Primitive operations for the construction and reorganization of minimally persistent formations

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Abstract

In this paper, we study the construction and transformation of two-dimensional persistent graphs. Persistence is a generalization to directed graphs of the undirected notion of rigidity. In the context of moving autonomous agent formations, persistence characterizes the efficacy of a directed structure of unilateral distances constraints seeking to preserve a formation shape. Analogously to the powerful results about Henneberg sequences in minimal rigidity theory, we propose different types of directed graph operations allowing one to sequentially build any minimally persistent graph (i.e., persistent graph with a minimal number of edges for a given number of vertices), each intermediate graph being also minimally persistent. We also consider the more generic problem of obtaining one minimally persistent graph from another, which corresponds to the on-line reorganization of an autonomous agent formation. We prove that we can obtain any minimally persistent formation from any other one by a sequence of elementary local operations such that minimal persistence is preserved throughout the reorganization process.

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1 Introduction

The recent progress in the field of autonomous agent systems has led to new problems in control theory [1,2] and graph theory [3,5,9]. By autonomous agent, we mean here any human controlled or unmanned vehicle that can move by itself and has a local intelligence or computing capacity, such as ground robots, air vehicles or underwater vehicles. The results derived in this paper concern mostly autonomous agents evolving in a two dimensional space.

Many applications require some inter-agent distances to be kept constant during a continuous move in order to preserve the shape of a multi-agent formation. In other words, some inter-agent distances are explicitly maintained constant so that all the inter-agent distances remain constant. The information structure arising from such a system can be efficiently modelled by a graph, where agents are abstracted by vertices and actively constrained inter-agent distances by edges. We assume here that those constraints are unilateral, i.e., that the responsibility for maintaining a distance is not shared by the two concerned agents but relies on only one of them while the other one is unaware of it. This asymmetry is modelled by the use of directed edges in the graph. The characterization of the directed information structures which can efficiently maintain the formation shape has begun to be studied under the name of “directed rigidity” [1, 3]. These works included several conjectures about minimal directed rigidity, i.e., directed rigidity with a minimal number of edges for a fixed number of vertices. In [5], Hendrickx et al. proposed a theoretical framework to analyze these issues, where the name of “persistence” was advanced in preference to “directed rigidity”, since the latter notion does not correspond to the immediate transposition of the undirected notion of rigidity to directed graph. The intuitive definition of persistence is the following: An information structure is persistent if, provided that each agent is trying to satisfy all the distance constraints for which it is responsible, all the inter-agent distances remain constant and as a result the formation shape is preserved. It is shown in [5] that persistence is actually the conjunction of two distinct notions: rigidity of the underlying undirected graph (i.e. the undirected graph obtained by ignoring the direction of the edges), and constraint consistence. Intuitively, rigidity means that, provided that all the prescribed distance constraints are satisfied during a continuous displacement, all the inter-agent distances remain constant, as shown in Figure 1. Constraint consistence of an information structure means that, provided that each agent is trying to satisfy all its distances constraints, all the agents actually succeed in doing so. In other words, no agent has an impossible task, as shown in the example in Figure 2. Observe that this last notion depends strongly on the directed structure of the graph, while rigidity only relies on its underlying undirected graph. An example of persistent graph is provided in Figure 3. Note that for agents evolving in a two-dimensional space, a purely combinatorial criterion to decide persistence is provided in [5].

In this paper, we focus on minimally persistent graphs, that are persistent graphs having a minimal number of edges (for a given number of vertices), and
Figure 1: Representation of (a) a non-rigid and (b) a rigid graph/formation. The solid structure in (a) can indeed be deformed to the dotted structure without breaking any distance constraint.

Figure 2: Representation of (a) a constraint consistent and (b) a non-constraint consistent (in 2 dimension) graph/formation. One can indeed see in (b) that for almost any uncoordinated continuous displacement of the agents 2 and 4 (which are unconstrained), the agent 3 is unable to move in such a way that it maintains its distances to all of 1, 2 and 4 constant. However, such a situation could not happen in graph (a).

Figure 3: Representation of a persistent graph, i.e., a rigid constraint consistent graph.
their connections with minimally rigid graphs. More particularly we analyze different ways to sequentially build minimally persistent graphs, analogously to the Henneberg sequences for the minimally rigid graphs [7, 10]. It has indeed long been known that every minimally rigid graph can be obtained from the complete graph on two vertices by a sequence of two basic operations, as detailed in Section 2. The natural extension of these operations to directed graphs [3] does not allow one to build all minimally persistent graphs, as remarked in [5] and reviewed later in Section 2. We consider here different possible additional operations that would help to achieve this purpose. We also consider the more generic problem of obtaining one persistent graph from another. From an autonomous agent point of view, this corresponds to an on-line reorganization of the agent formation. The subsequent analysis leads us then to the definition of different “distances” between persistent graphs (the distance between two graphs being the smallest number of operations needed to obtain one from the other). Note that although the notion of persistence has been also defined in three or higher dimensions [6, 12, 13], the present analysis only concerns two-dimensional persistence, i.e., the persistence of graphs representing the information structure of a formation evolving in a two-dimensional space. Extension to the three-dimensional case may be difficult; even for undirected graphs, Henneberg sequence theory is effectively incomplete.

In Section 2 we review the main properties of minimally rigid and minimally persistent graphs. We present the two basic undirected operations - vertex addition and edge splitting - involved in the Henneberg sequences, together with their natural extension to directed graphs. We show that although these directed operations preserve minimal persistence, they are not sufficient to build all minimally persistent graphs. This analysis is done by reasoning on reverse construction of persistent graphs using reverse operations. In Section 3 we show how the goal of building all minimally persistent graph can be reached by introducing a third local directed operation - edge reversal. We see that, unlike when building minimally rigid undirected graphs with Henneberg sequences, the required number of operations is not uniquely determined by the size of the graph. We show in Section 4 that this drawback can be avoided by using only directed operations equivalent to the vertex addition and the edge splittings from an undirected point of view. However, we prove that a set of such operations allowing one to build all minimally persistent graphs always contain at least one non-confined operation, i.e. an operation reversing the directions of (possibly several) edges that are not affected by the corresponding operation for undirected graphs. We provide then such a set of four operations, and analyze the relations between this set and the set of three operations treated in Section 3. Finally, this paper ends with the concluding remarks of Section 5.
2 Directed and undirected Henneberg sequences

In this section, we recall some results about (minimal) rigidity and (minimal) persistence. We also describe the Henneberg sequences for undirected graphs and show why their obvious adjustment to the directed case is not sufficient to build all minimally persistent graphs.

