THE LOW RANK APPROXIMATIONS AND RITZ VALUES IN LSQR FOR LINEAR DISCRETE ILL-POSED PROBLEMS

ZHONGXIAO JIA

Abstract. LSQR and its mathematically equivalent CGLS have been popularly used over the decades for large-scale linear discrete ill-posed problems, where the iteration number \( k \) plays the role of the regularization parameter. It has been long known that if the Ritz values in LSQR converge to the large singular values of \( A \) in natural order until its semi-convergence then LSQR must have the same the regularization ability as the truncated singular value decomposition (TSVD) method and can compute a 2-norm filtering best possible regularized solution. However, hitherto there has been no definitive rigorous result on the approximation behavior of the Ritz values in the context of ill-posed problems. In this paper, for severely, moderately and mildly ill-posed problems, we give accurate solutions of the two closely related fundamental and highly challenging problems on the regularization of LSQR: (i) How accurate are the low rank approximations generated by Lanczos bidiagonalization? (ii) Whether or not the Ritz values involved in LSQR approximate the large singular values of \( A \) in natural order? We also show how to judge the accuracy of low rank approximations reliably during computation without extra cost. Numerical experiments confirm our results.

Key words. Discrete ill-posed, LSQR iterate, TSVD solution, semi-convergence, Lanczos bidiagonalization, Ritz values, near best rank \( k \) approximation, Krylov subspace

AMS subject classifications. 65F22, 15A18, 65F10, 65F20, 65R32, 65J20, 65R30

1. Introduction. Consider the linear discrete ill-posed problem

\[
(1.1) \quad \min_{x \in \mathbb{R}^n} \| Ax - b \| \quad \text{or} \quad Ax = b, \quad A \in \mathbb{R}^{m \times n}, \quad b \in \mathbb{R}^m,
\]

where the norm \( \| \cdot \| \) is the 2-norm of a vector or matrix, and \( A \) is extremely ill-conditioned with its singular values decaying and centered at zero without a noticeable gap, and the right-hand side \( b = b_{\text{true}} + e \) is assumed to be contaminated by a Gaussian white noise \( e \), where \( b_{\text{true}} \) is noise-free and \( \| e \| < \| b_{\text{true}} \| \). Without loss of generality, we assume \( m \geq n \) since the results in this paper hold for the \( m \leq n \) case. (1.1) arises from many applications, such as image deblurring, signal processing, geophysics, computerized tomography, heat propagation, biomedical and optical imaging, and groundwater modeling, to name a few; see, e.g., [1, 7, 8, 26, 28, 29, 30, 31, 35]. Due to the noise \( e \) and the high ill-conditioning of \( A \), the naive solution \( x_{\text{naive}} = A^+ b \) of (1.1) is generally a meaningless approximation to the true solution \( x_{\text{true}} = A^+ b_{\text{true}} \). Therefore, regularization must be used to extract a good approximation to \( x_{\text{true}} \).

For a Gaussian white noise \( e \), throughout the paper, we always assume that \( b_{\text{true}} \) satisfies the discrete Picard condition \( \| A^+ b_{\text{true}} \| \leq C \) with some constant \( C \) for \( \| A^+ \| \) arbitrarily large [1, 10, 15, 16, 18, 20, 29]. Assume that \( Ax_{\text{true}} = b_{\text{true}} \). Then a dominating regularization approach is to solve the following problem

\[
(1.2) \quad \min_{x \in \mathbb{R}^n} \| Lx \| \quad \text{subject to} \quad \| Ax - b \| \leq \tau \| e \|
\]

with \( \tau > 1 \) slightly [18, 20], where \( L \) is a regularization matrix and its suitable choice is based on a-prior information on \( x_{\text{true}} \). If \( L \neq I \), (1.2) can be mathematically transformed into a standard-form problem [18, 20], i.e., a 2-norm filtering regularization problem. In this paper, we always take \( L = I \).

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†Department of Mathematical Sciences, Tsinghua University, 100084 Beijing, China. (jiazz@tsinghua.edu.cn)
The solutions of (1.1) and (1.2) can be analyzed by the means of the singular value decomposition (SVD) of $A$:

\[(1.3) \quad A = U \left( \begin{array}{c} \Sigma \\ 0 \end{array} \right) V^T,\]

where $U = (u_1, u_2, \ldots, u_m) \in \mathbb{R}^{m \times m}$ and $V = (v_1, v_2, \ldots, v_n) \in \mathbb{R}^{n \times n}$ are orthogonal, $
 \Sigma = \text{diag}(\sigma_1, \sigma_2, \ldots, \sigma_n) \in \mathbb{R}^{n \times n}$ with the singular values $\sigma_1 > \sigma_2 > \cdots > \sigma_n > 0$ assumed to be simple throughout the paper, and the superscript $T$ denotes the transpose of a matrix or vector.

From the SVD expansion $x_{true} = \sum_{i=1}^{n} \frac{u_i^T b_{true}}{\sigma_i} v_i$, the discrete Picard condition means that, on average, the Fourier coefficient $|u_i^T b_{true}|$ decays faster than $\sigma_i$, and it enables regularization to compute useful approximations to $x_{true}$. The following common model is used throughout Hansen’s books [18, 20] and the references therein as well as [27] and the current paper:

\[(1.4) \quad |u_i^T b_{true}| = \sigma_i^{1+\beta}, \quad \beta > 0, \quad i = 1, 2, \ldots, n.\]

For the Gaussian white noise $e$, its covariance matrix is $\eta^2 I$, the expected values $\mathcal{E}(\|e\|^2) = m\eta^2$ and $\mathcal{E}(\|u_i^T e\|) = \eta, \quad i = 1, 2, \ldots, n$, so that $\|e\| \approx \sqrt{m}\eta$ and $|u_i^T e| \approx \eta, \quad i = 1, 2, \ldots, n$; see, e.g., [18, p.70-1] and [20, p.41-2]. Under the condition (1.4), for large singular values, the signal term $|u_i^T b_{true}|/\sigma_i$ is dominant relative to the noise term $|u_i^T e|/\sigma_i$. Once $|u_i^T b_{true}| \leq |u_i^T e|$ from some $i$ onwards, the noise $e$ dominates $|u_i^T b|$ for small singular values and must be suppressed. The number $k_0$ satisfying

\[(1.5) \quad |u_{k_0}^T b| \approx |u_{k_0}^T b_{true}| > |u_{k_0+1}^T e| \approx \eta, \quad |u_{k_0+1}^T b| \approx |u_{k_0+1}^T e| \approx \eta\]

is called the transition point; see [18, p.70-1] and [20, p.42, 98].

The truncated SVD (TSVD) method [15, 18, 20] is a reliable and commonly used method for solving a small or medium sized (1.2). It solves a sequence of problems

\[(1.6) \quad \min \|x\| \quad \text{subject to} \quad \|A_k x - b\| = \min\]

starting with $k = 1$ onwards, where $A_k = U_k \Sigma_k V_k^T$ is the 2-norm best rank $k$ approximation to $A$ with $U_k = (u_1, \ldots, u_k)$, $V_k = (v_1, \ldots, v_k)$ and $\Sigma_k = \text{diag}(\sigma_1, \ldots, \sigma_k)$, and $\|A - A_k\| = \sigma_{k+1}$ [3, p.12]. The solution $x_{tsvd}^k = A_k^\dagger b$ to (1.6) is called the TSVD regularized solution, and the index $k$ plays the role of the regularization parameter.

For the Gaussian white noise $e$, it follows from [20, p.71,86-8,95] that $x_{tsvd}^k$ is the best TSVD solution. Moreover, it is known from [8, 18, 20, 35] that $x_{tsvd}^k$ is a 2-norm filtering best possible regularized solution of (1.1) when only deterministic 2-norm filtering regularization methods are taken into account. As a result, we can take $x_{tsvd}^k$ as the standard reference when assessing the regularization ability of a deterministic 2-norm filtering regularization method; for more general elaborations, see [27].

Over the decades Krylov solvers have been popularly used to solve a large (1.1). The methods project (1.1) onto a sequence of low dimensional Krylov subspaces and computes iterates from the subspaces to approximate $x_{true}$ [1, 8, 11, 12, 18, 20, 23]. Of them, the CGLS method [3], which implicitly applies the CG method to $A^T A x = A^T b$, and its mathematically equivalent LSQR algorithm [32] have been most commonly used. They are 2-norm filtering regularization methods, have general regularizing effects [1, 6, 11, 12, 13, 18, 20, 22, 23], and exhibit typical semi-convergence [31, p.89]:
the iterates converge to \( x_{true} \) in an initial stage; then the noise \( e \) starts to deteriorate the iterates so that they start to diverge from \( x_{true} \) and instead converge to \( x_{naive} \); see also [3, p.314], [4, p.733], [18, p.135] and [20, p.110].

It is important to stress two special practical cases. First, if the noise \( e \) is so small that all the \( |u_i^T b| \approx |u_i^T b_{true}| > \eta \), then \( k_0 = n \) in (1.5), meaning that \( x_{n}^{svd} = x_{naive} \) is the best approximation to \( x_{true} \). Second, if \( e \) is such that all the \( |u_i^T e| \approx \eta < \sigma_n \), that is, the noise level \( \|e\| \) is small relative to \( \sigma_n \), then the noise amplification is tolerable even without regularization and \( x_{naive} \) is a good approximation to \( x_{true} \), as has been noticed in [35, p.7]. Both cases show that for a given \( e \), if \( A \) is not ill conditioned enough, regularization does not play a role in the solution process.

Indeed, we have encountered such practical image deblurring problems resulting from two or three dimensional continuous ill-posed problems, e.g., the image deblurring problems \texttt{fanbeamtomo} of \( m = 61200 \), \( n = 14400 \) with the relative noise level \( \|e\|/\|b_{true}\| \leq 10^{-3} \) [2], \texttt{blur} [19], \texttt{parallelomo} [21] of order over ten thousands, three \texttt{GaussianBlur4XX} of \( m = n = 65536 \) [2]. We have found that the singular values \( \sigma_j \) of these matrices, on average, decay more slowly than \( O(\lambda^{-\alpha}) \) with \( \alpha < \frac{1}{2} \) and the ratios \( \sigma_1/\sigma_n \) are quite modest, i.e., \( O(10) \sim O(10^3) \). In view of conditioning, such problems with noise-free right-hand sides \( b_{true} \) seem to be well conditioned ordinary least squares problems or linear systems. We have observed that \( \|e\|/\|b_{true}\| \leq 10^{-3} \) for the former three ones or \( 5 \times 10^{-4} \) for the latter three ones leads to the best TSVD solutions \( x_{n}^{svd} = x_{naive} \). Therefore, there is no semi-convergence, and regularization plays no role for them. In this case, the mentioned Krylov iterative methods solve (1.1) in their standard manners as if they solved an ordinary other than ill-posed problem. On the other hand, if \( e \) is relatively bigger, say, 0.05, then the semi-convergence of the TSVD method and LSQR may occur.

It has long been well known [14, 18, 19, 20] and further addressed in [27] that provided that the Ritz values involved in LSQR approximate the large singular values of \( A \) in natural order until the occurrence of semi-convergence, the best regularized solution obtained by LSQR is as accurate as \( x_{n}^{svd} \). Unfortunately, as stressed by Hanke and Hansen [14], Hansen [19] and many others, e.g., Gazzola and Novati [10], a strict proof of the regularizing properties of conjugate gradients is extremely difficult and proving if the Ritz values converge in this order is a difficult task. In fact, up to now there has been no either general or specific rigorous result on the approximation behavior of the Ritz values.

As matter of fact, as we have observed in [27], the Ritz values converge to the large singular values of \( A \) in natural order for severely ill-posed problems but they may fail to do so at some iterations \( k \leq k^* \) for some moderately and mildly ill-posed problems, where \( k^* \) is the iteration at which the semi-convergence of LSQR occurs. In the latter case, the regularization of LSQR is much more involved, and hitherto nothing has been theoretically known on the accuracy of the best regularized solution by LSQR at semi-convergence, and a common belief seems that LSQR has the partial regularization. However, the numerical experiments in [27] have indicated that the best regularized solutions by LSQR are as accurate as \( x_{n}^{svd} \) even if the Ritz values fail to approximate the large singular values of \( A \) in natural order for some iterations \( k \leq k^* \). For the definition of severely, moderately and mildly ill-posed problems, we refer to [24]; also see [27] for a supplement.

