STEKLOV EIGENVALUE PROBLEM ON SUBGRAPHS OF INTEGER LATTICES

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Abstract. We study the eigenvalues of the Dirichlet-to-Neumann operator on a finite subgraph of the integer lattice \( \mathbb{Z}^n \). We estimate the first \( n + 1 \) eigenvalues using the number of vertices of the subgraph. As a corollary, we prove that the first non-trivial eigenvalue of the Dirichlet-to-Neumann operator tends to zero as the number of vertices of the subgraph tends to infinity.

1. Introduction

Given a bounded smooth domain \( \Omega \) in \( \mathbb{R}^n \), we consider the Steklov (eigenvalue) problem on \( \Omega \). For a smooth function \( \phi \) on \( \partial \Omega \), we denote by \( u_\phi \) the harmonic extension of \( \phi \) into \( \Omega \) which satisfies

\[
\begin{align*}
\Delta u_\phi(x) &= 0, \quad x \in \Omega, \\
u_\phi|_{\partial \Omega} &= \phi.
\end{align*}
\]

The Steklov problem on \( \Omega \) is the following eigenvalue problem,

\[
\Lambda \phi := \frac{\partial u_\phi}{\partial n} = \lambda \phi \quad \text{on} \quad \partial \Omega,
\]

where \( \n \) denotes the unit outward normal of \( \partial \Omega \). The operator \( \Lambda \) is called the Dirichlet-to-Neumann (DtN in short) operator, which is also known as the voltage-current map in physics, see e.g. Calderón’s problem \([\text{Cal06}]\). As a pseudo-differential operator, the eigenvalues of DtN operator \( \Lambda \), called Steklov eigenvalues, are discrete and can be ordered as

\[0 = \sigma_1(\Omega) \leq \sigma_2(\Omega) \leq \cdots \leq \sigma_k(\Omega) \leq \cdots \uparrow \infty.\]

The Steklov problem has been intensively studied for domains in Euclidean spaces and Riemannian manifolds in the literature, see e.g. \([\text{Wei54}, \text{Esc97}, \text{Esc99}, \text{Bro01}, \text{HP2008}, \text{WX09}, \text{SU10}, \text{CSG11}, \text{Has11}, \text{GP12}, \text{FL14}, \text{KKP14}, \text{GLS16}, \text{FS16}, \text{Mat17}, \text{FW17}, \text{YY17}]\). For \( n = 2 \), Weinstock \([\text{Wei54}]\) proved that for any planar simply-connected domain \( \Omega \) with analytic boundary,

\[
\sigma_2(\Omega) \leq \frac{2\pi}{\text{area}(\Omega)},
\]

where \( \text{area}(\Omega) \) denotes the area of \( \Omega \). This was improved by Hersch and Payne \([\text{HP68}]\) to the following

\[
\frac{1}{\sigma_2(\Omega)} + \frac{1}{\sigma_3(\Omega)} \geq \frac{\pi}{\text{area}(\Omega)}.
\]

The higher dimensional generalization was proved by Brock \([\text{Bro01}]\).

Theorem 1.1 (Theorem 3 in \([\text{Bro01}]\)). Let \( \Omega \) be a smooth domain in \( \mathbb{R}^n \). Then

\[
\sum_{i=2}^{n+1} \frac{1}{\sigma_i(\Omega)} \geq C(n)(\text{vol}(\Omega))^{\frac{1}{n}},
\]

where \( \text{vol}(\Omega) \) denotes the volume of \( \Omega \), and \( C(n) = n\omega_n^\frac{1}{n} \) (here \( \omega_n \) is the volume of the unit ball in \( \mathbb{R}^n \)).

In this paper, we study the Steklov problem on graphs and prove a discrete analog for Brock’s result. The DtN operator on a subgraph of a graph was introduced by \([\text{HHW17}, \text{HM17}]\) independently, see e.g. \([\text{HHW18}, \text{Per18}]\) for more results. A graph \( G = (V, E) \) consists of the set of vertices \( V \) and the set of edges \( E \). In this paper, we only consider simple, undirected graphs. Two vertices \( x, y \) are called neighbours,
denoted by \( x \sim y \), if there is an edge \( e \) connecting \( x \) and \( y \), i.e. \( e = \{x, y\} \in E \). Integer lattice graphs are of particular interest which serve as the discrete counterparts of \( \mathbb{R}^n \). We denote by \( \mathbb{Z}^n \) the set of integer \( n \)-tuples \( \mathbb{R}^n \). The \( n \)-dimensional integer lattice graph, still denoted by \( \mathbb{Z}^n \), is the graph consisting of the set of vertices \( V = \mathbb{Z}^n \) and the set of edges \( E = \{ \{x, y\} : x, y \in \mathbb{Z}^n, \sum_{i=1}^{n} |x_i - y_i| = 1 \} \).

