Hochschild cohomology for a class of some self-injective special biserial algebras of rank four

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Abstract

In this paper, we construct an explicit minimal projective bimodule resolution of a self-injective special biserial algebra $A_T$ ($T \geq 0$) whose Grothendieck group is of rank 4. As a main result, we determine the dimension of the Hochschild cohomology group $HH^i(A_T)$ of $A_T$ for $i \geq 0$, completely. Moreover we give a presentation of the Hochschild cohomology ring modulo nilpotence $HH^i(A_T)/N_{aT}$ of $A_T$ by generators and relations in the case where $T = 0$.

Keywords: Hochschild cohomology, self-injective algebra, Koszul algebra.

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1. Introduction

Let $\Gamma$ be the following circular quiver with four vertices 0, 1, 2, 3 and eight arrows $a_i, b_i$ for $i = 0, 1, 2, 3$:

```
0 ← a0 ← 1
   ↑     ↑
   a3  b3  b1  a1
   ↓     ↓
3 ← b2 ← 2
```

Denote the trivial path corresponding to the vertex $i$ by $e_i$ for $0 \leq i \leq 3$. We always consider the subscripts $i$ of $e_i, a_i$ and $b_i$ as modulo 4. Therefore the arrows $a_i$ and $b_i$ start at $e_i$ and end with $e_{i+1}$ for all $i \in \mathbb{Z}$. Paths are written from left to right.

Let $K$ be an algebraically closed field, and let $K\Gamma$ be the path algebra of $\Gamma$ over $K$. We set $x := \sum_{i=0}^{3} a_i \in K\Gamma$ and $y := \sum_{i=0}^{3} b_i \in K\Gamma$. Then, for integers $0 \leq i \leq 3$ and $j \geq 0$, the elements $e_i x^j$ and $e_j y^j$ are precisely the paths $a_0 a_{i+1} \cdots a_{i+j-1}$ and $b_0 b_{i+1} \cdots b_{i+j-1}$ of length $j$ respectively, so that $e_i x^j = e_i x^j e_{i+j} = x^j e_{i+j}$ and $e_j y^j = e_j y^j e_{i+j} = y^j e_{i+j}$ hold. Fix an integer $T \geq 0$, and let $I_T$ be the ideal in $K\Gamma$ generated by the elements $x y, x^{4T+2} + y^{4T+2}$ and $y x$, that is, $I_T := \langle xy, x^{4T+2} + y^{4T+2}, yx \rangle$. Define the algebra $A_T$ to be the quotient $K\Gamma/I_T$. We then see that the set $\{e_i x^j, e_j y^j | 0 \leq i \leq 3; 0 \leq j \leq 4T+2; 1 \leq l \leq 4T+1\}$ is a $K$-basis of $A_T$, so that $\dim_K A_T = 16(2T+1)$. Furthermore $A_T$ is a self-injective special biserial algebra, and hence is of tame representation type. In particular, if $T = 0$, then we see that $A_0$ is a Koszul algebra of radical cube zero (see Proposition 2.3).

The purpose of this paper is to investigate the Hochschild cohomology for $A_T$. The Hochschild cohomology groups and rings of algebras are important invariants in the representation theory of algebras, and have been studied by many researchers. However, in general, it is not easy to describe their structures, even if the given algebras are easier to deal with.

Recently, in the papers [2, 16, 12, 14], the Hochschild cohomology groups or rings of certain finite-dimensional self-injective algebras were described, where the authors provided projective bimodule resolutions by using certain sets $G^{\theta}$ found in [9]. These sets are also used in the papers [4, 5, 6] in constructing projective bimodule resolutions.

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In this paper, following this technique, we give a projective bimodule resolution of \( A_T \) for all \( T \geq 0 \), and then study its Hochschild cohomology.

One important problem in the study of Hochschild cohomology is to find necessarily and sufficient conditions for the Hochschild cohomology ring modulo nilpotence to be finitely generated as an algebra. So far it has been proved that the Hochschild cohomology rings modulo nilpotence for several classes of finite-dimensional algebras, such as group algebras \([3, 13]\), self-injective algebras of finite representation type \([7]\), monomial algebras \([8]\), are finitely generated. Also, some examples of infinitely generated Hochschild cohomology rings modulo nilpotence can be found in \([11, 12]\). However, a definitive answer to this problem has not yet been obtained. In this paper, we show that the Hochschild cohomology ring modulo nilpotence of \( A_T \) is finitely generated in the case where \( T = 0 \).

This paper is organized as follows: In Section 2, we construct sets \( G^n (n \geq 0) \) for the right \( A_T \)-module \( A_T/\tau_{A_T} \), and then give an explicit minimal projective bimodule resolution of \( A_T \). In Section 3, we find a \( K \)-basis of the Hochschild cohomology group \( \text{HH}'(A_T) \) (\( i \geq 0 \)) of \( A_T \) for all \( T \geq 0 \), and then determine its dimension, completely. In Section 4, we present a presentation by generators and relations of the Hochschild cohomology ring modulo nilpotence \( \text{HH}'(A_T)/\mathcal{N}_{A_T} \) in the Koszul case, \( T = 0 \), and show that \( \text{HH}'(A_T)/\mathcal{N}_{A_T} \) is finitely generated.

For every arrow \( e \in \Gamma \), we denote the origin by \( o(e) \) and the terminus by \( t(e) \). For simplicity we write \( \Theta_K \) as \( \Theta \). Moreover we denote the enveloping algebra \( A_T^\ast \otimes A_T \) by \( A_T^\ast \). Note that there is a natural one to one correspondence between the family of \( A_T-A_T^\ast \)-bimodules and that of right \( A_T^\ast \)-modules. We also denote the Jacobson radical of \( A_T \) by \( \tau_{A_T} \).

