Relativistic Hamiltonians in many-body theories

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Abstract

We discuss the description of a many-body nuclear system using Hamiltonians that contain the nucleon relativistic kinetic energy and potentials with relativistic corrections. Through the Foldy-Wouthuysen transformation, the field theoretical problem of interacting nucleons and mesons is mapped to an equivalent one in terms of relativistic potentials, which are then expanded at some order in $1/m_N$. The formalism is applied to the Hartree problem in nuclear matter, showing how the results of the relativistic mean field theory can be recovered over a wide range of densities.

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I. INTRODUCTION

In recent years considerable efforts have been devoted to the development of relativistic nuclear models, whose basis is usually a relativistic Lagrangian containing nucleons and mesons [1–3]. On the other hand, the traditional concept of interparticle potential has proved to be quite useful in nuclear physics and rather sophisticated techniques of calculation have been developed within the hamiltonian formalism.

It is clear that it would be useful to be able to reformulate the relativistic problem of interacting Dirac particles in terms of Pauli spinors and potentials, since this would allow one to employ the non-relativistic computational techniques. Furthermore, the high energies and high momentum transfers accessible at facilities such as Cebaf require calculations to be performed in regimes where the relativistic kinematics is certainly important: then, one may ask the question whether genuine relativistic dynamical effects are present and to which extent they may be reconstituted to a potential treatment.

A possible way of connecting the two approaches relies on Hamiltonians containing the nucleon relativistic kinetic energy and a potential obtained by expanding at some order in $1/m_N$ the nucleon-nucleon (NN) scattering amplitude. For instance, in Ref. [4] the quasielastic charge response of electron scattering has been evaluated in the relativistic Fermi gas model using the Bonn potential expanded up to $1/m_N^2$. A similar approach has been followed in Ref. [5], using a relativistic Hamiltonian in variational Monte Carlo calculations.

The aim of the present paper is to show how a systematic expansion of the nucleon-nucleon interaction in a many-body nuclear system can be constructed, using the well-known Foldy-Wouthuysen transformation. The many-body problem, expressed in terms of relativistic Green’s functions and meson-nucleon vertices, is transformed to an exactly equivalent one, in terms of modified Green’s functions and vertices. After projection on the positive energy states and expansion in powers of $1/m_N$ one recovers the familiar relativistic potentials.

This procedure gives a systematic way of dealing with $1/m_N$ expansions in a many-body context: it allows one to address, for instance, the issue of self consistency, i.e. the fact that the Dirac spinors depend on the nucleon mass, which is modified by the medium. In terms of relativistic Hamiltonians, this means that the potential, which affects the effective nucleon mass via the self-energy, should in turn depend on it. In a fully relativistic context this problem is rather simple to solve, whereas in a non-relativistic framework it is ill-defined.

In the following we shall not be concerned with antinucleons degrees of freedom: although this is an interesting issue in itself, we would like to maintain our initial commitment of trying to understand the nuclear phenomena including “simple” relativistic effects, such as kinetic energies and momentum-dependent potentials. Any failure in this program might of course open the way to more interesting relativistic effects, like, e.g., vacuum polarization.

The reader should also be aware of a potential source of inconsistency connected to retardation effects. In fact, the parameters of empirical NN potentials, such as the Bonn one, have usually been fixed assuming a static interaction. Although the inclusion of retardation in the formalism developed in the present paper is rather straightforward, when dealing with an empirical NN potential this may not be the most appropriate course to follow. Since in the following we have applied the formalism only to Hartree calculations, which are not affected by this problem, we have considered, for simplicity, static propagators for the
meson fields. The issue will have to be reconsidered when dealing, e.g., with random phase approximation calculations.

The paper is organized as follows: in Sect. II, the two most commonly used schemes for defining a relativistic potential are briefly sketched, employing $\sigma$ and $\omega$ exchange as examples. In Sect. III, the Foldy-Wouthuysen expansion is introduced in the nuclear many-body system, using nuclear matter for illustration, whereas in Sect. IV the Hartree problem is solved, comparing the results obtained with the relativistic Hamiltonians to the exact ones. Finally, in the last Section we present our conclusions.

