Electromagnetism and Gravitation

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Abstract

Modern experiment has shown that bulk matter exhibits quantum structure and that this structure is electromagnetic in character. Therefore, we treat the electromagnetic field as the source of gravitation. In this way, the theory of gravitation becomes consistent with the quantum theory of matter, which holds that electric charge (or ‘generalized charge’) is the most fundamental attribute of matter.

The following predictions of the theory may lie within reach of present-day methods in astronomy:

(1) any massive body generates a time-dependent gravitational field;

(2) there is a linear correlation between the gravitational red-shift of a stellar source and the energy of cosmic rays emitted by that source, given by \( \frac{\Delta \nu}{\nu_0} = \text{energy (eV)}/10^{27} \);

(3) the maximum energy of cosmic rays is \( 10^{27} \) eV;

(4) this limit is associated with an infinitely red-shifted stellar object—an “electrostatic black-hole”—at the potential \( c^2/G^{1/2} = 10^{27} \) volts.

Finally, the theory predicts that the gravitational potential near any charged elementary particle is many orders of magnitude greater than the Newtonian value.
I. Introduction

In the theory which follows, electromagnetism is taken to be the source of gravitation. The foundation for this hypothesis lies in the observed nature of matter itself. Modern experiment has shown that matter can no longer be described, in any fundamental way, by an amorphous ‘density of mass.’ Rather, the experiments reveal an intricate quantum structure which is largely electromagnetic in character. This profound change in our knowledge of matter has met with no corresponding change in the theory of the gravitational field. Consequently, a perplexing estrangement has replaced the close historical tie between gravity and matter. It is toward restoring this close tie that we propose to trace the origin of gravitation to the quantum of electric charge.

A development of this nature cannot take place within the context of Einstein’s theory of general relativity. According to that theory, electrostatic energy produces a repulsive gravitational field. (This follows from the Reissner-Nordstrom solution of Einstein’s field equations, for a free charged particle, at rest.) In order to arrive at the attractive gravitational field of experience, we must reformulate the problem of space and time. To this end, we abandon the four-dimensional vector basis of general relativity. At each point in space-time, we introduce a scalar basis $e_0$ and a three-dimensional vector basis $e_i$, $i = 1, 2, 3$. We begin by expressing the Lorentz transformation in terms of this scalar, three-vector basis. This was first accomplished by L. Silberstein in 1912 [1]. We then go on to eliminate relative motion from the theory, by suitably restricting the transformations of space and time. This gives rise to a covariant expression which is identified with the gravitational field. Thus, according to our theory, a gravitational field induces non-uniformity in space and time which ‘breaks the symmetry’ of Lorentz invariance. This field has well-defined energy, momentum, and stress at each point in space. Moreover, it has tensor character and cannot be arbitrarily transformed to zero. Finally, it interacts with electromagnetism via a tensor force and thereby justifies our unified treatment of the field.

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1Evidence for this may be found in any book on modern physics. On the one hand, there are the strong, electromagnetic, and weak interactions between the elementary particles of matter; on the other hand, some 40 orders of magnitude removed, there is the Newtonian gravitational interaction of ‘bulk’ matter.
II. Transformations of Space and Time

The quaternion formulation of special relativity was introduced by L. Silberstein in 1912–1913 [1]. In this work, Silberstein made use of the original quaternion basis due to Hamilton (1, i, j, k), where ij = −ji = k (cyclic) and i² = j² = k² = −1. Here, we adopt the system of Pauli operators \((\sigma_0, \sigma_1, \sigma_2, \sigma_3)\), where \(\sigma_1 \sigma_2 = -\sigma_2 \sigma_1 = i \sigma_3\) (cyclic) and \(\sigma_1^2 = \sigma_2^2 = \sigma_3^2 = \sigma_0\). The fundamental interval in this theory is a quaternion, i.e., a sum of scalar and three-vector displacements

\[
ds = c \, d\tau + \mathbf{d}x = \sigma_0 \, dx^0 + \sigma_i \, dx^i = \sigma_\mu \, dx^\mu \tag{1}
\]

The operator \(\sigma_0\) commutes with all operators \(\sigma_\mu\) and, in this way, time is distinguished as a scalar variable. The interval \(ds\) transforms according to the general rule

\[
\overline{ds} = a(ds)b \tag{2}
\]

where \(a\) and \(b\) are also quaternions. For example, the Lorentz transformation

\[
\begin{align*}
\overline{dx^0} & = \gamma(dx^0 - \frac{v}{c} dx^1) \tag{3} \\
\overline{dx^1} & = \gamma(dx^1 - \frac{v}{c} dx^0) \tag{4} \\
\overline{dx^2} & = dx^2 \tag{5} \\
\overline{dx^3} & = dx^3 \tag{6}
\end{align*}
\]

is given by (2), with [1,4]

\[
a = b = \pm \frac{1}{\sqrt{2}} (\sigma_0 \sqrt{\gamma + 1} - \sigma_1 \sqrt{\gamma - 1})
\]

\[
= \pm (\sigma_0 \cosh \frac{\xi}{2} - \sigma_1 \sinh \frac{\xi}{2}) \tag{7}
\]

and
\[ \gamma = \cosh \xi = \left( 1 - \frac{v^2}{c^2} \right)^{-1/2} \] (8)

