The axiom of spheres in Finsler geometry

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Abstract Here, an axiom of spheres in Finsler geometry is proposed and it is proved that if a Finslerian manifold satisfies the axiom of spheres then it is of constant flag curvature.

Keywords \((\kappa, \mu)\)-contact metric manifold · \(\eta\)-Einstein manifold · Quasi-conharmonically flat manifold · \(\phi\)-conharmonically flat manifold · \(\xi\)-conharmonically flat manifold · Conharmonically semi-symmetric manifold

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1 Introduction

In Riemannian geometry, E. Cartan defined an axiom of \(r\)-planes as follows. A Riemannian manifold \(M\) of dimension \(n \geq 3\) satisfies the axiom of \(r\)-planes, where \(r\) is a fixed integer \(2 \leq r < n\), if for each point \(p\) of \(M\) and any \(r\)-dimensional subspace \(S\) of the tangent space \(T_pM\) there exists an \(r\)-dimensional totally geodesic submanifold \(V\) containing \(p\) such that \(T_pV = S\). He proved that if \(M\) satisfies the axiom of \(r\)-planes for some \(r\), then \(M\) has constant sectional curvature (see [7]). The axiom of \(r\)-spheres in Riemannian geometry was proposed by Leung and Nomizu [9] as follows. For each point \(p\) of \(M\) and any \(r\)-dimensional subspace \(S\) of \(T_pM\), there is an \(r\)-dimensional umbilical submanifold \(V\) with parallel mean curvature vector field such that \(p \in V\) and \(T_pV = S\). They proved that if a Riemannian manifold \(M\) of dimension \(n \geq 3\) satisfies the axiom of \(r\)-spheres for some \(r; 2 \leq r < n\), then \(M\) has constant sectional curvature (see [9]). In [2], Akbar-Zadeh extends the Cartan's axiom of 2-planes to Finsler geometry as follows. A Finslerian manifold \(M\) of dimension \(n \geq 3\) satisfies the axiom of 2-planes if for each point \(p \in M\) and every subspace \(E_2\) of dimension two of \(T_pM\) there exists a totally geodesic surface \(S\) passing through \(p\) such that \(T_pS = E_2\). He proved that every Finsler manifold satisfying the axiom of 2-planes is of constant flag curvature (see [2], page 182).
Recently, a definition of circle in Finsler spaces is introduced by one of the present authors in a joint work with Shen [5]. Based on the definition of a circle we will show later that a connected submanifold of a Finsler manifold is an extrinsic sphere if and only if its circles coincide with circles of the ambient manifold. The proof will appear elsewhere.

The present authors in a previous work have proved that if a forward geodesically complete Finsler manifold admits a circle preserving change of metric then its indicatrix is conformally diffeomorphic to the Euclidean sphere (see [11]).

In the present work, we propose in a natural way, the following axiom of $r$-spheres in Finsler geometry.

**Axiom of $r$-spheres**

Let $(M, F)$ be a Finsler manifold of dimension $n \geq 3$. For each point $x$ in $M$ and any $r$-dimensional subspace $E_r$ of $T_xM$, there exists an $r$-dimensional umbilical submanifold $S$ with parallel mean curvature vector field such that $x \in S$ and $T_xS = E_r$.

We shall prove the following theorem.

**Theorem 1.1** If a Finsler manifold of dimension $n \geq 3$ satisfies the axiom of $r$-spheres for some $r$, $2 \leq r < n$, then $M$ has constant flag curvature.

## 2 Notations and preliminaries on Finsler submanifolds

Let $M$ be a real $n$-dimensional manifold of class $C^\infty$. We denote by $TM$ the tangent bundle of tangent vectors, by $p : TM \to M$ the fiber bundle of non-zero tangent vectors and by $p^*TM \to TM_0$ the pull back tangent bundle. Let $(x, U)$ be a local chart on $M$ and $(x^i, y^i)$ the induced local coordinates on $p^{-1}(U)$. A Finsler structure on $M$ is a function $F : TM \to [0, \infty)$, with the following properties:

