Arithmetical Properties of Finite Groups*

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Abstract

Let $G$ be a finite group and $Ch_i(G)$ some quantitative sets. In this paper we study the influence of $Ch_i(G)$ to the structure of $G$. We present a survey of author and his colleagues’ recent works.

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Let $G$ be a finite group and $Ch(G)$ be one of the following sets:

$Ch_1(G) = |G|$, that is, the order of $G$;

$Ch_2(G) = \pi_e(G) = \{o(g) \mid g \in G\}$, that is, the set of element orders of $G$;

$Ch_3(G) = cs(G) = \{|g^G| \mid g \in G\}$, that is, the set of conjugacy class sizes of $G$;

$Ch_4(G) = cd(G)$, that is, the set of irreducible character degrees of $G$.

Our aim is to study the structure of $G$ under certain arithmetical hypotheses of $Ch_i(G)$, $i = 1, 2, 3$ or 4.

Except the above quantitative sets, we may define $Ch_5(G)$ be the set of the maximal subgroup orders of $G$(see [25]), $Ch_6(G)$ be the set of Sylow normalizer orders of $G$(see [2]), and the other quantitative sets(for example, see [1]). In this paper we discuss the cases of $Ch_i(G)$, $i = 1, 2, 3$ or 4, especially for the cases of $i = 1, 2$.

**Question A** If $Ch(G)$ is fixed, what can we say about the structure of $G$?

For the set $Ch_i(G)$, $i = 2, 3$ or 4, we can define a graph $\Gamma_i(G)$ as follows: Its vertices are the primes dividing the numbers in $Ch_i(G)$; and two distinct vertices $p, q$ are connected if $pq | m$ holds for $m \in Ch_i(G)$.

**Question B** If we know the information of graph $\Gamma_i(G)$, $i = 2, 3$ or 4, what can we say about the structure of $G$?

In our characterization using the element orders, the graph $\Gamma_2(G)$(prime graph) and Gruenberg-Kegel theorem on groups with disconnected prime graphs(see [47]) play an important role.

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For any \( a \in Ch_i(G) \), \( i = 2, 3 \) or 4, for example, \( a \in Ch_2(G) \), we define \( M_2(a) \), the multiplicity of \( a \) in \( G \) as the number of elements of order \( a \). Also, \( DC_{3,2}(G) = |Ch_2'(G)| - |Ch_2(G)| \), the difference of conjugacy classes number \( |Ch_2'(G)| \) and same order classes number in \( G \). Furthermore, \( QC_{1,3}(G) = |G|/|Ch_3'(G)| \), the quotient of \( |G| \) and the number of conjugacy classes.

**Question C** If \( Ch(G) \) and \( M(a) \), for all \( a \in G \), are known what can we say about the structure of \( G \)? If \( DC(G) \) is "small", what can we say about the structure of \( G \)? If \( QC(G) \) is known what can we say about the structure of \( G \)?

For example, \( DC_{3,2}(G) = 0 \), that is, any elements of a finite group \( G \) with same order are conjugate. This is Syskin’s problem, and some group theorists have proved that \( G \cong 1, Z_2 \), or \( S_3 \) using the classification theorem of finite simple groups(see [12], [50]). In [11] we classified all finite groups of \( DC_{3,2}(G) = 1 \), and studied such finite groups in which elements of the same order outside the center are conjugate.(see [32]).

**Question D** Let \( b_i(G) = \max \{Ch_i(G)\}, \ i = 2, 3 \) or 4. If some information of \( b_i(G) \) is known, what can we say about the structure of \( G \)?

For Problem A we know there exists some very famous results for \( Ch_1(G) = |G| \), for example, **Sylow’s theorem**, the odd order theorem, and **Burnside’s** \( p^aq^b \) theorem etc.(see [3], [13] and some important bibliography listed in [14]).

**Burnside’s** \( p^aq^b \) **theorem** implies that if \( G \) is a non-Abelian simple group, then \( |\pi(G)| \geq 3 \).

A finite group \( G \) is called a \( K_n \)-group if \( |\pi(G)| = n \). There are only eight simple \( K_n \)-groups(see [17]). In [34] we classified all simple \( K_4 \)-groups, but we do not know whether the number of \( K_4 \)-groups is finite or not. This problem depends on the solutions of some special Diophantine equations(also see [4]).

