Algebraic Integration of Sigma Model Field Equations

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Abstract

We prove that the dualization algebra of the symmetric space coset sigma model is a Lie algebra and we show that it generates an appropriate adjoint representation which enables the local integration of the field equations yielding the first-order ones.

1 Introduction

The field content of the symmetric space coset sigma model consists of scalar fields that parametrize the target manifold which is a homogeneous and a Riemannian globally symmetric space. By doubling the field content via the introduction of higher order dual fields one can realize the theory through the construction of an enlarged coset. The origin of this method lies in the dualization of supergravity theories [1, 2] whose scalar sectors correspond to above mentioned type of sigma models. The most important element of this enlarged realization of mentioned theories is the construction of the dualized coset parametrizing algebra which for the case of the pure sigma model is a deformation of the original coset algebra. Although the geometrical construction of this extended formulation is not yet well known the dualized
algebra for a general coset sigma model with a symmetric target space is derived in \([3, 4]\). In these works the first-order field equations of the theory are also obtained as consistency conditions embedded in the method of dualization. However either of the works lack the direct algebraic connection of these first-order equations with the second-order ones which arise from the least action principle.

In this work we present a rigorous proof which shows that the dualized coset algebra obtained in \([3, 4]\) is indeed a Lie algebra. The main perspective of our proof will be to show that in the most general terms (for an arbitrary sigma model) the commutation relations of the dualized algebra which comes out to be a deformation of the ordinary coset algebra satisfies the Jacobi identities. Following this we will also discuss that being a Lie algebra the dualized coset algebra admits a natural adjoint representation for the original coset algebra which is a Lie subalgebra of the former. Finally we show that when one assumes this natural adjoint representation generated by the dualized algebra one can locally relate the first-order equations derived in \([3, 4]\) to the second-order field equations algebraically. Namely starting from the first-order equation anzats by applying an exterior derivative we will show that one obtains the second-order field equations under the special representation generated by the dualized coset algebra.

Section two which is a rather formal one inspects all the possible conditions of a generic dualized coset algebra thus it presents a complete proof of the Lie algebra structure of it for an arbitrary coset sigma model. Section three discusses the natural adjoint representation of the original coset algebra suggested within this scheme. The last section proves that the first-order equations which appear in \([3, 4]\) are indeed the true local ones which can be obtained from the second-order field equations of the theory by locally abolishing an exterior derivative when one chooses the above-mentioned particular representation.

2 The Dualized Coset Algebra

The dualized coset algebra of a generic symmetric space sigma model is derived in \([3, 4]\). It is generated by the set of generators

\[
\{H_i, E_\alpha, \tilde{H}_i, \tilde{E}_\alpha\}, \tag{2.1}
\]
where the first two set of generators correspond to a subset of the Cartan-Weyl basis of the global symmetry group of the sigma model Lagrangian which generate the solvable Lie subalgebra $s$. Here for $i = 1, \cdots, r$ the generators $H_i$ form a subset of the Cartan generators of the Lie algebra of the global symmetry group of the sigma model and $E_\alpha$ generate the root subspaces of the non-compact positive roots $\Delta_{nc}^+$. The last two set of generators are the duals of the former. The commutation relations of the dualized coset algebra that is generated by (2.1) can be given as

$$[H_i, H_j] = [H_i, \tilde{H}_j] = [E_\alpha, \tilde{H}_j] = [\tilde{E}_\alpha, \tilde{H}_j] = [\tilde{E}_\alpha, \tilde{E}_\beta] = 0,$$

$$[H_i, E_\alpha] = \alpha_i E_\alpha, \quad [E_\alpha, E_\beta] = N_{\alpha,\beta} E_{\alpha+\beta} \quad \text{if} \quad \alpha + \beta \in \Delta,$$

$$[E_\alpha, E_\beta] = 0 \quad \text{if} \quad \alpha + \beta \notin \Delta,$$

$$[E_\alpha, \tilde{E}_\alpha] = \frac{1}{4} \sum_{j=1}^{r} \alpha_j \tilde{H}_j, \quad [H_j, \tilde{E}_\alpha] = -\alpha_j \tilde{E}_\alpha,$$

$$[E_\alpha, \tilde{E}_\beta] = 0 \quad \text{if} \quad \alpha - \beta \notin \Delta,$$

or $\alpha - \beta \in \Delta$ but $\beta - \alpha \notin \Delta_{nc}^+$,

$$[E_\alpha, \tilde{E}_\beta] = N_{\alpha,-\beta} \tilde{E}_\gamma \quad \text{if} \quad \alpha - \beta \in \Delta,$$

$$\beta - \alpha \in \Delta_{nc}^+, \quad \text{and} \quad \alpha - \beta = -\gamma. \quad (2.2)$$

Here $\Delta$ corresponds to the roots of the Lie algebra of the global symmetry group of the sigma model. $\alpha_i$ are the root vector components and the real coefficients $N_{\alpha,\beta}$ are the structure constants corresponding to the commutation relations of the root subspace generators $E_\alpha$. We should remark that if $\alpha$ and $\beta$ are noncompact positive roots and if $\alpha + \beta \in \Delta$ then $\alpha + \beta$ must also be a noncompact positive root since if it is not then

$$[E_\alpha, E_\beta] = N_{\alpha,\beta} E_{\alpha+\beta} \notin s, \quad (2.3)$$
which causes a contradiction for the closure of the solvable Lie subalgebra. Now we will introduce the notation
\[
\{T_m\} \equiv \{H_i, E_\alpha\}, \quad \tilde{T}_m \equiv \{	ilde{H}_i, \tilde{E}_\alpha\},
\] (2.4)
so that the index \( m \) is split into two sets
\[
m = \underbrace{i, j, k, \ldots}_{\alpha, \beta, \gamma, \ldots}, \quad \underbrace{r, r + 1, r + 2, \ldots}_{\alpha, \beta, \gamma, \ldots}, \quad \text{dims}.
\]
In other words
\[
T_1 = H_1, \quad T_2 = H_2, \quad \ldots, \quad T_r = H_r,
\]
\[
\tilde{T}_1 = \tilde{H}_1, \quad \tilde{T}_2 = \tilde{H}_2, \quad \ldots, \quad \tilde{T}_r = \tilde{H}_r,
\]
\[
T_{r+1} = E_\alpha, \quad T_{r+2} = E_\beta, \quad \ldots,
\]
\[
\tilde{T}_{r+1} = \tilde{E}_\alpha, \quad \tilde{T}_{r+2} = \tilde{E}_\beta, \quad \ldots.
\] (2.5)
The commutation relations in (2.2) can more compactly be written as
\[
[H_i, H_j] = 0, \quad [H_i, E_\alpha] = \alpha_i E_\alpha, \quad [E_\alpha, E_\beta] = 0 \quad \text{if} \quad \alpha + \beta \notin \Delta,
\]
\[
[E_\alpha, E_\beta] = N_{\alpha, \beta} E_{\alpha + \beta} \quad \text{if} \quad \alpha + \beta \in \Delta,
\]
\[
[\tilde{T}_m, \tilde{T}_n] = 0, \quad [E_\gamma, \tilde{T}_m] = \tilde{f}_{\gamma m}^{n} \tilde{T}_n, \quad [H_j, \tilde{T}_m] = \tilde{g}_{jm}^{n} \tilde{T}_n,
\] (2.6)
where the structure constant matrices \( \tilde{f}_\gamma \) and \( \tilde{g}_j \) in partitioned form can be given as
\[
\tilde{f}_\gamma = \begin{pmatrix}
0 & 0 & \cdots & 0 \\
0 & \tilde{f}^{1}_{\gamma} & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & R_{\gamma}
\end{pmatrix}, \quad \tilde{g}_j = \begin{pmatrix}
0 & 0 & \cdots & 0 \\
0 & \tilde{g}^{1}_{j} & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & R_{\gamma}
\end{pmatrix},
\] (2.7)
\[ \tilde{g}_j = - \left( \begin{array}{c|c|c} \begin{array}{c} 0 \\ \hline 0 \\ \hline \alpha_j \\ \hline \beta_j \\ \hline \end{array} & 0 & \end{array} \right) \left( \begin{array}{c} \begin{array}{c} 0 \\ \hline \alpha \end{array} \\ \hline \beta \\ \hline \end{array} \right) \right)_{i=1,2,\ldots,r}^{i=1,2,\ldots,r}, \quad (2.8) \]

where we have defined

\[ R_\gamma = \left( \begin{array}{ccccccc} \alpha & \beta & \cdots & \kappa & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots \\ \hline 0 & \cdots & 0 & (\tau, \kappa := \gamma) & 0 & \cdots & 0 \\ \hline 0 & \cdots & 0 & \cdots & \cdots & \cdots & \cdots \\ \hline 0 & \cdots & 0 & \cdots & \cdots & \cdots & \cdots \\ \hline 0 & \cdots & 0 & \cdots & \cdots & \cdots & \cdots \\ \hline 0 & \cdots & 0 & \cdots & \cdots & \cdots & \cdots \\ \end{array} \right) \right)_{\alpha \beta \cdots \gamma \cdots \tau}^{\alpha \beta \cdots \tau}, \quad (2.9) \]

