English Translation of Chapter 9 of the book

Adams spectral sequence and stable homotopy groups of spheres

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Chapter 9
A sequence of new families in the stable homotopy groups of spheres

In this Chapter, we will state and prove the existence of a sequence of new families in the stable homotopy groups of spheres which is in base on several papers of the author (especially [7][8][9][24]). As a preminaries, in §1 we introduce some spectra which is closely related to Moore spectrum and Smith-Toda spectrum $V(1)$ and state some of their properties. In §2 we state and prove a general result on the convergence of $h_0\sigma$ and $h_0\sigma'$ for a pair of $a_0$-related elements $\sigma$ and $\sigma'$. (the generalization of [8] Theorem A). §3 is devoted to state and prove a general result on the convergence of $(i'i)_*(h_0\sigma)$ induces the convergence of $(i'i)_*(g_0\sigma)$ in the stable homotopy groups of Smith-Toda spectrum $V(1)$ (the generalization of [7] Theorem II).

In §4 we prove a pull back Theorem in the Adam spectral sequence and as a corollary of the main results in §2 and §4 , in §5 we obtain the convergence of a sequence of $h_0 h_n, h_0 b_n, h_0 h_n h_m, h_0 (h_n b_{m-1} - h_m b_{n-1})$ new families in the stable homotopy groups of spheres. §6 concerns with the convergence of a sequence of $h_0 \sigma \gamma_s, g_0 \sigma \gamma_s$-elements. In §7, we first prove $h_n$ Theorem and then obtain the third periodicity $\gamma_{p^n/s}$-families ([24] Theorem I and Theorem II). At last , in §8, the second periodicity $\beta_{tp^n/i,i+1}$-families in the stable homotopy groups of spheres are detected.

§1. Some spectra closely related to the Moore spectrum and Smith-Toda spectrum $V(1)$

Let $M$ be the Moore spectrum given by the cofibration

(9.1.1) \[ S \xrightarrow{p} S \xrightarrow{i} M \xrightarrow{j} \Sigma S \]

Let $\alpha : \Sigma^q M \rightarrow M$ be Adams map and $K$ be its cofibre given by the cofibration

(9.1.2) \[ \Sigma^q M \xrightarrow{\alpha} M \xrightarrow{i'} K \xrightarrow{j'} \Sigma^{q+1} M \]

The above spectrum $K$ which we briefly write as $K$ actually is the Smith-Toda spectrum $V(1)$ in Chapter 6 §2.
Now we introduce some spectra closely related to $S, M$ or $K$. Let $L$ be the cofibre of $\alpha_1 = j\alpha : \Sigma^{-1}S \to S$ given by the cofibration

\[ \Sigma^{-1}S \xrightarrow{\alpha_1} S \xrightarrow{j''} \Sigma^qS. \]

Let $Y$ be the cofibre of $i' : S \to K$ given by the cofibration

\[ S \xrightarrow{i'} K \xrightarrow{j'} \Sigma^qS. \]

$Y$ actually is the Toda spectrum $V(1\frac{1}{2})$, and it also is the cofibre of $j\alpha : \Sigma^qM \to \Sigma^qS$ given by the cofibration

\[ \Sigma^qM \xrightarrow{j\alpha} \Sigma S \xrightarrow{\pi} Y \xrightarrow{\pi} \Sigma^q+1M. \]

This can be seen by the following homotopy commutative diagram of $3 \times 3$-Lemma in the stable homotopy category (cf. Chapter 3 §7)

\[ \begin{array}{cccccc}
S & \xrightarrow{i'} & K & \xrightarrow{j'} & \Sigma^q+1M \\
\downarrow i & / & \uparrow i' & \downarrow \pi & / \uparrow \pi \\
M & \downarrow j & Y & \downarrow \epsilon \\
\Sigma^qM & \xrightarrow{j\alpha} & \Sigma S & \xrightarrow{\pi} & \Sigma S
\end{array} \]

Note that $\alpha_1 \cdot p = p \cdot \alpha_1 = 0$, then there exist $\pi \in [\Sigma^qS, L]$ and $\xi \in [L, S]$ such that $p = j'' \pi$ and $p = \xi i''$. Since $\pi q S = 0$, then $\pi q L \cong Z(p)\{\pi\}$. Moreover, $i''\xi i'' = i'' \cdot p = (p \land 1_L)i''$, then $p \land 1_L = i''\xi + \lambda \pi j''$ for some $\lambda \in Z(p)$. By composing $j''$ on the above equation we have $p \cdot j'' = j''(p \land 1_L) = \lambda j''\pi \cdot j'' = \lambda p \cdot j''$ so that $\lambda = 1$ and we have

\[ \lambda \pi j'' = j''(p \land 1_L) \]

By the following homotopy commutative diagram of $3 \times 3$-Lemma

\[ \begin{array}{cccccc}
\Sigma^qS & \xrightarrow{\pi} & \Sigma S & \xrightarrow{\alpha_1} & \Sigma S \\
\downarrow \pi & / \uparrow j'' & \downarrow i & / \uparrow j\alpha & \downarrow i'' \\
L & \xrightarrow{j''} & \Sigma^qM & \xrightarrow{\pi} & \Sigma^q+1L \\
\downarrow i'' & \uparrow \pi h & \downarrow \pi & / \uparrow \pi & \downarrow j & / \uparrow \pi \\
S & \xrightarrow{\pi} & \Sigma^{-1}Y & \xrightarrow{\pi} & \Sigma^q+1S
\end{array} \]

we obtain the following cofibration

\[ \Sigma^qS \xrightarrow{\pi} L \xrightarrow{\pi} \Sigma^{-1}Y \xrightarrow{\pi} \Sigma^q+1S \]

and there are equations $\pi h = i \cdot j''$, $\pi i'' = \pi$, $\pi \cdot j = i''j\alpha$. By $2\alpha ij\alpha = i\alpha^2 + \alpha^2 ij$ (cf. (6.5.3)), then we have $\alpha_1 \alpha_1 = 0$ and so there are $\phi \in [\Sigma^q-1S, L]$ and $(\alpha_1)_L \in [\Sigma^q-1L, S]$ such that

\[ j''\phi = \alpha_1 = (\alpha_1)_L \cdot i''. \]
Let $W$ be the cofibre of $\phi : \Sigma^{2q-1}S \to L$, then $W$ also is the cofibre of $(\alpha_1)_L : \Sigma^{q-1}L \to S$. This can be seen by the following homotopy commutative diagram of $3 \times 3$-Lemma

$$
\begin{array}{ccc}
\Sigma^{2q-1}S & \xrightarrow{\alpha_1} & \Sigma^q S \\
\downarrow \phi & \nearrow j'' & \downarrow \iota''
\end{array} \quad \begin{array}{ccc}
\Sigma^q L & \xrightarrow{(\alpha_1)_L} & S
\end{array}

(9.1.11)

that is, we have two cofibrations

$$
\begin{array}{ccc}
\Sigma^q S & \xrightarrow{i''} & W \\
\downarrow w'' & \nearrow u & \downarrow j''
\end{array} \quad \begin{array}{ccc}
\Sigma^{2q} S & \xrightarrow{(\alpha_1)_L} & \Sigma^q L
\end{array}

(9.1.12)

We write the Toda spectrum $V(\frac{1}{2})$ as $K'$, it is the cofibre of $jj' : \Sigma^{-1}K \to \Sigma^{q+1}S$ given by the cofibration

$$
\begin{array}{ccc}
\Sigma^{-1}K & \xrightarrow{jj'} & \Sigma^{q+1}S \\
\downarrow \alphaijj' & \nearrow x & \downarrow y
\end{array}

(9.1.14)

$K'$ also is the cofibre of $\alpha : \Sigma^q S \to M$ given by the cofibration

$$
\begin{array}{ccc}
\Sigma^q S & \xrightarrow{\alpha} & M \\
\downarrow \iota & \nearrow x & \downarrow y
\end{array}

(9.1.15)

This can be seen by the following homotopy commutative diagram of $3 \times 3$-Lemma

$$
\begin{array}{ccc}
\Sigma^q M & \xrightarrow{\alpha} & K' \\
\downarrow j & \nearrow \alpha & \downarrow z
\end{array} \quad \begin{array}{ccc}
\Sigma^{q+1} S & \xrightarrow{p} & \Sigma^{q+1} S
\end{array}

(9.1.16)

By $\alpha_1 \wedge 1_M = i j \alpha - \alpha i j$, let $\alpha' = \alpha_1 \wedge 1_K \in [\Sigma^{q-1}K, K]$, then $j'\alpha' = -(\alpha_1 \wedge 1_M)j' = \alpha i j' \in [\Sigma^{-2}K, M]$. By (9.1.16) we have $y \cdot z = p$, then $\alpha' \wedge 1_M = 1_{K'} \wedge p$. This is because $[K', M] = 0$ which can be seen by the following exact sequence induced by (9.1.15)

$$
0 = [\Sigma^{q+1} S, M] \xrightarrow{y^*} [K', M] \xrightarrow{v^*} [M, M] \xrightarrow{(\alpha i)_L^*}

$$

where $[M, M] \cong \mathbb{Z}_p\{1_M\}$ so that the above $(\alpha i)_L^*$ is monic. Moreover, by the following homotopy commutative diagram of $3 \times 3$-Lemma we know that the cofibre of $\alpha i j' : \Sigma^{-1}K \to \Sigma M$ is $K' \wedge M$ given by the following cofibration

$$
\begin{array}{ccc}
\Sigma^{-1}K & \xrightarrow{\alpha i j'} & \Sigma M \\
\downarrow \psi & \nearrow \rho & \downarrow \psi
\end{array} \quad \begin{array}{ccc}
K' \wedge M & \xrightarrow{\rho} & K
\end{array}

(9.1.17)
\[
\begin{array}{c}
\Sigma^{-1}K \xrightarrow{\alpha ij'j'} \Sigma M \xrightarrow{\nu} \Sigma K' \\
\downarrow j' j' \xrightarrow{\alpha i} \psi \xrightarrow{1_{K'} \wedge j}
\end{array}
\]

(9.1.18)

\[
\begin{array}{c}
\Sigma^{q+1}S \xrightarrow{\mu} \Sigma K' \wedge M \\
\downarrow y \xrightarrow{\rho} \psi \xrightarrow{1_{K'} \wedge j}
\end{array}
\]

From \((1_{K'} \wedge j)(v \wedge 1_M)\overline{m}_M = v(1_M \wedge j)\overline{m}_M = v = (1_{K'} \wedge j)\psi\) we have \((v \wedge 1_M)\overline{m}_M = \psi\) and \(d(\psi) \in [\Sigma^2 M, K' \wedge M] = 0\). Similarly, from \(m_K(x \wedge 1_M)(1_{K'} \wedge i) = m_K(1_{K'} \wedge i)x = \rho(1_{K'} \wedge i)\) we have \(\rho = m_K(x \wedge 1_M)\) and \(d(\rho) \in [\Sigma K' \wedge M, K] = 0\). Concludingly, up to sign we have

(9.1.19) \[\rho = m_K(x \wedge 1_M), \quad \psi = (v \wedge 1_M)\overline{m}_M, \quad d(\rho) = 0, d(\psi) = 0.\]

Let \(\alpha' = \alpha_1 \wedge 1_K \in [\Sigma^{q-1} K, K]\), where \(\alpha_1 = j\alpha i \in \pi_{q-1} S\), then \(j' \alpha' \alpha' = 0\) and so by (9.1.17), there exists \(\alpha'_{K' \wedge M} \in [\Sigma^{q-1} K, K' \wedge M]\) such that \(\rho \alpha'_{K' \wedge M} = \alpha'\). and \(d(\alpha'_{K' \wedge M}) \in [\Sigma^2 K, K' \wedge M] = 0\). Hence \(\rho \alpha'_{K' \wedge M} = \alpha'\) and \(d(\alpha'_{K' \wedge M}) = 0, d(\alpha'_{K' \wedge M}) = 0, d(\alpha_1 \wedge 1_M) = 0, d(\psi) = 0\) and \(d(\alpha'_{K' \wedge M}) = -\alpha_1 \wedge 1_M\), then by applying \(d\) to the above equation we have \(\lambda \psi(\alpha_1 \wedge 1_M) = 0\) so that \(\lambda = 0\). Concludingly we have

(9.1.20) \[\rho \alpha'_{K' \wedge M} = \alpha', \quad \alpha'_{K' \wedge M} = (vi \wedge 1_M)(\alpha_1 \wedge 1_M), \quad d(\alpha'_{K' \wedge M}) = 0, \quad d(\alpha'_{K' \wedge M}) = -\alpha'' \in [\Sigma^{q-2} K, K]\]

where we use \(d(\alpha'') = -\alpha'\) (cf. (6.5.5)).

**Proposition 9.1.21** Let \(p \geq 5, V\) be any spectrum and \(f : \Sigma^j K' \rightarrow V \wedge K\) be any map, then \(f \cdot z = 0 \in [\Sigma^{q+1} S, V \wedge K]\).

**Proof:** By Theorem 6.5.16 and Theorem 6.5.19, there is a commutative multiplication \(\mu : K \wedge K \rightarrow K\) such that \(\mu(j'j' \wedge 1_K) = 1_K = \mu(1_K \wedge j')\) and there is an injection \(\nu : \Sigma^{q+2} K \rightarrow K \wedge K\) such that \((jj' \wedge 1_K)\nu = 1_K\). Then by (9.1.14) we have \(z \wedge 1_K = (z \wedge 1_K)(jj' \wedge 1_K)\nu = 0\) and \(f \cdot z = (1_V \wedge \mu)(1_V \wedge K \wedge j'j')f \cdot z = (1_V \wedge \mu)(f \cdot z \wedge 1_K)j'i = 0\). Q.E.D.

By (9.1.6) we have \(\epsilon \cdot \overline{w} = p(\text{up to sign})\), Then it is easy to proof that \(\overline{w} \cdot \epsilon = (1_V \wedge p)\). By the following homotopy commutative diagram of 3 \(\times\) 3-Lemma in the stable homotopy category.
we know that the cofibre of $\alpha' i': \Sigma^q M \to \Sigma K$ is $Y \land M$ given by the following cofibration

\[(9.1.22)\quad \Sigma^q M \xrightarrow{\alpha' i'} \Sigma K \xrightarrow{r} \Sigma Y \xrightarrow{\gamma} \gamma \land Y \land M \xrightarrow{\gamma \land Y \land M} Y \land M \xrightarrow{\gamma} \Sigma^q M\]

By (9.1.10), $\alpha_1 = j'' \cdot \phi$, where $\phi \in \pi_{2q-1}L$. Then we have

\[(9.1.23)\quad \Sigma^q M \xrightarrow{\alpha' i'} \Sigma K \xrightarrow{(r \land 1_M)m_M} Y \land M \xrightarrow{m_M(\pi \land 1_M)} \Sigma^{q+1} M.

But (9.1.23) and (9.1.24) we have $\alpha_1 = m_M(\pi \land 1_M) \alpha_1 \land 1_M$. In addition, $\alpha_1 \land 1_M = (1_Y \land 1_M) (r \land 1_M) k_M (\pi \land 1_M)$ so that by (9.1.23) we have $\alpha_1 \land 1_M = (1_Y \land 1_M) (r \land 1_M) k_M (\pi \land 1_M)$.

Moreover we have, $(1_Y \land 1_M) (r \land 1_M) k_M (\pi \land 1_M)$, so that by (9.1.23) we have $\alpha_1 \land 1_M = (1_Y \land 1_M) (r \land 1_M) k_M (\pi \land 1_M)$.

Hence, we have the following relations

\[(9.1.25)\quad m_M(\pi \land 1_M) \alpha_1 \land 1_M = \alpha_1 \land 1_M = (1_Y \land 1_M) (r \land 1_M) k_M (\pi \land 1_M)\]

where $\alpha_1 \land 1_M \in \Sigma^{q+1} M, Y \land M \cap (ker d)$ and $\phi \in \pi_{2q-1}L$.

Now recall the ring spectrum properties of the spectrum $K$. By Theorem 6.5.16 and (6.5.17), there is a homotopy equivalence $K \land K = K \lor \Sigma L \land K \lor \Sigma^{q+2} K$ and there are projections and injections

\[(9.1.26)\quad \mu : K \land K \to K, \quad \nu_2 : K \land K \to \Sigma L \land K, \quad j'j' \land 1_K : K \land K \to \Sigma^{q+1} K\]

Such that (cf. Theorem 6.5.16, Theorem 6.5.19)
\[ \mu(i' i \land 1_K) = 1_K = \mu(1_K \land i' i), \quad (jj' \land 1_K) \nu = 1_K = (1_K \land jj') \nu, \]

\[ (i' i \land 1_K) \mu + \nu_2 \mu_2 + (jj' \land 1_K) \nu = 1_{K \land K}, \quad \mu_2 (i' i \land 1_K) = 0. \]

Hence, by (9.1.4), there exists \( \overline{\pi}_2 \in [Y \land K, \Sigma L \land K] \) such that \( \overline{\pi}_2 (r \land 1_K) = \mu_2 \) and \( d(\overline{\pi}_2) = 0 \in [Y \land K, L \land K] \), this can be obtained from \( d(\overline{\pi}_2 (r \land 1_K)) = d(\mu_2) = 0 \) (cf. Theorem 6.5.19(H)). By the first equation of (9.1.25), (9.1.23)(9.1.13) and the following homotopy commutative diagram (9.1.28) of 3 \times 3-Lemma we know that the cofibre of \( \alpha_{Y \land M} : \Sigma^{2q+1} M \to Y \land M \)

is \( \Sigma L \land K \) given by the following cofibration

\[ \Sigma^{2q+1} M \xrightarrow{\alpha_{Y \land M}} Y \land M \xrightarrow{\overline{\pi}_2} \Sigma L \land K \]

\[ \xrightarrow{i'} \Sigma K \xrightarrow{\alpha'} \Sigma L \land K \]

\[ \xrightarrow{(r \land 1_M) \overline{\pi}_K} \overline{\mu}_2 (1_Y \land i') \]

\[ \xrightarrow{\nu_2} Y \land M \]

\[ \xrightarrow{\nu_Y} \Sigma^{q+2} K \]

\[ \xrightarrow{(j'' \land 1_K)} \Sigma^{2q+1} M \xrightarrow{\alpha} \Sigma^{q+1} M \]

Since \( \epsilon \land 1_K = \mu(i' i \land 1_K)(\epsilon \land 1_K) = 0 \), then the cofibration (9.1.4) induces a split cofibration \( K \xrightarrow{i' \land 1_K} K \land K \xrightarrow{r \land 1_K} Y \land K \). That is, there is a homotopy equivalence \( K \land K = K \land Y \land K \) so that \( Y \land K = \Sigma L \land K \land \Sigma^{q+2} K \) and there are projections \( \overline{\pi}_2 : Y \land K \to \Sigma L \land K \), \( j\pi \land 1_K : Y \land K \to \Sigma^{q+2} K \) and injections \( \nu_Y : \Sigma^{q+2} K \to Y \land K \), \( \overline{\nu}_2 : \Sigma L \land K \to Y \land K \) such that \( \nu_Y = (r \land 1_K) \nu \) and

\[ (j\pi \land 1_K) \nu_Y = 1_K, \quad \overline{\pi}_2 \overline{\nu}_2 = 1_{L \land K}, \quad \nu_Y (j\pi \land 1_K) + \overline{\nu}_2 \overline{\pi}_2 = 1_{Y \land K}. \]

By (9.1.1)(9.1.15)(9.1.3) and homotopy commutative diagram of 3 \times 3-Lemma we can easily know that the cofibre of \( vi : S \to K' \) is \( \Sigma L \) given by the following cofibration

\[ S \xrightarrow{vi} K' \xrightarrow{k} \Sigma L \xrightarrow{\xi} \Sigma S \]

with relations \( \xi \cdot i'' = p \) so that \( \xi i'' \land 1_M = p \land 1_M = 0 \) and so \( \xi \land 1_M = \alpha(j'' \land 1_M) \). In addition, \( \xi i'' \land 1_K = p \land 1_K = 0 \) so that \( \xi \land 1_K \in (j'' \land 1_K)^{*} [\Sigma^q K, K] \) = 0. Then, the cofibration (9.1.30) induces a split cofibration \( K \xrightarrow{vi \land 1_K} K' \land K \land k_{L \land K} \land \Sigma L \land K \). That is to say, \( K' \land K \) splits into \( K \land \Sigma L \land K \) so that there is \( \nu_2' : \Sigma L \land K \to K' \land K \) such that \( (k \land 1_K) \nu_2' = 1_{L \land K} \) and \( \mu (x \land 1_K)(vi \land 1_K) = 1_K, (vi \land 1_K) \mu (x \land 1_K) + \nu_2'(k \land 1_K) = 1_{K' \land K} \). Moreover, \( x(1_{K'} \land \epsilon) = (1_K \land \epsilon)(x \land 1_Y) = 0 \in [\Sigma^{-1} K' \land Y, K] \). Hence, by (9.1.14) we have, \( 1_{K'} \land \epsilon = z \cdot \omega, \omega \in [K' \land Y, \Sigma^{q+2} S] \). We claim that \( K' \land Y \) splits into
\[\Sigma^{q+2}S \vee \Sigma L \wedge K, \text{ this can be seen by the following homotopy commutative diagram of } 3 \times 3-\text{Lemma in the stable homotopy category}\]

\[
\begin{array}{ccc}
K' \wedge Y & \xrightarrow{1_K \wedge \epsilon} & \Sigma K' \\
\downarrow \tilde{\nu} & \swarrow z & \downarrow \epsilon_{1_K \wedge \nu'} \\
\Sigma^{q+2}S & \rightarrow & \Sigma K' \wedge K \\
\downarrow \mu' \wedge \nu' & \swarrow \nu'_1 & \downarrow \epsilon_{1_K \wedge \nu'} \\
K & \xrightarrow{0} & \Sigma^2L \wedge K \\
\end{array}
\]

(9.1.31)

That is, we have a split cofibration \(\Sigma L \wedge K \xrightarrow{\tilde{\nu}} K' \wedge Y \xrightarrow{\nu} \Sigma^{q+2}S\) so that there are \(\tilde{\tau} : \Sigma^{q+2}S \rightarrow K' \wedge Y, \tilde{\mu}_2 : K' \wedge Y \rightarrow \Sigma L \wedge K\) such that

\[
\tilde{\nu} \cdot \tilde{\tau} = 1_S, \quad \tilde{\mu}_2\tilde{\nu}_2 = 1_{L \wedge K}, \quad \tilde{\tau}\tilde{\nu} + \tilde{\nu}_2\tilde{\mu}_2 = 1_{K' \wedge Y}.
\]

**Proposition 9.1.33** Let \(V\) be any spectrum, then there is a direct sum decomposition

\[
[S^*M, V \wedge K] = \kerd i' \oplus (\kerd)_{i'ij}
\]

where \(\kerd = [S^*K, V \wedge K] \cap (\kerd)\).

**Proof:** For any \(f \in [S^*M, V \wedge K]\) and where \(1_V \wedge (i' \wedge 1_K)\) the multiplication of the ring spectrum \(K\) such that \(1_V \wedge \mu(1_K \wedge i') = 1_K = \mu(1_K \wedge i')\) (cf. (9.1.26)). Then \(f = (1_V \wedge \mu)(i' \wedge 1_K) + f_2 \cdot j\) for some \(f_2 \in [S^*+1S, V \wedge K]\). It follows that \(f = (1_V \wedge \mu)(i' \wedge 1_K) + (1_V \wedge \mu)(f_2 \wedge 1_K)i'ij\) which proves the result, where \(d(f \wedge 1_K) = f \wedge d(1_K) = 0, d(1_V \wedge \mu) = 1_V \wedge d(\mu) = 0\) (cf. Theorem 6.5.19(G)). Q.E.D.

\[\xi_2. \text{ A general result on convergence of } a_0\text{-related elements}\]

From [12] p. 11 Theorem 1.2.14, there is a nontrivial secondary differential in the Adams spectral sequence \(d_2(h_n) = a_0b_{n-1}, n \geq 1\), where \(d_2 : \text{Ext}_A^1p^nq(Z_p, Z_p) \rightarrow \text{Ext}_A^3p^{n+1}q(Z_p, Z_p)\) is the secondary differential of the Adams spectral sequence. We call \(h_n \in \text{Ext}_A^1p^nq(Z_p, Z_p)\) and \(b_{n-1} \in \text{Ext}_A^2p^nq(Z_p, Z_p)\) is a pair of \(a_0\)-related elements. In this section, we prove a general result on convergence of \(a_0\)-related elements in the Adams spectral sequence of sphere spectrum and Moore spectrum.

**Definition 9.2.1** Let \(p \geq 7, s \leq 4\), and there is a nontrivial secondary differential of the Adams spectral sequence \(d_2(\sigma) = a_0\sigma', \) we call
σ ∈ Ext_A^{s,t} (Z_p, Z_p) and σ' ∈ Ext_A^{s+1,t} (Z_p, Z_p) is a pair of a₀-related elements. We have the following general result.

**The main Theorem A** (the generalization of [8] Theorem A) Let \( p \geq 7, s \leq 4, \sigma \) be the unique generator of Ext_A^{s,t} (Z_p, Z_p) and there is a nontrivial secondary differential \( d_2 (\sigma) = a_0 \sigma' \) in the ASS, where \( \sigma' \) is the unique generator (or the linear combination of the two generators) of Ext_A^{s+1,t} (Z_p, Z_p). Moreover, suppose that

(I) \( \text{Ext}_A^{s,qr-u} (Z_p, Z_p) = 0 (r = 2, 3, 4, u = 1, 2) \).

(II) \( \text{Ext}_A^{s+2,tq+u} (Z_p, Z_p) = 0, r = 2, 3, 4, u = -1, 0 \) or \( r = 3, 4, u = 1 \),

(III) \( \text{Ext}_A^{s+3,tq+ru} (Z_p, Z_p) = 0 (r = 1, 3, 4) \).

Then \( h_0 \sigma' \in \text{Ext}_A^{s+2,tq+u} (Z_p, Z_p) \) and \( i_*(h_0 \sigma) \in \text{Ext}_A^{s+1,tq+u} (H^* M, Z_p) \) are permanent cycles in the ASS.

To prove the main Theorem A, we need some preliminaries as follows.

For \( (\alpha_1) \in [\Sigma^{q-1} L, S] \) in (9.1.10) we have \( \alpha_1 \cdot (\alpha_1) \in [\Sigma^{2q-2} L, S] = 0 \) which is obtained from \( \pi_{r-2} S = 0 \) \((r = 2, 3)\). Then there is \( \phi \in [\Sigma^{2q-1} L, L] \) such that \( j'' \phi = (\alpha_1) \in [\Sigma^{q-1} L, S] \) and \( \phi \cdot \nu' \in \pi_{2q-1} L \). Since \( \pi_{r-1} S \) has unique generator \( \alpha_1 = j \alpha_i, \alpha_2 = j a^2 i \) for \( r = 1, 2 \) respectively and \( j'' \phi \cdot p = \alpha_1 \cdot p = 0 \), then \( \phi \cdot \nu' = \nu'' \alpha_2 \) (up to scalar). That is, \( \nu'' \pi_{2q-1} S \) also is generated by \( \phi' \), so that \( \pi_{2q-2} S \cong Z_p' \{ \phi \} \), for some \( s \geq 1 \). Hence, \( \overline{\phi} \) is \( \lambda \phi \) with \( \lambda \in Z_p \) and we have \( \lambda \phi' = (\alpha_1) \in [\Sigma^{q-1} L, S] \) and \( \phi' \cdot \nu'' = \nu'' \phi' = \nu'' a^2 \) (up to scalar). Moreover, \( (\alpha_1) \in [\Sigma^{2q-2} L, S] = 0 \) is because \( \pi_{r-2} S = 0 \) \((r = 3, 4)\), then, by (9.1.13), there is \( \overline{\phi} \in [\Sigma^{3q-1} L, W] \) such that \( u \overline{\phi} = \overline{\phi} \). Concludingly, we have elements \( \overline{\phi} \in [\Sigma^{2q-1} L, L] \), \( \overline{\phi} \in [\Sigma^{3q-1} L, W] \) such that
(9.2.2) \( j'' \bar{\phi} = (\alpha_1)_L, \quad \bar{\phi}'' = \lambda \phi, \lambda = 1 \pmod p, \quad \bar{\phi}_W = \bar{\phi}. \)

**Proposition 9.2.3** Let \( p \geq 7 \), then
(1) Up to nonzero scalar we have \( \phi \cdot p = i'' \alpha_2 = \pi \cdot \alpha_1 \neq 0, (\alpha_1)_L \cdot \pi = \alpha_2, \)
\( p \cdot (\alpha_1)_L = \alpha_2 \cdot j'' = (\alpha_1)_L \pi j'' \neq 0, \)
\[ \Sigma^{q-1} L, L \] has unique generator \( \bar{\phi} \) modulo some elements of filtration \( \geq 2. \)
(2) \( \bar{\phi}(p \wedge 1_L) \neq 0 \in [\Sigma^{2q} L, Y] \)
(3) \( \bar{\phi}(\pi \wedge 1_L)(p \wedge 1_L) \neq 0 \in [\Sigma^{3q} L, Y], j'' \bar{\phi}(\pi \wedge 1_L) = j \alpha^3 i \in \pi_{3q-1} S \)
(4) \( \pi_{4q} Y \) has unique generator \( \bar{\phi}(\pi \wedge 1_L) \pi \cdot \phi = 0. \)

**Proof:** (1) Since \( j'' \phi \cdot p = \alpha_1 \cdot p = 0 = j'' \pi \cdot \alpha_1, \) and \( \pi_{2q-1} S \cong Z_p \{\alpha_2\}, \) then \( \phi \cdot p = i'' \alpha_2 = \pi \cdot \alpha_1 \) (up to scalar). We claim that \( \phi \cdot p \neq 0, \) this can be proved as follows. Consider the following exact sequence
\[ Z_p \{ j \alpha^2 \} \cong [\Sigma^{2q} L, S] \xrightarrow{i''} [\Sigma^{2q} L, M] \xrightarrow{j''} [\Sigma^{q-1} M, S] \xrightarrow{(\alpha_1)_*}, \]
induced by (9.1.3). The right group has unique generator \( j \alpha \) satisfying \( (\alpha_1)_* j \alpha = j \alpha i \alpha = j \alpha \alpha i j \neq 0, \) then the above \( (\alpha_1)_* \) is monic, \( \text{im} j'' = 0 \) so that \( [\Sigma^{2q} L, M] \cong Z_p \{ i' i' \alpha^2 \} \). Suppose in contrast that \( \phi \cdot p = 0, \) then \( \phi \in i' [\Sigma^{2q-1} L, M] \) and so \( \phi = i'' j \alpha^2 i \), \( \alpha_1 = j'' \phi = j'' i' i' \alpha_2 = 0 \) which is a contradiction. This shows that \( \phi \cdot p \neq 0 \) so that the above scalar is nonzero (mod \( p \)).

The proof of the second result is similar. To prove the last result, let \( x \) be any element of \( [\Sigma^{2q-1} L, L] \), then \( j'' x \in [\Sigma^{q-1} L, S] \cong Z_p \{ (\alpha_1)_L \} \) for some \( s \geq 2 \). Hence, \( j'' x = \lambda j'' \phi \) with \( \lambda \in Z_p \) so that \( x = \lambda \phi + i'' x' \), where \( x' \in [\Sigma^{2q-1} L, S] \). Since \( x' i'' \in \pi_{2q-1} S \cong Z_p \{ j \alpha^2 i \} \) and \( \pi_{3q-1} S \cong Z_p \{ j \alpha^3 i \} \),
then \( x' \) is an element of filtration \( \geq 2 \) which shows the result.

(2) Suppose in contrast that \( \bar{\phi}(p \wedge 1_L) = 0 \), then by (9.1.9) we have \( \bar{\phi}(p \wedge 1_L) = \lambda \pi \cdot (\alpha_1)_L, \) where \( \lambda \in Z_p \). Since \( j'' \pi \wedge 1_M = p \wedge 1_M = 0, \) then \( \pi \wedge 1_M = (i'' \wedge 1_M) \alpha, \) and so \( \lambda \pi (\pi \wedge 1_M) i \cdot (\alpha_1)_L = \lambda (\pi \wedge 1_M) i \pi (\alpha_1)_L = 0. \) Moreover we have \( \lambda (i'' \wedge 1_M) \alpha i (\alpha_1)_L = \lambda (\pi \wedge 1_M) i (\alpha_1)_L = 0, \) then \( \lambda \alpha i (\alpha_1)_L \in (\alpha_1 \wedge 1_M)_* \{ \Sigma^{q} L, M \} \) and so \( \lambda \alpha i (\alpha_1)_L \in (\alpha_1 \wedge 1_M) (i'') \Sigma^{q} L, M \) = 0 which can be obtained by the following exact sequence
\[ \Sigma^{2q} L, M] \xrightarrow{(i'')^*} [\Sigma^{q} L, M] \xrightarrow{(i'')^*} [\Sigma^{q} M] \xrightarrow{(\alpha_1)_*} \]
induced by (9.1.3) where the right group has unique generator \( \alpha i \) satisfying
(α1)∗αi = αiαj αi ≠ 0 so that (i′′)∗[ΣqL, M] = 0. The above equation implies that λ′ = 0 so that we have φ(p ∧ 1L) = 0, this contradicts with the result in (1) on j′′φ(p ∧ 1L) = p · (α1)∗L ≠ 0. This shows that hφ(p ∧ 1L) ≠ 0.

(3) Since πr−2S = 0 (r = 2, 3, 4), then φ(1L ∧ α1) ∈ [Σ3q−2L, L] = 0, and so there is φ ∈ [Σ3q−1L ∧ L, L] such that φ(1L ∧ i′′) = φ. We first prove φ(π ∧ 1L)(p ∧ 1L)̸= 0. For otherwise, if it is zero, then φ(p · p = φ(π ∧ 1L)(p ∧ 1L)′′ = 0 so that 1φπ ∈ i∗[Σ3q−1M, L]. However, (j′′)∗[Σ3q−1M, L] ⊂ [Σ3q−1M, S] the last of which has unique generator jα2 satisfying (α1)∗(jα2) = jαiα2 ≠ 0, then (j′′)∗[Σ3q−1M, L] = 0 and so we have (α1)∗L = j′′φπ ∈ i∗(j′′)∗[Σ3q−1M, L] = 0, this contradicts with the result in (1).

Now suppose in contrast that hφ(π ∧ 1L)(p ∧ 1L) = 0, then by (9.1.9) we have, φ(π ∧ 1L)(p ∧ 1L) = π · ω, where ω ∈ [Σ3q−1L, S] satisfying ω′′ = λ1α2 for some λ1 ∈ Zp. It follows that (i′′ ∧ 1M)αiω = (1L ∧ i′′)π · ω = 0, then αiω ∈ (α1 ∧ 1M),[Σ3qL, M] and so λ1αiα2 = αiω′′ ∈ (α1 ∧ 1M),[Σ3qL, M] = (α1)*[Σ3q−1L, M] = 0. This shows that λ1 = 0 (since αiα2 = αiα2)i′′ ≠ 0.

Hence, ω = αiLα3 · i′′ and φ(π1L)(p ∧ 1L) = λ2π · jα3 · i′′ for some λ2 ∈ Zp. It follows that φ(p · p = φ(1L ∧ 1L)(p ∧ 1L)′′ = 0, then φπ ∈ i∗[Σ3q−1M, L] and so (α1)∗L = j′′φπ ∈ i∗(j′′)∗[Σ3q−1M, L] = 0. This contradicts with the result in (1) on (α1)∗L ≠ 0.

For the second result, by (9.1.9) we have π · j = i′′jα , then j′′φ(π ∧ 1L)π · j = j′′φ(π ∧ 1L)′′jα = j′′φπjα = (α1)∗Ljα = λ2jα = jα3 · i′′ (up to mod p nonzero scalar). Consequently we have j′′φ(π ∧ 1L)π = jα3 · i′′ (up to nonzero scalar). This is because π3q−1S ≡ Zp{α3} so that p∗π3q−1S = 0.

For the last result, we first prove φ(π ∧ 1L)π ≠ 0. For otherwise, if it is zero, then 0 = φ(π ∧ 1L)π · j = φ(π ∧ 1L)′′jα = φπjα and so α2jα = (α1)∗Ljα = j′′φπjα = 0 , this is a contradiction (since α2jα = jα2 · i′′ ≠ 0 ∈ [Σ3q−2M, S]). Now suppose instead that hφ(π ∧ 1L)π = 0, then by (9.1.9) and π3q−1S ≡ Zp{α3} we have φ(π ∧ 1L)π = λπ · jα3 · i = λi′′jα4 · i for some λ ∈ Zp and so j′′φ(π ∧ 1L)π = 0, this contradicts with the second result.

(4) Since (w)∗π3q−1Y ⊂ π3q−1M and the last of which has unique generator iα3 · i = i′′jφ(π ∧ 1L)π = wφ(π ∧ 1L)π (up to nonzero scalar) and π3q−1S ≡ Zp{α3 · i} such that (w)∗π3q−1S = 0, then π3q−1Y has unique generator hφ(π ∧ 1L)π. Moreover by (9.1.7) we have, hφ(π ∧ 1L)π · p = h(p ∧
Theorem A we have
\[ \text{Ext}^{s+1,tq} (H^* L, Z_p) = 0, \text{Ext}^{s+1,tq} (H^* L, H^* L) \cong Z_p \{ (\sigma')_L \} \] or
\[ Z_p \{ (\sigma'_1)_L, (\sigma'_2)_L \}, \] where \( L \) is the spectrum in (9.1.3) and there are relations
\[ (i')^*(\sigma')_L = (i')_*(\sigma') \quad \text{or} \quad (i'')^*(\sigma')_L = (i'')_*(\sigma') \quad \text{or} \quad (i'')^*(\sigma'_2)_L = (i'')_*(\sigma'_2). \]

**Proposition 9.2.4** Let \( p \geq 7 \), then under the supposition of the main
Theorem A we have
\[ \text{Ext}^{s+1,tq+q} (H^* L, Z_p) = 0, \text{Ext}^{s+1,tq} (H^* L, H^* L) \cong Z_p \{ (\sigma')_L \} \] or
\[ Z_p \{ (\sigma'_1)_L, (\sigma'_2)_L \}, \] where \( L \) is the spectrum in (9.1.3) and there are relations
\[ (i'')^*(\sigma')_L = (i'')_*(\sigma') \quad \text{or} \quad (i'')^*(\sigma'_2)_L = (i'')_*(\sigma'_2). \]

**Proof:** Consider the following exact sequence
\[ \text{Ext}^{s+1,tq+q} (Z_p, Z_p) \overset{i''}{\rightarrow} \text{Ext}^{s+1,tq} (H^* L, Z_p) \]
\[ \overset{j''}{\rightarrow} \text{Ext}^{s+1,tq} (H^* L, Z_p) \]
induced by (9.1.3). The right group has unique generator \( \sigma' \) or has two
generators \( \sigma'_1, \sigma'_2 \) satisfying \((\alpha)_*(\sigma') = h_0 \sigma' \neq 0 \) or \((\alpha)_*(\sigma'_1) = h_0 \sigma'_1 \neq 0, \)\((\alpha)_*(\sigma'_2) = h_0 \sigma'_2 \neq 0 \in \text{Ext}^{s+2,tq+q} (Z_p, Z_p) \) (cf. the supposition II),
then the above \((\alpha)_* \) is monic so that \( \text{im} \ j'' = 0. \)
Moreover, the left
group has unique generator \( h_0 \sigma = (\alpha)_*(\sigma) \), then \( \text{im} \ i'' = 0 \)
so that
\[ \text{Ext}^{s+1,tq+q} (H^* L, Z_p) = 0. \]
Look at the following exact sequence
\[ 0 = \text{Ext}^{s+1,tq+q} (H^* L, Z_p) \overset{(i'')^*}{\rightarrow} \text{Ext}^{s+1,tq} (H^* L, H^* L) \]
\[ \overset{(i'')^*}{\rightarrow} \text{Ext}^{s+1,tq} (H^* L, Z_p) \]
induced by (9.1.3). Since \( \text{Ext}^{s+1,tq} (Z_p, Z_p) \cong Z_p \{ \sigma' \} \) or \( Z_p \{ \sigma'_1, \sigma'_2 \} \) and
\( \text{Ext}^{s+1,tq-q} (Z_p, Z_p) = 0, \) then the right group has unique generator \((i'')_*(\sigma') \)
or has two generators \((i'')_*(\sigma'_1), (i'')_*(\sigma'_2), \) the image of which under \((\alpha)_* \)
is zero. Then, the result on the middle group is proved. Q.E.D.

**Proposition 9.2.5** Let \( p \geq 7 \), then under the supposition of the main
Theorem A we have
\( \text{Ext}^{s+3,tq+3q+1} (H^* L, Z_p) \cong Z_p \{ \tilde{\phi}_s \pi_s (\sigma'_1), \tilde{\phi}_s \pi_s (\sigma'_2) \} \) or has unique
generator \( \tilde{\phi}_s \pi_s \sigma' \)

\( \text{Ext}^{s+3,tq+3q+2} (H^* Y, H^* L) \cong Z_p \{ \tilde{\phi}_s \pi_s (\pi \wedge 1)_s (\sigma'_1)_L, \tilde{\phi}_s \pi_s (\pi \wedge 1)_s (\sigma'_2)_L \} \)
or has unique generator \( \tilde{\phi}_s \pi_s (\pi \wedge 1)_s (\sigma')_L, \) where \( \tilde{\phi} \in [\Sigma^{2q-1} L \wedge L] \) such that
\( \tilde{\phi} (1_L \wedge i''') = \tilde{\phi} \in [\Sigma^{2q-1} L \wedge L] \) (cf. Prop. 9.2.3(3)).

**Proof:** (1) Consider the following exact sequence
\[ \text{Ext}^{s+3,tq+3q+1} (Z_p, Z_p) \overset{(i'')^*}{\rightarrow} \text{Ext}^{s+3,tq+3q+1} (H^* L, Z_p) \]
\[ \overset{j''}{\rightarrow} \text{Ext}^{s+3,tq+3q+1} (Z_p, Z_p) \]
induced by (9.1.3). The left group is zero and the right group has unique
generator \( \tilde{\alpha}_2 \sigma' \) or has two generators \( \tilde{\alpha}_2 \sigma'_1, \tilde{\alpha}_2 \sigma'_2 \) (cf. the supposition III).
Note that $j\alpha i = (\alpha_1)L \cdot \pi = j''\phi \cdot \pi \in \pi_{2q-1}S$, (cf. Prop. 9.2.3), then $\bar{\alpha}_2\sigma'_1 = j_\pi \alpha_\pi \pi_\sigma (\sigma'_1) = j''\phi_\pi \pi_\sigma (\sigma'_1)$ and $\bar{\alpha}_2(\sigma'_2) = j''\phi_\pi \pi_\sigma (\sigma'_2)$ so that the result on the middle group follows.

(2) Consider the following exact sequence

$$0 = Ext^{s+3,tq+4q+1}_A (H^*L, Z_p) \xrightarrow{(j''*)} Ext^{s+3,tq+4q+2}_A (H^*Y, H^*L) \xrightarrow{\pi_*} Ext^{s+3,tq+3q+1}_A (H^*L, Z_p) \xrightarrow{(\alpha_1)_*}$$

induced by (9.1.3). By the supposition III, $Ext^{s+3,tq+rq+1}_A (Z_p, Z_p) = 0$ ($r = 3, 4$). By (1) and $\phi = \tilde{\phi}(1_L \wedge i'')$, the right group has unique generator $\phi_\pi \pi_\sigma (\sigma') = (i'')^*(\phi_\pi (\pi \wedge 1)L_\pi (\sigma')_L$ or has two generators $\bar{\phi}_\pi \pi_\sigma (\sigma'_1) = (i'')^*\phi_\pi (\pi \wedge 1)L_\pi (\sigma'_1)_L, \bar{\phi}_\pi \pi_\sigma (\sigma'_2) = (i'')^*\phi_\pi (\pi \wedge 1)L_\pi (\sigma'_2)_L$ the image of which under $(\alpha_1)_*$ is zero, then the middle group has unique generator $\bar{\phi}_\pi (\pi \wedge 1)L_\pi (\sigma'_1)_L$ or has two generators $\bar{\phi}_\pi (\pi \wedge 1)L_\pi (\sigma'_1)_L, \bar{\phi}_\pi (\pi \wedge 1)L_\pi (\sigma'_2)_L$.

Moreover, by $Ext^{s+3,tq+rq}_A (Z_p, Z_p) = 0$ (r = 2, 3) we know that $Ext^{s+3,tq+2q}_A (Z_p, H^*L) = 0$, then by (9.1.9), $Ext^{s+3,tq+3q+2}_A (H^*Y, H^*L) = \tilde{h}_s Ext^{s+3,tq+3q+1}_A (H^*L, H^*L)$ and the result follows as desired. Q.E.D.

**Proposition 9.2.6** Let $p \geq 7$, then under the supposition of the main theorem A we have

(1) $Ext^{s+2,tq+3q+1}_A (H^*Y, H^*L) = 0, Ext^{s+2,tq+3q+2}_A (H^*Y, Z_p) = 0$.

(2) $Ext^{s+1,tq+3q+r}_A (H^*Y, H^*L) = 0, r = 0, 1$.

**Proof:** (1) Consider the following exact sequence

$$Ext^{s+2,tq+3q}_A (H^*L, H^*L) \xrightarrow{(\tilde{h}_s)} Ext^{s+2,tq+3q+1}_A (H^*Y, H^*L)$$

induced by (9.1.9). By the supposition II on $Ext^{s+2,tq+rq}_A (Z_p, Z_p) = 0$ ($r = 2, 3, 4$) we know that the left group is zero. By the supposition II on $Ext^{s+2,tq+rq-1}_A (Z_p, Z_p) = 0 (r = 2, 3)$ also know that the right group is zero. Then the middle group is zero as desired.

For the second result, consider the following exact sequence

$$Ext^{s+2,tq+4q+1}_A (H^*L, Z_p) \xrightarrow{(\tilde{h}_s)} Ext^{s+2,tq+4q+2}_A (H^*Y, Z_p)$$

induced by (9.1.9). By the supposition II on $Ext^{s+2,tq+rq+1}_A (Z_p, Z_p) = 0$ ($r = 3, 4$) we know that the left group is zero. Similarly, the right group also is zero. Then the middle group is zero as desired.

(2) Consider the following exact sequence ($r = 0, 1$)
\[ \text{Ext}_A^{s+1, tq+3q+r-1}(H^*L, H^*L) \xrightarrow{(\mathfrak{m})_*} \text{Ext}_A^{s+1, tq+3q+r}(H^*Y, H^*L) \]
\[ \text{Ext}_A^{s+1, tq+2q+r-2}(Zp, H^*L) \]

induced by (9.1.9). By the supposition I on \( \text{Ext}_A^{s+1, tq+kq+r-1}(Zp, Zp) = 0 \)
\((k = 2, 3, 4, r = 0, 1)\) we know that the left group is zero. By the supposition II on \( \text{Ext}_A^{s+1, tq+kq+r-2}(Zp, Zp) = 0 \)
\((k = 2, 3, r = 0, 1)\) also know that the right group is zero, then the middle group is zero as desired. Q.E.D.

**Proposition 9.2.7** Let \( p \geq 7 \), then under the supposition of the main Theorem A we have

(1) \( \text{Ext}_A^{s+1, tq+3q}(H^*W, H^*L) \cong Z_p((\phi W)_*(\sigma)_L) \), where \( \phi W \in [\Sigma^{3q-1}L, W] \)
satisfying \( u\phi W = \phi \in [\Sigma^{2q-1}L, L] \) and \( (\sigma)_L \in \text{Ext}_A^{s+1q}(H^*L, H^*L) \)
such that \((i'')^*(\sigma)_L = (i'')^*((\alpha_1)_L)^*(\sigma) \) and \( \text{Ext}_A^{s+1, tq+2q}(Zp, Zp) \neq 0 \), then the right

group has unique generator \((\alpha_1)_L \), the left group is zero. By \((i'')^* \cdot \text{Ext}_A^{s+1, tq+q}(Zp, H^*L) \subset \text{Ext}_A^{s+1, tq+q}(pZp) \)
and the last of which has unique generator \( h_0 \sigma = (\alpha_1)^* \cdot (\sigma) = (i'')^*((\alpha_1)_L)^*(\sigma) \) and \( \text{Ext}_A^{s+1, tq+2q}(Zp, Zp) = 0 \), then the right

group has unique generator \((\alpha_1)_L \), the left group is zero. By \((i'')^* \cdot \text{Ext}_A^{s+1, tq+q}(Zp, H^*L) \subset \text{Ext}_A^{s+1, tq+q}(pZp) \)
and the last of which has unique generator \( h_0 \sigma = (\alpha_1)^* \cdot (\sigma) = (i'')^*((\alpha_1)_L)^*(\sigma) \) and \( \text{Ext}_A^{s+1, tq+2q}(Zp, Zp) = 0 \), then the right

group has unique generator \((\alpha_1)_L \), the left group is zero. By \((i'')^* \cdot \text{Ext}_A^{s+1, tq+q}(Zp, H^*L) \subset \text{Ext}_A^{s+1, tq+q}(pZp) \)
and the last of which has unique generator \( h_0 \sigma = (\alpha_1)^* \cdot (\sigma) = (i'')^*((\alpha_1)_L)^*(\sigma) \) and \( \text{Ext}_A^{s+1, tq+2q}(Zp, Zp) = 0 \), then the right

(2) Consider the following exact sequence
\( \text{Ext}_A^{s+1, tq+3q}(H^*L, H^*L) \xrightarrow{(j'')_*} \text{Ext}_A^{s+1, tq+3q}(H^*W, H^*L) \)
\( \text{Ext}_A^{s+1, tq+2q+q}(Zp, H^*L) \)

induced by (9.1.12). By the supposition I on \( \text{Ext}_A^{s+1, tq+rq}(Zp, Zp) = 0 \) \((r = 2, 3, 4)\) we know that the left group is zero. By \((i'')^* \cdot \text{Ext}_A^{s+1, tq+q}(Zp, H^*L) \subset \text{Ext}_A^{s+1, tq+q}(pZp) \)
and the last of which has unique generator \( h_0 \sigma = (\alpha_1)^* \cdot (\sigma) = (i'')^*((\alpha_1)_L)^*(\sigma) \) and \( \text{Ext}_A^{s+1, tq+2q}(Zp, Zp) = 0 \), then the right

group has unique generator \((\alpha_1)_L \), the left group is zero. By \((i'')^* \cdot \text{Ext}_A^{s+1, tq+q}(Zp, H^*L) \subset \text{Ext}_A^{s+1, tq+q}(pZp) \)
and the last of which has unique generator \( h_0 \sigma = (\alpha_1)^* \cdot (\sigma) = (i'')^*((\alpha_1)_L)^*(\sigma) \) and \( \text{Ext}_A^{s+1, tq+2q}(Zp, Zp) = 0 \), then the right

(2) Consider the following exact sequences
\( \text{Ext}_A^{s+1, tq+3q-1}(H^*L, H^*L) \xrightarrow{h_*} \text{Ext}_A^{s+1, tq+3q}(H^*Y, H^*L) \)
\( \text{Ext}_A^{s+1, tq+2q-2}(Zp, H^*L) \)

induced by (9.1.9) and (9.1.1) respectively. By the supposition I on \( \text{Ext}_A^{s+1, tq+rq-1}(Zp, Zp) = 0 \) \((r = 2, 3, 4)\) we know that the upper left group

is zero. By the supposition I on \( \text{Ext}_A^{s+1, tq+rq-2}(Zp, Zp) = 0 \) \((r = 2, 3)\), the

upper right group is zero. Then the upper middle group is zero as desired. Similarly, the lower middle also is zero. Q.E.D.
**Proposition 9.2.8** Let \( p \geq 7 \), then under the supposition (I)(III) of the main Theorem A we have

\[
\begin{align*}
& Ext_A^{s+3, t_q+2}(H^*M, Z_p) = 0, \\
& Ext_A^{s+1, t_q+1}(H^*M \wedge L, Z_p) \cong Z_p\{(i \wedge 1_L)_* \pi_*(\sigma)\}.
\end{align*}
\]

**Proof:** Consider the following exact sequence

\[
\begin{align*}
& Ext_A^{s+3, t_q+2}(Z_p, Z_p) \xrightarrow{j_*} Ext_A^{s+3, t_q+2}(H^*M, Z_p) \\
& \downarrow \quad \downarrow \pi_* \\
& Ext_A^{s+3, t_q+1}(Z_p, Z_p) \xrightarrow{p_*}
\end{align*}
\]

induced by (9.1.1). By the supposition III, the right group is zero or has unique generator \( a_0 \) which satisfies \( p_*(a_0) = a_0^2 \neq 0 \in Ext_A^{s+4, t_q+2}(Z_p, Z_p) \), then \( \text{im } j_* = 0 \). By the supposition III, the left group has unique generator \( a_0^2 \sigma' \) or has two generators \( a_0^2 \sigma'_1 = p_*(a_0 \sigma'_1), a_0^2 \sigma'_2 = p_*(a_0 \sigma'_2) \) so that we have \( \text{im } i_* = 0 \). Then, the middle group is zero as desired.

