THE BI-NORMAL FIELDS ON SPACELIKE SURFACES IN $\mathbb{R}^4_1$

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Abstract

A normal field on a spacelike surface in $\mathbb{R}^4_1$ is called bi-normal if $K^\nu$, the determinant of Weingarten map associated with $\nu$, is zero. In this paper we give a relationship between the spacelike pseudo-planar surfaces and spacelike pseudo-umbilical surfaces, then study the bi-normal fields on spacelike ruled surfaces and spacelike surfaces of revolution.

0 Introduction

Let $\alpha : I \to \mathbb{R}^3$ be a bi-regular parametric curve. Along this curve, the vector field defined by

$$\mathbf{b} = \frac{1}{|\alpha' \times \alpha''|} (\alpha' \times \alpha'')$$

is called the bi-normal field of $\alpha$. A bi-normal vector can be seen as a direction whose the corresponding height function has a degenerate (non-Morse) critical point.

Let $M$ be a regular surface in $\mathbb{R}^4_1$ (or $\mathbb{R}^4$) and $f_\nu$ be the height function on $M$ associated with a direction $\mathbf{v}$. By analogy with the case of curves in $\mathbb{R}^3$, a direction $\mathbf{v}$ is called a bi-normal direction of $M$ at a point $p$ if the height function $f_\nu$ has a degenerate singularity at $p$. The height function $f_\nu$ having a degenerate singularity means that its hessian is singular.

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Given a normal field \( \nu \) on \( M \) and denoted by \( S^\nu \) the shape operator associated with \( \nu \). The \( \nu \)-Gauss curvature \( K^\nu \) of \( M \) is defined by \( K^\nu = \det S^\nu \). The eigenvalues \( k^\nu_1 \) and \( k^\nu_2 \) of the shape operator \( S^\nu \) are called \( \nu \)-principal curvatures. The \( \nu \)-mean curvature of \( M \) is defined by
\[
H^\nu = \frac{1}{2} \text{trace} S^\nu = \frac{1}{2}(k^\nu_1 + k^\nu_2).
\]
A point \( p \in M \) is called \( \nu \)-umbilic if \( k^\nu_1(p) = k^\nu_2(p) = k \) and is called \( \nu \)-flat if \( k^\nu_1(p) = k^\nu_2(p) = 0 \). If there exists a normal field \( \nu \) on \( M \) such that \( M \) is \( \nu \)-umbilic (\( \nu \)-flat) then \( M \) is called pseudo-umbilical (pseudo-flat) surface. \( M \) is called umbilic if it is \( \nu \)-umbilic for all normal fields \( \nu \). \( M \) is called maximal if \( H^\nu = 0 \) for all normal fields \( \nu \).

It is easy to show that \( \nu(p) \) is bi-normal direction of \( M \) at \( p \) if \( \det S^\nu(p) = 0 \) i.e. either \( k^\nu_1(p) = 0 \) or \( k^\nu_2(p) = 0 \). Such a point is called \( \nu \)-planar and a direction belonged to the kernel of \( S^\nu(p) \) is said to be asymptotic. The normal field \( \nu \) of \( M \) is called bi-normal field if for each \( p \in M \), \( \nu(p) \) is bi-normal direction of \( M \) at \( p \). If there exists a bi-normal field on \( M \) then \( M \) is called pseudo-planar surface, in the case each normal field is bi-normal \( M \) is called planar surface. For everything concerning to these notions in more detail, we refer the reader to [9], [8], [16], [17], [19] and references therein.

The existence of bi-normal direction on surfaces in \( \mathbb{R}^4 \) has been studied by several authors (e.g. [4], [16], [17], [19], [21], [14], [22], ...). Little ([14], Theorem 1.3(b), 1969) showed that a surface whose all normal fields are bi-normal if and only if it is a ruled developable surface. In 1995, D.K.H. Mochida et. al ([16], Corollary 4.3 repeated at [21], (2010)) showed that a surface admitting two bi-normal fields if and only if it is strictly locally convex. These results was expanded to surfaces of codimension two in \( \mathbb{R}^{n+2} \) by them [17] in 1999. These methods are used later by M.C. Romero-Fuster and F. Sánchez-Brigas ([19], Theorem 3.4, 2002) to study the umbilicity of surfaces.

