PROPERTIES OF MULTIHOMOGENEOUS SPACES AND RELATION WITH T-VARIETIES

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ABSTRACT. We study multihomogeneous spaces corresponding to \( \mathbb{Z}^n \)-graded algebras over an algebraically closed field of characteristic 0 and their relation with the description of T-varieties.

INTRODUCTION

Algebraic varieties with torus actions, including but not limited to toric varieties, have been at the centre of much attention for the past few decades. During our research we came across various constructions around such varieties. This paper tries to relate two of these constructions under some hypothesis.

The first object of interest is a variety with an effective algebraic torus action. These were studied by various people, for example, as toric varieties over discrete valuation rings considered by Kempf, Knudsen, Mumford and Saint-Donat [11]; as a part of the general case of varieties with the action of a reductive group by Timashev [12]; and the case of \( \mathbb{C}^* \) actions on normal affine surfaces were studied by Flenner and Zaidenberg [8] to name a few. The theory was neatly generalized and combined into a single theory by Altmann, Hausen and Süss (see [1] for the affine case and [2] for the general case). The combinatorial descriptions of the geometric properties were studied extensively and are reported in the survey [3]. There has been quite a bit of activity in this area in the recent years.

Another concept which drew our attention was that of a multihomogeneous projective space defined by Brenner and Schröer [6]. These spaces are generalizations of weighted projective spaces and are divisorial schemes. Brenner and Schröer gave a criterion for a scheme of a finite type over a noetherian ring to be divisorial in terms of existence of an embedding of the scheme into a multihomogeneous space associated to a multigraded algebra [6, corollary 4.7]. Extending their work, Zanchetta [13] proved that the ambient multihomogeneous space can be chosen to be smooth. Some applications of this theory can be seen in Kanda [10].

This paper delves into the relationship between these two concepts. Digging a bit deeper, not surprisingly, GIT quotients play a role in both the theories. We try to follow this link as far as we could.

While studying and working with multihomogeneous spaces we proved some results generalizing similar results in weighted projective spaces (see, for example, [7] and [4]). A criterion for a twisted module, defined in a similar fashion as the twisted modules on projective varieties, to be a line bundle (theorem 3.9). Furthermore, in
multihomogeneous spaces, the points need not correspond to homogeneous prime ideals. This paper proves a criterion for this to happen (corollary 1.7).

Normal varieties along with an effective action of a torus $T$ are called $T$-varieties. Such varieties can be described by partially combinatorial data in the form of a semiprojective variety $Y$ and a proper polyhedral divisor on $Y$, which are generalization of usual $\mathbb{Q}$-divisors on $Y$ where rational linear combinations are replaced by formal sums of the divisors with polyhedral coefficients.

We show that $Y$ associated to an affine $T$-variety $X = \text{Spec } A$ is birational to a multihomogeneous space obtained as the Proj of $\text{Hom}(T, \mathbb{G}_m)$-graded ring $A$, where the grading is obtained by taking isotypical components under the torus action (see theorem 4.4). We end the paper by giving one criterion when this birational morphism is an isomorphism.

The paper is divided into 4 sections. Starting with a review of the theory of multihomogeneous spaces, the first section goes on to study some conditions under which the points in the multihomogeneous spaces correspond to homogeneous prime ideals. This is not true in general as remarked in [6, remark 2.3]. We end the section with a condition under which the multihomogeneous Proj will be normal.

The second section is a review of the theory of $T$-varieties. This section is just for clarity of exposition and fixing notation and does not contain any new results.

The third section defines and proves some results for twisted sheaves over multihomogeneous spaces. We end with some hypothesis under which the twisted sheaves are line bundles.

The last section deals with the question about when these constructions yield the same space. After studying some cases where this fails, we end with a theorem which describes some sufficient conditions under which they are isomorphic.

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1. Multihomogeneous Spaces

In this section we review the theory of multihomogeneous spaces. For more details regarding multihomogeneous spaces we refer to [6, section 2]. See also [4] for results on geometry of multigraded algebras and their properties.

**Definition 1.1.** Let $D$ be a finitely generated abelian group and

$$A = \bigoplus_{d \in D} A_d$$

be a $D$-graded ring. One says that $A$ is periodic if $D' = \{d \in D | \exists f \in A_d \cap A^x\}$, the subgroup of $D$ consisting of degrees of all the homogeneous invertible elements in $A$ is a finite index subgroup. A homogeneous element $f$ in a $D$-graded ring $A$ is said to be relevant if $A_f$ is periodic. For a relevant element $f$, note that the localization $A_f$ is $D$-graded. We shall denote the degree 0 part of $A_f$ by $A_{f(0)}$.

The following lemma by Brenner and Schröer is useful.

**Lemma 1.2.** ([6, lemma 2.1]) Let $D$ be a finitely generated abelian group and

$$A = \bigoplus_{d \in D} A_d$$

be a $D$-graded periodic ring. Then the projection $\text{Spec } A \to \text{Spec } (A_0)$ is a geometric quotient in GIT sense.
Definition 1.3. For $D$ and $A$ as in definition 1.1, the grading on $A$ corresponds to an action of the diagonalizable group scheme $	ext{Spec } A_0[D]$ on $	ext{Spec } A$. Let $Q$ be the quotient in the category of ringed spaces. Now for a relevant element $f$, consider the inclusion

$$D_+(f) = \text{Spec } A_{(f)} \subset Q.$$ 

One defines

$$\text{Proj}_{\text{MHS}} A = \bigcup_{f \in A} D_+(f) \subset Q.$$ 

Remark 1.4. The points in a multihomogeneous projective space $\text{Proj}_{\text{MHS}} A$ of a $D$-graded ring $A$ correspond to homogeneous ideals in $A$ which may not be prime (see [6, remark 2.3]). However, these ideals have the property that all the homogeneous elements in the complement form a multiplicatively closed set.