Here the non-zero entry in a certain column and a row is defined as

\[ (\tau, \kappa := \gamma) = N_{\gamma,-\kappa} \text{ if } \gamma - \kappa = -\tau, \quad \text{otherwise } (\tau, \kappa := \gamma) = 0. \quad (2.10) \]

In general

\[ (R_\gamma)^\alpha_\beta = N_{\gamma,-\beta} \text{ if } \gamma - \beta = -\alpha, \]

\[ (R_\gamma)^\alpha_\beta = 0 \text{ if } \gamma - \beta \neq -\alpha. \quad (2.11) \]

Before going further we will state some facts about the matrix \( R_\alpha \);

- if for a fixed column \( \beta, \alpha - \beta \notin \Delta \), or \( \alpha - \beta \in \Delta \) but for none of the rows \( \gamma, \alpha - \beta \neq -\gamma \) then the column \( \beta \) has all null entries,

- if again for a fixed column \( \beta, \alpha - \beta \in \Delta \), and if \( \alpha - \beta = -\gamma \) and \( \alpha - \beta = -\tau \Rightarrow \gamma = \tau \), thus at a column \( \beta \) if all the entries are not null then there is a single non-zero entry which is \( N_{\alpha,-\beta} \).
• if for a fixed row \( \gamma \), \( \nexists \) a column \( \beta \) such that \( \alpha - \beta = -\gamma \) then all the entries in \( \gamma \) are null. In other words if \( \alpha + \gamma \notin \Delta \) then \( \nexists \beta \) such that \( \alpha - \beta = -\gamma \) and the \( \gamma \) row is null. However if \( \alpha + \gamma \in \Delta \) then from our previous discussion \( \alpha + \gamma \in \Delta \) so \( \exists \) a column \( \beta \) such that \( \alpha - \beta = -\gamma \) and the row \( \gamma \) has a non-zero entry;

• if for a fixed row \( \gamma \), \( \exists \) two columns \( \beta \) and \( \tau \) such that \( \alpha - \beta = -\gamma \) and \( \alpha - \tau = -\gamma \Rightarrow \beta = \tau \), thus at a row \( \gamma \) if all the entries are not null then there is a single non-zero entry which is \( N_{\alpha,-\beta} \),

• we should state that the first and the third items are consistent that is to say if \( \alpha - \beta \neq -\gamma \) then this is valid from either the column or the row point of view,

• for the diagonal elements \( \gamma = \beta \) thus the condition \( \alpha - \beta = -\gamma \) implies that \( \alpha = \beta - \gamma = 0 \notin \Delta \). However since \( \alpha \in \Delta \) the condition \( \alpha - \beta = -\gamma \) can not be held for the diagonal elements therefore the diagonal elements must all be zero,

• if \( \alpha - \beta \notin \Delta \), or \( \alpha - \beta \in \Delta \) but \( \alpha - \beta \neq -\gamma \) for any \( \gamma \in \Delta_{nc}^+ \) for \( n \) times as \( \beta \) runs over \( \Delta_{nc}^+ \) then there are \( n \) zero columns. Also since there is a unique non-zero entry at columns and rows there must be a total number of \( \text{dim}\Delta_{nc}^+ - n \) non-zero entries in \( \text{dim}\Delta_{nc}^+ - n \) distinct columns and rows which denotes that there must also be \( n \) zero rows,

• on the other hand if \( \alpha + \gamma \notin \Delta \) for \( n \) rows as \( \gamma \) runs over \( \Delta_{nc}^+ \) then there are \( n \) zero rows. Upon the reasoning given in the previous item there are also \( n \) zero columns. Due to the consistency of the row and the column point of views these last two items are also consistent.

Although as a result of a standard dualization method the structure constants of the algebra (2.2) are derived in [3] and [4] it is not proven in either of these works that the algebra defined in (2.2) which can be called the dualization deformation of the coset algebra \( \Gamma \) of the sigma model at hand forms a Lie algebra. Therefore in this section we will prove that the algebra defined in (2.2) indeed is a Lie algebra. In general an \( m \)-dimensional Lie algebra is generated by \( m \) generators \( X_a \) such that

\[
[X_a, X_b] = C_{ab}^c X_c,
\]

(2.12)

\(^1\)Which is the solvable Lie subalgebra of the global symmetry group of the sigma model.
with $C_{ab}^c = -C_{ba}^c$ and
\[ [X_a, X] = 0. \quad (2.13) \]
The generators must also satisfy the Jacobi identities
\[ [X_a, [X_b, X_c]] + [X_b, [X_c, X_a]] + [X_c, [X_a, X_b]] = 0. \quad (2.14) \]

Thus our task is to show that the generators (2.1) whose structure constants are defined in (2.2) satisfy the Jacobi identities (2.14). At first glance if we choose $X_a = T_m$, $X_b = T_n$, $X_c = T_l$ from (2.6) we get
\[ [T_m, 0] + [T_n, 0] + [T_l, 0] = 0, \quad (2.15) \]
which is also satisfied. If we let $X_a = T_l$, $X_b = T_m$, $X_c = T_n$ then again from (2.6) we have
\[ [T_l, 0] - U_{ln}^t [T_m, T_l] + U_{lm}^t [T_n, T_l] = 0, \quad (2.16) \]
which is instantly satisfied due to the commutation of the dual generators.

In writing (2.16) we have defined
\[ [T_l, T_m] = U_{lm}^t T_t, \quad (2.17) \]
where the structure constants $U_{lm}^t$ can be read from (2.6). Now let us consider $X_a = T_l$, $X_b = T_n$, $X_c = T_m$. In this case from (2.14) after some algebra we find
\[ Z_{ln}^t U_{lm}^s = (U_l U_n - U_n U_l)^s_m, \quad (2.18) \]
where we have defined
\[ [T_l, T_n] = Z_{ln}^t T_t, \quad (2.19) \]
and we have introduced the structure constant matrices $(U_l)^s_m = U_{lm}^s$. Now we will prove that for the three distinct cases:

1. $T_l = H_j$, $T_n = H_k$,
2. $T_l = H_j$, $T_n = E_\gamma$,
3. $T_l = E_\gamma$, $T_n = E_x$, 


(2.18) is satisfied. For the first case \([H_j, H_k] = 0\) thus the LHS of (2.18) vanishes. From (2.17) and (2.6) the RHS of (2.18) becomes

\[
(\tilde{g}_j\tilde{g}_k - \tilde{g}_k\tilde{g}_j)^s_m.
\]

However from (2.8) we have

\[
\tilde{g}_j\tilde{g}_k = \begin{pmatrix}
\alpha_{ij} & \beta_{ij} \\
\beta_{ij} & \gamma_{ij}
\end{pmatrix},
\]

and similarly for \(\tilde{g}_k\tilde{g}_j\). Thus inserting these in (2.20) we see that the RHS of (2.18) also vanishes. Now for the second case (2.18) becomes

\[
Z^{t \gamma}_{j \gamma} U_{\gamma m} = (U_j U_{\gamma} - U_{\gamma} U_j)^s_m.
\]

After using the identifications

\[
Z^{i \gamma}_{j \gamma} = 0, \quad Z^{\beta \gamma}_{j \gamma} = 0 \quad \text{if} \quad \beta \neq \gamma,
\]

in (2.22) the matrix equality to be proven becomes

\[
\gamma_j \tilde{f}_{\gamma} = \tilde{g}_j \tilde{f}_{\gamma} - \tilde{f}_{\gamma} \tilde{g}_j.
\]

