Existence of solution to scalar BSDEs with weakly $L^{1+}$-integrable terminal values
Ying Hu, Shanjian Tang

To cite this version:
Ying Hu, Shanjian Tang. Existence of solution to scalar BSDEs with weakly $L^{1+}$-integrable terminal values. 2017. hal-01507371v2

HAL Id: hal-01507371
https://hal.archives-ouvertes.fr/hal-01507371v2
Preprint submitted on 16 Apr 2017

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L’archive ouverte pluridisciplinaire HAL, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d’enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.
Existence of solution to scalar BSDEs with weakly $L^{1+}$-integrable terminal values

Ying Hu* Shanjian Tang†
April 16, 2017

Abstract. In this paper, we study a scalar linearly growing BSDE with a weakly $L^{1+}$-integrable terminal value. We prove that the BSDE admits a solution if the terminal value satisfies some $\Psi$-integrability condition, which is weaker than the usual $L^p$ ($p > 1$) integrability and stronger than $L \log L$ integrability. We show by a counterexample that $L \log L$ integrability is not sufficient for the existence of solution to a BSDE of a linearly growing generator.

AMS Subject Classification: 60H10

Key Words Backward stochastic differential equation, weak integrability, terminal condition, dual representation.

1 Introduction

Consider the following Backward Stochastic Differential Equation (BSDE):

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s)ds - \int_t^T Z_s dW_s,$$

where the function $f : [0, T] \times \Omega \times \mathbb{R} \times \mathbb{R}^{1 \times d} \rightarrow \mathbb{R}$ satisfies

$$|f(s, y, z)| \leq \alpha_s + \beta|y| + \gamma|z|,$$

with $\alpha \in L^1(0, T), \beta \geq 0$ and $\gamma > 0$.

It is well known that if $\xi \in L^p$, with $p > 1$, then there exists a solution to BSDE (1.1), see e.g. [5, 4, 1]. The aim of this paper is to find a weaker integrability condition for the terminal value $\xi$, under which the solution still exists.

Set

$$\Psi_\lambda(x) = xe^{(\frac{1}{\lambda} \log(x+1))^{1/2}}, \quad (\lambda, x) \in (0, \infty) \times [0, \infty).$$

Our sufficient condition is: there exists $\lambda \in (0, \frac{\gamma}{\sqrt{T}})$ such that

$$\mathbb{E}[\Psi_\lambda(|\xi|)] < +\infty.$$
Remark 1.1 Note that the preceding $\Psi_\lambda$-integrability is stronger than $L^1$, weaker than $L^p$ for any $p > 1$, because for any $\varepsilon > 0$, we have,

$$x \leq x e^{(x + 1)/\varepsilon} \leq e^{\log(x + 1) + 1/x} \leq e^{1/x} x(x + 1)^\varepsilon, \quad x \geq 0.$$ 

Moreover, for any $p \geq 1$, there exists a constant $C_p > 0$ such that

$$x e^{(x + 1)/\varepsilon} \geq C_p x \log^p (x + 1).$$ 

We will see that even the condition

$$\mathbb{E}[\Psi_\lambda(|\xi|)] < +\infty$$ 

for a certain $\lambda > 1/\gamma$ (which implies that $|\xi| \log^p (|\xi| + 1) \in L^1$) is still too weak to ensure the existence of solution by giving a simple example in Example 2.3.

Note that if the generator $f$ is of sublinear growth with respect to $z$, i.e. there exists $q \in [0, 1)$,

$$|f(t, y, z)| \leq \alpha + \beta|y| + \gamma|z|^q,$$

then there exists a solution for $\xi \in L^1$, see [1].

Our method applies the dual representation of solution to BSDE with convex generator (see, e.g. [4, 6, 3]) in order to establish some a priori estimate and then the localization procedure of $\xi$, then there exists a solution for $\xi \in L^1$, see [1].

