Peeling and Multi-critical Matter Coupled to Quantum Gravity

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Abstract. We show how to determine the unknown functions arising when the peeling decomposition is applied to multi-critical matter coupled to two-dimensional quantum gravity and compute the loop-loop correlation functions. The results that $\eta = 2 + 2/(2K - 3)$ and $\nu = 1 - 3/2K$ agree with the slicing decomposition, and satisfy Fisher scaling.

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1 Introduction

One of the outstanding problems in the theory of two-dimensional quantum gravity is the effect of matter fields on the Hausdorff dimension. In models of discretized two-dimensional quantum gravity we define the grand canonical partition function for an ensemble of graphs (which for the moment we assume are triangulations) $\mathcal{G}$ by

$$Z(\mu) = \sum_{G \in \mathcal{G}} e^{-\mu |G|} w_G$$

(1)

where $|G|$ denotes the number of triangles in $G$, and $w_G$ the partition function of any matter fields in the theory on the graph $G$ (for an introduction to this material see for example [1]). To define the Hausdorff dimension [2, 3] we first define the geodesic distance $d_G(i, j)$ between two links $i$ and $j$ as the minimum number of triangles which must be traversed to get from the centre of one link to the centre of the other. Then we introduce the two-point function

$$H(r, \mu) = \sum_{G \in \mathcal{G}} e^{-\mu |G|} w_G \sum_{i, j \in G} \delta(d_G(i, j) - r)$$

(2)

We expect that $H$ has the asymptotic behaviour [3, 4]

$$H(r, \mu) \sim e^{-m(\mu)r}, \quad m(\mu)r >> 1,$$

$$\sim r^{1-\eta_g}, \quad m(\mu)^{-1} >> r >> 1,$$

(3)

where, as $\mu \to \mu_c$, the mass gap vanishes as

$$m(\mu) \sim (\mu - \mu_c)^{\nu_g}.$$  

(4)

In general it is also convenient to consider a more general correlation function between boundary loops of length $l_1$ and $l_2$; [2] is essentially the correlator for minimum length loops. Note that it follows from (2) that

$$\sum_r H(r, \mu) = \sum_{G \in \mathcal{G}} e^{-\mu |G||G|^2} w_G$$

$$\sim (\mu - \mu_c)^{-\gamma_{str}}$$

(5)

where, in unitary theories, $\gamma_{str}$ is the string susceptibility exponent, and inserting the form (3) we conclude that

$$\nu_g(2 - \eta_g) = \gamma_{str}$$

(6)

which is the Fisher scaling relation. At least in unitary theories the Hausdorff dimension $d_H$ is given by $d_H \nu_g = 1$ and has the geometrical meaning that in the
continuum limit the average volume is related to the geodesic size by \( \langle V \rangle \sim R^{d_H} \).

Analytic calculations of the scaling behaviour of the correlation functions \( \text{[2]} \) were first done by means of the slicing decomposition introduced by Kawai et al. \( \text{[2]} \) and then somewhat later Watabiki \( \text{[5]} \) introduced the peeling decomposition. For pure gravity (ie. \( w_G = 1, \gamma_{\text{str}} = -1/2 \)) both peeling and slicing decompositions keep track of the geodesic distance and give the same results which tell us directly that the Hausdorff dimension of the ensemble is 4.

When matter fields are introduced the situation becomes more complicated. The time scale, usually called the string time \( t \), introduced in the decompositions that have been formulated is no longer by construction the geodesic distance, nor indeed are the time scales for different decompositions necessarily equivalent. However the above discussion of correlation functions can be repeated in terms of the string time \( t \) instead of the geodesic distance \( r \) leading to another pair of exponents, \( \eta \) and \( \nu \), which are also expected to satisfy the Fisher scaling relation. For example in the \( c = -2 \) model the scaling with string time has been calculated completely by the peeling decomposition \( \text{[6]} \) with the result that \( \nu = \frac{1}{2} \) which would imply that \( d_H = 2 \) if the string time and geodesic distance are proportional. In fact high precision numerical calculations \( \text{[7]} \) find \( d_H = 3.58 \pm 0.04 \) in agreement with the formula \( \text{[8]} \)

\[
d_H = 2 \sqrt{\frac{25 - c + \sqrt{49 - c}}{25 - c + \sqrt{1 - c}}} \quad (7)
\]
derived using scaling arguments for diffusion in Liouville theory. For unitary matter complete calculations have not been made but it seems that \( \nu = |\gamma_{\text{str}}|/2 \text{[9]} \); the implied value of \( d_H \) is in contradiction with the results of numerical simulations which suggest that \( d_H \) is close to 4 \( \text{[10]} \) but are not in particularly good agreement with \( \text{[2]} \) either. It seems certain that when matter is present the string time and the geodesic distance have different scaling dimensions but the relation between them is unknown.

In this paper we will be concerned with the matrix models with a critical point corresponding to the \( (p, q) = (2, 2K - 1), K = 2, 3, 4, \ldots \) multi-critical models coupled to quantum gravity \( \text{[11, 12]} \). For \( K > 2 \) the ensemble of graphs \( \mathcal{G} \) now allows polygons with \( \{4, \ldots, 2K\} \) sides and there are \( K \) independent coupling constants, with polygons of order \( \{6, 10, 14, \ldots\} \) having negative weights which makes the models non-unitary. The partition function is again defined by

\[
\mathcal{Z}(\mu) = \sum_{G \in \mathcal{G}} e^{-\mu |\mathcal{G}|} w_G \quad (8)
\]
where now \( G \) denotes the number of polygons, \( \mu \) is the coupling constant conjugate to
the number of polygons, and $w_G$ depends on the remaining $K-1$ coupling constants of the theory. $K = 2$ corresponds to pure gravity; the second coupling constant which appears in this matrix model is conjugate to the length of the boundary of the graph. Since there are now in general many couplings, there are many correlation functions which are second derivatives with respect to the couplings so there are many susceptibilities. We define the susceptibility

$$\chi_\mu \equiv \frac{\partial^2 Z(\mu)}{\partial \mu^2} = \sum_{G \in G} e^{-\mu |G|} |G|^2 w_G$$

$$\sim (\mu - \mu_c)^{-\gamma}$$

(9)

where the exponent $\gamma$ is known to take the value $-K^{-1}$ at the multicritical point.

The string time, $t_G(i,j)$, separating two links is now defined as the minimum number of polygons which must be traversed to get from the centre of one link to the centre of the other and the two-point function is defined as

$$\mathcal{H}(t,\mu) = \sum_{G \in G} e^{-\mu |G|} \sum_{i,j \in G} \delta(t_G(i,j) - t).$$

(10)

We expect (and shall confirm) that, if all the couplings except $\mu$ are set to their values at the multi-critical point, $\mathcal{H}$ has the asymptotic behaviour

$$\mathcal{H}(t,\mu) \sim e^{-m(\mu)t}, \quad m(\mu)t >> 1,$$

$$\sim t^{1-\eta}, \quad m(\mu)^{-1} >> t >> 1,$$

(11)

where, as $\mu \to \mu_c$, the mass gap vanishes as

$$m(\mu) \sim (\mu - \mu_c)^\nu.$$  

(12)

and that the Fisher scaling relation (which may be obtained by similar manipulations as before)

$$\nu(2 - \eta) = \gamma$$

(13)

is satisfied. One could hold that $t$ is still the geodesic distance but this is slightly problematic for large $K$; it implies that all sides of a given polygon, no matter how many sides it has, are separated from one another by geodesic distance 1 and we shall argue in section 6 that the continuum limit of $t$ is not a sensible continuum geodesic distance. These models have been analyzed using the slicing decomposition in the same way as pure gravity \cite{[13]} and they have also been considered using the peeling decomposition in the scaling limit but the discretized equations have not been solved completely \cite{[3], [4]}. In this paper we will examine their peeling decomposition in detail and explain how to solve completely the non-trivial differential equations that
arise. We have two motivations for this; to check whether the results are indeed the same as for slicing, and the intrinsic interest of the method of solution.

This paper is structured as follows. In section 2 we briefly describe the standard peeling calculation for pure gravity and then derive the evolution equation for the multi-critical models. In section 3 we consider the $K = 2$ case and show that it always gives the standard pure gravity results. Then in section 4 we show in detail how to calculate the $\eta$ exponent for $K = 4$ and describe how the calculation extends to all higher even $K$. In section 5 we explain how to calculate $\nu$ for all even $K$ and in section 6 we give our conclusions.

