On Weyl cosmology in five dimensions and the cosmological constant

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In this talk notes we expose the possibility to induce the cosmological constant from extra dimensions, in a geometrical framework where our four-dimensional Riemannian space-time is embedded into a five-dimensional Weyl integrable space. In particular following the approach of the induced matter theory, we show that when we go down from five to four dimensions, we may recover in the context of the general theory of relativity, the induced energy momentum tensor of the induced matter theory plus a cosmological constant term, that is determined by the presence of the Weyl scalar field on the bulk.

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I. INTRODUCTION

Due to the recent discovery of cosmic acceleration there has been a renewed interest in the role the cosmological constant could play to explain the new data. For instance, there is strong evidence that the so-called dark energy might have a connection with the cosmological constant. Moreover, the present most popular model of cosmology, the Lambda-CDM model, tacitly assumes the existence of the cosmological constant [1]. Particle physics theorists have always argued in favour of the existence of the cosmological constant as a consequence of the energy density of the vacuum [2]. From the standpoint of cosmological theory it seems then desirable to have a justification of the cosmological constant on theoretical grounds. This quest has led some theoreticians to modify Einstein’s gravitational theory, these attempts going back to the works of Eddington and Schrödinger [3].

In this talk notes, based on our recent work [4], we present a new approach to this old question in the context of extra dimensions [5], [6], [9]. In particular in our proposal our ordinary spacetime is viewed as a hypersurface (the brane) embedded in a five-dimensional (5D) manifold (the bulk) which is described by a Weyl integrable geometry. Mathematical theorems regulate these embeddings, in particular, the Campbell-Magaard theorem [7] and its extensions specify the conditions under which the embeddings are possible [8]. In the case when the Weyl field depends only on the extra dimension, the embedded spacetime is Riemannian and general relativity holds in the brane [11], although the non-Riemannian character of the whole bulk propagates into the brane.

II. THE FORMALISM

Let us consider a five-dimensional space $M^5$ endowed with a metric tensor $(5) g$ and an integrable Weyl scalar field $\phi$. In local coordinates $\{y^a\}$ the five-dimensional line element can be written as $dS^2 = g_{ab}(y) dy^a dy^b$, where $g_{ab}$ are the components of $(5) g$. The simplest action that can be constructed for a Weylian theory of gravity in a five-dimensional vacuum is given by

$$(5) S = \int d^5 y \sqrt{|(5) g|} \left[ (5) R + \xi \phi^a_{;a} \right],$$

where $\xi$ is an arbitrary coupling constant, $\phi_a \equiv \phi_{;a}$ is the gauge vector associated to the Weyl field $\phi(y)$, $| (5) g |$ is the absolute value of the determinant of the metric $(5) g$, $(5) R$ is the Weylian Ricci scalar. The variation of the action [1] with respect to the tensor metric and the Weyl scalar field yields

$$(5) G_{ab} + \phi_{;ab} - (2\xi - 1) \phi_a \phi_b + \xi g_{ab} \phi_c \phi^c = 0,$$

$$\phi^a_{;a} + 2 \phi_a \phi^a = 0$$

where $(5) G_{ab}$ denotes the Einstein tensor calculated with the Weyl connection $(5) \Gamma^a_{bc} = (5) \left\{ a \right\} - \left\{ (1/2) \right\} f_{bc} \delta^a + \phi_a \delta^a_{bc} - g_{bc} \phi^a \right\}$ and $\left\{ a \right\}$ are the Christoffel symbols of Riemannian geometry. The equations (2) and (3) are the field equations of the five-dimensional Weyl gravitational theory and describes the dynamics of a five-dimensional bulk in vacuum. A better insight may be gained if we recast the field equations (2) and (3) into its Riemannian part plus the contribution
of the Weyl scalar field \(\Phi\). We are then led to

\[
\begin{align*}
(5) \tilde{G}_{ab} &= -\frac{1}{2} (6 - 5\xi) \left[ \phi_a \phi_b - \frac{1}{2} g_{ab} \phi_c \phi^c \right] = 0, \\
(5) \tilde{\Box} \phi &= 0,
\end{align*}
\]

where the tilde (\(\sim\)) is used to denote quantities calculated with the Riemannian part of the Weyl connection and \(\Box\) denotes the 5D d’Alembertian operator in the Riemannian sense. Now expressing the local coordinates \(\{y^a\}\) as \(\{x^\alpha, l\}\) and denoting by \(l\) the fifth (spacelike) coordinate, we choose for simplicity a line element in the form

\[
dS^2 = g_{\alpha\beta}(x, l) dx^\alpha dx^\beta - \Phi^2(x, l)dl^2.
\]

where the function \(\Phi^2(x, l)\) is the 5D analogue of the lapse function used in canonical general relativity, which supposes that spacetime may be foliated by a family of spacelike surfaces \(\Sigma\). As in the induced matter approach [9], with \(\Lambda(x)\) as an induced cosmological parameter, whereas if \(\Phi\) is constant then (15) reduces to an induced cosmological constant.