The LHS is

\[
\gamma_j \tilde{f}_{\gamma} = \begin{pmatrix}
\alpha_{ij} & \gamma_{ij} \\
\gamma_{ij} & R_{\gamma}
\end{pmatrix},
\]

and similarly for \(\tilde{g}_k\tilde{g}_j\). Thus inserting these in (2.20) we see that the RHS of (2.18) also vanishes. Now for the second case (2.18) becomes

\[
Z^{t \gamma}_{j \gamma} U_{\gamma m} = (U_j U_{\gamma} - U_{\gamma} U_j)^s_m.
\]

After using the identifications

\[
Z^{i \gamma}_{j \gamma} = 0, \quad Z^{\beta \gamma}_{j \gamma} = 0 \quad \text{if} \quad \beta \neq \gamma,
\]

in (2.22) the matrix equality to be proven becomes

\[
\gamma_j \tilde{f}_{\gamma} = \tilde{g}_j \tilde{f}_{\gamma} - \tilde{f}_{\gamma} \tilde{g}_j.
\]

The LHS is

\[
\gamma_j \tilde{f}_{\gamma} = \begin{pmatrix}
\alpha_{ij} & \gamma_{ij} \\
\gamma_{ij} & R_{\gamma}
\end{pmatrix},
\]
where
\[
\gamma_j R_{\gamma} = \begin{pmatrix}
\alpha & \beta & \cdots & \kappa & \cdots & \cdots & \cdots \\
0 & 0 & \cdots & \cdots & \cdots & \cdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots \\
0 & \cdots & 0 & (\tau, \kappa := \gamma) \gamma_j & 0 & \cdots & 0 \\
0 & \cdots & 0 & \cdots & \ddots & \ddots & \ddots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
0 & \cdots & 0 & 0 & \cdots & \ddots & \ddots \\
\cdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
0 & \cdots & 0 & 0 & \cdots & \ddots & \ddots \\
\cdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
\end{pmatrix}
\]
(2.26)

To calculate the RHS from (2.7) and (2.8) we first find that
\[
\tilde{g}_j \tilde{f}_{\gamma} = -\begin{pmatrix}
\begin{pmatrix}
0 \\
\vdots \\
0
\end{pmatrix}
\end{pmatrix}
\begin{pmatrix}
\begin{pmatrix}
0 \\
\vdots \\
0
\end{pmatrix}
\end{pmatrix}
\begin{pmatrix}
\begin{pmatrix}
0 \\
\vdots \\
0
\end{pmatrix}
\end{pmatrix}
\begin{pmatrix}
\begin{pmatrix}
0 \\
\vdots \\
0
\end{pmatrix}
\end{pmatrix}
\]
(2.27)

and
\[
\tilde{f}_{\gamma} \tilde{g}_j = \begin{pmatrix}
\begin{pmatrix}
0 \\
\vdots \\
0
\end{pmatrix}
\end{pmatrix}
\begin{pmatrix}
\begin{pmatrix}
0 \\
\vdots \\
0
\end{pmatrix}
\end{pmatrix}
\begin{pmatrix}
\begin{pmatrix}
0 \\
\vdots \\
0
\end{pmatrix}
\end{pmatrix}
\begin{pmatrix}
\begin{pmatrix}
0 \\
\vdots \\
0
\end{pmatrix}
\end{pmatrix}
\]
(2.28)
where we have introduced the matrices

$$R_{\gamma j} = \begin{pmatrix} \alpha & \beta & \ldots & \kappa & \ldots & \ldots & \alpha \\ 0 & 0 & \ldots & 0 & \ldots & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots & \ddots & \ddots & \vdots \\ 0 & \ldots & 0 & -\tau_{j, k} := \gamma & 0 & \ldots & 0 \\ 0 & \ldots & 0 & 0 & \ddots & \ddots & \ddots \\ \vdots & \vdots & \ddots & \vdots & \ddots & \ddots & \ddots \\ 0 & \ldots & 0 & 0 & \ldots & 0 & 0 \end{pmatrix}, \quad (2.29)$$

also

$$R'_{\gamma j} = \begin{pmatrix} \alpha & \beta & \ldots & \kappa & \ldots & \ldots & \alpha \\ 0 & 0 & \ldots & 0 & \ldots & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots & \ddots & \ddots & \vdots \\ 0 & \ldots & 0 & -\tau_{j, k} := \gamma & 0 & \ldots & 0 \\ 0 & \ldots & 0 & 0 & \ddots & \ddots & \ddots \\ \vdots & \vdots & \ddots & \vdots & \ddots & \ddots & \ddots \\ 0 & \ldots & 0 & 0 & \ldots & 0 & 0 \end{pmatrix}, \quad (2.30)$$

whose non-zero entries coincide in column and row with the non-zero entries of (2.26). Therefore the RHS of (2.24) becomes

$$\tilde{g}_{j} \tilde{f}_{\gamma} - \tilde{f}_{\gamma} \tilde{g}_{j} = \begin{pmatrix} \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix} & \begin{pmatrix} \frac{1}{4} \gamma_{1} \gamma_{j} \\ \vdots \\ \frac{1}{4} \gamma_{r} \gamma_{j} \end{pmatrix} \end{pmatrix}^{i=1,2,\ldots,r}_{\begin{pmatrix} \frac{1}{4} \gamma_{1} \gamma_{j} \\ \vdots \\ \frac{1}{4} \gamma_{r} \gamma_{j} \end{pmatrix}^{\alpha,\beta,\ldots,\gamma,\ldots}_{\alpha,\beta,\ldots,\gamma,\ldots}}$$

$$R_{\gamma j} - R'_{\gamma j}$$

(2.31)
We see that from (2.25) and (2.31) the three block-matrices except the one at the lower-rightmost corner are obviously equal to each other. On the other hand since the non-zero entries of the matrices (2.26), (2.29) and (2.30) coincide we must show that these entries in the lower-rightmost block-matrices on the LHS and the RHS of (2.24) are equal. The non-zero entries of $R_{\gamma j} - R'_{\gamma j}$ at the row $\tau$ and the column $\kappa$ are

\[-(\tau, \kappa := \gamma) \tau_j + (\tau, \kappa := \gamma) \kappa_j = -N_{\gamma, -\kappa} \tau_j + N_{\gamma, -\kappa} \kappa_j = N_{\gamma, -\kappa} (\kappa_j - \tau_j)\]  

(2.32)

However from (2.10) a non-zero entry exists if and only if $\gamma - \kappa = -\tau$. This root condition is also valid for the root vector components and we have $\kappa_j - \tau_j = \gamma_j$ thus the non-zero entries of $R_{\gamma j} - R'_{\gamma j}$ at the row $\tau$ and the column $\kappa$ become

\[N_{\gamma, -\kappa} \gamma_j = (\tau, \kappa := \gamma) \gamma_j,\]  

(2.33)

which are equal to the non-zero entries in (2.26) which is the LHS lower-rightmost matrix. Finally for the third case of $T_l = E_\gamma$, and $T_n = E_\lambda$ (2.18) yields the matrix equality to be proven

\[N_{\gamma \lambda} \tilde{f}_{\gamma + \lambda} = \tilde{f}_\gamma \tilde{f}_\lambda - \tilde{f}_\lambda \tilde{f}_\gamma.\]  

(2.34)

In order to obtain this we have used that

\[Z^i_{\gamma \lambda} = 0, \quad Z^\beta_{\gamma \lambda} = 0 \quad \text{if} \quad \gamma + \lambda \neq \beta;\]

\[Z^{\gamma + \lambda}_{\gamma \lambda} = N_{\gamma \lambda}, \quad U_\gamma = \tilde{f}_\gamma.\]  

(2.35)

We should state that if $\gamma + \lambda$ is not a root then the LHS is zero. If it is a root then as we have discussed before it must be in $\Delta_{nc}^+$ and in this case the LHS of (2.34) becomes

\[N_{\gamma \lambda} \tilde{f}_{\gamma + \lambda} = \begin{pmatrix} \begin{pmatrix} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{pmatrix} & \begin{pmatrix} \frac{1}{4} N_{\gamma \lambda} (\gamma_1 + \lambda_1) \\ \vdots \\ \frac{1}{4} N_{\gamma \lambda} (\gamma_r + \lambda_r) \end{pmatrix} \end{pmatrix} \begin{pmatrix} \tilde{f}_{\gamma + \lambda} \\ \tilde{f}_\gamma \\ \cdots \\ \tilde{f}_\lambda \end{pmatrix} = \begin{pmatrix} \begin{pmatrix} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{pmatrix} & \begin{pmatrix} \frac{1}{4} N_{\gamma \lambda} R_{\gamma + \lambda} \end{pmatrix} \end{pmatrix} \begin{pmatrix} \tilde{f}_{\gamma + \lambda} \\ \tilde{f}_\gamma \\ \cdots \\ \tilde{f}_\lambda \end{pmatrix}.\]  