The first section of this paper shows that there exist pseudo-planar surfaces are not pseudo-umbilic, defines the number of bi-normal fields on the pseudo-umbilical surfaces and gives some interesting corollaries.

In the second of this paper we show that each point on the spacelike ruled surfaces admits either one or all bi-normal directions, a spacelike ruled surface is pseudo-umbilic iff umbilic.

In the third section of this paper we show that the spacelike surfaces of revolution admit exactly two bi-normal fields whose asymptotic fields respectively are orthogonal. Therefore, they are pseudo-umbilic but not umbilic.

The final section of this paper shows that the number of bi-normal fields on the rotational spacelike surface whose meridians lie in two-dimension space are depended on the properties of its meridian.
1 Bi-normal Fields on Pseudo-umbilical Surfaces

For the surfaces in $\mathbb{R}^4$ Romero Fuster [19] showed that pseudo-umbilical surfaces are pseudo-planar; moreover, their two asymptotic fields are orthogonal. These results are also true for spacelike surfaces in $\mathbb{R}^4_1$, and I would like to show it here. Notice that there exist the pseudo-planar spacelike surfaces are not pseudo-umbilic, let see Example 1.2 and Example 1.3. We have the similar example for surfaces in $\mathbb{R}^4$.

The following theorem shows that the pseudo-umbilical spacelike surfaces are pseudo-planar and gives us the number of bi-normal fields on them.

**Theorem 1.1.** Let $M$ be a spacelike surface in $\mathbb{R}^4_1$. If $M$ is pseudo-umbilic (not pseudo-flat) then it admits either one or two bi-normal fields. Moreover, $M$ admits only one bi-normal field iff it is umbilic.

**Proof.** Suppose that $M$ is $\nu$-umbilic (not $\nu$-flat). Let $n$ be a normal field on $M$ such that $\{X_u, X_v, \nu, n\}$ is linearly independent and $k$ is $\nu$-principal curvature. Given a normal field $B$, then we have the following interpretation

$$B = \lambda \nu + \mu n,$$

where $\lambda, \mu$ are smooth functions on $M$. Suppose that the coefficients of the first fundamental form of $M$ satisfy

$$g_{11} = g_{22} = \varphi, g_{12} = 0,$$

then we have

$$S^B = \lambda S^\nu + \mu S^n = \lambda k \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \frac{\mu}{\varphi} \begin{bmatrix} b^n_{11} & b^n_{12} \\ b^n_{12} & b^n_{22} \end{bmatrix} = \begin{bmatrix} \frac{\mu}{\varphi} b^n_{11} + \lambda k & \frac{\mu}{\varphi} b^n_{12} \\ \frac{\mu}{\varphi} b^n_{12} & \frac{\mu}{\varphi} b^n_{22} + \lambda k \end{bmatrix}.$$  

Therefore,

$$K^B = \gamma^2 (b^n_{11} b^n_{22} - (b^n_{12})^2) + \lambda \gamma k (b^n_{11} + b^n_{22}) + \lambda^2 k^2,$$

$$K^B = 0 \Leftrightarrow \gamma^2 (b^n_{11} b^n_{22} - (b^n_{12})^2) + \lambda \gamma k (b^n_{11} + b^n_{22}) + \lambda^2 k^2 = 0,$$

where $\gamma = \frac{\mu}{\varphi}$. Since $\nu$ is not bi-normal, $\mu \neq 0$. Then the equation (1) can be rewrote by

$$\left(\frac{\lambda k}{\gamma}\right)^2 + (b^n_{11} + b^n_{22}) \frac{\lambda k}{\gamma} + b^n_{11} b^n_{22} - (b^n_{12})^2 = 0.$$  

It is from

$$(b^n_{11} - b^n_{22})^2 + 4(b^n_{12})^2 \geq 0$$  

that the equation (2) has at least one or at most two solutions. That means $M$ admits at least one or at most two bi-normal fields.

