Robust Bilateral Trade Mechanisms with Known Expectations

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Abstract

We study the design of revenue-maximizing bilateral trade mechanisms in the correlated private value environment. We assume the designer only knows the expectations of the agents' values, but knows neither the marginal distribution nor the correlation structure. The performance of a mechanism is evaluated in the worst-case over the uncertainty of joint distributions that are consistent with the known expectations. Among all dominant-strategy incentive compatible and ex-post individually rational mechanisms, we provide a complete characterization of the maxmin trade mechanisms and the worst-case joint distributions.

Keywords: Bilateral trade, mechanism design, information design, revenue maximization, correlated private values, max-min, worst-case, dominant strategy incentive compatible, ex-post individual rational, randomization, deterministic mechanism.

JEL Codes: C72, D82, D83.

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*I have benefited from discussions with Songzi Du and Joel Sobel.
1 Introduction

In this paper, we study the design of revenue-maximizing bilateral trade mechanisms when
the mechanism designer is poorly informed of the values of the buyer and the seller. The
mechanism designer’s revenue is the difference between what the buyer pays and what the
seller receives. We could think of this mechanism designer as the commercial designer of a
trading platform who charges a fee for transactions on the platform. We assume the designer
knows only the information about the expectations of the private values of the buyer and
the seller, but does not know the joint distribution of the private values\(^1\). The designer
evaluates any trading mechanism by the expected revenue in the worst case, over all possible
joint distributions consistent with the known expectations. The objective of the designer is to
find a trading mechanism that maximizes worst-case expected revenue among all dominant
strategy incentive compatible (DSIC) and ex-post individually rational (EPIR) mechanisms.

This study is motivated by several observations. First, the joint distribution is
statistically a high dimensional object, which may be hard to estimate. In contrast, the
expectations are two parameters, about which it may be relatively easier to form an educated
guess. Practically, it may fit into situations in which the designer knows little about the
agents. At a high level, this study follows the “Wilson doctrine” (Wilson (1987)) that
motivated the search for economic institutions not sensitive to unrealistic assumptions about
the information structure.

Our main results offer a complete characterization of the maxmin trade mechanisms and
the corresponding worst case joint distributions. Theorem 1 considers the symmetric case in
which the known expectations of value distributions of the buyer and the seller sum up to
the upper bound of the support, which is normalized to 1.\(^2\) In this case, Maxmin Trade
Mechanism (I) can be described as follows. Trade occurs with a positive probability if
and only if the difference between the reported values exceeds some threshold; the trading
probability is linear and strictly increasing in the difference between the reported values; in
addition, trade occurs with probability 1 if and only if the reported value of the buyer and the
seller is 1 and 0 respectively; finally, the transfer function is quadratic in the reported values
of the agents. The support of Worst-Case Joint Distribution (I) is a triangular subset

\(^1\)That is, the designer knows neither the marginal distributions nor the correlation structure except for
the expectations of the marginal distributions.

\(^2\)Note the lower the seller’s value, the higher his willingness to trade. Thus it is plausible to regard the
highest value seller as the lowest type seller. When the known expectations sum up to 1, the expectation of
buyer and the expectation of seller have the same distance from the lowest type buyer and the lowest type
seller respectively. Therefore we refer to this case as the symmetric case.
in the set of joint valuations, which is the same as the trading region\(^3\) of Maxmin Trade Mechanism (I). Its marginal distribution for the buyer is a combination of a uniform distribution on an interior interval of values and an atom on 1, while for the seller is a combination of a uniform distribution on an interior interval and an atom on 0; Its conditional distribution is some truncated generalized Pareto distribution. Theorem 2 considers the asymmetric case in which the known expectations of value distributions of the buyer and the seller sum up to a number other than 1. For the asymmetric case, Maxmin Trade Mechanism (II) can be described as follows. Trade occurs with a positive probability if and only if the difference between the weighted reported values exceeds some threshold; the trading probability is strictly increasing in the difference between some logarithmic functions of some linear transformation of reported values; in addition, trade occurs with probability 1 if and only if the reported value of the buyer and the seller is 1 and 0 respectively; finally, the transfer function is the sum of a logarithmic function and a linear function of the reported value of the agents. The support of Worst-Case Joint Distribution (II) is also a triangular subset in the set of joint valuations, which is the same as the trading region of Maxmin Trade Mechanism (II). Its marginal distribution for the buyer is some truncated generalized Pareto distribution with an atom on 1, while for the seller is some truncated generalized Pareto distribution with an atom on 0; Its conditional distribution is some truncated generalized Pareto distribution.

We take a constructive approach based on the saddle point property. Specifically, we reformulate the designer’s problem into a zero-sum game between the designer and Nature, who chooses a feasible joint distribution consistent with the known expectations to minimize expected revenue. Finding an optimal mechanism is equivalent to finding a saddle point of the zero-sum game.

We first consider the symmetric case in which the buyer’s value and the seller’s value sum up to 1. To form an educated guess about the saddle point, we begin with the trading region of the maxmin mechanism and the support of the worst case joint distribution. First in the maxmin mechanism, trade occurs with a positive probability if and only if the difference between the values of the buyer and the seller exceeds some threshold. The intuition behind this property can be summarized as follows. In the symmetric case, the difference between the private values of buyer and seller can be interpreted as the true value to the designer\(^4\). And the difference between the payment from buyer and the transfer to seller can be interpreted as the price charged by the designer. From the mechanism design literature, we learned that the

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\(^3\)We refer to the set of value profiles in which trade occurs with a positive probability as the trading region.

\(^4\)We may view buyer and seller as two departments of a company. The overall benefit to the company from trading between these two department is the difference between their private values.
revenue is generally higher if the designer exercises monopoly power, which corresponds to restricting trade by allowing trade to occur only when the true value exceeds some threshold in our environment. Second, the support of the worst-case joint distribution is the same as the trading region. The intuition behind this property can be summarized as follows. To the designer, the trades occurring outside of the support generate no revenue. However, to Nature, by allocating some probabilities outside of the trading region, there are two opposite effects: the upside is that this operation will reduce the overall probability on the trading region, which is potentially beneficial to the adversarial Nature, while the downside is that in order to respect the known expectation constraints, this operation will increase the probability of certain value profiles in the trading region, which is potentially detrimental to the adversarial Nature especially when the revenue from those value profiles are high. In the worst-case joint distribution, the tradeoff is resolved in favor of the downside in the saddle point and the trading region exactly coincides with the support of the worst-case joint distribution. Then, by the strong duality, the revenue from value profiles in the support is some linear function of the values. Intuitively, we expect the maxmin mechanism to exhibit a lot of indifference to various plausible joint distributions. When the revenue from value profiles in the support is some linear function, any plausible joint distribution will generate the same revenue as long as its support is contained in the support of the worst case joint distribution. Now with the help of envelop representation of the revenue, we are confronted with (essentially) a system of partial differential equations involving the trading probability. We can prove the trading probability is separable in the buyer’s value and the seller’s value. Therefore it can be written in the sum of two functions with only one argument. Then we take a guess and verify approach to solve for the closed-form of these two functions, which turn out to be linear.

For the construction of the worst-case joint distribution, we first derive a virtual representation of the expected revenue. That is, the expected revenue is equivalent to the inner product of the trading probability and the weighted virtual value, which will be defined in the main context. In the worst-case joint distribution, the weighted virtual value are positive only for the value profile in which the buyer’s value and the seller’s value is 1 and 0 respectively. Beside, the weighted virtual value are 0 for the other value profiles in the support. The intuition behind is that Maxmin Trade Mechanism (I) requires randomization (trade occurs with some probability greater than 0 but less than 1) for all interior value profiles. That is, the designer is indifferent between trade and no trade for these value profiles. Indeed, given the above properties, any trade mechanism will generate the same revenue and be a best response for the designer provided that it is a feasible and monotone trade mechanism in which trade occurs with a positive probability only for the
value profiles in the support and trade occurs with probability 1 for the value profile in which the buyer’s value and the seller’s value is 1 and 0 respectively. And it is easy to verify that Maxmin Trade Mechanism (I) is such a mechanism. The remaining issue is whether these properties can hold. We provide an affirmative answer by constructing such a joint distribution. Briefly, these properties implies a system of ordinary differential equations and partial differential equations, which can be solved by the guess and verify approach.

For the asymmetric case in which the buyer’s value and the seller’s value sum up to a number other than 1, trade occurs with a positive probability if and only if the difference between the weighted values of the buyer and the seller exceeds some threshold in the maxmin mechanism. The rough intuition behind this property is that it may be beneficial to the designer to attach different weights to the values of the buyer and the seller, since they have different eagerness to trade for the asymmetric case. The intuition for the threshold is the same as the aforementioned intuition that the revenue is generally higher if the designer exercises monopoly power. Second, the support of the worst-case joint distribution is the same as the trading region, based on the same intuitions. The remaining procedures for characterizing the maxmin mechanism and the worst-case joint distribution are similar and we delay the details to the main context.

Lastly, we restrict attention to deterministic DSIC and EPIR mechanisms and characterize the entire set of maxmin deterministic mechanisms as well as the worst-case joint distribution. Our finding is that any determinisitic DSIC and EPIR mechanism whose trade boundary contains two given value profiles and lies above (including) the line connecting the two given value profiles will be optimal; the worst-case joint distribution puts probability mass only on the two given value profiles and the value profile in which the buyer’s value and seller’s value is 1 and 0 respectively. Examples about the maxmin deterministic mechanisms includes linear trading, in which trade occurs if and only if the difference between the weighted values exceeds a threshold, and threshold trading, in which trade occurs if and only if the buyer’s value exceeds a threshold and the seller’s value falls below a threshold. Our construction is based on strong duality of linear programming. We first rule out mechanisms in which trade occurs for value profiles in which the value of the seller exceeds the value of the buyer. To do so, we note that the revenue from the four vertices value profiles are non-positive, implied by the monotonicity property of the mechanism. Then we show Nature can always put probability mass only on some of the four vertices value profiles, thus resulting in non-positive revenue guarantee. For the remaining

\footnote{The expectation may be viewed as a metric about the average eagerness to trade. The higher the average eagerness to trade, the higher the expectation of the buyer and the lower the expectation of the seller.}
mechanisms, we invoke strong duality and work on the dual maximization problem. We further propose a relaxation of the dual by omitting many constraints, resulting in a finite dimensional linear programming problem. We identify a greatest upper bound of the value of the relaxation and argue that the upper bound is attainable by constructing both the (class of) mechanisms and the worst-case joint distribution.

The remaining of the paper proceeds as follows. Section 2 provides a literature review. Section 3 presents the model. Section 4 characterizes our main results. Section 5 characterizes the class of maxmin deterministic mechanisms. Section 6 concludes. All proofs are in the Appendix.

2 Related Literature

This paper contributes to the literature of robust mechanism design. The closest related papers are Carrasco et al. (2018), Che (2020), Koçyiğit et al. (2020), Suzdaltsev (2020), Zhang (2021), Brooks and Du (2021).

Carrasco et al. (2018) study the revenue-maximizing selling mechanisms when the seller faced with a single buyer only knows the first $N$ moments of distribution, and solve the problem with a known expectation as a special case. They also find the optimal deterministic posted price for this special case. This paper can be viewed as a generalization of the special case of their model to the multidimensional bilateral trade setting, as our model is reduced to theirs when the expectation of the seller’s value is known to be 0. Their approach also essentially bases on duality. However, our setting requires us to verify a guess about the joint distribution with rather intricate correlation structure, while they need to verify a guess about the single dimensional distribution.

Che (2020) considers a model of auction design in which the auctioneer only knows the expectation of each bidder’s value, and characterizes the optimal random reserve prices for the second price auction. He further shows that it also achieves the greatest revenue guarantee within a class of competitive mechanisms. The constructive approach is similar, but one of the key assumptions differs. We do not restrict attention to any particular mechanism, but to the DSIC and EPIR mechanisms, which also does not coincide with the class of competitive mechanisms.

Suzdaltsev (2020) also considers exactly the same setting, but focuses on auction design and deterministic mechanisms. To wit, he considers an auctioneer who knows only the expectations of bidders’ values and shows that a linear version of Myerson’s optimal auction is optimal among all deterministic DSIC and EPIR mechanisms. We consider a bilateral trade model and also derive a result on maxmin deterministic DSIC and EPIR mechanisms.
(Theorem 3). In addition, both papers characterize the entire class of maxmin deterministic mechanisms based on strong duality.

Koçyiğit et al. (2020) considers an auction design model in which Nature chooses the worst-case joint distribution subject to symmetric expectations of bidder’s values. They find, among other results, that a highest-bidder lottery mechanism is optimal within the DSIC and EPIR mechanisms in which only the highest bidder is allocated the good. In contrast, we consider a broader class of mechanisms and do not have any restriction on the known expectations.