Proposition 1.5. Suppose $D$ is a free finitely generated $\mathbb{Z}$-module and $A = \bigoplus_{d \in D} A_d$ is a $D$-graded ring. Assume that we have a collection of relevant elements $F$ such that

$$\text{Proj}_{\text{MHS}} A = \bigcup_{f \in F} \text{Spec } A_{(f)}$$

and for each $f \in F$, \( \{ d \in D \mid d = \deg g \text{ for some homogeneous } g \in A_f^\times \} = D \). Then every point $p \in \text{Proj}_{\text{MHS}} A$ corresponds to a homogeneous prime in $A$.

Proof. Suppose $p \in \text{Spec } A_{(f)}$ for some relevant element $f \in A$. Then $A_f$ is periodic and

$$D' = \{ d \in D \mid d = \deg g \text{ for some homogeneous } g \in A_f^\times \}$$

is a free subgroup of $D$ of finite index. Define

$$A'_f = \bigoplus_{d \in D'} (A_f)_d.$$ 

It is easy to see that in this case, $A'_f = A_f[T_1^{\pm 1}, \ldots, T_r^{\pm 1}]$, where $r = \text{rank } D'$.

Note the primes $P \in A_{(f)}$ correspond to the primes $P[T_1^{\pm 1}, \ldots, T_r^{\pm 1}] \subset A'_f$. Now consider the diagram

$$\begin{array}{ccc}
A_f' & \hookrightarrow & A_f \\
\downarrow & & \downarrow \\
A_f & & A_f \\
\end{array}$$

It is easy to see that if $A'_f = A_f$, then the primes in $A_{(f)}$ would correspond to homogeneous primes in $A$ which do not contain $f$. The condition $A'_f = A_f$ holds whenever the hypothesis of the proposition is satisfied. \qed

Corollary 1.6. Under the hypothesis of proposition 1.5, the points in $D_+(f) \subset \text{Proj}_{\text{MHS}} A$ correspond to all homogeneous primes in $A$ which do not contain $f$.

Proof. This was mentioned in the proof of proposition 1.5 after the diagram. \qed

Corollary 1.7. Suppose $A$ is a $D$-graded ring generated over $A_0$ by a set

$$\{ a_1, \ldots, a_n \}$$

of homogeneous elements such that any $\mathbb{Z}$-linearly independent subset of

$$\{ \deg a_1, \ldots, \deg a_n \}$$
having rank $D$ elements is a basis for the abelian group $D$. In this case the hypothesis of proposition 1.13 holds and hence the points in $\text{Proj}_{\text{MH}}A$ will correspond to homogeneous prime ideals in the graded ring $A$.

**Remark 1.8.** The way $\text{Proj}_{\text{MH}}A$ is defined for a $D$-graded ring $A$, it can happen that $A$ has no relevant element and then $\text{Proj}_{\text{MH}}A = \emptyset$. If $A$ is a finitely generated algebra over $A_0$, one sufficient condition for the existence of relevant elements is that there exists a collection of homogeneous generators $\{x_i \mid 1 \leq i \leq r\}$ such that $\deg x_i | 1 \leq i \leq r$ generates a finite index subgroup in $D$. This condition is easy to check, for example, when $A$ is the polynomial ring over $\mathbb{C}$.

**Remark 1.9.** By [6, Lemma 2.1], the map $\text{Spec} A_f \to \text{Spec} A_{(f)}$, which is induced by the inclusion $A_{(f)} \hookrightarrow A_f$, is a geometric quotient.

By definition, the collection of affine open subschemes

$$\{D_+(f) \mid f \in A \text{ is homogeneous and relevant}\}$$

covers $\text{Proj}_{\text{MH}}A$. We state the following easy fact for subsequent use.

**Lemma 1.10.** With the notation as above, $D_+(f) \cap D_+(g) = D_+(fg) \subset \text{Proj}_{\text{MH}}A$.

**Proof.** This is implicit in [6, proposition 3.1]. Note that for relevant elements $f$ and $g$ in $A$, $\text{Spec} A_{(fg)} = \text{Spec} A_f \cap \text{Spec} A_g$ as subschemes of $\text{Spec} A$. Now $\text{Spec} A_{(fg)}$, $\text{Spec} A_{(f)}$ and $\text{Spec} A_{(g)}$ are geometric quotients (see remark 1.9) under the action of $\text{Spec} A_0[D]$ and hence $\text{Spec} A_{(fg)} = \text{Spec} A_{(f)} \cap \text{Spec} A_{(g)}$ considered as a subscheme of $\text{Proj}_{\text{MH}}A$. □

For later, we record two results of Brenner and Schröer regarding finiteness.

**Lemma 1.11** ([6], lemma 2.4). For a finitely generated abelian group $D$ and a $D$-graded ring $A$, the following are equivalent:

(i) The ring $A$ is noetherian.

(ii) $A_0$ is noetherian and $A$ is an $A_0$-algebra of finite type.

**Proposition 1.12** ([6], proposition 2.5). Suppose $A$ is a noetherian ring graded by a finitely generated abelian group $D$. Then the morphism $\varphi : \text{Proj}_{\text{MH}}A \to \text{Spec} A_0$ is universally closed and of finite type.

**Definition 1.13** ([6], page 10). Let $R$ be a ring, $M$ be a free abelian group of finite rank, and $N := \text{Hom}(M, \mathbb{Z})$ be dual of $M$. Let $X$ be an $R$-scheme and $T := \text{Spec} R[M]$ be the torus. A simplicial torus embedding of torus $T$ is $T$-equivariant open map $T \hookrightarrow X$ locally given by semigroup algebra homomorphisms $R[n^\sigma \cap M] \to R[T]$, where $\sigma$ is a strongly convex, simplicial cone in $N$.

**Remark 1.14.** If $X$ is a toric variety with torus $T$, then $X$ is a simplicial torus embedding of the torus $T$. There are other schemes which are simplicial torus embeddings of some torus. Homogeneous spectrum of multigraded polynomial algebras are examples of this type.

Let $D$ be an abelian group of finite type and $A = k[x_1, \ldots, x_n]$ be a $D$-graded polynomial $k$-algebra. Suppose the grading is given by a linear map $P : \mathbb{Z}^n \to D$ with finite co-kernel. Then we have the following sequence of abelian groups

$$0 \to M \to \mathbb{Z}^n \to D,$$

where $M$ is the kernel of $P$. 
Definition 2.1. (See, [3, section 1.1]) Suppose $M$ is a free $\mathbb{Z}$-module of finite rank and $T = \text{Spec } k[M]$ be the corresponding torus. An affine $T$-variety is a normal affine variety with an effective action of $T$.

