More on correlators and contact terms
in $\mathcal{N} = 4$ SYM at order $g^4$

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Abstract

We compute two-point functions of chiral operators $\text{Tr} \Phi^k$ for any $k$, in $\mathcal{N} = 4$ supersymmetric $SU(N)$ Yang-Mills theory. We find that up to the order $g^4$ the perturbative corrections to the correlators vanish for all $N$. The cancellation occurs in a highly non trivial way, due to a complicated interplay between planar and non planar diagrams.

In complete generality we show that this same result is valid for any simple gauge group.

Contact term contributions signal the presence of ultraviolet divergences. They are arbitrary at the tree level, but the absence of perturbative renormalization in the non singular part of the correlators allows to compute them unambiguously at higher orders. In the spirit of the AdS/CFT correspondence we comment on their relation to infrared singularities in the supergravity sector.

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1 Introduction

In this paper we continue the work presented in Ref. [1]. There we computed two-point functions of chiral operators \( \text{Tr}\Phi^3 \) in \( \mathcal{N} = 4 \) \( SU(N) \) supersymmetric Yang-Mills theory to the order \( g^4 \) in perturbation theory and proved that corrections vanish for all values of \( N \). In our perturbative approach, the fact that the nonrenormalization occurs for any value of \( N \), not only at leading order in the \( N \to \infty \) limit, was strictly connected to the vanishing of the colour combinatoric factor of the nonplanar diagrams. The open question was to see if the same pattern were true also for correlators of operators \( \text{Tr}\Phi^k \), with general \( k \).

We have found that for \( k > 3 \) the colour structure of nonplanar diagrams at order \( g^4 \) does not vanish. It is only a complicated cancellation between \( 1/N \) subleading contributions from planar diagrams and nonplanar ones which allows to obtain a complete all \( N \) nonrenormalization of the two-point correlators.

The colour structure identities we have used in our calculations can be generalized to arbitrary gauge groups. In so doing we have been able to prove that the nonrenormalization of the correlators up to order \( g^4 \) is valid for any group.

The other issue we focus on in this paper, is related to the presence of contact term contributions [2]. They arise in the computation of the correlators because the theory is affected by ultraviolet divergences. In order to control such infinities one has to choose a regularization scheme: at order \( g^0 \), i.e. at the tree level, the singularity of the two-point function needs to be subtracted and this leads to the introduction of arbitrary contact terms. At higher orders in \( g \) (we have computed the two-point functions explicitly up to the order \( g^4 \)) divergences cancel out and as a consequence, the local contact terms are determined unambiguously. Thus the contact terms in the two-point correlators are of the form

\[
[a + f(g^2, N)] \partial^p \delta^{(4)}(x - y)
\]

with \( a \) the arbitrary finite constant of the subtraction and \( p \) an integer depending on the dimensions of the operators. The function \( f(g^2, N) \) is in principle exactly computable order by order in \( g^2 \) for any finite \( N \), such that \( f(g^2 = 0, N) = 0 \).

According to the AdS/CFT prescription one should establish a correspondence between these terms and corresponding ones in the supergravity sector. There ambiguities arise due to the fact that the theory suffers from infrared divergences when approaching the boundary of the AdS space. Within the holographic viewpoint proposed in [3], the supergravity effective action, evaluated on a solution of the equations of motion with prescribed boundary conditions, becomes the generating functional for the conformal field theory in the large \( N \) limit. The bulk fields \( \phi \) evaluated at the boundary act as source terms for composite operators \( \mathcal{O} \) of the Yang-Mills theory

\[
\langle e^{\int d^4x \mathcal{O}(\phi_0)} \rangle_{\text{CFT}} = e^{S[\phi_0]}
\]

with the boundary action given by

\[
S[\phi_0] = \int d^4x \int d^4y \phi_0(x) \left[ \frac{1}{(x - y)^{2\Delta}} + b(\partial^2)^{\Delta - 2} \delta^{(4)}(x - y) \right] \phi_0(y)
\]
The local term, which acts as a regulator at short distances in (1.3), has an infrared origin in the 5d AdS supergravity expanded near the boundary [4, 5, 6]. The coefficient $b$ is a function of a mass scale parameter identified with the inverse of the IR 5d cutoff [7].

Now, comparing eq. (1.1) at small coupling with eq. (1.3) one should find that in the large $N$ limit, with the 't Hooft coupling $g^2N$ fixed but large

$$a + f(g^2, N) \to b_f$$

(1.4)

where $b_f$ is the arbitrary finite part of the subtraction in (1.3). In the context of our specific calculation we will show how this identification can be realized.

We present this paper as a sequel to the one referred in [1], since the techniques used here to perform the perturbative calculations of the correlators are the same as the ones used in [1]. Therefore in order to avoid lengthy repetitions in the main text, and at the same time to make this paper self-contained, we have recollected the main formulas and the rules of the game in the Appendices. In the next Section we simply remind the reader which are the quantities we have to focus on. Then in Section 3 we enter directly *in medias res*: we present the tree–level and the order $g^2$ result for the $<\text{Tr}(\Phi^1)^k\text{Tr}(\bar{\Phi}^1)^k>$ correlators. In Section 4 the order $g^4$ contributions are considered. For the relevant diagrams we compute the colour structure factors and the momentum integrals that one obtains after completion of the $D$-algebra. The various, complicated contributions give rise in the end to a complete cancellation for any finite $N$ [8]. This is in agreement with previous results [9, 10]. In Section 5 we show that the nonrenormalization properties we have found for the $SU(N)$ gauge group, are actually valid for general groups. Finally in Section 6 we concentrate on the evaluation of the contact terms and study what they correspond to in the supergravity sector.

## 2 Two-point functions of chiral operators

Our goal is to compute two-point correlators for $\mathcal{N} = 4$ supersymmetric $SU(N)$ Yang-Mills theory, perturbatively in $\mathcal{N} = 1$ superspace. The operators under consideration are the chiral primary operators in the $(0, k, 0)$ representation of the $SU(4)$ $R$–symmetry group. In a $\mathcal{N} = 1$ superspace description of the theory (see Appendix A for details) they are given by $\mathcal{O} = \text{Tr}(\Phi^{i_1}\Phi^{i_2}\ldots\Phi^{i_k})$, with flavor indices on the superfields symmetrized and traceless. In fact, in order to simplify matters as in Refs. [1, 4], we consider the $SU(3)$ highest weight superfield $\Phi^1$ and compute $<\text{Tr}(\Phi^1)^k\text{Tr}(\bar{\Phi}^1)^k>$. In this way the flavor combinatorics is avoided. At the same time we do not lose in generality since the $SU(3)$ transformations, which are invariances of the theory, allow to reconstruct all the other primary chiral correlators from the one above.