$M$ admits only one bi-normal field if and only if $b^n_{11} = b^n_{22}$ and $b^n_{12} = 0$. Which means that $M$ is $n$-umbilic. Then the Lemma 4.1 in [2] shows that $M$ is umbilic.

$\square$
The following example gives a spacelike surface admitting one bi-normal field but not pseudo-umbilic.

**Example 1.2.** Let $M$ be a surface given by following parameterization

$$X(u, v) = (\cos u(1 + v), \sin u(1 + v), \sinh u, \cosh u), \ u, v \in \mathbb{R}. \quad (4)$$

The coefficients of the first fundamental form of $M$ are determined by

$$g_{11} = \langle X_u, X_u \rangle = v^2 + 2 > 0, \ g_{12} = \langle X_u, X_v \rangle = 0, \ g_{22} = \langle X_v, X_v \rangle = 1.$$ 

Therefore, $M$ is a spacelike surface. Let $n = (n^1, n^2, n^3, n^4)$ be a normal field on $M$. That means

$$\langle X_u, n \rangle = 0 \iff n^1 \cos u + n^2 \sin u = 0, \quad (5)$$

$$\langle X_v, n \rangle = 0 \iff -n^1 \sin u(1 + v) + n^2 \cos u(1 + v) + n^3 \cosh u - n^4 \sinh u = 0. \quad (6)$$

Using (5) the coefficients of the second fundamental form associated with $n$ are

$$b_{11}^n = n^3 \sinh u - n^4 \cosh u, \ b_{12}^n = -n_1 \sin u + n_2 \cos u, \ b_{22}^n = 0.$$ 

We have

$$\det(b_{ij}^n) = (b_{12}^n)^2.$$ 

So,

$$K^n = 0 \iff -n^1 \sin u + n^2 \cos u = 0. \quad (7)$$

Connecting (5), (6) and (7) we imply that $n$ is a bi-normal field on $M$ if and only if

$$n^1 = n^2 = 0, \ n^3 \cosh u - n^4 \sinh u = 0.$$ 

That means $M$ admits only one bi-normal field

$$B = (0, 0, \sinh u, \cosh u).$$

It is a unit timelike normal field. Since

$$S^B = \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix},$$

$M$ is not $B$-flat.

On the other hand the $n$-principal curvatures are the solutions of the following equation

$$\det \left( (b_{ij}^n) - \lambda(g_{ij}) \right) = 0 \iff (v^2 + 2)\lambda^2 - \lambda b_{11}^n - (b_{12}^n)^2 = 0.$$ 

Therefore, $M$ is $n$-umbilic if and only if $b_{11}^n = b_{12}^n = 0$. Which doesn’t take place, by connecting (5), (6) and (7). So, $M$ is not pseudo-umbilic.
Even when \( M \) admits two bi-normal fields, it is not pseudo-umbilic. Let see the following example.

**Example 1.3.** Let \( M \) be a surface given by following parameterization

\[
X(u, v) = (e^{2u} \cos v, e^{2u} \sin v, e^{-u} \cosh v, e^{-u} \sinh v), \quad u > 1, \ v \in (0, 2\pi).
\]

It is easy to show that \( M \) is spacelike and \( \{n_1, n_2\} \) is a frame of the variable normal bundle, where

\[
n_1 = -\frac{1}{\sqrt{g_{11}}} (e^{-u} \cos v, e^{-u} \sin v, 2e^{2u} \cosh v, 2e^{2u} \sinh v),
\]

\[
n_2 = \frac{1}{\sqrt{g_{22}}} (-e^{-u} \sin v, e^{-u} \cos v, e^{2u} \sinh v, e^{2u} \cosh v).
\]

