Bethe–Salpeter equation in QCD

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Abstract

We extend to regular QCD the derivation of a confining $q\bar{q}$ Bethe–Salpeter equation previously given for the simplest model of scalar QCD in which quarks are treated as spinless particles. We start from the same assumptions on the Wilson loop integral already adopted in the derivation of a semirelativistic heavy quark potential. We show that, by standard approximations, an effective meson squared mass operator can be obtained from our BS kernel and that, from this, by $\frac{1}{m^2}$ expansion the corresponding Wilson loop potential can be reobtained, spin–dependent and velocity–dependent terms included. We also show that, on the contrary, neglecting spin–dependent terms, relativistic flux tube model is reproduced.

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In a preceding paper [1] we have derived a confining Bethe-Salpeter equation for the simplified model of a scalar QCD. In that we started from the same assumptions already used in the derivation of the semirelativistic potential for a heavy quark-antiquark system [2], [3]. The basic object was the Wilson loop integral

$$W = \frac{1}{3} \text{Tr} P \exp ig \left\{ \oint_{\Gamma} dx^\mu A_\mu \right\}, \quad (1.1)$$

where as usual the loop $\Gamma$ is supposed made by a quark world line ($\Gamma_1$), an antiquark world line ($\Gamma_2$) followed in the reverse direction and closed by two straight lines connecting the initial and the final points of the two world lines ($y_1, y_2$ and $x_1, x_2$); $A_\mu(x)$ denotes a colour matrix of the form $A_\mu(x) = \frac{1}{2} A^a_\mu \lambda^a$; $P_T$ prescribes the ordering along the loop and Tr denotes the trace of the above matrices; the expectation value stands for the functional integration on the gauge field alone.

The basic assumption was

$$i \ln W = i (\ln W)_{\text{pert}} + \sigma S_{\text{min}} \quad (1.2)$$

($\ln W)_{\text{pert}}$ being the perturbative contribution to $\ln W$ and $S_{\text{min}}$ the minimum area enclosed by $\Gamma$. Furthermore, we used for $S_{\text{min}}$ the straight line approximation, consisting in replacing $S_{\text{min}}$ with the surface spanned by the straight lines connecting equal time points on the quark and the antiquark worldlines. In practice we wrote

$$S_{\text{min}} \simeq \int_{t_i}^{t_f} dt \int_0^1 ds \left[ 1 - (s \frac{dz_1 T}{dt} + (1 - s) \frac{dz_2 T}{dt})^2 \right]^{\frac{1}{2}}, \quad (1.3)$$

where $t$ stands for the ordinary time in the center of mass frame, $z_1(t)$ and $z_2(t)$ for the quark and the antiquark positions at the time $t$, $\frac{dz_{1,2} T}{dt}$ denotes the transverse velocity $(\delta^{hk} - \frac{r_{hk}}{r^2}) \frac{dz_{1,2} T}{dt}$ of the particle and $r(t)$ the relative position $z_1(t) - z_2(t)$.

In ref.[1] an essential tool was the Feynman–Schwinger path–integral representation for the spinless one–particle propagator in an external field.
In this paper we want to extend the derivation to the case of the regular QCD with the lagrangian

\[ L = \sum_{f=1}^{N_f} \bar{\psi}_f (i\gamma^\mu D_\mu - m_f) \psi_f - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} + L_{GF} \]  

(1.4)

(where \( D_\mu = \partial_\mu - igA_\mu \) and \( L_{GF} \) is the gauge fixing term), in which the quarks are fermions and have spin and \( f \) is the flavour index.

To do this we find convenient to work in the second order formalism.

As usual the gauge invariant quark–antiquark Green function is given by

\[ G_{gi}^q(x_1, x_2, y_1, y_2) = \frac{1}{3} \langle 0 | T \bar{\psi}_2^c(x_2) U(x_2, x_1) \psi_1(x_1) \bar{\psi}_1(y_1) U(y_1, y_2) \psi_2^c(y_2) | 0 \rangle = \frac{1}{3} \text{Tr} (U(x_2, x_1) S_1(x_1, y_1; A) U(y_1, y_2) C^{-1} S_2(y_2, x_2; A) C) \]  

(1.5)

where \( c \) denotes the charge-conjugate fields, \( C \) is the charge-conjugation matrix, \( U \) the path-ordered gauge string

\[ U(b, a) = P_{ba} \exp \left( ig \int_a^b dx^\mu A_\mu(x) \right) \]  

(1.6)

(the integration path being the straight line joining \( a \) to \( b \)), \( S_1 \) and \( S_2 \) the quark propagators in the external gauge field \( A^\mu \) and obviously

\[ \langle f[A] \rangle = \frac{\int \mathcal{D}[A] M_f(A) f[A] e^{iS[A]} \} \int \mathcal{D}[A] M_f(A) e^{iS[A]} \} , \]  

(1.7)

\( S[A] \) being the pure gauge field action and \( M_f(A) \) the determinant resulting from the explicit integration on the fermionic fields (which however in practice we set equal to 1 in the adopted approximation).

The propagators \( S_1 \) and \( S_2 \) are supposed to be defined by the equation

\[ (i\gamma^\mu D_\mu - m)S(x, y; A) = \delta^4(x - y) \]  

(1.8)

and the appropriate boundary conditions.

Alternatively we can set

\[ S(x, y; A) = (i\gamma^\nu D_\nu + m) \Delta^\sigma(x, y; A) \]  

(1.9)
and have

$$(D_\mu D^\mu + m^2 - \frac{1}{2} g \sigma^{\mu\nu} F_{\mu\nu}) \Delta^q(x, y; A) = -\delta^q(x - y)$$ (1.10)

($$\sigma^{\mu\nu} = \frac{i}{2} [\gamma^\mu, \gamma^\nu]$$). Then, taking into account gauge invariance we can write

$$G^g_i(x_1, x_2; y_1, y_2) = (i\gamma^\mu \partial_{x_1\mu} + m_1)(i\gamma^\nu \partial_{x_2\nu} + m_2)H_4(x_1, x_2; y_1, y_2)$$ (1.11)

with

$$H_4(x_1, x_2; y_1, y_2) = -\frac{1}{3}\text{Tr}(U(x_2, x_1)\Delta^q(x_1, y_1; A)U(y_1, y_2)\tilde{\Delta}^q(x_2, y_2; -\tilde{A}))$$ (1.12)

and the tilde denotes transposition on the colour indices alone. What we shall show is that the ”second order” Green function $$H_4(x_1, x_2; y_1, y_2)$$ satisfies a Bethe-Salpeter type nonhomogeneous equation of the form

$$H_4(x_1, x_2; y_1, y_2) = H_2(x_1 - y_1) H_2(x_2 - y_2) - i \int d^4\xi_1 d^4\xi_2 d^4\eta_1 d^4\eta_2$$

$$H_2(x_1 - \xi_1) H_2(x_2 - \xi_2) I_4(\xi_1, \xi_2; \eta_1, \eta_2) H_4(\eta_1, \eta_2; y_1, y_2)$$ (1.13)

where $$H_2$$ stands for a kind of colour independent one particle dressed propagator and $$I_4$$ denotes the appropriate kernel which is obtained as an expansion in the strong coupling constant $$\alpha_s = \frac{g^2}{4\pi}$$ and in the string tension $$\sigma$$ (better in $$\sigma a^2$$, a being a characteristic length, typically the radius of the particular bound state).

