A spherical CR structure on the complement of the figure eight knot with discrete holonomy.

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Abstract

We describe a general geometrical construction of spherical CR structures. We construct then spherical CR structures on the complement of the figure eight knot and the Whitehead link. They have discrete holonomies contained in $PU(2,1,\mathbb{Z}[\omega])$ and $PU(2,1,\mathbb{Z}[i])$ respectively. These are the same ring of integers appearing in the real hyperbolic geometry of the corresponding links.

1 Introduction

One of the most important examples of hyperbolic manifolds is the complement of the figure eight knot. It was shown by Riley in \cite{R} that the fundamental group of that manifold had a discrete representation in $PSL(2,\mathbb{C})$. In fact he showed that there exists a representation contained in $PSL(2,\mathbb{Z}[\omega])$ where $\mathbb{Z}[\omega]$ is the ring of Eisenstein integers. On the other hand the construction by Thurston is based on gluing of ideal tetrahedra and that led to general constructions on a large family of 3-manifolds.

It is not known which hyperbolic manifolds admit a spherical CR structure. In fact very few constructions of spherical CR 3-manifolds with discrete holonomy exist at all. The only construction of such a structure on a 3-manifold (which is not a circle bundle) previous to this work is essentially for the Whitehead link and other manifolds obtained from it by Dehn surgery in \cite{S1, S2}.

We propose a geometrical construction by gluing appropriate tetrahedra adapted to CR geometry. In particular we prove in this paper that the complement of the figure eight knot has a spherical CR structure with discrete holonomy such that the holonomy of the boundary torus is parabolic and faithful (see Theorem 4.2 and Proposition 4.3). As another example we also construct a spherical structure on the complement of the Whitehead link which differs from \cite{S1} with discrete holonomy (Theorem 6.1). It is interesting to observe that we obtain representations of the fundamental groups of those link complements with values in $PU(2,1,\mathbb{Z}[\omega])$ and $PU(2,1,\mathbb{Z}[i])$, that is the same rings of integers of the complete structures in the case of real hyperbolic geometry.

There are two different aspects in the construction. The first is a very general method to construct CR manifolds by gluing which can be used to construct CR structures on many other (hyperbolic or not) manifolds. The second aspect is discreteness which in the real hyperbolic case is much simpler to decide than in the CR case because of the absence of a metric structure in the
latter. For the complement of the figure eight knot and the Whitehead link we show discreteness of the representation of the fundamental group by explicitly showing that the group is a subgroup of \( \text{PU}(2,1,\mathbb{Z}[\omega]) \) and \( \text{PU}(2,1,\mathbb{Z}[i]) \) respectively.

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## 2 Complex hyperbolic space

### 2.1 \( \text{PU}(2,1), \tilde{\text{PU}}(2,1) \) and the Heisenberg group

Let \( \mathbb{C}^{2,1} \) denote the complex vector space equipped with the Hermitian form

\[
\langle z, w \rangle = z_1\bar{w}_3 + z_2\bar{w}_2 + z_3\bar{w}_1.
\]

Consider the following subspaces in \( \mathbb{C}^{2,1} \):

\[

t _{\mathrm{V}_+} = \{ z \in \mathbb{C}^{2,1} : \langle z, z \rangle > 0 \}, \\
t _{\mathrm{V}_0} = \{ z \in \mathbb{C}^{2,1} \setminus \{0\} : \langle z, z \rangle = 0 \}, \\
t _{\mathrm{V}_-} = \{ z \in \mathbb{C}^{2,1} : \langle z, z \rangle < 0 \}.
\]

Let \( P : \mathbb{C}^{2,1} \setminus \{0\} \to \mathbb{CP}^2 \) be the canonical projection onto complex projective space. Then \( \mathbb{H}^2_\mathbb{C} = P(\mathrm{V}_-) \) equipped with the Bergman metric is complex hyperbolic space. The boundary of complex hyperbolic space is \( P(V_0) = \partial \mathbb{H}^2_\mathbb{C} \). The isometry group \( \tilde{\text{PU}}(2,1) \) of \( \mathbb{H}^2_\mathbb{C} \) comprises holomorphic transformations in \( \text{PU}(2,1) \), the unitary group of \( \langle \cdot, \cdot \rangle \), and anti-holomorphic transformations arising elements of \( \text{PU}(2,1) \) followed by complex conjugation.

The Heisenberg group \( \mathfrak{N} \) is the set of pairs \( (z,t) \in \mathbb{C} \times \mathbb{R} \) with the product

\[
(z,t) \cdot (z',t') = (z + z', t + t' + 2\text{Im} \, z'z).
\]

Using stereographic projection, we can identify \( \partial \mathbb{H}^2_\mathbb{C} \) with the one-point compactification \( \overline{\mathfrak{N}} \) of \( \mathfrak{N} \). The Heisenberg group acts on itself by left translations. Heisenberg translations by \( (0,t) \) for \( t \in \mathbb{R} \) are called vertical translations.

