Effects of weak self-interactions in a relativistic plasma on cosmological perturbations

Herbert Nachbagauer

Laboratoire de Physique Théorique ENSLAPP 
B.P. 110, F-74941 Annecy-le-Vieux Cedex, France

Anton K. Rebhan

DESY, Gruppe Theorie,
Notkestraße 85, D-22603 Hamburg, Germany

Dominik J. Schwarz 

Institut für Theoretische Physik, Technische Universität Wien,
Wiedner Hauptstraße 8-10/136, A-1040 Wien, Austria
(November 30, 1994)

Abstract

The exact solutions for linear cosmological perturbations which have been obtained for collisionless relativistic matter within thermal field theory are extended to a self-interacting case. The two-loop contributions of scalar $\lambda \varphi^4$ theory to the thermal graviton self-energy are evaluated, which give the $O(\lambda)$ corrections in the perturbation equations. The changes are found to be perturbative on scales comparable to or larger than the Hubble horizon, but the determination of the large-time damping behavior of subhorizon perturbations requires a resummation of thermally induced masses.

PACS numbers: 98.80.-k, 11.10.Wx, 52.60.+h

Typeset using REVTeX
The theory of linear perturbations of otherwise homogeneous and isotropic cosmological models \[1\] plays a central rôle in the problem of large-scale structure formation in the early universe. The conventional approach to their study in cosmological models involving more complicated forms of matter than a perfect fluid \[1\] is based on the coupled Einstein-Boltzmann equations \[2,3\], which are usually solved numerically. In Ref. \[7\], a novel framework has been developed which employs perturbative thermal field theory \[8\] to derive self-consistent and automatically gauge-invariant perturbation equations. In this formalism, the connection of the perturbed metric and the perturbed energy-momentum tensor is provided by the thermal graviton self-energy. At one-loop order, the leading high-temperature contributions to the latter have been obtained in Ref. \[9\] (see also Ref. \[10\]) in momentum space, and conformal invariance allows one to transform these results directly to a curved space with vanishing Weyl tensor. The one-loop high-temperature contributions describe relativistic collisionless matter interacting with the metric perturbations, and the resulting perturbation equations have been shown \[11\] to be equivalent to a gauge-invariant reformulation of the Einstein-Vlasov equations \[12\]. Moreover, analytic solutions for scalar, vector, and tensor perturbations have been found for a spatially flat Friedmann-Robertson-Walker (or, Einstein-de Sitter) cosmological model in the general case of a two-component system containing also a perfect fluid part \[13\], as well as for higher-dimensional models \[14\].

In this Letter, we investigate the effects of weak self-interactions on the exact solutions that were found for relativistic collisionless matter. For simplicity we consider scalar particles with \(\lambda \phi^4\) interaction Lagrangian. The leading self-interaction effects are described by two-loop corrections to the thermal graviton self-energy in the high-temperature limit. The corrections are such that the ensuing perturbation equations can still be solved exactly in terms of rapidly converging power series.

The background space-time of the Einstein-de Sitter cosmological model, which we shall consider, is given through the line element

\[
ds^2 = S^2(\tau) \left( -d\tau^2 + \delta_{ij}dx^i dx^j \right),
\]

with \(\tau\) being the conformal time which measures the size of the horizon in comoving coordinates, \(R_H = \tau\). A thermal distribution of relativistic (i.e., effectively massless) scalar particles with \(\lambda \phi^4\) self-interactions gives rise to a background energy-momentum tensor

\[
\tilde{T}^{\mu \nu} = u^{\mu} u^{\nu} (\tilde{E} + \tilde{P}) + \tilde{P} \delta^{\mu \nu}, \quad u^{\mu} = S \delta^{\mu}_0,
\]

with \(\tilde{E} = 3\tilde{P}\). Including order \(\lambda\), \(\tilde{T}^{\mu \nu}\) is determined by the diagrams of Fig. 1, which at finite temperature \(T\) yield

\[
\tilde{E}(S = 1) = \left( \frac{\pi^2}{30} - \frac{\lambda}{16} \right) T^4 + O(\lambda^{3/2}).
\]

The scale dependence of the mean energy density is given by \(\tilde{E}(S) = \tilde{E}(S = 1) S^{-4}\), with \(S \propto \tau\).