2 The Peeling Decomposition and Evolution Equations

We start by reviewing the calculation in [5] for the simplest pure gravity model which has matrix model potential

$$U(\phi) = \frac{1}{2} \phi^2 - \frac{1}{3} g \phi^3.$$  \hfill (14)

The matrix model generates the dual graphs to the triangulations $\mathcal{G}$ (see equation (1)) with $g = e^{-\mu}$ and $w_G = 1$. The Schwinger Dyson equation for connected Green’s functions is obtained by marking one external line and pulling it out to expose the vertex to which it is attached [15], see fig.(1). In the peeling decomposition we assign a time variable to this process; a single iteration advances $t$ by an amount $1/n$ so that we obtain

$$A_n(t + 1/n) = \delta_{n,2} + g A_{n+1}(t) + g \sum_{m=1}^{n} A_m(t) A_{n-m+1}(t).$$  \hfill (15)

We are interested in the loop-loop correlation function; suppose for the moment that at $t = 0$ the entry loop is a one-loop and form the quantity

$$G_n(t) = \frac{\delta A_n(t)}{\delta A_1(0)}.$$  \hfill (16)
which is the amplitude for an exit $n$-loop at time $t$ given an entry 1-loop at time $t = 0$. Differentiating (15) we obtain

$$G_n(t + 1/n) = gG_{n+1}(t) + 2g \sum_{m=1}^{n} A_m(t)G_{n-m+1}(t).$$

(17)

If we restrict to spherical topology then $A_m(t)$ may be replaced by the disk amplitude for $m$ legs, $A_m$, because the branch can never rejoin the main tube (see fig. 2). The next step is to approximate the time by a continuous variable to obtain the evolution equation

$$G_n(t) + \frac{1}{n} \frac{dG_n(t)}{dt} = gG_{n+1}(t) + 2g \sum_{m=1}^{n} A_mG_{n-m+1}(t)$$

(18)

with the initial condition that

$$G_n(0) = \frac{\delta A_n(0)}{\delta A_1(0)} = \delta_{n,1}. $$

(19)

Note that by differentiating (18) we can show iteratively that all the derivatives of $G_n(t)$ at $t = 0$ are finite. Defining the generating function

$$G(t, x) = \sum_{n=1}^{\infty} x^n G_n(t)$$

(20)

equation (18) becomes

$$-\frac{\partial G}{\partial t} = x \frac{\partial}{\partial x} \left( \frac{F(x)G}{x} \right)$$

(21)

where

$$F(x) = 2gA(x) + g - x,$$

(22)
and the form of the disk amplitude

\[ A(x) = \sum_{n=1}^{\infty} A_n x^n \]  

(23)
is known [9]. Note that the function \( F(x) \) contains just the universal scaling part of the amplitude \( A(x) \); it is given by

\[ F(x) = (f - x)(1 - 8fg - 4gx)^{\frac{1}{2}} \]  

(24)

where \( f \) is the root of the cubic

\[ (1 - 8fg)f^2 - g^2 = 0 \]  

(25)

which is positive and vanishes as \( g \to 0 \). It is then straightforward to solve the evolution equation which gives

\[ G(t, x) = \frac{x}{F(x)} U(t + J(x)) \]  

(26)

where

\[ \frac{dJ}{dx} = \frac{1}{F(x)} \]  

(27)

and the function \( U(y) \) is fixed by the initial conditions to satisfy

\[ 1 = \frac{1}{F(x)} U(J(x)). \]  

(28)

This leads to the scaling behaviour at large \( y \)

\[ U(y) \simeq (\mu - \mu_c)^{\frac{\nu}{\eta}} \frac{\cosh(\mu - \mu_c)^{\frac{\nu}{\eta}} y}{\sinh^{\frac{\nu}{2}}(\mu - \mu_c)^{\frac{\nu}{2}} y} \]  

(29)

where we have suppressed various constant factors; so we deduce \( \nu = \frac{1}{4} \) and \( \eta = 4 \) in agreement with the Fisher scaling relation [3].

Now we turn to the multi-critical models for which we will use the notation of reference [16]. The potential for the \( K \)-th multi-critical model is given by

\[ U(\Phi) = \Phi^2 + \sum_{p=1}^{K} g_p^K \Phi^{2p} \]  

(30)

where the couplings at the multicritical point are

\[ g_p^K = (-1)^{p-1} \frac{K!(p-1)!}{(K-p)!2p!} - \frac{1}{2} \delta_{p,1}. \]  

(31)
This time we will deal with disconnected graphs. The evolution equation follows from the Schwinger-Dyson equation, shown in fig. 3 just as in the pure gravity case; we obtain the evolution equation

\[ A_n(t + 1/n) = \sum_{j=0}^{n-2} A_{n-j-2}(t)A_j(t) - 2 \sum_{k=1}^{K} kg_k^K A_{n+2k-2}(t). \]  

(32)

Proceeding as before, but assuming that the entry loop is an \( m \)-loop and defining

\[ G_{n,m}(t) = \frac{\delta A_n(t)}{\delta A_m(0)}, \]  

(33)

we find

\[ G_{n,m}(t) + \frac{1}{n} \frac{dG_{n,m}(t)}{dt} = 2 \sum_{j=0}^{n-2} G_{j,m}(t)A_{n-j-2} - \sum_{k=1}^{K} 2kg_k^K G_{n+2k-2,m}(t) \]  

(34)

with the initial condition that

\[ G_{n,m}(0) = \delta_{n,m}. \]  

(35)

Again, note that by differentiating (34) we can show iteratively that all the derivatives of \( G_{n,m}(t) \) at \( t = 0 \) are finite.

Defining the generating function

\[ G_m(x, t) = \sum_{n=1}^{\infty} x^n G_{n,m}(t) \]  

(36)

we obtain the partial differential equation

\[ \frac{\partial G_m}{\partial t} = x \frac{\partial}{\partial x} \left\{ G_m \left( 2x^2 A(x) - 1 - \sum_{k=1}^{K} \frac{2kg_k^K}{x^{2k-2}} \right) \right\} - \sum_{k=1}^{K} 2kg_k^K \sum_{j=1}^{2k-3} G_{j,m}(t)(2k - 2 - j)x^{j-2k+2}. \]  

(37)
At this stage we should make several remarks. Firstly that one reason for dealing with the disconnected graphs is technical convenience; the required disk amplitudes are known and take a simple form \[16\], and the structure of the evolution equations is similar to the $\phi^3$ case. However it is also very clear that because the graph ensemble is disconnected there is no direct correspondence between the string time $t$ and a geodesic distance – the latter can only be sensibly defined on connected graphs. These equations are more difficult to solve than the pure gravity example reviewed above because of the presence of the \textit{a priori} unknown functions $G_{j,m}(t)$ which have to be determined by the self-consistency and analyticity properties of the solutions.

We will work our way through the problem in a number of steps. First we will study the solution in the $K = 2$ case and show that it gives the standard pure gravity results.

### 3 Universality in the $K = 2$ case

It is simpler to work at the multi-critical points initially and to compute the $\eta$ exponent directly at the critical point; we leave the $\nu$ exponent for section 5. For the $K = 2$ multi-critical model we have

\[
\frac{\partial G_m}{\partial t} = x \frac{\partial}{\partial x} \left( G_m (2x^2A(x) - 1 - 2g_1^2 - 4g_2^2x^{-2}) - 4g_2^2x^{-1}G_{1,m}(t) \right) \tag{38}
\]

where $g_1^2 = 1/2$, $g_2^2 = -1/12$ and $A(x)$ is given by

\[
A(x) = \frac{1}{x^2} + \frac{1}{6x^4} \left( (1 - 4x^2)^{\frac{3}{2}} - 1 \right). \tag{39}
\]

Thus $G_m$ satisfies

\[
\frac{\partial x^{-1}G_m}{\partial t} = \frac{\partial}{\partial x} \left( \frac{G_m (1 - 4x^2)^{\frac{3}{2}}}{3x^2} \right) + \frac{1}{3x^2} G_{1,m}(t) \tag{40}
\]

with the initial condition

\[
G_m(0, x) = x^m. \tag{41}
\]

The potential (30) is an even function of the fields and therefore $G_{n,m}(t)$, which is the amplitude for an entrance $m$-loop and exit $n$-loop, can only be non-zero of $n$ and $m$ are both odd or both even. Thus if $m$ is odd there is an unknown function on the r.h.s. of (10) whereas if $m$ is even there is no such problem; we will consider the even and odd cases separately.
### 3.1 Even $m$

When $m$ is even $G_{1,m}(t) = 0$ and we can solve (40) immediately to obtain

$$G_m(t, x) = \frac{x}{F(x)}U(t + J(x))$$

(42)

where

$$F(x) = \frac{(1 - 4x^2)^{\frac{3}{2}}}{3x},$$

$$\frac{dJ}{dx} = \frac{1}{F(x)},$$

$$J(x) = \frac{3}{4} \left( (1 - 4x^2)^{-\frac{1}{2}} - 1 \right),$$

(43)

and the function $U(y)$ is determined by the initial condition (41)

$$x_m = \frac{x}{F(x)}U(J(x)).$$

(44)

Thus we obtain

$$U(y) \sim \frac{1}{y^3}$$

(45)

for large $y$ and hence

$$G_{n,m}(t) \sim \frac{1}{t^3}$$

(46)

for even $n$ and $m$ so that $\eta = 4$ as expected for all these amplitudes.

