A second-order accurate numerical method for the space-time tempered fractional diffusion-wave equation

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Abstract
This paper focuses on providing the high order algorithms for the space-time tempered fractional diffusion-wave equation. The designed schemes are unconditionally stable and have the global truncation error $O(\tau^2 + h^2)$, being theoretically proved and numerically verified.

Keywords: Space-time tempered fractional diffusion-wave equation; Integro-differential equation; Numerical stability and convergence

1. Introduction
We study a second-order accurate numerical method in both space and time for the integro-differential equation whose prototype is, for $1 < \alpha, \gamma \leq 2, \lambda \geq 0$,

$$\frac{\partial}{\partial t} u(x, t) = I_t^{\gamma - 1, \lambda} \nabla_x^\alpha u(x, t) = \frac{1}{\Gamma(\gamma - 1)} \int_0^t (t - \tau)^{\gamma - 2} e^{-\lambda(t - \tau)} \nabla_x^\alpha u(x, \tau) d\tau, \quad (1.1)$$

with the initial condition $u(x, 0) = u_0(x)$, $x \in \Omega = (a, b)$ and the homogeneous Dirichlet boundary conditions, characterizing the propagation of wave with the tempered power law decay. Here the tempered fractional integral $I_t^{\beta, \lambda}$ with $\beta = \gamma - 1 > 0$ is defined as \[1, 4\]

$$I_t^{\beta, \lambda} u(x, t) = \frac{1}{\Gamma(\beta)} \int_0^t (t - \tau)^{\beta - 1} e^{-\lambda(t - \tau)} u(x, \tau) d\tau, \quad t > 0. \quad (1.2)$$

The Riesz fractional derivative with $\alpha \in (1, 2)$, is defined as \[10\]

$$\nabla_x^\alpha u(x, t) = -\kappa_\alpha (aD_x^\alpha + xD_b^\alpha) u(x, t) \quad \text{with} \quad \kappa_\alpha = \frac{1}{2 \cos(\alpha \pi/2)}, \quad (1.3)$$

$$aD_x^\alpha u(x, t) = \frac{1}{\Gamma(2 - \alpha)} \frac{\partial^2}{\partial x^2} \int_a^x (x - \xi)^{1 - \alpha} u(\xi, t) d\xi, \quad xD_b^\alpha u(x, t) = \frac{1}{\Gamma(2 - \alpha)} \frac{\partial^2}{\partial x^2} \int_x^b (\xi - x)^{1 - \alpha} u(\xi, t) d\xi.$$

It can be noted that, if $\lambda = 0$, (1.1) reduces to the following space-time fractional diffusion-wave equation \[8, 71832 \[24\].

$${}^tD_t^\gamma u(x, t) = \nabla_x^\alpha u(x, t) \quad \text{for} \quad 1 < \alpha, \gamma \leq 2.$$
Nonnenmacher (2002) investigate the physical background and implications of a space-and time-fractional diffusion and wave equations [10]. Recently, the numerical solutions of space-time fractional diffusion-wave equations and space fractional diffusion-wave equations are, respectively, discussed in [9] and [7]. However, it seems that achieving a second-order accurate scheme for (1.1) is not an easy task. This paper focuses on providing effective and highly accurate numerical algorithms for the space-time tempered fractional diffusion-wave equation (1.1). The designed schemes are unconditionally stable and have the global truncation error $O(\tau^2 + h^2)$, being theoretically proved and numerically verified. It can be easily extended to the problems discussed in [3] [4] [21].

The rest of the paper is organized as follows. In the next section, we propose the second-order algorithm to the model. In Sec. 3, we do the detailed theoretical analyses for the stability and convergence with the second order accuracy in both time and space directions for the derived schemes. To verify the theoretical results, especially the convergence orders, the extensive numerical experiments are performed in Sec. 4. The paper is concluded with some remarks in the last section.

