Optimal three-ball inequalities and quantitative uniqueness for the Lamé system with Lipschitz coefficients

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Abstract

In this paper we study the local behavior of a solution to the Lamé system with Lipschitz coefficients in dimension $n \geq 2$. Our main result is the bound on the vanishing order of a nontrivial solution, which immediately implies the strong unique continuation property. This paper solves the open problem of the strong uniqueness continuation property for the Lamé system with Lipschitz coefficients in any dimension.

1 Introduction

Assume that $\Omega$ is a connected open set containing 0 in $\mathbb{R}^n$ for $n \geq 2$. Let $\lambda(x)$ and $\mu(x)$ be Lamé coefficients in $C^{0,1}(\Omega)$ satisfying

\[
\begin{align*}
\mu(x) & \geq \delta_0 > 0, \\
\lambda(x) + 2\mu(x) & \geq \delta_0 > 0 \quad \forall \ x \in \Omega, \\
\|\mu(x)\|_{C^{0,1}(\Omega)} + \|\lambda(x)\|_{C^{0,1}(\Omega)} & \leq M_0.
\end{align*}
\]  

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The isotropic elasticity, which represents the displacement equation of equilibrium, is given by

\[
\text{div}(\mu(\nabla u + (\nabla u)^t)) + \nabla(\lambda \text{div} u) = 0 \quad \text{in } \Omega, \tag{1.2}
\]

where \(u = (u_1, u_2, \cdots, u_n)^t\) is the displacement vector and \((\nabla u)_{jk} = \partial_k u_j\) for \(j, k = 1, 2, \cdots, n\).

Results on the weak unique continuation for the Lamé system in \(\mathbb{R}^n, n \geq 2\), have been proved by Dehman and Robbiano for \(\lambda(x), \mu(x) \in C^\infty(\Omega)\) [3], Ang, Ikehata, Trong and Yamamoto for \(\lambda \in C^2(\Omega), \mu(x) \in C^3(\Omega)\) [2], Weck for \(\lambda(x), \mu(x) \in C^2(\Omega)\) [17], and Eller for \(\lambda(x), \mu(x) \in C^1(\Omega)\) [6]. As for the SUCP, it was proven by Alessandrini and Morassi [1] in the case of \(\lambda(x), \mu(x) \in C^{1,1}(\Omega)\) and \(n \geq 2\). Their proofs were based on ideas developed by Garofalo and Lin [4], [5]. When Lamé coefficients are Lipschitz, i.e., \(\lambda, \mu \in C^{0,1}(\Omega)\), the SUCP was established by the first and third authors in [11] for \(n = 2\). Later, the result of [11] was improved to \(\mu \in C^{0,1}(\Omega)\) and \(\lambda\) being measurable by Escauriaza [7]. In this work, we completely resolve the SUCP problem for (1.2) when \(\lambda, \mu \in C^{0,1}(\Omega)\) and \(n \geq 2\). It is important to remark that in the three or higher dimensions, the Lipschitz regularity assumption on the principal coefficients of the second order elliptic equation is the minimal requirement for the unique continuation property to hold [15]. Not only do we solve the SUCP for the Lamé system with the minimal regularity assumption, we also derive a quantitative form of the SUCP.

The ideas of our proof originate from our series papers on proving quantitative uniqueness for elliptic equations or systems by the method of Carleman estimates [12], [13], and [14]. In particular, the idea used in [14] plays a key role in our arguments here. Specifically, let us write (1.2) into a non-divergence form:

\[
\mu \Delta u + (\lambda + \mu) \nabla \text{div} u + \nabla \lambda \text{div} u + (\nabla u + (\nabla u)^t) \nabla \mu = 0. \tag{1.3}
\]

Letting \(p = \text{div} u\), taking divergence on (1.3), and using (1.3) for \(\Delta u\), yields

\[
(\lambda + 2\mu) \Delta p + (\nabla \lambda - \mu^{-1}\lambda \nabla \mu) \cdot \nabla p - (\mu^{-1}\nabla \mu \cdot \nabla \lambda) \text{div} u
\]
\[
-\mu^{-1}\nabla \mu \cdot ((\nabla u) + (\nabla u)^t) \nabla \mu + \text{div} \left( \nabla \lambda \text{div} u + (\nabla u + (\nabla u)^t) \nabla \mu \right)
\]
\[
= 0. \tag{1.4}
\]

From (1.3) and (1.4), we then obtain a system of equations with the Laplacian.
as the principal part, namely,

\[
\begin{aligned}
\Delta u + P_1(x, \partial) p + P_2(x, \partial) u &= 0, \\
\Delta p + Q_1(x, \partial) p + Q_2(x, \partial) u + \text{div} G(x, u) &= 0,
\end{aligned}
\]

(1.5)

where \( P_j(x, \partial), Q_j(x, \partial), j = 1, 2 \), are first order differential operators with at least essentially bounded coefficients and

\[
G(x, u) = (\lambda + 2\mu)^{-1} \left( \nabla \lambda \text{div} u + (\nabla u + (\nabla u)^t) \nabla \mu \right).
\]

Note that system (1.5) is not decoupled.

