ON BRAIDED POISSON AND QUANTUM INHOMOGENEOUS GROUPS

S. Zakrzewski

Department of Mathematical Methods in Physics, University of Warsaw
Hoża 74, 00-682 Warsaw, Poland

The well known incompatibility between inhomogeneous quantum groups and the standard $q$-deformation is shown to disappear (at least in certain cases) when admitting the quantum group to be braided. Braided quantum $ISO(p, N−p)$ containing $SO_q(p, N−p)$ with $|q| = 1$ are constructed for $N = 2p$, $2p + 1$, $2p + 2$. Their Poisson analogues (obtained first) are presented as an introduction to the quantum case.

1 Introduction

It is well known [1, 2] that the Lorentz part of any quantum (or Poisson) Poincaré group is triangular. This is in fact a general feature, which excludes the standard $q$-deformation from the context of inhomogeneous quantum groups [3]. In order to make the standard $q$-deformation compatible with inhomogeneous groups one has to consider some generalization of the notion of quantum (Poisson) group, such as, for example, a braided quantum (Poisson) group.

The notion of a braided Hopf algebra is due to S. Majid [4]. It is a natural generalization of the notion of a Hopf algebra when we replace the usual symmetric monoidal category of vector spaces by a braided one (the incorporation of *-structures is more controversial — we follow here the approach of [5]). A characteristic feature of this generalization is that the comultiplication is a morphism of algebras when the product algebra is considered with a crossed tensor product structure rather than the ordinary one.

On the Poisson level, it means that instead of ordinary Poisson groups $(G, \pi)$ (where $\pi$ is such a Poisson structure on $G$ that the group multiplication is a Poisson map from the usual product Poisson structure $\pi \oplus \pi$ on $G \times G$ to $\pi$ on $G$), we consider triples $(G, \pi, \pi\star)$, where $\pi$ is a Poisson structure on $G$ and $\pi\star$ is a bi-vector field on $G \times G$ of the cross-type (i.e. having zero both projections on $G$) such that

1. $\pi_{12} := \pi \oplus \pi + \pi\star$ is a Poisson structure on $G \times G$,

2. the group multiplication is a Poisson map from $\pi_{12}$ to $\pi$.

In the next section we shall construct such structures on the inhomogeneous orthogonal groups $ISO(p, p)$, $ISO(p, p + 1)$, $ISO(p, p + 2)$, with the homogeneous part being non-triangular (with standard Belavin-Drinfeld $r$-matrix).

In Sect. 3, similar result is obtained for the quantum case.
2 The Poisson case

In this section we discuss Poisson-Lie structures (possibly braided) on inhomogeneous orthogonal groups (in particular, on the Poincaré group). Let \( V \cong \mathbb{R}^N = \mathbb{R}^{p+(N-p)} \) be equipped with the standard scalar product \( \eta \) of signature \((p, N-p)\). Special linear transformations preserving \( \eta \) and the projection from \( G \) and the projection from \( \mathbb{R}^N \) to \( \mathbb{R}^p \) and \( \mathbb{R}^{N-p} \). Let us simplify the discussion to the case when \( r \) is triangular (hence non-standard). The problem now arises if a non-triangular \( c \) can be used to construct (at least) a braided Poisson \( G \).

Let us simplify the discussion to the case when \( r = c \) (note that then the inclusion \( H \subset G \) is also a Poisson map). The brackets have now the form

\[
\{h_1, h_2\} = rh_1h_2 - h_1h_2r,
\]

\[
\{x_1, h_2\} = rx_1h_2, \quad \{x_1, x_2\} = rx_1x_2.
\]

