EXISTENCE OF SOLUTION TO A NONLINEAR FIRST-ORDER
DYNAMIC EQUATION ON TIME SCALES

BENAOUMEUR BAYOUR, AHMED HAMMOUDI, DELFIM F. M. TORRES

Abstract. We prove existence of solution to a nonlinear first-order nabla dy-
namic equation on an arbitrary bounded time scale with boundary conditions,
where the right-hand side of the dynamic equation is a continuous function.

1. Introduction

In this work we prove existence of solution to the following system:

\[ \begin{align*}
  x'(t) &= f(t, x(t)), & t & \in \mathbb{T}_k, \\
  x(a) &= x(b).
\end{align*} \tag{1.1} \]

Here \( \mathbb{T} \) is an arbitrary bounded time scale, where we denote \( a := \min \mathbb{T}, b := \max \mathbb{T}, \)
\( \mathbb{T}_o = \mathbb{T} \setminus \{ a \} \), and \( f : \mathbb{T}_o \times \mathbb{R}^n \to \mathbb{R}^n \) is a continuous function. Problem (1.1) unifies
continuous and discrete problems. We use the notion of tube solution for system
(1.1), in the spirit of the works of Gilbert and Frigon [7–9]. This notion is useful
to get existence results for systems of differential equations of first order, as a
generalization of lower and upper solutions [3, 6, 10, 11]. Our main result provides
existence of solution to the nonlinear nabla boundary value problem (1.1).

The article is organized as follows. In Section 2 we review some basic de-
finitions and theorems regarding \( \nabla \)-differentiation and \( \nabla \)-integration on time scales, and
we prove some preliminary results. In Section 3 we introduce the notion of tube
solution for system (1.1) and we prove our main result (Theorem 3.3). We end with
Section 4 mentioning some directions for future work.

2. Preliminaries

A time scale \( \mathbb{T} \) is defined to be any nonempty closed subset of \( \mathbb{R} \). Then
the forward and backward jump operators \( \sigma, \rho : \mathbb{T} \to \mathbb{T} \) are defined by
\[ \sigma(t) = \inf \{ s \in \mathbb{T} : s > t \} \quad \text{and} \quad \rho(t) = \sup \{ t \in \mathbb{T} : s < t \}. \]
For \( t \in \mathbb{T} \), we say that \( t \) is left-scattered (respectively right-scattered) if \( \rho(t) < t \)
(respectively \( \sigma(t) > t \)); that \( t \) is isolated if it is left-scattered and right-scattered.
Similarly, if \( t > \inf(\mathbb{T}) \) and \( \rho(t) = t \), then we say that \( t \) is left-dense; if \( t < \sup(\mathbb{T}) \)
and \( \sigma(t) = t \), then we say that \( t \) is right-dense. Points that are simultaneously
left- and right-dense are called dense. If \( \mathbb{T} \) has a right-scattered minimum \( m \), then

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we define $T_\kappa := \mathbb{T} - \{m\}$; otherwise, we set $T_\kappa := \mathbb{T}$. The (backward) graininess $\nu : T_\kappa \to [0, +\infty]$ is defined by $\nu(t) := t - \rho(t)$.

**Definition 2.1** (See [119]). For $f : T \to \mathbb{R}^n$ and $t \in T_\kappa$, the nabla derivative of $f$ at $t$, denoted by $f^\nabla(t)$, is defined to be the number (provided it exists) with the property that given any $\epsilon > 0$ there is a neighborhood $U$ of $t$ such that

$$\|f(\rho(t)) - f(s) - f^\nabla(t)[\rho(t) - s]\| \leq \epsilon|\rho(t) - s|$$

for all $s \in U$. If $f$ is $\nabla$-differentiable at $t$ for every $t \in T_\kappa$, then $f : T \to \mathbb{R}^n$ is called the $\nabla$-derivative of $f$ on $T_\kappa$.

