Advanced computer algorithms to simulation of transient dynamics in solids and structures

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Abstract. The new approach of designing computation algorithms for simulation of transient dynamics in solids and structures is proposed. The algorithms are based on explicit finite-difference scheme built under special conditions imposed on the values of difference mesh parameters. The designed calculation devices possess a minimal influence of spurious effects of numerical dispersion that allows discontinuities and high-gradient components caused by wave fronts and fracture events to be accurately computed. A set of advanced computer algorithms and calculation examples of 1D/2D model and engineering problems are presented: (a) longitudinal impact onto a rod embedded into an elastic medium, (b) propagation of cylindrical and spherical waves with front discontinuities, (c) stress concentrations near a head surface of hard projectile indented into elastic solids and (d) transient wave dynamics and delamination in fiber reinforced composites.

1. Introduction

Precise calculation of wave fronts and high-frequency disturbances in solid and structure subjected by impacts is always of utmost importance for problems of numerical simulation of wave processes in solids and structures. Numerical solutions allow to obtain qualitative and quantitative evaluations of the process under study and to explain physical consequences. At the same time, explicit finite-difference scheme (EFDS) used as a rule in contemporary commercial computer hydro-codes come across specific obstacles, which do not allow accurately calculating wave gaps and high-gradient components of solution. Among them, the significant obstacle is the spurious effect caused by the Mesh Dispersion (MD) responsible for the emergence of high-frequency "parasite" oscillations damaged the computer solution.

The studies of the MD in initial-boundary hyperbolical problems have a long-standing history and extensive literature: see, e.g., classical works [1-4] and diverse approaches to the MD control presented in [5-10]. Among them we note the so-called Mesh Dispersion Minimization (MDM) technique, which allows drawbacks caused by the MD to be eliminated or significantly decreased. The MDM approach was initially used in [5] and upgraded in [8] to calculating 1D initial-boundary dynamic processes. In [11], the MDM-algorithm and related calculations were presented for the problem of unsteady diffraction of plane wave on cylindrical shell buried into the elastic medium. Application of MDM algorithms to engineering problems can be found in various fields of wave mechanics: diffraction of elastic waves on buried structures [12], impact driving of piles [13], dynamics and high-speed penetration of layered shields [14, 15], computer modeling in geodynamics and mining [16, 17].
calculations of impact-contact problems [18]. The MDM approach was upgraded in [19] to calculation of linear multidimensional initial-boundary problems, and in [20] – to a non-linear transient process. Note that despite certain achievements in the control of the MD effect and methods intended to its suppression, elimination of the MD or its minimization remains topical.

The algorithms proposed in the present work are constructed on the basis of the MDM technique that is a generalized concept of the Courant condition relating parameters of the finite-difference model to the phase wave velocity, which reflects properties of the material at hand. Difference representation of original differential equations exhibits some typical domains of influence, and the idea behind the MDM is to properly adjust these domains so as to improve convergence. To this end, one has to consider the phase velocities of high-frequency components of the continuous models, and to set the mesh parameters so that the phase velocities induced by them approximate the former as closely as possible. An important technical advantage of the MDM is that it utilizes the same difference mesh for both high-gradient and smoothed solution components.

The aim or the present study is primary to describe the new MDM algorithms developed for calculation of transient processes possessing the pronounced discontinued structure of perturbations and, secondary, to find new precise solutions of stress concentration and crack dynamic that are not obtained up to date.

2. One-dimensional problems

The initial basis of the MDM application is built for 1D problems [5]. First, we consider a set of related examples.

2.1. The wave equation

The simplest example is the wave equation:

\[ ii = c_0^2 u'', \]  

(1)

were \( u \equiv u(x,t) \) is displacement and \( c_0 \) is the sound speed in a waveguide.

We substitute in (1) the Fourier's steady-state representation,

\[ u = \exp \left[ i \omega (ot - qx) \right], \]

\( \omega \) is the frequency, \( q \) is the wave number, which being substituted into (1) results in the Dispersion Relation (DR) defining usually as \( \omega = \omega(q) \):

\[ \omega = c_0 q. \]  

(2)

There phase velocity, \( c = \omega/q = c_0 \), is constant and independent of \( q \) – dispersion is absent.

