the equation above and multiplying each side by $a^{2n+1}$, we get

$$a^{2(n+1)} \frac{q^{(n+1)}(a)}{q(a)} = -a^{2n+1}\varphi^{(n+1)}(a) - \sum_{\{k|j=1 \ldots j_k=n+1\}} c_k \prod_{j=1}^{n} \left( \frac{a^{2j}q^{(j)}(a)}{q(a)} \right)^{k_j}$$

$$- a(n+1) \sum_{\{k|j=1 \ldots j_k=n\}} d_k \prod_{j=1}^{n} \left( \frac{a^{2j}q^{(j)}(a)}{q(a)} \right)^{k_j}$$

and the right hand side converges in $\mathbb{R}$ as $a \to 0$ by our induction hypothesis, proving (2.15). To prove (2.16), we see that combining this result with Lemma 2.4 gives

$$\lim_{a \to 0} \frac{q^{(k)}(a)}{a^m} = \lim_{a \to 0} \frac{a^{2k}q^{(k)}(a)}{q(a)} \frac{q(a)}{a^{m-2k}} = 0$$

Lemma 2.3 Let $N \in \mathbb{N}^+$ and assume that $\lim_{a \to 0} \varphi^{(k)}(a)$ exists in $\mathbb{R}$, $\forall k \leq N$. Then, $\forall k \leq N$,

$$\lim_{a \to 0} \rho^{(k)}(a) = 0, \quad k \neq 0$$

$$\lim_{a \to 0} \rho(a) = 1$$

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By our assumptions, \[ \lim_{a \to q} \] so it suffices to prove that \[ \sigma_n(a) \] as \[ a \to 0 \] using (2.10) as

\[ \frac{d}{d \gamma} \] which, by Lemma 2.2 tends to 0 as \( a \to 0 \), as \( q^{(j)}(a) \to 0 \).

**Lemma 2.4** Let \( N \in \mathbb{N}^n \) and assume that \( \lim_{a \to 0} \varphi^{(k)}(a) \) exists in \( \mathbb{R} \), \( \forall k \leq N \). Then, \( \forall k \leq N \),

\[ \lim_{a \to 0} \sigma^{(k)}(a) = 0 \]

**Proof** We have

\[ \frac{d}{d \gamma} \int_{0}^{q(a)} t^n (1-t)^{b-2} \log^m t \, dt \to 0 \]

as \( q(a) \to 0 \) and \( (1-t)^{b-2} \log^m t \) is integrable near 0. Hence, \( \sigma(a) \to 0 \). For \( n > 0 \) we have

\[ \sigma^{(n)}(a) = \int_{0}^{q(a)} t^n (1-t)^{b-2} \log^{n+1} t \, dt + \sum_{k=1}^{n} [e^{-\varphi(a)}(1-q(a))^{b-2} q'(a) \log^k q(a)]^{(n-k)} \]

(2.17)

So, it suffices to prove that

\[ [e^{-\varphi(a)}(1-q(a))^{b-2} q'(a) \log^k q(a)]^{(l)} \to 0, \quad \forall k, l \leq N \]

By (2.9) we can write

\[ [e^{-\varphi(a)}(1-q(a))^{b-2} q'(a) \log^k q(a)]^{(l)} = \sum_{\{m: \sum_{j=1}^{k} m_j = l\}} c_m [e^{-\varphi(a)}]^{(m_1)} [(1-q(a))^{b-2}]^{(m_2)} [q'(a) \log^k q(a)]^{(m_3)} \]

By our assumptions, \( \lim_{a \to 0} [e^{-\varphi(a)}]^{(m_1)} \in \mathbb{R} \), and as in Lemma 2.3 \( [(1-q(a))^{b-2}]^{(m_2)} \) also converges. Finally, by (2.9), (2.10) and Lemma 2.2

\[ [q'(a) \log^k q(a)]^{(m)} = \sum_{\{n: \sum_{j=1}^{k+1} n_j = m\}} c_n q^{(n_1+1)}(a) \prod_{j=2}^{k+1} [\log q(a)]^{(n_j)} \]

\[ \sum_{\{n: \sum_{j=1}^{k+1} n_j = m\}} c_n q^{(n_1+1)}(a) \prod_{j=2}^{k+1} \sum_{\{r: \sum_{s=1}^{n_j} r_s = n_j\}} d_r \prod_{s=1}^{n_j} \left( \frac{q^{(s)}(a)}{q(a)} \right)^{r_s} \]

\[ \sum_{\{n: \sum_{j=1}^{k+1} n_j = m\}} c_n q^{(n_1+1)}(a) \prod_{j=2}^{k+1} \sum_{\{r: \sum_{s=1}^{n_j} r_s = n_j\}} d_r \prod_{s=1}^{n_j} \left( \frac{q^{(s)}(a)2^s}{q(a)} \right)^{r_s} \to 0 \]
which completes the proof of the Lemma. □