**Theorem 2.2** (See [5]). Assume $f : T \to \mathbb{R}^n$ and let $t \in T_\kappa$. The following holds:

1. If $f$ is $\nabla$-differentiable at $t$, then $f$ is continuous at $t$.
2. If $f$ is continuous at the left-scattered point $t$, then $f$ is $\nabla$-differentiable at $t$ with

$$f^\nabla(t) = \frac{f(t) - f(\rho(t))}{\nu(t)}.$$

3. If $t$ is left-dense, then $f$ is nabla differentiable at $t$ if and only if the limit

$$\lim_{s \to t} \frac{f(t) - f(s)}{t - s}$$

exists in $\mathbb{R}^n$. In this case,

$$f^\nabla(t) = \lim_{s \to t} \frac{f(t) - f(s)}{t - s}.$$  

4. If $f$ is nabla differentiable at $t$, then

$$f^\rho(t) = f(t) - \nu(t)f^\nabla(t),$$

where $f^\rho(t) := f(\rho(t))$.

**Theorem 2.3** (See [5]). Assume $f, g : T \to \mathbb{R}$ are nabla differentiable at $t \in T_\kappa$.

Then, (1) $f + g$ is nabla differentiable at $t$ and $(f + g)^\nabla(t) = f^\nabla(t) + g^\nabla(t)$;

2. $\alpha f$ is nabla differentiable at $t$ for every $\alpha \in \mathbb{R}$ and $(\alpha f)^\nabla(t) = \alpha f^\nabla(t)$;

3. $fg$ is nabla differentiable at $t$ and

$$(fg)^\nabla(t) = f^\nabla(t)g(t) + f^\rho(t)g^\nabla(t) = f(t)g^\nabla(t) + f^\nabla(t)g^\rho(t);$$

4. if $g(t)g^\rho(t) \neq 0$, then $\frac{f}{g}$ is nabla differentiable at $t$ and

$$\left(\frac{f}{g}\right)^\nabla(t) = \frac{f^\nabla(t)g(t) - f(t)g^\nabla(t)}{g(t)g^\rho(t)};$$

5. if $f$ and $f^\nabla$ are continuous, then

$$\left(\int_a^t f(t,s)\nabla s\right)^\nabla = f(\rho(t),t) + \int_a^t f^\nabla(t,s)\nabla s.$$  

**Theorem 2.4.** Let $W$ be an open set of $\mathbb{R}^n$ and $t \in T$ be a left-dense point. If $g : T \to \mathbb{R}^n$ is nabla-differentiable at $t$ and $f : W \to \mathbb{R}$ is differentiable at $g(t) \in W$, then $f \circ g$ is nabla-differentiable at $t$ with $(f \circ g)^\nabla(t) = (f^\rho(g(t)), g^\nabla(t))$. 

Example 2.5. Assume $\epsilon_k$:
(See [2]) a neighborhood $V \rightarrow U$ at this point, and there exists a neighborhood $C$ by if whose nabla-derivative is ld-continuous, is denoted by $\epsilon > 0$. We need to show that there exists a neighborhood $U$ of $t$ such that $\|g(t) - g(s) - g(t) - g(s)\| \leq \epsilon |t - s|$ for all $s \in U$. In addition, there exists a neighborhood $V \subset W$ of $g(t)$ such that $|f(g(t)) - f(g(y)) - f'(g(t))g(t)| \leq \epsilon' |g(t) - y|$ for all $y \in V$. Since function $g$ is $\nabla$-differentiable at $t$, it is also continuous at this point, and there exists a neighborhood $U_2$ of $t$ such that $g(s) \in V$ for all $s \in U_2$. Let $U := U_1 \cap U_2$. In this case $U$ is a neighborhood of $t$ and if $s \in U$, then
\[
|f(g(t)) - f(g(s)) - f'(g(t))g(t)(t - s)|
\leq |f(g(t)) - f(g(s)) - f'(g(t))g(t - s)|
+ |f'(g(t))g(t) - g'(s)|
\leq \epsilon' (|g'(s)| + \|f'(g(t))g(t) - g'(s) - g(t) - s\|)
\leq \epsilon' (1 + \|g(t) - s\|)
\leq \epsilon' (1 + \|g(t) - s\|)(t - s).
\]
Put $k = 1 + \|g(t)\| + \|f'(g(t))\|$ and the theorem is proved. \hfill \square

Example 2.5. Assume $x : T \rightarrow \mathbb{R}^n$ is nabla differentiable at $t \in T$. We know that $\|\cdot\| : \mathbb{R}^n \setminus \{0\} \rightarrow [0, +\infty]$ is differentiable if $t = \rho(t)$. It follows from Theorem 2.4 that
\[
\|x(t)\| = \langle x(t), x(t) \rangle = \|x(t)\| = \|x(t)\| = \|x(t)\|.
\]