2.1 Minimally rigid graphs

Note that in this section, all graphs are considered as undirected, but in the rest of this paper, they are always assumed to be directed. However, although all the definitions and results of this section are given for undirected graphs, they can also be applied to directed graphs. If $G$ is a directed graph, we call the underlying undirected graph of $G$ the undirected graph obtained by ignoring the directions of the edges of $G$.

The rigidity of a graph has the following intuitive meaning: Suppose that each vertex represents an agent in a formation, and each edge represents an inter-agent distance constraint enforced by an external observer. The graph is rigid if for almost every such structure, the only possible continuous moves are those which preserve every inter-agent distance. Note that this notion also represents the rigidity of a framework where the vertices correspond to joints and the edges to bars. For a more formal definition, the reader is referred to [5, 10]. In $\mathbb{R}^2$, there exists a combinatorial criterion to check if a given graph is rigid (Laman’s theorem [8, 11]). A minimally rigid graph is a rigid graph such that no edge can be removed without losing rigidity. From Laman’s Theorem, it is possible to deduce the following criterion:

\textbf{Proposition 1.} A graph $G = (V, E)$ ($|V| > 1$) is minimally rigid if and only if $|E| = 2|V| - 3$ and for all $E'' \subseteq E, E'' \neq \emptyset$, there holds $|E''| \leq 2|V(E'')| - 3$.

We say that a pair of unconnected vertices defines an implicit edge in a graph $G = (V, E)$ if their connection would create a subgraph $G' = (V', E')$ with $|E'| > 2|V'| - 3$. Intuitively, this means that the addition of such an edge would not improve the rigidity of the graph, i.e., the constraint that this edge would enforce is a linear combination of already present constraints. One can easily prove that two unconnected vertices define an implicit edge in a graph if and only if there is a minimally rigid subgraph containing both of them. By extension, we sometimes call an edge of a graph an \textit{explicit edge}. In a (minimally) rigid graph, every pair of vertices is connected by either an explicit or implicit edge. But, if one removes an (explicit) edge in a minimally rigid graph, the corresponding pair of vertices never defines an implicit edge in the graph obtained.

Let $j, k$ be two distinct vertices of a minimally rigid graph $G = (V, E)$. A \textit{vertex addition} operation consists in adding a vertex $i$, and connecting it to $j$ and $k$, as shown in Figure 4(a). One can see using Proposition 1 that this
operation preserves minimal rigidity. Moreover, if a vertex $i$ has degree 2 in a minimally rigid graph, one can always perform the inverse vertex addition operation by removing it (and its incident edges) and obtain a smaller minimally rigid graph.

Let $j, k, l$ be three vertices of a minimally rigid graph such that there is an edge between $j$ and $k$. An edge splitting operation consists in removing this edge, adding a vertex $i$ and connecting it to $j$, $k$ and $l$, as shown in Figure 4(b). This operation provably preserves minimal rigidity [10]. The reverse operation is less straightforward than the reverse vertex addition operation. Given a vertex $i$ connected to $j$, $k$ and $l$, the minimal rigidity of the graph is preserved if one removes $i$ and adds one edge among $(j, k)$, $(k, l)$ and $(l, j)$. However, one cannot always freely choose any one of these edges to add. One has indeed to make sure that the added edge does not already belong to the graph, and also that its addition does not create a subgraph $G' = (V', E')$ with $|E'| > 2|V'|-3$, i.e., that the pair of vertices does not define an implicit edge in the graph obtained after deletion of $i$. Figure 5 shows an example of such an unfortunate added edge selection. Suppose indeed that the vertex 5 is removed from the minimally rigid graph 5(a). The pair $(1, 4)$ does provably not define an implicit edge, and its addition leads thus to a minimally rigid graph, which is represented in Figure 5(b). However, if $(1, 6)$ is added instead of $(1, 4)$, the graph obtained contains a subgraph $G' = (V', E')$ with $V' = \{1, 2, 3, 6\}$ such that $6 = |E'| > 2|V'|-3 = 5$, as shown in Figure 5(c). The pair $(1, 6)$ defines thus an implicit edge. It is possible to prove that at least one among the three possible pairs of vertices does not define an actual nor an implicit edge [8, 10]. One can thus always perform a reverse edge splitting on any vertex with a degree 3.

A Henneberg sequence is a sequence of graphs $G_2, G_3, \ldots, G_{|V|}$ with $G_2$ be-
Figure 5: Example of unfortunate added edge selection in reverse edge splitting. After the removal of the vertex 5 from the minimally rigid graph (a), minimal rigidity can be preserved by the addition of the edge (1, 4) but not of (1, 6), as shown respectively on (b) and (c). The pair (1, 6) defines an implicit edge in the minimally rigid subgraph induced by 1, 2, 3 and 6.

A simple degree counting argument shows that every minimally rigid graph \( G_{|V|} = (V, E) \) with more than 2 vertices contains at least one vertex with degree 2 or 3. One can thus always perform either a reverse vertex addition or a reverse edge splitting operation and obtain a smaller minimally rigid graph on two vertices, which can only be \( K_2 \). It is straightforward to see that the sequence \( K_2 = G_2, G_3, \ldots G_{|V|} \) is then a Henneberg sequence. We have thus proved the following result [10]:

**Theorem 1.** Every minimally rigid graph on more than one vertex can be obtained as the result of a Henneberg sequence.

The result of Theorem 1 provides a way to exhaustively enumerate all minimally rigid graphs. One can thus use it to obtain an upper bound on the number of minimally rigid graph having a certain number of vertices. However, this only provides an upper bound for the Henneberg sequence allowing one to build a certain minimally rigid graph is usually not unique. The graph in Figure 1(b) can for example be obtained from \( K_2 \) by either two vertex additions or one vertex addition followed by one edge splitting.

### 2.2 Minimally persistent graphs

Consider a group of autonomous agents represented by vertices of a graph. To each of these agents, one assigns a (possibly empty) set of unilateral distance constraints represented by directed edges: the notation \((i, j)\) for a directed edge connotes that the agent \(i\) has to maintain its distance to \(j\) constant during any
continuous move. The persistence of the directed graph means that provided that each agent is trying to satisfy its constraints, the distance between any pair of connected or non-connected agents is maintained constant during any continuous move, and as a consequence the shape of the formation is preserved. A formal definition of persistence is given in [5].

In a two-dimensional space, an agent having only one distance constraint to satisfy can move on a circle centered on its neighbor, and has thus one degree of freedom. Similarly, an agent having no distance constraint to satisfy can move freely in the plane and has thus two degrees of freedom. We call the number of degrees of freedom of a vertex \( i \) the (generic) dimension of the set in which the corresponding agent can chose its position (all the other agents being fixed). It represents thus in some sense the decision power of this agent. The number of degrees of freedom of a vertex \( i \) is given by \( \max(0, 2 - d^+(i) + d^-(i)) \) (where \( d^+(i) \) and \( d^-(i) \) represent respectively the in- and out-degree of the vertex \( i \)).