The general strategy we have adopted for performing the actual calculation is outlined in Appendix A. Quite generally, at non–coincident points we can write the two-point function as

$$<\text{Tr}(\Phi^1)^k(z_1)\text{Tr}(\bar{\Phi}^1)^k(z_2)> = \frac{F(g^2, N)}{(x_1 - x_2)^{2k}}\delta^{(4)}(\theta_1 - \theta_2)$$

(2.1)
where \( z \equiv (x, \theta, \bar{\theta}) \). Away from short distance singularities, the \( x \)-dependence of the result is fixed by the conformal invariance of the theory, and \( F(g^2, N) \) is the function that we want to determine perturbatively in \( g^2 \). As we have done in Ref. [11] we compute loop integrals in momentum space and use dimensional regularization and minimal subtraction scheme to treat ultraviolet divergences. In \( n \) dimensions, with \( n = 4 - 2\epsilon \), the Fourier transform of a power factor \((x_1 - x_2)^{-2\nu}\) is given by

\[
\frac{1}{(x^2)^\nu} = 2^{n-2\nu} \pi^{\frac{n}{2}} \frac{\Gamma\left(\frac{n}{2} - \nu\right)}{\Gamma(\nu)} \int \frac{d^n p}{(2\pi)^n} \frac{e^{-ipx}}{(p^2)^\frac{n}{2} - \nu} \quad \nu \neq 0, -1, -2, \ldots
\]

\[
\nu \neq \frac{n}{2}, \frac{n}{2} + 1, \ldots
\]

(2.2)

The main advantage of such an approach is due to the fact that the \( x \)-space structure in (2.1), with a non–vanishing contribution to \( F(g^2, N) \), can be obtained simply by looking at the contributions that behave like \( 1/\epsilon \) from the singular factor \( \Gamma\left(\frac{n}{2} - \nu\right) = \Gamma(-\nu+2-\epsilon) \), \( \nu \geq 2 \) in (2.2). By analytic continuation one can write the general identity

\[
\int \frac{d^n p}{(2\pi)^n} \frac{e^{-ipx}}{(p^2)^{2-k+\alpha\epsilon}} = \frac{2^{2k-4}}{\pi^2} (-1)^k (k-1)! (k-2)! \alpha \frac{\epsilon}{(x^2)^{k-(\alpha+1)\epsilon}} [1 + \mathcal{O}(\epsilon)]
\]

(2.3)

Once the UV divergent terms are determined at a given order in \( g \), one can reconstruct the complete answer using (2.3).

The UV divergent terms have been computed using the method proposed in [12] and various techniques presented in [13, 14]. Infrared divergences have not been considered since the theory we are dealing with is conformally invariant.

Finally we emphasize that finite momentum space contributions to the correlators correspond in \( x \)-space to terms proportional to \( \epsilon \). These are the terms which give rise to contact terms [13]. We will come back to this point in Section 5.

3 At tree-level and order \( g^2 \)

At tree–level, the 2–point correlation function \( \langle \text{Tr}(\Phi^1)^k(z_1)\text{Tr}(\bar{\Phi}^1)^k(z_2) \rangle \) is given by the diagram in Fig. 1. The colour structure

\[
\text{Tr}(T_{a_1} \cdots T_{a_k}) \sum_\sigma \text{Tr}(T_{a_{\sigma(1)}} \cdots T_{a_{\sigma(k)}})
\]

(3.1)

which includes all possible permutations \( \sigma \) in the contractions of the scalar lines, is a \( k \)-degree polynomial in \( N \). In a double line representation for the colour indices, the \( N \)-leading power term corresponds to the planar double line graph associated to the diagram in Fig. 1, whereas nonplanar graphs give rise to subleading contributions.
Now using the result (B.5) for the momentum integrals to leading order in the $\epsilon$ expansion, we obtain

$$\frac{1}{\epsilon} \left[ \frac{1}{(4\pi)^2} \right]^{k-1} \frac{(-1)^k}{[(k-1)!]^2} (p^2)^{k-2-(k-1)\epsilon} \text{Tr}(T_{a_1} \cdots T_{a_k}) \sum_\sigma \text{Tr}(T_{\sigma(1)} \cdots T_{\sigma(k)}) \delta^{(4)}(\theta_1 - \theta_2)$$

(3.2)

In $x$-space, using eq. (2.3), the result can be rewritten as

$$< \text{Tr}(\Phi^1)^k(z_1) \text{Tr}(\Phi^1)^k(z_2) >_0$$

$$= \left( \frac{1}{4\pi^2} \right)^k \text{Tr}(T_{a_1} \cdots T_{a_k}) \sum_\sigma \text{Tr}(T_{\sigma(1)} \cdots T_{\sigma(k)}) \frac{1}{(x_1 - x_2)^{2k}} \delta^{(4)}(\theta_1 - \theta_2)$$

(3.3)

At order $g^2$ the only contribution to the correlator is given by the diagram in Fig. 2, with the insertion of a vector line.

Figure 2: $g^2$-order contribution to $< \text{Tr}(\Phi^1)^k \text{Tr}(\Phi^1)^k >$

From the two internal vertices $V_1$ in (A.6) we obtain the colour structure $f_{amb}f_{a'm'b'}$ which contracted with the colour matrices associated to the rest of the diagram gives

$$- \text{Tr}(T_{a_1} \cdots T_{a_k}) \sum_{i \neq j} \sum_{\sigma} \text{Tr}(T_{\sigma(i)} \cdots [T_{\sigma(i)}, T_m] \cdots [T_{\sigma(j)}, T_m] \cdots T_{\sigma(k)})$$

(3.4)
Here the sum is over all possible permutations of the external lines and eq. (C.1) has been used. The previous expression can be simplified by noticing that for any set of matrices $M_j$, $j = 1, \cdots, n$, and any matrix $P$ the following identity holds

$$\Sigma_{i=1}^n \text{Tr}(M_1 \cdots [M_i, P] \cdots M_n) = 0$$ \hspace{1cm} (3.5)

Thus (3.4) can be written as

$$\text{Tr}(T_{a_1} \cdots T_{a_k}) \sum_{\sigma} \sum_{i=1}^k \text{Tr}(T_{a_{\sigma(i)}} \cdots [T_{a_{\sigma(i)}}, T_m] \cdots T_{a_{\sigma(k)}})$$ \hspace{1cm} (3.6)

which, using the identity (C.6), can be reduced further to the expression in (3.1) up to a factor $2N$. Including the various factors from vertices and propagators, we finally have

$$(g^2N)^k \text{Tr}(T_{a_1} \cdots T_{a_k}) \sum_{\sigma} \text{Tr}(T_{a_{\sigma(1)}} \cdots T_{a_{\sigma(k)}})$$ \hspace{1cm} (3.7)

The momentum integral associated to the graph after completion of the $D$-algebra, is evaluated in (B.6). The final result (here we reinstate a factor $\frac{1}{(4\pi)^2}$ for each loop) is

$$12\zeta(3)(g^2N) \left[ \frac{1}{(4\pi)^2} \right]^k \frac{(-1)^{k-1}(k-1)}{[(k-1)!]^2} (p^2)^{k-2-k\epsilon} \text{Tr}(T_{a_1} \cdots T_{a_k}) \sum_{\sigma} \text{Tr}(T_{a_{\sigma(1)}} \cdots T_{a_{\sigma(k)}})$$ \hspace{1cm} (3.8)

In the limit $\epsilon \to 0$ this expression is finite, therefore it does not contribute to the correlation function at separate points. It gives rise to a contact term which will be discussed in detail in Section 5.