II. RELATIVISTIC NN POTENTIALS

In the literature one can find essentially two ways to derive a two-body potential expanded up to a given order in $1/m_N$.

In the first case one expands the relativistic scattering amplitude and interprets it as the matrix element of the Fourier transform of the potential (Born term) between Pauli reduced spin wave functions. We shall refer to it as “Breit reduction” [6].

An alternative procedure starts from the three-dimensional Blankenbecler-Sugar (BbS) reduction of the Bethe-Salpeter (BS) equation, which yields an amplitude satisfying elastic unitarity [7,8].

Both derivations are briefly sketched in the two following subsections. Here, to set the notation, let us define the momentum transfer $q$, the relative momentum in the initial state $p$ and the total momentum $P$ for the scattering from a two nucleon initial state with momenta $k_1$ and $k_2$ to a final state with momenta $k'_1$ and $k'_2$:

\begin{align}
q &= k_1 - k'_1 = k'_2 - k_2 \\
p &= \frac{1}{2} (k_1 - k_2) \\
P &= k_1 + k_2 = k'_1 + k'_2 .
\end{align}

We will be considering the amplitudes for the exchange of a scalar particle of mass $m_s$ (described by the scalar field $\phi$) and of a vector particle of mass $m_v$ (described by the vector field $v_\mu$), whose couplings to the nucleon are given by

\begin{align}
H_s &= g_s \overline{\Psi} \Psi \phi \\
H_v &= g_v \overline{\Psi} \gamma^\mu \Psi v_\mu ,
\end{align}

where $\Psi$ is the nucleon field and $\gamma^\mu$ are the Dirac matrices.

A. Breit reduction

The amplitudes for the exchange of a scalar meson, $M_s$, and of a vector meson, $M_v$, are given by:
\[ M_s = -\frac{g_s^2}{q^2 + m_s^2} \frac{1}{\mu(k')} u(k_1) \mu(k_2) u(k_2) \]  
(2.3a)

\[ M_v = \frac{g_v^2}{q^2 + m_v^2} \frac{\gamma^\mu}{\mu(k')} u(k_1) \gamma^\mu u(k_2) \]  
(2.3b)

where \( u(k) \) is the Dirac spinor,

\[ u(k) = \sqrt{\frac{E_k + m_N}{2m_N}} \left( \frac{\chi}{\sigma_k} \right) \]  
(2.4)

whereas \( \chi \) is the Pauli spinor and \( E_k = \sqrt{k^2 + m_N^2} \).

When expanded up to order \( k^2/m_N^2 \), the normalized Dirac spinor turns out to be

\[ u(k) = \left(1 - \frac{k^2}{8m_N^2} \right) \left( \frac{\chi}{\sigma_k} \right) + O \left( \left( \frac{k}{m_N} \right)^4 \right) \]  
(2.5)

Inserting this expression in (2.3) and using the standard relation between the Born amplitude and the spin matrix elements of the potential in momentum space, \( M = \chi_1^\dagger \chi_2 \chi_1 \chi_2 \), one gets

\[ V_s^{BR}(q, p, P) = -\frac{g_s^2}{q^2 + m_s^2} \left\{1 - \frac{P^2}{4m_N^2} - \frac{(2p - q)^2}{4m_N^2} \right\} \]  
(2.6a)\n
\[ V_v^{BR}(q, p, P) = \frac{g_v^2}{q^2 + m_v^2} \left\{1 - \frac{q^2}{4m_N^2} - \frac{P^2}{4m_N^2} + \frac{(2p - q)^2}{4m_N^2} \right\} \]  
(2.6b)\n
where the superscript BR stands for “Breit reduced” and \( S_{12} \) is the standard tensor operator \( S_{12} = 3(\sigma_1 \cdot q)(\sigma_2 \cdot q)/q^2 - \sigma_1 \cdot \sigma_2 \).

### B. Minimal relativity

A natural framework for relativistic two-nucleon potential scattering is given by the BS equation. This is a four-dimensional integral equation for the scattering amplitude, which is rather hard to solve even in the ladder approximation. Hence, many approximation schemes have been derived in order to obtain from it a more tractable three-dimensional, covariant equation, based on the requirement of relativistic elastic unitarity.

