Chain Decompositions of $q, t$-Catalan Numbers: Tail Extensions and Flagpole Partitions

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Abstract. This article is part of an ongoing investigation of the combinatorics of $q, t$-Catalan numbers $Cat_n(q,t)$. We develop a structure theory for integer partitions based on the partition statistics $\text{dinv}$, deficit, and minimum triangle height. Our goal is to decompose the infinite set of partitions of deficit $k$ into a disjoint union of chains $C_\mu$ indexed by partitions of size $k$. Among other structural properties, these chains can be paired to give refinements of the famous symmetry property $Cat_n(q,t) = Cat_n(t,q)$. Previously, we introduced a map that builds the tail part of each chain $C_\mu$. Our first main contribution here is to extend this map to construct larger second-order tails for each chain. Second, we introduce new classes of partitions called flagpole partitions and generalized flagpole partitions. Third, we describe a recursive construction for building the chain $C_\mu$ for a (generalized) flagpole partition $\mu$, assuming that the chains indexed by certain specific smaller partitions (depending on $\mu$) are already known. We also give some enumerative and asymptotic results for flagpole partitions and their generalized versions.

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1. Introduction

This article is the third in a series of papers developing the combinatorics of the $q, t$-Catalan numbers $Cat_n(q,t)$. We refer readers to Haglund’s monograph...
for background and references on $q,t$-Catalan numbers. Our motivating problem [1] is to find a purely combinatorial proof of the symmetry property $\text{Cat}_n(q,t) = \text{Cat}_n(t,q)$. It turns out that this symmetry is just one facet of an elaborate new structure theory for integer partitions. Each partition $\gamma$ has a size $|\gamma|$, a diagonal inversion count $\text{dinv}(\gamma)$, a deficit $\text{defc}(\gamma)$, and a minimum triangle height $\text{min}_\Delta(\gamma)$. Here, $\text{min}_\Delta(\gamma)$ is the smallest $n$, such that the Ferrers diagram of $\gamma$ is contained in the diagram of $\Delta_n = \langle n-1, n-2, \ldots, 3, 2, 1, 0 \rangle$, $\text{dinv}(\gamma)$ counts certain boxes in the diagram of $\gamma$, and $\text{defc}(\gamma)$ counts the remaining boxes (see Sect. 2.2 for details). The $q,t$-Catalan numbers can be defined combinatorially as

$$\text{Cat}_n(q,t) = \sum_{\gamma : \text{min}_\Delta(\gamma) \leq n} q^{\binom{n}{2}} - |\gamma| t^{\text{dinv}(\gamma)},$$  \hspace{1cm} (1.1)$$

To explain the term “deficit,” note that $|\gamma| = \text{dinv}(\gamma) + \text{defc}(\gamma)$ holds by definition. Therefore, the monomials in (1.1) indexed by partitions $\gamma$ with a given deficit $k$ are precisely the monomials of degree $\binom{n}{2} - k$ in $\text{Cat}_n(q,t)$. To prove symmetry of the full polynomial $\text{Cat}_n(q,t)$, it suffices to prove the symmetry of each homogeneous component of degree $\binom{n}{2} - k$. Fixing $k$ and letting $n$ vary, we are led to study the collection $\text{Def}(k)$ of all integer partitions $\gamma$ with $\text{defc}(\gamma) = k$. Informally, the structural complexity of these collections (stratified by the $\text{dinv}$ statistic) grows exponentially with $k$; see Theorem 2.20 for a more precise version of this remark.

### 1.1. Global Chains

The first paper in our series [6] introduced the idea of global chain decompositions for the collections $\text{Def}(k)$. One might observe that each term of degree $\binom{n}{2} - k$ in $\text{Cat}_n(q,t)$ has coefficient at most $p(k)$, the number of integer partitions of $k$ (cf. Theorem 1.3 of [7] and Theorem 2.20 below). This suggests the possibility of decomposing each set $\text{Def}(k)$ into a disjoint union of global chains $\mathcal{C}_\mu$ indexed by partitions $\mu$ with $|\mu| = k$, where each $\mathcal{C}_\mu$ should be an infinite sequence of partitions having constant deficit $k$ and consecutive $\text{dinv}$ values. In other words, each conjectural global chain should have the form $\mathcal{C}_\mu = (c_\mu(i) : i \geq i_0)$, where $\text{defc}(c_\mu(i)) = k = |\mu|$ and $\text{dinv}(c_\mu(i)) = i$ for all integers $i$ starting at some value $i_0$ that depends on $\mu$. The sequence of $\text{min}_\Delta$-values $(\text{min}_\Delta(c_\mu(i)) : i \geq i_0)$, which we call the $\text{min}_\Delta$-profile of the chain $\mathcal{C}_\mu$, has intricate combinatorial structure that is crucial to understanding the symmetry of $q,t$-Catalan numbers.

More specifically, we conjectured in [6] that the chains $\mathcal{C}_\mu$ (satisfying the above conditions) could be chosen to satisfy the following opposite property. Given a proposed chain $\mathcal{C}_\mu$, define

$$\text{Cat}_{n,\mu}(q,t) = \sum_{\gamma \in \mathcal{C}_\mu : \text{min}_\Delta(\gamma) \leq n} q^{\binom{n}{2}} - |\gamma| t^{\text{dinv}(\gamma)},$$  \hspace{1cm} (1.2)$$

which is the sum of only those terms in (1.1) arising from partitions $\gamma$ in the chain $\mathcal{C}_\mu$. We conjecture that there is a size-preserving involution $\mu \mapsto \mu^*$ (defined on the set of all integer partitions), such that for all integers $n > 0$, 

...
Cat_{n,\mu^*}(q, t) = Cat_{n,\mu}(t, q). Each pair \{\mu, \mu^*\} yields new small slices of the full \(q, t\)-Catalan polynomials (namely \(Cat_{n,\mu}(q, t) + Cat_{n,\mu^*}(q, t)\) if \(\mu \neq \mu^*\), or \(Cat_{n,\mu}(q, t)\) if \(\mu = \mu^*\) that are symmetric in \(q\) and \(t\).

**Example 1.1.** We constructed the global chains for \(\mu = \langle 6, 1 \rangle\) and \(\mu^* = \langle 3, 3, 1 \rangle\) in [3, Appendix 4.3]. All partitions in these chains have deficit \(k = 7\). The min\(_\Delta\)-profiles for \(C_\mu\) and \(C_{\mu^*}\) are shown here

\[
\begin{align*}
\text{min}_\Delta: & \quad 3456789101112131415161718 \ldots 252627 \ldots \\
\text{dinv:} & \quad 7877877877888899 \ldots 9109 \ldots \\
\text{min}_\Delta: & \quad 910778778878888888888888888 \\
\end{align*}
\]

In both cases, all values of min\(_\Delta\) not shown are at least 9. Taking \(n = 7\), we find \(\binom{n}{2} - k = 14\) and

\[
\begin{align*}
\text{Cat}_{\gamma,\mu}(q, t) &= q^{11}t^3 + q^9t^5 + q^8t^6 + q^7t^7 + q^5t^9 + q^4t^{10}; \\
\text{Cat}_{\gamma,\mu^*}(q, t) &= q^{10}t^4 + q^9t^5 + q^7t^7 + q^9t^8 + q^5t^9 + q^3t^{11} = \text{Cat}_{\gamma,\mu^*}(q, t). \\
\end{align*}
\]

Despite the apparent irregularity of these profiles, the same opposite property holds for all \(n\). The value \(n = 9\) is especially striking: here, \(\text{Cat}_{9,\mu^*}(q, t)\) and \(\text{Cat}_{9,\mu}(t, q)\) both equal \(\sum_{d=2}^{26} q^{29-d}t^d\) with \(q^{26}t^3\) omitted, due to the two displayed 10s in the min\(_\Delta\)-profiles.

### 1.2. Local Chains

The second paper in our series [3] introduced important technical tools called local chains, which guide our construction of global chains and greatly simplify the task of verifying the opposite property for given \(C_\mu\) and \(C_{\mu^*}\). The main idea is that each global chain should be the union of certain overlapping local chains whose min\(_\Delta\)-profiles have special relationships. In fact, we showed that any proposed global chain \(C_\mu\) can be decomposed into local chains in at most one way. The min\(_\Delta\)-profiles of the global chain and its local constituents can be distilled into lists of integers called the amh-vectors. We proved that chains \(C_\mu\) and \(C_{\mu^*}\) have the opposite property if the amh-vectors for these chains satisfy three easily checkable conditions (illustrated in the next example and fully explained in Sect. 5). These ideas enabled us to build all global chains and verify the opposite property for all deficit values \(k\) up to 11. These 195 chains are listed in [6, Appendix] for \(0 \leq k \leq 6\), in [3, Appendix 4.3] for \(k = 7, 8, 9\), and in the extended appendix [4] for \(k = 10, 11\). Our first paper [6, §2 and §4] also constructs two infinite families of chains, namely \(C_{(k)}\) for every \(k \geq 0\) and \(C_{(ab-a-1,a-1)}\) for every \(a, b \geq 2\). We showed in [6, §3 and §5] that the opposite property holds if we set \((k)^* = \langle k \rangle\) and \((ab-a-1,a-1)^* = \langle ab-b-1, b-1 \rangle\).

**Example 1.2.** Continuing Example 1.1, the amh-vectors for \(C_{(61)}\) (from [3, App. 4.3]) are \(a = (3, 5, 9, 27), m = (0, 2, 1, 0),\) and \(h = (7, 7, 7, 9)\). The amh-vectors for \(C_{(331)}\) are \(a^* = (2, 4, 7, 11), m^* = (0, 1, 2, 0),\) and \(h^* = (9, 7, 7, 7)\). The opposite property for \(C_{(61)}\) and \(C_{(331)}\) is verified by noting that \(m^*\) is the reversal of \(m\), \(h^*\) is the reversal of \(h\), and \(a + m + 7 + a_{5-i}^* = \binom{h_i}{2}\) for \(i = 1, 2, 3, 4\), where \(7 = \|\langle 61 \rangle\|\) is the deficit value.
1.3. The Successor Map

One important tool for building the chains $C_\mu$ is the successor map (called NU$_1$ in this paper, and called $\nu$ in [3,6]). For each partition $\gamma$ with deficit $k$ and dinv $i$, NU$_1(\gamma)$ (if defined) is a partition with deficit $k$ and dinv $i + 1$. We would like to build the entire global chain $C_\mu$ by repeatedly applying NU$_1$ to some starting partition $\nu_0(i_0)$. The trouble is that NU$_1$ is not defined for all partitions. There is a known set of NU$_1$-initial objects where NU$_1^{-1}$ is undefined, and there is a known set of NU$_1$-final objects where NU$_1$ is undefined (see Sect. 2.4 for details). Given any NU$_1$-initial partition $\gamma$ of deficit $k$, we obtain the NU$_1$-segment NU$_1^\gamma(\gamma)$ by starting at $\gamma$ and applying NU$_1$ as many times as possible. Each NU$_1$-segment is either infinite or terminates after finitely many steps at a NU$_1$-final object. (Note that a NU$_1$-segment cannot cycle back on itself, since dinv increases with each application of NU$_1$.)

For each partition $\mu$ of size $k$, we have constructed a specific NU$_1$-initial object TI($\mu$), called the tail-initiator partition indexed by $\mu$. The partition TI($\mu$) has deficit $k$ and generates an infinite NU$_1$-segment TAIL($\mu$), called the NU$_1$-tail indexed by $\mu$. Section 2.5 reviews the definitions and properties of TI($\mu$) and TAIL($\mu$) in more detail. The remaining NU$_1$-segments consist of finite chains called NU$_1$-fragments. For each deficit value $k$, the challenge is to assemble the huge number of NU$_1$-fragments and NU$_1$-tails of deficit $k$ to produce $p(k)$ global chains $C_\mu$ satisfying the opposite property.

Example 1.3. For the partitions $\mu = \langle 61 \rangle$ and $\mu^* = \langle 331 \rangle$ from Example 1.1, TI($\mu$) = $\langle 77654311 \rangle$ and TI($\mu^*$) = $\langle 544311 \rangle$. The chain $C_\mu$ is the union of the fragments NU$_1^\mu(\langle 5111111 \rangle)$, NU$_1^\gamma(\langle 3333 \rangle)$, NU$_1^\gamma(\langle 44422 \rangle)$, NU$_1^\gamma(\langle 554421 \rangle)$, and TAIL($\langle 61 \rangle) = NU_1^\gamma(\langle 77654311 \rangle)$. The chain $C_{\mu^*}$ is the union of fragments NU$_1^\gamma(\langle 21111111 \rangle)$, NU$_1^\gamma(\langle 32222 \rangle)$, NU$_1^\gamma(\langle 43331 \rangle)$, and TAIL($\langle 331 \rangle) = NU_1^\gamma(\langle 544311 \rangle)$. These chains come from [3, App. 4.3].

1.4. Extending the Successor Map

In the first part of this paper, we extend the map NU$_1$ by defining a new map NU$_2$ that acts on certain NU$_1$-final partitions. This extension causes many NU$_1$-fragments to coalesce, making it easier to assemble global chains. In particular, each original NU$_1$-tail starting at TI($\mu$) may now extend backwards to a new starting object TI$_2$($\mu$), called the second-order tail-initiator indexed by $\mu$. These generate longer second-order tails called TAIL$_2$($\mu$). We proceed to define and study flagpole partitions, which are partitions $\mu$ satisfying $|\mu| \leq 2\text{min}_\Delta(\text{TI}_2(\mu)) - 8$. We use the term “flagpole”, because, informally, flagpole partitions must end in many parts equal to 1 (see Remark 4.7 for a precise statement). Therefore, the Ferrers diagram of a flagpole partition (drawn in the English style, with the largest part on top) looks like a flag flying on a pole. A generalized version of flagpole partitions is introduced later (Sect. 8.1).

The principal results in the first half of this paper are as follows.

- Theorems 2.16, 2.17, and 2.18 give detailed structural information about the NU$_1$-tails, including the min$_\Delta$-profile of each tail and a precise description of the objects in each tail having a given value of min$_\Delta$. Theorem 2.21
shows that for each $k$, all partitions of deficit $k$ with sufficiently large $\text{dinv}$ belong to one of these tails.

- Theorem 3.6 shows that the maps $\nu_1$ and $\nu_2$ assemble to give a bijection $\nu_1$ (defined on a specific subcollection of integer partitions) that preserves deficit and increases $\text{dinv}$ by 1.

- Theorem 3.10 gives specific characterizations of which partitions appear in the second-order tails and which partitions are second-order tail initiators.

- Theorem 3.17 explicitly computes the second-order tails that start from objects $\text{TIL}_2(\mu)$ having a particular form. Among other structural facts, we show that the $\nu_1$-fragments comprising such tails are demarcated by the descents in the $\min\Delta$-profile.

- Theorem 4.4 characterizes flagpole partitions $\mu$ based on the form of the Dyck vector representing $\text{TIL}_2(\mu)$. This leads to an exact enumeration of flagpole partitions as a sum of partition numbers (Theorem 4.9).

We also did computer experiments to obtain an empirical comparison of the sizes of $\text{TAIL}(\mu)$ and $\text{TAIL}_2(\mu)$. Specifically, for $k \leq 30$, we divided the number of partitions in $\text{Def}(k)$ not in any $\text{TAIL}_2(\mu)$ by the number of partitions in $\text{Def}(k)$ not in any $\text{TAIL}(\mu)$. This ratio is always less than 0.38 and quite close to that value for $20 \leq k \leq 30$. This means that introducing $\nu_2$ causes more than 62% of the objects in the $\nu_1$-fragments to be absorbed into the second-order tails.

1.5. A Recursive Construction of Certain Chains

Our ultimate goal in this series of papers is to build all global chains $C_\mu$ by an elaborate recursive construction using induction on the deficit value $k$. Here is an outline of the construction we envisage. For the base case, all chains $C_\mu$ with (say) $|\mu| \leq 5$ have already been defined successfully. For the induction step, we consider a particular fixed value of $k \geq 6$. As the induction hypothesis, we assume that all chains $C_\lambda$ with $|\lambda| < k$ are already defined and satisfy various technical conditions (including the opposite property of $C_\lambda$ and $C_\lambda^*$ and other requirements on the $amh$-vectors). This information is used to build the chains $C_\mu$ for all partitions $\mu$ of size $k$ and to verify the corresponding technical conditions for these chains.

At this time, we cannot execute the entire recursive construction just outlined. However, for each particular flagpole partition $\mu$, we identify a specific list of needed partitions for $\mu$ (Definition 6.3). Assuming that $C_\rho$ and $C_\rho^*$ exist and have required structural properties for each needed partition $\rho$, we can explicitly define $\mu^*$, $C_\mu$, and $C_{\mu^*}$ and prove that the new chains satisfy the same properties. Section 6 provides all the details of this chain construction, and Theorem 6.4 proves that the construction works. Starting with a particular base collection of known chains, we can apply this construction repeatedly to augment the given collection with many new chains. Theorem 6.5 describes this process precisely. Section 8 extends these results to generalized flagpole partitions. Theorem 8.3 gives a lower bound on the number of generalized flagpole partitions of size $k$ and an asymptotic estimate based on this bound.
2. Preliminaries on Dyck Classes, Deficit, NU$_1$, and Tail Initiators

This section covers needed background material on quasi-Dyck vectors, the dinv and deficit statistics, the original NU$_1$ map, and the tail initiators TI($\mu$). Some new ingredients not found in earlier papers include: the representation of integer partitions by equivalence classes of quasi-Dyck vectors (Proposition 2.2); useful formulas for the deficit statistic (Proposition 2.3 and Lemma 2.5); an explicit description of how iterations of NU$_1$ act on binary Dyck vectors (Proposition 2.13); and a detailed characterization of the Dyck classes belonging to each NU$_1$-tail (Theorems 2.16, 2.17, and 2.18).

2.1. Quasi-Dyck Vectors and Dyck Classes

A quasi-Dyck vector (abbreviated QDV) is a sequence of integers ($v_1, v_2, \ldots, v_n$), such that $v_1 = 0$ and $v_{i+1} \leq v_i + 1$ for $1 \leq i < n$. A Dyck vector is a quasi-Dyck vector where $v_i \geq 0$ for all $i$. We often use word notation for QDVs, writing $v_1 v_2 \cdots v_n$ instead of $(v_1, v_2, \ldots, v_n)$. The notation $i^c$ always indicates $c$ copies of the symbol $i$, as opposed to exponentiation. For example, $0^3 12^2 (−1)^3 01^2 0$ stands for the QDV $(0, 0, 0, 1, 2, 2, −1, −1, −1, 0, 1, 1, 0)$.

A binary Dyck vector (abbreviated BDV) is a Dyck vector with all entries in $\{0, 1\}$. A ternary Dyck vector (abbreviated TDV) is a Dyck vector with all entries in $\{0, 1, 2\}$. For any integer $a$, write $a^- = a - 1$ and $a^+ = a + 1$. For any list of integers $A = a_1 \cdots a_n$, write $A^- = a_1^- \cdots a_n^-$ and $A^+ = a_1^+ \cdots a_n^+$.

Let $\lambda$ be an integer partition with $\ell = \ell(\lambda)$ positive parts $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_\ell$. The size of $\lambda$ is $|\lambda| = \sum_{i=1}^\ell \lambda_i$. We define $\lambda_i = 0$ for all $i > \ell$. For each integer $n > \ell$, we associate with $\lambda$ a quasi-Dyck vector of length $n$ by setting

$$QDV_n(\lambda) = (0 - \lambda_n, 1 - \lambda_{n-1}, 2 - \lambda_{n-2}, \ldots, i - \lambda_{n-i}, \ldots, n - 1 - \lambda_1).$$

(2.1)

Visually, we obtain this QDV by trying to embed the diagram of $\lambda$ in the diagram of $\Delta_n = (n - 1, \ldots, 2, 1, 0)$ and counting the number of boxes of $\Delta_n$ in each row (from bottom to top) that are not in the diagram of $\lambda$. However, we allow $\lambda$ to protrude outside $\Delta_n$, which leads to negative entries in the QDV. We define the minimum triangle height $\text{min}_\Delta(\lambda)$ to be the least $n > \ell(\lambda)$, such that the diagram of $\lambda$ does fit inside the diagram of $\Delta_n$. This is also the least $n$, such that $QDV_n(\lambda)$ has all entries nonnegative.

Example 2.1. Let $\lambda = (5441)$. Figure 1 shows the diagrams of $\lambda$ and $\Delta_n$ for $n = 5, 6, 7, 8$. We have $QDV_5(\lambda) = 00(−2)(−1)(−1)00$, $QDV_6(\lambda) = 011(−1)00$, $QDV_7(\lambda) = 0122011$, $QDV_8(\lambda) = 01233122$, and $\text{min}_\Delta(\lambda) = 7$.

Suppose $n > \ell(\lambda)$ and $QDV_n(\lambda) = v_1 v_2 \cdots v_n$. Then, $QDV_{n+1}(\lambda) = 0(v_1 v_2 \cdots v_n)^+$. Define $\sim$ to be the equivalence relation on the set of all QDVs generated by the relations $v_1 v_2 \cdots v_n \sim 0(v_1 v_2 \cdots v_n)^+$. Therefore, for all QDVs $y = y_1 \cdots y_n$ and $z = z_1 \cdots z_{n+k}$, $y \sim z$ if and only if $z = (0, 1, 2, \ldots, k-1, y_1 + k, y_2 + k, \ldots, y_n + k)$. Each equivalence class of $\sim$ is called a Dyck class. Let $[v]$ denote the Dyck class containing the QDV $v$. 
Proposition 2.2. The map sending each partition $\lambda$ to the Dyck class $\{QDV_n(\lambda) : n > \ell(\lambda)\}$ is a bijection from the set of all integer partitions onto the set of all Dyck classes.

Proof. First, the set $\{QDV_n(\lambda) : n > \ell(\lambda)\}$ is a Dyck class. This holds, since $QDV_{m+1}(\lambda) = 0QDV_m(\lambda)$ for all $m > \ell(\lambda)$, but the shortest QDV in this set (of length $\ell(\lambda) + 1$) cannot be equivalent to any shorter QDV, as the first symbol of a QDV must be 0. Therefore, we have a function $F$ mapping integer partitions to Dyck classes. We must show that this function is bijective. Given the Dyck class $[v]$, where $v$ is a QDV of length $n$ representing the equivalence class, define $\lambda_1 = n - 1 - v_n, \lambda_2 = n - 2 - v_{n-1},$ etc., and $\lambda_m = 0$ for all $m > n$. It is routine to check that $\lambda$ does not depend on the representative chosen, and that $\lambda$ is the unique integer partition satisfying $F(\lambda) = [v]$. □

Henceforth, we make no distinction between the partition $\lambda$ and its associated Dyck class, regarding the list of parts $\langle \lambda_1, \ldots, \lambda_\ell \rangle$ and the Dyck class $[v] = F(\lambda)$ as two notations for the same underlying object. For $n = \min_\Delta(\lambda)$, the Dyck vector $QDV_n(\lambda)$ is called the reduced Dyck vector for $\lambda$. The reduction of a QDV $w$ is the unique reduced Dyck vector $v$ with $[w] = [v]$. For example, the reduction of $012\cdots d$ is 0 for any $d \geq 0$; here, $[0]$ is the Dyck class representing the zero partition $\langle 0 \rangle$, which has $\min_\Delta(\langle 0 \rangle) = 1$.

2.2. Area, Dinv, and Deficit for Quasi-Dyck Vectors

Let $v = v_1v_2\cdots v_n$ be a quasi-Dyck vector. Define $\text{len}(v) = n$ (the length of the list $v$) and $\text{area}(v) = v_1 + v_2 + \cdots + v_n$. If $\lambda$ is a partition and $n > \ell(\lambda)$, then (2.1) shows that $|\lambda| + \text{area}(QDV_n(\lambda)) = |\Delta_n| = \binom{n}{2}$. 
The diagonal inversion statistic for a Dyck vector \( v \), written \( \text{dinv}(v) \), is the number of pairs \((i, j)\) with \( 1 \leq i < j \leq n \) and \( v_i - v_j \in \{0, 1\} \). To generalize this definition to all QDVs \( v \), we define \( v_k = k - 1 \) for all \( k \leq 0 \) and then set \( \text{dinv}(v) \) to be the number of pairs of integers \((i, j)\) with \( i < j \leq n \) and \( v_i - v_j \in \{0, 1\} \). Visually, we compute \( \text{dinv}(v) \) by looking at the infinite word \( \cdots(3)(2)(-1)v_1v_2 \cdots v_n \) and counting all pairs of symbols \( \cdots b \cdots b \cdots \) or \( \cdots (b + 1) \cdots b \cdots \). Suppose we replace \( v = v_1 \cdots v_n \) by the equivalent QDV \( w = w_1w_2 \cdots w_{n+1} = 0v_1^+ \cdots v_n^+ \). The infinite word for \( w \) is obtained from the infinite word for \( v \) by incrementing every entry, and therefore, \( \text{dinv}(w) = \text{dinv}(v) \). It follows that for all QDVs \( v \) and \( z \), \( v \sim z \) implies \( \text{dinv}(v) = \text{dinv}(z) \). Thus, \( \text{dinv} \) is constant on Dyck classes.

The deficit statistic for a QDV \( v = v_1v_2 \cdots v_n \) is \( \text{defc}(v) = \binom{\text{len}(v)}{2} - \text{area}(v) - \text{dinv}(v) \). Replacing \( v \) by \( w \) as above, \( \text{len} \) increases from \( n \) to \( n + 1 \), \( \binom{\text{len}(v)}{2} \) increases by \( n \), area increases by \( n \), and \( \text{dinv} \) does not change. Therefore, \( \text{defc}(w) = \text{defc}(v) \), so \( \text{defc} \) is constant on Dyck classes.

For a partition \( \lambda \) corresponding to a Dyck class \([v]\), we set \( \text{dinv}(\lambda) = \text{dinv}([v]) = \text{dinv}(v) \) and \( \text{defc}(\lambda) = \text{defc}([v]) = \text{defc}(v) \). Note that area is not constant on Dyck classes. For \( n > \ell(\lambda) \), we define \( \text{area}_n(\lambda) = \text{area}(\text{QDV}_n(\lambda)) \). For any such \( n \), \( \text{dinv}(\lambda) + \text{defc}(\lambda) + \text{area}_n(\lambda) = \binom{n}{2} = |\lambda| + \text{area}_n(\lambda) \), and hence, \( \text{dinv}(\lambda) + \text{defc}(\lambda) = |\lambda| \) for all \( \lambda \). (It can be shown that \( \text{dinv}(\lambda) \) is the number of cells \( c \) in the diagram of \( \lambda \) with \( \text{arm}(c) - \text{leg}(c) \in \{0, 1\} \), and \( \text{defc}(\lambda) \) counts the remaining cells, but we do not need these formulas here.) Define \( \text{area}_\Delta(\lambda) = \text{area}_n(\lambda) \) where \( n = \min_\Delta(\lambda) \); so \( \text{area}_\Delta(\lambda) \) is the area of the reduced Dyck vector for \( \lambda \). In Example 2.1, \( |\lambda| = 14 \), \( \text{dinv}(\lambda) = \text{dinv}(01220111) = 10 \), \( \text{defc}(\lambda) = \text{defc}(01220111) = 4 \), \( \min_\Delta(\lambda) = 7 \), \( \text{area}_5(\lambda) = -4 \), \( \text{area}_6(\lambda) = 1 \), \( \text{area}_7(\lambda) = 7 = \text{area}_\Delta(\lambda) \), and \( \text{area}_8(\lambda) = 14 \).

The next proposition gives a convenient alternate formula for computing \( \text{defc}(v) \).

**Proposition 2.3.** For any Dyck vector \( v = v_1v_2 \cdots v_n \), \( \text{defc}(v) \) is the number of pairs \((i, j)\), such that \( 1 \leq i < j \leq n \) and either \( v_i - v_j \geq 2 \) or there exists \( k < i \) with \( v_k = v_i < v_j \). In other words, \( \text{defc}(v) \) is the number of pairs of letters \( \cdots b \cdots c \cdots \) in the word \( v \), such that either \( b \geq c + 2 \), or \( b < c \) and the displayed \( b \) is not the leftmost occurrence of \( b \) in \( v \).

**Proof.** On one hand, we know \( \binom{n}{2} = \text{dinv}(v) + \text{defc}(v) + \text{area}(v) \). On the other hand, there are \( \binom{n}{2} \) pairs \((i, j)\) with \( 1 \leq i < j \leq n \). Each such pair satisfies exactly one of the following conditions: (a) \( v_i - v_j \in \{0, 1\} \); (b) \( v_i - v_j \geq 2 \); (c) for some \( k < i \), \( v_k = v_i < v_j \); (d) for all \( k < i \), \( v_k \neq v_i < v_j \). Pairs satisfying (a) are counted by \( \text{dinv}(v) \), while pairs satisfying (b) or (c) are the pairs mentioned in the lemma statement. Therefore, it suffices to prove that the number of pairs \((i, j)\) satisfying (d) is \( \text{area}(v) = \sum_{j=1}^n v_j \). Consider a fixed \( j \) with \( v_j = c \neq 0 \). The Dyck vector \( v \) has nonnegative integer entries, begins with 0, and consecutive entries may increase by at most 1 reading left to right. Therefore, \( c > 0 \), and each symbol 0, 1, 2, ..., \( c - 1 \) must occur at least once to the left of position \( j \) in \( v \). The leftmost occurrence of each symbol 0, 1, ..., \( c - 1 \) pairs with \( v_j = c \) to give a pair \((i, j)\) of type (d). Thus, we get exactly \( c = v_j \)
type (d) pairs \((i, j)\) for this fixed \(j\). The total number of type (d) pairs is 
\[
\sum_{j: v_j \neq 0} v_j = \text{area}(v),
\]
as needed. \(\square\)

Example 2.4. For all integers \(n, q \geq 0\), we claim 
\(\text{defc}(0^3 12^n 1^q) = 2(n + q + 1)\). Here, there are no pairs of symbols \(b \cdots c\) with \(b \geq c + 2\). We ignore the leftmost zero; the next 0 pairs with \(1 + n + q\) larger symbols to its right. The same is true of the third 0. Ignoring the leftmost 1, the remaining 1s do not pair with any larger symbols to their right (similarly for the 2s). Therefore, the total contribution to deficit is \(2(1 + n + q)\).