The $T$-varieties have a partial combinatorial description which we review below.

Definition 2.2. Suppose $Y$ is a semiprojective variety; i.e. an algebraic variety such that the $k$-algebra $\Gamma(Y, \mathcal{O}_Y)$ is finitely generated and $Y$ is projective over $Y_0 = \text{Spec } \Gamma(Y, \mathcal{O}_Y)$. Let $N$ be a finite rank free $\mathbb{Z}$-module and $M = \text{Hom}_\mathbb{Z}(N, \mathbb{Z})$. Then $T = \text{Spec } k[M]$ is a split torus over $k$. Fix a pointed (i.e. a strongly convex polyhedral) cone $\sigma \in N_\mathbb{Q} = N \otimes \mathbb{Z} \mathbb{Q}$. Let $\text{Poly}_\mathbb{R}^+(N_\mathbb{Q})$ be the collection of convex polyhedra $\Delta$ (i.e. finite intersection of closed half spaces) such that 

$$\text{tail}(\Delta) = \{ v \in N_\mathbb{Q} \mid v' + tv \in \Delta \forall v' \in \Delta, t \in Q_{\geq 0} \} = \sigma,$$
and let the group of $\sigma$-polyhedra, $\text{Pol}_\sigma(N_Q)$, be the Grothendieck group of $\text{Pol}^r(N_Q)$ (see [1] definition 1.2). The group of rational polyhedral Weil (respectively, Cartier) divisors with respect to $\sigma$ is defined as $\text{WDiv}_Q(Y, \sigma) := \text{Pol}_\sigma(N_Q) \otimes_{\mathbb{Z}} \text{WDiv}(Y)$ (respectively, $\text{CDiv}_Q(Y, \sigma) := \text{Pol}_\sigma(N_Q) \otimes_{\mathbb{Z}} \text{CDiv}(Y)$). To describe the integral polyhedral divisors, one considers those polyhedra which admit a decomposition as a Minkowski sum of a polytope with vertices in $N$ and $\sigma$ (see [1] definitions 1.1, 1.2 and 2.3). For an element $u \in \sigma^\vee$, one can define a linear evaluation functional $\text{eval}_u : \text{Pol}_\sigma(N_Q) \rightarrow \mathbb{Q}$ such that for any $\Delta \in \text{Pol}^r(N_Q)$, $\text{eval}_u(\Delta) = \min_{v \in \Delta} \langle u, v \rangle$. A Weil (respectively, Cartier) polyhedral divisor is an element of $\text{Pol}_\sigma(N_Q) \otimes_{\mathbb{Z}} \text{WDiv}(Y)$ (respectively, $\text{Pol}_\sigma(N_Q) \otimes_{\mathbb{Z}} \text{WDiv}(Y)$). Given a polyhedral divisor $\mathfrak{D} = \sum \Delta_D \otimes D$, and an $u \in \sigma^\vee$, one defines $\mathfrak{D}(u) = \sum \text{eval}_u(\Delta_D) D$.

By a pp-divisor (or a proper, polyhedral divisor) one means a polyhedral divisor $\mathfrak{D} \in \text{CDiv}_Q(Y, \sigma)$ such that it can be represented as $\mathfrak{D} = \sum \Delta_i \otimes D_i$ with $\Delta_i \in \text{Pol}_\sigma(N_Q)$ and effective divisors $D_i$ satisfying the following: for any $u \in \text{rel} \sigma^\vee$, $\mathfrak{D}(u)$ is a big divisor on $Y$; and for any $u \in \sigma^\vee$, $\mathfrak{D}(u)$ is semiequal (see [1] definition 2.7). The semigroup of all pp-divisors having tail cone $\sigma$ is denoted by $\text{PPDiv}_Q(Y, \sigma)$.

**Definition 2.3** (Weight cone). (See [1] introductory discussion, section 3, [5] section 2.) Given an affine variety $X = \text{Spec} A$ with an effective action of the torus $T = \text{Spec} k[M]$, suppose that the decomposition of $A$ into $\chi^m : T \rightarrow \mathbb{G}_m(k)$ semi-invariants is given by $A = \bigoplus_{m \in M} A_m$. Then the weight cone is the convex polyhedral cone $\omega \subset M_Q$ generated by the weight monoid $S = \{ m \in M \mid A_m \neq 0 \}$.

Altmann and Hausen prove the following theorem.

**Theorem 2.4** (AH08, Theorem 3.1 and 3.4). Given a normal, semiprojective variety $Y$, a lattice $N$, the dual lattice $M$, a pointed cone $\sigma \subset N_Q$, a pp-divisor $\mathfrak{D} \in \text{PPDiv}_Q(Y, \sigma)$, the affine scheme associated to $(Y, \mathfrak{D})$ is described as

$$X = \text{Spec} \Gamma \left( Y, \bigoplus_{u \in \sigma^\vee \cap M} \mathcal{O}_Y(\mathfrak{D}(u)) \right).$$

Then $X$ is a normal $T$-variety where $T = \text{Spec} k[M]$. Moreover given any normal affine $T$-variety $X = \text{Spec} A$ with weight cone $\omega \subset M_Q$, there is exists a normal semiprojective variety $Y$ and a pp-divisor $\mathfrak{D} \in \text{PPDiv}_Q(Y, \omega^\vee)$ such that the $T$-variety associated to $(Y, \mathfrak{D})$ is $X$.

**Remark 2.5.** Note that the variety $Y$ in theorem 2.4 is not uniquely determined, but is unique up to a birational class. For more details, see [1] corollary 8.12.