Zhang (2021) considers a model of auction design in which the auctioneer knows the marginal distribution of each bidder’s value but does not knows the correlation structure, and characterizes maxmin auctions among some general class of mechanisms under certain regularity conditions. The worst-case joint distributions are motivated by some property of some version of “virtual values” in both papers. However, the construction of the worst-case joint distribution requires us to solve for some partial differential equation in addition to ordinary differential equations. In addition, this paper offers a complete characterization for all primitives.

Brooks and Du (2021) consider informationally robust auction design in the interdependent value environment. They assume the auctioneer only knows the expectation of each bidder’s value, but does not know the distribution of values and higher order beliefs. The solution concept they use is what they refer to as strong maxmin solution. In contrast, our framework assumes values are known to the agents, and restrict attention to DSIC mechanisms, ruling out issues brought by higher order beliefs. Therefore, our methodology differs. However, in a high level, both papers rely on some version of virtual values to proceed the analysis. And interestingly, they characterize a proportional auction as a maxmin mechanism, which has similar features with the maxmin trade mechanism in our model for the symmetric case.

There are other papers seeking robustness to value distributions, e.g., Carrasco et al. (2019), Auster (2018), Bergemann and Schlag (2011), Bergemann and Schlag (2008). Carroll (2017), Giannakopoulos et al. (2020) and Chen et al. (2019) focus on the problem of selling multiple goods to a single buyer when the value distributions are unknown. A separate strand of papers focus on the case in which the designer does not have reliable information about the agents’ hierarchies of beliefs about each other while assuming the knowledge of the payoff environment, e.g., Bergemann and Morris (2005), Chung and Ely (2007), Chen and Li (2018), Bergemann et al. (2016, 2017, 2019), Du (2018), Brooks and Du (2020), Libgober and Mu (2018), Yamashita and Zhu (2018). Carroll (2019) provide an elaborate survey on various notions of robustness studied in the literature, e.g., robustness
to preferences, robustness to strategic behavior and robustness to interaction among agents.

There are other papers studying robust bilateral trade mechanisms. Wolitzky (2016) studies optimal mechanisms in terms of efficiency for bilateral trade when agents are maxmin expected utility maximizers, with similar ambiguity sets (that is, a buyer knows only the mean of a seller’s valuation, and vice versa). Bodoh-Creed (2012) also assumes the agents are maxmin expected utility maximizers, but focuses on revenue-maximizing bilateral trade mechanisms in their applications. Carroll (2016) studies bilateral trade mechanisms within the informationally robust framework with a focus on the expected surplus. In contrast, our paper considers a private-value environment, assumes that the designer has limited knowledge about the economic environment and focuses on revenue-maximizing mechanism design.

3 Preliminaries

We consider an environment where a single indivisible good is traded between two risk-neutral agents. One is the Seller (S), who holds the good initially, while the other is the Buyer (B), who does not hold the good initially. We denote by \( I = \{S, B\} \) the set of agents. Each agent \( i \) has private information about her valuation for the object, which is modeled as a random variable \( v_i \) with cumulative distribution function \( F_i \).

We use \( f_i(v_i) \) to denote the density of \( v_i \) in the distribution \( F_i \) when \( F_i \) is differentiable at \( v_i \); We use \( Pr_i(v_i) \) to denote the probability of \( v_i \) in the distribution \( F_i \) when \( F_i \) has a probability mass at \( v_i \). We denote \( V_i \) as the support of \( F_i \). We assume each \( V_i \) is bounded. Throughout, we assume common support, i.e., \( V_S = V_B \). As a normalization, we assume \( V_i = [0, 1] \). The joint support of \( F_i \) is denoted as \( V = [0, 1]^2 \) with a typical value profile \( v \). The joint distribution is denoted as \( F \).

The valuation profile \( v \) is drawn from a joint distribution \( F \). The designer only knows the expectations \( M_B \) and \( M_S \) for the private value of \( B \) and \( S \) respectively as well as the support, but does not know the joint distribution of the values of these two agents. Formally, we denote by

\[
\Pi(M_B, M_S) = \{ \pi \in \Delta V : \int v_B \pi(v) dv = M_B, \int v_S \pi(v) dv = M_S \}
\]

the collection of such joint distributions.

The designer seeks a dominant strategy incentive compatible (DSIC) and ex-post

\[^6\text{We do not make any assumption on the distributions of these random variables. It could be continuous, discrete, or any mixtures. Also we allow asymmetric distributions, that is, } F_S \text{ can be different from } F_B.\]

\[^7\text{That is, except for the expectations, the designer know neither the marginal distributions nor the correlation structure.}\]
individually rational (EPIR) mechanism. A direct mechanism\(^8\) \((q, t_B, t_S)\) is defined as a trading probability \(q : V \to [0, 1]\) and transfer functions \(t_i : V \to \mathbb{R}\).\(^9\) With slight abuse of notations, we assume each agent report \(v_i \in V_i\) to the designer. Upon receiving the reported profile \(v = (v_B, v_S)\), the buyer \(B\) gets the good with probability \(q(v)\) and pays \(t_B(v)\); the seller \(S\) holds the good with the remaining probability \(1 - q(v)\) and receives a payment of \(t_S(v)\). We use \(t(v) \equiv t_B(v) - t_S(v)\) to denote the difference between what \(B\) pays and what \(S\) receives. The set of all DSIC and EPIR mechanisms is denoted as \(\mathcal{D}\).

We are interested in the designer’s expected revenue in the dominant strategy equilibrium in which each agent truthfully reports her valuation of the good. The expected revenue of a DSIC and EPIR mechanism \((q, t_B, t_S)\) when the joint distribution is \(\pi\) is \(U((q, t_B, t_S), \pi) \equiv \int_{v \in V} \pi(v)t(v)dv\). The designer evaluates each such mechanism \((q, t_B, t_S)\) by its worst-case expected revenue over plausible joint distributions. The designer’s goal is to find a mechanism with the maximal worst-case revenue for a given pair of expectations \((M_B, M_S)\). Formally, the designer tries to find a mechanism \((q^*, t_B^*, t_S^*)\) that solves the following problem:

\[
(q^*, t_B^*, t_S^*) \in \arg \max_{(q, t_B, t_S) \in \mathcal{D}} \min_{\pi \in \Pi(M_S, M_B)} \int_{v \in V} \pi(v)t(v)dv
\]

s.t.

\[
v_Bq(v) - t_B(v) \geq 0 \quad \forall v \quad (EPIR_B)
\]
\[
v_Bq(v) - t_B(v) \geq v_Bq(v'_B, v_S) - t_B(v'_B, v_S) \quad \forall v, v'_B \quad (DSIC_B)
\]
\[
v_S(1 - q(v)) + t_S(v) \geq v_S \quad \forall v \quad (EPIR_S)
\]
\[
v_S(1 - q(v)) + t_S(v) \geq v_S(1 - q(v_B, v'_S)) + t_S(v_B, v'_S) \quad \forall v, v'_S \quad (DSIC_S)
\]
\[
0 \leq q(v) \leq 1 \quad \forall v \quad (Feasibility)
\]

**Remark 1.** We may consider a slightly more general mechanism in which we allow the designer to destroy part of the good uniformly, i.e., the sum of the final allocations to \(B\) and \(S\) could be some \(0 \leq a \leq 1\). However, we argue it is without loss of generality to assume \(a\) is exactly 1 for all \(v\). To see this, note the constraints in the above program becomes

\[
v_Bq(v) - t_B(v) \geq 0 \quad \forall v \quad (EPIR_B)
\]
\[
v_Bq(v) - t_B(v) \geq v_Bq(v'_B, v_S) - t_B(v'_B, v_S) \quad \forall v, v'_B \quad (DSIC_B)
\]

\(^8\)Since we restrict attention to DSIC mechanisms, the revelation principle holds and it is without loss of generality to focus on direct mechanisms.

\(^9\) \(q\) is the probability that \(B\) obtains the good, which can be interpreted as the trading probability in our environment. We allow randomization, which will play a crucial role in our analysis.
\[ v_S(a - q(v)) + t_S(v) \geq v_S \quad (EPIR'_S) \]
\[ v_S(a - q(v)) + t_S(v) \geq v_S(a - q(v_B, v'_S)) + t_S(v_B, v'_S) \quad \forall v, v'_S \quad (DSIC'_S) \]

\[ 0 \leq q(v) \leq a, 0 \leq a \leq 1 \quad \forall v \quad (Feasibility') \]

Given any \((a, q(v), t_B(v), t_S(v))\) satisfying the new constraints, we can inflate it to \((1, \frac{1}{a}q(v), \frac{1}{a}t_B(v), \frac{1}{a}t_S(v))\). Under the new mechanism, \((EPIR'_S)\) holds since the RHS of \((EPIR'_S)\) is greater and the LHS of \((EPIR'_S)\) remains the same. The other constraints also holds trivially. And the new mechanism achieves weakly better revenue for any joint distribution because it inflates the revenue. Thus, we can assume \(a = 1\). After we present the main result, we can also consider a model in which we allow the designer to destroy part of the good not necessarily uniformly.

4 Main Results

To facilitate the analysis, it will be useful to further simplify the problem. We will use the following proposition: its proof is standard but included in the Appendix for completeness. And all formal proofs are deferred to the Appendix.

**Proposition 1.** Maxmin Bilateral Trade Mechanisms have the following properties:

(i). \(q(v)\) is nondecreasing in \(v_B\) and nonincreasing in \(v_S\).

(ii). \(t_B(v) = v_Bq(v) - \int_0^{v_B} q(b, v_S)db\).

(iii). \(t_S(v) = 1 - (1 - q(v))v_S - \int_{v_S}^1 (1 - q(v_B, s))ds\).

(iv). \(t(v) = (v_B - v_S)q(v) - \int_0^{v_B} q(b, v_S)db - \int_{v_S}^1 q(v_B, s)ds\)

For the rest of the paper, we focus on the case in which \(M_B > M_S\). Otherwise the problem becomes trivial as the nature can always choose a distribution that the seller’s value is never below the buyer’s value, for instance, the joint distribution that put all probability mass on \((M_B, M_S)\). Thus, the revenue guarantee can not be positive, implied by EPIR. Then, trivially, No Trade Mechanism \((q(v) = t_S(v) = t_B(v) = 0 \text{ for any } v)\) is a maxmin mechanism. We summarize this observation as follows.

**Observation 1.** No Trade Mechanism is a maxmin mechanism if \(M_B \leq M_S\). And the revenue guarantee is 0.

In addition, we assume \(M_S > 0\) for the rest of the paper. If \(M_S = 0\), the problem becomes one-agent problem, which has been solved by Carrasco et al. (2018).
4.1 The Symmetric Case: $M_B + M_S = 1$

In this subsection, we characterize the maxmin mechanism when the two known expectations sum up to 1. We observe that the maxmin optimization problem can be interpreted as a two-player sequential zero-sum game. The two players are the designer and Nature. The designer first chooses a mechanism $(q, t_B, t_S) \in D$. After observing the designer’s choice of the mechanism, Nature chooses a joint distribution $\pi \in \Pi(M_B, M_S)$. The designer’s payoff is $U((q, t_B, t_S), \pi)$, and Nature’s payoff is $-U((q, t_B, t_S), \pi)$. Now instead of solving directly for such a subgame perfect equilibrium we can solve for a Nash equilibrium $((q^*, t_B^*, t_S^*), \pi^*)$ of the simultaneous move version of this zero-sum game, which corresponds to a saddle point of the payoff functional $U$, i.e.,

$$U((q^*, t_B^*, t_S^*), \pi) \geq U((q, t_B, t_S), \pi) \geq U((q, t_B, t_S), \pi^*)$$

for any $(q, t_B, t_S)$ and any $\pi$. The properties of a saddle point imply that the principal’s equilibrium strategy in the simultaneous move game, $(q^*, t_B^*, t_S^*)$, is also his maxmin strategy (i.e. his equilibrium strategy in the subgame perfect equilibrium of the sequential game).

We propose the following **Maxmin Trade Mechanism (I)** and **Worst-Case Joint Distribution (I)**. Formally, they are described as below.