4 At order $g^4$

The relevant diagrams, drawn in single line representation, are shown in Fig. 3 and 4. They group themselves into planar ($A$) and nonplanar ($B$) ones. If one were to use a double line representation for the colour indices from $T^a_{ij}$, then the single-line planar graphs ($A$) would give rise to two distinct types of graphs, double-line planar graphs ($A_1$) and nonplanar double-line graphs ($A_2$). Now from the $A_1$ graphs the colour structure would produce leading $N$ contributions, while from the $A_2$ graphs subleading contributions would arise. Obviously the single-line type ($B$) diagrams could only give structures subleading in $N$ since their double-line representation would be necessarily nonplanar.
For the correlators computed in [1], i.e. for the $k = 3$ case, the colour factor of the diagrams in the class (B) turned out to be identically zero. All the graphs in the class (A) gave rise to the same overall combinatorics, so that the final cancellation occurred as a cancellation among the various terms produced by the momentum integrations.

In the present case, $k > 3$, we will show that the situation is much more complicated and highly non trivial. All types of diagrams mentioned above have non vanishing colour structures. The diagrams (A) give contributions proportional to the colour structure (3.1) which, after multiplication by the momentum integrals, sum up to zero. However in addition, from the diagrams in Fig. 3c and 3d a new subleading structure arises which is of the same kind as the one from diagrams (B). Again, a nontrivial cancellation occurs among the diagrams in Fig. 3c, 3d and (B) due to the special structure of their momentum integrals.
Now for each of diagram we present the evaluation of the colour factor, while we list the corresponding momentum integrals in Appendix B. The final results are summarized at the end of the Section where the actual cancellation is discussed.

In Fig. 3a we have the insertion of a two–loop propagator correction [13, 1]

\[-2g^4 N^2 \Phi_a^i(p, \theta) \Phi_a^j(-p, \theta) \int \frac{d^n q \, d^n k}{k^2 q^2(k - q)^2(k - p)^2(p - q)^2} \]

\[= -2g^4 N^2 \Phi_a^i(p, \theta) \Phi_a^j(-p, \theta) \frac{1}{(p^2)^2} [6\zeta(3) + O(\epsilon)] \quad (4.1)\]

In this case the colour structure is easy to compute. Inserting all the coefficients from vertices, propagators and combinatorics we obtain

\[12\zeta(3)(g^2 N)^2 k \text{Tr}(T_{a_1} \cdots T_{a_k}) \sum_{\sigma} \text{Tr}(T_{a_{\sigma(1)}} \cdots T_{a_{\sigma(k)}}) \quad (4.2)\]

The momentum integral emerging after the $D$–algebra is given in eq. (B.7).

We now consider the diagram 3b where the $O(g^2)$ effective vertex [13] appears

\[\frac{g^3}{4} N i f_{abc} \Phi_a^i(q, \theta) \Phi_b^j(-p, \theta) \left(4D^\alpha \bar D^2 D_{\alpha} + (p + q)^{\alpha \dot{\alpha}} [D_{\alpha}, \bar D_{\dot{\alpha}}]\right) \int \frac{d^n k}{k^2(p - q)^2(k - q)^2} \quad (4.3)\]

Performing the $D$–algebra one easily realizes that only the first term in (4.3) gives rise to potentially divergent loop integrals. The second term produces finite contributions which might generate order $g^4$ contact terms. We concentrate on the $D^\alpha \bar D^2 D_{\alpha}$ term, since we will not compute contact contributions to $g^4$ order. The colour structure for this diagram is computed following the same procedure as in the case of the diagram in Fig. 2 (see eqs. (3.1, 3.7)). Inserting all the various factors we obtain

\[-4(g^2 N)^2 k \text{Tr}(T_{a_1} \cdots T_{a_k}) \sum_{\sigma} \text{Tr}(T_{a_{\sigma(1)}} \cdots T_{a_{\sigma(k)}}) \quad (4.4)\]

The corresponding momentum integral is given in eq. (B.3).

We now turn to the discussion of the colour structure for the diagram in Fig. 3c. The insertion of the two vector lines gives rise to the structure $f_{a_1 m b_1} f_{a_2 m n} f_{n p b_2} f_{a_3 p b_3}$. The contraction with scalar lines from the external vertices takes into account all possible permutations. Using eq. (C.1) we can write

\[\text{Tr}(T_{a_1} \cdots T_{a_k}) \sum_{\sigma} \sum_{i \neq j \neq l} \text{Tr}(T_{a_{\sigma(1)}} \cdots [T_{a_{\sigma(i)}}, T_{a_{\sigma(l)}}] \cdots [T_{a_{\sigma(j)}}, T_{a_{\sigma(k)}}] \cdots T_{a_{\sigma(k)}}) \quad (4.5)\]

This expression can be manipulated by making use of the identity (3.3) and it can be written as the sum of two terms

\[-\text{Tr}(T_{a_1} \cdots T_{a_k}) \sum_{\sigma} \sum_{j \neq l} \left\{ \text{Tr}(T_{a_{\sigma(1)}} \cdots [[T_{a_{\sigma(j)}}, T_{a_{\sigma(l)}}, T_{a_{\sigma(l)}}, T_{a_{\sigma(l)}}, T_{a_{\sigma(k)}}]) + \text{Tr}(T_{a_{\sigma(1)}} \cdots [[T_{a_{\sigma(j)}}, T_{a_{\sigma(l)}}, T_{a_{\sigma(l)}}, T_{a_{\sigma(l)}}, T_{a_{\sigma(k)}}]) \quad (4.6)\]
The first term can be further reduced by using the identities (C.6), (C.7) and (3.5) again. Inserting the factors from combinatorics, vertices and propagators the final expression for the colour structure can be written as the sum of a term leading in $N$ plus a subleading contribution

$$g^4 \text{Tr}(T_{a_1} \cdots T_{a_k}) \sum_{\sigma} \{2kN^2 \text{Tr}(T_{a_{\sigma(1)}} \cdots T_{a_{\sigma(k)}})$$

$$- \sum_{j \neq l} \text{Tr}(T_{a_{\sigma(1)}} \cdots [T_{a_{\sigma(j)}}, T_m], T_n] \cdots [T_{a_{\sigma(l)}}, T_n], T_m] \cdots T_{a_{\sigma(k)}}) \}$$

(4.7)

For $k = 3$ the second term vanishes and the above expression reduces to the one computed in Ref. \[1\]. However for $k > 3$ the subleading term is nonzero, as shown in Appendix C explicitly for the $k = 4$ case. For this diagram the loop–integral result can be read in eq. (B.9).

The colour factor for the diagram in Fig. 3 can be computed by exploiting the previous results. In fact, from the internal vertices $V_1$ and $V_3$ (see eq. (A.6)) we have the structure $f_{a_1m_1} f_{b_2n_2} f_{m_2a_2} f_{a_3p_3}$ which is identical to the one from the diagram 3c. Performing all the contractions and taking into account the coefficients from combinatorics, vertices and propagators we finally obtain

$$\frac{g^4}{2} \text{Tr}(T_{a_1} \cdots T_{a_k}) \sum_{\sigma} \{2kN^2 \text{Tr}(T_{a_{\sigma(1)}} \cdots T_{a_{\sigma(k)}})$$

$$- \sum_{j \neq l} \text{Tr}(T_{a_{\sigma(1)}} \cdots [T_{a_{\sigma(j)}}, T_m], T_n] \cdots [T_{a_{\sigma(l)}}, T_n], T_m] \cdots T_{a_{\sigma(k)}}) \}$$

(4.8)

The momentum integral for this graph is given in eq. (B.10).