A popular reduction procedure leads to the BbS equation, which, in the centre-of-mass (c.m.) frame, reads
\[ M(k_f, k_i) = V(k_f, k_i) + \int \frac{d^3k}{E_k} V(k_f, k) \frac{\Lambda^+_+(k)\Lambda^+_-(k)}{k^2 - k^2 + i\epsilon} M(k, k_i), \quad (2.7) \]

where \( V \) is the bare two-body potential, \( \Lambda^+_+ \) are the positive energy projection operators and \( k_i, k \) and \( k_f \) are the initial, intermediate and final c.m. momenta, respectively.

If one defines

\[
\tilde{M}(k_f, k_i) = \sqrt{\frac{m_N}{E_{k_f}}} M(k_f, k_i) \sqrt{\frac{m_N}{E_{k_i}}}, \quad (2.8a)
\]

\[
\tilde{V}(k_f, k_i) = \sqrt{\frac{m_N}{E_{k_f}}} V(k_f, k_i) \sqrt{\frac{m_N}{E_{k_i}}}, \quad (2.8b)
\]

Eq. (2.7) becomes identical to the non-relativistic Lippmann-Schwinger equation

\[
\tilde{M}(k_f, k_i) = \tilde{V}(k_f, k_i) + m_N \int d^3k \tilde{V}(k_f, k) \frac{\Lambda^+_+(k)\Lambda^+_-(k)}{k^2 - k^2 + i\epsilon} \tilde{M}(k, k_i). \quad (2.9)
\]

Prescription (2.8) is usually known as “minimal relativity” [7]. It can be generalized to a generic frame by introducing a factor \( (m_N/E)^{1/4} \) for each spinor in the amplitude, i.e.

\[
V_{MR}^\sigma(q, p, P) = \left[ \frac{m_N m_N m_N m_N}{E_{k_1} E_{k_1} E_{k_2} E_{k_2}} \right]^{1/4} V(q, p, P), \quad (2.10)
\]

where MR stands for “minimal relativity”.

Expanding again up to the order \( k^2/m_N^2 \) one gets, for the exchange of scalar and vector mesons, the following expressions

\[
V_{MR}^\sigma(q, p, P) = -\frac{g^2}{q^2 + m_s^2} \left\{ 1 - \frac{P^2}{8m_N^2} - \frac{(2p - q)^2}{8m_N^2} + \frac{q^2}{8m_N^2} + \right.
\]
\[
\left. -ip \cdot \left[ \frac{q \times (\sigma_1 + \sigma_2)}{4m_N^2} \right] - ip \cdot \left[ \frac{q \times (\sigma_1 - \sigma_2)}{8m_N^2} \right] \right\} \quad (2.11a)
\]

\[
V_{MR}^\sigma(q, p, P) = \frac{g^2}{q^2 + m_v^2} \left\{ 1 - \frac{q^2}{8m_N^2} - \frac{P^2}{8m_N^2} + \frac{3(2p - q)^2}{8m_N^2} + \right.
\]
\[
\left. +3ip \cdot \left[ \frac{q \times (\sigma_1 + \sigma_2)}{4m_N^2} \right] - ip \cdot \left[ \frac{q \times (\sigma_1 - \sigma_2)}{8m_N^2} \right] - \right.
\]
\[
\left. -\frac{q^2}{6m_N^2} \sigma_1 \cdot \sigma_2 + \frac{q^2}{12m_N^2} S_{12} \right\}. \quad (2.11b)
\]

Eq. (2.9) has been used in deriving the relativistic Bonn potential [9]. Hence, the non-relativistic expansion of this potential, useful in nuclear structure calculations, is given, for \( \sigma \) and \( \omega \) exchange, by Eq. (2.11).
III. FOLDY-WOUTHUYSEN EXPANSION

In this section we would like to rederive the previous expansions using the language of the Green’s functions and within the scheme devised by Foldy and Wouthuysen [10] (FW) to decouple the large and small components in the relativistic wave function.

Let us start by introducing the FW unitary transformation [11,12],

\[ T(k) = \sqrt{\frac{E_k + m_N}{2E_k}} \left( \mathbb{1} + \frac{\gamma \cdot k}{E_k + m_N} \right), \]  

and the associated FW nucleonic field operator,

\[ \Psi_{FW} = T \Psi, \quad \Psi_{FW}^\dagger = \overline{\Psi} T \]  

(having used the property \( \gamma^0 T^\dagger \gamma^0 = T \)).

