BEHAVIOR OF GAUSSIAN CURVATURE
AROUND NON-DEGENERATE SINGULAR POINTS
ON WAVE FRONTS

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Abstract. We define cuspidal curvature $\kappa_c$ along cuspidal edges in Riemannian 3-manifolds, and show that it gives a coefficient of the divergent term of the mean curvature function. Moreover, we show that the product $\kappa_H$ (called product curvature) of $\kappa_c$ and the limiting normal curvature $\kappa_\nu$ is an intrinsic invariant of the surface and is closely related to the boundedness of the Gaussian curvature. We also consider the limiting behavior of $\kappa_H$ when cuspidal edges accumulate to other singularities. As an application, several new geometric properties of limiting normal curvature $\kappa_\nu$ are shown.

Introduction

In [11], the behavior of Gaussian curvature $K$ along cuspidal edge singularities in Riemannian 3-manifolds was discussed. However, the existence of intrinsic invariants related to the boundedness of $K$ was not mentioned there. The purpose of this paper is to show the existence of such invariants for non-degenerate singular points on wave fronts. In fact, geometric invariants of cross cap singularities on surfaces are recently discussed by several geometers ([12, 13, 14, 15, 16, 17]). Also, Nuño-Ballesteros and the first author [8] investigate geometric properties of co-rank one singularities other than cross caps. After that, in a joint work [9] of the first two authors, a West-type normal form for cuspidal edges is given and geometric meanings of its coefficients are discussed. In this paper, we define limiting normal curvature $\kappa_\nu$, for an arbitrarily given co-rank one singular point on wave fronts, which is a generalization of $\kappa_\nu$ for cuspidal edges defined in [11]. (The definition of wave fronts are given in the first section.) Our first result is as follows:

**Theorem A.** Let $f : \mathcal{U} \to \mathbb{R}^3$ be a front, and $p \in \mathcal{U}$ a co-rank one singular point, where $\mathcal{U}$ is a domain in $\mathbb{R}^2$. Then the Gauss map $\nu : \mathcal{U} \to S^2$ of $f$ has a singularity at $p$ if and only if the limiting normal curvature $\kappa_\nu(p)$ is equal to zero, where $S^2$ denotes the unit sphere.

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The proof is given in the first section. We next define cuspidal curvature function $\kappa_c$ along cuspidal edges in Section 2 and show that it appears in the first coefficient of the divergent term of the mean curvature function. Then, we consider their product called ‘product curvature’

$$\kappa_{\Pi} := \kappa_{\nu}\kappa_c,$$

along cuspidal edges, and show that $\kappa_{\Pi}$ is an intrinsic invariant, i.e. determined only on the first fundamental form of the surface. Using $\kappa_{\Pi}$, we give a necessary and sufficient condition for the boundedness of the Gaussian curvature around cuspidal edges. We then discuss on the limiting behavior of $\kappa_{\Pi}$ when cuspidal edges accumulate to other singularities in Section 3. As a consequence, we get the following property of the limiting normal curvature:

**Theorem B.** Let $f : U \rightarrow (M^3, g)$ be a front, and $p \in U$ a non-degenerate singular point, where $U$ is a domain in $\mathbb{R}^2$ and $(M^3, g)$ is a Riemannian 3-manifold. Then the Gaussian curvature of $f$ is rationally bounded at $p$ if and only if the limiting normal curvature $\kappa_{\nu}(p)$ is equal to zero.

The definition of rational boundedness is given in Definition 2.3.

This assertion is a consequence of Corollary 2.9 and Theorem 3.2. The above two theorems yield the following assertion, which summarize the geometric properties of the limiting normal curvature:

**Corollary C.** Let $f : U \rightarrow \mathbb{R}^3$ be a front, and $p \in U$ a non-degenerate singular point, where $U$ is a domain in $\mathbb{R}^2$. Then the following three properties are equivalent:

1. The Gaussian curvature of $f$ is rationally bounded at $p$.
2. The limiting normal curvature at $p$ is equal to zero.
3. A singular point $p$ of $f$ is also a singular point of the Gauss map of $f$.

In [14, Lemma 3.25], the authors showed that the singular set of Gauss map coincides with the singular set of $f$ if $\log |K|$ is bounded. The equivalency of (1) and (3) is a refinement of it.

1. LIMITING NORMAL CURVATURE

Let $\Sigma^2$ be an oriented 2-manifold and $f : \Sigma^2 \rightarrow (M^3, g)$ a $C^\infty$-map into an oriented Riemannian 3-manifold $(M^3, g)$. A singular point of $f$ is a point at which $f$ is not an immersion. The map $f$ is called a frontal if for each $p \in \Sigma^2$, there exist a neighborhood $U$ of $p$ and a unit vector field $\nu$ along $f$ defined on $U$ such that $\nu$ is perpendicular to $df(a)$ for all tangent vectors $a \in TU$. Moreover, if $\nu : U \rightarrow TM^3$ gives an immersion, $f$ is called a (wave) front. We fix a frontal $f : \Sigma^2 \rightarrow M^3$.

**Definition 1.1.** A singular point $p \in \Sigma^2$ of the frontal $f$ is called non-degenerate if the exterior derivative of the function

$$\lambda := \det_g(f_u, f_v, \nu), \quad (f_u := df(\partial_u), \quad f_v := df(\partial_v))$$

does not vanish at $p$, where $(u, v)$ is a local coordinate system of $\Sigma^2$ at $p$, $\partial_u = \partial/\partial u$, $\partial_v = \partial/\partial v$, and where $\det_g$ is the Riemannian volume element of $(M^3, g)$. If the ambient space is the Euclidean 3-space $E^3$, then $\det_g$ can be identified with the usual determinant.
A non-degenerate singular point \( p \) of \( f \) is a co-rank one singular point, i.e., the kernel of \( df(p) \) is of one dimensional. Since \( \{ \lambda = 0 \} \) is the singular set, by the implicit function theorem, we can take a smooth curve \( \gamma(t) (|t| < \varepsilon) \) on \( \mathcal{U} \) as a parametrization of the singular set, where \( \varepsilon > 0 \). We may assume that \( \gamma(0) = p \). (We call \( \gamma \) the singular curve.) There exists a non-vanishing vector field \( \eta(t) \) along \( \gamma \) such that \( df(\gamma(t)) \) vanishes identically. We call \( \eta(t) \) a null vector field along \( \gamma \).

A non-degenerate singular point \( p \) is said to be of the first kind if \( \eta(0) \) is not proportional to \( \gamma'(0) := d\gamma/dt|_{t=0} \). Otherwise, it is said to be of the second kind.

**Definition 1.2.** A singular point \( p \in \Sigma^2 \) of a map \( f : \Sigma^2 \to M^3 \) is a cuspidal edge if the map germ \( f \) at \( p \) is right-left equivalent to \( (u,v) \mapsto (u,v^2,v^3) \) at the origin. A singular point \( p \) of \( f \) is a swallowtail (respectively, cuspidal cross cap) if \( f \) at \( p \) is right-left equivalent to \( (u,v) \mapsto (u,4v^3 + 2uv, 3v^4 + uv^2) \) (respectively, \( (u,v) \mapsto (u,v^2,uv^3) \)) at the origin. Here, \( f \) is considered as a map germ \( f : (\mathbb{R}^2,0) \to (\mathbb{R}^3,0) \) by taking local coordinates systems of \( \Sigma^2 \) and \( M^3 \) at \( p \) and \( f(p) \), respectively, and two map germs \( f_1 \) and \( f_2 \) are right-left equivalent if there exist diffeomorphism germs \( d_s : (\mathbb{R}^2,0) \to (\mathbb{R}^2,0) \) and \( d_t : (\mathbb{R}^3,0) \to (\mathbb{R}^3,0) \) such that \( d_t \circ f_1 = f_2 \circ d_s \) holds.

Figures of these singularities are drawn in Figure 1. There are criteria for these singularities.