**Proof of theorem 1** By Proposition 2 we have that \( \varphi(0) = -\log p \). For the first derivative, as \( \rho(0) = 1 \) and \( \sigma(0) = 0 \), and \( \varphi(0) = \psi(0) = -\log p \), we get from (2.10) that the limit \( \lim_{a \to 0} \varphi'(a) = \varphi'(0) \) exists and \( \varphi'(0) = \psi'(0) \). We proceed inductively. Let \( n \in \mathbb{N}^* \) and assume that \( \lim_{a \to 0} \varphi^{(k)}(a) \) exists and \( \varphi^{(k)}(0) = \psi^{(k)}(0) \) \( \forall k \leq n \). Differentiating (2.8) \( n \) times we get

\[
(e^{-\psi(a)})^{(n+1)} - (e^{-\psi(a)})^{(n+1)}\rho(a) - \sum_{k=0}^{n-1} (e^{-\psi(a)})^{(k+1)}\rho(a)^{(n-k)} = (b-1)\sigma^{(n)}(a)
\]

and by Lemmas 2.3 and 2.4 we get

\[
\lim_{a \to 0} (e^{-\psi(a)})^{(n+1)} = \lim_{a \to 0} (e^{-\psi(a)})^{(n+1)}
\]

which, by formula (2.10) and the induction hypothesis, gives that the limit \( \lim_{a \to 0} \varphi^{(n+1)}(a) =: \varphi^{(n+1)}(0) \) exists in \( \mathbb{R} \) and

\[
\sum_{\{k|\sum_{j=1}^{n+1} j_m = n+1\}} c_k e^{-\psi(0)} \prod_{j=1}^{n+1} \left( \frac{\psi^{(j)}(0)}{j!} \right)^{m_j} = \sum_{\{k|\sum_{j=1}^{n+1} j_m = n+1\}} c_k e^{-\psi(0)} \prod_{j=1}^{n+1} \left( \frac{\psi^{(j)}(0)}{j!} \right)^{m_j}
\]

and as by the induction hypothesis \( \varphi^{(j)}(0) = \psi^{(j)}(0) \) for \( j \leq n \), it gives

\[
\varphi^{(n+1)}(0) = \psi^{(n+1)}(0)
\]

which completes the induction. To prove (1.13), the fact that

\[
\log \Gamma(x + 1) - \log \Gamma(x) = \log x
\]

gives the functional relation for the polygamma function

\[
\Psi(k, x + 1) - \Psi(k, x) = \frac{(-1)^k k!}{x^{k+1}}
\]

hence

\[
\varphi^{(k+1)}(0) = \psi(k, b) - \psi(k, 1) = \sum_{n=1}^{b-1} (\Psi(k, n + 1) - \Psi(k, n)) = \sum_{n=1}^{b-1} \frac{(-1)^k k!}{n^{k+1}}
\]

**Proof of Corollary 1** The fact that \( \varphi \) and \( \psi \) have the same asymptotic expansion at 0 implies that an approximation of \( \varphi \) is

\[
\varphi(a) \sim \log \frac{\Gamma(a + b)}{\Gamma(a + 1) \Gamma(b)} - \log p \quad \text{as } a \to 0
\]
and the error decreases faster than any positive power of \( a \). This also implies that

\[
q(a) \sim \left( \frac{\Gamma(a + 1)\Gamma(b)}{\Gamma(a + b)} \right)^{1/a} p^{1/a} \quad \text{as } a \to 0
\]

in the sense that \( \forall n \in \mathbb{N}, \varepsilon > 0, \exists a_{n, \varepsilon} > 0 \) such that \( \forall a < a_{n, \varepsilon} \)

\[
e^{-\varepsilon a^n} \left( \frac{\Gamma(a + 1)\Gamma(b)}{\Gamma(a + b)} \right)^{1/a} p^{1/a} < q(a) < e^{\varepsilon a^n} \left( \frac{\Gamma(a + 1)\Gamma(b)}{\Gamma(a + b)} \right)^{1/a} p^{1/a}
\]

hence

\[
\lim_{a \to 0} \frac{q(a)}{p^{1/a}} = e^{-\gamma - \Psi(0, b)}
\]

(2.19)

