Analytical solution of narrow quantum rings with general Rashba and Dresselhaus spin-orbit couplings

J. M. Lia\textsuperscript{1} and P. I. Tamborenea\textsuperscript{1}

\textsuperscript{1}Departamento de Física and IFIBA, FCEN, Universidad de Buenos Aires, Ciudad Universitaria, Pab. I, C1428EHA Buenos Aires, Argentina

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We solve analytically the energy eigenvalue problem of narrow semiconductor quantum rings with a general spin-orbit term that includes as a special case the Rashba and Dresselhaus interactions acting simultaneously. The eigenstates and eigenenergies of the system are found for arbitrary values of the spin-orbit coupling constants without making use of approximations. The general eigenstates are expressed as products of a scalar Mathieu function and a spinor factor which is periodic or pseudo-periodic on the ring. Our general solution reduces to the previously found solutions for particular combinations of the Rashba and Dresselhaus couplings, like the well-studied cases of Rashba-only and of equal coupling constants.
I. INTRODUCTION

Semiconductor quantum rings (QR) are an elegant example of how fundamental quantum mechanical systems can be realized with current nanofabrication techniques.\textsuperscript{1} Their simple quasi-one-dimensional circular geometry lends itself perfectly to the study of orbital angular momentum of charge carriers. In addition, the spin degree of freedom can be brought into play thanks to the Rashba (RSOI) and Dresselhaus (DSOI) spin-orbit interactions, present and controllable in semiconductor nanostructures. Furthermore, the injection of angular momentum via excitation with twisted light can provide a way to initialize the system in a finite-angular-momentum state.\textsuperscript{2,3} Thus, the stage is set for an interesting angular momentum dynamics to take place, which could be studied and ultimately controlled for its use in quantum information processing. From a fundamental point of view, QR are an excellent scheme for studying spin-interference quantum mechanical effects.\textsuperscript{4–6}

It has been recognized that the simultaneous action of RSOI and DSOI in semiconductor nanostructures can lead to interesting and varied effects.\textsuperscript{7–9} Recently, several authors have investigated this combination of spin-orbit interactions for 2D and 1D quantum rings.\textsuperscript{10–12} Up to now, the necessary electronic structure—eigenvalues and eigenstates of the one-electron Hamiltonian in the presence of both RSOI and DSOI, and eventually also of an external magnetic field—used in transport and optical-properties studies has usually been obtained numerically. The only exception in which the analytic solution has also been found is the case with equal RSOI and DSOI coupling constants.\textsuperscript{7} On the other hand, the problem with RSOI alone in 1D QR has been solved analytically by Frustaglia and Richter\textsuperscript{13} while, to the best of our knowledge, no analytical solution to the problem with DSOI alone has been given in the literature. In this article we provide an analytical solution to the eigenvalue problem of the Hamiltonian of a quasi-1D QR with both RSOI and DSOI of arbitrary intensity.

Actually, the Hamiltonian of the DSOI in quantum wells is not unique since it depends on the orientation of the quantum well relative to the crystal axes.\textsuperscript{14} For this reason, we consider a generalized (linear-in-$k$) form for the spin-orbit coupling in quantum wells. The solution provided here applies to this general problem.

This article is organized as follows. In Section II we describe the quantum-ring system with the general spin-orbit interaction which includes the Rashba and Dresselhaus interactions. In Section III we provide the derivation of the analytical solution to the general spin-orbit problem. In Section IV A we briefly review the solution of the ring without spin-orbit interaction and in
Sections IV B and IV C we recover the known solutions of the ring with RSOI alone and with RSOI and DSOI with equal coupling constants, respectively. In Section V we explore the solution with general RSOI and DSOI acting simultaneously, and in Section VI we provide concluding remarks.

