A singular radial connection over $\mathbb{B}^5$ minimizing the Yang-Mills energy

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Abstract We prove that the pullback of the $SU(n)$-soliton of Chern number $c_2 = 1$ over $S^4$ via the radial projection $\pi : \mathbb{B}^5 \setminus \{0\} \to S^4$ minimizes the Yang-Mills energy under a topologically fixed boundary trace constraint. In particular this shows that stationary Yang-Mills connections in high dimension can have singular sets of codimension 5.

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1 Introduction

Let $G$ be a compact connected Lie group, $M$ be a compact Riemannian manifold and $E \to M$ be a vector bundle associated to the adjoint representation of a principal $G$-bundle over $M$. Following e.g. [2], let $\nabla = d + A$ locally represent a connection over $E$ in a given trivialization. Here $d$ represents the exterior derivative of $M$ and $A$ is a 1-form with coefficients in the Lie algebra $g$ of $G$. Let the Lie algebra valued 2-form representing the curvature of $\nabla$ be locally given by $F = dA + A \wedge A$. We recall that the Yang-Mills functional is defined in terms of the ad-invariant norm $| \cdot |$ on $g$ by

$$YM(\nabla) = \int_M |F|^2 d\text{vol}_M.$$
We are interested in the variational study of the functional $\mathcal{YM}$ and in particular in the structure of minimizers in dimension 5.

1.1 The framework in dimension 4

In [12,13] the analytic study of Yang-Mills connections on bundles $E \to M^4$ over 4-dimensional compact Riemannian manifolds $M$ individuated the following natural space of $su(n)$-valued connection forms:

$$A_{SU(n)}(M^4) := \left\{ A \in L^2, \ F_A \overset{\text{def}}{=} dA + A \wedge A \in L^2, \right. \\
\left. \text{and loc. } \exists g \in W^{1,2}(M^4, SU(n)) \text{ s.t. } A^g \in W^{1,2}_{\text{loc}} \right\},$$

where $A^g := g^{-1}dg + g^{-1}Ag$ is the formula representing the change of a connection form $A$ under a change of gauge $g$.

Recall (see the appendix of [5]) that a complete isomorphism invariant of $SU(2)$-bundles $E$ over 4-manifolds is the Chern number $c_2(E)$ which by Chern-Weil theory can be expressed as

$$c_2(E) = \frac{1}{8\pi^2} \int_{S^4} \text{tr}(F_A \wedge F_A).$$

The minimization of $\mathcal{YM}$ in 4 dimensions was studied in [13], where it was proved that the minimizer exists and the Chern number is preserved under the underlying weak convergence of connections. A special class of critical points for $\mathcal{YM}$ are minimizers with a fixed Chern number constraint. Such solutions have a special structure which arises as follows. The space of $su(n)$-valued 2-forms $\Omega^2$ splits into the $L^2$-orthogonal eigenspaces $\Omega^\pm$ of the Hodge star operator: $\Omega^2 \to \Omega^2$ and thus $F_A$ splits as $F_A = F_A^{\pm} + F_A^{-}$ with $*F_A^{\pm} = \pm F_A^{\pm}$. We may write

$$c_2(E) = \frac{1}{8\pi^2} \int_{S^4} \text{tr}(F_A \wedge F_A) = \frac{1}{8\pi^2} \int_{S^4} \left( |F_A^{+}|^2 - |F_A^{-}|^2 \right) d\text{vol}_{S^4}. \quad (1.1)$$

It follows then from Eq. (1.1) that any minimizing curvature must be anti-self-dual i.e. they satisfy $F_A^+ = 0$ (see [2,5]). In this case there holds

$$\text{tr}(F \wedge F) = |F|^2. \quad (1.2)$$

1.2 The framework in supercritical dimension 5

Several good properties break down at once in passing from the study of $\mathcal{YM}$ in dimension 4 to its study in dimension 5 (see the introduction of [9]) and the framework of the previous section must be changed if we look for a suitable functional analytic framework to study $\mathcal{YM}$ variationally in supercritical dimension. This framework appears in [9] and we describe it briefly here for the purpose of self-containedness. No further reference to the results of [9] is needed besides what stated in this subsection.

For two $L^2$ connection forms $A, A'$ over $\mathbb{R}^5$ we write $A \sim A'$ if there exists a gauge change $g \in W^{1,2}(\mathbb{R}^5, SU(n))$ such that $A' = A^g$. The class of all such $L^2$ connection forms $A'$ is denoted $[A]$. In [9] a class suited to the direct minimization of $\mathcal{YM}$ in 5 dimensions was defined as follows:

$$A_{SU(n)}(\mathbb{R}^5) := \left\{ [A] : A \in L^2, \ F_A \overset{\text{def}}{=} dA + A \wedge A \in L^2, \right. \\
\left. \forall p \in \mathbb{R}^5 \text{ a.e. } r > 0, \ \exists A(r) \in A_{SU(n)}(\partial B_r(p)) \right\},$$

where $A(r)$ satisfies $i^*_p A(r) \sim A(r)$. 

