Quantum Invariants of the Pairing Hamiltonian

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Quantum invariants of the orbit dependent pairing problem are identified in the limit where the orbits become degenerate. These quantum invariants are simultaneously diagonalized with the help of the Bethe ansatz method and a symmetry in their spectra relating the eigenvalues corresponding to different number of pairs is discussed. These quantum invariants are analogous to the well known rational Gaudin magnet Hamiltonians which play the same role in the reduced pairing case (i.e., orbit independent pairing with non degenerate energy levels). It is pointed out that although the reduced pairing and the degenerate cases are opposite of each other, the Bethe ansatz diagonalization of the invariant operators in both cases are based on the same algebraic structure described by the rational Gaudin algebra.

Keywords: Nuclear Pairing, Bethe Ansatz, Exact Solution, Quasispin, Gaudin Algebra, Integrability.

I. INTRODUCTION

Strong pair correlations are observed in fermionic many body systems which energetically favor large wave function overlaps. This phenomenon is known as pairing and it plays an important role in our understanding of many body physics (See Ref. [1] for a review). Historically, the physical significance of pairing was first realized with the microscopic theory of superconductivity developed by J. Bardeen, L. N. Cooper and J. R. Schrieffer (BCS) in 1957 [2]. Following the success of the BCS theory, the idea of pairing was carried over to other areas of physics as well. In particular, pairing now plays an essential role in the nuclear shell model as the residual interaction between nucleons and successfully recounts for various properties of atomic nuclei [3, 4].

In order to investigate the influence of pairing on nuclear properties, many authors have used exact analytical solutions of nuclear shell model which are available in some simplified cases. For example, in Ref. [5], Kerman considered pairs of nucleons coupled to angular momentum zero occupying a single orbit and introduced the quasi-spin formalism in which these pairs can be treated within suitable representations of the angular momentum algebra. He used this formalism to write down the exact energy eigenstates and to analyze the influence of pairing on the collective vibrations of nuclei. Quasi-spin formalism can also be extended to the case of several orbits in which case the quasi-spin angular momenta corresponding to different orbits commute with each other and the pairing term has the form of a coupling between these angular momenta. This observation establishes a direct link between the fermion pairing models and the interacting spin models (see Refs. [6, 7, 8] for reviews). The exact diagonalization of the later model was carried out by R. W. Richardson in 1962 in the case of the orbit-independent (i.e., reduced) pairing interaction [9]. Later, it was clarified that the exact solvability of the pairing Hamiltonian in the reduced pairing case can be understood in terms of a set of quantum invariants which commute with one another and also with the Hamiltonian [10, 11]. These invariants are called rational Gaudin magnet Hamiltonians since they stem from the work of M. Gaudin who was originally trying to find the largest set mutually commuting operators for a given system of interacting spins. In Ref. [10], Gaudin also showed that the pair creation

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and annihilation operators which are used in building the simultaneous eigenstates of the rational Gaudin magnet Hamiltonians form an algebra which is today known as the rational Gaudin algebra. It is worth mentioning that the rational Gaudin algebra is related to the rational solution of the classical Yang-Baxter equation which appears as an integrability condition in many contexts. As a result, the rational Gaudin magnet operators and the rational Gaudin algebra have found many other applications in physics (see Refs. \cite{6, 7, 8} for reviews and Refs. \cite{12, 13, 14, 15, 16} for some interesting applications). They have also been generalized to include other underlying algebraic structures (i.e., higher rank algebras, super-algebras and deformed algebras) besides the angular momentum algebra. There is an extensive literature on this subject and the interested reader may find the Refs. \cite{17, 18, 19, 20, 21, 22, 23} useful.

Although the Richardson-Gaudin solution is successfully used in nuclear physics, the assumption of reduced pairing sometimes proves to be too stringent. In many cases, the effective residual interactions between the nucleons are best described by a pairing force whose strength differs between the orbits\textsuperscript{1}. The Hamiltonian

\[ \hat{H} = \sum_{j} \sum_{m} \varepsilon_{j} a_{j m}^{\dagger} a_{j m} - |G| \sum_{j,j', m,m'} c_{j} c_{j'} a_{j m}^{\dagger} a_{j'm'}^{\dagger} a_{j'm'} a_{j m} \]  

(1)

is frequently used to describe such an orbit dependent pairing interaction. Here, \( j \) denotes the total angular momentum of an orbit and \( \varepsilon_{j} \) denotes its energy. The overall strength of the pairing term against the kinetic term is measured by the constant \( |G| \) which has the dimension of energy whereas the relative pairing strengths are measured by the dimensionless constants \( c_{j} \). Richardson-Gaudin scheme mentioned above applies to the special case of this Hamiltonian in which all \( c_{j} \)'s are the same whereas all single particle energy levels \( \varepsilon_{j} \) are different from one another (the reduced pairing case). The focus of this paper, however, is the opposite case in which all the \( c_{j} \)'s are different from one another and all single particle energies \( \varepsilon_{j} \) are the same (the degenerate case).

The problem described by Hamiltonian given in Eq. (1) is exactly solvable in both the reduced pairing case and the degenerate case. As mentioned above, in the reduced pairing case the solution was given by Richardson-Gaudin scheme and the corresponding quantum invariants are the rational Gaudin magnet Hamiltonians. In the degenerate case, the exact energy eigenvalues and eigenstates were obtained in a series of papers by Pan et al. \cite{25} and by Balantekin et al. \cite{26, 27} and the purpose of the present paper is to identify the corresponding quantum invariants in the degenerate case. In addition, it will be shown that the quantum invariants in the degenerate case can be simultaneously diagonalized with the help of the algebraic Bethe ansatz method. An interesting observation regarding the Bethe ansatz diagonalization is that although the reduced pairing and the degenerate cases are opposite of each other, the Bethe ansatz diagonalization of the invariant operators in both cases is connected with the rational Gaudin algebra.

