Coexistence of non-periodic attractors

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Abstract

The Hénon family contains a countable set of maps which have simultaneously two non-periodic attractors, i.e. two Cantor attractors or one Cantor attractor and a strange one. This holds in any general two-dimensional unfolding of a map with a strong homoclinic tangency. Moreover when one adds any number of parameters to the two-dimensional family, the two Cantor attractors start to move creating a codimension two lamination. This allows to find other type of coexistence of non-periodic attractors in the three dimensional quadratic Hénon-like family, such as three Cantor attractors or two Cantor attractors and a strange one.

1 Introduction

In order to understand the long term behavior of a dynamical system, one possible approach is to study the set where a lot of orbits spend most of the time. This set is called the attractor of the system. Moreover, as soon as an attractor is detected, one would like to know for which other parameters a similar attractor occurs in a family of systems of the same type, i.e. if the attractor appears for many parameters in the family. This says how much and in which form an attractor is stable. Attractive periodic orbits and hyperbolic attractors, for example, persist by changing parameters in an open set. They have the strongest form of stability.

We study here two dimensional unfoldings of maps with a strong homoclinic tangency, see Definition 2.10. What makes such families special is that, like in most systems with frictions, in a neighborhood of an homoclinic tangency, the first return map is an Hénon-like map, see [9]. The study of the local dynamics is reduced to the dynamics of an Hénon map or an Hénon-like map, see Section 7 for the precise definition. In the Hénon family and in other two-dimensional Hénon-like families the following attractors has been detected:

- there are maps, for an open set of parameters, having one attractive periodic orbit, a sink,

- there are maps, for a positive Lebesgue set of parameters, having a strange attractor

\[1\] See Definition 7.1
there are maps having a Cantor attractor and they form a smooth curve in parameter space, see [5].

A natural question is to ask if there are maps in the Hénon family having two or more of these attractors, if they are ”observable” in the family and in which form. The most natural strategy to find them is to start with a map having a periodic attracting orbit and try to find parameters in the open set where the periodic orbit persists which have also another attractor. This approach has been used originally by Newhouse in [8]. One difference from this classical construction is that we need to make a selection of parameters. This can be a very sophisticated procedure, which have been solved, in [2, 3, 10] using different strategies. In these papers the authors find parameters in the Hénon family corresponding to maps having either coexistence of periodic attractors (sinks), or periodic attractors and one non-periodic attractor. Here we solve the more delicate problem to find parameters in the Hénon family or in general in any two-dimensional unfoldings whose corresponding maps have two non-periodic attractors. Non-periodic attractors are much less stable than the periodic ones, they can be easily destroyed by changing parameters. Despite this we can prove that two Cantor attractors coexist, for a countable set of parameters.

**Theorem A.** Let \( F : \mathcal{P} \times M \to M \) be a real-analytic two dimensional unfolding of a map \( f \) with a strong homoclinic tangency, then there exists a countable set \( 2PD \subset \mathcal{P} \), such that, each map in \( 2PD \) has two period doubling Cantor attractors.

Our method combined with the method in [2], allows also to find parameters in which finitely many sinks and two Cantor attractors coexist.

**Theorem B.** Fix \( S \in \mathbb{N} \). Let \( F : \mathcal{P} \times M \to M \) be a real-analytic two dimensional unfolding of a map \( f \) with a strong homoclinic tangency, then there exists a countable set \( S2PD \subset \mathcal{P} \), such that, each map in \( S2PD \) has at least \( S \) sinks and two period doubling Cantor attractors.

In two-dimensional families the coexistence phenomena described in Theorem A and Theorem B appear in the parameter space in isolated points. However if you add any new number of parameters, these points start to move forming codimension two laminations. This says that the coexistence phenomena found are ”observable” in families with at least three parameters and they come in codimension two laminations. The coexistence of two Cantor attractor has a codimension two nature. This is quite surprising if one consider that a non-periodic attractor is by itself a very unstable object and it is easily destroyable by changing parameters. Here we prove that not only one, but even two Cantor attractors together survive in higher dimensional families and they form laminations.

**Theorem C.** Let \( M, \mathcal{P} \) and \( \mathcal{T} \) be real-analytic manifolds and \( F : (\mathcal{P} \times \mathcal{T}) \times M \to M \) a real-analytic family with dim(\( \mathcal{P} \)) = 2 and dim(\( \mathcal{T} \)) \geq 1. If \( F_0 : (\mathcal{P} \times \{\tau_0\}) \times M \to M \) is an unfolding of a map \( f_{\tau_0} \) with a strong homoclinic tangency, then the set of maps, \( 2PD_F \), with two period doubling Cantor attractors satisfy the following:

- \( 2PD_F \) contains a codimension 2 lamination \( L_F \),

- the leaves of \( L_F \) are real-analytic codimension 2 manifolds,

See Definition 4.1
- the two period doubling Cantor attractors persist along each leave of the lamination.

The same is true for finitely many sinks and two period doubling Cantor attractors. In families of dimension at least three they form codimension two laminations.

**Theorem D.** Let \( M, P \) and \( T \) be real-analytic manifolds and \( F : (P \times T) \times M \to M \) a real-analytic family with \( \dim(P) = 2 \) and \( \dim(T) \geq 1 \). Fix \( S \in \mathbb{N} \). If \( F_0 : (P \times \{\tau_0\}) \times M \to M \) is an unfolding of a map \( f_{\tau_0} \) with a strong homoclinic tangency, then the set of maps, \( S2PD_F \), with \( S \) sinks and two period doubling Cantor attractors satisfy the following:

- \( S2PD_F \) contains a codimension 2 lamination \( L_F \),
- the leaves of \( L_F \) are real-analytic codimension 2 manifolds,
- the \( S \) sinks and the two period doubling Cantor attractors persist along each leave of the lamination.

Theorem B and Theorem D apply in particular to the real Hénon family \( F : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}^2 \),

\[
F_{a,b}(x, y) = \left( a - x^2 - by, x \right)
\]

and polynomial maps of \( \mathbb{R}^2 \).

**Theorem E.** Fix \( S \in \mathbb{N} \). The real Hénon family contains a countable set \( S2PD \) of maps with at least \( S \) sinks and two period doubling Cantor attractors. Moreover the space \( \text{Poly}_d(\mathbb{R}^2) \) of real polynomials of \( \mathbb{R}^2 \) of degree at most \( d \), with \( d \geq 2 \), contains a codimension 2 lamination of maps with at least \( S \) sinks and two period doubling Cantor attractors. The leaves of the lamination are real-analytic. The \( S \) sinks and the two period doubling Cantor attractors persist along each leave of the lamination.

Actually, in the Hénon family, one can even find the coexistence of any number of sinks, one Cantor attractor and one strange attractor. In the parameter space, the set of points presenting this phenomenon has Hausdorff dimension at least one.

**Theorem F.** Fix \( S \in \mathbb{N} \). The set of Hénon maps with \( S \) sinks, one period doubling Cantor attractor and one strange attractor has Hausdorff dimension at least one.

Theorem E and Theorem F also hold for two-dimensional Hénon-like families. The lamination in Theorem E allows to find other coexistence phenomena in a three dimensional family of quadratic Hénon-like maps. A map of the plane \( \mathbb{R}^2 \) of the form

\[
F_{a,b,\tau}(x, y) = \left( a - x^2 - by + \tau y^2, x \right)
\]

is called a quadratic Hénon-like map.
**Theorem G.** Fix $S \in \mathbb{N}$. There are countably many quadratic Hénon-like maps with $S$ sinks and three period doubling Cantor attractors.