A graph is minimally persistent if it is persistent and if no edge can be removed without losing persistence. The following result provides a swift criterion to decide minimal persistence:

**Proposition 2.** [5] A graph is minimally persistent if and only if it is minimally rigid and no vertex has an out-degree larger than 2.

As a consequence of Proposition 2 the number of degrees of freedom of a vertex \( i \) in a minimally persistent graph is \( 2 - d^+(i) \). By Proposition 1 it follows after summation on all the vertices that the total number of degrees of freedom present in such a graph is always 3. This result is consistent with the intuition, there are indeed three degree of freedom to chose the position and orientation of a rigid body in a 2-dimensional space.

Let \( j, k \) be two distinct vertices of a minimally persistent graph \( G = (V, E) \). A directed vertex addition [3, 4] consists in adding a vertex \( i \) and two directed edges \((i, j)\) and \((i, k)\) as shown in Figure 6(a). Since it is a vertex addition operation, it preserves minimal rigidity. Besides, the added vertex has an out-degree 2 and the out-degree of the already existing vertices are unchanged. By Proposition 2 the directed vertex addition thus preserves the minimal persistence. Moreover, if a vertex has an out-degree 2 and an in-degree 0 in a minimally persistent graph, one can always perform a reverse (directed) vertex addition by removing it, and obtain a smaller minimally persistent graph.

Let \((j, k)\) be a directed edge in a minimally persistent graph and \( l \) a distinct vertex. A directed edge splitting [3, 4] consists in adding a vertex \( i \), an edge \((i, l)\), and replacing the edge \((j, k)\) by \((j, i)\) and \((i, k)\), as shown in Figure 6(b). Again, this operation preserves minimal rigidity since it is an edge splitting operation from an undirected point of view, and since the added vertex has an out-degree 2 and the out-degree of the already existing vertices are unchanged, it also preserves minimal persistence. But, unlike in the case of directed vertex
addition, the reverse operation cannot always be performed. Suppose indeed that we have a vertex \( i \) with out-degree 2 and in-degree 1, and call its neighbors \( j, k \) and \( l \). The reverse operation consists in removing \( i \) and its incident edges, and adding either \((j, k)\) or \((j, l)\) (note that \( k \) and \( l \) are interchangeable). Adding any other edge such as \((k, l)\) or \((l, k)\) would indeed prevent the operation from being out-degree preserving, and one could then not guarantee the minimal persistence of the graph obtained (by Theorem 2). But, it can happen that both pairs \((j, l)\) and \((j, k)\) are already connected by explicit or implicit edges. In such a case, minimal rigidity is only preserved by addition of an edge between \( k \) and \( l \), which as explained above may not preserve persistence.

We now show that the vertex addition and edge splitting operations do not allow one to grow all minimally persistent graphs from an initial seed. Consider the graph in Figure 7 (for \( n \geq 1 \)). One can verify by Theorem 1 that it is minimally rigid. Moreover, no vertex has an out-degree larger than 2; by Proposition 2 it is thus minimally persistent. Observe that no vertex has an in-degree 0; it is thus impossible to perform a reverse vertex addition operation. Moreover, only the vertex \( 2n \) satisfies the required conditions about the in- and out-degree in order to offer the possibility of removal by a reverse edge splitting operation, and one can verify that this operation cannot be performed due to the presence in the graph of the edges \((2n + 1, 2n - 1)\) and \((2n - 2, 2n - 1)\). Since this is true for any \( n \), we have an infinite class of graphs on which none of the two above defined reverse operations can be performed (Note that there provably exists other such infinite classes in which the graphs have one vertex with two degrees of freedom and one vertex with one degree of freedom instead of three vertices with one degree of freedom). As a consequence, it is not possible to build every minimally persistent graph by performing a sequence of directed vertex addition or edge splitting operations on some seed graph taken in a finite set of graphs. However, we have the following less powerful result (as argued in [5]).
Proposition 3. It is possible to assign directions to the edges of any minimally rigid graph such that the obtained directed graph is minimally persistent and can be obtained by performing a sequence of vertex additions and edge splittings on an initial graph of two vertices connected by one directed edge (called a “leader-follower seed”).

Proof. Let $G$ be a minimally rigid (undirected) graph. By Theorem 1, it can be obtained by performing a sequence of undirected vertex additions and edge splittings on $K_2$. By performing the same sequence of the directed version of these operations on an initial leader-follower seed, one obtains a directed graph having $G$ as underlying undirected graph. Moreover, since this initial seed is trivially minimally persistent (by Proposition 2), and since the directed versions of both vertex addition and edge splitting preserve minimal persistence, the obtained graph is minimally persistent.

In the following sections, we examine different possibilities of additional operations that allow the construction of all minimally persistent graphs. In order to avoid confusion, we shall sometimes refer to the directed version of vertex addition and edge splitting as standard vertex addition and standard edge splitting. We denote by $S$ the set consisting of these two operations and $S^{-1}$ the one consisting of their inverses (the same convention is used in the sequel for all the operations set). Note that it is always possible to perform an operation of $S$ on a minimally persistent graph, but we have seen that this is not true for operations of $S^{-1}$.

3 A purely directed operation

We introduce here a third persistence-preserving operation: the edge reversal. Unlike those of $S$, does not affect the underlying undirected graph. We then define to macro-operations which help us to prove that the edge reversal is sufficient to obtain any minimally persistent graph from any other one having the same underlying undirected graph, and show how this implies that this operation combined with those of $S$ is sufficient to obtain any minimally persistent graph from a unique initial seed.
3.1 Edge reversal

Let \((i, j)\) be an edge such that \(j\) as at least one degree of freedom, i.e., \(d^+(j) = 0\) or \(d^+(j) = 1\). The edge reversal operation consists in replacing the edge \((i, j)\) by \((j, i)\). As a consequence, one degree of freedom is transferred from \(j\) to \(i\). This operation is its auto-inverse and preserves minimal persistence since it does not affect the underlying undirected graph and the only increased out-degree \(d^+(j)\) remains no greater than 2. From an autonomous agent point of view \(j\) transfers its decision power (or a part of it) to \(i\).

3.2 Path reversal

Given a directed path \(P\) between a vertex \(i\) and a vertex \(j\) such that \(j\) has a positive number of degrees of freedom, a path reversal consists in reversing the directions of all the edges of \(P\). As a result, \(j\) loses a degree of freedom, \(i\) acquires one, and there is a directed path from \(j\) to \(i\). Moreover, the number of degrees of freedom of all the other vertices remain unchanged. Note that \(i\) and \(j\) can be the same vertex, in which case the path either has a trivial length 0 or is a cycle. In both of these situations, the number of degrees of freedom is preserved for every vertex.

