CONSTRAINT-PRECONDITIONED KRYLOV SOLVERS FOR
REGULARIZED SADDLE-POINT SYSTEMS

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Abstract. We consider the iterative solution of regularized saddle-point systems. When the leading block is symmetric and positive semi-definite on an appropriate subspace, Dollar, Gould, Schilders, and Wathen (2006) describe how to apply the conjugate gradient (CG) method coupled with a constraint preconditioner, a choice that has proved to be effective in optimization applications. We investigate the design of constraint-preconditioned variants of other Krylov methods for regularized systems by focusing on the underlying basis-generation process. We build upon principles laid out by Gould, Orban, and Rees (2014) to provide general guidelines that allow us to specialize any Krylov method to regularized saddle-point systems. In particular, we obtain constraint-preconditioned variants of Lanczos and Arnoldi-based methods, including the Lanczos version of CG, MINRES, SYMMLQ, GMRES(m) and DQGMRES. We also provide MATLAB implementations in hopes that they are useful as a basis for the development of more sophisticated software. Finally, we illustrate the numerical behavior of constraint-preconditioned Krylov solvers using symmetric and nonsymmetric systems arising from constrained optimization.

Key words. Regularized saddle-point systems, constraint preconditioners, Lanczos and Arnoldi procedures, Krylov solvers.

AMS subject classifications. 65F08, 65F10, 65F50, 90C20.

1. Introduction. We consider the iterative solution of the regularized saddle-point system

\[
\begin{bmatrix}
A & B^T \\
B & -C
\end{bmatrix}
\begin{bmatrix}
x \\
y
\end{bmatrix}
=
\begin{bmatrix}
b \\
0
\end{bmatrix},
\]

where \(A \in \mathbb{R}^{n \times n}\) may be nonsymmetric, \(C \in \mathbb{R}^{m \times m}\) is nonzero and symmetric, and \(B \in \mathbb{R}^{m \times n}\). We denote \(K\) the matrix of (1). There is no loss of generality in assuming that the last \(m\) entries of the right-hand side of (1) are zero, as discussed later.

A constraint preconditioner for (1) has the form

\[
P = \begin{bmatrix}
G & B^T \\
B & -C
\end{bmatrix},
\]

where \(G\) is an approximation to \(A\) such that (2) is nonsingular. When \(A\) is symmetric and has appropriate additional properties, a constraint preconditioner allows the application of CG even though \(K\) and \(P\) are indefinite (Dollar et al., 2006).

We are interested in the design of constraint-preconditioned versions of additional Krylov methods for (1), including methods that can be used when \(A\) is nonsymmetric.

We extend the work of Gould et al. (2014) on projected and constraint-preconditioned Krylov methods for saddle-point systems with \(C = 0\) by exploiting a suitable reformulation of (1) suggested by Dollar et al. (2006). We develop constraint-preconditioned

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variants of the Lanczos and Arnoldi basis-generation processes, and use them to
derive variants of Krylov solvers based on those processes. More generally, we provide
guidelines that can be also exploited to obtain constraint-preconditioned versions of
other Krylov methods not considered in this paper. Finally, we distribute MATLAB
implementations of the constraint-preconditioned methods discussed here as templates
for the development of more sophisticated numerical software.

Systems of type (1) arise in interior-point methods for constrained optimization
in the presence of inequality constraints or when regularization is used (Benzi, Golub,
and Liesen, 2005; D’Apuzzo, De Simone, and di Serafino, 2010; Friedlander and Orban,
2012). They also appear in Lagrangian approaches for variational problems with
equality constraints when the constraints are relaxed or a penalty term is applied
(Pestana and Wathen, 2015). In the above cases, $A$ is usually symmetric, but may also
be nonsymmetric—see Section 7, and often has additional properties, e.g., it accounts
for local convexity of the optimization problem. Regularized saddle-point systems with
nonsymmetric $A$ arise also from the stabilized finite-element discretization of Oseen
problems obtained by linearization, through Picard’s method, of the steady-state
Navier-Stokes equations governing the flow of a Newtonian incompressible viscous
fluid (Benzi et al., 2005).

Constraint preconditioners have widely demonstrated their effectiveness on saddle-
point systems, especially when the leading block is symmetric and enjoys additional
properties, such as being positive definite; much work has been carried out to develop,
analyze and approximate constraint preconditioners in this case, see, e.g., (Benzi et al.,
2005; D’Apuzzo et al., 2010; Gould et al., 2014; De Simone, di Serafino, and Morini,
2018) and the references therein.

The rest of this paper is organized as follows. Section 2 provides preliminary
results used in the sequel. In Section 3, we describe the constraint-preconditioned
Lanczos process and, in Section 4, we present variants of Krylov solvers based on it.
In Section 5, we describe the constraint-preconditioned Arnoldi process and associated
Krylov methods. In Section 6, we discuss implementation issues and provide details
on the MATLAB codes. In Section 7, we illustrate the numerical behavior of some
constraint-preconditioned solvers on regularized saddle-point systems, with symmetric
and nonsymmetric matrices, from constrained optimization. We conclude in Section 8.

Notation. Uppercase Latin letters ($A$, $B$, ...), lowercase Latin letters ($a$, $b$, ...),
and lowercase Greek letters ($\alpha$, $\beta$, ...) denote matrices, vectors and scalars,
respectively. The Euclidean norm is denoted $\| \cdot \|$. If $S = S^T$ is a positive definite matrix, the
$S$-norm is defined as $\|u\|_S^2 = u^T S u$. All vectors are column vectors. For brevity, we
use the MATLAB-like notation $[v; w]$ to represent the vector $[v^T \ w^T]^T$.

2. Preliminaries. We assume throughout that $K$ is nonsingular, which implies

$$\text{Null}(A) \cap \text{Null}(B) = \{0\} \quad \text{and} \quad \text{Null}(B^T) \cap \text{Null}(C) = \{0\}. \quad (3)$$

In general the converse is not true. A counterexample consists in taking

$$A = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.\$$

Benzi et al. (2005) and D’Apuzzo et al. (2010) give additional conditions that guarantee
nonsingularity of $K$. Note however that we do not require $B$ to have full rank or $C$ to
be positive (semi-)definite.
In order to develop constraint-preconditioned Krylov methods for (1), we specialize the basis-generation processes underlying those methods. We focus on the Lanczos (1950) and Arnoldi (1951) processes, which compute orthonormal bases of Krylov spaces associated with symmetric and general matrices, respectively. For reference, the standard Lanczos process is stated as Algorithm 4 in Appendix A. It is straightforward to apply our arguments to the Lanczos (1950) biorthogonalization process and its transpose-free variants (Brezinski and Redivo-Zaglia, 1998; Chan, de Pillis, and van der Vorst, 1998). We implicitly assume that $A = A^T$ when considering the Lanczos process.

Following Dollar et al. (2006), we reformulate (1) as follows. Assume that \( \text{rank}(C) = p \) and $C$ has been decomposed as \[ C = EFE^T, \]
where $F \in \mathbb{R}^{p \times p}$ is symmetric and nonsingular and $E \in \mathbb{R}^{m \times p}$. Then, by using the auxiliary variable \[ w = -FE^T y, \]
equation (1) may be written \[ \begin{bmatrix} A & B^T \\ F^{-1} & E^T \end{bmatrix} \begin{bmatrix} x \\ w \end{bmatrix} = \begin{bmatrix} b \\ 0 \end{bmatrix}, \]
which has a standard symmetric saddle-point form \[ \begin{bmatrix} M & N^T \\ N & 0 \end{bmatrix} \begin{bmatrix} g \\ y \end{bmatrix} = \begin{bmatrix} b_0 \\ 0 \end{bmatrix}, \]
The principles laid out by Gould et al. (2014) may now be applied to (6).

Note that (6) is nonsingular if and only if (1) is nonsingular, and therefore $N$ must have full rank. Because $g \in \text{Null}(N)$, there exists $\hat{d} \in \mathbb{R}^{n+p-m}$ such that \[ g = \begin{bmatrix} x \\ w \end{bmatrix} = Z \hat{d} = \begin{bmatrix} Z_1 \\ Z_2 \end{bmatrix} \hat{d}, \]
where the columns of $Z$ form a basis of $\text{Null}(N)$. The restriction of (6) to $\text{Null}(N)$ is \[ \begin{bmatrix} M & N^T \\ N & 0 \end{bmatrix} \begin{bmatrix} \tilde{M} \hat{x} \\ \hat{b} \end{bmatrix} = \begin{bmatrix} \tilde{b} \end{bmatrix}, \]
where \[ \begin{bmatrix} \tilde{M} = Z^T M Z = Z_1^T A Z_1 + Z_2^T F^{-1} Z_2, \\ x \\ w \end{bmatrix} = \begin{bmatrix} Z_1 \\ Z_2 \end{bmatrix} \hat{x}, \quad \begin{bmatrix} \hat{b} \end{bmatrix} = \begin{bmatrix} Z_1^T & Z_2^T \end{bmatrix} \begin{bmatrix} b \\ 0 \end{bmatrix} = Z_1^T b. \]

In a Krylov method for (9), it is appropriate to use a preconditioner of the form \[ \hat{P} = Z_1^T G Z_1 + Z_2^T F^{-1} Z_2. \]
If $G$ is suitable, the preconditioned method can be reformulated entirely in terms of full space quantities (Gould, Hribar, and Nocedal, 2001; Dollar et al., 2006; Gould et al., 2014). Following (Gould et al., 2014, Assumption 2.2), we require the following assumption.

\[ \text{Note that (4) will be only used for the purpose of deriving computational processes and need not be computed in practice.} \]
Assumption 2.1. The matrix
\[
\begin{bmatrix}
G \\
F^{-1}
\end{bmatrix}
\]
is symmetric and positive definite on \(\text{Null}(N)\).