In order to assess the regularization ability of a regularization method, the definition of full and partial regularization is introduced in [25, 27]. For the 2-norm filtering regularization problem (1.2), if a regularized solution is as accurate as \( x_{k_0}^{svd} \), then it is
called a 2-norm filtering best possible regularized solution. If a regularization method
\text{can compute such a best possible one, then it is said to have the full regularization.}
Otherwise, it is said to have only the partial regularization. By such a definition, a
natural and fundamental question is: \textit{Does LSQR have the full or partial regulariza-
tion for severely, moderately and mildly ill-posed problems?} This question was implicit
for CGLS in the survey paper [5] of Björck and Eldén and explicitly posed in [25, 27].

In [27], the author has established a general sin Θ theorem for the 2-norm dis-
tances between the underlying k dimensional Krylov subspace and the k dimensional
dominant right singular subspace of A, and derived accurate estimates for these dis-
tances for severely, moderately and mildly ill-posed problems, respectively. As has
been addressed in [27], these results are the first key and fundamental step towards
to answering the question of LSQR having the full or partial regularization. This
paper is a continuation of [27]. On the basis of [27], for these three kinds of ill-posed
problems, we will give accurate solutions of the two closely related fundamental and
highly challenging problems on the regularization of LSQR: (i) How accurate are the
low rank approximations generated by Lanczos bidiagonalization? (ii) Whether or
not the Ritz values involved in LSQR approximate the large singular values of A in
natural order? We establish accurate estimates for the accuracy of the low rank ap-
proximations and give definitive results on how the Ritz values converge. In addition,
notice that the accuracy of low rank approximations is computationally infeasible
for A large. We show how to judge it reliably during the Lanczos bidiagonalization
process without extra cost.

The paper is organized as follows. In Section 2, we describe the Lanczos bidiag-
onalization process and LSQR, and state some of the results in [27] that are used to
analyze the results in this paper. In Section 3, for severely and moderately problems
with suitable ρ > 1 and α > 1 we prove that the k-step Lanczos bidiagonalization
always generates a near best rank k approximation to A and the k Ritz values always
approximate the large singular values of A in natural order until the occurrence of
semi-convergence. For mildly ill-posed problems, we prove that the above results gen-
erally fail to hold. In Section 5, we establish a monotonic property of the accuracy of
rank k approximations generated by Lanczos bidiagonalization, derive bounds for the
decay rates of entries of the bidiagonal matrices generated by Lanczos bidiagonaliza-
tion, and show that they can be used to reliably judge the accuracy of the low rank
approximations. In Section 6, we report numerical experiments to confirm our results
and make some observations on the regularization of LSQR. Finally, we conclude the
paper and come to the conjecture that LSQR has the full regularization in Section 7.

In the paper, we use \( \mathcal{K}_k(C, w) = \text{span}\{w, Cw, \ldots, C^{k-1}w\} \) to denote the k di-

dimensional Krylov subspace generated by the matrix C and the vector w, and by \( I \)
and the bold letter 0 the identity matrix and the zero matrix with orders clear from
the context, respectively. For the matrix \( B = (b_{ij}) \), define \( |B| = (|b_{ij}|) \), and for
\( |C| = (|c_{ij}|) \), \( |B| \leq |C| \) means \( |b_{ij}| \leq |c_{ij}| \) componentwise.

2. The LSQR algorithm and the estimates for the distances between \( \mathcal{V}_k \)
and \( \text{span}\{V_k\} \). The LSQR algorithm is based on Lanczos bidiagonalization that
computes two orthonormal bases \( \{q_i\}_{i=1}^{k+1} \) and \( \{p_i\}_{i=1}^{k+1} \) of \( \mathcal{K}_k(A^TA, A^Tb) \) and \( \mathcal{K}_{k+1}(AA^T, b) \)
for \( k = 1, 2, \ldots, n \), respectively. For \( k = 1, 2, \ldots, n \), the k-step Lanczos bidiagonaliza-
tion process can be written in the matrix form

\begin{align}
AQ_k &= P_{k+1}B_k, \\
A^TP_{k+1} &= Q_kB_k^T + \alpha_kq_k+1(\epsilon_{k+1}^{(k+1)})^T, 
\end{align}
where $e_{k+1}$ is the $(k+1)$-th canonical basis vector of $\mathbb{R}^{k+1}$, $P_{k+1} = (p_1, p_2, \ldots, p_{k+1})$, $Q_k = (q_1, q_2, \ldots, q_k)$, and

$$B_k = \begin{pmatrix}
\alpha_1 & \alpha_2 & \cdots & \alpha_k \\
\beta_2 & \beta_3 & \cdots & \beta_{k+1}
\end{pmatrix} \in \mathbb{R}^{(k+1) \times k}.$$

It is known from (2.1) that

$$B_k = P_k^T A Q_k.$$  

We remark that the singular values of $B_k$, called the Ritz values of $A$ with respect to the left and right subspaces $\text{span}\{P_{k+1}\}$ and $\text{span}\{Q_k\}$, are all simple. It is easily justified that Lanczos bidiagonalization can be run to completion without breakdown since the starting vector $b$ has nonzero components in all the $u_i$ and the singular values of $A$ are assumed to be simple.

Write $V_R^k = K_k(A^T A, A^T b)$. At iteration $k$, LSQR solves the problem $\|Ax_{lsqr} - b\| = \min_{x \in V_R^k} \|Ax - b\|$ and computes the iterate $x_{lsqr}^k = Q_k y_{lsqr}^k$ with

$$y_{lsqr}^k = \arg \min_{y \in \mathbb{R}^k} \|B_k y - \beta_1 e_1^{(k+1)} \| = \beta_1 B_k^\dagger e_1^{(k+1)},$$

where $e_1^{(k+1)}$ is the first canonical basis vector of $\mathbb{R}^{k+1}$.

Note that $\beta_1 e_1^{(k+1)} = P_k^T b$. From (2.5) we have

$$x_{lsqr}^k = Q_k B_k^\dagger P_{k+1}^T b,$$

which is the minimum 2-norm solution to the perturbed problem that replaces $A$ in (1.1) by its rank $k$ approximation $P_{k+1} B_k Q_k^T$. In [27], a key point is that LSQR has been interpreted as solving

$$\min \|x\| \quad \text{subject to} \quad \|P_{k+1} B_k Q_k^T x - b\| = \min$$

for the regularized solutions $x_{lsqr}^k$ of (1.1) starting with $k = 1$ onwards. Therefore, LSQR is similar to the TSVD method and replaces the best rank $k$ approximation $A_k$ to $A$ by a rank $k$ approximation $P_{k+1} B_k Q_k^T$. Based on this connection, the author [27] has shown that the near best rank $k$ approximation of $P_{k+1} B_k Q_k^T$ to $A$ and the approximations of the $k$ singular values of $B_k$ to the large ones of $A$ in natural order for $k = 1, 2, \ldots, k_0$ are sufficient conditions for LSQR to have the full regularization.

Regarding the semi-convergence points $k^*$ and $k_0$ of LSQR and the TSVD method, the author [27] has proved the following basic property, which is useful to analyze some results and numerical experiments in this paper.

**Theorem 2.1.** The semi-convergence of LSQR must occur at some iteration

$$k^* \leq k_0.$$  

If the Ritz values $\theta_i^{(k)}$ do not converge to the large singular values of $A$ in natural order for some $k \leq k^*$, then $k^* < k_0$, and vice versa.
In terms of the canonical angles $\Theta(\mathcal{X}, \mathcal{Y})$ between two subspaces $\mathcal{X}$ and $\mathcal{Y}$ of equal dimension (cf. [33, p.74-5] and [34, p.43]), the author [27] has established accurate estimates on the accuracy of $V_k^R$ approximating the $k$ dimensional dominant right singular subspace $V_k = \text{span}\{V_k\}$ for severely, moderately and mildly ill-posed problems. Since these estimates play a central role in analyzing the results in the next three sections, we state them as Lemma 2.2 and Theorems 2.3–2.4. To this end, we introduce some notation that appeared in [27] and will be used in this paper. Define

\begin{equation}
\Delta_k = D_2 T_{k2} T_{k1}^{-1} D_1^{-1} \in \mathbb{R}^{(n-k)\times k},
\end{equation}

in which the matrices involved are

\[
D = \text{diag}(\sigma_j u_j^T b) \in \mathbb{R}^{n \times n}, \quad T_k = \begin{pmatrix}
1 & \sigma_1^2 & \cdots & \sigma_1^{2k-2} \\
1 & \sigma_2^2 & \cdots & \sigma_2^{2k-2} \\
\vdots & \vdots & \ddots & \vdots \\
1 & \sigma_n^2 & \cdots & \sigma_n^{2k-2}
\end{pmatrix}
\]

with the partitions

\[
D = \begin{pmatrix}
D_1 & 0 \\
0 & D_2
\end{pmatrix}, \quad T_k = \begin{pmatrix}
T_{k1} \\
T_{k2}
\end{pmatrix}
\]

and $D_1, T_{k1} \in \mathbb{R}^{k \times k}$. Then we have the following precise $\sin \Theta$ theorem on the 2-norm distance between $V_k^R$ and $V_k$.

**Lemma 2.2** ([27]). For $k = 1, 2, \ldots, n - 1$ we have

\begin{equation}
\| \sin \Theta(V_k, V_k^R) \| = \frac{\| \Delta_k \|}{\sqrt{1 + \| \Delta_k \|^2}}
\end{equation}

with $\Delta_k \in \mathbb{R}^{(n-k) \times k}$ defined by (2.8).

From the lemma it is direct to get

\begin{equation}
\| \tan \Theta(V_k, V_k^R) \| = \| \Delta_k \|.
\end{equation}

The following two theorems give estimates for $\| \Delta_k \|$ for the three kinds of ill-posed problems, which have been shown and numerically confirmed to be accurate in [27].

**Theorem 2.3** ([27]). Let the SVD of $A$ be as (1.3). Assume that (1.1) is severely ill-posed with $\sigma_j = O(\rho^{-2})$ and $\rho > 1$, $j = 1, 2, \ldots, n$. Then

\begin{equation}
\| \Delta_k \| \leq \frac{\sigma_{k+1}}{\sigma_k} \frac{\max_{2 \leq i \leq n} |u_i^T b|}{\min_{1 \leq i \leq k} |u_i^T b|} \left( 1 + O(\rho^{-2}) \right).
\end{equation}

**Theorem 2.4** ([27]). For a moderately or mildly ill-posed (1.1) with $\sigma_j = \zeta j^{-\alpha}$, $j = 1, 2, \ldots, n$, where $\alpha > \frac{1}{2}$ and $\zeta > 0$ is some constant, we have

\begin{align*}
\| \Delta_k \| & \leq \frac{\min_{2 \leq i \leq n} |u_i^T b|}{\max_{1 \leq i \leq k} |u_i^T b|} \sqrt{\frac{1}{2\alpha - 1}}, \\
\| \Delta_k \| & \leq \frac{\max_{k+1 \leq i \leq n} |u_i^T b|}{\min_{1 \leq i \leq k} |u_i^T b|} \sqrt{\frac{k^2}{4\alpha^2 - 1} + \frac{k}{2\alpha - 1} |L_{k1}(0)|}, \quad k = 2, 3, \ldots, n - 1.
\end{align*}
where

\[
|L_{k_1}(0)| = \max_{j=1,2,\ldots,k} |L_j^{(k)}(0)|, \quad |L_j^{(k)}(0)| = \prod_{i=1, i \neq j}^{k} \frac{\sigma_i^2}{|\sigma_j - \sigma_i|^2}, \quad j = 1, 2, \ldots, k.
\]

The author in [27] has derived estimates for \(|L_j^{(k)}(0)|, \quad j = 1, 2, \ldots, k\) and \(|L_{k_1}(0)|\) for \(k = 2, 3, \ldots, n - 1\) for the moderately and mildly ill-posed problems. It is shown that \(|L_{k_1}(0)| > 1\) becomes large soon for mildly ill-posed problems as \(k\) increases and it is \(1 + O(k)\) for moderately ill-posed problems with suitable \(\alpha > 1\).

3. The rank \(k\) approximation \(P_{k+1}B_kQ_k^T\) to \(A\). This section is devoted to the study of the quality of rank \(k\) approximations generated by Lanczos bidiagonalization, which replaces the highly ill-conditioned \(A\) in (1.1) and has fundamental effects on the regularization of LSQR. We first establish the following intermediate results, which play a key role in deriving our main results.

