THE NONLINEAR MEMBRANE ENERGY: VARIATIONAL
DERIVATION UNDER THE CONSTRAINT “det∇u ≠ 0”

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Résumé. Acerbi, Buttazzo et Percivale ont donné une définition variationnelle de l’énergie d’une corde non linéaire sous la contrainte “det∇u > 0” (voir [1]). Dans le même esprit, nous obtenons l’énergie d’une membrane non linéaire sous la contrainte plus simple “det∇u ≠ 0”.

Abstract. Acerbi, Buttazzo and Percivale gave a variational definition of the nonlinear string energy under the constraint “det∇u > 0” (see [1]). In the same spirit, we obtain the nonlinear membrane energy under the simpler constraint “det∇u ≠ 0”.

Key words. Dimensional reduction, Γ-convergence, relaxation, nonlinear membrane, determinant constraint.

1. Introduction

Consider an elastic material occupying in a reference configuration the bounded open set Σε ⊂ R³ given by

Σε := Σ × ]−ε/2, ε/2],

where ε > 0 is very small and Σ ⊂ R² is Lipschitz, open and bounded. A point of Σε is denoted by (x, x³) with x ∈ Σ and x³ ∈ ]−ε/2, ε/2[. Denote by W : M³×³ → [0, +∞) the stored-energy function supposed to be continuous and coercive, i.e.,

W(F) = +∞ if and only if detF = 0;

W(F) ≤ C|F|^p for all F ∈ M³×³ and some C > 0. In order to take into account the fact that an infinite amount of energy is required to compress a finite volume into zero volume², i.e.,

W(F) → +∞ as detF → 0,

where detF denotes the determinant of the 3 × 3 matrix F, we assume that:

(C₁) W(F) = +∞ if and only if detF = 0;

(C₂) for every δ > 0, there exists c₄ > 0 such that for all F ∈ M³×³,

if |detF| ≥ δ then W(F) ≤ c₄(1 + |F|^p).

Our goal is to show that as ε → 0 the three-dimensional free energy functional Eε : W¹,p(Σε; R³) → [0, +∞] (with p > 1) defined by

Eε(u) := 1/ε ∫ _Σε W(∇u(x, x³))dx dx³

converges in a variational sense (see Definition 2.1) to the two-dimensional free energy functional E_mem : W¹,p(Σ; R³) → [0, +∞] given by

E_mem(v) := ∫ _Σ W_mem(∇v(x))dx

In [9], Ben Belgacem announced to have obtained a variational definition of the nonlinear membrane energy under the constraint “det∇u > 0”. To our knowledge, his statement [9, Theorem 1] never was proved (see Remark 2.13).

²However, we do not prevent orientation reversal.
with \( W_{\text{mem}} : \mathbb{M}^{3 \times 2} \to [0, +\infty]. \) Usually, \( E_{\text{mem}} \) is called the nonlinear membrane energy associated with the two-dimensional elastic material with respect to the reference configuration \( \Sigma. \) Furthermore we wish to give a representation formula for \( W_{\text{mem}}. \)

Such a problem was studied by Le Dret and Raoult in [16] when \( W \) is of \( p \)-polynomial growth, i.e., \( W(F) \leq c(1 + |F|^p) \) for all \( F \in \mathbb{M}^{3 \times 3} \) and some \( c > 0, \) so that (1) is not satisfied. The distinguishing feature here is that \( W \) is not of \( p \)-polynomial growth.

An outline of the paper is as follows. The variational convergence of \( E_{\varepsilon} \) to \( E_{\text{mem}} \) as \( \varepsilon \to 0 \) as well as a representation formula for \( W_{\text{mem}} \) are given by Corollary 2.16 (see also Proposition 2.4). Corollary 2.16 is a consequence of Theorems 2.7 and 2.14. As Theorem 2.14 is proved in our previous article [6], the main result of the paper is Theorem 2.7. In fact, Theorem 2.14 is analogous to Theorem 2.12 established by Ben Belgacem in [10]. A comparison of these results is made in Sect. 2.3 (see also [6, Remark 2.6]). Theorem 2.7 is proved in Section 4: the principal ingredients being Theorem 2.8 (stated in Sect. 2.2 and whose proof is contained in [6]) and Theorem 3.4 (whose statement and proof are given in Section 3).

For the convenience of the reader, we recall the proofs of Theorems 2.8 and 2.14 in appendix.

2. Results

2.1. Variational convergence. As in [1], to accomplish our asymptotic analysis, we use the notion of convergence introduced by Anzellotti, Baldo and Percivale in [7] in order to deal with dimension reduction problems in mechanics. Let \( \pi = \{\pi_{\varepsilon}\}_{\varepsilon} \) be the family of maps \( \pi_{\varepsilon} : W^{1,p}(\Sigma; \mathbb{R}^3) \to W^{1,p}(\Sigma; \mathbb{R}^3) \) defined by

\[
\pi_{\varepsilon}(u) := \frac{1}{\varepsilon} \int_{-\frac{\varepsilon}{2}}^{\frac{\varepsilon}{2}} u(\cdot, x_3) dx_3.
\]

Definition 2.1. We say that \( E_{\varepsilon} \) \( \Gamma(\pi) \)-converges to \( E_{\text{mem}} \) as \( \varepsilon \to 0, \) and write \( E_{\varepsilon} \to \Gamma(\pi) E_{\text{mem}} \) if the following two assertions hold:

(i) for all \( v \in W^{1,p}(\Sigma; \mathbb{R}^3) \) and all \( \{u_{\varepsilon}\}_{\varepsilon} \subset W^{1,p}(\Sigma; \mathbb{R}^3), \)

\[
\text{if } \pi_{\varepsilon}(u_{\varepsilon}) \to v \text{ in } L^p(\Sigma; \mathbb{R}^3) \text{ then } E_{\text{mem}}(v) \leq \liminf_{\varepsilon \to 0} E_{\varepsilon}(u_{\varepsilon});
\]

(ii) for all \( v \in W^{1,p}(\Sigma; \mathbb{R}^3) \), there exists \( \{u_{\varepsilon}\}_{\varepsilon} \subset W^{1,p}(\Sigma; \mathbb{R}^3) \) such that:

\[
\pi_{\varepsilon}(u_{\varepsilon}) \to v \text{ in } L^p(\Sigma; \mathbb{R}^3) \text{ and } E_{\text{mem}}(v) \geq \limsup_{\varepsilon \to 0} E_{\varepsilon}(u_{\varepsilon}).
\]

In fact, Definition 2.1 is a variant of De Giorgi’s \( \Gamma \)-convergence. This is made clear by Lemma 2.3. Consider \( E_{\varepsilon} : W^{1,p}(\Sigma; \mathbb{R}^3) \to [0, +\infty] \) defined by

\[
E_{\varepsilon}(v) := \inf \left\{ E_{\varepsilon}(u) : \pi_{\varepsilon}(u) = v \right\}.
\]

Definition 2.2. We say that \( E_{\varepsilon} \) \( \Gamma \)-converges to \( E_{\text{mem}} \) as \( \varepsilon \to 0, \) and we write \( E_{\varepsilon} \to \Gamma \) \( E_{\text{mem}} \), if for every \( v \in W^{1,p}(\Sigma; \mathbb{R}^3), \)

\[
\left( \Gamma \liminf_{\varepsilon \to 0} E_{\varepsilon} \right)(v) = \left( \Gamma \limsup_{\varepsilon \to 0} E_{\varepsilon} \right)(v) = E_{\text{mem}}(v),
\]

where \( \left( \Gamma \liminf_{\varepsilon \to 0} E_{\varepsilon} \right)(v) := \inf \left\{ \liminf_{\varepsilon \to 0} E_{\varepsilon}(u_{\varepsilon}) : u_{\varepsilon} \to v \text{ in } L^p(\Sigma; \mathbb{R}^3) \right\} \) and \( \left( \Gamma \limsup_{\varepsilon \to 0} E_{\varepsilon} \right)(v) := \inf \left\{ \limsup_{\varepsilon \to 0} E_{\varepsilon}(u_{\varepsilon}) : u_{\varepsilon} \to v \text{ in } L^p(\Sigma; \mathbb{R}^3) \right\}. \)

For a deeper discussion of the \( \Gamma \)-convergence theory we refer to the book [13]. Definition 2.2 is equivalent to assertions (i) and (ii) in Definition 2.1 with “\( \pi(u_{\varepsilon}) \to v \)” replaced by “\( u_{\varepsilon} \to v \).” It is then obvious that

Lemma 2.3. \( E_{\text{mem}} = \Gamma(\pi) \lim_{\varepsilon \to 0} E_{\varepsilon} \) if and only if \( E_{\varepsilon} = \Gamma \lim_{\varepsilon \to 0} E_{\varepsilon}. \)
As in [1], suppose that the exterior loads derive from a potential \( \Psi : \Sigma_1 \times \mathbb{R}^3 \to \mathbb{R} \) given by \( \Psi((x, x_3), \zeta) := (\psi(x, x_3), \zeta) + |\zeta|^p \), where \( \psi : \Sigma_1 \to \mathbb{R}^3 \) is continuous and \( \langle \cdot, \cdot \rangle \) denotes the scalar product in \( \mathbb{R}^3 \), and define \( L_\varepsilon : W^{1,p}(\Sigma; \mathbb{R}^3) \to \mathbb{R} \) and 
\[ L_\varepsilon(u) := \frac{1}{\varepsilon} \int_{\Sigma} \Psi((x, x_3), u(x, x_3)) \, dx \]
by