A scalar product can be defined by introducing the conjugate basis \( \tilde{\sigma}_\mu = (\sigma_0, -\sigma_i) \) and conjugate interval [5]

\[ d\tilde{s} = \sigma_\mu dx^\mu = \sigma_0 dx^0 - \sigma_i dx^i \] (9)

It follows from the commutation relations that

\[ dsd\tilde{s} = (\sigma_0 dx^0 + \sigma_i dx^i)(\sigma_0 dx^0 - \sigma_j dx^j) = \sigma_0(dx^{02} - dx^{12} - dx^{22} - dx^{32}) \] (10)

Transformers \( a \) and \( b \) have unit magnitude

\[ a\tilde{a} = b\tilde{b} = \sigma_0 \] (11)

therefore, the scalar product is invariant

\[ \overline{d\tilde{s}} = a(ds)b\tilde{b}(d\tilde{s})\tilde{a} = dsd\tilde{s} \] (12)

It is convenient to form the symmetric products

\[ \frac{1}{2}(\sigma_\mu \tilde{\sigma}_\nu + \sigma_\nu \tilde{\sigma}_\mu) = \sigma_0 \eta_{\mu\nu} \] (13)

where

\[ \eta_{\mu\nu} = \begin{pmatrix} 1 & 0 \\ -1 & 0 \\ 0 & -1 \end{pmatrix} \] (14)

Then

\[ dsd\tilde{s} = \sigma_\mu \tilde{\sigma}_\nu dx^\mu dx^\nu \]
\[ = \frac{1}{2}(\sigma_\mu \tilde{\sigma}_\nu + \sigma_\nu \tilde{\sigma}_\mu) dx^\mu dx^\nu \]
\[ = \sigma_0 \eta_{\mu\nu} dx^\mu dx^\nu \] (15)
The above formalism may be extended in a straightforward manner to variable basis systems:

\[
    ds = c\,d\tau + d\mathbf{r}
    = e_0(x)\,dx^0 + e_i(x)\,dx^i
    = e_\mu(x)\,dx^\mu
\]  (16)

Here, we retain the scalar nature of time and the three-vector nature of space by setting

\[
    e_0(x) = e_0^0(x)\,\sigma_0
\]  (17)

\[
    e_i(x) = e_j^i(x)\,\sigma_j
\]  (18)

The conjugate interval is

\[
    d\tilde{s} = \tilde{e}_\mu\,dx^\mu = e_0\,dx^0 - e_i\,dx^i
\]  (19)

Symmetric products are found to yield scalar functions

\[
    \frac{1}{2}(e_\mu \tilde{e}_\nu + e_\nu \tilde{e}_\mu) = \sigma_0\,g_{\mu\nu}
\]  (20)

where

\[
    g_{\mu\nu} = \begin{pmatrix}
        g_{00} & 0 & 0 & 0 \\
        0 & 0 & 0 & 0 \\
        0 & 0 & g_{ij} & 0 \\
        0 & 0 & 0 & 0
    \end{pmatrix}
\]  (21)

and

\[
    g_{00} = e_0^0 e_0^0
\]  (22)

\[
    g_{ij} = - \left[ e_1^1 e_1^j + e_2^2 e_2^j + e_3^3 e_3^j \right]
\]  (23)

These metric coefficients yield the invariant product

\[
    ds\,d\tilde{s} = e_\mu \tilde{e}_\nu\,dx^\mu dx^\nu
    = \sigma_0\,g_{\mu\nu}\,dx^\mu dx^\nu
\]  (24)

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2The term “metric” will be used, even though component \(g_{00}\) has no foundation in geometry.
III. The Structure of Space and Time

The basis system \( e_\mu(x) \) varies from point to point in a given manifold, the vectors \( e_i \) changing in both magnitude and direction. This behavior is expressed in terms of coefficients \( Q^\mu_{\nu\lambda} \)

\[
\nabla_\nu e_\mu = e_\lambda Q^\lambda_{\mu\nu} \tag{25}
\]

The rate of change of the scalar basis must be of scalar form

\[
\nabla_\nu e_0 = e_0 Q^0_{0\nu} \quad (\nu = 0, 1, 2, 3) \tag{26}
\]

\[
Q^j_{0\nu} \equiv 0 \quad (j = 1, 2, 3) \tag{27}
\]

while the rate of change of the vector basis is a three-vector

\[
\nabla_\nu e_i = e_j Q^j_{i\nu} \tag{28}
\]

\[
Q^0_{i\nu} \equiv 0 \tag{29}
\]

Therefore, the \( Q^\mu_{\nu\lambda} \) are generally non-symmetric in lower indices.

