STACKY HAMILTONIAN ACTIONS AND SYMPLECTIC REDUCTION

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Abstract. We introduce the notion of a Hamiltonian action of an étale Lie group stack on an étale symplectic stack and establish versions of the Kirwan convexity theorem, the Mayer-Marsden-Weinstein symplectic reduction theorem, and the Duistermaat-Heckman theorem in this context.

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1. Introduction

The leaf space of a foliation on a smooth manifold can be interpreted as a differentiable stack. Many invariants of the foliation, such as its K-theory and cyclic homology, depend only on this stack. In this paper we are concerned with transversely symplectic foliations and with some properties of their associated stacks.

For instance, work of He [17], Ishida [19], Ratiu and Zung [32], and Lin and Sjamaar [24] shows that the convexity properties of the moment map in symplectic geometry hold for certain transversely symplectic foliations. Our first main result, Theorem 7.6, upgrades their results to a convexity theorem for Hamiltonian actions of étale Lie group stacks on étale symplectic stacks. To avoid technicalities, let us here state our theorem in the case where the Lie group stack is a stacky torus $T$. Our assertion is that if $T$ acts on a symplectic stack $X$ and if the action admits a moment map, then under a certain “cleanness” hypothesis the image of the

2010 Mathematics Subject Classification. 53D20 (57R30, 58H05).
moment map is a convex polytope. This statement extends the Atiyah-Guillemin-Sternberg convexity theorem, which holds in the case where $X$ is a symplectic manifold and $T$ an ordinary torus, but the stacky situation has an interesting new feature, namely that the normal fan of the moment polytope is not necessarily rational. More precisely, the normal vectors to the polytope are cocharacters of $T$, and the cocharacter group of a stacky torus is not a lattice, but a quasi-lattice in the Lie algebra. We call a pair consisting of a stacky torus and a convex polytope in the dual of its Lie algebra a \textit{stacky polytope}.

The second main result of this paper is Theorem 9.1, which generalizes the Mayer-Marsden-Weinstein symplectic reduction theorem to the setting of group stack actions on symplectic stacks. We give a necessary and sufficient condition for the reduction of a symplectic stack by a Hamiltonian action of an étale Lie group stack to be again a symplectic stack. The theorem holds under a regularity hypothesis on the moment map, but we make no assumption on the compactness of the group stack or the properness of the action. This generalizes a theorem of Lerman and Malkin [21, Theorem 3.13], who considered the case of a Hamiltonian action of a compact Lie group on a symplectic stack. In view of work of Pecharich [30] and Safronov [33] we expect that in the absence of any regularity assumptions the reduction of a symplectic stack by the action of a Lie group stack is a derived symplectic stack.

Our third main result is Theorem 10.3, which is an extension of the Duistermaat-Heckman theorem to Hamiltonian actions of stacky tori. The Duistermaat-Heckman theorem has two parts: (1) the variation of the reduced symplectic form is linear, and (2) the moment map image of the Liouville measure is piecewise polynomial. It is only the first part that we generalize here, leaving the second part for later.

These results require a basic theory of Hamiltonian actions of Lie group stacks, which we outline in Sections 6–8. Our starting point is the theory developed by Lerman and Malkin [21], which we extend in two respects: the stacks that we deal with are étale, but usually not separated, and the groups that act on them are themselves étale stacks. What we call a symplectic stack is a $0$-shifted symplectic 1-stack in the terminology of Pantev et al. [29], except that our stacks are defined over the category of differentiable manifolds instead of the category of algebraic schemes. See Getzler’s lecture notes [15] for an introduction to higher symplectic stacks over manifolds and for an explanation of how Weinstein’s symplectic groupoids [36] and Xu’s quasi-symplectic groupoids [37] are presentations of 1-shifted symplectic 1-stacks. We expect that some of our results, especially the reduction theorem, can be extended to higher symplectic stacks.

An illuminating example of stacky symplectic reduction is Prato’s construction of toric quasifolds [31], which predates many of these developments. We present (a slight extension of) her construction as a running example in order to show that every simple stacky polytope is the moment polytope of what we call a \textit{toric} symplectic stack. Toric symplectic stacks are a $C^\infty$ counterpart of the toric stacks of algebraic geometry, a comprehensive treatment of which was given by Gerashchenko and Satriano [14]. However, the two theories are very different. The correspondence between toric symplectic manifolds and nonsingular complex projective toric varieties established by Delzant [12] breaks down in the world of stacks, because $C^\infty$ stacky tori, which include such objects as the quotient of a
two-dimensional torus by a dense line, are seldom algebraic. We hope to return to this topic in a future paper.

We are grateful to Chenchang Zhu for her contribution to this project, which appears in Appendix C.

2. Notation and conventions

Unless otherwise noted, manifolds are $C^\infty$, second countable, and Hausdorff. Manifolds and smooth maps form the category $\text{Diff}$. Lie groups are required to have countably many components. The Lie algebra of vector fields on a manifold $X$ is denoted by $\text{Vect}(X)$. Given a smooth map $f: X \to Y$, two vector fields $v \in \text{Vect}(X)$ and $w \in \text{Vect}(Y)$ are $f$-related (notation: $v \sim_f w$) if $df(v_x) = w_{f(x)}$ for all $x \in X$. A Lie groupoid with object manifold $X_0$ and arrow manifold $X_1$ is denoted by $X_1 \rightrightarrows X_0$. Stacks over $\text{Diff}$ are written in boldface, $X$, $Y$, and so are their 1-morphisms $\phi: X \to Y$ and 2-morphisms $\alpha: \phi \Rightarrow \psi$. The classifying stack of a Lie groupoid $X_\star$ is denoted by $BX_\star$. We write equivalences of stacks as $X \cong Y$. We denote by $\star$ a terminal object in the 2-category of stacks.

3. Hamiltonian actions on presymplectic manifolds

This section is a brief exposition of the presymplectic convexity theorem [24, Theorem 2.2], which is the prototype of our stacky convexity theorem, Theorem 7.6. A presymplectic manifold is a smooth manifold $X$ equipped with a closed 2-form $\omega$ of constant rank. The kernel $\ker(\omega)$ defines an involutive distribution on $X$. The corresponding foliation of $X$ is called the null foliation, which we denote by $\mathcal{F}$. For $x \in X$, we write $\mathcal{F}(x)$ for the leaf of $\mathcal{F}$ containing $x$.

Consider a left action of a Lie group $G$ on a presymplectic manifold $(X, \omega)$. For $\xi \in \mathfrak{g} = \text{Lie}(G)$, denote by $\mathcal{F}_\xi$ the fundamental vector field of $\xi$ on $X$. Let

$$n(\mathcal{F}) := \{ \xi \in \mathfrak{g} \mid (\mathcal{F}_\xi)_x \in T_x \mathcal{F} \text{ for all } x \in X \}.$$

The subspace $n(\mathcal{F}) \subseteq \mathfrak{g}$ is an ideal in $\mathfrak{g}$, which we call the null ideal of $\mathcal{F}$, following [24]. Let $N(\mathcal{F}) \subseteq G$ be the connected immersed Lie subgroup of $G$ with Lie algebra $n(\mathcal{F})$, which we will call the null subgroup. The action of $G$ on $(X, \omega)$ is clean if

$$T_x(N(\mathcal{F}) \cdot x) = T_x(G \cdot x) \cap T_x \mathcal{F}$$

for all $x \in X$. For a Lie subalgebra $\mathfrak{h}$ of $\mathfrak{g}$, let

$$\text{ann}(\mathfrak{h}) = \{ \eta \in \mathfrak{g}^* \mid \langle \eta, \mathfrak{h} \rangle = 0 \} \cong (\mathfrak{g}/\mathfrak{h})^*$$

be the annihilator of $\mathfrak{h}$, with $\langle \cdot, \cdot \rangle: \mathfrak{g}^* \times \mathfrak{g} \to \mathbb{R}$ being the natural pairing. The action of $G$ on $(X, \omega)$ is Hamiltonian if there is a moment map, i.e. a map $\mu: X \to \mathfrak{g}^*$ that satisfies the following conditions:

(i) $d\mu^\xi = \iota_{\mathcal{F}_x} \omega$ for all $\xi \in \mathfrak{g}$, where $\mu^\xi(x) = \langle \mu(x), \xi \rangle$ denotes the component of $\mu$ along $\xi$;

(ii) $\mu$ intertwines the $G$ action on $X$ and the coadjoint action of $G$ on $\mathfrak{g}^*$;

(iii) $\mu(X) \subseteq \text{ann}(n(\mathcal{F}))$.

The tuple $(X, \omega, G, \mu)$ is then called a presymplectic Hamiltonian $G$-manifold. An isomorphism between two Hamiltonian $G$-manifolds $(X, \omega, G, \mu)$ and $(X', \omega', G, \mu')$ is a $G$-equivariant diffeomorphism $\phi: X \to X'$ which preserves the presymplectic structure and the moment map.
The conditions (i)–(iii) on the moment map are not independent. By [24, Proposition 2.9.1], if $G$ is compact and if $\mu: X \to g^*$ satisfies (i) and (ii), there exists $\lambda \in g^*$ fixed by the coadjoint action of $G$ such that $\mu + \lambda$ satisfies (i)–(iii).

3.1. Example. This example is drawn from [12] and [31]. Let $T$ be the circle $\mathbb{R}/\mathbb{Z}$, $G$ the $n$-dimensional torus $\mathbb{T}^n$, and $N \subseteq G$ an immersed Lie subgroup. Consider the Hamiltonian $G$-manifold $(X, \omega, G, \mu)$, where

$$X = \mathbb{C}^n, \quad \omega = \frac{1}{2\pi i} \sum_j dz_j \wedge d\bar{z}_j, \quad \mu(z) = \sum_j |z_j|^2 e_j^* + \lambda,$$

and $G$ acts on $\mathbb{C}^n$ in the standard way. Here $e_j^*$ is the dual of the standard basis of $\mathbb{R}^n = g$, and $\lambda \in g^*$ is in the open negative orthant. Let $\iota: n \to g$ be the inclusion of Lie algebras and $\iota^*: g^* \to n^*$ the dual projection. Then $(X_0, \omega_0, G, \mu_0)$ is a presymplectic Hamiltonian $G$-manifold, where

$$X_0 = (\iota^* \circ \mu)^{-1}(0), \quad \omega_0 = \omega|_{X_0}, \quad G = \mathbb{T}^n, \quad \mu_0 = \mu|_{X_0}.$$

Note that $\mu_0$ takes values in $\text{ann}(n)$. We will return to this example throughout the text.

The cleanness condition is essential for the following to be true.

3.2. Theorem (Lin and Sjamaar [24]). Let $(X, \omega, G, \mu)$ be a Hamiltonian presymplectic $G$-manifold, where $X$ is connected, and $G$ is compact and connected. Assume that the $G$-action is clean, and the moment map $\mu: X \to g^*$ is proper. Choose a maximal torus $T$ of $G$ and a closed Weyl chamber $C$ in $t^*$, where $t = \text{Lie}(T)$, and define $\Delta(X) = \mu(X) \cap C$.

(i) The fibers of $\mu$ are connected and $\mu: X \to \mu(X)$ is an open map.

(ii) $\Delta(X)$ is a closed convex polyhedral set.

(iii) $\Delta(X)$ is rational if and only if the null subgroup $N(\mathcal{F})$ of $G$ is closed.

4. Lie groupoids and differentiable stacks

This section is a summary of definitions and conventions. For more about Lie groupoids see e.g. Moerdijk and Mrčun [26] or Crainic and Moerdijk [11], and for the relationship between Lie groupoids and differentiable stacks see e.g. Behrend and Xu [3], Carchedi [9], Lerman [20], Metzler [25], Noohi [27], or Villatoro [35].

**Lie groupoids.** A Lie groupoid $X_\circ = (X_1 \xrightarrow{s} X_0)$ has structure maps $s$, $t$, $m$, $(\cdot)^{-1}$, and $u$ which are called source, target, multiplication, inversion, and the identity bisection, respectively. When two arrows $f$, $g \in X_1$ have $s(f) = t(g)$, they are composable. We typically write $f \circ g$ for multiplication $m(f, g)$ of two composable arrows. The object manifold $X_0$ is smooth (second countable, Hausdorff) manifold, and the arrow manifold $X_1$ is a smooth manifold which we do not assume to be Hausdorff or second countable. The maps $s$ and $t$ are surjective submersions, and the fibers of $s$ and $t$ are required to be Hausdorff and second countable. If all source fibers $s^{-1}(x)$ are connected (resp. simply connected), then $X_\circ$ is source-connected (resp. source-simply connected). For $x \in X_0$, the Lie group $\text{Iso}(x) = s^{-1}(x) \cap t^{-1}(x)$ of arrows starting and ending at $x$ is the isotropy group of $x$. The immersed submanifold $t(s^{-1}(x))$ of $X_0$ is the orbit of $x$. The set of orbits equipped with the quotient topology is the orbit space or coarse quotient space $X_0/X_1$ of the Lie groupoid.

If a Lie group $G$ acts on a manifold $X$, denote the action groupoid $G \ltimes X \xrightarrow{\mu} X$. We sometimes denote a Lie groupoid by its space of arrows; for instance in this
notation the action groupoid is \( G \ltimes X \). If \( X \) is a smooth manifold, we consider it as the identity Lie groupoid \( X \rightrightarrows X \).

The **Lie algebroid** of a Lie groupoid \( X \), is the vector bundle \( \text{Alg}(X) \) over \( X_0 \) given by

\[
\text{Alg}(X) = \{ w \in TX_{|\text{alg}(X)} \mid ds(w) = 0 \}.
\]

The anchor map is the vector bundle morphism \( \rho : \text{Alg}(X) \to TX_0 \) given by \( \rho = dt|_{\text{Alg}(X)} \). Sections of the Lie algebroid extend uniquely to left-invariant vector fields on \( X_1 \), and so the space of sections of \( \text{Alg}(X) \) carries a natural Lie bracket.

A **morphism of Lie groupoids** \( \phi : X \to Y \) is a smooth functor, i.e. a morphism of groupoids which is smooth on the manifolds of objects \( \phi_0 : X_0 \to Y_0 \) and on the manifolds of arrows \( \phi_1 : X_1 \to Y_1 \). For two morphisms \( \phi, \psi : X \to Y \), of Lie groupoids, a natural transformation \( \gamma : \phi \Rightarrow \psi \) is a smooth map \( \gamma : X_0 \to Y_1 \) with the property that for each \( x \in X_0 \), \( \gamma(x) \) is an arrow from \( \phi(x) \) to \( \psi(x) \) in \( Y \), and for every arrow \( f : x_1 \to x_2 \) in \( X \), the following diagram commutes in \( Y \):

\[
\begin{array}{ccc}
\phi(x_1) & \xrightarrow{\gamma(x_1)} & \psi(x_1) \\
\phi(f) & \downarrow & \psi(f) \\
\phi(x_2) & \xrightarrow{\gamma(x_2)} & \psi(x_2)
\end{array}
\]

Any natural transformation of Lie groupoids is a natural isomorphism. The \((2, 1)\)-category of Lie groupoids is denoted \textbf{LieGpd}.

Let \( \phi : X \to Y \) be a morphism of Lie groupoids. Then \( \phi \) is **essentially surjective** if the map \( t \circ \text{pr}_1 : Y_1 \times_{Y_0} X_0 \to Y_0 \), which sends \( (g, x) \mapsto t(g) \) is a surjective submersion. Here \( Y_1 \times_{Y_0} X_0 = Y_1 \times_{Y_0, X_0} X_0 \) means as usual the fibred product of \( Y_1 \) and \( X_0 \), which consists of all \( (g, x) \in Y_1 \times X_0 \) satisfying \( s(g) = \phi(x) \). The morphism \( \phi \) is **fully faithful** if the square

\[
\begin{array}{ccc}
X_1 & \xrightarrow{\phi} & Y_1 \\
\downarrow_{(s, t)} & & \downarrow_{(s, t)} \\
X_0 \times X_0 & \xrightarrow{\phi \times \phi} & Y_0 \times Y_0
\end{array}
\]

is a fibered product of manifolds. If \( \phi \) is both essentially surjective and fully faithful, we say that \( \phi \) is a **Morita morphism**. If there is a zigzag of Morita morphisms \( X \to Y \) and \( X \to Z \), then \( Y \) and \( Z \) are **Morita equivalent**. Morita equivalence is an equivalence relation on Lie groupoids.

4.1 **Definition.** Given a Lie groupoid \( Y \), and a smooth map \( \phi : X \to Y_0 \), one can form the **pullback groupoid** \( X_\phi = \phi^*(Y) \) by setting \( X_0 = X \) and

\[
X_1 := X \times_{X_0 Y_0} Y_1 \times_{Y_0 Y_0} X = \{ (x, y) \in X \times Y_1 \times X \mid \phi(x) = s(g) \text{ and } t(g) = \phi(y) \}.
\]

The pullback groupoid is a Lie groupoid whenever \( (\phi, \phi) : X \times X \to Y_0 \times Y_0 \) is transverse to \( (s, t) : Y_1 \to Y_0 \times Y_0 \). The map \( \phi \) then lifts to a Lie groupoid homomorphism \( \phi^*(Y) \to Y_\phi \), which is fully faithful, and which is essentially surjective if and only if \( t \circ \text{pr}_1 : Y_1 \times_{Y_0} X \to Y_0 \) is surjective.
Differentiable stacks. We consider stacks over the category $\text{Diff}$ of $(C^\infty, \text{Hausdorff}$ and second countable) manifolds, with the open cover Grothendieck topology. A \textit{differentiable stack} is a stack $X$ with a representable epimorphism called an \textit{atlas} $X \to X$ from a manifold $X$. We will allow for differentiable stacks for which $X \times_X X$ is non-Hausdorff, but we will always require that fibers of the canonical morphisms $X \times_X X \to X$ be Hausdorff. A \textit{morphism} of differentiable stacks is a morphism of the underlying weak presheaves on $\text{Diff}$. Differentiable stacks and their morphisms form a (strict) $(2,1)$-category $\text{DiffStack}$.

Recall the 2-functor $B$: $\text{LieGpd} \to \text{DiffStack}$ which takes a Lie groupoid $X$, to its associated \textit{classifying stack} $BX$. Explicitly, $BX$ is the stackification of the presheaf in groupoids $\text{Hom}(\cdot, X)$: $\text{Diff} \to \text{Gpd}$.

Under $B$, finite weak limits in $\text{LieGpd}$ are taken to finite weak limits in $\text{DiffStack}$. Essentially surjective Lie groupoid morphisms are taken to stack epimorphisms, and fully faithful Lie groupoid morphisms are taken to stack monomorphisms. In particular, Morita equivalences are taken to equivalences of stacks. Conversely, if $BX \approx BY$, then $X$ is Morita equivalent to $Y$. If $X \to X$ is an atlas, then $X \approx BX$, where $X$, is the Lie groupoid $X \times_X X \to X$. If $X \approx BX$, then $X$, \textit{presents} $X$.

5. Foliation groupoids and étale stacks

In this section we review basic facts about foliation groupoids and their associated stacks, and draw some elementary consequences. The sources are Crainic and Moerdijk [11] and Lerman and Malkin [21].

Foliation groupoids. Let $X = (X_1 \Rightarrow X_0)$ be a Lie groupoid. If all the isotropy groups of $X$ are discrete, $X$ is a \textit{foliation groupoid}. If the source map $s$ is étale (i.e. a local diffeomorphism) then $X$ is an \textit{étale groupoid}. Clearly, étale groupoids are foliation groupoids. If $X$, is a source-connected foliation groupoid, then the orbits form a (constant rank) foliation $\mathcal{F} = \mathcal{F}_X$, of $X_0$, the anchor map $\rho$: $\text{Alg}(X) \to TX_0$ is injective, and the image of $\rho$ is the tangent bundle $T\mathcal{F}$ of the foliation.

Conversely, let $(X, \mathcal{F})$ be a foliated manifold. A Lie groupoid $X$, over $X_0 = X$ with the property that the anchor map $\rho$: $\text{Alg}(X) \to TX$ is injective and has image equal to $T\mathcal{F}$ is said to \textit{integrate} the foliation $\mathcal{F}$. The integrations of $\mathcal{F}$ form a category, the objects of which are pairs $(X_\ast, \psi)$, where $X_\ast$ is a Lie groupoid integrating $\mathcal{F}$ and $\psi$: $\text{Alg}(X) \to T\mathcal{F}$ is an isomorphism of Lie algebroids, and the arrows $\phi$: $(X_\ast, \psi) \to (X_\ast', \psi')$ of which are morphisms $\phi$: $X_\ast \to X_\ast'$ that respect the maps $\psi$ and $\psi'$. There are source-connected foliation groupoids, the \textit{monodromy groupoid} $\text{Mon}(X, \mathcal{F})$, and the \textit{holonomy groupoid} $\text{Hol}(X, \mathcal{F})$, both of which integrate $\mathcal{F}$. For the construction of these groupoids, see e.g. [26, §5.2]. There is a Lie groupoid morphism $\text{hol}$: $\text{Mon}(X, \mathcal{F}) \to \text{Hol}(X, \mathcal{F})$ which is the identity map on the object manifold $X$ and sends an arrow in $\text{Mon}(X, \mathcal{F})$ to its holonomy action. The following theorem says that the category of source-connected integrations of $(X, \mathcal{F})$ is a preorder with the monodromy group as a greatest element and the holonomy groupoid as a least element.

5.1. Theorem (Crainic and Moerdijk [11, Proposition 1]). Let $(X, \mathcal{F})$ be a foliated manifold. For every source-connected Lie groupoid $X = (X_1 \Rightarrow X_0)$ over $X_0 = X$ integrating $\mathcal{F}$, there is a natural factorization of the holonomy morphism $\text{Mon}(X, \mathcal{F}) \to$
\( \text{Hol}(X, \mathcal{F}) \) into morphisms of Lie groupoids over \( X \),

\[
\text{Mon}(X, \mathcal{F}) \xrightarrow{\psi_X} X \xrightarrow{\text{hol}_X} \text{Hol}(X, \mathcal{F}).
\]

The maps \( \psi_X \) and \( \text{hol}_X \) are étale and surjective on the manifolds of arrows, and \( X \) is source-simply connected if and only if \( \psi_X \) is an isomorphism.

A Lie group bundle is a Lie groupoid where every arrow \( f \) has \( s(f) = t(f) \). Let \( X \) and \( X' \) be Lie groupoids with the same object manifold \( X_0 = X'_0 \), and let \( \psi: X \to X' \) be a Lie groupoid morphism which is the identity on \( X_0 \). The kernel of \( \psi \) is given by

\[
\ker(\psi) := \{ f \in X_1 \mid \psi(f) = u(s(f)) = u(t(f)) \}.
\]

If \( \psi \) is transverse to the identity bisection of \( X'_0 \), then \( \ker(\psi) \) is a Lie group bundle over \( X_0 \). For instance, the kernels of \( \psi_X \) and \( \text{hol}_X \) in Theorem 5.1 are Lie group bundles.

Let \((X, \mathcal{F})\) be a foliated manifold. A smooth map \( \phi: Y \to X \) is transverse to \( \mathcal{F} \) if it is transverse to each leaf of \( \mathcal{F} \). A transverse map \( \phi: Y \to X \) is complete if \( \phi(Y) \) intersects each leaf of \( \mathcal{F} \) at least once. (This extends the usual notion of a complete transversal, where \( \phi \) is an injective immersion and \( \dim Y = \text{codim} \mathcal{F} \).

**5.3. Lemma.** Let \( X \) be a Lie groupoid integrating a foliation \( \mathcal{F} \) on \( X_0 \) and let \( \phi_0: Y_0 \to X_0 \) be transverse to \( \mathcal{F} \). Then the pullback groupoid \( Y = \phi_0^*(X_0) \) is a foliation groupoid which integrates the foliation \( \phi_0^* \mathcal{F} \), and the induced morphism \( \phi: Y \to X \) is fully faithful. If \( \phi_0 \) is complete, then \( \phi \) is a Morita morphism.

**Sketch of proof.** This is well-known when \( \phi_0 \) is a transversal in the usual sense, in which case \( \phi_0^* \mathcal{F} \) is zero-dimensional and \( Y \) is étale; see e.g. Crainic and Moerdijk [11, Lemma 2]. The general case is proved in a similar way: since \( \phi_0 \) is transverse to \( \mathcal{F}, (\phi_0, \phi_0) \) is transverse to \( (s, t) \), so \( Y \), is a Lie groupoid, \( \phi_1: Y_1 \to X_1 \) given by \( \phi_1(x, f, y) = f \) smooth, \( \phi \) is fully faithful and, if \( \phi_0 \) is complete, essentially surjective.