### 2. The sets \( G^n \) and a projective bimodule resolution \((Q^*, \partial)\) of \( A_T \)

Let \( \Lambda = KQ/I \) be any finite-dimensional \( K \)-algebra with \( Q \) a finite quiver, and \( I \) an admissible ideal in \( KQ \), and let \( \tau_{A_T} \) be the Jacobson radical of \( \Lambda \). We denote by \( G^0 \) the set of all vertices of \( Q \), by \( G^1 \) the set of all arrows of \( Q \), and by \( G^2 \) a minimal set of uniform generators of \( I \). In \([9]\), Green, Solberg and Zacharia showed that, for each \( n \geq 3 \), there is a set \( G^n \) of uniform elements in \( KQ \) such that we have a minimal projective resolution \((P^*, \partial)\) of the right \( \Lambda \)-module \( \Lambda/\tau_{A_T} \) satisfying the following conditions:

(a) For \( n \geq 0 \), \( P^n = \bigoplus_{x \in G^n} t(x)\Lambda \).

(b) For \( x \in G^n \), there are unique elements \( r_y, s_z \in KQ \), where \( y \in G^{n-1} \) and \( z \in G^{n-2} \), such that \( x = \sum_{y \in G^{n-1}} r_y = \sum_{z \in G^{n-2}} s_z \).

(c) For \( n \geq 1 \), the differential \( d^n : P^n \to P^{n-1} \) is defined by \( d^n(x(\lambda)) := \sum_{y \in G^{n-1}} r_y t(x) \lambda \) for \( x \in G^n \) and \( \lambda \in \Lambda \), where \( r_y \) denotes the element in the expression (b).

In this section, we will construct sets \( G^n (n \geq 0) \) for the right \( A_T \)-module \( A_T/\tau_{A_T} \), and then use them to give a projective bimodule resolution \((Q^*, \partial)\) of \( A_T \).

#### 2.1. Sets \( G^n \) for \( A_T/\tau_{A_T} \)

First, in order to give sets \( G^n (n \geq 0) \) for \( A_T/\tau_{A_T} \), we introduce the following elements in \( K\Gamma \):

**Definition 2.1.** For \( 0 \leq i \leq 3 \), we put \( s_{i, 1} := e_i \). Furthermore, for \( n \geq 1 \), we inductively define the elements \( g^{(n)}_{k, j} \in K\Gamma \) for \( 0 \leq i \leq 3 \) and \( 0 \leq j \leq n \) as follows:

(a) If \( n = 2m + 1 \) with \( m \geq 0 \), then

\[
\begin{align*}
S_{k, i, j}^{2m+1} & := \begin{cases} 
2m \times 2m, & \text{if } i = 0, 2 \text{ and } j = 0 \\
2m \times 2m, & \text{if } i = 1, 3 \text{ and } j = 0 \\
2m \times 2m, & \text{if } i = 0, 2 \text{ and } 1 \leq j \leq m \\
2m \times 2m, & \text{if } i = 1, 3 \text{ and } 1 \leq j \leq m \\
2m \times 2m, & \text{if } i = 0, 2 \text{ and } m + 1 \leq j \leq 2m \\
2m \times 2m, & \text{if } i = 1, 3 \text{ and } m + 1 \leq j \leq 2m \\
2m \times 2m, & \text{if } i = 0, 2 \text{ and } j = 2m + 1 \\
2m \times 2m, & \text{if } i = 1, 3 \text{ and } j = 2m + 1.
\end{cases}
\end{align*}
\]
We get

We immediately see that these elements $g_{i,j}^n$ are uniform.

Now we put the set

$$G^2 := \{ g_{i,j}^n \mid 0 \leq i \leq 3; 0 \leq j \leq n \}$$

for all $n \geq 0$.

**Remark 2.2.** (a) For all $n \geq 0$, $0 \leq i \leq 3$ and $0 \leq j \leq n$, we have $\sigma(g_{i,j}^n) = e_i$. Also, if $i + n \equiv t \pmod{4}$, then $t(g_{i,j}^n) = e_i$.

(b) We get

$$G^0 = \{ e_i \mid 0 \leq i \leq 3 \},$$
$$G^1 = \{ a_i, b_i \mid 0 \leq i \leq 3 \},$$
$$G^2 = \{ e_i x y, e_i (x^{4^t+2} + y^{4^t+2}), e_{i+j} x \mid 0 \leq i \leq 3 \},$$

and so $G^2$ is a minimal set of generators of $I_T$.

It is not hard to check that these sets satisfy the conditions (a), (b), and (c) in the beginning of this section.

Now it can be seen that, for all $T \geq 0$, $A_T$ is a self-injective algebra. Moreover, if $T = 0$, then we have the following proposition.

**Proposition 2.3.** The algebra $A_0$ is a self-injective Koszul algebra.

**Proof.** If $T = 0$, then we notice that the resolution $(P^*, d)$ determined by (a), (b) and (c) above is a linear resolution of $A_0/\sigma A_0$, and hence $A_0$ is a Koszul self-injective algebra. 

2.2. A projective bimodule resolution of $A_T$

Now we give a projective bimodule resolution $(Q^*, \partial)$ for $A_T$. For simplicity we denote the element $\sigma(g_{i,j}^n) \otimes t(g_{i,j}^n)$ in $A_T \sigma(g_{i,j}^n) \otimes t(g_{i,j}^n) A_T$ by $a_{i,j}^n$ for $n \geq 0$, $0 \leq i \leq 3$, and $0 \leq j \leq n$. For $n \geq 0$, we define the projective $A_T$-$A_T$-bimodule $Q^n$ by

$$Q^n := \bigoplus_{g \in G^n} A_T \sigma(g) \otimes t(g) A_T \equiv \bigoplus_{i=0}^{3} \left[ \bigoplus_{j=0}^{n} A_T a_{i,j}^n A_T \right].$$

Furthermore we define the map $\partial^n$ as follows:

**Definition 2.4.** Define $\partial^0 : Q^0 \rightarrow A_T$ to be the multiplication map, and for $n \geq 1$ define $\partial^n : Q^n \rightarrow Q^{n-1}$ to be the homomorphism of $A_T$-$A_T$-bimodules determined by the following formulas. Here, we note that the lower left subscripts $i$ of $a_{i,j}^n$ are considered as modulo 4.
(a) If $n = 2m + 1$ for $m \geq 0$, then, for $0 \leq i \leq 3$ and $0 \leq j \leq 2m + 1$,

$$
\delta^{2m+1}(a_{i,j}^{2m+1}) :=
\begin{cases}
\alpha_{0,0}^{2m}x - xa_{0,1}^{2m} & \text{if } i = 0, 2 \text{ and } j = 0 \\
\alpha_{0,0}^{2m}y - ya_{0,1}^{2m} & \text{if } i = 1, 3 \text{ and } j = 0 \\
\alpha_{0,0}^{2m-1}T + \alpha_{0,1}^{2m}x - xa_{0,1}^{2m-1} - y^{d+1}a_{0,1}^{2m-1} & \text{if } i = 0, 2 \text{ and } 1 \leq j \leq m \\
\alpha_{0,0}^{2m-1}T + \alpha_{0,1}^{2m}y - ya_{0,1}^{2m-1} - x^{d+1}a_{0,1}^{2m-1} & \text{if } i = 1, 3 \text{ and } 1 \leq j \leq m \\
\alpha_{0,0}^{2m-1}T + \alpha_{0,1}^{2m}x - xa_{0,1}^{2m-1} - y^{d+1}a_{0,1}^{2m-1} & \text{if } i = 0, 2 \text{ and } m + 1 \leq j \leq 2m \\
\alpha_{0,0}^{2m-1}T + \alpha_{0,1}^{2m}y - ya_{0,1}^{2m-1} - x^{d+1}a_{0,1}^{2m-1} & \text{if } i = 1, 3 \text{ and } m + 1 \leq j \leq 2m \\
\alpha_{0,0}^{2m}y - xa_{0,1}^{2m} & \text{if } i = 0, 2 \text{ and } j = 2m + 1 \\
\alpha_{0,0}^{2m}x - xa_{0,1}^{2m} & \text{if } i = 1, 3 \text{ and } j = 2m + 1.
\end{cases}
$$

(b) If $n = 2m$ for $m \geq 1$, then, for $0 \leq i \leq 3$ and $0 \leq j \leq 2m$,

$$
\delta^{2m}(a_{i,j}^{2m}) :=
\begin{cases}
\alpha_{0,0}^{2m-1}y + xa_{0,1}^{2m-1} & \text{if } i = 0, 2 \text{ and } j = 0 \\
\alpha_{0,0}^{2m-1}x + ya_{0,1}^{2m-1} & \text{if } i = 1, 3 \text{ and } j = 0 \\
\alpha_{0,1}^{2m-1}T + \alpha_{0,1}^{2m}x - xa_{0,1}^{2m-1} - y^{d+1}a_{0,1}^{2m-1} & \text{if } i = 0, 2 \text{ and } 1 \leq j \leq m - 1 \\
\alpha_{0,1}^{2m-1}T + \alpha_{0,1}^{2m}y - ya_{0,1}^{2m-1} - x^{d+1}a_{0,1}^{2m-1} & \text{if } i = 1, 3 \text{ and } 1 \leq j \leq m - 1 \\
\sum_{s=0}^{T}x^{d}(a_{0,1}^{2m-1}x - xa_{0,0,1}^{2m-1} \gamma^{d+1}) + \sum_{s=0}^{T}x^{d}(a_{0,1}^{2m-1}y + ya_{0,1,1}^{2m-1} \gamma^{d+1}) & \text{if } i = 0, 2 \text{ and } j = m \\
\end{cases}
$$

It is straightforward to check that the composite $d^o \delta^{o+1}$ is zero for all $n \geq 0$, so that $(Q^o, \delta)$ is a complex of $A_T$-$A_T$-bimodules.

**Remark 2.5.** For $n \geq 0$, the map $G^n : A_T / \mathcal{A}_x \otimes_A A_T \to P^n$ determined by $G^n(\varepsilon(\delta^n)) \otimes_A A_T \to (\varepsilon(\delta^n)) (0 \leq i \leq 3; 0 \leq j \leq n)$ is an isomorphism of right $A_T$-modules, and this map makes the following diagram commutative:

$$
\begin{array}{ccc}
A_T / \mathcal{A}_x \otimes_A A_T & \xrightarrow{\delta^{o+1}} & A_T / \mathcal{A}_x \otimes_A A_T \\
\downarrow G^{o+1} & & \downarrow G^o \\
P^{o+1} & \xrightarrow{d^{o+1}} & P^o
\end{array}
$$

This shows that $(A_T / \mathcal{A}_x \otimes_A A_T, \delta)$ is isomorphic to $(P^o, d)$ as complexes and hence is a minimal projective resolution of $A_T / \mathcal{A}_x \otimes_A A_T (\simeq A_T / \mathcal{A}_x)$.

Now we have the following theorem. The proof is done with Remark 2.5 and by following [6] (and see also [14]), so we omit it.

**Theorem 2.6.** The complex $(Q^o, \delta)$ is a minimal projective bimodule resolution of $A_T$. 

4
3. Hochschild cohomology groups of $A_T$

In this section we find an explicit $K$-basis of the Hochschild cohomology group $\text{HH}^i(A_T)$ $(i \geq 0)$ by using the resolution $(Q^*, \partial)$ in Section 2 and then give the dimension of $\text{HH}^i(A_T)$, completely. Throughout this section we keep the notation from Section 2.

By applying the functor $\text{Hom}_{A_T}(-, A_T)$ to $(Q^*, \partial)$, we have the complex

$$0 \to \text{Hom}_{A_T}(Q^0, A_T) \xrightarrow{\text{Hom}_{A_T}(\partial^0, A_T)} \text{Hom}_{A_T}(Q^1, A_T) \xrightarrow{\text{Hom}_{A_T}(\partial^1, A_T)} \cdots.$$

Recall that, for $n \geq 0$, the $n$th Hochschild cohomology group $\text{HH}^n(A_T)$ of $A_T$ is defined to be the $K$-space $\text{HH}^n(A_T) := \text{Ext}_{A_T}^n(A_T, A_T) = \text{Ker} \text{Hom}_{A_T}(\partial^n, A_T)/\text{Im} \text{Hom}_{A_T}(\partial^n, A_T)$.