Maxmin Trade Mechanism (I)

Let $v = (v_B, v_S)$ be the reported value profile of the two agents. If $v_B - v_S \geq r$, then

$$q^*(v_B, v_S) = \frac{1}{1-r}(v_B - v_S - r)$$

$$t_B^*(v_B, v_S) = \frac{1}{2(1-r)}(v_B^2 - (v_S + r)^2)$$

$$t_S^*(v_B, v_S) = \frac{1}{2(1-r)}((v_B - r)^2 - v_S^2)$$

where $r = 1 - \sqrt{1 - (M_B - M_S)}$

Otherwise

$$q^*(v_B, v_S) = t_B^*(v_B, v_S) = t_S^*(v_B, v_S) = 0$$

Worst-Case Joint Distribution (I)

Let $\pi^*(v_B, v_S)$ denote the density of the value profile $(v_B, v_S)$ whenever the density exists. Let $Pr^*(v_B, v_S)$ denote the probability mass of the value profile $(v_B, v_S)$ whenever there is some probability mass on $(v_B, v_S)$. Let $V(I) := \{v|v_B - v_S \geq r\}$. Worst-Case Joint Distribution
(I) has the support $V(I)$ and is defined as follows:

$$\pi^*(v_B, v_S) = \begin{cases} \frac{2r^2}{(v_B - v_S)^3} & v_B - v_S \geq r, v_B \neq 1, v_S \neq 0 \\ \frac{r^2}{v_S} & v_B = 1, 0 < v_S \leq 1 - r \\ \frac{r^2}{v_B} & r \leq v_B < 1, v_S = 0 \end{cases}$$

$$Pr^*(1, 0) = r^2$$

Equivalently, **Worst-Case Joint Distribution (I)** can be described by worst-case marginal distributions and conditional distributions as follows:

For $B$, the worst-case marginal distribution is

$$\pi^*_B(v_B) = 1$$

for $r \leq v_B < 1$,

$$Pr^*_B(1) = r$$

That is, worst-case marginal distribution for $B$ is uniform distribution on $[r, 1)$ with a probability mass $r$ on 1.

For $S$, the worst-case marginal distribution is

$$\pi^*_S(v_S) = 1$$

for $0 < v_S \leq 1 - r$,

$$Pr^*_S(0) = r$$

That is, worst-case marginal distribution for $S$ is uniform distribution on $(0, 1 - r]$ with a probability mass $r$ on 0.

The conditional distribution of $v_S$ on $v_B$ is

$$\pi^*(v_S|v_B) = \frac{2r^2}{(v_B - v_S)^3}$$

for $r \leq v_B < 1, 0 < v_S \leq v_B - r$,

$$Pr^*(v_S = 0|v_B) = \frac{r^2}{v_B^2}$$

for $r \leq v_B < 1$;

$$\pi^*(v_S|v_B = 1) = \frac{r}{(1 - v_S)^2}$$
for $0 < v_S \leq 1 - r$,

$$Pr^*(v_S = 0|v_B = 1) = r$$

That is, the conditional distribution of $v_S$ on $v_B$ is certain generalized Pareto distribution on $(0, v_B - r]$ with some probability mass on 0. Likewise, the conditional distribution of $v_B$ on $v_S$ is certain generalized Pareto distribution on $[v_S + r, 1)$ with some probability mass on 1. The feature is similar and we omit the derivation.

**Definition 1** (Positive correlation for bivariate distribution). Let $Z = (X, Y)$ be a bivariate random vector. $Z$ exhibits positive correlation for $D_X$ and $D_Y$ if $F(X|Y = y)$ first order stochastically dominates $F(X|Y = y')$ for any $y > y', y, y' \in D_Y$ and $F(Y|X = x)$ first order stochastically dominates $F(Y|X = x')$ for any $x > x', x, x' \in D_X$.

**Remark 2.** Worst-Case Joint Distribution (I) exhibits positive correlation for $r \leq v_B < 1$ and $0 < v_S \leq 1 - r$.

**Theorem 1.** When $M_B + M_S = 1$, Maxmin Trade Mechanism (I) and Worst-Case Joint Distribution (I) form a Nash equilibrium. The revenue guarantee is $r^2$.

### 4.2 Illustration of Theorem 1

#### 4.2.1 Weighted Virtual Values

We begin our analysis by defining a generalized version of virtual values in our environment. We consider the problem that fixing any joint distribution $\pi$, the designer designs an optimal mechanism $(q, t_B, t_S)$. We denote the density of value profile $v = (v_i, v_j)$ as $\pi(v_i, v_j)$. We define $\pi_i(v_i) \equiv \int_{v_j} \pi(v_i, v_j)dv_j$. We denote the probability of $v_i$ conditional on $v_j$ as $\pi_i(v_i|v_j)$, the cumulative distribution function of $v_i$ conditional on $v_j$ as $\Pi_i(v_i|v_j) = \int_{s_i \leq v_i} \pi_i(s_i|v_j)ds_i$. We define $\Pi_i(v_i, v_j) \equiv \pi_j(v_j)\Pi_i(v_i|v_j) = \int_{s_i \leq v_i} \pi(s_i, v_j)ds_i$. An direct implication of Proposition 1 is that the expected revenue of $(q, t)$ under the correlation $\pi$ is

$$E[t_B(v) - t_S(v)] = \int v q(v)\Phi(v)dv$$

where

$$\Phi(v) = \pi(v)(v_B - v_S) - (\pi_S(v_S) - \Pi_B(v_B, v_S)) - \Pi_S(v_S, v_B)$$

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10To see this, note $F(v_S|v_B) = \frac{r^2}{(v_B - v_S)^2}$ is decreasing w.r.t. $v_B$ for $r \leq v_B < 1$. The positive correlation breaks when $v_B = 1$. Similarly, $F(v_B|v_S) = 1 - \frac{r^2}{(v_B - v_S)}$ is decreasing w.r.t. $v_S$ for $0 < v_S \leq 1 - r$. The positive correlation breaks when $v_S = 0$. 

13
Here $\Phi(v)$ is defined as the weighted virtual value\(^{11}\) when the value profile is $v$. Thus the problem of designing an optimal mechanism given a joint distribution can be viewed as maximizing the product of the trading probability and the weighted virtual values given that the trading probability is feasible and satisfies the monotonicity condition defined in Proposition 1.

### 4.2.2 Characterization of Maxmin Trade Mechanism (I)

We are now ready to illustrate Theorem 1. At a high level, we expect that our solution exhibits a lot of “indifference”, which is a general lesson from the robust mechanism design literature. In our environment, that means the maxmin mechanism should generate the same payoff for the designer across many plausible joint distributions and the worst-case joint distribution should generate the same payoff for the Nature across many feasible mechanisms.

We start with the illustration of Maxmin Trade Mechanism (I). As mentioned in the introduction, we form a simple and educated guess (A) that in the maxmin solution, trade occurs with positive probability if and only if the difference between the private values of $B$ and $S$ exceeds certain threshold, i.e., $v_B - v_S > r$. Second, note we can define the value profile $(1, 0)$ as the highest type in our environment since it has the maximal virtual value\(^{12}\) (suppose it has non-zero density or probability mass). Hence, in the maxmin solution, it is without loss of generality to assume (B) the trade probability is 1 when the value profile is $(1, 0)$\(^{13}\).

Now consider the Nature’s problem of finding a worst-case joint distribution $\pi$ to any mechanism $(q, t_B, t_S)$. We observe this is a semi-infinite dimensional linear program. We derive its dual program. By Theorem 3.12 in Anderson and Nash (1987), we establish the strong duality (We leave all the details to the Appendix). Then, by the Complementarity Slackness condition, we obtain the following lemma.

**Lemma 1.** If $\pi$ is a best response for the Nature to a given mechanism $(q, t_B, t_S)$, then there

\[^{11}\text{Note } \Phi(v) = \pi(v)(v_B - \frac{1 - \Pi_B(v_B|v_S)}{\pi_B(v_B|v_S)} - (v_S + \frac{\Pi_S(v_S|v_B)}{\pi_S(v_S|v_B)})) \text{ when } \pi(v) \neq 0. \text{ Here } \phi(v) \equiv v_B - \frac{1 - \Pi_B(v_B|v_S)}{\pi_B(v_B|v_S)} - (v_S + \frac{\Pi_S(v_S|v_B)}{\pi_S(v_S|v_B)}), \text{ is the virtual value in our environment, which is the difference between the conditional virtual values of } B \text{ and } S. \text{ However, it turns out the weighted virtual values is more convenient for our analysis because it is well defined even for } \pi(v) = 0. \text{ Henceforth we directly work with the weighted virtual values.}\]

\[^{12}\text{In our environment, high-value buyer and low-value seller are more willing to trade. Thus, a value profile with a high buyer’s value and a low seller’s value can be referred to as a “high type” in the traditional mechanism design literature.}\]

\[^{13}\text{This does not affect the monotonicity constraints as the value profile } (1, 0) \text{ is the highest type in our environment.}\]
exists some real numbers $\lambda_B, \lambda_S, \mu$ such that

\[ \lambda_B v_B + \lambda_S v_S + \mu \leq t(v) \quad \forall v \in V \quad (1) \]

\[ \lambda_B v_B + \lambda_S v_S + \mu = t(v) \quad \forall v \in \text{supp}(\pi) \quad (2) \]

We conjecture that the support of the worst-case joint distribution $\pi^*$ is the area in which $v_B - v_S \geq r$. Then together with (iv) in Proposition 1, (A) and (2), we obtain that for any $v_B - v_S \geq r$,

\[ \lambda_B v_B + \lambda_S v_S + \mu = (v_B - v_S)q^*(v) - \int_{v_S+r}^{v_B} q^*(b, v_S)db - \int_{v_S}^{v_B-r} q^*(v_B, s)ds \quad (3) \]

To solve for the trading probability, first we take first order derivatives with respect to $v_B$ and $v_S$ respectively, and we obtain

\[ (v_B - v_S) \frac{\partial q^*(v_B, v_S)}{\partial v_B} - \frac{\partial}{\partial v_B} \int_{v_S+r}^{v_B} q^*(v_B, s)ds = \lambda_B \quad (4) \]

\[ (v_B - v_S) \frac{\partial q^*(v_B, v_S)}{\partial v_S} - \frac{\partial}{\partial v_S} \int_{v_S}^{v_B} q^*(b, v_S)db = \lambda_S \quad (5) \]

Then, we take cross partial derivative, with some algebra, we obtain

\[ (v_B - v_S) \frac{\partial q^*(v_B, v_S)}{\partial v_B \partial v_S} = 0 \quad (6) \]

Thus, $q^*(v_B, v_S)$ is separable, which can be written as

\[ q^*(v_B, v_S) = f(v_B) + g(v_S) \quad (7) \]

Plugging (7) into (4) and (5), we obtain

\[ rf'(v_B) - (f(v_B) + g(v_B - r)) = \lambda_B \quad (8) \]

\[ rg'(v_S) + f(v_S + r) + g(v_S) = \lambda_S \quad (9) \]

Note both (8) and (9) involve the two functions $f$ and $g$. We guess (C) that $f(v_B) + g(v_B - r) = 0$ and $f(v_S + r) + g(v_S) = 0$ for any $v$, then we can easily solve (8) and (9), and we obtain

\[ f(v_B) = \frac{\lambda_B}{r} v_B + c_B \quad (10) \]
\[ g(v_S) = \frac{\lambda_S}{r} v_S + c_S \]  

In order for (C) to hold, we must have

\[ \lambda_B = -\lambda_S, c_B + c_S + \lambda_B = 0 \]  

(12)

Now plugging (10), (11) and (12) into (7), we obtain for any \( v_B - v_S \geq r \),

\[ q^*(v_B, v_S) = \frac{\lambda_B}{r} (v_B - v_S - r) \]  

(13)

Finally, using (B), i.e., \( q^*(1,0) = 1 \), we obtain \( \lambda_B = \frac{r}{1 - r} \), and therefore,

\[ q^*(v_B, v_S) = \frac{1}{1 - r} (v_B - v_S - r) \]  

(14)

### 4.2.3 Characterization of Worst-Case Joint Distribution (I)

Now we will illustrate the **Worst-Case Joint Distribution (I)**. As mentioned in the previous subsection, we expect the worst-case joint distribution to exhibit indifference to many mechanisms. We propose a guess that the worst-case joint distribution exhibits the property that the weighted virtual value is positive only for the highest type \((1,0)\), zero for the other value profiles in the support and weakly negative for value profiles outside the support\(^{14}\). Formally, we guess (D) that in the worst-case joint distribution, we have

\[ \Phi(1,0) > 0 \]  

(15)

\[ \Phi(v) = 0 \quad \forall v_B - v_S \geq r \quad \text{and} \quad v \neq (1,0) \]  

(16)

\[ \Phi(v) \leq 0 \quad \forall v_B - v_S < r \]  

(17)

Now if the joint distribution satisfies (15), (16) and (17), then any feasible and monotone mechanism in which trade occurs with some positive probability if and only if \( v_B - v_S > r \) and trade occurs with probability 1 when \((v_B, v_S) = (1,0)\) yields the same payoff to the Nature, and is optimal for the designer. Then, the only remaining issue is whether we can construct a plausible joint distribution satisfying (15), (16) and (17).