**Figure 1.** A cuspidal edge, a swallowtail and a cuspidal cross cap.

**Fact 1.3** ([4] Proposition 1.3], [5] Corollary 1.5] see also [12 Corollary 2.5]). Let \( f : \Sigma^2 \to (M^3,g) \) be a frontal and \( p \) a non-degenerate singular point. Take the singular curve \( \gamma(t) \) such that \( \gamma(0) = p \) and a null vector field \( \eta(t) \) along \( \gamma \). Then

1. \( f \) at \( p \) is a cuspidal edge if and only if \( f \) is a front, and \( \eta \) and \( \gamma' \) are linearly independent at \( p \).
2. \( f \) at \( p \) is a swallowtail if and only if \( f \) is a front, \( \eta \) and \( \gamma' \) are linearly dependent but \( (d/dt)\det(\gamma'(t),\eta(t))|_{t=0} \neq 0 \) holds, where \( \det \) denotes an area element of \( \Sigma^2 \), and
3. \( f \) at \( p \) is a cuspidal cross cap if and only if \( \eta \) and \( \gamma' \) are linearly independent at \( p \), \( \psi_{ccc}(0) = 0 \) and \( \psi'_{ccc}(0) \neq 0 \), where

\[
\psi_{ccc}(t) = \det_g \left( \hat{\gamma}'(t), \nu(\gamma(t)), (\nabla_{\nu'})(\gamma(t)) \right), \quad \hat{\gamma}(t) := f(\gamma(t)),
\]

\( \nu \) is the unit normal vector field, and \( \nabla \) is the Levi-Civita connection of \( (M^3,g) \).
About $\psi_{crr}$, the following lemma holds.

**Lemma 1.4** ([5, Corollary 1.7]). Let $f : \Sigma^2 \to M^3$ be a frontal and $p$ a singular point of the first kind. Then $f$ is a front at $p$ if and only if $\psi_{crr} \neq 0$ at $p$.

**Proof.** Since $\eta$ is a null vector and $p$ is a singularity of co-rank one of $f$, then $f$ is a front at $p$ if and only if $\nabla_\eta \nu(p) \neq 0$. Thus the necessity is obvious. Let us assume that $\psi_{crr} = 0$ at $p$. Since $p$ is a singular point of first kind, $\gamma'(t) \neq 0$ at $p$ holds. Thus $\nabla_\eta \nu$ is a linear combination of $\hat{\gamma}'(t)$ and $\nu$ at $p$. On the other hand, noticing that $\nabla$ is the Levi-Civita connection, it holds that

$$\langle \nabla_\eta \nu, f(\gamma(t))' \rangle = \langle \nu(\gamma(t))', (\nabla_\eta f)(\gamma(t)) \rangle = 0$$

and $\langle \nabla_\eta \nu, \nu \rangle = 0$ at $p$, where $\langle , \rangle$ is the inner product corresponding to the Riemannian metric $g$. Hence $\nabla_\eta \nu(p) = 0$ holds. □

By Fact 1.3, if $f$ is a front, then a non-degenerate singular point of the first kind gives a cuspidal edge. On the other hand, cuspidal cross caps are typical examples of non-degenerate singular points of the first kind when $f$ is a frontal but not a front. Swallowtails are singularities of the second kind. When $f$ is a front, a non-degenerate peak in the sense of [11, Definition 1.10] is a singular point of the second kind.

**Example 1.5.** Let $f : \mathbb{R}^2 \to \mathbb{R}^3$ be a map defined by

$$f(u, v) = (5u^4 + 2uv, v, 4u^5 + u^2v - v^2).$$

Then the singular set is $\{10u^3 + v = 0\}$, the null vector field is $\eta = \partial_u = \partial / \partial u$ and $d\nu(\partial_u) = (-1, 0, 0)$ holds, where $\nu$ is the unit normal vector field. Hence $f$ is a front and $0$ is a singular point of the second kind (namely, a non-degenerate peak) but not a swallowtail (see the left hand side of Figure 2).

On the other hand, let $f : \mathbb{R}^2 \to \mathbb{R}^3$ be a map defined by

$$f(u, v) = (u^2 + 2v, uv + 3uv, u^5 + 5u^3v).$$

Then the singular set is $\{v = 0\}$, the null vector field is $\eta = \partial_u - u\partial_v$, and $d\nu(0, 0) = 0$ hold. Hence $f$ is a frontal but not a front at $0$, and $0$ is a singular point of the second kind (see the right hand side of Figure 2).

![Figure 2. Singularities of the second kind](image-url)
Definition 1.6. Let \( p \) be a co-rank one singular point of a frontal \( f \). A local coordinate system \((U; u, v)\) centered at \( p \) is called admissible if \( f_u(p) = 0 \) and it is compatible with respect to the orientation of \( \Sigma^2 \).

We denote by \('\langle \cdot, \cdot \rangle'\) the inner product on \( M^3 \) induced by the metric \( g \), and \(|a| := \sqrt{\langle a, a \rangle} \) \((a \in TM^3)\). We fix a unit normal vector field \( \nu \) of \( f \). Take an admissible coordinate system at a co-rank one singular point \( p \). Then we define

\[
\kappa_\nu(p) := \frac{\langle f_{uu}(p), \nu(p) \rangle}{|f_u(p)|^2},
\]

which is called the limiting normal curvature. Here, we denote \( f_{uu} := \nabla_{\partial u}f_u \). By definition, \( \kappa_\nu(p) \) depends on the \( \pm \)-ambiguity of \( \nu \).

Proposition 1.7 (The continuity of the limiting normal curvature). Let \( p \) be a co-rank one singular point of \( f \). The definition of \( \kappa_\nu \) does not depend on a choice of the admissible coordinate system. Moreover, if \( p \) is non-degenerate and \( \gamma(t) \) is a singular curve such that \( \gamma(0) = p \), and if \( \gamma(t) \) \((t \neq 0)\) consists of singular points of the first kind, then it holds that

\[
\kappa_\nu(p) = \left( \lim_{t \to 0} \kappa_\nu(\gamma(t)) \right) = \lim_{t \to 0} \frac{\langle \hat{\gamma}''(t), \nu(\gamma(t)) \rangle}{|\hat{\gamma}'(t)|^2},
\]

where \( \hat{\gamma}(t) := f(\gamma(t)), \hat{\gamma}''(t) := \nabla_{\hat{\gamma}'(t)}\hat{\gamma}'(t) \).

Proof. Let \((U, V)\) be another admissible coordinate system. Then \( U_u(p) \neq 0 \) holds. Moreover,

\[
\langle f_{uu}, \nu \rangle = -\langle f_u, \nu_u \rangle = -U_u \langle f_U, U_u \nu_U + V_u \nu_V \rangle = -U^2 \langle f_U, \nu_U \rangle - U_u V_u \langle f_U, \nu_V \rangle
\]

holds at \( p \), where \( \nu_u = \nabla_{\partial_u} \nu \). Since

\[
\langle f_U, \nu_V \rangle = -\langle f_U V, \nu \rangle = -\langle f_{UV}, \nu \rangle = \langle f_V, \nu_U \rangle = 0,
\]

we have that

\[
\frac{\langle f_{uu}, \nu \rangle}{|f_u|^2} = -\frac{U^2}{U_u^2} \frac{\langle f_U, \nu_U \rangle}{|f_U|^2} = \frac{\langle f_{UV}, \nu \rangle}{|f_U|^2}
\]

at \( p \), which proves the first assertion. Take a coordinate system \((x, y)\) at \( p \) so that \( f_y(\gamma(t)) = 0 \) and \( |f_x(\gamma(t))| = 1 \) for all \( t \) \((\partial_x := \partial/\partial x \) does not necessarily point to the singular direction\). Then \((1.4)\) implies the continuity of the limiting normal curvature, that is, it holds that

\[
\lim_{t \to 0} \kappa_\nu(\gamma(t)) = \kappa_\nu(\gamma(0)).
\]

Now fix \( t \neq 0 \). Since \( \gamma(t) \) is a singular point of the first kind, we can take another coordinate system \((u, v)\) centered at \( \gamma(t) \) so that \( u \)-curve is a singular curve and \( \partial_v := \partial/\partial v \) is the null direction. Since \( f(u, 0) = \hat{\gamma}(u) \), we have that

\[
\kappa_\nu(\gamma(t)) = \frac{\langle \hat{\gamma}''(t), \nu(\gamma(t)) \rangle}{|\hat{\gamma}'(t)|^2}
\]

for \( t \neq 0 \). Then \((1.2)\) follows from \((1.4)\) and \((1.5)\). □
Remark 1.8. When \( p \) is a cuspidal edge, the limiting normal curvature is initially defined in [11]. Our definition is a generalization of it for non-degenerate singular points, which can be checked by comparing [12] and the original definition in [11]. Recently, Nuño-Ballesteros and the first author [8] defined a notion of umbilic curvature \( \kappa_u \) for corank one singular points in the Euclidean 3-space \( E^3 \). We remark that the unsigned limiting normal curvature \( \kappa_u := |\kappa_v| \) coincides with \( \kappa_u \) for cuspidal edges, see [9] for details.