By adding one more parameter to the Hénon family, one can find maps with finitely many sinks and three period doubling Cantor attractors. Surprisingly, this coexistence phenomenon has a codimension three nature. If, in three dimensional families, the maps with finitely many sinks and three period doubling Cantor attractors form points, in higher dimensional families they form a codimension three lamination. This is stated in the following theorem.

**Theorem H.** Let $S \in \mathbb{N}$. The space $\text{Poly}_d(\mathbb{R}^2)$, $d \geq 2$, of real polynomials of $\mathbb{R}^2$ of degree at most $d$ contains a codimension 3 lamination of maps with at least $S$ sinks and three period doubling Cantor attractors. The leaves of the lamination are real-analytic. The $S$ sinks and the three period doubling Cantor attractors persist along the leaves.

Finally, in the quadratic Hénon-like family, one can find also maps having simultaneously finitely many sinks, two Cantor attractors and one strange attractor. In the parameter space they have Hausdorff dimension at least one.

**Theorem I.** Fix $S \in \mathbb{N}$. The set of quadratic Hénon-like maps with $S$ sinks, two period doubling Cantor attractors and one strange attractor has Hausdorff dimension at least one.

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## 2 Preliminaries

The following well-known linearization result is due to Sternberg.

**Theorem 2.1.** Given $(\lambda, \mu) \in \mathbb{R}^2$, there exists $N (\lambda, \mu) \in \mathbb{N}$ such that the following holds. Let $M$ be a 2 dimensional $C^\infty$ manifold and let $f : M \to M$ be a diffeomorphism with saddle point $p \in M$, with unstable eigenvalue $|\mu| > 1$ and stable eigenvalue $\lambda$. If

$$\lambda \neq \mu^{k_1} \quad \text{and} \quad \mu \neq \lambda^{k_2}$$

for $k = (k_1, k_2) \in \mathbb{N}^2$ with $2 \leq |k| = k_1 + k_2 \leq N$, with $N$ large enough, then $f$ is $C^4$ linearizable.

**Definition 2.3.** Let $M$ be an 2-dimensional $C^\infty$ manifold and $f : M \to M$ a diffeomorphism with a saddle point $p \in M$. We say that $p$ satisfies the $C^4$ non-resonance condition if (2.2) holds.

**Theorem 2.4.** Let $M$ be a 2-dimensional $C^\infty$ manifold and $f : M \to M$ a diffeomorphism with a saddle point $p \in M$ which satisfies the $C^4$ non-resonance condition. Let $0 \in \mathcal{P} \subset \mathbb{R}^n$ and $F : M \times \mathcal{P} \to M$ a $C^\infty$ family with $F_0 = f$. Then, there exists a neighborhood $U$ of $p$ and a neighborhood $V$ of 0 such that, for every $t \in V$, $F_t$ has a saddle point $p_t \in U$ satisfying the $C^4$ non-resonance condition. Moreover $p_t$ is $C^4$ linearizable in the neighborhood $U$ and the linearization depends $C^4$ on the parameters.
The proofs of Theorem 2.1 and Theorem 2.4 can be found in [4, 6]. The following lemma is a direct consequence of Theorem 2.1.

**Definition 2.5.** Let $M$ be a 2-dimensional $C^\infty$ manifold and $f : M \to M$ a local diffeomorphism satisfying the following conditions:

1. $f$ has a saddle point $p$, with unstable eigenvalue $|\mu| > 1$ and stable eigenvalue $\lambda$,
2. $|\lambda||\mu|^3 < 1$,
3. $p$ satisfies the $C^4$ non-resonance condition,
4. $f$ has a non-degenerate homoclinic tangency, $q_1 \in W^u(p) \cap W^s(p)$,
5. $f$ has a transversal homoclinic tangency, $q_2 \in W^u(p) \cap W^s(p)$,
6. let $[p, q_2]^u \subset W^u(p)$ be the arc connecting $p$ to $q_2$, then there exist arcs $W^u_{loc,n}(q_2) = [q_2, u_n]^u \subset W^u(q_2)$ such that $[p, q_2]^u \cap [q_2, u_n]^u = \{q_2\}$ and
   \[
   \lim_{n \to \infty} f^n(W^u_{loc,n}(q_2)) = [p, q_2]^u,
   \]
7. there exist neighborhoods $W^u_{loc,n}(q_1) \subset W^u(q_1)$ such that
   \[
   \lim_{n \to \infty} f^n(W^u_{loc,n}(q_1)) = [p, q_2]^u,
   \]
8. there exists $N \in \mathbb{N}$ such that
   \[
   f^{-N}(q_1) \in [p, q_2]^u.
   \]

A map $f$ with these properties is called a map with a strong homoclinic tangency.

**Remark 2.6.** If the unstable eigenvalue is negative, $\mu < -1$, then (f6), (f7), and (f8) are redundant.

Following [9], we define now an unfolding of a map $f$ with a strong homoclinic tangency. Let $\mathcal{P} = [-r, r]^2$ with $r > 0$. Given a map $f$ with a strong homoclinic tangency, we consider a $C^\infty$ family $F : \mathcal{P} \times M \to M$ through $f$ with the following properties:

1. $F_{0,0} = f$,
2. $F_{t,a}$ has a saddle point $p(t,a)$ with unstable eigenvalue $|\mu(t,a)| > 1$ and stable one $\lambda(t,a)$,
3. let $\mu_{\max} = \max_{(t,a)} |\mu(t,a)|$, $\lambda_{\max} = \max_{(t,a)} |\lambda(t,a)|$ and assume
   \[
   \lambda_{\max}\mu_{\max}^3 < 1,
   \]
4. there exists a $C^2$ function $[-r, r] \ni t \mapsto q_1(t) \in W^u(p(t,0)) \cap W^s(p(t,0))$ such that $q_1(t)$ is a non-degenerate homoclinic tangency.
According to Theorem 2.4 we may assume without loss of generality that the family \( F \) is \( C^4 \) and for all \((t,a) \in [-r_0, r_0]^2\) with \(0 < r_0 < r\), \(F_{t,a}\) is linear on the ball \([-2, 2]^2\), namely
\[
F_{t,a} = \begin{pmatrix} \lambda(t,a) & 0 \\ 0 & \mu(t,a) \end{pmatrix}.
\]

Moreover the saddle point \( p(t,a) = (0,0) \) and the local stable and unstable manifolds satisfy:
- \( W^s_{\text{loc}}(0) = [-2, 2] \times \{0\} \)
- \( W^u_{\text{loc}}(0) = \{0\} \times [-2, 2] \)
- \( q_1(t) \in (0, 1] \times \{0\} \subset W^s_{\text{loc}}(0) \)
- \( q_2(t, a) \in \{0\} \times \left( \frac{1}{\mu}, 1 \right) \subset W^u_{\text{loc}}(0) \)
- there exists \( N \) such that \( f^N(q_3(t, a)) = q_1(t, a) \) where \( q_3(t, a) = (0, 1) \)
- \( Df^N_q(e_1) \notin T_q W^s(0) \) and points in the positive \( y \) direction.

The next lemma states that \( q_3 \) is contained in a curve of points whose vertical tangent vectors are mapped by \( DF^N \) to horizontal ones. The proof is the same as Lemma 2 in \[2\]. Let \((x, y)\) in a neighborhood of \( q_3 \) and consider the point
\[(X_{t,a}(x, y), Y_{t,a}(x, y)) = F^N_{t,a}(x, y)\].