Let \( \Omega \) be a finite subset of \( \mathbb{Z}^n \). We denote by
\[
\delta \Omega := \{ x \in \mathbb{Z}^n \setminus \Omega : \exists y \in \Omega, \text{s.t. } x \sim y \}
\]
the vertex boundary of \( \Omega \). Analogous to the continuous setting, one can define the DtN operator on \( \Omega \),
\[
\Lambda : \mathbb{R}^{\delta \Omega} \to \mathbb{R}^{\delta \Omega},
\]
where \( \mathbb{R}^{\delta \Omega} \) denotes the set of functions on \( \delta \Omega \), see Section 2 for details. In this paper, we denote by \(| \cdot |\) the cardinality of a set. Note that \( \Lambda \) can be written as a symmetric matrix whose eigenvalues are ordered as
\[
0 = \lambda_1(\Omega) \leq \lambda_2(\Omega) \leq \cdots \leq \lambda_N(\Omega), \quad \text{where } N = |\delta \Omega|.
\]

The following is our main result, which is a discrete analog of [Bro01, Theorem 3].

**Theorem 1.2.** Let \( \Omega \) be a finite subset of \( \mathbb{Z}^n \). Then
\[
\sum_{i=2}^{n+1} \frac{1}{\lambda_i(\Omega)} \geq C_1 |\Omega|^\frac{1}{n} - \frac{C_2}{|\Omega|^\frac{1}{n}},
\]
where \( \lambda_i(\Omega) \) are Steklov eigenvalues on \( \Omega \), \( C_1 = (64n^3\omega_n^\frac{1}{n})^{-1} \), and \( C_2 = \frac{1}{32n} \).

**Remark 1.3.** By this theorem, for \(| \Omega | \geq \left( \frac{C_2}{C_1} \right)^{n} = (4n^2\omega_n^\frac{1}{n})^{\frac{n}{n+1}} \),
\[
\sum_{i=2}^{n+1} \frac{1}{\lambda_i(\Omega)} \geq \frac{1}{2} C_1 |\Omega|^\frac{1}{n}.
\]

The main ingredient of the proof of Brock’s result is a weighted isoperimetric inequality, see Lemma 3.5 below, which depends on the rotational symmetry of the Euclidean spaces. As is well-known in the discrete theory, the symmetrization approaches don’t work on \( \mathbb{Z}^n \) due to the lack of rotational symmetry. In this paper, we follow Brock’s idea to bound the Steklov eigenvalues by the geometric quantities of the subset in the lattice \( \mathbb{Z}^n \), and then estimate these discrete quantities by their counterparts in the Euclidean space \( \mathbb{R}^n \), for which we can apply the symmetrization approach. Through this process we obtain the quantitative estimate of discrete Steklov eigenvalues, but lose the sharpness of the constants \( C_1 \) and \( C_2 \) in our result.

As a corollary, we obtain the estimate for the first non-trivial eigenvalue of the DtN operator.

**Corollary 1.4.** Let \( \Omega \) be a finite subset of \( \mathbb{Z}^n \). Then
\[
\lambda_2(\Omega) \leq \frac{n}{C_1 |\Omega|^\frac{1}{n} - C_2 |\Omega|^\frac{1}{n}},
\]
where \( C_1 \) and \( C_2 \) are the constants in Theorem 1.2.

This yields an interesting consequence that for any sequence of finite subsets in \( \mathbb{Z}^n \), \( \{\Omega_i\}_{i=1}^\infty \), satisfying \(|\Omega_i| \to \infty\),
\[\lambda_2(\Omega_i) \to 0, \quad i \to \infty.\]

The paper is organized as follows: In the next section, we recall some facts on DtN operators on graphs. In Section 3 we prove Theorem 1.2.
2. Preliminaries

Let \( G = (V, E) \) be a simple, undirected graph. For two subsets \( A, B \) of \( V \), we define
\[
E(A, B) := \{ \{x, y\} \in E : x \in A, y \in B, \text{ or vice versa} \},
\]
which is a set of edges connecting a vertex in \( A \) to another vertex in \( B \). For a subset \( \Omega \) of \( V \), we write \( \Omega^c := V \setminus \Omega \). The edge boundary of \( \Omega \) is defined as \( \partial \Omega := E(\Omega, \Omega^c) \), and we set \( \bar{\Omega} := \Omega \cup \delta \Omega \).

For a finite subset \( S \) of \( V \), we denote by \( R^S \) the set of functions on \( S \), by \( \ell^2(S) \) the Hilbert space of functions on \( S \) equipped with the inner product
\[
\langle \varphi, \psi \rangle_S = \sum_{x \in S} \varphi(x)\psi(x), \quad \varphi, \psi \in R^S.
\]

In the following, we will take \( S = \Omega, \delta \Omega, \bar{\Omega} \), respectively. For \( u \in R^{\bar{\Omega}} \), the Dirichlet energy of \( u \) is defined as
\[
D_\Omega(u) := \sum_{\{x, y\} \in E(\Omega, \bar{\Omega})} (u(x) - u(y))^2.
\]
The polarization of the Dirichlet energy is given by
\[
D_\Omega(u, v) := \sum_{\{x, y\} \in E(\Omega, \bar{\Omega})} (u(x) - u(y))(v(x) - v(y)), \quad \forall u, v \in R^{\bar{\Omega}}.
\]