We shall show that these brackets are not Poisson, unless \( r \) is triangular. It is convenient to check if the Jacobi identity is satisfied in a slightly more general case:

\[
\{h_1, h_2\} = rh_1h_2 - h_1h_2r, \quad \{x_1, h_2\} = wx_1h_2, \quad \{x_1, x_2\} = rx_1x_2,
\]
where $w \in \mathfrak{h} \otimes \mathfrak{h}$. Let $J(f_1, f_2, f_3) := \{\{f_1, f_2\}, f_3\} + \{\{f_2, f_3\}, f_1\} + \{\{f_3, f_1\}, f_2\}$ for any functions $f_1, f_2, f_3$. It is easy to check that

\[
J(h_1, h_2, h_3) = [[r, r]] h_1 h_2 h_3 - h_1 h_2 h_3 [[r, r]] \quad (7)
\]
\[
J(x_1, h_2, h_3) = (w_{12}, w_{13}) + [w_{12}, \frac{1}{2}] x_1 h_2 h_3 \quad (8)
\]
\[
J(x_1, x_2, h_2) = (w_{12} + w_{13}, w_{23}) x_1 x_2 h_2 \quad (9)
\]
\[
J(x_1, x_2, x_3) = [[r, r]] x_1 x_2 x_3 \quad (10)
\]

where $[[\cdot, \cdot]]$ is the bracket defined by Drinfeld: for any $\rho \in \mathfrak{h} \otimes \mathfrak{h}$,

\[
[[\rho, \rho]] := [\rho_{12}, \rho_{13}] + [\rho_{12}, \rho_{23}] + [\rho_{13}, \rho_{23}].
\]

If $w = r$, then the Jacobi identity holds provided $[[r, r]] = 0$ ($r$ triangular).

If $w = r + s$, where $s$ is a symmetric invariant element of $\mathfrak{h} \otimes \mathfrak{h}$ and $[[w, w]] = 0$ (i.e. $r$ is real-quasitriangular), then the Jacobi identity is satisfied, provided (10) is zero, i.e. the fundamental bivector field $r_V$ on $V$ (cf. [3]) is Poisson. We shall show that it is Poisson for almost all $N, p$, namely when $\mathfrak{h} = \mathfrak{so}(p, N - p)$ is absolutely simple. Indeed, in this case all invariant symmetric 2-tensors are proportional to the (Killing element)

\[
\tilde{s}^{jk} = \eta^{jk} \eta_{lm} - \delta^j_m \delta^k_l \quad (11)
\]

and all invariant elements of $\mathfrak{h} \otimes \mathfrak{h}$ are proportional to $\Omega := [[\tilde{s}, \tilde{s}]] = [\tilde{s}_{12}, \tilde{s}_{13}]$. From (11) we obtain

\[
\Omega^{abc} = \eta^{ab} \eta_{jl} \delta^c_j + \eta^{ac} \eta_{jl} \delta^b_j - \eta^{bc} \eta_{jl} \delta^a_j - \eta^{ac} \eta_{jl} \delta^b_j + \eta^{ac} \eta_{jl} \delta^b_j - \eta^{bc} \eta_{jl} \delta^a_j
\]

which yields $\Omega^{abc} x^a x^b x^c = 0$. For any classical $r$-matrix $r$ on $\mathfrak{h}$, $[[r, r]]$ must be proportional to $\Omega$ and therefore (10) is zero.

If $\mathfrak{h} = \mathfrak{so}(1, 3)$, all invariant symmetric 2-tensors are complex multiples of

\[
\tilde{s} = X_+ \otimes X_+ + X_- \otimes X_+ + \frac{1}{2} H \otimes H \quad (complex \ tensor \ product). \quad (12)
\]

We use here the embedding of the complex tensor product $\mathfrak{h} \otimes \mathbb{C} \mathfrak{h}$ into the real $\mathfrak{h} \otimes \mathfrak{h}$ as described in (11) $(X_+, X_-, H)$ is the standard complex basis of $\mathfrak{so}(1, 3) \cong \mathfrak{sl}(2, \mathbb{C})$ normalized as in (13); the reader should excuse the double use of the letter $H$). One can check easily that

\[
\tilde{s} = \tilde{M} \cdot \bar{M} - \bar{L} \cdot L, \quad -i\tilde{s} = \bar{M} \cdot \bar{L} + \bar{L} \cdot \bar{M}, \quad (13)
\]

where $M_i := \varepsilon_{ijk} e_k \otimes e^j$, $L_i = e_0 \otimes e^i + e_i \otimes e^0$ ($i, j, k = 1, 2, 3$) are standard generators of $\mathfrak{so}(1, 3)$ and therefore $\tilde{s}$ coincides with (11). All invariant 3-vectors are complex multiples of