Example 1.9. Consider a swallowtail

\[
f(u, v) = \left( u^4 - 4u^2v, u^3 - 3uv, \frac{v^2}{2} - v \right) + (u^2 - 2v)^2(a, b, 0) \quad (a, b \in \mathbb{R})
\]

in \( E^3 \). Then the singular set is \( \{ v = 0 \} \) and we can take a unit normal vector

\[
\nu(u, v) = \frac{\left( 3, -8u, -4(3(-1 + a)u^2 - 8bu^3 - 6av + 16buv) \right)}{\sqrt{9 + 64u^2 + 16(3(-1 + a)u^2 - 8bu^3 - 6av + 16buv)^2}}.
\]

Since \( f_u(0, 0) = 0 \) holds by (1.1), the limiting normal curvature at \( (0, 0) \) is

\[
\frac{\langle f_{uu}(0, 0), \nu(0, 0) \rangle}{\langle f_u(0, 0), f_v(0, 0) \rangle} = 8a.
\]

On the other hand, the limiting normal curvature on \( \tilde{\gamma}(u) = f(u, 0), u \neq 0 \) is

\[
\frac{\langle \tilde{\gamma}''(u), \nu(u, 0) \rangle}{|\tilde{\gamma}'(u)|^2} = 8a - \frac{64b}{3}u + O(u^2),
\]

where \( O(u^2) \) is the higher order term with respect to \( u^2 \).

Example 1.10. Consider a cone \( f(u, v) := (v \cos u, v \sin u, v^2 + v) \). Then the set of singular points is \( \{ v = 0 \} \), and then the limiting normal curvature at \( (u, 0) \) for a normal vector

\[
- \frac{(1 + 2v) \cos u, (1 + 2v) \sin u, -1)}{\sqrt{(1 + 2v)^2 + 1}}
\]

is \( 1/\sqrt{2} \).

We now prove Theorem A in the introduction.

(Proof of Theorem A) Let \( \nu \) be the Gauss map of \( f \). Take an admissible coordinate system \((u, v)\) at a given co-rank one singular point \( p \). By (1.3), \( f_u(p) \) is perpendicular to \( \nu_u(p) \). Since \( p \) is of co-rank one, \( f_u(p) \neq 0 \). \( p \) is a singular point of \( \nu \) if and only if \( \nu_u(p) \) is proportional to \( \nu_v(p) \). Since \( f \) is a front, \( \nu_v(p) \neq 0 \). Thus, \( p \) is a singular point of \( \nu \) if and only if \( f_u(p) \) is perpendicular to \( \nu_v(p) \), namely \( \kappa_v(p) \) is equal to zero.

We denote by \( S^3 \) the unit 3-sphere. For a front \( f : \Sigma^2 \to S^3 \), its unit normal vector field can be considered as a map \( \nu : \Sigma^2 \to S^3 \). We call this the Gauss map of \( f \). Using a modification of the above proof of Theorem A, we get the following:

Proposition 1.11. Let \( f : \Sigma^2 \to S^3 \) be a front, and \( p \in \Sigma^2 \) a co-rank one singular point. Then the Gauss map \( \nu : \Sigma^2 \to S^3 \) of \( f \) has a singularity at \( p \) if and only if the limiting normal curvature \( \kappa_v(p) \) is equal to zero.
The hyperbolic space
\[ H^3 := \{(t, x, y, z) \in \mathbb{R}^4_1 : t^2 - x^2 - y^2 - z^2 = 1, \ t > 0\} \]
of constant curvature $-1$ can be considered as a hyperboloid in the Lorentz-Minkowski 4-space $\mathbb{R}^4_1$. For fronts $f : \Sigma^2 \to H^3$, its unit normal vector fields can be considered as the Gauss map $\nu : \Sigma^2 \to S^3_1$, where
\[ S^3_1 := \{(t, x, y, z) \in \mathbb{R}^4_1 : t^2 - x^2 - y^2 - z^2 = -1, \} \]
is the de Sitter space. We also get the following:

**Proposition 1.12.** Let $f : \Sigma^2 \to H^3$ be a front, and $p \in \Sigma^2$ a co-rank one singular point. Then its Gauss map $\nu : \Sigma^2 \to S^3_1$ of $f$ has a singularity at $p$ if and only if the limiting normal curvature $\kappa_\nu(p)$ is equal to zero.

We now return to the general situation. Let $f : \Sigma^2 \to (M^3, g)$ be a frontal. Suppose that $p$ is of the first kind. Let $\kappa(t) = |\gamma''(t)|$ be the curvature function of $\gamma := f \circ \gamma$ in $M^3$, where $t$ is the arclength parameter of $\gamma$. Then $\kappa_\nu = \langle \gamma''(t), \nu \rangle$ gives the normal part of $\gamma''(t)$. On the other hand, we set
\[
\kappa_s := \text{sgn}(d\lambda(\eta)) \det_g \left( \langle \gamma'(t), \gamma''(t), \nu(\gamma(t)) \rangle \right)
\]
which is called the singular curvature, where $\eta(t)$ is a null vector field along $\gamma$ such that $\{\gamma', \eta\}$ is compatible to the orientation of $\Sigma^2$, and $\text{sgn}(d\lambda(\eta))$ is the sign of the function $d\lambda(\eta)$ at $p$. Since $\kappa_s$ gives the tangential component of $\gamma''(s)$, it holds that
\[
\kappa'' = \kappa_s^2 + \kappa_{\nu'}^2.
\]
When $p$ is a cuspidal edge, the singular curvature is defined in [11]. In this case, it was shown in [11] that $\kappa_s$ depends only on the first fundamental form (i.e. it is intrinsic) and we can prove the following assertion as an application of [11].

**Fact 1.13.** Let $p$ be a non-degenerate singular point of the first kind of a frontal $f$. If the Gaussian curvature function $K$ is bounded near $p$, then the limiting normal curvature $\kappa_\nu$ vanishes at $p$.

**Proof.** In the first paragraph of [11] Theorem 3.1, it was stated that the second fundamental form of a frontal $f$ vanishes at non-degenerate singular points if $K$ is bounded. In fact, the proof there needed only that $f$ is a frontal. By [11], $\kappa_\nu$ is a coefficient of the second fundamental form, and must vanish.

Fact 1.13 strongly suggests the existence of an intrinsic invariant related to the behavior of the Gaussian curvature, as a product of terms that are each individually not intrinsic. This is our motivation for introducing the ‘product curvature’.

For a non-degenerate singular point $p$ of the first kind, the derivatives of the limiting normal curvature and the singular curvature with respect to the arclength parameter $t$ of $\gamma(t) = f(\gamma(t))$
\[
\kappa_\nu'(p) := \left. \frac{d\kappa_\nu(t)}{dt} \right|_{t=0}, \quad \kappa_s'(p) := \left. \frac{d\kappa_s(t)}{dt} \right|_{t=0},
\]
where $\gamma(t)$ is the singular curve such that $p = \gamma(0)$, are called the derivate limiting normal curvature and the derivate singular curvature, respectively, which will be useful in the following sections,
2. Cuspidal curvature

Let \( \sigma(t) \) be a curve in the Euclidean plane \( \mathbb{E}^2 \) and suppose that \( t = 0 \) is a 3/2-cusp. Then the cuspidal curvature of the 3/2-cusp is given by (cf. [13] and [15])

\[
(2.1) \quad \tau := \frac{\text{det}(\sigma''(0), \sigma'''(0))}{|\sigma''(0)|^{3/2}}.
\]

In [15], the following formula was shown

\[
(2.2) \quad \tau = 2\sqrt{2} \lim_{t \to 0} \sqrt{|s(t)|} \kappa(t), \quad s(t) := \int_0^t |\sigma'(t)|dt,
\]

where \( \kappa(t) \) is the curvature function of \( \sigma(t) \).