3.1. A basis of $\text{Hom}_{A_T}(Q^i, A_T)$

We start with the following remark:

**Remark 3.1.** (a) For integers $n \geq 0$, $0 \leq i \leq 3$ and $0 \leq j \leq n$, if $n \equiv t \pmod{4}$, then, by Remark 2.2 (a), we get $\sigma(g_{i,j}^n)A_T t(g_{i,j}^n)$, hence $\sigma(g_{i,j}^n)A_T t(g_{i,j}^n)$ has a $K$-basis

$$\begin{align*}
\{e_i\} & \quad \text{if } t = 0 \text{ and } n \equiv 0 \pmod{4} \\
\{e_i^x \mid 0 \leq i \leq T \} & \quad \text{if } t \equiv 1 \pmod{4} \text{ and } n \equiv 0 \pmod{4} \\
\{e_i^x \mid 0 \leq i \leq T \} & \quad \text{if } n \equiv 1 \pmod{4} \\
\{e_i^y \} & \quad \text{if } t = 0 \text{ and } n \equiv 2 \pmod{4} \\
\{e_i^{y_{a+2}} \mid 0 \leq i \leq T \} & \quad \text{if } t \equiv 1 \pmod{4} \text{ and } n \equiv 2 \pmod{4} \\
\{e_i^{y_{a+3}} \mid 0 \leq i \leq T - 1 \} & \quad \text{if } t \equiv 1 \pmod{4} \text{ and } n \equiv 3 \pmod{4}.
\end{align*}$$

Moreover, $\sigma(g_{i,j}^n)A_T t(g_{i,j}^n) = \{0\}$, if $t = 0$ and $n \equiv 3 \pmod{4}$.

(b) For $n \geq 0$ the map $F : \bigoplus_{g \in G^n} \sigma(g)A_T t(g) \to \text{Hom}_{A_T}(Q^n, A_T)$ given by $(F(\sum_{g \in G^n} z_g))(a^n_{i,j}) = z_{g_{i,j}}$, where $z_g \in \sigma(g)A_T t(g)$ for $g \in G^n$, $0 \leq i \leq 3$ and $0 \leq j \leq n$, is an isomorphism of $K$-spaces.

We need the following maps.

**Definition 3.2.** Let $n \geq 0$ be an integer. For $0 \leq i \leq 3$, $0 \leq j \leq n$ and $0 \leq l \leq T$ if $n \equiv 3 \pmod{4}$ and $0 \leq l \leq T - 1$ if $n \equiv 3 \pmod{4}$, we define the maps $\beta_{i,j}^{a+t}, \gamma_{i,j}^{a+t} : Q^n \to A_T$ to be the homomorphisms of $A_T$-bimodules determined by

$$\beta_{i,j}^{a+t}(a^n_{i,j}) := \begin{cases} e_i x^{a+t} & \text{if } r = i, s = j \text{ and } n \equiv t \pmod{4} \text{ where } 0 \leq t \leq 3 \\ 0 & \text{otherwise} \end{cases}$$

and

$$\gamma_{i,j}^{a+t}(a^n_{i,j}) := \begin{cases} e_i y^{a+t} & \text{if } r = i, s = j \text{ and } n \equiv t \pmod{4} \text{ where } 0 \leq t \leq 3 \\ 0 & \text{otherwise} \end{cases}$$

for $0 \leq r \leq 3$ and $0 \leq s \leq n$, respectively.

Note that, for $t \geq 0$ and $0 \leq i \leq 3$, we get $\beta_{i,j}^{0} = \gamma_{i,j}^{0}$ for $0 \leq j \leq 4t$ and $\beta_{i,j}^{t+2} = -\gamma_{i,j}^{t+2}$ for $0 \leq j \leq 4t + 2$.

Then, by Remark 3.3, we immediately have a $K$-basis of $\text{Hom}_{A_T}(Q^n, A_T)$:
Lemma 3.3. Let \( n \geq 0 \) be an integer. Then

\[
\begin{align*}
\{ & \beta_{i,j}^{n,0} | 0 \leq i \leq 3; 0 \leq j \leq n \} & \text{if } n \equiv 0 \pmod{4} \text{ and } T = 0 \\
\{ & \gamma_{i,j}^{n,4u} | 0 \leq i \leq 3; 0 \leq j \leq n; 0 \leq l \leq T; 1 \leq u \leq T \} & \text{if } n \equiv 0 \pmod{4} \text{ and } T \geq 1 \\
\{ & \beta_{i,j}^{n,4u+1}; \gamma_{i,j}^{n,4u+1} | 0 \leq i \leq 3; 0 \leq j \leq n; 0 \leq l \leq T \} & \text{if } n \equiv 1 \pmod{4} \\
\{ & \beta_{i,j}^{n,4u+2}; \gamma_{i,j}^{n,4u+2} | 0 \leq i \leq 3; 0 \leq j \leq n; 0 \leq l \leq T - 1 \} & \text{if } n \equiv 2 \pmod{4} \text{ and } T = 0 \\
\{ & \beta_{i,j}^{n,4u+3}; \gamma_{i,j}^{n,4u+3} | 0 \leq i \leq 3; 0 \leq j \leq n; 0 \leq l \leq T - 1 \} & \text{if } n \equiv 3 \pmod{4} \text{ and } T \geq 1
\end{align*}
\]

gives a \( K \)-basis of \( \text{Hom}_{A_T}(Q^n,A_T) \). Moreover, \( \text{Hom}_{A_T}(Q^n,A_T) = \{0\} \), if \( n \equiv 3 \pmod{4} \) and \( T = 0 \).

In the rest of the paper, we consider the subscripts \( i \) of all maps \( \beta_{i,j}^{n,l} \) and \( \gamma_{i,j}^{n,l} \) as modulo 4.