**Remark 2.6.** Suppose $Y$ and $\mathfrak{D}$ are as in the first part of the theorem 2.4. Let $A = \bigoplus_{m \in \sigma^\vee} A_m$ where $A_m = \mathcal{O}_Y(\mathfrak{D}(m))$. Then, one can also consider the relative spectrum $\tilde{X} = \text{Spec}_Y A$. Then $A = \Gamma(Y, A)$. The scheme $\tilde{X}$ is a normal affine variety with an effective $T$ action such that $\pi: \tilde{X} \rightarrow Y$ is a good quotient. Furthermore there is a contraction morphism $r: \tilde{X} \rightarrow X$ which is proper, birational and $T$-equivariant.

The orbits are described using orbit cones which in turn define a GIT fan. We shall require this concept later on and hence we recall the associated definitions briefly.
Definition 2.7. (See [1, definition 5.1], [3, definition 2.1].) Let \( X = \text{Spec} \, A \) be a normal affine variety with an action of a torus \( T = \text{Spec} \, k[M] \). Suppose the action of the torus determines the decomposition \( A = \bigoplus_{m \in M} A_m \) into spaces of semi-invariants. For \( x \in X \), the orbit monoid is the submonoid \( S(x) \subset M \) defined as

\[
S(x) = \{ m \in M \mid \exists f \in A_m \text{ such that } f(x) \neq 0 \}.
\]

The orbit monoid generates a convex cone \( \omega(x) \subset M_Q \) called the orbit cone.

The set of \( \chi^m \) semistable points is defined as

\[
X_{ss}(m) = \{ x \in X \mid m \in \omega(x) \}.
\]

The GIT cone associated to \( m \in \omega \cap M \) is the intersection \( \lambda(m) := \bigcap_{x \in X; m \in \omega(x)} \omega(x) \). Suppose \( \omega \) is the weight cone for the torus action on \( X \). The collection of GIT cones \( \Lambda = \{ \lambda(m) \mid m \in \omega \cap M \} \) forms a quasi-fan in \( M_Q \) having \( \omega \) as its support. For brevity, we shall call this quasifan as a GIT fan.

Given a normal variety \( X = \text{Spec} \, A = \text{Spec} \, \bigoplus_{m \in M} A_m \) with an effective torus action, theorem 2.4 above ensures the existence of \((Y, D)\). We recall the description of \( Y \), as it will be useful in section 4. According to the theory in [3, section 2], \( X_{ss}(m) = X_{ss}(m') \) for \( m, m' \) belonging to the relative interior of a GIT cone \( \lambda \) of the GIT fan \( \Lambda \). Let \( X_{ss}(m) = X_{ss}(m) \) for some \( m \in \text{rel int} \lambda \). Then one also has that \( Y_m = X_{ss}(m) \// T = \text{Proj} \, \bigoplus_{r \in \mathbb{Z}} A_{rm} \). Thus, \( Y_m \)'s also depend only on the fan \( \lambda \) such that \( m \in \text{rel int} \lambda \), and hence are denoted by \( Y_{\lambda} \). If \( \lambda' \preceq \lambda \), then one has a birational morphism \( \varphi_{\gamma \lambda} : Y_{\lambda} \to Y_{\gamma} \). Putting everything together compatibly one has the following diagram which also defines \( Y \) (see [1, section 6]):

\[
\begin{array}{ccc}
X' & \rightarrow & X_{\lambda} & \rightarrow & X_{\gamma} & \rightarrow & X \\
\downarrow & & \downarrow & & \downarrow & & \\
Y' & \rightarrow & Y_{\lambda} & \rightarrow & Y_{\gamma} & \rightarrow & Y \\
\text{Normal(image}(X')) & \rightarrow & \\
\end{array}
\]

where \( X' = \lim X_{\lambda} \) and \( Y' = \lim Y_{\lambda} \). It is also known that \( Y \) is a good quotient of the torus action on \( X \).

In the previous paragraph, we constructed the \( Y \) in the pair \((Y, D)\) describing affine variety \( X \) with an effective action of the torus. The construction of the pp-divisor \( D \) is not relevant to this paper.

3. SOME RESULTS ABOUT MULTIHOMOGENEOUS SPACES

3.1. Sheaves associated to multigraded modules. Consider a finitely generated abelian group \( D \) and let \( A \) be a \( D \)-graded ring. Suppose \( M = \bigoplus_{d \in D} M_d \) is a \( D \)-graded \( A \)-module. Just as in the case of quasicoherent sheaves of modules over \( \text{Proj} \) of a \( \mathbb{N} \)-graded ring [3, definition before proposition 5.11, page 116], we can construct \( M \). We sketch some details to fix notation and show the similarities in the two setups.

In the construction of \( \text{Proj} \) of a \( \mathbb{Z} \)-graded module over a \( \mathbb{N} \)-graded ring, by the definition of \( \text{Proj} A \) the points correspond to homogeneous prime ideals in the defining graded ring which do not contain the whole of the irrelevant ideal. However
this is no longer true for a multihomogeneous space and a point \( P \) may correspond to an ideal which is not a prime.

Let \( A \) be a \( D \)-graded ring and \( M \) be a \( D \)-graded coherent module on \( A \) with the usual condition that \( A_\mu M_\nu \subseteq M_{\mu+\nu} \). Since the points \( p \) in \( \text{Proj}_{\mathcal{M}} A \) correspond to graded ideals \( I_p \) in \( A \) such that the homogeneous elements in the complement \( A \setminus I_p \) form a multiplicatively closed set, it is still true that the stalk of the structure sheaf at \( p \) is given by \( A(I_p) \) (see remark 3.3). One can now define \( \tilde{M} \) in the same way by associating to \( U \subseteq \text{Proj}_{\mathcal{M}} A \), the \( \mathcal{O}_{\text{Proj}_{\mathcal{M}}} A(U) \)-module of sections \( s : U \to \coprod_{p \in U} M(I_p) \) satisfying the usual condition that locally such \( s \) should be defined by a single element of the form \( m/a \) with \( m \in M \) and \( a \in A \) but not in any of the ideals \( I_p \). These modules are coherent under some mild conditions, as we state below.