Let \( p \) be a non-degenerate singular point of the first kind of a frontal \( f \), and \( \gamma(t) \) a singular curve such that \( \gamma(0) = p \). Let \( \eta \) be a non-vanishing vector field defined on a neighborhood of \( p \) such that \( \eta \) points to the null direction along \( \gamma \), which is called an extended null vector field. We set

\[
(2.3) \quad f_\eta := df(\eta), \quad f_{\eta\eta} := \nabla_\eta f_\eta, \quad f_{\eta\eta\eta} := \nabla_\eta f_{\eta\eta}.
\]

By definition, \( f_\eta \) vanishes along \( \gamma \). Let \((u,v)\) be an admissible coordinate system at \( p \). Since \( \partial_u \) and \( \eta \) are linearly independent at \( p \), we may consider \( \lambda \) in Definition 1.1 as \( \lambda = \text{det}_g(f_u, f_\eta, \nu) \). Since \( \lambda = 0 \) along \( \gamma \), the non-degeneracy of \( p \) yields that \( (2.3) \) \( 0 \neq \lambda_\eta = \text{det}_g(f_u, f_{\eta\eta}, \nu) \), namely, \( f_u \) and \( f_{\eta\eta} \) are linearly independent at the non-degenerate singular point. Define the exterior product \( \times_g \) so that

\[
(a \times_g b, c) = \text{det}_g(a, b, c)
\]

holds for each \( a, b, c \in T_p M^3 (q \in M^3) \). Since \( \hat{\gamma}'(t) \) is proportional to \( f_u \), \( \hat{\gamma}'(t) \times_g f_{\eta\eta}(\gamma(t)) \) does not vanish. Then we define the cuspidal curvature for singular points of the first kind as

\[
(2.4) \quad \kappa_\epsilon(t) := \frac{|\hat{\gamma}'(t)|^{3/2} \text{det}_g(\hat{\gamma}'(t), f_{\eta\eta}(\gamma(t)), f_{\eta\eta\eta}(\gamma(t)))}{|\hat{\gamma}'(t) \times_g f_{\eta\eta\eta}(\gamma(t))|^{5/2}},
\]

where \( \hat{\gamma} = f(\gamma(t)) \) and \( \eta \) is chosen so that \( \{\partial_u, \eta\} \) consists of a positively oriented frame on \( \Sigma^2 \). If \( t \) is the arclength parameter of \( \hat{\gamma} \), then

\[
\kappa_\epsilon'(p) := \frac{d\kappa_\epsilon(t)}{dt} \bigg|_{t=0}
\]

is called the derivate cuspidal curvature.

**Proposition 2.1.** Let \( f \) be a front in \( \mathbb{E}^3 \) and \( p \) a cuspidal edge, and let \( \gamma(t) \) be the singular curve such that \( \gamma(0) = p \). Then the intersection of the image of \( f \) and a plane \( P \) passing through \( f(p) \) perpendicular to \( \hat{\gamma}'(0) \) consists of a 3/2-cusp \( \sigma \) on \( P \). Then \( \kappa_\epsilon(0) \) coincides with the cuspidal curvature of \( \sigma \) at \( p \).

**Proof.** Without loss of generality we may assume that \( p = (0,0) \) and \( f(p) \) is the origin. We denote by \( \Gamma \) the intersection of the image of \( f \) and the plane \( P \) as in the statement. Let \((u,v)\) be an admissible coordinate system. Then we have that

\[
\Gamma = \left\{ f(u,v) : \langle f(u,v), f_u(0,0) \rangle = 0 \right\}.
\]
Since \( f_u(0, 0) \neq 0 \), by applying the implicit function theorem for \( \langle f(u, v), f_u(0, 0) \rangle = 0 \), there exists a \( C^\infty \)-function \( u = u(v) \) such that
\[
\langle f(u(v), v), f_u(0, 0) \rangle = 0 \quad \text{and} \quad u(0) = 0,
\]
and
\[
\sigma(t) := f(u(t), t)
\]
gives a parametrization of the set \( \Gamma \). Differentiating \((2.5)\), we have that
\[
\langle u'(v) f_u(u(v), v) + f_v(u(v), v), f_u(0, 0) \rangle = 0,
\]
where \( u' := du/dv \). Since \( f_v(0, 0) = 0 \), we have that
\[
u'(0) = 0.
\]
Differentiating \((2.7)\) again, one can get
\[
u''(0) = -\frac{\langle f_u(0, 0), f_v(0, 0) \rangle}{\langle f_u(0, 0), f_u(0, 0) \rangle}.
\]
On the other hand, differentiating \((2.6)\), \((2.8)\) yields that
\[
\sigma'(0) = 0,
\]
\[
\sigma''(0) = f_v(0, 0) + f_u(0, 0)u''(0),
\]
\[
\sigma'''(0) = f_vv(0, 0) + 3f_u(0, 0)u''(0) + f_u(0, 0)u'''(0),
\]
which imply that
\[
\det(f_u(p), \sigma''(0), \sigma'''(0)) = \det(f_u(p), f_vv(p), f_u(0, 0) + 3f_u(0, 0)u''(0))
\]
\[
= \det(f_u(p), f_vv(p), f_u(0, 0) + 3u''(0)\det(f_u(p), f_v(p), f_u(p))).
\]
Since \( p \) is not a cross cap, \( \det(f_u(p), f_vv(p), f_u(p)) \) vanishes by the well-known Whitney’s criterion of cross caps [18 Page 161 (b)]. Thus, we get the identity
\[
\det(f_u(p), \sigma''(0), \sigma'''(0)) = \det(f_u(p), f_vv(p), f_u(p)).
\]
By \((2.10)\) and \((2.9)\), we have that
\[
|\sigma''(0)|^2 = \langle f_vv(p), f_u(p) \rangle - \frac{\langle f_vv(p), f_u(p) \rangle^2}{\langle f_u(p), f_u(p) \rangle} = \frac{|f_u(p) \times f_vv(p)|^2}{|f_u(p)|^2}.
\]
By \((2.11)\), \((2.4)\), \((2.11)\) and \((2.12)\), we get the assertion. \( \square \)

**Remark 2.2.** In [9], a normal form of a cuspidal edge singular point in \( E^3 \) was given, and its proof can be applied to a given non-degenerate singular point \( p \) of the first kind without any modification. So there exists a local coordinate system \( (u, v) \) at \( p \) such that
\[
f(u, v) = \left( u, \frac{a(u)u^2 + v^2}{2}, \frac{b_0(u)u^2 + b_2(u)uv^2}{2} + \frac{b_3(u, v)v^3}{6} \right),
\]
where \( a, b_0, b_2 \) and \( b_3 \) are \( C^\infty \)-functions. It holds that
\[
\kappa_u(p) = a(0), \quad \kappa_v(p) = b_0(0), \quad \kappa_v(p) = b_3(0, 0).
\]
Moreover, we have that
\[
\kappa'_u(p) = b_0(0)b_2(0) + 3a'(0),
\]
\[
\kappa'_v(p) = -a(0)b_2(0) + 3b'_0(0), \quad \kappa'_v(p) = (b_3)_u(0, 0).
\]
In particular, since $\kappa_s$ is intrinsic, so is $\kappa'_s$, which gives a geometric meaning for the coefficient $a'(0)$. On the other hand,

$$k_t(p) := b_2(0), \quad k_i(p) := 3b'_0(0),$$
called the \textit{cusp-directional torsion} and the \textit{edge inflectional curvature}, are investigated in [9].

\textbf{Definition 2.3.} Let $p$ be a point of $\Sigma^2$, and $(\mathcal{U}; u, v)$ a coordinate neighborhood centered at $p$. We set $u = r \cos \theta$ and $v = r \sin \theta$, that is, $(r, \theta)$ is the polar coordinate system. A continuous function $\varphi$ defined on an open dense subset $O$ of $\mathcal{U}$ is called \textit{rationally bounded} at $p$ if there exists a $2\pi$-periodic continuous function $\lambda = \lambda(\theta)$ such that

- the zero set of $\lambda$ has no interior points,
- there exists $\varepsilon > 0$ such that the function $\lambda \varphi$ is a restriction of a continuous function $\psi$ defined on $\{(r, \theta); 0 \leq r < \varepsilon, 0 \leq \theta \leq 2\pi\}$.

Moreover, if there exists a constant $c$ such that $\psi(0, \theta) = c\lambda(\theta)$ for $\theta \in [0, 2\pi]$, then the function $\varphi$ is called \textit{rationally continuous} at $p$.

By definition, continuity implies rational continuity, and rational continuity implies rational boundedness. For example, for $a \in \mathbb{R} \setminus \{0\}$ and $k \geq 2$, we set

$$f(u, v) := (u, v^2, v^3 + au^k).$$

Then its Gaussian curvature function is given by

$$K(u, v) = 3ak(k - 1)\frac{u^{k-2}}{v} + O(1).$$

Hence $K$ is rationally bounded (resp. rationally continuous) if $k \geq 3$ (resp. $k \geq 4$).