**QED**

**Basic vector fields and forms.** Let \( X_* = (X_1 \triangleright X_0) \) be a foliation Lie groupoid integrating a foliation \( \mathcal{F} \) on \( X_0 \). The set

\[
\mathcal{L}(X_*) = \{ (v_0, v_1) \in \text{Vect}(X_0) \times \text{Vect}(X_1) \mid v_1 \sim_s v_0, v_1 \sim_t v_0 \}
\]

is a Lie subalgebra of \( \text{Vect}(X_0) \times \text{Vect}(X_1) \). The set

\[
\mathcal{S}(X_*) = \{ (v_0, v_1) \in \text{Vect}(\mathcal{F}) \times \text{Vect}(X_1) \mid v_1 \sim_s v_0, v_1 \sim_t v_0 \},
\]

where \( \text{Vect}(\mathcal{F}) \) denotes the Lie algebra of vector fields tangent to \( \mathcal{F} \), is an ideal of \( \mathcal{L}(X_*) \). A basic vector field on \( X_* \) is an element of the quotient Lie algebra

\[
\text{Vect}_{\text{bas}}(X_*) = \mathcal{L}(X_*)/\mathcal{S}(X_*).
\]

A basic differential form on \( X_* \) is a pair of differential forms \( (\zeta_0, \zeta_1) \in \Omega^*(X_0) \times \Omega^*(X_1) \) satisfying \( s^* \zeta_0 = \zeta_1 = t^* \zeta_0 \). The set of basic differential forms on \( X_* \) is the basic de Rham complex \( \Omega^*_{\text{bas}}(X_*) \). We have an exterior differential, contraction operators, and Lie derivatives on \( \Omega^*_{\text{bas}}(X_*) \) defined by

\[
d(\zeta_0, \zeta_1) := (d\zeta_0, d\zeta_1),
\]

\[
i_{(v_0, v_1)}(\zeta_0, \zeta_1) := (i_{v_0} \zeta_0, i_{v_1} \zeta_1),
\]

\[
\mathcal{L}_{(v_0, v_1)}(\zeta_0, \zeta_1) := (\mathcal{L}_{v_0} \zeta_0, \mathcal{L}_{v_1} \zeta_1)
\]

\[
(5.5)
\]
for basic forms \((\zeta_0, \zeta_1)\) and basic vector fields \((v_0, v_1)\). A Lie groupoid morphism \(\phi: X \to Y\) induces a pullback operator \(\phi^*: \Omega^*_\text{bas}(Y) \to \Omega^*_\text{bas}(X)\) defined by 
\[
\phi^*(\zeta_0, \zeta_1) = (\phi^*\zeta_0, \phi^*\zeta_1).
\]
If \(\phi\) and \(\psi\) are naturally isomorphic morphisms of Lie groupoids, then \(\phi^*(\zeta_0, \zeta_1) = \psi^*(\zeta_0, \zeta_1)\). We will usually identify \(\Omega^*_\text{bas}(X)\) with the subcomplex of \(\Omega^p(X_0)\) consisting of all \(\zeta\) satisfying \(s^*\zeta = t^*\zeta\).

5.6. Remark. As a consequence of Proposition 5.13 and Lemma 5.14 below, the basic de Rham complex and the operations on it are Morita invariant notions.

A differential form \(\zeta \in \Omega^k(X_0)\) is horizontal if \(t_{p(\zeta)}\zeta = 0\) for all sections \(\sigma\) of the Lie algebroid \(\text{Alg}(X_0)\), and infinitesimally invariant if \(\mathcal{L}_{p(\zeta)}\zeta = 0\) for all sections \(\sigma\) of \(\text{Alg}(X_0)\). Let \(\Omega^*_\text{bas}(X_0)\) be the set of all forms on \(X_0\) which are horizontal and infinitesimally invariant. The notions of horizontal, basic, and invariant forms are well-known in the context of action groupoids of Lie group actions, for which the next result is standard.

5.7. Lemma. Let \(X\) be a foliation groupoid. Then \(\Omega^*_\text{bas}(X_0) \subseteq \Omega^*_{\text{bas}}(X_0)\), with equality if \(X\) is source-connected.

Proof. Let \(\zeta \in \Omega^k(X_0)\) be a differential form and let \(\sigma\) be a section of the Lie algebroid \(\text{Alg}(X_0)\), with anchor map \(\rho = dt\text{Alg}(X_0)\). Let \(\sigma R\) denote the right-invariant vector field on \(X_1\) induced by \(\sigma\). Then \(\sigma R\) is \(s\)-related to the zero vector field on \(X_0\) because \(\sigma R\) is tangent to the source fibres, and \(\sigma R\) is \(t\)-related to the vector field \(\rho(\sigma)\) on \(X_0\).

Hence
\[
\mathcal{L}_{\sigma R} s^*\zeta = 0, \quad \mathcal{L}_{\sigma R} t^*\zeta = t^*\mathcal{L}_{\rho(\sigma)}\zeta.
\]
At the identity bisection \(u(X_0) \subseteq X_1\) the tangent bundle of \(X_1\) is a direct sum \(TX_1|_{u(X_0)} = \text{Alg}(X_0) \oplus u^*TX_0\). Let \(x \in X_0\). For \(\sigma_1, \sigma_2, \ldots, \sigma_k \in \text{Alg}(X_0)_{u(x)}\) and for \(w_1, w_2, \ldots, w_k \in T_xX_0\) we have
\[
\begin{align*}
(s^*\zeta)_{u(x)}(\sigma_1 + u, w_1, \sigma_2 + u, w_2, \ldots, \sigma_k + u, w_k) &= \zeta_x(s_\sigma_1 + w_1, s_\sigma_2 + w_2, \ldots, s_\sigma_k + w_k) \\
&= \zeta_x(w_1, w_2, \ldots, w_k),
\end{align*}
\]
\[
\begin{align*}
(t^*\zeta)_{u(x)}(\sigma_1 + u, w_1, \sigma_2 + u, w_2, \ldots, \sigma_k + u, w_k) &= \zeta_x(t, \sigma_1 + w_1, t, \sigma_2 + w_2, \ldots, t, \sigma_k + w_k) \\
&= \zeta_x(t, \rho(\sigma_1) + w_1, \rho(\sigma_2) + w_2, \ldots, \rho(\sigma_k) + w_k).
\end{align*}
\]
Now assume \(\zeta \in \Omega^*_{\text{bas}}(X_0)\). Then \(\mathcal{L}_{\rho(\sigma)}\zeta = 0\) by (5.8) and \(t_{p(\zeta)}\zeta = 0\) by (5.9). Conversely, suppose that \(\zeta \in \Omega^*_{\text{bas}}(X, \mathcal{F})\) and that \(X\) is source-connected. From horizontality and from (5.9) we obtain that \((s^*\zeta)_{u(x)} = (t^*\zeta)_{u(x)}\) for every \(x \in X_0\). From invariance and from (5.8) we obtain that \(\mathcal{L}_{\sigma R} s^*\zeta = \mathcal{L}_{\sigma R} t^*\zeta = 0\). The vector fields \(\sigma R\) span the subbundle \(\ker(ds)\) of \(TX_1\). The source fibre \(s^{-1}(x)\) is Hausdorff and connected, so we have \((s^*\zeta)_f = (t^*\zeta)_f\) for every \(f \in s^{-1}(x)\).

5.10. Remark. If we extend the definitions of \(\Omega^*_\text{bas}(X)\) and \(\Omega^*_{\text{bas}}(X_0)\) verbatim to arbitrary Lie groupoids \(X_0\), the previous lemma still holds.

0-Symplectic groupoids. Let \(X_0\) be a foliation groupoid with foliation \((X_0, \mathcal{F})\), and let \(\omega = (\omega_0, \omega_1) \in \Omega^2_\text{bas}(X_0)\). Then \(\omega\) is 0-symplectic if
\[
(i) \quad d\omega = 0, \quad \text{and}
\]
(ii) $\omega$ is nondegenerate; that is, contraction with $\omega$ induces a linear isomorphism $\text{Vec}_{bas}(X_\bullet) \cong \Omega^1_{bas}(X_\bullet)$. Equivalently, $\omega$ is nondegenerate iff $\ker(\omega_0) = T\mathcal{F}$, iff $\ker(\omega_1) = \ker(ds) + \ker(dt)$.

In this case the pair $(X_\bullet, \omega)$ is a 0-symplectic Lie groupoid. (The qualifier “0” refers to the fact that our symplectic forms are 0-shifted, as opposed to Weinstein’s symplectic groupoids [36], which are 1-shifted.)

5.11. Proposition. (i) Let $(X_\bullet, \omega)$ be a 0-symplectic groupoid with foliation $\mathcal{F}$. Then $(X_0, \omega_0)$ is a presymplectic manifold with $\ker(\omega) = T\mathcal{F}$.

(ii) Let $(X_\bullet, \omega)$ be a presymplectic manifold with null foliation $\mathcal{F}$. Let $X_\bullet$ be a source-connected foliation groupoid with Lie algebroid equal to $T\mathcal{F}$. Then $\omega$ is $X_\bullet$-basic and hence defines a 0-symplectic structure on $X_\bullet$.

Proof. (i) This follows immediately from the definition of presymplectic (Section 3) and 0-symplectic.

(ii) The form $\omega$ is horizontal with respect to $\mathcal{F}$. Since it is closed, it is also infinitesimally invariant. By Lemma 5.7 it is basic on $X_\bullet$. QED

5.12. Example. We revisit our Example 3.1. There are many foliation groupoids $X_\bullet$ that integrate $\ker(\omega_0)$. For instance, we can take $N \to \tilde{N}$ to be any étale Lie group homomorphism, let $\tilde{N}$ act on $X_0$ through this homomorphism, and take $X_\bullet$ to be the action groupoid $\tilde{N} \ltimes X_0$.

This groupoid is not source-connected unless $\tilde{N}$ is connected. Nevertheless, the presymplectic form $\omega_0$ is basic with respect to the $\tilde{N}$-action, so $X_\bullet$ is 0-symplectic with 0-symplectic form $(\omega_0, s^*\omega_0) \in \Omega^2_{bas}(X_\bullet)$. Possible choices of $\tilde{N}$ are $\tilde{N} = N$, or $\tilde{N} = \text{Lie}(N)$, the universal cover of the identity component of $N$. Another alternative is the surjective simply connected covering group $\tilde{N} = \pi^{-1}(N)$, where $\pi: \mathbb{R}^n \to \mathbb{T}^n$ is the projection.

Étale stacks. A differentiable stack $X$ is étale if any of the following equivalent conditions hold: (1) $X \cong BX$, for some foliation groupoid $X_\bullet$; (2) $X \cong BX$, for some étale groupoid $X_\bullet$; or (3) $X$ admits an étale atlas $X \to X$.

Let $X = BX_\bullet$, where $X_\bullet$ is an étale groupoid. Let $TX = BTX_\bullet$, and $A^k T^*X = B^k T^*X_\bullet$. Note that $A^k T^*X_\circ = A^k T^*X_1 \Rightarrow A^k T^*X_0$ is a Lie groupoid since the cotangent functor $T^*$ is covariant when restricted to étale maps. If $BX_\bullet = BX'$ for some other étale groupoid $X'_\bullet$, then there is an equivalence $TBX_\bullet \cong TBX'_\bullet$, and similarly for $T^*BX$ and its exterior powers. For an arbitrary étale stack $X_\bullet$, we may define $TX$ and $A^k T^*X$ by fixing an equivalence $X \cong BX_\bullet$, where $X_\bullet$ is an étale groupoid. Vector fields on $X$ are sections of $TX \to X$, and $k$-forms on $X$ are sections of $A^k T^*X \to X$.

A special feature of $X$ being étale is that the 2-vector space of vector fields $\text{Vec}(X)$ is equivalent to a vector space, see [18] and [21]. We will therefore identify sections which differ by a natural isomorphism. Similarly, the 2-vector space of $k$-forms $\Omega^k(X)$ on $X$ is equivalent to a vector space.

5.13. Proposition (Lerman and Malkin [21]). Let $X$ be an étale stack. An equivalence $\psi: BX_\bullet \xrightarrow{\sim} X$, where $X_\bullet$ is a foliation groupoid, induces equivalences

$$\psi^* : \text{Vec}(X) \xrightarrow{\sim} \text{Vec}_{bas}(X_\bullet) \quad \text{and} \quad \psi^* : \Omega^k(X) \xrightarrow{\sim} \Omega^k_{bas}(X_\bullet).$$

Under these equivalences, contractions, Lie derivatives, and the exterior derivative are given as in (5.5).
If $\zeta \in \Omega^k(X)$, then we say the associated element $\psi^*\zeta$ of $\Omega^k_{bas}(X)$ represents $\zeta$, and similarly for vector fields on $X$. The next statement says that we can pull back forms along stack morphisms.

5.14. Lemma. Let $\phi: X \to Y$ be a morphism of étale stacks, and let $\phi: X_\circ \to Y_\circ$ and $\phi': X'_\circ \to Y'_\circ$ be foliation groupoid morphisms so that $B\phi \cong B\phi' \cong \phi$. Let $\zeta \in \Omega^k_{bas}(Y)$ and $\zeta' \in \Omega^k_{bas}(Y'_\circ)$ both represent the same form $\zeta \in \Omega(Y)$. Then $\phi^*\zeta \in \Omega^k_{bas}(X)$ and $(\phi')^*\zeta' \in \Omega^k_{bas}(X'_\circ)$ represent the same form, denoted by $\phi^*\zeta \in \Omega(X)$.

Proof. We assert that there exists a 2-commutative diagram of Lie groupoids

\[
\begin{array}{ccc}
X' & \xrightarrow{a'} & X'' & \xrightarrow{a} & X \\
\downarrow{\phi'} & & \downarrow{\phi''} & & \downarrow{\phi} \\
Y' & \xrightarrow{b'} & Y'' & \xrightarrow{b} & Y
\end{array}
\]

where $a$, $a'$, $b$, and $b'$ are Morita morphisms. Granted this, let $\zeta'' \in \Omega^k_{bas}(Y'')$ represent $\zeta$. Then $\beta^*\zeta = \zeta'' = (\beta')^*\zeta'$ by Proposition 5.13. The 2-commutativity of (5.15) then gives $a'^*(\phi^*\zeta) = (\phi''^*)\zeta'' = (a')^*((\phi')^*\zeta')$, which proves the lemma.

To produce the diagram (5.15), choose atlases $X: X_0 \to X$, $X': X'_0 \to X$, $\psi: Y_0 \to Y$, $\psi': Y'_0 \to Y$, and define $Y''_0$ to be the fibre product $Y_0 \times_Y Y'_0$. Then $Y''_0$ is a manifold because atlases are representable morphisms. We form the cube

\[
\begin{array}{ccc}
X'' & \xrightarrow{\phi''} & Y''_0 \\
\downarrow{\phi_0} & & \downarrow{\beta_0} \\
X_0 & \xrightarrow{\alpha_0} & Y_0 \\
\downarrow{\alpha} & & \downarrow{\beta} \\
X & \xrightarrow{\phi} & Y \\
\downarrow{\psi} & & \downarrow{\psi'} \\
X' & \xrightarrow{\phi'} & Y' _0 \\
\end{array}
\]

where $X''_0$ is the weak pullback of the three squares that contain $Y$. Since these three squares are weak pullbacks, every face of the cube is a weak pullback of stacks. In particular, the top face is a pullback diagram of manifolds, so $X''_0$ is diffeomorphic to $X_0 \times_{Y_0} Y''_0$. The morphism $\chi'' = \chi \circ \alpha: X''_0 \to X$ is an atlas, so we can form a groupoid $X''_\circ = (X''_0 \Rightarrow X''_0)$ with arrow manifold $X''_1 = X''_0 \times_X X''_0$. The second dictionary lemma, [3, Lemma 2.5], applied to the map $\phi_0''$ and the stack morphism $\phi$, provides a lift of $\phi_0'$ to a groupoid morphism $\phi''': X'' \to Y''_0$ such that $B\phi'' \cong \phi$. Applying the lemma to the map $\alpha_0$ and the stack morphism $\text{id}_X$, we get a lift of $\alpha_0$ to a groupoid morphism $\alpha: X'_0 \to X$, such that $B\alpha \cong \text{id}_X$. Taking the map $\beta_0$ and the stack morphism $\text{id}_Y$, we get a lift of $\beta_0$ to a groupoid morphism $\beta: Y'' \to Y$, such that $B\beta \cong \text{id}_Y$. We have $B(\phi \circ \alpha) \cong \phi \cong B(\beta \circ \phi')$, so $\phi \circ \alpha \cong \beta \circ \phi'$ by the third dictionary lemma, [3, Lemma 2.6]. The morphisms $\alpha'$ and $\beta'$ and the 2-isomorphism $\phi' \circ \alpha' \cong \beta' \circ \phi''$ are defined similarly. QED
Symplectic stacks. Let X be an étale stack. A 2-form $\omega \in \Omega^2(X)$ is symplectic if $d\omega = 0$ and $\omega$ is nondegenerate, meaning the linear map $\text{Vect}(X) \to \Omega^1(X)$ given by contraction with $\omega$ is an isomorphism. The pair $(X, \omega)$ is a symplectic stack.

5.16. Proposition. Let $(X_\ast, \omega)$ be a 0-symplectic groupoid. Then $(B X_\ast, B \omega)$ is a symplectic stack, where $B \omega \in \Omega^2_{bas}(X_\ast)$ is the form corresponding to $\omega \in \Omega^2(B X_\ast)$ under the equivalence $\Omega^2(B X_\ast) \simeq \Omega^2_{bas}(X_\ast)$. Conversely, if $(X, \omega)$ is a symplectic stack and $X \simeq B X_\ast$, then $X_\ast$ is a 0-symplectic groupoid.

Proof. This is immediate from the definitions and Proposition 5.13. QED

From Proposition 5.11 and Proposition 5.16, we see that any presymplectic manifold $(X, \omega)$ gives rise to many symplectic stacks, each of which we may interpret as a stacky quotient of $X$ along the null foliation of $\omega$.

6. Lie 2-groups and Lie group stacks

This section starts with a review of Lie 2-groups, Lie group stacks, and their actions. We follow Baez and Lauda [1], Bursztyn, Noseda, and Zhu [8, §3.2], and Trentinaglia and Zhu [34], but make some minor modifications to the terminology to suit our purposes. We will refer to coherent group objects in a 2-category as weak group objects, and to group objects in the underlying 1-category as strict, or just plain, group objects. Of main interest to us are connected étale Lie group stacks, which according to [34] can always be strictified. We show that the action of an étale Lie group stack on an étale stack can also be strictified. We end with a discussion of basic features of étale Lie group stacks, such as the Lie 2-algebra, the adjoint action, the structure of compact étale Lie group stacks, and maximal stacky tori.

Lie 2-groups. A weak Lie 2-group is a coherent group object in the 2-category $\text{LieGpd}$ ([1, Definition 7.1]). A strict Lie 2-group is a group object in $\text{LieGpd}$, considered as a 1-category ([1, Definition 7.1]). We will refer to strict Lie 2-groups as simply Lie 2-groups.

More explicitly, a (strict) Lie 2-group is a Lie groupoid $G$, so that $G_1$ and $G_0$ are both Lie groups, and all the groupoid structure maps are Lie group homomorphisms. We will write $g \cdot h$ for the group product of $g, h \in G_1$, and $g \circ h$ for the groupoid product (composition) of composable $g, h \in G_1$. We denote the group identity of $G_1$ and $G_0$ by 1, and $u: G_0 \to G_1$ the groupoid identity bisection. We write $m: G_1 \times G_1 \to G_0$ for group multiplication in $G_\ast$, and note that $m$ is a Lie groupoid homomorphism. We use $(\cdot)^{-1}$ to denote both inverse with respect to the groupoid structure and the group structure on $G_\ast$; the meaning should be clear from context.

A Morita morphism of Lie 2-groups is a Morita morphism of the underlying Lie groupoids which preserves the group structure on both the manifolds of objects and arrows. Morita equivalence of Lie 2-groups is defined as a zigzag of Morita morphisms of Lie 2-groups.

The coarse quotient $G_0/G_1$ of a Lie 2-group $G$, is a (not necessarily Hausdorff) topological group, namely $G_0/G_1 = G_0/\ker(s)$. The coarse quotient is preserved under Morita equivalence.
An action of a Lie 2-group $G$, on a Lie groupoid $X$, is a Lie groupoid morphism $a : G \times X \to X$, with the property that the following diagrams 2-commute

$$
\begin{array}{ccc}
G \times X & \xrightarrow{a} & X \\
\downarrow{1 \times \text{id}} & = & \downarrow{\text{id} \times a} \\
X & \xrightarrow{a} & G \times X
\end{array}
$$

subject to some coherence conditions on the natural transformations; see for instance [8, § 3.2]. If the diagrams in (6.1) commute, the action is strict.

A strict action of a $G$, on $X$, is equivalent to the data of a morphism of Lie groupoids $G \times X \to X$, both of whose component maps

$$
\begin{align*}
G_0 \times X_0 & \to X_0: (g, x) \mapsto L(g)(x) = g \cdot x, \\
G_1 \times X_1 & \to X_1: (k, f) \mapsto L(k)(f) = k * f,
\end{align*}
$$

define Lie group actions.

6.3. Example. Consider the torus $G = T^n$ and its immersed subgroup $N$ of Examples 3.1 and 5.12. Let $\tilde{N} \to N$ be an étale homomorphism as in Example 5.12. Form the action groupoid $G_\ast := (\tilde{N} \ltimes G \rightrightarrows G)$, where $\tilde{N}$ acts on $G$ by left translations via $N$. Then $G_\ast$ has the structure of a Lie 2-group. The group structure on $\tilde{N} \ltimes G$ is the product group structure $\tilde{N} \times G$. Taking $X = (\tilde{N} \ltimes X_0 \rightrightarrows X_0)$, from the action of $G$ on $X_0$ we obtain an action $G \times X_0 \to X_0$, which on arrows is given by

$$(n, g) \cdot (n', x) = (nn', g \cdot x).$$

Lie 2-algebras. A strict Lie 2-algebra is a Lie groupoid $g_1 \rightrightarrows g_0$ so that each $g_i$ is a Lie algebra and the groupoid structure maps are Lie algebra homomorphisms. The Lie 2-algebra of a strict Lie 2-group $G_1 \rightrightarrows G_0$ is the Lie groupoid $\text{Lie}(G_\ast) := (g_1 \rightrightarrows g_0)$, where the Lie groupoid structure maps of $g_\ast$ are obtained by differentiating the structure maps of $G_\ast$ at the identity. In this case $G_\ast$ integrates $g_\ast$. Recall that the Lie algebroid of $G_\ast$ is denoted $\text{Alg}(G_\ast)$, not $\text{Lie}(G_\ast)$.

Morita morphisms and Morita equivalence of Lie 2-algebras are defined as for Lie 2-groups: Morita morphisms are structure-preserving morphisms which are Morita morphisms on the underlying Lie groupoid, and a Morita equivalence is a zigzag of Morita morphisms.

6.4. Lemma. (i) Let $\phi : G_\ast \to H_\ast$ be a Morita morphism of Lie 2-groups. Then $\text{Lie}(\phi) : g_\ast \to h_\ast$ is a Morita morphism of Lie 2-algebras.

(ii) Let $g$ and $g'$ be Lie algebroids, considered as Lie 2-algebras $g_1 \rightrightarrows g_0$ and $g'_1 \rightrightarrows g'_0$. If $g$ and $g'$ are Morita equivalent as Lie 2-algebras, then they are isomorphic as Lie algebroids.

Proof. (i) The functor “Lie” taking Lie groups to Lie algebras preserves fibered products and takes surjective submersions to surjective submersions. Therefore “Lie” preserves essential surjectivity and full faithfulness.

(ii) Let $\phi : h_\ast \to g$ be a Morita morphism of Lie 2-algebras. Then $\phi_0 : h_0 \to g$ is surjective, so $g \cong h_0 / \ker(\phi_0)$. Note that $t(\ker(s)) \subseteq \ker(\phi_0)$, since $\phi_0$ is a Lie groupoid morphism. Since $\phi_0$ is a Morita equivalence, we have

$$
\dim g = 2 \dim b_0 - \dim b_1 = \dim b_0 - \dim t(\ker(s)),
$$

and so $t(\ker(s)) = \ker(\phi_0)$. So $g \cong b_0 / t(\ker(s))$. Similarly, $g' \cong b_0 / t(\ker(s))$. QED
6.5. Example. The Lie 2-algebra of the 2-group \( G \), in Example 6.3 is \( \pi \ltimes g \to g \),
where \( \pi \ltimes g \) is isomorphic as a Lie algebra to the abelian Lie algebra \( \pi \oplus g \).

Crossed modules. A crossed module of Lie groups is a quadruple \((G, H, \partial, \alpha)\),
where \( G \) and \( H \) are Lie groups, \( \partial : H \to G \) is a Lie group homomorphism, and \( \alpha : G \to \text{Aut}(H) \) is an action of \( G \) on \( H \)
such that \( \alpha(g_1g_2) = \alpha(g_1)\alpha(g_2) \) for all \( g_1, g_2 \in G \). We will usually abbreviate \( \alpha(g)(h) \) to \( \delta h \). A morphism of crossed modules \((\psi_G, \psi_H) : (G, H, \partial, \alpha) \to (G', H', \partial', \alpha')\) is a pair of Lie group homomorphisms \( \psi_G : G \to G' \) and \( \psi_H : H \to H' \)
which commute with the structure maps and actions.

Recall the equivalence between crossed modules and strict 2-groups established by Brown and Spencer [7] (see also Baez and Lauda [1, §8.4]): a crossed module of Lie groups \((G, H, \partial, \alpha)\) gives rise to the Lie 2-group \( G \), with \( G_0 = G \) and \( G_1 = H \ltimes G \).
As a Lie groupoid, \( G \) is the action groupoid for the action of \( H \) on \( G \) by left translations of \( \partial(h) \). As a Lie group, \( H \ltimes G \) is the semidirect product of \( H \) and \( G \) with respect to the action \( \alpha : G \to \text{Aut}(H) \). Conversely, a 2-group \( G \) determines the crossed module \((G, H, \partial, \alpha)\): we have \( G_0 = G_0, H = \ker(\partial), \partial = t_H : H \to G \),
and \( \alpha \) is the conjugation action of \( G_1 \) on \( H \) composed with the identity bisection \( u : G_0 \to G_1 \). The Lie 2-algebra \( \text{Lie}(H \ltimes G) \) is bijective.