2.3. Some Deficit Calculations

For any finite list \(A\), let \(\text{len}(A)\) be the length of \(A\).

**Lemma 2.5.** (a) Let \(v = AB12^n\) be a Dyck vector where \(n \geq 1\) and \(A\) either has at least three 0s or has two 0s and at least two 1s. Then, 
\(\text{defc}(v) \geq 2\text{len}(B) + \text{defc}(A12^n)\).

(b) Let \(v = 00A0B12^n 1^q\) be a Dyck vector where \(n \geq 1, q \geq 0\), and \(A, B\) are lists that might be empty. Then, 
\(\text{defc}(v) \geq 2\text{len}(A) + 2\text{len}(B) + 2(n+q)+1\).

**Proof.** (a) We use the formula in Proposition 2.3 to justify the stated lower bound on \(\text{defc}(v)\). We first show that each symbol in \(B\) contributes at least 2 to \(\text{defc}(v)\). *Case 1: Assume \(A\) has at least three 0s. Each occurrence of 0 in \(B\) is not the leftmost 0 in \(v\) and contributes at least 2 (in fact, at least \(1 + n\)) to \(\text{defc}(v)\) by pairing with one of the symbols in the suffix \(12^n\). Each occurrence of a symbol \(c > 0\) in \(B\) pairs with the second and third 0s in \(A\) to contribute at least 2 to \(\text{defc}(v)\).*

*Case 2: Assume \(A\) has two 0s and at least two 1s. Each 0 in \(B\) contributes at least 2 to \(\text{defc}(v)\), as in Case 1. Each 1 in \(B\) (which is not the leftmost 1 in \(v\)) pairs with the second 0 in \(A\) and with each 2 in the suffix \(12^n\) to contribute at least 2 to \(\text{defc}(v)\). Each symbol \(c \geq 2\) in \(B\) pairs with the second 0 in \(A\) and the second 1 in \(A\) to contribute at least 2 to \(\text{defc}(v)\).

So far, we have found at least \(2 \text{len}(B)\) contributions to \(\text{defc}(v)\) coming from symbols in \(B\) pairing with other symbols. On the other hand, the subword \(A12^n\) of \(v\) is a Dyck vector. Any pair of symbols in this subword contributing to \(\text{defc}(A12^n)\) also contributes to \(\text{defc}(v)\). This proves (a).

(b) Arguing as in Case 1 of (a), we see that each symbol in \(B\) and each 0 in \(A\) contributes at least 2 to \(\text{defc}(v)\). Each symbol \(c \geq 2\) in \(A\) pairs with the zero just before \(A\) and the zero just after \(A\) in \(v\). Finally, each 1 in \(A\) (if any) pairs with the second 0 in \(v\), while each 1 in \(A\) except the leftmost 1 pairs with the rightmost 2 in \(v\). So far, symbols in \(A\) and \(B\) account for at least \(2 \text{len}(A) - 1 + 2 \text{len}(B)\) contributions to \(\text{defc}(v)\). When we delete \(A\) and \(B\) from \(v\), we get the subword \(0^3 12^n 1^q\). By Example 2.4, this subword has deficit \(2(n + q + 1)\), and all pairs contributing to this deficit also contribute to \(\text{defc}(v)\). \(\square\)

The next two lemmas will be used later to define the *antipode map*, which interchanges area and \(\text{dinv}\) for a restricted class of Dyck vectors.
Lemma 2.6. Let $E$ be a ternary Dyck vector and $S = 0E1$. Then, $\text{len}(S) = \text{len}(E) + 2$, $\text{area}(S) = \text{area}(E) + 1$, $\text{dinv}(S) = \text{dinv}(E) + \text{len}(E)$, and $\text{defc}(S) = \text{defc}(E) + \text{len}(E) > \text{defc}(E)$.

Proof. The formulas for $\text{len}(S)$ and $\text{area}(S)$ are clear. Each pair of symbols in $E$ that contribute to $\text{dinv}(E)$ also contribute to $\text{dinv}(S)$. We get additional contributions to $\text{dinv}(S)$ from the initial 0 pairing with each 0 in $E$, and from the final 1 pairing with each 1 and 2 in $E$. Since $E$ is ternary, there are exactly $\text{len}(E)$ such pairs. Therefore, $\text{dinv}(S) = \text{dinv}(E) + \text{len}(E)$. The formula for $\text{defc}(S)$ follows from the previous formulas using area + $\text{dinv} + \text{defc} = \binom{\text{len}}{2}$, or by an argument based on Proposition 2.3.

Lemma 2.7. Let $v$ and $z$ be Dyck vectors, such that $v = 00z^+$. Then, $\text{len}(v) = \text{len}(z) + 2$, $\text{area}(v) = \text{area}(z) + \text{len}(z)$, $\text{dinv}(v) = \text{dinv}(z) + 1$, and $\text{defc}(v) = \text{defc}(z) + \text{len}(z) > \text{defc}(z)$.

Proof. The formulas for $\text{len}(v)$ and $\text{area}(v)$ are clear. We know $\text{dinv}(0z^+) = \text{dinv}(z)$, since $z \sim 0z^+$. Preceding $0z^+$ with one more 0 adds 1 to $\text{dinv}$, since all symbols in $z^+$ are positive and the new 0 only pairs with the 0 immediately following it. This proves $\text{dinv}(v) = \text{dinv}(z) + 1$. As in Lemma 2.6, the formula for $\text{defc}(v)$ follows from the definition or via Proposition 2.3.

2.4. The Maps NU$_1$ and ND$_1$

We now review the definition and basic properties of the original NEXT-UP map NU$_1$ (called $\nu$ in [3,6]). For any integer partition $\gamma$, recall $\gamma_1$ is the first (longest) part of $\gamma$, and $\ell(\gamma)$ is the length (number of positive parts) of $\gamma$. The domain of NU$_1$ is the set $D_1 = \{\gamma : \gamma_1 \leq \ell(\gamma) + 2\}$. For $\gamma \in D_1$ of length $\ell$, define NU$_1(\gamma) = \langle \ell + \gamma_1^- \gamma_2^- \cdots \gamma_\ell^- \rangle$. The map NU$_1$ is not defined for partitions $\gamma$ outside the set $D_1$. Such $\gamma$ satisfy $\gamma_1 > \ell(\gamma) + 2$ and are called NU$_1$-final objects.

We now define the NEXT-DOWN map ND$_1$ (called $\nu^{-1}$ in [3,6]). The domain of ND$_1$ is the set $C_1 = \{\gamma : \gamma_1 \geq \ell(\gamma)\}$. For $\gamma \in C_1$ of length $\ell$, define ND$_1(\gamma) = \langle \gamma_1^+ \gamma_3^+ \cdots \gamma_{\ell+1}^+ \rangle$. The map ND$_1$ is not defined for partitions $\gamma$ outside the set $C_1$. Such $\gamma$ satisfy $\gamma_1 < \ell(\gamma)$ and are called NU$_1$-initial objects.

The following proposition summarizes some known properties of the maps NU$_1$ and ND$_1$; see [6, §2.1] for more details. Properties (a), (b), and (c) will be used frequently hereafter.

Proposition 2.8. (a) The map NU$_1 : D_1 \rightarrow C_1$ is a bijection with inverse ND$_1 : C_1 \rightarrow D_1$.
(b) For $\gamma \in D_1$, $\text{defc}(\text{NU}_1(\gamma)) = \text{defc}(\gamma)$ and $\text{dinv}(\text{NU}_1(\gamma)) = \text{dinv}(\gamma) + 1$.
(c) For $\gamma \in C_1$, $\text{defc}(\text{ND}_1(\gamma)) = \text{defc}(\gamma)$ and $\text{dinv}(\text{ND}_1(\gamma)) = \text{dinv}(\gamma) - 1$.
(d) For $\gamma \in D_1$, NU$_1$ acts on the Ferrers diagram of $\gamma$ by removing the leftmost column (containing $\ell(\gamma)$ boxes), then adding a new top row with $\ell(\gamma) + 1$ boxes.
(e) For $\gamma \in C_1$, ND$_1$ acts on the Ferrers diagram of $\gamma$ by removing the top row (containing $\gamma_1$ boxes), then adding a new leftmost column with $\gamma_1 - 1$ boxes.
(f) For $\gamma \in D_1$, the (finite or infinite) sequence $\gamma, \text{NU}_1(\gamma), \text{NU}_1^2(\gamma), \text{NU}_1^3(\gamma), \ldots$ contains no repeated entries.

Note that part (f) follows from part (b), since each object in the sequence in (f) must have a different value of dinv. Part (f) shows that iterating $\text{NU}_1$ can never produce a cycle of partitions.

Example 2.9. Given $\gamma = \langle 5441 \rangle$, we compute $\text{NU}_1(\gamma) = \langle 5433 \rangle$, $\text{NU}_1^2(\gamma) = \langle 54322 \rangle$, $\text{NU}_1^3(\gamma) = \langle 643211 \rangle$, and so on. On the other hand, $\text{ND}_1(\gamma) = \langle 5521 \rangle$, $\text{ND}_1^2(\gamma) = \langle 6321 \rangle$, and $\text{ND}_1^3(\gamma) = \langle 43211 \rangle$, which is an $\text{NU}_1$-initial object.

Since integer partitions correspond bijectively with Dyck classes (Proposition 2.2), the maps $\text{NU}_1$ and $\text{ND}_1$ can be viewed as well-defined functions acting on certain Dyck classes (those corresponding to the partitions in $D_1$ and $C_1$, respectively). We now describe a convenient formula for computing $\text{NU}_1([v])$ or $\text{ND}_1([v])$ by acting on a representative QDV $v = v_1 \cdots v_n$ for the Dyck class $[v]$. Define the leader of $v$ to be the largest $d \geq 0$, such that $v$ starts with the increasing sequence $012 \cdots d$. Call this first occurrence of $d$ the leader symbol of $v$. With this notation, the following rule is readily verified (Lemma 2.3 of [6] proves it for Dyck vectors, and the proof easily extends to QDVs).

Proposition 2.10. Let $v$ be a QDV of length $n > 1$ with leader $d$ and last symbol $v_n$.

(a) Suppose $v_2 \geq 0$. In the case $d > v_n + 2$, $[v]$ is a $\text{NU}_1$-final object and $\text{NU}_1([v])$ is not defined. In the case $d \leq v_n + 2$, $\text{NU}_1([v]) = [z]$ where $z$ is obtained from $v$ by deleting the leader symbol $d$ and appending $d - 1$.

(b) Suppose $v_n = s \geq -1$ and $[v] \neq [0]$. In the case $d < v_n$, $[v]$ is a $\text{NU}_1$-initial object and $\text{ND}_1([v])$ is not defined. In the case $d \geq v_n$, $\text{ND}_1([v]) = [z]$ where $z$ is obtained from $v$ by deleting $v_n$ and inserting $s + 1$ immediately after the leftmost $s$ in $v$. (When $s = -1$, this means putting a new 0 at the front of $v$.)

It follows that no Dyck class $[v]$ is both $\text{NU}_1$-initial and $\text{NU}_1$-final.

Example 2.11. We repeat Example 2.9 using Dyck vectors; here, $\gamma = \langle 5441 \rangle = \langle 0122011 \rangle$. In the following computation, the leader symbol of each QDV is underlined:

$$\begin{align*}
[0112222] & \xrightarrow{\text{ND}_1} [0122220] \xrightarrow{\text{ND}_1} [0122201] \xrightarrow{\text{ND}_1} \gamma \\
& = [0122011] \xrightarrow{\text{NU}_1} [0120111] \xrightarrow{\text{NU}_1} [0101111] \xrightarrow{\text{NU}_1} [0011110] \xrightarrow{\text{NU}_1} \cdots .
\end{align*}$$

We frequently need the fact that a reduced Dyck vector $v$ has $\min(\nu)_1([v]) = \text{len}(v)$. Using this and Proposition 2.10, we obtain the following.

Proposition 2.12. (a) Let $v$ be a reduced Dyck vector. If $v$ starts with 00, then $\text{NU}_1([v])$ is defined and $\min(\nu)_1(\text{NU}_1([v])) = \min(\nu)_1([v]) + 1$. If $v$ starts with 01 and $\text{NU}_1([v])$ is defined, then $\min(\nu)_1(\text{NU}_1([v])) = \min(\nu)_1([v])$.

(b) Suppose a Dyck vector $v$ starts with 0012 and ends with a positive symbol. Then, $[v]$ is a $\text{NU}_1$-initial object, $\text{NU}_1([v])$ is defined and is a $\text{NU}_1$-final object, and $\min(\nu)_1(\text{NU}_1([v])) = \min(\nu)_1([v]) + 1 = \text{len}(v) + 1$. 

Proposition 2.13. Let \( v = v_0v_2v_3 \cdots v_n \) be a binary Dyck vector of length \( n \). Starting at \([v]\) and applying \( \text{NU}_1 \) \( n \) times lead to \([v0]\) via the following chain of Dyck classes:

\[
[v] = [0v_0v_2 \cdots v_n] \xrightarrow{\text{NU}_1} [0v_2v_3 \cdots v_nv_3^-] \xrightarrow{\text{NU}_1} [0v_4 \cdots v_nv_2^-v_3^-]
\]

Starting at \([v]\), \( \text{NU}_1 \) \((\text{NU}_1)\) times lead to \([v0]\) via the following chain of Dyck classes:

\[
[v] \xrightarrow{\text{NU}_1} [0v_2v_3^- \cdots v_n^-] = [0v_2v_3 \cdots v_n] \xrightarrow{\text{NU}_1} [0v_2v_3 \cdots v_n0] = [v0].
\]

We call the intermediate vectors \( 0v_{k+1} \cdots v_nv_2^- \cdots v_k^- \) (where \( 1 \leq k \leq n \)) cycled versions of \( v \).

Proof. Let \( w = 0w_2w_3 \cdots w_n \) be a QDV with all \( w_i \) in \(-1, 0, 1\). Suppose \( w_2 \) is 0 or 1. By checking the two cases, we see that \( \text{NU}_1([w]) = [0w_3 \cdots w_nw_2^-] \). This observation justifies the links in (2.2) leading to \([0v_2v_3^- \cdots v_n^-]\). At this last stage, the representative QDV might have second symbol \(-1\), so we change to the new representative \( 0v_2v_3 \cdots v_n \) of length \( n + 1 \). We then apply the observation at the start of the proof once more to reach \([v0]\). \( \square \)

2.5. Tail Initiators and \( \text{NU}_1\)-Tails

Definition 2.14. Given a nonzero partition \( \mu = \langle r^n_1 \cdots 2^n_2 1^n_1 \rangle \) with \( n_r > 0 \), define \( B_\mu = 01^n_1 01^n_2 \cdots 01^n_r \). Note that every binary word starting with 0 and ending with 1 has the form \( B_\mu \) for exactly one such \( \mu \). When \( \mu = \langle 0 \rangle \), define \( B_\mu \) to be the empty word. For any partition \( \mu \), define the tail-initiator of \( \mu \) to be the Dyck class \( \text{TI}(\mu) = [0B_\mu] \). Define \( \text{TAIL}(\mu) \) to be the sequence of Dyck classes reachable from \( \text{TI}(\mu) \) by applying \( \text{NU}_1 \) zero or more times.

For example, \( \mu = \langle 33111 \rangle = \langle 3^2 2^1 1^3 \rangle \) has \( \text{TI}(\mu) = [001110011] \), which is the Dyck class identified with the partition \( \langle 76653211 \rangle \). The map \( \text{TI} \) is a bijection from the set of integer partitions onto the set of Dyck classes \( [v] \), such that the reduced representative \( v \) is either 0 or a BDV starting with 00 and ending with 1.

Proposition 2.15. For every partition \( \mu \), \( \text{TI}(\mu) \) is a \( \text{NU}_1 \)-initial object with the following statistics:

\[
\begin{align*}
(a) & \quad \min_\Delta(\text{TI}(\mu)) = \text{len}(0B_\mu) = \mu_1 + \ell(\mu) + 1. \\
(b) & \quad \text{defc}(\text{TI}(\mu)) = \text{defc}(0B_\mu) = |\mu|. \\
(c) & \quad \text{area}_\Delta(\text{TI}(\mu)) = \text{area}(0B_\mu) = \ell(\mu). \\
(d) & \quad \text{dinv}(\text{TI}(\mu)) = \text{dinv}(0B_\mu) = \left(\binom{\mu_1 + \ell(\mu) + 1}{2} - \ell(\mu) - |\mu|\right).
\end{align*}
\]

Proof. All statements are immediately verified for \( \mu = \langle 0 \rangle \) and \( \text{TI}(\mu) = [0] \). Now, consider a nonzero partition \( \mu = \langle r^n_1 \cdots 2^n_2 1^n_1 \rangle \) and \( B_\mu = 01^n_1 01^n_2 \cdots 01^n_r \). We have \( r = \mu_1 \) and \( n_1 + \cdots + n_r = \ell(\mu) \). Now, \( 0B_\mu \) is the reduced representative of the Dyck class \( \text{TI}(\mu) \), since \( 0B_\mu \) starts with 00. As \( 0B_\mu \) contains \( 1 + r \) zeroes, we have \( \min_\Delta(\text{TI}(\mu)) = \text{len}(0B_\mu) = \mu_1 + \ell(\mu) + 1 \). Using Proposition 2.3 to find \( \text{defc}(0B_\mu) \), each 1 in \( 1^n_i \) pairs with \( i \) preceding 0s (not including the leftmost 0). Therefore, \( \text{defc}(0B_\mu) = n_1 + 2n_2 + \cdots + rn_r = |\mu| \).

The area of \( 0B_\mu \) is \( n_1 + \cdots + n_r = \ell(\mu) \). The formula for \( \text{dinv}(0B_\mu) \) follows, since \( \text{dinv} + \text{defc} + \text{area} = \left(\binom{\text{len}}{2}\right) \). Finally, since \( 0B_\mu \) has leader \( d = 0 \) and last symbol 1, \( \text{TI}(\mu) \) is a \( \text{NU}_1 \)-initial object by Proposition 2.10(b). \( \square \)
Given any sequence $\mathcal{C}$ of partitions, the $\min_\Delta$ -profile of $\mathcal{C}$ is the numerical sequence obtained from $\mathcal{C}$ by replacing each term $\gamma$ by $\min_\Delta(\gamma)$.

**Theorem 2.16.** For each partition $\mu$, $\text{tail}(\mu)$ is the infinite sequence $(\text{nu}_1^n(\text{TI}(\mu))) : m \geq 0$, which consists of all $[z]$, such that $z$ is a cycled version of $0B_\mu 0^c$ for some $c \geq 0$. Letting $n = \min_\Delta(\text{TI}(\mu)) = \mu_1 + \ell(\mu) + 1$, the $\min_\Delta$ -profile of $\text{tail}(\mu)$ is $n^1(n+1)^n(n+2)^{n+1}\cdots(n+c+1)^{n+c}\cdots$. All objects in $\text{tail}(\mu)$ have deficit $|\mu|$.

**Proof.** The sequence $\text{tail}(\mu)$ has first entry $\text{TI}(\mu) = [0B_\mu]$, where the reduced representative $0B_\mu$ is a BDV of length $n = \min_\Delta(\text{TI}(\mu))$. By Proposition 2.13, applying $\nu_1$ $n$ times leads to the Dyck class $[0B_\mu 0]$. The $n$ Dyck classes following $[0B_\mu]$ all have $\min_\Delta = n+1$, as we see by inspection of (2.2). We now invoke Proposition 2.13 again, taking $v$ there to be the BDV $0B_\mu 0$ of length $n+1$. After $n+1$ applications of $\nu_1$, we reach $[0B_\mu 00]$, where the $n+1$ Dyck classes following $0B_\mu 0$ all have $\min_\Delta = n+2$. We proceed similarly. Having reached $[0B_\mu 0^c]$ for some $c \geq 0$, the tail continues to $[0B_\mu 0^{c+1}]$ in $n+c$ steps. By Proposition 2.13, the Dyck classes from $[0B_\mu 0^c]$ (inclusive) to $[0B_\mu 0^{c+1}]$ (exclusive) are precisely the classes $[z]$ where $z$ is a cycled version of $0B_\mu 0^c$. Moreover, the Dyck classes from $[0B_\mu 0^c]$ (exclusive) to $[0B_\mu 0^{c+1}]$ (inclusive) all have $\min_\Delta = n+c+1$. The first part of the theorem follows by induction on $c$. All objects in $\text{tail}(\mu)$ have deficit $|\mu|$ by Propositions 2.15(b) and 2.8 (b). □

**2.6. Further Structural Analysis of the $\nu_1$-Tails**

In our later work, we need an even more detailed version of the description of $\text{tail}(\mu)$ in Theorem 2.16. For each $j \geq 0$, let the $j$th plateau of $\text{tail}(\mu)$ consists of all $[z]$ in $\text{tail}(\mu)$ with $\min_\Delta([z]) = n + j$, where $n = \min_\Delta(\text{TI}(\mu))$. The $0$th plateau consists of $\text{TI}(\mu) = [0B_\mu]$ alone. For $j > 0$, the $j$th plateau consists of $n+j−1$ objects with consecutive dinv values, namely all objects strictly after $[0B_\mu 0^{j-1}]$ and weakly before $[0B_\mu 0^j]$ in $\text{tail}(\mu)$, as we saw in the proof of Theorem 2.16. The next result explicitly lists all such objects using reduced representatives for each Dyck class.

**Theorem 2.17.** For any nonzero partition $\mu$ and $j > 0$, the $j$th plateau of $\text{tail}(\mu)$ consists of the following Dyck classes, listed in order from lowest dinv to highest dinv:

(a) first, $[01Z+1^{j-1}Y]$ where $Y$ and $Z$ are nonempty strings such that $B_\mu = YZ$, listed in order from the shortest $Y$ to the longest $Y$;

(b) second, $[01^aB_\mu 0^b]$ where $a + b = j$, listed in order from $b = 0$ to $b = j$.

For $\mu = (0)$ and $j > 0$, the $j$th plateau of $\text{tail}(\emptyset)$ consists of $[01^a0^b]$ where $a + b = j$ and $b > 0$, listed in order from $b = 1$ to $b = j$.

**Proof.** This theorem follows from the calculation (2.2) applied to $v = 0B_\mu 0^{j-1}$. Initially, the symbols in $B_\mu$ cycle to the end of the list and decrement, one at a time, producing the Dyck classes $[0Z0^{j-1}Y] = [01Z+1^{j-1}Y]$ in the order listed in (a). When all symbols in $B_\mu$ have cycled to the end, we have reached $[00^{j-1}B_\mu] = [01^jB_\mu]$, which is the first Dyck class in (b). Applying $\nu_1 j$ more
times in succession gives the remaining objects in (b) in order, ending with $[0B_\mu 0^j]$. The special case $\mu = \langle \emptyset \rangle$ is different, because the objects in (a) do not exist and the Dyck vector $01^j$ is not reduced. Since $[01^j] = [0^j]$, this Dyck class belongs to plateau $j - 1$, not plateau $j$. Applying (2.2) to $v = 01^j$ proves the theorem in this case.

We now show that every Dyck class $[w]$ represented by a binary Dyck vector $w$ belongs to exactly one $\text{TAIL}(\mu)$, where $\mu$ can be easily deduced from $w$. This result also holds when $w$ is a ternary Dyck vector with a particular structure.

\textbf{Theorem 2.18.} (a) For each binary Dyck vector $w$, $[w] \in \text{TAIL}(\mu)$ for exactly one partition $\mu$.

(b) For each non-reduced ternary Dyck vector $w$, $[w] \in \text{TAIL}(\mu)$ for exactly one partition $\mu$.

(c) For each reduced ternary Dyck vector $w$ containing 2, $[w] \in \text{TAIL}(\mu)$ for some (necessarily unique) $\mu$ if and only if $w_1 = 0$ is the only 0 in $w$ before the last 2 in $w$.

\textbf{Proof.} Part (a) is true, since every binary string starting with 0 has the form given in Theorem 2.17(b) for exactly one choice of $\mu$, $a$, and $b$. Part (b) follows from (a), since a non-reduced TDV $w$ has the form $w = 0z^+$ for some BDV $z$, and $[w] = [0z^+] = [z]$. To prove (c), let $w$ be a reduced TDV containing 2. First, assume $[w] \in \text{TAIL}(\mu)$. Since Theorem 2.17 lists all reduced Dyck vectors representing Dyck classes in $\text{TAIL}(\mu)$, we must have $w = 01Z^+1^j1^{-1}Y$ for some $j > 0$ and some nonempty lists $Y$ and $Z$ with $B_\mu = YZ$. Every symbol of $Z^+$ is 1 or 2 and the last symbol is 2, while every symbol of $Y$ is 0 or 1 and $Y$ starts with 0. Thus, $w$ has only one 0 before the last 2. Conversely, assume $w$ has only one 0 before the last 2. Then, we can factor $w$ as $w = 01Z^+1^j1^{-1}Y$ by letting the last symbol of $Z^+$ be the last 2 in $w$, and choosing the maximal $j > 0$ to ensure $Y$ starts with 0. This 0 must exist, since $w$ is reduced with only one 0 before the last 2. We see that $YZ$ is a binary vector of the form $B_\mu$, so that $[w] \in \text{TAIL}(\mu)$ by Theorem 2.17(a).

\textbf{Example 2.19.} (a) The BDV $w = 011110101$ matches the form in Theorem 2.17(b) with $a = 4$, $b = 0$, $B_\mu = 0101$, so $\mu = \langle 21 \rangle$. Therefore $[w]$ is in plateau 4 of $\text{TAIL}(\langle 21 \rangle)$.

(b) The TDV $w = 01211221$ is not reduced; in fact, $[w] = [0100110]$. The binary representative matches Theorem 2.17(b) with $a = b = 1$, $B_\mu = 0011$, and $\mu = \langle 22 \rangle$. Therefore $[w]$ is in plateau 2 of $\text{TAIL}(\langle 22 \rangle)$.

(c) The TDV $w = 01122110$ is reduced with only one 0 before the last 2. This TDV matches the form in Theorem 2.17(a) with $Z^+ = 122$, $j = 3$, $Y = 0$, $B_\mu = YZ = 0011$, so $\mu = \langle 22 \rangle$. Therefore, $[w]$ is the first element in plateau 3 of $\text{TAIL}(\langle 22 \rangle)$.

Using a hard result from [8], we proved the following fact in Remark 2.3 of [3].
Theorem 2.20. For all \( k, d \geq 0 \), the number of integer partitions with deficit \( k \) and \( \text{dinv} \ d \) equals the number of integer partitions of size \( k \) with largest part at most \( d \). Hence, for all \( d \geq k \), there are exactly \( p(k) \) partitions with deficit \( k \) and \( \text{dinv} \ d \).

As a consequence, we now show that all but finitely many partitions of deficit \( k \) belong to one of the tail sequences \( \text{TAIL}(\mu) \).

Theorem 2.21. For all \( k \geq 0 \), there exists \( d_0(k) \), such that for all \( d \geq d_0(k) \), each partition with deficit \( k \) and \( \text{dinv} \ d \) appears in exactly one of the sequences \( \text{TAIL}(\mu) \) as \( \mu \) ranges over partitions of size \( k \).

Proof. Fix \( k \geq 0 \). As \( \mu \) ranges over all partitions of size \( k \), we obtain \( p(k) \) disjoint sequences \( \text{TAIL}(\mu) \), where \( \text{TAIL}(\mu) \) starts at \( \text{TI}(\mu) \) and \( \text{dinv} \) increases by 1 as we move along each sequence. Let \( d_0(k) \) be the maximum of \( \text{dinv}(\text{TI}(\mu)) \) over all partitions \( \mu \) of size \( k \). Fix \( d \geq d_0(k) \). Then, each sequence \( \text{TAIL}(\mu) \) contains a partition with \( \text{dinv} \ d \) and deficit \( |\mu| = k \). By Theorem 2.20, these sequences already account for all \( p(k) \) partitions with \( \text{dinv} \ d \) and deficit \( k \). Thus, each such partition must belong to one (and only one) of these sequences.

Here is a different proof not relying on Theorem 2.20. Fix \( k \geq 0 \) and let \( d_0(k) = \binom{k+4}{2} + 1 \). Assume \( [v] \) is a Dyck class with deficit \( k \) that belongs to none of the sequences \( \text{TAIL}(\mu) \). It suffices to prove that \( \text{dinv}(v) < d_0(k) \). We may choose \( v \) to be a reduced Dyck vector. By Theorem 2.18(a), \( v \) cannot be a binary vector, so \( v \) contains a 2. As \( v \) is reduced, \( v \) must contain at least two 0s.

Case 1: \( v \) contains a 3 to the left of the second 0 in \( v \). Then, we can write \( v = 0A3B0C \) where \( A \) and \( B \) contain no 0s. We use Proposition 2.3 to show that \( \text{defc}(v) \geq \text{len}(v) - 4 \). Each symbol \( x \geq 2 \) in \( A, B, \) or \( C \) pairs with the 0 following \( B \). Each \( x \leq 1 \) in \( B \) or \( C \) pairs with the 3 before \( B \). Each 1 in \( A \) except the leftmost 1 pairs with the 3 after \( A \). Thus, \( k = \text{defc}(v) \geq \text{len}(A) - 1 + \text{len}(B) + \text{len}(C) = \text{len}(v) - 4 \).

