STABILITY OF TRAVELLING WAVE SOLUTIONS TO THE SINE-GORDON EQUATION

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ABSTRACT. We give a geometric proof of spectral stability of travelling kink wave solutions to the sine-Gordon equation. For a travelling kink wave solution of speed $c \neq \pm 1$, the wave is spectrally stable. The proof uses the Maslov index as a means for determining the lack of real eigenvalues. Ricatti equations and further geometric considerations are also used in establishing stability.

1. INTRODUCTION

The sine-Gordon equation:

\[ u_{tt} = u_{xx} + \sin u \]  \hspace{1cm} (1)

has applications in many areas of physics and mathematics. It can be used to model magnetic flux propagation in long Josephson junctions: two ideal superconductors separated by a thin insulating layer [SCR76], [DDvGV03]. It can be thought of a model for mechanical vibrations of the so-called ‘ribbon pendulum’ - the continuum limit of a line of pendula each coupled to their nearest neighbor via Hooke’s law [BM07]. In biology, it has found applications in modeling the transcription and denaturation in DNA molecules [Sal91]. Further, it can be used to model propagation of a crystal dislocation, Bloch wall motion of magnetic crystals, propagation of “splay-wave” along a lipid membrane, and pseudo-spherical surfaces to name a few others (see [SCM73] and the references therein).

In this paper, we consider solutions of the form $v(x + ct, t)$ where $c$ is the (positive) speed of the traveling wave. Making the change of variable $z = x + ct$ and substituting into (1) gives

\[ c^2 v_{zz} + 2cv_{zt} + v_{tt} = v_{zz} + \sin v. \] \hspace{1cm} (2)

A travelling wave solution will be a $t$ independent solution $v(z)$ to (2). Thus it solves the (nonlinear) pendulum equation:

\[ (c^2 - 1)v_{zz} = \sin v. \] \hspace{1cm} (3)

A kink wave solution is a travelling wave solution to (2) corresponding to a heteroclinic orbit in the phase plane of (3).

Stability of a singularly perturbed kink wave solution was shown in [DDvGV03]. In [BM07], the Cauchy problem for the sine-Gordon equation in laboratory coordinates was studied using inverse scattering techniques. In this paper we take a more geometric approach determining the spectral stability of kink-wave solutions to equation (1) via geometric considerations and elementary methods of ODE theory.
1.1. The Maslov Index. Note: In this paper we will use the description of the Maslov index as in [RS93].

Let the matrix $J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ denote the standard symplectic structure on $\mathbb{R}^2$. A line $\ell$ passing through the origin is considered Lagrangian in the sense that for any $v_1, v_2 \in \ell$ the inner product of $v_1$ with $Jv_2$, $\langle v_1, Jv_2 \rangle = 0$. Let $\ell(t)$ be a curve of lines in $\mathbb{R}^2$. If $\ell(t)$ can be written as the span of the $2 \times 1$ matrix $\begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$, we will call the functions $x(t)$ and $y(t)$ a frame for $\ell(t)$.

Alternatively we can view $\ell(t)$ as a curve in $\mathbb{R}P^1 \approx S^1$, and if $\begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$ is a frame of $\ell$, then $\ell(t) = [x(t) : y(t)]$. Now let $a = [a_1 : a_2]$ be a fixed line in $\mathbb{R}^2$. Suppose that $\ell(t) : [t_0, t_n] \to \mathbb{R}P^1$, and $\ell(t) = a$ at $t_1, t_2, \ldots t_{n-1}$ with $t_0 < t_1 < t_2 < \ldots < t_{n-1} < t_n$. Suppose further that we have $\ell(t_i) \neq (0, 0)$ for all $t_i$. We define the crossing form, $\Gamma(\ell(t), a, t_i)$ of $\ell(t)$, with respect to $a$, at $t_i$ as:

$$\Gamma(\ell(t), a, t_i) = x(t_i)\dot{y}(t_i) - y(t_i)\dot{x}(t_i).$$

(4)

For a curve $\ell(t)$ in $\mathbb{R}P^1$ as above, we define the Maslov index, $\mu(\ell(t), a)$, as:

$$\mu(\ell(t), a) = \sum_{t_i} \text{sign} \left( \Gamma(\ell(t), a) \right).$$

(5)

The Maslov index defined in this way is a signed count of the number of times that $\ell(t)$, viewed as a curve in $S^1$ crosses the point $a$.

