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Chiral Floquet systems and quantum walks at half period

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We classify periodically driven quantum systems on a one-dimensional lattice, where the driving process is local and subject to a chiral symmetry condition. The analysis is in terms of the unitary operator at a half-period and also covers systems in which this operator is implemented directly, and does not necessarily arise from a continuous time evolution. The full-period evolution operator is called a quantum walk, and starting the period at half time, which is called choosing another timeframe, leads to a second quantum walk. We assume that these walks have gaps at the spectral points ±1, up to at most finite dimensional eigenspaces. Walks with these gap properties have been completely classified by triples of integer indices (arXiv:1611.04439). These indices, taken for both timeframes, thus become classifying for half-step operators. In addition a further index quantity is required to classify the half step operators, which decides whether a continuous local driving process exists. In total, this amounts to a classification by five independent indices. We show how to compute these as Fredholm indices of certain chiral block operators, show the completeness of the classification, and clarify the relations to the two sets of walk indices. Within this theory we prove bulk-edge correspondence, where second timeframe allows to distinguish between symmetry protected edge states at ±1 and −1 which is not possible with only one timeframe. We thus resolve an apparent discrepancy between our above mentioned index classification for walks, and indices defined (arXiv:1208.2143). The discrepancy turns out to be one of different definitions of the term ‘quantum walk’.

I. INTRODUCTION

Topological classifications of quantum systems in the presence of symmetries provide novel phases of matter beyond the Landau theory of symmetry breaking. Prominent examples in physics are the distinction between ordinary and topological insulators [16–18, 20, 26, 27] or the quantization of the Hall conductance in two-dimensional samples [6], but the theory extends to all dimensions and all symmetry classes of the tenfold way [1]. The key feature of these novel phases are their exceptional stability against all perturbations which preserve basic ingredients to the theory. From a physical point of view, an important consequence of this stability is the so-called bulk-boundary correspondence which allows to predict phenomena occurring near the boundary between two systems through their asymptotic properties.

Recently, the topological classification of lattice systems with discrete symmetries has become one of the cornerstones of a theory of topological quantum matter. Model systems beyond static Hamiltonians, in which topological phases are still relevant, come in two settings. On one hand they are Floquet systems with continuous time periodic driving [5, 7, 13, 14, 22, 24, 28–30] and, on the other, discrete time quantum walks [2, 4, 8–11, 21, 23]. These classes are closely related, because the Floquet evolution operator for a full period is a walk and, conversely many walks can be generated by an “effective Hamiltonian” or as a sequence of pulses in continuous time [12]. This correspondence was fruitful and led to the first results on topological properties of quantum walks [2, 4, 21, 23].

However, the two setups are fundamentally different in what it means for a system to “satisfy a symmetry”: In discrete time, the symmetries act directly on the walk unitary itself and map it either to itself or to its adjoint. In one spatial dimension, under additional assumptions on locality and spectral gaps, such systems are completely classified by three topological invariants, called the “symmetry indices” [8, 11], of which only one is non-trivial under the additional assumption of translation invariance [10]. In contrast, in periodically driven systems it is natural to impose the symmetries not only at the endpoint of one period but for the whole generating process. When the symmetry involves a time inversion this gives a special role to the half-period evolution operator. Clearly this suffices to construct the full-period, and since deformations of a driving process become deformations of the half-step unitary large parts of the classification can be done via these unitaries. Moreover, as a bonus for bringing these operators into the focus, one also covers discrete
“walk protocols” in which an overall walk is constructed from the half-step unitary in the same way as for continuous driving, however, without this operator itself being the result of a driving process. A standard example in this class is the so-called “split-step quantum walk”.

In this paper we analyze the topological classification of half-step unitary operators for chirally symmetric 1D quantum walks. These unitaries are required to satisfy the weak locality condition of [8–11] but are otherwise unconstrained. In particular, no translation invariance and no special involutive symmetry are assumed. Chiral symmetry enters only in the way the two half-periods of driving are related, and hence in the formula for building the full-period walk. Actually, there are two characteristic ways of building a Floquet operator for the entire process, depending on whether one partitions time into intervals between integer multiples of the period or into half-integer multiples of the period. Both Floquet operators, which are called the timeframes of the process [4], are readily expressed in terms of the half-step operator. They automatically satisfy chiral symmetry, but we also impose that both have essential gaps at the symmetry protected points ±1 in their spectrum. This allows us to express their symmetry indices in terms of Fredholm indices of the matrix blocks of the half-step unitary in the chiral eigenbasis. By construction, these indices are invariant under homotopies of half-step walks and integer-valued as expected for chiral symmetric systems. We identify an independent subset of five indices for half-step walks. By giving an explicit construction for paths between half-step walks with the same set of indices we prove the completeness of this index set.

This classification of half-step walks bridges the classification of single timeframes in [8–11] and that of two timeframes in [4, 25, 32]. We show, that even without assuming an underlying continuous driving process, there exists a half-step operator for every chiral symmetric walk. Thus, a subset of the indices of half-step operators completely classifies the set of chiral symmetric quantum walks. Fixing a second timeframe imposes additional restrictions and therefore, the full set of indices for the half-step operator becomes relevant. A key insight of the classification of single timeframes in [8–11] is that quantum walks admit compact perturbations that are not contractible along symmetric paths. We show here that such “non-gentle” perturbations cannot occur if one considers both timeframes together, see also [3]. Moreover, taking into account the second timeframe stabilizes the bulk-boundary correspondence proved in [8, 11] in the following sense: In the setting with only one timeframe the distribution of symmetry protected edge states to the symmetry invariant eigenvalues +1 and −1 depends on how the crossover between two bulks is designed and only their total number can be predicted. Examples of different edge state distributions for the same pair of bulks can easily be found already for split-step walks in [31]. Here we show that the second timeframe allows us to predict the number of +1 and −1 eigenvalues at interfaces between two bulks independently.

We noted that our classification does not require the half-step operator to arise from a continuous driving process. In fact, one of the five indices mentioned above measures exactly this: It vanishes iff the half-step unitary is continuously connected to the identity while respecting the locality condition. This index cannot be determined from either timeframe.

A point we touch on only peripherically, is the structure of the half-step walks. In many works (see e.g. [2, 4, 32]) the walk is given explicitly as a “protocol” sequence of coin operations, acting separately on each site, and state dependent shifts. This view is halfways between the others, because we can, on the one hand, think of the evolution up to the inversion point as a concrete realization of a half-step walk, and on the other hand the whole product of the protocol steps gives a unitary walk operator. The classification of protocols with fixed shift-coin skeletons can be important for experimental implementations where the protocol is fixed by the design of the experiment. However, since our results do not hinge on this concrete form we leave this task to future work [3].

This paper is organized as follows: In the next section we give a detailed account of different meanings in which classifications of chiral symmetric quantum walks were discussed and lay down the setting we study. Moreover, to make the present paper as self-contained as possible we summarize the main points of the topological classification of quantum walks. In Sect. III we express the symmetry indices of the two timeframes of chiral symmetric walks in terms of Fredholm indices of the chiral blocks of the half-step walk. Also, we explicitly construct such a half-step walk for every chiral symmetric walk. Taking locality requirements into account gives a total of ten indices for the half-step walk. However, the two timeframes are not independent which allows us to reduce this number to a set of five independent indices in Sect. V. Subsequently, in Sect. VI we clarify the relation between the classification of two timeframes via these half-step indices and the existing classification of single timeframes. In Sect. VII we prove completeness of the independent set of half-step indices by explicitly constructing a homotopy between two half-step walks with the same indices.
II. SETTING

A. Walks and Timeframes

We consider quantum walks with chiral symmetry, mostly in one dimension. This terminology has been used with two distinct meanings in the literature, and our aim is to clarify the connections.

(W) In [8, 10, 11] a “walk” is a unitary operator \( W \) on the system Hilbert space describing one step of a discrete time evolution.

(H) In [5] a “walk” is a process in continuous time, driven by a time dependent local Hamiltonian \( H(t) \), which is periodic in time, without loss with period 1. The walk operator is then the Floquet operator, defined by solving the Schrödinger equation over one period.

Taking into account the assumed chiral symmetry there are accordingly different requirements. On the one hand, in scenario (W) we impose the condition

\[
\gamma W \gamma^* = W^*. \tag{1}
\]

Since in this picture we do not distinguish how the operator was implemented, that is all we can say. If \( W = \exp(-iH) \) for some constant Hamiltonian \( H \), this is equivalent to \( \gamma H \gamma^* = -H \). However, when we take standpoint (H), it is natural to impose the symmetry also on the generating process, demanding that

\[
\gamma H(t) \gamma^* = -H(-t). \tag{2}
\]

This is easily understood, when \( \gamma \) arises as the product of a particle-hole symmetry and a time-reversal symmetry. It is also the condition which automatically implies the chiral symmetry for \( W \). This condition can also be imposed on shift-coin protocols, i.e. unitaries of the form \( S_1 C_1 \ldots S_n C_n \) where the \( S_i \) are (state-dependent) shift operators and the \( C_i \) are possibly position-dependent local coin operators. We will also consider such protocols without stepwise chiral symmetry \(^2\).

If we do impose (2), and combine this condition with the periodicity \( H(t+1) = H(t) \), we see that it is only necessary to specify \( H(t) \) for \( t \in [0,1/2] \), in order to know it for all times. Moreover, we can integrate the time evolution for half a step, and this evolution operator will be sufficient to compute the walk operator. Explicitly, if we denote by \( W(t) \) the solution of \( \partial_t W(t) = -iH(t)W(t) \), so that \( W(0) = 1 \) and, by definition \( W = W(1) \), we set \( F = W(1/2) \) and find

\[
W = \gamma F^* \gamma F \quad \text{and} \quad W' = F \gamma F^* \gamma. \tag{3}
\]

Here \( W' \) describes the same periodically driven process as \( W \), but with the origin shifted by half a period. These operators do not determine each other, but they are closely related, and have the same long-term iteration behaviour and the same spectrum. Vice versa, given a unitary operator \( F \) with suitable locality conditions we can construct chiral symmetric walks via (3). This determines the meaning of chiral symmetric quantum walk considered in this paper:

(F) A chiral symmetric “walk” is determined by a unitary operator \( F \) with suitable locality conditions via (3). We call \( F \) the half-step walk and \( W \) and \( W' \) the two timeframes of the walk.

It is one of the main results of this work that (F) is different from (H). This is understood from the observation that the half-step walk in scenario (F) is not necessarily continuously connected to the identity. Indeed, we show below that this is only the case if the Fredholm index of the half-step walk vanishes on a half-chain. This does not hold even for the split-step walk that is used as an example throughout the literature on the topological classification of quantum walks. Moreover, while every half-step walk determines two chiral symmetric timeframes via (3) the converse does not always hold. The existence of a half-step operator for a pair of chiral symmetric walks is addressed in Sect. III D.

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1 Taking instead each \( H(t) \) to be chiral \( (\gamma H(t) \gamma^* = -H(t)) \) would not do, since the product of chiral unitaries is not chiral due to the adjoint in the condition \( \gamma W \gamma^* = W^* \).

2 In experiments the chiral symmetry is in most cases not naturally observed, in contrast to the particle-hole symmetry in Fermionic systems and the time reversal in many systems in the absence of magnetic fields. It needs to be put in by hand into the design. So it may be adequate to demand the symmetry only of the net result.
B. Brief review of the classification of quantum walks

In this paragraph we review the topological classification of symmetric quantum walks in the setting \((W)\) on a one-dimensional chain given in \([8, 11]\). We provide the basic setup of this classification, define the invariants and discuss their stability against different classes of perturbations. This sets up the language within which we relate the different settings from Sect. II A in subsequent sections.

The quantum systems in \([8, 11]\) are defined on Hilbert spaces with one-dimensional cell structure, i.e.

\[
\mathcal{H} = \bigoplus_{x \in \mathbb{Z}} \mathcal{H}_x,
\]

where each cell \(\mathcal{H}_x\) is finite-dimensional. For every \(a \in \mathbb{Z}\) we denote by \(P_{\geq a}\) and \(P_{< a}\) the orthogonal projections onto the half-chains \(\mathcal{H}_{\geq a} = \bigoplus_{x \geq a} \mathcal{H}_x\) and \(\mathcal{H}_{< a} = \bigoplus_{x < a} \mathcal{H}_x\), respectively.

On Hilbert spaces of the form (4) quantum walks are described by unitary operators \(W\) which satisfy a suitable locality condition. The standing assumption in \([8]\) as well as in this paper is that the walks are essentially local in the sense that their commutator with half-space projections \([W, P_{\geq a}]\) is compact for every \(a \in \mathbb{Z}\). This condition generalizes the strict locality found in many papers on quantum walks where \([W, P_{\geq a}]\) vanishes outside of some finite region around \(a\).