### 3.2 Odd $m$

When $m$ is odd $G_{1,m}(t) \neq 0$ and the presence of the unknown function on the r.h.s. of (40) complicates matters; however this is more typical of the general multi-critical models than the even $m$ case and so we shall study it in some detail. Although the differential equation can of course still be solved in the $t$ domain it is more convenient to work with the Laplace transformed correlation functions

$$\overline{G}_{n,m}(s) = \int_0^\infty G_{n,m}(t)e^{-st}dt$$

$$\overline{G}_m(s, x) = \frac{1}{x} \int_0^\infty G_m(t, x)e^{-st}dt.$$  

(47)

Taking the Laplace transform of (40) we obtain the equation

$$s\overline{G}_m(s, x) - x^{m-1} = \frac{\partial}{\partial x} \left( \overline{G}_m(s, x)F(x) \right) + \frac{1}{3x^2}\overline{G}_{1,m}(s).$$

(48)
Integrating this differential equation we find

\[
\overline{G}_m(s, x) = \frac{1}{F(x)} \left( \frac{\overline{G}_{1,m}(s)}{3x} - e^{sJ(x)} \int_0^x dy e^{-sJ(y)} \left( y^{m-1} - \frac{s\overline{G}_{1,m}(s)}{(1 - 4y^2)^{\frac{3}{2}}} \right) \right). \tag{49}
\]

The function \( \overline{G}_m(s, x) \) has a power series expansion in \( x \)

\[
\overline{G}_m(s, x) = \sum_n x^{n-1} \overline{G}_{n,m}(s) \tag{50}
\]

which we expect from (49) and (43) to have finite radius of convergence \( \frac{1}{2} \); within the radius of convergence the coefficients are the Laplace transforms of the correlation functions. Unless the \( G_{n,m}(t) \) grow faster than exponentially at large \( t \), something we do not expect, their Laplace transforms \( \overline{G}_{n,m}(s) \) will have an asymptotic series representation at large \( s \). This series can be obtained by successive integration by parts of the definition (47); as we observed in section 2, \( G_{n,m}(t) \) and all its (finite order) derivatives are finite at \( t = 0 \) so we obtain the formal series

\[
\overline{G}_{n,m}(s) = \frac{1}{s} \sum_{k=0}^n \alpha_k \frac{1}{s^k}, \quad \alpha_0 = \delta_{n,m}. \tag{51}
\]

(Of course the \( \alpha_k \) depend on \( n \) and \( m \) but we will always suppress such dependence for clarity.) We will now show that imposing the condition that the coefficients in the \( x \) expansion of \( \overline{G}_m(s, x) \) behave like (51) at large \( s \) is sufficient to fix the unknown function \( \overline{G}_{1,m}(s) \).

It is more convenient to impose the condition on the integral of \( \overline{G}_m(s, x) \),

\[
\int_0^x \overline{G}_m(s, y)dy = \frac{x^m}{sm} - \frac{1}{s} e^{sJ(x)} \int_0^x dy e^{-sJ(y)} \left( y^{m-1} - \frac{s\overline{G}_{1,m}(s)}{(1 - 4y^2)^{\frac{3}{2}}} \right). \tag{52}
\]

Using (50), (51) and (52) the consistency condition is that

\[
g_m = e^{sJ(x)} \int_0^x dy e^{-sJ(y)} \left( y^{m-1} - \frac{s\overline{G}_{1,m}(s)}{(1 - 4y^2)^{\frac{3}{2}}} \right) \sim \frac{1}{s}. \tag{53}
\]

That is to say the small \( x \), large \( s \), expansion of \( g_m \) contains no terms \( O(1) \) or higher in \( s \). Substituting (51) and expanding the exponential in its Taylor series (which is of course allowed for all values of the argument) we get

\[
g_m = \int_0^x dy \sum_{p=0}^{\infty} \frac{s^p}{p!} (J(x) - J(y))^p \left( y^{m-1} - \frac{1}{(1 - 4y^2)^{\frac{3}{2}}} \sum_{k=0}^{\infty} \frac{\alpha_k}{s^k} \right). \tag{54}
\]
For \( x < \frac{1}{2} \) (54) clearly has a power series expansion in \( x \) so the requirement that there are no \( O(1) \) terms in \( s \) becomes

\[
C_0 = \int_0^x dy \left( y^{m-1} - \frac{1}{(1-4y^2)^{\frac{3}{2}}} \sum_{p=0}^\infty \frac{(J(x) - J(y))^p \alpha_p}{p!} \right) = 0. \tag{55}
\]

First we show that this constraint alone implies that all higher powers of \( s \) vanish as well; for \( O(s) \) we get

\[
C_1 = \int_0^x dy \left( J(x) - J(y) \right) \left( y^{m-1} - \frac{1}{(1-4y^2)^{\frac{3}{2}}} \sum_{p=0}^\infty \frac{(J(x) - J(y))^p \alpha_p}{p+1!} \right) = 0, \tag{56}
\]

and observe that

\[
\frac{dC_1}{dx} = \frac{dJ}{dx} C_0 = 0 \tag{57}
\]

provided \( x < \frac{1}{2} \). It follows that \( C_1(x) \) is a constant; but \( C_1(0) = 0 \) therefore

\[
C_1(x) = 0. \tag{58}
\]

Any pair \( C_k(x) \) and \( C_{k+1}(x) \), with \( k > 0 \) are related in the same way and so it is straightforward to proceed inductively to show that all \( C_{k>0}(x) \) are zero.

Taking the Laplace transform of (55) with respect to \( J \) we find that

\[
\int_0^\infty x(J)^m \frac{e^{-sJ}}{m} dJ = \overline{G}_{1,m}(s) \int_0^\infty \frac{e^{-sJ}}{3x(J)} dJ \tag{59}
\]

where \( x(J) \) is obtained by inverting (43). Thus \( \overline{G}_{1,m}(s) \) is determined; making the change of variables \( x = \frac{1}{2} \tanh \phi \) and integrating by parts we get

\[
\overline{G}_{1,m}(s) = \int_0^\infty \tanh^{m-1} \phi \frac{e^{-\frac{s}{2} \cosh \phi} \sech^2 \phi}{2^{m-1}s} \int_0^\infty \cosh \phi \frac{e^{-\frac{s}{2} \cosh \phi} d\phi}{d\phi}. \tag{60}
\]

It is clear that \( \overline{G}_{1,m}(s) \) is positive for all real, positive \( s \) and straightforward to check that its second derivative with respect to \( s \) diverges logarithmically as \( s \to 0 \); it follows that

\[
G_{1,m}(t) \sim \frac{1}{t^3} \tag{61}
\]

at large \( t \). The remaining integrals in (49) yield functions which are analytic in \( s \) in the neighbourhood of the origin provided \( x < x_c \). Thus we can conclude that

\[
G_{n,m}(t) \sim \frac{1}{t^3} \tag{62}
\]

for all odd \( n \) and \( m \). So the result \( \eta = 4 \) is universal for all amplitudes \( G_{n,m}(t) \) in agreement with every other calculation for pure gravity.
In fact $G_{1,m}(s)$ can be fixed more simply by examining (49). As $x \uparrow x_c$, $J(x)$ diverges and hence $G_{m}(s, x)$ grows faster than exponentially in $s$, which is impossible, unless

$$
\int_0^{\frac{1}{2}} dx e^{-s J(x)} \left( x^{m-1} - \frac{s G_{1,m}(s)}{(1 - 4x^2)^s} \right) = 0 \quad (63)
$$

which is the same condition as (50). We have explored the more indirect route because this will help in the multi-critical case.

4 The Multi-critical Models

The multi-critical evolution equation (37) may be written

$$
\frac{\partial x^{-1} G_m(x, t)}{\partial t} = \frac{\partial}{\partial x} \left\{ x^{-1} G_m(x, t) F(x) \right\}
- \sum_{k=1}^K 2kg_k^K \sum_{j=1}^{2k-3} G_{j,m}(t) (2k - 2 - j)x^{j-2k+1} \quad (64)
$$

where, taking $A(x)$ from [16], we have

$$
F(x) \equiv x \left( 2x^2 A(x) - 1 - \sum_{k=1}^K \frac{2kg_k}{x^{2k-2}} \right)
- 2Kg_K^K x^{3-2K} (1 - 4x^2)^{K-\frac{1}{2}}. \quad (65)
$$

The solution to (64) takes the form [17]

$$
x^{-1} G_m = \frac{1}{F(x)} U_0(t + J(x))
- \sum_{k=1}^K 2kg_k^K \sum_{j=1}^{2k-3} \frac{(2k - 2 - j)}{F(x)} \int_0^t d\tau G_{j,m}(\tau) U_{jk}(t - \tau + J(x)) \quad (66)
$$

where, as before,

$$
J(x) = \int_0^x \frac{dy}{F(y)} = \sum_{r=0}^{K-2} h^K_r (1 - 4x^2)^{r-K+\frac{3}{2}} - h^K_{K-1} \quad (67)
$$

and the coefficients $h^K_r$ are easy to compute. The function $U_0(y)$ is fixed by the initial conditions

$$
x^{m-1} = \frac{1}{F(x)} U_0(J(x)) \quad (68)
$$

and the functions $U_{jk}(y)$ by requiring that (66) is a solution of (64),

$$
x^{j-2k+1} = \frac{1}{F(x)} U_{jk}(J(x)). \quad (69)
$$
Because of the unknown functions \( G_{j,m}(t) \), (66) is not of course a complete solution. Note that \( J(x) \) is always an even function which diverges as \( |x| \to \frac{1}{2} \) and which is positive for even \( K \) and negative for odd \( K \). To determine the critical behaviour from the properties at large \( t \) we need to know the behaviour of the functions \( U_0(\tau) \) and \( U_{jk}(\tau) \) for large positive argument. For even \( K \) we see by considering (68) and (69) as \( |x| \to \frac{1}{2} \) that
\[
U_0(\tau) \xrightarrow{\tau \to \infty} \tau^{-\frac{2K-1}{2K-3}}, \quad U_{jk}(\tau) \xrightarrow{\tau \to \infty} \tau^{-\frac{2K-1}{2K-3}}. \tag{70}
\]
However for odd \( K \) only the large negative argument behaviour is determined. This phenomenon always occurs in calculations for the multi-critical models. Extrapolating the solution to positive time leads to a singularity at finite time; it seems to us quite likely that this is an artefact of the truncation of the original finite difference equation (32) into a first order differential equation (34) and that the solution may be stabilised by higher derivative terms. From now on we will concentrate on the even \( K \) models and start by studying the \( K = 4 \) case in detail.

