The Square Trees in the Tribonacci Sequence

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Abstract: The Tribonacci sequence $T$ is the fixed point of the substitution $\sigma(a, b, c) = (ab, ac, a)$. In this note, we get the explicit expressions of all squares, and then establish the tree structure of the positions of repeated squares in $T$, called square trees. Using the square trees, we give a fast algorithm for counting the number of repeated squares in $T[1, n]$ for all $n$, where $T[1, n]$ is the prefix of $T$ of length $n$. Moreover, we get explicit expressions for some special $n$ such as $n = t_m$ (the Tribonacci number) etc., which including some known results such as H. Mousavi and J. Shallit[6].

Key words: the Tribonacci sequence, kernel, square, gap sequence.

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1 Introduction

Let $A = \{a, b, c\}$ be a three-letter alphabet. The Tribonacci sequence $T$ is the fixed point beginning with $a$ of the substitution $\sigma(a, b, c) = (ab, ac, a)$. As a natural generalization of the Fibonacci sequence, $T$ has been studied extensively by many authors, see [1–6][7][8].

Let $\omega$ be a factor of $T$, denoted by $\omega \prec T$. Let $\omega_p$ be the $p$-th occurrence of $\omega$. If the factor $\omega$ and integer $p$ such that $\omega_p \omega_{p+1} \prec T$, we call $\omega_p \omega_{p+1}$ a square of $T$. As we know, $T$ contains no fourth powers. The properties of squares and cubes are objects of a great interest in mathematics and computer science etc.

We denote by $|\omega|$ the length of $\omega$. Denote $|\omega|_\alpha$ the number of letter $\alpha$ in $\omega$, where $\alpha \in A$. Let $\tau = x_1 \cdots x_n$ be a finite word (or $\tau = x_1 x_2 \cdots$ be a sequence). For any $i \leq j \leq n$, define $\tau[i, j] = x_i x_{i+1} \cdots x_j x_j$. By convenient, denote $\tau[i, i] = \tau[i, i] = x_i$, $\tau[i, i-1] = \varepsilon$ (empty word). Denote $T_m = \sigma^m(a)$ for $m \geq 0$, $T_m = \varepsilon, T_m = c$. Denote $t_m = |T_m|$ for $m \geq -2$, called the $m$-th Tribonacci number. Denote by $\delta_m$ the last letter of $T_m$ for $m \geq -1$.

Denote $A(n) = \#\{\omega, p : \omega_p \omega_{p+1} \prec T[1, n]\}$ the number of repeated squares in $T[1, n]$. In 2014, H. Mousavi and J. Shallit[6] gave expression of $A(t_m)$, which they proved by mechanical way. In [4], we give a fast algorithm for counting the number of repeated squares in each prefix of the Fibonacci sequence. In this note, we give a fast algorithm for counting $A(n)$ for all $n$. In Section 2, we establish the tree structure of the positions of repeated squares in $T$, called the square trees. Section 3 is devoted to give a fast algorithm for counting $A(n)$. As a special case, we get expression of $A(t_m)$ in Section 4.

The main tools of the paper are “kernel word” and “gap sequence”, which introduced and studied in [2]. We define the kernel numbers that $k_0 = 0, k_1 = k_2 = 1, k_m = k_{m-1} + k_{m-2} + k_{m-3} - 1$ for $m \geq 3$. The kernel word with order $m$ is defined as $K_1 = a, K_2 = b, K_3 = c, K_m = \delta_{m-1} T_m-3[1, k_m - 1]$ for $m \geq 4$. Using the property of gap sequence, we can determine the positions of all $\omega_p$, and then establish the square trees.

