Two-parameter quantum algebras, twin-basic numbers, and associated generalized hypergeometric series†

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Abstract

We give a method to embed the $q$-series in a $(p,q)$-series and derive the corresponding $(p,q)$-extensions of the known $q$-identities. The $(p,q)$-hypergeometric series, or twin-basic hypergeometric series (different from the usual bibasic hypergeometric series), is based on the concept of twin-basic number $[n]_{p,q} = (p^n - q^n)/(p - q)$. This twin-basic number occurs in the theory of two-parameter quantum algebras and has also been introduced independently in combinatorics. The $(p,q)$-identities thus derived, with doubling of the number of parameters, offer more choices for manipulations; for example, results that can be obtained via the limiting process of confluence in the usual $q$-series framework can be obtained by simpler substitutions. The $q$-results are of course special cases of the $(p,q)$-results corresponding to choosing $p = 1$. This also provides a new look for the $q$-identities.

1. Introduction

For the two-parameter quantum group $GL_{p,q}(2)$ the fundamental representation is given by the $T$-matrix,

$$T = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

whose elements satisfy the commutation relations

$$ab = p^{-1}ba, \quad cd = p^{-1}dc, \quad ac = q^{-1}ca, \quad bd = q^{-1}db,$$

$$bc = q^{-1}pcb, \quad ad - da = (p^{-1} - q)bc,$$

consistent with the equation

$$R(T \otimes I)(I \otimes T) = (I \otimes T)(T \otimes I)R, \quad (3)$$

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corresponding to the $R$-matrix

$$R = (pq)^{1/4} \begin{pmatrix} (pq)^{-1/2} & 0 & 0 & 0 \\ 0 & (p/q)^{-1/2} & 0 & 0 \\ 0 & 0 & (pq)^{-1/2} - (pq)^{1/2} & (p/q)^{1/2} \\ 0 & 0 & 0 & (pq)^{-1/2} \end{pmatrix}. \quad (4)$$

The two-parameter quantum algebra, $U_{p,q}(gl(2))$, dual to $GL_{p,q}(2)$, is generated by $\{Z, J_0, J_\pm\}$ satisfying the commutation relations

$$[Z, J_0] = 0, \quad [Z, J_\pm] = 0,$$

$$[J_0, J_\pm] = \pm J_\pm, \quad J_+ J_- - pq^{-1} J_- J_+ = \frac{p^{-2J_0} - q^{2J_0}}{p^{-1} - q}. \quad (5)$$

To realize this algebra (5), a $(p,q)$-oscillator algebra,

$$aa^\dagger - qa^\dagger a = p^{-N}, \quad [N, a] = -a, \quad [n, a^\dagger] = a^\dagger, \quad (6)$$

was introduced in [1] generalizing/unifying several forms of $q$-oscillator algebras well known in the earlier physics literature related to the representation theory of single-parameter quantum algebras. The algebra (6) is satisfied when

$$a^\dagger a = \frac{p^{-N} - q^N}{p^{-1} - q}, \quad aa^\dagger = \frac{p^{-(N+1)} - q^{N+1}}{p^{-1} - q}. \quad (7)$$

When $p = q$ or $p = 1$ the algebra (6) becomes two different versions of the $q$-oscillator algebra related to the representation theory of $U_q(sl(2))$. The relations (5 and (7) suggest immediately a generalization of the Heine $q$-number,

$$[n]_q = \frac{1 - q^n}{1 - q}, \quad (8)$$

to a $(p,q)$-number as

$$[n]_{p,q} = \frac{p^n - q^n}{p - q}. \quad (9)$$

If we define a $(p,q)$-derivative by

$$\hat{D}_{p,q} f(z) = \frac{f(pz) - f(qz)}{(p - q)z} \quad (10)$$

then

$$\hat{D}_{p,q} z^n = [n]_{p,q} z^{n-1}. \quad (11)$$

Several properties of this $(p,q)$-number (9), which we will now call as the twin-basic number, including the elements of $(p,q)$-calculus following from (10) were studied very briefly in [1]. For the sake of convenience, we shall denote $[n]_{p,q}$ simply as $[n]$ and omit the subscripts $p, q$ from other expressions also whenever the values of these twin-base parameters are clear from the context.

Around the same time as [1], Brodimas, et al. [2] and Arik, et al. [3] also, independently, introduced the $(p,q)$-number in the physics literature, but in a very much less detailed
manner. They also introduced the \((p, q)\)-oscillator and the \((p, q)\)-number in the same context of realization of \(U_{p,q}(gl(2))\). It is a surprising fact that around the same time, without any connection to the quantum group related mathematics/physics literature, Wachs and White [4] introduced the \((p, q)\)-number, defined as \((p^n - q^n)/(p - q)\), in the mathematics literature while generalizing the Sterling numbers, motivated by certain combinatorial problems (for further generalizations and applications in this direction see [5]). In physics literature, Katriel and Kibler [6] defined the \((p, q)\)-binomial coefficients and derived a \((p, q)\)-binomial theorem while discussing normal ordering for deformed boson operators obeying the algebra [5]. Smirnov and Wehrhahn [7] gave an operator, or noncommutative, version of such a \((p, q)\)-binomial theorem. Floreanini, Lapointe and Vinet [8] related the \((p, q)\)-differential identities into their \((p, q)\)-identities and in fact give a new look to the latter. Gelfand, et al. [9, 10] generalized the two-parameter deformed derivative \((p, q)\)-hypergeometric series 

\[ u = \frac{1}{z} \left( \frac{d}{dz} \right) f(z) \]  

where \(u(z)\) is an arbitrary entire function. This leads to a \(u\)-calculus and a unified exposition of the classical theory and the \(q\)-theory and results in new \(u\)-analogues of classical hypergeometric functions. The \((p, q)\)-hypergeometric series corresponds to the choice \(u(z) = (p^z - q^z)/(p - q)\). Generalizing the definition of \(r\Psi_{r-1}\) by Burban and Klimyk [11], one of us defined the general \((p, q)\)-hypergeometric series \(r\Phi_s\) and derived some related preliminary results [13]. Some applications of the \((p, q)\)-hypergeometric series in the context of representations of two-parameter quantum groups have been considered by Nishizawa [15] and Sahai and Srivastava [16].