II. RING SYSTEM

We consider a narrow homogeneous semiconductor QR of inner radius \( a \). We assume that the QR is doped with a conduction-band electron which is subject to spin-orbit interaction. We work in the envelope-function approximation and with a Hamiltonian of the form

\[
H = H_0 + H_R + H_D + H_\Delta,
\]

where \( H_0 \) is the envelope-function Hamiltonian without the spin-orbit interactions, \( H_R \) and \( H_D \) are respectively the well-known Rashba and Dresselhaus spin-orbit Hamiltonians

\[
H_R = \alpha(k_x \sigma_y - k_y \sigma_x),
\]

\[
H_D = \beta(k_x \sigma_x - k_y \sigma_y),
\]

and the last term

\[
H_\Delta = (\delta_{xx}k_y + \delta_{yx}k_x)\sigma_x
\]

represents deviations from these two interactions. In the expressions above, \( k_x = -i\partial_x \) and \( k_y = -i\partial_y \) are momentum operators in coordinate space, \( \sigma_x \) and \( \sigma_y \) are the Pauli matrices and \( \alpha, \beta, \delta_{xx} \) and \( \delta_{yx} \) are real constants that depend only on the properties of the system.

We set the QR on the \( xy \)-plane centered at the origin of coordinates and write

\[
H_0 = -\hbar^2\nabla^2/2m^* + V(r),
\]

with \( m^* \) the conduction-band effective mass and \( V(r) \) the confining potential that defines the QR. Switching to cylindrical coordinates and introducing the operators

\[
\partial^\pm = \partial_x \pm i\partial_y = e^{\pm i\phi} \left( \partial_r \pm \frac{i}{r} \partial_\phi \right),
\]

the Hamiltonians \( H_R, H_D \) and \( H_\Delta \) can be recast into the forms

\[
H_R = \alpha(\partial^- S^+ - \partial^+ S^-),
\]

\[
H_D = -i\beta(\partial^- S^+ + \partial^+ S^-),
\]

\[
H_\Delta = (\Delta^*\partial^- - \Delta\partial^+)(S^+ + S^-)
\]
where $\Delta = (\delta_{yx} + i\delta_{xx})/2$ and $S^\pm = \sigma_x \pm i\sigma_y$ are the spin raising (+) and lowering (−) operators.

We now take the limit of very narrow, quasi-one-dimensional QR and, following Ref. [15], assume that the radial-part contributions to the eigenfunctions of $H$, arising from the finite width of the ring, are adequately described by the lowest radial eigenfunction $R_0(r)$ of $H_0$. In line with this assumption, we replace factors $1/r$ and terms involving first-derivatives in $r$, appearing in both the azimuthal part of $H_0$ and the operators $\partial^\pm$, with the quantities $\langle 1/r \rangle_{R_0} \approx 1/a$ and $\langle \partial_r \rangle_{R_0} = \int R_0(r) \partial_r R_0(r) r dr \approx -1/2a$, respectively.

We thus arrive at the following effective one-dimensional ($\phi$-dependent) Hamiltonian:

$$H_{\phi} = -\epsilon_0 \partial_{\phi}^2 + [(\alpha - \Delta)S^- - (\Delta + i\beta)S^+] \frac{e^{i\phi}}{a} \left( i\partial_\phi - \frac{1}{2} \right) + [(\alpha - \Delta^*)S^+ - (\Delta^* - i\beta)S^-] \frac{e^{-i\phi}}{a} \left( i\partial_\phi + \frac{1}{2} \right);$$

where $\epsilon_0 = \hbar^2/2m^*a^2$.

We define the $2\pi$-periodic, anti-hermitian operator:

$$F(\phi) = \frac{i}{a} \left\{ e^{i\phi} [((\alpha - \Delta)S^- - (\Delta + i\beta)S^+] + e^{-i\phi} [((\alpha - \Delta^*)S^+ - (\Delta^* - i\beta)S^-] \right\}$$

in terms of which the Hamiltonian reads

$$H_{\phi} = -\epsilon_0 \partial_{\phi}^2 + F\partial_\phi + \frac{1}{2}(\partial_\phi F).$$