3.2. Maps \( \text{Hom}_{A_T}(\partial^n, A_T) \)

Now, by direct computations, we have the images of the basis elements in Lemma 3.3 under the map \( \text{Hom}_{A_T}(\partial^n, A_T) \):

Lemma 3.4. For \( m \geq 0 \), we have the following:

(a) For \( 0 \leq i \leq 3 \) and \( 0 \leq j \leq 4m + 1 \),

\[
\beta_{i,j}^{4m+1,0} \cdot \partial^{4m+2} = \begin{cases} 
\beta_{i,j}^{4m+2,0} + \beta_{i-1,j+1}^{4m+2,0} & \text{if } i = 0, 2 \text{ and } 0 \leq j \leq 2m - 1 \\
\beta_{i,j}^{4m+2,0} + \beta_{i-1,j}^{4m+2,0} & \text{if } i = 1, 3 \text{ and } 0 \leq j \leq 2m \\
(T + 1) \beta_{i,j}^{4m+2,0} + T \beta_{i+1,2m+1}^{4m+2,0} + (T + 1) \beta_{i,j}^{4m+2,0} & \text{if } i = 0, 2 \text{ and } j = 2m, \\
\text{or } i = 1, 3 \text{ and } j = 2m + 1 \\
\beta_{i,j}^{4m+2,0} + \beta_{i-1,j+1}^{4m+2,0} & \text{if } i = 0, 2 \text{ and } 2m + 1 \leq j \leq 4m + 1 \\
\beta_{i,j}^{4m+2,0} + \beta_{i-1,j}^{4m+2,0} & \text{if } i = 1, 3 \text{ and } 2m + 2 \leq j \leq 4m + 1
\end{cases}
\]

and

\[
\gamma_{i,j}^{4m+1,0} \cdot \partial^{4m+2} = \begin{cases} 
\gamma_{i,j}^{4m+2,0} + \gamma_{i-1,j+1}^{4m+2,0} & \text{if } i = 0, 2 \text{ and } 0 \leq j \leq 2m \\
\gamma_{i,j}^{4m+2,0} + \gamma_{i-1,j}^{4m+2,0} & \text{if } i = 1, 3 \text{ and } 0 \leq j \leq 2m - 1 \\
(T + 1) \gamma_{i,j}^{4m+2,0} + T \gamma_{i+1,2m+1}^{4m+2,0} + (T + 1) \gamma_{i,j}^{4m+2,0} & \text{if } i = 0, 2 \text{ and } j = 2m + 1, \\
\text{or } i = 1, 3 \text{ and } j = 2m \\
\gamma_{i,j}^{4m+2,0} + \gamma_{i-1,j+1}^{4m+2,0} & \text{if } i = 0, 2 \text{ and } 2m + 2 \leq j \leq 4m + 1 \\
\gamma_{i,j}^{4m+2,0} + \gamma_{i-1,j}^{4m+2,0} & \text{if } i = 1, 3 \text{ and } 2m + 1 \leq j \leq 4m + 1.
\end{cases}
\]

Moreover, for \( 1 \leq l \leq T, 0 \leq i \leq 3 \) and \( 0 \leq j \leq 4m + 1 \),

\[
\beta_{i,j}^{4m+1,0} \cdot \partial^{4m+2} = \begin{cases} 
0 & \text{if } i = 0, 2 \text{ and } 0 \leq j \leq 2m \\
\beta_{i,j}^{4m+2,0} + \beta_{i-1,j+1}^{4m+2,0} & \text{if } i = 1, 3 \text{ and } 0 \leq j \leq 2m \\
\beta_{i,j}^{4m+2,0} + \beta_{i-1,j}^{4m+2,0} & \text{if } i = 0, 2 \text{ and } 2m + 1 \leq j \leq 4m + 1 \\
0 & \text{if } i = 1, 3 \text{ and } 2m + 1 \leq j \leq 4m + 1
\end{cases}
\]

and

\[
\gamma_{i,j}^{4m+1,0} \cdot \partial^{4m+2} = \begin{cases} 
0 & \text{if } i = 0, 2 \text{ and } 0 \leq j \leq 2m \\
\gamma_{i,j}^{4m+2,0} + \gamma_{i-1,j+1}^{4m+2,0} & \text{if } i = 1, 3 \text{ and } 0 \leq j \leq 2m \\
0 & \text{if } i = 0, 2 \text{ and } 2m + 1 \leq j \leq 4m + 1 \\
\gamma_{i,j}^{4m+2,0} + \gamma_{i-1,j}^{4m+2,0} & \text{if } i = 1, 3 \text{ and } 2m + 1 \leq j \leq 4m + 1.
\end{cases}
\]
(b) For $0 \leq l \leq T - 1$ (so $T \geq 1$), $0 \leq i \leq 3$ and $0 \leq j \leq 4m + 2$,

$$
\begin{align*}
\beta_{i,j}^{4m+2,l,0} \delta_{4m+3} &= \begin{cases} 
\rho_{i,j}^{4m+3,l} & \text{if } i = 0, 2 \text{ and } 0 \leq j \leq 2m, \\
-\beta_{i-1,j}^{4m+3,l} & \text{if } i = 1, 3 \text{ and } 0 \leq j \leq 2m, \\
\rho_{i,2m+1}^{4m+3,l} - \rho_{i-1,2m+1}^{4m+3,l} & \text{if } i = 0, 2 \text{ and } j = 2m + 1, \\
-\beta_{i,2m+2}^{4m+3,l} & \text{if } i = 1, 3 \text{ and } j = 2m + 1, \\
\rho_{i,4m+3,l} - \rho_{i-1,4m+3,l} & \text{if } i = 0, 2 \text{ and } 2m + 2 \leq j \leq 4m + 2, \\
-\beta_{i,4m+3,l} & \text{if } i = 1, 3 \text{ and } 2m + 2 \leq j \leq 4m + 2
\end{cases}
\end{align*}
$$

and

$$
\gamma_{i,j}^{4m+2,l,0} \delta_{4m+3} = \begin{cases} 
-\gamma_{i,j}^{4m+3,l} & \text{if } i = 0, 2 \text{ and } 0 \leq j \leq 2m, \\
\gamma_{i,j}^{4m+3,l} & \text{if } i = 1, 3 \text{ and } 0 \leq j \leq 2m, \\
\gamma_{i,j} - \gamma_{i-1,j} & \text{if } i = 0, 2 \text{ and } j = 2m + 1, \\
\gamma_{i,j} - \gamma_{i-1,j} & \text{if } i = 1, 3 \text{ and } j = 2m + 1, \\
\gamma_{i,j} & \text{if } i = 0, 2 \text{ and } 2m + 2 \leq j \leq 4m + 2, \\
\gamma_{i-1,j} & \text{if } i = 1, 3 \text{ and } 2m + 2 \leq j \leq 4m + 2
\end{cases}
$$

Moreover, for $0 \leq i \leq 3$ and $0 \leq j \leq 4m + 2$, $\beta_{i,j}^{4m+2,T,0} \delta_{4m+3} = \gamma_{i,j}^{4m+2,T} \delta_{4m+3} = 0$.