The other main ingredient in \([8, 11]\) are discrete, involutive symmetries which act in each cell of \(\mathcal{H}\) as unitary or anti-unitary operators. Exploiting all possible combinations establishes the so-called tenfold way [1]. In this work we focus only on chiral symmetry represented on \(\mathcal{H}\) by a unitary operator \(\gamma\) with \(\gamma^2 = \mathbb{1}\). In scenario \((W)\) in Sect. II A, a quantum walk is called “chiral symmetric”\(^3\) if it satisfies \(\gamma W \gamma^* = W^*\) and is essentially gapped at the symmetry-invariant points \(\pm 1\), i.e. if its spectrum at \(\pm 1\) is finitely degenerate and isolated. Completely analogous a static Hamiltonian \(H\) is called “chiral symmetric” if it satisfies \(\gamma H \gamma^* = -H\) and is essentially gapped at 0.

These symmetry-invariant points \(\pm 1\) in the spectrum of chiral symmetric walks play a crucial role for the topological classification in \([8, 11]\): The corresponding eigenspaces are characterized by integers \(\text{i}_\pm(W) \in \mathbb{Z}\) that are stable against gentle perturbations, i.e. continuous perturbations that preserve chiral symmetry. Their sum

\[
\text{i}(W) = \text{i}_+(W) + \text{i}_-(W)
\]

is called the symmetry index of \(W\). Analogously, the symmetry-invariant 0-eigenspace of chiral symmetric static Hamiltonians \(H\) is characterized by the symmetry index \(\text{i}(H) \in \mathbb{Z}\). We emphasize that the classification of chiral symmetric walks in terms of \(\text{i}_\pm\) does not make use of the cell structure of \(\mathcal{H}\) and therefore applies to arbitrary chiral symmetric unitary operators.

Taking the cell structure of \(\mathcal{H}\) into account one can argue that a physically relevant classification is one which allows for statements like bulk-boundary correspondence, i.e. a relation between indices of infinite bulk systems and indices of crossover systems of bulks. Ideally, such a statement is independent of how the crossover is designed, which requires stability against local perturbations. While in the Hamiltonian case this stability comes for free since all local and, more generally, all compact perturbations are automatically gentle, a peculiarity of the unitary setting is that one can construct compact and even local perturbations that are not contractible along a symmetric path. Such non-gentle perturbations can alter the values of \(\text{i}_+(W)\) and \(\text{i}_-(W)\), while leaving their sum \(\text{i}(W)\) invariant.

To nevertheless find a classification for chiral symmetric walks \(W\) that does not depend on local properties in \([8]\) a new set of invariants is introduced in terms of the half-chain operators \(P_{< a}WP_{< a}\) and \(P_{\geq a}WP_{\geq a}\). These are not unitary anymore but merely essentially unitary on the respective half-chains, i.e. \(X^*X = \mathbb{1}\) and \(XX^* - \mathbb{1}\) are compact operators on \(P_{< a}\mathcal{H}\) and \(P_{\geq a}\mathcal{H}\) for \(X \in \{P_{< a}WP_{< a}, P_{\geq a}WP_{\geq a}\}\) due to essential locality of \(W\). In particular, their direct sum coincides with \(W\) asymptotically far to the left and to the right. Moreover, their imaginary parts \(\Im X = (X - X^*)/2i\) are self-adjoint, symmetric and essentially gapped at 0 \([8]\). Thus, the 0-eigenspaces of these half-space operators are characterized by \(\text{i}(W) = \text{i}(\Im P_{\geq a}WP_{\geq a})\) and similarly for \(\text{i}(W)\). These invariants are independent of \(a \in \mathbb{Z}\), they add up to the symmetry index of \(W\), i.e.

\[
\text{i}(W) = \text{i}_+(W) + \text{i}_-(W) = \text{i}(W),
\]

\(^3\) In \([8-11]\) quantum walks that satisfy these two conditions are called “admissible”.
and they turn out to be stable against gentle and compact perturbations preserving the symmetry (including non-gentle ones) \cite{8}. This stability allows for a proof of bulk-boundary correspondence: Any crossover $W$ between two bulks $W_L$ and $W_R$ which coincides with $W_L$ asymptotically far to the left and $W_R$ asymptotically far to the right hosts topologically protected eigenvalues whose combined degeneracy is lower bounded by $|\bar{s}_i(W)| = |\bar{s}_i(W_L) - \bar{s}_i(W_R)|$. However, it is not possible to predict how these eigenvalues are distributed between $+1$ and $-1$.

An important result of \cite{8} is the completeness of the classification of chiral symmetric walks in terms of the symmetry indices introduced above. Not only is the classification stable against gentle and non-gentle perturbations, but also the converse is true: Whenever two chiral symmetric walks share the values of the independent invariants $s_i, \bar{s}_i$ and $s_{i+}$ they can be deformed into each other along an admissible path.

The following theorem summarizes the results of \cite{8} relevant for the present paper:

**Theorem II.1.** Let $W$ be a chiral symmetric walk with essential gaps at $\pm 1$. Then:

1. The indices $\bar{s}_i(W)$ are integer valued and invariant under gentle perturbations.
2. The indices $s_i(W)$, $\bar{s}_i(W)$ and $\bar{s}_i(W)$ are integer valued and invariant under gentle and compact perturbations.
3. The index triple \{$s_i(W), \bar{s}_i(W), s_{i+}(W)$\} is independent and complete, and, moreover, every index combination in $\mathbb{Z}^3$ can be reached, i.e. for every element of $\mathbb{Z}^3$ there is a walk with corresponding index triple.

### III. THE HALF-STEP OPERATOR

In this section we begin the analysis of half-step operators $F$ as objects in their own right. It is clear from formula (3) that such an operator determines a pair of unitary operators $W$ and $W'$, and our standing assumption will be that these are chiral and essentially gapped in the sense of Sect. II.B. For the homotopy classification of such half-step operators we thus allow any deformation which preserves these properties. Since $W$ and $W'$ clearly depend continuously on $F$ this implies that the indices of the walks are homotopy invariants for $F$. But are there more? Thinking of $F$ as a more detailed description of the process, a finer description might be available for arbitrary $W$. Moreover, what are the conditions on $W$ and $W'$ for being related as in (3)?

We will begin with a setting, in which only the chiral symmetry is taken into account, i.e., essential locality is not yet imposed. All the statements in the current section will therefore be valid in any lattice dimension, or indeed without considering a local structure of any kind. Since some locality condition is usually part of the definition of quantum walks, we thus speak of unitary operators $W, W'$ and a half-space operator, reserving the term half-step walk for the setting with locality. The indices of essentially gapped chiral unitaries are then reduced to $s_{i+}(W)$, and the right index $\bar{s}_i$ is not even defined. However, in preparation for the right/left indices we distinguish two kinds of results: One kind, e.g., Lem. III.1 and Lem. III.3, is valid for essentially unitary $F$, and these can later be applied to half-space projections of $F$. The other kind, e.g. Lem. III.2, requires exact unitarity and typically involves the eigenspaces at $\pm 1$, assuming implicitly that the operator is normal and thus has a good spectral resolution.

We begin by writing $F$ as a block matrix with respect to the two eigenspaces of $\gamma$. Then in Sect. III.B we provide formulas for the indices of $W, W'$ in terms of Fredholm indices of the blocks of $F$. This will imply a necessary index relation for pairs $W, W'$, i.e., $s_{i\pm}(W) = \pm s_{i\pm}(W')$. The Fredholm Index of $F$ is likewise determined in a straightforward way by the indices of the blocks (Sect. III.C). But does a half-step operator always exist? There are two issues here, discussed in Sect. III.D. We first show that any essentially gapped chiral $W$ can be written in terms of a half-step operator. For pairs $W, W'$ to be related by (3) we have already established an index constraint. Moreover, two unitary operators that can be written as $U_1U_2$ and $U_2U_1$ in terms of other unitaries are always unitarily equivalent. This is clearly applies to $W, W'$ in (3), so $W$ and $W'$ have the same spectral data as well. In addition we will need that the intertwining unitary satisfies a condition involving the chiral symmetry. The resulting characterization of pairs $W, W'$ is given in Thm. III.6.

Essential locality is then added to the picture in Sect. IV. We will impose this as a condition on $F$, which certainly makes sense for an $F$ obtained by continuous driving, but also for more general protocols. Since $\gamma$ acts locally in each cell, $W$ and $W'$ are then also essentially local. For the existence results the inclusion of locality also works: The construction given in Sect. III.D naturally preserves essential locality. So all chiral walks can be written in terms of a half-step walk as well.
W | F
--|--
si | \text{ind} [A] - \text{ind} [B]
si | \text{ind} [A] - \text{ind} [B]
si' | \text{ind} [C] - \text{ind} [A]
si' | \text{ind} [C] - \text{ind} [A]

\text{si}_\pm = \frac{\text{si} + \text{si'}}{2}
\text{ind} [C] = -\text{ind} [B]

\text{ind} [A] + \text{ind} [D] = \text{ind} [B] + \text{ind} [C]
\text{ind} [A] + \text{ind} [D] = \text{ind} [B] + \text{ind} [C]

| \text{si} | \text{ind} [A] - \text{ind} [B] |
| \text{si} | \text{ind} [A] - \text{ind} [B] |
| \text{si'} | \text{ind} [C] - \text{ind} [A] |
| \text{si'} | \text{ind} [C] - \text{ind} [A] |

TABLE I. Index table relating the various symmetry indices of \( W \) and \( W' \) in the timeframe setting to (combinations of) Fredholm indices of the chiral blocks of the half-step operator \( F \). The first four lines follow from applying Lem. III.1 to the blocks of \( F \) and \( PF \), while the relation in the last line requires the unitarity of \( F \) and is given in Lem. III.2. The “only if” direction of the relation between \( \text{si}_\pm \) and the symmetry indices in different timeframes is derived in Cor. VI.1.

A. Chiral blocks

Since the chiral symmetry satisfies \( \gamma^2 = \mathbb{1} \) we can write it in terms of its eigenbasis as
\[
\gamma = \Gamma_+ - \Gamma_- = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix},
\]
where \( \Gamma_\pm \) denote the projections onto the \( \pm 1 \)-eigenspaces of \( \gamma \). It turns out to be useful to write the half-step operator \( F \) in block form with respect to the eigenbasis of \( \gamma \), i.e.,
\[
F = \begin{pmatrix} A & B \\ C & D \end{pmatrix}.
\]
Then we can express \( W \) and \( W' \) in (3) in terms of the \textbf{chiral blocks} \( A, B, C, D \) of \( F \), i.e.,
\[
W = \begin{pmatrix} A^* A - C^* C & A^* B - C^* D \\ D^* C - B^* A & D^* D - B^* B \end{pmatrix}
\text{ and } \quad W' = \begin{pmatrix} A A^* - B B^* & B D^* - A C^* \\ C A^* - D B^* & D D^* - C C^* \end{pmatrix}.
\]

B. Index relations for the half-step operator

Recall that a bounded operator \( X \) is called essentially invertible or \textbf{Fredholm} if there is a (bounded) operator \( X^I \) such that \( XX^I - \mathbb{1} \) and \( X^I X - \mathbb{1} \) are compact. If no further restrictions are assumed, Fredholm operators are completely classified by the integer-valued \textbf{Fredholm index}
\[
X \mapsto \text{ind} [X] = \dim \ker(X) - \dim \ker(X^*),
\]
which satisfies \( \text{ind} [X^*] = -\text{ind} [X] \) and \( \text{ind} [X_1 X_2] = \text{ind} [X_1] + \text{ind} [X_2] \) for \( X_1, X_2 \) Fredholm.

The following lemma relates the gap condition of \( W \) and \( W' \) to the essential invertibility of the chiral blocks of the half-step operator \( F \). Moreover, it provides an expression of the symmetry indices of \( W \) and \( W' \) in terms of Fredholm indices of the chiral blocks \( A, B, C \) and \( D \).

**Lemma III.1.** Let \( F \) be an essentially unitary operator in chiral block representation (8), and let \( W, W' \) be the essentially unitary operators determined from \( F \) via (3). Then the following are equivalent:

1. \( W \) is essentially gapped.
2. \( W' \) is essentially gapped.
3. \( A, B, C, D \) are Fredholm operators.

In this case
\[
\text{si}(W) = \text{ind} [A] - \text{ind} [B] = \text{ind} [C] - \text{ind} [D],
\]
\[
\text{si}'(W') = \text{ind} [C] - \text{ind} [A] = \text{ind} [D] - \text{ind} [B].
\]
Proof. (1)⇔(3): For the first part we can replace each operator by its image in the Calkin algebra. Then $F$, $W$ and $W'$ are exactly unitary. Item (1) means that $\pm 1$ are not in the spectrum of $W$, i.e., $\gamma F^* \gamma F \equiv 1 = \gamma F^* \gamma (F \equiv \gamma F \gamma)$ is invertible in the Calkin algebra. Since $\gamma F^* \gamma$ is unitary, this is saying that $(F \equiv \gamma F \gamma)$ are invertible. But these are just the versions of (8) with either the diagonal or the off-diagonal blocks set equal to zero. Clearly such an operator is invertible iff its non-zero blocks are. That is, $A, B, C, D$ must be invertible in the Calkin algebra, that is Fredholm operators.

(1)⇔(2): In the Calkin algebra $W' = FWF^*$ and the essential spectra of $W$ and $W'$ are the same.

