ADMISSIBLE BANACH FUNCTION SPACES FOR LINEAR DYNAMICS WITH NONUNIFORM BEHAVIOR ON THE HALF-LINE

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Abstract. For nonuniform exponentially bounded evolution families on the half-line we introduce a class of Banach function spaces on which we define nonuniform evolution semigroups. We completely characterize nonuniform exponential stability in terms of invertibility of the corresponding generators. We emphasize that in particular our results apply to all linear differential equations with bounded operator and finite Lyapunov exponent.

1. Introduction

A linear dynamics is called well-posed if we assume the existence, uniqueness and continuous dependence of solutions on initial data. In the case of the nonautonomous equation \( \frac{dx}{dt} = A(t) x \), well-posedness is equivalent to the existence of an evolution family that solves the equation (see, for instance, Proposition 9.3 in [7], p. 478), that is a family of bounded linear operators \( U = \{ U(t,s) \}_{t \geq s \geq 0} \) acting on the underlying Banach space \( X \), with properties:

\begin{align*}
(\text{e}_1) & \quad U(t,t) = \text{Id}, \ t \geq 0; \\
(\text{e}_2) & \quad U(t,\tau)U(\tau,s) = U(t,s), \ t \geq \tau \geq s \geq 0; \\
(\text{e}_3) & \quad \text{The map} \ (t,s) \mapsto U(t,s)x \ \text{is continuous for each} \ x \in X.
\end{align*}

If the linear operators \( A(t) \) are bounded, then well-posedness is guaranteed [6, Chapter 3].

In the autonomous case, the equation \( \frac{dx}{dt} = Ax \) is well-posed if and only if the linear operator \( A \) generates a \( C_0 \)-semigroup (Corollary 6.9 in [7], p. 151). We recall that a family of bounded linear operators \( T = \{ T(t) \}_{t \geq 0} \) acting on \( X \) is said to be a \( C_0 \)-semigroup if

\begin{align*}
(\text{s}_1) & \quad T(0) = \text{Id}; \\
(\text{s}_2) & \quad T(t)T(s) = T(t+s) \ \text{for} \ t, s \geq 0; \\
(\text{s}_3) & \quad \lim_{t \to 0^+} T(t)x = x \ \text{for every} \ x \in X.
\end{align*}

The (closed and densely defined) linear operator \( G : D(G) \subset X \to X \) defined by

\[ Gx = \lim_{t \to 0^+} \frac{T(t)x - x}{t} \]

is called the (infinitesimal) generator of the \( C_0 \)-semigroup \( T \). For more details on the theory of \( C_0 \)-semigroups we refer the reader to [7, 10].

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In the particular case of linear dynamics, the concept of uniform asymptotic stability (in the sense of Lyapunov) translates as uniform exponential stability. The evolution family $U$ is called uniform exponentially stable if $\|U(t, s)\| \leq Me^{-\alpha(t-s)}$, $t \geq s \geq 0$, for some constants $M, \alpha > 0$. Under such hypothesis it is possible to introduce a $C_0$-semigroup on some Banach function spaces, basically defined as

$$
(T(t)u)(s) = \begin{cases} 
U(s, s-t)u(s-t), & \text{if } s > t, \\
U(s, 0)u(0), & \text{if } 0 \leq s \leq t,
\end{cases}
$$

(1)

and it is called the (uniform) evolution semigroup. This construction reduces the study of a nonautonomous equation to the analysis of an autonomous one in the form $dy/dt = Gy$, where $G$ is the generator of the evolution semigroup. Thus, asymptotic behavior of an evolution family can be essentially characterized in terms of spectral properties of the generator of the corresponding evolution semigroup.

The theory of uniform behavior is too restrictive, and it is important to consider a more general view, such as nonuniform asymptotic stability. A serious motivation for weakening the notion of uniform exponential behavior lies in the ergodic theory. In this regard, a consistent contribution is due to Ya. Pesin, L. Barreira and C. Valls (we refer the reader to monographs [1, 3] and the references therein). Roughly speaking, while uniformity relates to the finiteness of the Bohl exponents [6, Chapter 3], nonuniformity analyses situations when the Lyapunov exponent is finite [3]. This type of argumentation is not at all of a formal type, as illustrated in our examples. The orbits of the evolution family in our Example 4.4 are all asymptotically stable, even if the evolution family is not uniform exponentially bounded. In this case it is impossible to construct the evolution semigroup. From another hand, Example 4.5 (Perron) delivers a uniform exponentially bounded evolution family which is not uniform exponentially stable, with all the orbits being asymptotically stable. In this case the evolution semigroup exists, but it does not furnish any kind of information about the (nonuniform) asymptotic behavior of the orbits. To sum up, in many situation the classical tool either does not exist or it is completely useless.

Our main goal is to emphasize an important difference between uniform and nonuniform behavior: while in the uniform case all the evolution families reflect into a unique output function space (of functions vanishing at 0 and infinity, for instance), in the nonuniform case there are infinitely many output function spaces, which are the admissible ones. They depend on each evolution family, and on each particular admissible exponent.

Let us point out that the theory of evolution semigroups and that of admissibility are quite similar. In fact, admissibility methods deal with couples of Banach function spaces, while in the case of evolution semigroups the spaces into each couple coincide. In this view, we prefer to use the term “admissible” throughout our paper. For a better understanding, we refer the reader to [8, 11] for uniform behavior, and to [4, 13] for the nonuniform setting.

We emphasize that the idea of connecting the nonuniform behavior to $C_0$-semigroups is already present in the recent paper [2], in the case of discrete families.

This paper is organized as follows.
In Section 2 we introduce the set of admissible exponents associated to a nonuniform exponentially bounded evolution family, and the class of the corresponding Banach function spaces that we also call admissible. We emphasize the connections with the conditions on the finiteness of the corresponding Bohl and Lyapunov exponents.

In Section 3 we generalize the notion of evolution semigroup for linear dynamics with nonuniform behavior, by shrinking the output space. We call this mathematical object a nonuniform evolution semigroup. Moreover, we point out the connection with the uniform case.

The last section presents an introspective study of nonuniformity via evolution semigroups. We characterize nonuniform exponential stability in terms of invertibility of the corresponding infinitesimal generators. In this regard, we introduce a new notion that we call quasi-negative exponent, specific to nonuniform behavior. As a consequence, we prove a spectral mapping theorem for nonuniform evolution semigroups.

We consider as our main result the statement in Theorem 4.6.

2. Admissible Banach function spaces

Throughout our paper X is a Banach space. We denote \( C(\mathbb{R}_+, X) \) the space of all continuous, \( X \)-valued functions defined on the half-line, and \( C_c,0(\mathbb{R}_+, X) \) is the space of all functions in \( C(\mathbb{R}_+, X) \) with compact support vanishing at 0. We also make use of the following notation:

\[
C_{00}(\mathbb{R}_+, X) = \{ u \in C(\mathbb{R}_+, X) : \lim_{t \to \infty} u(t) = u(0) = 0 \}.
\]

**Definition 2.1.** For any fixed \( \alpha \in \mathbb{R} \), the evolution family \( U \) is called \( \alpha \)-nonuniform exponentially bounded, if there exists a continuous map \( M_\alpha : \mathbb{R}_+ \to (0, \infty) \) such that

\[
\| U(t, s) \| \leq M_\alpha(s) e^{\alpha(t-s)}, \quad t \geq s \geq 0. \tag{2}
\]

If the above estimation holds for some \( \alpha < 0 \), then \( U \) is called \( \alpha \)-nonuniform exponentially stable. Each \( \alpha \) satisfying (2) is called an admissible exponent, and the set of all admissible exponents is denoted \( A(U) \).