2. Kink Waves

Travelling wave solutions to the sine-Gordon equation for which the quantity $c^2 - 1 < 0$ are called subluminal waves. When $c^2 - 1 > 0$ they are called superluminal waves. We have the following theorem:

Theorem 1. Kink wave solutions to equation (1) $u_{tt} = u_{xx} + \sin u$, are spectrally stable if $c^2 \neq 1$.

2.1. Subluminal kink waves. Until otherwise specified, the quantity $c^2 - 1 < 0$. We will focus primarily on the orbit that satisfies the following boundary conditions, though the analysis that follows will apply to all subluminal kink waves. Let $v(z)$ be a solution to (1) satisfying:

$$\lim_{z \to -\infty} v(z) = -\pi \text{ and } \lim_{z \to \infty} v(z) = \pi$$

(6)

Linearizing equation (2) about the kink wave solution $v$ gives:

$$(c^2 - 1)\varphi_{zz} + 2c\varphi_{zt} + \varphi_{tt} = (\cos v)\varphi.$$ 

(7)

Setting $\varphi_t = \psi$ we can rewrite (7) as:

$$\begin{pmatrix} \varphi \\ \psi \end{pmatrix}_t = \cos v \varphi - (c^2 - 1)\varphi_{zz} - 2c\psi_z$$

(8)

In order to establish spectral stability, we need to consider the eigenvalue problem of equation (8). Letting $\lambda$ be the eigenvalue parameter, we obtain:

$$(\cos v)\varphi - (c^2 - 1)\varphi_{zz} - 2c\psi_z = \lambda \begin{pmatrix} \varphi \\ \psi \end{pmatrix}$$

(9)
STABILITY OF TRAVELLING WAVES

We will consider (twice differentiable) $L^2$ perturbations, $\varphi$. This leads to the following eigenvalue condition: A function $\varphi$ is an eigenfunction of the linearized operator with eigenvalue $\lambda$ if

$$(c^2 - 1)\varphi_{zz} + 2c\lambda \varphi_z + (\lambda^2 - \cos v)\varphi = 0,$$

is satisfied, together with the condition that

$$\lim_{z \to \pm\infty} \varphi(z) = 0$$

The idea now is to reformulate the above eigenvalue condition in an equivalent, geometric way. Setting $w = (w_1, w_2)$, with $w_1 = \varphi$, $w_2 = \varphi_z$ and $'= \frac{d}{dz}$, we have:

$$w' = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}' = \begin{pmatrix} 0 & \frac{1}{c^2-1} \\ \frac{2c\lambda}{c^2-1} & -1 \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} := A(\lambda, z)w.$$  \hfill (11)

Also we set

$$A(\lambda) := \lim_{z \to \pm\infty} A(\lambda, z) = \begin{pmatrix} 0 & 1 \\ \frac{1}{c^2-1} & -\lambda \end{pmatrix}. $$ \hfill (12)

We remark that $A(\lambda)$ has an unstable and a stable subspace which we will denote by $\xi^u$ and $\xi^s$ respectively.

**Lemma 2** (Geometric version of the eigenvalue condition). Given that $\begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$ solves (11)\hfill (11)

$$\lim_{z \to \pm\infty} p_1 = 0$$ \hfill (13)

is equivalent to

$$\lim_{z \to -\infty} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \to \xi^u, \text{ and } \lim_{z \to \infty} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \to \xi^s.$$ \hfill (14)

**Proof.** We compactify the extended phase plane of (11) by introducing a new variable $z = z(\tau) = \tan(\frac{\pi}{2}\tau)$, where $\tau \in (-1, 1)$. Now the extended system of ODE’s becomes the (autonomous)

$$w' = A(\tau, \lambda)w$$

$$\tau' = \frac{2}{\pi} \cos^2 \frac{\pi}{2}$$ \hfill (15)