To study the unique continuation problem for (1.2), it suffices to consider that for (1.5) with \( p = \text{div} u \). To tackle this problem, we rely on suitable Carleman estimates. An important step is to handle the second equation of (1.5). The trick is to use a Carleman estimate with the divergence operator on the right hand side (see Lemma 2.4). This idea was first introduced in [9] and later used in [16] and [14]. In order to derive an upper bound on the vanishing order of a nontrivial solution to (1.2), it is also important to derive optimal three-ball inequalities.

We now state main results of the paper. Their proofs will be given in the subsequent sections. Assume that there exists \( 0 < R_0 \leq 1 \) such that \( B_{R_0} \subset \Omega \). Hereafter \( B_r \) denotes an open ball of radius \( r > 0 \) centered at the origin.

**Theorem 1.1** There exists a positive number \( \bar{R} < 1 \), depending only on \( n, M_0, \delta_0 \), such that if \( 0 < R_1 < R_2 < R_3 \leq R_0 \) and \( R_1/R_3 < R_2/R_3 < \bar{R} \), then

\[
\int_{|x|<R_2} |u|^2 dx \leq C \left( \int_{|x|<R_1} |u|^2 dx \right)^\tau \left( \int_{|x|<R_3} |u|^2 dx \right)^{1-\tau},
\]

(1.6)

for \( u \in H^1_{\text{loc}}(B_{R_0}) \) satisfying (1.2) in \( B_{R_0} \), where the constant \( C \) depends on \( R_2/R_3, n, M_0, \delta_0 \), and \( 0 < \tau < 1 \) depends on \( R_1/R_3, R_2/R_3, n, M_0, \delta_0 \). Moreover, for fixed \( R_2 \) and \( R_3 \), the exponent \( \tau \) behaves like \( 1/(-\log R_1) \) when \( R_1 \) is sufficiently small.

**Remark 1.2** We would like to emphasize that \( C \) is independent of \( R_1 \) and \( \tau \) has the asymptotic \( (-\log R_1)^{-1} \). These facts are crucial in deriving an vanishing order of a nontrivial \( u \) to (1.1). Due to the behavior of \( \tau \), the three-ball inequality is called optimal [8].
Theorem 1.3 Let \( u \in H^1(\Omega) \) be a nontrivial solution of (1.2), then there exist positive constants \( K \) and \( m \), depending on \( n, M_0, \delta_0 \) and \( u \), such that

\[
\int_{|x|<R} |u|^2 dx \geq KR^m
\]

for all \( R \) sufficiently small.

Remark 1.4 Based on Theorem 1.1, the constants \( K \) and \( m \) in (1.7) are explicitly given by

\[
K = \int_{|x|<R_3} |u|^2 dx
\]

and

\[
m = \tilde{C} \log \left( \frac{\int_{|x|<R_3} |u|^2 dx}{\int_{|x|<R_2} |u|^2 dx} \right),
\]

where \( \tilde{C} \) is a positive constant depending on \( n, M_0, \delta_0 \) and \( R_2/R_3 \).

2 Carleman estimates

In this section, we will derive two Carleman estimates. The first one is taken from [14]. Denote \( \varphi_\beta = \varphi_\beta(x) = \exp(-\beta \tilde{\psi}(x)) \), where \( \beta > 0 \) and \( \tilde{\psi}(x) = \log |x| + \log((\log |x|)^2) \). Note that \( \varphi_\beta \) is less singular than \( |x|^{-\beta} \). For simplicity, we denote \( \psi(t) = t + \log t^2 \), i.e., \( \psi(x) = \psi(\log |x|) \). From now on, the notation \( X \lesssim Y \) or \( X \gtrsim Y \) means that \( X \leq CY \) or \( X \geq CY \) with some constant \( C \) depending only on \( n \).

Lemma 2.1 [14] Lemma 2.1 There exist a sufficiently small \( r_0 > 0 \) depending on \( n \) and a sufficiently large \( \beta_0 > 1 \) depending on \( n \) such that for all \( u \in U_{r_0} \) and \( \beta \geq \beta_0 \), we have that

\[
\beta \int \varphi_\beta^2(\log |x|)^{-2} |x|^{-n}(|x|^2|\nabla u|^2 + |u|^2) dx \lesssim \int \varphi_\beta^2 |x|^{-n} |x|^4 |\Delta u|^2 dx, \quad (2.1)
\]

where \( U_{r_0} = \{ u \in C_0^\infty(\mathbb{R}^n \setminus \{0\}) : \text{supp}(u) \subset B_{r_0} \} \).

To prove the second Carleman estimate, we need some preparations. Firstly, we introduce polar coordinates in \( \mathbb{R}^n \setminus \{0\} \) by setting \( x = r \omega \), with
$r = |x|$, $\omega = (\omega_1, \cdots, \omega_n) \in S^{n-1}$. Using new coordinate $t = \log r$, we can see that

$$\frac{\partial}{\partial x_j} = e^{-t}(\omega_j \partial_t + \Omega_j), \quad 1 \leq j \leq n,$$

where $\Omega_j$ is a vector field in $S^{n-1}$. We could check that the vector fields $\Omega_j$ satisfy

$$\sum_{j=1}^n \omega_j \Omega_j = 0 \quad \text{and} \quad \sum_{j=1}^n \Omega_j \omega_j = n - 1.$$

Since $r \to 0$ iff $t \to -\infty$, we are mainly interested in values of $t$ near $-\infty$.