Evidently for each evolution family \( U \), the set \( A(U) \) is either a (semi) infinite interval, or empty. If \( A(U) \neq \emptyset \), then the evolution family \( U \) is called nonuniform exponentially bounded, and if \( A(U) \) contains negative admissible exponents, we say that \( U \) is nonuniform exponentially stable. We emphasize that our paper is devoted to the study of the nonuniform exponentially bounded evolution families, that is we only consider the case \( A(U) \neq \emptyset \). In the above terminology, whenever there exists a bounded map \( M_\alpha(s) \) satisfying (2) (which is equivalent to the existence of a constant one), we just replace the term “nonuniform” with “uniform”. Also, in such cases we call \( \alpha \) a strict (admissible) exponent, and we denote \( A_s(U) \) the set of all strict exponents.

**Remark 2.2.** Assume that the evolution family \( U \) is reversible (i.e. \( U(t, s) \) is invertible for all \( t \geq s \geq 0 \), \( U(s, t) \) denotes its inverse). If the Lyapunov exponent \( K_L \) is finite and not attained, then \( A(U) = (K_L, \infty) \) (in particular \( K_L = -\infty \) whenever \( A(U) = \mathbb{R} \)). Also if the Bohl exponent \( K_B \) is finite and not attained, then \( A_s(U) = (K_B, \infty) \). The intervals of admissibility are closed at their left endpoints whenever the Lyapunov or the Bohl exponents are attained.
Indeed, as

\[ K_L = \inf \{ \alpha \in \mathbb{R} : \text{there exists } M_\alpha > 0 \text{ with } \| U(t,0) \| \leq M_\alpha e^{\alpha t}, \ t \geq 0 \} \]

and is not attained, if \( \alpha \in \mathcal{A}(\mathcal{U}) \), replacing \( s = 0 \) in (2), one has

\[ \| U(t,0) \| \leq M_\alpha(0)e^{\alpha t}, \]

thus \( \alpha \in (K_L, \infty) \). For \( \alpha \in (K_L, \infty) \), assuming that \( \mathcal{U} \) is reversible, we get

\[ \| U(t,s) \| = \| U(t,0)U(0,s) \| \leq M_\alpha e^{\alpha t} \| U(0,s) \| = M_\alpha e^{\alpha s} \| U(0,s) \| e^{\alpha(t-s)}, \]

that implies \( \alpha \in \mathcal{A}(\mathcal{U}) \). The second statement can be proved similarly if we notice that

\[ K_B = \inf \{ \alpha \in \mathbb{R} : \text{there exists } M_\alpha > 0 \text{ with } \| U(t,s) \| \leq M_\alpha e^{\alpha(t-s)}, \ t \geq s \geq 0 \} . \]

Suppose that \( \mathcal{A}(\mathcal{U}) \neq \emptyset \), and let \( \alpha \in \mathcal{A}(\mathcal{U}) \). For each \( t \geq 0 \) and \( u \in \mathcal{C}(\mathbb{R}_+, X) \) we set

\[ \varphi_{\mathcal{U}t,\alpha}(t,u) = \sup_{\tau \geq t} \ e^{-\alpha(\tau-t)} \| U(\tau,t)u(t) \| . \quad (3) \]

Inequality (2) implies

\[ \| u(t) \| \leq \varphi_{\mathcal{U}t,\alpha}(t,u) \leq M_\alpha(t) \| u(t) \|, \ t \geq 0. \quad (4) \]

If in particular \( u(t) \equiv x \) for some \( x \in X \), we step over the norm on \( X \) defined in [4 Eq. (5)], precisely \( \| x \|_t = \sup_{\tau \geq t} e^{-\alpha(\tau-t)} \| U(\tau,t)x \| \). In this regard, we notice that the map \( \varphi_{\mathcal{U}t,\alpha} \) in (3) can be (indirectly) defined as \( \varphi_{\mathcal{U}t,\alpha}(t,u) = \| u(t) \|_t \).

The following result is essential in the sequel.

**Proposition 2.3.** The map \( \mathbb{R}_+ \ni t \mapsto \varphi_{\mathcal{U}t,\alpha}(t,u) \in \mathbb{R}_+ \) is continuous for each fixed \( \alpha \in \mathcal{A}(\mathcal{U}) \) and \( u \in \mathcal{C}(\mathbb{R}_+, X) \). In addition, for each \( u \in \mathcal{C}(\mathbb{R}_+, X) \) for which \( \lim_{t \to \infty} \varphi_{\mathcal{U}t,\alpha}(t,u) = 0 \), there exists (possibly not unique) \( t_u \geq 0 \) such that

\[ \sup_{t \geq 0} \varphi_{\mathcal{U}t,\alpha}(t,u) = \varphi_{\mathcal{U}t,\alpha}(t_u,u). \]

**Proof.** To prove the first statement we set \( V(t,s) = e^{-\alpha(t-s)}U(t,s) \). It follows that \( \mathcal{V} = \{ V(t,s) \}_{t,s \geq 0} \) is also an evolution family with \( \| V(t,s) \| \leq M_\alpha(s), \ t \geq s \geq 0 \). For fixed \( u \in \mathcal{C}(\mathbb{R}_+, X) \), \( t_0 \geq 0 \) and \( \varepsilon > 0 \), there exists \( \delta_1, \delta_2 > 0 \) such that

\[ |t-t_0| < \delta_1 \Rightarrow M_\alpha(t) \| u(t) - u(t_0) \| < \varepsilon/3, \]

\[ t_0 - \delta_2 < t \leq t_0 + \delta_2 \Rightarrow M_\alpha(t) \| u(t) - V(t,t_0)u(t_0) \| < \varepsilon/3. \]

Let \( \delta = \max \{ \delta_1, \delta_2 \} \) and choose \( t \geq 0 \) with \( |t-t_0| < \delta \). We only analyze the case \( t \geq t_0 \). For any \( \tau \geq t \) we have

\[ \| V(\tau,t)u(t) \| \leq \| V(\tau,t)(u(t) - u(t_0)) \| + \| V(\tau,t)u(t_0) - V(\tau,t_0)u(t_0) \| \]

\[ + \| V(\tau,t_0)u(t_0) \| \leq \| V(\tau,t) \| \| u(t) - u(t_0) \| + \| V(\tau,t) \| \| u(t_0) - V(t,t_0)u(t_0) \| \]

\[ + \| V(\tau,t_0)u(t_0) \| \leq M_\alpha(t) \| u(t) - u(t_0) \| + M_\alpha(t) \| u(t_0) - V(t,t_0)u(t_0) \| \]

\[ + \| V(\tau,t_0)u(t_0) \| \leq (2\varepsilon)/3 + \varphi_{\mathcal{U}t,\alpha}(t_u,u). \]
When taking the supremum with respect to $\tau \geq t$, we get
\[ \varphi_{\mathcal{U},\alpha}(t, u) - \varphi_{\mathcal{U},\alpha}(t_0, u) < \varepsilon. \]
Similarly one can prove that $\varphi_{\mathcal{U},\alpha}(t_0, u) - \varphi_{\mathcal{U},\alpha}(t, u) < \varepsilon$, that is the map $t \mapsto \varphi_{\mathcal{U},\alpha}(t, u)$ is continuous at $t_0$. The second statement follows from the continuity of the map in question, together with condition $\lim_{t \to \infty} \varphi_{\mathcal{U},\alpha}(t, u) = 0$. \hfill $\square$