For the second result, consider the following exact sequence

\[
\begin{align*}
& Ext_A^{s+1, t_q+1}(H^*L, Z_p) \xrightarrow{(j^!1_L)^*} Ext_A^{s+1, t_q+1}(H^*M \wedge L, Z_p) \\
& \downarrow \quad \downarrow \pi_* \\
& Ext_A^{s+1, t_q+1}(Z_p, Z_p) \xrightarrow{p_*}
\end{align*}
\]

induced by (9.1.1). By Prop. 9.2.4, the right group is zero. Since \( (j^!)_* \) \( Ext_A^{s+1, t_q+1}(H^*L, Z_p) \subset Ext_A^{s+1, t_q+1}(Z_p, Z_p) \cong Z_p\{a_0 \sigma = (j^!)_* \pi_*(\sigma)\} \) and \( Ext_A^{s+1, t_q+1}(Z_p, Z_p) = 0 \) then the left group has unique generator \( \pi_*(\sigma) \) and the result follows. Q.E.D.

Now we begin to prove the main Theorem A. The proof will be done by processing an argument processing in the Adams resolution of some spectra related to \( S \). Let

\[
\begin{align*}
& \cdots \xrightarrow{a_2} \Sigma^{-2} E_2 \xrightarrow{a_1} \Sigma^{-1} E_1 \xrightarrow{a_0} E_0 = S \\
& \downarrow \quad \downarrow \quad \downarrow \\
& \Sigma^{-2} KG_2 \xrightarrow{a_2} \Sigma^{-1} KG_1 \xrightarrow{K_0}
\end{align*}
\]

be the minimal Adams resolution of the sphere spectrum \( S \) which satisfies

1. \( E_s \xrightarrow{b_s} KG_s \xrightarrow{c_s} E_{s+1} \xrightarrow{a_s} S \) are cofibrations for all \( s \geq 0 \), which induce short exact sequences in \( Z_p \)-cohomology \( 0 \to H^* E_{s+1} \xrightarrow{c_s^*} H^* KG_s \xrightarrow{b_s^*} H^* E_s \to 0 \).

2. \( KG_s \) is a graded wedge sum of Eilenberg-Maclane spectra \( KZ_p \) of type \( Z_p \).

3. \( \pi_t KG_s \) are the \( E_1^{s,t} \)-terms of the Adams spectral sequence, \( (\bar{b}_s \bar{c}_{s-1})_s : \pi_t KG_{s-1} \to \pi_t KG_s \) is the \( d_1^{s-1,t} \)-differentials of the Adams spectral sequence.

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and $\pi_t KG_s \cong Ext^s_t A(Z_p, Z_p)$ (cf. [3] p.180).

Then, an Adams resolution of an arbitrary spectrum $V$ can be obtained by smashing $V$ to $(9.2.9)$. We first prove some Lemmas.

**Lemma 9.2.10** Let $p \geq 7$, then under the supposition of the main Theorem A we have

$$\bar{c}_{s+1} \cdot h_0 \sigma = (1_{E_{s+2}} \wedge \alpha_1) \kappa \quad \text{(up to scalar)}$$

where $\kappa \in \pi_{tq+1} E_{s+2}$ such that $\bar{c}_{s+1} \cdot \sigma = \bar{a}_{s+1} \cdot \kappa$ and $\bar{b}_{s+2} \cdot \kappa = a_0 \sigma' \in \pi_{tq+1} KG_{s+2} \cong Ext^{s+2, tq+1}_A(Z_p, Z_p)$.

**Proof:** The $d_1$-cycle $(1_{KG_{s+1}} \wedge i'')h_0 \sigma \in \pi_{tq+1} KG_{s+1} \wedge L$ represents an element in $Ext^{s+1, tq+q}_A(H^*L, Z_p)$ and this group is zero by Prop. 9.2.4, then it is a $d_1$-boundary and so $(\bar{c}_{s+1} \wedge 1_L)(1_{KG_{s+1}} \wedge i'')h_0 \sigma = 0$, $\bar{c}_{s+1} \cdot h_0 \sigma = (1_{E_{s+2}} \wedge \alpha_1)f''$ for some $f'' \in \pi_{tq+1} E_{s+2}$. It follows that $\bar{a}_{s+1} \cdot (1_{E_{s+2}} \wedge \alpha_1)f'' = 0$, then $\bar{a}_{s+1} \cdot f'' = (1_{E_{s+1}} \wedge j'')f''_2$ with $f''_2 \in \pi_{tq+q}(E_{s+1} \wedge L)$. The $d_1$-cycle $(\bar{b}_{s+1} \wedge 1_L)f''_2 \in \pi_{tq+q} KG_{s+1} \wedge L$ represents an element in $Ext^{s+1, tq+q}_A(H^*L, Z_p)$ and this group is zero, then $(\bar{b}_{s+1} \wedge 1_L)f''_2 = (\bar{b}_{s+1} \bar{c}_s \wedge 1_L)g''$ with $g'' \in \pi_{tq+q}(KG_{s} \wedge L)$. Hence, $f''_2 = (\bar{c}_s \wedge 1_L)g'' + (\bar{a}_{s+1} \wedge 1_L)f''_3$, for some $f''_3 \in \pi_{tq+q+1} E_{s+2} \wedge L$ and we have $\bar{a}_{s+1} \cdot f'' = \bar{a}_{s+1}(1_{E_{s+2}} \wedge j'')f''_3 + \bar{c}_s(1_{KG} \wedge j'')g'' = \bar{a}_{s+1}(1_{E_{s+2}} \wedge j'')f''_3 + \lambda \bar{c}_s \cdot \sigma = \bar{a}_{s+1}(1_{E_{s+2}} \wedge j'')f''_3 + \lambda \bar{a}_{s+1} \cdot \kappa$ for some $\lambda \in Z_p$. This is because $(1_{KG} \wedge j'')g'' \in \pi_{tq} KG_s \cong Ext^{s+1, tq}_A(Z_p, Z_p) \cong Z_p \{\sigma\}$. Then, $f'' = (1_{E_{s+2}} \wedge j'')f''_3 + \lambda \kappa + \bar{c}_{s+1} \cdot g''_3$ for some $g''_3 \in \pi_{tq+q} KG_{s+1}$ and so $\bar{c}_{s+1} \cdot h_0 \sigma = (1_{E_{s+1}} \wedge \alpha_1) \kappa$ (up to scalar). Q.E.D.

Since $\bar{h} \phi \cdot p = \bar{h} \psi j \alpha^2 i = 0$ (cf. Prop. 9.2.3(1) and (9.1.9)(9.1.5)), then $\bar{h} \phi = (1_Y \wedge j) \alpha_Y M$, where $\alpha_Y M \in [\Sigma^{2q+1} M, Y \wedge M]$. Let $\Sigma U$ be the cofibre of $\bar{h} \phi = (1_Y \wedge j) \alpha_Y M : \Sigma^{2q+1} S \to Y$ given by the cofibration

$$\Sigma^{2q} S \xrightarrow{\bar{h} \sigma} Y \xrightarrow{w_2} \Sigma U \xrightarrow{w_2} \Sigma^{2q+1} S$$

Moreover, $w_2(1_Y \wedge j) \alpha_Y M = \bar{w} \cdot j$, where $\bar{w} : \Sigma^{2q} S \to U$ whose cofibre is $X$ given by the cofibration $\Sigma^{2q} S \xrightarrow{\bar{w}} U \xrightarrow{\bar{v}} X \xrightarrow{j \bar{v}} \Sigma^{2q+1} S$. Then, $\Sigma X$ also is the cofibre of $\omega = (1_Y \wedge j) \alpha_Y M : \Sigma^{2q} M \to Y$ given by the cofibration

$$\Sigma^{2q} M \xrightarrow{(1_Y \wedge j) \alpha_Y M} Y \xrightarrow{\bar{w} w_2} \Sigma X \xrightarrow{\bar{w} \bar{v}} \Sigma^{2q+1} M$$

This can be seen by the following homotopy commutative diagram of $3 \times 3$-Lemma.
Since \( j\pi(\bar{h}\phi) = 0 \), then by (9.2.11) we have, \( j\pi = u_3w_2 \), for some \( u_3 \in [U, \Sigma^{q+1}S] \). Hence, the spectrum \( U \) in (9.2.11) also is the cofire of \( w\pi : \Sigma^qS \to W \) given by the cofibration
(9.2.14) \[ \Sigma^qS \xrightarrow{w\pi} W \xrightarrow{\overline{u}_3} U \xrightarrow{u_3} \Sigma^{q+1}S \]
This can be seen by the following homotopy commutative diagram of 3 × 3-Lemma in the stable homotopy category
(9.2.15) \[
\begin{array}{ccccccc}
\Sigma^{-1}Y & \xrightarrow{j\pi} & \Sigma^{q+1}S & \xrightarrow{w\pi} & \Sigma W \\
\downarrow{w_2} & & \downarrow{u_3} & & \downarrow{\pi} & & \downarrow{w} & & \downarrow{w_3} \\
U & & \Sigma L & & \Sigma U \\
\end{array}
\]
Moreover, by \( u_3\overline{w} = \alpha_1 \), the cofibre of \( \overline{uw}_3 : W \to X \) is \( \Sigma^{q+1}L \) given by the cofibration
(9.2.16) \[ W \xrightarrow{\overline{uw}_3} X \xrightarrow{w'} \Sigma^{q+1}L \xrightarrow{w'(\pi\wedge L)} \Sigma W \]
where \( w' \in [L \wedge L, W] \) such that \( w'(1_L \wedge i'') = w \). This can be seen by the following homotopy commutative diagram of 3 × 3-Lemma in the stable homotopy category
(9.2.17) \[
\begin{array}{ccccccc}
W & \xrightarrow{\overline{uw}_3} & X & \xrightarrow{jv} & \Sigma^{q+1}S \\
\downarrow{w_3} & & \downarrow{u} & & \downarrow{u''} & & \downarrow{j''} \\
U & & \Sigma^{q+1}L \\
\end{array}
\]

**Lemma 9.2.18**  
(1) Let \( \overline{\phi}_W \in [\Sigma^{3q-1}L, W] \) be the map in Prop. 9.2.7 such that \( u\overline{\phi}_W = \overline{\phi} \in [\Sigma^{2q-1}L, L] \), then
(1) \( \overline{uw}_3\overline{\phi}_W (p \wedge 1_L) \neq 0 \in [\Sigma^{3q-1}L, X] \).
(2) \( Ext_A^{s,tq+3q-1}(H^*X, H^*L) = 0 \), \( Ext_A^{s+1,tq+3q}(H^*X, H^*L) = (\overline{uw}_3)_* Ext_A^{s+1,tq+3q}(H^*W, H^*L) \).
Proof: (1) Suppose in contrast that \( \widetilde{uw}_3\phi_W(p \wedge 1_L) = 0 \), then by (9.2.16) and the result of Prop. 9.2.3(1) on \([\Sigma^{2q-1}L, L]\) we have
\[(9.2.19) \quad \phi_W(p \wedge 1_L) = \lambda w'(\pi \wedge 1_L)\phi \mod F_3[\Sigma^{3q-1}L, W] \]
for some \( \lambda \in \mathbb{Z}_p \), where \( F_3[\Sigma^{3q-1}L, W] \) denotes the subgroup of \([\Sigma^{3q-1}L, W]\) consisting by all elements of filtration \( \geq 3 \). Moreover, note that \( uw'(\pi \wedge 1_L) \in [L, L] \) and this group has two generators \((p \wedge 1_L), \pi j''\) which has filtration one, then \( uw'(\pi \wedge 1_L) = \lambda_1(p \wedge 1_L) + \lambda_2 \pi j'' \) with \( \lambda_1, \lambda_2 \in \mathbb{Z}_p \). By (9.1.13) we have \( \lambda_1 p \cdot (\alpha_1)_L + \lambda_2 (\alpha_1)_L \pi j'' = 0 \) so that \( \lambda_2 = \lambda_0 \lambda_1 \), here we use the equation \((\alpha_1)_L \pi j'' = -\lambda_0 p \cdot (\alpha_1)_L, \lambda_0 \neq 0 \in \mathbb{Z}_p \). Then, by composing \( u \) on (9.2.19) we have \( \widetilde{\phi}(p \wedge 1_L) = u_{\phi W}(p \wedge 1_L) = \lambda uw'(\pi \wedge 1_L)\phi = \lambda_1 \phi(p \wedge 1_L) + \lambda_0 \lambda_1 \pi j'' \phi \mod F_3[\Sigma^{3q-1}L, L] \) and so by (9.1.9) \( h_{\phi}(p \wedge 1_L) = \lambda_1 h_{\phi}(p \wedge 1_L) \mod F_3[\Sigma^{2q}L, Y] \). This implies that \( \lambda \lambda_1 = 1 \mod p \) (cf. the following Remark 9.2.20).

Hence we have \( \lambda \lambda_1 \lambda_0 \pi j'' \phi = 0 \mod F_3[\Sigma^{2q-1}L, L] \) and by the same reason as shown in the following Remark 9.2.20, this implies that \( \lambda \lambda_1 \lambda_0 = 0 \mod p \) which yields a contradiction.

(2) Consider the following exact sequence
\[
\begin{align*}
E_{t_A}^{s,tq+3q}(H^*Y, H^*L) & \xrightarrow{\widetilde{uw}_3} E_{t_A}^{s,tq+3q-1}(H^*X, H^*L) \\
& \xrightarrow{(\phi)} E_{t_A}^{s,tq+q-1}(H^*M, H^*L)
\end{align*}
\]
induced by (9.2.12). By Prop. 9.2.7(2), both sides of groups are zero, so that the middle group is zero as desired. Look at the following exact sequence
\[
\begin{align*}
E_{t_A}^{s+1,tq+3q}(H^*W, H^*L) & \xrightarrow{\widetilde{uw}_3} E_{t_A}^{s+1,tq+3q}(H^*X, H^*L) \\
& \xrightarrow{(u')} E_{t_A}^{s+1,tq+2q-1}(H^*L, H^*L)
\end{align*}
\]
induced by (9.2.16). By the supposition on \( E_{t_A}^{s+1,tq+rq-1}(Z_p, Z_p) = 0 \) \((r = 1, 2, 3)\) we know that the right group is zero and so the result follows.
Q.E.D.

Remark 9.2.20  Here we give an explanation on the reason why the coefficient in the equation \((1 - \lambda \lambda_1)h_{\phi}(p \wedge 1_L) = 0 \mod F_3[\Sigma^{2q}L, Y]\) must be zero \(\mod p\). For otherwise, if \(1 - \lambda \lambda_1 \neq 0 \mod p\), then \((1 - \lambda \lambda_1)h_{\phi}(p \wedge 1_L)\) must be represented by some nonzero element \(x \in E_{t_A}^{2q+2}(H^*Y, H^*L)\) in the ASS. However, it also equals to an element of filtration \( \geq 3 \), then \(x\) must be a \(d_2\)-boundary, that is, \(x = d_2(x') \in d_2 E_{t_A}^{2q+1}(H^*Y, H^*L) = 0\), this is because \(E_{t_A}^{2q+1}(H^*Y, H^*L) = Hom_{t_A}^{2q+1}(H^*Y, H^*L) = 0\) which is
obtained by $H^rL \neq 0$ only for $r = 0, q$. This is a contradiction so that we have $1 - \lambda \lambda_1 = 0 \pmod{p}$.

**Lemma 9.2.21** For the element $\kappa \in \pi_{tq+1}E_{s+2}$ in Lemma 9.2.19, it is known that $\bar{a}_{s+1} \cdot \kappa = \bar{c}_{s+1} \cdot \sigma$ and $\bar{b}_{s+2} \cdot \kappa = a_0 \sigma' \in \pi_{tq+1}KG_{s+2} \cong Ext_{\Lambda}^{s+2,tq+1}(Z_p, Z_p)$, then there exists $f \in \pi_{tq+3}E_{s+4} \wedge M$ and $g \in \pi_{tq+1}(KG_{s+1} \wedge M)$ such that

(A) \[ (1_{E_{s+2}} \cdot i)\kappa = (\bar{c}_{s+1} \wedge 1_M)g + (\bar{a}_{s+2} \bar{a}_{s+3} \wedge 1_M)f \]

and

(B) \[ (1_{E_{s+4}} \wedge (1_Y \wedge j)\alpha_{Y \wedge M})f \cdot (\alpha_1)_L = 0 \in [\Sigma^{tq+2}L, E_{s+4} \wedge Y], \]

where $\alpha_{Y \wedge M} \in [\Sigma^{2q+1}M, Y \wedge M]$ satisfying $(1_Y \wedge j)\alpha_{Y \wedge M} = \bar{h} \phi \in \pi_{2q}Y$.

**Proof:** Note that the $d_1$-cycle $(\bar{b}_{s+2} \wedge 1_M)(1_{E_{s+2}} \wedge i)\kappa \in \pi_{tq+1}KG_{s+2} \wedge M$ represents an element $i_*(a_0 \sigma') = i_*p_*(\sigma') = 0 \in Ext_{\Lambda}^{s+2,tq+1}(H^*M, Z_p)$ so that it is a $d_1$-boundary. That is, $(\bar{b}_{s+2} \wedge 1_M)(1_{E_{s+2}} \wedge i)\kappa = (\bar{b}_{s+2} \bar{c}_{s+1} \wedge 1_M)g$ for some $g \in \pi_{tq+1}KG_{s+1} \wedge M$. Then, by $Ext_{\Lambda}^{s+3,tq+2}(H^*M, Z_p) = 0$ (cf. Prop. 9.2.8) we have $(1_{E_{s+2}} \wedge i)\kappa = (\bar{c}_{s+1} \wedge 1_M)g + (\bar{a}_{s+2} \bar{a}_{s+3} \wedge 1_M)f$ for some $f \in \pi_{tq+3}E_{s+4} \wedge M$. This shows (A). For (B), by Prop. 9.2.3(1) we have $\phi \cdot p = i'' j \alpha^2 i$ (up to nonzero scalar), then $\bar{h} \phi \cdot p = \bar{h} i'' j \alpha^2 i = 0$ and so $\bar{h} \phi = (1_Y \wedge j)\alpha_{Y \wedge M}i$ for some $\alpha_{Y \wedge M} \in [\Sigma^{2q+1}M, Y \wedge M]$. Then, by composing $1_{E_{s+2}} \wedge (1_Y \wedge j)\alpha_{Y \wedge M}$ on the equation (A) we have

\((1_{E_{s+2}} \wedge \bar{h} \phi)\kappa = (1_{E_{s+2}} \wedge (1_Y \wedge j)\alpha_{Y \wedge M})i \kappa = \bar{a}_{s+2} \bar{a}_{s+3} \wedge 1_Y)(1_{E_{s+4}} \wedge (1_Y \wedge j)\alpha_{Y \wedge M})f \)

where $(1_Y \wedge j)\alpha_{Y \wedge M}$ induces zero homomorphism in $Z_p$-cohomology so that $(\bar{c}_{s+1} \wedge 1_Y)(1_{KG_{s+1}} \wedge (1_Y \wedge j)\alpha_{Y \wedge M})g = 0$.

By composing $(\alpha_1)_L$ on (9.2.22) we have $(\bar{a}_{s+2} \bar{a}_{s+3} \wedge 1_Y)(1_{E_{s+4}} \wedge (1_Y \wedge j)\alpha_{Y \wedge M})f \cdot (\alpha_1)_L = (\bar{c}_{s+2} \wedge 1_Y)g_1 = 0$.

By composing $(\alpha_1)_L$ on (9.2.22) we have $(\bar{a}_{s+2} \bar{a}_{s+3} \wedge 1_Y)(1_{E_{s+4}} \wedge (1_Y \wedge j)\alpha_{Y \wedge M})f \cdot (\alpha_1)_L = (\bar{c}_{s+2} \wedge 1_Y)g_1 = 0$.

where the $d_1$-cycle $g_1 \in [\Sigma^{tq+2}L, KG_{s+2} \wedge Y]$ represents an element in $Ext_{\Lambda}^{s+2,tq+3}(H^*Y, H^*L)$ and this group is zero (cf. Prop. 9.2.6(1)) so that it is a $d_1$-boundary and we have $(\bar{c}_{s+2} \wedge 1_Y)g_1 = 0$. Briefly write $(1_Y \wedge j)\alpha_{Y \wedge M} = \omega$ and let $V$ be the cofibre of $(1_Y \wedge (\alpha_1)_L)(\omega \wedge 1_L) = \omega \cdot (1_M \wedge (\alpha_1)_L) : \Sigma^{3q-1}M \wedge L \rightarrow Y$ given by the cofibration

\((9.2.23) \quad \Sigma^{3q-1}M \wedge L \xrightarrow{(1_Y \wedge (\alpha_1)_L)(\omega \wedge 1_L)} Y \xrightarrow{u_3} V \xrightarrow{u_4} \Sigma^{3q}M \wedge L \)

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It follows that \((\bar{a}_{s+3} \land 1_Y)(1_{E_{s+4}} \land 1_Y \land (\alpha_1)_L)(\omega \land 1_L)(f \land 1_L) = (\bar{a}_{s+3} \land 1_Y)(1_{E_{s+4}} \land 1_Y \land (\alpha_1 \circ Y \land M)f \cdot (\alpha_1)_L = 0\), then by (9.2.23) we have \((\bar{a}_{s+3} \land 1_M \land L)(f \land 1_L) = (1_{E_{s+3}} \land u_4)f_2\) for some \(f_2 \in [\Sigma^{q+3q+2}L, E_{s+3} \land V]\). Consequently, \((\bar{b}_{s+3} \land 1_M \land L)(1_{E_{s+3}} \land u_4)f_2 = 0\) so that we have

\[(9.2.24) \quad (\bar{b}_{s+3} \land 1_Y)f_2 = (1_{KG_{s+3}} \land w_4)g_2\]

with \(g_2 \in [\Sigma^{q+3q+2}L, KG_{s+3} \land X]\). Then, \((\bar{b}_{s+4} \bar{c}_{s+3} \land 1_Y)(1_{KG_{s+3}} \land w_4)g_2 = 0\) and so \((\bar{b}_{s+4} \bar{c}_{s+3} \land 1_Y)g_2 \in (1_{KG_{s+4}} \land (\alpha_1)_L(\omega \land 1_L))_{[\Sigma^*L, KG_{s+4} \land M \land L]} = 0\). That is, \(g_2\) is a \(d_1\)-cycle which represents an element \([g_2] \in E_{xt^A}^{s+3, t_4 + 3q + 2}(H^* Y, H^* L)\) and this group has two generators as shown in Prop. 9.2.5(2)), then we have

\[(9.2.25) \quad [g_2] = \bar{h}_{\omega} \bar{\phi}_s(\pi \land 1_L)_s(\lambda_1[\sigma_1' \land 1_L] + \lambda_2[\sigma_2' \land 1_L])\]

with \(\lambda_1, \lambda_2 \in Z_p\). By (9.2.24) we know that

\[(w_4)_s[g_2] \in E_{t+3}^{s+3, t_4 + 3q + 2}(V) = E_{xt^A}^{s+3, t_4 + 3q + 2}(H^* V, H^* L)\]

is a permanent cycle in the ASS. However, \((1_Y \land (\alpha_1)_L)(\omega \land 1_L)\) is a map of filtration 2, then the cofibration (9.2.23) induces an exact sequence in \(Z_p\)-cohomology which is split as \(A\)-module. That is, it induces a split exact sequence in the \(E_1\)-term of the ASS : \(E_1^{s+3, s'}(Y) \overset{(w_4)_s}{\longrightarrow} E_1^{s+3, s'}(V) \overset{(u_4)_s}{\longrightarrow} E_1^{s+3, s'-3q}(M \land L)\). It follows that it induces a split exact sequence in the \(E_r\)-term of the ASS for all \((r \geq 2)\)

\[(9.2.26) \quad E_r^{s+3, s'}(Y) \overset{(w_4)_s}{\longrightarrow} E_r^{s+3, s'}(V) \overset{(u_4)_s}{\longrightarrow} E_r^{s+3, s'-3q}(M \land L)\]

Then, \(d_r((w_4)_s[g_2]) = 0\) implies that \(d_r([g_2]) = 0\) \((r \geq 2)\). That is, (9.2.24) implies that \([g_2]\) also is a permanent cycle in the ASS. Since the secondary differential \(d_2([g_2]) = 0\) and \(d_2(\sigma') = a_0 \sigma'\) in which \(\sigma'\) is the linear combination of \(\sigma'_1, \sigma'_2\), then \(\lambda_1, \lambda_2\) linearly dependent. That is, (9.2.25) becomes

\([g_2] = \lambda_1 \bar{h}_{\omega} \bar{\phi}_s(\pi \land 1_L)_s[\sigma' \land 1_L]\).

Now we consider the case \(\lambda_1\) is nonzero or zero respectively.

If \(\lambda_1 \neq 0\), (9.2.24) implies \([g_2]\) and so \(\bar{h}_{\omega} \bar{\phi}_s(\pi \land 1_L)_s[\sigma' \land 1_L] \in E_2^{s+3, t_4 + 3q + 2}(Y) = E_{xt^A}^{s+3, t_4 + 3q + 2}(H^* Y, H^* L)\) is a permanent cycle in the ASS. Moreover, by \((\bar{a}_{s+3} \land 1_Y)(1_{E_{s+4}} \land (1_Y \land j)\alpha_Y \land M)f \cdot (\alpha_1)_L = 0\) we have

\[(1_{E_{s+4}} \land (1_Y \land j)\alpha_Y \land M)f \cdot (\alpha_1)_L = (\bar{c}_{s+3} \land 1_Y)g_3\]

for some \(d_1\)-cycle \(g_3 \in [\Sigma^{q+3q+2}L, KG_{s+3} \land Y]\) and it represents an element \([g_3] \in E_{xt^A}^{s+3, t_4 + 3q + 2}(H^* Y, H^* L)\) so that we have \([g_3] = \bar{h}_{\omega} \bar{\phi}_s(\pi \land 1_L)_s[\lambda_3[\sigma'_1 \land 1_L] + \lambda_4[\sigma'_2 \land 1_L]]\) for some \(\lambda_3, \lambda_4 \in Z_p\). By the above equa-
tion and \((1Y \wedge (\alpha_1)_L)(\omega \wedge 1_L)\) has filtration 2 we know that the secondary
differential \(d_2([g_3]) = 0\) so that by the similar reason as above, \(\lambda_3, \lambda_4\) is
linearly dependent. That is, \([g_3] = \lambda_3 \bar{h}_s \bar{\phi}_s(\pi \wedge 1_L)_s[\sigma' \wedge 1_L]\) so that we have
\((1_{E_{s+4}} \wedge (1Y \wedge (\alpha_1)_L)(\omega \wedge 1_L))(f \wedge 1_L) = (\bar{c}_{s+3} \wedge 1_Y)g_3 = 0\) and the result
follows.

If \(\lambda_1 = 0\), then \(g_2 = (\bar{b}_{s+3} \bar{c}_{s+2} \wedge 1_Y)g_4\) for some \(g_4 \in [\Sigma^{q+3q+2}L, KG_{s+2} \wedge 
Y]\) and (9.2.24) becomes \((\bar{b}_{s+3} \wedge 1_Y)f_2 = (\bar{b}_{s+3} \bar{c}_{s+2} \wedge 1_Y)(1_{KG_{s+2}} \wedge w_4)g_4\).
Consequently we have \(f_2 = (\bar{c}_{s+2} \wedge 1_Y)(1_{KG_{s+2}} \wedge w_4)g_4 + (\bar{a}_{s+3} \wedge 1_Y)f_3\) with \(f_3 \in [\Sigma^{q+3q+3}L, E_{s+4} \wedge V]\) and so \((\bar{a}_{s+3} \wedge 1_{M \wedge L})(f \wedge 1_L) = (1_{E_{s+4}} \wedge u_4)f_2 =
(\bar{a}_{s+3} \wedge 1_{M \wedge L})(1_{E_{s+4}} \wedge u_4)f_3\). Hence, \((f \wedge 1_L) = (1_{E_{s+4}} \wedge u_4)f_3 + (\bar{c}_{s+3} \wedge 1_{M \wedge L})g_5\) for some \(g_5 \in [\Sigma^{q+3q+3}L, KG_{s+3} \wedge M \wedge L]\) and so by (9.2.23) we have \((1_{E_{s+4}} \wedge (1Y \wedge (\alpha_1)_L)(\omega \wedge 1_L))(f \wedge 1_L) = (\bar{c}_{s+3} \wedge 1_Y)(1_{KG_{s+3}} \wedge (1Y \wedge (\alpha_1)_L)(\omega \wedge 1_L))g_5 = 0\) (this is because \((\alpha_1)_L\) induces zero homomorphism is \(Z_p\)-cohomology).
Q.E.D.

**Proof of the main Theorem A:** We will continue the argument in Lemma 9.2.21. Note that the spectrum \(V\) in (9.2.23) also is the cofibre of
\((1_M \wedge w^n)\bar{\psi} : X \twoheadrightarrow \Sigma^{2q}M \wedge W\) given by the cofibration
(9.2.27) \(X (1_M \wedge w^n)\bar{\psi} \Sigma^{2q}M \wedge W \xrightarrow{u_2} \Sigma X \)
this can be seen by the following homotopy commutative diagram of \(3 \times 3\)-
Lemma
\[(\Sigma^{3q-1}M \wedge L \quad \longrightarrow \quad Y \quad \longrightarrow \quad \Sigma X)
\]
\[\langle 1_M \wedge (\alpha_1)_L \rangle \quad \langle \omega \rangle \quad \langle w_4 \rangle \quad \langle u_5 \rangle \quad \langle \bar{\psi} \rangle \]

(9.2.28) \(\Sigma^{2q}M \quad V \quad \Sigma^{2q+1}M \)
\[\langle \bar{\psi} \rangle \quad \langle 1_M \wedge w^n \rangle \quad \langle w_5 \rangle \quad \langle u_4 \rangle \quad \langle 1_M \wedge (\alpha_1)_L \rangle \]
\(X \longrightarrow \Sigma^{2q}M \wedge W \xrightarrow{1_M \wedge u} \Sigma^{3q}M \wedge L \)

By Lemma 9.2.21(B) and (9.2.23), \(f \wedge 1_L = (1_{E_{s+4}} \wedge u_4)f_5\) for some \(f_5 \in [\Sigma^{q+3q+3}L, E_{s+4} \wedge V]\) and so by Lemma 9.2.21(A) we have
(9.2.29) \((\bar{a}_{s+2} \bar{a}_{s+3} \wedge 1_{M \wedge L})(1_{E_{s+4}} \wedge u_4)f_5 = (\bar{a}_{s+2} \bar{a}_{s+3} \wedge 1_{M \wedge L})(f \wedge 1_L)
= (1_{E_{s+4}} \wedge i \wedge 1_L)(g \wedge 1_L) - (1_{E_{s+4}} \wedge u_{4})f_5\)
It follows that \((\bar{a}_{s} \bar{a}_{s+1} \bar{a}_{s+3} \wedge 1_{M \wedge L})(1_{E_{s+4}} \wedge u_4)f_5 = 0\) and so \((\bar{a}_{s} \bar{a}_{s+1} \bar{a}_{s+2} \bar{a}_{s+3} \wedge 1_{M \wedge L})(f \wedge 1_L) = 0\) and by \(Ext_{A}^{s+1+r,tq+3q+r}(H^*Y, \quad \Sigma X)\)
$H^*L) = 0 (r = 0, 1$, cf. Prop. 9.2.6) we have $(\bar{a}_s \bar{a}_{s+1} \bar{a}_{s+2} \bar{a}_{s+3} \wedge 1_V)f_5 = (\bar{a}_s \bar{a}_{s+1} \\
\bar{a}_{s+2} \wedge 1_V)(1_{E_{s+3}} \wedge w_4)f_7$, with $f_7 \in [\Sigma^{q+3q+2}L, E_{s+3} \wedge Y]$. Consequently we have

(9.2.30) \[(\bar{a}_s+1 \bar{a}_{s+2} \bar{a}_{s+3} \wedge 1_V)f_5 \]

with $d_1$-cycle $g_6 \in [\Sigma^{q+3q+2}L, KG_s \wedge V]$ which represents an element $[g_6] \in Ext_{A}^{s, t q+3 q}(H^*V, H^*L)$. Note that the $d_1$-cycle $(\bar{b}_{s+3} \wedge 1_Y)f_7 \in [\Sigma^{q+3 q+2}L, KG_{s+3} \wedge Y]$ represents an element $[(\bar{b}_{s+3} \wedge 1_Y)f_7] \in Ext_{A}^{s, t q+3 q+2}(H^*Y, H^*L)$ which has two generators (cf. Prop. 9.2.5(2)), then $[(\bar{b}_{s+3} \wedge 1_Y)f_7] = \lambda^h \bar{\phi}_s(\pi \wedge 1_L)s[\sigma'_1 \wedge 1_L] + \lambda^h \bar{\phi}_s(\pi \wedge 1_L)s[\sigma'_2 \wedge 1_L]$ for some $\lambda', \lambda'' \in Z_P$. By the vanishes of the secondary differential : $0 = d_2[(\bar{b}_{s+3} \wedge 1_Y)f_7]$ we know that $\lambda', \lambda''$ is linearly dependent. Then we have

(9.2.31) \[ [(\bar{b}_{s+3} \wedge 1_Y)f_7] = \lambda^h \bar{\phi}_s(\pi \wedge 1_L)s[\sigma'_1 \wedge 1_L] \]

\[ \in Ext_{A}^{s+3. t q+3 q+2}(H^*Y, H^*L) \]

We claim that the scalar $\lambda$ in (9.2.31) is zero. This can be proved as follows.

The equation (9.2.30) means that the secondary differential of the ASS $d_2[g_6] = 0 \in E_2^{s+2. t q+3 q+1}(L, V) = Ext_A^{s+2. t q+3 q+1}(H^*V, H^*L)$, then $[g_6] \in E_3^{s, t q+3 q}(L, V)$ and

The third differential $d_3[g_6] = (w_4)_s[(\bar{b}_{s+3} \wedge 1_Y)f_7] \in E_3^{s+3. t q+3 q+2}(L, V)$

Note that $(\omega \wedge 1_L)(1_M \wedge (\alpha_1)_L)(i \wedge 1_L) \pi = (1_Y \wedge j) \gamma Y \wedge M i(\alpha_1)_L \pi = \bar{\phi}(\alpha_1)_L \pi = 0$, this is because the $\phi(\alpha_1)_L \in [\Sigma^{q-2}L, L] = 0$ which is obtained by $\pi_{r-2}S = 0 (r = 2, 3, 4)$. Hence, by (9.2.23), $(i \wedge 1_L) \pi = u_4 \tau$ with $\tau \in [\Sigma^{q}S, V]$ which has filtration 1. Moreover, $u_4 \tau \cdot \tau = (i \wedge 1_L) \pi \cdot \tau = 0$, then by Prop. 9.2.3(4), $\tau \cdot p = \lambda w_4 \bar{\phi}(\pi \wedge 1_L) \pi$ for some $\lambda \in Z_P$. This scalar $\lambda$ must be zero (mod $p$), this is because the left hand side of the equation has filtration 2 and the right hand side has filtration 3 (cf. Remark 9.2.20 and $Ext_A^{0. 4 q+1}(H^*V, Z_P) = 0$ which is obtained by $Ext_A^{0. 4 q+1}(H^*Y, Z_P) = 0 = Ext_A^{0, q+1}(H^*M \wedge L, Z_P)$). Consequently, by Prop. 9.2.3(4) we have $\tau \cdot p = 0$ and so $\tau = \tau \pi \in [\Sigma^{q}M, V]$. Since $(u_4)_s[\pi]^*[g_6] \in Ext_A^{s+1, t q+q+1}(H^*M \wedge L, Z_P) \cong Z_P \{ (i \wedge 1_L)s, (\gamma Y \wedge M i(\alpha_1)_L) \}$ (cf. Prop. 9.2.8), then $(u_4)_s[\pi]^*[g_6] = \lambda_0 i(\alpha_1, 1_L)s(\pi) = \lambda_0 (u_4)_s[\tau \pi s] \pi \sigma = \pi (u_4)_s[\tau \pi s] \pi \sigma$ for some $\lambda_0 \in Z_P$ and so by (9.2.23) we have $\pi^*[g_6] = \lambda_0 \tau \pi s, (\gamma Y \wedge M i(\alpha_1)_L) \in Ext_A^{s+1, t q+3 q+1}(H^*V, Z_P)$, this is because $Ext_A^{s+1, t q+3 q+1}(H^*Y, H^*L) = 0$ (cf. Prop. 9.2.6). By the supposition on $d_2[\sigma] = a_0 \sigma' =
Prop. 9.2.7(1), $\pi_*^*(\sigma') \in \text{Ext}_{A}^{s+2,tq+1}(Z_p, Z_p)$, we have $d_2i_*(\sigma) = 0$ so that $i_*(\sigma) \in E_3^{s+2,tq+1}(S, M)$. Moreover, $E_3^{s+3,tq+2}(S, M) = \text{Ext}_{A}^{s+3,tq+2}(H^*M, Z_p) = 0$ (cf. Prop. 9.2.8) then $E_3^{s+3,tq+2}(S, M) = 0$ so that the third differential $d_3i_*(\sigma) \in E_3^{s+3,tq+2}(S, M) = 0$. Since $\pi_*^*[g_6] = \lambda_0(\pi_*)i_*(\sigma) \in E_2^{s+1,tq+4q+1}(S, V)$, then $\pi_*^*[g_6] = \lambda_0(\pi_*)i_*(\sigma)) \in E_3^{s+1,1q+4q+1}(S, V)$ and so

$$d_3\pi_*^*[g_6] = \lambda_0d_3(\pi_*)i_*(\sigma)) = \lambda_0(\pi_*)d_3(i_*(\sigma)) = 0 \in E_3^{s+4,tq+4q+3}(S, V)$$

Hence, $(w_4)_*\pi_*^*[(\bar{b}_{s+3} \wedge 1\gamma)f_7] = d_3\pi_*^*[g_6] = 0 \in E_3^{s+4,tq+4q+3}(S, V)$. In addition, by the split exact sequence (9.2.26) we have $\pi_*^*[(\bar{b}_{s+3} \wedge 1\gamma)f_7] = 0 \in E_3^{s+4,tq+4q+3}(S, Y)$. Then, in the $E_2$-term, $\pi_*^*[(\bar{b}_5 \wedge 1\gamma)f_7]$ must be a $d_2$-boundary, that is

$$\pi_*^*[(\bar{b}_{s+3} \wedge 1\gamma)f_7] \in \delta d_2E_2^{s+2,tq+4q+2}(S, Y) = d_2\text{Ext}_{A}^{s+2,tq+4q+2}(H^*Y, Z_p) = 0$$

(cf. Prop. 9.2.6(1)). Hence, by (9.2.31), $\lambda'\bar{h}_s\phi_*(\pi \wedge 1L)_*\pi_*(\sigma') = 0$. This implies that the scalar $\lambda'$ is zero (cf. Prop. 9.2.9(3)) which shows the above claim.

So, (9.2.30) becomes $(\bar{a}_{s+1}\bar{a}_{s+2}\bar{a}_{s+3} \wedge 1\gamma)f_5 = (\bar{a}_{s+1}\bar{a}_{s+2}\bar{a}_{s+3} \wedge 1\gamma)(1_{E_{s+4}} \wedge w_4)f_8 + (\bar{c}_s \wedge 1\gamma)g_6$ for some $f_8 \in [\Sigma^{tq+3q+3}L, E_{s+4} \wedge Y]$. By composing $1_{E_{s+4}} \wedge u_5$ on the above equation we have $(\bar{a}_{s+1}\bar{a}_{s+2}\bar{a}_{s+3} \wedge 1\gamma)(1_{E_{s+4}} \wedge u_5)f_5 = (\bar{a}_{s+1}\bar{a}_{s+2}\bar{a}_{s+3} \wedge 1\gamma)(1_{E_{s+4}} \wedge \bar{u}w_2)f_8$ (cf. (9.2.28)), this is because $(1KG_s \wedge u_5)g_6 \in [\Sigma^{tq+3q-1}L, KG_s \wedge X]$ represents an element in $\text{Ext}_{A}^{s+2,tq+3q-1}(H^*X, H^*L) = 0$ (cf. Lemma 9.2.18(2)) so that it is a $d_1$-boundary and $(\bar{c}_s \wedge 1\gamma)(1KG_s \wedge u_5)g_6 = 0$. Consequently we have

$$\begin{align*}
(9.2.32) \quad (\bar{a}_{s+2}\bar{a}_{s+3} \wedge 1\gamma)(1_{E_{s+4}} \wedge u_5)f_5 \\
= (\bar{a}_{s+2}\bar{a}_{s+3} \wedge 1\gamma)(1_{E_{s+4}} \wedge \bar{u}w_2)f_8 + (\bar{c}_{s+1} \wedge 1\gamma)g_7
\end{align*}$$

for some $d_1$-cycle $g_7 \in [\Sigma^{tq+3q+1}L, KG_{s+1} \wedge X]$ such that $[g_7] \in \text{Ext}_{A}^{s+1,tq+3q}(H^*X, H^*L)$.

Now we prove $(\bar{c}_{s+1} \wedge 1\gamma)g_7 = 0$ as follows. By Lemma 9.2.18(2) and Prop. 9.2.7(1), $[g_7] = \lambda_3(\bar{u}w_3)_*(\pi_*(\sigma_0 \wedge 1L)$ and the equation (9.2.32) means that the secondary differential $d_2[g_7] = 0$. Since $d_2(\sigma) = a_0(\sigma') = p_*(\sigma') \in \text{Ext}_{A}^{s+2,tq+1}(Z_p, Z_p)$, then $\lambda_3(\bar{u}w_3)_*(\pi_*(\sigma_0 \wedge 1L) = d_2[g_7] = 0 \in \text{Ext}_{A}^{s+3,tq+3q+1}(H^*X, H^*L)$. By Lemma 9.2.18(1), this implies that $\lambda_3 = 0$ so that $g_7$ is a $d_1$-boundary and $(\bar{c}_{s+1} \wedge 1\gamma)g_7 = 0$.

Hence, (9.2.32) becomes $(\bar{a}_{s+2}\bar{a}_{s+3} \wedge 1\gamma \wedge W)(1_{E_{s+4}} \wedge u_5)f_5 = (\bar{a}_{s+2}\bar{a}_{s+3} \wedge 1\gamma \wedge 1\gamma)(1_{E_{s+4}} \wedge \bar{u}w_2)f_8$ and so by (9.2.28)(9.2.12) we have, $(\bar{a}_{s+2}\bar{a}_{s+3} \wedge 1M)(1_{E_{s+4}} \wedge 1\gamma \wedge 1\gamma)(1_{E_{s+4}} \wedge \bar{u}w_2)f_8 = 0$. On the other
hand, by composing \((1_{E^{s+2}} \wedge 1_M \wedge (\alpha_1)L)\) on the equation (9.2.29) we have
\[
(1_{E^{s+2}} \wedge i)\kappa \cdot (\alpha_1)L = (1_{E^{s+2}} \wedge 1_M \wedge (\alpha_1)L)(1_{E^{s+2}} \wedge i \wedge 1_L)(\kappa \wedge 1_L) = (\bar{a}_{s+2} \bar{a}_{s+3} \wedge 1_M)(1_{E^{s+4}} \wedge (1_M \wedge (\alpha_1)L)t_4)f_5 = 0.
\]

It follows that
\[
(9.2.33) \quad \kappa \cdot (\alpha_1)L = (1_{E^{s+2}} \wedge p)f_9
\]
for some \(f_9 \in [\Sigma^{q+q}L, E_{s+2}]\). Since \(\bar{b}_{s+2} \cdot \kappa = a_0\sigma' = p_*(\sigma') = Ext_A^{s+2, t_4+1}(Z_p, Z_p)\), then \(\kappa \cdot (\alpha_1)L\) lifts to a map \(\tilde{f} \in [\Sigma^{q+q+1}L, E_{s+3}]\) such that \(\bar{b}_{s+3} \cdot \tilde{f}\)
represents \(p_*((\alpha_1)L)_*[\sigma' \wedge 1_L] \neq 0 \in Ext_A^{s+3, t_4+q+1}(Z_p, H^*L)\) (cf. Prop. 9.2.3(1)). Then, by (9.2.33), \(p_*[\bar{b}_{s+2} \cdot f_9] = p_*((\alpha_1)L)_*[\sigma' \wedge 1_L]\) so that \([\bar{b}_{s+2} \cdot f_9] \in Ext_A^{s+2, t_4+q}(Z_p, H^*L)\) must equal to \(((\alpha_1)L)_*[\sigma' \wedge 1_L]\), this is because the group has two generators \(((\alpha_1)L)_*[\sigma'_1 \wedge 1_L], ((\alpha_1)L)_*[\sigma'_2 \wedge 1_L]\).

Write \(\xi_{n,s+2} = fg^\nu\), then
\[
(9.2.34) \quad \kappa \cdot \alpha_1 = (1_{E^{s+2}} \wedge p)\xi_{n,s+2}
\]
such that \(\bar{b}_{s+2} \cdot \xi_{n,s+2} = h_0\sigma' \in Ext_A^{s+2, t_4+q}(Z_p, Z_p)\) and by Lemma 9.2.10 we have \((\tilde{c}_{s+1} + 1_M)(1_{KG_{s+1}} \wedge i)h_0\sigma = (1_{E^{s+2}} \wedge i)\kappa \cdot \alpha_1 = 0\). This shows the second result of the main Theorem.

In addition, by (9.2.34) and Lemma 9.2.10(2), \(\bar{a}_0\bar{a}_1 \cdots \bar{a}_{s+1}(1_{E^{s+2}} \wedge p)\xi_{n,s+2} = 0\), this shows that \(\xi_n = \bar{a}_0\bar{a}_1 \cdots \bar{a}_{s+1} \cdot \xi_{n,s+2} \in \pi_{t_4+q-s-2}S\) is an element of order \(p\) and it is represented by \(h_0\sigma' \in Ext_A^{s+2, t_4+q}(Z_p, Z_p)\) in the ASS. Q.E.D.

**Remark 9.2.35** In the proof of the main Theorem A, we obtain a stronger result. By (9.2.33), \(\kappa \cdot (\alpha_1)L = (1_{E^{s+2}} \wedge p)f_9\), then \((1_{E^{s+2}} \wedge i)\kappa \cdot (\alpha_1)L = 0\), and so \((1_{E^{s+2}} \wedge 1_L \wedge i)(\kappa \wedge 1_L)\tilde{f}' = (1_{E^{s+2}} \wedge 1_L \wedge i)(\kappa \wedge 1_L)\tilde{f}'' = 0\), where \(\tilde{f}'' \in \pi_{t_4}L \wedge L\) such that \(((\alpha_1)L \wedge 1_L)\tilde{f}'' = 0\). It can be easily proved that \(\kappa \cdot 1_L)\tilde{f} = (\tilde{c}_{s+1} + 1_L)\sigma\phi, where \(\sigma\phi \in \pi_{t_4+q}(KG_{s+1} \wedge L)\) is a \(d_1\)-cycle which represents \((\phi)_*(\sigma) \in Ext_A^{s+1, t_4+2q}(H^*L, Z_p)\). Then we obtain that \((\tilde{c}_{s+1} + 1_L)\tilde{f}'_1 = (1_{KG_{s+1}} \wedge i)\tilde{f}'' = 0\). That is to say, \((1_L \wedge i)_*(\phi)_*(\sigma) \in Ext_A^{s+1, t_4+2q}(H^*L \wedge M, Z_p)\) is a permanent cycle in the ASS. Moreover, by (9.2.34) we have \(\xi_{n,s+2} = fg^\nu\), then \((1_{KG_{s+2}} \wedge \alpha_1)\xi_{n,s+2} = (1_{KG_{s+2}} \wedge \alpha_1)f_9g^\nu = f_9g^\nu \cdot \alpha_1 = 0\) and so \(\xi_{n,s+2} = (1_{E^{s+2}} \wedge f_9g^\nu)\tilde{f}_9\) with \(\tilde{f}_9 \in \pi_{t_4+2q}E_{s+2} \wedge L\). Since \(\xi_{n,s+2}\) is represented by \(h_0\sigma' = (j'')_*(\phi)_*(\sigma')\) in the ASS, then \(\tilde{f}_9\) is represented by \((\phi)_*(\sigma') \in Ext_A^{s+2, t_4+2q}(H^*L, Z_p)\) in the ASS. That is to say \((\phi)_*(\sigma') \in Ext_A^{s+2, t_4+2q}(H^*L, Z_p)\) is a permanent cycle in the ASS. This is a
stronger result obtained in the main Theorem A.

§3. A general result on convergence in the spectrum $V(1)$

In this section we will prove, under some suppositions, a general result on the convergence of $i'_* i^*(h_0\sigma) \in Ext_{A+tq}^{s+1,tq+q}(H^*V(1),Z_p)$ to the homotopy groups of the spectrum $V(1)$ can implies the convergence of $i'_* i^*(g_0\sigma) \in Ext_{A}^{s+2,tq+pq+2q}(H^*V(1),Z_p)$ in the ASS. We have the following main theorem.

The main Theorem B (generalization of [7] Theorem II) Let $p \geq 5, s \leq 4$, then under the supposition of the main Theorem B we have

(I) $Ext_{A}^{s+1,tq+r}(Z_p, Z_p) = 0$, for $r = 1, u = -1, 1, 2, 3$ or $r = 2, u = -1, 0, 1, 2, 3$.

$Ext_{A}^{s+1,tq}(Z_p, Z_p)$ is zero or has (one or two) generator $\sigma'$ satisfying $a_0\sigma' \neq 0$.

$Ext_{A}^{s+1,tq+r}(Z_p, Z_p) = 0$ for $r = -2, -1, 2, 3$ and has unique generator $a_0\sigma$ for $r = 1$ satisfying $a_0^2\sigma \neq 0$.

$Ext_{A}^{s,tq+q}(Z_p, Z_p) = 0$ or $Z_p\{h_0\tau\}, Ext_{A}^{s,tq+1}(Z_p, Z_p) = 0$ or $Z_p\{a_0\tau\}$.

$Ext_{A}^{s,tq+r+u}(Z_p, Z_p) = 0$, $r = 1, u = 1, 2, r = -1, u = -1, 0$.

(II) $i'_* i_*(h_0\sigma) \in Ext_{A}^{s+1,tq+q}(H^*K, Z_p)$ is a permanent cycle in the ASS, then $i'_* i_*(g_0\sigma) \in Ext_{A}^{s+2,tq+pq+2q}(H^*K, Z_p)$ also is a permanent cycle in the ASS and it converges to a nontrivial element in $\pi_{tq+pq+2q-s-2K}$.

To prove the main Theorem B, we need some knowledge on derivation of maps between $M$-module spectra and some lower dimensional Ext groups. These preminilaries will be used in the proof of the main Theorem B and especially in the proof of Theorem 9.3.9 below.

Prop. 9.3.0 Let $p \geq 5, s \leq 4$, then under the supposition of the main Theorem B we have

1) $Ext_{A}^{s+1,tq+r}(H^*M, H^*M) = 0$ for $r = 1, 2$.

2) $Ext_{A}^{s+1,tq+r}(Z_p, H^*M) = 0$ for $r = 0, 1$.

$Ext_{A}^{s,tq+r}(H^*M, Z_p) = 0$ for $r = 1, 2$.
Theorem B we have $\exists H = 0$ so that $\text{im} i = 0$. By the supposition, the right group is zero for $r = 3$ and has unique generator $a_0 \sigma = 0$. Then, the above $p_*$ is monic so that $\text{im} j = 0$. This shows that the middle group is zero which shows the first result.