### 4.1 \( K = 4 \), even \( m \)

As for the \( K = 2 \) model we will consider the cases of even and odd \( m \) separately. For even \( m \) we have
\[
\frac{\partial x^{-1}G_m(x,t)}{\partial t} = \frac{\partial}{\partial x} \left\{ x^{-1}G_m(x,t)F(x) \right\} + G_{2,m}(t) \left( \frac{4}{35x^5} - \frac{4}{5x^3} \right) + \frac{2}{35x^3}G_{4,m}(t) \tag{71}
\]
with
\[
F(x) = \frac{(1 - 4x^2)^{\frac{7}{2}}}{35x^5}. \tag{72}
\]
Taking the Laplace transform gives
\[
s \tilde{G}_m(s,x) - x^{m-1} = \frac{\partial}{\partial x} \left( \tilde{G}_m(s,x)F(x) \right) + \tilde{G}_{2,m}(s) \left( \frac{4}{35s^5} - \frac{4}{5s^3} \right) + \frac{2}{35s^3}\tilde{G}_{4,m}(s) \tag{73}
\]
where the transformed correlation functions are defined as in (47); integrating (73) we obtain
\[
\tilde{G}_m(s,x) = \frac{1}{F(x)} \left\{ \frac{1}{35x^4}\tilde{G}_{2,m}(s) + \frac{1}{35x^2}\tilde{G}_{4,m}(s) + \frac{1}{35}\tilde{G}_{6,m}(s) 
+ e^{sJ(x)} \int_0^x dy e^{-sJ(y)} \left( -y^{m-1} + \frac{s}{35y^5}F(y) \left( y^5\tilde{G}_{6,m}(s) + y^3\tilde{G}_{4,m}(s) + y\tilde{G}_{2,m}(s) \right) \right) \right\} \tag{74}
\]
where we have introduced the combinations
\[
\begin{align*}
\overline{G}^{C}_{4,m}(s) &= \overline{G}_{4,m}(s) - 14\overline{G}_{2,m}(s), \\
\overline{G}^{C}_{6,m}(s) &= 70\overline{G}_{2,m}(s) - 14\overline{G}_{4,m}(s) + \overline{G}_{6,m}(s).
\end{align*}
\]
Note the appearance of \(\overline{G}_{6,m}(s)\) in (74); this happens because \(F(x)\) is singular at \(x = 0\) which makes the evaluation of the limits of integration slightly non-trivial.

As before it is convenient to deal with
\[
\int_{0}^{x} \overline{G}_{m}(s,y)dy = \frac{x^m}{sm} + \frac{g_m}{s} \tag{76}
\]
where now
\[
g_m(s,x) = e^{sJ(x)} \int_{0}^{x} dy e^{-sJ(y)} \left\{ -y^{m-1} + \frac{s}{35y^5F(y)} \left( y^5\overline{G}^{C}_{6,m}(s) + y^3\overline{G}^{C}_{4,m}(s) + y\overline{G}_{2,m}(s) \right) \right\} \tag{77}
\]
and we deduce that
\[
g_m \sim \frac{1}{s} \tag{78}
\]
in order to fulfil the constraint that \(\overline{G}_{m}(s,x)\) has the correct small \(x\), large \(s\), expansion. In fact we can immediately determine \(\overline{G}_{6,m}(s)\) as explained at the end of section 2 by requiring that \(\overline{G}_{m}(s,x)\) does not grow faster than any exponential of \(s\) as \(x \uparrow \frac{1}{2}\) which implies that
\[
g_m(s, \frac{1}{2}) = 0 \tag{79}
\]
and so \(\overline{G}_{6,m}(s)\) is related to \(\overline{G}_{2,m}(s)\) and \(\overline{G}_{4,m}(s)\) which still have to be determined. Defining
\[
\begin{align*}
\overline{G}^{C}_{6,m}(s) &= \frac{1}{s} \sum_{k=0}^{\infty} \frac{\alpha_k}{s^k}, \\
\overline{G}^{C}_{4,m}(s) &= \frac{1}{s} \sum_{k=0}^{\infty} \frac{\beta_k}{s^k}, \\
\overline{G}_{2,m}(s) &= \frac{1}{s} \sum_{k=0}^{\infty} \frac{\gamma_k}{s^k},
\end{align*}
\]
(of course the \(\alpha\), \(\beta\) and \(\gamma\) coefficients depend on \(m\) but we have suppressed this to avoid clutter) and substituting into (74) we obtain the \(O(s^0)\) constraint
\[
C_0 = - \int_{0}^{x} dy \left\{ y^{m-1} - \frac{1}{35y^5F(y)} \sum_{k=0}^{\infty} \frac{(J(x) - J(y))^k}{k!} (y^5\alpha_k + y^3\beta_k + y\gamma_k) \right\} = 0.
\]
\]
\]
\[
\begin{align*}
\overline{G}^{C}_{6,m}(s) &= \frac{1}{s} \sum_{k=0}^{\infty} \frac{\alpha_k}{s^k}, \\
\overline{G}^{C}_{4,m}(s) &= \frac{1}{s} \sum_{k=0}^{\infty} \frac{\beta_k}{s^k}, \\
\overline{G}_{2,m}(s) &= \frac{1}{s} \sum_{k=0}^{\infty} \frac{\gamma_k}{s^k},
\end{align*}
\]
(80)
Identical manipulations to the $K = 2$ case show that $C_0 = 0$ ensures $C_{k>0} = 0$ also. There are in fact three separate constraints hidden in (81) which are sufficient to determine the $\alpha_k$, $\beta_k$ and $\gamma_k$ coefficients. To see this we proceed by writing everything in terms of $J$. $J(x)$ is an even function with a small $x$ expansion

$$J(x) = \frac{35}{6} x^6 + \frac{245}{4} x^8 + \ldots$$

so it follows that, by reverting the series,

$$\frac{x(J)^m}{m} = (J^{\frac{1}{2}})^{m/2} \sum_{k=0}^{\infty} a_k (J^{\frac{1}{2}})^k$$

where we take the real positive cube root of $J$ (remember that $m$ is even and again we suppress the $m$ dependence on $a_k$). Similarly we have

$$\int_0^x y dy \frac{1}{35y^5 F(y)} (J(x) - J(y))^k = \int_0^J \frac{dJ'}{35x(J')^4} (J - J')^k$$

$$= \int_0^J dJ' (J - J')^k J'^{-\frac{2}{3}} \sum_{l=0}^{\infty} b_l (J'^{\frac{1}{3}})^l$$

$$= \sum_{l=0}^{\infty} b_l J^{k+(l+1)/3} \frac{\Gamma(k+1)\Gamma\left(\frac{l+1}{3}\right)}{\Gamma(k+\frac{l+1}{3}+1)}$$

and

$$\int_0^x y^3 dy \frac{1}{35y^5 F(y)} (J(x) - J(y))^k = \sum_{l=0}^{\infty} c_l J^{k+(l+2)/3} \frac{\Gamma(k+1)\Gamma\left(\frac{l+2}{3}\right)}{\Gamma(k+\frac{l+2}{3}+1)}$$

We will not need explicit expressions for the coefficients $a_l$, $b_l$ and $c_l$. Substituting (83), (84) and (83) into $C_0$ we obtain

$$C_0(J) = - (J^{\frac{1}{2}})^{m/2} \sum_{k=0}^{\infty} a_k (J^{\frac{1}{2}})^k + \frac{1}{35} \sum_{k=0}^{\infty} \frac{\alpha_k J^{k+1}}{(k+1)!}$$

$$+ \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \gamma_k b_l J^{k+(l+1)/3} \frac{\Gamma\left(\frac{l+1}{3}\right)}{\Gamma(k+\frac{l+1}{3}+1)}$$

$$+ \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \beta_k c_l J^{k+(l+2)/3} \frac{\Gamma\left(\frac{l+2}{3}\right)}{\Gamma(k+\frac{l+2}{3}+1)}.$$  

Letting $N$ be an integer, (86) yields three conditions corresponding to terms $O(J^N)$, $O(J^{N+\frac{1}{3}})$ and $O(J^{N+\frac{2}{3}})$ respectively. The corresponding coefficients are easily extracted from (86) and we find

$$\frac{\alpha_{N-1}}{35} - a_{3N-m/2} \Gamma(N+1) + \sum_{L=1}^{N} \gamma_{N-L} b_{3L-1} \Gamma(L) + \sum_{L=1}^{N} \beta_{N-L} c_{3L-2} \Gamma(L) = 0,$$  