The organization of this paper is as follows: Section II is a brief review of the quasispin formalism and it also serves to introduce some notation. In Section III, a short review of the Richardson-Gaudin formalism and the rational Gaudin magnet Hamiltonians is presented. The main results of this paper, i.e., the quantum invariants in the degenerate case and their simultaneous diagonalization with the Bethe ansatz method are presented in Section IV. This section also contains a discussion about a symmetry in the spectra of these quantum invariants relating eigenvalues corresponding to different number of particles. In Section V, we consider the rational Gaudin algebra and point out its relationship with the Bethe ansatz diagonalization in both the reduced pairing and the degenerate cases. Section VI summarizes the main conclusions. The details of some of the Bethe ansatz calculations can be found in the Appendix.

\textsuperscript{1} See, for example, the treatment of the neutron pairs occupying the valence shell of Ni isotopes by Auerbach \cite{24}.
II. QUASI-SPIN FORMALISM AND THE EXACT SOLUTIONS OF THE PAIRING HAMILTONIAN

In the quasi-spin formalism, nucleon pairs coupled to angular momentum zero are created and annihilated at the level $j$ by the operators

$$S_j^+ = \sum_{m>0} (-1)^{j-m} a_j^\dagger m a_j^\dagger - m \quad S_j^- = \sum_{m>0} (-1)^{j-m} a_j - m a_j m,$$

respectively. Together with the operator

$$S_j^0 = \frac{1}{2} \sum_{m>0} \left( a_j^\dagger m a_j m + a_j^\dagger - m a_j - m - 1 \right),$$

they obey the well known angular momentum commutations relations

$$[S_j^+, S_j^-] = 2\delta_{jj'} S_j^0, \quad [S_j^0, S_j^\pm] = \pm \delta_{jj'} S_j^\pm.$$

As a result, one has an angular momentum algebra (the so called quasi-spin algebra) for each orbit $j$ such that those angular momenta corresponding to different orbits commute with one another. The pair number operator for the orbit $j$ is given by

$$\hat{N}_j = \frac{1}{2} \sum_{m>0} \left( a_j^\dagger m a_j m + a_j^\dagger - m a_j - m \right).$$

It is related to the operator $S_j^0$ given in Eq. (3) by the formula

$$S_j^0 = \hat{N}_j = \frac{\Omega_j}{2},$$

where $\Omega_j$ is the maximum number of pairs which can occupy the level $j$. Note that $j$ is always an half integer because of the spin-orbit coupling in the nuclei. As a result, if there are no unpaired particles at the level $j$, then

$$\Omega_j = j + \frac{1}{2}.$$

Also note that the pairing term in the Hamiltonian given in Eq. (1) does not act on the unpaired particles. If there is an unpaired particle at the level $j$, its effect will be i) to add a constant $\varepsilon_j$ to the Hamiltonian because of the kinetic term and ii) to reduce the maximum number of pairs which can occupy the level $j$ by one, i.e., to take $\Omega_j$ to $\Omega_j - 1$. But here it will be assumed that there are no unpaired particles in the system. In this case, Eq. (6) implies that

$$-\frac{\Omega_j}{2} \leq S_j^0 \leq \frac{\Omega_j}{2},$$

i.e., quasi-spin algebra corresponding to the level $j$ is realized in the $\Omega_j/2$ representation. Therefore, in addition to the physical angular momentum quantum number $j$, we also have the quasi-spin quantum number $\Omega_j/2$ for each level. The states

$$\left| \frac{\Omega_j}{2}, -\frac{\Omega_j}{2} \right\rangle \quad \text{and} \quad \left| \frac{\Omega_j}{2}, \frac{\Omega_j}{2} \right\rangle$$

respectively represent the situations in which i) the level $j$ is not occupied by any pairs and ii) it is maximally occupied by pairs. In the presence of several orbits with angular momenta $j_1, j_2, \ldots, j_n$, the state

$$|0\rangle = \left| \frac{\Omega_{j_1}}{2}, -\frac{\Omega_{j_1}}{2} \right\rangle \otimes \left| \frac{\Omega_{j_2}}{2}, -\frac{\Omega_{j_2}}{2} \right\rangle \otimes \cdots \otimes \left| \frac{\Omega_{j_n}}{2}, -\frac{\Omega_{j_n}}{2} \right\rangle$$

(10)
represents a shell which contains no pairs whereas the state
\[ |\bar{0}\rangle = \frac{1}{2}|\Omega_{j_1}, \frac{\Omega_{j_2}}{2}\rangle \otimes \frac{1}{2}|\Omega_{j_2}, \Omega_{j_2}'\rangle \otimes \cdots \otimes \frac{1}{2}|\Omega_{j_n}, \frac{\Omega_{j_n}}{2}\rangle \]  \hspace{1cm} (11)
represents a shell which is fully occupied by pairs.