(2.36)
where
\[
R_{\gamma+\lambda} = \begin{pmatrix}
\alpha & \beta & \cdots & \kappa & \cdots & \cdots \\
0 & 0 & \cdots & 0 & \cdots & \cdots \\
0 & \cdots & \cdots & \cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & \cdots & \cdots & \cdots & \cdots & \cdots \\
\end{pmatrix}
\]
(2.37)

Now after a non-straightforward matrix multiplication by using (2.7) we find that
\[
\tilde{f}_\gamma \tilde{f}_\lambda = \begin{pmatrix}
\begin{pmatrix}
0 & \cdots & \cdots & \cdots \\
\end{pmatrix}
\end{pmatrix}
\begin{pmatrix}
\begin{pmatrix}
\frac{n_1}{4}(\gamma, \theta := \lambda) & 0 & \cdots & \cdots \\
0 & \frac{n_2}{4}(\gamma, \theta := \lambda) & \cdots & \cdots \\
\vdots & \ddots & \ddots & \ddots \\
0 & \cdots & \cdots & \frac{n_r}{4}(\gamma, \theta := \lambda) \\
\end{pmatrix}
\end{pmatrix}
\begin{pmatrix}
\begin{pmatrix}
0 & \cdots & \cdots \\
\end{pmatrix}
\end{pmatrix}
\begin{pmatrix}
\begin{pmatrix}
\alpha & \beta & \cdots \\
\end{pmatrix}
\end{pmatrix}
\]
(2.38)

where \( \theta = \gamma + \lambda \) and if \( \gamma + \lambda \) is not a root then the upper-rightmost block matrix in (2.38) is zero since in this case there would be no column \( \theta \) in \( R_\lambda \) which would satisfy \( \theta = \gamma + \lambda \) so that the \( \gamma \) row of \( R_\lambda \) would be composed of all zero elements. In (2.38) we have defined
\[
R_{\gamma\lambda} = R_\gamma R_\lambda,
\]
(2.39)
which through (2.9) can explicitly be calculated as

\[
\begin{pmatrix}
\alpha & \beta & \cdots & \cdots & \kappa & 0 & \cdots & 0 \\
0 & 0 & \cdots & \cdots & 0 & \cdots & \cdots & 0 \\
& & & & & & & \\
& & & & & & & \\
& & & & & & & \\
& & & & & & & \\
& & & & & & & \\
& & & & & & & \\
0 & \cdots & 0 & 0 & (v, \kappa := \lambda)(\tau, v := \gamma) & 0 & \cdots & 0 \\
\end{pmatrix}
\]

\(\tau\)

\(R_{\gamma\lambda} =\)

In calculating \(R_{\gamma\lambda}\) we have efficiently made use of the properties of \(R_\alpha\) which we have itemized before. Apart from the values of its non-zero entries and where they are the matrix (2.40) also obeys all the characteristics of \(R_\alpha\). That is to say its diagonal elements are zero, it may have columns or rows whose elements are all zero but their multiplicity must be equal, only one non-zero element can exist in a row also only one non-zero element can exist in a column. The orientation of an entry at a column \(\kappa\) in (2.40) can be found as follows; one first solves the non-zero entry condition \(\lambda - \kappa = -v\) of \(R_\lambda\) for \(v\) then one solves the row \(\tau\) from the non-zero entry condition \(\gamma - v = -\tau\) of \(R_\gamma\). This is because in the matrix multiplication in (2.39) we multiply the unique non-zero element in the column \(\kappa\) which is at the row \(v\) of \(R_\lambda\) with the unique non-zero element in the column \(v\) which is at the row \(\tau\) of \(R_\gamma\) and write it in the column \(\kappa\) and the row \(\tau\) of \(R_{\gamma\lambda}\). Of course if the condition \(\lambda - \kappa = -v\) is not satisfied for any \(v\) then this generates a pair of a column and a row which are both null in \(R_\lambda\) and the \(\kappa\) column of \(R_{\gamma\lambda}\) would be null too. Also separately if the condition \(\gamma - v = -\tau\) is not satisfied for any \(\tau\) then this generates a pair of a column and a row which are also null in \(R_\gamma\) and the \(\kappa\) column of \(R_{\gamma\lambda}\) would again be null. These two cases may coexist. In addition in either of these cases there would also be a completely null row.
in $R_{\gamma \lambda}$. Now we can write the RHS of (2.34) as

$$\tilde{f}_\gamma \tilde{f}_\lambda - \tilde{f}_\lambda \tilde{f}_\gamma = \begin{pmatrix} \left( \begin{array}{cccc} 0 & & & \\ & \ddots & & \\ & & 0 & \\ \end{array} \right) & \left( \begin{array}{cccc} \gamma_1(\gamma,\theta:=\lambda) - \lambda_1(\lambda,\theta:=\gamma) \\ \vdots \\ \gamma_r(\gamma,\theta:=\lambda) - \lambda_r(\lambda,\theta:=\gamma) \\ \end{array} \right) \end{pmatrix}^{i \parallel i = 1, 2, \ldots, r}_{\alpha, \beta, \ldots, \alpha, \beta, \ldots} - \begin{pmatrix} \left( \begin{array}{cccc} 0 & & & \\ & \ddots & & \\ & & 0 & \\ \end{array} \right) & \left( \begin{array}{cccc} R_{\gamma \lambda} & & & \\ & \ddots & & \\ & & R_{\lambda \gamma} & \\ \end{array} \right) \end{pmatrix}^{i \parallel i = 1, 2, \ldots, r}_{\alpha, \beta, \ldots, \alpha, \beta, \ldots}$$

(2.41)

Before going further we should show that the non-zero entries of the matrices $R_{\gamma \lambda}$ and $R_{\lambda \gamma}$ indeed coincide. This can be seen as follows; if there is a non-zero entry at a row $\tau$ in $R_{\gamma \lambda}$ this means that

$$\lambda - \kappa = -v, \quad \gamma - v = -\tau,$$

(2.42)

if one adds these two root conditions side by side one finds that the non-zero entry must be at the column $\kappa = \gamma + \lambda + \tau$ which is certainly an element of $\Delta^+_n$; also if there is a non-zero entry at a row $\tau$ in $R_{\lambda \gamma}$ then we have

$$\gamma - \kappa = -\xi, \quad \lambda - \xi = -\tau,$$

(2.43)