Self-consistent perturbations of the metric \(g_{\mu \nu} = \tilde{g}_{\mu \nu} + \delta g_{\mu \nu}\) are found by varying the Einstein equations,
\[ \delta G^{\mu \nu} \equiv \delta (R^{\mu \nu} - \frac{1}{2} g^{\mu \nu} R) = -8\pi G \delta T^{\mu \nu} \quad (4) \]

with

\[ \delta T_{\mu \nu}(x) = \int d^4 y \frac{\delta T_{\mu \nu}(x)}{\delta g^{\alpha \beta}(y)} \delta g^{\alpha \beta}(y). \quad (5) \]

Hence, \( \delta T_{\mu \nu} \) is determined by the graviton self-energy

\[ \Pi_{\mu \nu \alpha \beta} \equiv \frac{\delta^2 \Gamma}{\delta g^{\mu \nu} \delta g^{\alpha \beta}} = \frac{1}{2} \frac{\delta (\sqrt{-g} T_{\mu \nu})}{\delta g^{\alpha \beta}}, \quad (6) \]

where \( \Gamma \) contains all contributions to the effective action besides the classical Einstein-Hilbert action.

\( \Gamma \) is a nonlocal functional of the metric. With a flat background metric, the graviton self-energy \( \Pi_{\mu \nu \alpha \beta} \) is most easily evaluated through momentum-space Feynman rules. In the high-temperature limit, \( \Gamma \) is conformally invariant \[7\] which makes it possible to directly transform the flat-space results to conformally flat cosmological models such as the Einstein-de Sitter one of Eq. (1) according to

\[ \Pi^{\mu \nu \rho \sigma}(x, x') \bigg|_{g=S^2 \eta} = S^{-2}(\tau) \int \frac{d^4 k}{(2\pi)^4} e^{ik(x-x')} \Pi^{\mu \nu \rho \sigma}(k) \bigg|_{\eta} S^{-2}(\tau'). \quad (7) \]

The one-loop contribution to \( \Pi^{\mu \nu \rho \sigma} \) has been evaluated in Ref. \[9\] at temperature \( T \gg k_0, k \), which is relevant for the case of cosmological perturbations [the momentum scale is set by the inverse Hubble radius \( \sim (T/m_{\text{Plank}})^T \)]. It describes collisionless matter, interacting solely with the metric perturbations. Since collisionless matter can sustain anisotropic pressure, this medium admits a rich spectrum of possible cosmological solutions. Those have been studied in great detail in Ref. \[11\] by two of the present authors.

In order to investigate the effects of weak self-interactions, one has to include higher loop corrections. With \( \lambda \phi^4 \) interactions, the order \( \lambda \) contributions are contained in the two-loop diagrams shown in Fig. 2. Evaluating again the high-temperature limit \( T \gg k_0, k \) (full details will be presented elsewhere) gives a contribution to \( \Pi^{\mu \nu \rho \sigma}(k) \) that again satisfies the Ward identity corresponding to conformal invariance of \( \Gamma \) which is necessary for the simple transformation law of Eq. (7).

Since we did not have to single out a particular gauge, the perturbed Einstein equations (4) involve only gauge invariant combinations of the perturbed metric components and can be given entirely in terms of Bardeen’s gauge invariant potentials \[15\]. Expanding the latter in Fourier modes (plane waves with respect to the conformally flat background of Eq. (1)) then leads to equations depending only on the conformal time \( \tau \), which can be combined with the wavelength of the metric perturbations \( \ell \equiv 2\pi/k \) into the dimensionless variable

\[ x \equiv k\tau = \frac{2\pi R_H}{\ell}. \quad (8) \]

\( x/\pi = 1 \) defines the point in conformal time when half a wavelength fits inside the growing Hubble horizon.
Because of the nonlocality of the effective action $\Gamma$, the resulting equations are integro-differential equations rather than ordinary differential equations, involving a convolution integral of $\Pi^{\mu\nu\rho\sigma}(x, x')$ with the metric perturbations.

