Investigating Exponential and Geometric Polynomials with Euler-Seidel Algorithm

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Abstract

In this paper we use Euler-Seidel matrices method to find out some properties of exponential and geometric polynomials and numbers. Some known results are reproved and some new results are obtained.

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1 Introduction.

This work is based on Euler-Seidel matrices method ([12]) which is related to algorithms, combinatorics and generating functions. This method is quite useful to investigate properties of some special numbers and polynomials.

In this work we use this method to find out some interesting results of exponential (or Bell) and geometric (or Fubini) polynomials and numbers. Although some results in this paper are known, this method provides different proofs as well as new identities.

We first consider a given sequence \((a_n)_{n \geq 0}\), and then determine the Euler-Seidel matrix corresponding to this sequence is recursively by the formulae

\[
\begin{align*}
a_n^0 &= a_n, \quad (n \geq 0), \\
a_n^k &= a_n^{k-1} + a_{n+1}^{k-1}, \quad (n \geq 0, \, k \geq 1)
\end{align*}
\]

where \(a_n^k\) represents the \(k\)th row and \(n\)th column entry. From relation \((1)\), it can be seen that the first row and the first column can be transformed into each other via the well known binomial inverse pair as:

\[
a_0^n = \sum_{k=0}^{n} \binom{n}{k} a_k^0,
\]
and
\[ a_n^0 = \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} a_k^0. \]  

(3)

Euler ([13]) deduced the following proposition which states a connection between the ordinary generating functions of the initial sequence \((a_n)_{n\geq0} = (a_n^0)_{n\geq0}\) and the first column \((a_0^0)_{n\geq0}\).

**Proposition 1 (Euler)** Let
\[ a(t) = \sum_{n=0}^{\infty} a_n^0 t^n \]
be the generating function of the initial sequence \((a_n^0)_{n\geq0}\). Then the generating function of the sequence \((a_0^0)_{n\geq0}\) is
\[ \pi(t) = \sum_{n=0}^{\infty} a_0^0 t^n = \frac{1}{1-t} a \left( \frac{t}{1-t} \right). \]  

(4)

A similar statement was proved by Seidel ([20]) with respect to the exponential generating function.

**Proposition 2 (Seidel)** Let
\[ A(t) = \sum_{n=0}^{\infty} \frac{a_n^0 t^n}{n!} \]
be the exponential generating function of the initial sequence \((a_n^0)_{n\geq0}\). Then the exponential generating function of the sequence \((a_0^0)_{n\geq0}\) is
\[ \overline{A}(t) = \sum_{n=0}^{\infty} \frac{a_0^0 t^n}{n!} = e^t A(t). \]  

(5)

Dumont ([12]), presented several examples of Euler-Seidel matrices mainly using Bernoulli, Euler, Genocchi, exponential (Bell) and tangent numbers. He also attempted to give the polynomial extension of Euler-Seidel matrices method. In [9], Dil et al obtained some identities on Bernoulli and allied polynomials by introducing polynomial extension of this matrices method. By choosing the initial sequence from the elements of \(\mathbb{Z}_p\) (\(p\) is prime), Dil and Kurt ([11]) were interested in the type of Euler-Seidel matrices on \(\mathbb{Z}_p\). [17] contains detailed study on harmonic and hyperharmonic numbers using Euler-Seidel matrices method. More than that the reader also can find in [17], some results on \(r\)-Stirling numbers and a new characterization of the Fibonacci sequence. In [10], Dil and Mező, presented another algorithm which depends on a recurrence relation and two initial sequences. Using the algorithm which is symmetric respect to the rows and columns, they obtained some relations between Lucas sequences and
incomplete Lucas sequences. Besides Fibonacci and Lucas numbers they also investigated hyperharmonic numbers.

In this paper we consider Euler-Seidel matrices method for some combinatorial numbers and polynomials. This method is relatively easier than most of combinatorial methods to investigate the structure of such numbers and polynomials.