The pairing Hamiltonian given in Eq. (11) can be written in terms of the quasi-spin operators given in Eqs. (2) and (3) as
\[ \hat{H} = \sum_j \varepsilon_j (2S^0_j + \Omega_j) - |G| \left( \sum_j c_j S^+_j \right) \left( \sum_{j'} c_{j'} S^-_{j'} \right). \]  \hspace{1cm} (12)
Note that the operator \( \sum_j c_j S^+_j \) in Hamiltonian (12) creates a pair of particles in such a way that \( c_j \) can be viewed as the probability amplitude that this pair is found at the level \( j \). For this reason the coefficients \( c_j \) are usually called occupation probability amplitudes and they are normalized as
\[ \sum_j c_j^2 = 1. \]  \hspace{1cm} (13)
Although an occupation probability amplitude is a complex number in general, the parameters \( c_j \) can be taken as real without loss of generality. Because if one \( c_j \) is complex, a unitary transformation can always be performed on the quasi-spin algebra corresponding to the level \( j \) to make that \( c_j \) real. Also note that the Hamiltonian in Eq. (12) contains a constant term \( \sum_j 2\varepsilon_j \Omega_j \) which comes from Eq. (6). This constant term is not dropped because it guarantees that the energy of the empty shell is zero, i.e.,
\[ \hat{H}|\bar{0}\rangle = 0. \]  \hspace{1cm} (14)
Using the commutators given in Eq. (4), one can show that the fully occupied shell \( |\bar{0}\rangle \) is also an eigenstate of the Hamiltonian with the energy
\[ \hat{H}|\bar{0}\rangle = \sum_j \left( 2\varepsilon_j - |G|c_j^2 \right) \Omega_j |\bar{0}\rangle. \]  \hspace{1cm} (15)
Unlike the empty shell \( |0\rangle \) and the fully occupied shell \( |\bar{0}\rangle \), the eigenstates of the pairing Hamiltonian corresponding to a partially occupied valance shell are unknown in the most general case. But, as mentioned in the Introduction, exact energies and eigenstates are known in the two opposite cases. Namely, the reduced pairing case characterized by
\[ \varepsilon_1 \neq \varepsilon_2 \neq \cdots \neq \varepsilon_n \]
\[ c_1 = c_2 = \cdots = c_n, \]  \hspace{1cm} (16)
and the degenerate case characterized by
\[ \varepsilon_1 = \varepsilon_2 = \cdots = \varepsilon_n \]
\[ c_1 \neq c_2 \neq \cdots \neq c_n. \]  \hspace{1cm} (17)
These solutions will be reviewed in the next two sections together with the corresponding quantum invariants. But before closing this section, mention must be made of a third case in which exact eigenstates of the pairing Hamiltonian are known. This solution is available in the presence of two orbits with unequal energies and unequal occupation probability amplitudes, i.e.,
\[ \varepsilon_1 \neq \varepsilon_2 \]
\[ c_1 \neq c_2. \]  \hspace{1cm} (18)
But this third case will not be considered in this paper. Because the main interest of this paper is the quantum invariants of the pairing Hamiltonian and in the case of a two level system we have only two quantum invariants which are simply the Hamiltonian itself and the total pair number operator.
III. REDUCED PAIRING AND THE GAUDIN MAGNET OPERATORS

In the reduced pairing case, described by Eq. (16), the pairing Hamiltonian given in Eq. (12) becomes

\[ \hat{H}_R = \sum_j \varepsilon_j \left(2S^0_j + \Omega_j\right) - |G|d \sum_{j,j'} S^+_j S^-_{j'} \]  (19)

Here \( d = 1/n \) is known as the level spacing and its appearance in the Hamiltonian is due to the normalization condition (13). Using a variational technique, Richardson showed in Ref. [9] that the eigenstates of the Hamiltonian given in Eq. (19) containing \( N \) pairs of particles are in the form

\[ J^+(\xi_1)J^+(\xi_2)\ldots J^+(\xi_N)|0\rangle \]  (20)

where the pair creation operators \( J^+(\xi) \) are given by

\[ J^+(\xi) = \sum_j \frac{S^+_j}{2\varepsilon_j - \xi} \]  (21)

and \( |0\rangle \) is the state with no pairs defined in Eq. (10). The values of the parameters \( \xi_1, \xi_2, \ldots, \xi_N \) which appear in Eq. (20) are to be determined by solving the system of equations

\[ \sum_j -\Omega_j/2 = -\frac{1}{2} |G|d + \sum_{l=1, l \neq k}^N \frac{1}{\xi_k - \xi_l} \]  (22)

simultaneously for \( k = 1, 2, \ldots, N \) (see Ref. [9]). These equations generally have several distinct solutions. For each one of these solutions we have an eigenstate in the form of Eq. (20) and the corresponding energy is given by

\[ E^{(N)} = \sum_{k=1}^N \xi_k. \]  (23)

The quantum invariants of the Hamiltonian in Eq. (19) are the rational Gaudin magnet Hamiltonians mentioned in the introduction [10, 11]. They are given by

\[ \hat{R}_j = S^0_j - |G|d \sum_{j' \neq j} \frac{\overrightarrow{S}_j \cdot \overrightarrow{S}_{j'}}{\varepsilon_j - \varepsilon_{j'}} \]  (24)

where \( \overrightarrow{S}_j \cdot \overrightarrow{S}_{j'} \) is defined as

\[ \overrightarrow{S}_j \cdot \overrightarrow{S}_{j'} = S^0_j S^0_{j'} + \frac{1}{2} \left(S^+_j S^-_{j'} + S^-_j S^+_{j'}\right). \]  (25)

The rational Gaudin magnet Hamiltonians mutually commute with one another and with the Hamiltonian \( \hat{H}_R \), i.e.,

\[ [\hat{R}_j, \hat{R}_{j'}] = 0 \quad [\hat{R}_j, \hat{H}_R] = 0 \]  (26)

for every \( j, j' = 1, 2, \ldots, n \). The Hamiltonian itself is not an independent invariant and it can be written in terms of the operators \( \hat{R}_j \) as

\[ \hat{H}_R = \sum_j \left(2\varepsilon_j - |G|d\right) \hat{R}_j + \frac{1}{2} |G|d \sum_{j \neq j'} \hat{R}_j \hat{R}_{j'} - |G|d \sum_j \overrightarrow{S}_j \cdot \overrightarrow{S}_j + \sum_j \varepsilon_j \Omega_j. \]  (27)

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2 Strictly speaking, the operators studied by Gaudin himself did not include the one body term \( S^0_j \).
Similarly, the total pair number operator can also be written in terms of the operators $\hat{R}_j$ as

$$\hat{N} = \sum_j \left( \hat{R}_j + \frac{\Omega_j}{2} \right)$$

As a result of Eq. (26), the eigenstates of the Hamiltonian are at the same time simultaneous eigenstates of rational Gaudin magnet operators as well. Let us denote the eigenvalues of the invariant $\hat{R}_j$ corresponding to the eigenstate with $N$ pairs by $E_j^{(N)}$. In other words,

$$\hat{R}_j |0\rangle = E_j^{(0)} |0\rangle$$

for the empty shell and

$$\hat{R}_j J^+(\xi_1) J^+(\xi_2) \ldots J^+(\xi_N) |0\rangle = E_j^{(N)} J^+(\xi_1) J^+(\xi_2) \ldots J^+(\xi_N) |0\rangle$$

for the eigenstates containing $N$ pairs described by Eqs. (20-22). The eigenvalues $E_j^{(0)}$ and $E_j^{(N)}$ are given by

$$E_j^{(0)} = -\frac{\Omega_j}{2} - \frac{|G|^d}{4} \sum_{j \neq j'} \frac{\Omega_j \Omega_j'}{\varepsilon_j - \varepsilon_{j'}}$$

and

$$E_j^{(N)} = E_j^{(0)} + \frac{|G|^d}{2} \sum_{k=1}^{N} \frac{\Omega_j}{2\varepsilon_j - \xi_k}$$

respectively.