The path reversal can easily be implemented with a sequence of edge reversals: Since \(j\) has a degree of freedom, one can reverse the last edge of the path, say \((k, j)\), such that \(j\) loses one degree of freedom while \(k\) acquires one. One can then iterate this operation along the path until \(i\), as shown in Figure 8. At the end, \(i\) has have an additional degree of freedom, \(j\) has lost one, and all the edges of the path have have been reversed. Note that the sequence of edge reversals can usually not be performed in another order, for the condition about the availability of a degree of freedom would not be satisfied. The final result would be the same, but all the intermediate graphs would not necessarily be minimally persistent.

We now show that this operation allows one to reposition the three degrees of freedom of a minimally persistent graph onto any chosen vertices (with at most two degrees of freedom on a single vertex). For this purpose, we need the following result (which is a particular case of a result available in [6, 13]):

**Proposition 4.** Let \(G\) be a minimally persistent graph, \(i\) and \(j\) two vertices of \(G\) with \(d^+(i) \geq 1\) and \(d^+(j) \leq 1\). Then, there is a directed path from \(i\) to \(j\).

**Proof.** Suppose (to obtain a contradiction) that we have a minimally persistent (and thus minimally rigid) graph \(G = (V, E)\), a vertex \(i\) with positive out-degree and a vertex \(j\) with a positive number of degrees of freedom such that there is no directed path connecting \(i\) to \(j\). Let \(V'' \subset V\) be the set of vertices that can be reached from \(i\), and \(E''\) the set of edges that leave vertices of \(V''\). Obviously, every edge of \(E''\) is incident to two vertices of \(V''\), and because \(d^+(i) > 0\), we have \(|V''| \geq 2\) and \(|E''| \geq 1\). Moreover, the sum on the vertices of \(V''\) of
the numbers of degrees of freedom (which we denote $F(V'')$) is smaller than 3. There are indeed only three degrees of freedom in a minimally persistent graph as explained in Section 2.2, and the vertex $j$ which has at least one of them does by hypothesis not belong to $V'$. Since every vertex has an out-degree smaller no greater than two in a minimally persistent graph (by Proposition 2), we have thus a subgraph $G'' = (V'', E'')$ such that

$$|E''| = \sum_{k \in V''} d^+(k, G'') = 2 |V''| - F(V'') > 2 |V''| - 3,$$

which by Proposition 4 is impossible for a subgraph of a minimally rigid graph. 

Let us now suppose that one wants to transfer a degree of freedom from some vertex $j$ to some vertex $i$ which has at most one degree of freedom (transferring a degree of freedom to a vertex that has already two degrees of freedom would indeed be impossible as there is no edge inwardly incident). By Proposition 4 there exists a directed path from $i$ to $j$. The transfer can then be done by reversing this path, which leaves the positions of all the other degrees of freedom unchanged. By doing this at most three times, one can thus reposition the three degrees of freedom onto any chosen vertices. As a consequence, we have the following result.

**Proposition 5.** Let $G_A$ and $G_B$ be two minimally persistent graphs having the same underlying undirected graph. By applying a sequence of at most three path reversals on $G_A$, it is possible to obtain a minimally persistent graph $G'_A$ in which every vertex has the same number of degrees of freedom as in $G_B$. 

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3.3 Cycle reversal

A cycle reversal consists in reversing all the edges of a directed cycle. This operation does not affect the number of degrees of freedom of any vertex nor the underlying undirected graph, and preserves therefore minimal persistence.

A cycle reversal on a minimally persistent graph can be implemented by a sequence of edge reversals. Let us indeed first suppose that there is a vertex $i$ in the cycle that has at least one degree of freedom. In that case, the cycle reversal is just a particular case of the path reversal, with $i = j$. We now assume that no vertex in the cycle has a degree of freedom. Let $l$ be a vertex in the cycle, and $m$ a vertex that does not belong to the cycle but has a degree of freedom. By Proposition 4 it follows that there exists a directed path from $l$ to $m$. Let $i$ be the last vertex in this path belonging to the cycle. There is trivially a path $P$ from $i$ to $m$ such that every other vertex of this path does not belong to the cycle. The implementation of a cycle reversal by three path reversals is then represented in Figure 9. One begins by reversing the path $P$ into $P'$ such that $i$ acquires a degree of freedom. As explained above, the cycle can then be reversed since it is a particular case of path reversal, and finally, one reverses the path $P'$ back to $P$ such that the degree of freedom acquired by $i$ is re-transmitted to $m$. Note that an alternative equivalent approach is to reverse the path from $l$ to $m$ containing $i$ and one part of the cycle, and then to reverse the newly created path from $m$ to $l$ containing $i$ and the other part of the cycle.

Remark 1. Both cycle reversal and path reversal are their auto-inverse, as is the case for edge reversal. Moreover, the fact that they can be implemented using only edge reversals is another way to show that they preserve minimal persistence.

We now prove that from any minimally persistent graph, one can obtain any other minimally persistent graph having the same underlying undirected graph and allocation of degrees of freedom by a sequence of cycle reversals. For this purpose, we need the following result.

Lemma 1. Let $G_A = (V, E_A)$ and $G_B = (V, E_B)$ be two graphs having the same underlying undirected graph and such that every vertex has the same out-degree
in both graphs. If an edge of \( G_A \) has the opposite direction to that in \( G_B \), it belongs to (at least) one cycle of such edges in \( G_A \).

Proof. Suppose that \((i_0, i_1) \in E_A \) and \((i_1, i_0) \in E_B \) (i.e., this edge has opposite directions in \( G_A \) and \( G_B \)); then there exists at least one vertex \( i_2 \neq i_0 \) such that \((i_1, i_2) \in E_A \) and \((i_2, i_1) \in E_B \). For if the contrary holds, we would have \( d^+(i_1, G_A) = d^+(i_1, G_B) - 1 \), which contradicts our hypothesis. Repeating this argument recursively, we obtain an (infinite) sequence of vertices \( i_0, i_1, i_2, \ldots \) such that for each \( j \geq 0 \), \((i_j, i_{j+1}) \in E_A \) and \((i_{j+1}, i_j) \in E_B \). Since there are only a finite number of vertices in \( V \), at least one of them will appear twice in this sequence. By taking the subsequence of vertices (and induced edges) appearing in the infinite sequence between any two of its occurrences we obtain then a cycle having the required properties.