Next we note that if we view the space $\mathbb{R}^2 \times (-1, 1)$ as a plane bundle over the segment $[-1, 1]$, the linearity of (11) means that linear subspaces of the fibres are preserved, so the flow defined by (15) on $\mathbb{R}^2 \times (-1, 1)$ induces a well defined flow on the cylinder $S^1 \times (-1, 1)$. Moreover, this flow can be continuously extended to a flow on $S^1 \times [-1, 1]$. Finally we note that on this cylinder, the points $\xi^u$ and $\xi^s$ are fixed points of the induced flow, with a one dimensional unstable and stable manifold respectively. The uniqueness of the stable and unstable manifolds concludes the proof of the lemma. \hfill \square

We have redefined the original eigenvalue condition as the existence of a heteroclinic orbit joining $\xi^u$ and $\xi^s$.

Now, as $c^2 - 1 < 0$, the matrix

$$A(\lambda, z) \to A(\lambda) = \begin{pmatrix} 0 & 1 \\ \frac{1}{c^2-1} & -\frac{2c\lambda}{c^2-1} \end{pmatrix}$$
has eigenvalues $\gamma_u$ and $\gamma_s$ corresponding to the unstable and stable subspaces.

$$\gamma_u = -c\lambda - \sqrt{\lambda^2 - (c^2 - 1)} \over c^2 - 1,$$
and
$$\gamma_s = -c\lambda + \sqrt{\lambda^2 - (c^2 - 1)} \over c^2 - 1.$$

(16)

and $(c^2 - 1)\gamma^2_{u,s} + 2c\lambda \gamma_{u,s} + (\lambda^2 + 1) = 0$. Observe:

$$\xi^u = <\begin{pmatrix} -c\lambda + \sqrt{\lambda^2 - (c^2 - 1)} \\ \lambda^2 + 1 \end{pmatrix} >$$
and
$$\xi^s = <\begin{pmatrix} -c\lambda - \sqrt{\lambda^2 - (c^2 - 1)} \\ \lambda^2 + 1 \end{pmatrix} > .$$

(17)

where $<\begin{pmatrix} a \\ b \end{pmatrix} >$ denotes the linear space spanned by the vector $\begin{pmatrix} a \\ b \end{pmatrix}$. For each fixed $\lambda$, let $\ell(z)$ be the set of lines in $\mathbb{R}^2$ that tend to the unstable subspace of $A$ at $-\infty$, under the flow of (11), that is:

$$\ell(z) = \left\{ <\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} > \mid \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \text{ solves (10) and } \to \xi^u, \text{ as } z \to -\infty \right\} .$$

(18)

The lines $\ell(z)$ are Lagrangian, and in fact the representation $\begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$ is a frame for the line $\ell$, so we can define the crossing form relative to the subspace $\xi^s$, $\Gamma(\ell(z), \xi^s, \lambda, z)$ as $\Gamma(\ell(z), \xi^s, \lambda, z) = w_1w'_2 - w_2w'_1$. Substituting as in equation (11) gives

$$\Gamma(\ell(z), \xi^s, \lambda, z) = \cos \nu - {\lambda^2 \over c^2 - 1}w_1^2 - {2c\lambda \over c^2 - 1}w_1w_2 - w_2^2$$

(19)

**Lemma 3.** The crossing form of $\ell(z)$, relative to the stable subspace at infinity, $\Gamma(\ell(z), \xi^s, \lambda, z)$ defined above is independent of $\lambda$. 

Proof. To evaluate the crossing form on $\xi^s$, we use the fact that on $\xi^s$ we have that $w_2 = \frac{\lambda^2 + 1}{-c\lambda + \sqrt{\lambda^2 - (c^2 - 1)}} w_1 = \frac{\lambda^2 + 1}{\gamma_s(c^2 - 1)} w_1$. This gives

$$
\Gamma = \omega^2 \left[ \frac{\cos v - \lambda^2}{(c^2 - 1)} - \frac{2c\lambda}{c^2 - 1} \frac{(\lambda^2 + 1)}{\gamma_s(c^2 - 1)} - \left( \frac{(\lambda^2 + 1)}{\gamma_s(c^2 - 1)} \right)^2 \right]
$$