**Theorem 3.1.** Let \(\Delta_k \in \mathbb{R}^{(n-k) \times k}\) and \(L_{k_1}(0)\) be defined as (2.8) and (2.14), and \(\Sigma_k = \text{diag}(\sigma_1, \sigma_2, \ldots, \sigma_k)\). Then for severely ill-posed problems we have

\[
\|\Sigma_k \Delta_k^T\| \leq \begin{cases} 
\sigma_1 \max_{1 \leq i, \sigma_i \leq 1} \frac{|u_i^T b|}{|u_i^T b|} \left(1 + O(\rho^{-2})\right) & \text{for } 1 \leq k \leq k_0, \\
\sigma_k \max_{1 \leq i, \sigma_i \leq 1} \frac{|u_i^T b|}{|u_i^T b|} \sqrt{k - k_0 + 1} \left(1 + O(\rho^{-2})\right) & \text{for } k_0 < k \leq n - 1,
\end{cases}
\]

and for moderately or mild ill-posed problems with the singular values \(\sigma_j = \zeta \alpha^{-j}\), \(\alpha > \frac{1}{2}\) and \(\zeta\) a positive constant we have

\[
\|\Sigma_k \Delta_k^T\| \leq \begin{cases} 
\frac{1}{\sqrt{2\alpha - 1}} & \text{for } k = 1, \\
\frac{\sqrt{k^2 - (4\alpha - 1)^2} + \frac{k^2}{2\alpha} - \frac{1}{2\alpha} |L_{k_1}(0)|}{\sqrt{k^2 - (4\alpha - 1)^2} + \frac{k^2}{2\alpha} - \frac{1}{2\alpha} |L_{k_1}(0)|} & \text{for } 1 < k \leq k_0, \\
\frac{kk_0}{\sqrt{k^2 - (4\alpha - 1)^2} + \frac{k^2}{2\alpha} - \frac{1}{2\alpha} |L_{k_1}(0)|} & \text{for } k_0 < k \leq n - 1.
\end{cases}
\]

**Proof.** It has been proved in [27] (cf. the proof of Theorem 4.2 there) that

\[
|\Delta_k| = |D_2T_{k+1}T_{k_1}^{-1}D_1^{-1}| \leq |L_{k_1}(0)||\tilde{\Delta}_k|
\]

with the definition \(L_{1}^{(1)}(0) = 1\), where

\[
|\Delta_k| = \left|\left(\sigma_{k+1}u_{k+1}^T b, \sigma_{k+2}u_{k+2}^T b, \ldots, \sigma_n u_n^T b\right)^T \left(\frac{1}{\sigma_1 u_1^T b}, \frac{1}{\sigma_2 u_2^T b}, \ldots, \frac{1}{\sigma_k u_k^T b}\right)\right|
\]

is a rank one matrix. Therefore, we have

\[
|\Delta_k \Sigma_k| \leq |L_{k_1}(0)| \left|\left(\sigma_{k+1}u_{k+1}^T b, \sigma_{k+2}u_{k+2}^T b, \ldots, \sigma_n u_n^T b\right)^T \left(\frac{1}{u_1^T b}, \frac{1}{u_2^T b}, \ldots, \frac{1}{u_k^T b}\right)\right|
\]

from which we obtain

\[
\|\Sigma_k \Delta_k^T\| = \|\Delta_k \Sigma_k\| \leq \|\Delta_k \Sigma_k\|
\]

\[
\leq |L_{k_1}(0)| \left(\sum_{j=k+1}^{n} \sigma_j^2 |u_j^T b|^2\right)^{1/2} \left(\sum_{j=1}^{k} \frac{1}{|u_j^T b|^2}\right)^{1/2}.
\]
For the severely ill-posed problems and moderately or mildly ill-posed problems, it has been proved in [27] (cf. the proofs of Theorems 4.2 and 4.4, respectively) that

\[ \left( \sum_{j=k+1}^{n} \sigma_j^2 |u_j^T b|^2 \right)^{1/2} = \sigma_{k+1} \max_{k+1 \leq i \leq n} |u_i^T b| (1 + \mathcal{O}(\rho^{-2})) \]

with \(1 + \mathcal{O}(\rho^{-2}) = 1\) for \(k = n - 1\) and

\[ \left( \sum_{j=k+1}^{n} \sigma_j^2 |u_j^T b|^2 \right)^{1/2} \leq \sigma_k \max_{k+1 \leq i \leq n} |u_i^T b| \sqrt{\frac{k}{2\alpha - 1}}, \]

respectively, from which and (3.5) we obtain (3.1) and (3.2) for \(k = 1\).

In order to bound \(\| \Sigma_k \Delta_k^T \|\) for \(k > 1\), we need to estimate \(\left( \sum_{j=1}^{k} \frac{|u_j^T b|^2}{|u_j^T b_j|^2} \right)^{1/2}\). Next we do this for severely and moderately or mildly ill-posed problems, respectively, for each kind of which we need to consider the cases \(k \leq k_0\) and \(k > k_0\) separately.

By the discrete Picard condition (1.4), (1.5) and the properties on \(e\), it is known from [18, p.70-1] and [20, p.41-2] that \(|u_i^T b| = |u_i^T b_{true}| = \sigma_i^{1+\beta} > \eta\) monotonically decreases with \(i = 1, 2, \ldots, k_0\), and becomes stabilized for \(i > k_0\) since \(|u_i^T b| \approx |u_i^T e|\) with the expected values \(\mathcal{E}(|u_i^T e|) = \eta\). Therefore, to present our derivation and results compactly and elegantly, in the later proof we will use the \textit{ideal equality}

\[ |u_i^T b| = |u_i^T b_{true}| = \sigma_i^{1+\beta}, \; i = 1, 2, \ldots, k_0 \]

by dropping the negligible \(|u_i^T e|\).

The case \(k \leq k_0\) for severely ill-posed problems: From (3.8), we have \(\min_{1 \leq i \leq k} |u_i^T b| = |u_k^T b| \leq |u_{j+1}^T b|\) for \(j = 1, 2, \ldots, k - 1\). Making use of (1.4) and (3.8), we obtain

\[
\sum_{j=1}^{k} \frac{1}{|u_j^T b|^2} = \frac{1}{\min_{1 \leq i \leq k} |u_i^T b|^2} \sum_{j=1}^{k} \min_{1 \leq i \leq k} \frac{|u_i^T b|^2}{|u_j^T b|^2} \\
\leq \frac{1}{\min_{1 \leq i \leq k} |u_i^T b|^2} \left( \frac{|u_k^T b|^2}{|u_k^T b|^2} + \sum_{j=1}^{k-1} \frac{|u_{j+1}^T b|^2}{|u_j^T b|^2} \right) \\
\leq \frac{1}{\min_{1 \leq i \leq k} |u_i^T b|^2} \left( 1 + \sum_{j=1}^{k-1} \frac{\sigma_j^{2\beta+1}}{\sigma_j^2} \right) \\
\leq \frac{1}{\min_{1 \leq i \leq k} |u_i^T b|^2} \left( 1 + \mathcal{O} \left( \sum_{j=1}^{k-1} \rho^{2(j-k)} \right) \right) \\
= \frac{1}{\min_{1 \leq i \leq k} |u_i^T b|^2} (1 + \mathcal{O}(\rho^{-2})).
\]

The case \(k > k_0\) for severely ill-posed problems: Exploiting the above result for
\[ k \leq k_0 \text{ and } \min_{1 \leq i \leq k} |u_i^T b| \leq |u_j^T b| \text{ for } j = k_0 + 1, \ldots, k \text{ for } k > k_0, \text{ we obtain} \]

\[
\sum_{j=1}^{k} \frac{1}{|u_j^T b|^2} = \frac{1}{\min_{1 \leq i \leq k} |u_i^T b|^2} \left( \sum_{j=1}^{k} \frac{\min_{1 \leq i \leq k} |u_i^T b|^2}{|u_j^T b|^2} + \sum_{j=k_0+1}^{k} \frac{\min_{1 \leq i \leq k} |u_i^T b|^2}{|u_j^T b|^2} \right) \\
\leq \frac{1}{\min_{1 \leq i \leq k} |u_i^T b|^2} \left( \sum_{j=1}^{k} \frac{\min_{1 \leq i \leq k} |u_i^T b|^2}{|u_j^T b|^2} + \sum_{j=k_0+1}^{k} \frac{\min_{1 \leq i \leq k} |u_i^T b|^2}{|u_j^T b|^2} \right) \\
\leq \frac{1}{\min_{1 \leq i \leq k} |u_i^T b|^2} \left( 1 + \mathcal{O} \left( \sum_{j=1}^{k_0} \rho^{2(j-k_0)} \right) \right) + k - k_0 \right) \\
= \frac{1}{\min_{1 \leq i \leq k} |u_i^T b|^2} \left( 1 + \mathcal{O}(\rho^{-2}) + k - k_0 \right) .
\]

Substitute the above two relations for the two cases into (3.5) and combine them with (3.6) and \(|\zeta_{k_0}^{(k)}(0)| = 1 + \mathcal{O}(\rho^{-2})\) proved in [27]. We then obtain (3.1).

The case \(k \leq k_0\) for moderately or mildly ill-posed problems: In a similar way to the above, we have

\[
\sum_{j=1}^{k} \frac{1}{|u_j^T b|^2} = \frac{1}{\min_{1 \leq i \leq k} |u_i^T b|^2} \sum_{j=1}^{k} \frac{\min_{1 \leq i \leq k} |u_i^T b|^2}{|u_j^T b|^2} \\
\leq \frac{1}{\min_{1 \leq i \leq k} |u_i^T b|^2} \left( 1 + \sum_{j=1}^{k-1} \frac{|u_{j+1}^T b|^2}{|u_j^T b|^2} \right) \\
\leq \frac{1}{\min_{1 \leq i \leq k} |u_i^T b|^2} \left( 1 + \sum_{j=1}^{k-1} \frac{\sigma_{j+1}^2}{\sigma_j^2} \right) \\
\leq \frac{1}{\min_{1 \leq i \leq k} |u_i^T b|^2} \left( 1 + \sum_{j=1}^{k-1} \left( \frac{j}{k} \right) ^{2\alpha} \right) \\
= \frac{1}{\min_{1 \leq i \leq k} |u_i^T b|^2} \left( 1 + \sum_{j=1}^{k-1} \frac{1}{k} \left( \frac{j}{k} \right) ^{2\alpha} \right) \\
\leq \frac{1}{\min_{1 \leq i \leq k} |u_i^T b|^2} \left( 1 + \frac{k}{\int_0^1 x^{2\alpha} \, dx} \right) \\
= \frac{1}{\min_{1 \leq i \leq k} |u_i^T b|^2} \left( 1 + \frac{k}{2\alpha + 1} \right) .
\]

The case \(k > k_0\) for moderately or mildly ill-posed problems: Exploiting the
and Theorem 3.1

Theorems for moderately and mildly ill-posed problems as we would have obtained a bound which not only does not decay but also increases to derive those accurate bounds to be presented. The subtlety to bound (3.12)

\[
\begin{aligned}
\sum_{j=1}^{k} \frac{1}{|u_j^T b|^2} &= \frac{1}{\min_{1 \leq i \leq k} |u_i^T b|^2} \left( \sum_{j=1}^{k_0} \frac{\min_{1 \leq i \leq k} |u_i^T b|^2}{|u_j^T b|^2} + \sum_{j=k_0+1}^{k} \frac{\min_{1 \leq i \leq k} |u_i^T b|^2}{|u_j^T b|^2} \right) \\
&\leq \frac{1}{\min_{1 \leq i \leq k} |u_i^T b|^2} \left( \sum_{j=1}^{k_0} \left( \frac{j}{k_0} \right)^{2\alpha} + k_0 \right) \\
&\leq \frac{1}{\min_{1 \leq i \leq k} |u_i^T b|^2} \left( 1 + \frac{k_0}{2\alpha + 1} + k_0 \right).
\end{aligned}
\]

Substitute the above two bounds for the two cases into (3.5) and combine them with (3.7). We then obtain (3.2).