A consequence of Assumption 2.1 is that (11) is symmetric and positive definite.

We enforce Assumption 2.1 throughout this paper to guarantee that Krylov methods for (9) give rise to corresponding full-space methods for (1). However, at least in principle, Assumption 2.1 is not always necessary, e.g., in Krylov methods based on the Arnoldi process.

The application of the preconditioner \(\hat{P}\), i.e., \(\tilde{u} = \hat{P}^{-1}\tilde{u}\), can be written as
\[
\begin{bmatrix}
\bar{u}_x \\
\bar{u}_w
\end{bmatrix}
= P_G
\begin{bmatrix}
u_x \\
u_w
\end{bmatrix},
\quad
P_G = Z\hat{P}^{-1}Z^T,
\quad
\begin{bmatrix}
\bar{u}_x \\
\bar{u}_w
\end{bmatrix}
= Z\tilde{u},
\quad
Z^T
\begin{bmatrix}
u_x \\
u_w
\end{bmatrix}
= \tilde{u}.
\]
Furthermore,
\[
P_G
\begin{bmatrix}
G \\
F^{-1}
\end{bmatrix}
\]
is an oblique projector into \(\text{Null}(N)\). Let \(\hat{L}\) be the lower triangular Cholesky factor of \(\hat{P}\) and let
\[
\hat{K} = \mathcal{K}\left(\hat{L}^{-1}\tilde{M}\hat{L}^{-T}, \hat{L}^{-1}(\hat{b} - \tilde{M}\tilde{x}_0)\right)
\]
be the Krylov space generated by the preconditioned reduced operator \(\hat{L}^{-1}\tilde{M}\hat{L}^{-T}\) and initial vector \(\hat{L}^{-1}(\hat{b} - \tilde{M}\tilde{x}_0)\), where \(\hat{b}\) is given in (10) and \(x_0 = Z_1\tilde{x}_0\), with \(Z_1\) defined in (8).

The computation of (12) can be obtained by solving
\[
\begin{bmatrix}
G \\
F^{-1}
\end{bmatrix}
\begin{bmatrix}
B^T \\
E^T
\end{bmatrix}
\begin{bmatrix}
\bar{u}_x \\
\bar{u}_w
\end{bmatrix}
= \begin{bmatrix}
u_x \\
u_w
\end{bmatrix},
\]
see, e.g., Gould et al. (2001), so that \(P_G\) could be expressed as
\[
P_G = \begin{bmatrix}
I & 0 & 0 \\
0 & I & 0
\end{bmatrix}
\begin{bmatrix}
G \\
F^{-1}
\end{bmatrix}
\begin{bmatrix}
B^T \\
E^T
\end{bmatrix}
^{-1}
\begin{bmatrix}
I & 0 \\
0 & I
\end{bmatrix}.
\]

We now apply Principles 2.1 and 2.2 of Gould et al. (2014) to the standard Lanczos basis-generation process for \(\hat{K}\), and obtain the projected Lanczos process outlined in Algorithm 1.

In Algorithm 1, the notation \(\|u\|_{\mathcal{P}}\) represents a measure of the deviation of \(u = [u_x; u_w]\) from \(\text{Null}(N)\) (Gould et al., 2014, Section 3). More precisely
\[
\|u\|_{\mathcal{P}}^2 := u_x^T\bar{u}_x + u_w^T\bar{u}_w,
\]
where \(\bar{u} = [\bar{u}_x; \bar{u}_w]\) is defined by (14). Note that \(\|u\|_{\mathcal{P}}\) is actually a seminorm and vanishes if and only if \([u_x; u_w]\) is orthogonal to \(\text{Null}(N)\).
Algorithm 1 Projected Lanczos Process

1: choose \([x_0; w_0]\) such that \(Bx_0 + Ew_0 = 0\) \hspace{1cm} \text{initial guess}
2: \(v_0,x = 0, \quad v_0,w = -w_0\) \hspace{1cm} \text{initial Lanczos vector}
3: \(u_{0,x} = b - Ax_0, \quad u_{0,w} = -F^{-1}w_0\)
4: \(\bar{u}_{1,x} = u_{1,x}, \quad \bar{u}_{1,w} = \bar{u}_{1,w}\) \hspace{1cm} \text{solution of (14) with right-hand side \([u_{0,x}; u_{0,w}; 0]\)}
5: \(v_{1,x} = \bar{u}_{1,x}, \quad v_{1,w} = \bar{u}_{1,w}\)
6: \(\beta_1 = \left(v_{1,x}^Tu_{0,x} + v_{1,w}^Tu_{0,w}\right)^{1/2}\)
7: if \(\beta_1 = 0\) then
8: \(v_{1,x} = \bar{v}_{1,x}/\beta_1, \quad v_{1,w} = \bar{v}_{1,w}/\beta_1\) \hspace{1cm} \|v_1\|_p = 1
9: end if
10: end while
11: while \(\beta_k \neq 0\) do
12: \(u_{k,x} = Av_{k,x}, \quad u_{k,w} = F^{-1}v_{k,w}\)
13: \(\alpha_k = v_{k,x}^Tu_{k,x} + v_{k,w}^Tu_{k,w}\)
14: \(\bar{u}_{k+1,x} = u_{k+1,x} - \alpha_kv_{k+1,x} - \beta_kv_{k,x}, \quad v_{k+1,x} = \bar{u}_{k+1,x} - \alpha_kv_{k+1,x} - \beta_kv_{k,x}\)
15: \(v_{k+1,w} = \bar{v}_{k+1,w} - \alpha_kv_{k+1,w} - \beta_kv_{k+1,w}\)
16: \(\beta_{k+1} = \left(v_{k+1,x}^Tu_{k,x} + v_{k+1,w}^Tu_{k,w}\right)^{1/2}\)
17: if \(\beta_{k+1} = 0\) then
18: \(v_{k+1,x} = \bar{v}_{k+1,x}/\beta_{k+1}, \quad v_{k+1,w} = \bar{v}_{k+1,w}/\beta_{k+1}\) \hspace{1cm} \|v_{k+1}\|_p = 1
19: end if
20: \(k = k + 1\)
21: end while

Conceptually, the Lanczos process corresponding to Algorithm 1 can be summarized as:

\[
\begin{bmatrix}
A & B^T

\end{bmatrix}
\begin{bmatrix}
V_{k,x}

\end{bmatrix}
\begin{bmatrix}
G & F^{-1} & B^T

\end{bmatrix}
\begin{bmatrix}
V_{k,w}

\end{bmatrix}
\begin{bmatrix}
T_k + \beta_{k+1}

\end{bmatrix}
\begin{bmatrix}
v_{k+1,x}
v_{k+1,w}

\end{bmatrix}
\begin{bmatrix}
v_k

\end{bmatrix}
\begin{bmatrix}
E^T

\end{bmatrix}
\begin{bmatrix}
Z_k

\end{bmatrix},
\]

where

\[
V_{k,x} = \begin{bmatrix} v_{1,x} & \ldots & v_{k,x} \end{bmatrix}, \quad V_{k,w} = \begin{bmatrix} v_{1,w} & \ldots & v_{k,w} \end{bmatrix}, \quad Z_k = \begin{bmatrix} \bar{z}_1 & \ldots & \bar{z}_k \end{bmatrix},
\]

and \(T_k\) is the usual Lanczos tridiagonal matrix. Provided that \([x_0; w_0] \in \text{Null}(N),\) (Gould et al., 2014, Theorem 2.2) guarantees that Algorithm 1 is well defined and equivalent to Algorithm 5 in Appendix A applied to (9)–(10) with preconditioner (11).

In Algorithm 1 and subsequent algorithms, we use the symbol “\(\mapsfrom\)” to assign to the vector on the left of the arrow the result of the external procedure on the right of the arrow.

In the next sections we show how the projected basis-generation procedures can be further reformulated by referring to the original system (1), thus avoiding the use of \(E\) and \(F\) and the factorization (4).

3. Constraint-Preconditioned Lanczos Process. If we define \(\bar{p}_k = \bar{u}_{k,x}\) for all \(k \geq 1,\) and

\[
t_k = E F u_{k,w}, \quad k = 0, 1, \ldots
\]
then (14) at line 14 of Algorithm 1 can be written as

\[
\begin{bmatrix}
G & B^T \\
B & -C
\end{bmatrix}
\begin{bmatrix}
p_{k+1} \\
\bar{z}_{k+1}
\end{bmatrix}
= \begin{bmatrix}
u_{k,x} \\
t_k
\end{bmatrix}.
\]

Assumption 2.1 occurs when the sum of the number of negative eigenvalues of the matrix of (17) and \( C \) is \( m \) (Dollar et al., 2006, Theorem 2.1), which may be verified if an inertia-revealing symmetric indefinite factorization is used to solve (17), such as that of Duff (2004).