Note that, given a \( D \)-graded \( A \)-module \( M = \bigoplus_{d \in D} M_d \) and an \( e \in D \), one can define a graded module \( M(e) \) where as \( A \)-modules \( M(e) = M, \) but \( M(e)_d = M_{d+e} \forall d \in D. \)

**Lemma 3.1.** Suppose \( D \) is a finitely generated abelian group and \( A \) is a \( D \)-graded integral noetherian ring. Then for \( X = \text{Proj}_{\mathcal{M}} A \), the following hold

(a) \( \mathcal{O}_X = \mathcal{O}_X \). This allows us to define
\[
\mathcal{O}_X(d) := \tilde{A}(d).
\]
\( \mathcal{O}_X(d) \) is a coherent sheaf.
(b) For a \( D \)-graded \( A \)-module \( M \), \( \tilde{M} \) is quasi-coherent and \( \tilde{M}|_{D_+(f)} \cong \tilde{M}(f) \)
for any relevant element \( f \in A \), where \( \tilde{M}(f) \) is the sheaf of modules over \( \text{Spec} A(f) \) corresponding to the module \( M(f) \), the degree zero elements in \( M(f) \). Moreover, \( \tilde{M} \) is coherent whenever \( M \) is finitely generated.
(c) The functor \( M \to \tilde{M} \) is an covariant exact functor from category of \( D \)-graded \( A \)-modules to category of quasi-coherent \( \mathcal{O}_X \)-modules, and commutes with direct limits and direct sums.

The proof follows almost by definition and is very similar to proof of [9] proposition 5.11. The proof of the next lemma is also evident.

**Remark 3.2.** Note that we have used \( M \) as a lattice as well as \( A \)-module. Meaning should be clear from context.

**Remark 3.3.** In general, the functor \( \tilde{*} \) is not faithful, even for projective varieties.

**Lemma 3.4.** Suppose \( D \) is a finitely generated abelian group and \( A \) is a \( D \)-graded algebra such that \( A = A_0[x_1, \ldots, x_r] \), where \( x_i \in A_d \) are homogeneous. Then \( \{d \in D \mid A_d \neq 0\} \) generate a finite index subgroup of \( D \) if and only if \( \{d_i \mid 1 \leq i \leq r\} \) does.

This lemma provides a way to ensure one of the points of the hypothesis in the theorem below.

**Theorem 3.5.** Suppose \( D \) is a free finitely generated abelian group and \( A = \bigoplus_{d \in D} A_d \) is a \( D \)-graded integral domain which is finitely generated by homogeneous elements \( x_1, \ldots, x_r \in A \) over the ring \( A_0 \). Also assume that for all \( k, 1 \leq k \leq r, \) the set \( \{d \in D \mid 1 \leq i \leq r, i \neq k\} \) generates a finite index subgroup of \( D. \) Let \( X = \text{Proj}_{\mathcal{M}} A \). Then \( \Gamma(X, \mathcal{O}_X(d)) \cong A_d. \) Furthermore, \( \mathcal{O}_X(d) \) is a reflexive sheaf.
Before proving the theorem, we observe a fact.

**Lemma 3.6.** With the notation as in theorem 3.5,

\[ X = \text{Proj}_{\text{MH}} A = \bigcup_{f: \text{ is relevant and is a monomial in } x_1, \ldots, x_r} D_+(f). \]

*Proof.* We shall prove this for \( D_+(f) \) for every relevant \( f \) and the lemma will follow. Suppose \( f = m_1 + \cdots + m_t \) where each \( m_i \) is a monomial. Any point \( p \) in \( D_+(f) \) corresponds to a homogeneous ideal \( P \) in \( A \) such that the set of homogeneous elements in \( A \setminus P \) is multiplicatively closed. Let \( H \) be the collection of all such homogeneous ideals.

\[ D_+(f) = \{ P \in H \mid f \not\in P \} \subset \bigcup_{i=1}^t \{ P \in H \mid m_i \not\in P \} = \bigcup_{i=1}^t D_+(m_i) \]

as was to be proved. \( \square \)

Now we return to the proof of the theorem.

*Proof of theorem 3.5.* Giving an element \( t \in \Gamma(X, \mathcal{O}_X(d)) \) is the same as giving a collection \( t_f \in D_+(f) = \text{Spec} A(f) \) for each relevant monomial \( f \) such that they agree on the pairwise intersections: \( D_+(f) \cap D_+(g) = D_+(fg) \) (see lemma 1.10).

Suppose \( t \in \Gamma(X, \mathcal{O}_X(d)) \). For each relevant monomial \( f \in A \) (which are enough to consider by lemma 3.6),

\[ t|_{D_+(f)} \in \mathcal{O}_X(d) \left( D_+(f) \right) = \widetilde{A(d)}(D_+(f)) = (A_f)_d, \]

the \( d \)-th component of the \( D \)-graded ring \( A_f \). Thus, for each such \( f \) write

\[ t|_{D_+(f)} = \frac{p_f}{f^{k_f}} \]

where \( \deg p_f - k_f \deg f = d \). Now since \( A \) is a domain, each \( A_f \subset A_{x_1, \ldots, x_r} \), and since the local expressions of \( t \) match over the intersections, \( t \) is of the form \( x_1^{a_1} \cdots x_r^{a_r} f' \) with \( f' \in A \). Since for each \( i, x_1 \cdots x_i \) is relevant, \( x_1^{a_1} \cdots x_r^{a_r} f' \in A_{x_1, \ldots, x_r} \) implies that \( a_i \geq 0 \). This proves that \( t \in A \) and therefore, \( t \in A_d \).

Since \( \text{Hom}_{\mathcal{O}_X}(\mathcal{O}_X(d), \mathcal{O}_X) = \mathcal{O}_X(-d) \) for all \( d \in D \), reflexivity of \( \mathcal{O}_X(d) \) is clear. \( \square \)

**Example 3.7.** The hypothesis of the above theorem is necessary. For example, consider the ring \( A = \mathbb{C}[X,Y,Z] \) with \( \mathbb{Z}^2 \)-grading given by

\[ \deg X = (0, 1) \quad \deg Y = (1, 0) = \deg Z \]

The scheme \( \text{Proj}_{\text{MH}} A \) is covered by two affines \( D_+(XY) \) and \( D_+(XZ) \). Now consider the module \( M = A((2,-1)) \). Consider the section \( YZ/X \) which is defined over both \( \tilde{M}(D_+(XY)) \) and \( \tilde{M}(D_+(XZ)) \). Therefore, \( YZ/X \in \Gamma(\text{Proj}_{\text{MH}} A, \tilde{M}) \), whereas \( A((2,-1)) = 0 \).
3.2. Line bundles on Multihomogeneous spaces. The reflexive coherent sheaves of modules $\mathcal{O}_X(d)$ will not be line bundles for every $d \in D$. We give a criterion for these to be line bundles generalizing the well-known similar results for weighted projective spaces. Before that we prove a short lemma.