$$
= \frac{\omega^2}{(c^2 - 1)} \left[ \frac{(\cos v - \lambda^2)\gamma_s^2(c^2 - 1) - 2c\lambda(\lambda^2 + 1)\gamma_s - (\lambda^2 + 1)^2}{\gamma_s^2(c^2 - 1)} \right]
$$

$$
= \frac{\omega^2}{(c^2 - 1)} \left[ \frac{\cos v\gamma_s^2(c^2 - 1) - (\lambda^2)\gamma_s^2(c^2 - 1) - 2c\lambda(\lambda^2 + 1)\gamma_s - (\lambda^2 + 1)^2}{\gamma_s^2(c^2 - 1)} \right]
$$

$$
+ \frac{\omega^2}{(c^2 - 1)} \left[ \frac{-\gamma_s^2(c^2 - 1) + \gamma_s^2(c^2 - 1)}{\gamma_s^2(c^2 - 1)} \right]
$$

$$
= \frac{\omega^2}{(c^2 - 1)} \left[ \frac{(\cos v + 1)\gamma_s^2(c^2 - 1) - (\lambda^2 + 1)((\lambda^2 - 1)\gamma_s^2 + 2c\lambda\gamma_s + (\lambda^2 + 1))}{\gamma_s^2(c^2 - 1)} \right]
$$

$$
= \frac{\omega^2}{(c^2 - 1)} \left[ \frac{(\cos v + 1)\gamma_s^2(c^2 - 1)}{\gamma_s^2(c^2 - 1)} \right]
$$

because $(c^2 - 1)\gamma_s^2 - 2c\lambda\gamma_s + (\lambda^2 + 1) = 0$, so we have that

$$
\Gamma(\ell(z), \xi^s, \lambda, z) = \frac{(\cos v + 1)\omega^2}{(c^2 - 1)}
$$

(20)

Which shows that $\Gamma$ is independent of $\lambda$.

Evaluating the limits and using lemma 3 we get that the Maslov index of the curve of Lagrangian subspaces $\ell(z)$, with respect to the stable subspace at infinity is:

$$
\mu(\ell(z), \xi^s, \lambda) = \sum_z \text{sign} \Gamma(\ell(z), \xi^s, \lambda, z)
$$

(21)

$$
= -\# \text{ of crossings that occur.}
$$

(22)

We next observe that only regular crossings occur. A crossing is called regular if $\ell(z)$ is non-zero. Equation (20) is clearly non-zero, except at $z = \pm \infty$.

At a regular crossing we have that $\ell(z, \lambda) = \left( \frac{w_1}{w_2} \right) = \xi^s$. The function $f(z, \lambda) = \frac{w_2}{w_1}$ is well defined at a regular crossing of the unstable manifold $\ell(z)$ with $\xi^s$. Moreover at a regular crossing we have that

$$
f(z, \lambda) = \frac{\lambda^2 + 1}{-c\lambda + \sqrt{\lambda^2 - (c^2 - 1)}},
$$

(23)
and that
\[ \frac{\partial}{\partial \zeta} f(z, \lambda) = \frac{\Gamma(\ell(z), \xi^*, \lambda, z)}{w^2} \neq 0. \] (24)
Thus we can use the implicit function theorem to write the location of the crossing in the \( z \) variable as a function of the parameter \( \lambda \), \( z = z(\lambda) \), further because equation (20) is independent of \( \lambda \), we can do this for all \( \lambda \).

When \( \lambda \gg 1 \), the system in (11) tends to
\[ \begin{pmatrix} w_1' \\ w_2' \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\frac{\lambda^2}{c^2-1} & -\frac{2\lambda}{c^2-1} \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} =: A(\lambda \gg 1) \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \] (25)
which has eigenvalues \( \gamma_{\pm} = \frac{-\lambda}{c^2-1} \). Which are real and as \( c^2 - 1 < 0 \), one is positive and one is negative. For large enough \( \lambda \) then, the number of crossings of \( \ell(z) \) relative to \( \xi^* \) is the same as the number of crossings as for the solution of the constant coefficient equation (25) tending towards the unstable subspace of the matrix \( A(\lambda \gg 1) \). Thus for large enough \( \lambda \) the number of crossings of the unstable manifold \( \ell(z) \) and \( \xi^* \) is equal to zero.