**Lemma 2.7.** There exist \( x_0, a_0 > 0 \), a \( C^2 \) function \( c : [-x_0, x_0] \times [-t_0, t_0] \times [-a_0, a_0] \to \mathbb{R} \) and a positive constant \( Q \) such that
\[
\frac{\partial Y_{t,a}}{\partial y}(x, c(x, t, a)) = 0
\]
and
\[
\frac{\partial^2 Y_{t,a}}{\partial y^2}(x, c(x, t, a)) \geq Q.
\]
Moreover
\[
|c(x, t, a) - c(0, t, a)| = O \left( |x| \right).
\]

**Definition 2.9.** Let \((t, a) \in [-t_0, t_0] \times [-a_0, a_0] \). We call the point
\[
c_{t,a} = (0, c(0, t, a))
\]
the primary critical point and
\[
z_{t,a} = F^N_{t,a}(c_{t,a}) = (z_x(t, a), z_y(t, a))
\]
the primary critical value of \( F_{t,a} \).

Observe that, near the saddle point, vertical vectors are expanding and horizontal ones are contracting. The critical point are defined to have the property that the expanding vertical vectors are sent to the contracting horizontal ones. See \[2\] for a more extensive discussion on the role of critical points.
Definition 2.10. A family \( F_{t,a} \) is called an unfolding of \( f \) if it can be reparametrized such that

\[
P_1: \quad z_y(t,0) = 0,
\]

\[
P_2: \quad \frac{\partial z_y(t,0)}{\partial a} \neq 0.
\]

Remark 2.11. Without lose of generality we may assume that if \( F \) is an unfolding then \( z_y(t,a) = a \), the primary critical value is at height \( a \).

Remark 2.12. A generic 2 dimensional family trough \( f \) can locally be re-parametrized to become an unfolding.

3 First return map

Consider the real Hénon family \( F : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}^2 \),

\[
F_{a,b}(x,y) = \left( a - x^2 - by, x \right),
\]

a two parameter family. We abuse the notation with respect to the general definition of unfolding, \( F_{t,a} \). The parameter curve \((a,0)\) in the Hénon family is not a curve of homoclinic tangencies. In particular,

\[
F_{a,0} = \left( a - x^2, x \right)
\]

is called the degenerate Hénon family. Given an unfolding \( F_{t,a} \) we are going to study the first return map to a rectangle in the phase space near the critical value. We are going to prove that the first return map, restricted to a well chosen box, is arbitrarily close to a degenerate Hénon map. This appears already in [9, 10]. In order to get a more quantitative version, we repeat here the argument. For the sequel we need in fact precise estimates on the dependence of first return map in terms of the parameters.

Fix an unfolding \( F \) and for each \((t, a) \in [-t_0, t_0] \times [-a_0, a_0]\) let

\[
\Gamma = \Gamma_{t,a} = \{ (x, c(x, t, a)) | x \in [-x_0, x_0] \}.
\]

In the next lemma, whose proof appears already in [2] (see Lemma 3), we build a curve \( a_n \) of points, in the parameter space, whose corresponding critical values are mapped after \( n \) steps into \( \Gamma \).

Lemma 3.2. For \( n \) large enough, there exists a \( C^2 \) function \( a_n : [-t_0, t_0] \to (0, a_0] \) such that

\[
F_{t,a_n(t)}^n \left( z_{(t,a_n(t))} \right) \in \Gamma_{t,a_n(t)}.
\]

Moreover

\[
\frac{da_n}{dt} = -n \frac{\partial \mu}{\partial t} \frac{1}{\mu_n+1} \left[ 1 + O(|\lambda_1|^n) \right].
\]

The following is a preliminary lemma.
Lemma 3.3. There exist \( x'_0 < x_0, a'_0 < a_0, b > 0 \) and \( Q > 0 \), such that, for all \((t, a) \in [-t_0, t_0] \times [-a'_0, a'_0] \) and for every \((x, y) \in \Gamma_{t, a} \) with \(|x| < |x'_0|\) the following holds. There exist \( A_{x,y}, B_{x,y} \neq 0 \) and \( C_{x,y} \neq 0 \) such that \( F^N_{t,a} \) in coordinates centered in \((x, y)\) and \( F^N_{t,a}(x, y) \) has the form

\[
F^N_{t,a} \left( \begin{array}{c} \Delta x \\ \Delta y \end{array} \right) = \left( \begin{array}{cc} A_{x,y} & B_{x,y} \\ C_{x,y} & 0 \end{array} \right) \left( \begin{array}{c} \Delta x \\ \Delta y \end{array} \right) + \left( \begin{array}{c} O \left( |\Delta x|^2 + |\Delta y|^2 \right) \\ Q_{x,y} \Delta y^2 + O \left( |\Delta x|^2 + |\Delta y|^3 + |\Delta x| |\Delta y| \right) \end{array} \right)
\]

where \( A_{x,y}, B_{x,y}, C_{x,y} \) are \( C^2 \) dependent on \( x \) and \( y \), \( Q_{x,y} > Q \), \( C_{x,y} > C > 0 \) and \(|B_{x,y}| > B > 0\).

Proof. The lemma gives the Taylor expansions of \( F^N_{t,a} \) when \((x, y) \in \Gamma_{t,a} \). It follows immediately from Lemma 2.7. In particular the horizontal tangency,

\[
DF^N_{t,a}(x, y) \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} B_{x,y} \\ 0 \end{pmatrix}
\]

is not degenerate for all \((x, y) \in \Gamma_{t,a} \). Let \( t \in [-t_0, t_0] \). Because \( F^N_{t,0}(q_3(t)) = q_1(t) \) is a non degenerate homoclinic tangency, we know that the vector \( B_{q_3(t)} \neq 0 \) and \( Q_{q_3(t)} > 0 \). By taking \(|a| < |a'_0|, |x| < |x'_0|\) small enough, the lower bounds on \( Q_{x,y} \) and \( B_{x,y} \) follow. The lower bound on \( C_{x,y} \) follows from \((f7)\). \(\square\)

Remark 3.4. Observe that the order symbols in Lemma 3.3 represent \( C^1 \) functions whose \( C^3 \) norm is given by the order.

Choose \( E > 0 \) and define, for \( n \) large enough,

\[
\mathcal{H}_n = \left\{ (t, a) \in [-t_0, t_0] \times [-a_0, a_0] \mid |a - a_n(t)| \leq \frac{E}{|\mu(t, a_n(t))|^{2n}} \right\}.
\]

Remark 3.5. Since \( \partial \mu / \partial a \) is bounded we have that for all \((t, \tilde{a}) \in \mathcal{H}_n\),

\[
\left( \frac{\left| \mu(t, a_n(t)) \right|}{\mu(t, \tilde{a})} \right)^n = 1 + O \left( \frac{n}{\mu(t, a_n(t))^{2n}} \right).
\]

For \((t, a) \in \mathcal{H}_n\), consider \( z_{t,a} = (z_x(t, a), z_y(t, a)) \) and the point

\[
s_{t,a} = (s_x(t, a), s_y(t, a)) \in \Gamma_n(t, a) = F^{-n}_{t,a} \left( \Gamma_{t,a} \right) \text{ with } s_x(t, a) = z_x(t, a),
\]

see Figure 1. When the dependence on the parameters is clear and not essential, we suppress the parameter indices. Observe that \( s \) exists because, by the \( \lambda \)-lemma, see [9], for \( n \) large enough, \( \Gamma_n \) extends to a global graph.

Take \((t, a) \in \mathcal{H}_n\) and denote by \( c_n(t) = c_{t,a_n(t)} \) and by \( z_n(t) = z_{t,a_n(t)} = (z_{n,x}(t), z_{n,y}(t)) \). When the choice of \( t \) clear we just use the notation \( c_n \) and \( z_n \). The following is a technical lemma showing the vertical distance between \( s \) and the image of any point in \( \Gamma(t, a) \) under the map \( F^N_{t,a} \).

