Mass-Energy-Momentum Radiation in Stueckelberg-Horwitz-Piron (SHP) Electrodynamics

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Abstract. In Stueckelberg-Horwitz-Piron (SHP) electrodynamics individual particles are not restricted to fixed mass shells. Particles and fields may exchange mass along with energy-momentum, while total mass-energy-momentum is conserved for the system. By extension, interacting particles may exchange mass through the fields, and much as a radio transmitter produces electromagnetic waves that carry power to a remote radio receiver, the corresponding SHP apparatus can, in principle, transport mass across spacetime. The conserved Noether current contains energy density and energy flux density into space (Poynting 3-vector) as in Maxwell theory, as well as two additional features of the electromagnetic field: the mass density and mass flux density into spacetime (Poynting 4-vector). In this paper we consider the radiation from a simple dipole antenna in SHP, and use the mass-energy-momentum tensor to compare the results with the standard treatment in Maxwell theory. We find that mass radiation can only take place when the net charge on the entire antenna oscillates along with the current. We show that much as an ideal amplifier drives current into the antenna as the source of the radiated energy, the oscillating net charge drives spacetime events into the antenna as the source of the radiated mass. This process illuminates the event-oriented nature of mass dynamics in SHP theory.

1. Introduction
As a classical field theory, Maxwell electrodynamics describes particles, which interact relativistically through U(1) gauge fields, and fields, which are induced by currents associated with particle worldlines in spacetime. In a slightly different manner, Stueckelberg-Horwitz-Piron (SHP) electrodynamics [1–6] describes individual spacetime events interacting relativistically through U(1) gauge fields, and the fields induced by currents associated with the construction of particle worldlines by events evolving through spacetime. As a canonical evolution theory, events in 4D spacetime are defined on an unconstrained 8D phase space

\[ (x^\mu(\tau), \dot{x}^\mu(\tau)) \quad \text{where} \quad \dot{x}^\mu = \frac{dx^\mu}{d\tau} \quad \text{and} \quad \mu, \nu = 0, 1, 2, 3 \quad (1) \]

such that each worldline is traced out as the coordinate-independent evolution parameter \( \tau \) advances monotonically. The absence of constraints among the four-velocities means that the system is not generally reparametrization invariant and so particle trajectories evolving dynamically are not \textit{a priori} restricted to fixed mass shells. In particular, events and gauge fields may exchange mass, although the total mass of the particles and fields is conserved. A self-interaction that tends to restore particle masses to their on-shell values has been described [7] and Horwitz [8] has shown that a particle constructed as a statistical ensemble of events will...
return to its equilibrium mass following collisions. Still, preliminary work indicates that a Gaussian wavepacket representing a free particle in SHP quantum mechanics may radiate mass, so considerable work remains in understanding these mass dynamics and the analytical techniques used to describe them.

In this paper we explore mass radiation by generalizing to SHP the standard treatment of the power radiated from a simple dipole antenna. Just as the Maxwell antenna radiates energy provided by the amplifier driving the oscillating current density, we find that the SHP antenna will generally radiate both mass and energy provided by the amplifier. However, when the oscillating current density is such that the total charge on the antenna remains fixed, mass radiation vanishes and power radiation reduces to the standard Maxwell expression. This result will be shown to illustrate the relationship between mass radiation and event creation.

Section 2 presents a brief overview of Stueckelberg-Horwitz-Piron electrodynamics, including the field equations and mass-energy-momentum tensor. In section 3 we briefly discuss some preliminary examples that are useful in understanding the mass-energy-momentum tensor: sourceless plane wave solutions of the pre-Maxwell field equations, the absence of radiation from a constant current in Maxwell theory, and the corresponding result in pre-Maxwell theory. Section 4 provides a brief review of the standard treatment in Maxwell theory of energy radiation from a simple dipole antenna. In section 5 we generalize this treatment to SHP theory and in section 6 we discuss the results.

2. Stueckelberg-Horwitz-Piron (SHP) electrodynamics

SHP may be seen as formalizing the distinction between two aspects of time, the chronological world time \( \tau \) that determines monotonic ordering of events, and the coordinate time \( x^0 \) that locates each event on the laboratory clock \([9]\). Given this distinction, the theory is obtained by generalizing nonrelativistic mechanics to relativistic form through the substitution of \( \tau \) for the Newtonian time \( t \), and Poincaré symmetry for Galilean symmetry. The free particle Lagrangian is then

\[
L = \frac{1}{2} M \dot{x}^\mu \dot{x}_\mu \implies M \ddot{x}^\mu = 0 \quad \text{so that defining} \quad p^\mu = \partial L / \partial \dot{x}_\mu \quad \text{leads to} \quad E = p^0 c = Mc^2 (dt/d\tau).
\]

This aspect of the formalism permits a unified description of particles (positive energy, future directed events with \( \dot{x}^0 > 0 \)) and antiparticles (negative energy, past directed events with \( \dot{x}^0 < 0 \)), with smoothly evolving pair processes possible under appropriate interactions in classical mechanics.

In analogy to \( x^0 = ct \), we introduce a notation for “fifth” scalar quantities

\[ x^5 = c_5 \tau \quad \dot{x}^5 = c_5 \quad \partial_5 = \frac{1}{c_5} \frac{\partial}{\partial \tau} \quad \alpha, \beta, \gamma, \delta, \epsilon = 0, 1, 2, 3, 5 \]

\[ \lambda, \mu, \nu, \rho = 0, 1, 2, 3 \]

(2)

and note that the action

\[
S = \int d\tau \left\{ \frac{1}{2} M \dot{x}^\mu \dot{x}_\mu + \frac{e}{c} \left[ \dot{x}^\mu a_\mu(x, \tau) + c_5 a_5(x, \tau) \right] \right\}
\]

\[
= \int d\tau \left\{ \frac{1}{2} M \dot{x}^\mu \dot{x}_\mu + \frac{e}{c} \dot{x}^5 a_5(x, \tau) \right\}
\]

(3)

is invariant under the 5D gauge transformation

\[
a_\alpha(x, \tau) \rightarrow a_\alpha'(x, \tau) = a_\alpha(x, \tau) + \partial_\alpha \Lambda(x, \tau).
\]

(4)

We rewrite the interaction term as

\[
\dot{x}^\alpha a_\alpha(x, \tau) \rightarrow \frac{1}{c} \int d^4 x \, j^\alpha(x, \tau) a_\alpha(x, \tau)
\]

(5)
where the event current
\[ j^\alpha(x, \tau) = c \dot{x}^\alpha(\tau) \delta^4(x - x(\tau)) \]  
(6)
is defined at each \( \tau \), with support restricted to the spacetime location of the event at \( x = x(\tau) \). This current is related to the standard divergenceless Maxwell current by
\[ J^\mu(x) = \int d\tau \, j^\mu(x, \tau) = c \int d\tau \, \dot{x}^\mu(\tau) \delta^4(x - x(\tau)), \]
(7)
an integration called concatenation [10], which can be seen as the sum at \( x \) of all contributions to the current by the events that trace out the worldline represented by \( J^\mu(x) \). Although the interaction term (5) exhibits a formal 5D symmetry — possibly \( O(3,2) \) or \( O(4,1) \), which contain \( O(3,1) \) as a subgroup — the physical symmetry is understood to be a vector+scalar representation of \( O(3,1) \).

We specify the dynamical action with electromagnetism [5, 11] as
\[ S = \int d\tau \left[ \frac{1}{2} M \dot{x}^\mu \dot{x}_\mu + \int d^4x \left\{ \frac{e}{c^2} j^\alpha(x, \tau) a_\alpha(x, \tau) - \frac{1}{4c} \int ds \, \Phi(\tau - s) f_{a\beta}(x, s) \right\} \right] \]
(8)
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(8)
where we have defined the field intensities
\[ f_{a\beta} = \partial_a a_\beta(x, \tau) - \partial_\beta a_a(x, \tau) \]
(9)
which are invariant under the 5D gauge transformation (4). Expanding the expression
\[ f^{a\beta} a_{a\beta} = f^{\mu\nu} f_{\mu\nu} + 2 f^{5\mu} f_{5\mu} = f^{\mu
u} f_{\mu
u} + 2 g^{55} f^{\mu}_5 f_{5\mu} \]
(10)
we may avoid interpreting \( g^{55} \) as an element of a 5D metric, and rather see its role as equivalent to the choice of sign for the vector contribution \( f_5^\mu f_5^\mu \) to the field energy. The field interaction kernel is
\[ \Phi(\tau) = \delta(\tau) - (\xi \lambda)^2 \delta''(\tau) = \int \frac{d\kappa}{2\pi} \left[ 1 + \left( \frac{\xi \lambda \kappa}{c} \right)^2 \right] e^{-i\kappa \tau} \]
(11)
where \( \lambda \) is a parameter with dimensions of time and
\[ \xi = \frac{1}{2} \left[ 1 + \left( \frac{c_5}{c} \right)^2 \right] \]
(12)
is chosen so that the low energy Lorentz force agrees with Coulomb’s law. Writing
\[ \varphi(\tau) = \lambda \int \frac{d\kappa}{2\pi} \frac{e^{-i\kappa \tau}}{1 + (\xi \lambda \kappa)^2} = \frac{1}{2\xi} e^{-|\tau|/\xi \lambda} \]
(13)
we see that
\[ \int \frac{d\tau}{\lambda} \varphi(\tau) = 1 \]
(14)
permitting the convolution in (8) to be inverted. Integrating by parts the nonlocal term \( \delta''(\tau) \) in (5), puts the kinetic part for the fields into the form
\[ \int d\tau \, ds \, f_{a\beta}(x, \tau) \Phi(\tau - s) f_{a\beta}(x, s) = \int d\tau \, f^{a\beta}(x, \tau) f_{a\beta}(x, \tau) \]
\[ + \int d\tau \left( \xi \lambda \right)^2 \partial_\tau f^{a\beta}(x, \tau) \partial_\tau f_{a\beta}(x, \tau) \]
(15)
which breaks any 5D symmetry \((x, \tau) \rightarrow (x', \tau')\) to \(O(3,1)\).