Consider real valued BSDEs which are equations of type (1.1), where $f(t, y, z)$ is a standard Brownian motion with values in $\mathbb{R}$, $\mathcal{F}$ is a filtration of the Brownian motion $W$ augmented by the $\mathbb{P}$-null sets of $\mathcal{F}$. The sigma-field of predictable subsets of $[0, T] \times \Omega$ is denoted by $\mathcal{P}$.

Let us close this introduction by giving the notations that we will use in all the paper. For the remaining of the paper, let us fix a nonnegative real number $T > 0$. First of all, $(W_t)_{t \in [0, T]}$ is a standard Brownian motion with values in $\mathbb{R}^d$ defined on some complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$. $(\mathcal{F}_t)_{t \geq 0}$ is the natural filtration of the Brownian motion $W$ augmented by the $\mathbb{P}$-null sets of $\mathcal{F}$. The sigma-field of predictable subsets of $[0, T] \times \Omega$ is denoted by $\mathcal{P}$.

Definition 1.2 By a solution to BSDE (1.1), we mean a pair $(Y_t, Z_t)_{t \in [0, T]}$ of predictable processes with values in $\mathbb{R} \times \mathbb{R}^{1 \times d}$ such that $\mathbb{P}$-a.s., $t \mapsto Y_t$ is continuous, $t \mapsto Z_t$ belongs to $L^2(0, T)$ and $t \mapsto g(t, Y_t, Z_t)$ belongs to $L^1(0, T)$, and $\mathbb{P}$-a.s. $(Y, Z)$ verifies (1.1).

By BSDE $(\xi, f)$, we mean the BSDE of generator $f$ and terminal condition $\xi$.

For any real $p \geq 1$, $\mathcal{S}^p$ denotes the set of real-valued, adapted and càdlàg processes $(Y_t)_{t \in [0, T]}$ such that

$$||Y||_{\mathcal{S}^p} := \mathbb{E} \left[ \sup_{0 \leq t \leq T} |Y_t|^p \right]^{1/p} < +\infty,$$

and $\mathcal{M}^p$ denotes the set of (equivalent class of) predictable processes $(Z_t)_{t \in [0, T]}$ with values in $\mathbb{R}^{1 \times d}$ such that

$$||Z||_{\mathcal{M}^p} := \mathbb{E} \left[ \left( \int_0^T |Z_s|^2 \, ds \right)^{p/2} \right]^{1/p} < +\infty.$$

The rest of the paper is organized as follows. Section 2 establishes a necessary and sufficient condition for the existence of solution to BSDE (1.1) for the typical form of generator $f(t, y, z) = \alpha_t + \beta y + \gamma |z|$. Section 3 gives the $\Psi_\lambda$ integrability condition for the existence of solution to BSDE (1.1) for $f(t, y, z) = \alpha_t + \beta y + \gamma |z|$. Section 4 is devoted to the sufficiency of the $\Psi_\lambda$ integrability condition for the existence of solution to BSDE (1.1) of the general linearly growing generator.
2  Typical Case

Let us first consider the following BSDE:

$$Y_t = \xi + \int_t^T (\alpha_s + \beta Y_s + \gamma |Z_s|)ds - \int_t^T Z_s dW_s,$$  \hspace{1cm} (2.2)

where $\alpha \in L^1(0,T)$, and $\beta \geq 0$ and $\gamma > 0$ are some real constants. We suppose further that the terminal condition $\xi$ is nonnegative. Note that if $Y$ is a solution belonging to class $D$, then as $e^{\beta Y_t}$ is a local supermartingale, it is a supermartingale, from which we deduce that $Y \geq 0$. In this subsection, we restrict ourselves to nonnegative solution.

For $\xi \in L^p (p > 1)$, BSDE (2.2) has a unique solution. It has a dual representation as follows (see, e.g. [4, 3])

$$Y_t = \text{ess sup}_{q \in A} \{ E_q[e^{\beta(T-t)\xi} \mid \mathcal{F}_t] \} + \int_t^T e^{\beta(s-t)}\alpha_s ds,$$  \hspace{1cm} (2.3)

where $A$ is the set of progressively measurable processes $q$ such that $|q| \leq \gamma$,

$$\frac{dQ^q}{dP} = M_t^q,$$

with

$$M_t^q = \exp \left\{ \int_0^t q_sdW_s - \frac{1}{2} \int_0^t |q_s|^2 ds \right\}, \quad t \in [0,T],$$

and $E_q$ is the expectation with respect to $Q^q$.