Unfortunately, (17) still appears to depend on \( F \) via (16). We now reformulate Algorithm 1 in terms of full-space quantities. Define the initial guess

\[
w_0 = -FE^T q_0,
\]

where \( q_0 \in \mathbb{R}^m \) is arbitrary (e.g., \( q_0 = 0 \)). Line 3 of Algorithm 1 and (16) yield

\[
u_{0,x} = r_0 = b - Ax_0, \quad u_{0,w} = E^T q_0, \quad t_0 = Cq_0.
\]

From here on, let us denote \( p_k = v_{k,x} \). At lines 4-5 of Algorithm 1, we compute \( p_1 = \bar{p}_1 \) and \( \bar{z}_1 \) from (14), which yields, in particular, \( u_{1,w} = FE^T(q_0 - \bar{z}_1) \). If we define

\[
s_1 = q_0 - \bar{z}_1, \quad q_1 = s_1,
\]

lines 5-6 of Algorithm 1 take the form

\[
\begin{align*}
(19a) & \quad p_1 = \bar{p}_1 \\
(19b) & \quad v_{1,w} = FE^T q_1, \\
(19c) & \quad \beta_1 = \left( p_1 u_{0,x} + q_1 C q_1 \right) \frac{1}{2} = \left( p_1 u_{0,x} + q_1 T t_0 \right) \frac{1}{2}.
\end{align*}
\]

We then normalize by dividing \( p_1 \) and \( q_1 \) by \( \beta_1 \). Lines 12–13 of Algorithm 1 and (16) give

\[
\begin{align*}
(20a) & \quad u_{1,w} = E^T q_1, \\
(20b) & \quad t_1 = C q_1, \\
(20c) & \quad \alpha_1 = \frac{1}{4} \left( p_1 u_{1,x} + q_1 T t_1 \right) = \frac{1}{4} \left( p_1 A p_1 + q_1 T C q_1 \right).
\end{align*}
\]

We now compute \( p_2 \) and \( v_{2,w} \) from lines 15–16 of Algorithm 1 with \( k = 1 \), i.e., we compute \( \bar{p}_2 \) and \( \bar{z}_2 \) from (17), and note that \( \bar{u}_{2,w} = FE^T(q_1 - \bar{z}_2) \). Thus, by setting

\[
s_2 = q_1 - \bar{z}_2, \quad q_2 = s_2 - \alpha_1 q_1 - \beta_1 q_0,
\]

we obtain from lines 2 and 15–16 of Algorithm 1 together with (19b), (20a) and (20b):

\[
\begin{align*}
p_2 & = \bar{p}_2 - \alpha_1 p_1 - \beta_1 p_0 \\
v_{2,w} & = FE^T s_2 - \alpha_1 FE^T q_1 - \beta_1 FE^T q_0 = FE^T q_2 \\
\beta_2 & = \left( p_2 A p_1 + q_2 C q_1 \right) \frac{1}{2} = \left( p_2 u_{1,x} + q_2 T t_1 \right) \frac{1}{2}.
\end{align*}
\]

Then, according to line 19, \( p_2 \) must be divided by \( \beta_2 \), and we do the same with \( q_2 \). An induction argument shows that for all \( k \geq 1 \)

\[
\begin{align*}
u_{k,w} & = E^T q_k, \\
t_k & = C q_k, \\
\alpha_k & = \frac{1}{4} \left( p_k u_{k,x} + q_k T t_k \right) = \frac{1}{4} \left( p_k A p_k + q_k T C q_k \right),
\end{align*}
\]
where \( q_k \) has been normalized by \( \beta_k \). Furthermore, letting
\[
s_{k+1} = q_k - \bar{z}_{k+1}, \quad q_{k+1} = s_{k+1} - \alpha_k q_k - \beta_k q_{k-1},
\]
we obtain
\[
p_{k+1} = \bar{p}_{k+1} - \alpha_k p_k - \beta_k p_{k-1}
\]
and
\[
v_{k+1,w} = FE^\top q_{k+1},
\]
\[
\beta_{k+1} = (p_{k+1}^T A p_k + q_{k+1}^T C q_k)^{\frac{1}{2}} = (p_{k+1}^T u_{k,x} + q_{k+1}^T t_k)^{\frac{1}{2}}.
\]
We divide \( p_{k+1} \) and \( q_{k+1} \) by \( \beta_{k+1} \) to obtain the vectors to be used at the next iteration.

Thus, if we rename \( u_{k,x} \) as \( u_k \), we obtain Algorithm 2.

---

**Algorithm 2** Constraint-Preconditioned Lanczos Process

1. choose \([x_0; q_0]\) such that \( B x_0 - C q_0 = 0\) \hspace{1cm} initial guess
2. \( p_0 = 0\) \hspace{1cm} initial Lanczos vector
3. \( u_0 = b - A x_0, \quad t_0 = C q_0\)
4. \([\bar{p}_1; \bar{z}_1] \leftarrow \) solution of (17) with right-hand side \([u_0; -t_0]\)
5. \( p_1 = \bar{p}_1\)
6. \( s_1 = q_0 - \bar{z}_1, \quad q_1 = s_1\)
7. \( \beta_1 = (p_1^T u_0 + q_1^T t_0)^{\frac{1}{2}}\)
8. if \( \beta_1 \neq 0 \) then
9. \( p_1 = p_1/\beta_1, \quad q_1 = q_1/\beta_1\)
10. end if
11. \( k = 1\)
12. while \( \beta_k \neq 0 \) do
13. \( u_k = A \bar{p}_k, \quad t_k = C q_k\)
14. \( \alpha_k = p_k u_k + q_k t_k\)
15. \([\bar{p}_{k+1}; \bar{z}_{k+1}] \leftarrow \) solution of (17) with right-hand side \([u_k; -t_k]\)
16. \( p_{k+1} = \bar{p}_{k+1} - \alpha_k p_k - \beta_k p_{k-1}\)
17. \( s_{k+1} = q_k - \bar{z}_{k+1}, \quad q_{k+1} = s_{k+1} - \alpha_k q_k - \beta_k q_{k-1}\)
18. \( \beta_{k+1} = (p_{k+1}^T u_k + q_{k+1}^T t_k)^{\frac{1}{2}} = (p_{k+1}^T A p_k + q_{k+1}^T C q_k)^{\frac{1}{2}}\)
19. if \( \beta_{k+1} \neq 0 \) then
20. \( p_{k+1} = p_{k+1}/\beta_{k+1}, \quad q_{k+1} = q_{k+1}/\beta_{k+1}\)
21. end if
22. \( k = k + 1\)
23. end while

The above transformations can be condensed in the following principle, which summarizes the conversion a of projected process into a constraint-preconditioned process.

**Principle 1.**

1. Basis vectors \( v_{k+1,x} \) are unchanged;
2. Basis vectors \( v_{k+1,w} \) have the form \( F E^\top q_{k+1} \), where \( q_{k+1} \) is defined by
\[
s_{k+1} = q_k - \bar{z}_{k+1}, \quad q_1 = s_1,
\]
\[
q_{k+1} = s_{k+1} - \alpha_k q_k - \beta_k q_{k-1}, \quad (k \geq 1),
\]
and where \( \bar{z}_{k+1} \) results from the solution of (17);
3. Inner products of the form $v_i^T w_j, w_k$ become $q_i^T C q_j = q_i^T t_j$.

Theorem 1 summarizes the equivalence between the two formulations.

**Theorem 1.** Let $E$ and $F$ be as defined in (4) and $G$ chosen to satisfy Assumption 2.1. Let $q_0 \in \mathbb{R}^m$ be arbitrary. Then, Algorithm 1 with starting guesses $x_0 \in \mathbb{R}^n$ and $w_0 = -FE^T q_0$, such that $Bx_0 + EW_0$, is equivalent to Algorithm 2 with starting guesses $x_0$ and $q_0$. In particular, for all $k$, the vectors $v_{k, x}$ and $v_{k, w}$, and the scalars $\alpha_k$ and $\beta_k$ in Algorithm 1 are equal to the vectors $p_k$ and $FE^T q_k$, and to the scalars $\alpha_k$ and $\beta_k$ in Algorithm 2, respectively.

Note that Algorithm 2 does not contain references to $E$ and $F$. The variable $s_k$ is used only to improve readability. Assumption 2.1 guarantees that Algorithm 2 is well posed because it is equivalent to Algorithm 1, which, in turn, is equivalent to the standard Lanczos process for building an orthonormal basis of (13). The main advantages of Algorithm 2 are that it works directly with the formulation (1) and it only requires storage for three vectors of size $n + m$ and $\{[p_k : q_k], [u_k : t_k], and [\tilde{p}_k : \tilde{z}_k]\}$, as opposed to the same number of vectors of size $n + p + m$ for Algorithm 1.

We call Algorithm 2 the Constraint-Preconditioned Lanczos (CP-Lanczos) process because of its similarity to a Lanczos process for building an orthonormal basis of a Krylov space associated with the preconditioned operator $P^{-1}M$, even though the latter appears nonsymmetric.

4. Constraint-Preconditioned Lanczos-Based Krylov Solvers. We may exploit Theorem 1 and use Algorithm 2 to derive a constraint-preconditioned version of any Krylov method based on the Lanczos process. To this aim, we must understand how the update of the $k$-th iterate $x_k : w_k$ in a Krylov method based on Algorithm 1 translates into the update of the $k$-th iterate $x_k : y_k$ in the version of that Krylov method based on Algorithm 2. In the following, the former and the latter version of the Krylov method are referred to as projected-Krylov (P-Krylov) and constraint-preconditioned-Krylov (CP-Krylov), respectively.

Because the initial guess $q_0 = [x_0 : y_0]$ of P-Krylov applied to (6) must lie in Null($N$), CP-Krylov must be initialized with $[x_0 : y_0]$ such that

$$\begin{align*}
Bx_0 - Cy_0 & = 0.
\end{align*}$$

Our first result states a property of Algorithm 2 that follows from a specific $q_0$.