A EFDS analog of (1) possessing an explicit grid stencil of "cross" type is written as follows:

\[ \left[ u_{m+1}^k - 2u_m^k + u_{m-1}^k / (\Delta x^2) \right]\{ \text{approx } ii \} = c_0^2 \left[ (u_{m+1}^{k+1} - 2u_m^k + u_{m-1}^k) / (\Delta x^2) \right]\{ \text{approx } u'' \}, \]  

(3)

where \( m \) and \( k \) are coordinates of the current grid node ( \( m = 0, \pm 1, \pm 2, \ldots, \) \( k = 0, 1, 2, \ldots, \) \( x = m\Delta x, \) \( t = k\Delta t \)). Substitution in (4) of the Fourier's representation,

\[ u_m^k = U \exp [i(\omega k \Delta t - qm\Delta x)], \]

results in following dispersion relations – the discrete analog of (2):

\[ \omega = (2/\Delta t) \arcsin \left[ \varphi \sin \left( q\Delta x/2 \right) \right], \quad c = \left[ 2/(q\Delta t) \right] \arcsin \left[ \varphi \sin \left( q\Delta x/2 \right) \right], \]

where \( \varphi = c_0 \Delta t / \Delta x \) is the Courant number. Commonly speaking, Eqn. (4) determine the existing of dispersion: the MD of scheme (3). Only long waves (\( \lambda = 2\pi/q \to \infty, q \to 0 \)) turn out be dispersionless: the limiting phase velocity corresponds to that in the continual problem: \( c = c_0 \).
In the considered case, the aim of the MDM is to assign mesh steps resulting in MD suppression. It can be seen, that relation \( \Delta x = c_o \Delta t \) (\( \phi = 1 \)) resulting to the following MDM-algorithm for calculation of the wave equation:

\[
u_{m+1}^{k+1} = u_m^k + u_{m+1}^k - u_{m}^{k+1},
\]

where the MD is eliminated: the same dispersion relation (2) \( \omega = c_o q \) is proved for scheme (3).

First two modes of \( \omega (q) \) are depicted in Figure 1 (a) for a set values of \( \phi \). The smaller \( \phi \), the stronger influence of mesh dispersion. Comparison of the MDM solution \( (\phi = 1) \) with a conventional one \( (\phi = 0.5) \) is shown in Figure 1 (b) and (c) for the wave propagation problem: a thin elastic rod is loaded at the left end by the Heaviside step, \( dx/dx = H(t) \) at \( x = 0 \). Steps \( \Delta x \), \( \Delta t \), and speed \( c_o \) are taken as measurement units. One can see that the MDM solution \( (\phi = 1) \) coincides with the analytical one (the moving step), while the spurious mesh oscillation spoils the solution in case \( \phi = 0.5 \).

![Figure 1](image-url)

Figure 1. (a) mesh dispersion in the EFDS for equation (4), I and II are mode numbers; propagation of Heaviside’s step strain vs. time: (b) \( \phi = 1 \) and (c) \( \phi = 0.5 \); (d) snapshots of pulse stress form at \( t = 200 \); \( \phi = 1 \) (MDM) – solid lines and \( \phi = 0.5 \) – the dotted curve.

2.2. The Klein-Gordon equation

The next example is equation,

\[
\ddot{u} = c_o^2 u'' - gu \quad (g \geq 0),
\]

that is the linear Klein-Gordon equation being originally introduced in quantum physics (see, e.g., [21]) and used at several stages of the wave analysis in classic physics and mechanics. There Eqn. (5) means to describe axial wave propagation in a thin straight rod upon an elastic foundation of constant stiffness \( g \).

Let \( u \equiv u(x,t) \) be the displacement of rod cross-sections, \( g \) be the foundation rigidity, and speed \( c_o \) be the measurement unit. Eqn. (5) possesses the wave dispersion:

\[
\omega = q \sqrt{1 + g/q^2}.
\]

The MDM scheme is built using the nonlocal three-point stencil for term \( gu \) \((x,t)\) in (5):

\[
gu \sim g(u_{i+1}^{k+1} + 2u_i^k + u_{i-1}^k)/4.
\]

With such an approximation, the explicit finite difference analog of Eqn. (5) is as follows:

\[
u_i^{k+1} = 2u_i^k - u_i^{k-1} + \phi^2[(u_{i+1}^{k} - 2u_i^k + u_{i-1}^k) + g(u_{i+1}^k + 2u_i^k + u_{i-1}^k)/4].
\]