\( \gamma \) being the Euler’s constant. The RHS of the above inequality may be rewritten as

\[
\frac{q(a)}{p^{1/a}} < \left( \frac{\Gamma(a + 1)\Gamma(b)}{\Gamma(a + b)} \right)^{1/a} + \varepsilon' a^n \left( \frac{\Gamma(a + 1)\Gamma(b)}{\Gamma(a + b)} \right)^{1/a}
\]

close to 0 and for an \( \varepsilon' > \varepsilon \), and the LHS

\[
\left( \frac{\Gamma(a + 1)\Gamma(b)}{\Gamma(a + b)} \right)^{1/a} - \varepsilon a^n \left( \frac{\Gamma(a + 1)\Gamma(b)}{\Gamma(a + b)} \right)^{1/a} < \frac{q(a)}{p^{1/a}}
\]

Hence

\[
\frac{q(a)}{p^{1/a}} = \left( \frac{\Gamma(a + 1)\Gamma(b)}{\Gamma(a + b)} \right)^{1/a}
\]

with a remainder term vanishing faster than any power of \( a \) at 0. The rest comes from considering

\[
\left( \frac{\Gamma(a + 1)\Gamma(b)}{\Gamma(a + b)} \right)^{1/a} = \exp \left( \frac{1}{a} \log \frac{\Gamma(a + 1)\Gamma(b)}{\Gamma(a + b)} \right)
\]

along with Faà di Bruno formula.

\[
\square
\]

3 Asymptotics at \( \infty \)

3.1 The operator \( D \)

To find the asymptotic expansion at infinity, the previous technique has to be adjusted accordingly. First, we introduce the differential operator \( D \) defined by

\[
Df(a) = a^2 \hat{c} f(a)
\]
It satisfies the product rule

\[ D(fg)(a) = g(a)Df(a) + f(a)Dg(a) \]  (3.2)

and the composition rule

\[ D(f \circ g)(a) = f'(g(a))Dg(a) \]

The last two relations combined give us the Faa di Bruno formula for \( D^n(f \circ g)(a) \)

\[ D^n(f \circ g)(a) = \sum_{\{m \in \mathbb{N}_0^n\mid \sum_{j=1}^n m_j = n\}} \frac{n!}{m_1!m_2! \ldots m_n!} f^{(|m|)}(g(a)) \prod_{j=1}^n \left( \frac{D^{j}g(a)}{j!} \right)^{m_j} \]  (3.3)

where \( |m| = \sum_{j=1}^n m_j \). Also, we have the two-arguments composition rule

\[ Df(a, \varphi(a)) = D_1f(a, \varphi(a)) + D_2f(a, \varphi(a)) \]  (3.4)

where \( D_1f(a, b) = a^2(\varphi_1f)(a, b) \), \( \varphi_1 \) denoting differentiation with respect to the first variable of a multivariate function, i.e. in our case \( D_1f(a, \varphi(a)) = a^2(\varphi_1f)(a, \varphi(a)) \). Furthermore, we remark that it acts on monomials, for \( m \in \mathbb{Z} \), by

\[ Da^m = ma^{m+1} \]

and by induction

\[ D^n a^m = (m)_na^{m+n} \]

The operator \( D \) can be used to deal with asymptotic expansions at infinity. To see this, intuitively, starting from the formal power series

\[ f(x) = c_0 + \frac{c_1}{x} + \frac{c_2}{x^2} + \frac{c_3}{x^3} + \ldots \]

one can get

\[ D^n f(x) = \sum_{k=n}^{\infty} (-1)^n \frac{k!}{(n - k)!x^{k-n}} c_k \]

If certain conditions apply and it is possible to take limits to \( \infty \), all but the first term of the sum vanish and we get

\[ \lim_{x \to \infty} D^n f(x) = (-1)^n n! c_n \]

This is rigorously treated in Lemma 1.1 which is proved below:

**Proof of Lemma 1.1** To show i), we notice that if for a function \( f \) we have \( \lim_{x \to \infty} f(x) = a_0 \in \mathbb{R} \) and \( D f(x) = a_1 + a_2/x + a_3/x^2 + \ldots + a_{k+1}/x^k + O(1/x^{k+1}) \), then by integrating we get that \( f(x) = a_0 - a_1/x - a_2/2x^2 - a_3/3x^3 + \ldots + a_{k+1}/kx^{k+1} + O(1/x^{k+2}) \). Next, we see that, under the assumptions of the first part of the lemma, we have that \( \lim_{x \to \infty} D^m f(x) = a \in \mathbb{R} \) and
\[ \lim_{x \to \infty} D^n f(x) = b \in \mathbb{R}. \]  This implies that \( D^{n-1} f(x) = a + O(1/x) \). Applying this observation inductively to find the asymptotic expansions of lower powers of \( D \) proves the first part of the Lemma. For the second part, we notice that as the derivatives admit asymptotic expansions, these can be obtained by differentiating the asymptotic expansion of the original function. In the same way, we may apply the operator \( D \) to the original asymptotic expansion, as \( D^k \) can be expressed as a combination of operators \( \partial^l \) for \( l \leq k \), and take limits to \( \infty \) to prove the second part. 