The pairing Hamiltonian in Eq. (19) is only one of the exactly solvable models which can be built using the rational Gaudin magnet Hamiltonians. Various other linear or nonlinear combinations of rational Gaudin magnet operators can be used to build other useful exactly solvable models (see, for example, Refs. [13, 14, 16]).

IV. INTEGRABILITY IN THE DEGENERATE CASE

In the case of several orbits having the same energy but different occupation probability amplitudes, i.e., when the conditions in Eq. (17) are satisfied, the first term in the pairing Hamiltonian given in Eq. (12) becomes a constant which is proportional to the total number of pairs in the shell. Discarding this term, one can write the Hamiltonian as

$$\hat{H}_D = -|G|^d (\sum_j c_j S_j^+) (\sum_{j'} c_{j'} S_{j'}^-).$$

(33)

Exact eigenvalues and eigenstates of the Hamiltonian given in Eq. (33) were obtained in Refs. [25, 26, 27]. The purpose of this paper is to introduce the corresponding quantum invariants, i.e., the set of operators which commute with one another and with the Hamiltonian in Eq. (33).

The fact that the rational Gaudin magnet operators given in Eqs. (24) mutually commute with one another is independent of the values of the parameters $\varepsilon_j$. Naturally, one can try to replace the parameters $\varepsilon_j$ in the Gaudin operators with some arbitrary functions of $c_j$ and try to determine the form of these functions so
that the new operators commute with the Hamiltonian in Eq. (33) as well. It turns out, however, that such as course of action does not yield the quantum invariants of the Hamiltonian given in Eq. (33)\(^3\).

In order to find the invariant operators one can consider general number conserving operators in the form

\[
\hat{P}_j = A_j S_j^0 + B_j S_j^+ S_j^- + \sum_{j' \neq j} D_{jj'} S_{j'}^0 S_j^0 + \sum_{j' \neq j} F_{jj'} \left( S_j^+ S_{j'}^- + S_j^- S_{j'}^+ \right)
\]  

(34)

where \(A_j, B_j, D_{jj'},\) and \(F_{jj'}\) are some arbitrary coefficients. The condition that the above operators commute with one another and with the Hamiltonian in Eq. (33) gives us the allowed values of these coefficients. A straightforward calculation shows that the desired operators are given by

\[
\hat{P}_j = -S_j^+ S_j^- + 2 \sum_{j' \neq j} \frac{c_{jj'}^2}{c_{jj'}^2 - c_j^2} S_{j'}^0 S_j^0 + \sum_{j' \neq j} \frac{c_j c_{jj'}}{c_{jj'}^2 - c_j^2} \left( S_j^+ S_{j'}^- + S_j^- S_{j'}^+ \right).
\]  

(35)

These operators mutually commute with one another

\[
[\hat{P}_j, \hat{P}_{j'}] = 0
\]  

(36)

for every \(j\) and \(j'\). They also commute with the Hamiltonian given in Eq. (33) and with the total pair number operator:

\[
[\hat{P}_j, \hat{H}_D] = 0 \quad \quad [\hat{P}_j, \hat{N}] = 0.
\]  

(37)

The Hamiltonian \(\hat{H}_D\) and the total number operator \(\hat{N}\) are not independent invariants but they are related to the operators \(\hat{P}_j\) by the formulas

\[
G \sum_j c_j^2 \hat{P}_j = \hat{H}_D
\]  

(38)

and

\[
\sum_j \hat{P}_j = \hat{N}^2 - \hat{N} \left( \sum_j \Omega_j + 1 \right) + \frac{1}{4} \sum_{j, j' \neq j} \Omega_j \Omega_{j'}.
\]  

(39)

As a result of Eqs. (36) and (37), the invariants \(\hat{P}_j\) have the same eigenstates as the pairing Hamiltonian \(\hat{H}_D\) given in Eq. (33). These eigenstates were given in Refs. \([25, 26, 27]\) with the help of the Bethe ansatz method \([29]\). In what follows, the corresponding eigenvalues of the invariant operators \(\hat{P}_j\) will be presented. A summary of the results of this Section can be found in Table I.

Following Refs. \([25, 26, 27]\), let us introduce the pair creation and annihilation operators

\[
S^+(x) = \sum_j \frac{c_j}{1 - c_j^2 x} S_j^+, \quad S^-(x) = \sum_j \frac{c_j}{1 - c_j^2 x} S_j^-.
\]  

(40)

Here, \(x\) is a complex variable and \(S_j^\pm\) are the quasispin operators introduced in Eq. (2). The Hamiltonian in Eq. (33) itself can be written in terms of these operators as

\[
\hat{H}_D = -|G| \hat{S}^+(0) \hat{S}^-(0).
\]  

(41)

\(^3\) In addition to the rational Gaudin magnet Hamiltonians, Gaudin also studied the so called trigonometric and hyperbolic Gaudin magnet Hamiltonians which also mutually commute with one another. Similar to the rational case, however, one cannot choose the parameters of trigonometric or hyperbolic Gaudin magnet Hamiltonians so as to make them commute with the Hamiltonian in Eq. (33).
The eigenstates of the Hamiltonian $\hat{H}_D$ which are also simultaneous eigenstates of the invariants $\hat{P}_j$ can be written in terms of the pair creation and annihilation operators in Eq. (40). Below, these eigenstates which were obtained in Refs. [25, 26, 27] will be reviewed in the order of increasing number of pairs and the corresponding eigenvalues of the invariant operators $\hat{P}_j$ will be given.