Case 2: All symbols in \( v \) before the second 0 are at most 2. Here, we can write \( v = 0A4B2C \) where every symbol in \( A \) is 1 or 2. Note that the displayed 2 after \( B \) must exist, either because the Dyck vector \( v \) contains a 3 after the second 0 or (when \( v \) is ternary) by Theorem 2.18(c). Here, each \( x \geq 2 \) in \( A \) or \( B \) pairs with the 0 between \( A \) and \( B \). Each \( x \leq 1 \) in \( A \) or \( B \) (except the leftmost 1) pairs with the 2 after \( B \). Each 0 in \( C \) pairs with the 2 before \( C \), while other symbols in \( C \) pair with the 0 before \( B \). We again have \( k = \text{defc}(v) \geq \text{len}(A) + \text{len}(B) - 1 + \text{len}(C) = \text{len}(v) - 4 \).

In both cases, \( \text{dinv}(v) \leq \binom{\text{len}(v)}{2} \leq \binom{k+4}{2} < d_0(k) \). \( \square \)

3. Extending the Map \( \text{NU}_1 \)

This section extends the function \( \text{NU}_1 \) to act on certain \( \text{NU}_1 \)-final objects. Using the inverse of this extended map, each infinite \( \text{NU}_1 \)-tail (starting at \( \text{TI}(\mu) \), say) can potentially be extended backward to a new starting point called \( \text{TI}_2(\mu) \). This leads to the concept of flagpole partitions in the next section.
3.1. Two Rules Extending \( \text{NU}_1 \)

The next definition gives two new rules that extend \( \text{NU}_1 \).

**Definition 3.1.**

(a) Assume \( h \geq 2 \) and \( A = A_1 \cdots A_s \) is a list of integers such that \( A = \emptyset \), or all \( A_i \leq 2 \) and \( A_s \geq 0 \) and \( A_{i+1} \leq A_i + 1 \) for all \( i < s \). Define \( \text{NU}_2([012^hA(-1)^{h-1}]) = [00^{h-1}1A^h] \).

(b) Assume \( k \geq 1 \) and \( B = B_1 \cdots B_s \) is a list of integers, such that \( B = \emptyset \), or all \( B_i \leq 2 \) and \( B_1 \leq 1 \) and \( B_s \geq -1 \) and \( B_{i+1} \leq B_i + 1 \) for all \( i < s \).

Define \( \text{NU}_2([012^kB(-1)^k]) = [00^kB01^k] \).

(c) Let \( D_2 \) be the set of Dyck classes matching one of the input templates \([012^hA(-1)^{h-1}]\) or \([012^kB(-1)^k]\) in (a) and (b). Let \( C_2 \) be the set of Dyck classes matching one of the output templates \([0^h1A^h]\) or \([0^{k+1}B01^k]\) in (a) and (b).

**Example 3.2.** \( \text{NU}_2([01222(-1)001(-1)(-1)]) = [00012(-1)00111] \) by letting \( h = 3 \) and \( A = 2(-1)001 \) in rule (a). Also, \( \text{NU}_2([012211(-1)(-1)(-1)]) = [00011(-1)011] \) by letting \( k = 2 \) and \( B = 11(-1) \) in rule (b).

Recall (Proposition 2.8) that \( \text{NU}_1 \) is a well-defined bijection from the domain \( D_1 \) onto the codomain \( C_1 \), where \( D_1 \) and \( C_1 \) are defined at the beginning of Sect. 2.4.

**Lemma 3.3.** The rules in 3.1(a) and (b) specify a well-defined bijection \( \text{NU}_2 : D_2 \to C_2 \) with an inverse called \( \text{ND}_2 : C_2 \to D_2 \). Moreover, \( D_2 \) is disjoint from \( D_1 \), and \( C_2 \) is disjoint from \( C_1 \).

**Proof.** A given Dyck class has at most one representative \( v \) ending in \(-1\), which is the only representative that rules (a) and (b) might apply to. We claim that rules (a) and (b) cannot both apply to such a \( v \). On one hand, since \( A \) cannot end in \(-1\), the number of 2s at the start of the subword \( 2^hA \) is strictly greater than the number of \(-1\)s at the end of \( v \) when rule (a) applies. On the other hand, since \( B \) cannot start with 2, the number of 2s at the start of \( 2^kB \) is not greater than the number of \(-1\)s at the end of \( v \) when rule (b) applies. The conditions on \( A \) and \( B \) ensure that the outputs of the two rules are valid Dyck classes. This shows that \( \text{NU}_2 \) is a well-defined function mapping the domain \( D_2 \) into the codomain \( C_2 \).

We define the inverse \( \text{ND}_2 \) to \( \text{NU}_2 \) by reversing the rules in Definition 3.1. For example,

\[
\text{ND}_2([0001(-1)001111]) = [01222(-1)001(-1)(-1)] \quad \text{and} \quad \text{ND}_2([000011]) = [0122(-1)(-1)].
\]

Reasoning similar to the previous paragraph shows that \( \text{ND}_2 \) is a well-defined function mapping \( C_2 \) into \( D_2 \). On one hand, a Dyck class has at most one representative \( v \) beginning with \( 00 \). On the other hand, the inverse of rule (a) applies only when the number of \( 1\)s at the end of \( v \) weakly exceeds the number of \( 0\)s at the start of \( v \), while the inverse of rule (b) applies only when the number of initial \( 0\)s strictly exceeds the number of final \( 1\)s. Thus, the two inverse rules can never both apply to the same object. Since \( \text{ND}_2 \) clearly inverts
NU₂, we conclude that \( \text{NU}_2 : D_2 \to C_2 \) is a well-defined bijection with inverse \( \text{ND}_2 : C_2 \to D_2 \).

Each input \([012^h A(-1)^{-1}]\) to rule (a) is a \( \text{NU}_1 \)-final object, since the leader 2 exceeds the last symbol \(-1\) by more than 2 (Proposition 2.10(a)). Similarly, each input to rule (b) is a \( \text{NU}_1 \)-final object. This shows that \( D_1 \) and \( D_2 \) are disjoint. Next, each output \([0^h 1 A 1^h]\) for rule (a) is a \( \text{NU}_1 \)-initial object, since the leader 0 is less than the last symbol 1 (Proposition 2.10(b)). Likewise, each output for rule (b) is a \( \text{NU}_1 \)-initial object. Therefore, \( C_1 \) and \( C_2 \) are disjoint.

The next lemma shows that \( \text{NU}_2 \) has the required effect on the dinv and deficit statistics.

**Lemma 3.4.** Acting by \( \text{NU}_2 \) increases dinv by 1 and preserves deficit.

**Proof.** Let \( v = 012^h A(-1)^{-1} \) and \( v' = 00^{h-1} 1 A 1^h \) be the input and output representatives appearing in rule 3.1(a). For each \( s \), let \( n_s(A) \) be the number of copies of \( s \) in the list \( A \). We have \( \text{len}(v) = 2h + 1 + \text{len}(A) = \text{len}(v') \) and \( \text{area}(v) = h + 2 + \text{area}(A) = \text{area}(v') + 1 \). Next, we show \( \text{dinv}(v') = \text{dinv}(v) + 1 \).

We compute \( \text{dinv}(v) \) by starting with \( \text{dinv}(01A) \) and adding contributions involving symbols in the subwords \( 2^h \) or \( (-1)^{h-1} \). Recall the convention \( v_k = k - 1 \) for all \( k \leq 0 \); we must count pairs \( \cdots b \cdots b \cdots \) or \( \cdots (b + 1) \cdots b \cdots \) in the extended word where one (or both) of the displayed symbols comes from the subwords \( 2^h \) or \( (-1)^{h-1} \). We get \( \binom{h}{2} \) contributions from pairs of 2s in \( 2^h \) and \( \binom{h-1}{2} \) contributions from pairs of \(-1\)s in \((-1)^{h-1}\). Each 2 in \( 2^h \) contributes nothing when compared to the earlier symbols \( \cdots (-2)(-1)01 \) or the later symbols \((-1)^{h-1}\). Comparing each 2 in \( 2^h \) to later symbols in \( A \) gives \( h \) new contributions \( n_1(A) + n_2(A) \). Next, the \( h - 1 \) copies of \(-1\) in \((-1)^{h-1}\) each contribute 1 (comparing to the initial \(-1\) and 0) and \( n_{-1}(A) + n_0(A) \) (comparing to symbols in \( A \)). The total is

\[
\text{dinv}(v) = \text{dinv}(01A) + \binom{h}{2} + \binom{h-1}{2} + h(n_1(A) + n_2(A)) + (h - 1)(2 + n_{-1}(A) + n_0(A)).
\]

Similarly, isolating contributions from \( 0^{h-1} \) and \( 1^h \) in \( v' \), we find

\[
\text{dinv}(v') = \text{dinv}(01A) + \binom{h-1}{2} + \binom{h}{2} + (h - 1)(1 + n_{-1}(A) + n_0(A)) + n_0(A) + h(1 + n_1(A) + n_2(A)).
\]

Comparing the expressions, we get \( \text{dinv}(v') = \text{dinv}(v) + 1 \). It follows that:

\[
\text{defc}(v') = \left( \frac{\text{len}(v')}{2} \right) - \text{area}(v') - \text{dinv}(v')
= \left( \frac{\text{len}(v)}{2} \right) - (\text{area}(v) - 1) - (\text{dinv}(v) + 1) = \text{defc}(v).
\]

A similar proof works for rule 3.1(b). Now \( v = 012^kB(-1)^k \), \( v' = 00^kB01^k \), \( \text{len}(v) = 2k + 2 + \text{len}(B) = \text{len}(v') \), and \( \text{area}(v) = k + 1 + \text{area}(B) = \).
area($v') + 1$. Isolating the dinv contributions of $12^k$ and $(-1)^k$ in $v$, and the

dinv contributions of $0^k$ and $01^k$ in $v'$, we get
\[
\text{dinv}(v') = \text{dinv}(0B) + 2 \binom{k}{2} + n_0(B) + n_1(B) + kn_1(B) + n_2(B) \\
+ n_0(B) + n_{-1}(B)) + 2k + 1 = \text{dinv}(v) + 1.
\]

Therefore, $\text{defc}(v') = \text{defc}(v)$ holds here, too. \qed

We can now combine the bijections $\text{NU}_1$ and $\text{NU}_2$ to obtain the extended

version of the successor map.

**Definition 3.5.** Let $D = D_1 \cup D_2$ and $C = C_1 \cup C_2$. Define the extended

next-up map $\text{NU} : D \to C$ by $\text{NU}((\gamma)) = \text{NU}_1(\gamma)$ for $\gamma \in D_1$ and $\text{NU}((\gamma)) = \text{NU}_2(\gamma)$ for $\gamma \in D_2$. Define the extended next-down map $\text{ND} : C \to D$ by $\text{ND}((\gamma)) = \text{ND}_1(\gamma)$ for $\gamma \in C_1$ and $\text{ND}((\gamma)) = \text{ND}_2(\gamma)$ for $\gamma \in C_2$.

The next theorem summarizes the crucial properties of the extended maps.

**Theorem 3.6.** (a) The map $\text{NU} : D \to C$ is a well-defined bijection with in-

verse $\text{ND} : C \to D$.

(b) For $\gamma \in D$, $\text{defc}(\text{NU}((\gamma)) = \text{defc}(\gamma)$ and $\text{dinv}(\text{NU}((\gamma)) = \text{dinv}(\gamma) + 1$.

(c) For $\gamma \in C$, $\text{defc}(\text{ND}((\gamma)) = \text{defc}(\gamma)$ and $\text{dinv}(\text{ND}((\gamma)) = \text{dinv}(\gamma) - 1$.

(d) For $\gamma \in D$, the (finite or infinite) sequence $\gamma, \text{NU}((\gamma)), \text{NU}^2(\gamma), \text{NU}^3(\gamma), \ldots$

contains no repeated entries.

**Proof.** Part (a) follows from Proposition 2.8(a) and Lemma 3.3. In particular,

the combination of $\text{NU}_1$ and $\text{NU}_2$ (resp. $\text{ND}_1$ and $\text{ND}_2$) is a well-defined function,

since the domains $D_1$ and $D_2$ (resp. $C_1$ and $C_2$) are disjoint. Parts (b) and (c)

follow from Proposition 2.8(b) and (c) and Lemma 3.4. Part (d) follows from part (b), since each object in the sequence in (d) must have a different value of dinv. \qed

Part (d) of the theorem assures us that starting at some partition $\gamma$ and

iterating $\text{NU}$ can never produce a cycle of partitions. Instead, we either get a

finite sequence (called a $\text{NU}$-fragment if $\gamma \notin C$) or an infinite sequence (called

a $\text{NU}$-tail if $\gamma \notin C$). A similar remark applies to iterations of $\text{ND}$, but here the

sequence must be finite, since dinv cannot be negative.

### 3.2. Second-Order Tail Initiators

**Definition 3.7.** Given an integer partition $\mu$, the second-order tail initiator of $\mu$

is the Dyck class $\text{TI}_2(\mu)$ obtained by starting at $\text{TI}(\mu)$ and iterating $\text{ND}$ as

many times as possible. Since $\text{ND}$ decreases dinv, this iteration must termi-

nate in finitely many steps. The second-order tail indexed by $\mu$ is $\text{TILI}_2(\mu) = 
\{\text{NU}^m(\text{TI}_2(\mu)) : m \geq 0\}$. All objects in this tail have deficit $|\mu|$.

**Example 3.8.** The following table shows $\mu$, $\text{TI}(\mu)$, and $\text{TI}_2(\mu)$ for all partitions

of size 4 or less

\[
\begin{array}{|c|c|c|c|c|c|c|c|}
\hline
\mu & 0 & 1 & 2 & 3 & 4 & 5 & 6 \\
\text{TI}(\mu) & 0 & 001 & 0011 & 00111 & 001111 & 0011111 & 00111111 \\
\text{TI}_2(\mu) & 0 & 001 & 0011 & 00111 & 001111 & 0011111 & 00111111 \\
\hline
\end{array}
\]


The entry for \( \mu = \langle 211 \rangle \) is computed as follows:

\[
\text{TI}(\langle 211 \rangle) = [001101] \xrightarrow{ND_2} [01211(1)] \xrightarrow{ND_1} [001211]
\]

Some further values (found with a computer) are

\[
\text{TI}_2(\langle 2111 \rangle) = [0012221], \text{TI}_2(\langle 11111 \rangle) = [0012222], \quad \text{TI}_2(\langle 321 \rangle) = [0012121],
\]

\[
\text{TI}_2(\langle 3111 \rangle) = [0012212], \text{TI}_2(\langle 322111 \rangle) = [001221222], \text{TI}_2(\langle 4321 \rangle) = [001212121].
\]

(3.1)

Although \( \text{TI}(\mu) \) is built from \( \mu \) by a simple explicit formula (see Definition 2.14), we do not know any analogous formula for \( \text{TI}_2(\mu) \). However, we can characterize the set of all Dyck classes \( \text{TI}_2(\mu) \) as \( \mu \) ranges over all partitions. We also prove an explicit criterion for when a Dyck class belongs to some second-order tail \( \text{TAIL}_2(\mu) \).

**Definition 3.9.** A QDV \( v \) is a cycled ternary Dyck vector if and only if \( v \) is a ternary Dyck vector or \( v = A(B^-) \) for some ternary Dyck vectors \( A, B \). Equivalently, a QDV \( v \) is a cycled TDV if and only if every \( v_i \) is in \( \{-1, 0, 1, 2\} \) and there do not exist \( j < k \) with \( v_j = -1 \) and \( v_k = 2 \).

**Theorem 3.10.** (a) A Dyck class \( [w] \) belongs to \( \text{TAIL}_2(\mu) \) for some partition \( \mu \) if and only if \( [w] = [v] \) for some cycled ternary Dyck vector \( v \).

(b) A Dyck class \( [w] \) has the form \( \text{TI}_2(\mu) \) for some partition \( \mu \) if and only if \( [w] = [v] \) for some ternary Dyck vector \( v \) matching one of these forms:

- **Type 1:** \( v = 01^m0X2^n \) where \( n \geq 1 \) and \( 0 \leq m \leq n \) and \( X \) does not end in 2.
- **Type 2:** \( v = 0^nY21^m \) where \( n \geq 2 \) and \( 0 < m < n \) and \( Y \) does not begin with 0.
- **Type 3:** \( v = 0^n1^n \) or \( v = 0^n1^{n-1} \) where \( n \geq 2 \), or \( v = 0 \).

**Proof.** Let \( T \) be the set of cycled ternary Dyck vectors, and let \( S \) be the set of vectors of type 1, 2, 3 described above. Note that \( S \subseteq T \).

**Step 1:** We show that for all \( v \in T \), there exists \( z \in T \) with \( \text{NU}([v]) = [z] \). Fix \( v \in T \), and consider cases based on the initial symbols in \( v \). In the case \( v = 00R \), Proposition 2.10(a) gives \( \text{NU}([v]) = [z] \), where \( z = 0R(1) \) is in \( T \). In the case \( v = 01R \) where \( R \) does not start with 2, \( \text{NU}([v]) = [z] \) where \( z = 0R0 \) is in \( T \). In the case \( v = 0(-1)R \), we must have every entry of \( R \) in \( \{-1, 0, 1\} \). Then, \( [v] = [010R^+] \) where the new input representative is a TDV satisfying the previous case, so the result holds. In the case \( v = 012R \) where the last symbol of \( R \) is at least 0, \( \text{NU}([v]) = [z] \) where \( z = 01R1 \) is in \( T \). The final case is that \( v = 012^aR(1)^b \) for some \( a, b > 0 \), where we can choose \( a \) and \( b \), so \( R \) does not begin with 2 and does not end with \( -1 \). If \( a > b \), then rule 3.1(a) applies with \( h = b + 1 \leq a \) and \( A = 2^{a-b-1}R \). We get \( \text{NU}([v]) = [z] \) for \( z = 0^{b+1}2^{a-b-1}R1^{b+1} \), which is easily seen to be in \( T \). If \( a \leq b \), then rule 3.1(b) applies with \( k = a \) and \( B = R(1)^{b-a} \). Here, we get \( \text{NU}([v]) = [z] \) for \( z = 0^{a+1}R(1)^{b-a}1^{a+1} \), which is also in \( T \).

**Step 2:** We show that for all \( v \in S \), \( \text{ND}([v]) \) is not defined. Fix \( v \in S \). Since \( \text{ND}([0]) \) is undefined, we may assume \( v \neq 0 \). Checking each type, we
see that the leader of \( v \) is always less than the last symbol, so \( \text{ND}_1([v]) \) is not defined (Proposition 2.10(b)). Next, consider the \( \text{ND}_2 \) rules. If \( v \) is type 1 with \( m = 0 \), neither rule in 3.1 applies, because no representative of \([v]\) starts with 00 and ends with 1. If \( v \) is type 1 with \( m > 0 \), note that \([v] = [0^m(-1)X^{-1}]\). Rule (a) does not apply, since \( 0^m \) is not followed by 1, while rule (b) does not apply, since \( m \leq n \). If \( v \) is type 2, rule (a) does not apply, since \( n > m \), while rule (b) does not apply, since the final 1s in \( v \) are preceded by 2, not 0. If \( v \) is type 3 with \( v \neq 0 \), rule (a) does not apply, because \( v \) starts with too many 0s, while rule (b) does not apply, because \( v \) starts with too few 0s. (Observe that when \( A \) or \( B \) is empty, the inputs to the two \( \text{ND}_2 \) rules are \([0^k1^{h+1}]\) and \([0^{k+21}]\).)

**Step 3:** We show that for all \( v \in \mathcal{T} \), either \([v] = [v']\) for some \( v' \in \mathcal{S} \) or \( \text{ND}([v]) = [z] \) for some \( z \in \mathcal{T} \). Fix \( v \in \mathcal{T} \) and consider cases based on the last symbol of \( v \). The conclusion holds if \([v] = [0]\), since \( 0 \in \mathcal{S} \), so assume \([v] \neq [0]\). In the case \( v = 0R(-1), \text{ND}_1([v]) = [z] \) where \( z = 00R \) is in \( \mathcal{T} \). In the case \( v = 0R0, \text{ND}_1([v]) = [z] \) where \( z = 01R \) is in \( \mathcal{T} \). In the case \( v = 01R1, \text{ND}_1([v]) = [z] \) where \( z = 012R \) is in \( \mathcal{T} \). In the case \( v = 0(-1)1R1, [v] = [v'] \) where \( v' = 010R^+2 \) is a type 1 vector in \( \mathcal{S} \) with \( m = 1 \) (note \( R \) cannot contain 2 here). In the case \( v = 00 \cdots 1 \), we can write \( v = 0^aR^b1^c \) where \( a \geq 2, b \geq 1, R \) does not start with 0 or 2, and \( R \) does not end with 1 or 2. If \( a \leq b \) and \( R \) starts with 1, then rule 3.1(a) for \( \text{ND}_2 \) applies and yields an output representative in \( \mathcal{T} \). If \( a > b \) and \( R \) ends with 0, then the same outcome holds using rule 3.1(b). If \( a \leq b \) and \( R \) starts with 1, then \([v] = [v']\) where \( v' = 01^aR^+2b \) is a type 1 vector in \( \mathcal{S} \) with \( m = a \) (note 2 cannot appear in \( R \)). If \( a > b \) and \( R \) ends with 2, then \( v \) is a type 2 TDV in \( \mathcal{S} \) (note \(-1 \) cannot appear in \( R \)). If \( R \) is empty, then rule 3.1(a) applies if \( a < b \), rule 3.1(b) applies if \( a > b + 1 \), and \( v \) is type 3 if \( a = b \) or \( a = b + 1 \). In the final case where \( v \) ends in 2, \( v \) must be a TDV. If \( v = 01R2 \), then we reduce to a previous case by noting \([v] = [w]\) where \( w = 0R^\cdots \in \mathcal{T} \). If \( v = 00R2 \), then \( v \) is a type 1 vector in \( \mathcal{S} \) with \( m = 0 \).

**Step 4:** We prove the “if” parts of Theorem 3.10. Fix arbitrary \( v \in \mathcal{T} \). By iteration of Step 1, the NU-segment \( U = \{\text{NU}^m([v]) : m \geq 0\} \) is infinite, and every Dyck class in \( U \) is represented by something in \( \mathcal{T} \). Because \( U \) is infinite, it contains Dyck classes with arbitrarily large divn. By Theorem 2.21, \( U \) must overlap one of the original tails \( \text{TAIL}(\mu) \) for some \( \mu \). This forces \( U \) to be a subsequence of the new tail \( \text{TAIL}_2(\mu) \). If the \( v \) we started with is in \( \mathcal{S} \), then Step 2 forces \([v]\) to be the initial object in \( \text{TAIL}_2(\mu) \), namely \( \text{TI}_2(\mu) \).

**Step 5:** We prove the “only if” parts of Theorem 3.10. Fix a partition \( \mu \). Note that \( \text{TI}(\mu) \) is a Dyck class represented by a binary Dyck vector \( v \), which belongs to \( \mathcal{T} \). Applying \( \text{ND} \) to \([v]\) repeatedly, we get a finite sequence ending at \( \text{TI}_2(\mu) \). By Steps 2 and 3, we must have \( \text{TI}_2(\mu) = [u] \) for some \( u \in \mathcal{S} \). Now, by Step 1, every Dyck class in \( \text{TAIL}_2(\mu) \) is represented by something in \( \mathcal{T} \). □

### 3.3. Computation of Some NU-Chains

In this section, we compute detailed information about the NU-chain obtained by iterating NU starting at a Dyck class with reduced representative of the form
$v = 00A^+B$ for some binary vectors $A$ and $B$. Since this $\nu$-chain is already understood if $v$ itself is binary (see Theorem 2.18(a) and Theorem 2.17), we assume here that $A^+$ ends in 2. In particular, the results of this section let us explicitly compute the $\nu$-chain leading from $T_2(\mu)$ to $T(\mu)$ in the case where $T_2(\mu)$ has reduced representative $v$ of the indicated form. Referring to the classification in Theorem 3.10(b), our analysis applies to type 1 vectors $v$, such that $m = 0$ and $X$ does not contain 0, as well as to type 2 vectors $v$, such that $n = 2$, $m = 1$, and $Y$ does not contain 0. (Type 3 vectors are binary and thus present no problems.) Theorem 3.10(b) also shows that not every Dyck class $[00A^+B]$ has the form $T_2(\mu)$; for simple counterexamples, consider $v = 00Y21^m$ where $Y$ contains only 1s and 2s and $m \geq 2$.

**Definition 3.11.** Let $v = 00A^+B$ be a Dyck vector where $A$ and $B$ are binary vectors with $A^+$ ending in 2. Define $S_0(v) = v$. Let $S_1(v), S_2(v), \ldots, \text{and} S_j(v)$ be the reduced Dyck vectors for all the $\nu_1$-initial objects appearing in the $\nu$-chain $(\nu^j([v])) : i > 0)$, listed in the order they appear in the chain (in increasing order of div). Note that $J$ must be finite, since the $\nu$-chain either terminates or enters some infinite $\nu_1$-tail $\text{tail}(\mu)$ (Theorem 2.21). Let $S_j(v) = \text{len}(S_j(v)) = \min_{\Delta}([S_j(v)])$ for $0 \leq j \leq J$. We write $L_j$ for $L_j(v)$ when $v$ is understood from context.

We are most interested in the case where $[v]$ itself is $\nu_1$-initial, which occurs if and only if $B$ is empty or ends in 1 (Proposition 2.10b). In this case, the $\nu$-chain starting from $[v]$ consists of $\nu_1$-chains starting at each $[S_j(v)]$, linked together by $\nu_2$-steps arriving at each $[S_j(v)]$ with $j > 0$. The next lemmas show how to compute $S_j(v), L_j(v)$, and the $\min_{\Delta}$-profile of the $\nu$-chain starting at $[v]$.

**Lemma 3.12.** Let $v = 0012^{m_1}12^{m_2} \cdots 12^{m_s}B$ be a Dyck vector of length $L = \min_{\Delta}(v)$ where $s \geq 1$, $m_i \geq 0$ for all $i$, $m_s > 0$, and $B$ is a binary vector.

(a) If $m_1 = 0$, then $\nu^1([v]) = 0012^{m_2} \cdots 12^{m_s}B01$, which is reached by applying $\nu_1 L$ times. The $\min_{\Delta}$-profile of $[v], \nu^1([v]), \ldots, \nu^L([v])$ is $L(L + 1)L$.

(b) If $m_1 = 1$, then $\nu^2([v]) = 0012^{m_2} \cdots 12^{m_s}B01$, which is reached by applying $\nu_1$ and then $\nu_2$. The $\min_{\Delta}$-profile of $[v], \nu([v]), \nu^2([v])$ is $L, L + 1, 1$.

(c) If $m_1 \geq 2$, then $\nu^2([v]) = 0012^{m_1-2}12^{m_2} \cdots 12^{m_s}B11$, which is reached by applying $\nu_1$ and then $\nu_2$. The $\min_{\Delta}$-profile of $[v], \nu([v]), \nu^2([v])$ is $L, L + 1, 1$.

**Proof.** (a) We can apply $\nu_1$ to $[v] = [0012^{m_2} \cdots 12^{m_s}B]$ repeatedly, using Proposition 2.10(a). The first two steps give $\nu_1([v]) = 0112^{m_2} \cdots 12^{m_s} B(-1)$ and $\nu_1^2([v]) = 012^{m_2} \cdots 12^{m_s}B(-1)10$. The next $m_2$ applications of $\nu_1$ remove the $m_2$ copies of 2 from $12^{m_2}$, one at a time, and put $m_2$ copies of 1 at the end. Next, the 1 from $12^{m_2}$ is removed and a 0 is added to the end. At this point, $\nu_1^{3+m_2}([v]) = 012^{m_2} \cdots 12^{m_s}B(-1)012^{m_2}0$. This pattern now continues: in the next $m_2 + 1$ steps, $\nu_1$ gradually removes $12^{m_3}$ from the front and adds $1^{m_2}0$ to the end. Eventually, we reach $[0B(-1)012^{m_2}01^{m_3} \cdots 1^{m_s}0]$. Next, $\nu_1$ removes each
Remark 3.13. For each application of $\text{nu}$ in Lemma 3.12, doing $\text{nu}_1$ weakly increases the value of $\min_\Delta$, while doing $\text{nu}_2$ strictly decreases the value of $\min_\Delta$. This remark allows us to identify the $\text{nu}_1$-initial objects in $\text{nu}$-chains built from iteration of Lemma 3.12, simply by finding descents in the $\min_\Delta$-profile of such an $\text{nu}$-chain. More precisely, for all objects $\text{nu}^i([v])$ mentioned in the lemma with $i > 0$, $\text{nu}^i([v])$ is $\text{nu}_1$-initial if and only if $\text{nu}^i([v])$ is reached by a $\text{nu}_2$-step if and only if $\min_\Delta(\text{nu}^{i-1}([v])) > \min_\Delta(\text{nu}^i([v]))$.

