Exact CutFEM Polynomial Integration

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Abstract

The implementation of discontinuous functions occurs in many of today’s state of the art partial differential equation solvers. In finite element methods this poses an inherent difficulty: there are no quadrature rules readily available, when integrating functions whose discontinuity falls in the interior of the element. Many approaches to this issue have been developed in recent years, among them is the equivalent polynomial technique. This method replaces the discontinuous function with a polynomial, potentially allowing for the integration to occur over the entire domain, rather than integrating over complex subdomains. Although, eliminating the issues involved with discontinuous function integration, the equivalent polynomial tactic introduces its own set of problems. In particular, either adaptivity is required to capture the discontinuity or error is introduced when regularization of the discontinuous function is implemented. In the current work we eliminate both of these issues. The results of this work provide exact algebraic expressions for subdomain and interface polynomial integration, where the interface represents the boundary of the cut domain. We also provide algorithms for the implementation of these expressions for standard finite element shapes in one, two, and three dimensions, along with a hypercube of arbitrary dimension.

Keywords: CutFEM integration, Equivalent Polynomial, Polylogarithm, Interface integration

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1. Introduction

Partial differential equation (PDE) solvers are ubiquitous among many engineering and applied mathematics practitioners. Today, there are many PDE solvers which employ a discontinuous function, especially in the context of fluid dynamics problems. These methods make use of discontinuous functions in order to distinguish different domains and to ensure no extrinsic contributions are incurred while utilizing an arbitrary discontinuity. A few particular extensions of the Finite Element Method (FEM) utilizing discontinuous functions are CutFEM or Extended FEM (XFEM), generalized FEM (GFEM), and nonlocal FEM. In XFEM and GFEM, an enrichment function, e.g. the Heavyside function, is employed to distinguish different domains defined by a common interface, alleviating cumbersome remeshing techniques [1, 2]. The nonlocal FEM implements a kernel function, defined by the step function, that ensures nonlocal contributions are
zero outside of some specified region \([3, 4, 5]\). Venturing outside of the FEM, an example of a method which also employs discontinuous functions is the Volume of Fluid (VOF) method. The VOF method makes use of the characteristic function to determine what portion of a cell is occupied by a fluid \([6, 7, 8]\). From the aforementioned methods, one can see the important role of discontinuous functions in many of today’s PDE solvers, all of which would benefit from an accurate and efficient way of dealing with the integration of a discontinuous function.

Discontinuous function integration can be cast as integration over several disjoint subdomains involving continuous functions, i.e. the region over which the integration is to occur can be broken up into multiple subdomains where only continuous functions are defined. However, the boundary defining the subdomains is rarely trivial and traditional integration schemes are not practical. Even invoking the divergence theorem in such cases has proven to be intractable for even simple geometries and discontinuities, as seen in \([9, 10]\), since integration over the subdomain boundaries must be performed. There has also been work devoted to moment fitting approaches, such as in \([9]\), which also rely on the divergence theorem. Another approach, which depends on the convexity of the region of integration, is presented in \([11]\). Although in general, it cannot be expected that the region of integration is convex.

The most common approach to discontinuous function integration is the use of an adaptive algorithm, i.e. an algorithm which uses a grid refinement technique in order to better capture the discontinuity and produce a more accurate approximation to the integral. Adaptive methods still require extensive information about the boundaries of the subdomains and typically lead to high computational costs, ultimately slowing down the numerical PDE scheme. There have been several recent developments that deal with the issue of discontinuous function integration, avoiding expensive adaptive methods. Among these are the use of equivalent polynomials \([12, 13, 14, 10]\), more specifically the use of a polynomial that replaces the discontinuous function in the integrand and yields an equivalent integral. Equivalent polynomial methods allow for integration of continuous functions over an entire region without the difficulty of discontinuous functions and, for line/plane discontinuities, the high computation cost of adaptive quadrature methods.

The equivalent polynomial method was first introduced in \([10]\), where equivalent polynomials were found analytically for simple geometries and discontinuities\([15, 16, 17]\). The ideas introduced in \([10]\) were limited to lower order elements, e.g. linear triangles and bi-linear quadrilaterals, as a consequence of using the divergence theorem to analytically calculate the coefficients of the equivalent polynomial. The difficulty behind this method is introduced when analytical integration is applied to a generic discontinuity, since integration must be carried out on two sub-domains. Therefore, information about the discontinuity must be implemented as the region in which the analytical integration takes place. This causes severe restrictions when the dimension increases resulting in impractical discontinuity considerations, even when the discontinuity is a hyperplane. This work was further extendend in \([14]\).

The work in \([14]\) builds on the core idea presented in \([10]\), i.e. the idea of replacing a discontinuous function with an equivalent polynomial. The limitation of the work done in \([10]\) is overcome by using a reg-
ularized Heaviside function, which approaches the Heaviside function in the limit, in place of the Heaviside function. This regularized Heaviside function is continuous and differentiable for any value of the regularization parameter $\rho$. The use of the regularized Heaviside function allows one to perform analytical integration over the entire domain and then take the limit of the resulting expression when deriving the equivalent polynomial coefficients. The extended work in [14] creates a more robust method by eliminating the need for analytical integration over arbitrary sub-domains created by the discontinuity. This method requires equality between the integral of the regularized Heaviside function multiplied by some monomial and the integral of the equivalent polynomial multiplied by the same monomial. The highest degree of the monomial and the dimension dictate the size of the linear sysem that needs to be solved in order to recover the coefficients of the equivalent polynomial. Since the equation for the discontinuity appears in the regularized Heaviside function, the equivalent polynomial coefficients will be dependent upon the discontinuity, as well as the regularization parameter. Large values of the regularization parameter can then be taken to approximate the Heaviside function. Automation of this method relies on numerical libraries in the calculation of the polylogarithm function which naturally arises from the integration of the regularized Heaviside function. As a consequence of using the regularized Heaviside function, one is left with expressions that involve a linear combination of polylogarithm functions of various orders. The two sources of error arising from the use of equivalent polynomials, as mentioned in [14], are the numerical evaluation of the polylogarithm and round-off error introduced by large values of the regularization parameter $\rho$.

In [12] the concept of equivalent polynomials was extended to incorporate Legendre polynomials, which give rise to very beneficial properties. The main idea is to represent the equivalent and element shape polynomials with Legendre polynomials. The properties of Legendre polynomials are then utilized to allow for analytical integration over specified squares in 2-D or cubes in 3-D. Hence the error incurred from this method is produced by a spacetree refinement algorithm for complex discontinuities. It is stated in [12] that the analytical integration results are the same as those in [14] for a line or plane discontinuity; however, the implementation of the equivalent Legendre polynomials for the specified discontinuities lacks the ease of algorithmic automation for the analytical integration.

The method we propose utilizes a closed form expression for the analytical equivalent polynomial integral for simple discontinuities. More specifically line, plane, and hyperplane discontinuities. We provide algebraic expressions for analytical subdomain and interface integrals. The proposed method further extends the method in [14] by considering the regularized Heaviside function as a special case of the polylogarithm function and utilizing asymptotic properties of the polylogarithm function. This leads to the elimination of the regularization parameter and numerical evaluation of the polylogarithm, which in turn eliminates both sources of error introduced by equivalent polynomials as mentioned in [14]. We also give an alternative expression for the closed forms which are not susceptible to round-off errors. By utilizing the asymptotic properties of the polylogarithm and the relationship the Heavyside function shares with the Dirac distribution, we provide closed form expressions for analytical interface integrals. Similar to the method given by [14],
the proposed method eliminates the need to integrate over a specified, often times intractable, subdomain \( \Omega_i \). We provide closed form expression for triangles and squares in dimension 2, and for the cube, prism, and tetrahedron in dimension 3. We also provide the pseudo-codes to evaluate integrals on hypercubes cut by hyperplanes for any dimension. Additional pseudo-code is given for the triangle, the tetrahedron and the prism. A great deal of attention has been spent in providing algorithms which avoid overflow with regard to computer arithmetic. All closed form expressions are algebraic and easy to implement, since the need for subdomain integration and the use of expensive numerical libraries has been eliminated.

The outline of this paper is as follows. In Section 2, we discuss the properties of the polylogarithm which are implemented in the paper. In Section 3, the closed form expressions for the different elements are given. Also, pseudo code for the n-dimensional cube can be found in this section. Lastly, in Section 4, we provide some useful notes on practical implementation of the equivalent polynomials. All the alghorithms developed in this article are implemented in FEMuS \[18\], an in-house open-source finite element C++ library built on top of PETSc \[19\] and publicly available on GitHub.

2. Preamble

The polylogarithm, \( \text{Li}_s(z) \) where \( s, z \in \mathbb{C} \) with \(|z|<1\), can be defined as

\[
\text{Li}_s(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^s},
\]

or in integral form as

\[
\text{Li}_s(z) = \frac{z}{\Gamma(s)} \int_0^\infty \frac{x^{s-1}}{e^x - z} \, dx,
\]

by analytic continuation, where \( \Gamma(s) \) is the gamma function. The integral representation of \( \text{Li}_s(z) \) is analytic for \( z \in \mathbb{C} \setminus [1, \infty) \) and \( \Re(s) > 0 \) \[20, 21\]. When the above integral is replaced with an appropriate complex contour integral we can consider \( s \in \mathbb{Z}^- \cup \{0\} \)\[21\]. For the purpose of this paper we will only consider polylogarithms of the form \( \text{Li}_s(w) \), where \( s \in \{-1, 0, 1, \ldots\} \) and \( w \in \mathbb{R} \). All identities in this paper are used when \( \text{Li}_s(w) \) is well defined.

Two useful properties used throughout this paper are \( \text{Li}_s(-e^w) = -F_{s-1}(w) \), where \( F_{s-1}(w) \) is the Complete Fermi-Dirac integral and \( \frac{dL_{s-1}(-e^\mu)}{d\mu} = L_{s-1}(-e^\mu) \) \[20, 22\]. Therefore results of this paper should be implemented with consideration of the aforementioned properties. The following results are a direct consequence of these properties, as can be seen in \[20, 22, 21\].

**Proposition 2.1.** For \( s = 0, 1, 2, \ldots \),

\[
\lim_{t \to \infty} \text{Li}_s(a) := \lim_{t \to \infty} \frac{\text{Li}_s(-\exp(at))}{t^s} = \begin{cases} 
-0.5 & \text{if } s = 0 \text{ and } a = 0 \\
\frac{a^s}{s!} & \text{if } a > 0 \\
0 & \text{otherwise}
\end{cases}.
\]
Proof. Case 1: \( s = a = 0 \). In this case we have

\[
\lim_{t \to \infty} \text{Li}_0(0) = \lim_{t \to \infty} \text{Li}_0(-1) = \frac{-1}{1 + 1} = -\frac{1}{2}.
\]

Case 2: By induction on \( s \), with \( a > 0 \).