(i) $F$ is differentiable $C^\infty$ on $TM_0$;
(ii) $F$ is positively homogeneous of degree one in $y$, that is, $F(x, \lambda y) = \lambda F(x, y)$, for all $\lambda > 0$;
(iii) The Finsler metric tensor $g$ defined by the Hessian matrix of $F^2$, $(g_{ij}) = \left(\frac{1}{2} \frac{\partial^2}{\partial y^i \partial y^j} F^2\right)$, is positive definite on $TM_0$. A Finsler manifold is a pair $(M, F)$ consisting of a differentiable manifold $M$ and a Finsler structure $F$ on $M$. We denote by $TTM_0$, the tangent bundle of $TM_0$ and by $\rho$, the canonical linear mapping $\rho : TTM_0 \to p^*TM$, where, $\rho = p_*$. There is the horizontal distribution $HTM$ such that we have the Whitney sum $TTM_0 = HTM_0 \oplus VTM_0$. This decomposition permits to write a vector field $\hat{X} \in \chi(TM_0)$ into the horizontal and vertical parts in a unique manner, namely $\hat{X} = H\hat{X} + V\hat{X}$. In the sequel, we decorate the vector fields on $TM_0$ by hat, i.e. $\hat{X}$ and $\hat{Y}$ and the corresponding sections of $p^*TM$ by $X = \rho(\hat{X})$ and $Y = \rho(\hat{Y})$, respectively, unless otherwise specified (see [2]). For all $X \in p^*TM$ we denote by $hX$ the horizontal lift of $X$ defined by the bundle morphism $\beta : p^*TM \to HTM$ where, $\beta(\frac{\partial}{\partial x}) = \frac{\partial}{\partial x}$ (see [1]). For another approach on geometry of Finslerian manifolds, one can refer to [6].
2.1 Finsler geometry of submanifolds

Let \((M, F)\) be a Finsler manifold and \(S \subset M\) a \(k\)-dimensional submanifold defined by the immersion \(i : S \rightarrow M\). We identify any point \(x \in S\) by its image \(i(x)\) and any tangent vector \(X \in T_x S\) by its image \(i_*(X)\), where \(i_*\) is the linear mapping. Thus \(T_x S\) becomes a sub-space of \(T_x M\). Let \(TS_0\) be the fiber bundle of non-zero tangent vectors on \(S\). \(TS_0\) is a sub-vector bundle of \(TM_0\) and the restriction of \(p\) to \(TS_0\) is denoted by \(q : TS_0 \rightarrow S\). We denote by \(\hat{T}(S) = i^*TM\), the pull back induced vector bundle of \(TM\) by \(i\). The Finslerian metric \(g\) on \(TM_0\) induces a Finslerian metric on \(TS_0\), where, we denote it again by \(g\). At a point \(x = qz \in S\), where \(z \in TS_0\), the orthogonal complement of \(T_{qz}S\) in \(T_{qz}M\) is denoted by \(N_{qz}S\), namely, \(T_x(S) = T_x(S) \oplus N_{qz}S\), where \(T_x(S) \cap N_{qz}S = 0\). We have the following decomposition:

\[
q^*TS = q^*TS \oplus N,
\]

(2.1)

where, \(N\) is called the normal fiber bundle. If \(TTS_0\) is the tangent vector bundle to \(TS_0\), we denote by \(\varrho\), the canonical linear mapping \(\varrho : TTTS_0 \rightarrow q^*TS\). Let \(\hat{X}\) and \(\hat{Y}\) be the two vector fields on \(TS_0\). For \(z \in TS_0\), \((\nabla_{\hat{X}}\hat{Y})_z\) belongs to \(T_{qz}S\). Attending to (2.1) we have

\[
\nabla_{\hat{X}}\hat{Y} = \nabla_{\hat{X}}\hat{Y} + \alpha(\hat{X},\hat{Y}), \quad \hat{Y} = \varrho(\hat{Y}), \quad \hat{X} = \varrho(\hat{X}),
\]

(2.2)

where, \(\nabla\) is the covariant derivative of Cartan connection and \(\alpha(\hat{X},\hat{Y})\) the second fundamental form of the submanifold \(S\). It belongs to \(N\) and is bilinear in \(\hat{X}\) and \(\hat{Y}\).

It results from (2.2) that the induced connection \(\nabla\) is a metric compatible covariant derivative with respect to the induced metric \(g\) in the vector bundle \(q^*TS \rightarrow TS_0\).