**Theorem 2.1** Let us suppose that $\xi \geq 0$. Then BSDE (2.2) admits a solution $(Y, Z)$ such that $Y \geq 0$ if and only if there exists a locally bounded process $\bar{Y}$ such that

$$\text{ess sup}_{q \in A} \{ E_q[e^{\beta(T-t)\xi} \mid \mathcal{F}_t] \} + \int_t^T e^{\beta(s-t)}\alpha_s ds \leq \bar{Y}_t.$$

**Proof.** If BSDE (2.2) admits a solution $(Y, Z)$ such that $Y \geq 0$, then we define a sequence of stopping times

$$\sigma_n = T \wedge \inf \{ t \geq 0 : |Y_t| > n \},$$

with the convention that $\inf \emptyset = +\infty$.

As $W^q_t = W_s - \int_0^s q_r dr$ is a Brownian motion under $Q^q$, we have

$$Y_{t \wedge \sigma_n} = Y_{\sigma_n} + \int_{t \wedge \sigma_n}^{\sigma_n} (\alpha_s + \beta Y_s + \gamma |Z_s| - Z_s q_s)ds - \int_{t \wedge \sigma_n}^{\sigma_n} Z_s dW^q_s.$$

Applying Itô’s formula to $e^{\beta s} Y_s$, we deduce

$$e^{\beta (t \wedge \sigma_n)} Y_{t \wedge \sigma_n} = e^{\beta \sigma_n} Y_{\sigma_n} + \int_{t \wedge \sigma_n}^{\sigma_n} e^{\beta s} (\alpha_s + \gamma |Z_s| - Z_s q_s)ds - \int_{t \wedge \sigma_n}^{\sigma_n} e^{\beta s} Z_s dW^q_s,$$

from which we obtain

$$E_q[e^{\beta (\sigma_n - t \wedge \sigma_n)} Y_{\sigma_n} + \int_{t \wedge \sigma_n}^{\sigma_n} e^{\beta(s-t \wedge \sigma_n)} \alpha_s ds \mid \mathcal{F}_t] \leq Y_{t \wedge \sigma_n}.$$