(c) For $0 \leq l \leq T - 1$ (so $T \geq 1$), $0 \leq i \leq 3$ and $0 \leq j \leq 4m + 3$,

$$
\beta_{i,j}^{4m+3,l,0} \delta_{4m+4} = \begin{cases} 
\rho_{i,j}^{4m+4,l+1} & \text{if } i = 0, 2 \text{ and } 0 \leq j \leq 2m + 1, \\
\beta_{i,j}^{4m+4,l+1} & \text{if } i = 1, 3 \text{ and } 0 \leq j \leq 2m + 1, \\
\beta_{i,j}^{4m+4,l+1} & \text{if } i = 0, 2 \text{ and } 2m + 2 \leq j \leq 4m + 3, \\
\beta_{i-1,j}^{4m+4,l+1} & \text{if } i = 1, 3 \text{ and } 2m + 2 \leq j \leq 4m + 3
\end{cases}
$$

and

$$
\gamma_{i,j}^{4m+3,l,0} \delta_{4m+4} = \begin{cases} 
0 & \text{if } i = 0, 2 \text{ and } 0 \leq j \leq 2m + 1, \\
\gamma_{i,j}^{4m+4,l+1} & \text{if } i = 1, 3 \text{ and } 0 \leq j \leq 2m + 1, \\
\gamma_{i,j}^{4m+4,l+1} & \text{if } i = 0, 2 \text{ and } 2m + 2 \leq j \leq 4m + 3, \\
\gamma_{i-1,j}^{4m+4,l+1} & \text{if } i = 1, 3 \text{ and } 2m + 2 \leq j \leq 4m + 3
\end{cases}
$$

(d) For $0 \leq l \leq T$, $0 \leq i \leq 3$ and $0 \leq j \leq 4m$,

$$
\beta_{i,j}^{4m,0,l} \delta_{4m+1} = \gamma_{i,j}^{4m,0,l} \delta_{4m+1} = \begin{cases} 
\rho_{i,j}^{4m+1,l} & \text{if } i = 0, 2 \text{ and } 0 \leq j \leq 2m - 1, \\
-\beta_{i,j}^{4m+1,l} & \text{if } i = 1, 3 \text{ and } 0 \leq j \leq 2m - 1, \\
\rho_{i,j}^{4m+1,l} & \text{if } i = 0, 2 \text{ and } j = 2m, \\
-\beta_{i,j}^{4m+1,l} & \text{if } i = 1, 3 \text{ and } j = 2m, \\
\rho_{i,j}^{4m+1,l} - \rho_{i-1,j}^{4m+1,l} + \gamma_{i,j}^{4m+1,l} - \gamma_{i-1,j}^{4m+1,l} & \text{if } i = 0, 2 \text{ and } 2m + 1 \leq j \leq 4m, \\
-\beta_{i,j}^{4m+1,l} & \text{if } i = 1, 3 \text{ and } 2m + 1 \leq j \leq 4m
\end{cases}
$$

and also, for $1 \leq l \leq T$, $0 \leq i \leq 3$ and $0 \leq j \leq 4m$,

$$
\beta_{i,j}^{4m,l} \delta_{4m+1} = \begin{cases} 
\rho_{i,j}^{4m+1,l} & \text{if } i = 0, 2 \text{ and } 0 \leq j \leq 2m - 1, \\
-\beta_{i,j}^{4m+1,l} & \text{if } i = 1, 3 \text{ and } 0 \leq j \leq 2m - 1, \\
\rho_{i,j}^{4m+1,l} - \rho_{i-1,j}^{4m+1,l} & \text{if } i = 0, 2 \text{ and } j = 2m, \\
-\beta_{i,j}^{4m+1,l} & \text{if } i = 1, 3 \text{ and } j = 2m, \\
\rho_{i,j}^{4m+1,l} - \rho_{i-1,j}^{4m+1,l} + \gamma_{i,j}^{4m+1,l} - \gamma_{i-1,j}^{4m+1,l} & \text{if } i = 0, 2 \text{ and } 2m + 1 \leq j \leq 4m, \\
-\beta_{i,j}^{4m+1,l} & \text{if } i = 1, 3 \text{ and } 2m + 1 \leq j \leq 4m
\end{cases}
$$
Corollary 3.6. The proof of this lemma follows from easy computations, and thus we omit it.

3.3. A basis of $\text{Im Hom}_{A^t}(\mathcal{P}, A^r)$

Now, by using Lemma 3.4 we have a $K$-basis of $\text{Im Hom}_{A^t}(\mathcal{P}, A^r)$ for $n \geq 1$:

Lemma 3.5. For $m \geq 0$, we have

(a) If $T = 0$, then $\text{Im Hom}_{A^t}(\mathcal{P}^{m+1}, A_0) = \{0\}$.

(b) If $T > 0$, then $\left\{ Y_i^1, Y_i^{1,0}, Y_{i,j}^{1,0}, Y_{i,j}^1, Y_{i,j}^{2,0}, Y_{i,j}^2, Y_{i,j}^{3,0}, Y_{i,j}^3, Y_{i,j}^{4,0}, Y_{i,j}^4, Y_{i,j}^{m+1,0}, Y_{i,j}^{m+1} \right\}$ is a $K$-basis of $\text{Im Hom}_{A^t}(\mathcal{P}^{m+1}, A_0)$.

The proof of this lemma follows from easy computations, and so we omit it.

3.3. A basis of $\text{Im Hom}_{A^t}(\mathcal{P}, A^r)$

As an immediate consequence, we get the dimension of $\text{Im Hom}_{A^t}(\mathcal{P}, A^r)$ for $n \geq 1$:

Corollary 3.6. For $T \geq 0$, $m \geq 0$, and $0 \leq r \leq 3$,

$$\dim_K \text{Im Hom}_{A^t}(\mathcal{P}^{m+r}, A^r) = \begin{cases} 16Tm & \text{if } r = 0 \text{ (where } m \neq 0) \\ 2T(8m + 3) + 3(4m + 1) & \text{if } r = 1 \\ 8T(2m + 1) + 4(3m + 2) & \text{if } r = 2 \\ 4T(2m + 1) + 3(4m + 3) & \text{if } r = 3 \end{cases}$$
3.4. A basis of \( \text{Ker} \text{Hom}_{A_T}(\partial^m, A_T) \)

By using Lemma 3.3, we also have a \( K \)-basis of \( \text{Ker} \text{Hom}_{A_T}(\partial^m, A_T) \) for \( n \geq 1 \):

**Lemma 3.7.** For \( m \geq 0 \), we have

(a) If \( T = 0 \), then \( \text{Ker} \text{Hom}_{A_T}(\partial^{4m+4}, A_T) = \{0\} \).