**Empty shell:** The empty shell $|0\rangle$ given in Eq. (10) obeys

$$\hat{P}_j|0\rangle = E_j^{(0)}|0\rangle \quad (42)$$

where $E_j^{(0)}$ is given by

$$E_j^{(0)} = \frac{\Omega_j^2}{2} \sum_{j' \neq j} \frac{\Omega_{j'}}{1 - c_j^2/c_{j'}^2}. \quad (43)$$

**Eigenstates with $N = 1$:** The eigenstates with one pair of particles fall in two classes. The state

$$\hat{S}^+(0)|0\rangle \quad (44)$$

is an eigenstate where $\hat{S}^+(0)$ is obtained by putting $x = 0$ in the operator given in Eq. (40). This state was first suggested by Talmi in Ref. [30] and was shown to be an eigenstate of a class of Hamiltonians including the Hamiltonian in Eq. (33).

In addition to the state in Eq. (44), the state

$$\hat{S}^+(x)|0\rangle \quad (45)$$

is also an eigenstate if $x$ is a solution of the Bethe ansatz equation

$$\sum_j -\frac{\Omega_j/2}{1/c_j^2 - x} = 0. \quad (46)$$

The eigenvalues of the operators $\hat{P}_j$ corresponding to the eigenstates described above will be denoted by $\lambda_j^{(1)}$ and $\mu_j^{(1)}$, respectively, i.e.,

$$\hat{P}_j \hat{S}^+(0)|0\rangle = \lambda_j^{(1)} \hat{S}^+(0)|0\rangle \quad (47)$$

$$\hat{P}_j \hat{S}^+(x)|0\rangle = \mu_j^{(1)} \hat{S}^+(x)|0\rangle. \quad (48)$$

The eigenvalues $\lambda_j^{(1)}$ and $\mu_j^{(1)}$ can easily be computed using the commutators given in Eq. (41) together with Eqs. (40) and (46) as follows (see the Appendix)

$$\lambda_j^{(1)} = E_j^{(0)} - \Omega_j \quad (49)$$

$$\mu_j^{(1)} = E_j^{(0)} - \frac{\Omega_j}{1 - c_j^2 x}. \quad (50)$$

Note that the eigenstate in Eq. (44) is unique whereas the eigenstate in Eq. (45) represents several eigenstates. Because in general the Bethe ansatz equation (46) has more than one solutions and for each one of them we have an eigenstate in the form of Eq. (45). As a result, Eq. (48) also represents several eigenvalue-eigenstate equations.

**Eigenstates for $2 \leq N \leq N_{\text{max}}/2$:** The results given above can be generalized to the states corresponding to a shell which is at most half full. Let $N_{\text{max}} = \sum_j \Omega_j$ denote the maximum number of pairs which can occupy the shell in consideration. Then for $2 \leq N \leq N_{\text{max}}/2$ the results obtained for one pair generalizes as follows: The state

$$\hat{S}^+(0)\hat{S}^+(z_1) \ldots \hat{S}^+(z_{N-1})|0\rangle \quad (51)$$
which has \( N \) pairs of particles is an eigenstate if the parameters \( z_k \) are all different from one another and obey the following system of Bethe ansatz equations

\[
\sum_j \frac{-\Omega_j/2}{1/c_j^2 - z_m} = \frac{1}{z_m} + \sum_{k=1}^{N-1} \frac{1}{z_m - z_k},
\]

for every \( m = 1, 2, \ldots N - 1 \). In addition, the state

\[
\hat{S}^+(x_1)\hat{S}^+(x_2)\ldots\hat{S}^+(x_N)|0\rangle
\]

which also has \( N \) pairs of particles is an eigenstate if the parameters \( x_k \) are all different from one another and satisfy the following system of Bethe ansatz equations:

\[
\sum_j \frac{-\Omega_j/2}{1/|e_j|^2 - x_m} = \sum_{k=1}^N \frac{1}{x_m - x_k},
\]

for every \( m = 1, 2, \ldots, N \). Note that the states given in Eqs. (51) and (53) can be thought of as the generalizations of the states in Eqs. (44) and (45), respectively.

The Bethe ansatz equations given in Eqs. (52) and (54) have more than one solutions. Each one of these solutions gives us an eigenstate in the form of Eqs. (51) and (53), respectively. One should keep in mind, however, that the states in Eqs. (51) and (53) are invariant under the permutations of the parameters and so are the corresponding Bethe ansatz equations. Consequently, solutions of the Bethe ansatz equations which differ only by a reordering of the variables should be counted as one solution.

The eigenvalues of the operators \( \hat{P}_j \) corresponding to the eigenstates given in Eqs. (51) and (53) above will be denoted by \( \lambda_j^{(N)} \) and \( \mu_j^{(N)} \), respectively, i.e.,

\[
\hat{P}_j \hat{S}^+(0)\hat{S}^+(z_1)\ldots\hat{S}^+(z_{N-1})|0\rangle = \lambda_j^{(N)} \hat{S}^+(0)\hat{S}^+(z_1)\ldots\hat{S}^+(z_{N-1})|0\rangle
\]

\[
\hat{P}_j \hat{S}^+(x_1)\hat{S}^+(x_2)\ldots\hat{S}^+(x_N)|0\rangle = \mu_j^{(N)} \hat{S}^+(x_1)\hat{S}^+(x_2)\ldots\hat{S}^+(x_N)|0\rangle
\]

These eigenvalues can be computed with the help of Eqs. (4), (40), (52) and (54). The results are given by (see the Appendix for the details)

\[
\lambda_j^{(N)} = E_j^{(0)} - \frac{\sum_{k=1}^{N-1} \Omega_j}{1 - c_j^2 z_k},
\]

\[
\mu_j^{(N)} = E_j^{(0)} - \frac{\sum_{k=1}^N \Omega_j}{1 - c_j^2 x_k}
\]

**Eigenstates for** \( N_{max}/2 < N \): In order to write down the eigenstates and eigenvalues corresponding to a shell which is more than half full, we introduce the operator

\[
\hat{T} = \exp \left( -i\pi \sum_j \frac{\hat{S}^+_j + \hat{S}^-_j}{2} \right)
\]