Many properties of 2-groups are more conveniently stated in terms of crossed modules. For instance, a Morita morphism of crossed modules is a morphism
\[(\phi_G, \phi_H) : (G, H, \partial, \alpha) \to (G', H', \partial', \alpha')\]
that induces group isomorphisms in “homology” \( \ker(\partial) \cong \ker(\partial') \) and \( \text{coker}(\partial) \cong \text{coker}(\partial') \). The kernel and the cokernel of \( \partial \) are Morita invariants of a crossed module \((G, H, \partial, \alpha)\). Note that the cokernel \( G/\partial(H) \) is the coarse quotient group of the corresponding 2-group. We omit the proof of the following, which is a straightforward exercise in the differential geometry of fibered products and elementary Lie theory.

6.6. Lemma. Let \( \phi : G \to G' \) be a morphism of Lie 2-groups, and let \((\phi_G, \phi_H)\) be the associated morphism of crossed modules. Then the following are equivalent:
(i) \( \phi \) is a Morita morphism of Lie 2-groups;
(ii) \((\phi_G, \phi_H)\) is a Morita morphism of crossed modules;
(iii) \( G' = \partial'(H')\phi_G(G) \) and the map \( \phi_H : H \to H' \rtimes G' \to G \) is bijective.

6.7. Remark. We record three special cases of the lemma. Let \((G, H, \partial, \alpha)\) be a crossed module. First, suppose we are given a closed subgroup \( G' \) of \( G \) with the property \( \partial(H)G' = G \). Then we can form the restricted crossed module \((G', H', \partial', \alpha')\) with \( H' = \partial^{-1}(G') \), and the inclusion \((G, H, \partial, \alpha)\) is a Morita morphism. Second, given a Lie group extension (e.g. a covering homomorphism) \( \phi : \tilde{G} \to G \) we can form the pullback extension \( \tilde{H} = H \ltimes \tilde{G} \) of \( H \). Define \( \tilde{\partial} : \tilde{H} \to \tilde{G} \)
by \( \tilde{\partial}(h, \tilde{g}) = \phi(g) \) and a \( \tilde{G} \)-action \( \tilde{\alpha} \) on \( \tilde{G} \) by \( \tilde{\alpha}(h, \tilde{g}) = (\phi(h)\tilde{g}, \tilde{g}^\alpha\tilde{g}^{-1}) \). Then \((\tilde{G}, \tilde{H}, \tilde{\partial}, \tilde{\alpha})\) is a crossed module and \( \phi \) induces a Morita morphism \((\tilde{G}, \tilde{H}, \tilde{\partial}, \tilde{\alpha}) \to (G, H, \partial, \alpha)\).
The third case is the second case run in reverse: given a closed normal subgroup \( N \) of \( G \) that is contained in \( \partial(H) \), we can form the quotient crossed module \((\tilde{G}, \tilde{H}, \tilde{\partial}, \tilde{\alpha})\) with \( \tilde{G} = G/N \) and \( \tilde{H} = H/\partial^{-1}(N) \), and we have a Morita morphism \((G, H, \partial, \alpha) \to (\tilde{G}, \tilde{H}, \tilde{\partial}, \tilde{\alpha})\).
We can also reformulate the notion of a Lie 2-group action in terms of the associated crossed module of Lie groups \((G, H, \partial, \alpha)\): a \(G_*\)-action consists of three smooth actions,

\[
\begin{align*}
(6.8) & \quad G \times X_0 \to X_0: \quad (g, x) \mapsto L(g)(x) = g \cdot x, \\
(6.9) & \quad G \times X_1 \to X_1: \quad (g, f) \mapsto L(g)(f) = g \ast f, \\
(6.10) & \quad H \times X_1 \to X_1: \quad (h, f) \mapsto L(h)(f) = h \ast f,
\end{align*}
\]

satisfying the following compatibility conditions: for all \(g \in G\) the pair of maps 
\(L(g): X_0 \to X_0\) and \(L(g): X_1 \to X_1\) is an endofunctor on \(X_*\), i.e.

\[
\begin{align*}
(6.11) & \quad g \ast u(x) = u(g \cdot x), \quad g \ast s(f) = s(g \ast f), \quad g \ast t(f) = t(g \ast f), \\
(6.12) & \quad g \ast (f_1 \circ f_2) = (g \ast f_1) \circ (g \ast f_2),
\end{align*}
\]

for all \(x \in X_0\) and \(f, f_1, f_2 \in X_1\) for which \(f_1 \circ f_2\) is defined; for all \(h \in H\) the map \(L(h): X_1 \to X_1\) is a natural transformation from the identity functor to the functor \(L(\partial(h))\), i.e.

\[
\begin{align*}
(6.13) & \quad s(h \ast f) = s(f), \quad t(h \ast f) = \partial(h) \cdot t(f), \\
(6.14) & \quad h \ast f = (h \ast u(t(f)) \circ f = (\partial(h) \ast f) \circ (h \ast u(s(f))),
\end{align*}
\]

for all \(f \in X_1\); and \((8h) = L(g) \circ L(h) \circ L(g)^{-1}\), i.e.

\[
(6.15) \quad g \ast (h \ast f) = 8h \ast (g \ast f)
\]

for all \(g \in G\), \(h \in H\), and \(f \in X_1\). Conversely, if a crossed module \((G, H, \partial, \alpha)\) acts on \(X_*\), the corresponding action of \(G\) on \(X_*\) is given by

\[
\begin{align*}
G_0 \times X_0 & \to X_0, \quad (g, x) \mapsto g \cdot x, \\
G_1 \times X_1 & \to X_1, \quad ((h, g), x) \mapsto h \ast (g \cdot x),
\end{align*}
\]

where we write \(G_0 = G\) and \(G_1 = H \rtimes_\alpha G\).

The next lemma records a simple consequence of these axioms. We use that every element \(\xi \in \mathfrak{g}\) gives birth to two vector fields, namely the vector field \(\xi_{X_1}\) induced via the \(G\)-action on \(X_1\) and the vector field \(\xi_{X_0}\) induced via the \(G\)-action on \(X_0\). Similarly, every \(\eta \in \mathfrak{h}\) gives birth to three vector fields: the vector field \(\eta_{X_1}\) induced via the \(H\)-action on \(X_1\), and the two vector fields \(\partial(\eta)_{X_1}\) and \(\partial(\eta)_{X_0}\) induced by the element \(\partial(\eta) \in \mathfrak{g}\) via the \(G\)-actions on \(X_1\) and \(X_0\).

**6.16. Lemma.** Let \(G_*\) be a strict Lie 2-group acting on a Lie groupoid \(X_*\). Let \((G = G_0, H, \partial, \alpha)\) be the crossed module associated with \(G_*\). Let \(\eta \in \mathfrak{h}\). The vector field \(\eta_{X_1}\) is tangent to the source fibres, right-invariant, and \(t\)-related to the vector field \(\partial(\eta)_{X_0}\). The vector field \(\eta_{X_1} - \partial(\eta)_{X_1}\) is tangent to the target fibres, left-invariant, and \(s\)-related to the vector field \(\partial(\eta)_{X_0}\).

**Proof.** Let \(h \in H\). Property (6.13) can be restated as \(s \circ L(h) = s\) and \(t \circ L(h) = L(\partial(h)) \circ t\). In other words, \(s: X_1 \to X_0\) is \(H\)-invariant and \(t: X_1 \to X_0\) is equivariant with respect to the homomorphism \(\partial: H \to G\). It follows that \(\eta_{X_1}\) is tangent to the fibres of \(s\) and that the vector fields \(\eta_{X_1}\) and \(\partial(\eta)_{X_0}\) are \(t\)-related. Let \(f \in X_1\) have source \(s(f) = x\) and target \(t(f) = y\). Right composition with \(f\), \(R(f)(f') = f' \circ f\), defines a map \(R(f): s^{-1}(y) \to s^{-1}(x)\), and it follows from (6.14) that

\[
L(h)(f) = R(f)(L(h)(u(y)))).
\]

Differentiating this identity with respect to \(h\) yields \(\eta_{X_1, f} = R(f)(\eta_{X_1, u(y)})\), which tells us that \(\eta_{X_1}\) is right-invariant. This proves the first assertion. The second
assertion is proved similarly, by considering the element \( k = \partial(h)^{-1}t = h\partial(h)^{-1} \in G_1 = H \rtimes G \) and the left composition map \( L(f): t^{-1}(y) \rightarrow t^{-1}(x) \), and by noting the properties
\[
s \circ L(k) = L(\partial(h)^{-1}) \circ s, \quad t \circ L(k) = t, \quad L(k)(f) = L(f)(L(k)(u(x))),
\]
which follow from (6.13) and (6.14).

QED

6.17. Example. The crossed module associated to the Lie 2-group \( G \) of Example 6.3 is the homomorphism \( \tilde{N} \rightarrow G \) obtained by composing the map \( \tilde{N} \rightarrow N \) with the inclusion \( N \rightarrow G \). The action of \( G \) on \( \tilde{N} \) is trivial.

**Foliation 2-groups.** Let \( G \) be a Lie 2-group with crossed module \((G = G_0, H, \partial, \alpha)\). The subgroup \( \partial(H) \) of \( G_0 \) is normal, so
\[
\text{Lie}(t)(\ker(\text{Lie}(s))) = \text{Lie}(\partial)(b)
\]
is an ideal of \( g \), and \( g/\text{Lie}(\partial)(b) \) is a Lie algebra. We call \( G \) a foliation 2-group if any of the following equivalent conditions hold: (1) \( G \) is a foliation groupoid; (2) the homomorphism \( \partial: H \rightarrow G \) has discrete kernel; or (3) the homomorphism \( \text{Lie}(\partial): h \rightarrow g \) is injective. When \( G \) is a foliation Lie 2-group, we will consider \( \mathfrak{h} \) as an ideal of \( g \). A foliation 2-group \( G \) is effective if \( \partial \) is injective. Being a foliation 2-group is a Morita invariant property, and so is being effective. We call \( G \) an étale 2-group if either of the following equivalent conditions holds: (1) \( G \) is an étale groupoid; or (2) \( H \) is discrete.

We require some basic structural results on foliation 2-groups. The first result says that the Lie 2-algebra of a foliation 2-group \( G \) is equivalent to the quotient Lie algebra \( g/\mathfrak{h} \), and that \( g/\mathfrak{h} \) is isomorphic to the Lie algebra of left-invariant basic vector fields on \( G \). We denote by \( \xi_L \) the left-invariant vector field on \( G \) induced by \( \xi \in g \). A basic vector field \([v_0, v_1] \in \text{Vect}_{\text{bas}}(G)\) is left-invariant if it is represented by a pair of left-invariant vector fields \((v_0, v_1) \in \text{Vect}(G_0) \times \text{Vect}(G_1)\).

6.18. Lemma. Let \( G \) be a foliation 2-group with crossed module \((G = G_0, H, \partial, \alpha)\), where \( G_0 = G \) and \( G_1 = H \rtimes G \). Let \( l = g/\mathfrak{h} \) be the quotient Lie algebra and \( \pi: g \rightarrow l \) the quotient map.

(i) **The map**

\[
\begin{array}{ccc}
g_1 & \xrightarrow{\text{Lie}(t)} & l \\
\downarrow & & \downarrow \\
g_0 & \xrightarrow{\pi} & l
\end{array}
\]

**is a Morita morphism of Lie 2-algebras.**

(ii) **A Morita morphism of 2-groups \( G \rightarrow G' \) induces an isomorphism \( g/\mathfrak{h} \rightarrow g'/\mathfrak{h}' \).**

(iii) The map \( f: g \rightarrow \text{Vect}(G_0) \times \text{Vect}(G_1) \) defined by \( f(\xi) = (\xi_L, (\text{Lie}(u)\xi)_L) \) descends to a Lie algebra embedding \( g/\mathfrak{h} \hookrightarrow \text{Vect}_{\text{bas}}(G) \), whose image is the Lie algebra of left-invariant basic vector fields of \( G \).

**Proof.** For simplicity, we write \( s = \text{Lie}(s) \) for the source map of \( g_s \), and similarly for the other structure maps.

(i) Since \( \pi \) and \( t \) are Lie algebra homomorphisms, to show that \((\pi, \pi \circ t)\) is a homomorphism of Lie 2-algebras it suffices to show \( t(\xi) - s(\xi) \in t(\ker s) = \mathfrak{h} \) for
all $\xi \in g_1$. Indeed, $\xi - u(s(\xi)) \in \ker s$ and

$$(t - s)(\xi) = (t - s)(\xi - u(s(\xi))) = t(\xi - u(s(\xi))).$$

Next we show that $(\pi, \pi \circ t)$ is a Morita morphism. Essential surjectivity is automatic since $\pi$ is a surjective linear map. For full faithfulness it is enough to show that the canonical map $(s, t): g_1 \to g_0 \times_{\ker h} g_0$ is a linear isomorphism. To show the map is injective assume $s(\xi) = t(\xi) = 0$. Then $\xi \in \ker(\partial: h \to g_0)$, which is 0 since $G_*$ is assumed to be a foliation groupoid. So $\xi = 0$ and the map is injective.

Surjectivity follows from counting dimensions, as in the proof of Lemma 6.4.

(ii) This follows from Lemma 6.4.

(iii) For $\xi \in g$ we have $s(u(\xi)) = t(u(\xi)) = \xi_L$, so $f(\xi) \in \mathfrak{L}(G_*)$ is as in (5.4). Let $(\xi_0, \xi_1) \in g_0 \times g_1$. Then the pair $(\xi_{0L}, \xi_{1L}) \in \text{Vect}(G_0) \times \text{Vect}(G_1)$ is in $\mathfrak{L}(G_*)$ if and only if $s(\xi_1) = t(\xi_1) = \xi_0$ if and only if $\xi_1 = u(\xi_0)$. The pair $(\xi_{0L}, (u(\xi_0)L)) \in \text{Vect}(G_0) \times \text{Vect}(G_1)$ is in $\mathfrak{L}(G_*)$ if and only if $\xi_0 \in h$, because the foliation of $G$ is given by the $h$-orbits. It follows that $f$ induces an isomorphism from $g/h$ onto the subalgebra of left-invariant elements of $\text{Vect}_{\text{bas}}(G_*)$.

QED

We will henceforth identify the Lie 2-algebra $\text{Lie}(G_*)$ of a foliation 2-group with the quotient Lie algebra $g/h$.

We call a foliation 2-group $G_*$ $\text{base-connected}$ (resp. $\text{base-simply connected}$) if $G_0$ is connected (resp. simply connected). We call $G_*$ $\text{base-connected}$ if it is base-connected and base-simply connected. In the remainder of this section we will without further mention apply Lemma 6.6, which allows us to define Morita morphisms of 2-groups in terms of their crossed modules. We will also use the restriction, extension, and quotient Morita morphisms of Remark 6.7.

6.19. **Lemma.** Every foliation 2-group is Morita equivalent to a base-simply connected étale 2-group.

**Proof.** Let $(G, H, \partial, \alpha)$ be a crossed module with $\ker(\partial)$ discrete. We must show that $(G, H, \partial, \alpha)$ is Morita equivalent to a crossed module $(G', H', \partial', \alpha')$ with $G'$ simply connected and $H'$ discrete. Let $\phi: \hat{G} \to G$ be a surjective étale homomorphism with $\hat{G}$ simply connected (which exists even if $G$ is not connected; see [6, Corollary 5.6]), and let $\hat{H} = H \times_G \hat{G}$ be the pullback cover. The extension $(\hat{G}, \hat{H}, \hat{\partial}, \hat{\alpha}) \to (G, H, \partial, \alpha)$ is a Morita morphism. Let $N$ be the identity component of $\hat{\partial}(\hat{H})$. Then $N$ is the connected immersed normal subgroup of $\hat{G}$ with Lie algebra $h$. It follows from [5, Proposition III.6.14] that $N$ is closed and the quotient $G' = \hat{G}/\hat{N}$ is simply connected. So we can form the quotient crossed module $(G', H', \partial', \alpha')$ with $H' = \hat{H}/\hat{\partial}^{-1}(N)$, and we have a Morita morphism $(\hat{G}, \hat{H}, \hat{\partial}, \hat{\alpha}) \to (G', H', \partial', \alpha')$. In conclusion we have defined a zigzag of Morita morphisms

$$
(G, H, \partial, \alpha) \longrightarrow (\hat{G}, \hat{H}, \hat{\partial}, \hat{\alpha}) \longrightarrow (G', H', \partial', \alpha')
$$

with $G'$ simply connected and $H'$ discrete.

QED

For any foliation 2-group $G_*$, the quotient $G_0/G_1 = G/\partial(\partial)$ (i.e. the largest Hausdorff quotient of the coarse quotient group $G_0/G_1$) is a Lie group, which is a Morita invariant of $G_*$. Compactness properties of this Lie group translate into compactness properties of the Lie algebra of $G_*$. 
6.20. **Lemma.** Let $G_*$ be an étale 2-group. If the Lie group $G_0/G_1$ is compact, then the Lie algebra of $G_*$ is compact.

**Proof.** Let $(G, H, \partial, \alpha)$ be the crossed module of $G_*$. We may assume without loss of generality that $G$ is connected. Let $N$ be the immersed subgroup $\partial(H)$ of $G$, let $\bar{N}$ be its closure, and let $\bar{n} = \text{Lie}(N)$ and $\bar{\bar{n}} = \text{Lie}(\bar{N})$. Since $N$ is 0-dimensional and normal, it is central in $G$, so $\bar{N}$ is central in $G$, so $\bar{\bar{n}}$ is a central subalgebra of $\mathfrak{g}$. Since $K = G/\bar{N}$ is compact Lie and its Lie algebra is $\mathfrak{g}/\bar{\bar{n}}$, the central extension of Lie algebras $\bar{\bar{n}} \hookrightarrow \bar{n} \twoheadrightarrow \mathfrak{k}$ is split. (To produce a splitting, start with an arbitrary linear right inverse $s$ of the projection $\bar{\bar{n}} \twoheadrightarrow \mathfrak{k}$. The adjoint action of $G$ on $\mathfrak{g}$ descends to an action of $K$, and the map $\bar{s} = \int_K \text{Ad}_k \circ \text{Ad}_k^{-1} \, dk$, where $dk$ is normalized Haar measure on $K$, is a Lie algebra splitting of $\mathfrak{p}$.) This shows that $\mathfrak{g} \cong \bar{\bar{n}} \oplus \mathfrak{k}$ is compact.

These lemmas can be made more precise when the coarse quotient $G_0/G_1$ is connected, as follows.

6.21. **Lemma.** Let $G_*$ be a foliation 2-group. The following conditions are equivalent.

(i) $G_0/G_1$ is connected;

(ii) $G_0/\bar{G}_1$ is connected;

(iii) the inclusion $G_*' \hookrightarrow G_*$ is a Morita morphism, where $G_*'$ is the full subgroupoid obtained by restricting $G_*$ to the identity component $G_0'$ of $G_0$;

(iv) $G_*$ is Morita equivalent to a base-1-connected étale 2-group.

**Proof.** Let $(G, H, \partial, \alpha)$ be the crossed module of $G_*$. The equivalence $(i) \iff (ii)$ is straightforward.

(i) $\implies$ (iii): If $G/\partial(H)$ is connected, then $G = \partial(H)G'$, where $G'$ is the identity component of $G$. Therefore the inclusion $(G', H', \partial', \alpha') \to (G, H, \partial, \alpha)$ is a Morita morphism, where $(G', H', \partial', \alpha')$ is the restricted crossed module with $H' = \partial^{-1}(H \cap G') = H \times_G G'$.

(iii) $\implies$ (iv): Given the Morita morphism of crossed modules $(G', H', \partial', \alpha') \to (G, H, \partial, \alpha)$, we let $\bar{G}$ be the universal cover of $G'$, and as in the proof of Lemma 6.19 we obtain a base-1-connected crossed module $(\bar{G}, \bar{H}, \bar{\partial}, \bar{\alpha})$ and a Morita morphism $(\bar{G}, \bar{H}, \bar{\partial}, \bar{\alpha}) \to (G', H', \partial', \alpha')$. Next we let $N$ the identity component of $\bar{\partial}(\bar{H})$, we form the quotient crossed module $(G'', H'', \partial'', \alpha'')$ with $G'' = \bar{G}/N$, and we obtain a zigzag of Morita morphisms

$$(G, H, \partial, \alpha) \leftarrow (G', H', \partial', \alpha') \leftarrow (\bar{G}, \bar{H}, \bar{\partial}, \bar{\alpha}) \rightarrow (G'', H'', \partial'', \alpha''),$$

where $G''$ is 1-connected and $H''$ is discrete.

(iv) $\implies$ (i): If $G_* \cong G_*'$ with $G_*'$ base-connected, then $G/\partial(H) = G''/\partial''(H'')$ is connected.

A foliation 2-group $G_*$ is of **compact type** if the Lie algebra $\mathfrak{g}$ is a compact Lie algebra and the coarse quotient $G_0/G_1$ is a compact topological space. We call $G_*$ **base-compact** if $G$ is compact.

6.22. **Proposition.** Let $G_*$ be a foliation 2-group. The following conditions are equivalent.

(i) $G_0/G_1$ is compact and connected;

(ii) $G_0/\bar{G}_1$ is compact and connected;

(iii) $G_*$ is Morita equivalent to a base-1-connected étale 2-group of compact type;
(iv) $G_*$ is Morita equivalent to a base-compact and base-connected étale 2-group.

Proof. Let $(G, H, \partial, \alpha)$ be the crossed module of $G_*$. The implication (i) $\implies$ (ii) is obvious.

(ii) $\implies$ (iii): Suppose $G/\partial(H)$ is compact and connected. By Lemma 6.21 we may assume without loss of generality that $G_*$ is base-1-connected and étale. It then follows from Lemma 6.20 that the Lie algebra $\mathfrak{g}$ is compact.

(iii) $\implies$ (iv): Suppose that $G_*$ is base-1-connected, étale, and of compact type. Then $G$ is isomorphic to the product $E \times K$ of a vector group $E$ and a 1-connected compact Lie group $K$. Let $\text{pr}_E : G \rightarrow E$ be the projection and let $N_E = \text{pr}_E(N)$, where $N = \partial(H)$. Then $\text{pr}_E$ induces a surjection $G/N \rightarrow E/N_E$, so $E/N_E$ is compact, so $N_E$ contains a basis $e_1, e_2, \ldots, e_k$ of $E$. Choose $n_i \in N$ with $\text{pr}_E(n_i) = e_i$; then the subgroup $L$ of $N$ generated by the $n_i$ is a discrete cocompact normal subgroup of $G$ isomorphic to $\mathbb{Z}^k$. Then $G' = G/L$ is compact and connected, and we can form the quotient crossed module $(G', H' = H/\partial^{-1}(L), \partial', \alpha')$, which is Morita equivalent to $(G, H, \partial, \alpha)$. The associated 2-group $G'$ is Morita equivalent to $G_*$ and is base-compact, base-connected, and étale.

(iv) $\implies$ (i): If $G_* \simeq G'$ with $G'$ base-compact and base-connected, then $G/\partial(H) = G'/\partial'(H')$ is compact and connected.

QED

6.23. Example. The Lie 2-group $G_*$ of Example 6.3 is a foliation 2-group of compact type. Its Lie 2-algebra $\mathfrak{n} \ltimes \mathfrak{g} \cong \mathfrak{g}$ (see Example 6.5) is Morita equivalent to the abelian Lie algebra $\mathfrak{g}/\mathfrak{n}$.

Lie group stacks. A weak Lie group stack is a coherent group object in the 2-category DiffStack. A strict Lie group stack is a group object in DiffStack, considered as a 1-category. The 2-functor $\mathbf{B}$ takes weak (resp. strict) Lie 2-groups to (resp. strict) Lie group stacks. We will usually abbreviate strict Lie group stack to Lie group stack.

A weak homomorphism of weak Lie group stacks $\phi : G \rightarrow G'$ is a weak monoidal functor in the sense of [1, §2]: a morphism $\phi$ of the underlying differentiable stacks, plus two 2-isomorphisms $\phi_m : \phi \circ m \Rightarrow m' \circ (\phi \times \phi)$ and $\phi_1 : \phi \circ 1 \Rightarrow 1'$ (where $m$ and $1$ denote multiplication and the unit, respectively), which are required to satisfy the coherence conditions stated on [1, p. 430]. Two weak Lie group stacks $G$ and $G'$ are equivalent, written $G \simeq G'$, if there are weak homomorphisms $\phi : G \rightarrow G'$ and $\phi' : G' \rightarrow G$ so that $\phi \circ \phi'$ and $\phi' \circ \phi$ are isomorphic to the identity on $G'$ and $G$, respectively. In particular, equivalent weak Lie group stacks are equivalent as differentiable stacks.

A weak Lie group stack $G$ is compact, resp. connected, if the underlying topological space (coarse moduli space) is compact, resp. connected. Our focus on the strict case is justified by the following strictification theorem, which says that a weak Lie group stack $G$ that is connected and étale is equivalent to a strict Lie group stack and has a particularly nice atlas.