Using (3.2) one can define a FW Green’s function

\[ iG_{FW}(x, y) \equiv \langle \Psi_0 | T \left[ \Psi_{FW}(x) \overline{\Psi_{FW}}(y) \right] | \Psi_0 \rangle = T iG(x, y) T. \]  

In momentum space, and for the case of the relativistic Fermi gas, the latter reads

\[ G_{0FW}(k, k_0) \equiv T(k) G_0(k, k_0) T(k) = P_+ \left[ \frac{\vartheta(k - k_F)}{k_0 - E_k + i\varepsilon} + \frac{\vartheta(k_F - k)}{k_0 - E_k - i\varepsilon} \right] - P_- \frac{1}{k_0 + E_k - i\varepsilon}, \]  

where \( P_\pm \equiv (\mathbb{1} \pm \gamma^0)/2 \) are operators that project out the large/small components in the wave function and \( k_F \) is the Fermi momentum.

It is evident that one gets two pieces, acting separately on the large and small components, since the transformation (3.1) has washed out all the operators inducing a mixing (i.e. the \( \gamma_i \)'s).

It is also clear from the definition (3.3) that one can always redefine any given Feynman diagram in terms of \( G_{FW} \) and a transformed interaction (see Fig. 1).

For a typical meson-exchange interaction one has, for instance,

\[ V = g^2 \Gamma_1 D(q) \Gamma_2, \]  

where \( g \) is the coupling constant, \( D(q) \) the meson propagator and \( \Gamma_i \) the vertex operators (e.g., \( \mathbb{1} \) or \( \gamma^\mu \)).

Then, the FW interaction would read

\[ V_{FW} = g^2 T^\dagger(k_{1}) \Gamma_1 T^\dagger(k_1) D(q) T^\dagger(k_{2}) \Gamma_2 T^\dagger(k_2). \]  

We make now a further simplification, neglecting in the following the contribution of antinucleons:
$$G_{0}^{FW}(k, k_0) \rightarrow G_{0(+)}^{FW}(k, k_0) = P_{+} \left[ \frac{\vartheta(k - k_F)}{k_0 - E_k + i\varepsilon} + \frac{\vartheta(k_F - k)}{k_0 - E_k - i\varepsilon} \right].$$  \hspace{1cm} (3.7)

It is a simple matter to move the operators $P_{+}$ from the Green’s functions to the potential, defining

$$V_{(+)FW}^{FW} = g^2 P_{+} \mathcal{T}^{\dagger}(k'_1) \Gamma_1 \mathcal{T}^{\dagger}(k_1) P_{+} D(q) P_{+} \mathcal{T}^{\dagger}(k'_2) \Gamma_2 \mathcal{T}^{\dagger}(k_2) P_{+}. \hspace{1cm} (3.8)$$

The many-body problem is then formally equivalent to the standard non-relativistic one, in terms of a potential $V_{(+)FW}^{FW}$ and the Green’s function

$$G_{0(+)}(k, k_0) = \left[ \frac{\vartheta(k - k_F)}{k_0 - E_k + i\varepsilon} + \frac{\vartheta(k_F - k)}{k_0 - E_k - i\varepsilon} \right], \hspace{1cm} (3.9)$$

which is identical to the non-relativistic one, apart from the energy-momentum relation.

For the case of scalar and vector meson exchange, the FW reduced potentials have the following form:

$$V_{s(+)}^{FW} = - \frac{g_s^2}{q^2 + m_s^2} \sqrt{\frac{(E_{k'_1} + m_N)(E_{k_1} + m_N)(E_{k'_2} + m_N)(E_{k_2} + m_N)}{16 E_{k'_1} E_{k_1} E_{k'_2} E_{k_2}}} \times \left\{ 1 - \frac{k_1 \cdot k'_1}{(E_{k'_1} + m_N)(E_{k_1} + m_N)} - \frac{i k'_1 \cdot (k_1 \times \sigma_1)}{(E_{k'_1} + m_N)(E_{k_1} + m_N)} \right\},$$