Certain of the coefficients \( Q^\mu_{\nu\lambda} \) have tensor character. These are determined in the following way. A given observer may record a time measurement \( d\tau \) and vector displacement \( dr \) with respect to basis \( e_\mu \):

\[
c\,d\tau = e_0 \, dx^0, \quad dr = e_i \, dx^i \tag{30}
\]

This observer is free to change the vector basis \( e_i \rightarrow e'_i \), in which case

\[
\,dr' = e'_i \, dx'^i \quad \text{where} \quad dr' = dr \tag{31}
\]

He may also change the scalar basis \( e_0 \rightarrow e'_0 \) (rate of time coordinate clocks), without affecting the length of time involved

\[
c\,d\tau' = e'_0 \, dx'^0 \quad \text{where} \quad d\tau' = d\tau \tag{32}
\]

The new time coordinate is independent of space coordinate labels, \( x^{0'} = x^{0'}(x^0) \), and the new coordinates are independent of clock rates, \( x^{i'} = \quad \)

\footnote{In the literature, Cartan’s notation is often used, \( de_\mu = e_\lambda Q^\lambda_{\mu\nu} \, dx^\nu \), even though the \( e_\mu \) are not generally functions and the \( de_\mu \) are not differentials. Here, we set \( de_\mu = (\nabla_\nu e_\mu) dx^\nu \) so that (25) follows.}
$x'^i(x^j)$. In short, since no relative motion is involved, the transformation reduces to

$$e_0' = \frac{\partial x^0}{\partial x'^0} e_0$$

(33)

$$e_i' = \frac{\partial x^i}{\partial x'^j} e_j$$

(34)

Applying these restricted transformations to the $Q^\mu_{\nu\lambda}$, we find

$$Q^0_{00}' = \frac{\partial x^0}{\partial x'^0}Q^0_{00} + \frac{\partial x^0'}{\partial x'^0} \frac{\partial^2 x^0}{\partial x'^0\partial x'^0}$$

(35)

$$Q^0_{0k}' = \frac{\partial x^a}{\partial x'^k}Q^0_{0a}$$

(36)

$$Q^i_{j0}' = \frac{\partial x^i}{\partial x'^j} \frac{\partial x^0}{\partial x'^j} + \frac{\partial x^0'}{\partial x'^j} Q^i_{j0}$$

(37)

$$Q^i_{jk}' = \frac{\partial x^i}{\partial x'^a} \frac{\partial x^c}{\partial x'^k} Q^i_{bc} + \frac{\partial x^c}{\partial x'^a} \frac{\partial^2 x^a}{\partial x'^j\partial x'^k}$$

(38)

Functions $Q^0_{0k}$ and $Q^i_{j0}$ transform as tensors and thus have invariant significance for our observer. Defining

$$Q^\mu_{[\nu\lambda]} = Q^\mu_{\nu\lambda} - Q^\mu_{\lambda\nu}$$

(39)

we obtain the complete array of tensor components

$$Q^0_{[0k]} = Q^0_{0k} - Q^0_{k0} = Q^0_{0k}$$

(40)

$$Q^0_{[jk]} = Q^0_{jk} - Q^0_{kj} = 0$$

(41)

$$Q^i_{[j0]} = Q^i_{j0} - Q^i_{0j} = Q^i_{j0}$$

(42)

$$Q^i_{[jk]} = Q^i_{jk} - Q^i_{kj}$$

(43)

The vectors $e_i$ form the basis of a time-dependent, three-dimensional geometry, which is characterized by metric components $g_{ij}$. The metric changes from point to point in the spatial manifold according to the formula

$$\frac{\partial g_{ij}}{\partial x^k} = g_{ia}Q^a_{jk} + g_{ja}Q^a_{ik}$$

(44)