We give an affirmative answer by taking a constructive approach. We start from constructing the joint distribution for the boundary value profiles, i.e., either \( v_B = 1 \) or \( v_S = 0 \). Assume \( Pr^*(1,0) = m \). Consider value profiles \((v_B,0)\) in which \( r \leq v_B < 1 \). Define

\(^{14}\)By the definition of the weighted virtual values, the weighted virtual values are negative for value profiles outside the support.
\[ S^*(v_B, 0) \equiv \int_{[v_B, 1]} \pi^*(b, 0) db + Pr^*(1, 0) \text{ for } r \leq v_B < 1; \quad S^*(1, 0) \equiv Pr^*(1, 0) = m. \]

Then we have \( \pi^*(v_B, 0) = -\frac{\partial S^*(v_B, 0)}{\partial v_B} \) for \( r \leq v_B < 1 \). Since the weighted virtual values for value profiles \( (v_B, 0) \) in which \( r \leq v_B < 1 \) are zeroes, we obtain for \( r \leq v_B < 1 \),

\[ \pi^*(v_B, 0)(v_B - 0) - S^*(v_B, 0) = 0 \quad (18) \]

Note \( (18) \) is a simple ordinary differential equation, to which the solution is

\[ S^*(v_B, 0) = \frac{m}{v_B}, \quad \pi^*(v_B, 0) = \frac{m}{v_B^2} \quad \forall r \leq v_B < 1 \quad (19) \]

Then consider value profiles \( (1, v_S) \) in which \( 0 < v_S \leq 1 - r \). Define \( S^*(1, v_S) \equiv \int_{[0, v_S]} \pi^*(1, s) ds + Pr^*(1, 0) \) for \( 0 < v_S \leq 1 - r \). Then we have \( \pi^*(1, v_S) = \frac{\partial S^*(1, v_S)}{\partial v_S} \) for \( 0 < v_S \leq 1 - r \). Since the weighted virtual values for value profiles \( (1, v_S) \) in which \( 0 < v_S \leq 1 - r \) are zeroes, we obtain for \( 0 < v_S \leq 1 - r \),

\[ \pi^*(1, v_S)(1 - v_S) - S^*(1, v_S) = 0 \quad (20) \]

Note \( (20) \) is also a simple ordinary differential equation, to which the solution is

\[ S^*(1, v_S) = \frac{m}{1 - v_S}, \quad \pi^*(1, v_S) = \frac{m}{(1 - v_S)^2} \quad \forall 0 < v_S \leq 1 - r \quad (21) \]

Now we will construct the joint distribution for the interior value profiles in the support, i.e., \( v_B - v_S \geq r \) and \( v_B \neq 1, v_S \neq 0 \). Define \( S^*(v_B, v_S) \equiv \int_{[v_B, 1]} \pi^*(b, v_S) db + \pi^*(1, v_S) \) for \( v_B - v_S \geq r \) and \( v_B \neq 1, v_S \neq 0 \). Then we have \( \pi^*(v_B, v_S) = -\frac{\partial S^*(v_B, v_S)}{\partial v_B} \) for \( v_B - v_S \geq r \) and \( v_B \neq 1, v_S \neq 0 \). Since the weighted virtual values for value profiles \( (v_B, v_S) \) in which \( v_B - v_S \geq r \) and \( v_B \neq 1, v_S \neq 0 \) are zeroes, we obtain for \( v_B - v_S \geq r \) and \( v_B \neq 1, v_S \neq 0 \),

\[ \pi^*(v_B, v_S)(v_B - v_S) - S^*(v_B, v_S) - \int_{[0, v_S]} \pi^*(v_B, s) ds - \pi^*(v_B, 0) = 0 \quad (22) \]

Note \( (22) \) is a (second order) partial differential equation. By taking the cross partial derivative, we find \( S^*(v_B, v_S) \) is not separable. We take the guess and verify approach to solve for the PDE. We guess that for \( v_B - v_S \geq r \) and \( v_B \neq 1, v_S \neq 0 \),

\[ S^*(v_B, v_S) = \frac{m}{(v_B - v_S)^2} \quad (23) \]

Then the LHS of \( (22) \) is \( \frac{2m}{(v_B - v_S)^3}(v_B - v_S) - \frac{m}{(v_B - v_S)^2} - \int_{[0, v_S]} \frac{2m}{(v_B - s)^2} ds - \frac{m}{v_B^2} \), which can be shown to be 0 with some algebra. Thus, we verified the guess.
To solve for \( m \), we use the fact that \( \pi^*(v) \) is a distribution. We note the marginal distribution for \( S \) is \( \pi^*_S(v_S) = S(v_S + r, v_S) = \frac{m}{(v_S + r - v_S)^2} = \frac{m}{r^2} \) for \( 0 < v_S \leq 1 - r \) and \( \pi^*_S(v_S = 0) = S(r, 0) = \frac{m}{r} \). Since the integration is 1, we obtain

\[
\frac{m}{r} + \frac{m}{r^2} \cdot (1 - r) = 1
\]

Thus, we obtain \( m = r^2 \).

So far we have constructed **Worst-Case Joint Distribution (I)**. The final step is to make sure that **Worst-Case Joint Distribution (I)** satisfies the mean constraints, which will allow us to solve for the monopoly reserve \( r \). Given the marginal distributions for \( S \) and \( B \), we have the following mean constraints,

\[
r \cdot 1 + \int_r^1 t \, dt = M_B \tag{25}
\]

\[
r \cdot 0 + \int_0^{1-r} t \, dt = M_S \tag{26}
\]

Summing up (25) and (26), we obtain \( M_B + M_S = 1 \), which is the special case we are considering. Thus, in the special case where \( M_B + M_S = 1 \), we have a (unique) solution \( r = 1 - \sqrt{1 - (M_B - M_S)} \).

**Remark 3.** The symmetric case may be a reasonable assumption for situations in which the designer knows both sides have similar eagerness to trade.

### 4.3 The Asymmetric Case: \( M_B + M_S \neq 1 \)

We now turn to the asymmetric case in which \( M_B + M_S \neq 1 \). We follow the same approach as in the symmetric case. Indeed, the characterization for the symmetric case provides us with good intuitions about the solution to the general case, which will be made clear shortly. We propose the following **Maxmin Trade Mechanism (II)** and **Worst-Case Joint Distribution (II)**. Formally, they are described as below.

**Maxmin Trade Mechanism (II)**
Let \( v = (v_B, v_S) \) be the reported value profile of the two agents. If \( r_2 v_B - (1 - r_1) v_S \geq r_1 r_2 \), then

\[
q^*(v_B, v_S) = \frac{1}{\ln \frac{1-r_2}{1-r_1}} \left( \ln \left( (1 - \frac{r_2}{1-r_1}) v_B + \frac{r_1 r_2}{1-r_1} \right) - \ln \left( \frac{1-r_1}{r_2} - 1 \right) v_S + r_1 \right)
\]

\[
t^*_B(v_B, v_S) = -\frac{r_1 r_2}{(1-r_1-r_2) \ln \frac{1-r_2}{1-r_1}} \left( \ln \left( (1 - \frac{r_2}{1-r_1}) v_B + \frac{r_1 r_2}{1-r_1} \right) - \ln \left( \frac{1-r_1}{r_2} - 1 \right) v_S + r_1 \right)
\]
where \( r_1, r_2 \) is the unique solution to the following equations:

\[
M_B = \int_{r_1}^{1} \frac{r_1(1-r_2)}{(1-r_1-v_B) + \frac{r_1}{1-r_1}} v_B dv_B + r_1 := H_1(r_1, r_2) \tag{27}
\]

\[
M_S = \int_{0}^{r_2} \frac{r_1(1-r_2)}{(1-r_1-v_S + r_1)^2} v_S dv_S := H_2(r_1, r_2) \tag{28}
\]

Otherwise

\[
q^*(v_B, v_S) = t^*_B(v_B, v_S) = t^*_S(v_B, v_S) = 0
\]

**Worst-Case Joint Distribution (II)**

Let \( \pi^*(v_B, v_S) \) denote the density of the value profile \((v_B, v_S)\) whenever the density exists.

Let \( Pr^*(v_B, v_S) \) denote the probability mass of the value profile \((v_B, v_S)\) whenever there is some probability mass on \((v_B, v_S)\). Let \( V(II) := \{ v | v_2v_B - (1-r_1)v_S \geq r_1r_2 \} \). Worst-Case Joint Distribution (II) has the support \( V(II) \) and is defined as follows:

\[
\pi^*(v_B, v_S) = \begin{cases} 
\frac{2r_1(1-r_2)}{(v_B-v_S)^2} & r_2v_B - (1-r_1)v_S \geq r_1r_2, v_B \neq 1, v_S \neq 0 \\
\frac{r_1(1-r_2)}{(1-v_S)^2} & v_B = 1, 0 < v_S \leq r_2 \\
\frac{r_1(1-r_2)}{v_B} & r_1 \leq v_B < 1, v_S = 0 
\end{cases}
\]

\[
Pr^*(1, 0) = r_1(1-r_2)
\]

Equivalently, **worst-case joint distribution (II)** can be described by worst-case marginal distributions and conditional distributions as follows:

For \( B \), the worst-case marginal distribution is

\[
\pi^*_B(v_B) = \frac{r_1(1-r_2)}{(1-r_1-v_B) + \frac{r_1r_2}{1-r_1}}
\]

for \( r_1 \leq v_B < 1 \),

\[
Pr^*_B(1) = r_1
\]
That is, worst-case marginal distribution for $B$ is some generalized Pareto distribution on $[r_1, 1)$ with a probability mass $r_1$ on 1.

For $S$, the worst-case marginal distribution is

$$\pi^*_S(v_S) = \frac{r_1(1 - r_2)}{r_2 v_S + r_1}$$

for $0 < x \leq r_2$,

$$Pr^*_S(0) = 1 - r_2$$

That is, worst-case marginal distribution for $S$ is some generalized Pareto distribution on $(0, r_2]$ with a probability mass $1 - r_2$ on 0.

The conditional distribution can be easily derived from the joint distribution and the marginal distributions. It can be seen that the conditional distribution is some generalized Pareto distribution with a probability mass on either 0 or 1, which share the same feature with that in the symmetric case, therefore we omit the description of the conditional distribution.

**Remark 4.** Worst-Case Joint Distribution (II) exhibits positive correlation for $r_1 \leq v_B < 1$ and $0 < v_S \leq r_2$.\(^{15}\)

**Lemma 2.** For any given $M_B + M_S \neq 1$, there is a unique solution $r_1, r_2$ to (27) and (28).

**Theorem 2.** When $M_B + M_S \neq 1$, Maxmin Trade Mechanism (II) and Worst-Case Joint Distribution (II) form a Nash equilibrium. The revenue guarantee is $r_1(1 - r_2)$.

### 4.3.1 Characterization of Maxmin Trade Mechanism (II)

As mentioned in the introduction, we guess that (A') that in the maxmin solution, trade occurs with positive probability if and only if the difference between the weighted private values of $B$ and $S$ exceeds certain threshold, i.e., $r_2 v_B - (1 - r_1) v_S \geq r_1 r_2$. We further conjecture that the support of the worst-case joint distribution $\pi^*$ is the area in which $r_2 v_B - (1 - r_1) v_S \geq r_1 r_2$. Then together with (iv) in Proposition 1, (A') and (2), we obtain that for any $r_2 v_B - (1 - r_1) v_S \geq r_1 r_2$,

$$\lambda_B v_B + \lambda_S v_S + \mu = (v_B - v_S) q(v_B, v_S) - \int_{r_1 v_S + r_1}^{v_B} q(b, v_S) db - \int_{v_S}^{r_2} (v_B - r_1) q(v_B, s) ds \quad (29)$$

\(^{15}\)To see this, note $F(v_S|v_B) = \frac{(1-r_1-r_2 v_B + r_2 v_S)^2}{(v_B-v_S)^2}$ is decreasing w.r.t. $v_B$ for $r_1 \leq v_B < 1$. The positive correlation breaks when $v_B = 1$. Similarly, $F(v_B|v_S) = 1 - \frac{(1-r_1-r_2 v_B + r_2 v_S)^2}{(v_B-v_S)^2}$ is decreasing w.r.t. $v_S$ for $0 < v_S \leq r_2$. The positive correlation breaks when $v_S = 0$.\)
To solve for the trading probability, first we take first order derivatives with respect to $v_B$ and $v_S$ respectively, and we obtain

$$(v_B - v_S) \frac{\partial q^*(v_B, v_S)}{\partial v_B} - \frac{\partial}{\partial v_B} \int_{v_S}^{r_2} q(v_B, s) ds = \lambda_B$$  \hspace{1cm} (30)$$

$$(v_B - v_S) \frac{\partial q^*(v_B, v_S)}{\partial v_S} - \frac{\partial}{\partial v_S} \int_{v_S}^{v_B - r_1 v_S + r_1} q(b, v_S) db = \lambda_S$$  \hspace{1cm} (31)$$

Then, we take cross partial derivative, with some algebra, we obtain

$$(v_B - v_S) \frac{\partial q^*(v_B, v_S)}{\partial v_B \partial v_S} = 0$$  \hspace{1cm} (32)$$

