Average coherence and its typicality for random mixed quantum states

Lin Zhang

Institute of Mathematics, Hangzhou Dianzi University, Hangzhou 310018, People’s Republic of China

E-mail: godyalin@163.com and linyz@zju.edu.cn

Received 21 July 2016, revised 27 January 2017
Accepted for publication 20 February 2017
Published 14 March 2017

Abstract

The Wishart ensemble is a useful and important random matrix model used in diverse fields. By realizing induced random mixed quantum states as a Wishart ensemble with fixed unit trace, using matrix integral technique we give a fast track to the average coherence for random mixed quantum states induced via partial-tracing of the Haar-distributed bipartite pure states. As a direct consequence of this result, we get a compact formula for the average subentropy of random mixed states. These compact formulae extend our previous work.

Keywords: random quantum state, quantum coherence, Wishart ensemble, entropy, subentropy

1. Introduction

Quantum coherence, due to the superposition rule, is an important ingredient in quantum information processing, and plays a pivotal role in such diverse fields as quantum thermodynamics [2, 28] and quantum biology [16, 17, 26]. Quantum coherence is the basis of single particle interferometry; it gives coherence the status of a resource and makes necessary the development of a solid framework allowing assessment and quantification of this property [1]. Quantum entanglement, due to the tensor product structure of composite quantum systems, is another fundamental feature of quantum mechanics. It is also a necessary resource in many quantum information processing tasks, such as superdense coding, quantum teleportation etc [32]. Recently, researchers have contributed much effort to connecting quantum coherence with entanglement and quantum discord—a kind of quantum correlation containing entanglement as a proper subset. Streltsov et al [31] have connected quantum coherence with entanglement, and have shown that any degree of coherence with respect to some reference basis can be converted to entanglement via incoherent operations. Ma et al [18]
have proven that the creation of quantum discord with multipartite incoherent operations is bounded by the amount of quantum coherence consumed in its subsystems during the process.

In recent years, many efforts have been made in the research of quantum correlations of random quantum states [8, 34]. In quantum information theory, many quantities such as quantum entanglement and the diagonal entropy of a density matrix [6] have been proved very useful. In particular, the typicality of some quantity can reduce the computational complexity involved in its determination [9]. For example, the typicality entanglement of pure bipartite states sampled randomly according to the uniform Haar measure provides an explanation for the equal a priori postulate of statistical physics [7]. To the knowledge of the author, the distribution of entanglement among two subsystems of a large quantum system has been a subject of interest among physicists and mathematicians for a long time, and many interesting results have been obtained—but, however, similar consideration for quantum coherence is still missing. In the present work, we are concerned with the statistical behavior of coherence of a subsystem of a large quantum system. Specifically, although the authors of [37] make an attempt to calculate the average coherence of an induced random mixed state ensemble [38] by brute force, the calculation is very complicated and also tedious. This motivates finding a more compact approach to the typicality of quantum coherence. Luckily, we find a simple approach to get a compact formula for the average coherence quickly. Although the topic of the present paper has already been investigated and some results obtained [37], the method used here is new and very different from that of [37]. The authors of [37] proceeded by calculating the average subentropy first, and then deriving the average coherence by using the formula obtained for the average subentropy. In obtaining the final compact forms for the average subentropy and the average coherence, they have shown some ingenious combinatorial identities (we can see this from the very recent published version). The difference here, however, is that we calculate the average coherence directly (by a simpler method than that used in [37]) and get a compact form for the average subentropy as a by-product. Based on this elegant formula for the average subentropy, we get the fact that as the dimension of the system to be considered increases, the average subentropy of random mixed states approaches to the maximum value of the subentropy, which is attained for the maximally mixed state.

Let us fix some notations before proceeding. For a given density matrix \( \rho \) (i.e. non-negative square matrix of or \( m \) with unit trace), its von Neumann entropy is defined as 
\[
S(\rho) := -\text{Tr}(\rho \ln \rho),
\]
where \( \ln \rho \) is in the sense of the functional calculus of \( \rho \). In fact, \( S(\rho) = -\sum \lambda_j(\rho) \ln \lambda_j(\rho) \), where \( \lambda_j(\rho) \) stands for the \( j \)-th eigenvalue of \( \rho \). Quantum relative entropy of coherence (in short quantum coherence in the present paper) in a state \( \rho \) is given by [1]:
\[
\mathcal{S}(\rho) = S(\rho_{\text{diag}}) - S(\rho).
\]
In this paper, we will calculate exactly the average coherence \( \mathcal{S} \) for random mixed quantum states. Then the typicality of coherence is obtained immediately.

The paper is organized as follows. In section 2, We recall the Dirac delta function and its extension to the matrix delta function. Then, we derive the distribution of the diagonal part of the Wishart ensemble—as the marginal distribution of matrix elements of the Wishart ensemble—in section 3, and, by realizing the induced random mixed quantum states as a Wishart ensemble with fixed unit trace, we obtain the distribution of the diagonal part of induced random mixed quantum states. In this section, we also calculate the average entropy of the diagonal part of random mixed quantum states. We present our main results (i.e. theorems 4.1, 4.3, and 4.6) in section 4. In section 5, we discuss the average coherence of the mixing of random mixed quantum states. Finally, we give concluding remarks in section 6.
2. Matrix delta function

Although the matrix delta function has already been used in the literature, there is no formal and rigorous treatment, to my best knowledge. For the reader’s convenience, we will give complete details along this line. We recall that the Dirac delta function $\delta(x)$ is defined by

$$
\delta(x) = \begin{cases} +\infty, & \text{if } x = 0; \\ 0, & \text{if } x \neq 0. 
\end{cases}
$$

(2.1)

The Fourier integral representation of the Dirac delta function

$$
\delta(x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{i\alpha x} \, d\alpha \quad (x \in \mathbb{R})
$$

(2.2)

can be extended to the matrix case.

**Definition 2.1 (The matrix delta function).**

(i) For an $m \times n$ complex matrix $Z = [z_{ij}]$, the matrix delta function $\delta(Z)$ is defined as

$$
\delta(Z) := \prod_{i=1}^{m} \prod_{j=1}^{n} \delta(\text{Re}(z_{ij}))\delta(\text{Im}(z_{ij})).
$$

(2.3)

(ii) For an $m \times m$ Hermitian complex matrix $X = [x_{ij}]$, the matrix delta function $\delta(X)$ is defined as

$$
\delta(X) := \prod_{i<j} \delta(x_{ij}) \prod_{i=j} \delta(\text{Re}(x_{ij}))\delta(\text{Im}(x_{ij})).
$$

(2.4)

From the above definition, we see that the matrix delta function of a complex matrix is equal to the product of one-dimensional delta functions over the independent real and imaginary parts of this complex matrix. The following proposition is very important in this paper.

**Proposition 2.2 (The Fourier integral representation of the matrix delta function).** For an $m \times m$ Hermitian complex matrix $X$, we have

$$
\delta(X) = \frac{1}{2\pi^{m^2}} \int e^{\text{Tr}(TX)} [dT],
$$

(2.5)

where $T = [t_{ij}]$ is also an $m \times m$ Hermitian complex matrix, and $[dT] := \prod_{j} dt_{ij} \prod_{i<j} d\text{Re}(t_{ij})d\text{Im}(t_{ij})$.