The last potential contributions from diagrams of type (A) are shown in Figures 3e–3h. The loop integral resulting from the D–algebra on the diagram 3e given in eq. (B.11), is $\mathcal{O}(\epsilon)$ and then it does not contribute to the correlation function. The diagrams in Figures 3f, 3g and 3h, exactly as for $k = 3$, only contribute to finite terms.

We now study the nonplanar diagrams (B) in Figure 4.

The combinatorial and colour factors, and the loop integrals for the two diagrams 4a and 4b are identical. Thus we concentrate on one of them, e.g. 4b. From the internal $V_1$ vertices with vector lines contracted as in figure, the colour factor which arises is $f_{a_1m_1} f_{npb_1} f_{a_2pq} f_{qmb_2}$. When connected with the rest of the diagram it gives

$$\frac{g^4}{2} \sum_{\sigma} \sum_{j \neq l} \text{Tr}(T_{a_{\sigma(1)}} \cdots [T_{a_{\sigma(j)}}, T_m], T_n] \cdots [T_{a_{\sigma(l)}}, T_n], T_m] \cdots T_{a_{\sigma(k)}})$$

(4.9)

where combinatorial factors have been included. We note that the previous expression has the same form as the subleading contribution already present in the diagrams 3c and 3d (see eqs. (1.7, 4.8)). As previously mentioned and proven in Appendix C, this term is zero for $k = 3$ but in general nonvanishing when $k > 3$. The corresponding loop diagram arising after the D–algebra is given in eq. (B.8).
The last nonplanar graph to be considered is the one in Fig. 4c. As in the \( k = 3 \) case \[\text{[4]}\], its colour coefficient vanishes. A simple way to prove this is to notice that from the internal vertices one obtains

\[
f_{a_1m_1}f_{a_2n_1}f_{a_3p_1}f_{mnp}
\] (4.10)

which is antisymmetric under the exchange \( a_1 \leftrightarrow a_2 \) and \( b_1 \leftrightarrow b_2 \). When multiplied by all possible permutations of colour matrices from the external vertices it gives a zero result.

We now collect all the divergent contributions we have computed at order \( g^4 \). For each diagram we list the final result obtained as a product of the colour and combinatorial factors times the results from momentum integrations. We define

\[
P_k \equiv \text{Tr}(T_{a_1} \cdots T_{a_k}) \sum_{\sigma} \text{Tr}(T_{a_{\sigma(1)}} \cdots T_{a_{\sigma(k)}})
\] (4.11)

\[
Q_k \equiv \text{Tr}(T_{a_1} \cdots T_{a_k}) \sum_{\sigma} \sum_{j \neq l} \text{Tr}(T_{a_{\sigma(1)}} \cdots [T_{a_{\sigma(j)}}, T_m], T_n] \cdots [T_{a_{\sigma(l)}}, T_m], T_n] \cdots T_{a_{\sigma(k)}})
\]

Factorizing an overall common coefficient

\[
\frac{1}{\varepsilon} \left( g^2 N \right)^2 \left[ \frac{1}{(4\pi)^2} \right]^{k+1} \frac{(-1)^k(k-1)}{[(k-1)!]^2(k+1)} (\rho^2)^{k-2-(k+1)\varepsilon}
\] (4.12)

the various contributions are

- Diagram 3a:

\[12k\zeta(3) P_k\] (4.13)

- Diagram 3b:

\[-24k\zeta(3) P_k\] (4.14)

- Diagram 3c:

\[12k\zeta(3) - 40k\zeta(5)\] \( P_k + \frac{1}{N^2}[20\zeta(5) - 6\zeta(3)]Q_k\) (4.15)

- Diagram 3d:

\[40k\zeta(5) P_k - \frac{1}{N^2}20\zeta(5)Q_k\] (4.16)

- Diagrams 4a + 4b:

\[\frac{1}{N^2}6\zeta(3) Q_k\] (4.17)

The leading structure \( P_k \) appears in the diagrams 3a and 3b with a coefficient proportional to the \( \zeta(3) \) Riemann function, whereas in the diagram 3d it is proportional to \( \zeta(5) \). The same pattern repeats itself for the non-leading structure \( Q_k \): in the diagrams 4a and 4b it is multiplied by \( \zeta(3) \), whereas in the diagram 3d it is proportional to \( \zeta(5) \). Both structures appear in the diagram 3c and they contribute with such a coefficient to cancel the rest
of the divergent terms. In conclusion, only finite contributions survive. As emphasized at the beginning this amounts to say that the correlation function is not renormalized at order $g^4$, up to contact terms.

We stress that the complete cancellation of divergent contributions for general $k$ occurs for any finite $N$.

The results in (4.13-4.17) if restricted to $k = 3$, reproduce the results discussed in Ref. [1]. In this case the subleading structure $Q_k$ vanishes, so that the proof of non-renormalization for the correlator with general $k$ cannot be implemented from the $k = 3$ example.

5 General gauge groups

Here we want to show that the nonrenormalization properties of the correlators proven in the two previous Sections for the $SU(N)$ super Yang-Mills theory, actually hold for general simple gauge groups. To this end it is sufficient to realize that, using (C.3) and (C.5) which are valid for any group, we can generalize the identities in (C.6) and (C.7) to the following ones

$$[[T_a, T_m], T_m] = k_1 T_a$$

and

$$[[[T_a, T_m], T_n], T_m] = \frac{k_1}{2} [T_a, T_n].$$

At the order $g^2$ this enables us to replace $2N$ with $k_1$ in (3.7) and (3.8), thus obtaining a result which depends only on the Casimir of the adjoint representation of the gauge group. In any event to this order the momentum integral gives a finite result which does not affect the correlator at non coincident points.

In the same way one can analyze the general situation at the next perturbative $g^4$ order. For the propagator and vertex insertions which appear in the graphs of Fig. 3a, 3b we use the result as in Ref. [13], i.e. we set $2N \rightarrow k_1$ in (3.7) and in (3.8). For the rest of the diagrams we substitute (C.6), (C.7) with (5.1), (5.2) respectively. Once this operation is performed consistently everywhere in the various formulas of Section 4 the final, complete cancellation of all the corrections is achieved following exactly the same pattern and the same steps as in the $SU(N)$ case.

6 Contact terms in the AdS/CFT correspondence

Now we wish to focus our attention on the two-point correlators for $SU(N)$ when the two points approach each other. In the limit $x_1 \rightarrow x_2$ the expression in (2.1) becomes singular and needs to be regulated. Within the dimensional regularization approach we have adopted, this short-distance singularity is signaled by $1/\epsilon$ poles, according to the general identity

$$\frac{1}{(x^2)^{k-\kappa}} \sim -\frac{\pi^2}{2^{2k-4}(k-1)!(k-2)!} \frac{1}{\epsilon} (\partial^2)^{k-2}\delta^{(n)}(x) \quad \text{for } \epsilon \to 0$$

\[ (6.1) \]
For the two–point function of the operator Tr(Φ^1)^k these short-distance UV divergences manifest themselves already at tree–level (see (3.3)). In order to obtain a well defined function at coincident points we perform a subtraction and define in configuration space

\[< \text{Tr}(\Phi_1^1(z_1)\text{Tr}(\Phi_1^1(z_2))>_{\text{reg}} \equiv \delta(4)(\theta_1 - \theta_2) \times \]

\[\lim_{\epsilon \to 0} \frac{\mathcal{P}_k}{(4\pi)^{2k}} \left[ \frac{1}{(x^2)^{k-1}} + (-1)^k(\mu^2)^{(1-k)\epsilon} \frac{2^{4-2k} \pi^2}{[(k-1)!]^2} \left( \frac{1}{\epsilon} + \gamma \right) (\partial^2)^{k-2} \delta^{(n)}(x_1 - x_2) \right] \] (6.2)

where \(\mathcal{P}_k\) has been defined in (4.12) and \(\mu\) is the mass scale of dimensional regularization. The coefficient \(\gamma\) corresponds to an arbitrary finite subtraction and generates a scheme-dependent finite contact term.