Regarding the factor \(\frac{\max_{k+1 \leq i \leq n} |u_i^T b|}{\min_{1 \leq i \leq k} |u_i^T b|}\), based on (1.4) and the properties of \(e\) described in the introduction, we can easily justify (cf. [27]) that

\[
\begin{aligned}
\max_{k+1 \leq i \leq n} |u_i^T b| &\approx \frac{|u_{k+1}^T b|}{|u_k^T b|} \approx \sigma_{k+1}^{1+\beta} \sigma_k^{1+\beta} < 1, \ k = 1, 2, \ldots, k_0, \\
\max_{k+1 \leq i \leq n} |u_i^T b| &\approx \frac{|u_{k+1}^T b|}{|u_k^T b|} \approx \frac{\eta}{\eta} = 1, \ k = k_0 + 1, \ldots, n - 1.
\end{aligned}
\]

(3.1) and (3.2) indicate that \(\|\Sigma_k \Delta_k^T\|\) decays swiftly as \(k\) increases up to \(k_0\) for severely and moderately ill-posed problems. Trivially, we have

\[
\sigma_k \|\Delta_k\| \leq \|\Sigma_k \Delta_k^T\| \leq \sigma_1 \|\Delta_k\|.
\]

By carefully comparing the accurate estimates for \(\|\Sigma_k \Delta_k^T\|\) with those for \(\|\Delta_k\|\) in Theorems 2.3–2.4, for \(k \leq k_0\), Theorem 3.1 indicates that

\[
\sigma_k \|\Delta_k\| \leq \|\Sigma_k \Delta_k^T\| \approx \sigma_k \|\Delta_k\|,
\]

that is, the true \(\|\Sigma_k \Delta_k^T\|\) approximately attains its sharpest lower bound, and it is impossible to improve the estimate and get a smaller one. In contrast, for \(k = 2, 3, \ldots, k_0\), if we had estimated it by its simple upper bound

\[
\|\Sigma_k \Delta_k^T\| \leq \|\Sigma_k\| \|\Delta_k^T\| = \sigma_1 \|\Delta_k\|,
\]

we would have obtained a bound which not only does not decay but also increases for moderately and mildly ill-posed problems as \(k\) increases. Such bound is useless to derive those accurate bounds to be presented. The subtlety to bound \(\|\Sigma_k \Delta_k^T\|\) consists in deriving (3.3) and (3.5) and bounding \(\|\Sigma_k \Delta_k^T\|\) as a whole.

Making use of Theorems 2.3–2.4 and Theorem 3.1, in what follows we prove that, at iteration \(k\), Lanczos bidiagonalization generates a near best rank \(k\) approximation to \(A\) and the \(k\) Ritz values \(\theta_i^{(k)}\) approximate the large singular values \(\sigma_i\) of \(A\) in natural order for severely or moderately ill-posed problems with suitable \(\rho > 1\) or \(\alpha > 1\) for \(k \leq k_0\), but these two results fail to hold for mildly ill-posed problems for some \(k \leq k^*\). By Theorem 2.1, this means that \(k^* < k_0\) for mildly ill-posed problems and for severely or moderately ill-problems with \(\rho > 1\) or \(\alpha > 1\) not enough.

Define

\[
\gamma_k = \|A - P_{k+1}B_k Q_k^T\|,
\]

and the following theorem.
which measures the accuracy of the rank $k$ approximation $P_{k+1}B_kQ_k^T$ to $A$. We will introduce the precise definition of near best rank $k$ approximation to $A$ later on.

**Theorem 3.2.** Let $|L_k^{(k)}(0)|$ be defined by (2.14). Then we have

$$
(3.13) \quad \sigma_{k+1} \leq \gamma_k \leq \sqrt{1 + \eta_k^2 \sigma_{k+1}}
$$

with

$$
(3.14) \quad \eta_k \leq \begin{cases} 
\xi_k \frac{\max_{1 \leq i \leq n} |u_i^T b|}{\min_{1 \leq i \leq k} |u_i^T b|} (1 + \mathcal{O}(\rho^{-2})) & \text{for } 1 \leq k \leq k_0, \\
\xi_k \frac{\max_{1 \leq i \leq n} |u_i^T b|}{\min_{1 \leq i \leq k} |u_i^T b|} \sqrt{k - k_0 + 1} (1 + \mathcal{O}(\rho^{-2})) & \text{for } k_0 < k \leq n - 1
\end{cases}
$$

for severely ill-posed problems with $\sigma_i = \mathcal{O}(\rho^{-i})$, $i = 1, 2, \ldots, n$ and

$$
(3.15) \quad \eta_k \leq \begin{cases} 
\xi_k \frac{\max_{1 \leq i \leq n} |u_i^T b|}{\min_{1 \leq i \leq k} |u_i^T b|} \sqrt{\frac{1}{2\alpha - 1}} & \text{for } k = 1, \\
\xi_k \frac{\max_{1 \leq i \leq n} |u_i^T b|}{\min_{1 \leq i \leq k} |u_i^T b|} \sqrt{\frac{k}{4\alpha - 1} + \frac{k - k_0 + 1}{\alpha - 1}} |L_k^{(k)}(0)| & \text{for } 1 < k \leq k_0, \\
\xi_k \frac{\max_{1 \leq i \leq n} |u_i^T b|}{\min_{1 \leq i \leq k} |u_i^T b|} \sqrt{\frac{k_0}{4\alpha - 1} + \frac{k(k - k_0 + 1)}{\alpha - 1}} |L_k^{(k)}(0)| & \text{for } k_0 < k \leq n - 1
\end{cases}
$$

for moderately or mildly ill-posed problems with $\sigma_i = \zeta_i^{-\alpha}$, $i = 1, 2, \ldots, n$, where $\xi_k = \sqrt{\left(\frac{\|\Delta_k\|}{1 + \|\Delta_k\|^2}\right)^2 + 1}$ for $\|\Delta_k\| < 1$ and $\xi_k \leq \frac{\sqrt{\alpha}}{2}$ for $\|\Delta_k\| \geq 1$.

**Proof.** Since $A_k$ is the best rank $k$ approximation to $A$ and $\|A - A_k\| = \sigma_{k+1}$, the lower bound in (3.13) holds trivially. Next we prove the upper bound.

From (2.1), we obtain

$$
(3.16) \quad \gamma_k = \|A - P_{k+1}B_kQ_k^T\| = \|A - A_kQ_k^T\| = \|A(I - Q_kQ_k^T)\|.
$$

From the proof of Lemma 2.2 (cf. Lemma 4.1 [27]), we obtain

$$
V_k^R = K_k(A^TA^Tb) = span\{Q_k\} = span\{Z_k\},
$$

where the orthonormal $Q_k$ is generated by Lanczos bidiagonalization and

$$
(3.17) \quad \hat{Z}_k = (V_k + V_k^\perp \Delta_k)(I + \Delta_k^T \Delta_k)^{-\frac{1}{2}}.
$$

Therefore, the orthogonal projector onto $V_k^R$ is $Q_kQ_k^T = \hat{Z}_k \hat{Z}_k^T$. Keep in mind that $A_k = U_k \Sigma_k V_k^T$. It is direct to justify that $(U_k \Sigma_k V_k^T)(A - U_k \Sigma_k V_k^T) = 0$ for $k = 1, 2, \ldots, n - 1$. Exploiting this and noting that $\|I - \hat{Z}_k \hat{Z}_k^T\| = 1$ and $V_k^TV_k^\perp = 0$ for
Define the function

\[
(\lambda_k) \quad \text{and} \quad \kappa_k \leq \sigma_k
\]

for \( k = 1, 2, \ldots, n - 1 \), we get from (3.16) and (3.17) that

\[
\gamma_k^2 = \| (A - U_k \Sigma_k V_k^T + U_k \Sigma_k V_k^T)(I - Z_k Z_k^T) \|^2 \\
= \max_{\|y\| = 1} \| (A - U_k \Sigma_k V_k^T + U_k \Sigma_k V_k^T)(I - Z_k Z_k^T)y \|^2 \\
= \max_{\|y\| = 1} \| (A - U_k \Sigma_k V_k^T)(I - Z_k Z_k^T)y \|^2 + \| U_k \Sigma_k V_k^T(I - Z_k Z_k^T)y \|^2 \\
= \max_{\|y\| = 1} \left( \| (A - U_k \Sigma_k V_k^T)(I - Z_k Z_k^T)y \|^2 + \| U_k \Sigma_k V_k^T(I - Z_k Z_k^T)y \|^2 \right) \\
\leq \| (A - U_k \Sigma_k V_k^T)(I - Z_k Z_k^T) \|^2 + \| U_k \Sigma_k V_k^T(I - Z_k Z_k^T) \|^2 \\
\leq \sigma_k^2 + \| \Sigma_k V_k^T(I - Z_k Z_k^T) \|^2 \\
= \sigma_k^2 + \| \Sigma_k V_k^T(I - (V_k + V_k^T \Delta_k)(I + \Delta_k \Delta_k)^{-1}(V_k + V_k^T \Delta_k)^T) \|^2 \\
= \sigma_k^2 + \| \Sigma_k V_k^T(I + \Delta_k \Delta_k)^{-1}(V_k + V_k^T \Delta_k)^T \|^2 \\
= \sigma_k^2 + \| \Sigma_k (I + \Delta_k \Delta_k)^{-1}(I + \Delta_k \Delta_k)^{-1}(V_k + V_k^T \Delta_k)^T \|^2 \\
= \sigma_k^2 + \| \Sigma_k (I + \Delta_k \Delta_k)^{-1}\Delta_k \Delta_k \| \| (I + \Delta_k \Delta_k)^{-1}(V_k + V_k^T \Delta_k)^T \|^2 \\
\leq \sigma_k^2 + \| \Sigma_k (I + \Delta_k \Delta_k)^{-1}\Delta_k \Delta_k \|^2 + \| \Sigma_k (I + \Delta_k \Delta_k)^{-1}\Delta_k \Delta_k \|^2 \\
= \sigma_k^2 + \epsilon_k.
\]

(3.18)

We estimate \( \epsilon_k \) accurately below. To this end, we need to use two key identities and some results related. By the SVD of \( \Delta_k \), it is direct to justify that

\[
(I + \Delta_k \Delta_k)^{-1}\Delta_k \Delta_k = \Delta_k \Delta_k(I + \Delta_k \Delta_k)^{-1}
\]

and

\[
(I + \Delta_k \Delta_k)^{-1}\Delta_k \Delta_k = \Delta_k \Delta_k(I + \Delta_k \Delta_k)^{-1}.
\]

Define the function \( f(\lambda) = \frac{\lambda}{1 + \lambda^2} \) with \( \lambda \in [0, \infty) \). Since the derivative \( f'(\lambda) = \frac{1 - \lambda^2}{(1 + \lambda^2)^2} \), \( f(\lambda) \) is monotonically increasing for \( \lambda \in [0, 1] \) and decreasing for \( \lambda \in [1, \infty) \), and the maximum of \( f(\lambda) \) over \( \lambda \in [0, \infty) \) is \( \frac{1}{2} \), which attains at \( \lambda = 1 \). Based on these properties and exploiting the SVD of \( \Delta_k \), we obtain

\[
\| \Delta_k(I + \Delta_k \Delta_k)^{-1} \| = \frac{\| \Delta_k \|}{1 + \| \Delta_k \|^2}
\]

for \( \| \Delta_k \| \leq 1 \) and

\[
\| \Delta_k(I + \Delta_k \Delta_k)^{-1} \| \leq \frac{1}{2}
\]

for \( \| \Delta_k \| \geq 1 \). (Note: in this case, since \( \Delta_k \) may have at least one singular value smaller than one, we do not have the expression (3.21)). It then follows from (3.18), (3.21), (3.22) and \( \| (1 + \Delta_k \Delta_k)^{-1} \| \leq 1 \) that

\[
\epsilon_k^2 = \| \Sigma_k \Delta_k(I + \Delta_k \Delta_k)^{-1} \|^2 + \| \Sigma_k \Delta_k(I + \Delta_k \Delta_k)^{-1} \|^2 \\
\leq \| \Sigma_k \Delta_k \|^2 \| \Delta_k(I + \Delta_k \Delta_k)^{-1} \|^2 + \| \Sigma_k \Delta_k \|^2 \| (I + \Delta_k \Delta_k)^{-1} \|^2 \\
\leq \| \Sigma_k \Delta_k \|^2 \| (I + \Delta_k \Delta_k)^{-1} \|^2 + \| \Sigma_k \Delta_k \|^2 \\
= \| \Sigma_k \Delta_k \|^2 \left( \frac{\| \Delta_k \|^2}{1 + \| \Delta_k \|^2} + 1 \right) = \epsilon_k^2 \| \Sigma_k \Delta_k \|^2
\]

(3.23)
for \( ||\Delta_k|| < 1 \) and
\[
e_k \leq ||\Sigma_k \Delta_k^T|| \sqrt{||\Delta_k (I + \Delta_k^T \Delta_k)^{-1}||^2 + 1} = \xi_k ||\Sigma_k \Delta_k^T|| \leq \frac{\sqrt{5}}{2} ||\Sigma_k \Delta_k^T||
\]
for \( ||\Delta_k|| \geq 1 \). Replace \( ||\Sigma_k \Delta_k^T|| \) by its bounds (3.1) and (3.2) in the above, insert the resulting bounds for \( e_k \) into (3.18), and let \( e_k = \eta_k \sigma_{k+1} \). Then we obtain the upper bound in (3.13) with \( \eta_k \) satisfying (3.14) and (3.15) for severely and moderately or mildly ill-posed problems, respectively.