Twenty years ago, we studied such finite groups in which every non-identity element has prime order, i.e. finite EPO groups and got an interesting result: \( G \cong A_5 \) if and only if \( \pi_e(G) = \{1, 2, 3, 5\} \)(That is, we may characterize the integral property only using the local property, see [24]).

After this, we developed such characterizations for all finite simple groups using the ”two orders”, and for the finite nonsolvable groups only using the ”set of elements orders”(or ”spectrum”). That is, we researched the following characterizations of two kinds.

(1) Characterizing all finite simple groups unitization using only the two sets \( Ch_1(G) \) and \( Ch_2(G) \).

We have finished the above works except \( B_n, C_n \) and \( D_n \) (n even)(see [25], [26], [27], [28], [39], [8] and [23]).

(2) Many finite simple groups are characterizable using only the set \( Ch_2(G) \).

The most recent version of the latter characterization is presented in Table 1 of [30].

Let \( |G| = n \) and \( f(n) \) denote the numbers of \( G \), pairwise non-isomorphic, such that \( |G| = n \). It is easy to see that \( f(n) = 1 \) if and only if \( (n, \varphi(n)) = 1 \), where \( \varphi(n) \) is a Euler function of \( n \). Also the solutions for \( f(n) = 2, 3, \) and 4 are found(see [23]).

The following question is posed: For any integer \( k \), is there a solution for \( f(n) = k \)?

Now we consider substituting \( |G| \) by the set \( \pi_e(G) \). \( \pi_e(G) \) is a set of some positive integers. Similarly, for a set \( \Gamma \) of positive integers, let \( h(\Gamma) \) be the number of isomorphic classes of finite group \( G \) such that \( \pi_e(G) = \Gamma \).
If \( \Gamma = \pi_e(G) \), then we have \( h(\pi_e(G)) \geq 1 \). If \( \Gamma = \{1, 2, 3, 5\} \), then \( h(\Gamma) = 1 \). Conversely, which groups \( G \) satisfy \( h(\pi_e(G)) = 1 \)? Such groups are called **characterizable groups** or **recognizable groups**.

With respect to characterizable groups, summarizing many scholars’ works, we have the following results:

**Theorem 1** The following groups are characterizable groups:

(a) Alternating groups \( A_n \), where \( n = 5, 16, p, p + 1, p + 2, \) and \( p \geq 7 \) is a prime; Symmetric group \( S_n \), where \( n = 7, 9, 11, 12, 13, 14, 19, 20, 23, 24 \).

(b) Simple groups of Lie type \( L_2(q), q \neq 9 \), \( L_4(2^m) \), \( L_n(2^m), n = 2^k > 8 \), \( F_4(2^m) \), \( U_3(2^m), m \geq 2 \), series of simple groups of Suzuki-Ree type \( Sz(2^{3m+1}) \), \( 2G_2(3^{2m+1}) \), \( 2F_4(2^{2m+1}) \); \( S_4(3^{2m+1})(m > 0) \), \( G_2(3^m) \); \( L_3(T) \), \( L_4(3) \), \( L_5(2) \), \( L_6(2) \), \( L_7(2) \), \( L_8(2) \), \( U_3(9) \), \( U_3(11) \), \( U_4(3) \), \( U_6(2) \), \( O_6^-(2) \), \( O_{10}^- \), \( 3D_4(2) \), \( G_2(4) \), \( G_2(5) \), \( F_4(2) \), \( F_4(2)^\prime \), \( E_6(2) \); and non-solvable groups \( PGL_2(p^m), m > 1, p^m \neq 9 \), \( L_2(9)2_3 \) (\( \cong M_{10} \)), \( L_3(4)2_1 \).

(c) All sporadic simple groups except \( J_2 \).

extended the characterization of the above some groups from finite groups to periodic groups.