Define the inversion in the \( x \)-axis in \( \mathbb{C} \subset \mathfrak{N} \) by

\[
\iota_x : (z,t) \mapsto (\bar{z},-t).
\]

All these actions extend trivially to the compactification \( \overline{\mathfrak{N}} \) of \( \mathfrak{N} \) and represent transformations in \( \tilde{\text{PU}}(2,1) \) acting on the boundary of complex hyperbolic space (see [3]).

A point \( p = (z,t) \) in the Heisenberg group and the point \( \infty \) are lifted to the following points in \( \mathbb{C}^{2,1} \):

\[
\hat{p} = \begin{bmatrix}
-|z|^2+it \\
z \\
1
\end{bmatrix}
\quad \text{and} \quad \infty = \begin{bmatrix}
1 \\
0 \\
0
\end{bmatrix}.
\]

Given any three points \( p_1, p_2, p_3 \) in \( \partial \mathbb{H}^2_\mathbb{C} \) we define Cartan’s angular invariant \( \mathcal{A} \) as

\[
\mathcal{A}(p_1, p_2, p_3) = \arg(-\langle \hat{p}_1, \hat{p}_2 \rangle \langle \hat{p}_2, \hat{p}_3 \rangle \langle \hat{p}_3, \hat{p}_1 \rangle).
\]

In the special case where \( p_1 = \infty, p_2 = (0,0) \) and \( p_3 = (z,t) \) we simply get \( \tan(\mathcal{A}) = t/|z|^2 \).
2.2 $\mathbb{R}$-circles, $\mathbb{C}$-circles and $\mathbb{C}$-surfaces

There are two kinds of totally geodesic submanifolds of real dimension 2 in $\mathbb{H}^2_\mathbb{C}$: complex lines in $\mathbb{H}^2_\mathbb{C}$ are complex geodesics (represented by $\mathbb{H}^1_\mathbb{C} \subset \mathbb{H}^2_\mathbb{C}$) and Lagrangian planes in $\mathbb{H}^2_\mathbb{C}$ are totally real geodesic 2-planes (represented by $\mathbb{H}^2_\mathbb{R} \subset \mathbb{H}^2_\mathbb{C}$). Each of these totally geodesic submanifolds is a model of the hyperbolic plane.

Consider complex hyperbolic space $\mathbb{H}^2_\mathbb{C}$ and its boundary $\partial \mathbb{H}^2_\mathbb{C}$. We define $\mathbb{C}$-circles in $\partial \mathbb{H}^2_\mathbb{C}$ to be the boundaries of complex geodesics in $\mathbb{H}^2_\mathbb{C}$. Analogously, we define $\mathbb{R}$-circles in $\partial \mathbb{H}^2_\mathbb{C}$ to be the boundaries of Lagrangian planes in $\mathbb{H}^2_\mathbb{C}$.

Proposition 2.1 (see [G]) In the Heisenberg model, $\mathbb{C}$-circles are either vertical lines or ellipses, whose projection on the $z$-plane are circles.

Finite $\mathbb{C}$-circles are determined by a centre $M = (z = a + ib, c)$ and a radius $R$. They may also be described using polar vectors in $P(V_+)$ (see Goldman [G] page 129).

If we use the Hermitian form $\langle \cdot, \cdot \rangle$, a finite chain with centre $(a + ib, c)$ and radius $R$ has polar vector (that is the orthogonal vector in $\mathbb{C}^2.1$ to the plane determined by the chain).

\[
\begin{bmatrix}
R^2 - a^2 - b^2 + ic^2 \\
a + ib \\
1
\end{bmatrix}
\]

Given two points $p_1$ and $p_2$ in Heisenberg space, we write $[p_1, p_2]$ for a choice of one of the two segments of $\mathbb{C}$-circle joining them. The choice will be determined from the context.

Definition 2.2 A $\mathbb{C}$-triangle determined by three points $[p_0, p_1, p_2]$ is a triangular surface determined by segments of $\mathbb{C}$-circles joining $p_0$ to each point a segment of $\mathbb{C}$-circle $[p_1, p_2]$.

Observe that, in principle, there are four smooth triangular surfaces canonically associated to $[p_0, p_1, p_2]$. Each of those triangles could be part of a $\mathbb{C}$-sphere (see [FZ]).