Proposition 6. Let \( G_A = (V, E_A) \) and \( G_B = (V, E_B) \) be two minimally persistent graphs having the same underlying undirected graph and such that every vertex has the same number of degrees of freedom in both of them. Then it is possible to obtain \( G_B \) from \( G_A \) by a sequence of at most \(|E_A|/3 = |E_B|/3 \) cycle reversals.

Proof. Suppose that \( G_A \neq G_B \), and let \( E_o \) denote the set of edges of \( G_A \) that do not have the same direction as in \( G_B \). Since both graphs have the same underlying undirected graph and since all the vertices have the same out-degrees in both of them it follows from Lemma 1 that there exists a cycle of edges of \( E_o \). \(|E_o| \) is thus strictly decreased by reversing this cycle. Doing this recursively leads then to \(|E_o| = 0 \), i.e., to a situation where \( G_A = G_B \). Moreover, since every cycle has at least three edges. The number of cycle reversals is at most \(|E|/3 \)

3.4 Obtaining all minimally persistent graphs using three primitive operations

Using the results of the two previous subsections, we can now show the following Proposition.

Proposition 7. By applying a sequence of edge reversals to a given minimally persistent graph, it is possible to obtain any other minimally persistent graph having the same underlying undirected graph. Moreover, all the intermediate graphs are then minimally persistent.

Proof. Since both path reversal and cycle reversal can be implemented by a sequence of edge reversals (which preserves minimal persistence), this result is a direct consequence of Propositions 5 and 6.

From an autonomous agent formation perspective, suppose that a reorganization of the distance constraints distribution needs to be performed, and that this reorganization preserves the structure of constraints from an undirected point of view, i.e., the reorganization is solely one involving changes of some
directions. Proposition 7 implies that this can be done by a sequence of local degree of freedom transfers, in such a way that during all the intermediate stages, the formation shape is guaranteed to be maintained as a result of persistence being preserved.

Let $T$ be the set of operations containing vertex addition, edge splitting, and edge reversal. A leader-follower seed is a minimally persistent graph on two vertices. It contains only one edge, leaving a vertex called “the follower”, and arriving at the other one, called “the leader”. The next theorem states that one can obtain any minimally persistent graph from an initial leader-follower seed using only operations of $T$.

**Theorem 2.** Every minimally persistent graph can be obtained by applying a sequence of operations of $T$ to an initial leader-follower seed. Moreover, all the intermediate graphs are minimally persistent.

**Proof.** Consider a minimally persistent graph $G$. This graph is also minimally rigid. By Proposition 5, there exists thus a (possibly different) minimally persistent graph having the same underlying undirected graph that can be obtained by performing a sequence of operations of $S \subset T$ on an initial leader-follower seed. By Proposition 7, $G$ can then be obtained by applying a sequence of edge reversals on this last graph. Moreover, since all the operations of $T$ preserve minimal persistence, all the intermediate graphs are minimally persistent.

As an illustration of Theorem 2, consider the graph $G$ represented in the right hand side of Figure 10(c), which is an instantiation of the graph of Figure 7 with $n = 2$. As explained in Section 2.2, it cannot be obtained by applying a vertex addition or an edge splitting on a smaller minimally persistent graph. However, by Theorem 2, it can be obtained by applying a sequence of operations of $T$ on an initial leader-follower seed. Let us take 1 and 2 as respectively leader and follower of this initial seed. One can begin by adding 3, 4 and 5 using three vertex additions as shown in Figure 10(a). The graph obtained has the same underlying undirected graph as $G$, but the degrees of freedom are not allocated to the same vertices. By reversing the path $P (\{5, 4, 2, 1\})$ (using a sequence of edge reversals), one can then transfer one degree of freedom from 1 to 5 as shown in Figure 10(b) such that in the obtained graph, all the vertices have the same number of degrees of freedom as in $G$. As stated in Proposition 6, any edge of this graph that does not have the same direction as in $G$ belongs to a cycle of such edges. The only such cycle here is $C$. By reversing it (using a sequence of edge reversals), one finally obtains the graph $G$, as shown in Figure 10(c). Note that consistently with Theorem 2 all the intermediate graphs are minimally persistent.

Theorem 2 also proves that it is always possible to obtain a leader-follower pair from any minimally persistent graph by applying an appropriate sequence of operations of $T^{-1}$. This can be also stated as follows:

**Theorem 3.** Let $G$ be a minimally persistent graph. By applying a (possibly empty) sequence of edge reversals on $G$, it is always possible to obtain a mini-
Figure 10: Example of obtaining of a minimally persistent graph by applying a sequence of operations of $T$ on a leader-follower seed. The graph $G$ is obtained from the leader-follower seed by (a) three vertex additions, (b) the reversal of the path $P$ and (c) of the cycle $C$.

A ** mally persistent graph on which at least one operation of $S^{-1}$ (i.e, reverse edge splitting or reverse vertex addition) can be performed.

Starting from a minimally persistent graph, one can thus first use operations of $T^{-1}$ to obtain a leader-follower pair, and then use operations of $T$ to obtain any other minimally persistent graph. This method is generally not optimal in terms of the number of operations. However, the argument proves the following corollary.

**Corollary 1.** Every minimally persistent graph can be transformed into any other minimally persistent graph using only operations of $T \cup T^{-1}$.

This result allows us to define a distance on the minimally persistent graphs (on more than one vertex) by saying that the distance between two of them is the minimal number of operations of $T \cup T^{-1}$ needed to transform one into the other. Propositions 3, 5 and 6 imply that the distance between two graphs is quadratically bounded by their size, the quadratic character coming from the cycle reversing operations (the others requires only a linear number of operations). However, a better bound is likely to exist.

**Remark 2.** Observe that the three operations of $T$ are relatively basic ones and are performed locally. They could thus easily be implemented in a local way on an autonomous agent formation. It might be however possible to improve this
Figure 11: Implementation of the edge splitting by a vertex addition and an edge reorientation. The vertex \( i \) is first added with two out-going edges by vertex addition, and the edge \((j, k)\) is then reoriented and becomes \((j, i)\).