6.24. Theorem (Trentinaglia and Zhu [34, Theorem 5.13]). Let $G$ be a weak Lie group stack. Then the following properties are equivalent.

(i) $G$ is connected and étale;

(ii) $G$ is equivalent to a connected étale strict Lie group stack;

(iii) $G$ is equivalent to the classifying stack $BG$, of a base-1-connected étale 2-group $G_*$;

(iv) there exist a 1-connected Lie group $G$ and an étale atlas $\psi : G \rightarrow G$ which is a weak homomorphism.
An equivalence $BG_s \to G$, where $G_s$ is a Lie 2-group and $G$ a Lie group stack, is a presentation of $G$. We will often use the following case of the strictification theorem.

**6.25. Corollary.** The following conditions on a weak Lie group stack $G$ are equivalent:

1. $G$ is compact, connected, and étale;
2. there exists a presentation $BG_s \cong G$, where $G_s$ is a base-compact, base-connected, étale 2-group;
3. there exist a compact connected Lie group $G$ and an étale atlas $\psi : G \to G$ which is a weak homomorphism.

**Proof.** Combine Theorem 6.24 with Proposition 6.22, using the fact that the coarse moduli space of $G$ is isomorphic to the coarse quotient of a presenting 2-group.

QED

**Presentations of equivalent Lie group stacks.** The following result states that the fibered product of two strict Lie group stacks (if it exists as a differentiable stack) is a weak Lie group stack. The proof is given in Appendix C.

**6.26. Theorem.** Let $G \to H$ and $G' \to H$ be weak homomorphisms of (strict) Lie group stacks, and assume that the fibered product of stacks $K = G \times_H G'$ is a differentiable stack. Then $K$ is naturally a weak Lie group stack, and the projections $K \to G$ and $K \to G'$ are weak homomorphisms.

We will deduce from this that if the classifying stacks of two Lie 2-groups are equivalent as Lie group stacks, then the two Lie 2-groups are Morita equivalent as Lie 2-groups. Let $G$ be a Lie group stack and suppose that we are given two presentations

$$\psi : BG_s \to G, \quad \psi' : BG'_s \to G,$$

where $G_s$ and $G'_s$ are Lie 2-groups. Composing with the quotient maps $G_0 \to BG_s$ and $G'_0 \to BG'_s$, we get two atlases $\psi_0 : G_0 \to G$ and $\psi'_0 : G'_0 \to G'$, both of which are weak homomorphisms. Because atlases are surjective representable submersions, the fibered product $K_0 = G_0 \times_{G_s} G'_0$ is (equivalent to) a manifold, and the projections $K_0 \to G_0$ and $K_0 \to G'_0$ are surjective submersions. On the other hand, by Theorem 6.26, $K_0$ is also a weak Lie group stack. It follows that $K_0$ is a Lie group. Moreover, the composition $K_0 \to G_0 \to G$ is an atlas which is a weak homomorphism. Repeating the argument we see that

$$K_1 = K_0 \times_{G_s} K_0 \cong G_1 \times_{G_s} K_0 \times_{G'_0} G'_1$$

is likewise a Lie group, and that the Lie groupoid $K_\ast = (K_1 \triangleright K_0)$ is a Lie 2-group equipped with two Morita morphisms of 2-groups $K_\ast \to G_s$ and $K_\ast \to G'_s$. We denote $K_\ast$ by $G_s \times_{G_s} G'_s$. This proves the first part of the next statement.

**6.28. Proposition.** Let $G$ be a (strict) Lie group stack. Given two presentations (6.27) by Lie 2-groups $G_s$ and $G'_s$, let $K_\ast$ be the Lie 2-group $G_s \times_{G_s} G'_s$.

1. $\psi$ and $\psi'$ induce Morita morphisms of Lie 2-groups $K_\ast \to G_s$ and $K_\ast \to G'_s$.
2. The maps $K_0 \to G_0$ and $K_0 \to G'_0$ are surjective submersions.
3. If $G$ is compact, connected, and étale, and if $G_s$ and $G'_s$ are of compact type, then $K_\ast$ is of compact type.
Proof. (ii) This follows from (i) and Lemma 6.4.

(iii) It follows from (i) that $K_0/K_1 \cong G_0/G_1$ is compact. Let $G$ be a 1-connected Lie group and $\chi: G \to G$ an étale atlas which is a weak homomorphism as in the strictification theorem, Theorem 6.24(iv). It follows from (ii) and from Lemma 6.18 that $\psi$ and $\psi'$ induce isomorphisms of Lie algebras $g_0/b \cong \text{Lie}(G)$ and $g'_0/b' \cong \text{Lie}(G)$. Therefore the Lie algebra $t'_0 := g_0 \times_{\text{Lie}(G)} t'_0$ is compact. We claim that $t'_0$ is isomorphic to $t_0 = \text{Lie}(K_0)$. Let $G_0 = G_0 \times_G G$ and $G'_0 = G'_0 \times_G G$. Consider the cube

$$
\begin{array}{c}
\tilde{K}_0 \\
\downarrow \\
\tilde{G}_0 \\
\downarrow \\
K_0
\end{array}
\quad
\begin{array}{c}
\quad \\
\downarrow \\
\quad \\
\downarrow \\
G
\end{array}
\quad
\begin{array}{c}
\quad \\
\downarrow \\
\quad \\
\downarrow \\
\tilde{G}'_0
\end{array}
\quad
\begin{array}{c}
\quad \\
\downarrow \\
\quad \\
\downarrow \\
\quad
\end{array}
\quad
\begin{array}{c}
\quad \\
\downarrow \\
\quad \\
\downarrow \\
G'
\end{array}
$$

where $\tilde{K}_0$ is the weak limit of the three squares which contain $G$. Since these squares are all weak pullbacks, every face of the cube is a weak pullback of stacks and we have $\tilde{K}_0 \cong \tilde{G}_0 \times_G \tilde{G}'_0$. Since $\chi: G \to G$ is étale, the vertical maps are all étale. In particular, $t_0 = \text{Lie}(K_0) \cong \text{Lie}(\tilde{K}_0)$. We then have

$$
t_0 \cong \text{Lie}(K_0) \cong \text{Lie}(\tilde{G}_0 \times_G \tilde{G}'_0) \cong \text{Lie}(\tilde{G}_0) \times_{\text{Lie}(G)} \text{Lie}(\tilde{G}'_0) \cong g_0 \times_{\text{Lie}(G)} t'_0 \cong t'_0
$$

as was claimed.

QED

6.29. Remark. Even if the 2-groups $G_*$ and $G'_*$ in part (iii) are base-connected, the fibred product $K_*$ is not necessarily base-connected. However, by Lemma 6.21(iii) $K_*$ has a Morita equivalent full subgroupoid $K'_*$ which is a sub-Lie 2-group, base-connected, and the maps $K'_0 \to G_0$ and $K'_0 \to G'_0$ are still surjective submersions.

The Lie algebra of an étale Lie group stack. Let $G$ be a connected étale Lie group stack. By Theorem 6.24 there exists a presentation $\psi: BG \to G$, where $G$ is a base-connected foliation 2-group. By Lemma 6.4 and Proposition 6.28(ii), a different choice of presentation $\psi': BG' \to G$ gives rise to an isomorphism $\text{Lie}(G_*) \cong \text{Lie}(G'_*)$, so $\text{Lie}(G)$ only depends on this choice up to isomorphism. If we pick $G$, to be étale as in Theorem 6.24(iii), we have $\text{Lie}(G) \cong \text{Lie}(BG_*) = g_0$. The embedding $\text{Lie}(G_*) \to \text{Vect}_{\text{bas}}(G_*)$ of Lemma 6.18(iii) fits into a commutative square

$$
\begin{array}{c}
\text{Lie}(G_*) \longrightarrow \text{Vect}_{\text{bas}}(G_*) \\
\downarrow \cong \\
\text{Lie}(G'_*) \longrightarrow \text{Vect}_{\text{bas}}(G'_*)
\end{array}
$$

and so gives rise to a well-defined Lie algebra embedding $\text{Lie}(G) \to \text{Vect}(G)$. The element $\xi_L \in \text{Vect}(G)$ corresponding to $\xi \in \text{Lie}(G)$ is the left-invariant vector
field induced by \( \xi \). An equivalence \( \phi: G \to G' \) of Lie group stacks induces an isomorphism of their Lie algebras, which we denote by

\[
\operatorname{Lie}(\phi): \operatorname{Lie}(G) \xrightarrow{\phi} \operatorname{Lie}(G').
\]

Let \( \xi \in \operatorname{Lie}(BG) = \mathfrak{g}_0/\mathfrak{b} \) and let \( \pi : \mathfrak{g}_0 \to \mathfrak{g}_0/\mathfrak{b} \) be the projection. A basic vector field \((\xi^0, \xi^1) \in \operatorname{Vect}_{\text{bas}}(G)\) with the property

\[
\pi \circ \operatorname{Lie}(s)(\xi^1) = \pi \circ \operatorname{Lie}(t)(\xi^1) = \pi(\xi^0) = \xi
\]

is said to represent \( \xi \). Every \( \xi \in \operatorname{Lie}(BG) \) has a representative in \((\xi^0, \xi^1) \in \operatorname{Vect}_{\text{bas}}(G)\), namely take any \( \xi^0 \in \pi^{-1}(\xi) \), and set \( \xi^1 = \operatorname{Lie}(u)(\xi^0) \).

**Actions of Lie group stacks.** An action of a (strict) Lie group stack \( G \) on a differentiable stack \( X \) is a map of stacks \( a: G \times X \to X \) so that the diagrams

\[
\begin{array}{ccc}
G \times X & \xrightarrow{a} & X \\
\downarrow{\text{id} \times \text{id}} & & \downarrow{a} \\
X & \xrightarrow{\text{id} \times a} & G \times X
\end{array}
\]

2-commute, subject to the same coherence conditions on the natural isomorphisms as in the groupoid case (6.1). If the diagrams commute, the action is strict. The 2-functor \( B \) takes (strict) actions of Lie 2-groups to (strict) actions of Lie group stacks.

If there are actions of \( G \) on \( X \) and \( X' \) given by morphisms \( a: G \times X \to X \) and \( G \times X' \to X' \), then a morphism \( \phi: X \to X' \) is \( G \)-equivariant if there is a 2-isomorphism \( \phi_2: \phi \circ a \Rightarrow a' \circ (\text{id} \times \phi) \) that satisfies the coherence conditions stated in [8, Definition 3.20]. Actions of connected étale Lie group stacks, like the stacks themselves, can be strictified in the sense of the following statement, which generalizes [21, Proposition 3.2]. The proof is in Appendix A.

**6.33 Theorem.** Let \( G \) be a connected étale Lie group stack acting on an étale differentiable stack \( X \). For every base-connected Lie 2-group \( G \), with \( BG \simeq G \) (the existence of which is guaranteed by Theorem 6.24) there exists a Lie groupoid \( X \), with \( BX \simeq X \) so that

(i) \( G \) acts strictly on \( X \);

(ii) identifying \( BG \simeq G \), the equivalence \( BX \simeq X \) is \( G \)-equivariant.

Theorem 6.24 and Theorem 6.33 together justify our focus on strict actions of foliation Lie 2-groups on foliation groupoids.

**Fundamental vector fields.** Let \( G \) be a connected étale Lie group stack, and fix a presentation \( G \simeq BG \). Let \( X \) be an étale stack and \( a: G \times X \to X \) an action. Let \( \xi \in \operatorname{Lie}(G) \). Define the fundamental vector field \( \xi_X \in \operatorname{Vect}(X) \) as

\[
\xi_X: X \simeq \star \times X \xrightarrow{\text{id} \times \text{id}} G \times X \xrightarrow{\xi \times \text{id}^0} TG \times TX \simeq T(G \times X) \xrightarrow{Ta} TX,
\]

where \( \xi_L: G \to TG \) is the left-invariant vector field determined by \( \xi \), the equivalence \( TG \times TX \simeq T(G \times X) \) follows from [18, Example 4.6], and \( Ta \) is the tangent morphism of the action \( a \) defined in [18, §3.1]. It is easy to check that \( \xi_X \) is a section of \( TX \to X \).

**6.35 Proposition.** Let a foliation Lie 2-group \( G \) act strictly on a foliation groupoid \( X \), and let \( \xi = (\xi^0, \xi^1) \in \mathfrak{g}_0 \times \mathfrak{g}_1 \).
(i) If $\text{Lie}(s)(\xi^1) = \xi^0$, then $\xi^1_{X_1}$ is $s$-related to $\xi^0_{X_0}$, and similarly for the target map. In particular, if $\text{Lie}(s)(\xi^1) = \text{Lie}(t)(\xi^1) = \xi^0$ then the pair induces a basic vector field on $X_*$:

$$\xi_{X_*} := (\xi^0_{X_0}, \xi^1_{X_1}) \in \text{Vect}_{\text{bas}}(X_*) .$$

We will call this the fundamental vector field of $\xi$ on $X_*$.

(ii) Let $\xi^0 \in g_0$. Then $\xi^1 = \text{Lie}(u)(\xi^0)$ is the unique element of $g_1$ with $\text{Lie}(s)(\xi^0) = \text{Lie}(t)(\xi^1) = \xi^0$.

(iii) Let $\xi^0 \in h = \text{Lie}(t)(\ker(\text{Lie}(s)))$ and $\xi^1 = \text{Lie}(u)(\xi^0)$. Then $(\xi^0_{X_0}, \xi^1_{X_1})$ is equivalent to $(0, 0)$ in $\text{Vect}_{\text{bas}}(X_*)$.

(iv) Put $X = BX$, and $G = BG_*$. Let $\xi \in \text{Lie}(G)$ and pick a representative $(\xi^0, \xi^1) \in \text{Vect}_{\text{bas}}(G_*)$ as in (6.31). Then the fundamental vector field $\xi_{X} \in \text{Vect}(X)$ is represented by $(\xi^0_{X_0}, \xi^1_{X_1}) \in \text{Vect}_{\text{bas}}(X_*)$.

**Proof.** (i) We must check that $\xi^1_{X_1}$ is $s$-related to $\xi^0_{X_0}$; this follows from $\text{Lie}(s)(\xi^1) = \xi^0$ and the definition of the fundamental vector field on a manifold. The same holds for the target map.

(ii) If $\text{Lie}(s)(\xi^1) = \text{Lie}(t)(\xi^1) = \xi^0$, then

$$\xi^1 \in \text{Lie}(u)(\xi^0) + \ker(\text{Lie}(s)) \cap \ker(\text{Lie}(t)).$$

Since $G_*$ is a foliation Lie 2-group, $\ker(\text{Lie}(s)) \cap \ker(\text{Lie}(t)) = \{0\}$, so $\xi^1 = \text{Lie}(u)(\xi^0)$.

(iii) Let $\xi^0 \in h$, and pick $\xi^0 \in \ker(\text{Lie}(s))$ so that $\text{Lie}(t)(\xi^0) = \xi^0$. Then by part (i), $\xi^0_{X_1}$ is parallel to the $s$-fibers of $X$, and is $t$-related to $\xi^0_{X_0}$. So $\xi^0_{X_0}$ is tangent to the foliation $\mathcal{F}$. Therefore, in $\text{Vect}_{\text{bas}}(X_*)$, the pair $(\xi^0_{X_0}, \xi^1_{X_1})$ is equivalent to $(0, 0)$.

(iv) This follows from the definition (6.34) of the fundamental vector field, and [18, Theorem 2.8].

**QED**

**The adjoint action.** The objects and arrows of a Lie 2-group $G_*$ each have the adjoint action on their Lie algebra

$$(6.36) \quad \text{Ad}: \ G_1 \times g_1 \to g_1, \quad \text{Ad}: \ G_0 \times g_0 \to g_0 .$$

By naturality of the adjoint action, these two actions together define a Lie groupoid morphism $\text{Ad}: \ G_0 \times g_0 \to g_0$, which is the *adjoint action* of $G_*$ on $\text{Lie}(G_*) = g_*$. A homomorphism of Lie 2-groups $G \to G'$ takes the adjoint action of $G_*$ on $g_0$ to the adjoint action of $G'_*$ on $g'_0$. In particular, Morita morphisms of Lie 2-groups preserve the adjoint action.

If $(G, H, \partial, \alpha)$ is the crossed module associated to $G_*$, then $\partial(H)$ is a normal subgroup of $G = G_0$. So the adjoint action of $G_0$ on preserves $\text{Lie}(\partial)(h) \subseteq g_0$. Similarly, the adjoint action of $G_1$ preserves $\ker(\text{Lie}(s)) + \ker(\text{Lie}(t))$. Identifying $g_0/b \cong g_1/(\ker(\text{Lie}(s)) + \ker(\text{Lie}(t)))$, we get an action of $G_0$ on $g_0/b$. If $G_*$ is a foliation Lie 2-group, the Morita map $(\pi, \pi \circ d\ell)$ of Lemma 6.18 is $G_*$-equivariant and the adjoint action can be written

$$\text{Ad}: \ G_* \times g_0/b \to g_0/b .$$

If $G = BG_*$ is an étale Lie group stack presented by $G_*$, we have the action obtained by applying the 2-functor $B$:

$$\text{Ad}: \ G \times \text{Lie}(G) \to \text{Lie}(G) .$$

Morita invariance of the adjoint action $\text{Ad}: \ G \times g_* \to g_*$, together with Proposition 6.28, gives that the adjoint action of $G$ depends on the presentation only up
to isomorphism of Lie(G). Once we have the adjoint action of G, on g₀/b, we can
dualize to define the coadjoint action of G, on ann(h) ⊆ g₀. Applying the 2-functor
B gives us the coadjoint action

\[ Ad^*: G \times \text{Lie}(G)^* \rightarrow \text{Lie}(G)^*. \]

This description depends on a choice of presentation of G only up to isomorphism
of Lie(G).

**Stacky tori.** Stacky tori play an analogous role to that of compact tori in the theory
of compact Lie groups. A 2-torus is a Lie 2-group which is Morita equivalent to a
foliation 2-group G, with the property that G₀ is a torus. A stacky torus is an étale
Lie group stack equivalent to BGₓ, where Gₓ is a 2-torus.

**6.37. Lemma.** Suppose that Gₓ is a foliation 2-group and that G₀ is connected and abelian.
Then G₁ is abelian and the action \( \alpha: G₀ \rightarrow \ker(s) \) is trivial.

**Proof.** Let \( g \in G \) and \( h \in H \). Then \( \partial(s h) = g \partial(h) g^{-1} = \partial(h) \) because G is abelian.
Therefore \( \partial(s h^{-1}) = 1 \), i.e. \( s h^{-1} \in \ker(\partial) \). In other words, for each \( h \in H \) the
map \( f(g) = s h^{-1} \) maps G to \( \ker(\partial) \). But Gₓ is a foliation groupoid, so \( \ker(\partial) \) is
discrete, and G is connected, so \( f \) is constant. Thus \( s h^{-1} = 1 h^{-1} = h^{-1} = 1 \), so \( s h = h \), i.e. the action \( \alpha: G \rightarrow \text{Aut}(H) \) is trivial. Hence for all \( h, h' \in H \) we have
\( h' = \alpha(h) h' = h h' \), i.e. \( h' = h h' \). It follows that \( G₁ = H \rightarrow G \) is abelian. QED

In the setting of Lemma 6.37 we will often denote the crossed module \((H, H, \partial, \alpha)\)
simply by \( \partial: H \rightarrow G \) or \( H \rightarrow G \). A quasi-lattice is a crossed module \( \partial: A \rightarrow E \),
where \( E \) is (the additive group of) a finite-dimensional real vector space and \( A \) is
a countable discrete abelian group, and where the image \( \partial(A) \) is required to span
\( E \) as a vector space. (This generalizes a definition of Prato [31], who assumes the
map \( \partial: A \rightarrow E \) to be injective.)

Our next result is a stacky analogue of the familiar fact that a torus is isomorphic
to the quotient of its Lie algebra by the exponential map. Recall from [34, § 5] that
the fundamental group \( \pi₁(G) \) of a Lie group stack is the set of equivalence classes
of maps \( S¹ \rightarrow G \) based at the identity of \( G \), modulo homotopy.

**6.38. Proposition.** Let \( G \) be a stacky torus and let \( Gₓ \) be a 2-torus with \( G \simeq B Gₓ \). The
crossed module of \( Gₓ \), is Morita equivalent to a quasi-lattice \( \partial: A \rightarrow E \). This quasi-lattice
is isomorphic to \( \pi₁(G) \rightarrow \text{Lie}(G) \), and hence is uniquely determined up to isomorphism
by \( G \).

**Proof.** We may assume that \( Gₓ \) is étale and that \( G₀ \) is a torus. By Lemma 6.37, \( G₁ \)
is abelian and \( G₀ \) acts trivially on \( G₁ \). We obtain our Morita equivalence from two
weak equivalences as in the diagram

\[
\begin{align*}
H & \leftarrow \tilde{H} \rightarrow A \\
\partial & \downarrow \partial \downarrow \partial \\
G \exp & \leftarrow \mathfrak{g} \rightarrow E.
\end{align*}
\]

Here \( \tilde{H} \) is the fibred product

\[ H \times_G \mathfrak{g} = \{ (h, \xi) \mid \partial(h) = \exp(\xi) \} \]
and $E$ is the quotient $E = g/h$. The kernel of the exponential map $\exp: g \to G$ is isomorphic to $\pi_1(G) = \text{Hom}(U(1), G)$, the fundamental group of $G$, and we have a short exact sequence

$$\pi_1(G) \hookrightarrow \tilde{H} \to H,$$

which shows that $\tilde{H}$ is an extension of $H$ by $\pi_1(G)$. The group $\tilde{H}$ contains a copy $\tilde{h}$ of $h$, namely the image of the embedding $h \to \tilde{H}$ which sends $\eta$ to $(\exp(\eta), \text{Lie}(\partial)(\eta))$. We have $\text{Lie}(\tilde{H}) \cong \tilde{h}$. We define $A = \tilde{H}/\tilde{h}$ to complete the diagram. Then $\text{Lie}(A) = 0$, so $A$ is discrete. The weak equivalence between $H \to G$ and $A \to E$ gives us a group isomorphism $G/\partial(H) \cong E/\partial(A)$, and hence a surjection $G \to E/\partial(A)$, which implies that $\partial(A)$ generates $E$ as a vector space, because $G$ is compact. This proves the existence of the quasi-lattice $\partial: A \to E$.

The uniqueness is proved as follows. Since $H \to G$ is weakly equivalent to $A \to E$, the crossed module of Lie algebras $h \to g$ is weakly equivalent to $\text{Lie}(A) \to \text{Lie}(E)$, where $\text{Lie}(A) = 0$ and $\text{Lie}(E) = E$. Therefore $E \cong \text{Lie}(G)$. The equivalence $G \cong [E/A]$ gives a fibration of stacks $A \to E \to G$ in the sense of Noohi [28, §5], and hence a long exact homotopy sequence, which yields $A \cong \pi_1(G)$. QED

**Maximal stacky tori.** Let $G$ be a compact connected étale Lie group stack. By the strictification theorem, Corollary 6.25, $G$ is equivalent to $BG_*$, where $G_*$ is a base-connected foliation 2-group of compact type. Let $(G, H, \partial, \alpha)$ be the crossed module associated to $G_*$. Select a maximal abelian subalgebra $t$ of $g = \text{Lie}(G_0)$, and let $T \subseteq G_0$ be the connected subgroup with Lie algebra $t$. Let $(T, H_T = \partial^{-1}(T), \partial, \alpha)$ be the restriction of the crossed module to $T$. By Lemma 6.37, the closed subgroup $H_T$ of $H$ is abelian and the action of $T$ on $H_T$ is trivial. The Lie 2-group $T_* : T_1 \to T_0$ with $T_0 = T$ and $T_1 = H_T \rtimes T$ is a 2-torus. Let $T = BT_*$. The morphism $T \hookrightarrow G$ induced by the inclusion $T_* \hookrightarrow G_*$ is a maximal stacky torus of $G$.

6.39. **Lemma.** Let $(\psi_G, \psi_H): (G, H, \partial, \alpha) \to (G', H', \partial', \alpha')$ be Morita morphism of crossed modules. Assume the Lie algebras $g$ and $g'$ are compact, and assume $\psi_G: G \to G'$ is surjective. Let $t$ and $t'$ be maximal abelian Lie subalgebras of $g$ and $g'$, respectively, chosen so that $t'$ contains $\text{Lie}(\psi_G)(t)$. Let $T \subseteq G$ and $T' \subseteq G'$ be the connected subgroups with Lie algebras $t$ and $t'$, respectively. Then the restriction

$$(\psi_T, \psi_{H_T}) := (\psi_G|_T, \psi_H|_{H_T}): (T, H_T, \partial, \alpha) \to (T', H'_T, \partial', \alpha')$$

is a Morita morphism.

**Proof.** Let us first show that $\psi_T: T \to T'$ is surjective. Since $g$ and $g'$ are compact, they decompose into $g = [g, g] \oplus z(g)$ and $g' = [g', g'] \oplus z(g')$, where $[g, g]$ and $[g', g']$ are the derived subalgebras of $g$ and $g'$, respectively, and $z(g)$ and $z(g')$ are the centers of $g$ and $g'$, respectively. Since $\psi_G$ is surjective, $\text{Lie}(\psi_G)$ maps $z(g)$ into $z(g')$. If $[g, g] = \bigoplus g_i$ is a decomposition into simple subalgebras, then $[g', g'] = \bigoplus g'_i$, where $g'_i = \text{Lie}(\psi_G)(g_i) \cong g_i$ or $g'_i = 0$. It follows that $\text{Lie}(\psi_G)(t)$ is a maximal abelian subalgebra of $g'$, and hence $\psi_T(T) = T'$.