Thus, $q^*(v_B, v_S)$ is separable, which can be written as (with abuse of notations)

$$q^*(v_B, v_S) = f(v_B) + g(v_S)$$  \hspace{1cm} (33)$$

Plugging (33) into (30) and (31), we obtain

$$(((1 - \frac{r_2}{1 - r_1})v_B + \frac{r_1 r_2}{1 - r_1}) f'(v_B) - \frac{r_2}{1 - r_1} (f(v_B) + g(\frac{r_2}{1 - r_1} (v_B - r_1)))) = \lambda_B$$  \hspace{1cm} (34)$$

$$(((\frac{1 - r_1}{r_2} - 1)v_S + r_1) g'(v_S) + \frac{1 - r_1}{r_2} (f(\frac{1 - r_1}{r_2} v_S + r_1) + g(v_S)) = \lambda_S$$  \hspace{1cm} (35)$$

Note both (34) and (35) involve the two functions $f$ and $g$. We guess (C') that $f(v_B) + g(\frac{r_1 r_2}{1 - r_1} (v_B - r_1)) = 0$ and $f(\frac{1 - r_1}{r_2} v_S + r_1) + g(v_S) = 0$ for any $v$, then we can easily solve (34) and (35), and we obtain

$$f(v_B) = \frac{(1 - r_1) \lambda_B}{1 - r_1 - r_2} \ln (((1 - \frac{r_2}{1 - r_1})v_B + \frac{r_1 r_2}{1 - r_1}) + c_B$$  \hspace{1cm} (36)$$

$$g(v_S) = \frac{r_2 \lambda_S}{1 - r_1 - r_2} \ln (((\frac{1 - r_1}{r_2} - 1)v_S + r_1) + c_S$$  \hspace{1cm} (37)$$

Observe that

$$g(\frac{r_2}{1 - r_1} (v_B - r_1)) = \frac{r_2 \lambda_S}{1 - r_1 - r_2} \ln (((1 - \frac{r_2}{1 - r_1})v_B + \frac{r_1 r_2}{1 - r_1}) + c_S$$

Then, in order for (C') to hold, we must have

$$(1 - r_1) \lambda_B = -r_2 \lambda_S, c_B + c_S = 0$$  \hspace{1cm} (38)$$
Now plugging (36),(37) and (38) into (33), we obtain for any \( r_2v_B - (1 - r_1)v_S \geq r_1r_2, \)

\[
q^*(v_B, v_S) = \frac{(1 - r_1)}{1 - r_1 - r_2} \left( \ln \left( (1 - \frac{r_2}{1 - r_1})v_B + \frac{r_1r_2}{1 - r_1} \right) - \ln \left( (1 - r_1 \frac{r_2}{1 - r_1})v_S + r_1 \right) \right)
\]

Finally, using (B), i.e., \( q^*(1,0) = 1 \), we obtain \( \lambda_B = \frac{(1-r_1-r_2)}{(1-r_1) \ln \frac{r_2}{r_1}} \), and therefore,

\[
q^*(v_B, v_S) = \frac{1}{\ln \frac{r_2}{r_1}} \left( \ln \left( (1 - \frac{r_2}{1 - r_1})v_B + \frac{r_1r_2}{1 - r_1} \right) - \ln \left( (1 - r_1 \frac{r_2}{1 - r_1})v_S + r_1 \right) \right)
\]

4.3.2 Characterization of Worst-Case Joint Distribution (II)

Just as the characterization for the special case, we guess that for the general case, the worst-case joint distribution also exhibits the property that the weighted virtual value is positive only for the highest type \((1,0)\), zero for the other value profiles in the support and weakly negative for value profiles outside the support. The construction procedure for the joint distribution is exactly the same. Therefore we omit it. However, note here the marginal distributions no longer have uniform distribution part since \( v_B - v_S \) is no longer constant on the line boundary due to different weights for \( B \) and \( S \). We start from the derivation for the marginal distribution of \( S \). \( \pi^*_S(v_S) = S(\frac{1-r_1}{r_2}v_S + r_1, v_S) = \frac{m}{(\frac{1-r_1}{r_2}-1)v_S + r_1} \) for \( 0 < v_S \leq r_2 \) and \( \pi^*_S(v_S = 0) = S(r_1, 0) = \frac{m}{r_1} \). Since the integration is 1, we obtain

\[
\frac{m}{r_1} + \int_0^{r_2} \frac{m}{(\frac{1-r_1}{r_2}-1)v_S + r_1} \, dv_B = 1
\]

With some algebra, we obtain \( m = r_1(1 - r_2) \). The final step is to make sure that Worst-Case Joint Distribution (II) satisfies the mean constraints, which will allow us to solve for the \( r_1, r_2 \). Given the marginal distributions for \( S \) and \( B \), we have a system of two equations (27) and (28). Lemma 2 states the solution exists and is unique for the general case, details of which are left to the Appendix.

Remark 5. We can now consider a general model in which the designer can destroy the good not necessarily uniformly. To wit, the sum of the final allocation \( q_B(v_B, v_S) \) to \( B \) and \( q_S(v_B, v_S) \) to \( S \) does not exceed 1. Formally, the constraints now becomes

\[
v_Bq_B(v) - t_B(v) \geq 0 \quad \forall v \quad \text{(EPIR}_B\text{)}
\]

\[
v_Bq_B(v) - t_B(v) \geq v_Bq_B(v', v_S) - t_B(v_B, v_S) \quad \forall v, v' \quad \text{(DSIC}_B\text{)}
\]

\[
v_Sq_S(v) + t_S(v) \geq v_S \quad \forall v \quad \text{(EPIR}_S\text{)}
\]
\[ v_S q_S(v) + t_S(v) \geq v_S q_S(v_B, v'_S) + t_S(v_B, v'_S) \quad \forall v, v'_S \quad (DSIC'_g) \]

\[ q_B(v) + q_S(v) \leq 1 \quad \forall v \quad (Feasibility") \]

We argue the solution to the above problem coincides with our main results. To see this, first note a simple adaption of Proposition 1 yields a virtual representation of the revenue for the above problem:

\[ E_{\pi} t = E_{\pi}[q_B\phi_B + q_S\phi_S] - 1 \]

where \( \phi_B(v) = v_B - \frac{1-H_B(v_B|v_S)}{\pi_B(v_B|v_S)} \), \( \phi_S(v) = v_S + \frac{H_S(v_S|v_B)}{\pi_S(v_S|v_B)} \). Given the constructed joint distribution in the main results, \( \phi_B = \phi_S > 0 \) for any interior value profile except for the highest type \((1,0)\), in which \( 1 = \phi_B(1,0) > \phi_S(1,0) = 0 \). It is easy to see the constructed trade mechanisms remain optimal.

## 5 Deterministic Mechanisms

In this section, we restrict attention to deterministic DSIC and EXIR trade mechanisms and characterize the maxmin trade mechanisms in this class of mechanisms. This section is motivated by practical concerns. To wit, deterministic mechanisms are easier to understand and more practical than randomized mechanisms in many situations, e.g., when the agents do not trust the randomization device. Note that Proposition 1 still holds, with the additional property that \( q(v) \) is either 0 or 1 for any \( v \).

We begin with a definition, which is useful for exposition.

**Definition 2.** Trade boundary of a given deterministic DSIC and EXIR trade mechanism \((q, t_B, t_S)\) is a set of value profiles \( \mathcal{B} := \{ v = (v_B, v_S) | q(v) = 1 \text{ if } \exists \bar{v} \text{ s.t. } v_B \geq \bar{v}_B, v_S < \bar{v}_S \text{ or } v_B > \bar{v}_B, v_S \leq \bar{v}_S; q(v) = 0 \text{ if } \exists \bar{v} \text{ s.t. } v_B \leq \bar{v}_B, v_S \geq \bar{v}_S \} \).

We observe the trading boundary exhibits a monotone property, which is summarized below.

**Observation 2.** If \( \bar{v}, \bar{v}' \in \mathcal{B} \) and \( \bar{v}_B > \bar{v}'_B \), then \( \bar{v}_S \geq \bar{v}'_S \).

The main idea is as follows. We divide all possible deterministic DSIC and EXIR trade mechanisms into four classes according to the trade boundary. By strong duality, we can work on the dual program. We propose a relaxation of the dual program by omitting a lot

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\[ ^{16}\text{For technical reasons, we assume the trading probability on the trade boundary is 0. This is to have a minimization problem for Nature. Otherwise we have to replace min with inf. See also in Carrasco et al. (2018).} \]

\[ ^{17}\text{To see this, since } \bar{v}' \in \mathcal{B} \text{ and } \bar{v}_B > \bar{v}'_B, q(v_B, \bar{v}_S') = 1. \text{ Then by definition, } \bar{v}_S \geq \bar{v}'_S. \]
of constraints. The merit of doing so is to have a finite dimensional linear programming problem. Then we derive an upper bound of the value of the relaxation for each class. We identify the greatest upper bound and then show that the greatest upper bound can be achieved by constructing the deterministic mechanisms and the worst-case joint distribution.

**Theorem 3.** When \( \sqrt{M_S} + \sqrt{1 - M_B} < 1 \), any deterministic satisfying the following properties is a maxmin deterministic mechanism:

(i). \( (1 - \sqrt{1 - M_B}, 0) \in B, (1, \sqrt{M_S}) \in B \).

(ii). \( B \) is above (including) the line boundary \( \sqrt{M_S}v_B - \sqrt{1 - M_B}v_S = \sqrt{M_S}(1 - \sqrt{1 - M_B}) \).

(iii). Transfers are characterized by Proposition 1.

The worst-case joint distribution put point mass \( \sqrt{1 - M_B}, \sqrt{M_S} \) and \( 1 - \sqrt{1 - M_B} - \sqrt{M_S} \) on value profile \( (1 - \sqrt{1 - M_B}, 0), (1, \sqrt{M_S}) \) and \( (1, 0) \) respectively. The revenue guarantee is \( (1 - \sqrt{M_S} - \sqrt{1 - M_B})^2 \); When \( \sqrt{M_S} + \sqrt{1 - M_B} \geq 1 \), no trade is optimal.

That is, we characterize the whole class of maxmin deterministic mechanisms. The worst-case joint distribution is discrete, and is the same across the mechanisms in this class. Now we provide examples of the trading rules of some maxmin deterministic mechanisms.

**Example 1.** Linear Trading: trade occurs with probability 1 if and only if \( \sqrt{M_S}v_B - \sqrt{1 - M_B}v_S > \sqrt{M_S}(1 - \sqrt{1 - M_B}) \).

**Example 2.** Threshold Trading: trade occurs with probability 1 if and only if \( v_B > 1 - \sqrt{1 - M_B} \) and \( v_S < \sqrt{M_S} \).

### 6 Concluding Remarks

In this paper, we provide a complete characterization of the maxmin trade mechanisms and the worst-case joint distributions when the designer knows only the expectations of the values, among all DSIC and EPIR mechanisms. The maxmin trade mechanisms are novel, featuring either linear randomization for the symmetric case or logarithmic-linear randomization for the asymmetric case. In addition, the revenue guarantee is positive as long as the expectation of the buyer’s value exceeds the expectation of the seller’s value. The key step in the construction of the worst-case joint distributions is to obtain a system of differential equations from properties about the weighed virtual value. The construction method may be of independent interest and useful for other design problems, e.g., multidimensional Bayesian persuasion, and even more general robust optimization problems.
7 Appendix

A Proofs for Section 4

A.1 Proof of Proposition 1

(i) $q(v_B, v_S)$ is nondecreasing in $v_B$ and nonincreasing in $v_S$:

Dominant strategy incentive compatibility for a type $v_B$ of $B$ requires that for any $v_S$ and $v'_B \neq v_B$:

$$v_B q(v_B, v_S) - t_B(v_B, v_S) \geq v_B q(v'_B, v_S) - t_B(v'_B, v_S)$$

DSIC also requires that:

$$v'_B q(v'_B, v_S) - t_B(v'_B, v_S) \geq v_B q(v_B, v_S) - t_B(v_B, v_S)$$

Adding the two inequalities, we have that:

$$(v_B - v'_B)(q(v_B, v_S) - q(v'_B, v_S)) \geq 0$$

It follows that $q(v_B, v_S) \geq q(v'_B, v_S)$ whenever $v_B > v'_B$.

Similarly, dominant strategy incentive compatibility for a type $v_S$ of $S$ requires that for any $v_B$ and $v'_S \neq v_S$:

$$v_S(1 - q(v_B, v_S)) + t_B(v_B, v_S) \geq v_S(1 - q(v_B, v'_S)) + t_S(v_B, v'_S)$$

DSIC also requires that:

$$v'_S(1 - q(v_B, v'_S)) + t_S(v_B, v'_S) \geq v'_S(1 - q(v_B, v_S)) + t_S(v_B, v_S)$$

Adding the two inequalities, we have that:

$$(v_S - v'_S)(q(v_B, v_S) - q(v_B, v'_S)) \geq 0$$

It follows that $q(v_B, v_S) \leq q(v_B, v'_S)$ whenever $v_S > v'_S$.