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Let $\phi$ be a smooth $su(n)$-valued connection 1-form over $\partial \mathbb{B}^5$. We have from [9] that $A_{SU(n)}(\mathbb{B}^5)$ is the strong $L^2$-closure of the following space of more regular connections:

$$\mathcal{R}^\infty(\mathbb{B}^5) := \left\{ F \text{ corresponding to some } [A] \in A_{SU(n)}(\mathbb{B}^5) \text{ s.t. } \exists k, \exists a_1, \ldots, a_k \in \mathbb{B}^5, \quad F = F_\nabla \text{ for a smooth connection } \nabla \text{ on some smooth } SU(n)-\text{bundle } E \rightarrow \mathbb{B}^5 \setminus \{a_1, \ldots, a_k\} \right\}.$$ 

In [9] the class $A_{SU(n)}^\phi(\mathbb{B}^5)$ of weak connections with trace $\phi$ was introduced. A characterization of such class is as the strong closure of connection classes $[A] \in R^\infty(\mathbb{B}^5)$ which satisfy $i^{a_5}_* A \sim \phi$. The main results of [9] can be combined into the following theorem:

**Theorem 1.1** (Main results of [9]) A minimizer of

$$\inf \left\{ \|FA\|_{L^2(\mathbb{B}^5)} : [A] \in A_{\phi}^\infty(\mathbb{B}^5) \right\}$$

exists and each minimizer is smooth outside a set of isolated singular points.

A question which arised naturally is whether the partial regularity as in Theorem 1.1 is optimal. The results of the present work insure that this is indeed the case, i.e. for better regularity one must include more hypotheses on the structure of the curvatures under consideration. Some intriguing candidates for such hypotheses can be extrapolated from [11] and [3].

1.3 Optimality of Theorem 1.1 and main result of the paper

The above optimality question is related to a conjecture was formulated by Gang Tian in [11]. This states that the singular set of $F$ would have Hausdorff dimension at most $n - 6$ in the case of so-called admissible $\Omega$-anti-self-dual curvatures $F$. These are curvatures satisfying $F = -\Omega \wedge \dot{F}$ for a smooth closed $(n - 4)$-form $\Omega$ in dimension $n$ under the further requirement of $F$ being admissible, i.e. that the underlying connection be locally smooth outside of a $(n - 4)$-dimensional rectifiable set. A natural question is to ask for examples of stationary or energy-minimizing connection classes which show that the $\Omega$-anti-self-dual requirement is necessary.

To the author’s knowledge, in the literature no proof is available that stationary curvatures $F$ having a singular set of Hausdorff codimension greater or equal than $5$ exist. This situation is similar to the one taking place in the theory of harmonic maps precedently to Hardt, Lin and Wang’s celebrated paper [6] where it was proved that the map $x/|x| : \mathbb{B}^3 \rightarrow S^2$ minimizes, for $2 \leq p < 3$, the $p$-the norm of the gradient among maps whose boundary trace is equal to the identity.

The main goal of this paper is to fill this gap and show that a result similar in spirit to [6] holds for the case of Yang-Mills minimization in dimension 5. We use a very symmetric instanton over $S^4 = \partial \mathbb{B}^5$. Let the group be $G = SU(n)$ and endow $S^4$ with the standard metric. We denote by

$$F_{S^4} = dA_{S^4} + A_{S^4} \wedge A_{S^4}$$

the anti-self-dual instanton on $S^4$ minimizing the Yang-Mills energy on associated $SU(n)$-bundles $E \rightarrow S^4$ under fixed second Chern number constraint $c_2(E) = 1$:

$$A \in \arg\min \left\{ \int_{S^4} |FA|^2 d\text{vol}_{S^4} \left| \quad A \text{ is loc. } W^{1,2}, \quad \frac{1}{8\pi^2} \int_{S^4} \text{tr}(FA \wedge FA) = 1 \right. \right\}.$$ 

With this notation we can state our main theorem:
Theorem 1.2 (Main result) Consider the radial pullback $A_{rad} := \left( \frac{x}{|x|} \right)^* A_{S^4}$. Then $A_{rad}$ is a minimzer for the problem

$$\min \left\{ \int_{B^5} |F_A|^2 dvol_{B^5} : [A] \in \mathcal{A}_{SU(n)}(B^5), \left[ i_A^* \partial B^5 \right] = [A_{S^4}] \right\}.$$ 

Note that for the above minimizer $[A]$ there holds

$$d \left( \text{tr} (F_A \wedge F_A) \right) = 8\pi^2 \delta_0. \quad (1.3)$$

Thus the minimizer presents a topological singularity, which cannot be eliminated by local gauge changes. In particular there does not exist a local gauge near the origin where the connection form $A$ becomes $W_{1,2}$, else the Chern–Simons formula $\text{tr} (F_A \wedge F_A) = d \left( \text{tr} (dA \wedge A + \frac{2}{3} A \wedge A \wedge A) \right)$ valid for smooth forms would stay true by approximation also for our $A$, contradicting (1.3).