2 The square trees

In [3], we determined the three cases of squares with kernel $K_m$ (i.e., the maximal kernel word occurring in these squares is $K_m$). For $m \geq 4$ and $p \geq 1$, we denote

$$A(m, p) = pt_{m-1} + |T[1, p-1]|t_{m-2} + |T[1, p-1]|t_{m-3} + |T[1, p-1]|t_{m-2}.$$
By Property 6.1 in [5], we have the position of the last letter of the p-th occurrence of \( K_m \) that 
\[ P(K_m,p) = \Lambda(m,p) + k_m - 1. \]
Thus we can define three sets for \( m \geq 4 \) and \( p \geq 1 \), which contain the three cases of squares, respectively.

\[
\begin{align*}
\langle 1, K_m, p \rangle &= \{ P(\omega, p) : \text{Ker}(\omega\omega) = K_m, |\omega| = t_{m-1}, \omega \prec T \} \\
&= \{ \Lambda(m,p) + t_{m-1}, \cdots, \Lambda(m,p) + k_{m+3} - 2 \}; \\
\langle 2, K_m, p \rangle &= \{ P(\omega, p) : \text{Ker}(\omega\omega) = K_m, |\omega| = t_{m-4} + t_{m-3}, \omega \prec T \} \\
&= \{ \Lambda(m,p) + t_{m-3} + t_{m-4}, \cdots, \Lambda(m,p) + k_{m+2} - 2 \}; \\
\langle 3, K_m, p \rangle &= \{ P(\omega, p) : \text{Ker}(\omega\omega) = K_m, |\omega| = t_{m-4} - k_{m-3}, \omega \prec T \} \\
&= \{ \Lambda(m,p) + k_m - 1, \cdots, \Lambda(m,p) + 2t_{m-4} - 1 \}.
\end{align*}
\]

For \( m \geq 4 \) and \( p \geq 1 \), we consider the sets

\[
\begin{align*}
\Gamma_{1,m,p} &= \{ \Lambda(m,p) + k_{m+2} - 1, \cdots, \Lambda(m,p) + k_{m+3} - 2 \}; \\
\Gamma_{2,m,p} &= \{ \Lambda(m,p) + k_{m+1} - 1, \cdots, \Lambda(m,p) + k_{m+2} - 2 \}; \\
\Gamma_{3,m,p} &= \{ \Lambda(m,p) + k_m - 1, \cdots, \Lambda(m,p) + k_{m+1} - 2 \}.
\end{align*}
\]

Obviously, \( \langle 1, K_m, p \rangle \) (resp. \( \langle 2, K_m, p \rangle \), \( \langle 3, K_m, p \rangle \)) contains the several maximal (resp. maximal, minimal) elements of \( \Gamma_{1,m,p} \) (resp. \( \Gamma_{2,m,p} \), \( \Gamma_{3,m,p} \)). Moreover \( \max \Gamma_{2,m,p} + 1 = \min \Gamma_{1,m,p} \) and \( \max \Gamma_{3,m,p} + 1 = \min \Gamma_{2,m,p} \). Using Lemma 6.4 in [5], comparing minimal and maximal elements in these sets below, we have

\[
\begin{align*}
\Gamma_{1,m,p} &= \Gamma_{3,m,p} - P(a,p)+1 \cup \Gamma_{2,m,p} - P(b,p)+1 \cup \Gamma_{1,m,p} - P(a,p)+1, \quad m \geq 4; \\
\Gamma_{2,m,p} &= \Gamma_{3,m,p} - P(b,p)+1 \cup \Gamma_{2,m,p} - P(b,p)+1 \cup \Gamma_{1,m,p} - P(b,p)+1, \quad m \geq 5; \\
\Gamma_{3,m,p} &= \Gamma_{3,m,p} - P(c,p)+1 \cup \Gamma_{2,m,p} - P(c,p)+1 \cup \Gamma_{1,m,p} - P(c,p)+1, \quad m \geq 6.
\end{align*}
\]