**Proof.** Indeed, we know that

$$
\text{Tr}(TX) = \sum_{j=1}^{m} t_{ij}x_{ij} + \sum_{i \neq j} (t_{ij}x_{ij}) = \sum_{j=1}^{m} \text{Re}(t_{ij})\text{Re}(x_{ij}) + \sum_{1 \leq i < j \leq m} (t_{ij}x_{ij} + t_{ij}^*x_{ij})
$$

$$
= \sum_{j=1}^{m} t_{ij}x_{ij} + \sum_{1 \leq i < j \leq m} 2(\text{Re}(t_{ij})\text{Re}(x_{ij}) + \text{Im}(t_{ij})\text{Im}(x_{ij})),
$$

implying that

L Zhang
J. Phys. A: Math. Theor. 50 (2017) 155303

L Zhang
\[
\int e^{\text{Tr}(TX)} \, dT = m \prod_{j=1}^{m} \int \exp(i t_j x_j) \, dt_j \prod_{1 \leq i < j \leq m} \int \exp(\text{Re}(t_j)(2\text{Re}(x_{ij})) \, dt_j \int \exp(\text{Im}(t_j)(2\text{Im}(x_{ij})) \, dt_j
\]
\[
= \prod_{j=1}^{m} 2\pi \delta(x_j) \times \prod_{1 \leq i < j \leq m} \pi \delta(\text{Re}(x_{ij})) \pi \delta(\text{Im}(x_{ij}))
\]
\[
= (2\pi)^m (\pi^2)^{m^2} \prod_{j=1}^{m} \delta(x_j) \prod_{i<j} \delta(\text{Re}(x_{ij})) \delta(\text{Im}(x_{ij}))
\]
\[
= 2^{2m^2} \pi^m \delta(X).
\]

Thereby we get the desired identity.

**Remark 2.3.** Indeed, since
\[
\text{Tr}(T^{\text{off}}X^{\text{off}}) = \sum_{i<j} 2(\text{Re}(t_{ij}) \text{Re}(x_{ij}) + \text{Im}(t_{ij}) \text{Im}(x_{ij}))
\]
and
\[
[dT^{\text{off}}] = \prod_{i<j} d\text{Re}(t_{ij}) d\text{Im}(t_{ij}),
\]
it follows that
\[
\int [dT^{\text{off}}] \exp(i \text{Tr}(T^{\text{off}}X^{\text{off}}))
\]
\[
= \prod_{i<j} \int d\text{Re}(t_{ij}) \exp(i \text{Re}(t_{ij})(2\text{Re}(x_{ij}))) \int d\text{Im}(t_{ij}) \exp(i \text{Im}(t_{ij})(2\text{Im}(x_{ij})))
\]
\[
= \prod_{i<j} 2\pi \delta(2\text{Re}(x_{ij})) \times 2\pi \delta(2\text{Im}(x_{ij})) = \prod_{i<j} \pi \delta(\text{Re}(x_{ij})) \pi \delta(\text{Im}(x_{ij}))
\]
\[
= \pi^{2m} \prod_{i<j} \delta(\text{Re}(x_{ij})) \delta(\text{Im}(x_{ij})) = \pi^{m(m-1)} \delta(X^{\text{off}}).
\]

From the above discussion, we see that (2.5) can be separated into the two identities below:
\[
\delta(X^{\text{diag}}) = \frac{1}{(2\pi)^{m^2}} \int [dT^{\text{diag}}] e^{i \text{Tr}(T^{\text{diag}}X^{\text{diag}})},
\]
\[
\delta(X^{\text{off}}) = \frac{1}{\pi^{m(m-1)}} \int [dT^{\text{off}}] e^{i \text{Tr}(T^{\text{off}}X^{\text{off}})}.
\]

### 3. Wishart ensemble

We use the notation \( x \sim N(\mu, \sigma^2) \) to indicate a Gaussian random real variable \( x \) with mean \( \mu \) and variance \( \sigma^2 \). Let \( Z \) denote an \( m \times n \) complex random matrix \([10, 20]\). These
elements are independent complex random variables subject to $\mathcal{N}(0, 1) = N(0, \frac{1}{2}) + iN(0, \frac{1}{2})$ with Gaussian densities:

$$\frac{1}{\pi} \exp(-|z|^2),$$  

(3.1)

where $\text{Re}(z)$, $\text{Im}(z)$ are independent and identically distributed (i.i.d.) Gaussian random real variables with mean 0 and variance $\frac{1}{2}$.

**Definition 3.1. (Wishart matrices, [12]).** With $m \times n$ random matrices $Z$ specified as above, define the complex Wishart ensemble as consisting of matrices $W = ZZ^\dagger$. The matrices $W = ZZ^\dagger$ are referred to as (uncorrelated) **Wishart matrices**.

As chosen previously, $m \leq n$ for definiteness. The probability distribution followed by $Z$ is

$$\varphi(Z) \propto \exp(-\text{Tr}(ZZ^\dagger)).$$  

(3.2)

Indeed, let $Z = [z_{ij}]$ be a complex random matrix, where $z_{ij} = \text{Re}(z_{ij}) + \sqrt{-1} \text{Im}(z_{ij})$ with $\text{Re}(z_{ij}), \text{Im}(z_{ij}) \sim N(0, \frac{1}{2})$. The probability distribution of $Z$ is just the joint distribution of all matrix elements $z_{ij}$ of $Z$. Thus

$$\varphi(Z) = \prod_{i=1}^{m} \prod_{j=1}^{n} \text{Pr}(z_{ij}) = \prod_{i=1}^{m} \prod_{j=1}^{n} \frac{1}{\pi} \exp(-|z|^2) = \frac{1}{\pi^{mn}} \exp\left(-\sum_{i=1}^{m} \sum_{j=1}^{n} |z_{ij}|^2 \right).$$

Thus, we have

$$\varphi(Z) = \frac{1}{\pi^{mn}} \exp(-\text{Tr}(ZZ^\dagger)).$$  

(3.3)

Then the distribution of Wishart matrices $W$ is given by

$$Q(W) = \int [dZ] \delta(W - ZZ^\dagger) \varphi(Z),$$  

(3.4)

where $[dZ] = \prod_{i=1}^{m} \prod_{j=1}^{n} dz_{ij}$ and $dz = d\text{Re}(z)d\text{Im}(z)$ for $z \in \mathbb{C}$. With the matrix delta function, we can rewrite the expression (3.4) as [5]:

$$Q(W) = \frac{1}{2^m \pi^{mn}} \int [dT] \frac{\exp(i\text{Tr}(TW))}{\text{det}^n(I + iT)}. $$  

(3.5)

### 3.1. Distribution of the diagonal part of a Wishart ensemble

Now break up $W = [w_{ij}]$ as $W = W^{\text{diag}} + W^{\text{off}}$, where $W^{\text{diag}}$ and $W^{\text{off}}$ are the diagonal part and off-diagonal part of $W$ respectively. Clearly this decomposition is orthogonal with respect to the Hilbert–Schmidt inner product in operator space. The distribution of diagonal entries of Wishart ensemble can be calculated via the following integral:

$$q(W^{\text{diag}}) = \int Q(W)[dW^{\text{off}}]$$

$$= \frac{1}{(2\pi)^m} \int [dT] \frac{\exp(i\text{Tr}(T^{\text{diag}}W^{\text{diag}}))}{\text{det}^n(I + iT)} \left[ \frac{1}{\pi^{m(m-1)}} \int [dW^{\text{off}}] \exp(i\text{Tr}(T^{\text{off}}W^{\text{off}})) \right]$$

$$= \frac{1}{(2\pi)^m} \int [dT] \delta(T^{\text{off}}) \frac{\exp(i\text{Tr}(T^{\text{diag}}W^{\text{diag}}))}{\text{det}^n(I + iT)}. $$