We note that the curvature form $F_{rad} := F_{A_{rad}}$ of Theorem 1.2 is $\Omega$-anti-self-dual with respect to the radial 1-form

$$\Omega = \sum_{k=1}^5 \frac{x_k}{|x|} dx_k = dr$$

outside the origin. In other words we have $F_{rad} \wedge \Omega = - * F_{rad}$. The form $\Omega$ is closed in the sense of distributions, however it is not smooth. Therefore it does not fully enter the setting presented in [11] and because of the smoothness requirement on $\Omega$ Conjecture 2 of [11] remains still open.

We note that the above minimizer could be thought of as the natural candidate for generic tangent cones to Yang-Mills minimizers. Therefore it is natural to ask whether like for minimizing harmonic maps [1] also in this case tangent cones are all modelled on a single bundle up to symmetries. It was proved by B. Yang that tangent cones are unique [14], but the classification question is still open. See [7] for another attempt.

1.4 Minimizing $L^1$ vector fields with defects

We note that to an $L^2$-integrable $su(n)$-values 2-form $F$ defined on $B^5$ we may associate an $L^1$ vector field $X$ by requiring the duality formula

$$\langle \phi, X \rangle = \langle \text{tr} (F \wedge F), * \phi \rangle = \int_{B^5} \text{tr} (F \wedge F) \wedge \phi$$

to hold for all smooth 1-forms $\phi$ on $\mathbb{B}^5$. Through the pointwise inequality

$$|\text{tr} (F \wedge F)| \leq |F|^2$$

(1.4)

we also deduce that

$$\|X\|_{L^1(B^5)} = \|\text{tr} (F \wedge F)\|_{L^1(B^5)} \leq \|F\|^2_{L^2(B^5)}. \quad (1.5)$$

Note that the curvature form $A_{rad}$ of Theorem 1.2 realizes the pointwise equality in (1.4) and thus also in (1.5). We will deduce our main theorem from the following result, which is of independent interest.

Theorem 1.3 The vector field $X_{rad}(x) = |S^4|^{-1} \frac{x}{|x|^5}$ minimizes the $L^1$-norm among vector fields $X \in L^1(B^5, \mathbb{R}^5)$ which are locally smooth outside some finite subset $\Sigma \subset \mathbb{B}^5$, satisfy...
for some integers $d_x$ and $X \cdot v_{S^4} \equiv 1$ where $v$ is the interior normal vector field to $S^4$.

This result is proved using similar tools as in [8], i.e. Smirnov’s decomposition for 1-currents [10] and a combinatorial result based on the maxflow-mincut theorem.

2 Proof of Theorem 1.3

2.1 Smirnov’s decomposition and combinatorial reduction

We start by recalling a version of Smirnov’s result [10], which allows to reduce the larger step of the proof of Theorem 1.3 to a combinatorial argument. We formulate the result in the case of vector fields with divergences of special form as in Theorem 1.3.

We recall the following definitions and notations:

- An arc in $\mathbb{R}^5$ is a rectifiable curve which has an injective parameterization $\gamma : [0, 1] \to \mathbb{R}^5$. To an arc we may associate a one dimensional current whose action on a smooth 1-form $\alpha \in D^1(\mathbb{R}^5)$ is given via $\langle [\gamma], \alpha \rangle := \int_{\gamma} \alpha$. We also call an arc the functional $[\gamma]$. Similarly to an $L^1$ vector field $X$ over $\mathbb{R}^5$ we associate the one dimensional current whose action is given via $\langle X, \alpha \rangle := \int_{\mathbb{R}^5} \alpha(X(x))dx$.
- The space of all arcs $[\gamma]$ is topologized with the weak topology. Note that the functions $s([\gamma]), e([\gamma])$ which to an arc assign its starting and ending point respectively, are Borel measurable. The variation measure of such $[\gamma]$ is denoted by $\|\gamma\|$ and its total variation is the length $\|\gamma\|([\mathbb{B}^5]) = \ell(\gamma)$. The boundary $\partial[\gamma]$ is given by the difference of Dirac masses $\delta_{e([\gamma])} - \delta_{s([\gamma])}$, and its variation measure is $\delta_{e([\gamma])} + \delta_{s([\gamma])}$.
- For a $A \in L^1(\mathbb{B}^5, \mathbb{R}^3)$, we denote $\partial A$ the boundary of the corresponding one dimensional current. It corresponds to the distributional divergence of $A$. We say that two such $L^1$ vector fields $B, C$ with divergences being Radon measures of finite total variation decompose a vector field $A$ if $A(x) = B(x) + C(x), |A(x)| = |B(x)| + |C(x)|$ for a.e. $x$. We say that a vector field $X$ is acyclic if for each such decomposition with $\partial C = 0$ there holds $C = 0$. Note that any minimizer as in Theorem 1.3 must be acyclic since if we had a decomposition $X = B + C$ as above with $\partial C = 0, C \neq 0$ then $B$ would be a competitor to $X$ of strictly smaller $L^1$ norm.