For \( s = 1 \) we have

\[
\lim_{t \to \infty} \text{Li}_1(a) = \lim_{t \to \infty} \frac{\text{Li}_1(-\exp(at))}{t} = \lim_{t \to \infty} -\frac{\ln(1 + e^{at})}{t} = -a.
\]

For \( s = k - 1 \): assume

\[
\lim_{t \to \infty} \frac{\text{Li}_{k-1}(-e^{at})}{t^{k-1}} = -\frac{a^{k-1}}{(k-1)!}.
\]

Then for \( s = k \) we have

\[
\lim_{t \to \infty} \frac{\text{Li}_k(-e^{at})}{t^k} = \lim_{t \to \infty} a \frac{\text{Li}_{k-1}(-e^{at})}{t^{k-1}} = -\frac{a^k}{k!},
\]

where we have used the property \( \frac{d\text{Li}_s(-e^\mu)}{d\mu} = \text{Li}_{s-1}(-e^\mu) \).

Case 3: For any other case, i.e., \( a \leq 0, s \neq 0 \), the terms of the appropriate series expansion for the Fermi-Dirac integral vanish when the limit is taken inside the series, when the series converges uniformly[20].

\[
\square
\]

Proposition 2.2. Let \( a \neq 0, m = 0, 1, 2, \ldots \) and \( s = -1, 0, 1, \ldots \), then

\[
\int x^m \text{Li}_s(-\exp(ax)) \, dx = \sum_{i=1}^{m+1} \frac{m!}{(m+1-i)!} x^{m+1-i} \frac{\text{Li}_{s+i}(-\exp(ax))}{a^i} + C.
\]

Proof. In the domain of interest, where the polylogarithm function converges uniformly, we use the following identities

\[
\frac{d\text{Li}_s(-e^\mu)}{d\mu} = \text{Li}_{s-1}(-e^\mu),
\]

and

\[
\text{Li}_s(-e^\mu) = \int \text{Li}_{s-1}(-e^\mu) d\mu + C.
\]

For ease of notation we drop the constant \( C \) in the proof.

For \( m = 0 \), we get

\[
\int \text{Li}_s(-\exp((ax + d)t)) \, dx = \frac{\text{Li}_{s+1}(-\exp((ax + d)t))}{at} = \sum_{i=1}^{1} (-1)^{i+1} 0! \frac{\text{Li}_{s+i}(-\exp((ax + d)t))}{(0 - i + 1)!(ta)^i i^{i+1}}.
\]
For \( m = k \) assume

\[
\int x^k \text{Li}_{s+1}(-\exp((ax + d)t)) \, dx = \sum_{i=1}^{k+1} \frac{k! (-1)^{i+1}}{(k+1-i)!} x^{k+1-i} \text{Li}_{s+1+i}(-\exp((ax + d)t)) (at)^i.
\]

Then for \( m = k + 1 \) we have

\[
\int x^{k+1} \text{Li}_{s}(-\exp((ax + d)t)) \, dx
\]

\[
= x^{k+1} \int \text{Li}_{s}(-\exp((ax + d)t)) \, dx - (k+1) \int x^k \frac{\text{Li}_{s+1}(-\exp((ax + d)t))}{at} \, dx
\]

\[
= \frac{x^{k+1}}{at} \text{Li}_{s+1}(-\exp((ax + d)t)) - (k+1) \sum_{i=1}^{k+1} \frac{k! (-1)^{i+1}}{(k+1-i)!} x^{k+1-i} \frac{\text{Li}_{s+i+1}(-\exp((ax + d)t))}{(at)^{i+1}}
\]

\[
= \sum_{i=1}^{k+2} \frac{(k+1)! (-1)^{i+1}}{(k+2-i)!} x^{k+2-i} \text{Li}_{s+i+1}(-\exp((ax + d)t)) (at)^{i+1}.
\]
Li₀ and from the dominated convergence theorem [23], i.e.,

\[
\int_D P_m(x) \mathbb{U}(n \cdot x + d) dx
\]

\[
= \int_D P_m(x) \left( - \lim_{t \to \infty} \text{Li}_0(-\exp((n \cdot x + d)t)) \right) dx
\]

\[
= - \lim_{t \to \infty} \int_D P_m(x) \text{Li}_0(-\exp((n \cdot x + d)t)) dx.
\]

Eq. (2) follows from Eq. (1) and the following integral equality

\[
\int_{D_1} P_m(x) dx = \int_D P_m(x) (1 - \mathbb{U}(-(n \cdot x + d))) dx.
\]

Equality (3) follows from the integral equality

\[
\int_{\Gamma} P_m(x) d\mu = \int_D P_m(x) \delta(n \cdot x + d) dx,
\]

and from the weak convergence of \( t \text{Li}_{-1}(-\exp((n \cdot x + d)t)) \) to the the Dirac distribution \( \delta \) [24, 25], i.e.,

\[
\int_D P_m(x) \delta(n \cdot x + d) dx
\]

\[
= - \lim_{t \to \infty} \int_D P_m(x) t \text{Li}_{-1}(-\exp((n \cdot x + d)t)) dx.
\]

Eq. (4) follows from Eq. (3) and the following integral equality

\[
\int_D P_m(x) \delta(n \cdot x + d) dx = \int_D P_m(x) \delta(-(n \cdot x + d)) dx.
\]

Remark 2.1. In proving (3), we assumed that the interface \( \Gamma \) is tangential to the boundary of \( D \) only on a set of measure zero. Such distinction is needed, since in this situation the Dirac function, centered on \( \partial D \) and aligned with the normal direction, would be only half contained within \( D \), thus contributing only for half to the interface integral. In all the applications we are going to consider next, \( D \) will only be a convex domain with piece-wise flat boundaries. In doing so, \( \Gamma \) is either completely tangential or never tangential to \( \partial D \). This allows us to compute the interface integral also in the tangential case (the boundary integral) by doubling the value of the computed integral in (3).

Corollary 2.1. For \( a \neq 0 \) the following definite integral is given by

\[
I_1 = - \lim_{t \to \infty} \frac{1}{L_t} \int_{-L_t}^{L_t} x^m \text{Li}_s(-\exp((ax + d)t)) \, dx
\]

\[
= \sum_{i=1}^{m+1} \frac{m!}{(m + 1 - i)!} \frac{1}{(-a)^i} \lim_{s \to i} \text{Li}_{s+i}(a + d) - \frac{m!}{(-a)^{m+1}} \lim_{s \to m+1} \text{Li}_{s+m+1}(d).
\]
Proof. The proof follows directly from combining Propositions 2.1 and 2.2.

Definition 2.1. For \( a \neq 0 \), let

\[
I_2 = \sum_{i=0}^{s} \frac{(-a)^{s-i}(a+d)^i}{i!(m+1+s-i)!}.
\]  (6)

Remark 2.2. In the next proposition we will show that for \( s \geq 0 \) and positive arguments Eq. (5) is equivalent to Eq. (6). In computer arithmetic Eq. (5) suffers from overflow for \( d \gg |a| > 0 \) and \( \alpha \to 0 \), because of the presence of the \( 1/\alpha \) terms in the sums. Proposition 2.4 will show all these terms actually simplify after expanding the definition of \( \text{limLi} \) for positive arguments.

Remark 2.3. In Eq. (5), for \( d \geq 0 \) and/or \( |d| \sim |a| \), and \( \alpha \to 0 \), either the arguments of the polylogarithm functions are non positive, or if positive they are of the same order of \( \alpha \). In the first case the contribution of their limits is zero. In the second case using the definition of \( \text{limLi} \) with positive argument one would get

\[
\left| \text{limLi}_{s+i}(|\sim a|) \right| \sim \left| (\sim a)^{s+i} \right| \sim |a|^s
\]

for all \( i \), thus all terms in the sum would have comparable size, and since \( a \) does not appear in the denominator it no longer contributes to an overflow issue, for small \( a \).

Proposition 2.4.

For \( s \geq 0 \), \( a \neq 0 \), \( d > 0 \) and \( a+d > 0 \), Eq. (5) is equivalent to Eq. (6).

Proof. First note that the conditions \( a \neq 0 \), \( d > 0 \) and \( a+d > 0 \) are equivalent to \( \frac{-d}{a} \notin [0, 1] \).

In proving the proposition one simply needs to apply integration by parts and utilize Proposition 2.1. For a fixed \( s \geq 0 \) and \( a \neq 0 \) we have

\[
-\lim_{t \to \infty} \frac{1}{t^s} \int_0^1 x^m \text{Li}_s(-\exp((ax+d)t)) dx
= -\lim_{t \to \infty} \frac{1}{t^s} \left( \frac{x^{m+1}}{m+1} \text{Li}_s(-\exp((ax+d)t) \bigg|_0^1 - \frac{at}{m+1} \int_0^1 x^{m+1} \text{Li}_{s-1}(-\exp((ax+d)t)) dx \right)
= \frac{(a+d)^s}{s!(m+1)} - \frac{a}{m+1} \left( -\lim_{t \to \infty} \frac{1}{t^{s-1}} \int_0^1 x^{m+1} \text{Li}_{s-1}(-\exp((ax+d)t)) dx \right)
= \frac{(a+d)^s m!}{s!(m+1)!} - \frac{a(a+d)^{s-1}m!}{s!(m+1)!} - \frac{a^2}{(m+1)(m+2)} \left( -\lim_{t \to \infty} \frac{1}{t^{s-2}} \int_0^1 x^{m+2} \text{Li}_{s-2}(-\exp((ax+d)t)) dx \right)
= \ldots
= \sum_{i=0}^{s} \frac{m! (-a)^{s-i}(a+d)^i}{i!(m+1+s-i)!} - \frac{(-a)^{s+1}m!}{(m+2+s)} \left( -\lim_{t \to \infty} \int_0^1 x^{m+2+s} t \text{Li}_{-1}(-\exp((ax+d)t)) dx \right)
= \sum_{i=0}^{s} \frac{m! (-a)^{s-i}(a+d)^i}{i!(m+1+s-i)!}.
\]
Where we have used the weak convergence of $t \text{Li}_{-1}\left(\exp((-ax + d)t)\right)$ to the Dirac distribution $\delta$ with $-\frac{d}{a} \not\in [0, 1]$.