\textbf{Remark 2.4.} The above definitions of rational boundedness and continuity do not depend on the choice of local coordinate systems. In fact, it is a special case of the following definition: Let $p$ be a point of $n$-manifold $M$ and $\pi : \hat{M}_p \to M$ the blowing up of $M$ at $p$. Let $S$ be a subset of $M$. A function $\varphi : S \to \mathbb{R}$ is called \textit{rationally bounded} if there exist a neighborhood $\mathcal{U}$ of $p$ and a continuous function $\lambda : \pi^{-1}(\mathcal{U}) \to \mathbb{R}$ such that

- $\mathcal{U} \cap S$ is an open dense subset of $\mathcal{U}$,
- the zero set of the restriction of $\lambda$ into $\pi^{-1}(p)$ has no interior points in $\pi^{-1}(p)$,
- there exists a continuous function $\psi$ on $\pi^{-1}(\mathcal{U})$ such that $(\varphi \circ \pi) \lambda$ is the restriction of $\psi$.

Moreover, if there exists a constant $c$ such that $\psi = c\lambda$ holds on $\pi^{-1}(p)$, then $\varphi$ is called \textit{rationally continuous}.

\textbf{Definition 2.5.} An admissible coordinate system $(u, v)$ at a singular point $p$ of the first kind of a frontal $f$ is called \textit{adapted} if it satisfies the following properties along the $u$-axis

(a) $|f_u| = 1$,
(b) $f_v = 0$, in particular, the singular set coincide with the $u$-axis,
(c) $\{f_u, f_{vv}, \nu\}$ is an orthonormal basis which is compatible with respect to the orientation of $M^3$.
The existence of adapted coordinate system was shown in [11, Lemma 3.2]. (In fact, the proof of [11, Lemma 3.2] does not use the assumption that $f$ is a front.) Since $\langle f_{uv}, \nu \rangle = -\langle f_{u}, \nu \rangle = 0$ along the $u$-axis, (c) is equivalent to the condition that $\langle f_{u}, f_{vv} \rangle = (f_{u}, f_{v})_{v} = 0$ and $\langle f_{vv}, f_{vv} \rangle = \frac{1}{2} (f_{v}, f_{v})_{vv} = 1$ along the $u$-axis up to an orientation. So the definition of adapted coordinate system depends only on the first fundamental form of $f$. If $(U, V)$ be another adapted coordinate system at $p$, then it holds that $V_{\nu}(p) = \pm 1$.

We fix an adapted coordinate system $(u, v)$ of a frontal $f : \Sigma^{2} \to M^{3}$. Then $f_{1} := f_{u}$ and $f_{2} := f_{v}$ give smooth vector fields around the origin of the $(u, v)$-plane. We now set

$$
\hat{g}_{ij} := \langle f_{i}, f_{j} \rangle, \quad \hat{h}_{ij} := -\langle f_{i}, \nu \rangle (i, j = 1, 2),
$$

where $\nu_{1} := \nu_{u}$ and $\nu_{2} := \nu_{v}$. Then the mean curvature function $H$ of $f$ is given by

$$
H = \frac{\hat{g}_{11} \hat{h}_{22} - 2 v \hat{g}_{12} \hat{h}_{21} + v \hat{g}_{22} \hat{h}_{11}}{2 v (\hat{g}_{11} \hat{g}_{22} - (\hat{g}_{12})^{2})},
$$

namely $\hat{H} := v H$ is a $C^{\infty}$-function of $u, v$ such that $\hat{H} = \frac{1}{2} \hat{g}_{11} \hat{h}_{22} + O(v)$, where $O(v)$ is the higher order term with respect to $v$. By definition, $f_{2}(u, 0) = f_{vv}(u, 0)$. Then by (a) and (c) we have $\hat{g}_{11}(u, 0) = \hat{g}_{22}(u, 0) = 1$. Differentiating $f_{u} = v \varphi$ ($\varphi := f_{2}$), we have that

$$
f_{uv} = \varphi + v \varphi_{v}, \quad f_{vv} = 2 \varphi_{v} + v \varphi_{vv}.
$$

By (c) $\nu = f_{u} \times g f_{vv}$ holds at $(u, 0)$. Then we have

$$
\kappa_{c} = \det g(f_{u}, f_{vv}, f_{vv}) = \langle \nu, f_{vv} \rangle = 2 \langle \nu, \varphi_{v} \rangle = -2 \langle \nu, \varphi \rangle = 2 \hat{h}_{22}
$$

along the $u$-axis. So it holds that

$$
4 \hat{H}(u, 0) = \kappa_{c}(u) = \kappa_{c}(p) + u \kappa'_{c}(p) + O(u^{2}),
$$

which yields the following assertion:

**Proposition 2.6.** Let $p$ be a singular point of the first kind of a frontal $f$. Then the mean curvature function $H$ is bounded at $p$ if and only if $\kappa_{c}$ vanishes on a neighborhood of $p$ in the singular set (cf. [11, Corollary 3.5]). Moreover, $H$ is rationally bounded (resp. rationally continuous) if and only if $\kappa_{c}(p) = 0$ (resp. $\kappa'_{c}(p) = 0$).

Next, we discuss on the Gaussian curvature $K$, which is given by

$$
K := \frac{\hat{h}_{11} \hat{h}_{22} - \epsilon \hat{h}_{21}^{2}}{v (\hat{g}_{11} \hat{g}_{22} - \hat{g}_{12}^{2})}.
$$

In particular, $\hat{K} := v K$ is a $C^{\infty}$-function of $u, v$. Since $\kappa_{\nu}(u, 0) = \hat{h}_{11}(u, 0)$, we have that

$$
2 \hat{K}(u, 0) = \kappa_{11}(u) = \kappa_{11}(p) + \kappa'_{11}(p) u + O(u^{2}),
$$

where

$$
\kappa_{11}(u) := \kappa_{\nu}(u) \kappa_{c}(u) \quad \text{and} \quad \kappa'_{11}(p) := \kappa'_{\nu}(p) \kappa_{c}(p) + \kappa_{\nu}(p) \kappa'_{c}(p).
$$

We call $\kappa_{11}$ and $\kappa'_{11}$ the product curvature and the derive product curvature, respectively. Since the adapted coordinate system is intrinsic, we get the following.
Theorem 2.7. The two curvatures $\kappa_\Pi$ and $\kappa'_\Pi$ are both intrinsic invariants\(^1\). Moreover, the Gaussian curvature $K$ is rationally bounded (resp. rationally continuous) if and only if $\kappa_\Pi = 0$ (resp. $\kappa_\Pi = \kappa'_\Pi = 0$) holds at $p$. Furthermore, $K$ is bounded on a neighborhood $U$ of $p$ if and only if $\kappa_\Pi$ vanishes along the singular curve in $U$.

By [11, Corollary 3.5], $\kappa_c(p) \neq 0$ holds if $p$ is a cuspidal edge. In fact, the following lemma holds.

**Lemma 2.8.** Let $f$ be a frontal and $p$ a singular point of the first kind. Then $\kappa_c$ is proportional to $\psi_{ccr}$ on the singular set.

**Proof.** On an adapted coordinate system, $\kappa_c = \det g(f_u, f_{vv}, f_{vvv})(u,0)$. On the other hand, since $f_v(u,0) = 0$ holds, $f_{uv}(u,0) = 0$. Thus it holds that

$$\psi_{ccr}(u) = \det g(f_u, f_{vv}, f_{vvv})(u,0) = \det g(f_u, f_{uv} \times g f_{vv}, f_u \times g f_{vvv})(u,0).$$

Hence the assertion holds. Here, note that the tensor fields $\det g$ and $\langle , \rangle$ are parallel with respect to $\nabla$. □

**Corollary 2.9.** Let $p$ be a cuspidal edge. Then $K$ is rationally bounded (resp. rationally continuous) near $p$ if and only if $\kappa_\nu = 0$ (resp. $\kappa'_\nu = 0$) holds at $p$.

**Remark 2.10.** In [11], it was pointed out that the Gaussian curvature of a front $f$ takes opposite signs on the left and right hand sides of the singular curve if $\kappa_\nu \neq 0$. This follows immediately from the formula $vK = \frac{1}{2} \kappa_\Pi + O(u,v)$.

By Lemma 2.8 and Fact 1.3 (3), if $p$ is a cuspidal cross cap, then $\kappa_c(p) = 0$ and $\kappa'_c(p) \neq 0$ hold. So we have:

**Corollary 2.11.** Let $p$ be a cuspidal cross cap. Then $K$ is rationally bounded. Moreover it is rationally continuous around $p$ if and only if $\kappa_\nu(p) = 0$.