At the lowest order in $$\alpha_s$$ and $$\sigma a^2$$ we can write in the momentum space

$$\hat{I}(p'_1, p'_2; p_1, p_2) = \hat{I}_{\text{pert}}(p'_1, p'_2; p_1, p_2) + \hat{I}_{\text{conf}}(p'_1, p'_2; p_1, p_2)$$ (1.14)

($$p'_1 + p'_2 = p_1 + p_2$$) with

$$\hat{I}_{\text{pert}} = 16\pi \frac{4}{3} \alpha_s \{D_{\rho\sigma}(Q) q_1^\rho q_2^\sigma - \frac{i}{4} \sigma_1^{\mu\nu}(\delta^\rho_{\mu} Q_{\nu} - \delta^\rho_{\nu} Q_{\mu}) q_2^\sigma D_{\rho\sigma}(Q)$$

$$+ \frac{i}{4} \sigma_2^{\mu\nu}(\delta^\rho_{\mu} Q_{\nu} - \delta^\rho_{\nu} Q_{\mu}) q_1^\rho D_{\rho\sigma}(Q) + \frac{1}{16} \sigma_1^{\mu_1\nu_1} \sigma_2^{\mu_2\nu_2} (\delta^\rho_{\mu_1} Q_{\nu_1} - \delta^\rho_{\nu_1} Q_{\mu_1})(\delta^\rho_{\mu_2} Q_{\nu_2} - \delta^\rho_{\nu_2} Q_{\mu_2}) D_{\rho\sigma}(Q)\} + \ldots$$ (1.15)

and
\[ \hat{I}_{\text{conf}} = \int d^3r \ e^{iQ \cdot r} J(r, q_1, q_2) \]  

with

\[
J(r, q_1, q_2) = \frac{2\sigma_r}{q_{10} + q_{20}} \left[ q_{20}^2 \sqrt{q_{10}^2 - q_{1T}^2} + q_{10}^2 \sqrt{q_{20}^2 - q_{1T}^2} + \frac{q_{10}q_{20}^2}{q_{1T}} \left( \arcsin \frac{q_{1T}}{q_{10}} + \arcsin \frac{q_{1T}}{q_{20}} \right) + 2\sigma \frac{q_{10}q_{20}q_{1\nu}r^k}{r\sqrt{q_{10}^2 + q_{1T}^2}} + 2\sigma \frac{q_{10}q_{20}q_{2\nu}r^k}{r\sqrt{q_{20}^2 - q_{1T}^2}} + \ldots \right]
\]

In (1.15)–(1.17) we have set

\[
q_1 = \frac{p_1' + p_1}{2}, \quad q_2 = \frac{p_2' + p_2}{2}, \quad Q = p_1' - p_1 = p_2 - p_2',
\]

\[ D(q) \] denotes the gluon free propagator and the center of mass system is understood \( q_1 = -q_2 = q, \) \( q_1^k = (\delta^{bk} - \delta^{h^k})q^k. \)

Eqs. (1.13)-(1.17) are the basic results of this paper.

Notice that Eq. (1.15) corresponds to the usual ladder approximation in this second order formalism, while (1.17) reduces to Eq. (1.8) of [1] when the spin dependent terms are neglected. Notice also that instead of (1.13) one could have written the homogenous equation

\[
\hat{\Phi}_P(k') = -i \int \frac{d^4k}{(2\pi)^2} \hat{H}_2(\eta_1 P + k') \hat{H}_2(\eta_2 P - k') \hat{I}(k', k; P) \Phi_P(k)
\]

which is more appropriate for the bound state problem. In this equation \( \eta_j = \frac{m_j}{m_1 + m_2} P \) denotes the total momentum \( p_1 + p_2, \) \( k \) the relative momentum \( \eta_2 p_1 - \eta_1 p_2 \) \( (q_j = \eta_j P + \frac{k + k'}{2} \) and in the CM frame \( q = \frac{k + k'}{2} \), \( \Phi_P(k) \) is the ordinary Bethe–Salpeter wave function in the momentum space.

From (1.19) by replacing \( \hat{H}_2(p) \) with the free propagator \( \frac{\gamma}{p^2 - m^2} \) and performing an appropriate instantaneous approximation on \( \hat{I} \) [consisting in setting \( Q_0 = 0, j_0 = \frac{w_j + w_j'}{2} \) or \( p_{j0} = p_{j0}' = \frac{w_j + w_j'}{2} \) or \( k_0 = k_0' = \eta_2 \frac{w_j + w_1}{2} - \eta_1 \frac{w_j' + w_2}{2} \) and \( P_0 = \frac{1}{2}(w_j + w_1 + w_j' + w_2) \); with \( w_j = \sqrt{m_j^2 + k^2}, w_j' = \sqrt{m_j^2 + k'^2} \) one can obtain an effective square mass operator for the mesons (in the CM frame \( P = 0, P = (m_B, 0) \) )
\[ M^2 = M_0^2 + U \] (1.20)

with \( M_0 = \sqrt{m_1^2 + k^2} + \sqrt{m_2^2 + k^2} \) and

\[ \langle k'|U|k \rangle = \frac{1}{(2\pi)^3} \sqrt{\frac{w_1' + w_2'}{2w_1'w_2'}} I_{\text{inst}}(k', k) \sqrt{\frac{w_1 + w_2}{2w_1w_2}}. \] (1.21)

The quadratic form of Eq. (1.20), obviously derives from the second order character of the formalism we have used. It should be mentioned that for light mesons this form seems phenomenologically favoured with respect to the linear one.

In more usual terms one can also write

\[ M = M_0 + V \] (1.22)

with

\[ \langle k'|V|k \rangle = \frac{1}{(2\pi)^3} \frac{1}{4\sqrt{w_1w_2w_1'w_2'}} I_{\text{inst}}(k', k) + \ldots \] (1.23)

where the dots stand for higher order terms in \( \alpha_s \) and \( \sigma a^2 \) and kinematical factors equal to 1 on the energy shell have been neglected. In the limit of small \( \frac{p^2}{m^2} \) the potential \( V \) as given by (1.20)–(1.22) reproduces the semirelativistic potential of ref. [2], [3]. Similarly, if we neglect in \( V \) the spin dependent terms and the coulombian one, we reobtain the hamiltonian of the relativistic flux tube model [4] with an appropriate ordering prescription [3]. However we have not yet completely understood the relation between the spin dependent terms we obtain and those appearing in the relativistic flux tube model with “fermionic ends” recently proposed [5].

Finally we want to mention that a result directly in hamiltonian form (1.22) but strictly related to our one has been obtained by Simonov and collaborators [6,7].

The paper is organized in this way. In Sect. II we discuss the Feynman–Schwinger representation for the one quark propagator in an external field, in Sect. III and IV we study the corresponding representation for \( H_4 \) and derive the BS equation for such quantity, in Sect. V we introduce the effective mass operator and consider its semirelativistic limit and its relation with the flux tube model. Finally in Sect. VI we summarize the results and make some additional remarks.
II. THE FEYNMAN-SCHWINGER REPRESENTATION

The solution of Eq.(1.10) can be expressed in terms of path integral as (Feynman-Schwinger representation)

\[
\Delta^\sigma(x, y; A) = -\frac{i}{2} \int_0^\infty ds \exp \frac{is}{2} \left( -D_\mu D^\mu - m^2 + \frac{1}{2} g\sigma^{\mu\nu} F_{\mu\nu} \right)
\]

\[
= -\frac{i}{2} \int_0^\infty ds \int_y^x Dz P_{xy} T_{xy} \exp \left[ \int_0^s d\tau \left( -\frac{1}{2} (m^2 + \dot{z}^2) + gA_\rho(z) \dot{z}^\rho + \frac{g}{4} \sigma^{\mu\nu} F_{\mu\nu}(z) \right) \right]
\] (2.1)

where the path integral is understood to be extended over all paths \( z^\mu = z^\mu(\tau) \) connecting \( y \) with \( x \) and expressed in terms of a parameter \( \tau \) with \( 0 \leq \tau \leq s \), \( \dot{z} \) stands for \( \frac{dz}{d\tau} \), the “functional measure” is assumed to be defined as

\[
Dz = \left( \frac{1}{2\pi i \epsilon} \right)^{2N} d^4 z_1 \ldots d^4 z_{N-1},
\] (2.2)

\( P_{xy} \) and \( T_{xy} \) prescribe the ordering along the path from right to left respectively of the colour matrices and of the spin matrices.