2.2 Shape operator or Weingarten formula

Let \(S\) be an immersed submanifold of \((M, F)\). For any \(\hat{X} \in \chi(TS_0)\) and \(W \in \Gamma(N)\) we set

\[
\nabla_{\hat{X}}W = -A_W\hat{X} + \nabla^+_{\hat{X}}W,
\]

(2.3)

where, \(A_W\hat{X} \in \Gamma(q^*TS)\) and \(\nabla^+_{\hat{X}}W \in \Gamma(N)\) and we have partially used notations of [4]. It follows that \(\nabla^+\) is a linear connection on the normal bundle \(N\). We also consider the bilinear map \(A : \Gamma(N) \otimes \Gamma(TTTS_0) \rightarrow \Gamma(q^*TS), A(W, \hat{X}) = A_W\hat{X}\).

For any \(W \in \Gamma(N)\), the operator \(A_W : \Gamma(TTTS_0) \rightarrow \Gamma(q^*TS)\) is called the shape operator or the Weingarten map with respect to \(W\). Finally, (2.3) is said to be the Weigarten formula for the immersion of \(S\) in \(M\). We have \(g(\alpha(h\hat{X}, Y), W) = g(A_Wh\hat{X}, Y)\), where, \(g\) is the Finslerian metric of \(M\), \(X, Y \in \Gamma(q^*TS)\) and \(h\hat{X}\) is the horizontal lift of \(X\) (see [1]).

2.3 Totally umbilical submanifolds in Finsler spaces

The mean curvature vector field \(\eta\) of the isometric immersion \(i : S \rightarrow M\) is defined by

\[
\eta = \frac{1}{n} tr_g \alpha(h\hat{X}, Y),
\]

(2.4)
where, \( X, Y \in \Gamma(q^*TS) \) and \( h^X \) is the horizontal lift of \( X \) (see [1]). We say that the mean curvature vector field \( \eta \) is parallel in all directions if \( \nabla_{X^h} \eta = 0 \) for all \( X \in \Gamma(q^*TS) \).

**Definition 2.1 ([1])** A submanifold of a Finsler manifold is said to be totally umbilical, or simply umbilical, if it is equally curved in all tangent directions.

More precisely, let \( i : S \rightarrow M \) be an isometric immersion. Then \( i \) is called totally umbilical if there exists a normal vector field \( \xi \in N \) along \( i \) such that its second fundamental form \( \alpha \) with values in the normal bundle satisfies

\[
\alpha(h^X, Y) = g(X, Y)\xi,
\]

for all \( X, Y \in \Gamma(q^*TS) \), where \( h^X \) is the horizontal lift of \( X \). Equivalently, \( S \) is umbilical in \( M \) if \( A_W = g(W, \xi)I \) for all \( W \in \Gamma(N) \) where, \( I \) is the identity transformation (see [1]). To give an example of a totally umbilical submanifold in Finsler space, we refer to a theorem on totally umbilical submanifolds given in [8]. There is shown that if \( (M^{n+1}, \tilde{\alpha} + \tilde{\beta}) \) is a Randers space, where \( \tilde{\alpha} \) is an Euclidean metric and \( \tilde{\beta} \) is a closed 1-form, then any complete and connected n-dimensional totally umbilical submanifold of \( (M^{n+1}, \tilde{\alpha} + \tilde{\beta}) \) must be either a plane or an Euclidean sphere. The latter case happens only when there exist a point \( \tilde{x}_0 \) and a function \( \lambda(\tilde{x}) \) on \( M^{n+1} \) such that \( \tilde{\beta} = \lambda(\tilde{x})d(||\tilde{x} - \tilde{x}_0||_\tilde{\alpha}^2) \) and the sphere is centered at \( \tilde{x}_0 \).

**Example 2.1 ([8])** Let \( (\tilde{M}^{n+1}, \tilde{F}) \) be a Randers space with \( \tilde{F} = \tilde{\alpha} + \tilde{\beta} \), where,

\[
\tilde{\alpha} = \sqrt{\sum_{k=1}^{n+1} (\tilde{y}^k)^2}, \quad \tilde{\beta} = \sum_{k=1}^{n+1} \frac{b\tilde{x}^k d\tilde{x}^k}{\sqrt{\sum_{k=1}^{n+1} (\tilde{x}^k)^2}}, \quad \forall (\tilde{x}, \tilde{y}) \in T\tilde{M}_0,
\]