2. High order schemes for the space-time tempered fractional diffusion-wave equation

Let the mesh points $x_i = ih$ for $i = 0, 1, \ldots, M$, and $t_n = n\tau$ for $n = 0, 1, \ldots, N$, where $h = b/M$ and $\tau = T/N$ are the uniform space stepsize and time steplength, respectively. Denote $u^n_l$ as the numerical approximation to $u(x_l, t_n)$. Nowadays, there are already several types of high order discretization schemes for the Riemann-Liouville space fractional derivatives [3] [4] [10] [13] [20] [22]. Here, we utilize the second-order formula [3] to approximate the Riesz fractional derivative [12], that is

$$\nabla_x^\alpha u(x, t) \big|_{x=x_i} = -\frac{\kappa_{\alpha}}{\Gamma(4-\alpha)h^\alpha} \sum_{j=1}^{M-1} w_{i,j}^\alpha u(x_j, t) + O(h^2) \quad (2.1)$$

with $i = 1, \ldots, M - 1$, where

$$w_{i,j}^\alpha = \begin{cases} w_{i-j+1}^\alpha, & j < i - 1, \\ w_{i-j}^\alpha + w_{i-j+1}^\alpha, & j = i - 1, \\ 2w_{i-j}^\alpha, & j = i, \\ w_{i-j}^\alpha + w_{i-j+1}^\alpha, & j = i + 1, \\ w_{i-j-1}^\alpha + w_{i-j}^\alpha, & j > i + 1, \end{cases} \quad \text{and} \quad w_{m}^\alpha = \begin{cases} 1, & m = 0, \\ -4 + 2^3 - \alpha, & m = 1, \\ 6 - 2^5 - \alpha + 3^3 - \alpha, & m = 2, \\ (m+1)^3 - \alpha - 4m^3 - \alpha + 6(m-1)^3 - \alpha - 4(m-2)^3 - \alpha + (m-3)^3 - \alpha, & m \geq 3. \end{cases}$$

Further denoting $u^n = \left[ u_0^n, u_1^n, \ldots, u_{M-1}^n \right]^T$, from (2.1), then we obtain

$$\nabla_h^\alpha u^n = -\frac{\kappa_{\alpha}}{\Gamma(4-\alpha)h^\alpha} \sum_{j=1}^{M-1} w_{i,j}^\alpha u_j^n \quad \text{and} \quad \nabla_h^\alpha u^n = -\frac{\kappa_{\alpha}}{\Gamma(4-\alpha)h^\alpha} A_\alpha u^n, \quad (2.2)$$

where the matrix

$$A_\alpha = \left[ \begin{array}{cccccccc} 2w_1^\alpha & w_0^\alpha & w_2^\alpha & w_3^\alpha & \cdots & w_{M-2}^\alpha & w_{M-1}^\alpha \\ w_0^\alpha + w_1^\alpha & 2w_1^\alpha & w_0^\alpha + w_2^\alpha & w_3^\alpha & \cdots & w_{M-2}^\alpha \\ w_0^\alpha & w_1^\alpha & 2w_1^\alpha & w_0^\alpha + w_2^\alpha & \cdots & w_{M-2}^\alpha \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ w_{M-2}^\alpha & w_0^\alpha + w_1^\alpha & \cdots & \cdots & \cdots & w_{M-2}^\alpha \\ w_{M-1}^\alpha & w_0^\alpha + w_1^\alpha & \cdots & \cdots & \cdots & w_{M-2}^\alpha \end{array} \right]. \quad (2.3)$$