Then, using similar arguments to those in [1, proof of Proposition 3.1 p. 141 and proof of Theorem 2.1 p. 145], we obtain

**Proposition 2.4.** Assume that \( E_\varepsilon \) in (2) \( \Gamma(\pi) \)-converges to \( E_{\text{mem}} \) in (3) as \( \varepsilon \to 0 \), and consider \( \{u_\varepsilon\}_\varepsilon \subset W^{1,p}(\Sigma; \mathbb{R}^3) \) such that

\[ E_\varepsilon(u_\varepsilon) + L_\varepsilon(u_\varepsilon) = \inf \left\{ E_\varepsilon(u) + L_\varepsilon(u) : u \in W^{1,p}(\Sigma; \mathbb{R}^3) \right\} \to 0 \text{ as } \varepsilon \to 0. \]

Then, \( \{\pi_\varepsilon(u_\varepsilon)\}_\varepsilon \) is weakly relatively compact in \( W^{1,p}(\Sigma; \mathbb{R}^3) \) and each of its cluster points \( \bar{v} \) satisfies

\[ E_{\text{mem}}(\bar{v}) + L_{\text{mem}}(\bar{v}) = \min \left\{ E_{\text{mem}}(v) + L_{\text{mem}}(v) : v \in W^{1,p}(\Sigma; \mathbb{R}^3) \right\}. \]

The method used in this paper for passing from (2) to (3) was initiated by Anza Hafsa in [2, 3] (see also Mandallena [17, 18] and Anza Hafsa-Mandallena [4, 5] for the relaxation case). It first consists of studying the \( \Gamma \)-convergence of \( E_\varepsilon \) as \( \varepsilon \to 0 \) (see Sect. 2.2), and then establishing an integral representation for the corresponding \( \Gamma \)-limit (see Sect. 2.3).

### 2.2. \( \Gamma \)-convergence of \( E_\varepsilon \) as \( \varepsilon \to 0 \).

From now on, given a bounded open set \( D \subset \mathbb{R}^2 \) with \( |\partial D| = 0 \), we denote by \( \text{Aff}(D; \mathbb{R}^3) \) the space of all continuous piecewise affine functions from \( D \) to \( \mathbb{R}^3 \), i.e., \( v \in \text{Aff}(D; \mathbb{R}^3) \) if and only if \( v \) is continuous and there exists a finite family \( (D_i)_{i \in I} \) of open disjoint subsets of \( D \) such that \( |\partial D_i| = 0 \) for all \( i \in I \), \( |D \setminus \bigcup_{i \in I} D_i| = 0 \) and for every \( i \in I \), \( \nabla v(x) = \zeta_i \) in \( D_i \) with \( \zeta_i \in M^{3 \times 2} \) (where \( | \cdot | \) denotes the Lebesgue measure in \( \mathbb{R}^2 \)).

**Remark 2.5.** From Ekeland-Temam [14], we know that \( \text{Aff}^{ET}(D; \mathbb{R}^3) \) is strongly dense in \( W^{1,p}(D; \mathbb{R}^3) \), where \( \text{Aff}^{ET}(D; \mathbb{R}^3) \) is defined as follows: \( v \in \text{Aff}^{ET}(D; \mathbb{R}^3) \) if and only if \( v \) is continuous and there exists a finite family \( (D_i)_{i \in I} \) of open disjoint subsets of \( D \) such that \( |\partial D_i| = 0 \) for all \( i \in I \), \( |D \setminus \bigcup_{i \in I} D_i| = 0 \) and for every \( i \in I \), the restriction of \( v \) to \( D_i \) is affine. As \( \text{Aff}^{ET}(D; \mathbb{R}^3) \subset \text{Aff}(D; \mathbb{R}^3) \subset W^{1,p}(D; \mathbb{R}^3) \), it is clear that \( \text{Aff}(D; \mathbb{R}^3) \) is also strongly dense in \( W^{1,p}(D; \mathbb{R}^3) \). (Note that the fact of considering \( \text{Aff}(D; \mathbb{R}^3) \) instead of \( \text{Aff}^{ET}(D; \mathbb{R}^3) \) plays an important role in our analysis, see Remarks A.2 and A.9.)

Let \( \mathcal{E} : W^{1,p}(\Sigma; \mathbb{R}^3) \to [0, +\infty] \) be defined by

\[ \mathcal{E}(v) := \begin{cases} \int_{\Sigma} W_0(\nabla v(x)) \, dx & \text{if } v \in \text{Aff}(\Sigma; \mathbb{R}^3) \\ +\infty & \text{otherwise} \end{cases} \]

where, as in [16], \( W_0 : M^{3 \times 2} \to [0, +\infty] \) is given by

\[ W_0(\xi) := \inf_{\zeta \in \mathbb{R}^3} W(\xi \mid \zeta) \]

with \( (\xi \mid \zeta) \) denoting the element of \( M^{3 \times 3} \) corresponding to \( (\xi, \zeta) \in M^{3 \times 2} \times \mathbb{R}^3 \). (As \( W \) is coercive, it is easy to see that \( W_0 \) is coercive, i.e., \( W_0(\xi) \geq C|\xi|^p \) for all \( \xi \in M^{3 \times 2} \) and some \( C > 0 \).) Note that conditions (C1) and (C2) imply \( W_0 \) is not of \( p \)-polynomial growth. In fact, we have

**Lemma 2.6.** Denote by \( \xi_1 \land \xi_2 \) the cross product of vectors \( \xi_1, \xi_2 \in \mathbb{R}^3 \).
The main result of the paper is the following. We need the concepts of quasiconvex envelope and rank-one convex envelope. By Theorem 2.12 and 2.14, we can assert that there exists \( c_3 > 0 \) such that for all \( \xi = (\xi_1 | \xi_2) \in M^{3 \times 2} \),

\[
\text{if } |\xi_1 \wedge \xi_2| \geq \delta \text{ then } W_0(\xi) \leq c_3(1 + |\xi|^p).
\]

Proof. (i) Given \( \xi = (\xi_1 | \xi_2) \), if \( W_0(\xi_1 | \xi_2) < +\infty \) (resp. \( W_0(\xi_1 | \xi_2) = +\infty \)) then \( W(\xi | \zeta) < +\infty \) (resp. \( W(\xi | \zeta) = +\infty \)) for some \( \zeta \in \mathbb{R}^3 \) (resp. for all \( \zeta \in \mathbb{R}^3 \)), and so \( \xi_1 \wedge \xi_2 \neq 0 \) (resp. \( \xi_1 \wedge \xi_2 = 0 \)) by (C1).

(ii) Let \( \delta > 0 \) and let \( \xi = (\xi_1 | \xi_2) \) be such that \( |\xi_1 \wedge \xi_2| \geq \delta \). Setting \( \zeta := \frac{\xi_1 \wedge \xi_2}{|\xi_1 \wedge \xi_2|} \), we have \( \det(\xi | \zeta) \geq \delta \), and using (C2) we can assert that there exists \( c_3 > 0 \), which does not depend on \( \xi \), such that \( W_0(\xi) \leq c_3(1 + |\xi|^p) \).

Assume furthermore that

\( \xi_1 \wedge \xi_2 = 0 \) for all \( \xi \in M^{3 \times 2} \) and all \( \zeta \in \mathbb{R}^3 \).

The main result of the paper is the following.

**Theorem 2.7.** Under (C1), (C2) and (C3), we have \( \Gamma \)-lim_{\varepsilon \to 0} \( \mathcal{E}_\varepsilon = \overline{\mathcal{E}} \) and \( \overline{\mathcal{E}} : W^{1,p}(\Sigma; \mathbb{R}^3) \to [0, +\infty) \) given by

\[
\overline{\mathcal{E}}(v) := \inf \left\{ \liminf_{n \to +\infty} \mathcal{E}(v_n) : W^{1,p}(\Sigma; \mathbb{R}^3) \ni v_n \to v \text{ in } L^p(\Sigma; \mathbb{R}^3) \right\}.
\]

The proof of Theorem 2.7 is established in Section 4. It uses Theorem 3.4 (see Section 3) and Theorem 2.8 whose proof is contained in [6].

**Theorem 2.8.** If (C2) holds then \( \mathcal{I} = \overline{\mathcal{I}} = \mathcal{I} : W^{1,p}(\Sigma; \mathbb{R}^3) \to [0, +\infty) \) given by

\[
\mathcal{I}(v) := \inf \left\{ \liminf_{n \to +\infty} \int_\Sigma W_0(\nabla v_n(x)) dx : W^{1,p}(\Sigma; \mathbb{R}^3) \ni v_n \to v \text{ in } L^p(\Sigma; \mathbb{R}^3) \right\}.
\]

**Remark 2.9.** Theorem 2.7 can be applied when \( W : M^{3 \times 3} \to [0, +\infty) \) is given by

\[
W(F) := h(\det |F|) + |F|^p,
\]
where \( h : [0, +\infty] \to [0, +\infty] \) is a continuous functions such that:

- \( h(t) = +\infty \) if and only if \( t = 0 \);
- for every \( \delta > 0 \), there exists \( r_3 > 0 \) such that \( h(t) \leq r_3 \) for all \( t \geq \delta \).