The second result can be obtained immediately by the first result.

(2) Consider the following exact sequence $(r = 0, 1)$

$$Ex_t^{s+1, t+1}(Z_p, Z_p) \xrightarrow{j^*} Ex_t^{s+1, t+1}(Z_p, H^* M) \xrightarrow{i^*} Ex_t^{s+1, t+1}(Z_p, Z_p) \xrightarrow{p^*}$$

induced by (9.1). By the supposition, the left group is zero for $r = 1$ and has unique generator $a_0 \sigma = p^*(\sigma)$ for $r = 0$ so that $\text{im} j = 0$. The right group is zero for $r = 0$ or has (one or two) generator $\sigma'$ satisfying $p^*(\sigma') = a_0 \sigma' \neq 0$. Then $\text{im} i^* = 0$ so that the middle group is zero as desired. The proof of the second result is similar. Q.E.D.

Proposition 9.3.1 Let $p \geq 5, s \leq 4$, then under the supposition of the main Theorem B we have

(1) $Ex_t^{s, t}(H^* M, H^* M) \cong Z_p \{\tilde{\alpha}\}$ satisfying $i^*(\tilde{\alpha}) = i_*(\sigma) \in Ex_t^{s, t}(H^* M, Z_p)$. $j^*(\tilde{\sigma}) = j^*(\sigma) \in Ex_t^{s, t-1}(Z_p, H^* M)$.

(2) $Ex_t^{s+1, t+1}(H^* M, H^* M) \cong Z_p \{\alpha_1, \alpha_2, \alpha_3\}$.

(3) $Ex_t^{s+1, t+q+1}(H^* M, H^* M) \cong Z_p \{\alpha_4, \alpha_5\}$.

(4) $Ex_t^{s+1, t+q+1}(H^* K, H^* M) \cong Z_p \{\alpha_6, \alpha_7\}$.

where $\alpha_1 = 0$, $S \to S$ and $\alpha_2 = 0$. $Ex_t^{s, t}(H^* M, H^* M) \to Ex_t^{s+1, t+q+1}(H^* M, H^* M)$ is the connecting (or boundary) homomorphism induced by $\alpha : \Sigma^2 M \to M$.

Proof: (1) Consider the following exact sequence

$$0 = Ex_t^{s, t+1}(H^* M, Z_p) \xrightarrow{j^*} Ex_t^{s, t+1}(H^* M, H^* M) \xrightarrow{i^*} Ex_t^{s, t}(H^* M, Z_p) \xrightarrow{p^*}$$

induced by (9.1). The right group has unique generator $i_*(\sigma)$, this is because $Ex_t^{s, t}(Z_p, Z_p) = 0$ for $r = 1$ and has unique generator $\sigma$ for $r = 0$. Moreover, $p^* i_*(\sigma) = i_*(a_0 \sigma) = i_*(a_0 \sigma) = 0$, then the
middle has unique generator $\tilde{\sigma}$ such that $\iota^* (\tilde{\sigma}) = i_\ast (\sigma)$. This shows the result and the second relation can be similarly proved.

(2) By the supposition, $\text{Ext}^{s+1, lq+q}(\mathbb{Z}_p, \mathbb{Z}_p)$ has unique generator $h_0 \sigma = j_\ast \alpha_\ast i_\ast (\sigma) = j_\ast \alpha_\ast i^* (\tilde{\sigma})$, then the result follows by the following exact sequence

$$\text{Ext}^{s+1, lq+q+1}(\mathbb{H}^s, \mathbb{Z}_p) \xrightarrow{\iota^*} \text{Ext}^{s+1, lq+q}(\mathbb{H}^s, \mathbb{H}^s)$$

induced by (9.1.1), where the right group has unique generator $i^* (ij)_\ast \alpha_\ast (\tilde{\sigma}) = (ij)_\ast \alpha_\ast i_\ast (\sigma)$ satisfying $p^* (ij)_\ast \alpha_\ast i_\ast (\sigma) = 0$ and the left group has unique generator $\alpha_\ast i_\ast (\sigma) = i^* \alpha_\ast (\tilde{\sigma})$.

(3) Consider the following exact sequence

$$\text{Ext}^{s+2, lq+q+2}(\mathbb{H}^s, \mathbb{Z}_p) \xrightarrow{\iota^*} \text{Ext}^{s+1, lq+q+1}(\mathbb{H}^s, \mathbb{H}^s) \mathbb{Z}_p \xrightarrow{p^*} \text{Ext}^{s+1, lq+q+1}(\mathbb{H}^s, \mathbb{H}^s)$$

induced by (9.1.1). The left group is zero, this is because by the supposition, $\text{Ext}^{s+1, lq+q+r}(\mathbb{Z}_p, \mathbb{Z}_p) = 0$ for $r = 1, 2, 3$. The right group has unique generator $(\alpha i)_\ast (\sigma) = i^* \alpha_\ast (\tilde{\sigma})$, this is because $\text{Ext}^{s+1, lq+q+r}(\mathbb{Z}_p, \mathbb{Z}_p)$ is zero for $r = 1$ and has unique generator $h_0 \sigma = j_\ast (\alpha i)_\ast (\sigma)$ for $r = 0$. Since $p^* (\alpha i)_\ast (\sigma) = (\alpha i)_\ast p_\ast (\sigma) = 0$, then the middle group has unique generator $\alpha_\ast (\tilde{\sigma})$ as desired. Moreover we have $\alpha_\ast (\tilde{\sigma}) = \alpha^\ast (\tilde{\sigma})$, this is because $i^* j_\ast \alpha_\ast (\tilde{\sigma}) = j_\ast \alpha_\ast i_\ast (\sigma) = h_0 \sigma = (ja i)^\ast (\sigma) = i^* j_\ast \alpha^\ast (\tilde{\sigma})$.

(4) Consider the following exact sequence

$$\text{Ext}^{s+1, lq+q}(\mathbb{H}^s, \mathbb{H}^s) \xrightarrow{\iota^*} \text{Ext}^{s+1, lq+q}(\mathbb{H}^s, \mathbb{H}^s) \xrightarrow{\alpha^*} \text{Ext}^{s+1, lq-1}(\mathbb{H}^s, \mathbb{H}^s)$$

induced by (9.1.2). By the supposition, $\text{Ext}^{s+1, lq-r}(\mathbb{Z}_p, \mathbb{Z}_p) = 0$ for $r = 1, 2$ and has unique generator $\sigma'$ for $r = 0$, then the right group has unique generator $(ij)^\ast (\tilde{\sigma}')$ satisfying $\alpha_\ast (ij)^\ast (\tilde{\sigma}') = j^\ast \alpha_\ast i_\ast (\sigma') \neq 0 \in \text{Ext}^{s+2, lq+q}(\mathbb{H}^s, \mathbb{H}^s)$. Hence, $\text{Ext}^{s+1, lq+q}(\mathbb{H}^s, \mathbb{H}^s) = \iota^\ast \text{Ext}^{s+1, lq+q}(\mathbb{H}^s, \mathbb{H}^s)$ has unique generator $(ij)_\ast \alpha_\ast \tilde{\sigma} = i^\ast (\alpha_1 \wedge \mathbb{M})_\ast (\tilde{\sigma})$, this is because $(\alpha_1 \wedge \mathbb{M})_\ast (\tilde{\sigma}) = (ij)_\ast \alpha_\ast (\tilde{\sigma}) - \alpha_\ast (ij)_\ast (\tilde{\sigma})$ which is obtained by $\alpha_1 \wedge \mathbb{M} = i^\ast \alpha - \alpha i j$. Q.E.D.

**Proposition 9.3.2** Let $p \geq 5, s \leq 4$, then under the supposition of the main Theorem B we have

(1) $\text{Ext}^{s+1, lq+2q+q}(\mathbb{H}^s, \mathbb{H}^s) = 0, r = 0, 1, 2,$
\[ Ext_A^{s+1,tq+2q+1}(H^*K, Z_p) = 0. \]

(2) \[ Ext_A^{s+1,tq+q+r}(H^*K, Z_p) = 0, \quad r = 1, 2, 3, \]
\[ Ext_A^{s+1,tq+q+r}(H^*K, H^*M) = 0, \quad r = 1, 2. \]

(3) \[ Ext_A^{s+1,tq+q}(H^*K, H^*K) \cong Z_p\{(h_0\sigma)'\} \] with \((i'\sigma)(h_0\sigma)' = (i'j\alpha)_s(\tilde{\sigma}).\]

**Proof:**

(1) Consider the following exact sequence
\[ Ext_A^{s+1,tq+2q+r}(H^*M, H^*M) \xrightarrow{j_*} Ext_A^{s+1,tq+2q+r-1}(H^*M, H^*M) \xrightarrow{a_*} \]
induced by (9.1.2). The left group is zero by the supposition on \( Ext_A^{s+1,tq+2q+u}(Z_p, Z_p) = 0 \) for \( u = -1, 0, 1, 2 \). The right group has unique generator \((ij)_s\alpha_s(\tilde{\sigma})\) for \( r = 0 \) and is generated by two generators \((ij)_s\alpha_s(\tilde{\sigma}) \), \((ij)_s\alpha_s(\tilde{\sigma})\) for \( r = 1 \). Moreover, the right group has unique generator \((ij)_s\alpha_s(\tilde{\sigma})\) for \( r = 2 \) (cf. Prop. 9.3.1(3)). We claim that (i) \( \alpha_s(ij)_s\alpha_s(\tilde{\sigma}) \neq 0 \).

(ii) \( \alpha_s[\lambda_1(ij)_s\alpha_s(\tilde{\sigma}) + \lambda_2\alpha_s(ij)_s(\tilde{\sigma})] \neq 0. \) (iii) \( \alpha_s\alpha_s(\tilde{\sigma}) \neq 0. \) Then the above \( \alpha_s \) is monic and so \( im j_* = 0 \). This shows \( Ext_A^{s+1,tq+q+r}(H^*K, H^*M) = 0 \) with \( r = 0, 1, 2 \) and consequently we have \( Ext_A^{s+1,tq+2q+1}(H^*K, Z_p) = 0. \)

To prove the claim, recall from the supposition that \( \tilde{\alpha}_2\sigma = j_s\alpha_s\alpha_s(i_s(\sigma)) \neq 0 \in Ext_A^{s+2,tq+2q+1}(Z_p, Z_p) \), then \( i_s(\tilde{\alpha}_2\sigma) \neq 0 \in Ext_A^{s+2,tq+2q+1}(H^*M, Z_p) \), this is because \( Ext_A^{s+1,tq+2q}(Z_p, Z_p) = 0 \) from the supposition. In addition, we also have \( j_s i_s(\alpha_2\sigma) \neq 0 \in Ext_A^{s+2,tq+2q}(H^*M, H^*M) \), this is because \( Ext_A^{s+1,tq+2q}(H^*M, Z_p) = 0. \) Hence, by \( 2 \alpha i j \alpha = i j \alpha^2 + \alpha^2 i j \) we have
\[
(9.3.3) \quad \alpha_s(ij)_s(\tilde{\sigma}) = j_s\alpha_s(ij)_s\alpha_s(\tilde{\sigma}) \]
\[
= \frac{1}{2} j_s(ij)_s\alpha_s\alpha_s(i_s(\sigma)) = \frac{1}{2} j_s(i_s(\alpha_2\sigma)) \neq 0
\]
This shows the claim (i). For the claim (ii),
\[
\alpha_s[\lambda_1(ij)_s\alpha_s(\tilde{\sigma}) + \lambda_2\alpha_s(ij)_s(\tilde{\sigma})] \neq 0
\]
, this is because this two terms is linearly independent which can be obtained from \((ij)_s\alpha_s(\tilde{\sigma}) \neq 0 \) (cf. (9.3.3)). The claim (iii) is immediate, this is because \( i_s(ij)_s\alpha_s(\tilde{\sigma}) = j_s\alpha_s\alpha_s(i_s(\sigma)) \neq 0. \)

(2) Consider the following exact sequence \((r = 1, 2, 3)\)
\[ Ext_A^{s+1,tq+q+r}(H^*M, Z_p) \xrightarrow{j_*} Ext_A^{s+1,tq+q+r}(H^*K, Z_p) \]
\[ \xrightarrow{a_*} Ext_A^{s+1,tq+r-1}(H^*M, Z_p) \]
induced by (9.1.2). The left group is zero for \( r = 2, 3 \) which can be obtained from the supposition of \( Ext_A^{s+1,tq+q+u}(Z_p, Z_p) = 0 \) \(( u = 1, 2, 3)\). The left group has unique generator \( \alpha_s i_s(\sigma) \) for \( r = 1 \), then in any case we
have in $i'_s = 0$. The right group is zero for $r = 2, 3$ (cf. Prop. 9.3.0) and has unique generator $i_s(\sigma')$ for $r = 1$ which satisfies $\alpha_s i_s(\sigma') \neq 0 \in Ext^{s+2, t_q+q+1}_A(H^*M, Z_p)$ so that $j'_s = 0$ and the result follows.

(3) Consider the following exact sequence

$$Ext^{s+1, t_q+q+1}_A(H^*K, H^*M) \xrightarrow{\psi'} Ext^{s+1, t_q+q}_A(H^*K, H^*K)$$

$$\xrightarrow{(i')^*} Ext^{s+1, t_q+q}_A(H^*K, H^*M)$$

induced by (9.1.2). The left group is zero by (1) and the right group has unique generator $i'_s(\sigma)$ (cf. Prop. 9.3.1(4)) which satisfies $\alpha^* i'_s(\sigma)$.

$\alpha_s(\sigma) = i'_s(\sigma), \alpha_s \alpha^*(\sigma) = i'_s(\sigma)$.

This is because $i'ij\alpha^2 = 2i' \alpha \alpha - i'\alpha^2 ij = 0 \in \Sigma^{2q-1}M, K$. Then the result follows. Q.E.D.

**Proposition 9.3.4** Let $p \geq 5, s \leq 4$, then under the supposition of the main Theorem B we have

$$Ext^{s+1, t_q+q-1}_A(H^*K, H^*K) \cong Z_p\{h_0\sigma''\}$$

satisfying $(i')^*(h_0\sigma'') = i'_s(\sigma), (\alpha_1 \land 1_M), (\bar{\sigma}).$

**Proof:** Consider the following exact sequence

$$Ext^{s+1, t_q+2q}_A(H^*K, H^*M) \xrightarrow{\psi} Ext^{s+1, t_q+q-1}_A(H^*K, H^*M)$$

$$\xrightarrow{(i')^*} Ext^{s+1, t_q+q-1}_A(H^*K, H^*M)$$

induced by (9.1.2). The left group is zero by Prop. 9.3.2(1) and similar to that in Prop. 9.3.1, the right group has unique generator $(ij)'_s(\sigma), (\bar{\sigma})$ which satisfies $\alpha^* (ij)'_s(\sigma), (\bar{\sigma}) = (ij)'_s(\sigma), (\bar{\sigma}).$

$0 \in Ext^{s+1, t_q+2q}_A(H^*K, H^*M)$, this is because $i'ij(\alpha_1 \land 1_M)\alpha = 0 \in \Sigma^{2q-2}M, K$. Then the result follows. Q.E.D.

**Proposition 9.3.5** Let $p \geq 5, s \leq 4$, then under the supposition of the main Theorem B we have

$$Ext^{s+1, t_q+q+1}_A(H^*K' \land M, H^*M) \cong Z_p\{\psi (ij)_s(\bar{\sigma}), \psi (ij)_s(\bar{\sigma})\}.$$

where $\psi : \Sigma M \to K' \land M$ is the map in (9.1.17).

**Proof:** Consider the following exact sequence

$$Ext^{s+1, t_q+q}_A(H^*M, H^*M) \xrightarrow{\psi} Ext^{s+1, t_q+q+1}_A(H^*K' \land M, H^*M)$$

$$\xrightarrow{\psi} Ext^{s+1, t_q+q+1}_A(H^*K, H^*M) = 0$$

induced by (9.1.17). The result follows immediately form Prop. 9.3.1(2) and Prop.9.3.2.(Note: By the supposition, similar to that given in Prop. 9.3.2(2), we can prove that $Ext^{s+q+1}_A(H^*K, H^*M) = 0$ so that the above $\psi$ is monic). Q.E.D.
Proposition 9.3.6  Let $p \geq 5$, $s \leq 4$, then under the supposition of the main Theorem B we have
\[ E^{s,tq}_A(H^*K, H^*K) \cong Z_p\{\sigma') \} \text{ satisfying } (i')^*(\sigma') = (i')_*(\tilde{\sigma}). \]

Proof: Consider the following exact sequence
\[ E^{s,tq}_A(H^*K, H^*K) \to E^{s,tq}_A(H^*K, H^*K) \]
induced by (9.1.2). Since $j'_*E^{s,tq+q+1}_A(H^*K, H^*M) \subset E^{s,tq}_A(H^*M, H^*M) \cong Z_p\{\tilde{\sigma}\}$ and $\alpha_*(\tilde{\sigma}) \neq 0 \in E^{s+1,tq+q+1}_A(H^*M, H^*K)$, then $i'_*E^{s,tq+q+1}_A(H^*M, H^*K) = i'_*E^{s,tq+q+1}_A(H^*K, H^*M)$. Moreover, by the supposition on $E^{s,tq+q+1}_A(Z_p, Z_p) = 0$, $r = 1, 2,$ and $E^{s,tq+q}_A(Z_p, Z_p)$ is zero or $\cong Z_p\{h_0\sigma''\}$ we have $E^{s,tq+q+1}_A(H^*M, H^*M) \cong Z_p\{\alpha_*(\sigma'')\}$, then $E^{s,tq+q+1}_A(H^*K, H^*H) = (i')_*E^{s,tq+q+1}_A(H^*K, H^*M) = 0$. On the other hand, it is easily seen that $E^{s,tq}_A(H^*K, H^*M)$ has unique generator $i'_*(\tilde{\sigma})$ which satisfies $\alpha^*_i i'_*(\tilde{\sigma}) = i'_*\alpha_*(\tilde{\sigma}) = 0$. Then the result follows. Q.E.D.

Then, $\alpha''$ induces a boundary homomorphism (or connecting homomorphism) $(\alpha'')^* : E^{s,tq}_A(H^*K, H^*K) \to E^{s+1,tq+q+1}_A(H^*K, H^*K)$. Since $\alpha'' i' = i' \alpha ij$, then $(i')^*(\alpha'')^*(\sigma') = (i'\alpha ij)(\alpha_1 \land 1_M)^*(\sigma') = (i'\alpha ij)(\alpha_1 \land 1_M)^*(\sigma') = (i')^*(\alpha^*_i i'_*(\tilde{\sigma}) = i'_*\alpha_*(\tilde{\sigma}) = (i')^*(h_0\sigma)''$ (cf. Prop. 9.3.4). Then we have
\[ (h_0\sigma)' = (\alpha''^*)^*(\sigma') \in E^{s+1,tq+q+1}_A(H^*K, H^*K) \]
is because the above $(i')^*$ is monic which can be obtained by $E^{s+1,tq+2q}_A(H^*K, H^*M) = 0$ (cf. Prop. 9.3.2).

After finishing the above preminlaries, we now turn to prove the following Theorem 9.3.9. It is proved by some argument processing in the Adams resolution (9.2.9) of some spectra related to the sphere spectrum $S$.

Theorem 9.3.9  Let $p \geq 5$, $s \leq 4$, then under the supposition of the main Theorem B we have $(\tilde{\sigma}_{s+1} \land 1_K)(h_0\sigma)' = 0$, where $(h_0\sigma)' \in [\Sigma^{tq+q-1}K, KG_{s+1} \land K]$ is a $d_1$-cycle which represents the unique generator $(h_0\sigma)'$ of $E^{s+1,tq+q-1}_A(H^*K, H^*K)$ (cf. Prop. 9.3.4).

Before proving Theorem 9.3.9, we first prove the following Lemma.
**Lemma 9.3.10** Let $p \geq 5$, $s \leq 4$, then under the supposition of the main Theorem B we have

1. $(\bar{c}_{s+1} \land 1_K)(h_0\sigma)^n = (1_{E_{s+2}} \land \alpha^n)(\kappa \land 1_K)$,
2. $(\bar{c}_{s+1} \land 1_K)(h_0\sigma \land 1_K) = (1_{E_{s+2}} \land \alpha_1 \land 1_K)(\kappa \land 1_K)$

where $\kappa \in \pi_{tq+1}E_{s+2}$ such that $\bar{a}_{s+1}\kappa = \bar{c}_s\sigma$ with $\sigma \in \pi_{tq}KG_s$.

**Proof.** Recall that $X$ is the cofibre of $\alpha'' : \Sigma^{q-2}K \to K$ given by the cofibration (9.3.7). Since $(h_0\sigma)^n \in [\Sigma^{tq+q-1}K, KG_{s+1} \land K]$ represents $(h_0\sigma)^n = (\alpha''\cdot)(\sigma') \in Ext^{s+1,tq+q-1}_A(H^*K, H^*K)$, then $(h_0\sigma)^n u \in [\Sigma^{tq}X, KG_{s+1} \land K]$ is a $d_1$-boundary so that $(\bar{c}_{s+1} \land 1_K)(h_0\sigma)^n u = 0$ and $(\bar{c}_{s+1} \land 1_K)(h_0\sigma)^n = f'\alpha''$ for some $f' \in \Sigma^{q+1}K, E_{s+2} \land K)$. It follows that $(\bar{a}_{s+1} \land 1_K)f'\alpha'' = 0$ and so $(\bar{a}_{s+1} \land 1_K)f' = f_2'w$ with $f_2' \in [\Sigma^{tq}X, E_{s+1} \land K]$. Then, $(\bar{b}_{s+1} \land 1_K)f_2' = 0$ and $(\bar{a}_{s+1} \land 1_K)f_2' = g'\cdot u$ for some $g' \in [\Sigma^{q+q-1}K, KG_{s+1} \land K]$. $g'$ is a $d_1$-cycle, this is because $(\bar{b}_{s+2}\bar{c}_{s+1} \land 1_K)g' = g''_2\alpha''$ (with $g''_2 \in [\Sigma^{q+1}K, KG_{s+2} \land K]$) $= 0$ since $\alpha''$ induces zero homomorphism in $Z_p$-cohomology. Then, by Prop. 9.3.4 and (9.3.8), $g'$ represents $(h_0\sigma)^n = (\alpha''\cdot)(\sigma') \in Ext^{s+1,tq+q-1}_A(H^*K, H^*K)$ and so $g' \cdot u$ is a $d_1$-boundary, that is $g'\cdot u = (\bar{b}_{s+1}\bar{c}_s \land 1_K)g_3'$ with $g_3' \in [\Sigma^{tq}X, KG_{s} \land K]$. Then $(\bar{b}_{s+1} \land 1_K)f_2' = (\bar{b}_{s+1}\bar{c}_s \land 1_K)g_3'$ and so $f_2' = (\bar{c}_s \land 1_K)g_3' + (\bar{a}_{s+1} \land 1_K)f_3'$ for some $f_3' \in [\Sigma^{tq}X, E_{s+2} \land K]$ and we have $(\bar{a}_{s+1} \land 1_K)f' = f_2'w = (\bar{c}_s \land 1_K)g_3'w + (\bar{a}_{s+1} \land 1_K)f_3'w$. Clearly, $g_3'w \in [\Sigma^{tq}K, KG_{s} \land K]$ is a $d_1$-cycle which represents $Ext^{s,tq}_A(H^*K, H^*K) \cong Z_p(\sigma')$ (cf. Prop. 9.3.6). Then $g_3'w = \sigma \land 1_K$ (up to scalar and modulo $d_1$-boundary), where $\sigma \in \pi_{tq}KG_s \cong Ext^{s,tq}_A(Z_p, Z_p)$.

So we have $(\bar{a}_{s+1} \land 1_K)f' = (\bar{c}_s \land 1_K)(\sigma \land 1_K) + (\bar{a}_{s+1} \land 1_K)f_3'w = (\bar{a}_{s+1} \land 1_K)(\kappa \land 1_K) + (\bar{a}_{s+1} \land 1_K)f_3'w$, where $\kappa \in \pi_{tq+1}E_{s+2}$ satisfying $\bar{a}_{s+1}\kappa = \bar{c}_s\sigma$. It follows that $f' = \kappa \land 1_K + f_3'w + (\bar{c}_{s+1} \land 1_K)g_4'$ for some $g_4' \in [\Sigma^{tq+1}K, KG_{s+1} \land K]$ and we have $(\bar{c}_{s+1} \land 1_K)(h_0\sigma)^n = f'\alpha'' = (\kappa \land 1_K)\alpha'' = (1_{E_{s+2}} \land \alpha^n)(\kappa \land 1_K)$.

This shows (1). The proof of (2) is similar. Q.E.D.

**Proof of Theorem 9.3.9** At first, by the supposition of the main Theorem B on $i_\ast i_\ast (h_0\sigma) \in Ext^{s+1,tq+q}_A(H^*K, Z_p)$ in a permanent cycle in the ASS we have $(\bar{c}_{s+1} \land 1_K)(h_0\sigma \land 1_K) = 0$. The there exists $\eta'_{n,s+1} \in [\Sigma^{q+q}K, E_{s+1} \land K]$ such that $(\bar{b}_{s+1} \land 1_K)\eta'_{n,s+1} = (h_0\sigma \land 1_K)$.

By Lemma 9.3.10, it suffices to prove $(1_{E_{s+2}} \land \alpha^n)(\kappa \land 1_K) = 0$. Note that, by $\bar{a}_{s+1}\kappa = \bar{c}_s\sigma$ we have $\bar{a}_{s+1}(1_{E_{s+2}} \land \alpha_1)\kappa = \bar{c}_s(1_{KG_s} \land \alpha_1)\sigma = 0$ and 31
so \((1_{E_{s+2}} \wedge \alpha_1)k = \tilde{c}_{s+1}(h_0\sigma)\) (up to scalar), this is because \(\pi_{q+q}KG_{s+1} \cong \text{Ext}^{s+1, lq+q}(Z_p, Z_p) \cong Z_p\{h_0\sigma\}\). Then, by Lemma 9.3.10 we have

\[
\text{(9.3.11)} \quad (1_{E_{s+2}} \wedge \alpha_1 \wedge 1_K)(k \wedge 1_K) = (\tilde{c}_{s+1} \wedge 1_K)(h_0\sigma \wedge 1_K) = 0.
\]

Moreover, by (9.1.20) we have \((1_{E_{s+2}} \wedge \rho\alpha'_{K' \wedge M})(k \wedge 1_K)i' = (1_{E_{s+2}} \wedge \alpha')(k \wedge 1_K)i' = 0\). Then, by (9.1.17), \((1_{E_{s+2}} \wedge \alpha'_{K' \wedge M})(k \wedge 1_K)i' = (1_{E_{s+2}} \wedge (v \wedge 1_M)\overline{m}_M)f)\) for some \(f \in \Sigma^{q+q-1}M, E_{s+2} \wedge K' \wedge M\) and so \((1_{E_{s+2}} \wedge i')f = (1_{E_{s+2}} \wedge (\rho(1_{K'} \wedge ij)\alpha'_{K' \wedge M})(k \wedge 1_K)i') = (1_{E_{s+2}} \wedge \alpha')(k \wedge 1_K)i' = (1_{E_{s+2}} \wedge \alpha')(k \wedge 1_K)i'ij = 0\). Hence \(f = (1_{E_{s+2}} \wedge \alpha)f_2\) for some \(f_2 \in \Sigma^{q-1}M, E_{s+2} \wedge M\) and we have \((1_{E_{s+2}} \wedge (x \wedge 1_M)\alpha'_{K' \wedge M})(k \wedge 1_K)i' = (1_{E_{s+2}} \wedge (i' \wedge 1_M)\overline{m}_M\alpha)f_2 = 0\) and \((1_{E_{s+2}} \wedge (x \wedge 1_M)\alpha'_{K' \wedge M})(k \wedge 1_K)\rho(v \wedge 1_M) = (1_{E_{s+2}} \wedge (x \wedge 1_M)\alpha'_{K' \wedge M})(k \wedge 1_K)\rho(v \wedge 1_M)\overline{m}_M(j \wedge 1_M)\) = 0, this is because \(\rho(v \wedge 1_M)\overline{m}_M = 0\), \(\rho(vi \wedge 1_M) = i'\). Then we have

\[
\text{(9.3.12)} \quad (1_{E_{s+2}} \wedge (x \wedge 1_M)\alpha'_{K' \wedge M})(k \wedge 1_K)\rho = f_3(y \wedge 1_M)
\]

with \(f_3 \in \Sigma^{q+q+1}M, E_{s+2} \wedge K' \wedge M \cap (\ker d)(\text{cf. (9.1.15) and Cor. 6.4.15})\). It follows that

\[
\text{(9.3.13)} \quad (\tilde{a}_{s+1} \wedge 1_K)f_3 = f_4(\alpha i \wedge 1_M)
\]

with \(f_4 \in \Sigma^{q+q}M \wedge M, E_{s+2} \wedge K' \wedge M \cap (\ker d)(\text{cf. (9.1.15) and Cor. 6.4.15})\). Note that the d_1-cycle \((\tilde{b}_{s+1} \wedge 1_M)(1_{E_{s+1}} \wedge jj' \wedge 1_M)f_4 \in \Sigma^{q-2}M \wedge M, KG_{s+1} \wedge M) \cong Z_p\{(\sigma' \wedge 1_M)ij(j \wedge 1_M)\}, \) then \((\tilde{b}_{s+1} \wedge 1_M)(1_{E_{s+1}} \wedge jj' \wedge 1_M)f_4 = \lambda \cdot (\sigma' \wedge 1_M)ij(j \wedge 1_M)\) and by applying the derivation \(d\) we have \(\lambda \cdot (\sigma' \wedge 1_M)(j \wedge 1_M) = 0\) and this implies that \(\lambda = 0\). That is to say \((\tilde{b}_{s+1} \wedge 1_M)(1_{E_{s+1}} \wedge jj' \wedge 1_M)f_4 = 0\), then \((\tilde{b}_{s+1} \wedge 1_K \wedge M)f_4 = (1_{KG_{s+1}} \wedge x \wedge 1_M)g\) with d_1-cycle \(g \in \Sigma^{q+q}M \wedge M, E_{s+1} \wedge K' \wedge M \cap (\ker d)(\text{cf. Cor. 6.4.15})\).

By Theorem 6.4.3, \(g = g(i \wedge 1_M)m_M + g\overline{m}_M(j \wedge 1_M)\). Now we claim that \(g(i \wedge 1_M) = \lambda_1(1_{KG_{s+1}} \wedge vi \wedge 1_M)(h_0\sigma \wedge 1_M)\) and \(g\overline{m}_M = \lambda_2(1_{KG_{s+1}} \wedge (v \wedge 1_M)\overline{m}_M(h_0\sigma \wedge 1_M)\) (mod d_1-boundary), where \(\lambda_1, \lambda_2 \in Z_p\).

To prove the claim, note that the d_1-cycle \(g(i \wedge 1_M)\) represents an element \([g(i \wedge 1_M)] \in \text{Ext}^{s+1, lq+q}(H^*K' \wedge M, H^*M)\) and \([1_{KG_{s+1}} \wedge (\rho(vi \wedge 1_M))(h_0\sigma \wedge 1_M)\] (cf. Prop. 9.3.2(4)). Then \((1_{KG_{s+1}} \wedge \rho)g(i \wedge 1_M) = \lambda_1(1_{KG_{s+1}} \wedge \rho(vi \wedge 1_M))(h_0\sigma \wedge 1_M) + (\tilde{b}_{s+1} \tilde{c}_s \wedge 1_K)g_2\) for some \(g_2 \in \Sigma^{q+q+1}M, KG_{s+1} \wedge M\). Since \((1_{KG_{s+1}} \wedge jj' \alpha')g_2 = 0\), then \(g_2 = (1_{KG_{s+1}} \wedge \rho)g_3\) with \(g_3 \in [\Sigma^{q+q}M, KG_{s+1} \wedge K' \wedge M]\). Then \(g(i \wedge 1_M) = \lambda_1(1_{KG_{s+1}} \wedge vi \wedge 1_M)(h_0\sigma \wedge 1_M) + (\tilde{b}_{s+1} \tilde{c}_s \wedge 1_K \wedge M)g_3 + (1_{KG_{s+1}} \wedge \psi)g_4\) with \(g_4 \in [\Sigma^{q+q-1}M, KG_{s+1} \wedge M] \cong Z_p\{(h_0\sigma \wedge 1_M)ij\}\) and so \(g_4 = \lambda'(h_0\sigma \wedge 1_M)ij\)
for some $\lambda' \in Z_p$. However, $d(i \wedge 1_M) = 0$ and $d(g) = 0$ this implies that $d(g(i \wedge 1_M)) = 0$, then, by applying the derivation $d$ to the above equation we have $(1_{KG_{s+1}} \wedge \psi)(d(g_4) + (\bar{b}_{s+1} \bar{c}_s \wedge 1_{K' \wedge M})d(g_3)) = 0$, that is $\lambda'(1_{KG_{s+1}} \wedge \psi)(h_0 \sigma \wedge 1_M) = (\bar{b}_{s+1} \bar{c}_s \wedge 1_{K' \wedge M})d(g_3)$ and this means that the scalar $\lambda' = 0$, this is because $\psi[h_0 \sigma \wedge 1_M] \neq 0 \in Ext_{A}^{t+1,t+q+1}(H^*K' \wedge M, H^*M)$ (cf. Prop. 9.3.5). This shows that $g(i \wedge 1_M) = \lambda_1(1_{KG_{s+1}} \wedge \psi)(h_0 \sigma \wedge 1_M) \pmod{d_1}$-boundary. In addition, by $d(m_M) \in \Sigma^2M, M \wedge M \cong \Sigma^2M, M + \Sigma M, M$ = 0, then similarly we have $g(m_M) = \lambda_2(1_{KG_{s+1}} \wedge \psi)(h_0 \sigma \wedge 1_M) \pmod{d_1}$-boundary. This proves the above claim.

Then, modulo $d_1$-boundary we have

\[(9.3.14) \quad g = g(i \wedge 1_M)m_M + g(m_M)(j \wedge 1_M)\]

\[= \lambda_1(1_{KG_{s+1}} \wedge \psi)(h_0 \sigma \wedge 1_M)m_M + \lambda_2(1_{KG_{s+1}} \wedge \psi)(h_0 \sigma \wedge 1_M)(j \wedge 1_M)\]

\[= \lambda_1(h_0 \sigma \wedge 1_{K'} \wedge M)(v_i \wedge 1_M)m_M + \lambda_2(h_0 \sigma \wedge 1_{K'} \wedge M)(v_i \wedge 1_M)m_M (j \wedge 1_M)\]

We claim that

\[(9.3.15) \quad \text{The scalar in (9.3.14) } \lambda_1 = \lambda_2.\]

This will be proved in the last. Then, $g = \lambda_1(1_{KG_{s+1}} \wedge v \wedge 1_M)(h_0 \sigma \wedge 1_M \wedge 1_M)$ and so we have $(\bar{b}_{s+1} \wedge 1_{K \wedge M})f_4 = (1_{KG_{s+1}} \wedge x \wedge 1_M)g = \lambda_1(1_{KG_{s+1}} \wedge i' \wedge 1_M)(h_0 \sigma \wedge 1_M \wedge 1_M) = \lambda_1(h_0 \sigma \wedge 1_k \wedge 1_M)(i' \wedge 1_M) + (\bar{b}_{s+1} \bar{c}_s \wedge 1_{K \wedge M})g_5 = \lambda_1(\bar{b}_{s+1} \wedge 1_{K \wedge M})(\bar{e}_{s+1} \wedge 1_M)(i' \wedge 1_M) + (\bar{b}_2 \bar{e}_1 \wedge 1_{K \wedge M})g_5$ and $f_4 = \lambda_1(\eta_{s+1} \wedge 1_{K \wedge M}) + (\bar{c}_s \wedge 1_{K \wedge M})g_5 + (\bar{a}_{s+1} \wedge 1_{K \wedge M})f_5$ with $f_5 \in \Sigma^{q+1}M, M \wedge M + \Sigma M, M$. It follows that $(\bar{a}_{s+1} \wedge 1_{K \wedge M})f_5 = f_4(\alpha \wedge 1_M) = (\bar{a}_{s+1} \wedge 1_{K \wedge M})f_5(\alpha \wedge 1_M)$ and so $f_3 = f_5(\alpha \wedge 1_M) + (\bar{e}_{s+1} \wedge 1_{K \wedge M})g_6$ for some $g_6 \in \Sigma^{q+2q+1}M, E_{s+2} \wedge K \wedge M$. So $(1_{E_{s+2}} \wedge \alpha')(\kappa \wedge 1_K)(i' \wedge 1_M) = (1_{E_{s+2}} \wedge 1_{K \wedge j})(i' \wedge 1_M)\alpha'_{K \wedge M}(\kappa \wedge 1_K)(i' \wedge 1_M) = (1_{E_{s+2}} \wedge 1_{K \wedge j})f_3(y \wedge 1_M) = (\bar{e}_{s+1} \wedge 1_K)(1_{KG_{s+1}} \wedge 1_K \wedge j)g_6(y \wedge 1_M) = 0$, this is because the $d_1$-cycle $(1_{KG_{s+1}} \wedge 1_K \wedge j)g_6 \in \Sigma^{q+2q+1}M, KG_{s+1} \wedge K$ represents an element in $E_{xt}^{t+1,t+q+1}(H^*K, H^*M) = 0$ (cf. Prop. 9.3.2(1)).

It follows from (9.1.11) that $(1_{E_{s+2}} \wedge \alpha')(\kappa \wedge 1_K) = f_6(\alpha ij j')$ for some $f_6 \in \Sigma^{q+q+1}M, E_{s+2} \wedge K$ and $(\bar{a}_{s+1} \wedge 1_K)f_6(\alpha ij j') = (\bar{a}_{s+1} \wedge 1_K)(1_{E_{s+2}} \wedge \alpha')(\kappa \wedge 1_K) = (\bar{c}_1 \wedge 1_K)(1_{KG_{s+1}} \wedge \alpha')(\kappa \wedge 1_K) = 0$. Then, by (9.1.14) we have $(\bar{a}_{s+1} \wedge 1_K)f_6(\alpha i = f_7z$ with $f_7 \in \Sigma^{t+q+1}K', E_{s+1} \wedge K$. Moreover, by Prop. 9.1.21, $f_7z = 0$, then $f_6(\alpha = (\bar{c}_{s+1} \wedge 1_K)g_7$ for some $g_7 \in \pi_{t+2q+1}(KG_{s+1} \wedge K)$ and so $(1_{E_{s+2}} \wedge \alpha')(\kappa \wedge 1_K) = f_6(\alpha ij j') = (\bar{c}_{s+1} \wedge 1_K)g_7(\alpha ij j') = 0$, this is because the $d_1$-cycle $g_7 \in \pi_{t+2q+1}(KG_{s+1} \wedge K)$ represents an element in
\[ Ext_{A}^{s+1,p^n+q+2q+1}(H^* K, Z_p) = 0. \] This shows the result of the Theorem and the remaining work is to prove the claim (9.3.15).

To prove (9.3.15), Note that by Theorem 6.4.3 and (9.1.15) we have
\[ (v \wedge 1_M)(\overline{m}_M(\alpha_1 \wedge 1_M) = (v \wedge 1_M)(\overline{m}_M(j \wedge 1_M)(\alpha i \wedge 1_M) = -(v \wedge 1_M)(i \wedge 1_M)\overline{m}_M(\alpha i \wedge 1_M) = -(v \wedge 1_M)\alpha. \] Similarly we have \( \alpha(j\overline{m}_M = -(\alpha_1 \wedge 1_M)\overline{m}_M(\overline{m}_M \wedge 1_M) \), where \( \overline{m}_M : Y \to \Sigma^{q+1} M \) and \( v : \Sigma M \to K' \) are the map (9.1.5)(9.1.15).

Then, modulo \( d_1 \)-boundary we have
\[ (1_{KG_{s+1}} \wedge v i \wedge 1_M)(\overline{h_0\sigma}) = -(1_{KG_{s+1}} \wedge v \wedge 1_M)\overline{m}_M(\overline{h_0\sigma} \wedge 1_M) \]
\[ (\overline{h_0\sigma})(j\overline{m}_M = -(\overline{h_0\sigma} \wedge 1_M)\overline{m}_M(\overline{m}_M \wedge 1_M) \]
where \( \overline{h_0\sigma} \in [\Sigma^{q+1} M, KG_{s+1} \wedge M] \) is a \( d_1 \)-cycle which represents \( \alpha_*(\tilde{\alpha}) \in Ext_{A}^{s+1,q+1}(H^* M, H^* M) \). So, by (9.3.14), modulo \( d_1 \)-boundary we have
\[ g(\overline{m}_M \wedge 1_M) = \lambda_1(1_{KG_{s+1}} \wedge v \wedge 1_M)(\overline{h_0\sigma} \wedge 1_M \wedge 1_M)(\overline{m}_M \wedge 1_M) + (\lambda_2 - \lambda_1) \]
\[ (1_{KG_{s+1}} \wedge v \wedge 1_M)\overline{m}_M(\overline{h_0\sigma} \wedge 1_M)(j\overline{m}_M \wedge 1_M) \]
\[ = (\lambda_1 - \lambda_2)(1_{KG_{s+1}} \wedge v i \wedge 1_M)(\overline{h_0\sigma})(j\overline{m}_M \wedge 1_M) \]
this is because \( (1_{KG_{s+1}} \wedge v)(\overline{h_0\sigma} \wedge 1_M j\overline{m}_M \wedge 1_M) = (1_{KG_{s+1}} \wedge v)(\overline{h_0\sigma})(j\overline{m}_M \wedge 1_M) = 0 \) (mod \( d_1 \)-boundary). On the other hand, modulo \( d_1 \)-boundary we have
\[ g(\overline{m}_M \wedge 1_M) = \lambda_2(1_{KG_{s+1}} \wedge v \wedge 1_M)(\overline{h_0\sigma} \wedge 1_M \wedge 1_M)(\overline{m}_M \wedge 1_M) + (\lambda_1 - \lambda_2)(1_{KG_{s+1}} \wedge v i \wedge 1_M)(\overline{h_0\sigma} \wedge 1_M)(j\overline{m}_M \wedge 1_M) \]
\[ = (\lambda_2 - \lambda_1)(1_{KG_{s+1}} \wedge v i \wedge 1_M)\overline{h_0\sigma}(j\overline{m}_M \wedge 1_M) \]
Moreover, \( (1_{KG_{s+1}} \wedge v i \wedge 1_M)\overline{h_0\sigma}(j\overline{m}_M \wedge 1_M) \) represents a nonzero element in the \( \text{Exr} \) group, this is because \( (1_{KG_{s+1}} \wedge (1_K \wedge i))(\overline{h_0\sigma})(j\overline{m}_M \wedge 1_M)(\overline{m}_M \wedge 1_M) = (1_{KG_{s+1}} \wedge v i')(\overline{h_0\sigma})ij' = (1_{KG_{s+1}} \wedge v i')(\overline{h_0\sigma})(j\overline{m}_M \wedge 1_M)ij' \) represents a nonzero element in the \( \text{Exr} \) group. Then, by comparison to the above two equations we have \( \lambda_1 - \lambda_2 = \lambda_2 - \lambda_1 \) so that \( \lambda_1 = \lambda_2 \). This shows the claim (9.3.15). Q.E.D.

**Remark** In the last of section 4, we will also give another proof of Theorem 9.3.9.

**Proof of the main Theorem B** By Theorem 9.3.9, there exists
\[ (\eta_{n,s+1})'' \in [\Sigma^{q+q-1} K, E_{s+1} \wedge K] \] such that \( (\partial_4 \wedge 1_K)(\eta_{n,s+1})'' = (h_0\sigma)'' \in [\Sigma^{q+q-1} K, KG_{s+1} \wedge K] \). Let \( \eta_n'' = (a_0 \cdots a_n \wedge 1_K)(\eta_{n,s+1})'' \in [\Sigma^{q+q-s-2} K, K] \) and consider the map
\[ (\eta_n'')\beta i'i \in \pi_{tq+pq+2q-s-2} K \]
where $\beta \in [\Sigma(p+1)qK,K]$ is the known second periodicity element which has filtration 1. Since $(\eta_n)^{''}$ is represented by $(h_0\sigma)^{''} \in Ext_{A}^{s+1,tq+q-1}(H^*K,H^*K)$ in the ASS, then $(\eta_n)^{''}\beta_i' \in \pi_{tq+pq+2q-s-2}K$ is represented by $(\beta i')^*(h_0\sigma)^{''} = (\beta i')^*(\alpha''\phi)(\sigma) = \alpha''\beta_s(i')_*(\sigma) \in Ext_{A}^{s+2,tq+pq+2q}(H^*K,\pi_{tq+pq+2q}(H^*K,\pi_{tq+pq+2q}(H^*K,Z_p))$. By [14] Theorem 3.2 and [15] Theorem 5.2 we know that $\alpha''\beta_i' \in \pi_{tq+pq+2q}K$ is represented by $\alpha''\beta_s(i')_*(1) = (i')_*(g_0) \in Ext_{A}^{2,tq+pq+2q}(H^*K,Z_p)(up to nonzero scalar) in the ASS so that $(\eta_n)^{''}\beta_i'$ is represented by $\alpha''\beta_s(i')_*(\sigma) = (i')_*(g_0\sigma) \in Ext_{A}^{s+2,tq+pq+2q}(H^*K,Z_p)$. Q.E.D.

Using the stronger result of the main Theorem A which is stated in the Remark 9.2.35, the result of the main Theorem B also can be obtained by the following main Theorem B'.

**The main Theorem B'** Let $\sigma \in Ext_{A}^{s,tq}(Z_p,Z_p), \sigma' \in Ext_{A}^{s+1,tq}(Z_p,Z_p)$ be a pair of $a_0$-related elements, that is, there is a secondary differential $d_2(\sigma) = a_0\sigma'$. Suppose that all the supposition of the main Theorem A hold, then $(i')_*(g_0\sigma) \in Ext_{A}^{s+2,tq+pq+2q}(H^*K,Z_p), (i')_*(g_0\sigma') \in Ext_{A}^{s+3,tq+pq+2q}(H^*K,Z_p)$ are permanent cycles in the ASS.

**Proof** Let $\phi\sigma \in \pi_{tq+pq}KG_{s+1}, \phi\sigma' \in \pi_{tq+pq}KG_{s+2}$ be $d_1$-cycles which represent $\phi_s(\sigma) \in Ext_{A}^{s+1,tq+2q}(H^*L,Z_p), \phi_s(\sigma') \in Ext_{A}^{s+2,tq+2q}(H^*L,Z_p)$ respectively. By the stronger result of the main Theorem A (cf. Remark 9.2.35) we have $(\bar{c}_{s+2} \wedge 1_L)\phi\sigma' = 0, (\bar{c}_{s+1} \wedge 1_L)(1_{KG_{s+1}} \wedge 1_L \wedge i)\phi\sigma = 0$. Then $(\bar{c}_{s+2} \wedge 1_L)(\phi\sigma' \wedge 1_K) = 0$ and by using the multiplication of the ring spectrum $K$ we have $(\bar{c}_{s+1} \wedge 1_L)(\phi\sigma \wedge 1_K) = 0$. In addition, $(h_0\sigma)^{''} = (1_{KG_{s+1}} \wedge \overline{\Delta})(\phi \wedge 1_K), (h_0\sigma')^{''} = (1_{KG_{s+2}} \wedge \overline{\Delta})(\phi \wedge 1_K), this is because $\overline{\Delta}(\phi \wedge 1_K) = \alpha^{''} \in [\Sigma^{q-2}K,K]$. Then we have $(\bar{c}_{s+1} \wedge 1_K)(h_0\sigma)^{''} = 0, (\bar{c}_{s+2} \wedge 1_K)(h_0\sigma')^{''} = 0$. The remaining steps is similar to that given in the proof of the main Theorem B. Q.E.D.

§4. A general result on pull back convergence of $h_0\sigma$

In this section, we will prove that, under some suppositions, the convergence of the element $(1_L \wedge i)_*\phi_s(\sigma) \in Ext_{A}^{s+1,tq+2q}(H^*L \wedge M,\pi_{tq+pq}(Z_p,Z_p))$ can be pull backed to obtain the convergence of $h_0\sigma \in Ext_{A}^{s+1,tq+q}(Z_p,Z_p)$ in the stable homotopy groups of spheres. We have the following main Theorem.
The main Theorem C (generalization of [24] Theorem A) Let $p \geq 5, s \leq 4$ and suppose that

\begin{enumerate}[(a)]
\item $\text{Ext}_{A}^{s,q}(Z_{p},Z_{p}) \cong Z_{p}\{\sigma\}$, $\text{Ext}_{A}^{s+1,tq+q}(Z_{p},Z_{p}) \cong Z_{p}\{h_{0}\sigma\}$
\item $\text{Ext}_{A}^{s+2,tq+2q+1}(Z_{p},Z_{p}) \cong Z_{p}\{\alpha_{2}\sigma\}$ satisfying $a_{0}^{2}\sigma \neq 0$.
\end{enumerate}

(b) $\text{Ext}_{A}^{s+1,tq+u}(Z_{p},Z_{p}) \cong Z_{p}\{a_{0}\sigma\}$ for $u = 1$ and is zero for $u = 2, 3$.

Let $\text{Ext}_{A}^{s+1,tq}(Z_{p},Z_{p})$ be zero or has (one or two) generator $\sigma'$ such that (both) satisfies

$h_{0}\sigma' \neq 0, a_{0}\sigma' \neq 0,$

$\text{Ext}_{A}^{s+1,tq+q+u}(Z_{p},Z_{p}) = 0$ for $r = -1, 2, 3, u = -2, -1, 0, 1, 2, 3$

or

for $r = 1, u = -2, -1, 1, 2, 3$

(c) $\text{Ext}_{A}^{s,tq+u}(Z_{p},Z_{p}) = 0$ for $u = -1, 1, 2, 3$

$\text{Ext}_{A}^{s,tq+q+u}(Z_{p},Z_{p}) = 0$ for $r = -2, -1, 1, 2, u = -2, -1, 0, 1, 2, 3$

II $(1L \cap i)_{*}(\phi)_{*}(\sigma) \in \text{Ext}_{A}^{s+1,tq+2q}(H^{*}L \wedge M, Z_{p})$ is a permanent cycle in the ASS, then $(\alpha)_{*}(\sigma) \in \text{Ext}_{A}^{s+1,tq+q+1}(H^{*}M, Z_{p})$ also is a permanent cycle in the ASS so that $h_{0}\sigma = j_{*}(\alpha)_{*}(\sigma) \in \text{Ext}_{A}^{s+1,tq+q}(Z_{p},Z_{p})$ converges to an element in $\pi_{q+q-s-1}S$ of order $p$.

Note that the supposition (I) of the main Theorem C contains the supposition I of the main Theorem B, then some results on Ext groups in §3 also hold under the supposition of the main Theorem C. Before proving the main Theorem C, we first recall the properties of some spectra related to $K$ and $M$ and prove some results on low dimensional Ext groups.