On the other side, it is well known that, as a consequence of a variation in the path \( z^\mu(\tau) \rightarrow z^\mu(\tau) + \delta z^\mu(\tau) \) respecting the extreme points, one finds

\[
\delta \{ P_{xy} \exp ig \int_0^s d\tau \dot{z}^\mu(\tau) A_\mu(z) \} =
\]

\[
= ig \int_0^s \delta S^{\mu\nu}(z(\tau)) P_{xy} \left\{ - F_{\mu\nu}(z(\tau)) \exp \int_0^s d\tau' \dot{z}^\mu(\tau') A_\mu(z(\tau')) \right\}
\] (2.3)

with \( \delta S^{\mu\nu}(z) = \frac{1}{2} (dz^\mu \delta z^\nu - dz^\nu \delta z^\mu) \). So one can naturally write

\[
T_{xy} \exp \left( -\frac{1}{4} \int_0^s d\tau \sigma^{\mu\nu} \frac{\delta}{\delta S^{\mu\nu}(z)} \right) \left( P_{xy} \exp ig \int_0^s d\tau' \dot{z}^\mu(\tau') A_\mu(z(\tau')) \right)
\]

\[
= T_{xy} P_{xy} \exp ig \int_0^s d\tau [\dot{z}^\mu(\tau) A_\mu(z(\tau)) + \frac{1}{4} \sigma^{\mu\nu} F_{\mu\nu}(z(\tau))]
\] (2.4)

and Eq.(2.1) can be rewritten as

\[
\Delta^\sigma(x, y; A) = -\frac{i}{2} \int_0^s d\tau \int_y^x Dz P_{xy} T_{xy} S_0^s \exp \left[ \int_0^s d\tau \left[ -\frac{1}{2} (m^2 + \dot{z}^2) + ig\dot{z}^\mu A_\mu(\bar{z}) \right] \right]
\] (2.5)

with

\[
S_0^s = \exp \left[ -\frac{1}{4} \int_0^s d\tau \sigma^{\mu\nu} \frac{\delta}{\delta S^{\mu\nu}(\bar{z})} \right]
\] (2.6)
In (2.6) it is understood that $\bar{z}^\mu(\tau)$ has to be put equal to $z^\mu(\tau)$ after the action of $S_{0}^{s_{1}}$. Alternatively, it is convenient to write $\bar{z} = z + \zeta$, assume that $S_{0}$ acts on $\zeta(\tau)$ with $\delta S_{\mu\nu}(z) = \frac{1}{2}(dz^\mu \delta \zeta^\nu - dz^\nu \delta \zeta^\mu)$ and set eventually $\zeta = 0$.

### III. THE FEYNMAN–SCHWINGER REPRESENTATION

Replacing (2.5) in (1.12) we obtain

$$H_{4}(x_{1}, x_{2}; y_{1}, y_{2}) = \frac{i}{2} \int_{t}^{t'} \int_{0}^{1} ds_{1} \int_{0}^{1} ds_{2} \int_{y_{1}}^{x_{1}} Dz_{1} \int_{y_{2}}^{x_{2}} Dz_{2} T_{x_{1}y_{1}} T_{x_{2}y_{2}} S_{0}^{s_{1}} S_{0}^{s_{2}}$$

$$\exp \left( \frac{-i}{2} \left\{ \int_{0}^{s_{1}} d\tau_{1}(m_{1}^{2} + \dot{z}_{1}^{2}) + \int_{0}^{s_{2}} d\tau_{2}(m_{2}^{2} + \dot{z}_{2}^{2}) \right\} \right)$$

$$\frac{1}{3} \langle \text{TrP} \exp(i) \{ \oint_{\Gamma} dz_{\mu} A_{\mu}(\bar{z}) \} \rangle$$

(3.1)

where now $\bar{z} = \bar{z}_{j} = z_{j} + \zeta_{j}$ on $\Gamma_{1}$ and $\Gamma_{2}$; $\bar{z} = z$ on the end lines $x_{1}x_{2}$ and $y_{2}y_{1}$ and the final limit $\zeta_{j} \rightarrow 0$ is again understood.

Then, let us try to be more explicit concerning Eq. (1.2) and (1.3). For the first term in (1.2) we have, at the lowest order of perturbation theory,

$$i(\ln W)_{\text{pert}} = i \ln \left( \frac{1}{3} \text{TrP} \exp ig \oint_{\Gamma} dz_{\mu} A_{\mu}(z) \right)_{\text{pert}} = \frac{4}{3} g^{2} \int_{0}^{s_{1}} d\tau_{1} \int_{0}^{s_{2}} d\tau_{2} D_{\mu\nu}(z_{1} - z_{2}) \dot{z}_{1}^{\mu} \dot{z}_{2}^{\nu}$$

$$- \frac{2}{3} g^{2} \int_{0}^{s_{1}} d\tau_{1} \int_{0}^{s_{1}} d\tau_{1} D_{\mu\nu}(z_{1} - z_{1}') \dot{z}_{1}^{\mu} \dot{z}_{1}'^{\nu} - \frac{2}{3} g^{2} \int_{0}^{s_{2}} d\tau_{2} \int_{0}^{s_{2}} d\tau_{2} D_{\mu\nu}(z_{2} - z_{2}') \dot{z}_{2}^{\mu} \dot{z}_{2}'^{\nu} + \ldots$$

(3.2)

On the other side, for the second one in general we have to write

$$S_{\text{min}} = \int_{t_{i}}^{t_{f}} dt \int_{0}^{1} ds \left[ - \left( \frac{\partial u^{\mu}}{\partial t} \frac{\partial u^{\mu}}{\partial t} \right) - \left( \frac{\partial u^{\mu}}{\partial s} \frac{\partial u^{\mu}}{\partial s} \right) + \left( \frac{\partial u^{\mu}}{\partial t} \frac{\partial u^{\mu}}{\partial s} \right)^{2} \right]$$

(3.3)

$x^{\mu} = u^{\mu}(t, s)$ being the equation of the minimal surface with contour $\Gamma$. Let us assume that for fixed $t$ and for $s$ varying from 0 to 1, $u^{\mu}(s, t)$ describes a line connecting a point on the quark world line $\Gamma_{1}$ with one on the antiquark world line $\Gamma_{2}$,

$$u^{\mu}(1, t) = z_{1}^{\mu}(\tau_{1}(t)), \quad u^{\mu}(0, t) = z_{2}^{\mu}(\tau_{2}(t))$$

(3.4)

Obviously (3.3) is invariant under reparametrization, so a priori $t$ and $s$ could be everything. In particular if $\Gamma_{1}$ and $\Gamma_{2}$ never go backwards in time, $t$ can be choosen as the ordinary time
$u^0(s, t) \equiv t$. For the moment let us assume that this is the case. Then $\tau_1(t)$ and $\tau_2(t)$ are specified by the equation

$$z^0_1(\tau_1) = z^0_2(\tau_2) \quad (3.5)$$

and we can set

$$L = \int_0^1 ds \{ - \left( \frac{\partial u^\mu}{\partial t} \frac{\partial u^\mu}{\partial t} \right) \left( \frac{\partial u^\mu}{\partial s} \frac{\partial u^\mu}{\partial s} \right) + \left( \frac{\partial u^\mu}{\partial t} \frac{\partial u^\mu}{\partial s} \right)^2 \}^{\frac{1}{2}}. \quad (3.6)$$

Obviously $L$ cannot depend only on on the extremal points $z_1(\tau_1)$ and $z_2(\tau_2)$ but has to depend even on the shape of the world lines at least in a neighbourhood of such points. So, we can think of it as a function of all derivatives in $\tau_1$ and $\tau_2$ and write $L = L(z_1, z_2, \dot{z}_1, \dot{z}_2, \ldots)$.

Finally (3.3) can be rewritten as

$$S_{\text{min}} = \int dz^0_1 \int dz^0_2 \delta(z^0_1 - z^0_2) L(z_1, z_2, \dot{z}_1, \dot{z}_2, \ldots) = \int d\tau_1 \int d\tau_2 \delta(z^0_1 - z^0_2) z^0_1 z^0_2 L(z_1, z_2, \dot{z}_1, \dot{z}_2, \ldots). \quad (3.7)$$

and in principle this expression can be considered a good approximation even if the world lines contain pieces going backwards in time. In fact, in such a case if we fix e.g. $\tau_1$, (3.5) has more than one solution in $\tau_2$ and if $\Gamma_1$ and $\Gamma_2$ are not too much irregular in space (otherwise $S_{\text{min}}$ is large and the weight of the loop is small) the minimal surface can be reconstructed as the algebraic sum of various pieces of surface.

In the straight line approximation we must choose

$$u^0(s, t) = t$$

$$u^k(s, t) = sz^k_1(\tau_1(t)) + (1 - s)z^k_2(\tau_2(t)) \quad (3.8)$$

and we have

$$z^0_1 \dot{z}^0_2 L = \sigma |z'_1 - z'_2| \int_0^1 ds \{ \dot{z}^0_1 \dot{z}^0_2 - (s\dot{z}^0_1 \dot{z}^0_2 + (1 - s)\dot{z}^0_2 \dot{z}^0_2) \}^{\frac{1}{2}} \quad (3.9)$$

which introduced in (3.7) becomes equivalent to Eq.(1.3). The important point concerning (3.7) with (3.9) is that it has the same general form as (3.2). However we stress that the
approximation (3.8) must be performed only \textit{after} that the application of the operators $S_0^s$ and $S_0^{s2}$ has been performed.