Fatou’s lemma yields that

$$E_q[e^{\beta(T-t)\xi} \mid \mathcal{F}_t] + \int_t^T e^{\beta(s-t)}\alpha_s ds \leq Y_t.$$
On the other hand, if there exists a locally bounded process $\bar{Y}$ such that
\[
\text{ess sup}_{q \in A} \{\mathbb{E}[e^{\beta(t-t)} \xi | \mathcal{F}_t] \} + \int_t^T e^{\beta(s-t)} \alpha_s ds \leq \bar{Y}_t,
\]
then we construct the solution by use of a localization method (see e.g. [2]). We describe this method here for completeness. Let $(Y^n, Z^n)$ be the unique solution in $S^2 \times M^2$ of the following BSDE
\[
Y^n_t = \xi 1_{\{t \leq n\}} + \int_t^T (\alpha_s + \beta Y^n_s + \gamma |Z^n_s|) ds - \int_t^T Z^n_s dW_s.
\]
By comparison theorem, $Y^n$ is nondecreasing with respect to $n$. Moreover, setting $q^n_s = \gamma \text{ sgn}(Z^n_s)$, we obtain
\[
Y^n_t = \mathbb{E}[e^{\beta(T-t)} \xi 1_{\{t \leq n\}} | \mathcal{F}_t] + \int_t^T e^{\beta(s-t)} \alpha_s ds \leq \mathbb{E}[e^{\beta(T-t)} \xi | \mathcal{F}_t] + \int_t^T e^{\beta(s-t)} \alpha_s ds \leq \bar{Y}_t.
\]
Set
\[
\tau_k = T \land \inf \{t \geq 0 : \bar{Y}_t > k\},
\]
and
\[
Y^n_k(t) = Y^n_{t \land \tau_k}, \quad Z^n_k(t) = Z^n_{t \land \tau_k}.
\]
Then $(Y^n_k, Z^n_k)$ satisfies
\[
Y^n_k(t) = Y^n_k(T) + \int_t^T 1_{s \leq \tau_k} (\alpha_s + \beta Y^n_k(s) + \gamma |Z^n_k(s)|) ds - \int_t^T Z^n_k(s) dW_s. \tag{2.4}
\]
For fixed $k$, $Y^n_k$ is nondecreasing with respect to $n$ and remains bounded by $k$. We can now apply the stability property of BSDE with bounded terminal data (see e.g. Lemma 3, page 611 in [2]). Setting $Y_k(t) = \sup_n Y^n_k(t)$, there exists $Z_k$ such that $\lim_n Z^n_k = Z_k$ in $M^2$ and
\[
Y_k(t) = \sup_n Y^n_k(t) + \int_t^{\tau_k} (\alpha_s + \beta Y_k(s) + \gamma |Z_k(s)|) ds - \int_t^{\tau_k} Z_k(s) dW_s. \tag{2.5}
\]
Finally, noting that
\[
Y_{k+1}(t \land \tau_k) = Y_k(t \land \tau_k), \quad Z_{k+1}1_{t \leq \tau_k} = Z_k1_{t \leq \tau_k},
\]
we conclude the existence of solution $(Y, Z)$.

\begin{remark}
Consider the case $d = 1$. If BSDE (2.2) admits a solution $(Y, Z)$ such that $Y \geq 0$, by taking $q = \gamma$ and $q = -\gamma$, we deduce that both $\xi e^{W_T}$ and $\xi e^{-W_T}$ are in $L^1(\Omega)$, which implies that $\xi e^{W_T} \in L^1(\Omega)$, as
\[
\xi e^{\gamma W_T} \leq \xi e^{W_T} + \xi e^{-W_T}.
\]
\end{remark}

\begin{example}
Let us set $d = 1$, $T = 1$, $\beta = 0$, $\gamma = 1$, $\mu \in (0, 1)$, and
\[
\xi = e^{|W_1^\mu + \frac{1}{2}\mu^2|} - 1.
\]

\end{example}
In this case, BSDE (2.2) does not admit a solution \((Y, Z)\) such that \(Y \geq 0\), as \(\xi e^{[W_1]}\) does not belong to \(L^1(\Omega)\) by the following direct calculus:

\[
\mathbb{E}[\xi e^{[W_1]}] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} (e^{\frac{1}{2}|x|^2 - \mu|x| + \frac{1}{2}\mu^2 - 1})e^{\frac{1}{2}|x|e^{-\frac{1}{2}|x|^2}}dx = +\infty.
\]

Whereas it is straightforward to see that for any \(p \geq 1\), \(\xi \log^p(\xi + 1) \in L^1(\Omega)\). For \(\lambda > 1\), consider the following terminal condition

\[
\xi = e^{\frac{1}{2}W_2 - \mu|W_1| + \frac{1}{2}\mu^2} - 1, \quad \mu \in (\frac{1}{\sqrt{\lambda}}, 1).
\]

We have \(\Psi_\lambda(\xi) \in L^1\) by the following straightforward calculus:

\[
\mathbb{E}[\Psi_\lambda(\xi)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} (e^{\frac{1}{2}|x|^2 - \mu|x| + \frac{1}{2}\mu^2 - 1})e^{\frac{1}{2}|x| - \mu|x|e^{-\frac{1}{2}|x|^2}}dx < +\infty,
\]

while BSDE (2.2) has no solution, in view of the fact that \(\xi e^{[W_1]}\) does not belong to \(L^1(\Omega)\).