Example 3.14. Consider the input $v = 0012221122$, which matches the template in Lemma 3.12 with $m_1 = 3$, $m_2 = 0$, $m_3 = 2$, $B = \emptyset$, and $L = 10$. By Lemma 3.12(c), $\text{nu}^2([v]) = 0012112211$. Now, Lemma 3.12(b) applies to input 0012112211, giving $\text{nu}^4([v]) = 0011222110$. Now, Lemma 3.12(a) applies to input 0011222110 of length 10, giving $\text{nu}^{14}([v]) = 001221110101$. Finally, one more application of Lemma 3.12(c) gives $\text{nu}^{16}([v]) = 001111010111 = [0B_\mu] = \text{TI}(\mu)$ for $\mu = 3321^4$. The $\min_\Delta$-profile of the chain from $[v]$ to $\text{TI}(\mu)$ is 10, 11, 10, 11, 10, 11, 10, 12, 11. Based on the descents in this profile, the $\text{nu}_1$-initial objects in this chain are $\text{nu}^i([v])$ for $i = 0, 2, 4, 16$. Therefore, $S_0(v) = v$, $S_1(v) = 0012112211$, $S_2(v) = 0011222110$, $S_3(v) = 001111010111$, $L_0(v) = L_1(v) = L_2(v) = 10$, and $L_3(v) = 11$.

Iterating Lemma 3.12 gives the following result.

Lemma 3.15. Let $v = 0012^m_112^m_2 \cdots 12^m_s B$ be a Dyck vector of length $L = \min_\Delta(v)$ where $s \geq 1$, $m_i \geq 0$ for all $i$, $m_s > 0$, and $B$ is a binary vector.

(a) If $m_1$ is odd, then $\text{nu}^{m_1+1}([v]) = [0012^{m_2}_2 \cdots 12^{m_s}_s B 1^{m_1-1}011]$. The $\min_\Delta$-profile of $\text{nu}^i([v]) : 0 \leq i \leq m_1 + 1$ is $L$ followed by $(m_1 + 1)/2$ copies of $L + 1, L$.

(b) If $m_1$ is even and $s = 1$, then $\text{nu}^{m_1}([v]) = [001B_1^{m_1}]$, which is $\text{TI}(\mu)$ for some $\mu$. The $\min_\Delta$-profile of $\text{nu}^i([v]) : 0 \leq i \leq m_1$ is $L$ followed by $m_1/2$ copies of $L + 1, L$. 

symbol of $B$ and puts the corresponding decremented symbol at the end. Since $L - 1$ symbols of $v$ have now cycled to the end, we have reached $\text{nu}^{L-1}([v]) = [0(1)01^{m_2}01^{m_3}0 \cdots 1^{m_s}0B^-]$. All powers $\text{nu}^i([v])$ computed so far have representatives of length $L$ with smallest entry $-1$, which implies $\min_\Delta(\text{nu}^i([v])) = L + 1$ for $1 \leq i < L$. The reduced representatives for $\text{nu}^i([v])$ all begin with $01$ and have length $L + 1$. In particular, $\text{nu}^{L-1}([v]) = [01012^m_112^{m_3}1 \cdots 2^{m_s}1B]$. Using this representative, we can do $\text{nu}_1$ one more time to reach $\text{nu}^{L}([v]) = [01012^{m_2}12^{m_3} \cdots 12^{m_s}_s 1B0]$, which also has $\min_\Delta$ equal to $L + 1$.

(b) Given $m_1 = 1$, $\text{nu}([v]) = \text{nu}_1([v]) = [012112^{m_2} \cdots 12^{m_s}_s B(-1)]$. As in (a), this Dyck class has $\min_\Delta$ equal to $L + 1$. We continue by applying $\text{nu}_2$ [namely, rule 3.1(b) with $k = 1$] to get $\text{nu}_2^2([v]) = [012^{m_2} \cdots 12^{m_s}_s B01]$, which has length and $\min_\Delta$ equal to $L$.

(c) Given $m_1 \geq 2$, $\text{nu}([v]) = \text{nu}_1([v]) = [012^m_212^m_2 \cdots 12^{m_s}_s B(-1)]$, which has $\min_\Delta$ equal to $L + 1$. We continue by applying $\text{nu}_2$ (rule 3.1(a) with $h = 2$) to get $\text{nu}_2^2([v]) = [012^{m_1-1}_12^{m_2} \cdots 12^{m_s}_s B1^2]$, which has length and $\min_\Delta$ equal to $L$. 

\(\Box\)
(c) If $m_1$ is even and $s > 1$, then $\text{NU}^{m_1+L}([v]) = [0012^{m_2} \ldots 12^{m_s} 1B1^{m_1}]$. The $\min_{\Delta}$-profile of $(\text{NU}^i([v]) : 0 \leq i \leq m_1 + L)$ is $L$ followed by $m_1/2$ copies of $L + 1, L$, followed by $(L + 1)^L$.

(d) For all $\text{NU}^i([v])$ mentioned in (a) through (c) with $i > 0$, $\text{NU}^i([v])$ is $\text{NU}_1$-initial if and only if $\min_{\Delta}(\text{NU}^{i-1}([v])) > \min_{\Delta}(\text{NU}^i([v]))$, which occurs precisely for the even $i \leq m_1 + 1$.

(e) The $\text{NU}_1$-initial objects in (d) are the powers $\text{NU}^i([v]) = [0012^{m_1-i}12^{m_2} \ldots 12^{m_s} 1B1]$ for even $i \leq m_1$ in cases (a) through (c), along with $\text{NU}^{m_1+1}([v])$ in case (a).

Proof. To prove (a), first apply Lemma 3.12(c) a total of $(m_1 - 1)/2$ times. The net effect is to remove $m_1 - 1$ copies of 2 from $12^{m_1}$ and add $1^{m_1-1}$ to the end. Now, $m_1$ has been reduced to 1, so Lemma 3.12(b) applies. We do $\text{NU}$ twice more, removing 12 from the front and adding 01 to the end. The claims about $\min_{\Delta}$ also follow from Lemma 3.12. Part (b) follows similarly, by applying Lemma 3.12(c) $m_1/2$ times. At this point, all 2s have been removed (since $s = 1$), so we have reached a binary Dyck vector representing some TI($\mu$). In part (c), we find $\text{NU}([v]), \ldots, \text{NU}^{m_1}([v])$ using Lemma 3.12(c). Since $m_1$ has now been reduced to 0 but another 2 still remains, we can find the next $L$ powers using Lemma 3.12(a). Part (d) follows from Remark 3.13, since all $\text{NU}$-powers computed in parts (a) through (c) were found using Lemma 3.12. Part (e) follows from part (d) and iteration of Lemma 3.12(c).

Example 3.16. Given $v = 001222212211121221$, let us compute the $\text{NU}$-chain starting at $[v]$, the $\min_{\Delta}$-profile of this chain, and the reduced vectors $S_j(v)$.

To start, apply Lemma 3.15 to input $v$, so $L = 18$, $s = 6$, $m_1 = 4$, $m_2 = 2$, $m_3 = m_4 = 0$, $m_5 = 1$, $m_6 = 2$, and $B = 1$. Part (c) of the lemma says $\text{NU}^{22}([v]) = [001221111221\text{60}]$, and the $\min_{\Delta}$-profile of the $\text{NU}$-chain from $[v]$ to $\text{NU}^{22}([v])$ is 18, 19, 18, 19, 18, 19, 18, 19. To continue, apply the lemma to input 00122111122160, so now $L = 19$, $s = 5$, $m_1 = 2$, $m_2 = m_3 = 0$, $m_4 = 1$, $m_5 = 2$, and $B = 1\text{60}$. Part (c) applies again, giving $\text{NU}^{22+21}([v]) = [00111221221\text{70110}]$, and the $\min_{\Delta}$-profile from $\text{NU}^{23}([v])$ through $\text{NU}^{43}([v])$ is 20, 19, 20, 19. At the next stage, the input to the lemma is 0011122122170110, with $L = 20$, $s = 4$, $m_1 = m_2 = 0$, $m_3 = 1$, $m_4 = 2$, and $B = 1\text{70110}$. We reach $\text{NU}^{63}([v]) = [0011221\text{801100}]$ and append 2120 to the $\min_{\Delta}$-profile. Next, we continue to $\text{NU}^{84}([v]) = [001221\text{901100}]$ and append 2221 to the $\min_{\Delta}$-profile. At the next step, Lemma 3.15(a) applies, taking us to $\text{NU}^{86}([v]) = [001221\text{90110000}]$ and appending 23, 22 to the $\min_{\Delta}$-profile. Finally, we use part (b) of the lemma, reaching $\text{NU}^{88}([v]) = [0011\text{9011000011}]$ and appending 23, 22 to the $\min_{\Delta}$-profile. We have reached the Dyck class $[0B_\mu] = \text{TI}(\mu)$ where $\mu = (6^221^{10})$.

The full $\min_{\Delta}$-profile from $[v]$ to $\text{TI}(\mu)$ is 18, 19, 18, 19, 18, 19, 18, 19, 18, 19, 20, 19, 21, 20, 22. We find the $S_j(v)$ by applying Lemma 3.15(e) to each of the input vectors used in the previous paragraph. We obtain $S_0(v) = v$, $S_1(v) = 001221221112122111$, $S_2(v) = 001122111212211111$, etc.
Theorem 3.17. Let $v = 0012^n 12^{n_1} \cdots 12^{n_r} C$ be a Dyck vector where $r \geq 0$, $n_i \geq 0$ for all $i$, $n_r > 0$, and $C$ is a binary vector. For $0 \leq i \leq r + 1$, let $p_i$ be the number of even integers in the list $n_0, \ldots, n_{i-1}$.

(a) The NU-chain starting at $[v]$ contains all Dyck classes $[v^{(0)}] = [v], [v^{(1)}], \ldots, [v^{(r+1)}]$, where
\[
v^{(i)} = 0012^{n_i} \cdots 12^{n_r} 1^{p_i} C 1^{2[n_0/2]} 01^{n_0 \mod 2} \cdots 1^{2[n_{i-1}/2]} 01^{n_{i-1} \mod 2}
\]
for $0 \leq i \leq r$, and
\[
v^{(r+1)} = 001^{p_{r+1}} C 1^{2[n_0/2]} 01^{n_0 \mod 2} \cdots 1^{2[n_{r-1}/2]} 01^{n_{r-1} \mod 2} C 1^{2[n_r/2]} (01)^{n_r \mod 2}.
\]

We have $[v^{(r+1)}] = TI(\mu)$ for some partition $\mu$.

(b) Put $L = \text{len}(v^{(i)}) = \min_\Delta(v^{(i)})$. For $i \leq r$, the $\min_\Delta$-profile of the part of the NU-chain from $[v^{(i)}]$ through $[v^{(i+1)}]$ always starts $L, (L + 1, L)^{[n_i/2]}$; this is followed by $(L + 1)^2$ if $n_i$ is even and $i < r$. For $i = r + 1$, the $\min_\Delta$-profile of the NU-chain from $TI(\mu)$ onward is $L(L + 1)^L(L + 2)^{L+1} \cdots$.

(c) For all $i > 0$, $\text{NU}^i([v])$ is a NU-1-initial object if and only if $\min_\Delta(\text{NU}^{i-1}([v])) > \min_\Delta(\text{NU}^i([v]))$.

(d) The reduced vectors $S_j(v)$ representing the NU-1-initial objects in the NU-chain starting at $[v]$ are: $v$ itself, if $C$ is empty or ends in 1; $v^{(i)}$, for each $i > 0$, such that $n_{i-1}$ is odd; and all vectors
\[
0012^{n_i-2c} 12^{n_{i+1}} \cdots 12^{n_r} 1^{p_i} C 1^{2[n_0/2]} 01^{n_0 \mod 2} \cdots 1^{2[n_{i-1}/2]} 01^{n_{i-1} \mod 2} 1^{2c},
\]
where $0 \leq i \leq r$ and $0 < 2c \leq n_i$. The final vector (3.3) is $S_j(v)$.

(e) Each $S_j(v)$ except $S_j(v)$ contains a 2. For $j > 0$, each $S_j(v)$ is a reduced Dyck vector of the form 00X1 with X ternary. No other reduced representatives of classes in the chain from $\text{NU}^i([v])$ to $[S_j(v)]$ have this form. For any Dyck class $\delta$ in the NU-chain starting at $[v]$, $\min_\Delta(\delta) < \min_\Delta(\text{NU}(\delta))$ if and only if the reduced Dyck vector for $\delta$ begins with 00.

(f) The $\min_\Delta$-profile from $[S_j(v)]$ to just before $[S_{j+1}(v)]$ is a prefix of $L_j(L_j + 1)^{L_j+1} \cdots$. Each $[S_j(v)]$ is immediately preceded (if $j > 0$) and followed by an object with $\min_\Delta$ equal to $L_j + 1$. Hence, $L_0 \leq L_1 \leq \cdots \leq L_J$.

(g) For $0 \leq j < J$, $\text{area}(S_{j+1}(v)) = \text{area}(S_j(v)) - 2$.

Proof. We prove (a) and (b) by induction on $i$, by iterating Lemma 3.15. The NU-chain starts at $[v] = [v^{(0)}]$. For the induction step, fix $i$ with $0 \leq i \leq r$, and assume the NU-chain has reached $[v^{(i)}]$. Lemma 3.15 applies taking $v$ there to
be \( v^{(i)} \), taking \( m_1, \ldots, m_s \) there to be \( n_i, \ldots, n_r \), and taking \( B \) there to be the part of \( v^{(i)} \) starting with \( 1^p \). The lemma describes how repeated action by \( \nu \) removes \( 12^n \) from the front of \( v^{(i)} \) and adds new symbols further right. Assume \( i < r \) first. If \( n_i \) is odd, then we add \( 1^{n_i-1}01 \) on the right end [Lemma 3.15(a)]. If \( n_i \) is even (possibly zero), then we add a 1 immediately after \( 2^n \), and add \( 1^n0 \) on the right end [Lemma 3.15(c)]. Since \( p_{i+1} = p_i \) for \( n_i \) odd and \( p_{i+1} = p_i + 1 \) for \( n_i \) even, we obtain the correct vector \( v^{(i+1)} \) in both cases. The analysis for \( i = r \) is similar, but now we need Lemma 3.15(b) if \( n_r \) is even. In that case, we still add a new 1 before \( 1^p \) (in agreement with \( p_{r+1} = p_r + 1 \)), but we only add \( 1^n \) (not \( 1^n0 \)) at the right end. This explains the term \( (01)_{r} \mod 2 \) in (3.3). Part (b) of the theorem follows (in all cases just discussed) from the descriptions of the \( \min_\Delta \)-profiles in Lemma 3.15. Since \( v^{(r+1)} \) is a binary vector starting with 00 and ending with 1, \([v^{(r+1)}]\) = \( T\iota(\mu) \) for some partition \( \mu \). The \( \min_\Delta \)-profile of the \( \nu \)-tail starting here is given by Theorem 2.16.

Parts (c) and (d) of the theorem follow from parts (d) and (e) of Lemma 3.15. The first two statements in part (e) of the theorem follow from the explicit description in 3.17(d). The next claim in (e) follows by checking that the output of each \( \nu_1 \)-step in the proof of Lemma 3.12 never has reduced representative 00X1 with \( X \) ternary. The last assertion in (e) is verified similarly, also using Theorem 2.17 to check those \( \delta \) in \( \text{TAIL}(\mu) \). Part (f) follows from parts (b) and (c), recalling that \( L_j = \text{len}(S_j(v)) = \min_\Delta([S_j(v)]) \), since \( S_j(v) \) is reduced.

Part (g) also follows from the computations in Lemma 3.12. If part (b) or part (c) of that lemma applies to input \( S_j(v) \), then \( S_{j+1}(v) \) is the reduced representative of \( \nu^2([S_j(v)]) \). By that lemma and Theorem 3.6, \( L_{j+1} = L_j \), \( \text{dinv}(S_{j+1}(v)) = \text{dinv}(S_j(v)) + 2 \), \( \text{defc}(S_{j+1}(v)) = \text{defc}(S_j(v)) \), and \( \text{area}(S_{j+1}(v)) = \text{area}(S_j(v)) - 2 \). On the other hand, suppose Lemma 3.12(a) applies to input \([S_j(v)]\) for \( c > 0 \) successive times, which happens when an even \( m_i \) has been reduced to zero and is followed by \( m_{i+1} = \cdots = m_{i+c-1} = 0 < m_{i+c} \). In this situation, the \( \min_\Delta \)-profile mentioned in (f) is the prefix \( L_j(L_j+1)^{L_j} \cdots (L_j+c)^{L_j+c-1}(L_j+c+1) \), and the next object has reduced representative \( S_{j+1}(v) \) with length \( L_{j+1} = L_j + c \). Going from \( S_j(v) \) to \( S_{j+1}(v) \), we see that the deficit has not changed, the length has increased from \( L_j \) to \( L_j + c \), and \( \text{dinv} \) has increased by \( L_j + (L_j+1) + \cdots + (L_j+c-1) + 2 = \binom{L_j+c}{2} - \binom{L_j}{2} + 2 \). Using \( \text{area} + \text{dinv} + \text{defc} = \binom{\text{len}}{2} \), it follows that \( \text{area}(S_{j+1}(v)) = \text{area}(S_j(v)) - 2 \). \( \square \)

We work through a detailed example illustrating Theorem 3.17 in Sect. 6.5.

4. Flagpole Partitions

The following strange-looking definition will be explained by Lemma 4.2.

**Definition 4.1.** A flagpole partition is an integer partition \( \mu \), such that \( |\mu| + 8 \leq 2\min_\Delta(T\iota_2(\mu)) \).
Thus, (a) is false for $2 \lambda$.

$\lambda$ has the form $00v_1v_2\ldots v_n$ with $v_r \geq 1$ for all $r$. Then, $v_r$ does not begin with $00$.

If $m \geq 2$, then (using $\lambda = \langle b, 1^c \rangle$) is a nonzero hook. Define $v(\lambda, a, 0) = 0012^{a}B^{+}_{\lambda}$ and $v(\lambda, a, 1) = 0012^{a-1}B^{+}_{\lambda}$ for all partitions $\lambda$ and integers $a \geq 2$. For example, $v(33111), 3, 0) = 0012212221$ and $v((4421), 3, 1) = 001221212122122$.

Lemma 4.2. For all Dyck vectors $v$ listed in Theorem 3.10(b), the following conditions are equivalent:

(a) $\text{defc}(v) + 8 \leq 2 \text{len}(v)$;
(b) there exists a partition $\lambda$ and an integer $a \geq a(0)$ with $v = v(\lambda, a, 0)$ or $v = v(\lambda, a, 1)$.

Proof. We look at seven cases based on the possible forms of $v$.

Case 1. $v = 01^{n}0X^{2}$ is type 1 where $0 < m \leq n$ and $X$ has at least two 1s. Then, $v$ has the form $AB12^n$, where $A = 01^{n}0^1$ has at least two 0s and at least two 1s. By Lemma 2.5(a), $\text{defc}(v) \geq 2 \text{len}(B) + \text{defc}(A12^n)$. By Proposition 2.3, $\text{defc}(A12^n) = \text{defc}(01^{n}0^{1}12^n) = r(2 + n) + (m + 1)n \geq 2r + 2n + mn$. Since $m \leq n$, we get

$$\text{defc}(v) + 8 \geq 2 \text{len}(B) + 2r + 2n + 6 + (m^2 + 1) \geq 2\text{len}(B) + r + n + 3 + m = 2\text{len}(v).$$

Thus, condition 4.2(a) is false for $v$, and condition 4.2(b) is also false, since $v$ does not begin with $00$.

Case 2. $v = 01^{n}0X^{2}$ is type 1 where $0 < m \leq n$ and $X$ has only one 1. Then, $v$ has the form $01^{m}0^{1}12^n$, $\text{len}(v) = m + r + n + 2$, and $\text{defc}(v) = r(1 + n) + mn$. Here, $\text{defc}(v) + 8 - 2 \text{len}(v)$ simplifies to $(m - 2)(n - 2) + (n - 1)r$. If $m \geq 2$, then (using $n \geq m \geq 2$), we get $(m - 2)(n - 2) + (n - 1)r > 0$. If $m = 1$, this expression becomes $(n - 1)(r - 1) + 1 > 0$, which is also positive. Thus, (a) is false for $v$, and (b) is false, since $v$ does not begin with $00$.

Case 3. $v = 00X^{2}$ is type 1 where $m = 0$ and $X$ has at least one 0. Then, $v$ has the form $00A0B2^n$, $\text{len}(v) = \text{len}(A) + \text{len}(B) + n + 4$, and Lemma 2.5(b) gives $\text{defc}(v) \geq 2 \text{len}(A) + 2 \text{len}(B) + 2n + 1$. Therefore, $\text{defc}(v) + 8 \geq 2 \text{len}(v)$, and (a) is false for $v$. Condition (b) is also false, since $v$ has too many 0s.

Case 4. $v = 00X^{2}$ is type 1 where $m = 0$ and $X$ contains no 0. Then, there exist a partition $\lambda$ and positive integers $c, a$, such that $v = 001^{c}2^{a}B^{+}_{\lambda}$. We compute $\text{len}(v) = 2 + c + a + \lambda_1 + \ell(\lambda)$. Using Proposition 2.3 to find $\text{defc}(v)$, the second 0 in $v$ contributes $\text{len}(v) - 2$, each 1 in $1^c$ except the
first contributes \(a + \ell(\lambda)\), and the 1s in \(B^+_{\lambda}\) pair with later 2s to contribute \(n_1 + 2n_2 + \cdots + rn_r = |\lambda|\). In total, we get

\[
8 + \text{defc}(v) = \text{len}(v) + 6 + (c - 1)(a + \ell(\lambda)) + |\lambda|.
\] (4.1)

Consider the subcase \(c \geq 2\). Here, condition (b) is false for \(v\), since \(v\) begins with 0011. On the other hand, because \(a \geq 1\), we have \((c - 2)a > c - 4\), and hence, \(6 + (c - 1)a > 2 + c + a\). By (4.1)

\[
8 + \text{defc}(v) > \text{len}(v) + 2 + c + a + \ell(\lambda) + |\lambda| \geq 2\text{len}(v),
\] (4.2)

so condition (a) is false for \(v\). In the subcase \(c = 1\), condition (b) is true for \(v\) if \(a \geq a_0(\lambda)\). On the other hand, using (4.1) with \(c = 1\), (a) is true for \(v\) if \(6 + |\lambda| \leq \text{len}(v)\) if \(6 + |\lambda| \leq 3 + a + \lambda_1 + \ell(\lambda)\) if \(a \geq a_0(\lambda)\). Thus, (a) and (b) are equivalent in this subcase.

**Case 5.** \(v = 0^nY21^m\) is type 2 and contains at least three 0s. Then, \(v = 00.A0B12p1^m\) where \(p, m > 0\), and \(\text{len}(v) = \text{len}(A) + \text{len}(B) + 4 + p + m\). Lemma 2.5(b) gives \(\text{defc}(v) + 3 \geq 2\text{len}(A) + 2\text{len}(B) + 2(p + m) + 9 > 2\text{len}(v)\). Therefore, conditions (a) and (b) are equivalent in this subcase.

**Case 6.** \(v = 0^nY21^m\) is type 2 with exactly two 0s, which forces \(n = 2\) and \(m = 1\). Therefore, there exist a partition \(\lambda\) and integers \(c \geq 1, a \geq 2\) with \(v = 001c2a - 1B^+_{\lambda}1\). Similarly to Case 4, we compute \(\text{len}(v) = 2 + c + a + \lambda_1 + \ell(\lambda)\) and

\[
8 + \text{defc}(v) = \text{len}(v) + 6 + (c - 1)(a - 1 + \ell(\lambda)) + |\lambda|.
\] (4.3)

In the subcase \(c \geq 2\), (b) is false for \(v\), since \(v\) begins with 0011. On the other hand, \((c - 2)(c - 2) \geq 0\) in this subcase, so \(6 + (c - 1)(a - 1) > 2 + c + a\). Using this in (4.3) yields (4.2), so (a) is false for \(v\). In the subcase \(c = 1\), (b) is true for \(v\) if \(a \geq a_0(\lambda)\). By (4.3) with \(c = 1\), (a) is true for \(v\) if \(6 + |\lambda| \leq \text{len}(v)\) if \(a \geq a_0(\lambda)\) (as in Case 4). Therefore, (a) and (b) are equivalent in this subcase.

**Case 7.** \(v\) is a type 3 vector. Then, condition (b) is false for \(v\), since \(v\) contains no 2. If \(v = 0^n1^n\) with \(n \geq 2\), then \(\text{defc}(v) = (n - 1)n, \text{len}(v) = 2n,\) and it is routine to check \((n - 1)n + 8 > 2n\). If \(v = 0^n1^{n-1}\) with \(n \geq 2\), then \(\text{defc}(v) = (n - 1)^2, \text{len}(v) = 2n - 1,\) and \((n - 1)^2 + 8 > 2n - 1\) holds. If \(v = 0\), then \(\text{defc}(v) = 0, \text{len}(v) = 1,\) and \(8 > 2\) holds. Therefore, condition (a) is false for all type 3 vectors \(v\).

**Remark 4.3.** As seen in the proof, \(v(\lambda, a, 0)\) and \(v(\lambda, a, 1)\), both have length \(L = a + 3 + \lambda_1 + \ell(\lambda)\) and deficit \(a + 1 + \lambda_1 + \ell(\lambda) + |\lambda| = L + |\lambda| - 2\). We also saw that \(a \geq a_0(\lambda)\) if and only if \(L \geq |\lambda| + 6\). Since \(\text{area}(v(\lambda, a, 1)) = \text{area}(v(\lambda, a, 0)) - 1\), it follows that \(\text{dinv}(v(\lambda, a, 1)) = \text{dinv}(v(\lambda, a, 0)) + 1\). Thus, \(v(\lambda, a, 0)\) and \(v(\lambda, a, 1)\) have \(\text{dinv}\) of opposite parity. More precisely, one readily checks that

\[
\text{area}(v(\lambda, a, \epsilon)) = 2L - \lambda_1 - 5 - \epsilon \quad \text{and} \quad \text{dinv}(v(\lambda, a, \epsilon)) = \left(\frac{L}{2}\right) - 3L - |\lambda| + \lambda_1 + 7 + \epsilon.
\] (4.4)

**Theorem 4.4.** A partition \(\mu\) is a flagpole partition if and only if there exist a partition \(\lambda\) and an integer \(a \geq a_0(\lambda)\), such that \(\text{TI}_2(\mu) = [v(\lambda, a, 0)]\) or \(\text{TI}_2(\mu) = [v(\lambda, a, 1)]\).
Proof. Given any partition \( \mu \), we know \( \text{TI}_2(\mu) = [v] \) for some vector \( v \) listed in Theorem 3.10(b). Since these \( v \) are all reduced, \( \min_\Delta(\text{TI}_2(\mu)) = \text{len}(v) \). Also, \( \text{defc}(v) = \text{defc}(\text{TI}_2(\mu)) = \text{defc}(\text{TI}(\mu)) = |\mu| \). The theorem now follows from Definition 4.1 and Lemma 4.2. \( \square \)

Definition 4.5. For any integer partition \( \mu \), such that \( \text{TI}_2(\mu) = [v(\lambda, a, \epsilon)] \), we call \( \lambda \) the flag type of \( \mu \) and write \( \lambda = ftype(\mu) \).

4.2. Representations of Flagpole Partitions

Theorem 4.4 leads to some useful representations of flagpole partitions involving the flag type and other data.

Lemma 4.6. Let \( F \) be the set of flagpole partitions, and let \( G \) be the set of triples \((\lambda, a, \epsilon)\), where \( \lambda \) is any integer partition, \( a \) is an integer with \( a \geq a_0(\lambda) \), and \( \epsilon \) is 0 or 1. There is a bijection \( \Phi : F \to G \), such that \( \Phi(\mu) = (\lambda, a, \epsilon) \) if and only if \( \text{TI}_2(\mu) = [v(\lambda, a, \epsilon)] \).

Proof. For a given flagpole partition \( \mu \), there exists \((\lambda, a, \epsilon) \in G \) with \( \text{TI}_2(\mu) = [v(\lambda, a, \epsilon)] \) by Theorem 4.4. This triple is uniquely determined by \( \mu \), since no two Dyck vectors in Theorem 3.10(b) are equivalent. Therefore, \( \Phi \) is a well-defined function from \( F \) into \( G \). To see \( \Phi \) is bijective, fix \((\lambda, a, \epsilon) \in G \). Then, \([v(\lambda, a, \epsilon)] = \text{TI}_2(\mu) \) for some partition \( \mu \) by Theorem 3.10, and \( \mu \) is a flagpole partition by Theorem 4.4. Thus, \( \Phi \) is surjective. Since we can recover \( \text{TI}(\mu) \) and \( \mu \) itself from \( \text{TI}_2(\mu) \), \( \Phi \) is injective. \( \square \)

Remark 4.7. Theorem 3.17 provides an explicit formula for \( \mu = \Phi^{-1}(\lambda, a, \epsilon) \). Apply the theorem to the vector \( v = v(\lambda, a, \epsilon) \) representing \( \text{TI}_2(\mu) \). This \( v \) has the required form \( v = 0012^{a_0}12^{n_1} \cdots 12^{n_r}C \), where \( n_0 = a - \epsilon \), \( n_i \) is the number of \( \text{s} \)s in \( \lambda \) for \( 1 \leq i \leq r \), and \( C = 1^\epsilon \). From (3.3), the reduced vector \( 0B_\mu \) for \( \text{TI}(\mu) \) is given explicitly as

\[
001^{p+\epsilon}1^{2[(a-\epsilon)/2]}01^{(a-\epsilon) \mod 2}1^{2[n_1/2]}01^{n_1 \mod 2} \cdots 1^{2[n_{r-1}/2]}01^{n_{r-1} \mod 2} \mod 2,
\]

where \( p \) is the number of even integers in the list \( a - \epsilon, n_1, \ldots, n_r \). We can then recover \( \mu \) from \( B_\mu \) via Definition 2.14. In particular, the number of parts in \( \mu \) equal to 1 is the length of the first block of \( \text{s} \)s in (4.5), namely \( p + \epsilon + 2[(a-\epsilon)/2] \), which is at least \( a - 1 \geq a_0(\lambda) - 1 \). Informally, this shows that a flagpole partition \( \mu \) must end in many \( \text{s} \), so that the Ferrers diagram of \( \mu \) looks like a flag flying on a pole.