In the following subsections we shall compute the asymptotic expansion of \( \varphi \) using the operator \( D \). We start with the equation

\[
\int_0^{\varphi(a)} \tau(a; s) ds = (1 - p) \frac{\Gamma(b) \Gamma(a) a^b}{\Gamma(a + b)}
\]  (3.5)

Where

\[
\tau(a; s) = e^{-s}(a - ae^{-s})^{-b-1}
\]  (3.6)

Our method consists of acting and iterating the operator \( D \) on (3.5) and taking the limits to \( \infty \) on both sides. So we have to see how \( D \) acts on \( \tau \) and on the right hand side.

### 3.2 Asymptotics of the RHS

To study the right hand side of the equation (3.5), we study the asymptotics of the ratio

\[
\frac{\Gamma(a) a^b}{\Gamma(a + b)}
\]  (3.7)

In [13], Tricomi and Erdelyi derived an asymptotic expansion for such ratios of Gamma functions, in terms of a generalisation of Nörlund Polynomials, which in our special case it may be expressed as

\[
\frac{\Gamma(a) a^b}{\Gamma(a + b)} \sim \sum_{n \geq 0} \frac{\Gamma(1 - b)}{\Gamma(1 - (b + n))} \frac{B_n^{(1-b)}}{n! a^n}, \quad x \to \infty
\]

which by the reflection formula for the Gamma function can be rewritten as

\[
\frac{\Gamma(a) a^b}{\Gamma(a + b)} \sim \sum_{n = 0}^{\infty} \frac{(-1)^n}{n!} (b)_n \frac{B_n^{(1-b)}}{a^n}
\]  (3.8)

We shall prove the following Lemma:

**Lemma 3.1** For \( n \in \mathbb{N} \), we have

\[
\lim_{x \to \infty} D^n \left( (1 - p) \frac{\Gamma(b) \Gamma(a) a^b}{\Gamma(a + b)} \right) = (1 - p) \Gamma(b + n) B_n^{(1-b)}
\]  (3.9)
The coefficients of the asymptotic expansion (3.8), by Lemma 1.1, can be used to give the limit in (3.9), if the derivatives of the ratio also admit asymptotic expansions. Hence we shall find these asymptotic expansions of the derivatives, and also a different expression for the coefficients in the asymptotic expansion of the ratio (3.7) on the way.

The tool we shall work with is the operator $D$ and its Faà di Bruno formula eq3.3. We denote the logarithmic derivative of the ratio (3.7) by $V_{p/a} ^{b} := \log \frac{\Gamma(x+h)}{\Gamma(x)} = \log a + \log \Gamma(x) - \log \Gamma(x+h)$ (3.10)

A classic result on the asymptotic expansion of $\log \Gamma$ is the following, see [9, 5.11.8], for fixed $h \in \mathbb{C}$,

$$\log \Gamma(x+h) \sim \log \sqrt{2\pi x} + \left(x + h - \frac{1}{2}\right) \log x - x + \sum_{n \geq 2} \frac{B_n(h)}{n(n-1)} x^{1-n}, \quad x \to +\infty$$

(3.11)

which has the nice property that it can also be differentiated, and give us asymptotic expansions of polygamma functions. This implies also that the derivatives of $V$ admit asymptotic expansions. We have, asymptotically,

$$V(a) \sim \sum_{n \geq 2} \frac{B_n - B_{n-1}(b)}{n(n-1)} x^{1-n} = \sum_{n \geq 1} \frac{B_{n+1} - B_{n+1}(b)}{n(n+1)} x^{1-n}$$

We have, then, by Lemma 1.1 that

$$\lim_{a \to \infty} D^n V(a) = (-1)^n n! \frac{B_{n+1} - B_{n+1}(b)}{n(n+1)}$$

(3.12)

Acting $D$ $n$ times on ratio (3.7) we get

$$D^n \left( \frac{\Gamma(a)}{\Gamma(a+b)} \right) = D^n e^{V(a)} = e^{V(a)} \sum_{\{m \in \mathbb{N}^n | \sum_{j=1}^n j m_j = n\}} \frac{n!}{m_1! m_2! \cdots m_n!} \prod_{j=1}^n \left( \frac{D^j V(a)}{j!} \right)^m_j$$

(3.13)

and taking limits we end up with

$$\lim_{a \to \infty} D^n \left( \frac{\Gamma(a)}{\Gamma(a+b)} \right) = \sum_{\{m \in \mathbb{N}^n | \sum_{j=1}^n j m_j = n\}} \frac{(-1)^n n!}{m_1! m_2! \cdots m_n!} \prod_{j=1}^n \left( \frac{B_{j+1} - B_{j+1}(b)}{j(j+1)} \right)^m_j$$

Hence, by Lemma 1.1, the derivatives of the ratio (3.7) admit asymptotic expansions at infinity, and these can be given by differentiating the asymptotic expansion (3.8).