In the case of scalar metric perturbations, there are two independent equations for the two gauge invariant potentials $\Phi_H$ and $\Phi_A$ of Bardeen [15]. Using instead the combinations

$$\Phi = \frac{1}{2}\Phi_H, \quad \Pi = -(\Phi_H + \Phi_A),$$

which are related to the gauge-invariant (cf. [15]) density contrast $\epsilon_m = x^2\Phi/3$ and to the anisotropic pressure $\pi_{\text{anis}} = x^2\Pi$, one equation (the 00-component of Eq. (4)) is of the form

$$(x^2 - 3)\Phi(x) + 3x\Phi'(x) - 6\Pi(x) = -12 \int dx' K(x - x')(\Phi + \Pi)'(x'),$$

while the second is an ordinary differential equation by virtue of the vanishing of the trace of the energy-momentum tensor in our ultrarelativistic setting,

$$\Phi'' + \frac{4}{x}\Phi' + \frac{1}{3}\Phi + \frac{2}{x}\Pi' - \frac{2}{3}\Pi = 0.$$  

(11)

All the remaining components of the perturbed Einstein equations (4) turn out to be combinations of Eqs. (10) and (11).

The one-loop contribution to the kernel $K$ is determined by the discontinuity of the one-loop graviton self-energy $\bar{\Pi}^{0000}(k_0, k)$ in Eq. (7). It reads $\bar{\Pi} (\omega = k_0/k)$

$$K_1(x - x') = \frac{i}{2\pi} \int_{-\infty+i\epsilon}^{+\infty+i\epsilon} d\omega e^{-i\omega(x-x')} \frac{1}{2} \ln \frac{\omega + 1}{\omega - 1}$$

$$= j_0(x - x')\theta(x - x'),$$

(12)

with $j_0(x) \equiv \sin(x)/x$. The step function gives an upper bound of $x$ in the integral in Eq. (10); a lower bound at some initial $x_0$ can be adopted by adding an inhomogeneous term $\sum_{n=0}^{\infty} \gamma_n K^{(n)}(x - x_0)$ to the convolution integral in Eq. (10), encoding the infinite number of initial conditions one has to impose by specifying the likewise infinite number of moments of the initial particle distribution [14].

The two-loop contributions have more complicated structure, but can still be evaluated exactly,

$$K_2(x - x') = \frac{5i\lambda}{32\pi^3} \int_{-\infty+i\epsilon}^{+\infty+i\epsilon} d\omega e^{-i\omega(x-x')}$$

$$\times \left[ \omega \ln^2 \frac{\omega + 1}{\omega - 1} - \ln \frac{\omega + 1}{\omega - 1} - \frac{2\omega}{\omega^2 - 1} \right]$$

$$= \frac{5\lambda}{8\pi^2}(2\kappa' - j_0 - \cos)(x - x')\theta(x - x'),$$

(13)

where we introduced
\[ \kappa(x) \equiv \frac{1}{x} [\sin(x) \text{Si}(2x) + \cos(x) \{ \text{Ci}(2x) - \gamma - \ln(2x) \}] \]

\[ = 2 \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m+1}}{(2m+2)!} \sum_{j=0}^{m} \frac{1}{2j+1}, \]  

(14)

with Si and Ci being the sine and cosine integral \[16\]. In Eq. (13), there are two contributions which differ in form from the one-loop kernel: the first term in the integrand, which arises exclusively from the last diagram in Fig. 2, and the last term, which is due solely to the third diagram.

With the explicit result for \( K_2 \), which has a well-behaved power series representation, the same methods as in Ref. [7,11] can be used to solve the coupled equations (10) and (11). Since \( K_2(x) \propto x^4 + O(x^6) \), \( K = K_1 + K_2 \) still satisfies

\[ K(0_+) = 1, \quad K(0_+) + 3K''(0_+) = 0, \]  

(15)

as did \( K_1 \). Thereby the conditions [11] on the initial data (summarized by the set of \( \gamma_n \)) following from demanding regularity at \( x = x_0 = 0 \) are left unchanged.