**2.3. Integral representation of \( \overline{\mathcal{E}} \).** Our framework leads us to deal with relaxation of nonconvex integral functionals which are not of \( p \)-polynomial growth. Such relaxation problems were studied in Ben Belgacem [10] and Anza Hafsa-Mandallena [6] (see also Carbone-De Arcangelis [11] for the scalar case). To state the integral representation theorems obtained in these papers (see Theorems 2.12 and 2.14), we need the concepts of quasiconvex envelope and rank-one convex envelope.

**Definition 2.10.** Let \( f : M^{3 \times 2} \to [0, +\infty) \) be a Borel measurable function.

(i) We say that \( f \) is quasiconvex if for every \( \xi \in M^{3 \times 2} \), every bounded open set \( D \subset \mathbb{R}^2 \) with \( |\partial D| = 0 \) and every \( \phi \in W^{1,\infty}_0(D; \mathbb{R}^3) \),

\[
f(\xi) \leq \frac{1}{|D|} \int_D f(\xi + \nabla \phi(x)) dx.
\]

(ii) By the quasiconvex envelope of \( f \), we mean the unique function (when it exists) \( Qf : M^{3 \times 2} \to [0, +\infty] \) such that:

- \( Qf \) is Borel measurable, quasiconvex and \( Qf \leq f \);
- for all \( g : M^{3 \times 2} \to [0, +\infty] \), if \( g \) is Borel measurable, quasiconvex and \( g \leq f \), then \( g \leq Qf \).
(Usually, for simplicity, we say that $Qf$ is the greatest quasiconvex function which less than or equal to $f$.)

(iii) We say that $f$ is rank one convex if for every $\alpha \in [0, 1]$ and every $\xi, \xi' \in M^{3 \times 2}$ with $\text{rank}(\xi - \xi') = 1$,

$$f(\alpha \xi + (1 - \alpha)\xi') \leq \alpha f(\xi) + (1 - \alpha)f(\xi').$$

(iv) By the rank one convex envelope of $f$, that we denote by $Rf$, we mean the greatest rank one convex function which less than or equal to $f$.

**Remark 2.11.** It is well known that if $f$ is quasiconvex and continuous then $f$ is rank one convex. This is false for a general Borel measurable $f$ (see [8, Example 3.5]).

2.3.1. *Ben Belgacem’s theorem.* In [10, Section 5.1] Ben Belgacem asserts that if $W_0$ satisfies ($\mathcal{C}_2$) then $RW_0$ is of $p$-polynomial growth, so that is $Q[RW_0]$. (As $W_0$ is coercive, it is easy to see that $RW_0$ is coercive.) Then, using his main result [10, Theorem 3.1], he obtains

**Theorem 2.12.** If ($\mathcal{C}_2$) holds then for every $v \in W^{1,p}(\Sigma; \mathbb{R}^3)$,

$$\mathcal{E}(v) = \int_\Sigma Q[RW_0](\nabla v(x))dx.$$

According to Remark 2.11, in Theorem 2.12 we cannot know if $Q[RW_0] = QW_0$. In fact, under ($\mathcal{C}_2$) the latter equality holds (see Remark 2.15).

**Remark 2.13.** In [9, Theorem 1] Ben Belgacem announced to have established the $\Gamma(\pi)$-convergence of $E_\varepsilon$ to $E_{\text{mem}}$ as $\varepsilon \to 0$ under the two (more physical) conditions:

- ($\mathcal{C}_1$) $W(F) = +\infty$ if and only if $\det F \leq 0$;
- ($\mathcal{C}_2$) for every $\delta > 0$, there exists $c_\delta > 0$ such that for all $F \in M^{3 \times 3}$,

$$\text{if } \det F \geq \delta \text{ then } W(F) \leq c_\delta (1 + |F|^p).$$

In [10], which is the paper corresponding to the note [9], the statement [9, Theorem 1] is not proved. To our knowledge, under ($\mathcal{C}_1$) and ($\mathcal{C}_2$) the problem of passing from (2) to (3) by using $\Gamma(\pi)$-convergence is still open.

2.3.2. *An alternative theorem.* Define $ZW_0 : M^{3 \times 2} \to [0, +\infty]$ by

$$ZW_0(\xi) := \inf \left\{ \int_Y W_0(\xi + \nabla \phi(y))dy : \phi \in \text{Aff}_0(Y; \mathbb{R}^3) \right\}.$$  

with $Y := [0, 1]^2$ and $\text{Aff}_0(D; \mathbb{R}^3) := \{ \phi \in \text{Aff}(Y; \mathbb{R}^3) : \phi = 0 \text{ on } Y \}$. (As $W_0$ is coercive, it is easy to see that $ZW_0$ is coercive.) In [6], under ($\mathcal{C}_2$), we prove that $ZW_0$ is of $p$-polynomial growth and continuous (see Propositions A.3 and A.1(iii)), and that $ZW_0$ is the quasiconvex envelope of $W_0$, i.e., $ZW_0 = QW_0$ (see Propositions A.5). Theorem 2.14 is contained in [6] (for the convenience of the reader, we give the proof in appendix).

**Theorem 2.14.** If ($\mathcal{C}_2$) holds then for every $v \in W^{1,p}(\Sigma; \mathbb{R}^3)$,

$$\mathcal{E}(v) = \int_\Sigma QW_0(\nabla v(x))dx.$$  

**Remark 2.15.** If ($\mathcal{C}_2$) holds then $Q[RW_0] = QW_0$. Indeed, by Proposition A.3, $ZW_0(\xi) \leq c(1 + |\xi|^p)$ for all $\xi \in M^{3 \times 2}$ and some $c > 0$. Then $ZW_0$ is finite, and so $ZW_0$ is continuous by Proposition A.1(iii). It follows that $ZW_0 = QW_0$ (see the proof of Proposition A.5). Thus $QW_0$ is continuous, hence $QW_0$ is rank-one convex (see Remark 2.11), and the result follows.
2.4. $\Gamma(\pi)$-convergence of $E_\varepsilon$ to $E_{\text{mem}}$ as $\varepsilon \to 0$. According to Lemmas 2.3 and 2.6(ii), a direct consequence of Theorems 2.7 and 2.14 is the following.

**Corollary 2.16.** Let assumptions (C1), (C2) and (C3) hold. Then as $\varepsilon \to 0$, $E_\varepsilon$ in (2) $\Gamma(\pi)$-converge to $E_{\text{mem}}$ in (3) with $W_{\text{mem}} = QW_0$.

3. Representation of $\mathcal{E}$

The goal of this section is to show Theorem 3.4. To this end, we begin by proving three lemmas. From now on, we set

$$
\text{Aff}_*(\Sigma; \mathbb{R}^3) := \left\{ v \in \text{Aff}(\Sigma; \mathbb{R}^3) : \partial_1 v(x) \wedge \partial_2 v(x) \neq 0 \text{ a.e. in } \Sigma \right\},
$$

where $\partial_1 v(x)$ (resp. $\partial_2 v(x)$) denotes the partial derivative of $v$ at $x = (x_1, x_2)$ with respect to $x_1$ (resp. $x_2$). By definition, to every $v \in \text{Aff}_*(\Sigma; \mathbb{R}^3)$ there corresponds a finite family $(V_i)_{i \in I}$ of open disjoint subsets of $\Sigma$ such that:
- $|\partial V_i| = 0$ for all $i \in I$;
- $|\Sigma \setminus \bigcup_{i \in I} V_i| = 0$;
- for every $i \in I$, $\nabla v(x) = \xi_i$ in $V_i$ with $\xi_i = (\xi_i,1 | \xi_i,2) \in \mathbb{M}^{3 \times 2}$;
- $\xi_{i,1} \wedge \xi_{i,2} \neq 0$ for all $i \in I$.

**Lemma 3.1.** If (C1) holds then $\text{dom} \mathcal{E} = \text{Aff}_*(\Sigma; \mathbb{R}^3)$, where $\text{dom} \mathcal{E}$ is the effective domain of $\mathcal{E}$.

*Proof.* It is a direct consequence of Lemma 2.6(i). \hfill $\Box$

Given $v \in \text{Aff}_*(\Sigma; \mathbb{R}^3)$, for every $i \in I$ and every integer $j \geq 1$, we consider the subsets $U_{i,j}^-$ and $U_{i,j}^+$ of $\mathbb{R}^3$ given by

$$
U_{i,j}^- := \left\{ \xi \in \mathbb{R}^3 : \det(\xi_i | \xi) \leq -\frac{1}{j} \right\} \text{ and } U_{i,j}^+ := \left\{ \xi \in \mathbb{R}^3 : \det(\xi_i | \xi) \geq \frac{1}{j} \right\}.
$$

Here are some elementary properties of these sets:

(P1) both $U_{i,j}^-$ and $U_{i,j}^+$ are nonempty convex subsets of $\mathbb{R}^3$;

(P2) $U_{i,j}^- \cup U_{i,j}^+ = \left\{ \xi \in \mathbb{R}^3 : |\det(\xi_i | \xi)| \geq \frac{1}{j} \right\}$;

(P3) $U_{i,1}^- \subset U_{i,2}^- \subset \cdots \subset U_{i,j}^- \subset \cdots \subset U_{i,j}^+ = \left\{ \xi \in \mathbb{R}^3 : \det(\xi_i | \xi) < 0 \right\}$;

(P4) $\bigcap_{i \in I} U_{i,j}^- \subset U_{i,1}^- \subset U_{i,2}^- \subset \cdots \subset U_{i,j}^- = \left\{ \xi \in \mathbb{R}^3 : \det(\xi_i | \xi) > 0 \right\}$.