Remark 3.1 In the proceeding proof, we find two different ways to express the asymptotic expansion of the ratio of gamma functions, which implies a relation
between Nørlund, Bernoulli and Bell polynomials we could not trace in the literature,

\[ (b)_nB_n^{(1-b)} = B_n(B_2(b) - B_2, B_3(b) - B_3, \ldots, B_{n+1}(b) - B_{n+1}) \quad (3.14) \]

and using the fact that

\[ \sum_{k=1}^{j+1} B_{j-k+1} \frac{(j - 1)!}{k!(j + 1 - k)!} b^k = B_{j+1}(b) - B_{j+1} \]

we get

\[ (b)_nB_n^{(1-b)} = \sum_{\{m \in \mathbb{N}^n | \sum_{j=1}^n jm_j = n\}} \frac{n!}{m_1!m_2! \ldots m_n!} \prod_{j=1}^n \left( \sum_{k=1}^{j+1} B_{j-k+1} \frac{(j - 1)!}{k!(j + 1 - k)!} b^k \right)^m_j \]

3.3 Asymptotics of the LHS

We shall first study the asymptotic behaviour of \( \tau \), defined in \( (3.6) \), through the following Lemma.

Lemma 3.2 We have the limits

\[ \lim_{a \to \infty} D^n \tau(a; s) = B_n^{(1-b)} e^{-s} s^{b-1+n} \quad (3.15) \]

Proof We have that

\[ \tau(a; s) = e^{-s} (a - ae^{-s/a})^{b-1} = e^{-s} \left( \frac{1}{1 - e^{-s/a}} \right)^{1-b} = e^{-s} s^{b-1} \left( \frac{-s}{e^{-s/a} - 1} \right)^{1-b} \]

We may write, in terms of Nørlund polynomials, by \( (3.6) \),

\[ \tau(a; s) = e^{-s} s^{b-1} \sum_{k=0}^{\infty} B_k^{(1-b)} (-1)^k s^k \]

and

\[ D^n \tau(a; s) = e^{-s} s^{b-1} \sum_{k=n}^{\infty} B_k^{(1-b)} (-1)^{k+n} s^k \]

and thus we get

\[ \lim_{a \to \infty} D^n \tau(a; s) = B_n^{(1-b)} e^{-s} s^{b-1+n} \]

\[ \square \]
Acting $D$ on the left hand side of (3.5) gives the expression

$$D \int_0^{\varphi(a)} \tau(a; s) ds = D\varphi(a)\tau(a; s) + \int_0^{\varphi(a)} D\tau(a; s) ds$$

and hence by induction, iterating $D$ totally $n$ times,

$$D^n \int_0^{\varphi(a)} \tau(a; s) ds = \sum_{k=0}^{n-1} D^k(D\varphi(a)D_1^{n-k-1}\tau(a; \varphi(a))) + \int_0^{\varphi(a)} D^n\tau(a; s) ds$$

We shall study the terms

$$D^k(D\varphi(a)D_1^{n-k-1}\tau(a; \varphi(a))) = \sum_{j=0}^{k} \binom{k}{j} D^{k-j+1}\varphi(a)D^j[D_1^{n-k-1}\tau(a; \varphi(a))]$$

and as

$$D^j[D_1^{n-k-1}\tau(a; \varphi(a))] = D^{j-1}[D_1^{n-k}\tau(a; \varphi(a)) + D\varphi(a)D_1^{n-k-1}\hat{e}_2\tau(a; \varphi(a))]$$

it is important to study the terms defined as

$$d(k, m, n) := \lim_{a \to \infty} D^m[D_1^k \hat{e}_2^{\tau(a; \varphi(a))}]$$

In other words, we will compute, recursively, the limits of these terms for $a \to \infty$. We note that, as $D^n\tau(a; s)$ is an analytic function of $s$ in some disc around 0, as seen by its power series, we can interchange differentiation wrt the second variable and the limit for $a \to \infty$, as we know that $\varphi(a)$ converges to a finite limit, provided that the convergence for $a \to \infty$ is locally uniform, which indeed is (an argument: as $a \to \infty$, the radius of convergence of the power series increase, so taking a compact set and assuming $a$ large enough, we can use the convergence of the sequence of power series to prove this result). We have

$$D^m[D_1^k \hat{e}_2^{\tau(a; \varphi(a))}] = D^{m-1}[D_1^{k+1} \hat{e}_2^{\tau(a; \varphi(a))} + D\varphi(a)D_1^{k+1}\hat{e}_2^{\tau(a; \varphi(a))}]$$

hence we get the recursive relation

$$d(k, m, n) = d(k, m - 1, n + 1) + \sum_{j=0}^{m-1} \binom{m-1}{j} \varphi_{m-j}d(k + 1, j, n)$$