To prove the index formulas we consider the null space of $W$.

\[ \Im W = (W - W^*)/(2i) = \frac{1}{i} \begin{pmatrix} 0 & A^*B - C^*D \\ D^*C - B^*A & 0 \end{pmatrix} = \begin{pmatrix} 0 & X \\ -X^* & 0 \end{pmatrix} \] (13)

Even when $W$ is not exactly unitary, and not diagonalizable, this operator is hermitian, and its eigenspaces are well defined. Its kernel consists of the vectors $(\psi_+, \psi_-)$ with $\psi_+ \in \ker X^*$ and $\psi_- \in \ker X$, so $\text{si}(\Im W) = \dim \ker X^* - \dim \ker X = -\text{ind } [X]$, see [10, Lemma 3.8]. Since by essential unitarity $A^*B + C^*D$ is compact, the Fredholm operator $2iA^*B$ is a compact perturbation of $X$, and has the same index as $X$, namely $\text{ind } [A^*B] = \text{ind } [B] - \text{ind } [A]$. \hfill \Box

The statement of the this lemma remains true when all “essentially”-s are dropped, i.e. in the exactly unitary setting the exact gappedness of $W$ or $W'$ is equivalent to the exact invertibility of the chiral blocks.

In contrast, the following lemma requires exact unitarity and allows us to express $\text{si}_+$ and $\text{si}_-$ in terms of Fredholm indices of the chiral blocks of $F$.

Lemma III.2. Let $F$ be a unitary operator in chiral block representation (8), and let the corresponding unitary operators $W, W'$ from (3) be essentially gapped. Then, the symmetry indices can be defined separately for the eigenspaces at $\pm 1$, and we get

\[
\begin{align*}
\text{si}_+(W) &= \text{ind } [C] = -\text{ind } [B] = \text{si}_+(W') \\
\text{si}_-(W) &= \text{ind } [A] = -\text{ind } [D] = -\text{si}_-(W').
\end{align*}
\] (14, 15)

Proof. The unitarity condition $F^*F = 1$ implies that

\[
\begin{align*}
A^*A + C^*C &= 1 = B^*B + D^*D \\
A^*B + C^*D &= 0 = B^*A + D^*C.
\end{align*}
\] (16)

Now suppose that $\psi \in \ker A$. Then $D^*C\psi = 0$, i.e., $C\psi \in \ker D^*$. In the same way we get 4 inclusions and another 4 from $FF^* = 1$.

\[
\begin{align*}
C \ker A &\subset \ker D^* \\
B \ker D &\subset \ker A^* \\
A \ker C &\subset \ker B^* \\
D \ker B &\subset \ker C^*.
\end{align*}
\]

Since $C$ is isometric on ker $A$, this embedding is isometric, and there is a complementary inclusion making ker $A$ and ker $D^*$ unitarily isomorphic. Extending this idea to the other inclusions it follows that

\[
\begin{align*}
\dim \ker A^* &= \dim \ker D \\
\dim \ker B^* &= \dim \ker C.
\end{align*}
\] (17)

Note that this implies $\text{ind } [A] = -\text{ind } [D]$ and $\text{ind } [C] = -\text{ind } [B]$.

As in the proof of Lem. III.1 we consider the $+1$-eigenspace of $W$ as the kernel of $F - \gamma F \gamma$, which is the off-diagonal part of $F$. A vector writing in chiral components as $(\psi_+, \psi_-)$ lies in this kernel iff $\psi_- \in \ker B$ and $\psi_+ \in \ker C$. Hence $\text{si}_+(W) = \dim \ker C - \dim \ker B = \text{ind } [C]$. Similarly, for the $-1$-eigenspace we need the kernel of the diagonal part of $F$, getting $\text{si}_-(W) = \dim \ker A - \dim \ker D = \text{ind } [A]$. The computations for $W'$ are analogous. \hfill \Box
C. The Fredholm index of $F$

Clearly, if the half-step operator $F$ is unitary its Fredholm index vanishes. If $F$ is merely essentially unitary we can express its Fredholm index in terms of the Fredholm indices of its chiral blocks:

**Lemma III.3.** Let $F$ be an essentially unitary half-step operator. Then its Fredholm index is given by

$$\text{ind } [F] = \text{ind } [B] + \text{ind } [C] = \text{ind } [A] + \text{ind } [D].$$

(19)

**Proof.** By essential unitarity of $F$, $A^*B + C^*D$ is compact. Moreover, since the half-step operator constitutes essentially gapped walks $W$ and $W'$, the chiral blocks are Fredholm operators, i.e. for each $X \in \{ A, B, C, D \}$, there is an $X^I$, such that $XX^I - \mathbb{1}$ and $X^I X - \mathbb{1}$ are compact. Hence, $D = -C^* I A^* B + K' = -C C^* I A^* B + K$, for some compact operators $K$ and $K'$. This allows the following factorisation:

$$F = \begin{pmatrix} A & B \\ C & -C C^* I A^* B + K \end{pmatrix} = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & C \end{pmatrix} \begin{pmatrix} A & \mathbb{1} \\ \mathbb{1} & -C C^* I A^* \end{pmatrix} \begin{pmatrix} \mathbb{1} & 0 \\ 0 & B \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & K \end{pmatrix},$$

(20)

All three factors are Fredholm operators, thus the Fredholm index of this expression is given by the sum of their Fredholm indices. The first and the third factor have the same indices as $C$ and $B$, respectively. Moreover, the Fredholm index of the middle factor vanishes. Indeed, let $M$ be any positive operator, then, for every $A$

$$\ker \left( \begin{pmatrix} A & \mathbb{1} \\ \mathbb{1} & -MA^* \end{pmatrix} \right) = \{(\varphi, \psi) | A \varphi + \psi = \varphi - MA^* \psi = 0\} \quad (21)$$

$$= \{(MA^* \psi, \psi) | (\mathbb{1} + MA^*) \psi = 0\} \quad (22)$$

$$= \{0\}, \quad (23)$$

where the last equality follows from strict positivity of $\mathbb{1} + MA^*$. A similar reasoning shows that also the kernel of the adjoint is trivial. Since $C^* C^I$ is positive, this guarantees, that the middle factor in (20) has trivial Fredholm index. Thus,

$$\text{ind } [F] = \text{ind } [F + K] = \text{ind } [B] + \text{ind } [C],$$

(24)

and the second equation in (19) follows from (11). \hfill \Box

D. Existence of $F$

A question of primary interest is whether for every chiral symmetric unitary $W$ there exists a half-step operator $F$ such that $W = \gamma F^* \gamma F$. In this paragraph we answer this question in the affirmative and, moreover, identify conditions which guarantee that a pair of chiral symmetric unitaries is a pair of timeframes for the same half-step operator $F$. Before diving into these existential questions, we examine the uniqueness of a given $F$ when only one of the timeframes is kept fixed.

**Lemma III.4.** Let $F_0$ be a half-step operator for $W$, i.e. $W = \gamma F_0^* \gamma F_0$. Then any other $F$ is also a half-step operator for $W$ if and only if $F = UF_0$, with $\gamma U = U \gamma$. In the chiral eigenbasis this means

$$F = \begin{pmatrix} U_+ & 0 \\ 0 & U_- \end{pmatrix} F_0.$$

(25)

**Proof.** From $\gamma F^* \gamma F = \gamma F_0^* \gamma F_0$ we get

$$F^* \gamma F = F_0^* \gamma F_0 \quad \iff \quad \gamma FF_0^* = F_0 F^* \gamma.$$

(26)

Hence, $F = (FF_0^*)F_0 = UF_0$ with $\gamma U = U \gamma$. The statement for $W'$ follows analogously. \hfill \Box
In the following we write $W$ in the chiral eigenbasis as
\[
W = \begin{pmatrix} \alpha & \beta \\ -\beta^* & \delta \end{pmatrix},
\] (27)
where the relation between the off-diagonal blocks is due to the chiral symmetry, which also forces $\alpha$ and $\delta$ to be self-adjoint [10, Lemma 3.7]. Moreover, we denote by $P_{\pm}$ be the projections onto the $\pm 1$-eigenspaces of $W$ and by $\mathcal{K} = (P_+ + P_-)^* \mathcal{H}$ the part of the Hilbert space on which $W$ is properly gapped.

**Proposition III.5.** Let $W$ be a chiral symmetric essentially gapped unitary. Then there exists a half-step operator $F$, such that
\[
W = \gamma F^* \gamma F.
\] (28)

**Proof.** We prove the existence of $F$ by explicitly constructing it. To this end we write $\mathcal{H}$ as
\[
\mathcal{H} = P_- \mathcal{H} \oplus P_+ \mathcal{H} \oplus \mathcal{K}.
\] (29)
The finite dimensional $\pm 1$-eigenspaces of $W$ can then further be split into
\[
P_- \mathcal{H} = \ker(\alpha + 1) \oplus \ker(\delta + 1) =: \ker A \oplus \ker D
\] (30)
\[
P_+ \mathcal{H} = \ker(\alpha - 1) \oplus \ker(\delta - 1) =: \ker C \oplus \ker B
\] (31)
where the direct sums refer to a splitting with respect to $(\Gamma_+ \oplus \Gamma_-) \mathcal{H}$ and we wrote $\ker(\alpha \pm 1)$ instead of $\ker(\alpha \pm \Gamma_+)$ and $\ker(\delta \pm 1)$ instead of $\ker(\delta \pm \Gamma_-)$ in order to streamline notation. The second equalities identify the obtained spaces with the kernels of the matrix blocks of an $F$, which is to be constructed. They follow from arguments similar to those in the proof of Lem. III.1. Since we did not fix a second timeframe, which would fix the co-kernels, i.e. the kernels of $A^*, B^*, C^*$ and $D^*$, we are free to choose these finite dimensional subspaces. We do this by choosing some finite dimensional unitaries, such that
\[
C^- \ker A =: \ker D^*
\] (32)
\[
A^+ \ker C =: \ker B^*
\] (33)
which is certainly possible due to the equality of the respective dimensions according to the proof of Lem. III.1 and the fact, that on both side, the kernels are pairwise orthogonal. Denoting by $\mathcal{K}'$ the complement of the co-kernels, we define a unitary $V$ via $V \mathcal{K} = \mathcal{K}'$, where we choose $V$ in a way, such that $V (\Gamma_{\pm}|_{\mathcal{K}}) V^* = \Gamma_{\pm}|_{\mathcal{K}'}$, which guarantees
\[
V \gamma_{\mathcal{K}} V^* = \gamma_{\mathcal{K}'}.
\] (34)
For the choice above note, that $\Gamma_{\pm}|_{\mathcal{K}}$ and $\Gamma_{\pm}|_{\mathcal{K}'}$ are well defined, since $P_{\pm}$ and $\Gamma_{\pm}$ commute pairwise. From the proof of Lem. III.1 we also know, that $F$ has to map the kernels of $A, B, C, D$ isometrically onto the corresponding co-kernels. Hence, we can proceed by constructing separate unitaries $F = F_- \oplus F_+ \oplus F_{\mathcal{K}}$ with
\[
F_- : \ker(A) \oplus \ker(D) \to \ker(A^*) \oplus \ker(D^*)
\] (35)
\[
F_+ : \ker(C) \oplus \ker(B) \to \ker(B^*) \oplus \ker(C^*)
\] (36)
\[
F_{\mathcal{K}} : \mathcal{K} \to \mathcal{K}'.
\] (37)
For $F_{\pm}$ we can make use of the choices we already made: Setting
\[
F_- = \begin{pmatrix} 0 & B^- \\ C^- & 0 \end{pmatrix}
\text{ we get } \gamma F_-^* \gamma' F_- = -P_-
\] (38)
as needed. Note, that one has to keep track on which subspace $\gamma$ is given during the product, which we indicated by distinguishing between $\gamma = \text{diag}(\mathds{1}_{\ker A}, -\mathds{1}_{\ker D})$ and $\gamma' = \text{diag}(\mathds{1}_{\ker A^*}, -\mathds{1}_{\ker D^*})$, which are different for non-vanishing $\ker A \neq \ker A^*$. 