thus again adding side by side gives us the column of the non-zero entry which also becomes $\kappa = \gamma + \lambda + \tau$. Therefore if both matrices have a non-zero entry at a row (which are unique) their difference also has a non-zero entry at that row which is also unique. If either $R_{\gamma \lambda}$ or $R_{\lambda \gamma}$ has a zero row then again $R_{\gamma \lambda} - R_{\lambda \gamma}$ has a unique non-zero entry at that row. Also if both of the matrices $R_{\gamma \lambda}$ and $R_{\lambda \gamma}$ have coinciding zero-rows then $R_{\gamma \lambda} - R_{\lambda \gamma}$ will have zero elements in that row. These facts show us that the matrix $R_{\gamma \lambda} - R_{\lambda \gamma}$ has zero diagonal elements and if it has a non-zero entry at a row then that entry must be unique. However on the other hand $R_{\gamma \lambda} - R_{\lambda \gamma}$ may have zero rows too. The matrix $R_{\gamma \lambda} - R_{\lambda \gamma}$ also has unique non-zero entries at its columns if they exist. This is due to two facts; firstly if both matrices have non-zero entries at a row then as we have discussed above they coincide and since both matrices have unique entries in a column their difference will have a unique entry at the corresponding column, secondly if one of the matrices has a zero $\tau$ row but the other’s $\tau$ row is not zero, the one which has a non-zero $\tau$ row will have a unique entry at the column $\kappa = \gamma + \lambda + \tau$, also it can be seen from
the conditions (2.42) and (2.43) that the other one which has the zero \( \tau \) row must have zero elements in the column \( \kappa = \gamma + \lambda + \tau \) as if it has a non-zero element in the column \( \kappa = \gamma + \lambda + \tau \) at a different row say \( \phi \) than \( \tau \) this would imply \( \kappa = \gamma + \lambda + \phi \) which would contradict with \( \kappa = \gamma + \lambda + \tau \) which is obtained through the addition of the non-zero-entry existence conditions of the first matrix whose \( \tau \) row is not composed of zero elements. On the other hand \( \tau = \phi \) would contradict with the assumption that the second matrix in question has zero \( \tau \) row. Thus its \( \kappa = \gamma + \lambda + \tau \) column must be a null-column. These two facts denote that if there is a non-zero entry in a column of \( R_{\gamma \lambda} - R_{\lambda \gamma} \) then it must be unique. For this reason if \( R_{\gamma \lambda} - R_{\lambda \gamma} \) has \( n \) zero rows then it also has \( n \) zero columns. We immediately see that the block matrices \( R_{\gamma + \lambda} \) and \( R_{\gamma \lambda} - R_{\lambda \gamma} \) which are on the LHS and the RHS of (2.34) respectively obey similar properties like zero diagonal elements and unique non-zero row or column entries if they exist. If at a fixed row \( \tau \) there is a non-zero entry (which is unique) in \( R_{\gamma + \lambda} \) then from (2.37) we deduce that it must be at the column \( \kappa = \gamma + \lambda + \tau \). Also following our discussion above at a fixed row \( \tau \) if \( R_{\gamma \lambda} - R_{\lambda \gamma} \) has a non-zero entry which is also unique then it must also be at the column \( \kappa = \gamma + \lambda + \tau \) too. On the other hand at a fixed row \( \tau \) if \( R_{\gamma + \lambda} \) does not have a non-zero entry this means that there exists no column \( \kappa \) which would satisfy \( \gamma + \lambda - \kappa = -\tau \). In this case there can not be a non-zero entry at the row \( \tau \) on the RHS in \( R_{\gamma \lambda} - R_{\lambda \gamma} \) as if it exists from either (2.42) or (2.43) we must have \( \kappa = \gamma + \lambda + \tau \) which would contradict with the assumed impossibility of this root condition at the \( \tau \) row of \( R_{\gamma + \lambda} \) on the LHS. In summary we conclude that: i) if both \( R_{\gamma + \lambda} \) and \( R_{\gamma \lambda} - R_{\lambda \gamma} \) have a non-zero entry at a row \( \tau \) they must coincide, ii) if \( R_{\gamma + \lambda} \) does not have a non-zero entry at a row \( \tau \) then \( R_{\gamma \lambda} - R_{\lambda \gamma} \) can not have a non-zero entry at the same row. Therefore all the entries of \( R_{\gamma + \lambda} \) and \( R_{\gamma \lambda} - R_{\lambda \gamma} \) coincide and we may question the equality of the lower rightmost block matrices on the LHS and the RHS of (2.34) whose entries coincide. However we should state that we leave dealing with the case of a non-zero entry at a row \( \tau \) in \( R_{\gamma + \lambda} \) but all zero entries at the row \( \tau \) of \( R_{\gamma \lambda} - R_{\lambda \gamma} \) for later. We will prove that in this case the entry on the LHS must be also zero due to structure constant conditions of the Cartan-Weyl basis.

\(^2\)Again from our discussion about the structure of \( R_{\gamma \lambda} - R_{\lambda \gamma} \) we know that this is valid at a row \( \tau \) for either of the cases when both \( R_{\gamma \lambda} \) and \( R_{\lambda \gamma} \) contribute a non-zero entry, and when only one of them contributes.

\(^3\)This can happen either if for none of the columns \( \kappa, \gamma + \lambda - \kappa \in \Delta \) or \( \gamma + \lambda - \kappa \neq -\tau \) if there exists a \( \kappa \) such that \( \gamma + \lambda - \kappa \in \Delta \).
Now if we take a look at the non-zero entries of the upper-rightmost block matrix of (2.41) which are on the column $\gamma + \lambda$ we have

$$\frac{\gamma_i(\gamma, \theta := \lambda) - \lambda_i(\lambda, \theta := \gamma)}{4} = \frac{\gamma_i N_{\lambda,-\theta} - \lambda_i N_{\gamma,-\theta}}{4}.$$  (2.44)

As we have discussed before if $\gamma + \lambda$ is not a root then the upper-rightmost block matrix of (2.41) will be zero which will be equal to the upper-rightmost block matrix of the LHS of (2.34) which will again be zero owing to the vanishing of $N_{\gamma,\lambda}$. However if $\gamma + \lambda$ is a root then through our previous discussion about the closure of the coset algebra it must be in $\Delta_{nc}^+$ and then there exists a root $\gamma + \lambda = \theta \in \Delta_{nc}^+$ such that $\lambda - \theta = -\gamma$ and $\gamma - \theta = -\lambda$. Therefore in this case the non-zero entries given in (2.44) do exist. For the root generators of a Cartan-Weyl basis if $\alpha + \beta + \gamma = 0$ then we have

$$N_{\alpha\beta} = N_{\beta\gamma} = N_{\gamma\alpha}.$$  (2.45)

Since for the non-zero entries (2.44) $\gamma + \lambda - \theta = 0$ we have $N_{\lambda,-\theta} = N_{\gamma,\lambda}$ and $N_{\gamma,-\theta} = -N_{-\theta,\gamma} = -N_{\gamma,\lambda}$. Thus the non-zero entries in (2.44) become

$$\frac{\gamma_i N_{\lambda,-\theta} - \lambda_i N_{\gamma,-\theta}}{4} = \frac{N_{\gamma\lambda}(\gamma_i + \lambda_i)}{4},$$  (2.46)

which are equal to the non-zero entries in the upper-rightmost block matrix of (2.36) which are also on the column $\gamma + \lambda$.

Now we will come back to the question of the equality of the lower-rightmost block matrices on the LHS and the RHS of (2.34). Firstly let us assume that $\gamma + \lambda$ is not a root as we have mentioned before in this case directly from (2.14) the LHS of (2.34) is zero. If on the RHS in $R_{\gamma\lambda} - R_{\lambda\gamma}$ at a row $\tau$ and a column $\kappa$ both of the root conditions (2.42) and (2.43) hold and an entry exists then it must be

$$N_{\lambda,-\kappa} N_{\gamma,-\nu} - N_{\gamma,-\kappa} N_{\lambda,-\xi}.$$  (2.47)

However from the root conditions (2.42) and (2.43) of entry existence, also from the identity (2.45) we have

$$N_{\gamma,-\nu} = N_{\tau\gamma}, \quad N_{\lambda,-\xi} = N_{\tau\lambda}.$$  (2.48)

Thus the entry becomes

$$N_{\lambda,-\kappa} N_{\tau\gamma} - N_{\gamma,-\kappa} N_{\tau\lambda}.$$  (2.49)
In general if for four roots $\alpha + \beta + \gamma + \delta = 0$ and if none of the pairs sum up to zero then the Cartan-Weyl basis structure constants obey the identity

$$N_{\alpha\beta}N_{\gamma\delta} + N_{\beta\gamma}N_{\alpha\delta} + N_{\gamma\alpha}N_{\beta\delta} = 0. \quad (2.50)$$

We have shown that if one adds the entry existence conditions (2.42) and (2.43) side by side one gets the root condition $\gamma + \lambda + \tau - \kappa = 0$. Now if we apply (2.50) then we have

$$N_{\gamma\lambda}N_{\tau,-\kappa} + N_{\lambda\tau}N_{\gamma,-\kappa} + N_{\tau\gamma}N_{\lambda,-\kappa} = 0. \quad (2.51)$$