(3.19)

where $\varphi_j = \lim_{a \to \infty} D^j\varphi(a)$, assuming that the limit is already known, and the boundary conditions

$$d(k, 0, n) = \lim_{a \to \infty} D_1^k \hat{e}_2^{\tau(a; \varphi(a))} = \lim_{a \to \infty} \hat{e}_2^{D_1^k\tau(a; \varphi(a))}$$

$$= B_n(1-b) \sum_{j=0}^{k} \binom{k}{j} (-1)^{k-j}(b + n - j)e^{-\gamma_n\hat{e}_b^{k+1}+n-j}$$
As for the integral term, we have

\[
\lim_{a \to \infty} \int_0^{\varphi(a)} D^n \tau(a; s) ds = B_n^{(1-b)} \int_0^{\gamma_b} e^{-s} s^{b-1+n} ds
\]  

(3.20)

and

\[
\int_0^{\gamma_b} e^{-s} s^{b-1+n} ds = - \sum_{k=0}^{n-1} (b + n - k) e^{-\gamma_b} s^{b-1+n-k} + (b)_n (1 - p) \Gamma(b)
\]  

(3.21)

by repeated integrations by parts and the fact that \( \int_0^{\gamma_b} e^{-s} s^{b-1} ds = (1 - p) \Gamma(b) \).

We have got, then, for the left hand side that

\[
\lim_{a \to \infty} D^n \int_0^{\varphi(a)} (a; s) ds
\]

\[
= \lim_{a \to \infty} \sum_{k=0}^{n-1} D^k (D \varphi(a) D^{n-k-1} f(a; \varphi(a))) + \lim_{a \to \infty} \int_0^{\varphi(a)} D^n f(a; s) ds
\]

\[
= \lim_{a \to \infty} \sum_{k=0}^{n-1} \sum_{j=0}^{k} \binom{k}{j} D^{k-j} \varphi(a) D^j [D_1^{n-k-1} f(a; \varphi(a))]
\]

\[
- B_n^{(1-b)} \sum_{k=0}^{n-1} (b + n - k) e^{-\gamma_b} s^{b-1+n-k} + (b)_n (1 - p) \Gamma(b) B_n^{(1-b)}
\]

\[
= \sum_{k=0}^{n-1} \sum_{j=0}^{k} \binom{k}{j} \varphi_{k-j+1} d(0, j, n-k-1)
\]

\[
- B_n^{(1-b)} \sum_{k=0}^{n-1} (b + n - k) e^{-\gamma_b} s^{b-1+n-k} + (1 - p) \Gamma(b + n) B_n^{(1-b)}
\]

\[
= \varphi_n d(0, 0, 0) + \sum_{j=1}^{n-1} \binom{n-1}{j} \varphi_{n-j} d(0, j, 0)
\]

\[
+ \sum_{k=0}^{n-2} \sum_{j=0}^{k} \binom{k}{j} \varphi_{k-j+1} d(0, j, n-k-1)
\]

\[
- B_n^{(1-b)} \sum_{k=0}^{n-1} (b + n - k) e^{-\gamma_b} s^{b-1+n-k} + (1 - p) \Gamma(b + n) B_n^{(1-b)}
\]

We notice that the term \((1 - p) \Gamma(b + n) B_n^{(1-b)}\) cancels exactly with the right hand side.
3.4 Conclusion

Proof of Theorem 2 Summing up, using the normalisation \( \delta = \frac{d}{e^{-\gamma_b \gamma_b}} \), we are left with

\[
\varphi_n = -\sum_{j=1}^{n-1} \binom{n-1}{j} \varphi_{n-j} \delta(0, j, 0) - \sum_{k=0}^{n-2} \sum_{j=0}^{k} \binom{k}{j} \varphi_{k-j+1} \delta(0, j, n-k-1) + B_n^{(1-b)} \sum_{k=0}^{n-1} (b + n - k) \gamma_{b}^{n-k}
\]

and \( \delta \) and \( \varphi \) also satisfying the recursive relation, by (3.19),

\[
\delta(k, m, n) = \delta(k, m-1, n+1) + \sum_{j=0}^{m-1} \binom{m-1}{j} \varphi_{m-j} \delta(k+1, j, n)
\]

We have the initial conditions

\[
\varphi_0 = \gamma_b
\]

\[
\delta(k, 0, n) = B_n^{(1-b)} \sum_{j=0}^{k} \binom{k}{j} (-1)^{k-j} (b + n - j) \gamma_{b}^{n-j}
\]

To prove Corollary 2 we need the following lemma.