Lemma 3.6. Let \((t, a) \in \mathcal{H}_n\), \( c' = (c'_{x}, c'_{y}) \in \Gamma(t, a) \) with \(|c'_x| \leq 3 \lambda (t, a)^n \) and \( F^N_{t,a}(c') = (F^N_{t,a}(c')_x, F^N_{t,a}(c')_y) \) then, for \( n \) large enough,

\[
|F^N_{t,a}(c')_y - s_y| = |a - a_n(t)| + O \left( \frac{n}{\mu(t, a)^{3n}} \right).
\]
Figure 1: Location of the point $s$

Proof. Observe that

$$|F_{t,a}^N(c')_y - s_y| = |F_{t,a}^N(c')_y - z_y + z_y - z_{n,y} + z_{n,y} - s_y|,$$  

(3.7)

$$|F_{t,a}^N(c')_y - z_y| = O \left( \lambda (t,a)^n \right)$$  

(3.8)

and

$$|z_y - z_{n,y}| = |a - a_n(t)|.$$  

(3.9)

It is left to estimate $|z_{n,y} - s_y|$. Because $|\Gamma_{t,a} - \Gamma_{t,a_n(t)}|_{C^1} = O \left( |a - a_n(t)| \right)$, it follows that

$$\mu (t,a)^n s_y = \mu (t,a_n(t))^n z_{n,y} + O \left( |a - a_n(t)| \right).$$

As consequence, using the fact that $(t,a) \in \mathcal{H}_n$, i.e. $|a - a_n(t)| \leq O \left( 1/\mu (t,a)^2n \right)$ and Remark 3.5 we have

$$s_y = \frac{\mu (t,a_n(t))^n}{\mu (t,a)^n} z_{n,y} + O \left( \frac{|a - a_n(t)|}{\mu (t,a)^n} \right)$$

$$= \left( 1 + O \left( \frac{n}{\mu (t,a_n(t))^{2n}} \right) \right) z_{n,y} + O \left( \frac{1}{\mu (t,a)^{3n}} \right).$$

Finally,

$$|z_{n,y} - s_y| = O \left( \frac{n}{\mu (t,a_n(t))^{2n}} \right) \cdot O \left( \frac{1}{\mu (t,a_n(t))^{n}} \right) + O \left( \frac{1}{\mu (t,a)^{3n}} \right)$$

$$= O \left( \frac{n}{\mu (t,a)^{3n}} \right).$$  

(3.10)

where we used Lemma 3.2. The lemma follows by combining (3.7), (3.8), (3.9), (3.10) and using $(f2)$. □
Take \((t,a) \in \mathcal{H}_n\). For \(v, h > 0\) we define a box in the phase space, centered around \(s\), where we will study the first return map, see Figure 2

\[
B_{v,h}^n(t,a) = \left\{ (x,y) | |x - s_x| \leq \frac{h}{|\mu(t,a)|^{2n}}, |y - s_y| \leq \frac{v}{|\mu(t,a)|^{2n}} \right\}.
\]

Consider now the rescaled first return map to the box \(B_{v,h}^n(t,a)\),

\[
r_{F_{t,a}} = \varphi_{t,a}^{-1} \circ F_{t,a}^N \circ F_{t,a}^n \circ \varphi_{t,a}
\]

where the rescaling \(\varphi_{t,a} : [-1,1] \times [-1,1] \mapsto B_{v,h}^n(t,a)\) is defined as

\[
\varphi_{t,a}(x,y) = \left( \frac{h}{|\mu(t,a)|^{2n}} y + s_x, \frac{v}{|\mu(t,a)|^{2n}} x + s_y \right).
\]  

(3.11)

Observe that \(\varphi_{t,a}^{-1} : B_{v,h}^n(t,a) \mapsto [-1,1] \times [-1,1]\) is given by

\[
\varphi_{t,a}^{-1}(x,y) = \left( \frac{|\mu(t,a)|^{2n}}{h} y - \frac{|\mu(t,a)|^{2n}}{v} s_y, \frac{|\mu(t,a)|^{2n}}{h} x - \frac{|\mu(t,a)|^{2n}}{v} s_x \right).
\]

Because \(\varphi_{t,a}^{-1}\) extends to a map from \(\mathbb{R}^2\) to \(\mathbb{R}^2\), \(r_{F_{t,a}}\) is well defined also when the image of \(F_{t,a}^N \circ F_{t,a}^n \circ \varphi_{t,a}\) is not contained in \(B_{v,h}^n(t,a)\).

In the sequel, in order to make the presentation cleaner, we suppress the dependence on the parameters, for example \(B_{v,h}^n = B_{v,h}^n(t,a), \varphi = \varphi_{t,a}, \lambda = \lambda(t,a)\) and \(\mu = \mu(t,a)\).

Using the notation from Lemma 3.3, we fix \(v = 1/Q\) and \(h = B/Q\) where \(B = B_{F^n(s)}\) and \(Q = Q_{F^n(s)}\). The following lemma shows the shape of the rescaled first return map to the box \(B_{v,h}^n(t,a)\). A similar statement appears already in [9, 10].
Lemma 3.12. Fix $\beta \in [-E, E]$ and let $(t, a) \in \mathcal{H}_n$ such that $a = a_n(t) + \beta/|\mu(t, a_n(t))|^{2n}$. Then, for $n$ large enough and for all $(x, y) \in [-1, 1] \times [-1, 1]$, we have

$$r_{F_{t,a}} \left( \begin{array}{c} x \\ y \end{array} \right) = \left( \begin{array}{c} x^2 + Q\beta + CB \left( |\lambda| |\mu| \right)^n y + O \left( n |\mu|^{-n} \right) \\ x + O \left( (|\lambda| |\mu|)^n \right) \end{array} \right).$$

Proof. By the expression of the map $\phi$, the fact that $F^n$ is linear, the expression for $F^N$ given in Lemma 3.3 and the expression of $\varphi^{-1}$, we have

$$r_{F_{t,a}} \left( \begin{array}{c} x \\ y \end{array} \right) = \left( \begin{array}{c} x^2 + CB \left( |\lambda| |\mu| \right)^n y + Q |\mu|^{2n} \left[ F^N(c')_x - s_x \right] \\ x + BQ^{-1} |\mu|^n \left[ F^N(c')_x - s_x \right] \end{array} \right) + \left( \begin{array}{c} O \left( |\lambda|^{2n} y^2 \right) + O \left( (|\lambda|^n |x| y) + O \left( (|\mu|^{-n} x^3 \right) \\ O \left( (|\lambda|^n y) + O \left( (|\lambda|^2 |\mu|^{-1})^n y^2 \right) + O \left( (|\mu|^{-n} x^2 \right) \end{array} \right) \right)$$

(3.13)

where $c' = f^n(s) \in \Gamma$ and $F^N(c') = (F^N(c')_x, F^N(c')_y)$. Observe now that

$$F^N(c')_x - s_x = O \left( |\lambda|^n \right)$$

(3.14)

and by Lemma 3.6

$$|F^N(c')_y - s_y| = |a - a_n(t)| + O \left( \frac{n}{\mu^{3n}} \right) = \frac{\beta}{|\mu|^{2n}} + O \left( \frac{n}{\mu^{3n}} \right)$$

(3.15)

The lemma follows by plugging (3.14) and (3.15) in (3.13) and by using (f2). \(\square\)

In the sequel we prove that, if one starts with a real-analytic unfolding $F$, then the first return map to an appropriately chosen rectangle can be normalized to obtain a map arbitrarily close to a degenerate Hénon map, see (3.1).