Now fix a \( \lambda \) and a location of a crossing \( z_1(\lambda) \) say. Because the number of crossings tends to zero as \( \lambda \) increases, and because of the implicit function theorem assertion above, we have that for some finite \( \lambda_1 \), \( \lim_{\lambda \to \lambda_1} z_1(\lambda) = \infty \). But this is exactly the geometric reformulation of the eigenvalue condition. Moreover, the implicit function theorem implies that we have a unique value, \( z_1(\lambda) \), for this crossing for each \( \lambda \), and the locations of different crossings (say \( z_2(\lambda) \)) cannot intersect. Thus as \( \lambda \) increases, the number of crossings will decrease monotonically, as will the Maslov index. This last paragraph is summed up in the following corollary.

**Corollary 4.** If \( N_1 \) is the number of times \( \ell(z) \) crosses \( \xi^* \) when \( \lambda = \lambda_1 \), as \( z \) ranges from \( -\infty \) to \( \infty \) and \( N_2 \) is the number of times \( \ell(z) \) crosses \( \xi^* \) when \( \lambda = \lambda_2 \), then \( |N_1 - N_2| = \) the number of eigenvalues of the linearized operator (3) in \( (\lambda_1, \lambda_2) \).

Lastly, we show that \( \mu(\ell(z), \xi^*, 0, z) = 0 \). This shows that there are no crossings when \( \lambda = 0 \) and so there are no crossings for \( \lambda \in (0, \infty) \). To see this observe that when \( \lambda = 0 \), equation (11) reduces to the equation of variations of the pendulum equation (3). Thus the unstable manifold of equation (11) is just the tangent line to the heteroclinic orbit, joining \((-\pi, 0)\) to \((\pi, 0)\) in the phase plane of (11). This orbit in the phase plane is given by:
\[ v_z = \sqrt{\frac{2 \cos v + 1}{1 - c^2}} \] (26)
and the slope of the tangent line is given by
\[ \frac{dv_z}{dv} = \frac{-\sin v}{\sqrt{(c^2 - 1)(2 \cos v + 2)}}. \] (27)
In order for a crossing to occur, the tangent line must be parallel to \( \xi^* \). This means that
\[ \frac{-\sin v}{\sqrt{(c^2 - 1)(2 \cos v + 2)}} = \frac{-1}{\sqrt{-(c^2 - 1)}} \] (28)
But this only happens when \( v = \pm \pi \). But \( v = \pm \pi \) is a critical point in the phase plane. That is, there is no \( z \in (-\infty, \infty) \) where this can happen, so there can be no crossings, so \( \mu(\ell(z), \xi^*, 0, z) = 0 \). This shows that there are no crossings for \( \lambda \in (0, \infty) \).
Lastly we note that the crossing form argument works in the negative \( \lambda \) direction and that the asymptotic behaviour of the system in (11) for \( \lambda \ll -1 \) is the same as that for \( \lambda \gg 1 \). Thus we have no real eigenvalues \( \lambda \neq 0 \).

2.2. Complex Eigenvalues. We now return our attention to equation (10) to show that there are no complex eigenvalues \( \lambda = p + iq \) where \( p > 0 \). Without loss of generality we can assume that \( q > 0 \) so \( \lambda = re^{i\theta} \) where \( r > 0 \) and \( \theta \in (0, \frac{\pi}{2}) \). Rewriting (10) as:

\[
\frac{w''}{c^2 - 1} + \frac{2c\lambda}{c^2 - 1} w' + \left( \frac{\lambda^2 - \cos(v(z))}{c^2 - 1} \right) w = 0, \tag{29}
\]

we make the substitution

\[
\psi = we^{\left( \frac{c\lambda}{c^2 - 1} z \right)} \tag{30}
\]

and re-write (29) as

\[
\psi'' = \left( \cos v - \frac{\lambda^2}{c^2 - 1} + \frac{c^2 \lambda^2}{(c^2 - 1)^2} \right) \psi \tag{31}
\]