**III. GENERAL ANALYTIC SOLUTION**

Our goal is to solve the spinor eigenvalue problem

$$H_{\phi} \eta(\phi) = E\eta(\phi).$$

We propose a factorised solution of the form $\eta(\phi) = f(\phi)\chi(\phi)$, where $f$ is a complex-valued scalar function and $\chi(\phi)$ a complex-valued spinor, both to be determined. Inserting the proposed solution into Eq. (13), we get

$$\epsilon_0 (f''\chi + f'\chi') + (2\epsilon_0 f' - F\chi)f' - Ff\chi' + \left( -\frac{F'}{2} + E \right) f\chi = 0,$$

where the primes denote derivatives with respect to $\phi$. As we have imposed no restriction on the form or properties of the factors $f(\phi)$ and $\chi(\phi)$, aside from the basic requirement of being smooth functions of $\phi$, we can conveniently pick $\chi$ from among the solutions to the equation

$$2\epsilon_0 \chi' - F(\phi)\chi = 0.$$
Assuming that $\chi(\phi)$ is not identically zero, this choice reduces Eq. (14) to a differential equation for $f(\phi)$ alone:

$$
\epsilon_0 f'' + \left(-\frac{|\Gamma|}{2a^2\epsilon_0} \cos(2\phi + \phi_G) + \frac{|\alpha - \Delta|^2 + |\Delta + i\beta|^2}{4a^2\epsilon_0} + E\right) f = 0
$$

(16)

where $\Gamma = (\alpha - \Delta)(\Delta + i\beta)$ and $\phi_G = \text{Arg } \Gamma$.

It is important to bear in mind that the separation of Eq. (13) into an equation for each of the factors in $\eta(\phi) = f(\phi)\chi(\phi)$ does not indicate that any product of solutions to Eqs. (15) and (16) form an eigenstate of $H_{\phi}$. Indeed, both factors $f(\phi)$ and $\chi(\phi)$ are still related through the energy eigenvalue $E$, which enters Eq. (16) as a parameter with no restriction other than being real, and is ultimately determined by imposing the condition that its associated eigenstate $\eta(\phi)$ be single-valued on $\phi$.

Let us study each of the Eqs. (15) and (16) separately. It can be shown (see Appendix A) that, irrespective of the quantities $\alpha, \beta, \Delta$ and $\epsilon_0$, the former always has a set of two pointwise orthonormal solutions $\chi_{\mu s}(\phi)$ which satisfy the pseudo-periodic property

$$
\chi_{\mu s}(\phi + 2\pi) = e^{i2\pi s\mu}\chi_{\mu s}(\phi),
$$

(17)

where $s = \pm 1$, $0 \leq \mu \leq 1/2$ and $e^{i2\pi \mu}$ is a characteristic Floquet multiplier of Eq. (15). The latter, in turn, can be recast into the general form of the well-known and extensively studied Mathieu equation

$$
(p - 2q \cos(2\phi)) f + f'' = 0,
$$

(18)

by applying the translation $2\phi + \phi_G \rightarrow 2\phi$ and defining the dimensionless parameters

$$
2q = \frac{|\Gamma|}{2a^2\epsilon_0} = \frac{|(\alpha - \Delta)(\Delta + i\beta)|}{2a^2\epsilon_0^2},
$$

$$
p = \frac{|\alpha - \Delta|^2 + |\Delta + i\beta|^2}{4a^2\epsilon_0^2} + \frac{E}{\epsilon_0}.
$$

(19)

It can be seen, on the one hand, that parameter $q$ is always real and that it is completely determined by the quantities $\epsilon_0, \alpha, \beta$ and $\Delta$ that define $H_{\phi}$. On the other hand, the parameter $p$, unlike $q$, depends on the energy and therefore it can be chosen freely, provided that it remains a real quantity.