The Laplace operator on \( \Omega \) is defined as
\[
\Delta : R^{\bar{\Omega}} \rightarrow R^\Omega
\]
\[
\Delta u(x) := \sum_{y \in V, y \sim x} (u(y) - u(x)).
\]
The exterior normal derivative on vertex boundary set \( \delta \Omega \) is defined as
\[
\frac{\partial}{\partial n} : R^{\delta \Omega} \rightarrow R^{\delta \Omega}
\]
\[
n(x) := \sum_{y \in \Omega, y \sim x} (u(x) - u(y)), \quad x \in \delta \Omega.
\]
The Dirichlet-to-Neumann (DtN in short) operator is defined as
\[
\Lambda : R^{\delta \Omega} \rightarrow R^{\delta \Omega}
\]
\[
\varphi \mapsto \Lambda \varphi := \frac{\partial u_\varphi}{\partial n},
\]
where \( u_\varphi \) is the harmonic function on \( \Omega \) whose Dirichlet boundary condition on \( \delta \Omega \) is given by \( \varphi \). We recall the well-known Green’s formula.

**Theorem 2.1** (Lemma 2.1 in [HHW17]). For \( \varphi \in R^{\delta \Omega} \),
\[
D_\Omega(u_\varphi) = -\langle \Delta u_\varphi, u_\varphi \rangle_\Omega + \langle \frac{\partial u_\varphi}{\partial n}, u_\varphi \rangle_{\delta \Omega}.
\]

**Proposition 2.1.** The multiplicity of trivial eigenvalue 0 of the DtN operator \( \Lambda \) is equal to the number of connected components of the graph \((\bar{\Omega}, E(\Omega, \bar{\Omega}))\).

**Proof.** Assume that \( \varphi \in R^{\delta \Omega} \) is an eigenfunction corresponding to the eigenvalue 0 of the DtN operator \( \Lambda \), then by Green’s formula [24],
\[
D_\Omega(u_\varphi) = -\langle \Delta u_\varphi, u_\varphi \rangle_\Omega + \langle \frac{\partial u_\varphi}{\partial n}, u_\varphi \rangle_{\delta \Omega} = 0.
\]
Thus \( u_\varphi \) is constant on every connected component of \((\bar{\Omega}, E(\Omega, \bar{\Omega}))\). This yields the result. \(\square\)
Let \( \lambda \) be an eigenvalue of the DtN operator \( \Lambda \) satisfying
\[
\frac{\partial u_\varphi}{\partial n} = \lambda u_\varphi \quad \text{on} \quad \partial \Omega.
\]
Then by Green formula (2.1) we get
\[
D_\Omega(u_\varphi) = \lambda(u_\varphi, u_\varphi)_{\delta \Omega} = \lambda \sum_{z \in \delta \Omega} \varphi^2(z).
\]
The following variational principle is useful, see e.g. [Ban80, pp.99].

**Theorem 2.2.** Let \( \Omega \) be a finite subset of \( \mathbb{Z}^n \). Then
\[
\sum_{i=2}^{n+1} \frac{1}{\lambda_i(\Omega)} = \max \left\{ \sum_{i=2}^{n+1} \sum_{z \in \delta \Omega} v_i^2(z) : v_i \in \mathbb{R}^{\Omega}, D_\Omega(v_i, v_j) = \delta_{ij}, \sum_{z \in \delta \Omega} v_i(z) = 0, \ 2 \leq i, j \leq n + 1 \right\},
\]
where \( \delta_{ij} = \begin{cases} 1, & i = j, \\ 0, & i \neq j. \end{cases} \)

3. The proof of Theorem 1.2

In this section we prove the main theorem. Let \( \mathbb{Z}^n \) be the \( n \)-dimensional integer lattice graph, and
\[
e_i := (0, \cdots, 0, 1, 0, \cdots, 0), \quad 1 \leq i \leq n,
\]
be the standard basis of \( \mathbb{R}^n \). We denote by
\[
Q = \{ x = (x_1, x_2, \cdots, x_n) \in \mathbb{R}^n : |x_i| \leq \frac{1}{2}, \ \forall 1 \leq i \leq n \}
\]
the unit cube centered at the origin in \( \mathbb{R}^n \), and by
\[
Q_i := Q \cap \{ x \in \mathbb{R}^n : x_i = 0 \}, \quad 1 \leq i \leq n,
\]
the \((n-1)\)-dimensional unit cube in \( x_i \)-hyperplane. For any edge \( \tau = \{x, y\} \) in \( \mathbb{Z}^n \), there is a unique \( e_i \) such that \( y - x = e_i \) or \(-e_i\), and we define
\[
\tau^\perp := \frac{1}{2}(x + y) + Q_i.
\]
That is, \( \tau^\perp \) is an \((n-1)\)-dimensional unit cube centered at the point \( \frac{1}{2}(x + y) \) parallel to \( x_i \)-hyperplane. Note that such a cube \( \tau^\perp \) is one-to-one corresponding to \( \tau \).