3 Tetrahedra

To copy the tetrahedra of the conformal case we start with 4 points such that each triple of points up to a sign has a fixed Cartan’s invariant. The edges of the tetrahedron could be segments of either $\mathbb{R}$-circles or $\mathbb{C}$-circles and the faces should be adapted later to that one skeleton. In this paper we will use $\mathbb{C}$-circles and $\mathbb{C}$-triangles.

3.1 CR triples of points

We first describe triples of points in the standard spherical CR sphere. They are classified up to $PU(2, 1)$ in the following proposition.

Proposition 3.1 ([C], see [G]) The Cartan invariant classifies triples of points up to $PU(2, 1)$.

A natural way to obtain a triangle is then to join the 3 points by $\mathbb{C}$-circles. As in spherical geometry, for each pair of points there are two choices of circular segments joining them.
3.1.1 CR tetrahedra

For a general tetrahedra we have 4 Cartan invariants corresponding to each triple of points. But one of them is determined by the others in view of the cocycle condition (see [G] pg. 219):

\[-A(x_2, x_3, x_4) + A(x_1, x_3, x_4) - A(x_1, x_2, x_4) + A(x_1, x_2, x_3) = 0.\]

As a special case of tetrahedra we have the following.

**Proposition 3.2** If three triples of four points are contained in \(\mathbb{R}\)-circles (\(\mathbb{C}\)-circles), the four points are contained in a common \(\mathbb{R}\)-circle (\(\mathbb{C}\)-circle).

**Proof.** We will prove the result on \(\mathbb{R}\)-circles, the other case being easier. From the cocycle relation, each triple is contained in an \(\mathbb{R}\)-circle as \(A = 0\) for all triples. Without loss of generality, we can suppose that three of the points are \(\infty, [0,0], [1,0]\) in Heisenberg coordinates. The fourth point is in an \(\mathbb{R}\)-circle containing \(\infty, [0,0]\) on one hand, so it is in the plane \(t = 0\). On the other hand, it should be in an \(\mathbb{R}\)-circle passing through \([1,0]\) and \(\infty\), that is in the contact plane at \([1,0]\). The intersection of both planes is precisely the \(x\)-axis. \(\square\)

**Definition 3.3** A tetrahedron is a configuration of four points and a choice of edges, that is a choice of \(\mathbb{C}\)-circle segments joining each pair of points.

**Definition 3.4** A symmetric tetrahedron is a configuration of four points with an anti-holomorphic symmetry and a choice of \(\mathbb{C}\)-circle segments joining each pair of points.

By normalizing the coordinates of the four points we can assume that they are given by

\[ p_1 = \infty \quad p_2 = 0 \quad q_1 = (1,t) \quad q_2 = (z, s|z|^2) \]

**Lemma 3.5** (cf. [W]) The configuration of four points \(p_1, p_2, q_1, q_2\) has a \(\mathbb{Z}_2\) anti-holomorphic symmetry exchanging \(p_1, p_2\) and \(q_1, q_2\) if and only if \(t = s\).

**Proof.** A simple proof follows writing the general form of an anti-holomorphic transformation permuting \(\infty\) and 0. It is given by

\[ (z, t) \rightarrow \left( -\frac{\bar{z}}{|\lambda|^2(|z|^2 + it)}, \frac{t}{|\lambda|^4(|z|^4 + t^2)} \right), \]

where \(\lambda \in \mathbb{C}^\ast\). Imposing that the points \(q_1\) and \(q_2\) are permuted then gives the result. \(\square\)

In that case \(\mathcal{A}(p_1, p_2, q_1) = \mathcal{A}(p_1, p_2, q_2)\) and \(\mathcal{A}(p_1, q_1, q_2) = \mathcal{A}(p_2, q_1, q_2)\).

**Lemma 3.6** For configurations with \(\mathbb{Z}_2\) symmetry as above, \(\mathcal{A}(p_1, p_2, q_1) = \mathcal{A}(p_1, q_1, q_2)\) if and only if \(t g(\mathcal{A}(p_1, p_2, q_1)) = t = \frac{\text{Im} z}{1 - \text{Re} z}\).

**Proof.** A simple computation shows that

\[ \mathcal{A}(p_1, p_2, q_1) = \text{arg}(|z|^2(1 + it)/2) = \text{arctg}(t) \]
and
\[ A(p_1, q_1, q_2) = \arg(|z|^2(1 + it)/2 + (1 - 2\bar{z} - it)/2) = \arctg \frac{t(|z|^2 - 1) + 2\text{Im} z}{|z|^2 + 1 - 2\text{Re} z}. \]

The proposition follows by equating the two formulas and solving for \( t \).

\[ \square \]

**Definition 3.7** We call a symmetric tetrahedron regular if the configuration of four points satisfies
\[ A(p_1, p_2, q_1) = A(p_1, p_2, q_4) = A(p_1, q_1, q_2) = A(p_2, q_1, q_2). \]

In that case \( s = t \) and \( t = \frac{\text{Im} z}{1-\text{Re} z} \).