**Theorem 2.1** (Decomposition of vector fields, [10]) Assume $X$ is an acyclic $L^1$ vector field over $\mathbb{B}^5$ such that $\text{div} X$ is a measure of finite total variation. We may then find a finite Borel measure over the space of arcs such that the following hold for all smooth 1-forms $\alpha$ and for all smooth functions $f$ over $\mathbb{B}^5$:

$$\langle X, \alpha \rangle = \int \langle [\gamma], \alpha \rangle d\mu(\gamma), \quad (2.1)$$

$$\|X\|, f = \int \langle \|\gamma\|, f \rangle d\mu(\gamma), \quad (2.2)$$

$$\langle \text{div} X, f \rangle = \int \langle \delta_{e([\gamma])} - \delta_{s([\gamma])}, f \rangle d\mu(\gamma), \quad (2.3)$$

$$\|\text{div} X\|, f = \int \langle \delta_{e([\gamma])} + \delta_{s([\gamma])}, f \rangle d\mu(\gamma). \quad (2.4)$$
Fig. 1 We present here some of the arcs in the Smirnov decomposition of a vector field $X$ as in Theorem 1.3, including some of the points supporting $\text{div}X$ and some of the curves in the support of $\mu$ as in Theorem 2.1. Note that the arcs are actually oriented by the directions of the vector field $X$.

In other words the vector field $X$ decomposes as a superposition of arcs without cancellations whose boundaries decompose $\text{div}X$ without cancellations. See Fig. 1.

Note that for a vector field $X$ as in Theorem 1.3 for $\mu$-a.e. arc $\gamma$ the starting point $s(\gamma)$ is in one of the points $x$ such that $d_x < 0$ in the expression of $\text{div}X \lrcorner\mathbb{B}^5$ and the end point $e(\gamma)$ is either in one of the points $x$ with $d_x > 0$ or on the boundary $\partial B^5$. We may, for $x, y \in \mathbb{B}^5$, consider as in [8] the Borel sets of the form

$$C_{x,y} = \{\gamma \mid s(\gamma) = x, e(\gamma) = y\} \text{ with } d_x < 0, d_y > 0$$

(2.5)

and

$$C_{\partial,y} = \{\gamma \mid s(\gamma) \in \partial \mathbb{B}^5, e(\gamma) = y\}, \text{ with } d_y > 0,$$

(2.6)

where $d_x$ are the integers appearing in (1.6). We observe that the above $\mu$ measurable sets $C_{*,*}$ form a finite partition of $\mu$-almost all of $\text{spt}(\mu)$ and that given a function

$$\alpha : \{C_{x,y}\} \cup \{C_{\partial,y}\} \to [-1, 1]$$

the measure

$$\mu_{\alpha} := \sum_{x,y} \alpha(C_{x,y}) \mu \llcorner C_{x,y} + \sum_{x} \alpha(C_{\partial,y}) \mu \llcorner C_{\partial,y}$$

(2.7)

also gives an $L^1$ vector field $X_{\alpha}$ with finite variation divergence which is defined via an equation like (2.1) and satisfies (2.2), (2.3) but not necessarily (2.4). The measure $\mu_{X_{\alpha}}$ obtained applying Theorem 2.1 to $X_{\alpha}$ is supported on curves which are concatenations of curves in the support of $\mu_{\alpha}$.

2.2 Combinatorial results

We now fix the notations for a combinatorial structure which will be associated to data $X, \mu$ given above. For an illustration of the next definition and of the significance of the function $w$ see Fig. 2 and Table (1).