In the present work we shall deal only with the \((p, q)\)-hypergeometric series as defined in [13]. We introduce a method of application of the \((p, q)\)-series to convert the various well known \(q\)-identities into their \((p, q)\)-analogues; after the conversion the resulting \((p, q)\)-identities offer more choices for symbolic manipulations transcending the applications of the original \(q\)-identities and in fact give a new look to the latter.

2. Twin-basic hypergeometric series \(r\Phi_s\)

Let us recall some basic definitions from the theory of \(q\)-hypergeometric series [17]. The \(q\)-shifted factorial is given by

\[ (a; q)_n = \begin{cases} 1, & n = 0, \\ (1 - a)(1 - aq)(1 - aq^2) \cdots (1 - aq^{n-1}), & n = 1, 2, \ldots. \end{cases} \]

With

\[ (a_1, a_2, \ldots, a_k; q)_n = (a_1; q)_n (a_2; q)_n \cdots (a_k; q)_n. \]  

the \(q\)-hypergeometric series, or the basic hypergeometric series, \(r\phi_s\) is defined as

\[ r\phi_s(a_1, a_2, \ldots, a_r; b_1, b_2, \ldots, b_s; q, z) = \sum_{n=0}^{\infty} \frac{(a_1, a_2, \ldots, a_r; q)_n}{(b_1, b_2, \ldots, b_s; q)_n (q; q)_n} \left((-1)^n q^{n(n-1)/2}\right)^{1+s-r} z^n. \]
Let us now call the $(p, q)$-number \([9]\) as twin-basic number and define the twin-basic analogues of \([13]\) and \([14]\) as follows:

\[
((a, b); (p, q))_n = \begin{cases} 
1, & n = 0, \\
(a - b)(ap - bq)(ap^2 - bq^2) \ldots (ap^{n-1} - bq^{n-1}), & n = 1, 2, \ldots .
\end{cases}
\]

\[
((a_{1p}, a_{1q}); (a_{2p}, a_{2q}), \ldots , (a_{mp}, a_{mq}); (p, q))_n \\
= ((a_{1p}, a_{1q}); (p, q))_n ((a_{2p}, a_{2q}); (p, q))_n \ldots ((a_{mp}, a_{mq}); (p, q))_n.
\]

Note that

\[
(a; q)_n = ((1, a); (1, q))_n.
\]

Then, the $(p, q)$-analogue of \([13]\), the $(p, q)$-hypergeometric series, or the twin-basic hypergeometric series, can be defined as

\[
r \Phi_s((a_{1p}, a_{1q}), \ldots , (a_{xp}, a_{rq}); (b_{1p}, b_{1q}), \ldots , (b_{sp}, b_{sq}); (p, q), z) \\
= \sum_{n=0}^{\infty} \frac{((a_{1p}, a_{1q}); (p, q))_n ((a_{xp}, a_{rq}); (p, q))_n \ldots ((a_{mp}, a_{mq}); (p, q))_n}{((b_{1p}, b_{1q}); (p, q))_n ((b_{sp}, b_{sq}); (p, q))_n \ldots ((s, q); (p, q))_n} \\
\times ((-1)^n (q/p)^{(n-1)/2})^{-s-r} z^n,
\]

with $|q/p| < 1 \,[13]$. Though, generally, we shall assume $0 < q < p$, $p$ and $q$ can also take other values if there is no problem with convergence of the particular series involved in a result. When $a_{1p} = a_{2p} = \ldots = a_{rp} = b_{1p} = b_{2p} = \ldots = b_{sp} = 1$, $a_{1q} = a_1, a_{2q} = a_2, \ldots , a_{rq} = a_r$, $b_{1q} = b_1, b_{2q} = b_2, \ldots , b_{s,q} = b_s$, and $p = 1$, $r \Phi_s \longrightarrow r \phi_s$. Special interesting choices for $(p, q)$, from the point of view of quantum groups, are $(q^{-1/2}, q^{1/2})$, $(q^{-1}, q)$ and, more generally, $(p^{-1}, q)$. Throughout the paper we shall assume $|z| < 1$. Also, we shall assume all the parameters to be generic, with nonzero values, unless specified otherwise. While referring to the classical results of the $q$-series we shall use the standard notations as in \([17]\) (see also \([21]\)). Often, the parameter doublets $(a_p, a_q)$, $(b_p, b_q)$, etc., will be denoted by different symbols according to the convenience of the situation and such notations should be clear from the context.

Let us recall the definition of a bibasic hypergeometric series with two bases $q$ and $q_1 \,[9, 10]$ (see also \([17]\)):

\[
\mathcal{F}(a, c; b, d; q, q_1, z) = \\
\sum_{n=0}^{\infty} \frac{(a; q)_n (c; q_1)_n}{(b; q)_n (d; q_1)_n (q; q)_n} \\
\times ((-1)^n (q^{n(n-1)/2})^{s+r} (-1)^s q_1^{(n-1)/2})^{s_1+r_1} z^n,
\]

where $\mathfrak{a} = (a_1, a_2, \ldots , a_r), \mathfrak{b} = (b_1, b_2, \ldots , b_s), \mathfrak{c} = (c_1, c_2, \ldots , c_{r_1})$, and $\mathfrak{d} = (d_1, d_2, \ldots , d_{s_1})$. It is clear that in \([20]\) the two unconnected bases $q$ and $q_1$ are regarded are assigned partially to different numerator and denominator parameters whereas in the twin-basic hypergeometric series \([19]\) the twin base parameters $p$ and $q$ are inseparable and assigned to all the numerator and denominator parameter doublets.