Varying the action with respect to \(\dot{x}^\mu\) leads to the Lorentz force \([12]\)

\[
M \ddot{x}^\mu (\tau) = \frac{e}{c} f^\mu_a (x, \tau) x^a (\tau) = \frac{e}{c} \left[ f^\mu_\nu (x, \tau) x^\nu (\tau) + c_5 f_{55}^\mu (x, \tau) \right]
\]

\[
\frac{d}{d\tau} (-\frac{1}{2} M \dot{x}^2) = -M \dot{x}^\mu \dot{x}_\mu = \frac{e c_5}{c} f_{55} \dot{x}^\mu
\]

(16)

where (17) exhibits the possibility of mass exchange between events and fields. Varying the action with respect to \(a_\alpha (x, \tau)\) produces the field equations \([4, 11]\)

\[
\partial_\beta f^{\alpha \beta} (x, \tau) = \frac{e}{c} \dot{f}^\alpha (x, \tau) \quad \partial_\alpha f_{\beta \gamma} + \partial_\gamma f_{\alpha \beta} + \partial_\beta f_{\alpha \gamma} = 0
\]

(18)

whose source

\[
f_\beta^\alpha (x, s) = \int ds \, \varphi (\tau - s) \, f^\alpha (x, s)
\]

(19)

where we used (14) to replace \(\Phi (\tau)\) in the field term on LHS with \(\varphi (\tau)\) in the current term on the RHS. These field equations are formally similar to Maxwell’s equations in 5D, and are called pre-Maxwell equations. Rewriting the field equations in 4-vector and scalar components, they take the form

\[
\partial_\mu f^{5\mu} = \frac{e}{c} F^\mu_\varphi = \frac{e c_5}{c} \rho_\varphi \quad \partial_\mu f_{\varphi \nu} + \partial_\nu f_{\mu \varphi} + \partial_\varphi f_{\mu \nu} = 0
\]

(20)

\[
\partial_\nu f^{\mu \nu} - \frac{1}{c^5} \frac{\partial}{\partial \tau} f^{5\mu} = \frac{e}{c} f^{\mu}_\varphi \quad \partial_\nu f_{\mu \nu} - \partial_\mu f_{5\nu} + \frac{1}{c^5} \frac{\partial}{\partial \tau} f_{\mu \nu} = 0
\]

(21)

which may be compared with the 3-vector form of Maxwell equations

\[
\nabla \cdot \mathbf{E} = \frac{e}{c} j^0 = \varrho \quad \nabla \cdot \mathbf{B} = 0
\]

(22)

\[
\nabla \times \mathbf{B} - \frac{1}{c} \frac{\partial}{\partial t} \mathbf{E} = \frac{e}{c} \mathbf{J} \quad \nabla \times \mathbf{E} + \frac{1}{c} \frac{\partial}{\partial t} \mathbf{B} = 0
\]

(23)

with \(f^{5\mu}\) playing the role of the vector electric field and \(f^{\mu \nu}\) playing the role of the magnetic field. For our purposes it is convenient to have the 3-vector and scalar forms

\[
(\mathbf{e})^i = f^{0i} \quad (\mathbf{b})_i = \varepsilon_{ijk} f^{jk} \quad (\mathbf{e})^i = f^{5i} \quad e^0 = f^{50}
\]

(24)

for which the field equations split into four generalizations of the 3-vector Maxwell equations

\[
\nabla \cdot \mathbf{e} - \frac{1}{c^5} \frac{\partial}{\partial \tau} \epsilon^0 = \frac{e}{c} f^0_\varphi = \varrho_\varphi \quad \nabla \cdot \mathbf{b} = 0
\]

(25)

\[
\nabla \times \mathbf{b} - \frac{1}{c} \frac{\partial}{\partial t} \mathbf{e} - \frac{1}{c^5} \frac{\partial}{\partial \tau} \mathbf{e} = \frac{e}{c} f^\varphi_\mathbf{e} \quad \nabla \times \mathbf{e} + \frac{1}{c} \frac{\partial}{\partial t} \mathbf{b} = 0
\]

and three new equations for the fields \(\mathbf{e}\) and \(e^0\)

\[
\nabla \cdot \mathbf{e} + \frac{1}{c} \frac{\partial}{\partial t} e^0 = \frac{e}{c} \dot{f}^5_\mathbf{e} = \frac{e c_5}{c} \rho_\varphi \quad \nabla \cdot \mathbf{e} - s^5 s^5 \frac{1}{c^5} \frac{\partial}{\partial \tau} \mathbf{e} = 0
\]

\[
\nabla e^0 + \frac{1}{c} \frac{\partial}{\partial t} e^0 + s^5 s^5 \frac{1}{c^5} \frac{\partial}{\partial \tau} e^0 = 0
\]

(26)
The field equations (18) can be combined to find the wave equation
\[
\left( \partial^\mu \partial_\mu + g^{55} \frac{1}{c^2} \partial^2 \right) f^{\alpha\beta} = \partial^\alpha j^\beta - \partial^\beta j^\alpha
\] (27)
for which the principal part Green’s function \[13\]
\[
G_P(x, \tau) = -\frac{1}{2\pi} \delta(x^2) \delta(\tau) - \frac{c_5}{2\pi^2} \partial x^2 \partial x^2 \theta(-g^{55} g_{\alpha\beta} x^\alpha x^\beta) \frac{1}{\sqrt{-g^{55} g_{\alpha\beta} x^\alpha x^\beta}}
\] (28)
includes a Maxwell term with support on the lightcone at simultaneous \(\tau\), and a term with timelike or spacelike support, depending on \(g^{55}\). In most cases, field contributions from the second term in (28) can be ignored in the far field compared to contributions from the first term. It follows from (27) that this approximation is equivalent to dropping terms on the order \(\delta''(\tau)/c^2\) and this exclusion should be maintained consistently.

The connection between SHP electrodynamics and standard Maxwell theory can be established through the concatenation integral defined for the current in \[7\]. Under the equilibrium boundary conditions
\[
\rho_\phi(x, \tau) \xrightarrow{\tau \to \pm\infty} 0 \quad f^{5\mu}(x, \tau) \xrightarrow{\tau \to \pm\infty} 0
\] (29)
iintegration over worldline permits the identification
\[
\begin{align*}
\partial_v f^{\mu\nu} - \frac{1}{c^5} \frac{\partial}{\partial \tau} f^{5\nu} &= \frac{e}{c} f^{\mu\nu} \\
\partial_\mu f_{\nu\rho} + \partial_v f_{\rho\mu} + \partial_\rho f_{\mu\nu} &= 0 \\
\partial_\alpha j^\alpha &= 0
\end{align*}
\] (30)
where
\[
A^\mu(x) = \int \frac{d\tau}{\lambda} a^\mu(x, \tau) \quad F^{\mu\nu}(x) = \int \frac{d\tau}{\lambda} f^{\mu\nu}(x, \tau) \quad j^\mu(x) = \int \frac{d\tau}{\lambda} j^\mu(x, \tau).
\] (31)
Similarly, concatenation of \[28\] produces
\[
\int d\tau G_P(x, \tau) = -\frac{1}{2\pi} \delta(x^2)
\] (32)
recovering the Maxwell Green’s function.

The translation invariance of the action leads to a conserved Noether current \[4\] in the form
\[
\partial_\alpha T_{\phi}^{\alpha\beta} = -\frac{e}{c^2} f^{\beta\alpha} j_\alpha \quad T_{\phi}^{\alpha\beta} = \frac{1}{c} \left( f_{\phi}^{\alpha\gamma} f^{\beta\gamma} - \frac{1}{4} f_{\phi}^{\beta\gamma} f^{\alpha\gamma} g_{\alpha\beta} \right)
\] (33)
where \(T_{\phi}^{\alpha\beta}\) is interpreted as the mass-energy-momentum tensor and
\[
f_{\phi}^{\alpha\beta}(x, \tau) = \int \frac{ds}{\lambda} \Phi(\tau - s) f^{\alpha\beta}(x, s)
\] (34)
is the convolved field strength appearing in the action (8). In terms of the 3-vector fields, the mass-energy-momentum tensor components are

\[ T_{00}^{\Phi} = \frac{1}{2c} \left[ \mathbf{e} \cdot \mathbf{e}_\Phi + \mathbf{b} \cdot \mathbf{b}_\Phi + g_{55} (\mathbf{e} \cdot \mathbf{e}_\Phi + e^0_e^0_\Phi) \right] \]

\[ T_{0i}^{\Phi} = \frac{1}{c} \left( \mathbf{e} \times \mathbf{b}_\Phi + g_{55} e^0_\Phi \right)^i \]

\[ T_{50}^{\Phi} = \frac{1}{c} \mathbf{e} \cdot \mathbf{e}_\Phi \]

\[ T_{5i}^{\Phi} = \frac{1}{c} \left( \mathbf{e} \times \mathbf{b}_\Phi + c^0_\Phi \right)^i \]

\[ T_{55}^{\Phi} = \frac{1}{2c} \left[ \mathbf{e} \cdot \mathbf{e}_\Phi - e^0_\Phi + g_{55} (\mathbf{e} \cdot \mathbf{e}_\Phi - \mathbf{b} \cdot \mathbf{b}_\Phi) \right]. \]

Integrating the \( \beta = 5 \) component of the conservation law (33) over spacetime and using (6) for the event current leads to

\[ \frac{1}{c_5} \frac{d}{d\tau} \int d^4x \ T_{55}^{\Phi} = -\frac{e}{c} f^{5\mu}(x(\tau), \tau) \dot{x}_\mu(\tau) \]

for the event trajectory \( x(\tau) \). Using the Lorentz force (17) on the RHS this becomes

\[ \frac{d}{d\tau} \left[ \int d^4x \ T_{55}^{\Phi} + g_{55} \left( -\frac{1}{2} M x^2 \right) \right] = 0 \]

expressing conservation of the combined mass of particles and fields [12], with \( T_{55}^{\Phi} \) interpreted as the spacetime density of mass associated with the fields, in analogy to \( T_{00}^{\Phi} \) as energy density. It is precisely this interpretation that will be studied in the following sections.

3. Preliminary examples

Before studying the classical antenna we examine the interpretation of \( T_{\alpha\beta}^{\Phi} \) by considering plane waves and the fields from a uniform current.