**Definition 2.2** ($X$-graph) Consider a finite set of vertices $V$ of which a special vertex $\partial \in V$ is highlighted, a graph on $V$, i.e. a subset $E \subset V \times V$ such that $(a, b) \in E \Rightarrow (b, a) \in E$, and a weight function $w : E \to \mathbb{R}$ such that $w(a, b) = -w(b, a)$ for $(a, b) \neq (\partial, \partial)$ and
We illustrate an X-flow on the left; to see the way in which the $w$ function corresponds to arrows and weights compare to Table 1. On the right there are three situations which are forbidden by the definition of an X-flow: in the first case not all arrows point the same way thus condition 3 is not fulfilled; in the second case the non-loop arrows near the vertex $\partial$ point towards it, thus the corresponding weights $w$ are positive and condition 2 will not be fulfilled; in the last case the integer flux condition 1 is not fulfilled

| $\partial$ | $a$ | $b$ | $c$ | $d$ | $e$ |
|-----------|-----|-----|-----|-----|-----|
| $\partial$ | 0.2 | 0.3 | 0.7 | .  | .   |
| $a$  | -0.3 | .  | .  | -0.25 | .  | -0.45 |
| $b$  | -0.7 | .  | .  | -0.1 | .  | -0.2 |
| $c$  | .  | 0.25 | 0.1 | .  | 0.65 | .   |
| $d$  | .  | .  | -0.65 | .  | -0.35 | .   |
| $e$  | .  | 0.45 | 0.2 | .  | 0.35 | .   |

The entry $(x, y)$ in the table corresponds to the value of $w(x, y)$ and dots represent edges not present in the graph

$w(\partial, \partial) \geq 0$. We define the flux function:

$$fl(v) := \frac{\sum\{w(a, v) : a \in V, (a, v) \in E\}}{\sum\{w(a, \partial) : a \in V \setminus \{\partial\}, (a, \partial) \in E\}}$$

for $v \in V \setminus \{\partial\}$, for $v = \partial$.

We call such data $(V, E, w, \partial)$ an X-graph if

1. We have integer fluxes: $\forall v \in V$, $fl(v) \in \mathbb{Z}$.
2. $fl(\partial) = -1$.
3. For all $v \in V \setminus \{\partial\}$ we have that all terms in the sum defining $fl(v)$ have the same sign.

We denote $V^+$ the vertices for which the sign in the last point is positive and $V^-$ those for which it is negative. Given $x, y \in V$ such that $(x, y), (y, x) \in E$ we say that $(x, y)$ is a directed edge if $w(x, y) > 0$.

We associate an X-graph to a vector field $X$ as in Theorem 1.3 by defining

$$V := \{x : d_x \neq 0\} \cup \{\partial\},$$

$$E := \{(x, y), (y, x), (\partial, y), (y, \partial) : C_{x,y}, C_{\partial,y} \text{ are as in (2.5),(2.6)}\}$$

and for $\ast \in V$

$$w(\ast, y) := \mu(C_{\ast,y}), \ w(y, \ast) := -\mu(C_{\ast,y})$$

if $C_{\ast,y}$ appears in (2.5) or (2.6).
We leave the verification of the properties as in Definition 2.2 to the reader. Note that in this case we obtain property (3) of Definition 2.2 also for \( v = \partial \) and we have \( \partial \in V^-, w(\partial, \partial) = 0 \).

To facilitate our inductive proofs, we also need the following definition which is a modification of that of an \( X \)-graph.

**Definition 2.3** (\( \bar{X} \)-graph) Consider \((V, E, w, \bar{\partial})\) as in Definition 2.2 except that \( \bar{\partial} \subset V \) may now contain more than one vertex, and that we require \( w(a, b) \) for all edges \((a, b) \in E\) including with \( a \) or \( b \) in \( \bar{\partial} \). We say that \((V, E, w, \bar{\partial})\) form an \( \bar{X} \)-graph if

1. \( \forall v \in V \setminus \bar{\partial}, \ fl(v) \in \mathbb{Z} \).
2. \( \sum_{v \in \bar{\partial}} fl(v) = 0 \).
3. For \( v \in V \setminus \bar{\partial} \) all terms in the sum defining \( fl(v) \) have the same sign.

Note that as a consequence of the fact that \( w(a, b) = -w(b, a) \) even for \( a, b \in \bar{\partial} \) it follows that a \( \bar{X} \)-graph has no loops, unlike \( X \)-graphs, who were allowed to have loops on the boundary.

A tool in our combinatorial construction will be the maxflow-mincut theorem, to state which we recall the definition of a (combinatorial) flow.

**Definition 2.4** (\( X \)-flows, \( \bar{X} \)-flows and cuts) Let \((V, E, w, \partial)\) be an \( X \)-graph and fix a vertex \( a^+ \in V^+ \). A function \( f : E \to \mathbb{R} \) such that \( f(a, b) = -f(b, a) \) for \((a, b) \neq (\partial, \partial)\) and \( f(\partial, \partial) \geq 0 \) is an \( X \)-flow if the following properties hold:

1. \( |f(a, b)| \leq |w(a, b)| \) for all \((a, b) \in E\).
2. For all \( v \in V \setminus \{\partial, A^+\} \) there holds \( \sum \{f(a, v) : (a, v) \in E\} = 0 \).
3. For all \((x, \partial), (a^+, y) \in E \) there holds \( \text{sgn}(f(x, \partial)) = \text{sgn}(w(x, \partial)) \) and \( \text{sgn}(f(a^+, y)) = \text{sgn}(w(a^+, y)) \).

