AN ERSATZ EXISTENCE THEOREM FOR FULLY NONLINEAR PARABOLIC EQUATIONS WITHOUT CONVEXITY ASSUMPTIONS

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Abstract. We show that for any uniformly parabolic fully nonlinear second-order equation with bounded measurable “coefficients” and bounded “free” term in the whole space or in any cylindrical smooth domain with smooth boundary data one can find an approximating equation which has a continuous solution with the first and the second spatial derivatives under control: bounded in the case of the whole space and locally bounded in case of equations in cylinders. The approximating equation is constructed in such a way that it modifies the original one only for large values of the second spatial derivatives of the unknown function. This is different from a previous work of Hongjie Dong and the author where the modification was done for large values of the unknown function and its spatial derivatives.

1. Introduction

This article is a natural continuation of [2] and is written in the same framework. We are given a function \( H(u, t, x) \),

\[
\begin{align*}
\begin{array}{c}
\quad u = (u', u'') , \\
\quad u' = (u'_0, u'_1, ..., u'_d) \in \mathbb{R}^{d+1} , \\
\quad u'' \in \mathbb{S} , \\
\quad (t, x) \in \mathbb{R}^{d+1},
\end{array}
\end{align*}
\]

where \( \mathbb{S} \) is the set of symmetric \( d \times d \) matrices, and we are dealing with some modifications of the parabolic equation

\[
\partial_t v(t, x) + H[v](t, x) := \partial_t v(t, x) + H(v(t, x), Dv(t, x), D^2v(t, x), t, x) = 0
\]

in subdomains of \((0, T) \times \mathbb{R}^d\), where \( T \in (0, \infty) \),

\[
\mathbb{R}^d = \{ x = (x_1, ..., x_d) : x_1, ..., x_d \in \mathbb{R} \},
\]

\[
\begin{align*}
\partial_t & = \frac{\partial}{\partial t} , \\
D^2u & = (D_{ij}u) , \\
Dv & = (D_iu) , \\
D_i & = \frac{\partial}{\partial x_i} , \\
D_{ij} & = D_iD_j.
\end{align*}
\]

As in [2] we are looking for a uniformly elliptic operator \( P[v] \) given by a convex positive-homogeneous of degree one function \( P \) independent of \((t, x)\)
such that the boundary-value problem we are interested in for the equation

$$\partial_t v + \max(H[v], P[v] - K) = 0$$ (1.2)

would be solvable in the classical sense (a.e.) for any constant $K > 0$. However, unlike [2] we do not allow $P[v]$ to depend on $v$ and its first derivatives, so that $P(u, t, x) = P(u'')$. A big advantage of this approach is that we do not need Lipschitz continuity of $H(u, t, x)$ with respect to $u'$ but rather not faster than linear growth of $H(u', 0, t, x)$ as $|u'| \to \infty$. Actually, our results even in the particular case of $H$ independent of $u'$ play a major role in paper [12] aimed at proving that $L^p$-viscosity solutions of (1.1) are in $C^{1+\alpha}$ provided that “the main coefficients” of $H$ are in VMO.

Solvability theory for uniformly nondegenerate parabolic equations like (1.1) and its elliptic counterparts in Hölder classes of functions is well developed in case $H$ is convex or concave in $u''$ (see, for instance, [4], [6], [13]). In case this condition is abandoned N. Nadirashvili and S. Vlăduţ [14] gave an example of elliptic fully nonlinear equation which does not admit classical (or even $C^{1+\alpha}$ viscosity) solution. For that reason the interest in Sobolev space theory became even more justifiable. In [10] the author proved the first existence (and uniqueness) result for fully nonlinear elliptic equations under relaxed convexity assumption for equations with VMO “coefficients”. Previously, M. G. Crandall, M. Kocan, and A. Święch [1] established the solvability in local Sobolev spaces of the boundary-value problems for fully nonlinear parabolic equations and N. Winter [15] established the solvability in the global $W^2_p$-space of the associated boundary-value problem in the elliptic case. In the solvability parts of these two papers $H$ is assumed to be convex in $u''$ and, basically, have continuous “coefficients” (actually, it is assumed to be uniformly sufficiently close to the ones having continuous “coefficients”).

There is also a quite extensive a priori estimates side of the story (not involving the solvability) for which we refer the reader to [1], [2], [15] and the references therein.

Apart from Theorems 2.1 about the solvability of equations in the whole space and 2.2 about that in cylinders, which are proved in Sections 5 and 6, respectively, Theorem 2.3 proved in Section 7 is also one of our main results. Roughly speaking, it says that as $K \to \infty$ the solutions of (1.2) converge to the maximal $L^p_{d+1}$-viscosity solution of (1.1). The existence of the maximal $L^p$-viscosity solution for elliptic case was proved in [5]. We provide a method which in principle allows one to find it.

Finally, Section 2 contains our main results, Section 3 is devoted to reducing Theorem 2.1 to a simpler statement, and in the rather long Section 4 we prepare necessary tools in order to be able to prove our main results by using finite-difference approximations.
2. Main results

Fix some constants $\delta \in (0,1)$ and $K_0 \in [0, \infty)$. Set
\[ S_\delta = \{ a \in S : \delta |\xi|^2 \leq a_{ij} \xi_i \xi_j \leq \delta^{-1} |\xi|^2, \ \forall \xi \in \mathbb{R}^d \}, \]
where and everywhere in the article the summation convention is enforced.

Assumption 2.1. (i) The function $H(u, t, x)$ is measurable with respect to $(u', t, x)$ for any $u''$, Lipschitz continuous in $u''$, and at all points of differentiability of $H$ with respect to $u''$ we have $H_{u''} \in S_\delta$.
(ii) The number $\bar{H} := \sup_{u', t, x} (|H(u', 0, t, x)| - K_0 |u'|) (\geq 0)$ is finite,
(iii) There is an increasing continuous function $\omega(r), r \geq 0$, such that $\omega(0) = 0$ and
\[ |H(u', u'', t, x) - H(v', u'', t, x)| \leq \omega(|u' - v'|) \]
for all $u, v, t, x$.

By Theorem 3.1 of [8] there exists a set
\[ \Lambda = \{ l_1, ..., l_m \} \subset \mathbb{Z}^d, \tag{2.1} \]
$m = m(\delta, d) \geq d$, chosen on the sole basis of knowing $\delta$ and $d$ and there exists a constant $\hat{\delta} = \hat{\delta}(\delta, d) \in (0, \delta/4]$ such that:
(a) We have $l_i = e_i$ and
\[ e_i \pm e_j \in \{ l_1, ..., l_m \} = \{ -l_1, ..., -l_m \} \]
for all $i, j = 1, ..., d$, where $e_1, ..., e_d$ is the standard orthonormal basis of $\mathbb{R}^d$;
(b) There exist real-analytic functions $\lambda_1(a), ..., \lambda_m(a)$ on $S_{\delta/4}$ such that for any $a \in S_{\delta/4}$
\[ a = \sum_{k=1}^{m} \lambda_k(a) l_k l_k^*, \quad \hat{\delta}^{-1} \geq \lambda_k(a) \geq \hat{\delta}, \ \forall k. \tag{2.2} \]

Introduce
\[ \mathcal{P}(z'') = \max_{\hat{\delta}^{-2} \leq a_k \leq 2\hat{\delta}^{-1}} \sum_{k=1}^{m} a_k z''_k, \tag{2.3} \]
and for $u'' \in S$ define
\[ P(u'') = \mathcal{P}(\langle u'' l_1, l_1 \rangle, ..., \langle u'' l_m, l_m \rangle), \]
where $\langle \cdot, \cdot \rangle$ is the scalar product in $\mathbb{R}^d$. Naturally, by $P[v]$ we mean a differential operator constructed as in (1.1).
Theorem 2.2. Let $N$ (in particular, $\Omega$ only on $\alpha \in \mathbb{R}$),\(\cdots\)ary condition $v_{\alpha}$

\[
P_{ij} \xi_i \xi_j \geq \min_{\delta/2 \leq a_k \leq \delta/2} \sum_{k=1}^{m} a_k \langle \xi, l_k \rangle^2 \geq (\delta/2) \sum_{k=1}^{d} \langle \xi, l_k \rangle^2 = (\delta/2)\|\xi\|^2.
\]

It follows that there exists $\bar{\delta} \in (0,1)$ depending only on $\delta, d$, such that the function $F_K = \max(H, P - K)$ is Lipschitz continuous with respect to $u''$ and at each point of its differentiability with respect to $u''$ we have $F_{K_{u''}} \in \mathcal{S}_{\bar{\delta}}$.

Let $\Omega$ be an open bounded subset of $\mathbb{R}^d$ with $C^2$ boundary. We denote the parabolic boundary of the cylinder $\Omega_T = (0, T) \times \Omega$ by

\[
\partial' \Omega_T = (\partial \Omega_T) \setminus \{(0) \times \Omega\}.
\]

Below, for $\alpha \in (0, 1)$, the parabolic spaces $C^{\alpha/2, \alpha}$ and elliptic spaces $C^\alpha$ are usual Hölder spaces. These spaces are provided with natural norms.

**Theorem 2.1.** Let $K > 0$ be a fixed constant, $g \in W_{1,2}^{1,2}(\Omega_T) \cap C(\bar{\Omega}_T)$. Suppose that Assumption 2.1 is satisfied. Then equation (1.2) in $\Omega_T$ with boundary condition $v = g$ on $\partial' \Omega_T$ has a solution $v \in C(\Omega_T) \cap W_{1,2}^{1,2}(\Omega_T)$. In addition, for all $i, j$, and $p \in (d + 1, \infty)$,

\[
|v|, |D_i v|, \rho|D_{ij} v|, |\partial_t v| \leq N(H + K + \|g\|_{W_{1,2}^{1,2}(\Omega_T)}) \quad \text{in} \quad \Omega_T \quad (a.e.),
\]

\[
\|v\|_{W_p^{1,2}(\Omega_T)} \leq N_p(H + K + \|g\|_{W_p^{1,2}(\Omega_T)}),
\]

\[
\|v\|_{C^{\alpha/2, \alpha}(\Omega_T)} \leq N(H + \|g\|_{C^{\alpha/2, \alpha}(\Omega_T)}),
\]

where

\[
\rho = \rho(x) = \text{dist}(x, \mathbb{R}^d \setminus \Omega),
\]

$\alpha \in (0, 1)$ is a constant depending only on $d$ and $\delta$, $N$ is a constant depending only on $\Omega, T, K_0$, and $\delta$, whereas $N_p$ only depends on $\Omega$, $T$, $K_0$, $\delta$, and $p$ (in particular, $N$ and $N_p$ are independent of $\omega$).

Here is our second main result.

**Theorem 2.2.** Let $K > 0$ be a fixed constant, and $g \in W_{1,2}^{2}(\mathbb{R}^d)$. Then equation (1.2) in $Q_T := (0, T) \times \mathbb{R}^d$ (a.e.) with terminal condition $v(T, x) = g(x)$ has a solution $v \in W_{1,2}^{1,2}(Q_T) \cap C(Q_T)$. In addition,

\[
|v|, |Dv|, |D^2 v|, |\partial_t v| \leq N(H + K + \|g\|_{W_{1,2}^{1,2}(\mathbb{R}^d)}) \quad \text{in} \quad Q_T \quad (a.e.),
\]

\[
\|v\|_{C^{\alpha/2, \alpha}(Q_T)} \leq N(H + \|g\|_{C^{\alpha}(\mathbb{R}^d)}),
\]

where $\alpha \in (0, 1)$ is a constant depending only on $d$ and $\delta$ and $N$ is a constant depending only on $T$, $K_0$, $d$, and $\delta$.

Before stating our third main result introduce the following.
Assumption 2.2. The function $H$ is a nonincreasing function of $u'_p$, which is continuous with respect to $u'_p$ uniformly with respect to other variables, and is Lipschitz continuous with respect to $(u'_1, \ldots, u'_d)$ with constant independent of other variables.