We know that the tempered fractional integral [12] has the second-order approximation [4]

$$I_t^{\beta,\lambda} u(x, t) |_{t=t_n} = \frac{1}{\Gamma(\beta)} \int_0^{t_n} (t_n - s)^{\beta - 1} e^{-\lambda(t_n - s)} u(x, s) ds = \tau^\beta \sum_{k=0}^{n} I_k^{\beta} u(x, t_{n-k}) + O(\tau^2), \quad (2.4)$$

where $I_k^{\beta}$ are the coefficients of the Taylor expansions of the generating function

$$I^{\beta} (z) = \left( 1 - \frac{z}{e^{\lambda \tau}} \right)^{-\beta} \left( 1 + \frac{1}{2} \left( 1 - \frac{z}{e^{\lambda \tau}} \right) \right)^{-\beta} = \sum_{k=0}^{\infty} I_k^{\beta} z^k \quad (2.5)$$
with
\[ l_k^\beta = e^{-\lambda_k\tau} \left( \frac{3}{2} \right)^{-\beta} \sum_{m=0}^{\beta} 3^{-m} m^{-\beta} \frac{1}{\Gamma(\gamma - 1)} \int_0^{t_n+\frac{\tau}{2}} (t_n' + \frac{\tau}{2} - \tau)^{2\gamma - 2} \nabla_x^\alpha u(x, \tau) d\tau + f(x, t_n+\frac{\tau}{2}). \]

Without loss of generality, we suppose \( u(\cdot, 1) \) with the zero initial value \( u(\cdot, 0) = 0 \) and add a force term \( f(x, t) \) on the right side of (1.1). Considering (1.1) at the point \((x, t_{n+\frac{\tau}{2}})\), there exists
\[ \frac{\partial}{\partial t} u(x, t_{n+\frac{\tau}{2}}) = \frac{1}{\Gamma(\gamma - 1)} \int_0^{t_n+\frac{\tau}{2}} (t_n' + \frac{\tau}{2} - \tau)^{2\gamma - 2} \nabla_x^\alpha u(x, \tau) d\tau + f(x, t_{n+\frac{\tau}{2}}). \]

According to (2.8) and (2.4) and taking \( \beta = \gamma - 1 \), we can write (2.7) as
\[ u(x, t_{n+1}) - u(x, t_n) = \frac{\tau^\beta}{2} \sum_{k=0}^{\beta} l_k^\beta \nabla_x^\alpha (u(x, t_{n+1-k}) + u(x, t_{n-k})) + \tau f(x, t_{n+\frac{\tau}{2}}) + R_i^{n+1}. \]

with the residual term
\[ |R_i^{n+1}| \leq C_u (\tau^2 + h^2), \]
and \( C_u \) is a positive constant independent of \( \tau \) and \( h \). Then the full discretization of (2.3) has the following form
\[ u_{i+1}^n - u_i^n = \frac{\tau^\gamma}{2} \sum_{k=0}^{\gamma} l_k^\gamma \nabla_x^\alpha (u_{i+\gamma-k}^n + u_{i-k}^n) + \tau f_{i+\frac{\tau}{2}}^{n+1}. \]

### 3. Stability and convergence

In this section, we prove that the scheme (2.11) is unconditionally stable and convergent in discrete \( L^2 \) norm. Denote the grid functions \( u^n = \{u_i^n[0 \leq i \leq M, n \geq 0]\} \) and \( v^n = \{v_i^n[0 \leq i \leq M, n \geq 0]\} \); and
\[ (u^n, v^n) = \sum_{i=1}^{M-1} u_i^n v_i^n, \quad ||u^n|| = (u^n, u^n)^{1/2}. \]

**Lemma 3.1** (2.23). Let \( A_\alpha \) be given in (2.23) with \( 1 < \alpha < 2 \). Then there exists an operator \( A^\alpha \) satisfies
\[ -(A_\alpha u, u) > 0 \quad \text{and} \quad -(A_\alpha u, v) = (A^\alpha u, A^\alpha v). \]

**Lemma 3.2.** Let \( l_k^\beta \) be defined by (2.26) with \( \beta = \gamma - 1 \). Then \( l_k^\beta \geq 0, \forall k \geq 0 \).