It is easy to see that

$$\frac{\partial^2}{\partial x_j \partial x_\ell} = e^{-2t}(\omega_j \partial_t - \omega_j + \Omega_j)(\omega_\ell \partial_t + \Omega_\ell), \quad 1 \leq j, \ell \leq n.$$

and, therefore, the Laplacian becomes

$$e^{2t} \Delta = \partial_t^2 + (n-2)\partial_t + \Delta_\omega,$$

where $\Delta_\omega = \sum_{j=1}^n \Omega_j^2$ denotes the Laplace-Beltrami operator on $S^{n-1}$. We recall that the eigenvalues of $-\Delta_\omega$ are $k(k + n - 2), k \in \mathbb{N}$, and the corresponding eigenspaces are $E_k$, where $E_k$ is the space of spherical harmonics of degree $k$. Let

$$\Lambda = \sqrt{\frac{(n-2)^2}{4} - \Delta_\omega},$$

then $\Lambda$ is an elliptic first-order positive pseudodifferential operator in $L^2(S^{n-1})$. The eigenvalues of $\Lambda$ are $k + \frac{n-2}{2}$ and the corresponding eigenspaces are $E_k$ which represents the space of spherical harmonics of degree $k$. Hence

$$\Lambda = \Sigma_{k \geq 0} (k + \frac{n-2}{2})\pi_k,$$

(2.2)

where $\pi_k$ is the orthogonal projector on $E_k$. Denote

$$L^\pm = \partial_t + \frac{n-2}{2} \pm \Lambda.$$

Then it follows that

$$e^{2t} \Delta = L^+ L^- = L^- L^+.$$

We first recall a Carleman estimate proved in [14, Lemma 2.2].
Lemma 2.2 There exists a sufficiently small number \( t_0 < 0 \) depending on \( n \) such that for all \( u \in V_{t_0}, \beta > 1 \), we have that
\[
\sum_{j+|\alpha| \leq 1} \beta^{1-2(j+|\alpha|)} \int t^{-2} \phi_\beta^2 \partial_t^j \Omega^\alpha u^2 dt d\omega \lesssim \int \int \phi_\beta^2 |L^- u|^2 dt d\omega, \tag{2.3}
\]
where \( V_{t_0} = \{ u(t, \omega) \in C_0^\infty((\infty, t_0) \times S^{n-1}) \} \).

Next, we need an auxiliary Carleman estimate.

Lemma 2.3 There exists a sufficiently small number \( t_1 < -2 \) depending on \( n \) such that for all \( u \in V_{t_1}, g = (g_0, g_1, \cdots, g_n) \in (V_{t_1})^{n+1} \) and \( \beta > 1 \), we have that
\[
\beta \int \int \phi_\beta^2 |u|^2 dt d\omega \lesssim \int \int \phi_\beta^2 (|L^+ u + \partial_t g_0 + \sum_{j=1}^n \Omega_j g_j|^2 + \beta \| g \|^2) dt d\omega. \tag{2.4}
\]

Proof. We shall prove this lemma following the lines of [16, Lemma 2.2]. By defining \( u = e^{\beta \psi(t)} v \) and \( g = e^{\beta \psi(t)} h \), (2.4) is equivalent to
\[
\beta \int \int |v|^2 dt d\omega \lesssim \int \int (|L^+_\beta v + (\partial_t + \beta \psi') h_0 + \sum_{j=1}^n \Omega_j h_j|^2 + \beta \| h \|^2) dt d\omega. \tag{2.5}
\]
where \( L^+_\beta v = e^{-\beta \psi(t)} L^+(e^{\beta \psi(t)} v) \) and \( h = (h_0, \cdots, h_n) \). By direct computations, we obtain that
\[
L^+_\beta v + (\partial_t + \beta \psi') h_0 + \sum_{j=1}^n \Omega_j h_j
= \partial_t v + \beta v + 2\beta t^{-1} v + \frac{(n-2)}{2} v + \Lambda v + (\partial_t + \beta \psi') h_0 + \sum_{j=1}^n \Omega_j h_j
= T_\beta v + 2\beta t^{-1} v + (\partial_t + \beta \psi') h_0 + \sum_{j=1}^n \Omega_j h_j \tag{2.6}
\]
with \( T_\beta v = \partial_t v + \beta v + \frac{(n-2)}{2} v + \Lambda v \).