Let

\[
\Delta(\alpha, \beta)f(z) = \alpha f(qz) - \beta f(pz).
\]
Thus, we can write, formally,

$$\Delta f(z) = \Delta_{(1,1)} f(z) = f(qz) - f(pz),$$  \hspace{1cm} (22)

it may be noted that

$$\hat{D} f(z) = \frac{\Delta f(z)}{\Delta z}.$$  \hspace{1cm} (23)

Then it is seen that $r, \Phi_s$ satisfies the $(p, q)$-difference equation

$$\left( \Delta \prod_{i=1}^s \Delta(b_{iq}/q,b_{ip}/p) \right) r, \Phi_s = \left( z \prod_{i=1}^r \Delta(a_{iq},a_{ip}) \right) r, \Phi_s \left( (q/p)^{1+s-r} \right).$$  \hspace{1cm} (24)

When $a_{1p} = a_{2p} = \ldots = a_{rp} = b_{1p} = b_{2p} = \ldots = b_{sp} = 1$, $a_{1q} = a_{1}, a_{2q} = a_{2}, \ldots, a_{rq} = a_{r}$, $b_{1q} = b_{1}, b_{2q} = b_{2}, \ldots, b_{s, q} = b_{s}$, and $p = 1$ this equation reduces to the $q$-difference equation satisfied by $r, \phi_s$.

Let us now construct a method to embed the usual $r, \phi_s$-series in the $r, \Phi_s$-series. To this end, we note

$$((l, a, lb); (p, q))_n = l^n ((a, b); (p, q))_n,$$

for any arbitrary nonzero $l$, and

$$(b/a; q/p)_n = a^{-n} p^{-n(n-1)/2} ((a, b); (p, q))_n.$$  \hspace{1cm} (26)

Thus, we can write, formally,

$$r, \phi_s(a_{1q}/a_{1p}, a_{2q}/a_{2p}, \ldots, a_{rq}/a_{rp}; b_{1q}/b_{1p}, b_{2q}/b_{2p}, \ldots, b_{sq}/b_{sp}; q/p, z)$$

$$= \left\{ \begin{array}{ll}
  r, \Phi_s((a_{1p}, a_{1q}), \ldots; (a_{rp}, a_{rq}); (b_{1p}, b_{1q}), \ldots; (b_{sp}, b_{sq}); (p, q), \mu z) & \text{if } s = r - 1, \\
  s+1, \Phi_s((a_{1p}, a_{1q}), \ldots; (a_{rp}, a_{rq}); (0, 1), \ldots; (0, 1); (b_{1p}, b_{1q}), \ldots; (b_{sp}, b_{sq}); (p, q), \mu z) & \text{if } s > r - 1, \\
  r, \Phi_{r-1}((a_{1p}, a_{1q}), \ldots; (a_{rp}, a_{rq}); (b_{1p}, b_{1q}), \ldots; (b_{sp}, b_{sq}); (0, 1), \ldots; (0, 1); (p, q), \mu z) & \text{if } s < r - 1, \\
\end{array} \right.$$  \hspace{1cm} (27)

assuming that the given $r, \phi_s$-series is convergent or terminating. Hence any well behaved $\phi$-series can be written as a $\Phi$-series. But, the converse is not true, in general; in the general case, when $p \neq 1$, this is possible only for an $r, \Phi_{r-1}$. To see this, it is enough to look at $0, \Phi_0$:

$$0, \Phi_0(-; -; (p, q), z) = \sum_{n=0}^{\infty} \frac{(-1)^n (q/p)^{n(n-1)/2}}{((p, q); (p, q))_n} z^n = \sum_{n=0}^{\infty} \frac{(-1)^n (\rho/p)^{n(n-1)/2}}{\rho; \rho^n} (z/p)^n,$$

with $\mu = \frac{b_{1p}b_{2p}... b_{sp}}{a_{1p}a_{2p}... a_{rp}}$, \hspace{1cm} (28)

which shows that $0, \Phi_0$ becomes a $\phi$-series if and only if $p = 1$. Similarly, one is easily convinced that a generic $r, \Phi_s$-series cannot be identified within the class of $\phi$-series unless $p = 1$ or $s = r - 1$ (the first case in the above equation (27)). It is thus clear that the $(p, q)$-series is a larger structure in which the $q$-series gets embedded. Also, note that in
the usual theory of $\phi$-series there is no direct analogue for the choice $a_{ip} = 0$ or $b_{ip} = 0$, for any $i$, permissible, in general (of course, subject to conditions of convergence and so on), in the case of the $(p, q)$-series; to obtain a corresponding result in the case of the $\phi$-series one will have to resort to the limit process of confluence, namely, replacing $z$ by $z/a_r$ and taking the limit $a_r \to \infty$. As an example consider the following. As is well known, in the definition of the usual $q$-hypergeometric series \([15]\), presence of the factor \((-1)^n q^{n(n-1)/2} (1 + s - r)\) (absent in the earlier literature \([18, 19, 20]\)) leads to the useful relation

$$\lim_{a_r \to \infty} r\phi_s(z/a_r) = r^{-1} \phi_s(z). \quad (29)$$

For the $(p, q)$-hypergeometric series \([19]\) the corresponding property is:

$$\lim_{a_ip \to \infty} r\Phi_s(z/a_{rq}) = r\Phi_s((a_{1p}, a_{1q}), \ldots, (a_{(r-1)p}, a_{(r-1)q}))(0, 1);$$

$$(b_{1p}, b_{1q}), \ldots, (b_{sp}, b_{sq}); (p, q), z)$$

$$\lim_{a_ip \to \infty} r\Phi_s(z/a_{rp}) = r\Phi_s((a_{1p}, a_{1q}), \ldots, (a_{(r-1)p}, a_{(r-1)q}))(1, 0);$$