This property is important since, with recourse to Mathieu’s equation theory, it can be shown that if $q \in \mathbb{R}$ and $\nu$ is chosen real, then there exist a real $p(\nu; q)$ and a solution $f_{\nu}(\phi; q)$ to Eq. (18) associated with it that satisfies the pseudo-periodic property

$$
f_{\nu}(\phi + 2\pi; q) = e^{2\pi i \nu} f_{\nu}(\phi; q).
$$

(20)
This freedom in choosing $\nu$ suggests that any single-valued (i.e., periodic) eigenstate of $H_\phi$ may be assembled from a pseudo-periodic spinor $\chi_{\mu s}(\phi)$ and a Mathieu function $f_\nu(\phi; q)$ by conveniently choosing the latter so that their product $f_\nu(\phi + \phi_T/2; q)\chi_{\mu s}(\phi)$ satisfies

$$f_\nu\left(\phi + \frac{\phi_T}{2} + 2\pi; q\right)\chi_{\mu s}(\phi + 2\pi) = \left[e^{2\pi i \nu}f_\nu\left(\phi + \frac{\phi_T}{2}; q\right)\right]\left[e^{2\pi i s\mu}\chi_{\mu s}(\phi)\right] = f_\nu\left(\phi + \frac{\phi_T}{2}; q\right)\chi_{\mu s}(\phi).$$  \hspace{1cm} (21)

This requirement can be met by picking $\nu = -s(\mu - m)$, with $m \in \mathbb{Z}$. The integer $m$ takes into account the fact that only the fractional part $-s\mu$ of the Floquet exponent $\nu$ is unique, since adding an integer to it leaves its corresponding Floquet multiplier invariant.

The relation between $\nu$ and $\mu$ thus defines a set of Mathieu functions $f_{-s\mu + sm}(\phi; q)$ which can be shown to be orthonormal (see Appendix B) on $0 \leq \phi \leq 2\pi$. It also determines the energy spectrum $E/\epsilon_0$ in terms of $\epsilon_0$ and the SO coupling constants through the associated set of values for the parameter $p$, $p(-s\mu + sm; q)$.

In order to give a concrete expression $\eta_{\mu s, m}(\phi)$, we separate the pure periodic case where $\mu = 0$ for which the solutions to Mathieu’s equation can be chosen to have well-defined parity with respect to $\phi$, from the pseudo-periodic ones where $\mu \neq 0$. We thus write, for the former

$$\eta_{\pm, m}(\phi; q) = f_m\left(\phi + \frac{\phi_T}{2}\right)\chi_{\pm}(\phi), \hspace{1cm} (22)$$

where $\chi_{\pm}(\phi)$ are two orthonormal solutions to Eq. [15] and

$$f_m\left(\phi + \frac{\phi_T}{2}; q\right) = \frac{1}{\sqrt{\pi}}\begin{cases} ce_m\left(\phi + \frac{\phi_T}{2}; q\right) & m \geq 0 \\ -i se_{-m}\left(\phi + \frac{\phi_T}{2}; q\right) & m < 0, \end{cases} \hspace{1cm} (23)$$

with $ce_m(\phi; q)$ and $se_{-m}(\phi; q)$ the even and odd Mathieu functions of integer order, respectively. The expressions for the latter cases are, in turn,

$$\eta_{\mu \pm, m}(\phi) = \frac{1}{\sqrt{2\pi}} me_{\mp(\mu - m)}\left(\phi + \frac{\phi_T}{2}; q\right)\chi_{\mu \pm}(\phi). \hspace{1cm} (24)$$

In both the periodic and pseudo-periodic cases, the normalization of each eigenstate depends only on the scalar factor $f_{-s\mu + sm}(\phi; q)$, as it can be seen by computing the product $\eta_{\mu s, m}(\phi)\dagger\eta_{\mu s, m}(\phi)$ and recalling that the spinors $\chi_{\mu s}(\phi)$ are pointwise orthonormal.