Thus we establish the recursive relations for any \( \Gamma_{1,m,p} \) (\( m \geq 4 \)), \( \Gamma_{2,m,p} \) (\( m \geq 5 \)) and \( \Gamma_{3,m,p} \) (\( m \geq 6 \)). By the relation between \( \Gamma_{i,m,p} \) and \( \langle i, K_m, p \rangle \), we get the tree structure of the positions of repeated squares in \( T \), called the square trees.

\[
\begin{align*}
\pi_1(1, K_m, p) &= \langle 3, K_{m-1}, \hat{a} \rangle \cup \langle 2, K_{m-1}, \hat{a} \rangle \cup \langle 1, K_{m-1}, \hat{a} \rangle, m \geq 4; \\
\pi_2(2, K_m, p) &= \langle 3, K_{m-2}, \hat{b} \rangle \cup \langle 2, K_{m-2}, \hat{b} \rangle \cup \langle 1, K_{m-2}, \hat{b} \rangle, m \geq 5; \\
\pi_3(3, K_m, p) &= \langle 3, K_{m-3}, \hat{c} \rangle \cup \langle 2, K_{m-3}, \hat{c} \rangle \cup \langle 1, K_{m-3}, \hat{c} \rangle, m \geq 6.
\end{align*}
\]

Here we denote \( P(\alpha, p) + 1 \) by \( \hat{\alpha} \) for short, \( \alpha \in \{a,b,c\} \).

On the other hand, for \( m \geq 4 \) and \( i \in \{1,2,3\} \), each \( \langle i, K_m, 1 \rangle \) belongs to the square trees. Moreover \( \langle i, K_m, \hat{\alpha} \rangle \) (resp. \( \langle i, K_m, \hat{b} \rangle \), \( \langle i, K_m, \hat{c} \rangle \)) is subset of \( \pi_1(1, K_{m+1}, p) \) (resp. \( \pi_2(2, K_{m+2}, p) \), \( \pi_3(3, K_{m+3}, p) \)). Notice that \( \mathbb{N} = \{1\} \cup \{P(a,p) + 1\} \cup \{P(b,p) + 1\} \cup \{P(c,p) + 1\} \), the square trees contain all \( \langle i, K_m, p \rangle \), i.e. all squares in \( T \). Fig. 4 shows some examples.

### 3 Algorithm: the numbers of repeated squares in \( T[1, n] \)

Denote \( a(n) = \#\{ (\omega, p) : \omega \omega \omega \omega + 1 \succ T[1, n] \} \) the number of squares ending at position \( n \). By Proposition 3.1 we can calculate \( a(n) \), and otherwise calculate \( A(n) \) by \( A(n) = \sum_{i=4}^{n} a(i) \). For \( m \geq 4 \), since \( k_m = \frac{t_m - 2t_{m-1} + t_{m-2} + 1}{2} \),

\[
\begin{align*}
\Gamma_{1,m,1} &= \{ t_m + 2t_{m-1} - t_{m-2} - 1, \ldots, t_m + 2t_{m-1} + t_{m-2} - 3 \}; \\
\Gamma_{2,m,1} &= \{ -t_m + 4t_{m-1} - t_{m-2} - 1, \ldots, t_m + 2t_{m-1} - t_{m-2} - 3 \}; \\
\Gamma_{3,m,1} &= \{ t_m + t_{m-2} - 1, \ldots, -t_m + 4t_{m-1} - t_{m-2} - 3 \}.
\end{align*}
\]
Figure 1: (a)-(c) are square trees from root $\langle 1, K_6, 1 \rangle$, $\langle 2, K_7, 1 \rangle$, $\langle 3, K_7, 1 \rangle$, respectively.
(a) \( \Phi_m = \frac{m}{37}(-5t_m + 14t_{m-1} + 4t_{m-2}) + \frac{1}{14}(67t_m - 166t_{m-1} + 5t_{m-2}) + \frac{1}{4} \).