We also remind the reader a definition from [1] according to which we say that a function $u(t, x)$ is an $L_p$-viscosity subsolution of (1.1) in $\Omega_T$ if for any $(t_0, x_0) \in \Omega_T$ and any $\phi \in W^{1,2}_{p,loc}(\Omega_T)$ for which $u - \phi$ is continuous at $(t_0, x_0)$ and attains a local maximum at $(t_0, x_0)$, we have

$$\lim \text{ess sup}_{r \to 0} \left[ \partial_t \phi(t, x) + H(u(t, x), D\phi(t, x), D^2\phi(t, x), t, x) \right] \geq 0,$$

where

$$C_r(t_0, x_0) = (t_0, t_0 + r^2) \times \{ x \in \mathbb{R}^d : |x - x_0| < r \}.$$

In a natural way one defines $L_p$-viscosity supersolution and calls a function an $L_p$-viscosity solution if it is an $L_p$-viscosity supersolution and an $L_p$-viscosity subsolution. The reader is referred to [1] for numerous properties of $L_p$-viscosity solutions.

Observe that under Assumption 2.2 the solutions $v = v_K$ constructed in Theorem 2.1 for each $K$ are unique and decrease as $K \to \infty$.

Theorem 2.3. Suppose that Assumptions 2.1 and 2.2 are satisfied. Then, as $K \to \infty$, $v_K$ converges uniformly on $\Omega_T$ to a continuous function $v$ which is an $L_{d+1}$-viscosity solutions of (1.2) in $\Omega_T$ with boundary condition $v = g$ on $\partial' \Omega_T$. Furthermore, $v$ is the maximal $L_{d+1}$-viscosity subsolution of class $C(\Omega_T)$ of this problem.

3. Reduction of Theorem 2.1 to a simpler statement

Denote by $C^{1,2}(\Omega_T)$ the set of functions $g(t, x)$ such that $g, Dg, D^2g, \partial_t g \in C(\Omega_T)$. The norm in $C^{1,2}(\Omega_T)$ is introduced in an obvious way.

Lemma 3.1. Suppose that the assertions of Theorem 2.1 hold true if $g \in C^{1,2}(\Omega_T)$ and, in addition to Assumption 2.1, for any $s, t \in \mathbb{R}$, $x, y \in \mathbb{R}^d$, $u = (u', u'')$, and $v = (v', v'')$,

$$|H(u, t, x) - H(u, s, y)| \leq N'(|t - s| + |x - y|)(1 + |u|),$$

$$|H(u', t, x) - H(v', t, x)| \leq N'|u' - v'|$$

where $N'$ is independent of $t, s, x, y, u,$ and $v$. Then the assertions of Theorem 2.1 hold true without these additional assumptions as well.

Proof. First we suppose that the assertions of Theorem 2.1 hold true with $g$ as there but under the additional assumption that (3.1) and (3.2) hold.

Note that

$$|H(u, t, x)| \leq |H(u, t, x) - H(u', 0, t, x)| + K_0|u'| + \bar{H} \leq \bar{H} + N(K_0, d, \delta)|u|.$$

Then let $B_1$ be the open unit ball in $\mathbb{R}^{d+1}$ centered at the origin. Take a nonnegative $\zeta \in C_0^\infty(B_1)$, which integrates to one and introduce $H_n(u, t, x)$
as the convolution of \( H(u, t, x) \) and \( n^{d+1}\zeta(nt, nx) \) performed with respect to \((t, x)\). Observe that \( H_n \) satisfies Assumption 2.1 with the same constant \( \delta \), whereas

\[
|H_n(u, t, x) - H_n(u, s, y)| \leq n|B_1|(|t - s| + |x - y|) \sup\limits_z |H(u, z)| \sup\limits_z |D\zeta|,
\]

where \( |B_1| \) is the volume of \( B_1 \), and (3.1) (with \( N' \) of course depending on \( n \)) is satisfied due to (3.3).

Next, define \( H_n(u, t, x) \) as the convolution of \( H_n(u, x) \) and \( n^{d+1}\zeta(nu', t, x) \) performed with respect to \( u' \). Obviously, for each \( n \), \( H_n \) satisfies (3.1) with a constant \( N' \). Furthermore, for any \( k = 0, \ldots, d \)

\[
H_{u_k}^n(u', u'', t, x) = \frac{n}{d+1} \int_{\mathbb{R}^{d+1}} H_n(u' - v'/n, u'', t, x) \zeta_{u_k}(v') dv',
\]

It follows that

\[
|H_{u_k}^n(u, t, x)| \leq n\omega(1/n)|B_1| \sup\limits_z |D\zeta|,
\]

so that \( H_n \) also satisfies (3.2).

Now by assumption there exist solutions \( v^n \in C(\overline{\Omega}_T) \cap W^{1,2}_{\infty, \text{loc}}(\Omega_T) \) of

\[
\partial_t v^n + \max(H^n[v^n], P[v^n] - K) = 0 \quad (3.4)
\]

in \( \Omega_T \) (a.e.) with boundary condition \( v^n = g \), for which estimates (2.4), (2.5), and (2.6) hold with \( v^n \) in place of \( v \) with the constants \( N \) and \( N_p \) from Theorem 2.1 and with

\[
\bar{H}^n = \sup_{u', t, x} (|H^n(u', 0, t, x)| - K_0|u'|) \quad (\leq \bar{H} + K_0 n^{-1})
\]

in place of \( \bar{H} \). Furthermore, being uniformly bounded and uniformly continuous, the sequence \( \{v^n\} \) has a subsequence uniformly converging to a function \( v \), for which (2.4), (2.5), and (2.6), of course, hold and \( v \in C(\overline{\Omega}_T) \cap W^{1,2}_{\infty, \text{loc}}(\Omega_T) \). For simplicity of notation we suppose that the whole sequence \( v^n \) converges.

Observe that

\[
\partial_t v^m + \bar{H}^n_K[v^m] \geq 0 \quad (3.5)
\]

in \( \Omega_T \) (a.e.) for all \( m \geq n \), where

\[
\bar{H}_K^n(u, t, x) := \sup_{k \geq n} \max(H^k(v^k(t, x), Dv^k(t, x), u'', t, x), P(u'') - K).
\]

In light of (3.5) and the fact that the norms \( ||v^n||_{W^{1,2}_p(\Omega_T)} \) are bounded, by Theorem 3.5.9 of [6] we have

\[
\partial_t v + \bar{H}^n_K[v] \geq 0 \quad (3.6)
\]

in \( \Omega_T \) (a.e.).
Now we notice that by embedding theorems $Dv^k$ are locally uniformly continuous in $\Omega_T$ and this and the convergence $v^n \to v$ implies by a standard fact of calculus that $Dv^k$ converge to $Dv$ locally uniformly in $\Omega_T$. Also

$$|H_k(u, t, x) - H_k(u, t, x)| \leq \omega(1/k),$$

$$|H_k(v^k(t, x), Dv^k(t, x), D^2v(t, x), t, x) - H_k(v(t, x), Dv(t, x), D^2v(t, x), t, x)|$$

$$\leq \omega(|v^k - v|(t, x) + |Dv^k - Dv|(t, x)),$$

which along with what was said above implies that

$$\partial_t v + \hat{H}_K^n[v] \geq -\varepsilon_n$$

(3.7)

in $\Omega_T$ (a.e.), where the functions $\varepsilon_n \to 0$ in $\Omega_T$ (even locally uniformly) and

$$\hat{H}_K^n(u, t, x) := \sup_{k \geq n} \max(H_k(u, t, x), P(u^{''}) - K).$$

Then we notice that by the Lebesgue differentiation theorem for any $u$

$$\lim_{n \to \infty} \hat{H}_K^n(u, t, x) = \max(H(u, t, x), P(u) - K)$$

(3.8)

for almost all $(t, x)$. Since for any bounded set $\Gamma$ in the range of $u$, $\hat{H}_K^n(u, t, x)$ are uniformly continuous on $\Gamma$ uniformly with respect to $(t, x)$ and $n$, there exists a subset of $\Omega_T$ of full measure such that (3.8) holds on this subset for all $u$.

We conclude that in $\Omega_T$ (a.e.)

$$\partial_t v + \max(H[v], P[v] - K) \geq 0.$$

(3.9)

The opposite inequality is obtained by considering

$$\inf_{k \geq n} \max(H_k(v^k(t, x), Dv^k(t, x), u^{''}, t, x), P(u^{''}) - K).$$

The fact that it suffices to prove Theorem 2.1 under the additional assumption that $g \in C^{1,2}(\bar{\Omega}_T)$ is proved by mollifying $g$ and using a very simplified version of the above arguments. The lemma is proved. □

Next, we show that one may assume that $H$ is boundedly inhomogeneous with respect to $u^{''}$ (in the sense described in Lemma 3.2 below). Introduce

$$P_0(u) = P_0(u^{''}) = \max_{a \in \mathbb{S}_{\delta/2}} a_{ij}u_{ij}^{''},$$

where the summation is performed before the maximum is taken. It is easy to see that $P_0[u]$ is Pucci’s operator:

$$P_0(u) = -(\delta/2) \sum_{k=1}^{d} \lambda_k^{-}(u^{''}) + 2\delta^{-1} \sum_{k=1}^{d} \lambda_k^{+}(u^{''}),$$

where $\lambda_1(u^{''}), \ldots, \lambda_d(u^{''})$ are the eigenvalues of $u^{''}$ and $a^{\pm} = (1/2)(|a| \pm a)$. Observe that

$$P(u) = \max_{\delta/2 \leq a_k \leq 2\delta^{-1}} \sum_{i,j=1}^{m} a_{ik}a_{kj}u_{ij}^{''}.$$
Moreover, owing to property (b) in Section 2, the collection of matrices
\[ \sum_{k=1}^{m} a_k l_k^* \]
such that \( \delta \leq a_k \leq \delta^{-1} \), \( k = 1, \ldots, m \), covers \( S_{\delta/4} \). Hence,
\[ P(u) \geq -\left(\frac{\delta}{4}\right) d \sum_{k=1}^{d} \lambda_k(u'') + 4\delta \sum_{k=1}^{d} \lambda_k^+(u'') \]
\[ \geq P_0(u) + \left(\frac{\delta}{4}\right) \sum_{k=1}^{d} |\lambda_k(u'')|. \quad (3.10) \]

In particular, \( P_0 \leq P \) and therefore,
\[ \max(H, P - K) = \max(H_K, P - K), \]
where \( H_K = \max(H, P_0 - K) \). It is easy to see that the function \( H_K \) satisfies Assumption 2.1 (i) with \( \delta/2 \) in place of \( \delta \), satisfies Assumption 2.1 (iii) with the same function \( \omega \), and
\[ |H_K(u', 0, t, x)| \leq |H(u', 0, t, x)| \leq K_0 |u'| + \bar{H}, \]
so that the number
\[ \bar{H}_K := \sup_{u',t,x} (|H_K(u', 0, t, x)| - K_0 |u'|) \quad (\leq \bar{H}) \]
is finite and Assumption 2.1 (ii) is also satisfied. Also observe that \( H_K \) satisfies (3.1) and (3.2) with the same constant \( N' \).

To continue we note the following.

**Lemma 3.2.** There is a constant \( \kappa > 0 \) depending only on \( \delta \) and \( d \) such that
\[ H \leq P_0 - \kappa |u''| + K_0 |u'| + \bar{H}_K, \quad (3.11) \]
\[ H_K \leq P - \kappa |u''| + K_0 |u'| + \bar{H}_K, \quad (3.12) \]
where
\[ \bar{H}_K := \sup_{u',t,x} (H^+(u', 0, t, x) - K_0 |u'|) \leq \bar{H}_K. \]

Furthermore, \( H_K \) is boundedly inhomogeneous with respect to \( u'' \) in the sense that at all points of differentiability of \( H_K(u, t, x) \) with respect to \( u'' \)
\[ |H_K(u, t, x) - H_{Ku''}(u, t, x)u''_i| \leq N(K_0 + 1)(\bar{H}_K + K + |u'|), \quad (3.13) \]
where \( N \) depends only on \( d \) and \( \delta \).

**Proof.** To prove (3.11) fix \( u', t, x \) and denote by \( D_H \) the set in \( S \) of points at which \( H(u', u''), t, x) \) is differentiable with respect to \( u'' \). Since \( H \) is Lipschitz continuous with respect to \( u'' \), by Rademacher’s theorem, the set \( D_H \) has full measure. By Fubini’s theorem the sets of full measure contain almost entirely almost any ray, so that for almost any \( u'' \) the set of \( s \in [0, 1] \)
We have proved that for almost any $u$ where $H$ holds. Since $H$ is differentiable with respect to $v'$ such that $s$ points $H$ is differentiable with respect to $v''$ at $v'' = 0$. Hence by calculus at those points $s$ which have full measure we obtain

$$\frac{\partial}{\partial s} H(u', su'', t, x) = u''_i H_{ij}(u', su'', t, x),$$

which along with (3.14) and the assumption that $H_{u''} \in S_\delta$ shows that for almost all $u''$

$$H(u, t, x) = H(u', 0, t, x) + a_{ij}u''_{ij},$$

where $a = (a_{ij}) \in S_\delta$ is defined by

$$a = \int_0^1 H_{u''}(u', su'', t, x) \, ds.$$ 

We have proved that for almost any $u''$ there exists an $a \in S_\delta$ such that (3.15) holds. Since $H$ is continuous with respect to $u''$ and $S_\delta$ is a compact set, (3.15) holds for any $u''$ with an appropriate $a \in S_\delta$. We basically repeated part of the proof of Lemma 2.2 in [9].