Substituting (3.2) and (3.7) in (3.1) we obtain

$$
H_4(x_1, x_2, y_1, y_2) = \left(\frac{1}{2}\right)^2 \int_0^\infty ds_1 \int_0^\infty ds_2 \int_{-x_1}^{x_1} Dz_1 \int_{y_2}^{x_2} Dz_2 T_{x_1y_1} T_{x_2y_2} S_0^s S_0^{s2} \exp \left\{ -\frac{1}{2} \int_0^{s_1} d\tau_1 (m_1^2 + \dot{z}_1^2) - \frac{1}{2} \int_0^{s_2} d\tau_2 (m_2^2 + \dot{z}_2^2) + \right\}
$$

$$
+ \frac{2}{3} g^2 \int_0^{s_1} d\tau_1 \int_0^{s_2} d\tau_2 D_{\mu\nu}(\dot{z}_1 - \dot{z}_1') \dot{z}_1^\mu \dot{z}_1^\nu + \frac{2}{3} g^2 \int_0^{s_2} d\tau_2 \int_0^{s_2} d\tau_2' D_{\mu\nu}(\dot{z}_2 - \dot{z}_2') \dot{z}_2^\mu \dot{z}_2^\nu
$$

$$
- \int_0^{s_1} d\tau_1 \int_0^{s_2} d\tau_2 E(\dot{z}_1, \dot{z}_2, \dot{\dot{z}}_1, \dot{\dot{z}}_2, \ldots) \right\}, \quad (3.10)
$$

where we have set

$$
E(\dot{z}_1, \dot{z}_2, \dot{\dot{z}}_1, \dot{\dot{z}}_2, \ldots) = \frac{4}{3} g^2 D_{\mu\nu}(z_1 - z_2) \dot{z}_1^\mu \dot{z}_2^\nu +
$$

$$
+ \sigma \delta(\dot{z}_{10} - \dot{z}_{20}) \dot{z}_{10} \dot{z}_{20} L(z_1, z_2, \dot{z}_1, \dot{z}_2, \ldots). \quad (3.11)
$$

Now let us denote the quantity in curly bracket in (3.10) by $S_4$ and perform a Legendre transformation by introducing the momenta $p_{j\mu} = -\frac{\delta S_4}{\delta \dot{z}_j^\mu}$ (in this the various quantities $z_j$, $\dot{z}_j$, $\ddot{z}_j$, \ldots are assumed to be treated as independent)

$$
p_{\mu 1} = \dot{z}_1^\mu + \frac{4}{3} g^2 \int_0^{s_1} d\tau_1' D_{\mu\nu}(\dot{z}_1 - \dot{z}_1') \dot{z}_1' + \int_0^{s_2} d\tau_2' \frac{\partial E(\dot{z}_1, \dot{z}_2', \ddot{z}_1, \ddot{z}_2, \ldots)}{\partial \dot{z}_1'}
$$

$$
p_{\mu 2} = \dot{z}_2^\mu + \frac{4}{3} g^2 \int_0^{s_2} d\tau_2' D_{\mu\nu}(z_2 - z_2') \dot{z}_2' + \int_0^{s_2} d\tau_1' \frac{\partial E(\dot{z}_1', \dot{z}_2', \ddot{z}_1', \ddot{z}_2', \ldots)}{\partial \dot{z}_2'}. \quad (3.12)
$$

Eq. (3.12) cannot be inverted in a closed form with respect to $\dot{z}_1$ and $\dot{z}_2$. However, we can do this by an expansion in $\alpha_s = \frac{g^2}{4\pi}$ and $\sigma a^2$ and at the lowest order we have

$$
\dot{z}_1^\mu = p_1^\mu - \frac{4}{3} g^2 \int_0^{s_1} d\tau_1' D_{\mu\nu}(\dot{z}_1 - \dot{z}_1') \dot{p}_1^\nu - \int_0^{s_2} d\tau_2' \frac{\partial E(\dot{z}_1, \dot{z}_2', \ddot{p}_1, \ddot{p}_1', \ldots)}{\partial \dot{p}_1^\mu} + \ldots
$$

$$
\dot{z}_2^\mu = p_2^\mu - \frac{4}{3} g^2 \int_0^{s_2} d\tau_2' D_{\mu\nu}(\dot{z}_2 - \dot{z}_2') \dot{p}_2^\nu - \int_0^{s_2} d\tau_1' \frac{\partial E(\dot{z}_1', \dot{z}_2', \ddot{p}_1', \ddot{p}_2, \ldots)}{\partial \dot{p}_2^\mu} + \ldots \quad (3.13)
$$

with

$$
\ddot{p}_j^\mu = \ddot{p}_j^\mu + \dddot{p}_j^\mu. \quad (3.14)
$$

In conclusion we find
\[ H_4(x_1, x_2, y_1, y_2) = \left( \frac{1}{2} \right)^2 \int_0^\infty ds_1 \int_0^\infty ds_2 \int_y^1 dz_1 Dp_1 \int_y^{x_1} dz_2 Dp_2 T_{x_1 y_1} T_{x_2 y_2} \] 

where

\[ S_0^{s_1} S_0^{s_2} \exp \left\{ \int_0^{s_1} d\tau_1 K_1 + \int_0^{s_2} d\tau_2 K_2 - \int_0^{s_1} d\tau_1 \int_0^{s_2} d\tau_2 E(z_1, z_2, \bar{p}_1, \bar{p}_2, \ldots) + \ldots \right\}. \tag{3.15} \]

includes the self-interacting term. Notice that now in \( S_0^{s_1} \) it must be understood \( \delta S^{\mu\nu}(z_j) = \frac{1}{2} d\tau_j (p_j^\mu \delta \xi_j^\nu - p_j^\nu \delta \xi_j^\mu) + \ldots \)

Eq. (3.15) is the starting point for the derivation of our Bethe–Salpeter equation.

**IV. THE HOMOGENEOUS BETHE–SALPETER EQUATION**

In Eq. (3.15) we proceed along the same line followed in Ref. [1].

Applying to the interaction term \( E \) the identity

\[ \exp \int_0^s d\tau A(\tau) = 1 + \int_0^s d\tau A(\tau) \exp \left( \int_0^\tau d\tau' A(\tau') \right) \tag{4.1} \]

we have

\[ H_4(x_1, x_2; y_1, y_2) = \left( \frac{1}{2} \right)^2 \int_0^\infty ds_1 \int_0^\infty ds_2 \int_y^1 dz_1 Dp_1 \int_y^{x_1} dz_2 Dp_2 \]

\[ T_{x_1 y_1} T_{x_2 y_2} S_0^{s_1} S_0^{s_2} \exp \left[ \int_0^{s_1} d\tau_1 K_1 + \int_0^{s_2} d\tau_2 K_2 \right] - i \int_0^{s_1} d\tau_1 \int_0^{s_2} d\tau_2 E(z_1, z_2, \bar{p}_1, \bar{p}_2, \ldots) \]

\[ \times \exp \left\{ \int_0^{s_1} d\tau_1 K_1 + \int_0^{s_2} d\tau_2 K_2 - \int_0^{s_1} d\tau_1' \int_0^{s_2} d\tau_2' E(z_1', z_2', \bar{p}_1', \bar{p}_2', \ldots) \right\}. \tag{4.2} \]