$$V_{v(+)}^{FW} = - \frac{g_v^2}{q^2 + m_v^2} \sqrt{\frac{(E_{k'_1} + m_N)(E_{k_1} + m_N)(E_{k'_2} + m_N)(E_{k_2} + m_N)}{16 E_{k'_1} E_{k_1} E_{k'_2} E_{k_2}}} \times \left\{ 1 - \frac{k_2 \cdot k'_2}{(E_{k'_2} + m_N)(E_{k_2} + m_N)} - \frac{i k'_2 \cdot (k_2 \times \sigma_2)}{(E_{k'_2} + m_N)(E_{k_2} + m_N)} \right\}. \hspace{1cm} (3.10a)$$
\[ \times \left\{ 1 + \frac{k_1 \cdot k'_1}{(E_{k'_1} + m_N)(E_{k_1} + m_N)} + \frac{i k'_1 \cdot (k_1 \times \sigma_1)}{(E_{k'_1} + m_N)(E_{k_1} + m_N)} \right\} \]
\[ \times \left\{ 1 + \frac{k_2 \cdot k'_2}{(E_{k'_2} + m_N)(E_{k_2} + m_N)} + \frac{i k'_2 \cdot (k_2 \times \sigma_2)}{(E_{k'_2} + m_N)(E_{k_2} + m_N)} \right\} \]
\[ + \left\{ i \left( \frac{k'_1}{E_{k'_1} + m_N} - \frac{k_1}{E_{k_1} + m_N} \right) \times \sigma_1 - \left( \frac{k_1}{E_{k_1} + m_N} + \frac{k'_1}{E_{k'_1} + m_N} \right) \right\} \cdot \left\{ i \left( \frac{k_2}{E_{k_2} + m_N} - \frac{k'_2}{E_{k'_2} + m_N} \right) \times \sigma_2 + \left( \frac{k_2}{E_{k_2} + m_N} + \frac{k'_2}{E_{k'_2} + m_N} \right) \right\} \}. \quad (3.10b) \]

It is then a straightforward matter to check, by expanding these expressions up to second order in the inverse nucleon mass, that one gets the $V^{BR}$ potentials of Eq. (2.6). On the other hand, if one performs the expansions dropping in (3.10) the big square root containing kinematical factors $(E_k + m_N)/2E_k$, then one recovers the $V^{MR}$ potentials of Eq. (2.11).

The kinematical factors of the minimal relativity prescription (2.8) have been introduced in order to reduce the BS equation to a non-relativistic Lippmann-Schwinger equation: hence, it would appear natural to use the non-relativistic expression for the kinetic energy in doing calculations with the $V^{MR}$ potentials. Note, however, that in the literature both the non-relativistic [7,13] and the relativistic [14,15] kinetic energies have been employed.

**IV. RELATIVISTIC HARTREE CALCULATIONS**

We have seen that the FW transformation leads to a new Green’s function, in which the small and the large components are decoupled. Simultaneously, we define the potentials according to the prescription (3.8). In this way we are able to operate within a framework that clearly resembles the non-relativistic one: the FW Green’s function, projected upon the large components, has indeed the same structure of the familiar non-relativistic Green’s function (see (3.9)), apart from the energy of the fermion, which is now fully relativistic. On the other hand, the potentials obtained from a non-relativistic expansion of $V^{FW}$ reproduce the ones obtained following other procedures (for example, see [7,13]).

Let us now go further and apply the FW transformation to the relativistic Hartree approximation for the nucleon Green’s function, assuming again that only scalar and vector isoscalar mesons are exchanged among the nucleons. This is the basis of many relativistic nuclear structure calculations. We will compare the results in the present approach with those obtained in the mean field theory (MFT), i.e. the Hartree approximation in which antinucleon contributions to the self-energies are neglected.