Remark 3.1. For severely ill-posed problems, from Theorem 2.3 and (3.9)–(3.10) we approximately have
\[
\begin{align}
(3.24) \quad ||\Delta_k|| \leq \frac{\sigma_{k+1}^{2+\beta}}{\sigma_k^{2+\beta}} (1 + O(\rho^{-2})) \sim \rho^{-2+\beta}, & \quad k = 1, 2, \ldots, k_0, \\
(3.25) \quad ||\Delta_k|| \leq \frac{\sigma_{k+1}^{2+\beta}}{\sigma_k^{2+\beta}} (1 + O(\rho^{-2})) \sim \rho^{-1}, & \quad k = k_0 + 1, \ldots, n - 1,
\end{align}
\]
from which and the definition of \( \xi_k \) it follows that
\[
\xi_k (1 + O(\rho^{-2})) = 1 + O(\rho^{-2})
\]
for both \( k \leq k_0 \) and \( k > k_0 \). Therefore, from (3.14), for \( k \leq k_0 \) we obtain
\[
(3.26) \quad \eta_k \leq \xi_k \frac{|u_{k+1}^T b|}{|u_k^T b|} (1 + O(\rho^{-2})) = \frac{|u_{k+1}^T b|}{|u_k^T b|} (1 + O(\rho^{-2})) = \rho^{1-\beta} < 1
\]
by dropping the smaller \( O(\rho^{-3+\beta}) \). From (3.10), for \( k > k_0 \) we obtain
\[
(3.27) \quad \eta_k \leq \xi_k \frac{\max_{k+1 \leq i \leq n} |u_i^T b|}{\min_{1 \leq i \leq k} |u_i^T b|} \sqrt{k - k_0 + 1} (1 + O(\rho^{-2})) \sim \sqrt{k - k_0 + 1}.
\]

Remark 3.2. From (3.13), (3.14) and (3.26), for severely ill-posed problems and \( k \leq k_0 \) we have
\[
1 < \sqrt{1 + \eta_k^2} < 1 + \frac{1}{2} \eta_k^2 \leq 1 + \frac{1}{2} \frac{\sigma_{k+1}^{2(1+\beta)}}{\sigma_k^{2(1+\beta)}} \sim 1 + \frac{1}{2} \rho^{-2(1+\beta)},
\]
which and (3.13) indicate that \( \gamma_k \) is an accurate approximation to \( \sigma_{k+1} \). Thus, the rank \( k \) approximation \( P_{k+1} B_k Q_k^T \) is as accurate as the best rank \( k \) approximation \( A_k \) within the factor \( \sqrt{1 + \eta_k^2} \approx 1 \) for suitable \( \rho > 1 \) and \( k \leq k_0 \). In contrast, (3.13) and (3.27) shows that \( \gamma_k \) is a marginally less accurate approximation to \( \sigma_{k+1} \) for \( k > k_0 \).

Remark 3.3. For the moderately or mildly ill-posed problems with \( \sigma_i = \zeta i^{-\alpha} \), from the estimate (3.15) for \( \eta_k \), for \( k \leq k_0 \) we approximately have
\[
(3.28) \quad \frac{\sigma_k}{\sigma_{k+1}} ||\Delta_k|| \leq \eta_k \leq \frac{\sqrt{5}}{2} \frac{\sigma_k}{\sigma_{k+1}} ||\Delta_k||.
\]
Therefore, based on Theorems 2.4–2.4, it is known that \( P_{k+1} B_k Q_k^T \) is almost as accurate as \( A_k \) for suitable \( \alpha > 1 \) and \( k \leq k^* \) with \( k^* \) not large.

Remark 3.4. For both severely and moderately ill-posed problems, we notice that the situation is not so satisfying for increasing \( k > k_0 \). But at that time, a possibly
big $\eta_k$ does not do harm to our regularization purpose since Theorem 2.1 shows that the semi-convergence of LSQR must occur at some iteration $k^* \leq k_0$, and LSQR is stopped once its semi-convergence is practically identified.

Remark 3.5. For mildly ill-posed problems, the situation is fundamentally different. We have $\sqrt{\frac{k^2}{4n^2} + \frac{k}{2a}} > 1$ and $|L_k^{(j)}(0)| > 1$ considerably as $k$ increases up to $k_0$ because of $\frac{1}{2} < \alpha \leq 1$, leading to $\eta_1 > 1$ substantially. This means that for some iterations $k \leq k^*$, $\gamma_k$ may be substantially bigger than $\sigma_{k+1}$ and can well lie between $\sigma_k$ and $\sigma_1$. In this case, the rank $k$ approximation $P_k B_k Q_k^T$ is considerably less accurate than the best rank $k$ approximation $A_k$.

Remark 3.6. This theorem is different from Theorem 2.7 in [25]. There are several subtle treatments in the proof of Theorem 3.2, and ignoring or missing any one of them would make it impossible to obtain accurate estimates for $\gamma_k$. The first is to treat $\|U_k \Sigma_k V_k^T (I - \hat{Z}_k \hat{Z}_k^T)\|$ as a whole. If we amplified it by

$$
\|U_k \Sigma_k V_k^T (I - \hat{Z}_k \hat{Z}_k^T)\| \leq \|\Sigma_k\| \|V_k^T (I - \hat{Z}_k \hat{Z}_k^T)\| = \sigma_1 \sin \Theta(V_k, V_k^R),
$$

as in the proof of Theorem 2.7 in [25], we would obtain a too large overestimate, which is almost constant for severely ill-posed problems for $k = 1, 2, \ldots, k^*$ and increases with $k$ for moderately and mildly ill-posed problems. The second is the use of (3.19) and (3.20). The third is the extraction of $\|\Sigma_k \Delta_k^T\|$ from (3.23) other than amplify it to $\|\Sigma_k\| \|\Delta_k\| = \sigma_1 \|\Delta_k\|$. The fourth is accurate estimates for $\|\Sigma_k \Delta_k^T\|$; see (3.1) and (3.2) in Theorem 3.1. For example, without using (3.19) and (3.20), we would obtain

$$
\epsilon_k^2 \leq \|\Sigma_k\|^2 \|I + \Delta_k^T \Delta_k\|^{-1} \Delta_k^T \Delta_k\|^2 + \|\Sigma_k\|^2 \|I + \Delta_k^T \Delta_k\|^{-1} \Delta_k^T \Delta_k\|^2
$$

$$
= \sigma_1^2 \left( \frac{\|\Delta_k\|^2}{1 + \|\Delta_k\|^2} \right)^2 + \sigma_1^2 \|I + \Delta_k^T \Delta_k\|^{-1} \Delta_k^T \Delta_k\|^2.
$$

From (3.22) and the previous estimates for $\|\Delta_k\|$, such bound is too pessimistic, and it even does not decrease and become small as $k$ increases, while our estimates for $\epsilon_k = \eta_k \sigma_{k+1}$ in Theorem 3.2 are optimal and can decay swiftly with $k$.

As it will turn out in the next section, there are intimate relationships between the quality of the rank $k$ approximation $P_{k+1} B_k Q_k^T$ and the approximation behavior of the Ritz values $\vartheta_i^{(k)}$, $i = 1, 2, \ldots, k$. To be precise, we introduce the following definition of a near best rank $k$ approximation to $A$: By definition (3.12), the rank $k$ matrix $P_{k+1} B_k Q_k^T$ is called a near best rank $k$ approximation to $A$ if it satisfies

$$
\sigma_{k+1} \leq \gamma_k < \frac{\sigma_k + \sigma_{k+1}}{2},
$$

that is, $\gamma_k$ lies between $\sigma_{k+1}$ and $\sigma_k$ and is closer to $\sigma_{k+1}$.

Based on Theorem 3.2, for the severely and moderately or mildly ill-posed problems with the model singular values $\sigma_i = \zeta \rho^{-i}$ and $\sigma_i = \zeta i^{-\alpha}$, $i = 1, 2, \ldots, n$ where $\rho > 1$ and $\alpha > \frac{1}{2}$, we next derive the sufficient conditions on $\rho$ and $\alpha$ that guarantee that $P_{k+1} B_k Q_k^T$ is a near best rank $k$ approximation to $A$ for $k = 1, 2, \ldots, k^*$.

**Theorem 3.3.** For a given (1.1), $P_{k+1} B_k Q_k^T$ is a near best rank $k$ approximation to $A$ if

$$
\sqrt{1 + \eta_k^2} < \frac{1}{2} \frac{\sigma_k}{\sigma_{k+1}} + \frac{1}{2}.
$$
Furthermore, \( P_{k+1}B_kQ_k^T \) is a near best rank \( k \) approximation to \( A \) if \( \rho > 2 \) for the severely ill-posed problems with \( \sigma_i = \zeta \rho^{-i}, \ i = 1, 2, \ldots, n \) or \( \alpha \) satisfies

\[
(3.31) \quad 2 \sqrt{1 + \eta_k^2} - 1 < \left( \frac{k + 1}{k} \right)^\alpha, \ k = 1, 2, \ldots, k^*
\]

for the moderately and mildly ill-posed problems with \( \sigma_i = \zeta i^{-\alpha} \) and \( \alpha > \frac{1}{2}, \ i = 1, 2, \ldots, n \).

Proof. By (3.13), we see that \( \gamma_k \leq \sqrt{1 + \eta_k^2} \sigma_{k+1} \). Therefore, \( P_{k+1}B_kQ_k^T \) is a near best rank \( k \) approximation to \( A \) in the sense of (3.29) provided that

\[
\sqrt{1 + \eta_k^2} \sigma_{k+1} < \frac{\sigma_k + \sigma_{k+1}}{2},
\]

from which (3.30) follows.

From (3.26), for the severely ill-posed problems with \( \sigma_i = \zeta \rho^{-i}, \ i = 1, 2, \ldots, n \) we have

\[
(3.32) \quad \sqrt{1 + \eta_k^2} < 1 + \frac{1}{2} \eta_k^2 < 1 + \frac{1}{2} \rho^{-2(1 + \beta)} < 1 + \rho^{-1}, \ k = 1, 2, \ldots, k^*.
\]

Since \( \sigma_k/\sigma_{k+1} = \rho \), (3.30) holds provided that

\[
1 + \rho^{-1} < \frac{1}{2} \rho + \frac{1}{2},
\]

i.e., \( \rho^2 - \rho - 2 > 0 \), solving which for \( \rho \) we get \( \rho > 2 \). For the moderately or mildly ill-posed problems with \( \sigma_i = \zeta i^{-\alpha} \) and \( \alpha > \frac{1}{2}, \ i = 1, 2, \ldots, n \), we have \( \frac{\sigma_k}{\sigma_{k+1}} = (\frac{k+1}{k})^\beta \), from which and (3.30) it is direct to obtain (3.31). \( \square \)

Remark 3.7. For severely ill-posed problems with the model singular values \( \sigma_i = \zeta \rho^{-i} \) and \( k = 1, 2, \ldots, k^* \), one always obtains a near best rank \( k \) approximation \( P_{k+1}B_kQ_k^T \) to \( A \) provided \( \rho > 2 \).

Remark 3.8. For the moderately ill-posed problems with \( \sigma_i = \zeta i^{-\alpha} \), on the one hand, for each fixed \( k \leq k^* \), there must be \( \alpha > 1 \) such that (3.31) holds since, by (3.15), its left-hand side decreases to zero and the right-hand side increases to infinity with respect to \( \alpha \); the bigger \( k \) is, the bigger \( \alpha > 1 \) is required. Therefore, there exists a suitable \( \alpha > 1 \) that guarantees that \( P_{k+1}B_kQ_k^T \) is a near best rank \( k \) approximation to \( A \) for all \( k \leq k^* \). On the other hand, for a given problem (1.1) with \( \alpha > 1 \) fixed, the smaller \( k \) is, the more easily (3.31) is met as the left-hand side decreases and the right-hand side increases with respect to \( k \). As a consequence, it is more possible that \( P_{k+1}B_kQ_k^T \) is a near best rank \( k \) approximations to \( A \) for \( k \) small and it may not be a near best rank \( k \) approximation at some iterations \( k \leq k^* \) if \( \alpha \) is not big enough.

Remark 3.9. For the mildly ill-posed problems with \( \sigma_i = \zeta i^{-\alpha} \), Theorem 2.4 has shown that \( \| \Delta_k \| \) is generally not small and can be large as \( k \) increases up to \( k^* \). From (3.28), we see that the size of \( \eta_k \) is comparable to \( \| \Delta_k \| \). Note that \( (\frac{k+1}{k})^\alpha \leq 2 \) for \( \frac{1}{2} < \alpha \leq 1 \) and any \( k \geq 1 \). Consequently, (3.31) may be satisfied only for \( k \) very small and \( \alpha \) not close to \( \frac{1}{2} \), and it cannot be met generally as \( k \) increases. Hence \( P_{k+1}B_kQ_k^T \) may be a near best rank \( k \) approximation to \( A \) no longer soon as \( k \) increases.