For any finite subset \( \Omega \subset \mathbb{Z}^n \), we denote
\[
\partial \Omega^\perp := \{ \tau^\perp : \tau \in \partial \Omega \}.
\]
Note that for any \( \tau \in \partial \Omega \), there is a unique end-vertex of the edge \( \tau \) belonging to \( \delta \Omega \). We define a mapping
\[
P : \partial \Omega^\perp \to \mathbb{R}^{\delta \Omega}
\]
\[
\tau^\perp \mapsto P(\tau^\perp),
\]
where \( P(\tau^\perp) \) is the end-vertex of \( \tau \) in \( \delta \Omega \).

**Lemma 3.1.** For any \( \tau_1, \tau_2 \in \partial \Omega \) satisfying \( P(\tau_1^\perp) \neq P(\tau_2^\perp) \), we have
\[
\int_{\tau_1^\perp}^{\tau_2^\perp} |s - t|^2 d\mathcal{H}^{n-1}(s) d\mathcal{H}^{n-1}(t) \leq C_3 \cdot |P(\tau_1^\perp) - P(\tau_2^\perp)|^2,
\]
where \( C_3 = 4n \), and \( \mathcal{H}^{n-1} \) denotes the \((n-1)\)-dimensional Hausdorff measure in \( \mathbb{R}^n \).
For our purposes, we will use the geometry of $\hat{\Omega}$, which is a domain in $\mathbb{C}_r$. By the definition $\tau_1^\perp = \frac{1}{2}(x_1 + y_1) + Q_{e_1}$. By the symmetry, without loss of generality we may assume that $P(\tau_1^\perp) = x_1$. For any $s \in \tau_1^\perp$, we write $s = \frac{1}{2}(x_1 + y_1) + q_i$, where $q_i \in Q_{e_1}$. Then
\[ |s - P(\tau_1^\perp)| = |\frac{1}{2}(x_1 + y_1) + q_i - x_1| = \frac{|e_1|}{2} + q_i \leq \frac{\sqrt{n}}{2}. \]

Similarly we get for any $t \in \tau_2^\perp$
\[ |t - P(\tau_2^\perp)| \leq \frac{\sqrt{n}}{2}. \]

Since $|P(\tau_1^\perp) - P(\tau_2^\perp)| \geq 1$, by the triangle inequality, for any $s \in \tau_1^\perp, t \in \tau_2^\perp$ we have
\[ |s - t| \leq |s - P(\tau_1^\perp)| + |P(\tau_1^\perp) - P(\tau_2^\perp)| + |P(\tau_2^\perp) - t| \]
\[ \leq (\sqrt{n} + 1)|P(\tau_1^\perp) - P(\tau_2^\perp)|. \]

Hence
\[ \int_{\tau_1^\perp} \int_{\tau_2^\perp} |s - t|^2 d\mathcal{H}^{n-1}(s)d\mathcal{H}^{n-1}(t) \leq (\sqrt{n} + 1)^2|P(\tau_1^\perp) - P(\tau_2^\perp)|^2 \]
\[ \leq 4n|P(\tau_1^\perp) - P(\tau_2^\perp)|^2 \]

This proves the lemma. $\square$

For any $r > 0$, we denote by
\[ Q_r(x) := \{ y \in \mathbb{R}^n : \max_{1 \leq i \leq n} |y_i - x_i| \leq \frac{r}{2} \} \]
the cube of side length $r$ centered at $x$. For any subset $\Omega \subset \mathbb{Z}^n$, we construct a domain $\hat{\Omega}$ in $\mathbb{R}^n$ as follows
\[ \hat{\Omega} := \bigcup_{x \in \Omega} Q_1(x). \]

Note that the topological boundary of $\hat{\Omega}$, denoted by $\partial(\hat{\Omega})$, is one-to-one corresponding to $\partial\Omega^\perp$, i.e.
\[ (3.1) \]
\[ \partial(\hat{\Omega}) = \bigcup_{\tau \in \partial\Omega} \tau^\perp. \]

For our purposes, we will use the geometry of $\hat{\Omega}$, which is a domain in $\mathbb{R}^n$, to estimate that of $\Omega$, a subset in $\mathbb{Z}^n$.

We give the definition of “bad” boundary vertices on $\partial\Omega$.

**Definition 3.1** ("Bad" boundary vertex). We call $x \in \partial\Omega$ a "bad" boundary vertex if it is isolated by the set $\Omega$, i.e. for any $y \sim x$ in $\mathbb{Z}^n$, $y \in \Omega$. The set of “bad” boundary vertices is denoted by $\delta\Omega$.

**Proposition 3.1.** For any finite subset $\Omega$ in $\mathbb{Z}^n$,
\[ |\delta\Omega| \leq |\Omega|. \]

**Proof of Proposition 3.1.** We define a mapping
\[ F : \delta\Omega \rightarrow \Omega \]
\[ x \mapsto F(x) := x + e_1. \]

It is easy to see that $F$ is injective, which yields $|\delta\Omega| = |F(\delta\Omega)| \leq |\Omega|$. This proves the proposition. $\square$

The next example shows that the above estimate is sharp.