In Theorem B, the assumption that $f$ is a front is crucial. The Gaussian curvature of the following cuspidal cross cap is rationally bounded but $\kappa_\nu \neq 0$ holds.

**Example 2.12.** Let us consider a cuspidal cross cap defined by

$$f(u,v) = (u, v^2, uv^3 + u^2).$$

Then

$$\nu(u,v) = \frac{1}{\sqrt{4 + 4(2u + v^3)^2 + 9u^2v^2}} \left( -2(2u + v^3), 3uv, 2 \right)$$

gives a unit normal vector. By a direct calculation, one can see that $\kappa_\nu(0,0) = 2$ and the Gaussian curvature $K$ is

$$K(u,v) = \frac{12(2u - 3v^3)}{v \left( 4 + 16u^2 + 9u^2v^2 + 16uv^3 + 4v^6 \right)^2}.$$ 

This is rationally bounded around $(0,0)$.

---

\(^1\) This might be considered as a variant of Gauss’ Theorema Egregium.
Remark 2.13. If the ambient space is $E^3$, we can take the normal form as in Remark 2.2. Then the $C^\infty$-function $K := vK(u,v)$ satisfies

$$2\bar{K} = \kappa_{II} + \kappa_{II} u - v \left(2(b_2)^2 + \frac{\kappa_2^2}{2} - \frac{8(b_3)_{\nu} \kappa_{II}}{3}\right) + O(u^2 + v^2).$$

If $p$ is a cuspidal edge and $K \geq 0$ near $p$, then $\kappa_{II} = 0$ and thus $0 \leq 4K = -4b_2^2 - \kappa_2^2$ holds. So we have $\kappa_2 \leq 0$, which reproves the second assertion of [11, Theorem 3.1] in the special case that the ambient space is $E^3$.

Remark 2.14. It is well-known that the Gaussian curvature $K$ is the product of two principal curvatures $\lambda_1$ and $\lambda_2$ which is an intrinsic invariant of surfaces, although principal curvatures are not. So one can expect that there might exist a suitable modification of the shape operator along the cuspidal edge singularities so that its eigenvalues are equal to $\kappa_\nu$ and $\kappa_c$. In fact, we can accomplish to give such a modification of the shape operator using the adapted coordinate system as follows: We fix a frontal $f$ in $(M^3, g)$. Let $(u,v)$ be an adapted coordinate system centered at a non-degenerate singular point $p$ of the first kind. We denote by $\nu_p^\perp$ the subspace of $T_p M^3$ which is orthogonal to the unit normal vector $\nu(p)$ at $p$. Take an orthonormal basis $\{e_1, e_2\}$ of $\nu_p^\perp$ and define a matrix

$$A_p := -\begin{pmatrix} \langle u, e_1 \rangle & \langle u, e_2 \rangle \\ \langle v, e_1 \rangle & \langle v, e_2 \rangle \end{pmatrix}.$$  

Then the eigenvalues of $A_p$ do not depend on a choice of $\{e_1, e_2\}$. So we may set $e_1 := f_u$ and $e_2 := f_v$. Then one can easily check that $e_1, e_2$ are eigenvectors, and the eigenvalues of $A_p$ are equal to $\kappa_\nu(p)$ and $\kappa_c(p)/2$ respectively, using (1.5) and (2.3).

It is classically known that regular surfaces in $E^3$ admit non-trivial isometric deformations, and it might be interesting to discuss on the existence of such deformations at cuspidal edge singularities: Let $\xi(s)$ ($|s| < 1$) be a regular curve on the unit sphere $S^2(\subset E^3)$ with the arclength parameter $s$, and let $a(s)$ ($|s| < 1$) be a positive valued function. Then

$$f_{a,\xi}(u,v) := \left(\int_0^u a(s) \xi(s) \, ds \right) + v\xi(u)$$

gives a developable surface with singularities on $u$-axis. Then [24] yields $\kappa_c = -2\mu_g/\sqrt{a}$, where $\mu_g$ is the geodesic curvature of the spherical curve $\xi$. As pointed out in [4], moving $\xi$ so that $\mu_g$ varies, we get an isometric deformation of $f_{a,\xi}$ so that $\kappa_c$ changes. Thus $\kappa_c$ is not an intrinsic invariant. Unfortunately, this method cannot produce any isometric deformations of cuspidal edges changing $\kappa_\nu$, since any ruled cuspidal edges have vanishing limiting normal curvature. So the following problem seems interesting:

**Problem.** *Do there exist isometric deformations of an arbitrarily given cuspidal edge in $E^3$ which change $\kappa_\nu$?*

3. **Singularities of the second kind.**

We fix a frontal $f : \Sigma^2 \rightarrow M^3$. Let $p \in \Sigma^2$ be a non-degenerate singular point of the second kind.
Definition 3.1. A local coordinate system \((u, v)\) at \(p = (0, 0)\) is called adapted at \(p\) if it is compatible with respect to the orientation of \(\Sigma^2\), and the following conditions holds.

(i) \(f_u(p) = 0\),
(ii) the \(u\)-axis gives a singular curve,
(iii) \(|f_u(p)| = 1\).

Let \((U, V)\) be another adapted coordinate system, then the condition \(f_u(p) = 0\) and \((\text{iii})\) yield that
\[
V_v(p) = \pm 1.
\]

We fix an adapted coordinate system \((u, v)\) at \(p = (0, 0)\) and take a null vector field \(\eta\) along the \(u\)-axis. Then there exists a smooth function \(\varepsilon = \varepsilon(u)\) such that
\[
\eta = \partial_u + \varepsilon(u) \partial_v \quad (\varepsilon(0) = 0).
\]

We can extend this \(\eta\) as a vector field defined on a neighborhood of the origin. Since \(f_\eta = f_u + \varepsilon(u)f_v\) vanishes on the \(u\)-axis, there exists a \(C^\infty\)-function \(\psi\) such that \(f_\eta(u, v) = \psi\psi_0(u, v)\). Since \(f_u(0, 0) = f_\eta(0, 0) = \psi\psi_0(0, 0)\), we have that \(\psi(0, 0) = f_{uv}(0, 0)\). Since \(\lambda(u, 0) = 0\), the non-degeneracy of \(p\) yields that
\[
0 \neq \lambda_v = \det g(f_{uv}, f_v, \nu) = \det g(\psi, f_u, \nu)
\]
at \(p\), which implies that \(\psi(0, 0) \neq 0\). We now set
\[
g_{ij} := \langle f_{u_i}, f_{u_j} \rangle, \quad h_{ij} := -\langle f_{u_i}, \nu_{u_j} \rangle \quad (i, j = 1, 2),
\]
where \(u_1 = u\) and \(u_2 = v\). Then it holds that
\[
g_{11} = \langle \psi \varepsilon - f_v, \nu \varepsilon f_v \rangle, \quad g_{12} = \langle \psi \varepsilon f_v, f_v \rangle, \quad g_{22} = \langle f_v, f_v \rangle,
\]
which yields that
\[
g_{11}g_{22} - (g_{12})^2 = v^2(\langle \psi \varepsilon f_v, \nu \varepsilon f_v \rangle - \langle \psi f_v, \nu f_v \rangle)^2 = v^2 \det f_v \times \nu f_v.
\]

On the other hand,
\[
\langle f_u + \varepsilon f_v, \nu_{u_i} \rangle = \langle f_\eta, \nu_{u_i} \rangle = v \langle \psi, \nu_{u_i} \rangle \quad (i = 1, 2)
\]
holds, namely, we have
\[
-h_{11} - \varepsilon h_{12} = v \langle \psi, \nu_u \rangle, \quad -h_{12} - \varepsilon h_{22} = v \langle \psi, \nu_v \rangle.
\]
So we have that
\[
h_{12} = -v \langle \psi, \nu_v \rangle - \varepsilon h_{22}, \quad h_{11} = -v \langle \psi, \nu_u \rangle + v \langle \psi, \nu_v \rangle + \varepsilon^2 h_{22}.
\]
Since the mean curvature \(H\) of \(f\) is expressed as
\[
2vH = \frac{g_{11}h_{22} - 2g_{12}h_{12} + g_{22}h_{11}}{g_{11}g_{22} - (g_{12})^2}
\]
then \(\hat{H} := vH\) is a \(C^\infty\)-function of \(u, v\). So we can expand
\[
2\hat{H}(u, 0) = \kappa_H(p) + \kappa'_H(p)u + O(u^2).
\]
Thus, \(H\) is rationally bounded (resp. rationally continuous) if and only if \(\kappa_H(p) = 0\) (resp. \(\kappa_H(p) = \kappa'_H(p) = 0\)). By (3.1), \(\kappa_H(p)\) is a geometric invariant, but \(\kappa'_H(p)\)
In particular, the product curvature 
\[ \kappa_H(p) = \frac{-\langle \psi(p), \nu_u(p) \rangle}{|\psi(p) \times g f_v(p)|^2} = -\frac{\langle f_{uv}(p), \nu_u(p) \rangle}{|f_{uv}(p) \times g f_v(p)|^2}. \]

The right hand side of (3.2) is independent of the choice of an adapted coordinate system.