Closed analytical expressions for the spinors $\chi_{\pm\mu}(\phi)$ and the energy spectrum $E/\epsilon_0$ can only be obtained for a handful of special cases, some of which have already been completely\cite{13} or partially\cite{17} solved. Nevertheless, expansions of $E/\epsilon_0$ in powers of $q$ are known\cite{10,18,19} for both the periodic and
pseudo-periodic cases. For brevity, we only reproduce here the first few terms given in Ref. [17] for the latter case that approximate the spectrum when $|q|$ is small compared to unity.

$$E_m(\mu, q) = -\frac{\alpha^2 + \beta^2}{4\epsilon_0 a^2} + \epsilon_0 (\mu - m)^2 + \frac{\epsilon_0 q^2}{2(\mu - m)^2 - 2} + O(q^4). \quad (25)$$

It is worth noticing that although neither of these terms have singularities when $\mu \neq 0$, this series may not be suitable for numerical estimations of the spectrum when $\mu \to 0$, as it converges slowly even for small $|q|$. In those cases, other and more accurate methods are available.[16][17][19]

It stems from the expansion in Eq. (25) that, at least to fourth order in $|q|$, the spectrum does not depend on the sign of the quantity $\mu - m$. This remarkable property actually holds to any order in $|q|$ and implies that the eigenstates $\eta_{\mu\pm,m}(\phi; q)$ of $H_{\phi}$ in the cases for which $0 < \mu$ are degenerate. Similarly, in the case $\mu = 0$ the eigenstates $\eta_{\pm,m}(\phi; q)$ share the same energy, since the spectrum is fundamentally independent of the spinor $\chi_{\pm}(\phi)$. In every case, this degeneracy is the two-fold Kramers’ degeneracy that arises from the time-reversal symmetry of the total Hamiltonian $H$.[20]

Furthermore, it can be shown by direct computation that, up to a constant phase, the eigenstates $\eta_{\mu+,m}(\phi; q)$ and $\eta_{-m}(\phi; q)$ are one the time-reversed state of the other.

In the next Section we revisit some of the special cases in which analytical expressions for either the eigenstate $\eta_{\mu\pm,m}(\phi; q)$ or the spinor part $\chi_{\mu s}(\phi)$ can be found. It is our purpose to show that the general method developed above effectively reproduces these well-known results.

IV. SPECIAL CASES

A. Electron in the conduction band without SO

In this case, the SO interaction is completely absent. The Hamiltonian that describes the electron in the conduction band therefore reduces to

$$H_0 = \frac{\epsilon_0}{\hbar^2} L_z^2 = -\epsilon_0 \partial_{\phi}^2; \quad (26)$$

which can be obtained by setting $\alpha = \beta = \Delta = 0$ in $H_{\phi}$. Eigenfunctions of $H_0$ that are single-valued on the ring can be readily obtained from the rightmost equality and are of the form $e^{im\phi} \chi$, where $m \in \mathbb{Z}$ and $\chi$ is a constant spinor, as $H_0$ is independent of spin. The scalar factors $e^{im\phi}$ in this case are also periodic solutions to Mathieu’s equation.[16]

It can also be seen that the operator $F(\phi)$ vanishes in this case and that Eq. (15) is reduced to $\chi' = 0$. Solutions to this equation are constant spinors which can be thought of as $2\pi$-periodic
functions of $\phi$ and therefore correspond to the case for which $\mu = 0$. This result is thus consistent with the periodicity of the eigenstates of $H_0$.

The spectrum of $H_0$ can be readily obtained and takes the form $\epsilon_0 m^2$, which coincides with the zeroth-order term in $q$ in the expansion [25].

B. Rashba only

In this case, only the Rashba interaction is considered and the corresponding Hamiltonian is obtained by setting $\beta = \Delta = 0$ in $H_0$. In a previous work [13] it has been shown the eigenvalue problem [13] is exactly solvable and that the eigenvectors have, in the basis of eigenstates of $\sigma_z$, the general structure,

$$\eta_m(\phi) = e^{im\phi} \begin{pmatrix} c^\uparrow \\ c_\downarrow e^{i\phi} \end{pmatrix},$$  \hspace{1cm} (27)

with $m \in \mathbb{Z}$ and $c^\uparrow, c_\downarrow \in \mathbb{C}$ constants dependent on the parameters of the system. This eigenstate can be decomposed into a product of a scalar function and a spinor, both pseudo-periodic in $\phi$, by rewriting it as follows