For each $\alpha \in \mathcal{A}(\mathcal{U})$ we set
\[ \mathcal{C}(\mathcal{U}, \alpha) = \left\{ u \in C(\mathbb{R}_+, X) : \lim_{t \to \infty} \varphi_{\mathcal{U},\alpha}(t, u) = \varphi_{\mathcal{U},\alpha}(0, u) = 0 \right\}. \]
It is a kind of straightforward argument to verify that $\mathcal{C}(\mathcal{U}, \alpha)$ is a Banach (function) space equipped with the norm
\[ \| u \|_{\mathcal{U},\alpha} = \sup_{t \geq 0} \varphi_{\mathcal{U},\alpha}(t, u), \]
called the admissible Banach function space corresponding to the evolution family $\mathcal{U}$ and the admissible exponent $\alpha \in \mathcal{A}(\mathcal{U})$. Eq. (4) also implies
\[ C_{c,0}(\mathbb{R}_+, X) \subset \mathcal{C}(\mathcal{U}, \alpha) \subset C_0(\mathbb{R}_+, X). \]

Let us remark that if $\alpha \in \mathcal{A}(\mathcal{U})$, $u \in C(\mathbb{R}_+, X)$ and $\beta \geq \alpha$, then $\beta \in \mathcal{A}(\mathcal{U})$, $\varphi_{\mathcal{U},\beta}(t, u) \leq \varphi_{\mathcal{U},\alpha}(t, u)$. Moreover,
\[ \mathcal{C}(\mathcal{U}, \alpha) \subset \mathcal{C}(\mathcal{U}, \beta) \text{ and } \| u \|_{\mathcal{U},\beta} \leq \| u \|_{\mathcal{U},\alpha}. \]

3. NONUNIFORM EVOLUTION SEMIGROUPS

In this section we introduce the concept of nonuniform evolution semigroup, emphasizing the connections with the uniform case.

**Theorem 3.1.** Each $\alpha \in \mathcal{A}(\mathcal{U})$ defines a $C_0$-semigroup $T_\alpha = \{T_\alpha(t)\}_{t \geq 0}$ on $\mathcal{C}(\mathcal{U}, \alpha)$ by setting
\[ (T_\alpha(t)u)(s) = \begin{cases} U(s, s-t)u(s-t), & \text{if } s > t, \\ 0, & \text{if } 0 \leq s \leq t. \end{cases} \]
Moreover, the following estimation holds
\[ \| T_\alpha(t)u \|_{\mathcal{U},\alpha} \leq e^{\alpha t} \| u \|_{\mathcal{U},\alpha}, \quad u \in \mathcal{C}(\mathcal{U}, \alpha). \]

**Proof.** Evidently $T_\alpha(0) = I_d$ and $T_\alpha(t)T_\alpha(s) = T_\alpha(t+s)$, for all $t, s \geq 0$. It remains to prove that the map $T_\alpha(t)$ in (6) is well defined on $\mathcal{C}(\mathcal{U}, \alpha)$, and the norm $\| T_\alpha(t)u - u \|_{\mathcal{U},\alpha} \to 0$ as $t \to 0_+$, for each fixed $u \in \mathcal{C}(\mathcal{U}, \alpha)$. Pick $u \in \mathcal{C}(\mathcal{U}, \alpha)$ and $t \geq 0$. For arbitrary $s \geq t$ we have
\[ \varphi_{\mathcal{U},\alpha}(s, T_\alpha(t)u) = \sup_{\tau \geq s} e^{-\alpha(\tau-s)} \| U(\tau, s-t)u(s-t) \| \]
\[ = e^{\alpha t} \sup_{\tau \geq s} e^{-\alpha(\tau-(s-t))} \| U(\tau, s-t)u(s-t) \| \]
\[ \leq e^{\alpha t} \varphi_{\mathcal{U},\alpha}(s-t, u), \]
therefore $\varphi_{\mathcal{U},\alpha}(s, T_\alpha(t)u) \to 0$ as $s \to \infty$, which leads to $T_\alpha(t)u \in \mathcal{C}(\mathcal{U}, \alpha)$. Notice that the above estimation also proves inequality (7). We now show that the space $C_{c,0}(\mathbb{R}_+, X)$ is dense in $\mathcal{C}(\mathcal{U}, \alpha)$ with respect to the norm $\| \cdot \|_{\mathcal{U},\alpha}$. For any fixed
$u \in C(U, \alpha)$ and any non-negative integer $n \in \mathbb{N}$, let us consider a continuous function $\alpha_n : \mathbb{R}_+ \to [0, 1]$ such that
\[
\alpha_n(t) = 1, \text{ for all } t \in [0, n], \text{ and } \alpha_n(t) = 0, \text{ for all } t \geq n + 1.
\]
Putting $u_n = \alpha_n u$, we notice that $u_n \in C_{c,0}(\mathbb{R}_+, X)$. We claim that the limit
\[
\lim_{n \to \infty} \| u_n - u \|_{U, \alpha} = 0.
\]
Indeed, as $t \to \infty$, $\varphi_{U, \alpha}(t, u) = 0$, it follows that for each $\varepsilon > 0$ there exists $\delta > 0$ such that for $t > \delta$ we have $\varphi_{U, \alpha}(t, u) < \varepsilon / 2$. Set $n_0 = [\delta] + 1$ and choose $n \geq n_0$. The definition of map $\alpha_n$ readily implies
\[
\| u_n - u \|_{U, \alpha} = \sup_{t \geq n} \varphi_{U, \alpha}(t, u_n - u) \leq \sup_{t \geq n} \varphi_{U, \alpha}(t, u) < \varepsilon,
\]
which concludes the claim.

For the second statement, pick $u \in C_{c,0}(\mathbb{R}_+, X)$. There exist $a, b \geq 0$, $a < b$ such that $\text{supp}(T_\alpha(t) u - u) \subset [a, b]$, for sufficiently small $t \geq 0$. For such $t$ we have
\[
\| T_\alpha(t) u - u \|_{U, \alpha} = \sup_{s \geq 0} \varphi_{U, \alpha}(s, T_\alpha(t) u - u)
\]
\[
\leq \sup_{s \geq 0} M_\alpha(s) \| (T_\alpha(t) u)(s) - u(s) \|
\]
\[
= \sup_{s \in \text{supp}(T_\alpha(t) u - u)} M_\alpha(s) \| (T_\alpha(t) u)(s) - u(s) \|
\]
\[
\leq K_\alpha \sup_{s \in \text{supp}(T_\alpha(t) u - u)} \| (T_\alpha(t) u)(s) - u(s) \|,
\]
where $K_\alpha = \max M_\alpha(s)$. Using standard arguments (ex. [12]), one can easily prove that $\| T_\alpha(t) u - u \|_{U, \alpha} \to 0$ as $t \to 0_+$, and this completes the proof. \qed

The $C_0$-semigroup $T_\alpha$ defined above is called the \textit{nonuniform evolution semigroup} associated to the evolution family $U$ and the admissible exponent $\alpha$. We denote $G_{U, \alpha}$ its generator.