It follows that

$$H(u, t, x) \leq (H^+(u', 0, t, x) - K_0 |u'|) + K_0 |u'| + \sup_{a \in S_\delta} a_{ij}u''_{ij}.$$ 

Here the first term on the right is less than $\bar{H}_+$ by definition and the last term equals

$$-\delta \sum_{k=1}^d \lambda_k^-(u'') + \delta^{-1} \sum_{k=1}^d \lambda_k^+(u'') = P_0(u) - (\delta/2) \sum_{k=1}^d \lambda_k^-(u'') - \delta^{-1} \sum_{k=1}^d \lambda_k^+(u'')$$

$$\leq P_0(u) - (\delta/2) \sum_{k=1}^d |\lambda_k(u'')|. $$

This certainly implies (3.11).

Estimate (3.12) now also follows since $P_0 \leq P$. To prove (3.13) note that if

$$\kappa |u''| \geq K_0 |u'| + \bar{H}_+(+) + K,$$

then by (3.11)

$$H(u, t, x) \leq P_0(u) - \kappa |u''| + K_0 |u'| + \bar{H}_+(+) \leq P_0(u) - K,$$

so that $H_K(u, t, x) = P_0(u) - K$ and the left-hand side of (3.13) is just $K$ owing to the fact that $P_0$ is positive homogeneous of degree one. On the other hand, if the opposite inequality holds in (3.16), then it follows from

$$|H_K(u, t, x)| \leq |H_K(u, t, x) - H_K(u', 0, t, x)| + |H_K(u', 0, t, x)|$$
connection with (2.1)). Recall that $K$ is a second order derivative along $l$, $H$ is satisfied. First we show that one can rewrite (3.1) and (3.2) hold with a constant $N$, also recall that $u, v$ respect to $H$.

After that it only remains to notice that

$$H(u', 0, t, x) \leq \max(H(u', 0, t, x), -K) = H_K(u', 0, t, x),$$

$$H^+(u', 0, t, x) \leq |H_K(u', 0, t, x)|, \quad \bar{H}_{(+)} \leq \bar{H}_K.$$ The lemma is proved.

This lemma shows that in the rest of the proof of Theorem 2.1 we may assume that not only Assumption 2.1 is satisfied with $\delta/2$ in place of $\delta$ and (3.1) and (3.2) hold with a constant $N'$, but also at all points of differentiability of $H$ with respect to $u$

$$|H(u, \gamma, x) - H_{\gamma ij}(u, \gamma, x)u_{\gamma ij}^\prime| \leq K'_{\gamma}(|\bar{H}| + K |u'|),$$

where $K'_{\gamma} = N(\delta, d)(K_0 + 1)$ and

$$H \leq P - \kappa |u''| + K_0|u'| + \bar{H}, \quad (3.18)$$

where $\kappa$ is the constant from Lemma 3.2.

As a result of the above arguments we see that to prove Theorem 2.1 it suffices to prove the following.

**Theorem 3.3.** Suppose that $g \in C^{1,2}(\Omega_T)$ and Assumption 2.1 is satisfied with $\delta/2$ in place of $\delta$. Also assume that (3.17) holds at all points of differentiability of $H(u, t, x)$ with respect to $u$. Finally, assume that estimates (3.1) and (3.2) with a constant $N'$ and (3.18) hold for any $t, s \in \mathbb{R}$, $x, y \in \mathbb{R}^d$, and $u, v$. Then the assertions of Theorem 2.1 hold true.

4. **Some auxiliary results**

In this section the assumptions of Theorem 3.3 are supposed to be satisfied. First we show that one can rewrite $H[v]$ in such a way that only pure second order derivatives along $l_k$'s of $v$ enter ($l_k$'s are introduced in connection with (2.1)). Recall that $K_0'$ is introduced after (3.17), define

$$I = [-K_0', K_0'], \quad J = [-2K_0', 2K_0'], \quad C'' = I \times S_{\delta/2}, \quad B'' = J \times S_{\delta/4},$$

and also recall that $H_{ij} \in \mathbb{S}_{\delta} at all points of differentiability of $H$ with respect to $u''$. In terminology of [8] this means that for any $(t, x)$

$$H(\cdot, t, x) \in \mathcal{H}_{C''} \subset \mathcal{H}_{B''}.$$ Next, for $u', (t, x) \in \mathbb{R}^{d+1}$, and $y' \in \mathbb{S}$ introduce

$$B(u', y'', t, x) = \{(f, t') \in B'' : (\bar{H} + K|u'|)f + l_{ij}y_{ij}'' \leq H(u', y'', t, x)\}.$$ As follows from [8] or from the properties of $H$, the sets $B(u', y'', t, x)$ are closed and nonempty. We now recall (2.2) and for $u', (t, x) \in \mathbb{R}^{d+1}, and
$z'' \in \mathbb{R}^m$ ($m$ is the same as in (2.2) and $z''$ in this section is a vector rather than a matrix) define

$$
\mathcal{H}(u', z'', t, x) = \inf_{y'' \in \mathcal{S}(f,l'') \in B(u',y'',t,x)} \left[ (\tilde{H} + K + |u'|) f + \lambda_{k}(l'')z''_{k} \right].
$$

By Theorem 5.2 and Corollary 5.3 of [8] (modified in an obvious way by replacing $1+|u'|$ with $\tilde{H} + K + |u'|$), the function $\mathcal{H}$ is measurable, Lipschitz continuous with respect to $z''$ with constant independent of $(u',t,x)$,

$$
H(u,t,x) = \mathcal{H}(u', \langle u'' l_1, l_1 \rangle, \ldots, \langle u'' l_m, l_m \rangle, t, x) \quad (4.1)
$$

for all values of arguments, where $l_k$ are taken from (2.1), and at all points of differentiability of $\mathcal{H}$ with respect to $z''$ we have

$$
D_{z''} \mathcal{H}(u', z'', t, x) \in \left[ \tilde{\delta}, \tilde{\delta}^{-1} \right]^m, \quad (4.2)
$$

$$
(\tilde{H} + K + |u'|)^{-1} \left[ \mathcal{H}(u', z'', t, x) - \langle z'', D_{z''} \mathcal{H}(u', z'', t, x) \rangle \right] \in J, \quad (4.3)
$$

$$
|\mathcal{H}(u', z'', t, x) - \mathcal{H}(u', z'', s, y)| \leq N(|t-s| + |x-y|)(1 + |z''| + |u'|), \quad (4.4)
$$

where $N$ is a constant independent of $u', z'', t, s, y$.

We also need the following result in which assumption (3.2) is crucial.

**Lemma 4.1.** The function $\mathcal{H}$ is locally Lipschitz continuous with respect to $u'$ and at all points of its differentiability with respect to $u'$ we have

$$
|\mathcal{H}_{u'}(u', z'', t, x)| \leq N(1 + |u'| + |z''|),
$$

where the constant $N$ is independent of $(u', z'', t, x)$.

**Proof.** The reader might find many similarities of the argument below with the proof of Theorem 4.6 of [8]. It suffices to show that there exist constants $N, \varepsilon_0 > 0$ such that for all $u', v', (t,x) \in \mathbb{R}^{d+1}$, $z'' \in \mathbb{R}^m$, and $y'' \in \mathcal{S}$, with $|v'| \leq \varepsilon_0$, we have

$$
\max_{(f,l'') \in B(u',v',y'',t,x)} [ (\tilde{H} + K + |u'|) f + \lambda_{k}(l'')z''_{k} ] \\
\leq N(1 + |u'| + |z''|) + N|v'|(1 + |u'| + |z''|). \quad (4.5)
$$

For simplicity of notation we drop the arguments $t,x$ below. Fix $u', v', y''$. Inequality (4.3) shows that there is $(f_0, l''_0) \in C''$ such that

$$
(\tilde{H} + K + |u'|) f_0 + l''_{0ij} y''_{ij} = H(u', y'').
$$

For $t \in [0,1]$ and $(f, l'') \in B''$ define

$$
f_t(f) = (1-t)f_0 + tf, \quad l''_t(l'') = (1-t)l''_0 + tl''
$$

and observe that since $C''$ lies in the interior of $B''$, for any $t \in [0,1]$

$$
(f_t(f) - K'_0(1-t), l''_t(l'')) \in B''.
$$

Now if $(f, l'') \in B(u',v',y'')$, then

$$
I := (\tilde{H} + K + |u'|)[f_t(f) - K'_0(1-t)] + l''_{ij}(l'')y''_{ij} \\
t[1 + |u'| - |u'| + |v'|] f
$$

for all $t \in [0,1]$. The desired inequality (4.5) now follows from

$$
\max_{(f,l'') \in B(u',v',y'',t,x)} [ (\tilde{H} + K + |u'|) f + \lambda_{k}(l'')z''_{k} ] \\
\leq N(1 + |u'| + |z''|) + N|v'|(1 + |u'| + |z''|).
$$
+(1 - t)H(u', y'') - K'_0(1 - t)(\tilde{H} + K + |u'|).

Here the first term on the right is by definition less than tH(u' + v', y'') \leq tH(u', y'') + N'v', the second one is less that 2K'_0v'. Hence

\[ I \leq H(u', y'') + (N' + 2K'_0)|v'| - K'_0(1 - t)(\tilde{H} + K) \leq H(u', y''), \]

provided that

\[ (N' + 2K'_0)|v'| \leq K'_0(1 - t)(\tilde{H} + K). \quad (4.6) \]

In particular, for those v' and t we have \((f_t(f) - K'_0(1 - t), l''(l'')) \in B(u', y'')\) so that

\[ J := \max_{(f, l'' \in B(u', y''))} [(\tilde{H} + K + |u'|)l'' + \lambda_k(l''(l''))z''_k] \]

\[ \leq \max_{(f, l'' \in B(u', y''))} [(\tilde{H} + K + |u'|)f + \lambda_k(l''(l''))z''_k]. \]

Furthermore, \(\lambda_k\) are Lipschitz continuous and \(|\lambda_k(l''(l'')) - \lambda_k(l'')| \leq N(1 - t)\), where \(N\) depends only on \(\delta, d\), and the Lipschitz constants of \(\lambda_k\). Also \(|f_t(f) - f| \leq 4K'_0(1 - t)\). It follows that

\[ J \geq -N(1 - t)(1 + |u'| + |z''|) + \max_{(f, l'' \in B(u', y''))} [(\tilde{H} + K + |u'|)f + \lambda_k(l''(l''))z''_k] \]

\[ \geq -N(1 - t)(1 + |u'| + |z''|) - 2K'_0|v'| \]

\[ + \max_{(f, l'' \in B(u', y''))} [(\tilde{H} + K + |u' + v'|)f + \lambda_k(l''(l''))z''_k], \]

where \(N\) is independent of \(u', v', z'', y''\) (and \((t, x)\)). We thus have obtained (4.5) with \(N(1 - t)(1 + |u'| + |z''|) + N|v'\) in place of \(N|v'|(1 + |u'| + |z''|)\) provided that (4.6) holds. After taking (here we use that \(K > 0\))

\[ \varepsilon_0 = K'_0(\tilde{H} + K)/(N' + 2K'_0), \quad (0 < \varepsilon_0 < 1) \]

we come to the original form of (4.5) and the lemma is proved.

\[ \square \]

Having representation (4.1) and having in mind finite-differences make it natural to use the following "monotone" approximations of \(H[v]\) and \(P[v]\) with finite difference operators. For \(h > 0\) and vectors \(l\) introduce

\[ T_{h,l} \phi(x) = \phi(x + hl), \quad \delta_{h,l} = h^{-1}(T_{h,l} - 1), \quad \Delta_{h,l} = h^{-2}(T_{h,l} - 2 + T_{h, -l}). \]

Also set (recall that \(\mathcal{P}\) is introduced in (2.3))

\[ \mathcal{H}_K = \max(\mathcal{H}, \mathcal{P} - K), \quad P[h][v](t, x) = \mathcal{P}(\Delta_h v(t, x)), \]

where

\[ \Delta_h v = (\Delta_{h,t_1} v, ..., \Delta_{h,t_n} v). \]

Similarly we introduce

\[ H_h[v][t, x] = \mathcal{H}(v(t, x), \delta_h v(t, x), \Delta_h v(t, x)), \]

where

\[ \delta_h v = (\delta_{h,e_1} v, ..., \delta_{h,e_d} v), \]

and \(H_{K,h}[v] = \max(H_h[v], P_h[v] - K)\).
Owing to (4.1) we have $H(u', 0, t, x) = H(u', 0, t, x)$ which in light of (4.2) and Assumption 2.1 (ii) yields the following.