15
\[-a_{3N+1-m/2} \Gamma(N + \frac{4}{3}) + \sum_{L=0}^{N} \gamma_{N-L} b_{3L} \Gamma(L + \frac{1}{3}) + \sum_{L=1}^{N} \beta_{N-L} c_{3L-1} \Gamma(L + \frac{1}{3}) = 0, \quad (88)\]
\[-a_{3N+2-m/2} \Gamma(N + \frac{5}{3}) + \sum_{L=0}^{N} \gamma_{N-L} b_{3L+1} \Gamma(L + \frac{2}{3}) + \sum_{L=0}^{N} \beta_{N-L} c_{3L} \Gamma(L + \frac{2}{3}) = 0. \quad (89)\]

Now suppose that \(\gamma_0 \ldots \gamma_{N-1}\) and \(\beta_0 \ldots \beta_{N-1}\) are known; from (87) we can obtain \(\alpha_0 \ldots \alpha_{N-1}\). Then (88) contains only these known coefficients together with \(\gamma_N\) which is thus determined. Now (89) determines \(\beta_N\). Knowing \(\gamma_0 \ldots \gamma_{N-1}\) and \(\beta_0 \ldots \beta_{N-1}\) we now determine \(\alpha_N\) from (87). Proceeding in this way all the coefficients, and thus the right hand sides of (80), can be obtained iteratively. To analyse the asymptotic behaviour of the correlation functions we need to know the behaviour of \(G_{2,m}(s)\) etc at small \(s\). Resumming the relations (87), (88), and (89), and using the integral representation
\[\Gamma(n) = \int_0^\infty d\xi \xi^{n-1} e^{-\xi s}, \quad (90)\]
we get
\[0 = \frac{G_{6,m}(s)}{35s} - \int_0^\infty d\xi e^{-\xi s} \sum_{N=1}^\infty a_{3N-m/2} \xi^N + G_{2,m}(s) \int_0^\infty d\xi e^{-\xi s} \sum_{N=1}^\infty b_{3N-1} \xi^{N-1} \]
\[+ G_{4,m}(s) \int_0^\infty d\xi e^{-\xi s} \sum_{N=1}^\infty c_{3N-2} \xi^{N-1}, \quad (91)\]
\[0 = -\int_0^\infty d\xi e^{-\xi s} \sum_{N=0}^\infty a_{3N+1-m/2} \xi^{N+\frac{1}{3}} + G_{2,m}(s) \int_0^\infty d\xi e^{-\xi s} \sum_{N=0}^\infty b_{3N} \xi^{N-\frac{2}{3}} \]
\[+ G_{4,m}(s) \int_0^\infty d\xi e^{-\xi s} \sum_{N=1}^\infty c_{3N-1} \xi^{N-\frac{2}{3}}, \quad (92)\]
\[0 = -\int_0^\infty d\xi e^{-\xi s} \sum_{N=0}^\infty a_{3N+2-m/2} \xi^{N+\frac{2}{3}} + G_{2,m}(s) \int_0^\infty d\xi e^{-\xi s} \sum_{N=0}^\infty b_{3N+1} \xi^{N-\frac{1}{3}} \]
\[+ G_{4,m}(s) \int_0^\infty d\xi e^{-\xi s} \sum_{N=0}^\infty c_{3N} \xi^{N-\frac{1}{3}}, \quad (93)\]
These three equations determine the unknown functions that we need.

Now consider the equation
\[J(x) = \xi \quad (94)\]
where \(\xi\) is real and positive; it can be rewritten as the quintic equation for \(u \equiv x^2\)
\[(1 + \frac{24}{7} \xi^2 (1 - 4u)^5 - (30u^2 - 10u + 1)^2 = 0. \quad (95)\]
When $\xi = 0$ it reduces to

$$-4u^3(10 - 95u + 256u^2) = 0$$

(96)

which has a complex conjugate pair of roots

$$u = \frac{95 \pm i9\sqrt{15}}{512},$$

(97)

and three roots vanishing when $\xi = 0$; any integer power of these roots, which we denote by $u_{1,2,3}$, has a series expansion

$$(u_1(\xi))^M = (\xi^{\frac{1}{3}})^M \sum_{n=0}^{\infty} D_{Mn}(\xi^{\frac{1}{3}})^n,$$

$$(u_2(\xi))^M = (\omega \xi^{\frac{1}{3}})^M \sum_{n=0}^{\infty} D_{Mn}(\omega \xi^{\frac{1}{3}})^n,$$

$$(u_3(\xi))^M = (\omega^2 \xi^{\frac{1}{3}})^M \sum_{n=0}^{\infty} D_{Mn}(\omega^2 \xi^{\frac{1}{3}})^n,$$

(98)

where $\omega$ is a complex cube root of unity and $\xi^{\frac{1}{3}}$ is the real cube root of $\xi$.

Now take linear combinations of (91)- (93) with the coefficients being different powers of $\omega$ to obtain

$$0 = \int_0^\infty d\xi e^{-\xi} \left\{ u_i(\xi)^{\frac{1}{3}m} - \frac{1}{35} \left( \frac{G_{2,m}(s)}{u_i(\xi)^2} + \frac{G_{4,m}(s)}{u_i(\xi)} + \frac{G_{6,m}(s)}{u_i(\xi)^3} \right) \right\}$$

(99)

with $i = 1, 2, 3$.

As $\xi \to \infty$ all the roots of (95) converge on $u = \frac{1}{4}$ like

$$u = \frac{1}{4} \left( 1 - \sigma \frac{c}{\xi^{\frac{1}{3}}} + \ldots \right)$$

(100)

where $\sigma$ is a fifth root of unity and $c$ is a constant. The flow of the roots in the complex plane as $\xi$ varies is shown in fig.4. We see that $u_1$ flows into the $\xi = \infty$ point as (100) with $\sigma = 1$; and that the complex conjugate pair $u_{2,3}$ flow into the pair with $\sigma = e^{\pm i\frac{2\pi}{5}}$. Note that there are no degenerate roots of (94) in the interval $\xi = (0, \infty)$ (this follows from the fact that $F$ is regular in this interval).

Now define the function $J$ by

$$J(u = x^2) = J(x)$$

(101)

and make the change of variable

$$\xi = J(u_i)$$

(102)
Figure 4: The flow of the roots of (95) in the complex $u$ plane; the arrows denote the direction of increasing $\xi$.

in (99). The constraint equations then take their final form

$$0 = \int_{P_i} d\xi e^{-s\mathcal{J}(\xi)} \left\{ u^2 \mathcal{G}_{2,m}(s) + \frac{1}{35} \left( \mathcal{G}_{4,m}(s) + \mathcal{G}_{6,m}(s) \right) \right\}$$

(103)

where the contours $P_i$ in the complex $u$-plane are shown in fig. (4). Note that the $i = 1$ constraint is the same as (79) after an integration by parts and that these equations are guaranteed to have a unique solution by the argument immediately following (87)-(89).

To find the asymptotic large $t$ dependence of the correlation functions it suffices to find the leading non-trivial $s$-dependence of their Laplace transforms at small $s$; for this purpose we need the integrals in (103) up to and including $O(s^{5/2})$. Replacing $\mathcal{J}$ by its explicit form we need

$$I_i^F = \frac{35}{2} \int_{P_i} du u^2 (1-4u)^{-7/2} F(u) \exp(-s(A(1-4u)^{-\frac{3}{2}} + B(1-4u)^{-\frac{1}{2}} + C(1-4u)^{-\frac{1}{2}}))$$

(104)

where $F(u) = \left\{ u^{-2}, u^{-1}, u^{\frac{3}{2}m} \right\}$, $A = 7/64$, $B = -35/96$ and $C = 35/64$ and we have dropped a factor $\exp(7s/24)$ which cancels in (103). Now make the change of variables

$$z = s(1-4u)^{-\frac{5}{2}}.$$

(105)

Then all the integrals in (102) are linear combinations of

$$\mathcal{H}_k = s^{(2k-7)/5} \int_{P_i'} dz z^{-2(k-1)/5} \exp(-Az) \exp \left( -B s^{\frac{3}{2}} z^{\frac{3}{2}} + C s^{\frac{1}{2}} z^{\frac{1}{2}} \right)$$

(106)
Figure 5: Integration contours $P'_{1,2}$ for (106) in the complex $z$ plane. All contours start at $z = s$ and go out to infinity; the cut is for the fractional powers of $z$. The contour $P'_3$ is simply the complex conjugate of $P'_2$.

where $k = 1, 2, \ldots$. The contours $P'_i$ in the complex $z$-plane are shown in fig. 5. The second exponential in (106) is then expanded in its Taylor series, and the $P'_2, P'_3$ contours collapsed onto the branch cut as shown in fig. 6; all the integrals then become

$$
G_{2,m}(s) = \frac{35}{2^{7+m}}(m^2 + 6m + 8)(1 + 2\cos \frac{4\pi}{5}) \left(\frac{7s}{64}\right)^{\frac{4}{5}} + O(s^\frac{4}{5}),
$$

$$
G_{4,m}(s) = \frac{35}{2^{6+m}}(3m - 2)(m^2 + 6m + 8)(1 + 2\cos \frac{4\pi}{5}) \left(\frac{7s}{64}\right)^{\frac{4}{5}} + O(s^\frac{4}{5}),
$$

$$
G_{6,m}(s) = \frac{35}{2^{6+m}}(15m^2 - 50m + 64)
+ \frac{875}{2^{7+m}} \left(\frac{7s}{64}\right)^{\frac{4}{5}} + O(s^\frac{4}{5}).
$$

These expressions have the expected properties; they are positive and decreasing functions of $s$ for small positive $s$. It follows from (107) that $G_{2,m}(t), G_{4,m}(t)$ and
\( G_{6,m}(t) \) all have the same asymptotic behaviour at large \( t \) namely

\[
G_{j,m}(t) \sim t^{-\frac{7}{5}} \tag{108}
\]

for \( j = 2, 4, 6 \) and hence the exponent \( \eta = \frac{12}{5} \). To determine the behaviour of the higher correlation functions it is simplest to return to (74). All the integrals yield functions of \( s \) which are analytic in some neighbourhood of the origin provided \( x < \frac{1}{2} \) and so we can conclude that every coefficient of the \( x \) expansion of \( G_m(x, t) \) behaves the same way and that (108) is valid for all (even) \( j \) and \( m \).