(b) \( \sum a(\Gamma_{i,4,1}) = 1 \) where \( i \in \{1, 2, 3\} \), and

\[
\begin{aligned}
\sum a(\Gamma_{1,m,1}) &= \frac{m}{37}(4t_m - 9t_{m-1} + 10t_{m-2}) + \frac{1}{14}(19t_m + 36t_{m-1} - 169t_{m-2}) - \frac{1}{4}; \\
\sum a(\Gamma_{2,m,1}) &= \frac{m}{37}(10t_m - 6t_{m-1} - 19t_{m-2}) + \frac{1}{14}(-189t_m + 156t_{m-1} + 331t_{m-2}) - \frac{1}{4}; \\
\sum a(\Gamma_{3,m,1}) &= \frac{m}{37}(-19t_m + 29t_{m-1} + 13t_{m-2}) + \frac{1}{14}(237t_m - 358t_{m-1} - 157t_{m-2}) + \frac{3}{4}.
\end{aligned}
\]

(c) \( \sum_{j=4}^{m-1} \Phi_j = \frac{m}{37}(13t_m - 10t_{m-1} + 5t_{m-2}) + \frac{2}{14}(-8t_m + 8t_{m-1} - 7t_{m-2}) + \frac{m}{4} + 2. \)

\[
\begin{aligned}
A(\max \Gamma_{3,m,1}) &= \frac{m}{37}(-25t_m + 48t_{m-1} + 31t_{m-2}) + \frac{1}{14}(173t_m - 294t_{m-1} - 213t_{m-2}) + \frac{m+11}{4}; \\
A(\max \Gamma_{2,m,1}) &= \frac{m}{37}(-5t_m + 36t_{m-1} - 7t_{m-2}) + \frac{1}{14}(-8t_m + 69t_{m-1} + 59t_{m-2}) + \frac{m+10}{4}; \\
A(\max \Gamma_{1,m,1}) &= \frac{m}{37}(3t_m + 18t_{m-1} + 13t_{m-2}) + \frac{1}{14}(3t_m - 102t_{m-1} - 51t_{m-2}) + \frac{m+9}{4}.
\end{aligned}
\]

Figure 2: These properties can be proved easily by induction, where (a) and (c) hold for \( m \geq 4 \), (b) and (d) hold for \( m \geq 5 \).

**Proposition 3.1.** \( a([8]) = [1] \), \( a([9, 10]) = [0, 1] \), \( a([11, \cdots, 14]) = [0, 0, 0, 1] \), \( a([15, 16]) = [1, 1] \), \( a([17, \cdots, 20]) = [0, 0, 1, 1] \), \( a([28, \cdots, 31]) = [1, 1, 1, 1] \),

\[
\begin{aligned}
a(\Gamma_{1,m,1}) &= [a(\Gamma_{3,m-1,1}), a(\Gamma_{2,m-1,1}), a(\Gamma_{1,m-1,1})] + \left[ \underbrace{0, \ldots, 0}_{t_{m-2}-k_{m+1}}, \underbrace{1, \ldots, 1}_{k_{m-1}} \right]; \\
a(\Gamma_{2,m,1}) &= [a(\Gamma_{3,m-2,1}), a(\Gamma_{2,m-2,1}), a(\Gamma_{1,m-2,1})] + \left[ \underbrace{0, \ldots, 0}_{t_{m-3}-k_{m+1}}, \underbrace{1, \ldots, 1}_{k_{m-1}} \right]; \\
a(\Gamma_{3,m,1}) &= [a(\Gamma_{3,m-3,1}), a(\Gamma_{2,m-3,1}), a(\Gamma_{1,m-3,1})] + \left[ \underbrace{1, \ldots, 1}_{t_{m-4}-k_{m+1}}, \underbrace{0, \ldots, 0}_{k_{m-3}-1} \right].
\end{aligned}
\]