Lemma 3.3 Let \( f \) have asymptotic expansion

\[
f(x) = \sum_{k=0}^{N} \frac{a_k}{k! x^k} + r(x)
\]

where \( r(x) = \mathcal{O}(1/x^{N+1}) \). Then,

\[
e^{f(x)} = e^{a_0} + e^{a_0} \sum_{k=1}^{N} \frac{B_k(a_1, a_2, \ldots, a_k)}{k! x^k} + \mathcal{O}(1/x^{N+1})
\]
Proof. We have

\[ f(x) = \sum_{k=0}^{N} \frac{a_k}{k!} x^k + r(x) = e^{f(x) - \sum_{k=0}^{N} \frac{a_k}{k!} x^k} = e^{r(x)} = 1 + O(1/x^{N+1}) \]

\[ \Rightarrow e^{f(x)} = e^{\sum_{k=0}^{N} \frac{a_k}{k!} x^k} + O(1/x^{N+1}) = e^{a_0} \prod_{k=1}^{N} e^{\frac{a_k}{k!} x^k} + O(1/x^{N+1}) \]

\[ = e^{a_0} \prod_{n=0}^{N} \left( 1 + \sum_{m=1}^{\lfloor \frac{n+1}{2} \rfloor - 1} \frac{a_k^m}{m!k!^{m}x^{km}} + O(1/x^{N+1}) \right) + O(1/x^{N+1}) \]

\[ = e^{a_0} \sum_{n=0}^{N} B_n(a_1, a_2, ..., a_n) \frac{n! x^n}{n!} + O(1/x^{N+1}) \]

where the last equality is derived by a combinatorial argument, the coefficient of \(1/x^n\) being the sum of products of the form \(\prod_{k=1}^{N} \frac{a_k^m}{m!k!^{m}x^{km}}\) such that \(\sum_{k=1}^{n} km_k = n\), which defines the complete Bell polynomials.

Proof of Corollary 2. The proof is an immediate consequence of Theorem 2 and the foregoing Lemma.

4 Relations between Bell, Bernoulli and Nørlund Polynomials

In the course of trying to find the asymptotic expansion of \(\varphi \) at \(\infty\), using Faà di Bruno formulas, we encountered identities between Bell polynomials and Nørlund polynomials, that we have not been able to trace in the literature, hence we state them in this section as a separate result.

Proposition 3. Let \(c \in \mathbb{C}\). Then, the Nørlund polynomial \(B_n^{(c)}\) can be expressed as

\[ B_n^{(c)} = \sum_{\{m \in \mathbb{N}^n : \sum_{j=1}^{n} m_j = n\}} \frac{n!}{m_1!m_2! \ldots m_n!} \prod_{j=1}^{n} \left( \frac{(-1)^{j+1} c B_j}{j!j} \right)^{m_j} \quad (4.1) \]

or, phrased in terms of Bell polynomials \(B_n\),

\[ B_n^{(c)} = B_n(cB_1, -cB_2/2, 0, -cB_4/4, 0, \ldots, -cB_n/n), \quad n > 1 \quad (4.2) \]

Moreover, we have that

\[ (c-n)_n B_n^{(c)} = (-1)^n B_n(B_2(c) - B_2, -B_3(c) - B_3, \ldots, (-1)^{n+1} B_{n+1}(c) - B_{n+1}) \quad (4.3) \]
Proof The last equation is derived by Remark 3.1 and the symmetries $B_n(1 - x) = (-1)^n B_n(x)$ and $(1 - c)_n = (-1)^n (c - n)_n$. For the rest, by [14.6] we have

$$B^{(c)}_n = \lim_{z \to 0} \partial^n \left[ \left( \frac{z}{e^z - 1} \right)^c \right]$$

By using Faà di Bruno formula we get

$$\partial^n \left[ \left( \frac{z}{e^z - 1} \right)^c \right] = \partial^n \left( e^{c \log \frac{z}{e^z - 1}} \right) = \left( \frac{z}{e^z - 1} \right)^c \sum_{\{m \in \mathbb{N}^n | \sum_{j=1}^n m_j = n\}} \frac{n!}{m_1! m_2! \ldots m_n!} \prod_{j=1}^n \left( \frac{c}{j!} \partial^j \left( \log \frac{z}{e^z - 1} \right) \right)^{m_j}$$

and we have the limit

$$\lim_{z \to 0} \left( \frac{z}{e^z - 1} \right)^c = 1$$

and

$$-z \left( \frac{e^z - 1}{z} \right)' = -\frac{ze^z}{e^z - 1} + 1 = \sum_{n=1}^{\infty} (-1)^n + B_n \frac{z^n}{n!}$$