Similarly, with

$$F_+ = \left( \begin{array}{cc} A^+ & 0 \\ 0 & D^+ \end{array} \right)$$

we get

$$\gamma F_+^* \gamma' F_+ = P_+$$

(39)

where in this case $\gamma$ and $\gamma'$ have the same form in both subspaces. Note, that we are completely free in the choice of the finite dimensional unitaries $A^+, D^+, B^-$ and $C^-$. This leaves us with the task of finding an appropriate $F_\mathcal{K}$, which we simply denote by $F$ in the following construction. Instead of directly considering $F: \mathcal{K} \to \mathcal{K}'$, define $\bar{F}: \mathcal{K} \to \mathcal{K}$ via

$$\bar{F} = \frac{1}{\sqrt{2}} \left( \begin{array}{c} \sqrt{1 + \alpha} V_\beta \sqrt{1 - \delta} \\ -V_\beta^* \sqrt{1 - \alpha} \sqrt{1 + \delta}. \end{array} \right)$$

(40)

which is well defined, because on the complement of $\text{ker}(\alpha \pm 1)$ and $\text{ker}(\delta \pm 1)$, $1 \pm \alpha$ and $1 \pm \delta$ are positive and $\beta$ is invertible, providing a unique polar isometry $V_\beta$. The unitarity of $F$ then follows from the unitarity conditions of $W$. Using $\sqrt{1 + \alpha} V_\beta = V_\beta \sqrt{1 - \delta}$ and $V_\beta \sqrt{1 - \delta}^* = \sqrt{1 + \alpha} V_\beta = \beta$, which also follow from the unitarity of $W$, we find, that $\bar{F}$ is a square root for $W$ on $\mathcal{K}$, i.e. $\bar{F}^2 = W_\mathcal{K}$. Moreover, abbreviating $\gamma_\mathcal{K} = \gamma$ and $\gamma_{\mathcal{K}'} = \gamma'$, we get $\gamma F \gamma = \bar{F}^*$, i.e. $\bar{F}$ itself is chiral symmetric, which implies $\gamma \bar{F}^* \gamma \bar{F} = W_\mathcal{K}$. Setting $F = V \bar{F}$, with $V$ from above, indeed gives

$$\gamma F^* \gamma' F = \gamma \bar{F}^* (V^* \gamma' V) \bar{F} = \gamma \bar{F}^* \gamma \bar{F} = W_\mathcal{K}.$$  

(41)

Combining the three operators, we get $F = F_- \oplus F_+ \oplus F_\mathcal{K}: \mathcal{H} \to \mathcal{H}$, with $W = \gamma F^* \gamma F$.

The block structure of $F = F_- \oplus F_+ \oplus F_\mathcal{K}$ with respect to the eigenspaces of $W$ is not just a special example, which serves the proof of the proposition, but is actually valid for every given $F$. This standard form will be established in general later in Sect. VII. Note, that the existence of an $F$ for every $W$ hinges on the infinite dimension of the Hilbert space. For finite systems an additional condition, namely $\text{si}_-(W) = 0$ is necessary for the existence of $F$ (see also [3]).

Having established the existence of a half-step operator $F$ for any chiral symmetric unitary $W$, we now focus on the possible second time frames. By Lem. III.4 we know, that $W$ fixes $F$ only up to multiplication with a $\gamma$-commuting unitary from the left. However, this typically changes the second timeframe which raises the question whether a given second chiral symmetric unitary $W'$ lies in the orbit of $U F_0 \gamma F_0^* \gamma U^*$. In other words: Which conditions do $W$ and $W'$ need to fulfill, in order to be considered as time frames of each other?

**Theorem III.6.** Let $W$ and $W'$ be two chiral symmetric essentially gapped unitaries for the same chiral symmetry $\gamma$. Then the following are equivalent:

1. There is a half-step operator $F$, such that

$$W = \gamma F^* \gamma F \quad \text{and} \quad W' = F \gamma F^* \gamma.$$

2. $W$ and $W'$ fulfill

$$\text{si}_\pm(W) = \pm \text{si}_\pm(W')$$

and there exists a unitary $U$ with

$$W' = U W U^* \quad \text{and} \quad U \gamma_\mathcal{K} U^* = \gamma_{\mathcal{K}'}$$

where $\gamma_\mathcal{K}$ and $\gamma_{\mathcal{K}'}$ denote the chiral symmetry restricted to $\mathcal{K}$ and $\mathcal{K}'$. I.e., in chiral eigenbasis, $U$ is a block diagonal mapping between the parts of the Hilbert space, where $W$ and $W'$ are gapped.

**Proof.** (2)$\Rightarrow$(1): This proof direction will be quite similar to the proof of Prop. III.5, with the difference, that the second time frame fixes the kernels of $A^*, B^*, C^*$ and $D^*$. The identifications are now given by

$$\begin{align*}
\ker A &= \ker(\alpha + 1), & \ker B &= \ker(\delta - 1), & \ker C &= \ker(\alpha - 1), & \ker D &= \ker(\delta + 1), \\
\ker A^* &= \ker(\alpha' + 1), & \ker B^* &= \ker(\alpha' - 1), & \ker C^* &= \ker(\delta' - 1), & \ker D^* &= \ker(\delta' + 1).
\end{align*}$$
Thereby, the necessary dimension-equalities in (17) are guaranteed by the unitary equivalence and the index condition (43). Indeed, the unitary equivalence of $W$ and $W'$ guarantees rank $P_\pm = \text{rank} P'_\pm$, i.e.
\[
\dim \ker(\alpha \pm 1) + \dim \ker(\delta \pm 1) = \dim \ker(\alpha' \pm 1) + \dim \ker(\delta' \pm 1).
\] (45)
Moreover, using (30), we can express the symmetry indices of $W$ as $\text{si}_\pm(W) = \dim \ker(\alpha \mp 1) - \dim \ker(\delta \mp 1)$ and similarly for $W'$. The index condition (43) then reads
\[
\dim \ker(\alpha \mp 1) - \dim \ker(\delta \mp 1) = \pm (\dim \ker(\alpha' \mp 1) - \dim \ker(\delta' \mp 1)).
\] (46)
Combining (46) with (45), we conclude
\[
\begin{align*}
\dim \ker(\alpha + 1) & = \dim \ker(\delta' + 1) & \dim \ker(\alpha - 1) & = \dim \ker(\alpha' - 1) \\
\dim \ker(\delta + 1) & = \dim \ker(\alpha' + 1) & \dim \ker(\delta - 1) & = \dim \ker(\delta' - 1),
\end{align*}
\] in accordance with (17), together with the identifications displayed above.

We can now use the construction of $F_\pm$ from above, noting, that (38) and (39) yield $-P'_\pm$ and $P'_\pm$, when we evaluate $F_\pm \gamma F_\pm^* \gamma'$ instead of $\gamma F_\pm^* \gamma' F_\pm$. For $F_K$ we again use the same ansatz via $\tilde{F}$. By assumption, $U$, restricted to $K \rightarrow K'$, fulfills $U^* \gamma U = \gamma$. Hence, setting $F = U \tilde{F}$ gives $\gamma F^* \gamma F = W_K$ as before. Moreover, since $\tilde{F}$ is a chiral symmetric square root of $W_K$, i.e. $W_K = \gamma \tilde{F}^* \gamma \tilde{F} = \tilde{F} \gamma \tilde{F}^* \gamma = \tilde{F}^2$, we also get
\[
F \gamma F^* \gamma' = U \tilde{F} \gamma \tilde{F}^* (U^* \gamma U^*) U^* = U \tilde{F} \gamma \tilde{F}^* U = U W_K U^* = W_{K'}.
\] (48)
(1)$\Rightarrow$(2): The index relation directly follows from (11), (12) and (14). For the unitary equivalence, we need to show, that the unitary equivalence of $W$ and $W'$ via $F$ also guarantees the existence of a unitary $U$, with the extra condition in (44). Between the $\pm$-eigenspaces we can just take $U = F$. On their complement, however, we need to fulfil $U \gamma U^* = \gamma'$, which $F$ certainly doesn’t. So let us restrict the considerations to $K$ and $K'$. In order to streamline the notation a bit, we drop the $K$ and $K'$ suffixes and indicate the space we are currently in by writing e.g. $\gamma'$ instead of $\gamma$. On $K$ we have $W = \gamma F^* \gamma F = \gamma \tilde{F}^* \gamma \tilde{F}$, with $\tilde{F}$ from the construction above. From this we get $F \tilde{F}^* \gamma' = \gamma' F \tilde{F}^*$, which in turn implies
\[
W' = F \gamma F^* \gamma' = (F \tilde{F}^*) \gamma \tilde{F}^* (F \tilde{F}^*)^* \gamma' = (F \tilde{F}^*) \gamma \tilde{F}^* (F \tilde{F}^*)^* \gamma'
\] (49)
\[
= (F \tilde{F}^*) \tilde{F} \gamma \tilde{F}^* (F \tilde{F}^*)^* \gamma',
\] (50)
\[
= (F \tilde{F}^*) W (F \tilde{F}^*)^*,
\] (51)
where in the last step we again used that $\tilde{F}$ is a chiral symmetric square root for $W_K$. Hence, combining $U = F (P_+ + P_- + \tilde{F}^*)$, we get the unitary equivalence $W' = U W U^*$, with a unitary intertwining $\gamma_K$ and $\gamma_{K'}$, as needed.

Note, that the condition on $U$ in (44) might be an artefact of the proof strategy. In fact, it is always met in a timeframed setting and we could not find a counterexample of two unitarily equivalent walks, which are chiral symmetric for the same symmetry, where (44) is violated. Hence, the extra condition might well be redundant.

IV. INTRODUCING LOCALITY

So far we considered half-step operators $F$ and the corresponding $W$ and $W'$ in (3) without taking the spatial structure of the underlying Hilbert space into account. The above results are thus valid on arbitrary separable Hilbert spaces. However, our goal is to classify physical systems obeying a locality condition on Hilbert spaces of the form (4). Our standing assumption will be that $F$, and consequently the timeframe unitaries $W, W'$ in (3) are essentially local. This means that for some (hence for all) $a \in \mathbb{Z}$ the commutator $[F, P_{2a}]$ with the half-space projection $P_{2a}$ is a compact operator. We denote by $A_{\text{loc}} \subset B(H)$ the set of operators satisfying this condition. It is easy to see that $A_{\text{loc}}$ is a norm closed operator algebra, which makes some of the constructions below very easy. Let us collect some of the salient features.
Proposition IV.1. Let $F \in \mathcal{A}_{\text{loc}}$ be unitary, and denote by $P = P_{\geq a}$ some half-space projection. Then

(1) $\text{PFP}$ is an essentially unitary operator on $PH$, and thus has a Fredholm index, which we denote by

$$\tilde{\text{ind}}[F] := \text{ind}[\text{PFP}] = \dim \ker PFP - \dim \ker PF^*P \in \mathbb{Z}. \quad (52)$$

(2) $\tilde{\text{ind}}[\cdot]$ is norm continuous on the unitary group of $\mathcal{A}_{\text{loc}}$, and satisfies the product rule

$$\tilde{\text{ind}}[F_1 F_2] = \tilde{\text{ind}}[F_1] + \tilde{\text{ind}}[F_2].$$

(3) The following are equivalent:

(a) $\tilde{\text{ind}}[F] = 0$

(b) There is a continuous unitary path $[0,1] \ni t \mapsto F(t) \in \mathcal{A}_{\text{loc}}$ such that $F(0) = I$ and $F(1) = F$.

(c) $F$ results from a Hamiltonian driving, i.e., it is connected to the identity by a path satisfying the differential equation $\partial_t F(t) = iH(t)F(t)$, where $H(t) = H(t)^* \in \mathcal{A}_{\text{loc}}$, $\|H(t)\|$ is bounded and $t \mapsto H(t)$ is measurable or, alternatively, piecewise constant.

The index $\tilde{\text{ind}}[\cdot]$ was first introduced in [19] for banded unitaries as a kind of net information flow across the point 0. In that case and, more generally, when $[F,P]$ is even a Hilbert-Schmidt operator, it can be expressed by a simple formula, which, however, obscures somewhat that it evaluates to an integer. When $P_x$ is the projection onto the cell $H_x$, it is

$$\tilde{\text{ind}}[F] = \sum_{x,y: \ x \leq 0 \leq y} \text{tr}(P_x F^{*} P_y F) - \text{tr}(P_y F^{*} P_x F). \quad (53)$$

In any case, this is well-defined and finite for band matrices. Similarly, if a Hamiltonian $H$ is a banded matrix, with a maximal length $L$ such that $P_x H P_y = 0$ for $|x - y| > L$, it is also essentially local, and thus produces unitary evolutions $F = \exp(itH) \in \mathcal{A}_{\text{loc}}$ with vanishing index. Since the exponential function is not a finite degree polynomial, $F$ will not be a banded matrix itself. This is why essential locality is a more manageable condition than bandedness, especially for the construction of suitable continuous paths. In this spirit, the key statement in (3) is that vanishing index indeed completely captures the idea of the existence of an essentially local driving.

Sketch of proof of Prop. IV.1. The main ideas have been explained elsewhere in increasing generality in [8, 15], so we mostly just sketch the new features.

(1),(2): $PF^*P$ is an essential inverse for $\text{PFP}$, because $P - PF^*PFP = PF^*[F,P]P$ is compact. Continuity and the product formula thus follow from general properties of the Fredholm index.

(3c)$\Leftrightarrow$(3b): The Picard-Lindelöf iteration for the differential equation lives entirely in the set of norm continuous functions $[0,1] \rightarrow \mathcal{A}_{\text{loc}}$. The necessary Lipshitz condition follows from the boundedness of $\|H(t)\|$, so we can conclude the existence of a continuous solution $F_t$. Conversely, suppose that there is a continuous path. We can then find intermediate points $0 = t_0 < t_1 < \cdots < t_r = 1$ so that $\|F(t_{k+1}) - F(t_k)\| < 2$. We claim that on each such interval we can find a constant effective Hamiltonian $H_k$ so that $F(t_{k+1}) = \exp(i(t_{k+1} - t_k)H_k)F(t_k)$. By multiplying from the right with $F(t_k)^*$ this reduces to the general statement that in any C*-algebra $\mathcal{A}$, a unitary $U \in \mathcal{A}$ with $\|U - 1\| < 2$ can be written as $\exp(iH)$ with $H \in \mathcal{A}$ and $\|H\| < \pi$. This is, however, just an application of the functional calculus. When $\|U - 1\| < 2$, $-1$ is not an eigenvalue of $U$, so on the spectrum of $U$ the logarithm function with a branch cut on the negative real axis is continuous, so we can set $H = -i \log U \in \mathcal{A}$.