3. LSI: Line Segment Integral on $[0, 1]$, with $a \neq 0$

For fix $s = -1, 0, 1, \ldots$, we want to evaluate integrals in the form

$$\text{LSI}_s^m(a, d) = -\lim_{t \to \infty} \frac{1}{t^s} \int_0^1 x^m \text{Li}_s(-\exp((ax + d)t)) \, dx.$$  

From Eq. (5)

$$\text{LSI}_s^m(a, d) := \sum_{i=1}^{m+1} \frac{m!}{(m+1-i)!} \frac{1}{(-a)^i} \text{limLi}_{s+i}(a + d) - \frac{m!}{(-a)^{m+1}} \text{limLi}_{s+m+1}(d). \quad (7)$$

For all $a, d \in \mathbb{R}$, with $|a| > 0$, we have the subdomain integral

$$\text{LSI}_0^m(a, d) = \int_0^1 x^m U(ax + d) \, dx, \quad (8)$$

and, for $a^2 = 1$, the interface integral

$$\text{LSI}_{-1}^m(a, d) = \int_0^1 x^m \delta(ax + d) \, dx. \quad (9)$$

For all $|a| > 0$ we also have the explicit point evaluation formula

$$\text{LSI}_{-1}^m(a, d) = \begin{cases} \frac{1}{|a|} \left(-\frac{d}{a}\right)^m & \text{if } 0 < -\frac{d}{a} < 1 \\ \frac{1}{2|a|} \left(-\frac{d}{a}\right)^m & \text{if } -\frac{d}{a} = 0 \text{ or } -\frac{d}{a} = 1, \\ 0 & \text{elsewhere} \end{cases} \quad (10)$$

with the assumption that $0^0 = 1$. That is the case for $m = 0$ and $d = 0$. Although equivalent to Eq. (7), for $s = -1$, Eq. (10) is generally faster to compute and does not suffer from overflow in computer arithmetic. Also, for $s \geq 0$, $d > 0$, and $a + d > 0$ we replace Eq. (7) with the equivalent Eq. (6) to avoid overflow. The pseudo-code for the line segment integration is given in Algorithm 1.

Remark 3.1. The formula for $\text{LSI}_{-1}^m(a, d)$ halves the value of the interface integral if the point $-d/a$ is one of the two boundary points. This happens because half of the Dirac distribution falls outside the line segment, thus it does not contribute to the integral value. If this is not the desired behavior, and the boundary integral
should account for the whole value, the definition of $\text{LSI}^{-1}_{m-1}(a, d)$ should be replaced by

$$
\text{LSI}^{-1}_{m-1}(a, d) = \begin{cases} 
\frac{1}{|a|} \left( -\frac{d}{a} \right)^m & \text{if } 0 \leq -\frac{d}{a} \leq 1 \\
0 & \text{elsewhere}
\end{cases},
$$

again with the assumption that $0^0 = 1$.

**Algorithm 1** Pseudo-code for integration on the line segment $[0, 1]$ with $a \neq 0$ and $s = -1, 0, 1, \ldots$. For $s = -1$ and $|a| = 1$ it corresponds to the interface integral. For $s = 0$ it corresponds to the subdomain integral.

```plaintext
1: function LINE_SEGMENT_INTEGRATION(a, d, m, s)
2: if $s = -1$ then
3: return $\text{LSI}^{-1}_{m-1}(a, d)$ from Eq. (10)
4: else
5: if $d \leq 0$ or $a + d \leq 0$ then
6: return $\sum_{i=1}^{m+1} \frac{m!}{(m + 1 - i)! (-a)^i} \lim_{t \to \infty} \text{Li}_{s+i}(a + d) - \frac{m!}{(-a)^{m+1}} \lim_{t \to \infty} \text{Li}_{s+m+1}(d)$
7: else
8: return $\sum_{i=0}^{s} \frac{m!}{i! (m + 1 + s - i)!} (-a)^{s-i} (a + d)^i$
9: end if
10: end if
11: end function
```

### 3.1. SQI: Square Integral $[0, 1]^2$, with $a^2 + b^2 > 0$

Fix $s = -1, 0, 1, \ldots$, we want to evaluate integrals in the form

$$
\text{SQI}^{mn}_s(a, b, d) = -\lim_{t \to \infty} \frac{1}{t^s} \int_0^1 \int_0^1 x^m y^n \text{Li}_s(-\exp((ax + by + d)t)dy\,dx).
$$

We will first consider the case when the interface $\Gamma$ is parallel to either the square sides, and then all the remaining cases.

If $a = 0$ the iterated integral can be split in the product of 2 integrals

$$
\text{SQI}^{mn}_s(0, b, d) = \int_0^1 x^m dx \left( -\lim_{t \to \infty} \frac{1}{t^s} \int_0^1 y^n \text{Li}_s(-\exp((by + d)t)dy \right) = \frac{1}{m+1} \text{LSI}^n_s(b, d).
$$

Similarly, if $b = 0$

$$
\text{SQI}^{mn}_s(a, 0, d) = \frac{1}{n+1} \text{LSI}^m_s(a, d).
$$

If both $a$ and $b$ are different from zero, after the integration of the inner integral we get

$$
\text{SQI}^{mn}_s(a, b, d) = -\lim_{t \to \infty} \int_0^1 x^m \left( -\sum_{j=1}^{n+1} \frac{n!}{(n + 1 - j)! (-b)^j} \frac{\text{Li}_{s+j}(-\exp((ax + b + d)t)}{t^{s+j}} \right).
$$
\[ + \frac{n!}{(-b)^{n+1}} \frac{\text{Li}_{s+n+1}(- \exp((ax + d)t))}{t^{s+n+1}} \, dx \]

\[ = -\sum_{j=1}^{n+1} \frac{n!}{(n+1-j)!} \frac{1}{(-b)^j} \text{LSI}_{s,j}^n(a, b + d) + \frac{n!}{(-b)^{n+1}} \text{LSI}_{s+n+1}^n(a, d), \]

(12)

Then, for all \( a, b, d \in \mathbb{R} \), such that \( a^2 + b^2 > 0 \), we have the subdomain integral

\[ \text{SQI}_{0}^{mn}(a, b, d) = \int_{0}^{1} \int_{0}^{1} x^m y^n U(ax + by + d) \, dy \, dx, \]

and, for \( a^2 + b^2 = 1 \), the interface integral

\[ \text{SQI}_{1}^{mn}(a, b, d) = \int_{0}^{1} \int_{0}^{1} x^m y^n \delta(ax + by + d) \, dy \, dx. \]

These formulas are general and versatile: they work regardless of where the line \( ax + by + d = 0 \) intersects the square domain, and the orientation of the unit step function follows the orientation of the normal \( \langle a, b \rangle \).

Remark 3.2. In the special cases \( \text{SQI}_{-1}^{mn}(a, 0, d) \) (or \( \text{SQI}_{-1}^{mn}(0, b, d) \)), with \(-d/a = 0 \) or \( 1 \) (or \(-b/d = 0 \) or \( 1 \)), the corresponding line \( ax + d = 0 \) (or \( by + d = 0 \)) overlaps with one of the sides of the square. Depending on which definition is used for \( \text{LSI}_{-1}^m \), either Eq. (10) or Eq. (11), one is left with half the boundary integral or the entire boundary integral, respectively, over the specified side of the square. This is also the case for the cube and the hypercube we are going to consider next.

3.2. CBI: Cube integral on \([0, 1]^3\), with \( a^2 + b^2 + c^2 > 0 \).

Fix \( s = -1, 0, 1, \ldots \), we want to evaluate integrals in the form

\[ \text{CBI}_{s}^{mn\alpha}(a, b, c, d) = -\lim_{t \to \infty} \frac{1}{t^s} \int_{0}^{1} \int_{0}^{1} \int_{0}^{1} x^m y^n z^\alpha \, \text{Li}_s(- \exp((ax + by + cz + d)t)) \, dz \, dy \, dx. \]

For \( c = 0 \) the above integral reduces to the square case, i.e.

\[ \text{CBI}_{s}^{mn\alpha}(a, b, 0, d) = \frac{1}{\alpha} \text{SQI}_{s}^{mn}(a, b, d). \]

If \( c \neq 0 \) and both \( a = 0 \) and \( b = 0 \)

\[ \text{CBI}_{s}^{mn\alpha}(0, 0, c, d) = \frac{1}{m+1} \frac{1}{n+1} \text{LSI}_{\alpha}^o(c, d). \]

(13)

For all other cases, after integrating in \( z \) we get

\[ \text{CBI}_{s}^{mn\alpha}(a, b, c, d) = -\lim_{t \to \infty} \int_{0}^{1} \int_{0}^{1} x^m y^n \left( -\sum_{k=1}^{o+1} \frac{o!}{(o+1-k)!} \frac{1}{(-c)^k} \frac{\text{Li}_{s+k}(- \exp((ax + by + c + d)t))}{t^{s+k}} \right) \, dy \, dx \]

\[ + \frac{o!}{(-c)^{o+1}} \frac{\text{Li}_{s+o+1}(- \exp((ax + by + d)t))}{t^{s+k}} \, dy \, dx. \]
\[ = -\sum_{k=1}^{o+1} \frac{o!}{(o+1-k)!} \frac{1}{(-c)^k} \text{SQI}_{s+k}^{m+n}(a, b, c + d) + \frac{o!}{(-c)^{o+1}} \text{SQI}_{s+o+1}^{m+n}(a, b, d). \]  

(14)

The cases \( a = 0 \) or \( b = 0 \) are handled by the square integrals as described in the previous section.

Then, for all \( a, b, c, d \in \mathbb{R} \), with \( a^2 + b^2 + c^2 > 0 \), we have the subdomain integral

\[ \text{CBI}_{s}^{m+n}(a, b, c, d) = \int_0^1 \int_0^1 \int_0^1 x^my^nz^\alpha \ U(ax + by + cz + d) \ dz \ dy \ dx, \]

and, for \( a^2 + b^2 + c^2 = 1 \), the interface integral

\[ \text{CBI}_{s-1}^{m+n}(a, b, c, d) = \int_0^1 \int_0^1 \int_0^1 x^my^nz^\alpha \ \delta(ax + by + cz + d) \ dz \ dy \ dx. \]

It is remarkable how such simple formulas can handle all possible intersections between the cube and the plane. Moreover, they can be easily extended to evaluate corresponding integrals on hypercubes cut by hyperplanes for any dimension.