**Lemma 4.2.** For all values of arguments

$$H \leq P - (\delta/2) \sum_{k=1}^{m} |z_k''| + K_0|u'| + \bar{H}.$$ 

Introduce $B$ as the smallest closed ball containing $\Lambda$ (recall its definition (2.1)) and set

$$\Omega_h = \{x \in \Omega : x + hB \subset \Omega\} = \{x : \rho(x) > \lambda h\},$$

where $\lambda$ is the radius of $B$.

For $h > 0$ such that $\Omega_h \neq \emptyset$ consider the equation

$$\partial_t v + H_{K,h}[v] = 0 \text{ in } [0, T] \times \Omega_h$$

with boundary condition

$$v = g \text{ on } \big([T] \times \Omega_h\big) \cup \big([0, T] \times (\bar{\Omega} \setminus \Omega_h)\big).$$

In view of Picard’s method of successive iterations, for any $h > 0$, there exists a unique bounded solution $v = v_h$ of (4.7)–(4.8). Furthermore, $\partial_t v_h(t, x)$ is bounded and is continuous with respect to $t$ for any $x$. A solution of (1.2), whose existence is claimed in Theorem 3.3, will be obtained as the limit of a subsequence of $v_h$ as $h \downarrow 0$. Therefore, we need to have appropriate bounds on $\partial_t v_h$ and the first- and second-order differences in $x$ of $v_h$.

Below in this section by $h_0$ and $N$ with occasional indices we denote various (finite positive) constants depending only on $\Omega$, $\{l_1, \ldots, l_m\}$, $d$, $K_0$, $T$, and $\delta$, unless specifically stated otherwise.

Denote

$$\Lambda_1 = \Lambda, \quad \Lambda_{n+1} = \Lambda_n + \Lambda, \quad n \geq 1, \quad \Lambda_\infty = \bigcup_n \Lambda_n, \quad \Lambda_{h_\infty} = h\Lambda_\infty.$$ 

Observe that the set of points in $\Lambda_{h_\infty}$ lying in any bounded domain is finite since the $l_i$’s have integral coordinates.

We need a particular case of Theorem 4.3 of [2]. Let $Q^o$ be a nonempty subset of $(0, T) \times \Lambda_{h\infty}$, which is open in the relative topology of $(0, T) \times \Lambda_{h\infty}$. We introduce $\hat{Q}^o$ as the set of points $(t_0, x_0) \in (0, T) \times \Lambda_{h\infty}$ for each of which there exists a sequence $t_n \uparrow t_0$ such that $(t_n, x_0) \in Q^o$. Observe that $Q^o \subset \hat{Q}^o$. Also define

$$Q = \hat{Q}^o \cup \{(t, x + h\Lambda) : (t, x) \in Q^o\}. \quad (4.9)$$

For $x \in \Lambda_{h\infty}$ we denote by $Q^o_{[x]}$ the $x$-section of $Q^o$: \{t : (t, x) \in Q^o\}. Assume that

$$Q^o \subset G := \{(t, x) \in \Omega^h_T : (\delta/2) \sum_{k=1}^{m} |\Delta_{h,k}v_h(t, x)| > \bar{H} + K + K_0(|v_h(t, x)| + M_h(t, x))\}.$$ \quad (4.10)
where

$$M_h(t,x) = \sum_{k=1}^{m} |\delta_{h,l_k} v_h(t,x)|,$$

so that, owing to Lemma 4.2, \(\partial_t v_h + P_h[v_h] - K \leq 0\) in \([0,T] \times \Omega^h\) and

$$\partial_t v_h + P_h[v_h] = K \quad \text{in} \quad Q^h. \quad (4.11)$$

Also observe that, owing to the continuity of \(v_h\) in \(t\), \(G \cap [(0,T) \times \Lambda^h_{\infty}]\) is open in the relative topology of \(\mathbb{R} \times \Lambda^h_{\infty}\).

To proceed with estimating \(\partial_t v_h\) and second-order differences of \(v_h\) we introduce the following. Take a function \(\eta \in C^\infty(\mathbb{R}^d)\) with bounded derivatives, such that \(|\eta| \leq 1\) and set \(\zeta = \eta^2\),

$$|\eta'(x)|_h = \sup_k |\delta_{h,l_k} \eta(x)|, \quad |\eta''(x)|_h = \sup_k |\Delta_{h,l_k} \eta(x)|,$$

$$||\eta'||_h = \sup_{\Lambda^h_{\infty}} |\eta'|_h, \quad ||\eta''||_h = \sup_{\Lambda^h_{\infty}} |\eta''|_h.$$  

Here is a particular case of Theorem 4.3 of [2] we need.

**Lemma 4.3.** Assume that \(Q \subset [0,T] \times \Omega^h\). Then there exists a constant \(N = N(m,\delta) \geq 1\) such that on \(Q^h\) for any \(k = 1, \ldots, m\)

$$\zeta^2((\Delta_{h,l_k} v_h)^-) \leq \sup_{Q,Q^h} \zeta^2((\Delta_{h,l_k} v_h)^-) + N(||\eta''||_h + ||\eta'||_h^2)\bar{W}_k,$$

where

$$\bar{W}_k = \sup_Q (|\delta_{h,l_k} v_h|^2 + |\delta_{h,-l_k} v_h|^2).$$

To investigate \(v_h\) near the boundary we need part of Lemma 8.8 of [8].

**Lemma 4.4.** For any constants \(\delta_0, N_0 \in (0,\infty)\) there exists a constant \(N\), depending only on \(\delta_0, N_0, \Omega\), and there exists a function \(\Psi \in C^2(\bar{Q})\) such that \(N \rho \geq \Psi \geq \rho\) in \(\Omega\) and for all sufficiently small \(h\) on \(\Omega^h\)

$$\sum_{j=1}^{m} a_j \Delta_{h,t_j} \Psi + N_0 \sum_{j=1}^{m} |\delta_{h,t_j} \Psi| \leq -1,$$

whenever \(\delta_0^{-1} \geq a_j \geq \delta_0\).

**Remark 4.1.** Actually, the inequality \(N \rho \geq \Psi\) and the exact dependence of \(N\) on the data are not claimed in the statement of Lemma 8.8 of [8]. These assertions follow directly from the proof. It may be also worth noting that which \(h\) are sufficiently small depend on the modulus of continuity of the second-order derivatives of \(\Psi\) which are defined by the continuity properties of the second-order derivatives of functions defining \(\partial \Omega\).

**Lemma 4.5.** There are constants \(h_0 > 0\) and \(N\) such that for all \(h \in (0,h_0]\)

$$|v_h - g| \leq N(\tilde{H} + K + \|g\|_{C^{1,2}(\bar{Q}_h)})\rho, \quad (4.12)$$

$$|\partial_t v_h| \leq N(\bar{M}_h + \tilde{H} + K + \|g\|_{C^{1,2}(\bar{Q}_h)}) \quad (4.13)$$
on $\bar{\Omega}_T$, where $\bar{M}_h := \sup_{[0,T] \times \Omega^h} M_h$.

Proof. To prove (4.12) observe that by Hadamard’s formula (cf. (3.15))

$0 = \partial_t v_h + H_{K,h}[v_h] = \partial_t v_h + \max\left(\mathcal{H}(v_h, \delta_h v_h, \Delta_h v_h, t, x), P_h[v_h] - K\right) - \max\left(\mathcal{H}(v_h, \delta_h v_h, 0, t, x), -K\right)
+ \max\left(\mathcal{H}(v_h, \delta_h v_h, 0, t, x), -K\right)
= \partial_t v_h + \sum_{k=1}^m a_k \Delta_h,t_k v_h + f(v_h, \delta_h v_h, t, x),
(4.14)

where $a_k$ are some functions satisfying $\hat{\delta}/2 \leq a_k \leq 2\hat{\delta}^{-1}$ and, owing to (4.3), $f(v_h, \delta_h v_h, t, x)$ satisfies

$|f| \leq N_1(\bar{H} + K) + |v_h| + \sum_{k=1}^d |\delta_h,e_k v_h|),
(4.15)$

where $N_1 = N(d)K_0'$. This property of $f$ implies that there exist functions $b_k$, $k = 1, ..., d$, $c$, and $\theta$ with values in $[-N_1, N_1]$ such that

$f(v_h, \delta_h v_h, t, x) = cv_h + \sum_{k=1}^d b_k \delta_h,e_k v_h + \theta(\bar{H} + K),

so that $w_h(t, x) := v_h(t, x) \exp(N_1 t)$ satisfies

$\partial_t w_h + L_h w_h + \theta(\bar{H} + K)e^{N_1 t} = 0
in [0, T] \times \Omega^h$, where

$L_h w := \sum_{k=1}^m a_k \Delta_h,t_k w + \sum_{k=1}^d b_k \delta_h,t_k w + (c - N_1)w.

After that (4.12) for $h$ small enough follows in a standard way from the maximum principle and the properties of $\Psi$ from Lemma 4.4. To be more specific observe that for a constant $N_2$ we have on $[0, T] \times (\Omega^h \cap \Lambda_{\infty}^h)$ that

$\partial_t (ge^{N_1 t}) + L_h (ge^{N_1 t}) \leq N_2 \|g\|_{C^{1,2}(\bar{\Omega}_T)} =: N_3.$

Furthermore, $c - N_1 \leq 0$ and for an appropriate choice of $\delta_0, N_0$ and $N_4 = N_3 + N_1(\bar{H} + K) \exp(N_1 T)$

$\partial_t (N_4 \Psi) + L_h (N_4 \Psi) + N_3 + \theta(\bar{H} + K)e^{N_1 t} \leq 0
in [0, T] \times (\Omega^h \cap \Lambda_{\infty}^h).$ Hence, the function

$u_h = (v_h - g)e^{N_1 t} - N_4 \Psi
satisfies

$\partial_t u_h + L_h u_h \geq 0
in [0, T] \times (\Omega^h \cap \Lambda_{\infty}^h).$ Since the set $\Omega^h \cap \Lambda_{\infty}^h$ has only finite number of points it follows from the maximum principle that

$u_h \leq \max\{u_h^+(T, x) : x \in \Omega \cap \Lambda_{\infty}^h\}$
\[ u_h = (v_h - g)e^{N_1 t} - N_4 \Psi \leq 0 \]

in \([0, T] \times (\Omega^h \cap \Lambda^h)\) and, owing to an obvious possibility of translations, in \([0, T] \times \Omega^h\). By using (4.8) one more time we see that, actually,

\[ v_h - g \leq N_4 \Psi \]

in \(\bar{\Omega}_T\). This yields the needed estimate of \(v_h - g\) from above. Similarly one obtains it from below as well.

Passing to (4.13) and having in mind translations and the continuity of \(\partial_t v_h\) with respect to \(t\) we see that it suffices to prove (4.13) on \((0, T) \times (\bar{\Omega} \cap \Omega^h)\). Recall that \(G\) is defined in (4.10) and introduce

\[ Q^o = \{(0, T) \times [\Omega^h \cap \Lambda^h]\} \cap G. \]

Since \(v_h\) satisfies (4.8), estimate (4.13) obviously holds on \((0, T) \times (\bar{\Omega} \setminus \Omega^h)\).

On \((0, T) \times [\Omega^h \cap \Lambda^h] \setminus Q^o\), we have

\[ \left(\frac{\delta}{2}\right) \sum_k |\Delta h,l_k v_h| \leq \bar{H} + K + K_0(|v_h| + M_h), \]

which together with (4.12), (4.14), and (4.15) implies that (4.13) holds on \((0, T) \times [\bar{\Omega} \cap \Lambda^h] \setminus Q^o\). Therefore, it remains to establish (4.13) on \(Q^o\) assuming that \(Q^o \neq \emptyset\).