Next we show that $H_T = \partial^{-1}(T) \cong T \times_{T'} (\partial')^{-1}(T')$. Since $(\psi_G, \psi_H)$ is a Morita morphism, we may identify $H = G \times_{G'} H'$ and $H_T = T \times_{T'} H'$. Assume that $t \in T$ and $h' \in H'$ with $\psi_T(t) = \partial'(h')$. Then $\partial'(h') \in T'$, so $h' \in (\partial')^{-1}(T')$. Thus $H_T = T \times_{T'} (\partial')^{-1}(T')$. The result now follows from Lemma 6.6. QED
The conjugation action $C : G \times G \to G$ is a strict action of $G$ on itself. Composing this action with a categorical point $g : \star \to G$ gives an equivalence $C_g : G \to G$, called conjugation by $g$.

6.40. Corollary. Let $G$ be a compact connected étale Lie group stack. Then, up to equivalence and conjugation by $G$, the choice of maximal stacky torus $T$ of $G$ does not depend on a choice of equivalence $G \simeq B G$.

Proof. This follows from Lemma 6.39, Proposition 6.28, and the uniqueness up to conjugation of the maximal torus of a compact group. QED

By Corollary 6.40, we see that a maximal stacky torus $T$ is maximal in the sense that, if $T' \to T$ is a sub-Lie group stack and $T'$ is a stacky torus, then there is some $g : \star \to G$ so that $T' \to G$ factors through the morphism $T \to G \xrightarrow{\ell_{\star}} G$.

7. Hamiltonian actions on groupoids and stacks

In this section we introduce Hamiltonian actions in the 2-categories of Lie groupoids and of differentiable stacks. Our notion of Hamiltonian actions extends that of Hamiltonian Lie group actions on stacks defined by Lerman and Malkin [21] in two ways: we allow our group objects to be étale Lie group stacks, and we allow our stacks to be non-separated. We show that every presymplectic Hamiltonian action can be integrated, in several different ways, to a Hamiltonian action of a foliation groupoid on a 0-symplectic groupoid (Theorem 7.2). The classifying functor $B$ takes any such Hamiltonian groupoid to a Hamiltonian stack. The converse is also true: any Hamiltonian stack arises from a Hamiltonian groupoid (Theorem 7.4). We then show that, for compact Lie group stacks, the moment map image is an invariant of a stacky Hamiltonian action and obtain the stacky convexity theorem (Theorem 7.6).

Hamiltonian actions on 0-symplectic Lie groupoids. Let $(X_*, \omega)$ be a 0-symplectic Lie groupoid, let $G_*$ be a foliation 2-group, and let $(G, H, \partial, a)$ be the associated crossed module. We assume the coarse quotient group $G_0 / G_1$ to be connected. A strict action $a : G_0 \times X_0 \to X_*$ is Hamiltonian if there is a morphism of Lie groupoids called the moment map

$$\mu = (\mu_0, \mu_1) : X_0 \to (g_0 / h)^* \cong \text{ann}(h) \subseteq g_0^*$$

which satisfies the following conditions:

(i) Let $\xi = (\xi^0, du(\xi^0)) \in g_0 \times g_1$, and let $\xi_{X_*}$ be the fundamental vector field of $\xi$ on $X_*$, as in Proposition 6.35. Then $d\mu_0 \xi = \iota_{\xi_{X_*}} \omega$, or, equivalently, $d\mu_0 \xi = \iota_{\xi_{X_*}} \omega_0$.

(ii) $\mu$ intertwines the $G_*$ action on $X_0$ and the $G_*$-action on ann$(h)$ coming from the coadjoint action, as defined in (6.36). Equivalently, $\mu_0$ intertwines the $G_0$-action on $X_0$ and the $G_0$-action on ann$(h)$.

In this case, the tuple $(X_*, \omega, G_*, \mu)$ is a Hamiltonian $G_*$-groupoid.

7.1. Example. The actions of Example 6.3 are Hamiltonian, with moment map $(\mu_0, \mu_0 \circ s)$ which was described in Example 3.1.
From manifolds to Lie groupoids. Let \((X, \omega, G, \mu)\) be a Hamiltonian presymplectic \(G\)-manifold with null foliation \(\mathcal{F} = \ker(\omega)\) and null ideal \(n = n(\mathcal{F})\). We will show we can integrate these data to a Hamiltonian Lie groupoid \((X_0, \omega, G_0, \mu)\), where \(X_0\) is a foliation groupoid that integrates \((X, \mathcal{F})\), and \(G_0\) is a foliation 2-group that integrates the Lie 2-algebra \(g_0 = (n \rtimes g \Rightarrow g)\). As usual we can integrate both \((X, \mathcal{F})\) and \(g_0\) in a number of different ways, but we have to integrate them in a compatible manner. The following theorem shows that the monodromy groupoid of \(X\) and the source-simply connected integration of \(g_0\) always work. If the action of \(G\) on the monodromy groupoid descends to an action on the holonomy groupoid, then we can take \(X_0\) to be the holonomy groupoid and \(G_0 = (N \rtimes G \Rightarrow G)\), where \(N = N(\mathcal{F})\) is the null subgroup, i.e., the immersed subgroup of \(G\) generated by \(n\).

7.2. Theorem. Let \(G\) be a Lie group and \((X, \omega, G, \mu)\) a presymplectic Hamiltonian \(G\)-manifold with null foliation \(\mathcal{F}\) and null ideal \(n\).

(i) There exists a source-simply-connected Lie 2-group \(G_\ast\) with object group \(G_0 = G\) and Lie 2-algebra \(\text{Lie}(G_\ast) = n \rtimes g\). This Lie 2-group is unique up to a unique isomorphism that induces the identity map of \(G\) and of \(n \rtimes g\).

(ii) Let \(X\) be a source-connected Lie groupoid over \(X_0 = X\) integrating \(\mathcal{F}\) and let \(\psi = \psi_{X_0} : \text{Mon}(X, \mathcal{F}) \to X\) be the universal morphism as in Theorem 5.2. There exists a \(G_\ast\)-action on \(X\), that extends the action of \(G_0 = G\) on \(X\) if and only if \(\ker(\psi)\) is preserved by the \(G\)-action, where \(\ker(\psi)\) is as in (5.2). This \(G_\ast\)-action is unique and it is Hamiltonian with respect to the 0-symplectic structure on \(X\), determined by \(\omega\).

(iii) Assume that the \(G\)-action preserves the kernel of the holonomy homomorphism \(\text{hol}: \text{Mon}(X, \mathcal{F}) \to \text{Hol}(X, \mathcal{F})\). Then the \(G_\ast\)-action on \(\text{Hol}(X, \mathcal{F})\) given by (ii) descends to a Hamiltonian action of the Lie 2-group \(N \rtimes G \Rightarrow G\), where \(N \subseteq G\) is the null subgroup.

Proof. (i) Let \(H\) be the universal cover of \(N\). We interpret elements of \(H\) as homotopy classes relative to endpoints of paths \(\nu: [0, 1] \to N\) starting at \(\nu(0) = 1\). Define the homomorphism \(\partial: H \to G\) by \(\partial(\nu) = \nu(1)\) and the action \(\alpha: G \to \text{Aut}(H)\) by \(\alpha(g) (\nu) = \delta(\nu) = [\tau \mapsto g(\tau)g^{-1}]\). This defines a crossed module \((G, H, \partial, \alpha)\) and hence a Lie 2-group \(G_\ast\) with \(G_0 = G\), simply connected source fibre \(\ker(s) = H\), and \(\text{Lie}(G_\ast) = n \rtimes g\). The uniqueness of \(G_\ast\) follows from the uniqueness of the simply connected group \(H\).

(ii) We start by showing that the \(G\)-action on \(X\) extends in at most one way to a \(G_\ast\)-action on \(X\). The \(G\)-action on \(X\) preserves the foliation \(\mathcal{F}\) and therefore induces an action on \(\text{Lie}(X_0) = T\mathcal{F}\) by Lie algebroid automorphisms. Since \(X_0\) is source-connected, by Lie’s theorems for Lie groupoids (see [26, §6.3]) there can exist no more than one \(G\)-action on \(X_0\) that is compatible with the action on \(\text{Lie}(X_0)\). We show that the \(H\)-action on \(X_1\) is unique by showing that the action of its Lie algebra \(\mathfrak{h}\) is unique. Let \(\eta \in \mathfrak{h}\). By assumption \(\partial: \mathfrak{h} \to g\) is an isomorphism onto \(n\), so the vector field \(\partial(\eta)_X\) is tangent to the foliation, and therefore \(\partial(\eta)_X = \rho(\sigma(\eta))\) for a unique section \(\sigma(\eta)\) of \(\text{Lie}(X_0) = T\mathcal{F}\). Let \(\sigma(\eta)_K\) be the right-invariant vector field on \(X_1\) determined by \(\sigma(\eta)\). By Lemma 6.16 we must have \(\sigma(\eta)_K = 0\). So the \(\mathfrak{h}\)-action on \(X_1\) is determined by the \(G\)-action on \(X\).

Next we show the existence of a \(G_\ast\)-action on \(X_\ast\). First consider the case of \(M_\ast = \text{Mon}(X, \mathcal{F})\), the monodromy groupoid of \(\mathcal{F}\). An element of \(M_\ast\) is a leafwise homotopy class relative to endpoints of a path \(\gamma: [0, 1] \to M_0 = X\) in a leaf of \(\mathcal{F}\). Let \([\nu] \in H\). The homotopy class of the path \(\nu \cdot \gamma: \tau \mapsto \nu(\tau) \cdot \gamma(\tau)\) depends only
on the homotopy classes $[\nu] \in H$ and $[\gamma] \in M_1$, so we have a well-defined action of $H$ on $M_1$ given by $[\nu] * [\gamma] = [\nu \cdot \gamma]$. Similarly, the homotopy class of the path $g \cdot \gamma: \tau \mapsto g \cdot \gamma(\tau)$ depends only on $g \in G$ and on the homotopy class $[\gamma] \in M_1$, so we have a well-defined action of $G$ on $M_1$ given by $g * [\gamma] = [g \cdot \gamma]$. The actions of $G$ on $M_0$ and on $M_1$ and the action of $H$ on $M_1$ satisfy the rules (6.11)–(6.15), and therefore combine to an action of $G$, on $M_0$.

Now consider the general case of a source-connected groupoid $X$, integrating $(X, \mathcal{F})$ and for which the kernel of $\psi: M_0 \to X$, is preserved by $G$. Then the $G$-action on $M_1$ descends to a $G$-action on $X_1 = M_1/\ker(\psi)$. As we saw in the discussion of uniqueness, the $G$-action on $X_0$ determines an action of $G$ on $X_1$. For each $\eta \in \mathfrak{h}$ the vector field $\eta_{X_1}$ is complete because it lifts to the complete vector field $\eta_{M_0}$. Hence, by the Lie-Palais theorem, the $G$-action on $X_1$ integrates to an $H$-action. The conditions (6.11)–(6.15) hold because they hold on $M_0$.

Finally, the moment map $\mu$ is invariant on leaves of the null foliation, so it defines a map of Lie groupoids $\mu: X, \rightarrow \text{ann}(\mathfrak{n})$. The map $\mu_0$ is $G$-equivariant by assumption.

(iii) It suffices to show that the kernel of the homomorphism $\partial: H \to G$ acts trivially on the arrows of $\text{Hol}(X, \mathcal{F})$. If $[\nu] \in \ker(\partial)$, then $\nu$ is a loop based at the identity of $N$. Let $S$ be a local transversal to $x \in X$, then $\text{hol}([\nu \cdot u(x)])$ is the germ of the holonomy action of the path $v(\tau) \cdot x: [0, 1] \times \{x\} \to X$ on $S$. This path extends to $v(\tau) \cdot S: [0, 1] \times S \to X$, and so the holonomy action on $S$ is just $v(1) \cdot S = v(0) \cdot S$, and therefore $\text{hol}([\nu \cdot u(x)]) = u(x)$. For $\gamma \in \text{Mon}(X, \mathcal{F})$, applying (6.14) gives

$$[\nu] \cdot \text{hol}([\gamma]) = \text{hol}([\nu \cdot \gamma]) = \text{hol}([\nu \cdot u(t(g))] \circ [\gamma]) = \text{hol}([\gamma]),$$

which proves the claim.

QED

**Hamiltonian actions on stacks.** Let $(X, \omega)$ be a symplectic stack, and let $G$ be a connected étale Lie group stack. An action $a: G \times X \to X$ is Hamiltonian if there is a map of stacks called the moment map $\mu: X \to (\text{Lie}(G))^\circ$, where $(\text{Lie}(G))^\circ$ is the linear dual of $\text{Lie}(G)$. We require that

(i) $d\mu^\xi = i_\xi \omega$ for all $\xi \in \text{Lie}(G)$, and

(ii) $\mu$ is $G$-equivariant with respect to the coadjoint action of $G$ on $\text{Lie}(G)^\circ$.

In this case $(X, \omega, G, \mu)$ is a Hamiltonian $G$-stack.

### 7.3 Definition.

An equivalence of Hamiltonian $G$-stacks

$$\phi: (X, \omega, G, \mu) \simeq (X', \omega', G', \mu')$$

is a pair $(\phi_X, \phi_G)$, where $\phi_X: (X, \omega) \simeq (X', \omega')$ is an equivalence of symplectic stacks, and $\phi_G: G \simeq G'$ is an equivalence of Lie group stacks, subject to the following conditions. The equivalence $\phi_G$ determines an action of $G'$ on $X$, which is

$$G' \times X \xrightarrow{\phi_X \times \text{id}} G \times X \xrightarrow{a} X,$$

where $\phi_G^{-1}$ is a weak inverse of $\phi_G$. The tuple $(X, \omega, G', \text{Lie}(\phi_G)^\circ \circ \mu)$ is a Hamiltonian $G'$-stack, where $\text{Lie}(\phi_G)$ is as in (6.30). We then require that

(i) $\phi_X$ is $G'$-equivariant, and

(ii) $\mu' \circ \phi_X \simeq \text{Lie}(\phi_G)^\circ \circ \mu$.

The preceding sections establish the following bijection up to equivalence between Hamiltonian stacks and groupoids.

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**STACKY HAMILTONIAN ACTIONS AND SYMPLECTIC REDUCTION**

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7.4. Theorem. (i) Let \((X_\ast, \omega, G_\ast, \mu)\) be a Hamiltonian \(G_\ast\)-groupoid. Then the associated stack \((B X_\ast, B \omega, B G_\ast, B \mu)\) is a Hamiltonian \(G\)-stack.

(ii) Let \((X, \omega, G, \mu)\) be a Hamiltonian \(G\)-stack. For every base-connected Lie 2-group \(G\), with \(B G_\ast \cong G\), there exists a Hamiltonian \(G_\ast\)-groupoid \((X_\ast, \omega, G_\ast, \mu)\) so that \((B X_\ast, B \omega, B G_\ast, B \mu)\) is equivalent to \((X, \omega, G, \mu)\) as a Hamiltonian \(G\)-stack.

Proof. (i) This follows from Proposition 5.16 and the definitions of Hamiltonian groupoids and stacks.

(ii) Let \(X_\ast\) and \(G_\ast\) be Lie groupoids presenting \(X\) and \(G\) as in Theorem 6.33, so that \(G_\ast \times X_\ast \rightarrow X_\ast\) is a strict action presenting \(a\). It follows from Proposition 5.16 that \(X_\ast\) is 0-symplectic Lie groupoid. The map \(X_0 \rightarrow X \xrightarrow{\mu} \text{Lie}(G)^*\) gives a moment map for the \(G_\ast\) action on \(X_\ast\), and \(X_\ast\) is a Hamiltonian \(G_\ast\)-groupoid. Finally, \(B X_\ast\) is equivalent to \(X\) as a Hamiltonian stack by construction.

The stacky moment body. Let \((X, \omega, G, \mu)\) be a Hamiltonian \(G\)-stack, where \(G\) is compact and connected. Fix a presentation \(G \cong \text{BG}_\ast\), where \(G_\ast\) is of compact type and base-connected. Choose a maximal stacky torus \(T\) of \(G\). Let \(g = \text{Lie}(G)\) and \(t = \text{Lie}(T)\). Then \(t \subseteq g\) is a maximal abelian subalgebra, and is in a natural way a direct summand of \(g\), so we can identify \(t^*\) with a subspace of \(g^*\). By Proposition 6.38, \(T\) is presented by the quasi-lattice \(\pi_1(T) \rightarrow t\). Choose a Weyl chamber \(C\) of \(t^*\). Let \(\mu(X) \subseteq \text{Lie}(G)^*\) be the set \(\mu(X) = \mu(X_0) \subseteq \text{Lie}(G)^*\) for any presentation \(B X_\ast \cong X\); then picking a different presentation of \(X\) leaves \(\mu(X)\) unchanged. Define the stacky moment body of \((X, \omega, G, \mu)\) to be the pair \((\Delta(X), T)\), where \(\Delta(X) = \mu(X) \cap C\). An equivalence of stacky moment bodies \((\Delta, T) \cong (\Delta', T')\) is an equivalence \(\phi_T : T \xrightarrow{\phi_T} T'\) so that \(\text{Lie}\(\phi_T)^*\(\Delta'\) = \Delta\).

7.5. Proposition. Let \(G\) be compact and connected and let \((X, \omega, G, \mu)\) be a Hamiltonian \(G\)-stack.

(i) Up to equivalence, the stacky moment body of \(X\) is independent of the choice of presentation of \(G\).

(ii) If \(\phi : (X, \omega, G, \mu) \cong (X', \omega', G', \mu')\) is an equivalence of Hamiltonian stacks, then the stacky moment body of \(X\) is equivalent to the stacky moment body of \(X'\).

Proof. (i) Fix a presentation \(G \cong \text{BG}_\ast\), where \(G_\ast\) is of compact type. If we choose a different presentation \(G \cong \text{BG}'_\ast\) then by Corollary 6.40, the maximal stacky torus \(T\) of \(G\) is equivalent to a maximal stacky torus \(T'\) of \(G'\). Let \(\phi_T : T \cong T'\) be this equivalence; then \(\text{Lie}(\phi_G)\) and \(\text{Lie}(\phi_T)\) are linear isomorphisms. We can pick a Weyl chamber \(C'\) of \(\text{Lie}(T')^*\) so that \(\phi_T\) gives an equivalence between the stacky moment body of \((X, \omega, G \cong \text{BG}_\ast, \mu)\) and that of \((X, \omega, G \cong \text{BG}'_\ast, \mu)\).

(ii) By part (i) we can assume we have fixed the same presentation \(G \cong G' \cong \text{BG}_\ast\). Then \(\mu(X) = \mu'(X')\) by condition (ii) of Definition 7.3.

We can now rephrase the main result of [24] in the language of stacks.

7.6. Theorem. Let \((X, \omega, G, \mu)\) be a Hamiltonian \(G\)-stack. If \((X, \omega, G, \mu)\) is equivalent to \((B X_\ast, B \omega, B G_\ast, B \mu)\), where \(X_0\) and \(G_0\) are compact and connected, and the action of \(G_0\) on \(X_0\) is clean, then \(\Delta(X)\) is a closed convex polyhedral set.

Proof. This follows from Proposition 7.5 and Theorem 3.2.

7.7. Example. Let \((X_\ast, \omega, G_\ast, \mu)\) be the Hamiltonian groupoid of Example 7.1. The Lie 2-group \(G\) presents a stacky torus \(G\). Let \((X, \omega, G, \mu)\) be the Hamiltonian
G-stack presented by \((X_\ast, \omega, G_\ast, \mu)\). Referring back to Example 3.1, we see that the image of the moment map \(\mu_0: X_0 \to \text{ann}(n)\) is the polyhedron

\[ P = \{ \eta \in \text{ann}(n) | \langle a_i, \eta \rangle \geq \lambda_i \text{ for } 1 \leq i \leq n \}. \]

Here \(a_i \in g/n\) is the image of the standard basis vector \(e_i \in g = \mathbb{R}^n\) under the projection \(g \to g/n\). We conclude that the stacky moment body is \(\Delta(X) = (P, G)\).

Following Prato [31] we call \(X\) a \textit{toric quasifold} associated with the stacky polyhedron \((P, G)\).

7.8. \textbf{Remark.} Lerman and Tolman [22] show that compact symplectic toric orbifolds (which can be thought of as certain proper Hamiltonian stacks, as in [21]) are classified by their moment polytopes which have positive integer labels attached to their faces. Let \(M\) be a toric \(T^k\)-orbifold and let \(n\) be the number of faces of the moment polytope \(\Delta\) of \(M\). The labels of \(\Delta\) can be thought of as defining a homomorphism \(\mathbb{Z}^n \to \mathbb{Z}^k\). More generally, we can consider homomorphisms \(\mathbb{Z}^n \to A\) labeling the stacky moment polytope of a compact Hamiltonian \(G\)-stack, where \(A \to E\) is a quasi-lattice presenting a stacky torus \(G\). We will expand on this perspective in future work, and hope to interpret the results of [22], [31], and [32] in a stacky context.

8. \textbf{Leafwise transitivity}

In this section we prove two basic structural results about 2-group actions on groupoids, with an eye toward the symplectic reduction theorem and the Duistermaat-Heckman theorem. We introduce the notion of a leafwise transitive Lie 2-group action. We show that if \((X_\ast, \omega, G_\ast, \mu)\) is a Hamiltonian \(G_\ast\)-groupoid, then a regular fiber of \(\mu\) is Morita equivalent to a Lie groupoid with a locally leafwise transitive \(G_\ast\)-action (Proposition 8.1). If \(G_\ast\) is a 2-torus acting leafwise transitively on a foliation groupoid \(X_\ast\), we show that \(X_\ast\) is isomorphic to an action groupoid (Proposition 8.4).

\textbf{Notation.} Throughout this section \(X\) denotes a foliation groupoid and \(G\) denotes a foliation 2-group acting on \(X\). We denote by \(\mathcal{F}\) the foliation of \(X_0\) defined by \(X_\ast\), and by \((G, H, \partial, \alpha)\) the crossed module associated to \(G_\ast\).

\textbf{Leafwise transitive and regular actions.} Let \(x \in X_0\) and \(h \in H\). Let \(f = h \ast \mu(x) \in X_1\). It follows from (6.13) that \(s(f) = x\) and \(t(f) = \partial(h) \cdot x\). This shows that the orbit of \(x\) under the \(\partial(H)\)-action is contained in the \(X_\ast\)-orbit of \(x\). We call the \(G_\ast\)-action on \(X\) \textit{leafwise transitive} if this inclusion is an equality, in other words if \(\partial(H) \cdot x = t(s^{-1}(x))\) for all \(x \in X_0\). We call the action \textit{locally leafwise transitive} if for every \(x \in X_0\) the image of the map \(h \mapsto T_x X_0\) defined by \(\eta \mapsto (\partial(\eta)_{X_0}) x\) is equal to \(T_x \mathcal{F}\). A leafwise transitive action is locally leafwise transitive. Conversely, if the action is locally leafwise transitive and if additionally \(G_\ast\) and \(X\) are both source-connected, then \(\partial(H) \cdot x = \mathcal{F}(x) = t(s^{-1}(x))\) for all \(x \in X_0\), so in particular the action is leafwise transitive. We call the \(G_\ast\)-action on \(X\) \textit{regular} under the following conditions: the \(G_\ast\)-action is locally leafwise transitive; the \(G\)-action on \(X_0\) is free; and if \(h \in H\) satisfies \(h \ast f = f\) for any \(f \in X_1\), then \(h \in \ker(\partial)\).
Regular form of the zero fibre. Let \((X_\bullet, \omega_\bullet, G_\bullet, \mu)\) be a Hamiltonian \(G_\bullet\)-groupoid. The zero fibre of \(\mu\) is the subgroupoid \(Z_\bullet = \mu^{-1}(0) = (\mu_1^{-1}(0) \Rightarrow \mu_0^{-1}(0))\) of \(X_\bullet\). We say that \(0\) is a regular value of \(\mu\) if \(0 \in (g_0/h)^\perp\) is a regular value of \(\mu_0: X_0 \to (g_0/h)^\perp\). By \([21, \S\ 3.9]\), \(Z_\bullet\) is a Lie subgroupoid of \(X_\bullet\) if \(0\) is a regular value. A regular form of \(Z_\bullet\) is a pair \((R_\bullet, \phi)\) with the following properties: \(R_\bullet\) is a foliation groupoid equipped with a regular \(G_\bullet\)-action, \(\phi: \mathcal{F}_\bullet \to Z_\bullet\), is a \(G_\bullet\)-equivariant Morita morphism, and the \(G\)-orbits of \(R_0\) are the leaves of the null foliation of the form \(\phi_0^*\omega_0 \in \Omega^2(R_0)\).

8.1. Proposition. Assume that \(0\) is a regular value of \(\mu\). Then there exists a regular form \((R_\bullet, \phi)\) of the zero fibre \(Z_\bullet = \mu^{-1}(0)\).