$$(b_{1p}, b_{1q}), \ldots, (b_{sp}, b_{sq}); (p, q), z). \quad (30)$$

Let us also note down the converse of \((27)\) in the case $s = r - 1$:

$$r\Phi_{r-1}((a_{1p}, a_{1q}), \ldots, (a_{rp}, a_{rq}); (b_{1p}, b_{1q}), \ldots, (b_{r-1p}, b_{r-1q}); (p, q), z)$$

$$= r\phi_{r-1}(a_{1q}/a_{1p}, a_{2q}/a_{2p}, \ldots, a_{rq}/a_{rp};$$

$$b_{1q}/b_{1p}, b_{2q}/b_{2p}, \ldots, b_{r-1q}/b_{r-1p}; q/p, z/\mu). \quad (31)$$

Another set of relations often useful are

$$\frac{(b/a; q/p)_\infty}{(d/c; q/p)_\infty} = \lim_{N \to \infty} \frac{(b/a; q/p)_N}{(d/c; q/p)_N}$$

$$= \lim_{N \to \infty} \frac{a^{-N} p^{-N(N-1)/2} ((a, b); (p, q))_N}{c^{-N} p^{-N(N-1)/2} ((c, d); (p, q))_N}$$

$$= \frac{((c, bc/a); (p, q))_\infty}{((c, d); (p, q))_\infty}$$

$$= \frac{((a, b); (p, q))_\infty}{((a, ad/c); (p, q))_\infty}, \quad (32)$$

and its obvious generalizations containing several factors in the numerator and denominator.

Manipulations using the above relations take the usual $q$-identities to $(p, q)$-identities. The original $q$-identities are, of course, special cases corresponding to the choice $a_{1p} = a_{2p} = \ldots a_{rp} = b_{1p} = b_{2p} = \ldots = b_{r-1p} = 1$, and $p = 1$. We shall consider a few examples below.

3. $(p, q)$-Binomial theorem

The usual $q$-binomial theorem is

$$1\phi_0(a; -; q, z) = \frac{(az; q)_\infty}{(z; q)_\infty}. \quad (33)$$
The \((p, q)\)-analogue of this is given by
\[
{}_{1} \Phi_{0}((a, b); -; (p, q), z) = \frac{((p, bz); (p, q))_{\infty}}{((p, az); (p, q))_{\infty}}.
\] (34)

**Proof**: Let us rewrite (33) as
\[
{}_{1} \phi_{0}(b/a; -; q/p, \zeta) = \frac{(b\zeta/a; q/p)_{\infty}}{(\zeta; q/p)_{\infty}}.
\] (35)

Using (27) and (32), we have
\[
{}_{1} \Phi_{0}((a, b); -; (p, q), p\zeta/a) = \frac{((a, b\zeta/p); (p, q))_{\infty}}{((a, a\zeta); (p, q))_{\infty}}.
\] (36)

Now, taking \(\zeta = za/p\), we get
\[
{}_{1} \Phi_{0}((a, b); -; (p, q), z) = \frac{((a, abz/p); (p, q))_{\infty}}{((a, a^2z/p); (p, q))_{\infty}}.
\] (37)

Using the arguments of (25) and (32), by pulling out powers of \(a/p\) in the numerator and denominator of the r.h.s., the \((p, q)\)-binomial theorem (34) follows.

The usual \(q\)-binomial theorem (33) is recovered when \(a = 1\) and \(p = 1\). The \((p, q)\)-binomial theorem obtained in \([11]\) is a special case of (34) corresponding to the specific choice \((a, b) = (q^{-a/2}, p^{a/2})\) and \((p, q) = (q^{-1/2}, p^{1/2})\). An interesting feature of the \((p, q)\)-binomial theorem (34) may be noted here. The product \(\prod_{i=1}^{n} {}_{1} \Phi_{0}((a_{ip}, a_{iq}); -; (p, q), z)\) is seen to be an invariant under the group of independent permutations of the \(p\)-components \((a_{1p}, a_{2p}, \ldots, a_{np})\) and the \(q\)-components \((a_{1q}, a_{2q}, \ldots, a_{nq})\). This product has value 1 if the \(n\)-tuple of \(p\)-components \((a_{1p}, a_{2p}, \ldots, a_{np})\) is related to the \(n\)-tuple of \(q\)-components \((a_{1q}, a_{2q}, \ldots, a_{nq})\) by a mere permutation.

For \(n = 2\) this result implies that
\[
{}_{1} \Phi_{0}((a, b); -; (p, q), z) {}_{1} \Phi_{0}((b, a); -; (p, q), z) = 1.
\] (38)

A special case of this relation is
\[
{}_{1} \Phi_{0}((1, 0); -; (1, q), z) {}_{1} \Phi_{0}((0, 1); -; (1, q), z) = 1.
\] (39)

Recognizing that
\[
{}_{1} \Phi_{0}((1, 0); -; (1, q), z) = \sum_{n=0}^{\infty} \frac{1}{(q; q)_{n}} z^{n} = e_{q}(z),
\]
\[
{}_{1} \Phi_{0}((0, 1); -; (1, q), z) = \sum_{n=0}^{\infty} \frac{q^{n(n-1)/2}}{(q; q)_{n}} (-z)^{n} = E_{q}(-z),
\] (40)

where \(e_{q}(z)\) and \(E_{q}(z)\) are the canonical \(q\)-exponentials, the well known relation
\[
e_{q}(z) E_{q}(-z) = 1,
\] (41)
follows from (39). It should be noted that, while in the usual q-theory \[e_q(z)\] is \[1\phi_0(0; -; q, z)\] and \[E_q(z)\] is \[0\phi_0(-; -; q, -z)\], in the \((p, q)\)-series formalism both \(e_q(z)\) and \(E_q(z)\) belong to the same \(1\Phi_0\)-series. This result suggests the natural definitions

\[
e_{p,q}(z) = 1\Phi_0((1, 0); -(p, q), z) = \sum_{n=0}^{\infty} \frac{p^{n(n-1)/2}}{((p, q); (p, q))^n} z^n, \quad (42)
\]

\[
E_{p,q}(z) = 1\Phi_0((0, 1); -(p, q), -z) = \sum_{n=0}^{\infty} \frac{q^{n(n-1)/2}}{((p, q); (p, q))^n} z^n, \quad (43)
\]

for the \((p, q)\)-exponentials such that

\[
e_{p,q}(z)E_{p,q}(-z) = 1. \quad (44)
\]