### 3.1.2 Parameters of Ideal Tetrahedra

To each vertex of a tetrahedra we associate the complex coordinates of the three vertical lines obtained when we place that vertex at \( \infty \). That gives us four euclidean triangles. Consider the following configuration of points
\[ p_1 = \infty \quad p_2 = 0 \quad q_1 = (1, t) \quad q_2 = (z, s|z|^2). \]

There are many possible choices for edges. We will choose the infinite edges to be the halves of the vertical \( \mathbb{C} \)-circles which tend to \(+\infty\). The other edges will be clear from the context. In particular for
\[ p_1 = \infty \quad p_2 = 0 \quad q_1 = (1, \sqrt{3}) \quad q_2 = \left(\frac{1}{2} + \frac{i\sqrt{3}}{2}, \sqrt{3}\right), \]
we have Figure 3.1.2. The invariant of the triangle determined by the points \((p_2, q_1, q_2)\) at the line determined by \( p_2 \) is \( z \). In order to obtain the invariant of the triangle determined by the triple \((q_2, p_1, p_2)\) at \( q_2 \) we use the complex inversion
\[ I(z, t) = \left(\frac{z}{|z|^2 - it}, \frac{-t}{|z|^4 + t^2}\right) \]
to move the point \( q_1 \) to \( \infty \). We proceed in the same manner for the other points.

We consider Figure 2 to describe the parameters of a tetrahedron. Note that, contrary to the ideal tetrahedron in real hyperbolic geometry, the euclidean invariant at each vertex is not the same. The following proposition follows immediately from the considerations above by a simple calculation.

**Proposition 3.8** For a tetrahedron given by
\[ p_1 = \infty \quad p_2 = 0 \quad q_1 = (1, t) \quad q_2 = (z, s|z|^2) \]
then \( z_1 = z, z'_1 = \frac{i + t}{2(i + s)}, \quad \tilde{z}_1 = \frac{z + i - \tilde{z}(i + s)}{(z - 1)(t - i)} \) and \( \tilde{z}'_1 = \frac{1 - (i + t) + \tilde{z}(i + s)}{2(z - 1)(t - i)}. \) Where, as usual, \( z_2 = \frac{1}{1 - z_1} \) and \( z_3 = 1 - \frac{1}{z_1} \) and so on.

\[ \tan A(p_1, p_2, q_1) = t \]
\[ \tan A(p_1, q_1, q_2) = \frac{|z_1|^2 s - t + 2\text{Im} z_1}{|z_1 - 1|^2} \]
and \( \tan A(p_1, p_2, q_2) = s \).
Here, the three Cartan invariants are independent. We also have the following relations

\[ \tilde{z} = \frac{z(z' - 1)(t + i)}{z'(z - 1)(t - i)} \text{ and } z' = \frac{(z' - 1)(i + s)}{(z - 1)(i - s)}. \]

Therefore

\[ t = i \frac{zz' - z - \tilde{z}z' + \tilde{z}z'}{-zz' + z - \tilde{z}z' + \tilde{z}z'} \text{ and } s = i \frac{z' - 1 - z' + z'}{-z' + 1 - z' + z'}. \]

For the symmetric tetrahedra, the situation is simpler:

**Corollary 3.9** For a symmetric tetrahedron given by

\[ p_1 = \infty \quad p_2 = 0 \quad q_1 = (1, t) \quad q_2 = (z, t|z|^2) \]

then \( z_1 = z, z'_1 = z/|z|^2, \tilde{z}_1 = z \frac{(z - 1)(1 - it)}{(z - 1)(1 + it)} \) and \( \tilde{z}'_1 = \tilde{z}_1/|\tilde{z}_1|^2 \). Where, as usual, \( z_2 = \frac{1}{1 - z_1} \) and \( z_3 = 1 - \frac{1}{z_1} \).

\[ tgA(p_1, p_2, q_1) = -\frac{i z_1 \tilde{z}_1 - z_1 - z_1 \tilde{z}_1 + \tilde{z}_1}{z_1 \tilde{z}_1 - z_1 + z_1 \tilde{z}_1 - \tilde{z}_1} = t \]

\[ tgA(p_1, q_1, q_2) = -\frac{i z_1 \tilde{z}_1 + z_1 - z_1 \tilde{z}_1 - \tilde{z}_1}{z_1 \tilde{z}_1 - z_1 + z_1 \tilde{z}_1 - \tilde{z}_1} = t(|z|^2 - 1) + 2\text{Im}z \]