3.1. Sourceless fields

From the wave equation (27) for \( j^\alpha(x, \tau) = 0 \) we may write the field in terms of the Fourier transform [12]

\[ f^{\alpha\beta}(x, \tau) = \frac{1}{(2\pi)^5} \int d^5k \ e^{ik \cdot x} f^{\alpha\beta}(k) = \frac{1}{(2\pi)^5} \int d^4k \ dk \ e^{i(k \cdot x - k_0x^0 + g_{55}5k \tau)} f^{\alpha\beta}(k, \kappa) \]

where

\[ \kappa = k^5 = g_{55}k_5 \]

is understood to represent the mass carried by the plane wave, much as \( k^0 \) and \( \mathbf{k} \) represent energy and 3-momentum. This interpretation is supported by the wave equation which imposes the 5D constraint

\[ k^\alpha k_\alpha = k^2 - (k^0)^2 + g_{55} \kappa^2 = 0 \quad \Rightarrow \quad g_{55} \kappa^2 = (k^0)^2 - \mathbf{k}^2 \]

expressing \( \kappa \) in terms of the difference between energy and momentum. Under concatenation, the field becomes

\[ F^{\alpha\beta}(x) = \int \frac{d\tau}{\lambda} f^{\alpha\beta}(x, \tau) = \int \frac{d^4k}{(2\pi)^4} e^{ik \cdot x^\nu} \left( \frac{1}{\lambda c_5} f^{\alpha\beta}(k, 0) \right) = \int \frac{d^4k}{(2\pi)^4} e^{ik \cdot x^\nu} F^{\alpha\beta}(k) \]
and recovers the 4D mass-shell constraint \( k^\mu k_\mu = 0 \) for the Maxwell field. In the transform domain, the sourceless pre-Maxwell equations take the form

\[
\begin{align*}
\mathbf{k} \cdot \mathbf{e} - g_{55} \kappa e^0 &= 0 \\
\mathbf{k} \cdot \mathbf{b} &= 0 \\
\mathbf{k} \cdot \mathbf{e} - k^0 e^0 &= 0 \\
\mathbf{k} \times \mathbf{e} - k^0 \mathbf{b} &= 0 \\
\mathbf{k} \times \mathbf{b} &= 0 \\
\mathbf{e} - \kappa \mathbf{e} + k^0 \mathbf{e} - k^0 \mathbf{e}^0 &= 0
\end{align*}
\]

which can be solved by taking \( \mathbf{e}_\parallel \) and \( \mathbf{e}_\perp \) as independent 3-vector polarizations, and writing

\[
\begin{align*}
\mathbf{e}_\parallel = g_{55} \frac{\kappa}{k^0} \mathbf{e}_\parallel \\
\mathbf{e}_\perp = \frac{\kappa}{k^0} \mathbf{e}_\perp \\
e^0 = \frac{1}{k^0} \mathbf{k} \cdot \mathbf{e}_\parallel \\
\mathbf{b} = \frac{1}{k^0} \mathbf{k} \times \mathbf{e}_\perp
\end{align*}
\]

for the remaining fields. Unlike Maxwell plane waves, for which \( \mathbf{E}, \mathbf{B}, \) and \( \mathbf{k} \) are mutually orthogonal, the pre-Maxwell electric fields \( \mathbf{e} \) and \( \mathbf{e}_\perp \) have both transverse and longitudinal components. When \( \kappa \to 0 \), we find that \( \mathbf{e}, \mathbf{b}, \) and \( \mathbf{k} \) become mutually orthogonal and \( \mathbf{e}_\perp \) becomes a decoupled longitudinal polarization parallel to \( \mathbf{k} \).

We use (11) to write the convolved field as

\[
f_{\Phi}^{\alpha \beta}(x, \tau) = \int \frac{ds}{\lambda} \Phi(\tau - s) f_{\Phi}^{\alpha \beta}(x, s) = \frac{1}{(2\pi)^5} \int d^4 k \int d^4 \kappa \mathcal{E}^{i(k \cdot x - k^0 x^0 + g_{55} \kappa \kappa \tau)} f_{\Phi}^{\alpha \beta}(k, \kappa)
\]

where

\[
f_{\Phi}^{\alpha \beta}(k, \kappa) = \frac{1 + (\xi \lambda c_5 \kappa)^2}{\lambda} f_{\Phi}^{\alpha \beta}(k, \kappa)
\]

introduces a multiplicative factor that will appear once in each field bilinear of \( T_{\Phi}^{\alpha \beta} \). Using (35), the energy density is

\[
T_{\Phi}^{00} = \frac{1}{2c} \left[ \mathbf{e} \cdot \mathbf{e}_\Phi + \mathbf{b} \cdot \mathbf{b}_\Phi + g_{55} (\mathbf{e} \cdot \mathbf{e}_\Phi + e^0 \mathbf{e}_\Phi) \right] = \frac{1}{c} \left( e^2_\perp + g_{55} e^2_\parallel \right) \frac{1 + (\xi \lambda \kappa)^2}{\lambda}
\]

which, since \( e^2_\perp = \frac{1}{2} (e^2_\perp + b^2) \), is equivalent to the energy density in Maxwell theory

\[
\rho^{00} = \frac{1}{2c} (E^2 + B^2)
\]

with the addition of the independent polarization \( e_\parallel \). The mass density is found to be

\[
T_{\Phi}^{55} = \frac{\kappa^2}{ck^0} \left( e^2_\perp + g_{55} e^2_\parallel \right) \frac{1 + (\xi \lambda \kappa)^2}{\lambda} = \frac{\kappa^2}{k^0} T_{\Phi}^{00}
\]

expressing energy density scaled by the squared mass-to-energy ratio for the field. The energy flux — the standard Poynting 3-vector — is

\[
T_{\Phi}^{0} \rightarrow T_{\Phi}^{0} = \frac{\mathbf{k}}{k^0} T_{\Phi}^{00}
\]

expressing the energy density \( T_{\Phi}^{00} \) flowing uniformly in the direction of the momentum normalized to energy. We note that for a free particle

\[
\frac{\mathbf{k}}{k^0} = \frac{\mathbf{p}}{E/c} = \frac{1}{c} \frac{M \, dx/d\tau}{M \, dt/d\tau} = \frac{v}{c}
\]
which will not generally be a unit vector unless $\kappa = 0$, as it must be for Maxwell plane waves. The mass flux vector — a second Poynting 3-vector — can be written

$$T^{5i}_{\Phi} \rightarrow T^5_{\Phi} = \frac{k_i}{k} T^{55}_{\Phi}$$

expressing the mass density $T^{55}_{\Phi}$ flowing uniformly in the direction of the momentum normalized to mass. Finally,

$$T^{50}_{\Phi} = \frac{k^0}{k} T^{55}_{\Phi} = \frac{k}{k^0} T^{00}_{\Phi}$$

so that $T^{5\mu}_{\Phi}$ can be written as

$$T^{5\mu}_{\Phi} = \frac{k^\mu}{k} T^{55}_{\Phi} = \frac{k k^\mu}{k^2} T^{00}_{\Phi}$$

expressing the mass density $T^{55}_{\Phi}$ flowing in the direction of the normalized 4-momentum. In this sense, $T^{50}_{\Phi}$ represents the flow of mass into the time direction. We notice that when $k \rightarrow 0$, as is the case for Maxwell plane waves, $k/k^0$ becomes a unit vector and $T^{5\mu}_{\Phi} = 0$, so that mass density and flow vanish. The interpretation of plane waves carrying energy and momentum (energy flux) uniformly to infinity is thus seen to generalize to mass flow, where mass is best understood through $[10]$ as the non-identity of energy and momentum.

Suppose that a plane wave of this type impinges on a test particle in its rest frame, described by $\mathbf{x}^a(\tau) = (c\tau, 0, c_5 \tau)$. Since $\dot{x} = 0$, the wave will interact with the event through the Lorentz force $[16]$ and $[17]$ as

$$M \ddot{x}^\mu (\tau) = \frac{e}{c} \left[ \mathcal{f}^\mu_0(x, \tau) \dot{x}^0 (\tau) + c_5 \mathcal{f}^\mu_5(x, \tau) \right] \quad \frac{d}{d\tau} \left( -\frac{1}{2} M \dot{x}^2 \right) = -\frac{e c_5}{c} g_{55} \dot{e}^0 \dot{x}^0$$

which for $e_\parallel \neq 0 \Rightarrow k \cdot e_\parallel \neq 0$ becomes

$$\ddot{t} = -g_{55} e \frac{c_5}{M c^2 k^0} \mathbf{k} \cdot e_\parallel \quad \ddot{x} = \frac{e}{M} \left[ e_\perp \left( 1 + \frac{c_5}{c} \frac{k}{k^0} \right) + e_\parallel \left( \frac{c_5}{c} + g_{55} \frac{k}{k^0} \right) \right]$$

$$\frac{d}{d\tau} \left( -\frac{1}{2} M \dot{x}^2 \right) = -g_{55} e c_5 \frac{1}{k^0} \mathbf{k} \cdot e_\parallel$$

showing that the incident plane wave will initially accelerate the test event in such a way as to transfer mass. If the plane wave is a far field approximation to the radiation field of an accelerating charge, then the resulting picture describes the transfer of mass by the radiation field between charged events. As the test event accelerates, the equations of motion become more complex

$$\ddot{t} = \frac{e}{M c^2} \left[ \left( g_{55} \frac{k}{k^0} e_\parallel + e_\perp \right) \cdot \dot{x} - g_{55} c_5 \frac{1}{k^0} \mathbf{k} \cdot e_\parallel \right]$$

$$\ddot{x} = \frac{e}{M} \left[ e_\perp \left( \dot{t} + \frac{c_5}{c} \frac{k}{k^0} \right) + \frac{1}{k^0 c} \left[ (\dot{x} \cdot e_\perp) \mathbf{k} - (\dot{x} \cdot \mathbf{k}) e_\perp \right] + e_\parallel \left( \frac{c_5}{c} + g_{55} \frac{k}{k^0} \right) \right]$$

$$\frac{d}{d\tau} \left( -\frac{1}{2} M \dot{x}^2 \right) = -e c_5 g_{55} \left[ \frac{1}{k^0} \mathbf{k} \cdot e_\parallel \dot{t} + \frac{1}{c} \left( e_\parallel + \frac{k}{k^0} e_\perp \right) \cdot \dot{x} \right]$$

and depend on the relative orientations of the fields $e_\perp$ and $e_\parallel$, and the phase

$$\exp\{i(\mathbf{k} \cdot \mathbf{x} - k_0 \dot{x}^0 + g_{55} c_5 \kappa \tau)\}.$$
3.2. Energy flux from a uniformly moving charge in Maxwell electrodynamics