**Lemma 3.8.** Suppose $A$ is a $D$-graded ring for a finitely generated free abelian group $D$, generated as an $A_0$-algebra by homogeneous elements $x_1, \ldots, x_r$. Suppose $A^x = A_0^x$. Assume that $f$ is a relevant monomial in $A$. Suppose $d \in D_f$, where $D_f$ is the sublattice of $D$ generated by

$$\{ \deg a \mid a \text{ divides } f^N \text{ for some } N > 0 \}.$$ 

Then there is a monomial $m$ in $x_1, \ldots, x_r$ and $k \in \mathbb{N} \cup \{ 0 \}$ such that $\deg(m/f^k) = d$ and $m \mid f^N$ for some $N > 0$.

**Proof.** Suppose $f = x_1^{\alpha_1} \cdots x_r^{\alpha_r}$. Then $D_f$ is generated by $\{ \deg x_1, \ldots, \deg x_r \}$. Then for $d \in D_f$, there exists integers $a_1, \ldots, a_s$ such that $d = \sum_{i=1}^s a_i d_i$. Consider the element $a = x_1^{\alpha_1} \cdots x_r^{\alpha_r}$. Let $I = \{ k \mid 1 \leq k \leq s, a_k < 0 \}$. Note that $\prod_{j \in I} x_i^{-a_j} / f^M$ for some $M > 0$. Let $b \in A$ be such that

$$\prod_{j \in I} x_i^{-a_j} b = f^M.$$

Then

$$a = \frac{\prod_{j \in I} x_i^{-a_j} b}{f^M}.$$

This completes the proof by taking $m = \prod_{j \notin I} x_i^{a_j} b$. $\square$

**Theorem 3.9.** Suppose $X = \text{Proj}_{MH} A$ is a multihomogeneous space defined for a $D$-graded integral domain $A = \bigoplus_{d \in D} A_d$ generated by homogeneous elements $x_1, \ldots, x_r$ over $A_0$. Moreover assume that $A_0$ is a field and $A^x = A_0^x$. Let $d \in D_f$ (see lemma 3.8) for every relevant element $f \in A$. Then $\mathcal{O}_X(d)$ is a line bundle.

**Proof.** By lemma 3.6 we can consider an open cover of $X$ given by relevant monomials. Fix a $d$ such that $d \in D_f$ for all relevant $f$. And fix an $f$ which is a relevant monomial. On $D_+(f)$,

$$\mathcal{O}_X(d)|_{D_+(f)} = A_d \stackrel{\sim}{\longrightarrow} (A(d))_{(f)}.$$

by lemma 3.1(b). We claim that $A(d)_{(f)} \cong A_{(f)}$. Note that 1 $\in A(d)$ has degree $-d$, which belongs to $D_f$ by hypothesis. Thus by lemma 3.8 we can find an $m$ such that $m \mid f^N$ for some $N$ and $\deg(m/f^k) = -d$ for some $k$. This implies $m/f^k$ is invertible in $A_f$ and $\deg f^k/m = d$. Now it is evident that for any element of the form $\prod_{i=1}^r x_i^{a_i}/f^r$ in $A(d)_{(f)},$

$$\deg_A \prod_{i=1}^r x_i^{a_i}/f^r = 0 \iff \deg_A \prod_{i=1}^r x_i^{a_i}/f^r = d \iff \deg_A \prod_{i=1}^r x_i^{a_i}/f^r = 0$$

and thus $\prod_{i=1}^r x_i^{a_i}/f^r \in A_f$. Since $m/f^k$ is invertible in $A_f$, this gives an isomorphism of $A_{(f)}$-modules. This proves that $\mathcal{O}_X(d)$ is a line bundle. $\square$

**Example 3.10.** In case of a weighted projective space, $P = \text{Proj} \mathbb{C}[x_0, \ldots, x_n]$ with $\deg x_i = d_i$, theorem 3.9 reduces to saying $\mathcal{O}_P(d)$ is a line bundle if and only if $d$ is divisible by each of the $d_i$'s. This is well known [Delorme, remark 1.8].
4. A relation between a Multihomogeneous space and a $T$-variety

To study the relationship, we need a couple of assumptions. We shall explore them one by one.

**Assumption 4.1.** Let $D \cong \mathbb{Z}^r$ for a natural number $r$ and suppose $A = \bigoplus_{d \in D} A_d$ be a multigraded, noetherian, integral domain such that $A_0 = k$, where $k$ is an algebraically closed field of characteristic 0.

**Assumption 4.2.** In this section, $Y$ always refers to the variety constructed in equation (2.8) where, following the notation in assumption 4.1, Spec $A$ is considered as a $T$-variety under the action of Spec $k[D]$.

**Lemma 4.3.** Suppose $\Lambda$ is the GIT fan (see definition 2.7) associated to the $T = \text{Spec } k[D]$ action on $X = \text{Spec } A$ induced by the $D$-grading. Suppose $\lambda$ is a full-dimensional cone in the quasi-fan $\Lambda$. Then there exists $u \in \text{rel int } \lambda$ such that $A_u$ contains a relevant element.

**Proof.** By the definition of a quasi-fan, each of the rays $\rho \in \lambda(1)$ is also an orbit cone and hence there exists an $u_\rho \in \rho \cap D$ such that $A_{u_\rho} \neq \{0\}$.