**Example 3.2.** For $R \in \mathbb{N}$, set
\[ \Omega_R := \{(x, y) \in \mathbb{Z}^2 : |x| = R \text{ or } |y| = R\} \cup \{(x, y) \in \mathbb{Z}^2 : |x| \leq R - 1, \ |y| \leq R - 1 \text{ s.t. } x + y \text{ is odd}\}. \]

Then
\[ \lim_{R \rightarrow \infty} \frac{|\delta\Omega_R|}{|\Omega_R|} = 1. \]
For any subset $\Omega \subset \mathbb{Z}^n$, any vertex in $V \setminus \bar{\Omega}$ is called an exterior vertex of $\Omega$. For simplicity, we write $\Omega^e := V \setminus \bar{\Omega}$ for the set of exterior vertices of $\Omega$. This yields the decomposition

$$\mathbb{Z}^n = \Omega \sqcup \delta \Omega \sqcup \Omega^e,$$

where $\sqcup$ denotes the disjoint union.

**Lemma 3.2.** Let $\Omega$ be a finite subset of $\mathbb{Z}^n$. For any $x \in \delta \Omega \setminus \delta^* \Omega$,

$$|Q_2(x) \cap \delta \Omega| \geq 2.$$

**Proof.** For any $x \in \delta \Omega$, set $N(x) := \{y \in \mathbb{Z}^n : y \sim x\}$. Without loss of generality, we may assume that $N(x) \cap \delta \Omega = \emptyset$, otherwise the lemma follows trivially. By the assumption $x \notin \delta^* \Omega$,

$$N(x) \cap \Omega^c \neq \emptyset, \text{ and } N(x) \cap \Omega \neq \emptyset.$$

We may choose $y \in N(x) \cap \Omega^c$ and $z \in N(x) \cap \Omega$ such that

$$|y - z| = \sqrt{2}.$$

Set $w := y + z - x$. Then

$$w \in Q_2(x), \ y \sim w, \ z \sim w.$$

Note that any path in $\mathbb{Z}^n$ connecting a vertex in $\Omega^c$ and a vertex in $\Omega$ contains a vertex in $\delta \Omega$. Applying this to the path $y \sim w \sim z$, we get $w \in \delta \Omega$. This proves the lemma. \hfill $\Box$

For our purposes, we classify pairs of boundary edges, $\partial \Omega \times \partial \Omega$, into “good” pairs and “bad” pairs.

**Definition 3.3.** A pair of boundary edges $(\tau_1, \tau_2) \in \partial \Omega \times \partial \Omega$ is called “good” if $P(\tau_1^+) \neq P(\tau_2^+)$ or $P(\tau_1^+) = P(\tau_2^+) \notin \delta^* \Omega$. Otherwise, it is called “bad”. We denote by $(\partial \Omega)^G$ the set of “good” pairs of boundary vertices, and by $(\partial \Omega)^B$ the set of “bad” pairs of boundary vertices.

By the definition,

$$(3.2) \quad \partial \Omega \times \partial \Omega = (\partial \Omega)^G \sqcup (\partial \Omega)^B$$

and

$$(\partial \Omega)^B = \{(\tau_1, \tau_2) \in \partial \Omega \times \partial \Omega \mid P(\tau_1^+) = P(\tau_2^+) \in \delta^* \Omega\}.$$

**Definition 3.4 (The multiplicity of a mapping).** Let $T : A \to B$ be a mapping. The multiplicity of the mapping $T$ is defined as

$$m_T := \sup_{b \in B} |\{T^{-1}(b)\}|.$$

We estimate the number of “bad” pairs of boundary vertices by the number of “bad” vertices.

**Proposition 3.2.** Let $\Omega$ be a finite subset of $\mathbb{Z}^n$. Then

$$|(\partial \Omega)^B| \leq 4n^2|\delta^* \Omega|.$$

**Proof.** We define a mapping

$$H : (\partial \Omega)^B \to \delta^* \Omega$$

$$(\tau_1, \tau_2) \mapsto H(\tau_1, \tau_2) = P(\tau_1^+).$$

We estimate the multiplicity of the mapping $H$. Fix any $x \in \delta^* \Omega$. For any $(\tau_1, \tau_2) \in (\partial \Omega)^B$ such that

$$H(\tau_1, \tau_2) = x,$$

so that $x$ is a common end-vertex of $\tau_1$ and $\tau_2$. The number of possible pairs $(\tau_1, \tau_2)$ satisfying this property is at most $4n^2$. Hence $m_T \leq 4n^2$. This proves the proposition. \hfill $\Box$
We define a mapping on “good” pairs of boundary vertices as

\[ (\tau_1, \tau_2) \mapsto f(\tau_1, \tau_2) = (f_1(\tau_1, \tau_2), f_2(\tau_1, \tau_2)) = \begin{cases} (P(\tau_1^+), P(\tau_2^+)), & \text{if } P(\tau_1^+) \neq P(\tau_2^+) \\ (P(\tau_1^+), z), & \text{if } P(\tau_1^+) = P(\tau_2^+) \notin \delta'\Omega, \end{cases} \]

where \( z \in Q_2(P(\tau_1^+)) \cap \delta\Omega \) such that \( z \neq P(\tau_1^+) \).