Denote \( \Phi_m = \sum a(\Gamma_{3,m,1}) + \sum a(\Gamma_{2,m,1}) + \sum a(\Gamma_{1,m,1}) \). The immediately corollaries of Proposition 3.1 are \( \sum a(\Gamma_{1,m,1}) = \Phi_{m-1} + k_m - 1 \), \( \sum a(\Gamma_{2,m,1}) = \Phi_{m-2} + k_m - 1 \), \( \sum a(\Gamma_{3,m,1}) = \Phi_{m-3} + t_m - 4 - k_{m-3} + 1 \). Moreover, for \( m \geq 7 \),

\[
\Phi_m = \Phi_{m-1} + \Phi_{m-2} + \Phi_{m-3} + \frac{-3t_m + 6t_{m-1} + t_{m-2} - 1}{2}.
\]

By induction, we can prove the 4 properties in Fig 2.

Obversely we can calculate \( A(n) \) by \( A(n) = \sum_{i=1}^{n} a(i) \). But when \( n \) large, this method is complicated. Now we turn to give a fast algorithm. For any \( n \geq 52 \), let \( m \) such that \( n \in \Gamma_{3,m,1} \cup \Gamma_{2,m,1} \cup \Gamma_{1,m,1} = \{ \frac{t_m + t_m - 2}{2}, \ldots, \frac{t_m + t_m + t_m - 2}{2} \} \). We already determine the expression of \( A(\max \Gamma_{i,m,1}) \) for \( i \in \{1, 2, 3\} \), \( m \geq 5 \). In order to calculate \( A(n) \), we only need to calculate \( \sum_{i=\min \Gamma_{i,m,1}}^{n} a(i) \).

**Algorithm.** Step 1. For \( n \leq 51 \), calculate \( \sum_{i=\min \Gamma_{i,m,1}}^{n} a(i) \) by Property 3.1.

Step 2. For \( n \geq 52 \), find the \( m \) and \( i \) such that \( n \in \Gamma_{i,m,1} \), then \( m \geq 7 \). We calculate \( \sum_{i=\min \Gamma_{i,m,1}}^{n} a(i) \) by the properties in Fig 3.

Step 3. Calculate \( A(\min \Gamma_{i,m,1} - 1) \) by the Property (d) in Fig 2.

Step 4. \( A(n) = A(\min \Gamma_{i,m,1} - 1) + \sum_{i=\min \Gamma_{i,m,1}}^{n} a(i) \).

4 Expression: the numbers of repeated squares in \( T_m \)

Since \( \theta_m^6 \leq t_m < \theta_m^7 \) and \( \theta_{m-1}^6 \leq t_m - t_{m-1} < \theta_{m-1}^7 \) for \( m \geq 7 \), see Fig 3.

\[
\begin{aligned}
\sum_{i=\min \Gamma_{1,m,1}}^{t_m} a(i) - \sum_{i=\min \Gamma_{1,m-1,3}}^{t_{m-3}} a(i) &= \sum a(\Gamma_{3,m-1,1}) + \sum a(\Gamma_{3,m-3,1}) + \sum a(\Gamma_{2,m-3,1}) + 2t_m - 2t_{m-1} - 3t_{m-2} + 1 \\
&= \frac{m}{37}(-19t_m + 29t_{m-1} + 13t_{m-2}) + \frac{1}{14}(347t_m - 622t_{m-1} - 47t_{m-2}) + \frac{9}{4}.
\end{aligned}
\]
(a) \( n \in \Gamma_{3,m,1} = \left\{ \frac{t_m + t_{m-2} - 1}{2}, \ldots, \frac{t_m + 4t_{m-1} + t_{m-2} - 3}{2} \right\} \) for \( m \geq 7 \). Denote