**Lemma 3.2.** Given $v \in \text{Aff}_*(\Sigma; \mathbb{R}^3)$, there exist $j_v \geq 1$ and two subsets $I^- \text{ and } I^+$ of $I$, with $I^- \cup I^+ = I$ and $I^- \cap I^+ = \emptyset$, such that for all $j \geq j_v$,

$$
\left( \bigcap_{i \in I^-} U_{i,j}^- \right) \cap \left( \bigcap_{i \in I^+} U_{i,j}^+ \right) \neq \emptyset.
$$

*Proof.* For every $i \in I$, define the hyperplane $H_i$ of $\mathbb{R}^3$ by $H_i := \left\{ \xi \in \mathbb{R}^3 : \det(\xi_i | \xi) = 0 \right\}$. It is obvious that $\bigcup_{i \in I} H_i \neq \mathbb{R}^3$, and so there exists $\xi \in \mathbb{R}^3$ such that $\det(\xi_i | \xi) \neq 0$ for all $i \in I$. Taking (P2) into account, we deduce the existence of an integer $j_v \geq 1$ for which $\xi \in \bigcap_{i \in I^-} (U_{i,j_v}^- \cup U_{i,j_v}^+)$. Hence, there are two subsets $I^-$ and $I^+$ of $I$, with $I^- \cup I^+ = I$ and $I^- \cap I^+ = \emptyset$, such that $(\bigcap_{i \in I^-} U_{i,j_v}^-) \cap (\bigcap_{i \in I^+} U_{i,j_v}^+) \neq \emptyset$, and the lemma follows by using (P3) and (P4). \hfill $\Box$

Setting $V := \cup_{i \in I} V_i$, for every $j \geq j_v$, with $j_v$ given by Lemma 3.2, we define $\Gamma_{j_v} : \Sigma \to \mathbb{R}^3$ by

$$
\Gamma_{j_v}(x) := \begin{cases} 
U_{i,j}^- & \text{if } x \in V_i \text{ with } i \in I^- \\
U_{i,j}^+ & \text{if } x \in V_i \text{ with } i \in I^+ \\
\bigcap_{i \in I^-} U_{i,j}^- \cap \bigcap_{i \in I^+} U_{i,j}^+ & \text{if } x \in \Sigma \setminus V.
\end{cases}
$$
The nonlinear membrane energy

(8) inf

function such that

From (C

reduced to prove that

Proof. It is obvious that

Prove then the converse inequality. By Lemma 3.2, (∩

it is easy to verify that

Clearly,

In the sequel, given Γ : Σ ⊃ R^3, we set

C(Σ; Γ) := \{ φ ∈ C(Σ; R^3) : φ(x) ∈ Γ(x) a.e. in Σ \},

where C(Σ; R^3) denotes the space of all continuous functions from Σ to R^3.

Lemma 3.3. Given v ∈ Aff_*(Σ; R^3) and j ≥ j_v, if (C_2) holds, then

\[ \inf_{\phi \in C(\Sigma; \Gamma_i)} \int_{\Sigma} W(\nabla v(x) \mid \phi(x))dx = \inf_{\phi \in C(\Sigma; \Gamma_i)} \int_{\Sigma} W(\nabla v(x) \mid \zeta)dx. \]

Proof. Fix any n ≥ 1. Consider α_n : Σ → R given by α_n(x) := h(\text{dist}(x, Σ \setminus V)), where dist(x, Σ \setminus V) := inf \{ |x - y| : y ∈ Σ \setminus V \} and h : [0, +∞) → [0, 1] is a continuous function such that h(0) = 0 and h(t) = 1 for all t ≥ 1. Define φ_n : Σ → R by

\[ φ_n(x) := (1 - α_n(x))\tilde{c} + α_n(x)\zeta_i. \]

Clearly, φ_n is continuous and φ_n(x) ∈ Γ_i(x) for all x ∈ Σ since Γ_i(x) is convex, and so φ_n ∈ C(Σ; Γ_i). Using (C_2) we deduce that sup_{n≥1} W(\nabla v(\cdot) \mid φ_n(\cdot)) ∈ L^1(Σ).

Recalling that W is continuous and taking (5) into account, it is easy to see that

\[ \lim_{n → +∞} \int_{\Sigma} W(\nabla v(x) \mid φ_n(x))dx = \inf_{\zeta ∈ Γ_i(x)} \int_{\Sigma} W(\nabla v(x) \mid \zeta)dx \]

by Lebesgue’s dominated convergence theorem, and the proof is complete. □

For every j ≥ j_v, we define Λ^j_i : Σ → R^3 by

\[ Λ^j_i(x) := \begin{cases} U^-_{i,j} \cup U^+_{i,j} & \text{if } x ∈ V_i \\ Γ_i(x) & \text{if } x ∈ Σ \setminus V. \end{cases} \]

Here is our (non integral) representation theorem for E.

Theorem 3.4. If (C_1), (C_2) and (C_3) hold, then for every v ∈ domE,

\[ E(v) = \inf_{j ≥ j_v} \inf_{\phi ∈ C(\Sigma; Λ_i)} \int_{\Sigma} W(\nabla v(x) \mid \phi(x))dx. \]

Proof. By Lemma 3.1, domE = Aff_*(Σ; R^3). Fix v ∈ Aff_*(Σ; R^3) and denote by \( \hat{E}(v) \) the right-hand side of (6). It is easy to verify that \( E(v) ≤ \hat{E}(v) \). We are thus reduced to prove that

\[ \hat{E}(v) ≤ E(v). \]

From (C_3) we see that for every j ≥ j_v and every x ∈ Σ,

\[ \inf_{\zeta ∈ Γ_i(x)} W(\nabla v(x) \mid \zeta) = \inf_{\zeta ∈ Λ_i(x)} W(\nabla v(x) \mid \zeta). \]
Noticing that $\Gamma_{\varepsilon}(x) \subset \Lambda_{\varepsilon}(x)$ for all $x \in \Sigma$ and using Lemma 3.3 together with (8), we obtain
\begin{equation}
\hat{E}(v) \leq \inf_{j \geq j_0} \left[ \inf_{\zeta \in \Lambda_{\varepsilon}(x)} \int_{\Sigma} W(\nabla v(x) \mid \zeta) \, dx \right].
\end{equation}

On the other hand, $\inf_{\zeta \in \Lambda_{\varepsilon}(x)} W(\nabla v(x) \mid \zeta) \in L^1(\Sigma)$ by (C2), and from (P3) and (P4) we deduce that if $x \in V$ then $\Lambda_{\varepsilon}(x) \subset \Lambda_{\varepsilon}^{j_{v+1}} \subset \cdots \subset \bigcup_{j \geq j_0} \Lambda_{\varepsilon}(x)$ with $\bigcup_{j \geq j_0} \Lambda_{\varepsilon}(x) = \{ \zeta \in \mathbb{R}^3 : \det(\nabla v(x) \mid \zeta) \neq 0 \}$. Hence $\inf_{\zeta \in \Lambda_{\varepsilon}(x)} W(\nabla v(x) \mid \zeta)_{j \geq j_0}$ is non-increasing and for every $x \in V$,
\begin{equation}
\inf_{j \geq j_0} \inf_{\zeta \in \Lambda_{\varepsilon}(x)} W(\nabla v(x) \mid \zeta) = W_0(\nabla v(x)),
\end{equation}
and (7) follows from (9) and (10) by using the monotone convergence theorem. \qed

4. PROOF OF THEOREM 2.7

In this section we prove Theorem 2.7. Since $\Gamma-\lim \inf_{\varepsilon \to 0} E_\varepsilon \leq \Gamma-\lim \sup_{\varepsilon \to 0} E_\varepsilon$, we only need to show that:
\begin{enumerate}[(a)]
\item $\bar{E} \leq \Gamma-\lim \inf_{\varepsilon \to 0} E_\varepsilon$;
\item $\Gamma-\lim \sup_{\varepsilon \to 0} E_\varepsilon \leq \bar{E}$.
\end{enumerate}
In the sequel, we follow the notation used in Section 3.