By (9.1.27), \((1Y \wedge j)\alpha_{Y \wedge M} \wedge 1M)\bar{M} = \alpha_{Y \wedge M} \wedge 1M \), (9.2.12) and the following homotopy commutative diagram of $3 \times 3$-Lemma

$$
\begin{array}{c}
X \wedge M \overset{m_{M}(\bar{\psi} \wedge 1M)}{\longrightarrow} \Sigma^{2q}M \overset{0}{\longrightarrow} \Sigma^{2q+2}M \\
\Sigma^{2q}M \overset{\psi \wedge 1M}{\longrightarrow} m_{M} \overset{\Sigma L \wedge K}{\longrightarrow} \Sigma^{2q+2}M \wedge M \\
\Sigma^{2q+1}M \overset{Y \wedge M}{\longrightarrow} \Sigma L \wedge K \wedge \Sigma^{2q+2}M \wedge M \\
\Sigma X \wedge M \overset{\Sigma X \wedge M}{\longrightarrow} \Sigma X \wedge M \\
\end{array}
$$

(9.4.1)

we know that the cofibre of $m_{M}(\bar{\psi} \wedge 1M) : X \wedge M \to \Sigma^{2q}M$ is $\Sigma L \wedge K$ given by the following cofibration

$$
\begin{array}{c}
X \wedge M \overset{m_{M}(\bar{\psi} \wedge 1M)}{\longrightarrow} \Sigma^{2q}M \overset{\phi \wedge 1M}{\longrightarrow} \Sigma L \wedge K \overset{u'}{\longrightarrow} \Sigma X \wedge M \\
\end{array}
$$

(9.4.2)

Since $(1L \wedge i')(\phi \wedge 1M)m_{M}(\bar{\psi} \wedge 1M) = 0$, then by $[\Sigma^{-q-1}X \wedge M, L \wedge M] \cap$
(kerd) \cong \mathbb{Z}_p\{u'' \land 1_M\}$ and (9.1.2) we have $(\phi \land 1_M)m_M(\tilde{\psi} \land 1_M) = (1_L \land \alpha)(u'' \land 1_M)$ (up to nonzero scalar). Since $(\phi \land 1_K)i'\alpha = 0$, then by (9.4.2), there exists $\alpha_{X,M} \in [\Sigma^{3q}M, X \land M]$ such that $m_M(\tilde{\psi} \land 1_M)\alpha_{X,M} = \alpha$. In addition, $m_M(\tilde{\psi} \land 1_M)\alpha_{X,M}m_M(\tilde{\psi} \land 1_M) = \alpha m_M(\tilde{\psi} \land 1_M) = m_M(\tilde{\psi} \land 1_M)(1_X \land \alpha)$ so that by (9.4.2) we have $\alpha_{X,M}m_M(\tilde{\psi} \land 1_M) = 1_X \land \alpha$ modulo $(u')_*[\Sigma^qX \land M, L \land K] = 0$, this is because $[\Sigma^qL \land K, L \land K] = 0$ and $[\Sigma^{3q}M, L \land K] = 0$. Conclusively we have

(9.4.3) \[ (\phi \land 1_M)m_M(\tilde{\psi} \land 1_M) = (1_L \land \alpha)(u'' \land 1_M), \]

\[ \alpha_{X,M}m_M(\tilde{\psi} \land 1_M) = 1_X \land \alpha \]

The cofibre of the map $\alpha_{X,M} : \Sigma^{3q}M \to X \land M$ is $W \land K$ given by the cofibration

(9.4.4) \[ \Sigma^{3q}M \overset{\alpha}{\to} \Sigma^{2q}M \overset{(\phi \land K)'i'}{\to} \Sigma L \land K \]

\[ \downarrow \alpha_{X,M} \downarrow m_M(\tilde{\psi} \land 1_M) \downarrow \phi \land 1_K \]

\[ X \land M \overset{\phi \land 1_K}{\to} \Sigma^{2q}K \]

\[ \downarrow u' \downarrow \mu_{X,M} \downarrow j''u \land 1_K \downarrow j' \]

\[ \downarrow \phi \land 1_K \downarrow \Sigma^{3q+1}M \]

By (9.2.13), $ijm_M(\tilde{\psi} \land \tilde{u} \land 1_M) = iju_2 \land 1_M = (u_2 \land 1_M)(1_U \land ij) = m_M(\tilde{\psi} \land \tilde{u} \land 1_M)(1_U \land ij) = m_M(\tilde{\psi} \land 1_M)(1_X \land i)\tilde{u} \land 1_M$, then we have $ijm_M(\tilde{\psi} \land 1_M) = m_M(\tilde{\psi} \land 1_M)(1_X \land i) + \lambda(j\tilde{\psi} \land 1_M)$ for some $\lambda \in \mathbb{Z}_p$. It follows that $\lambda j(j\tilde{\psi} \land 1_M) = -jm_M(\tilde{\psi} \land 1_M)(1_X \land i) = -j\tilde{\psi}(1_X \land i) = j(j\tilde{\psi} \land 1_M)$ and so $\lambda = 1$. In addition, $i'(\alpha_1 \land 1_M)m_M(\tilde{\psi} \land 1_M) = (j'' \land 1_K)(1_L \land i')(\phi \land 1_M)m_M(\tilde{\psi} \land 1_M) = 0$, then by (9.1.23) we have $m_M(\tilde{\psi} \land 1_M) = m_M(\tilde{\psi} \land 1_M)\psi_{X,M}$, where $\psi_{X,M} \in [\Sigma^{-q+1}X \land M, Y \land M]$.

In addition, $[\Sigma^{-q+1}X \land M, Y \land M] \cong \mathbb{Z}_p\{\psi_{X,M}\}$, this can be obtained from $[\Sigma^{-2q}X \land M, M] \cong Z_p\{m_M(\tilde{\psi} \land 1_M)\}$, (9.1.23) and $[\Sigma^{-q}X \land M, K] = 0$. Then, by $j''(j''u \land 1_K) \cdot \mu_{X,M} = 0$ and (9.1.27) we have $(u \land 1_K)\mu_{X,M} = \overline{\psi_{X,M}} = m_M(\tilde{\psi} \land 1_M)$ (up to nonzero scalar). Conclusively we have

(9.4.6) \[ j\tilde{\psi} \land 1_M = ijm_M(\tilde{\psi} \land 1_M) - m_M(\tilde{\psi} \land 1_M)(1_X \land i), \]

\[ (u \land 1_K)\mu_{X,M} = \overline{\psi_{X,M}} \]

\[ [\Sigma^{-q+1}X \land M, Y \land M] \cong \mathbb{Z}_p\{\psi_{X,M}\}, \]

\[ m_M(\tilde{\psi} \land 1_M) = m_M(\tilde{\psi} \land 1_M), \]
By the following homotopy commutative diagram of $3 \times 3$-Lemma

\[
\begin{array}{ccc}
L \wedge K & \xrightarrow{(1_X \wedge j)u'} & \Sigma X & \xrightarrow{1_X \wedge p} & \Sigma X \\
\downarrow u' & & \downarrow 1_X \wedge j & & \downarrow \omega & & \downarrow \bar{u}w_2 \\
X \wedge M & \xrightarrow{1_X \wedge i} & m_M(\psi \wedge 1_M) & \xrightarrow{(1_Y \wedge j)\alpha_Y \wedge M} & \bar{\tau}_2(1_Y \wedge i') \\
X & \xrightarrow{\bar{\nu}} & \Sigma^2qM & \xrightarrow{(\phi \wedge 1_K)q} & \Sigma L \wedge K
\end{array}
\]

we know that the cofibre of $(1_X \wedge j)u' : L \wedge K \to \Sigma X$ is $Y$ given by the cofibration

(9.4.7) $L \wedge K \xrightarrow{(1_X \wedge j)u'} \Sigma X \xrightarrow{\omega} Y \xrightarrow{\bar{\tau}_2(1_Y \wedge i')} \Sigma L \wedge K$

In addition, by the commutativity of the above rectangle we have

(9.4.8) $\omega \wedge 1_M = \alpha_Y \wedge M m_M(\psi \wedge 1_M)$.

**Proposition 9.4.9** Under the supposition (I) of the main theorem C we have

1. $\Ext_A^{s+1,tq+r}(H^*K, H^*M) = 0$ for $r = 1, 2$,
2. $\Ext_A^{s+1,tq+r+1}(H^*K, H^*K) = 0$ for $r = -1, 0, 1, 2$.

**Proof:** (1) By the supposition, $\Ext_A^{s+1,tq+r-1}(Z_p, Z_p) = 0$ for $r = -1, 0, 1, 2, 3$, then $(j'_*) \Ext_A^{s+1,tq+r}(H^*K, Z_p) \subset \Ext_A^{s+1,tq+r-1}(H^*M, Z_p) = 0$ for $r = 1, 2, 3$ and so $\Ext_A^{s+1,tq+r+1}(H^*K, Z_p) = (i'_*) \Ext_A^{s+1,tq+r+1}(H^*M, Z_p) = 0$ for $r = 1, 2, 3$ (cf. Prop. 9.3.0(1)) and the result follows.

(2) Consider the following exact sequence ($r = -1, 0, 1, 2$)

\[
0 = \Ext_A^{s+1,tq+(r+1)+2}(H^*K, H^*M) \xrightarrow{(j')^*} \Ext_A^{s+1,tq+r+1}(H^*K, H^*K) \xrightarrow{(i'_*)} \Ext_A^{s+1,tq+r+1}(H^*K, H^*M)
\]

induced by (9.1.2). The right group is zero for $r = 0, 1, 2$ (cf. (1) and Prop. 9.3.2(1)(2)) and also is zero for $r = -1$ which is obtained by the supposition on $\Ext_A^{s+1,tq-r-1}(Z_p, Z_p) = 0$ for $r = -1, 0, 1, 2$. The left group is zero for $r = -1, 0, 1$ (cf. (1) and Prop. 9.3.2). The left group also is zero for $r = 2$, this is because $\Ext_A^{s+1,tq+r+u}(Z_p, Z_p) = 0$ for $r = 2, 3, u = 0, 1, 2, 3$ by the supposition. Then the middle group is zero as desired. Q.E.D.

**Proposition 9.4.10** Under the supposition (I) of the main theorem C we have

1. $\Ext_A^{s+1,tq+r+1}(H^*W \wedge K, H^*X \wedge M) = 0$.  

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(2) \( \text{Ext}^{s+1,tq+2q+1}(H^*Y, H^*M) \cong Z_p\{((1Y \wedge j)\alpha Y \wedge M)_*(\tilde{\sigma})\} \),
\( \text{Ext}^{s+1,tq+q}(H^*Y, H^*Y) \cong Z_p\{((1\wedge j)\alpha Y \wedge M)_*(\tilde{\sigma})\} \).
(3) \( \text{Ext}^{s+1,tq+3q}(H^*X, H^*M) \cong Z_p\{((1X \wedge j)\alpha X \wedge M)_*(\tilde{\sigma})\} \).

Proof: (1) Consider the following exact sequence

\[ 0 = \text{Ext}^{s+1,tq+q+1}(H^*W \wedge K, H^*Y \wedge M) \xrightarrow{(\varpi)_*} \text{Ext}^{s+1,tq+q+1}(H^*W \wedge K, H^*X \wedge M) \xrightarrow{(\psi)_*} \text{Ext}^{s+1,tq+q+1}(H^*W \wedge K, H^*Y \wedge M) \]

induced by (9.4.2). The right group is zero by Prop. 9.4.9(2) and (9.1.12)(9.1.3). The left group also is zero by \( \text{Ext}^{s+1,tq+q+1}(H^*K, H^*M) = 0 \) (for \( r = 1, 2, 3 \))(cf. the proof of Prop. 9.4.9(2)). Then the middle group is zero as desired.

(2) Since \( \varpi((1Y \wedge j)\alpha Y \wedge M) \in [\Sigma^{-1}M, M] \cong Z_p\{ij\alpha, \alpha ij\} \), then \( \varpi((1Y \wedge j)\alpha Y \wedge M) = \lambda_1ij\alpha + \lambda_2\alpha ij \) where the scalar \( \lambda_1, \lambda_2 \in Z_p \) satisfy \( \lambda_1ij\alpha + \lambda_2\alpha ij = 0 \). Consider the following exact sequence

\[ \xrightarrow{(\varpi)_*} \text{Ext}^{s+1,tq+q}(H^*M, H^*M) \xrightarrow{(j\alpha)_*} \text{Ext}^{s+1,tq+q+1}(H^*Y, H^*M) \]

induced by (9.1.5). The left group is zero which can be obtained by the supposition on \( \text{Ext}^{s+1,tq+q+k}(Z_p, Z_p) = 0 \) (for \( k = 0, 1 \)). By Prop. 9.3.1(2), the right group has two generators \( (ij)_*, (j\alpha)_* \) and \( (ij)_*, (\alpha j)_* \). Then \( (\varpi)_* \), \( \text{Ext}^{s+1,tq+q+1}(H^*Y, H^*M) \) has unique generator \( (\varpi)_*((1Y \wedge j)\alpha Y \wedge M)_*(\tilde{\sigma}) \) so that the first result follows. For the second result, consider the following exact sequence

\[ \xrightarrow{(\varpi)_*} \text{Ext}^{s+1,tq+q}(H^*M, Z_p) \xrightarrow{(j\alpha)_*} \text{Ext}^{s+1,tq+q+1}(H^*Y, Z_p) \]

induced by (9.1.5). By the supposition, the left group has unique generator \( h_0\sigma = (j\alpha)_*(\sigma) \) so that im \((\varpi)_* = 0 \). The right group is zero or (one or two) generator \( i_*(\sigma') \) such that \( (j\alpha)_*i_*(\sigma') = h_0\sigma' \neq 0 \). Then the middle group is zero and so the second result follows.

(3) Since \( \tilde{\psi}(1X \wedge j)\alpha X \wedge M \in [\Sigma^{-1}M, M] \cong Z_p\{ij\alpha, \alpha ij\} \), then \( \tilde{\psi}(1X \wedge j)\alpha X \wedge M = \lambda_3ij\alpha + \lambda_4\alpha ij \), where the scalar \( \lambda_3, \lambda_4 \in Z_p \) satisfy \( \lambda_3(1Y \wedge j)\alpha Y \wedge Mij\alpha + \lambda_4(1Y \wedge j)\alpha Y \wedge M\alpha ij = 0 \). Then, similar to that in (2), \( (\tilde{\psi})_* \), \( \text{Ext}^{s+1,tq+3q}(H^*X, H^*M) \) has unique generator \( ((1X \wedge j)\alpha X \wedge M)_*(\tilde{\sigma}) \) so that \( \text{Ext}^{s+1,tq+3q}(H^*X, H^*M) \) has unique generator \( ((1X \wedge j)\alpha X \wedge M)_*(\tilde{\sigma}) \), this is because \( \text{Ext}^{s+1,tq+3q+1}(H^*Y, H^*M) = 0 \) which can be obtained by the
supposition (I)(b) on $\text{Ext}_{A}^{s+1,tq+rq+k}(Z_{p}, Z_{p}) = 0$ for $r = 1, 2, k = -1, 0, 1, 2$.

Q.E.D.

Proposition 9.4.11  Under the supposition (I) of the main Theorem C we have

1.  $\text{Ext}_{A}^{s,tq-2q}(H^{*} M, H^{*} X \wedge M) \cong Z_{p}\{m_{M}(\tilde{\psi} \wedge 1_{M})^{*}(\tilde{\sigma})\}$.
2.  $\text{Ext}_{A}^{s,tq+rq+u}(H^{*} K, H^{*} M) = 0$ for $r = -1, 1, 2, 3, u = 0, 1, 2$

$\text{Ext}_{A}^{s,tq}(H^{*} K, H^{*} K) \cong Z_{p}\{\sigma_{1}\}^{*}(\tilde{\sigma})$ satisfying $(\tilde{\sigma})^{*}(\tilde{\sigma}) = (\tilde{\sigma})^{*}(\tilde{\sigma})$.

3.  $\text{Ext}_{A}^{s,tq}(H^{*} L \wedge K, H^{*} L \wedge K) \cong Z_{p}\{\sigma_{2}\}$

satisfying $(\tilde{\sigma})^{*}(\tilde{\sigma}) = (\tilde{\sigma})^{*}(\tilde{\sigma})$.

$\text{Ext}_{A}^{s,tq+rq+u}(H^{*} L \wedge K, H^{*} M) = 0$ for $r = 1, 2, 3, u = 0, 1, 2,$

4.  $\text{Ext}_{A}^{s,tq+rq+u}(H^{*} W \wedge K, H^{*} M) = 0$ for $r = 1, 2, 3, u = 0, 1, 2,$

$\text{Ext}_{A}^{s,tq+q}(H^{*} W \wedge K, H^{*} X \wedge M) = 0$

Proof:  (1) Consider the following exact sequence

$\text{Ext}_{A}^{s,tq}(H^{*} M, H^{*} M \wedge M) (\tilde{\psi} \wedge 1_{M})^{*} \text{Ext}_{A}^{s,tq-2q}(H^{*} M, H^{*} X \wedge M)$

induced by (9.2.12). By the supposition on $\text{Ext}_{A}^{s,tq-rq+u}(Z_{p}, Z_{p}) = 0$ with $r = 1, 2, u = 0, 1, 2$ and the degree of the top cell of $Y \wedge M$ is $q + 3$ we know that the right group is zero. Since $(m_{M})^{*} \text{Ext}_{A}^{s,tq}(H^{*} M, H^{*} M \wedge M) \subset \text{Ext}_{A}^{s,tq+1}(H^{*} M, H^{*} M) = 0$ (cf. Prop. 9.3.0(2)), then the left group has unique generator $(m_{M})^{*}(\tilde{\sigma})$ and so the result follows.

(2) Consider the following exact sequence $(r = -1, 1, 2, 3, u = 0, 1, 2)$

$\text{Ext}_{A}^{s,tq+rq+u}(H^{*} M, H^{*} M) (\tilde{\psi})^{*} \text{Ext}_{A}^{s,tq+rq+u}(H^{*} K, H^{*} M)$

induced by (9.1.2). The left group is zero for $r = -1, 1, 2, 3, u = 0, 1, 2$, this is obtained from the supposition I(c) on $\text{Ext}_{A}^{s,tq+rq+k}(Z_{p}, Z_{p}) = 0$ (for $r = -1, 1, 2, 3, k = -1, 0, 1, 2, 3$). By the supposition and Prop. 9.3.0(2), the right group is zero except for $r = 1, u = 0, 1$ it has unique generator $(ij)_{*}(\tilde{\sigma})$ or $\tilde{\sigma}$ respectively. However, it satisfies $\alpha_{*}(ij)_{*}(\tilde{\sigma}) \neq 0, \alpha_{*}(\tilde{\sigma}) \neq 0$ then, the middle group is zero as desired. Consider the following exact sequence

$0 = \text{Ext}_{A}^{s,tq+rq+k}(H^{*} K, H^{*} M) (\tilde{\psi})^{*} \text{Ext}_{A}^{s,tq}(H^{*} K, H^{*} K)$

induced by (9.1.2). The left group is zero as shown above. The right group has unique generator $(i'j)_{*}(\tilde{\sigma})$, this is because $(i'j)_{*}\text{Ext}_{A}^{s,tq}$
(H^*K, H^*M) \subset \text{Ext}^{s,tq}_{\Lambda}(H^*M, H^*M) = 0 \text{ and } \text{Ext}^{s,tq}_{\Lambda}(H^*M, H^*M) \cong Z_p\{\tilde{\sigma}\}. \text{ Then the middle has unique generator } \sigma_K \text{ as desired.}

(3) Consider the following exact sequence \((r = -1, 0)\)
\[
\begin{align*}
\text{Ext}^{s,tq+(r+1)q}_{\Lambda}(H^*K, H^*K) & \xrightarrow{(i'\wedge 1K)^*} \text{Ext}^{s,tq+rq}_{\Lambda}(H^*K, H^*L \wedge K) \\
\end{align*}
\]
induced by (9.1.3). The left group is zero for \(r = 0\), this is because by (2) \((i')^*\text{Ext}^{s,tq}_{\Lambda}(H^*K, H^*K) \subset \text{Ext}^{s,tq+q}_{\Lambda}(H^*K, H^*M) = 0 \text{ and } \text{Ext}^{s,tq+2q+1}_{\Lambda}(H^*K, H^*M) = 0. \) Moreover, by (2), the left group has unique generator \(\sigma_K\) for \(r = -1.\) The right group is zero \(r = -1\), this is because by (2) \((i')^*\text{Ext}^{s,tq-q}_{\Lambda}(H^*K, H^*K) \subset \text{Ext}^{s,tq+q}_{\Lambda}(H^*K, H^*M) = 0 \text{ and } \text{Ext}^{s,tq+1}_{\Lambda}(H^*K, H^*M) = 0. \) The right group has unique generator \(\sigma_K\) for \(r = 0\) which satisfies \((\alpha_1 \wedge 1K)^*(\sigma_K) \neq 0 \in \text{Ext}^{s+1,tq+q}_{\Lambda}(H^*K, H^*K), \) this is because \((i')^*(\alpha_1 \wedge 1K)^*(\sigma_K) = (\alpha_1 \wedge 1M)^*(i')^*(\sigma_K) = (\alpha_1 \wedge 1M)^*(\sigma) = (\tilde{\sigma}^*)^*(\alpha_1 \wedge 1M)^*(\sigma) \neq 0 \in \text{Ext}^{s+1,tq+q}_{\Lambda}(H^*K, H^*M). \) Then the middle group is zero for \(r = 0\) and has unique generator \((i'' \wedge 1K)^*(\sigma_K)\) for \(r = -1\) so that the first result can be obtained by the following exact sequence
\[
0 = \text{Ext}^{s,tq}_{\Lambda}(H^*K, H^*L \wedge K) \xrightarrow{(i''\wedge 1K)^*} \text{Ext}^{s,tq}_{\Lambda}(H^*L \wedge K, H^*L \wedge K) \\
\text{Ext}^{s,tq}_{\Lambda}(H^*L \wedge K, H^*L \wedge K) \xrightarrow{(\alpha_1 \wedge 1K)^*}
\]
induced by (9.1.3). For the second result, look at the following exact sequence \((r = 1, 2, 3, u = 0, 1, 2)\)
\[
0 = \text{Ext}^{s,tq+rq+u}_{\Lambda}(H^*K, H^*M) \xrightarrow{(i''\wedge 1K)^*} \text{Ext}^{s,tq+rq+u}_{\Lambda}(H^*L \wedge K, H^*M) \\
\text{Ext}^{s,tq+rq+u}_{\Lambda}(H^*L \wedge K, H^*M) \xrightarrow{(\alpha_1 \wedge 1K)^*}
\]
induced by (9.1.3). By (2), the left group is zero for \(r = 1, 2, 3, u = 0, 1, 2\) and the right group also is zero for \(r = 2, 3, u = 0, 1, 2.\) By Prop. 9.3.0 and the supposition , the right group also is zero for \(r = 1, u = 1, 2.\) For \(r = 1, u = 0,\) The right group has unique generator \((i')^*(\tilde{\sigma})\) which satisfies \((\alpha_1 \wedge 1K)^*(i')^*(\tilde{\sigma}) \neq 0. \) Then the middle group is zero for \(r = 1, 2, 3, u = 0, 1, 2.\)

(4) Consider the following exact sequence \((r = 1, 2, 3, u = 0, 1, 2)\)
\[
0 = \text{Ext}^{s,tq+rq+u}_{\Lambda}(H^*L \wedge K, H^*M) \xrightarrow{(i''\wedge 1K)^*} \text{Ext}^{s,tq+rq+u}_{\Lambda}(H^*W \wedge K, H^*M) \\
\text{Ext}^{s,tq+rq+u}_{\Lambda}(H^*W \wedge K, H^*M) \xrightarrow{(\phi \wedge 1K)^*}
\]
induced by (9.1.12). By (3), the left group is zero for \(r = 1, 2, 3, u = 0, 1, 2.\) By (2), the right group is zero for \(r = 1, 3, u = 0, 1, 2\) and by Prop. 9.3.0
and the supposition, it also is zero for $r = 2, u = 1$. For $r = 2, u = 0$, the right group has unique generator $(i')_{u} (\sigma)$ which satisfies \((\phi \wedge 1_{K})_{u} (i')_{u} (\sigma) \neq 0 \in Ext^{s+1, t+2q}_{A}(H^{*}L \wedge K, H^{*}M)\). Then the middle group is zero for $r = 1, 2, 3, u = 0, 1, 2$ as desired.

Since \((\bar{\nu}w_{2}w \wedge 1_{M})^{*} Ext^{s,t+q}_{A}(H^{*}W \wedge K, H^{*}X \wedge M) \subset Ext^{s,t+q}_{A}(H^{*}W \wedge K, H^{*}M) = 0\), then, by (9.1.5), \((\bar{\nu}w_{2}w \wedge 1_{M})^{*} Ext^{s,t+q}_{A}(H^{*}W \wedge K, H^{*}X \wedge M) = (\bar{\nu} \wedge 1_{M})^{*} Ext^{s,t+q+2}^{1}(H^{*}W \wedge K, H^{*}M \wedge M) = 0\). and by using (9.2.12) we know that \(Ext^{s,t+q+2}^{1}(H^{*}W \wedge K, H^{*}X \wedge M) = (\bar{\psi} \wedge 1_{M})^{*} Ext^{s+3}^{1,q}(H^{*}W \wedge K, H^{*}M \wedge M) = 0\). Q.E.D.

The proof of the main Theorem C will be done by some argument processing in the Adams resolution (cf. 9.2.9) of some spectra related to the sphere spectrum $S$. Before proving the main Theorem C, we first prove the following Lemmas.

**Lemma 9.4.12** Under the supposition (I)(II) of the main Theorem C we have

1. Let $\tilde{h}_{0} \sigma \in [\Sigma^{q+1} M, KG_{s+1} \wedge M]$ be a $d_{1}$-cycle which represents \(\alpha_{*} (\bar{\sigma}) \in Ext^{s+1, t+q+1}_{A}(H^{*}M, H^{*}M)\), then \((\bar{c}_{s+1} \wedge 1_{M})^{*} \tilde{h}_{0} \sigma = (1_{E_{s+2}} \wedge \alpha) (\bar{\kappa} \wedge 1_{M}) \) (up to scalar), where \(\kappa \in \pi_{t+1} E_{s+2} \) such that \(\bar{a}_{s+1} \cdot \kappa = \bar{c}_{s} \cdot \sigma \) and \(\sigma \in \pi_{t+1} KG_{s} \cong Ext^{s+1}_{A}(Z_{p}, Z_{p})\).

2. \((1_{E_{s+2}} \wedge \phi \wedge 1_{M}) (\kappa \wedge 1_{M}) = 0, (1_{E_{s+2}} \wedge \alpha_{1} \wedge 1_{M}) (\kappa \wedge 1_{M}) = 0\).

**Proof:** (1) Since \((1_{KG_{s+1}} \wedge i)^{*} \tilde{h}_{0} \sigma \) is a $d_{1}$-boundary, then \((\bar{c}_{s+1} \wedge 1_{K}) (1_{KG_{s+1}} \wedge i')^{*} \tilde{h}_{0} \sigma = 0\) so that \((\bar{c}_{s+1} \wedge 1_{M})^{*} \tilde{h}_{0} \sigma = (1_{E_{s+2}} \wedge \alpha) f' \) for some \(f' \in [\Sigma^{q+1} M, E_{s+2} \wedge M]\). It follows that \((\bar{a}_{s+1} \wedge 1_{M}) (1_{E_{s+2}} \wedge \alpha) f' = 0\) and so \((\bar{a}_{s+1} \wedge 1_{M}) f' = (1_{E_{s+1}} \wedge j') f_{2}' \) with \(f_{2}' \in [\Sigma^{q+1} M, E_{s+1} \wedge K]\). The $d_{1}$-cycle \((\bar{b}_{s+1} \wedge 1_{K}) f_{2}' \) represents an element in \(Ext^{s+1, t+q+1}_{A}(H^{*}K, H^{*}M)\) and this group is zero by Prop. 9.3.2(2), then \((\bar{b}_{s+1} \wedge 1_{K}) f_{2}' = (\bar{b}_{s+1} \wedge 1_{K}) g_{0}' \) for some \(g_{0}' \in [\Sigma^{q+1} M, KG_{s} \wedge K]\). Consequently we have, \(f_{2}' = (\bar{c}_{s} \wedge 1_{K}) g_{0}' \) so \(f_{3}' = (\bar{a}_{s+1} \wedge 1_{K}) f_{3}' \) for some \(f_{3}' \in [\Sigma^{q+2} M, E_{s+2} \wedge K]\) and so \((\bar{a}_{s+1} \wedge 1_{M}) f' = (\bar{a}_{s+1} \wedge 1_{M}) (1_{E_{s+2}} \wedge j') f_{3}' + (\bar{c}_{s} \wedge 1_{M}) (1_{KG_{s}} \wedge j') g_{0}' = (\bar{a}_{s+1} \wedge 1_{M}) (1_{E_{s+2}} \wedge j') f_{3}' + (\bar{c}_{s} \wedge 1_{M}) (1_{KG_{s}} \wedge j') g_{0}' \) where the $d_{1}$-cycle \((1_{KG_{s}} \wedge j') g_{0}' \in [\Sigma^{q} M, KG_{s} \wedge M]\) represents an element in \(Ext^{s,t+q}_{A}(H^{*}M, H^{*}M)\) and this group has unique generator \(\tilde{\sigma}\) so that it equals to \(\sigma \wedge 1_{M} \) (mod $d_{1}$-boundary). Hence we have \(f' = (1_{E_{s+1}} \wedge j') f_{3}' + (\kappa \wedge 1_{M}) + \frac{1}{2} \).
\((c_{s+1} \land M) \tilde{g}_1\) for some \(\tilde{g}_1 \in [\Sigma^{q+1}M, KG_{s+1} \land M]\) and so \((c_{s+1} \land M) h_0 \sigma = (1_{E_{s+1}} \land \alpha) f' = (1_{E_{s+1}} \land \alpha)(\kappa \land 1_M)\) which shows the result.

(2) Since \(\text{Ext}^{s+1,tq+2q}_A(Z_p, Z_p)\) is zero for \(r = 2\) and has unique generator \(h_0 \sigma = (j'' \psi)(\phi)_*(\sigma)\) for \(r = 1\), then \(\text{Ext}^{s+1,tq+2q}_A(H^*L, Z_p) \cong Z_p\{((\phi)_*(\sigma))\} \text{ and } \text{Ext}^{s+1,tq+2q}_A(H^*W, Z_p) = 0\). By this and a similar proof as given in (1) we know that \((1_{E_{s+1}} \land \phi)(\tilde{c}_{s+1} \land 1_L)\sigma \phi\) (up to scalar), where \(\sigma \phi \in \pi_{tq+2q}(KG_{s+1} \land L)\) is a \(d_1\)-cycle which represents \((\phi)_*(\sigma) \in \text{Ext}^{s+1,tq+2q}_A(H^*L, Z_p)\). Then, by the supposition (II) of the main Theorem C we have \((1_{E_{s+1}} \land \phi \land 1_M)(\kappa \land 1_M) = (c_{s+1} \land 1_{L \land M})(\sigma \phi \land 1_M) = 0\) so that the result follows.

Q.E.D.

**Lemma 9.4.13** Under the supposition (I) of the main Theorem C we have

1. \(\text{Ext}^{s,tq}_A(H^*X \land M, H^*X \land M) \cong Z_p\{[\sigma \land 1_{X \land M}]\}.

2. For any \(d_1\)-cycle \(g_0 \in [\Sigma^{tq+2}X, KG_{s+1} \land X]\), \(g_0 = \chi'(h_0 \sigma \land 1_X)\) (mod \(d_1\)-boundary) with \(\chi' \in Z_p\) and \((\psi_{X \land M}, h_0 \sigma \land 1_{X \land M}) \neq 0 \in \text{Ext}^{s,tq+1}_A(H^*Y \land M, H^*X \land M).

**Proof** (1) Consider the following exact sequence

\[
\text{Ext}^{s,tq}_A(H^*L \land K, H^*M) \xrightarrow{\tilde{g}_1} \text{Ext}^{s+1,tq+2q}_A(H^*L \land K, H^*X \land M) \xrightarrow{(1_{L \land K}, \tilde{g}_1)} \text{Ext}^{s,tq}_A(H^*L \land K, H^*L \land K)
\]

induced by (9.4.2). By Prop. 9.4.11(3), the left group is zero and the right group has unique generator \(\sigma_{L \land K}\) which satisfies \(((1_{L \land K}, \tilde{g}_1))([\sigma \land 1_K]) \neq 0 \in \text{Ext}^{s+1,tq+2q}_A(H^*L \land K, H^*M)\), this is because \((j'' \land 1_K)_*(1_{L \land K}) = (1_{L \land K})_*([\sigma \land 1_K])\) \((1_{L \land K}, \tilde{g}_1) = (1_{L \land K}, \tilde{g}_1, \sigma_{L \land K}) = ((1_{L \land K}, \tilde{g}_1))([\sigma \land 1_K]) \neq 0 \in \text{Ext}^{s+1,tq+2q}_A(H^*L, H^*M)\). The middle group is zero. Look at the following exact sequence

\[
\text{Ext}^{s,tq}_A(H^*L \land K, H^*X \land M) \xrightarrow{\tilde{g}_1} \text{Ext}^{s,tq}_A(H^*X \land M, H^*X \land M) \xrightarrow{\text{Ext}^{s,tq}_A(H^*M, H^*X \land M)}
\]

induced by (9.4.2). As shown above, the left group is zero. By Prop. 9.4.11(1), the right group has unique generator \(m_M([\tilde{\psi} \land 1_M]) = m_M([\tilde{\psi} \land 1_M]) \neq 0\) so that the result follows.
Then the middle group has unique generator \([\sigma \wedge 1_{X \wedge M}]\) as desired.

(2) Note that \((\tilde{\psi})_*(\tilde{uw}_2)^*\text{Ext}^{s+1,tq+q}_A(H^*X, H^*X) \subset \text{Ext}^{s+1,tq-q-1}_A(H^*M, H^*Y)\). Similar to that in Prop. 9.3.0(1), by the supposition we know that \(\text{Ext}^{s+1,tq}_A(H^*M, H^*M)\) is zero or has (one or two) generator \(\tilde{\sigma}'\), then \(\text{Ext}^{s+1,tq-q-1}_A(H^*M, H^*Y)\) is zero or has (one or two) generator \((\overline{w})^*(\tilde{\sigma}')\) and it satisfies \(((1_Y \wedge j)\alpha_{Y \wedge M})_*(\overline{w})^*(\tilde{\sigma}') = ((1_Y \wedge j)\alpha_{Y \wedge M})_*[\sigma' \wedge 1_Y] = (1_Y \wedge \alpha_1)_*[\sigma' \wedge 1_Y] = [h_0\sigma' \wedge 1_Y] \neq 0\), then \((\tilde{\psi})_*(\tilde{uw}_2)^*\text{Ext}^{s+1,tq+q}_A(H^*X, H^*X) = 0\) and so we have \((\tilde{uw}_2)^*\text{Ext}^{s+1,tq+q}_A(H^*X, H^*X) = (\tilde{uw}_2)^*\text{Ext}^{s+1,tq+q}_A(H^*Y, H^*Y) = 0\), this is because \(\text{Ext}^{s+1,tq+q}_A(H^*Y, H^*Y) \cong Z_p(((1_Y \wedge j)\alpha_{Y \wedge M})_*(\overline{w})^*(\tilde{\sigma}))\) (cf. Prop. 9.4.10(2)). Then \(\text{Ext}^{s+1,tq+q}_A(H^*X, H^*X) = (\tilde{\psi})_*(\tilde{uw}_2)^*\text{Ext}^{s+1,tq+q}_A(H^*X, H^*X)\) and it has unique generator \((\tilde{\psi})^*((1_X \wedge j)\alpha_{X \wedge M})_*(\tilde{\sigma}) = ((1_X \wedge j)\alpha_{X \wedge M})_*[\sigma \wedge 1_M] = (1_X \wedge j)_*[\sigma \wedge 1_M] = [h_0\sigma \wedge 1_X]\) (cf. Prop. 9.4.10(3)). Then the first result follows. For the second result, by (9.4.6), the \(d_1\)-cycle \((1_{KG_{s+1}} \wedge m_3(M))\psi_{X \wedge M})_*(h_0\sigma \wedge 1_{X \wedge M}) = (1_{KG_{s+1}} \wedge m_3(M))\psi_{X \wedge M})_*(h_0\sigma \wedge 1_{X \wedge M}) = (1_{KG_{s+1}} \wedge m_3(M))\psi_{X \wedge M})_*(h_0\sigma \wedge 1_{X \wedge M}) = (1_{KG_{s+1}} \wedge m_3(M))\psi_{X \wedge M})_*(h_0\sigma \wedge 1_{X \wedge M}) = (h_0\sigma \wedge 1_{X \wedge M}) = (h_0\sigma \wedge 1_{X \wedge M}) = m_3(M)_*(\tilde{\psi} \wedge 1_M)^*(\alpha_1 \wedge 1_M)_*(\tilde{\sigma}) \neq 0\) so that the second result follows. Q.E.D.

**Proof of the main Theorem C**  By Lemma 9.4.12(1), it suffices to prove \((\tilde{e}_{s+1} \wedge 1_M)\tilde{h}_0\sigma = (1_{E_{s+2} \wedge 1})_*(\tilde{h}_0\sigma) = 0\). The proof is divided into the following two steps.

**Step 1**  To prove \((\kappa \wedge 1_{X \wedge M})_*(1_X \wedge \alpha) = 0\).

By (9.4.3), \((\phi \wedge 1_{X \wedge M})m_3(M)_*(\tilde{\psi} \wedge 1_M) = (u'' \wedge 1_M)_*(1_X \wedge \alpha)\), then by Lemma 9.4.12(2) we have \((1_{E_{s+2} \wedge 1_{X \wedge M}})_*(1_{E_{s+2} \wedge 1_X \wedge \alpha}_*(\kappa \wedge 1_{X \wedge M}) = (1_{E_{s+2} \wedge 1_{X \wedge M}})_*(\kappa \wedge 1_X \wedge \alpha)_*(\tilde{h}_0\sigma) = 0\). Moreover, by (9.2.16) we have \((1_{E_{s+2} \wedge 1_X \wedge \alpha}_*(\kappa \wedge 1_{X \wedge M}) = (1_{E_{s+2} \wedge \tilde{u}w_3 \wedge 1_M})f\) for some \(f \in [\Sigma^{q+1}X \wedge M, E_{s+2} \wedge W \wedge M] \cap (\text{kerd})\) (cf. Cor. 6.4.15). By composing \((1_{E_{s+2} \wedge 1_X \wedge \iota' i \wedge 1_M})\) on the above equation we have \((1_{E_{s+2} \wedge \tilde{u}w_3 \wedge 1_{K \wedge M}})_*(1_{E_{s+2} \wedge 1_W \wedge \iota' i \wedge 1_M})f = (1_{E_{s+2} \wedge 1_X \wedge (\iota' i \wedge 1_M)_*(\kappa \wedge 1_{X \wedge M}) = (1_{E_{s+2} \wedge 1_X \wedge (\tilde{h}_0\sigma \wedge 1_{X \wedge M})_*(\kappa \wedge 1_{X \wedge M}) = 0\) in Lemma 9.4.12(2). Consequently, by (9.2.16), \((1_{E_{s+2} \wedge 1_L \wedge \iota' i \wedge 1_M})f = (1_{E_{s+2} \wedge 1_L \wedge \iota' i \wedge 1_{K \wedge M})f_2 = 0\) for some \(f_2 \in [\Sigma^{q+1}X \wedge M, E_{s+2} \wedge L \wedge K \wedge M]_2\) this is because \(\pi \wedge 1_K = 0\). Then by (9.1.4) we have \(f = (1_{E_{s+2} \wedge 1_W \wedge \iota' i \wedge 1_M})f_3 = (1_{E_{s+2} \wedge 1_W \wedge \alpha M_3(M)}_*(\tilde{h}_0\sigma) = 0\).
f_3 \in [\Sigma^{q+q+2}X \wedge M, E_{s+2} \wedge W \wedge Y \wedge M] \cap (\ker d) \quad (\text{cf. Cor. 6.4.15}) \quad \text{and so}

(9.4.14) \quad (1_{E_{s+2}} \wedge 1_X \wedge \alpha)(\kappa \wedge 1_X \wedge M)

= (1_{E_{s+2}} \wedge \bar{w}u \wedge 1_M)(1_{E_{s+2}} \wedge 1_W \wedge \alpha m_M(1_M \wedge \bar{\pi}))f_3

= (1_{E_{s+2}} \wedge \alpha X \wedge M(j''u \wedge 1_M))(1_{E_{s+2}} \wedge 1_W \wedge m_M(1_M \wedge \bar{\pi}))f_3 \quad (\text{cf. (9.4.3)})

By (9.4.14),

(\bar{a}_{s+1} \wedge 1_{X \wedge M})(1_{E_{s+2}} \wedge \bar{w}u \wedge 1_M)(1_{E_{s+2}} \wedge 1_W \wedge \alpha m_M(1_M \wedge \bar{\pi}))(1_X \wedge \alpha)(\sigma \wedge 1_X \wedge M) = 0, \text{this is because } \alpha \text{ induces zero homomorphism in } Z_p\text{-cohomology. Then, by (9.2.16) and } w'(\pi \wedge 1_L) \wedge 1_M = (w \wedge 1_M)(1_L \wedge \alpha) \text{ we have}

(9.4.15) \quad (\bar{a}_{s+1} \wedge 1_{W \wedge M})(1_{E_{s+2}} \wedge 1_W \wedge \alpha m_M(1_{W \wedge 1_M}))f_3

= (1_{E_{s+1}} \wedge 1_W \wedge \alpha)(w \wedge 1_M))f_5

with f_5 \in [\Sigma^{q}X \wedge M, E_{s+1} \wedge L \wedge M] \cap (\ker d) \quad (\text{cf. Cor. 6.4.15})).

By (9.4.15)(9.12),

(\bar{a}_{s+1} \wedge 1_{W \wedge M})(1_{E_{s+2}} \wedge 1_W \wedge m_M(1_{W \wedge 1_M}))f_3 = (1_{E_{s+2}} \wedge w \wedge 1_M)f_5 + (1_{E_{s+2}} \wedge 1_W \wedge j')f_6 \text{ for some } f_6 \in [\Sigma^{q+q+1}X \wedge M, E_{s+1} \wedge W \wedge M] \cap (\ker d) \quad (\text{cf. Prop. 6.5.26}). \text{ Since } (1_W \wedge \alpha)(w = w(1_L \wedge \alpha) = w \cdot \phi j'' = 0, \text{ then } w = (1_W \wedge j'')(\psi W), \text{ where } \psi W \in [\Sigma^{q}L, W \wedge L]. \text{ So we have}

w \wedge 1_M = (1_W \wedge j'')(\psi W) \wedge 1_M = (1_W \wedge m_M(1_{W \wedge 1_M}))(1_W \wedge \tilde{h})\psi W \wedge 1_M.

Hence, 

\(-\bar{a}_{s+1} \wedge 1_{W \wedge Y \wedge M})f_3 = (1_{E_{s+2}} \wedge 1_W \wedge \kappa \wedge 1_M)\tilde{\psi}W \wedge 1_M f_5 + (1_{E_{s+1}} \wedge 1_W \wedge (1_Y \wedge \kappa)\kappa \wedge 1_M)\tilde{\psi}W \wedge 1_M f_6 + (1_{E_{s+1}} \wedge 1_L \wedge (1_Y \wedge \kappa)\kappa \wedge 1_M)\tilde{\psi}W \wedge 1_M f_7 \text{ by Prop. 6.5.26,}

f_7 = f_6(1_X \wedge i') + f_6(1_X \wedge i' \wedge i) \text{, where } f_6 \in [\Sigma^{q+q+1}X \wedge K, E_{s+1} \wedge W \wedge K] \cap (\ker d) \text{ and } f_9 \in [\Sigma^{q+q+1+1}X \wedge K, E_{s+1} \wedge W \wedge K] \cap (\ker d). \text{ Since } d((1_Y \wedge \kappa)\kappa \wedge 1_M)\tilde{\psi}W \wedge 1_M f_6 = (1_{E_{s+1}} \wedge 1_W \wedge (r \wedge 1_M)\tilde{\psi}W \wedge 1_M)\tilde{\psi}W \wedge 1_M f_7 = (1_{E_{s+1}} \wedge 1_W \wedge (1_Y \wedge \kappa)\kappa \wedge 1_M)\tilde{\psi}W \wedge 1_M f_6 \text{ by applying the derivation } d \text{ using Theorem 6.4.8(1)} \quad \text{we have} \quad (1_{E_{s+1}} \wedge 1_W \wedge (r \wedge 1_M)\tilde{\psi}W \wedge 1_M)\tilde{\psi}W \wedge 1_M f_6 - (1_{E_{s+1}} \wedge 1_W \wedge (1_Y \wedge \kappa)\kappa \wedge 1_M)\tilde{\psi}W \wedge 1_M f_9(1_X \wedge i') \quad \text{is 0} \quad \text{(Note: } f_6 \text{ has odd degree) and so}

(9.4.16) \quad -(\bar{a}_{s+1} \wedge 1_{W \wedge Y \wedge M})f_3 = (1_{E_{s+1}} \wedge 1_W \wedge \tilde{h})\psi W \wedge 1_M f_5

+ (1_{E_{s+1}} \wedge 1_W \wedge (1_Y \wedge \kappa)\kappa \wedge 1_M)\tilde{\psi}W \wedge 1_M f_6 + (1_{E_{s+1}} \wedge 1_W \wedge (1_Y \wedge \kappa)\kappa \wedge 1_M)\tilde{\psi}W \wedge 1_M f_7 \quad (1_X \wedge i')

= (1_{E_{s+1}} \wedge 1_W \wedge (r \wedge 1_M)\tilde{\psi}W \wedge 1_M)\tilde{\psi}W \wedge 1_M f_6(1_X \wedge i)

Note that the d_1-cycle

(\tilde{b}_{s+1} \wedge 1_{W \wedge K}) f_6 \in [\Sigma^{q+q+1}X \wedge M, KG_{s+1} \wedge W \wedge K] \cap (\ker d) \text{ represents an element in } Ext_{A}^{s+1,q+1}(H \wedge W \wedge K, H \wedge X \wedge M) \text{ and by Prop. 9.4.10(1)} \text{ this group is zero, then}

(\tilde{b}_{s+1} \wedge 1_{W \wedge K}) f_6 = (\tilde{\psi}W \wedge 1_M)\tilde{\psi}W \wedge 1_M f_6 \text{ for some } g \in [\Sigma^{q+q+1}X \wedge M, KG_{s+1} \wedge W \wedge K] \cap (\ker d) \quad (\text{cf. Prop. 6.5.26}) \text{ and so } f_6 = (\tilde{\psi}W \wedge 1_M)\tilde{\psi}W \wedge 1_M f_6 \text{ with } f' \in [\Sigma^{q+q+2}X \wedge M, E_{s+2} \wedge W \wedge K] \cap (\ker d) \quad (\text{cf. Prop. 6.5.26}). \text{ Then we}
have

\[(9.4.17)\quad -(\bar{a}_{s+1} \land 1_W \land Y \land M) f_3 = (1_{E_{s+1}} \land (1_W \land \tilde{h}) \psi_W \land 1_M) f_5 \]
\[(+ (\bar{a}_{s+1} \land 1_W \land Y \land M)(1_{E_{s+2}} \land 1_W \land (1_Y \land i)r) f') \]
\[+ (e_s \land 1_W \land Y \land M)(1_{KGs} \land 1_W \land (1_Y \land i) r) g \]
\[+ (a_{s+1} \land 1_W \land Y \land M)(1_{E_{s+2}} \land 1_W \land (r \land 1_M) \overline{m}_K) f'(1_X \land i j) \]
\[- (e_s \land 1_W \land Y \land M)(1_{KGs} \land 1_W \land (r \land 1_M) \overline{m}_K) g(1_X \land i j) \]
\[+ (1_{E_{s+1}} \land 1_W \land (r \land 1_M) \overline{m}_K) f_8 (1_X \land \epsilon') \]

Let \(P\) be the cofibre of \((1_W \land \tilde{h}) \psi_W : \Sigma^{q+1} L \to W \land Y\) given by the cofibration

\[(9.4.18)\quad \Sigma^{q+1} L \to (1_W \land \tilde{h}) \psi_W W \land Y \xrightarrow{w_5} P \xrightarrow{u_5} \Sigma^{q+2} L \]

Then the cofibre of \(w_5(1_W \land r) : W \land K \to P\) is \(\Sigma X\) given by the cofibration

\[(9.4.19)\quad W \land K \xrightarrow{w_5(1_W \land r)} V \xrightarrow{w_6} \Sigma X \xrightarrow{u_6} \Sigma W \land K \]

This can be seen by the following homotopy commutative diagram of \(3 \times 3\)-Lemma

\[
\begin{array}{ccc}
W \land K & \xrightarrow{w_5(1_W \land r)} & P \\
\swarrow 1_W \land r & \wedge & \wedge & \searrow u'' \\
W \land Y & \xrightarrow{w_5} & W \land \epsilon & \xrightarrow{\bar{u}w_3} & u_6 \\
\Sigma^{q+1} L & \xrightarrow{w'(\pi \land 1_L)} & \Sigma W & \xrightarrow{1_W \land \epsilon'} & \Sigma W \land K \\
\end{array}
\]

Note that \(u_6 = \mu_{X \land M}(1_X \land i)\), then by composing \((\bar{b}_{s+1} \land 1_P)(1_{E_{s+1}} \land w_5 \land j)\) on the left hand side of \((9.4.17)\) and composing \((1_X \land i)\) on the right hand side we have \((\bar{b}_{s+1} \land 1_P)(1_{E_{s+1}} \land w_5(1_W \land r)) f_8(1_X \land \epsilon') = 0\) and so \((\bar{b}_{s+1} \land 1_W \land K) f_8(1_X \land \epsilon') = (1_{KG_{s+1}} \land u_6) g_0 = (1_{KG_{s+1}} \land \mu_{X \land M}(1_X \land i)) g_0 = (1_{KG_{s+1}} \land \mu_{X \land M}) (g_0 \land 1_M)(1_X \land i)\) with \(d_1\)-cycle \(g_0 \in [\Sigma^{q+q} X, KG_{s+1} \land X]\). Moreover , by Lemma 9.4.13(2), \(g_0 = \lambda_1(h_0 \sigma \land 1_X)\) (mod \(d_1\)-boundary), where \(\lambda_1 \in Z_p\). On the other hand, by applying the derivation \(d\) to \((\bar{b}_{s+1} \land 1_W \land K) f_8(1_X \land \epsilon') = (1_{KG_{s+1}} \land \mu_{X \land M})(g_0 \land 1_M)(1_X \land i)\) we have

\[(9.4.20)\quad (\bar{b}_{s+1} \land 1_W \land K) f_8(1_X \land \epsilon') = (1_{KG_{s+1}} \land \mu_{X \land M})(g_0 \land 1_M),\]
\[g_0 = \lambda_1(h_0 \sigma \land 1_X) \in [\Sigma^{q+q} X, KG_{s+1} \land X] \quad \text{(mod } d_1\text{-boundary)}\]
Consider the following commutative diagram of exact sequences

\[ \Sigma^{q+1}L \wedge M \xrightarrow{u \wedge 1_M} \Sigma^{q+1}W \wedge M \xrightarrow{j''u \wedge 1_M} \Sigma^{3q+1}M \xrightarrow{\phi \wedge 1_M} \Sigma^{q+2}L \wedge M \]

\[ \uparrow 1_{L \wedge M} \quad \uparrow 1_{W \wedge M} \quad \uparrow (\mathfrak{w} \wedge \Sigma^{q+1}M) \quad \uparrow u_7 \quad \uparrow 1_{L \wedge M} \]

\[ \Sigma^{q+1}L \wedge M \xrightarrow{(1W \wedge \mathfrak{h})\psi_W \wedge 1_M} W \wedge Y \wedge M \xrightarrow{w_5 \wedge 1_M} P \wedge M \xrightarrow{w_5 \wedge 1_M} \Sigma^{q+2}L \wedge M \]

of the cofibrations (9.1.12)(9.4.18). Since the left rectangle homotopy commutes then there exists \( u_7 \in [\Sigma^{-3q-1}P \wedge M, M] \) such that all the above rectangle homotopy commute. That is we have

\[(9.4.21)\]

\[ u_7(w_5 \wedge 1_M) = (j''u \wedge 1_M)(1W \wedge m_M(\mathfrak{w} \wedge 1_M)), \]

\[ (\phi \wedge 1_M)u_7 = \pm \ u_5 \wedge 1_M \]

where \( u_7 \in [\Sigma^{-3q-1}P \wedge M, M] \). By the above two equations, we have the following homotopy commutative diagram of \( 3 \times 3 \)-Lemma in which we use the cofibrations (9.2.12)(9.4.18)(9.1.23)

\[ \begin{array}{ccc}
P \wedge M & \xrightarrow{u_5 \wedge 1_M} & \Sigma^{q+2}L \wedge M \\
\downarrow u_7 & \nearrow \phi \wedge 1_M & \downarrow (1W \wedge \mathfrak{h})\psi_W \wedge 1_M \\
\Sigma^{q+1}M & \xrightarrow{j''u \wedge 1_M} & \Sigma^{3q+2}M \\
\downarrow j''u \wedge 1_M & \nearrow (\phi_W \wedge 1_K)'' & \downarrow (1W \wedge (r \wedge m_M)\mathfrak{w}_K) \\
\Sigma^{q+2}W \wedge M & \xrightarrow{w_5 \wedge 1_M} & \Sigma P \wedge M
\end{array} \]