\[
\Omega = [[\tilde{s}, \tilde{s}]] = X_+ \wedge H \wedge X_+ \quad (complex \ products; \ we \ use \ \bigwedge^3 \mathfrak{h} \subset \bigwedge^3 \mathfrak{h}) \quad (14)
\]
Since $\Omega x_1x_2x_3 = 0$ and $(i\Omega)x_1x_2x_3 \neq 0$ (Ex. 3.3 of [8]), $r_V$ is Poisson only if $[r, r]$ is (real) proportional to $\Omega$. It means that if $r_- = i\lambda X_+ \wedge X_-$ (the only possibility of non-triangular $r$, up to automorphism; the notation of [8]), then $[r, r] = [r_-, r_-] = \lambda^2 \Omega$, hence $\lambda^2$ must be real, i.e. $\lambda$ real or imaginary (cf. [8]).

Now we turn to the question of real-quasitriangularity. From Thm. 3.3 of [8] it follows that real-quasitriangular (not triangular) $r$-matrices exist only in the following three cases of $so(p, N-p)$:

- $so(p, p)$, $so(p, p+1)$ (real split cases) and $so(p, p+2)$.

For $so(1,1+2)$ in fact every $r$-matrix is real-quasitriangular (with suitable $s$). If it is not triangular, then, up to automorphism, $r_- = i\lambda X_+ \wedge X_-$ and $[r, r] = \lambda^2 \Omega$, whereas $[[s, s]] = -\lambda^2 \Omega$ for $s = i\lambda \bar{s}$, hence $[[r + s, r + s]] = [[r, r]] + [[s, s]] = 0$.

Concluding, for real-quasitriangular $r$ such that $r_V$ is Poisson, we have a natural Poisson structure $\pi$ on $G$ defined by (8), which generalizes $\pi_r$. This structure is not multiplicative (for $s \neq 0$). It differs from the multiplicative structure $\pi_r$ only by the following brackets:

$$\{h_1, h_2\}_s = 0, \quad \{x_1, h_2\}_s := sx_1h_2, \quad \{h_1, h_2\}_s = 0. \quad (15)$$

Denoting by $\Delta$ the comultiplication: $\Delta h = hh'$, $\Delta x = xhx'$ (the primed functions refer to the second copy of $G$), we obtain

$$\{\Delta h_1, \Delta h_2\}_s = \Delta\{h_1, h_2\}_s, \quad \{\Delta x_1, \Delta h_2\}_s = \Delta\{x_1, h_2\}_s,$$

but

$$\{\Delta x_1, \Delta x_2\}_s - \Delta\{x_1, x_2\}_s = \{\Delta x_1, \Delta x_2\}_s = (s - Ps)x_1h_2x_2', \quad (16)$$

where $P$ is the permutation in the tensor product. It is therefore natural to look for cross-term \{\cdot, \cdot\}_s which is nontrivial only between $x$ and $x'$. With such an assumption, $(G, \pi, \pi_\otimes)$ will be a braided Poisson group if $\{\Delta x_1, \Delta x_2\}_s + \{\Delta x_1, \Delta x_2\}_\otimes = 0$, i.e.

$$(s - Ps)x_1h_2x_2' + h_2\{x_1, x_2\}_\otimes + h_1\{x_1', x_2\}_\otimes = 0. \quad (17)$$

Consider first the generic $s$ which is proportional to $(\bar{s})$: $s = \nu \bar{s}$. Since $\bar{s} - Ps = I - P$, (17) is equivalent to

$$\nu(x_1h_2x_2' - x_2h_1x_1') = h_2\{x_2', x_1\}_\otimes - h_1\{x_1', x_2\}_\otimes,$$

which is satisfied by

$$\{x_2', x_1\}_\otimes = \nu x_1x_2' \quad \text{(more explicitly: } \{(x')^k, x^j\}_\otimes = \nu x^j(x')^k). \quad (19)$$