4.1. Proof of (a). Let $v \in W^{1,p}(\Sigma; \mathbb{R}^3)$ and let \{v_\varepsilon\}_\varepsilon \subset W^{1,p}(\Sigma; \mathbb{R}^3)$ be such that $v_\varepsilon \to v$ in $L^p(\Sigma; \mathbb{R}^3)$. We have to prove that
\begin{equation}
\liminf_{\varepsilon \to 0} E_\varepsilon(v_\varepsilon) \geq \bar{E}(v).
\end{equation}
Without loss of generality we can assume that $\sup_{\varepsilon > 0} E_\varepsilon(v_\varepsilon) < +\infty$. To every $\varepsilon > 0$ there corresponds $u_\varepsilon \in \pi_{\varepsilon}^{-1}(v_\varepsilon)$ such that
\begin{equation}
E_\varepsilon(u_\varepsilon) = \int_{\Sigma} W(\partial_1 u_\varepsilon(x, x_3) \mid \partial_2 u_\varepsilon(x, x_3) \mid \frac{1}{\varepsilon} \partial_3 u_\varepsilon(x, x_3)) \, dx \, dx_3.
\end{equation}
Defining $\hat{u}_\varepsilon : \Sigma_1 \to \mathbb{R}^3$ by $\hat{u}_\varepsilon(x, x_3) := u_\varepsilon(x, x_3)$ we have
\begin{equation}
E_\varepsilon(u_\varepsilon) = \int_{\Sigma_1} W(\partial_1 \hat{u}_\varepsilon(x_3) \mid \partial_2 \hat{u}_\varepsilon(x_3) \mid \frac{1}{\varepsilon} \partial_3 \hat{u}_\varepsilon(x_3)) \, dx \, dx_3.
\end{equation}
Using the coercivity of $W$, we deduce that $\|\partial_2 \hat{u}_\varepsilon\|_{L^p(\Sigma_1; \mathbb{R}^3)} \leq c_\varepsilon$ for all $\varepsilon > 0$ and some $c > 0$, and so $\|\hat{u}_\varepsilon - v_\varepsilon\|_{L^p(\Sigma_1; \mathbb{R}^3)} \leq c\varepsilon$ by Poincaré–Wirtinger’s inequality, where $c\varepsilon > 0$ is a constant which does not depend on $\varepsilon$. It follows that $\hat{u}_\varepsilon \to v$ in $L^p(\Sigma_1; \mathbb{R}^3)$. For $x_3 \in [-\frac{1}{2}, \frac{1}{2}]$, let $w_{x_3}^{\varepsilon} \in W^{1,p}(\Sigma; \mathbb{R}^3)$ given by $w_{x_3}^{\varepsilon}(x) := \hat{u}_\varepsilon(x, x_3)$. Then (up to a subsequence) $w_{x_3}^{\varepsilon} \to v$ in $L^p(\Sigma; \mathbb{R}^3)$ for a.e. $x_3 \in [-\frac{1}{2}, \frac{1}{2}]$. Taking (12) and (13) into account and using Fatou’s lemma, we obtain
\begin{equation}
\liminf_{\varepsilon \to 0} E_\varepsilon(v_\varepsilon) \geq \int_{\frac{1}{4}}^{\frac{3}{4}} \left( \liminf_{\varepsilon \to 0} \int_{\Sigma} W_0(\nabla w_{x_3}^{\varepsilon}(x)) \, dx \right) \, dx_3,
\end{equation}
and so $\liminf_{\varepsilon \to 0} E_\varepsilon(v_\varepsilon) \geq \bar{E}(v)$, and (11) follows by using Theorem 2.8. \qed

4.2. Proof of (b). By Lemma 3.1, $\text{dom}E = \text{Aff}_*(\Sigma; \mathbb{R}^3)$. As $\Gamma-\lim sup_{\varepsilon \to 0} E_\varepsilon$ is lower semicontinuous with respect to the strong topology of $L^p(\Sigma; \mathbb{R}^3)$ (see [13, Proposition 6.8 p. 57]), it is sufficient to prove that for every $v \in \text{Aff}_*(\Sigma; \mathbb{R}^3)$,
\begin{equation}
\limsup_{\varepsilon \to 0} E_\varepsilon(v) \leq E(v).
\end{equation}
Given $v \in \text{Aff}_*(\Sigma; \mathbb{R}^3)$, fix any $j \geq j_0$ (with $j_0$ given by Lemma 3.2) and any $n \geq 1$. Using Theorem 3.4 we obtain the existence of $\phi \in C(\Sigma; \Lambda_{j_0})$ such that
\begin{equation}
\int_{\Sigma} W(\nabla v(x) \mid \phi(x)) \, dx \leq E(v) + \frac{1}{n}.
\end{equation}
By Stone-Weierstrass’s approximation theorem, there exists \( \{ \phi_k \}_{k \geq 1} \subset C^\infty(\Sigma; \mathbb{R}^3) \) such that
\[
\phi_k \to \phi \text{ uniformly as } k \to +\infty.
\]

We claim that:
\[
\begin{align*}
(c_1) & \quad |\det(\nabla v(x) | \phi_k(x))| \geq \frac{1}{2j} \quad \text{for all } x \in V, \text{ all } k \geq k_v \text{ and some } k_v \geq 1; \\
(c_2) & \quad \lim_{k \to +\infty} \int_\Sigma W(\nabla v(x) | \phi_k(x))dx = \int_\Sigma W(\nabla v(x) | \phi(x))dx.
\end{align*}
\]
Indeed, setting \( \mu_v := \max_{i \in I} [\xi_{i,1} \wedge \xi_{i,2}] (\mu_v > 0) \) and using (16), we deduce that there exists \( k_v \geq 1 \) such that for every \( k \geq k_v \),
\[
\sup_{x \in \Sigma} |\phi_k(x) - \phi(x)| < \frac{1}{2j\mu_v}.
\]

Let \( x \in V_i \) with \( i \in I \), and let \( k \geq k_v \). As \( \phi \in C(\Sigma, \Lambda_i) \) we have
\[
|\det(\xi_i | \phi_k(x))| \geq \frac{1}{j} - |\det(\xi_i | \phi_k(x) - \phi(x))|.
\]

Noticing that \( |\det(\xi_i | \phi_k(x) - \phi(x))| \leq |\xi_{i,1} \wedge \xi_{i,2}| |\phi_k(x) - \phi(x)| \), from (17) and (18) we deduce that \( |\det(\xi_i | \phi_k(x))| \geq \frac{1}{2j} \), and (c1) is proved. Combining (c1) with (c2) we see that \( \sup_{k \geq k_v} W(\nabla v(\cdot) | \phi_k(\cdot)) \in L^1(\Sigma) \). As \( W \) is continuous we have
\[
\lim_{k \to +\infty} W(\nabla v(x) | \phi_k(x)) = W(\nabla v(x) | \phi(x)) \quad \text{for all } x \in V, \quad \text{and (c2) follows by using Lebesgue’s dominated convergence theorem, which completes the claim.}
\]

Fix any \( k \geq k_v \) and define \( \theta : [0, \frac{1}{2}] \to \mathbb{R} \) by \( \theta(x_3) := \min_{i \in I} \inf_{x \in \Sigma} |\det(\xi_i + x_3 \nabla \phi_k(x) | \phi_k(x))| \). Clearly \( \theta \) is continuous. By (c1) we have \( \theta(0) \geq \frac{1}{2j} \), and so there exists \( \eta_v \in [0, \frac{1}{2}] \) such that \( \theta(x_3) \geq \frac{1}{4j} \) for all \( x_3 \in ]-\eta_v, \eta_v[ \). Let \( u_k : \Sigma \to \mathbb{R} \) be given by \( u_k(x, x_3) := v(x) + x_3 \phi_k(x) \). From the above it follows that
\[
(c_3) \quad |\det \nabla u_k(x, \varepsilon x_3)| \geq \frac{1}{2j} \quad \text{for all } \varepsilon \in ]0, \eta_v[ \text{ and all } (x, x_3) \in V \times ]-\frac{1}{2}, \frac{1}{2}].
\]

As in the proof of (c2), from (c3) together with (c2) and the continuity of \( W \), we obtain
\[
\lim_{\varepsilon \to 0} E_\varepsilon(u_k) = \lim_{\varepsilon \to 0} \int_{\Sigma} W(\nabla u_k(x, \varepsilon x_3))dx = \int_{\Sigma} W(\nabla v(x) | \phi_k(x))dx.
\]

For every \( \varepsilon > 0 \) and every \( k \geq k_v \), since \( \pi_\varepsilon(u_k) = v \) we have \( E_\varepsilon(v) \leq E_\varepsilon(u_k) \).

Using (19), (c2) and (15), we deduce that
\[
\limsup_{\varepsilon \to 0} E_\varepsilon(v) \leq E(v) + \frac{1}{n},
\]
and (14) follows by letting \( n \to +\infty. \)

**Appendix A. Representation of \( \Sigma \)**

Theorems 2.8 and 2.14 are contained in [6]. For the convenience of the reader, we give the proofs in this appendix.

**A.1. Preliminary results.** Throughout this appendix we will use Proposition A.1 which gives three interesting properties of \( ZW_0 : \mathbb{M}^{3 \times 2} \to [0, +\infty] \) defined by (4). The proof can be adapted from Fonseca [15, Lemma 2.16, Lemma 2.20, Theorem 2.17 and Proposition 2.3] (the detailed verification is left to the reader).

**Proposition A.1.**

(i) For every bounded open set \( D \subset \mathbb{R}^2 \) with \( |\partial D| = 0 \) and every \( \xi \in \mathbb{M}^{3 \times 2} \),
\[
ZW_0(\xi) = \inf \left\{ \frac{1}{|D|} \int_D W_0(\xi + \nabla \phi(y))dy : \phi \in \text{Aff}_0(D; \mathbb{R}^3) \right\}.
\]
(ii) For every bounded open set $D \subset \mathbb{R}^2$ with $|\partial D| = 0$, every $\xi \in \mathbb{M}^{3\times 2}$ and every $\phi \in \text{Aff}_0(D; \mathbb{R}^3)$,
\[ ZW_0(\xi) \leq \frac{1}{|D|} \int_D ZW_0(\xi + \nabla \phi(x))dx. \]

(iii) If $ZW_0$ is finite then $ZW_0$ is continuous.