For the case of \( h(\pi_e(G)) = \infty \) we have

**Theorem 2** If all the minimal normal subgroups of \( G \) are elementary (especially \( G \) is solvable), or \( G \) is one of the following: \( A_6 \), \( A_{10} \), \( L_3(3) \), \( U_3(3) \), \( U_3(5) \), \( U_3(7) \), \( U_4(2) \), \( U_5(2) \); \( S_4(q)(q \neq 3^{2m+1} \) and \( m > 0) \), then \( h(\pi_e(G)) = \infty \).

In the case of \( k \)-recognized groups, we have found infinite pairs of 2-recognizable groups as follows:

(a) \( L_3(q) \), \( L_3(q)\langle \theta \rangle \), where \( \theta \) is a graph automorphism of \( L_3(q) \) of order 2, \( q = 5, 29, 41, \) or \( q \equiv \pm 2(\text{mod} 5) \) and \( 6,(q - 1)/2 \) = 2 (see [28], [31] and some references listed in [31]);

(b) \( L_3(9) \), \( L_3(9)2_1 \) (see [13]);

(c) \( S_6(2) \), \( O_6^+(2) \) (see [28] and [10]);

(d) \( O_7(3) \), \( O_7^+(3) \) (see [10]);

(e) \( L_6(3) \), \( L_6(3)\langle \theta \rangle \), where \( \theta \) is a graph automorphism of \( L_6(3) \) of order 2 (see [13]);

(f) \( U_4(5) \), \( U_4(5)\langle \gamma \rangle \), where \( \gamma \) is a graph automorphism of \( U_4(5) \) of order 2 (see [13]).

For any \( r > 0 \), \( h(\pi_e(L_3(7^r))) = r + 1 \), and these \( r + 1 \) groups are \( L_3(7^r)\langle \rho \rangle \), where \( \rho \) is a field automorphism of \( L_3(7^r) \), \( k = 0, 1, 2, \ldots, r \) (see [49]).

**Problem 1** For the cases of \( B_n(q) \) and \( C_n(q) \), \( q \) odd, we have \( |B_n(q)| = |C_n(q)| \), and \( B_n(q) \) is not isomorphic to \( C_n(q) \). How to distinguish them using the set \( \pi_e(G) \)?

**Problem 2** Find new characterizable simple groups within a particular class of finite simple groups.

Considering the independent number of the prime graph A.V. Vasil’ev and E.P. Vdovin find recently a new approach which makes possible to study the case of a finite simple group with the connected prime graph (see [45], also see [14] and [10] for its application).

**Problem 3** Whether or not there exist two section-free finite groups \( G_1 \) and \( G_2 \) such that \( \pi_e(G_1) = \pi_e(G_2) \) and \( h(\pi_e(G_1)) \) is finite?

**Problem 4** For a enough large positive integer \( n \), is alternating group \( A_n \) characterizable?

**Problem 5** For any \( n \geq 3 \), does \( h(L_n(2)) = 1 \) (see [15])?

In 1987, when the author communicated the first characterization (that is, Characterizing all finite simple groups unitization using only the two sets \( |G| \) and \( \pi_e(G) \)) with Prof. J.G.
Thompson, he put forward the following questions and conjectures in his letters:

For any finite group $G$ and any integer $d > 0$, let $G(d) = \{x \in G; x^d = 1\}$. Two finite groups $G_1$ and $G_2$ are of the same order type if and only if $|G_1(d)| = |G_2(d)|$, $d = 1, 2, \cdots$ (That is, $Ch_2(G_1) = Ch_2(G_2)$ and $M_2(G_1) = M_2(G_2)$).

**Problem 6 (J.G. Thompson)** Suppose $G_1$ and $G_2$ are groups of the same order type. If $G_1$ is solvable, is $G_2$ necessarily solvable?

The problem that the solvability of groups in which the number of elements whose orders are largest are given, induced by Thompson’s problem, interested many Chinese group-theory specialists. They proved the following results:

**Theorem 4** Let $G$ be a finite group and $b_2'(G)$ be the number of elements of maximal order in $G$. If $b_2'(G) = \text{odd}$, $32, 2p, 4p, 6p, 8p, 10p, 2p^2, 2p^3, 2pq$ ($p, q$ are primes), or $b_2'(G) = \varphi(k)$, where $\varphi(k)$ is the Euler function of maximal order $k$, then $G$ is solvable except $G \cong S_5$.