The proposition above shows that the set of symmetric tetrahedra is parametrized by a strictly pseudoconvex CR hypersurface in \( \mathbb{C} \times \mathbb{C} \), namely, solving for \( t \), we obtain the equation

\[ |z_1| = |\tilde{z}_1|. \]
Proposition 3.10 If the special symmetric tetrahedron is given by
\[ p_1 = (0, t), \quad p_2 = (0, -t), \quad q_1 = (1, 0), \quad q_2 = (e^{i\theta}, 0) \]
then \( z_1 = e^{i\theta} \) and \( \tilde{z}_1 = \frac{(t+i)^2}{(t-i)^2} \). Where, as usual, \( z_2 = \frac{1}{1-z_1} \) and \( z_3 = 1 - \frac{1}{z_1} \).

In the regular symmetric case there is only one complex parameter which should be compared to the parameter for real hyperbolic tetrahedra:

Proposition 3.11 Regular symmetric tetrahedra are parametrized by the complex number \( z_1 = \tilde{z}_1 = z \) with \( \text{Re} z \neq 1 \). In the coordinates above, \( \text{tg}(A(p_1, p_2, q_1)) = \text{tg}(A(p_1, q_1, q_2)) = t = \frac{\text{Im} z}{1-\text{Re} z} \).

This proposition shows that if a real hyperbolic ideal triangulation has modular invariants for its tetrahedra contained in a line \( \frac{\text{Im} z}{1-\text{Re} z} = \text{constant} \) gives rise to representations of the fundamental group of the the manifold into \( \text{PU}(2,1) \).

As a last observation, the moduli for a tetrahedron can be expressed using other invariants as the Koranny-Reimann cross-ratio and Cartan’s invariant.

### 3.1.3 The standard special tetrahedron

We make \( \omega = e^{-i\pi/3} \) and \( t = 2 + \sqrt{3} \) for a special tetrahedra. Using the formulas above we obtain

Lemma 3.12 If \( p_1 = (0, 2 + \sqrt{3}), \quad p_2 = (0, -(2 + \sqrt{3})), \quad q_1 = (\omega, 0), \quad q_2 = (1, 0) \) then in the parameters above \( z_1 = \tilde{z}_1 = \bar{\omega} \). Moreover the tetrahedron is symmetric and \( A(q_1, q_2, p_2) = \frac{\pi}{3} \) and \( A(p_1, q_2, p_2) = -\frac{\pi}{3} \).
3.1.4 Another special tetrahedron

We make $p_1 = (0, 1 + \sqrt{2}), p_2 = (0, -(1 + \sqrt{2})), q_1 = (1, 0)$ and $q_2 = (i, 0)$. We obtain the following

Lemma 3.13 For $T_w = [p_1, p_2, q_1, q_2]$ as above $z_1 = \bar{z}_1 = i$.

3.2 Fundamental lemma for special symmetric tetrahedra

We define the procedure of filling the faces from the one skeleton of the tetrahedra in such a way that the 2-skeleton will be $\mathbb{Z}_2$-invariant:

Definition 3.14 The diverging $\mathbb{C}$-rays procedure is the definition of the 2-skeleton by taking $\mathbb{C}$-segments from $p_1$ to the edges $[q_1, q_2], [q_2, p_2]$ and $\mathbb{C}$-segments from $p_2$ to the edges $[q_1, q_2], [q_1, p_1]$.

Observe that the rays start from $p_1$ or $p_2$ and not from $q_1$ or $q_2$.

Lemma 3.15 The special symmetric tetrahedron defined by the procedure of diverging $\mathbb{C}$-rays is homeomorphic to a tetrahedron.

Proof. We make one of the vertexes go to infinity keeping the other on the vertical axis. The other pair of points is in an orthogonal $\mathbb{C}$-circle. They correspond to the normalization $p_1 = \infty$, $p_2 = 0$, $q_1 = (1, t_3)$ and $q_2 = (e^{i\theta}, t_3)$. The computations are easy and show that the faces don’t intersect.

4 Gluing the standard tetrahedron: figure eight knot

Theorem 4.1 There exists a spherical $\mathbb{C}R$-structure on the complement of the figure eight knot with discrete holonomy.