3.3. HCI: Hypercube Integration on \([0,1]^{\dim}\), with \( \mathbf{n} = \langle a_1, a_2, \ldots, a_{\dim} \rangle, \| \mathbf{n} \| > 0 \) and \( \mathbf{m} = \langle m_1, m_2, \ldots, m_{\dim} \rangle \).

We are seeking integrals in the form

\[ \text{HCI}_{s}^{m}(\mathbf{n}, d) = -\lim_{t \to \infty} \frac{1}{t^s} \int_{[0,1]^{\dim}} \prod_{i=1}^{\dim} x_i^{m_i} \ \text{Li}_s(-\exp((\mathbf{n} \cdot \mathbf{x} + d)t)dx, \]

where we assume \( |a_i| \leq |a_{i+1}| \). However, if this is not the case, one can perform a reordering of the normal coefficients due to the symmetry of the domain and the integrand. Define \( \dim_0 \in \mathbb{N}_0 \) with \( \dim_0 \leq \dim \) such that \( \dim_0 \) is an upper bound for the indices corresponding to all the \( a_i = 0 \ \forall i < \dim_0 \). Define \( \dim' := \dim - \dim_0, \ m' := \langle m_{\dim_0+1}, \ldots, m_{\dim} \rangle \) and \( \mathbf{n}' := \langle a_{\dim_0+1}, \ldots, a_{\dim} \rangle \). Then

\[ \text{HCI}_{s}^{m}(\mathbf{n}, d) = \prod_{i=1}^{\dim_0} \frac{1}{1 + m_i} \text{HCI}_{s}^{m'}(\mathbf{n}', d). \]

Then, dropping the \( ' \) superscript, the problem reduces to evaluating integrals in the form

\[ \text{HCI}_{s}^{m}(\mathbf{n}, d) = -\lim_{t \to \infty} \frac{1}{t^s} \int_{\text{HC}_{\dim}} \prod_{i=1}^{\dim} x_i^{m_i} \ \text{Li}_s(-\exp((\mathbf{n} \cdot \mathbf{x} + d)t)dx, \]

with \( |a_i| \leq |a_{i+1}| \) and \( a_1 \neq 0 \). Following the same integration strategy used for the square and the cube, with \( m = m_{\dim} \) and \( a = a_{\dim} \), we obtain the following recursive formula

\[ \text{HCI}_{B}^{m}(\mathbf{n}, d) = -\sum_{i=1}^{m+1} \frac{m!}{(m+1-i)!} \frac{1}{(-a)^i} \text{HCI}_{A}^{m}(\mathbf{n}^{-}, a + d) \]

\[ + \frac{m!}{(-a)^{m+1}} \text{HCI}_{A}^{m}(\mathbf{n}^{-}, d), \]

(15)
where \( \mathbf{m}^- = (m_1, \ldots, m_{\text{dim}-1}) \) and \( \mathbf{n}^- = (a_1, \ldots, a_{\text{dim}-1}) \). This formula is recursively applied until dimension 1, where the line segment integration formula, LSI, is used. At each level of integration two contributions occur, one that involves a sum and a single term. The most expensive terms to compute are the ones involving a summation, with each one of them requiring the computation of

\[
\lim \text{Li}_{s+k} \left( \sum_{i=1}^{\text{dim}} a_i + d \right),
\]

for some \( k \geq \text{dim} \). It is then desirable to have

\[
\sum_{i=1}^{\text{dim}} a_i + d < 0,
\]

so that all the \( \lim \text{Li} \) contributions vanish. From Proposition 2.3, changing the sign of the normal without any contribution is only allowed for \( s = -1 \), hence

\[
\text{HCI}_{\text{B}}^{m}_{-1,\text{dim}}(\mathbf{n}, d) = \text{HCI}_{\text{B}}^{m}_{-1,\text{dim}}(-\mathbf{n}, -d).
\]

Similarly to Remarks 2.2 and 2.3, the \( \text{HCI}_{\text{B}} \) formula also suffers from overflow in computer arithmetic when

\[
\sum_{i=1}^{\text{dim}} a_i + d >> |a_{\text{dim}}|.
\]

To overcome these difficulties we introduce the alternative formula

\[
\text{HCI}_{\text{C}}^{m}_{s,\text{dim}}(\mathbf{n}, d) = \sum_{i=0}^{s} \frac{m!}{(m + 1 + i)!} (-a)^i \text{HCI}_{\text{A}}^{m^-}_{s-i,\text{dim}-1}(\mathbf{n}^-, a + d)
\]

\[
+ \frac{m!}{(m + s + 1)!} (-a)^{s+1} \text{HCI}_{\text{A}}^{m^*}_{-1,\text{dim}}(\mathbf{n}, d),
\]

(16)

where \( m^* = (m_1, m_2, \ldots, m_{\text{dim}-1}, m + s + 1) \). This formula is obtained by utilizing the identity \( \frac{d\text{Li}_{s+1}(-e^p)}{dp} = \text{Li}_{s+1}(-e^p) \) and by recursive use of integration by parts, increasing the monomial power and reducing the polylogarithm order \( s \) until it reaches \(-1\). More specifically, \( \text{HCI}_{\text{C}} \) follows the idea in Proposition 2.4, where an equivalent closed form expression is given in which \( a \) does not appear in the denominator. Note that in \( \text{HCI}_{\text{C}} \)

\[
\text{HCI}_{\text{A}}^{m^*}_{-1,\text{dim}}(\mathbf{n}, d) = \text{HCI}_{\text{A}}^{m^*}_{-1,\text{dim}}(-\mathbf{n}, -d),
\]

which permits choosing the optimal sign for the normal \( \mathbf{n} \).

The pseudo-code for general dimension \( \text{dim} \geq 1 \) is given in Algorithms 2 and 3. In Algorithm 2, the contributions of each component with a zero coefficient \( a_i \) are handled first. Algorithm 3 is then called to compute the contributions from all the remaining components. The recursive nature of the algorithm follows
from the patterns developed in the HCI_A, HCI_B, and HCI_C formulas.

Note that Algorithm 2 also handles the case $n = 0$. Although this case was excluded here, it will be needed later when integrating on the prism.

Algorithm 2

Pseudo-code for the integration on the hypercube $[0, 1]^d$ cut by the hyperplane $n \cdot x + d = 0$ with $n = (a_1, a_2, \ldots, a_{\dim})$, $m = (m_1, m_2, \ldots, m_{\dim})$ and $s = -1, 0, 1, \ldots$. For $s = -1$ and $\|n\| = 1$ it corresponds to the interface integral. For $s = 0$ it corresponds to the subdomain integral.

1: function HYPERCUBE_INTEGRATION($dim$, $n$, $d$, $m$, $s$)
2: HCI = 1
3: for $i = 1, \ldots, dim$ do
4: if $a_i = 0$ then
5: \[ HCI = \frac{1}{m_i + 1} \]
6: Remove the $i$-th component of $n$ and $m$
7: $dim = dim - 1$
8: $i = i - 1$
9: end if
10: end for
11: if $dim > 0$ then
12: Sort $n$, and accordingly $m$, from the smallest to the largest coefficient in magnitude
13: return $HCI \ast HYPERCUBE_INTEGRATION_A(dim, n, d, m, s)$
14: else
15: return $-HCI \ast \lim_{t \rightarrow \infty} Li_s(d)$
16: end if
17: end function

4.

4.1. TRI: Triangle Integration on \[
\begin{cases}
0 \leq x \leq 1 \\ 0 \leq y \leq 1 - x
\end{cases}
\]

with $a^2 + b^2 > 0$

To ease the computation we choose a non-standard polynomial bases, namely $(1 - x)^m y^n$. We then seek integrals in the form

\[
\text{TRI}^{m,n}_{-1}(a, b, d) = -\lim_{t \rightarrow \infty} \frac{1}{t^s} \int_{\text{TRI}} (1 - x)^m y^n \exp((ax + by + d)t) dA. \tag{17}
\]

Then, for all $a, b, d \in \mathbb{R}$ such that $a^2 + b^2 > 0$, the subdomain integral is given by

\[
\text{TRI}^{m,n}_0(a, b, d) = \int_{\text{TRI}} (1 - x)^m y^n U(ax + by + d) dA,
\]

and, for $a^2 + b^2 = 1$, the interface integral is given by

\[
\text{TRI}^{m,n}_1(a, b, d) = \int_{\text{TRI}} (1 - x)^m y^n \delta(ax + by + d) dA.
\]
Algorithm 3 Pseudo-code for the integration on the hypercube $[0,1]^d$ cut by the hyperplane $n \cdot x + d = 0$ with $n = (a_1,a_2,\ldots,a_{\text{dim}})$, $a_1 \neq 0$ and $|a_i| \leq |a_{i+1}|$ for all $i = 1,\ldots,dim - 1$, $m = (m_1,m_2,\ldots,m_{\text{dim}})$, and $s = -1,0,1,\ldots$. For $s = -1$ and $|n| = 1$ it corresponds to the interface integral. For $s = 0$ it corresponds to the subdomain integral.

1: function HypercubeIntegrationA($dim$, $n$, $d$, $m$, $s$)
2:   if $dim = 1$ then
3:     return LineSegmentIntegration($a_1 d$, $m_1$, $s$)
4:   end if
5:   sum = $\sum_{i=1}^{dim} a_i + d$
6:   if $s = -1$ then
7:     if $sum \leq 0$ then
8:       return HypercubeIntegrationB($dim$, $n$, $d$, $m$, $-1$)
9:     else
10:    return HypercubeIntegrationB($dim$, $-n$, $-d$, $m$, $-1$)
11:  end if
12: else
13:    if $sum \leq |a_{\text{dim}}|$ then
14:      return HypercubeIntegrationB($dim$, $n$, $d$, $m$, $s$)
15:    else
16:      return HypercubeIntegrationC($dim$, $n$, $d$, $m$, $s$)
17:    end if
18: end if
19: end function

1: function HypercubeIntegrationB($dim$, $n$, $d$, $m$, $s$)
2:   $m = m_{\text{dim}}$; $a = a_{\text{dim}}$
3:   Remove the last component of $n$ and $m$
4:   return $-\sum_{i=1}^{m+1} \frac{m!}{(m+1-i)!} \frac{1}{(-a)^i} \text{HypercubeIntegrationA}(dim-1, n, a+d, m, s+i)$
5:     $+ \frac{m!}{(-a)^{m+1}} \text{HypercubeIntegrationA}(dim-1, n, d, m, s+m+1)$
6: end function

1: function HypercubeIntegrationC($dim$, $n$, $d$, $m$, $s$)
2:   $m = m_{\text{dim}}$; $a = a_{\text{dim}}$; $m_{\text{dim}} = m_{\text{dim}} + s + 1$
3:   HCI = $\frac{m!}{(m+s+1)!} (-a)^{s+1} \text{HypercubeIntegrationA}(dim, n, d, m, -1)$
4:   Remove the last component of $n$ and $m$
5:   HCI += $\sum_{i=0}^{s} \frac{m!}{(m+i+1)!} (-a)^i \text{HypercubeIntegrationA}(dim-1, n, a+d, m, s-i)$
6:   return HCI
7: end function
In Eq. (17), changing variables and renaming constants as follows

\[ x' = 1 - x, \quad y = y, \quad a' = -a, \quad b' = b, \quad d' = d + a, \]

yields

\[
- \lim_{t \to \infty} \frac{1}{t^s} \int_0^1 \int_0^{x'} x'^m y^n \text{Li}_s(- \exp((a'x' + b'y' + d')t)) dy' dx'.
\]

Dropping the ‘ superscript, for a fixed \( s = -1, 0, 1, \ldots, \) the problem reduces to evaluating integrals in the form

\[
\text{TRI}_A^{mn}(a, b, d) = - \lim_{t \to \infty} \frac{1}{t^s} \int_0^1 \int_0^x x'^m y^n \text{Li}_s(- \exp((ax + by + d)t)) dy dx.
\]

(18)

First, we will consider the three separate cases where the interface \( \Gamma \) is parallel to one of the triangle edges.