Proof. Let \(\mathcal{F}_0\) be the restriction of the foliation \(\mathcal{F}\) to \(Z_0\). Let \(D\) be the subbundle of \(TZ_0\) spanned by \(T\mathcal{F}_0\) and the fundamental vector fields of \(G\). Then \(\text{rank}(D) = \text{dim}(T\mathcal{F}) + \text{dim} g - \text{dim} h\) is constant because \(0\) is a regular value of \(\mu\). Let \(v \in \Gamma(T\mathcal{F})\) and \(\xi \in g\). The identity

\[
t([\xi, v], v) = (\mathcal{L}_{\xi}v)\omega_0 = \mathcal{L}_{\xi}v \cdot \omega_0 - v \cdot \mathcal{L}_{v}\xi \omega_0 = 0 - 0 = 0
\]

shows that \([\xi, v], v\) \(\in \Gamma(T\mathcal{F})\), so \(D\) is involutive. Therefore \(D = T\mathcal{F}\) for a unique foliation \(\mathcal{D}\) of \(Z_0\). This is the foliation denoted by \(g \triangleleft \mathcal{F}_0\) in \([23, \S\ 2]\); its leaves are the \(G\)-orbits \(G \cdot \mathcal{F}_0(x)\) of leaves of \(\mathcal{F}_0\). Let \(S \hookrightarrow Z_0\) be a complete transversal of \(\mathcal{D}\) and define \(\phi_0: G \times S \to Z_0\) by \(\phi_0(g, s) = g \cdot s \in Z_0\). Then \(\phi_0\) is transverse to \(\mathcal{F}_0\) and is complete, so it follows from Lemma 5.3 that the pullback groupoid \(R_\bullet = \phi_0^* Z_\bullet\) is a Lie groupoid and that the induced morphism \(\phi: R_\bullet \to Z_\bullet\) is a Morita morphism. The object manifold of \(R_\bullet\) is \(R_0 = G \times S\). Elements of \(R_1\) are tuples \((r, (j, x), f, (j', x')) \in R_0 \times Z_1 \times R_0\) where \(s(f) = j \cdot x\) and \(t(f) = j' \cdot x'\). Using the action of the crossed module \((G, H, \partial, \alpha)\) on \(Z_0\), we define an action on \(R_\bullet\) as follows.

\[
\begin{align*}
G \times R_0 & \longrightarrow R_0: \quad g \cdot (j, x) = (gj, x) \\
G \times R_1 & \longrightarrow R_1: \quad g \ast ((j, x), f, (j', x')) = ((gj, x), g \ast f, (gj', x')) \\
H \times R_1 & \longrightarrow R_1: \quad h \ast ((j, x), f, (j', x')) = ((j, x), h \ast f, \partial(h)j', x')).
\end{align*}
\]

These actions satisfy the conditions (6.11)–(6.15) and so determine a strict action of \(G\) on \(R_\bullet\). The morphism \(\phi\) is \(G_\bullet\)-equivariant, the \(G\)-action on \(R_0\) is free, and if \(h \in H\) fixes any tuple \(((j, x), f, (j', x')) \in R_1\), then \(\partial(h) = 1\). To check that the action of \(G\) on \(R_\bullet\) is locally leafwise transitive, we count dimensions. Since the \(G\)-action on \(R_0\) is free, the \(H\)-action \(y \mapsto \partial(h) \cdot y\) is locally free. Hence for every \(y \in R_0\) the infinitesimal orbit map \(h \mapsto T_y R_0\) is injective, so the dimension of its image is equal to \(\text{dim} h\). On the other hand, by Lemma 6.16 the image is contained in the fibre \(y\) of the Lie algebroid \(\text{Alg}(R_\bullet)\), which has rank equal to \(\text{dim} R_1 - \text{dim} R_0 = \text{dim} h\). This proves that the \(G_\bullet\)-action on \(R_\bullet\) is regular. Since \(0\) is a regular value of \(\mu\), the foliation \(\mathcal{D}\) of \(Z_0\) is transversely symplectic, so \((S, \omega_0|_S)\) is a symplectic manifold, and \((R_0, \phi_0^* \omega_0)\) is a presymplectic manifold with null foliation given by the \(G\)-orbits in \(R_0\).

QED

Foliations groupoids versus action groupoids. Recall that a Lie group bundle is a Lie groupoid where every arrow \(g\) has \(s(g) = t(g)\). For a group \(K\) acting on a manifold \(X\), let \(K_x \subset K\) denote the stabilizer of a point \(x \in X\).

8.2. Lemma. Let \(K\) be a 1-connected Lie group acting locally freely on a manifold \(X\) and let \(\mathcal{F}\) be the foliation of \(X\) into \(K\)-orbits. Let \(L = \ker(K \to \text{Diff}(X))\) be the kernel of the \(K\)-action. Assume that the set \(X_L = \{x \in X \mid K_x = L\}\) is dense in \(X\).
(i) The monodromy groupoid \( \text{Mon}(X, \mathcal{F}) \) is isomorphic to the action groupoid \( K \ltimes X \).

(ii) The holonomy groupoid \( \text{Hol}(X, \mathcal{F}) \) is isomorphic to the action groupoid \( (K/L) \ltimes X \).

(iii) Every source-connected Lie groupoid \( X_1 \Rightarrow X_0 = X \) integrating \( \mathcal{F} \) is isomorphic to one of the form \( (K \ltimes X)/Z \), where \( Z \rightarrow X \) is an open Lie group subbundle of the Lie group bundle \( L \times X \rightarrow X \); and \( X_1 \) is Hausdorff if and only if \( Z \) is closed.

Proof. The Lie groupoids \( K \ltimes X \) and \( (K/L) \ltimes X \) are source-connected, have Lie algebra isomorphic to \( T \mathcal{F} \), and \( K \ltimes X \) is source-1-connected. Therefore \( K \ltimes X \) is isomorphic to \( \text{Mon}(X, \mathcal{F}) \), which proves (i).

Let \( X \), be any source-connected Lie groupoid with base \( X_0 = X \) and Lie algebroid \( T \mathcal{F} \). By Theorem 5.1 the holonomy morphism \( \text{Mon}(X, \mathcal{F}) = K \ltimes L \rightarrow \text{Hol}(X, \mathcal{F}) \) factors as two surjective étale maps:

\[
\text{hol}: K \ltimes L \xrightarrow{\psi^{-1}} X, \xrightarrow{\text{hol}_{X_0}} \text{Hol}(X, \mathcal{F}).
\]

Let us first take \( X_0 = (K/L) \ltimes X \). To establish (ii) we need to prove that \( \text{hol}_{X_0} \) is an isomorphism, which amounts to showing that \( \ker(\text{hol}) = \ker(\psi_{X_0}) \).

Let \( H \) be the kernel of the \( \psi_{X_0} \) is the Lie group bundle \( L \times X \). Consider any pair \( (g, x_0) \in K \ltimes X \) contained in the kernel of \( \text{hol} \). The relation on \( X \) is defined by the action of \( H \) that \( x_0 = x_0 \) and, for a sufficiently small section \( S \) of the foliation \( \mathcal{F} \) at \( x_0 \), every \( x \in S \), the points \( x \) and \( g \cdot x \) are in the same plaque of the foliation near \( S \). The map \( \Phi: K \times S \rightarrow X \) that sends \( (g, x) \) to \( g \cdot x \) is étale at \( (1_k, x_0) \), so, after shrinking \( S \) if necessary, \( \Phi \) restricts to a diffeomorphism \( \phi: U \times S \rightarrow V \), where \( U \) is a neighbourhood of \( 1_k \in K \) and \( V \) a neighbourhood of \( x_0 \in X \). The inverse \( \phi^{-1} \) is a foliation chart at \( x_0 \). The slice \( g \cdot x \) of \( \mathcal{F} \) is transverse to the orbit \( K \cdot x_0 \) at \( x_0 \) and therefore \( (S) \) is small enough \( \phi^{-1}(g \cdot S) \subseteq U \times S \) is the graph of a unique smooth function \( f: S \rightarrow U \). Since \( x \) and \( g \cdot x \) are in the same plaque we have \( \phi^{-1}(g \cdot x) \in U \times \{x\} \), which is equivalent to \( g \cdot x = f(x) \cdot x \). Hence \( f(x) \in g^{-1}K_x \) for all \( x \in S \). The set \( X_L \) being preserved by the \( K \)-action and dense in \( X \), intersects \( S \) in a dense set. Thus \( f(x) \in g^{-1}L \) for a dense set of \( x \in S \). Since \( L \) is discrete, the smooth map \( f \) is constant on \( S \). It follows that \( f(x) = f(x_0) = 1_k \) for \( x \in S \), in other words that \( g \in L \) and \( (g, x_0) \in L \times X \). Thus \( \ker(\text{hol}) = L \times X = \ker(\psi_{X_0}) \), which proves (ii).

Returning to a general Lie groupoid \( X \), integrating \( \mathcal{F} \), as in the diagram (8.3), we see that \( X_0 = (K \ltimes X)/Z \), where \( Z = \ker(\psi_{X_0}) \) is a group subbundle of the Lie group bundle \( \ker(\text{hol}) = L \times X \). The Lie group bundles \( Z \) and \( L \times X \) are manifolds of dimension equal to that of \( X \), and therefore \( Z \) is open in \( L \times X \). The equivalence relation on \( K \ltimes X \) defined by the action of \( Z \) is closed if and only if \( Z \) is closed. This proves (iii).

QED

Specializing to the abelian case gives the following result.

8.4. Proposition. Let \( G \) be a 2-torus. Assume that \( G \) and \( X \) are source-connected. Also assume that \( X_0 \) is connected, \( X_1 \) is Hausdorff, the action of \( G \) on \( X \) is leafwise transitive, and the action \( H \xrightarrow{\mathcal{F} \to \text{Diff}(X_0)} H \) on \( X_0 \) is locally free. Then \( X \) is isomorphic to the action groupoid \( H/Z \ltimes X_0 \), where \( Z \) is a subgroup of the discrete subgroup \( \ker(H \rightarrow \text{Diff}(X_0)) \) of \( H \). The action of \( G_1 \equiv H \times G_0 \) on \( X_1 \equiv H/Z \times X_0 \) is given by \( h(g \cdot x) = (hZ, g \cdot x) \).

Proof. Assume first that \( H \) is 1-connected. Put \( X = X_0 \). Let \( L_G \) be the kernel of the action \( G \rightarrow \text{Diff}(X) \) and \( L_H = \partial^{-1}(L_G) \) the kernel of the action \( H \rightarrow \text{Diff}(X) \). Since
$H$ acts locally freely, $L_H$ is a discrete subgroup of $H$. Let $X_{L_G} = \{ x \in X \mid G_x = L \}$ and $X_{L_H} = \{ x \in X \mid H_x = L \}$. Then $X_{L_G} \subseteq X_{L_H}$. Since $G$ is a torus, it follows from the principal orbit type theorem (see e.g. [5, §IX.9, Theoreme 2]) that $X_{L_G}$ is dense in $X$. Hence $X_{L_H}$ is dense in $X$. Therefore we can apply Lemma 8.2(iii) to the $H$-action on $X$. An open and closed Lie group subbundle of $\ker(\text{hol}) = L_H \times X$ is a trivial bundle $Z \times X$ for some subgroup $Z$ of $L_H$, so we see that $X_1 = H/Z \times X_0$. The identification $\text{Mon}(X, \mathcal{F}) \cong H \ltimes X$ is $G_x$-equivariant with respect to the action of $G_1 = H \ltimes G$ on $H \ltimes X$ given by $(h, g) \cdot (k, x) = (hk, g \cdot x)$. This action descends to the action $(h, g) \ast (kZ, x) = (hkZ, g \cdot x)$ on $H/Z \times X_0 \cong X_1$. If $H$ is not simply connected, we can apply the previous argument to the crossed module $\tilde{H} \to G$, where $\tilde{H}$ is the universal cover of $H$, to get that $X_1 \cong \tilde{H}/\tilde{Z} \times X_0$, where $\tilde{Z}$ is a subgroup of $\tilde{Z}$. But the action of $\tilde{H}$ on $X_1$ descends to an action of $H$, so $N = \ker(\tilde{H} \to H)$ is a subgroup of $\tilde{Z}$. Write $Z = \tilde{Z}/N$; then $H/Z \times X \cong \tilde{H}/\tilde{Z} \times X$. QED

9. Symplectic reduction

The main result of this section is Theorem 9.1, which is an extension of the Meyer-Marsden-Weinstein symplectic reduction theorem to the 2-category of Hamiltonian $G$-stacks, where $G$ is an étale Lie group stack. The theorem is valid under the assumption that $0$ is a regular value of the moment map $\mu: X \to \text{Lie}(G)^*$, which ensures that the zero fibre $\mu^{-1}(0)$ is a differentiable stack. The group stack $G$ is not required to be compact or separated, nor is it required to act freely or properly. A quotient stack $\mu^{-1}(0)/G$ then exists and is symplectic, provided that a “second-order” freeness condition is fulfilled. This second-order condition holds automatically if $G$ is equivalent to a Lie group. If it fails, the quotient does not exist as a 1-stack, although it might still exist as a higher-order stack. To keep the size of this paper within reasonable limits we have omitted any discussion of reduction at nonzero levels. We draw the reader’s attention to the recent preprint [2], which handles a special case of our situation, namely symplectic reduction of toric quasifolds by stacky tori.

Notation and conventions. In this section $G$ denotes a connected étale Lie group stack and $(X, \omega, G, \mu)$ a Hamiltonian $G$-stack. Throughout we assume $0 \in \text{Lie}(G)^*$ to be a regular value of $\mu$ in the sense of [21, §3.9], and we let $i: Z \to X$ be the inclusion of the fibre $Z = \mu^{-1}(0)$. A symplectic reduction (at 0) of $X$ is a triple $(Y, p, \omega_Y)$ consisting of an étale stack $Y$, a stack morphism $p: Z \to Y$ which is a principal $G$-bundle in the sense of [8, Definition 3.24], and a symplectic form $\omega_Y \in \Omega^2(Y)$ with the property $p^* \omega_Y = i^* \omega$. As a consequence of [8, Theorem 5.2] and Remark B.4, if a symplectic reduction exists it is unique up to equivalence.

Symplectic reduction. The goal of this section is to prove the following theorem, which provides a necessary and sufficient condition for a symplectic reduction to exist. The theorem is formulated in terms of a presentation of the Hamiltonian stack. We choose a base-connected Lie 2-group $G$, presenting $G$, which exists by the strictification theorem (Theorem 6.24), and we let $(G, H, \partial, \alpha)$ be the crossed module of $G$. We also choose a Hamiltonian $G_{\ast}$-groupoid $(X_{\ast}, \omega, G_{\ast}, \mu)$ presenting $(X, \omega, G, \mu)$, which exists by Theorem 7.4(ii). Then the fibre $Z_{\ast} = \mu^{-1}(0)$ presents the stack $Z = \mu^{-1}(0)$. Next we choose a regular form $(\partial_{\ast}, \phi)$ of $Z_{\ast}$, which exists by Proposition 8.1. The group $H$ acts locally freely on $R_{\ast}$.  


9.1. **Theorem.** A symplectic reduction of $X$ exists if and only if $H$ acts freely on $R_1$. If $H$ acts freely on $R_1$, the 0-symplectic groupoid $(G \times^H R_1, \alpha^{red})$ defined in Lemma 9.2 below presents a symplectic reduction of $X$.

We will give the proof after establishing some auxiliary results. The notation will be as in the statement of the theorem.

9.2. **Lemma.** Assume that $H$ acts freely on $R_1$. Then the orbit space $R_1/H$ is a (not necessarily Hausdorff) manifold and the projection $p: R_1 \rightarrow R_1/H$ is a principal $H$-bundle. The associated bundle $G \times^H R_1$ with fibre $G$ is the arrow manifold of a foliation groupoid $G \times^H R_1 = (G \times^H R_1 \Rightarrow R_0)$. The presymplectic form $\phi^* \omega_0 \in \Omega^2(R_0)$ defines a 0-symplectic form $\alpha^{red}$ on $G \times^H R_1$.

**Proof.** The source map $s: R_1 \rightarrow R_0$ is $H$-invariant. Since the action of $G$, on $R_1$, is locally leafwise transitive, the kernel of $(ds)_x$ is precisely the span of the fundamental vector fields at $x$. The first assertion now follows from Lemma 9.3 below. The associated bundle $G \times^H R_1$ is the quotient of $G \times R_1$ by the action $h \cdot (g, f) = (g \phi(h^{-1}), h \cdot f)$. Using the local trivializations of $R_1 \rightarrow R_1/H$, one gives $G \times^H R_1$ a smooth manifold structure which makes the projection $G \times R_1 \rightarrow G \times^H R_1$ a surjective submersion. Let $[g, f] \in G \times^H R_1$ denote the equivalence class of the pair $(g, f) \in G \times R_1$. We define the groupoid $G \times^H R_1$ as follows. For $[g, f], [g', f'] \in G \times^H R_1$ and $x \in R_0$ put

$$s([g, f]) = s(f), \quad t([g, f]) = g \cdot t(f), \quad u(x) = [1, u(x)],$$

$$[g, f] \cdot [g', f'] = [g g', ((g')^{-1} \cdot f) \circ f'],$$

$$[g, f]^{-1} = [g^{-1}, (g \cdot f)^{-1}] = [g^{-1}, g \cdot f^{-1}].$$

It follows from (6.11)–(6.15) that these structure maps are well defined. Because $G$ is connected, the form $\alpha^{red}_0 = \phi^* \omega_0$ is $G$-invariant. Since the action of $H$ preserves source fibers, the form $\phi^* \omega_1$ is $H$-basic, and so descends to a form $\alpha^{red}_1$ on $G \times^H R_1$. The pair $\alpha^{red} = (\omega^{red}_0, \alpha^{red}_1)$ is a basic form on the groupoid $G \times^H R_1$. Since the $G$-orbits of $R_0$ are the leaves of the null foliation of $\omega^{red}_0$, the form $\alpha^{red}$ is 0-symplectic.

Lemma 9.2 makes use of the following fact, which is part of [26, Lemma 5.5].

9.3. **Lemma.** Let $X$ be a (possibly non-Hausdorff) manifold and let $G$ be a Lie group with a smooth free action $a: G \times X \rightarrow X$. There is a (necessarily unique) smooth (possibly non-Hausdorff) manifold structure on orbit space $X/G$ such that the quotient map $X \rightarrow X/G$ is a principal $G$-bundle, if and only if there exist a (possibly non-Hausdorff) manifold $Y$ and a smooth map $f: X \rightarrow Y$ which is $G$-invariant and satisfies $\ker(d(f))_x = T_x(G \cdot x)$ for all $x \in X$.

The next proposition completes one direction of the proof of Theorem 9.1.

9.4. **Proposition.** The Lie groupoid morphism $\psi: R_1 \rightarrow G \times^H R_1$, defined by $\psi_0 = \id_{R_0}$ and $\psi_1(f) = [1_G, f]$ is a principal $G$-bundle in the sense of Definition B.1, and satisfies $\psi^* \omega^{red} = \phi^* \omega$.

**Proof.** The statement $\psi^* \alpha^{red} = \phi^* \omega$ holds because $\psi^* \alpha^{red}_0 = \omega^{red}_0 = \phi^* \omega_0 \in \Omega^2(R_0)$. To show that $\psi$ is a principal $G$-bundle we will verify conditions (i)–(iii) of Definition B.1. Condition (i) is obvious: $\psi$ is essentially surjective because $\psi_0 = \id_{R_0}$. To
First we show that \( \tau \) be the map is a surjective submersion. Since the action of \( H \) \( G \), the action. Recall that \( G_0 = G \) and \( G_1 = H \rtimes_n G \), and define \( \gamma: \psi \circ \text{pr}_2 \Rightarrow \psi \circ a \) to be the map \( \gamma: G \times R_0 \to G \times^H R_1, \quad (g, x) \mapsto [g, u(x)]. \)

Let \( f \in R_1 \) be an arrow in \( R \), from \( x \) to \( y \), and let \( ((h, g), f) \in G_1 \times R_1 \) be an arrow from \( s((h, g), f) = (g, x) \) to \( t((h, g), f) = (\partial(h)g, y) \). Then the following diagram in \( G \times^H R \), commutes:

\[
\begin{array}{c}
\psi \circ \text{pr}(g, x) = x & \xrightarrow{\gamma(g, x) = [g, u(x)]} & \psi \circ a(g, x) = g \cdot x \\
\psi \circ \text{pr}(h, g, f) = [1, f] & \downarrow & \psi \circ a(h, g, f) = [1, h \cdot (g \cdot f)] \\
\psi \circ \text{pr}(\partial(h)g, y) = y & \xrightarrow{\gamma(\partial(h)g, y) = [\partial(h)g, u(y)]} & \partial(h)g \cdot y.
\end{array}
\]

Thus \( \gamma \) is a natural transformation. It is automatically a natural isomorphism, because \( \text{LieGpd} \) is a \((2, 1)\)-category. The higher coherence conditions on \( \gamma \) are verified in a similar way. This shows that \( \psi \) is \( G_r \)-invariant. To check condition (iii) we first describe the groupoid

\[
P_r := R_r \times^{(u)}_{G \times^H R_r} R_r,
\]

which is a weak fibered product as described in Appendix B. The object and arrow manifolds are

\[
P_0 = R_0 \times_{R_0} (G \times^H R_1) \times_{R_0} R_0 \equiv G \times^H R_1,
\]

\[
P_1 = R_1 \times_{R_0} (G \times^H R_1) \times_{R_0} R_1.
\]

The source and target maps of \( P_r \) can be written

\[
s(r, [g, f], f') = [g, f], \quad t(r, [g, f], f') = [g, (g^{-1} \cdot f') \circ f \circ r^{-1}].
\]

We must show that the canonical morphism \( \tau = (\text{pr}_2, a): G_r \times R_r \to P_r \) is a Morita morphism. On objects this is the map \( \tau_0: G_0 \times R_0 \to P_0 \) given by

\[
\tau_0(g, x) = \gamma(g, x) = [g, u(x)],
\]

and on arrows this is the map \( \tau_1: (H \rtimes_n G) \times R_1 \to P_1 \) given by

\[
\tau_1((h, g), f) = (f, \gamma(s((h, g), f)), h \cdot (g \cdot f)) = (f, [g, u(s(f))], h \cdot (g \cdot f)).
\]

First we show that \( \tau \) is essentially surjective, i.e. the map

\[
t \circ \text{pr}_1: P_1 \times_{R_0 \times \tau_0} (G \times R_0) \to P_0
\]

\[
((r, [g, u(s(r))], f'), (g, u(s(r)))) \mapsto [g, (g^{-1} \cdot f') \circ r^{-1}]
\]

is a surjective submersion. Since the action of \( H \) on \( R_1 \) preserves source fibers and is free, whenever \([g, u(x)] = [g', u(x')] \in G \times^H R_1\) we must have \( g = g' \) and \( x = x' \). It follows that we have a diffeomorphism \( \lambda: P_1 \times_{R_0} (G \times R_0) \equiv (R_1 \times G) \times_{R_0} R_1 \) given by

\[
\lambda((r, [g, u(s(r))], f'), (g, u(s(r)))) = (r, g, f'),
\]

where

\[
(R_1 \times G) \times_{R_0} R_1 = \{(r, g, f') \in R_1 \times G \times R_1 \mid g \cdot s(r) = s(r') \}.
\]
Under the isomorphism \( f \) the map \( \tau \circ \text{pr}_1 \) becomes the map \( \kappa : (R_1 \times G) \times_{R_0} R_1 \to P_0 = G \times^H R_1 \) given by

\[
\kappa(r, g, r') = [g, (g^{-1} \ast r') \circ r^{-1}].
\]

For surjectivity, let \([g, f] \in P_0\). Then \((f^{-1}, g, g \ast u(t(f))) \in (R_1 \times G) \times_{R_0} R_1\) and \(\kappa(f^{-1}, g, g \ast u(t(f))) = [g, f]\). For submersivity, note that there is a \(G\)-action on \((R_1 \times G) \times_{R_0} R_1\) given by

\[ g \cdot (r, g', r') = (g \ast r, g'g^{-1}, r') \]

and also a \(G\)-action on \(P_0 = G \times^H R_1\) given by

\[ g \cdot [g', f] = [g'g^{-1}, g \ast f]. \]

The map \(\kappa\) is \(G\)-equivariant. So it suffices to show that, for each \([1, f] \in P_0\) and for any \((r, 1, r') \in (R_1 \times G) \times_{R_0} R_1\) with \(r' = r\), there is a local section \(\sigma \colon P_0 \to (R_1 \times G) \times_{R_0} R_1\) of \(\kappa\) with \(\sigma[1, f] = (r, 1, r')\). But this follows from the fact that the multiplication map of \(R_1\) is a submersion. Therefore \(\tau\) is essentially surjective. Next we show \(\tau\) is fully faithful. Consider the fibred product

\[ M = ((G \times R_0) \times (G \times R_0)) \times_{P_0 \times P_0} P_1 \]

with respect to the maps \(\tau_0 \circ \tau_0 : (G \times R_0) \times (G \times R_0) \to P_0 \times P_0\) and \((s, t) : P_1 \to P_0 \times P_0\). A typical element of \(M\) is a tuple \([g, s(r)], (g', x), (r, [g, u(s(r))], r') \in (G \times R_0) \times (G \times R_0) \times P_1\) satisfying

\[
[g', u(x)] = t(r, [g, u(s(r))], r') = [g, (g^{-1} \ast r') \circ r^{-1}],
\]

where \(x = t(r)\) because \(h \ast u(x) = (g^{-1} \ast r') \circ r^{-1}\) for some \(h \in H\), and because the \(H\)-action preserves the \(s\)-fibers. The universal property of the fibred product \(M\) yields a canonical map \(\chi : (H \times_a G) \times R_1 \to M\) given by

\[
\chi(h, g, f) = ((g, s(f)), (\partial(h)g, t(f)), (f, [g, u(s(f))], h \ast (g \ast f))).
\]

We must show \(\chi\) is a diffeomorphism. For \(r, r' \in R_1\) in the same \(H\)-orbit, let \(\delta(r, r')\) be the unique element \(h \in H\) satisfying \(hr = r'\). The map \(\delta : R_1 \times_{R_1/H} R_1 \to H\) is smooth, because \(R_1 \to R_1/H\) is a principal \(H\)-bundle. Define \(\zeta : M \to (H \times_a G) \times R_1\) by

\[
\zeta((g, s(r)), (g', x), (r, [g, u(s(r))], r')) = (\delta(g \ast r, r'), g, r).
\]

We assert that \(\zeta\) is the inverse of \(\chi\). Indeed,

\[
(\zeta \circ \chi)(h, g, f) = \zeta((g, s(f)), (\partial(h)g, t(f)), (f, [g, u(s(f))], h \ast (g \ast f)))
\]

\[
= (\delta(g \ast f, h \ast (g \ast f)), g, f)
\]

\[
= (h, g, f),
\]

and

\[
(\chi \circ \zeta)((g, s(r)), (g', t(r)), (r, [g, u(s(r))], r'))
\]

\[
= \chi(\delta(g \ast r, r'), g, r)
\]

\[
= ((g, s(r)), (\partial(\delta(g \ast r, r'))g, t(r)), (r, [g, u(s(r))], \delta(g \ast r, r') \ast (g \ast r)))
\]

\[
= ((g, s(r)), (\partial(\delta(g \ast r, r'))g, t(r)), (r, [g, u(s(r))], r')).
\]
It remains to show that \( \partial(\delta(g \ast r, r'))g = g' \). We have \( \partial(\delta(g \ast r, r'))g \cdot t(r) = t(r') \) by (6.13). And \( g' \cdot t(r) = t(r') \) by the definition of \( M \). Since \( G \) acts freely on \( R_0 \), we have \( \partial(\delta(g \ast r, r'))g = g' \). So \( \zeta = \chi^{-1} \), and \( \tau \) is fully faithful.