4. The approximation properties of the Ritz values \( \theta_i^{(k)} \). In this section, we make an in-depth analysis on the approximation behavior of the Ritz values \( \theta_i^{(k)} \). This problem has been highly concerned since the use of LSQR in the context of ill-posed problems, but has remained open.
Starting with Theorem 3.2, we prove that, under certain sufficient conditions on \( p \) and \( \alpha \) for the severely and moderately ill-posed problems with \( \sigma_i = \xi \rho^{-i} \) and \( \sigma_i = \xi i^{-\alpha} \), respectively, the \( k \) Ritz values \( \theta_i^{(k)} \) approximate the large singular values \( \sigma_i \) of \( A \) in natural order for \( k = 1, 2, \ldots, k^* \), which means that the semi-convergence of LSQR occurs at iteration \( k^* = k_0 \) and no Ritz value smaller than \( \sigma_{k_0 + 1} \) appears before \( k \leq k_0 \). Combining this result with Theorem 3.3, we come to the definitive conclusion that LSQR has the full regularization for these two kinds of problems for suitable \( \rho > 1 \) and \( \alpha > 1 \). On the other hand, we will show that for some \( k \leq k^* \) the Ritz values generally do not approximate the large singular values of \( A \) in natural order for severely or moderately ill-posed problems with \( \rho > 1 \) or \( \alpha > 1 \) not enough and mildly ill-posed problems, which, by Theorem 2.1, means that \( k^* < k_0 \).

**Theorem 4.1.** Assume that (1.1) is severely ill-posed with \( \sigma_i = \xi \rho^{-i} \) and \( \rho > 1 \) or moderately and mildly ill-posed with \( \sigma_i = \xi i^{-\alpha} \), \( i = 1, 2, \ldots, n \), and let the Ritz values \( \theta_i^{(k)} \) be labeled as \( \theta_1^{(k)} > \theta_2^{(k)} > \cdots > \theta_n^{(k)} \). Then

\[
0 < \sigma_i - \theta_i^{(k)} \leq \sqrt{1 + \eta_k^2} \sigma_{k+1}, \quad i = 1, 2, \ldots, k.
\]

For \( k = 1, 2, \ldots, k^* \), if \( \rho \geq 1 + \sqrt{2} \) or \( \alpha \) satisfies

\[
1 + \sqrt{1 + \eta_k^2} < \left( \frac{k + 1}{k} \right)^{\alpha},
\]

then the \( k \) Ritz values \( \theta_i^{(k)} \) strictly interlace the first large \( k + 1 \) singular values of \( A \):

\[
\sigma_{i+1} < \theta_i^{(k)} < \sigma_i, \quad i = 1, 2, \ldots, k,
\]

which means that the \( \theta_i^{(k)} \) approximate the first \( k \) large \( \sigma_i \) in natural order.

**Proof.** Note that, for \( k = 1, 2, \ldots, k^* \), the \( k \) Ritz values \( \theta_i^{(k)} \) are just the nonzero singular values of \( P_{k+1}B_kQ_k^T \), whose other \( n - k \) singular values are zeros. We write

\[
A = P_{k+1}B_kQ_k^T + (A - P_{k+1}B_kQ_k^T).
\]

Since \( ||A - P_{k+1}B_kQ_k^T|| = \gamma_k \), by the Mirsky’s theorem of singular values [34, p.204, Theorem 4.11] we have

\[
|\sigma_i - \theta_i^{(k)}| \leq \sqrt{1 + \eta_k^2} \sigma_{k+1}, \quad i = 1, 2, \ldots, k.
\]

Since the singular values of \( A \) are simple and \( b \) has components in all the left singular vectors \( u_1, u_2, \ldots, u_n \) of \( A \), Lanczos bidiagonalization can be run to completion without breakdown, producing \( P_{n+1}, Q_n \) and the lower bidiagonal \( B_n \in \mathbb{R}^{(n+1)\times n} \) such that

\[
P^T A Q_n = \begin{pmatrix} B_n & \mathbf{0} \end{pmatrix}
\]

with \( P = (P_{n+1}, \hat{P}) \in \mathbb{R}^{m \times m} \) and \( Q_n \in \mathbb{R}^{n \times n} \) being orthogonal and the diagonals \( \alpha_i \) and subdiagonals \( \beta_{i+1}, i = 1, 2, \ldots, n \), of \( B_n \) being positive.\(^1\) Notice that the singular values

\[\xi \rho^{-i} \leq \sigma_i \leq \xi i^{-\alpha}, \quad i = 1, 2, \ldots, n\]

\(^1\) If \( m = n \), it is easily justified that \( \beta_{n+1} = 0 \), Lanczos bidiagonalization generates the orthogonal matrices \( P_n, Q_n \) and the \( n \times n \) lower bidiagonal \( B_n \) with the positive diagonals \( \alpha_i \) and subdiagonals \( \beta_i \). This does not affect the derivation and results followed, and we only need to replace \( P_{n+1} \) by \( P_n \).
values of $B_k$, $k = 1, 2, \ldots, n$, are all simple and $B_k$ consists of the first $k$ columns of $B_n$ with the last $n - k$ zero rows deleted. Applying the Cauchy’s strict interlacing theorem [34, p.198, Corollary 4.4] to the singular values of $B_k$ and $B_n$, we have

\begin{equation}
\sigma_{n-k+i} < \theta^{(k)}_i < \sigma_i, \ i = 1, 2, \ldots, k.
\end{equation}

Therefore, (4.4) becomes

\begin{equation}
0 < \sigma_i - \theta^{(k)}_i \leq \gamma_k \leq \sqrt{1 + \eta^2_k} \sigma_{k+1}, \ i = 1, 2, \ldots, k,
\end{equation}

which proves (4.1).

For $i = 1, 2, \ldots, k$, notice that $\rho^{-(k+i)} - (1 + \rho^{-1}) \leq 1$. Then from (4.7), (3.32) and $\sigma_i = \zeta \rho^{-i}$ we obtain

\begin{equation*}
\theta^{(k)}_i \geq \sigma_i - \gamma_k \geq \sigma_i - (1 + \rho^{-1}) \sigma_{k+1}
= \zeta \rho^{-i} - (1 + \rho^{-1}) \rho^{-(k+i)}
= \zeta \rho^{-i} \left( (1 + \rho^{-1}) \rho^{-k+i} \right)
\geq \zeta \rho^{-i} \left( (1 + \rho^{-1}) \right)
\geq \zeta \rho^{-i} = \sigma_{i+1},
\end{equation*}

provided that $\rho - (1 + \rho^{-1}) = \rho - \rho^{-1} - 1 \leq 1$. Solving the inequality gives $\rho \geq 1 + \sqrt{2}$. Together with the upper bound of (4.6), we have proved (4.3).

For the moderately and mildly ill-posed problems with $\sigma_i = \zeta i^{-\alpha}, \ i = 1, 2, \ldots, n$ and $k = 1, 2, \ldots, k^*$, we get

\begin{equation*}
\theta^{(k)}_i \geq \sigma_i - \gamma_k \geq \sigma_i - \sqrt{1 + \eta^2_k} \sigma_{k+1}
= \zeta i^{-\alpha} - \zeta \sqrt{1 + \eta^2_k (k+1)^{-\alpha}}
= \zeta (i+1)^{-\alpha} \left( \frac{i+1}{k+1} \right)^{\alpha} - \sqrt{1 + \eta^2_k} \left( \frac{i+1}{k+1} \right)^{\alpha}
\geq \zeta (i+1)^{-\alpha} = \sigma_{i+1},
\end{equation*}

i.e., (4.3) holds, provided that $\eta_k$ and $\alpha > 1$ satisfy

\begin{equation*}
\left( \frac{i+1}{k+1} \right)^{\alpha} - \sqrt{1 + \eta^2_k} \left( \frac{i+1}{k+1} \right)^{\alpha} > 1,
\end{equation*}

which means that

\begin{equation*}
\sqrt{1 + \eta^2_k} \leq \left( \frac{i+1}{k+1} \right)^{\alpha} - 1 = \left( \frac{k+1}{i+1} \right)^{\alpha} = \left( \frac{k+1}{i} \right)^{\alpha} - \left( \frac{k+1}{i} \right)^{\alpha}, \ i = 1, 2, \ldots, k.
\end{equation*}

It is easily justified that the above right-hand side monotonically decreases with respect to $i = 1, 2, \ldots, k$, whose minimum attains at $i = k$ and equals $(\frac{k+1}{i})^\alpha - 1$, from which we obtain the condition (4.2).

**Remark 4.1.** For each $k \leq k^*$, since the left-hand side of (4.2) tends to two and its right-side tends to infinity with respect to $\alpha > 1$, there must be $\alpha > 1$ such that (4.2) holds. Comparing Theorem 3.3 with Theorem 4.1, we find out that, as far as the severely and moderately ill-posed problems are concerned, Lanczos bidiagonalization
generates the near best rank $k$ approximations $P_{k+1}B_kQ_k^T$ to $A$, and the singular values $\theta_i^{(k)}$ of $B_k$ approximate the large singular values $\sigma_i$ of $A$ in natural order for suitable $p > 1$ and $\alpha > 1$ for $k = 1, 2, \ldots, k^*$. On the other hand, for a given problem with $\alpha > 1$, the smaller $k$ is, the more easily the condition (4.2) is met, and thus the more possible is for the $\theta_i^{(k)}$ to approximate the large singular values $\sigma_i$ in natural order. The $\theta_i^{(k)}$ may not approximate the large singular values $\sigma_i$ in natural order at some iterations $k \leq k^*$ if $\alpha > 1$ is not enough.

Remark 4.2. For the mildly ill-posed problems $\sigma_i = \zeta i^{-\alpha}$, the sufficient condition (4.2) for (4.3) is never met because its left-hand side is always bigger than two but the right-hand side $\left(\frac{k+1}{k}\right)^{\alpha} \leq 2$ for any $k \geq 1$ and $\frac{1}{2} < \alpha \leq 1$. This indicates that it is hard that the $\theta_i^{(k)}$ approximate the $k$ large singular values $\sigma_i$ in natural order.

5. Monotonicity of $\gamma_k$, decay rates of $\alpha_k$ and $\beta_{k+1}$ and their practical importance. In this section, we present a number of results on $\gamma_k$ and the decay rates of $\alpha_k$ and $\beta_{k+1}$, and highlight their implications and practical importance.

Theorem 5.1. With the notation defined previously, the following results hold:

\begin{equation}
\alpha_{k+1} < \gamma_k \leq \sqrt{1 + \eta_k^2 \sigma_{k+1}}, \quad k = 1, 2, \ldots, n - 1, \tag{5.1}
\end{equation}

\begin{equation}
\beta_{k+2} < \gamma_k \leq \sqrt{1 + \eta_k^2 \sigma_{k+1}}, \quad k = 1, 2, \ldots, n - 1, \tag{5.2}
\end{equation}

\begin{equation}
\frac{\alpha_{k+1} \beta_{k+2}}{2} \leq \frac{(1 + \eta_k^2) \sigma_{k+1}^2}{2}, \quad k = 1, 2, \ldots, n - 1, \tag{5.3}
\end{equation}

\begin{equation}
\gamma_{k+1} < \gamma_k, \quad k = 1, 2, \ldots, n - 2. \tag{5.4}
\end{equation}

Proof. From (4.5), since $P$ and $Q_n$ are orthogonal matrices, we have

\begin{equation}
\gamma_k = \| A - P_{k+1}B_kQ_k^T \| = \| P^T (A - P_{k+1}B_kQ_k^T)Q_n \| \tag{5.5}
\end{equation}

\begin{equation}
= \left\| \begin{pmatrix} B_n & 0 \end{pmatrix} - (I, 0)^T B_k (I, 0) \right\| = \| G_k \| \tag{5.6}
\end{equation}

with

\begin{equation}
G_k = \begin{pmatrix}
\alpha_{k+1} & \alpha_{k+2} & \alpha_{k+3} & \cdots \\
\beta_{k+2} & \beta_{k+3} & \cdots & \alpha_n \\
\beta_{n+1} & \cdots & \alpha_n & \cdots & \beta_{n+1}
\end{pmatrix} \in \mathbb{R}^{(n-k+1) \times (n-k)} \tag{5.7}
\end{equation}

resulting from deleting the $(k+1) \times k$ leading principal matrix $B_k$ of $B_n$ and the first $k$ zero rows and columns of the resulting matrix. For $k = 1, 2, \ldots, n - 1$ we have

\begin{equation}
\alpha_{k+1}^2 + \beta_{k+2}^2 = \| G_k e_1 \|^2 \leq \| G_k \| \leq \gamma_k^2, \tag{5.8}
\end{equation}

which proves the lower bounds in (5.1) and (5.2) since $\alpha_{k+1} > 0$ and $\beta_{k+2} > 0$. Furthermore, from (3.13), we obtain the upper bounds in (5.1) and (5.2). Noting that

\[ 2\alpha_{k+1} \beta_{k+2} \leq \alpha_{k+1}^2 + \beta_{k+2}^2 = \gamma_k^2, \]

we prove (5.3).
By $\gamma_k = ||G_k||$ and (5.7), observe that $\gamma_{k+1} = ||G_{k+1}||$ equals the 2-norm of the submatrix deleting the first column of $G_k$. Applying the Cauchy’s strict interlacing theorem to the singular values of this submatrix and $G_k$, we obtain (5.4).