For \( v \in C_0^\infty(\mathbb{R} \times S^{n-1}) \), we denote \( \hat{v} \) its Fourier transformation with respect to \( t \), then it follows from (2.2) that
\[
T_\beta v(t, \omega) = (2\pi)^{-1} \sum_{k \geq 0} \int_{-\infty}^\infty e^{i\sigma} (i\sigma + \beta + k + n - 2) \pi_k \hat{v}(\sigma, \omega) d\sigma. \tag{2.7}
\]
It is easily seen that $T_\beta$ is invertible whose inverse is given by

$$T^{-1}_\beta v(t, \omega) = (2\pi)^{-1} \sum_{k \geq 0} \int_{-\infty}^{\infty} e^{i\sigma t} (i\sigma + \beta + k + n - 2)^{-1} \pi_k \hat{v}(\sigma, \omega) d\sigma. \quad (2.8)$$

From (2.7), (2.8) and Plancherel’s theorem, we have for $u \in C_0^\infty (\mathbb{R} \times \mathbb{S}^{n-1})$ that

$$\begin{cases}
\|T_\beta u\|_{L^2(\mathbb{R} \times \mathbb{S}^{n-1})} \geq \beta\|u\|_{L^2(\mathbb{R} \times \mathbb{S}^{n-1})}, \\
\beta\|T^{-1}_\beta u\|_{L^2(\mathbb{R} \times \mathbb{S}^{n-1})} \leq \|u\|_{L^2(\mathbb{R} \times \mathbb{S}^{n-1})}, \\
\|T^{-1}_\beta (\sum_{j+|\alpha| \leq 1} \partial_t^j \Omega^\alpha u)\|_{L^2(\mathbb{R} \times \mathbb{S}^{n-1})} \lesssim \|u\|_{L^2(\mathbb{R} \times \mathbb{S}^{n-1})}.
\end{cases} \quad (2.9)$$

Combining (2.6) and (2.9), we get that

$$\|L^+_\beta v + (\partial_t + \beta \psi') h_0 + \sum_{j=1}^{n} \Omega^j h_j\|_{L^2(\mathbb{R} \times \mathbb{S}^{n-1})}$$

$$= \|T_\beta v + 2\beta t^{-1} v + (\partial_t + \beta \psi') h_0 + \sum_{j=1}^{n} \Omega^j h_j\|_{L^2(\mathbb{R} \times \mathbb{S}^{n-1})}$$

$$\geq \beta\|v + T^{-1}_\beta (2\beta t^{-1} v + (\partial_t + \beta \psi') h_0 + \sum_{j=1}^{n} \Omega^j h_j)\|_{L^2(\mathbb{R} \times \mathbb{S}^{n-1})}$$

$$\geq \beta\|v\|_{L^2(\mathbb{R} \times \mathbb{S}^{n-1})} - \beta\|T^{-1}_\beta (2\beta t^{-1} v + (\partial_t + \beta \psi') h_0 + \sum_{j=1}^{n} \Omega^j h_j)\|_{L^2(\mathbb{R} \times \mathbb{S}^{n-1})}$$

$$\geq \beta\|v\|_{L^2(\mathbb{R} \times \mathbb{S}^{n-1})} - 2\beta\|t^{-1} v\|_{L^2(\mathbb{R} \times \mathbb{S}^{n-1})}$$

$$- \beta\|T^{-1}_\beta ((\partial_t + \beta \psi') h_0 + \sum_{j=1}^{n} \Omega^j h_j)\|_{L^2(\mathbb{R} \times \mathbb{S}^{n-1})}$$

$$\geq \beta\|v\|_{L^2(\mathbb{R} \times \mathbb{S}^{n-1})} - C\beta\|h\|_{L^2(\mathbb{R} \times \mathbb{S}^{n-1})}. \quad (2.10)$$

In deriving (2.10), we have used the fact that $v \in V_{t_1}$ with $t_1 < -2$. Now dividing $\sqrt{\beta}$ on both sides of (2.10) and squaring the new inequality, we have
that
\[
\beta \int \int |v|^2 dt \omega 
\]
\[
\lesssim \int \int \beta^{-1} (|L^+_\beta v + (\partial_t + \beta \psi') h_0 + \sum_{j=1}^n \Omega_j h_j|^2 + \beta^2 \|h\|^2) dt \omega 
\]
\[
\lesssim \int \int (|L^+_\beta v + (\partial_t + \beta \psi') h_0 + \sum_{j=1}^n \Omega_j h_j|^2 + \beta \|h\|^2) dt \omega. \tag{2.11}
\]

The proof is complete. \(\square\)

Now we are ready to prove our second Carleman estimate.

**Lemma 2.4** There exist a sufficiently small number \(r_1 > 0\) depending on \(n\) and a sufficiently large number \(\beta_1 > 3\) depending on \(n\) such that for all \(w \in U_{r_1}\) and \(f = (f_1, \cdots, f_n) \in (U_{r_1})^n\), \(\beta \geq \beta_1\), we have that
\[
\int \varphi^2_\beta (\log |x|)^2 (|x|^{4-n} |\nabla w|^2 + |x|^{2-n} |w|^2) dx 
\]
\[
\lesssim \int \varphi^2_\beta (|x|^2 \Delta w + |x| \div f)^2 + \beta \|f\|^2 |x|^{-n} dx, \tag{2.12}
\]
where \(U_{r_1}\) is defined as in Lemma 2.1.