\textbf{Remark 3.2.} If the Banach function spaces $C(U, \alpha)$ and $C(U, \beta)$ coincide for some admissible exponents $\alpha, \beta \in A(U)$, then $T_\alpha = T_\beta$ and $G_{U, \alpha} = G_{U, \beta}$.

While in the uniform case an evolution family defines a unique evolution semigroup $T$ on $C_{00}(\mathbb{R}_+, X)$, in the nonuniform case there are infinitely many nonuniform evolution semigroups $T_\alpha$ defined on $C(U, \alpha)$, $\alpha \in A(U)$.

The next proposition emphasizes the connection between the concepts of uniform and nonuniform evolution semigroup.

\textbf{Proposition 3.3.} For $\alpha \in A(U)$, the Banach spaces $C(U, \alpha)$ and $C_{00}(\mathbb{R}_+, X)$ coincide if and only if $\alpha$ is a strict exponent. In this case, the nonuniform evolution semigroup $T_\alpha$ coincides with the (uniform) evolution semigroup $T$ on $C_{00}(\mathbb{R}_+, X)$.

\textbf{Proof.} \textit{Necessity.} Let us assume that $C(U, \alpha) = C_{00}(\mathbb{R}_+, X)$, for some $\alpha \in A(U)$. If $\alpha$ is not a strict exponent, then for each positive integer $n \in \mathbb{N}^*$ there exist $t_n \geq s_n \geq 0$ and $x_n \in X$ with $\| x_n \| = 1$, such that
\[
\| U(t_n, s_n)x_n \| > ne^{\alpha(t_n - s_n)}.
\]
If the sequence $\{s_n\}_{n \in \mathbb{N}}$ is bounded, say $s_n \leq k$, then inequality
\[
n < e^{-\alpha(t_n - s_n)} \| U(t_n, s_n)x_n \| \leq M_\alpha(s_n) \leq \sup_{0 \leq s \leq k} M_\alpha(s)\]
leads to a contradiction, $n < \sup_{0 \leq s \leq k} M_{\alpha}(s)$ for all $n \in \mathbb{N}^*$. Thus, without loss of
generality one can always assume that the sequence $\{s_n\}_{n \in \mathbb{N}^*}$ is strictly increasing,
unbounded and let us put $t_0 = s_0 = 0$, $x_0 = 0$. Setting $y_n = \frac{s_n}{\sqrt{n}}$, $n \geq 1$ and $y_0 = 0$, one gets
e^{-\alpha(t_n-s_n)} \| U(t_n, s_n)y_n \| \geq \sqrt{n}, n \in \mathbb{N}.
Consider $u_y : \mathbb{R}_+ \to X$ by
\[ u_y(s) = \frac{s(y_{n+1} - y_n)}{s_{n+1} - s_n} + \frac{s_{n+1}y_n - s_ny_{n+1}}{s_{n+1} - s_n}, \text{ if } s \in [s_n, s_{n+1}], n \in \mathbb{N}. \]
We notice that $u_y(s_n) = y_n$ for each $n \in \mathbb{N}$, that results in $u_y \in C_{00}(\mathbb{R}_+, X)$. From estimation
\[ \varphi_{\mathcal{U}, \alpha}(s_n, u_y) \geq e^{-\alpha(t_n-s_n)} \| U(t_n, s_n)y_n \| \geq \sqrt{n}, \]
we deduce $u_y \notin C(\mathcal{U}, \alpha)$, that is $C(\mathcal{U}, \alpha) \neq C_{00}(\mathbb{R}_+, X)$, which is a contradiction.

Sufficiency. If $\alpha$ is a strict exponent, then there exists $M_\alpha > 0$ such that
\[ \| U(t, s) \| \leq M_\alpha e^{\alpha(t-s)} \text{ for } t \geq s \geq 0. \]
Assumption $u \in C_{00}(\mathbb{R}_+, X)$ implies $\varphi_{\mathcal{U}, \alpha}(t, u) \leq M_\alpha \| u(t) \| \to 0$ as $t \to \infty$, hence $u \in C(\mathcal{U}, \alpha)$. \hfill \Box

The following result is taken from the theory of $C_0$-semigroups (see [10] Theorem 2.4 or [2] Lemma 1.3):

**Lemma 3.4.** Let $\mathcal{T} = \{T(t)\}_{t \geq 0}$ be a $C_0$-semigroup on a Banach space $E$, and $G$
its infinitesimal generator. If $x, y \in E$, then $x \in D(G)$ and $Gx = y$ if and only if
\[ T(t)x - x = \int_0^t T(\xi)y d\xi, \quad t \geq 0. \]

Let us substitute $E = C(\mathcal{U}, \alpha), x = u, y = -f$ and $G = G_{\mathcal{U}, \alpha}$. This rewrites as follows: if $u, f \in C(\mathcal{U}, \alpha)$, then $u \in D(G_{\mathcal{U}, \alpha})$ and $G_{\mathcal{U}, \alpha}u = -f$ if and only if
\[ u(t) = \int_0^t U(t, \xi)f(\xi) d\xi, \quad t \geq 0. \tag{8} \]

We notice that Lemma 1.1 in [9] uses the same source (Lemma 3.4) to present its similar conclusions.

4. Criteria for nonuniform exponential stability

**Lemma 4.1.** If there exists $\alpha \in \mathcal{A}(\mathcal{U}), \alpha < 0$ (i.e. $\mathcal{U}$ is $\alpha$-nonuniform exponentially stable), then the generator $G_{\mathcal{U}, \alpha}$ is invertible and
\[ \left\| G_{\mathcal{U}, \alpha}^{-1}f \right\|_{\mathcal{U}, \alpha} \leq -\frac{1}{\alpha} \| f \|_{\mathcal{U}, \alpha}. \tag{9} \]

**Proof.** For each $f \in C(\mathcal{U}, \alpha)$ we define
\[ u_f(t) = \int_0^t U(t, \xi)f(\xi) d\xi, \quad t \geq 0. \]
From $f = 0 \Rightarrow u_f = 0$ we deduce that $G_{\mathcal{U}, \alpha}$ is a one-to-one map on $C(\mathcal{U}, \alpha)$. To prove that $G_{\mathcal{U}, \alpha}$ is invertible, one needs to show first that $u_f \in C(\mathcal{U}, \alpha)$ for each
\( f \in \mathcal{C}(\mathcal{U}, \alpha) \). From estimation
\[
\varphi_{\mathcal{U}, \alpha}(t, u_f) \leq \sup_{\tau \geq t} e^{-\alpha(t-\tau)} \int_{0}^{t} \| U(\tau, \xi)f(\xi) \| \, d\xi
\]
\[
\leq \sup_{\tau \geq t} e^{-\alpha(t-\tau)} \int_{0}^{t} e^{\alpha(\tau-\xi)} \varphi_{\mathcal{U}, \alpha}(\xi, f) \, d\xi
\]
\[
= \int_{0}^{t} e^{\alpha(t-\xi)} \varphi_{\mathcal{U}, \alpha}(\xi, f) \, d\xi,
\]
as \( \lim_{t \to \infty} \varphi_{\mathcal{U}, \alpha}(t, f) = 0 \), one gets \( \lim_{t \to \infty} \varphi_{\mathcal{U}, \alpha}(t, u_f) = 0 \), therefore \( u_f \in \mathcal{C}(\mathcal{U}, \alpha) \). We conclude that \( G_{\mathcal{U}, \alpha} \) is algebraically invertible and \( -u_f = G_{\mathcal{U}, \alpha}^{-1} f \). Furthermore, inequality
\[
\varphi_{\mathcal{U}, \alpha}(t, u_f) \leq \int_{0}^{t} e^{\alpha(t-\xi)} \varphi_{\mathcal{U}, \alpha}(\xi, f) \, d\xi
\]
yields
\[
\| u_f \|_{\mathcal{U}, \alpha} \leq \| f \|_{\mathcal{U}, \alpha} \int_{0}^{t} e^{\alpha(t-\xi)} \, d\xi \leq -\frac{1}{\alpha} \| f \|_{\mathcal{U}, \alpha},
\]
that is \( G_{\mathcal{U}, \alpha}^{-1} \) is bounded and estimation (\ref{estimation}) holds. \( \square \)