(b) If \( T > 0 \), then \( \left\{ \gamma_{i_1_0}^{4m+3}, \gamma_{i_2_0}^{4m+3}, \gamma_{i_3_0}^{4m+3}, \gamma_{i_4_0}^{4m+3}, \gamma_{i_5_0}^{4m+3} \right\} \) is a \( K \)-basis of \( \text{Ker} \text{Hom}_{A_T}(\partial^{4m+4}, A_T) \).

(c) By using Lemma 3.4, we also have a \( \text{Ker} \text{Hom}_{A_T}(\partial^{4m+1}, A_T) \).

(d) If \( T = 0 \), then \( \left\{ \gamma_{i_1_0}^{4m+1}, \gamma_{i_2_0}^{4m+1}, \gamma_{i_3_0}^{4m+1}, \gamma_{i_4_0}^{4m+1}, \gamma_{i_5_0}^{4m+1} \right\} \) is a \( K \)-basis of \( \text{Ker} \text{Hom}_{A_T}(\partial^{4m+2}, A_T) \).

The proof of this lemma follows from easy computations, and thus we omit it.

By the lemma above, we immediately have the dimension of \( \text{Ker} \text{Hom}_{A_T}(\partial^r, A_T) \) for \( n \geq 1 \):

**Corollary 3.8.** For \( T \geq 0, m \geq 0, \) and \( 0 \leq r \leq 3 \),

\[
\dim_K \text{Ker} \text{Hom}_{A_T}(\partial^{4m+r}, A_T) = \begin{cases} 
16Tm & \text{if } r = 0 \text{ (where } m \neq 0) \\
2T(8m + 1) + 4m + 1 & \text{if } r = 1 \\
8T(2m + 1) + 4(5m + 2) & \text{if } r = 2 \\
8T(2m + 1) + 20m + 7 & \text{if } r = 3 \\
2T(8m + 5) + 4(4m + 3) & \text{if } r = 3.
\end{cases}
\]

3.5. The Hochschild cohomology group of \( A_T \)

Now, by Lemmas 3.5 and 3.7, we have a \( K \)-basis of the Hochschild cohomology group \( HH^r(A_T) \) for \( n \geq 0 \):

**Proposition 3.9.** For \( T \geq 0 \) and \( m \geq 0 \),

(a) If \( T = 0 \), then \( \left\{ \sum_{i_0=0}^3 \beta_{i_0, i_0}^{4m,0} \mid 0 \leq j \leq 4m \right\} \) is a \( K \)-basis of \( HH^{4m}(A_T) \).
commutative graded algebra. We denote by \( HH^A \) the Hochschild cohomology ring of \( A \). The Yoneda product in \( HH^A \) is a commutative monoid, which acts on \( HH^A \) as a graded ring. We only deal with the Koszul self-injective algebra \( A \) where the center of \( HH^A \) is a K-basis. There is an isomorphism \( HH^A \cong HH^A \otimes \mathbb{K} \) and \( \mathbb{K} \) is the isomorphism of commutative graded algebras.

This proposition provides us with the main result in this paper.

\[ (b) \text{ If } T > 0, \text{ then } \]

\[ \left( 0 \right) \left( 0 \right), \text{ is a K-basis of } HH^{4m+1}(A) \).

\[ (2) \left( 0 \right) \left( 0 \right), \text{ is a K-basis of } HH^{4m+2}(A) \).

\[ (3) \left( 0 \right) \left( 0 \right), \text{ is a K-basis of } HH^{4m+3}(A) \).

This proposition provides us with the main result in this paper.

**Theorem 3.10.** For \( T \geq 0 \) and \( n \geq 0 \), the dimension of \( HH^n(A_T) \) is given as follows: For \( m \geq 0 \) and \( 0 \leq r \leq 3 \),

\[ \text{dim}_K HH^{4m+r}(A_T) = \begin{cases} 2T + 4m + 1 & \text{if } r = 0 \\ 2T + 8m + 5 & \text{if } r = 1 \\ 2T + 4(2m + 1) & \text{if } r = 2 \\ 2T + 4m + 3 & \text{if } r = 3 \end{cases} \]

**Remark 3.11.** There is an isomorphism \( HH^0(A_T) \rightarrow Z(A_T) \); \( \phi \mapsto \phi(\sum_i e_i \otimes e_i) \) of algebras, where \( Z(A_T) \) is the centre of \( A_T \). Hence, by Proposition 3.12(a)(1), \( Z(A_T) = K[x, y]/(x^2 + 1, xy, y + 1) \).

### 4. Hochschild cohomology ring modulo nilpotence of \( A_T \) for \( T = 0 \)

Throughout this section, we keep the notation from the previous sections, and suppose that \( T = 0 \), namely, we only deal with the Koszul self-injective algebra \( A_0 \). For simplicity, we denote the algebra \( A_0 \) by \( A \).

Recall that the Hochschild cohomology ring of \( A \) is defined as the graded ring \( HH^A := \bigoplus_{i \geq 0} HH^i(A) = \bigoplus_{i \geq 0} \text{Ext}_A^i(A, A) \) with the Yoneda product. Let \( N_A \) be the ideal generated by all homogeneous nilpotent elements in \( HH^A \). Note that \( N_A \) is a homogeneous ideal in \( HH^A \). The purpose in this section is to find generators and relations of the Hochschild cohomology ring modulo nilpotence, \( HH^A/N_A \), of \( A \). It is known that \( HH^A/N_A \) is a commutative graded algebra. We denote by \( HH^A/N_A \) the graded subalgebra \( \bigoplus_{i \geq 0} HH^i(A)/N_A \) of \( HH^A \), and by \( \times \) the Yoneda product in \( HH^A \).