For \(p = 1\), \(e_{1,q}(z)\) and \(E_{1,q}(z)\) become \(e_q(z)\) and \(E_q(z)\) respectively.

For \(n = 3\) the above general result and the relation \(38\) imply

\[
1\Phi_0((u, v); -(p, q), z)1\Phi_0((v, w); -(p, q), z) = 1\Phi_0((u, w); -(p, q), z).
\quad (45)
\]

Now, if we take \(u = 1, v = a, w = ab\) and \(p = 1\) then this equation \(45\) is just the well known product formula for \(1\phi_0\), namely,

\[
1\phi_0(a; -; q, z)1\phi_0(b; -; q, az) = 1\phi_0(ab; -; q, z), \quad (46)
\]

in view of the relation \(31\). Thus we get a new way of looking at the product formula \(46\) within the \((p, q)\)-series formalism.

4. \((p, q)\)-Binomial coefficient

The definition

\[
\left[ \begin{array}{c}
\frac{n}{k} \\
\end{array} \right]_{p,q} = \frac{((p, q); (p, q))^n}{((p, q); (p, q))^k((p, q); (p, q))_{n-k}}, \quad k = 0, 1, \ldots, n, \quad (47)
\]

provides a natural generalization of the \(q\)-binomial coefficient. In terms of the \((p, q)\)-number the \((p, q)\)-binomial coefficient (written without the subscript \(p, q\)) becomes

\[
\left[ \begin{array}{c}
\frac{n}{k} \\
\end{array} \right] = \frac{[n]!}{[k]![n-k]!}, \quad (48)
\]

where, as usual,

\([n]! = [n][n-1]\ldots[2][1], \quad [0]! = 1. \quad (49)
\]

Then, the result

\[
1\Phi_0((p^n, q^n); -(p, q), z) = \sum_{k=0}^{\infty} \left[ \begin{array}{c}
\frac{n-1+k}{k} \\
\end{array} \right] z^k
\]

\[
= \frac{p^{n(n+1)/2}}{((p, p^n z); (p, q))^n} = \left\{ \sum_{k=0}^{n} \left[ \begin{array}{c}
\frac{n}{k} \\
\end{array} \right] (pq)^k(k-1/2)(-z)^k \right\}^{-1}, \quad (50)
\]
follows by taking $a = p^n$ and $b = q^n$ in (38). The relation (50) is obviously a generalization of the result

$$1\Phi_0(q^n; -; q, z) = \sum_{k=0}^{\infty} \left[ \frac{n - 1 + k}{k} \right]_q z^k = \frac{1}{(z; q)_n}$$

$$= \left\{ \sum_{k=0}^{n} \left[ \frac{n}{k} \right]_q q^{k(k-1)/2} (-z)^k \right\}^{-1}. \quad (51)$$

If we take $p = 0$ in (50) we get, correctly of course,

$$\sum_{k=0}^{\infty} (q^n z)^k = \sum_{k=0}^{\infty} \frac{1}{1 - q^n z}. \quad (52)$$

It should be noted that there is no analogue for the choice $p = 0$ in the usual $q$-series formalism. We can also take the limit $p \rightarrow q \neq 1$. Then, the equation (50) takes the form

$$1F_0(n; -; q^{-1}z) = \sum_{k=0}^{\infty} \left( \frac{n - 1 + k}{k} \right) (q^{-1}z)^k$$

$$= (1 - q^{-1}z)^{-n} = \left\{ \sum_{k=0}^{n} \left( \frac{n}{k} \right) (-q^{-1}z)^k \right\}^{-1}. \quad (53)$$

Thus, it is seen that, though a $(p, q)$-identity may be derived starting with a $q$-identity, the $(p, q)$-identity offers more choices for manipulations. If we choose $(p, q) = (q^{-1}, q)$, then, the identity (50) becomes

$$1\Phi_0((q^{-n}, q^n); -; (q^{-1}, q), z) = \sum_{k=0}^{\infty} \left[ \frac{n - 1 + k}{k} \right]_q z^k$$

$$= \frac{q^{-n(n+1)/2}}{((q^{-1}, zq^{-n}) ; (q^{-1}, q))_n} = \left\{ \sum_{k=0}^{n} \left[ \frac{n}{k} \right]_q (-z)^k \right\}^{-1}, \quad (54)$$

with

$$\left[ \frac{n}{k} \right]_q^{-1} = \frac{((q^{-1}, q); (q^{-1}, q))_n}{((q^{-1}, q); (q^{-1}, q))_k((q^{-1}, q); (q^{-1}, q))_{n-k}}, \quad k = 0, 1, \ldots, n. \quad (55)$$

which should be relevant in the context of quantum groups.