**Proof.** According to (2.26) and (2.24), the desired results are obtained. \( \square \)

**Lemma 3.3.** Let \( l_k^\gamma \) be defined by (2.26) with \( \beta = \gamma - 1, 1 < \gamma \leq 2 \). Then for any positive integer \( N \) and real vector \((v_i^0, v_i^1, \ldots, v_i^N) \in \mathbb{R}^{N+1}\), it holds that
\[ \sum_{n=0}^{N} \left( \sum_{k=0}^{n} l_k^\gamma v_i^{n-k} \right) v_i^n \geq 0, \quad i = 1, 2, \ldots, M - 1. \]

**Proof.** By the mathematical induction method, we can prove that
\[ \sum_{n=0}^{N} \left( \sum_{k=0}^{n} l_k^\gamma v_i^{n-k} \right) v_i^n = V_i L^\beta V_i^T, \quad i = 1, 2, \ldots, M - 1, \]
where
\[ V_i = (v_i^0, v_i^1, \ldots, v_i^{N-1}, v_i^N) \]
and the real symmetric matrix

\[
L^\beta = \begin{bmatrix}
n_0 & n_1 & n_2 & \cdots & n_{N-1} \\
n_1 & n_0 & n_1 & \cdots & n_{N-2} \\
n_2 & n_1 & n_0 & \cdots & n_{N-3} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
n_{N-1} & n_{N-2} & n_{N-3} & \cdots & n_0
\end{bmatrix}
\]  
(3.1)

Next we prove that the real symmetric matrix \(L^\beta\) defined in (3.1) is positive semi-definite. With \(J = \sqrt{-1}\), we know that the generating function \([2, \text{p. 12-14}]\) of \(L\) is

\[
f(\beta, x) = \frac{1}{2} \sum_{k=0}^{\infty} n_k e^{Jkx} + \frac{1}{2} \sum_{k=0}^{\infty} n_k e^{-Jkx} = \sum_{k=0}^{\infty} n_k \cos(kx) = \frac{1}{2} \beta \cos(e^{Jx}) + \frac{1}{2} \beta \cos(e^{-Jx})
\]

\[
= \frac{1}{2} \left(1 - \frac{e^{Jx}}{e^{\lambda x}}\right)^{-\beta} \left(1 + \frac{1}{2} \left(1 - \frac{e^{Jx}}{e^{\lambda x}}\right)\right) - \frac{1}{2} \left(1 - \frac{e^{-Jx}}{e^{\lambda x}}\right)^{-\beta} \left(1 + \frac{1}{2} \left(1 - \frac{e^{-Jx}}{e^{\lambda x}}\right)\right) - \beta\). 
\]  
(3.2)

Since \(f(\beta, x)\) is an even function and \(2\pi\)-periodic continuous real-valued functions defined on \([-\pi, \pi]\), we just need to consider its principal value on \([0, \pi]\). Next we prove that \(f(\beta, x)\) defined in (3.2) is nonnegative. Denoting \(d = e^{\lambda x} \geq 1\), we have

\[
\left(1 - \frac{e^{\pm Jx}}{e^{\lambda x}}\right)^{-\beta} = d^\beta \left((d - \cos x)^2 + \sin^2 x\right)^{-\frac{\beta}{2}} \cos^\beta (\theta_1 + \theta_2); \\
\left(1 + \frac{1}{2} \left(1 - \frac{e^{\pm Jx}}{e^{\lambda x}}\right)\right)^{-\beta} = (2d)^\beta \left((3d - \cos x)^2 + \sin^2 x\right)^{-\frac{\beta}{2}} \cos^\beta (\theta_1 + \theta_2);
\]

where

\[
\theta_1 = \arctan(\frac{\sin x}{d - \cos x}) \quad \text{and} \quad \theta_2 = \arctan(\frac{\sin x}{3d - \cos x}).
\]

It yields

\[
f(\beta, x) = (2d^2)^\beta \left((d - \cos x)^2 + \sin^2 x\right)^{-\frac{\beta}{2}} \cos^\beta (\theta_1 + \theta_2).
\]