Performing explicitly the \(\epsilon \to 0\) limit in (6.2) a residual dependence on the mass scale survives from the (divergent) counter–contact term. Therefore, the subtraction of the infinity at tree–level necessarily introduces a mass scale in the regularized correlation function which breaks conformal invariance \([16]\). On the other hand, the scheme dependent finite term proportional to \(\gamma\) is independent of \(\mu\) and it does not affect conformal invariance.

Now we discuss the appearance of this type of terms in the perturbative computation of the two–point correlator. We have performed our loop calculations in momentum space, with the Fourier transformation given in the basic formula (2.3). Using (2.3) and the general identity in (6.1) we can write

\[\lim_{\epsilon \to 0} \int \frac{d^n p}{(2\pi)^n} \frac{e^{-ipx}}{2^{-k+\alpha\epsilon}} = (-1)^{k+1} \alpha (\partial^2)^{k-2} \delta^{(4)}(x) \] (6.3)

i.e. any finite contribution in momentum space gives rise to finite contact terms in the correlation functions.

In general, the presence of contact terms at any loop order is related to the UV regularization and renormalization procedure. Different subtraction conditions correspond to different finite counterterms which eventually contribute to contact terms in the correlation functions of the theory. In \(n = 4 - 2\epsilon\) dimensions, one has to choose a particular regularization for evaluating the integrals

\[\int d^4p f(p) \to G(\epsilon) \int d^n p f(p) \] (6.4)

where \(G(\epsilon)\) is a regular function near \(\epsilon = 0\), with \(G(0) = 1\). A given prescription has to be used in the computation of both the effective action and physical quantities like correlation functions or scattering amplitudes. By expanding \(G(\epsilon)\) in powers of \(\epsilon\) one can

1 A convenient choice is \(G(\epsilon) = (4\pi)^{-\epsilon} \Gamma(1-\epsilon) (G\text{–scheme \([14]\)})\) which cancels irrelevant terms proportional to the Euler constant, \(\log 4\pi\) and \(\zeta(2)\) Riemann function in the \(\epsilon\)–expansion of \(\Gamma\) functions which appear in the calculation of multiloop integrals.
easily realize that in multiloop integrals with only simple pole divergences the coefficients of the divergent terms do not depend on the particular choice of the \( G \)–function, whereas they might depend on the regularization scheme in multiloop integrals with higher order divergences. In any case, a scheme dependence is always present in the finite part of \( any \) divergent diagram. It follows that in the evaluation of correlation functions, different choices of the \( G \)–function, i.e. different regularization prescriptions, give rise in general to different finite quantum contact terms. However, if a nonrenormalization theorem holds, then the contact terms become independent of the regularization prescription and they are unambiguously computable.

Indeed let us exemplify for simplicity the case in which at a given loop order the perturbative contribution to a correlation function contains at most a simple pole \( 1/\epsilon \) divergence

\[
\left( \frac{a}{\epsilon} + b + O(\epsilon) \right) G(\epsilon)
\]

(6.5)

By expanding \( G(\epsilon) = 1 + c\epsilon + \cdots \), in the limit \( \epsilon \to 0 \) we obtain the divergent scheme independent contribution \( a/\epsilon \) and the finite term \((b + ca)\) which contains a scheme dependence through the coefficient \( c \) from \( G(\epsilon) \). However, if there is no perturbative renormalization, i.e. \( a = 0 \), the finite term is uniquely determined.

In our specific case, we have shown that up to the \( g^4 \)–order the two–point function \(< \text{Tr}(\Phi^k(z_1))\text{Tr}(\bar{\Phi}^k(z_2)) >\) is not perturbatively renormalized and we are in the situation where the finite contact terms are uniquely determined. The term at order \( g^2 \) can be easily inferred from (3.8) by using the identity (6.3). In the limit \( \epsilon \to 0 \) we find

\[
12\zeta(3)(g^2N)\frac{k^2 - 1}{[(k-1)!]^2} \left( \frac{\mathcal{P}_k}{(4\pi)^{2k}} \right) \left( \partial^2 \right)^{k-2}(\delta(4)(x_1 - x_2)\delta(4)(\theta_1 - \theta_2)
\]

(6.6)

The same procedure can be applied in order to compute the contact term at order \( g^4 \). One should keep track of all finite contributions from the momentum integrals and then use (3.3) to obtain the result in configuration space. We have not performed the explicit calculation but in general we expect to find a nonvanishing result.

We note that, since these loop contact terms come from finite contributions in the \( \epsilon \)–expansion, in the four dimensional limit no dependence on the scale \( \mu \) survives. Therefore these terms do not affect the conformal invariance properties of the theory. As discussed in Ref. [16] the breaking of conformal invariance can only be ascribed to the UV divergences in the correlation function.

If the non–renormalization theorem holds for any finite \( N \) at all orders in perturbation theory, one could determine unambiguously the finite contact terms loop by loop, and generate a contribution of the form

\[
[a + f(g^2, N)] \left( \partial^2 \right)^{k-2}(\delta(4)(x_1 - x_2)\delta(4)(\theta_1 - \theta_2)
\]

(6.7)

where the arbitrary tree–level coefficient \( a \) is given by

\[
a \equiv \gamma(-1)^k \frac{\mathcal{P}_k}{(4\pi)^{2k}} \frac{2^{4-2k}\pi^2}{[(k-1)!]^2}
\]

(6.8)
Our result shows that the complete answer for the two-point functions of the theory necessarily contains contact terms, unless we were to choose a non-minimal subtraction at short distances. This would amount to the subtraction of a finite term with a coefficient 

$$a = - f(g^2, N).$$

The SL(2,Z) invariance of the theory would require $f(g^2, N)$ to be a modular function. However, following the arguments in [17], if loop contact terms were present in the final result they would necessarily break the $U(1)_Y$ invariance that the theory inherits from the 5d supergravity. It might be possible to establish a correspondence with $U(1)$–breaking terms in the supergravity action [18].

According to the AdS/CFT conjecture, the generating functional of the regularized correlation functions in the large $N$ limit is the semiclassical 5d supergravity action with IR divergences suitably subtracted [19]. At the boundary, i.e. with the IR cut–off removed, it has the form (1.3). In particular, the arbitrary finite contact term in (6.7) corresponds to an arbitrary finite subtraction in 5d

$$S_c[\phi_0] = b_f \int d^4 x \phi_0(x)(\partial^2)^{k-2}\phi_0(x)$$

(6.9)

related to the particular IR regularization scheme chosen for the supergravity action. In other words, in the large $N$ limit with $g^2N \gg 1$ we should find

$$[a + f(g^2, N)] \rightarrow b_f$$

(6.10)

If in this limit $f(g^2, N)$ is not vanishing, we might conclude that either coupling dependent local terms appear in supergravity, or one is forced to choose a particular subtraction scheme in the Yang–Mills sector.