For \( b = 0 \)

\[
\text{TRI}_B^{mn}(a, 0, d) = - \lim_{t \to \infty} \frac{1}{t^s} \int_0^1 \int_0^x x'^m y^n \text{Li}_s(- \exp((ax + d)t)) dy dx
\]

\[
= - \lim_{t \to \infty} \frac{1}{t^s} \int_0^1 \frac{1}{n + 1} \text{Li}_s(- \exp((ax + d)t)) dx
\]

\[
= \frac{\text{LSI}_s^{mn+1}(a, d)}{n + 1}.
\]

(19)

For \( a = 0 \)

\[
\text{TRI}_B^{mn}(0, b, d) = - \lim_{t \to \infty} \frac{1}{t^s} \frac{1}{m + 1} \int_0^1 (y^n - y^{n+m+1}) \text{Li}_s(- \exp((by + d)t)) dy
\]

\[
= \frac{\text{LSI}_s^m(b, d) - \text{LSI}_s^{mn+1}(b, d)}{m + 1}.
\]

(20)

For \( a + b = 0 \)

\[
\text{TRI}_B^{mn}(a, -a, d) = - \lim_{t \to \infty} \frac{1}{t^s} \int_0^1 \int_0^x x'^m y^n \text{Li}_s(- \exp((ax - ay + d)t)) dy dx
\]

\[
= - \lim_{t \to \infty} \frac{1}{t^s} \int_0^1 n! \left( \sum_{i=1}^{n+1} \frac{(-1)^{i-1}}{(n + 1 - i)!} x^{m+n+1-i} \text{Li}_{s+i}(- \exp(\frac{(ax - ay + d)t}{-at})) \right) dx
\]

\[
= n! \left( \sum_{i=1}^{n+1} \frac{1}{a} \frac{\text{Li}_{s+i}(d)}{(n + 1 - i)!} \int_0^1 x^{m+n+1-i} dx + \left( \frac{1}{a} \right)^{n+1} \text{LSI}_{s+n+1}^m(a, d) \right)
\]

\[
= n! \left( \sum_{i=1}^{n+1} \frac{1}{a} \frac{\text{Li}_{s+i}(d)}{(n + 1 - i)! (m + n + 2 - i)} + \left( \frac{1}{a} \right)^{n+1} \text{LSI}_{s+n+1}^m(a, d) \right).
\]

(21)
Next, we consider the remaining cases where the interface \( \Gamma \) is not parallel to one of the triangle edges. For \( a \neq 0 \), \( b \neq 0 \) and \( a + b \neq 0 \)

\[
\text{TRI}_{s}^{m,n}(a, b, d) = -\lim_{t \to \infty} \frac{1}{t^s} \int_{0}^{1} \int_{0}^{x} x^m y^n \text{Li}_s(-\exp((ax + by + d)t)) \, dx \, dy
\]

\[
= -\lim_{t \to \infty} \frac{1}{t^s} \int_{0}^{1} n! \left( \sum_{j=1}^{n+1} \frac{(-1)^{j-1}}{(n+1-j)!} x^{m+n+1-j} \frac{\text{Li}_{s+j}(-\exp((a+b)x + d)t)}{(bt)^j} \right) dx
\]

\[
= -\sum_{j=1}^{n+1} \frac{n!}{(-b)^j(n+1-j)!} \text{LSI}_{s+j}^{m+n+1-j}(a + b, d) + \frac{n!}{(b)^{n+1}} \text{LSI}_{s+n+1}^{m}(a, d). \tag{22}
\]

Alternatively, the same integral could be evaluated by reversing the order of integration. Specifically,

\[
\text{TRI}_{s}^{m,n}(a, b, d) = -\lim_{t \to \infty} \frac{1}{t^s} \int_{0}^{1} \int_{y}^{1} x^m y^n \text{Li}_s(-\exp((ax + by + d)t)) \, dx \, dy
\]

\[
= -\lim_{t \to \infty} \frac{1}{t^s} \int_{0}^{1} m! \left( \sum_{j=1}^{m+1} \frac{(-1)^{j-1}}{(m+1-j)!} y^n \frac{\text{Li}_{s+j}(-\exp(by + a + d)t)}{(at)^j} \right)
\]

\[
= \sum_{j=1}^{m+1} \frac{m!}{(m+1-j)!(-a)^j} \left( \text{LSI}_{s+j}^{m+n+1-j}(a + b, d) - \text{LSI}_{s+j}^{n}(b, a + d) \right). \tag{23}
\]

In the limit for \( b \to 0 \), with \( |a| > M > 0 \), Eq. (22) may suffer from overflow. Similarly, in the limit for \( a \to 0 \), with \( |b| > M > 0 \), Eq. (23) may suffer from overflow. The choice of which formula to use, Eq. (22) or Eq. (23), should take into consideration the magnitude of \( a \) and \( b \).

**Remark 4.1.** In Eq. (22), for \( a + b + d \leq 0 \) the summation within the \( \text{LSI}_{s+j}^{m+n+1-j}(a + b, d) \) terms vanishes. This is due to \( \lim_{x \to 0+} \text{Li}_{s+1+i}(x) = 0 \), with \( s \geq -1 \), \( i \in \mathbb{Z}^+ \), and non positive argument \( x \). Specifically, for \( a + b + d \leq 0 \), Eqs. (22) and (23) reduce to

\[
\text{TRI}_{s}^{m,n}(a, b, d) = n! \left( \lim_{x \to 0+} \text{Li}_{s+m+n+2}(x) \sum_{j=1}^{n+1} \frac{(m + n + 1 - j)!}{(n+1-j)!} \left( -\frac{a}{b} \right)^j \right)
\]

\[
+ \frac{\text{LSI}_{s+n+1}^{m}(a, d)}{(-b)^{n+1}}), \tag{24}
\]

and

\[
\text{TRI}_{s}^{m,n}(a, b, d) = m! \left( -\lim_{x \to 0+} \text{Li}_{s+m+n+2}(x) \sum_{j=1}^{m+1} \frac{(m + n + 1 - j)!}{(m+1-j)!} \left( -\frac{a}{b} \right)^j \right)
\]
which are less expensive to compute. For \( s = -1 \) and \( a + b + d > 0 \), we can still take advantage of this reduction by changing the sign of the normal and utilizing Proposition 2.3. Namely,

\[
\text{TRI}^m \text{BR}^{-1}(a, b, d) = \text{TRI}^m \text{BR}^{-1}(-a, -b, -d).
\]

A similar reasoning can be extended to the limiting cases \( \text{TRI}^m(a, 0, d) \), \( \text{TRI}^m(0, b, d) \) and \( \text{TRI}^m(a, -a, 0) \), when \( a + d \leq 0 \), \( b + d \leq 0 \), and \( d \leq 0 \), respectively. However, special attention should be used if \( s = -1 \) and \( a + b + d = 0 \), since for this case the first terms in the “supposedly vanishing” sums would be \( \lim Li_0(0) = -0.5 \neq 0 \). Rewriting the three reduced formulas in a conservative way, always including the first term in the sum, leads to

\[
\text{TRI}^m \text{BR}^s(a, 0, d) = \frac{1}{n + 1} \left( -\lim Li_{s+1}^{(a+d)} + (m + n + 1)! \frac{(-1)^{m+n+1} \lim Li_{s+m+n+2}^{(d)}}{a^{m+n+2}} \right),
\]

\[
\text{TRI}^m \text{BR}^s(0, b, d) = \frac{1}{m + 1} \left( n! \frac{(-1)^{m} \lim Li_{s+n+1}^{(d)}}{b^{n+1}} - (m + n + 1)! \frac{(-1)^{m+n+1} \lim Li_{s+m+n+2}^{(d)}}{b^{m+n+2}} \right),
\]

\[
\text{TRI}^m \text{BR}^s(a, -a, d) = \lim Li_{s+1}^{(d)} + n! m! \sum_{i=1}^{m+1} \frac{(-1)^i \lim Li_{s+n+1+i}^{(a+d)}}{(m + 1 - i)! a^{n+1+i}},
\]

which hold for \( a + b + d \leq 0 \) and \( s \geq -1 \).

We also include the two alternative formulas below. These are obtained by utilizing the property \( \frac{d \text{Li}_s(-e^\mu)}{d \mu} = \text{Li}_{s-1}(-e^\mu) \) and by recursive use of integration by parts, increasing the monomial power and reducing the polylogarithm order \( s \) until it reaches \(-1\). Namely, for \( s \geq 0 \) and \( b \neq 0 \),

\[
\text{TRI}^m \text{C}^s(a, b, d) = \frac{n!}{(n + s + 1)!} (-b)^{s+1} \text{TRI}^{m,n+s+1}_A(a, b, d)
\]

\[
+ \sum_{i=0}^{s} \frac{n!}{(n + i + 1)!} (-b)^i \text{LSI}^{m+n+i+1}_{s-i}(a + b, d),
\]

and, for \( s \geq 0 \) and \( a \neq 0 \),

\[
\text{TRI}^m \text{C}^s(a, b, d) = \frac{n!}{(m + s + 1)!} (-a)^{s+1} \text{TRI}^{m+s+1,n}_A(a, b, d)
\]

\[
+ \sum_{i=0}^{s} \frac{n!}{(m + i + 1)!} (-a)^i \left( \text{LSI}^{n}_{s-i}(b, d + a) - \text{LSI}^{m+n+i+1}_{s-i}(a + b, d) \right).
\]

For \( a + b + d > \max(\|a\|, |b|) \), the combination of Remark 4.1 and Eqs. (29)-(30) yields a formulation which
protections against overflow for \( a \to 0 \) and/or \( b \to 0 \). In particular, the calls to the integrals in Eq. (29) and Eq. (30) can be replaced by

\[
\text{TRI}_{A_{m,n}+s+1}(a, b, d) \quad \text{and} \quad \text{TRI}_{A_{m,n}+s+1,n}(a, b, d),
\]

respectively, for which \((-a) + (-b) + (-d) < 0\).