(9.4.22)

Then there is a cofibration

\[(9.4.23)\]

\[ \Sigma^{3q-1}M \xrightarrow{(\phi_W \wedge 1_K)''} W \wedge K \]

\[ \Sigma^{-1}P \wedge M \xrightarrow{w_7} \Sigma^{q+1}M \]

in which \( \phi_W \in [\Sigma^{3q-1}S, W] \) such that \( u \cdot \phi_W = \phi \in [\Sigma^{2q-1}S, L] \). Since \((\phi \wedge 1_K)'' \cdot u_7 = (u \cdot \phi_W \wedge 1_K)'' \cdot u_7 = 0 \), then by (9.4.2) we have

\[(9.4.24)\]

\[ u_7 = m_M(\mathfrak{w} \wedge 1_M)u_8 \]

where \( u_8 \in [\Sigma^{-q-1}P \wedge M, X \wedge M] \). On the other hand, by (9.4.8), \((\omega \wedge 1_M)u_8(u_5(1W \wedge r) \wedge 1_M) = \alpha_{Y, A, M}m_M(\mathfrak{w} \wedge 1_M)u_8(u_5(1W \wedge r) \wedge 1_M) = \alpha_{Y, A, M}u_7 \]

\( (u_5(1W \wedge r) \wedge 1_M) = \alpha_{Y, A, M}u_7 \)

Then, by (9.4.7), \( u_8(u_5(1W \wedge r) \wedge 1_M) = (\omega \wedge 1_M)u_7(1W \wedge r) \wedge 1_M = \Delta_1 \)

with \( \Delta_1 \in [\Sigma^{q-1}W \wedge K \wedge M, L \wedge K \wedge M] \cap (ker d) \). By composing \( \mu_{X, A, M}(1X \wedge i) \wedge 1_M \) on the above equation and using (9.4.19) we have \((\omega \wedge 1_M)u_7(1W \wedge r) \wedge 1_M = 0 \) and so by (9.4.7)(9.4.6), \( \Delta_1(\mu_{X, A, M}(1X \wedge i) \wedge 1_M) = (\mathfrak{w}_2(1Y \wedge i')) \wedge 1_M \)

\( \psi_{X, A, M} \) and then \( (j''u \wedge 1_K \wedge M) \Delta_1(\mu_{X, A, M}(1X \wedge i) \wedge 1_M) = (j''u \wedge 1_K \wedge M)(i' \wedge 1_M) \)

\( \psi_{X, A, M} = (i' \wedge 1_M) \)

\( (i' \wedge 1_M) \)

\[ (j''u \wedge 1_K \wedge M) \Delta_1(\mu_{X, A, M}(1X \wedge i) \wedge 1_M) = (j''u \wedge 1_K \wedge M)(i' \wedge 1_M) \]

\[ (i' \wedge 1_M) \]

\[ (j''u \wedge 1_K \wedge M) \Delta_1(\mu_{X, A, M}(1X \wedge i) \wedge 1_M) = (j''u \wedge 1_K \wedge M)(i' \wedge 1_M) \]

\[ (i' \wedge 1_M) \]

\[ (j''u \wedge 1_K \wedge M) \Delta_1(\mu_{X, A, M}(1X \wedge i) \wedge 1_M) = (j''u \wedge 1_K \wedge M)(i' \wedge 1_M) \]

\[ (i' \wedge 1_M) \]

\[ (j''u \wedge 1_K \wedge M) \Delta_1(\mu_{X, A, M}(1X \wedge i) \wedge 1_M) = (j''u \wedge 1_K \wedge M)(i' \wedge 1_M) \]

\[ (i' \wedge 1_M) \]

\[ (j''u \wedge 1_K \wedge M) \Delta_1(\mu_{X, A, M}(1X \wedge i) \wedge 1_M) = (j''u \wedge 1_K \wedge M)(i' \wedge 1_M) \]

\[ (i' \wedge 1_M) \]

\[ (j''u \wedge 1_K \wedge M) \Delta_1(\mu_{X, A, M}(1X \wedge i) \wedge 1_M) = (j''u \wedge 1_K \wedge M)(i' \wedge 1_M) \]

\[ (i' \wedge 1_M) \]

\[ (j''u \wedge 1_K \wedge M) \Delta_1(\mu_{X, A, M}(1X \wedge i) \wedge 1_M) = (j''u \wedge 1_K \wedge M)(i' \wedge 1_M) \]

\[ (i' \wedge 1_M) \]

\[ (j''u \wedge 1_K \wedge M) \Delta_1(\mu_{X, A, M}(1X \wedge i) \wedge 1_M) = (j''u \wedge 1_K \wedge M)(i' \wedge 1_M) \]

\[ (i' \wedge 1_M) \]

\[ (j''u \wedge 1_K \wedge M) \Delta_1(\mu_{X, A, M}(1X \wedge i) \wedge 1_M) = (j''u \wedge 1_K \wedge M)(i' \wedge 1_M) \]

\[ (i' \wedge 1_M) \]

\[ (j''u \wedge 1_K \wedge M) \Delta_1(\mu_{X, A, M}(1X \wedge i) \wedge 1_M) = (j''u \wedge 1_K \wedge M)(i' \wedge 1_M) \]

\[ (i' \wedge 1_M) \]

\[ (j''u \wedge 1_K \wedge M) \Delta_1(\mu_{X, A, M}(1X \wedge i) \wedge 1_M) = (j''u \wedge 1_K \wedge M)(i' \wedge 1_M) \]

\[ (i' \wedge 1_M) \]

\[ (j''u \wedge 1_K \wedge M) \Delta_1(\mu_{X, A, M}(1X \wedge i) \wedge 1_M) = (j''u \wedge 1_K \wedge M)(i' \wedge 1_M) \]

\[ (i' \wedge 1_M) \]

\[ (j''u \wedge 1_K \wedge M) \Delta_1(\mu_{X, A, M}(1X \wedge i) \wedge 1_M) = (j''u \wedge 1_K \wedge M)(i' \wedge 1_M) \]

\[ (i' \wedge 1_M) \]

\[ (j''u \wedge 1_K \wedge M) \Delta_1(\mu_{X, A, M}(1X \wedge i) \wedge 1_M) = (j''u \wedge 1_K \wedge M)(i' \wedge 1_M) \]
= 0. Consequently we have \((j'' \land 1_{K \land M}) \Delta_1 (\mu_{X \land M}(\bar{w}_2 \land 1_M) \land 1_M) \in (1_Y \land j \land 1_M)^*[\Sigma^{-2q}Y \land M, K \land M] = 0\), this is because the degree of the top cell of \(Y \land M\) is \(q + 3\). Then \((j'' \land 1_{K \land M}) \Delta_1 (\mu_{X \land M} \land 1_M) \in (\bar{\psi} \land 1_{M \land M})*[M \land M, K \land M]\) and so \((j''(j''u \land 1_K) \land 1_M) \Delta_1 = \lambda(j''(j''u \land 1_K) \land 1_M)\) with \(\lambda \in Z_p\), this is because \(\Delta_2 \in [M \land M, M \land M] \cap (\kerd) \cong Z_p\{1_{M \land M}\}\). Hence, 

\[
(\bar{\psi} \land 1_M)u_8(w_5(1_W \land i) r) \land 1_M) = (\bar{\psi}(1_X \land j)u_1 \land 1_M) \Delta_1 = (j''(j''(j''u \land 1_K) \land 1_M) \Delta_1 = \lambda(j''(j''u \land 1_K) \land 1_M)\) and by \((9.4.21)(9.4.24)\) we know that \(\lambda = 1\) so that

\[
(9.4.25) \quad m_M(\bar{\psi} \land 1_M)u_8(w_5 \land 1_M)(1_W \land (1_Y \land i) r) = j''(j''u \land 1_K)
\]

\[
= (j'' \bar{\psi} \land 1_M)u_8(w_5 \land 1_M)(1_W \land (r \land 1_M) \bar{\sigma} K),
\]

\[
(j'' \bar{\psi} \land 1_M)u_8(w_5 \land 1_M)(1_W \land (1_Y \land i) r) = i jj''(j''u \land 1_K)
\]

where we use \((jj'' \land 1_M) \bar{\sigma} K = j''\) in the above equation. By composing \((1_{E_{s+1}} \land u_8(w_5 \land 1_M))\) (it has odd degree on \((9.4.17)\) we have

\[
(9.4.26) \quad (\bar{a}_{s+1} \land 1_{X \land M})(1_{E_{s+2}} \land u_8(w_5 \land 1_M))f_3
\]

\[
= -(\bar{a}_{s+1} \land 1_{X \land M})(1_{E_{s+2}} \land u_8(w_5 \land 1_M))(1_W \land (1_Y \land i)r)f''
\]

\[
-\bar{\lambda}(\bar{a}_{s+1} \land 1_{X \land M})(1_{E_{s+2}} \land u'(u \land 1_K))f''(1_X \land ij)
\]

\[
+ (\bar{\epsilon}_{s} \land 1_{X \land M})(1_{K \land M} \land u_8(w_5 \land 1_M)(1_W \land (1_Y \land i)r)g
\]

\[
- \bar{\lambda}(\bar{a}_{s+1} \land 1_{X \land M})(1_{K \land M} \land u_8(w_5 \land 1_M)(1_W \land (r \land 1_M) \bar{\sigma} K)g(1_X \land ij)
\]

\[
+ \bar{\lambda}(1_{E_{s+1}} \land u'(u \land 1_K))f_8(1_X \land i')
\]

where we use \(u_8(w_5 \land 1_M)(1_W \land (r \land 1_M) \bar{\sigma} K) = \bar{\lambda}u'(u \land 1_K)\), for some nonzero \(\bar{\lambda} \in Z_p\). Moreover, by \((9.4.20)(9.4.6)\), \((\bar{b}_{s+1} \land 1_{L \land K})(1_{E_{s+1}} \land u \land 1_K)f_8(1_X \land i') = (1_{K \land M} \land (u \land 1_K) \mu_{X \land M})(g_0 \land 1_M) = (1_{K \land M} \land \bar{\sigma}_2(1_Y \land i') \psi_{X \land M})(\bar{g}_0 \land 1_M) = \lambda_1(1_{K \land M} \land \bar{\sigma}_2(1_Y \land i') \psi_{X \land M})(h_0 \sigma \land 1_{X \land M}) = \lambda_1(h_0 \sigma \land 1_{L \land K}) \bar{\sigma}_2(1_Y \land i') \psi_{X \land M} (\bmod d_1\text{-boundary}). \) Then \([(\bar{b}_{s+1} \land 1_{L \land K})(1_{E_{s+1}} \land u \land 1_K)f_8(1_X \land i')]_1 = \lambda_1(\phi \land 1_K)*[\sigma \land 1_{L \land K}] \bar{\sigma}_2(1_Y \land i') \psi_{X \land M} = \lambda_1(\phi \land 1_K)*[\sigma \land 1_{L \land K}] \bar{\sigma}_2(1_Y \land i') \psi_{X \land M} = \lambda_1(\phi \land 1_K)*[\sigma \land 1_{L \land K}] \bar{\sigma}_2(1_Y \land i') \psi_{X \land M} = \lambda_1(\phi \land 1_K)*[\sigma \land 1_{L \land K}] \bar{\sigma}_2(1_Y \land i') \psi_{X \land M} = 0 \in Ext_{\Lambda \land H}^{s+1,t}(H^*L \land K, H^*X \land M).\) That is we have \((\bar{b}_{s+1} \land 1_{L \land K})f_8(1_X \land i') = (\bar{\epsilon}_{s+1} \land \bar{a}_{s} \land 1_{L \land K})g_3\) with \(g_3 \in [\Sigma^{q}X \land M, KG \land L \land K] \cap (\kerd)\) (cf. Prop. 9.5.26) and so \((1_{E_{s+1}} \land u \land 1_K)f_8(1_X \land i') = (\bar{\epsilon}_{s+1} \land \bar{a}_{s} \land 1_{L \land K})g_3 + (\bar{a}_{s+1} \land 1_{L \land K})f'_2\) with \(f'_2 \in [\Sigma^{q+1}X \land M, E_{s+2} \land L \land K] \cap (\kerd)\) (cf. Prop. 6.5.26). Hence, \((9.4.26)\) becomes

\[
(9.4.27) \quad (\bar{a}_{s+1} \land 1_{X \land M})(1_{E_{s+2}} \land u_8(w_5 \land 1_M))f_3
\]

\[
= -(\bar{a}_{s+1} \land 1_{X \land M})(1_{E_{s+2}} \land u_8(w_5 \land 1_M)(1_W \land (1_Y \land i)r))f''
\]

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\[- \overline{\lambda}(a_{s+1} \wedge 1_{X \wedge M})(1_{E_{s+2}} \wedge u'(u \wedge 1_K))f'(1_X \wedge ij) \]
\[+ (\overline{\epsilon}_s \wedge 1_{X \wedge M})(1_{KG} \wedge u_8(w_5 \wedge 1_M)(1_{W} \wedge (1_Y \wedge i) r))g \]
\[- (\overline{\epsilon}_s \wedge 1_{X \wedge M})(1_{KG} \wedge u_8(w_5 \wedge 1_M)(1_{W} \wedge (r \wedge 1_M)\overline{\mu}_K))g(1_X \wedge ij) \]
\[+ \overline{\lambda}(\overline{\epsilon}_s \wedge 1_{X \wedge M})(1_{KG} \wedge u')g_3 + \overline{\lambda}(a_{s+1} \wedge 1_{X \wedge M})(1_{E_{s+2}} \wedge u')f'_2' \]

By (9.4.27), \((1_{KG} \wedge u_8(w_5 \wedge 1_M)(1_{W} \wedge (1_Y \wedge i) r))g - (1_{KG} \wedge u_8(w_5 \wedge 1_M)(1_{W} \wedge (r \wedge 1_M)\overline{\mu}_K))g(1_X \wedge ij) + \overline{\lambda}(1_{KG} \wedge u')g_3 \in [\Sigma^q X \wedge M, KG \wedge X \wedge M]\]
is a \(d_1\)-cycle which represents an element in \(\operatorname{Ext}^{q+1}_{A}(H^*X \wedge M, H^*X \wedge M) \cong Z_p\{[\sigma \wedge 1_{X \wedge M}]\}\) (cf. Lemma 9.4.13). Then we have

\begin{equation} \tag{9.4.28} \label{eq:9.4.28}
(1_{KG} \wedge u_8(w_5 \wedge 1_M)(1_{W} \wedge (1_Y \wedge i) r))g + \overline{\lambda}(1_{KG} \wedge u')g_3 \\
- (1_{KG} \wedge u_8(w_5 \wedge 1_M)(1_{W} \wedge (r \wedge 1_M)\overline{\mu}_K))g(1_X \wedge ij) \\
= \lambda_0(\sigma \wedge 1_{X \wedge M}) \mod d_1\text{-boundary}. 
\end{equation}

Now we consider the cases of \(\lambda_0 \neq 1\) or \(\lambda_0 = 1\) separately.

If \(\lambda_0 \neq 1\), then by (9.4.27) and \(\overline{\epsilon}_s \cdot \sigma = a_{s+1} \cdot \kappa\) we have

\[(1_{E_{s+2}} \wedge u_8(w_5 \wedge 1_M))f_3 = -(1_{E_{s+2}} \wedge u_8(w_5 \wedge 1_M)(1_{W} \wedge (1_Y \wedge i) r))f' \]
\[- \lambda(1_{E_{s+2}} \wedge u'(u \wedge 1_K))f'(1_X \wedge ij) + \lambda(1_{E_{s+2}} \wedge u')f'_2 \]
\[+ \lambda_0(\kappa \wedge 1_{X \wedge M}) + (\overline{\epsilon}_s \wedge 1_{X \wedge M})g_4 \]
with \(g_4 \in [\Sigma^q X \wedge M, KG \wedge X \wedge M]\) and by composing \((1_{E_{s+2}} \wedge 1_X \wedge \alpha) = (1_{E_{s+2}} \wedge \alpha_X \wedge m_M(\overline{\psi} \wedge 1_M))\) we obtain that \((1_{E_{s+2}} \wedge 1_X \wedge \alpha)(\kappa \wedge 1_{X \wedge M}) = (1_{E_{s+2}} \wedge \alpha_X \wedge (j''u \wedge 1_M)(1_{W} \wedge m_M(\overline{\mu} \wedge 1_M))f_3 = (1_{E_{s+2}} \wedge \alpha_X \wedge m_M(\overline{\psi} \wedge 1_M)u_8(w_5 \wedge 1_M))f_3 = \lambda_0(1_{E_{s+2}} \wedge 1_X \wedge \alpha)(\kappa \wedge 1_{X \wedge M})\) so that the result of the step 1 follows.

If \(\lambda_0 = 1\), then by composing \((1_{KG} \wedge m_M(\overline{\psi} \wedge 1_M))\) on (9.4.28) and using (9.4.25) we have \((1_{KG} \wedge j''(j''u \wedge 1_K))g = (1_{KG} \wedge m_M(\overline{\psi} \wedge 1_M)u_8(w_5 \wedge 1_M)(1_{W} \wedge (1_Y \wedge i) r))g = (\sigma \wedge 1_{X \wedge M})m_M(\overline{\psi} \wedge 1_M)\) (mod \(d_1\)-boundary). Moreover, by composing \((1_{KG} \wedge j'' \overline{\psi} \wedge 1_M)\) on (9.4.28) and using (9.4.25) we have

\[(1_{KG} \wedge j'' \overline{\psi} \wedge 1_M)(\sigma \wedge 1_{X \wedge M}) \]
\[= (1_{KG} \wedge (j'' \overline{\psi} \wedge 1_M)u_8(w_5 \wedge 1_M)(1_{W} \wedge (1_Y \wedge i) r))g \]
\[- (1_{KG} \wedge (j'' \overline{\psi} \wedge 1_M)u_8(w_5 \wedge 1_M)(1_{W} \wedge (r \wedge 1_M)\overline{\mu}_K))g(1_X \wedge ij) \]
\[+ \lambda(1_{KG} \wedge (j'' \overline{\psi} \wedge 1_M))g_3 \]
\[= (1_{KG} \wedge i(j''(j''u \wedge 1_K))(u \wedge 1_K))g \]
\[- (1_{KG} \wedge j''(j''u \wedge 1_K))g(1_X \wedge ij) + \lambda(1_{KG} \wedge j''(j''u \wedge 1_K))g_3 \]
\[= (1_{KG} \wedge i(j'' \wedge 1_M)m_M(\overline{\psi} \wedge 1_M) - (\sigma \wedge 1_{X \wedge M})m_M(\overline{\psi} \wedge 1_M)(1_X \wedge ij) \]
\[+ \lambda(1_{KG} \wedge j''(j''u \wedge 1_K))g_3 \]
\[= (1_{KG} \wedge j'' \overline{\psi} \wedge 1_M)(\sigma \wedge 1_{X \wedge M}) + \lambda(1_{KG} \wedge j''(j''u \wedge 1_K))g_3 \]
\[\text{by (9.4.6)} \]

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(mod $d_1$-boundary), then $(1_{KG_s} \land j'(j'' \land 1_K))g_3 = 0$ and so $g_3 = (1_{KG_s} \land \overline{w}_3(1_Y \land i'))g_5$ (mod $d_1$-boundary) for some $g_5 \in \Sigma^{tq+q+1}X \land M, KG_s \land Y \land M$. So, by (9.4.6)(9.4.20) $(1_{KG_{s+1}} \land \overline{w}_2(1_Y \land i'))\psi_{X \land M})(g_0 \land 1_M) = (1_{KG_{s+1}} \land (u \land 1_K)\mu_{X \land M})(g_0 \land 1_M) = (\overline{b}_{s+1} \land 1_{LA_K})(1_{E_{s+1}} \land u \land 1_K)f_8(1_X \land i') = (\overline{b}_{s+1} \land 1_{LA_K})g_3 = (\overline{b}_{s+1} \land 1_{LA_K})(1_{KG_s} \land \overline{w}_2(1_Y \land i'))g_5$ so that $(1_{KG_{s+1}} \land \psi_{X \land M})(g_0 \land 1_M) = (\overline{b}_{s+1} \land 1_{LA_K})g_5$, this shows $\lambda_1(\psi_{X \land M})*[h_0 \sigma \land 1_{X \land M}] = (\psi_{X \land M})*[g_0 \land 1_M] = 0 \in \text{Ext}_A^{s,1,tq+1}(H^*Y \land M, H^*X \land M)$ and by Lemma 9.4.13(2) we have $\lambda_1 = 0$. Then $[g_0 \land 1_M] = 0$ and so $(\overline{b}_{s+1} \land 1_{W \land K})f_8(1_X \land i') = (\overline{b}_{s+1} \land 1_{W \land K})g_6$ for some $g_6 \in \Sigma^{tq+q}X \land M, KG_s \land W \land K$ and $f_8(1_X \land i') = (\overline{c}_s \land 1_{W \land K})g_6 + (\overline{a}_{s+1} \land 1_{W \land K})f'_3$ with $f'_3 \in \Sigma^{tq+q+1}X \land M, E_{s+2} \land W \land K$. Then, by composing $(1_{E_{s+1}} \land w_5 \land 1_M)$ on (9.4.17) we have

$$-(\overline{a}_{s+1} \land 1_{P \land M})(1_{E_{s+2}} \land w_5 \land 1_M)f_3 = (\overline{a}_{s+1} \land 1_{P \land M})(1_{E_{s+2}} \land (w_5 \land 1_M)(1_W \land (1_Y \land i)r)f' + (\overline{a}_{s+1} \land 1_{P \land M})(1_{E_{s+2}} \land (w_5 \land 1_M)(1_W \land (r \land 1_M)\overline{w}_K)f'(1_X \land ij) + (\overline{a}_{s+1} \land 1_{P \land M})(1_{E_{s+2}} \land (w_5 \land 1_M)(1_W \land (r \land 1_M)\overline{w}_K)f'_3 + (\overline{c}_s \land 1_{P \land M})g_7$$

where the $d_1$-cycle $g_7 = (1_{KG_s} \land (w_5 \land 1_M)(1_W \land (1_Y \land i)r)g - (1_{KG_s} \land (w_5 \land 1_M)(1_W \land (r \land 1_M)\overline{w}_K)g(1_X \land ij) + (1_{KG_s} \land (w_5 \land 1_M)(1_W \land (r \land 1_M)\overline{w}_K)g_6 \in \Sigma^{tq+q+1}X \land M, KG_s \land P \land M$ which represents an element in $\text{Ext}_A^{s,tq+q+1}(H^*P \land M, H^*X \land M)$. However, this group is zero, this can be obtained by the following exact sequence

$$0 = \text{Ext}_A^{s,tq+q}(H^*W \land K, H^*X \land M) \xrightarrow{(w_5 \land 1_M)(1_W \land (r \land 1_M)\overline{w}_K)^*} \text{Ext}_A^{s,tq+q+1}(H^*P \land M, H^*X \land M) \xrightarrow{(w_5)_{tq+q+1}} \text{Ext}_A^{s,tq+q-2q}(H^*M, H^*X \land M) \xrightarrow{(1_W \land i')(1_w \land K)^*}$$

induced by (9.4.23), where the left group is zero by Prop. 9.4.11(4) and by Prop. 9.4.11(1) the right group has unique generator $m_M(\overline{\psi} \land 1_M)^*(\overline{\delta})$, which satisfies $((1_W \land i')(1_w \land K)^*)m_M(\overline{\psi} \land 1_M)^*(\overline{\delta}) \neq 0 \in \text{Ext}_A^{s,1,tq+q}(H^*W \land K, H^*X \land M)$.

Then, $(\overline{c}_s \land 1_{P \land M})g_7 = 0$ and so $-(1_{E_{s+2}} \land w_5 \land 1_M)f_3 = (1_{E_{s+2}} \land (w_5 \land 1_M)u_8(1_W \land (1_Y \land i)r)f' - (1_{E_{s+2}} \land (w_5 \land 1_M)(1_W \land (r \land 1_M)\overline{w}_K)f'(1_X \land ij) + (1_{E_{s+2}} \land (w_5 \land 1_M)(1_W \land (r \land 1_M)\overline{w}_K))f'_3 + (\overline{a}_{s+1} \land 1_{P \land M})g_8$ for some $g_8 \in \Sigma^{tq+q+2}X \land M, KG_{s+1} \land P \land M$. By composing $(1_{E_{s+2}} \land \alpha_{X \land M} \land u_7)$ we have $(1_{E_{s+2}} \land 1_X \land \alpha)(\kappa \land 1_{X \land M}) = (1_{E_{s+2}} \land \alpha_{X \land M}(j''u \land 1_M)(1_W \land m_M(\overline{\mu} \land
1_M))f_3 = (1_{E_{s+2}} \wedge \alpha_{X\wedge M} \cdot u_7(w_5 \wedge 1_M))f_3 = 0$. This shows the result of step 1.

**Step 2** To prove $(\tilde{c}_{s+1} \wedge 1_M)\overline{h_0}\sigma = (\kappa \wedge 1_M)\alpha = 0$.

By (9.4.3)(9.4.4), $\mu_{X\wedge M}(1_X \wedge \alpha i) = \mu_{X\wedge M}\alpha_{X\wedge M}\overline{\psi} = 0$ and so by (9.1.15) $\mu_{X\wedge M} = \mu_{X\wedge K'}(1_X \wedge v)$, where $\mu_{X\wedge K'} \in [X \wedge K', W \wedge K]$. We claim that $X \wedge K'$ splits into $W \wedge K \vee \Sigma Y$, that is, there is a split cofibration $\Sigma Y \to X \wedge K' \to W \wedge K$, this can be seen by the following homotopy commutative diagram of $3 \times 3$-Lemma and using $(1_Y \wedge j)\alpha_{Y\wedge M} = r(1_K \wedge \alpha_1)$.

\[
\begin{array}{ccc}
X \wedge M & \xrightarrow{\mu_{X\wedge M}} & W \wedge K \\
\downarrow 1_X \wedge v & & \downarrow 0 \\
\mu_{X\wedge K'} & \xrightarrow{\psi} & \Sigma Y \\
\uparrow \tau_{X\wedge K'} & & \uparrow \tau_{X\wedge K'} \\
\Sigma Y & \xrightarrow{\psi_1} & \Sigma X \wedge M \\
\end{array}
\]

Hence, there is a split cofibration $\Sigma Y \xrightarrow{\tau_{X\wedge K'}} X \wedge K' \xrightarrow{\mu_{X\wedge K'}} W \wedge K$ and so there are $\nu_{X\wedge K'} : X \wedge K' \to \Sigma Y$ and $\tilde{\nu}_{X\wedge K'} : W \wedge K \to X \wedge K'$ such that $\nu_{X\wedge K'} \cdot \tau_{X\wedge K'} = 1_Y$, $\mu_{X\wedge K'} \cdot \tilde{\nu}_{X\wedge K'} = 1_{W\wedge K}$, $\tau_{X\wedge K'} \cdot \nu_{X\wedge K'} + \tilde{\nu}_{X\wedge K'} \cdot \mu_{X\wedge K'} = 1_{X\wedge K'}$.

By the result of step 1 we have $(\kappa \wedge 1_{M\wedge X\wedge K'})(\alpha \wedge 1_{X\wedge K'}) = 0$, then $(\kappa \wedge 1_{M\wedge Y})(\alpha \wedge 1_Y) = (1_{E_{s+2}} \wedge 1_M \wedge \mu_{X\wedge K'})(\kappa \wedge 1_{M\wedge X\wedge K'})(\alpha \wedge 1_{X\wedge K'})(1_{M} \wedge \tau_{X\wedge K'}) = 0$. Moreover, by using the splitness in (9.1.32) we have $(\tilde{c}_{s+1} \wedge 1_M)\overline{h_0}\sigma = (\kappa \wedge 1_M)\alpha = (1_{E_{s+2}} \wedge 1_M \wedge \tilde{\nu})(\kappa \wedge 1_{M\wedge Y\wedge K'})(\alpha \wedge 1_{Y\wedge K'})(1_{M} \wedge \tau_{X\wedge K'}) = 0$ which shows the main Theorem C. Q.E.D.

**Remark.** In the proof of the main Theorem C, We only use the supposition (II) for our geometric input to obtain that $(1_{E_{s+2}} \wedge \phi \wedge 1_M)(\kappa \wedge 1_M)m_M(\overline{\psi} \wedge 1_M) = 0$. Then, the geometric supposition (II) of the main Theorem C can be weakened to be the supposition on $m_M(\overline{\psi} \wedge 1_M)^s(\phi \wedge 1_M)^s(\tilde{\sigma}) \in Ext^{s+1,td}(H^L \wedge M, H^X \wedge M)$ is a permanent cycle in the ASS.

Using some new cofibrations in this section, we also can give an alternative proof of Theorem 9.3.9( and so the main Theorem B). We first do
some preminerals.

Since \( \alpha' \alpha'i' = 0 \), then by (9.1.23), there exists \( \alpha''_{Y \wedge M} \in [\Sigma^{q-2}Y \wedge M, K] \) such that \( \alpha''_{Y \wedge M}(r \wedge 1_M)\overline{\alpha''} = \alpha' \). By applying the derivation \( d, d(\alpha''_{Y \wedge M})(r \wedge 1_M)\overline{\alpha''} = -d(\alpha') = 0 \) and so \( d(\alpha''_{Y \wedge M}) \in (m_M(\overline{\alpha} \wedge 1_M))^*[\Sigma^{q+2}M, K] = 0 \). \( \alpha''_{Y \wedge M}(1_Y \wedge i)r \in [\Sigma^{q-2}K, K] \cong Z_p \{ \alpha'' \} \) and so \( \alpha''_{Y \wedge M}(1_Y \wedge i)r = \lambda \alpha'' \) for some \( \lambda \in Z_p \). Note that \( d((1_Y \wedge i)r) = (r \wedge 1_M)d(1_K \wedge i) = (r \wedge 1_M)\overline{\alpha''} \), then by applying the derivation \( d \), we have \( \alpha' = \alpha''_{Y \wedge M}(r \wedge M)\overline{\alpha''} = \lambda d(\alpha'') = -\lambda \alpha' \) and so \( \lambda = -1 \). By (9.1.8), \( hi'' = \overline{\alpha} \), \( ri'' = \overline{\alpha} \cdot j \) (up to sign), then \( (r \wedge 1_M)\overline{\alpha''}i'' = -(ri' \wedge 1_M)\overline{\alpha''} = \pm (\overline{\alpha} \wedge 1_M) = \pm (hi'' \wedge 1_M) \) and so \( \alpha''_{Y \wedge M}(hi'' \wedge 1_M) = \lambda_0 \alpha''_{Y \wedge M}(r \wedge 1_M)\overline{\alpha''} = \lambda_0 \alpha'i' = \lambda_0 i'((\alpha_1)Li'' \wedge 1_M) \) and we have \( \alpha''_{Y \wedge M}(h \wedge 1_M) = \lambda_0 i'((\alpha_1)L \wedge 1_M) \), where \( \lambda_0 = \pm 1 \). On the other hand, \( i'((\alpha_1)L \wedge 1_M)(1_L \wedge j') = i'((\alpha_1 \wedge 1_M))j' = i'(ij_\alpha - \alpha_1 j') = 0 \), then \( i'((\alpha_1)L \wedge 1_M)(1_L \wedge j') = \lambda' \alpha''(j'' \wedge 1_K) \) with \( \lambda' \in Z_p \). By composing the map \( \Delta \) in Theorem 6.5.18 we have \( \lambda' \alpha'i'j''j' = \lambda' \alpha''i'j''j' = \lambda' \alpha''(j'' \wedge 1_K) \Delta = i''((\alpha_1)L \wedge 1_M)(1_L \wedge j'') \Delta = -i''((\alpha_1)Li'' \wedge 1_M)ij'' = -\alpha'ij'' \) so that \( \lambda' = -1 \). Conclusively, there is \( \alpha''_{Y \wedge M} \in [\Sigma^{q-2}Y \wedge M, K] \) such that

\[
\text{(9.4.29)} \quad \alpha''_{Y \wedge M}(r \wedge 1_M)\overline{\alpha''} = \alpha', \quad \alpha''_{Y \wedge M}(1_Y \wedge i)r = -\alpha'', \quad d(\alpha''_{Y \wedge M}) = 0, \quad \alpha''_{Y \wedge M}(h \wedge 1_M) = \lambda_0 i'((\alpha_1)L \wedge 1_M), \quad i''((\alpha_1)L \wedge 1_M)\overline{\alpha''} = -\alpha''(j'' \wedge 1_K)
\]

where \( \lambda_0 = \pm 1 \).

Note that the cofibre of \( \alpha''_{Y \wedge M} : \Sigma^{q-2}Y \wedge M \to K \) is \( X \wedge M \) given by the cofibration

\[
\text{(9.4.30)} \quad \Sigma^{q-2}Y \wedge M \xrightarrow{\alpha''_{Y \wedge M}} K \xrightarrow{u'(i''(\overline{\alpha'} \wedge 1_K))} X \wedge M \xrightarrow{\psi_X \wedge M} \Sigma^{q-1}Y \wedge M
\]

and the above map \( \psi_X \wedge M \in [X \wedge M, \Sigma^{q-2}Y \wedge M] \) and \( u' \in [L \wedge K, X \wedge M] \) is just the map in (9.4.2) and (9.4.6). This can be seen by the equation \( m_M(\overline{\alpha} \wedge 1_M)\psi_X \wedge M = m_M(\psi_1 \wedge 1_M) \) in (9.4.6),(9.4.2) and the following homotopy commutative diagram of \( 3 \times 3 \)-Lemma

\[
\text{(9.4.31)} \quad \begin{array}{cccccccc}
X \wedge M & m_M(\overline{\alpha' \wedge 1_M}) & \Sigma^{2q}M & \alpha'' i' & \Sigma^{q+1}K & \\
\downarrow \psi_X \wedge M & \downarrow m_M(\overline{\alpha' \wedge 1_M}) & \downarrow (\phi \wedge 1_K) i' & \downarrow j'' \wedge 1_K & \\
\Sigma^{q-1}Y \wedge M & m_M(\overline{\alpha'' \wedge 1_M}) & \Sigma L \wedge K & \Sigma^{q}Y \wedge M & \\
\downarrow (r \wedge 1_M)\overline{m_K} & \downarrow \psi_X \wedge M & \downarrow i'' & \downarrow \psi_X \wedge M & \\
\Sigma^{q}K & \alpha' & \Sigma K & u'(i'' \wedge 1_K) & \Sigma X \wedge M & \\
\end{array}
\]

and by this we have the following relation

\[
\text{(9.4.32)} \quad \psi_X \wedge M u' = (r \wedge 1_M)\overline{\alpha''}(j'' \wedge 1_K).
\]
Proposition 9.4.33 Let \( p \geq 5 \) and \( V \) be any spectrum, then for any map \( f \in [\Sigma^*K,V \wedge K] \) we have \((1_V \wedge \alpha')d(f) = d(f)\alpha' = 0.\)

Proof: By (6.5.12), \( \alpha \wedge 1_K = \overline{m}_K\alpha' m'_K \), where \( m'_K = m_K T : M \wedge K \to K \), \( \overline{m}_K = Tm_K : \Sigma K \to M \wedge K \). \( d(f)\alpha' m'_K = (1_V \wedge m'_K)(T' \wedge 1_K)(1_M \wedge f)\overline{m}_K\alpha' m'_K = (1_V \wedge m'_K)(T' \wedge 1_K)(1_M \wedge f)(\alpha \wedge 1_K) = (1_V \wedge m'_K)(T' \wedge 1_K)(\alpha \wedge 1_V \wedge 1_K)(1_M \wedge f) = 0 \), where \( T' : M \wedge V \to V \wedge M \) is the switching map. Q.E.D.

Proposition 9.4.34 Under the supposition (I) of the main Theorem B we have

1. \( \text{Ext}^{s,tq-1}_{A}(H^*K, H^*K) = 0. \)
2. \( \text{Ext}^{s,tq}_{A}(H^*Y \wedge M, H^*K) \) has unique generator \((1_Y \wedge i)_*r_\ast(\sigma \wedge 1K). \)

Proof: (1) Consider the following exact sequence
\[
\text{Ext}^{s,tq+q}_{A}(H^*M, H^*M) \xrightarrow{(ij)_*} \text{Ext}^{s,tq+q}_{A}(H^*K, H^*M) \xrightarrow{(j')_*} \text{Ext}^{s,tq-1}_{A}(H^*M, H^*M) \xrightarrow{\alpha_*} \]

induced by (9.1.2). By the supposition (I), the right group has unique generator \( j^* i_\ast(\sigma) \) which satisfies \( \alpha_* j^* i_\ast(\sigma) = (ij)_* \alpha_* (\tilde{\sigma}) \neq 0 \). Then \( \text{im}(j'_*) = 0 \). By the supposition (I), the left group is zero or has two generators \((ij)_* \alpha_*(\tilde{\sigma}'), (ij)_* \alpha_*(\tilde{\sigma})' \) (this can be obtained by a similar proof as given in Prop. 9.3.1(2)), then \( \text{Ext}^{s,tq+q}_{A}(H^*K, H^*M) = (i)_* \text{Ext}^{s,tq+d}_{A}(H^*M, H^*M) \) is zero or has unique generator \((i'_*)_*(\alpha \wedge 1M)_*(\tilde{\tau}) \). Look at the following exact sequence
\[
\text{Ext}^{s,tq+q}_{A}(H^*K, H^*M) \xrightarrow{(j')_*} \text{Ext}^{s,tq-1}_{A}(H^*K, H^*K) \xrightarrow{(i)_*} \text{Ext}^{s,tq-1}_{A}(H^*M, H^*M) \xrightarrow{\alpha_*} \]

induced by (9.1.2). By the supposition (I), the right group has unique generator \( j^*(i'_i)_\ast(\sigma) \) which satisfies \( \alpha^* j^*(i'_i)_\ast(\sigma) = (i'_i)_*(ij)_* \alpha_*(\tilde{\sigma}) \neq 0 \in \text{Ext}^{s+1,tq+q}_{A}(H^*K, H^*M) \) so that \( \text{im}(i'_i) = 0 \). The left group is zero or has unique generator \((i'_i)_*(\alpha_\ast 1M)_*(\tilde{\tau})', \) \( \text{im}(j'_i) = 0 \) and the middle group is zero as desired.

2. For any \( g \in \text{Ext}^{s,tq}_{A}(H^*Y \wedge M, H^*K),m_M(\pi \wedge 1M)_\ast(g) \in \text{Ext}^{s,tq-q-1}_{A}(H^*M, H^*K) \equiv Z_p\{j'_\ast(\bar{\sigma})\} \), this can be obtained from \( \text{Ext}^{s,tq}_{A}(H^*M, H^*M) \equiv Z_p\{\bar{\sigma}\} \) in Prop. 9.3.0(2) and \( \text{Ext}^{s,tq-q-1}_{A}(H^*M, H^*M) = 0 \), where the last is obtained by the supposition (I) on \( \text{Ext}^{s,tq-q+_1}_{A}(Z_p, Z_p) = 0 \) for \( u = 0, -1, 1 \). Then \( (m_M(\pi \wedge 1M)_\ast(g) = \lambda'(j'_\ast[\sigma \wedge 1M] = \lambda'((1K,G_\ast \wedge j'_\ast(\sigma \wedge 1K)) = \lambda'(j'_\ast[\sigma \wedge 1K]) = \lambda'(m_M(\pi \wedge \lambda') = 53}
with $\lambda$ $E$ $H$ Hence, $\bar{\alpha}$ so by (9.4.30) we have $\bar{\alpha}$ $E$ $1$ induces zero homomorphism in $1$ $\lambda$ $\bar{\alpha}$ (9.4.36)

(9.4.35)

By (9.4.29)(9.4.35), $(1_{E_{s+2}} \land i''((\alpha_1)_{L} \land (1_{L} \land j'))f = -\lambda_0(1_{E_{s+2}} \land i''((\alpha_1)_{L} \land (1_{L} \land j'))f = -\lambda_0(1_{E_{s+2}} \land \alpha'')(\bar{\alpha} \land 1_{K})$, where $\lambda_0 = \pm 1$. That is we have

(9.4.36) $(1_{E_{s+2}} \land \alpha''_{Y \land M}\bar{\alpha})((1_{L} \land j'))f = -\lambda_0(1_{E_{s+2}} \land \alpha'')(\bar{\alpha} \land 1_{K})$

where $\lambda_0 = \pm 1$

It follows that $(\bar{\alpha}_{s+1} \land 1_{K})(1_{E_{s+2}} \land \alpha''_{Y \land M}\bar{\alpha})((1_{L} \land j'))f = -\lambda_0(1_{E_{s+2}} \land 1_{K})((1_{E_{s+2}} \land \alpha'')(\bar{\alpha} \land 1_{K}) = 0$ since $\alpha''$ induces zero homomorphism in $Z_p$-cohomology. Then, by (9.4.30), $(\bar{\alpha}_{s+1} \land 1_{Y \land M})(1_{E_{s+2}} \land \bar{\alpha})((1_{L} \land j'))f = (1_{E_{s+2}} \land \bar{\alpha})((1_{L} \land j'))f = 0$ and so by (9.4.30) we have $(\bar{\alpha}_{s+1} \land 1_{X \land M})f = (1_{K_{s+1}} \land \bar{\alpha})((1_{L} \land j'))g$, with $d_1$-cycle $g \in [\Sigma^{q+q-1}, K, GS_{s+1} \land K]$ and this $d_1$-cycle represents an element in $Ext^{s+1,1}_{A}(H^*K, H^*K) \cong Z_p((h_0 \sigma)''$. Then $[\bar{\alpha}] = \lambda((h_0 \sigma)'' = \lambda')(\bar{\alpha})_{s}[\sigma \land 1_{K}]$ for some $\lambda' \in Z_p$ so that

$$[[\bar{\alpha}_{s+1} \land 1_{Y \land M})f = (u''((\alpha_1)_{L} \land (1_{L} \land j'))s[\alpha''_{Y \land M}]((1_{Y} \land i)\lambda_{s}[\sigma \land 1_{K}] = 0$$

Hence, $(\bar{\alpha}_{s+1} \land 1_{X \land M})f = (\bar{\alpha}_{s+1} \land 1_{X \land M})g_2$ for some $g_2 \in [\Sigma^{q+q-1}, K, GS_{s} \land X \land M]$ and so $f_2 = (\bar{\alpha}_{s+1} \land 1_{X \land M})g_2$ with $f_3 = [\Sigma^{q+q-1}, K, E_{s+2} \land X \land M]$ and we have

$$(\bar{\alpha}_{s+1} \land 1_{Y \land M})(E_{s+2} \land (\bar{\alpha} \land 1_{M}))(L_{1} \land j')f = (\bar{\alpha}_{s+1} \land 1_{Y \land M})(E_{s+2} \land \bar{\alpha})((1_{L} \land j'))f = (\bar{\alpha}_{s+1} \land 1_{Y \land M})(E_{s+2} \land \bar{\alpha})((1_{L} \land j'))f = (\bar{\alpha}_{s+1} \land 1_{Y \land M})(E_{s+2} \land \bar{\alpha})((1_{L} \land j'))f + \lambda(\bar{\alpha}_{s+1} \land 1_{Y \land M})(E_{s+2} \land (1_{L} \land j'))f((\alpha_1)_{L} \land (1_{L} \land j'))f = 0$$

with $\lambda \in Z_p$, where the $d_1$-cycle $(1_{K_G} \land \bar{\alpha})((1_{L} \land j'))f = [\Sigma^{q}, K, GS_{s} \land X \land M]$
represents an element $\lambda((1_Y \wedge i)r)_*[\sigma \wedge 1_K] \in Ext^{\alpha_{1q}}_A(H^*Y \wedge M, H^*K)$ (cf. Prop. 9.4.34(2)) and so it equals to $\lambda(1_{KG_s} \wedge (1_Y \wedge i)r)(\sigma \wedge 1_K)$ (mod $d_1$-boundary). Then we have $(1_{E_{s+2}} \wedge (\bar{h} \wedge 1_M)(1_L \wedge j'))f = (1_{E_{s+2}} \wedge \psi_{X \wedge M})f_3 + \lambda(1_{E_{s+2}} \wedge (1_Y \wedge i)r)(\kappa \wedge 1_K) + (\bar{c}_{s+1} \wedge 1_Y \wedge M)g_3$ for some $g_3 \in [\Sigma^{q+1}K, KG_{s+1} \wedge Y \wedge M]$. By composing $1_{E_{s+2}} \wedge 1_Y \wedge \alpha$ and using (9.4.8) we have $(1_{E_{s+2}} \wedge \alpha \wedge 1_M)f_3 = (1_{E_{s+2}} \wedge \alpha_Y \wedge M(P \wedge 1_M))(\bar{\psi} \wedge 1_M)f_3 = (1_{E_{s+2}} \wedge (1_Y \wedge \alpha)\psi_{X \wedge M})f_3 = -\lambda(1_{E_{s+2}} \wedge (1_Y \wedge \alpha)r)(\kappa \wedge 1_K) = -\lambda(1_{E_{s+2}} \wedge (r \wedge 1_M)\bar{m}_K\alpha')(\kappa \wedge 1_K) = 0$ and by (9.4.7) we have $f_3 = (1_{E_{s+2}} \wedge 1_M)(u' \wedge 1_M)f_4$ with $f_4 \in [\Sigma^{q+1}K, E_{s+2} \wedge L \wedge K \wedge M]$. That is we have

(9.4.37) \[(1_{E_{s+2}} \wedge (\bar{h} \wedge 1_M)(1_L \wedge j'))f = (1_{E_{s+2}} \wedge \psi_{X \wedge M}((1_X \wedge j)u' \wedge 1_M))f_4 \]
\[+\lambda(1_{E_{s+2}} \wedge (1_Y \wedge i)r)(\kappa \wedge 1_K) + (\bar{c}_{s+1} \wedge 1_Y \wedge M)g_3 \]
\[= (1_{E_{s+2}} \wedge \psi_{X \wedge M}u')f_5 + (1_{E_{s+2}} \wedge \psi_{X \wedge M}(1_X \wedge i)j)u'(1_L \wedge m_K)f_4 \]
\[+\lambda(1_{E_{s+2}} \wedge (1_Y \wedge i)r)(\kappa \wedge 1_K) + (\bar{c}_{s+1} \wedge 1_Y \wedge M)g_3 \]

where we use $f_4 = (1_{E_{s+2}} \wedge (1_L \wedge \bar{m}_K)(1_L \wedge j))f_4 + (1_{E_{s+2}} \wedge (1_{L \wedge K} \wedge i)(1_L \wedge m_K)f_4$ and write $(1_{E_{s+2}} \wedge 1_L \wedge K \wedge j)f_4 = f_5$.

By composing $1_{E_{s+2}} \wedge \alpha'' \wedge \alpha'' \wedge \alpha'' \wedge \alpha'' \wedge \alpha'' \wedge 1_M$ on (9.4.37) and using (9.4.36)(9.4.30) we have

$-\lambda_0(1_{E_{s+2}} \wedge \alpha'' \wedge \alpha'' \wedge \alpha'' \wedge \alpha'' \wedge \alpha'' \wedge 1_M)(\bar{h} \wedge 1_M)f = (1_{E_{s+2}} \wedge \alpha_Y \wedge M(1_Y \wedge i))(\kappa \wedge 1_K) = -\lambda(1_{E_{s+2}} \wedge \alpha'' \wedge \alpha'' \wedge \alpha'' \wedge \alpha'' \wedge \alpha'' \wedge 1_M)(1_Y \wedge i)r)(\kappa \wedge 1_K) = -\lambda(1_{E_{s+2}} \wedge \alpha'' \wedge \alpha'' \wedge \alpha'' \wedge \alpha'' \wedge \alpha'' \wedge 1_M)(1_Y \wedge i)r)(\kappa \wedge 1_K).$ If $\lambda \neq \lambda_0$, then $(1_{E_{s+2}} \wedge \alpha'' \wedge \alpha'' \wedge \alpha'' \wedge \alpha'' \wedge \alpha'' \wedge 1_M)(\bar{h} \wedge 1_M) = 0$ and the Theorem follows. So, we suppose that $\lambda = \lambda_0$.

By (9.1.8) we have $\bar{m}_k = i \cdot j'' \wedge 1_M$ so that $m_M(\bar{m} \wedge 1_M)(\bar{h} \wedge 1_M) = j'' \wedge 1_M$ (up to sign). Then, what happen is either $m_M(\bar{m} \wedge 1_M)(\bar{h} \wedge 1_M) = \lambda_0(j'' \wedge 1_M)$ or $m_M(\bar{m} \wedge 1_M)(\bar{h} \wedge 1_M) = -\lambda_0(j'' \wedge 1_M)$. Now we consider this two cases separately.

**Case 1** $m_M(\bar{m} \wedge 1_M)(\bar{h} \wedge 1_M) = \lambda_0(j'' \wedge 1_M)$.

In this case, by composing $1_{E_{s+2}} \wedge m_M(\bar{m} \wedge 1_M)$ on (9.4.37) we have

(9.4.38) \[\lambda_0(1_{E_{s+2}} \wedge \alpha'' \wedge \alpha'' \wedge \alpha'' \wedge \alpha'' \wedge \alpha'' \wedge 1_M)(\bar{h} \wedge 1_M)f = (1_{E_{s+2}} \wedge \alpha'' \wedge \alpha'' \wedge \alpha'' \wedge \alpha'' \wedge \alpha'' \wedge 1_M)(1_Y \wedge i)r)(\kappa \wedge 1_K) \]
\[= \lambda_0(1_{E_{s+2}} \wedge (j'' \wedge 1_M)(1_L \wedge j'))f = (1_{E_{s+2}} \wedge \alpha'' \wedge \alpha'' \wedge \alpha'' \wedge \alpha'' \wedge \alpha'' \wedge 1_M)(\bar{h} \wedge 1_M)(1_L \wedge j')f \]
\[= (1_{E_{s+2}} \wedge m_M(\bar{m} \wedge 1_M)\psi_{X \wedge M}(1_X \wedge i)j)u'(1_L \wedge m_K)f_4 \]
\[+\lambda_0(1_{E_{s+2}} \wedge \alpha'' \wedge \alpha'' \wedge \alpha'' \wedge \alpha'' \wedge \alpha'' \wedge 1_M)(1_KG_{s+1} \wedge \alpha'' \wedge m_M(\bar{m} \wedge 1_M))g_3 \]
\[= -(1_{E_{s+2}} \wedge \alpha'' \wedge \alpha'' \wedge \alpha'' \wedge \alpha'' \wedge \alpha'' \wedge 1_M)(1_L \wedge m_K)f_4 \]
\[+\lambda_0(1_{E_{s+2}} \wedge \alpha'' \wedge \alpha'' \wedge \alpha'' \wedge \alpha'' \wedge \alpha'' \wedge 1_M)(1_KG_{s+1} \wedge \alpha'' \wedge m_M(\bar{m} \wedge 1_M))g_3 \]
\[\text{and so} \quad (1_{E_{s+2}} \wedge \alpha'' \wedge \alpha'' \wedge \alpha'' \wedge \alpha'' \wedge \alpha'' \wedge 1_M)(1_L \wedge m_K)f_4 = (\bar{c}_{s+1} \wedge 1_M)(1_KG_{s+1} \wedge m_M(\bar{m} \wedge 1_M))g_3 \]
where we use $m_M(\pi \land 1_M)\psi_{X \land M}(1_X \land ij)u' = m_M(\tilde{\psi} \land 1_M)(1_X \land ij)u' = -(j\tilde{\psi} \land 1_M)u' = -j'(j'' \land 1_K)$ which is obtained by (9.4.6) and the right rectangle of the diagram (9.4.1). Moreover, by applying the derivation $d$ to the equation (9.4.37) we have

\begin{equation}
(9.4.39) \quad (\iota_{E+2} \land (\tilde{h} \land 1_M)(1_L \land j')(i'' \land 1_K))f' = (1_{E+2} \land \psi_{X \land M}u')d(f_5)
\end{equation}

\begin{align*}
&+ (1_{E+2} \land \psi_{X \land M}(1_X \land ij)u')d((1_{E+2} \land 1_L \land m_K)f_4) \\
&+ (1_{E+2} \land \psi_{X \land M}(1_L \land m_K))f_4 - \lambda_0(1_{E+2} \land (r \land 1_M)\overline{m}_K)(\kappa \land 1_K) \\
&+ (\iota_{s+1} \land 1_{Y \land M})d(g_3)
\end{align*}

By (9.4.32) we have $\psi_{X \land M}u' = (r \land 1_M)\overline{m}_K(j'' \land 1_K)$ so that $(j\pi \land 1_M)\psi_{X \land M}u' = j'(j'' \land 1_K)$. Then, by composing $1_{E+2} \land \phi \cdot j\overline{\pi} \land 1_M$ on (9.4.39), it becomes

\begin{equation}
(9.4.40) \quad \lambda_0(1_{E+2} \land (\phi \land 1_M)j')(\kappa \land 1_K) = (1_{E+2} \land (\phi \land 1_M)j'(j'' \land 1_K))d(f_5)
\end{equation}

\begin{align*}
&+ (1_{E+2} \land (\phi \land 1_M)ijj'(j'' \land 1_K))d((1_{E+2} \land 1_L \land m_K)f_4) = 0 \\
\end{align*}

here we use $(1_{E+2} \land (\phi \cdot j\overline{\pi} \land 1_M)\psi_{X \land M}u'(1_L \land m_K))f_4 = (1_{E+2} \land j'(j'' \land 1_K)(1_L \land m_K))f_4 = (\iota_{s+1} \land 1_{L \land M})(1_{KG_{s+1}} \land (\phi \land 1_M)m_M(\pi \land 1_M)g_3 = 0$ and by $1_L \land \alpha_1 = \phi \cdot j''(\text{up to nonzero scalar})$ we obtain that $(1_{E+2} \land (\phi \land 1_M)j'(j'' \land 1_K))d(f_5) = (1_{E+2} \land (1_L \land j'\alpha'))d(f_5) = 0$ (cf. Prop. 9.4.33) and so the first term of the right hand side of (9.4.40) is zero. The second term of the right hand side of (9.4.40) is zero by the same reason.