Recall that (4.11) holds. Furthermore, every \(x\)-section of \(Q^o\) is the union of open intervals on which \(\partial_t v_h\) is Lipschitz continuous by virtue of (4.11). By subtracting the left-hand sides of (4.11) evaluated at points \(t\) and \(t + \varepsilon\), then transforming the difference by using Hadamard’s formula (as in (3.15)), and finally dividing by \(\varepsilon\) and letting \(\varepsilon \to 0\), we get that there exist functions \(a_k\) such that \(\hat{\delta}/2 \leq a_k \leq 2\hat{\delta}^{-1}\) and on every \(x\)-section of \(Q^o\) (a.e.) we have

\[ \partial_t (\partial_t v_h) + a_k \Delta h,l_k (\partial_t v_h) = 0. \]

By Lemma 4.2 of [2] this yields

\[ \sup_{Q^o} |\partial_t v_h| \leq \sup_{(0, T) \times [\bar{\Omega} \cap \Lambda^h] \setminus Q^o} |\partial_t v_h|, \]

which implies (4.13) on \(Q^o\). The lemma is proved. \(\square\)

Remark 4.2. The fact that the first-order differences enter the right-hand side of (4.13) reflects a big difference between settings in this paper and in [2] and [8] where it was possible to estimate the first-order differences on the account of having them in \(P\) and then requiring from the start Lipschitz continuity of \(H\) with respect to \(u'\). In our situation the first-order differences will also enter estimates of the second order differences and then will
be excluded from the right-hand sides by using interpolation, which is somewhat more delicate than usual because we could not obtain global estimates of the second-order differences and only get estimates blowing up near the boundary.

**Lemma 4.6.** There are constants $h_0 > 0$ and $N$ such that for all $h \in (0, h_0]$ and $r = 1, \ldots, m$

$$\left(\rho - 6\lambda h\right)|\Delta_{h,i,v}h| \leq N(\bar{M}h + \bar{H} + K + \|g\|_{C^{1,2}(\Omega_T)})$$ (4.17)
on the compact set $[0, T] \times \Omega^h$ (remember that $\lambda$ is the radius of $B$).

**Proof.** As in the proof of Lemma 4.5 we will focus on proving (4.17) in $((0, T) \times [\Omega^h \cap \Lambda^h_\infty]) \cap G$.

For $Q$ from (4.9), obviously, $Q \subset [0, T] \times \Omega^h$. Next, if $t \in (0, T)$, and $x \in \Omega^h \cap \Lambda^h_\infty$ is such that $(t, x) \notin Q^o$, then either $x \notin \Omega^{3h}$, so that $\rho(x) \leq 3\lambda h$ and (4.17) holds, or else $x \in \Omega^{3h}$ but (4.16) is valid, in which case (4.17) holds again.

Thus we need only prove (4.17) on $Q^o$ assuming, of course, that $Q^o \neq \emptyset$.

We know that (4.11) holds and the left-hand side of (4.11) is nonpositive in $Q \setminus Q^o$. To proceed further observe a standard fact that there are constants $\mu_0 \in (0, 1]$ and $N \in [0, \infty)$ depending only on $\Omega$ such that for any $\mu \in (0, \mu_0)$ there exist $\eta_\mu \in C^\infty_0(\Omega)$ satisfying

$$\eta_\mu = 1 \text{ on } \Omega^{2\mu}, \quad \eta_\mu = 0 \text{ outside } \Omega^\mu,$$

$$|\eta_\mu| \leq 1, \quad |D\eta_\mu| \leq N/\mu, \quad |D^2\eta_\mu| \leq N/\mu^2.$$ (4.18)

By Lemma 4.3 on $Q^o \cap \Omega^{2\mu}$

$$\left[(\Delta_{h,i,v}h)_{\eta_\mu}^{-1}\right]^2 \leq \sup_{Q \setminus Q^o} \eta_\mu[(\Delta_{h,i,v}h)_{\eta_\mu}^{-1}]^2 + N\mu^{-2}\bar{M}_h^2.$$ (4.19)

While estimating the last supremum we will only concentrate on $h_0 \leq \mu_0/3$ and $\mu \in [3h, \mu_0]$, when $\eta_\mu = 0$ outside $\Omega^{3h}$. In that case, for any $(s, y) \in Q \setminus Q^o$, either $y \notin \Omega^{3h}$ implying that

$$\eta_\mu[(\Delta_{h,i,v}h)_{\eta_\mu}^{-1}]^2(s, y) = 0,$$

or $y \in \Omega^{3h} \cap \Lambda^h_\infty$ but (4.16) holds at $(s, y)$, or else $(y \in \Omega^{3h} \cap \Lambda^h_\infty$ and $(s, y) \notin Q^o$ and) there is a sequence $s_n \uparrow s$ such that $(s_n, y) \in Q^o$.

The third possibility splits into two cases: 1) $s = T$, 2) $s < T$. In case 1 we have

$$|\Delta_{h,i,v}h(s, y)| = |\Delta_{h,i,v}g(s, y)| \leq N\|g\|_{C^{1,2}(\bar{\Omega}_T)}.$$ (4.20)

In case 2, estimate (4.16) holds by the definition of $Q^o$. 

It follows that as long as \( h \in (0, h_0], (t, x) \in Q^0 \cap \Omega^{2\mu}_T \), and \( \mu \in [3h, \mu_0] \) we have

\[
(\Delta_{h,t}v_h)^-(t, x) \leq N\mu^{-1}(\bar{H} + K + \|g\|_{C^{1,2}(\Omega_T)} + \bar{M}_h). \tag{4.19}
\]

If \((t, x) \in Q^0 \) and \( x \) is such that \( \rho(x) \geq 6\lambda h \), take \( \mu = \mu_0 \wedge (\rho(x)/(2\lambda)) \), which is bigger than \( 3h \) for \( h \in (0, h_0] \) since \( h_0 \leq \mu_0/3 \). In that case also \( \rho(x) \geq 2\lambda \mu \), so that \( x \in \Omega^{2\mu} \) and we conclude from (4.19) that

\[
(\Delta_{h,t}v_h)^-(t, x) \leq N\mu^{-1}(\bar{H} + K + \|g\|_{C^{1,2}(\Omega_T)} + \bar{M}_h).
\]

Furthermore, still in case \( \mu = \mu_0 \wedge (\rho(x)/(2\lambda)) \), as is easy to see, there is a constant \( N \), depending only on \( \lambda, \mu_0 \), and the diameter of \( \Omega \), such that \( \mu^{-1} \leq N\rho^{-1}(x) \). Therefore,

\[
\rho(x)(\Delta_{h,t}v_h)^-(t, x) \leq N(\bar{H} + K + \|g\|_{C^{1,2}(\Omega_T)} + \bar{M}_h),
\]

\[
(\rho(x) - 6\lambda h)(\Delta_{h,t}v_h)^-(t, x) \leq N(\bar{H} + K + \|g\|_{C^{1,2}(\Omega_T)} + \bar{M}_h)
\]

for \( (t, x) \in Q^0 \) such that \( \rho(x) \geq 6\lambda h \). However, the second relation here is obvious for \( \rho(x) \leq 6\lambda h \).

As a result of all the above arguments we see that

\[
(\rho - 6\lambda h)(\Delta_{h,t}v_h)^- \leq N(\bar{H} + K + \|g\|_{C^{1,2}(\Omega_T)} + \bar{M}_h) \tag{4.20}
\]

holds in \( (0, T) \times [\Omega^h \cap \Lambda^h_{\infty}] \) for any \( r \) whenever \( h \in (0, h_0] \).

Finally, since \( \partial_t v_h + P_h[v_h] \leq K \) in \( (0, T) \times \Omega^h \), we have that

\[
2\hat{\delta}^{-1} \sum_r (\Delta_r v_h)^+ \leq -\partial_t v_h + (\hat{\delta}/2) \sum_r (\Delta_r v_h)^- + K,
\]

which after being multiplied by \( \rho - 6h \) along with (4.20) and (4.13) leads to (4.17) on \( (0, T) \times [\Omega^h \cap \Lambda^h_{\infty}] \). Thus, as is explained at the beginning of the proof, the lemma is proved. \( \square \)

Our final estimates hinge on the first-order difference estimates.

**Lemma 4.7.** There is a constant \( N \) such that for all sufficiently small \( h > 0 \) the estimates

\[
|v_h|, |\partial_t v_h|, |\delta_{h, li} v_h|, (\rho - 6\lambda h)|\Delta_{h, li} v_h| \leq N(\bar{H} + K + \|g\|_{C^{1,2}(\Omega_T)}) \tag{4.21}
\]

hold in \( [0, T] \times \Omega^h \) for all \( k \).

**Proof.** The first estimate in (4.21) is obtained in Lemma 4.5. Owing to Lemmas 4.5 and 4.6, the remaining estimates would follow if we can prove that

\[
|\delta_{h, li} v_h| \leq N(\bar{H} + K + \|g\|_{C^{1,2}(\Omega_T)}) \tag{4.22}
\]

in \( [0, T] \times \Omega^h \) for all \( k \).

We are going to use interpolation inequalities. Note that if we have a function \( u(i) \) on a set \( -r + 1, \ldots, 0, 1, \ldots, r \), where \( r \geq 2 \) is an integer, which satisfies

\[
u(i + 1) - 2u(i) + u(i - 1) \geq -N_1 \tag{4.23}\\
\]
for $i = -r + 2, ..., r - 1$, where $N_1$ is a constant, then

$$u(i + 1) - u(i) \geq u(i) - u(i - 1) - N_1.$$  

It follows that $w(i) := u(i + 1) - u(i) + N_1i$ is an increasing function of $i = -r + 1, ..., r - 1$. In particular,

$$u(1) - u(0) = w(0) \leq \frac{1}{r - 1} \sum_{i=1}^{r-1} w(i)$$

$$= \frac{1}{r - 1} \sum_{i=1}^{r-1} (u(i + 1) - u(i) + N_1i) = \frac{1}{r - 1} (u(r) - u(1)) + \frac{1}{2} N_1r.$$  

On the other hand,

$$u(1) - u(0) \geq \frac{1}{r - 1} \sum_{i=-r+1}^{-1} (u(i + 1) - u(i) + N_1i)$$

$$= \frac{1}{r - 1} (u(0) - u(-r + 1)) - \frac{1}{2} N_1r.$$  

It follows that

$$|u(1) - u(0)| \leq \frac{1}{2} N_1r + \frac{2}{r - 1} \max\{|u(i)| : i = -r + 1, ..., r\},$$

and for any function $w$ (use that $(r - 1)^{-1} \leq 2r^{-1}$ for $r \geq 2$)

$$|w(1) - w(0)| \leq \frac{r}{2} \max_{|i| \leq r} |w(i + 1) - 2w(i) + w(i - 1)| + \frac{4}{r} \max_{|i| \leq r} |w(i)|. \tag{4.24}$$  

Now fix an $\varepsilon \in (0, 1]$ and set

$$n(\varepsilon) = 10/\varepsilon.$$  

Observe that if $x \in \Omega^{n(\varepsilon)h}$ and we take $r = [(\varepsilon \rho(x) - 6\lambda h)/(2\lambda h)^{-1}]$ ([a] is the integer part of a), then $r \geq 2$ and

$$\varepsilon [\rho(x + i\lambda h_k) - 6\lambda h] \geq r \lambda h$$  \hspace{1cm} \tag{4.25}

for $|i| \leq r$ since $\rho(x + i\lambda h_k) \geq \rho(x) - \lambda rh$ and

$$\varepsilon \rho(x) - (1 + \varepsilon) r \lambda h \geq \varepsilon \rho(x) - 2r \lambda h \geq 6\lambda h.$$  

In particular, $x + i\lambda h_k \in \Omega^h$ for $|i| \leq r$ and it makes sense applying (4.24) to $w(i) = v_h(t, x + i\lambda h_k) - g(t, x + i\lambda h_k)$ with $t \in (0, T)$, which yields

$$|\delta_{h,k}(v_h - g)(t, x)| \leq \frac{1}{2} rh \max_{|i| \leq r} |\Delta_{h,k}(v_h - g)(t, x + i\lambda h_k)|$$

$$+ \frac{4}{r h} \max_{|i| \leq r} |(v_h - g)(t, x + i\lambda h_k)|. \tag{4.26}$$  