**Proof.** Replacing \(\beta\) by \(\beta + 2\) in (2.12), we see that it suffices to prove
\[
\int \varphi^2_\beta (\log |x|)^{-2} (|x|^2 |\nabla w|^2 + |w|^2) |x|^{-n} dx 
\]
\[
\lesssim \int \varphi^2_\beta (|x|^2 \Delta w + |x| \div f)^2 + \beta \|f\|^2 |x|^{-n} dx. \tag{2.13}
\]
Working in polar coordinates and using the relation \(e^{2t} \Delta = L^+ L^-\), (2.13) is equivalent to
\[
\sum_{j+\alpha \leq 1} \int \int t^{-2} \varphi^2_\beta |\partial_t^j \Omega^\alpha w|^2 dt \omega 
\]
\[
\lesssim \int \int \varphi^2_\beta (|L^+ L^- w + \partial_t (\sum_{j=1}^n \omega_j f_j) + \sum_{j=1}^n \Omega_j f_j|^2 + \beta \|f\|^2) dt \omega. \tag{2.14}
\]
Applying Lemma 2.3 to $u = L^{-}w$ and $g = (\sum_{j=1}^{n} \omega_j f_j, f_1, \cdots, f_n)$ yields
\[
\beta \int \int \varphi_\beta^2 |L^{-}w|^2 dt d\omega \\
\lesssim \int \int \varphi_\beta^2 (|L^+L^{-}w + \partial_t (\sum_{j=1}^{n} \omega_j f_j) + \sum_{j=1}^{n} \Omega_j f_j|^2 + \beta \|f\|^2) dt d\omega. \quad (2.15)
\]
Now (2.14) is an easy consequence of (2.3) and (2.15).

\section{Proof of Theorem 1.1 and Theorem 1.3}

This section is devoted to the proofs of Theorem 1.1 and 1.3. To begin, we state an interior estimate for the Lamé system (1.2). For fixed $a_3 < a_1 < a_2 < a_4$, there exists a constant $\tilde{C}$ such that
\[
\int_{a_1 r < |x| < a_2 r} |x|^{\alpha} |D_\alpha u|^2 dx \leq \tilde{C} \int_{a_3 r < |x| < a_4 r} |u|^2 dx, \quad |\alpha| \leq 2 
\]
for all sufficiently small $r$. Estimate (3.1) can be proved by repeating the arguments of Corollary 17.1.4 in [10]. Plugging $p = \text{div } u$ in (3.1) yields
\[
\int_{a_1 r < |x| < a_2 r} |x|^{\alpha} |D_\alpha p|^2 dx \leq \tilde{C} \int_{a_3 r < |x| < a_4 r} |u|^2 dx, \quad |\alpha| \leq 1. \quad (3.2)
\]

To proceed the proof, let us first consider the case where $0 < R_1 < R_2 < R < R_0$. The constant $R$ will be determined later. Since $u \in H^1_{\text{loc}}(B_{R_0})$, the elliptic regularity theorem for (1.2) implies $u \in H^2_{\text{loc}}(B_{R_0})$. Therefore, to use estimate (2.1), we simply cut-off $u$. So let $\chi(x) \in C_0^\infty(\mathbb{R}^n)$ satisfy $0 \leq \chi(x) \leq 1$ and
\[
\chi(x) = \begin{cases} 
0, & |x| \leq R_1/e, \\
1, & R_1/2 < |x| < eR_2, \\
0, & |x| \geq 3R_2, 
\end{cases}
\]
where $e = \exp(1)$. We first choose a small $R$ such that $R \leq \min\{r_0, r_1\}/3 = \tilde{R}_0$, where $r_0$ and $r_1$ are constants appeared in (2.1) and (2.12). Hence $\tilde{R}_0$ depends on $n$. It is easy to see that for any multiindex $\alpha$
\[
\begin{cases} 
|D_\alpha \chi| = O(R_1^{-|\alpha|}) \text{ for all } R_1/e \leq |x| \leq R_1/2, \\
|D_\alpha \chi| = O(R_2^{-|\alpha|}) \text{ for all } eR_2 \leq |x| \leq 3R_2. 
\end{cases} \quad (3.3)
\]
Applying (2.1) to $\chi u$ gives

$$C_1 \beta \int (\log |x|)^{-2} \varphi^2_\beta |x|^{-n} (|x|^2 |\nabla (\chi u)|^2 + |\chi u|^2) dx \leq \int \varphi^2_\beta |x|^{-n} |x|^4 |\Delta (\chi u)|^2 dx.$$  

(3.4)

From now on, $C_1, C_2, \ldots$ denote general constants whose dependence will be specified whenever necessary. Next we want to apply (2.12) to $w = \chi p$ and $f = |x| \chi G$. Since $u \in H^2_{\text{loc}}$ and $p = \text{div} u \in H^1_{\text{loc}}$, in view of the second equation of (1.5), by the standard limiting argument, (2.12) holds true for $(w,f)$ above. Thus, we get that

$$C_2 \int \varphi^2_\beta (\log |x|)^2 (|x|^{4-n} |\nabla (\chi p) + |x|^{2-n} |\chi p|^2) dx$$

$$\leq \int \varphi^2_\beta (\log |x|)^4 |x|^{2-n} |x|^2 \Delta (\chi p) + |x| \text{div} (|x| \chi G)|^2 dx$$

$$+ \beta \int \varphi^2_\beta (\log |x|)^4 |x|^{2-n} |x| \chi G|^2 dx.$$  

(3.5)