\( b \) is a constant and \( 0 < \lvert b \rvert < 1 \). One can see that \( d\tilde{\beta} = 0 \). Let

\[
M = \{ \tilde{x} \in \tilde{M}^{n+1} : \sum_{k=1}^{n+1} (\tilde{x}^k - \tilde{x}_0^k)^2 = r^2 \},
\]

and \( f : (M, F) \rightarrow (\tilde{M}^{n+1}, \tilde{F}) \) be an isometric immersion where \( F = \alpha + \beta \) such that

\[
\alpha = \sqrt{\sum_{k=1}^{n+1} \frac{\partial f^k}{\partial x^i} \frac{\partial f^k}{\partial y^j} (\tilde{x}_0^i)^2}, \quad \beta = \sum_{k=1}^{n+1} \frac{\partial f^k}{\partial x^i} \frac{b\tilde{x}^k y^i}{\sqrt{\sum_{k=1}^{n+1} (\tilde{x}^k)^2}},
\]

where, \( (x, y) \in TM_0 \). It is obvious that \( \sum_{k=1}^{n+1} (f^k(x) - \tilde{x}_0^k) \frac{\partial f^k}{\partial x^i} = 0 \) hence if \( \tilde{x}_0 = 0 \) then \( \beta = 0 \). On the other hand from theorem mentioned above we see that \( (M, F) \) is a totally umbilical submanifold of \( (\tilde{M}^{n+1}, \tilde{F}) \) if \( \tilde{x}_0 = 0 \). Therefore if \( \tilde{x}_0 = 0 \), the Euclidean sphere \( (M, \alpha) \) is a totally umbilical submanifold of Randers space \( (M^{n+1}, \tilde{\alpha} + \tilde{\beta}) \). For more details on totally umbilical Finsler submanifolds one can refer to [10].

**Remark 2.1** Let \( i : S \rightarrow M \) be an isometric immersion. If \( S \) is totally umbilical then the normal vector field \( \xi \) is equal to the mean curvature vector field \( \eta \).
2.4 Codazzi equation for Finsler submanifolds

Consider a vector field $\hat{X} \in \Gamma(TM_0)$. We have locally $\hat{X} = X^i \frac{\delta}{\delta x^i} + \hat{X}^i \frac{\partial}{\partial y^i}$, where, $\{\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^i}\}$ are horizontal and vertical bases of $TM$. Then one can define $Q : \Gamma(TM_0) \to \Gamma(TM_0); Q \hat{X} := \hat{X}^i \frac{\delta}{\delta x^i} + X^i \frac{\partial}{\partial y^i}$. By means of the Cartan connection $\nabla$ on $(M,F)$ and the operator $Q$, one can define a linear connection on the manifold $TM_0$ by $D_XY := \nabla_XY + Q\nabla_XQ(H\hat{Y})$, for all $\hat{X}, \hat{Y} \in \Gamma(TM_0)$. $D$ is called the associated linear connection to $\nabla$ on $TM_0$. The torsion tensor field $T^D$ of $D$ is given by $T^D(X,Y) := \tau(\hat{X}, \hat{Y}) + Q(\nabla_XQ(H\hat{Y}) - \nabla_\hat{Y}Q(H\hat{X}) - H[\hat{X}, \hat{Y}])$, where, $\tau$ is the torsion tensor field of $\nabla$ (see [4]). Let $R$ be the $hh$-curvature tensor of the Cartan connection $\nabla$, $\nabla$ the induced connection on the submanifold $S$, $D$ the associated linear connection to the induced connection $\nabla$ and $\nabla^*$ the linear connection on the normal bundle $N$. Let $A$ be the shape operator. One can define a covariant derivative $\nabla'$ of $A$ as follows (see [4]).

$$ (\nabla'_{\hat{X}} A)(W, \hat{Y}) := \nabla_{\hat{X}}(AW \hat{Y}) - A_{\hat{X} W} \hat{Y} - A_W(D_X \hat{Y}), $$

(2.6) for any $\hat{X}, \hat{Y} \in \Gamma(TTS_0)$ and $W \in \Gamma(N)$. The A-Codazzi equation for the Finsler submanifold $S$ with respect to the connection $\nabla$ on Finsler manifold $(M,F)$ is written

$$ g(R(X,Y)W, Z) = g((\nabla'_{HY}A)(W, H\hat{X}) - (\nabla'_{H\hat{X}}A)(W, H\hat{Y}), Z) $$

$$ - g(A_W(T^D(H\hat{X}, H\hat{Y})), Z), $$

(2.7) where, $W \in \Gamma(N)$, $X = g(\hat{X})$, $Y = g(\hat{Y})$ and $X,Y,Z \in \Gamma(q^*TS)$ (see [4], page 84).