Consider a real-analytic unfolding $F_{t,a}$ of a strong homoclinic tangency, say with $(t, a) \in (-1, 1) \times (-1, 1)$. Let $\beta$ such that $a = a_n(t) + \beta/|\mu(t, a_n(t))|^{2n}$. We may assume that this family extends to an holomorphic family of the form

$$\mathbb{D} \times \mathbb{D} \ni (t, a) \mapsto F_{t,a} : U \times U \to \mathbb{C}^2$$

where $U$ is a domain in $\mathbb{C}$. There is a local holomorphic change of coordinates such that the saddle point becomes $(0, 0)$, the local stable manifold contains the unit disc in the $x$-axis, and the local unstable manifold contains the unit disc in the $y$-axis. Moreover, the restriction of the map to the invariant manifolds is linearized, that is

$$F(x, 0) = (\lambda x, 0) \text{ and } F(0, y) = (0, \mu y).$$

From now on we will study the map $F^{n+N}$ in this new coordinates. From [2] there exists an analytic function $t \mapsto sa_n(t)$ such that, in the parameter $(t, sa_n(t))$ the periodic point, $p_t$, called the strong sink of period $n + N$, has trace zero. The graph of the real part of this function is contained in $\mathcal{H}_n$. Let $(t, a) \in \mathcal{H}_n$ and consider the strong sink $p_t = (p_{t,x}, p_{t,y})$. Take the box $B_{t,a}$ centered in $p_t$ in the new coordinates of the same size of the the previous box $B^n_{v,h}$. Consider now the rescaled first return map to the box $B_{t,a}$,

$$pHF_{t,a} = \phi_{t,a}^{-1} \circ F_{t,a}^N \circ F_{t,a}^n \circ \phi_{t,a}$$

where $\phi_{t,a} : [-1, 1] \times [-1, 1] \mapsto B_{t,a}$ is defined as

$$\phi_{t,a} \left( \begin{array}{c} x \\ y \end{array} \right) = \left( \frac{\frac{h}{|\mu(t,a)|} y + p_{t,x}}{|\mu(t,a)|^{2n} x + p_{t,y}} \right).$$
Observe that \( \mathcal{H}_n \ni (t, a) \rightarrow pHF_{t,a} \) is a real-analytic family. For \( n \) large enough, each \( pHF_{t,a} \) extends holomorphically to \( \mathbb{D} \times \mathbb{D} \).

Let \( h_{t,a} \) be a smooth linearization of \( F_{t,a} \) and let \( \tilde{F}_{t,a} = h_{t,a} \circ F_{t,a} \circ h_{t,a}^{-1} \) be the corresponding smooth unfolding of a strong homoclinic tangency. Let \( \varphi_{t,a} \) be the linearization defining \( r\tilde{F}_{t,a} \), see (3.11) and Lemma 3.12. Observe that

\[
D\phi_{t,a} \circ D\varphi_{t,a}^{-1} = id. \tag{3.16}
\]

**Lemma 3.17.** Let \( \tilde{p}_{t,a} = (\tilde{p}_x(t,a),\tilde{p}_y(t,a)) = h_{t,a}(p_t) \). Then

\[
|\tilde{p}_x(t,a) - s_x(t,a)| = O\left(\frac{1}{|\mu(t,a_n(t))|^2n}\right),
|\tilde{p}_y(t,a) - s_y(t,a)| = O\left(\frac{n}{|\mu(t,a_n(t))|^{3n}}\right).
\]

**Proof.** Fix \((t,a) \in \mathcal{H}_n\) and let \( \Delta a = a - sa_n(t) \). Then \( \Delta a = O(1/\mu^{2n}) \). Let \( \Delta h_x = \tilde{p}_x(t,a) - \tilde{p}_s(t,sa_n(t)) \). Because the linearization \( h_{t,a} \) depends smoothly on the parameters we have that

\[
\Delta h_x = O\left(\frac{1}{\mu^{2n}}\right). \tag{3.18}
\]

Moreover, the linearization has the form \( h_y(x,y,t,a) = A(x,y,t,a)y \) where \( A(x,y,t,a) \) is a smooth function. Let \( \Delta h_y = \tilde{p}_y(t,a) - \tilde{p}_y(t,a_n(t)) \). Then

\[
\Delta h_y = O\left(\frac{1}{\mu^n}\right) \Delta a = O\left(\frac{1}{\mu^{3n}}\right). \tag{3.19}
\]

Without lose of generality we may assume that \( s_x(t,a) = 1 \). Let \( \Delta s = s_y(t,a) - s_y(t,sa_n(t)) \). Because \( s \in \Gamma_n \) and by Lemma 12 in [2] we have

\[
\Delta s = \frac{\partial \Gamma_n}{\partial a} \Delta a = O\left(\frac{n}{\mu^{3n}}\right). \tag{3.20}
\]

The lemma follows by comparing (3.18), the fact that \( s_x(t,a) = 1 \), (3.19) and (3.20). \( \square \)

Let \( r\tilde{p}_{t,a} = \varphi_{t,a}^{-1}(\tilde{p}_{t,a}) \). From Lemma 3.17

\[
|r\tilde{p}_{t,a}| = O\left(\frac{n}{|\mu(t,a_n(t))|^{n}}\right). \tag{3.21}
\]

Observe that \( r\tilde{p}_{t,a} \) depends \( C^1 \) smoothly on the parameters. Let \( \psi_{t,a} \) be the translation over \( r\tilde{p}_{t,a} \) and define

\[ pr\tilde{F}_{t,a} = \psi_{t,a}^{-1} \circ r\tilde{F}_{t,a} \circ \psi_{t,a}. \]

From Lemma 3.12 we know

\[
pr\tilde{F}_{t,a} \left(\begin{array}{c} x \\ y \end{array}\right) = \left(\begin{array}{c} x^2 + Q\beta + CB(|\lambda|\mu)^ny + O(n|\mu|^{-n}) \\ x + O(n|\mu|^{-n}) \end{array}\right). \tag{3.22}
\]

Consider now the diffeomorphism

\[ ph_{t,a} = \psi_{t,a}^{-1} \circ r\tilde{F}_{t,a} \circ \psi_{t,a}. \]

Without lose of generality we can take the domain of linearization to be an arbitrarily small neighborhood of the saddle point. This implies that the linearization \( h_{t,a} \) is arbitrarily \( C^1 \) close to identity. Moreover, because of (3.16) and (3.21) we have that \( ph_{t,a} \) is \( C^1 \) close to identity. Hence, by construction, the real-analytic family

\[ pH_{t,a} = ph_{t,a}^{-1} \circ pr\tilde{F}_{t,a} \circ ph_{t,a} \]
is $C^1$ close to $pr\tilde{F}_{t,a}$. As consequence

$$pHF_{t,a}(x, y) = \left(\frac{x^2 + Q\beta + u_{t,a}}{x + v_{t,a}}\right)$$

where $u_{t,a}$ and $v_{t,a}$ are real-analytic functions which are arbitrarily close to zero in $C^1$. Let

$$\xi_{t,a}(x, y) = \left(-x - v_{t,a}, -y\right).$$

Then

$$\xi_{t,a}^{-1}(x, y) = \left(-x + \tilde{v}_{t,a}, -y\right)$$

where $\tilde{v}_{t,a}$ is a real-analytic function arbitrarily close to zero in $C^1$. Define $HF_{t,a} = \xi_{t,a} \circ pHF_{t,a} \circ \xi_{t,a}^{-1}$. The previous discussion gives that the normalized maps $HF$ are Hénon-like maps, see [5]. This is stated in the following proposition.