By imposing nine symmetry conditions
\begin{align*}
Q^i_{jk} &= Q^i_{kj} \quad (45) \\
\text{this formula can be inverted in the usual way to yield} \\
Q^i_{jk} &= \frac{1}{2} g^{ia} \left( \frac{\partial g_{ja}}{\partial x^k} + \frac{\partial g_{ak}}{\partial x^j} - \frac{\partial g_{ik}}{\partial x^a} \right) \quad (46) \\
\text{The time-dependence of the metric is expressed in terms of coefficients } Q^i_{j0}: \\
\frac{\partial g_{ij}}{\partial x^0} &= g_{ia} Q^a_{j0} + g_{ja} Q^a_{i0} \quad (47) \\
\text{In order to invert this formula, we impose three conditions} \\
g_{ia} Q^a_{j0} &= g_{ja} Q^a_{i0} \quad (48) \\
\text{obtaining} \\
Q^i_{j0} &= \frac{1}{2} g^{ia} \frac{\partial g_{ja}}{\partial x^0} \quad (49) \\
\text{Finally, the function } g_{00} \text{ changes according to} \\
\frac{\partial g_{00}}{\partial x^\mu} &= 2 g_{00} Q^0_{0\mu} \quad (50) \\
\text{which immediately gives} \\
Q^0_{0\mu} &= \frac{1}{2} g^{00} \frac{\partial g_{00}}{\partial x^\mu} \quad (51) \\
\text{These formulae express all 28 independent coefficients } Q^\mu_{\nu\lambda} \text{ in terms of the} \\
\text{28 derivatives } \frac{\partial g_{\mu\nu}}{\partial x^\lambda}. \\
\text{We can state these results more concisely by noting that conditions (48),} \\
\text{together with (45), are equivalent to the equation} \\
g_{\mu\alpha} Q^\alpha_{[\nu\lambda]} + g_{\nu\alpha} Q^\alpha_{[\lambda\mu]} + g_{\lambda\alpha} Q^\alpha_{[\mu\nu]} = 0 \quad (52) \\
\text{We now begin with the general formula} \\
\frac{\partial g_{\mu\nu}}{\partial x^\lambda} &= g_{\mu\alpha} Q^\alpha_{\nu\lambda} + g_{\nu\alpha} Q^\alpha_{\mu\lambda} \quad (53) \\
\text{in the combination} 
\end{align*}
\[
\left( \frac{\partial g_{\nu\mu}}{\partial x^\lambda} + \frac{\partial g_{\mu\lambda}}{\partial x^\nu} - \frac{\partial g_{\lambda\nu}}{\partial x^\mu} \right) = 
\]
\[
g_{\nu\rho} Q^\rho_{\mu\lambda} + g_{\mu\rho} Q^\rho_{\lambda\nu} - g_{\lambda\nu} Q^\rho_{\mu\rho} - g_{\nu\rho} Q^\rho_{\lambda\mu} = 2 g_{\mu\rho} Q^\rho_{\nu\lambda} + g_{\mu\rho} Q^\rho_{[\lambda\nu]} + g_{\nu\rho} Q^\rho_{[\mu\lambda]} + g_{\lambda\rho} Q^\rho_{[\mu\nu]} \quad (54)
\]
and then make use of (52) to find
\[
\left( \frac{\partial g_{\nu\mu}}{\partial x^\lambda} + \frac{\partial g_{\mu\lambda}}{\partial x^\nu} - \frac{\partial g_{\lambda\nu}}{\partial x^\mu} \right) = 2 g_{\mu\rho} Q^\rho_{\nu\lambda} + 2 g_{\lambda\rho} Q^\rho_{[\mu\nu]} \quad (55)
\]
Raise index \( \mu \) and define the Christoffel symbols
\[
\Gamma^\mu_{\nu\lambda} = \frac{1}{2} g^{\mu\alpha} \left( \frac{\partial g_{\nu\alpha}}{\partial x^\lambda} + \frac{\partial g_{\alpha\lambda}}{\partial x^\nu} - \frac{\partial g_{\lambda\nu}}{\partial x^\alpha} \right) \quad (56)
\]
in order to obtain the formula
\[
\Gamma^\mu_{\nu\lambda} = Q^\mu_{\nu\lambda} + g^{\mu\alpha} g_{\lambda\beta} Q_{\alpha\beta} \quad (57)
\]

IV. Conservation of Energy-Momentum

The density of energy, momentum, and stress is represented by
\[
T = e_\mu \otimes e_\nu T^{\mu\nu} \quad (58)
\]
where \( T^{\mu\nu} = T^{nu}. \) Consider the product
\[
(\sqrt{-g} T, dV) = e_\mu \sqrt{-g} T^{\mu\nu} dV_\nu = e_0 \sqrt{-g} T^{00} dV_0 + e_0 \sqrt{-g} T^{0j} dV_j + e_i \sqrt{-g} T^{ij} dV_j \quad (59)
\]
The first term in this expression is a scalar, which gives the amount of energy in region \( dV_0 \)
\[
dE = e_0 \sqrt{-g} T^{00} dV_0 \quad (60)
\]
while the momentum is given by the three-vector
\[
c d\mathbf{P} = e_i \sqrt{-g} T^{i0} dV_0 \quad (61)
\]
The remaining terms represent the flow of energy and momentum through surfaces $dV_j$.

The divergence theorem for energy-momentum is \[6\]

\[
\oint e_{j} \sqrt{-g} T^{j\mu} dV = \int \left\{ e_{\mu} \frac{\partial \sqrt{-g} T^{\mu}_{\nu}}{\partial x^{\nu}} + (\nabla_{\nu} e_{\mu}) \sqrt{-g} T^{\mu\nu} \right\} d^{4}x
\]

Energy-momentum is conserved if

\[
\text{div} T^{\mu\nu} = \frac{1}{\sqrt{-g}} \frac{\partial \sqrt{-g} T^{\mu\nu}}{\partial x^{\nu}} + Q^{\mu}_{\nu\lambda} T^{\nu\lambda} = 0 \quad (63)
\]

The divergence takes on a more useful form, if we replace $Q^{\mu}_{\nu\lambda}$ by means of \(57\):

\[
\text{div} T^{\mu\nu} = \left\{ \frac{1}{\sqrt{-g}} \frac{\partial \sqrt{-g} T^{\mu\nu}}{\partial x^{\nu}} + \Gamma^{\mu}_{\nu\lambda} T^{\nu\lambda} \right\} + g^{\mu\alpha} Q_{\rho\gamma} T_{\rho\gamma}(64)
\]

(A semicolon indicates the covariant derivative defined in terms of the Christoffel symbols.)