In addition to regular solutions, there exist also ones that are singular as \( x \to 0 \). In the collisionless case those behave as \( \Phi, \Pi \sim x^{-5/2} \cos[\sqrt{27/20 \ln(x)}] \), thus showing oscillations even on superhorizon scales [17,18]. This behavior is modified by the weak self-interactions, to wit,

\[ \Phi, \Pi \sim x^{-5/2} \cos[\sqrt{\frac{27}{20} - \frac{2\lambda}{\pi^2} \ln(x)}], \]  

(16)

up to a constant shift of phase.

In Ref. [7], it was pointed out that the coupled system of Eqs. (10) and (11) can be solved exactly for \( x_0 = 0 \), yielding power series representations with infinite radius of convergence. This still holds true after adding the self-interaction term \( K_2 \). For the particular initial condition determined by \( \gamma_n = 0, n \geq 1 \), the first few terms in the regular solutions are given by

\[ \Phi(x) = \gamma_0 \left( \frac{28 - 10\lambda/\pi^2}{5} \right. \]

\[ - \frac{-4644 - 3735\lambda/\pi^2 + 350(\lambda/\pi^2)^2}{315(54 - 5\lambda/\pi^2)} x^2 \pm \cdots \right), \]

\[ \Pi(x) = \gamma_0 \left( \frac{-4 - 5\lambda/\pi^2}{5} \right. \]

\[ + \frac{2808 - 5445\lambda/\pi^2 + 700(\lambda/\pi^2)^2}{630(54 - 5\lambda/\pi^2)} x^2 \mp \cdots \right). \]

(17)

With \( \lambda = 1 \), which corresponds to a prefactor \( \approx 0.06 \) in the last line of Eq. (13), the density contrast \( \epsilon_m = x^2 \Phi/3 \) is plotted in Fig. 3 by the dashed line and compared with the collisionless case (full line). On superhorizon scales, \( x \ll 1 \), the solutions are dominated by the constant modes in \( \Phi \) and \( \Pi \). The amount of anisotropic pressure associated with a given density contrast is determined by \( \Pi/\Phi \), which is seen to be reduced with increasing \( \lambda \), in
agreement with intuitive expectation, since the collision-dominated case of a perfect fluid has $\Pi \equiv 0$.

For large $x \gg 1$, the asymptotic behavior of the density contrast $\epsilon_m(x)$ is proportional to the one of $K(x)$. In the collisionless case $K(x) = K_1(x) \sim \sin(x)/x$, whereas in the other extreme case of a perfect fluid $\epsilon_m \sim \sin(x/\sqrt{3})$. With weak self-interactions, the asymptotic behavior is indeed modified towards weaker damping (see Fig. 3 and 4), which ceases completely for sufficiently large $x$ where $K(x) \sim \lambda \cos(x)$. However, there the correction $K_2$ is overtaking the lowest order term $K_1$, which signals a possible breakdown of perturbation theory.

Inspecting the analytic structure that gives rise to the different asymptotic behavior of $K_2$, one finds that the discontinuity in the integrand of Eq. (13) is singular on the light-cone $\omega = \pm 1$. There is a logarithmically singular term which is responsible for $\kappa'(x) \sim \sin(x) \ln(x)/x$, but there are even poles at $\omega = \pm 1$, which contribute the term involving the $\cos(x)$. However, the scalar particles, whose rest masses are negligible at sufficiently high temperature, acquire thermal masses $m = \sqrt{\lambda T}$. A resummation of these those will remove the singularities at $\omega = \pm 1$, which are associated with the masslessness of the scalar particles.

The logarithmically singular integrand of Eq. (13) will be regularised at $\omega = \pm 1$, but since the logarithmic singularity is integrable anyway, the resummed result will not change dramatically; the poles at $\omega = \pm 1$, however, disappear completely after a resummation of the thermal masses.