In the rest frame of a Maxwell charge, the potentials are

\[ A^0(x) = \frac{e}{4\pi R} \quad A = 0 \]  

(61)

leading to fields

\[ E(x) = -\nabla A^0 = \frac{e}{4\pi R^2} \hat{R} \quad B(x) = \nabla \times A = 0 \]  

(62)

so the energy density and Poynting vector are

\[ T^{00} = \frac{1}{2} (E^2 + B^2) = \frac{1}{2} \left( \frac{e}{4\pi R^2} \right)^2 \quad T = E \times B = 0. \]  

(63)

The vanishing Poynting vector is understood to mean that a static charge does not transfer energy, and in particular does not radiate. To sharpen this point, we step outside the rest frame to consider a uniformly moving charge along the \( \hat{x} \)-axis with \( \tau \)-dependent position

\[ r = u\tau = (\gamma ct, \gamma \beta ct, 0, 0) = \gamma ct \hat{t} + \gamma \beta ct \hat{x} \]  

(64)

where

\[ \gamma = \frac{1}{\sqrt{1 - \beta^2}} \]  

(65)

as usual. From an observation point

\[ x = (0, a, b, 0) = a\hat{x} + b\hat{y} \]  

(66)

the field strengths are easily found to be

\[ E = -e\gamma \frac{a\hat{x} + b\hat{y}}{4\pi (a^2\gamma^2 + b^2)^{3/2}} \quad B = -e\gamma \beta \frac{b \hat{x} \times \hat{y}}{4\pi (a^2\gamma^2 + b^2)^{3/2}} \]  

(67)

so that the nonzero Poynting vector

\[ T = E \times B = \frac{(e\gamma)^2 b\beta}{(4\pi)^2 (a^2\gamma^2 + b^2)^{3/2}} (b\hat{x} - a\hat{y}) \]  

(68)

represents energy transfer along the path of the moving charge. To see that the uniformly moving charge does not radiate — transfer energy that is lost to the charge — we enclose the path along the \( \hat{x} \)-axis in an infinitely long cylinder of radius \( \rho \) with surface area element

\[ dS = \rho (\cos \phi \hat{y} + \sin \phi \hat{z}) d\phi \, da. \]  

(69)

Summing the total energy flux through the cylinder leads to

\[ \int (E \times B) \cdot dS = -r \int_{0}^{2\pi} \cos \phi \, d\phi \int_{-\infty}^{\infty} da \frac{ab (e\gamma)^2 \beta}{(4\pi)^2 (a^2\gamma^2 + b^2)^{3/2}} = 0 \]  

(70)

showing that although the field varies in time at any fixed observation point, with nonzero Poynting vector representing energy flux at that point, the symmetry (odd parity) of the field configuration (both radially and along the direction of the current) means that no net energy is radiated away by the uniformly moving charge.
3.3. Energy flux from a “static” charge in SHP electrodynamics

In SHP theory, an event in its rest frame is in uniform motion in the time direction, and so the position is given by the 4-vector and scalar

\[ X(\tau) = (ct, 0) \quad X^5(\tau) = c_5 \tau \]  

(71)

with current components are

\[ j_\phi(x, \tau) = (c \delta^3(x) \phi(t - \tau), 0) \quad j^5(x, \tau) = \frac{c_5}{c} j^5(x, \tau). \]  

(72)

Using the leading term of Green’s function (28) leads to the potential

\[ a^0(x, \tau) = \frac{e}{4\pi R^2} (\phi(\tau - \tau_R) - \frac{R}{c} \phi'(\tau - \tau_R)) \hat{R} \quad a(x, \tau) = 0 \quad a^5(x, \tau) = \frac{c_5}{c} a^0(x, \tau) \]  

(73)

where \( \tau_R = t - R/c \) is the retarded time at an observation point \((ct, x)\) and \( R = |x| \). The field strengths are easily found to be

\[ e = -\nabla a^0 = \frac{e}{4\pi R^2} \left( \phi(\tau - \tau_R) - \frac{R}{c} \phi'(\tau - \tau_R) \right) \hat{R} \quad b = \nabla \times a = 0 \]  

\[ e^0 = g^{55} \frac{1}{c_5} \frac{\partial}{\partial \tau} a^0 + \frac{1}{c} \frac{\partial}{\partial t} a^5 = -\frac{e}{4\pi R} \left( g^{55} - \frac{c_5^2}{c^2} \right) \frac{\phi'}{c_5} \quad e = -\nabla a^5 = \frac{c_5}{c} e. \]  

(74)

Some care is required in treating the convolved fields \( f^{\alpha\beta}_\Phi \). The factor

\[ \phi(\tau - \tau_R) - \frac{R}{c} \phi'(\tau - \tau_R) \]  

(75)

contained in \( e \) will be replaced under convolution with \( \Phi(\tau) \) by

\[ \delta(\tau - \tau_R) - \frac{R}{c} \delta'(\tau - \tau_R) \]  

(76)

so that the bilinear \( e \cdot e_\phi \) contains the product

\[ \left[ \phi(\tau - \tau_R) - (R/c) \phi'(\tau - \tau_R) \right] \left[ \delta(\tau - \tau_R) - (R/c) \delta'(\tau - \tau_R) \right] = \frac{1}{2c_5} \delta(\tau - \tau_R) \]  

(77)

where we use

\[ \phi(\tau) = \frac{1}{2c_5} e^{-|\tau|/\xi \lambda} \quad \Rightarrow \quad \phi(0) = \frac{1}{2c_5}, \quad \phi'(0) = 0 \]  

(78)

and discard terms containing \( \phi''(\tau) \) under the approximation discussed in connection with the Green’s function (28). The remaining \( \delta \)-function in (77) can be handled by integrating over \( x^0 \), so that the energy and mass densities are found to be

\[ \int dx^0 T^{00} = \left( 1 + \frac{c_5^2}{c^2 g^{55}} \right) \frac{1}{2} \left( \frac{e}{4\pi R^2} \right)^2 \]  

(79)

\[ \int dx^0 T^{55} = g^{55} \left( 1 + \frac{c_5^2}{c^2 g^{55}} \right) \frac{1}{2} \left( \frac{e}{4\pi R^2} \right)^2. \]  

(80)
As in the Maxwell case, the Poynting 3-vectors contain cross products with $b = 0$, but also terms containing $\epsilon_0$, which vanishes because $\phi'(0) = 0$. Therefore

$$T^0 = \frac{1}{c} g_{55} e^0 e_\Phi = g_{55} \frac{c_5}{c} e^0 e_\Phi = g_{55} \frac{c_5}{c} T^5 = 0$$

(81)

so that the mass and energy flux into space vanishes and there is no radiation. Nevertheless, the mass flux into the time direction is

$$\int dx^0 T^{50} = c \left( \frac{e}{4\pi R^2} \right)^2$$

(82)

representing the flow of the event, and the corresponding mass density, from the past to the future, along the path of the event.

4. Radiation from a simple dipole antenna in Maxwell theory

The radiation from an oscillating current density is both a practical model in communications engineering and a textbook example of radiation from an accelerating charge. In order to compare the treatments of this problem in Maxwell theory and SHP, we begin with a brief review of the standard derivation [14] of radiation from a dipole antenna, which assumes a separable current density oscillating according to

$$J(x,t) = J(x) e^{i\omega t}$$

where the second expresses represents current conservation. As an aside, we examine possible considerations leading to separability. In a microscopic approach to the antenna problem, we would pose a collection of oscillating charges with position 4-vectors

$$X_\nu(\tau) = \left( ct_\nu(\tau) , a_\nu e^{i\omega \tau} \right)$$

(84)

for which we may assume nonrelativistic motion so that $t_\nu(\tau) = t = \tau$ for each particle. The Maxwell current for this collection is

$$J(x,t) = \sum_\nu \int d\tau c X_\nu(\tau) \delta^4(x - X_\nu(\tau))$$

$$= \sum_\nu \int d\tau i\omega c a_\nu e^{i\omega \tau} \delta (ct - c\tau) \delta^3(x - a_\nu e^{i\omega \tau})$$

$$= \left[ \sum_\nu i\omega a_\nu \delta^3(x - a_\nu e^{i\omega \tau}) \right] e^{i\omega t}.$$  

(85)

We now replace the term in square brackets with its time average over one cycle of oscillation $T = 2\pi / \omega$ and write

$$J(x,t) \simeq \left[ \frac{1}{T} \int_0^T dt \sum_\nu i\omega a_\nu \delta^3(x - a_\nu e^{i\omega t}) \right] e^{i\omega t} = J(x) e^{i\omega t}.$$  

(86)

Thus, $J(x)$ approximates the time-dependent current density by a time averaged static configuration in space, rendering the antenna problem tractable.
Using the current (83) and the 4D Green’s function $D(x) = -\delta \left( x^2 \right) / 2\pi$, the vector potential is

$$ A(x, t) = \frac{1}{4\pi} \int d^3x' dt' \frac{1}{|x - x'|} \delta \left( t - t' + \frac{|x - x'|}{c} \right) J(x') e^{i\omega t'} $$

$$ = \frac{1}{4\pi} \int d^3x' e^{-ik|x-x'|} \frac{1}{|x - x'|} J(x') e^{i\omega t} $$

$$ = A(x) e^{i\omega t} $$

(87)

where $k = \omega / c = 2\pi / \lambda$. Now, writing $x = r\hat{r}$, we make the far field approximation

$$ R = |x - x'| = \left( r^2 + (x')^2 - 2r\hat{r} \cdot x' \right)^{1/2} \simeq r - \hat{r} \cdot x' $$

(88)

and the dipole approximation

$$ |k\hat{r} \cdot x'| < kd = \frac{2\pi d}{\lambda} \ll 1 \Rightarrow e^{ik\hat{r} \cdot x'} \simeq 1$$

(90)

$$ \frac{r - \hat{r} \cdot x'}{c} \simeq \frac{r}{c} \left( 1 - \hat{r} \cdot \frac{d}{r} \right) \simeq \frac{r}{c} $$