2.5 Sectional and flag curvatures

Let $G_2(M)$ be the fiber bundle of 2-planes on $M$. Denote by $\pi^{-1}G_2(M) \to SM$ the fiber induced on $SM$ by $\pi : SM \to M$, where $SM$ is the unit sphere bundle. Let $P \in \pi^{-1}G_2(M)$ be a 2-plane generated by vectors $X, Y \in T_xM$ linearly independent at $x = \pi z \in M$ where, $z \in SM$. By means of $hh$-curvature tensors of Berwald and Cartan connection Akbar-Zadeh defined two sectional curvatures denoted by $K_1$ and $K_2$ respectively. Here in this work we are dealing with Cartan connection and related sectional curvature $K_2 : \pi^{-1}G_2(M) \to \mathbb{R}$ defined by

$$ K_2(z, X, Y) = \frac{g(R(X,Y)Y, X)}{\|X\|^2 \|Y\|^2 - g(X,Y)^2}, $$

(2.8) where, $R$ is the $hh$-curvature tensor of Cartan connection. The scalar $K_2$ is called the sectional curvature at $z \in SM$. If the vector field $Y$ is replaced by the canonical section $\nu$ then sectional curvature is called flag curvature and does not depend on the choice of connection. If we denote the flag curvature by $K$ then we have $K_2(z, \nu, X) = K(z, \nu, X)$, where, $\nu$ is the canonical section (see [2], page 156).

Akbar-Zadeh as a generalization of Schur’s theorem has proved the following theorem.

**Theorem 2.2** ([2]) $K_2(z, P)$ is independent of 2-plane $P(X,Y)$ ($\dim M > 2$) if and only if the curvature tensor $R$ of the Cartan connection satisfies $R(X,Y)Z = K[g(Y,Z)X - g(X,Z)Y]$, where $K$ is a constant and $X, Y, Z \in T_xM$. 

887
3 Main results

Lemma 3.1 Let \((M,F)\) be a Finsler manifold of dimension \(n \geq 3\) satisfying the axiom of \(r\)-spheres for some \(r, 2 \leq r < n\), then \(g(R(X,Y)Z,X) = 0\), where, \(X,Y,Z \in T_xM\) are three orthonormal vectors.

Proof. Let \((M,F)\) be a Finsler manifold which satisfies the axiom of \(r\)-spheres. Consider the Cartan connection \(\nabla\) on the pull back bundle \(p^*TM\), the induced connection \(\nabla\) on \(S\) and the normal connection \(\nabla^\perp\) on normal bundle. Let \(X, Y\) and \(Z\) be the three orthonormal vectors at \(x = pz, z \in TM_0\). Consider the \(r\)-dimensional subspace \(E_r\) of \(T_xS\) which is normal to \(Z\) and contains \(X\) and \(Y\). By assumption there exists an \(r\)-dimensional umbilical submanifold \(S\) with parallel mean curvature vector field \(\eta\) such that \(x \in S\) and \(T_xS = E_r\). It is well known for every point \(x\) in a Finsler manifold there is a sufficiently small neighborhood \(U\) on \(M\) such that every pair of points in \(U\) can be joined by a unique minimizing geodesic, see for instance [3], page 160. Hence there is a specific neighborhood \(U\) of \(x\) such that for each point \(u \in U\) there exists a unique minimizing geodesic from \(x\) to \(u\). Let \(W_u \in N_uS\) be the normal vector at \(u\) which is parallel to \(Z\) with respect to the normal connection \(\nabla^\perp\) along the geodesic from \(x\) to \(u\) in \(U\). The Finslerian metric \(g\) on \(TM_0\) defined by \(F\), induces a Finslerian metric on \(T_{M_0}\), where we denote it again by \(g\). By means of metric compatibility of Cartan connection, along each geodesic \(\gamma\) from \(x\) to any point in \(U\) we have

\[
\frac{d}{dt} g(W, \eta) = g(\nabla_{\dot{\gamma}} W, \eta) + g(W, \nabla_{\dot{\gamma}} \eta),
\]

where, \(h^\gamma\) is the horizontal lift of the tangent vector field \(\dot{\gamma}\). By means of the Weingarten formula (2.3), rewrite (3.1) as follows

\[
\frac{d}{dt} g(W, \eta) = g(-A_W(h^\gamma), \eta) + g(\nabla^\perp_{\dot{\gamma}} W, \eta) + g(W, -A_\eta(h^\gamma) + \nabla^\perp_{\dot{\gamma}} \eta)
\]

\[
= g(-A_W(h^\gamma), \eta) + g(\nabla^\perp_{\dot{\gamma}} W, \eta) + g(W, -A_\eta(h^\gamma)) + g(W, \nabla^\perp_{\dot{\gamma}} \eta).
\]  