### 4.2 \( K = 4, \text{ odd } m \)

The method is very similar to the even \( m \) case. After Laplace transforming the evolution equation and solving for \( \overline{G}_m(s, x) \) we obtain

\[
\overline{G}_m(s, x) = \frac{1}{F(x)} \left\{ \frac{1}{35x^5} \overline{G}_{1,m}(s) + \frac{1}{35x^3} \overline{G}_{3,m}(s) + \frac{1}{35x} \overline{G}_{5,m}(s) + e^{sJ(x)} \int_0^x dy e^{-sJ(y)} \left( -y^{m-1} \right) + \frac{s}{35y^5 F(y)} \left( y^4 \overline{G}_{5,m}^C(s) + y^2 \overline{G}_{3,m}^C(s) + \overline{G}_{1,m}(s) \right) \right\} \tag{109}
\]

The consistency conditions can be cast in the form

\[
0 = \int_{P_1} du \frac{dJ}{du} e^{-sJ(u)} \left\{ u^m - \frac{1}{35} \left( \frac{\overline{G}_{1,m}(s)}{u^\frac{7}{2}} + \frac{\overline{G}_{3,m}^C(s)}{u^\frac{5}{2}} + \frac{\overline{G}_{5,m}(s)}{u^\frac{1}{2}} \right) \right\} \tag{110}
\]

This time we need the integrals

\[
I_i^F = \frac{35}{2} \int_{P_1} du \frac{u^2}{u^\frac{7}{2}} (1 - 4u)^{-7/2} \{ u^{-2}, u^{-1}, 1, u^\frac{1}{2}(m+1) \} e^{-sJ(u)} \tag{111}
\]

To the required order in \( s \) these can be calculated by expanding the \( u^{-\frac{1}{2}} \) factor about \( u = \frac{1}{4} \); then term by term the integrals are just the \( H_k \). The resulting expressions for the singular terms in \( s \) are simple but the constants appear as infinite sums; fortunately the constants are only needed for the sub-leading \( s \) dependence so this method suffices (the reason for this is explained in the next sub-section). The final expressions for the correlation functions are very similar to those for even \( m \)

\[
\overline{G}_{1,m}(s) = \frac{35}{28+m_m} (m^2 + 4m + 3)
\]

\[
+ \frac{175}{6.2^{2+m_m} \Gamma(\frac{4}{5})} (m^3 + 9m^2 + 23m + 15)(1 + 2 \cos \frac{4\pi}{5}) \left( \frac{7s}{64} \right)^{\frac{2}{5}} + O(s^4),
\]

20
\[ G_{3,m}(s) = \frac{35}{2^{7+m}m} (3m^2 + 4m + 1) + \frac{175}{2^{8+m}m} \Gamma\left(\frac{4}{5}\right) (m^3 + 9m^2 + 23m + 15)(1 + 2 \cos \frac{4\pi}{5}) \left(\frac{7s}{64}\right)^{\frac{2}{5}} + O(s^{\frac{4}{5}}), \]
\[ G_{5,m}(s) = \frac{35}{2^{7+m}m} (15m^2 - 20m + 29) + \frac{875}{2^{8+m}m} \Gamma\left(\frac{4}{5}\right) (m^3 + 9m^2 + 23m + 15)(1 + 2 \cos \frac{4\pi}{5}) \left(\frac{7s}{64}\right)^{\frac{2}{5}} + O(s^{\frac{4}{5}}). \]

(112)

Thus we can conclude that the exponent \( \eta \) always takes the value 12/5 in the \( K = 4 \) model.

4.3 General even \( K \)

The method is very similar to the \( K = 4 \) case. After Laplace transforming the evolution equation and solving for \( G_m(s, x) \) we obtain

\[
G_m(s, x) = \frac{1}{F(x)} \left\{ -2K g^K \sum_{p=2}^{2K-2} x^{1-p} G^{C}_{2K-p-1,m}(s) + e^{sJ(x)} \int_0^x dy e^{-sJ(y)} \left(-y^{m-1} - \frac{2K g^K s}{y^{2K-3} F(y)} \sum_{p=1}^{2K-2} y^{p-1} G^{C}_{p,m}(s) \right) \right\} \tag{113}
\]

where

\[
G^{C}_{2K-p-1,m}(s) = \sum_{k=2}^{K} \frac{k g^K g^K}{K} G^{C}_{2k-p-1,m}(s). \tag{114}
\]

Note that for given \( m \) (either odd or even) half of the \( G^{C}_{p,m}(s) \) in (113) are automatically zero so there are \( K - 1 \) undetermined functions. Proceeding as before we next examine the roots of the equation

\[
J(x) = \xi \tag{115}
\]

which can be rewritten as a degree \( 2K - 3 \) polynomial equation for \( u \equiv x^2 \) taking the generic form

\[
(1 + a \xi)^2 (1 - 4u)^{2K-3} - (1 + b_1 u + \ldots + b_{K-2} u^{K-2})^2 = 0 \tag{116}
\]

where \( a, b_1, \ldots b_{K-2} \) are constants. There are \( K - 1 \) roots which vanish at \( \xi = 0 \) and \( K - 2 \) roots which do not; the \( K - 1 \) vanishing roots correspond to the \( K - 1 \)
functions which have to be determined. As $\xi \to \infty$ all the roots converge on $u = 1/4$ like
\[ u = \frac{1}{4} \left( 1 - \frac{c}{\xi^{2K-3}} + \ldots \right) \] (117)
where $c$ is a constant and $\sigma$ is a $(2K - 3)$rd root of unity. The roots which vanish at $\xi = 0$ flow into
\[ \sigma = \exp \left( i \frac{4\pi n}{2K-3} \right), \quad n = -\frac{K}{2} + 1, \ldots, 0, \ldots \frac{K}{2} - 1. \] (118)

From now on we will use the integer $n$ to label the roots. The consistency conditions become
\[ 0 = \int_{P_n} du \frac{d\mathcal{J}}{du} e^{-s\mathcal{J}(u)} \left\{ u^{2m} + 2K^2 g_K \sum_{p=1}^{2K-2} \frac{\mathcal{C}_{p,m}(s)}{u^{(2K-2-p)/2}} \right\} \] (119)
where the $K - 1$ paths $P_n$ follow in the complex plane the roots which vanish at the origin. Note that in the case of $K = 4$ the conditions for even $m$, (103), and for odd $m$, (110), can be merged into the form (119); for any fixed $m$ we have $K - 1$ equations for $K - 1$ unknowns.

Restricting ourselves to even $m$ all the integrals in (119) can be written as linear combinations of
\[ \mathcal{K}_p = \int_{P'_n} du (1 - 4u)^{p-K+\frac{1}{2}} \exp \left( sh_{K-1}^K - s \sum_{r=0}^{K-2} h_r^K (1 - 4u)^{r-K+\frac{3}{2}} \right). \] (120)
Note that the factor $\exp(sh_{K-1}^K)$ cancels out in (119) so from now on we drop it. After the substitution
\[ z = s(1 - 4u)^{-K+\frac{3}{2}} \] (121)
we obtain
\[ 2(2K - 3)\mathcal{K}_p = s^{-1+\frac{2p}{2K-3}} \int_{P'_n} dz z^{-\frac{2p}{2K-3}} \exp \left( -h_0^K z - \sum_{r=1}^{K-2} h_r^K s \frac{2r}{2K-3} z^{1-\frac{2r}{2K-3}} \right) \] (122)
where the contours $P'_n$ encircle the origin $n$ times before heading off to real positive infinity, see fig.(7a). As we will demonstrate shortly all these integrals are needed up to and including the first positive power of $s$ and so they can be separated into three classes;

1. $p < K - 1$; the integrals are divergent as $s \to 0$ but the singularities of the integrand are integrable at $z = 0$ so the contour can be deformed as shown in fig.(7b). The integral from the starting point to the first blob is given by
\[ 2(2K - 3)\mathcal{K}_p^{fin} = \frac{2K - 3}{2p - 2K + 3} + O(s). \] (123)
Figure 7: a) A contour \( P' \) with \( n = 3 \) and b) the deformed version of it. The branch cut on the positive real axis is suppressed for clarity.