2 Definitions and Notation.

Now we give a summary about some special numbers and polynomials which we will need later.

Stirling numbers of the second kind.

Stirling numbers of the second kind \( \left\{ \binom{n}{k} \right\} \) are defined by means of generating functions as follows ([1], [6]):

\[
\sum_{n=0}^{\infty} \left\{ \binom{n}{k} \right\} \frac{x^n}{n!} = \frac{(e^x - 1)^k}{k!}.
\] (6)

Exponential polynomials.

Exponential polynomials (or single variable Bell polynomials) \( \phi_n(x) \) are defined by ([2], [18], [19]) as

\[
\phi_n(x) := \sum_{k=0}^{n} \binom{n}{k} x^k.
\] (7)

We refer [5] for comprehensive information on exponential polynomials.

The first few exponential polynomials are:

\[
\begin{align*}
\phi_0(x) &= 1, \\
\phi_1(x) &= x, \\
\phi_2(x) &= x + x^2, \\
\phi_3(x) &= x + 3x^2 + x^3, \\
\phi_4(x) &= x + 7x^3 + 6x^3 + x^4.
\end{align*}
\] (8)

Exponential generating function of exponential polynomials is given by (6),

\[
\sum_{n=0}^{\infty} \phi_n(x) \frac{t^n}{n!} = e^{e^t-1}.
\] (9)

The well known exponential numbers (or Bell numbers) \( \phi_n \) ([3], [4], [7], [21]) are obtained by setting \( x = 1 \) in (7), i.e

\[
\phi_n := \phi_n(1) = \sum_{k=0}^{n} \binom{n}{k}.
\] (10)
The first few exponential numbers are:

$$\phi_0 = 1, \phi_1 = 1, \phi_2 = 2, \phi_3 = 5, \phi_4 = 15.$$  \hfill (11)

Readers might also consult the lengthy bibliography of Gould \([15]\), where several papers and books are listed about exponential numbers.

Following recurrence relations that we reprove with Euler-Seidel matrices method hold for exponential polynomials \([19]\),

$$\phi_{n+1}(x) = x \left( \phi_n(x) + \phi'_n(x) \right)$$  \hfill (12)

and

$$\phi_{n+1}(x) = x \sum_{k=0}^{n} \binom{n}{k} \phi_k(x).$$  \hfill (13)

Geometric polynomials and numbers.

Geometric polynomials (also known as Fubini polynomials) are defined as follows \([4]\):

$$F_n(x) = \sum_{k=0}^{n} \binom{n}{k} k! x^k.$$  \hfill (14)

By setting \(x = 1\) in (14) we obtain geometric (or preferential arrangement-, or Fubini-) numbers \(F_n\), \([8,16]\) as

$$F_n := F_n(1) = \sum_{k=0}^{n} \binom{n}{k} k!.$$  \hfill (15)

The exponential generating functions of geometric polynomials is given by \([4]\)

$$\frac{1}{1 - x (e^t - 1)} = \sum_{n=0}^{\infty} F_n(x) \frac{t^n}{n!}.$$  \hfill (16)

Let us give a short list of these polynomials and numbers as follows

| \(F_n(x)\) |
|---|
| \(F_0(x) = 1,\) |
| \(F_1(x) = x,\) |
| \(F_2(x) = x + 2x^2,\) |
| \(F_3(x) = x + 6x^2 + 6x^3,\) |
| \(F_4(x) = x + 14x^2 + 36x^3 + 24x^4,\) |

and

$$F_0 = 1, \quad F_1 = 1, \quad F_2 = 3, \quad F_3 = 13, \quad F_4 = 75.$$  \hfill (17)

Geometric and exponential polynomials are connected in \([4]\) by the relation

$$F_n(x) = \int_{0}^{\infty} \phi_n(x\lambda) e^{-\lambda} d\lambda.$$  \hfill (17)

Now we will state our results.
3 Results obtained by matrix method.