(3b)$\Leftrightarrow$(3a): The trivial direction follows by continuity and $\tilde{\text{ind}}[\mathbb{I}] = 0$. For the converse, let $\tilde{\text{ind}}[F] = 0$. We then construct a path in two stages: First we connect $F$ to a unitary $F'$, for which exactly commutes with $P$, i.e., $F' = F_L \oplus F_R$ with unitaries $F_L$ on $PH$ and $F_L$ on $(\mathbb{I} - P)PH$. We then separately connect these unitaries to the respective identities. This automatically preserves (exact) essential locality, and is possible by the measurable functional calculus, i.e., by keeping the spectral family of the operator fixed and merely deforming all eigenvalues continuously to 1. No spectral gap or branch cut is needed here, and a possible eigenvalue at $-1$ can be deformed to $+1$ along with the top or the bottom half circle. The norm continuity of the path only requires that the deformation of each spectral value is continuous. It remains to construct a continuous decoupling of $F$ to $F'$. This is discussed at great length in [8, Theorem VII.4] and will not be reproduced here.
Hence, by Prop. IV.1 (3), the index \( \text{ind } [F] \) distinguishes the settings \( (H) \) and \( (F) \) in Sect. II A. This means, that there is no reason in the first place, that the “usual” classification via effective Hamiltonians gives the correct invariants for quantum walk protocols like the split-step walk, which has a non-trivial invariant \( \text{ind } [F] = -1 \) (see also the discussion below and the example later on).

An important consequence of essential locality of the half-step operator \( F \) is the essential locality of its chiral blocks \( A, B, C, D \): It implies that their right Fredholm indices are well-defined. Moreover, since the half-chain projection \( \text{PF} \) of an essentially local \( F \) is essentially unitary on the half-chain the above results which are valid in the merely essentially unitary setting yield statements about systems confined to a half-chain after replacing the Fredholm indices by right Fredholm indices.

In this way, Lem. III.1 on the one hand expresses the symmetry indices \( \text{s}(W) \) and \( \text{s}(W') \) of the walks in (3) in terms of the Fredholm indices \( \text{ind } [A], \text{ind } [B], \text{ind } [C] \) and \( \text{ind } [D] \). On the other, by applying it to the essentially unitary half-chain walks \( PW'P \) and \( PW''P \) it also determines the right symmetry indices \( \text{sr}(W) \) and \( \text{sr}(W') \) in terms of the right Fredholm indices \( \text{ind } [A], \text{ind } [B], \text{ind } [C] \) and \( \text{ind } [D] \), i.e.

\[
\text{s}(W) = \text{ind } [A] - \text{ind } [B] = \text{ind } [C] - \text{ind } [D]
\]

(54)

\[
\text{s}(W') = \text{ind } [C] - \text{ind } [A] = \text{ind } [D] - \text{ind } [B].
\]

(55)

Similarly, Lem. III.3 expresses the right Fredholm index of \( F \) in terms of \( \text{ind } [A], \text{ind } [B], \text{ind } [C] \) and \( \text{ind } [D] \). This result is closely related to [5], in which [5, Eq.(14)] is just (19) applied to half-chain walks with the additional assumption \( \text{ind } [F] = 0 \), and the invariants \( \nu_{1} \) and \( \nu_{2} \) defined in [5, Eq.(17)] correspond to \( -\text{ind } [B] \) and \( -\text{ind } [D] \). Unfortunately, the assumption \( \text{ind } [F] = 0 \) does not hold for typical examples of halfstep operators of quantum walks (e.g. for the split-step walk, see (84)), so that a direct application of the resulting index formulas in [25] is, at first sight, problematic. However, it can be justified in our framework for settings in which two blocks are joined (see Sect. VI).

In contrast to Lem. III.3, Lem. III.2 requires exact unitarity of \( F, W \) and \( W' \), and has therefore nothing to say about the corresponding half-chain operators. This in accordance with the observation that the symmetry indices like \( \text{s}_1 \) are neither invariant under compact nor under gentle perturbations [8]. We collect the connections between the symmetry indices of the quantum walks \( W \) and \( W' \) and the Fredholm indices \( \text{ind } [\cdot] \) and \( \text{ind } [\cdot] \) of the matrix blocks of \( F \) in Table I.

Let us finally address the locality of the half-step operator \( F \) constructed from a given chiral symmetric \( W \) in the proof of Prop. III.5.

**Scholium IV.2.** Let \( W \) be a chiral symmetric walk. Then the half-step operator constructed in the proof of Prop. III.5 can be chosen essentially local.

To show this, we only need to consider \( F_K \), since \( F_+ \) and \( F_- \) have finite rank and can therefore be neglected. First note, that the \( \bar{F} \) in (40) is given entirely in terms continuous functions of matrix blocks of \( W \) or their polar isometries. Therefore it is essentially local by the following lemma:

**Lemma IV.3.** Let \( X \in \Lambda_{\text{loc}} \) be a Fredholm operator. Then its polar isometry \( U_X \), as well as its absolute value \( |X| = \sqrt{X^*X} \) are essentially local.

**Proof.** On the one hand \( A_{\text{loc}} \) is norm-closed, so the essential locality of \( |X| \) follows via the Weierstraß-theorem. On the other, let \( P \) be a half-chain projection on \( \mathcal{H} \). Then, by \( [X, P] = [U_X X, P] = U_X [X, P] + [U_X, P] X \), \( [U_X, P] X \) is the difference of compact operators. Moreover, since \( X \) is Fredholm, the only way for \( [U_X, P] X \) to be compact is that \( [U_X, P] \) is compact, i.e \( U_X \) is essentially local.

It remains to show, that the unitary \( V : K \to K' \) in the proof of Prop. III.5 can be chosen essentially local, which boils down to the following lemma:

**Lemma IV.4.** Let \( \mathcal{H} \) be a Hilbert space with a one-dimensional lattice structure, as in (4). Moreover, let \( Q_1 \) and \( Q_2 \) be two finite rank projections on \( \mathcal{H} \). Then there exists an essentially local partial isometry \( \Lambda \), with

\[
\Lambda^* \Lambda = \mathbb{I} - Q_1 \quad \text{and} \quad \Lambda \Lambda^* = \mathbb{I} - Q_2.
\]

(56)

In order to see how this applies to \( V : K \to K' \), let \( \mathcal{H}_\pm = \Gamma_\pm \mathcal{H} \) and similarly for \( \mathcal{K}_\pm \) and \( \mathcal{K}'_\pm \). Then, since \( \mathcal{K}_\pm \) and \( \mathcal{K}'_\pm \) are the complements of finite dimensional subspaces of \( \mathcal{H}_\pm \), by Lem. IV.4 we find essentially local isometries \( V_+ \) and \( V_- \), such that

\[
V_+ V_+^* \mathcal{H}_\pm = \mathcal{K}_\pm \quad \text{and} \quad V_- V_-^* \mathcal{H}_\pm = \mathcal{K}'_\pm.
\]

(57)
Considering $V_{\pm}$ as mappings between $\mathcal{K}_\pm$ and $\mathcal{K}_\pm'$ they become unitary and define the desired essentially local unitary $V$ via

$$V = \begin{pmatrix} V_+ & 0 \\ 0 & V_- \end{pmatrix}.$$

(58)

**Proof of Lem. IV.4.** The proof can be reduced to the construction of an essentially local partial isometry $\Pi$, with $\Pi^*\Pi = I - N$ and $\Pi\Pi^* = I$, for an arbitrary finite rank projection $N$. Then, with $\Pi_1$ and $\Pi_2$ being such isometries for $Q_1$ and $Q_2$, respectively, the statement follows from setting $\Lambda = \Pi_2^*\Pi_1$.

To construct $\Pi$, let $n = \text{rank } N$ and

$$\Pi_M = (I - P) + PS^*nP,$$

(59)

where $P = P_{>0}$ is the projection onto $\bigoplus_{x>0} \mathcal{H}_x$ and $S = \sum_{x,i} |x + \delta_{1i}, i\rangle \langle x, i|$ denotes the partial bilateral shift which shifts the first basis vector in each cell $\mathcal{H}_x = \text{span}\{ |x, i\rangle : i = 1, \ldots, d_x \}$ to the right. Sandwiching $S$ with $P$, we get the unilateral shift, with

$$\text{e}^{(PS^*P)(PSP)} = P \quad \text{and} \quad \text{e}^{(PSP)(PS^*P)} = P - |1, 1\rangle \langle 1, 1|.$$

(60)

Extending these properties to $S^n$, we get

$$\Pi_M\Pi_M^* = (I - P) + P = I$$

and

$$\Pi_M^*\Pi_M = (I - P) + (PS^*nP)(PS^*nP) = (I - P) + (P - M) = I - M,$$

(62)

with $M = \sum_{k=1}^n |k, 1\rangle \langle k, 1|$. $\Pi_M$ is defined as the identity on one half of the chain and as a unilateral shift on the other, wherefore it is automatically essentially local.

Since $\text{rank } M = n = \text{rank } N$, there exists a unitary $U$ which identifies the subspaces $N\mathcal{H}$ and $M\mathcal{H}$, i.e. $N = U^* MU$, and acts like the identity on their complement. Being a finite rank perturbation of the identity, $U$ is clearly essentially local. Hence, $\Pi = \Pi_M U$ is essentially local with

$$\Pi^*\Pi = U^*\Pi_M^*\Pi_M U = U^*(I - M)U = I - N \quad \text{and} \quad \Pi\Pi^* = \Pi_M\Pi_M^* = I.$$

(63)

□

**Assumption IV.5.** From now on we consider only half-step operators $F$ that are essentially local, i.e. half-step walks. This is equivalent to the essential locality of the walks $W$ and $W'$ corresponding to $F$ via (3).

**V. COMPLETE SET OF INDICES**

In the previous section we defined ten indices for half-step walks $F$: On the one hand, there are the Fredholm indices $\text{ind } [\cdot]$ of $F$ and its four chiral blocks $A, B, C$ and $D$. On the other, there are the right Fredholm indices $\tilde{\text{ind }} [\cdot]$ of these five operators. However, we also saw that these ten indices are not independent of each other. In the following we identify a subset of five indices, show that it is independent, and prove its completeness in the sense that two half-step walks $F_1$ and $F_2$ are homotopic in the set of half-step walks if and only if all five indices coincide.

The results of the previous sections imply the following dependencies between the ten indices of $F$: By (11) or (12) in Lem. III.1 we can drop the index of one of the matrix blocks, w.l.o.g. $\text{ind } [D]$. Since the lemma is formulated for merely essentially unitary half-step operators, we can also drop $\tilde{\text{ind }} [D]$. Furthermore, by (14) in Lem. III.2, we can drop $\tilde{\text{ind }} [C]$. This lemma requires exact unitarity and hence has nothing to say about right Fredholm indices. Finally, Lem. III.3, which is again valid also for merely essentially unitary half-step operators, allows us to drop $\text{ind } [F]$ and $\tilde{\text{ind }} [F]$. Thus:

**Scholium V.1.** Let $F$ be a unitary half-step operator. Then the following set of integer-valued indices is independent:

$$\{ \text{ind } [A], \text{ind } [B], \tilde{\text{ind }} [A], \tilde{\text{ind }} [B], \tilde{\text{ind }} [C] \}.$$

(64)

In the following, we take this as the standard set of independent indices for $F$. 


The independence of the five indices in (64) is proved by defining a generating example which allows us to realize every index combination in $\mathbb{Z}^5$:

**Example V.2.** Let $S$ be the unilateral shift on $\ell^2(\mathbb{Z})$ with $\text{ind}[S] = 0$ and $\overrightarrow{\text{ind}}[S] = -1$, and consider the unitary operators

$$ U(n,m) = \frac{1}{\sqrt{2}} \begin{pmatrix} S^n & S^m \\ -S^{-m} & S^{-n} \end{pmatrix} \quad \text{and} \quad T(k) = \begin{pmatrix} S^k & 0 \\ 0 & 1 \end{pmatrix}, \quad n,m,k \in \mathbb{Z}. \quad (65) $$

It is straightforward to see that $\text{ind}[U(n,m)] = 0$ for all $m,n \in \mathbb{Z}$. Thus, by [8, Theorem VII.4] $U$ can be decoupled gently, i.e. there exists a chiral symmetric homotopy $U(m,n) \mapsto U_L(n_L,m_L) \oplus U_R(n_R,m_R)$, where $U_L(n_L,m_L)$ and $U_R(n_R,m_R)$ are exactly unitary quantum walks on the left and the right half-chain, respectively. We define the generating example as

$$ F(n_L,m_L,n_R,m_R,k) = T(k)(U_L(n_L,m_L) \oplus U_R(n_R,m_R)). \quad (66) $$

Using $\text{ind}[S] = 0$, $\overrightarrow{\text{ind}}[S] = -1$ and $\text{ind}[S^a(P^bP^cP^d) \oplus PS^eP] = b-c$ we can determine the index tuple (64) of $F$:

$$ \{\text{ind}[A], \text{ind}[B], \overrightarrow{\text{ind}}[A], \overrightarrow{\text{ind}}[B], \overrightarrow{\text{ind}}[C]\} = \{n_L - n_R, m_L - m_R, -n_R - k, -m_R - k, m_R\}, \quad (67) $$

i.e. the indices of $F$ may be calculated by applying the integer matrix

$$ M = \begin{pmatrix} 1 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & -1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix} \quad (68) $$

to the parameter vector $(n_L,m_L,n_R,m_R,k)$. Clearly $|\det M| = 1$ and, moreover, its inverse is also an integer matrix. Hence, for any possible combination of indices in (64) we can construct an $F$ via (66) with parameters obtained by applying $M^{-1}$ to this index set.