The coefficients of the second fundamental form associated with \( n_1 \) and \( n_2 \) are

\[
b_{11}^{n_1} = -\frac{6e^u}{\sqrt{g_{11}}}, \ b_{12}^{n_1} = 0, \ b_{22}^{n_1} = -\frac{e^u}{\sqrt{g_{11}}},
\]

\[
b_{11}^{n_2} = 0, \ b_{12}^{n_2} = \frac{3e^u}{\sqrt{g_{22}}}, \ b_{22}^{n_2} = 0.
\]

Therefore, both \( n_1 \) and \( n_2 \) are not bi-normal. Fore each normal field \( n \) on \( M \) we have the following interpretation

\[
n = n_1 + \mu n_2 \tag{8}
\]

and

\[
(b_{ij}^n) = \begin{bmatrix}
   b_{11}^{n_1} & \mu b_{12}^{n_1} \\
   \mu b_{12}^{n_2} & b_{22}^{n_2}
\end{bmatrix}.
\]

So

\[
K^n = \frac{b_{11}^{n_1}b_{22}^{n_1} - \mu^2 (b_{12}^{n_2})^2}{g_{11}g_{22}}.
\]

Since \( b_{11}^{n_1}b_{22}^{n_1} > 0 \) and \( b_{12}^{n_2} \neq 0 \), \( M \) admits exactly two bi-normal fields.

On the other hand the \( n \)-principal curvatures of \( M \) are solutions of the following equation

\[
\det \left( (b_{ij}^n) - \lambda(g_{ij}) \right) = 0 \iff g_{11}g_{22}\lambda^2 + \left( e^u \sqrt{g_{11}} + \frac{6e^u g_{22}}{\sqrt{g_{11}}} \right) \lambda + \frac{6e^{2u} g_{11}}{g_{11}} - \frac{9e^{2u} \mu^2}{g_{22}} = 0, \tag{9}
\]

where \( \lambda \) is the variable. Since

\[
\frac{1}{g_{11}} \left( (2e^{4u} - 7e^{-2u})^2 \right) + 36g_{11}\mu^2 > 0, \ \forall \mu,
\]

for each normal field \( n \), \( M \) is not \( n \)-umbilic. It means that \( M \) is not pseudo-umbilic.
**Corollary 1.4.** If $M$ is umbilic then $M$ is pseudo-flat.

**Corollary 1.5.** Let $M$ be a surface contained in the a pseudo-sphere (Hyperbolic or de Sitter). Then the following statements are equivalent.

1. $M$ is umbilic,
2. $M$ admits only one bi-normal field,
3. $M$ is contained in a hyperplane.

**Corollary 1.6.** The following statements are equivalent.

1. $M$ is locally umbilic.
2. $M$ is locally contained in the intersection of a Hyperbolic (or de Sitter) with a hyperplane.
3. $M$ locally admits only one bi-normal field $B$ and $\nu$-umbilic (not $\nu$-flat), for some normal field $\nu$.

**Corollary 1.7.** If $M$ admits only one bi-normal field $B$ (not $B$-flat) then it is not pseudo-umbilic.

**Remark 1.8.** The results in [19] are also true for the spacelike surfaces in $\mathbb{R}^4_1$. So that the following statements are equivalent:

1. $M$ has two everywhere defined orthogonal asymptotic fields,
2. $M$ is pseudo-umbilic,
3. The normal curvature of $M$ vanishes at every point,
4. All points of $M$ are semi-umbilic.

## 2 Bi-normal Fields on Ruled Spacelike Surface in $\mathbb{R}^4_1$

The notion of ruled surface in $\mathbb{R}^4$ have been introduced by Lane in [13]. It is similar to ruled (spacelike) surface in $\mathbb{R}^4_1$ and can be introduced by the similar way. A surface $M$ in $\mathbb{R}^4_1$ is called ruled if through every point of $M$ there is a straight line that lies on $M$. We have a local parameterization of $M$

$$\mathbf{X}(u, t) = \alpha(t) + uW(t), \ t \in I \subset \mathbb{R}, u \in \mathbb{R},$$
where $\alpha(t)$ is a differential curve in $\mathbb{R}^4$ and $W(t)$ is a smooth vector field along $\alpha(t)$.