Next Lemma is crucial in the sequel, and generalizes the implication \( (ii) \Rightarrow (i) \) from Theorem 2.2 in \cite{9}, for nonuniform exponentially bounded families. Obviously our techniques, based on induction, are of a completely different type (otherwise the methods and constructions used in the bibliographic source we refer to cannot apply in the nonuniform setting).

**Lemma 4.2.** If the generator \( G_{\mathcal{U}, \alpha} \) is invertible for some \( \alpha \in \mathcal{A}(\mathcal{U}) \), then \( \mathcal{U} \) is nonuniform exponentially stable.

**Proof.** Without loss of generality we may assume that \( \alpha \geq 0 \). Suppose that the operator \( G_{\mathcal{U}, \alpha} : D(G_{\mathcal{U}, \alpha}) \subset \mathcal{C}(\mathcal{U}, \alpha) \rightarrow \mathcal{C}(\mathcal{U}, \alpha) \) is invertible and put
\[
c = c(\alpha) = \| G_{\mathcal{U}, \alpha}^{-1} \|.
\]

For each positive integer \( n \in \mathbb{N}^* \) we denote \( \theta_n = \ln \left( \frac{n}{e-1} \right) \rightarrow 0 \). For fixed \( t > s \geq 0 \) and \( n \) large enough such that \( s + \theta_n \leq t \leq n \), let us consider a continuous function \( \alpha_n : \mathbb{R}_+ \rightarrow [0, 1] \) with
\[
\alpha_n(\xi) = \begin{cases}
0, & \text{if } 0 \leq \xi \leq s, \\
1, & \text{if } s + \theta_n \leq \xi \leq n, \\
0, & \text{if } \xi \geq n + \theta_n.
\end{cases}
\]

**Step 1.** We prove that
\[
\| U(t, s) \| \leq (c\alpha + 1)M_\alpha(s), \text{ for } t \geq s \geq 0.
\]
Evidently (\ref{estimate}) holds for \( \alpha = 0 \). Assume now that \( \alpha > 0 \) and fix \( t > s \geq 0 \), and \( x \in \mathcal{X} \). For \( n \) large enough we define
\[
f_n(\xi) = \begin{cases}
\alpha_n(\xi)e^{-\alpha(\xi-s)}U(\xi, s)x, & \text{if } \xi > s, \\
0, & \text{if } 0 \leq \xi \leq s.
\end{cases}
\]
We have \( f_n \in C_{c, 0}(\mathbb{R}_+, \mathcal{X}) \) and thus \( f_n \in \mathcal{C}(\mathcal{U}, \alpha) \). We claim that
\[
\| f_n \|_{\mathcal{U}, \alpha} \leq M_\alpha(s) \| x \|.
\]
Indeed, if $\xi \leq s$, as $f_n(\xi) = 0$, one has $\varphi_{u,\alpha}(\xi, f_n) = 0$, meanwhile for $\xi > s$ the following estimation holds:

$$
\varphi_{u,\alpha}(\xi, f_n) = \sup_{\tau \geq \xi} e^{-\alpha(\tau - \xi)} \| U(\tau, s)x \| \alpha_n(\xi)e^{-\alpha(\xi - s)} \leq \sup_{\tau \geq \xi} e^{-\alpha(\tau - s)} \| U(\tau, s)x \| \leq M_\alpha(s) \| x \|
$$

Putting $u_n = G_{u,\alpha}^{-1}(-f_n)$ one gets

$$
\begin{align*}
  u_n(t) &= \int_0^t U(t, \xi)f_n(\xi)d\xi \\
  &= \int_s^{s + \theta_n} \alpha_n(\xi)e^{-\alpha(\xi - s)}d\xi U(t, s)x + \int_{s + \theta_n}^t \alpha_n(\xi)e^{-\alpha(\xi - s)}d\xi U(t, s)x \\
  &= I_n U(t, s)x + \frac{1}{\alpha} \left[ e^{-\alpha\theta_n} - e^{-\alpha(t - s)} \right] U(t, s)x.
\end{align*}
$$

Here $I_n = \int_s^{s + \theta_n} \alpha_n(\xi)e^{-\alpha(\xi - s)}d\xi$. Inequality $0 \leq I_n \leq \frac{1}{\alpha} (1 - e^{-\alpha\theta_n})$ leads to

$$
\begin{align*}
  \frac{1}{\alpha} e^{-\alpha\theta_n} \| U(t, s)x \| &\leq \| u_n(t) \| + \frac{1}{\alpha} (1 - e^{-\alpha\theta_n}) \| U(t, s)x \| + \frac{1}{\alpha} e^{-\alpha(t - s)} \| U(t, s)x \| \\
  &\leq \| u_n \|_{\mathcal{U},\alpha} + \frac{1}{\alpha} (1 - e^{-\alpha\theta_n}) \| U(t, s)x \| + \frac{1}{\alpha} M_\alpha(s) \| x \| \\
  &\leq c \| f_n \|_{\mathcal{U},\alpha} + \frac{1}{\alpha} (1 - e^{-\alpha\theta_n}) \| U(t, s)x \| + \frac{1}{\alpha} M_\alpha(s) \| x \| \\
  &\leq \frac{\alpha + 1}{\alpha} M_\alpha(s) \| x \| + \frac{1}{\alpha} (1 - e^{-\alpha\theta_n}) \| U(t, s)x \|.
\end{align*}
$$

Now inequality (10) results immediately when letting $n \to \infty$.