Remark A.2. In [15], Fonseca proved that $ZW_0 : \mathbb{M}^{3\times 2} \to [0, +\infty]$ defined by
\[ ZW_0(\xi) := \inf \left\{ \int_Y W_0(\xi + \nabla \phi(y))dy : \phi \in W_0^{1,\infty}(Y; \mathbb{R}^3) \right\}, \]
where $W_0^{1,\infty}(Y; \mathbb{R}^3) := \{ \phi \in W^{1,\infty}(Y; \mathbb{R}^3) : \phi = 0 \text{ on } Y \}$, satisfies the three properties:

(j) ([15, Lemma 2.16]) for every bounded open set $D \subset \mathbb{R}^2$ with $|\partial D| = 0$ and every $\xi \in \mathbb{M}^{3\times 2}$,
\[ ZW_0(\xi) = \inf \left\{ \frac{1}{|D|} \int_D W_0(\xi + \nabla \phi(y))dy : \phi \in W_0^{1,\infty}(D; \mathbb{R}^3) \right\}; \]

(jj) ([15, Lemma 2.20]) for every bounded open set $D \subset \mathbb{R}^2$ with $|\partial D| = 0$, every $\xi \in \mathbb{M}^{3\times 2}$ and every $\phi \in \text{Aff}_0^{ET}(D; \mathbb{R}^3) := \{ \phi \in \text{Aff}^{ET}(D; \mathbb{R}^3) : \phi = 0 \text{ on } D \}$ (with $\text{Aff}^{ET}(D; \mathbb{R}^3)$ defined in Remark 2.5),
\[ ZW_0(\xi) \leq \frac{1}{|D|} \int_D ZW_0(\xi + \nabla \phi(x))dx; \]

(jjj) ([15, Theorem 2.17 and Proposition 2.3]) if $ZW_0$ is finite then $ZW_0$ is continuous.

The proof of (j) requires Vitali’s covering theorem. Thus, by an examination of the details, we see that Proposition A.1(i) can be established by following the same method as in [15] if $\text{Aff}_0(D; \mathbb{R}^3)$, where $D \subset \mathbb{R}^2$ is a bounded open set such that $|\partial D| = 0$, satisfies the “stability” condition:

(S) for every $\phi \in \text{Aff}_0(D; \mathbb{R}^3)$, every bounded open set $E \subset \mathbb{R}^2$ with $|\partial E| = 0$ and every finite or countable family $(a_i + \alpha_i E)_{i \in I}$ of disjoint subsets of $D$ with $a_i \in \mathbb{R}^3$, $\alpha_i > 0$ and $|D \setminus \cup_{i \in I}(a_i + \alpha_i E)| = 0$, the function $v : D \to \mathbb{R}^3$
defined by
\[ v(x) = \alpha_i \phi \left( \frac{x - a_i}{\alpha_i} \right) \quad \text{if } x \in a_i + \alpha_i E \]
belongs to $\text{Aff}_0(D; \mathbb{R}^3)$.

In fact, $\text{Aff}_0(D; \mathbb{R}^3)$ has this property, and so Proposition A.1(i) holds. Ben Belgacem was the first to point out the importance of considering a “good” space of continuous piecewise affine functions. In a similar context (see [10]), he introduced the space $\text{Aff}^V(D; \mathbb{R}^3)$ of Vitali continuous piecewise affine functions as follows: $\phi \in \text{Aff}^V(D; \mathbb{R}^3)$ if and only if $\phi$ is continuous and there exists a finite or countable family $(O_i)_{i \in I}$ of disjoint open subsets of $D$ such that $|\partial O_i| = 0$ for all $i \in I$, $|D \setminus \cup_{i \in I} O_i| = 0$, and $\phi(x) = \xi_i + x + a_i$ if $x \in O_i$, where $a_i \in \mathbb{R}^3$, $\xi_i \in \mathbb{M}^{3\times 2}$ and Card$\{\xi_i : i \in I\}$ is finite (setting $D_i := \{ x \in \cup_{i \in I} O_i : \nabla \phi(x) = \xi_i \}$ for all $i \in I$, we see that Card$\{D_i : i \in I\}$ is finite, and so $\text{Aff}^V(D; \mathbb{R}^3) \subset \text{Aff}(D; \mathbb{R}^3)$). Clearly, $\text{Aff}_0^V(D; \mathbb{R}^3) := \{ \phi \in \text{Aff}^V(D; \mathbb{R}^3) : \phi = 0 \text{ on } D \}$ satisfies (S). Ben Belgacem then proved (j) replacing “$W_0^{1,\infty}$” by “$\text{Aff}_0^V$”. As noticed by him, since $\text{Aff}_0^{ET}(D; \mathbb{R}^3)$ does not satisfy (S), if we consider “$\text{Aff}_0^{ET}$” instead of “$W_0^{1,\infty}$”, (j) seems to be false. Moreover, as the proofs of (jj) and (jjj) need (j), if we replace “$W_0^{1,\infty}$” by “$\text{Aff}_0^{ET}$”, we are no longer sure that (jj) and (jjj) are true. However, Ben Belgacem also showed that these properties remain valid if we consider “$\text{Aff}_0^V$” instead of
\text{“}W^{1,\infty}\text{” and “}\text{Aff}_{0,0}^{\text{ET}}\text{”}. As in [10], by carefully checking, we see that the proofs given in [15] can be adapted to establish Proposition A.1(ii) and (iii).

To prove Theorems 2.8 and 2.14 we will need the following proposition.

**Proposition A.3.** If (\(\mathcal{C}_2\)) holds then \(Z_{W_0}(\xi) \leq c(1 + |\xi|^p)\) for all \(\xi \in \mathbb{M}^{3 \times 2}\) and some \(c > 0\).

To show Proposition A.3 we need the following lemma.

**Lemma A.4.** If (\(\mathcal{C}_2\)) holds then for every \(\delta > 0\), there exists \(r_\delta > 0\) such that for every \(\xi = (\xi_1 \mid \xi_2) \in \mathbb{M}^{3 \times 2}\),

\[
\text{if } \min\{|\xi_1 + \xi_2|, |\xi_1 - \xi_2|\} \geq \delta \text{ then } Z_{W_0}(\xi) \leq r_\delta (1 + |\xi|^p).
\]

**Proof.** Let \(\delta > 0\) and \(\xi = (\xi_1 \mid \xi_2) \in \mathbb{M}^{3 \times 2}\) be such that \(\min\{|\xi_1 + \xi_2|, |\xi_1 - \xi_2|\} \geq \delta\).

Then, one of the three possibilities holds:

- (i) \(|\xi_1 \wedge \xi_2| \neq 0\);
- (ii) \(|\xi_1 \wedge \xi_2| = 0\) with \(\xi_1 \neq 0\);
- (iii) \(|\xi_1 \wedge \xi_2| = 0\) with \(\xi_2 \neq 0\).

Set \(D := \{(x_1, x_2) \in \mathbb{R}^2 : x_1 - 1 < x_2 < x_1 + 1 \text{ and } -x_1 - 1 < x_2 < 1 - x_1\}\) and, for each \(t \in \mathbb{R}\), define \(\varphi_t \in \text{Aff}_0(D; \mathbb{R})\) by

\[
\varphi_t(x_1, x_2) := \begin{cases} 
-tx_1 + t(x_2 + 1) & \text{if } (x_1, x_2) \in \Delta_1 \\
(t(1 - x_1) - tx_2 & \text{if } (x_1, x_2) \in \Delta_2 \\
x_1 + t(1 - x_2) & \text{if } (x_1, x_2) \in \Delta_3 \\
x_2 + tx_2 & \text{if } (x_1, x_2) \in \Delta_4
\end{cases}
\]

with

\[
\Delta_1 := \{(x_1, x_2) \in D : x_1 \geq 0 \text{ and } x_2 \leq 0\}; \\
\Delta_2 := \{(x_1, x_2) \in D : x_1 \geq 0 \text{ and } x_2 \geq 0\}; \\
\Delta_3 := \{(x_1, x_2) \in D : x_1 \leq 0 \text{ and } x_2 \geq 0\}; \\
\Delta_4 := \{(x_1, x_2) \in D : x_1 \leq 0 \text{ and } x_2 \leq 0\}.
\]

Consider \(\phi \in \text{Aff}_0(D; \mathbb{R}^3)\) given by

\[
\phi := (\varphi_{\nu_1}, \varphi_{\nu_2}, \varphi_{\nu_3}) \text{ with } \begin{cases} 
\nu = \frac{\xi_1 \wedge \xi_2}{|\xi_1 \wedge \xi_2|} & \text{if (i) is satisfied} \\
|\nu| = 1 \text{ and } \langle \xi_1, \nu \rangle = 0 & \text{if (ii) is satisfied} \\
|\nu| = 1 \text{ and } \langle \xi_2, \nu \rangle = 0 & \text{if (iii) is satisfied}
\end{cases}
\]

(\(\nu_1, \nu_2, \nu_3\) are the components of the vector \(\nu\)). Then,

\[
\xi + \nabla \phi(x) = \begin{cases} 
\langle \xi_1 - \nu \mid \xi_2 + \nu \rangle & \text{if } x \in \text{int}(\Delta_1) \\
\langle \xi_1 - \nu \mid \xi_2 - \nu \rangle & \text{if } x \in \text{int}(\Delta_2) \\
\langle \xi_1 + \nu \mid \xi_2 - \nu \rangle & \text{if } x \in \text{int}(\Delta_3) \\
\langle \xi_1 + \nu \mid \xi_2 + \nu \rangle & \text{if } x \in \text{int}(\Delta_4)
\end{cases}
\]