It can be readily shown that the approximation order of Eqs. (7) is \((\Delta t)^2 + (\Delta x)^2\), the same as in the case of conventional approximation. For Eqn. (7), the dispersion relation acquires the following form:
\[ c = \pm \frac{2}{q \varphi} \arcsin \left( \varphi \sqrt{\frac{\sin^2 \frac{q}{2} + \frac{\varphi}{4} \cos^2 \frac{q}{2}}}{2} \right). \]  

(8)

If we set \( \varphi = 1 \) in (8) and examine the velocity of extremely short waves of length \( l = 2 \ (q = \pi) \), then there is revealed that such waves propagate in the discrete model with the same velocity \( c = 1 \) as infinitely short waves \( [l \to 0 \ (q \to \infty)] \) in the continual model. So, as in the free waveguide above, \( MD \) is completely eliminated over the entire discrete spectrum.

Compare the computation results related to a transient problem for different values of \( \varphi \). We have calculated the pulse propagation in the case of a semi-infinite system \( (x \geq 0) \) described by Eqn. (5) together with zero initial conditions and the boundary condition \( u' = \sigma(0,t) = -H(t_0 - t) \) at \( x = 0 \) where \( t_0 \) is the pulse length (parameters of the rod serve as measurement units).

**Figure 2.** Snapshots of stress pulse propagation in a straight rod upon elastic foundation \((g = 0.01)\): \((a)\) – the conventional algorithm; \((b)\) – the MDM algorithm

The results depicted in Figure 2 are calculated at \( \Delta x = 1, \ g = 0.01, \ t_0 = 50 \) for two values \( \varphi = 0.9 \) and \( \varphi = 1 \). One can see the essential distortion of the solution at \( \varphi = 0.9 \), while the MDM scheme \((\varphi = 1)\), results in the exact solution in the discrete set of mesh nodes.

2.3. **Inhomogeneous Klein-Gordon equation**

Consider now problem (5) with variable stiffness: \( g = g(x) \). The introduced above three-point approximation (6) within MDM-algorithms functions also in the case of the inhomogeneous foundation. Although the dispersion equation is absent here, the use of the so-called method of frozen coefficients can lead to the goal: if we denoted \( G = \max_x |g(x)| \) and change variables as \( \bar{x} = x \sqrt{G}, \ \bar{t} = t \sqrt{G} \) then the MDM-algorithm (7) with \( \varphi = 1 \) is used.

2.4. **Nonlinear Klein-Gordon equation**

Consider wave propagation processes in a semi-infinite thin rod \((x \geq 0)\) upon a nonlinear foundation. Let boundary and initial conditions be the same that used above, in p. 1.2. The corresponded initial-boundary problem is formulated as follows:

\[ \ddot{u} - c^2 u'' + G(u) u = 0, \ \varepsilon(0,t) = u'(0,t) = -F(t), \ u(x,0) = \dot{u}(x,0) = 0, \]

\[ \varepsilon(0,t) = u'(0,t) = -F(t), \ u(x,0) = \dot{u}(x,0) = 0, \]  

(9)
There $G(u)$ is positive odd function of bounded variation. Consider example where, without loss of generality, function $G(u)$ has the following kind:

$$G(u) = gu\left[1 + g_0u^2\right],$$

which is chosen as an initial part of the Taylor's expansion of function $G(u)$. If $g_0 = 0$, the linear case is explored above.

There is no analytical solution of problem (10), therefore investigate it numerically. Our aim is to design the MDM algorithm for calculation of problem (9)-(10) and to reveal the influence of nonlinearity on the wave propagation process by the computer simulation. Conducted tests show that the MDM representation,

$$G(u) \Rightarrow gU\left(1 + g_0U^2\right), \quad U = \frac{u_{i+1}^k + 2u_i^k + u_{i-1}^k}{4},$$

together with condition $\phi=1$, results in the following dispersionless MDM-algorithm:

$$u_i^{k+1} = u_{i+1}^k + u_{i-1}^k - u_i^{k-1} + gU\left(1 + g_0U^2\right),$$

allowing linear and nonlinear problems to be calculated on the same mesh and by the same accuracy.