It follows from (9.4.40) that $(1_{E+2} \land (\phi \land 1_K)(\kappa \land 1_K) = (1_{E+2} \land (1_L \land \mu(i'' \land 1_K))(\phi \land 1_K)(\kappa \land 1_K)(1_L \land m_K))f_4 = (1_{E+2} \land (\phi \land 1_K)(\kappa \land 1_K)(1_L \land m_K))f_4 = -2\lambda_0(1_{E+2} \land j')(\kappa \land 1_K)$ and by composing $1_{E+2} \land \phi \cdot j\overline{\pi} \land 1_M$ on (9.4.39) we have

\begin{align*}
(\lambda_0 - 2\lambda_0)(1_{E+2} \land (\phi \land 1_K))f_4 \\
= \lambda_0(1_{E+2} \land (\phi \land 1_K)) + (1_{E+2} \land (\phi \land 1_K))j'(j'' \land 1_K)(1_L \land m_K))f_4 \\
= (1_{E+2} \land (\phi \land 1_K))j'(j'' \land 1_K))d(f_5) \\
+ (1_{E+2} \land (\phi \land 1_K)ijj'(j'' \land 1_K))d((1_{E+2} \land 1_L \land m_K)f_4) = 0
\end{align*}

so that the Theorem follows by the same reason. Q.E.D.

\section{5} A sequence of $h_0\sigma$ new families in the stable homotopy groups of spheres
In this section, the convergence of a sequence of $h_0\sigma$ and $h_0\sigma'$ new families will be derived by the main Theorem A in §2 and the main Theorem C in §4, where $\sigma$ and $\sigma'$ is a pair of $a_0$-related elements.

**Theorem 9.5.1** Let $p \geq 7, n \geq 2$, then

$$h_0n \in Ext^2_A p^\sigma q^\sigma (Z_p, Z_p), \quad h_0b_{n-1} \in Ext^3_A p^\sigma q^\sigma q^\sigma (Z_p, Z_p)$$

are permanent cycles in the ASS and they converge in the ASS to homotopy elements of order $p$ in $\pi_p q^\sigma q-2 S, \pi_p q^\sigma q-3 S$ respectively.

**Proof**: By [12] Theorem 1.2.14 we have $d_2(h_n) = a_0b_{n-1} \in Ext^3_A p^\sigma q^\sigma+1 (Z_p, Z_p), n \geq 1$, where $d_2 : Ext^1_A p^\sigma q^\sigma (Z_p, Z_p) \to Ext^3_A p^\sigma q^\sigma+1 (Z_p, Z_p)$ is a secondary differential in the ASS. That is, $h_n$ and $b_{n-1}$ is a pair of $a_0$-related elements so that the main Theorem A can apply to $(\sigma, \sigma') = (h_n, b_{n-1}), (s, tq) = (1, p^\sigma q)$. We only need to check the supposition (I)(II)(III) of the main Theorem A hold for $(\sigma, \sigma') = (h_n, b_{n-1}), (s, tq) = (1, p^\sigma q)$. On the other hand, from some results on $Ext^s_A(Z_p, Z_p)$ in [17] we know that the following hold.

$$Ext^4_A p^\sigma q^\sigma+2q+1 (Z_p, Z_p) = 0 (r = 1, 3, 4),$$

$$Ext^4_A p^\sigma q^\sigma+4q+2 (Z_p, Z_p) = 0 (r = 2, 3),$$

$$Ext^4_A p^\sigma q^\sigma+2q+3 (Z_p, Z_p) \cong Z_p \{\alpha_2 b_{n-1}\},$$

$$Ext^4_A p^\sigma q^\sigma+3 (Z_p, Z_p) \cong Z_p \{a_2 b_{n-1}\}, Ext^4_A p^\sigma q^\sigma+1 (Z_p, Z_p) = 0.$$

That is, the supposition (III) of the main Theorem A hold for $(\sigma, \sigma') = (h_n, b_{n-1}), (s, tq) = (1, p^\sigma q)$. Then, by the main Theorem A we obtain that $h_0b_{n-1} \in Ext^3_A p^\sigma q^\sigma q^\sigma (Z_p, Z_p), i_s(h_0h_n) \in Ext^2_A p^\sigma q^\sigma q^\sigma (H^*M, Z_p)$ are permanent cycles in the ASS. By Remark 9.2.35, the main Theorem A also obtains that $(1_L \wedge i)_s \phi_s(h_n) \in Ext^2_A p^\sigma q^\sigma+2q (H^*L, Z_p)$ is a permanent cycle in the ASS so that the main Theorem C can apply to obtain the result of the Theorem, this is because by knowledge on the $A_p$-base of $Ext^s_A(Z_p, Z_p)$ for $s = 1, 2, 3$ we can easy to see that the supposition (I) of the main Theorem C hold for $(\sigma, \sigma', s, tq) = (h_n, b_{n-1}, 1, p^\sigma q)$. Q.E.D.

Now we apply the main Theorem A and the main Theorem C to $(\sigma, \sigma') = (h_n h_m, h_n b_{m-1} - h_m b_{n-1}), (s, tq) = (2, p^\sigma q + p^m q)$ to obtain another sequence of $h_0\sigma$ families in the stable homotopy groups of spheres. For checking the supposition (I)(II)(III) of the main Theorem A, we first
prove the following Proposition.

**Proposition 9.5.2** Let \( p \geq 7, n \geq m + 2 \geq 4, tq = p^n q + p^m q, \) then

1. \( \text{Ext}_A^{4tq + p}(Z_p, Z_p) = 0 \) for \( r = 2, 3, 4, u = -1, 0 \) or \( r = 3, 4, u = 1, \)
\[
\text{Ext}_A^{4tq + p}(Z_p, Z_p) \cong Z_1[h_0 h_{n-1}, h_0 h_m, b_{m-1}].
\]
\[
\text{Ext}_A^{4tq}(Z_p, Z_p) \cong Z_1(b_{n-1} b_{m-1}),
\]
\[
\text{Ext}_A^{4tq + 1}(Z_p, Z_p) \cong Z_1(a_0 h_{n-1}, a_0 h_m, b_{m-1}).
\]

2. \( \text{Ext}_A^{5tq + 1}(Z_p, Z_p) = 0 \) for \( r = 1, 3, 4, \)
\[
\text{Ext}_A^{5tq + 1}(Z_p, Z_p) \cong Z_1 \{a_0 b_{n-1}, a_0 h_{n-1}, a_0 h_m, b_{m-1}\},
\]
\[
a_0^2 b_{n-1} b_{m-1} \neq 0 \in \text{Ext}_A^{5tq + 2}(Z_p, Z_p).
\]

**Proof:** By Theorem 5.5.3, there is a May spectral sequence (MSS) \( \{E_n^{i,s}, d_r\} \) which converges to \( \text{Ext}_A^{i,s}(Z_p, Z_p) \) and whose \( E_1 \)-term is
\[
E_1^{i,s} = E(h_{i,j} \mid i > 0, j \geq 0) \otimes P(b_{i,j} \mid i > 0, j \geq 0) \otimes P(a_i \mid i \geq 0),
\]
where \( E \) denotes the exterior algebra and \( P \) denotes a polynomial algebra,
\[
h_{i,j} \in E_1^{i,j} Z_{p^{i-1}} p^{s-1}, \quad b_{i,j} \in E_1^{i,j} Z_{p^{i+1}} p^{s-1}, \quad a_i \in E_1^{i,j} Z_{p^{i+1}} p^{s-1}.
\]
Consider the following second degrees \((\text{mod } p^n q)\) of the generators in the \( E_1^{i,s} \)-term, where \( 0 \leq i \leq n, n \geq m + 2 \geq 4 \)
\[
\begin{align*}
| h_{s,i} | &= (p^{s+i-1} + \cdots + p^i q) \pmod{p^n q}, \quad 0 \leq s + i - 1 < n, \\
&= (p^{n+1} + \cdots + p^i q) \pmod{p^n q}, \quad 0 \leq s + i - 1 = n, \\
| b_{s,i-1} | &= (p^{s+i-1} + \cdots + p^i q) \pmod{p^n q}, \quad 1 \leq s + i - 1 < n, \\
&= (p^{n+1} + \cdots + p^i q) \pmod{p^n q}, \quad 1 \leq s + i - 1 = n, \\
| a_{i+1} | &= (p^i + \cdots + 1) q + 1 \pmod{p^n q}, \quad 1 \leq i < n, \\
| a_{i+1} | &= (p^{n-1} + \cdots + 1) q + 1 \pmod{p^n q}, \quad i = n.
\end{align*}
\]
For degree \( k = tq + rq + u \) such that \( 0 \leq r \leq 4, -1 \leq u \leq 2 \) we have \( k \equiv p^m q + rq + u \pmod{p^n q} \). Then, for \( 3 \leq w \leq 5 \), \( E_1^{w, tq + rq + u} \) has no such generators which have one of the above elements as a factor, this is because such a generator will have second degree \((c_n p^{n-1} + \cdots + c_1 p + c_0) q + d \pmod{p^n q}\), where \( c_i \neq 0 (1 \leq i \leq m - 1 \text{ or } m < i < n) \), \( 0 \leq c_i < p, l = 0, \ldots, n, 0 \leq d \leq 5 \). In addition, the second degree \( | b_{1,i-1} | = p^i q \pmod{p^n q}(1 \leq i \leq n), | h_{1,i} | = p^i q \pmod{p^n q}(0 \leq i \leq n) \). Then, exclude the above factor and the factor which has second degree \( \geq tq + pq \), we know that the
only possibility of the factor of the generators in $E_1^{w, tq+ru, *}$ are $a_1, a_0, h_{1,0}$, $h_{1,n}, h_{1,m}, b_{1,n-1}, b_{1,m-1}$.

Then, by degree reasons we have

$$E_1^{tq+ru+1, *} = 0 \text{ for } r = 3, 4, \quad E_1^{tq+ru, *} = 0 \text{ for } r = 2, 3, 4, u = -1, 0$$

$$E_1^{tq, *} = Z_p\{b_{1, n-1}b_{1, m-1}\}, \quad E_1^{tq+1, *} = Z_p\{a_0h_{1,n}b_{1,m-1}, a_0h_{1,m}b_{1,n-1}\};$$

$$E_1^{tq+2, *} = Z_p\{a_0^2h_{1,n}h_{1,m}\}, \quad E_1^{tq+2q+1, *} = Z_p\{h_{1,0}a_1h_{1,n}h_{1,m}\}.$$
Theorem 9.5.3 Let \( p \geq 7, n \geq m + 2 \geq 4 \), then

\[
h_0 h_n h_m \in \text{Ext}_A^{3,p^n q+p^m q+q}(Z_p, Z_p),
\]
\[
h_0(h_n b_{m-1} - h_m b_{n-1}) \in \text{Ext}_A^{4,p^n q+p^m q+q}(Z_p, Z_p)
\]
are permanent cycles in the ASS and they converge to homotopy elements of order \( p \) in \( \pi_{p^n q+p^m q+q-3}S \) and \( \pi_{p^n q+p^m q+q-4}S \) respectively.

Proof: By [12]p.11 Theorem 1.2.14, there is a nontrivial secondary differential \( d_2(h_n) = a_0 b_{n-1}(n \geq 1) \) and it follows that \( d_2(h_n h_m) = d_1(h_n) h_m + (-1)^{1+p^n q} h_n d_2(h_m) = a_0 h_m b_{n-1} - a_0 h_n b_{m-1} \). That is, \( (h_n h_m, h_m b_{n-1} - h_n b_{m-1}) \) is a pair of \( a_0 \)-related elements. By applying the main Theorem A to \( (\sigma, \sigma') = (h_n h_m, h_m b_{n-1} - h_n b_{m-1}) \), \( (s, tq) = (2, p^n q + p^m q) \) we have \( h_0(h_m b_{n-1} - h_n b_{m-1}) \in \text{Ext}_A^{4+p^n q+p^m q+q}(Z_p, Z_p) \) and \( i_s(h_0 h_n h_m) \in \text{Ext}_A^{3,p^n q+p^m q+q}(Z_p, Z_p) \) are permanent cycles in the ASS, this is because by knowledge of \( Z_p \)-base of \( \text{Ext}_A^{s,*}(Z_p, Z_p) \) for \( s \leq 3 \) we know that the supposition (I)(II)(III) of the main Theorem A hold. By Remark 9.2.35, the main Theorem A also obtains that \( (1_L \wedge i)_s \phi_s(h_n h_m) \in \text{Ext}_A^{3+p^n q+p^m q+2q}(H^* L, Z_p) \) is a permanent cycle in the ASS so that by the main Theorem C , the result of the Theorem follows. This is because the supposition (I) of the main Theorem C hold by the knowledge of the \( Z_p \)-base of \( \text{Ext}_A^{s,*}(Z_p, Z_p) \) for \( s = 1, 2, 3 \). Q.E.D.

From Theorem 9.5.1 and Theorem 9.5.3, we obtain four families of \( h_0 \sigma \) new families. In fact, there are many pairs of \( a_0 \)-related elements so that we can expect to obtain some other sequence of \( h_0 \sigma \) new families in the stable homotopy groups of spheres. We have the following conjectures.

Conjecture 9.5.4 Let \( p \geq 7, n \geq 3 \), then there is a secondary differential \( d_2(g_n) = a_0 l_n \in \text{Ext}_A^{3,p^n+1 q+2p^n q+1}(Z_p, Z_p), n \geq 3 \) (up to nonzero scalar) and

\[
h_0 g_n \in \text{Ext}_A^{3,p^n+1 q+2p^n q+q}(Z_p, Z_p),
\]
\[
h_0 l_n \in \text{Ext}_A^{4,p^n+1 q+2p^n q+q}(Z_p, Z_p)
\]
are permanent cycles in the ASS and they converge to homotopy elements of order \( p \) in \( \pi_{p^n+1 q+2p^n q+q-3}S \) and \( \pi_{p^n+1 q+2p^n q+q-4}S \) respectively, where \( g_n \in \text{Ext}_A^{2,p^n+1 q+2p^n q}(Z_p, Z_p), l_n \in \text{Ext}_A^{3,p^n+1 q+2p^n q}(Z_p, Z_p) \).

Conjecture 9.5.5 Let \( p \geq 7, n \geq 3 \), then there is a secondary dif-
ferential \( d_2(k_n) = a_0 h_n \in Ext_A^{1,2p^{n+1}+p^nq+1}(Z_p, Z_p) \) (up to nonzero scalar), \( n \geq 3 \) and
\[
\begin{align*}
\hat{h}_0 k_n & \in Ext_A^{3,2p^{n+1}+p^q+q}(Z_p, Z_p) \\
\hat{h}_0 i'_n & \in Ext_A^{1,2p^{n+1}+p^nq+q}(Z_p, Z_p)
\end{align*}
\]
are permanent cycles in the ASS and they converge to homotopy elements of order \( p \) in \( \pi_{2p^{n+1}+p^nq+q-3}S \) and \( \pi_{2p^{n+1}+p^nq+q-4}S \), where \( k_n \in Ext_A^{2,2p^{n+1}+p^nq}(Z_p, Z_p), i'_n \in Ext_A^{3,2p^{n+1}+p^nq}(Z_p, Z_p) \).

**Remark 9.5.6** By [10][25], there is Thom map \( \Phi : Ext_{BP,BP}^{s,t}(BP_s, BP_t) \to Ext_A^{s,t}(Z_p, Z_p) \) (\( s = 2, 3 \)) such that \( \Phi(\beta_{p^n-1/p^n-1}) = h_0 h_n, \Phi(\beta_{p^n-1/p^n-1}) = b_{n-1}, \Phi(\beta_{p^n-1/p^n-1}) = b_{n-1} - h_n b_{m-1}, \Phi(\gamma_{p^n-2/p^n-2} - \beta_{p^n-1/p^n-1}) = h_0 h_n h_m. \) Then, the \( h_0 h_n, h_0 b_{n-1}, h_0(h_n b_{n-1} - h_n b_{m-1}), h_0 h_n h_{m-1}-map obtained by Theorem 9.5.1 and Theorem 9.5.3 are represented by \( \beta_{p^n-1/p^n-1} \) + other terms \( \in Ext_{BP,BP}^{2(p^nq+q)}(BP_s, BP_t), \alpha_1 \beta_{p^n-1/p^n-1} + other terms \in Ext_{BP,BP}^{3(p^nq+q)}(BP_s, BP_t), \beta_{p^n-1/p^n-1} + other terms \in Ext_{BP,BP}^{3(p^nq+q)}(BP_s, BP_t), \) \( \gamma_{p^n-2/p^n-2} - \beta_{p^n-1/p^n-1} \) + other terms \( \in Ext_{BP,BP}^{4(p^nq+q)}(BP_s, BP_t) \) respectively in the Adams-Novikov spectral sequence.

§6. A sequence of \( h_0 \sigma \tilde{g}_s, g_0 \sigma \tilde{g}_s \) new families in the stable homotopy groups of spheres

In this section, we use the main Theorem B to obtain \( i'_s i_s(g_0 h_n), i'_s i_s(g_0 b_{n-1}), i'_s i_s(g_0 h_n h_m), i'_s i_s(g_0(h_n b_{m-1} - b_m b_{n-1})) \) et al converge to the corresponding nontrivial homotopy elements in the homotopy groups of Smith-Toda spectrum \( V(1) \). In base of these results, we obtain a sequence of \( g_0 \sigma \tilde{g}_s, h_0 \sigma \tilde{g}_s \) new families in the stable homotopy groups of spheres.

**Theorem 9.6.1** Let \( p \geq 5, n \geq 2 \), then
\[
\begin{align*}
i'_s i_s(g_0 h_n) & \in Ext_A^{3,p^nq+pq+2q}(H^*K, Z_p) \\
i'_s i_s(g_0 b_{n-1}) & \in Ext_A^{4,p^nq+pq+2q}(H^*K, Z_p)
\end{align*}
\]
are permanent cycles in the ASS and they converge to the corresponding homotopy element in \( \pi_{p^nq+pq+2q-3}K, \pi_{p^nq+pq+2q-4}K \) respectively.

**Proof:** We first apply the main Theorem B to \( (\sigma, s, tq) = (h_n, 1, p^nq) \). By Theorem 9.5.1, the supposition (II) of the main Theorem B holds. More-
over, by knowledge on the $Z_p$-base of $\text{Ext}_A^{s,n}(Z_p, Z_p)$ for $s = 1, 2, 3$ we know that the supposition (I) of the main Theorem B holds, then the first result of the Theorem follows by the main Theorem B.

For the second result, we apply the main Theorem B to $(\sigma, s, tq) = (b_{n-1}, 2, p^nq)$. Similarly by Theorem 9.5.1, the supposition (II) of the main Theorem B holds. Morever, by knowledge on the $Z_p$-base of $\text{Ext}_A^{s,n}(Z_p, Z_p)$ for $s = 2, 3$ and some result in [17] on $\text{Ext}_A^{4,n}(Z_p, Z_p)$ we know that the supposition (I) of the main Theorem B holds. Then, the second result also follows by the main Theorem B. Q.E.D.

**alternative Proof:** It is known from the proof of Theorem 9.5.1 that the supposition (I)/(II)/(III) if the main Theorem A hold for $(\sigma, \sigma') = (h_n, b_{n-1}), (s, tq) = (1, p^nq)$. Then, applying the main Theorem B' in §3 to $(\sigma, \sigma') = (h_n, b_{n-1}), (s, tq) = (1, p^nq)$ we obtain the two results of the Theorem. Q.E.D.

**Theorem 9.6.2** Let $p \geq 7, n \geq m + 2 \geq 4$, then

\[ i'_* i_*(g_0 h_n h_m) \in \text{Ext}_A^{4,n+p^nq+p^nq+pq+2q}(H^* K, Z_p) \]

\[ i'_* i_*(g_0 (h_n b_{m-1} - h_m b_{n-1})) \in \text{Ext}_A^{5,n+p^nq+p^nq+pq+2q}(H^* K, Z_p) \]

are permanent cycles in the ASS and they converge to nontrivial homotopy elements in $\pi_{p^nq+p^nq+pq+2q-4} K, \pi_{p^nq+p^nq+pq+2q-5} K$ respectively.

**Proof:** We first apply the main Theorem B to $(\sigma, s, tq) = (h_n, h_{m, 2, p^nq+p^nq})$. By Theorem 9.5.3, the supposition (II) of the main Theorem B holds. By knowledge on the $Z_p$-base of $\text{Ext}_A^{s,n}(Z_p, Z_p)$ for $s = 2, 3$ and some result in [17] on $\text{Ext}_A^{4,n}(Z_p, Z_p)$ we know that the supposition (I) of the main Theorem B also holds. Then the first result follows by the main Theorem B. Moreover, we apply the main Theorem B to $(\sigma, s, tq) = (h_n b_{m-1} - h_m b_{n-1}, 3, p^nq+p^nq)$. Similarly by Theorem 9.5.3, the supposition (II) of the main Theorem B holds. By knowledge on the $Z_p$-base of $\text{Ext}_A^{s,n}(Z_p, Z_p)$ for $s = 3, 4$ and the result on $\text{Ext}_A^{5,n+p^nq+p^nq+2q+1}(Z_p, Z_p) \cong Z_p \{\tilde{\sigma}_2 h_n b_{m-1}, \tilde{\sigma}_2 h_m b_{n-1}\}$ in Prop. 9.5.2 we know that the supposition (I) of the main Theorem B holds. Then, the second result follows immediately by the main Theorem B. Q.E.D.

**alternative Proof:** It is known from the proof of Theorem 9.5.2 that the supposition (I)/(II)/(III) of the main Theorem A hold for $(\sigma, \sigma') = (h_n, h_{m, b_{m-1} - h_m b_{n-1}}), (s, tq) = (2, p^nq+p^nq)$. Then by applying the main Theorem B' in §3 to $(\sigma, \sigma') = (h_n, h_{m, b_{m-1} - h_m b_{n-1}}), (s, tq) = (2, p^nq+p^nq)$ we obtain the two results of the Theorem. Q.E.D.
respectively. Then, the products $g$ is represented by $(i_2)_*(g_0 h_n) \in \text{Ext}_A^{s+3, p^n q + p q + s + 3}(Z_p, Z_p)$, converge to the following homotopy element of order $p$ in the stable homotopy groups of spheres, where $\sigma = h_n, b_{n-1}, h_n m$, or $h_n b_{m-1} - h_m b_{n-1}$.

**Theorem 9.6.3** Let $p \geq 7, n \geq 3, 3 \leq s < p$, then the products
\[
g_0 h_n \tilde{\gamma}_s \neq 0 \in \text{Ext}_A^{s+3, p^n q + p q + s + 3}(Z_p, Z_p)
\]
are permanent cycles in the ASS and they converge to the corresponding homotopy elements of order $p$ in the stable homotopy groups of spheres.

**Proof:** By Theorem 9.6.1, there is a nontrivial $f \in \pi_{p^n q + p q + 2 q - 3} K$ such that it is represented by $i'_* i_*(g_0 h_n) \in \text{Ext}_A^{s+3, p^n q + p q + 2 q}(K^* K, Z_p)$ in the ASS. Let $\tilde{f} = j_1 j_2 j_3 \gamma^* i_3 f$ be the following composition ($tq = p^n q + p q + 2 q - 3$)
\[
\tilde{f}: \Sigma^q S \xrightarrow{\cdot} V(1) \xrightarrow{i_3} V(2) \xrightarrow{\gamma^*} \Sigma^{-s(p^2 q + p q + q)} \Sigma^2 \text{Ext}_A^{s+3, p^n q + p q + s + 3}(Z_p, Z_p)
\]
Since $f$ is represented by $(i_2)_*(i_1)_*(g_0 h_n) \in \text{Ext}_A^{s+3, p^n q + p q + 2 q}(K^* K, Z_p)$ in the ASS, then the above $\tilde{f}$ is represented by
\[
c = (j_1 j_2 j_3)_*(\gamma^*)_*(i_3 i_2 i_1)_*(g_0 h_n) \in \text{Ext}_A^{s+3, p^n q + s(p^2 + p + 1)q + s - 3}(Z_p, Z_p)
\]
By knowledge of Yoneda products we know that the above element $c$ is just the products $g_0 h_n \tilde{\gamma}_s \in \text{Ext}_A^{s+3, p^n q + s(p^2 + p + 1)q + s - 3}(Z_p, Z_p)$. Then, to obtain the first result, it suffices to prove the product $g_0 h_n \tilde{\gamma}_s$ is nonzero in the Ext group and it is not a $d_r$-boundary in the ASS, that is, we still need to prove $\text{Ext}_A^{s+3-r, p^n q + s(p^2 + p + 1)q + s - 2-r}(Z_p, Z_p)$ is zero for $r \geq 2$. We may prove this two facts by an argument in the May spectral sequence. By degree reasons, $h_n, g_0, \tilde{\gamma}_s$ is represented by $h_{1,n}, h_{2,0} h_{1,0}, h_{2,1} h_{1,2} h_{3,0} a_{3}^{s-3} \in E_1^{s,*}$ in the MSS respectively. Then, the products $g_0 h_n \tilde{\gamma}_s$ is represented by
\[
h_{1,n} h_{2,0} h_{1,0} h_{2,1} h_{1,2} h_{3,0} a_{3}^{s-3} \in E_1^{s+3, p^n q + s(p^2 + p + 1)q + s - 3,*}
\]
in the MSS and so we can do some computation in the degree to prove $E_1^{s+2, p^n q + s(p^2 + p + 1)q + s - 3,*} = 0$ and $E_1^{s+3-r, p^n q + s(p^2 + p + 1)q + s - 2-r,*} = 0(r \geq 2)$
so that the first result follows. We leave this computation to the reader. The proof and computation for the second result is similar. Q.E.D.

By using Theorem 9.6.2, Theorem 9.5.1 and Theorem 9.5.3, similar to that given in the proof of Theorem 9.6.3, we can obtain the following Theorem 9.6.4–9.6.6.

**Theorem 9.6.4** Let \( p \geq 7, n \geq m + 2 \geq 5, 3 \leq s < p \), then
\[
g_0 h_n h_m \tilde{\gamma}_s \neq 0 \in Ext^s_A\pi^{p^n q + p^n q + s(p^2 + p + 1)q + s-3}(Z_p, Z_p)
\]
\[
g_0 (h_n b_{m-1} - h_m b_{n-1}) \tilde{\gamma}_s \neq 0 \in Ext^s_A\pi^{p^n q + p^n q + s(p^2 + p + 1)q + s-3}(Z_p, Z_p)
\]
are permanent cycles in the ASS and they converge to the corresponding homotopy elements of order \( p \) in the stable homotopy groups of spheres.

**Theorem 9.6.5** Let \( p \geq 7, n \geq m + 2 \geq 3, 3 \leq s < p \), then the products
\[
h_0 h_n \tilde{\gamma}_s \neq 0 \in Ext^s_A\pi^{s+2 p^n q + s p^2 q + (s-1)(p+1)q + s-3}(Z_p, Z_p)
\]
\[
h_0 b_{n-1} \tilde{\gamma}_s \neq 0 \in Ext^s_A\pi^{s+3 p^n q + s p^2 q + (s-1)(p+1)q + s-3}(Z_p, Z_p)
\]
are permanent cycles in the ASS and they converge to the corresponding elements of order \( p \) in the stable homotopy groups of spheres.

**Theorem 9.6.6** Let \( p \geq 7, n \geq m + 2 \geq 5, 3 \leq s < p \), then the products
\[
h_0 h_n h_m \tilde{\gamma}_s \neq 0 \in Ext^s_A\pi^{s+3 p^n q + p^n q + s p^2 q + (s-1)(p+1)q + s-3}(Z_p, Z_p)
\]
\[
h_0 (h_n b_{m-1} - h_m b_{n-1}) \tilde{\gamma}_s \neq 0 \in Ext^s_A\pi^{s+4 p^n q + p^n q + s p^2 q + (s-1)(p+1)q + s-3}(Z_p, Z_p)
\]
are permanent cycles in the ASS and they converge to the corresponding homotopy elements of order \( p \) in the stable homotopy groups of spheres.

**Remark 9.6.7** The new families obtained in Theorem 9.6.5 and Theorem 9.6.6
are the composition products of \( h_0 h_n \)-element, \( h_0 b_{n-1} \)-element in Theorem 9.5.1, \( h_0 h_n h_m \)-element, \( h_0 (h_n b_{m-1} - h_m b_{n-1}) \)-element in Theorem 9.5.3 and \( \gamma_s = j_1 j_2 j_3 \gamma^s i_3 i_2 i_1 \in \pi^{s p^2 q + (s-1)p q + (s-2)q - 3}S \). However, the new families obtained in Theorem 9.6.3 and Theorem 9.6.4 are indecomposable elements in the stable homotopy groups of spheres, that is, they are not compositions of some other elements of lower degrees in the stable homotopy groups of spheres. This is because \( g_0 \in Ext^2_A\pi^{2pq + 2q}(Z_p, Z_p) \) dies in the ASS, that is, it support a nontrivial differential in the Adams spectral sequence: \( d_2(g_0) = b_0 \alpha_2 \) (up to nonzero scalar) \( \in Ext^4_A\pi^{4pq + 2q + 1}(Z_p, Z_p) \) which can be
easily proved as follows. Since $\tilde{\alpha}_2, b_0$ converge in the ASS to $\alpha_2 = j\alpha^2i, \beta_1 = j\beta'(i \in \pi_sS$, then the composition products of $\beta_1\alpha_2 \in \pi_{pq}+2q-3S$ must be represented by $b_0\tilde{\alpha}_2 \in Ext^A_{pq+2q+1}(Z_p, Z_p)$ in the ASS. However, it is easily seen that $\beta_1\alpha_2 = j\beta'i j\alpha^2i = 0$ and $b_0\tilde{\alpha}_2 \neq 0 \in Ext^A_{pq+2q+1}(Z_p, Z_p)$, then $b_0\tilde{\alpha}_2$ must be a $d_s$-boundary. By degree reason, the only possibility is $b_0\tilde{\alpha}_2 = d_2(g_0)$ (up to nonzero scalar).

**Conjecture 9.6.8** By the conjecture 9.5.4–9.5.5, we can conjecture that, for $p \geq 7, n \geq 3, 3 \leq s < p$, the products $h_0g_n\tilde{\gamma}_s, h_0l_n\tilde{\gamma}_s, g_0g_n\tilde{\gamma}_s, g_0l_n\tilde{\gamma}_s, h_0k_n\tilde{\gamma}_s, h_0l'_n\tilde{\gamma}_s, g_0k_n\tilde{\gamma}_s, g_0l'_n\tilde{\gamma}_s$ are permanent cycles in the ASS and they converge to the corresponding homotopy elements of order $p$ in the stable homotopy groups of spheres. In addition, all results or conjectures in this section also hold when we replace the products with $\tilde{\beta}_s(2 \leq s < p)$. That is, we can obtain a sequence of $h_0\sigma\tilde{\beta}_s, g_0\sigma\tilde{\beta}_s$-elements, where $\tilde{\beta}_s = (j_1j_2)_{s}\tilde{\beta}_s(i_1i_2)_{s}(1) \in Ext^A_{pq+(s-1)q+s-2}(Z_p, Z_p), 2 \leq s < p$.

§7. Third periodicity families in the stable homotopy groups of spheres

In this section, we will first prove the convergence of $h_n$-elements in the homotopy groups of Smith-Toda spectrum $V(1)$ and in base of this we obtain the convergence of third periodicity $\gamma_{pq/s}$ families ($1 \leq s \leq p^n - 1$) in the Adams-Novikov spectral sequence.

**Theorem 9.7.1** ([9] Theorem II) Let $p \geq 5, n \geq 0$,

$$h_n \in Ext^{1p^nq}_{BP,BP}(BP_s, BP_sK)$$

be the element represented by $[p^n_1]$ in the cobar complex. Then this $h_n$ is a permanent cycle in the Adams-Novikov spectral sequence and it converges to a nontrivial homotopy element in $\pi_{pqq-1}K$.

The proof of the above $h_n$-Theorem will be the main content of this section. By Theorem 8.1.6(b)(ii), there is a relation

$$h_n = c_2(p^{n-2}) + v^{pq-2}_s h_{n-2} \in Ext^{1p^nq}_{BP,BP}(BP_s,BP_sK)$$

By [10], p.502 Cor. 7.8, the image of $v^{pq-2}_s c_2(p^{n-2})(p^{n-2} > s \geq 1)$ under the boundary homomorphism (or connecting homomorphism)

$$j^*_s : Ext^{1s}_{BP,BP}(BP_s, BP_sK) \to Ext^{2s}_{BP,BP}(BP_s, BP_sM)$$

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and
\[ j_* : \text{Ext}^{2,*}_{BP,BP}(BP_*, BP_* M) \to \text{Ext}^{3,*}_{BP,BP}(BP_*, BP_*) \]
is just the third periodicity family \( \gamma_{p^n-2/p^n-2-s} \neq 0 \in \text{Ext}^{3,*}_{BP,BP}(BP_*, BP_*) \). Then, by Theorem 9.7.1, the relation (9.7.2) and Theorem 7.3.2, we can obtain the following convergence Theorem of third periodicity families in the stable homotopy groups of spheres immediately.

**Theorem 9.7.3** ([9] Theorem 1) Let \( p \geq 5, n \geq 1 \) and \( 1 \leq s \leq p^n-1 \), then the following third periodicity family
\[ \gamma_{p^n/s} \in \text{Ext}^{3,*}_{BP,BP}(BP_*, BP_*) \]
is a permanent cycle in the ASS and it converge to an element of order \( p \) in \( \pi_* S \) which has degree \( p^{n+2}q + (p^n-s)(p+1)q - q - 3 \).

To prove Theorem 9.7.1, we first prove the following weaker Theorem.

**Theorem 9.7.4** ([9] Theorem 4.1) Let \( p \geq 5, n \geq 0, h_n \in \text{Ext}^{1,p^n q}_A(H^* K, Z_p) \) be the element represented by \( \xi p^n \) in the cobar complex, then
\[ i'_* i_* (h_n) \in \text{Ext}^{1,p^n q}_A(H^* K, Z_p) \]
is a permanent cycle in the ASS and it converge to a nontrivial homotopy element in \( \pi_{p^n q-1} K \).

The proof of Theorem 9.7.4 will be the main content of the rest of this section. The proof need some preminilaries on low dimensional Ext groups and an argument processing in the Adams resolution of some spectra related to \( S \). We first prove some results on Ext groups.

**Theorem 9.7.5** Let \( p \geq 3, n \geq 2, \) , then
\begin{enumerate}
  \item \( \text{Ext}^{s,p^n q+r}_A(H^* K, Z_p) = 0 \) for \( s = 2, 3, r = 1, 2, \)
  \( \text{Ext}^{s,p^n q+1}_A(H^* K, H^* M) = 0, \)
  \( \text{Ext}^{3,p^n q+q}_A(H^* K, H^* K) \cong Z_p \{ (h_0 b_{n-1})' \}. \)
  \item \( \text{Ext}^{s-1,p^n q+q+s-3}_A(H^* Y, Z_p) = 0 \) for \( s = 1, 2, 3, \)
  \item \( \text{Ext}^{1,p^n q}_A(H^* K, H^* Y) = 0, \)
\end{enumerate}
where \( Y \) is the spectrum in the cofibration (9.1.4).

**Proof**: (1) Consider the following exact sequence \( (s = 1, 2, 3, r = 1, 2) \)
\[ \text{Ext}^{s,p^n q+r}_A(Z_p, Z_p) \to \text{Ext}^{s,p^n q+r}_A(H^* M, Z_p) \to \text{Ext}^{s,p^n q+r-1}_A(Z_p, Z_p) \]
induced by (9.1.1). By knowledge of \( Z_p \)-base of \( \text{Ext}^{s,*}_A(Z_p, Z_p) \) for \( s = 1, 2, 3 \) we know that the right group is zero except for \( (s, r) = (1, 1), (2, 1), (2, 2), (3, 2) \) it has unique generator \( h_n, b_{n-1}, a_0 h_n, a_0 b_{n-1} \). However, \( p_*(h_n) = a_0 h_n \neq \)
0, \pi_s(b_{n-1}) = a_0b_{n-1} \neq 0, \pi_s(a_0h_n) = a_0^2h_n \neq 0, \pi_s(a_0b_{n-1}) = a_0^2b_{n-1} \neq 0,
\text{then the above } \pi_s \text{ is monic and so } \text{im } j_s = 0. \text{ In addition, the left group is zero except for } (s, r) = (2, 1), (3, 1), (3, 2) \text{ it has unique generator } a_0h_n = p_s(h_n), a_0b_{n-1} = p_s(b_{n-1}), a_0^2h_n = p_s(a_0h_n) \text{ respectively. Then we have im } i_s = 0 \text{ and obtain that } \text{Ext}_A^{s,p^q+r}(H^sM, Z_p) = 0 \text{ for } s = 1, 2, 3, r = 1, 2.

Look at the following exact sequence \((s, 2, 3, r = 1, 2)\)

\[0 = \text{Ext}_A^{s,p^q+r}(H^sM, Z_p) \xrightarrow{i_s^*} \text{Ext}_A^{s,p^q+r}(H^sK, Z_p)\]

induced by \((9.1.2)\). The left group is zero as shown above. The right group also is zero, this is because \(\text{Ext}_A^{s,p^q+r}(Z_p, Z_p) = 0 \) for \(s = 2, 3, r = 1, 2, 3\) (cf. Chap. 5). Then, the middle group is zero for \(s = 2, 3, r = 1, 2\) and so \(\text{Ext}_A^{s,p^q+1}(H^sK, H^sM) = 0 \) \((s = 2, 3)\).

For the last result, consider the following exact sequence

\[\text{Ext}_A^{s,p^q+2+1}(H^sK, H^sM) \xrightarrow{(i')^*} \text{Ext}_A^{s,p^q+q}(H^sK, H^sK)\]

\[\xrightarrow{(i')^*} \text{Ext}_A^{s,p^q+q}(H^sK, H^sM) \xrightarrow{\alpha^*} \]

induced by \((9.1.2)\). The left group is zero by Prop. \(9.3.2(1)\) and the right group has unique generator \(i'_s(\alpha_1 \wedge 1_M)_{s}(\tilde{b}_{n-1})\) (cf. Prop. \(9.3.1\)) such that \(\alpha^*(i'_s)_{s}(\alpha_1 \wedge 1_M)_{s}(\tilde{b}_{n-1}) = 0\). Then, the middle group has unique generator \((h_0b_{n-1})'\) such that \((i')^*(h_0b_{n-1})' = i'_s(\alpha_1 \wedge 1_M)(\tilde{b}_{n-1})\). Q.E.D.

(2) The result is obvious for \(s = 1\). For \(s = 2, 3\), consider the following exact sequence

\[\xrightarrow{(i')^*} \text{Ext}_A^{s-1,p^q+q+s-3}(H^sK, Z_p) \xrightarrow{r_s} \text{Ext}_A^{s-1,p^q+q+s-3}(H^sY, Z_p)\]

\[\xrightarrow{r_s} \text{Ext}_A^{s,p^q+q+s-3}(Z_p, Z_p) \xrightarrow{(i')^*}\]

induced by \((9.1.4)\). The left group is zero for \(s = 2\), this is because \(\text{Ext}_A^{1,t}(Z_p, Z_p) = 0 \) for \(t = -1, -2 \) (mod \(q\)). The left group has unique generator \((i'^*s)(h_0h_n)\) for \(s = 3\) so that \(\text{im } r_s = 0\). The right group is zero for \(s = 2\) and has unique generator \(h_0b_{n-1} \) for \(s = 3\) which satisfies \((i'^*s)(h_0b_{n-1}) \neq 0\), then im \(\epsilon_s = 0\) and so the middle group is zero for \(s = 1, 2, 3\).

(3) Observe the following exact sequence

\[0 = \text{Ext}_A^{0,p^q}(H^sK, Z_p) \xrightarrow{\epsilon'^*} \text{Ext}_A^{1,p^q}(H^sK, H^sY)\]

\[\xrightarrow{\epsilon'^*} \text{Ext}_A^{1,p^q}(H^sK, H^sK) \xrightarrow{(i')^*}\]

induced by \((9.1.4)\). The left group clearly is zero and the right group has unique generator \((h_n)'\) (cf. Prop. \(9.3.6\)) which satisfies \((i'^*)(h_n) =
(i'') \neq 0 \in Ext_A^{2,p^nq}(H^* K, Z_p)$, then the middle group is zero as desired. Q.E.D.

**Prop. 9.7.6** Let $p \geq 3, n \geq 2$, then

1. $\text{Ext}_A^{2,p^nq}(Z_p, H^* M) = 0, \text{Ext}_A^{3,p^nq+1}(Z_p, H^* M) = 0$.
2. $\text{Ext}_A^{2,p^nq}(Z_p, H^* K) = 0, \text{Ext}_A^{3,p^nq+1}(Z_p, H^* K) = 0$.
3. $\text{Ext}_A^{2,p^nq+q-u}(Z_p, H^* K) = 0$ for $u = 0, 1$.

**Proof:** (1) Consider the following exact sequences

\[ \text{Ext}_A^{2,p^nq+1}(Z_p, Z_p) \xrightarrow{\dot{j}^*} \text{Ext}_A^{2,p^nq}(Z_p, H^* M) \xrightarrow{\dot{i}^*} \text{Ext}_A^{2,p^nq}(Z_p, Z_p) \xrightarrow{\dot{\alpha}} \]
\[ \text{Ext}_A^{2,p^nq+2}(Z_p, Z_p) \xrightarrow{\dot{j}^*} \text{Ext}_A^{2,p^nq+1}(Z_p, H^* M) \xrightarrow{\dot{i}^*} \text{Ext}_A^{2,p^nq+1}(Z_p, Z_p) \xrightarrow{\dot{\alpha}} \]

induced by (9.1.1). The upper left group has unique generator $a_0 h_n$ which satisfies $j^*(a_0 h_n) = j^* p^*(h_n) = 0$ and the upper right group has unique generator $b_{n-1}$ satisfying $p^*(b_{n-1}) = a_0 b_{n-1} \neq 0 \in \text{Ext}_A^{3,p^nq+1}(Z_p, Z_p)$ (cf. Theorem 5.4.1), then we have $\text{Ext}_A^{2,p^nq}(Z_p, H^* M) = 0$. The lower left group has unique generator $a_0^2 h_n$ satisfying $j^*(a_0^2 h_n) = j^* p^*(a_0 h_n) = 0$ and the lower right group has unique generator $a_0 b_{n-1}$ such that $p^*(a_0 b_{n-1}) = a_0^2 b_{n-1} \neq 0 \in \text{Ext}_A^{4,p^nq+2}(Z_p, Z_p)$ (cf. Prop. 9.5.2(2)) , then $\text{Ext}_A^{3,p^nq+1}(Z_p, H^* M) = 0$.

(2) Consider the following exact sequences

\[ 0 = \text{Ext}_A^{2,p^nq+q+1}(Z_p, H^* M) \xrightarrow{(\dot{j}')^*} \text{Ext}_A^{2,p^nq}(Z_p, H^* K) \]
\[ \xrightarrow{(\dot{i}')^*} \text{Ext}_A^{2,p^nq}(Z_p, H^* M) = 0 \]
\[ 0 = \text{Ext}_A^{3,p^nq+q+2}(Z_p, H^* M) \xrightarrow{(\dot{j}')^*} \text{Ext}_A^{3,p^nq+1}(Z_p, H^* K) \]
\[ \xrightarrow{(\dot{i}')^*} \text{Ext}_A^{3,p^nq+1}(Z_p, H^* M) = 0 \]

induced by (9.1.2). Both two right groups are zero by (1) and both two left groups are also zero, this is because $\text{Ext}_A^{2,p^nq+q+r}(Z_p, Z_p) = 0$ for $r = 1, 2$ (cf. Chapter 5) and $\text{Ext}_A^{3,p^nq+q+r}(Z_p, Z_p) = 0$ for $r = 2, 3$ (cf. Theorem 5.4.1), then the result follows.

(3) Consider the following exact sequence

\[ \xrightarrow{\alpha^*} \text{Ext}_A^{2,p^nq+2q}(Z_p, H^* M) \xrightarrow{(\dot{j}')^*} \text{Ext}_A^{2,p^nq+q-1}(Z_p, H^* K) \]
\[ \xrightarrow{(\dot{i}')^*} \text{Ext}_A^{2,p^nq+q-1}(Z_p, H^* M) \xrightarrow{\alpha^*} \]

induced by (9.1.2). The left group is zero, this is because $\text{Ext}_A^{2,p^nq+2q+r}(Z_p, Z_p)$
$= 0$ for $r = 0, 1$ (cf. Chapter 5). The right group has unique generator $j^*(h_0h_n)$ since $\Ext^2_A p^nq+q-1(Z_p, Z_p) = 0$ and $\Ext^2_A p^nq+q(Z_p, Z_p) \cong Z_p\{h_0h_n\}$. In addition, we claim that $\alpha^*j^*(h_0h_n) = \frac{1}{2}j^*(\tilde{\alpha}_2h_n) \neq 0 \in \Ext^3_A p^nq+2q(Z_p, H^*M)$. To prove this, it suffices to prove $\alpha^*j^*(h_0) = \frac{1}{2}j^*(\tilde{\alpha}_2) \in \Ext^2_A p^nq+q(Z_p, H^*M)$. Since $i^*\alpha^*j^*(h_0) = \alpha^1(h_0) = h_0^* = 0$, then $\alpha^*j^*(h_0) = \lambda j^*(\tilde{\alpha}_2)$ for some scalar $\lambda \in Z_p$. Since both sides of the equation detect the corresponding homotopy elements, then the relation $\alpha_1j^* = \frac{1}{2}\alpha_2j$ implies $\lambda = \frac{1}{2}$. This shows the above claim and so the above $\alpha^*$ is monic, $\im (i')^* = 0$ and we have $\Ext^2_A p^nq+q-1(Z_p, H^*K) = 0$. The proof of the case for $u = 0$ is similar.

For the second result, consider the following exact sequence

$$\Ext^3_A p^nq+2q+1(Z_p, H^*M) (j')^* \Ext^3_A p^nq+q(Z_p, H^*K) \alpha^* \Ext^2_A p^nq+q(Z_p, H^*K) \sim \Ext^3_A p^nq+q(Z_p, H^*M) \alpha^*$$

induced by (9.1.2). The left group has unique generator $j_*\alpha_*\alpha_*\tilde{h}_n = j_*\alpha_*\alpha^*(\tilde{h}_n)$, this is because $\Ext^3_A p^nq+2q+1(Z_p, Z_p)$ has unique generator $\tilde{\alpha}_2h_n = j_*\alpha_*\alpha_*i_*(h_n) = \alpha^*j_*\alpha_*\alpha_*\tilde{h}_n$ and $\Ext^3_A p^nq+2q+2(Z_p, Z_p) = 0$ (cf. Theorem 5.4.1), then $\im (j')^* = 0$. The right group has unique generator $j_*\alpha_*\tilde{b}_n-1$ since $\Ext^3_A p^nq+q(Z_p, Z_p)$ has unique generator $h_0b_{n-1} = j_*\alpha_*i_*(b_{n-1}) = j_*\alpha_*i^*(\tilde{b}_{n-1})$ and $\Ext^3_A p^nq+q+1(Z_p, Z_p) = 0$ (cf. Theorem 5.4.1). In addition, $\alpha^*j_*\alpha_*\tilde{b}_{n-1} = j_*\alpha_*\alpha_*\tilde{b}_{n-1} \neq 0 \in \Ext^3_A p^nq+2q+1(Z_p, H^*M)$, this is because $i^*j_*\alpha_*\alpha_*\tilde{b}_{n-1} = \tilde{\alpha}_2b_{n-1} \neq 0$ (cf. Prop. 9.5.2(2)). Then the above $\alpha^*$ is monic, $\im (i')^* = 0$ and so the middle group is zero as desired. Q.E.D.

**Proposition 9.7.7** Let $p \geq 3, n \geq 2$, then

1. $\Ext^2_A p^nq,Z_p, H^*X) = 0$, $\Ext^3_A p^nq+1(Z_p, H^*X) = 0$.
2. $\Ext^3_A p^nq+q(H^*X, H^*K) \cong Z_p\{w_s(h_0b_{n-1})\}$.
3. $\Ext^1_A p^nq+q-1(H^*X, Z_p) \cong Z_p\{\tau_s(h_n)\}$,

where $X$ is the spectrum in the cofibration (9.3.7), $\tau : \Sigma^{q-1}S \to X$ is a map satisfying $ur = i'i : S \to K$ which is obtained by $\alpha^*i'i = 0$ and (9.3.7).

**Proof:** (1) Consider the following exact sequences

$$0 = \Ext^2_A p^nq+q-1(Z_p, H^*K) \xrightarrow{u^*} \Ext^2_A p^nq(Z_p, H^*X)$$

$$\xrightarrow{w^*} \Ext^2_A p^nq(Z_p, H^*K) = 0$$

$$0 = \Ext^3_A p^nq+q(Z_p, H^*K) \xrightarrow{u^*} \Ext^3_A p^nq+1(Z_p, H^*X)$$

$$\xrightarrow{w^*} \Ext^3_A p^nq+1(Z_p, H^*K) = 0$$

induced by (9.3.7). By Prop. 9.7.6(2)(3), Both sides four groups are zero so
that the result follows.

(2) We first claim that \( Ext^s_{\mathbb{A}}(H^*K, H^*K) = 0 \) for \( s = 2, 3 \), then the result follows by the following exact sequence

\[
\alpha^* \xrightarrow{(\alpha'')^*} Ext^3_{\mathbb{A}}(H^*K, H^*M) \xrightarrow{(\beta')^*} Ext^4_{\mathbb{A}}(H^*K, H^*K)
\]

induced by (9.3.7), where the left group has unique generator \((h_0b_{n-1})'\) (cf. Prop. 9.7.5(1)). To prove the above claim, consider the following exact sequence

\[
\alpha^* \xrightarrow{(\alpha'')^*} Ext^3_{\mathbb{A}}(H^*K, H^*K) \xrightarrow{(\beta')^*} Ext^4_{\mathbb{A}}(H^*K, H^*K)
\]

induced by (9.1.2). The right group is zero for \( s = 2, 3 \) (cf. Prop. 9.7.5(1)) and the left group is zero by Prop. 9.3.2(2). This shows the above claim.