The density of electromagnetic energy, momentum, and stress is given by

\[
T^{\mu\nu}_{\text{em}} = F^{\mu}_{\alpha} F^{\alpha\nu} + \frac{1}{4} g^{\mu\nu} F_{\alpha\beta} F^{\alpha\beta} \quad (65)
\]

In order to investigate conservation, we calculate

\[
\text{div} T^{\mu\nu}_{\text{em}} = T^{\mu\nu}_{\text{em}} + g^{\mu\alpha} Q_{\rho\gamma} T_{\rho\gamma e-m}(66)
\]

which, by virtue of Maxwell’s equation, becomes

\[
\text{div} T^{\mu\nu}_{\text{em}} = F^{\mu}_{\alpha} J^{\alpha} + g^{\mu\alpha} Q_{\rho\gamma} T_{\rho\gamma e-m}(67)
\]

Therefore, electromagnetic energy-momentum is not conserved in regions free of charged matter, due to the interaction term in $Q_{\rho\gamma}$. In this way, the gravitational field $Q_{\rho\gamma}$ becomes physically manifest.

Performing a similar calculation with the matter tensor
we find the covariant derivative
\[ T_{\mu
u;M} = \rho c^2 u^\mu \frac{\partial u^\nu}{\partial x^\nu} + c^2 u^\mu \frac{1}{\sqrt{-g}} \frac{\partial \sqrt{-g} \rho u^\nu}{\partial x^\nu} + \rho c^2 \Gamma^\mu_{\nu\lambda} u^\nu u^\lambda \] (69)

The second term is zero, if rest mass is conserved. Taken together with (67), we obtain the divergence
\[ \text{div} \left( T_{\mu\nu}^M + T_{\mu\nu}^e - m \right) = \rho c^2 \left( u^\nu \frac{\partial u^\mu}{\partial x^\nu} + \Gamma^\mu_{\nu\lambda} u^\nu u^\lambda \right) + F^\mu_\alpha J^\alpha \] (70)

The first expression, set equal to zero, gives the equation of motion for charged matter
\[ \rho c^2 \left( u^\nu \frac{\partial u^\mu}{\partial x^\nu} + \Gamma^\mu_{\nu\lambda} u^\nu u^\lambda \right) + F^\mu_\alpha J^\alpha = 0 \] (71)

and we are left with the gravitational interaction
\[ \text{div} \left( T_{\mu\nu}^M + T_{\mu\nu}^e - m \right) = g^{\mu_\alpha} Q^\rho_{[\eta\alpha]} (T^\eta_{\rho M} + T^\eta_{\rho e - m}) \] (72)

This interaction immediately unites electromagnetism and matter with gravitation. Moreover, the interaction has tensor character, which implies that the gravitational field must have real dynamical content (energy, momentum, stress) in each region of space. This is established in the following section.

V. Gravitational Field Equations

In forming a Lagrangian suitable for gravitation, we begin with the invariant expression
\[ \frac{c^4}{8\pi G} g^{\alpha\beta} Q^\rho_{[\eta\alpha]} Q^\eta_{[\rho\beta]} \] (73)

\((G = 6.67 \times 10^{-8} \text{ cm}^3/\text{g} \cdot \text{sec}^2)\). The coefficient is chosen such as to yield the gravitational energy tensor
\[
T_{\mu\nu}^{\text{grav}} = \frac{c^4}{4\pi G} \left\{ g^{\mu\alpha} g^{\nu\beta} Q_{\alpha\beta}^\rho Q_\rho - \frac{1}{2} g^{\mu\nu} g^{\alpha\beta} Q_{\alpha\beta}^\rho Q_\rho \right\} \quad (74)
\]

The non-zero components of \(Q_{[\nu\lambda]}^\mu\) were determined in section III:

\[
Q_{[0k]}^0 = Q_{0k}^0 = \frac{1}{2} g^{00} \frac{\partial g_{00}}{\partial x^k} \quad (75)
\]

\[
Q_{[j0]}^i = Q_{j0}^i = \frac{1}{2} g^{ij} \frac{\partial g_{ij}}{\partial x^0} \quad (76)
\]

Consider the special case of a weak gravitational field,

\[
g_{00} = 1 + \frac{2}{c^2} \psi \quad (77)
\]

\[
g_{ij} = -\delta_{ij} \quad (78)
\]

where \(\psi\) is the Newtonian potential. Substituting into (74), we obtain the weak-field limit of \(T_{\mu\nu}^{\text{grav}}\):

\[
T_{\text{grav}}^{00} = \frac{1}{8\pi G} (\nabla \psi)^2 \quad (79)
\]

\[
T_{\text{grav}}^{ij} = \frac{1}{4\pi G} \left\{ \delta^{il} \delta^{jm} (\nabla_l \psi) (\nabla_m \psi) - \frac{1}{2} \delta^{ij} (\nabla \psi)^2 \right\} \quad (80)
\]

\(T_{\text{grav}}^{00}\) is the positive definite energy density of the gravitational field. \(T_{\text{grav}}^{ij}\) is the stress tensor of Newtonian gravitation [7]. The stress is compressive along the gravitational lines of force, with tension acting between the lines of force.