Note that, in the third equality, we use the fact that
\[
\delta(T^{\text{off}}) = \frac{1}{\pi^{(m-1)n}} \int [dW^{\text{off}}] \exp(i\text{Tr}(W^{\text{off}} T^{\text{off}})).
\] (3.6)

Now we have obtained that
\[
q(W^{\text{diag}}) = \frac{1}{(2\pi)^m} \int [dT] \delta(T^{\text{off}}) \frac{\exp(i\text{Tr}(T^{\text{diag}} W^{\text{diag}}))}{\det^n(I + iT)}. \tag{3.7}
\]

Next, we calculate the integral in (3.7). To this end, denote
\[
\mathcal{I}_{m,n}(W_m^{\text{diag}}) := \int [dT_m] \delta(T_m^{\text{off}}) \frac{\exp(i\text{Tr}(T_m^{\text{diag}} W_m^{\text{diag}}))}{\det^n(I_m + iT_m)}. \tag{3.8}
\]

and we follow the approach taken by Janik [13], partition \( T_m \) as \( 2 \times 2 \) block matrix:
\[
T_m = \begin{bmatrix}
T_{m-1} & |uangle \\
\langle u| & t_{mm}
\end{bmatrix}.
\]

Thus
\[
I_m + iT_m = \begin{bmatrix}
I_{m-1} + iT_{m-1} & i|uangle \\
\langle u| & 1 + it_{mm}
\end{bmatrix}.
\]

By employing the Schur determinant formula [36], we get that
\[
\det(I_m + iT_m) = \det(I_{m-1} + iT_{m-1}) \left| iu_{mm} + 1 + \langle u| (I_{m-1} + iT_{m-1})^{-1} |u\rangle \right|.
\] (3.9)

Apparently,
\[
[dT_m] = [dT_{m-1}][du]dt_{mm}, \tag{3.10}
\]

\[
\delta(T_m^{\text{off}}) = \delta(T_{m-1}^{\text{off}}) \delta(u), \tag{3.11}
\]

\[
\exp(i\text{Tr}(T_m^{\text{diag}} W_m^{\text{diag}})) = \exp(i\text{Tr}(T_{m-1}^{\text{diag}} W_{m-1}^{\text{diag}})) \exp(iu_{mm}w_{mm}). \tag{3.12}
\]

Now (3.8) can be transformed into the following form:
\[
\mathcal{I}_{m,n}(W_m^{\text{diag}}) = \int [dT_{m-1}] \delta(T_{m-1}^{\text{off}}) \frac{\exp(i\text{Tr}(T_{m-1}^{\text{diag}} W_{m-1}^{\text{diag}}))}{\det^n(I_{m-1} + iT_{m-1})} \int [du] \delta(u) \\
\times \int dt_{mm} \left( iu_{mm} + 1 + \langle u| (I_{m-1} + iT_{m-1})^{-1} |u\rangle \right). \tag{3.13}
\]

Using the following complex integral formula:
\[
\int_{-\infty}^{+\infty} dx \frac{e^{ix}}{(x-a)^n} = \frac{2\pi i}{\Gamma(n)} x^{n-1} e^{-ia} \tag{3.14}
\]
or
\[
\int_{-\infty}^{+\infty} dx \frac{e^{ix}}{(ix+a)^n} = \frac{2\pi}{\Gamma(n)} x^{n-1} e^{-a}, \tag{3.15}
\]
we have
\[ \int dq_{nm} \frac{\exp(i u_{nm} w_{nm})}{(u_{nm} + 1 + \langle u \mid \mathbb{I}_{m-1} + i T_{m-1}^{-1} \rangle |u\rangle)^n} \pi \] (3.16)
\[ = \frac{1}{\pi} \int_0^{\infty} dq_{nm} \frac{\exp(i u_{nm} w_{nm})}{(u_{nm} - i(1 + \langle u \mid \mathbb{I}_{m-1} + i T_{m-1}^{-1} \rangle |u\rangle)^n} \] (3.17)
\[ = \frac{1}{\pi} \frac{2 \pi i^n}{\Gamma(n)} w_{nm}^{n-1} \exp[-w_{nm}(1 + \langle u \mid \mathbb{I}_{m-1} + i T_{m-1}^{-1} \rangle |u\rangle)], \] (3.18)
that is,
\[ \int dq_{nm} \frac{\exp(i u_{nm} w_{nm})}{(u_{nm} + 1 + \langle u \mid \mathbb{I}_{m-1} + i T_{m-1}^{-1} \rangle |u\rangle)^n} \pi \] (3.19)
\[ = \frac{2 \pi}{\Gamma(n)} w_{nm}^{n-1} e^{-w_{nm}} \exp(-w_{nm}\langle u \mid \mathbb{I}_{m-1} + i T_{m-1}^{-1} \rangle |u\rangle). \] (3.20)

Then
\[ \mathcal{I}_{m,n}(W_{nm}^{\text{diag}}) = \frac{2 \pi}{\Gamma(n)} w_{nm}^{n-1} e^{-w_{nm}} \int [d\mathbf{T}_{m-1}] \delta(\mathbf{T}_{m-1}) \frac{\exp(i \text{Tr}(\mathbf{T}_{m-1}^{\text{diag}} W_{nm}^{\text{diag}}))}{\det(\mathbf{I}_{m-1} + i \mathbf{T}_{m-1}^{-1})} \times \int [du] \delta(u) \exp(-w_{nm}\langle u \mid \mathbb{I}_{m-1} + i T_{m-1}^{-1} \rangle |u\rangle). \] (3.21)

Therefore
\[ \mathcal{I}_{m,n}(W_{nm}^{\text{diag}}) = \frac{2 \pi}{\Gamma(n)} w_{nm}^{n-1} e^{-w_{nm}} \int [d\mathbf{T}_{m-1}] \delta(\mathbf{T}_{m-1}) \frac{\exp(i \text{Tr}(\mathbf{T}_{m-1}^{\text{diag}} W_{nm}^{\text{diag}}))}{\det(\mathbf{I}_{m-1} + i \mathbf{T}_{m-1}^{-1})}, \] (3.22)
where we have used the fact that
\[ \int [du] \delta(u) \exp(-w_{nm}\langle u \mid \mathbb{I}_{m-1} + i T_{m-1}^{-1} \rangle |u\rangle) = 1. \] (3.23)
That is,
\[ \mathcal{I}_{m,n}(W_{nm}^{\text{diag}}) = \frac{2 \pi}{\Gamma(n)} w_{nm}^{n-1} e^{-w_{nm}} \mathcal{I}_{m-1,n}(W_{m-1}^{\text{diag}}). \] (3.24)

By this, we finally get
\[ \mathcal{I}_{m,n}(W_m^{\text{diag}}) = \frac{(2 \pi)^n}{\Gamma(n)^m} \prod_{j=1}^m w_{ji}^{n-1} \exp\left(- \sum_{j=1}^m w_{ij}\right). \] (3.25)

In summary, we have the distribution of the diagonal part of the Wishart ensemble, which can be presented as the following proposition.