(3) Since \( \alpha'' \beta'i = 0 \), then, by (9.3.7), there is \( \tau \in \pi_{q-1}X \) such that \( u\tau = \beta'i \). Consider the following exact sequence

\[
0 = Ext^1_{\mathbb{A}}(H^*K, Z_p) \xrightarrow{w_*} Ext^1_{\mathbb{A}}(H^*K, Z_p)
\]

induced by (9.3.7). The left group is zero since \( Ext^1_{\mathbb{A}}(Z_p, Z_p) = 0 \) for \( t \equiv -1, -2 \mod q \). The right group has unique generator \((i'1)_*(h_n)\) which satisfies \( \alpha''(i'1)_*(h_n) = 0 \), then the middle group has unique generator \( \tau_*(h_n) \) such that \( u_\tau \tau_*(h_n) = (i'1)_*(h_n) \). Q.E.D.

Since \( u_\tau \cdot p = \beta'i \cdot p = 0 \), then by (9.3.7) we have \( \tau \cdot p = \beta'i\alpha_1 \) (uo to nonzero scalar), this is because \( \pi_{q-1}K \cong Z_p\{i'1(\alpha_1)\} \). Then, by \( Ext^2_{\mathbb{A}}(H^*K, Z_p) = 0 \) (cf. Prop. 9.7.5(1)) and the Ext exact sequence induced by (9.3.7) we have

\[
\tau_*(a_0b_{n-1}) = \tau_*(p_+(b_{n-1})) = w_*(i'1)_*(\alpha_1)_*(b_{n-1}) = w_*(i'1)_*(h_0b_{n-1}) \neq 0 \in Ext^3_{\mathbb{A}}(H^*K, Z_p).
\]

Proposition 9.7.9 \hspace{1cm} Let \( p \geq 3, n \geq 2 \), then

1. \( Ext^1_{\mathbb{A}}(H^*K, H^*K) = 0 \), \( Ext^1_{\mathbb{A}}(H^*K, H^*X) = 0 \).
2. \( Ext^1_{\mathbb{A}}(H^*K, H^*X) \cong Z_p\{u_*(h_n)\}' \).
3. \( Ext^2_{\mathbb{A}}(H^*X, Z_p) \cong Z_p\{w_*(i'1)_*(h_0h_n)\} \).
4. \( Ext^2_{\mathbb{A}}(H^*K, H^*K) = 0 \).

Proof: (1) Consider the following exact sequence

\[
\alpha^* \xrightarrow{(\alpha'')^*} Ext^3_{\mathbb{A}}(H^*K, H^*M) \xrightarrow{(\beta')^*} Ext^4_{\mathbb{A}}(H^*K, H^*K)
\]
induced by (9.1.2). The right group is zero since $\operatorname{Ext}_A^{1,p^nq+q-1}(H^*M, H^*M) = 0$, $\operatorname{Ext}_A^{1,p^nq+q-1}(H^*M, H^*M) = 0$ which is obtained by $\operatorname{Ext}_A^{1,t}(Z_p, Z_p) = 0$ for $t = -1, -2$ (mod $q$) and $\operatorname{Ext}_A^{1,p^nq+q+t}(Z_p, Z_p) = 0$ for $t = -1, 0, 1$. The left group also is zero since $\operatorname{Ext}_A^{1,p^nq+q-1}(H^*M, H^*M) = 0$ and $\operatorname{Ext}_A^{1,p^nq+2q}(H^*M, H^*M) = 0$ which is obtained by the same reason as above. Then the middle group is zero.

The second result follows by the following exact sequence

$$0 = \operatorname{Ext}_A^{1,p^nq+q-1}(H^*K, H^*K) \xrightarrow{w^*} \operatorname{Ext}_A^{1,p^nq}(H^*K, H^*X)$$

induced by (9.3.7), where the right group has unique generator $(h_n)'$ which satisfies $(\alpha^n)'(h_n)' = (h_0 h_n)^n \neq 0 \in \operatorname{Ext}_A^{2,p^nq+q-1}(H^*K, H^*K)$ (cf. (9.3.8)).

(2) Consider the following exact sequence

$$\xrightarrow{(\gamma)^*} \operatorname{Ext}_A^{1,p^nq+q}(H^*K, H^*M) \xrightarrow{(\gamma')^*} \operatorname{Ext}_A^{1,p^nq+q+1}(H^*K, H^*K)$$

induced by (9.1.2). The left group is zero since $\operatorname{Ext}_A^{1,p^nq+q+1}(H^*M, H^*M) = 0$ and $\operatorname{Ext}_A^{1,p^nq+2}(H^*M, H^*M) = 0$ which is obtained by $\operatorname{Ext}_A^{1,p^nq+q+t}(Z_p, Z_p) = 0$ for $t = 0, 1, 2$ and $\operatorname{Ext}_A^{1,p^nq+t}(Z_p, Z_p) = 0$ for $t = 1, 2$. The right group also is zero since $\operatorname{Ext}_A^{1,p^nq-2q}(H^*M, H^*M) = 0$ and $\operatorname{Ext}_A^{1,p^nq+1}(H^*M, H^*M) = 0$. Then we have $\operatorname{Ext}_A^{1,p^nq+q+1}(H^*K, H^*K) = 0$.

The desired result can be obtained by the following exact sequence

$$\xrightarrow{(\alpha^n)^*} \operatorname{Ext}_A^{1,p^nq}(H^*K, H^*K) \xrightarrow{w^n} \operatorname{Ext}_A^{1,p^nq+1}(H^*K, H^*X) \xrightarrow{\alpha^n} \operatorname{Ext}_A^{1,p^nq+1}(H^*K, H^*K) = 0$$

induced by (9.3.7), where the left group has unique generator $(h_n)'$ (cf. Prop. 9.3.6).

(3) Consider the following exact sequence

$$\operatorname{Ext}_A^{1,p^nq+1}(H^*K, Z_p) \xrightarrow{\alpha^n} \operatorname{Ext}_A^{2,p^nq+q}(H^*K, Z_p) \xrightarrow{w^n} \operatorname{Ext}_A^{2,p^nq+q}(H^*X, Z_p) \xrightarrow{u^n} \operatorname{Ext}_A^{2,p^nq+1}(H^*K, Z_p) = 0$$

induced by (9.3.7). The right group is zero by Prop. 9.7.5(1) and the left group has unique generator $(i' i)_n(h_0 h_n)$ since $\operatorname{Ext}_A^{2,p^nq+q}(Z_p, Z_p) \cong Z_p[1]$ and $\operatorname{Ext}_A^{2t}(Z_p, Z_p) = 0$ for $t = -1, -2$ (mod $q$). In addition, $\operatorname{Ext}_A^{1,p^nq+1}(H^*K, Z_p) = 0$, this is because $\operatorname{Ext}_A^{1,p^nq+1}(H^*M, Z_p) = 0$ (cf. the proof of Prop. 9.7.5(1)) and $\operatorname{Ext}_A^{1,p^nq+q}(H^*M, Z_p) = 0$ (cf Chapter 5), then $w_n$ is monic so
that the result follows.

(4) Consider the following exact sequence

\[ 0 = \text{Ext}_A^{2p^nq+1}(H^*K, H^*K) \xrightarrow{w} \text{Ext}_A^{2p^nq+1}(H^*X, H^*K) \xrightarrow{w} \text{Ext}_A^{2p^nq+2}(H^*K, H^*K) = 0 \]

induced by (9.3.7). The left group is zero as pointed out in the proof of Prop. 9.7.7(2). The right group also is zero since \( \text{Ext}_A^{2p^nq+3}(H^*K, H^*M) = 0 \) and \( \text{Ext}_A^{2p^nq+2}(H^*K, H^*M) = 0 \) which is obtained by \( \text{Ext}_A^{2p^nq+t} \)

\( (Z_p, Z_p) = 0 \) for \( t = 2, 3 \) and \( \text{Ext}_A^{2p^nq-r+t}(Z_p, Z_p) = 0 \) for \( r = 1, 2, t = 1, 2, 3 \). Then the middle group is zero as desired. Q.E.D.

Now we proceed to prove the main Theorem 9.7.4 in this section. The proof will be done by some argument processing in the Adams resolution (9.2.9). We first prove the following Proposition and Lemmas.

**Proposition 9.7.10** Let \( p \geq 5, n \geq 2, (h_0h_n)'' \in [\Sigma^{p^nq+q-1}K, KG_2 \wedge K] \) be \( d_1 \)-cycle which represents the element \( (h_0h_n)'' = (\alpha')^*(h_n)' \) in \( \text{Ext}_A^{2p^nq+q-1}(H^*K, H^*K) \)(cf. (9.3.8)). Then there exist \( \eta_{n,2}' \in [\Sigma^{p^nq+q-1}K, E_2 \wedge K] \) and \( (\eta_{n,2}')_Y \in [\Sigma^{p^nq+q-1}Y, E_2 \wedge K] \) such that \( (\bar{b}_2 \wedge 1_K)(\eta_{n,2}')_Y \cdot r = (h_0h_n)'' \in [\Sigma^{p^nq+q-1}K, KG_2 \wedge K] \) where \( r : K \rightarrow Y \) is the map in (9.1.4).

**Proof**: Applying Theorem 9.3.9 to \( (\sigma, s, tq) = (h_n, b_n-1), (s, tq) = (1, p^nq) \), we have \( (\bar{c}_2 \wedge 1_K)(h_0h_n)'' = 0 \). Then there exists \( \eta_{n,2}' \in [\Sigma^{p^nq+q-1}K, KG_2 \wedge K] \) such that \( (\bar{b}_2 \wedge 1_K)(\eta_{n,2}' = (h_0h_n)'' \in [\Sigma^{p^nq+q-1}K, KG_2 \wedge K] \). For the second result, note that \( (h_0h_n)'' \in [\Sigma^{p^nq+q-1}S, KG_2 \wedge K] = 0 \), this is because \( \pi_{p^nq+q-t-1}KG_2 \cong \text{Ext}_A^{2p^nq+q-t-1}(Z_p, Z_p) = 0 \) for \( t = 0, 1, u = 1, 2, 3 \). Then there is \( (h_0h_n)''_Y \in [\Sigma^{p^nq+q-1}Y, KG_2 \wedge K] \) such that \( (h_0h_n)'' = (h_0h_n)''_Y \cdot r \), where \( r : K \rightarrow Y \) is the map in (9.1.4). Then by Theorem 9.3.9 we have \( (\bar{c}_2 \wedge 1_K)(h_0h_n)''_Y \cdot r = 0 \) and so, by the cofibration (9.1.4), \( (\bar{c}_2 \wedge 1_K)(h_0h_n)''_Y = f_0' \epsilon = (1_{E_2} \wedge \epsilon \wedge 1_K)(1_Y \wedge f_0') = 0 \), for some \( f_0' \in [\Sigma^{p^nq+q}S, E_3 \wedge K] \), where we use \( \epsilon \wedge 1_K = \mu(\nu'i \wedge 1_K)(\epsilon \wedge 1_K) = 0 \) which is obtained by (9.1.26) and the cofibration (9.1.4). Hence, there is \( (\eta_{n,2}')_Y \in [\Sigma^{p^nq+q-1}Y, E_2 \wedge K] \) such that \( (\bar{b}_2 \wedge 1_K)(\eta_{n,2}')_Y = (h_0h_n)''_Y \in [\Sigma^{p^nq+q-1}Y, KG_2 \wedge K] \). Q.E.D.

**Lemma 9.7.11** Let \( p \geq 5, n \geq 2 \) and \( (\eta_{n}')_Y = (\alpha_0\alpha_1 \wedge 1_K)(\eta_{n,2}')_Y \in [\Sigma^{p^nq+q-3}Y, K] \) be the map obtained in Prop. 9.7.10, then \( w(\eta_{n}')_Y \cdot r = \)

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\[ \lambda' w(\zeta_{n-1} \land 1_{K}) + (a_0 \bar{a}_2 \bar{a}_3 \land 1_{X}) f''_1 \] for some \( f''_1 \in [\Sigma^{p^n+q+1} K, E_4 \land X] \) and nonzero \( \lambda' \in Z_p \), where \( \zeta_{n-1} \in \pi^{p^n+q-3} S \) is the element obtained in Theorem 9.5.1 which is represented by \( h_0 b_{n-1} \in Ext{}_A^{3,p^n+q}(Z_p, Z_p) \) in the ASS.

**Proof**: By Prop. 9.7.10 and (9.3.8)(9.3.7), \( (\bar{b}_2 \land 1_{X})(1_{E_2} \land w)(\eta''_{n,2}) \cdot r = (1_{KG_2} \land w)(h_0 b_{n})'' = (\bar{b}_2 \bar{c}_1 \land 1_{X}) g'' \) with \( g'' \in [\Sigma^{p^n+q-1} K, KG_1 \land X] \), then

\[
(1_{E_2} \land w)(\eta''_{n,2}) \cdot r = (\bar{c}_1 \land 1_{X}) g'' + (\bar{a}_2 \land 1_{X}) f''_0
\]

for some \( f''_0 \in [\Sigma^{p^n+q+1} K, E_3 \land X] \). The \( d_1 \)-cycle \( (\bar{b}_3 \land 1_{X}) f''_0 \in [\Sigma^{p^n+q+1} K, KG_3 \land X] \) represents an element in \( Ext_A^{3,p^n+q}(H^*X, H^*K) \) and this group has unique generator \( w_*[h_0 b_{n-1} \land 1_{K}] \) (cf. Prop. 9.7.7(2)), then

\[
(\bar{b}_3 \land 1_{X}) f''_0 = \lambda'(1_{KG_3} \land w)(h_0 b_{n-1} \land 1_{K}) + (\bar{b}_2 \bar{c}_2 \land 1_{X}) \bar{g}_0
\]

with \( \lambda' \in Z_p \) and \( \bar{g}_0 \in [\Sigma^{p^n+q+1} K, KG_2 \land X] \), where we use \( (\bar{b}_3 \land 1_{K})(\zeta_{n-1,3} \land 1_{K}) = h_0 b_{n-1} \land 1_{K} \) (cf. Theorem 9.5.1). Then \( f''_0 = \lambda'(1_{E_3} \land w)(\zeta_{n-1,3} \land 1_{K}) + (\bar{c}_2 \land 1_{X}) \bar{g}_0 + (\bar{a}_3 \land 1_{X}) f''_1 \) for some \( f''_1 \in [\Sigma^{p^n+q+1} K, E_4 \land X] \) and so we have \( (\bar{a}_2 \land 1_{X}) f''_0 = \lambda'(\bar{a}_2 \land 1_{X})(1_{E_3} \land w)(\zeta_{n-1,3} \land 1_{K}) + (\bar{a}_2 \bar{a}_3 \land 1_{X}) f''_1 \) and (9.7.12) becomes

\[
(1_{E_3} \land w)(\eta''_{n,2}) \cdot r = (\bar{c}_1 \land 1_{X}) g''
\]

with \( g'' \in [\Sigma^{p^n+q-1} K, KG_1 \land X] \), \( f''_1 \in [\Sigma^{p^n+q+1} K, E_4 \land X] \) and \( \lambda' \in Z_p \).

To prove the Lemma, it suffices to prove the scalar \( \lambda' \) in (9.7.13) is nonzero. Suppose in contrast that \( \lambda' = 0 \), then by (9.7.13)(9.1.4) we have

\[
(\bar{a}_2 \bar{a}_3 \land 1_{X}) f''_1 i' i = -(\bar{c}_1 \land 1_{X}) g'' i' i
\]

This will yield a contradiction as shown below.

Note that the \( d_1 \)-cycle \( g'' i' i \in \pi^{p^n+q-1} KG_1 \land X \) represents an element in \( Ext_A^{1,p^n+q-1}(H^*X, Z_p) \cong Z_p\{\tau_*(h_n)\} \) (cf. Prop. 9.7.7(3)). Then \( g'' i' i = \lambda_0(1_{KG_1} \land \tau)(h_n) \), where \( h_n \in \pi^{p^n+q} KG_1 \cong Ext_A^{1,p^n}(Z_p, Z_p) \) and \( \lambda_0 \in Z_p \). Consequently, (9.7.14) becomes

\[
(\bar{a}_2 \bar{a}_3 \land 1_{X}) f''_1 i' i = -\lambda_0(\bar{c}_1 \land 1_{X})(1_{KG_1} \land \tau)(h_n)
\]

The equation (9.7.15) means the secondary differential \(-\lambda_0 d_2(\tau_*(h_n)) = 0 \). However, by [12] p.11 Theorem 1.2.14, \( d_2(h_n) = a_0 b_{n-1} \neq 0 \in Ext_A^{3,p^n+1}(Z_p, Z_p) \), where \( d_2 : Ext_A^{1,p^n}(Z_p, Z_p) \to Ext_A^{3,p^n+1}(Z_p, Z_p) \) is the secondary differential in the ASS. This implies that \( d_2(\tau_*(h_n)) = \tau_*(a_0 b_{n-1}) = w_*(i' i)_*(h_0 b_{n-1}) \neq 0 \in Ext_A^{3,p^n+q}(H^*X, Z_p) \) (cf. (9.7.8)).

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This shows that \( \lambda_0 = 0 \) and so by (9.7.15) we have \((\bar{a}_2\bar{a}_3 \land 1_X)f_1''i = 0\).

It follows that \((\bar{a}_3 \land 1_X)f''_1i\) = \((\bar{c}_2 \land 1_X)g''_2 = 0\), where the \(d_1\)-cycle \(g''_2 \in \pi_{p^q+q}KG_2 \land X\) represents an element in \(\text{Ext}^2_{\mathcal{A}}(p^q+q)(H^*X,\mathbb{Z}_p) \cong Z_p\langle w_s(i^*i)\rangle\) \((h_0h_n)\} \ (\text{cf. Prop. 9.7.9}(3)) and the generator of this group is a permanent cycle in the ASS \(\text{cf. Theorem 9.5.1) so that we have \((\bar{c}_2 \land 1_X)g''_2 = 0\). Then \(f''_1i = (\bar{c}_3 \land 1_X)g''_3 = (\bar{c}_3 \land 1_X)g''_3i\) for some \(g''_3 \in \pi_{p^q+q+1}KG_3 \land X\) and \(g''_3 \in [\Sigma^{p^q+q+1}1,KG_3 \land X]\), this is because \(g''_3 \cdot \epsilon = 0\) which is obtained by the fact that \(\epsilon : Y \to \Sigma S\) induces zero homomorphism in \(Z_p\)-cohomology.

Consequently we have \(f''_1 = (\bar{c}_3 \land 1_X)g''_3 + f''_2r\) with \(f''_2 \in [\Sigma^{p^q+q+1}1,Y, \mathcal{E}_4 \land X]\) and \((\bar{a}_2\bar{a}_3 \land 1_X)f_1'' = (\bar{a}_2\bar{a}_3 \land 1_X)f''_2r\). Hence, if \(\lambda' = 0\), (9.7.13) becomes

\[
(1_{E_2} \land w)(\eta''_{n,2})Y \cdot r = (\bar{a}_2\bar{a}_3 \land 1_X)f''_2r + (\bar{c}_3 \land 1_X)g''_3r
\]

where \(g''_3 \in [\Sigma^{p^q+q+1}Y,KG_2 \land X]\) such that \(g''_3r = g''\). Moreover, by the above equation we have

\[
\text{(9.7.16)} \quad (1_{E_2} \land w)(\eta''_{n,2})Y = (\bar{a}_2\bar{a}_3 \land 1_X)f''_2 + (\bar{c}_1 \land 1_X)g''_3 + f''_3\epsilon
\]

with \(f''_3 \in \pi_{p^q+q}E_2 \land X\). Since \(\epsilon : Y \to \Sigma S\) induces zero homomorphism in \(Z_p\)-cohomology , then the right hand side of (9.7.16) has filtration \( \geq 3 \).

However, \((\eta''_{n,2})Y = (\bar{a}_0\bar{a}_1 \land 1_K)(\eta''_{n,2})Y\) has filtration 2, this is because it is represented by \((h_0h_n)\}_{Y} \in \text{Ext}^2_{\mathcal{A}}(p^q+q+1)(H^*K,H^*Y)\) in the ASS. Moreover, by the following exact sequence

\[
0 = \text{Ext}^1_{\mathcal{A}}(p^q(H^*K,H^*Y) \xrightarrow{\alpha''_n} \text{Ext}^2_{\mathcal{A}}(p^q+q-1)(H^*K,H^*Y) \xrightarrow{\omega} \text{Ext}^2_{\mathcal{A}}(p^q+q-1)(H^*X,H^*Y) \xrightarrow{\alpha''_*})
\]

induced by (9.3.7) we know that \(\omega_n(h_0h_n)_{Y} \neq 0\), where the left group is zero by Prop. 9.7.5(3). That is to say, \((1_{E_2} \land w)(\eta''_{n,2})Y\) has filtration 2 which is represented by \(\omega_n(h_0h_n)_{Y}\) in the ASS. This shows that the equation (9.7.16) is a contradiction so that the scalar \(\lambda'\) must be nonzero. Q.E.D.

**Lemma 9.7.17** Let \(w : K \to X\) be the map in the cofibration (9.3.7) and \(W\) is the cofibre of \(wi : S \to X\) given by the cofibration \(S \xrightarrow{wi} X \xrightarrow{wi} W \xrightarrow{u_1} \Sigma S\), then

\[
\begin{align*}
(1) \quad & \text{Ext}_{\mathcal{A}}^{s-1} (p^q+q+s-3) (H^*W,\mathbb{Z}_p) = 0 \text{ for } s = 1,3 \text{ and has unique generator } (w_1)_* (a)_{S}(h_n) = (\tau)^*(w_1)_*[h_n \land 1_X] \text{ for } s = 2. \\
(2) \quad & \text{Ext}_{\mathcal{A}}^{s} (p^q) (H^*W, H^*X) \cong \mathbb{Z}_p \langle (w_1)_*(h_n)_{X} = (w_1)_*[h_n \land 1_X]\rangle. \\
(3) \quad & (w_1)_*[a_0b_{n-1} \land 1_X] \neq 0 \in \text{Ext}_{\mathcal{A}}^{3}(p^q+1)(H^*W, H^*X).
\end{align*}
\]
Proof : (1) Note that $W$ also is the cofibre of $ra'' : \Sigma^{q-2}K \to Y$, this can be seen by the following homotopy commutative diagram of $3 \times 3$-Lemma.

\[
\begin{array}{ccc}
S & \xrightarrow{w'_{i}} & X \\
\downarrow i' & & \downarrow w \\
\downarrow w_1 & & \downarrow w_2 \\
K & & \Sigma^{q-1}K \\
\downarrow \alpha' & & \downarrow u_2 \\
\Sigma^{q-2}K & \xrightarrow{ra''} & Y \\
\end{array}
\]

That is, we have a cofibration $\Sigma^{q-2}K \xrightarrow{ra''} Y \xrightarrow{u_2} W \xrightarrow{u_2} \Sigma^{q-1}K$ and it induces the following exact sequence

\[
\text{Ext}_A^{s-1,p^n(q+s-3)}(H^*Y, Z_p) \xrightarrow{(w_2)_*} \text{Ext}_A^{s-1,p^n(q+s-3)}(H^*W, Z_p) \xrightarrow{(u_2)_*} \text{Ext}_A^{s-1,p^n(q+s-2)}(H^*K, Z_p) \xrightarrow{(ra'')_*}.
\]

The left group is zero for $s = 1, 2, 3$ (cf. Prop. 9.7.5(2)). The right group also is zero for $s = 1, 3$ (cf. Prop. 9.7.5(1)) and has unique generator $(i')_*(h_n) = u_*\tau_*(h_n) = (u_2)_*(w_1)_*(\tau)_*(h_n)$ for $s = 2$. Then the result follows.

(2) Consider the following exact sequence

\[
\text{Ext}_A^{1,p^n(q)}(H^*X, H^*X) \xrightarrow{(w_1)_*} \text{Ext}_A^{1,p^n(q)}(H^*W, H^*X) \xrightarrow{(u_1)_*} \text{Ext}_A^{2,p^n(q)}(Z_p, H^*X) = 0.
\]

The right group is zero by Prop. 9.7.7(1) and by Prop. 9.7.9(2)(1) we know that the left group has unique generator $(h_n)'_X = [h_n \wedge 1_X]$ which satisfies $u_*(h_n)'_X = u^*(h_n)' \in \text{Ext}_A^{1,p^n(q+1)}(H^*K, H^*X)$. Then the middle group has unique generator $(w_1)_*(h_n)'_X$.

(3) Since $\alpha'' : \Sigma^{q-2}K \to K$ is not an $M$-module map, then, as the cofibre of $\alpha''$, the spectrum $X$ is not an $M$-modul spectrum, that is, the map $p \wedge 1_X \neq 0 \in [X, X]$. So $[a_0 \wedge 1_X] = (p \wedge 1_X)_* [\tau \wedge 1_X] \neq 0 \in \text{Ext}_A^{1,1}(H^*X, X^*X)$ (where $\tau$ is the unit in $\pi_0 KG_0$), and so $[a_0 b_{n-1} \wedge 1_X] = (p \wedge 1_X)_* [b_{n-1} \wedge 1_X] = [a_0 \wedge 1_X][b_{n-1} \wedge 1_X] \neq 0 \in \text{Ext}_A^{3,p^n(q+1)}(H^*X, X^*X)$ which can be obtained by knowledge of Yoneda products and $a_0 b_{n-1} \neq 0 \in \text{Ext}_A^{3,p^n(q+1)}(Z_p, Z_p)$. Note to the following exact sequence

\[
0 = \text{Ext}_A^{3,p^n(q+1)}(Z_p, H^*X) \xrightarrow{(w'_{i})_*} \text{Ext}_A^{3,p^n(q+1)}(H^*X, H^*X)
\]

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where the left group is zero by Prop. 9.7.7(1), then \((w_1)_*\) is monic and so the result follows. Q.E.D.

**Remark** The result on \([a_0b_{n-1} \wedge 1_X] \neq 0\) in Lemma 9.7.17(3) also can be proved by some computation in Ext groups as follows. Suppose in contrast that \((p \wedge 1_X)_*[b_{n-1} \wedge 1_X] = [a_0b_{n-1} \wedge 1_X] = 0\), then by (9.1.1), \([b_{n-1} \wedge 1_X] = (j \wedge 1_X)_*(x_1)\) with \(x_1 \in Ext_A^{2p^q+1}(H^*M \wedge X, H^*X)\). Recall that \(X\) is the spectrum in (9.3.7), then we have \(w^*(1_M \wedge u)_*(x_1) \in Ext_A^{2p^q-q+1}(H^*M \wedge K, H^*K) = 0\) which can be obtained by \(Ext_A^{2p^q+2}(H^*M \wedge K, H^*M) = 0\). By the Ext exact sequence induced by (9.3.7) we have \((1_M \wedge u)_*(x_1) \in w^*Ext_A^{2p^q+1}(H^*M \wedge K, H^*K)\). However, this group has unique generator \((\alpha''h)_{(b_{n-1})}'\), then \((1_M \wedge u)_*(x_1) = \lambda u^*(\alpha''h)_{(b_{n-1})}'\) for some \(\lambda \in Z_p\). By applying \((1_M \wedge \alpha'')_*\) we have \(\lambda u^*(1_M \wedge \alpha'')_{(b_{n-1})}' = 0\) and so \(\lambda(1_M \wedge \alpha'')_{(b_{n-1})}' \in (\alpha'')^*\)ExtA^{2p^q+1}(H^*M \wedge K, H^*K). Then \(\lambda(\alpha_1 \wedge 1_K)_*(b_{n-1})' = \lambda(m_K)_*(1_M \wedge \alpha')_{(b_{n-1})}' = 0\) which shows that \(\lambda = 0\). So \(x_1 = (1_M \wedge w)_*(x_2)\) for some \(x_2 \in Ext_A^{2p^q+1}(H^*M \wedge K, H^*X)\). Similarly we can prove that \(w^*(x_2) = 0\). Then \(x_1 \in u^*(1_M \wedge w)_*Ext_A^{2p^q+q}(H^*M \wedge K, H^*K)\). However, \(Ext_A^{2p^q+q}(H^*M \wedge K, H^*K)\) has two generators \((i \wedge 1_K)_*(h_0h_n)'\), \((\alpha''h)_{(b_{n-1})}'\) and \(u^*(\alpha''h)_{(h_n)''} = 0\), then we have \(x_1 = u^*(1_M \wedge w)_*(i \wedge 1_K)_*(h_0h_n)'\) (up to scalar) and so \([b_{n-1} \wedge 1_X] = (j \wedge 1_X)_*(x_1) = 0\). This is a contradiction and shows that \([a_0b_{n-1} \wedge 1_X] \neq 0 \in Ext_A^{3p^q+1}(H^*X, X^*X)\).

**Proof of Theorem 9.7.4**: The result for \(n = 0, 1\) is well-known, then we assume that \(n \geq 2\). By Lemma 9.7.11 and (9.1.4) we have

\[
\begin{align*}
(9.7.18) \quad \lambda'w'i'\zeta_{n-1} = \lambda'w'(\zeta_{n-1} \wedge 1_K)i'i = -(\bar{a}_0\bar{a}_1\bar{a}_2\bar{a}_3 \wedge 1_W)f''_1'i'.
\end{align*}
\]

Moreover, by the cofibration in Lemma 9.7.17 we have \((a_0\bar{a}_1\bar{a}_2\bar{a}_3 \wedge 1_W)(1_{E_1} \wedge w_1)f''_1'i' = 0\) and so \((a_0\bar{a}_1\bar{a}_2\bar{a}_3 \wedge 1_W)(1_{E_1} \wedge w_1)f''_1'u = k'r'\) for some \(k \in \Sigma^{p^q-2}X, W\), where \(i'i = u\tau : \Sigma^{-1}S \overset{\tau}{\rightarrow} X \overset{u}{\rightarrow} \Sigma^{-1}K\) which is obtained by \(\alpha''i'i = 0\) and (9.3.7) and \(\tau' : X \rightarrow X'\) is the map in the following cofibration

\[
(9.7.19) \quad \Sigma^{-1}S \overset{\tau}{\rightarrow} X \overset{i'}{\rightarrow} X' \overset{\tau''}{\rightarrow} \Sigma S.
\]

We claim that \(k \in \Sigma^{p^q-2}X', W\) has filtration \(\geq 4\), this can be proved
as follows.

By Lemma 9.7.17(1) and (9.7.18) we have $(\tau^\nu)^* \text{Ext}_A^{s-1,p^{n}q+s-3}(H^*W, Z_p) = 0 \subset \text{Ext}_A^{s,p^{n}q+s-2}(H^*W, H^*X')$ and so $(\tau^\nu)^* : \text{Ext}_A^{s,p^{n}q+s-2}(H^*W, H^*X') \to \text{Ext}_A^{s,p^{n}q+s-2}(H^*W, H^*X)$ for $s = 1, 2, 3$ is monic. Then, the fact that $k\tau'$ has filtration $\geq 4$ implies that $k \in [\Sigma p^{n}q-2 X', W]$ also has filtration $\geq 4$. This shows the above claim and so $k = (\bar{a}_0\bar{a}_1\bar{a}_2\bar{a}_3 \wedge 1 W)k_3$ for some $k_3 \in [\Sigma p^{n}q+2 X', E_4 \wedge W]$ and $(\bar{a}_0\bar{a}_1\bar{a}_2\bar{a}_3 \wedge 1 W)(1_{E_4} \wedge w_1)f_{1''}^u = (\bar{a}_0\bar{a}_1\bar{a}_2\bar{a}_3 \wedge 1 W)k_3\tau'$. It follows that

(9.7.20) \((\bar{a}_0\bar{a}_1 \wedge 1 W)(1_{E_4} \wedge w_1)f_{1''}^u = (\bar{a}_0\bar{a}_1 \wedge 1 W)k_3\tau' + (\bar{c}_1 \wedge 1 W)\bar{g}\)

where the $d_1$-cycle $\bar{g} = \lambda_1(1_{KG_1} \wedge w_1)(h_n \wedge 1_X) \in [\Sigma p^{n}q X, KG_1 \wedge W]$ with $\lambda_1 \in Z_p$ which is obtained by Lemma 9.7.17(2).

The equation (9.7.20) means that the differential $d_2(\lambda_1(w_1)_*[h_n \wedge 1 X]) = 0$. However, $d_2((w_1)_*[h_n \wedge 1 X]) = (w_1)_*[a_0b_{n-1} \wedge 1 X] \neq 0$ (cf. Lemma 9.7.17(3)). Then the scalar $\lambda_1 = 0$ and we have $\bar{g} = 0 = (\bar{a}_0\bar{a}_3 \wedge 1 W)(1_{E_4} \wedge w_1)f_{1''}^u = (\bar{a}_0\bar{a}_3 \wedge 1 W)k_3\tau'$ and $(\bar{a}_2\bar{a}_3 \wedge 1 K)(1_{E_4} \wedge u)f_{1''}^i\bar{i} = (\bar{a}_2\bar{a}_3 \wedge 1 K)(1_{E_4} \wedge u_2w_1)f_{1''}^u\bar{w} = 0$. Consequently we have $(\bar{a}_3 \wedge 1 K)(1_{E_4} \wedge u)f_{1''}^i\bar{i} = (\bar{c}_2 \wedge 1 K)\bar{g}_2 = 0$, this is because the $d_1$-cycle $\bar{g}_2 \in \pi_{p^{n}q+1}KG_2 \wedge K$ represents an element in $\text{Ext}_A^{2,p^{n}+1}(H^*K, Z_p) = 0$ (cf. 9.7.5(1)). Then, $(1_{E_4} \wedge u)f_{1''}^i\bar{i} = (\bar{c}_3 \wedge 1 K)\bar{g}_3$ for some $\bar{g}_3 \in [\Sigma p^{n}q+2 S, KG_3 \wedge K]$. Since $(1_{KG_3} \wedge \alpha''')\bar{g}_3 = 0$, then $\bar{g}_3 = (1_{KG_3} \wedge u)\bar{g}_4$ for some $\bar{g}_4 \in [\Sigma p^{n}q+q+1 S, KG_3 \wedge X]$ and so we have

(9.7.18) \(\lambda' w_1\bar{i}^1\zeta_{n-1} = -(\bar{a}_0\bar{a}_1\bar{a}_2\bar{a}_3 \wedge 1 X)f_{1''}^i\bar{i} = -(\bar{a}_0\bar{a}_1\bar{a}_2\bar{a}_3 \wedge 1 X)(1_{E_4} \wedge w)\bar{f}_2\)

and by (9.3.7), $\lambda' w_1\bar{i}^1\zeta_{n-1} = -(\bar{a}_0\bar{a}_1\bar{a}_2\bar{a}_3 \wedge 1 K)\bar{f}_2 + \alpha''w_n$ with $w_n \in \pi_{p^{n}q-1}K$. Since $\lambda' w_1\bar{i}^1\zeta_{n-1}$ is a map of filtration 3 which is represented by $\lambda'(\bar{i}^1)_{*(h_0b_{n-1} \wedge 1)} \in \text{Ext}_A^{2,p^{n}q+q}(H^*K, Z_p)$ in the ASS, then $\alpha'''w_n$ has filtration 3 and so $w_n \in \pi_{p^{n}q-1}K$ has filtration $\leq 2$. However, by Prop. 9.7.5(1) we have $\text{Ext}_A^{2,p^{n}q+1}(H^*K, Z_p) = 0$, then $w_n \in \pi_{p^{n}q-1}K$ must be represented by the unique generator $(\bar{i}^1)_{*(h_n)} \in \text{Ext}_A^{1,p^{n}q}(H^*K, Z_p)$ (up to nonzero scalar). This shows the Theorem. Q.E.D.

**Remark** The element $w_n \in \pi_{p^{n}q-1}K$ obtained in Theorem 9.7.4 can be extended to $(w_n)' \in [\Sigma p^{n}q-1 K, K]$ such that $(w_n)' = w_n$. Then, $(w_n)'$
is represented by \((h_n)′ \in \text{Ext}^1_{A_B}(H^*K, H^*K)\) in the ASS and \(\alpha''(\omega_n)'\), \((\omega_n)'\alpha'' \in [\Sigma_p^{n+2q-3}K, K]\) is represented by \(\alpha''(h_n)' = (\alpha'')^*(h_n)' = (h_0h_n)'' \in \text{Ext}^2_{A_B}(H^*K, H^*K)\). By Theorem 9.7.4 and Lemma 9.7.11 we have \(\alpha''(\omega_n)' = (\omega_n)'\alpha'' + \lambda'\zeta_{n-1} \wedge 1_K\) (modulo higher filtration).

By [10] p.511, there is a map \(\phi_\ast: BP_*BP \to A_*\) such that \(t_n \mapsto \) the conjugate of \(\xi_n\), where \(A_* = E[\tau_0, \tau_1, \tau_2, \ldots] \otimes P[\xi_1, \xi_2, \ldots]\) is the dual of the Steenrod algebra \(A\). Then \(\phi_\ast\phi\) induces the Thom map \(\Phi: \text{Ext}^1_{BP_*BP}(BP_*, BP_K) \to \text{Ext}^1_{H^*K, Z}^1(H^*K, Z_p)\) such that the image of \(h_n \in \text{Ext}^1_{BP_*BP}(BP_*, BP_K)\) is \(\Phi(h_n) = (\iota')_\ast(h_n) \in \text{Ext}^1_{H^*K, Z}^1(H^*K, Z_p)\). Then, the element \(\omega_n \in \pi_p^q-1K\) obtained in Theorem 9.7.4 is represented by \(h_n + \) other terms in \(\text{Ext}^1_{BP_*BP}(BP_*, BP_K)\) in the Adams-Novikov spectral sequence. To know what the elements in the other terms, we first prove the following Lemma.

**Lemma 9.7.21** By degree reason, \(\text{Ext}^1_{BP_*BP}(BP_*, BP_K)\) is generated (additively) by the following \(v_2\)-torsion elements \(c_2(p^{n-2})\) and \(v_2\)-torsion free elements \(h_n, v_2^{p^{n-2}(p-1)}h_{n-2}, v_2ap^i h_i\), where \(i \geq 0, a_i = (p^{2k} - 1)/(p + 1)\), \(n - i = 2k \geq 4\). In addition, there is a relation \(h_n = c_2(p^{n-2}) + v_2^{p^{n-2}(p-1)}h_{n-2} \in \text{Ext}^1_{BP_*BP}(BP_*, BP_K)\).

**Proof:** By [19] Theorem 1.1 and 1.5, \(\text{Ext}^1_{BP_*(BP_*)}(BP_*, BP_K)\) is a \(Z_p[v_2]\)-module which is generated by \(v_2\)-torsion elements \(c_2(ap^s)\) and \(v_2\)-torsion free elements \(w_2, h_i\), where \(a \neq 0\) (mod \(p\)), \(s \geq 0\) and \(i \geq 0\). Moreover, the internal degree \(|h_i| = p^ih_i, |c_2(ap^s)| = ap^s(p^2 + p + 1)q - q(ap^s)(p + 1)q\) and \(|w_2| = (p^2 + 1)^2q\).

Since \(v_2^bw_2 \equiv 0 \pmod{(p + 1)q}\), then \(|v_2^bw_2| = p^nq\). If \(v_2^bh_i \equiv p^nq\), then \(b(p + 1) + q = p^nq, b(p + 1) = p^i(p^{n-i} - 1)\) and so \((p^{n-i} - 1)\) must be divisible by \(p + 1\). Hence \(b = 0\) and \(i = n\) or \(b = a_ip^i\) with \(a_i = (p^{2k} - 1)/(p + 1)\) and \(n - i = 2k \geq 2\). Then \(h_n, v_2^{p^{n-2}(p-1)}h_{n-2}\) and \(v_2^ap^ih_i(0 \leq i < n - 2)\) are the only torsion free elements of \(\text{Ext}^1_{BP_*(BP_*)}(BP_*, BP_K)\).

If \(|v_2^bc_2(ap^s)| = p^nq\), then \(p^nq = ap^s(p^2 + p + 1)q - (q(ap^s) - b)(p + 1)q\), \(ap^s(p^2 + p + 1) = p^n + (q(ap^s) - b)(p + 1)\) and so the right hand side must be divisible by \(p^2 + p + 1\). So we have

\[
(9.7.22) \quad ap^s = p^{n-2} + \frac{(q(ap^s) - b - p^{n-2})(p + 1)}{p^2 + p + 1}.
\]

We claim that \(s \leq n - 2\) which will be proved below, then \(q(ap^s) - b\) must be divisible by \(p^s\). However, by [19] p.132, \(q(ap^s) = p^s\) for \(a = 1\)
and \(q(ap^s) = p^s + \text{other terms for } a \geq 2\). Then, the only possibility is \(q(ap^s) = p^s, a = 1, b = 0\) and \(s = n - 2\). That is to say, the only \(v_2\)-torsion elements in \(\text{Ext}^{1,p^n q}(BP_s, BP_s K)\) is \(c_2(p^{n-2})\).

Now we prove the above claim. Suppose in contrast that \(s \geq n - 1\), then, by (9.7.22) we have \(\frac{(q(ap^s) - b - p^{n-2})(p+1)}{p^2 + p+1} = ap^s - p^{n-2} \geq p^s - p^{n-2}\),

\[
2p^s > q(ap^s) - b - p^{n-2} \geq \frac{(p^s - p^{n-2})(p^2 + p+1)}{p+1} > p^{s+1} - p^{n-1}
\]

and this is a contradiction which shows the above claim. Q.E.D.

**Proof of Theorem 9.7.1**  For the Thom \(\Phi : \text{Ext}_{BP_s BP}^{1,p^n q}(BP_s, BP_s K) \to \text{Ext}_{BP_s}^{1,p^n q}(H_s K, Z_p)\) we have \(\Phi(h_n) = (i'\xi)_s(h_n)\). By this we know that the element \(\omega_n \in \pi_{p^n q - 1} K\) obtained in Theorem 9.7.4 is represented by \(h_n + (\text{other terms}) \in \text{Ext}_{BP_s BP}^{1,p^n q}(BP_s, BP_s K)\) in the Adams-Novikov spectral sequence. By Lemma 9.7.21, the other terms are the linear combination of \(v_2^{n-2(p-1)}h_{n-2}\) and \(v_2^{a_p} h_i\), where \(i \geq 0, n - i = 2k \geq 4\) and \(a_i = (p^k - 1)/(p+1)\). Let \(\beta \in [\Sigma^{(p+1)}q K, K]\) be the known \(v_2\)-map, then \(i'\alpha_1 \in \pi_{q-1} K\) and \(i'j'\beta i' \in \pi_{pq-1} K\) is represented by \(h_0, h_1 \in \text{Ext}_{BP_s BP}^{1,p^n q}(BP_s, BP_s K)\) respectively. That is, \(h_0, h_1 \in \text{Ext}_{BP_s BP}^{1,p^n q}(BP_s, BP_s K)\) are permanent cycles in the Adams-Novikov spectral sequence. Suppose inductively that \(h_i \in \text{Ext}_{BP_s BP}^{1,p^n q}(BP_s, BP_s K)\) for \(i \leq n - 1(n \geq 2)\) are permanent cycles in the Adams-Novikov spectral sequence. Since \(\omega_n \in \pi_{p^n q - 1} K, \omega_{n+1} \in \pi_{p^n q + 1 - 1} K\) are represented by the linear combination of

\[
h_{n+1} + v_2^{a_p} h_{n-1}\]

\[
h_{n+1} + v_2^{a_p} h_{n-1}\]

then \(h_{n+1} \in \text{Ext}_{BP_s BP}^{1,p^n q}(BP_s, BP_s K)\) also are permanent cycles. This completes the induction and the result of the Theorem follows. Q.E.D.

**Conjecture 9.7.22**  Theorem 9.7.4 can be generalized to be the following general result. Let \(p \geq 5, s \leq 4\), \(\text{Ext}_{A}^{s,tq}(Z_p, Z_p) \cong Z_p\{x\}, \text{Ext}_{A}^{s+1,tq+q}(Z_p, Z_p) \cong Z_p\{h_0 x\}\) and some supposition on vanishes of some Ext groups. If the secondary differential \(d_2(x) = a_0 x' \in \text{Ext}_{A}^{s+2,tq+1}(Z_p, Z_p)\) with \(x' \in \text{Ext}_{A}^{s+1,tq}(Z_p, Z_p)\), that is, \(x\) and \(x'\) is a pair of \(a_0\)-related elements, then there exists \(\omega \in \pi_{tq - s} K\) such that \(i'\xi = a'' \cdot \omega \) (mod \(F^{s+2} P_s K\)) and \(\omega \in \pi_{tq - s} K\) is represented by \((i'\xi)_s(x) \in \text{Ext}_{A}^{s,tq}(H_s K, Z_p)\)
in the ASS, where $\xi \in \pi_{tq+q-s-2}S$ is the homotopy element which is repre-
sented by $h_0 x' \in \text{Ext}^{s+2, tq+q}_{A_{\text{ASS}}} (\mathbb{Z}_p, \mathbb{Z}_p)$ in the ASS and $F^{s+2} \pi_* K$ denotes the
group consisting of all elements in $\pi_* K$ filtration $\geq s+2$.

§8. Second periodicity families in the stable homotopy groups of
spheres

By Theorem 8.1.2 in chapter 8, $\text{Ext}_{\text{BP}, \text{BP}}^1 (\text{BP}, \text{BP})$ is generated by
$\alpha_{t p^n/n+1} (n \geq 0, p$ not divisible by $t \geq 1$) and It was proved by Novikov that
all these first periodicity families converge to the im $J \subset \pi_* S$. In this sec-
tion, using the $h_0 h_{n+1}$-element obtained in Theorem 9.5.1 and the elements
$\beta_{p/r}, 1 \leq r \leq p-1$ and $\beta_{tp/r}, t \geq 2, 1 \leq r \leq p$ as our geometric input,
we prove the following Theorem on the convergence of second periodicity
families $\beta_{tp^n/r}$ in the Adams-Novikov spectral sequence.

**Theorem 9.8.1** Let $p \geq 5, n \geq 1, 1 \leq s \leq p^n-1$ if $t \geq 1$ is not
divisible by $p$ or $1 \leq s \leq p^n$ if $t \geq 2$ is not divisible by $p$ , then The elements
$\beta_{tp^n/s} \in \text{Ext}_{\text{BP}, \text{BP}}^2 (\text{BP}, \text{BP})$ in Theorem 8.1.3 are permanent cycles in the Adams-Novikov spectral se-
quence and they converge to the corresponding homotopy elements of order
$p$ in $\pi_{tp^n(p+1)q-sq-2} S$.

We will prove Theorem 9.8.1 in case $t \geq 1$ or $t \geq 2$ separately. The proof
will be done by some arguments processing in the cannical Adams-Novikov
resolution. We first do some preminilaries as follows.

Let $M$ be the Moore spectrum whose $\text{BP}$-homology are $\text{BP}^*/(p)$. Let $\alpha : \Sigma^q M \to M$ be the Adams map which induces $\text{BP}$-
homomorphisms are $v_1 : \text{BP}^*/(p) \to \text{BP}^*/(p)$. Let $K_\tau$ be the cofibre of
$\alpha^\tau : \Sigma^q M \to M$ given by the cofibration

$$
\Sigma^q M \xrightarrow{\alpha^\tau} M \xrightarrow{i^\tau} K_\tau \xrightarrow{j^\tau} \Sigma^{q+1} M
$$

(9.8.2)

The cofibration (9.8.2) induces a short exact sequence of $\text{BP}$-homology

$$
0 \to \text{BP}^*/(p) \xrightarrow{\alpha^\tau} \text{BP}^*/(p) \to \text{BP}^*/(p) \to 0
$$

Recall from §5 in chapter 6, $K_\tau$ is a $M$-module spectrum and we have the following derivations

$$
(9.8.3) \quad d(i^\tau) = 0, \quad d(j^\tau) = 0, \quad d(\alpha) = 0, \quad d(ij) = -1_M.
$$
Moreover, the cofibre of $i'_s j'_r : K_r \to \Sigma^{rq+1} K_s$ is $\Sigma K_{r+s}$ given by the cofibration

$$\Sigma^{rq} K_s \xrightarrow{\psi_{s+r}} K_{s+r} \xrightarrow{\nu_{s+r}} K_r \xrightarrow{i'_s j'_r} \Sigma^{rq+1} K_s \tag{9.8.4}$$

This can be seen by the following homotopy commutative diagram of $3 \times 3$-Lemma

$$\begin{array}{ccc}
\Sigma^{rq} K_s & \xrightarrow{i'_s j'_r} & \Sigma^{rq+1} K_s \\
\downarrow j'_r & \swarrow & \downarrow j'_s \\
\Sigma^{rq+1} M & \xrightarrow{\alpha^s} & \Sigma K_{r+s} \\
\downarrow \alpha^s & \swarrow & \downarrow \rho_{r+s, r} \\
\Sigma^{(r+s)q+2} M & \xrightarrow{i'_r} & \Sigma K_r 
\end{array} \tag{9.8.5}$$

Moreover, the cofibration (9.8.4) induces a short exact sequence of $BP_s$-homology

$$0 \to BP_*/(p, v_i^s) \xrightarrow{v_i^r} BP_*/(p, v_i^{s+r}) \to BP_*/(p, v_r^i) \to 0$$

and by the homotopy commutative diagram (9.8.5), we have the following relations

$$\begin{align*}
\psi_{s, s+r} i'_s &= \alpha^s j'_s, \\
j'_r \rho_{s+r, r} &= \alpha^s j'_s, \\
p_{s+r, s} i'_s &= \alpha^s j'_s, \\
p_{s+r, r} i'_s &= i'_r. \tag{9.8.6}
\end{align*}$$

**Proposition 9.8.7** Let $p \geq 5$ and $f \in [\Sigma^f K_r, S]$ be any map, then $f = j j'_r \tilde{f}$ for some $\tilde{f} \in [\Sigma^{r+q+2} K_r, K_r]$.

**Proof**: By Theorem 6.5.16(A) in chapter 6, there is $\nu_r : \Sigma^{rq+2} K_r \to K_r \wedge K_r$ such that $(j j'_r \wedge 1_{K_r}) \nu_r = 1_{K_r}$. Let $K'_r$ be the cofibre of $j j'_r : \Sigma^{-1} K_r \to \Sigma^{rq+1} S$ given by the cofibration $\Sigma^{-1} K_r \xrightarrow{jj'_r} \Sigma^{rq+1} S \xrightarrow{\nu_r} K'_r \to K_r$, then $z_r \wedge 1_{K_r} = (z_r j j'_r \wedge 1_{K_r}) \nu_r = 0 \in [\Sigma^{rq+1} K_r, K'_r \wedge K_r]$. Consequently, $z_r f = (1_{K_r} \wedge f) (z_r \wedge 1_{K_r}) = 0$ and so $f = j j'_r \tilde{f}$ for some $\tilde{f} \in [\Sigma^{r+q+1} K_r, K_r]$.

Q.E.D.

Let

$$\begin{align*}
\ldots & \xrightarrow{\tilde{a}_2} \Sigma^{-2} \tilde{E}_2 & \xrightarrow{\tilde{a}_1} \Sigma^{-1} \tilde{E}_1 & \xrightarrow{\tilde{a}_0} \tilde{E}_0 = S \\
\Sigma^{-2} BP \wedge \tilde{E}_2 & \xrightarrow{\tilde{b}_2} \Sigma^{-1} BP \wedge \tilde{E}_1 & BP \wedge \tilde{E}_0 = BP
\end{align*} \tag{9.8.8}$$

be the canonical Adams-Novikov resolution of the sphere spectrum $S$, where $\tilde{E}_s \xrightarrow{\tilde{b}_s} BP \wedge \tilde{E}_s \xrightarrow{\tilde{c}_s} \tilde{E}_{s+1} \xrightarrow{\tilde{a}_s} \Sigma \tilde{E}_s$ are cofibrations for all $s \geq 0$.