(91)

leading to

$$ A(x) = \frac{1}{4\pi} \int d^3x' \frac{e^{-ik|x-x'|}}{|x - x'|} J(x') \simeq \frac{e^{-ikr}}{4\pi r} \int d^3x' J(x') $$

(92)

Integrating by parts and using current conservation, this becomes

$$ B = \nabla \times A = \frac{Id}{4\pi} \left( -ik \right) \frac{e^{-ikr}}{r} \left( 1 + \frac{1}{ikr} \right) \hat{r} \times \hat{d} \simeq \frac{\omega k e^{-ikr}}{4\pi r} \hat{r} \times \hat{p} $$

(93)

where we used $\lambda \ll r \Rightarrow 1/kr \ll 1$. The electric field follows from Maxwell’s equations as

$$ E = \frac{1}{i\omega} \nabla \times B = \frac{1}{4\pi} \left( k^2 e^{-ikr} \left( \hat{r} \times \hat{p} \right) \hat{r} + \left| 3\hat{r} (\hat{r} \cdot \hat{p}) - \hat{p} \right| \left( \frac{1}{r^3} + \frac{ik}{r^2} e^{-ikr} \right) \right) $$

so that in the far field

$$ E \simeq \frac{k^2 e^{-ikr}}{4\pi} \left( \hat{r} \times \hat{p} \right) \hat{r} = \frac{k}{\omega} B \times \hat{r} $$

(94)

Expressing the direction of oscillation in radial coordinates as

$$ \hat{d} = \hat{z} = \cos \theta \hat{r} - \sin \theta \hat{\theta} $$

(95)

leads to expressions for the fields

$$ B = \frac{\omega k e^{-ikr}}{4\pi r} \sin \theta \hat{\phi} $$

$$ E = \frac{\omega k e^{-ikr}}{4\pi r} \sin \theta \hat{\theta} $$

(96)
and the time-averaged average Poynting vector

\[ \bar{\theta} = \frac{1}{2} \mathbf{E} \times \mathbf{B} = \frac{1}{2} \left( \frac{k}{4\pi} \right)^2 \frac{(\omega d)^2}{r^2} \sin^2 \theta \hat{r} \]  

(97)

from which the total radiated power is found by integrating over the surface of a sphere of radius \( r \)

\[ \bar{P} = \frac{1}{2} \int d\Omega r^2 \hat{r} \cdot \left[ \left( \frac{k}{\omega} \mathbf{B} \times \hat{r} \right) \times \mathbf{B} \right] = \frac{1}{2} \int_0^{2\pi} d\phi \int_0^\pi d\theta \left( \frac{\omega k}{4\pi |\mathbf{p}|} \right)^2 \sin^3 \theta = \frac{k^2 (\omega d)^2}{12\pi}. \]  

(98)

This nonzero integral is understood to describe the power invested by the amplifier in accelerating the charges to produce a radiation field (identifiable by its \( 1/r \) dependence) and transferred away through any closed surface enclosing the antenna.

5. The oscillating dipole in SHP theory

In this section we follow the essential steps taken in the previous section in deriving the radiated power, adapting them as necessary to the microscopic dynamics of SHP.

5.1. Modeling the oscillating current

Carrying over the approximations used to write the Maxwell current (83) from the previous section requires some care, because the microscopic dynamics in SHP impose certain additional restrictions on the 5D pre-Maxwell current.

First, the time averaging (86) that renders the Maxwell problem tractable does not produce a clear separation into a space-dependent \( J(x) \) and time-dependent oscillation for the pre-Maxwell event current. The microscopic SHP current for a collection of nonrelativistic charged events is

\[ j(x, t, \tau) = c \sum_n X_n(\tau) \delta^4(x - X_n(\tau)) = \sum_n \frac{i\omega}{c} a_n e^{i\omega t} \delta(t - \tau) \delta^3(x - a_n e^{i\omega t}) \]  

(99)

from which the exact potential takes the complicated form

\[ a_\phi(x, \tau) = \frac{i\omega}{c} \sum_n a_n e^{i\omega t} \frac{e}{4\pi |x - a_n e^{i\omega t}|} \phi \left( t - \frac{|x - a_n e^{i\omega t}|}{c} - \tau \right). \]  

(100)

But the presence of \( \delta(t - \tau) \) in the current (99) as an unintegrated distribution renders time averaging meaningless. Proceeding (twice) with an averaging procedure would lead to

\[ j(x, t, \tau) \approx \left\{ \begin{array}{ll} \frac{1}{T} \sum_n i\omega a_n \delta^3(x - a_n e^{i\omega t}) e^{i\omega t} \approx J(x) e^{i\omega t}, & t \text{ average} \\ \frac{1}{T} \sum_n i\omega a_n \delta^3(x - a_n e^{i\omega t}) e^{i\omega t} \approx J(x) e^{i\omega t}, & \tau \text{ average} \end{array} \right. \]  

(101)

which should be equivalent because \( t = \tau \). However, integrating the Green’s function (28) with the \( \tau \)-independent current \( J(x) e^{i\omega t} \) recovers the wavelike Maxwell potential found in (87), while integrating with the \( t \)-independent current \( J(x) e^{i\omega t} \) leads to an oscillating Coulomb potential with no wavelike behavior. Since equivalent currents should not induce qualitatively different potentials, we conclude that time averaging is not a useful approach here.
Second, the component \( j^5 (x, \tau) \) represents an event density in spacetime and must be non-negative. For example, given a collection of events as in (84), the event density is

\[
j^5 (x, \tau) = c \sum_{n=1}^{N} X^5_n (\tau) \delta^4 (x - X_n (\tau)) = cc^5 \sum_{n=1}^{N} \delta^4 (x - X_n (\tau)) \geq 0
\]  

(102)

and the total number of events is

\[
\frac{1}{cc^5} \int d^4x \, j^5 (x, \tau) = N.
\]  

(103)

In the general case, conservation of the 5-current

\[
\partial_{\beta} f^{\alpha \beta} (x, \tau) = \frac{\epsilon}{c} j^\alpha (x, \tau) \implies \partial_{\alpha} j^\alpha (x, \tau) = 0
\]  

(104)

implies continuity in the form

\[
\frac{1}{c^5} \frac{d}{d\tau} \int d^4x \, j^5 (x, \tau) = c \frac{dN}{d\tau} = - \int d^4x \, \partial_{\mu} j^\mu (x, \tau)
\]  

(105)

so that total event number is conserved unless the four-current \( j^\mu (x, \tau) \) includes an external contribution — such as an ideal amplifier driving an antenna — whose divergence does not vanish at the boundary of spacetime.

Third, the component \( j^0 (x, \tau) \) must remain non-negative in order to eliminate pair processes. For a single event, \( j^0 (x, \tau) < 0 \) is associated with \( t < 0 \) through

\[
j^0 (x, \tau) = c^2 \frac{t}{(\tau - t)} \delta^4 (x - X (\tau))
\]  

(106)

and is characteristic of an antiparticle. We therefore take the oscillation as a perturbation \( \rho (x) \) to a static background current density \( \rho_0 (x) \) that preserves these requirements on the event current.

In analogy to the approximation used in the Maxwell approach, and in light of the relationship between \( j^0 \) and \( j^5 \) seen in (72), we pose a current of the form

\[
j^0 (x, \tau) = c \left[ \rho_0 (x) + \rho (x) e^{i\omega t} \right] \phi (\tau - t)
\]  

(107)

\[
j (x, \tau) = J (x) e^{i\omega t} \phi (\tau - t)
\]  

(108)

\[
j^5 (x, \tau) = \frac{c^5}{c} j^0 (x, \tau) = c^5 \left[ \rho_0 (x) + \rho (x) e^{i\omega t} \right] \phi (\tau - t)
\]  

(109)

where \( \phi (\tau - t) \) expresses a correlation between \( t \) and \( \tau \), inserted by hand instead of using a time averaging procedure. In this sense, the replacement

\[
j (x, t, \tau) \rightarrow J (x) e^{i\omega t} \phi (\tau - t)
\]  

(110)

may be less precise than the comparable approximation in Maxwell theory, and we must be attentive to artifacts introduced by the model. In analogy to (13) we choose

\[
\phi (\tau - t) = \frac{1}{2\sigma} e^{-|\tau - t|/\sigma}
\]  

(111)
which imposes a correlation \( t - \tau \sim \sigma \) through

\[
\phi (\tau - t) \rightarrow \begin{cases} 
\text{strong correlation:} & \sigma \to 0 \Rightarrow \phi (\tau - t) \to \delta (\tau - t) \Rightarrow t = \tau \\
\text{weak correlation:} & \sigma \to \text{large} \Rightarrow t - \tau \text{ evenly distributed.}
\end{cases}
\]

Notice that in the strong correlation limit, the potential found from the Green’s function \( (28) \) is

\[
a (x, \tau) = \frac{e}{2\pi} \int d^3x' d(c\tau') \delta \left( (x - x')^2 - c^2 (t - t')^2 \right) J (x') e^{i\omega t} \delta (\tau - t')
\]

\[
= \frac{e}{4\pi c} e^{i\omega \tau} \int d^3x' \frac{1}{|x - x'|} \delta \left( \tau - t + \frac{|x - x'|}{c} \right) J (x')
\]

(113)

describing a Coulomb-like potential that oscillates in \( \tau \) simultaneously across spacetime, rather than a wave propagating with phase \((kr - \omega t)\). This suppression of the expected wavelike behavior can be characterized by the dimensionless parameter

\[
\frac{1}{\omega \sigma} = \frac{T}{2\pi \sigma} = \text{antenna period} / \text{correlation time}
\]

(114)

which we take to be small but greater than zero. We note the useful properties

\[
\phi (-\tau) = \phi (\tau) \quad \int d\tau \phi (\tau) = 1 \quad \phi' (\tau) = -\frac{\epsilon (\tau)}{\sigma} \phi (\tau)
\]

(115)

and the Fourier transform

\[
\phi (\tau - t) = \int \frac{d\omega}{2\pi} \Phi (\omega) e^{i\omega (\tau - t)} \quad \Phi (\omega) = \frac{1}{1 + (\sigma \omega)^2}.
\]

(116)