Like as in the case of the mean curvature, \( \tilde{K} := vK \) is a \( C^\infty \)-function of \( u, v \) satisfying
\[ \tilde{K}(u, 0) = \tilde{H}(u, 0) \frac{h_{22}(u, 0)}{| f_v(u, 0) |^2} = \tilde{H}(u, 0) \kappa_v(u) \]
because of (1.1), where \( \kappa_v(u) \) is the limiting normal curvature defined in Section 1 (see Proposition 1.7), where \( K \) is the Gaussian curvature of \( f \). Then we have
\[ \tilde{K}(u, 0) = \tilde{H}(u, 0) \left( \kappa_v(p) + \kappa_p(u) + O(u^2) \right), \]

Since \( \kappa_p(u) = d\kappa_v(u, 0)/du \) depends on the parameter \( u \), we consider the co-vector
\[ \omega_v(p) := \kappa_p'(u)du \in T^*_p \Sigma^2 \]
instead, which does not depend on the choice of a parameter of the singular curve \( \gamma \). In [11, Corollary 3.5], it is proved that \( \tilde{H}(0, 0) \neq 0 \) when \( f \) is a front, that is, \( \kappa_H(p) \) does not vanish. By (3.3), we get the following:

**Theorem 3.2.** Let \( f \) be a front and \( p \) its non-degenerate singular point of the second kind. Then the Gaussian curvature \( K \) is rationally bounded (resp. rationally continuous) if and only if \( \kappa_v(p) = 0 \) (resp. \( \kappa_v(p) = 0 \) and \( \omega_v(p) = 0 \)). Moreover, \( K \) is bounded on a neighborhood \( U \) of \( p \) if and only \( \kappa_v \) vanishes along the singular curve in \( U \).

**Definition 3.3.** An adapted coordinate system \( (u, v) \) at \( p \) is called strongly adapted if \( f_{uv} \) is perpendicular to \( f_v \) at \( p \).

**Lemma 3.4.** A strongly adapted coordinate system exists at \( p \).

**Proof.** For an adapted coordinate system \( (u, v) \), the new coordinate system \( (U, V) \) defined by \( U = u \) and \( V = u - \langle f_{uv}(p), f_v(p) \rangle_{uv} \), gives a strongly adapted coordinate system. \( \square \)

**Theorem 3.5.** Let \( f \) be a frontal and \( p \) its non-degenerate singular point of the second kind, and let \( \gamma(t) \) be the singular curve such that \( \gamma(0) = p \). If \( \gamma(t) \) \( t \neq 0 \) is a singular point of the first kind, then it holds that
\[ \kappa_H(p) = \lim_{t \to 0} \frac{\kappa_v(\gamma(t))}{2|\kappa_v(\gamma(t))|^{1/2}}. \]

In particular, the product curvature \( \kappa_H(\gamma(t)) \) does not converge to \( \kappa_v(p)\kappa_H(p) (= \tilde{K}(0, 0)) \).

**Proof.** Let \( (u, v) \) be a strongly adapted coordinate system and take the null vector field as \( \eta = \partial_u + \varepsilon(u)\partial_v \), where \( \varepsilon(u) \neq 0 \) for \( u \neq 0 \) and \( \varepsilon(0) = 0 \). Since \( \psi(p) = f_{uv}(p) \) is perpendicular to \( f_v(p) \), (3.2) reduces to
\[ \kappa_H(p) = \frac{\langle \psi, \nu_u(p) \rangle}{|\psi|^2}. \]
By [11] Page 501, we have that
\[ \lim_{u \to 0} |\varepsilon(u)\kappa_s(u)| = |\det_g(f_v(p), f_{uv}(p), \nu)| = |\det_g(f_v(p), \psi, \nu)| = |\psi|. \]

On the other hand, we have that
\[ f_{\eta \eta} = \psi_{\eta \eta} = \psi_{\eta \eta} v_{\eta} + \psi v_{\eta \eta}. \]

Since \( v_{\eta} = v_{u} + \varepsilon v_{v} = \varepsilon \), we have that
\[ f_{\eta \eta}(u, 0) = \psi_{\varepsilon} + \psi v_{\eta \eta}. \]

Since \( \xi := f_v(u, 0) \) is proportional to \( f_u(u, 0) \) (resp. orthogonal to \( \psi \)), (2.4) reduces to
\[ \kappa_c = \frac{|\det_g(\xi, \psi, 2\psi_{\varepsilon})|}{|\psi_{\varepsilon}|^5/2} = 2\frac{|\det_g(\xi, \psi, \psi_{\eta})|}{\sqrt{|\varepsilon| |\psi|^3/2}} = \frac{2}{\sqrt{|\varepsilon|}} \]
along \( \gamma(u) = (u, 0) \). Thus it holds that (cf. [2.4])
\[ \lim_{t \to 0} \frac{\kappa_c(\gamma(t))}{|\kappa_c(\gamma(t))|^{1/2}} = \frac{\lim_{t \to 0} \sqrt{|\varepsilon(u)|\kappa_c(u)}}{\lim_{t \to 0} |\varepsilon(u)\kappa_s(u)|} = 2\kappa_H(p), \]
which proves the assertion. \( \square \)

We now assume that \( p \) is a swallowtail singularity of \( f \). Then \( f_{uu}(p) = -\varepsilon'(0)f_v(p) \) and by Fact [13] (2), \( \varepsilon'(0) \neq 0 \), it holds that \( \tilde{\gamma}''(0) \neq 0 \).

**Proposition 3.6.** Let \( f : \Sigma^2 \to E^3 \) be a front, and \( p \in \Sigma^2 \) a swallowtail singularity. Then the following identity holds
\[ (\tau_s :=) 2\sqrt{2} \lim_{t \to 0} \sqrt{|t| |\kappa_s(\gamma(t))|} = \frac{|\det(\tilde{\gamma}''(0), \tilde{\gamma}'''(0), \nu)|}{|\tilde{\gamma}''(0)|^{5/2}}, \]
where \( t \) is the arclength parameter of \( \tilde{\gamma} \). We call \( \tau_s \) the limiting singular curvature at \( p \). As a consequence, \( \tau_s \) is an intrinsic invariant.

Remark that \( \kappa_s \) diverges to \( -\infty \) at a swallowtail ([11] Corollary 1.14]), only the absolute value of \( \tau_s \) is meaningful.

**Proof.** Take a strongly adapted coordinate system \( (U; (u, v)) \) and let \( t = t(u) \) be the arclength parameter of \( \tilde{\gamma} = f(u, 0) \). Since the tensor fields \( \det_g \) and \( \langle \cdot, \cdot \rangle \) are parallel, we have
\[ \lim_{u \to 0} \frac{\det_g(\tilde{\gamma}', \tilde{\gamma}'', \nu(\gamma))}{u^2} = \lim_{u \to 0} \frac{\det_g(\tilde{\gamma}', \tilde{\gamma}'', \nu(\gamma))'}{2} = \lim_{u \to 0} \frac{\det_g(\tilde{\gamma}', \tilde{\gamma}'', \nu(\gamma))'}{2} + \det_g(\tilde{\gamma}', \tilde{\gamma}'', \nu(\gamma)'). \]

and
\[ \lim_{u \to 0} \frac{\langle \tilde{\gamma}', \tilde{\gamma}' \rangle}{u^2} = \lim_{u \to 0} \frac{\langle \tilde{\gamma}', \tilde{\gamma}' \rangle'}{2} = \langle \tilde{\gamma}', \tilde{\gamma}' \rangle_{u=0}. \]