Definition 2.6. A function $f : T \rightarrow \mathbb{R}^n$ is called ld-continuous provided it is continuous at left-dense points in $T$ and its right-sided limits exist (finite) at right-dense points in $T$. The set of all ld-continuous functions $f : T \rightarrow \mathbb{R}^n$ is denoted by $C_{ld}(T, \mathbb{R}^n)$. The set of functions $f : T \rightarrow \mathbb{R}^n$ that are nabla-differentiable and whose nabla-derivative is ld-continuous, is denoted by $C_{ld}^n(T, \mathbb{R}^n)$. It is known that if $f$ is ld-continuous, then there is a function $F$ such that $F' = f$. In this case,
\[
\int_a^b f(t)\nabla t := F(b) - F(a).
\]

Theorem 2.7 (See [2]). Assume $a, b, c \in T$. Then
(1) $\int_a^b f(t) + g(t)\nabla t = \int_a^b f(t)\nabla t + \int_a^b g(t)\nabla t$;
(2) $\int_a^b kf(t)\nabla t = k \int_a^b f(t)\nabla t$;
(3) $\int_a^b f(t)\nabla t = -\int_b^a f(t)\nabla t$;
(4) $\int_a^b f(t)\nabla t = \int_a^c f(t)\nabla t + \int_c^b f(t)\nabla t$;
(5) $\int_a^b f\nabla (t) g(t)\nabla t = f(t) g(t) |t|^b_a - f|a^b_a f^p(t) g(t)\nabla (t)\nabla t$.

Theorem 2.8 (See [2]). The following inequalities hold:
\[
\left| \int_a^b f(t) g(t)\nabla t \right| \leq \int_a^b |f(t) g(t)\nabla t| \leq \left( \max_{\sigma(a) \leq t \leq b} |f(t)| \right) \int_a^b |g(t)|\nabla t.
\]

Definition 2.9 (See [3]). For $\epsilon > 0$, the (nabla) exponential function $\hat{e}_\epsilon(\cdot, t_0) : T \rightarrow \mathbb{R}$ is defined as the unique solution to the initial value problem
\[
x\nabla (t) = \epsilon x(t), \quad x(t_0) = 1.
\]
More explicitly, the exponential function \( \hat{e}_x(t, t_0) : \mathbb{T} \to \mathbb{R} \) is given by the formula

\[
\hat{e}_x(t, t_0) = \exp \left( \int_{t_0}^{t} \hat{\xi}(\nu(s)) \nabla s \right),
\]

where for \( h \geq 0 \) we define \( \hat{\xi}(h) \) as

\[
\hat{\xi}(h) = \begin{cases} 
\epsilon & \text{if } h = 0, \\
-\log(1-h) h & \text{otherwise}.
\end{cases}
\]

**Proposition 2.10.** If \( g \in C^1(\mathbb{T}_n, \mathbb{R}^n) \), then function \( x : \mathbb{T} \to \mathbb{R}^n \) defined by

\[
x(t) = \hat{e}_1(t, b) \left[ \frac{\hat{e}_1(a, b)}{\hat{e}_1(a, b) - 1} \int_{(a, b, \mathbb{R})} g(s) \nabla s - \int_{(a, b, \mathbb{R})} \frac{g(s)}{\hat{e}_1(\rho(s), b)} \nabla s \right]
\]

is solution to the problem

\[
\begin{align*}
x^\nabla(t) - x(t) &= g(t), \quad t \in \mathbb{T}_n, \\
x(a) &= x(b).
\end{align*}
\]

**Proof.** We check (2.1) for each pair \((x_i, g_i), i \in \{1, 2, \ldots, n\}\), by direct calculation. To simplify notation, we omit the indices \( i \) and we write

\[
k = \frac{\hat{e}_1(a, b)}{\hat{e}_1(a, b) - 1} \int_{(a, b, \mathbb{R})} g(s) \nabla s.
\]