Since $\lambda$ is full dimensional, $|\lambda(1)| \geq \dim \lambda$ and hence ($\lambda$ be a strongly convex polyhedral cone) $\{u_\rho | \rho \in \lambda(1)\}$ is a spanning set of $D$ over $\mathbb{Q}$. Choose a homogeneous $f_\rho \in A_{u_\rho}$ for each $\rho$ and consider $f = \prod_{\rho \in \lambda(1)} f_\rho$.

We claim that $f$ is relevant. This follows as once $f$ is inverted, the degrees of units in $A_f$ contains $\{\pm u_\rho | \rho \in \lambda(1)\}$ and hence $[D : D_f] < \infty$, where $D_f$ is defined in the statement of theorem 3.9. $\square$

**Theorem 4.4.** Under the assumption 4.1, the torus $T = \text{Spec } k[D]$ acts on $X = \text{Spec } A$ giving $X$ a structure of a $T$-variety which, suppose, is represented by $(Y, \mathcal{D})$. Then $Y$ and $\text{Proj}_{\text{MH}} A$ are birational.

**Proof.** Let $\Lambda$ be the GIT fan and $\lambda$ be a cone of maximal dimension. Choose a relevant $f$ using lemma 4.3 such that $\deg f \in \text{rel int } \lambda$. Suppose $u = \deg f$. Note that $\text{Spec } A_f \hookrightarrow \text{Spec } A$ is a $T$-equivariant embedding. On the other hand, consider $X_{ss}(u) \cap \text{Spec } A_f$. Clearly both being open irreducible subsets of $\text{Spec } A$, they are birational. Now the result follows from the following commutative diagram:

$$
\begin{array}{ccc}
X_{ss} & \hookrightarrow & X' = \text{Spec } A_f \\
\downarrow & & \downarrow \\
Y & \hookrightarrow & Y_{\lambda} \\
\downarrow & & \downarrow \\
U & \hookrightarrow & \text{Proj}_{\text{MH}} A
\end{array}
$$

where the first two vertical maps are geometric quotients (by remark 3.9). The rightmost vertical map restricted to the complement of the irrelevant subscheme is a geometric quotient. Note that $Y \hookrightarrow Y_{\lambda}$ is birational follows from [1] lemma 6.1. This proves that $Y$ and $\text{Proj}_{\text{MH}} A$ are birational. $\square$

In the rest of this section, we shall explore conditions under which they become isomorphic.

**Remark 4.5.** It is not always true that $Y$ and $\text{Proj}_{\text{MH}} A$ considered above are isomorphic. For example, take a divisorial variety which does not admit an ample line bundle, but does admit a family of ample line bundles. Such a variety corresponds to a multihomogeneous space which is not projective. But the corresponding $Y$ will be projective by construction.
Assumption 4.6. Suppose $\lambda = \omega$, i.e. the GIT fan contains only one full dimensional cone and its faces. Assume that $A$ is generated by $\bigcup_{u \in R} A_u$ where $R = \bigcup_{\rho \in \lambda(1)} \rho$.

Proposition 4.7. Assume 4.1 and 4.6. Assume that $\omega$ is simplicial and $A$ is generated by $\{ f_\rho | \rho \in \lambda(1) \}$ such that $\deg f_\rho \in \rho \cap D$. Then $Y$, as constructed in equation 2.8, and $\Proj_{\text{MH}} A$ are isomorphic. In fact, both of them are projective.

Proof. Under the given conditions, there exists a collection of relevant monomials $\prod_{\rho \in \omega(1)} f_\rho$ which have degree $u = nu'$ where $u' = \sum_\rho u_\rho$, $n \in \mathbb{N}$ and

$$\Proj_{\text{MH}} A = \bigcup \mathcal{D}_+ \left( \prod_{\rho \in \omega(1)} f_\rho \right)$$

Consider $A_u = \bigoplus_{n \geq 0} A_{nu}$. It is generated by $A_u = (A_u)_1$. Therefore, $\Proj_{\text{MH}} A = \Proj A_u \cong Y$ (see [1] 6.1).

Remark 4.8. In the special case when $A = k[X_1, \ldots, X_n]$ with deg $X_i \in \mathbb{Z}^d$, the affine space becomes a $T$-variety with the action of a $d$-dimensional torus. Assume that this action is effective. Then then we know that the $Y$ one gets from the description of the $T$-variety is normal and projective. It is difficult to characterize these further.

Corollary 4.9. The hypothesis of proposition 4.7 holds if and only if $\Proj_{\text{MH}} A$ is a product of weighted projective spaces. Thus, $Y$ and $\Proj_{\text{MH}} A$ constructed above are isomorphic if and only if $\Proj_{\text{MH}} A$ is a product of weighted projective spaces.

Proof. In the case of projective spaces and weighted projective spaces, the weight cone is the only full dimensional cone in the GIT fan. Also, if $X$ and $Y$ are varieties where the weight cones are the only full dimensional cones in their GIT fans, then the same is true for $X \times Y$.

The other direction follows easily.

We can not weaken the hypothesis of the above proposition 4.7. Here is an example of an affine toric variety $X$ and a subtorus $T$ such that corresponding varieties $Y$ and $\Proj_{\text{MH}} A$, where $A$ is the algebra of global sections of $X$, are not isomorphic.

Example 4.10 ([1], example 11.1). Take the affine toric variety $X = k^4$ associated to the canonical cone $\delta := (\mathbb{Z}_{\geq 0})^4$ in $N_X = \mathbb{Z}^4$ and consider the subtorus $T := k^{*2}$ action on $X$ given in standard coordinates by the embedding $t = (t_1, t_2) \rightarrow (t_1^4, t_1^3, t_2, t_1^{12} t_2^{-1})$. Then we have the following short exact sequence of lattices:

$$0 \rightarrow N_T \xrightarrow{F} N_X \xrightarrow{P} N_Y \rightarrow 0,$$

where $N_T$ is the lattice of one parameter subgroups of $T$ and $N_Y := N_X / N_T$ is the quotient lattice. We shall also consider a section $s : N_X \rightarrow N_T$. The linear maps are described by

$$F = \begin{bmatrix} 4 & 0 \\ 3 & 0 \\ 0 & 1 \\ 12 & -1 \end{bmatrix}, \quad P = \begin{bmatrix} 3 & 0 & -1 & -1 \\ 0 & 4 & -1 & -1 \end{bmatrix} \quad \text{and} \quad s = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$
Let $\Sigma_Y$ be the coarsest fan in $(N_Y)_\mathbb{Q}$ generated by $P(\delta_0)$ where $\delta_0$ are faces of $\delta$. The maximal cones of $\Sigma_Y$ are given by

$$\sigma_1 = \langle (1, 0), (0, 1) \rangle, \quad \sigma_2 = \langle (0, 1), (-1, -1) \rangle \quad \text{and} \quad \sigma_3 = \langle (-1, -1), (1, 0) \rangle.$$ 