In Figure 3 (a,b), snapshots of linear and nonlinear waves patterns of strain are compared at $t=250$. Fig. 3 (b) shows the same results in an extended scale along the $x$ axis near the wave front. The related results show that the fundamental difference in front zone in linear and nonlinear solutions is not found (despite of relatively huge value of $g_0$). This at first sight surprising result can be explained by the fact that the package of high frequency oscillations, generating in the front zone, propagates together with the front of the velocity $c_0$, while the wave package related to the presence of the foundation (and, in this way, the nonlinearity) moves behind the front zone.

2.5. Spatial wave propagation problems

Consider now problems of propagation of center-symmetric (spherical) and axisymmetric (cylindrical) waves in an ideal compressible liquid described by the one-dimensional equation (the bulk compression modulus, the liquid density and sound velocity serve as measurement units):

$$\ddot{\phi} = \phi^s + \left[s/(r - r_0)\right]\phi',$$

where $\phi \equiv \phi(r,t)$ is the potential of the radial velocity in cylindrical ($s=1$) or spherical ($s=2$).
coordinate systems, \( r_0 \) is the radius of the internal transient source. The wave velocity and the pressure are expressed as \( v = \partial \phi / \partial r \) and \( P = -\partial \phi / \partial t \), respectively. We introduce cylindrical/spherical difference mesh of steps \( \Delta r \) and \( \Delta t \). Eqn. (12) under the requirement \( \phi = 1 (\Delta r = \Delta t = 1) \) turn out be expressed by the following MDM-algorithm:

\[
u^{k+1}_m = u^{k+1}_m - u^{k-1}_m + \left[ s/(r_0 + m\Delta r) \right] (u^{k-1}_m - u^{k+1}_m) / 2.
\]

(13)

Consider zero initial conditions and the Heaviside step pressure functioning on cavity surfaces:

\[ P(r_0, t) = H(t) \]

Pressure snapshots obtained from (13) are shown in Figure 4.

![Figure 4. Snapshots of pressure in cylindrical (s = 1) and spherical (s = 2) systems.](image)

Note that the numerical solution in the spherical case coincides with the well known analytical one (see, e.g., [22]), while for the cylindrical case, only the asymptotic solution of the closed form exists in the front vicinity. So, obtained here the MDM numerical solution can be considered as adequate to the exact one.

The algorithms presented in Section 2 can be served as principal parts of the basis to designing MDM-algorithms intended for calculation of different transient processes in solids and structures.

3. Impact indentation of a hard projectile into elastic solids

3.1. MDM algorithms for equations of dynamic theory of elasticity

Plane and axi-symmetric dynamics problems of elastic media we will consider together:

\[
u = c_1^2 u_{xx} + \left[ c_2^2 u_{yy} + \kappa (u_{xy} - \kappa u_y) \right] + (1 - c_2^2) v_{yy},
\]

\[
v = c_2^2 v_{yy} + c_1^2 \left( v_{xx} + \kappa v_{xy} \right) + (1 - c_2^2) \left( u_{xy} + \kappa u_y \right),
\]

\[ c_1 = \sqrt{\left( \lambda + 2\mu \right)/\rho}, \quad c_2 = \sqrt{\mu/\rho}, \]

(14)

where \( y \) is the vertical coordinate. In the plane problem, \( \kappa = 0 \), \( x \) and \( u(x, y, t) \) are the horizontal coordinate and displacement, while in the axi-symmetric problem, \( \kappa = 1/x \), \( x \) and \( v(x, y, t) \) – are the radial coordinate and displacement; \( c_1 \) and \( c_2 \) are longitudinal and shear velocities, \( \lambda \) and \( \mu \) are Lame’s parameters and \( \rho \) – is the density of the solid. Besides, in transient problems, external forces, boundary and initial conditions are to be added to system (14).