At last we include the degenerate case when both \( a = 0 \) and \( b = 0 \) for \( s \geq 0 \), which was excluded because of the constraint \( a^2 + b^2 > 1 \). This case is needed for external calls made by higher dimensional objects, such as the tetrahedron and prism, for which the normal \( n = (a, b, c) \) could take the form \( n = (0, 0, c) \). After integration

\[
\text{TRI}_{A_{mn}(0, 0, d)} = -\lim_{t \to \infty} \frac{1}{t^s} \int_0^1 \int_0^x x^m y^n \text{Li}_s(-\exp(dt)) \, dy \, dx = -\frac{\text{Li}_s^{mn}(d)}{(n+1)(m+n+2)}. \tag{31}
\]

The pseudo-code for the triangle integration is given in Algorithms 4 and 5. Algorithm 4 evaluates the integral in Eq. (17) on the triangle \( \{(x, y) : x \in [0, 1], y \in [0, 1-x]\} \). It calls the function \text{TRIANGLE\_INTEGRATION\_A} in Algorithm 5, which evaluates the transformed integral in Eq. (18) on the triangle \( \{(x, y) : x \in [0, 1], y \in [0, x]\} \). \text{TRIANGLE\_INTEGRATION\_A} handles the degenerate case \( a = b = 0 \) and sorts the different \( s \)-cases. For each case it ensures that the reduced integration function, \text{TRIANGLE\_INTEGRATION\_BR}, is called only for \( a + b + d \leq 0 \). For \( 0 < a + b + d \leq \max(|a|, |b|) \), the function \text{TRIANGLE\_INTEGRATION\_B} is called, otherwise the alternative function \text{TRIANGLE\_INTEGRATION\_C} is used. The recursive calls follow from the patterns developed in Eqs. (29) and (30). Every time the line segment integration formula, LSI, is needed the function \text{LINE\_SEGMENT\_INTEGRATION} in Algorithm 1 is called.

\begin{algorithm}[h]
1: \textbf{function} \text{TRIANGLE\_INTEGRATION}(a, b, d, m n, s)
2: \hspace{1cm} \text{return} \text{TRIANGLE\_INTEGRATION\_A}(-a, b, d + a, m, n, s)
3: \textbf{end function}
\end{algorithm}

4.2. TTI: Tetrahedron Integration on \[
\begin{align*}
0 \leq x \leq 1 \\
0 \leq y \leq 1 - x \\
0 \leq z \leq 1 - x - y
\end{align*}
\]

To ease the computation in the case of the tetrahedron, we choose different polynomial bases depending on the magnitude of the coefficients \( a, b, \) and \( c \).
Algorithm 5 Pseudo-code for the integration of Eq. (18) on the triangle \( \{(x, y) : x \in [0, 1], y \in [0, x]\} \) cut by the line \( ax + by + d = 0 \) with \( \mathbf{n} = (a, b) \), \( \mathbf{m} = (m, n) \) and \( s = -1, 0, 1, \ldots \). For \( s = -1 \) and \( \|\mathbf{n}\| = 1 \) it corresponds to the interface integral. For \( s = 0 \) and \( \|\mathbf{n}\| > 0 \) it corresponds to the subdomain integral.

1: function TRIANGLE\_INTEGRATION\_A(a, b, d, m n, s)
2:     if \( b = 0 \) and \( a = 0 \) then return \( \text{TRI}_{s}^{m n}(0,0,d) \) from Eq. (31)
3:     end if
4:     if \( s = -1 \) then
5:         if \( a + b + d \leq 0 \) then return TRIANGLE\_INTEGRATION\_BR(a, b, d, m, n, -1)
6:         else return TRIANGLE\_INTEGRATION\_BR(-a, -b, -d, m, n, -1)
7:     end if
8:     else
9:         if \( a + b + d \leq 0 \) then return TRIANGLE\_INTEGRATION\_BR(a, b, d, m, n, s)
10:        else if \( a + b + d \leq \max(|a|, |b|) \) then return TRIANGLE\_INTEGRATION\_B(a, b, d, m, n, s)
11:       else return TRIANGLE\_INTEGRATION\_C(a, b, d, m, n, s)
12:     end if
13: end if
14: end function

1: function TRIANGLE\_INTEGRATION\_B(a, b, d, m n, s)
2:     if \( b = 0 \) then return \( \text{TRI}_{B}^{m n}(a,0,d) \) from Eq. (19)
3:     else if \( a = 0 \) then return \( \text{TRI}_{B}^{m n}(0,a,d) \) from Eq. (20)
4:     else if \( a + b = 0 \) then return \( \text{TRI}_{B}^{m n}(a,-a,d) \) from Eq. (21)
5:     else
6:         if \( |a| \leq |b| \) then return \( \text{TRI}_{B}^{m n}(a,b,d) \) from Eq. (22)
7:         else return \( \text{TRI}_{B}^{m n}(a,b,d) \) from Eq. (23)
8:     end if
9: end if
10: end function

1: function TRIANGLE\_INTEGRATION\_BR(a, b, d, m n, s)
2:     if \( b = 0 \) then return \( \text{TRI}_{BR}^{m n}(a,0,d) \) from Eq. (26)
3:     else if \( a = 0 \) then return \( \text{TRI}_{BR}^{m n}(0,a,d) \) from Eq. (27)
4:     else if \( a + b = 0 \) then return \( \text{TRI}_{BR}^{m n}(a,-a,d) \) from Eq. (28)
5:     else
6:         if \( |a| \leq |b| \) then return \( \text{TRI}_{BR}^{m n}(a,b,d) \) from Eq. (24)
7:         else return \( \text{TRI}_{BR}^{m n}(a,b,d) \) from Eq. (25)
8:     end if
9: end if
10: end function

1: function TRIANGLE\_INTEGRATION\_C(a, b, d, m n, s)
2:     if \( |a| \leq |b| \) then return
3:         \( \sum_{i=0}^{s} \frac{n!}{(n+i+1)!}(b^{i} LSI_{s-i}^{m+n+i+1}(a+b,d) + \frac{n!}{(n+s+1)!}(b^{s+1} \text{TRIANGLE\_INTEGRATION\_A}(a,b,d,m,n+s+1,-1))
4:     else return
5:         \( \sum_{i=0}^{s} \frac{m!}{(m+i+1)!}(a^{i} (LSI_{s-i}^{m}(b,d+a) - LSI_{s-i}^{m+n+i+1}(a+b,d)) + \frac{m!}{(m+s+1)!}(a^{s+1} \text{TRIANGLE\_INTEGRATION\_A}(a,b,d,m+s+1,n,-1))
6:     end if
7: end function
Let \( m_1 = \max(|a+b|, |c-b|) \), \( m_2 = \max(|b+c|, |a-c|) \) and \( m_3 = \max(|c+a|, |b-a|) \), with the constraints \( a^2 + b^2 + c^2 > 0 \) and \( \max(m_1, m_2, m_3) > 0 \).

For \( m_1 \geq \max(m_2, m_3) \), we evaluate integrals in the form

\[
\text{TTI}_{s}^{m_1\alpha}(a, b, c, d) = - \lim_{t \to \infty} \frac{1}{t^s} \iiint_{\text{TET}} (x + y + z)^m (y + z)^n z^o \text{Li}_s(- \exp((ax + by + cz + d)t)) \, dV, \tag{32}
\]

else, for \( m_2 > m_3 \), we evaluate integrals in the form

\[
\text{TTI}_{s}^{m_2\alpha}(a, b, c, d) = - \lim_{t \to \infty} \frac{1}{t^s} \iiint_{\text{TET}} x^m (x + y + z)^n (z + x)^o \text{Li}_s(- \exp((ax + by + cz + d)t)) \, dV, \tag{33}
\]

else we evaluate integrals in the form

\[
\text{TTI}_{s}^{m_3\alpha}(a, b, c, d) = - \lim_{t \to \infty} \frac{1}{t^s} \iiint_{\text{TET}} (x + y)^m y^n (x + y + z)^o \text{Li}_s(- \exp((ax + by + cz + d)t)) \, dV. \tag{34}
\]

We make the following change of variables and constant renaming

- for Eq. (32),

\[
x' = x + y + z, \quad y' = y + z, \quad z' = z, \quad m' = m, \quad n' = n, \quad o' = o, \quad a' = a, \quad b' = b + a, \quad c' = c - b, \quad d' = d,
\]

- for Eq. (33),

\[
x' = x + y + z, \quad y' = z + x, \quad z' = x, \quad m' = n, \quad n' = o, \quad o' = m, \quad a' = b, \quad b' = c + b, \quad c' = a - c, \quad d' = d,
\]

- for Eq. (34),

\[
x' = x + y + z, \quad y' = x + y, \quad z' = y, \quad m' = o, \quad n' = m, \quad o' = n, \quad a' = c, \quad b' = a + c, \quad c' = b - a, \quad d' = d,
\]

always obtaining the same integral

\[
- \lim_{t \to \infty} \frac{1}{t^s} \int_{0}^{1} \int_{0}^{x'} \int_{0}^{y'} x'^{m'} y'^{n'} z'^{o'} \text{Li}_s(- \exp((a' x' + b' y' + c' z' + d')t)) \, dz' \, dy' \, dx'.
\]

Dropping the ' superscript, for a fixed \( s = -1, 0, 1, \ldots \), the problem reduces to find integrals in the form

\[
\text{TTI}_{s}^{m\alpha}(a, b, c, d) = - \lim_{t \to \infty} \frac{1}{t^s} \int_{0}^{1} \int_{0}^{x} \int_{0}^{y} x^m y^n z^o \text{Li}_s(- \exp((ax + by + cz + d)t)) \, dz \, dy \, dx, \tag{35}
\]

where \( \max(|b|, |c|) = \max(m_1, m_2, m_3) > 0 \).
For $|b| \leq |c|$, after integrating in $z$

\[
TTIB^m_{s}x_{n}\langle a, b, c, d \rangle = - \lim_{t \to \infty} \frac{1}{t^s} \int_{0}^{1} \int_{0}^{x} x^m y^n \left( \sum_{i=1}^{o+1} \frac{\exp((-1)^{i-1}(o+1-i)!b_{i+1}(a+b+c+d)t)}{(ct)^i} \right) dy dx.
\]