$$\eta_m(\phi) = e^{i(m-\mu)\phi} \begin{pmatrix} \chi^\uparrow \\ \chi_\downarrow e^{i\phi} \end{pmatrix},$$  \hspace{1cm} (28)

with $\mu \in \mathbb{R}$ to be determined. Since the parameter $q$ vanishes in this case, the factor $e^{i(m-\mu)\phi}$ can be seen to be a solution of Eq. (18) provided that

$$(m - \mu)^2 = \frac{E}{\epsilon_0} + \frac{\alpha^2}{4\epsilon_0^2 a^2}. \hspace{1cm} (29)$$

Inserting now the spinor part into Eq. (15) we get

$$\frac{c^\uparrow}{c_\downarrow} = \frac{\alpha}{2\epsilon_0 a \mu} = \frac{2\epsilon_0 a (\mu + 1)}{\alpha} \hspace{1cm} (30)$$

which can be solved for $\mu$ to yield

$$\mu_{\pm} = -\frac{1}{2} \pm \frac{1}{2} \sqrt{1 + \frac{\alpha^2}{\epsilon_0^2 a^2}}. \hspace{1cm} (31)$$

These exponents can be rewritten as $\mu_{\pm} = \pm \mu + n_{\pm}$ in terms of a characteristic exponent $\mu$, which satisfies $|\mu| \leq 1/2$, and a pair of integers $n_{\pm}$, if the former and the integer $n_+$ are defined through the equality

$$\mu_+ - \mu_- = \sqrt{1 + \frac{\alpha^2}{\epsilon_0^2 a^2}} = 2n_+ + 1 + 2\mu. \hspace{1cm} (32)$$
Equating the above expression to (29) and rearranging terms, the energy spectrum is obtained

\[ E = \left( m + \frac{1}{2} \right)^2 \mp \left( m + \frac{1}{2} \right) \sqrt{1 + \frac{\alpha^2}{\epsilon_0^2} + \frac{1}{4}}. \] (33)

Notice that this expression can also be obtained from the first two terms of the expansion given in Eq. (25) which are independent of \(|q|\) and correspond to the exact form of the spectrum in this limit \(q = 0\).

The constants \(\epsilon_0\) and \(\alpha^2/\epsilon_0^2 a^2\) can be seen to correspond respectively to \(\hbar \omega_0/2\) and \(Q_R\) as defined in Ref. [13]. Moreover, \(m\) can be decomposed into \(m = \lambda n\), with \(\lambda = \pm 1\) and \(n \in \mathbb{N}_0\), and the constants \(c_\uparrow\) and \(c_\downarrow\) can be seen to depend, through \(\mu\), only on the parameters of the system. In turn, the factor \(\mp (m + 1/2)\) in Eq. (33) can be rewritten as \(s|m + 1/2|\), where \(s = \pm 1\).

Finally, it is worth noting that the same steps can be taken to arrive at a similar solution in the pure Dresselhaus case \(\alpha = 0 \neq \beta\).

C. Case \(|\alpha| = |\beta|\)

It has been shown in a previous work\cite{7} that, in each of these cases \((\alpha = \pm \beta)\), a conserved quantity appears which is associated to an equilibrium orientation of the spin with respect to the axis of the ring. These quantities can be used to obtain analytical expressions for the spinors \(\chi_{\mu\pm}(\phi)\) in both cases.

In what follows, we derive these expressions and from their periodicity we deduce the corresponding Mathieu functions of the eigenstates of \(H_\phi\). For brevity and clarity, we concentrate on the case \(\alpha = \beta\), but there is actually no restriction to applying the same procedure to the \(\alpha = -\beta\) case as well.