Now, using the method of Ref. [1] and having in mind (3.11), (3.6) and (3.10) one finds (see Appendix for details)

\[ \frac{\delta}{\delta S^{\mu\nu}(z_1')} \int_0^{s_1} d\tau_1' \int_0^{s_2} d\tau_2' E(z_1', z_2', \bar{p}_1', \bar{p}_2', \ldots) = \]

\[ = \int_0^{s_2} d\tau_2' \left[ \frac{4}{3} g^2 (\partial_\mu D^{\mu\nu}(z_1 - z_2') - \partial_{\mu'} D^{\nu\sigma}(z_1 - z_2')) p_2' + \sigma \delta(z_{10} - z_{20}) \right. \]

\[ \left. \frac{p_{1\nu}(z_{1\mu} - z_{2\mu}' - p_{1\mu}(z_{1\nu} - z_{2\nu}')}{\sqrt{(p_{10}^2 - \bar{p}_1^2)(z_1 - z_2)^2 + (p_1 \cdot (z_1 - z_2'))^2 + \ldots} \right] \tag{4.3} \]

and a similar result, with a minus sign of difference in front, for the derivative in \( \frac{\delta}{\delta S^{\mu\nu}(z_2')} \).
Furthermore
\[ \frac{\delta^2}{\delta S^\mu\nu(z_1)\delta S^\rho\sigma(z_1')} \int_0^{s_1} d\tau_1'' \int_0^{s_2} d\tau_2'' E = \frac{\delta^2}{\delta S^\mu\nu(z_2)\delta S^\rho\sigma(z_2')} \int_0^{s_1} d\tau_1'' \int_0^{s_2} d\tau_2'' E = 0 \]
(4.4)

but
\[ \frac{\delta^2}{\delta S^\mu\nu_1(z_1)\delta S^\mu\nu_2(z_2)} \int_0^{s_1} d\tau_1'' \int_0^{s_2} d\tau_2'' E = \frac{4}{3} g^2 (\delta_1^\mu_1 \partial_{\nu_1} - \delta_2^\mu_2 \partial_{\nu_2})(\delta_1^\nu_1 \partial_{\mu_1} - \delta_2^\nu_2 \partial_{\mu_2}) D_{\rho\sigma}(z_1 - z_2) \]
(4.5)

Then, taking into account the relation
\[ e^A Be^{-A} = \sum_{n=0}^{\infty} \frac{1}{n!} [A, [A, \ldots [A, B], \ldots]], \]
(4.6)

and specifically \([4, 4]\), we have
\[ S_0^0 S_0^{s_2} \int_0^{s_1} d\tau_1 \int_0^{s_2} d\tau_2 E(\bar{z}_1, \bar{z}_2, \bar{p}_1, \bar{p}_2, \ldots)(S_0^0 S_0^{s_2})^{-1} = \]
\[ = (1 - \frac{1}{4} \int_0^{s_1} ds_1' \sigma_1^{\mu\nu} \frac{\delta}{\delta S^{\mu\nu_1}(z_1')})(1 - \frac{1}{4} \int_0^{s_2} ds_2' \sigma_2^{\mu\nu} \frac{\delta}{\delta S^{\mu\nu_2}(z_2')}) \]
\[ \int_0^1 d\tau_1 \int_0^1 d\tau_2 E(\bar{z}_1, \bar{z}_2, \bar{p}_1, \bar{p}_2, \ldots) = R(z_1, z_2, p_1, p_2) \]
(4.7)

with
\[ R = R_{\text{pert}} + R_{\text{conf}} \]
(4.8)

\[ R_{\text{pert}} = -\frac{4}{3} g^2 \left\{ D_{\rho\sigma}(z_1 - z_2)p_1^\rho p_2^\sigma \right\} \]
\[ -\frac{1}{4} \sigma_1^{\mu\nu}(\delta_1^\mu \partial_1^\nu - \delta_2^\mu \partial_2^\nu) D_{\rho\sigma}(z_1 - z_2)p_1^\sigma \]
\[ -\frac{1}{4} \sigma_2^{\mu\nu}(\delta_1^\mu \partial_1^\nu - \delta_2^\mu \partial_2^\nu) D_{\rho\sigma}(z_1 - z_2)p_2^\rho \]
\[ + \frac{1}{16} \sigma_1^{\mu\nu_1} \sigma_2^{\mu\nu_2}(\delta_1^\mu \partial_{\nu_1} - \delta_2^\mu \partial_{\nu_2})(\delta_1^\nu \partial_{\mu_1} - \delta_2^\nu \partial_{\mu_2}) D_{\rho\sigma}(z_1 - z_2) \]
(4.9)

and
\[ R_{\text{conf}} = \sigma (z_10 - z_{20}) \left\{ |z_1 - z_2| \int_0^1 ds \sqrt{p_{10}^2 p_{20}^2 - [sp_{1T}p_{20} + (1 - s)p_{2T}p_{10}]^2} \right. \]
\[ -\frac{1}{4} p_{20} \sigma_1^{\mu\nu_1} p_{1\mu}(z_1 - z_{20}) - p_{1\mu}(z_1 - z_{20}) \]
\[ + \frac{1}{4} p_{10} \sigma_2^{\mu\nu_2} p_{2\nu}(z_1 - z_{20}) - p_{2\nu}(z_1 - z_{20}) \right\}. \]
(4.10)

Finally setting
\[ H_2(x - y) = \frac{-i}{2} \int_0^\infty ds \int_y^x DzD\rho S_0^* \exp i \int_0^s d\tau K \] (4.11)

Eq.(1.18) can be written

\[
H_4(x_1, x_2; y_1, y_2) = H_2(x_1 - y_1)H_2(x_2 - y_2) + 
- \frac{i}{4} \int_0^\infty ds_1 \int_0^\infty ds_2 \int_{y_1}^{x_1} Dz_1 Dp_1 \int_{y_2}^{x_2} Dz_2 Dp_2 T_{x_1y_1} T_{x_2y_2} \int_0^{s_1} d\tau_1 \int_0^{s_2} d\tau_2 R(z_1, z_2, p_1, p_2)
\]

\[ S_0^1 S_0^2 \exp i \left\{ \int_0^{s_1} d\tau_1' K_1' + \int_0^{s_2} d\tau_2' K_2' - i \int_0^{\tau_1} d\tau_1 \int_0^{\tau_2} d\tau_2 E(z_1', z_2', p_1, p_2, \ldots) \right\}. \] (4.12)

At this point it is necessary to take explicitly into account the discrete form of (4.12).

If we take

\[
P \exp [ig \oint_\Gamma dz^\mu A_\mu(z)] = P \prod_\Gamma U(z_n, z_{n-1}) \exp ig \sum_\Gamma (z^\mu_n - z^\mu_{n-1})A_\mu(z_n + z_{n-1})^{-1} \] (4.13)

(as required by a gauge invariant definition of the integral on the gluon field) we have

\[
H_4(x_1, x_2; y_1, y_2) = H_2(x_1 - y_1)H_2(x_2 - y_2) - \frac{i}{4} \varepsilon^2 \sum_{N_1=0}^\infty \sum_{N_2=0}^\infty \frac{1}{(2\pi)^{N_1+N_2}} \int d^4p_{11} d^4z_{11} \ldots d^4z_{1N_1-1} d^4p_{1N_1} \int d^4p_{21} d^4z_{21} \ldots d^4z_{2N_2-1} d^4p_{2N_2-1} T_{x_1y_1} T_{x_2y_2}
\]

\[
\sum_{R_1=1}^{N_1-1} \sum_{R_2=1}^{N_2-1} R(\frac{z_{1R_1} + z_{1R_1-1}}{2}, \frac{z_{2R_2} + z_{2R_2-1}}{2}, p_{1R}, p_{2R})
\]

\[ S_0^1 S_0^2 \exp i \left\{ \sum_{j=1}^{2N_i} \sum_{j'=1}^{2N_i'} \left[ - p_{jn}(z_{jn} - z_{jn-1}) + \varepsilon(p_{jn}^2 - m_j^2) - \frac{4}{3} g^2 \varepsilon^2 \sum_{n'=1}^{n-1} D_\mu(\frac{z_{jn'} + z_{jn'-1}}{2} - \frac{z_{jn'} + z_{jn'-1}}{2})p_{jn'}^\mu p_{jn'}^\nu \right]
\]

\[-\varepsilon^2 \sum_{n_1=1}^{R_1-1} \sum_{n_2=1}^{R_2} E(\frac{z_{1n_1} + z_{1n_1-1}}{2}, \frac{z_{2n_2} + z_{2n_2-1}}{2}, p_{1n_1}, p_{2n_2}, \ldots) \} \] (4.14)