Also notice that for $x \in \Omega^{n(\varepsilon)h}$

$$2r \lambda h \geq \varepsilon \rho(x) - 8\lambda h, \hspace{1cm} 10\lambda h \leq \varepsilon \rho(x), \hspace{1cm} 10r \lambda h \geq \varepsilon \rho(x),$$
Estimates (4.25) and (4.27) allow us to derive from (4.26) that
\begin{equation}
\rho(x + ihl_k) \leq \rho(x) + r\lambda h \leq r\lambda(10\lambda\varepsilon^{-1} + \lambda) \leq r\lambda 11\varepsilon^{-1}.
\end{equation}

(4.27)

Estimates (4.25) and (4.27) allow us to derive from (4.26) that
\[
|\delta_{h,l_k}(v_h - g)(t, x)| \leq N\varepsilon\max_{|i| \leq r} \rho(x + ihl_k) - 6\lambda h \|\Delta_{h,l_k}(v_h - g)(t, x + ihl_k)\|
\]
\[
+ N\varepsilon^{-1} \max_{|i| \leq r} \rho(x + ihl_k)^{-1}|(v_h - g)(t, x + ihl_k)|,
\]

which along with Lemmas 4.5 and 4.6 shows that for all sufficiently small \(h\), \(\varepsilon \in (0, 1]\), and \(x \in \Omega^{\varepsilon,h}\)
\[
|\delta_{h,l_k}(v_h - g)(t, x)| \leq N\varepsilon(M_h + \bar{H} + K + \|g\|_{C^{1,2}(\bar{\Omega}_T)})
\]
\[
+ N\varepsilon^{-1}(\bar{H} + K + \|g\|_{C^{1,2}(\bar{\Omega}_T)}).
\]

Hence, for all sufficiently small \(h\) we have
\[
M_h = \sup_{[0,T] \times \Omega^h} \sum_{k=1}^{m} |\delta_{h,l_k}v_h| \leq N_1\varepsilon(M_h + \bar{H} + K + \|g\|_{C^{1,2}(\bar{\Omega}_T)})
\]
\[
+ N\varepsilon^{-1}(\bar{H} + K + \|g\|_{C^{1,2}(\bar{\Omega}_T)}) + \sup_{[0,T] \times (\Omega^h \setminus \Omega^{\varepsilon,h})} \sum_{k=1}^{m} |\delta_{h,l_k}(v_h - g)|,
\]

where the last term is dominated by
\[
N_1\varepsilon(M_h + \bar{H} + K + \|g\|_{C^{1,2}(\bar{\Omega}_T)}),
\]
in light of (4.12). To finish proving (4.22) it now remains only pick and fix \(\varepsilon \in (0, 1]\) so that \(N_1\varepsilon \leq 1/2\). The lemma is proved. \(\square\)

5. Proof of Theorem 3.3

In contrast with the proofs in [2] and [9] of the statements similar to Theorem 3.3, here the proof consists of two parts. The first part goes indeed very much like in [2] and [9] but only in case that \(H\) is independent of \(u'_0\). This happens because while getting uniform in \(h\) estimates of the modulus of continuity of \(v_h\), we apply a finite-difference operator \(T_{h,l} - 1\) to the equation and obtain an equation for \((T_{h,l} - 1)v_h\) with coefficients controlled by \(v_h\), \(\delta_hv_h\), and \(\Delta_hv_h\). This is harmless if the coefficient of \((T_{h,l} - 1)v_h\) turns out to be bounded. Observe that this coefficient is basically the derivative of \(H\) with respect to \(u'_0\) and it is indeed under control in the situation of [2] and [9] or when \(\Omega = \mathbb{R}^d\). Note that in the estimate of this coefficient the second-order differences of \(v_h\) enter (see Lemma 4.1) and in the case of bounded domain the estimate blows up near the boundary. That is why we first prove Theorem 3.3 when \(H\) is independent of \(u'_0\), so that we can set \(u'_0 = 0\) in \(\mathcal{H}\) and then we forget about \(\mathcal{H}\) and prove Theorem 3.3 in full generality by using the Banach fixed point theorem.

Here is an estimate of the modulus of continuity of \(v_h\) useful in the particular case that \(H\) is independent of \(u'_0\). In the following lemma (4.4) plays
a crucial role and in (4.4) only the Lipschitz continuity in $x$ is needed. By the way, notice that as is easy to see all the results in Section 4 are valid for the solution $v^0_h$ of the equation

$$\partial_t v + \mathcal{H}^0_h(\delta_h v, \Delta_h v, t, x) = 0$$

in $[0, T] \times \Omega^h$ with the same boundary condition (4.8), where

$$\mathcal{H}^0_h(\delta_h v, \Delta_h v, t, x) = \max(\mathcal{H}(0, \delta_h v, \Delta_h v, t, x), P_h[v] - k)$$

**Lemma 5.1.** There are constants $h_0 > 0$ and $M$ and there is a function $\omega_1(h), h > 0$, such that $\omega_1(0^+) = 0$ and for all $h \in (0, h_0], t \in [0, T]$, and $x, y \in \Omega$, we have

$$|v^0_h(t, x) - v^0_h(t, y)| \leq M(|x - y| + \omega_1(h)).$$

**Proof.** We closely follow the main idea of the proof of Corollary 2.7 of [11] which is about elliptic equations. Fix an $l \in \mathbb{R}^d$ such that $|l| \leq 1$ and define

$$w_h(t, x) = v^0_h(t, x + hl) - v^0_h(t, x).$$

This function is well defined in $[0, T] \times \Omega^h$ (since $\lambda > 1$). Then observe that in $[0, T] \times \Omega^h$

$$0 = \partial_tw_h(t, x) + I_h(t, x) + J_h(t, x) + K_h(t, x),$$

where

$$I_h(t, x) = \mathcal{H}^0_h(\delta_h v^0_h(t, x + hl), \Delta_h v^0_h(t, x + hl), t, x + hl) - \mathcal{H}^0_h(\delta_h v^0_h(t, x + hl), \Delta_h v^0_h(t, x), t, x + hl),$$

$$J_h(t, x) = \mathcal{H}^0_h(\delta_h v^0_h(t, x + hl), \Delta_h v^0_h(t, x), t, x + hl) - \mathcal{H}^0_h(\delta_h v^0_h(t, x), \Delta_h v^0_h(t, x), t, x + hl),$$

$$K_h(t, x) = \mathcal{H}^0_h(\delta_h v^0_h(t, x), \Delta_h v^0_h(t, x), t, x + hl) - \mathcal{H}^0_h(\delta_h v^0_h(t, x), \Delta_h v^0_h(t, x), t, x).$$

As a few times in the past Hadamard’s formula allows us to conclude that there exist functions $a_{hk}(t, x), k = 1, \ldots, m,$ such that $\delta/2 \leq a_{hk} \leq 2\delta^{-1}$ and in $[0, T] \times \Omega^h$

$$I_h = a_{hk}\Delta_ht_kw_h.$$

According to Lemma 4.1 in $[0, T] \times \Omega^h$ we have

$$|J_h| \leq N \sum_{k=1}^d |\delta_{h,ek} w_h| \left[1 + \sum_{k=1}^m |\Delta_ht_kw^0_h(t, x)| \right]$$

$$+ \sum_{k=1}^d \left( |\delta_{h,ek} w^0_h(t, x)| + |\delta_{h,ek} w^0_h(t, x + hl)| \right).$$
where and below by $N$ with occasional indices we denote constants independent of $h$. As far as $K_h$ is concerned, by (4.4)

$$|K_h| \leq Nh\left[1 + \sum_{k=1}^{d} |\delta_{h,e_k}v_k^0(t,x)| + \sum_{k=1}^{m} |\Delta_{h,l_k}v_k^0(t,x)| \right].$$

The above estimates of $J_h$ and $K_h$ along with Lemma 4.7 show that

$$J_h = b_{hk}\delta_{h,e_k}w_h, \quad K_h = f_hh$$

with appropriate functions $b_{hk}, f_h$ which satisfy the inequality

$$\sum_{k=1}^{d} |b_{hk}| + |f_h| \leq N_1/\rho$$

in $[0, T] \times \Omega^{10h}$ for sufficiently small $h > 0$. Thus,

$$\partial_tw_h + a_{hk}\Delta_{h,l_k}w_h + b_{hk}\delta_{h,e_k}w_h + f_hh = 0 \quad (5.2)$$

in $[0, T] \times \Omega^{10h}$ for sufficiently small $h > 0$. We take $\varepsilon \geq 10h$ and notice that, due to (4.12) and the fact that $w_h(T, x) = g(T, x + hl) - g(T, x)$, on

$$\{(T) \times \Omega^h\} \cup ([0, T] \times \Omega^h \setminus \Omega^\varepsilon) \quad (5.3)$$

we have $|w_h| \leq N_2\varepsilon$, where the constant $N_2$ is independent of $\varepsilon$ (and $h$). It follows that the function

$$w_h(t, x) = w_h(t, x) - N_2\varepsilon$$

is negative on (5.3) and on $[0, T] \times \Omega^\varepsilon$ satisfies (5.2). On the other hand, the function $u = e^{N_1(T-t)/\varepsilon}h$ is nonnegative on (5.3) and is as easy to check on $[0, T] \times \Omega^\varepsilon$ satisfies

$$\partial_tu + a_{hk}\Delta_{h,l_k}u + b_{hk}\delta_{h,e_k}u + f_hu \leq 0.$$

By the maximum principle in $[0, T] \times \Omega^h$, if $\varepsilon \geq 10h$, then

$$w_h \leq N_2\varepsilon + e^{N_1(T-t)/\varepsilon}h.$$ 

In other words if $x, y \in \Omega$, $|x - y| \leq h$, and one of $x$ or $y$ is in $\Omega^h$ and $t \in [0, T]$, then

$$|v_h^0(t, x) - v_h^0(t, y)| \leq \min_{\varepsilon \geq 10h} [N_2\varepsilon + e^{N_1T/\varepsilon}h] =: \omega_2(h). \quad (5.4)$$

Obviously, $\omega_2(0+) = 0$ and if both $x, y \in \Omega \setminus \Omega^h$ and $|x - y| \leq h$, then

(5.1) holds with $\omega_1(h) = \omega_2(h) + N_1h$, where $N_1$ responsible for the boundary condition is independent of $h$, $t$, $x$, and $y$.

In case $|x - y| \geq h$ and $h$ is sufficiently small, owing to the smoothness of $\Omega$, one can find points $x^1, ..., x^n \in h\mathbb{Z}^d \cap \Omega$, such that $|x - x^1|, |x^n - y| \leq kh$, $x^{i+1} - x^i \in \{\pm e_1, ..., \pm e_d\}$ for $i = 1, ..., n-1, n \leq N|x - y|$, and $k \in \{1, 2, ..., \}$, where $N$ and $k$ depend only on $\Omega$. Then one derives (5.1) from the above result and from estimate (4.21) which, in particular, gives an estimate of $v_h^0(t, x^{i+1}) - v_h^0(t, x^i)$. The lemma is proved. \hfill $\square$
Proof of Theorem 3.3. First we assume that $H$ is independent of $u'_0$. Then in what concerns the first assertion of Theorem 2.1 and estimates (2.4) one derives them in the same way as Theorem 5.2 in [2] is proved relying on the properties of $v^h_0$.

In the general case we use the Banach fixed point theorem. To start we take a Lipschitz continuous with respect to $(t, x)$ function $w(t, x)$ defined in $\Omega_T$ and equal to $g$ on the parabolic boundary of this set, and introduce the function

$$H^w(u, t, x) = H(w(t, x), u', u'', t, x).$$

Obviously, $H^w$ satisfies Assumption 2.1 with $\delta/2$ in place of $\delta$ and $\bar{H}^w \leq \bar{H} + K_0 \bar{w}$ in place of $\bar{H}$, where

$$\bar{w} = \sup_{\Omega_T} |w(t, x)|.$$ 

The function $H^w$ also satisfies (3.17) and (3.18) if we replace $\bar{H}$ with $\bar{H} + (K_0 + 1) \bar{w}$. Finally, $H^w$ satisfies (3.2) (with the same $N'$) and (3.1) (with a different one).