A ruled surface $M$ is called developable if its Gaussian curvature identifies zero.

It is from $X_u = W(t), X_t(0, t) = \alpha'(t)$ and $M$ is spacelike that both $W(t)$ and $\alpha'(t)$ are spacelike. We can assume that $|W| = |\alpha'| = 1$ and $\langle W, \alpha' \rangle = 0$.

The coefficients of the first fundamental form of $M$ are

$$g_{11} = \langle X_u, X_u \rangle = \langle \alpha', \alpha' \rangle + 2t \langle \alpha', W' \rangle + t^2 \langle W', W' \rangle,$$

$$g_{12} = \langle X_u, X_t \rangle = 0, \quad g_{22} = \langle X_t, X_t \rangle = 1.$$

Since $M$ is spacelike, $\langle W', W' \rangle > 0$.

Let $n$ be a normal field on $M$, the coefficients of the second fundamental form associated $n$ are defined as following

$$b^n_{11} = \langle X_{uu}, n \rangle = \langle \alpha'', n \rangle + t \langle W'', n \rangle, \quad b^n_{12} = \langle X_{ut}, n \rangle = \langle W', n \rangle,$$

$$b^n_{22} = \langle X_{tt}, n \rangle = 0.$$

Therefore,

$$S^n = (g_{ij})^{-1}.(b^n_{ij}) = \frac{1}{g_{11}} \begin{bmatrix} b^n_{11} & b^n_{12} \\ b^n_{12} & g_{11} & 0 \end{bmatrix}, \quad K^n = -\frac{(b^n_{12})^2}{g_{11}}. \quad (11)$$

So,

$$K^n = 0 \iff b^n_{12} \iff \langle W', n \rangle = 0. \quad (12)$$

The following proposition gives us the number of bi-normal directions at each point on a ruled surface.

**Proposition 2.1.** Let $M$ be a ruled spacelike surface given by (11), we then have:

1. at the point such that $\{\alpha', W, W'\}$ is linearly dependent each normal vector is bi-normal direction;

2. at the point such that $\{\alpha', W, W'\}$ is linearly independent $M$ admits only one bi-normal direction.

3. $M$ is pseudo-umbilic if and only if umbilic.

**Proof.**

1. Since

$$\langle \alpha', W \rangle = 0, \quad \langle W', W \rangle = 0$$

and $\{\alpha', W, W'\}$ is linearly dependent, $W' \in T_p M$. Therefore, by using (12), we imply that each normal vector on $M$ is bi-normal direction.
2. Since
\[ K \mathbf{n} = 0 \iff \begin{cases} 
\langle \mathbf{n}, \mathbf{X}_u \rangle = 0, \\
\langle \mathbf{n}, \mathbf{X}_v \rangle = 0, \\
\langle \mathbf{n}, W' \rangle = 0,
\end{cases} \iff \begin{cases} 
\langle \mathbf{n}, \alpha' \rangle = 0, \\
\langle \mathbf{n}, W \rangle = 0, \\
\langle \mathbf{n}, W' \rangle = 0,
\end{cases} \]
\( \mathbf{n} \) is an unit bi-normal direction if and only if
\[ \mathbf{n} = \frac{\alpha' \wedge W \wedge W'}{|\alpha' \wedge W \wedge W'|}. \]
It is followed from the fact that \( \alpha', W, W' \) are spacelike that the unique unit bi-normal direction on \( M \) is timelike.