*basic character using for example an operations such as an edge reorientation, i.e., an operation consisting in changing the arrival vertex of an edge. As shown in Figure 11, a vertex addition operation and an edge reorientation operation can indeed implement an edge splitting operation which could thus be discarded. However, this would require an efficient and simple criterion to determine when such an edge reorientation operation could be performed.*

4 An alternative set of four primitive operations

As explained in Section 3, every minimally persistent graph can be obtained by applying a sequence of operations belonging to \( T \) on an initial leader-follower seed, in such a way that all the intermediate graphs are minimally persistent. However, unlike in the case of an undirected Henneberg sequence (see Section 2), the number of vertices in the final graph does not determine uniquely the required number of intermediate graphs, but only an upper bound on it (see Section 3). In this section, we focus on sets of operations equivalent to those of \( S \) from an undirected point of view and that allow one to build all minimally persistent graphs (the number of intermediate graphs being thus uniquely determined by the number of vertices of the final graph since each operation adds one vertex). It is proved that those sets always contain at least one operation involving the reversal of edges that are not affected by the corresponding operation for undirected graphs. We then provide such a set \( \mathcal{A} \) of four types of operations and show how it allows one to build any minimally persistent graph \( G = (V, E) \) by applying \(|V| - 2\) operations to an initial leader-follower seed. Finally we study the relations between the two sets \( \mathcal{A} \) and \( T \).

4.1 Necessary involvement of external edges.

In the sequel, we adopt the terms *generalized vertex addition* and *generalized edge splitting* for any operation which is equivalent to a vertex addition or an edge splitting from an undirected point of view. Such an operation is said to be *confined* if it only affects edges that are involved in the corresponding undirected operation. For example, all the operation of \( S \) are confined, while the
edge reversal operation defined in Section 3.1 is not.

Suppose that one wants to remove a vertex (without losing persistence) from the provably minimally persistent graph represented in Figure 12 using a generalized reverse edge splitting or reverse vertex addition. The only ones that can be removed are those with a label “+”, and due to their total degree, this could only be done by a generalized reverse edge splitting operation. Suppose now that one wants to use a confined version of this operation. One would then remove one of the vertices with a label “+” and connect two of its neighbors by a directed edge. Observe that among the three pairs of neighbors of any vertex with a label “+”, two are already connected, and the last pair contains two vertices with an out-degree 2. Adding an edge between a pair of neighbors of the removed vertex without reversing the direction of any other edge would thus imply the presence of either a vertex with out-degree 3 (which by Theorem 2 is impossible in a minimally persistent graph) or of a cycle of length 2 (which by Proposition 1 cannot appear in a minimally rigid graph). This removal should therefore be performed by a non-confined reverse generalized edge splitting. The following result is thus proved.

**Proposition 8.** If a set exists of generalized vertex additions and edge splittings allowing one to build all minimally persistent graphs from an initial leader-follower seed, such a set must always contain a non-confined edge splitting.

The existence of confined operations that would not be equivalent to vertex addition or edge splitting, but that would however preserve minimal persistence and allow one to build all minimally persistent graphs with \(|V|\) vertices in \(|V|−2\) operations (starting with a leader-follower seed) remains an open question. Note that such operations would have to be proved to preserve minimal rigidity.

### 4.2 Description of a set \(\mathcal{A}\) of four primitive operations

We define here a new set \(\mathcal{A}\) of four operations. The first two are the standard vertex additions and edge splitting as described in Section 2.2, which implies that \(\mathcal{S} \subset \mathcal{A}\). The two others are atypical versions of these.

Let \(j, k\) be two vertices of a minimally persistent graph such that \(j\) has at least one degree of freedom. The *atypical vertex addition* operation consists in adding the vertex \(i\), the edges \((j, i)\) and \((i, k)\), as shown in Figure 13(a). As a result, \(j\) loses a degree of freedom, and \(i\) appears with one. The *reverse atypical vertex addition* operation consists in removing a vertex with in- and out-degree 1.

**Proposition 9.** Atypical vertex addition and reverse atypical vertex addition preserve minimal persistence.

*Proof.* Since these operations are respectively a generalized vertex addition and a reverse generalized vertex addition, they preserve minimal rigidity as explained
Figure 12: A minimally persistent graph no vertex of which can be removed (without losing persistence) by a reverse (generalized) vertex addition or a confined (generalized) reverse edge splitting. The symbol “*” represents one degree of freedom. Vertices that are candidate to be removed by a reverse generalized edge splitting are labelled “+”.

Moreover, the reverse atypical vertex addition does not increase the out-degree of any vertex, while the atypical vertex addition only increases by one an out-degree that is smaller than 2. In both situations the graph obtained after performing the operation does not contain any vertex with an out-degree larger than 2 and is thus minimally persistent (by Proposition 2).

Let $j$, $k$ and $l$ be three vertices of a minimally persistent graph such that there is a (simple) directed path from $j$ to $k$ and $(k, l) \in E$. The atypical edge splitting operation consists in removing $(k, l)$, adding a vertex $i$ and the edges $(j, i)$, $(i, k)$ and $(i, l)$, and reversing the direction of all the edges belonging to the path from $j$ to $k$, as represented in Figure 13(b).

Proposition 10. Atypical edge splitting preserves minimal persistence.

Proof. This operation is a generalized edge splitting and thus preserves minimal rigidity. Since it does not affect the out-degree of any already existing vertex and adds a vertex with out-degree 2, it also preserves minimal persistence (by Proposition 2).

Consider a vertex $i$ with out-degree 2 and in-degree 1 in a minimally persistent graph, and call its neighbors $j$, $k$ and $l$ as in Figure 13(b). Suppose that in the graph obtained after deletion of $i$, there is a path from $k$ (or equivalently $l$) to $j$ and the pair $(k, l)$ is not connected by an implicit nor an explicit edge. The reverse atypical edge splitting consists then in removing $i$, reversing all the edges of the path from $k$ to $j$ to obtain a path from $j$ to $k$, and adding the edge $(k, l)$.

Proposition 11. Reverse atypical edge splitting preserves minimal persistence.
Proof. From an undirected point of view, this operation consists in removing one vertex incident to three edges, and then connecting a pair of unconnected vertices that does not define an implicit edge in the intermediate graph. It thus preserves minimal rigidity. Moreover, it does not affect the out-degree of any remaining vertex. It follows from Proposition 2 that reverse atypical edge splitting preserve minimal persistence.

The conditions in which the reverse atypical edge splitting can be performed are not always easy to check. However, the following result holds:

Lemma 2. In a minimally persistent graph, a vertex with in-degree 1 and out-degree 2 can always be removed by either a reverse standard edge splitting or a reverse atypical edge splitting.

Proof. Consider a minimally persistent graph $G = (V, E)$ and a vertex $i \in V$ with $d^+(i) = 2, d^-(i) = 1$. We call its neighbors $j, k$ and $l$ such that $(j, i), (i, k), (i, l) \in E$, as in Figure 14(a).