Finally we notice that perturbative contact terms in the two–point functions give rise in general to coupling dependent contact–type contributions in higher–point correlators, as well as enter the definition of multitrace operators through the point–splitting regularization [4].

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A The basic rules for the computation of the two-point correlators

In $\mathcal{N} = 1$ superspace the action of $\mathcal{N} = 4$ supersymmetric Yang-Mills theory can be written in terms of one real vector superfield $V$ and three chiral superfields $\Phi^i$ (we follow the notations in [11])

\begin{equation}
S[J, \bar{J}] = \int d^8 z \operatorname{Tr} \left( e^{-gV} \Phi^i e^{gV} \Phi^i \right) + \frac{1}{2g^2} \int d^6 z \operatorname{Tr} W^a W_a \\
+ \frac{ig}{3!} \int d^6 z \epsilon_{ijk} \Phi^i [\Phi^j, \Phi^k] + \frac{ig}{3!} \int d^6 \bar{z} \epsilon_{ijk} \Phi^i [\bar{\Phi}^j, \bar{\Phi}^k] \\
+ \int d^6 z JO + \int d^6 \bar{z} \bar{J} \bar{O}
\end{equation} (A.1)

where $W_a = i \bar{D}^2 (e^{-gV} D_a e^{gV})$, and $V = V^a T^a$, $\Phi_i = \Phi^a_i T^a$, $T^a$ being $SU(N)$ matrices in the fundamental representation. We have added to the classical action source terms for the chiral primary operators generically denoted by $O$.

We define the generating functional in Euclidean space

\begin{equation}
W[J, \bar{J}] = \int \mathcal{D} \Phi \mathcal{D} \bar{\Phi} \mathcal{D} V \ e^{S[J, \bar{J}]}
\end{equation} (A.2)

so that for $O = \operatorname{Tr}(\Phi^1)^k$ the two-point function is given by

\begin{equation}
< \operatorname{Tr}(\Phi^1)^k(z_1) \operatorname{Tr}(\Phi^1)^k(z_2) > = \frac{\delta^2 W}{\delta J(z_1) \delta \bar{J}(z_2)} \bigg|_{J=\bar{J}=0}
\end{equation} (A.3)

where $z \equiv (x, \theta, \bar{\theta})$. We use perturbation theory to evaluate the contributions to $W[J, \bar{J}]$ which are quadratic in the sources, i.e.

\begin{equation}
W[J, \bar{J}] \rightarrow \int d^4 x_1 \ d^4 x_2 \ d^4 \theta \ J(x_1, \theta, \bar{\theta}) \ F(g^2, N) \frac{F(g^2, N)}{(x_1 - x_2)^2k} \bar{J}(x_2, \theta, \bar{\theta})
\end{equation} (A.4)

The $x$-dependence of the result is fixed by the conformal invariance of the theory, and $F(g^2, N)$ is the function to be determined.

In order to obtain the result in (A.4) one has to consider all the two-point diagrams from $W[J, \bar{J}]$ with $J$ and $\bar{J}$ on the external legs. First one evaluates all factors coming from combinatorics and colour structures of a given diagram. Then one performs the superspace $D$-algebra following standard techniques (see for example [11]), and reduces the result to a multi-loop integral.

The quantization procedure of the classical action in (A.1) requires the introduction of a gauge fixing (we work in Feynman gauge) and corresponding ghost terms. The ghost superfields only couple to the vector multiplet and are not interesting for our calculation. In momentum space we have the superfield propagators

\begin{equation}
< V^a V^b > = -\frac{\delta^{ab}}{p^2} \quad \quad < \Phi^a_i \Phi^b_j > = \delta_{ij} \frac{\delta^{ab}}{p^2}
\end{equation} (A.5)
The vertices are read directly from the interaction terms in (A.1), with additional $\bar{D}^2$, $D^2$ factors for chiral, antichiral lines respectively. The ones that we need are the following

\[
V_1 = ig f_{abc} \delta^{ij}_i \Phi^a_i V^b_j \Phi^c_j \quad V_2 = -\frac{i}{2} g f_{abc} \bar{D}^2 D^a V^b D_\alpha V^c \\
V_3 = \frac{g^2}{2} \delta^{ij} f_{admn} f_{bcma} V^a V^b \bar{D}^2 \Phi_i \Phi^c_j \\
V_4 = -\frac{g}{3!} \epsilon^{ijk} f_{abc} \Phi^a_i \Phi^b_j \Phi^c_k \quad V_4 = -\frac{g}{3!} \epsilon^{ijk} f_{abc} \Phi^a_i \Phi^b_j \Phi^c_k
\]  

(A.6)

All the calculations are performed in $n$ dimensions with $n = 4 - 2\epsilon$ and in momentum space. We have used the method of uniqueness [12] which is particularly efficient for the computation of massless Feynman integrals of a single variable.

## B Relevant integrals in momentum space

In this Appendix we list the relevant multiloop integrals which have been used in the course of our calculation.

As described in Section 5, in dimensional regularization $n = 4 - 2\epsilon$, one has to choose a particular prescription for the regularized integrals. In our case the integrals have at the most $1/\epsilon$ divergences so that they do not depend on the particular choice of the $G$-function, as far as divergent contributions are concerned. In order to simplify the calculation, we forget about $(2\pi)^2$ factors at intermediate stages and reinsert at the end a $1/(4\pi)^2$ factor for each loop. Having this in mind we now list the relevant integrals and their $\epsilon$ expansion.

The basic integrals from which all our results can be deduced are the following ones:

At one loop

\[
I_1 = \int \frac{d^n k}{(k^2)^{\alpha}[(p - k)^2]^{\beta}} = \frac{\Gamma(\alpha + \beta - \frac{n}{2}) \Gamma(\frac{n}{2} - \beta) \Gamma(\frac{n}{2} - \alpha)}{\Gamma(\alpha) \Gamma(\beta) \Gamma(n - \alpha - \beta)} \frac{1}{(p^2)^{\alpha + \beta - \frac{n}{2}}} 
\]  

(B.1)

At two loops

\[
I_2 = \int \frac{d^n k \, d^n l}{k^2 l^2 (k - l)^2 (p - k)^2 (p - l)^2} = \frac{1}{(p^2)^{1+2\epsilon}} [6 \zeta(3) + O(\epsilon)]
\]  

(B.2)

At four loops

\[
I_4 = \int \frac{d^n k \, d^n l \, d^n r \, d^n s}{k^2 l^2 (k - l)^2 (r - k)^2 (s - l)^2 (r - s)^2 (p - r)^2 (p - s)^2} = \frac{1}{\epsilon} \frac{1}{(p^2)^{4\epsilon}} [5 \zeta(5) + O(\epsilon)]
\]  

(B.3)

and

\[
\tilde{I}_4 = \int \frac{d^n k \, d^n l \, d^n r \, d^n s \, r^2 (p - l)^2}{k^2 l^2 (k - l)^2 (r - k)^2 (r - l)^2 (s - l)^2 (r - s)^2 (p - r)^2 (p - s)^2} = \frac{1}{\epsilon} (p^2)^{1-4\epsilon} \left[ \frac{5}{2} \zeta(5) - \frac{3}{4} \epsilon(3) + O(\epsilon) \right]
\]  

(B.4)
(the explicit evaluation of the last integral was reported in Ref. [1]).