$$\Rightarrow \log \frac{z}{e^z - 1} = \sum_{n=1}^{\infty} (-1)^{n+1} B_n \frac{z^n}{n!}$$

hence

$$\lim_{z \to 0} \partial^j \left( \log \frac{z}{e^z - 1} \right) = (-1)^{j+1} \frac{B_j}{j}$$

and thus, summing up,

$$B^{(c)}_n = \lim_{a \to 0} \partial^n \left[ \left( \frac{z}{e^z - 1} \right)^c \right] = \sum_{\{m \in \mathbb{N}^n | \sum_{j=1}^n m_j = n\}} \frac{n!}{m_1! m_2! \ldots m_n!} \prod_{j=1}^n \left( \frac{(-1)^{j+1} c B_j}{j!} \right)^{m_j}$$

which concludes the proposition.

\[\square\]

Appendix

In this appendix, we provide code in Maple and Sage for computing the terms of asymptotic expansion of $\varphi$ and $q$ at infinity recursively.
A Maple code

In the first algorithm, the procedure \texttt{phiinf(n)} computes what we define as \( \phi_n \) in Theorem 2.

Algorithm A.1

```maple
nor1:=proc(n,c);
    if n=0 then
        return 1;
    else
        return (-1)^n*CompleteBellB(n,seq(-c*bernoulli(j)/j,j=1..n));
    end if;
end proc;
phiinf:=proc(n) option remember;
    if n=0 then
        return gamma[b];
    end if;
    for k from 1 to n-1 do
        phiinf(k);
    od;
    return expand(-add(binomial(n-1,j)*phiinf(n-j)*delta(0,j,0),j=1..n-1)
                 -add(add(binomial(k,j)*phiinf(k-j+1)*delta(0,j,n-k-1),j=0..k,k=0..n-2)*nor1(n,1-b)*add(pochhammer(b+n-k,k)*phiinf(0)^(n-k),k=0..n-1));
end proc;
delta:=proc(k,m,n) option remember;
    if m=0 then
        return simplify(nor1(n,1-b)*add(binomial(k,j)*(-1)^(k-j)*pochhammer(b+n-j,j)*phiinf(0)^(n-j),j=0..k));
    end if;
    return simplify(delta(k,m-1,n)+add(binomial(m-1,j)*phiinf(m-j)*delta(k+1,j,n),j=0..m-1));
end proc;
```

In the second algorithm, the procedure \texttt{qinf(n)} computes the nth coefficient of the asymptotic expansion of \( q \) in Corollary 2.

Algorithm A.2
\[ q_{\inf}(b, n) := \begin{cases} 1 & \text{if } n = 0 \\ \frac{\text{CompleteBellB}(n, \sum (-1)^{k+1} \phi_{\inf}(k), k=0..n-1)}{n!} & \text{else} \end{cases} \]

### B Sage code

The function \( \phi_{\inf}(n) \) computes what we define as \( \varphi_n \) in Theorem 2.

#### Algorithm B.1

```python
gamma_b = var('gamma_b')
b = var('b')
def norlund(n, c):
    if n == 0:
        return 1
    else:
        return (-1)^n * sum(bell_polynomial(n, k)([-c*bernoulli(j)/j for j in [1..n-k+1]]) for k in [1..n])
@CachedFunction
def phiinf(n):
    if n == 0:
        return gamma_b
    else:
        return expand(-sum(binomial(n-1, j) * phiinf(n-j) * delta(0, j, 0) for j in [1..n-1]) - sum(sum(binomial(k, j) * phiinf(k-j+1) * delta(0, j, n-k-1) for j in [0..k]) for k in [0..n-2]) + norlund(n, 1-b) * sum(rising_factorial(b+n-k, k) * phiinf(0)^(n-k) for k in [0..n-1]))
@CachedFunction
def delta(k, m, n):
    if m == 0:
        return simplify(norlund(n, 1-b) * sum(binomial(k, j) * (-1)^(k-j) * rising_factorial(b+n-j, j) * phiinf(0)^(n-j) for j in [0..k]))
    else:
        return simplify(delta(k, m-1, n+1) * sum(binomial(m-1, j) * phiinf(m-j) * delta(k+1, j, n))
```
The function $q_{\text{inf}}(n)$ computes the $n$th coefficient of the asymptotic expansion of $q$ in Corollary 2.

**Algorithm B.2**

```python
def qinf(n):
    return sum(bell_polynomial(n, j)(
               [(-1)**(k+1)*phiinf(k)
               for k in [0..n-j]]) for j in [1..n])/factorial(n)
```

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