Let us assume that $i$ cannot be removed by a reverse standard edge splitting, i.e., that $j$ is connected to both $k$ and $l$ by an explicit or implicit edge in $G \setminus \{i\}$. As already mentioned in Section 2.1, $k$ and $l$ are in such a case never connected by an implicit nor an explicit edge in $G \setminus \{i\}$, and the graph obtained by connecting them after removing $i$ from $G$ is therefore minimally rigid. It remains to prove the existence of a directed path from $k$ or $l$ to $j$ (note that $k$ and $l$ are interchangeable) in order that an edge splitting operation can be applied. For this purpose, we are going to construct a minimally persistent graph $G'$ close to $G$ and in which $j$ has a degree of freedom. As explained below, Proposition 2 guarantees then the existence of a directed path from either $k$ or $l$ to $j$. It will be proved that this implies the existence of such a path in $G \setminus \{i\}$. 

Figure 13: Representation of the atypical (a) vertex addition operation and (b) edge splitting operation, both belonging to $A$. The symbol "*" represents one degree of freedom.
Consider a vertex $p$ having at least one degree of freedom in $G$. Since $d^+(i) = 2$ in $G$, Proposition 4 guarantees the existence (still in $G$) of a (cycle-free) directed path from $i$ to $p$. Without loss of generality, let us assume that the second vertex of this path is $k$ (it has indeed to be a neighbor of $i$, and $k$ and $l$ are interchangeable). There exists thus a directed path $P$ from $k$ to $p$ to which $i$ does not belong. We build $G'$ by reversing the path $P$ (which becomes $P'$), removing $i$ and adding the edge $(k, l)$, as shown in Figure 14(a) and (b).

As already mentioned, any graph obtained by removing $i$ and connecting $k$ to $l$ is minimally rigid. Moreover, after the reversal of $P$ and addition of $(k, l)$, $p$ loses one degree of freedom, i.e., its out-degree in $G'$ is increased by one with respect to its out-degree in $G$ (which is smaller than two). On the other hand, $j$ acquires a degree of freedom. No vertex has thus an out-degree larger than 2 in $G'$, which is therefore minimally persistent.

In $G'$, $k$ has by construction a positive out-degree, and $j$ has at least one degree of freedom. By Proposition 4, there exists thus a (cycle-free) directed path $Q$ in $G'$ from $k$ to $j$. In order to prove the existence of such a path in $G \setminus \{i\}$, we consider three cases. Observe that the only edges of $G'$ that do not exist in $G \setminus \{i\}$ are $(k, l)$ and those of $P'$ (which exist but with the opposite direction).

- $Q$ and $P'$ have no common edge, $(k, l) \notin Q$: In that case, the path $Q$ also exists in $G \setminus \{i\}$.

- $Q$ and $P'$ have no common edge, $(k, l) \in Q$: Since the (simple) path $Q$ does not contain any cycle, $(k, l)$ must be the first edge of $Q$. By removing this edge, one obtains a directed path from $l$ to $j$ having no intersection with $P'$ and that does not contain $(k, l)$. This path exists thus also in $G \setminus \{i\}$.

- $Q$ and $P'$ have some common edge(s): Let $q$ be the last vertex of $Q$ that also belongs to $P'$. The edges and vertices of $Q$ which are after $q$ constitute a directed path from $q$ to $j$. By definition, it does not intersect $P'$, and

![Figure 14: Representation of different paths and graph involved in Lemma 2. (b) shows the minimally persistent graph $G'$ obtained from the graph $G$ represented in (a) by removing $i$, adding $(k, l)$ and reversing $P$. (c) shows then the path $Q$ (in $G'$) from $k$ to $j$ in a case where it has a non empty intersection with $P'$.](image-url)
it does not contain \((k, l)\) for this would mean that \(Q\) contains a cycle. It exists therefore also in \(G\). Moreover, since \(q\) belongs to the path \(P'\) in \(G'\), it belongs to the path \(P\) in \(G \setminus \{i\}\), and there is thus a directed path from \(k\) to \(q\) in \(G \setminus \{i\}\). By taking the union of the path from \(k\) to \(q\) and the one from \(q\) to \(j\), one obtains then a directed path from \(k\) to \(j\) in \(G \setminus \{i\}\), as shown in Figure 14(c).

In any of these three cases, there is thus a directed path from \(k\) or \(l\) to \(j\) in \(G \setminus \{i\}\). As explained above, this implies that one can perform the reverse atypical edge splitting on \(i\) if one cannot perform the reverse standard one.

### 4.3 Obtaining all minimally persistent graphs using \(A\)

Let \(G = (V, E)\) be a minimally persistent and therefore minimally rigid graph with more than two vertices. By Proposition 1, there holds \(4|V| - 6 = |E| = \sum_{i \in V} d^-(i) + d^+(i)\). Moreover, it can be shown (using Proposition 1) that such a graph never contains any vertex with a total degree smaller than 2. A counting argument shows then that it always contain at least one vertex with either \((d^-, d^+) = (0, 2), (1, 1)\) or \((1, 2)\). In the first two cases, one can perform on this vertex a reverse standard or atypical vertex addition, while in the last case, it follows from Lemma 2 that either a reverse standard edge splitting or a reverse atypical one can be performed. It is thus always possible to obtain a minimally persistent graph \(G' = (V', E')\) with \(|V'| = |V| - 1\) by performing an operation of \(A^{-1}\) on \(G\). Doing this recursively, one can obtains after \(|V| - 2\) operation a minimally persistent graph on two vertices, i.e., a leader-follower seed. The reverse sequence of operations allows then one to obtain \(G\) from this seed. Since the operations of \(A\) and \(A^{-1}\) preserve minimal persistence, all the intermediate graph of such a sequence are persistent. We have thus proved the following theorem:

**Theorem 4.** Every minimally persistent graph \(G = (V, E)\) \(||V| > 1\) can be obtained by performing \(|V| - 2\) operations of \(A\) on an initial leader-follower seed. Moreover, all the intermediate graphs are minimally persistent.

Using the same argument as for Corollary 1, we obtain the following result.

**Corollary 2.** Every minimally persistent graph can be transformed into any other minimally persistent graph using only operations of \(A \cup A^{-1}\).

As in Section 3.4, one can use the set \(A \cup A^{-1}\) to define a distance on the minimally persistent graphs (on more than one vertex). But, as a consequence of Theorem 4, the distance between the graph \(G = (V, E)\) and \(G' = (V', E')\) is never greater than \(|V| + |V'| - 4\).

**Remark 3.** The non-confined character of the atypical edge splitting makes it more complicated to implement in an autonomous agent formation. It can indeed involve the direction reversal of a potentially large number of edges that are
not involved in the corresponding operation for undirected graph. Proposition 8 states that there always is such a non-confined operation in a set of generalized vertex addition and edge splitting operations which allows one to build all minimally persistent graphs. However, the example in Figure 12 only requires one edge to be reversed, and no example was found yet where it was necessary to reverse more than one edge. There might thus exist a set of operations having the same properties as $\mathcal{A}$ (with respect to the building of all minimally persistent graphs) but in which the non-confined operation only involves the reversal of a number of edges bounded independently of $|V|$.