**Proposition 3.23.** Let $F : \mathcal{P} \times M \to M$ be a real-analytic two dimensional unfolding of a map $f$ with a strong homoclinic tangency. Fix $\beta \in [-E, E]$ and let $(t, a) \in \mathcal{H}_n$ such that $a = a_n(t) + \beta/|\mu(t, a_n(t))|^{2^n}$. Then, for $n$ large enough and for all $(x, y) \in [-1, 1] \times [-1, 1],

$$HF_{t,a}(x, y) = \left(-x^2 - Q\beta + u_{t,a}\right)$$

defines an Hénon-like family where $u_{t,a}$ is arbitrarily close to zero and it extends holomorphically to $\mathbb{D} \times \mathbb{D}$.

Fix $t \in [-t_0, t_0]$ and consider the one-parameter real-analytic family

$$[-E, E] \ni \beta \mapsto vHF_\beta = HF_{t,a_n(t) + \beta/|\mu(t, a_n(t))|^{2^n}}.$$

See Figure 3. Observe that, for all $\beta$, $vHF_\beta$ is a strongly dissipative Hénon-like map and it is contained in the bigger space of Hénon-like maps, see [5].

By the form of the map $vHF_\beta$ and as corollary of Proposition 3.23 we find the following.

**Lemma 3.24.** By chosing the return box close enough to the saddle point and for $n$ large enough, the family $[-E, E] \ni \beta \mapsto vHF_\beta$ is arbitrarily close to the degenerate Hénon family. In particular, if $\beta = 0$, then $vHF_0$ is arbitrarily close to the degenerate Hénon map

$$vHF_0 \sim \left(-x^2\right)$$

and if $\beta = -E$, with $E = 2/\min_{(t, a_n(t))} Q$, then $R_{-E}$ is arbitrarily close to the degenerate Hénon map

$$vHF_{-E} \sim \left(2 - x^2\right).$$

In particular $vHF_0$ has entropy $0$ and $vHF_{-E}$ has entropy $\log 2$. 


In this section we prove the existence of a curve $PD_n$ contained in $\mathcal{H}_n$ such that, the maps corresponding to the points in $PD_n$ have a period doubling Cantor attractor. We start by recalling the following definition.

**Definition 4.1.** Let $M$ be a manifold and $f : M \to M$. An invariant Cantor set $A \subset M$ is called a period doubling Cantor attractor of $f$ if $f|A$ is conjugated to a 2-adic adding machine and there is a neighborhood $M \supset U \supset A$ such that the orbit of almost every point in $U$ accumulates at $A$.

**Remark 4.2.** A period doubling Cantor attractor has zero topological entropy. It carries a unique invariant probability measure. Strongly dissipative Hénon-like maps at the boundary of chaos have period doubling Cantor attractors, see [5].

In the space of Hénon-like maps, there exists a codimension one manifold $PD$ of maps which have a period doubling Cantor attractor, see [5].

**Proposition 4.3.** The one-parameter family $\beta \mapsto vHF_\beta$ intersects the $PD$ manifold of maps with a period doubling Cantor attractor in a unique point. Moreover the intersection is transversal.

**Proof.** From [5], the degenerate Hénon family crosses the manifold $PD$ transversally in a unique point. Because, by Lemma [3.24], the one-parameter family $\beta \mapsto vHF_\beta$ is arbitrarily close to the degenerate Hénon family in the $C^1$ topology, the proposition follows. \qed

We define the period doubling curve $PD_n$ as

$$PD_n = \{(t,a) \in \mathcal{H}_n \mid HF_{t,a} \text{ has a period doubling Cantor attractor}\}.$$

The transversality in Proposition [4.3] gives the following, see Figure 4.
Proposition 4.4. Let $F : \mathcal{P} \times M \to M$ be a real-analytic two dimensional unfolding of a map $f$ with a strong homoclinic tangency, then the period doubling curve $PD_n$ is the graph of a real-analytic function.

5 Coexistence of sinks and non-periodic attractors

In this section we prove that each real-analytic two-dimensional unfolding contains countably many maps with two period doubling Cantor attractors. As consequence we also prove the existence of maps with finitely many sinks and two period doubling Cantor attractor.

Given any family $F$ of real-analytic maps, we define the set $2PD_F$ as the set of parameters having two period doubling Cantor attractors and the set $S2PD_F$ as the set of parameters having at least $S$ sinks and two period doubling Cantor attractors.

Theorem A. Let $F : \mathcal{P} \times M \to M$ be a real-analytic two dimensional unfolding of a map $f$ with a strong homoclinic tangency, then there exists a countable set $2PD \subset 2PD_F$, such that, each map in $2PD$ has two period doubling Cantor attractors.

Remark 5.1. Observe $2PD$ consists of isolated points. However the closure of $2PD$ contains a curve of homoclinic tangencies.

Proof. By Proposition 4 in [2], there are countably many curves $b_{n,n_0} : [t_{n,n_0}^-, t_{n,n_0}^+] \to \mathbb{R}$ such that, the maps corresponding to points in these curves have a secondary homoclinic tangency. Moreover, for all $E > 0$ and for $n$ large enough,

$$b_{n,n_0}(t_{n,n_0}^+) > a_n(t) + \frac{E}{|\mu(t, a_n(t))|^{2n}}$$

and

$$b_{n,n_0}(t_{n,n_0}^-) < a_n(t) - \frac{E}{|\mu(t, a_n(t))|^{2n}}.$$ 

As consequence the curves $b_{n,n_0}$ cross the strip $\mathcal{H}_n$ and in particular the curve $PD_n$. Let now

$$\mathcal{P}_{n,n_0} = \left\{(t,a) \in b_{n,n_0}^{-1}(\mathcal{H}_n) \times [-a_0,a_0] \mid |a - a_n(t)| \leq \frac{E}{|\mu(t, a_n(t))|^{2n}}\right\}.$$
Figure 5: Intersections of curves of period doubling attractors

Proposition 5 in [2] says that $F: \mathcal{P}_{n,n_0} \times M \mapsto M$ can be reparametrized to become an unfolding. We apply now Sections 3 and 4 to the restricted unfolding and we get new curves $PD_m^{(n,n_0)}$, corresponding to maps with a period doubling Cantor attractor which, for $m$ large enough, accumulate to the curve $b_{n,n_0}$. Because $b_{n,n_0}$ intersects the curve $PD_n$, then also the curve $PD_m^{(n,n_0)}$ intersects $PD_n$ and the intersection point corresponds then to a map with 2 period doubling Cantor attractors, see Figure 5.

Theorem B. Fix $S \in \mathbb{N}$. Let $F: \mathcal{P} \times M \mapsto M$ be a real-analytic two dimensional unfolding of a map $f$ with a strong homoclinic tangency, then there exists a countable set $S2PD \subset S2PD_F$, such that, each map in $S2PD$ has at least $S$ sinks and two period doubling Cantor attractors.

Proof. This is a consequence of Theorem A in [2] and Theorem A in this paper. It is enough, to stop the inductive procedure in the proof of Theorem A in [2] at the step $S$. At this moment there is a rectangle in the parameter space which is crossed diagonally by a curve of secondary homoclinic tangencies, see Figure 5. The family restricted to this rectangle is an unfolding of a map with a strong homoclinic tangency given by this curve and all maps have $S$ sinks, see Proposition 5 in [2]. In particular the curve of homoclinic tangencies is real-analytic and the parameters rectangle can be reparametrized in an analytic way to obtain a real-analytic unfolding. By applying Theorem A to this family one gets the stated theorem.