3. Since \( M \) admits only one bi-normal field, it is followed Theorem 1.1 that \( M \) is pseudo-umbilic iff umbilic.

\[ \square \]

**Remark 2.2.**
1. The Proposition 2.1 is also true for the ruled surfaces in \( \mathbb{R}^4 \).

2. Using the Gauss equation we can show that the Gaussian curvature of a spacelike surface in \( \mathbb{R}_1^4 \) can be defined by sum of \( K^{e_1} \) and \( K^{e_2} \), where \( \{e_1, e_2\} \) is an orthogonal frame of normal bundle of surface. Therefore, a ruled spacelike surface is developable iff \( \{\alpha', W, W'\} \) is linearly dependent.

3. Similarly the results on the surfaces in \( \mathbb{R}^4 \) (see [14]), it is easy to show that if a spacelike surface \( M \) is planar and the causal character of its ellipse curvature (see [9]) is invariant then \( M \) is a ruled developable surface.

Lane [13] showed that if a ruled surface in \( \mathbb{R}^4 \) is minimal then it is contained in a hyperplane and of course it is either plane or helicoid. We have the same results for the maximal ruled spacelike surfaces in \( \mathbb{R}_1^4 \). That means a ruled spacelike surface in \( \mathbb{R}_1^4 \) is maximal if and only if it is maximal in a spacelike hyperplane.

### 3 Bi-normal Fields on Spacelike Surfaces of Revolution

Let \( C \) be a spacelike curve in \( \text{span}\{e_1, e_2, e_4\} \) parametrized by arc-length,
\[ z(u) = (f(u), g(u), 0, \rho(u)), \quad \rho(u) > 0, \quad u \in I. \]
The orbit of \( C \) under the action of the orthogonal transformations of \( \mathbb{R}^4 \) leaving the spacelike plane \( Oxy \),

\[
A_s = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & \cosh v & \sinh v \\
0 & 0 & \sinh v & \cosh v \\
\end{bmatrix}, \ v \in \mathbb{R},
\]

is a surface given by

\[
[RH] \quad X(u, v) = (f(u), g(u), \rho(u) \sinh v, \rho(u) \cosh v), \ u \in I, \ v \in \mathbb{R}. \quad (13)
\]

The coefficients of the first fundamental form of \([RH]\) are

\[
g_{11} = (f'(u))^2 + (g'(u))^2 - (\rho'(u))^2 = 1, \ g_{12} = 0, \ g_{22} = (\rho(u))^2 > 0.
\]

It follows that \([RH]\) is a spacelike surface, which is called the spacelike surface of revolution of hyperbolic type in \( \mathbb{R}^4 \). From now on we always assume that \( f' \neq 0, g' \neq 0 \) and \( \rho' \neq 0 \).

**Proposition 3.1.** Suppose that \( f'g'' - f''g' \neq 0 \), we then have:

(a) \([RH]\) admits exactly two bi-normal fields and its asymptotic fields are orthogonal,

(b) There exists only one normal field \( \nu \) satisfying that \([RH]\) is \( \nu \)-umbilic.

**Proof.** (a) Let \( n = (n_1, n_2, n_3, n_4) \) be a normal field on \( M \), we have

\[
\langle X_u, n \rangle = 0, \ \langle X_v, n \rangle = 0.
\]

That means

\[
n_1 f' + n_2 g' + n_3 \rho' \sinh v - n_4 \rho \cosh v = 0, \quad \rho(n_3 \cosh v - n_4 \sinh v) = 0. \quad (14)
\]

Since (14),

\[
b_{12}^n = \langle X_{uv}, n \rangle = \rho' (n_3 \cosh v - n_4 \sinh v) = 0.
\]

Therefore,

\[
\det(S^n) = \frac{b_{11}^n b_{22}^n}{\rho^2}, \quad (15)
\]

where \( b_{ij}^n \) are the coefficients of the second fundamental form associated with \( n \) of \([RH]\).