From Theorem 2.3 we have that

\[
x^\nabla(t) - x(t) = \hat{e}_1(t, b)k - \hat{e}_1(t, b) \int_{(a, b, \mathbb{R})} g(s) \nabla s + \hat{e}_1(\rho(t), b) \frac{g(t)}{\hat{e}_1(\rho(t), b)} - \hat{e}_1(\rho(t), b)k + \hat{e}_1(\rho(t), b) \int_{(a, b, \mathbb{R})} g(s) \nabla s = g(t)
\]

for all \( t \in \mathbb{T}_n \). It is easy to verify that \( x(a) = x(b) \). \( \square \)

**Lemma 2.11.** Let \( r \in C^1_{id}(\mathbb{T}, \mathbb{R}^n) \) be a function such that \( r^\nabla(t) < 0 \) for all \( t \in \{t \in \mathbb{T}_n : r(t) > 0\} \). If \( r(a) \geq r(b) \), then \( r(t) \leq 0 \) for all \( t \in \mathbb{T} \).

**Proof.** Suppose that there exists a \( t \in \mathbb{T} \) such that \( r(t) > 0 \). Then there exists a \( t_0 \in \mathbb{T} \) such that \( r(t_0) = \max_{t \in \mathbb{T}} (r(t) > 0) \). If \( r(t_0) < t_0 \), then

\[
r^\nabla (t_0) = \frac{r(\rho(t_0)) - r(t_0)}{\rho(t_0) - t_0} \geq 0,
\]

which contradicts the hypothesis. If \( t_0 > a \) and \( t_0 = \rho(t_0) \), then there exists an interval \([t_1, t_0]\) such that \( r(t) > 0 \) for all \( t \in [t_1, t_0] \). Thus

\[
\int_{t_1}^{t_0} r^\nabla(s) \nabla s = r(t_0) - r(t_1) < 0,
\]

which contradicts the maximality of \( r(t_0) \). Finally, if \( t_0 = a \), then by hypothesis \( r(b) \geq r(a) \) gives \( r(a) = r(b) \). Taking \( t_0 = a \), one can check that \( r(a) \leq 0 \) by using previous steps of the proof. The lemma is proved. \( \square \)
3. Main Result

In this section we prove existence of solution to problem (1.1). A solution of this problem is a function \( x \in C^1_{\text{id}}(\mathbb{T}, \mathbb{R}^n) \) satisfying (1.1). Let us recall that \( \mathbb{T} \) is bounded with \( a = \min \mathbb{T} \) and \( b = \max \mathbb{T} \). We introduce the notion of solution tube for problem (1.1) as follows.

**Definition 3.1.** Let \( (v, M) \in C^1_{\text{id}}(\mathbb{T}, \mathbb{R}^n) \times C^1_{\text{id}}(\mathbb{T}, [0, +\infty]) \). We say that \( (v, M) \) is a tube solution of (1.1) if

\[
\begin{align*}
\text{(1)} & \quad \langle x - v(t), f(t, x(t)) - v\nabla(t) \rangle + M(t)\|x - v(t)\| \leq M(t)M\nabla(t) \quad \text{for every } t \in \mathbb{T}_\kappa, \\
\text{(2)} & \quad v\nabla(t) = f(t, v(t)) \quad \text{and} \quad \|x - v(t)\| - M\nabla(t) < 0 \quad \text{for every } t \in \mathbb{T}_\kappa \quad \text{such that} \quad M(t) = 0; \\
\text{(3)} & \quad \|v(a) - v(b)\| \leq M(a) - M(b).
\end{align*}
\]

Let \( T(v, M) := \{ x \in C^1_{\text{id}}(\mathbb{T}, \mathbb{R}^n) : \|x(t) - v(t)\| \leq M(t) \quad \text{for every } t \in \mathbb{T} \} \). We consider the following problem:

\[
x\nabla(t) - x(t) = f(t, \hat{x}(t)) - \hat{x}(t), \quad t \in \mathbb{T}_\kappa, \\
x(a) = x(b),
\]

where

\[
\hat{x}(t) = \begin{cases} 
\frac{M(t)}{\|x - v(t)\|}(x(t) - v(t)) + v(t) & \text{if } \|x - v(t)\| > M(t), \\
\text{otherwise}.
\end{cases}
\]