(\(\text{int}(E)\) denotes the interior of the set \(E\)). Taking Proposition A.1(i) into account, it follows that

\[
Z_{W_0}(\xi) \leq \frac{1}{4} \left( W_0(\xi_1 - \nu \mid \xi_2 + \nu) + W_0(\xi_1 - \nu \mid \xi_2 - \nu) \\
+ W_0(\xi_1 + \nu \mid \xi_2 - \nu) + W_0(\xi_1 + \nu \mid \xi_2 + \nu) \right).
\]

But \(|\langle \xi_1 - \nu \mid \xi_2 + \nu \rangle|^2 = |\xi_1 \wedge \xi_2 + (\xi_1 + \xi_2) \wedge \nu|^2 = |\xi_1 \wedge \xi_2|^2 + |\xi_1 + \xi_2 \wedge \nu|^2 \geq |\xi_1 + \xi_2 \wedge \nu|^2\), and so

\[
|\langle \xi_1 + \nu \mid \xi_2 - \nu \rangle| \geq |\langle \xi_1 + \xi_2 \wedge \nu \rangle| = |\xi_1 + \xi_2|.
\]

Similarly, we obtain:

- \(|\langle \xi_1 - \nu \mid \xi_2 - \nu \rangle| \geq |\xi_1 - \xi_2|\); \\
- \(|\langle \xi_1 + \nu \mid \xi_2 - \nu \rangle| \geq |\xi_1 + \xi_2|\);
Consider \( \varphi \in \text{Aff}_0(Y; \mathbb{R}) \) by

\[
\varphi_t(x_1, x_2) := \begin{cases}
tx_2 & \text{if } (x_1, x_2) \in \Delta_1 \\
t(1 - x_1) & \text{if } (x_1, x_2) \in \Delta_2 \\
t(1 - x_2) & \text{if } (x_1, x_2) \in \Delta_3 \\
tx_1 & \text{if } (x_1, x_2) \in \Delta_4
\end{cases}
\]

with

\[
\begin{align*}
\Delta_1 & := \{(x_1, x_2) \in Y : x_2 \leq x_1 \leq -x_2 + 1\}; \\
\Delta_2 & := \{(x_1, x_2) \in Y : -x_1 + 1 \leq x_2 \leq x_1\}; \\
\Delta_3 & := \{(x_1, x_2) \in Y : -x_2 + 1 \leq x_1 \leq x_2\}; \\
\Delta_4 & := \{(x_1, x_2) \in Y : x_1 \leq x_2 \leq -x_1 + 1\}.
\end{align*}
\]

Consider \( \phi \in \text{Aff}_0(Y; \mathbb{R}^3) \) given by

\[
\phi := (\varphi_{\nu_1}, \varphi_{\nu_2}, \varphi_{\nu_3}) \quad \text{with} \quad
\begin{cases}
\nu = \frac{\langle \xi_1 \land \xi_2 \rangle}{|\xi_1 \land \xi_2|} & \text{if (i) is satisfied} \\
|\nu| = 1 & \text{if (ii) is satisfied} \\
|\nu| = 1 \text{ and } \langle \xi_1, \nu \rangle = 0 & \text{if (iii) is satisfied} \\
|\nu| = 1 \text{ and } \langle \xi_2, \nu \rangle = 0 & \text{if (iv) is satisfied}
\end{cases}
\]

\((\nu_1, \nu_2, \nu_3)\) are the components of the vector \( \nu \). Then,

\[
\xi + \nabla \phi(x) = \begin{cases}
(\xi_1 + \xi_2 + \nu) & \text{if } x \in \text{int}(\Delta_1) \\
(\xi_1 - \nu \land \xi_2) & \text{if } x \in \text{int}(\Delta_2) \\
(\xi_1 \land \xi_2 - \nu) & \text{if } x \in \text{int}(\Delta_3) \\
(\xi_1 + \nu \land \xi_2) & \text{if } x \in \text{int}(\Delta_4)
\end{cases}
\]

(\text{where int}(E) denotes the interior of the set E). Taking Proposition A.1(ii) into account, it follows that

\[
ZW_0(\xi) \leq \frac{1}{7} \left( ZW_0(\xi_1 \land \xi_2 + \nu) + ZW_0(\xi_1 - \nu \land \xi_2) \\
+ ZW_0(\xi_1 \land \xi_2 - \nu) + ZW_0(\xi_1 + \nu \land \xi_2) \right).
\]
But $|\xi_1 + (\xi_2 + \nu)|^2 = |(\xi_1 + \xi_2) + \nu|^2 = |\xi_1 + \xi_2|^2 + |\nu|^2 = |\xi_1 + \xi_2|^2 + 1 \geq 1$, hence $|\xi_1 + (\xi_2 + \nu)| \geq 1$. Similarly, we obtain $|\xi_1 - (\xi_2 + \nu)| \geq 1$, and so

$$\min\{|\xi_1 + (\xi_2 + \nu)|, |\xi_1 - (\xi_2 + \nu)|\} \geq 1.$$  

In the same manner, we have:

$$\min\{|\xi_1 - \nu| + \xi_2, |(\xi_1 - \nu) - \xi_2|\} \geq 1;$$
$$\min\{|\xi_1 + (\xi_2 - \nu)|, |\xi_1 - (\xi_2 - \nu)|\} \geq 1;$$
$$\min\{|(\xi_1 + \nu) + \xi_2, |(\xi_1 + \nu) - \xi_2|\} \geq 1.$$

Using Lemma A.4 it follows that

$$\mathcal{Z}W_0(\xi_1 | \xi_2 + \nu) \leq r_1 (1 + |(\xi_1 | \xi_2 + \nu)|^p)$$
$$\leq r_1 2^p(1 + |(\xi_1 | \xi_2)|^p + |0 | \nu|^p)$$
$$\leq r_1 2^{p+1}(1 + |\xi|^p).$$

Similarly, we obtain:

$$\mathcal{Z}W_0(\xi_1 - \nu | \xi_2) \leq r_1 2^{p+1}(1 + |\xi|^p);$$
$$\mathcal{Z}W_0(\xi_1 | \xi_2 - \nu) \leq r_1 2^{p+1}(1 + |\xi|^p);$$
$$\mathcal{Z}W_0(\xi_1 + \nu | \xi_2) \leq r_1 2^{p+1}(1 + |\xi|^p),$$

and, from (21), we conclude that $\mathcal{Z}W_0(\xi) \leq r_1 2^{p+1}(1 + |\xi|^p)$. □

The next proposition will be used in the proof of Theorem 2.14.

**Proposition A.5.** If $(\mathcal{T}_2)$ holds then $\mathcal{Z}W_0 = QW_0 = Q[\mathcal{Z}W_0]$.  

**Proof.** By Proposition A.3, $\mathcal{Z}W_0(\xi) \leq c(1 + |\xi|^p)$ for all $\xi \in \mathbb{M}^{3 \times 2}$ and some $c > 0$. Then $\mathcal{Z}W_0$ is finite, and so $\mathcal{Z}W_0$ is continuous by Proposition A.1(iii). Recall the (classical) theorem:

**Theorem A.6** (Dacorogna [12]). If $f : \mathbb{M}^{3 \times 2} \to [0, +\infty]$ is finite and continuous then $\mathcal{Z}f = Qf$.

By Theorem A.6 we have $\mathcal{Z}[\mathcal{Z}W_0] = Q[\mathcal{Z}W_0]$. But $\mathcal{Z}[\mathcal{Z}W_0] = \mathcal{Z}W_0$ by Proposition A.1(ii), hence $\mathcal{Z}W_0 = Q[\mathcal{Z}W_0]$. Thus $\mathcal{Z}W_0$ is quasiconvex and $\mathcal{Z}W_0 \leq W_0$. On the other hand, noticing that $\mathcal{Z}g = g$ whenever $g$ is quasiconvex, we see that if $g$ is quasiconvex and $g \leq W_0$ then $g \leq \mathcal{Z}W_0$. According to Definition 2.10(ii), it follows that $\mathcal{Z}W_0 = QW_0$. □

A.2. **Proof of Theorems 2.8 and 2.14.** We begin by proving Proposition A.7 which will play an essential role in the proof of Theorems 2.8 and 2.14.

**Proposition A.7.** $\mathcal{T} = \mathcal{J}$ with $\mathcal{J} : W^{1,p}(\Sigma; \mathbb{R}^3) \to [0, +\infty]$ given by

$$\mathcal{J}(v) := \inf \left\{ \liminf_{n \to +\infty} \int_{\Sigma} \mathcal{Z}W_0(\nabla v_n(x))dx : \text{Aff}(\Sigma; \mathbb{R}^3) \ni v_n \to v \text{ in } L^p(\Sigma; \mathbb{R}^3) \right\}.$$

To prove Proposition A.7 we need the following lemma.