Example 4.8. Given \( \lambda = \langle 4433111 \rangle \), let us find \( \mu = \Phi^{-1}(\lambda, 10, 0) \). Here, \( a = 10 \), \( \epsilon = 0 \), \( \text{TI}_2(\mu) = [0012^{10}12221122122] \), \( n_0 = 10, n_1 = 3, n_2 = 0, n_3 = 2, n_4 = 2, C = 0, \) and \( p = 4 \). Using (4.5), we get \( 0B_\mu = 001^{4}(10^{10})(1010)(0)(110)(11) = 001^{4}01^{2}0101^{3}01^{2} \). Therefore, \( \mu = \langle 52^{4}2^{1}2^{1}1^{4} \rangle \). When computing \( \Phi^{-1}(\lambda, 10, 1) \), we get \( n_0 = 9, C = 1, p = 3 \), \( 0B_\mu = 001^{3}(1801)(1101)(0)(110)(11) = 001^{3}01^{2}0101^{3}01^{2} \), and the answer is \( \langle 52^{4}2^{3}1^{2}1^{4} \rangle \). More generally, for any \( a \geq a_0(\lambda) = 9 \), we see that \( \Phi^{-1}(\lambda, a, 0) \) is \( \langle 52^{4}2^{3}1^{2}a \rangle \) when \( a \) is even and is \( \langle 52^{4}2^{3}1^{2}a \rangle \) when \( a \) is odd. Also, \( \Phi^{-1}(\lambda, a, 1) \) is \( \langle 52^{4}2^{3}1^{2}a \rangle \) when \( a \) is odd and is \( \langle 52^{4}2^{3}1^{2}a \rangle \) when \( a \) is even. In particular, for all \( b \geq 13 \),...
Theorem 4.9. The number of flagpole partitions of size $n$ is $\sum_{j=0}^{\lfloor(n-4)/2\rfloor} 2p(j)$, where $p(j)$ is the number of integer partitions of size $j$.

Proof. Suppose $\mu$ is a flagpole partition of size $n$ and $\Phi(\mu) = (\lambda, a, \epsilon)$. Then, $TI_2(\mu)$ has deficit $n$ and is represented by $v(\lambda, a, \epsilon)$. By Remark 4.3, $\text{defc}(v(\lambda, a, \epsilon)) = a + 1 + \lambda_1 + \ell(\lambda) + |\lambda|$. Since $a \geq a_0(\lambda)$, the smallest possible value of $\text{defc}(v(\lambda, a, \epsilon))$ is $2|\lambda| + 4$. Thus, $n \geq 2|\lambda| + 4$ and $|\lambda| \leq (n-4)/2$. By reversing this argument, we can construct each flagpole partition of $n$ by making the following choices. Pick an integer $j$ with $0 \leq j \leq (n-4)/2$, and pick $\lambda$ to be any of the $p(j)$ partitions of $j$. Pick the unique integer $a \geq a_0(\lambda)$, such that $a + 1 + \lambda_1 + \ell(\lambda) + |\lambda| = n$. Pick $\epsilon$ to be 0 or 1 (two choices). Finally, define $\mu = \Phi^{-1}(\lambda, a, \epsilon)$. The number of ways to make these choices is $\sum_{0 \leq j \leq (n-4)/2} 2p(j)$. \hfill $\Box$

Remark 4.10. Let $f(n)$ be the number of flagpole partitions of size $n$. It is known [9, (5.26)] that $\sum_{j \leq n} p(j) = \Theta\left(n^{-1/2} \exp(\pi \sqrt{2n/3})\right)$. Using this and Theorem 4.9, we get $f(n) = \Theta\left(n^{-1/2} \exp(\pi \sqrt{n/3})\right)$. Hardy and Ramanujan [5] proved that $p(n) = \Theta\left(n^{-1} \exp(\pi \sqrt{2n/3})\right)$. Therefore, $f(n) = \Theta\left(p(n)^{1/2} n^{(\sqrt{2}-1)/2}\right)$.

The following variation of the bijection $\Phi$ will help us construct global chains indexed by flagpole partitions.

Lemma 4.11. Let $F$ be the set of flagpole partitions, and let $H$ be the set of triples $(\lambda, L, \eta)$, where $\lambda$ is an integer partition, $L$ is an integer with $L \geq |\lambda|+6$, and $\eta$ is 0 or 1. There is a bijection $\Psi : F \rightarrow H$ given by

$$\Psi(\mu) = (\text{ftype}(\mu), \min_{\Delta}(TI_2(\mu)), \text{dinv}(TI_2(\mu)) \mod 2). \tag{4.6}$$

Proof. Given a flagpole partition $\mu \in F$, we know $\Phi(\mu) = [v(\lambda, a, \epsilon)]$ for a unique partition $\lambda = \text{ftype}(\mu)$, $a \geq a_0(\lambda)$, and $\epsilon \in \{0, 1\}$, namely for $(\lambda, a, \epsilon) = \Phi(\mu)$. Since $v = v(\lambda, a, \epsilon)$ is a reduced Dyck vector, $\min_{\Delta}(TI_2(\mu)) = \text{len}(v)$. By Remark 4.3 and the definition of $a_0(\lambda)$, $\text{len}(v) = a + 3 + \lambda_1 + \ell(\lambda) \geq |\lambda|+6$. Thus, $\Psi$ is a well-defined function mapping into the codomain $H$.

To see that $\Psi$ is invertible, consider $(\lambda, L, \eta) \in H$. Define $a = L - 3 - \lambda_1 - \ell(\lambda)$, and note $L \geq |\lambda|+6$ implies $a \geq a_0(\lambda)$. Since $\text{dinv}(v(\lambda, a, 1)) = \text{dinv}(v(\lambda, a, 0)) + 1$, there is a unique $\epsilon \in \{0, 1\}$ with $\text{dinv}(v(\lambda, a, \epsilon)) = \eta$. Now, let $\mu = \Phi^{-1}(\lambda, a, \epsilon)$ be the unique flagpole partition with $TI_2(\mu) = [v(\lambda, a, \epsilon)]$. It is routine to check that the map $(\lambda, L, \eta) \mapsto \mu$ defined in this paragraph is the two-sided inverse of $\Psi$. \hfill $\Box$

Example 4.12. Let us find $\Psi(\mu)$ for $\mu = \langle 322111 \rangle$. From (3.1), $TI_2(\mu) = [001221222] = [v(111), 2, 0]$. Since $\min_{\Delta}(TI_2(\mu)) = 9$ and $\text{dinv}(TI_2(\mu)) = 14$ is even, $\Psi(\mu) = \langle 111, 9, 0 \rangle$.

\[\langle 55443221^b \rangle\) and $\langle 55443221^{b-2} \rangle$ are flagpole partitions of flag type $\lambda$. A similar pattern holds for other choices of $\lambda$.

Next, we use the bijection $\Phi$ to enumerate flagpole partitions.

Next, we compute \( \Psi^{-1}(\langle 0 \rangle, L, 0) \) for each \( L \geq 6 \). For \( a \geq a_0(\langle 0 \rangle) = 3 \), we have \( v(\langle 0 \rangle, a, \epsilon) = 0012^{a-\epsilon}1^\epsilon \). By (3.3), \( \Phi^{-1}(\langle 0 \rangle, a, \epsilon) \) is \( \langle 21^{a-1} \rangle \) if \( a - \epsilon \) is odd and \( \langle 1^{a+1} \rangle \) if \( a - \epsilon \) is even. Since we need \( v(\langle 0 \rangle, a, \epsilon) \) to have length \( L \), we take \( a = L - 3 \). Thus, \( \Psi^{-1}(\langle 0 \rangle, L, 0) \) and \( \Psi^{-1}(\langle 0 \rangle, L, 1) \) are \( \langle 1^{L-2} \rangle \) and \( \langle 21^{L-4} \rangle \) in some order. Using (4.4) to compute \( \text{dinv}(v(\langle 0 \rangle, a, \epsilon)) \), one readily checks that \( \Psi^{-1}(\langle 0 \rangle, L, 0) \) is \( \langle 1^{L-2} \rangle \) when \( L \mod 4 \in \{0,1\} \) and is \( \langle 21^{L-4} \rangle \) when \( L \mod 4 \in \{2,3\} \).

5. Review of the Local Chain Method

This section reviews the local chain method from [3], which gives a convenient way to prove that two proposed global chains \( C_\mu \) and \( C_{\mu^*} \) satisfy the opposite property \( \text{Cat}_{n,\mu^*}(t,q) = \text{Cat}_{n,\mu}(q,t) \) for all \( n > 0 \). (Recall the definition of \( \text{Cat}_{n,\mu} \) from (1.2).) The idea is to distill the \( \text{min}_\Delta \)-profiles of \( C_\mu \) and \( C_{\mu^*} \) into three finite lists of integers called the \( \text{amh} \)-vectors for \( C_\mu \) and \( C_{\mu^*} \). The infinitely many conditions in the opposite property can be verified through a simple finite computation on the \( \text{amh} \)-vectors. First, we define more carefully what a “proposed global chain” must look like.

**Definition 5.1.** Suppose we are given specific partitions \( \mu \) and \( \mu^* \) of the same size \( k > 0 \) (with \( \mu^* = \mu \) allowed) and two sequences of partitions \( C_\mu \) and \( C_{\mu^*} \). We say that \( C_\mu \) and \( C_{\mu^*} \) have basic required structure iff the following conditions hold:

(a) \( C_\mu \) is a sequence \( (c_\mu(i) : i \geq i_0(\mu)) \) where \( c_\mu(i) \) is a partition with deficit \( k(= |\mu|) \) and \( \text{dinv} i \).

(b) \( C_{\mu^*} \) is a sequence \( (c_{\mu^*}(i) : i \geq i_0(\mu^*)) \) where \( c_{\mu^*}(i) \) is a partition with deficit \( k(= |\mu^*|) \) and \( \text{dinv} i \).

(c) \( C_\mu \) starts at \( \text{dinv} \) value \( i_0(\mu) = \ell(\mu^*) \), and \( C_{\mu^*} \) starts at \( \text{dinv} \) value \( i_0(\mu^*) = \ell(\mu^*) \).

(d) \( C_\mu \) ends with the sequence \( \text{tail}(\mu) \), and \( C_{\mu^*} \) ends with the sequence \( \text{tail}(\mu^*) \).

(e) If \( \mu \neq \mu^* \), then \( C_\mu \) and \( C_{\mu^*} \) are disjoint. If \( \mu = \mu^* \), then \( C_\mu = C_{\mu^*} \).

Assume \( C_\mu \) and \( C_{\mu^*} \) have basic required structure. We now review the definition of the \( \text{amh} \)-vectors for these chains. Recall that the \( \text{min}_\Delta \)-profile of \( C_\mu \) is the sequence of integers \( (p_i : i \geq i_0(\mu)) \) where \( p_i = \text{min}_\Delta(c_\mu(i)) \) for each \( i \). Define the descent set \( \text{Des}(\mu) \) to be the set consisting of \( i_0(\mu) \) and all \( i > i_0(\mu) \) with \( p_i > p_{i-1} \). Since the \( \text{min}_\Delta \)-values in \( \text{tail}(\mu) \) form a weakly increasing sequence (Theorem 2.16), \( \text{Des}(\mu) \) is a finite set. The \( a \)-vector for \( \mu \) is the list \((a_1, a_2, \ldots, a_N)\) of members of \( \text{Des}(\mu) \) written in increasing order. The \( h \)-vector for \( \mu \) is \((h_1, h_2, \ldots, h_N)\), where \( h_i = p_{a_i} \) for \( 1 \leq i \leq N \). The \( m \)-vector for \( \mu \) is \((m_1, m_2, \ldots, m_N)\), where \( m_i \geq 0 \) is the largest integer, such that \( p_{a_i} = p_{a_i+1} = \cdots = p_{a_i+m_i} \). This definition means that the \( i \)th ascending run of the \( \text{min}_\Delta \)-profile of \( C_\mu \) starts at \( \text{dinv} \) index \( a_i \) with \( m_i + 1 \) copies of \( h_i \) followed by a different value. We similarly define the \( a \)-vector \((a_1^*, \ldots, a_N^*)\) for \( \mu^* \), the \( h \)-vector \((h_1^*, \ldots, h_N^*)\) for \( \mu^* \), and the \( m \)-vector \((m_1^*, \ldots, m_N^*)\) for...
\( \mu^* \). The next definition lists the conditions needed to decompose \( C_\mu \) into local chains.

**Definition 5.2.** Assume \( C_\mu \) and \( C_{\mu^*} \) have basic required structure. We say that chain \( C_\mu \) has **local required structure** iff the following conditions hold:

(a) \( a_N = \text{dinv}(\text{TII}(\mu)) \).

(b) For \( 1 \leq i < N \), the \( i \)th ascending run of the \( \min_\Delta \)-profile of \( C_\mu \) is some prefix of the staircase sequence \( h_i \), \( h_i + 1 \), \( h_i + 2 \), \( h_i + 3 \), \( h_i + 2 \), \ldots that includes at least one copy of \( h_i + 1 \).

We use analogous conditions to define the local required structure for \( C_{\mu^*} \). Condition (a) means that the last ascending run of the \( \min_\Delta \)-profile for \( C_\mu \) corresponds to \( \text{TII}(\mu) \). In other words, \( \text{TII}(\mu) \) must have a smaller \( \min_\Delta \) value than the preceding object (if any) in \( C_\mu \). By Theorems 2.16 and 5.1 (d), it is guaranteed that \( m_N = 0 \) and condition (b) holds for \( i = N \) (with the prefix being the entire infinite staircase sequence).

**Definition 5.3.** Assume \( C_\mu \) and \( C_{\mu^*} \) have basic and local required structure. We say \( C_\mu \) and \( C_{\mu^*} \) **satisfy the amh-hypotheses** iff the following conditions hold:

(a) The \( h \)-vector for \( C_{\mu^*} \) is the reverse of the \( h \)-vector for \( C_\mu \) (forcing \( N^* = N \)).

(b) The \( m \)-vector for \( C_{\mu^*} \) is the reverse of the \( m \)-vector for \( C_\mu \).

(c) For \( 1 \leq i \leq N \), \( a_i + m_i + k + a_{N+1-i} = \binom{h_i}{2} \), where \( k = |\mu| = |\mu^*| \).

**Theorem 5.4.** [3, Thm. 3.10 and Sec. 4]. **Assume** \( C_\mu \) and \( C_{\mu^*} \) **have basic and local required structure and satisfy the amh-hypotheses. Then, for all** \( n > 0 \),

\[ \text{Cat}_{n,\mu^*}(t, q) = \text{Cat}_{n,\mu}(q, t) \]

All chains we have constructed previously (see [3, Appendix] and [4]) satisfy some additional conditions that we need for the recursive construction in Sect. 6. We list these conditions next.

**Definition 5.5.** Assume \( C_\mu \) and \( C_{\mu^*} \) have basic required structure. We say \( C_\mu \) has **extra required structure** iff the following conditions hold:

(a) The \( h \)-vector \( (h_1, \ldots, h_N) \) is a weakly decreasing sequence followed by a weakly increasing sequence.

(b) For \( i < N \), all values in the \( i \)th ascending run of the \( \min_\Delta \)-profile for \( C_\mu \) are at most \( 1 + \max(h_i, h_{i+1}) \).

(c) For \( i \geq i_0(\mu) \), \( \min_\Delta(c_\mu(i)) < \min_\Delta(c_\mu(i + 1)) \) iff the reduced Dyck vector for \( c_\mu(i) \) is 0 or begins with 00.

(d) \( C_\mu \) contains \( \text{TII}_2(\mu) \) (not just \( \text{TII}(\mu) \)).

We make an analogous definition for \( C_{\mu^*} \). Condition (c) is guaranteed for objects \( c_\mu(i) \) in \( \text{TII}(\mu) \) by Theorem 2.17 (note \( \mu \neq \langle 0 \rangle \) here). If \( \text{TII}_2(\mu) = [v] \) where \( v \) has the form studied in Theorem 3.17, then condition (c) is also guaranteed for objects \( c_\mu(i) \) in \( \text{TII}_2(\mu) \) by part (e) of that theorem.

**Example 5.6.** Let \( \mu = \langle 1^5 \rangle = \mu^* \), and define \( C_\mu = C_{\mu^*} \) as the union of the \( \text{NU}_1 \)-segments starting at \([0012332], [0012222] = \text{TII}_2(\mu), [0012211], \) and \([0011111] \)
= \text{TI}(\mu). Each Dyck class listed here is a NU_1-initial object with deficit 5 = |\mu|.
By computing the NU_1-segments, we see that they assemble to give a chain of partitions with consecutive dinv values starting at 5 = \ell(\mu^*) = \ell(\mu), namely
\[ C_{\langle 5 \rangle} : [0012332], [01234430], [0012222], [01233330], [0012211], [01233220], \text{ followed by } \text{TAIL}(\langle 1^5 \rangle). \]
Thus, \( C_\mu \) has basic required structure. \( C_\mu \) has min\(\Delta\)-profile 78787878798109 \ldots, a-vector \( (5, 7, 9, 11) \), m-vector \( (0, 0, 0, 0) \), and h-vector \( (7, 7, 7, 7) \). We see that \( C_\mu \) has local required structure by inspection of the min\(\Delta\)-profile [in particular, 5.2(a) holds since \( a_N = 11 = \text{dinv(TI(\mu)))} \]. The first two amh-hypotheses are true, since the m-vector and h-vector are palindromes (equal to their own reversals). We check amh-hypothesis (c) for \( i = 1, 2, 3, 4 \) by computing
\[
5 + 0 + 5 + 11 = 7 + 0 + 5 + 9 = 9 + 0 + 5 + 7 = 11 + 0 + 5 + 5 = 21 = \binom{7}{2}.
\]
By Theorem 5.4, \( \text{Cat}_{n, \mu}(t, q) = \text{Cat}_{n, \mu}(q, t) \) for all \( n > 0 \). It is also routine to check that \( C_\mu \) has the extra required structure.

Example 5.7. In Sect. 6, we perform an elaborate construction to build proposed chains for \( \mu = \langle 531^4 \rangle \) and \( \mu^* = \langle 3321^4 \rangle \), which are partitions of size \( k = 12 \). As described in detail later, \( C_\mu \) contains \( \text{TAIL}_2(\mu) \) and has min\(\Delta\)-profile given by

\[
11, 12, (10, 11)^7, 10, 11^{10}, 12, 11, 12^{11}, 13, 12, 13^{12}, 14^{13}, 15^{14}, \ldots,
\]
which is the concatenation of (6.9), (6.3), and (6.6). \( C_{\mu^*} \) contains \( \text{TAIL}_2(\mu^*) \) and has min\(\Delta\)-profile given by

\[
12, 13, 11, 12, (10, 11)^7, 10, 11^{10}, 12, 11, 12^{11}, 13^{12}, 14^{13}, \ldots,
\]
which is the concatenation of (6.10), (6.5), and (6.8). \( C_\mu \) starts at dinv index 7 = \ell(\mu^*), and \( C_{\mu^*} \) starts at dinv index 6 = \ell(\mu). The specific NU_1-initial objects used to make \( C_\mu \) are different from the NU_1-initial objects used to make \( C_{\mu^*} \), so \( C_\mu \) and \( C_{\mu^*} \) are disjoint chains. All the NU_1-initial objects used have deficit \( k = 12 \). Thus, \( C_\mu \) and \( C_{\mu^*} \) have basic required structure and extra structure condition (d).

The m-vectors for \( C_\mu \) and \( C_{\mu^*} \) are identically 0. The a-vector and h-vector for \( C_\mu \) are
\[
a = (7, 9, 11, 13, 15, 17, 19, 21, 23, 35, 48),
\]
\[
h = (11, 10, 10, 10, 10, 10, 10, 10, 10, 11, 12).
\]
The a-vector and h-vector for \( C_{\mu^*} \) are
\[
a^* = (6, 8, 10, 12, 14, 16, 18, 20, 22, 24, 36),
\]
\[
h^* = (12, 11, 10, 10, 10, 10, 10, 10, 10, 10, 11).
\]
It is now routine to verify that \( C_\mu \) and \( C_{\mu^*} \) have local required structure and satisfy the amh-hypotheses. For instance, we check amh-hypothesis (c) for \( i = 1, 2, 3 \) and \( i = 11 \) as follows:
7 + 0 + 12 + 36 = 55 = \binom{11}{2}; \quad 9 + 0 + 12 + 24 = 45 = \binom{10}{2};

11 + 0 + 12 + 22 = 45 = \binom{10}{2}; \quad 48 + 0 + 12 + 6 = 66 = \binom{12}{2}.

By Theorem 5.4, \text{Cat}_{n,\mu^*}(t, q) = \text{Cat}_{n,\mu}(q, t) for all \( n > 0 \). We can also verify the extra required structure for these chains, using the specific objects constructed below to verify 5.5(c) for objects \( c_{\mu}(i) \) preceding the second-order tails. All of these verifications are special cases of general results to be proved later.

6. Constructing Global Chains Indexed by Certain Flagpole Partitions

6.1. Statement of Results

Our ultimate goal (not yet achieved in this paper) is to define \( \mu^* \) and chains \( C_{\mu} \) and \( C_{\mu^*} \) for every integer partition \( \mu \) and to prove that all structural conditions and \textit{amh}-hypotheses in Sect. 5 are true. We hope to reach this goal recursively, constructing chains indexed by partitions \( \mu \) of a given size \( k \) by referring to previously built chains \( C_{\lambda} \) for various partitions \( \lambda \) of size less than \( k \). This section achieves a limited version of this construction, building \( C_{\mu} \) and \( C_{\mu^*} \) for all flagpole partitions \( \mu \) of one particular size \( k \), assuming that all chains \( C_{\lambda} \) indexed by \textit{all} partitions \( \lambda \) of size less than \( k \) are already available. In fact, we prove a much sharper conditional result by keeping track of exactly which smaller chains \( C_{\lambda} \) are needed to build \( C_{\mu} \) and \( C_{\mu^*} \), for each specific flagpole partition \( \mu \). To state our findings, we first define precisely what we mean by a “collection of previously built chains.”

**Definition 6.1.** A \textit{chain collection} is a triple \( \mathfrak{C} = (\mathcal{P}, I, \mathcal{C}) \) satisfying these conditions:

(a) \( \mathcal{P} \) is a collection of integer partitions containing (at a minimum) all partitions of size at most 5.

(b) \( I : \mathcal{P} \to \mathcal{P} \) is a size-preserving involution with domain \( \mathcal{P} \), written \( I(\lambda) = \lambda^* \) for \( \lambda \in \mathcal{P} \).

(c) \( \mathcal{C} \) is a function with domain \( \mathcal{P} \) that maps each \( \lambda \) in \( \mathcal{P} \) to a sequence of partitions \( C_{\lambda} = (c_{\lambda}(i) : i \geq i_0(\lambda)) \).

(d) For each \( \lambda \in \mathcal{P} \), \( C_{\lambda} \) and \( C_{\lambda^*} \) have basic required structure, local required structure, extra required structure, and satisfy the \textit{amh}-hypotheses (as defined in Sect. 5).

(e) The chains \( C_{\lambda} \) (for \( \lambda \in \mathcal{P} \)) are pairwise disjoint.

For example, our prior work [3,4] defines \( I(\lambda) = \lambda^* \) and constructs chains \( C_{\lambda} \) and \( C_{\lambda^*} \) (with all required properties) for \( \lambda \) in the set \( \mathcal{P} \) of all integer partitions of size at most 11. Thus, we have a chain collection \( (\mathcal{P}, I, \mathcal{C}) \) for this choice of \( \mathcal{P} \), which can be taken as a “base case” for the entire recursive construction. The minimalist base case takes \( \mathcal{P} \) to be the collection of all
partitions $\lambda$ of size at most 5. The chains for these $\lambda$ are not hard to construct and verify, even without a computer. Example 5.6 illustrates this process for $\lambda = (1^5)$.

Let $\mathcal{C} = (\mathcal{P}, I, \mathcal{C})$ be a fixed chain collection. We can use this collection to define $\mu^*$, $\mathcal{C}_\mu$, and $\mathcal{C}_{\mu^*}$ (with all required properties) for certain flagpole partitions $\mu$. The construction for a specific $\mu$ only requires us to know $\rho^*$, $\mathcal{C}_{\rho^*}$, and $\mathcal{C}_{\rho^*}$ for a specific list of smaller partitions $\rho$ (depending on $\mu$). One of these partitions is the flag type $\lambda$ of $\mu$ (Definition 4.5), but there could be others. Before we can describe these others, we must discuss how to compute $\mu^*$.

**Definition 6.2.** Let $\mathcal{C} = (\mathcal{P}, I, \mathcal{C})$ be a chain collection, and let $\mu$ be a flagpole partition. We say $I$ extends to $\mu$ iff $|\mu| > |\rho|$ for all $\rho \in \mathcal{P}$, and $\lambda = \text{ftype}(\mu)$ belongs to $\mathcal{P}$. In this case, define $\mu^*$ as follows. Let the reduced Dyck vector for $\text{TI}_2(\mu)$ has length $L$ and area $A$. Define $\mu^* = \Psi^{-1}(\lambda^*, L, A \mod 2)$, where $\Psi$ is the bijection (4.6).

In Sect. 6.3, we show that this definition does give a well-defined flagpole partition $\mu^*$ of the same size as $\mu$ with $\text{ftype}(\mu^*) = \text{ftype}(\mu)^*$ and $\mu^{**} = \mu$.

**Definition 6.3.** Let $\mathcal{C} = (\mathcal{P}, I, \mathcal{C})$ be a chain collection, and let $\mu$ be a flagpole partition, such that $I$ extends to $\mu$ (so $\mu^*$ is defined). Let $\text{TI}_2(\mu) = [V]$ and $\text{TI}_2(\mu^*) = [V^*]$ where $V$ and $V^*$ are reduced. Recall the vectors $S_j(V)$ and $S_j(V^*)$ from Definition 3.11, which (for $j > 0$) are reduced TDVs of the form $S_j(V) = 0E_j1$ and $S_j(V^*) = 0E_j^*1$ (Theorem 3.17(e)). The needed partitions for $\mu$ are: (a) the flag type $\lambda$ of $\mu$; and (b) each partition $\rho$, such that $\text{TAIL}(\rho)$ contains $[E_j]$ or $[E_j^*]$ for some $j > 0$.

Since $\lambda \in \mathcal{P}$, all objects mentioned in this definition are explicitly computable. In particular, because each $E_j$ and $E_j^*$ are a TDV, $[E_j]$ and $[E_j^*]$ belong to some $\text{TAIL}_2(\rho)$ [Theorem 3.10(a)], and we know these tails for every partition $\rho$. In fact, Remark 7.3 proves that each $[E_j]$ and $[E_j^*]$ are in some $\text{TAIL}(\rho)$, so that $\rho$ itself is easily found from Theorem 2.17, as in Example 2.19.

We can now state the main results of this section.

**Theorem 6.4.** Let $\mathcal{C} = (\mathcal{P}, I, \mathcal{C})$ be a chain collection, and let $\mu$ be a flagpole partition, such that $I$ extends to $\mu$. Assume all needed partitions for $\mu$ belong to $\mathcal{P}$. Then, we can explicitly construct chains $\mathcal{C}_\mu$ and $\mathcal{C}_{\mu^*}$ having basic required structure, local required structure, extra required structure, and satisfying the amh-hypotheses. Therefore, $\text{Cat}_{n,\mu^*}(t, q) = \text{Cat}_{n,\mu}(q, t)$ for all $n > 0$.

By applying the construction to flagpole partitions in increasing order of size, we can pass from a given initial chain collection $\mathcal{C}$ to a larger one that, intuitively, consists of all chains that can be built from $\mathcal{C}$ using the methods described here. The next theorem makes this precise.

**Theorem 6.5.** Let $\mathcal{C} = (\mathcal{P}, I, \mathcal{C})$ be a chain collection, such that $k_0 = \max_{\rho \in \mathcal{P}} |\rho|$ is finite. Recursively define $\mathcal{C}^k = (\mathcal{P}^k, I^k, \mathcal{C}^k)$ for all $k \geq k_0$, as follows. Let $\mathcal{C}^{k_0} = \mathcal{C}$. Fix $k > k_0$ and assume that $\mathcal{C}^k_{k-1}$ has been defined.Enlarge $\mathcal{P}^k_{k-1}$ to
By adding all \( \mu \) satisfying: (i) \( \mu \) is a flagpole partition of size \( k \); (ii) \( I^{k-1} \) extends to \( \mu \); and (iii) all partitions needed for \( \mu \) are in \( P^{k-1} \). Extend \( I^{k-1} \) to \( I^k \) and \( C^{k-1} \) to \( C^k \) using the construction in Theorem 6.4 to define \( \mu^* \), \( C_\mu \), and \( C_{\mu^*} \) for each newly added \( \mu \). Let \( C = (P', I', C') \) be the union of the increasing sequence \( (C^k) \). Then, every \( \mathcal{C}^k \) and \( \mathcal{C}' \) is a chain collection.

Section 8 extends these theorems to generalized flagpole partitions.

### 6.2. Outline of Construction

In the rest of Sect. 6, we give the constructive proof of Theorem 6.4. Some technical proofs are delayed until Sect. 7. The construction is quite intricate, so we illustrate each step with a running example where \( \mu = \langle 5314 \rangle \). Here is an outline of the main ingredients in the construction. Throughout the rest of Sect. 6, we fix a chain collection \( \mathcal{C} = (P, I, C) \) and a flagpole partition \( \mu \) of size \( k \), such that \( I \) extends to \( \mu \). Thus, \( |\rho| < k \) for all \( \rho \in P \), and the flag type \( \lambda \) of \( \mu \) is in \( P \). Because \( \lambda^* \) is known, we can compute \( \mu^* \) (Definition 6.2); Sect. 6.3 supplies the required details. At this point, we can explicitly compute the needed partitions for \( \mu \) (Definition 6.3). If they all belong to \( P \), we may proceed with the following construction.