\[
\begin{align*}
\theta^1_m &= \min \Gamma_{3,m,1} = \frac{t_m + t_{m-2} - 1}{2}, \\
\theta^2_m &= \min \Gamma_{3,m,1} + \Gamma_{3,m-3,1} = \frac{-5t_m + 10t_{m-1} + 3t_{m-2} - 1}{2}, \\
\theta^3_m &= \min \Gamma_{3,m,1} + \Gamma_{3,m-3,1} + |\Gamma_{2,m-1} - 3t_{m-2} - 1|, \\
\eta^1_m &= \min \Gamma_{3,m,1} + t_{m-4} - k_{m-3} + 1 = -2t_m + 5t_{m-1}, \\
\eta^2_m &= \max \Gamma_{3,m,1} + 1 = \min \Gamma_{2,m,1} = \frac{-t_m + 4t_{m-1} + t_{m-2} - 1}{2}.
\end{align*}
\]

Obviously, \( \theta^3_m < \eta^1_m < \eta^2_m \) for \( m \geq 7 \), and \( \min \Gamma_{3,m,1} - \min \Gamma_{3,m-3,1} = t_{m-1} \). By Property 3.1 we have: for \( n \geq 52 \), let \( m \) such that \( n \in \Gamma_{3,m,1} \), then \( m \geq 7 \) and

\[
\sum_{i=\min \Gamma_{3,m,1}} \aleph(i) = \left\{ \begin{array}{ll}
\sum_{i=\min \Gamma_{3,m,1}} a(i) + n - \min \Gamma_{3,m,1} + 1, & \text{if } \theta^3_m \leq n < \theta^2_m; \\
\sum_{i=\min \Gamma_{3,m,1}} a(i) + n - \min \Gamma_{3,m,1} + 1, & \text{if } \theta^2_m \leq n < \theta^1_m; \\
\sum_{i=\min \Gamma_{3,m,1}} a(i) + n - \min \Gamma_{3,m,1} + 1, & \text{if } \theta^1_m \leq n.
\end{array} \right.
\]

(b) \( n \in \Gamma_{2,m,1} = \left\{ \frac{-t_m + 4t_{m-1} + t_{m-2} - 3}{2}, \ldots, \frac{t_m + 2t_{m-1} - t_{m-2} - 3}{2} \right\} \) for \( m \geq 6 \). Denote

\[
\begin{align*}
\theta^5_m &= \min \Gamma_{2,m,1} + |\Gamma_{3,m-2,1}| = \frac{3t_m - 5t_{m-1} - 1}{2}, \\
\eta^5_m &= \min \Gamma_{2,m,1} + t_{m-3} - k_{m} + 1 = 2t_{m-1} - t_{m-2}, \\
\theta^6_m &= \min \Gamma_{2,m,1} + |\Gamma_{3,m-3,1}| + |\Gamma_{2,m-2,1}| = \frac{3t_m - 2t_{m-1} - t_{m-2} - 1}{2}, \\
\theta^7_m &= \max \Gamma_{2,m,1} + 1 = \min \Gamma_{2,m,1} = \frac{t_m + 2t_{m-1} + t_{m-2} - 1}{2}.
\end{align*}
\]

Obviously, \( \theta^5_m < \theta^6_m \) for \( m \geq 6 \), and \( \min \Gamma_{2,m,1} - \min \Gamma_{3,m-3,1} = t_{m-1} \). By Property 3.1 we have: for \( n \geq 32 \), let \( m \) such that \( n \in \Gamma_{2,m,1} \), then \( m \geq 6 \) and

\[
\sum_{i=\min \Gamma_{2,m,1}} \aleph(i) = \left\{ \begin{array}{ll}
\sum_{i=\min \Gamma_{2,m,1}} a(i) + n - \min \Gamma_{2,m,1} + 1, & \text{if } \theta^7_m \leq n < \theta^6_m; \\
\sum_{i=\min \Gamma_{2,m,1}} a(i) + n - \min \Gamma_{2,m,1} + 1, & \text{if } \theta^6_m \leq n < \theta^5_m; \\
\sum_{i=\min \Gamma_{2,m,1}} a(i) + n - \min \Gamma_{2,m,1} + 1, & \text{if } \theta^5_m \leq n.
\end{array} \right.
\]