However since we assume the case when $\gamma + \lambda$ is not a root $N_{\gamma\lambda} = 0$ and we have

$$-N_{\tau\lambda}N_{\gamma,-\kappa} + N_{\tau\gamma}N_{\lambda,-\kappa} = 0, \quad (2.52)$$

whose LHS is exactly equal to (2.49). This proves the equality of the lower rightmost block matrices on the LHS and the RHS of (2.34) when $\gamma + \lambda$ is not a root and when (2.42) and (2.43) both hold. As a second case when $\gamma + \lambda$ is not a root if at least one of the root conditions is not satisfied in both (2.42) and (2.43) then of course the lower rightmost block matrix on the RHS of (2.34) will be zero as well as the LHS one. On the other hand if both of the root conditions are satisfied in (2.42) but at least one condition is not satisfied in (2.43) then again we have $\gamma + \lambda + \tau - \kappa = 0$ and the identity (2.51) holds. This shows that a case in which one of the conditions holds but the other one does not hold in (2.43) would be contradictory as in this case one can take the difference of $\gamma + \lambda + \tau - \kappa = 0$ with the holding root condition to show that the second condition in (2.43) must also hold. Now $\gamma + \lambda + \tau - \kappa = 0$ can be written as $\gamma - \kappa = -(\lambda + \tau)$. This shows that if $\gamma - \kappa \in \Delta$ then $-(\lambda + \tau) \in \Delta$ also $\lambda + \tau \in \Delta$. Since $\lambda, \tau \in \Delta_{nc}^{+}$ we have $\lambda + \tau \in \Delta_{nc}^{+}$. Thus it can not be true that if $\gamma - \kappa \in \Delta$ there does not exist any $\xi \in \Delta_{nc}^{+}$ which would satisfy $\gamma - \kappa = -\xi$ in (2.43) as in this case $\xi$ is nothing but $\lambda + \tau$. Therefore when (2.42) holds the only possible conditions of the non-existence of (2.43) are; $\gamma - \kappa \notin \Delta$ and $\lambda - \xi \notin \Delta$ or $\lambda - \xi \in \Delta$ but there exists no $\tau \in \Delta_{nc}^{+}$ such that $\lambda - \xi = -\tau$. Thus in this case the entry in $R_{\gamma\lambda} - R_{\lambda\gamma}$ at the row $\tau$ and the column $\kappa$ becomes

$$N_{\lambda,-\kappa}N_{\gamma,-\nu}, \quad (2.53)$$

Since (2.42) holds from the identities (2.45) we again have $N_{\gamma,-\nu} = N_{\tau\gamma}$ so that (2.53) can be written as

$$N_{\lambda,-\kappa}N_{\tau\gamma}. \quad (2.54)$$
However since $N_{\gamma,-\kappa} = 0$ and we assume that $\gamma + \lambda$ is not a root giving $N_{\gamma\lambda} = 0$ from (2.51) we have

$$N_{\tau\gamma}N_{\lambda,-\kappa} = 0,$$

which proves the equality of the lower rightmost block matrices on the LHS and the RHS of (2.34) when $\gamma + \lambda$ is not a root and when (2.42) holds but (2.43) does not hold. Now instead if (2.43) holds but (2.42) does not hold then a similar reasoning and analysis denotes that in this case the only possible conditions of the non-existence of (2.42) are; $\lambda - \kappa \notin \Delta$ and $\gamma - \nu \notin \Delta$ or $\gamma - \nu \in \Delta$ but there exists no $\tau \in \Delta_{n_{s}}^{+}$ such that $\gamma - \nu = -\tau$. Thus in this case the entry in $R_{\gamma\lambda} - R_{\lambda\gamma}$ at the row $\tau$ and the column $\kappa$ is

$$-N_{\gamma,-\kappa}N_{\lambda,-\xi}. \quad (2.56)$$

The conditions in (2.43) hold thus again from the identities (2.43) we have $N_{\lambda,-\xi} = N_{\tau\lambda}$ so that (2.56) becomes

$$-N_{\gamma,-\kappa}N_{\tau\lambda}. \quad (2.57)$$

In this case since $N_{\lambda,-\kappa} = 0$ and again $\gamma + \lambda$ is not a root giving $N_{\gamma\lambda} = 0$ from (2.51) we have

$$-N_{\tau\lambda}N_{\gamma,-\kappa} = 0. \quad (2.58)$$

By considering all the possible cases we have completed the proof of the equality of the lower rightmost block matrices on the LHS and the RHS of (2.34) when $\gamma + \lambda$ is not a root. Our next task will be to perform a similar proof for the case when $\gamma + \lambda$ is a root. We have already mentioned that if $R_{\gamma+\lambda}$ does not have a non-zero entry at a row $\tau$ then $R_{\gamma\lambda} - R_{\lambda\gamma}$ can not have a non-zero entry at the same row and we have shown that the entries of $R_{\gamma+\lambda}$ and $R_{\tau\gamma} - R_{\lambda\gamma}$ coincide. Thus for the following we will assume that there exists an entry at the row $\tau$ in $R_{\gamma+\lambda}$ which means that the root condition $\gamma + \lambda - \kappa = -\tau$ holds due to (2.37). Thus in this case again (2.51) is valid. We will start with the case in which the root conditions in (2.42) and (2.43) are both satisfied so that $R_{\gamma\lambda}$ and $R_{\lambda\gamma}$ both have entries at the row $\tau$ and the column $\kappa$. Then from (2.34) at the row $\tau$ and the column $\kappa$ the equality of the coinciding entries of the lower rightmost block matrices on the LHS and the RHS to be proven becomes

$$N_{\gamma\lambda}N_{\gamma+\lambda,-\kappa} \equiv N_{\lambda,-\kappa}N_{\gamma,-\nu} - N_{\gamma,-\kappa}N_{\lambda,-\xi}. \quad (2.59)$$
Existence of the root conditions (2.42) and (2.43) again allows the usage of the identity (2.45) and (2.59) becomes

\[ N_{\gamma+\lambda,-\kappa} \equiv N_{\lambda,-\kappa}N_{\tau\gamma} - N_{\gamma,-\kappa}N_{\tau\lambda}. \]  

(2.60)

Now since \( \gamma + \lambda - \kappa + \tau = 0 \) from (2.45) we have

\[ N_{\gamma+\lambda,-\kappa} = N_{-\kappa,\tau}. \]  

(2.61)

Therefore (2.60) can be written as

\[ 0 \equiv N_{\gamma\lambda}N_{\tau,-\kappa} + N_{\lambda,-\kappa}N_{\tau\gamma} + N_{\gamma,-\kappa}N_{\lambda\tau}. \]  

(2.62)

However this equality holds due to the validity of (2.51) which is the desired result. The next step is to show that the equality in (2.59) holds when the root conditions (2.42) and (2.43) are partially satisfied or not satisfied at all.

If we refer to our previous root condition analysis which we have done for (2.42) and (2.43) when we have discussed the cases when \( \gamma + \lambda \) is not a root we can conclude that the following three cases are the only ones which are not contradictory with the condition \( \gamma + \lambda - \kappa + \tau = 0 \) which comes from the existence of the lower rightmost block matrix entry on the LHS of (2.34):

- both of the root conditions in (2.42) hold, in addition \( \gamma - \kappa \notin \Delta \), and \( \lambda - \xi \notin \Delta \) or \( \lambda - \xi \in \Delta \) but there exists no \( \tau \in \Delta_{nc}^+ \) such that \( \lambda - \xi = -\tau \),
- both of the root conditions in (2.43) hold, in addition \( \lambda - \kappa \notin \Delta \), and \( \gamma - \nu \notin \Delta \) or \( \gamma - \nu \in \Delta \) but there exists no \( \tau \in \Delta_{nc}^+ \) such that \( \gamma - \nu = -\tau \),
- \( \gamma - \kappa \notin \Delta \), and \( \lambda - \xi \notin \Delta \) or \( \lambda - \xi \in \Delta \) but there exists no \( \tau \in \Delta_{nc}^+ \) such that \( \lambda - \xi = -\tau \), in addition \( \lambda - \kappa \notin \Delta \), and \( \gamma - \nu \notin \Delta \) or \( \gamma - \nu \in \Delta \) but there exists no \( \tau \in \Delta_{nc}^+ \) such that \( \gamma - \nu = -\tau \).