**Proposition 3.2 (The distribution of the diagonal parts of the Wishart ensemble).** The distribution of the diagonal parts of the Wishart ensemble is given by the following formula:
\[ q(W_{\text{diag}}) = \frac{1}{\Gamma(n)^m} \left( \prod_{j=1}^{m} w_{jj}^{n-1} \right) \exp \left( -\sum_{j=1}^{m} w_{jj} \right). \] (3.26)

**Remark 3.3.** In the above process, we calculate from first principles the joint distribution of diagonal parts of Wishart matrices. What we emphasize here is that the distribution of diagonal parts of Wishart matrices is the marginal distribution of the distribution of Wishart matrices. Of course, we can derive this result directly from the definition of Wishart matrices.

### 3.2. The Wishart ensemble and random mixed quantum states

For the mathematical treatment of a quantum system, one usually associates with it a Hilbert space whose vectors describe the states of that system. In our situation, we associate with \(A\) and \(B\) two complex Hilbert spaces \(\mathcal{H}_A\) and \(\mathcal{H}_B\) with respective dimensions \(m\) and \(n\), which are assumed here to be such that \(m \leq n\). In these settings, the vectors of the spaces \(\mathcal{H}_A\) and \(\mathcal{H}_B\) describe the states of the systems \(A\) and \(B\). Those of the tensorial product \(\mathcal{H}_A \otimes \mathcal{H}_B\) (of dimension \(mn\)) then describe the states of the combined system \(AB\).

It will be helpful throughout this paper to make use of a simple correspondence between the linear operator spaces \(L_{\mathcal{X}, \mathcal{Y}}\) and \(\mathcal{Y} \otimes \mathcal{X}\), for given complex Euclidean spaces \(\mathcal{X}\) and \(\mathcal{Y}\). We define the mapping

\[ \mathcal{L}, () \rightarrow \mathcal{Y} \otimes \mathcal{X} \]

to be the linear mapping that represents a change of bases from the standard basis of \(L_{\mathcal{X}, \mathcal{Y}}\) to the standard basis of \(\mathcal{Y} \otimes \mathcal{X}\). Specifically, in the Dirac notation, this mapping amounts to flipping a bra to a ket; we define

\[ \langle i \mid i \rangle = |i \rangle \langle i| = |ij\rangle, \]

at this point the mapping is determined for every \(M = \sum_{i,j} M_{ij} |i \rangle \langle j| \in L_{\mathcal{X}, \mathcal{Y}}\) by linearity [32]. For convenience, we denote \(\text{vec}(M) := |M\rangle\). Clearly \(\text{Tr}_\mathcal{X}(|M\rangle\langle N|\rangle) = MN^\dagger\) for \(M, N \in L_{\mathcal{X}, \mathcal{Y}}\). In this problem, we assume that \(\mathcal{X} = \mathcal{H}_A = \mathbb{C}^m\) and \(\mathcal{Y} = \mathcal{H}_B = \mathbb{C}^n\).

For Wishart matrices \(W\) on \(\mathbb{C}^m\), it can be considered as the reduced state of a purified random vector \(|Z\rangle\) with random coordinates \(z_{ij}, i = 1, \ldots, m; j = 1, \ldots, n\), the probability distribution of which being the uniform distribution on the unit sphere of \(\mathbb{C}^m \otimes \mathbb{C}^n\). That is, \(W = ZZ^\dagger = \text{Tr}_\mathcal{X}(|Z\rangle \langle Z|\rangle)\). Since any random mixed quantum states can be generated by Wishart matrices with fixed trace one, it follows that the distribution of random mixed quantum states is given by the following via \(\rho = \frac{W}{\text{Tr}(W)}\):

\[ P(\rho) \propto \delta(1 - \text{Tr}(W))Q(W). \] (3.27)

The distribution of random mixed quantum states is given by the following:

\[ P(\rho) \propto \delta(1 - \text{Tr}(\rho)) \int [dZ] \delta(\rho - ZZ^\dagger) \varphi(Z). \] (3.28)

That is [38],

\[ P(\rho) \propto \delta(1 - \text{Tr}(\rho)) \det^{m-\text{Tr}(\rho)}(\rho). \] (3.29)

In view of this, we get
Proposition 3.4. The distribution of diagonal parts \( \rho_{\text{diag}} \) of random mixed quantum states is given by the following:

\[
p(\rho_{\text{diag}}) = \int [d\rho_{\text{off}}] p(\rho) = \frac{\Gamma(mn)}{\Gamma(n)^m} \left( 1 - \sum_{j=1}^{m} \rho_{jj} \right) \prod_{j=1}^{m} \rho_{jj}^{n-1}.
\] (3.30)

We check this directly. A random reduced quantum state \( \rho \), obtained by partial tracing a Haar-distributed bipartite state \( |\in\rangle \otimes Z_{mn}^{CC} \), can be expressed via a Wishart matrix as follows:

\[
\rho = \frac{W}{\text{Tr}(W)} = \frac{1}{t} W,
\]

where \( W = ZZ^\dagger \) for \( Z = [z_{ij}] \) is an \( m \times n \) matrix with independent Gaussian complex entries, and \( t := \text{Tr}(W) = \text{Tr}(ZZ^\dagger) \). Then \( t\rho = W = ZZ^\dagger \), i.e.

\[
W_i = t\rho_i = \sum_{j=1}^{n} |j\rangle \langle j| = \sum_{j=1}^{n} [(\text{Re}(z_{ij}))^2 + (\text{Im}(z_{ij}))^2],
\]

where \( \text{Re}(z_{ij}), \text{Im}(z_{ij}) \sim N(0, \frac{1}{2}) \), leading to the following: for all \( i = 1, \ldots, m \),

\[
p(W_i) = \frac{W_{ii}^{n-1}}{2^{n} \Gamma(n)} e^{-\frac{1}{2} W_{ii}},
\]

leading in turn to the following:

\[
p(t) = \frac{t^{mn-1}}{2^{mn} \Gamma(mn)} e^{-\frac{1}{2} t}. \]

Let us perform the following change of variables: \( (W_1, \ldots, W_{mn}) \mapsto (\rho_{11}, \ldots, \rho_{m-1,m-1}, t) \). The Jacobian of the transformation [19] is

\[
\left| \frac{\partial(W_1, \ldots, W_{mn})}{\partial(\rho_{11}, \ldots, \rho_{m-1,m-1}, t)} \right| = t^{m-1}.
\]

That is

\[
dW_1 \cdots dW_{mn} = t^{m-1} dr \rho_{11} \cdots dr_{m-1,m-1}.
\]

Furthermore,

\[
p(W_1) \cdots p(W_{mn}) dW_1 \cdots dW_{mn}
\]

\[
= p(t_{11}) \cdots p(t_{mm}) t^{m-1} dr \rho_{11} \cdots dr_{m-1,m-1}
\]

\[
= \frac{(t_{11})^{n-1}}{2^n \Gamma(n)} e^{-\frac{1}{2} t_{11}} \cdots \frac{(t_{mm})^{n-1}}{2^n \Gamma(n)} e^{-\frac{1}{2} t_{mm}} t^{m-1} dr \rho_{11} \cdots dr_{m-1,m-1}
\]

\[
= \frac{t^{mn-m}}{2^{mn} \Gamma(n)^m} \frac{1}{2^n} \prod_{j=1}^{m} \rho_{jj}^{n-1} \times t^{m-1} dr \rho_{11} \cdots dr_{m-1,m-1}.
\]

Finally we get
\begin{align*}
p(W_{11} \cdots W_{mn})dW_{11} \cdots dW_{mn} &= p(W_{11}, \ldots, W_{mn})dW_{11} \cdots dW_{mn} \\
&= p(t_{11}, \ldots, t_{mn})n^{-1}d\rho_{11} \cdots d\rho_{m-1,m-1} \\
&= \frac{e^{-\frac{1}{2}t^tCt}}{2^m(mn)!} \prod_{j=1}^m \rho_{jj}^{n-1}d\rho_{11} \cdots d\rho_{m-1,m-1}.
\end{align*}

Taking the integration with respect to \( t \) gives rise to the marginal distribution—the distribution of the diagonal elements which is the symmetric Dirichlet distribution [19]:

\[
p(\rho_{\text{diag}}) := p(\rho_{11}, \ldots, \rho_{mn}) = \frac{\Gamma(mn)}{\Gamma(n)^m} \delta \left( 1 - \sum_{j=1}^m \rho_{jj} \right) \prod_{j=1}^m \rho_{jj}^{n-1}.
\]

The following result, although beyond our goal in the present paper, is recorded here for independent interest. It deals with the exact analytical relationship between the joint distributions of diagonal entries and eigenvalues of the same invariant ensemble.