### 3 Sufficient Condition

Let us now look for a sufficient condition for the existence of a locally bounded process \(Y\) such that

\[
\text{ess sup}_{q \in A}\{\mathbb{E}_q[e^{\beta(T-t)}\xi|\mathcal{F}_t]\} + \int_t^T e^{\beta(s-t)}\alpha_s ds \leq Y_t.
\]

For \(\lambda > 0\), define the functions \(\Phi_\lambda\) and \(\Psi_\lambda\):

\[
\Phi_\lambda(x) = e^{\frac{1}{2}\lambda}\log^2(x), \quad x > 0,
\]

\[
\Psi_\lambda(y) = ye^{\frac{\lambda}{2}\log(y+1)}/, \quad y \geq 0.
\]

Then we have

**Proposition 3.1** For any \(x \in \mathbb{R}\) and \(y \geq 0\), we have

\[
e^xy \leq \Phi_\lambda(e^x) + e^\frac{\lambda}{2}\Psi_\lambda(y).
\]

**Proof.** Set

\[
z = \left(\frac{2}{\lambda} \log(y + 1)\right)^{1/2} \geq 0,
\]

then

\[
y = e^{\frac{1}{2}z^2} - 1.
\]

It is sufficient to prove that for any \(x \in \mathbb{R}\) and \(z \geq 0\),

\[
e^{\frac{1}{2}\lambda z^2 - x} + \left(e^{\frac{1}{2}z^2} - 1\right)(e^{z+\frac{\lambda}{2}-x} - 1) \geq 0.
\]

It is evident to see that the above inequality holds when \(z + \frac{2}{\lambda} - x \geq 0\).

Consider the case \(z + \frac{2}{\lambda} - x < 0\). Then \(x > z + \frac{2}{\lambda} > 0\). Hence

\[
e^{\frac{1}{2}\lambda z^2 - x} + \left(e^{\frac{1}{2}z^2} - 1\right)(e^{z+\frac{\lambda}{2}-x} - 1)
\]

\[
= e^{\lambda(z+\frac{1}{\lambda})^2-\frac{1}{\pi\lambda}} + e^{\frac{1}{2}z^2 + z + \frac{\lambda}{2} - x} + 1 - e^{z+\frac{\lambda}{2}-x} - e^{\frac{1}{2}z^2}
\]

\[
\geq e^{\lambda(z+\frac{1}{\lambda})^2-\frac{1}{\pi\lambda}} - e^{\frac{1}{2}z^2}
\]

\[
\geq 0.
\]
Proposition 3.2 Let $0 < \lambda < \frac{1}{\gamma T}$. For any $q \in \mathcal{A}$,
\[
\mathbb{E}[\Phi_\lambda(e^\int_t^T q_s dW_s)|\mathcal{F}_t] \leq \frac{1}{\sqrt{1 - \lambda \gamma^2(T - t)}}.
\]

**Proof.** Firstly, by use of Girsanov’s lemma, for $\theta \in \mathbb{R}$,
\[
\mathbb{E}[\mathbb{E}[e^{\theta \int_t^T q_s dW_s}|\mathcal{F}_t] = \mathbb{E}[e^{\mathbb{E}[\mathbb{E}[e^{\theta \int_t^T q_s dW_s}|\mathcal{F}_t] - \frac{\theta}{2} \int_t^T |q_s|^2 ds} e^{\frac{\theta}{2} \int_t^T |q_s|^2 ds}|\mathcal{F}_t]
\leq e^{\frac{\theta^2}{2}(T - t)}.
\]

Then we apply
\[
e^{\frac{\lambda^2}{2}} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{\sqrt{\lambda y} - \frac{y^2}{2}} dy
\]
to deduce that
\[
\mathbb{E}[\Phi_\lambda(e^\int_t^T q_s dW_s)|\mathcal{F}_t] = \mathbb{E}[e^{\frac{\lambda}{2}(\int_t^T q_s dW_s)}|\mathcal{F}_t] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \mathbb{E}[e^{\sqrt{\lambda y} \int_t^T q_s dW_s - \frac{y^2}{2}}|\mathcal{F}_t] dy
\leq \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{\frac{(\sqrt{\lambda y})^2}{2}(T - t) - \frac{y^2}{2}} dy = \frac{1}{\sqrt{1 - \lambda \gamma^2(T - t)}}.
\]

Applying the above two propositions, we deduce the following sufficient condition.