Since the background density \( \rho_0(x) \) is independent of \( t \) and \( \tau \), conservation of the 5D current becomes

\[
0 = \frac{1}{c} \frac{\partial}{\partial t} J^0 + \nabla \cdot j + \frac{1}{c_5} \frac{\partial}{\partial \tau} \mathcal{J}^5 = \rho (x) \frac{\partial}{\partial \tau} \left[ e^{i\omega \tau} \phi (\tau - t) \right] + \nabla \cdot J (x) e^{i\omega \tau} \phi (\tau - t)
\]

\[
+ \rho (x) e^{i\omega \tau} \frac{\partial}{\partial \tau} \phi (\tau - t)
\]

\[
= i\omega \rho (x) e^{i\omega \tau} \phi (\tau - t) - \rho (x) e^{i\omega \tau} \phi' (\tau - t)
\]

\[
+ \nabla \cdot J (x) e^{i\omega \tau} \phi (\tau - t) + \rho (x) e^{i\omega \tau} \phi' (\tau - t)
\]

\[
= [i\omega \rho (x) + \nabla \cdot J (x)] e^{i\omega \tau} \phi (\tau - t)
\]

(117)

so that \( \rho (x) \) and \( J (x) \) are related as in the Maxwell problem through

\[
\rho (x) = -\frac{1}{i\omega} \nabla \cdot J (x) \quad \rightarrow \quad e \int d^3x J (x) = -e \int d^3x \nabla \cdot \mathcal{J} = i\omega \int d^3x x e\rho (x).
\]

(118)

The total charge at time \( \tau \) is found from the spacetime integral

\[
Q (\tau) = \frac{e}{c} \int d^3x J^0 (x, \tau) = \frac{e}{c} \int d^4x c \left[ \rho_0 (x) + \rho (x) e^{i\omega \tau} \right] \phi (\tau - t)
\]

\[
= \int d^3x e\rho_0 (x) \int dt \phi (\tau - t) + \int d^3x e\rho (x) \int dt e^{i\omega \tau} \phi (\tau - t)
\]

\[
= \int d^3x e\rho_0 (x) + \int d^3x e\rho (x) \Phi (\omega) e^{i\omega \tau}
\]

\[
= Q_0 + Q \Phi (\omega) e^{i\omega \tau}
\]

(119)
where we write
\[ Q_0 = \int d^3 x \, e \rho_0 (x) \quad Q = \int d^3 x \, e \rho (x) \]  
\[ (120) \]
and used (115) and (116). The total charge in spacetime takes the form of a background charge with an oscillating (externally driven) perturbation. Similarly, the total current density is
\[ J (\tau) = \frac{e}{c} \int d^4 x \, J(x) \, e^{i\omega t} \phi (\tau - t) = \Phi (\omega) \, e^{i\omega \tau} \int d^3 x \, e \mathbf{J}(x) = \Phi (\omega) \, i \omega \mathbf{p} e^{i\omega \tau} \]  
\[ (121) \]
representing an oscillating dipole.

5.2. Induced potential and field strengths
Using the leading term in the Green’s function (28)
\[ G_{\text{Maxwell}} = -\frac{1}{2\pi} \delta(x^2) \delta(\tau) \]  
\[ (122) \]
the induced potential is
\[ a^a (x, \tau) = \frac{e}{c} \int d^3 x' \, \frac{1}{4\pi |x - x'|} \, j^a \left( c \left( t - \frac{|x - x'|}{c} \right), x', \tau \right) \]  
\[ (123) \]
with components
\[ a^0 (x, \tau) = e \int d^3 x' \, \frac{\rho_0 (x')}{4\pi |x - x'|} \phi \left( \tau - t + \frac{|x - x'|}{c} \right) + e \int d^3 x' \, \frac{\rho (x')}{4\pi |x - x'|} \, e^{i\omega \left( \tau - t + \frac{|x - x'|}{c} \right)} \phi \left( \tau - t + \frac{|x - x'|}{c} \right) \]  
\[ (124) \]
\[ a^a (x, \tau) = \frac{e}{c} \int d^3 x' \, \frac{J(x')}{4\pi |x - x'|} \, e^{i\omega \left( \tau - t + \frac{|x - x'|}{c} \right)} \phi \left( \tau - t + \frac{|x - x'|}{c} \right) \]  
\[ (125) \]
\[ a^5 (x, \tau) = \frac{c_5}{c} a^0 (x, \tau) \]  
\[ (126) \]
Applying the far field and dipole approximations these potentials can be evaluated as
\[ a^0 (x, \tau) \approx \frac{1}{4\pi r} \phi \left( \tau - t + \frac{r}{c} \right) \int d^3 x' \, e \rho_0 (x') + \frac{1}{4\pi r} e^{i\omega t} e^{-ikr} \phi \left( \tau - t + \frac{r}{c} \right) \int d^3 x' \, e \rho (x') \]  
\[ = \frac{Q_0}{4\pi r} + \frac{Q}{4\pi r} e^{-i(kr - \omega t)} \phi \left( \tau - t + \frac{r}{c} \right) \]  
\[ (127) \]
\[ a^a (x, \tau) \approx \frac{1}{4\pi r} e^{-ikr} e^{i\omega t} \frac{e}{c} \int d^3 x' \, \mathbf{J}(x') \phi \left( \tau - t + \frac{r}{c} \right) \]  
\[ = \frac{ik}{4\pi r} e^{-i(kr - \omega t)} \phi \left( \tau - t + \frac{r}{c} \right) \]  
\[ (128) \]
\[ a^5 (x, \tau) = \frac{c_5}{c} a^0 (x, \tau) \]  
\[ (129) \]
from which the components of \( f^{\alpha\beta}(x, \tau) \) are found by taking appropriate derivatives. We define the spherical wave factor

\[
\chi(x, \tau) = \frac{e^{-i(kr - \omega t)}}{4\pi r} \phi\left(\tau - t + \frac{r}{c}\right)
\]  

and split the field strengths into a spacetime factor and polarization factors, as

\[
b = \nabla \times a = \hat{b} \chi \quad \quad \quad \quad \quad \hat{b} = -ikld \left(1 + \frac{\varepsilon_R}{\omega \sigma}\right) \hat{r} \times \hat{d}
\]

\[
e = -\frac{1}{c} \frac{\partial}{\partial t} a - \nabla a^0 = \frac{Q_0}{4\pi r^2} \hat{r} + \hat{e} \chi \quad \quad \quad \hat{e} = ik \left(1 + \frac{\varepsilon_R}{\omega \sigma}\right) \left(Q \hat{r} - l d \hat{d}\right)
\]

\[
e = g_{55} \frac{1}{c_5} \frac{\partial}{\partial t} a - \frac{c_5}{c} \nabla a^0 = \frac{c_5}{c} \frac{Q_0}{4\pi r^2} \hat{r} + \hat{\epsilon} \chi \quad \quad \quad \hat{\epsilon} = ik \left[\frac{c_5}{c} \left(1 + \frac{\varepsilon_R}{\omega \sigma}\right) - g_{55} \frac{\varepsilon_R}{c_5 \omega \sigma} l d \hat{d}\right]
\]

\[
e^0 = g_{55} \frac{1}{c_5} \frac{\partial}{\partial t} a^0 - \frac{1}{c} \frac{\partial}{\partial t} \dot{a}^0 = \hat{\epsilon}^0 \chi \quad \quad \quad \hat{\epsilon}^0 = ik \left[\frac{c_5}{c} \left(1 + \frac{\varepsilon_R}{\omega \sigma}\right) - g_{55} \frac{\varepsilon_R}{c_5 \omega \sigma}\right] Q
\]

where we used

\[
\phi' (\tau) = -\frac{\varepsilon (\tau)}{\sigma} \phi (\tau) \quad \quad \frac{1}{k r} \ll 1 \quad \quad p = \frac{1}{i \omega} l d \hat{d}
\]

and write \( \varepsilon_R = \varepsilon(\tau - t + R/c) \). We drop the static Coulomb terms produced by \( Q_0 \), as these do not contribute to radiation. Since we saw in \( 112 \) and \( 113 \) that \( \omega \sigma \sim \sigma \) small tends to suppress wavelike behavior, we take

\[
1 + \frac{\varepsilon_R}{\omega \sigma} \approx 1
\]

but note \( 5 \) that since \( c_5 / c \ll 1 \) we leave factors

\[
\frac{c}{c_5} \frac{\varepsilon_R}{\omega \sigma}
\]

unchanged. Taking the orientation of the antenna to be \( \hat{d} = \hat{z} \) the polarizations then simplify to

\[
\hat{e} \approx ik \left(Q \hat{r} - l d \hat{z}\right) \quad \quad \hat{b} \approx -ikld \hat{r} \times \hat{z}
\]

\[
\hat{\epsilon}^0 \approx ik \left[\frac{c_5}{c} - g_{55} \frac{\varepsilon_R}{c_5 \omega \sigma}\right] Q \quad \quad \hat{\epsilon} \approx ik \left[\frac{c_5}{c} Q \hat{r} - g_{55} \frac{\varepsilon_R}{c_5 \omega \sigma} l d \hat{d}\right]
\]

and we notice that terms containing \( 1 / \omega \sigma \) appear only in the components of \( \hat{\epsilon}^\mu \). Such terms are artifacts of modeling the time correlation by \( \phi (\tau - t) \), and can be understood as the contribution to the fields required to impose this correlation across spacetime. As was seen for plane waves, these fields will accelerate a test event initially at rest through the Lorentz force

\[
\ddot{r} = \frac{e}{M c^2} \left(\hat{e} \cdot \dot{x} - g_{55} c^2 \epsilon_0^0\right) = -g_{55} \frac{e c_5}{M c^2} e_0^0
\]

\[
\ddot{x} = \frac{e}{M} \left[\hat{e} \dot{t} + \frac{1}{c} \hat{b} \times \dot{x} + \frac{c_5}{c} \epsilon\right] = \frac{e}{M} \left[\hat{e} + \frac{c_5}{c} \epsilon\right]
\]

\[
\frac{d}{d\tau} (-\frac{1}{2} M x^2) = -e g_{55} c_5 \left(e_0^0 \dot{t} - \epsilon \cdot \dot{x}(\tau)\right) = -e g_{55} c_5 e_0^0
\]