If we neglect in the exponent the “non planar” terms \( \sum_{n=R_1+1}^{N_1} \sum_{n'=1}^{R_1} D_\mu p_{jn}^\mu p_{jn'}^\nu \) and \( \sum_{n_1=1}^{R_1} \sum_{n_2=R_2+1}^{N_2} D_\mu p_{1n}^\mu p_{2n_2}^\nu \), corresponding in the continuous to the quantity

\[
\int_0^{s_j} d\tau_j' \int_0^{\tau_j} d\tau_j'' D_\mu(\tau_j' - \tau_j'')p_{j'}^\mu p_{j''}^\nu \] (4.15)

and

\[
\int_0^{\tau_1} d\tau_1' \int_0^{s_2} d\tau_2' D_\mu(\tau_1' - \tau_2')p_{1}^\mu p_{2}^\nu, \] (4.16)
Eq. (1.14) can be written

\[
H_4(x_1, x_2, y_1, y_2) = H_2(x_1 - y_1)H_2(x_2 - y_2) - \frac{i}{4} \varepsilon^4 \sum_{R_1=1}^{\infty} \sum_{R_2=1}^{\infty} \sum_{N_1=R_1+1}^{\infty} \sum_{N_2=R_2+1}^{\infty} \int_{y_1}^{x_1} Dz_1 Dp_1 \int_{y_2}^{x_2} Dz_2 Dp_2
\]

\[
\frac{1}{(2\pi)^4} \int d^4z_1 R_1 d^4p_1 R_1 d^4z_1 R_1 - \frac{1}{(2\pi)^4} \int d^4z_2 R_2 d^4p_2 R_2 d^4z_2 R_2 - 1 \int_{y_1}^{x_1} Dz_1 Dp_1 \int_{y_2}^{x_2} Dz_2 Dp_2
\]

\[
T_{x_1z_1R} T_{x_2z_2R} S_{R_1}^{e_1} S_{R_2}^{e_2} \exp i \left\{ \sum_{j=1}^{2} \sum_{n=R_j+1}^{N_1} \left[ -p_{jn}(z_{jn} - z_{jn-1}) + \varepsilon(p_{jn}^2 - m_j^2) - \frac{4}{3} g^2 \varepsilon^2 - \sum_{n'=R_j+1}^{N_1} D_{\mu\nu} p_{jn} \eta^\nu \right] \right\}
\exp \left[-i \sum_j p_j R(z_j R_j - z_{jR_j-1}) R(\frac{z_1 R_1 + z_1 R_1 - 1}{2}, \frac{z_2 R_2 + z_2 R_2 - 1}{2}, p_{1R}, p_{2R}) \right]
\]

\[
T_{z_1R_1} T_{z_2R_2} S_0^{e_1} S_0^{e_2} \exp i \left\{ \sum_{j=1}^{2} \sum_{n=R_j+1}^{N_1} \left[ -p_{jn}(z_{jn} - z_{jn-1}) + \varepsilon(p_{jn}^2 - m_j^2) - \sum_{n'=R_j+1}^{N_1} D_{\mu\nu} p_{jn} \eta^\nu \right] \right\}
\exp \left[ \sum_{n_1=1}^{R_1-1} \sum_{n_2=1}^{R_2-1} E(\frac{z_1 n_1 + z_1 n_1 - 1}{2}, \frac{z_2 n_2 + z_2 n_2 - 1}{2}, p_{1n_1}, p_{2n_2}) \right] \]

(4.17)

which, going back to the continuous corresponds to Eq.(1.13) with

\[
I(\xi_1, \xi_2, \eta_1, \eta_2) = -4i \int \frac{d^4 k_1 d^4 k_2}{(2\pi)^8} R(\frac{\xi_1 + \eta_1}{2}, \frac{\xi_2 + \eta_2}{2}, k_1, k_2) \exp \{ (-i)(\xi_1 - \eta_1) k_1 + (\xi_2 - \eta_2) k_2) \}
\]

(4.18)

In conclusion, taking the Fourier transform

\[
(2\pi)^4 \delta(p'_1 + p'_2 - p_1 p_2) \hat{I}(p'_1, p'_2; p_1, p_2) = -4i \int d^4 \xi_1 d^4 \xi_2 \int d^4 \eta_1 d^4 \eta_2 \int \frac{d^4 k_1}{(2\pi)^4} \frac{d^4 k_2}{(2\pi)^4} e^{i(p'_1 - k_1) \xi_1 + i(p'_2 - k_2) \xi_2} R(\frac{\xi_1 + \eta_1}{2}, \frac{\xi_2 + \eta_2}{2}, k_1, k_2) e^{-i(p_1 - k_1) \eta_1 - i(p_2 - k_2) \eta_2}
\]

(4.19)

we obtain (1.14)–(1.17).

V. EFFECTIVE MASS OPERATOR

As we mentioned, for bound states Eq. (1.13) can be replaced by

\[
\Phi_P(k') = -i \int \frac{d^4 k}{(2\pi)^4} \hat{H}_2(\eta_1 P + k') \hat{H}_2(\eta_2 P - k') \hat{I}(k', k; P) \Phi_P(k)
\]

(5.1)

\[(P = p_1 + p_2, k = \eta_2 p_1 - \eta_1 p_2, k' = \eta_2 p'_1 - \eta_1 p'_2) and in the center of mass frame P = 0, P_0 = \sqrt{s}, p_1 = -p_2 = k, p'_1 = -p'_2 = k'.\]
Let us recall the definition of the instantaneous kernel given in Sec. 1 and consider the approximation consisting in replacing in (5.1) \( \hat{I}(k', k; P) \) by \( \hat{I}_{\text{inst}}(k, k') \) and in substituting \( \hat{H}_2(p) \) with the free propagator \( \frac{-i}{p^2 + m^2} \). Further let us introduce the reduced wave function

\[
\varphi_p(k') = \sqrt{\frac{2w_1(k)w_2(k')}{w_1(k') + w_2(k')}} \int_{-\infty}^{\infty} dk'_0 \Phi_P(k').
\]  

and integrate over \( k_0 \) and \( k'_0 \) using

\[
\int dk'_0 \frac{1}{(k'_0 + \eta m_B)^2 - k'^2 - m_1^2 + i\varepsilon} \frac{1}{(-k'_0 + \eta m_B)^2 - k'^2 - m_2^2 + i\varepsilon} = -\pi i \frac{w'_1 + w'_2}{w'_1 w'_2} \frac{1}{m_B^2 -(w'_1 + w'_2)^2}
\]  

we have

\[
(w_1(k') + w_2(k'))^2 \varphi_{m_B}(k') + \int \frac{d^3k}{(2\pi)^3} \frac{w_1(k') + w_2(k')}{2w_1(k')w_2(k')} \hat{I}_{\text{inst}}(k', k) \frac{w_1(k) + w_2(k)}{2w_1(k)w_2(k)} \varphi_{m_B}(k) = m_B^2 \varphi_{m_B}(k')
\]  

from which Eq.(1.21) immediately follows.

The linear potential of Eq.(1.22) is then given by

\[
\langle k' | V | k \rangle = \frac{1}{w'_1 + w'_2 + w_1 + w_2} \langle k' | U | k \rangle + \ldots
\]  

from which (neglecting kinematical factors becoming equal to one for \( |k| = |k'| \)) we obtain Eq.(1.23). By performing a \( \frac{1}{m^2} \) expansion on Eq. (1.23) we find the \( qq \) potential at \( \frac{1}{m^2} \) order

\[
\langle k' | V | k \rangle = -\frac{4}{3} \frac{\alpha_s}{2\pi^2 Q^2} \frac{1}{\pi^2 Q^1} - \frac{4}{3} \frac{\alpha_s}{2\pi^2 m_1 m_2 Q^2} [q^2 - \frac{(q \cdot Q)^2}{Q^2}]
- \frac{4}{3} i\alpha_s \langle k' | \frac{1}{2m_1} \frac{\alpha_1 \cdot r}{r^3} \frac{1}{2m_2} \frac{\alpha_2 \cdot r}{r^3} | k \rangle + \frac{4}{3} \frac{\alpha_s}{m_1 m_2} \frac{1}{2} \frac{k'_1 + k'_3}{2} (\sigma^1_1 + \sigma^2_2) \langle k' | \frac{k'_r}{r^3} | k \rangle
+ \frac{4}{3} \frac{\alpha_s}{m_1 m_2} \frac{1}{r^3} (k' \cdot r) w_1 w_2 (\sigma^h_1 \sigma^k_2) \langle k' | \frac{r_k}{r^3} | k \rangle
- \frac{\sigma}{6} \left( \frac{1}{m_1^2} + \frac{1}{m_2^2} - \frac{1}{m_1 m_2} \right) \langle k' | q_1^2 r | k \rangle
- \frac{\sigma}{2} \frac{\varepsilon_{hkl}}{2} \frac{k'_1 + k'_3}{2} \left( \frac{\sigma^h_1}{m_1^2} + \frac{\sigma^k_2}{m_2^2} \right) \langle k' | \frac{r_k}{r} | k \rangle + \frac{\sigma i}{2} \langle k' | - \frac{1}{m_1} \frac{\alpha_1 \cdot r}{r} + \frac{1}{m_2} \frac{\alpha_2 \cdot r}{r} | k \rangle + \ldots
\]  