In the MFT the vector field shifts the energy of the nucleon, while the scalar field dresses the mass. It is worth stressing that this property follows directly from the Lorentz structure of the self-energy: the scalar self-energy, which is a scalar under Lorentz transformations, transforms as the mass, while the vector self-energy transforms as the time component of a four-vector. An important related issue in a Hartree calculation is connected to self-consistency. Indeed, in a relativistic framework the spinors depend on the nucleon mass, which is modified in the medium and should then be calculated self-consistently. This is rather straightforward in the MFT, because of the above mentioned Lorentz structure of
the self-energy. In a non-relativistic calculation, based on an effective potential expanded at some order in \( \frac{k^2}{m^2} \), the dependence on the nucleon mass has been shifted to the potential: however, in this case one can no longer rely on general symmetry arguments and the FW expansion provides a systematic way to deal with self-consistency.

Let us start by writing the Dyson equation for the inverse Hartree Green’s function

\[
G_H(k, k_0)^{-1} = G_0(k, k_0)^{-1} - \Sigma_H,
\]

where \( G_0(k, k_0)^{-1} = \gamma^0 k_0 - \gamma \cdot k - m_N \) is the inverse of the free Green’s function, whereas \( \Sigma_H \) is the Hartree self-energy due to the exchange of \( \sigma \) and \( \omega \), i.e.,

\[
\Sigma_H \equiv \Sigma^s_H - \gamma_0 \Sigma^v_H
\]

\[
= i \frac{g_0^2}{m^2} \int \frac{d^4k}{(2\pi)^4} \text{Tr} [G_H(k, k_0)] e^{ik_0\eta} - i \frac{g_v^2}{m^2} \int \frac{d^4k}{(2\pi)^4} \gamma_0 \text{Tr} [\gamma^0 G_H(k, k_0)] e^{ik_0\eta}.
\]

The solution of (4.1) is clearly:

\[
G_H(k, k_0)^{-1} = \gamma^0 \mathfrak{T}_0 - \gamma \cdot k - m_N^*,
\]

with

\[
\mathfrak{T}_0 = k_0 + \Sigma^v_H
\]

\[
m_N^* = m_N + \Sigma^s_H
\]

being the nucleon energy shifted by the vector field and the nucleon mass dressed by the scalar field, respectively. These relations, together with Eq. (4.2), define the self-consistent Hartree solution in the MFT.

Applying the FW procedure, one can introduce in the same way a Dyson equation for the FW Hartree Green’s function:

\[
G_{HFW}^F(k, k_0)^{-1} = \mathcal{T}_{H}^F(k)\mathcal{T}(k) G_{HFW}^F(k, k_0)^{-1} \mathcal{T}(k)\mathcal{T}_{H}^\dagger(k) - \mathcal{T}_{H}^\dagger(k)\Sigma_H \mathcal{T}_{H}^\dagger(k),
\]

where \( G_{HFW}^F(k, k_0)^{-1} = \gamma^0 k_0 - E_k \) is the inverse of the free FW Green’s function, while \( \mathcal{T}_H(k) \) is the operator (3.1), evaluated for a nucleon with dressed mass \( m_N^* \). The solution to this equation can again be written as

\[
G_{HFW}^F(k, k_0)^{-1} = \gamma^0 \mathfrak{T}_0 - E_k^*,
\]

with \( E_k^* = \sqrt{k^2 + m_N^{*2}} \) and \( \mathfrak{T}_0 \) and \( m_N^* \) still given by Eq. (4.4), that is, at this stage one recovers the results of MFT.

Now, in order to make contact with the effective potentials of the previous Sections, one has to project away the small components and to express the self-energies in terms of the potentials. Starting again from Eq. (4.5), the left hand side becomes

\[
P_+ G_{HFW}^F(k, k_0)^{-1} P_+ = P_+(\mathfrak{T}_0 - E_k^*) \equiv P_+ G_{H(+)}(k, k_0)^{-1},
\]

whereas the last term on the right hand side can be rewritten as
\[ P_+ \mathcal{T}_H^1(k) \Sigma_H \mathcal{T}_H^1(k) P_+ = -i \int \frac{d^4p}{(2\pi)^4} P_+ \text{Tr} \left[ V_{H(+)}^\text{FW}(k, p) G_{H(+)}^\text{FW}(p, p_0) \right] e^{ip_0\eta} \]

\[ = P_+ \left( \frac{m_N^* \Sigma^s_H - \Sigma^v_H}{E_k} \right) \]

\[ \equiv P_+ \Sigma_{H(+)}^\text{FW} . \quad (4.8) \]