Adding a thermal mass term to the Lagrangian and subtracting it again as a higher-order counter-term has the effect of replacing all the propagators in Fig. 2 by massive ones and subtracting out the third and fourth diagram, which were precisely the ones contributing the pole term to Eq. (13). This resummation of mass insertions replaces the leading-order result $K_1(x) = j_0(x)\theta(x)$ by

$$K_1^{\text{res}}(x) = -\theta(x)\int_{-1}^{1} d\omega e^{-i\omega x}$$

$$\times \int_{m/\sqrt{1-\omega^2}}^{\infty} dp \frac{d}{d\omega} \left( \frac{1}{\exp(p/T) - 1} \right) \left/ \left( \frac{8\pi^4 T^4}{15} \right) \right.$$ up to terms whose amplitude is down by factors of $\lambda$. With a non-zero $m = \sqrt{\lambda T}$, the integrand is now seen to vanish at $\omega = \pm 1$, whereas before it was a constant; in fact, it vanishes together with all its derivatives, but rapidly recovers the bare one-loop value away from the light-cone. This is a negligible effect for small $x$, but the large $x$ behavior is dominated by the (non-analytic) integration region $|\omega| \approx 1$.

Eq. (18) can be evaluated by a Mellin transform [19] which yields

$$K_1^{\text{res}}(x) = \theta(x)\sqrt{\frac{\pi}{2}} \frac{15}{8\pi^4} \times$$

$$\sum_{k=0}^{\infty} \lambda^{k/2} \frac{(-1)^{1+k}(k-4)\zeta(k-3)}{(2\pi)^{k-4}\Gamma(1+k/2)} \left( \frac{2}{x} \right)^{\frac{1+k}{2}} J_{\frac{1+k}{2}}(x),$$

where in the term with $k = 4$ one has to substitute $(k-4)\zeta(k-3) \to 1$. For small $x$,
\[ K_1^{\text{res.}}(x) = \theta(x) \left( j_0(x) - \frac{5\lambda}{8\pi^2} \cos(x) \right) + O(\lambda^{3/2}) \]  

(20)

is a good approximation; for \( x \gg 1 \), on the other hand, the complete function \( K_1^{\text{res.}}(x) \) turns out to decay even slightly faster than \( j_0(x) \), oscillating with a reduced phase velocity

\[ v = 1 - \frac{5\lambda}{8\pi^2} + O(\lambda^{3/2}). \]  

(21)

A first approximation to a fully resummed calculation is therefore to change

\[ K_1(x) \propto j_0(x) \rightarrow j_0((1 - \frac{5\lambda}{8\pi^2})x), \]

which reduces

\[ K_2(x) \rightarrow \frac{5\lambda}{4\pi^2}(\kappa' - j_0)(x)\theta(x) \]

up to higher orders in \( \lambda \). However, the latter of the relations in Eq. (15) is now violated at the higher order \( O(\lambda^2) \), which drastically restricts the solvability of the perturbation equations, so further modifications are needed. A natural solution is to change the phase velocity in \( K_2 \) by the same amount as in \( K_1 \) and to adjust the prefactor in \( K_2 \) appropriately. This leads in a unique manner to the remarkably simple result

\[ K^{\text{res.}}(x) = \left( j_0 + \frac{1 - v^2}{v^2}(\kappa' - j_0) \right) (vx)\theta(x). \]  

(22)

Up to and including order \( \lambda \), this is the same as \( K_1 + K_2 \). But now the asymptotic behavior of \( \epsilon_m(x) \) for large \( x \), which is linked to \( K(x) \), is \( \sim \kappa'(vx) \sim \sin(vx) \ln(x)/x \). Thus there is still strong damping of subhorizon-scale oscillations, which is only logarithmically weaker than in the collisionless case. Moreover, the phase velocity \( v \) is now smaller than 1, as one may expect from the comparison with the strongly collisional case of a perfect radiation fluid, where (undamped) acoustic waves oscillate with \( v = 1/\sqrt{3} \). In Fig. 4 the effects of the above resummation are shown for \( x/\pi \gtrsim 1 \).