From the previous integrals we can derive all the results needed for our calculation. By repeated use of (B.1) we obtain for the $g^0$–order diagram

$$\int \frac{d^n q_1 \cdots d^n q_{k-1}}{q_1^2 (q_2 - q_1)^2 (q_3 - q_2)^2 \cdots (p - q_{k-1})^2} = \frac{1}{\epsilon} \frac{(-1)^k}{[(k-1)!]^2} (p^2)^{k-2-(k-1)\epsilon} + O(1) \quad (B.5)$$

For the $g^2$–order (finite) diagram we have

$$\int d^n q_2 \cdots d^n q_{k-1} \frac{-q_2^2}{(q_3 - q_2)^2 \cdots (p - q_{k-1})^2} \int \frac{d^n k}{k^2 l^2 (k - l)^2 (q_2 - k)^2 (q_2 - l)^2} = \frac{(-1)^{k-1}(k-1)}{[(k-1)!]^2 k} 12 \zeta(3) (p^2)^{k-2-k\epsilon} + O(\epsilon) \quad (B.6)$$

In order to obtain this result, one first performs the $k$ and $l$ integrations with the help of eq. (B.2), then the other integrals are computed with the use of eq. (B.1).

At $g^4$–order, the integrals emerging after having performed the $D$–algebra are the following ones:

For the graph 3a, by using eq. (B.1) one obtains

$$\int \frac{d^n q_1 \cdots d^n q_{k-1}}{(q_1^2)^{1+2\epsilon}(q_2 - q_1)^2 (q_3 - q_2)^2 \cdots (p - q_{k-1})^2} = \frac{1}{\epsilon} \frac{(-1)^k(k-1)}{[(k-1)!]^2(k+1)} (p^2)^{k-2-(k+1)\epsilon} + O(1) \quad (B.7)$$

For the graphs 3b, 4a and 4b, it is easy to deduce

$$\int d^n r \ d^n q_2 \cdots d^n q_{k-1} \frac{1}{(q_2 - r)^2 (q_3 - q_2)^2 \cdots (p - q_{k-1})^2} \int \frac{d^n k}{k^2 l^2 (k - l)^2 (r - k)^2 (r - l)^2} = \frac{1}{\epsilon} \frac{(-1)^k(k-1)}{[(k-1)!]^2(k+1)} 6 \zeta(3) (p^2)^{k-2-(k+1)\epsilon} + O(1) \quad (B.8)$$

by first evaluating the two–loop integrals in $k$ and $l$ with the help of eq. (B.2), and then applying eq. (B.1).

The integral relevant for the graph 3c is

$$\int d^n q_3 \cdots d^n q_{k-1} \frac{1}{(q_4 - q_3)^2 \cdots (p - q_{k-1})^2} \int d^n k \ d^n l \ d^n r \ d^n s \ r^2 (q_3 - l)^2 \frac{r^2 (q_3 - r)^2 (q_3 - s)^2}{k^2 l^2 (k - l)^2 (r - k)^2 (r - l)^2 (s - l)^2 (r - s)^2 (q_3 - r)^2 (q_3 - s)^2} = \frac{1}{\epsilon} \frac{(-1)^k(k-1)}{[(k-1)!]^2(k+1)} [6 \zeta(3) - 20 \zeta(5)] (p^2)^{k-2-(k+1)\epsilon} + O(1) \quad (B.9)$$

by first evaluating the four–loop integral with momenta $k, l, r, s$ with use of eq. (B.4), and then performing the rest of the integrations using eq. (B.1).
Finally, the momentum integral for the graph 3d is given by

\[
\int d^n q_3 \cdots d^n q_{k-1} \left( \frac{-q_3^2}{(q_4 - q_3)^2} \cdots \frac{(p - q_{k-1})^2}{(q_4 - q_{k-1})^2} \right) \\
\int \frac{d^n k \ d^n l \ d^n r \ d^n s}{k^2 l^2 (k - l)^2 (r - k)^2 (s - l)^2 (r - s)^2 (q_3 - r)^2 (q_3 - s)^2} \\
= \frac{1}{\epsilon} \frac{(-1)^k (k - 1)}{[(k - 1)!]^2 (k + 1)} 40 \zeta(5) (p^2)^{k-2} (k+1) + O(1) \tag{B.10}
\]

Here, one first performs the \( k, l, r, s \) integrations with eq. (B.3), and then uses eq. (B.1).

For the graph 3e the momentum integral produced after completion of the D–algebra is of order \( \epsilon \). Indeed, it is given by

\[
\int d^n q_4 \cdots d^n q_{k-1} \left( \frac{1}{(q_5 - q_4)^2} \cdots \frac{(p - q_{k-1})^2}{(q_4 - q_{k-1})^2} \right) \\
\int \frac{d^n k \ d^n l \ d^n t}{k^2 l^2 (k - l)^2 (r - k)^2 (s - l)^2 (r - s)^2 (q_4 - r)^2 (q_4 - s)^2} \\
= \epsilon \frac{(-1)^k (k - 1)}{[(k - 1)!]^2 (k + 1)} 144 \zeta(3) (p^2)^{k-2} (k+1) + O(\epsilon^2) \tag{B.11}
\]

This result can be obtained by using first eq. (B.2) for the \( k, l \) and \( s, t \) two-loop integrals, and then eq. (B.1) for the remaining integrations.

## C Colour structures

In this Appendix we give our conventions and a series of useful identities involving the group generators. Moreover we show that the colour structure (4.9) of the nonplanar graphs 4a and 4b is nonvanishing, by evaluating it explicitly in the \( k = 4 \) case.

For a general simple Lie algebra we have

\[
[T_a, T_b] = i f_{abc} T_c \tag{C.1}
\]

where \( T_a \) are the generators and \( f_{abc} \) the structure constants. The matrices \( T_a \)'s are normalized as

\[
\text{Tr}(T_a T_b) = k_2 \delta_{ab} \tag{C.2}
\]

We have also

\[
f_{amn} f_{bmn} = k_1 \delta_{ab} \tag{C.3}
\]

From the Jacobi identity one obtains

\[
f_{abm} f_{cdn} + f_{cbm} f_{dam} + f_{dbm} f_{acm} = 0. \tag{C.4}
\]

which in turn allows to write

\[
f_{am} f_{bmn} f_{cmt} = \frac{1}{2} k_1 f_{abc} \tag{C.5}
\]
Now we specialize our formulas to the gauge group $SU(N)$: we have $k_1 = 2k_2N$ and we choose a normalization in (C.2) such that $k_2 = 1$. The generators $T_a$, $a = 1, \ldots, N^2 - 1$, in the fundamental representation of $SU(N)$ are $N \times N$ traceless matrices. For $SU(N)$ the relations in (C.3) and (C.5) can be written as

$$[[T_a, T_m], T_m] = 2N T_a$$

(C.6)

$$[[[T_a, T_m], T_n], T_m] = N [T_a, T_n].$$

(C.7)