**Theorem 4.1.** There is the following isomorphism of commutative graded algebras:

\[ HH^A/N_A \cong HH^e(A) \]

\[ = K[z_j, z_1, z_2, z_3, z_4]/(z_0z_2 - z_1^2, z_0z_3 - z_1z_2, z_0z_4 - z_2^2, z_0z_4 - z_3z_2, z_2z_4 - z_3^2) \]

with \( z_j \) in degree 4 for \( 0 \leq j \leq 4 \). Therefore \( HH^A/N_A \) is finitely generated as an algebra.
Proof. First we establish the second isomorphism. For \(0 \leq j \leq 4\) let \(z_j := \sum_{i=0}^3 a_i^j 0 \in HH^4(A)\), and for \(k \geq 0\) define the homomorphism of \(A\)-\(A\)-bimodules \(\sigma^k_{j} : Q^{k+4} \rightarrow Q^k\) by

\[
\sigma^k_{j,t} \mapsto \begin{cases} 
0 & \text{if } s = j + t \text{ for some integer } t \text{ with } 0 \leq t \leq k \\
0 & \text{otherwise.}
\end{cases}
\]

Then \(\sigma^k_{j}\) are liftings of \(z_j\), namely, \(z_j = \sigma^0_{j} \phi^0\) and \(\sigma^k_{j} \phi^k = \phi^{k+1} \sigma^k_{j} = \phi^{k+1} \sigma^0_{j}\) hold for \(l \geq 0\). Thus, for integers \(0 \leq u, v \leq 4\), it follows that the composite \(z_u \sigma^4_{u} : Q^8 \rightarrow A\) is given by

\[
\sigma^8_{u,t} \mapsto \begin{cases} 
eq & \text{if } s = u + v \\
0 & \text{otherwise.}
\end{cases}
\]

Note that the product \(z_u \times z_v \in HH^8(A)\) is represented by this map.

Now, for each positive integer \(t\), let \(i_t\) be an integer with \(0 \leq i_t \leq 4\). Then, for any \(w \geq 2\), it can be shown by induction on \(w\) that the product \(z_{i_1} \times \cdots \times z_{i_w}\) is represented by the map

\[
Q^{4w} \rightarrow A; \ a_{i_{w},t} \mapsto \begin{cases} e_r & \text{if } s = \sum_{p=1}^w i_p \\
0 & \text{otherwise.}
\end{cases}
\]

This tells us that \(HH^8(A)\) is generated by \(z_0, \ldots, z_4 \in HH^8(A)\). Also, for two products \(z_{j_1} \times \cdots \times z_{j_k}\) and \(z_{k_1} \times \cdots \times z_{k_l}\) in \(HH^8(A)\) (where \(u \geq 1\) and \(0 \leq j_p, k_p \leq 4\) for \(1 \leq p \leq u\)), we have that \(z_{j_1} \times \cdots \times z_{j_k} = z_{k_1} \times \cdots \times z_{k_l}\) if and only if \(\sum_{p=1}^u j_p = \sum_{p=1}^u k_p\). This gives the relation \(z_{j_1}z_{j_2} - z_{j_1}z_{j_1}\) for \(0 \leq i, j, k, l \leq 4\) such that \(i + j = k + l\), and then, considering all possible elements in \(HH^4(A)\), we get the second isomorphism.

Now, using the second isomorphism, we easily see that all elements in \(HH^4(A)\) are nilpotent. Furthermore, for \(m \geq 0\) and \(l = 1, 2, 3\), the images of all basis elements of \(HH^{4m+1}(A)\) of Proposition 3.9 (a) are in \(\tau_A\), so that, by Proposition 4.4, \(HH^{4m+1}(A)\) is contained in \(N_A\). Hence we have the first isomorphism. Therefore the proof of the theorem is completed.

Finally we consider the graded centre of the Ext algebra \(E(A) := \bigoplus_{i \geq 0} \text{Ext}^i_{A}(A/\tau_A, A/\tau_A)\) of \(A\). Recall that the graded centre \(Z_{gpl}(E(A))\) of \(E(A)\) is the subring of \(E(A)\) generated by all homogeneous elements \(x\) in \(E(A)\) such that \(xy = (-1)^{|x||y|}yx\) for all homogeneous elements \(y \in E(A)\), where \(|x|\) and \(|y|\) denote the degree of \(x\) and \(y\), respectively. Let \(N'_A\) be the ideal of \(Z_{gpl}(E(A))\) generated by all homogeneous nilpotent elements. Since \(A\) is a Koszul algebra, we know from [1] that \(Z_{gpl}(E(A))/N'_A \cong HH^4(A)/N_A\) as graded rings. Therefore, by Theorem 4.1 we have the following:

**Corollary 4.2.** There is the following isomorphism of commutative graded rings:

\[
Z_{gpl}(E(A))/N'_A \cong HH^4(A)
\]

with \(z_j\) in degree 4 for \(0 \leq j \leq 4\).

**Remark 4.3.** We notice that our algebra \(A_T\) \((T \geq 0)\) belongs to the class of more general algebras \(B_{k,s}\) defined as follows: For \(s \geq 1\), let \(\Delta_s\) be the quiver
and for $k \geq 2$ let $J_k := \langle xy, x^k + y^k, yx \rangle \subseteq K\Lambda$, where we put $x := \sum_{i=0}^{r-1} a_i \in K\Lambda$ and $y := \sum_{i=0}^{r-1} b_i \in K\Lambda$. Define the algebra $B_{k,s} := K\Lambda / J_k$. Then $B_{k,s}$ is a self-injective algebra.

The results in this paper provide the computations of the Hochschild cohomology groups of $B_{k,s}$ for $s = 4$ and $k = 4T + 2$ ($T \geq 0$). On the other hand, if $s = 2$ and $k = 2m$ ($m \geq 1$), then we know from [14] generators and relations of the Hochschild cohomology ring of $B_{2m,2}$. (In fact $B_{2m,2}$ is isomorphic to the algebra $\Lambda_N$ discussed in [14], where $N = m$ and the quiver of $\Lambda_N$ has 2 vertices.) For the other cases, the computations of the Hochschild cohomologies seem to be unknown.

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