(3.2)

Since \(-A_W(h^\gamma)\) and \(-A_\eta(h^\gamma)\) belong to \(T_xS\) and on the other hand \(\eta\) and \(W\) are normal to \(T_xS\) we have \(g(-A_W(h^\gamma), \eta) = g(W, -A_\eta(h^\gamma)) = 0\). By assumption the submanifold \(S\) has parallel mean curvature vector field, that is, \(\nabla^\perp_{\dot{\gamma}} \eta = 0\), hence \(g(W, \nabla^\perp_{\dot{\gamma}} \eta) = 0\).

By definition the vector \(W\) is parallel along the geodesic \(\gamma\) with respect to the normal connection \(\nabla^\perp\), i.e. \(\nabla^\perp_{\dot{\gamma}} W = 0\), hence \(g(\nabla^\perp_{\dot{\gamma}} W, \eta) = 0\). Therefore by means of (3.2) we have \(\frac{d}{dt} g(W, \eta) = 0\) and \(g(W, \eta) = \lambda\) is constant along each geodesic. Keeping in mind \(S\) is a totally umbilical submanifold of \(M\), we have \(A_W = g(W, \eta)I = \lambda I\) at every point of \(U\). Rewriting (2.6) for the horizontal lift \(h^\gamma X\) of \(X\) leads

\[
(\nabla^\gamma_{h^\gamma X}) (W, \dot{\gamma}) = (\nabla^\gamma_{h^\gamma X}) (A_W) (\dot{\gamma}) - A_W (\nabla^\perp_{h^\gamma X} W) \dot{\gamma},
\]

(3.3)

where, we have put, \((\nabla^\gamma_{h^\gamma X}) (A_W) (\dot{\gamma}) := \nabla^\gamma_{h^\gamma X} (A_W) \dot{\gamma} - A_W (D_{h^\gamma X} \dot{\gamma})\) which can be considered as a covariant derivative of \(A_W\). Plugging \(A_W = \lambda I\) in the last equation leads

\[
\nabla^\perp_{h^\gamma X} A_W = 0.
\]

(3.4)
Similarly for the horizontal lift $hY$ of $Y$ we have
\[ \nabla^h_{Y} A_W = 0. \] (3.5)

On the other hand, by means of metric compatibility of Cartan connection and the fact that $g(W, \eta)$ is constant we have $g(\nabla_{\bar{S}X} W, \eta) + g(W, \nabla_{\bar{S}X} \eta) = 0$. By means of the Weingarten formula (2.3) the last equation leads
\[ g(-A_W(h\bar{X}), \eta) + g(\nabla^h_{\bar{S}X} W, \eta) + g(W, -A_\eta(h\bar{X})) + g(W, \nabla^h_{\bar{S}X} \eta) = 0. \] (3.6)

Since $A_W(h\bar{X})$ and $A_\eta(h\bar{X})$ belong to $T_xS$ and on the other hand $\eta$ and $W$ are normal to $T_xS$ we have $g(-A_W(h\bar{X}), \eta) = g(W, -A_\eta(h\bar{X})) = 0$. By assumption the submanifold $S$ has parallel mean curvature vector field, that is, $\nabla^h_{\bar{S}X} \eta = 0$, hence $g(W, \nabla^h_{\bar{S}X} \eta) = 0$. Therefore (3.6) reduces to $g(\nabla^h_{\bar{S}X} W, \eta) = 0$. By non-degeneracy of the metric tensor $g$ at $x \in S$ we have
\[ \nabla^h_{\bar{S}X} W = 0. \] (3.7)

Similarly at $x \in S$ for vector $Y$ we obtain
\[ \nabla^h_{\eta} W = 0. \] (3.8)

Therefore plugging (3.4), (3.5), (3.7) and (3.8) in (3.3) at $x \in S$ we obtain $\nabla^h_{\bar{S}X} A = \nabla^h_{\eta} A = 0$. Now the Codazzi equation (2.7) implies
\[ g(R(X, Y)W, X) = -g(A_W(T^D(h\bar{X}, h\bar{Y})), X). \] (3.9)