We call the vertex \( a^+ \) the sink of the \( X \)-flow \( f \). The value of the \( X \)-flow \( f \) is by definition the number \( \text{val}(f) := \sum \{f(\partial, y) : (\partial, y) \in E, y \neq \partial\} \). We say that the edge \((a, b)\) is saturated by \( f \) if \( |f(a, b)| = |w(a, b)| \). If all edges with an end equal to \( v \in V \) are saturated, we say that \( f \) saturates \( v \).

If \((V, E, w, \bar{\partial})\) is a \( \bar{X} \)-graph then we define a \( \bar{X} \)-flow as above, except that \( f(a, b) = -f(b, a) \) will be required to hold for all edges \((a, b)\) with no exception.

For a given vertex \( a^+ \in V^+ \) a cut between \( \partial \) and \( a^+ \) of the \( X \)-graph \((V, E, w, \partial)\) is a subset \( S \subset E \) such that for every path \((\partial := a_0, a_1, (a_1, a_2), \ldots, (a_k, a_{k+1} := a^+) \) such that \((a_i, a_{i+1}) \in E \) for all \( i = 0, \ldots, k \) there exists an index \( i \) such that either \((a_i, a_{i+1}) \in S \) or \((a_{i+1}, a_i) \in S \). The value of a cut \( S \) is by definition the number \( \text{val}(S) := \sum \{|w(x, y)| : (x, y) \in S\} \).

We say that a \( X \)-flow (or a \( \bar{X} \)-flow) \( f \) saturates the cut \( S \) if it saturates all edges of \( S \).

Note that for a \( X \)-flow the following are equivalent:

(a) \( f \) saturates \( \partial \);
(b) \( f = w \) on all edges with an end in \( \partial \);
(c) \( f \) has value 1.

Our main combinatorial result is the following:

**Proposition 2.5** (existence of a saturating \( X \)-flow) Let \((V, E, w, \partial)\) be an \( X \)-graph. Then we may find a \( X \)-flow \( f \) which saturates \( \partial \).

We will need the following result present in [8], of which we present a different proof:
Proposition 2.6 (existence of a saturating $X$-flow, [8]) Let $(V'', E'', w'', \tilde{\delta})$ be a $\tilde{X}$-graph with the bound
\[
\sum \{|w''(a, b)| : a \in \tilde{\delta}, (a, b) \in E\} < 1. \tag{2.8}
\]
Then there exists a $\tilde{X}$-flow $f''$ saturating $\tilde{\delta}$.

Proof of Proposition 2.6 We proceed by induction on the number of non-boundary vertices $\#(V'' \setminus \tilde{\delta})$. In the case $\#(V'' \setminus \tilde{\delta}) = 0$ we may take $f'' = w''$ and we conclude.

Suppose that the statement is true when $\#(V'' \setminus \tilde{\delta}) < n$. We may prove it for the case $\#(V'' \setminus \tilde{\delta}) = n$ as follows.

Up to reducing to the connected components, we may assume that the underlying graph $(V'', E'')$ is connected.

Applying the maxflow-mincut theorem [4] we obtain the existence of a $\tilde{X}$-flow of maximum value $f$ and of a minimum value cut $S$ saturated by $f$. Then $S$ separates the graph $(V'', E'')$ into two connected components $V_1 \supset \tilde{\delta}^+$, $V_2 \supset \tilde{\delta}^-$. We obtain $\tilde{X}$-graphs $(V'_i, E_i, w, \tilde{\delta}_i)$, $i = 1, 2$ with $\#(V'_i \setminus \tilde{\delta}) < n$, if we take $V'_i := V_i \cup \{\text{ends of edges in } S\}$, $E_i := [E'' \cap (V_i \times V_i)] \cup S$, $w_i := w''|_{E_i}$ for $i = 1, 2$.

The properties of a $\tilde{X}$-graph are all easy to verify for the $(V'_i, E_i, w, \tilde{\delta}_i)$ except perhaps for property 2. in Definition 2.3, i.e. the fact that the total flux through the boundaries are zero. To prove this we use the bound (2.8) and the integrality condition in the definition of a $\tilde{X}$-graph.