Combining (3.4) and (3.5), we obtain that

$$\beta \int_{R/2 < |x| < eR_2} (\log |x|)^{-2} \varphi^2_\beta |x|^{-n} (|x|^2 |\nabla u|^2 + |u|^2) dx$$

$$+ \int_{R/2 < |x| < eR_2} (\log |x|)^2 \varphi^2_\beta |x|^{-n} (|x|^4 |\nabla p|^2 + |x|^2 |p|^2) dx$$

$$\leq \beta \int \varphi^2_\beta (\log |x|)^{-2} |x|^{-n} (|x|^2 |\nabla (\chi u)|^2 + |\chi u|^2) dx$$

$$+ \int (\log |x|)^2 \varphi^2_\beta |x|^{-n} (|x|^4 |\nabla (\chi p)|^2 + |x|^2 |\chi p|^2) dx$$

$$\leq C_3 \int \varphi^2_\beta |x|^{-n} |x|^4 |\Delta (\chi u)|^2 dx$$

$$+ C_3 \int (\log |x|)^4 \varphi^2_\beta |x|^{-n} |x|^2 \Delta (\chi p) + |x|^2 \text{div} (|x| \chi G)|^2 dx$$

$$+ \beta C_3 \int (\log |x|)^4 \varphi^2_\beta |x|^{-n} |x|^2 |\chi G|^2 dx.$$  

(3.6)
By (1.1), (1.5), and estimates (3.3), we deduce from (3.6) that

\[
\begin{align*}
\beta \int_{R_1/2 < |x| < eR_2} (\log |x|)^{-2} \varphi_\beta^2 |x|^{-n} (|x|^2 |\nabla u|^2 + |u|^2) dx \\
+ \int_{R_1/2 < |x| < eR_2} (\log |x|)^2 \varphi_\beta^2 |x|^{-n} (|x|^4 |\nabla p|^2 + |x|^2 |p|^2) dx \\
\leq C_4 \int_{R_1/2 < |x| < eR_2} \varphi_\beta^2 |x|^{-n} |x|^4 (|\nabla u|^2 + |\nabla p|^2) dx \\
+ C_4 \int_{R_1/2 < |x| < eR_2} (\log |x|)^4 |x|^2 \varphi_\beta^2 |x|^{-n} (|x|^4 |\nabla u|^2 + |x|^2 |\nabla p|^2) dx \\
+ C_4 \beta \int_{R_1/2 < |x| < eR_2} (\log |x|)^4 |x|^2 \varphi_\beta^2 |x|^{-n} |\nabla u|^2 dx \\
+ C_4 \int_{\{R_1/\epsilon \leq |x| \leq R_1/2\} \cup \{eR_2 \leq |x| \leq 3R_2\}} (\log |x|)^4 \varphi_\beta^2 |x|^{-n} |\tilde{U}(x)|^2 dx \\
+ C_4 \beta \int_{\{R_1/\epsilon \leq |x| \leq R_1/2\} \cup \{eR_2 \leq |x| \leq 3R_2\}} (\log |x|)^4 \varphi_\beta^2 |x|^{-n} |\tilde{U}(x)|^2 dx.
\end{align*}
\]

(3.7)

where \( |\tilde{U}(x)|^2 = |x|^4 |\nabla p|^2 + |x|^2 |p|^2 + |x|^2 |\nabla u|^2 + |u|^2 \) and the positive constant \( C_4 \) only depends on \( n, M_0, \delta_0 \).

Now letting \( R \) small enough, say \( R < \tilde{R}_1 \), such that \( 2C_4 (\log(eR))^6 (eR)^2 \leq 1 \) and \( (\log(eR))^2 \geq 2C_4 \), then the first three terms on the right hand side of (3.7) can be absorbed by the left hand side of (3.7). Also, it is easy to check that there exists \( \tilde{R}_2 > 0 \), depending on \( n \), such that for all \( \beta > 0 \), both \( (\log |x|)^{-2} |x|^{-n} \varphi_\beta^2 (|x|) \) and \( (\log |x|)^4 |x|^{-n} \varphi_\beta^2 (|x|) \) are decreasing functions in \( 0 < |x| < \tilde{R}_2 \). So we choose a small \( R < \tilde{R}_3 \), where \( \tilde{R}_3 = \min\{\tilde{R}_2/3, \tilde{R}_1, \tilde{R}_0\} \). It is clear that \( \tilde{R}_3 \) depends on \( n, M_0, \delta_0 \). With the choices described above, we obtain from (3.7) that
\[ R_2^{-n}(\log R_2)^{-2} \varphi_\beta^2(R_2) \int_{R_1/2 < |x| < R_2} |u|^2 \, dx \leq \int_{R_1/2 < |x| < eR_2} (\log |x|)^{-2} \varphi_\beta^2 |x|^{-n} |u|^2 \, dx \]
\[ \leq C_5 \int_{\{R_1/e \leq |x| \leq R_1/2\} \cup \{eR_2 \leq |x| \leq 3R_2\}} (\log |x|)^4 \varphi_\beta^2 |x|^{-n} |\tilde{U}|^2 \, dx \]
\[ \leq C_5 \int_{\{R_1/e \leq |x| \leq R_1/2\}} \int_{\{eR_2 \leq |x| \leq 3R_2\}} (\log (eR_2))^4 (eR_2)^{-n} \varphi_\beta^2 (eR_2) \int_{\{R_1/e \leq |x| \leq R_1/2\}} + C_5 \int_{\{eR_2 \leq |x| \leq 3R_2\}} |\tilde{U}|^2 \, dx. \quad (3.8) \]