[4] It is worth emphasizing that in driving Eqs. (52), all the parameters \( z_k \) were assumed to be different from each another and similarly for the Eqs. (54). As a result, any solution which contradicts this assumption should be discarded. In fact, it can be shown that if any two parameters in the state (51) are equal to each other, then this state cannot be an eigenstate. A similar result is also valid for the state in Eq. (53).
TABLE I: Summary of the energy eigenvalues and the eigenstates of the pairing Hamiltonian in the degenerate limit, given in Eq. (33) and its quantum invariants given in Eq. (35). Here, \( N_{\text{max}} \) denotes the maximum number of pairs which can occupy the shell.

| State | Bethe Ansatz Equation | Eigenvalue of \( \hat{P}_j \) | Eigenvalue of \( \hat{H}_D \) |
|-------|----------------------|------------------|------------------|
| | | $E_j^{(0)} = \frac{\Omega_j}{2} \sum_{j' \neq j} \frac{\Omega_{j'}}{1 - c_{j'}^2 / c_j^2}$ | 0 |
| Empty shell: \( N = 0 \) | | | |
| State | Bethe Ansatz Equation | Eigenvalue of \( \hat{P}_j \) | Eigenvalue of \( \hat{H}_D \) |
|-------|----------------------|------------------|------------------|
| | | $E_j^{(0)} = \Omega_j / (1 - c_j^2 x_j)$ | 0 |
| One pair of particles in the shell: \( N = 1 \) | | | |
| State | Bethe Ansatz Equation | Eigenvalue of \( \hat{P}_j \) | Eigenvalue of \( \hat{H}_D \) |
|-------|----------------------|------------------|------------------|
| | | $E_j^{(0)} - \Omega_j$ | $- |G| \sum_j c_j^2 \Omega_j$ |
| At most half full Shell: \( N \leq N_{\text{max}} / 2 \) | | | |
| State | Bethe Ansatz Equation | Eigenvalue of \( \hat{P}_j \) | Eigenvalue of \( \hat{H}_D \) |
|-------|----------------------|------------------|------------------|
| | | $E_j^{(0)} - \Omega_j - \sum_{k=1}^{N-1} \Omega_j / (1 - c_j^2 x_k)$ | $- |G| \sum_j c_j^2 \Omega_j + |G| \sum_{k=1}^{N-1} 2 / 2$ |
| More than half full Shell: \( N_{\text{max}} / 2 < N \) | | | |
| State | Bethe Ansatz Equation | Eigenvalue of \( \hat{P}_j \) | Eigenvalue of \( \hat{H}_D \) |
|-------|----------------------|------------------|------------------|
| | | $E_j^{(0)} - \Omega_j - \sum_{k=1}^{N-1} \Omega_j / (1 - c_j^2 x_k)$ | $- |G| \sum_j c_j^2 \Omega_j + |G| \sum_{k=1}^{N-1} 2 / 2$ |
| Full Shell \( N = N_{\text{max}} \) | | | |
| State | Bethe Ansatz Equation | Eigenvalue of \( \hat{P}_j \) | Eigenvalue of \( \hat{H}_D \) |
|-------|----------------------|------------------|------------------|
| | | $E_j^{(0)} - \Omega_j$ | $- |G| \sum_j c_j^2 \Omega_j$ |
following Ref. [27]. This operator transforms the empty shell given in Eq. (10) into the fully occupied shell given in Eq. (11), i.e.,

\[ \hat{T}^\dagger |0\rangle = |\bar{0}\rangle. \] (60)

In addition, it also transforms the pair creation operators into pair annihilation operators and visa versa:

\[ \hat{T}^\dagger S_j^+ T = S_j^- \quad \hat{T}^\dagger S_j^z T = -S_j^z \] (61)

Therefore, the operator \( \hat{T}^\dagger \) transforms a state containing \( N \) particle pairs into states containing \( N \) hole-pairs. However, there is no particle-hole symmetry in the problem described by the invariants \( \hat{P}_j \) as can be easily verified by showing that

\[ [\hat{P}_j, \hat{T}] \neq 0. \] (62)

On the other hand, it is easy to show that the operator

\[ \hat{B} = \hat{T}^\dagger S^- (0) \] (63)

commutes with the invariants \( \hat{P}_j \):

\[ [\hat{P}_j, \hat{B}] = 0 \quad [\hat{H}_D, \hat{B}] = 0. \] (64)

This tells us that if \( |\psi\rangle \) is a simultaneous eigenstate of the operators \( \hat{P}_j \) so is \( \hat{B}|\psi\rangle \) unless \( |\psi\rangle \) is annihilated by \( \hat{B} \). It is easy to see from the definition of \( \hat{B} \) that if the state \( |\psi\rangle \) has \( N \) particle-pairs, then the state \( \hat{B}|\psi\rangle \) has \( N - 1 \) hole-pairs. Because the operator \( \hat{B} \) first annihilates one pair and then replaces the remaining \( N - 1 \) particle-pairs with hole-pairs. Consider, for example, the eigenstates with one pair of particles given in Eqs. (44) and (45). One can easily show, using Eqs. (41), (60) and (61) that

\[ \hat{B} \hat{S}^+ (0)|0\rangle \propto |\bar{0}\rangle \] (65)

where \( |\bar{0}\rangle \) is the state which is maximally occupied by pairs defined in Eq. (11). Consequently, the states \( \hat{S}^+ (0)|0\rangle \) and \( |\bar{0}\rangle \) have the same eigenvalues given by Eqs. (49) and (50). On the other hand, the state in Eq. (45) which also has one pair of particles is annihilated by the operator \( \hat{B} \):

\[ \hat{B} \hat{S}^+ (x)|0\rangle = 0 \] (66)