We begin by computing the commutator

\[
\left[ \alpha S^- - i\beta S^+, \alpha S^+ + i\beta S^- \right] = (\alpha^2 - \beta^2),
\] (34)

which vanishes when \(|\alpha| = |\beta|\). This shows that, in those cases, the operators \(\alpha S^- - i\beta S^+\) and its hermitian conjugate can be simultaneously diagonalised. In the basis of eigenstates of the Pauli matrix \(\sigma_z\), \(\{\chi_\uparrow, \chi_\downarrow\}\), their eigenvectors are given by

\[
\xi_\pm = \frac{1}{\sqrt{2}} \left[ \chi_\uparrow \pm e^{i\pi/4} \chi_\downarrow \right],
\] (35)

and the eigenvalues of \(S^- - iS^+\) by \(\lambda_\pm = \pm e^{-i\pi/4}\). We thus propose and insert into Eq. (15) a solution of the form

\[ \chi_{\mu\pm}(\phi) = g(\phi) \xi_\pm \] (36)
with $g(\phi)$ is a complex-valued scalar function, and get

$$g'\xi_\pm = \frac{i\alpha}{2\epsilon_0 a} \left[ e^{i\phi}(S^- - iS^+) + e^{-i\phi}(S^+ + iS^-) \right] g\xi_\pm = \pm i\alpha \epsilon_0 \cos(\phi - \phi_0) g\xi_\pm$$

(37)

where $\phi_0 = \pi/4$. This equation can be readily integrated to yield

$$\chi_\pm(\phi) = \exp \left( \pm \frac{i\alpha}{a\epsilon_0} \sin(\phi - \phi_0) \right) \xi_\pm = \exp \left( \mp \frac{i\alpha}{\sqrt{2}a^2\epsilon_0} (x - y) \right) \xi_\pm$$

(38)

where the last equality is obtained by noting that if $x, y \in \mathbb{R}$ are defined as $x + iy = ae^{i\phi}$, then $a\sin(\phi - \pi/4) = (y - x)/\sqrt{2}$. Written as in Eq. (38), the expressions of $\chi_\pm(\phi)$ can be seen to correspond to those given Ref. [7].

The $2\pi$-periodicity of $\chi_\mu(\phi)$ show that the degenerate states in this are of form given in (22),

$$\eta_{\pm,m}(\phi; q) = f_m \left( \phi + \frac{\pi}{4} \right) \exp \left( \pm \frac{i\alpha}{a\epsilon_0} \sin \left( \phi - \frac{\pi}{4} \right) \right) \frac{1}{\sqrt{2}} \left( \pm e^{i\pi/4} \right)$$

(39)

where $f_m(\phi + \pi/4; q)$ are defined in Eq. (23).

The energy of each eigenstates depends on $q = (\alpha/2\epsilon_0 a)^2$ and the order $m$ of the Mathieu function $f_m(\phi; q)$. For the ground states $m = 0$ in particular, the spectrum can be expanded as a power series in $q$ that, up to the fourth-order, results

$$\frac{E}{\epsilon_0} = -2q - \frac{1}{2} q^2 + \frac{7}{128} q^4 + O(q^6).$$

(40)

V. GENERAL RASHBA+DRESSELHAUS CASE

In this section we drop the term $H_\Delta$ by setting $\Delta = 0$ and concentrate on those cases that correspond to the presence of both RSOI and DSOI with coupling constants of arbitrary strength. We analyze quantitatively the behavior of the eigenspinors $\eta_{\mu,\pm}(\phi; q)$ and their energy spectrum by integrating Eq. (15) for different combinations of the dimensionless parameters $\bar{\alpha} = \alpha/2\epsilon_0 a$ and $\bar{\beta} = \beta/2a\epsilon_0$.

In Figs. 1 and 2 we set $\bar{\alpha} = 1$ and compute for different values of $\bar{\beta}$ the amplitude squared of the projections onto the eigenstates of $\sigma_z$ and the graphical representation of the Bloch vector associated with the solution $\chi_{\mu,\pm}(\phi)$.