On the other hand we have

\[
\langle X_u, X_u \rangle = 1 \Rightarrow \langle X_{uu}, X_u \rangle = 0, \quad \langle X_u, X_v \rangle = 0 \Rightarrow \langle X_{uv}, X_v \rangle = -\langle X_u, X_{uv} \rangle = 0.
\]
So, \{X_u, X_v, X_{uu}\} is linearly independent. Therefore, \(b_1^n = 0\) if and only if \(n\) is parallel to \(B_1 = X_u \wedge X_v \wedge X_{uu}\).

It is easy to show that \(b_2^n = 0\) if and only if \(n\) is parallel to \(B_2 = (-g', f', 0, 0)\). \(X_v\) then is asymptotic field associated with \(B_2\).

Since \(f'g'' - f''g' \neq 0\), \(B_1, B_2\) are linearly independent. So, [RH] admits exactly two bi-normal fields.

(b) Using base \{\(X_u, X_v\)\} for tangent planes of [RH], we have

\[
S^{B_1} = \begin{bmatrix} 0 & 0 \\ 0 & -f'g'' + f''g' \end{bmatrix}, \quad S^{B_2} = \begin{bmatrix} f'g'' - f''g' & 0 \\ 0 & 0 \end{bmatrix}.
\]

Therefore, [RH] is \(\nu\)-umbilic, where \(\nu = B_1 - B_2\). Remark 1.8 shows that the normal curvature of [RH] identifies zero, [RH] has two orthogonal asymptotic fields everywhere, and [RH] is semi-umbilic.

**Remark 3.2.**

(a) If \(f'g'' - f''g' = 0\) then \(M\) is contained in a hyperplane.

(b) It is similar to the spacelike surfaces of revolution of elliptic type.

### 4  Bi-normal Fields on The Rotational Spacelike Surfaces Whose Meridians Lie in Two-dimension planes

Romero Fuster et. al [17] showed that there always an open region of a generic, compact 2-manifold in \(\mathbb{R}^4\) all whose points admit at least one bi-normal direction and at most \(n\) of them. This result is not true in the general case. This section gives a class of spacelike surfaces whose points can admit none, one, two or infinite bi-normal directions. It is similar to them on \(\mathbb{R}^4\).

Let \(C\) be a spacelike curve contained in \(\text{span}\{e_1, e_3\}\) and parametrized by

\[
r(u) = (f(u), 0, g(u), 0), \quad u \in I,
\]

and

\[
A = \begin{bmatrix} \cos \alpha v & -\sin \alpha v & 0 & 0 \\ \sin \alpha v & \cos \alpha v & 0 & 0 \\ 0 & 0 & \cosh \beta v & \sinh \beta v \\ 0 & 0 & \sinh \beta v & \cosh \beta v \end{bmatrix}, \quad v \in [0, 2\pi),
\]
such that

\[ \alpha^2 f^2(u) - \beta^2 g^2(u) > 0, \]

be a orthogonal transformations of \( \mathbb{R}^4 \), where \( u \in J \subset \mathbb{R} \) and \( \alpha, \beta \) are positive constants.

The orbit of \( C \) under the action of the orthogonal transformations \( A \) is a surface \([RS]\) given by

\[ X(u, v) = (f(u) \cos \alpha v, f(u) \sin \alpha v, g(u) \cosh \beta v, g(u) \sinh \beta v). \quad (16) \]

The coefficients of the first fundamental form of \([RS]\) are

\[ g_{11} = (f')^2 + (g')^2 > 0, \quad g_{12} = 0, \quad g_{22} = \alpha^2 f^2 - \beta^2 g^2 > 0. \]

That means \([RS]\) is spacelike. It is called rotational spacelike surface whose meridians lie in two-dimension planes of type I.