Let us define the operator \( T_\delta : C(\mathbb{T}, \mathbb{R}^n) \to C(\mathbb{T}, \mathbb{R}^n) \) by

\[
T_\delta(x)(t) = \hat{e}_1(t, b) \left[ \frac{\hat{e}_1(a, b)}{\hat{e}_1(a, b) - 1} \int_{(a, b) \cap \mathbb{T}} \frac{f(s, \hat{x}(s)) - \hat{x}(s)}{\hat{e}_1(\rho(s), b)} \nabla s 
- \int_{(t, b) \cap \mathbb{T}} \frac{f(s, \hat{x}(s)) - \hat{x}(s)}{\hat{e}_1(\rho(s), b)} \nabla s \right].
\]

**Proposition 3.2.** If \( (v, M) \in C^1_{\text{id}}(\mathbb{T}, \mathbb{R}^n) \times C^1_{\text{id}}(\mathbb{T}, [0, +\infty]) \) is a tube solution of (1.1), then \( T_\delta : C(\mathbb{T}, \mathbb{R}^n) \to C(\mathbb{T}, \mathbb{R}^n) \) is compact.

**Proof.** We first prove the continuity of the operator \( T_\delta \). Let \( \{x_n\}_{n \in \mathbb{N}} \) be a sequence of \( C(\mathbb{T}, \mathbb{R}^n) \) converging to \( x \in C(\mathbb{T}, \mathbb{R}^n) \). By Theorem 2.8

\[
\|T_\delta(x_n)(t) - T_\delta(x)(t)\|
\leq (1 + c)\|\hat{e}_1(t, b)\| \left\| \int_{(a, b) \cap \mathbb{T}} \frac{f(s, \hat{x}_n(s)) - f(s, \hat{x}(s)) - (\hat{x}_n(s) - \hat{x}(s))}{\hat{e}_1(\rho(s), b)} \nabla s \right\|
\leq \frac{k(1 + c)}{M} \left( \int_{(a, b) \cap \mathbb{T}} \|f(s, \hat{x}_n(s)) - f(s, \hat{x}(s))\| + \|\hat{x}_n(s) - \hat{x}(s)\| \nabla s \right),
\]

where \( k := \max_{t \in \mathbb{T}} |\hat{e}_1(t, b)|, \quad M := \min_{t \in \mathbb{T}} |\hat{e}_1(t, b)|, \quad \text{and} \quad c := \|\frac{\hat{e}_1(a, b)}{\hat{e}_1(a, b) - 1}\| \). Since there is a constant \( R > 0 \) such that \( \|\hat{x}\|_{C(\mathbb{T}, \mathbb{R}^n)} < R \), there exists an index \( N \) such that \( \|\hat{x}_n\|_{C(\mathbb{T}, \mathbb{R}^n)} < R \) for all \( n > N \). Thus \( f \) is uniformly continuous on \( \mathbb{T}_\kappa \times B_R(0) \). Therefore, for \( \epsilon > 0 \) given, there is a \( \delta > 0 \) such that for all \( x, y \in \mathbb{R}^n \), where

\[
\|x - y\| < \delta < \frac{\epsilon M}{2k(1 + c)(b - a)},
\]
one has
\[ \|f(s, y) - f(s, x)\| < \frac{\epsilon M}{2k(1 + c)(b - a)}. \]

By assumption, for all \( s \in \mathbb{T}_\kappa \) it is possible to find an index \( \hat{N} > N \) such that \( \|\hat{x}_n - \hat{x}\|_{C(\mathbb{T}, \mathbb{R}^n)} < \delta \) for \( n > \hat{N} \). In this case,
\[ \|\mathbf{T}_p(x_n)(t) - \mathbf{T}_p(x)(t)\| \leq \frac{2k(1 + c)}{M} \int_{(a, b) \cap \mathbb{T}} \|f(s, \hat{x}_n(s))\| \nabla s + \frac{\epsilon M}{2k(1 + c)(b - a)} \nabla s \leq \epsilon. \]