The set of invariants of half-step walks identified above is not only independent but also stable against homotopic deformations of $F$:

**Lemma V.3.** The half-step indices $\{\text{ind}[A], \text{ind}[B], \overrightarrow{\text{ind}}[A], \overrightarrow{\text{ind}}[B], \overrightarrow{\text{ind}}[C]\}$ are invariant under homotopies of $F$ which do not close the essential gaps of $W$ and $W'$.

**Proof.** Any homotopy of $F$ must also be a homotopy of $A,B,C$ and $D$ and keeping the essential gap condition for $W$ and $W'$ is equivalent to $A,B,C$ and $D$ being Fredholm operators along any allowed path. Hence, since $\text{ind}[X]$ and $\overrightarrow{\text{ind}}[X]$ for $X \in \{A,B,C,D\}$ are defined as Fredholm indices, they are constant along allowed paths. \qed

Importantly, the converse of this stability also holds, i.e. the independent set of indices in (64) is complete:

**Theorem V.4.** Let $F_1$ and $F_2$ be two half-step walks with the same indices $\{\text{ind}[A], \text{ind}[B], \overrightarrow{\text{ind}}[A], \overrightarrow{\text{ind}}[B], \overrightarrow{\text{ind}}[C]\}$. Then $F_1$ and $F_2$ are homotopic in the set of half-step walks.

Due to its length we delay the proof of this theorem until Sect. VII. Before, in the next section we connect the indices of half-step walks to the symmetry indices of timeframed quantum walks.

**VI. CONNECTION TO THE CLASSIFICATION OF TIMEFRAMED WALKS**

**A. Index connections**

Having identified a complete set of indices for half-step walks, we investigate their connection to the walk indices of the two timeframes $W$ and $W'$. Without further restrictions each of these walks is completely
characterized by the index set \( \{ s_i, s_i', s_i+ \} \), respectively \( \{ s_i', s_i', s_i' + \} \) [8]. Clearly, the walk indices are not in one-to-one correspondence with the index set for \( F \), since \( W \) and \( W' \) are not independent. For example, (14) implies that \( s_i+ (W) = \pm s_i+ (W') \), which is equivalent to \( s_i+ (W) = (s_i(W) + s_i(W') / 2 \). Hence, also \( s_i+ (W) \) itself is no longer independent, but determined by the other walk indices (compare also Thm. III.6). Apart from that restriction, however, the walk indices are independent, as the following result shows.

**Corollary VI.1.** Two index triples \( \{ s_i, s_i', s_i+ \} \) and \( \{ s_i', s_i', s_i' + \} \) have representatives \( W \) and \( W' \) according to (3) if and only if

\[
\text{Cor} = \frac{s_i+ (W') + s_i (W')} {2} = \frac{s_i(W) + s_i(W')}{2}.
\]

**Proof.** The “only if” part of the statement follows from Thm. III.6. For the converse direction let \( \{ s_i, s_i', s_i+ \} \) and \( \{ s_i', s_i', s_i' + \} \) be two index triples and assume (69) to hold. Then the generating example defined in (66) with parameters

\[
(n_L, m_L, n_R, m_R, k) = \left( n, \frac{s_i-\bar{s}_i}{2} + n, \frac{s_i'-\bar{s}_i}{2} + \bar{s}_i + n, \bar{s}_i - s_i' - \bar{s}_i + s_i' - 2n \right)
\]

gives \( W \) and \( W' \) with the desired indices where \( n \in \mathbb{Z} \) is arbitrary. By (69) \( s_i + s_i' \) is even, such that either \( s_i \) and \( s_i' \) are both even or both odd. In any case, they differ by an even number which guarantees that \( (s_i' - s_i) / 2 = s_i' - s_i' \) in the parameter choice (70) is an integer. \( \square \)

By this result \( s_i+ = s_i' \) are determined by \( s_i \) and \( s_i' \) and, moreover, the latter two differ by an even number. Hence, there are only four independent indices in terms of \( W \) and \( W' \), with the further restriction, that \( s_i' = 2l - s_i \), for \( l = s_i+ \in \mathbb{Z} \), whereas there are five independent indices for \( F \). This poses the question to what extend the classification of half-step walks in terms of \( \{ \text{ind} [A], \text{ind} [B], \text{ind} [A], \text{ind} [B], \text{ind} [C] \} \) is finer than that of timeframed walks in terms of \( \{ \text{si}(W), \text{si}(W), \text{si}(W'), \text{si}(W') \} \).

Before we answer this question, consider the following example:

**Example VI.2.** A modification of a half-step walk \( F \) which does not change \( W \) is to multiply \( F \) with a unitary operator which is diagonal in the chiral eigenbasis, see Lem. III.4. Consider for example the modification

\[
F \mapsto \begin{pmatrix} S^m & 0 \\ 0 & S^n \end{pmatrix} F,
\]

which leads to \( \{ \text{ind} [A], \text{ind} [B], \text{ind} [C] \} \mapsto \{ \text{ind} [A] - m, \text{ind} [B] - m, \text{ind} [C] - n \} \) and does not change the \( \text{ind}[\cdot]\)-indices. Some choices of the parameters \( m, n \) can be detected by the walk indices through

\[
\text{si}(W) - \text{si}(W') = 2 \text{ind} [A] - \text{ind} [C] - \text{ind} [B] = 2 \text{ind} [A] - \text{ind} [F].
\]

while others cannot: Choosing \( m = n = \ell \) does not change \( s_i - s_i' \), while \( m = 2\ell, n = 0 \) induces a change of \( -2\ell \). Yet, by Prop. IV.1 both choices imply \( \text{ind} [F] \to \text{ind} [F] - 2\ell \).

This observation has an important consequence: On the level of the walks \( W \) and \( W' \) it is not always possible to decide whether they stem from a periodically driven continuous time evolution. If the time evolution is continuous, i.e. if \( F \) is the time-ordered exponential of some Hamiltonian \( H(s) \), the half-step operator has to satisfy \( \text{ind} [F] = 0 \), by item (3) in Prop. IV.1.

The example suggests that \( \text{ind} [F] \) might serve as the missing index, which is indeed the case.

**Lemma VI.3.** Let \( W \) and \( W' \) be timeframed quantum walks as in (3) with the half-step walk \( F \). Then \( \{ \text{si}(W), \text{si}(W), \text{si}(W'), \text{si}(W'), \text{ind} [F] \} \) is a complete set of indices.

Note, that differently from the index set (64), the walk indices, together with \( \text{ind} [F] \) are not pairwise independent, since for example \( \text{si}(W) \) and \( \text{si}(W') \) have to differ by an even number by Cor. VI.1.

**Proof.** To prove this, we relate the index set above to the set of independent indices in (64), which we already know to be complete from Thm. V.4. From Sect. III we know, how to obtain the walk indices and \( \text{ind} [F] \) from the indices of the corresponding half-step walk. Since the latter are complete, we can also already
conclude, that every valid combination of the former can be reached. Hence, it remains to verify, that the converse is also true. From Lem. III.1 we get
\begin{equation}
\text{ind}[A] = \frac{\text{si}(W) - \text{si}(W')}{2} \quad \text{and} \quad \text{ind}[B] = -\frac{\text{si}(W) + \text{si}(W')}{2}.
\end{equation}
Since by Cor. VI.1 \(\text{si}(W)\) and \(\text{si}(W')\) differ by an even number, the right hand sides are always integers. Moreover, every combination \(\{\text{ind}[A], \text{ind}[B]\} \in \mathbb{Z}^2\) can be reached this way. Further, from (72) and similar expressions we get
\begin{align}
\text{ind}[A] &= \frac{1}{2} \text{ind}[F] + \frac{\text{si}(W) - \text{si}(W')}{2}, \\
\text{ind}[B] &= \frac{1}{2} \text{ind}[F] - \frac{\text{si}(W) + \text{si}(W')}{2}, \\
\text{ind}[C] &= \frac{1}{2} \text{ind}[F] + \frac{\text{si}(W) + \text{si}(W')}{2},
\end{align}
Again, by the relation between the combinations of \(\text{si}(W)\) and \(\text{si}(W')\) and \(\text{ind}[F]\), these are always integers. And one can again check, that every combination \(\{\text{ind}'[A], \text{ind}'[B], \text{ind}'[C]\} \in \mathbb{Z}^3\) can be realized.
Hence, there is a one to one correspondence between the complete index set (64) and the valid index combinations of walk indices, together with \(\text{ind}[F]\).

B. Bulk-edge correspondence in the timeframed setting

Bulk-edge correspondence for quantum walks was rigorously proved in the setting with one timeframe. More concretely, it was showed in [8, Corollary IV.3] that for a walk \(W\) which coincides with bulk walks \(W_L\) and \(W_R\) far to the left and far to the right, respectively,
\begin{equation}
\text{si}(W) = \text{si}(W_R) + \text{si}(W_L) = \text{si}(W_R) - \text{si}(W_L),
\end{equation}
where for the second equality the bulks have to have proper gaps. Thus, whenever \(W_L\) and \(W_R\) are in different topological phases, (77) gives a lower bound on the number of symmetry protected eigenvalues of the crossover \(W\). However, the theory in [8] does not predict whether these eigenvalues are at +1 or −1: This depends on \(\text{si}_-(W)\) which cannot be inferred from the asymptotic indices in (77) and, moreover, can be changed by non-gentle perturbations.

Taking into account the second timeframe we consider a situation with two chiral symmetric timeframed crossovers \(W\) and \(W'\) which correspond to \(W_R\) and \(W'_R\) far to the right, respectively, and to \(W_L\) and \(W'_L\) far to the left. Then, plugging (77) for \(W\) and \(W'\) into (69) we obtain
\begin{equation}
2 \text{si}_+(W) = 2 \text{si}_+(W') = \text{si}(W_R) + \text{si}(W'_R) - (\text{si}(W_L) + \text{si}(W'_L)),
\end{equation}
and
\begin{equation}
2 \text{si}_-(W) = -2 \text{si}_-(W') = \text{si}(W_R) - \text{si}(W'_R) - (\text{si}(W_L) - \text{si}(W'_L)).
\end{equation}
Thus, the second timeframe stabilizes the symmetry protected eigenvalues of the crossover and allows to attribute them to +1 and −1. These formulas agree with those in [4].

If additionally \(W_R, W'_R\) and \(W_L, W'_L\) are timeframed with \(F_R\) and \(F_L\), respectively, the above implies
\begin{equation}
\text{si}_+(W) = \text{ind}[C_R] - \text{ind}[C_L] = -(\text{ind}[B_R] - \text{ind}[B_L]),
\end{equation}
and
\begin{equation}
\text{si}_-(W) = \text{ind}[A_R] - \text{ind}[A_L] = -(\text{ind}[D_R] - \text{ind}[D_L]),
\end{equation}
where we used (54), (55) and (19) in combination with \(\text{ind}[F_R] = \text{ind}[F_L]\), which is a necessary condition for the existence of a unitary crossover between \(F_R\) and \(F_L\) [15].

The above formulas confirms those in [5], where the systems under considerations were treated by continuously driven Floquet time evolutions. Moreover, we generalized them to systems with \(\text{ind}[F] ≠ 0\), which indirectly validates [25], where, in the appendix, the invariants from [5] were applied to systems with \(\text{ind}[F] ≠ 0\). The results above are also in line with those in [4], where a walk was defined by a fixed shift-coin skeleton.
C. Example: No bridges for timeframed split-step walks

The stability of the independent set of indices for half-step operators in Lem. V.3 has an important consequence for the existence of “bridges over troubled gap closings” \cite{10} in the split-step walk. This walk is defined on $f_2(\mathbb{Z}) \otimes \mathbb{C}^2$ as

$$W(\theta_1, \theta_2) = R(\theta_1/2)S_rR(\theta_2)S_rR(\theta_1/2),$$

where $R(\theta) = I \otimes R_2(\theta)$ rotates the internal degree of freedom homogeneously by the angle $\theta$ around the $y$-axis, and $S_r$ is the right shift of the spin-up vectors whereas $S_l$ shifts spin-down vectors to the left. This walk is chiral symmetric with $\gamma = I \otimes \sigma_1$, and up to local basis changes its timeframes are identified via (3) by the half-step walk

$$F = R(\theta_2/2)S_rR(\theta_1/2).$$

It is known that in the setting $(W)$ in Sect. II A, i.e. if we restrict considerations to $W$ only and do not take the half-step operator $F$ and the other timeframe $W'$ into account, bridges exist between patches of equal indices in the phase diagram after regrouping neighbouring cells \cite[Sec. 3.1.3]{10}. However, in the setting $(F)$ where the other timeframe is taken into account this is no longer true:

**Corollary VI.4.** There is no bridge over the troubled gap closing for the split-step walk which leaves the gaps of $W$ and $W'$ open.