In the previous equation we have introduced for brevity \( V_{H(+)}^\text{FW}(k, p) \equiv V_{(+)}^\text{FW}(q = 0, (k - p)/2, k + p) \). Then, Eq. (4.5) becomes

\[ \bar{k}_0 - E_k^* = (k_0 + \Sigma^v_H) - \frac{k^2 + m_N^*(m_N + \Sigma^s_H)}{E_k} , \quad (4.9) \]

where we have gathered separately the terms shifting the energy and the terms dressing the mass. The self-consistency condition is still given by Eq. (4.4b), as one can easily check. However, we can now reformulate it in terms of the FW potentials, namely

\[ m_N^* = m_N - i \frac{E_k^*}{m_N} \int \frac{d^4p}{(2\pi)^4} \text{Tr} \left[ V_{H(+)}^\text{FW}(k, p)^* G_{H(+)}^\text{FW}(p, p_0) \right] e^{ip_0\eta} . \quad (4.10) \]

If we expand the potential up to \( O(1/m_N^4) \), then we obtain an approximate self-consistency equation for the effective mass: using the \( \sigma \) potential of Eq. (3.10a), that is, after expansion, the \( V_{s\text{BR}}^\text{BR} \) potential of Eq. (2.6a), we get

\[ m_N^* = m_N - \rho g_s^2 m_s^2 \left( 1 - \frac{3\ell_F^2}{10m_N^*} \right) , \quad (4.11) \]

where \( \rho = 2k_F^3/3\pi^2 \) is the density of nuclear matter.

The exact expression \( (4.4b) \) and the approximate one \( (4.11) \) may be compared with the standard non-relativistic formula for \( m_N^* \):

\[ m_N^* = \left( 1 + \frac{m_N}{m_N \partial \Sigma / \partial k} \right) . \quad (4.12) \]

Note that from this formula one gets a constant effective mass in the Hartree approximation only when the interaction is quadratic in the non-local part, whereas \( (4.11) \) yields a constant \( m_N^* \) at any order. Using the \( V_{s\text{BR}}^\text{BR} \) and the \( V_{s\text{MR}}^\text{MR} \) potentials of Sect. II one gets

\[ m_N^* = \left( 1 + \frac{\rho g_s^2 m_N}{2m_N m_s^2} \right) \left( \frac{g_s^2}{m_s^2} + \frac{g_v^2}{m_v^2} \right) , \quad (4.13a) \]

Breit reduced

\[ m_N^* = \left( 1 + \frac{\rho g_s^2 m_N}{2m_N m_s^2} \right) \left( \frac{g_s^2}{m_s^2} + \frac{g_v^2}{m_v^2} \right) , \quad (4.13b) \]

Minimal relativity.

It is apparent that while the \( V_{s\text{BR}}^\text{BR} \) potential complies with the relativistic symmetry requirements (i.e., only \( \sigma \) exchange dresses the mass), in the case of \( V_{s\text{MR}}^\text{MR} \) both \( \sigma \) and \( \omega \) exchange
are contributing. We display in Fig. 2 $m^*_N$ as a function of $k_F$ for two choices of the coupling constants: one set is taken from the MFT \[1\] (panels (a) and (c)), where they are fixed in order to reproduce the saturation point of nuclear matter, and the other from the Bonn potential \[9\] (panels (b) and (d)), which is fitted to the NN scattering data. The values of the parameters can be found in Table I. The largest difference is, of course, in the vector meson coupling, which is artificially reduced in the MFT in order to simulate the effect of short-range correlations.

In the case of the two upper panels ((a) and (b)), Eqs. (4.4b) and (4.11) have been solved dropping the requirement of self-consistency, i.e. utilizing the bare mass $m_N$ instead of $m^*_N$ in the right hand side of these equations. In the lower panels ((c) and (d)) the fully self-consistent solutions have been given. In the non-relativistic cases we have always applied

\[1\]Note that the Bonn potential contains also form factors, depending on the momentum transfer, which could be easily accommodated in the formalism. However, since in the Hartree self-energy ($q = 0$) they produce only a slight rescaling of the coupling constants, we shall neglect them in the following discussion, which has only illustrative purposes.
### Table I. Masses and coupling constants of the σ and ω mesons in the MFT [1] and in the Bonn potential (BP) [9].

|       | $m_\sigma$ (MeV) | $g_\sigma^2/4\pi$ | $m_\omega$ (MeV) | $g_\omega^2/4\pi$ |
|-------|------------------|-------------------|------------------|-------------------|
| MFT   | 550              | 7.29              | 783              | 10.84             |
| BP    | 550              | 7.7823            | 783              | 20                |

Eq. (4.13).