**Remark 4.** It is possible to show that among the four operations of $\mathcal{A}$ (resp. $\mathcal{A}^{-1}$), none can be removed without being replaced by some alternative new operation if the operation set is to produce all minimally persistent graphs (resp. contain for each minimally persistent graph an operation that can be performed on it). For each operation of $\mathcal{A}^{-1}$, one can indeed find a graph where none of the three other operations can be performed. However, the set of generalized vertex additions and edge splittings that we present here is just one among the several sets that we have found allowing one to build all minimally persistent graphs. It offers the advantage that its non-confined operation has a more local character than those contained in the other sets (which are not described here).

### 4.4 Relations between $\mathcal{A}$ and $\mathcal{T}$

We examine here the relation between the two sets $\mathcal{A}$ and $\mathcal{T}$. Let $\mathcal{X}$ and $\mathcal{Y}$ be two sets of operations. We say that $\mathcal{X} \leq \mathcal{Y}$ if all the operations of $\mathcal{X}$ can be implemented by a sequence of operations of $\mathcal{Y}$. If $\mathcal{X} \leq \mathcal{Y}$ and $\mathcal{X} \geq \mathcal{Y}$, we say that $\mathcal{X} = \mathcal{Y}$. If $\mathcal{X} \leq \mathcal{Y}$ and $\mathcal{X} \neq \mathcal{Y}$, we say that $\mathcal{X} < \mathcal{Y}$. One can see that $\mathcal{X}^{-1} \leq \mathcal{Y}^{-1}$ if and only if $\mathcal{X} \leq \mathcal{Y}$.

**Lemma 3.** An atypical vertex addition can be implemented using one standard vertex addition and one edge reversal.

*Proof.* Let $k$ and $j$ be two distinct vertices in a minimally persistent graph such that $k$ has at least one degree of freedom. Instead of adding a vertex $i$ and the edge $(i, j)$ and $(k, i)$ (atypical vertex addition), one can equivalently add the vertex $i$ and the edges $(i, j)$ and $(i, k)$ (standard vertex addition), and then reverse the edge $(i, k)$ (edge reversal). Note that this edge reversal can be performed because $k$ has a degree of freedom. $\Box$

**Lemma 4.** An atypical edge splitting can be implemented using one standard edge splitting and one or more edge reversal(s).

*Proof.* Let $j$, $k$ and $l$ be three vertices of a minimally persistent graph $G$ satisfying the conditions required to perform an atypical edge splitting (see Section 4.2), and let $G'$ be the graph obtained by performing an atypical edge splitting on $(k, l)$ in $G$ as represented in Figure 13(b). By performing a standard edge splitting on $(k, l)$ in $G$ such that the added vertex is also connected to $j$, one
obtains a minimally persistent graph having the same underlying undirected graph as \( G' \). It is then a consequence of Proposition 7 that this last graph can be obtained by a sequence of edge reversals.

**Proposition 12.** \( A < T \), and equivalently \( A^{-1} < T^{-1} \)

**Proof.** Observe first that all the operations of \( A \) increase the number of vertices in the graph, while edge reversal does not. Thus edge reversal cannot be implemented by a sequences of operations of \( A \), and \( A \not\geq T \).

Since the operations of \( S \) (standard vertex additions and edge splitting) belong to both \( A \) and \( T \), and since by Lemmas 3 and 4, the operations of \( A \setminus S \) (atypical vertex addition and atypical edge splitting) can be implemented using operations of \( T \), we have \( A \leq T \), which together with \( A \not\geq T \) implies that \( A < T \).

Since the set \( T \) of operations is more powerful than \( A \), Theorem 4 is a stronger result than Theorem 2. However, if we look at the sets containing both normal and inverse operations, the results are different. Suppose indeed that a graph \( G' \) is obtained by performing an operation of \( T \cup T^{-1} \) on a minimally persistent graph \( G \). Since both graphs are minimally persistent, Corollary 2 implies that \( G' \) can also be obtained by applying a sequence of operations of \( A \cup A^{-1} \) on \( G \). Any operation of \( T \cup T^{-1} \) can thus be implemented by a sequence of operations of \( A \cup A^{-1} \). Conversely, any operation of \( A \cup A^{-1} \) can be implemented by a sequence of operations of \( T \cup T^{-1} \). We have thus shown the following result:

**Proposition 13.** \( A \cup A^{-1} = T \cup T^{-1} \)

Both sets \( A \) and \( T \) allow one to enumerate exhaustively all minimally persistent graphs. However, as explained in Section 2.1, since the sequence that can build a certain minimally persistent graph is not unique, this enumeration can allow one to compute an upper bound on the number of minimally persistent graphs having a certain number of vertices, but not their exact number.

5 Conclusions and future work

In this paper, we have extended the Henneberg sequence concept to directed graphs. From an autonomous agent point of view, this provides a systematic approach to sequentially obtain or reorganize a minimally persistent agent formation. We also exposed some natural restrictions to these extensions, the main one being the impossibility of building all minimally persistent graphs using only confined generalized vertex additions or edge splittings.

We proposed two sets of operations, each of which allows one to obtain any minimally persistent graph from a leader-follower seed. The first one (\( T \) in Section 3) contains the two standard vertex additions and edge splittings already introduced in [3] and a purely directed operation (i.e. a neutral operation from
an undirected point of view). The second set (\( A \) in Section 4) contains, in addition to the two standard operations, two atypical versions of them, among which one is not a confined operation. It involves indeed the reversal of a path of undetermined length in the graph. However, it is still an open question to know if similar results could be obtained using operations involving a number of reversals fixed or bounded independently of the size of the graph. Note that for the second set, the number of operations required to build a minimally persistent graphs is uniquely determined by the size of the graph.

From an autonomous agent point of view, it would be useful to study how these various operations could be performed in a decentralized way in order to efficiently construct or reorganize a formation. For this purpose, the operations of \( T \) could be preferred for their simplicity and because the successive modifications are only local modifications. This study could also imply the development of an optimal algorithm (using one of the two sets of operations) to reorganize a persistent formation. An improvement could come from the use of an even more simple operation such as edge reorientation, which would consist in changing the arrival point of an edge. However, the conditions under which minimal rigidity is preserved by this operation are not known yet.

As a final remark, note that we have only focused on the transformations of minimally persistent graph into other minimally persistent graphs. Several practical issues concerning autonomous agent formations arise relating to the merging of such graphs, or their repair after the loss of some vertices or edges. It would thus be worthwhile to study these particular problems as well.

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