By the above the equation

$$\partial_t v + \max(H^w(Dv, D^2v, t, x), P[v] - K) = 0$$

in $\Omega_T$ with boundary condition $v = g$ on $\partial \Omega_T$ has a solution $v^w \in C(\bar{\Omega}_T) \cap W^{1,2}_{\infty,\text{loc}}(\Omega_T)$. In addition,

$$|v^w|, |Dv^w|, |\rho|D^2v^w|, |\partial_t v^w| \leq N(\bar{H} + \bar{w} + K + \|g\|_{W^{1,2}_{\infty,\text{loc}}(\Omega_T)})$$

in $\Omega_T$ (a.e.), where $N$ is a constant depending only on $\Omega$, $T$, $K_0$, and $\delta$. Due to the Lipschitz continuity of $H^w$ and parabolic Alexandrov maximum principle, the solution is unique, so that the notation $v^w$ is valid.

Next,

$$H^w(Dv^w, D^2v^w, t, x) = H(0, Dv^w, D^2v^w, t, x) + cw,$$

where

$$c = \frac{1}{w}[H(w, Dw^w, D^2w^w, t, x) - H(0, Dw^w, D^2w^w, t, x)] \quad (0^{-1}0 := 0)$$

and owing to (3.2) we have $|c| \leq N'$. As has already been seen before (cf. the proof of Lemma 4.5) this allows us to write

$$\partial_t v^w + a_{ij}D_{ij}v^w + b_iD_iv^w + c'w + f = 0,$$

where $a$ is an $S_\delta$-valued function ($\delta$ is introduced in Remark 2.1), $|b| \leq K_0$, $|c'| \leq |c| \leq N'$, $|f| \leq \bar{H} + K$. By the maximum principle

$$|v^w(t, x)| \leq N' \int_t^T \sup_{x \in \Omega} |w(x, s)| \, ds + T(\bar{H} + K) + \sup_{\Omega_T} |g|$$

in $\Omega_T$. It follows that if

$$|w(t, x)| \leq (T(\bar{H} + K) + \sup_{\Omega_T} |g|)e^{N'(T-t)} =: \hat{w}(t),$$
then the same inequality holds for $v^w$.

We now introduce $S$ as the subset of $C(\Omega_T) \cap W^{1,2}_{\infty,\text{loc}}(\Omega_T)$ of functions $w$ such that $|w| \leq \bar{w}$ and

$$|w|, |Dw|, |\rho|D^2w|, |\partial_tw| \leq N(\bar{H} + \bar{w}(0) + K + \|g\|_{W^{1,2}_{\infty,\text{loc}}(\Omega_T)}),$$

in $\Omega_T$ (a.e.), where $N$ is the constant from (5.6). Obviously $S$ is a closed set and the mapping $R : w \to Rw := v^w$ maps $S$ into $S$. Furthermore, if $u, w \in S$, then

$$H(u, DRu, D^2Ru) - H(w, DRw, D^2w) = a_{ij}D_{ij}(Ru - Rw) + b_iD_i(Ru - Rw) + c(u - w),$$

where $a$ is an $S_{ij}^{1/2}$-valued function, and due to (3.2) also $|b| \leq N', |c| \leq N'$ (we allow ourselves the liberty to use the same letters $a, b, c$ for objects which may be different). Hence

$$\partial_t(Ru - Rw) + a_{ij}D_{ij}(Ru - Rw) + b_iD_i(Ru - Rw) + c(u - w) = 0.$$  

By the maximum principle it follows that

$$|(Ru - Rw)(t, x)| \leq N' \int_0^T \sup_{x \in \Omega}(u - w)(s, x) \, ds$$

in $\Omega_T$, which implies that there exists an integer $n$ such that $R^n$ is a contraction of $S$. By the Banach fixed point theorem there exists $v \in S$ such that $Rv = v$.

In particular, this proves the first assertion of Theorem 2.1 in the general case and in light of (5.6) shows that to prove (2.4) it only remains to prove that

$$\sup_{\Omega_T} |v| \leq e^{K_0T}(T\bar{H} + \sup_{\Omega_T} |g|). \quad (5.7)$$

Take $F_K$ from Remark 2.1 and notice that since $|F_K(u', 0, t, x)| \leq \bar{H} + K_0|u'|$, there exist functions $b_1, ..., b_d$, $c$, and $f$ such that

$$|b_i|, |c| \leq K_0, \quad |f| \leq \bar{H},$$

$$F_K(v(t, x), Dv(t, x), 0, t, x) = b_i(t, x)D_i v(t, x) + c(t) v(t, x) + f(t, x),$$

so that

$$0 = \partial_t v(t, x) + F_K[v](t, x) - F_K(v(t, x), Dv(t, x), 0, t, x) + b_i(t, x)D_i v(t, x) + c(t) v(t, x) + f(t, x),$$

$$\partial_t v + a_{ij}D_{ij}v + b_i(t, x)D_i v(t, x) + c(t) v(t, x) + f(t, x) = 0, \quad (5.8)$$

where $(a_{ij})$ is an $S_{ij}^{1/2}$-valued function. By the maximum principle

$$|v(t, x)| \leq K_0 \int_0^T \sup_{x \in \Omega} |v(s, x)| \, ds + T\bar{H} + \sup_{\Omega_T} |g|,$$

and Gronwall’s inequality yields (5.7).

To prove (2.5) observe that

$$\max(H(v(t, x), Dv(t, x), u'', t, x), P(u'') - K) = P(u'') + G(u'', t, x),$$

where $P(v) = P(u'')$.

$$f(t, x) = \mathcal{L}(v, Dv, D^2v, 0, t, x).$$

$$F_K[v](t, x) = f(t, x).$$

$$H(u, DRu, D^2Ru) - H(w, DRw, D^2w) = a_{ij}D_{ij}(Ru - Rw) + b_iD_i(Ru - Rw) + c(u - w).$$
where
\[ G(u'', t, x) = (H(v(t, x), Dv(t, x), u'', t, x) - P(u'') + K)^+ - K. \]

Furthermore, in light of (2.4) and (3.18)
\[ |G(u'', t, x)| \leq (H(v(t, x), Dv(t, x), u'', t, x) - P(u'') + K)^+ + K \]
\[ \leq \bar{H} + K_0 (|v(t, x)| + |Dv(t, x)|) + 2K \leq N, \tag{5.9} \]
where \( N \) is a constant like the right-hand side of (2.4). Then set
\[ G(t, x) = G(D^2v(t, x), t, x) \]
and observe that our function \( v \) satisfies the equation
\[ \partial_t u(t, x) + P(D^2 u(t, x)) + G(t, x) = 0 \tag{5.10} \]

Since \( P \) is convex with respect to \( u'' \) and \( G(t, x) \) is bounded, due to Theorem 1.1 of [3] there is a unique solution \( u \in W^{1,2}_p(\Omega_T) \) of (5.10) with boundary condition \( u = g \) on \( \partial\Omega_T \). By uniqueness of \( W^{1,2}_{d+1,\text{loc}}(\Omega_T) \cap C(\bar{\Omega}_T) \) solutions we obtain \( u = v \in W^{1,2}_p(\Omega_T) \). This allows us to apply a priori estimates from Theorem 1.1 of [3] and along with (5.9) proves (2.5).

Finally, (2.6) follows from classical results (see, for instance, [6], [13]) since \( v \) satisfies (5.8). The theorem is proved. \( \square \)

6. Proof of Theorem 2.2

As in Section 3 we easily reduce proving Theorem 2.2 to proving the following.

**Theorem 6.1.** Suppose that \( g \in C^2(\mathbb{R}^d) \) and Assumption 2.1 is satisfied with \( \delta/2 \) in place of \( \delta \). Also assume that (3.17) holds at all points of differentiability of \( H(u, t, x) \) with respect to \( u \). Finally, assume that estimates (3.1) and (3.2) with a constant \( N' \) and (3.18) hold for any \( t, s \in \mathbb{R}, x, y \in \mathbb{R}^d, \) and \( u, v \). Then the assertions of Theorem 2.2 hold true.

To prove Theorem 6.1 consider the equation
\[ \partial_t v + H_{K,h}[v] = 0 \quad \text{in} \quad [0, T] \times \mathbb{R}^d \tag{6.1} \]
with terminal condition
\[ v(T, x) = g(x) \quad \text{on} \quad \mathbb{R}^d \tag{6.2} \]

In view of Picard’s method of successive approximations for any \( h > 0 \) there exists a unique bounded solution \( v = v_h \) of (6.1)–(6.2). Furthermore, \( \partial_t v_h \) is bounded and continuous with respect to \( t \) for any \( x \).

We need a version of Lemma 4.2 of [2] for unbounded domains, in which \( Q^0, \tilde{Q}^0, Q \) are generic objects described in Section 4 before assumption (4.10) was made.
Lemma 6.2. Let \((a, b, c)(t, x)\) be a bounded \(\mathbb{R}^m \times \mathbb{R}^d \times \mathbb{R}\)-valued function on \(\mathbb{R}^{d+1}\) satisfying \(a_k \geq 0\) and \(h b_k \leq a_k\). Also let \(h > 0\) be small enough for the arguments in the proof to go through. Let \(v(t, x)\) be a bounded function in \(Q\) which is absolutely continuous with respect to \(t\) on each open interval belonging to \(Q^0\) (if it is nonempty) and for any \(x \in \Lambda^h\), satisfying

\[
\partial_t v + L v := \partial_t v + \sum_{k=1}^m a_k \Delta_{h,l_k} v + \sum_{k=1}^d b_k \delta_{h,l_k} v - c v = -\eta \tag{a.e.}
\]

on \(Q^0\), where \(\eta = \eta(t, x)\) is a bounded function. Redefine \(v\) if necessary for \((t, x) \in \hat{Q} \setminus Q^0\) so that

\[
v(t, x) = \lim_{s \uparrow t, (s, x) \in \hat{Q}} v(s, x). \]

Then in \(Q^0\) we have

\[
v \leq T e^{\bar{c}T} \sup_{Q^0} \eta v + e^{\bar{c}T} \sup_{Q \setminus Q^0} v^+,
\]

where \(\bar{c} = \sup c^-,\)

Proof. First, as in [2] we reduce the general case to the one where \(c \geq 0\). Then, by considering

\[
v(t, x) - (T - t) \sup_{Q^0} \eta^+ - \sup_{Q \setminus Q^0} v^+,
\]

we reduce the general case to the one with \(\eta \leq 0\) and \(v \leq 0\) on \(Q \setminus Q^0\).

Observe that for \(\zeta(x) = \cosh |x|\) we have

\[
|D\zeta| + |D^2 \zeta| \leq N' \zeta,
\]

where \(N'\) depends only on \(d\). It follows that for a different \(N', h \in (0, 1)\), and \(k = 1, \ldots, m\)

\[
|\delta_{h,l_k} \zeta| + |\Delta_{h,l_k} \zeta| \leq N' \zeta.
\]

Hence, the bounded function \(w := v\zeta^{-1}\) satisfies

\[
-\eta = \partial_t (w\zeta) + L(w\zeta) = \zeta \partial_t w + \zeta \sum_{k=1}^m a_k \Delta_{h,l_k} w
\]

\[
+ \zeta \sum_{k=1}^m a_k [c_{-k} \delta_{h,-l_k} w + c_k \delta_{h,l_k} w] + \zeta \sum_{k=1}^d b_k \delta_{h,l_k} w + \zeta \bar{c} w
\]

where \(c_{\pm k} = \zeta^{-1} \delta_{h,\pm l_k} \zeta, \bar{b}_k = b_k \zeta^{-1} T_{h,l_k} \zeta,\)

\[
\bar{c} = -c + \zeta^{-1} \sum_{k=1}^m \Delta_{h,l_k} \zeta + \zeta^{-1} \sum_{k=1}^d b_k \delta_{h,l_k} \zeta.
\]

It follows that for any constant \(\lambda > 0\) we have

\[
\partial_t (w e^{\lambda(T-t)}) + \sum_{k=1}^m a_k \Delta_k(w e^{\lambda(T-t)}) + \sum_{k=1}^d \bar{b}_k \delta_{h,l_k}(w e^{\lambda(T-t)})
\]
For $\lambda$ sufficiently large and $h$ sufficiently small we have $\bar{c} - \lambda \leq 0$ and the coefficients in (6.3) satisfy other conditions of Lemma 4.2 of [2] which guarantee that the finite-difference operator involved in the left-hand side of (6.3) obeys the maximum principle, that is

$$-h\bar{b}_k + h2a_k|c_k| \leq 2a_k$$

for all $k$ which is true if $h$ is sufficiently small. This allows us to conclude that for any $R \in (0, \infty)$ on $Q^o \cap [(0, T) \times B_R]$ we have

$$w(t, x)e^{\lambda(T-t)} \leq \sup\{w^+(s, x)e^{\lambda(T-s)} : (s, x) \in Q, |x| \geq R\}.$$ 

Here the right-hand side goes to zero as $R \to \infty$ since $|w| = |v|\zeta^{-1}$ and $v$ is bounded. Hence $w \leq 0$ and this proves the lemma.