with \( q = \frac{k + k'}{2} \), \( Q = k' - k \).
Passing to the coordinate representation we may also write

\[ V = \frac{4}{3} \alpha_s \frac{1}{r} + \sigma r + \frac{4}{3} \alpha_s \left\{ \frac{1}{2 \alpha_s} \left( \delta^{hk} + \frac{r^h r^k}{r^2} \right) q^h q^k \right\}_W \]

\[ -\frac{4}{3} i \alpha_s \left( \frac{1}{2 m_1} \frac{\alpha_1}{r^3} - \frac{1}{2 m_2} \frac{\alpha_2}{r^3} \right) + \frac{4}{3} \alpha_s \left( \frac{\sigma_1 + \sigma_2}{3} \right) \cdot (r \times q) \]

\[ + \frac{1}{3 m_1 m_2} \left\{ 3 \left( \frac{\sigma_1 \cdot r}{r^5} - \frac{\sigma_1 \cdot \sigma_2}{r^3} \right) + \frac{4}{3} \alpha_s \frac{2 \pi}{3} \left( \frac{\sigma_1 \cdot \sigma_2}{3} \right) \delta^3(r) \right\}_W \]

\[ - \frac{\sigma}{6} \left( \frac{1}{m_1} + \frac{1}{m_2} \right) \{q^2 r\}_W \]

\[ - \frac{\sigma}{2} \left( \frac{\sigma_1}{m_1^2} + \frac{\sigma_2}{m_2^2} \right) \cdot \left( \frac{r}{r^3} \times q \right) - \frac{\sigma}{2} \left( \frac{\alpha_1}{m_1} \frac{r}{r} - \frac{\alpha_2}{m_2} \frac{r}{r} \right) \]

(5.7)

where now \( q \) stands for the momentum operator. Now, by performing a Foldy–Wouthuysen tranformation with generator

\[ S = i \frac{\alpha_1}{m_1} \cdot q - i \frac{\alpha_2}{m_2} \cdot q \]

(5.8)

we end up with the \( \frac{1}{m^2} \) potential which coincides with the Wilson loop potential:

\[ V = -\frac{4}{3} \alpha_s \frac{1}{r} + \sigma r \]

\[ - \frac{1}{2 m_1 m_2} \left\{ \frac{4}{3} \alpha_s \left( \delta^{hk} + \frac{r^h r^k}{r^2} \right) p^h_1 p^k_2 \right\}_W \]

\[ - \sum_{j=1}^{2} \frac{1}{6 m_j} \left\{ \sigma r p^2_j \right\}_W - \frac{1}{6 m_1 m_2} \left\{ \sigma r p^2_{1T} \cdot p^2_{2T} \right\}_W \]

\[ \frac{1}{8} \left( \frac{1}{m_1^2} + \frac{1}{m_2^2} \right) \nabla^2 \left( -\frac{4}{3} \alpha_s \frac{1}{r} + \sigma r \right) + \]

\[ + \frac{1}{2} \left( \frac{4}{3} \alpha_s \frac{\sigma}{r^3} - \frac{\sigma}{r} \right) \left[ \frac{1}{m_1^2} S_1 \cdot (r \times p_1) - \frac{1}{m_2^2} S_2 \cdot (r \times p_2) \right] \]

\[ + \frac{4}{m_1 m_2} \left\{ \frac{2 \pi}{3} \delta^3(r) S_1 \cdot S_2 \right\}_W \]

\[ + \frac{1}{m_1 m_2} \frac{4}{3} \alpha_s \left[ \frac{3}{r^3} (S_1 \cdot r)(S_2 \cdot r) - S_1 \cdot S_2 \right] \]

\[ + \frac{8 \pi}{3} \delta^3(r) S_1 \cdot S_2 \]

(5.9)

with \( \hat{r} = (r/r) \) and the symbol \{ \} \_W stands for the Weyl ordering prescription among momentum and position variables.

VI. CONCLUSIONS

In conclusion, under the assumption (1.2) and (1.3) for the evaluation of the Wilson loop integral, we have derived a quark-antiquark Bethe-Salpeter (BS) equation from QCD,
extending a preceding result obtained for spinless quarks. The assumptions are the same previously used for the derivation of a semirelativistic heavy quark potential and the technique is strictly similar. The kernel is constructed as an expansion in $\alpha_s$ and $\sigma a^2$ and at the lowest order is given by equations (1.14)-(1.18).

The BS equation that has been obtained is a second order one, analogous in some way to the iterated Dirac equation. Correspondently, by instantaneous approximation, an effective square mass operator can be obtained from (1.19) which is given by (1.20) and (1.21).

At the lowest order in $\alpha_s$ and $\sigma a^2$ even a linear mass operator can be written with a potential $V$ given by (1.23). Neglecting the spin dependent terms in $V$ the hamiltonian for the relativistic flux tube model comes out. On the contrary by a $\frac{1}{m}$ expansion and an appropriate Foldy-Wouthuysen transformation the ordinary semirelativistic potential is reobtained.

In equation (1.13) or (1.19) a colour independent dressed quark propagator appears which is defined by equations (4.11) and (3.16). Notice that only the perturbative expansion gives contribution to this quantity.

Few additional remarks are in order.

First of all, notice that the result does not depend strictly from equation (1.3) or (3.9) but from the possibility of writing the interaction term as an integral on the world lines of the quark and the antiquark, as evidenced in (3.10). Multiple integrations of the same type would be admissible, as it occurs for the perturbative contribution, but dependence of the integrand on higher derivatives in the parameters $\tau_1$ and $\tau_2$ would not enable to carry on the argument. We have no actual justification that $i \ln W$ is in general of the desired form, we observe however that this quantity is obviously independent of the parametrization. For an example of inclusion of higher order perturbative terms see Ref. [1].

A second point concerns the significance of the lowest order BS kernel we have derived. As the analysis in terms of potentials show, the inclusion of terms in $\alpha_s$ is essential for an understanding of the fine and the hyperfine structure. For what concerns the importance of $\sigma^2$ contributions an indication can be obtained considering the corresponding terms in the
relativistic tube flux model. Neglecting the coulombic terms and in the equal mass case the c.m. hamiltonian for such model at the $\sigma^2$ order can be written

$$H_{cm} = 2\sqrt{m^2 + q^2} + \frac{\sigma r}{2} \left[ \frac{\sqrt{m^2 + q^2}}{|q_T|} \arcsin \frac{|q_T|}{\sqrt{m^2 + q^2}} + \frac{\sqrt{m^2 + q^2}}{m^2 + q^2} \right] +$$

$$+ \frac{\sigma^2 r^2}{16q_T^2 \sqrt{m^2 + q^2}} \left[ \frac{\sqrt{m^2 + q^2}}{|q_T|} \arcsin \frac{|q_T|}{\sqrt{m^2 + q^2}} - \sqrt{m^2 + q^2} \right]$$

(6.1)

To better appreciate the relative magnitude of the two potential terms let us consider e.g. the case of small $q_T$ (small angular momentum) in which the above equation becomes simply

$$H_{cm} = 2\sqrt{m^2 + q^2} + \sigma r + \frac{\sigma^2 r^2}{16\sqrt{m^2 + q^2}}.$$  

(6.2)

Then, taking into account that $a \sim 1/(\sigma m)^{\frac{3}{2}}$, $q \sim 1/a$, and assuming typically $\sigma = 0.17\text{GeV}^2$, $m_u = 0.35\text{GeV}$, $m_c = 1.7\text{GeV}$, $m_b = 5\text{GeV}$ we find that the last term in (6.2) is of the order of the 5%, 0.8%, 0.2% of the preceding one for the $u\bar{u}$, $c\bar{c}$, $b\bar{b}$ systems respectively. This would correspond to contributions to the mass of the meson of about 20, 2, 0.2 MeV. The inclusion of the coulombic term would reduce $a$ and improve the result. In the $u\bar{u}$ case e.g. it would amount to cut the above contribution by a factor 2. Therefore only in this last case the $\sigma^2$ terms would be of any significance.