When \(x = 0\), according to Lemma 3.2 and Eq. (3.2), we have \(f(\beta, x) = \sum_{k=0}^{\infty} n_k^2 \geq 0\). When \(x = \pi\), using (3.2) and (2.5), we have

\[
f(\beta, x) = \sum_{k=0}^{\infty} n_k \cos(k\pi) = \sum_{k=0}^{\infty} n_k \cos k\pi = \left(1 + \frac{1}{e^{\lambda x}}\right)^{-\beta} \left(1 + \frac{1}{2} \left(1 + \frac{1}{e^{\lambda x}}\right)\right) - \beta > 0.
\]

Next we consider \(x \in (0, \pi)\). Using \(\tan(\frac{\pi}{2}) = \frac{\sin x}{1 - \cos x} \geq \frac{\sin x}{3 - \cos x} := \tan \theta_3 \geq 0\) and

\[
0 \leq \tan(\theta_1) = \frac{\sin x}{d - \cos x} \leq \frac{\sin x}{1 - \cos x} = \tan(\frac{\pi}{2} - \frac{x}{2}), \quad 0 \leq \tan(\theta_2) = \frac{\sin x}{3d - \cos x} \leq \frac{\sin x}{3 - \cos x} = \tan(\theta_3),
\]

we get \(0 \leq \theta_1 + \theta_2 \leq \frac{\pi}{2} - \frac{x}{2} + \theta_3 \leq \frac{\pi}{2}\). Hence \(f(\beta, x) \geq 0\) for \(\beta \in [-1, 1]\).

From the Grenander-Szego theorem \([3, \text{p. 13-14}]\), it implies that \(L^\beta\) is a real symmetric positive semi-definite matrix. The proof is completed. \(\square\)

**Theorem 3.1.** The difference scheme (2.11) with \(1 < \alpha, \gamma < 2\) is unconditionally stable.

**Proof.** Let \(\tilde{u}_i^n (i = 0, 1, \ldots, M; n = 0, 1, \ldots, N)\) be the approximate solution of \(u_i^n\), which is the exact solution of the difference scheme (2.11). Putting \(e_i^n = u_i^n - \tilde{u}_i^n\), and denoting \(e^n = [e_0^n, e_1^n, \ldots, e_M^n]\), then from (2.11) we obtain the following perturbation equation

\[
e_i^{n+1} - e_i^n = \frac{\gamma}{2} \sum_{k=0}^{n} \beta_k \nabla h (e_i^{n+1-k} + e_i^{n-k}). \quad (3.3)
\]
Multiplying (3.3) by \( h \left( e_i^{n+1} + e_i^n \right) \) and summing up for \( i \) from 1 to \( M - 1 \), we have

\[
\| e^{n+1} \|^2 - \| e^n \|^2 = \frac{\tau \gamma}{2} \sum_{k=0}^{\gamma} l_k^\beta \left( \nabla^\alpha_h (e^{n+1-k} + e^{n-k}), e^{n+1} + e^n \right).
\]

Summing up for \( n \) from 0 to \( N \) on both sides of the above equation, it yields

\[
\| e^{N+1} \|^2 - \| e^0 \|^2 = \frac{\tau \gamma}{2} \sum_{n=0}^{N} \sum_{k=0}^{\gamma} l_k^\beta \left( \nabla^\alpha_h (e^{n+1-k} + e^{n-k}), e^{n+1} + e^n \right).
\]

(3.4)

According to (2.2) and Lemmas 3.1 3.3 we get

\[
\frac{\tau \gamma}{2} \sum_{n=0}^{N} \sum_{k=0}^{\gamma} l_k^\beta \left( \nabla^\alpha_h (e^{n+1-k} + e^{n-k}), e^{n+1} + e^n \right) = \frac{|\kappa|}{2h^\alpha} \sum_{n=0}^{N} \sum_{k=0}^{\gamma} l_k^\beta \left( A \left( e^{n+1-k} + e^{n-k} \right), e^{n+1} + e^n \right).
\]

(3.5)

Using (3.4) and (3.5), for any positive integer \( N \), it yields \( \| e^N \| \leq \| e^0 \| \). The proof is completed.