**Proof of Theorem 9.1.** If \( H \) acts freely on \( R_1 \), Propositions 9.4 and B.3 show that a symplectic reduction of \( X \) exists. Conversely, suppose that \( H \) does not act freely on \( R_1 \). It follows from [8, Theorem 5.2] that \( BR_1 \to S \) is a principal \( BG_1 \)-bundle over some stack \( S \) if and only if the Lie groupoid

\[
Y := (R_0 \times R_0) \times_{\mathbb{R}^*} (G_1 \times \mathbb{R}_1)
\]

is Morita equivalent to a manifold. Here the weak fibered product is taken over the canonical map \( R_0 \times R_0 \to R_1 \times R_1 \), and the projection-action map \( pr_2 \times a : G_1 \times \mathbb{R}_1 \to R_1 \times \mathbb{R}_1 \). We will show that \( Y \), has non-trivial isotropy groups and so cannot be Morita equivalent to a manifold. Choose \( f \in R_1 \) and \( 1 \neq h \in H \) so that \( h \ast f = f \). Let \( x = s(f) \) and \( y = t(f) \). Consider the point

\[ z = ((x, x), (f, f), (1, y)) \in (R_0 \times R_0) \times_{\mathbb{R}_1 \times \mathbb{R}_1} (R_1 \times R_1) \times_{\mathbb{R}_0 \times \mathbb{R}_0} (G_1 \times R_0). \]

Let \( k \in Y_1 \) be the arrow

\[ k = ((x, x), (f, f), (h, 1), u(y))) \in (R_0 \times R_0) \times_{\mathbb{R}_0 \times \mathbb{R}_0} (R_1 \times R_1) \times_{\mathbb{R}_0 \times \mathbb{R}_0} (G_1 \times R_1), \]

where we identify \( G_1 = H \times a \cdot G \). Then \( s(k) = z \) and

\[
t(k) = ((x, x), (u(y), h \ast u(y)) \circ (f, f), (\partial(h), y)) = ((x, x), (f, f), (\partial(h), y)).
\]

But since \( R_1 \) is a regular form and \( h \ast f = f \), we have \( \partial(h) = 1 \). So \( t(k) = z \). Therefore the isotropy group of \( z \) is nontrivial.

QED

**Theorem 9.1** gives new information even in the context of ordinary symplectic reduction.

**9.5. Corollary.** Let \( G \) be a Lie group and \((X, \omega)\) a symplectic manifold on which \( G \) acts in a Hamiltonian fashion with moment map \( \mu : X \to \mathfrak{g}^* \). If \( 0 \) is a regular value of \( \mu \), then \( \mu^{-1}(0)/G \) is a symplectic stack.

An orbifold is a proper étale stack. If in Corollary 9.5 we make the extra assumption that \( G \) acts properly on \( \mu^{-1}(0) \), then the stack \( \mu^{-1}(0)/G \) is proper, so we obtain the following familiar Meyer-Marsden-Weinstein reduction theorem.

**9.6. Corollary.** Let \( G \) be a Lie group and \((X, \omega)\) a symplectic manifold on which \( G \) acts in a Hamiltonian fashion with moment map \( \mu : X \to \mathfrak{g}^* \). If \( 0 \) is a regular value of \( \mu \) and if \( G \) acts properly on \( \mu^{-1}(0) \), then \( \mu^{-1}(0)/G \) is a symplectic orbifold. If in addition the action of \( G \) on \( \mu^{-1}(0) \) is free, then \( \mu^{-1}(0)/G \) is a symplectic manifold.

**9.7. Example.** Consider \( \mathbb{C}^n \) with its standard symplectic form \( \omega \) and the standard action of the torus \( G = T^n \) as in Example 3.1. Let \( N \subseteq G \) an immersed Lie subgroup and \( \iota' \circ \mu : \mathbb{C}^n \to \mathfrak{n}^* \) the \( N \)-moment map with zero fibre \( X_0 \). Let \( \tilde{N} \to N \) a covering homomorphism as in Example 6.3. Let \((X_0, \omega)\) be the \( 0 \)-symplectic groupoid of Example 7.1 and let \((X, \omega)\) be the associated symplectic stack. There is an obvious morphism of groupoids \( p \) from (the identity groupoid of) \( X_0 \) to the action groupoid \( X = \tilde{N} \times X_0 \). The associated morphism of stacks \( p : X_0 \to X \) is a principal \( \tilde{N} \)-bundle in the sense of [8]. By definition the pullback of \( \omega \) to \( X_0 \) is equal to the presymplectic form \( \omega_0 \) on \( X_0 \). We conclude that the toric quasifold \((X, \omega)\) is the symplectic reduction of \( \mathbb{C}^n \) with respect to \( \tilde{N} \). The symplectic stack
(X, ω) is a symplectic orbifold if and only if N is a closed subgroup of G and the covering N → N is finite. The action of G = T^n on C^n descends to an action of the quotient Lie group stack G = G/Ñ on X, which is nothing other than the G-action defined in Example 7.1.

10. The Duistermaat-Heckman theorem

In this section we prove the analogue of the Duistermaat-Heckman theorem for Hamiltonian G-stacks, where G is a stacky torus. The Duistermaat-Heckman theorem has two parts: (1) the variation of the reduced symplectic form is linear, and (2) the moment map image of the Liouville measure is piecewise polynomial. It is only the first part that we generalize here; when the Hamiltonian stack is not proper, it is unclear how to integrate the Liouville measure along fibers of the moment map in a canonical, Morita-invariant fashion. (See [10] for a treatment of measures and densities on differentiable stacks.)

The following version of the Duistermaat-Heckman theorem was obtained by Guillemin and Sternberg by applying the coisotropic embedding theorem to the zero fiber of the moment map. Our approach is to generalize this formulation to Hamiltonian G-stacks. We focus our attention on the situation when we have a presentation of our G-stack X by a Hamiltonian groupoid with a leafwise transitive action (Theorem 10.3). We end by discussing how this result extends to a more general situation (Remark 10.15).

10.1. Theorem (Duistermaat-Heckman [13], Guillemin-Sternberg [16]). Let G be a torus and let (M, ω, G, µ) be a connected Hamiltonian G-manifold, where µ is a proper map. Let U be an open neighborhood of 0 ∈ g* which consists of regular values of µ, and let Z = µ^{-1}(0). Let θ ∈ Ω^2(Z) ⊗ g be a connection form for the locally free action of G on Z, and define the 1-form γ ∈ Ω^1(Z × g*) by

\[ γ(v|_p, w|_p) = \langle β, θ|_p(v|_p) \rangle. \]  

Then:

(i) After possibly shrinking U, there is an isomorphism of Hamiltonian G-manifolds

\[ (µ^{-1}(U), ω|_{µ^{-1}(U)}, G, µ|_{µ^{-1}(U)}) \cong (Z × U, ω|_Z + dγ, G, pr_2), \]

where the G-action on Z × U is t · (z, u) = (t · z, u).

(ii) If the action of G on Z is free, there is a symplectic isomorphism of reduced spaces

\[ (µ^{-1}(u)/G, ω^{red}(u)) \cong (µ^{-1}(0)/G, ω^{red}(0) + ⟨u, Γ⟩), \]

where ω^{red}(u), ω^{red}(0) denote the symplectic forms on the reduced spaces at u and 0, and Γ ∈ Ω^2(Z/G) ⊗ g is the curvature 2-form for the principal G-bundle Z → Z/G.

(iii) The de Rham cohomology class [⟨u, Γ⟩] varies linearly with u and does not depend on the choice of connection θ or the choices involved in constructing the isomorphism in (i).

We expand on the second part of (iii). In constructing the isomorphism in (i), one views µ: µ^{-1}(U) → U as a G-invariant locally trivial fiber bundle. To construct a trivialization µ^{-1}(U) ≅ Z × U, one chooses a G-invariant connection on this fiber bundle and a smooth contraction of U to 0. The horizontal lifts of this contraction gives a G-equivariant diffeomorphism from µ^{-1}(0) × Z to µ^{-1}(U). The isotropy class of this diffeomorphism is independent of the contraction and of the connection on µ^{-1}(U).
10.3. Theorem. Let \( G \) be a stacky torus, and let \((X, \omega, G, \mu)\) be a Hamiltonian \( G \)-stack presented by \((X_\ast, \omega, G_\ast, \mu)\). Denote by \( \partial: H \to G \) the crossed module of \( G_\ast \). Assume the following:

(a) \( X_1 \) is Hausdorff and \( 0 \in \text{Lie}(G)^\ast \) is a regular value of \( \mu \);
(b) \( G_0 \) is a torus and the action of \( H \) on \( X_0 \) is locally free;
(c) The action of \( G_\ast \) on \( X_\ast \) is leafwise transitive;
(d) The moment map \( \mu_0: X_0 \to \text{ann}(\mathfrak{h}) \) is proper, and \( X_0 = X \) is connected.

Then:

(i) There is an open neighborhood \( U \) of 0 and an isomorphism of Hamiltonian \( G \)-stacks

\[
(\mu^{-1}(U), \omega|_{\mu^{-1}(U)}, G, \mu) \cong (\mu^{-1}(0) \times U, \omega|_{\mu^{-1}(0)} + d\gamma|_{Z \times U}, G, \text{pr}_2),
\]

where \( d\gamma|_{Z \times U} \) depends on a choice and is described in (10.8) below.

(ii) If the symplectic reduction of \( X \) exists at 0, then it exists at all points of \( U \). For \( u \in U \), there is an equivalence of symplectic stacks of the reduced spaces

\[
(\mu^{-1}(u)/G, \omega^{\text{red}(u)}) \cong (\mu^{-1}(0)/G, \omega^{\text{red}(0)} + \Gamma)
\]

where \( \omega^{\text{red}(u)}, \omega^{\text{red}(0)} \) denote the symplectic forms on the reduced spaces at \( u \) and 0, respectively, and \( \Gamma \in \Omega^2(\mu^{-1}(0)/G) \) is described before (10.9) below.

(iii) The form \( \Gamma \) varies linearly with \( u \in \text{Lie}(G)^\ast \), and, after fixing the presentation \( X_\ast \) of \( X_\ast \), its cohomology with respect to the complex \( \Omega^1(\mu^{-1}(0)/G) \) does not depend on the choices involved.

Proof. For simplicity we will assume that \( H = N = N(\mathcal{F}) \), the null subgroup of \( \mathcal{F} \); the proof carries to other \( H \) with minor changes. Then \( \mathfrak{h} = \mathfrak{n} = \mathfrak{n}(\mathcal{F}) \), the null ideal, and \( \mu_0(X_0) \subseteq \text{ann}(\mathfrak{n}) \).

10.4. Lemma. (i) There is an isomorphism of Lie groupoids \( X_\ast \cong (N/L \ltimes X) \rightrightarrows X \), where \( L \) is a discrete subgroup of \( N \), and where the action of \( G_\ast \) is given as in Proposition 8.4. Under this isomorphism, the 0-symplectic form \( \omega = (\omega_0, \omega_1) \) can be written \( (\omega_0, 0 \oplus \omega_0) \).

(ii) There is an open set \( U \) containing 0 which consists of regular values of \( \mu_0 \).

(iii) The action of \( G \) is then locally free on \( V := \mu_0^{-1}(U) \).

Proof. Item (i) follows immediately from Proposition 8.4 and the condition \( \omega_1 = \mathfrak{s}^* \omega_0 \). Item (ii) holds because \( \mu_0 \) is proper. Item (iii) follows from (ii) because the action of \( N \) is locally free on \( X_0 \).

Recall the notion of the symplectization of a presymplectic manifold. For the presymplectic manifold \((X, \omega_0)\), let \( T^* \mathcal{F} \) be the vector bundle dual to \( T \mathcal{F} \), and let \( \text{pr}: T^* \mathcal{F} \to X \) be the bundle projection. By choosing a \( G \)-invariant metric on \( X \), one can embed \( j: T^* \mathcal{F} \subseteq T^* X \). Let \( \tilde{\omega} \) be the standard symplectic form on the cotangent bundle \( T^* X \), and let \( \Omega = \text{pr}^* \omega_0 + j^* \tilde{\omega} \) be the 2-form on \( T^* \mathcal{F} \). Then \( \Omega \) is symplectic near the zero section \( X \to T^* \mathcal{F} \), which is a coisotropic embedding of \( X \). There is a moment map \( \Psi = \text{pr}^* \mu_0 + j^* \tilde{\mu} \) for the \( G \)-action on \( T^* \mathcal{F} \), where \( \tilde{\mu}: T^* X \to \mathfrak{g}^* \) is the standard moment map for the \( G \)-action on \( T^* X \) given by

\[
\tilde{\mu}^\xi(y) = \langle y, \xi|_X \rangle, \quad \text{for } y \in T^*_X X.
\]

The germ at \( X \) of \( T^* \mathcal{F} \) is called the symplectization of \( X \). In [24] it is shown that in the leafwise transitive case, fibers of the moment map \( \mu_0: X \to \text{ann}(\mathfrak{h}) \subseteq \mathfrak{g}^* \) are fibers of \( \Psi: T^* \mathcal{F} \to \mathfrak{g}^* \).
10.5. Lemma. The restriction of the moment map \( \Psi: T^*\mathcal{F}|_V \to \mathfrak{g}^* \) to \( V = \mu_0^{-1}(U) \) is proper.

Proof. Choose a splitting \( \mathfrak{g} = \mathfrak{f} \oplus \mathfrak{n} \). Because the action of \( G \) is locally free on \( V \), the tangent bundle can be split \( TV \cong \mathfrak{f} \oplus \mathfrak{n} \oplus (TU/\mathfrak{g}) \), and because the action is leafwise transitive, \( T\mathcal{F}|_V \) is isomorphic to the trivial bundle with fiber \( \mathfrak{n} \). Let us choose the \( G \)-invariant metric on \( X \) so that \( \mathfrak{f} \) is orthogonal to \( \mathfrak{n} \), as follows. Fix a metric on \( TV \), by (1) choosing a basis \( \xi_i \) of \( \mathfrak{f} \) and a basis \( \nu_j \) of \( \mathfrak{n} \) and declaring the corresponding sections of \( TV \) to be orthonormal, and by (2) extending to get a metric on \( TV \) by choosing an arbitrary smooth metric on \( TV/\mathfrak{g} \). Averaging this metric over \( G \) gives a \( G \)-invariant metric on \( V \) where \( \mathfrak{f} \) is orthogonal to \( \mathfrak{n} \), as desired.

Now, consider the moment map \( \Psi \). Under the splitting \( \mathfrak{g}^* = \mathfrak{f}^* \oplus \mathfrak{n}^* \), the description of \( \mu_0 \) and the choice of splitting gives

\[
pr^*\mu_0(T^*\mathcal{F}|_V) = U \oplus 0, \quad j^*\tilde{\mu}(T^*\mathcal{F}|_V) = 0 \oplus \mathfrak{n}^*.
\]

Because the action of \( N \) is locally free, the restriction of \( j^*\tilde{\mu} \) to a fiber \( T^*\mathcal{F}|_x \) is a linear isomorphism to \( \mathfrak{n}^* \).

To show that the restriction of \( \Psi \) is proper, it is enough to show that the preimage of \( D \times E \subseteq \mathfrak{f}^* \oplus \mathfrak{n}^* \) in \( T^*\mathcal{F}|_U \) is compact, where \( D \subseteq \mathfrak{f}^* \) and \( E \subseteq \mathfrak{n}^* \) are compact. Indeed, from the description of \( \Psi \) above, the preimage of \( D \times E = \text{homomorphism to a fiber bundle over } \mu_0^{-1}(D) \) with fiber \( E \). Since \( \mu_0 \) is proper this space is compact.

QED

We can then apply Theorem 10.1 to the symplectization of \( V \subseteq X \). Let \( Z = \mu_0^{-1}(0) = \Psi^{-1}(0) \), and let

\[
\theta \in \Omega^1(Z) \otimes \mathfrak{g}
\]

be a connection form for the action of \( G \) on \( Z \), and let \( \gamma \in \Omega^1(Z \times \mathfrak{g}^*) \) be as in Theorem 10.1. After possibly shrinking \( U \), there is an open neighborhood \( U' \) of 0 in \( \mathfrak{g}^* \) so that \( U' \cap \text{ann}(\mathfrak{g}) = U \) and an isomorphism of Hamiltonian \( G \)-manifolds

\[
(\Psi^{-1}(U'), \Omega|_{\Psi^{-1}(U')}, G, \Psi|_{\Psi^{-1}(U')}) \cong (Z \times U', \omega_0|Z + d\gamma|Z \times U, G, pr_1).
\]

We require that we choose the contraction of \( U' \) described after Theorem 10.1 so that it restricts to a contradiction of \( U = U' \cap \text{ann}(\mathfrak{g}) \). Restricting the isomorphism to \( \Psi^{-1}(U) = \mu_0^{-1}(U) = V \) gives an isomorphism of presymplectic Hamiltonian \( G \)-manifolds

\[
F: (V, \omega_0|_V, G, \mu_0) \cong (Z \times U, \omega_0|Z + (d\gamma)|Z \times U, G, pr_2).
\]

By Lemma 10.4 (i) we can lift this to an isomorphism of Hamiltonian \( G \)-groupoids

\[
F: (X|_V, \omega, G, \mu) \cong (X|_Z \times U, \omega|Z + (d\gamma)|Z \times U, G, pr_2).
\]

Note that, by (10.7), the form \( (d\gamma)|Z \times U \in \Omega^2(Z \times U) \) determines an element of \( \Omega^2_{bas}(X|_Z \times U) \equiv \Omega^2(BX|_Z \times U) \) which we have also denoted by \( (d\gamma)|Z \times U \) in (10.8). Applying Theorem 7.4(i) proves (i).

The first statement of (ii) follows immediately. For the second, since \( \omega|Z \) and \( \omega|Z + (d\gamma)|Z \times U \) descend to symplectic forms on the stack \( \mu^{-1}(0)/G \), it follows that \( (d\gamma)|Z \times U \) descends to some

\[
\Gamma \in \Omega^2(\mu^{-1}(0)/G).
\]

This proves (ii). Finally, that \( \Gamma \) varies linearly with \( u \in \text{Lie}(G)^* \) is obvious from the definition. It remains to check that its cohomology class does not depend on the
choice of connection 1-form \( \theta \in \Omega^1(Z) \otimes g \) for the action of \( G \) on \( Z \) or on the choice of isomorphism \( X_s|_V \cong X_s|_Z \times U \).

10.10. Lemma. Let \((R_\bullet, \phi)\) be a regular form of the zero fibre \( Z_\bullet = \mu^{-1}(0) \) as in Proposition 8.1 and let \( G \times \mathbb{H} R_\bullet \) be the 0-symplectic groupoid of Lemma 9.2. Identify \( X_s|_V \cong Z \times U \) as in (10.8). If \( \eta \in \Omega^k_{bas}(Z, \times U) \) is \( G \)-invariant, then \( \phi^* \eta \) descends to an element of \( \Omega^k_{bas}(G \times \mathbb{H} R_\bullet) \cong \Omega^k(\mu^{-1}(0)/G) \).

Proof. Consider the form \( \phi^* \eta = (\phi^1_0 \eta_0, \phi^1_1 \eta_1) \in \Omega^k_{bas}(R_\bullet) \). We will show that \( (\phi^1_0 \eta_0, 0 \oplus \phi^1_1 \eta_1) \in \Omega^k_{bas}(G \times \mathbb{H} R_1) \Rightarrow R_0 \). Consider the maps \( \text{pr}_2 \circ (\text{id} \times s) : G \times R_1 \rightarrow R_0, \quad a \circ (\text{id} \times t) : G \times R_1 \rightarrow R_0 \),

which under the projection \( G \times R_1 \rightarrow G \times \mathbb{H} R_1 \) descend to the source and target maps of the groupoid \( G \times \mathbb{H} R_1 \). It suffices to show that

\[
(\text{pr}_2 \circ (\text{id} \times s))^\ast(\phi^1_0 \eta_0) = 0 \oplus \phi^1_1 \eta_1 = (a \circ (\text{id} \times t))^\ast(\phi^1_0 \eta_0).
\]

Since \( \eta \) is basic, the first equality is obvious, and the second follows immediately from the fact that \( \eta_0 \) is \( G \)-invariant and that \( \phi \) is \( G_\bullet \)-equivariant. QED

Now let \( \theta' \in \Omega^1(Z) \otimes g \) be another connection form. Then \( \theta - \theta' \in \Omega^1(Z) \otimes g \) is a \( G \)-invariant horizontal 1-form. If \( \gamma' \) is related to \( \theta' \) as in (10.2) then \( \gamma - \gamma' \in \Omega^1(Z \times g^\ast) \) is \( G \)-basic. By the description of \( X \), in Lemma 10.4 we consider \( (\gamma - \gamma')|_{Z \times U} \) as a \( G \)-invariant element of \( \Omega^1_{bas}(X_s|_Z \times U) \). So from Lemma 10.10 the restriction of \( \gamma - \gamma' \) to \( X_s|_Z \times \{u\} \) descends to an element of \( \Omega^1(\mu^{-1}(0)/G \times \{u\}) \). The form \( \Gamma' \in \Omega^2(\mu^{-1}(0)/G \times \{u\}) \) determined by \( \gamma' \) then differs from \( \Gamma \) by an exact form.

Let us now assume we choose a different isomorphism \( F' : V \equiv Z \times U \) from (10.7). Then \( F' \circ F^{-1} : Z \times U \rightarrow Z \times U \) is \( G \)-equivariant and isotropic the identity. So

\[
(F' \circ F^{-1})^\ast(\omega_0|_Z + (d\gamma)|_{Z \times U}) - (\omega_0|_Z + (d\gamma)|_{Z \times U}) = d\eta,
\]

where \( \eta \in \Omega^1(Z \times U) \) is \( G \)-invariant. We view \( \eta \) as a \( G \)-invariant element of \( \Omega^1_{bas}(X_s|_Z \times U) \) as before and apply Lemma 10.10. This proves (iii). QED

10.11. Remark. The cohomology class \([\Gamma]\) of Theorem 10.3(iii) is independent of the choice of presentation of \( X \). We omit the verification of this fact.

10.12. Example. Consider the previous theorem in the context of Examples 7.7 and 9.7. We have \( X_0 = (t^* \mu^{-1}(0), X_1 = \tilde{N} \times X_0 \), and \( G_\ast = \tilde{N} \times X_0 \), with moment map \( \mu : X_\ast \rightarrow \text{ann}(\eta) \). Let \( U \subseteq \text{ann}(\eta) \) be a small neighborhood of \( 0 \). Note that the action of \( G_\ast \) on \( X_\ast \) satisfies the assumptions of Theorem 10.3. The reduced spaces over points of \( U \) are then equivalent as smooth stacks to the reduced space \( B(\mu^{-1}(0))/BG_\ast \), which here is just equivalent to a point.