From (50), let us take

$$p^{-n(n+1)/2}((p, p^n z); (p, q))_n = \sum_{k=0}^{n} \left[ \frac{n}{k} \right] (pq)^{k(k-1)/2} (-1)^k s_q s_p^{-k}. \quad (56)$$

Using (26) and taking $z = \zeta_q / \zeta_p$, we can rewrite (56) as

$$((p\zeta_p, p^n\zeta_q); (p, q))_n = \sum_{k=0}^{n} \left[ \frac{n}{k} \right] p^{n(n+1)+k(k-1)/2} q^{k(k-1)/2} (-1)^k s_q s_p^{n-k}. \quad (57)$$
Now, renaming \( p_\zeta q \) and \( p^n \zeta q \) as \( a \) and \( b \), respectively, we get

\[
((a, b); (p, q))_n = \sum_{k=0}^{n} \binom{n}{k} (-1)^k p^{(n-k)(n-k-1)/2} q^{k(k-1)/2} a^{n-k} b^k. \tag{58}
\]

The \((p, q)\)-binomial theorem derived in [6], using the recursion relations of the \((p, q)\)-binomial coefficients, corresponds to \((58)\) with the notations \( a = l, b = -x \).

An operator, or noncommutative, form of the \(q\)-binomial theorem is known [17]: If \( x \) and \( y \) are noncommuting variables such that \( xy = qyx \), \( q \) commutes with \( x \) and \( y \), and the associative law holds, then

\[
(x + y)^n = \sum_{k=0}^{n} \binom{n}{k} y^k x^{n-k} = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}. \tag{59}
\]

A \((p, q)\)-extension of this result is derived in [7], in a specific context of a quantum group. This result can be stated in a general form as follows:

\[
(ax + by)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k y^k x^{n-k} = \sum_{k=0}^{n} \binom{n}{k} b^{n-k} a^k x^k y^{n-k}, \tag{60}
\]

where \( ab = p^{-1} ba, xy = qyx \), all other commutators among the variables \( \{a, b, x, y\} \) vanish, \( p \) and \( q \) commute with \( \{a, b, x, y\} \), and the associative law holds. Proof of \((60)\) follows by replacing in \((59)\) \( q \) by \( q/p \) and \((x, y)\) by \((ax, by)\), and reexpressing the result in terms of \( p, q, a, b, x, \) and \( y \). In deriving the second part of \((60)\) one has to use the formula

\[
((a, b); (p, q))_n = (-1)^n a^n b^n (pq)^{n(n-1)/2}((a^{-1}, b^{-1}); (p^{-1}, q^{-1}))_n. \tag{61}
\]

5. \((p, q)\)-Heine transformation for \(2\Phi_1\)

The Heine transformation of the \(2\Phi_1\) series, namely,

\[
2\Phi_1(a, b; c; q, z) = \frac{(b, az; q)_\infty}{(c, z; q)_\infty} 2\Phi_1(c/b, z; az; q, b), \tag{62}
\]

has the following \((p, q)\)-analogue:

\[
2\Phi_1((a, b); (c, d); (e, f); (p, q), z) = \frac{((ce, de), (pe, bcz); (p, q))_\infty}{((ce, cf), (pe, acz); (p, q))_\infty} \times 2\Phi_1((de, cf), (pe, acz); (pe, bcz); (p, q), p/ce), \tag{63}
\]

Proof: By the Heine transformation \((62)\)

\[
2\Phi_1(b/a, d/c; f/e; q/p, \zeta) = \frac{(d/c, b\zeta/a; q/p)_\infty}{(f/e, \zeta; q/p)_\infty} 2\Phi_1(cf/de, \zeta; b\zeta/a; q/p, d/c). \tag{64}
\]

Using \((27)\) and following arguments of the type used in \((32)\) we can rewrite this equation as

\[
2\Phi_1((a, b); (c, d); (e, f); (p, q), pe\zeta/ac) = \frac{((e, f), (ac/e, ac\zeta/e); (p, q))_\infty}{((e, f), (a, b\zeta); (p, q))_\infty} 2\Phi_1((de, cf), (1, \zeta); (a, b\zeta); (p, q), pa/ce). \tag{65}
\]
Now, taking $\zeta = acz/pe$, we get

$$\begin{align*}
2\Phi_1((a, b), (c, d); (e, f); (p, q), z) &= \\
&= \frac{((ce, de), (pe, bcz); (p, q))_\infty}{(ce, cf), (pe, acz); (p, q))_\infty} \\
2\Phi_1((de, cf), (pe, acz); (p, q), p/ce),
\end{align*}$$

thus, arriving at the $(p, q)$-Heine transformation formula (63) for $2\Phi_1$.

Setting $a = 0$, $b = c = e = 1$, relabeling $d$ as $a$ and $f$ as $b$, and taking $p = 1$, in (63) we obtain the transformation

$$\begin{align*}
1\phi_1(a; b; q, z) &= \frac{a, z; q)_\infty}{(b; q)_\infty} 2\phi_1(0, b/a; z; q, a), \tag{67}
\end{align*}$$

which can be directly derived from the $q$-Heine transformation formula (62) by using the limiting process of confluence, namely, replacing $z$ by $z/a$ and taking the limit $a \to \infty$, and then relabeling the parameters. Now, taking $z = b/a$ in (67) one obtains, using the $q$-binomial theorem, the summation formula

$$\begin{align*}
1\phi_1(a; b, q, b/a) &= \frac{(b/a; q)_\infty}{(b; q)_\infty}, \tag{68}
\end{align*}$$

which can also be obtained from the $(p, q)$-Gauss sum (70), given below, with the same choice of parameters.