**Theorem 3.3** Let us suppose that there exists $\lambda \in (0, \frac{1}{\gamma T})$ such that
\[
\mathbb{E}[\Psi_\lambda(\xi)] < +\infty.
\]
Then
\[
\text{ess sup}_{q \in \mathcal{A}} \{ \mathbb{E}[\mathbb{E}[e^{\beta(T - t)} \xi|\mathcal{F}_t]] + \int_t^T e^{\beta(s - t)} \alpha_s ds \leq \bar{Y}_t, \quad (3.6)
\]
with
\[
\bar{Y}_t = e^{\beta(T - t)} \left( \frac{1}{\sqrt{1 - \lambda \gamma^2(T - t)}} + e^{\frac{\lambda}{2} \mathbb{E}[\Psi_\lambda(\xi)|\mathcal{F}_t]} \right) + \int_t^T e^{\beta(s - t)} \alpha_s ds,
\]
and (2.2) admits a solution $(Y, Z)$ such that
\[
Y_t \leq \bar{Y}_t.
\]

**Proof.** Applying the above two propositions, we deduce
\[
\mathbb{E}[\mathbb{E}[\mathbb{E}[e^{\beta(T - t)} \xi|\mathcal{F}_t]] + \int_t^T e^{\beta(s - t)} \alpha_s ds \leq \bar{Y}_t,
\]

Then we get (3.6) and the rest follows from Theorem 2.1.
4 General Case

Consider the following BSDE:

\[ Y_t = \xi + \int_t^T f(s, Y_s, Z_s) ds - \int_t^T Z_s dW_s, \]

(4.7)

where \( f \) satisfies

\[ |f(s, y, z)| \leq \alpha_s + \beta|y| + \gamma|z|, \]

(4.8)

with \( \alpha \in L^1(0, T), \beta \geq 0 \) and \( \gamma > 0 \).

**Theorem 4.1** Let \( f \) be a generator which is continuous with respect to \((y, z)\) and verifies (4.8), and \( \xi \) be a terminal condition. Let us suppose that there exists \( \lambda \in (0, \frac{1}{\gamma^2}) \) such that

\[ \mathbb{E}[\Psi_\lambda(|\xi|)] < +\infty. \]

Then BSDE (4.7) admits a solution \((Y, Z)\) such that

\[ |Y_t| \leq e^{\beta(T-t)} \left( \frac{1}{\sqrt{1 - \lambda \gamma^2(T-t)}} + e^{\frac{2}{\lambda} \mathbb{E}[\Psi_\lambda(|\xi|)] \mathbb{E}[F_t]} \right) + \int_t^T e^{\beta(s-t)} \alpha_s ds. \]

**Proof.** Let us fix \( n \in \mathbb{N}^* \) and \( p \in \mathbb{N}^* \) and set \( \xi_{n,p} = \xi^+ \wedge n - \xi^- \wedge p \). Let \((Y_{n,p}, Z_{n,p})\) be the unique solution in \( S^2 \times M^2 \) of the BSDE \((|\xi_{n,p}|, f)\). Set

\[ \tilde{f}(s, y, z) = \alpha_s + \beta y + \gamma|z|, \]

and \((\tilde{Y}_{n,p}, \tilde{Z}_{n,p})\) be the unique solution in \( S^2 \times M^2 \) of the BSDE \((|\xi_{n,p}|, \tilde{f})\).