This proves the continuity of \( \mathbf{T}_p \). We now show that the set \( \mathbf{T}_p(C(\mathbb{T}, \mathbb{R}^n)) \) is relatively compact. Consider a sequence \( \{y_n\}_{n \in \mathbb{N}} \) of \( \mathbf{T}_p(C(\mathbb{T}, \mathbb{R}^n)) \) for all \( n \in \mathbb{N} \). It exists \( x_n \in C(\mathbb{T}, \mathbb{R}^n) \) such that \( y_n = \mathbf{T}_p(x_n) \). From Theorem 2.8 one has
\[ \|\mathbf{T}_p(x_n)(t)\| \leq \frac{k(1 + c)}{M} \left( \int_{(a, b) \cap \mathbb{T}} \|f(s, \hat{x}_n(s))\| \nabla s + \int_{(a, b) \cap \mathbb{T}} \|\hat{x}_n(s)\| \nabla s \right). \]

By definition, there is an \( R > 0 \) such that \( \|\hat{x}_n(s)\| \leq R \) for all \( s \in \mathbb{T} \) and all \( n \in \mathbb{N} \). Function \( f \) is compact on \( \mathbb{T}_\kappa \times B_R(0) \) and we deduce the existence of a constant \( A > 0 \) such that \( \|f(s, \hat{x}_n(s))\| \leq A \) for all \( s \in \mathbb{T}_\kappa \) and all \( n \in \mathbb{N} \). The sequence \( \{y_n\}_{n \in \mathbb{N}} \) is uniformly bounded. Note also that
\[ \|\mathbf{T}_p(x_n)(t_2) - \mathbf{T}_p(x_n)(t_1)\| \leq B\|\hat{e}_1(t_2, b) - \hat{e}_1(t_1, b)\| \]
\[ + k \int_{(a, b) \cap \mathbb{T}} \frac{f(s, \hat{x}_n(s)) - \hat{x}_n(s)}{\hat{e}_1(\rho(s), b)} \nabla s \leq B\|\hat{e}_1(t_2, b) - \hat{e}_1(t_1, b)\| + \frac{k(A + R)}{M} |t_2 - t_1| \]

for \( t_1, t_2 \in \mathbb{T} \), where \( B \) is a constant that can be chosen such that it is higher than
\[ \sup_{n \in \mathbb{N}} \left\| \frac{\hat{e}_1(a, b)}{\hat{e}_1(a, b) - 1} \int_{(a, b) \cap \mathbb{T}} \frac{f(s, \hat{x}_n(s)) - \hat{x}_n(s)}{\hat{e}_1(\rho(s), b)} \nabla s + \int_{(t, b) \cap \mathbb{T}} \frac{f(s, \hat{x}_n(s)) - \hat{x}_n(s)}{\hat{e}_1(\rho(s), b)} \nabla s \right\|. \]

This proves that the sequence \( \{y_n\}_{n \in \mathbb{N}} \) is equicontinuous. It follows from the Arzelà-Ascoli theorem, adapted to our context, that \( \mathbf{T}_p(C(\mathbb{T}, \mathbb{R}^n)) \) is relatively compact. Hence \( \mathbf{T}_p \) is compact. \( \square \)

**Theorem 3.3.** If \( (v, M) \in C^1_{ld}(\mathbb{T}, [0, +\infty]) \times C^1_{ld}(\mathbb{T}, \mathbb{R}^n) \) is a tube solution of (1.1), then problem (1.1) has a solution \( x \in C^1_{ld}(\mathbb{T}, \mathbb{R}^n) \cap \mathbf{T}(v, M) \).