One has only to check that $\pi \oplus \pi + \pi_\otimes$ is a Poisson bracket on $G \times G$, but this is true:

\[
\begin{align*}
J(x_1, x_2, x_3') &= \{rx_1x_2, x_3'\} + \{x_2x_3', x_1\} - \{x_3'x_1, x_2\} \\
&= 2r_1x_2x_3' + r_{21}x_2x_1x_3' - x_2x_1x_3' + x_3'x_2x_1 - r_{12}x_3'x_1x_2 = 0, \\
J(x_1, x_2', h_3) &= \{x_1x_2', h_3\} + \{-w_{13}x_1h_3, x_2'\} = w_{13}x_1h_3x_2' - w_{13}x_1x_2' h_3 = 0
\end{align*}
\]
\[ \{\cdot,\cdot\} \text{ denotes the full bracket on } G \times G \text{ defined by } \pi \otimes \pi + \pi_{\text{br}}. \]

In the Lorentz case \( h = so(1,3) \), apart from the generic case \( s = \nu \tilde{s} \), one has to consider also the case when \( s = \nu i \tilde{s} \). Using formula (13) for \( i \tilde{s} \), it is easy to see that \( i \tilde{s} - P i \tilde{s} = 2i \tilde{s} \) and (17) has no solutions. Thus the case of real \( \lambda \) in \( r_+ = i \lambda X^+ \wedge X_- \), which corresponds to real \( q \) in the quantum case (in particular, quantum double of \( SU_q(2) \)), is excluded. It means that from the list of \( r \)-matrices on \( so(1,3) \) in [7], only combinations of \( (X^+ \wedge X_- - JX^+ \wedge JX_-) \) and \( JH \wedge H \) fall in our scheme.

Finally, it is interesting to note that

1. the one-parameter group of automorphisms of \( G \) (dilations),
\[ t(h, x) := (h, e^t x) \quad \text{for } t \in \mathbb{R}, \]
   preserves \( \pi \) (because (6) is homogeneous in \( x \)),

2. the braiding bivector field \( \pi_{\text{br}} \) described by (19) is nothing else but the antisymmetrization of the fundamental tensor field on \( G \times G \) obtained by the action of the real-quasitriangular element
\[ \nu e_1 \otimes e_1 \in \mathbb{R} \otimes \mathbb{R} \quad (e_1 \text{ is the basic vector of } \mathbb{R}). \]

Similar property is satisfied by the cobracket \( \delta \) on \( g \), obtained by linearization of \( \pi \) at the group unit. It follows that \( (g, \delta) \) is an example of a braided-Lie bialgebra [9] (in the category of modules over quasitriangular \( \mathbb{R} \)). \( (G, \pi) \) will certainly be an example of a braided Poisson-Lie group, when the theory presented in [9] will be extended from Lie algebras to Lie groups.

3 **The quantum case**

Real-(co)quasitriangular quantum \( SO(p, p) \) and \( SO(p, p + 1) \) are introduced in [10] and \( SO(p, p + 2) \) in [11]. They all can be described by relations of the form
\[ W h_1 h_2 = h_2 h_1 W, \quad h_1 h_2 \eta = \eta, \quad \eta' h_1 h_2 = \eta', \quad h = h^*, \quad (20) \]
where
\[ \hat{W} = PW = q^{P(+) - q^{-1} P(-)} + q^{1-N} P(0) \quad (21) \]
is the standard \( R \)-matrix for the orthogonal series (here \( P(+) \), \( P(-) \) and \( P(0) \) are the spectral projections corresponding to symmetric (traceless), antisymmetric and proportional to the metric elements of \( V \otimes V \) with \( |q| = 1 \) and \( \eta' (\eta) \) is a deformed covariant (contravariant) metric. For \( q = 1 + i \varepsilon + \ldots \) we have \( W = I + i \varepsilon w + \ldots \), where \( w \) satisfies the classical Yang Baxter equation. To the skew-symmetric classical \( r \)-matrix \( r = (w - w_{21})/2 \) there corresponds the involutive intertwiner
\[ \hat{R} := I - 2P(-), \quad R = P \hat{R} = I + i \varepsilon r + \ldots \]
(note that \( R \) can be used instead of \( W \) in [21]).
Passing to the inhomogeneous group (1), we expect that the commutation relations for \(g\) should be

\[
\mathcal{R}g_1g_2 = g_2g_1\mathcal{R}, \quad \text{where} \quad \mathcal{R} = \begin{pmatrix} R & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}
\] (22)