By comparison theorem,

\[ |Y_{n,p}^i| \leq |\tilde{Y}_{n,p}^i|. \]

Setting \( q_{n,p} = \gamma \text{ sgn}(Z_{n,p}) \), we obtain,

\[
\begin{align*}
|Y_{n,p}^i| &\leq |\tilde{Y}_{n,p}^i| \\
&= E_{q_{n,p}} \left[ e^{\beta(T-t)} |\xi_{n,p}| \right] F_t + \int_t^T e^{\beta(s-t)} \alpha_s ds.
\end{align*}
\]

From inequality (3.6),

\[ |Y_{n,p}^i| \leq \tilde{Y}_t, \]

with

\[ \tilde{Y}_t = e^{\beta(T-t)} \left( \frac{1}{\sqrt{1 - \lambda \gamma^2(T-t)}} + e^{\frac{2}{\lambda} \mathbb{E}[\Psi_\lambda(|\xi|)] \mathbb{E}[F_t]} \right) + \int_t^T e^{\beta(s-t)} \alpha_s ds. \]

Moreover, \( Y_{n,p}^i \) is nondecreasing with respect to \( n \) and nonincreasing with respect to \( p \). Once again, we apply the localization method as follows to conclude the existence of solution.

Set

\[ \tau_k = T \wedge \inf\{t \geq 0 : \tilde{Y}_t > k\}, \]

and

\[ Y_{k,n,p}^i(t) = Y_{n,p}^{i \wedge \tau_k}, \quad Z_{k,n,p}^i(t) = Z_{n,p}^i 1_{t \leq \tau_k}. \]

Then \((Y_{k,n,p}^i, Z_{k,n,p}^i)\) satisfies

\[ Y_{k,n,p}^i(t) = Y_{k,n,p}^i(T) + \int_t^T 1_{s \leq \tau_k} f(s, Y_{k,n,p}^i(s), Z_{k,n,p}^i(s)) ds - \int_t^T Z_{k,n,p}^i(s) dW_s, \]

(4.9)

\[ 7 \]
For fixed \( k \), \( Y_{n,p}^k \) is nondecreasing with respect to \( n \) and nonincreasing with respect to \( p \), and remains bounded by \( k \). We can now apply the stability property of BSDEs with bounded terminal data. Setting \( Y_k(t) = \inf_p \sup_n Y_{n,p}^k \), there exists \( Z_k \) in \( \mathcal{M}^2 \) such that \( \lim_p \lim_n Z_{n,p}^k = Z_k \) in \( \mathcal{M}^2 \) and

\[
Y_k(t) = \inf_p \sup_n Y_{n,p}^k + \int_t^{\tau_k} f(s, Y_k(s), Z_k(s)) ds - \int_t^{\tau_k} Z_k(s) dW_s. \tag{4.10}
\]

Finally, noting that

\[
Y_{k+1}(t \wedge \tau_k) = Y_k(t \wedge \tau_k), \quad Z_{k+1}1_{t \leq \tau_k} = Z_k1_{t \leq \tau_k},
\]

we conclude the existence of solution \( (Y, Z) \).

\[
\square
\]

References

[1] P. Briand, B. Delyon, Y. Hu, E. Pardoux and L. Stoica, \( L^p \) solutions of backward stochastic differential equations. *Stochastic Process. Appl.* 108 (2003), no. 1, 109-129.

[2] P. Briand and Y. Hu, BSDE with quadratic growth and unbounded terminal value. *Probab. Theory Related Fields* 136 (2006), no. 4, 604-618.

[3] F. Delbaen, Y. Hu and A. Richou, On the uniqueness of solutions to quadratic BSDEs with convex generators and unbounded terminal conditions. *Ann. Inst. Henri Poincaré Probab. Stat.* 47 (2011), no. 2, 559-574.

[4] N. El Karoui, S. Peng and M. C. Quenez, Backward stochastic differential equations in finance. *Math. Finance* 7 (1997), no. 1, 1-71.

[5] E. Pardoux and S. Peng, Adapted solution of a backward stochastic differential equation. *Systems Control Lett.* 14 (1990), no. 1, 55-61.

[6] S. Tang, Dual representation as stochastic differential games of backward stochastic differential equations and dynamic evaluations. *C. R. Math. Acad. Sci. Paris* 342 (2006), no. 10, 773-778.