6 Laminations

Let $(t,a(t)) \in PD_n$, then $a(t) = a_n(t) + \Delta a(t)$ with corresponding $\beta(t) = \Delta a(t)|\mu(t,a_n(t))|^{2n}$. Consider the one parameter family $t \mapsto hHF_t$ where

$$hHF_t = vHF_{(t,a(t))}(x,y) = \left(-x^2 - Q\Delta a(t)|\mu(t,a_n(t))|^{2n} + u\right)$$

and for all $t$, $hHF_t$ has a period doubling Cantor attractor. By changing $t$ we get a one-parameter Hénon-like family in the $PD$ manifold, a real-analytic curve.
Proposition 6.1. For \( m \) large enough, the curve \( PD_m^{(n,n_0)} \) crosses the curve \( PD_n \) transversally and the angle is larger than \( V\lambda_{min}^n \) where \( V \) is a uniform constant and \( \lambda_{min} = \min_{(t,a)} |\lambda(t,a)| \).

Proof. Consider the families \( t \mapsto hHF_t \) and

\[
P_t \left( \begin{array}{c} x \\ y \end{array} \right) = \left( -x^2 - Q\Delta a(t)\mu(t,a_n(t))^{2n} \right) \frac{y}{x}
\]

where \( t \mapsto hHF_t \) is a real-analytic family in the space of Hénon-like maps and \( P_t \) is degenerate quadratic part. Observe that \( t \mapsto \text{dist}_{C^0} (hHF_t, P_t) \) is a smooth function uniformly bounded in \( C^1 \). Let now \( a_\infty \) in the degenerate Hénon family be the map with a period doubling Cantor attractor. Then

\[
(\text{dist}_{C^0} (P_t, F_{a_\infty}))^2 = (Q\Delta a(t)\mu(t,a_n(t))^{2n} + a_\infty)^2.
\]

Because the degenerate Hénon family crosses transversally the \( PD \) manifold, then

\[
0 = (\text{dist}_{C^0} (hHF_t, PD))^2 = (Q\Delta a(t)\mu(t,a_n(t))^{2n} + a_\infty + \omega(t))^2
\]

where \( \omega(t) \) is a uniformly bounded \( C^1 \) function. By differentiating this last equation, we get

\[
0 = \frac{dQ}{dt} \Delta a(t)\mu(t,a_n(t))^{2n} + 2nQ\Delta a(t)\mu(t,a_n(t))^{2n-1} \frac{d\mu}{dt} + Q\mu(t,a_n(t))^{2n} \frac{d\Delta a}{dt} + \frac{d\omega(t)}{dt} = O(n) + Q\mu(t,a_n(t))^{2n} \frac{d\Delta a}{dt} + \frac{d\omega(t)}{dt} = O(1)
\]

where we used the fact that \( \Delta(a) = O(\mu^{-2n}) \) and that \( d\omega(t)/dt = O(1) \). As consequence \( \frac{d\Delta a}{dt} = O(n\mu^{-2n}) \) and in particular, by Lemma 3.2

\[
\frac{dPD_n}{dt} = \frac{da_n}{dt} + \frac{d\Delta a}{dt} = -n \frac{d\mu}{dt} \frac{1}{\mu^{n+1}} \left[ 1 + O \left( |\mu|^{-n} \right) \right].
\]

By Proposition 4 in [2],

\[
\frac{db_{n,n_0}}{dt} = \frac{da_n}{dt} + Vn\lambda^n + O \left( |\lambda|^n \right)
\]

where \( V \) is uniformly away from zero, \( \theta < 1/2 \) and \( |\lambda|^{2\theta} |\mu|^3 > 1 \). In particular the curve \( b_{n,n_0} \) crosses the curve \( PD_n \) transversally and the angle is larger than \( V\lambda^n \) where \( V \) is a uniform constant. The proposition follows by using the fact that, for \( m \) large enough, the curve \( PD_m^{(n,n_0)} \) accumulate in \( C^1 \) on the curve \( b_{n,n_0} \). \( \square \)

Remark 6.2. Observe that, by Proposition 6.1, the angle formed by the intersection of \( PD_m^{(n,n_0)} \) with the curve \( PD_n \) is larger than \( V\lambda_{max}^n \), for \( n \geq n_0 \). By Remark 10 in [2], \( n_0 \) and \( V \) are locally constant, i.e. they depends continuously on the family.
Theorem C. Let $M, P$ and $T$ be real-analytic manifolds and $F : (P \times T) \times M \to M$ a real-analytic family with dim$(P) = 2$ and dim$(T) \geq 1$. If $F_0 : (P \times \{\tau\}) \times M \to M$ is an unfolding of a map $f_{\tau_0}$ with a strong homoclinic tangency, then the set of maps, $2PD_F$, with two period doubling Cantor attractors satisfy the following:

- $2PD_F$ contains a codimension 2 lamination $L_F$,
- the leaves of $L_F$ are real-analytic codimension 2 manifolds,
- the two period doubling Cantor attractors persist along each leave of the lamination.

Proof. Observe that there exists a small neighborhood $\tau_0 \in U \subset T$ and a real-analytic function $U \ni \tau \mapsto f_\tau$ such that, for all $\tau \in U$, $f_\tau$ has a strong homoclinic tangency and the family $F_\tau : (P \times \{\tau\}) \times M \to M$ is an unfolding of $f_\tau$. In the unfolding $F_{\tau_0}$, there is a curve $PD^{(a, a_0)}_m$ which intersects transversally the curve $PD_n$ in the point $(t_0, \tau_0, a_0)$. From Proposition 6.1 we get a lower bound for the angle between these curves which is independent of the parameter $\tau$, see Remark 6.2. This transversality implies that this intersection persists in a neighborhood of $\tau_0$ as the graph of a real-analytic function. The uniform lower bound of the angle implies that this function extends globally. In particular the two period doubling attractors at $(t_0, \tau_0, a_0)$ have their smooth continuation in all unfoldings $F_\tau$ for any given $\tau \in U$, creating a codimension 2 leave of the lamination. Because $PD_m$ and $PD_n$ are real-analytic curves, the leaves of the lamination are also real-analytic manifolds.

A similar proof as the one of Theorem C, gives the following.

Theorem D. Let $M, P$ and $T$ be real-analytic manifolds and $F : (P \times T) \times M \to M$ a real-analytic family with dim$(P) = 2$ and dim$(T) \geq 1$. Fix $S \in \mathbb{N}$. If $F_0 : (P \times \{\tau_0\}) \times M \to M$ is an unfolding of a map $f_{\tau_0}$ with a strong homoclinic tangency, then the set of maps, $S2PD_F$, with $S$ sinks and two period doubling Cantor attractors satisfy the following:

- $S2PD_F$ contains a codimension 2 lamination $L_F$,
- the leaves of $L_F$ are real-analytic codimension 2 manifolds,
- the $S$ sinks and the two period doubling Cantor attractors persist along each leave of the lamination.

7 The Hénon family

Consider the real Hénon family $F : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}^2$,

$$F_{a,b} \left( \begin{array}{c} x \\ y \end{array} \right) = \left( \begin{array}{c} a - x^2 - by \\ x \end{array} \right),$$

a two parameter family. Because the Hénon family is an unfolding of a map with a strong homoclinic tangency, see the proof of Theorem C in [2], we can apply Theorem B, and Theorem D to get a lamination of maps with $S$ sinks and two period doubling
Cantor attractors in the space of polynomial maps, see Theorem E. Actually we can start with any two parameter polynomial family which is an unfolding of a map with a strong homoclinic tangency and we get laminations scattered throughout the space of polynomial maps.