in such a way as to transfer mass to the event.
The real parts of these expressions, we designate includes all terms containing $1/\delta S$ all of which have spacetime dependence $\phi$ where the common spacetime factor $(130)$ contains the spherical wave and the time correlation bimodal field combinations of the type $r$ large and note that these functions drop off as $1/\delta T$, as is characteristic of radiation fields. Using these functions, the components of $T_{\sigma}^{\alpha \beta}$ are

$$
T_{00}^{00} = \frac{1}{2} \left[ (Q + \frac{1}{c} \cos \theta)^2 + 2 (1 d)^2 \left( 1 - (\cos \theta)^2 \right) + 2 g_{55} \left( \frac{c}{c} \right)^2 Q^2 \right] S(x, \tau)
$$

$$
T_{0}^{0} = \left[ 1 d (Q - \frac{1}{c} \cos \theta) \hat{z} + \left( 1 d (1d - Q \cos \theta) + g_{55} \left( \frac{c}{c} \right)^2 \right) \hat{r} \right] S(x, \tau)
$$

$$
T_{0}^{5} = \left( \frac{c}{c} \right)^2 (Q - \frac{1}{c} \cos \theta) S(x, \tau) \hat{f} = T_{50}^{0} \hat{f}
$$

$$
T_{5}^{5} = \frac{1}{2} g_{55} (Q - \frac{1}{c} \cos \theta)^2 S(x, \tau)
$$

all of which have spacetime dependence $S(x, \tau)$. The components of $T_{\sigma}^{\alpha \beta}$ are

$$
T_{c}^{0} = \frac{1}{2} \left[ \left( \frac{c}{c} \right)^2 \left( (1d)^2 + Q^2 \right) C(x, \tau) - \frac{\epsilon R}{\omega \sigma} Q \left( Q + 1d \cos \theta \right) X(x, \tau) \right]
$$

$$
T_{c}^{5} = -\frac{\epsilon R}{\omega \sigma} \left[ 1d \left( X(x, \tau) + g_{55} \frac{c}{c} C(x, \tau) \right) \hat{z} + Q X(x, \tau) \hat{f} \right]
$$

$$
T_{5}^{5} = g_{55} \frac{\epsilon R}{\omega \sigma} 1d (1d - Q \cos \theta) X(x, \tau)
$$

$$
T_{5}^{5} = \frac{\epsilon R}{\omega \sigma} 1d (Q - 1d \cos \theta) \hat{z} + \frac{\epsilon R}{\omega \sigma} \left( (1d)^2 - Q^2 \right) \hat{f} X(x, \tau)
$$

$$
T_{5}^{5} = \frac{1}{2} \left[ \left( \frac{c}{c} \right)^2 \left( (1d)^2 - Q^2 \right) C(x, \tau) - g_{55} \frac{\epsilon R}{\omega \sigma} Q (Q - 1d \cos \theta) X(x, \tau) \right]
$$
whose spacetime dependence is determined by \( C(x, \tau) \) and \( X(x, \tau) \) and is thus out of phase with \( T^{a\beta}_0 \).

As expected from the transfer of mass made possible by the fields \( \epsilon^\mu \) in (135) and seen explicitly in (136), and we find a nonzero mass density \( T^{55} \) and mass flux \( T^{00} \) and \( T^{5j} \) into time and space. Moreover, integrating over a sphere of radius \( r \), the net mass flux into space will be of the form

\[
P = \int d\Omega \ r^2 \ \hat{r} \cdot T^5_0 \]

\[
= \int d\Omega \ r^2 \ \hat{r} \cdot \left[ \frac{Q \ c_5}{c} (Q - Id (\cos \theta)) S (x, \tau) \hat{r} \right] \]

\[
= Q \ c_5 r^2 k^2 \left( \frac{\phi (\tau - t + \frac{r}{c})}{4\pi r} \right)^2 \sin^2 (kr - \omega t) \int d\Omega \ [Q - Id \cos \theta] \]

\[
= \frac{k^2 c_5 Q^2}{4\pi c} \left( \phi (\tau - t + \frac{r}{c}) \right)^2 \sin^2 (kr - \omega t) \]

and thus nonzero wherever \( \tau \simeq t - r/c \). Just as the energy radiated by a Maxwell dipole antenna must be provided by the amplifier that drives the oscillating current density, the mass radiated by an SHP antenna is continuously provided by an amplifier that creates events and drives them into the antenna.

5.4. Neutral antenna

We consider two scenarios for generating a current density in a split dipole antenna. In the first scenario, shown in Figure 1a, the current source simultaneously drives charged events into the left and right antenna segments, so that the total charge in each segment fluctuates around its neutral value by \( \pm Q \) over one cycle of oscillation. In the second scenario, shown in Figure 1b, the alternating current source removes charge \( Q \) from one segment and drives charge \( Q \) into the other segment, so that the antenna remains neutral at all times.

![Figure 1a: Scenario 1](image1.png)

![Figure 1b: Scenario 2](image2.png)

Taking the current along the z-axis in an antenna of length \( d \), the charge density is described by

\[
\rho (x) = \begin{cases} 
\delta (x) \delta (y) \rho_z (z), & -\frac{d}{2} \leq z \leq \frac{d}{2} \\
0, & \text{otherwise} 
\end{cases}
\]

(142)

where

\[
\rho_z (-z) = \begin{cases} 
+\rho_z (z), & \text{Scenario 1} \\
-\rho_z (z), & \text{Scenario 2} 
\end{cases}
\]

(143)

and so the current density is related through

\[
\rho (x) = -\frac{1}{i\omega} \nabla \cdot J (x)
\]

(144)
and has opposite parity. Evaluating the dipole moment, we find

\[
1 d \hat{d} = i \omega e \int d^3x \, \mathbf{x} \rho(\mathbf{x}) = i e \omega \hat{\mathbf{z}} \int_0^\frac{\pi}{2} dz \, z (\rho_z(z) - \rho_z(-z)) = \left\{ \begin{array}{ll}
0 & \text{Scenario 1} \\
2 i e \omega \int_0^\frac{\pi}{2} dz \, \rho_z(z) & \text{Scenario 2} 
\end{array} \right. \tag{145}
\]

restating the inherent condition that a nonzero dipole moment requires an asymmetric charge distribution. Discounting Scenario 1, we find that in Scenario 2 the total charge on the antenna

\[
Q_{\text{total}} = \int d^3x \, e \rho_0(\mathbf{x}) + \int d^3x \, e \rho(\mathbf{x}) = Q_0 + e \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} dz \, \rho_z(z) = Q_0 \tag{146}
\]
is just the background charge, and the total number of events in the antenna

\[
\frac{1}{cc_5} N = \frac{1}{cc_5} \int d^4x \, j^5(x, \tau) = \int d^3x \tau \left[ \rho_0(\mathbf{x}) + \rho(\mathbf{x}) e^{i \omega \tau} \right] \phi(\tau - t) = \frac{Q_0}{e} \tag{147}
\]
is also given by the constant background charge.

The system simplifies considerably in Scenario 2 with \( Q = 0 \). The fields reduce to

\[
\hat{\mathbf{e}} = -ik Id \hat{\mathbf{z}} \quad \hat{\mathbf{b}} = -ik Id \hat{\mathbf{r}} \times \hat{\mathbf{z}} \quad \hat{\mathbf{e}}^0 = 0 \quad \hat{\mathbf{e}} = -ig55 \frac{e}{c5 i \omega} Id \hat{\mathbf{z}} \tag{148}
\]

so that the effect of the waves on a test event at rest reduce to

\[
\ddot{t} = -g55 \frac{e c5}{M c^2} e^0 = 0 \quad \ddot{x} = - \frac{e}{M} ik Id \hat{\mathbf{z}} \left[ 1 + g55 \frac{e R}{c5 i \omega} \right] \tag{149}
\]

\[
\frac{d}{dt} \left( -\frac{1}{2} M \dot{x}^2 \right) = -e g55 c5 e^0 = 0 \tag{150}
\]

which does not involve transfer of mass. The components of become \( T^{0\beta}_0 \)

\[
T^{00}_0 = (Id)^2 \left( 1 - \frac{1}{2} \cos^2 \theta \right) S(x, \tau) \\
T^{05}_0 = (Id)^2 \left( - \cos \theta \hat{\mathbf{z}} + \hat{\mathbf{f}} \right) S(x, \tau) \\
T^{50}_0 = 0 \\
T^{30}_0 = 0 \\
T^{55}_0 = \frac{1}{2} g55 (Id)^2 \cos^2 \theta S(x, \tau) \tag{151}
\]

describing no transfer of mass into space or time.
The components of $T^\alpha_\beta$ also simplify to

$$T^0_\alpha = \frac{1}{2} \left( \frac{c}{c_s \omega'} \right)^2 (Id)^2 C (x, \tau) = \frac{1}{2} \left( \frac{c}{\omega c_5} \phi' \right)^2 (Id)^2 C (x, \tau)$$

$$T^5_\alpha = 0$$

$$T^5_\sigma = -\frac{c}{c_s} g_{55} \varepsilon_R \phi' (Id)^2 X (x, \tau) = \frac{c}{\omega c_5} g_{55} \phi' (Id)^2 X (x, \tau)$$

$$T^5_\sigma = T^{50} (\hat{z} - \hat{r})$$

$$T^{55}_\sigma = \frac{1}{2} \left( \frac{c}{c_s \omega'} \right)^2 (Id)^2 C (x, \tau) = \frac{1}{2} \left( \frac{c}{\omega c_5} \phi' \right)^2 (Id)^2 C (x, \tau)$$

where we used (116) to replace $-\varepsilon_R/\phi$ with $\phi'/\phi$. These expressions involve no transfer of energy but do describe nonzero transfer of mass into space and time directions. Once again, we understand this transfer as an artifact of the time correlation model that enters through the derivative of $\phi (\tau - t)$, rather than an inherent feature of radiation from an oscillating charge. In particular, all of the nonzero terms in the expression for mass conservation contain $\phi'$, so that these terms are separately conserved among themselves. To see this we write the mass component of (33) as

$$\partial_\alpha T^{55} = -\frac{1}{c} f^{5s} j_\alpha \rightarrow \frac{1}{c} \frac{\partial}{\partial t} T^{50} + \nabla \cdot T^5 + \frac{1}{c_s} \frac{\partial}{\partial \tau} T^{55} = -\frac{1}{c} \epsilon^\mu j_\mu$$

which becomes

$$\frac{1}{c} \frac{\partial}{\partial t} T^{50} + \nabla \cdot T^5 + \frac{1}{c_s} \frac{\partial}{\partial \tau} (T^{50} + T^{55}) = -\frac{1}{c} \epsilon^\mu j_\mu$$

because $T^{50}_0 = T^5_0 = 0$. We also write the field as $\epsilon_\mu^\nu$ because it is seen in (148) to contain the factor $\varepsilon_R/\omega'$. Finally, we note that because $T^{55}_0$ depends on $\tau$ only through the factor of $\phi^2$ in $S (x, \tau)$, the derivative $\partial_\tau T^{55}_0$ must similarly contain $\phi'$. Thus, each term in (154) enters through the derivative of the time correlation model, and these terms are conserved among themselves with no corresponding energy transfer.