**Lemma A.8.** If $v \in \text{Aff}(\Sigma; \mathbb{R}^3)$ then

$$\mathcal{Z}(v) \leq \int_{\Sigma} \mathcal{Z}W_0(\nabla v(x))dx.$$  

**Proof.** Let $v \in \text{Aff}(\Sigma; \mathbb{R}^3)$. By definition, there exists a finite family $(V_i)_{i \in I}$ of open disjoint subsets of $\Sigma$ such that $|\partial V_i| = 0$ for all $i \in I$, $|\Sigma \setminus \cup_{i \in I} V_i| = 0$ and, for every $i \in I$, $\nabla v(x) = \xi_i$ in $V_i$ with $\xi_i \in \mathbb{M}^{3 \times 2}$. Given any $\delta > 0$ and any $i \in I$, we consider $\phi_i \in \text{Aff}(Y; \mathbb{R}^3)$ such that

$$\int_{Y} \mathcal{W}_0(\xi_i + \nabla \phi_i(y))dy \leq \mathcal{Z}W_0(\xi_i) + \frac{\delta}{|\Sigma|}.$$  

(23)
Fix any integer \( n \geq 1 \). By Vitali’s covering theorem, there exists a finite or countable family \( (a_{i,j} + \alpha_{i,j}Y)_{j \in J_i} \) of disjoint subsets of \( V_i \), where \( a_{i,j} \in \mathbb{R}^2 \) and \( 0 < \alpha_{i,j} < \frac{1}{n} \), such that \( |V_i \setminus \bigcup_{j \in J_i} (a_{i,j} + \alpha_{i,j}Y)| = 0 \) (and so \( \sum_{j \in J_i} \alpha_{i,j}^2 = |V_i| \)).

Define \( \psi_n : \Sigma \to \mathbb{R}^3 \) by

\[
\psi_n(x) := \alpha_{i,j} \phi_i \left( \frac{x - a_{i,j}}{\alpha_{i,j}} \right) \quad \text{if} \quad x \in a_{i,j} + \alpha_{i,j}Y.
\]

Since \( \phi_i \in \text{Aff}_0(Y; \mathbb{R}^3) \), there exists a finite family \( (Y_{i,l})_{l \in L_i} \) of open disjoint subsets of \( Y \) such that \( |\partial Y_{i,l}| = 0 \) for all \( l \in L_i \), \( |Y \setminus \bigcup_{l \in L_i} Y_{i,l}| = 0 \) and, for every \( l \in L_i \),

\[
\nabla \phi_i(y) = \zeta_{i,l} \text{ in } Y_{i,l} \quad \text{with} \quad \zeta_{i,l} \in \mathbb{M}^{3 \times 2}.
\]

Set \( U_{i,l,n} := \bigcup_{j \in J_i} a_{i,j} + \alpha_{i,j}Y_{i,j,l} \), then \( |\partial U_{i,l,n}| = 0 \) for all \( i \in I \) and all \( l \in L_i \), \( |\Sigma \setminus \bigcup_{i \in I} U_{i,l,n}| = 0 \) and, for every \( i \in I \) and every \( l \in L_i \), \( \nabla \psi_n(x) = \zeta_{i,l} \) in \( U_{i,l,n} \), and so \( \psi_n \in \text{Aff}_0(\Sigma; \mathbb{R}^3) \).

On the other hand, \( \|\psi_n\|_{L^\infty(\Sigma; \mathbb{R}^3)} \leq \frac{1}{n} \max_{i \in I} \|\phi_i\|_{L^\infty(Y; \mathbb{R}^3)} \) and \( \|\nabla \psi_n\|_{L^\infty(\Sigma; \mathbb{R}^3)} \leq \max_{i \in I} c \max_{i \in I} \|\phi_i\|_{L^\infty(Y; \mathbb{R}^3)} \), hence (up to a subsequence) \( \psi_n \rightharpoonup 0 \) in \( W^{1,\infty}(\Sigma; \mathbb{R}^3) \), where \( \rightharpoonup \) denotes the weak* convergence in \( W^{1,\infty}(\Sigma; \mathbb{R}^3) \). Consequently, \( \psi_n \to 0 \) in \( W^{1,p}(\Sigma; \mathbb{R}^3) \), and so (up to a subsequence) \( \psi_n \to 0 \) in \( L^p(\Sigma; \mathbb{R}^3) \). Moreover,

\[
\int_{\Sigma} W_0(\nabla v(x) + \nabla \psi_n(x)) \, dx = \sum_{i \in I} \int_{V_i} W_0(\xi_i + \nabla \psi_n(x)) \, dx
\]

\[
= \sum_{i \in I} \sum_{j \in J_i} \alpha_{i,j}^2 \int_Y W_0(\xi_i + \nabla \phi_i(y)) \, dy
\]

\[
= \sum_{i \in I} |V_i| \int_Y W_0(\xi_i + \nabla \phi_i(y)) \, dy.
\]

As \( v + \psi_n \in \text{Aff}(\Sigma; \mathbb{R}^3) \) and \( v + \psi_n \to v \) in \( L^p(\Sigma; \mathbb{R}^3) \), from (23) we deduce that

\[
\overline{E}(v) \leq \liminf_{n \to +\infty} \int_{\Sigma} W_0(\nabla v(x) + \nabla \psi_n(x)) \, dx \leq \sum_{i \in I} |V_i| Z W_0(\xi_i) + \delta
\]

\[
= \int_{\Sigma} Z W_0(\nabla v(x)) \, dx + \delta,
\]

and (22) follows. \( \square \)

**Remark A.9.** As the proof of Lemma A.8 requires Vitali’s covering theorem, if we consider “AffET” (with “AffET” defined in Remark 2.5) instead of “Aff”, Lemma A.8 seems to be false. However, Lemma A.8 remains valid if we replace “Aff” by “AffV” (with “AffV” defined in Remark A.2).

**Proof of Proposition A.7.** Clearly \( J \leq \overline{E} \). We are thus reduced to prove that

\[
(24) \quad \overline{E} \leq J.
\]

Fix any \( v \in W^{1,p}(\Sigma; \mathbb{R}^3) \) and any sequence \( v_n \to v \) in \( L^p(\Sigma; \mathbb{R}^3) \) with \( v_n \in \text{Aff}(\Sigma; \mathbb{R}^3) \). Using Lemma A.8 we have

\[
\overline{E}(v_n) \leq \int_{\Sigma} Z W_0(\nabla v_n(x)) \, dx \quad \text{for all } n \geq 1.
\]

Thus,

\[
\overline{E}(v) \leq \liminf_{n \to +\infty} \overline{E}(v_n) \leq \liminf_{n \to +\infty} \int_{\Sigma} Z W_0(\nabla v_n(x)) \, dx,
\]

and (24) follows. \( \square \)

**Proof of Theorem 2.8.** By Proposition A.3, \( Z W_0(\xi) \leq c(1 + |\xi|^p) \) for all \( \xi \in \mathbb{M}^{3 \times 2} \) and some \( c > 0 \). Then \( Z W_0 \) is finite, and so \( Z W_0 \) is continuous by Proposition A.1(iii). As \( \text{Aff}(\Sigma; \mathbb{R}^3) \) is strongly dense in \( W^{1,p}(\Sigma; \mathbb{R}^3) \), we deduce that for every \( v \in W^{1,p}(\Sigma; \mathbb{R}^3) \),

\[
J(v) = \inf \left\{ \liminf_{n \to +\infty} \int_{\Sigma} Z W_0(\nabla v_n(x)) \, dx : W^{1,p}(\Sigma; \mathbb{R}^3) \ni v_n \to v \text{ in } L^p(\Sigma; \mathbb{R}^3) \right\},
\]
and so $\mathcal{J} \leq \mathcal{I}$. But $\mathcal{I} \leq \mathcal{F}$ and $\mathcal{F} = \mathcal{J}$ by Proposition A.7, hence $\mathcal{F} = \mathcal{I}$.

**Proof of Theorem 2.14.** An analysis similar to that of the proof of Theorem 2.8 shows that $ZW_0$ is continuous, $ZW_0(\xi) \leq c(1 + |\xi|^p)$ for all $\xi \in M^{3 \times 2}$ and some $c > 0$, and

$$\mathcal{E}(v) = \inf \left\{ \liminf_{n \to \infty} \int_{\Sigma} ZW_0(\nabla v_n(x)) \, dx : W^{1-p}(\Sigma; \mathbb{R}^3) \ni v_n \to v \text{ in } L^p(\Sigma; \mathbb{R}^3) \right\}.$$ 

Recall the (classical) integral representation theorem:

**Theorem A.10** (Dacorogna [12]). Let $f : M^{3 \times 2} \to [0, +\infty]$ be a Borel measurable function and let $\mathcal{F} : W^{1,p}(\Sigma; \mathbb{R}^3) \to [0, +\infty]$ be defined by

$$\mathcal{F}(v) := \inf \left\{ \liminf_{n \to \infty} \int_{\Sigma} f(\nabla v_n(x)) \, dx : W^{1,p}(\Sigma; \mathbb{R}^3) \ni v_n \to v \text{ in } L^p(\Sigma; \mathbb{R}^3) \right\}.$$ 

If $f$ is continuous and $C|\xi|^p \leq f(\xi) \leq C(1 + |\xi|^p)$ for all $\xi \in M^{3 \times 2}$ and some $c, C > 0$, then for every $v \in W^{1,p}(\Sigma; \mathbb{R}^3)$,

$$\mathcal{F}(v) = \int_{\Sigma} Qf(\nabla v(x)) \, dx.$$ 

Noticing that $ZW_0$ is coercive, from Theorem A.10 it follows that for every $v \in W^{1,p}(\Sigma; \mathbb{R}^3)$,

$$\mathcal{E}(v) = \int_{\Sigma} Q[ZW_0](\nabla v(x)) \, dx.$$ 

Moreover, $Q[ZW_0] = QW_0$ by Proposition A.5, and the proof is complete.

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