As was shown in [19] for rectangular coordinates, the explicit difference analogue of (14) transformed into MDM scheme with equalities \( \Delta x = \Delta y = \Delta \) for spatial mesh steps and \( \Delta t = \Delta / c_1 \) for the temporal
step. Besides, we have built a special difference approximation of derivations in (14). For temporal derivations, first and mixed spatial derivations, the standard central differences are used,
\[ \ddot{w} \sim \frac{(w_{i,j}^{k+1} - 2w_{i,j}^k + w_{i,j}^{k-1})}{(\Delta t)^2}, \quad \dot{w}_x \sim \frac{(w_{i+1,j}^{k+1} - w_{i-1,j}^{k+1})}{2\Delta t}, \quad \dot{w}_y \sim \frac{(w_{i,j+1}^{k+1} - w_{i,j}^{k+1})}{2\Delta t}, \]
\[ w_{x,y} = \frac{(w_{i+1,j+1}^k - w_{i-1,j-1}^k - w_{i+1,j-1}^k + w_{i-1,j+1}^k)}{4\Delta^2}, \]  
(15)
where \( w \sim \{u,v\} \), while for second spatial derivatives \( w_{xx} \) and \( w_{yy} \), we use the structure like to the mentioned above three-point approximation. In the first equation (14) we assume
\[ u_{rr}^* \sim \left( u_{i,j+1}^k - 2u_{i,j}^k + u_{i,j-1}^k \right)/\Delta^2, \quad v_{rr}^* \sim \left( v_{i,j+1}^k + 2v_{i,j}^k + v_{i,j-1}^k \right)/4, \quad \phi_{i,j}^k = \left( v_{i,j+1}^k - 2v_{i,j}^k + v_{i,j-1}^k \right)/\Delta^2, \]
\[ v_{zz}^* \sim \left( v_{i,j+1}^k - 2v_{i,j}^k + v_{i,j-1}^k \right)/\Delta^2, \quad u_{zz}^* \sim \left( U_{i,j+1}^k + 2U_{i,j}^k + U_{i,j-1}^k \right)/4, \quad U_{i,j}^k = \left( u_{i+1,j}^k - 2u_{i,j}^k + u_{i-1,j}^k \right)/\Delta^2. \]
After substituting approximations (15), (16) into Eqns. (14), we obtain the desired MDM-algorithms. Note that the approximation order of the obtained algorithm remains the same as for its the conventional analog: \((\Delta t)^2 + 2\Delta^2\). Looking ahead let’s say that wave fronts propagating in parallel to coordinate axes are precisely calculated as in the one-dimensional MDM-scheme above.

### 3.2. Problem formulation of the penetration event

We have presented the formulation of the considered impact-contact problem together for plane and axisymmetrical problems (see Figure 5) as follows: at time \( t = 0 \), a rigid indenter of mass \( M \) and velocity \( V_0 \) is normally impacted onto the free surface \( y = 0 \) of the elastic layer of thickness \( h \) and begins to press into it. We refer the layer to plane/cylindrical coordinates \( x, y(r, y) : -\infty < x < \infty / 0 \leq r < \infty \), and \( 0 \leq y \leq h \). The value of \( V_0 \) is assumed to be much less than speeds of elastic waves, and the indentation depth is so small that the linear model could be justified. So, a linear problem of theory elasticity with the boundary conditions valid for the undisturbed upper surface of the layer \( y = 0 \).

![Figure 5. The geometry of the problem](image)

We add to equations (14) zero initial conditions and the following boundary conditions:
\[ y = 0: \quad \sigma_{yy} = 0; \quad \psi(x, 0, t) = t\dot{Z}(t) \left( x| \leq l / r \leq R \right), \quad \sigma_{yy} = 0 \left( |x| > l / r > R \right) \]
\[ y = h: \quad \sigma_{yy} = 0, \quad v = 0. \]
(16)
where \( Z(t) \) is the current indentation depth obtained from the equation of indenter motion,
\[ \ddot{Z}(t) = -\frac{F(t)}{M}, \quad Z(0) = 0, \quad \dot{Z}(0) = V_0, \]
(17)
while for normalized resistant force \( F(t) \) – drag we have the following expressions:
\[
F(t) = \frac{1}{l} \int_0^1 \sigma_{yy} (x, 0, t) \, dx \quad \text{or} \quad F(t) = \frac{1}{2R^2} \int_0^R \sigma_{yy} (r, 0, t) \, dr.
\]

(18)

In examples below, the measurement units are: \( l/R, \, c_1, \, \rho \), while \( c_2 = 0.5 \) is everywhere taken. We consider compressive stresses as positive values.