Simplifying and using the triangle integration formula yields

\[
TTIB^m_{s}x_{n}\langle a, b, c, d \rangle = - \sum_{i=1}^{o+1} \frac{\exp((-1)^{i-1}(o+1-i)!b_{i+1}(a+b+c+d)t)}{(ct)^i} \frac{1}{(o+1-i)!(-c)^i} T_{R}^{m,n+o+1-1}(a, b, c, d) + \frac{\exp((-1)^{i-1}(o+1-i)!b_{i+1}(a+b+c+d)t)}{(ct)^i} \frac{1}{(o+1-i)!(-c)^i} T_{R}^{m,n+1-1,0}(a, b, c, d).
\]

For $|c| < |b|$, we reverse the order of integration and after simplification get

\[
TTIB^m_{s}x_{n}\langle a, b, c, d \rangle = - \lim_{t \to \infty} \frac{1}{t^s} \int_{0}^{1} \int_{0}^{x} x^m y^n z^o Li_s(-\exp((a+b+cz+d)t)) dy dz dx
\]

\[
= \sum_{i=1}^{n+1} \frac{n!}{(n+1-i)!(-b)^i} \left( T_{R}^{m,n+o+1-1}(a, b, c, d) - T_{R}^{m+1,n+1-1,0}(a, b, c, d) \right).
\]

All limiting cases, are left to be handled by the triangle integration formula as described in the previous section.

For $s \geq 0$ and $a + b + c + d > \max(|b|, |c|)$, we also include the alternative formulas below. These are obtained by utilizing the property $\frac{d}{dx} Li_s(-e^x) = Li_{s-1}(-e^x)$ and by recursive use of integration by parts, increasing the monomial power and reducing the polylogarithm order $s$ until it reaches $-1$. Namely, For $|b| \leq |c|

\[
TTIC^m_{s}x_{n}\langle a, b, c, d \rangle = \frac{\exp((-1)^{i-1}(o+1-i)!b_{i+1}(a+b+c+d)t)}{(ct)^i} \frac{1}{(o+1-i)!(-c)^i} T_{R}^{m,s+1}(a, b, c, d)
\]

\[
+ \sum_{i=0}^{s} \frac{\exp((-1)^{i-1}(o+1-i)!b_{i+1}(a+b+c+d)t)}{(ct)^i} \frac{1}{(o+1-i)!(-c)^i} T_{R}^{m,n+o+1}(a, b, c, d),
\]

otherwise

\[
TTIC^m_{s}x_{n}\langle a, b, c, d \rangle = \frac{\exp((-1)^{i-1}(o+1-i)!b_{i+1}(a+b+c+d)t)}{(ct)^i} \frac{1}{(o+1-i)!(-c)^i} T_{R}^{m,n+o+1}(a, b, c, d)
\]
\[
+ \sum_{i=0}^{s} \frac{n!}{(n+i+1)!} (-b)^i \left( \text{TRI}_{A_{s-i}}^{m+n+i+1,o} (a+b, c, d) - \text{TRI}_{A_{s-i}}^{m,n,o+i+1} (a+b, c, d) \right).
\] (40)

The pseudo-code for the integration over the tetrahedron is given in Algorithms 6 and 7. Every time the triangle integration formula TRI is needed, the function TRIANGLE_INTEGRATION_A in Algorithm 5 is called.

**Algorithm 6** Pseudo-code for the integration of Eqs. (32)-(34) on the tetrahedron \{ \(x, y, z\) : \(x \in [0,1], y \in [0,1-x], z \in [0,1-x-y]\}\} cut by the plane \(ax + by + cz + d = 0\) with \(n = \langle a, b, c, \rangle, \|n\| > 0\), \(m = \langle m, n, o \rangle\) and \(s = -1, 0, 1, \ldots\). For \(s = -1\) and \(\|n\| = 1\) it corresponds to the interface integral. For \(s = 0\) it corresponds to the subdomain integral.

1: function TETRAHEDRON_INTEGRATION(a, b, c, d, m, n, o, s)
2: \(m_1 = \max(|a+b|, |c-b|)\), \(m_2 = \max(|b+c|, |a-c|)\) and \(m_3 = \max(|c+a|, |b-a|)\)
3: if \(m_1 \geq \max(m_2, m_3)\) then
4: return TETRAHEDRON_INTEGRATION_A(a, a+b, c-b, d, m, n, o, s)
5: else if \(m_2 \geq m_2\) then
6: return TETRAHEDRON_INTEGRATION_A(b, b+c, a-c, d, m, n, o, s)
7: else
8: return TETRAHEDRON_INTEGRATION_A(c, c+a, b-a, d, o, m, n, s)
9: end if
10: end function

4.3. PRI: Prism Integration on \[\begin{align*}
0 & \leq x \leq 1 \\
0 & \leq y \leq 1 - x, \text{ with } a^2 + b^2 + c^2 > 0 \\
-1 & \leq z \leq 1
\end{align*}\]

The implementation of a polynomial basis whose elements are given by \((1-x)^m y^n z^o\), allows for computational simplicity when considering integrals in the form

\[
\text{PRI} = -\lim_{t \to \infty} \frac{1}{t^s} \int \int \int_{\text{PRI}} (1-x)^m y^n \left( \frac{1+z}{2} \right)^o \text{Li}_s(-\exp((ax+by+cz+d)t)) \frac{dV}{2}.
\] (41)

By using the following transformation

\[x' = 1 - x, \ y' = y, \ z' = \frac{1+z}{2}, \quad a' = -a, b' = b, c' = 2c, \ d' = d + a - c\]

we obtain

\[-\lim_{t \to \infty} \frac{1}{t^{s+1}} \int_0^1 \int_0^{x'} \int_0^{x'} x'^m y'^n z'^o \text{Li}_s(-\exp((a'x' + b'y' + c'z' + d')t)) \, dz' \, dy' \, dx'.\]

Dropping the ' superscript, for a fixed \(s = -1, 0, 1, \ldots\), the problem reduces to integrals in the form

\[
\text{PRI}_{A_s}^{mno} (a, b, c, d) = -\lim_{t \to \infty} \frac{1}{t^s} \int_0^1 \int_0^{x} \int_0^{x} x'^m y'^n z'^o \text{Li}_s(-\exp((ax+by+cz+d)t)) \, dz \, dy \, dx.
\]
Algorithm 7 Pseudo-code for the integration of Eq. (35) on the tetrahedron \( \{(x, y, z) : x \in [0,1], y \in [0,x], z \in [0,y]\} \) cut by the plane \( ax + by + cz + d = 0 \) with \( n = (a, b, c) \), either \( b \neq 0 \) or \( c \neq 0 \), \( m = (m, n, o) \) and \( s = -1,0,1, \ldots \). For \( s = -1 \) and \( ||n|| = 1 \) it corresponds to the interface integral. For \( s = 0 \) it corresponds to the subdomain integral.

1: function TetrahedronIntegration_A(a, b, c, d, m, n, o, s)
2:     \( sum = a + b + c + d \)
3:     if \( s = -1 \) then
4:         if \( sum \leq 0 \) then
5:             return TetrahedronIntegration_B(a, b, c, d, m, n, o, -1)
6:         else
7:             return TetrahedronIntegration_B(-a, -b, -c, -d, m, n, o, -1)
8:     end if
9:     else
10:        if \( sum \leq \max(|b|, |c|) \) then
11:           return TetrahedronIntegration_B(a, b, c, d, m, n, o, s)
12:        else
13:           return TetrahedronIntegration_C(a, b, c, d, m, n, o, s)
14:        end if
15:     end if
16: end function

1: function TetrahedronIntegration_B(a, b, c, d, m, n, o, s)
2:     if \( |b| \leq |c| \) then return TTI_B from Eq. (37)
3:     else return TTI_B from Eq. (38)
4: end if
5: end function

1: function TetrahedronIntegration_C(dim, n, d, m, s)
2:     if \( |b| \leq |c| \) then return
3:         \( \sum_{i=0}^{s} \frac{o!}{(o+i+1)!} (-c)^i \text{TRI}_{m,n+o+i+1}^A(a, b + c, d) \)
4:         \( + \frac{o!}{(o+s+1)!} (-c)^{s+1} \text{TetrahedronIntegration}_A(a, b, c, d, m, n, o + s + 1, -1) \)
5:     else return
6:         \( \sum_{i=0}^{s} \frac{n!}{(n+i+1)!} (-b)^i \left( \text{TRI}_{m+n+i+1,0}^A(a + b, c, d) - \text{TRI}_{m,n+o+i+1}^A(a, b + c, d) \right) \)
7:         \( + \frac{n!}{(n+s+1)!} (-b)^{s+1} \text{TetrahedronIntegration}_A(a, b, c, d, m, n + s + 1, o, -1) \)
8:     end if
9: end function
For $|c| \geq \max(|a|, |b|)$, after integrating in $z$ we get

$$\text{PRI}_{s}^{m,n,o}(a, b, c, d) = - \lim_{t \to \infty} \frac{1}{t^{s+1}} \int_{0}^{1} \int_{0}^{1} \int_{0}^{1} x^{m} y^{n} z^{o} \text{Li}_{s+1}(- \exp((ax + by + cz + dt)) \text{ dy dx dz}}$$

$$= - \sum_{i=1}^{o+1} \frac{a!}{(o + 1 - i)!} \frac{1}{(-c)^{o+1}} \text{ TRI}_{s+1}^{m,n,o}(a, b, c + d) + \frac{a!}{(-c)^{o+1}} \text{ TRI}_{s+o+1}^{m,n}(a, b, d). \quad (42)$$

For $|b| \geq |a|$, after integrating first in $y$ and simplifying we have

$$\text{PRI}_{s}^{m,n,o}(a, b, c, d) = - \lim_{t \to \infty} \frac{1}{t^{s+1}} \int_{0}^{1} \int_{0}^{1} \int_{0}^{1} x^{m} y^{n} z^{o} \text{Li}_{s}(- \exp((ax + by + cz + dt)) \text{ dy dz dx}}$$

$$= - \sum_{i=1}^{n+1} \frac{n!}{(n + 1 - i)!} \frac{1}{(-b)^{n+1}} \text{ HCI}_{s+i,2}^{m+n+1-i,o}((a + b, c, d) + \frac{n!}{(-b)^{n+1}} \text{ HCI}_{s+n+1,2}^{m,o}((a, c), d). \quad (43)$$

Lastly, for all other cases, after integrating first in $x$ and simplifying we obtain

$$\text{PRI}_{s}^{m,n,o}(a, b, c, d) = - \lim_{t \to \infty} \frac{1}{t^{s+1}} \int_{0}^{1} \int_{0}^{1} \int_{0}^{1} x^{m} y^{n} z^{o} \text{Li}_{s}(- \exp((ax + by + cz + dt)) \text{ dx dy dz}}$$

$$= \sum_{i=1}^{m+1} \frac{m!}{(m + 1 - i)!} \frac{1}{(-a)^{m+1}} \left( - \text{HCI}_{s+i,2}^{m,o}((b, c), a + d) + \text{HCI}_{s+i,2}^{m+n+1-i,o}((a + b, c), d) \right). \quad (44)$$

All limiting cases are left to be handled by the triangle and the hypercube integration formulas previously described.