We wish to consider now complex valued solutions to (31). We let \( \eta = \frac{\psi'}{\psi} \) and consider the induced flow on a chart of \( \mathbb{C}P^1 \), where \( \psi \neq 0 \):

\[
\eta' = \frac{\psi'' \psi - \psi' \psi'}{\psi^2} = \frac{\cos v - \lambda^2}{c^2 - 1} + \frac{c^2 \lambda^2}{(c^2 - 1)^2} - \eta^2. \tag{32}
\]

If we set \( \eta = \alpha + i\beta \), and consider the imaginary part of the vector field of (32) on the real axis, that is those points where \( \eta = \alpha \), we have

\[
\beta' \big|_{\beta=0} = \frac{-2pq}{c^2 - 1} + \frac{2c^2 pq}{(c^2 - 1)^2} = \frac{2pq}{(c^2 - 1)^2} > 0 \quad \text{if } p, q > 0. \tag{33}
\]

thus the flow of (32) on the real axis of \( \mathbb{C}P^1 \) is always pointing in the positive imaginary direction.

By considering \( \frac{w_2}{w_1} \) from equation (11), we now interpret the eigenvalue conditions, for the eigenvalues that we are interested in.

**Lemma 5.** An eigenvalue \( \lambda \) is a value of \( \lambda \) where there exists a heteroclinic orbit of the flow induced by (11), on (a chart of) \( \mathbb{C}P^1 \), i.e. \( \lambda \) an eigenvalue means that there is an orbit from \( \lim_{z \to -\infty} \frac{w_2}{w_1} = \gamma_u \) to \( \gamma_s = \lim_{z \to \infty} \frac{w_2}{w_1} \). Under the transformation given in (30) we have that

\[
\eta = \frac{w_2}{w_1} + \frac{c\lambda}{c^2 - 1}, \tag{34}
\]

and so \( \gamma_u \) goes to \( \eta_u \) and \( \gamma_s \) goes to \( \eta_s \) where

\[
\eta_u = \frac{-\sqrt{\lambda^2 - (c^2 - 1)}}{c^2 - 1}, \quad \text{and} \quad \eta_s = \frac{\sqrt{\lambda^2 - (c^2 - 1)}}{c^2 - 1}. \tag{35, 36}
\]

We claim that for \( \lambda = p + iq \) with \( p \) and \( q \) both positive, that \( \eta_u \) has a positive imaginary part and \( \eta_s \) has a negative imaginary part. Thus as \( \beta'|_{\beta=0} > 0 \), from above, there can be no orbit joining the two critical points, and so no eigenvalues with positive real part. Thus all kink waves satisfying \( c^2 - 1 < 0 \) are stable.
To see that $\eta_u$ has a positive imaginary part, write $\lambda = re^{i\theta}$ and $\eta_u = R_u e^{i \theta_u}$. Since we are assuming that $\lambda$ has positive imaginary part $\theta \in (0, \frac{\pi}{2})$. By equation (35) we have that

$$\theta_u = \arctan \left( \frac{r^2 \sin(2\theta)}{r^2 \cos(2\theta) + (1 - c^2)} \right).$$

(37)

Since $\theta_u$ will be the same for all points on the line $re^{i\theta}$, and since $c^2 - 1 < 0$, we can assume without loss of generality that $r^2 = (1 - c^2)$. Thus substituting into equation (37) we have:

$$\theta_u = \arctan \left( \frac{2 \sin \theta \cos \theta}{\cos^2(\theta) - \sin^2(\theta) + 1} \right) = \theta.$$

(38)

The exact same calculation shows that $\eta_s$ has a negative imaginary part.