**Step 2.** For all $k \in \mathbb{N}$ the following holds

$$
\| U(t, s) \| \leq \frac{c^k k!}{(t - s)^k} (\alpha + 1) M_\alpha(s), \ t > s \geq 0. \tag{11}
$$

Step 1 implies that inequality (11) holds for $k = 0$. Assume that (11) holds for some $k \in \mathbb{N}$. For fixed $t > s \geq 0$, $x \in X$ and sufficiently large $n$ we consider

$$
g_{n, k}(\xi) = \begin{cases} 
  \alpha_n(\xi)(\xi - s)^k U(\xi, s)x, & \text{if } \xi > s, \\
  0, & \text{if } 0 \leq \xi \leq s.
\end{cases}
$$

Since $g_{n, k} \in C_{c, 0}(\mathbb{R}_+, X)$, it follows that $g_{n, k} \in C(\mathcal{U}, \alpha)$. For $0 \leq \xi \leq s$ we notice that $\varphi_{u,\alpha}(\xi, g_{n, k}) = 0$, and if $\xi > s$ we have

$$
\begin{align*}
  \varphi_{u,\alpha}(\xi, g_{n, k}) &= \sup_{\tau \geq \xi} e^{-\alpha(\tau - \xi)} \| U(\tau, \xi)g_{n, k}(\xi) \| \\
  &= \sup_{\tau \geq \xi} e^{-\alpha(\tau - \xi)} \alpha_n(\xi)(\xi - s)^k \| U(\tau, s)x \| \\
  &\leq \sup_{\tau \geq \xi} (\tau - s)^k \| U(\tau, s)x \| \\
  &\leq c^k k! (\alpha + 1) M_\alpha(s) \| x \|,
\end{align*}
$$

that results in

$$
\| g_{n, k} \|_{\mathcal{U}, \alpha} \leq c^k k! (\alpha + 1) M_\alpha(s) \| x \|.
$$
If \( u_{n,k} = G^{-1}(-g_{n,k}) \), then
\[
  u_{n,k}(t) = \int_0^t U(t, \xi) g_{n,k}(\xi) d\xi
  = I_{n,k} U(t, s) x + \int_{s+\theta_n}^t (\xi - s)^k d\xi U(t, s) x \\
  = I_{n,k} U(t, s) x + \frac{1}{k+1} [(t-s)^{k+1} - \theta_n^{k+1}] U(t, s) x,
\]
where \( I_{n,k} = \int_s^{s+\theta_n} \alpha_n(\xi)(\xi - s)^k d\xi \leq \frac{1}{k+1} \theta_n^{k+1} \). Let us estimate
\[
  \frac{(t-s)^{k+1}}{k+1} \| U(t, s) x \|
  \leq \| u_{n,k}(t) \| + \frac{1}{k+1} \theta_n^{k+1} \| U(t, s) x \| + I_{n,k} \| U(t, s) x \|
  \leq c^{k+1} (c\alpha + 1) M\alpha(s) \| x \| \| U(t, s) x \|.
\]

Letting \( n \to \infty \), we deduce that \( \| U(t, s) \| \leq c^{k+1} (c\alpha + 1) M\alpha(s) \| x \| \| U(t, s) x \| \).

**Step 3.** Pick \( \delta \in (0, 1) \). Multiplying \( \| U(t, s) \| \leq c^{k+1} (c\alpha + 1) M\alpha(s) \| x \| \| U(t, s) x \| \) by \( \delta^k \) and summing with respect to \( k \in \mathbb{N} \) one easily gets
\[
  \| U(t, s) \| \leq \frac{c\alpha + 1}{1 - \delta} M\alpha(s) e^{-\frac{\delta}{M\alpha(s)} (t-s)}, \quad t \geq s \geq 0,
\]
that ends our proof. \( \square \)

**Definition 4.3.** We say that the admissible exponent \( \alpha \in \mathcal{A}(U) \) is quasi-negative if \( \mathcal{C}(U, \alpha) = \mathcal{C}(U, -\nu) \), for some admissible exponent \( -\nu < 0 \).

It follows that each negative admissible exponent is quasi-negative. We also notice that the set of all quasi-negative exponents is an interval. According to Proposition 3.3, the evolution family \( U \) is uniform exponentially stable if and only if it has strict quasi-negative exponents. In the nonuniform case the situation is more complicate, as illustrated in the below examples.

**Example 4.4.** For \( t \geq s \geq 0 \) we set
\[
  E(t, s) = s(2 + \sin s) - t(2 + \sin t),
\]
and let \( U(t, s)x = e^{E(t,s)x}, x \in X. \) We claim that the evolution family \( U \) is nonuniform exponentially stable, but not uniform exponentially bounded. In fact, \( \mathcal{A}(U) = [-1, \infty) \) and \( \mathcal{C}(U, \alpha) = \mathcal{C}(U, -1) \) for all \( \alpha \in \mathcal{A}(U) \), that is all admissible exponents are quasi-negative.

Indeed, let us notice that
\[
  E_1(t, s) = E(t, s) + t - s = s(1 + \sin s) - t(1 + \sin t) \leq s + s \sin s = f_1(s),
\]
and as \( E_1(2n\pi + 3\pi/2, s) = f_1(s) \) for \( n \) sufficiently large, one obtains
\[
  \sup_{t \geq s} E_1(t, s) = f_1(s).
\]

It follows that \(-1 \in \mathcal{A}(U) \). If \( \varepsilon > 0 \), then for \( E_\varepsilon(t, s) = E(t, s) + (1 + \varepsilon)(t - s) = s(1 - \varepsilon + \sin s) - t(1 - \varepsilon + \sin t), \)
we get
\[ E_c(2n\pi + 3\pi/2, s) = s (1 - \varepsilon + \sin s) + \varepsilon (2n\pi + 3\pi/2) , \]
which implies that \( \sup_{t \geq s} E_c(t, s) = \infty \), and consequently \( 1 - \varepsilon \notin A(\mathcal{U}) \). Thus, \( A(\mathcal{U}) = [-1, \infty) \). Since the map \( f_1 \) is unbounded, the evolution family \( \mathcal{U} \) is not uniform exponentially bounded. For any fixed \( \alpha \in A(\mathcal{U}) \) let us define
\[ E_\alpha(t, s) = E(t, s) - \alpha (t - s) = s (2 + \alpha + \sin s) - t (2 + \alpha + \sin t) , \]
and put \( f_2(s) = \sup_{t \geq s} E_\alpha(t, s) \). For \( u \in C(\mathbb{R}_+, X) \) we have
\[ \varphi_{\mathcal{U}, \alpha}(s, u) = \sup_{t \geq s} e^{-\alpha(t-s)} \| U(t, s) u(s) \| = \sup_{t \geq s} E_\alpha(t, s) \| u(s) \| = e^{f_2(s)} \| u(s) \| , \]
and similarly \( \varphi_{\mathcal{U}, -1}(s, u) = e^{f_1(s)} \| u(s) \| \). For any fixed \( s \geq 0 \) we denote \( n_s = \lfloor \frac{1}{2\pi} - \frac{3}{4} \rfloor \), that implies \( t_s = 2(n_s + 1)\pi + 3\pi/2 \in (s, s + 2\pi) \). Let us remark that
\[ f_1(s) - f_2(s) \leq f_1(s) - E_\alpha(t_s, s) = (1 + \alpha) (t_s - s) \leq 2\pi (1 + \alpha) . \]
Gathering the above identities and estimation, one gets
\[ \varphi_{\mathcal{U}, -1}(s, u) = e^{f_1(s)} \| u(s) \| = e^{f_1(s)} e^{f_2(s)} \varphi_{\mathcal{U}, \alpha}(s, u) \leq e^{2\pi(1 + \alpha)} \varphi_{\mathcal{U}, \alpha}(s, u). \]
We conclude that \( C(\mathcal{U}, \alpha) = C(\mathcal{U}, -1) \).

For the evolution family in the next example the situation is completely different. It has no admissible exponents both positive and quasi-negative at the same time. Besides, it is uniform exponentially bounded.