Since \( |\kappa_s(u)| = |\det_g(\tilde{\gamma}'(u), \tilde{\gamma}''(u), \nu(\gamma(u)))| |\tilde{\gamma}'(u)|^{-3} \), we have
\[ \lim_{u \to 0} \frac{|u||\kappa_s(u)|}{|\tilde{\gamma}'|^3} = \lim_{u \to 0} \frac{\det_g(\tilde{\gamma}', \tilde{\gamma}'', \nu(\gamma))}{|u|^2 |\tilde{\gamma}'|^3} = \frac{\det_g(\tilde{\gamma}', \tilde{\gamma}'', \nu(\gamma))}{2 |\tilde{\gamma}'|^3} \]
On the other hand,
\[ \lim_{u \to 0} \frac{t(u)}{u^2} = \lim_{u \to 0} \frac{|\dot{\gamma}'(u)|}{2u} = \frac{|\dot{\gamma}''(0)|}{2} \]
holds. Thus we have
\[ \lim_{u \to 0} \frac{\sqrt{|t(u)|}}{|u|} = \frac{\sqrt{|\dot{\gamma}''(0)|}}{\sqrt{2}}. \]
Hence
\[ 2\sqrt{2} \lim_{t \to 0} \sqrt{|t||\kappa_s(t)|} = 2\sqrt{2} \lim_{u \to 0} \frac{|\det g(\dot{\gamma}', \dot{\gamma}'', \nu(\gamma))(u)|}{u^2} \frac{\sqrt{|t|}}{u} \frac{|\dot{\gamma}'(u)|^3}{u^3} \]
\[ = 2\sqrt{2} \frac{|\det g(\dot{\gamma}'', \dot{\gamma}'''(0), \nu(\gamma))|}{2} \frac{|\dot{\gamma}''(0)|}{|\dot{\gamma}''(0)|^3/2} \bigg|_{u=0} \]
proves the assertion.

**Proposition 3.7.** Let \( f : \Sigma^2 \to \mathbf{E}^3 \) be a front, and \( p \in \Sigma^2 \) a swallowtail singularity. Let \( P \) be the tangential plane of \( f \) at \( f(p) \) (that is, the plane passing through \( f(p) \) which is orthogonal to \( \nu(p) \)), and \( \sigma \) the orthogonal projection of \( \dot{\gamma} := f \circ \gamma \) to the plane \( P \). Then \( \tau_s \) coincides with the absolute value of cuspidal curvature of the curve \( \sigma \) in the plane \( P \).

**Proof.** Take a strongly adapted coordinate \((U; (u, v))\). Then it holds that \( \sigma(u) = \dot{\gamma}(u) - \langle \dot{\gamma}(u), \nu(p) \rangle \nu(p) \). Then since \( |\nu(p)| = 1 \), the absolute value of cuspidal curvature of \( \sigma \) is
\[ \frac{|\det(\dot{\gamma}''(0), \dot{\gamma}'''(0), \nu(p))|}{|\sigma''(0)|^{5/2}}. \]
Since \( \dot{\gamma}''(u) = f_{uu}(u, 0) \) and \( f_u(p) = 0 \), we have \( \langle f_{uu}(p), \nu(p) \rangle = -\langle f_u(p), \nu_u(p) \rangle = 0 \). Thus \( |\sigma''(p)| = |\dot{\gamma}''(p)| \) holds. Hence we have the assertion. \( \square \)

Using Theorem 3.3, we then set
\[ \tau_c := \sqrt{|\tau_s||\kappa_H(p)|} = \frac{2\sqrt{2}}{\sqrt{2}} \lim_{t \to 0} \sqrt{|t|^{1/4} \kappa_c(\gamma(t))}, \]
where \( t \) is the arclength parameter of \( \gamma \), and call it the *limiting cuspidal curvature*. Noticing (1.1), we have the following.

**Corollary 3.8.** If \( f \) be a front and \( p \) is a swallowtail, then the following identity holds:
\[ \sqrt{|\tau_s||\kappa_H(p)|} = \kappa_c(p)\tau_c. \]
In particular, the right hand side is intrinsic.

**Example 3.9.** Consider a swallowtail
\[ f(u, v) = \left( u^4 - 4u^2v, u^3 - 3uv, \frac{u^2}{2} - v \right) + (u^2 - 2v)^2(a, b, 0) \quad (a, b \in \mathbf{R}) \]
in \( \mathbf{E}^3 \). Then we have
\[ 2\dot{H}(u, 0) = -\frac{8}{9} + O(u^2), \]
\[ \dot{K}(u, 0) = -\frac{64a}{9} + \frac{512b}{27}u + O(u^2) \]
and
\[
\kappa_c(u) = \frac{-16}{3\sqrt{3}u} + O(u^{3/2}), \quad \kappa_s(u) = \frac{-3}{u} - 8b + O(u).
\]
Moreover, we have \(\tau_s = \lim_{t \to 0} \sqrt{t||\kappa_s(t)||} = 6\), since the arclength \(t(u)\) of \(f(u, 0)\) is \(t(u) = u^2/2 + O(u^3)\). On the other hand, the absolute value of the cuspidal curvature of \(\sigma(u) = (u^3(1 + bu), u^2/2)\) in the \(yz\)-plane is 6. Thus the limiting cuspidal curvature \(\tau_c\) is \(-8\sqrt{6}/9\).

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References

[1] F. S. Dias and F. Tari, On the geometry of the cross-cap in the Minkowski 3-space, preprint, 2012. Available from www2.icmc.usp.br/~faridtari/Papers/DiasTari.pdf.
[2] T. Fukui and M. Hasegawa, Fronts of Whitney umbrella—a differential geometric approach via blowing up, J. Singul., 4 (2012), 35–67.
[3] T. Fukui and M. Hasegawa, Height functions on Whitney umbrellas, to appear in RIMS Kôkyûroku Bessatsu, 38 (2013).
[4] R. Garcia, C. Gutierrez, and J. Sotomayor, Lines of principal curvature around umbilics and Whitney umbrellas, Tohoku Math. J., 52 (2000), 163–172.
[5] S. Fujimori, K. Saji, M. Umehara and K. Yamada, Singularities of maximal surfaces, Math. Z., 259 (2008), 827–848.
[6] M. Hasegawa, A. Houda, K. Naokawa, M. Umehara, and K. Yamada, Intrinsic invariants of cross caps, to appear in Selecta Mathematica, arXiv:1207.3853v2.
[7] M. Kokubu, W. Rossman, K. Saji, M. Umehara, and K. Yamada, Singularities of flat fronts in hyperbolic 3-space, Pacific J. Math., 221 (2005), 303–351.
[8] L. F. Martins and J. J. Nuño-Ballesteros, Contact properties of surfaces in \(\mathbb{R}^3\) with corank 1 singularities, preprint, 2012. Available from www.uv.es/nano/Preprints/Nuño-Martins.pdf.
[9] L. F. Martins and K. Saji, Geometric invariants of cuspidal edges, preprint, 2013. Available from www.ibilce.unesp.br/Home/Departamentos/Mate\mbox{m}atica/Singularidades/martins-saji-geometric.pdf.

[10] R. Oset Sinha and F. Tari, Projections of surfaces in $\mathbb{R}^4$ to $\mathbb{R}^3$ and the geometry of their singular images, preprint, 2012. Available from www2.icmc.usp.br/~faridtari/Papers/OsetTariSingularSurfaces.pdf.

[11] K. Saji, M. Umehara, and K. Yamada, The geometry of fronts, Ann. of Math., 169 (2009), 491–529.

[12] K. Saji, M. Umehara, and K. Yamada, $A_k$ singularities of wave fronts, Math. Proc. Cambridge Philos. Soc., 146 (2009), 731–746.

[13] K. Saji, M. Umehara, and K. Yamada, The duality between singular points and inflection points on wave fronts, Osaka J. Math. 47 (2010), 591–607.

[14] K. Saji, M. Umehara and K. Yamada, Coherent tangent bundles and Gauss-Bonnet formulas for wave fronts, Journal of Geometric Analysis (2012) 22:383-409. DOI 10.1007/s12220-010-9193-5.

[15] S. Shiba and M. Umehara, The behavior of curvature functions at cusps and inflection points, Diff. Geom. Appl., 30 (2012), 285–299.

[16] F. Tari, On pairs of geometric foliations on a cross-cap, Tohoku Math. J., 59 (2007), 233–258.

[17] J. West, The differential geometry of the cross-cap, Ph. D. thesis, Liverpool Univ. 1995.

[18] H. Whitney, The general type of singularity of a set of $2n-1$ smooth functions of $n$ variables, Duke Math. J., 10 (1943), 161–172.

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