Similar issues arise also in the case of vector (rotational) as well as tensor perturbations (primordial gravitational waves). For small \( x \), weak self-interactions affect mainly the singular solutions, and the same shift of frequency of the superhorizon oscillation occurs as in Eq. (16): for large \( x \), vector and tensor perturbations differ from the scalar case in that their asymptotic behavior is not changed as much. Subhorizon-scale vector perturbations still have vorticity decaying \( \sim \sin(x)/x \), and resummation changes only the phase velocity; tensor perturbations are stable against the kind of resummation performed above, and have the same asymptotic behavior as the collisionless ones, which indeed is already the same as in the perfect fluid case. All these findings as well as a generalisation to more-component systems will be presented in detail in a forthcoming publication.

**ACKNOWLEDGMENTS**

This work was supported by the Austrian “Fonds zur Förderung der wissenschaftlichen Forschung (FWF)” under projects no. P9005-PHY and P10063-PHY, and by the EEC Programme “Human Capital and Mobility”, contract CHRX-CT93-0357 (DG 12 COMA).
REFERENCES

[1] E. Lifshitz, Zh. Eksp. Teor. Fiz. 16, 587 (1946); E. Lifshitz and I. Khalatnikov, Adv. Phys. 12, 185 (1963).
[2] P. J. E. Peebles and J. T. Yu, Astrophys. J. 162, 815 (1970).
[3] J. Ehlers, in General Relativity and Gravitation, edited by R. K. Sachs (Academic Press, New York, 1971); J. M. Stewart, Non-equilibrium Relativistic Kinetic Theory, (Springer-Verlag, New York, 1971).
[4] J. M. Stewart, Astrophys. J. 176, 323 (1972).
[5] P. J. E. Peebles, Astrophys. J. 180, 1 (1973).
[6] J. R. Bond and A. S. Szalay, Astrophys. J. 274, 443 (1983).
[7] U. Kraemmer and A. Rebhan, Phys. Rev. Lett. 67, 793 (1991); A. Rebhan, Nucl. Phys. B369, 479 (1992).
[8] N. P. Landsman and Ch. G. van Weert, Phys. Rep. 145, 141 (1987).
[9] A. Rebhan, Nucl. Phys. B351, 706 (1991).
[10] F. T. Brandt, J. Frenkel, and J. C. Taylor, Nucl. Phys. B374, 169 (1992); A. P. de Almeida, F. T. Brandt and J. Frenkel, Phys. Rev. D 49, 4196 (1994).
[11] M. Kasai and K. Tomita, Phys. Rev. D 33, 1576 (1986).
[12] A. Rebhan, Astrophys. J. 392, 385 (1992).
[13] D. J. Schwarz, Int. J. Mod. Phys. D 3, 265 (1994).
[14] J. M. Bardeen, Phys. Rev. D 22, 1882 (1980).
[15] M. Abramowitz and I. A. Stegun (eds.), Handbook of Mathematical Functions (Dover Publ., New York, 1965).
[16] A. V. Zakharov, Zh. Eksp. Teor. Fiz. 77, 434 (1979).
[17] E. T. Vishniac, Astrophys. J. 257, 456 (1982).
[18] B. Davies, Integral Transforms and Their Applications, (Springer-Verlag, New York, 1978).
FIGURES

FIG. 1. One- and two-loop diagrams for the energy-momentum tensor. Wavy lines denote external gravitons and straight lines scalar particles.

FIG. 2. The one- and two-loop diagrams for the graviton self-energy up to and including $O(\lambda)$. Only the last diagram contributes to the $\kappa'$-term in $K_2$. 
FIG. 3. The density contrast $|\epsilon_m(x)|$ (in arbitrary normalization). The full line shows the collisionless situation, the dashed line includes self-interactions with $\lambda = 1$. 

|epsilon_m|
FIG. 4. In addition to the lines in Fig. 3, the dotted line shows the solution with the resummed kernel ($\lambda = 1$).