**Proposition 3.5. (Derivative principle, [21]).** Let \( Z \) be a random matrix drawn from a unitarily invariant random matrix ensemble, \( \varphi_Z \) the joint eigenvalue distribution for \( Z \) and \( p_Z \) the joint distribution of the diagonal elements of \( Z \). Then

\[
\varphi_Z(\lambda) = \frac{1}{\prod_{k=1}^n k!} \Delta(\lambda)\Delta(-\frac{\partial}{\partial \lambda})p_Z(\lambda), \tag{3.31}
\]

where \( \Delta(\lambda) = \prod_{i<j}(\lambda_i - \lambda_j) \) is the Vandermonde determinant and \( \Delta(-\frac{\partial}{\partial \lambda}) \) the differential operator \( \prod_{i<j}\left(\frac{\partial}{\partial \lambda_i} - \frac{\partial}{\partial \lambda_j}\right) \).

### 3.3. Average entropy of the diagonal entries of random density matrices

In what follows, we calculate the average entropy of the diagonal part of random density matrices under the distribution of random density matrices subject to (3.28). Specifically, we will calculate the following integral:

\[
S_{m,n}^D = \int[d\rho_{\text{diag}}]S(p(\rho)) = \int[d\rho_{\text{diag}}]S(p_{\text{diag}}) \int[d\rho_{\text{off}}]p(\rho) \\
= \int[d\rho_{\text{diag}}]S(p_{\text{diag}})p(\rho_{\text{diag}}), \tag{3.32}
\]

where

\[
P(\rho) \propto \delta(1 - \text{Tr}(\rho)) \int[dZ]\delta(\rho - ZZ^t)\varphi(Z)
\]

for \( \varphi(Z) = \frac{1}{\pi^m} \exp(-\text{Tr}(ZZ^t)) \). We have the following result:

**Proposition 3.6.** The average diagonal entropy of random mixed quantum states, induced by Haar-distributed bipartite pure states on \( \mathbb{C}^n \otimes \mathbb{C}^n \), is given by the following:

\[
S_{m,n}^D = H_{mn} - H_n, \tag{3.33}
\]

where \( H_k := \sum_{j=1}^k \frac{1}{j} \) is the \( k \)th harmonic number for positive integer number \( k \).
Proof. According the distribution of diagonal parts of random mixed quantum states, we have

\[
\mathbb{S}_{m,n}^D = \int \mathcal{S} (\rho_{\text{diag}}) \rho(\rho_{\text{diag}}) d\rho_{\text{diag}}
\]

\[
= \frac{\Gamma(mn)}{\Gamma(n)^m} \int \left( -\sum_{j=1}^{m} \rho_{jj} \ln \rho_{jj} \right) \delta \left( 1 - \sum_{j=1}^{m} \rho_{jj} \right) \prod_{j=1}^{m} \rho_{jj}^{n-1} d\rho_{jj}
\]

\[
= -m \frac{\Gamma(mn)}{\Gamma(n)^m} \int (\rho_{11} \ln \rho_{11}) \delta \left( 1 - \sum_{j=2}^{m} \rho_{jj} \right) \prod_{j=2}^{m} \rho_{jj}^{n-1} d\rho_{jj}.
\]

Then it can be rewritten as

\[
\mathbb{S}_{m,n}^D = -m \frac{\Gamma(mn)}{\Gamma(n)^m} \int_0^1 d\rho_{11} \rho_{11}^{n-1} \rho_{11} \int_0^\infty \cdots \int_0^\infty \delta \left( 1 - \rho_{11} - \sum_{j=2}^{m} \rho_{jj} \right) \prod_{j=2}^{m} \rho_{jj}^{n-1} d\rho_{jj}.
\] (3.34)

Now denote

\[
F(t) = \int_0^\infty \cdots \int_0^\infty \delta \left( t - \sum_{j=2}^{m} \rho_{jj} \right) \prod_{j=2}^{m} \rho_{jj}^{n-1} d\rho_{jj}.
\]

Then via \( y_j = s \rho_{j+1,j+1} \), where \( j = 1, \ldots, m-1 \), performing Laplace transform \((t \mapsto s)\) to \( F(t) \) [33], we get

\[
\tilde{F}(s) = \int_0^\infty \cdots \int_0^\infty \exp \left( -s \sum_{j=2}^{m} \rho_{jj} \right) \prod_{j=2}^{m} \rho_{jj}^{n-1} d\rho_{jj}
\]

\[
= s^{-m(m-1)} \int_0^\infty \cdots \int_0^\infty \prod_{j=1}^{m-1} e^{-y_j} d\rho_{jj}
\]

\[
= s^{-m(m-1)} \prod_{j=1}^{m-1} \int_0^\infty y_j^{n-1} e^{-y_j} dy_j = s^{-m(m-1)} \prod_{j=1}^{m-1} \Gamma(n),
\]

that is, \( \tilde{F}(s) = \Gamma(n)^{m-1} s^{-m(m-1)} \). This implies

\[
F(t) = \frac{\Gamma(n)^{m-1}}{\Gamma(n(m-1))} t^{n(m-1)-1}.
\]

Therefore

\[
\mathbb{S}_{m,n}^D = -m \frac{\Gamma(mn)}{\Gamma(n)^m} \int_0^1 d\rho_{11} \rho_{11}^{n} \ln \rho_{11} F(1-\rho_{11})
\]

\[
= -m \frac{\Gamma(mn)}{\Gamma(n)^m} \frac{\Gamma(n)^{m-1}}{\Gamma(n(m-1))} \int_0^1 x^n \ln x (1-x)^{n(m-1)-1} dx
\]

\[
= -m \frac{\Gamma(mn)}{\Gamma(n)^m} \frac{\Gamma(n)^{m-1}}{\Gamma(n(m-1))} \frac{\partial B}{\partial \alpha} \bigg|_{\alpha,\beta = (m+1,n(m-1))},
\]
where \( B(\alpha, \beta) = \int_0^1 x^{\alpha-1}(1-x)^{\beta-1}dx = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} \) and

\[
\frac{\partial B}{\partial \alpha} = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}(\psi(\alpha) - \psi(\alpha + \beta)),
\]

where \( \psi(\alpha) := \frac{d}{d\alpha} \ln \Gamma(\alpha) \). Letting \((\alpha, \beta) = (n + 1, n(m - 1))\) give rises to

\[
S_{m,n}^D = -m \frac{\Gamma(mn)}{\Gamma(n)^m} \frac{\Gamma(n)^m - 1}{\Gamma(n(m - 1))} \frac{\Gamma(n + 1)\Gamma(n(m - 1))}{\Gamma(mn + 1)}(\psi(n + 1) - \psi(mn + 1)), \tag{3.35}
\]

where \( \psi(n + 1) = H_n - \gamma_{\text{Euler}} \) for Euler’s constant \( \gamma_{\text{Euler}} \approx 0.57722 \) [35]. That is,

\[
\bar{S}_{m,n} = \psi(mn + 1) - \psi(n + 1) = H_{mn} - H_n. \tag{3.36}
\]

We are done.