(c) \( n \in \Gamma_{1,m,1} = \left\{ \frac{t_m + 2t_{m-1} - t_{m-2} - 1}{2}, \ldots, \frac{t_m + 3t_{m-1} + t_{m-2} - 3}{2} \right\} \) for \( m \geq 5 \). Denote

\[
\begin{align*}
\theta^8_m &= \min \Gamma_{1,m,1} + |\Gamma_{3,m-1,1}| = \frac{t_m + 3t_{m-1} - 2}{2}, \\
\theta^9_m &= \min \Gamma_{1,m,1} + |\Gamma_{3,m-1,1}| + |\Gamma_{2,m-1,1}| = \frac{t_m + 4t_{m-1} + 3t_{m-2} - 1}{2}, \\
\eta^8_m &= \min \Gamma_{1,m,1} + t_{m-2} - k_{m-1} + 1 = 2t_{m-1}, \\
\eta^9_m &= \max \Gamma_{1,m,1} + 1 = \min \Gamma_{1,m,1} = \frac{t_m + 2t_{m-1} + t_{m-2} - 1}{2}.
\end{align*}
\]

Obviously, \( \theta^8_m < \eta^8_m < \eta^9_m \) for \( m \geq 5 \), and \( \min \Gamma_{1,m,1} - \min \Gamma_{3,m-1,1} = t_{m-1} \). By Property 3.1 we have: for \( n \geq 21 \), let \( m \) such that \( n \in \Gamma_{1,m,1} \), then \( m \geq 5 \) and

\[
\sum_{i=\min \Gamma_{1,m,1}} \aleph(i) = \left\{ \begin{array}{ll}
\sum_{i=\min \Gamma_{1,m,1}} a(i), & \text{if } \theta^9_m \leq n < \theta^8_m; \\
\sum_{i=\min \Gamma_{1,m,1}} a(i) + n - \min \Gamma_{1,m,1}, & \text{if } \theta^8_m \leq n < \theta^9_m; \\
\sum_{i=\min \Gamma_{1,m,1}} a(i) + n - \min \Gamma_{1,m,1}, & \text{if } \theta^9_m \leq n.
\end{array} \right.
\]

Figure 3: (a)-(c) show the three cases of recursive relations between \( \sum_{i=\min \Gamma_{k,m,k}} \aleph(i) \) and \( \sum_{i=\min \Gamma_{k,m,k}} a(i) \), where \( k, t \in \{1, 2, 3\} \), respectively. These relations are derived directly from the square trees (the tree structure of the positions of repeated squares). Using them, we can calculate \( \sum_{i=\min \Gamma_{k,m,k}} a(i) \) fast, and give a fast algorithm for \( A(n) \).
For $m \geq 7$, by induction, $\sum_{i=\min \Gamma_{1,m,1}}^{t_m} a(i)$ is equal to

$$\frac{m}{44}(23t_m - 38t_{m-1} - 3t_{m-2}) + \frac{1}{44}(-65t_m + 164t_{m-1} - 105t_{m-2}) + \frac{3m}{4} - \frac{9}{4}.$$ 

Since $\min \Gamma_{1,m,1} - 1 = \max \Gamma_{2,m,1}$, $A(t_m) = A(\max \Gamma_{2,m,1}) + \sum_{i=\min \Gamma_{1,m,1}}^{t_m} a(i)$. By the properties in Fig.3 we can prove Theorem 21 in H. Mousavi and J. Shallit [6] in a novel way: for $m \geq 3$,

$$A(t_m) = \frac{m}{22}(9t_m - t_{m-1} - 5t_{m-2}) + \frac{1}{44}(-81t_m + 26t_{m-1} + 13t_{m-2}) + m + \frac{1}{4}.$$ 

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