(ii) $t_B(v_B, v_S) = v_B q(v_B, v_S) - \int_0^{v_B} q(b, v_S)db$

Fix $v_S$, Define

$$U_B(v_B) = v_B q(v_B, v_S) - t_B(v_B, v_S)$$
By the first two inequalities in (i), we get

$$(v_B' - v_B)q(v_B, v_S) \leq U_B(v_B') - U_B(v_B) \leq (v_B' - v_B)q(v_B', v_S)$$

Dividing throughout by $v_B' - v_B$ (suppose $v_B' > v_B$):

$$q(v_B, v_S) \leq \frac{U_B(v_B') - U_B(v_B)}{v_B' - v_B} \leq q(v_B', v_S)$$

As $v_B \uparrow v_B'$, we get:

$$\frac{dU_B(v_B)}{dv_B} = q(v_B, v_S)$$

Then we get

$$t_B(v_B, v_S) = v_Bq(v_B, v_S) - \int_0^{v_B} q(b, v_S) db - U_B(0)$$

Note $U_B(0) \geq 0$ by the ex post IR constraint. If $U_B(0) > 0$, then we can reduce it to 0 so that we can increase the payment from $B$ for all value profiles and the value of the problem will be strictly greater. Thus, for any maxmin solution, $U_B(0) = 0$ and $t_B(v_B, v_S) = v_Bq(v_B, v_S) - \int_0^{v_B} q(b, v_S) db$

(iii) $t_S(v) = 1 - (1 - q(v))v_S - \int_{v_S}^1 (1 - q(v_B, s))ds$:

Similarly, Fix $v_B$, Define

$$U_S(v_S) = v_S(1 - q(v_B, v_S)) + t_S(v_B, v_S)$$

By the fourth and fifth inequalities in (i), we get

$$(v_S' - v_S)(1 - q(v_B, v_S)) \leq U_S(v_S') - U_S(v_S) \leq (v_S' - v_S)(1 - q(v_B, v_S'))$$

Dividing throughout by $v_S' - v_S$ (suppose $v_S' > v_S$):

$$1 - q(v_B, v_S) \leq \frac{U_S(v_S') - U_S(v_S)}{v_S' - v_S} \leq 1 - q(v_B, v_S')$$

As $v_S \uparrow v_S'$, we get:

$$\frac{dU_S(v_S)}{dv_S} = 1 - q(v_B, v_S)$$

Then we get

$$t_B(v_B, v_S) = U_S(1) - v_S(1 - q(v_B, v_S)) - \int_{v_S}^1 q(v_B, s) ds$$

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Note \( U_S(1) \geq 1 \) by the ex post IR constraint. If \( U_S(1) > 1 \), then we can reduce it to 1 so that we can decrease the payment to \( S \) for all value profiles and the value of the problem will be strictly greater. Thus, for any maxmin solution, \( U_S(1) = 1 \) and \( t_S(v) = 1 - (1 - q(v))v_S - \int_{v_S}^1 (1 - q(v_B, s))ds \).

(iv) \( t(v) \equiv t_B(v) - t_S(v) = (v_B - v_S)q(v) - \int_0^{v_B} q(b, v_S)db - \int_{v_S}^1 q(v_B, s)ds \): This is implied by (ii) and (iii).

**A.2 Proof of Lemma 1**

Given a DSIC and EPIR mechanism \((q, t_B, t_S)\), the (P) primal minimization problem of Nature is as follows (with dual variables in the bracket):

\[
(Primal) \quad \min_{F \in \Pi(M_B, M_S)} \int t(v) dF
\]

s.t.

\[
\int v_B dF = M_B \quad (\lambda_B)
\]

\[
\int v_S dF = M_S \quad (\lambda_S)
\]

\[
\int dF = 1 \quad (\mu)
\]

It has the following (D) dual maximization problem:

\[
(Dual) \quad \max_{\lambda_B, \lambda_S, \mu \in \mathbb{R}} \lambda_B M_B + \lambda_S M_S + \mu
\]

s.t.

\[
\lambda_B v_B + \lambda_S v_S + \mu \leq t(v) \quad (dF)
\]

Note that the value of (P) is bounded by 1 as \( t(v) \leq 1 \). In addition, the trivial joint distribution that put all probability mass on the point \((M_B, M_S)\) is in the interior of the primal cone. Then by Theorem 3.12 in Anderson and Nash (1987), strong duality holds. Then, by the Complementarity Slackness, (2) holds. And (1) is implied by feasibility constraints of (D).
A.3 Proof of Theorem 1

We already illustrated the main idea and main steps in Section 4. Now we summarize them and give a formal argument. We will prove the proposed pair forms a Nash Equilibrium.

(i): **Maxmin Trade Mechanism (I)** is a best response to **Worst-Case Joint Distribution (I)**: Note Worst-Case Joint Distribution (I) satisfies (18), (20), (22). Also note there is a point mass on the value profile (1,0). Thus (15), (16) and (17) hold. Then any feasible and monotone mechanism in which trade occurs with some positive probability if and only if \( v_B - v_S > r \) and trade occurs with probability 1 when \((v_B, v_S) = (1,0)\) is a best response for the designer. It is easy to see that Maxmin Trade Mechanism (I) is such a mechanism.

(ii): **Worst-Case Joint Distribution (I)** is a best response to **Maxmin Trade Mechanism (I)**: we use the duality theory to show (ii). First note that by (24), (25) and (26), all the three constraints in (P) holds. By the weighted virtual value representation, the value of \( P \) given Worst-Case Joint Distribution (I) and Maxmin Trade Mechanism (I) is simply \( Pr(1,0) \times (1-0) = r^2 \). Second, note by (3) and \( \lambda_B = \frac{r}{1-r} > 0 \) and \( \lambda_S = -\frac{r}{1-r} < 0 \), the constraints in (D) hold for all value profiles. To see this, note for any value profile \( v = (v_B, v_S) \) outside the support of Worst-Case Joint Distribution (I),

\[
\lambda_B v_B + \lambda_S v_S + \mu < \lambda_B r + \lambda_S 0 + \mu = 0
\]

for any value profile \( v = (v_B, v_S) \) inside the support of Worst-Case Joint Distribution (I), the constraints trivially holds. Also (3) is the Complementarity Slackness. Finally, the value of \( D \) given the constructed \( \lambda_B, \lambda_S, \mu \) is \( \lambda_B M_B + \lambda_S M_S + \mu \), which, by (25), (26) and some algebra, is equal to \( r^2 \). By the linear programming duality theory, (ii) holds and the revenue guarantee is \( r^2 \).

A.4 Proof of Lemma 2

We start from establishing the following four claims regarding some properties of the functions \( H_1(r_1, r_2) \) and \( H_2(r_1, r_2) \), which will play a crucial role in establishing Lemma 2.

**Claim 1.** Fix any \( 0 < r_1 \leq 1 \), \( H_2(r_1, r_2) \) is strictly increasing w.r.t. \( r_2 \) for \( r_2 \in [0,1) \). In addition, fix any \( 0 < r_1 \leq 1 \), as \( r_2 \uparrow 1 \), \( H_2(r_1, r_2) \to 1 \).

**Proof of Claim 1.** Note when \( 0 < r_1 \leq 1 \),

\[
H_2(r_1, r_2) = \frac{r_1(1-r_2)r_2^2}{(1-r_1-r_2)^2} \ln \frac{1-r_2}{r_1} - \frac{r_1r_2^2}{1-r_1-r_2}
\]  

(42)
Now taking first order derivative w.r.t. $r_2$ to (42), with some algebra, we obtain
\[
\frac{\partial H_2(r_1, r_2)}{\partial r_2} = \frac{r_1 r_2}{(1 - r_1 - r_2)^2}((2 - 3r_2 + \frac{2r_2(1 - r_2)}{1 - r_1 - r_2}) \ln \frac{1 - r_2}{r_1} - 2(1 - r_1)) \tag{43}
\]

Then to show the first part of Claim 1, it suffices to show that for any $r_2 \in (0, 1)$
\[
(2 - 3r_2 + \frac{2r_2(1 - r_2)}{1 - r_1 - r_2}) \ln \frac{1 - r_2}{r_1} - 2(1 - r_1) > 0 \tag{44}
\]

Let $b \equiv \frac{1 - r_2}{r_1}$, then $b \in (0, 1) \cup (1, \infty)$. Plugging $r_2 = 1 - br_1$ into (44), it suffices to show that for any $b \in (0, 1) \cup (1, \infty)$
\[
(3br_1 - 1 + \frac{2b(1 - br_1)}{b - 1}) \ln b - 2(1 - r_1) > 0 \tag{45}
\]

By slight rewriting (45), it suffices to show that for any $b \in (0, 1) \cup (1, \infty)$
\[
\frac{b + 1}{b - 1} \ln b - 2 + (\frac{b^2 - 3b}{b - 1} \ln b + 2)r_1 > 0 \tag{46}
\]

Then, it suffices to show that for any $b \in (0, 1) \cup (1, \infty)$, the following two inequalities hold
\[
\frac{b + 1}{b - 1} \ln b - 2 > 0 \tag{47}
\]
\[
\frac{b^2 - 3b}{b - 1} \ln b + 2 > 0 \tag{48}
\]

Now to prove (47), it suffices to show that $f(b) := \ln b - \frac{2(b - 1)}{b + 1} > 0$ for $b \in (1, \infty)$ and $f(b) < 0$ for $b \in (0, 1)$. Taking first order derivative to $f(b)$, we obtain
\[
f'(b) = \frac{(b - 1)^2}{b(b + 1)^2} \tag{49}
\]

Therefore, $f(b)$ is strictly increasing. Note $f(1) = 0$. Thus, we proved (47). To prove (48), it suffices to show that $g(b) := (b^2 - 3b) \ln b + 2(b - 1) > 0$ for $b \in (1, \infty)$ and $g(b) < 0$ for $b \in (0, 1)$. Taking first order derivative to $g(b)$, we obtain
\[
g'(b) = (2b - 3) \ln b + b - 1 \tag{50}
\]
Now taking derivative again, we obtain
\[ g''(b) = 2 \ln b - \frac{3}{b} + 3 \] (51)

Note \( g''(b) \) is strictly increasing and \( g''(1) = 0 \). This implies that \( g'(b) \) is minimized at \( b = 1 \).

Note \( g'(1) = 0 \). This implies that \( g(b) \) is strictly increasing. Finally, note \( g(1) = 0 \). This implies that (48) holds.

So far we have shown the first part of Claim 1. For the second part of Claim 1, we note by the L’Hopita rule, we have
\[ \lim_{x \to 0} x \ln x = \lim_{x \to 0} \frac{\ln x}{\frac{1}{x}} = \lim_{x \to 0} -1/x^2 = 0 \] (52)

Then, the first term of (42) goes to 0 as \( r_2 \uparrow 1 \), and we obtain
\[ \lim_{r_2 \uparrow 1} H_2(r_1, r_2) = 0 - \frac{r_1}{1 - r_1 - 1} = 1 \] (53)

\[ \square \]

**Claim 2.** Fix any \( 0 < r_2 < 1 \), \( H_2(r_1, r_2) \) is strictly increasing w.r.t. \( r_1 \) for \( r_1 \in [0, 1] \).