Note that similar integrals like the one in (3.32) have been considered recently with motivation from machine learning, see [22]. This result is very interesting, compared with Page’s formula [4], stating the average entropy of a subsystem given by \( S_{m,n} = H_{mn} - H_n \frac{m-1}{2n} \) when \( m \leq n \). With this result, we can give the average relative entropy of coherence for random mixed quantum states, obtained recently in the paper [37], in the following section.

### 4. Main results

In this section, we will present our main result about quantum coherence, stating the average relative entropy of coherence for random mixed quantum states can be given by the following compact formula (see also in [37]). Note that what we emphasize here is the method used for deriving this elegant formula.

**Theorem 4.1 (Average coherence).** For random mixed states of dimension \( m \) sampled from induced measures obtained via partial tracing of Haar distributed bipartite pure states of dimension \( mn \) where \( m \leq n \), the average relative entropy of coherence is given by the following compact form

\[
\bar{\mathcal{E}}_{m,n} = \frac{m - 1}{2n}. \tag{4.1}
\]

**Proof.** Now the distribution of random mixed quantum states is given by

\[
\mathbf{P}(\rho) \propto \delta(1 - \text{Tr}(\rho))\text{det}^{m-n}(\rho). \tag{4.2}
\]

Under this distribution, we calculate the average relative entropy of coherence as follows:

\[
\bar{\mathcal{E}}_{m,n} = \int [d\rho] \mathbf{P}(\rho) \mathcal{E}_{m,n}(\rho) = \int [d\rho] \mathbf{P}(\rho)S(\rho_{\text{diag}}) = \int [d\rho] \mathbf{P}(\rho)S(\rho)
\]

\[
= S_{m,n}^D - \bar{S}_{m,n} = (H_{mn} - H_n) - \left( H_{mn} - H_n - \frac{m - 1}{2n} \right) = \frac{m - 1}{2n}. \tag{4.4}
\]
Note here that we used the fact that
\[
S_{m,n} = \int [d\rho] P(\rho) S(\rho) = H_{mn} - H_n - \frac{m - 1}{2n}
\]  
(4.5)
which is called Page’s average entropy formula, conjectured in [25], and proven in [4, 29, 30].

**Remark 4.2.** For \( m = n \), we see that the average coherence is given by \( \frac{m-1}{2m} \), which is approaching to \( \frac{1}{2} \) when \( m \to \infty \). The asymptotic value \( \frac{1}{2} \) of the average coherence is confirmed by Puchała et al using tools from free probability theory [27].

We already shown that the distribution of random mixed quantum states is given by
\[
P(\rho) \propto (1 - \text{Tr}(\rho)) \det^{n-m}(\rho).
\]
By the spectral decomposition of \( \rho \), we have \( \rho = U \Lambda U^\dagger \) with \( \Lambda = \text{diag}(\lambda_1, \ldots, \lambda_m) \), where \( \lambda_i \neq \lambda_j \) for distinct indices \( i \) and \( j \). Since this distribution \( P(\rho) \) is unitary-invariant, noting that
\[
[d\rho] \propto \prod_{1 \leq i < j \leq m} (\lambda_j - \lambda_i)^2 [d\Lambda] d\mu_{\text{Haar}}(U),
\]
it follows that the joint distribution of the eigenvalues of random mixed quantum states is given by [38]
\[
P_{m,n}(\Lambda) \propto \delta\left(1 - \sum_{j=1}^m \lambda_j\right) \prod_{1 \leq i < j \leq m} (\lambda_j - \lambda_i)^2 \prod_{j=1}^m \lambda_j^{n-m}.
\]
In what follows, we reconsider the calculation of the average coherence of random mixed quantum states:
\[
\overline{\rho}_{m,n} = \int [d\rho] P(\rho) \rho_{m,n}(\rho) = \int [d\Lambda] P_{m,n}(\Lambda) \left[ \int d\mu_{\text{Haar}}(U) \rho_{m,n}(U \Lambda U^\dagger) \right].
\]
Clearly \( \rho_{m,n}(U \Lambda U^\dagger) = S((U \Lambda U^\dagger)_{\text{diag}}) - S(\Lambda U U^\dagger) = S((U \Lambda U^\dagger)_{\text{diag}}) - S(\Lambda) \), which implies that
\[
\overline{\rho}_{m,n} = \int [d\Lambda] P_{m,n}(\Lambda) \overline{\rho}_{m,n}^{\text{iso}}(\Lambda),
\]
where
\[
\overline{\rho}_{m,n}^{\text{iso}}(\Lambda) = \int S((U \Lambda U^\dagger)_{\text{diag}}) d\mu_{\text{Haar}}(U) - S(\Lambda).
\]
Earlier in the study of upper bounds and lower bounds of classical accessible information, one obtains that the average diagonal entropy of isospectral quantum state with a fixed spectrum \( \Lambda = \{ \lambda_1, \ldots, \lambda_m \} \) is given by [14, 15]:
\[
\int S((U \Lambda U^\dagger)_{\text{diag}}) d\mu_{\text{Haar}}(U) = H_m - 1 + Q(\Lambda),
\]  
(4.6)
where
\[
Q(\Lambda) := - \sum_{i=1}^m \frac{\lambda_i^m \ln \lambda_i}{\prod_{j \in \hat{i}}(\lambda_i - \lambda_j)} \overset{\hat{i} := \{1, \ldots, m\} \setminus \{i\}}{\infty} \tag{4.7}
\]
which is called subentropy [3, 15]. From this, we see that the average relative entropy of coherence of isospectral quantum states of fixed spectrum \( \lambda = \{ \lambda_1, \ldots, \lambda_n \} \) is obtained easily

\[
\mathcal{E}^{iso}_{\text{m,n}}(\Lambda) = \int (U \Lambda U^\dagger )_{\text{diag}} \, d \mu_{\text{Haar}}(U) - S(\Lambda) = H_m - 1 + Q(\Lambda) - S(\Lambda). 
\]  

(4.8)

Now

\[
\mathcal{Q}_{\text{m,n}} = \int [d \Lambda] P_{\text{m,n}}(\Lambda) \mathcal{E}^{iso}_{\text{m,n}}(\Lambda) = H_m - 1 + \int [d \Lambda] P_{\text{m,n}}(\Lambda) Q(\Lambda) - \int [d \Lambda] P_{\text{m,n}}(\Lambda) S(\Lambda).
\]  

(4.9)

By using Page’s formula for the average von Neumann entropy:

\[
\int [d \Lambda] P_{\text{m,n}}(\Lambda) S(\Lambda) = S_{\text{m,n}} = H_m - H_n - \frac{m - 1}{2n}.
\]  

(4.10)

If we denote

\[
\mathcal{Q}_{\text{m,n}} := \int [d \rho] P(\rho) Q(\rho),
\]  

(4.11)

then

\[
\mathcal{Q}_{\text{m,n}} = \int [d \Lambda] P_{\text{m,n}}(\Lambda) \int d \mu_{\text{Haar}}(U) Q(U \Lambda U^\dagger) = \int [d \Lambda] P_{\text{m,n}}(\Lambda) Q(\Lambda),
\]  

(4.12)

then—using the result of the average relative entropy of coherence for random mixed quantum states—we get the following result, which may be of independent interest later in the investigation of quantum information theory.