**Remark 5.1.** The strict decreasing property (5.4) of $\gamma_k$ and the lower bounds for $\gamma_k$ in (5.1)–(5.3) hold unconditionally for any general $A$, independent of the degree of ill-posedness of (1.1).

**Remark 5.2.** It is generally impractical to compute $\gamma_k$ for $A$ large. However, the proofs of (5.1) and (5.2) indicate that $\alpha_{k+1} + \beta_{k+2}$ decays as fast as $\gamma_k$. Hence, strikingly, we can reliably judge the decay rates of $\gamma_k$ by those of $\alpha_{k+1} + \beta_{k+2}$ during computation without extra cost.

**Remark 5.3.** For severely and moderately ill-posed problems with suitable $\rho > 1$ and $\alpha > 1$, based on the previous results, (5.1) and (5.2) show that $\alpha_{k+1} + \beta_{k+2}$ decays as fast as $\sigma_{k+1}$. For mildly ill-posed problems, since generally $\eta_k > 1$ considerably, $\alpha_{k+1} + \beta_{k+2}$ decays more slowly than $\sigma_{k+1}$.

6. **Numerical experiments.** We choose several 1D and 2D ill-posed problems and a random ill-posed problem with the prescribed singular values in the toolboxes [9, 19, 21]. Table 1 lists the test problems, which take default parameter(s) and include, severely, moderately and mildly ill-posed problems as well as the ones with the singular values decaying more slowly than those of the mildly ill-posed problems for given $m$ and $n$. The random mildly ill-posed problem regut.m [17, 19] is constructed with the prescribed singular values $\sigma_i = i^{-0.6}$ and the left and right singular vectors $u_i$, $v_i$, $i = 1, 2, \ldots, n$ having exactly $i - 1$ sign changes, and we set $x_{true} = ones(n, 1)$ and generate the noise-free right hand side $b_{true} = Ax_{true}$.

We mention that it is hard to find available 2D real-life suitable severely and moderately ill-posed problems for justifying our results. Gazzola, Hansen and Nagy [9] have presented a number of 2D test problems, where the image deblurring problem PRblurgauss, the inverse diffusion problem PRdiffusion and the nuclear magnetic resonance (NMR) relaxometry problem PRnmr are severely ill-posed. But the latter two matrices are only available as a function handle, for which we cannot compute their SVDs. Setting the parameter options.BlurLevel='severe', we have computed the SVD of PRblurgauss with $m = n = 10000$ and found $\sigma_1 / \sigma_{1500} \approx 1.99 \times 10^{14} = O(\frac{1}{\text{mach}})$. Unfortunately, we have found out that about half of the first 1500 large singular values are genuinely or numerically multiple. For example, among the first 40 singular values, $\sigma_3, \sigma_6, \sigma_8, \sigma_{11}, \sigma_{13}, \sigma_{15}, \sigma_{17}, \sigma_{19}, \sigma_{22}, \sigma_{24}, \sigma_{26}, \sigma_{28}, \sigma_{30}, \sigma_{33}, \sigma_{35}, \sigma_{37}$ and $\sigma_{39}$ are multiple. Therefore, these problems are either unsuitable or inaccessible for our propose. Meanwhile, there is no 2D moderately ill-posed problem, and all the other problems are mildly ill-posed in [9]. We will test the 2D mildly ill-posed problems PRblurrotation and PRblurspeckle of $m = n = 14400$, which simulate a spatially variant rotational motion blur around the center of the image and spatially invariant blurring caused by atmospheric turbulence, respectively.

For our propose of justifying the sharpness of our estimates, it is enough to test any severely, moderately and mildly ill-posed problems. In the meantime, for each test problem we compare the accuracy of the best LSQR regularized solution $x_{k_{LSQR}}$ with that of the best TSVD solution $x_{k_{TSVD}}$. In addition, we attempt to show that for the 2D problems blur, fanbeamtomo and paralleltomo whose singular values decay more slowly than those of a mildly ill-posed problem for given $m$ and $n$, LSQR and the TSVD method behave as if these problems were ordinary ones if the noise level

$$\varepsilon = \frac{\|e\|}{\|b_{true}\|}$$
is fairly small, e.g., $10^{-3}$, that is, the noise $e$ does not play any effect in regularization, and the best regularized solution by the TSVD method is simply $x^{\text{TSVD}}_n = x^{\text{naive}}_n = A^\dagger b$, and LSQR works in its regular way and ultimately finds a converged approximation to $x^{\text{naive}}_n$. In this case, no semi-convergence occurs, and the discrete ill-posed problems are actually ordinary ones; see the elaboration in the introduction. However, we will report that if $\varepsilon$ is relatively larger, e.g., 0.05, then the semi-convergence of LSQR and the TSVD method occurs.

Keep in mind that for an underlying linear compact operator equation $Kf = g$, the singular values of the operator $K$ must decay at least as fast as $O(k^{-\alpha})$, $k = 1, 2, \ldots, \infty$ with $\alpha > \frac{1}{2}$; see, e.g., [14, 18]. Therefore, provided that such a continuous problem is discretized finely enough by a suitable scheme, the resulting linear discrete ill-posed problem (1.1) will inherit such property, that is, the singular values $\sigma_k$, $k = 1, 2, \ldots, n$ of $A$ ultimately decay faster than $O(k^{-1/2})$ when $n$ is sufficiently large; otherwise, the solution norm $\|x^{\text{true}}\|$ of (1.1) is unbounded when $n \to \infty$. For the 2D image deblurring problems **blur** [19], **fanbeamtomo** and **paralleltomo** [21], we have found that their singular values $\sigma_k$ decay slowly and considerably more slowly than $O(10^{3})$, very modest! We will report precise details later. This indicates that the discretizations are not sufficiently fine. In Table 1, we take $n = 150^2$, $120^2$, $100^2$ for **blur**, **fanbeamtomo** and **paralleltomo**. For **blur**, it means that the 2D rectangular domain is discretized into $150^2$ cells; for **fanbeamtomo** and **paralleltomo**, they mean that the 2D domains are divided into 120 and 100 equally spaced intervals in both dimensions, which create $120^2$ and $100^2$ cells, respectively. Obviously, the discretizations are not fine enough, though the discrete problems are seemingly already large.

| Problem           | Description                        | Ill-posedness | $m \times n$ |
|-------------------|------------------------------------|---------------|--------------|
| shaw              | 1D image restoration model         | severe        | $10240 \times 10240$ |
| gravity           | 1D gravity surveying problem       | severe        | $10240 \times 10240$ |
| heat              | Inverse heat equation              | moderate      | $10240 \times 10240$ |
| deriv2            | Computation of second derivative   | moderate      | $10000 \times 10000$ |
| regutm            | Random ill-posed problem           | mild          | $10000 \times 10000$ |
| PRblurrrotation   | 2D image deblurring problem        | mild          | $14400 \times 14400$ |
| blur              | 2D image restoration               | unknown       | $22500 \times 22500$ |
| fanbeamtomo       | 2D fan-beam tomography problem     | unknown       | $61200 \times 14400$ |
| paralleltomo      | 2D tomography problem              | unknown       | $25380 \times 10000$ |

For each example, we use the code of [19, 21] to generate $A$, $x^{\text{true}}$ and $b^{\text{true}}$. In order to simulate the noisy data, we generate Gaussian white noise vectors $e$ such that the relative noise levels $\varepsilon$ equal some prescribed values. To simulate exact arithmetic, we use full reorthogonalization in Lanczos bidiagonalization. All the computations are carried out in Matlab R2017b on the Intel Core i7-4790k with CPU 4.00 GHz processor and 16 GB RAM with the machine precision $\varepsilon_{\text{mach}} = 2.22 \times 10^{-16}$ under the Microsoft Windows 8 64-bit system.

Our numerical experiments consist of three subsections. The first two subsections are devoted to the justification of our results, and the third subsection pay special attention to the regularization behavior and ability of LSQR.
6.1. The accuracy $\gamma_k$ of rank $k$ approximations and the approximation properties of the Ritz values $\theta_i^{(k)}$. We first investigate the accuracy $\gamma_k$ of rank $k$ approximations and the approximation behavior of $k$ Ritz values $\theta_i^{(k)}$ for the first five test problems shaw, gravity, heat, deriv2 and regutm. Given $\varepsilon = 10^{-3}$, for each problem and $k = 1, 2, \ldots$ up to some iteration bigger than the semi-convergence point $k^*$, each of Figures 1–5 depicts the curves of $\gamma_k$ and $\sigma_{k+1}$, the locations of the $k$ Ritz values $\theta_i^{(k)}$ and the first $k + 1$ large singular values $\sigma_i$ of $A$, and the decay curves of $\gamma_k$ and the sum $\alpha_{k+1} + \beta_{k+2}$. We exhibit them by (a), (b) and (c) in each figure, respectively. Figures 1 (d)–5 (d) draw the semi-convergence processes of LSQR and the TSVD method. We point out that for different $\varepsilon = 10^{-2}$ and 10$^{-4}$ we have obtained similar results and observed the same phenomena. So we omit details on them. Separately, Figure 6 draws the results on PRblurrotation with $\varepsilon = 10^{-2}$.

For the severely ill-posed problems shaw and gravity, their singular values decay fast, and the $\sigma_1/\sigma_k$ achieve the level of $\frac{1}{\varepsilon_{\text{mach}}}$ for $k = 21 \ll n$ and $k = 53 \ll n$, respectively. From Figures 1–2 (d), we see that the semi-convergence of TSVD and LSQR occurs at the same steps $k_0 = k^* = 7$ and the relative errors

$$\frac{\|x_{k_0}^{\text{tsvd}} - x_{\text{true}}\|}{\|x_{\text{true}}\|} \quad \text{and} \quad \frac{\|x_{k^*}^{\text{lsqr}} - x_{\text{true}}\|}{\|x_{\text{true}}\|}$$

of the best regularized solutions $x_{k_0}^{\text{tsvd}}$ and $x_{k^*}^{\text{lsqr}}$ are essentially the same for each problem, meaning LSQR has the full regularization. From Figures 1 (a)–2 (a), we observe
that the $\gamma_k$ are very close to $\sigma_{k+1}$ and are almost indispensable for $k = 1, 2, \ldots, k^*$. This indicates that the $P_{k+1}B_kQ_k^T$ are near best rank $k$ approximations to $A$ at least until the semi-convergence of LSQR, confirming Theorems 3.2 and 3.3, which states that Lanczos bidiagonalization can generate near best rank $k$ approximations for suitable $\rho > 1$ until the semi-convergence of LSQR. Figures 1 (b)–2 (b) tell us that, for shaw, the $k$ Ritz values $\theta_i^{(k)}$ approximate the first $k$ large singular values of $A$ in natural order, or interlace the first $k + 1$ large ones, at least until the semi-convergence of LSQR. This confirms Theorem 4.1. Furthermore, we can see from Figure 1 (a)–(b) that the near best approximations and the approximations of $\theta_i^{(k)}$ in this order are valid at least until $k = 16 > k^*$ and $k = 17 > k^*$, respectively, but for gravity both of them are valid only until $k = 13 > k^*$, smaller than those for shaw, as is seen from Figure 2 (a)-(b). This is not surprising because the singular values of gravity do not decay as fast as those of shaw though both of them are severely ill-posed.