The portion of the contour between the two blobs gives no contribution and the leg stretching out to infinity gives

\[
2(2K - 3)K_p^{\text{div}} = (se^{-i2\pi n})^{-1 + \frac{2p}{2K - 3}} \int_0^\infty dRR^{-\frac{2p}{2K - 3}} \exp \left( -h_0^K R - \sum_{r=1}^{K-2} h_r^K R \left( \frac{s e^{-i2\pi n}}{R} \right)^{\frac{2r}{2K - 3}} \right).
\]

To obtain the integral up to the desired order the second exponential factor can be Taylor expanded and then integrated term by term.

2. \( p = K - 1 \); after an integration by parts the above construction can be used and we get

\[
2(2K - 3)K_{K-1} = (2K - 3) \left( 1 - (h_0^K s e^{-i2\pi n})^{\frac{1}{2K - 3}} \Gamma \left( \frac{2K - 4}{2K - 3} \right) \right) + \ldots. \tag{125}
\]

3. \( p > K - 1 \); the leading non-analytic term is of higher order than we need to consider so

\[
2(2K - 3)K_p = \frac{2K - 3}{2p - 2K + 3}. \tag{126}
\]

Now observe that in all these integrals \( s \) always appears with a factor \( e^{-i2\pi n} \) and that there are no other phase factors. Thus all of the constraint equations \(^{119}\) can be obtained from the \( n = 0 \) case by making the replacement \( s \rightarrow se^{-i2\pi n} \). They can thus be written in the form

\[
\mathcal{F}_0(se^{-i2\pi n}) + \sum_{p=1}^{K-1} \mathcal{F}_p(se^{-i2\pi n})G_{2p,m}^C(s) = 0 \tag{127}
\]
where
\[ F_p(w) = w^{-1} \left( \sum_{q=0}^{K-1} f_{p,q} w^{2q} + f_{p,K} w + \ldots \right). \] (128)

We have truncated the expansion of \( F_p(w) \) in anticipation of the following. The equations (127) can now be written in the matrix form
\[ \Omega D f + s^{2(K-3)} f_{0,K-1} \omega + \Omega \bar{D} \bar{f} \bar{G} + s^{2(K-3)} \bar{f} \bar{G} = 0 \] (129)

where
\[ \Omega_{pq} = \exp\left(i2\pi(-K/2 + p)(-1 + 2q/(2K - 3))\right), \]
\[ D_{pq} = \delta_{pq}s^{2(p-1)/(2K-3)-1}, \]
\[ f_p = f_{0,p-1} + f_{0,K} s \delta_{p,1}, \]
\[ \omega_p = \exp\left(i2\pi(-K/2 + p)/(2K - 3)\right), \]
\[ \bar{f}_{pq} = f_{p,q-1} + f_{q,K} s \delta_{p,1}, \]
\[ \bar{f}_{pq} = \exp\left(i2\pi(-K/2 + p)/(2K - 3)\right) f_{q,K-2} \]
\[ \bar{G}_p = \bar{G}_{2p,m}. \] (130)

Now \( D \) is clearly non-singular and it is straightforward to check that \( \Omega \) is non-singular. There does not seem to be any simple way of writing the elements of \( f \) for general \( K \) but we expect it too is non-singular. Then the leading order solution for \( G \) is a constant vector; furthermore the next term in the solution is \( O(s^{2/(2K-3)}) \).

We have already shown by explicit solution that this is indeed what happens for \( K = 4 \); using Maple we have also checked it for \( K = 6, 8, 10 \). It follows that
\[ \eta = \frac{4(K-1)}{2K-3} \] (131)

for the correlation functions \( \bar{G}_{p,m}^C, p = 2, 4, \ldots 2(K-1) \). The relationship (114) between \( \bar{G}_{p,m}^C \) and \( \bar{G}_{p,m} \) is non-singular so this conclusion applies also to \( \bar{G}_{p,m}^C, p = 2, 4, \ldots 2(K-1) \). Finally we note as usual that the integrals in (113) are analytic functions of \( s \) in the neighbourhood of the origin provided \( x < 1/2 \) so the conclusion extends to all the correlation functions.

5 The \( \nu \) exponent

To find \( \nu \) we need to study the scaling behaviour as the multi-critical point is approached and to do this consider the modified couplings
\[ \tilde{g}_1^K = g_1^K, \]
\[ \tilde{g}_p^K = (1 - \Delta)g_p^K, \quad p \geq 2. \] (132)
In the graphical expansion the power of $1 - \Delta$ is the number of vertices in the graph so $\Delta$ is related to $\mu$ in (8) by

$$1 - \Delta = \exp(\mu - \mu_c). \quad (133)$$

By definition the multi-critical point is attained as $\Delta \to 0$. The disk amplitude $A(x)$ can still be calculated by exploiting the connection with topological gravity [16]; the topological potential is

$$V(z) = \Delta - Kz\Delta + z + (1 - \Delta)(1 - z)^K \quad (134)$$

and the disk amplitude

$$A(\Delta, x) = -\int_{C} \frac{dz}{2\pi i} (1 - 4zx^2)^{-\frac{1}{2}}(1 - V'(z)) \log \frac{z - V(z)}{z} \quad (135)$$

where the contour encircles the branch cut of the logarithm. The branch points are at $z = 0$ and

$$z = \frac{z_c}{4} \approx 1 - \epsilon_K \Delta^{1/K} \quad (136)$$

where $\epsilon_K$ is a constant. Collapsing the contour onto the cut gives

$$A(\Delta, x) = K \int_{0}^{z_c/4} dy (1 - 4yx^2)^{-\frac{1}{2}} \left( \Delta + (1 - \Delta)(1 - y)^{K-1} \right) \quad (137)$$

Integrating we find that

$$F(x) \equiv x \left( 2x^2 A(\Delta, x) - 1 - \sum_{k=1}^{K} \frac{2k g^K}{x^{2k-2}} \right)$$

$$= -xK\Delta(1 - z_c x^2)^{\frac{1}{2}} - x(1 - \Delta) \sum_{k=1}^{K} \frac{2k g^K}{x^{2k-2}} (\epsilon_K \Delta^{1/K})^{K-k}(1 - z_c x^2)^{k-\frac{1}{2}} \quad (138)$$

which reproduces (65) when $\Delta = 0$.

At small $x$, even for finite $\Delta$ we still get the leading behaviour

$$F(x) \sim x^{3-2K} \quad (139)$$

which implies that

$$J(x) = \int_{0}^{x} \frac{dy}{F(y)} \sim x^{2K-2}. \quad (140)$$

Thus the considerations described in detail for $K = 4$ in section 4 up to equation (93) go through as before – the only difference is that all the various coefficients...
are now functions of $\Delta$. However the contours $P_i$ are modified; when we make the change of variable (102) 

$$\xi = J(u_i)$$  \hspace{1cm} (141)$$
the $\xi = \infty$ endpoint of the contour occurs at the value of $u_i$ where $J$ diverges. The points where $J$ diverges are of course determined by the zeros of $F$. Defining $w$ through 

$$x^{-2} = z_c + \epsilon K \Delta^{1/K} w$$  \hspace{1cm} (142)$$
we get using (138)

$$J(w) = \left( \frac{\epsilon K \Delta^{1/K}}{4(1 - \Delta)} \right)^{\frac{3}{2} - K} \int_{\infty}^{w} \frac{w^{-\frac{1}{2}}(z_c + \epsilon K \Delta^{1/K} w)^{-\frac{1}{2}}}{\sum_{k=1}^{K} k g_k^K w^{k-1}} \, dw. \quad (143)$$

The term $K \Delta$ in the denominator is sub-leading; its only effect is to shift the zeros by an amount $O(\Delta^{1/K})$ and we discard it to obtain 

$$J(w) = \left( \frac{\epsilon K \Delta^{1/K}}{4(1 - \Delta)} \right)^{\frac{3}{2} - K} \int_{\infty}^{w} \frac{w^{-\frac{1}{2}}(z_c + \epsilon K \Delta^{1/K} w)^{-\frac{1}{2}}}{\sum_{k=1}^{K} k g_k^K w^{k-1}} \, dw. \quad (144)$$

Note that the denominator has precisely $K - 1$ simple zeroes; one is real and positive and the others come in complex conjugate pairs. Each zero is the end-point of one of the $K - 1$ contours, see fig.(8). Since we will only need the leading scaling behaviour

![Figure 8: The contours for $K = 4$ at finite $\Delta$.](image)

we can also approximate $(z_c + \epsilon K \Delta^{1/K} w)$ by $z_c$ in (144) so that 

$$J(w) = \left( \frac{\epsilon K \Delta^{1/K}}{4(1 - \Delta)(-K g_K^K)} \right)^{\frac{3}{2} - K} \int_{\infty}^{w} \frac{w^{-\frac{1}{2}}}{\prod_{k=1}^{K-1} (w - \overline{w}_k)} \, dw$$

$$= \left( \frac{\epsilon K \Delta^{1/K}}{4(1 - \Delta)(-K g_K^K)} \right)^{\frac{3}{2} - K} \sum_{k=1}^{K-1} R_k \int_{w}^{\infty} \frac{w^{-\frac{1}{2}}}{w - \overline{w}_k} \, dw \quad (145)$$
where $w_k$ is a zero of the denominator of (144) and $R_k$ the corresponding residue.