Let $S^\pm$ be the set of edges in $S$ for which $f''(a, b) = \pm w''(a, b)$. We have $\text{val}(f) < 1/2$ as a consequence of (2.8) and of the fact that in the original $\tilde{X}$-graph the total flux through $\tilde{\delta}$ was zero. Since $f''$ saturates $S$ we have

\[
1/2 > \text{val}(S) = \sum_{(a,b)\in S} |w''(a, b)| = \text{val}(S^+) + \text{val}(S^-).
\]

From the integrality condition 1. in Definition 2.3 and from the fact that $S$ disconnects the underlying graph $(V'', E'')$ we deduce that the total flux through $\tilde{\delta}_i$ must be an integer for $i = 1, 2$. At the same time the absolute value of this flux is bounded by $\text{val}(\tilde{\delta}^+) + \text{val}(S) < 1$. Therefore it must be zero, and condition 2. in Definition 2.3 is verified.

We may then apply the inductive hypotheses and obtain flows $f_1, f_2$ saturating $\tilde{\delta}_1, \tilde{\delta}_2$. In particular these flows coincide on $S$ and they extend to $\tilde{X}$-flow $f''$ over the initial $\tilde{X}$-graph.

The fact that $f''$ saturates $\tilde{\delta}$ follows from the fact that $f_i$ saturates $\tilde{\delta}_i$ for $i = 1, 2$. \hfill $\square$

Proof of Theorem 2.5 We proceed by induction on the number of vertices $\#V$. For $\#V = 2$ it suffices to take $f = w$. Let now $n > 2$, assume that the thesis is true whenever $\#V < n$, and consider the case $\#V = n$. Choose as a sink a vertex $a^+ \in V^+$. Consider a $X$-flow $f$ with sink $a^+$ of maximal value. By the maxflow-mincut theorem [4] there exists a cut $S$ realizing the minimum of possible values of cuts between $\partial$ and $a^+$ and such that $f$ saturates $S$ and the value of $f$ equals the value of $S$.

If $\text{val}(f) = 1$ then $f$ saturates $\partial$ and we conclude. If $\text{val}(f) = 0$ then $S$ can be taken to be empty, thus $a^+, \partial$ are in different connected components of the $X$-graph. In this case we may remove the connected component of $a^+$ and reduce to the case $\#V < n$. We then conclude by inductive hypothesis.

Consider now the remaining case when the value of $f$ is in $]0, 1[$. Let $S^\pm$ be the sets of edges in $S$ for which $\text{sgn}(w) = \pm$ and let $S^\pm := \sum_{S^\pm} |w(e)|$. We then conclude from the definition of an $X$-graph and from the fact that $f$ saturates $S$ that
We then replace the $X$-graph $(V, E, \partial, w)$ with the $X$-graph $(V', E', \partial, w')$ defined as follows:

- $V' \subset V$ consists of all vertices in the connected component of $\partial$ with respect to the cut $S$.
- $E'$ consists of all edges in $E \cap (V' \times V')$ and of new edges of the form $(v', \partial), (\partial, v')$ where $(v', x) \in S, v' \in V'$.
- $w'$ is defined to be equal to $w$ on $E \cap V' \times V'$, while for $(v', x) \in S, v' \in V'$ we define $w'(v', \partial) = -w'(\partial, v') = w(v', x)$.

We see that the properties as in the definition of an $X$-graph are trivially valid at vertices $v' \in V \setminus \{\partial\}$ while at $\partial$ they are still valid due to (2.9).

Since $\#V' < \#V$ we may apply the inductive hypothesis to $(V', E', \partial, w')$ and find a $X$-flow $f'$ with sink $b^+ \in (V')^+$ which saturates $\partial$. We then extend it to a $X$-flow $\tilde{f}$ on the original $X$-graph $(V, E, \partial, w)$ as follows.

Note that we may define a $\tilde{X}$-graph $(V'', E'', w'', \tilde{\partial})$ by defining

$$V'' := (V \setminus V') \cup \{\text{ends of edges in } S\},$$
$$E'' := \left[ E \cap (V'' \times V'') \right] \cup S,$$
$$w'' := w|_{E''},$$
$$\tilde{\partial} := \{\text{ends of edges in } S\} \setminus V'.$$

By applying Proposition 2.6 to this $\tilde{X}$-graph we may find a flow $f''$ on it saturating $\tilde{\partial}$. In particular this flow coincides with $f'$ on $\tilde{\partial}$ and the extension $\tilde{f}$ of $f'$ via $f''$ is well-defined and is an $X$-flow, saturating $\partial$, as desired.

See Fig. 3 for the corresponding picture for vector fields $X$ as in Theorem 1.3.

2.3 Proof of Theorem 1.3

Proof By the method described immediately after Definition 2.2 we associate an $X$-graph to the vector field $X$ as in the statement Theorem 1.3 and to the measure $\mu$ corresponding to its Smirnov decomposition. We then apply Proposition 2.5 to this $X$-graph, obtaining a flow $f$.