Proof. We use the same identifications that Thurston used in his construction for a hyperbolic real structure on the figure eight knot. That is, two tetrahedra with the identifications given in Figure 4. We realize the two tetrahedra in the Heisenberg space gluing a pair of sides. The side pairings transformations are shown in Figure 4 where the two tetrahedra are represented with a common side (here we introduce the point $q_3 = (\bar{\omega}, 0)$). They are determined by their action on three points and are defined by:

$$
\begin{align*}
g_1 : (q_2, q_1, p_1) &\rightarrow (q_3, p_2, p_1) \\
g_2 : (p_2, q_1, q_2) &\rightarrow (p_1, q_3, q_2) \\
g_3 : (q_1, p_2, p_1) &\rightarrow (q_2, p_2, q_3)
\end{align*}
$$

In order to define in a compatible way the faces we join the vertex $p_1$ to each point in the edge $[q_1, q_2]$ with segments of $\mathbb{C}$-circles and use $q_1$ to define the corresponding face. In the same manner, we define the other two pairs of identified faces. We verify that the faces are compatible and that the structure is well defined around the edges. This follows because the triangles at the vertexes are equilateral. The drawings in Figure 5 show the faces.

The rest of the proof concerns information about the holonomy of the structure, we divide it in several subsections.
Figure 3: A schematic view of the standard ideal tetrahedron in the Heisenberg group

Figure 4: Identifications on the tetrahedra.
Figure 5: Identification of two tetrahedra to obtain the figure eight knot in coordinates of the Heisenberg group: two views.
4.1 Discreteness of the representation

Recall from above and Figure 4 the side pairing transformations of the two tetrahedron with a common side:

\[ g_1 : (q_2, q_1, p_1) \rightarrow (q_3, p_2, p_1) \]
\[ g_2 : (p_2, q_1, q_2) \rightarrow (p_1, q_3, q_2) \]
\[ g_3 : (q_1, p_2, p_1) \rightarrow (q_2, p_2, q_3) \]

We conjugate each generator by the map

\[ \gamma : (\infty, 0, [1, -\sqrt{3}]) \rightarrow (p_1, q_2, q_1) \]

and obtain after some computation the following matrices in \( SU(2,1,\mathbb{Z}[\omega]) \), the Eisenstein-Picard group (see [FP]):

\[ G_1 = \begin{pmatrix} 1 & \omega & -\omega \\ 0 & 1 & -\bar{\omega} \\ 0 & 0 & 1 \end{pmatrix} \]
\[ G_2 = \begin{pmatrix} 1 & 1 & -\omega \\ -1 & 0 & -\bar{\omega} \\ -\bar{\omega} & \omega & 1 \end{pmatrix} \]
\[ G_3 = \begin{pmatrix} 1 & 1 & -\omega \\ -\omega & \bar{\omega} & -1 - \bar{\omega} \\ -\bar{\omega} & 0 & 1 + \omega \end{pmatrix} \]

Note that \( G_1, G_3 \) are parabolic and \( G_2 \) is elliptic.

**Theorem 4.2** The fundamental group of the complement of the figure eight knot has a discrete representation in \( PU(2,1) \).

**Proof.** As the generators are in \( SU(2,1,\mathbb{Z}[\omega]) \), the group is discrete. \( \square \)

4.2 Holonomy of the torus link

Refering to Figure 6, the holonomy of the torus link at the vertex can be computed following the identifications of the triangles forming the link. Starting with the triangle on the right of the first tetrahedron we obtain the generators

\[ H_1 = G_1^{-1}G_3G_1^{-1}G_2G_3^{-1}G_1G_3^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ -2\bar{\omega} & 1 & 0 \\ -2\omega - 1 & 2\omega & 1 \end{pmatrix} \]
\[ H_2 = G_2^{-1}G_1 = \begin{pmatrix} 1 & 0 & 0 \\ \bar{\omega} & 1 & 0 \\ -\omega & -\omega & 1 \end{pmatrix} \]

Figures 6 and 7 shows how to compute those elements. It turns out that they are parabolic and independent:

**Proposition 4.3** The holonomy of the torus link is faithful and parabolic.

Moreover \( H_1 \) is a Heisenberg translation by \([2\omega, 2\sqrt{3}]\) and \( H_2 \) is a Heisenberg translation by \([-\omega, \sqrt{3}]\). Therefore the holonomy is generated by \([-\omega, \sqrt{3}] \) \( H_2 \) and a vertical translation \([0, 4\sqrt{3}] \) \( (H_1H_2^2) \). Of course, that should be the case as the group is commutative and discrete.
Figure 6: The triangulation of the torus link

Figure 7: Computation of the holonomy at the vertex
4.3 Relation to Eisenstein-Picard group

In [FP] we proved that the Eisenstein-Picard Group $PU(2,1,\mathbb{Z}[\omega])$ is generated by

\[ P = \begin{bmatrix} 1 & 1 & -\omega \\ 0 & -\omega & +\omega \\ 0 & 0 & 1 \end{bmatrix}, \quad Q = \begin{bmatrix} 1 & 1 & -\omega \\ 0 & -1 & 1 \\ 0 & 0 & 1 \end{bmatrix}. \]

and

\[ I = \begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix}. \]

In this section we identify the generators of the holonomy in terms of these generators. The information is contained in the following proposition. We first state a lemma whose proof is a simple computation after a guess obtained by identifying the translational part of each parabolic element.