Now suppose that $w$ lies on the contour $P_l$ which terminates at $w_l$; when $w$ is close to the end-point we get

$$J(w) = \frac{(\epsilon_K \Delta^{1/K})^{3/2-K}}{4(1-\Delta)(-Kg_K)} \left( \frac{R_l}{\sqrt{w_l}} \log \frac{\sqrt{w} + \sqrt{w_l}}{\sqrt{w} - \sqrt{w_l}} + Q_l \right)$$

where $Q_l$ is the accumulated finite contribution from all the other poles in (145). Note that the branch cut for the square roots is taken down the negative real axis so that $\text{Re} \sqrt{w_l} > 0$ for all $l$. Inverting (146) we find that

$$w = w_l \left( \frac{T_l \exp \left( \Delta^{1-3/2K} S_l J \right) + 1}{T_l \exp \left( \Delta^{1-3/2K} S_l J \right) - 1} \right)^2$$

where $S_l$ and $T_l$ are (complex) constants of $O(1)$. It turns out that the quantity $R_l/\sqrt{w_l}$ is never pure imaginary and thus $\text{Re} S_l \neq 0$; except for the single real root, $S_l$ is not purely real either and hence the correlation functions have some oscillatory behaviour which is typical of a non-unitary theory. The formula (147) is valid for $J \to \infty$; expanding we get

$$w = w_l \left( 1 + \sum_{n=1}^{\infty} \psi_n^{(l)} \exp \left( -n \Delta^{1-3/2K} \tilde{S}_l J \right) \right)$$

where

$$\tilde{S}_l = S_l, \quad \text{if} \quad \text{Re} S_l > 0,$$

$$= -S_l, \quad \text{if} \quad \text{Re} S_l < 0.$$ (149)

Using (142) and (148) we see that the conditions (119) can be written in the form

$$0 = \frac{1}{s} (z_c + \epsilon_K \Delta^{1/K} w_l)^{m/2} + \sum_{n=1}^{\infty} \frac{A_{l}^{(n)}}{s + n \Delta^{1-3/2K} S_l}$$

$$+ 2Kg_K \left( \frac{1}{s} \left[ \frac{M^{(0)} G}{l} \right] + \sum_{n=1}^{\infty} \left[ \frac{M^{(n)} G}{l} \right] \right)$$

where $A_{l}^{(n)}$ and $M^{(n)}$ are collections of coefficients and

$$M^{(0)}_{pq} = (z_c + \epsilon_K \Delta^{1/K} w_p) K^{-1-q}. \quad \text{(151)}$$

The apparent singularity in (150) at $s = 0$ is of course not present provided $M^{(0)}$ is invertible. $M^{(0)}$ is of the Vandermonde form and therefore

$$\det M^{(0)} = (\epsilon_K \Delta^{1/K})^{(K-1)(K-2)/2} \prod_{p>q} (\overline{w}_p - \overline{w}_q). \quad \text{(152)}$$
Since none of the roots are degenerate $M^{(0)}$ is invertible. Therefore the first non-analyticity occurs at $s \sim -\Delta^{1-3/2K}$ and hence the mass gap exponent is

$$\nu = 1 - \frac{3}{2K}$$

(153)

6 Discussion

We have found that $\nu = 1 - 3/2K$ and $\eta = 2 + 2/(2K - 3)$ and therefore, since $\gamma = -1/K$, the Fisher scaling relation (13) is indeed satisfied. The results show that the functions which are initially undetermined in the peeling calculation do not in fact change the conclusions one would draw simply by ignoring their contribution in (66). The value of $\eta$ agrees with that obtained by slicing [13].

To compare $\nu$ it is necessary to examine the continuum limit for the perturbation (132) away from the multi-critical point. The multi-critical models have $K$ independent coupling constants so there are $K$ independent directions along which the multi-critical point can be approached; if chosen appropriately these directions correspond in the continuum limit to operators of (length) dimensions $\{1, 2, 3, \ldots K\}$ [12]. The procedure for taking the continuum limit is explained in [1]. By definition boundary loops with $l$ legs in the dual graph (ie which are $l$ links long) have continuum length $L = la$ where $a$ is the length of one link. This implies that the generating function variable $x$ conjugate to $l$ must be related to a continuum quantity $X$ by $x = x_c e^{-aX}$ where $x_c$ is the radius of convergence of the disk amplitude (138). We can construct a non-trivial continuum disk amplitude $A_c(\tilde{\Lambda}, X)$ provided that

$$\Delta \sim \tilde{\Lambda} a^K$$

(154)

where $\tilde{\Lambda}$ is some continuum coupling; we obtain

$$A(\Delta, x) = A_0(\Delta, x) - a^{K-1/2}A_c(\tilde{\Lambda}, X) + \ldots$$

(155)

where $A_0$ is the analytic (non-universal) part of the disk amplitude. Note that $\tilde{\Lambda}$ is not the cosmological constant; in the continuum theory the volume $V$ must have dimension (length)$^2$ and therefore the cosmological constant $\Lambda$ must correspond to whichever direction in the space of lattice couplings leads in the continuum limit to an operator of dimension 2. It is straightforward to check that this is accomplished by the modified couplings

$$\bar{g}_p^K = g_p^K - \Delta(-1)^{p-1}(K - 2)!(p - 1)!/(K - p - 2)!2p!, \quad p \leq K - 2$$

$$\bar{g}_p^K = g_p^K, \quad p = K, K - 1.$$ 

(156)
The weight for polygons of $2p$ sides is modified by a $p$-dependent factor and so the lattice quantity whose continuum limit is the volume is a complicated object with polygons of different number of sides weighted in different ways; the volume is not the number of polygons.

Using the scaling (154) for $\Delta$ in the large $t$ behaviour of the two-point function we find that
\[ \exp(-\Delta^\nu t) = \exp(-\tilde{\Lambda}^\nu (ta^{K\nu})) = \exp(-\tilde{\Lambda}^\nu T) \] (157)
and hence we will get a consistent non-trivial scaling limit provided the continuum string time scales as $T \sim ta^{K\nu}$. This also agrees with [13] (where what we call the string time is called the geodesic distance).

The structure of these multi-critical surfaces seems slightly bizarre. Recall that the coupling $\Delta$ is conjugate to the number of polygons which therefore behaves roughly like $\Delta^{-1}$ for typical surfaces in the ensemble. On the other hand from (157) we have that the typical $t$ must be roughly $\Delta^{-\nu}$ and therefore
\[ \langle \#\text{polygons} \rangle \sim \langle t \rangle^{\frac{1}{\nu}} \] (158)
When $K$ gets large $\nu \to 1$ and so these surfaces have an almost linear structure when viewed in terms of polygons. However, as we discussed in section 1, the polygons can be very large so that, for example, the number of links traversed in a typical cycle can also be very large and it does not follow that the surfaces in the continuum limit are at all one-dimensional. It does however cast doubt on the idea that it makes sense to identify the string time $T$ with a continuum geodesic distance.

Another manifestation of this can be seen by considering a perturbation in terms of the cosmological constant $\Lambda$. Then we expect that the two point function behaves as
\[ \exp(-\Lambda^{\frac{K\nu}{2}} T) \] (159)
(it is a straightforward calculation along the lines of section 5 but using the couplings (156) to check this). The average volume of the system will be $\langle V \rangle \sim \Lambda^{-1}$; on the other hand from (159) the string time extent must be of order $T_s \sim \Lambda^{-\frac{K\nu}{2}}$ and hence $\langle V \rangle \sim T_s^{\frac{2}{K\nu}}$. Interpreting $T$ as a geodesic distance leads to the conclusion that the Hausdorff dimension is $d_H = \frac{2}{K\nu}$ which vanishes as $K \to \infty$ and is less than two for all $K > 2$. However we can look at this another way; in [13] the probability distribution for the length $L$ of exit loops at time $T$ given a point-like entrance loop at $T = 0$ was calculated. At large but finite $K$ the distribution shows that there

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¹One might worry that the negative weights appearing in the multi-critical models cause some cancellation which alters this conclusion. We have checked explicitly that for $K = 4$ at least this does not happen.
is typically one macroscopic exit loop of length $L \sim T^{\frac{1}{K\nu}}$ but that as $K \to \infty$ the distribution function becomes a delta-function. Using this result we can relate the volume to the typical boundary length as $K \to \infty$ and find

$$V \sim L^2$$

which is typical of a smooth flat surface of Hausdorff dimension 2. This is the behaviour we expect because $K \to \infty$ corresponds to central charge $c \to -\infty$ where semi-classical properties are recovered; the volume and loop length have highly anomalous behaviour relative to the string time but exactly what we expect relative to each other. Note that our discussion above implies that the macroscopically large boundary loops are often made from a finite number of polygons with diverging number of sides.

The multi-critical models are non-unitary and have negative central charge, and so it is plausible that in the continuum theory the Hausdorff dimension is given by (7) which works well for the $c = -2$ model. As $c \to -\infty$ (7) gives $d_H = 2$ and so it is clear that the geodesic distance implicit in the derivation of this formula (which is done directly in the continuum using scaling arguments and Liouville theory) cannot be equivalent to the continuum limit of the string time, $T$. Since we do not know what the relationship between these distance measures really is, it is not clear what calculation in the discretized formulation would be needed to check (7) for general $K$; in any case it must be highly non-trivial because (7) gives irrational values for $d_H$ when $K > 2$ whereas discretized calculations are almost sure to give rational values for exponents.

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