**Corollary** For the above cases, Thompson’s problem is affirmative.

**Problem 7 (J.G. Thompson’s conjecture)** Let $G, H$ be two finite groups with $Ch_3(G) = Ch_3(H)$ ($cs(G) = cs(H)$). If $H$ is a non-Abelian simple group and $Z(G) = 1$, then $G \cong H$.

G.Y. Chen has introduced the concept of order components and proved that Thompson’s conjecture holds if $H$ is a finite simple group at least three prime graph components(see $[7]$).

Now we consider the case of "small" difference number $DC(G)$.

**Theorem 5** Let $G$ be a finite group. Then $DC_3,2(G) = 1$ (i.e., $G$ have one and only one same order class containing two conjugacy classes of $G$) if and only if $G \cong A_5, L_2(7), S_5, S_4, A_4, D_{10}, Z_3, Z_4, Hol(Z_5)$ or $[Z_3]Z_4$ (see $[1]$).

**Theorem 6** Let $G$ be a finite group.

1. If $G$ is non-Abelian, then $QC_{1,3}(G) \geq 8/5$, and $QC_{1,3}(G) = 8/5$ if and only if $G = P \times A$, where $A$ is abelian with odd order and $P$ is a specific non-Abelian 2-group.
2. If $Z(G) = 1$, then $QC_{1,3}(G) \geq 2$, and $QC_{1,3}(G) = 2$ if and only if $G \cong S_3$.
3. If $G$ is non-Abelian simple, then $QC_{1,3}(G) \geq 12$, and $QC_{1,3}(G) = 12$ if and only if $G \cong A_5$ (see $[4]$).

Some papers improved the above result (see $[51]$ and $[10]$).

For the case of $Ch_3(G)$ and $Ch_4(G)$, we may pose the similar problems. For example, which positive integers set can become $Ch_3(G)$ or $Ch_4(G)$ for some groups $G$? Also, we may define the corresponding graphs. In $[26]$, M.L. Lewis presented an integral overview. The following results are just related with the author's joint works.

**Theorem 7** (a) ($[23]$) If $G$ is a finite group with $|cs(G)| = 2$, then (1) $G = P \times A$ with $P$ a $p$-group and $N$ Abelian. (2) $P/Z(P)$ has exponent $p$. Also, if $dl(P) \leq 2$, then $c(P) \leq 3$.

(b) ($[2]$) Let $G$ be a finite group. Then $|cd(G)| = 2$ if and only if one of the following is true: (1) $G$ possesses a normal and Abelian subgroup $N$ with $|G : N| = q$, $q$ is a prime. (2) $G/Z(G)$ is a Frobenius group with kernel $(G/Z(G))/Z(G)$ and a cyclic complement. (3) $G = P \times A$, where $A$ is an abelian and $P$ is a $p$-group with $|cd(P)| = 2$.

For the case (a) it is proved that the nilpotent class of $G$ is at most 3 (see $[19]$).

**Theorem 8** (a) ($[21, 22, 23]$) If $|cs(G)| \leq 3$, then $G$ is solvable; If $G$ is simple with $|cs(G)| = 4$, then $G \cong PSL(2, 2^m)$ (and conversely); If $G$ is simple with $|cs(G)| = 5$, then $G \cong PSL(2, q)$, where $q$ is an odd prime power greater than 5 (and conversely).

(b) ($[18]$) If $|cd(G)| = k \leq 3$, then $G$ is solvable. If $G$ is solvable and $|cd(G)| = k \leq 4$, then $dl(G) \leq k$. 


If \( \text{cd}(G) = \{1, m, n, mn\} \), then \( G \) is solvable and one of the following is true: (1) \( dl(G) \leq 3 \). (2) \( \text{cd}(G) = \{1, 3, 13, 39\} \). (3) \( \text{cd}(G) = \{1, p^a, p^b, p^{a+b}\} \).

In [27], the authors classified nonsolvable groups with four irreducible character degrees. Furthermore, it is proved that if \( \text{cs}(G) = \{1, m, n, mn\} \) and \((m, n) = 1\) then \( G \) is solvable (see [5]).

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