Integrating the energy Poynting vector over the surface of a sphere of radius $\tau$ we must evaluate

$$\mathbf{r} \cdot T^0 = (Id)^2 \mathbf{r} \cdot \left( -\cos \theta \hat{z} + \hat{r} \right) S (x, \tau)$$

$$= (Id)^2 \left( -\cos^2 \theta + 1 \right) S (x, \tau)$$

$$= (Id)^2 \sin^2 \theta S (x, \tau)$$

(155)

to find the instantaneous radiated power

$$P = \int d\Omega \ r^2 (Id)^2 S (x, \tau) \sin^2 \theta$$

$$= (Id)^2 r^2 k^2 \left( \frac{\phi (\tau - t + \xi)}{4\pi r} \right)^2 \sin^2 (kr - \omega t) \int_0^{2\pi} d\phi \int_0^\pi d\theta \sin^3 \theta$$

$$= (Id)^2 k^2 \left( \frac{\phi (\tau - t + \xi)}{4\pi} \right)^2 \sin^2 (kr - \omega t) \frac{8\pi}{3}$$

$$= \frac{k^2 (Id)^2}{6\pi} \left( \phi (\tau - t + \xi) \sin (kr - \omega t) \right)^2.$$  

(156)
Since we have assumed that $1/\omega \sigma$ is small, we may take $\phi (\tau - t + \frac{r}{c})$ as effectively constant over one cycle of the wave, so that the average radiated power over one cycle is

$$P \simeq \frac{k^2(1d)^2}{6\pi} \left( \phi \left( \tau - t + \frac{r}{c} \right) \right)^2 \frac{1}{T} \int_0^T dt \left( \sin (kr - \omega t) \right)^2 = \frac{k^2(1d)^2}{12\pi} \left( \phi \left( \tau - t + \frac{r}{c} \right) \right)^2$$

which agrees with (98) up to the factor of $\phi^2$. The neutral antenna radiates energy in agreement with the Maxwell result and radiates no mass (leaving aside the derivatives of the arbitrarily chosen function $\phi$).

6. Discussion

As a canonical classical field theory, Stueckelberg-Horwitz-Piron (SHP) defines events in 4D spacetime on an unconstrained 8D phase space. It is therefore unsurprising, from an abstract point of view, that interactions may involve the transfer of both four-vector quantities and scalar quantities formed from four-vectors. Nevertheless, the concrete statement that electromagnetic fields and particles may exchange both energy-momentum and mass seems to demand further elaboration.

In standard formulations of relativistic mechanics [15] the a priori constraint associated with parameterization by proper time

$$-c^2 ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu = \eta_{\mu\nu} \dot{x}^\mu (s) \dot{x}^\nu (s) ds^2 \implies \dot{x}^2 = -c^2$$

restricts the degrees of freedom carried by the massless electromagnetic field in such a way that its squared energy and squared momentum are equal. This restriction is reflected in the fact that only three components of the Lorentz force, usually taken as 3-velocity, are independent, while the fourth component, (3-velocity)$^2 \sim$ energy, is dependent on the other three. In SHP theory, however, the proper time interval

$$-c^2 ds^2(\tau) = \eta_{\mu\nu} dx^\mu dx^\nu = \eta_{\mu\nu} \dot{x}^\mu (\tau) \dot{x}^\nu (\tau) d\tau^2$$

is a dynamical scalar quantity dependent on the four unconstrained components of velocity. Therefore, four components of the Lorentz force, usually taken as 4-velocity, are independent, and while the fifth component, (4-velocity)$^2 \sim$ mass, is dependent, determined by whatever the difference between squared energy and squared momentum turns out to be in any given interaction. As a result, the mass of an individual particle (a “fifth”, scalar quantity) is only conserved under physical conditions for which certain other “fifth” quantities vanish: $a^5, \partial_5 a^\mu$, and $\partial_5 \tilde{\phi} = -\partial_\mu j^\mu$. From a general potential found from the Green’s function (28)

$$a^\alpha (x, \tau) = -\frac{e}{c} \int d^4x' d\tau' G_p \left( x - x', \tau - \tau' \right) j_\phi (x', \tau')$$

we may conclude that $a^5(x, \tau) = 0$ requires $\tilde{\phi}(x, \tau) = 0$, and that $\partial_5 a^\mu(x, \tau) = 0$ requires $\partial_5 j^\mu(x, \tau) = 0$. These requirements, together with $\int d\tau G_p (x, \tau) = -\delta (x^2)/2\pi$, imply that mass conservation constrains $j^\mu(x, \tau) = j^\mu(x)$ to be a divergenceless Maxwell current.

In the absence of these restrictions, a simple plane wave impinging on a test particle may transfer momentum, energy, and mass, where the total 4-vector and scalar quantities are conserved for the particle and field, and the independence of four momentum components is expressed in the 5D mass shell condition (10). As seen in (56), for any fixed value of $k$, the mass transfer becomes more pronounced as $\kappa^2$, the difference between $k_0^2$ and $k^2$, grows, supporting the interpretation of $\kappa$ as representing the mass of the electromagnetic field. For these plane waves, the mass-energy-momentum tensor is composed of energy density $T^{00}$, mass density
$T^{55}$, a Poynting 3-vector $T^0i = (k^i / k^0)T^{00}$ representing energy flux into space, and a Poynting 4-vector $T^5\mu = (k^\mu / \kappa)T^{55}$ representing mass flux into spacetime.

A radiating particle may lose both energy and mass, whether or not the radiation is absorbed through subsequent interactions. Electromagnetic fields, in Maxwell and SHP theory alike, can be split into a retarded part, associated with uniform inertial motion, that falls off as $1/R^2$, and a radiation part, associated with acceleration, that falls off as $1/R$. The corresponding flux densities therefore split into retarded and radiative parts with $1/R^4$ and $1/R^2$ dependence, respectively. Since a closed surface in 3D has area element $R^2d\Omega$, the total flux from the retarded part decreases rapidly at long distances, while the radiation flux can propagate to infinity. Examples of these dependencies were seen in the Poynting vectors (68) for uniform motion and (97) for accelerated motion. But the symmetries of the field configurations are equally significant. For uniform motion, the Poynting vector is antisymmetric with respect to the path of the particle, so that the total flux vanishes identically at any distance, preventing energy loss by radiation regardless of the $R$-dependence. But the Poynting vector for the oscillating charge is directed radially outward at any observation point, so that the radiative energy flux will be nonzero at any finite distance, even if the flux were to fall off more strongly than $1/R^2$.

In the case of the simple SHP dipole antenna, the model we used to simplify the oscillating current density presents two opposite scenarios. In Scenario 1, the amplifier alternates between injecting and withdrawing charge $Q$ symmetrically into the antenna, with current $I = 0$ across the antenna. Under these conditions, we see from (139) that the field possesses both mass and energy density, each producing radiative flux. In Scenario 2, the antenna remains electrically neutral with $Q = 0$, while the amplifier shifts charge back and forth across the antenna segments characterized by an alternating current with magnitude $I$. Under these conditions, expressions (151) indicates that the field possesses both mass and energy density, producing radiation of energy equivalent to the Maxwell result, but no radiation of mass.

The absence of mass radiation in Scenario 2 can be attributed to the absence of event injection into the antenna. As was seen in (147), when the net injected charge $Q$ vanishes, the total number of events in the antenna is given by the background charge, with no contribution from the oscillating current. This association of event injection with mass radiation can also be understood through the continuity equation, which relates the event number $N$ to the divergence of the 4-current. Rewriting (105) in the form

$\Delta N = N(\infty) - N(-\infty) = -\frac{1}{c} \int d\tau \int d^4x \, \partial_\mu j^\mu(x, \tau)$

we generally interpret the spacetime integral of the total divergence as the current crossing the outer boundary of spacetime at infinity. We expect this integral to vanish on qualitative grounds unless the current includes an external component, such as an ideal amplifier injecting charged events into the system. Indeed, changing the order of integration and using the boundary conditions (29) we find

$\Delta N = -\frac{1}{c} \int d^4x \, \partial_\mu \int d\tau \, j^\mu(x, \tau) = -\frac{1}{c} \int d^4x \, \partial_\mu J^\mu(x) = 0$

expressing the net zero change in event number in Maxwell theory as the equilibrium limit of SHP. However, for the current defined in (107) and (108), and recalling (117), equation (105)
becomes
\[
\frac{dN}{d\tau} = -\frac{1}{c} \int d^4x \partial_\mu j^\mu (x, \tau) \\
= -\frac{1}{c} \int d^4x \left\{ i\omega \rho (x) e^{i\omega t} \phi (\tau - t) - \rho (x) e^{i\omega t} \phi (\tau - t) + \nabla \cdot J (x) e^{i\omega t} \phi (\tau - t) \right\} \\
= \int d^3x \rho (x) \int dt e^{i\omega t} \phi (\tau - t) \\
= \frac{Q}{e} e^{i\omega \Phi (\omega)} e^{i\omega \tau} \\
= \frac{d}{d\tau} Q (\tau)
\]
(163)

where \(Q (\tau)\) is the total spacetime charge given in (119). As expected, integrating over \(\tau\) takes \(e^{i\omega \tau} \rightarrow \delta (\omega)\) so that \(\Delta N = 0\). However, at any finite \(\tau\) the injection of charge \(Q \neq 0\) into the antenna is equivalent to a 4-current with non-vanishing divergence, and is thus equivalent to an oscillating event injection, leading to mass radiation.

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