10.13. Remark. Let us consider the case of an arbitrary Hamiltonian \( G \)-stack \((X, \omega, G, \mu)\), where \( G \) is a stacky torus. Assume that \( X \) satisfies the hypotheses (a), which is Morita invariant (see [26, Proposition 5.13]). The hypotheses (b) and (c) are not Morita invariant, but let us assume that we are given a presentation of \( G \) by an \( s \)-connected, base-connected Lie 2-group \( G_0 \), with \( G_0 \) a torus. Consider the Hamiltonian groupoid \((X_s, \omega, G_s, \mu)\) presenting \( X \) as in Theorem 7.4. Then on a neighborhood \( U \) of \( 0 \) consisting of regular values of \( \mu \), a slight variant of the construction of Proposition 8.1 gives a Hamiltonian \( G_s \)-groupoid \( Y \), which is Morita equivalent to \( \mu^{-1}(U) \), and which is in regular form with respect to the action of \( G_\ast \). In this case, the hypotheses (b) are satisfied. The hypothesis (c) is
almost satisfied, in the sense that the action is locally leafwise transitive even if it is not leafwise transitive. If \( Y \) satisfies the hypotheses (d), then let us take the \( s \)-connected subgroupoid \( Y^{(s)} \) of \( Y \). Then \( Y^{(s)} \) is a Hamiltonian \( G \)-groupoid, and it satisfies the hypotheses of Theorem 10.3. The map of Hamiltonian \( G \)-stacks \( BY^{(s)} \to BY \) induced by the inclusion has fibers of dimension 0. By Theorem 9.1, the reduction of \( X \) at a point \( u \in U \subseteq \text{Lie}(G)^r \) exists if and only if the reduction of \( BY^{(s)} \) exists at \( u \). In that case, there is naturally a map of the reduced spaces over \( u \), which is symplectic and has fibers of dimension 0.

Appendix A. Strictification of stacky actions

This appendix contains the proof of the following result, which is Theorem 6.33 in the main text.

A.1. Theorem. Let \( G \) be a connected étale Lie group stack acting on an étale differentiable stack \( X \). For every base-connected Lie 2-group \( G \), with \( BG \cong G \) (the existence of which is guaranteed by Theorem 6.24) there exists a Lie groupoid \( X \), with \( BX \cong X \) so that

(i) \( G \) acts strictly on \( X \);

(ii) identifying \( BG \cong G \), the equivalence \( BX \cong X \) is \( G \)-equivariant.

Proof. Let \( G \) be a base connected étale Lie 2-group which has \( BG \cong G \); we identify \( BG \cong G \). Let \( Y \) be a Lie groupoid with \( BY \cong X \). Let \( X_0 := G_0 \times Y_0 \), and consider the map

\[
\begin{align*}
\xymatrix{ b : X_0 = G_0 \times Y_0 \ar[r] & G \times X \ar[r]^\alpha & X. }
\end{align*}
\]

A.2. Proposition. The map \( b \) is a representable epimorphism and a submersion. So \( b \) is an atlas for \( X \) and \( BX \cong X \), where \( X_1 = X_0 \times_X X_0 \).

Proof. Consider the 2-cartesian square

\[
\begin{align*}
\xymatrix{ (G_0 \times Y_0) \times_X Y_0 \ar[r] \ar[d]^{pr_1} & Y_0 \ar[d]^p \\
G_0 \times Y_0 \ar[r] & G \times X \ar[r]^\alpha & X }
\end{align*}
\]

where \( p : Y_0 \to X \) is the atlas and \( pr_1 : (G_0 \times Y_0) \times_X Y_0 \to G_0 \times Y_0 \) is the canonical (up to isomorphism) map out of the fibered product. Note that \( (G_0 \times Y_0) \times_X Y_0 \) is (equivalent to) a manifold, since \( p \) is representable. By Lemmas 2.2 and 2.3 of [3], to show that \( \alpha \) is an atlas it is enough to show that \( \alpha \) is a surjective submersion.

Let \( p \in (G_0 \times Y_0) \times_X Y_0 \) with \( \tilde{a}(p) = x \) and \( pr_1(p) = (g, y) \). To show that \( \tilde{a} \) is a submersion, we will show that \( \tilde{a} \) admits local sections, meaning for any \( p \) as above there is a neighborhood \( U \) of \( x \) and a section \( \sigma : U \to (G_0 \times Y_0) \times_X Y_0 \) of \( \tilde{a} \) so that \( \sigma(x) = p \). In the following diagram, each square is 2-cartesian

\[
\begin{align*}
\xymatrix{ (\{g\} \times X_0) \times_X X_0 \ar[d] \ar[r] & (G_0 \times X_0) \times_X X_0 \ar[r] \ar[d] & X_0 \ar[d]^p \\
\{g\} \times X_0 \ar[r] & G_0 \times X_0 \ar[r]^\alpha & X }
\end{align*}
\]

and \( p \in (\{g\} \times X_0) \times_X X_0 \). Assume that the bottom row

\[
\begin{align*}
\xymatrix{ \{g\} \times X_0 \ar[r] & G_0 \times X_0 \ar[r] & X }
\end{align*}
\]

(A.3)
is an atlas for $X$, then since being an epimorphism and being a submersion are local properties, the top row
\[(g') \times X_0) \times X_0 \rightarrow (G_0 \times X_0) \times X_0 \rightarrow X_0\]
is a surjective submersion. In particular, $\tilde{a}$ is surjective. And there is a local section $\sigma$ of the top row sending $x$ to $p$, which is also a local section of $\tilde{a}$. So $\tilde{a}$ is a submersion.

It remains to show that (A.3) is an atlas. For $g \in G_0$, applying the functor $B$ gives us the categorical point $g : \ast \rightarrow G$. Then $g$ determines a map of stacks
\[L_g : \ast \times X \rightarrow \ast \times X \xrightarrow{g \times \text{id}} G \times X \xrightarrow{\tilde{a}} X\]
In fact, by the axioms for the action of $G$ on $X$, the map $L_g$ is an equivalence of stacks, with (weak) inverse $L_{g^{-1}}$. Then the composition (A.3) is naturally isomorphic to $L_g \circ p : X_0 \rightarrow X$. Since $L_g$ is an equivalence and $p$ is an atlas, we have that (A.3) is an atlas. This completes the proof.

QED

Without loss of generality assume $BX_0 = X$. Each square of the following diagram 2-commutes:

\[
\begin{array}{ccc}
G_0 \times G_0 \times Y_0 & \xrightarrow{m \times \text{id}} & G_0 \times Y_0 \\
\downarrow & & \downarrow \\
G \times G \times X & \xrightarrow{m \times \text{id}} & G \times X \\
\downarrow \text{id} \times a & & \downarrow a \\
G \times X & \xrightarrow{a} & X.
\end{array}
\]

There is a Lie groupoid morphism $A : G_0 \times X_0 \rightarrow X_0$, where $A_0 = M \times \text{id}$ and $BA \equiv a$. Then $A_0$ determines an action of $G_0$ on $X_0$. We write $A_0(g, x) = g \cdot x$, for $g \in G_0$ and $x \in X_0$, and $A_1(h, y) = h \cdot y$, for $h \in G_1$ and $y \in X_1$. We will show that $A$ is naturally isomorphic to a strict action $\tilde{A} : G_0 \times X_0 \rightarrow X_0$ with $\tilde{A}_0 = A_0$. Note that despite our notation, $A_0$ is not in general an action.

There are coherence conditions on the action $a$ as in [8], and we must write these as coherence conditions on $A$. To do this, we make use of the following: Let $F, F'$ be morphisms of Lie groupoids and let $\alpha : BF \Rightarrow BF'$ be a natural isomorphism in the category of stacks. Then there is a unique natural isomorphism $\alpha : F \Rightarrow F'$ in the category of Lie groupoids so that $\alpha = \beta \alpha$. Indeed, recall that there is an equivalence of bicategories between differentiable stacks and Hilsum-Skandalis bibundles. And, by [4, Theorem 2.8], if there is an equivalence between two bibundles which are associated to morphisms of Lie groupoids, then the equivalence itself must come from a unique natural isomorphism between these two Lie groupoid morphisms. Putting these two facts together gives the claim.

The coherence conditions on $a$ then translate as follows. There are natural isomorphisms
\[
\begin{align*}
\beta : A \circ (M \times \text{id}) & \Rightarrow A \circ (\text{id} \times A); \\
\beta : G_0 \times G_0 \times X_0 & \Rightarrow X_1; \\
\epsilon : A \circ (1 \times \text{id}) & \Rightarrow \text{id}; \\
\epsilon : X_0 & \Rightarrow X_1,
\end{align*}
\]
where $1: \star \to G$, is the 2-group unit, and we write

$$1 \times \text{id}: X_\ast \equiv \star \times X_\ast \to G \times X_\ast.$$ 

There are higher coherence conditions on $\epsilon$ and $\beta$. For $k, h, g \in G_0$ and $x \in X_0$, they are as follows.

(i) $(u(g) \cdot \epsilon(x)) \circ \beta(g, 1, x) = u(g \cdot x)$

(ii) $\epsilon(g \cdot x) \circ \beta(1, g, x) = u(g \cdot x)$

(iii) $\beta(k, h, g \cdot x) \circ \beta(kh, g, x) = (u(k) \cdot \beta(h, g, x)) \circ \beta(k, hg, x)$.

Since $A_0$ is an action, we have that $\epsilon(x)$ is an arrow from $x$ to $x$ and $\beta(h, g, x)$ is an arrow from $(hg) \cdot x$ to $(hg) \cdot x$.

A.4. Lemma. In the situation as above, for $k, j \in G_0$ and $x \in X_0$, one has

$$\beta(k, 1, j \cdot x) = \beta(k, j, x).$$

As a consequence, by applying coherence condition (i) we have

$$u(k) \cdot \epsilon(j \cdot x) = \beta(k, 1, j \cdot x)^{-1} = \beta(k, j, x)^{-1}. \tag{A.5}$$

Proof. Fix $x \in X_0$. Define smooth maps $\gamma_1, \gamma_2: G_0 \times G_0 \to s^{-1}(x) \cap t^{-1}(x)$ as follows.

$$\gamma_1(k, j) = u((kj)^{-1}) \cdot \beta(k, 1, j \cdot x)$$

$$\gamma_2(k, j) = u((kj)^{-1}) \cdot \beta(k, j, x).$$

Then, since $G_0$ is connected and $X_\ast$ has discrete isotropy groups, the maps $\gamma_1$ and $\gamma_2$ are constant. Moreover, $\gamma_1(1, 1) = 1 : \beta(1, 1, x) = \gamma_2(1, 1)$, so $\gamma_1 = \gamma_2$ and thus

$$u((kj)^{-1}) \cdot \beta(k, 1, j \cdot x) = u((kj)^{-1}) \cdot \beta(k, j, x)$$

for all $k, j \in G_0$. Therefore,

$$u(kj) \cdot (u((kj)^{-1}) \cdot \beta(k, 1, j \cdot x)) = u(kj) \cdot (u((kj)^{-1}) \cdot \beta(k, j, x))$$

Applying the natural transformation $\beta$ to both sides, one finds

$$\beta(kj, (kj)^{-1}, kj \cdot x) \circ (1 \cdot \beta(k, 1, j \cdot x)) \circ \beta(kj, (kj)^{-1}, kj \cdot x)^{-1}$$

$$\quad = \beta(kj, (kj)^{-1}, kj \cdot x) \circ (1 \cdot \beta(k, j, x)) \circ \beta(kj, (kj)^{-1}, kj \cdot x)^{-1}.$$

So,

$$1 \cdot \beta(k, 1, j \cdot x) = 1 \cdot \beta(k, j, x).$$

Applying the natural transformation $\epsilon$ to both sides, one has

$$\epsilon(kj \cdot x)^{-1} \circ \beta(k, 1, j \cdot x) \circ \epsilon(kj \cdot x) = \epsilon(kj \cdot x)^{-1} \circ \beta(k, j \cdot x) \circ \epsilon(kj \cdot x).$$

Therefore, $\beta(k, 1, j \cdot x) = \beta(k, j, x)$. QED

Now define $\tilde{A}: G_\ast \times X_\ast \to X$, by

$$\tilde{A}_0(g, x) = A_0(g, x) = g \cdot x$$

$$\tilde{A}_1(k, f) = \epsilon(t(k) \cdot t(f)) \circ (k \cdot f) \circ \epsilon(s(k) \cdot s(f))^{-1}.$$ 

We write $\tilde{A}_1(k, f) = k \circ f$. That $\tilde{A}$ is a Lie groupoid morphism is easy to check. The smooth map $\epsilon \circ M: G_0 \times X_0 \to X_0$ is a natural isomorphism from $A$ to $\tilde{A}$. It remains to check that $\tilde{A}_1: G_1 \times X_1 \to X_1$ is an action. First,

$$1 \circ f = \epsilon(t(f)) \circ (1 \cdot f) \circ \epsilon(s(f))^{-1} = f$$
since $e$ is a natural isomorphism $A \circ (1 \times \text{id}) \Rightarrow \text{id}$. Next,
\[
k \circ (j \circ f) = k \circ (e(t(j) \cdot t(f)) \circ (j \cdot f) \circ e(s(j) \cdot s(f))^{-1})
\]
\[
= e(t(kj) \cdot t(f)) \circ F \circ e(s(kj) \cdot s(f))^{-1},
\]
where $F = k \cdot (e(t(j) \cdot t(f)) \circ (j \cdot f) \circ e(s(j) \cdot s(f))^{-1})$. On the other hand,
\[
kj \circ f = e(t(kj) \cdot t(f)) \circ [kj \cdot f] \circ e(s(kj) \cdot s(f))^{-1}.
\]
We must then show that
\[(A.6) \quad F = kj \cdot f.
\]
We compute
\[
F = [u(t(k)) \circ k \circ u(s(k))] \cdot [e(t(j) \cdot t(f)) \circ (j \cdot f) \circ e(s(j) \cdot s(f))^{-1}] \\
(A.7) = [u(t(k)) \cdot e(t(j) \cdot t(f))] \circ [k \cdot (j \cdot f)] \circ [u(s(k)) \cdot e(s(j) \cdot s(f))^{-1}],
\]
since $A$ is a groupoid morphism. From $(A.5)$, one has that $(A.7)$ is equal to
\[
\beta(t(k), t(j), t(f))^{-1} \circ [k \cdot (j \cdot f)] \circ \beta(s(k), s(j), s(f)).
\]
But since $\beta$ is a natural transformation $A \circ (M \times \text{id}) \Rightarrow A \circ (\text{id} \times A)$ this is just $kj \cdot f$.
So we have found $(A.6)$, and proved the theorem.
QED

Appendix B. Weak fibered products and principal bundles

Bursztyn, Noseda, and Zhu [8] have introduced the notion of a principal bundle in the 2-category of differentiable stacks. In this appendix we briefly introduce the analogous notion in the 2-category of Lie groupoids, and compare the two. Let $\phi : X^* \to Z^*$ and $\psi : Y^* \to Z^*$ be Lie groupoid morphisms. The weak fibered product is the topological groupoid $X \underset{Z^*}{\times} Y$, whose space of objects is
\[
X_0 \times_{Z_0} Z_1 \times_{Z_0} Y_0 = \{ (x, k, y) \in X_0 \times Z_1 \times Y_0 \mid \phi(x) = s(k), \phi(y) = t(k) \},
\]
and whose space of arrows is
\[
X_1 \times_{Z_0} Z_1 \times_{Z_0} Y_1 = \{ (f, k, g) \in X_1 \times Z_1 \times Y_1 \mid \phi(s(f)) = s(k), \psi(s(g)) = t(k) \}.
\]
If either $\phi$ or $\psi$ is a submersion, the weak fibered product is a Lie groupoid, and it is a weak limit in $\text{LieGpd}$. For more details, including the groupoid structure maps of the weak fibered product, see [26, § 5.3]. For a description of weak fibered products in the 2-category $\text{DiffStack}$, as well as a detailed discussion of weak limits in general, see e.g. [9, §1.2]. The 2-functor $B : \text{LieGpd} \to \text{DiffStack}$ preserves finite weak limits, so it takes weak fibered products of Lie groupoids to weak fibered products of stacks.

B.1. Definition. Let $G_*$ be a Lie 2-group, $X_*$ and $Y_*$ Lie groupoids, and $a : G_* \times X_* \to X_*$ a strict action. A Lie groupoid morphism $\psi : X_* \to Y_*$ is $G_*$-invariant if there is a natural isomorphism $\gamma : \psi \circ pr_2 \Rightarrow \psi \circ a$ making the following diagram 2-commute:
\[
\begin{array}{ccc}
G_* \times X_* & \xrightarrow{a} & X_* \\
pr_2 \downarrow & & \downarrow \psi \\
X_* & \xrightarrow{\psi} & Y_*
\end{array}
\]
The 2-isomorphism $\gamma$ is required to satisfy two higher coherence conditions, as in [8, Definition 3.22]. A Lie groupoid morphism $\psi: X \to Y$, is a principal $G_\ast$-bundle if

(i) $\psi$ is essentially surjective;
(ii) $\psi$ is $G_\ast$-invariant;
(iii) the canonical morphism $G_\ast \times X \to X \times_{Y_\ast} X$, is a Morita morphism.

B.2. Example. Let $G$ be a Lie group (viewed as the identity 2-group $G \rightrightarrows G$) and $X$ a $G$-manifold (viewed as the identity groupoid $X \rightrightarrows X$). Then $\psi: X \to G \ltimes X$ defined by $\psi(x) = (1, x)$ is a principal $G$-bundle.

B.3. Proposition. Let $\psi: X_\ast \to Y_\ast$, be a principal $G_\ast$-bundle. Then $B\psi: BX \to BY$, is a principal $BG_\ast$-bundle in the sense of [8, Definition 3.24].

Proof. Since $\psi$ is essentially surjective, the map $B\psi$ is an epimorphism of stacks. To check the conditions of [8, Definition 3.24], it suffices to note that $B$ is a 2-functor which preserves weak fibered products, and show that there is an atlas $M \to BX$, so that the composition $M \to BX \to BY$, is representable. Indeed, we can take $M = X_0$. Then $BY \simeq B\psi_\ast(Y_\ast)$, where $\psi_\ast(Y_\ast)$ is the pullback groupoid (see Definition 4.1), and the composition $X_0 \to BX \to BY$, is $\psi^\ast(Y_\ast) \simeq BY$, is representable. QED

B.4. Remark. Let $\psi: X_\ast \to Y_\ast$, be a principal $G_\ast$-bundle. The canonical morphism $X_\ast \to \psi^\ast(Y_\ast)$ is the identity on the manifold of objects $X_0$, and therefore induces an injective morphism $\Omega_{bas}(\psi^\ast(Y_\ast)) \to \Omega_{bas}(X_\ast)$. By Remark 5.6 $\Omega_{bas}(\psi^\ast(Y_\ast))$ is isomorphic to $\Omega_{bas}(X_\ast)$, so we conclude that the pullback map of basic forms $\psi^\ast: \Omega_{bas}(Y_\ast) \to \Omega_{bas}(X_\ast)$ is an injection.

APPENDIX C. Weak fibered products of Lie group stacks (by C. ZHU)

This appendix is devoted to the proof of the following result, which is Theorem 6.26 in the main text.

C.1. Theorem. Let $G \to H$ and $G' \to H$ be weak homomorphisms of (strict) Lie group stacks, and assume that the fibered product of stacks $K = G \times_H G'$ is a differentiable stack. Then $K$ is naturally a weak Lie group stack, and the projections $K \to G$ and $K \to G'$ are weak homomorphisms.

Proof. It suffices to prove the statement object-wise. That is, for an object $U \in Diff$, the groupoid $K(U) = G(U) \times_H^{(w)} G'(U)$ is a weak 2-group. Because $K$ is assumed to be a differentiable stack, it will then automatically be a Lie group stack.

Let us denote

$$G = (G_1 \rightrightarrows G_0) := G(U)$$
$$G' = (G'_1 \rightrightarrows G'_0) := G'(U)$$
$$H = (H_1 \rightrightarrows H_0) := H(U)$$

and the maps between them $\phi: G \to H$ and $\phi': G' \to H$. We now define the group structure maps on $G \times_H^{(w)} G'$. In what follows, we refer the reader to [34] for the necessary background.

Multiplication. The multiplication

$$\tilde{m}: (G \times_H^{(w)} G') \times (G \times_H^{(w)} G') \to (G \times_H^{(w)} G') \times (G \times_H^{(w)} G')$$

is essentially given by $m \times m'$, where $m$ and $m'$ are the group multiplication maps for $G$ and $G'$ respectively. We denote by $\cdot$ the multiplications $m, m', \bar{m}$, etc. Then on the level of objects of $G \times_H^{(w)} G'$, define:

$$(g_0^1, h_1^1, g_0'^1) \cdot (g_2^0, h_1^0, g_0'^0) := (g_0^1 \cdot g_0'^1, a'^{-1} \circ (h_1^1 \cdot h_1^0) \circ a, g_0'^0 \cdot g_0'^0),$$

where $a : \phi_0(g_0^1 \cdot g_0'^0) \to \phi_0(g_0^1) \cdot \phi_0(g_0'^0)$ comes from the 2-isomorphism data of the weak homomorphism $\phi$; and similarly for $a'$. The middle element in $H_1$ is then obtained by the following composition

$$(\phi_0(g_0^1) \cdot \phi_0(g_0'^0)) \rightarrow \phi_0(g_0^1 \cdot g_0'^0) \rightarrow \phi_0(g_0^1) \cdot \phi_0(g_0'^0) \rightarrow \phi_0(g_0^1) \cdot \phi_0(g_0'^0).$$

On the level of morphisms, define

$$(g_1^1, h_1^1, g_1'^1) \cdot (g_2^2, h_2^0, g_2'^0) := (g_1^1 \cdot g_1'^1, a'^{-1} \circ (h_1^1 \cdot h_1^0) \circ a, g_1'^0 \cdot g_1'^0),$$

where $s(g_1^1, h_1^1, g_1'^1) = (g_1^1, h_1^1, g_1'^1)$ and $s(g_2^2, h_2^0, g_2'^0) = (g_2^2, h_2^0, g_2'^0)$.

**Ass ociator.** The associator $\hat{A}$ for $\bar{m}$ is a natural transformation, whose value at $(g_0^1, g_0^0, g_0^3), (h_1^1, h_1^0, h_1^0), (g_1^1, g_1^0, g_1^0)$ is given by the following element in $(G \times_H^{(w)} G')_{1'}$:

$$(g_0^1 \cdot g_0^3) \cdot g_0^0, (a'^{-1} \circ (h_1^1 \cdot h_1^0) \circ a, g_0^1 \cdot g_0'^0).$$

$$(\hat{A})^{-1} \circ (h_1^1 \cdot h_1^0) \circ a, g_0^1 \cdot g_0'^0,$$

where $\hat{a} = a'^{1,23} \circ (a^{1,2} \times \text{id})$ and $\hat{a} = a'^{1,23} \circ (\text{id} \times a^{2,3})$, and similarly for $\hat{a}'$ and $\hat{a}'$.

It is an arrow between these two objects thanks to the compatibility of $\alpha$ and the associator $A$, and that of $\alpha'$ and $A'$.

**Naturality of $\hat{A}$** comes from that of $A$ and $A'$. The pentagon identity for $\hat{A}$ is implied by that of $A$ and $A'$.

**Identity.** Similarly, we define the identity map $\hat{I} : * \to G \times_H^{(w)} G'$ to be essentially $(1, 1')$. More precisely, $\hat{I}_0 := (1_0, \beta^{-1} \circ \beta, 1'_0) \in (G \times_H^{(w)} G')_{1_0}$, where we have

$$(\phi_0(1_0) \rightarrow 1_0 \beta \rightarrow \phi_0(1'_0)).$$

Here $\beta$ and $\beta'$ come from the 2-isomorphism data in the definition of $\phi$ and $\phi'$. Then $\hat{I}_1 := (1_1, \beta^{-1} \circ \beta, 1'_1)$. The 2-morphisms involving $\hat{I}$ for $G \times_H G'$ is similarly defined by that for 1 and 1', and higher coherence of these 2-morphisms is similarly implied by that for $G$ and $G'$.

**Inverse.** The inverse map $\hat{i} : G \times_H^{(w)} G' \to G \times_H^{(w)} G'$ is essentially $(i, i')$. More precisely, on objects

$$(i_0(g_0, h_1, g_0') := (i_0(g_0), g' \circ i_1(h_1) \circ \gamma, i'_0(g_0')), \)$$

where

$$(\phi_0(i_0(g_0)) \gamma \rightarrow i_0(\phi_0(g_0)) \rightarrow i_0(\phi_0(g_0)) \gamma \rightarrow \phi_0.$$ And on arrows

$$(i_1(g_1, h_1, g_1') := (i_1(g_1), g' \circ h_1 \circ \gamma, i'_1(g_1)).$$
Here, $\gamma$ is uniquely determined by $\alpha$ and $\beta$ and satisfies certain coherence laws; see [1, §6]. The 2-morphisms involving $\tilde{i}$ for $G \times_H G'$ are similarly defined by those for $i$ and $i'$, and higher coherence of these 2-morphisms is similarly implied by that for $G$ and $G'$.

**Projections.** From the construction of the group structure maps $\tilde{m}, \tilde{1}, \tilde{i}$ for $K$, we see that the projections to $G$ and $G'$ are naturally Lie group stack morphisms. In fact, the 2-morphisms $\alpha$ and $\beta$ are the identity. QED

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