\( \square \)

**Theorem 3.2.** Let \( u(x_i, t_n) \) be the exact solution of (1.1) with \( 1 < \alpha, \gamma \leq 2 \), and \( u_i^n \) the solution of the finite difference scheme (2.11). Then

\[
\| u(x_i, t_n) - u_i^n \|_2 \leq 2C_u b^2 T (r^2 + h^2), \quad i = 0, 1, \ldots, M; \quad n = 0, 1, \ldots, N,
\]

where \( C_u \) is defined by (2.10) and \( (x_i, t_n) \in (0, b) \times (0, T) \) with \( N \tau \leq T \).

**Proof.** Denote \( e_i^n = u(x_i, t_n) - u_i^n \) and \( e^n = [e_0^n, e_1^n, \ldots, e_M^n]^T \). Subtracting (2.11) from (2.9) and using \( e^0 = 0 \), we obtain

\[
e_i^{n+1} - e_i^n = \frac{\tau \gamma}{2} \sum_{k=0}^{\gamma} l_k^\beta \left( \nabla^\alpha_h (e_i^{n+1-k} + e_i^{n-k}) + R_i^{n+1} \right)
\]

(3.6)

Multiplying (3.3) by \( h \left( e_i^{n+1} + e_i^n \right) \) and summing up for \( i \) from 1 to \( M - 1 \), we have

\[
\| e^{n+1} \|^2 - \| e^n \|^2 = \frac{\tau \gamma}{2} \sum_{k=0}^{\gamma} l_k^\beta \left( \nabla^\alpha_h (e^{n+1-k} + e^{n-k}), e^{n+1} + e^n \right) + \left( R_i^{n+1}, e^{n+1} + e^n \right).
\]

Replacing \( n \) with \( s \), there exists

\[
\| e^{s+1} \|^2 - \| e^s \|^2 = \frac{\tau \gamma}{2} \sum_{j=0}^{s} l_j^\beta \left( \nabla^\alpha_h (e^{s+1-j} + e^{s-j}), e^{s+1} + e^s \right) + \left( R^{s+1}, e^{s+1} + e^s \right).
\]

Summing up for \( s \) from 0 to \( n \) and using (3.3), there exists

\[
\| e^{n+1} \|^2 = \frac{\tau \gamma}{2} \sum_{s=0}^{n} \sum_{j=0}^{s} l_j^\beta \left( \nabla^\alpha_h (e^{s+1-j} + e^{s-j}), e^{s+1} + e^s \right) + \sum_{s=0}^{n} \left( R^{n+1}, e^{s+1} + e^s \right) \leq \sum_{s=0}^{n} \left( R^{n+1}, e^{s+1} + e^s \right).
\]
Using (2.10) and above inequality and the Cauchy-Schwarz inequality, it yields

$$
\|e^{n+1}\|^2 \leq h \sum_{s=0}^{n} \sum_{i=1}^{M-1} |R_{i}^{s+1} \cdot (|e_{i}^{s+1}| + |e_{i}^{s}|) \\
\leq C_{u}T(\tau^2 + h^2) \sum_{i=1}^{M-1} h (|e_{i}^{s+1}| + |e_{i}^{s}|) \\
\leq C_{u}T(\tau^2 + h^2) \left( \sum_{i=1}^{M-1} \sqrt{h^2} \right)^{1/2} \left[ \left( \sum_{i=1}^{M-1} \sqrt{h}\epsilon_{i}^{s+1} \right)^2 + \left( \sum_{i=1}^{M-1} \sqrt{h}\epsilon_{i}^{s} \right)^2 \right]^{1/2} \\
\leq C_{u}b^{1/2}T(\tau^2 + h^2) \left( \|e^{s+1}\| + \|e^{s}\| \right) \\
\leq 2\sigma C_{u}b^{1/2}T(\tau^2 + h^2),
$$