**Proof.** For definiteness, we consider the same patches of the phase diagram of the split-step walk as in \cite{10} where $\vec{s}(W) = 0$. After flattening the band structure of the walks, the parameters determining these patches are $\theta_1 = 0, \theta_2 = \pi/2$ and $\theta_1 = 0, \theta_2 = -\pi/2$, respectively. The phase diagram of the second timeframe $W'$ is the same as the one for $W$, but rotated by $90^\circ$, yielding $\vec{s}(W') = -1$. Therefore, a homotopy between the two patches on the level of both timeframes would imply a homotopy of $W'$ between patches of different indices $\vec{s}(W') = \pm 1$. Hence, no such homotopy exists.

Let us also investigate the present scenario directly on the level of the half-step walk $F$. In chiral eigenbasis we have

$$F = R(\theta_2/2 - \pi/4)S_rR(\theta_1/2 + \pi/4).$$

Hence, by (84), the corresponding half-step walks for the two parameter pairs $\theta_1 = 0, \theta_2 = \pi/2$ and $\theta_1 = 0, \theta_2 = -\pi/2$ are given by

$$F_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} S & -S \\ 1 & 1 \end{pmatrix}, \quad F_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -S & S \end{pmatrix},$$

FIG. 1. Harlequin for the half-step operator (84). For the split-step walk we have $\text{ind}[A] = \text{ind}[B] = 0$ for all parameters, wherefore, the legend only lists the three non-constant indices $(\text{ind}[A], \text{ind}[B], \text{ind}[C])$. Through the constraint $\text{ind}[F] = \text{ind}[B] + \text{ind}[C] = -1$ these three indices determine the two indices $\vec{s}$ and $\vec{s}'$. 
FIG. 2. The non-existence of bridges over troubled gap closings: While in the timeframe on the left a bridge exists [10], in the other timeframe the phase diagram is rotated by $\pi/2$. Therefore, the walks which were in the same phase in the first timeframe are now in different phases which renders the existence of the bridge impossible.

with index tuples

$$F_1 : \{0, 0, -1, -1, 0\} \quad F_2 : \{0, 0, 0, 0, -1\}. \quad (86)$$

If only one timeframe, namely $W$, is considered, we are free to multiply $F_1$ from the left as in Lem. III.4, with $m = 1, n = -1$ without changing $W$. This leaves invariant $\vec{s}_1$, but changes $\vec{s}_1'$. Therefore, when just one timeframe is under consideration (as in [10]), the bridge over the troubled gap closing exists on the level of walks. However, by Thm. V.4, there is no path on the level of half-step operators $F_i$, without such adjustment. In [10] such bridges in general exist after regrouping the cell-structure, i.e. considering pairs of neighbouring cells as a single cell. However, since all indices of $F$ are Fredholm indices they are invariant under regroupings of the cell structure of the Hilbert space. Thus, bridges do not exist even after regrouping.

While there are no bridges over troubled gap closings, other bridges between patches of the harlequin with the same index combinations of the corresponding half-step walks can be defined explicitly. For example, a bridge between the $F_1$ in (85) and $F_3$ with $\theta_1 = \pi, \theta_2 = \pi/2$, i.e.

$$F_3 = \frac{1}{\sqrt{2}} \begin{pmatrix} -S & -S \\ 1 & -1 \end{pmatrix} \quad (87)$$

is given by

$$[0, 1] \ni t \mapsto F_t = \frac{1}{\sqrt{2}} \begin{pmatrix} e^{i\pi t} S & -S \\ 1 & e^{-i\pi t} \end{pmatrix}. \quad (88)$$

Note however, that in this paper we only consider chiral symmetry. The path $F_t$ between $F_1$ and $F_3$ above breaks particle-hole symmetry of the split-step walk. Taking particle-hole symmetry into account a regrouping of cells similar to that described in [10] is necessary to find bridges that respect both symmetries.

D. Compact perturbations of half-step walks

The classification in [8] is not only stable against homotopies, but also includes stability against compact perturbations that are not continuously contractible along a symmetric path, so-called non-gentle perturbations. In fact, the index $s_i^+$ indicates, whether a compact perturbation can be contracted gently or not. In this section, we investigate how compact perturbations influence the index set for $F$.

**Lemma VI.5.** Let $F$ be a half-step walk and let $W$ and $W'$ be the corresponding timeframed walks. Then every compact perturbation of $F$ is a gentle perturbation of $W$ and $W'$.

**Proof.** Clearly, every compact perturbation of $F$ is a compact perturbation of $W$ and $W'$. Moreover, the chiral symmetry of $W$ and $W'$ is preserved by arbitrary compact perturbations of $F$. Since any chiral symmetric compact perturbation leaves the symmetry indices $s_i(W)$ and $s_i(W')$ invariant, also $s_i^+(W) = s_i^+(W')$ is invariant by Cor. VI.1. Hence, by [8, Thm. VI.4] in conjunction with [8, Lem. VI.3] the induced perturbations of $W$ and $W'$ are gentle.

The converse of this result has an important consequence regarding non-gentle perturbations of chiral symmetric quantum walks: For every $W = \gamma F^* \gamma F$ we know from Prop. III.5 that also every perturbed walk $\tilde{W} = VW$ possesses a half-step walk $\tilde{F}$. However, if the perturbation is non-gentle, the converse of Lem. VI.5 implies that $\tilde{F}$ cannot be a compact perturbation of $F$. We illustrate that with a simple example:
Example VI.6. Consider the split-step walk in (82) with \((\theta_1, \theta_2) = (0, \pi/2)\), i.e. \(W = \mathbb{I} \otimes (-i\sigma_2)\) which is chiral symmetric for \(\gamma = \mathbb{I} \otimes \sigma_1\). Its half-step walk is given by \(F = R(\pi/4)S_\gamma\). A typical non-gentle perturbation of \(W\) is to replace the coin at \(x = 0\) by \(\sigma_1\) [8, 31]. On the level of the half-step walk, this can be achieved, by modifying \(S_\gamma\) on a half-chain: With \(S_\gamma = (P_{<0} + P_{\geq 0} \gamma P_{\geq 0}) S_\gamma (P_{<0} + P_{\geq 0} \gamma P_{\geq 0})\), \(\tilde{F} = R(\pi/4)S_\gamma\) is a half-step operator for \(\tilde{W}\). Clearly, \(F - \tilde{F}\) is not compact.

For finite systems, e.g. systems with periodic boundary conditions this implies the following: Since in such systems there are no non-compact perturbations, a single non-gentle perturbation renders impossible the existence of a half-step walk \(F\). On the other hand, every large-scale modification of \(F\) needed to implement a local non-gentle perturbation of \(W\) produces another local non-gentle perturbation somewhere else on the ring. These two non-gentle perturbations are globally gentle. For a detailed discussion of non-gentle perturbations on finite systems see also [3].

VII. PROOF OF THEOREM V.4

In this section we prove the completeness result in Thm. V.4 in several steps: First we introduce a standard form for half-step walks \(F\) which facilitates the proof. Then, we construct a homotopy which connects each half-step walk with one whose timeframed walks have essential spectrum only at \(\pm i\). Finally, we assemble the proof of Thm. V.4 in the last part of this section.

A. Standard form

The \(F\) we constructed in the proofs of Prop. III.5 and Thm. III.6 had a special structure, based on the eigenspace decompositions of \(W\) and \(W'\). This structure was not just an artefact of the proof technique, but can actually be raised to a standard form for every given half-step operator \(F\). Let \(\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_- = \mathcal{H}_+ \oplus \mathcal{H}_- \) be the decomposition of \(\mathcal{H}\) into the \(\gamma\)-eigenspaces. Similar to the splitting in the proofs mentioned above these eigenspaces can be further decomposed in two different ways, namely

\[
\mathcal{H}_+ = \ker A \oplus \ker C \oplus \Gamma_+ \mathcal{K} = \ker A^* \oplus \ker B^* \oplus \Gamma_+ \mathcal{K}', \tag{89}
\]

\[
\mathcal{H}_- = \ker D \oplus \ker B \oplus \Gamma_- \mathcal{K} = \ker D^* \oplus \ker C^* \oplus \Gamma_- \mathcal{K}', \tag{90}
\]

where by Lem. III.1 ker \(X\) is finite dimensional for \(X = A, B, C, D\) due to the essential gap condition of the corresponding walks. The orthogonality of the direct summands is guaranteed by unitarity of \(F\) and \(F^*\) in (16). For example, ker \(A\) and ker \(C\) are orthogonal by \(A^* A + C^* C = \mathbb{I}\). Reordering the direct summands in (89) and (90) and using the arguments in the proof of Lem. III.2, \(F\) takes the following form:

\[
F: (\ker A \oplus \ker D) \oplus (\ker C \oplus \ker B) \oplus \mathcal{K} \to (\ker A^* \oplus \ker D^*) \oplus (\ker B^* \oplus \ker C^*) \oplus \mathcal{K}' \tag{91}
\]

\[
F = \begin{pmatrix}
0 & B^- & A^+ \\
C^- & 0 & D^+
\end{pmatrix} \oplus \begin{pmatrix}
A_K & B_K \\
C_K & D_K
\end{pmatrix}. \tag{92}
\]

Here, \(A^+: \ker C \to \ker B^*, B^-: \ker D \to \ker A^*, C^-: \ker A \to \ker D^*, D^+: \ker B \to \ker C^*\) are finite dimensional unitaries because they are isometries between spaces of the same dimension, see (17) and (18). \(A_K, B_K, C_K\) and \(D_K\) on the other hand are invertible operators on the remaining infinite dimensional Hilbert space. However, note that the finite dimensional unitaries \(A^+, B^-, C^-\) and \(D^+\) do not necessarily have to be of the same size. Indeed, according to Lem. III.2 the difference in dimension of \(A^+\) and \(D^+\) constitutes the index \(\delta_{\pm} (W) = \delta_{\pm} (W')\), whereas that of \(B^-\) and \(C^-\) determines \(\delta_{\pm} (W) = -\delta_{\pm} (W')\). For exactly gapped walks \(W\) and \(W'\) the finite dimensional blocks vanish, and we get \(\mathcal{K}_+ = \mathcal{K}_+ = \mathcal{H}_+\) and \(\mathcal{K}_- = \mathcal{K}_- = \mathcal{H}_-\).

B. The flattening construction

For the classification of quantum walks in [8] it turned out to be useful to flatten their spectrum, i.e. to continuously deform it to an operator, whose spectrum is contained in \(\{\pm 1, \pm i\}\) with only finitely degenerated eigenvalues at \(\pm 1\). We call such operators essentially flat band. In the current setting, flattening
the spectrum of one of the timeframe walks automatically implies that the other timeframe is also essentially flatband by unitary invariance of the spectrum. The essential flatband condition has the following consequences for \( F \):

**Lemma VII.1.** Let \( F \) be the half-step walk with corresponding walks \( W \) and \( W' \). Then the following are equivalent:

1. \( W \) and \( W' \) are essentially flatband.
2. \( \sqrt{2}X \) is essentially unitary for \( X \in \{A, B, C, D\} \).

**Proof.** It turns out easier to formulate the conditions in the Calkin algebra, as we get rid of the adjective “essential” in this way without loss of generality. In the Calkin algebra \( W \) and \( W' \) are exactly flat-band, which means, that \( W = -W^* \), and the same for \( W' \). But this is equivalent to \( A^*A = C^*C, B^*B = D^*D, AA^* = BB^* \) and \( DD^* = CC^* \) by (9), which, by unitarity of \( F \) in (16), is equivalent to \( \sqrt{2}X \) being unitary for \( X \in \{A, B, C, D\} \).

As for Lem. III.1 this lemma also holds when we drop all “essentially”s. Similarly, it does not make use of the locality of the involved operators. However, for the above definition of essentially flatband operators exact unitarity is a crucial assumption as it allows one to speak of “finitely degenerate eigenvalues” at \( \pm 1 \). Relaxing this definition to mere essential unitarity does not make sense since essentially unitary operators are not normal in general. To be able to nevertheless speak of essentially gapped essentially unitary operators, we could instead take the finite dimensionality of the kernels of \( W \) as a defining property. This would allow us to define essentially flatband essentially unitary operators as essentially gapped essentially unitary operators whose essential spectrum consists of \( \pm 1 \).

However, a much simpler choice is to just take Lem. VII.1 also in the essentially unitary case, i.e. we call essentially unitary operators \( W \) and \( W' \) in (3) essentially flatband if the chiral blocks of the corresponding half-step operator are essentially unitary up to a factor of \( 1/\sqrt{2} \). Moreover, for the sake of brevity we occasionally call a half-step operator \( F \) essentially flatband whenever the corresponding \( W \) and \( W' \) are.