For the first case we have to question

\[ N_{\gamma\lambda}N_{\gamma+\lambda,-\kappa} \equiv N_{\lambda,-\kappa}N_{\gamma,-\nu}. \]  

(2.63)

However this equation is the same with (2.59) if we use \( N_{\gamma,-\kappa} = 0 \) in (2.59) which is the characteristic feature of the first case. Bearing in mind the

\(^4\)This is a case which we have postponed to deal with before.
identity \( N_{\gamma,-\nu} = N_{\tau\gamma} \) (since (2.42) is satisfied) for this first case the proof of the equality in (2.63) coincides with the one we have performed for the previous case following (2.59). The other symmetrical case namely the second one leads to

\[ N_{\gamma\lambda}N_{\gamma+\lambda,-\kappa} \overset{?}{=} -N_{\gamma,-\kappa}N_{\lambda,-\xi}. \]

(2.64)

Again this equation is the same with (2.59) if one lets \( N_{\lambda,-\kappa} = 0 \) in (2.59) which is the characteristic feature of the second case. Upon the insertion of \( N_{\lambda,-\xi} = N_{\tau\lambda} \) (since (2.43) is satisfied) the proof of (2.64) again coincides with the one following (2.59). For the last item which is the combination of the first and the second ones one has to show

\[ N_{\gamma\lambda}N_{\gamma+\lambda,-\kappa} \overset{?}{=} 0. \]

(2.65)

One can obtain this equality from (2.59) by using \( N_{\tau,-\kappa} = 0 \) and \( N_{\lambda,-\kappa} = 0 \) which are the characteristics of the third case. Thus also for this case the proof of the equality of (2.63) comes automatically from the previous one following (2.59). Therefore with this last case we have shown that when a non-zero entry at a row \( \tau \) in \( R_{\gamma+\lambda} \) exists but the \( \tau \) row of \( R_{\gamma\lambda} - R_{\lambda\gamma} \) consists of zero elements the entry on the LHS of (2.34) must be also zero due to structure constant conditions of the Cartan-Weyl basis. By this we have completed the proof of the equality in (2.34) for all the possible cases which may arise. In conclusion, we can state that together with our previous results we have proven that the algebra structure given in (2.2) which is a deformation of the solvable Lie subalgebra of the global symmetry group of the sigma model obeys the Jacobi identities (2.14) thus it defines a Lie algebra.

### 3 The Adjoint Representation

In this section we will show that the dualized coset algebra given in (2.2) contains an adjoint representation for the subalgebra \( s \) which is generated by the original coset generators \( \{T_m\} \equiv \{H_i, E_\alpha\} \). This subalgebra is nothing but the original coset algebra of the sigma model which is the solvable subalgebra of the Lie algebra of the global symmetry group. The adjoint representation we mention exists due to the general scheme

\[ [\{T_m\}, \{\tilde{T}_n\}] \subset \{\tilde{T}_n\}, \]

(3.1)
of the structure of the dualized coset algebra (2.2). Now to display this representation let us consider the linear map
\[ f : s \rightarrow gl(S, \mathbb{R}), \]  
(3.2)
where \( S \) is the dimension of \( s \). Similar to the general adjoint representation of a generic Lie algebra we assume \( f \) is such that
\[ f(T_i) = U_i, \]  
(3.3)
where \( U_i \) is the \( S \times S \) matrix whose entries are \( U_{im} \) which are the structure constants defined in (2.17). Linearity and (3.3) defines the action of \( f \) on entire \( s \). This linear map becomes an algebra homomorphism if
\[ f([M, N]) = [f(M), f(N)], \]  
(3.4)
for all \( M, N \in s \). If one inserts \( M = M'T_i \) and \( N = N'kT_k \) in (3.4) one sees that (3.4) holds if
\[ f([T_i, T_n]) = [f(T_i), f(T_n)]. \]  
(3.5)
By using (2.19), and (3.3), also the fact that \( f \) is assumed to be a linear map the equality (3.5) which is in question can be written as
\[ Z'_{ln}U'^{s}_{tm} = (U_iU_n - U_nU_i)^s_{m}. \]  
(3.6)
However this equality is the same with the Jacobi identity (2.18) which we have exactly proven to hold when we showed that the dualized coset algebra (2.2) is a Lie algebra in the previous section. Thus we can conclude that \( f \) whose action on the basis \( \{T_m\} \) is defined via (3.3) is an algebra homomorphism and it forms a \( S \times S \) matrix representation for the coset algebra \( s \) which is a subalgebra in (2.2). We may state that the dualized coset algebra (2.2) which is a deformation of its subalgebra \( s \) and whose Lie algebra structure is proven in the previous section generates a natural adjoint representation for the original coset algebra \( s \).

4 The First-order Sigma Model Field Equations

We will now show that the representation presented in the last section enables one to derive the first-order field equations of the symmetric space.
sigma model. The first-order field equations of the sigma models with symmetric space coset target manifolds firstly appeared in [3, 4] as a result of the dualized coset construction of these theories. However they were formally generated in those works as consistency conditions within the dualization of the theory. Here we will algebraically prove that they correspond to the equations which would be obtained by a local integration of second-order field equations. In other words we will show that if one chooses the representation mentioned in the previous section then one can obtain the second-order field equations of the symmetric space sigma model by taking the exterior derivative of the first-order ones derived in [3, 4]. Thus our starting point is adopting from [3, 4] the set of equations

\[ * \Psi = (-1)^D e^\Gamma e^A \Lambda, \]  

where \( D \) is the dimension of the base manifold and the S-dimensional column vectors \( \Psi \) and \( A \) have the components

\[ \Psi^i = \frac{1}{2} d\phi^i, \quad \text{for} \quad i = 1, \ldots, r, \]

\[ \Psi^{\alpha+r} = e^{\frac{1}{2}r \chi^\alpha} \Omega^{\alpha\gamma} d\chi^\gamma, \quad \text{for} \quad \alpha = 1, \ldots, S - r, \]

\[ A^i = \frac{1}{2} d\phi^i, \quad \text{for} \quad i = 1, \ldots, r, \quad \text{and} \quad A^{\alpha+r} = d\chi^\alpha, \quad \text{for} \quad \alpha = 1, \ldots, S - r. \]

Here \( \phi^i \) and \( \chi^\gamma \) are the scalar fields to be solved which parametrize the coset space target manifold of the sigma model and we should state that \( \alpha \) stands both for the non-compact positive roots and their corresponding enumeration. Also \( \phi^i \) and \( \chi^\alpha \) are arbitrary \((D-2)\)-forms which emerge from the dualization of the coset map within the dualized coset realization of the theory. In (4.1) \( \Gamma(\phi^i) \) and \( \Lambda(\chi^\beta) \) are \( S \times S \) matrix functionals with components

\[ \Gamma^k_n = \frac{1}{2} \phi^i g^k_{in}, \quad \Lambda^k_n = \chi^\alpha f^k_{\alpha n}, \]  

where the matrices \( f_{\alpha} \) and \( g_i \) are defined in (2.7) and (2.8) respectively. Considering the definitions (2.6) and (2.17) under the adjoint representation
of \( s \) which we have proved to exist in the previous section we can immediately see the identification

\[
e^R e^A = e^\frac{1}{2} \phi^i g_i e^\chi^\alpha f_\alpha = e^\frac{1}{2} \phi^i H_i e^\chi^\alpha E_\alpha = \nu, \tag{4.4}
\]

where \( \nu \) is the coset representative of the sigma model \cite{3, 4}. Thus when the representation defined in (3.2) is chosen which sends

\[
H_i \longrightarrow \tilde{g}_i, \quad E_\alpha \longrightarrow \tilde{f}_\alpha, \tag{4.5}
\]

the set of first-order equations (4.1) can be written as

\[
* \tilde{\Psi} = (-1)^D \nu \tilde{A}. \tag{4.6}
\]

Now if we take the exterior derivative of both sides we get

\[
d(* \tilde{\Psi}) = (-1)^D d \nu \tilde{A}, \tag{4.7}
\]

where we have used \( dA^m = 0 \). Since (4.6) is a vector equation it can be written as

\[
\tilde{A} = (-1)^D \nu^{-1} * \tilde{\Psi}. \tag{4.8}
\]

Inserting this into (4.7) we get

\[
d(* \tilde{\Psi}) = d \nu \nu^{-1} * \tilde{\Psi}. \tag{4.9}
\]

In this equation we readily realize that

\[
G = d \nu \nu^{-1}, \tag{4.10}
\]

is the Cartan-form induced by the coset map \( \nu \) and it is explicitly calculated in \cite{4}. It reads

\[
G = \frac{1}{2} d \phi^i H_i + e^{\frac{1}{2} \phi^i} \Omega^\beta d \chi^\alpha E_\beta, \tag{4.11}
\]

where

\[
\Omega = \sum_{m=0}^{\infty} \frac{\omega^m}{(m+1)!} \tag{4.12}
\]

\[= (e^\omega - I)^{-1}. \]