Finally let us come to the problem of the type of confinement, which has been largely discussed in the literature. By this terminology it is usually meant the tentative assumption of a BS (first order) confining kernel of the instantaneous form

$$\hat{I}_{conf} = (2\pi)^3 \Gamma \frac{\sigma}{\pi^2} \frac{1}{Q^4},$$

(6.3)

or even the covariant counterpart of it

$$\hat{I}_{conf} = -(2\pi)^3 \Gamma \frac{\sigma}{\pi^2} \frac{1}{Q^4},$$

(6.4)

where $\Gamma$ is a combination of Dirac matrices. Typically the cases $\Gamma = 1$ (scalar confinement), $\Gamma = \gamma_1^0 \gamma_2^0$ (vectorial confinement) or a combination of them have been considered.

Eq. (6.4) is immediately ruled out by the fact that, even if formally it corresponds to (6.3) (by instantaneous approximation), actually, due to the strong infrared singularity, it
gives results very different from (6.3) [8]. As well known, Eq. (6.3) with $\Gamma = 1$ was motivated by the fact that it reproduces the static potential $\sigma r$ and the spin dependent potential as obtained in the Wilson loop context. This choice, however, gets both into phenomenological and theoretical difficulties:

1. it gives a first order velocity dependent relativistic correction to the potential which differs from the Wilson loop one [2] and does not seem to agree with the heavy meson data [9],

2. it does not reproduce straight line Regge trajectories [10].

Complementary objections can be moved to (6.3) with $\Gamma = \gamma_1 \gamma_2$.

On the contrary, even if we have not yet attempted calculations directly with the kernel established in this paper, very encouraging results have been obtained in the context of the relativistic flux tube model [3], of the dual QCD [11] and of the effective relativistic hamiltonian [7], formalisms that are all strictly related to our one. Therefore the complicated momentum dependence appearing in (1.16)-(1.17) seems essential to understand both the light and the heavy meson phenomenology.

**APPENDIX A: APPENDIX**

We want to prove Eq.(4.3).

Let us first consider the confinement part and rewrite Eq. (3.3) as

$$S_{\text{min}} = \int_{t_i}^{t_f} dt \int_0^1 ds \, S(u)$$

with

$$S(u) = \left[ - \left( \frac{\partial u^\mu}{\partial t} \frac{\partial u_\mu}{\partial t} \right) \left( \frac{\partial u^\nu}{\partial s} \frac{\partial u_\nu}{\partial s} \right) + \left( \frac{\partial u^\mu}{\partial t} \frac{\partial u_\mu}{\partial s} \right)^2 \right]^{\frac{1}{2}}.$$  \hspace{1cm} (A2)

Being $x^\mu = u^\mu(s,t)$, the equation of the minimal surface $u^\mu$ enclosed by the loop must be the solution of the Euler equations.
\[
\frac{\partial}{\partial s} \frac{\partial S}{\partial \left( \frac{\partial u^\mu}{\partial s} \right)} + \frac{\partial}{\partial t} \frac{\partial S}{\partial \left( \frac{\partial u^\mu}{\partial t} \right)} = 0
\]  

(A3)
satisfying the contour conditions \( u^\mu(1, t) = z_1^\mu(\tau_1(t)) \), \( u^\mu(0, t) = z_2^\mu(\tau_2(t)) \). Then, considering an infinitesimal variation of the world line of the quark 1, \( z_1^\mu(t) \rightarrow z_1^\mu(t) + \delta z_1^\mu(t) \), even \( u^\mu(s, t) \) must change, \( u^\mu(s, t) \rightarrow u^\mu(s, t) + \delta u^\mu(s, t) \) and one has

\[
\delta S_{\text{min}} = \int_{t_i}^{t_f} dt \int_0^1 ds \left[ \frac{\partial S}{\partial \left( \frac{\partial u^\mu}{\partial s} \right)} \frac{\partial}{\partial s} \delta u^\mu + \frac{\partial S}{\partial \left( \frac{\partial u^\mu}{\partial t} \right)} \frac{\partial}{\partial t} \delta u^\mu \right] = 0
\]  

(A4)

where \( \delta z_1^\mu(t) \) is assumed to vanish out of a small neighbourhood of a specific value of \( t \).

Finally taking into account that

\[
\delta u^\mu(1, t) = \delta z_1^\mu(1) , \quad \frac{\partial u^\mu(1, t)}{\partial t} = \dot{z}_1^\mu(t)
\]  

(A5)

one obtains

\[
\delta S_{\text{min}} = \int_{t_i}^{t_f} dt \int_0^1 ds \left[ \frac{\partial S}{\partial \left( \frac{\partial u^\mu}{\partial s} \right)} \frac{\partial}{\partial s} \delta u^\mu + \frac{\partial S}{\partial \left( \frac{\partial u^\mu}{\partial t} \right)} \frac{\partial}{\partial t} \delta u^\mu \right] = 0
\]  

(A4)

where \( \delta z_1^\mu(t) \) is assumed to vanish out of a small neighbourhood of a specific value of \( t \).

Finally taking into account that

\[
\delta u^\mu(1, t) = \delta z_1^\mu(1) , \quad \frac{\partial u^\mu(1, t)}{\partial t} = \dot{z}_1^\mu(t)
\]  

(A5)

one obtains

\[
\delta S_{\text{min}} = \int_{t_i}^{t_f} dt \int_0^1 ds \left[ \frac{\partial S}{\partial \left( \frac{\partial u^\mu}{\partial s} \right)} \frac{\partial}{\partial s} \delta u^\mu + \frac{\partial S}{\partial \left( \frac{\partial u^\mu}{\partial t} \right)} \frac{\partial}{\partial t} \delta u^\mu \right] = 0
\]  

(A4)

and more explicitly

\[
\frac{\delta S_{\text{min}}}{\delta S^\mu\nu(z_1)} = \frac{(\frac{\partial u^\mu}{\partial s})_1 \dot{z}_{1\nu} - (\frac{\partial u^\mu}{\partial s})_1 \dot{z}_{1\mu}}{[\dot{z}_1^2 (\frac{\partial u^\mu}{\partial s})_1 + (\dot{z}_1 (\frac{\partial u^\mu}{\partial s})_1)^2]^\frac{1}{2}}
\]  

(A7)

Then, in the straight line approximation we have

\[
\frac{\partial u^\mu}{\partial s} = z_{1\mu} - z_{2\mu} = r^\mu
\]  

(A8)

and

\[
\frac{\partial S_{\text{min}}}{\delta S^\mu\nu(z_1)} = \frac{r^\mu \dot{z}_{1\nu} - r^\nu \dot{z}_{1\mu}}{[-\dot{z}_1^2 r^2 + (\dot{z}_1 \cdot r)^2]^\frac{1}{2}}
\]  

(A9)
Using Eq. (A9) (having substituted the velocities with the momenta), Eqs. (3.11) and (3.6), we obtain the second term in (4.3).

Let us come to the perturbative part. Consider a variation $z_1 \to z_1 + \delta z_1$, then

$$\delta_1 \int d\tau_1 \int d\tau_2 \dot{z}_1^\rho D_{\rho \sigma}(z_1 - z_2) \dot{z}_2^\sigma =$$

$$= \int d\tau_1 \int d\tau_2 [\delta \dot{z}_1^\rho D_{\rho \sigma}(z_1 - z_2) + \dot{z}_1^\rho \delta z_2^\nu \partial_\nu D_{\rho \sigma}(z_1 - z_2)] \dot{z}_2^\sigma =$$

$$= \int \delta S^{\mu \nu} \int d\tau_2 [\partial_\nu D_{\rho \sigma}(z_1 - z_2) - \partial_\rho D_{\nu \sigma}(z_1 - z_2)] \dot{z}_2^\sigma$$  \hspace{1cm} (A10)

and so

$$\frac{\delta}{\delta \dot{u}^{\mu}(z_1)} \int d\tau_1 \int d\tau_2 p_1^\mu D_{\rho \sigma}(z_1 - z_2) p_2^\sigma = \int d\tau_2 (\delta_2^\mu \partial_1^\nu - \delta_1^\mu \partial_2^\nu) D_{\rho \sigma}(z_1 - z_2) p_2^\sigma$$  \hspace{1cm} (A11)

and then we recover the first term in (4.3).
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