Now we concentrate on the explicit evaluation of some colour factors in the case $k = 4$. In particular we want to show that the nonplanar colour structure of graphs $4a$ and $4b$ is nonvanishing, in contradistinction to the case $k = 3$ (see formula (A.18) in Appendix A of [1]). The relation which allows to deal with products of $T_a$'s is the following

$$T_{ij}^a T_{kl}^a = \left( \delta_{ij} \delta_{jk} - \frac{1}{N} \delta_{ij} \delta_{kl} \right).$$

(C.8)

From (C.8) one can obtain several useful formulas

$$\text{Tr}(T_a T_b T_c T_d) \text{Tr}(T_a T_b T_c T_d) = \frac{1}{N^2} (N^2 - 1)(N^2 + 3)$$

(C.9)

$$\text{Tr}(T_a T_b T_c T_d) \text{Tr}(T_a T_b T_c T_d) = -\frac{1}{N^2} (N^2 - 1)(N^2 - 3)$$

(C.10)

$$\text{Tr}(T_a T_b T_c T_d) \text{Tr}(T_a T_b T_c T_d) = k_1^2 \frac{1}{N^2} (N^2 - 1)(N^4 - 3N^2 + 3)$$

(C.11)

the last one being a planar type contribution. Moreover, by noticing that

$$f_{abc} = -i \text{Tr} ([T_a, T_b] T_c)$$

(C.12)

one can compute

$$\text{Tr}(T_{c_1} T_{a_1} T_{c_2} T_{a_2}) f_{c_1 m b_1} f_{c_2 m b_2} = -(\delta_{b_1 a_1} \delta_{b_2 a_2} + \delta_{b_2 a_1} \delta_{b_1 a_2})$$

(C.13)

$$\text{Tr}(T_{c_1} T_{c_2} T_{a_1} T_{a_2}) f_{c_1 m b_1} f_{c_2 m b_2} = \delta_{b_1 b_2} \delta_{a_1 a_2} + \text{Tr}(T_{b_1} T_{b_2} T_{a_1} T_{a_2})$$

(C.14)

Consider now the nonplanar colour structure for the graphs $4a$ and $4b$

$$Q_k \equiv \text{Tr}(T_{a_1} \cdots T_{a_k}) \sum_{\sigma} \sum_{i \neq j} \text{Tr} \left( T_{a_{\sigma(1)}} \cdots [T_{a_{\sigma(i)}}, T_{m}], T_n] \cdots [T_{a_{\sigma(j)}}, T_n], T_m] \cdots [T_{a_{\sigma(k)}}, T_n] \right)$$

(C.15)

As already mentioned $Q_3 = 0$. In general however it does not vanish. In the case $k = 4$ by exploiting the various symmetries of this structure, it can be brought to the more manageable expression

$$Q_4 = 32 \left[ 2\text{Tr}(T_{a_1} T_{a_2} T_{b_3} T_{b_4}) + \text{Tr}(T_{a_1} T_{b_3} T_{a_2} T_{b_4}) \right] f_{a_1 m n} f_{m p c_1} f_{a_2 p r} f_{r n c_2}
\times \left[ \text{Tr}(T_{c_1} T_{c_2} T_{b_3} T_{b_4}) + \text{Tr}(T_{c_1} T_{b_3} T_{c_2} T_{b_4}) + \text{Tr}(T_{c_1} T_{c_2} T_{b_4} T_{b_3}) \right]$$

(C.16)
By using the Jacobi identity \((\text{C.4})\) for the four \(f\) structure
\[
f_{a_1nm}f_{mpc}f_{a_2pc}f_{rnc} = -Nf_{a_1mc_1}f_{a_2mc_2} + f_{a_1pm}f_{a_2pr}f_{c_1nm}f_{c_2nr}
\]
and using equations \((\text{C.13}, \text{C.14})\) and \((\text{C.9, C.10})\), it is straightforward to obtain
\[
Q_4 = 192(N^2 - 1)(2N^2 - 3) \quad \text{(C.18)}
\]
As claimed the result is nonvanishing.

Now, as a check, we want to verify in the \(k = 4\) case that the colour structure of the diagrams 3c and 3d is given by the sum of a term proportional to the tree level structure \(P_k\) and a term proportional to the nonplanar structure \(Q_k\).

The tree level colour structure
\[
P_k \equiv \text{Tr}(T_{a_1} \cdots T_{a_k}) \sum_{\sigma} \text{Tr}(T_{a_{\sigma(1)}} \cdots T_{a_{\sigma(k)}}) \quad \text{(C.19)}
\]
for \(k = 4\) can be reduced to
\[
P_4 = 4 \text{Tr}(T_a T_b T_c T_d) [\text{Tr}(T_a T_b T_c T_a) + 4 \text{Tr}(T_a T_b T_d T_c) + \text{Tr}(T_a T_b T_c T_d)]
\]
and then, with \((\text{C.9–C.11})\), to
\[
P_4 = 4 \frac{1}{N^2} (N^2 - 1)(N^4 - 6N^2 + 18) \quad \text{(C.21)}
\]
Now consider the structure from the diagrams 3c and 3d
\[
R_k \equiv \text{Tr}(T_{a_1} \cdots T_{a_k}) \times 
\sum_{\sigma} \sum_{i \neq j \neq l} \text{Tr} \left( T_{a_{\sigma(1)}} \cdots [T_{a_{\sigma(i)}}, T_m] \cdots [T_{a_{\sigma(j)}}, T_n] \cdots [T_{a_{\sigma(l)}}, T_n] \cdots T_{a_{\sigma(k)}} \right) \quad \text{(C.22)}
\]
Setting \(k = 4\) it can be written as
\[
R_4 = 16 [\text{Tr}(T_{a_1} T_{a_2} T_{a_3} T_{b}) + \text{Tr}(T_{a_1} T_{b} T_{a_2} T_{a_3}) + \text{Tr}(T_{a_1} T_{a_2} T_{b} T_{a_3}) + \text{Tr}(T_{a_1} T_{a_3} T_{a_2} T_{b}) \\
+ \text{Tr}(T_{a_1} T_{b} T_{a_3} T_{a_2}) + \text{Tr}(T_{a_1} T_{a_3} T_{b} T_{a_2})]/f_{a_1mc_1}f_{a_2nc_2}f_{a_3rc_3}[\text{Tr}(T_{c_1} T_{c_2} T_{c_3} T_{t}) \\
+ \text{Tr}(T_{c_1} T_{c_2} T_{c_3} T_{c_3}) + \text{Tr}(T_{c_1} T_{c_2} T_{t} T_{c_3}) + \text{Tr}(T_{c_1} T_{c_3} T_{t} T_{c_2}) + \text{Tr}(T_{c_1} T_{t} T_{c_3} T_{c_2})
\]
Making use of the equations \((\text{C.9, C.14})\), it is a lengthy but straightforward calculation to show that
\[
R_4 = 32(N^2 - 1)(N^4 - 18N^2 + 36) \quad \text{(C.24)}
\]
At this point it is immediate, from eqs. \((\text{C.18}), (\text{C.21})\) and \((\text{C.24})\), to see that
\[
R_4 = 8N^2P_4 - Q_4 \quad \text{(C.25)}
\]
in accordance with the manipulations leading to eq. \((4.7)\).
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