**Theorem E.** Fix $S \in \mathbb{N}$. The real Hénon family contains a countable set $S2PD$ of maps with at least $S$ sinks and two period doubling Cantor attractors. Moreover the space $\text{Poly}_d(\mathbb{R}^2)$ of real polynomials of $\mathbb{R}^2$ of degree at most $d$, with $d \geq 2$, contains a codimension 2 lamination of maps with at least $S$ sinks and two period doubling Cantor attractors. The leaves of the lamination are real-analytic. The $S$ sinks and the two period doubling Cantor attractors persist along each leave of the lamination.

With the aim of proving the existence of maps in the Hénon family with finitely many sinks, one period doubling Cantor attractor and one strange attractor we need the following definition.

**Definition 7.1.** Let $M$ be a manifold and $f : M \to M$. An open set $U \subset M$ is called a trapping region if $f(U) \subset U$. An attractor in the sense of Conley is

$$\Lambda = \bigcap_{j \geq 0} f^j(U).$$

The attractor $\Lambda$ is called topologically transitive if it contains a dense orbit. If $\Lambda$ contains a dense orbit which satisfies the Collet-Eckmann conditions, i.e. there exist a point $z$, a vector $v \in T_z M$ and a constant $\kappa > 0$ such that

$$|Df^n(z)v| \geq e^{\kappa n} \text{ for all } n > 0,$$

then $\Lambda$ is called a strange attractor.

**Theorem F.** Fix $S \in \mathbb{N}$. The set of Hénon maps with $S$ sinks, one period doubling Cantor attractor and one strange attractor has Hausdorff dimension at least one.

**Proof.** We stop the inductive procedure in the proof of Theorem A in [2] at the step $S$. At this moment there is a rectangle in the parameter space crossing a curve of homoclinic tangencies. The family restricted to this rectangle is an unfolding of a map with a strong homoclinic tangency given by this curve and all maps have $S$ sinks, see Proposition 5 in [2]. In particular the curve of homoclinic tangencies is real-analytic and the parameters rectangle can be reparametrized in an analytic way to obtain a real-analytic unfolding. We apply to this family the construction of sections 3 and 4 and we get a curve $SPD_m$ of maps with $S$ sinks and one period doubling Cantor attractor. This curve intersects transversally secondary tangency curves of the type $b_{n,n_0}$, see Proposition 4 in [2]. In particular, this curve is a one-parameter analytic unfolding of a homoclinic tangency. The construction is similar as in the proof of Theorem A. According to [4] and [7], such a curve contains a set of positive measure of maps with a strange attractor. \qed
8 The quadratic Hénon-like family

A map of the plane $\mathbb{R}^2$ of the form
\[
F_{a,b,\tau} \left( \begin{array}{c} x \\ y \end{array} \right) = \left( \begin{array}{c} a - x^2 - by + \tau y^2 \\ x \end{array} \right)
\]
(8.1)
is called a quadratic Hénon-like map.

Theorem G. Fix $S \in \mathbb{N}$. There are countable many quadratic Hénon-like maps with $S$ sinks and three period doubling Cantor attractors.

Proof. By Proposition 7 in [2], there are in the Hénon family two analytic curves $a_{1,2} : [b_-, b_+] \mapsto \mathbb{R}$ such that each point on $a_1$ has a tangency at $z_1$ and each point on $a_2$ has a tangency at $z_2 \neq z_1$. Moreover $a_1$ and $a_2$ intersect transversally in a point having two tangencies and they have an analytic extension to $[b_-, b_+] \times [-\tau_0, \tau_0]$ for some $\tau_0 > 0$. Denote the graphs of the extensions of $a_{1,2}$ also by $a_{1,2}$. The transversality of the intersection between $a_1$ and $a_2$ at $\tau = 0$ implies that the graph of $a_1$ and $a_2$ intersects in an analytic curve. Along this curve there are two strong homoclinic tangencies. We can locally reparametrize the quadratic Hénon-like family near $(a_1(b_0), b_0, 0)$ to obtain a family $F_{t,a,\tau}$ such that, the maps $F_{t,a,0}$ are the maps in the graph of $a_2$ with a strong homoclinic tangency at $z_2$ and the maps $F_{t,0,\tau}$ are the maps in the graph of $a_1$ with a strong homoclinic tangency at $z_1$. The curve $F_{t,0,0}$ consists of the maps in the intersection of the two graphs having two strong homoclinic tangencies.

By the same argument as in the proof of Theorem D in [2], one can prove that the family $F_{t,a,\tau}$ is an unfolding of the strong homoclinic tangency near the point $z_2$. According to Theorem B the family $(t, a) \mapsto F_{t,a,0}$ contains a countable set $S2PD$ of parameters of maps $S$ sinks and two period doubling attractors. This set accumulates at a segment of parameters $(t, 0, 0)$. According to Theorem D, each point $m \in S2PD$ is contained in an analytic curve $\gamma_m : \tau \mapsto (t(\tau), a(\tau), \tau)$. These curves are pairwise disjoint forming a lamination. Each curve is a one-parameter unfolding of the homoclinic tangency near $z_2$ with $\tau = 0$.

Consider one of the curves $\gamma_m$. For $n \geq 1$ large enough there exits, according to Proposition 3.12 or [3], a rectangle $Q_n$ near $z_2$ and an interval $[\tau_n^0, \tau_n^1]$ such that the family
\[
[t_n^0, t_n^1] \ni \tau \mapsto F_{\gamma_m(\tau)}^n|Q_n
\]
after an analytic change of coordinates becomes a one-parameter family of Hénon-like maps which is arbitrarily $C^1$ close to the degenerated Hénon family
\[
F_{a,0} \left( \begin{array}{c} x \\ y \end{array} \right) = \left( \begin{array}{c} a - x^2 \\ x \end{array} \right).
\]
At $\tau_n^0$ the map is exponentially close to $F_{0,0}$, a map with a sink and at $\tau_n^1$ the map is arbitrarily close to $F_{3,0}$, a map with a fully developed horse shoe, see Lemma 3.24.

According to [5], for $n \geq 1$ large enough, there is a unique $\tau \in [\tau_n^0, \tau_n^1]$ such that $F_{\gamma_m(\tau)}^n|Q_n$ has a period doubling Cantor attractor. Namely, at $\tau$ the curve $[\tau_n^0, \tau_n^1] \ni \tau \mapsto F_{\gamma_m(\tau)}^n|Q_n$ crosses transversally a codimension one manifold of maps which have a period doubling Cantor attractor. \qed
Observe that, by the proof Theorem F, the codimension 2 laminations of maps with $S$ sinks and two period doubling attractors intersect transversally the codimension one manifold of maps with a period doubling Cantor attractor. This implies that the coexistence of finitely many sinks and three period doubling Cantor attractors is a codimension three phenomenon. This is stated in the following theorem.

**Theorem H.** Let $S \in \mathbb{N}$. The space $\text{Poly}_d(\mathbb{R}^2)$, $d \geq 2$, of real polynomials of $\mathbb{R}^2$ of degree at most $d$ contains a codimension 3 lamination of maps with at least $S$ sinks and three period doubling Cantor attractors. The leaves of the lamination are real-analytic. The $S$ sinks and the three period doubling Cantor attractors persist along the leaves.

**Theorem I.** Fix $S \in \mathbb{N}$. The set of quadratic Hénon-like maps with $S$ sinks, two period doubling Cantor attractors and one strange attractor has Hausdorff dimension at least one.

**Proof.** Consider the curves $\gamma_m$ introduced in the proof of Theorem F. The maps in these curves have $S$ sinks and two period doubling Cantor attractor. Moreover, they are one-parameter analytic unfoldings of a homoclinic tangency. According to [1] and [7] each curve contains a set of positive measure of maps with a strange attractor.

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