**Remark 3.5.** In the definition of the mapping \( f \), the existence of \( z \in Q_2(P(\tau_1^+)) \cap \delta\Omega \) in the second case, which is not necessarily unique, is guaranteed by Lemma \[3.3\]. Any choice of such a vertex \( z \) will be sufficient for our purposes.

**Lemma 3.3.** Let \( \Omega \) be a finite subset of \( \mathbb{Z}^n \). Then for any \((\tau_1, \tau_2) \in (\partial \Omega)_g^2\),

\[
\int_{\tau_1^+} \int_{\tau_2^+} |s - t|^2 d\mathcal{H}^{n-1}(s)d\mathcal{H}^{n-1}(t) \leq C_3 \cdot |f_1(\tau_1, \tau_2) - f_2(\tau_1, \tau_2)|^2,
\]

where \( C_3 \) is the constant in Lemma \[3.3\].

**Proof of Lemma 3.3.** By Lemma \[3.1\], it suffices to consider the case that \( P(\tau_1^+) = P(\tau_2^+) \neq \delta'\Omega \). Set \( X := P(\tau_1^+) = P(\tau_2^+) \). Then \( \tau_1 \cap \tau_2 = X \). By the triangle inequality, for any \( s \in \tau_1^+, t \in \tau_2^+ \), we have

\[ |s - t| \leq |s - X| + |X - t| \leq \sqrt{n}. \]

Then \( \int_{\tau_1^+} \int_{\tau_2^+} |s - t|^2 d\mathcal{H}^{n-1}(s)d\mathcal{H}^{n-1}(t) \leq n \). Obviously, since \( |f_1(\tau_1, \tau_2) - f_2(\tau_1, \tau_2)| = |P(\tau_1^+) - z| \geq 1 \),

\[ \int_{\tau_1^+} \int_{\tau_2^+} |s - t|^2 d\mathcal{H}^{n-1}(s)d\mathcal{H}^{n-1}(t) \leq n \cdot |f_1(\tau_1, \tau_2) - f_2(\tau_1, \tau_2)|^2 \leq C_3 \cdot |f_1(\tau_1, \tau_2) - f_2(\tau_1, \tau_2)|^2. \]

This proves the lemma. \( \Box \)

By the definition of the multiplicity of \( f \), we have the following proposition.

**Proposition 3.3.**

\[
\sum_{(\tau_1, \tau_2) \in (\partial \Omega)_g^2} |f_1(\tau_1, \tau_2) - f_2(\tau_1, \tau_2)|^2 \leq m_f \cdot \sum_{z \in \delta \Omega} \sum_{\omega \in \delta \Omega} |z - \omega|^2.
\]

**Lemma 3.4.** Let \( \Omega \) be a finite subset of \( \mathbb{Z}^n \). Then

\[ m_f \leq 8n^2. \]

**Proof.** Fix \((x, y) \in f((\partial \Omega)_g^2)\). Let \((\tau_1, \tau_2) \in (\partial \Omega)_g^2\) such that \( f(\tau_1, \tau_2) = (x, y) \). We estimate the number of possible pairs \((\tau_1, \tau_2)\) satisfying the above property. By the definition of \( f \) in \[3.3\], \( f_1(\tau_1, \tau_2) = P(\tau_1^+) = x \), which yields that \( x \) is an end-vertex of the edge \( \tau_1 \). Hence the number of possible choices of \( \tau_1 \) is at most \( 2n \). To determine the edge \( \tau_2 \), it is divided into the following cases.

**Case 1.** \( P(\tau_1^+) \neq P(\tau_2^+) \). By \[3.3\],

\[ f_2(\tau_1, \tau_2) = P(\tau_2^+) = y. \]

Hence \( y \) is an end-vertex of the edge \( \tau_2 \). So that the number of possible choices of \( \tau_2 \) is at most \( 2n \).

**Case 2.** \( P(\tau_1^+) = P(\tau_2^+) \notin \delta'\Omega \). In this case, \( P(\tau_2^+) = x \) and \( x \) is an end-vertex of the edge \( \tau_2 \). Hence the number of possible choices of \( \tau_2 \) is at most \( 2n \).

Combining the above cases, the number of possible choices of \( \tau_2 \) is at most \( 4n \). This yields that

\[ m_f \leq 2n \times 4n = 8n^2. \]

This proves the lemma. \( \Box \)

The following lemma is a version of weighted isoperimetric inequality by symmetrization approaches, see e.g. [BBMP99], which is crucial in the proof of Brock’s result in \( \mathbb{R}^n \). For any \( R > 0 \), we denote by \( B_R \) the ball of radius \( R \) centered at the origin in \( \mathbb{R}^n \).
Lemma 3.5. Let $U$ be a bounded domain in $\mathbb{R}^n$ with Lipschitz boundary. Let $R > 0$ such that $\mathcal{H}^n(B_R) = \mathcal{H}^n(U)$. Let $g \in C([0, +\infty))$ be nonnegative, nondecreasing and suppose that

$$(g(z_{\lambda}) - g(0))z^{1-\frac{1}{n}}, \ z \geq 0$$

is convex. Then

$$\int_{\partial U} g(|x|)dS \geq \int_{\partial B_R} g(|x|)dS = n\omega_n g(R)R^{n-1}.$$ 

Now we are ready to prove our main result.