For $s \geq 0$ and $a + b + c + d > \max(|a|, |b|, |c|)$, we also include the alternative formulas below. These are obtained by utilizing the property $\frac{d}{d \mu} \text{Li}_{s}(-e^{\mu}) = \text{Li}_{s-1}(-e^{\mu})$ and by recursive use of integration by parts, increasing the monomial power and reducing the polylogarithm order $s$ until it reaches $-1$. Namely, for $|c| \geq \max(|a|, |b|)$, we utilize the formula

$$\text{PRI}_{s}^{m,n,o}(a, b, c, d) = \frac{o!}{(o + s + 1)!} (-c)^{s+1} \text{PRI}_{s}^{m,n,o+1}(a, b, c, d)$$

$$+ \sum_{i=0}^{s} \frac{o!}{(o + i + 1)!} (-c)^{i} \text{ TRI}_{s+i}^{m,n}(a, b, c + d), \quad (45)$$

and for $|b| \geq |a|$ we implement

$$\text{PRI}_{s}^{m,n,o}(a, b, c, d) = \frac{n!}{(n + s + 1)!} (-b)^{s+1} \text{PRI}_{s}^{m,n+1,o}(a, b, c, d)$$

$$+ \sum_{i=0}^{s} \frac{n!}{(n + i + 1)!} (-b)^{i} \text{HCI}_{s+i}^{m+n+i+1,o}((a + b, c), d). \quad (46)$$
For any other case we employ

\[ PRI_{s,m,n,o}^{a,b,c,d} = \frac{m!}{(m + s + 1)!} (-b)^{s+1} PRI_{s-1}^{m+s+1,n,o} (a, b, c, d) \]
\[ + \sum_{i=0}^{s} \frac{m!}{(m+i+1)!} (-a)^i \left( HCI_{s+i,2}^{(n,o)} (b, c, a + d) - HCI_{s-i,2}^{(m+n+i+1,o)} (a + b, c, d) \right). \] (47)

The pseudo-code for integration over the prism is given in Algorithms 8 and 9. Every time the triangle integration formula TRI and the hypercube integration formula HCI are used, the functions TRIANGLE_INTEGRATION_A and HYPERCUBE_INTEGRATION in Algorithm 5 are called.

**Algorithm 8** Pseudo-code for the integration of Eq. (41) on the prism \( \{(x, y, z) : x \in [0, 1], y \in [0, 1-x], z \in [-1, 1]\} \) cut by the plane \( a x + b y + c z + d = 0 \) with \( n = (a, b, c), \|n\| > 0, m = (m, n, o) \) and \( s = -1, 0, 1, \ldots \). For \( s = -1 \) and \( \|n\| = 1 \) it corresponds to the interface integral. For \( s = 0 \) it corresponds to the subdomain integral.

1: function PRISM_INTEGRATION(a, b, c, d, m, n, o, s)
2: return PRISM_INTEGRATION_A(-a, b, 2c, d + a - c, m, n, o, s)
3: end function

5. Note on the equivalent polynomial

The equivalent polynomial problem can be stated as follows: find the equivalent polynomial coefficients \( c \), such that \( M c = f_o \), where

\[ f_o = - \lim_{t \to \infty} t^{-s} \left( \begin{array}{c} \int_{\Omega} b_{o,0}(x) Li_s(-\exp((n \cdot x)t)) \ dx \\ \int_{\Omega} b_{o,1}(x) Li_s(-\exp((n \cdot x)t)) \ dx \\ \vdots \\ \int_{\Omega} b_{o,L}(x) Li_s(-\exp((n \cdot x)t)) \ dx \end{array} \right) \]

and

\[ M = \left( \begin{array}{cccc} \int_{\Omega} b_{o,0}(x) b_{o,0}(x) \ dx & \int_{\Omega} b_{o,1}(x) b_{o,0}(x) \ dx & \cdots & \int_{\Omega} b_{o,L}(x) b_{o,0}(x) \ dx \\ \int_{\Omega} b_{o,0}(x) b_{o,1}(x) \ dx & \int_{\Omega} b_{o,1}(x) b_{o,1}(x) \ dx & \cdots & \int_{\Omega} b_{o,L}(x) b_{o,1}(x) \ dx \\ \vdots & \vdots & \ddots & \vdots \\ \int_{\Omega} b_{o,0}(x) b_{o,L}(x) \ dx & \int_{\Omega} b_{o,1}(x) b_{o,L}(x) \ dx & \cdots & \int_{\Omega} b_{o,L}(x) b_{o,L}(x) \ dx \end{array} \right), \]

with \( s = -1 \) or 0. Here \( b_o \) is the basis of the polynomial space. Then, the equivalent polynomial is given by \( p(x) = c_o^T \cdot b_o \). In order to avoid an ill-conditioned Gram matrix \( M \), we implement orthogonal polynomials, via Gram-Schmidt orthogonalization, using the \( L^2 \) inner product[26, 27]. This yields the following relation
**Algorithm 9** Pseudo-code for the integration on the prism \(\{(x, y, z): x \in [0, 1], y \in [0, x], z \in [0, 1]\}\) cut by the plane \(ax + by + cz + d = 0\) with \(n = (a, b, c)\), \(\|n\| > 0\), \(m = (m, n, o)\) and \(s = -1, 0, 1, \ldots\). For \(s = -1\) it corresponds to the interface integral. For \(s = 0\) it corresponds to the subdomain integral.

1: function PRISM\_INTEGRATION\_A\((a, b, c, d, m, n, o, s)\)
2: \(\text{sum} = a + b + c + d\)
3: if \(s = -1\) then
4: \(\text{return}\) PRISM\_INTEGRATION\_B\((a, b, c, d, m, n, o, -1)\)
5: else
6: \(\text{return}\) PRISM\_INTEGRATION\_B\((-a, -b, -c, -d, m, n, o, -1)\)
7: end if
8: else
9: if \(\text{sum} \leq 0\) then
10: \(\text{return}\) PRISM\_INTEGRATION\_B\((a, b, c, d, m, n, o, s)\)
11: else
12: \(\text{return}\) PRISM\_INTEGRATION\_C\((a, b, c, d, m, n, o, s)\)
13: end if
14: end if
15: end function

1: function PRISM\_INTEGRATION\_B\((a, b, c, d, m, n, o, s)\)
2: if \(|c| \geq \max(|a|, |b|)|\) then \(\text{return}\) PRI\(_B\) from Eq. (42)
3: else if \(|b| > |a|\) then \(\text{return}\) PRI\(_B\) from Eq. (43)
4: else \(\text{return}\) PRI\(_B\) from Eq. (44)
5: end if
6: end function

1: function PRISM\_INTEGRATION\_C\((\text{dim}, n, d, m, s)\)
2: if \(|c| \geq \max(|a|, |b|)|\) then \(\text{return}\)
3: \(\sum_{i=0}^{s} \frac{n!}{(m+i+1)!}(-b)^i \text{T}_{\text{RI}_{s-i,2}}(a, b, c, d)
4: \)
5: else if \(|b| \geq |a|\) then \(\text{return}\)
6: \(\sum_{i=0}^{s} \frac{n!}{(n+i+1)!}(-b)^i \text{H}_{\text{CI}_{s-i,2}}^{(m+n+i+1, o)}((a+b, c), d)
7: \)
8: else \(\text{return}\)
9: \(\sum_{i=0}^{s} \frac{m!}{(m+i+1)!}(-a)^i \left(\text{H}_{\text{CI}_{s-i,2}}^{(n, o)}((b, c), a+d) - \text{H}_{\text{CI}_{s-i,2}}^{(m+n+i+1, o)}((a+b, c), d]\right)
10: \) + \(\sum_{i=0}^{s} \frac{m!}{(m+s+1)!}(-a)^i \text{PRISM\_INTEGRATION\_A}(a, b, c, d, m+s+1, n, o, -1)
11: end if
12: end function
for basis elements: \( \mathbf{b}_n = A \mathbf{b}_o \), where the components in the new basis, \( \mathbf{b}_n \), are a linear combination of the components in the old basis, \( \mathbf{b}_o \). The matrix \( A \) is an \( L \times L \) lower triangular matrix, where \( L \) is the dimension of the space spanned by the basis vector \( \mathbf{b}_o \).

The implementation of equivalent polynomial using an orthonormal basis yields

\[
I \mathbf{c}_n = \mathbf{f}_n = A \mathbf{f}_o,
\]

resulting in

\[
p(x) = (\mathbf{c}_n)^T \mathbf{b}_n = \mathbf{f}_o^T A^T A \mathbf{b}_o(x).
\]

Note that the term \( A^T A \mathbf{b}_o(x) \) is independent of the hyperplane cut and can be evaluated off-line. Instead \( \mathbf{f}_o \) changes and has to be recalculated for every new cut.

6. Conclusion

The many closed form algebraic expressions provided in the current work can easily be implemented into numerous PDE solvers when discontinuous functions are implemented. Although the discontinuities we considered were points, lines, and planes, one could utilize a local refinement algorithm to reduce the error generated by approximating the discontinuity. We have eliminated the need to consider complicated subdomains while simultaneously eliminating any error produced by a regularization parameter and polylogarithm approximation. We provide exact formulas for cumbersome subdomain and interface integrals, along with the associated algorithms. These closed forms were designed with floating point arithmetic in mind. The results of this work provide one with the tools to eliminate many of the problems posed by discontinuous function integration.

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