Using the standard form introduced above, we now construct a flatband deformation of walks of the form (3) directly on the level of the half-step walk \( F \):

**Lemma VII.2.** Let \( F \) be a half-step walk. Then there is a continuous path \( t \mapsto F_t, t \in [0, 1] \), such that \( F_0 = F \) and \( F_1 \) constitutes essentially flatband walks \( W_1 \) and \( W'_1 \).

**Proof.** By Lem. VII.1, the essentially flatband condition on timeframe walks is equivalent to essential unitarity of the chiral blocks of the corresponding half-step walk (up to a factor of \( 1/\sqrt{2} \)). In order to achieve this essential unitarity, we first restrict considerations to the part of the half-step walk where each block is invertible, i.e. to the complement of the kernels and cokernels of the chiral blocks. In the standard form (92) this means that we only have to deal with the rightmost block. This reduces the flattening to the task of transforming the invertible blocks to exactly unitary operators up to a factor of \( 1/\sqrt{2} \), without destroying the unitarity of the half-step walk along the way.

Using the polar decomposition, the upper left chiral block \( A \) of \( F \) can be written as \( A = U_A|A| \), with a unique unitary polar isometry \( U_A \) and the absolute value \( |A| = \sqrt{A^*A} \). We can also write the remaining blocks in this way and, moreover, using the unitarity conditions of \( F \) in (16) we can express all absolute values in terms of \( A^*A \). This gives

\[
\begin{align*}
A &= U_A \sqrt{A^*A} \\
C &= U_C \sqrt{1 - A^*A} \\
B &= U_B \sqrt{1 - U_B^* U_A A^* A U_A^* U_B} \\
D &= U_D \sqrt{U_B^* U_A A^* A U_A^* U_B},
\end{align*}
\]

(93) (94)

where we used \( AA^* + BB^* = I \) and \( XX^* = U_X X^* X U_X^* \) for \( B \), \( A^*A + C^*C = I \) for \( C \) and \( D^*D + B^*B = I \) for \( D \). The remaining unitarity conditions all boil down to \( U_D = -U_C U_A^* U_B \), which guarantees the unitarity of the flatband half-step walk

\[
F^\oplus = \begin{pmatrix} U_A & U_B \\
U_C & U_D \end{pmatrix}/\sqrt{2}.
\]

(95)

In particular, the unitarity of \( F \) is independent of \( A^*A \), given that \( 0 < A^*A < I \), which holds because \( A \) is invertible, \( A^*A + C^*C = I \) and \( C^*C > 0 \). Hence, we can deform \( F \) into \( F^\oplus \) by constructing a continuous
path between $A^* A$ and $\mathbb{I}/2$. A particularly simple path is given by linearly interpolating between the two operators $t \mapsto t(\mathbb{I}/2) + (1 - t)A^* A$.

Taking into account the remaining summands in (92), note that the two left summands of $F$ are already in flat-band form, since their blocks are either zero or unitary. Hence, the above construction works also in this case, with the only difference that the polar isometries of $A, B, C, D$ now might have finite dimensional kernels and cokernels. These, however, only constitute the finite dimensional $\pm 1$-eigenspaces of $W$ and $W'$ and are left invariant by construction.

So far, we did not address the essential locality of the half-step walk. However, the path $F_t$ is constructed in a way that respects this property, given that $F$ is essentially local. Indeed, by Lem. IV.3, the result of the flattening construction above is essentially local. Moreover, since we only continuously deformed $|A|$, every $F_t$ is essentially local. 

This does not only proof the existence of a flattening path for every half-step walk: It does so by explicitly describing the construction of the flatband half-step walk $F^\circ$ through simply replacing the chiral blocks by their polar isometries.

C. Completeness

Before we address completeness for the classification of $F$, we state the following theorem which will be important in the proof. It states that for essentially local unitaries the right Fredholm index $\widetilde{\operatorname{ind}}[\cdot]$ is complete, and thereby extends the analogous result for strictly local walks which was proved in [15]. The index $\widetilde{\operatorname{ind}}[\cdot]$ was generalized to essentially local operators in [8]. However, the question whether $\widetilde{\operatorname{ind}}[\cdot]$ is complete for this larger set of essentially local operators was not considered.

Theorem VII.3. Let $U$ and $V$ be essentially local unitaries on the one-dimensional lattice. Then $U$ and $V$ are homotopic along an essentially local path if and only if $\widetilde{\operatorname{ind}}[U] = \widetilde{\operatorname{ind}}[V]$.

Proof. Let $\widetilde{\operatorname{ind}}[U] = n$. Then $U = S^{-n}(S^n U)$ and $\widetilde{\operatorname{ind}}[S^n U] = 0$, where $S^n$ denotes the shift by $n$ cites. By Prop. IV.1, $S^n U$ is homotopic to $\mathbb{I}$ and hence, $U$ is homotopic to $S^{-n}$. This is true for both, $U$ and $V$, and hence we can continuously connect them, via $S^{-n}$.

The multiplication with shift operators is a standard technique which we also use to alter the Fredholm indices of the chiral blocks of half-step walks in the proof of completeness below. In slight abuse of notation we write $S$ also for the conditional shift on $\mathcal{H}$ of the form (4) which shifts a one-dimensional subspace of each $\mathcal{H}_x$ to the right and leaves the complement invariant.

Having established completeness for the right Fredholm index $\widetilde{\operatorname{ind}}[\cdot]$ for essentially local unitary operators, allows us to approach the proof of completeness of the independent set of indices $\{\operatorname{ind}[A], \operatorname{ind}[B], \widetilde{\operatorname{ind}}[A], \widetilde{\operatorname{ind}}[B], \widetilde{\operatorname{ind}}[C]\}$ of half-step walks. We start with showing that all half-step walks $F$ with trivial indices are connected to the reference operator

$$F_0 = \frac{1}{\sqrt{2}} \begin{pmatrix} \mathbb{I} & \mathbb{I} \\ \mathbb{I} & \mathbb{I} - \mathbb{I} \end{pmatrix}.$$ \hspace{1cm} (96)

Lemma VII.4. Let $F$ be a half-step walk with trivial indices, i.e. $\{\operatorname{ind}[A], \operatorname{ind}[B], \widetilde{\operatorname{ind}}[A], \widetilde{\operatorname{ind}}[B], \widetilde{\operatorname{ind}}[C]\} = 0 \in \mathbb{Z}^5$. Then $F$ is homotopic to $F_0$ in (96).

Proof. Consider $F$ in the standard form (92). We begin with modifying the first direct summand in (92) which consists of two finite-dimensional units $B^{-}$ and $C^{-}$ on the off-diagonals. In general, these unitaries are not necessarily of the same size. However, $\operatorname{ind}[A] = 0$ implies $\dim \ker A = \dim \ker A^* = \dim \ker D$, where the last equality follows from (17). Hence, $B^{-}$ and $C^{-}$ act on equivalent finite dimensional Hilbert spaces.

$$\begin{pmatrix} 0 & B^{-} \\ C^{-} & 0 \end{pmatrix} = \begin{pmatrix} B^{-} & 0 \\ 0 & C^{-} \end{pmatrix} \begin{pmatrix} 0 & \mathbb{I} \\ \mathbb{I} & 0 \end{pmatrix} \sim \begin{pmatrix} B^{-} & 0 \\ 0 & C^{-} \end{pmatrix} \begin{pmatrix} \mathbb{I} & \mathbb{I} \\ \mathbb{I} & -\mathbb{I} \end{pmatrix} / \sqrt{2} = \begin{pmatrix} B^{-} & B^{-} \\ C^{-} & -C^{-} \end{pmatrix} / \sqrt{2},$$ \hspace{1cm} (97)

where “$\sim$” indicates a homotopy. Using $\operatorname{ind}[B] = 0$ we treat the second summand analogously. The chiral blocks in the third summand in (92) are invertible operators on an infinite dimensional Hilbert space which
can be made unitary up to a factor of $1/\sqrt{2}$ via the flattening procedure in Lem. VII.2. Then, we are left with a half-step walk of the form (92), where each summand consists of 4 blocks which are unitary up to a factor of $1/\sqrt{2}$. Undoing the rearrangement in (91) gives a half-step walk $F$, whose chiral blocks consist of $1/\sqrt{2}$ times a unitary, which we again denote by $A, B, C$ and $D$.

Since the modifications so far affected $F$ only via a finite rank perturbations and homotopies, its indices are unchanged. We can therefore deform $A, B$ and $C$ to $\mathbf{1}/\sqrt{2}$ according to Thm. VII.3. $D$ is automatically taken care of, by keeping $D = -2CA^*B$ along the path, which guarantees unitarity. □

Having constructed the homotopy to $F_0$ for every $F$ with trivial indices $0 \in \mathbb{Z}^5$, allows us to assemble the proof of the completeness result for the classification of half-step walks $F$ in terms of the indices \{\text{ind}[A], \text{ind}[B], \text{ind}[A], \text{ind}[B], \text{ind}[C]\}:

**Proof of Thm. VII.4.** We divide the proof into two steps: First we assume both $F_i$ to correspond to gapped walks $W_i$ and $W'_i$ for $i = 1, 2$, respectively. Then, by Lem. III.1, $A_i, B_i, C_i$ and $D_i$ are invertible with trivial \text{ind}[\cdot] indices. Similarly to the proof of Thm. VII.3, we can multiply the $F_i$ with appropriate shift combinations from the left and the right such that the \text{ind}[\cdot] indices of the resulting $\tilde{F}_i$ vanish. Explicitly, let \{a, b, c\} = \{\text{ind}[A], \text{ind}[B], \text{ind}[C]\} and consider

$$
\tilde{F}_i = \begin{pmatrix} S^a & 0 \\ 0 & S^c \end{pmatrix} \begin{pmatrix} A_i & B_i \\ C_i & D_i \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & S^{-b-a} \end{pmatrix} .
$$

(98)

By \text{ind}[S^{\text{ind}[\cdot]}X] = 0 we get \text{ind}[\tilde{A}_i] = \text{ind}[\tilde{B}_i] = \text{ind}[\tilde{C}_i] = 0 for $i = 1, 2$. Hence, by Lem. VII.4, $\tilde{F}_i, i = 1, 2$ are homotopic to $F_0$. Undoing the manipulation (98) after deforming to $F_0$ by multiplying with the respective inverses we constructed homotopies of $F_1$ and $F_2$ to the same operator

$$
F = \begin{pmatrix} S^{-a} & S^{-b} \\ S^{-c} & -S^{a-b-c} \end{pmatrix} /\sqrt{2}.
$$

(99)

In the general problem where $A_i, B_i, C_i$ and $D_i$ are merely Fredholm we have to take into account the first and the second summands of $F_1$ and $F_2$ in the standard form (92). To the third summands of $F_1$ and $F_2$ we can apply the modification (98) and thereby assume the summands of the resulting $\tilde{F}_1$ and $\tilde{F}_2$ to be equal to $F_0$. If we choose the bases for the finite dimensional kernels and co-kernels of $A_i, B_i, C_i$ and $D_i$ appropriately, the $\tilde{F}_i$ then take the form

$$
\tilde{F}_i = \begin{pmatrix} 0 & \mathbf{1}_{d(D_i)} \\ \mathbf{1}_{d(A_i)} & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & \mathbf{1}_{d(C_i)} \\ \mathbf{1}_{d(B_i)} & 0 \end{pmatrix} \oplus 1/\sqrt{2} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} ,
$$

(100)

where $d(X) = \dim \ker X$. Since \text{ind}[A_i] = d(C_i) - d(B_i) and \text{ind}[A_1] = \text{ind}[A_2], d(C_i)$ and $d(B_i)$ differ by the same amount for both $i = 1, 2$. Without loss of generality, let $d(C_2) - d(C_1) = d(B_2) - d(B_1) = n > 0$. Then we can “extract” a finite dimensional block

$$
\frac{1}{\sqrt{2}} \begin{pmatrix} \mathbf{1}_n & \mathbf{1}_n \\ \mathbf{1}_n & -\mathbf{1}_n \end{pmatrix}
$$

(101)

from the right summand of $\tilde{F}_1$, continuously deform it to the identity and associate it with the middle summand. This enlarges the dimensions of $\ker C_1$ and $\ker B_1$ by $n$ such that $d(C_1) = d(C_2)$ and $d(B_1) = d(B_2)$. Since we only changed $\tilde{F}_1$ on a finite dimensional subspace, this perturbation is essentially local. After rearranging the matrix blocks appropriately, the two middle $2 \times 2$ matrix blocks of $\tilde{F}_1$ and $\tilde{F}_2$ coincide. The same procedure applies to the left summand in the standard form and leads to $d(A_1) = d(A_2)$ and $d(D_1) = d(D_2)$ by deforming a finite dimensional block from the right infinite dimensional summand to $\sigma_n \otimes \mathbf{1}_m$ with the appropriately chosen $m$.

This construction continuously connects $\tilde{F}_1$ and $\tilde{F}_2$ in the set of essentially local half-step walks, and by undoing the left and right multiplication in (98) on the right summand we indeed obtain $F_1 \sim F_2$. □
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