Although we define Euler-Seidel matrices as matrices of numbers, we can consider entries of these matrices as polynomials also (see [9]). Thus, the generating functions that we mention in the statement of Seidel’s proposition, turn out to be two variables generating functions. Therefore from now on when we consider these generating functions as exponential generating functions of numbers we use the notations $A(t)$ and $\overline{A}(t)$, otherwise for the polynomial case we use the notations $A(t,x)$ and $\overline{A}(t,x)$. Using these notations relation (5) becomes

$$\overline{A}(t,x) = e^t A(t,x).$$

(18)

3.1 Results on Stirling numbers of the second kind.

Setting the initial sequence of an Euler-Seidel matrix as the sequence of Stirling numbers of the second kind, i.e. $a^0_n = \left\{ \frac{n}{m} \right\}$ where $m$ is a fixed nonnegative integer, we get the exponential generating function of the first row as

$$A(t) = \left( e^t - 1 \right)^m.$$ 

Thus we obtain from (2) the result

$$a^0_n = \sum_{k=0}^{n} \binom{n}{k} \left\{ \frac{k}{m} \right\}.$$ 

(19)

Now we proceed as follows; by using Proposition (2) we obtain another form of "$a^0_n$" and then we combine it with (19).

Equation (5) yields

$$\overline{A}(t) = \sum_{n=0}^{\infty} a^0_n \frac{t^n}{n!} = e^t \left( \frac{e^t - 1}{m!} \right)^m$$

which can equally well be written

$$\overline{A}(t) = \frac{d}{dt} \left( \frac{e^t - 1}{m+1} \right)^{m+1}.$$ 

(20)

Comparison of coefficients of $t^n$ in equation (19) yields the following result:

$$a^0_n = \left\{ \frac{n+1}{m+1} \right\}.$$ 

(21)

Thus we obtain the following proposition.

Proposition 3 Let $m$ be a nonnegative integer then we have,
\[
\sum_{k=0}^{n} \binom{n}{k} \binom{k}{m} = \binom{n+1}{m+1}
\]

and
\[
\sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} \binom{k+1}{m+1} = \binom{n}{m}.
\]

**Proof.** The equalities (19) and (21) constitute the first result stated in proposition. Then equation (23) directly follows from (3) and (22). □

Relations (22) and (23) can be found in [14], respectively as the equations (6.15) and (6.17) on page 265.

### 3.2 Results on exponential polynomials and numbers.

#### 3.2.1 Exponential numbers.

Let us construct an Euler-Seidel matrix with the initial sequence \((a_0^n)_{n \geq 0} = (\phi_n)_{n \geq 0}\). Then we get the following Euler-Seidel matrix

\[
\begin{pmatrix}
1 & 1 & 2 & 5 & 15 & 52 & \cdots \\
2 & 3 & 7 & 20 & \cdots \\
5 & 10 & 27 & \cdots \\
15 & 37 & \cdots \\
52 & \cdots \\
\cdots
\end{pmatrix}
\]

From this matrix we observe that \(a_0^n = \phi_{n+1}\). Now we prove this observation using generating functions.

\((a_0^n)_{n \geq 0} = (\phi_n)_{n \geq 0}\), from which it follows that

\[A(t) = e^{e^t-1}.\]

Equation (5) permits us to write

\[\overline{A}(t) = e^{e^t+t-1} = \frac{d}{dt} \left( e^{e^t-1} \right) = \sum_{n=0}^{\infty} \phi_{n+1} \frac{t^n}{n!}.\]

(24)

Comparison of the coefficients of both sides in (24) gives the desired result

\[a_0^n = \phi_{n+1}.\]

(25)

This leads to the following proposition.

**Proposition 4** We have

\[\phi_{n+1} = \sum_{k=0}^{n} \binom{n}{k} \phi_k\]
\[
\phi_n = \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} \phi_{k+1}.
\tag{27}
\]

**Proof.** Equation (26) follows from (2) and (25). Hence considering (3) and (26) together we obtain (27).

The identity (26) can be found in [14] on page 373 and (27) is inverse binomial transform of identity (26).