**Lemma 1.** Let Algorithm 2 be initialized with $x_0 \in \mathbb{R}^n$ and $q_0 \in \text{Null}(C)$. Then, for all $k \geq 0$,

$$\begin{align*}
Bp_k + Cq_k & = 0.
\end{align*}$$

**Proof.** We proceed by induction. For $k = 0$, (22) holds because $p_0 = 0$ and $q_0 \in \text{Null}(C)$. For $k = 1$, $p_1 = \tilde{p}_1$, $q_1 = q_0 - \tilde{z}_1 = -\tilde{z}_1$, and (17) and our assumption that $q_0 \in \text{Null}(C)$ yield

$$\begin{align*}
Bp_1 + Cq_1 & = B\tilde{p}_1 - C\tilde{z}_1 = -t_0 = -Cq_0 = 0.
\end{align*}$$

Assume (22) holds for any index $j \leq k$. Lines 16–17 of Algorithm 2, (16), (17), and our induction assumption imply that

$$\begin{align*}
Bp_{k+1} + Cq_{k+1} & = B\tilde{p}_{k+1} + C(q_k - \tilde{z}_{k+1}) - \alpha_k(Bp_k + Cq_k) - \beta_k(Bp_{k-1} + Cq_{k-1}) \\
& = B\tilde{p}_{k+1} + C(q_k - \tilde{z}_{k+1}) \\
& = B\tilde{p}_{k+1} - C\tilde{z}_{k+1} + t_k = 0,
\end{align*}$$

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which establishes (22).

An interesting property of the CP-Lanczos process is that it is equivalent to formally applying the standard Lanczos process to system (1) with preconditioner (2), where by “formal application”, we mean that the Lanczos process is applied blindly as if $P$ were positive definite. Such formal application is stated as Algorithm 6 in Appendix A. The equivalence with Algorithm 2 is stated in the next result, which parallels (Gould et al., 2014, Theorem 2.2).

**Theorem 2.** Let Algorithm 6 be initialized with $x_0 \in \mathbb{R}^n$ such that $Bx_0 = 0$, $q_0 = 0 \in \mathbb{R}^m$, and Algorithm 6 be initialized with the same $x_0$ and $y_0 \in \mathbb{R}^m$ such that (21) is satisfied. Then, for all $k \geq 0$, $v_{k,x} = p_k$ and $v_{k,y} = -q_k$, where $[v_{k,x}; v_{k,y}]$ is the $k$-th Lanczos vector generated in Algorithm 6, and $p_k$ and $q_k$ are the $k$-th Lanczos vectors generated in Algorithm 2. In addition, the scalars $\alpha_k$ and $\beta_k$, computed at each iteration are the same in both algorithms.

**Proof.** We proceed by induction. The result holds for $k = 0$ because $[v_{0,x}; v_{0,y}] = [0; 0] = [p_0; -q_0]$. With $q_0 = 0$, Algorithm 2 initializes $u_0 = b - Ax_0$ and $t_0 = 0$. Because (21) is satisfied, Algorithm 6 initializes $r_{0,x} = u_0 - B^T y_0$ and $r_{0,y} = 0$. Thus, $[v_{1,x}; v_{1,y}]$ solves (17) with right-hand side $[u_0 - B^T y_0; 0]$. By (Gould et al., 2014, Theorem 2.1, item 2), $[v_{1,x}; v_{1,y}]$ equivalently solves (17) with right-hand side $[v_0; 0]$, and therefore, $[v_{1,x}; v_{1,y}]$ at line 4 of Algorithm 6 is equal to $[p_1; z_1]$. Lines 5–6 of Algorithm 2 subsequently set $p_1 = \bar{p}_1 = v_{1,x}$ and $q_1 = s_1 = q_0 - z_1 = -v_{1,y}$.

With $q_0 = 0$, line 7 of Algorithm 2 computes $\beta_1 = (p_1^T u_0)^{\frac{1}{2}}$. We take the inner product of the second row of (17) with $z_1 = -q_1$ and note that $t_0 = 0$, and obtain $z_1^T B p_1 = z_1^T C z_1 = q_1^T C$. Similarly, we take the inner product of the first row of (17) with $p_1$ and substitute $z_1^T B p_1$ to obtain $p_1^T u_0 = p_1^T G p_1 + q_1^T C$, so that $\beta_1$ is the same as that computed at line 5 of Algorithm 6. We have established that the result also holds for $k = 1$.

At a general iteration $k$, Algorithm 2 sets $u_k = A p_k$, $t_k = C q_k$ and computes $\alpha_k = p_k^T u_k + q_k^T t_k = p_k^T A p_k + q_k^T C q_k$. By Lemma 1, $q_k^T B p_k + q_k^T C = 0$, so that $\alpha_k = p_k^T A p_k - 2 q_k^T B p_k - q_k^T C$. Under the recurrence assumption that $v_{k,x} = p_k$ and $v_{k,y} = -q_k$, this expression of $\alpha_k$ is the same as that computed at line 12 of Algorithm 6.

At line 15 of Algorithm 2, we compute $[\bar{p}_{k+1}; z_{k+1}]$ from (17), or, equivalently, as the solution to

$$
\begin{bmatrix}
G & B^T \\
B & -C
\end{bmatrix}
\begin{bmatrix}
\bar{p}_{k+1} \\
z_{k+1} - q_k
\end{bmatrix}
= 
\begin{bmatrix}
A p_k \\
0
\end{bmatrix}.
$$

In view of Lemma 1, our recurrence assumption, and (Gould et al., 2014, Theorem 2.1, item 2), line 13 of Algorithm 6 computes $[v_{k+1,x}; v_{k+1,y}]$ as the solution to the same system as above. Therefore, at that point in each algorithm $v_{k+1,x} = \bar{p}_{k+1}$ and $v_{k+1,y} = z_{k+1} - q_k = -s_{k+1}$. The vector updates at lines 16–17 of Algorithm 2 together with those at line 14 of Algorithm 6 show that $v_{k+1,x} = p_{k+1}$ and $v_{k+1,y} = -q_{k+1}$. Our recurrence assumption and Lemma 1 yield $B v_{k,x} - C v_{k,y} = 0$ and $B v_{k+1,x} - C v_{k+1,y} = 0$.

Finally, Algorithm 6 sets

$$
\beta_{k+1}^2 = v_{k+1,x}^T u_{k,x} + v_{k+1,y}^T u_{k,y}
= v_{k+1,x} A v_{k,x} + (B v_{k+1,x} - C v_{k+1,y})^T v_{k,y} + v_{k+1,y} B v_{k,x}
= v_{k+1,x} A v_{k,x} + v_{k+1,y} C v_{k,x},
$$

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which is the same value computed in Algorithm 2.

Theorem 2 shows that Algorithm 2 may be summarized as

$$
\begin{bmatrix}
A & B^T \\
B & -C
\end{bmatrix}
\begin{bmatrix}
P_k \\
-Q_k
\end{bmatrix}
= \begin{bmatrix}
G & B^T \\
B & -C
\end{bmatrix}
\left(\begin{bmatrix}
P_k \\
-Q_k
\end{bmatrix}T_k + \beta_{k+1}
\begin{bmatrix}
p_{k+1} \\
-q_{k+1}
\end{bmatrix}e_k^T, \right)
$$

provided that $Bx_0 = 0$ and $q_0 = 0$, where $T_k$ is the same as in Algorithm 1, and

$$
P_k = [p_1 \ldots p_k], \quad Q_k = [q_1 \ldots q_k].
$$

A consequence of Theorem 2 is that any CP-Krylov method is formally equivalent to the corresponding standard Krylov method applied to system (1) with preconditioner (2).

**Corollary 1.** Let Algorithm 2 be initialized with $x_0 \in \mathbb{R}^n$ such that $Bx_0 = 0$, $q_0 = 0 \in \mathbb{R}^m$, and Algorithm 6 be initialized with the same $x_0$ and $y_0 \in \mathbb{R}^m$ such that (21) is satisfied. The k-th approximate solution of (1) computed by any Lanczos-based CP-Krylov method coincides with the k-th approximate solution obtained by formally applying the standard version of the same method to (1) with preconditioner (2).

Although Corollary 1 states that standard Lanczos-based methods can be safely applied to (1) with preconditioner (2) and an appropriate starting point, Algorithm 2 reduces the computational effort by never requiring products with $B$ or $B^T$. Only products with $A$ and $C$ are necessary. On the other hand, thanks to Theorem 2, specialized implementations of the standard Lanczos-based methods can be developed by exploiting the equalities $Bp_k + Cq_k = 0$ and $Bx_k - Cy_k = 0$, thus saving matrix-vector products. The computation involving $s_{k+1}$ can be carried out, for example, as the update $q_{k+1} = q_k - \tilde{z}_{k+1} - \beta_k q_{k-1}$ followed by $q_{k+1} = q_{k-1} - \alpha_k q_k$, or $s_{k+1}$ can overwrite $\tilde{z}_{k+1}$. Finally, once (2) has been factorized, storing $B$ is no longer necessary, and this can be used to free memory if needed.

A consequence of Theorem 2 is a formal equivalence between the iterates generated by Lanczos-based methods applied by way of Algorithm 2 and Algorithm 6. This equivalence requires a re-interpretation of the optimality conditions associated with the Krylov method.

Consider, e.g., MINRES (Paige and Saunders, 1975). The residual associated with iterate $[x_k; w_k; y_k]$ generated by P-MINRES, with $w_k = -FE^T y_k$, is

$$
r_{P,k} = \begin{bmatrix}
r_{P,k,x} \\
r_{P,k,w}
\end{bmatrix} = \begin{bmatrix}
b \\
0
\end{bmatrix} - \begin{bmatrix}
A & B^T \\
F^{-1} & E^T
\end{bmatrix} \begin{bmatrix}
x_k \\
w_k
\end{bmatrix} = \begin{bmatrix}
b - Ax_k - B^T y_k \\
0
\end{bmatrix},
$$

where we used the fact that $Bx_k + Ew_k = 0$ for all $k$. This residual corresponds to the residual at iterate $[x_k; y_k]$ generated by CP-MINRES:

$$
r_{CP,k} = \begin{bmatrix}
b \\
0
\end{bmatrix} - \begin{bmatrix}
A & B^T \\
B & -C
\end{bmatrix} \begin{bmatrix}
x_k \\
y_k
\end{bmatrix} = \begin{bmatrix}
b - Ax_k - B^T y_k \\
0
\end{bmatrix},
$$

where we exploited the fact that $Bx_k - Cy_k = 0$ for all $k$, which comes from $Bx_k + Ew_k = 0$ and $w_k = -FE^T y_k$.