**Proof of Claim 2.** Note when \( 0 < r_2 < 1 \), (42) holds. Now taking first order derivative w.r.t. \( r_2 \) to (42), with some algebra, we obtain
\[ \frac{\partial H_2(r_1, r_2)}{\partial r_1} = \frac{(1 - r_2)r_2^2}{(1 - r_1 - r_2)^2}((1 + \frac{2r_1}{1 - r_1 - r_2})\ln \frac{1 - r_2}{r_1} - 2) \] (54)

Then it suffices to show that for any \( r_1 \in (0, 1) \)
\[ (1 + \frac{2r_1}{1 - r_1 - r_2})\ln \frac{1 - r_2}{r_1} - 2 > 0 \] (55)

Let \( b \equiv \frac{1 - r_2}{r_1} \), then \( b \in (0, 1) \cup (1, \infty) \). Plugging \( r_2 = 1 - br_1 \) into (55), it suffices to show that for any \( b \in (0, 1) \cup (1, \infty) \)
\[ \frac{b + 1}{b - 1} \ln b - 2 > 0 \] (56)

which is exactly (47) and has been shown in the Proof of Claim (1).

\[ \square \]

**Claim 3.** Fix any \( r_2 \in [0, 1] \), \( H_1(r_1, r_2) \) is strictly increasing w.r.t. \( r_1 \). In addition, for any \( r_2 \in [0, 1] \), as \( r_1 \uparrow 1 \), \( H_1(r_1, r_2) \rightarrow 1 \).
Proof of Claim 3. Note when \( r_2 = 1 \), \( H_1(r_1, r_2) = r_1 \). Then both parts of Claim 3 trivially holds when \( r_2 = 1 \). When \( r_2 = 0 \), \( H_1(r_1, r_2) = r_1 - r_1 \ln r_1 \). Taking derivative w.r.t. \( r_1 \), we have \( \frac{\partial H_1(r_1, r_2)}{\partial r_1} = -\ln r_1 \). By L’Hopital Rule, \( \lim_{r_1 \uparrow 1} H_1(r_1, r_2) = 1 - 0 = 1 \). Thus, Claim 3 holds when \( r_2 = 0 \). When \( 0 < r_2 < 1 \),

\[
H_1(r_1, r_2) = (1 - r_2) r_1 (1 - r_1)^2 \ln \frac{1 - r_2}{r_1} - \frac{r_1 r_2 (1 - r_1)}{1 - r_1 - r_2} + r_1
\]

(57)

Now taking first order derivative w.r.t. \( r_1 \) to (42), with some algebra, we obtain

\[
\frac{\partial H_1(r_1, r_2)}{\partial r_1} = \frac{(1 - r_1)(1 - r_2)}{(1 - r_1 - r_2)^2} \left( (1 - 3r_1 + \frac{2r_1(1 - r_1)}{1 - r_1 - r_2}) \ln \frac{1 - r_2}{r_1} - 2r_2 \right)
\]

(58)

Then to show the first part of Claim 1, it suffices to show that for any \( r_1 \in (0, 1) \)

\[
(1 - 3r_1 + \frac{2r_1(1 - r_1)}{1 - r_1 - r_2}) \ln \frac{1 - r_2}{r_1} - 2r_2 > 0
\]

(59)

Let \( b \equiv \frac{1 - r_2}{r_1} \), then \( b \in (0, 1) \cup (1, \infty) \). Plugging \( r_2 = 1 - br_1 \) into (59), it suffices to show that for any \( b \in (0, 1) \cup (1, \infty) \)

\[
(1 - 3r_1 + \frac{2(1 - r_1)}{b - 1}) \ln b - 2(1 - br_1) > 0
\]

(60)

By slight rewriting (60), it suffices to show that for any \( b \in (0, 1) \cup (1, \infty) \)

\[
\frac{b + 1}{b - 1} \ln b - 2 + \left( -\frac{3b - 1}{b - 1} \ln b + 2b \right) r_1 > 0
\]

(61)

Then, it suffices to show that for any \( b \in (0, 1) \cup (1, \infty) \), the following two inequalities hold

\[
\frac{b + 1}{b - 1} \ln b - 2 > 0
\]

(62)

\[
-\frac{3b - 1}{b - 1} \ln b + 2b > 0
\]

(63)

Note (62) is exactly (47), which has been shown in the Proof of Claim 1. To prove (63), it suffices to show that (with abuse of notations) \( g(b) := (1 - 3b) \ln b + 2b(b - 1) > 0 \) for \( b \in (1, \infty) \) and \( g(b) < 0 \) for \( b \in (0, 1) \). Taking first order derivative to \( g(b) \), we obtain

\[
g'(b) = 4b - 3 \ln b + \frac{1}{b} - 5
\]

(64)
Now taking derivative again, we obtain
\[ g''(b) = \frac{(4b + 1)(b - 1)}{b^2} \]
(65)

Note \( g''(b) > 0 \) when \( b > 1 \), \( g''(b) < 0 \) when \( b < 1 \) and \( g''(1) = 0 \). This implies that \( g'(b) \) is minimized at \( b = 1 \). Note \( g'(1) = 0 \). This implies that \( g(b) \) is strictly increasing. Finally, note \( g(1) = 0 \). This implies that (63) holds.

So far we have shown the first part of Claim 3. For the second part of Claim 3, it trivially holds when \( r_2 \in (0, 1) \) since the first two terms of (57) goes to 0 trivially as \( r_1 \) goes to 1.

\[ \square \]

Claim 4. Fix any \( 0 < r_1 < 1 \), \( H_1(r_1, r_2) \) is strictly increasing w.r.t. \( r_2 \) for \( r_2 \in [0, 1) \). In addition, fix any \( 0 < r_1 < 1 \), as \( r_2 \uparrow 1 \), \( H_1(r_1, r_2) \to 1 \).

Proof of Claim 4. Note when \( 0 < r_1 < 1 \), (57) holds. Now taking first order derivative w.r.t. \( r_2 \) to (57), with some algebra, we obtain
\[ \frac{\partial H_1(r_1, r_2)}{\partial r_2} = \frac{(1 - r_1)^2 r_1}{(1 - r_1 - r_2)^2} \left( -1 + \frac{2(1 - r_2)}{1 - r_1 - r_2} \ln \frac{1 - r_2}{r_1} - 2 \right) \]
(66)

Then it suffices to show that for any \( r_2 \in (0, 1) \)
\[ (-1 + \frac{2(1 - r_2)}{1 - r_1 - r_2}) \ln \frac{1 - r_2}{r_1} - 2 > 0 \]
(67)

Let \( b \equiv \frac{1 - r_2}{r_1} \), then \( b \in (0, 1) \cup (1, \infty) \). Plugging \( r_2 = 1 - br_1 \) into (67), it suffices to show that for any \( b \in (0, 1) \cup (1, \infty) \)
\[ \frac{b + 1}{b - 1} \ln b - 2 > 0 \]
(68)

which is exactly (47) and has been shown in the Proof of Claim 1.

So far we have shown the first part of Claim 4. For the second part of Claim 4, using L’Hopita rule and the same argument in the Proof of Claim 1, the first term of (57) goes to 0 as \( r_2 \uparrow 1 \), and we obtain
\[ \lim_{r_2 \uparrow 1} H_1(r_1, r_2) = 0 - \frac{r_1(1 - r_1)}{1 - r_1 - 1} + r_1 = 1 \]
(69)

We are now ready to prove Lemma 2. Fix any \( 1 \geq M_B > M_S > 0 \). By Claim 3, Claim 4 and the Inverse Function Theorem, for any \( 0 \leq r_2 < 1 \), there exists a strictly decreasing
function $F$ such that $r_1 = F(r_2)$ is a solution to (27); By Claim 1, Claim 2 and the Inverse Function Theorem, for any $0 < r_1 \leq 1$, there exists a strictly decreasing function $G$ such that $r_2 = G(r_1)$ is a solution to (28). Thus it suffices to prove that there exist $0 < r_2 < 1$ such that

$$G(F(r_2)) = r_2$$  \hspace{1cm} (70)$$

Note $G(F(\cdot))$ is a strictly increasing function. Also note $G(F(0)) \in (0, 1)$ since $F(0) \in (0, 1]$ and $G(r_1) \in (0, 1)$ when $r_1 \in (0, 1]$. Now, by the Intermediate Value Theorem, it suffices to show that there exists some $0 < r_2 < 1$ such that

$$G(F(r_2)) \leq r_2$$  \hspace{1cm} (71)$$

This is equivalent to showing there is some $0 < r_2 < 1$ such that

$$F(r_2) \geq G^{-1}(r_2)$$  \hspace{1cm} (72)$$

since $G$ is strictly decreasing. By Claim 3, this is equivalent to showing that there is some $0 < r_2 < 1$ such that

$$H_1(G^{-1}(r_2), r_2) \leq M_B$$  \hspace{1cm} (73)$$

Let $\epsilon \equiv M_B - M_S > 0$. We observe a relationship between the two functions $H_1$ and $H_2$ when $0 < r_1 \leq 1$ and $0 < r_2 < 1$:

$$H_1(r_1, r_2) - H_2(r_1, r_2) = \left(\frac{(1 - r_1)^2}{r_2^2} - 1\right)H_2(r_1, r_2) + r_1(2 - r_1)$$  \hspace{1cm} (74)$$

Note when $r_2 \uparrow 1$, $G^{-1}(r_2) \to 0$. To see this, suppose not, then by Claim 1, $H_2(G^{-1}(r_2), r_2) \to 1$ when $r_2 \uparrow 1$, a contradiction to $H_2(G^{-1}(r_2), r_2) = M_S < 1$. Then by the equation (74), as $r_2 \uparrow 1$,

$$H_1(G^{-1}(r_2), r_2) - M_S = H_1(G^{-1}(r_2), r_2) - H_2(G^{-1}(r_2), r_2)$$
$$= \left(\frac{(1 - G^{-1}(r_2))^2}{r_2^2} - 1\right)H_2(G^{-1}(r_2), r_2) + G^{-1}(r_2)(2 - G^{-1}(r_2))$$
$$= \left(\frac{(1 - G^{-1}(r_2))^2}{r_2^2} - 1\right)M_S + G^{-1}(r_2)(2 - G^{-1}(r_2))$$
$$\to \left(\frac{(1 - 0)^2}{1^2} - 1\right)M_S + 0(2 - 0)$$
$$= 0$$
This implies that there exists some $0 < r_2 < 1$ such that

$$|H_1(G^{-1}(r_2), r_2) - M_S| \leq \frac{\epsilon}{2} \quad (75)$$

Note (75) implies (73) as $H_1(G^{-1}(r_2), r_2) \leq M_S + \frac{\epsilon}{2} < M_S + \epsilon = M_B$. Finally, the uniqueness of the solution is implied by that $G(F(r))$ is strictly increasing w.r.t. to $r$ and thus can only cross the function $y(r) := r$ once.

A.5 Proof of Theorem 2

We already illustrated the main idea and main steps in Section 4. Now we summarize them and give a formal argument. We will prove the proposed pair forms a Nash Equilibrium.

(i): **Maxmin Trade Mechanism (II)** is a best response to **Worst-Case Joint Distribution (II)**: Note by construction, Worst-Case Joint Distribution (II) exhibits the property that the weighted virtual value is positive only for the highest type $(1,0)$, zero for the other value profiles in the support and negative for value profiles outside the support. Then any feasible and monotone mechanism in which trade occurs with some positive probability if and only if $r_2 v_B - r_1 v_S > r_1 r_2$ and trade occurs with probability 1 when $(v_B, v_S) = (1,0)$ is a best response for the designer. It is easy to see that Maxmin Trade Mechanism (II) is such a mechanism.

(ii): **Worst-Case Joint Distribution (II)** is a best response to **Maxmin Trade Mechanism (II)**: we use the duality theory to show (ii). First note that by (41), (27) and (28), all the three constraints in (P) holds. By the weighted virtual value representation, the value of P given Worst-Case Joint Distribution (I) and Maxmin Trade Mechanism (I) is simply $Pr(1,0) \times (1 - 0) = r_1 (1 - r_2)$. Second, note by (29) and $\lambda_B = \frac{1 - r_1 - r_2}{(1 - r_1) \ln \frac{1 - r_1}{r_1}} > 0$ (this holds no matter what the sign of $1 - r_1 - r_2$ is ) and $\lambda_S = -\frac{1 - r_1 - r_2}{r_2 \ln \frac{1 - r_1}{r_1}} < 0$, the constraints in (D) hold for all value profiles. To see this, note for any value profile $v = (v_B, v_S)$ outside the support of Worst-Case Joint Distribution (II),

$$\lambda_B v_B + \lambda_S v_S + \mu < \lambda_B r_1 + \lambda_S 0 + \mu = 0$$

For any value profile $v = (v_B, v_S)$ inside the support of Worst-Case Joint Distribution (II), the constraints trivially holds. Also (29) is the Complementarity Slackness. Third, the value of $D$ given the constructed $\lambda_B, \lambda_S, \mu$ is $\lambda_B M_B + \lambda_S M_S + \mu$, which, by (27), (28), (42), (57) and some algebra, is equal to $r_1 (1 - r_2)$. Finally, by Lemma 2, the solution to (27) and (28) exists. By the linear programming duality theory, (ii) holds and the revenue guarantee is
\[ r_1(1 - r_2). \]

**B Proof for Section 5**

**B.1 Proof of Theorem 3**

*Step 1: Narrow down the search to a class of mechanisms*

We divide all deterministic, DSIC and EPIR trade mechanisms into the following four classes:

*Class 1:* the trade boundary touches on the value profiles \((r_1, 1)\) and \((0, r_2)\) for some \(0 \leq r_1 \leq 1, 0 \leq r_2 \leq 1\).

*Class 2:* the trade boundary touches on the value profiles \((0, r_1)\) and \((1, r_2)\) for some \(0 \leq r_1 \leq 1, 0 \leq r_2 \leq 1\).

*Class 3:* the trade boundary touches on the value profiles \((r_1, 0)\) and \((r_2, 1)\) for some \(0 \leq r_1 \leq 1, 0 \leq r_2 \leq 1\).

*Class 4:* the trade boundary touches on the value profiles \((r_1, 0)\) and \((1, r_2)\) for some \(0 \leq r_1 \leq 1, 0 \leq r_2 \leq 1\).