Some solutions of the above field equations have been worked out in detail by Novello and collaborators considering different geometric settings [12]. Motivated by the recently result [11], if the Weyl scalar field depends only on the extra coordinate \(l\), then each leaf of the foliation \(l = \text{const}\) (which represents our 4D spacetime) has a Riemannian character and can be locally and isometrically embedded in a five-dimensional Weylian space whose metrical properties are given by (4). In view of the above let us assume that \(\phi = \phi(l)\), i.e the Weyl scalar field depends only on the extra coordinate \(l\). In this case the field equations (7), (8) and (9) become

\[
\begin{align*}
(5) \tilde{G}_{\alpha\beta} &= -\frac{1}{2} (6 - 5\xi) \left[ \phi_a \phi_b - \frac{1}{2} g_{ab} \phi_c \phi^c \right] = 0, \\
(5) \tilde{G}_{\alpha l} &= 0, \\
(5) \tilde{G}_{ll} &= \frac{1}{4} (6 - 5\xi) \left[ \phi^2_l + \Phi^2 \phi^{\gamma}_{\gamma} \right] = 0.
\end{align*}
\]

On a particular hypersurface \(\Sigma_0\) the induced line element will be given by \(dS^2_{\Sigma_0} = h_{\alpha\beta}(x) dx^\alpha dx^\beta\), where \(h_{\alpha\beta}(x) = g_{\alpha\beta}(x, l_0)\) is the induced metric on \(\Sigma_0\). From the Gauss-Codazzi equations it is easy to show (See, for instance [13]) that the induced dynamics on the hypersurface \(\Sigma_0\) is governed by the four-dimensional field equations

\[
(4) \tilde{G}_{\alpha\beta} = T^{(IM)}_{\alpha\beta} + \Lambda(x) h_{\alpha\beta},
\]

where \(T^{(IM)}_{\alpha\beta}\) is the energy momentum tensor obtained in the induced matter approach [4], which has the form

\[
T^{(IM)}_{\alpha\beta} = (\Phi_{\alpha||\beta}/\Phi) + (1/2\Phi^2) \left\{ \tilde{\Box} \phi_{\alpha||\beta} - g^{\lambda\mu} \tilde{\Box} g_{\beta\mu} \phi^\lambda + (1/2) g^{\mu\nu} \tilde{\Box} g_{\beta\mu} g^{\gamma\nu} \phi^\gamma + (1/4) g_{\alpha\beta} \left[ g^{\mu\nu} \tilde{\Box} g_{\mu\nu} + (g^{\mu\nu} g_{\mu\nu})^2 \right] \right\},
\]

with the bars (||) denoting covariant derivative in a Riemannian sense and the star (*) denoting derivative with respect to the fifth coordinate \(l\), and the function \(\Lambda(x)\) is given by

\[
\Lambda(x) = \frac{1}{4} (6 - 5\xi) \Phi^{-2} \phi^2_l \big|_{l=l_0}.
\]

The induced energy-momentum tensor \(T^{(IM)}_{\alpha\beta}\) can be obtained even if the bulk is Riemannian, but the interesting fact here is that the function \(\Lambda(x)\) is a new contribution depending directly on the Weyl scalar field. It is worth mentioning that when the lapse function \(\Phi\) depends only on the time then \(\Lambda(t)\) can be interpreted as an induced cosmological parameter, whereas if \(\Phi\) is constant then (15) reduces to an induced cosmological constant.

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1 We shall adopt the convention \(\text{diag}(+ - - -)\) for the signature of \(g_{\alpha\beta}\).
III. AN EXAMPLE

As an application of the formalism let us show some interesting results. To follow the details in deriving this results the reader may check our work [4]. When we consider a 5D space in the form of a triple warped product manifold [14] with metric given by

$$dS^2 = dt^2 - a^2(t)dr^2 - e^{2F(t)}dl^2,$$

(16)

where \(dr^2 = \delta_{ij}dx^idx^j\) is the three-dimensional Euclidian line element, \(t\) represents the cosmic time for co-moving observers, \(F(t)\) is a well-behaved real function and \(a(t)\) is the cosmological scale factor, the 4D induced line element on \(\Sigma_0\) becomes a FRW one. The metric function \(F(t)\) is determined by the system (10) to (13) and when a power law expanding universe \(a(t) \sim t^p\) is regarded, it gives \(F(t) = \ln(Bt^\gamma)\) where \(B\) is a constant and \(\gamma = (1/2 - p) + (1/2)\sqrt{1 - 32p^2 + 16p}\). To have real values for the power \(\gamma\) that are compatible with an expanding universe \((p > 0)\), the values of \(p\) must range in the interval \(0 < p \leq (1/4) + \sqrt{6}/8\). For (16) the equation (13) gives a Weyl scalar field \(\phi(t) = C_1l + C_2\), with \(C_1\) and \(C_2\) integration constants. Consequently using (15) we can induce a variable cosmological “constant” \(\Lambda(t) = (\frac{C_1}{l})^2 (6 - 5\xi)B_1^{-2}t^{-2\gamma}\). Clearly for \(\gamma > 0\) we have a decaying cosmological constant which may be a good candidate to explain the present period of accelerated expansion of the universe.

IV. FINAL REMARKS

In this talk notes we have exposed the idea of generating a cosmological constant, or rather, a cosmological parameter, from extra dimensions. Although this has already been investigated in the context of induced matter theory, the novelty of our approach is to regard the same problem in a more general setting, i.e by assuming the geometry of the embedding space to have a Weylian character. Two comments are in order: Firstly, the embedding space has a prescribed dynamics; secondly, the embedding does not affect the Riemannian geometry of the spacetime. These features depend on the fact that the Weyl field is assumed to be integrable and depending only on the extra dimension. Finally, by setting up a simple “toy model” our intention is to call attention to the richness of non-Riemannian geometries, in particular to the Weyl integrable manifolds, as a way of providing new degrees of freedom that might play a role in the theoretical framework of higher-dimensional embedding theories of spacetime. We believe that in this context issues such as the nature of the cosmological constant, dark energy and other important questions may be investigated from an entirely new point of view.

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