It follows from (3.1) and (3.2) that
\[ R_2^{-2\beta-n}(\log R_2)^{-4\beta-2} \int_{R_1/2 < |x| < R_2} |u|^2 \, dx \]
\[ \leq C_6 (\log (eR_2))^4 (eR_2)^{-n} \varphi_\beta^2 (eR_2) \int_{\{R_1/4 \leq |x| \leq R_1\}} |u|^2 \, dx \]
\[ + C_6 (\log (eR_2))^4 (eR_2)^{-n} \varphi_\beta^2 (eR_2) \int_{\{2R_2 \leq |x| \leq 4R_2\}} |u|^2 \, dx \]
\[ = C_6 (\log (R_1/e))^{-4\beta+4} (R_1/e)^{-2\beta-n} \int_{\{R_1/4 \leq |x| \leq R_1\}} |u|^2 \, dx \]
\[ + C_6 (\log (eR_2))^4 (eR_2)^{-n} \varphi_\beta^2 (eR_2) \int_{\{2R_2 \leq |x| \leq 4R_2\}} |u|^2 \, dx. \quad (3.9) \]

Replacing \(2\beta + n\) by \(\beta\), (3.9) becomes
\[ R_2^{-\beta}(\log R_2)^{-2\beta+2n-2} \int_{R_1/2 < |x| < R_2} |u|^2 \, dx \leq C_7 (\log (R_1/e))^{-2\beta+2n+4} (R_1/e)^{-\beta} \int_{\{R_1/4 \leq |x| \leq R_1\}} |u|^2 \, dx \]
\[ + C_7 (\log (eR_2))^4 (eR_2)^{-n} \varphi_\beta^2 (eR_2) \int_{\{2R_2 \leq |x| \leq 4R_2\}} |u|^2 \, dx. \quad (3.10) \]

Dividing \(R_2^{-\beta}(\log R_2)^{-2\beta+2n-2}\) on the both sides of (3.10) and if \(\beta \geq n + 2\),
we have that
\[
\int_{R_2/2 < |x| < R_2} |u|^2 dx \
\leq C_8 (\log R_2)^6 (e R_2 / R_1)^\beta \int_{|x| \leq R_1} |u|^2 dx \\
+ C_8 (\log R_2)^6 (1/e)^\beta [(\log R_2 / \log (e R_2))^2]^{\beta - n-2} \int_{|x| \leq 2 R_2} |u|^2 dx \\
\leq C_8 (\log R_2)^6 (e R_2 / R_1)^\beta \int_{|x| \leq R_1} |u|^2 dx \\
+ C_8 (\log R_2)^6 (4/5)^\beta \int_{|x| \leq 4 R_2} |u|^2 dx.
\] (3.11)

In deriving the second inequality above, we use the fact that
\[
\frac{\log R_2}{\log (e R_2)} \to 1 \quad \text{as} \quad R_2 \to 0,
\]
and thus
\[
\frac{1}{e} \cdot \frac{\log R_2}{\log (e R_2)} < \frac{4}{5}
\]
for all \( R_2 < \tilde{R}_4 \), where \( \tilde{R}_4 \) is sufficiently small. We now take \( \tilde{R} = \min\{\tilde{R}_3, \tilde{R}_4\} \), which depends on \( n, M_0, \delta_0 \).

Adding \( \int_{|x| < R_1/2} |u|^2 dx \) to both sides of (3.11) leads to
\[
\int_{|x| < R_2} |u|^2 dx \leq C_9 (\log R_2)^6 (e R_2 / R_1)^\beta \int_{|x| \leq R_1} |u|^2 dx \\
+ C_9 (\log R_2)^6 (4/5)^\beta \int_{|x| \leq 1} |u|^2 dx. \] (3.12)

It should be noted that (3.12) holds for all \( \beta \geq \tilde{\beta} \) with \( \tilde{\beta} \) depending only on \( n, M_0, \delta_0 \). For simplicity, by denoting
\[
E(R_1, R_2) = \log(e R_2 / R_1), \quad B = \log(5/4),
\]
(3.12) becomes
\[
\int_{|x| < R_2} |u|^2 dx \
\leq C_9 (\log R_2)^6 \{ \exp(E\beta) \int_{|x| < R_1} |u|^2 dx + \exp(-B\beta) \int_{|x| < 1} |u|^2 dx \}. \] (3.13)
To further simplify the terms on the right hand side of (3.13), we consider two cases. If \( \int_{|x|<R_1} |u|^2 dx \neq 0 \) and

\[
\exp (E \tilde{\beta}) \int_{|x|<R_1} |u|^2 dx < \exp (-B \tilde{\beta}) \int_{|x|<1} |u|^2 dx,
\]
then we can pick a \( \beta > 3 \beta \) such that

\[
\exp (E\beta) \int_{|x|<R_1} |u|^2 dx = \exp (-B\beta) \int_{|x|<1} |u|^2 dx.
\]

Using such \( \beta \), we obtain from (3.13) that

\[
\int_{|x|<R_2} |u|^2 dx 
\leq 2C_9 \log R_2 \exp (E\beta) \int_{|x|<R_1} |u|^2 dx 
= 2C_9 \log R_2 \left( \int_{|x|<R_1} |u|^2 dx \right)^{\frac{E}{E+B}} \left( \int_{|x|<1} |u|^2 dx \right)^{\frac{E}{E+B}} \cdot (3.14)
\]

If \( \int_{|x|<R_1} |u|^2 dx = 0 \), then letting \( \beta \to \infty \) in (3.13) we have \( \int_{|x|<R_2} |u|^2 dx = 0 \) as well. The three-ball inequality obviously holds.