(this corresponds to \(r\) given in (3) when \(a = 0, b = 0\)). Using the form of \(g_1g_2\) as in (3), we obtain

\[
Rh_1h_2 = h_2h_1R, \quad x_2h_1 = Rh_1x_2, \quad h_2x_1 = Rx_1h_2, \quad x_2x_1 = Rx_1x_2. \quad (23)
\]

The two equalities in the middle are equivalent, due to the involutivity of \(\hat{R}\). The last equality provides defining relations for the quantum orthogonal vector space [10, 11]. These relations are consistent: the corresponding algebra of polynomials has the classical size. Also the first equality gives consistent relations in this sense.

It remains to check the consistency of the ‘cross-relations’ with other ones. From

\[
R_{12}R_{13}R_{23}h_1h_2x_3 = x_3h_2h_1R_{12} = R_{23}R_{13}R_{12}h_1h_2x_3, \quad (24)
\]

\[
R_{12}R_{13}R_{23}h_1x_2x_3 = x_3x_2h_1 = R_{23}R_{13}R_{12}h_1x_2x_3, \quad (25)
\]

it follows that \(R\) should satisfy the Yang Baxter equation, hence \(q = 1\) (the triangular case). As in the Poisson case, we postulate then a modification of (23) as follows:

\[
Rh_1h_2 = h_2h_1R, \quad x_2h_1 = W'h_1x_2, \quad x_2x_1 = Rx_1x_2, \quad (26)
\]

with some matrix \(W'\). Instead of (24)–(25), we have now

\[
R_{12}W'_{13}W'_{23}h_1h_2x_3 = x_3h_2h_1R_{12} = W'_{23}W'_{13}R_{12}h_1h_2x_3, \quad (24')
\]

\[
W'_{12}W'_{23}h_1x_2x_3 = x_3x_2h_1 = R_{23}W'_{13}W'_{12}h_1x_2x_3. \quad (25')
\]

For the consistency of different ways of ordering, we postulate that

\[
W'_{12}W'_{13}W'_{23} = W'_{23}W'_{13}W'_{12} \quad \text{and} \quad \hat{R} (\text{or } P(-)) \text{ is a function of } \hat{W}' = PW'.
\] (27)

This is fulfilled if \(\hat{W}'\) a scalar multiple of \(\hat{W}\) (it is also possible that \(\hat{W}'\) is a scalar multiple of \(\hat{W}^{-1}\); this corresponds to the change \(s \mapsto -s\) in the Poisson case). The scalar coefficient is not arbitrary, due to the following two conditions:

1. From the reality requirement \((h^* = h, x^* = x)\) it follows that \(x_2h_1 = W'h_1x_2\) implies \(h_1x_2 = \overline{W'}x_2h_1\), hence \(x_2h_1 = W'\overline{W'}x_2h_1\) and we have to assume that

\[
W'\overline{W'} = I. \quad (28)
\]
2. Since \( x_3 \eta_{12} = x_3 h_1 h_2 \eta_{12} = W'_{13} W'_{23} h_1 h_2 x_3 \eta_{12} = W'_{13} W'_{23} \eta_{12} \), we have also the following condition of compatibility of \( W' \) with the metric:

\[
W'_{13} W'_{23} \eta_{12} = \eta_{12}. \tag{29}
\]

Both conditions are satisfied by \( W' = W \) (another solution, \( W' = -W \), has no proper classical limit). The first condition follows from

\[
\overline{W(q)} = W(\overline{q}) = W(q^{-1}) = W(q)^{-1}
\]

(cf. [14]; recall that \( |q| = 1 \)). The second coincides with formula (2.21) in [12]. Thus, in the sequel we set \( W' = W \).