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such that $\tilde{E}_0 = S, \tilde{b}_s = \tau \wedge 1_{\tilde{E}_s}$ ($s > 0$) and $\tilde{b}_0 = \tau : S \to BP$ is the injection of the bottom cell. Then $\pi_i BP \wedge \tilde{E}_s$ is the $E^s_t$-term of the Adams-Novikov spectral sequence, $(\tilde{b}_{s+1} \varepsilon_c)_* : \pi_i BP \wedge \tilde{E}_s \to \pi_i BP \wedge \tilde{E}_{s+1}$ are the $d^s_t$-differential and

$$E^s_2 = \text{Ext}^{s\times t}_{BP \wedge BP}(BP_s, BP_t) \Rightarrow (\pi_{t-s} S)_p$$

**Proposition 9.8.9** Let $p \geq 3, r \geq 1, s \geq 0$ and $\tilde{E}_s$ be the spectrum in the Adams-Novikov resolution (9.8.8), $\wedge^s BP = BP \wedge \cdots \wedge BP$ be the smash products of $s$ copies of $BP$, then $(BP \wedge \tilde{E}_s)^*, (BP \wedge \tilde{E}_s)^*(M), (BP \wedge \tilde{E}_s)^*(K_r)$ are the direct summand of $(\wedge^{s+1} BP)^*, (\wedge^{s+1} BP)^*(M), (\wedge^{s+1} BP)^*(K_r)$ and we have $[\Sigma^t M, BP \wedge \tilde{E}_s] = (BP \wedge \tilde{E}_s)^{-t}(M) = 0$ for $t \neq -1$ (mod q), $[\Sigma^t K_r, BP \wedge \tilde{E}_s] = (BP \wedge \tilde{E}_s)^{-t}(K_r) = 0$ for $t \neq -2$ (mod q).

**Proof:** We first consider the $BP^*$-cohomology. It is known that $\pi_i BP = BP^t_i = BP^{t-t}$, then $BP^* = Z(p)[v_1, v_2, \cdots ]$, where $|v_i| = -2(p^i - 1)$ and $I_n = (p, v_1, \cdots , v_{n-1}, (p, v_1^t)$ is the invariant ideal of $BP^*$. Clearly, there are two exact sequences on $BP^*$-cohomology as follows

$$0 \to BP^* \xrightarrow{p} BP^* \xrightarrow{\rho_0} BP^*/(p) \to 0$$

where $\rho_0, \rho_1$ are the projections.

Note that $(\wedge^s BP)^* = \pi_*(\wedge^s BP) = BP_*(\wedge^{s-1} BP) = BP_s BP \otimes \cdots \otimes BP_s BP$ with $s - 1$ copies of $BP_s BP$ and $s \geq 2$. Then we have the following short exact sequences ($s \geq 1$)

$$0 \to (\wedge^s BP)^* \xrightarrow{p} (\wedge^s BP)^*/(p) \to 0$$

For any $f \in [\Sigma^t M, \wedge^s BP] = (\wedge^s BP)^{-t}(M)$, if $t \neq 0$ (mod q), then by the sparseness of $(\wedge^s BP)^* = BP_s BP \otimes \cdots \otimes BP_s BP$ (that is, $BP_s BP = 0$ for $r \neq 0$ (mod q)) we have $f_i \in (\wedge^s BP)^{-t} = 0$; if $t = 0$ (mod q), then $f_i$ is an element of order $p$ in $Z(p)$-module $(\wedge^s BP)^{-t}$ so that we have $f_i = 0$. This shows that $i^* = 0 : (\wedge^s BP)^*(M) \to (\wedge^s BP)^*$. Similarly we have $(i^t_r)^* = 0 : (\wedge^s BP)^*(K_r) \to (\wedge^s BP)^*(M)$ ($r \geq 1$). Then, the cofibration (9.1.1) (9.8.2) induces respectively the following short exact sequences for
all \( r \geq 1 \)

\[
0 \to (\wedge^s BP)^* \xrightarrow{p} (\wedge^s BP)^* \xrightarrow{j^*} (\wedge^s BP)^*(M) \to 0
\]

\[
0 \to (\wedge^s BP)^*(M) \xrightarrow{(a^r)^*} (\wedge^s BP)^*(M) \xrightarrow{(j^r)^*} (\wedge^s BP)^*(K_r) \to 0
\]

where the degrees \(| j^* | = -1, | (j^r)^* | = -(rq + 1) \). By comparison to the above two short exact sequences with (9.8.10) we have

\[\text{(9.8.11)}\]

\[
(\wedge^s BP)^*(M) \cong \Sigma(\wedge^s BP)^*/(p),
\]

\[
(\wedge^s BP)^*(K_r) \cong \Sigma^q(\wedge^s BP)^*/(p, v_1).
\]

Let \( \mu : BP \wedge BP \to BP \) be the multiplication of the ring spectrum \( BP \) and \( \tau : S \to BP \) be the injection of the bottom cell, then we have

\[\mu(1_{BP} \wedge \tau) = 1_{BP} = \mu(\tau \wedge 1_{BP})\]

so that the cofibration \( \tilde{E}_{s-1} \xrightarrow{b_{s-1}} BP \wedge \tilde{E}_{s-1} \xrightarrow{\tilde{E}_s \rightarrow 1_{BP} \wedge \tilde{E}_{s-1}} \) induces a split short exact sequence

\[BP \wedge \tilde{E}_{s-1} \xrightarrow{1_{BP} \wedge b_{s-1}} BP \wedge \tilde{E}_{s-1} \xrightarrow{1_{BP} \wedge \tilde{E}_{s-1}} BP \wedge \tilde{E}_{s-1}
\]

this is because \((\mu \wedge 1_{\tilde{E}_{s-1}})(1_{BP} \wedge \tilde{b}_{s-1}) = (\mu \wedge 1_{\tilde{E}_{s-1}})(1_{BP} \wedge \tau \wedge 1_{\tilde{E}_{s-1}}) = 1_{BP \wedge \tilde{E}_{s-1}}\). That is to say, \( BP \wedge \tilde{E}_{s-1} \) is the direct summand of \( BP \wedge BP \wedge \tilde{E}_{s-1} \) and by induction we have \( BP \wedge \tilde{E}_s \) is the direct summand of \( \wedge^{s+1}BP \). Hence,

\[
(BP \wedge \tilde{E}_s)^*, (BP \wedge \tilde{E}_s)^*(M), (BP \wedge \tilde{E}_s)^*(K_r)
\]

are the direct summands of \( (\wedge^{s+1}BP)^*, (\wedge^{s+1}BP)^*/(p), (\wedge^{s+1}BP)^*/(p, v_1) \) respectively and the last result can be obtained by (9.8.11). Q.E.D.

**Proposition 9.8.12**

Let \( p \geq 3, n \geq 1 \), then

\[\text{Ext}^0_{BP, BP}BP_{*(p+1)q}(BP_{*}, BP_*/(p, v_1^{p^n-1}))\]

is generated additively by the generators \( v_2^n, v_1^{p^n-p^n-2r} c_1(t_r p^{n-2r}) ( r \geq 1 ) \), where \( t_r = (p^{2r+1}+1)/(p+1) \) and \( c_1(a^p) \) is the generator in Theorem 8.1.7 in chapter 8 which has degree \( sp^a(p+1)q \).

**Proof**: By Theorem 8.1.7, the desired generators are of the form \( v_2^n c_1(a^p) \) with degrees \( bq + ap^a(p+1)q = p^n(p+1)q, a \geq 1 \) is not divisible by \( p, 0 \leq b < p^n-1 \) and \( b \geq \max \{0, p^n-1-q_1(a^p)\} \), where \( q_1(a^p) = p^s \)

if \( a = 1, q_1(a^p) = p^s + p^{s-1} - 1 \) if \( a \geq 2 \) is not divisible by \( p \).

If \( b = 0 \), then the generator is \( v_2^n \). Since \( b < p^n-1 \), then \( s < n \) and \( b \equiv 0 \pmod{p^s} \) and so \( b \geq p^n-1-q_1(a^p) \geq p^n-p^{n-1} \) if \( b \geq 1 \). Let \( b = (p-1)p^{n-1}+c_{n-2}p^{n-2}+\cdots+c_sp^s \) be the p-adic expansion of \( b \) such that
0 \leq c_i \leq p-1. By b \geq (p^n-1) - (p^s + p^{s-1} - 1) or b \geq (p^n-1) - p^s we have
c_{n-2}p^{n-2} + \cdots + c_bp^s \geq p^{n-1} - p^s - p^{s-1} or p^{n-1} - p^s - 1. Consequently we have
c_{n-2} = \cdots = c_s = p-1. On the other hand, b is divisible by p + 1, then (p-1) - c_{n-2} + \cdots + (-1)^{n-1-s}c_s = 0 so that n-1-s must be odd.
Let n-1-s = 2r-1, then we have s = n - 2r, b = p^n - p^{n-2r} as desired and a = (p^{2r+1} + 1)/(p + 1). Q.E.D.

**Proposition 9.8.13** Let p \geq 3, n \geq 1, then

(1) $\text{Ext}^2BP,BP_{BP}(BP_*, BP_*)$ is generated additively by the generators

$\beta_{p^n/p^{n-1}}, \beta_{t_n/p^{n-2r}/p^{n-2r-1}}$ for all $r \geq 1$, where $t_r = (p^{2r+1} + 1)/(p + 1)$.

(2) $\text{Ext}^1BP,BP_{BP}(BP_*, BP_*)$ is generated additively by the generators

$\beta'_p/p^{n-1}, \beta'_t/p^{n-2r}/p^{n-2r-1}$ for all $r \geq 1$, where $t_r = (p^{2r+1} + 1)/(p + 1)$ and

$\beta'_p/s$ is the generator in $\text{Ext}^1BP,BP_{BP}(BP_*, BP_*)$ such that $j_*(\beta'_p/s) = \beta_{tp^n/s}$ in $\text{Ext}^2BP,BP_{BP}(BP_*, BP_*)$.

**Proof**: By Theorem 8.1.3 in chapter 8, $\text{Ext}^{2p}BP,BP_{BP}(BP_*, BP_*)$ is generated additively by the generators $\beta_{ap^{p,c+1}} \in \text{Ext}^{2ap^{p,c+1}}BP,BP_{BP}(BP_*, BP_*)$,

where $s \geq 0, a \geq 1$ is not divisible by $p$, $b \geq 1, c \geq 0$ and subject to

(i) $b \leq s$ if $a = 1$.

(ii) $p^c | b \leq p^{s-c} + p^{s-c-1} - 1$

(iii) $p^{s-c-1} + p^{s-c-2} - 1 < b$ if $p^{c+1} | b$,

and $\beta_{ap^{s,b}} = \beta_{ap^{s,b}}$. Then, for $\beta_{ap^{s,b,c+1}} \in \text{Ext}^{2ap^{s,b,c+1}}BP,BP_{BP}(BP_*, BP_*)$ we have $ap^{s}(p+1)q-bq = p^{n+1}q + q = p^n(p+1)q - (p^n-1)q$ so that $ap^{s}(p+1)q + (p^n-1-b)q = p^n(p+1)q$. Similar to that in the proof of Prop. 9.8.12 we have $a = 1, s = n, b = p^n - 1$ or $a = (p^{2r+1} + 1)/(p + 1), b = p^{n-2r} - 1, s = p^{n-2r}$ ($r \geq 1$) and consequently $c = 0$. This shows (1) and the proof of (2) is similar. Q.E.D.

After finishing the proof of the above Proposition, we proceed to prove Theorem 9.8.1 in case $t \geq 1$. The proof will be done by some argument processing in the Adams-Novikov resolution of some spectra and using the $h_0h_{n+1}$-map in Theorem 9.5.1 as our geometric input. We first prove the following Lemma.

**Lemma 9.8.14** If $g''$ is the element in $\pi_{p^n(p+1)q}BP$ such that $\tilde{b}_1\tilde{c}_0g''j$

$\tilde{j}_{p^{n-1}q} = 0 \in \Sigma^{p^{n+1}q+q-2}K_{p^{n-1}}, BP \wedge E_1$], then there exists $\bar{g} = px_1 + v_{p^{n-1}q}x_2 \in \pi_{p^n(p+1)q}BP$ such that $\tilde{b}_1\tilde{c}_0(g'' - \bar{g}) = 0 \in \pi_{p^n(p+1)q}BP \wedge \tilde{E}_1$, where

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where \( x_1, x_2 \) is some elements in \( \pi_*BP \).

**Proof**: Let \( \mu : BP \wedge BP \to BP \) be the multiplication of the ring spectrum \( BP \), then \( \mu(\tilde{b}_0 \wedge 1_{BP}) = 1_{BP} = \mu(1_{BP} \wedge \tilde{b}_0) \), where \( \tilde{b}_0 = \tau : S \to BP \) is the injection of the bottom cell as stated above. Then we have the following split cofibration

\[
BP \xrightarrow{1_{BP} \wedge \tilde{b}_0} BP \wedge BP \xrightarrow{1_{BP} \wedge \tilde{c}_0} BP \wedge \tilde{E}_1 \xrightarrow{1_{BP} \wedge \tilde{a}_0} \Sigma BP
\]

and there is \( \mu' : BP \wedge \tilde{E}_1 \to BP \wedge BP \) such that \((1_{BP} \wedge \tilde{b}_0)\mu + \mu'(1_{BP} \wedge \tilde{c}_0) = 1_{BP} \wedge \tilde{b}_0BP).

By \( \check{b}_1 \check{c}_0 g'' j_{j_p^n - 1} = 0 \) we have \( \check{c}_0 g'' j_{j_p^n - 1} = \check{a}_1 g' \) with \( g' \in [\Sigma^{p+1} q + q - 1 K_{p^n}, \tilde{E}_2] \). Then \((1_{BP} \wedge \check{c}_0 g'' j_{j_p^n - 1}) = (1_{BP} \wedge \check{a}_1)(1_{BP} \wedge g') = 0 \) so that \(1_{BP} \wedge g'' j_{j_p^n - 1} = [(1_{BP} \wedge \check{b}_0)\mu + \mu'(1_{BP} \wedge \check{c}_0)](1_{BP} \wedge g'' j_{j_p^n - 1}) = (1_{BP} \wedge \check{b}_0)\mu(1_{BP} \wedge g'' j_{j_p^n - 1}) \) and we have

\[\tag{9.8.15} (\check{b}_0 \wedge 1_{BP})g'' j_{j_p^n - 1} = (1_{BP} \wedge g'' j_{j_p^n - 1})(\check{b}_0 \wedge 1_{K_{p^n}}) = (1_{BP} \wedge \check{b}_0)(1_{BP} \wedge \check{c}_0) g'' j_{j_p^n - 1}.\]

Note that \((\check{b}_0 \wedge 1_{BP}) \ast (1_{BP} \wedge \check{b}_0) : BP \to BP \ast BP\) are the right and left unit \( \eta_R, \eta_L : BP \to BP \ast BP\) respectively, then by (9.8.15) we have \( \eta_R(g'') = \eta_L(g'') \mod (p, v_1^{p^n - 1}) \). This means that \((p, v_1^{p^n - 1}, g'')\) is a \( BP \ast \) invariant ideal, or equivalently, \( g'' \in Ext_0^{p(p+1)} q(BP, BP \ast (p, v_1^{p^n - 1})) \). Then by Prop. 9.8.12 we have

\[\tag{9.8.16} g'' = \lambda_2 v_2^p + \Sigma_\lambda v_1^{p^n - 2r} \check{c}_1(t_r p^{n-2r}) + px_1 + v_1^{p^n - 1} x_2 \in BP, \]

where \( 1 \leq \lambda, \lambda_r \leq p - 1, t_r = (p^{2r+1} + 1)/(p+1) \) and \( x_1, x_2 \) are some elements in \( BP \ast \).

Let \( \tilde{g} = px_1 + v_1^{p^n - 1} x_2 \), then \( (\check{b}_0 \wedge 1_{BP})(g'' - \tilde{g}) = \eta_R(g'' - \tilde{g}) = \eta_L(g'' - \tilde{g}) \) so that \( \check{b}_1 \check{c}_0 (g'' - \tilde{g}) = (1_{BP} \wedge \check{c}_0)(\check{b}_0 \wedge 1_{BP})(g'' - \tilde{g}) = 0 \). Q.E.D.

**Proof of Theorem 9.8.1 in case \( t \geq 1 \)** By Theorem 9.5.1, there is \( \tilde{h}_{n+1} \in \pi_{p^{n+1} q + q - 1} M \) such that \( \eta_{n+1} = j \tilde{h}_{n+1} \in \pi_{p^{n+1} q + q - 2} S \) is represented in the ASS by \( h_0 h_{n+1} \in Ext^2_{BP, BP}(Z_p, Z_p) \). By Theorem 8.15 in chapter 8, \( \Phi(\beta_{p^n/p^{n-1}}) = h_0 h_{n+1}, \) where \( \Phi : Ext^2_{BP, BP}(BP, BP \ast) \to Ext^2_{BP}(Z_p, Z_p) \) is the Thom map. Then \( \eta_{n+1} = j \tilde{h}_{n+1} \) is represented by \( \beta_{p^n/p^{n-1}} + x \in Ext^2_{BP, BP}(BP, BP \ast) \) in the Adams-Novikov spectral sequence, where \( x = \)
\[
\Sigma_{r \geq 1} \lambda_r \beta_{t_r p^{n-2r}/p^{n-2r-1}} \text{ with } \lambda_r \in \mathbb{Z}(p) \text{ (cf. Prop. 9.8.13). Moreover, } \tilde{\eta}_{n+1} \in \pi_{p^{n+1}q+q-1}M \text{ is represented by } \beta_{p^n/p^{n-1}}' + x' + i_*(y) \text{ in the Adams spectral sequence, where } y \in Ext^1_{BP, BP}(BP_*, BP_*) \text{, and } x' = \Sigma_{r \geq 1} \lambda_r \beta_{t_r p^{n-2r}/p^{n-2r-1}}' \text{, } \beta_{tp^n/s} \text{ are the elements in } Ext^1_{BP, BP}(BP_*, BP_*(M)) \text{ such that } j_*(\beta_{tp^n/s}) = \beta_{tp^n/s} \in Ext^2_{BP, BP}(BP_*, BP_*) . \text{ It is known that all the generators in } Ext^1_{BP, BP}(BP_*, BP_*) \text{ are permanent cycles in the Adams-Novikov spectral sequence , then there exists } \tilde{f} \in \pi_{p^{n+1}q+q-1}M \text{ such that it is represented by } \beta_{p^n/p^{n-1}}' + x' \text{. In addition, } \tilde{f} \text{ can be extended by } f \in [\Sigma^{p^n+1}q+q-1M, M] \cap \ker d \text{ such that } \tilde{f} = f_i. \text{ Recall from (9.8.8)}
\]
\[
\cdots \tilde{a}_2^{\wedge 1^M} \Sigma^{-2} E_2 \wedge M \tilde{a}_1^{\wedge 1^M} \Sigma^{-1} E_1 \wedge M \tilde{a}_0^{\wedge 1^M} E_0 \wedge M = M \downarrow \tilde{b}_2 \wedge 1_M \downarrow \tilde{b}_1 \wedge 1_M \downarrow \tilde{b}_0 \wedge 1_M
\]
\[
\Sigma^{-2} BP \wedge E_2 \wedge M \Sigma^{-1} BP \wedge E_1 \wedge M \downarrow \tilde{b}_1 \wedge 1_M \downarrow \tilde{b}_0 \wedge 1_M
\]
is the Adams-Novikov resolution of the Moore spectrum \(M\). Then \(fi \in \pi_{p^{n+1}q+q}(E_1 \wedge M) \) with \(f_1 \in [\Sigma^{p^n+1}q+qM, E_1 \wedge M] \cap \ker d \) such that \(\tilde{a}_0 \wedge 1_M f_1 i = f_i \) and the \(d_1\)-cycle \((\tilde{b}_1 \wedge 1_M) f_1 i \in \pi_{p^{n+1}q+q}BP \wedge E_1 \wedge M \) represents \(\beta_{p^n/p^{n-1}}' + x' \in Ext^1_{BP, BP}(BP_*(M)) \). By applying \(d\) to the equation \((\tilde{a}_0 \wedge 1_M f_1 i) = fi \) we have \((\tilde{a}_0 \wedge 1_M) f_1 = f_i \).

Since \((\tilde{a}_0 \wedge 1_M) f_1 i = f_i \in \pi_{p^{n+1}q+q-1}M \) is represented by \(\beta_{p^n/p^{n-1}}' + x' \in Ext^1_{BP, BP}(BP_*(M)) \) in the Adams-Novikov spectral sequence and \(v_1^{-1}(\beta_{p^n/p^{n-1}}' + x') = 0 \), then \((\tilde{a}_0 \wedge 1_M) f_1 i \alpha^{p^n-1} i = f_1 i \alpha^{p^n-1} i \) has \(BP\)-filtration \(> 1 \) so that \((\tilde{b}_1 \wedge 1_M) f_1 i \alpha^{p^n-1} i \) is a \(d_1\)-boundary and it equals to \((\tilde{b}_1 \tilde{c}_0 \wedge 1_M) g_i \) for some \(g \in [\Sigma^{p^{n}(p+1)q}M, BP \wedge M] \). Hence, 
\[
(\tilde{b}_1 \wedge 1_M) f_1 i \alpha^{p^n-1} = (\tilde{b}_1 \tilde{c}_0 \wedge 1_M) g_i \text{, this is because } \pi_{p^{n}(p+1)q+1}BP \wedge E_1 \wedge M = 0 \text{ which is obtained by the sparseness fo } BP_*(E_1 \wedge M). \text{ Consequently we have}
\]
\[
f_1 i \alpha^{p^n-1} = (\tilde{c}_0 \wedge 1_M) g + (\tilde{a}_1 \wedge 1_M) f_2 \text{ with } f_2 \in [\Sigma^{p^{n}(p+1)q+1}M, E_2 \wedge M] \text{ and}
\]
\[
(1_{E_2} \wedge j) f_1 i \alpha^{p^n-1} = \tilde{c}_0 g'' j + \tilde{a}_1 (1_{E_2} \wedge j) f_2 \text{ with } g'' \in \pi_{p^{n}(p+1)q}BP , \text{ where } g'' j = (1_{BP} \wedge j) g \text{ is because } (1_{BP} \wedge j) g i \in \pi_{p^{n}(p+1)q-1}BP = 0. \text{ In addition, by } \tilde{b}_2 (1_{E_2} \wedge j) f_2 \in [\Sigma^{p^{n}(p+1)q+1}M, BP \wedge E_2] = 0 \text{ and } [\Sigma^{p^{n}(p+1)q+1}M, BP \wedge E_3] = 0 \text{ (cf. Prop. 9.8.9), then } (1_{E_2} \wedge j) f_2 = \tilde{a}_2 \tilde{a}_3 f_3 \text{ for some } f_3 \in [\Sigma^{p^{n}(p+1)q+2}M, E_4] \text{ and we have}
\]
\[
(9.8.17) \quad (1_{E_1} \wedge j) f_1 i \alpha^{p^n-1} = \tilde{c}_0 g'' j + \tilde{a}_1 \tilde{a}_2 \tilde{a}_3 f_3.
\]

By (9.8.17) we have \(\tilde{b}_1 \tilde{c}_0 g'' j f_1 p^{n-1} = 0\), then by Lemma 9.8.14, there is
\[ g = px_1 + v_1^{p^n-1} x_2 \in \pi_{p^n(p+1)q} BP \] with \( x_1, x_2 \in \pi_s BP \) such that \( \tilde{b}_1 \tilde{c}_0 (g^n - \tilde{g}) = 0. \) Consequently, \( \tilde{c}_0 (g^n - \tilde{g}) = \tilde{a}_1 f_4 \) for some \( f_4 \in \pi_{p^n(p+1)q+1} \tilde{E}_2 \) and 
\[ f_4 = \tilde{a}_2 \tilde{a}_3 f_5 \] with \( f_5 \in \pi_{p^n(p+1)q+3} \tilde{E}_4 \) which is obtained by the sparseness of \( \pi_s BP \cap \tilde{E}_s. \) So, (9.8.17) becomes

\[ (9.8.18) \quad (1_{\tilde{E}_1} \wedge j) f_1 \alpha^{p^n-1} = \tilde{c}_0 \tilde{g} j + \tilde{a}_1 \tilde{a}_2 \tilde{a}_3 f_5 j + \tilde{a}_1 \tilde{a}_2 \tilde{a}_3 f_3. \]

Note that \( \tilde{g} = px_1 + v_1^{p^n-1} x_2, \) then, \( \tilde{g} j j'_{p^n-1} = 0 \in BP^s (K^{p^n-1}) \cong \Sigma^{-(p^n-1)q-2} BP^s / \langle p, v_1^{p^n-1} \rangle \) so that \( \tilde{g} j = \tilde{g} \alpha^{p^n-1} \) for some \( \tilde{g} \in [\Sigma^{p^n+1} q + q-1 M, BP] \). Consequently, by (9.8.4), the equation (9.8.18) becomes

\[ (9.8.19) \quad (1_{\tilde{E}_1} \wedge j) f_1 j'_1 \rho_{p^n,1} = (1_{\tilde{E}_1} \wedge j) f_1 \alpha^{p^n-1} j'_{p^n} = \tilde{c}_0 \tilde{g} \alpha^{p^n-1} j'_{p^n} + \tilde{a}_1 \tilde{a}_2 \tilde{a}_3 (f_5 j + f_3) j'_{p^n} = \tilde{c}_0 \tilde{g} j'_1 \rho_{p^n,1} + \tilde{a}_1 \tilde{a}_2 \tilde{a}_3 (f_5 j + f_3) j'_{p^n} \]

Moreover, by (9.8.4)(9.8.6) we have \( \tilde{a}_1 \tilde{a}_2 \tilde{a}_3 (f_5 j + f_3) j'_{p^n-1} = \tilde{a}_1 \tilde{a}_2 \tilde{a}_3 (f_5 j + f_3) j'_{p^n} \psi_{p^n-1} = 0 \) and so \( (f_5 j + f_3) j'_{p^n-1} = 0 \), this is because \( [\Sigma^{p^n+1} q + q+1 M, BP] \cap \tilde{E}_2 = 0 \) for \( r = -1, 0, 1 \) (cf. Prop. 9.8.9). This shows that \( (f_5 j + f_3) = f_6 \alpha^{p^n-1} \) with \( f_6 \in [\Sigma^{p^n+1} q + q+2 M, \tilde{E}_4] \). Hence, the equation (9.8.19) becomes

\[ (9.8.20) \quad (1_{\tilde{E}_1} \wedge j) f_1 j'_1 \rho_{p^n,1} = \tilde{c}_0 \tilde{g} j'_1 \rho_{p^n,1} + \tilde{a}_1 \tilde{a}_2 \tilde{a}_3 f_6 j'_{p^n} + \tilde{c}_0 \tilde{g} j'_1 \rho_{p^n,1} + \tilde{a}_1 \tilde{a}_2 \tilde{a}_3 f_6 j'_1 \rho_{p^n,1} \]

and by (9.8.6) we have

\[ (9.8.21) \quad (1_{\tilde{E}_1} \wedge j) f_1 j'_1 = \tilde{c}_0 \tilde{g} j'_1 + \tilde{a}_1 \tilde{a}_2 \tilde{a}_3 f_6 j'_1 + \epsilon j'_{p^n-1} j'_1 \]

for some \( \epsilon \in [\Sigma^{p^n+1} q + q-1 K^{p^n-1}, \tilde{E}_1] \). By composing \( \tilde{a}_0 \) to (9.8.21) we have

\[ j f j'_1 = \tilde{a}_0 \tilde{a}_1 \tilde{a}_2 \tilde{a}_3 f_6 j'_1 + \tilde{a}_0 \epsilon j'_{p^n-1} j'_1 = \tilde{a}_0 \tilde{a}_1 \tilde{a}_2 \tilde{a}_3 f_6 j'_1 + j j'_{p^n-1} \gamma j'_{p^n-1} j'_1 \]

This is because \( \tilde{a}_0 \epsilon = j j'_{p^n-1} \tau \) for some \( \tau \in [\Sigma^{p^n+1} q + 1 K^{p^n-1}, K^{p^n-1}] \) (cf. Prop. 9.8.2). Consequently we have

\[ (9.8.22) \quad j f i = \tilde{a}_0 \tilde{a}_1 \tilde{a}_2 \tilde{a}_3 f_6 i + j j'_{p^n-1} \tau j'_{p^n-1} i + f_7 \alpha i \]

with \( f_7 \in [\Sigma^{p^n+1} q-2 M, S] \).

We claim that the map \( f_7 \alpha i \) in (9.8.22) has filtration \( \geq 3 \) so that by (9.8.22) we obtain that \( j j'_{p^n-1} \epsilon j'_{p^n-1} i \in \pi_{p^n+1 q + q-2} S \) is represented by \( \tilde{h}_0 h_{n+1} \in Ext^2_{A_2} \pi_{p^n+1 q + q-2} (Z_p, Z_p) \) in the Adams spectral sequence. This claim will be proved in the last.

Then, \( j j'_{p^n-1} \epsilon j'_{p^n-1} i \) is represented by \( \lambda_0 \beta_\infty \pi_{p^n-1} + \Sigma_{r \geq 1} \lambda_r \beta_\infty \pi_{p^n-2r} \pi_{p^n-2r-1} \in Ext^2_{BP, BP} (BP_*, BP_*) \) in the Adams-Novikov spectral sequence so that by Prop. 9.8.12 we know that \( \tilde{c}'_{p^n-1} i \in \pi_{p^n(p+1)q} K^{p^n-1} \) is represented by 

\[ v_2^n + \Sigma_{r \geq 1} \lambda_r v_1^{n-2r} c_1 (t_r p^{n-2r}) \in Ext^0_{BP, BP} (BP_*, BP_*) K^{p^n-1} \], where
$\lambda_r \in \mathbb{Z}_p$ and $t_r = (p^{2r+2}+1)/(p+1)$.

By [22] Theorem C.D, it is known that $v_2^{bp} \in Ext_{BP_*BP}^0(BP_*, BP_*)$, for $t \geq 1, 1 \leq r \leq p-1$, is a permanent cycle in the Adams-Novikov spectral sequence. Suppose inductively that $v_2^{bp} \in Ext_{BP_*BP}^0(BP_*, BP_*)$ (for $t \geq 1, 1 \leq r \leq p^s-1$ and $s \leq n-1$) is a permanent cycle in the Adams-Novikov spectral sequence, then we know that $v_1^{p^n-p^{n-2r}}\c_1(t,p^{n-2r}) \in Ext_{BP_*BP}^0(p^{n+1}q)$ $(BP_*, BP_*, K_{p^{n+1}})$ also is a permanent cycle for all $r \geq 1$. Moreover, by the representation of the above $\bar{c}_1\epsilon_{r-1}i$ we obtain that $v_2^{bp} \in Ext_{BP_*BP}^0(p^{n+1}q)(BP_*, BP_*, K_{p^{n+1}})$ is a permanent cycle. Hence, by (9.8.6), there exists $k \in \Sigma^p_{n+1}K_{p^{n+1}, K_{p^{n+1}}}$ such that the induced $BP_*$-homomorphism $k_*= v_2^{bp}$. In addition, the map $p_{n-1,r} : K_{p^{n-1}} \rightarrow K_r$ in (9.8.4) for all $r \leq p^n-1$ is a projection, then $p_{n-1,r}k^i\epsilon_{r-1}i \in \pi_{tp^n(p+1)q}K_{r}$ is represented by $v_2^{bp} \in Ext_{BP_*BP}^0(BP_*, BP_*, K_r)$ in the Adams-Novikov spectral sequence. This completes the induction and $jji\rho_{p^n-1,r}k^i\epsilon_{r-1}i \in \pi_*S$ is just the $\beta_{tp^n/r}$-element of the Theorem.

Now our remaining work is to prove the above claim. We turn to an argument in the ASS and let $A$ be the mod $p$ Steenrod algebra. By $Ext_{A}^1(p^{n+1}q-1(Z_p, H^*M) \cong Z_p\{j^*(h_{n+1})\}$ and the result on $\beta_{p^n/p^n} \in Ext_{BP_*BP}^2(p^{n+1}q)(BP_*, BP_*)$ support a nontrivial differential in the Adams-Novikov spectral sequence in [12] p.106 Theorem 5.4.8(i), we know that

(9.8.23) $j^*(h_{n+1}) \in Ext_{A}^1(p^{n+1}q-1(Z_p, H^*M)$ dies in the ASS

Then, the map $f_7 \in \Sigma^p_{n+1}q-2M, S$ in (9.8.22) has filtration $\geq 2$ in the ASS and so $f_7\alpha_i$ has filtration $\geq 3$. Moreover, by (9.8.21) we know that $jji\epsilon_{r-1}i$ and $jf \in \pi_{p^n+1}q, 2S$ must have the same filtration so that it is represented by $h_{0h_{n+1}} \in Ext_{A}^2(p^{n+1}q+1q)(Z_p, Z_p)$ in the ASS. This shows the above claim and the Theorem is proved. Q.E.D.

Remark 9.8.24 We give a detail proof of the result in (9.8.23) as follows. It will be done by some argument processing in the Adams resolution (9.2.9). Suppose in contrast that the map $j^*h_{n+1} \in Ext_{A}^1(p^{n+1}q-1(Z_p, H^*M)$ is a permanent cycle in the ASS, then we have $\bar{c}_1h_{n+1}: j = 0$, where $h_{n+1} \in \pi_{p^n+1}KG_1 \cong Ext_{A}^1(p^{n+1}q)(Z_p, Z_p)$. Consequently $\bar{c}_1h_{n+1} = \bar{f} : p$ for some $\bar{f} \in \pi_{p^n+1}qE_2$. On the other hand, $\bar{b}_2\bar{f} \in \pi_{p^n+1}qKG_2 \cong Ext_{A}^2(p^{n+1}q)(Z_p, Z_p) \cong Z_p\{b_n\}$ so that we have $\bar{b}_2\bar{f} = \lambda \cdot b_n$ with $\lambda \in Z_p$. However, the scalar $\lambda$ must be zero, this is because $b_n$ support a nontrivial differential $d_{2p-1}(b_n) = 88$
$d_{p-1} \Phi(\beta_{p,n}/p^n) = \Phi d_{p-1}(\beta_{p,n}/p^n) = \Phi(\alpha_1 \beta_{p,n-1}/p^{n-1}) = h_0 b_{p,n-1} \neq 0$ (cf. [12] p.206 Theorem 5.4.8(i) ). Hence $\tilde{f} = a_2 \tilde{f}_1$ for some $\tilde{f}_1 \in \pi_{p^{n+1}q+1}E_3$ and we have $\tilde{c}_1 h_{n+1} = a_2 \tilde{a}_3 \tilde{f}_2$ with $\tilde{f}_2 \in \pi_{p^{n+1}q+2}E_4$. This means that the secondary differential $d_2(h_{n+1}) = 0$ which contradicts with the following known nontrivial differential $d_2(h_{n+1}) = a_0 b_n \neq 0 \in Ext^3_{A}(\pi_{p^{n+1}q+1}(Z_p, Z_p)$ (cf. [12] p.11 Theorem 1.2.14). So we have $\tilde{c}_1 h_{n+1} \cdot j \neq 0$ and so (9.8.23) holds.

Now we proceed to prove Theorem 9.8.1 in case $t \geq 2$. We first prove the following Lemmas and Propositions.

**Lemma 9.8.25** Let $p \geq 3$ and $v_1 x \in Ext^{0, tp^s(p+1)q}(BP_s, BP_sK_p^n)$, then $v_1 x = \sum_{r=1}^{[n/2]} \lambda_r v_1^{n-p^{r-2}} \tilde{c}_1(a_r p^{n-2r})$, where $\lambda_r \in Z_p$, $a_r = (tp^{2r+1} + tp^{2r} - p^{2r} + 1)/(p + 1)$ and $\tilde{c}_1(ap^s)$ is the generator in Theorem 8.1.7 in chapter 6 which has degree $ap^s(p + 1)q$.

**Proof**: By Theorem 8.1.7 in chapter 8, $v_1 x$ is a linear combination of the following generators $v_1^{q} \tilde{c}_1(ap^s)$, where $a \geq 1$ is not divisible by $p$, $1 \leq b < p^n$, $b \geq max\{0, p^n - q_1(ap^s)\}$ and $q_1(ap^s) = p^s$ if $a = 1$, $q_1(ap^s) = p^s + p^{s-1} - 1$ if $a \geq 2$.

By degree reasons we have $bq + ap^s(p + 1)q = tp^n(p + 1)q$, then $s < n$, $b \geq p^n - (p^s + p^{s-1} - 1) > 0$ and so $b \geq p^n - p^{n-1}$ if $s \leq n - 2$. If $s = n - 1$, then $b$ is divisible by $p^{n-1}(p + 1)$ so that $b \geq p^n + p^{n-1}$. So, in any case we have $b \geq p^{n-1}(p - 1)$ and the remaining steps is similar to that given in the proof of Prop. 9.8.12. Q.E.D.

**Proposition 9.8.26** Let $r > s$ and $\rho_{r,s} : K_r \to K_s$ be the map in (9.8.4), then $d(\rho_{r,s}) = i_s^* \xi j_r^*$ with $\xi \in [\Sigma^{q+1} M, M] \cap kerd$.

**Proof**: By (9.8.6)(9.8.3) we have $j_r^* d(\rho_{r,s}) = d(j_r^* \rho_{r,s}) = d(\alpha^{-1} j_r^*) = 0$, then $d(\rho_{r,s}) = i_s^* \xi j_r^*$ for some $\xi \in [\Sigma K_r, M]$ and $\xi = \xi j_r^*$ with $\xi \in [\Sigma^{q+1} M, M]$,

this is because $\xi i_r^* \in [\Sigma M, M] = 0$. By Theorem 6.4.14 in chapter 6, we may assume $\xi = \xi_1 + \xi_2 i j$ with $\xi_1, \xi_2 \in ker d \cap [\Sigma^s M, M]$. Then $d(\rho_{r,s}) = i_s^* \xi_1 j_r^* + i_s^* \xi_2 i j j_r^*$ and by applying the derivation $d$ using (9.8.3) we have $i_s^* \xi_2 j_r^* = 0$. Consequently we have $i_s^* \xi_2 = \xi_3 \alpha^r = 0$, this is because $\xi_3 \in [\Sigma^2 M, K_s]$ = 0. Then $d(\rho_{r,s}) = i_s^* \xi_1 j_r^*$ with $\xi_1 \in ker d \cap [\Sigma^{q+1} M, M]$. Q.E.D.

**Proof of Theorem 9.8.1 in case $t \geq 2$**: By Theorem 9.8.1 in case $t \geq 1$, there exists $f' \in [\Sigma^{q+1} K_a, K_a]$ such that the induced $BP_s$-
homomorphism \( f'_s = \psi^{p^n} \), where we briefly write \( p^n - 1 \) as \( a \). By Theorem 6.5.22 in chapter 6, we may assume \( f \in Mod_s \), this is because the components of \( f' \) in \( Der_s \) and \( Mod_s \) induce zero \( BP_s \)-homomorphism.

Write \( \delta' = i'_s j'_s \in [\Sigma^{s-q-1} K_s, K_s] \). By Theorem 6.5.23 in chapter 6, \( \delta' f' + f' \delta' \in Mod_s \) and this group is a commutative subring of \( [\Sigma^s K_s, K_s] \). Then we have \( f' (\delta' f' + f' \delta') = (\delta' f' + f' \delta') f' \) or equivalently, \( (f')^2 \delta - \delta (f')^2 = 2 (f')^2 \delta' - f' f' \delta' \). By induction we have \( (f')^s \delta' - \delta (f')^s = s ((f')^s \delta' - (f')^{s-1} \delta f') \), \( s \geq 1 \). That is

\[
(9.8.27) \quad s \cdot (f')^{s-1} \delta' f' = (\delta f')^s + (s-1) (f')^s \delta', \quad s \geq 1
\]

Let \( \rho_{a,1} : K_a \to K_1 \) be the projection in (9.8.4), then by Theorem in chapter 6, \( \rho_{a,1}(f')^s i'_a t \in \pi_s K_1 \) can be extended to \( k_s \in Mod_s \subset [\Sigma^{p^n (p+1)q} K_1, K_1] \) such that \( \rho_{a,1}(f')^s i'_a = k_s i'_a \) and \( (k_s)_s = i'_2 \). Since \( j'_1 k_s i'_a = \alpha^{-1} j'_a (f')^s i'_a \) and \( (i'_1 j'_1 k_s - k_s i'_1 j'_1) \in Mod_s \), then \( i'_1 j'_1 k_s - k_s i'_1 j'_1 \neq 0 \) and so \( i'_1 j'_1 k_s = k_s i'_1 j'_1 \). By applying the derivation \( d \) to the equation \( \rho_{a,1}(f')^s i'_a \delta = k_s i'_a \delta \) (where we write \( \delta = ij \)) we have

\[
(9.8.28) \quad \rho_{a,1}(f')^s i'_a = k_s i'_a - d(\rho_{a,1})(f')^s i'_a \delta = k_s i'_a - i'_1 \alpha j'_a (f')^s i'_a \delta
\]

where \( \xi \in [\Sigma^{aq+1} M, M] \cap ker(d) \) (cf. Prop. 9.8.26). Let \( t \geq 2 \) is not divisible by \( p \), then by (9.8.27)-(9.8.28) we have \( i'_1 j'_a (f')^s i'_a = \rho_{a,1} i'_a j'_a (f')^s i'_a = t \cdot \rho_{a,1} (f')^s i'_a f' i'_a = t \cdot k^{t-1} i'_1 j'_a (f')^s i'_a - t \cdot i'_1 j'_a (f')^s i'_a \xi j'_a f' i'_a \) and

\[
(9.8.29) \quad i'_1 j'_a f' i'_a = t \cdot k^{t-1} i'_1 j'_a f' i'_a,
\]

where we write \( \phi = (f')^s i'_a + t \cdot (f')^{s-1} i'_a \xi j'_a f' i'_a \).

Let \( X \) be the cofibre of \( i'_1 j'_a f' i'_a : \Sigma^{p^n+1+q-1} M \to K_1 \) given by the cofibration in the upper row of the following homotopy commutative diagram

\[
\begin{align*}
\Sigma^{-1} X & \xrightarrow{u} \Sigma^{p^n+1+q-1} M & i'_1 j'_a f' i'_a & \xrightarrow{u} K_1 & X \\
\Sigma^{-m-1} K_{a+1} & \xrightarrow{\rho_{a+1}^{p^n+1}} \Sigma^{-m-1} K_a & i'_1 j'_a & \xrightarrow{\Sigma^{-m-aq} K_1} \Sigma^{-m} K_{a+1}
\end{align*}
\]

(9.8.30)

Note that the above middle rectangle is homotopy commutative by (9.8.29), then there exists \( \tilde{\phi} \) such that all the above rectangles commute up to homotopy.

By \( w i'_1 j'_a f' i'_a = 0 \) we have \( w i'_1 j'_a f' = y j'_a \) with \( y \in [\Sigma^{p^n (p+1)q} M, X] \) so that \( uy j'_a = 0 \) and \( uy = \lambda \cdot \alpha^a \) for some \( \lambda \in [M, M] \cong Z_p \{1_M\} \), that is we have
(9.8.31) \[ w_i'j_a'f' = y_j'a' , \quad uy = \lambda \cdot \alpha^a \]

On the other hand, \( j_a'(f')^s \psi_1.a_1' = j_a'(f')^s \alpha^a - 1 = \alpha^a - 1 j_a'(f')^s \alpha - 1 = j_1'\rho_a.1 \]

(9.8.32) \[ y = wk_1'i_1 + w_i_1' \eta + z \alpha \]

which is obtained by (9.8.31).

We claim that

(9.8.33) \[ \tilde{\phi} z \alpha \in \pi_{tp^n(p+1)q+a}K_{a+1} \text{ has } BP \text{ filtration} > 0 \]

This will be proved in the last. Then \( \tilde{\phi} y_i = t \cdot \psi_{1,a+1}k_{i-1}k_1'i_1 \) (modulo higher filtration) is represented by \( t \cdot v_1^{p^n}v_2^{p^n} \in Ext_{BP,BP}^{0,*}(BP_s,BP_sK_{a+1}) \) in the Adams-Novikov spectral sequence.

Hence, by (9.8.31)(9.8.30)(9.8.28)(9.8.27) we have \( \tilde{\phi} y_i = \tilde{\phi} w_i_1\psi_1.a_1' \equiv t \cdot \psi_{1,a+1}k_{i-1}k_1'i_1 \psi_1.a_1' \equiv \psi_{1,a+1}k_{i-1}k_1'i_1 \psi_{1,a+1}k_{i-1}k_1'i_1 \tilde{\phi} y_i = t \cdot \psi_{1,a+1}k_{i-1}k_1'i_1 \psi_{1,a+1}k_{i-1}k_1'i_1 \)

By the claim (9.8.33), \( \tilde{\phi} y_i \) is represented by \( t \cdot v_1^{p^n}v_2^{p^n} \) in the Adams-Novikov spectral sequence, then \( \tilde{\phi} z \alpha \) is represented by \( v_1^{p^n}v_2^{p^n} \) and so \( \tilde{\phi} z \alpha \) is realizable in \( \Sigma [\Sigma^{p^n(p+1)q}M,K_{a+1}] \).

By [20], if \( t \geq 2 \) is not divisible by \( p \) and \( 1 \leq r \leq p, v_2^{p^n} \in Ext_{BP,BP}^{0,*}(BP_s,BP_sK_r) \) is a permanent cycle in the Adams-Novikov spectral sequence. Suppose inductively that \( v_2^{p^n} \in Ext_{BP,BP}^{0,*}(BP_s,BP_sK_r) \) are permanent cycles for all \( t \geq 2 \) is not divisible by \( p, 1 \leq r \leq p^s \) and \( s \leq n - 1 \). Then, it is easily seen that \( v_1^{p^n-p^{n-2r}c_1(a_r p^{n-2r})} \) is realizable in \( \Sigma^{tp^n(p+1)q}K_{a+1}, K_{a+1} \) so that the above by the induction hypothesis we know that \( v_1x \) also is a permanent cycle. So, \( v_2^{p^n} \in Ext_{BP,BP}^{0,*}(BP_s,BP_sK_{a+1}) \) is a permanent cycle in the Adams-Novikov spectral sequence and there exists \( h \in \pi_{tp^n(p+1)q}K_{a+1} \) such that the induced \( BP_s \)-homomorphism \( h_* = v_2^{p^n} \).

Hence, for \( 1 \leq r \leq a + 1 = p^n, j_a'j_{a+1}, h \in \pi_{tp^n(p+1)q-rq-2}S \) is just the \( h_{tp^n/s} \)-element of the Theorem.

Now our remaining work is to prove the claim (9.8.33). Recall as known above that \( j_a'j_{a+1}i \in \pi_sM \) is represented by \( \beta_{tp^n/p^n-1} \in Ext_{BP,BP}^{1,*}(BP_s, \)
$BP_*M$ in the Adams-Novikov spectral sequence and $\beta_{p^n/p^n-1}' = v_1 \beta_{p^n/p^n}'$, then $(i'_1)(\beta_{p^n/p^n-1}') = 0 \in Ext^1_{BP, BP}(BP_*, BP_*, K_1)$ and so $i'_1j'_a f'_{i'_a}i' \in \pi_* K_1$ has $BP$-filtration $\geq q + 1$. Then, in the Adams-Novikov resolution of the spectrum $K_1$, $i'_1j'_a f'_{i'_a}i'$ can be lifted to $\kappa \in \pi_* E_{q+1} \wedge K_1$ such that $(\tilde{a}_0 \wedge 1 K_1) \cdots (\tilde{a}_q \wedge 1 K_1) \kappa = i'_1j'_a f'_{i'_a}i'$. Since $K_1$ is an $M$-module spectrum, then $\kappa = \kappa' \cdot i$ with $\kappa' \in (\Sigma^* M, E_{q+1} \wedge K_1)$. Consequently we have $i'_1j'_a f'_{i'_a}i' = (\tilde{a}_0 \wedge 1 K_1) \cdots (\tilde{a}_q \wedge 1 K_1) \kappa' + \sigma j$ with $\sigma \in [\Sigma^{p^n+1}q+q, K_1]$. Note that $(\tilde{b}_0 \wedge 1 K_1) \sigma = 0$ so that $\sigma$ can be lifted to $\sigma' \in \pi_* E_{q+1} \wedge K_1$ such that $(\tilde{a}_0 \wedge 1 K_1) \cdots (\tilde{a}_q \wedge 1 K_1) \sigma' = \sigma$. So we have $i'_1j'_a f'_{i'a} = (\tilde{a}_0 \wedge 1 K_1) \cdots (\tilde{a}_q \wedge 1 K_1) (\kappa' + \sigma j)$. By this we know that the following short exact sequence induced by the cofibration in the top row of (9.8.30) is a split exact sequence of $BP_*BP$-comodule:

$$0 \rightarrow BP_* K_1 \xrightarrow{u_*} BP_* X \xrightarrow{w_*} BP_* M \rightarrow 0$$

where $|w_*| = -(p^n+1)q$

Moreover, this splitness also hold in the following $Ext_{BP, BP}$-stage:

$$0 \rightarrow Ext^0_{BP, BP} K_1 \xrightarrow{u_*} Ext^0_{BP, BP} X \xrightarrow{w_*} Ext^0_{BP, BP} M \rightarrow 0$$

That is to say, there is an invariant $BP_*$-homomorphism $u' : Ext^0_{BP, BP} K_1 \rightarrow Ext^0_{BP, BP} X$ and $w' : Ext^0_{BP, BP} X \rightarrow Ext^0_{BP, BP} M$ such that $u'u_* = 1_{Ext^0_{BP, BP} K_1}, w_*w' = 1_{Ext^0_{BP, BP} M}$ and $u_*u' + w'w_* = 1_{Ext^0_{BP, BP} X}$, where we briefly write $Ext^0_{BP, BP}(BP_*, BP_* X)$ as $Ext^0_{BP, BP} X$.

To prove the claim (9.8.33), suppose in contrast that $\bar{\phi}z_{0, i} \in \pi_* K_{n+1}$ has $BP$-filtration 0, then, by (9.8.32), it is represented by $\lambda \nu_{1}^{(p-1)q} \nu_{2}^{p^n} \in Ext^0_{BP, BP} K_{n+1}$ in the Adams-Novikov spectral sequence, where $\lambda \neq 0 \in Z_p$. Then $zi \in \pi_p(p^n+q+a-1)q X$ must have $BP$ filtration 0 and it is represented by some $x \in Ext^0_{BP, BP} X$ and $(\bar{\phi})_*(v_1 x) = \lambda \cdot \nu_{1}^{(p-1)q} \nu_{2}^{p^n}$. However,

$$x = u_* u'(x) + w_* w_*(x) = u' w_*(x)$$

, this is because by degree reason we have $u'(x) \in Ext^0_{BP, BP} (p^n+q)(a-1)q K_1 = 0$. Then

$$x = \lambda' w'(v_1^r)$$

for some $\lambda' \in Z_p, r q = tp^n(p+1)q + (a-1)q - p^n+1 - q$ since $w_*(x) \in Ext^0_{BP, BP} M \cong Z_p\{v_1^r\}$, Then

$$\lambda \nu_{1}^{(p-1)q} \nu_{2}^{p^n} = (\bar{\phi})_*(v_1 x) = \lambda' (\bar{\phi})_* w'(v_1^{r+1})$$

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Moreover, since $w'(v_1^{r+1})$ is belong to $Ext^0_*X$ the summand which is isomorphic to $Ext^0_*M$ and $Ext^0_*M$ is a trivial $\mathbb{Z}[v_2]$-module, then we have

$$0 = v_2^n \cdot (\tilde{\varphi})_* w'(v_1^{r+1}) = \lambda \cdot v_1^{(t+1)p^n} \in Ext^0_*K_{a+1}$$

This is a contradiction and then shows the claim(9.8.33). Q.E.D.

After finishing the proof of Theorem 9.8.1 on second periodicity elements in the stable homotopy groups of spheres, we state the following Theorem on further result on second periodicity families in the stable homotopy groups of spheres without proof. The proof is done in base on the result of Theorem 9.8.1 and using some properties of the spectrum $M(p^r,v_1^{ap^r})$ which is the geometric realization of $BP_*/(p^r,v_1^{ap^r})$. The details of the proof can be seen in [23] §3.

**Theorem 9.8.34** Let $p \geq 5$. $j = cp^i \leq p^{n-i} - 1$ if $t \geq 1$ ($cp^i \leq p^{n-i}$ if $t \geq 2$), then the element

$$\beta_{p^n/j,i+1} \in Ext^2_{BP_*BP}(BP_*BP)$$

is a permanent cycle in the Adams-Novikov spectral sequence and it converges to the corresponding homotopy element of order $p^{i+1}$ in $\pi_*$.$\Sigma$.

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