On the other hand, if

\[
\exp (-B\tilde{\beta}) \int_{|x|<1} |u|^2 dx \leq \exp (E\tilde{\beta}) \int_{|x|<R_1} |u|^2 dx,
\]
then we have

\[
\int_{|x|<R_2} |u|^2 dx 
\leq \left( \int_{|x|<1} |u|^2 dx \right)^{\frac{B}{E+B}} \left( \int_{|x|<1} |u|^2 dx \right)^{\frac{E}{E+B}} 
\leq \exp (B\tilde{\beta}) \left( \int_{|x|<R_1} |u|^2 dx \right)^{\frac{B}{E+B}} \left( \int_{|x|<1} |u|^2 dx \right)^{\frac{E}{E+B}} \cdot (3.15)
\]

Putting together (3.14), (3.15), and setting \( C_{10} = \max\{2C_9 \log R_2 \exp (\tilde{\beta} \log(5/4))\} \), we arrive at

\[
\int_{|x|<R_2} |u|^2 dx \leq C_{10} \left( \int_{|x|<R_1} |u|^2 dx \right)^{\frac{B}{E+B}} \left( \int_{|x|<1} |u|^2 dx \right)^{\frac{E}{E+B}} \cdot (3.16)
\]
It is readily seen that \( \frac{B}{E+B} \approx (\log(1/R_1))^{-1} \) when \( R_1 \) tends to 0.

Now for the general case, we consider \( 0 < R_1 < R_2 < R_3 \leq \bar{R} \), where \( \bar{R} \) is given as above. By scaling, i.e. defining \( \tilde{u}(y) := u(R_3 y), \tilde{\lambda}(y) := \lambda(R_3 y) \) and \( \tilde{\mu}(y) = \mu(R_3 y) \), (3.16) becomes

\[
\int_{|y|<R_2/R_3} |\tilde{u}(y)|^2 \, dy \leq C_{11} \left( \int_{|y|<R_1/R_3} |\tilde{u}(y)|^2 \, dy \right)^\tau \left( \int_{|y|<1} |\tilde{u}(y)|^2 \, dy \right)^{1-\tau},
\]

where
\[
\tau = B / [E(R_1/R_3, R_2/R_3) + B],
\]
\[
C_{11} = \max\{2C_9 \log(R_2/R_3)^6, \exp (\tilde{\beta} \log(5/4))\}.
\]

Note that \( C_{11} \) is independent of \( R_1 \). Restoring the variable \( x = R_3 y \) in (3.17) gives

\[
\int_{|x|<R_2} |u|^2 \, dx \leq C_{11} \left( \int_{|x|<R_1} |u|^2 \, dx \right)^\tau \left( \int_{|x|<R_3} |u|^2 \, dx \right)^{1-\tau}.
\]

The proof of Theorem 1.1 is complete.

We now turn to the proof of Theorem 1.3. We fix \( R_2, R_3 \) in Theorem 1.1 and define

\[
\tilde{u}(x) := u(x) / \sqrt{\int_{|x|<R_2} |u|^2 \, dx}.
\]

Note that \( \int_{|x|<R_2} |\tilde{u}|^2 \, dx = 1 \). From the three-ball inequality (1.2), we have that

\[
1 \leq C \left( \int_{|x|<R_3} |\tilde{u}|^2 \, dx \right)^\tau \left( \int_{|x|<R_3} |\tilde{u}|^2 \, dx \right)^{1-\tau}.
\]

Raising both sides by \( 1/\tau \) yields that

\[
\int_{|x|<R_3} |\tilde{u}|^2 \, dx \leq \left( \int_{|x|<R_3} |\tilde{u}|^2 \, dx \right) \left( \int_{|x|<R_3} |\tilde{u}|^2 \, dx \right)^{1/\tau}.
\]

In view of the formula for \( \tau \), we can deduce from (3.19) that

\[
\int_{|x|<R_3} |\tilde{u}|^2 \, dx \leq \left( \int_{|x|<R_1} |\tilde{u}|^2 \, dx \right) \left( 1/R_1 \right)^\tilde{C} \log(f_{|x|<R_4}|\tilde{u}|^2 \, dx),
\]

where \( \tilde{C} \) is a positive constant depending on \( n, M_0, \delta_0 \) and \( R_2/R_3 \). Consequently, (3.20) is equivalent to

\[
\left( \int_{|x|<R_3} |u|^2 \, dx \right) R_1^n \leq \int_{|x|<R_1} |u|^2 \, dx
\]

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for all $R_1$ sufficiently small, where

$$m = \tilde{C} \log \left( \frac{\int_{|x|<R_3} |U|^2 dx}{\int_{|x|<R_2} |U|^2 dx} \right).$$

We now end the proof of Theorem 1.3.

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