Returning to the general case, we introduce arbitrary variations of the seven fields \(g^{\mu\nu} = (g^{00}, g^{ij})\) and calculate

\[
\delta \int \frac{c^4}{8\pi G} g^{\alpha\beta} Q_{[\alpha\beta]}^\rho Q_\rho \sqrt{-g} \, d^4x = \delta \int \frac{c^4}{8\pi G} \left\{ g^{00} Q_{m0}^l Q_{0l}^m + g^{lm} Q_{00}^l Q_{0m}^0 \right\} \sqrt{-g} \, d^4x \]

\[
= \delta \int \frac{c^4}{8\pi G} \left\{ \frac{1}{4} g^{00} g^{l0} g_{m0} \frac{\partial g_{lm}}{\partial x^0} \frac{\partial g_{00}}{\partial x^0} + \frac{1}{4} g^{lm} g^{00} g_{00} \frac{\partial g_{00}}{\partial x^0} \frac{\partial g_{00}}{\partial x^0} \right\} \sqrt{-g} \, d^4x
\]
\[
\begin{align*}
&= \int \left\{ \frac{c^4}{8\pi G} \frac{1}{\sqrt{-g}} \frac{\partial}{\partial x^l} (\sqrt{-g} g^{lm} Q_0^0) + \frac{1}{2} T_{\text{grav}}^0 \right\} g_{00} \delta g^{00} \sqrt{-g} d^4 x \\
+ \int \left\{ \frac{c^4}{8\pi G} \frac{1}{\sqrt{-g}} \frac{\partial}{\partial x^0} (\sqrt{-g} g^{00} Q_0^j) + \frac{1}{2} T_{\text{grav}}^i \right\} g_{ij} \delta g^{lj} \sqrt{-g} d^4 x \quad (81)
\end{align*}
\]

We have made use of these formulae

\[
\delta \sqrt{-g} = -\frac{1}{2} \sqrt{-g} g_{\mu\nu} \delta g^{\mu\nu} \quad (82)
\]

\[
\delta g_{\mu\nu} = -g_{\mu\alpha} g_{\nu\beta} \delta g^{\alpha\beta} \quad (83)
\]

\[
\delta \left( \frac{\partial g_{\mu\nu}}{\partial x^k} \right) = \frac{\partial}{\partial x^k} (\delta g_{\mu\nu}) \quad (84)
\]
as well as integration by parts; variations \( \delta g^{\mu\nu} \) have been set equal to zero at the limits of integration. The first term in (81) will yield Poisson’s equation

\[
\nabla^2 \psi = 4\pi G \rho \quad (85)
\]

if we introduce the matter tensor \( T_{M}^{\mu\nu} \) (68). This is accomplished by means of the variation

\[
\delta \int - \left\{ \rho c^2 (g_{\mu\nu} u^\mu u^\nu)^{1/2} - \rho c^2 \right\} \sqrt{-g} d^4 x = \frac{1}{2} \int T_{\mu\nu} M \delta g^{\mu\nu} \sqrt{-g} d^4 x
\]

\[
= \frac{1}{2} \int \left\{ T_{00}^0 g_{00} \delta g^{00} + T_{ij}^0 g_{ij} \delta g^{lj} \right\} \sqrt{-g} d^4 x \quad (86)
\]
The final expression to be introduced is the energy tensor of electromagnetism [8]

\[
\delta \int - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} \sqrt{-g} d^4 x = \frac{1}{2} \int T_{\mu\nu} e_m \delta g^{\mu\nu} \sqrt{-g} d^4 x
\]

\[
= \frac{1}{2} \int \left\{ T_{0e}^0 g_{00} \delta g^{00} + T_{ij}^i g_{ij} \delta g^{lj} \right\} \sqrt{-g} d^4 x \quad (87)
\]

where \( T_{e-m}^{\mu\nu} \) is given by (65).