The chain \( C_\mu \) consists of three parts, called the antipodal part, the bridge part, and the tail part. We construct each part of the chain by building specific \( \nu_1 \)-initial objects that generate \( \nu_1 \)-segments comprising the chain. The tail part of \( C_\mu \) is \( \text{TAIL}_2(\mu) \), which we already built in Sect. 3.2. The bridge part of \( C_\mu \) (introduced in Sect. 6.4) consists of two-element \( \nu_1 \)-segments starting at partitions \( [M_i(\mu)] \) made by adding a new leftmost column to particular partitions in the known chain \( C_\lambda \), where \( \lambda = \text{ftype}(\mu) \). The tail part and bridge part of \( C_{\mu^*} \) are defined similarly, using \( \lambda^* = \text{ftype}(\mu^*) \).

The antipodal part of each chain is the trickiest piece to build. We must first identify the \( \nu_1 \)-initial objects in \( \text{TAIL}_2(\mu) \) and \( \text{TAIL}_2(\mu^*) \) (Theorem 3.17 and Sect. 6.5), which have reduced Dyck vectors called \( S_j(V) \) and \( S_j(V^*) \), respectively. In Sect. 6.6, we introduce the antipode map \( \text{Ant} \); this map interchanges area and dinv but only acts on a restricted class of Dyck vectors. Applying \( \text{Ant} \) to the vectors \( S_j(V^*) \) from the tail part of \( C_{\mu^*} \) produces \( \nu_1 \)-initial objects \( [A_j^*] \) that generate the antipodal part of \( C_{\mu^*} \) (see Sect. 6.7). Similarly, the Dyck classes \( [\text{Ant}(S_j(V))] \) generate the antipodal part of \( C_{\mu^*} \). Figure 2 may help visualize the overall construction.

To finish, we must assemble all the \( \nu_1 \)-segments and compute the min\( \Delta \)-profiles and \( \text{amh} \)-vectors for the new chains \( C_\mu \) and \( C_{\mu^*} \). Then, we verify all the required structural properties and \( \text{amh} \)-hypotheses from Sect. 5. The opposite property stated in Theorem 6.4 then follows from Theorem 5.4.

Our running example \( \mu = \langle 5314 \rangle \) has size \( k = 12 \), flag type \( \lambda = \langle 31 \rangle \), \( \lambda^* = \langle 22 \rangle \), and \( \mu^* = \langle 33214 \rangle \) (see Sect. 6.3 for details). The needed partitions for \( \mu \) are the flag type \( \lambda = \langle 31 \rangle \) along with the following partitions (computed in Example 6.13): \( \langle 22 \rangle \), \( \langle 21 \rangle \), \( \langle 3 \rangle \), and \( \langle 2 \rangle \). All needed partitions \( \rho \) have size at most 4 and are therefore in \( P \). For later reference, we list \( \rho, \rho^*, C_\rho \), and \( C_{\rho^*} \) for each needed \( \rho \) here.
For the discussion of $\mu^*$, which is frequently used later. Our proposed definition of $\mu^*$ is $\Psi^{-1}(\lambda^*, L, A \mod 2)$. Lemma 6.7 shows $\mu^*$ is well-defined. Let $V^*$ be the reduced Dyck vector for $\text{TI}_2(\mu^*)$. Then, $\text{len}(V^*) = \text{min}_\Delta(\text{TI}_2(\mu^*)) = L$, and we let $D^* = \text{dinv}(V^*)$ and $A^* = \text{area}(V^*)$.

Figure 2. Structure of the chains $C_\mu$ (right side) and $C_{\mu^*}$ (left side)

For our running example $\mu = \langle 531, 4 \rangle$, we compute $V = 0012212112 = v(\lambda, 2, 0)$ where $\lambda = \langle 31 \rangle$, $L = 10$, $D = 21$, $A = 12$, and $\Psi(\mu) = (31, 10, 1)$. From (6.1), we know $\lambda^* = \langle 22 \rangle$. (This is the only fact about $\lambda$ needed to compute $\mu^*$.) Since $A \mod 2 = 0$, $\mu^* = \Psi^{-1}(\langle 22 \rangle, 10, 0) = \langle 3321^4 \rangle$. Then, $V^* = 0012221122 = v(\lambda^*, 3, 0)$, $D^* = 20$, and $A^* = 13$. 

6.3. Defining $\mu^*$ for Flagpole Partitions

For the discussion of $\mu^*$ in this subsection, we assume $\mathcal{C} = (\mathcal{P}, I, \mathcal{C})$ is a fixed chain collection; $\mu$ is a fixed flagpole partition of size $k$, where $k > |\rho|$ for all $\rho \in \mathcal{P}$ [hence, $k \geq 6$ by 6.1(a)]; and the flag type of $\mu$ is in $\mathcal{P}$. We introduce notation that is used throughout Sects. 6 and 7. Let $\lambda = \text{ftype}(\mu)$, and let $V = v(\lambda, a, \epsilon)$ be the reduced Dyck vector for $\text{TI}_2(\mu)$. Let $L = \text{len}(V) = \text{min}_\Delta(\text{TI}_2(\mu))$, $D = \text{dinv}(V)$, and $A = \text{area}(V)$. Recall (Sect. 4.2) that $\Psi(\mu) = (\lambda, L, D \mod 2)$. Note $\text{defc}(V) = \text{defc}(\text{TI}_2(\mu)) = \text{defc}(\text{TI}(\mu)) = |\mu| = k$. Remark 4.3 gives

$$\text{defc}(V) = k = |\lambda| + L - 2,$$

which is frequently used later. Our proposed definition of $\mu^*$ is $\mu^* = \Psi^{-1}(\lambda^*, L, A \mod 2)$. Lemma 6.7 shows $\mu^*$ is well-defined. Let $V^*$ be the reduced Dyck vector for $\text{TI}_2(\mu^*)$. Then, $\text{len}(V^*) = \text{min}_\Delta(\text{TI}_2(\mu^*)) = L$, and we let $D^* = \text{dinv}(V^*)$ and $A^* = \text{area}(V^*)$. 

For our running example $\mu = \langle 531^4 \rangle$, we compute $V = 0012212112 = v(\lambda, 2, 0)$ where $\lambda = \langle 31 \rangle$, $L = 10$, $D = 21$, $A = 12$, and $\Psi(\mu) = (31, 10, 1)$. From (6.1), we know $\lambda^* = \langle 22 \rangle$. (This is the only fact about $\lambda$ needed to compute $\mu^*$.) Since $A \mod 2 = 0$, $\mu^* = \Psi^{-1}(\langle 22 \rangle, 10, 0) = \langle 3321^4 \rangle$. Then, $V^* = 0012221122 = v(\lambda^*, 3, 0)$, $D^* = 20$, and $A^* = 13$. 

$$\rho = \langle 31 \rangle : C_\rho = \text{NU}_1^4((2211)) \cup \text{NU}_1^4((443111)),$$

$$\rho = \langle 22 \rangle : C_\rho = \text{NU}_1^4((211^4)) \cup \text{NU}_1^4((3221)),$$

$$\rho = \langle 21 \rangle : C_\rho = \text{NU}_1^4((3111)) \cup \text{NU}_1^4((3311)),$$

$$\rho = \langle 111 \rangle : C_\rho = \text{NU}_1^4((2111)) \cup \text{NU}_1^4((3211)),$$

$$\rho = \langle 3 \rangle : C_\rho = \text{NU}_1^4((1111)) \cup \text{NU}_1^4((222)) \cup \text{NU}_1^4((3321)),$$

$$\rho = \langle 2 \rangle : C_\rho = \text{NU}_1^4((1111)) \cup \text{NU}_1^4((221)),$$
Example 6.6. We know ⟨0⟩* = ⟨0⟩. By Example 4.12, ⟨1^k⟩* is either ⟨1^k⟩ or ⟨21^k−2⟩.

**Lemma 6.7.** Assume μ satisfies the hypothesis in Definition 6.2. Then, μ* is a well-defined flagpole partition with |μ*| = |μ| and μ** = μ. Moreover, D* ≡ A (mod 2) and A* ≡ D (mod 2).

**Proof.** From Lemma 4.11, Ψ(μ) = (λ, L, D mod 2) where L ≥ |λ| + 6 ≥ 6. Since λ ∈ P, we know that λ* is already defined, λ* ∈ P, |λ*| = |λ|, and L ≥ |λ*| + 6. Thus, (λ*, L, A mod 2) does belong to the codomain of the bijection Ψ, and so, μ* is a well-defined flagpole partition. Since V* and V both have length L

|μ*| = defc(V*) = |λ*| + L − 2 = |λ| + L − 2 = defc(V) = |μ| = k.

Next we check D* ≡ A (mod 2), A* ≡ D (mod 2), and μ** = μ. By definition of μ*, Ψ(μ*) = (λ*, L, A mod 2), so D* ≡ A (mod 2) by definition of Ψ. Now, since defc(V*) = defc(V) = k and dinv + area + defc = \(\binom{\text{len} + 2}{2}\)

\[D + A = \left(\frac{L}{2}\right) - k = D^* + A^*. \quad (6.2)\]

Because D* ≡ A (mod 2), we also have A* ≡ D (mod 2). Finally, the definition of the involution gives Ψ(μ**) = (λ**, L, A* mod 2). Since λ** = λ (by 6.1(b)) and A* ≡ D (mod 2), Ψ(μ**) = (λ, L, D mod 2) = Ψ(μ). Since Ψ is one-to-one, μ** = μ follows. □

In the situation of the lemma, we really can extend the involution I by setting I(μ) = μ* and I(μ*) = μ without conflicting with any previously defined values I(ρ), since μ and μ* have size strictly larger than ρ and ρ* by hypothesis.

**Lemma 6.8.** Assume I extends to a flagpole partition μ (necessarily of size at least 6). Then, D ≥ A*.

This lemma is proved in Sect. 7.1.

6.4. The Bridge Part of C_μ

The bridge part of C_μ consists of two-element nu_1-segments \([M_i(μ)]\), nu_1 ([M_i(μ)]) with dinv(M_i(μ)) = i, for all i ∈ \{A^*, A^*+2, A^*+4, ..., D-4, D-2\}. The bridge part is empty if D = A*. To define M_i(μ), we require the following lemma, proved in Sect. 7.2.

**Lemma 6.9.** Assume I extends to μ, so μ = ftype(μ) is in P. For each i ∈ \{A^*, A^*+2, ..., D-4, D-2\}, there exists a unique object γ = c_λ(i − 1) in the known chain C_λ, such that dinv(γ) = i − 1 and defc(γ) = |λ|. Moreover, min_Δ(γ) ≤ L − 2, and z = QDV_{L−2}(γ) starts with 01 and contains a 2.

For i in \{A^*, A^*+2, ..., D−2\}, define M_i(μ) as follows. Take γ = c_λ(i − 1) and z = QDV_{L−2}(γ) as in the lemma, and let M_i(μ) = 00z+. Visually, the Ferrers diagram for the partition [M_i(μ)] is obtained from the diagram for γ by adding a new leftmost column containing L − 1 boxes. See Fig. 3 for an example.
Lemma 6.10. Assume $I$ extends to $\mu$. Each $M_i(\mu)$ is a reduced Dyck vector of length $L$ starting with $0012$, ending with a positive symbol, and containing a $3$. $M_i(\mu)$ has $\text{dinv}_i$ and deficit $k = |\mu|$. $[M_i(\mu)]$ is an $\text{NU}_1$-initial object with $\min_\Delta$ equal to $L$, while $\text{NU}([M_i(\mu)])$ is a $\text{NU}_1$-final object with $\min_\Delta$ equal to $L + 1$.

Proof. By Lemma 6.9, $z$ is a Dyck vector of length $L - 2$ starting with $01$ and containing a $2$. Since $M_i(\mu)$ has $\text{dinv} i$ and deficit $k = |\mu|$, $[M_i(\mu)]$ is an $\text{NU}_1$-initial object with $\min_\Delta$ equal to $L$, while $\text{NU}([M_i(\mu)])$ is a $\text{NU}_1$-final object with $\min_\Delta$ equal to $L + 1$.

For our example $\mu = \langle 531^4 \rangle$, let us compute $M_i(\mu)$ for $i = 13, 15, 17, 19$ (this range comes from $A^* = 13$ and $D = 21$). Here, $\lambda = \text{ftype}(\mu) = \langle 31 \rangle$. We look up each $c_\lambda(i - 1)$ from (6.1), find the representative $z$ of length $L - 2 = 8$, and then form $M_i(\mu) = 00z^+$. The results appear in the following table:

| $i$  | $\gamma = c_{\langle 31 \rangle}(i - 1)$ | $z = \text{QDV}_8(\gamma)$ | $M_i(\mu)$ |
|------|----------------------------------------|-----------------------------|-------------|
| 13   | 0112010                               | 0012334232                  | 0012334232  |
| 15   | 0101001                               | 01212112                   | 012323223   |
| 17   | 01211210                              | 012121120                  | 00123232321 |
| 19   | 01121010                              | 012112100                  | 012232121   |

The $\min_\Delta$-profile for the bridge part of $C_\mu$ is

$$\begin{pmatrix}
\text{dinv} : 13 & 14 & 15 & 16 & 17 & 18 & 19 & 20 \\
\min_\Delta : 10 & 11 & 10 & 11 & 10 & 11 & 10 & 11
\end{pmatrix}. \quad (6.3)
$$

This part ends just before $\text{dinv}$ index $D = 21$, which is where $\text{TAIL}_2(\mu)$ begins at $[V]$. In fact, if we take $i = D$ in the definition of $M_i(\mu)$, we find that $M_D(\mu) = V$ (see Remark 7.2 for a proof).

For $\mu^* = \langle 3321^4 \rangle$, we perform a similar calculation using $\lambda^* = \langle 22 \rangle$ and $i = 12, 14, 16, 18$ (since $A^{**} = A = 12$ and $D^* = 20$). The results are shown here.
\begin{align*}
i & = c_{(22)}(i - 1) \\
\gamma & = c_{(22)}(i - 1) \\
z & = QDV_8(\gamma) \\
M_i(\mu^*) & = 001234432200123223330012232232001232211.
\end{align*}

The \( \min_{\Delta} \)-profile for the bridge part of \( C_{\mu^*} \) is
\[
\begin{bmatrix}
\text{dinv} & : & 12 & 13 & 14 & 15 & 16 & 17 & 18 & 19 \\
\text{min}_{\Delta} & : & 10 & 11 & 10 & 11 & 10 & 11 & 10 & 11
\end{bmatrix}.
\]

The next proposition summarizes the key properties of the bridge part.

**Proposition 6.11.** Assume \( I \) extends to \( \mu \). The bridge part of \( C_\mu \) (resp. \( C_{\mu^*} \)) is a sequence of partitions in \( \text{Def}(k) \) indexed by consecutive \( \text{dinv} \) values from \( A^* \) to \( D - 1 \) (resp. \( A \) to \( D^* - 1 \)). The \( \min_{\Delta} \)-profile of the bridge part of \( C_\mu \) (and \( C_{\mu^*} \)) consists of \( (D - A^*)/2 = (D^* - A)/2 \) copies of \( L, L + 1 \).

Note that \( (D - A^*)/2 = (D^* - A)/2 \) follows from (6.2).

### 6.5. \( NU_1 \)-Initial Objects in the Tail Part

The construction of the antipodal part of \( C_{\mu^*} \) begins with the \( NU_1 \)-initial objects in \( \text{TAL}_2(\mu) \) (and similarly for \( C_\mu \) and \( \text{TAL}_2(\mu^*) \)), as shown in Fig. 2. Since \( \mu \) is a flagpole partition, Theorem 3.17 applies to \( V = v(\lambda, a, \epsilon) \). That theorem explicitly describes the \( NU_1 \)-chain from \( [V] = \text{TAL}_2(\mu) \) to \( \text{TAL}(\mu) \), including the \( \min_{\Delta} \)-profile of this chain and the \( NU_1 \)-initial objects along the way. Recall that the reduced Dyck vectors for these \( NU_1 \)-initial Dyck classes, listed in increasing order of \( \text{dinv} \), are \( S_0(V), S_1(V), \ldots, S_J(V) \). Here, \( J \) depends on \( \mu \). The corresponding vectors for \( \mu^* \) are \( S_0(V^*), S_1(V^*), \ldots, S_J(V^*) \).

Consider our running example \( \mu = \langle 531^4 \rangle \). We start at the reduced Dyck vector for \( \text{TAL}_2(\mu) \), which is \( V = 0012212112 \) with \( \text{dinv}(V) = D = 21 \) and \( \text{len}(V) = \min_{\Delta}(\text{dinv}(V)) = L = 10 \). Apply Theorem 3.17 with \( n_0 = 2, n_1 = 1, n_2 = 0, n_3 = 1, \) and \( C = \emptyset \). The vectors \( v^{(i)} \) in part (a) of the theorem are \( v^{(0)} = v, v^{(1)} = 0012121110, v^{(2)} = 0012111001, v^{(3)} = 00121110010, \) and \( v^{(4)} = 001111001001 = 0B_\mu \). These vectors have \( \text{dinv} \) values \( 21, 33, 35, 46, 48 \) (in order). Using part (b) of the theorem for \( 0 \leq i \leq 4 \), the complete \( \min_{\Delta} \)-profile of \( \text{TAL}_2(\mu) \) is
\[
\begin{bmatrix}
\text{dinv} & : & 21 & 22 & 23 \cdots & 34 & 35 & \cdots & 47 & 48 \cdots & \cdots \\
\text{min}_{\Delta} & : & 10 & 11 & 10 \cdots & 12 & 11 & 12 & 12 & 13 & 14 & 13 \cdots
\end{bmatrix}.
\]

By part (c) of the theorem, the underlined values correspond to the \( NU_1 \)-initial objects \( [S_J(V)] \). By part (d) of the theorem, we have \( S_0(V) = V = 0012212112, S_1(V) = 0011121211, S_2(V) = v^{(2)} = 0011211001, \) and \( S_3(V) = v^{(4)} = 001111001001 \). We can see that the structural properties promised in Theorem 3.17(e), (f), and (g) do hold for the specific vectors we have computed. In particular, \( \text{area}(S_0(V)) = 12, \text{area}(S_1(V)) = 10, \text{area}(S_2(V)) = 8, \) and \( \text{area}(S_3(V)) = 6 \).

Following the same procedure for \( \mu^* = \langle 3321^4 \rangle \) (using Theorem 3.17 or Example 3.14), we obtain:
$S_0(V^*) = V^* = 0012221122$, $S_1(V^*) = 0012112211$, $S_2(V^*) = 0011221101$, $S_3(V^*) = 00111101011 = 0B_\mu^*$.

(6.7)

Here, $\text{area}(S_j(V^*)) = A^* - 2j$, which is $13, 11, 9, 7$ for $j = 0, 1, 2, 3$. The $\text{min}_\Delta$-profile of $\text{TAIL}_2(V^*)$ is

$$\left[ \text{dinv} : 20 21 22 23 24 \cdots 35 36 \cdots \cdots \cdots \right],$$

$\text{min}_\Delta : 10 11 10 11 10 12 11 12 11 13 12 \cdots$ \quad (6.8)

The next proposition summarizes the fundamental properties of the tail part of $C_\mu$, which follow from Theorem 3.17. Here and below, let $L_j = \text{len}(S_j(V)) = \text{min}_\Delta([S_j(V)])$ and $D_j = \text{dinv}(S_j(V))$ for $0 \leq j \leq J$. The analogous quantities for $\mu^*$ are $L_j^*$ and $D_j^*$ for $0 \leq j \leq J^*$. Note that we have $L_0 = L_1 = L$, because $S_0(V) = V = v(\lambda, a, c)$ starts with 0012.

**Proposition 6.12.** Assume $\mu$ is any flagpole partition.

(a) The tail part of $C_\mu$, namely $\text{TAIL}_2(\mu)$, is an infinite sequence of partitions in $\text{Def}(k)$ indexed by consecutive $\text{dinv}$ values starting at $D$.

(b) The $\text{min}_\Delta$-profile for $\text{TAIL}_2(\mu)$ consists of weakly ascending runs starting at $\text{dinv}$ indices $D_j$ (for $0 \leq j \leq J$) corresponding to the $\nu_1$-initial objects $[S_j(V)]$. The $\text{min}_\Delta$-profile of the $j$th ascending run is a prefix of $L_j^1(L_j+1)L_j(L_j+2)^L_j+1 \cdots$, where the prefix length is at least 2 and (for $j < J$) the prefix ends with one copy of $L_j+1 + 1$. The $L_j$ weakly increase from $L_0 = L_1 = L$ to $L_J = \text{min}_\Delta(\text{TI}(\mu))$.

(c) We have $S_0(V) = V$ and $S_j(V) = 0B_\mu^*$; each $S_j(V)$ has the form 00X1 with $X$ ternary; and area($S_j(V)$) = $A - 2j$ for $0 \leq j \leq J$.

When $\mu^*$ is defined, analogous results hold for the tail part of $C_{\mu^*}$ (using $D^*, L_j^*, D_j^*, J^*$, and $V^*$).

### 6.6. The Antipode Map

We now define the *antipode map* $\text{Ant}$, which acts on certain ternary Dyck vectors that begin with 00 and end with 1. Intuitively, this map plays the following role in the overall construction. As shown in Fig. 2, we will apply $\text{Ant}$ to the reduced representatives $S_j(V^*)$ of the $\nu_1$-initial objects in $\text{TAIL}_2(\mu^*)$ to determine the $\nu_1$-initial objects in the antipodal part of $C_\mu$. To compute $\text{Ant}(S)$, we must already know two chains $C_\rho$ and $C_{\rho^*}$ with the opposite property, where $\rho$ depends on $S$.

Let $S$ be a TDV of deficit $k$ of the form $S = 00X1$. Then, $S = 0E1$, where $E = 0X$ is a ternary Dyck vector. Let $L' = \text{len}(S) = \text{min}_\Delta([S])$, $D' = \text{dinv}(S)$, and $A' = \text{area}(S)$. We compute $\text{Ant}(S)$ as follows. Since $E$ is a TDV, $[E]$ belongs to $\text{TAIL}_2(\rho)$ for a unique partition $\rho$ (Theorem 3.10(a)). If $\rho \notin \mathcal{P}$, then $\text{Ant}(S)$ is not defined. If $\rho \in \mathcal{P}$, look up $\gamma = c_{\rho^*}(A' - 1)$ in $C_{\rho^*}$. ($\gamma$ is the unique object in $C_{\rho^*}$ with $\text{dinv}(\gamma) = A' - 1$). Let $z = \text{QDV}_{L'-2}(\gamma)$, and define $\text{Ant}(S) = 0z^+$.

Lemma 6.14(a) shows this construction makes sense, but first we look at some examples.

**Example 6.13.** Continuing our running example, we now compute $\text{Ant}(S_j(V^*))$ and $\text{Ant}(S_j(V))$ for $j = 1, 2, 3$. First, consider $S = S_1(V^*) = 0012112211$. 
We have $S = 0E1$ with $E = 01211221$. The Dyck class $[E] = [01211221] = [0100110]$ belongs to plateau 2 of $\text{Tail}(22) \subseteq C_{(22)}$ by Example 2.19(b). The Dyck class $[E]$ has $\min_{\Delta} \leq 8 = \text{len}(E)$, $\text{area}_{\delta} = 10$, and $\text{dinv} = 14$. Here, $\rho = \langle 22 \rangle$, and the induction hypothesis (6.1) provides us with $\rho^* = \langle 31 \rangle$ and the chain $C_{(31)}$. To continue, we find the unique object $\gamma = c_{(31)}(10)$ in $C_{(31)}$ with $\text{dinv} = 10$, which (by the opposite property) is guaranteed to have $\min_{\Delta} \leq 8$ and $\text{area}_{\delta} = 14$. From (6.1), we find $\gamma = (6332) = [0121120]$. Then, $z = \text{QDV}_8(\gamma) = 01232231$, and $\text{Ant}(S) = 00z^+ = 0012343342$. $\text{Ant}(S)$ is a reduced Dyck vector with len $= 10$, area $= 22 = \text{dinv}(S)$, and $\text{dinv} = 11 = \text{area}(S)$.

Second, consider $S = S_2(V^*) = 0011221101$. Here, $E = 01122110$, $[E]$ belongs to plateau 3 of $\text{Tail}(22) \subseteq C_{(22)}$ [Example 2.19(c)], and $[E]$ has $\text{area}_{\delta} = 8$, $\text{dinv} = 16$, and $\min_{\Delta} \leq 8$. For this $S$, we again have $\rho = \langle 22 \rangle$ and $\rho^* = \langle 31 \rangle$. From (6.1), we look up $\gamma = c_{(31)}(8) = (642) = [01232310]$, so $z = \text{QDV}_8(\gamma) = [01234321]$. Note that $\min_{\Delta}(\gamma) \leq 8$, $\text{dinv}(\gamma) = 8$, and $\text{area}_{\delta}(\gamma) = 16$. Here, $\text{Ant}(S) = 0012345432$, which is a reduced Dyck vector with $\min_{\Delta} = 10$, area $= 24 = \text{dinv}(S)$, and $\text{dinv} = 9 = \text{area}(S)$.

Third, consider $S = S_3(V^*) = 00111101011$. Here $E = 0111110101$, $[E]$ belongs to plateau 4 of $\text{Tail}(21) \subseteq C_{(21)}$ [Example 2.19(a)], and $[E]$ has $\text{area}_{\delta} = 6$, $\text{dinv} = 27$, and $\min_{\Delta} \leq 9$. For this $S$, we have $\rho = \langle 21 \rangle$ and $\rho^* = \langle 111 \rangle$. From (6.1), we find $\gamma = c_{(111)}(6) = (441) = [012201]$ and $z = \text{QDV}_9(\gamma) = 012345534$. So $\text{Ant}(S) = 00123456645$, which is a reduced Dyck vector with $\min_{\Delta} = 11$, area $= 36 = \text{dinv}(S)$, and $\text{dinv} = 7 = \text{area}(S)$.

We compute each $\text{Ant}(S_j(V))$ similarly. For $S = S_1(V) = 0011211211$, we have $E = 01121121$, $[E] = [0010010] \in \text{Tail}(31)$, $\rho = \langle 31 \rangle$, $\rho^* = \langle 22 \rangle$, $\gamma = c_{(22)}(9) = (53221) = [000110]$, $z = 01222323$, and $\text{Ant}(S) = 0012333443$. For $S = S_2(V) = 00112111001$, we have $E = 011211100$, $[E] \in \text{Tail}(3)$, $\rho = \langle 3 \rangle = \rho^*$, $\gamma = c_{(3)}(7) = (5221) = [011120]$, $z = 012344453$, and $\text{Ant}(S) = 00123455564$. For $S = S_3(V) = 001111001001$, we have $E = 01111001001$, $[E] \in \text{Tail}(2)$, $\rho = \langle 2 \rangle = \rho^*$, $\gamma = c_{(2)}(5) = (43) = [01200]$, $z = 0123456755$, and $\text{Ant}(S) = 001234567666$.

**Lemma 6.14.** Let $S = 00X1 = 0E1$ be a TDV of deficit $k$, length $L'$, area $A'$, and $\text{dinv} D'$, such that $[E] \in \text{Tail}_2(\rho)$ and $\rho \in \mathcal{P}$.

(a) There exists a unique partition $\gamma \in C_{\rho^*}$ with $\text{dinv}(\gamma) = A' - 1$ and $\min_{\Delta}(\gamma) \leq L' - 2$. So $\text{Ant}(S)$ is well defined.

(b) $\text{Ant}(S)$ is a reduced Dyck vector with deficit $k$, length $L'$, area $D'$, and $\text{dinv} A'$.

(c) The Dyck class $[\text{Ant}(S)]$ is a NU1-initial object with $\min_{\Delta}([\text{Ant}(S)]) = L'$.  

**Proof.** We have $L' \geq 3$. By Lemma 2.7, $|\rho| = \text{defc}(E) = k - (L' - 2) < k$, $\text{dinv}(E) = D' - (L' - 2)$, and $\text{area}(E) = A' - 1$. Because $\rho \in \mathcal{P}$, $\rho^*$ is defined, and we know $C_{\rho}$ and $C_{\rho^*}$ satisfy the opposite property. Now, $[E]$ is an object in $C_{\rho}$ with $\min_{\Delta}([E]) \leq \text{len}(E) = L' - 2$, $\text{dinv}([E]) = D' - (L' - 2)$, and $\text{area}_{L'-2}([E]) = A' - 1$. Therefore, the opposite property guarantees the existence of a unique $\gamma \in C_{\rho^*}$, namely $\gamma = c_{\rho^*}(A' - 1)$, such that $\min_{\Delta}(\gamma) \leq L' - 2$, $\text{dinv}(\gamma) = A' - 1 = \text{area}(E)$, and $\text{area}_{L'-2}(\gamma) = D' - (L' - 2) = \text{dinv}(E)$.  

Therefore, \( z = \text{QDV}_{L'-2}(\gamma) \) is a Dyck vector (not just a QDV) with length \( L' - 2 \), area \( D' - (L' - 2) \), and \( \text{dinv} A' - 1 \). Thus, \( \text{Ant}(S) = 00z^+ \) is a well-defined reduced Dyck vector beginning with 00. By Lemma 2.7, \( \text{len}(\text{Ant}(S)) = L' \), area(\( \text{Ant}(S) \)) = \( D' \), \( \text{dinv}(\text{Ant}(S)) = A' \), and hence, \( \text{defc}(\text{Ant}(S)) = k \). Since \( \text{Ant}(S) = 00z^+ \) has leader 0 and a positive final symbol, \( [\text{Ant}(S)] \) is a \( \nu_1 \)-initial object by Proposition 2.10(b).