**Theorem 4.3 (Average subentropy).** For random mixed states of dimension \( m \) sampled from induced measures obtained via partial tracing of Haar distributed bipartite pure states of dimension \( mn \) where \( m \leq n \), the average relative entropy of coherence is given by

\[
\mathcal{Q}_{\text{m,n}} = 1 + H_m - H_n.
\]  

(4.13)

**Remark 4.4.** Note that in [37] (please find the meaning of corresponding notations therein), the authors have obtained that

\[
\mathcal{I}(n - m + 1, 1) = \frac{1}{mn} \sum_{k=0}^{m-1} (-1)^k \Gamma(m + n - k) \Gamma(m - k) \Gamma(n - k) \psi(mn + 1) - \psi(m + n - k).
\]  

(4.14)

In our notation here, \( \mathcal{I}(n - m + 1, 1) = \mathcal{Q}_{\text{m,n}} = 1 + H_m - H_n - H_n \). The above compact form of the average subentropy of random mixed quantum states can be rewritten as

\[
\mathcal{Q}_{\text{m,n}} = (1 - \gamma_{\text{Euler}}) - (a_m + a_n - a_{mn}),
\]  

(4.15)

where \( a_k := H_k - \ln k - \gamma_{\text{Euler}} \). We see that \( \lim_{k \to \infty} a_k = 0 \). Moreover, since \( m \leq n \), it follows that \( \lim_{m \to \infty} (a_m + a_n - a_{mn}) = 0 \). Therefore we obtain that

\[
\lim_{m \to \infty} \mathcal{Q}_{\text{m,n}} = 1 - \gamma_{\text{Euler}}.
\]  

(4.16)
Here we have given an analytical proof of the fact that the subentropy is a nonlinear function of random mixed quantum states; in higher dimensional space, the average subentropy approaches the maximal value of the subentropy where it is taken at only maximally mixed state. This amounts to saying that in higher dimensional space the following identity holds approximately

\[ Q\left(\int \rho P(\rho) [d\rho]\right) \simeq \int Q(\rho) P(\rho) [d\rho]. \]  

(4.17)

**Remark 4.5.** We make remark here about approximations of \( S_{mn} \) and \( Q_{mn} \). We know that \( S_{\text{max}} = \ln m \) and \( Q_{\text{max}} = (1 - \gamma_{\text{Euler}}) - a_m = 1 - \gamma_{\text{Euler}} \) [15]. Then

\[ S_{\text{max}} - S_{mn, n} = \frac{m - 1}{2n} + (a_n - a_{mn}), \]

(4.18)

\[ Q_{\text{max}} - Q_{mn, n} = a_n - a_{mn}. \]

(4.19)

Thus for a fixed ratio \( \frac{m}{n} \), \( S_{\text{max}} - S_{mn, n} \) approaches a nonzero constant, whereas \( Q_{\text{max}} - Q_{mn, n} \) approaches zero. However, this might not be too surprising, since \( S_{\text{max}} = \ln m \) grows indefinitely with \( m \), whereas \( Q_{\text{max}} \) is a constant. Thus the relative errors of \( S_{\text{max}} \) and of \( Q_{\text{max}} \) as approximations for the mean values \( S_{mn, n} \) and \( Q_{mn, n} \) both tend to zero for large \( m \):

\[ \frac{S_{\text{max}} - S_{mn, n}}{S_{\text{max}}} = \frac{m - 1}{2n \ln m} + \frac{a_n - a_{mn}}{\ln m} \sim \frac{m}{\ln m} \cdot \frac{1}{2n}, \]

(4.20)

\[ \frac{Q_{\text{max}} - Q_{mn, n}}{Q_{\text{max}}} = \frac{a_n - a_{mn}}{1 - \gamma_{\text{Euler}} - \frac{a_m}{1 - \gamma_{\text{Euler}}}} \sim \frac{1}{2n}. \]

(4.21)

For fixed \( m \), both of these go to zero when \( n \) is taken to infinity. However, for fixed ratios \( \frac{m}{n} \), the relative error of \( S_{\text{max}} \) goes to zero more slowly (as the inverse of the logarithm of \( m \) or \( n \)) than the relative error of \( Q_{\text{max}} \) (as the inverse of \( m \) or \( n \)), so in that sense for large \( m \) and fixed \( \frac{m}{n} \), \( Q_{\text{max}} \) is a relatively better approximation for \( Q_{mn, n} \) than \( S_{\text{max}} \) is for \( S_{mn, n} \).

The typicality of coherence is already established in [37] without the closed-form of the average coherence. For completeness, we include it here.

**Theorem 4.6 (Typicality of coherence).** Let \( \rho \) be a random mixed state on an \( m \)-dimensional Hilbert space, where \( m \geq 3 \), induced via partial-tracing of the Haar-distributed bipartite pure states on \( mn \)-dimensional Hilbert space. Then for all positive scalars \( \varepsilon > 0 \), the coherence \( \mathcal{C}_{mn, n}(\rho) \) of \( \rho \) satisfies the following inequality:

\[ \mathbb{P}\left\{ \left| \mathcal{C}_{mn, n}(\rho) - \frac{m - 1}{2n} \right| > \varepsilon \right\} \leq 2 \exp\left( -\frac{mn\varepsilon^2}{144\pi^3 \ln 2(\ln m)^2} \right). \]

(4.22)

From the above result, we can see that the entropy difference \( S(\rho_{\text{diag}}) - S(\rho) \) is centered around the fraction \( \frac{m - 1}{2n} \), except a set of exponentially small probability whenever the dimension of the system under consideration is large enough. This explains quantitatively why the diagonal part of random mixed quantum states is more disordered than the eigenvalue.
5. Extension to mixing of random mixed quantum states

In this section, we consider the following problem: we choose two arbitrary Haar-distributed bipartite states $|\psi_1\rangle$ and $|\psi_2\rangle$ from $\mathbb{C}^m \otimes \mathbb{C}^n (m \leq n)$. Choose uniformly a weight $w \in [0, 1]$. There exist two $m \times n$ complex matrices $Z_1, Z_2$ such that $\sqrt{w}|\psi_1\rangle = |Z_1\rangle$, $\sqrt{1-w}|\psi_2\rangle = |Z_2\rangle$. We can form a new state $w|\psi_1\rangle\langle\psi_1|+(1-w)|\psi_2\rangle\langle\psi_2|$ from $|\psi_1\rangle\langle\psi_1|$ and $|\psi_2\rangle\langle\psi_2|$. By partial tracing over the second system space $\mathbb{C}^n$, we get a random mixed quantum state $\sigma = Z_1Z_1^* + Z_2Z_2^*$. In fact, a random quantum state ensemble $\rho_{\text{mix}}$ can be realized as a Wishart matrix ensemble.