As was shown in [23] for the plane problem, normal stresses in the vicinity of the singular point for the semi-infinite indenter rise asymptotically as \( \sigma_{yy} (X, t) \bigg|_{y \to 0} \sim \ln t/\sqrt{1 - X^2} \quad (t \to \infty) \), where \( X \) is the distance from the indenter edge where \( X = 1 \). The root dependence in the dominator is the same as in the static problem, while the logarithmic rise with time describes the dynamic effect.

3.3. Results of computer modeling

With the formulation above, we have precisely obtained of stress fields developed with time along with the indentation process including stress peaks in singular points. Some examples of stress distributions in the case of the unbounded elastic halfspace and the constant velocity of indentation (\( \dot{Z}(t) = V_0 = 1 \)) corresponding to the infinite projectile mass, are presented in Figures 6 and 7, while the influence of reflections in the case of finite value of \( h \) can be seen in Figure 8. The accounting the resistance to penetration and indenter deceleration is shown in Figure 9.

Apart from the main aim – to obtain and analyze the stress pattern – these results are intended to show the advantage of the MDM device to the analysis of discontinuous patterns. As to the mentioned above conventional EFDS, its results in the vicinities of the singularities, as calculation show, have nothing in common with the presented below in Figures 6-8.

Figure 6. The plane problem. Distribution of stresses \( \sigma_{yy} (x, y) \) and \( \sigma_{xx} (x, y) \) along \( x \) in cross-sections: \( y = 0, 0.25 \) and \( 0.5 \) (bold, thin and dashed curves) at \( t = 0.5, 1, 2 \)

Figure 7. The axisymmetric problem: Distributions of normal stresses \( \sigma_{yy} (r, y) \) along the radial coordinate at time moments \( t = 0.5, 1.0 \) and \( 2.0 \). Curve 1 corresponds to the interface \( y = 0 \), curves 2, 3 and 4 correspond to cross-sections \( y = 0.05, 0.1 \) and \( 1.0 \)

In Figure 8, the stresses \( \sigma_{yy} (x,0) \) in a set points in the impact plane of the plane indenter \( x = 0, 0.5, 0.9, 0.95, 0.99 \) and \( x = 1 \) (the singular point) together with drag \( F \) vs. time are shown in cases of indentation into the half plane (a) and the slab (b). In the case (a), the distribution of stresses in the edge vicinity approaches with time to the asymptotic form [23]. Influence of the singular point \( x = 1 \) results in a
significant growth of stress amplitudes with time, while reflections significantly intensify this process. Sharp discontinuities can be seen at moments of incoming of reflected waves to the contact line. Due to the assumed infinite mass of the indenter, the overall dynamic compression of a slab of finite thickness becomes prevailing with time that the logarithmic growth is transformed into a quasi-linear growth wherein contributions of reflection gaps bit by bit decrease. This process the more noticeable the closer the current x-section to the singular point.

Figure 8. The plane problem: (a) half-plane, (b) layer of h = 1. Black curves are normal stresses $\sigma_{yy}$ $(x,0)$ vs. time, red lines – drag

In the case of the finite mass, the influence of singularities results in the significant but finite rise of stresses – see Figure 9. The greater the mass, the greater stresses.

Figure 9. The contact stresses $\sigma_{yy} (x,0)$ vs. time for several values of the indenter mass M. Curves 1, 2, 3, 4 and 5 corresponded to points $x = 0, 0.5, 0.9, 0.95 \text{ and } 1.0$; curves with solid circles are normalized indentation velocities, while curves with solid squares show drag

The data presented in Figure 9 make it possible to evaluate the applicability of the calculation results in practice. Suppose, for example, $c_1 = 5000 m/s$, and the linear model can be applicable at stresses $\sim 10^{-3} (\lambda + 2\mu)$. Then the above results for $t \leq 1$ can be used if $V_0 \leq 100 m/s$, while in the case for $t \leq 2$ – if $V_0 \leq 50 m/s$.