Then the toric variety $Y$ is $\mathbb{P}^2$ and there exist a pp-divisor $D$ over $Y = \mathbb{P}^2$ such that the $T$-variety $(X, T)$ is represented by the pair $(Y, D)$.

Now the algebra of global sections $\mathcal{O}_Y$ can be computed by the deg map in the following short exact sequence

$$0 \to M_Y \xrightarrow{\rho} M_X \xrightarrow{\tilde{F}} M_T \to 0,$$

where

\[
\begin{bmatrix}
3 & 0 \\
0 & 4 \\
-1 & -1 \\
-1 & -1 \\
\end{bmatrix}
\quad \text{and} \quad
\begin{bmatrix}
4 & 3 & 0 & 12 \\
0 & 0 & 1 & -1 \\
\end{bmatrix}.
\]

Let $I = \{1, 2, 3, 4\}$ be an index set. Then $\deg x_1 = (4, 0), \deg x_2 = (3, 0), \deg x_3 = (0, 1)$ and $\deg x_4 = (12, -1)$ in $M_T$. Let $\rho_i : M_X \to \mathbb{Z}, i \in I$ be the projections and $\rho_i := pr_i|_{M_X} \in N_Y$. Then we have four rays $\rho_1 = (1, 0)\mathbb{R}, \rho_2 = (0, 1)\mathbb{R}$ and $\rho_3 = \rho_4 = (-1, -1)\mathbb{R}$ generated by primitive vectors. Then by remark 1.16 a monomial $f = x_i x_j \in A$ where $i, j \in I$ is relevant if and only if the cone $\sigma_f = \langle \rho_i : i \in I \text{ and } x_i \nmid f \rangle$ is simplicial. Therefore, one can compute that

\[
\operatorname{Proj}_{\mathbb{A}^1} A = \bigcup_{f=x_i x_j \text{ relevant}} D_+(f)
\]

\[= D_+(x_3 x_4) \cup D_+(x_1 x_3) \cup D_+(x_2 x_3) \cup D_+(x_1 x_4) \cup D_+(x_2 x_4)\]

Note that

\[Y = \mathbb{P}^2 = D_+(x_3 x_4) \cup D_+(x_1 x_3) \cup D_+(x_2 x_3)\]

\[= D_+(x_3 x_4) \cup D_+(x_1 x_4) \cup D_+(x_2 x_4)\]

Therefore the multihomogeneous space $\operatorname{Proj}_{\mathbb{A}^1} A$ is union of two copies of $\mathbb{P}^2$ glued along open subcheme $D_+(x_3 x_4)$. However the canonical map in [14] identifies $Y$ with either $D_+(x_3 x_4) \cup D_+(x_1 x_3) \cup D_+(x_2 x_3)$ or $D_+(x_3 x_4) \cup D_+(x_1 x_4) \cup D_+(x_2 x_4)$ in $\operatorname{Proj}_{\mathbb{A}^1} A$. And hence the map in [14] is not an isomorphism. The weight cone $\omega$, generated by $(0, 1)$ and $(12, -1)$, is simplicial. The isomorphism fails to hold because the cone $\omega$ is not a GIT cone.

**References**

[1] Klaus Altmann and Jürgen Hausen. Polyhedral divisors and algebraic torus actions. *Math. Ann.*, 334(3):557–607, 2006.

[2] Klaus Altmann, Jürgen Hausen, and Hendrik Süß. Gluing affine torus actions via divisorial fans. *Transform. Groups*, 13(2):215–242, 2008.

[3] Klaus Altmann, Nathan Owen Iten, Lars Petersen, Hendrik Süß, and Robert Vollmert. The geometry of T-varieties. In *Contributions to algebraic geometry*, EMS Ser. Congr. Rep., pages 17–69. Eur. Math. Soc., Zürich, 2012.

[4] Ivan Arzhantsev, Ulrich Derenthal, Jürgen Hausen, and Antonio Laface. *Cox rings*, volume 144 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 2015.

[5] Florian Berchtold and Jürgen Hausen. GIT equivalence beyond the ample cone. *Michigan Math. J.*, 54(3):483–515, 2006.

[6] Holger Brenner and Stefan Schröer. Ample families, multihomogeneous spectra, and algebraization of formal schemes. *Pacific J. Math.*, 208(2):209–230, 2003.
[7] Igor Dolgachev. Weighted projective varieties. In Group actions and vector fields (Vancouver, B.C., 1981), volume 956 of Lecture Notes in Math., pages 34–71. Springer, Berlin, 1982.

[8] Hubert Flenner and Mikhail Zaidenberg. Normal affine surfaces with C*-actions. Osaka J. Math., 40(4):981–1009, 2003.

[9] Robin Hartshorne. Algebraic Geometry. Springer, 1977.

[10] Ryo Kanda. Non-exactness of direct products of quasi-coherent sheaves. Doc. Math., 24:2037–2056, 2019.

[11] G. Kempf, Finn Faye Knudsen, D. Mumford, and B. Saint-Donat. Toroidal embeddings. I. Lecture Notes in Mathematics, Vol. 339. Springer-Verlag, Berlin-New York, 1973.

[12] D. A. Timashëv. Classification of G-manifolds of complexity 1. Izv. Ross. Akad. Nauk Ser. Mat., 61(2):127–162, 1997.

[13] Ferdinando Zanchetta. Embedding divisorial schemes into smooth ones. J. Algebra, 552:86–106, 2020.

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