**Example 4.5.** For \( t \geq s \geq 0 \) we set
\[ E(t, s) = s \left( \sqrt{2} + \sin \ln s \right) - t \left( \sqrt{2} + \sin \ln t \right) , \]
and let
\[ E_\alpha(t, s) = E(t, s) - \alpha (t - s) = s \left( \alpha + \sqrt{2} + \sin \ln s \right) - t \left( \alpha + \sqrt{2} + \sin \ln t \right) , \]
for fixed \( \alpha \in \mathbb{R} \). We consider the evolution family \( U(t, s) x = e^{E(t,s)x}, t \geq s \geq 0, x \in X \).

We claim that \( A(\mathcal{U}) = [1 - \sqrt{2}, \infty) \), the quasi-negative admissible exponents are only the negative ones, and \( \mathcal{U} \) is uniform exponentially bounded.

If \( \alpha < 1 - \sqrt{2} \), then \( \alpha = 1 - \sqrt{2} - \varepsilon \), for some \( \varepsilon > 0 \). Let us notice that
\[ E_{1 - \sqrt{2} - \varepsilon}(t, s) = s (1 - \varepsilon + \sin \ln s) - t (1 - \varepsilon + \sin \ln t) . \]
For \( t_n = e^{2n\pi + \sqrt{2}} \), \( n \in \mathbb{N} \), we have \( E_{1 - \sqrt{2} - \varepsilon}(t_n, 0) = \varepsilon t_n \to \infty \), which implies that \( \alpha \notin A(\mathcal{U}) \).

If \( \alpha \in [1 - \sqrt{2}, 0) \), then there exists \( \varepsilon \in [0, \sqrt{2} - 1) \) such that \( \alpha = 1 - \sqrt{2} + \varepsilon \).
In this case we have
\[ E_{1 - \sqrt{2} + \varepsilon}(t, s) = s (1 + \varepsilon + \sin \ln s) - t (1 + \varepsilon + \sin \ln t) \leq s (1 + \sin \ln s) = f_1(s) . \]
Thus, \( \alpha \in A(\mathcal{U}) \). We conclude that \( A(\mathcal{U}) = [1 - \sqrt{2}, \infty) \).
For $\alpha \geq 0$ let us put $f_\alpha (t) = t \left( \alpha + \sqrt{2} + \sin t \right)$, $t \geq 0$. The derivative is

$$f'_\alpha (t) = \alpha + \sqrt{2} \left[ 1 + \sin \left( \frac{\pi}{4} + \ln t \right) \right] \geq 0.$$  

Thus,

$$E_\alpha (t, s) = f_\alpha (s) - f_\alpha (t) = f'_\alpha (\theta_{t,s}) (s - t) \leq 0,$$

and so $\| U (t, s) \| \leq e^{\alpha (t-s)}$. This means that $U$ is uniform exponentially bounded.

If $\alpha \in (1 - \sqrt{2}, 0)$, as $f_1 (s)$ is unbounded, the inclusion $C(U, \alpha) \subset C_0 (\mathbb{R}_+, X)$ is strict. In this case the admissible exponent $\alpha$ is not quasi-negative.

We consider the next theorem as the main result of the paper.

**Theorem 4.6.** Let $\alpha \in \mathcal{A} (U)$. The generator $G_{U, \alpha}$ is invertible if and only if $\alpha$ is a quasi-negative admissible exponent.

**Proof.** The sufficiency follows directly from Lemma 4.1 and Remark 3.2. For the necessity, assume that $G_{U, \alpha}$ is invertible. According to Lemma 4.2 $U$ is nonuniform exponentially stable. Choose $\nu > 0$ with $-\nu \notin \mathcal{A} (U)$, $-\nu \neq \inf \mathcal{A} (U)$. It suffices to consider the case $\alpha \geq 0$. For each fixed $s \geq 0$ and $x \in X \setminus \{ 0 \}$ we construct the map $\tilde{u}_{s,x} \in C (\mathbb{R}_+, X)$ given by

$$\tilde{u}_{s,x} (\xi) = \begin{cases} e^{\nu (\xi - s)} U (\xi, s) x, & \text{if } \xi > s; \\ x, & \text{if } 0 \leq \xi \leq s. \end{cases} \quad (12)$$

Pick $\varepsilon > 0$ such that $-\nu - \varepsilon \notin \mathcal{A} (U)$. For $t > s$ we have

$$\varphi_{U, \alpha} (t, \tilde{u}_{s,x}) = \sup_{\tau \geq t} e^{-\alpha (\tau - t)} e^{\nu (\tau - s)} \| U (\tau, s) x \| \leq \sup_{\tau \geq t} e^{\nu (\tau - s)} e^{-(\nu + \varepsilon)(\tau - s)} M_{-\nu - \varepsilon} (s) \| x \| \leq e^{-\varepsilon (t-s)} M_{-\nu - \varepsilon} (s) \| x \|,$$

therefore $\lim_{t \to \infty} \varphi_{U, \alpha} (t, \tilde{u}_{s,x}) = 0$. Proposition 2.3 implies that the below set is nonempty:

$$\Lambda_{s,x} = \left\{ t \geq 0 : \sup_{\xi \geq 0} \varphi_{U, \alpha} (\xi, \tilde{u}_{s,x}) = \varphi_{U, \alpha} (t, \tilde{u}_{s,x}) \right\}.$$

As the map $\xi \mapsto \varphi_{U, \alpha} (\xi, \tilde{u}_{s,x})$ is continuous, it follows that $t_{s,x} = \inf \Lambda_{s,x} \in \Lambda_{s,x}$.

We have the alternative:

(A1) There exists $\nu > 0$, $-\nu \notin \mathcal{A} (U)$, $-\nu \neq \inf \mathcal{A} (U)$ such that for each $s \geq 0$ and $x \in X \setminus \{ 0 \}$, $t_{s,x} \in \left[ s, s + \frac{1}{\nu} \right]$.

(A2) For any sufficiently small $\nu > 0$, $-\nu \notin \mathcal{A} (U)$, $-\nu \neq \inf \mathcal{A} (U)$, there exists $s \geq 0$ and $x \in X \setminus \{ 0 \}$ with $t_{s,x} > s + \frac{1}{\nu}$.

Assume that (A1) holds. In this case, for all $t \geq s$ and $x \in X \setminus \{ 0 \}$ we have

$$\| \tilde{u}_{s,x} (t) \| \leq \varphi_{U, \alpha} (t, \tilde{u}_{s,x}) \leq \varphi_{U, \alpha} (t_{s,x}, \tilde{u}_{s,x}) = \sup_{\tau \geq t_{s,x}} e^{-\alpha (\tau - t_{s,x})} e^{\nu (\tau - s)} \| U (\tau, s) x \|. \quad (13)$$
Therefore, for all $s \geq 0$ and $x \in X$ we have
\[
\sup_{t \geq s} e^{-\alpha(t-s)} \|U(t, s)x\| \leq K \sup_{t \geq s} e^{-\alpha(t-s)} \|U(t, s)x\|
\]
where $K = e^{\frac{1}{2}(\alpha + \nu)}$. It turns out that
\[
\varphi_{U, \alpha}(s, u) \leq \varphi_{U, -\nu}(s, u) \leq K \varphi_{U, \alpha}(s, u), \ s \geq 0, \ u \in C(\mathbb{R}_+, X).
\]
We conclude that $C(U, \alpha) = C(U, -\nu)$, as Banach spaces, thus $\alpha$ is quasi-negative.