Gathering terms together and setting coefficients of \( \delta g^{00} \) and \( \delta g^{ij} \) equal to zero, we arrive at the field equations of gravitation

\[
\frac{c^4}{4\pi G} \frac{1}{\sqrt{-g}} \frac{\partial}{\partial x^l} (\sqrt{-g} g^{lm} Q_0^0) + T_0^0 = 0 \quad (88)
\]
\[
\frac{c^4}{4\pi G} \frac{1}{\sqrt{-g}} \frac{\partial}{\partial x^0} (\sqrt{-g} g^{00} Q^0_i) + T^i_j = 0
\] (89)

\(T^{\mu\nu}\) represents the total energy, momentum, and stress

\[
T^{\mu\nu} = T^{\mu\nu}_{\text{grav}} + T^{\mu\nu}_{M} + T^{\mu\nu}_{e-m}
\] (90)

The corresponding invariant equation is

\[
\frac{c^4}{4\pi G} \frac{1}{\sqrt{-g}} \frac{\partial}{\partial x^\nu} (\sqrt{-g} Q^\nu) + T = 0
\] (91)

where

\[Q_\mu \equiv Q^\rho_{[\alpha\mu]}\] (92)

VI. The Gravitational Field of an Electron

According to our hypothesis, it is the electromagnetic nature of matter which gives rise to gravitation. Thus, our central problem is to determine the field of a discrete charged particle, at rest. The field equations admit an exact solution for this case, of the form

\[
g_{\mu\nu} = \begin{pmatrix}
    e^\nu & -e^\lambda & 0 \\
    -e^\lambda & -r^2 & 0 \\
    0 & 0 & -r^2 \sin^2 \theta
\end{pmatrix}
\] (93)

where \(\nu\) and \(\lambda\) are functions of \(r\) which vanish at infinity. The electromagnetic energy tensor is [9]

\[
T^{\mu\nu}_{e-m} = \frac{q^2}{8\pi r^4} \begin{pmatrix}
    e^\nu & -e^\lambda & 0 \\
    -e^\lambda & r^2 & 0 \\
    0 & 0 & r^2 \sin^2 \theta
\end{pmatrix}
\] (94)

where \(q\) is the electric charge in ordinary c.g.s. units (stat-coulomb). The components of the gravitational energy tensor are found from (74) and (93) to be
\[ T_{\mu\nu}^{\text{grav}} = \frac{c^4}{32\pi G} e^{-\lambda \nu/2} \begin{pmatrix} e^\nu & 0 \\ e^\lambda & -r^2 \\ 0 & -r^2 \sin^2 \theta \end{pmatrix} \] (95)

(A prime indicates differentiation with respect to \( r \)). Finally, we have

\[
\frac{\partial}{\partial x^l} (\sqrt{-g} g^{lm} Q_{0m}) = \frac{d}{dr} (\sqrt{-g} g^{11} Q_{01}) = -\frac{1}{2} r^2 \sin \theta e^{(\nu - \lambda)/2} \left( \nu'' + \frac{2}{r} \nu' + \frac{1}{2} \nu'^2 - \frac{1}{2} \nu' \lambda' \right)
\] (96)

Substitution into (88) and (89) yields two distinct equations

\[
\nu'' + \frac{2}{r} \nu' + \frac{1}{4} \nu'^2 - \frac{1}{2} \nu' \lambda' - \frac{Gq^2}{c^4 r^4} e^\lambda = 0 \quad (97)
\]

\[
-\frac{1}{4} \nu'^2 + \frac{Gq^2}{c^4 r^4} e^\lambda = 0 \quad (98)
\]

Adding the two equations gives

\[
\nu'' + \frac{2}{r} \nu' - \frac{1}{2} \nu' \lambda' = 0
\]

or

\[
d (\ln \nu') + \frac{2}{r} dr = \frac{1}{2} d\lambda \quad (99)
\]

Upon integration, we find an equation of the form (98). In order to arrive at a unique solution, we set

\[
\lambda = -\nu \quad (100)
\]

That is, we invoke a reciprocal relationship between time dilatation and length contraction along the direction of the field. Equation (98) now gives

\[
e^{\nu/2} d\nu = \frac{2G^{1/2}q}{c^2 r^2} dr \quad (101)
\]

This equation integrates to

\[
e^{\nu} = e^{-\lambda} = \left( 1 - \frac{G^{1/2}q}{c^2 r} \right)^2 \quad (102)
\]
the constant of integration being chosen such that $e^\nu = e^{-\lambda} = 1$ at infinity. The components of the metric tensor are

$$
\begin{pmatrix}
\left(1 - \frac{G^{1/2}q}{c^2 r}\right)^2 & 0 \\
-\left(1 - \frac{G^{1/2}q}{c^2 r}\right)^{-2} & -r^2 \\
0 & -r^2 \sin^2 \theta
\end{pmatrix}
$$

(103)

The gravitational field of an electron is given by setting $q = e = 4.8 \times 10^{-10}$ stat-coulomb in (103). Noting that

$$
G^{1/2}e/c^2 = 1.4 \times 10^{-34} \text{cm}
$$

(104)

the field may be represented by a gravitational potential

$$
\psi = -\frac{G^{1/2}e}{r}
$$

(105)

This potential is some 21 orders of magnitude greater than the Newtonian value, $-Gm/r$. It acts in those regions of space which are subject to the electrostatic field of the electron.