4. Simulation of dynamic delamination in fiber reinforced composites

The process of stress concentration and stepwise crack propagation at fiber-adhesive interfaces is numerically simulated. Note, some aspect of delamination phenomena in layered structures were discussed in [24], while the simultaneous fracture of fibers and matrix was considered in [25]. Below, the fracture of the composite caused by intense shear stresses at interfaces is studied. The problem statement corresponds to the following conditions: the material is stretched along the fibers at infinity by a constant tensile stress $\sigma_x$ (see Figure 10); here $E$ and $G$ are the Young module in fibers and the
shear module in adhesive respectively, $\rho$ and $\rho_a$ are densities, $h$ and $H$ are thicknesses. At zero moment of time ($t = 0$), one of the fibers (say it be fiber of number 0) starts to fail due to some defects; The fractured fiber starts to unload, and the intact ones start to load up due to action of shear stress waves propagating in the adhesive in the right and in the left. Along with this process, fibers thought to be intact, while the delamination events can happen depending on the strength of the adhesive.

The used model of the fiber dynamics describes the one-dimensional wave process in a thin rod embedded into adhesive, which represented as inertial bonds perceived shear stresses (tension-compression stresses in bonds are neglected). Such a theoretical treatment of the components performance can be justified by the fact that the shear modulus of adhesive is much less than that of fiber (see e.g. [26]), while their stretches have roughly the same level due to the cohesion of the fibers and the adhesive. When maximal stresses reached in adhesive do not exceed the strength limits ($\tau_m < \tau'$), the adhesive remains intact. The crack propagation is investigated on the basis of linear elastic fracture mechanics: if $\tau_m \geq \tau'$, fracture in adhesive is initiated and then propagated deep into the composite up to their stop due to continuous scattering of the initial impact energy with time.

Let $E$ and $G$ be the Young module in fibers and the shear module in adhesive respectively, $\rho$ and $\rho_a$ be densities, $h$ and $H$ be thicknesses of components (see figure 10), $c = \sqrt{E/\rho}$ and $c_a = \sqrt{G/\rho_a}$ be sound speeds in fibers and adhesive, respectively. Fiber constants $E, \rho, h$ serve as measurement units.

**4.1. Mathematical formulation**

In the mathematical sense, we met a non-linear hyperbolic problem possessing non-classical boundary conditions. Due to the natural symmetry, a quarter of plane $x, y$ is considered in the calculation algorithm (let it be $x \geq 0, y \geq 0$). Displacements and strains in fiber at the static state ($t < 0$) are

$$ u_m(y) = y \sigma_m/E, \quad \varepsilon_m(y) = \sigma_m/E; \quad u_m(X, y) = 0 \quad (m = 0, \pm 1, \pm 2, \ldots),$$

where $\varepsilon_m(y) = \partial u_m/\partial y$ is the strain in $m^{th}$ fiber ($m \neq 0$), and the fracture event of fiber $m = 0$ at $t = 0$ changes (19) by adding condition $\varepsilon_0(0, t) = 0$:

$$\varepsilon_0(0) = 0; \quad \varepsilon_m(y) = \sigma_m/E \quad (m \neq 0, y \neq 0), \quad \varepsilon_m(X, y) = 0 \quad (m = 0, \pm 1, \pm 2, \ldots).$$

Reformulate the problem for the additional dynamic state subtracting the static strains (19) from (20). Then boundary conditions for strains in fibers are the following:

$$y = 0: \quad \varepsilon_m(0, t) = (\partial u_m/\partial y)_{y=0} - \sigma_m/E, \quad \varepsilon_m(0, t) = (\partial u_m/\partial y)_{y=0} = 0.$$  

The motion of fibers is described by the system of 1D wave equations
\[ \rho \ddot{u}_m = E\nu u^"_{m,y} + \tau^r_m(y) - \tau^l_m(y), \quad m = 0,\pm1,\pm2,\ldots \]  \hspace{1cm} (22)

where \( \tau^r_m \) and \( \tau^l_m \) correspond to reactive shear forces at the fiber-adhesive interface on the right and the left, respectively:

\[ \tau^r_m = -G\nu\frac{\partial u'}{\partial x} \bigg|_{x=0}, \quad \tau^l_m = G\nu\frac{\partial u'}{\partial x} \bigg|_{x=H} \quad (m > 0) \]  \hspace{1cm} (23)

while displacements in adhesive described by wave equations

\[ \frac{\partial^2 \nu_m}{\partial x^2} = c^2 \frac{\partial^2 \nu_m}{\partial t^2} \quad (0 \leq X \leq H), \quad m = 0,1,2,\ldots \]  \hspace{1cm} (24)

with the following boundary conditions:

\[ \nu_m(0,y,t) = u_m(y,t), \quad \nu_m(H,y,t) = u_{m+1}(y,t). \]  \hspace{1cm} (25)