Assume that (A2) holds. For sufficiently large $n \in \mathbb{N}^*$ such that $-\frac{1}{n} \notin \mathcal{A}(U)$, we put $s_n \geq 0, \ x_n \in X \setminus \{0\}$ and $t_n = t_n x_n \in C(\mathbb{R}_+, X)$. Let us define the $C^1$-map $\psi_n : \mathbb{R}_+ \to \mathbb{R}_+$ by
\[
\psi_n(t) = \begin{cases}
\frac{1}{n}e^{\frac{1}{n}(t-s_n)}, & \text{if } t > t_n;
\frac{1}{n}e^{\frac{1}{n}(t-s_n)}(1 - \frac{t - s_n}{t_n}), & \text{if } s_n - \delta_n < t \leq t_n;
\frac{1}{n}t^2 + b_n t + c_n, & \text{if } s_n < t \leq s_n + \delta_n;
0, & \text{if } 0 \leq t \leq s_n.
\end{cases}
\]
The constants $a_n, b_n, c_n \in \mathbb{R}$ and $\delta_n \in (s_n, t_n)$ are determined by condition $\psi_n \in C^1(\mathbb{R}_+)$. We also define $u_n, f_n : \mathbb{R}_+ \to X$ by
\[
u_n(t) = \begin{cases}
\psi_n(t) U(t, s_n)x_n, & \text{if } t > s_n;
u_n(t) U(t, s_n)x_n, & \text{if } 0 \leq t \leq s_n;
0, & \text{if } 0 \leq t \leq s_n.
\end{cases}
\]
Evidently $u_n, f_n \in C(U, \alpha)$ and $G_{U, \alpha} u_n = -f_n$. Notice that for $t \in [0, t_n]$ we have
\[
\varphi_{U, \alpha}(t, u_n) = \sup_{\tau \geq t} e^{-\alpha(\tau-t)} \psi_n(t) \|U(\tau, s_n)x_n\|
\]
\[
\leq \sup_{\tau \geq t} e^{-\alpha(\tau-t)} e^{\frac{1}{n}(t-s_n)} \|U(\tau, s_n)x_n\|
\]
\[
= \varphi_{U, \alpha}(t, \tilde{u}_{s_n, x_n}),
\]
where $\tilde{u}_{s_n, x_n}$ is defined in (12). If $t \geq t_n$, as $\varphi_{U, \alpha}(t, u_n) = \varphi_{U, \alpha}(t, \tilde{u}_{s_n, x_n})$, it follows that $\| u_n \|_{U, \alpha} = \varphi_{U, \alpha}(t_n, \tilde{u}_{s_n, x_n})$, and similarly one gets $\| f_n \|_{U, \alpha} = \frac{1}{n} \varphi_{U, \alpha}(t_n, \tilde{u}_{s_n, x_n})$. These identities imply
\[
\frac{\| u_n \|_{U, \alpha}}{\| f_n \|_{U, \alpha}} = n,
\]
which leads to $\sup_{f \in C(U, \alpha)} \frac{\| G_{U, \alpha}^{-1} f \|_{U, \alpha}}{\| f \|_{U, \alpha}} \geq n$. Thus, $G_{U, \alpha}^{-1}$ is not bounded, which is false, and eventually only alternative (A1) holds, that proves the claim.
As in \cite{9}, the spectral mapping theorem is to be deduced from exponential stability.

**Corollary 4.7** (The Spectral Mapping Theorem). For each $\alpha \in A(\mathcal{U})$, the nonuniform evolution semigroup $T_\alpha = \{ T_\alpha(t) \}_{t \geq 0}$ satisfies the identity

$$e^{t\sigma(G_{\mathcal{U},\alpha})} = \sigma(T_\alpha(t)) \setminus \{ 0 \}, \ t \geq 0.$$  

Moreover, $\sigma(G_{\mathcal{U},\alpha})$ is a left half-plane and $\sigma(T_\alpha(t)), \ t \geq 0$, is a disc.

**Proof.** If $\lambda \in \mathbb{C}$, then we consider the rescaled evolution family

$$U_\lambda(t,s) = e^{-\lambda(t-s)}U(t,s).$$

For arbitrary $\alpha \in A(\mathcal{U}), \ t \geq 0$ and $u \in C(\mathbb{R}_+, X)$ one has

$$\varphi_{\mathcal{U},\alpha-Re\lambda}(t,u) = \sup_{\tau \geq t} e^{-(\alpha-Re\lambda)(\tau-t)} \| U_\lambda(\tau,t)u(t) \|$$

$$= \sup_{\tau \geq t} e^{-\alpha(\tau-t)} \| U(\tau,t)u(t) \| = \varphi_{\mathcal{U},\alpha}(t,u).$$

We deduce that the admissible sets $A(\mathcal{U}) = A(\mathcal{U}) - Re\lambda$, and the admissible Banach spaces $C(\mathcal{U},\alpha-Re\lambda) = C(\mathcal{U},\alpha)$. From another hand, the definition of the generator implies that $D(G_{\mathcal{U},\alpha}) = D(G_{\mathcal{U},\alpha-Re\lambda})$ and

$$G_{\mathcal{U},\alpha-Re\lambda} = G_{\mathcal{U},\alpha} - \lambda Id. \quad (13)$$

Let $\lambda \in \rho(G_{\mathcal{U},\alpha})$, that is $G_{\mathcal{U},\alpha-Re\lambda}$ is invertible. Choose $\mu \in \mathbb{C}$ with $Re\mu \geq Re\lambda$. We only have two options:

- (B1) $Re\mu > \alpha$;
- (B2) $Re\lambda \leq Re\mu \leq \alpha$.

In case (B1), Theorem 4.6 and formula (13) imply that $G_{\mathcal{U},\alpha-Re\mu}$ is invertible, consequently $\mu \in \rho(G_{\mathcal{U},\alpha})$.

Assume that (B2) holds. According to the same Theorem 4.6 as $G_{\mathcal{U},\alpha-Re\lambda}$ is invertible, there exists $\nu > 0$ with $C(\mathcal{U},\alpha-Re\lambda) = C(\mathcal{U},-\nu)$, and $G_{\mathcal{U},\alpha-Re\lambda} = G_{\mathcal{U},-\nu}$. Using Eq. (13), we successively have

$$G_{\mathcal{U},\alpha-Re\mu} = G_{\mathcal{U},\alpha-Re\lambda} + (\lambda - \mu) Id$$

$$= G_{\mathcal{U},-\nu} + (\lambda - \mu) Id$$

$$= G_{\mathcal{U},-\nu-Re\lambda-Re\mu}.$$  

Since $-\nu + Re\lambda - Re\mu < 0$, then $G_{\mathcal{U},-\nu-Re\lambda-Re\mu}$ is invertible, hence $G_{\mathcal{U},\alpha-Re\mu}$ is also invertible. This shows that $\mu \in \rho(G_{\mathcal{U},\alpha})$, which implies that the spectrum $\sigma(G_{\mathcal{U},\alpha})$ is a left half-plane. For the rest of the proof we refer the reader to \cite{9}. \hfill \Box

We now recover the result from Theorem 2.2 in \cite{9}, valid in the particular case of uniform exponential stability.

**Corollary 4.8.** Let $\mathcal{U}$ be a uniform exponentially bounded evolution family. Then $\mathcal{U}$ is uniform exponentially stable if and only if the generator of the corresponding evolution semigroup is invertible.
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