Choosing \( \{n_1, n_2\} \) is an orthonormal frame field on \([RS]\), where

\[
\begin{align*}
n_1 &= \frac{1}{\sqrt{(f')^2 + (g')^2}} (g' \cos \alpha v, g' \sin \alpha v, -f' \cosh \beta v, -f' \sinh \beta v), \\
n_2 &= \frac{1}{\sqrt{\alpha^2 f^2 - \beta^2 g^2}} (-\beta g \sin \alpha v, \beta g \cos \alpha v, \alpha f \sinh \beta v, \alpha f \cosh \beta v),
\end{align*}
\]

then the coefficients of the second fundamental form associated to \( n_1 \) and \( n_2 \) are defined by

\[
\begin{align*}
b_{11}^{n_1} &= \frac{f'' - f' g''}{\sqrt{(f')^2 + (g')^2}}, \quad b_{12}^{n_1} = 0, \quad b_{22}^{n_1} = -\frac{\beta^2 f' g + \alpha^2 f g'}{\sqrt{(f')^2 + (g')^2}}, \\
&\quad b_{11}^{n_2} = 0, \quad b_{12}^{n_2} = \frac{\alpha \beta (f g - f g')}{\sqrt{\alpha^2 f^2 - \beta^2 g^2}}, \quad b_{22}^{n_2} = 0,
\end{align*}
\]

respectively.

Let \( B \) be a normal field on \([RS]\), we have

\[ B = \lambda n_1 + \mu n_2, \]

where \( \lambda, \mu \) are smooth functions on \([RS]\). Then we have

\[ (b^B_{ij}) = \lambda(b_{ij}^{n_1}) + \mu(b_{ij}^{n_2}) = \begin{bmatrix} \lambda b_{11}^{n_1} & \mu b_{12}^{n_1} \\ \mu b_{12}^{n_2} & \lambda b_{22}^{n_2} \end{bmatrix}. \]

So,

\[ K^B = \frac{\lambda^2 b_{11}^{n_1} b_{22}^{n_1} - \mu^2 (b_{12}^{n_2})^2}{((f')^2 + (g')^2) (\alpha^2 f^2 - \beta^2 g^2)}. \]

Therefore,
\(\mathbf{n}_2\) is bi-normal if and only if \(f = cg\), where \(c\) is constant satisfying \(\alpha^2 - c\beta^2 > 0\). Then \(b_{22}^{n_2} = 0\). So, \(\mathbf{n}_1\) is also bi-normal. And it is easy to show that [RS] is a planar. That means [RS] is planar if and only if \(C\) is a line passing through the origin.

\(\mathbf{n}_1\) is bi-normal if and only if either

\[
\alpha^2fg' + \beta^2f'g = 0, \quad (17)
\]

where \(c, c_1\) are constant. In this case, if \(c_1 \neq 0\) then [RS] admits only one bi-normal field that is \(\mathbf{n}_1\). Therefore, [RS] admits only one bi-normal field if and only \(\mathbf{n}_1\) is bi-normal and \(\mathbf{n}_2\) is not bi-normal. Which takes place if and only if (17) is true and \(c_1 \neq 0\).

For example

\[
X(u, v) = (u \cos v, u \sin v, \cosh v, \sinh v), \quad u > 1, \quad v \in (0, 2\pi).
\]

[RS] does not admit any bi-normal field if and only if

\[
-(f''g' - f'g'')(\beta^2f'g + \alpha^2fg') < 0 \quad \text{and} \quad \alpha \beta (f'g - g'f) \neq 0.
\]

For example

\[
X(u, v) = (u^2 \cos v, u^2 \sin v, u \cosh v, u \sinh v), \quad u > 1, \quad v \in (0, 2\pi).
\]

[RS] admits exactly two bi-normal fields if and only if

\[
-(f''g' - f'g'')(\beta^2f'g + \alpha^2fg') > 0 \quad \text{and} \quad \alpha \beta (f'g - g'f) \neq 0.
\]

For example

\[
X(u, v) = (e^{2u} \cos v, e^{2u} \sin v, e^{-u} \cosh v, e^{-u} \sinh v), \quad u > 1, \quad v \in (0, 2\pi).
\]

It is similar to the rotational spacelike surfaces whose meridians lie in two-dimension planes of type II. This result is also true for the rotational surfaces whose meridians lie in two-dimension planes in \(\mathbb{R}^4\).

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