**Corollary 6.3.** There exists a constant $N$ depending only on $d$ and $K_0$ such that for all sufficiently small $h$ we have

$$|v_h| \leq Ne^{NT}(\bar{H} + K + \sup|g|).$$

This corollary is obtained from Lemma 6.2 by repeating the first part of the proof of Lemma 4.5.

**Lemma 6.4.** There exists a constant $N$ depending only on $d, \delta, T,$ and $K_0$ such that for all sufficiently small $h$ we have

$$|\partial_t v_h| \leq N(\bar{H} + K + \|g\|_{C^2(\mathbb{R}^d)} + \sup_{(0,T)\times\mathbb{R}^d} \sum_{k=1}^{m} |\delta_h, t_k v_h|),$$

$$\sum_{k=1}^{m} |\Delta_h, t_k v_h| \leq N(\bar{H} + K + \|g\|_{C^2(\mathbb{R}^d)} + \sup_{(0,T)\times\mathbb{R}^d} \sum_{k=1}^{m} |\delta_h, t_k v_h|)$$

on $(0, T) \times \mathbb{R}^d$.

**Proof.** One proves (6.5) in the same way as (4.13) with the only difference that instead of Lemma 4.2 of [2] one uses Lemma 6.2.

In case of (6.6) we add to (4.11) the fact that the left-hand side of (4.11) is nonpositive outside

$$Q^o := \{(t, x) \in (0, T) \times \Lambda^h : (\hat{\delta}/2) \sum_{k=1}^{m} |\Delta_h, t_k v_h(t, x)| > \bar{H} + K + K_0(|v_h(t, x)| + M_h(t, x))\},$$

Hence, for any $r \in \{1, ..., m\}$ on $Q^o$ there exist functions $a_k$ satisfying $\hat{\delta}/2 \leq a_k \leq 2\hat{\delta}^{-1}$ such that on every $x$-section of $Q^o$ (a.e.) we have

$$\partial_t(\Delta_h, t_r v_h) + a_k \Delta_h, t_k (\Delta_h, t_r v_h) \leq 0.$$
It follows by Lemma 6.2 that in $Q^o$

\[
(\Delta_{h,l}^+ v_h)^{-} \leq \sup_{(0,T) \times \Lambda^h_n \setminus Q^o} (\Delta_{h,l}^- v_h)^{+}.
\]

Now the continuity of $\Delta_{h,l} v_h$ with respect to $t$ and the definition of $Q^o$ show that $(\Delta_{h,l}^- v_h)^{-}$ is dominated by the right-hand side of (6.6). Then equation (4.14) combined with estimates (4.15), (6.5), and (6.4) allow us to conclude that also $(\Delta_{h,l}^+ v_h)^+$ is dominated by the right-hand side of (6.6). This proves the lemma. □

Our next step is to exclude $|\delta_{h,l} v_h|$ from the right-hand side of (6.5) and (6.6) by using interpolation, that is by using (4.24), which for $w(i) = v_h(t, x + i h l_k)$, where $(t, x) \in (0, T) \times \mathbb{R}^d$, $h < 1$, and integer $r \geq 2$ yields that

\[
|\delta_{h,l} v_h(t, x)| \leq \frac{1}{2} \frac{r h}{m} \max_{|i| \leq r} |\Delta_{h,l} v_h(t, x + i h l_k)| + \frac{4}{r h} \max_{|i| \leq r} |v_h(t, x + i h l_k)|.
\]

In light of the arbitrariness of $r \geq 2$ and (6.4) and (6.6) we conclude that for any $\varepsilon \geq 2 h$

\[
|\delta_{h,l} v_h| \leq N \varepsilon^{-1} (\bar{H} + K + \sup |g|) + N \varepsilon (\bar{H} + K + \|g\|_{C^2(\mathbb{R}^d)}) + \sup_{(0,T) \times \mathbb{R}^d} \sum_{k=1}^{m} |\delta_{h,l} v_h|.
\]

It follows that for all sufficiently small $h$ we have

\[
\sup_{(0,T) \times \mathbb{R}^d} \sum_{k=1}^{m} |\delta_{h,l} v_h| \leq N (\bar{H} + K + \|g\|_{C^2(\mathbb{R}^d)}), \quad (6.7)
\]

\[
\sup_{(0,T) \times \mathbb{R}^d} (|v_h| + |\partial_t v_h| + \sum_{k=1}^{m} |\Delta_{h,l} v_h|) \leq N (\bar{H} + K + \|g\|_{C^2(\mathbb{R}^d)}). \quad (6.8)
\]

Observe that, in contrast with (4.21), (6.8) yields a global estimate of $\Delta_{h,l,k} v_h$. This allows us to repeat the proof of Lemma 5.1 without excluding $u_0'$ from $H_K$ and in place of (5.2) obtain

\[
\partial_t w_h + a_{hk} \Delta_{h,l,k} w_h + b_{hk} \delta_{h,e_k} w_h + c_{hk} w_h + f_h = 0 \quad (6.9)
\]

in $(0, T) \times \mathbb{R}^d$, which implies the following.

**Corollary 6.5.** There is a constant $M$, which may depend on $N'$, such that for all $h > 0$, $t \in (0, T]$, and $x, y \in \mathbb{R}^d$, we have

\[
|v_h(t, x) - v_h(t, y)| \leq M (|x - y| + h).
\]

After that one finishes the proof of Theorem 2.2 in the same way as Theorem 3.3 is proved, of course, dropping the part of the proof dealing with the fixed point argument.
7. Proof of Theorem 2.3

By the maximum principle $v_K$ decreases as $K$ increases. Estimate (2.6) guarantees that $v_K$ converges uniformly to a function $v \in C(\Omega_T)$. To prove that $v$ is an $L_{d+1}$-viscosity solution we need the following, in which

$$C_r = (0, r^2) \times B_r, \quad C_r(t, x) = (t, x) + C_r.$$  

**Lemma 7.1.** There is a constant $N$ depending only on $d$, $\delta$, and the Lipschitz constant of $H$ with respect to $(u'_1, \ldots, u'_d)$ such that for any $r \in (0, 1]$ and $C_r(t, x)$ satisfying $C_r(t, x) \subset \Omega_T$ and $\phi \in W^{1,2}_{d+1}(C_r(t, x))$ we have on $C_r(t, x)$ that

$$v \leq \phi + N r^{d/(d+1)} \| (\partial_t \phi + H[\phi])^+ \|_{L_{d+1}(C_r(t, x))} + \max_{\partial^C C_r(t, x)} (v - \phi)^+. \quad (7.1)$$

$$v \geq \phi - N r^{d/(d+1)} \| (\partial_t \phi + H[\phi])^- \|_{L_{d+1}(C_r(t, x))} - \max_{\partial^C C_r(t, x)} (v - \phi)^-. \quad (7.2)$$

**Proof.** Observe that

$$-\partial_t \phi - \max(H[\phi], P[\phi] - K) = -\partial_t \phi - \max(H[\phi], P[\phi] - K) + \partial_t v_K + \max(H[v_K], P[v_K] - K)$$

$$= \partial_t (v_K - \phi) + a_{ij} D_{ij} (v_K - \phi) + b_i D_i (v_K - \phi) - c(v_K - \phi),$$

where $a = (a_{ij})$ is a $d \times d$ symmetric matrix-valued function whose eigenvalues are in $[\delta, \delta^{-1}]$, $b_i$ are bounded functions, and $c \geq 0$. It follows by Lemma 2.1 and Remark 1.1 of [7] with

$$u = v_K - \phi - \max_{\partial^C C_r(t, x)} (v_K - \phi)^+$$

that for $r \in (0, 1]$

$$v_K \leq \phi + \max_{\partial^C C_r(t, x)} (v_K - \phi)^+$$

$$+ N r^{d/(d+1)} \| (\partial_t \phi + \max(H[\phi], P[\phi] - K))^+ \|_{L_{d+1}(C_r(t, x))}, \quad (7.3)$$

where the constant $N$ is of the type described in the statement of the present lemma. We obtain (7.1) from (7.3) by letting $K \to \infty$. In the same way (7.2) is established. The lemma is proved.

Now we can prove that $v$ is an $L_{d+1}$-viscosity solution. Let $(t_0, x_0) \in \Omega_T$ and $\phi \in W^{1,2}_{d+1,loc}(\Omega_T)$ be such that $v - \phi$ attains a local maximum at $(t_0, x_0)$ and $v(t_0, x_0) = \phi(t_0, x_0)$. Then for $\varepsilon > 0$ and all small $r > 0$ for

$$\phi_{\varepsilon, r}(t, x) = \phi(t, x) + \varepsilon (|x-x_0|^2 + t-t_0 - r^2)$$

we have that

$$\max_{\partial^C C_r(t_0, x_0)} (v - \phi_{\varepsilon, r})^+ = 0.$$  

Hence, by Lemma 7.1

$$d r^2 = (v - \phi_{\varepsilon, r})(t_0, x_0) \leq N r^{d/(d+1)} \| (\partial_t \phi_{\varepsilon, r} + H[\phi_{\varepsilon, r}])^+ \|_{L_{d+1}(C_r(t_0, x_0))},$$

$$N r^{-d+1} \| (\partial_t \phi_{\varepsilon, r} + H[\phi_{\varepsilon, r}])^+ \|_{L_{d+1}(C_r(t_0, x_0))} \geq \varepsilon^{d+1}.$$
By letting \( r \downarrow 0 \) and using the continuity of \( H(u, t, x) \) in \( u_0' \), which is assumed to be uniform with respect to other variables, we obtain

\[
N \lim_{r \downarrow 0} \text{ess sup}_{C_r(t_0, x_0)} (\partial_t \phi_\varepsilon + H[\phi_\varepsilon]) \geq \varepsilon.
\] (7.4)

where \( \phi_\varepsilon = \phi + \varepsilon(|x - x_0|^2 + t - t_0) \). Finally, observe that \( v \) is continuous by construction, \( \phi \) is locally continuous by embedding theorems, and \( H(u, t, x) \) is continuous with respect to \( u \) uniformly with respect to \( (t, x) \) by assumption. Then letting \( \varepsilon \downarrow 0 \) in (7.4) proves that \( v \) is an \( L_{d+1} \)-viscosity subsolution. The fact that it is also an \( L_{d+1} \)-viscosity supersolution is proved similarly on the basis of (7.2).

Finally, we prove that \( v \) is the maximal continuous \( L_{d+1} \)-viscosity subsolution. Let \( u \) be an \( L_{d+1} \)-viscosity subsolution of (1.1) of class \( C(\bar{\Omega}_T) \). Then, as is easy to see, for any \( K \geq 0 \), \( u - v_K \) is an \( L_{d+1} \)-viscosity subsolution of

\[
\partial_t w + F[w] = -h_K,
\]

where \( F[w] := H[w + v_K] - H[v_K] \), so that \( F[0] = 0 \), and \( h_K(t, x) = H[v_K] \). Since \( h_K \leq 0 \), we conclude by Proposition 2.6 of [1] that, if, additionally, \( u = g \) on \( \partial \Omega_T \), then \( u - v_K \leq 0 \) in \( \Omega_T \). Now it only remains to let \( K \to \infty \). The theorem is proved.

Remark 7.1. As follows from [1] continuous \( L_{d+1} \)-viscosity subsolutions \( u \) of (1.1) satisfy (7.1) with \( u \) in place of \( v \) for any \( \phi \in W^{1,2}_{d+1}(C_r(t, x)) \) whenever \( r \in (0, 1] \) and \( C_r(t, x) \subset \Omega_T \). Therefore, this relation can be taken as an equivalent definition of what \( L_{d+1} \)-viscosity subsolutions are. A nice feature of (1.1) is that it is satisfied for any \( \phi \in W^{1,2}_{d+1}(C_r(t, x)) \) iff it is satisfied for any \( \phi \in C^{1,2}(\bar{C}_r(t, x)) \).

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