In each computation from Example 6.13, the Dyck class \([E]\) always appeared in some \( \text{TAL}(\rho) \), not just in \( \text{TAL}_2(\rho) \). Remark 7.3 proves that this always happens for \( S = S_j(V) \) or \( S = S_j(V^*) \), which are the only cases of interest below. This fact lets us quickly compute \( \rho \) from \( S_j(V) \) using Theorem 2.17, as illustrated in Example 6.13. Each such \( \rho \) is one of the needed partitions for \( \mu \).

6.7. The Antipodal Parts of \( \mathcal{C}_\mu \) and \( \mathcal{C}_{\mu^*} \)

Assume \( I \) extends to \( \mu \) and all needed partitions for \( \mu \) are in \( \mathcal{P} \). Then, we can define antipodal vectors \( A_j = \text{Ant}(S_j(V)) \) for \( 1 \leq j \leq J \) and \( A_j^* = \text{Ant}(S_j(V^*)) \) for \( 1 \leq j \leq J^* \). The antipodal part of \( \mathcal{C}_\mu \) consists of two-element \( \nu_1 \)-segments \( [A_j^*], \nu_1([A_j^*]) \), taken in order from \( j = J^* \) down to \( j = 1 \) (see Fig. 2). Similarly, the antipodal part of \( \mathcal{C}_{\mu^*} \) consists of \( \nu_1 \)-segments \( [A_j], \nu_1([A_j]) \), taken in order from \( j = J \) down to \( j = 1 \). We check that everything works in Proposition 6.16 after considering some examples.

Example 6.15. Take \( \mu = \langle 531^4 \rangle \) and \( \mu^* = \langle 3321^4 \rangle \), so \( J = J^* = 3 \). In Example 6.13, we computed \( A_1^* = 0012343342, \ A_2^* = 0012345432, \ A_3^* = 001234566645 \). The antipodal part of \( \mathcal{C}_\mu \) is

\[
[A_2^*, \nu_1([A_2^*]), [A_3^*, \nu_1([A_3^*]), [A_1^*, \nu_1([A_1^*])],
\]

where each \( [A_j^*] \) is \( \nu_1 \)-initial and each \( \nu_1([A_j^*]) \) is \( \nu_1 \)-final. The \( \min_\Delta \)-profile for the antipodal part of \( \mathcal{C}_\mu \) is

\[
\begin{bmatrix}
\text{dinv} & 7 & 8 & 9 & 10 & 11 & 12 \\
\text{min}_\Delta & 11 & 12 & 10 & 11 & 10 & 11
\end{bmatrix}.
\]

This antipodal part ends at \( \text{dinv} \) index 12, while the bridge part of \( \mathcal{C}_\mu \) starts at \( \text{dinv} \) index \( A^* = 13 \).

From Example 6.13, we also have \( A_1 = 0012333443, \ A_2 = 00123455564, \) and \( A_3 = 001234567866 \). The antipodal part of \( \mathcal{C}_{\mu^*} \) is

\[
[A_3, \nu_1([A_3]), [A_2, \nu_1([A_2]), [A_1, \nu_1([A_1])],
\]

which consists of three two-element \( \nu_1 \)-segments. The \( \min_\Delta \)-profile for the antipodal part of \( \mathcal{C}_{\mu^*} \) is

\[
\begin{bmatrix}
\text{dinv} & 6 & 7 & 8 & 9 & 10 & 11 \\
\text{min}_\Delta & 12 & 13 & 11 & 12 & 10 & 11
\end{bmatrix}.
\]

This antipodal part ends at \( \text{dinv} \) index 11, while the bridge part of \( \mathcal{C}_{\mu^*} \) starts at \( \text{dinv} \) index \( A^{**} = A = 12 \).

Proposition 6.16. Assume \( I \) extends to \( \mu \) and all needed partitions for \( \mu \) are in \( \mathcal{P} \).
(a) Each $A_j$ is a well-defined reduced Dyck vector with length $L_j$ that starts with 00 and ends with a positive symbol. In fact, $A_j$ starts with 0012 and contains a 3.

(b) Each $[A_j]$ is an $NU_1$-initial object with $\min_\Delta = L_j$. Each $NU_1([A_j])$ is an $NU_1$-final object with $\min_\Delta = L_j + 1$.

(c) $\delta fc([A_j]) = k$ and $\delta n v([A_j]) = A - 2j$.

(d) Parts (a), (b), and (c) are true replacing $A_j$ by $A_j^*$, $L_j$ by $L_j^*$, and $A$ by $A^*$.

(e) The antipodal part of $C_\mu$ is a sequence of partitions in $\text{Def}(k)$ indexed by consecutive dinv values from $\ell(\mu^*)$ to $A^* - 1$. The $\min_\Delta$-profile for the antipodal part of $C_\mu$ is $L_j^*, L_{j^*}^* + 1, \ldots, L_2^*, L_2^* + 1, L_1^*, L_1^* + 1$.

(f) The antipodal part of $C_{\mu^*}$ is a sequence of partitions in $\text{Def}(k)$ indexed by consecutive dinv values from $\ell(\mu)$ to $A - 1$. The $\min_\Delta$-profile for the antipodal part of $C_{\mu^*}$ is $L_j, L_j + 1, \ldots, L_2, L_2 + 1, L_1, L_1 + 1$.

**Proof.** Proposition 6.12(c) shows that each $S_j(V)$ is a valid input to $\text{Ant}$. The first sentence of part (a) follows from the definition of $\text{Ant}$. The proof that $A_j$ must start with 0012 and contain a 3 is rather technical and is postponed to Sect. 7.3. Part (b) follows from part (a) and Proposition 2.12(b).

For part (c), Lemma 6.14(b) and Proposition 6.12(c) imply $\delta fc(A_j) = k$ and $\delta n v(A_j) = \text{area}(S_j) = A - 2j$. Part (d) follows by replacing $\mu$ (and associated quantities) by $\mu^*$. By Proposition 2.15(c), $S_{j^*}(V^*) = 0B_{\mu^*}$ has area $\ell(\mu^*)$.

Part (e) follows from this observation and part (c). Similarly, part (f) follows from (d) and the formula $\text{area}(S_j(V)) = \text{area}(0B_\mu) = \ell(\mu)$. \qed

### 6.8. Proofs of the Main Theorems

We can now complete the proof of Theorem 6.4. Assume that $\mu$ satisfies the hypotheses of that theorem. Then, we can build the chains $C_\mu$ and $C_{\mu^*}$ by combining the tail parts, bridge parts, and antipodal parts described in the previous subsections. We need only verify that these chains have basic required structure, local required structure, and extra required structure, and satisfy the $amh$-hypotheses.

The antipodal part of $C_\mu$ consists of partitions with deficit $k = |\mu|$ and consecutive dinv values from $\ell(\mu^*)$ to $A^* - 1$ [Proposition 6.16(c) and (e)]. The bridge part of $C_\mu$ consists of partitions with deficit $k$ and consecutive dinv values from $A^*$ to $D - 1$ (Proposition 6.11). The tail part of $C_\mu$ is $\text{Tail}_2(\mu)$, which consists of partitions with deficit $k$ and dinv values from $D = \text{dinv}(Tl_2(\mu))$ onward. Analogous statements hold for $C_{\mu^*}$. These comments prove the basic structure conditions 5.1(a) through (d) and extra condition 5.5(d). In Sect. 7.4, we prove that $C_\mu$ and $C_{\mu^*}$ are disjoint (have no terms in common) when $\mu \neq \mu^*$.

Next, we show that the $h$-vector for $C_\mu$ is

\[
(L_{j^*}^*, \ldots, L_2^*, L_1^*, L^{(D - A^*)/2}, L_0, L_1, L_2, \ldots, L_J).
\]  

(6.11)

The ascending runs for the $\min_\Delta$-profile within the three parts of $C_\mu$ are given in Propositions 6.11, 6.12, and 6.16, but we must still check that a new ascending run begins at the start of the bridge part and the tail part. Since
that the sequences \((a, \ldots, w, h, \ldots)\) all show that the \(m\)-vector for \(C_\mu\) has all entries 0.

By analogous reasoning, the \(h\)-vector for \(C_{\mu^*}\) is

\[
(L_J, \ldots, L_2, L_1, L^{(D^*-A^*)/2}, L_0^*, L_1^*, L_2^*, \ldots, L_J^*). 
\]

Since \((D - A^*)/2 = (D^* - A)/2\) and \(L_0 = L = L_0^*\), the two \(h\)-vectors are reversals of each other and \(amh\)-hypothesis 5.3(a) holds. Now that we know the \(amh\)-vectors for \(C_\mu\) and \(C_{\mu^*}\) all have the same length \(N\), 5.3(b) follows, since both \(m\)-vectors are \(0^N\).

To check 5.3(c), we look at cases based on the Dyck class \([v]\) in \(C_\mu\) with \(dinv(v) = a_i\). By our determination of the \(h\)-vector of \(C_\mu\), \([v]\) is one of the \(nu_1\)-initial objects in \(C_\mu\). Recall that \(dinv(v) + defc(v) + area(v) = (\text{len}(v))\) holds for all Dyck vectors \(v\). First consider the tail case where \(v = S_j(\mu)\) for some \(j\) between 1 and \(J\). Then, \(a_i = dinv(v), m_i = 0, k = defc(v), a_{N+1-i}^* = dinv(\text{Ant}(S_j(\mu))) = area(S_j(\mu)) = area(v)\), and \(h_i = \min(v) = \text{len}(v)\). Therefore, 5.3(c) holds. In the special case where \(v = S_0(\mu) = V\), we have \(a_i = dinv(V) = D, m_i = 0, k = defc(V), \) and \(h_i = L = \text{len}(V)\). If the bridge parts are nonempty, then \(a_{N+1-i}^*\) is the dinv index of the first object in the bridge of \(C_{\mu^*}\), namely \(A = area(V)\). If the bridge parts are empty, then \(a_{N+1-i}^*\) is the dinv index of \(S_0(\mu^*) = V^*\), namely \(D^*\), but \(D^* = A\) since the bridge is empty. In all these situations, 5.3(c) holds, since \(D + k + A = (\frac{L}{2})\) by definition of \(defc(V)\).

Next, consider the bridge case where \(v = M_{D-2j}(\mu)\) for some \(j > 0\). Then, \(a_i = dinv(v) = D - 2j, m_i = 0, k = defc(v), \) and \(h_i = L\). We find \(a_{N+1-i}^* = A + 2j\) by counting up from the beginning of the bridge part of \(C_{\mu^*}\) (Proposition 6.11). Again, 5.3(c) holds, since \((D - 2j) + 0 + k + (A + 2j) = D + k + A = (\frac{L}{2})\).

Next, consider the antipodal case where \(v = \text{Ant}(S_j(\mu^*))\) for some \(j\) between 1 and \(J^*\). Then, \(a_i = dinv(v) = area(S_j(\mu^*)), m_i = 0, k = defc(v) = defc(S_j(\mu^*)), a_{N+1-i}^* = dinv(S_j(\mu^*)), \) and \(h_i = \min(v) = L^*_j = \text{len}(S_j(\mu^*)).\) Therefore, 5.3(c) holds here, as well. The opposite property for \(C_\mu\) and \(C_{\mu^*}\) now follows from Theorem 5.4.

To finish the proof, we check the first three extra structural conditions for \(C_\mu\). Condition 5.5(a) follows from (6.11) and the fact (Proposition 6.12(b)) that the sequences \((L_j)\) and \((L'_j)\) are weakly increasing with \(L_0 = L_0^* = L\). Similarly, Proposition 6.12(b) implies condition 5.5(b) for the ascending runs in \(\text{Tail}_2(\mu)\). Condition (b) is immediate for the runs in the antipodal and bridge parts, which all have length 2 and \(\min\)-profiles of the form \(L', L' + 1\). Finally, condition 5.5(c) follows from Theorem 2.17 for \(\text{Tail}(\mu)\), Theorem 3.17(c) for the rest of \(\text{Tail}_2(\mu)\), and Proposition 2.12(b) for the antipodal and bridge parts.
Now, Theorem 6.5 follows readily from Theorem 6.4 using induction on \( k \). The only condition not already checked is 6.1(e), which requires the chains in all the new chain collections to be pairwise disjoint. We prove this (for each value of \( k \) in the inductive construction) in Sect. 7.4. Since objects in chains for different values of \( k \) have different deficits, the final collection \( \mathcal{C}' \) also satisfies 6.1(e).

7. The Remaining Proofs

7.1. Proof of Lemma 6.8.

We use the notation \( \mu, \lambda, L, D, A^* \) introduced in Sect. 6.3. Assume \( D < A^* \), so \( D - A^* \) is a negative even integer by Lemma 6.7. We first show that \( L \in \{6, 7, 8\} \) and \( \lambda = \langle 0 \rangle \). Using (4.4), we have

\[
D - A^* \geq \left( \frac{L}{2} \right) - 3L - |\lambda| + \lambda_1 + 7 - (2L - \lambda_1^* - 5).
\]

Now, \( L \geq |\lambda| + 6 \geq 6 \), since \( \mu \) is a flagpole partition. Using this inequality to eliminate \(-|\lambda|\), we find

\[
D - A^* \geq \left( \frac{L}{2} \right) - 5L - L + 6 + \lambda_1 + 12 + \lambda_1^* = p(L) + \lambda_1 + \lambda_1^*,
\]

where \( p(L) = (L^2 - 13L + 36)/2 \). Now \( p(6) = p(7) = -3 \), \( p(8) = -2 \), and \( p(L) \geq 0 \) for all \( L \geq 9 \). Therefore, \( L \) must be 6, 7, or 8. Suppose, to get a contradiction, that \( \lambda^* \neq \langle 0 \rangle \). Then, \( \lambda^* \neq \langle 0 \rangle \), so \( \lambda_1 + \lambda_1^* \geq 2 \), so \( D - A^* \geq -3 + 2 = -1 \). This is impossible, since \( D - A^* \) is negative and even. Thus, \( \lambda = \langle 0 \rangle \).

We now know that the assumption \( D < A^* \) is possible only if \( \text{Tl}_2(\mu) = [V] \), where \( V \) is one of the six vectors \( v(\langle 0 \rangle, a, \epsilon) = 0012^{a-\epsilon}1^\epsilon \) with \( a \in \{3, 4, 5\} \) and \( \epsilon \in \{0, 1\} \). The following table computes \( \mu, D, A, \mu^*, A^* \) for each such \( V \). We see that \( D < A^* \) occurs in the first three rows only. But these cases are ruled out, because \( |\mu| \geq 6 \) by Definition 6.1(a).

| \( V \)   | \( \mu \)   | \( D \) | \( A \) | \( \mu^* \) | \( A^* \) |
|----------|-------------|--------|--------|------------|--------|
| 001222   | \langle 2^1 \rangle | 4      | 7      | \langle 1^4 \rangle | 6      |
| 001221   | \langle 1^4 \rangle | 5      | 6      | \langle 2^1 \rangle | 7      |
| 0012222  | \langle 1^5 \rangle | 7      | 9      | \langle 1^5 \rangle | 9      |
| 0012221  | \langle 2^13 \rangle | 8      | 8      | \langle 2^13 \rangle | 8      |
| 00122222  | \langle 2^14 \rangle | 11     | 11     | \langle 2^14 \rangle | 11     |
| 00122221  | \langle 1^6 \rangle | 12     | 10     | \langle 1^6 \rangle | 10     |
7.2. Proof of Lemma 6.9.
We must prove that for all \( i \in \{ A^*, A^* + 2, \ldots, D - 4, D - 2 \} \), \( \gamma = c_\lambda(i - 1) \) exists, \( \min_\Delta(\gamma) \leq L - 2 \), and \( z = \text{qdvl}_{L - 2}(\gamma) \) starts with 01 and contains a 2. When \( \min_\Delta(\gamma) \leq L - 3 \), the conclusion about \( z \) follows if \( \text{qdvl}_{L - 3}(\gamma) \) contains a 1, or equivalently \( \gamma \neq [0^{L - 3}] \).

Since \( \lambda \in P \), the chain \( C_\lambda \) already exists and starts at dinv index \( \ell(\lambda^*) \). To show that \( c_\lambda(i - 1) \) exists for all \( i \) in the given range, it suffices to prove \( A^* - 1 \geq \ell(\lambda^*) \). Using (4.4) and the bounds \( L \geq |\lambda| + 6 = |\lambda^*| + 6, |\lambda^*| \geq \lambda_1^* \), and \( |\lambda^*| \geq \ell(\lambda^*) \), we have the stronger bound
\[
A^* - 1 \geq 2L - \lambda_1^* - 7 \geq 2|\lambda^*| + 5 - \lambda_1^* \geq \ell(\lambda^*) + 5.
\]

Recall that \( \text{TI}_2(\mu) = [v(\lambda, a, \epsilon)] \) where \( v(\lambda, a, \epsilon) = 0012^{a - \epsilon}B^1_\lambda 1^\epsilon \) has length \( L \) and dinv \( D \). By Lemma 2.7 and Theorem 2.17(b), \( w = 01^{a - \epsilon}B_\lambda 0^\epsilon \) has dinv \( D - 1 \), has length \( L - 2 \), and belongs to plateau \( a \) of \( \text{tail}(\lambda) \subseteq C_\lambda \). This means that \( c_\lambda(D - 1) = [w] \). Every object \( c_\lambda(i - 1) \) considered here is a partition appearing in the chain \( C_\lambda \) an even positive number of steps before \( [w] \). Therefore, the needed conclusion follows from the next lemma.

**Lemma 7.1.** Assume \( I \) extends to \( \mu \) and \( \lambda = \text{ftype}(\mu) \). Let \( \gamma \) be any partition in the chain \( C_\lambda \) at least two steps before \( [w] = c_\lambda(D - 1) \), where \( w = 01^{a - \epsilon}B_\lambda 0^\epsilon \) has dinv \( D - 1 \), has length \( L - 2 \), and belongs to plateau \( a \) of \( \text{tail}(\lambda) \subseteq C_\lambda \). Then, \( \gamma \) satisfies one of these conditions: (a) \( \min_\Delta(\gamma) \leq L - 3 \) and \( \gamma \neq [0^{L - 3}] \); (b) \( \min_\Delta(\gamma) = L - 2 \) and \( z = \text{qdvl}_{L - 2}(\gamma) \) starts with 01 and contains a 2.

**Proof.** First, consider the case where \( \gamma \) is not in \( \text{tail}(\lambda) \). Let \( \lambda \) have h-vector \( (h_1, \ldots, h_N) \). For some \( j < N \), \( \min_\Delta(\gamma) \) is a value in the \( j \)th ascending run of the \( \min_\Delta \)-profile for \( C_\lambda \). By extra structure conditions 5.5(a) and (b), we deduce \( \min_\Delta(\gamma) \leq 1 + \max(h_1, h_N) \). Since the last ascending run of the \( \min_\Delta \)-profile corresponds to \( \text{tail}(\lambda) \), we have \( h_N = \min_\Delta(\text{TI}(\lambda)) \). By hypothesis 5.3(a), \( h_1 \) is the last entry in the h-vector for \( \lambda^* \), so that \( h_1 = \min_\Delta(\text{TI}(\lambda^*)) \). Proposition 2.15(a) shows \( \min_\Delta(\text{TI}(\lambda)) = \lambda_1 + \ell(\lambda) + 1 \leq |\lambda| + 2 \), and similarly \( \min_\Delta(\text{TI}(\lambda^*)) \leq |\lambda^*| + 2 = |\lambda| + 2 \). Since we know \( |\lambda| + 6 \leq L \), we finally get
\[
\min_\Delta(\gamma) \leq 1 + \max(\min_\Delta(\text{TI}(\lambda^*))), \min_\Delta(\text{TI}(\lambda^*)) \leq |\lambda| + 3 \leq L - 3. \tag{7.1}
\]

Here, \( \gamma \) cannot be \([0^{L - 3}] \), since \([0^{L - 3}] \) appears only in \( \text{tail}(\langle 0 \rangle) \) by Theorem 2.17, and \( \gamma \) is in \( C_\lambda \) outside \( \text{tail}(\lambda) \). Therefore, (a) holds.

Next, consider the case where \( \gamma \) is in \( \text{tail}(\lambda) \) and \( \lambda \neq \langle 0 \rangle \) (so that \( \gamma \neq [0^{L - 3}] \) and \( w \) must be reduced). Since \( \min_\Delta \) values weakly increase as we move forward through the tail (Theorem 2.16), we have \( \min_\Delta(\gamma) \leq \min_\Delta([w]) = \text{len}(w) = L - 2 \). If \( \gamma \) appears in the tail before plateau \( a \), then \( \min_\Delta(\gamma) \leq L - 3 \), so (a) holds. Suppose \( \gamma \) appears in plateau \( a \) before \([w] \), so that \( \min_\Delta(\gamma) = L - 2 \). Because \( \epsilon \) is 0 or 1, \([w] \) is the first or second Dyck class listed in Theorem 2.17(b). Since \( \gamma \) precedes \([w] \) in the tail by at least 2, \( \gamma \) must be one of the Dyck classes listed in Theorem 2.17(a). Then, \( z \) is one of the reduced vectors listed there, which all begin with 01 and contain a 2, since \( B_\lambda \) begins with 0 and ends with 1. Therefore, (b) holds.

Finally, consider the two special cases where \( \gamma \in \text{tail}(\langle 0 \rangle) \). If \( \epsilon = 0 \), then \([w] = [01^a] = [0^a] = [0^{L - 3}] \). Since \( \gamma \) appears before \([w] \) in the tail,
minhesion(\gamma) \leq L - 3$ and $\gamma$ is not $[0^{L-3}] = [w]$. Therefore, (a) holds. If $\epsilon = 1$, then $w = 01a^{-1}$ has length $L - 2$. $[w]$ is the first object in plateau $a$ of $\text{TAIL}(0)$, and the immediate predecessor of $[w]$ is $[0^a] = [0^a] = [0^{L-3}]$ (Theorem 2.17). Because $\gamma$ precedes $[w]$ by at least 2, $\min_{\Delta}(\gamma) \leq L - 3$ and $\gamma \neq 0^{L-3}$. Therefore, (a) holds.

\textbf{Remark 7.2.} We now show that $M_D(\mu) = V$ using the definition of $M_i(\mu)$ from §6.4. We saw above that $c_{\lambda}(D - 1) = [w]$ where $w = 01a^{-\epsilon}B_{\lambda}0^\epsilon$ has length $L - 2$. Therefore, $M_D(\mu) = 00w^+ = v(\lambda, a, \epsilon) = V$.

\textbf{7.3. Proof of Proposition 6.16(a)}

We must prove that each $A_j = \text{Ant}(S_j(V))$ starts with 0012 and contains a 3. Fix $j$ between 1 and $J$, and write $S_j(V) = 0E1$. As in Sect. 6.6, let $\rho$ be the unique partition with $[E]$ in $\text{TAIL}_2(\rho)$, let $\gamma$ be the unique object in $C_{\rho^*}$ with $\text{dinv}(\gamma) = \text{area}(E) = A - 2j - 1$ [see Proposition 6.12(c)], and let $z = QDV_{L - 2}(\gamma)$. Since $A_0 = 0z^+$, it suffices to prove: either $\min_{\Delta}(\gamma) \leq L - 3$ and $\gamma \neq 0^{L-3}$; or else $\min_{\Delta}(\gamma) = L - 2$, $z$ starts with 01, and $z$ contains a 2. Recall $S_0(\mu) = V = 0012n^0B_{\lambda^*}1^c$ where $n_0 = a - \epsilon \geq 1$, $\lambda = \text{ftype}(\mu)$, and $\epsilon$ is 0 or 1.

\textbf{Case 1:} Assume $1 \leq j \leq \lfloor n_0/2 \rfloor$. Here, $S_j(V) = 0012n_0^{-2j}B_{\lambda}1^c2^j$ by (3.4), so $L_j = \text{len}(S_j(V)) = \text{len}(V) = L$. Also, $E = 012n_0^{-2j}B_{\lambda}1^c2^j - 1$ has length $L - 2$ and area $A - 2j - 1$. We see that $[E] = [012n_0^{-2j}B_{\lambda}0^c2^j - 1]$ belongs to $\text{TAIL}(\lambda)$ by Theorem 2.17(b). Now, $\gamma = c_{\lambda^*}(A - 2j - 1)$ is an object in $C_{\rho^*}$ that is at least two steps before $c_{\lambda^*}(D^* - 1)$, since $j > 0$ and $A \leq D^*$ (by Lemma 6.8 for $\mu^*$). The required conclusions now follow from Lemma 7.1 (applied to $\mu^*$ and $\lambda^* = \text{ftype}(\mu^*)$), recalling from §6.3 that $V$ and $V^*$ both have length $L$.

\textbf{Case 2:} Assume $\lfloor n_0/2 \rfloor < j < J$. The description of $S_j(\mu)$ in Theorem 3.17(d) shows that $E = 01X^+2W$ for some binary vectors $X$ and $W$. $W$ must contain a 0, since for these $j$, the value of $i$ in (3.2) and (3.4) must be at least 1. Therefore, $E$ is reduced. We further claim that $W$ starts with $1^{c-10}$ for some $c \geq 2$. Formulas (3.2) and (3.4) show that the last 2 in $E$ is followed by $1^{n_0+\epsilon+2}[n_0/2]$. If $\epsilon = 1$, then this string of 1s is nonempty. If $\epsilon = 0$, then $n_0/2 = a/2 \geq 1$ (since $a \geq 2$), and again the string of 1s is nonempty. Theorem 2.17(a) now shows that $[E]$ belongs to the $c$th plateau of $\text{TAIL}(\rho)$ and that $\rho = 0$. Let $s$ be the number of objects in this plateau weakly following $[E]$, and let $n_0 = \min_{\Delta}(\text{TI}(\rho)) = \rho_1 + \ell(\rho) + 1$. By Theorem 2.16, $s \leq n_0 + c - 1$. Since $E$ is reduced, $L_j - 2 = \text{len}(E) = \min_{\Delta}([E]) = n_0 + c$.

Because $\rho$ is a needed partition for $\mu$, we know $\rho \in \mathcal{P}$, so $C_{\rho^*}$ and $C_{\rho^*}$ satisfy the opposite property. For each $n > 0$, let $C_{\rho^*}^{\leq n}$ be the finite set of $\gamma$ in $C_{\rho^*}$ with $\min_{\Delta}(\gamma) \leq n$; define $C_{\rho^*}^{\leq n}$ similarly. We use dinv to order these sets, so “the second largest object in $C_{\rho^*}^{\leq n}$” refers to the object with the second largest dinv value.

Take $n = n_0 + c$. The $c$th plateau of $\text{TAIL}(\rho)$ consists of objects with $\min_{\Delta} = n$, while objects in all later plateaus have $\min_{\Delta} > n$. Therefore, $[E]$ is the $s$th largest object in $C_{\rho^*}^{\leq n}$, and the $s$ largest objects have consecutive dinv
values. Recall from the proof of Lemma 6.14 that $\gamma$ is obtained from $[E]$ by invoking the opposite property $\text{Cat}_{n, \rho^*}(t, q) = \text{Cat}_{n, \rho}(t, q)$ for this value of $n$ (namely $n = L_j - 2 = n_0 + c$). Thus, $\gamma$ must be the $s$th smallest object in $C_{\rho^*}^{\leq n}$, where the $s$ smallest objects have consecutive dinv values. Hypothesis 5.3(a) shows that the smallest object in $C_{\rho^*}$, namely $\delta = c_{\rho^*}(\ell(\rho))$, has $\min_{\Delta}(\delta) = \min_{\Delta}(\text{TI}(\rho)) = n_0 \leq n$. Therefore, $\gamma$ must be $c_{\rho^*}(\ell(\rho) + s - 1)$.

Now, we prove $\min_{\Delta}(\gamma) \leq L_j - 3$. Apply the opposite property again, with $n - 1$ instead of $n$. The largest objects in $C_{\rho^*}^{\leq n-1}$ are the objects in plateaus $0$ through $c - 1$ of $\text{T} \text{A} \text{I} (\rho)$, which have consecutive dinv values. Because $c \geq 2$ and $s \leq n_0 + c - 1$, there are at least $s$ such objects (plateau $0$ contributes $1$ and plateau $c - 1$ contributes $n_0 + c - 2$). Therefore, the $s$ smallest objects in $C_{\rho^*}^{\leq n-1}$ have consecutive dinv values. Once again, the smallest object in $C_{\rho^*}$, namely $\delta$, has $\min_{\Delta}(\delta) = n_0 \leq n - 1$. Therefore, the $s$th smallest object in $C_{\rho^*}^{\leq n-1}$ is $c_{\rho^*}(\ell(\rho) + s - 1) = \gamma$. Thus, $\min_{\Delta}(\gamma) \leq n - 1 = L_j - 3$, as needed. Now, $\rho^* \neq (0)$, since $\rho \neq (0)$, and $[0^{L_j - 3}] \in \text{T} \text{A} \text{I} ((0)) \subseteq C((0))$. Therefore, $\gamma \neq [0^{L_j - 3}]$, since these objects belong to different $\text{NU}_1$-tails, which are pairwise disjoint.