We may apply the arguments of Gould et al. (2014, Section 3) to conclude that P-MINRES, and hence CP-MINRES, minimizes the deviation of $[r_{P,k,x}; 0]$ from the
range space of $N$, i.e., as in (15),

$$\|r_{P,k}\|_P^2 = (b - Ax_k - B^Ty_k)^T h_k, \tag{23}$$

where

$$\begin{bmatrix} G & B^T \\ F^{-1} & E^T \\ B & -C \end{bmatrix} \begin{bmatrix} h_k \\ f_k \\ l_k \end{bmatrix} = \begin{bmatrix} b - Ax_k - B^Ty_k \\ 0 \\ 0 \end{bmatrix}. \tag{24}$$

Because $h_k \in \text{Null}(B)$, we also have

$$\|r_{P,k}\|_P^2 = (b - Ax_k)^T h_k. \tag{25}$$

Equivalently, $h_k$ may be computed from

$$\begin{bmatrix} G & B^T \\ B & -C \end{bmatrix} \begin{bmatrix} h_k \\ l_k \end{bmatrix} = \begin{bmatrix} b - Ax_k \\ 0 \end{bmatrix}. \tag{26}$$

Because of its residual norm minimization property, CP-MINRES is appropriate to solve saddle-point systems in a linesearch inexact-Newton context, where we seek to reduce the residual of the Newton-like equations (1) in an appropriate space.

The same reasoning applies to the constraint-preconditioned version of any Lanczos-based Krylov method. For example, Paige and Saunders (1975) derive the conjugate gradient method of Hestenes and Stiefel (1952) directly from the Lanczos process. The nullspace variant of the constraint-preconditioned version, Lanczos CP-CG, generates iterates $\hat{x}_k$ so as to minimize the energy norm of the error, i.e.,

$$\|\hat{e}_k\|_M^2 = \hat{e}_k^T \hat{M} \hat{e}_k,$$

where $\hat{e}_k = \hat{x}_k - \hat{x}_*$, and $\hat{x}_*$ is the exact solution of (9). The definitions (10) yield

$$\|\hat{e}_k\|_M^2 = (\hat{x}_k - \hat{x}_*)^T Z^T M Z (\hat{x}_k - \hat{x}_*)$$

$$= (x_k - x_*)^T A (x_k - x_*) + (w_k - w_*)^T F^{-1} (w_k - w_*)$$

$$= (x_k - x_*)^T A (x_k - x_*) + (y_k - y_*)^T C (y_k - y_*), \tag{30}$$

where we used again the relationship $w_k = -F E^T y_k$ between iterates of P-CG and CP-CG. For Lanczos CP-CG to be applicable, $\hat{M}$ must be positive definite, which occurs when the sum of the number of negative eigenvalues of $K$ and $C$ is $m$ (Dollar et al., 2006, Theorem 2.1).

We can derive a “traditional” CP-CG implementation by applying the usual transformations to the Lanczos CP-CG. The result coincides with the implementation of Dollar et al. (2006), although the latter authors assume that $B$ has full row rank for specific purposes. It is also equivalent to that of Cafieri, D’Apuzzo, De Simone, and di Serafino (2007) for (1) with positive definite $C$. The above suggests that CP-CG is appropriate to solve saddle-point systems in constrained optimization where (1) is used to minimize a quadratic model of a penalty function and sufficient decrease of this quadratic model is sought, such as in trust-region methods.

Our last example considers SYMMLQ (Paige and Saunders, 1975), which does not require $\hat{M}$ to be positive definite but, like CG, requires (1) to be consistent. Its
constraint-preconditioned version, CP-SYMMLQ, computes \([x_k ; y_k]\) so as to minimize
the error in a norm defined by the preconditioner, i.e.,
\[
\tilde{e}_k^T \hat{P}^{-1} \tilde{e}_k = (\hat{x}_k - \hat{x}_*)^T (Z^T P Z)^{-1} (\hat{x}_k - \hat{x}_*) \\
= (\hat{x}_k - \hat{x}_*)^T Z^T Z(Z^T P Z)^{-1} Z^T Z(\hat{x}_k - \hat{x}_*) \\
= \begin{bmatrix} x_k - x_* \\ w_k - w_* \end{bmatrix}^T P G \begin{bmatrix} x_k - x_* \\ w_k - w_* \end{bmatrix},
\]
where we used similar identifications as above and assumed, without loss of generality,
that \(Z\) has orthonormal columns. In other words, if we define
\[
(25)
\begin{bmatrix}
G & B^T \\
F^{-1} & E^T \\
B & E \\
\end{bmatrix}
\begin{bmatrix}
e_x \\
e_w \\
\end{bmatrix} = \begin{bmatrix} x_k - x_* \\ w_k - w_* \end{bmatrix},
\]
then
\[
\tilde{e}_k^T \hat{P}^{-1} \tilde{e}_k = (x_k - x_*)^T e_x + (w_k - w_*)^T e_w = e_x^T G e_x + e_w^T F^{-1} e_w.
\]
By (5) and (25), there exists a vector \(e_y\) such that \(e_w = -F e_x \), and thus \(e_w^T F^{-1} e_w = e_y^T C e_y\). The second block row of (25) premultiplied by \(E\) yields \(E F (w_k - w_*) - C \bar{e} = -C e_y\), so that (25) can be written as
\[
\begin{bmatrix}
G & B^T \\
B & -C \\
\end{bmatrix}
\begin{bmatrix}
e_x \\
e_y \\
\end{bmatrix} = \begin{bmatrix} x_k - x_* \\ 0 \end{bmatrix}.
\]
Finally, CP-SYMMLQ minimizes
\[
\tilde{e}_k^T \hat{P}^{-1} \tilde{e}_k = e_x^T G e_x + e_y^T C e_y.
\]

5. Constraint-Preconditioned Arnoldi Process and Associated Krylov Solvers. A constraint-preconditioned version of the Arnoldi process can be derived by reasoning as in Section 3, obtaining Algorithm 3. The equivalence between the projected version (Algorithm 7 in Appendix A) and the constraint-preconditioned version is stated in Theorem 3, which is akin to Theorem 1.

**Theorem 3.** Let \(E\) and \(F\) be as defined in (4) and \(G\) chosen to satisfy Assumption 2.1. Let \(q_0 \in \mathbb{R}^m\) be arbitrary. Then, Algorithm 7 in Appendix A with starting guesses \(x_0 \in \mathbb{R}^m\) and \(w_0 = -F e_x q_0\), such that \(B x_0 + E w_0 = 0\), is equivalent to Algorithm 3 with starting guesses \(x_0\) and \(q_0\). In particular, for all \(k\), the vectors \(v_{k,x}\) and \(v_{k,w}\) and the scalars \(h_{i,k}\) in Algorithm 7 are equal to the vectors \(p_k\) and \(F E^T q_k\), and to the scalars \(h_{i,k}\) in Algorithm 3, respectively.

As in the case of the Lanczos process, the CP-Arnoldi process is equivalent to applying the corresponding standard Arnoldi process to system (1) with preconditioner (2) (see Algorithm 8 in Appendix A), as stated in the next theorem.

**Theorem 4.** Let Algorithm 3 be initialized with \(x_0 \in \mathbb{R}^n\) such that \(B x_0 = 0\),

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Algorithm 3 Constraint-Preconditioned Arnoldi Process

1: choose \([x_0 ; q_0]\) such that \(Bx_0 - Cq_0 = 0\)
2: \(p_0 = 0\)
3: \(u_0 = b - Ax_0\), \(t_0 = Cq_0\)
4: \([\tilde{p}_1 ; \tilde{z}_1] \leftarrow \) solution of (17) with right-hand side \([u_0 ; -t_0]\)
5: \(p_1 = \tilde{p}_1\)
6: \(q_1 = q_0 - \tilde{z}_1\)
7: \(h_{1,0} = (p_1^T u_0 + q_1^T t_0)^{\frac{1}{2}}\)
8: \(\text{if } h_{1,0} \neq 0 \text{ then}\)
9: \(p_1 = p_1/h_{1,0}\), \(q_1 = q_1/h_{1,0}\)
10: \(\text{end if}\)
11: \(k = 1\)
12: \(\text{while } h_{k,k-1} \neq 0 \text{ do}\)
13: \(u_k = Ap_k\), \(t_k = Cq_k\)
14: \([\tilde{p}_{k+1} ; \tilde{z}_{k+1}] \leftarrow \) solution of (17) with right-hand side \([u_k ; -t_k]\)
15: \(p_{k+1} = \tilde{p}_{k+1}\)
16: \(q_{k+1} = q_k - \tilde{z}_{k+1}\)
17: \(\text{for } i = 1, \ldots, k \text{ do}\)
18: \(h_{i,k} = (p_i^T u_k + q_i^T t_k)^{\frac{1}{2}} = p_i^T Ap_k + q_i^T Cq_k\)
19: \(p_{k+1} = p_{k+1} - h_{i,k} p_i\)
20: \(q_{k+1} = q_{k+1} - h_{i,k} q_i\)
21: \(\text{end for}\)
22: \(h_{k+1,k} = (p_{k+1}^T u_k + q_{k+1}^T t_k)^{\frac{1}{2}} = (p_{k+1}^T Ap_k + q_{k+1}^T Cq_k)^{\frac{1}{2}}\)
23: \(\text{if } h_{k,k+1} \neq 0 \text{ then}\)
24: \(p_{k+1} = p_{k+1}/h_{k,k+1}\), \(q_{k+1} = q_{k+1}/h_{k,k+1}\)
25: \(\text{end if}\)
26: \(k = k + 1\)
27: \(\text{end while}\)

Theorem 4 allows us to develop a constraint-preconditioned variant of any Krylov method based on the Arnoldi process, using a starting guess satisfying (21). Such variants are equivalent to their standard counterparts preconditioned with (2), but are computationally cheaper, as in the case of Lanczos-based methods. Furthermore, CP-Krylov versions of optimal Arnoldi-based Krylov methods preserve the minimization properties of these methods in the sense explained in Section 4. For example, the constraint-preconditioned version of GMRES (Saad and Schultz, 1986) minimizes the norm of the deviation of the residual from Range(\(N\)) similarly to MINRES.