**Lemma 4.4** The holonomy of the torus link is given by

\[ H_1 = I(QP^{-1}Q(PQ^{-1})^{-2})^2I \]

and

\[ H_2 = IPQ^{-1}P^2Q^{-1}I \]

From the lemma and a computation we obtain the generators of the group.

**Proposition 4.5** $G_1 = PQ^{-1}P^2Q^{-1}$, $G_2 = I(PQ^{-1}P^2Q^{-1})^{-1}IPQ^{-1}P^2Q^{-1}$ and $G_3 = AIH_2IA^{-1}$ where $A = PQ^{-2}(PQ^{-1})^2I(QP^{-1})^2P$

5 Equations along the edges

We refer again to the parametrization of tetrahedra using $z_i, z'_i$ and $\bar{z}_i$. In this section we obtain the general equations for gluing two tetrahedra according to the scheme in Figure 8.

The first set of equations concerns the compatibility of Cartan's invariants of each of the four triples of points in the tetrahedron:

\[ A \leftrightarrow A' \implies k(p_1, p_2, q_1) = k(\hat{p}_1, \hat{p}_2, \hat{q}_2) \implies t = \hat{s} \]

\[ B \leftrightarrow B' \implies k(p_1, q_1, q_2) = k(\hat{q}_1, \hat{q}_2, \hat{p}_2) \implies s|z|^2 - t + 2\text{Im }z = \text{function of } \hat{z}, \hat{t}, \hat{s} \]

\[ C \leftrightarrow C' \implies k(p_1, p_2, q_2) = k(\hat{q}_1, \hat{q}_2, \hat{p}_1) \implies s = \frac{s|\hat{z}|^2 - \hat{t} + 2\text{Im }\hat{z}}{|\hat{z} - 1|^2} \]

\[ D \leftrightarrow D' \implies k(p_2, q_1, q_2) = k(\hat{p}_2, \hat{q}_1, \hat{p}_1) \implies \text{function of } z, t, s = \hat{t} \]

The function of $z$, $t$ and $s$ above is

\[
\frac{2(s-t)\text{Re }z + 2(1+ts)\text{Im }z + t(1+s^2)|z|^2 - s(1+t^2)}{|(s-i)z + i - t|^2}.
\]

There are three independent equations, the fourth one being a consequence of the cocycle condition. We choose the first and the last two equations. From the first equation, $s$ is determined by $t$. From the last, $\hat{t}$ is determined by $z$, $t$ and $s$. Substituting in the third equation we obtain $s$ as a function of $z$, $\hat{z}$ and $t$. That gives a 5 parameter family of a couple of tetrahedra with compatible Cartan’s invariants under the gluing scheme.
5.1 Symmetric tetrahedra

**Proposition 5.1** If two symmetric tetrahedra are glued following the scheme to obtain the complement of the figure eight knot then they are both regular. In that case, a couple of regular tetrahedra is parametrized by a hypersurface in the variables $z$ and $\dot{z}$.

**Proof.** For a symmetric tetrahedron $s = t$. From the equations above we obtain that the four triples have the same Cartan’s invariant, therefore they are regular. In this case we have

$$t = \frac{\text{Im} \ z}{1 - \text{Re} \ z} = \frac{\text{Im} \ \dot{z}}{1 - \text{Re} \ \dot{z}}$$

In order to have a coherent gluing of the tetrahedra along the edges we have to impose the following equations (where we changed the dot notation for a different variable $w$) two for each cycle of edges, corresponding to the two end points of each cycle:

$$z_1 w_1 \bar{z}_2' w_3 z_2 \bar{w}_1 = 1$$
$$z_1' w_1' \bar{z}_2 z_3' \bar{w}_1 = 1$$
$$z_3 w_3 \bar{z}_2' w_2' \bar{z}_1 = 1$$
$$\bar{z}_3' w_3' w_2 \bar{z}_1' \bar{w}_2 = 1$$

The product of the four equations is clearly 1, so only three of the equations are independent. Using
the relations between the invariants we simplify to

\[(z_2 - 1)\tilde{z}'_2 (w_1 - 1)\tilde{w}_1 = 1\]
\[(z'_2 - 1)\tilde{z}_2 (\tilde{w}_1' - 1)w'_1 = 1\]
\[(\tilde{z}_1 - 1)z_3 (\tilde{w}_3 - 1)\tilde{w}'_2 = 1\]
\[(\tilde{z}'_1 - 1)z'_3 (w'_3 - 1)w_2 = 1\]

**Proposition 5.2** The only symmetric tetrahedra with identifications as the scheme above giving the eight knot complement is the one obtained in the previous section.