where $\sigma = \max_{0 \leq s \leq n+1} ||e^{s}||$. Taking the maximum over $n$ on both sides of above equation, there exists $\sigma^2 \leq 2\sigma C_{u}b^{1/2}T(\tau^2 + h^2)$, which leads to $\sigma \leq 2C_{u}b^{1/2}T(\tau^2 + h^2)$. Hence

$$
\|e^{n}\| \leq \max_{0 \leq s \leq n+1} ||e^{s}|| \leq 2C_{u}b^{1/2}T(\tau^2 + h^2).
$$

The proof is completed.

4. Numerical Results

Consider the integro-differential equation (1.1) on a finite domain $0 < x < 1$, $0 < t \leq 1/2$. Without loss of generality, we add a force term $f(x, t)$ on the right side of (1.1). Then the forcing function is

$$
f(x, t) = (3e^{-\lambda t} - \lambda e^{-\lambda t^3}) x^2(x - 1)^2 + \frac{\Gamma(4)}{2\Gamma(3+\gamma) \cos(\alpha\pi/2)} e^{-\lambda t^2+\gamma} \\
\times \left[ \frac{x^{2-\alpha} + (1-x)^{2-\alpha}}{\Gamma(3-\alpha)} - \frac{x^{3-\alpha} + (1-x)^{3-\alpha}}{\Gamma(4-\alpha)} + 24 \frac{x^{4-\alpha} + (1-x)^{4-\alpha}}{\Gamma(5-\alpha)} \right],
$$

the initial condition $u(x, 0) = 0$, and the boundary conditions $u(0, t) = u(1, t) = 0$. And (1.1) has the exact solution

$$
u(x, t) = e^{-\lambda t^3} x^2(1-x)^2.
$$

Table 1: The maximum errors and convergence orders for (2.11) with $h = \tau$ and $\lambda = 0.1$.

| $\tau$ | $\gamma = 2, \alpha = 1.5$ | Rate | $\gamma = 1.3, \alpha = 1.7$ | Rate | $\gamma = 1.7, \alpha = 1.3$ | Rate |
|-------|----------------|------|----------------|------|----------------|------|
| 1/20  | 5.2886e-05    | 1.91 | 6.352e-06      | 1.96 | 1.2539e-05     | 1.92 |
| 1/40  | 1.4084e-05    | 1.91 | 2.4815e-06     | 1.98 | 3.2206e-05     | 1.96 |
| 1/80  | 3.6352e-06    | 1.95 | 6.2446e-07     | 1.99 | 8.1607e-07     | 1.98 |
| 1/160 | 9.2322e-07    | 1.98 | 6.2446e-07     | 1.99 | 8.1607e-07     | 1.98 |

Table 1 shows the maximum errors at time $T = 1/2$ with $h = \tau$; and the numerical results confirm that the scheme (2.11) has the global truncation error $O(\tau^2 + h^2)$.

5. Conclusion

With numerical experiments and detailed theoretical analysis, we construct the second-order schemes for the space-time tempered fractional diffusion-wave equation. The corresponding algorithms, theoretical and numerical results can also be extended to the problems discussed in [7, 8, 24].