By assumption $A_W = g(W, \eta) I$. Thus we have
\[ g(A_W(T^D(h\bar{X}, h\bar{Y})), X) = g(T^D(h\bar{X}, h\bar{Y}), X) g(W, \eta). \] (3.10)

Plugging (3.10) in (3.9) we obtain
\[ g(R(X, Y)W, X) + g(T^D(h\bar{X}, h\bar{Y}), X) g(W, \eta) = 0. \] (3.11)

The first term $g(R(X, Y)W, X)$ is symmetric with respect to $Y$ and $W$ (see [2]). By means of the fact that $\eta$ is normal to $S$ we have $g(Y, \eta) = 0$. Therefore, we conclude $g(T^D(h\bar{X}, h\bar{Y}), X) g(W, \eta) = g(T^D(h\bar{X}, h\bar{W}), X) g(Y, \eta) = 0$. Thus (3.11) becomes $g(R(X, Y)W, X) = 0$. Hence for orthonormal vectors $X, Y \in T_xS$ and $Z \in N_xS$ we have $g(R(X, Y)Z, X) = 0$. This completes the proof. $\Box$

**Lemma 3.2** Let $(M, F)$ be a Finsler manifold of dimension $n \geq 3$. If $g(R(X, Y)Z, X) = 0$ whenever $X, Y$ and $Z$ are three orthonormal tangent vectors of $M$, then $M$ has constant flag curvature.

**Proof.** If we put $Y' = \frac{(Y + Z)}{\sqrt{2}}$, $Z' = \frac{(Y - Z)}{\sqrt{2}}$, then since $X, Y$ and $Z$ are orthonormal, the vectors $X, Y'$ and $Z'$ are again orthonormal. By means of assumption $g(R(X, Y')Z', X) = 0$. By replacing $Y'$ and $Z'$ we obtain
\[ g(R(X, Y)Y, X) = g(R(X, Z)Z, X). \] (3.12)

From which we can conclude from (2.8), $K_2(z, X, Y) = K_2(z, X, Z)$. Thus the sectional curvature $K_2$ does not depend on the 2-plane $P(X, Y)$. By generalization of Schur’s Theorem 2.2, $M$ has constant sectional curvature and hence constant flag curvature. This completes the proof. $\Box$

889
Proof of Theorem 1.1. Let \((M, F)\) be a Finsler manifold which satisfies the axiom of \(r\)-spheres. By means of Lemmas 3.1 and 3.2 we conclude that \(M\) has constant flag curvature. \(\Box\)

References

1. Abatangelo, L.M. – On totally umbilical submanifolds of a locally Minkowski manifold, Collect. Math., 43 (1992), 151–175.
2. Akbar-Zadeh, H. – Initiation to Global Finslerian Geometry, North-Holland Mathematical Library, 68, Elsevier Science B.V., Amsterdam, 2006.
3. Bao, D.; Chern, S.-S.; Shen, Z. – An introduction to Riemann-Finsler geometry, Graduate Texts in Mathematics, 200, Springer-Verlag, New York, 2000.
4. Bejancu, A.; Farran, H.R. – Geometry of Pseudo-Finsler Submanifolds, Kluwer Academic Publishers, Dordrecht, 2000.
5. Bidabad, B.; Shen, Z. – Circle-preserving transformations in Finsler spaces, Publ. Math. Debrecen, 81 (2012), 435–445.
6. Bucataru, I.; Miron, R. – Finsler-Lagrange Geometry. Applications to Dynamical Systems, Editura Academiei Romne, Bucharest, 2007.
7. Cartan, É. – Leçons sur la Géométrie des Espaces de Riemann, (French) 2d ed. Gauthier-Villars, Paris, 1946.
8. He, Q.; Yang, W.; Zhao, W. – On totally umbilical submanifolds of Finsler spaces, Ann. Polon. Math., 100 (2011), 147–157.
9. Leung, D.S.; Nomizu, K. – The axiom of spheres in Riemannian geometry, J. Differential Geometry, 5 (1971), 487–489.
10. Li, J. – Umbilical hypersurfaces of Minkowski spaces, Math. Commun., 17 (2012), no. 1, 63–70.
11. Sedaghat, M.K.; Bidabad, B. – On a class of complete Finsler manifolds, Internat. J. Math., 26 (2015), 1550091, 9 pp.