Obtaining constraint-preconditioned versions of GMRES(\(\ell\)) and DQGMRES is straightforward, by restarting and truncating the CP-Arnoldi basis generation process, respectively, as in the standard case (Saad, 2003). Note that DQGMRES with memory 2, i.e., with orthogonalization of each Arnoldi vector against the two previous vectors only, is equivalent to CP-MINRES in exact arithmetic when \(A\) is symmetric. In finite precision arithmetic, DQGMRES with a larger memory may dampen the loss.
of orthogonality among the Lanczos vectors and act as a local reorthogonalization
procedure, although we did not observe significant differences in Section 7.

Dollar (2007, Theorems 4.1 and 4.3) establishes that if $C$ is positive semi-definite
of rank $p$, $P^{-1}K$ has an eigenvalue at 1 of multiplicity $2m - p$, while the remaining
$n - m + p$ eigenvalues are defined by a generalized eigenvalue problem. A remark after
(Dollar, 2007, Theorem 4.1) states that Assumption 2.1 ensures that all eigenvalues are
real. In addition, the dimension of the Krylov space is at most $\min(n - m + p + 2, n + m)$.

Inspection reveals that Dollar’s proofs of those results do not use the fact that $A$ is
symmetric; the results hold for general $A$. Loghin (2017) establishes similar results on
the eigenvalues of non-regularized saddle-point matrices for general $A$ and general $G$.
Clustering eigenvalues accelerates convergence of nonsymmetric Krylov solvers in
many practical cases, although the convergence behavior of such solvers is not fully
characterized by the eigenvalues (Greenbaum, Pták, and Strakoš, 1996).

6. Implementation Issues. We implemented the constraint-preconditioned
variants of the Lanczos-CG, MINRES, SYMMLQ, GMRES($\ell$) and QGMRES methods
for (1) in a MATLAB library named cpkrylov. For completeness, we also included in
the library an implementation of the CP-CG method in the form given by Dollar et al.
(2006). We think that cpkrylov can be useful as a basis for the development of more
sophisticated numerical software.

All solvers are accessed via a common interface exposed by the main driver
reg_cpkrylov(), which performs pre-processing operations, calls the requested solver,
performs post-processing operations, and returns solutions and statistics to the user.

cpkrylov is freely available from github.com/optimizers/cpkrylov.

Because $A$ is never required as an explicit matrix, we allow the user to supply it as
an abstract linear operator as implemented in the Spot linear operator toolbox. Spot
allows us to use the familiar matrix notation with operators for which a representation
as an explicit matrix is unavailable or inefficient. This affords the user flexibility in
defining $A$ while keeping the implementation of the various Krylov methods as readable
as if $A$ were a matrix.

Gould et al. (2014) observe that the numerical stability of projected Krylov solvers
depends on keeping $[x_k; w_k]$ in $\text{Null}(N)$. While the iterates lie in the nullspace in exact
arithmetic, $[x_k; w_k]$ may have a non-negligible component in $\text{Range}(N^T)$ because of
roundoff error. In turn, the stability of CP-Krylov solvers depends on how accurately
$x_k; y_k$ satisfies

$$Bx_k - Cy_k = 0.$$  

Gould et al. (2001) suggest to increase the accuracy by applying iterative refinement
after solving (17) with a direct method. In cpkrylov, the constraint preconditioner
$P_G$ is implemented as a Spot operator $P$ such that writing $z = P*r$, where $z = [z_1 ; z_2]$ and $r = [r_1 ; r_2]$, corresponds to solving

$$\begin{bmatrix} G & B^T \\ B & -C \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} r_1 \\ r_2 \end{bmatrix},$$

and performing iterative refinement if requested by the user or if the residual norm
of (26) exceeds a given tolerance. By default, the matrix of (26) is factorized by
way of MATLAB’s $\text{ldl()}$. Spot allows us to separate the implementation of the
preconditioner from that of other phases of solvers, so that future extensions to the
The Newton step requires the solution of a linear system, which can be cast into the form (1) by an inexpensive elimination of variables. The corresponding matrix takes the form

\[
A \quad B^T
\]

subject to

\[
J_1 x + J_2 s + \delta r = b, \quad s \geq 0,
\]

where \(S = \text{diag}(s)\), \(Z = \text{diag}(z)\), with \(z\) the complementary variable of \(s\), and the block \(A\) is symmetric positive definite. Recently, unreduced KKT systems have attracted the interest of researchers because of their better spectral properties, especially as the IP iterate approaches a solution of the optimization problem (Greif, Moulding, and Orban, 2014; Morini, Simoncini, and Tani, 2016). Among the possible unreduced
matrices, we consider

\[
K_3 = \begin{bmatrix}
A & B^T \\
B & -C 
\end{bmatrix}
\]

which is a permutation of the matrix in the first system of (Orban, 2015a, page 3). All the linear systems from (Orban, 2015b) correspond to IP iterations 0, 5 and 10, where a larger iteration number yields a more ill-conditioned system. The differences between the matrices at the various iterations reside in the values of the vectors \(s\) and \(z\), and of the regularization parameters \(\rho\) and \(\delta\). The latter were set to \(\rho = \delta = 1\) at iteration 0, \(\rho = \delta = 10^{-5}\) at iteration 5, and \(\rho = \delta = 10^{-8}\) at iteration 10. On some CUTEst quadratic problems the IP method satisfied its stopping criterion before iteration 10 or even iteration 5; therefore, not all problems have three associated systems.

Here we show results concerning the CUTEst problems reported in the first column of Table 1. For three of them the IP method did not reach iteration 10; altogether, we considered 36 linear systems for each matrix type, i.e., \(K_2\) or \(K_3\). For each problem, we report in Table 1 the size of the matrices, their density (computed as the ratio between the number of nonzero entries and the total number of entries), the range of their condition numbers (obtained by using the \texttt{condest}() MATLAB function), and the associated IP iterations. We see that for six problems the condition number of \(K_3\) varies in a smaller range than that of \(K_2\). Note that we simply provide the condition number of \(K_3\) as an indication of the difficulty of solving a system with this matrix and the accuracy that may be expected of a solution.

We run CP-CG and CP-MINRES on the systems with matrix \(K_2\), and CP-DQGMRES and CP-GMRES(\(\ell\)) on the systems with matrix \(K_3\). We set \(\ell = 100\) and used the same value for the memory parameter of CP-DQGMRES, i.e., the number of Arnoldi vectors to be stored in the truncated CP-Arnoldi process. We denote this solver CP-DQGMRES(100). The leading block \(G\) of the constraint preconditioner (2) was set as the diagonal of \(A\). In (27) we set \(\epsilon_a = \epsilon_r = 10^{-6}\); we also fixed a maximum number of 1500 iterations. One iterative refinement step was performed during the application of the constraint preconditioner. The experiments were carried out on a 3.1 GHz Intel Core i7 processor with 16 GB of RAM, 8 MB of L3 cache and the

\begin{table}
\begin{tabular}{|c|c|c|c|c|c|}
\hline
Problem & size & dens % & cond & size & dens % & cond & IP iters \\
\hline
cvxqp1_s & 550 & 7.3e-1 & \(O(10^{10} - 10^{10})\) & 750 & 5.0e-1 & \(O(10^5 - 10^{10})\) & 0, 5, 10 \\
cvxqp1_m & 5500 & 7.4e-2 & \(O(10^3 - 10^{10})\) & 750 & 5.1e-2 & \(O(10^4 - 10^{11})\) & 0, 5, 10 \\
cvxqp1_l & 55000 & 7.4e-3 & \(O(10^4 - 10^{13})\) & 7500 & 5.1e-3 & \(O(10^5 - 10^{13})\) & 0, 5, 10 \\
cvxqp2_s & 525 & 7.4e-1 & \(O(10^5 - 10^{10})\) & 750 & 5.0e-1 & \(O(10^5 - 10^{10})\) & 0, 5, 10 \\
cvxqp2_m & 5250 & 7.4e-1 & \(O(10^4 - 10^{10})\) & 750 & 5.1e-2 & \(O(10^4 - 10^{10})\) & 0, 5, 10 \\
cvxqp2_l & 52500 & 7.5e-1 & \(O(10^5 - 10^{10})\) & 7500 & 5.1e-2 & \(O(10^5 - 10^{10})\) & 0, 5, 10 \\
cvxqp3_s & 575 & 7.2e-1 & \(O(10^5 - 10^{10})\) & 775 & 5.0e-1 & \(O(10^5 - 10^{10})\) & 0, 5, 10 \\
cvxqp3_m & 5750 & 7.3e-2 & \(O(10^5 - 10^{10})\) & 7750 & 5.0e-2 & \(O(10^5 - 10^{12})\) & 0, 5, 10 \\
cvxqp3_l & 57500 & 7.3e-3 & \(O(10^5 - 10^{10})\) & 77500 & 5.0e-3 & \(O(10^5 - 10^{13})\) & 0, 5, 10 \\
mosasqp1 & 8900 & 3.4e-2 & \(O(10 - 10^5)\) & 12100 & 2.5e-2 & \(O(10 - 10^5)\) & 0, 5 \\
mosasqp2 & 3900 & 9.6e-2 & \(O(10 - 10^5)\) & 5400 & 6.6e-2 & \(O(10 - 10^5)\) & 0, 5 \\
stepq1 & 22537 & 2.5e-2 & \(O(10^5 - 10^{13})\) & 30731 & 1.6e-2 & \(O(10^5 - 10^{13})\) & 0, 5, 10 \\
stepq2 & 22537 & 2.5e-2 & \(O(10^5 - 10^{13})\) & 30731 & 1.6e-2 & \(O(10^5 - 10^{13})\) & 0, 5 \\
\hline
\end{tabular}
\caption{Details on the saddle-point matrices used in the experiments.}
\end{table}
macOS 10.14.6 operating system, using MATLAB R2018b. The execution times were measured in seconds, by using the `tic` and `toc` MATLAB commands.