Note that \( \nu^{-1} \) exists by definition.
Here the \((S-r)\times(S-r)\) matrix \(\omega\) has the components
\[
\omega^\gamma_\beta = \chi^\alpha K^\gamma_{\alpha\beta},
\]
with \(K^\gamma_{\alpha\beta}\) defined as
\[
[K^\gamma_{\alpha\beta}, E_\alpha, E_\beta] = K^\gamma_{\alpha\beta} E_\gamma.
\]
Since we chose the representation generated by \((3.3)\) we can write \((4.11)\) as
\[
\mathcal{G} = \frac{1}{2} d\phi \tilde{g}_i + e^{\frac{1}{2} \beta_i \phi^i} \Omega^\alpha_\beta d\chi^\alpha \tilde{f}_\beta.
\]
If we insert this back in \((4.9)\) we obtain (in component form)
\[
d(\ast \Psi^m) = \left( \frac{1}{2} d\phi \tilde{g}_i \ast \Psi^i_\beta + e^{\frac{1}{2} \beta_i \phi^i} \Omega^\alpha_\beta d\chi^\alpha \ast \tilde{f}_\beta \right) \land \ast \Psi^n.
\]
For \(1 \leq m \leq r\) \((4.16)\) yields
\[
\frac{1}{2} d(\ast d\phi^i) = \frac{1}{2} d\phi^j \land \tilde{g}^i_{jn} \ast \Psi^n + e^{\frac{1}{2} \beta_k \phi^k} \Omega^\alpha_\beta d\chi^\alpha \land \tilde{f}^i_{\beta n} \ast \Psi^n.
\]
From \((2.8)\) the first term on the RHS vanishes and if we split the sum in the second term on the index \(n\) then we have
\[
\frac{1}{2} d(\ast d\phi^i) = e^{\frac{1}{2} \beta_k \phi^k} \Omega^\beta_\alpha d\chi^\alpha \land \left( \tilde{f}^j_{\beta j} \ast \Psi^j + \tilde{f}^j_{\beta,\gamma+r} \ast \Psi^{\gamma+r} \right).
\]
Now from \((2.7)\) again the first term on the RHS vanishes. Further index splitting in the sum on the remaining term gives
\[
\frac{1}{2} d(\ast d\phi^i) = e^{\frac{1}{2} \beta_k \phi^k} \Omega^\beta_\alpha d\chi^\alpha \land \left( \tilde{f}^j_{\beta j} \ast \Psi^{\beta+r} + \sum_{\kappa \neq \beta} \tilde{f}^j_{\beta,\kappa+r} \ast \Psi^{\kappa+r} \right).
\]
Due to \((2.7)\) the second sum on the RHS also vanishes. By reading\(^7\) the value of \(\tilde{f}^j_{\beta,\beta+r}\) from \((2.7)\) and also by using \((4.2)\) we finally get
\[
d(\ast d\phi^i) = \frac{1}{2} \sum_{\alpha,\beta,\gamma \in \Delta^{+}} \beta_i e^{\frac{1}{2} \beta_k \phi^k} \Omega^\beta_\alpha d\chi^\alpha \land e^{\frac{1}{2} \beta_j \phi^j} \Omega^\gamma_\beta d\chi^\gamma.
\]
\(^7\)The reader should pay attention that the first term inside the parentheses on the RHS of \((4.19)\) is not a sum but a single term.
where \( i \) is the free index. The set of equations in (4.20) are exactly the second-order dilaton field equations of the sigma model which are derived in [3, 4, 5, 6]. Now on the other hand if we consider (4.16) for \( m > r \) then we have

\[
d (e_{\frac{1}{2}}^{\beta \phi} \Omega_\alpha^\beta d\chi^\alpha) = \frac{1}{2} d\phi^i \tilde{g}_{i \mu} + e_{\frac{1}{2}}^{\kappa \phi} \Omega_\alpha^\kappa d\chi^\alpha \tilde{f}_{\mu +}^\beta + \Psi^\alpha. \tag{4.21}
\]

In this equation \( \beta \) is the free index. Similar to our calculation above again due to (2.7) and (2.8) after eliminating the vanishing terms on the RHS we get

\[
d (e_{\frac{1}{2}}^{\beta \phi} \Omega_\alpha^\beta d\chi^\alpha) = -\frac{1}{2} \beta_i d\phi^i \wedge \Psi^{\beta +} + e_{\frac{1}{2}}^{\kappa \phi} \Omega_\alpha^\kappa d\chi^\alpha \tilde{f}_{\kappa \gamma +}^\beta \wedge \Psi^{\gamma +}. \tag{4.22}
\]

Furthermore by using (4.2) and (2.7) we finally have

\[
d (e_{\frac{1}{2}}^{\beta \phi} \Omega_\alpha^\beta d\chi^\alpha) = -\frac{1}{2} \beta_i d\phi^i \wedge (e_{\frac{1}{2}}^{\beta \phi} \Omega_\alpha^\beta \ast d\chi^\alpha) + \sum_{\kappa - \theta = -\beta} e_{\frac{1}{2}}^{\kappa \phi} \Omega_\alpha^\kappa d\chi^\alpha \wedge (N_{\kappa - \theta} e_{\frac{1}{2}}^{\theta \phi} \Omega_\alpha^\theta \ast d\chi^\sigma), \tag{4.23}
\]

where on the RHS in the second term the sum is on the index \( \kappa \in \Delta^+_nc \) (again \( \beta \) is the free index) and due to (2.10) the index \( \theta \in \Delta^+_nc \) (if the corresponding root exists in \( \Delta^+_nc \) when one fixes \( \beta \) and \( \kappa \)) must be chosen according to the root condition stated above.

As the index (the root) \( \beta \) runs in \( \Delta^+_nc \) these equations are the second-order axion field equations of the sigma model which are derived in [3, 4, 5, 6].

## 5 Conclusion

We have presented a rigorous proof which denotes that the dualized algebra of the coset sigma model with globally Riemannian symmetric target space is indeed a Lie algebra. Although the commutation relations of the dualized coset algebra were derived in [4] their Lie algebra structure was not proved.

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8Of course from (2.10) if such a root \( \theta \) does not exist in \( \Delta^+_nc \) then that term for a particular choice of \( \beta \) and \( \kappa \) is zero. We should also state that there is no sum on the index \( \theta \) in (4.23) instead if it exists \( \theta \) becomes fixed when \( \beta \) and \( \kappa \) are chosen.
in that work. By showing that the structure constants or the commutators indeed satisfy the Jacobi identities we have justified the Lie algebra notion of the dualized algebra which is an extension of the original coset algebra of the sigma model. Later we have also mentioned that the dualized coset algebra which contains the ordinary one in it and therefore which can be considered as a deformation admits an adjoint representation for the original coset algebra. Finally under this special representation we have shown that the second-order field equations can be obtained by differentiating the first-order equations which appeared in [3,4] as consistency conditions of the dualization construction. Therefore we have proved that these consistency conditions are the true algebraic first-order equations of the corresponding sigma model.

As it can be inferred from the sequence of its sections this paper aims to show that under a special representation generated by a duality algebra the second-order Euler-Lagrange equations can be integrated to obtain first-order field equations. These first-order equations are already derived (in other words suggested) in [3,4]. However only via this work they are proven to be the algebraically correct ones since in [3,4] they appeared as a consistency condition embedded within the dualization of the theory. On the other hand in this work we have proven that if one applies an exterior derivative on these first-order equations one gets the correct second-order field equations which are the Euler-Lagrange ones. Depending on the analysis given here we can easily state that the dualization of a theory apart from its enlarged geometrical construction is an efficient way of inventing the correct representation of the coset algebra so that this representation leads to an integration of the field equations. In other words within the dualized theory the original coset algebra is implemented in a Lie algebra deformation of it (the dualized coset algebra) in such a way that the generated adjoint representation becomes an appropriate one in which the integration of the field equations exists. We have shown that the representation which enables to construct the first-order field equations depends on the Lie algebra structure of the dualized coset algebra which is a special and a non-trivial extension of the original one. In [3,4] this algebra and the first-order field equations were derived from a partially geometrical point of view. Here by showing the legacy of the Lie algebra structure and accordingly the adjoint representation we have algebraically complemented the achievements of [3,4]. Thus our exact proof additionally justifies the correctness of the first-order field equations of the symmetric space sigma model. This is an essential result as it enables
the reduction of order of the second-order partial differential field equations of the sigma model which is an important ingredient in supergravity as well as string theory also in QFT.

From the analysis point of view another essential result of the present work can be considered as the generation of an extended Lie algebra structure starting form a solvable Lie subalgebra of a Lie algebra. In fact the arguments of section two can easily be generalized and depending on the complete and the formal proof presented in section two we can state that every subalgebra of the Borel subalgebra of a Lie algebra sits in another Lie algebra with a doubled dimension.

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