### 3.2.2 Exponential polynomials.

Setting the initial sequence of an Euler-Seidel matrix as the sequence of exponential polynomials, i.e. \((a_0^n)_{n \geq 0} = (\phi_n (x))_{n \geq 0}\) we get following Euler-Seidel matrix,

\[
\begin{pmatrix}
1 & x & x^2 & x^3 & \cdots \\
1 + x & 2x + x^2 & 2x^2 + x^3 & \cdots \\
1 + 3x + x^2 & 4x + 5x^2 + x^3 & \cdots \\
1 + 7x + 6x^2 + x^3 & \cdots \\
\vdots
\end{pmatrix}
\]

It seems that, \(x a_0^n = \phi_{n+1} (x)\). Now we prove this fact.

With the aid of Proposition (4), we can write

\[
\mathcal{A} (t, x) = e^t e^x (e^t - 1) = \frac{1}{x} \frac{d}{dt} x (e^t - 1).
\tag{28}
\]

Comparision of the coefficients of the both sides in (28) gives the desired result:

\[x a_0^n = \phi_{n+1} (x)\] (29)

from which the next proposition follows.

**Proposition 5** We have

\[
\phi_{n+1} (x) = x \sum_{k=0}^{n} \binom{n}{k} \phi_k (x)
\tag{30}
\]

and

\[
x \phi_n (x) = \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} \phi_{k+1} (x)
\tag{31}
\]

**Proof.** Proof is like that of Proposition (4). Here we give a new proof of the equation (12) by using Euler-Seidel matrices method.

It is clear that equations (30) and (31) are the generalizations of equations (26) and (27), respectively.

Now with the help of generating functions technique we derive some relations for exponential polynomials.

Firstly we give a new proof of the equation (12).
**Proposition 6** Let \( \phi'_n(x) \) denote the derivative of the \( n \)th exponential polynomial \( \phi_n(x) \), with respect to the variable \( x \). Then the equation

\[
\phi_{n+1}(x) = x \left( \phi_n(x) + \phi'_n(x) \right)
\]  

holds.

**Proof.** Deriving both sides of the equation (32) respect to the \( x \) we get

\[
\sum_{n=0}^{\infty} \frac{\phi'_n(x)}{n!} t^n = e^x(e^t-1) - e^x(e^t-1)
\]

which combines with (28) to give

\[
\sum_{n=0}^{\infty} \frac{\phi'_n(x)}{n!} t^n = A(t,x) - A(t,x).
\]

Then the last equation gives the desired result by comparing coefficients. \( \blacksquare \)

**Corollary 7** Exponential polynomials and their derivatives satisfy following symmetric equation

\[
\sum_{k=0}^{n-1} \binom{n}{k} (-1)^{n-k} \phi_k(x) = \sum_{k=1}^{n} \binom{n}{k} (-1)^{n-k-1} \phi'_k(x).
\]  

**Proof.** Employing (32) in the equation (31), we obtain (33). \( \blacksquare \)

### 3.3 Results on geometric polynomials and numbers

This part of our work contains some relations on geometric numbers and polynomials, most of which seems to be new.

#### 3.3.1 Geometric numbers

Let us set the initial sequence of an Euler-Seidel matrix as the sequence of geometric numbers, i.e. \( \left(a^n_k\right)_{n \geq 0} = \left(F_n\right)_{n \geq 0} \). Then we have

\[
\begin{bmatrix}
1 & 1 & 3 & 13 & 75 & \cdots \\
2 & 3 & 7 & 20 & \cdots \\
6 & 10 & 27 & \cdots \\
26 & 37 & \cdots \\
150 & \cdots \\
\end{bmatrix}
\]

Considering the first row and the first column, we observe that \( a^0_0 = 2F_n, n \geq 1 \). We need a proof of this fact.
Proposition (2) permits us to write
\[ A(t) = \sum_{n=0}^{\infty} \frac{a_n^n t^n}{n!} = \frac{e^t}{2 - e^t} = 2 - \frac{1}{2 - e^t} - 1 = \sum_{n=1}^{\infty} 2F_n \frac{t^n}{n!} + 1. \]

Now comparison of the coefficients of the both sides gives \( a_0^n = 2F_n \) where \( n \geq 1 \).