In Figure 1 we compare CP-CG and CP-MINRES using Dolan and Moré (2002) performance profiles, with the number of iterations and the execution time as performance metrics. The execution time only includes the time for solving the linear system, since the time for building the preconditioner is the same for the two solvers. For completeness, we provide a brief description of the performance profiles. Let us consider a set of algorithms \( \{ A_i \mid i = 1, \ldots, n_A \} \) and a set of test problems \( \{ T_j \mid j = 1, \ldots, n_T \} \). Let \( S_{T_j, A_i} \geq 0 \) be a statistic corresponding to the solution of \( T_j \) by \( A_i \), and suppose that the smaller the statistic the better the algorithm. Furthermore, let \( S_{T_j} = \min \{ S_{T_j, A_i} \mid i = 1, \ldots, n_A \} \). The performance profile of algorithm \( A_i \) is defined as

\[
\pi_i(\chi) = \frac{\text{size}\{ T_j \mid S_{T_j, A_i}/S_{T_j} < \chi \}}{\text{size}\{ T_j \mid j = 1, \ldots, n_T \}}, \quad \chi \geq 1,
\]

where the ratio \( S_{T_j, A_i}/S_{T_j} \) is set to \( +\infty \) if \( A_i \) fails in solving \( T_j \). Thus \( \pi_i(1) \) gives the percentage of problems for which \( A_i \) is the best, while the percentage of problems that are successfully solved by \( A_i \) is \( \lim_{\chi \to +\infty} \pi_i(\chi) \).

![Figure 1: CP-CG vs. CP-MINRES on systems with matrix \( K_2 \): performance profiles in terms of iterations (left) and solve time (right).](image)

We see that CP-MINRES performs slightly better than CP-CG in terms of both iterations and solve time. This agrees with the fact that CP-MINRES minimizes the residual seminorm used in the stopping criterion, while CP-CG does not, and confirms that CP-MINRES is appropriate to solve saddle-point systems in linesearch inexact-Newton contexts. Furthermore, both methods were able to solve all the systems with the required accuracy. A closer examination of the results reveals that CP-CG and CP-MINRES performed their largest number of iterations, 1,288 and 1,045, respectively, on cvxqp2_1 at IP iteration 5. On all other systems, CP-CG and CP-MINRES performed at most 861 and 735 iterations, respectively.

In order to provide some details on the behavior of CP-CG and CP-MINRES, in Figure 2 we depict the histories of \( \| r_{P,k} \|/\| P \| \) for both solvers applied to the systems corresponding to cvxqp1_1 and stcp1 at IP iterations 10 and 5, respectively. We see that CP-MINRES is more efficient than CP-CG at reducing the residual when the problem requires a larger number of iterations. We verified that this is a general behavior.
In Figure 3 we compare CP-GMRES(100) with CP-DQGMRES(100) in terms of iterations and solve time.

CP-GMRES(100) appears more efficient than CP-DQGMRES(100). Neither manages to solve mosarqp1 at IP iteration 5 within 1,500 iterations. CP-GMRES(100) solves cvxqp2_l at IP iteration 5 in 1,319 iterations and all other systems in fewer than 933 iterations. CP-DQGMRES(100) requires between 1,248 and 1,447 iterations on three systems and fewer than 793 on the rest.

CP-DQGMRES(100) and CP-GMRES(100) performed fewer than 100 iterations, and therefore were one and the same method in principle, on about 47% of the systems. On those 47% of systems, they performed the same number of iterations. CP-DQGMRES(100) performed fewer iterations than CP-GMRES(100) on 22%, and more iterations on 28% of the systems. Closer inspection of the results revealed that in the former case the numbers of iterations are generally close, while in the latter the number of iterations of CP-GMRES(100) is often substantially smaller. Examples of this behavior are given in Figure 4, showing the convergence histories for the systems corresponding to cvxqp3_l and cvxqp1_m at IP iterations 10 and 5, respectively.
The solve time profiles show that CP-DQGMRES(100) was generally slower than CP-GMRES(100), although the times are essentially the same when the two solvers performed less than 100 iterations. This behavior seems to indicate that the implementation of CP-DQGMRES can be improved.

In order to evaluate the effectiveness of our constraint preconditioner on a saddle-point matrix with nonsymmetric leading block, we also applied the standard MATLAB gmres() with restart equal to 100, without preconditioner, to the systems with matrix $K_3$. We set the tolerance for the stopping criterion equal to $10^{-6}$ and fixed a maximum number of 1500 iterations. gmres() was able to reach the required accuracy in the residual only for 10 systems, all corresponding to IP iteration 0. Since the residual norm computed by the standard GMRES solver with no preconditioner is different from that computed by CP-GMRES, we also checked the accuracy of the computed solution, by using the 2-norm relative error with respect to the solution obtained by solving the saddle-point system with the \ (backslash) MATLAB operator. The error of the standard GMRES solver is larger than the error of CP-GMRES for all problems but stcqp2 at IP iteration 0 (both methods achieve the required accuracy in their residual norms), and is much larger in all the cases where the stopping criterion is not satisfied.

Finally, in Figure 5 we compare CP-MINRES applied to the systems with matrix $K_2$ with CP-GMRES(100) applied to the systems with matrix $K_3$, by using performance profiles of the number of iterations and the total execution time, i.e., the time for setting up the preconditioner plus the time for solving the system. CP-MINRES is always more efficient than CP-GMRES(100) in terms of both metrics on our test set.

8. Discussion. We extended the approach of Gould et al. (2014) to saddle-point systems with regularization and provided principles from which to derive constraint-preconditioned iterative methods. The resulting methods are conceptually equivalent to standard iterative methods applied to a reduced system in a way that preserves their properties, including quantities that increase or decrease monotonically at each iteration. Specifically, we discussed constraint-preconditioned versions of the CG-Lanczos, MINRES, SYMMLQ, GMRES(ℓ) and DQGMRES methods, showing that they preserve the properties of the corresponding standard methods in a suitable reduced Krylov space. We illustrated our approach on methods based on the Lanczos and Arnoldi processes, but it applies equally to other processes, including those
of Golub and Kahan (1965), Saunders, Simon, and Yip (1988), and the unsymmetric Lanczos (1952) bi-orthogonalization process. We also implemented these methods in a MATLAB library, named `cpkrylov`, which provides a basis for the development of more sophisticated numerical software.

An open question related to constraint preconditioners concerns the best way to reduce their computational cost. Inexact constraint preconditioners have been developed and analyzed, based on approximations of the Schur complement of the leading block of the constraint preconditioner or on other approximations (Lukšan and Vlček, 1998; Perugia and Simoncini, 2000; Durazzi and Ruggiero, 2003; Bergamaschi, Gondzio, Venturin, and Zilli, 2007; Sesana and Simoncini, 2013). Preconditioner updating techniques, producing inexact and exact constraint preconditioners, have been also proposed in order to reduce the cost of solving sequences of saddle-point systems (Bellavia, De Simone, di Serafino, and Morini, 2015, 2016; Fisher, Gratton, Gürol, Trémolet, and Vasseur, 2016; Bergamaschi, De Simone, di Serafino, and Martinez, 2018). It must be noted, however, that the inexact constraint preconditioners considered so far generally do not produce preconditioned vectors lying in the nullspace of $N$, which is a key issue to obtain CP-preconditioned methods methods for (1) equivalent to suitably preconditioned Krylov methods for (9). On the other hand, inexact preconditioners have proven effective in reducing the computational time for the solution of large-scale saddle-point systems. A further possibility for lowering the cost of constraint preconditioners is to apply them inexactly using an iterative method. Of course, preserving the property of obtaining preconditioned vectors lying in the nullspace of $N$ is still a main issue. To the best of our knowledge, this approach has not been yet addressed in the literature.

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Appendix A. Standard Lanczos and Arnoldi Processes. For reference we state the standard Lanczos process in plain and preconditioned versions, and the full-space Lanczos process for (1) with preconditioner (2). We also state the projected and full-space Arnoldi processes.