We have shown that there can be no homoclinic orbit on the chart of $\mathbb{CP}^1$ parametrized by $w_2/w_1$. To see that there can be no such orbit on the other chart, we have that off the origin, the charts are transformed into each other via $\eta \rightarrow \frac{1}{\eta}$ for $\zeta \in \mathbb{C} \setminus 0$. Thus $\eta_u \rightarrow \frac{1}{\eta_u}$ which will have a negative imaginary part. Likewise $\frac{1}{\eta_s}$ will have a positive imaginary part. If we write $\zeta = \frac{\psi}{\psi'} = \theta + i\zeta$ and consider the flow on this chart induced by (10)

$$\zeta' = \left( \frac{(\psi')^2 - \psi \psi''}{\psi'^2} \right) = 1 - \left( \frac{\cos v - \lambda^2}{c^2 - 1} + \frac{c^2 \lambda^2}{(c^2 - 1)^2} \right) \zeta^2.$$

(39)

Now letting $\zeta = \sigma + i\tau$ and considering only the imaginary part of the flow on $\mathbb{C}$ given by (39) restricted to where $\tau = 0$ we have that:

$$r' \big|_{\tau=0} = \frac{-2pqc^2\sigma^2}{(c^2 - 1)^2} - \frac{2pq\sigma^2}{c^2 - 1} = -\frac{2pq\sigma^2}{(c^2 - 1)^2} < 0 \quad \text{if } p, q > 0.$$

(40)

Thus the flow of (32) on the real axis of (this chart of) $\mathbb{CP}^1$ is always pointing in the negative imaginary direction, and so there can be no heteroclinic orbit connecting $\frac{1}{\eta_u}$ to $\frac{1}{\eta_s}$; and hence there are no eigenvalues $\lambda$ with positive real part to equation (9).

The same argument can be run in for $\lambda = p + iq$, with $p$ and $q$ both negative to show that the flow on the real axis is pointing in the wrong direction to allow for an orbit joining $\eta_u$ to $\eta_s$ as well. Just as in the previous case, one makes a Liouville transformation and then tracks the location of the stable and unstable subspaces on both charts of $\mathbb{CP}^1$. Thus we conclude that there are no eigenvalues off the imaginary axis.

2.3. Superluminal kink-waves. We now return our attention to the superluminal eigenvalue problem equation (9):

$$(\cos v)\varphi - (c^2 - 1)\varphi_{zz} - 2c\psi_z = \lambda \begin{pmatrix} \varphi \\ \psi \end{pmatrix}.$$

(41)

We are interested in the values of $\lambda \in \mathbb{C}$ for which there are solutions to (41) which decay to zero as $z \rightarrow \pm \infty$. We first investigate the limiting cases $\lim_{z \rightarrow \pm \infty}$, which becomes the constant coefficient ODE

$$\varphi'' - 2c\lambda \varphi - (\lambda^2 + 1)\varphi = 0.$$

(42)
which can easily be solved for any value of $\lambda \in \mathbb{C}$. The characteristic exponents $r_{1,2}$ are given by:

$$r_{1,2} = \frac{c\lambda \pm \sqrt{\lambda^2 + (1 - c^2)}}{1 - c^2} \quad (43)$$

By taking limits as $c \to 1^+$, we obtain that the signs of the real parts of $r_{1,2}$ are equal. In fact they will be the opposite of the sign of the real part of $\lambda$. We have:

$$\text{sgn} \left( \Re \left( \lim_{c \to 1^+} r_1 \right) \right) = -\text{sgn} \left( \Re (2\lambda) \right) \quad \text{and,} \quad \text{sgn} \left( \Re \left( \lim_{c \to 1^+} r_2 \right) \right) = -\text{sgn} \left( \Re (\lambda + \frac{1}{\lambda}) \right)$$

Moreover, by differentiating each in the $c$ variable, we obtain that the sign of the derivative of the characteristic exponents $\frac{\partial r_{1,2}}{\partial c}$ is the same as the sign of the real part of $\lambda$. Thus we can conclude that in the case of $c^2 - 1 > 0$, we have that for the asymptotic cases, the eigenvalues of $A(\lambda)$ have the same sign. This means that as $z \to -\infty$ there can be no unstable orbit of the construction in lemma 2 and therefore there can be no heteroclinic connection to the stable orbit at $\infty$. Thus we conclude that there is no point spectrum when $c^2 - 1 > 0$ is positive. This concludes the proof of theorem 1.

**References**

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