Proof of Theorem 1.2. For $1 \leq k \leq n$, we denote by $E_k$ the edges in $E(\Omega, \hat{\Omega})$ parallel to $e_k$, and by $|E_k|$ the cardinality of $E_k$. Using the coordinate functions of $\mathbb{R}^n$, we define functions $u_i \in \mathbb{R}^{2n}$ for $2 \leq i \leq n+1$ as follows,

$$u_i(z) := |E_{i-1}|^{-\frac{1}{2}}(z_{i-1} - \bar{z}_{i-1}),$$

where

$$\bar{z}_{i-1} := \frac{1}{|\delta \Omega|} \sum_{z \in \delta \Omega} z_{i-1}.$$ 

Then we get

$$D_\Omega(u_i, u_j) = \delta_{ij}, \sum_{z \in \delta \Omega} u_i(z) = 0, \ 2 \leq i, j \leq n+1.$$ 

Hence by the variational principle, Theorem 2.2,

$$\sum_{i=2}^{n+1} \frac{1}{\lambda_i(\Omega)} \geq \sum_{i=2}^{n+1} \sum_{z \in \delta \Omega} u_i^2(z) = \sum_{i=2}^{n+1} \sum_{z \in \delta \Omega} \frac{|z_{i-1} - \bar{z}_{i-1}|^2}{|E_{i-1}|} \geq \frac{1}{|E(\Omega, \Omega)|} \sum_{i=2}^{n+1} \sum_{z \in \delta \Omega} |z_{i-1} - \bar{z}_{i-1}|^2.$$

Note that

$$\sum_{z \in \delta \Omega} \sum_{\omega \in \delta \Omega} |z - \omega|^2 = \sum_{z \in \delta \Omega} \sum_{\omega \in \delta \Omega} \sum_{i=2}^{n+1} |z_{i-1} - \omega_{i-1}|^2 = \sum_{i=2}^{n+1} \left(2|\delta \Omega| \sum_{z \in \delta \Omega} z_{i-1}^2 - 2 \sum_{z \in \delta \Omega} z_{i-1}^2 \right)$$

$$= 2|\delta \Omega| \sum_{i=2}^{n+1} \left(\sum_{z \in \delta \Omega} z_{i-1}^2 - \frac{(\sum_{z \in \delta \Omega} z_{i-1})^2}{|\delta \Omega|} \right) = 2|\delta \Omega| \sum_{z \in \delta \Omega} \sum_{i=2}^{n+1} (z_{i-1}^2 - 2z_{i-1} \cdot \bar{z}_{i-1} + \bar{z}_{i-1}^2)$$

$$= 2|\delta \Omega| \sum_{z \in \delta \Omega} \sum_{i=2}^{n+1} (z_{i-1} - \bar{z}_{i-1})^2.$$ 

Hence we get

$$\sum_{i=2}^{n+1} \frac{1}{\lambda_i(\Omega)} \geq \frac{1}{2|\delta \Omega|} \cdot \frac{1}{|E(\Omega, \Omega)|} \cdot \sum_{z \in \delta \Omega} \sum_{\omega \in \delta \Omega} |z - \omega|^2.$$ 

In the following, we compare the right hand side of the above inequality, a discrete summation, with a continuous quantity,

$$\int_{\partial \Omega} \int_{\partial \Omega} |s - t|^2 d\mathcal{H}^{n-1}(s) d\mathcal{H}^{n-1}(t).$$

By 3.1 and the decomposition of pairs of boundary edges into “good” and “bad” ones, 3.2, we have

$$\int_{\partial \Omega} \int_{\partial \Omega} |s - t|^2 d\mathcal{H}^{n-1}(s) d\mathcal{H}^{n-1}(t) = \sum_{(\tau_1, \tau_2) \in \partial \Omega \times \partial \Omega} \int_{\tau_1}^t \int_{\tau_2}^t |s - t|^2 d\mathcal{H}^{n-1}(s) d\mathcal{H}^{n-1}(t)$$

$$= \int_{\partial \Omega} \int_{\partial \Omega} |s - t|^2 d\mathcal{H}^{n-1}(s) d\mathcal{H}^{n-1}(t).$$
By the equations (3.6) and (3.7),

\[
I = \sum_{(\tau_1, \tau_2) \in (\partial \Omega)^2} \int_{\tau_1^+} \int_{\tau_2^+} |s - t|^2 d\mathcal{H}^{n-1}(s) d\mathcal{H}^{n-1}(t) \leq C_3 \cdot m \sum_{x \in \partial \Omega} \sum_{\omega \in \partial \Omega} |z - \omega|^2.
\]