Note by (i) of Proposition 1, for each class, all the value profiles to the right and below the trade boundary has trade probability of 1. Then by (iv) of Proposition 1, we can show the revenue from the four vertices \((0,0), (0,1), (1,0), (1,1)\) will never be strictly positive for *Class 1, Class 2 and Class 3*. To see this, note for *Class 1*: \(t(0,0) = 0 - r_2 = -r_2 \leq 0, t(0,1) = 0, t(1,0) = (1 - 0) \cdot 1 - 1 = -1 < 0, t(1,1) = (1 - 1) \cdot 1 - (1 - r_1) \cdot 1 = -(1 - r_1) \leq 0\); for *Class 2*: \(t(0,0) = 0 - r_1 = -r_1 \leq 0, t(0,1) = 0, t(1,0) = (1 - 0) \cdot 1 - 1 - r_2 = -r_2 \leq 0, t(1,1) = 0\); for *Class 3*: \(t(0,0) = 0, t(0,1) = 0, t(1,0) = (1 - 0) \cdot 1 - (1 - r_1) - 1 = -(1 - r_1) \leq 0, t(1,1) = 0 - (1 - r_2) = -(1 - r_2) \leq 0\).

Now when \(M_B + M_S \leq 1\), consider the joint distribution that put point masses \(M_B, M_S\) and \(1 - M_B - M_S\) on the value profiles \((1,0), (0,1)\) and \((0,0)\) respectively. It is easily to verify that this is a plausible joint distribution and the revenue under this joint distribution cannot be strictly positive; when \(M_B + M_S \geq 1\), consider the joint distribution that put point masses \(1 - M_S, 1 - M_B\) and \(M_B + M_S - 1\) on the value profiles \((1,0), (0,1)\) and \((0,0)\) respectively. It is easily to verify that this is a plausible joint distribution and the revenue under this joint distribution cannot be strictly positive. Therefore, we can focus attention on *Class 4* only.

*Step 2: Identify an upper bound of the revenue guarantee*

We propose a relaxation of (D) by omitting many constraints. Specifically, the only remaining constraints are for the four vertices \((0,0), (1,0), (0,1)\) and \((1,1)\) and the value
profiles \((r_1, 0)\) and \((1, r_2)\). Formally, we have the following relaxed problem \((D')\):

\[
\max_{\lambda_B, \lambda_S, \mu \in \mathbb{R}} \lambda_B M_B + \lambda_S M_S + \mu
\]

s.t.

\[
\begin{align*}
\mu & \leq 0 \quad (76) \\
\lambda_B r_1 + \mu & \leq 0 \quad (77) \\
\lambda_B + \lambda_S r_2 + \mu & \leq 0 \quad (78) \\
\lambda_S + \mu & \leq 0 \quad (79) \\
\lambda_B + \lambda_S + \mu & \leq 0 \quad (80) \\
\lambda_B + \mu & \leq r_1 - r_2 \quad (81)
\end{align*}
\]

Note the value of \((D')\) (denoted by \(\text{val}(D')\)) is weakly greater than the value of \((D)\). Now we are trying to find a greatest upper bound of the value of \((D')\) across \(r_1, r_2\) and argue it is attainable by constructing the mechanism and the joint distribution. We discuss four cases:

**Case 1:** \(\lambda_B \leq 0, \lambda_S \leq 0\). Note then by \((76)\), \(\text{val}(D') \leq 0\) for any \(r_1, r_2\).

**Case 2:** \(\lambda_B \geq 0, \lambda_S \geq 0\). Note then by \((76), (80)\) and \(M_B > M_S\),

\[
\begin{align*}
\lambda_B M_B + \lambda_S M_S + \mu & \leq (\lambda_B + \lambda_S) M_B + \mu \\
& = (\lambda_B + \lambda_S + \mu) M_B + \mu (1 - M_B) \\
& \leq 0
\end{align*}
\]

Thus, \(\text{val}(D') \leq 0\) for any \(r_1, r_2\).

**Case 3:** \(\lambda_B \leq 0, \lambda_S \geq 0\). By the same argument as in Case 2, \(\text{val}(D') \leq 0\) for any \(r_1, r_2\).

**Case 4:** \(\lambda_B \geq 0, \lambda_S \leq 0\). We will restrict attention to \(r_1 \geq r_2\), otherwise by the previous argument, the revenue guarantee cannot be strictly positive. Now we are left with \((77), (78)\) and \((81)\) as they imply the other three constraints. Note at least one of \((77), (78)\) and \((81)\) is binding, otherwise we can increase the value of \((D')\) by increasing \(\lambda_B\) by a small amount.

We thus discuss three situations:

(a) \(\lambda_B r_1 + \mu = 0\). We plug \(\lambda_B = -\frac{\mu}{r_1}\) into \((78)\) and \((81)\), and we obtain

\[
\lambda_S \leq \frac{1 - r_1}{r_1 r_2} \mu \quad (82)
\]
\[
\mu \geq -\frac{r_1(r_1 - r_2)}{1 - r_1}
\] 

(83)

Then we have

\[
\lambda_B M_B + \lambda_S M_S + \mu = -\frac{\mu}{r_1} M_B + \lambda_S M_S + \mu \\
\leq -\frac{\mu}{r_1} M_B + \frac{1 - r_1}{r_1 r_2} \mu M_S + \mu \\
= \left( -\frac{M_B}{r_1} + 1 + \frac{1 - r_1}{r_1 r_2} M_S \right) \mu \\
\leq \max\{0, \frac{r_1 - r_2}{1 - r_1} M_B - \frac{r_1 - r_2}{r_2} M_S - \frac{r_1(r_1 - r_2)}{1 - r_1} \}
\]

(b) : \( \lambda_B + \mu = r_1 - r_2 \).

We plug \( \lambda_B = r_1 - r_2 - \mu \) into (77) and (78), and we obtain

\[
\lambda_S \leq -\frac{r_1 - r_2}{r_2} \mu 
\]

(84)

\[
\mu \leq -\frac{r_1(r_1 - r_2)}{1 - r_1} 
\]

(85)

Then we have

\[
\lambda_B M_B + \lambda_S M_S + \mu = (r_1 - r_2 - \mu) M_B + \lambda_S M_S + \mu \\
= (r_1 - r_2) M_B + \lambda_S M_S + (1 - M_B) \mu \\
\leq \frac{r_1 - r_2}{1 - r_1} M_B - \frac{r_1 - r_2}{r_2} M_S - \frac{r_1(r_1 - r_2)}{1 - r_1}
\]

(c) : \( \lambda_B + \lambda_S r_2 + \mu = 0 \).

We plug \( \lambda_B = -\mu - \lambda_S r_2 \) into (77) and (81), and we obtain

\[
\lambda_S \geq -\frac{r_1 - r_2}{r_2} 
\]

(86)

\[
\mu \leq \frac{r_1 r_2}{1 - r_1} \lambda_S 
\]

(87)

Then we have

\[
\lambda_B M_B + \lambda_S M_S + \mu = (-\lambda_S r_2 - \mu) M_B + \lambda_S M_S + \mu \\
= (M_S - r_2 M_B) \lambda_S + (1 - M_B) \mu \\
\leq \max\{0, \frac{r_1 - r_2}{1 - r_1} M_B - \frac{r_1 - r_2}{r_2} M_S - \frac{r_1(r_1 - r_2)}{1 - r_1} \} 
\]
Let \( K(r_1, r_2) := \frac{r_1 - r_2}{1 - r_1} M_B - \frac{r_1 - r_2}{r_2} M_S - \frac{r_1(r_1 - r_2)}{1 - r_1} \). We are now solving \( \max_{r_1 \geq r_2} K(r_1, r_2) \).

Taking first order derivative w.r.t. \( r_1 \), we obtain

\[
\frac{\partial K(r_1, r_2)}{\partial r_1} = \frac{1}{(1 - r_1)^2} ((1 - r_2)M_B - \frac{M_S}{r_2} (1 - r_1)^2 - r_1(2 - r_1) - r_2)
\]  

(88)

Let \( Q(r_1, r_2) := (1 - r_2)M_B - \frac{M_S}{r_2} (1 - r_1)^2 - r_1(2 - r_1) - r_2 \). Note fixing \( r_2 \), \( Q(r_1, r_2) \) is decreasing w.r.t. \( r_1 \) when \( r_2 \leq r_1 \leq 1 \). Note

\[
Q(r_2, r_2) = (1 - r_2)M_B - \frac{M_S}{r_2} (1 - r_1)^2 - r_2(1 - r_2) + r_2
\]

\[
= (1 - r_2)(M_B + M_S - (\frac{M_S}{r_2} + r_2))
\]

Note that \( M_B + M_S - (\frac{M_S}{r_2} + r_2) \leq M_B + M_S - 2\sqrt{M_S} \). Therefore, if \( M_B + M_S - 2\sqrt{M_S} \leq 0 \), \( Q(r_1, r_2) \leq 0 \) for any \( r_2 \) and \( r_1 \in [r_2, 1] \). Then \( K(r_1, r_2) \) is maximized at \( r_1 = r_2 \), whose value is 0. If \( M_B + M_S - 2\sqrt{M_S} > 0 \), solving \( Q(r_1^*, r_2) = 0 \), we obtain (we ignore the other solution which exceeds 1)

\[
r_1^* = 1 - \sqrt{\frac{(1 - r_2)(1 - M_B)}{1 - \frac{M_S}{r_2}}} \text{ (90)}
\]

If \( r_1^* \leq r_2 \), then again \( K(r_1, r_2) \) is maximized at \( r_1 = r_2 \), whose value is 0. Now if

\[
r_1^* > r_2 \text{ (90)}
\]

which, by some algebra, is equivalent to

\[
\sqrt{(1 - r_2)(1 - \frac{M_S}{r_2})} > \sqrt{1 - M_B} \text{ (91)}
\]

then \( K(r_1, r_2) \) is maximized at \( r_1 = r_1^* \), whose value, by some algebra, is

\[
(\sqrt{(1 - r_2)(1 - \frac{M_S}{r_2})} - \sqrt{1 - M_B})^2 \text{ (92)}
\]

Then (92) is maximized at \( r_2 = \sqrt{M_S} \). Then we have \( r_1^* = 1 - \sqrt{1 - M_B} \). Then (90) is equivalent to

\[
1 - \sqrt{M_S} > \sqrt{1 - M_B} \text{ (93)}
\]

which, by some algebra, is equivalent to \( M_B + M_S - 2\sqrt{M_S} > 0 \). Thus, we have found the solution \( r_1 = 1 - \sqrt{1 - M_B}, r_2 = \sqrt{M_S} \) when \( M_B + M_S - 2\sqrt{M_S} > 0 \). And \( K(r_1, r_2) = (1 - \sqrt{M_S} - \sqrt{1 - M_B})^2 \).
Step 3: Show the upper bound is attainable

The last step is to construct deterministic trade mechanisms whose revenue guarantee is \((1 - \sqrt{M_B - M_S - (1 - M_B)})^2\) when \(M_B + M_S - 2\sqrt{M_S} > 0\). Consider any deterministic mechanism satisfying (i), (ii) and (iii) in Theorem 3. Let \(\lambda_B = \frac{1 - \sqrt{1 - M_B - M_S}}{\sqrt{1 - M_B}}, \lambda_S = -\frac{1 - \sqrt{1 - M_B - M_S}}{\sqrt{M_S}}, \mu = -\frac{(1 - \sqrt{1 - M_B - M_S})(1 - \sqrt{1 - M_B})}{\sqrt{1 - M_B}}\). We will argue they are feasible for the original dual problem (D).

Note first the constraint for the value profile (1,0) hold with equality. Then the constraints hold for any interior value profile. The reason is that the constraint is the most stringent for the value profile (1,0) by the monotonicity of the trade boundary. To see this, note the constraint for any interior value profile \((v_B, v_S)\) is equivalent to

\[
\lambda_B v_B + g(v_B) + \lambda_S v_S - f(v_S) + \mu \leq 0
\]  

(94)

where \((v_B, g(v_B))\) and \((f(v_S), v_S)\) are in the trade boundary. Since \(\lambda_B > 0, \lambda_S < 0, g\) and \(f\) are nondecreasing, the LHS of (94) is maximized at (1,0). For the value profiles outside (including) the boundary, the constraints also hold if (ii) holds. To see this, note given the constructed \(\lambda_B, \lambda_S, \mu\), we have \(\lambda_B v_B + \lambda_S v_S + \mu = 0\) for the value profiles \((1 - \sqrt{1 - M_B}, 0)\) and \((1, \sqrt{M_S})\). Then, by linearity, we have \(\lambda_B v_B + \lambda_S v_S + \mu = 0\) for any value profiles on the line boundary mentioned in Theorem 3. Therefore, if (ii) holds, the constraints also holds for value profiles outside (including) the boundary. Finally, we calculate the value of (D) under the constructed dual variables, which, by some algebra, is exactly \((1 - \sqrt{M_S} - \sqrt{1 - M_B})^2\).

For the joint distribution in Theorem 3, we first can easily verify all probability masses add up to 1. Second, given the mechanisms satisfying the three properties and the joint distribution, the value of (P) is, by some algebra, \((1 - \sqrt{M_S} - \sqrt{1 - M_B})^2\). This finishes the proof.
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