6. $(p, q)$-Gauss sum

Using the Heine transformation (62) one obtains the $q$-Gauss sum

$$\begin{align*}
2\Phi_1((a, b), (c, d); (e, f); (p, q), pf/bd) &= \\
&= \frac{(c/a, c/b; q)_\infty}{(c, c/ab; q)_\infty}, \quad |c/ab| < 1. \tag{69}
\end{align*}$$

The $(p, q)$-Gauss sum takes the form

$$\begin{align*}
2\Phi_1((a, b), (c, d); (e, f); (p, q), pf/bd) &= \\
&= \frac{((bc, af), (de, cf); (p, q))_\infty}{((e, f), (bde, acf); (p, q))_\infty}, \quad |acf/bde| < 1. \tag{70}
\end{align*}$$

Proof. Let $z = pf/bd$ in the $(p, q)$-Heine transformation formula (63). The result is

$$\begin{align*}
2\Phi_1((a, b), (c, d); (e, f); (p, q), pf/bd) &= \\
&= \frac{((ce, de), (pe, pcf/d); (p, q))_\infty}{((ce, cf), (pe, pacf/bd); (p, q))_\infty} \\
&\quad \times 2\Phi_1((de, cf), (pe, pacf/bd); (pe, pcf/d); (p, q), p/ce), \\
&= \frac{((ce, de), (bde, bcf); (p, q))_\infty}{((ce, cf), (bde, acf); (p, q))_\infty} \\
&\quad \times 2\Phi_1((de, cf), (pe, pacf/bd); (pe, pcf/d); (p, q), p/ce). \tag{71}
\end{align*}$$
Note that
\[
2\Phi_1((de, cf), (pe, pacf/bd); (pe, pcf/fd); (p, q), p/ce) = 2\Phi_1((de, cf), (bde, acf); (bde, bcf); (p, q), p/ce) = 1\Phi_0((bde, acf); (p, q)/; (be, af); (p, q))_\infty = ((be, af); (p, q))_\infty = ((be, bde/c); (p, q))_\infty,
\]
in view of the \((p, q)\)-binomial theorem and (82). Hence,
\[
2\Phi_1((a, b), (c, d); (e, f); (p, q), pf/bd) = ((ce, de), (bde, bcf), (be, af); (p, q))_\infty = ((ce, ef), (bde, acf), (be, bcf); (p, q))_\infty = ((ce, ef), (bde, acf), (c, d); (p, q))_\infty = ((ce, ef), (be, af); (p, q))_\infty = ((ce, ef), (bde, acf); (p, q))_\infty.
\]
Thus, the \((p, q)\)-Gauss sum (70) is derived.

The identity
\[
\sum_{n=0}^{\infty} \frac{q^{n^2}}{(q, qz; q)_n} z^n = \frac{1}{(qz; q)_\infty},
\]
is usually obtained from the \(q\)-Gauss sum (69) by setting \(c = qz\) and letting \(a \to \infty\) and \(b \to \infty\). It should be noted that this identity follows immediately from the \((p, q)\)-Gauss sum (70) by mere substitution \(a = c = 0, b = d = e = 1, f = qz\) and \(p = 1\).

Another useful form of (70) is
\[
2\Phi_1((a, 1), (b, c); (d, \sigma c); (p, q), \sigma p) = \frac{((d, \sigma ac), (d, \sigma b); (p, q))_\infty}{((d, \sigma ac), (d, \sigma ab); (p, q))_\infty}, \quad |\sigma ab/d| < 1.
\]
Now, substituting in (75) \(a = c = 0, b = d = 1, \sigma = \sqrt{q}z\) and \(p = 1\), one gets another well-known identity
\[
\sum_{n=0}^{\infty} \frac{(-1)^n q^{n^2/2}}{(q; q)_n} z^n = (\sqrt{q}z; q)_\infty,
\]
which is usually obtained from the \(q\)-Gauss sum (69) by setting \(c = \sqrt{q}bz\) and then letting \(b \to 0\) and \(a \to \infty\). These examples illustrate the usefulness of the \((p, q)\)-series formalism even for the treatment of the usual \(q\)-series.

7. \((p, q)\)-Ramanujan sum

Let us assume the obvious \((p, q)\)-generalizations of the basic notations and definitions associated with bilateral \(q\)-hypergeometric series. Thus, we write
\[
((a, b); (p, q))_n = \frac{1}{((ap^{-n}, bq^{-n}); (p, q))_n}.
\]
\[
\frac{1}{(ap^{-1} - bq^{-1})(ap^{-2} - bq^{-2}) \cdots (ap^{-n} - bq^{-n})}
\]
\[
= \frac{(-pq/ab)^n(pq)^{n(n-1)/2}}{((p/a, q/b); (p, q))_n},
\]

and

\[
1 \Psi_1((a, b); (c, d); (p, q), z) = \sum_{n=-\infty}^{\infty} \frac{((a, b); (p, q))_n z^n}{((c, d); (p, q))_n},
\]

\[
= \sum_{n=0}^{\infty} \frac{((a, b); (p, q))_n z^n}{((c, d); (p, q))_n} + \sum_{n=1}^{\infty} \frac{((p/c, q/d); (p, q))_n}{((p/a, q/b); (p, q))_n} \left(\frac{cd}{abz}\right)^n.
\]

One can show that

\[
1 \Psi_1((a, b); (c, d); (p, q), z) = 1 \psi_1(b/a; d/c; q/p, za/c)
\]

where \(1 \psi_1\) is the usual bilateral \(q\)-series. Then, using the Ramanujan sum,

\[
1 \psi_1(a; b; q, z) = \frac{(q, b/a, az, q/az; q)_{\infty}}{(b, q/a, z, b/az; q)_{\infty}}, \quad |b/a| < |z| < 1,
\]

one can show that the \((p, q)\)-analogue of the Ramanujan sum is

\[
1 \Psi_1((a, b); (c, d); (p, q), z)
\]

\[
= \frac{((p, q), (bc, ad), (c, bz), (pbz, qc); (p, a))_{\infty}}{((c, d), (pb, qa), (c, az), (pbz, pd); (p, a))_{\infty}}, \quad |ad/bc| < |z| < 1.
\]

To obtain the \((p, q)\)-analogue of the Jacobi triple product identity from this the steps are:

(i) \((a, b) \rightarrow (1/a, 1/b), z \rightarrow zb/a, (ii) d = 0, (p, q) \rightarrow (p^2, q^2), z \rightarrow zq/p, (iii) b \rightarrow 0,\)

and (iv) \((p, q) \rightarrow (\sqrt{p}, \sqrt{q}).\) The result is:

\[
\sum_{n=-\infty}^{\infty} (-1)^n(q/p)^{n^2/2}(z/ac)^n
\]

\[
= \frac{((p, q), (\sqrt{pca}, \sqrt{qz}), (\sqrt{p}, \sqrt{qca}; (p, q))_{\infty}}{((p, 0), (\sqrt{pca}, 0), (\sqrt{p}, 0); (p, q))_{\infty}},
\]

which is same as the well known \(q\)-result with the replacements \(q \rightarrow q/p\) and \(z \rightarrow z/ac.\)

The usual Jacobi triplet product identity can also be obtained in a simpler way directly from \(1 \Psi_1\) by letting \(a = d = 0, b = c = 1, p = 1\) and \(z \rightarrow z\sqrt{q}.\)

Taking \(ac = 1\) in (82), we can also write the \((p, q)\)-analogue of the Jacobi triple product, for \(q < p, |z| < 1,\) as

\[
\sum_{n=-\infty}^{\infty} (-1)^n(q/p)^{n^2/2}z^n
\]

\[
= \prod_{n=1}^{\infty} \frac{(p^n - q^n)(p^{n-1/2} - q^{-1/2}z)(p^{n-1/2}z - q^{-1/2})}{p^{2n-1}z^n}.
\]
The Euler identity follows from the $1\Psi_1$-sum by taking $a = d = 0$, $b = c = 1$, $(p, q) \rightarrow (1, q^3)$, and $z = q$:

$$\sum_{n=\infty} (-1)^n q^{(3n^2-n)/2} = (q; q)_\infty.$$  \hfill (84)

8. $(p, q)$-Special functions

Let us now make some brief observations on the $(p, q)$-generalizations of the $q$-special functions. First let us consider an example. It is seen that

$$\left[ \begin{array}{c} n \\ k \end{array} \right]_{p,q} = \left[ \begin{array}{c} n \\ n-k \end{array} \right]_{p,q} = p^{k(n-k)} \left[ \begin{array}{c} n \\ k \end{array} \right]_{q/p}.$$  \hfill (85)

The continuous $q$-Hermite polynomial is given by

$$H_n(x|q) = \sum_{k=0}^{n} \left[ \begin{array}{c} n \\ k \end{array} \right]_q e^{i(n-2k)\theta}, \quad x = \cos \theta.$$  \hfill (86)

We may define a continuous $(p, q)$-Hermite polynomial as

$$H_n(x|p, q) = \sum_{k=0}^{n} \left[ \begin{array}{c} n \\ k \end{array} \right]_{p,q} e^{i(n-2k)\theta}, \quad x = \cos \theta.$$  \hfill (87)

In view of the relation (85) it is found that $H_n(x|p, q)$ is not just $H_n(x|(q/p))$ with a rescaling of $x$: e.g., letting $(p, q) \rightarrow (q^2, q^3)$ one would get a two-parameter family of generalized continuous $q$-Hermite polynomials, say $\{ H_n^{(\alpha, \beta)}(x|q) \}$ with the usual $H_n(x|q)$ identified as $H_n^{(0,1)}(x|q)$. This is in contrast to the case of $r\Phi_r$ which can always be identified, as already noted (see (27) and (51)), with an $r\Phi_{r-1}$; in this sense, $r\Phi_{r-1}$ may be considered a trivial generalization - examples in this category would be the $(p, q)$-generalizations of $q$-Krawtchouk polynomials, $q$-Meixner polynomials, $q$-Racah polynomials, $q$-Askey-Wilson polynomials, $q$-Jacobi polynomials, $q$-Hahn polynomials, $q$-Charlier polynomials, continuous $q$-ultraspherical polynomials, etc... However, such generalizations are also of interest from the point of view of physical applications: one example of such a situation is the study of the Clebsch-Gordon coefficients of the two-parameter quantum algebra $U_{p,q}(gl(2))$ - a simple relation between the CG-coefficients of $U_{p,q}(gl(2))$ and $U_q(sl(2))$ exists \cite{22} which must be due to the connection between the CG-coefficients of $U_q(sl(2))$ and $3\phi_2$ (see, e.g., \cite{23}). $(p, q)$-generalizations of gamma and beta functions are straightforward \cite{11}. Besides the continuous $q$-Hermite polynomials, there are several examples for which the $(p, q)$-generalization is nontrivial: discrete $q$-Hermite polynomials, $q$-Laguerre polynomials, $q$-Bessel functions $(J_{p/2}(x|q))$, etc... We hope to return to these topics elsewhere.

9. Conclusion

We have shown that it is profitable to study the $(p, q)$-hypergeometric series, or the twin-basic hypergeometric series, following naturally from the extension of the $q$-number $(1 - q^n)/(1 - q)$ to the twin-basic number $(p^n - q^n)/(p - q)$. In particular, we have studied the $(p, q)$-analagous of the $q$-binomial theorem, $q$-binomial coefficient, Heine transformation for $2\phi_1$, Gauss sum for $2\phi_1$, and the Ramanujan sum for $1\psi_1$. Further, we have made
some brief observations on the \((p, q)\)-generalizations of the \(q\)-special functions. In general, we have noted that many of the \(q\)-results can be generalized directly to \((p, q)\)-results and once we have the \((p, q)\)-results the \(q\)-results can be obtained more easily by mere substitutions for the parameters instead of any limiting process as required in the usual \(q\)-theory. We believe that a detailed study of the \((p, q)\)-hypergeometric series, or the twin-basic hypergeometric series, should be very interesting.

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