It is easy to see that the comultiplication preserves first two relations in (26), for instance \( \Delta x_2 \Delta h_1 \) equals

\[
(x_2 + h_2 x'_2) h'_1 = W h_1 x_2 h'_1 + h_2 h_1 W h'_1 x'_2 = W h_1 h'_1 x_2 + W h_1 h'_1 x'_2 = W \Delta h_1 \Delta x_2.
\]

This will be true also for a nontrivial braiding of the type

\[
x'_2 x_1 = B x_1 x'_2, \tag{30}
\]

which on the other hand may be used to remove the inconsistency related to the preservation of the third relation: \( P(-) x_1 x_2 = 0 \). We shall find now the condition under which \( P(-) \Delta x_1 \Delta x_2 = 0 \). The first two terms in

\[
\Delta x_1 \Delta x_2 = (x_1 + h_1 x'_1)(x_2 + h_2 x'_2) = x_1 x_2 + h_1 x'_1 h_2 x'_2 + x_1 h_2 x'_2 + h_1 x'_1 x_2
\]

are annihilated by \( P(-) \) (second, because \( P(-) h_1 h_2 x'_2 = h_1 h_2 P(-) x'_1 x'_2 = 0 \)). The sum of the last two terms is equal

\[
(W h_1 x_2 x'_1 + h_1 x'_1 x_2)^{jk} = W_{ab} h^a_c x^b x^c + h^l_1 B_{bc} x^b x^c = (W_{ab} h^a_c + \delta^a_b B_{bc}) h^l_1 x^b x^c,
\]

hence our condition is

\[
P^{-1}_{12}(W_{12} + B_{23}) = 0. \tag{31}
\]

If

\[
P(-)(W + \sigma I) = 0 \quad \text{for some } \sigma, \tag{32}
\]

then \( B = \sigma I \) is a solution of our problem and the non-trivial cross-relations are the following: \( x'^j x^k = \sigma x^k x'^j \). We call (32) the spectral condition. Taking into account that \( P(-) \) is a projection and a function of \( W \), it means that \( P(-) \) is a spectral projection of \( W \) corresponding to a single eigenvalue. This is of course satisfied for (26), with \( \sigma = q^{-1} \).

We conclude that relations (26) with \( W' = W \) and braiding

\[
x'^j x^k = q^{-1} x^k x'^j \tag{33}
\]

define a braided quantum \( ISO(p, N - p) \), which contains \( SO_q(p, N - p) \).
References

[1] P. Podles and S.L. Woronowicz, On the classification of quantum Poincaré groups, Commun. Math. Phys. 178 (1996), 61–82.

[2] S. Zakrzewski, Poisson structures on the Poincaré group, Commun. Math. Phys. 185 (1997), 285–311.

[3] P. Podles and S.L. Woronowicz, On the structure of inhomogeneous quantum groups, Commun. Math. Phys. 185 (1997), 325–358.

[4] S. Majid, Algebras and Hopf algebras in braided categories, Lect. Notes in Pure and Appl. Math. 158 (1994), 55–105.

[5] S.L. Woronowicz, Example of a braided locally compact group, in: Quantum Groups, Formalism and Applications, Proceedings of the XXX Winter School on Theoretical Physics 14–26 February 1994, Karpacz, J. Lukierski, Z. Popowicz, J. Sobczyk (eds.), Polish Scientific Publishers PWN, Warsaw 1995, pp. 155–171.

[6] S. Zakrzewski, Phase spaces related to standard classical r-matrices, J. Phys. A: Math. Gen. 29 (1996) 1841–1857.

[7] S. Zakrzewski, Poisson structures on the Lorentz group, Lett. Math. Phys. 32 (1994), 11–23.

[8] M. Cahen, S. Gutt and J. Rawnsley, Some remarks on the classification of Poisson Lie groups, in: Symplectic geometry and quantization, Y. Maeda, H. Omori, A. Weinstein (Eds.), Contemp. Math. 179 (1994), pp. 1–16.

[9] S. Majid, Braided-Lie bialgebras, preprint DAMTP/96-98, q-alg/9703004.

[10] L. D. Faddeev, N. Yu. Reshetikhin and L. A. Takhtajan, Quantization of Lie groups and Lie algebras, Algebra i analiz 1 (1989), 178–206 (in Russian).

[11] E. Celeghini, R. Giachetti, A. Reyman, E. Sorace and M. Tarlini, $SO_q(n+1,n-1)$ as a real form of $SO_q(2n,C)$, Lett. Math. Phys. 23 (1991), 45–49.

[12] P. Podles, Complex quantum groups and their real representations, Publ. RIMS, Kyoto Univ. 28 (1992), 709–745.