This can be easily verified using the commutators [41] and the Bethe ansatz equation [46]. Similarly, if we act on the eigenstate with \( N \) pairs of particles \((2 \leq N \leq N_{\text{max}}/2)\) given in Eq. (51), we find that

\[ \hat{B} \hat{S}^+ (0) \hat{S}^+ (z_1) \ldots \hat{S}^+ (z_{N-1})|0\rangle \propto \hat{S}^- (z_1) \ldots \hat{S}^- (z_{N-1})|0\rangle \] (67)

As a result the state

\[ \hat{S}^- (z_1) \ldots \hat{S}^- (z_{N-1})|0\rangle \] (68)

which has \( N-1 \) pairs of holes, has the same eigenvalues as the state in Eq. (51), i.e., those given in Eqs. (57) and (58). On the other hand, the state in Eq. (58) which also has \( N \) pairs of particles \((2 \leq N \leq N_{\text{max}}/2)\) is annihilated by the operator \( \hat{B} \)

\[ \hat{B} \hat{S}^+ (x_1) \hat{S}^+ (x_2) \ldots \hat{S}^+ (x_N)|0\rangle = 0 \] (69)

as can be verified by using the commutators [41] and the Bethe ansatz equations given in Eqs. (54).

The eigenstates and the eigenvalues of the invariants \( \hat{P}_j \) described in this section are summarized in Table II. This table also contains the corresponding eigenvalues of the pairing Hamiltonian \( \hat{H}_D \) given in Eq. (33) obtained from the eigenvalues of its invariants by using the formula (38). For example, the eigenvalue of \( \hat{H}_D \) corresponding to the state in Eq. (11) is given by

\[ |G| \sum_j c_j^2 \lambda_j^{(1)}, \] (70)

e etc . . . The energy eigenvalues of the Hamiltonian \( \hat{H}_D \) obtained from Eq. (38) are in agreement with those obtained earlier in Refs. [23, 26, 27].
V. RATIONAL GAUDIN ALGEBRA

Rational Gaudin algebra naturally appears from the pair creation and annihilation operators used in building the simultaneous eigenstates of the rational Gaudin magnet Hamiltonians. Formally, it is defined as an infinite dimensional algebra whose generators \( J^+(\lambda), J^-(\lambda), J^0(\lambda) \) depend on a complex variable \( \lambda \) and obey the commutation relations

\[
\begin{align*}
[J^+(\lambda), J^-(\mu)] & = 2 \frac{J^0(\lambda) - J^0(\mu)}{\lambda - \mu}, \\
[J^0(\lambda), J^\pm(\mu)] & = \pm \frac{J^\pm(\lambda) - J^\pm(\mu)}{\lambda - \mu},
\end{align*}
\]

for \( \lambda \neq \mu \). Commutators of \( J^+(\lambda), J^-(\lambda) \) and \( J^0(\lambda) \) at the same value of the complex parameter are given by taking the limit \( \mu \to \lambda \). We are usually interested, however, with the finite dimensional (unfaithful) realizations of this algebra. For example, it is not difficult to show that for any set of real parameters \( \alpha_j \) which are all different from one another, the operators

\[
\begin{align*}
J^+(\alpha_j; \lambda) & = \sum_j \frac{S^+_j}{\alpha_j - \lambda} \\
J^-(\alpha_j; \lambda) & = \sum_j \frac{S^-_j}{\alpha_j - \lambda} \\
J^0(\alpha_j; \lambda) & = \sum_j \frac{S^0_j}{\alpha_j - \lambda}
\end{align*}
\]

form a realization of the rational Gaudin algebra given in Eqs. (71). Note that in Eqs. (72) we explicitly stated the dependence of the generators on the parameters \( \alpha_j \). For different values of these parameters, one obtains different realizations of the rational Gaudin algebra.

The realization of the rational Gaudin algebra given in Eqs. (72) is related to the pairing problem in both the reduced pairing case and the degenerate case for different values of the parameters \( \alpha_j \). By taking \( \alpha_j = 2\varepsilon_j \) in Eq. (72), one obtains the operators given in Eq. (21) which are used in building the eigenstates in the reduced pairing case, i.e.,

\[
\begin{align*}
J^+(2\varepsilon_j; \lambda) & = \sum_j \frac{S^+_j}{2\varepsilon_j - \xi} \\
J^-(2\varepsilon_j; \lambda) & = \sum_j \frac{S^-_j}{2\varepsilon_j - \xi} \\
J^0(2\varepsilon_j; \lambda) & = \sum_j \frac{S^0_j}{2\varepsilon_j - \xi}
\end{align*}
\]

The pair creation operators given in Eq. (40) can also be obtained from a realization of the rational Gaudin algebra although a change of basis is necessary. By taking \( \alpha_j = 1/c_j \) in Eqs. (72) one obtains

\[
\begin{align*}
J^+(1/c_j; \lambda) & = \sum_j \frac{S^+_j}{1/c_j - \lambda} \\
J^-(1/c_j; \lambda) & = \sum_j \frac{S^-_j}{1/c_j - \lambda} \\
J^0(1/c_j; \lambda) & = \sum_j \frac{S^0_j}{1/c_j - \lambda}
\end{align*}
\]

and then the operators in Eq. (40) can be written as a linear combination of these operators, i.e.,

\[
S^\pm(x) = \frac{J^\pm(1/c_j; \sqrt{x}) + J^\pm(1/c_j; -\sqrt{x})}{2}
\]

Rational Gaudin algebra defined in Eqs. (71) appears in connection with various integrable models as mentioned in the Introduction. It is often the case that the step operators which are used in building the eigenstates live in one of its realizations. Here we see that the integrability of the pairing Hamiltonian in the reduced pairing and the degenerate pairing cases can be studied in two different realizations of the rational Gaudin algebra, namely those given in Eqs. (73) and (74), respectively.

VI. CONCLUSIONS

In this paper, we have obtained the quantum invariants of the pairing Hamiltonian in the degenerate case and simultaneously diagonalized them with the help of the algebraic Bethe ansatz method. Although in
order to obtain the eigenvalues one should first solve a system of Bethe ansatz equations which are nonlinear and coupled to each other, solving them usually proves to be much more convenient then a direct numerical diagonalization method. Exact analytical methods for solving the Bethe ansatz equations also exist in some simplified cases (see, for example, Refs. [16, 26, 31]). The quantum invariants obtained in this paper are the counterparts of the well known rational Gaudin magnet Hamiltonians which play the same role in the reduced pairing case. It is worth mentioning that since the quantum invariants are mutually commuting operators, they can be used to build various other integrable models besides the ones considered in this paper.