Then we may summarize the results so far obtained in:

**Proposition 8**

\[ 2F_n = \sum_{k=0}^{n} \binom{n}{k} F_k \text{ or equally } F_n = \sum_{k=0}^{n-1} \binom{n}{k} F_k \quad (34) \]

and

\[ F_n = 2 \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} F_k. \quad (35) \]

### 3.3.2 Geometric polynomials

Let us set the initial sequence of an Euler-Seidel matrix as the sequence of geometric polynomials, i.e. \( (a_n^0)_{n \geq 0} = (F_n(x))_{n \geq 0} \). Thus we obtain from (5) the result

\[ A(t, x) = \sum_{n=0}^{\infty} F_n(x) t^n = \frac{1}{1 - x (e^t - 1)} \]

and

\[ \overline{A}(t, x) = \frac{e^t}{1 - x (e^t - 1)}. \quad (36) \]

Then, derivative respect to the \( t \) yields

\[ \overline{A}(t, x) = \left[ \frac{1}{x} - (e^t - 1) \right] \frac{d}{dt} A(t, x) \]

which can equally well be written

\[ \overline{A}(t, x) = \sum_{n=0}^{\infty} \left[ \frac{F_{n+1}(x)}{x} + F_n(x) \right] t^n \frac{n!}{n!}. \]

The next equation follows by equating coefficients of \( \frac{t^n}{n!} \) in the preceding equation,

\[ a_0^n = \frac{F_{n+1}(x)}{x} - \sum_{k=1}^{n} \binom{n}{k-1} F_k(x). \quad (37) \]

We can now prove:
Proposition 9 \( F_n(x) \) geometric polynomials satisfy the following recurrence relation

\[
F_n(x) = x \sum_{k=0}^{n-1} \binom{n}{k} F_k(x).
\]

(38)

Proof. In view of (2), equation (37) shows validity of (38).  

Remark 10 As a special case, we get (34) by setting \( x = 1 \) in (38).

Corollary 11

\[
F_{n+1}(x) = \frac{x}{1+x} \sum_{k=0}^{n} \binom{n}{k} [F_k(x) + F_{k+1}(x)].
\]

(39)

Now we give two more recurrence relations for geometric polynomials.

Proposition 12 Let \( F'_k(x) \) denotes the first derivative of \( F_k(x) \), respect to the variable \( x \). Then we have

\[
F_{n+1}(x) = x \sum_{k=0}^{n} \binom{n}{k} [F_k(x) + xF'_k(x)].
\]

(40)

Proof. We may use (17) and (30) to conclude that,

\[
F_{n+1}(x) = x \sum_{k=0}^{n} \binom{n}{k} \int_0^\infty \phi_k(x\lambda) \lambda e^{-\lambda} d\lambda
\]

from which it follows that

\[
F_{n+1}(x) = x \sum_{k=0}^{n} \binom{n}{k} \sum_{r=0}^{k} \binom{k}{r} (r+1)!x^r.
\]

This can equally well be written by means of derivative as

\[
F_{n+1}(x) = x \frac{d}{dx} \sum_{k=0}^{n} \binom{n}{k} F_k(x)
\]

which completes the proof.  

We have the following symmetric relations between geometric polynomials and their derivatives.

Corollary 13

\[
\sum_{k=0}^{n} \binom{n}{k} xF'_k(x) = \sum_{k=1}^{n} \binom{n}{k-1} F_k(x)
\]

(41)

Proof. Combining results of Proposition 9 and Proposition 12 gives (41).
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