**Proof.** By Proposition 3.2 \( \mathbf{T}_p \) is compact. It has a fixed point by Schauder’s fixed point theorem. Proposition 2.10 implies that this fixed point is a solution to problem 2.10. Then it suffices to show that for every solution \( x \) of 2.10 one has \( x \in \mathbf{T}(v, M) \). Consider the set \( A = \{ t \in \mathbb{T}_\kappa : \|x(t) - v(t)\| > M(t) \} \). If \( t \in A \) is left dense, then by virtue of Example 2.8 we have
\[ \langle \|x(t) - v(t)\| - M(t)\rangle^\nabla = \frac{\langle x(t) - v(t), x^n(t) - v^n(t)\rangle}{\|x(t) - v(t)\|} - M^n(t). \]
If \( t \in A \) is left scattered, then
\[
(\|x(t) - v(t)\| - M(t))^\vee = \|x(t) - v(t)\|^\vee - M^\vee(t)
\]
\[
= \frac{\|x(t) - v(t)\|^2 - \|x(t) - v(t)\|\|x(\rho(t)) - v(\rho(t))\|}{\nu(t)\|x(t) - v(t)\|} - M^\vee(t)
\]
\[
\leq \frac{\langle x(t) - v(t), x(t) - v(t) - x(\rho(t)) + v(\rho(t)) \rangle}{\nu(t)\|x(t) - v(t)\|} - M^\vee(t)
\]
\[
= \langle x(t) - v(t), x^\vee(t) - v^\vee(t) \rangle - M^\vee(t)
\]
\[
= \frac{\langle x(t) - v(t), f(\tilde{x}(t)) - v^\vee(t) \rangle}{\|x(t) - v(t)\|} + \frac{\langle x(t) - v(t), -\tilde{x}(t) + x(t) \rangle}{\|x(t) - v(t)\|} - M^\vee(t)
\]
\[
= \frac{\langle \tilde{x}^\vee(t) - v(t), f(\tilde{x}(t)) - v^\vee(t) \rangle}{M(t)} - M(t) + \|x(t) - v(t)\| - M^\vee(t)
\]
\[
\leq \frac{M(t)\|x(t) - v(t)\|}{M(t)} - M(t) + \|x(t) - v(t)\| - M^\vee(t)
\]
\[
= -M(t) < 0.
\]
In addition, if \( M(t) = 0 \), then
\[
(\|x(t) - v(t)\| - M(t))^\vee = \frac{\langle x(t) - v(t), f(\tilde{x}(t)) - v^\vee(t) \rangle}{\|x(t) - v(t)\|} - M^\vee(t)
\]
\[
\leq \frac{\langle x(t) - v(t), f(\tilde{x}(t)) - v^\vee(t) \rangle}{\|x(t) - v(t)\|} + \|x(t) - v(t)\| - M^\vee(t) < 0.
\]
If we set \( r(t) := \|x(t) - v(t)\| - M(t) \), then \( r^\vee(t) < 0 \) for every \( t \in \{t \in \mathbb{T}_\kappa, r(t) \geq 0\} \).
Moreover, since \((v, M)\) is a tube solution of \((3.1)\), one has
\[
r(a) - r(b) \leq \|v(a) - v(b)\| - (M(a) - M(b)) \leq 0
\]
and thus the hypotheses of Lemma \(2.11\) are satisfied, which proves the theorem. \( \square \)

**Example 3.4.** Consider the following boundary value problem on time scales:
\[
x^\Delta(t) = a_1\|x(t)\|^2x(t) - a_2x(t) + a_3\varphi(t), \quad t \in \mathbb{T}_\kappa,
\]
\[
x(a) = x(b),
\]
where \( a_1, a_2, a_3 \geq 0 \) are nonnegative real constants chosen such that \( a_2 \geq a_1 + a_3 + 1 \) and \( \varphi: \mathbb{T}_\kappa \rightarrow \mathbb{R}^n \) is a continuous function satisfying \( \|\varphi(t)\| = 1 \) for every \( t \in \mathbb{T}_\kappa \).
It is easy to check that \((v, m) \equiv (0, 1)\) is a tube solution. By Theorem 3.3 problem \((3.2)\) has a solution \( x \) such that \( \|x(t)\| \leq 1 \) for every \( t \in \mathbb{T} \).
4. Conclusion and Future Work

We proved existence of a solution to a nonlinear first-order nabla dynamic equation on time scales. For that the notion of tube solution is used, in the spirit of the works of Frigon and Gilbert \cite{7,9}. Our results can be improved by using $\nabla$-Caratheodory functions $f$ on the right-hand side of equation (1.1), which are not necessarily continuous. For that one needs to define a proper Sobolev space and related nabla concepts. This is under investigation and will be addressed elsewhere.

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\end{enumerate}

Benaoumeur Bayour
University of Chlef, B. P. 151, Hay Es-salem Chlef, Algeria
E-mail address: b.benaoumeur@gmail.com

Ahmed Hammoudi
Laboratoire de Mathématiques, Université de Ain Témouchent
B. P. 89, 46000 Ain Témouchent, Algeria
E-mail address: hymmed@hotmail.com

Delfim F. M. Torres
Center for Research and Development in Mathematics and Applications (CIDMA)
Department of Mathematics, University of Aveiro, 3810-193 Aveiro, Portugal
E-mail address: delfim@ua.pt