Algorithm 4 Lanczos Process for $Ax = b$

1: choose $x_0$
2: $v_0 = 0, k = 1$
3: $r_0 = b - Ax_0$
4: $v_1 = r_0$
5: $\beta_1 = (v_1^T r_0)^{\frac{1}{2}}$
6: if $\beta_1 \neq 0$ then
7: \hspace{1em} $v_1 = v_1 / \beta_1$
8: \hspace{1em} $|v_1|_2 = 1$
9: end if
10: $k = 1$
11: while $\beta_k \neq 0$ do
12: \hspace{1em} $u_k = Av_k$
13: \hspace{1em} $\alpha_k = u_k^T v_k$
14: \hspace{1em} $v_{k+1} = u_k - \alpha_k v_k - \beta_k v_{k-1}$
15: \hspace{1em} $\beta_{k+1} = (v_{k+1}^T u_k)^{\frac{1}{2}}$
16: \hspace{1em} if $\beta_{k+1} \neq 0$ then
17: \hspace{2em} $v_{k+1} = v_{k+1} / \beta_{k+1}$
18: \hspace{2em} $|v_{k+1}|_2 = 1$
19: \hspace{1em} end if
20: \hspace{1em} $k = k + 1$
21: end while

Algorithm 5 Lanczos Process for $Ax = b$ with Preconditioner $J = J^T > 0$

1: choose $x_0$
2: $v_0 = 0$
3: $r_0 = b - Ax_0$
4: solve $Jv_1 = r_0$
5: $\beta_1 = (v_1^T r_0)^{\frac{1}{2}}$
6: if $\beta_1 \neq 0$ then
7: \hspace{1em} $v_1 = v_1 / \beta_1$
8: \hspace{1em} $|v_1|_J = 1$
9: end if
10: $k = 1$
11: while $\beta_k \neq 0$ do
12: \hspace{1em} $u_k = Av_k$
13: \hspace{1em} $\alpha_k = u_k^T v_k$
14: \hspace{1em} solve $Jv_{k+1} = u_k$
15: \hspace{1em} $v_{k+1} = v_{k+1} - \alpha_k v_k - \beta_k v_{k-1}$
16: \hspace{1em} $\beta_{k+1} = (v_{k+1}^T u_k)^{\frac{1}{2}}$
17: \hspace{1em} if $\beta_{k+1} \neq 0$ then
18: \hspace{2em} $v_{k+1} = v_{k+1} / \beta_k$
19: \hspace{2em} $|v_{k+1}|_J = 1$
20: \hspace{1em} end if
21: \hspace{1em} $k = k + 1$
22: end while
Algorithm 6 Full-Space Lanczos Process for (1) with Preconditioner (2)

1: choose \( [x_0; y_0] \) such that \( Bx_0 - Cy_0 = 0 \)
2: initialize
\[
\begin{bmatrix}
  v_{0,x} \\
  v_{0,y}
\end{bmatrix} = \begin{bmatrix} 0 \\
  0
\end{bmatrix}
\]
3: set
\[
\begin{bmatrix}
  r_{0,x} \\
  r_{0,y}
\end{bmatrix} = \begin{bmatrix} b \\
  0
\end{bmatrix} - \begin{bmatrix} A & B^T \\
  B & -C
\end{bmatrix} \begin{bmatrix} x_0 \\
  y_0
\end{bmatrix} = \begin{bmatrix} b - Ax_0 - B^Ty_0 \\
  0
\end{bmatrix}
\]
4: obtain \( v_1 \) as the solution of
\[
\begin{bmatrix} G & B^T \\
  B & -C
\end{bmatrix} \begin{bmatrix} v_{1,x} \\
  v_{1,y}
\end{bmatrix} = \begin{bmatrix} r_{0,x} \\
  r_{0,y}
\end{bmatrix}
\]
5: \( \beta_1 = (v^T_1 r_0)^{\frac{1}{2}} = (v^T_1 r_0)^{\frac{1}{2}} 
\]
6: if \( \beta_1 \neq 0 \) then
7: \( v_1 = v_1 / \beta_1 \)
8: end if
9: \( k = 1 \)
10: while \( \beta_k \neq 0 \) do
11: compute
\[
\begin{bmatrix}
  u_{k,x} \\
  u_{k,y}
\end{bmatrix} = \begin{bmatrix} A & B^T \\
  B & -C
\end{bmatrix} \begin{bmatrix} v_{k,x} \\
  v_{k,y}
\end{bmatrix}
\]
12: \( \alpha_k = u^T_k v_k = v^T_{k,x} A v_{k,x} + 2v^T_{k,x} B^T v_{k,y} - v^T_{k,y} C v_{k,y} \)
13: obtain \( v_{k+1} \) as the solution of
\[
\begin{bmatrix} G & B^T \\
  B & -C
\end{bmatrix} \begin{bmatrix} v_{k+1,x} \\
  v_{k+1,y}
\end{bmatrix} = \begin{bmatrix} u_{k,x} \\
  u_{k,y}
\end{bmatrix}
\]
14: \( v_{k+1} = v_{k+1} - \alpha_k v_k - \beta_k v_{k-1} \)
15: \( \beta_{k+1} = (v^T_{k+1} u_{k+1})^{\frac{1}{2}} = (v^T_{k+1} u_{k+1} + v^T_{k+1} u_{k+1})^{\frac{1}{2}} \)
16: if \( \beta_{k+1} \neq 0 \) then
17: \( v_{k+1} = v_{k+1} / \beta_{k+1} \)
18: end if
19: \( k = k + 1 \)
20: end while
Algorithm 7 Projected Arnoldi Process

1: choose \([x_0; w_0]\) such that \(Bx_0 + Ew_0 = 0\)
2: \(v_{0,x} = 0,\ v_{0,w} = -w_0\)
3: \(u_{0,x} = b - Ax_0,\ u_{0,w} = -F^{-1}w_0\)
4: \([\tilde{u}_{1,x}; \tilde{u}_{1,w}; \tilde{z}_1]\) ← solution of (14) with right-hand side \([u_{0,x}; u_{0,w}; 0]\)
5: \(v_{1,x} = \tilde{u}_{1,x},\ v_{1,w} = \tilde{u}_{1,w}\)
6: \(h_{1,0} = (v_{1,x}^Tu_{0,x} + v_{1,w}^Tu_{0,w})^{\frac{1}{2}}\)
7: if \(h_{1,0} \neq 0\) then
8: \(v_{1,x} = v_{1,x}/h_{1,0},\ v_{1,w} = v_{1,w}/h_{1,0}\)
9: end if
10: \(k = 1\)
11: while \(h_{k,k-1} \neq 0\) do
12: \(u_{k,x} = Av_{k,x},\ u_{k,w} = F^{-1}v_{k,w}\)
13: \([\tilde{u}_{k+1,x}; \tilde{u}_{k+1,w}; \tilde{z}_{k+1}]\) ← solution of (14) with right-hand side \([u_{k,x}; u_{k,w}; 0]\)
14: \(v_{k+1,x} = \tilde{u}_{k+1,x},\ v_{k+1,w} = \tilde{u}_{k+1,w}\)
15: for \(i = 1, \ldots, k\) do
16: \(h_{i,k} = v_{i,x}^Tu_{k,x} + v_{i,w}^Tu_{k,w}\)
17: \(v_{k+1,x} = v_{k+1,x} - h_{i,k}u_{i,x},\ v_{k+1,w} = v_{k+1,w} - h_{i,k}u_{i,w}\)
18: end for
19: \(h_{k+1,k} = (v_{k+1,x}^Tu_{k+1,x} + v_{k+1,w}^Tu_{k+1,w})^{\frac{1}{2}}\)
20: if \(h_{k+1,k} \neq 0\) then
21: \(v_{k+1,x} = v_{k+1,x}/h_{k+1,k},\ v_{k+1,w} = v_{k+1,w}/h_{k+1,k}\)
22: end if
23: \(k = k + 1\)
24: end while
Algorithm 8 Full-Space Arnoldi Process for (1) with Preconditioner (2)

1: choose \( x_0 ; y_0 \) such that \( Bx_0 - Cy_0 = 0 \)
2: initialize
\[
\begin{bmatrix}
v_{0,x} \\
v_{0,y}
\end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}
\]
3: set
\[
\begin{bmatrix}
r_{0,x} \\
r_{0,y}
\end{bmatrix} = \begin{bmatrix} b \\ 0 \end{bmatrix} - \begin{bmatrix} A & B^T \\ B & -C \end{bmatrix} \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} = \begin{bmatrix} b - Ax_0 - B^T y_0 \\ 0 \end{bmatrix}
\]
4: obtain \( v_1 \) as the solution of
\[
\begin{bmatrix} G & B^T \\ B & -C \end{bmatrix} \begin{bmatrix} v_{1,x} \\
v_{1,y}
\end{bmatrix} = \begin{bmatrix} r_{0,x} \\
r_{0,y}
\end{bmatrix}
\]
5: \( h_{1,0} = (v_{1,x}^T r_{0,x})^{\frac{1}{2}} = (v_{1,x}^T r_{0,x})^{\frac{1}{2}} \)
6: if \( h_{1,0} \neq 0 \) then
7: \( v_1 = v_1 / \beta_1 \)
8: end if
9: \( k = 1 \)
10: while \( h_{k,k-1} \neq 0 \) do
11: compute
\[
\begin{bmatrix}
u_{k,x} \\
u_{k,y}
\end{bmatrix} = \begin{bmatrix} A & B^T \\ B & -C \end{bmatrix} \begin{bmatrix} v_{k,x} \\
v_{k,y}
\end{bmatrix}
\]
12: obtain \( v_{k+1} \) as the solution of
\[
\begin{bmatrix} G & B^T \\ B & -C \end{bmatrix} \begin{bmatrix} v_{k+1,x} \\
v_{k+1,y}
\end{bmatrix} = \begin{bmatrix} u_{k,x} \\
u_{k,y}
\end{bmatrix}
\]
13: for \( i = 1, \ldots, k \) do
14: \( h_{i,k} = v_{i,x}^T u_k = v_{i,x}^T A v_{k,x} + 2v_{i,x}^T B^T v_{k,y} - v_{i,y}^T C v_{k,y} \)
15: \( v_{k+1} = v_{k+1} - h_{i,k} v_i \)
16: end for
17: \( h_{k+1,k} = (v_{k+1,x} u_k)^{\frac{1}{2}} = (v_{k+1,x} u_k + v_{k+1,y} u_k)^{\frac{1}{2}} \)
18: if \( h_{k+1,k} \neq 0 \) then
19: \( v_{k+1} = v_{k+1} / h_{k+1,k} \)
20: end if
21: \( k = k + 1 \)
22: end while