VII. An Electrostatic Red-Shift in Stars

In the previous section, it was shown that a spherically symmetric electrostatic field produces a gravitational potential given by

$$
\psi = -\frac{G^{1/2}q}{r}
$$

(106)

where $q$ is the absolute value of electric charge. This potential, in turn, will produce a red-shift of light given by

$$
\frac{\Delta \nu}{\nu_0} = -\frac{1}{c^2} \psi = \frac{G^{1/2}q}{c^2 r}
$$

(107)

where $\nu_0$ is the light frequency in the absence of gravitation, i.e., at $r \to \infty$. In this section, we discuss a plausible observation of this red-shift and its correlation with ordinary electrodynamic processes.
For purposes of our discussion, consider the following hypothetical stellar object, in which a spherically symmetric charge separation has occurred. Neutrons and protons, held strongly in the core, are surrounded by electrons which have been forced away by thermal energy and mutual repulsion. Under the simplest assumption, all electrons lie in a spherical shell of fixed radius. This model may be treated in terms of our theory as follows. Each neutron produces a Newtonian potential

$$\psi_n = -\frac{Gm}{r} = -\frac{10^{-31}}{r}$$

($m = 1.7 \times 10^{-24} \text{g}$), while each proton, according to (106) produces the much greater potential

$$\psi_p = -\frac{G^{1/2}e}{r} = -\frac{10^{-13}}{r}$$

Ignoring the neutron contribution, we write the gravitational potential (106) in the form

$$\psi = -\frac{G^{1/2}\phi}{r}$$

(110)

where $\phi$ is the total electrostatic potential. A ray of light, being emitted from the core and passing through potential $\phi$, will suffer a red-shift (107) given by

$$\frac{\Delta \nu}{\nu_0} = \frac{G^{1/2}}{c^2} \phi$$

(111)

This linear relation holds true over a very wide range of electrostatic potential (see below). In addition to light, one would expect positively charged particles (cosmic rays) to be emitted from such a stellar object. Particle energy is commensurate with the electrostatic potential. Thus, according to (111), one should observe a linear correlation between the red-shift of light and the energy of cosmic rays.

The exact solution to the electrostatic problem (103) takes the form
\[
g_{\mu\nu} = \begin{pmatrix}
\left(1 - \frac{G^{1/2}}{c^2} \phi\right)^2 & 0 \\
-(1 - \frac{G^{1/2}}{c^2} \phi)^{-2} & -r^2 \\
0 & -r^2 \sin^2 \theta
\end{pmatrix}
\]  
\tag{112}

Component \(g_{00}\) gives the frequency relation
\[
\frac{\nu}{\nu_0} = g_{00}^{-1/2} = \left(1 - \frac{G^{1/2}}{c^2} \phi\right)^{-1}
\]  
\tag{113}

and red-shift formula
\[
\frac{\Delta \nu}{\nu_0} = \frac{G^{1/2} \phi}{1 - \frac{G^{1/2}}{c^2} \phi}
\]  
\tag{114}

According to this formula, an infinite red-shift corresponds to a potential of
\[
c^2/G^{1/2} = 3.5 \times 10^{24} \text{ stat-volts} = 10^{27} \text{ volts}
\]  
\tag{115}

It follows that cosmic ray energy can be no greater than \(10^{27} \text{ eV}\).

It would be of interest to search for evidence of a correlation between the gravitational red-shift of a discrete source and the energy of cosmic rays emitted by that source. This may be within reach of modern experimental methods in astronomy. Perhaps of greater significance is the predicted upper limit to cosmic ray energy, \(10^{27} \text{ eV}\). This limit is associated with an infinitely red-shifted stellar object, an “electrostatic black-hole,” at a potential of \(10^{27} \text{ volts}\). The astronomers have found the maximum energy of cosmic rays to be at least \(10^{20} \text{ eV}\). This compares favorably with the theoretical value.

VIII. Concluding Remarks

According to equation (89), any flow of momentum (i.e., stress) will give rise to a time-dependent gravitational field. This applies, in particular, to
the field generated by a stationary uncharged mass such as the Sun. In a first
order approximation to this problem, all stress terms may be ignored; (88)
then yields Poisson’s equation and the static Newtonian potential. How-
ever, in second order, the theory predicts the existence of time-dependent
perturbations in the Newtonian field, due to the presence of material and
gravitational stress. Recently discovered resonant oscillations in the Sun
show that the solar field is, indeed, time-dependent. It remains to be seen
whether this provides corroboration for our theory.

The above behavior contrasts with that of the field surrounding a sta-
tionary electric charge (section VI). Here, the electrostatic stresses are equal
and opposite to those of the gravitational field. Therefore, the net flow of
momentum is everywhere zero, and the solution is time-independent. This
result underscores the significance of electric charge in our theory. Moreover,
it calls to mind the central role played by electric charge and ‘generalized
charge’ in elementary particle theory. The enormous strength of the elec-
tron’s gravitational potential may point toward an eventual link with the
physics of elementary particles.

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