We also pointed out that the integrability of the pairing Hamiltonian in both the reduced pairing and the degenerate cases is connected with the rational Gaudin algebra. The generalizations of the rational Gaudin algebra to different underlying algebraic systems (such as higher order Lie algebras, quantum algebras or superalgebras) have been used to study the reduced pairing model and the related rational Gaudin magnet Hamiltonians in more general frameworks. The question then naturally arises whether or not one can do the same generalization for the degenerate pairing model and the related quantum invariants too. For example can the invariant operators of the degenerate pairing given in Eq. (35) be generalized to other algebraic systems and then used to study different integrable many body systems? The answer of this questions goes beyond the scope of this paper and will be considered elsewhere.

**APPENDIX A: OBTAINING THE EIGENSTATES WITH BETHE ANSATZ METHOD**

The simultaneous eigenstates and eigenvalues of the degenerate the operators given in Eq. (35) can be obtained using the method of algebraic Bethe ansatz. In this method, one first constructs a Bethe ansatz state [29] which includes some undetermined parameters and then substitutes this state into the eigenvalue-eigenstate equation \( \hat{P}_j |\psi\rangle = E_j |\psi\rangle \). The requirement that the Bethe ansatz state obeys the eigenvalue-eigenstate equation yields a set of equations called the equations of Bethe ansatz, whose solutions determine the values of the parameters in the Bethe ansatz state. For example, in order to obtain the eigenstates with one pair of particles, one can start from a generic state in the form

\[
S^+(x)|0\rangle \tag{A1}
\]

where \( S^+(x) \) is defined in Eq. (40). Using the commutators given in Eq. (4), one can show that the action of the operator \( \hat{P}_j \) on such a state is given by

\[
\hat{P}_j S^+(x)|0\rangle = \left( E_j^{(0)} - \frac{\Omega_j}{1 - c_j^2 x} \right) S^+(x)|0\rangle + \left( x \sum_{j'} \frac{\Omega_{j'}}{1/c_{j'}^2 - x} \right) \frac{c_j S_j^+}{1 - c_j^2 x}|0\rangle. \tag{A2}
\]

Clearly, if we choose \( x \) in such a way that the second term on the right hand side of Eq. (A2) vanishes, i.e., if

\[
x \sum_j \frac{\Omega_j}{1/c_j^2 - x} = 0, \tag{A3}
\]

then Eq. (A2) becomes an eigenvalue-eigenstate equation. One way is to satisfy Eq. (A3) is to take \( x = 0 \) in which case Eq. (A2) yields the eigenvalue-eigenstate equation given in Eq. (47). Alternatively one can choose \( x \) in such a way that it satisfies the Bethe ansatz equation

\[
\sum_j \frac{\Omega_j}{1/c_j^2 - x} = 0 \tag{A4}
\]

in which case, Eq. (A2) yields the eigenvalue-eigenstate equation given in Eq. (48).
These results can be easily generalized to a state in the form
\[ S^+ (x_1) S^+ (x_2) \ldots S^+ (x_N) |0 \] (A5)
which has \( N \) pairs of particles where \( N \leq N_{\text{max}}/2 \). The parameters \( x_1, x_2, \ldots, x_N \) are in general complex and they are all different from one another. In fact, by acting on the state given in Eq. (A5) with the operators \( \hat{P}_j \) one can easily show that if any two of the two parameters \( x_1, x_2, \ldots, x_N \) are the same, then the state in Eq. (A5) cannot be an eigenstate. If, on the other hand, the parameters \( x_1, x_2, \ldots, x_N \) are all different from one another, then by acting on the state given in Eq. (A5) with the operators \( \hat{P}_j \) given in Eq. (35), we find

\[
\hat{P}_j S^+ (x_1) S^+ (x_2) \ldots S^+ (x_N) |0 \right) = \left( E_j^{(0)} - \sum_{k=1}^{N} \frac{\Omega_j}{1 - c_j^2 x_k} \right) S^+ (x_1) S^+ (x_2) \ldots S^+ (x_N) |0 \right) (A6)
\]

Clearly, Eq. (A6) becomes an eigenvalue-eigenstate equation for all the operators \( \hat{P}_j \) if we choose the variables \( x_1, x_2, \ldots, x_N \) so as to satisfy

\[
x_k \left( \sum_{l=1}^{N} \frac{2}{x_k - x_l} + \sum_{j} \frac{\Omega_j}{1/c_j^2 - x_k} \right) = 0 \quad (A7)
\]

for every \( k = 1, 2, \ldots, N \).

If the variables \( x_1, x_2, \ldots, x_N \) are all different from zero, then they clearly have to obey

\[
\sum_{l=1}^{N} \frac{2}{x_k - x_l} + \sum_{j} \frac{\Omega_j}{1/c_j^2 - x_k} = 0 \quad (A8)
\]

and in this case, Eq. (A6) yields the eigenvalue-eigenstate equation given in Eq. (50).

If, on the other hand, one of the parameters, say \( x_1 \), is chosen to be zero (we cannot choose more than one \( x_k \) to be zero since the parameters must be different from one another) then the Bethe ansatz equation (A7) is automatically satisfied for \( k = 1 \). In this case, the remaining parameters \( x_2, \ldots, x_N \) are to be found by solving the \( N - 1 \) equations

\[
\sum_{j} \frac{\Omega_j}{1/c_j^2 - x_k} + \sum_{l=2}^{N} \frac{2}{x_k - x_l} = 0 \quad (A9)
\]

for \( k = 2, 3, \ldots, N \). If we rename the remaining \( N - 1 \) variables as \( x_2 = z_1, x_3 = z_2, \ldots, x_N = z_{N-1} \), then Eq. (A6) becomes the eigenvalue-eigenstate equation given in Eq. (55).

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