Defining $\alpha = \chi_{|w|\neq 0} f/w$ we construct a measure $\mu_{\alpha}$ as in (2.7). This measure on arcs gives an $L^1$ vector field $X_{\alpha}$ which in turn decomposes by Smirnov’s Theorem 2.1, via a different measure $\mu'$ this time, into a superposition of arcs $\gamma$ such that

$$s(\gamma) = \partial B^5,$$
$$e(\gamma) = a^+,$$

where the point $a^+$ is the one where the charge corresponding to the sink of the flow $f$ is located.

Note that from the remarks at the beginning of Sect. 2.1 the lengths of such arcs give the $L^1$ norm of the vector field $X_{\alpha}$ and we have

$$\int_{B^5} |X_{\alpha}| = \int |\gamma| d\mu'(\gamma) \leq \int |\gamma| d\mu(\gamma) = \int |X|.$$

The inequality is due to the fact that since $|\alpha| \leq 1$ (i.e. we only decreased the absolute values of weights of curves) the new vector field $X_{\alpha}$ satisfies pointwise a.e. $x$ the inequality $|X_{\alpha}|(x) \leq |X|(x)$. Note however that while $\text{div}(X_{\alpha}) = 8\alpha + |S^4|^{-1} H^4 \ll S^4$ remains
We represent a possible situation in the last step of the proof of Proposition 2.5, seen at the level of the vector fields $X$ as in Theorem 1.3. On the left we represent the boundary $\partial B^5$ and on the right we have a positive charge corresponding to a possible sink of the saturating flow of Proposition 2.5. Some arcs corresponding to the saturating flow are represented by directed lines. These arcs are obtained as concatenations of arcs corresponding to the original vector field $X$, with the same orientation (continuous parts) or reversed orientation (dashed parts). The continuous vertical wiggly line in the center of the figure represents a minimal cut. We enlarging the cut with the dashed part of that line would make it non-minimal.

valid in the sense of distributions, we don’t have anymore the information that $X_\alpha$ is locally smooth on $\mathbb{B}^5 \setminus \{a^+\}$.

For $\mu'$-a.e. $\gamma \in \text{spt}(\mu')$ we have

$$\ell(\gamma) \geq |s([\gamma]) - e([\gamma])|,$$

thus if we denote the arc corresponding to a segment from $a$ to $b$ by $[\gamma] = [a, b]$ then we may write

$$\int_{\mathbb{B}^5} |X_\alpha| = \int \ell(\gamma) d\mu'(\gamma) \geq \int |s([\gamma]) - e([\gamma])| d\mu'(\gamma) = \int_{\mathbb{S}^4} |x - a^+| d\mathcal{H}^4(x). \quad (2.11)$$

Note that

$$\frac{\partial}{\partial t} \bigg|_{t=0} \int_{\mathbb{S}^4} |x - (a^+ + tb)| d\mathcal{H}^4(x) = -b \cdot \int_{\mathbb{S}^4} \frac{x - a^+}{|x - a^+|} d\mathcal{H}^4(x),$$

thus the minimum of (2.11) is achieved when $a^+$ satisfies

$$\frac{1}{|\mathbb{S}^4|} \int_{\mathbb{S}^4} \frac{x - a^+}{|x - a^+|} = 0,$$
i.e. precisely for $a^+ = 0$. The conclusion of Theorem 1.3 now follows from (2.10), (2.11). \hfill \Box

3 Proof of Theorem 1.2

**Proof** We see from (1.4), (1.5) that for the Poincaré-Hodge dual vector field $X$ to $\text{tr}(F \wedge F)$ there holds $\|X\|_{L^1} \leq \|F\|_{L^2}$ with equality in the case of the curvature form $F_{rad}$ as in Theorem 1.2. From the fact that $R^\infty(\mathbb{B}^5)$ is dense in $A_{SU(n)}(\mathbb{B}^5)$ it follows that the infimum of the $L^2$ norm of the curvature given by Theorem 1.1 in the case of the boundary trace $A_{\mathcal{S}^4}$, is equal to the infimum of the same functional over $[A] \in R^\infty(\mathbb{B}^5)$ with trace $\sim A_{\mathcal{S}^4}$. In particular we have that

$$\int_{\mathbb{B}^5} |F_{rad}|^2 = \int_{\mathbb{B}^5} |X_{rad}| = \inf \left\{ \int |X| : X \text{ as in Theorem 1.3} \right\} \leq \inf \left\{ \int |F_A|^2 : [A] \in R^\infty(\mathbb{B}^5), \ i^*_{\partial \mathbb{B}^5} A \sim A_{\mathcal{S}^4} \right\} = \min \left\{ \int |F_A|^2 : [A] \in A_{SU(n)}^{A_{\mathcal{S}^4}}(\mathbb{B}^5) \right\}.$$ 

In particular, since $F_{rad}$ is a competitor in the minimization on the last row, all inequalities above must be equalities and Theorem 1.2 follows. \hfill \Box

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