Let $\mathcal{E}_1$ the random mixed quantum state ensemble obtained by partial tracing over the second system space $\mathbb{C}^n$ of Haar-distributed bipartite pure states; $\mathcal{E}_2$ arbitrary probabilistic mixing of two random chosen quantum states;...+$\mathcal{E}_k$ arbitrary probabilistic mixing of $k$ random chosen quantum states, where $k$ is an arbitrary positive integer. Let $\mathcal{E}_{m,n}(\mathcal{E}_k)$ be the average coherence of ensemble $\mathcal{E}_k$ over $\mathbb{C}^m \otimes \mathbb{C}^n$. With these notations, we see that random state $\sigma$ is from ensemble $\mathcal{E}_2$, thus

$$\mathbb{E}_{m,n}(\mathcal{E}_2) = \int d\sigma P(\sigma) (S(\sigma) - S(\sigma)) = \frac{m-1}{4n}. \quad (5.3)$$

Therefore we can summarize our main results in the present paper as:

$$\mathbb{E}_{m,n}(\mathcal{E}_k) = \frac{1}{k} \cdot \frac{m-1}{2n} = \frac{1}{k} \mathbb{E}_{m,n}(\mathcal{E}_1). \quad (5.4)$$

Note that (4.1) is here just $\mathbb{E}_{m,n}(\mathcal{E}_1)$, a special case where $k = 1$ of (5.4). We see that mixing of random states changes the distribution of diagonal parts and eigenvalues as well. A possible interpretation of (5.4) is: for fixed $m$, $n$, when mixing time $k$ is larger, the average coherence is less. This suggests also that if one would like to enhance coherence of quantum states, then one needs to distill the coherent part by dropping the incoherent part of quantum states.

We also see that (5.4) confirms in the probabilistic sense that the convexity requirement for monotone coherence is reasonable. That is, mixing of quantum states decreases coherence.

6. Concluding remarks

In this paper we have analyzed the properties of the reduced density matrices obtained from a suitable ensemble of pure states; we have spent very few pages to extend our results concerning the statistical behavior of quantum coherence and subentropy that were obtained by much effort in [37]. The main contributions of this paper are that we give a new approach to obtaining the compact formulae for the average coherence and the average subentropy. In future research, we will study the distribution function of quantum coherence in order to get more elaborate results on coherence via the method used in [23, 24]. We hope that the methods and results in this paper can provide new light over the related problems in quantum information theory.
Acknowledgments

LZ would like to thank Don N Page for his comments on the approximations for the average entropy and subentropy. This work is supported by Natural Science Foundation of Zhejiang Province of China (LY17A010027) and by National Natural Science Foundation of China (Nos.11301124 & 61673145).

References

[1] Baumgratz T, Cramer M and Plenio M B 2014 Quantifying coherence Phys. Rev. Lett. 113 140401
[2] Brand ao F, Horodecki M, Ng N, Oppenheim J and Wehner S 2015 The second laws of quantum thermodynamics Proc. Natl Acad. Sci. USA 112 3275
[3] Datta N, Dorlas T, Jozsa R and Benatti F 2014 Properties of subentropy J. Math. Phys. 55 062203
[4] Foong S K and Kano S 1994 Proof of Page’s conjecture on the average entropy of a subsystem Phys. Rev. Lett. 72 1148
[5] Forrester P J 2010 Log-Gases and Random Matrices (London Mathematical Society Monographs) (Princeton, NJ: Princeton University Press)
[6] Giraud O, García-Mata I 2016 Average diagonal entropy in non-equilibrium isolated quantum systems Phys. Rev. E 94 012122
[7] Goldstein S, Lebowitz J L, Tumulka R and Zanghì N 2006 Canonical typicality Phys. Rev. Lett. 96 050403
[8] Hall M J W 1998 Random quantum correlations and density operator distributions Phys. Lett. A 242 123
[9] Hayden P, Leung D W, Winter A 2006 Aspects of Generic Entanglement Commun. Math. Phys. 265 95–117
[10] Hiai F and Petz D 2000 The Semicircle Law, Free Random Variables and Entropy (Providence, RI: American Mathematical Society)
[11] Hoskins R F 2016 Delta Functions: an Introduction to Generalised Functions (Oxford: Woodhead Publishing Limited) (2nd edn 2009)
[12] James A T 1964 Distributions of matrix variates and latent roots derived from normal samples Ann. Math. Stat. 35 475–501
[13] Janik R A and Nowak M A 2003 Wishart and anti-Wishart random matrices J. Phys. A: Math. Gen. 36 3629
[14] Jones K R W 1991 Riemann-Liouville integration and random distributions on hyperspheres J. Phys. A: Math. Gen. 24 1237–44
[15] Jozsa R, Robb D and Wootters W K 1994 Lower bound for accessible information in quantum mechanics Phys. Rev. A 49 668
[16] Levi F and Mintert F 2014 A quantitative theory of coherent delocalization New J. Phys. 16 033007
[17] Lloyd S 2011 Quantum coherence in biological systems J. Phys.: Conf. Ser. 302 012037
[18] Ma J, Yadin B, Girolami D, Vedral V and Gu M 2016 Converting coherence to quantum correlations Phys. Rev. Lett. 116 160407
[19] Mathai A M 1997 Jacobians of Matrix Transformations and Functions of Matrix Arguments (Singapore: World Scientific)
[20] Mehta M L 2004 Random Matrices 3rd edn (Amsterdam: Elsevier)
[21] Mejía J, Zapata C, Botero A 2017 The difference between two random mixed quantum states: exact and asymptoticspectral analysis J. Phys. A: Math. Theor. 50 025301
[22] Montúfar G F and Rauh J 2014 Scaling of model approximation errors and expected entropy distances Kybernetika 50 234–45
[23] Nadal C, Majumdar S N and Vergassola M 2010 Phase transitions in the distribution of bipartite entanglement of a random pure state Phys. Rev. Lett. 104 110501
[24] Nadal C, Majumdar S N and Vergassola M 2011 Statistical distribution of quantum entanglement for a random bipartite state J. Stat. Phys. 142 403
[25] Page D N 1993 Average entropy of a subsystem Phys. Rev. Lett. 71 1291
[26] Plenio M B and Huelga S F 2008 Dephasing-assisted transport: quantum networks and biomolecules New J. Phys. 10 113019
[27] Puchala Z, Pawela L, Życzkowski K 2016 Distinguishability of generic quantum states Phys. Rev. A 93 062112
[28] Rodríguez-Rosario C A, Frauenheim T and Aspuru-Guzik A 2013 Thermodynamics of quantum coherence (arXiv:1308.1245)
[29] Sánchez-Ruiz J 1995 Simple proof of Page’s conjecture on the average entropy of a subsystem Phys. Rev. E 52 5653
[30] Sen S 1996 Average entropy of a quantum subsystem Phys. Rev. Lett. 77 1
[31] Streltsov A, Singh U, Dhar H S, Bera M N and Adesso G 2015 Measuring quantum coherence with entanglement Phys. Rev. Lett. 115 020403
[32] Watrous J 2016 Theory of quantum information https://cs.uwaterloo.ca/ watrous/TQI/
[33] Williams J 1973 Laplace Transforms (London: George Allen & Unwin Ltd)
[34] Wootters W K 1990 Random quantum states Found. Phys. 20 1365
[35] Young R M 1991 Euler’s constant Math. Gaz. 75 187–90
[36] Zhang F-Z 2005 The Schur Complement and its Applications (Berlin: Springer)
[37] Zhang L, Singh U, Pati A K 2017 Average subentropy, coherence and entanglement of random mixed quantum states Ann. Phys. 377 125–46
[38] Życzkowski K and Sommers H J 2001 Induced measures in the space of mixed quantum states J. Phys. A: Math. Gen. 34 7111