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References

[1] Á. Cartea, D. del-Castillo-Negrete, Fluid limit of the continuous-time random walk with general Lévy jump distribution functions, Phys. Rev. E 76 (2007) 041105.
[2] R.H. Chan, X.Q. Jin, An Introduction to Iterative Toeplitz Solvers, SIAM, 2007.
[3] M.H. Chen, W.H. Deng, Fourth order accurate scheme for the space fractional diffusion equations, SIAM J. Numer. Anal. 52 (2014) 1418-1438.
[4] M.H. Chen, W.H. Deng, Discretized fractional substantial calculus ESAIM: Math. Mod. Numer. Anal. 49 (2015) 373-394.
[5] M.H. Chen, Y.T. Wang, X. Cheng, W.H. Deng, Second-order LOD multigrid method for multidimensional Riesz fractional diffusion equation, BIT. 54 (2014) 623-647.
[6] E. Cuesta, Ch. Lubich, C. Palencia, Convolution quadrature time discretization of fractional diffusion-wave equations, Math. Comput. 75 (2006) 673-696.
[7] K.Y. Deng, M.H. Chen, T.L. Sun, A weighted numerical algorithm for two and three dimensional two-sided space fractional wave equations, Appl. Math. Comput. 257 (2015) 264-273.
[8] Y. Fujita, Cauchy problems of fractional order and stable processes, Japan J. Appl. Math. 7 (1990) 459-476.
[9] M. Garg, P. Manohar, Matrix method for numerical solution of space-time fractional diffusion-wave equations with three space variables, Afr. Mat. 25 (2014) 161-181.
[10] Z.P. Hao, Z.Z. Sun, W.R. Cao, A fourth-order approximation of fractional derivatives with its applications, J. Comput. Phys. 281 (2015) 787-805.
[11] C.C. Ji, Z.Z. Sun, A high-order compact finite difference shcemes for the fractional sub-diffusion equation, J. Sci. Comput. 64 (2015) 959-985.
[12] F. Liu, M. Meerschaert, R. McGough, P. Zhuang, Q. Liu, Numerical methods for solving the multi-term time-fractional wave-diffusion equation, Fract. Calc. Appl. Anal. 16 (2013) 9-25.
[13] Ch. Lubich, Discretized fractional calculus, SIAM J. Math. Anal. 17 (1986) 704-719.
[14] F. Mainardi, Y. Luchko, G. Pagnini, The fundamental solution of the space-time fractional diffusion equation, Fract. Calc. Appl. Anal. 4 (2001) 153-192.
[15] W. McLean, K. Mustapha, A second-order accurate numerical method for a fractional wave-equation, Numer. Math. 105 (2007) 481-510.
[16] R. Metzler, T.F. Nomenmacher, Space-and time-fractional diffusion and wave equations, fractional Fokker-Planck equations, and physical motivation, Chem. Phys. 284 (2002) 67-90.
[17] K. Mustapha, W. Mclean, Superconvergence of a discontinuous galerkin method for fractional diffusion and wave equations, SIAM J. Numer. Anal. 51 (2013) 491-515.
[18] M.D. Ortigueira, Riesz potential operators and inverses via fractional centred derivatives, Int. J. Math. Math. Sci. 62 (2006) 48391.
[19] I. Podlubny, Fractional Differential Equations, New York, 1999.
[20] E. Sousa, C. Li, A weighted finite difference method for the fractional diffusion equation based on the Riemann-Liouville drivative, Appl. Numer. Math. 90 (2015) 22-37.
[21] Sun, Z.Z., Wu, X.N.: A fully discrete difference scheme for a diffusion-wave system. Appl. Numer. Math. 56, 193-209 (2006).
[22] W.Y. Tian, H. Zhou, W.H. Deng, A class of second order difference approximations for solving space fractional diffusion Equations, Math. Comp. 84 (2015) 1703-1727.
[23] P.D. Wang, C.M. Huang, An energy conservative difference shceme for the nonlinear fractional Schrödinger equations, J. Comput. Phys. 293 (2015) 238-251.
[24] J.Y. Yang, J.F. Huang, D.M. Liang, Y.F. Tang, Numerical solution of fractional diffusion-wave equation based on fractional multistep method, Appl. Math. Modell. 38 (2014) 3652-3661.
[25] F.H. Zeng, Second-order stable finite difference schemes for the time-fractional diffusion-wave equation, J. Sci. Comput. 65 (2015) 411-430.