For any \((\tau_1, \tau_2) \in (\partial \Omega)^2\), the inequality (3.4) yields

\[
\int_{\tau_1^+} \int_{\tau_2^+} |s - t|^2 d\mathcal{H}^{n-1}(s) d\mathcal{H}^{n-1}(t) \leq n.
\]

Combining this with Proposition 3.2, we get

\[
II \leq 4n^3 |\delta' \Omega|.
\]

Hence, we obtain

\[
(3.6) \quad \int_{\partial (\Omega)} \int_{\partial (\Omega)} |s - t|^2 d\mathcal{H}^{n-1}(s) d\mathcal{H}^{n-1}(t) = I + II \leq C_3 \cdot m \sum_{x \in \partial \Omega} \sum_{\omega \in \partial \Omega} |z - \omega|^2 + 4n^3 |\delta' \Omega|.
\]

By the calculation,

\[
\int_{\partial (\Omega)} \int_{\partial (\Omega)} |s - t|^2 d\mathcal{H}^{n-1}(s) d\mathcal{H}^{n-1}(t) = 2\mathcal{H}^{n-1}(\partial (\Omega)) \int_{\partial (\Omega)} |s - c|^2 d\mathcal{H}^{n-1}(s)
\]

where \(c = \frac{1}{\mathcal{H}^{n-1}(\partial (\Omega))} \int_{\partial (\Omega)} s d\mathcal{H}^{n-1}(s)\) is the barycenter of \(\partial (\Omega)\). We denote by \(\Omega_0 := \hat{\Omega} - c\) the translation of the domain \(\hat{\Omega}\) by a vector \(-c\). It is easy to see that the barycenter of the boundary of \(\Omega_0\) is the origin, so that

\[
\int_{\partial (\Omega)} |s - c|^2 d\mathcal{H}^{n-1}(s) = \int_{\delta \Omega_0} |s|^2 d\mathcal{H}^{n-1}(s).
\]

Since \(\mathcal{H}^n(\Omega_0) = \mathcal{H}^n(\hat{\Omega}) = |\Omega|\), we choose \(R > 0\), such that

\[
\mathcal{H}^n(B_R) = \omega_n R^n = |\Omega| = \mathcal{H}^n(\Omega_0).
\]

By Lemma 3.5 taking \(U = \Omega_0\) and \(\eta(t) = t^2\), we get

\[
\int_{\partial \Omega_0} |s|^2 d\mathcal{H}^{n-1}(s) \geq \int_{\partial B_R} |s|^2 d\mathcal{H}^{n-1}(s) = n \omega_n R^{n+1} = n |\Omega| R.
\]

By the above estimates, noting that \(\mathcal{H}^{n-1}(\partial (\Omega)) = |\partial \Omega|\), we obtain

\[
(3.7) \quad \int_{\partial (\Omega)} \int_{\partial (\Omega)} |s - t|^2 d\mathcal{H}^{n-1}(s) d\mathcal{H}^{n-1}(t) \geq 2n |\partial \Omega| \cdot |\Omega| R.
\]

By the equations (3.6) and (3.7),

\[
\sum_{x \in \partial \Omega} \sum_{\omega \in \partial \Omega} |z - \omega|^2 \geq \frac{2n (|\partial \Omega| \cdot |\Omega| R - 2n^2 |\delta' \Omega|)}{C_3 \cdot m}.
\]

Note that

\[
|\delta' \Omega| \leq |\delta \Omega| \leq |\partial \Omega| \leq 2n |\Omega|,
\]

and

\[
2 |E(\Omega, \hat{\Omega})| = 2n \cdot |\Omega| + |\partial \Omega|.
\]

This yields that

\[
|E(\Omega, \hat{\Omega})| = n |\Omega| + \frac{1}{2} |\partial \Omega| \leq 2n |\Omega|.
\]
By (3.5) and (3.8),
\[ \sum_{i=2}^{n+1} \frac{1}{\lambda_i(\Omega)} \geq \frac{2n(|\partial \Omega| \cdot |\Omega| - 2n^2|\delta \Omega|)}{2|\partial \Omega| \cdot |E(\Omega, \Omega)| C_3 \cdot m_f} \geq \frac{n(|\partial \Omega| \cdot |\Omega| - 2n^2|\delta \Omega|)}{|\partial \Omega| \cdot 2n|\Omega| C_3 \cdot m_f} \]
\[ \geq \frac{1}{2C_3 \cdot m_f} \left( R - \frac{2n^2}{|\Omega|} \right) = \frac{1}{2C_3 \cdot m_f} \left( (\omega_n^{-1} |\Omega|)^{\frac{1}{n}} - \frac{2n^2}{|\Omega|} \right). \]

By Lemma 3.4, we get
\[ \sum_{i=2}^{n+1} \frac{1}{\lambda_i(\Omega)} \geq C_1 |\Omega|^{\frac{1}{n}} - C_2 |\Omega|^\frac{1}{n}, \]
where \( C_1 = (64n^3 \omega_n^{-1})^{-1}, C_2 = \frac{1}{2n^2}. \) This proves the theorem. \( \square \)

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