Case 3: Assume $j = J$, so $E$ is $B_\mu$ with its final $1$ removed. First, assume $\mu \neq \langle 1^k \rangle$, so $B_\mu$ contains two $0$s and $E$ is reduced. By (3.3), $E$ is a binary Dyck vector beginning with $0^1c0$, where $c = p_{r+1} + \epsilon + 2|n_0/2| \geq 1$. We claim that either $c \geq 2$, or $c = 1$ and $E$ ends in $0$. This holds, because $a \geq 2$ and $c = 1$ imply $n_0 = a - \epsilon < 2$, $a = 2$, $\epsilon = 1$, $p_{r+1} = 0$, $n_r$ is odd, $0B_\mu$ given in (3.3) ends in $01$, and so $E$ ends in $0$. By the claim and Theorem 2.17(b), $[E]$ is in $\text{T} \text{A} \text{I} (\rho)$ in plateau $2$ or higher. We can now repeat the proof from Case 2 to see that $\min_{\Delta}(\gamma) \leq L_j - 3$. To see $\gamma \neq [0^{L_j - 3}]$, note that $E$ is a reduced BDV of length $L_j - 2$, so $\text{dinv}(\gamma) = \text{area}(E) \leq L_j - 4$. However, $\text{dinv}([0^{L_j - 3}]) = (L_j - 3) > L_j - 4$, since $L_j \geq L \geq 6$. (To see why $L \geq 6$, note from Example 3.8 that the only vectors $V = v(\lambda, a, \epsilon)$ with $\text{len}(V) < 6$ represent $\text{T} \text{I} \text{I}_2(\mu)$ for $\mu = \langle 111 \rangle$ and $\mu = \langle 21 \rangle$, but we are assuming $|\mu| \geq 6$.)

To finish Case 3, consider $\mu = \langle 1^k \rangle$. Here, $E = 01^{k-1}$ is not reduced and has length $k = L_j - 2$, area $k - 1$, and $\text{dinv} (k-1)$. Therefore, $\gamma$ is the unique object in $C((0)) = \text{T} \text{A} \text{I} ((0))$ having $\text{dinv} k - 1$. By Theorem 2.17, $[E] = [00^{k-2}]$ has $\min_{\Delta} = k - 1$ and is the last object in plateau $k - 2$ of $\text{T} \text{A} \text{I} ((0))$. Now, $k - 1 < (k-2)$ for all $k \geq 5$, so $\gamma$ appears strictly before $[E]$ in $\text{T} \text{A} \text{I} ((0))$. This means $\min_{\Delta} = \min_{\Delta}([E] = k - 1 = L_j - 3$, and moreover, $\gamma \neq [0^{L_j - 3}] = [E]$.

Remark 7.3. Letting $S_j(V) = 0E1$, we proved in all three cases that $[E]$ must belong to $\text{T} \text{A} \text{I} (\rho)$, not just $\text{T} \text{I} \text{I}_2(\rho)$.

7.4. Proof that Chains are Disjoint

This section completes the proofs of Theorems 6.4 and 6.5 by verifying that conditions 5.1(e) and 6.1(e) hold for all newly constructed chains $C_\mu$ indexed by $\mu$ of size $k$. In fact, we prove the stronger result that all such new chains $C_\mu$, along with the partial chains $\text{T} \text{I} \text{I}_2(\xi)$ for all other $\xi$ of size $k$, are pairwise disjoint.
Because the maps \( \text{NU} \) and \( \text{TI}_2 \) are one-to-one and each \( \text{TAIL}_2(\xi) \) is the \( \text{NU} \)-segment starting at \( \text{TI}_2(\xi) \), all second-order tails are pairwise disjoint. We must show that the bridge parts and antipodal parts of the various chains \( C_{\mu} \) do not overlap with each other or any second-order tail. Since \( \text{NU}_1 \) is a bijection and all parts are unions of \( \text{NU}_1 \)-segments, it suffices to analyze the \( \text{NU}_1 \)-initial objects \([M_i]\) and \([A_j^*]\) in the bridge part and antipodal part of \( C_{\mu} \).

**Step 1.** We show that \([M_i]\) and \([A_j^*]\) do not belong to any set \( \text{TAIL}_2(\xi) \). By Lemma 6.10 and Proposition 6.16, each \( M_i \) and \( A_j^* \) is a reduced Dyck vector starting with 00 and containing a 3. Examining Definition 3.9, we see that the reduction of a cycled ternary Dyck vector cannot have this form. Step 1 now follows from Theorem 3.10(a).

**Step 2.** We show that the bridge parts of the chains \( C_{\mu} \) do not overlap. It suffices to show that \( \mu \) can be recovered uniquely from any \( \text{NU}_1 \)-initial object \( \gamma = [M_i] \) defined in the construction of the bridge part of \( C_{\mu} \). Given such a \( \gamma \), we first obtain \( M_i \) as the unique reduced Dyck vector representing \( \gamma \). By Lemma 6.10, \( M_i \) must have length \( L \) and \( \text{dinv} \) \( i \) for some \( i \equiv D \) (mod 2). By definition, \( M_i = 00z^+ \) where \( [0z^+] = [z] = c_\lambda (i - 1) \) with \( \lambda = \text{ftype}(\mu) \). We deduce \( \lambda \) by finding the unique chain \( \gamma \lambda \) containing \([z] \). (This chain must already be known, given that \( C_{\mu} \) was successfully constructed.) Finally, since \( \Psi(\mu) = (\lambda, L, D \text{ mod } 2) \), we recover \( \mu \) by computing \( \mu = \Psi^{-1}(\lambda, L, i \text{ mod } 2) \).

**Step 3.** We show that the antipodal parts of the chains \( C_{\mu} \) do not overlap. It suffices to show that \( \mu \) can be recovered uniquely from any \( \text{NU}_1 \)-initial object \( \delta = [A_j^*] \) used in the construction of the antipodal part of \( C_{\mu} \). We recall the definition of \( A_j^* \) from Sect. 6.6. Starting with \( S = S_j(V^*) \), we write \( S = 0E1 \), find the unique \( \rho \) with \([E] \in C_{\rho} \), compute \( \gamma = c_{\rho^*}(\text{area}(S) - 1) \), let \( z \) be the representative of \( \gamma \) of length \( \text{len}(E) \), and set \( A_j^* = 00z^+ \). We recover all these quantities from \( \delta \) as follows. First, \( A_j^* \) is the unique reduced representative of the Dyck class \( \delta \). Then, \( L_j^* \) (the length of \( S \)) is \( \text{len}(A_j^*) \). Dropping the first 0 from \( A_j^* \), we can find the Dyck class \([0z^+] = [z] = \gamma \). We recover \( \rho^* \) by finding the unique chain \( C_{\rho^*} \) containing \( \gamma \) (this chain and its partner \( C_{\rho} \) must already be known, given that \( C_{\rho} \) was successfully constructed). Now, \([E] \) must be the unique object in \( C_{\rho} \) with \( \text{dinv}([E]) = \text{area}(z) \), since the proof of Lemma 6.14 shows that both sides must equal \( \text{dinv}(S) - (L_j^* - 2) \). Next, \( E \) itself is the representative of \([E] \) of length \( L_j^* - 2 \), and then, \( S_j(V^*) = 0E1 \). Finally, we recover \( \mu^* \) (and hence \( \mu \)) by finding the unique second-order tail \( \text{TAIL}_2(\mu^*) \) containing \([S_j(V^*)] \). One way to do this is to follow the NU-chain from \( S_j(V^*) \) until the Dyck class \( \text{TI}(\mu^*) = [0B_{\mu^*}] \) is reached. Then, \( \mu^* \) can be read off from the binary representative \( 0B_{\mu^*} \), and \( \mu \) is \( \mu^{**} \).

**Step 4.** We introduce a variation of the antipode map, called \( \text{Ant}' \), that is an involution interchanging area and \( \text{dinv} \) and preserving length, deficit, and \( \text{min}_{\Delta} \) (compare to Lemma 6.14(b)). Inputs to \( \text{Ant}' \) are certain vectors \( 00z^+ \) where \( z \) is a Dyck vector. Suppose \( 00z^+ \) has length \( \ell \), area \( a \), \( \text{dinv} \) \( d \), and deficit \( k \). By Lemma 2.7, \( z \) must have length \( \ell - 2 \), area \( a - (\ell - 2) \), \( \text{dinv} \) \( d - 1 \), and deficit \( k - (\ell - 2) < k \). Suppose \([z] \) belongs to a known chain \( C_{\rho} \) with known opposite chain \( C_{\rho^*} \), \( \gamma = c_{\rho^*}(a - 1) \) exists, and \( \gamma \) is represented by a Dyck vector \( w \) of length \( \ell - 2 \). In this situation, define \( \text{Ant}'(00z^+) = 00w^+; \)
otherwise, \( \text{Ant}'(00z^+) \) is not defined. Note that \( \text{len}(w) = \ell - 2 = \text{len}(z) \), \( \text{defc}(w) = |\rho'| = |\rho| = \text{defc}(z) \), and \( \text{dinv}(w) = \text{dinv}(\gamma) = a - 1 \). It follows that \( \text{area}(w) = d - (\ell - 2) \) since \( \text{area}(w) + \text{dinv}(w) = \left(\frac{\text{len}(w)}{2}\right) - \text{defc}(w) = \text{area}(z) + \text{dinv}(z) \). Now, Lemma 2.7 shows \( 00w^+ \) has length \( \ell \), area \( d \), \( \text{dinv} a \), and deficit \( k \), as needed. Applying \( \text{Ant}' \) to input \( 00w^+ \), we see (using \( \rho^{**} = \rho \) and \( [z] = c_\rho(d - 1) \)) that \( \text{Ant}'(00w^+) = 00z^+ \). Therefore, \( \text{Ant}' \) is an involution on its domain.

Step 5. We show that for \( i \in \{A^*, A^* + 2, \ldots, D - 2, D\} \), \( \text{Ant}' \) interchanges \( M_i(\mu) \) and \( M_{A+D-i}(\mu^*) \). From (6.2), \( A + D = \left(\frac{D}{2}\right) - k = A^* + D^* \), so \( i \) is in the given range if and only if \( A + D - i \in \{A, A + 2, \ldots, D^* - 2, D^*\} \). By Lemma 6.10 and Remark 7.2, \( M_i(\mu) \) and \( M_{A+D-i}(\mu^*) \) are well-defined Dyck vectors of length \( L \). Also, for \( \lambda = \text{ctype}(\mu) \), \( M_i(\mu) = 00z^+ \) where \( z \) is the length \( L - 2 \) representative of \( c_i(i - 1) \); and \( M_{A+D-i}(\mu^*) = 00w^+ \) where \( w \) is the length \( L - 2 \) representative of \( c_{A^*}(A + D - i - 1) \). Comparing these expressions to the definition of \( \text{Ant}'(00z^+) \) in Step 4, we need only check that \( \text{area}(M_i(\mu)) = A + D - i \). This holds, since Lemma 6.10 and (6.2) give \( \text{area}(M_i(\mu)) = \left(\frac{D}{2}\right) - k - i = A + D - i \).

Step 6. Let \( A_j = \text{Ant}(S_j(V)) \). We show that if \( \text{Ant}'(A_j) \) is defined, then \( \text{Ant}'(A_j) = S_j(V) \). Recall how \( A_j \) is computed. Let \( S_j(V) \) have length \( L' \), area \( A' = A - 2j \), and \( \text{dinv}(D') \). Write \( S_j(V) = 0E1 \), where \( E \) is a TDV, \( \text{len}(E) = L' - 2 \), \( \text{dinv}(E) = D' - (L' - 2) \), and (by Remark 7.3) \( [E] \) belongs to \( \text{Tail}(\rho) \) for some \( \rho \). Then, \( A_j = 00z^+ \), where \( \text{len}(z) = L' - 2, \text{dinv}(z) = A' - 1 \), and \( [z] \in C_\rho \). We proved \( \text{area}(A_j) = \text{dinv}(S_j(V)) = D' \). Assume \( \text{Ant}'(A_j) \) is defined and equals \( 00w^+ \). This means that \( w \) is a Dyck vector of length \( L' - 2 \), such that \( [w] \in C_\rho \) and \( \text{dinv}(|w|) = D' - 1 = \text{dinv}([E]) + L' - 3 \). Because \( [E] \) is in \( \text{Tail}(\rho) \), \( [w] \) must be \( \text{nul}_{L' - 3}([E]) \).

We claim that \( E \) is not reduced. Otherwise, \( [E] \) belongs to a plateau of \( \text{Tail}(\rho) \) consisting of \( L' - 3 \) (or fewer) objects with \( \text{min}_A \) equal to \( \text{min}_A(E) = \text{len}(E) = L' - 2 \) (Theorem 2.16). But then, \( [w] = \text{nul}_{L' - 3}([E]) \) has \( \text{min}_A([w]) > L' - 2 = \text{len}(w) \), which contradicts \( w \) being a Dyck vector. Therefore, the TDV \( E \) is not reduced, say \( E = 0Y^+ \) where \( Y \) is a BDV of length \( L' - 3 \). By Proposition 2.13, \( [w] = \text{nul}_{L' - 3}([Y]) = [Y0] \). Since \( \text{len}(w) = L' - 2 = \text{len}(Y0) \), we get \( w = Y0 \) and \( \text{Ant}'(A_j) = 00w^+ = 00Y^+1 = 0E1 = S_j(V) \).

Step 7. We show that for any two flagpole partitions \( \mu \neq \nu \), such that \( C_\mu \) and \( C_\nu \) have been constructed, the bridge part of \( C_\mu \) does not overlap the antipodal part of \( C_\nu \). To get a contradiction, assume there exist \( i \in \{A^*, A^* + 2, \ldots, D - 2\} \) and \( j > 0 \) with \( v = M_i(\mu) = A'_j \), where \( A'_j = \text{Ant}(S_j(V')) \) for \( V' \) the reduced representative of \( \text{Til}_2(\nu^*) \). By Step 5, \( \text{Ant}'(v) = M_{A+D-i}(\mu^*) \). Since \( \text{Ant}'(v) \) is defined, Step 6 shows that \( \text{Ant}'(v) = S_j(V') \). We have now contradicted Step 1, since \( S_j(V') \in \text{Tail}_2(\nu^*) \), while \( M_{A+D-i}(\mu^*) \) \( \text{not} \) in \( \text{Tail}_2(\nu^*) \). The index \( i = A^* \) is special. For this \( i \), (6.2) and Remark 7.2 imply \( M_{A+D-i}(\mu^*) = M_{D^*}(\mu^*) = V^* \). However, \( V^* \) is in \( \text{Tail}_2(\mu^*) \) and thus not in \( \text{Tail}_2(\nu^*) \), since \( \mu^* \neq \nu^* \). These contradictions prove Step 7.
8. Generalized Flagpole Partitions

We know that \( \mu \) is a flagpole partition if \( \text{TI}_2(\mu) = [v(\lambda, a, \epsilon)] \) for sufficiently large \( a \) (namely, \( a \geq a_0(\lambda) \), which is equivalent to \( v(\lambda, a, \epsilon) \) having length \( L \geq |\lambda| + 6 \)). Examining the constructions of Sects. 6 and 7, we see that the condition \( L \geq |\lambda| + 6 \) was used only three times: showing that \( \mu^* \) is well defined in Lemma 6.7; proving \( D \geq A^* \) in Sect. 7.1; and checking our claims about bridge generators in Sect. 7.2. By making minor modifications to these three proofs, we can extend the chain constructions for flagpole partitions to a larger class of partitions called generalized flagpole partitions. Informally, these new partitions arise by replacing the lower bound \( a_0(\lambda) \) by a smaller number (often as small as 2). We give the formal definition next, and then discuss the changes needed for the three proofs.

8.1. Definition of Generalized Flagpole Partitions

**Definition 8.1.** Let \( \rho \mapsto \rho^* \) be a size-preserving involution defined on some collection \( \mathcal{P} \) of partitions. Suppose \( \mu \) is a partition of size \( k \), such that \( \text{TI}_2(\mu) = [V] \) where \( V = v(\lambda, a, \epsilon) \) has length \( L \). We say \( \mu \) is a *generalized flagpole partition* (relative to the given involution) if and only if \( \lambda \in \mathcal{P} \) and

\[
L \geq 5 + \lambda_1 + \ell(\lambda) \quad \text{and} \quad L \geq 5 + \lambda_1^* + \ell(\lambda^*). \quad (8.1)
\]

Since \( L = a + 3 + \lambda_1 + \ell(\lambda) \) (Remark 4.3), \( \mu \) is a generalized flagpole partition iff \( \lambda \in \mathcal{P} \) and \( a \geq 2 \) and \( a \geq 2 + \lambda_1^* + \ell(\lambda^*) - \lambda_1 - \ell(\lambda) \). Note that this condition reduces to \( a \geq 2 \) when \( \lambda = \lambda^* \).

**Example 8.2.** For any chain collection \( (\mathcal{P}, I, C) \), we found that \( \lambda = \langle 3321^4 \rangle \) has \( \lambda^* = \langle 531^4 \rangle \) in Sect. 6.3. Since \( \lambda_1 + \ell(\lambda) = 10 \) and \( \lambda_1^* + \ell(\lambda^*) = 11 \), every \( \mu \), such that \( \text{TI}_2(\mu) = [v(\langle 3321^4 \rangle, a, \epsilon)] \) for some \( a \geq 3 \) (equivalently, \( L \geq 16 \)) is a generalized flagpole partition. For \( \mu \) to be a flagpole partition, we would need \( a \geq a_0(\lambda) = 5 \) (equivalently, \( L \geq 18 \)). Similarly, every \( \mu \) such that \( \text{TI}_2(\mu) = [v(\langle 531^4 \rangle, a, \epsilon)] \) for some \( a \geq 2 \) (equivalently, \( L \geq 16 \)) is a generalized flagpole partition. For \( \mu \) to be a flagpole partition, we would need \( a \geq 4 \) (equivalently, \( L \geq 18 \)). The difference in the bounds on \( a \) becomes more dramatic when the diagrams of \( \lambda \) and \( \lambda^* \) have many cells outside the first row and column.

For all generalized flagpole partitions \( \mu \), \( V \) starts with 0012. Thus, the analysis of \( \text{TAIL}_2(\mu) \) in Sect. 6.5 still applies. The next theorem is an unconditional result not relying on the assumed existence of any chain collections.

**Theorem 8.3.** Let \( \lambda \mapsto \lambda^* \) be any size-preserving involution on the set of all integer partitions. Let \( g(k) \) be the number of generalized flagpole partitions of size \( k \) (defined using this involution). (a) For all \( k \)

\[
g(k) \geq \sum_{j=1}^{k-1} 2 \max \left\{ 0, p(j) - 2 \sum_{i=k-3-j}^{j} q_i(j) \right\}. \quad (8.2)
\]

(b) For any real \( c < 1 \) and all sufficiently large \( k \), \( g(k) > p(k)^c \).
See Sect. 8.5 for the proof. Remark 4.10 gives the corresponding asymptotic enumeration of flagpole partitions.

**Theorem 8.4.** Theorems 6.4 and 6.5 hold with flagpole partitions replaced by generalized flagpole partitions throughout.

For Theorem 6.4, generalized flagpole partitions are defined relative to the involution $I$ in the given chain collection. For Theorem 6.5, generalized flagpole partitions of size $k$ are defined relative to the involution $I^k$ on $P^k$. Both theorems follow from the constructions and proofs already given, after making the modifications in the next three subsections.

### 8.2. Defining $\mu^*$ for Generalized Flagpole Partitions

We modify the bijection $\Psi$ from Lemma 4.11 as follows. Let $F_k$ be the set of generalized flagpole partitions of size $k$. Let $H_k$ be the set of triples $(\lambda, L, \eta)$, such that $\lambda$ is an integer partition in $P$ of size less than $k$, $L = k + 2 - |\lambda|$, $L$ satisfies (8.1), and $\eta \in \{0, 1\}$. Given $\mu \in F_k$, say $TI_2(\mu) = [V]$ where $V = v(\lambda, a, \epsilon)$ has length $L$, dinv $D$, and area $A$. Recall (Remark 4.3) that $k = \text{defc}(V) = |\lambda| + L - 2$. Define $\Psi_k : F_k \rightarrow H_k$ by $\Psi_k(\mu) = (\lambda, L, D \mod 2)$.

The proof of Lemma 4.11 shows that $\Psi$ is a bijection; we need only replace the old condition $L \geq |\lambda| + 6$ by (8.1).

Furthermore, (8.1) ensures that $\lambda^*, L, A \mod 2$ also belongs to the codomain $H_k$. Therefore, we may define $\mu^*$ to be the unique object in $F_k$ with $\Psi_k(\mu^*) = (\lambda^*, L, A \mod 2)$. The rest of the proof of Lemma 6.7 goes through with no changes.

### 8.3. Proving $D \geq A^*$ for Generalized Flagpole Partitions

In Sect. 7.1, we used $L \geq |\lambda| + 6$ to eliminate $|\lambda|$, in the estimate

$$D - A^* \geq \binom{L}{2} - 5L + 12 + \lambda_1 + \lambda_1^* - |\lambda|. \quad (8.3)$$

We give a modified estimate here using (8.1). First, consider the case $\lambda \neq \langle 0 \rangle$. The diagram of $\lambda$ fits in a rectangle with $\ell(\lambda)$ rows and $\lambda_1$ columns, so $|\lambda| \leq \lambda_1 \ell(\lambda)$. Moreover, $\lambda_1 \ell(\lambda) \leq \max(\lambda_1^2, \ell(\lambda)^2) \leq \lambda_1^2 + \ell(\lambda)^2$. Since $L \geq \lambda_1 + \ell(\lambda) + 5$, we get

$$(L - 5)^2 \geq \lambda_1^2 + 2\lambda_1 \ell(\lambda) + \ell(\lambda)^2 \geq 3|\lambda|.$$

Thus, $-|\lambda| \geq -(L - 5)^2/3$. We also have $\lambda_1 + \lambda_1^* \geq 2$, since $\lambda \neq \langle 0 \rangle$. Using these estimates in (8.3) and simplifying, we get $D - A^* \geq (L^2 - 13L + 34)/6$. This polynomial in $L$ exceeds $-2$ for all $L$, so the even integer $D - A^*$ must be nonnegative.

If $\lambda = \langle 0 \rangle$, then (8.3) becomes $D - A^* \geq \binom{5}{2} - 5L + 12$. Here, $D - A^* \leq -2$ is possible only for $5 \leq L \leq 8$. The exceptional cases $L = 6, 7, 8$ were already examined in the table in §7.1. If $L = 5$, then $|\mu| = |\lambda| + L - 2 = 3$, but we know from Definition 6.1(a) that $|\mu| \geq 6$. 
8.4. Modified Bridge Analysis

We modify two calculations in Sect. 7.2 where the old assumption $L \geq |\lambda| + 6$ was used. To prove $A^* - 1 \geq \ell(\lambda^*)$, use (4.4) and the second part of (8.1) to get

$$A^* - 1 \geq 2L - \lambda^*_1 - 7 \geq 3 + \lambda^*_1 + 2\ell(\lambda^*) > \ell(\lambda^*).$$

Since $\min_\Delta(T\lambda(\lambda)) = \lambda_1 + \ell(\lambda) + 1$ and $\min_\Delta(T\lambda(\lambda^*)) = \lambda^*_1 + \ell(\lambda^*) + 1$, the bound (7.1) becomes

$$\min_\Delta(\gamma) \leq \max(2 + \lambda_1 + \ell(\lambda), 2 + \lambda^*_1 + \ell(\lambda^*)) \leq L - 3.$$

8.5. Proof of Theorem 8.3

Let $g(k)$ be the number of generalized flagpole partitions of size $k$, relative to a size-preserving involution $\lambda \mapsto \lambda^*$ on the set of all integer partitions. Fix a real $c < 1$. We prove $g(k) > p(k)^c$ for all sufficiently large $k$. It suffices to consider $c > 1/2$. For any nonzero partition $\lambda$, let $h(\lambda) = \lambda_1 + \ell(\lambda) - 1$, which is the longest hook-length in the diagram of $\lambda$. For $0 < i \leq j$, let $q_i(j)$ be the number of partitions $\lambda$ of size $j$ with $h(\lambda) = i$. We begin by proving the bound (8.2).

Fix $j$ between 1 and $k - 1$, and consider a fixed partition $\lambda$ of size $j$. The Dyck vector $v(\lambda, a, \epsilon)$ has deficit $k$ iff the length $L$ of this vector (which is a constant plus $a$) satisfies $L = k + 2 - |\lambda| = k + 2 - j$. The corresponding partition $\mu = T\lambda_2^{-1}([v(\lambda, a, \epsilon)])$ is a generalized flagpole partition iff $k + 2 - j \geq h(\lambda) + 6$ and $k + 2 - j \geq h(\lambda^*) + 6$ [by (8.1)]. Therefore, for each partition $\lambda$ of size $j$, such that $h(\lambda) \leq k - 4 - j$ and $h(\lambda^*) \leq k - 4 - j$, we obtain two generalized flagpole partitions of size $k$ (since $\epsilon$ can be 0 or 1).

Let $P$ be the set of partitions of size $j$. $P$ is the disjoint union of the sets $A = \{\lambda \in P : h(\lambda) \leq k - 4 - j\}$ and $B = \{\lambda \in P : h(\lambda) \geq k - 3 - j\}$. The partitions in $P$ are paired by the involution $\lambda \mapsto \lambda^*$. For each $\lambda \in A$ that is paired with some $\lambda^* \in A$, we obtain 2 generalized flagpole partitions of size $k$. In the worst case, every partition in $B$ pairs with something in $A$. Then, we would still have at least $|A| - |B| = |P| - 2|B|$ partitions in $A$ that pair with something in $A$. Therefore, we get at least $2\max(0, |P| - 2|B|)$ generalized flagpole partitions of size $k$ from this choice of $j$. Now, $|P| = p(j)$ and $|B| = \sum_{i=k-3-j}^j q_i(j)$. Summing over $j$ gives (8.2).

Next, we estimate $q_i(j)$. To build the diagram of a partition counted by $q_i(j)$, first select a corner hook of size $i$ (consisting of the first row and column of the diagram) in any of $i$ ways. Then, fill in the remaining cells of the diagram with some partition of $j-i$. Not every such partition fits inside the chosen hook, but we get the bound $q_i(j) \leq ip(j - i)$. Therefore, (8.2) becomes $g(k) \geq \sum_{j=1}^{k-1} \max\left\{0, 2p(j) - \sum_{i=k-3-j}^j 4ip(j - i)\right\}$. We prove $g(k) > p(k)^c$ (if $k$ is large enough) by finding a single index $j$, such that

$$2p(j) - \sum_{i=k-3-j}^j 4ip(j - i) > p(j) > p(k)^c.$$
We claim $j = \lceil kc \rceil$ will work. Recall the Hardy–Ramanujan estimate $p(k) = \Theta \left( k^{-1} \exp(\pi \sqrt{2k/3}) \right)$. We have $p(j) = \Theta \left( (ck)^{-1} \exp(\pi \sqrt{2ck/3}) \right)$ and $p(k)^c = \Theta \left( k^{-c} \exp(\pi \sqrt{2c^2k/3}) \right)$. Since $c > c^2$, $p(j) > p(k)^c$ for large enough $k$.

Next, we show $\sum_{i=3-j}^{j} \frac{j}{i}(j-i) < p(j)$ for large $k$. There are $j-(k-4-j) = 2j-k+4 \leq k+4$ summands, and each summand is at most $4jp(j-(k-3-j)) \leq 4kp(2j-k+3)$. Therefore, it suffices to show $(k+4)4kp(2j-k+3) < p(j)$ for large $k$. Using $j = \lceil kc \rceil$, we compute

\[ p(j)/p(2j-k+3) = \Theta \left( \exp \left[ \frac{\pi \sqrt{2kc/3} - \pi \sqrt{(2c-1)2k/3 + 2}}{k/3} \right] \right). \]

Now, it is routine to check that for $A > B > 0$, any $C$, and any polynomial $f(k)$, $\exp(\sqrt{Ak} - \sqrt{Bk+C}) > f(k)$ for large enough $k$. This follows by taking logs, dividing by $\sqrt{k}$, and using L’Hospital’s Rule to take the limit as $k$ goes to infinity. Since $0 < 2c-1 < c$, we get $p(j)/p(2j-k+3) > 4k(k+4)$ for large $k$, as needed.

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**References**

[1] Maria Monks Gillespie, Two $q, t$-symmetry problems in symmetric function theory, Open Problems in Algebraic Combinatorics blog (online at http://samuelfhopkins.com/OPAC/files/opacblog_master.pdf), pp. 38–41.

[2] J. Haglund, The $q, t$-Catalan Numbers and the Space of Diagonal Harmonics, with an Appendix on the Combinatorics of Macdonald Polynomials, AMS University Lecture Series (2008).

[3] S. Han, K. Lee, L. Li, and N. Loehr, Chain decompositions of $q, t$-Catalan numbers via local chains, Ann. Comb. 24 (2020), 739–765.

[4] S. Han, K. Lee, L. Li, and N. Loehr, Extended Appendix of [3], online at sites.google.com/oakland.edu/li2345/code-and-data

[5] G. H. Hardy and S. Ramanujan, Asymptotic formulae in combinatory analysis, Proc. London Math. Soc. 17 (1918), 75–115.

[6] K. Lee, L. Li, and N. Loehr, A combinatorial approach to the symmetry of $q, t$-Catalan numbers, SIAM J. Discrete Math. 32 (2018), 191–232.

[7] K. Lee, L. Li, and N. Loehr, Limits of modified higher $q, t$-Catalan numbers, Electron. J. Combin. 20(3) (2013), research paper P4, 23 pages (electronic).
[8] N. Loehr and G. Warrington, A continuous family of partition statistics equidistributed with length, J. Combin. Theory Ser. A 116 (2009), 379–403.

[9] A. M. Odlyzko, Asymptotic enumeration methods, in Handbook of Combinatorics Vol. 2, edited by R. L. Graham, M. Groetschel, and L. Lovasz, Elsevier (1995), 1063–1229.

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