First of all, one should notice the remarkably good agreement between the exact expression (4.13) for $m_N^*$ and the approximate one (4.11) (in particular, in the non-self-consistent instance the two curves are practically indistinguishable). The non-relativistic approximation fails badly at the standard nuclear matter density and above. Note, however, that when the Fermi gas is applied to relatively light nuclei an effective lower value for $k_F$ should be employed. For instance, in Ref. [4] an analysis of the charge response has been performed, using the expanded Bonn potential, for $^{12}$C at $k_F = 225$ MeV/c. In that case, the non-relativistic $m_N^*$ differs from the exact one by less than 3% (Fig. 2d).

The dependence on the parameters is rather mild: while this is expected for the relativistic calculations and for $V^{BR}$, which depend only on the rather stable σ coupling constant (see Table I), it turns out that also the dependence on $g_\omega^2$ in $V^{MR}$ is not dramatic.

It is worth noticing that in spite of the fact that the effective mass stemming from $V^{MR}$ violates the relativistic symmetry requirements, the numerical values turn out to be rather close to the ones obtained from $V^{BR}$: the contribution of the σ meson is reduced to make room for the ω meson. We compare in Fig. 3 the effective masses (4.13), showing also the individual contribution of σ and ω in the case of $V^{MR}$.

Coming back to the approximate relativistic expression (4.11), it is clear from Figs. 2c and 2d that the expansion breaks above $k_F \approx 300$ MeV/c. While this is enough for most nuclear matter calculations, it is worth pointing out that the range of validity can be increased by considering higher order contributions. Note that for momentum space calculations, such as those of Ref. [4], the use of a potential expanded up to an arbitrary order does not pose any major problem.

### V. CONCLUSIONS

In this paper we have presented a systematic way of dealing with $1/m_N$ expansions in a relativistic many-body nuclear system. The formalism has been applied to the Hartree problem, showing the importance of accounting, at the same order in the expansion, also for the condition of self-consistency. Treating the expanded potentials strictly non-relativistically can lead to substantial discrepancies with the relativistic calculation, depending on the density. On the other hand, a correct treatment of the self-consistency condition considerably enlarges the range of densities where the relativistic potentials are applicable, already at order $1/m_N^2$.

A word of caution should be spent about the use of an empirical NN interaction, such as the Bonn potential: as we have seen, the minimal relativity prescription, which is used in deriving the potential, does not appear to be consistent with a relativistic dispersion relation...
FIG. 3. The effective mass as a function of the Fermi momentum for the non-relativistic models (4.13a) (solid) and (4.13b) (dot-dashed) using the Bonn potential set of parameters. The effective mass stemming from $V^{MR}$ when only the $\sigma$ (dotted) or the $\omega$ (dashed) mesons are employed is also shown.

for the kinetic energy and, furthermore, it generates contributions to the nucleon effective mass stemming from vector meson exchange. On the other hand, using the Bonn potential parameters with a “Breit reduced” potential would not be, strictly speaking, consistent. While this may not be a major problem in practice (at least in the case of $m_N^*$ the dependence on the choice of parameters is rather mild), a better option might turn out to be the one of fixing the parameters on some nuclear properties.

The results presented here indicate that at the mean field level it is possible to map, with high accuracy, the many-body relativistic field theoretical problem to an equivalent one, expressed within the standard hamiltonian formulation. The next step would be, of course, to verify whether such a correspondence is valid also for correlations beyond the mean field, like, e. g., those described by the random phase approximation (RPA). Calculations based on both non-relativistic [16] and relativistic [4] Hamiltonians have in fact shown the importance of the exchange terms in the RPA series, whereas fully relativistic calculations have always been performed keeping only the direct (ring) contributions.
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