Then the following additional relations in expressions of reactive forces (23) and boundary conditions (25) are:

\[ \tau_m^r(\xi^+,y,t) \geq \tau^r \Rightarrow \tau_{m+}^r = \tau; \quad t > \tau_{m+}^r \quad : \quad \tau_m^r(\xi^+,y,t) = 0, \quad \partial \nu_m^r(\xi^+,y,t) / \partial X = 0, \]  \hspace{1cm} (26)

where \( \xi^+ = 0, \xi^- = H \), while indices “*” at \( \tau^*_m \) denote right and left interfaces, respectively.

4.2. The MDM calculation algorithm

It is evident that each possible scenario of wave-fracture pattern is saturated by reflected waves with discontinuities appeared due to adhesive cracking. Our goal is to calculate such processes as precise as possible. Below we present the practical calculation device based on the MDM technique allowing this goal to be reached. Let mesh step in adhesive be \( \Delta x \).

Then the MDM condition are \( \Delta t = \Delta y = 1 \), \( \Delta x = c_\alpha \), and the adhesive dynamics, as the analogue of Eqn. (24), described by the following MDM-algorithm:

\[ v_{m,j,i}^{s+1} = v_{m,j,i}^{s+1} + v_{m,j,i-1}^{s-1} - v_{m,j,i}^{s-1} \quad (0 \leq i \leq s = H/\Delta x), \]  \hspace{1cm} (27)

while the difference analogue of continual Eqn. (22) becomes as the MDM-algorithm:

\[ u_{m,j}^{s+1} = u_{m,j}^{s+1} + u_{m,j-1}^{s-1} - u_{m,j}^{s-1} + \kappa F, \quad F = \left( v_{m,j,i}^{s+1} + v_{m,j,i}^{s-1} \right) \quad (m > 0), \quad F = 2v_{i,j}^k \quad (m = 0), \]  \hspace{1cm} (28)

\[ \kappa = GH / (E\nu \mu), \quad \mu = \alpha x + \Delta y, \quad \alpha = \rho \Delta x / \rho H. \]

Let us turn to examples. First, the shear stress pattern is in the interfaces \( X = 0 \) and \( X = H \) at \( y = 0 \) shown in Figure 11 (right image). It is convenient to relate shear stresses to its static limit, \( \tau_s \), in the corresponding linear problem [27]. We note that peak amplitudes (the peaks appears with the period \( t = 2\pi H c_\alpha \) due to reflections of shear waves) can be much more than \( \tau_s \) and do not change with time although their timelife decreases). These peaks play the main role in the fracture initiation.

**Figure 11.** (right image) Shear stresses in adhesive (\( \alpha = 2 \)) and (left image) Delamination pattern at \( \alpha = 20 \)

In Figure 11 (left image), the development of shear cracks along \( y \) occurs in two interfaces: \( j = 0 - (X = 0) \) and
shear waves propagated in the adhesive. Fiber reinforced composite plates
at the beginning of indentation processes into elast
obtained for a set of linear and non-linear 1D transient wave problems.
The precise parametric analysis is conducted of the contact-impact problem of hard projectile indentation into elastic media. The results of the stress development with time in the vicinity of the projectile head at the beginning of indentation processes can be used in practical calculations.
4. The MDM-algorithms built for exploring the stress concentration and progressive delamination of fiber reinforced composite plates allow one to analyze fracture development caused by discontinuities of shear waves propagated in the adhesive.

5. Conclusions
1. Designed MDM-algorithms allows step-wise character of wave processes to be precisely revealed.
2. The precise MDM-solutions described front propagation patterns in media and structures are obtained for a set of linear and non-linear 1D transient wave problems.
3. The parametric analysis is conducted of the contact-impact problem of hard projectile indentation into elastic media. The results of the stress development with time in the vicinity of the projectile head at the beginning of indentation processes can be used in practical calculations.
4. The MDM-algorithms built for exploring the stress concentration and progressive delamination of fiber reinforced composite plates allow one to analyze fracture development caused by discontinuities of shear waves propagated in the adhesive.

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