For the moderately ill-posed heat, we find $\sigma_1/\sigma_{900} = 8.3 \times 10^{15}$; for deriv2, we find with $\sigma_1/\sigma_n = 1.2 \times 10^8$. We can observe from Figure 3 (d) and Figure 4 (d) that the semi-convergence of LSQR occurs earlier than that of TSVD, i.e., $k^* < k_0$. Theorem 2.1 states that in this case the $k$ Ritz values $\theta_i^{(k)}$ must not approximate the large singular values of $A$ in natural order, or they do not interlace the first $k + 1$ large ones for some $k \leq k^*$. This is indeed true, and such phenomena occur from $k = 4$ and 8 onwards for heat and deriv2, respectively, as is seen clearly from Figure 3 (b) and Figure 4 (b). These are in accordance with Theorem 4.1, where, for a given moderately ill-posed problem, the sufficient condition (4.2) for the approximations in
this order fails to meet as \( k \) increases up to some point since (3.15) and the previous results show that the left-hand side of (4.2) monotonically increases, while its right-hand side monotonically decreases with respect to \( k \). In the meantime, Figure 3 (a) and Figure 4 (a) illustrate that Lanczos bidiagonalization generates near best rank \( k \) approximations to \( A \) for \( k \) no more than 4 and 9 for heat and deriv2, respectively, after which we cannot obtain near best rank \( k \) approximations to \( A \) any longer. These confirm Theorems 3.2–3.3, which show that the rank \( k \) approximation becomes poorer as \( k \) increases and it is no more a near best one when \( k \) increases up to some point smaller than \( k^* \) since the sufficient condition (3.31) fails to fulfill.

We now check the random mildly ill-posed regutm. Figure 5 (d) shows that the semi-convergence of LSQR occurs much more early than that of TSVD, that is, \( k^* = 26 \ll k_0 = 1527 \). Since the singular values \( \sigma_k \) decay to zero for \( n \) sufficiently large but substantially more slowly than the singular values of shaw, gravity, heat and deriv2, Theorem 3.2 and the previous analysis tell us that \( \gamma_k \) deviates from \( \sigma_{k+1} \) quickly as \( k \) increases. Actually, the condition (3.31) may hold only for \( k \) very small. Figure 5 (a) justifies our theory, and from it we see that \( P_{k+1}B_kQ_k^T \) is not a near best rank \( k \) approximation for \( k = 3 \) onwards by noticing that \( \gamma_3 \) is closer to \( \sigma_3 \) other than \( \sigma_4 \). Simultaneously, the Ritz values \( \theta_i^{(k)} \) fail to interlace the first \( k + 1 \) large singular values of \( A \) for \( k = 3 \) onwards, as indicated by Figure 5 (b). This demonstrates Theorem 3.3 and Theorem 4.1, where the sufficient condition (3.31) for near best rank \( k \) approximations may be satisfied only for \( k \) very small and the sufficient condition (4.2) for the approximations in this order is not satisfied, which
implies that the \( k \) Ritz values hardly approximate the large singular values in natural order even for \( k \) very small.

Finally, we look into the results on PRblurrotation. It is seen from Figure 6 (a) that \( P_{k+1}B_kQ_k^T \) is not a near best rank \( k \) approximation from the first iteration \( k = 1 \) upwards and \( \gamma_k \) is considerably bigger than \( \sigma_{k+1} \) for \( k = 1, 2, \ldots, 20 \). Figure 6 (b) indicates that the \( k \) Ritz values \( \theta_i^{(k)} \) never interlace the first \( k + 1 \) large \( \sigma_i \) for given \( k = 1, 2, \ldots, 10 \), and \( \theta_i^{(k)} < \sigma_{k+1} \) substantially. Figure 6 (d) shows that LSQR computes its best regularized solution \( x_{ks}^{lsqr} \) at \( k^* = 40 \), much more early than the TSVD method, which obtains its best solution \( x_{k0}^{tsvd} \) at \( k_0 = 5793 \).

6.2. Decay properties of \( \gamma_k \) and \( \alpha_{k+1} + \beta_{k+2} \). In Figures 1 (c)–5 (c), we depict the decay curves of \( \gamma_k \) and \( \alpha_{k+1} + \beta_{k+2} \). From them, we see that \( \alpha_{k+1} + \beta_{k+2} \) behaves very similar to \( \gamma_k \) and matches \( \gamma_k \) very well, and their decay curves highly resemble, independent of the degree of ill-posedness. Therefore, we can use \( \alpha_{k+1} + \beta_{k+2} \) to reliably determine the decay rate of \( \gamma_k \). We also see that \( \gamma_k \) monotonically decreases with respect to \( k \). These results justify Theorem 5.1 and the first two remarks followed.

Figure 6 (c) shows that \( \gamma_k \) monotonically decays very slowly and \( \gamma_1 = 1.1374 \) and \( \gamma_{50} = 1.1227 \), respectively. It is due to the scale of vertical ordinate that, at first glance, gives one an illusion that \( \alpha_{k+1} + \beta_{k+2} \) deviates from \( \gamma_k \) very much. As a matter of fact, the \( \alpha_{k+1} + \beta_{k+2} \) estimate the \( \gamma_k \) quite accurately, and the minimum and maximum of \( \alpha_{k+1} + \beta_{k+2} \) are 0.9507 and 1.2576, respectively. These confirm our theoretical results.
LOW RANK APPROXIMATIONS AND RITZ VALUES

The indices $k$ and $k+1$

0.25
0.3
0.35
0.4
0.45
0.5
0.55
0.6
0.65
0.7

The comparison of $k$ and $k+1$ for $\text{regutm}$ with relative noise level $0.001$

(a)

The numbers $k$ and $k+1$ of Ritz values and singular values

0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1

$k$ Ritz values and first $k+1$ large singular values

$\text{regutm}$ of $n=10,000$ with relative noise level $0.001$ and $x_{\text{true}} = \text{ones}(n, 1)$

(b)

The values of $k$ and $k+1$ for $\text{regutm}$ with $x_{\text{true}} = \text{ones}(n, 1)$

(c)

The semi-convergence process of LSQR and TSVD for $\text{regutm}$

(d)

Fig. 5. $\text{regutm}$ of $m = n = 10000$ with the singular values $\sigma_k = \frac{1}{k}$, $x_{\text{true}} = \text{ones}(n, 1)$ and the relative noise level $10^{-3}$.

6.3. Observations on the regularization ability of LSQR. We now report the results on the 2D problems $\text{blur}$, $\text{fanbeamtomo}$ and $\text{paralletomo}$ in Table 1. Particularly, we investigate the regularization behavior and ability of LSQR on them and all the previous problems.

We give some details in Figures 7–9. Although the orders $m$ and $n$ are already tens of thousands, the ratios $\sigma_1/\sigma_n$ on these three problems are only 31.5, 2472 and 408.8, and their singular values are far from small and are not yet clustered at zero, which, intuitively, do not satisfy the definition of a discrete ill-posed problem since the ratio $\sigma_1/\sigma_n$ is modest. Therefore, the existing regularization theory does not suit well for such practical problems.

Indeed, it is not prompt to regard such problems as discrete ill-posed ones because, in the context of solving least squares problems or linear systems, such problems are quite well conditioned and at least not ill conditioned at first glance. Nevertheless, the situation is subtle, and we will have more findings. With different relative noise levels $\varepsilon$, the TSVD method and LSQR may exhibit very different behavior. For $\varepsilon = 10^{-3}$, we have observed that the best TSVD regularized solutions are simply $x_{\text{tsvd}} = x_{\text{naive}} = A^+b = x_{\text{true}}$, as indicated by Figure 7 (b)–Figure 9 (b). This means that $\varepsilon$ does not exert influence on regularization and we have solved the problems as if they are ordinary ones. LSQR treats them as ordinary ones and solves them in its regular way too; it is seen from Figure 7 (a)–Figure 9 (a) that the solution errors decrease until they stabilize and no semi-convergence occurs.
For blur and paralleltomo with a larger \( \varepsilon = 10^{-2} \) and fanbeamtomo with \( \varepsilon = 0.05 \), the situation changes. The noises critically affect the solution processes, and both TSVD and LSQR exhibit semi-convergence phenomena, as we see from (c)-(d) in Figures 7–9. We have observed that TSVD takes many \( k_0 \) SVD dominant components to form the best regularized solutions but LSQR uses much fewer \( k^* \) iterations to obtain the best regularized solutions with the same accuracy as TSVD does.

We also see from Figure 1 (d)–Figure 5 (d) that LSQR takes \( k^* \leq k_0 \) iterations to compute the best regularized solutions \( x_{k^*}^{\text{lsqr}} \) as accurately as \( x_{k_0}^{\text{tsvd}} \) for the five test problems shaw, gravity, heat, deriv2 and regutm that have different degrees of ill-posedness; the weaker is the degree of ill-posedness, the smaller \( k^* \) is relative to \( k_0 \). For \( k^* = k_0 \), we have established the rigorous regularization theory in this paper and [27] and shown that LSQR has the full regularization; for \( k^* < k_0 \), the full or partial regularization is not yet revealed theoretically. However, beyond one’s common expectation, as we have seen from Figures 3 (d)–9 (d), the experiments on all the other ill-posed problems show that the best regularized solutions \( x_{k^*}^{\text{lsqr}} \) by LSQR are as accurate as the best solutions \( x_{k_0}^{\text{tsvd}} \) by the TSVD method. It is worthwhile to notice that both \( \varepsilon = 10^{-3} \) and 0.05 are practical. For these problems, the experiments have demonstrated that LSQR has the full regularization. Numerical experiments on a number of 2D mildly ill-posed image deblurring problems and tomography problems from [9] have also demonstrated that LSQR has the full regularization [27].

7. Conclusions. For the large-scale (1.1), iterative solvers are generally the only viable approaches. Of them, the mathematically equivalent LSQR and CGLS...
are most popularly used Krylov iterative solvers for general purposes. They have
general regularizing effects and exhibit semi-convergence. It has long been known
that if the Ritz values converge to the large singular values of $A$ in natural order until
the occurrence of semi-convergence of LSQR then the regularized solution $x_{lsqr}^{k^*}$ is as
accurate as the TSVD solution $x_{tsvd}^{k_0}$, that is, LSQR has the full regularization.

For severely and moderately ill-posed problems, we have proved that, with suit-
able $\rho > 1$ or $\alpha > 1$, a $k$-step Lanczos bidiagonalization produces a near best rank
$k$ approximation of $A$ and the $k$ Ritz values approximate the first $k$ large singular
values of $A$ in natural order until the semi-convergence of LSQR, so that LSQR has
the full regularization. But for moderately ill-posed problems with $\alpha > 1$ not enough
and mildly ill-posed problems, we have proved that the above results generally do not
hold for some $k \leq k^*$. These results have given accurate and definitive solutions of
the highly concerned and challenging problems on the convergence behavior of Ritz
values for the three kinds of ill-posed problems.

We have proved that the accuracy $\gamma_k$ of rank $k$ approximation generated by Lancer-
zos bidiagonalization monotonically increases with $k$. We have also derived bounds
for the diagonals and subdiagonals of bidiagonal matrices generated by Lanczos bidi-
agonalization. Particularly, we have proved that they decay as fast as the singular
values of $A$ for severely or moderately ill-posed problems with suitable $\rho > 1$ or
$\alpha > 1$. These bounds are of theoretical and practical importance since we have shown
that $\alpha k+1 + \beta k+2$ can be used to reliably judge the decay rate of $\gamma_k$ of the rank $k$
approximations during computation without extra cost.
We have made illuminating numerical experiments and confirmed our theory. In addition, we have investigated the regularizing effects of LSQR and the TSVD method on some practical discrete problems arising from 2D image continuous deblurring problems. The 2D test problems, though large scale, are quite well conditioned. We have found that LSQR and the TSVD methods work as if they solved ordinary linear systems, in which a noise $e$ with practical level $\varepsilon$ may not play a role in regularization; if $e$ is larger, the two methods have semi-convergence phenomena. In any case, the best regularized solutions obtained by LSQR and the TSVD methods essentially have the same accuracy, meaning that LSQR has the full regularization.

As the numerical experiments in this paper and [27] have demonstrated, LSQR has the full regularization for all the test problems in [9, 19, 21], independent of the degree of ill-posedness. These draw us to the conjecture that LSQR has the full regularization for any kind of ill-posed problem with the discrete Picard condition satisfied.

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The convergence of LSQR for parallel tomography with relative noise 0.001

(a)

The convergence of TSVD for parallel tomography with relative noise 0.001

(b)

The semi-convergence process of LSQR for parallel tomography with relative noise 0.01

(c)

The semi-convergence process of TSVD for parallel tomography with relative noise 0.01

(d)

Fig. 9. Parallel tomography of $m = 25380$, $n = 10000$ with $\frac{\sigma_1}{\sigma_n} = 408.8$ and $\varepsilon = 10^{-3}$ and $10^{-2}$.
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