**Proof.** If the tetrahedra are regular \(z_1 = \tilde{z}_1\). We obtain then

\[\tilde{z}'_2 = \frac{1}{1 - \tilde{z}'_1} = \frac{1}{1 - \frac{\tilde{z}_1}{z_1}} = 1 - \tilde{z}_2.\]

The second equation becomes \(\frac{-z_1}{1 - z_1^2} = 1\) and the last one \(\frac{1 - z_1}{\tilde{z}_1 - 1} = 1\). From those two equations follows that \(w^2_1 + \tilde{w}_1 = 0\) which has the unique solution \(w_1 = e^{i\pi/3}\). \(\square\)

### 6 Gluing special tetrahedra: The Whitehead Link

Using the other special tetrahedra defined in 3.1.4 we obtain the complement of the Whitehead link. It suffices to observe that we can glue four tetrahedra as in Thurston forming an octahedra with dihedral angles equal to \(\pi/2\). We make \(p_1 = (0, 1 + \sqrt{2}), p_2 = (0, -(1 + \sqrt{2})), q_1 = (1, 0)\) and \(q_2 = (i, 0)\). We have \(z_1 = \tilde{z}_1 = i\) with \(\mathbb{R}(p_1, q_1, q_2) = \pi/4\). We want to show completeness. Define \(q_3 = (-1, 0)\) and \(q_4 = (-i, 0)\).

The generators of the group are given by

\[g_A : [p_1, q_1, q_2] \rightarrow [q_2, q_3, p_2]\]
\[g_B : [p_1, q_2, q_3] \rightarrow [q_4, p_2, q_3]\]
\[g_C : [p_1, q_3, q_4] \rightarrow [q_4, q_1, p_2]\]
\[g_D : [p_1, q_4, q_1] \rightarrow [q_2, p_2, q_1]\]

conjugating the generators above with the mapping

\([p_1, q_1, q_2] \rightarrow [\infty, 0, (1, 1)]\)

we obtain the following matrices in \(SU(2, 1)\) representing the generators:

\[G_1 = \begin{bmatrix}
1 & 0 & -i \\
-1 - i & 1 & -1 + i \\
-1 - i & 1 - i & i
\end{bmatrix}\]

\[G_2 = \begin{bmatrix}
1 & 1 - i & -1 + i \\
-1 - i & -1 & 1 - i \\
-1 + i & 1 + i & -1 - 2i
\end{bmatrix}\]
Figure 9: The Whitehead link complement

$G_3 = \begin{bmatrix}
 i & 1 + i & -i \\
 1 - i & -1 - 2i & 2i \\
 -1 - i & -3 + i & 3 + 2i
\end{bmatrix}$

$G_4 = \begin{bmatrix}
 -i & 0 & 0 \\
 -1 + i & -1 & 0 \\
 -1 + i & -1 + i & -i
\end{bmatrix}$

$G_1$ and $G_3$ have trace $2 + i$ and therefore are loxodromic, $G_2$ and $G_4$ have trace $-1 - 2i$ and are elliptic of order four.

We obtained the following

Theorem 6.1 The representation of the fundamental group of the Whitehead link complement generated by $G_1, G_2, G_3, G_4$ is in $PU(2,1, \mathbb{Z}[i])$ and is therefore discrete.

6.1 Holonomy

There are two tori. We use the notation as in [Ra]. We compute their holonomy as in the case of the figure eight knot. The first torus has holonomy generated by

$H_1 = G_3^{-1}G_1^{-1} = \begin{bmatrix}
 -1 - 6i & -6 - 4i & 2 + 4i \\
 -4 + 6i & 1 + 8i & 2 - 4i \\
 2 + 4i & 4 + 2i & -1 - 2i
\end{bmatrix}$ and $H_2 = G_2$. 

16
Observe that $H_1$ is parabolic but $H_2$ is elliptic. The other torus has holonomy generated by

$$H_1' = G_3 G_1^{-2} G_3 = \begin{bmatrix} 5 & 2 - 6i & -4 \\ -8 - 4i & -7 + 8i & 6 + 2i \\ -8 + 8i & 8 + 12i & 5 - 8i \end{bmatrix} \quad \text{and} \quad H_2' = \text{Id.}$$

Here $H_1'$ is parabolic. Note that the holonomy of that torus is not faithful.

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