In Fig. 3 we study the dependence of the Floquet exponent $\mu$ on $\bar{\beta}$ on them first by setting one parameter to unity and letting the other vary continuously, and then by choosing a range of realistic values for the parameters $\alpha_R = \alpha/\hbar$ and $\beta_D = \beta/\hbar$. 
Figure 1. Behavior of the amplitude squared of the projections of the spinor $\chi_{\mu+}(\phi)$ onto the eigenstates of $\sigma_z$ for different values of $\beta/2a\epsilon_0$, while keeping $\alpha/2a\epsilon_0 = 1$. In all four cases, it can be seen that the spinor is normalized to unity for all $0 \leq \phi \leq 2\pi$. In the special case when $\beta = 0$, the amplitude of each of the components becomes independent of $\phi$, as the exact solution in the pure Rashba case predicts (see Sec. IV B).

Finally, in Fig. 4 we analyze the dependence of the spectrum on the parameters $\alpha_R$ and $\beta_D$ for values in the same range explored in Fig. 3. We also compute the density of the ground and excited states, $|\eta_{\mu+,0}(\phi,q)|^2$ and $|\eta_{\mu+,1}(\phi,q)|^2$, respectively.

VI. CONCLUSION

We studied the energy eigenvalue problem of a charge in narrow, quasi-one-dimensional semiconductor quantum rings in which the effects of spin-orbit interaction are taken into account. We considered the usual expressions for the Rashba and Dresselhaus SOI in this kind of geometry and included an ad-hoc term that takes into account deviations from these two important interactions. We found a factorization of the problem that does not make use of approximations and can be applied to derive expressions for both the eigenstates of the full Hamiltonian and the energy spectrum. In Sec. III we showed that each eigenstate can be written as a product of spinor
Figure 2. Bloch vector representation of the solution $\chi_{\mu^+}(\phi)$ to Eq. (15) for different values of $\alpha$ and $\beta$. In the pure Rashba case $\beta = 0$, the orientation of the vector with respect to the axis of the ring is constant though its direction is not. It is in the case $|\alpha| = |\beta|$, that both its orientation and direction become independent of $\phi$. In these two cases the spin behaves as expected\textsuperscript{7,13} (see Secs. IV B and IV C).

function, which is a Floquet solution to Eq. (15), and a scalar Mathieu function whose Floquet multiplier is chosen conjugate to that of the spinor factor. This relation is important because the real parts of the Floquet exponents associated with this multiplier determine the orders of the Mathieu functions and the spectrum of energies. With recourse to Mathieu’s equation theory, we found that the eigenstates so obtained are at least doubly degenerate. This degeneracy is also present in the special cases discussed in Secs. IV A, IV B and IV C and is accounted for by the time-reversal invariance of the Hamiltonian. Finally, we showed in the aforementioned sections that the obtained expressions of the eigenstates and the energy spectrum reduce to those already known in the literature when appropriate limits are taken.
Figure 3. Left: dependence of $\mu$ on $\bar{\beta}$ while keeping $\bar{\alpha} = 1$. Notice that $\mu$ reaches the special value $\mu = 0$ at the points $\bar{\alpha} = \pm \bar{\beta}$, as it is predicted by the analytical solution to the problem in these cases (see Sec. IV C). The sharp change in the behavior of $\mu$ for values of $\bar{\beta}$ in the range $0.9 \leq |\bar{\beta}| \leq 1$ seems to suggest the existence of another two zeros. This, however, can be ruled out by numerically estimating both minima.

A similar behavior is obtained if $\bar{\beta}$ is kept fixed and $\bar{\alpha}$ is allowed to vary. Right: behavior of $\mu$ as a function of $\alpha_R$ and $\beta_D$ for realistic values of these parameters and for a ring of radius 200 Å with a conduction-band effective mass of $0.1m_0$ where $m_0$ is the bare electron mass.

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Appendix A

Provided that the angular variable $\phi$ is identified with “time”, the anti-hermitian property of the operator $F(\phi)$ allows Eq. (15) to be interpreted as a time-dependent Schrödinger equation

$$i\chi' = \frac{i}{2\epsilon_0} F(\phi)\chi.$$  \hspace{1cm} (A1)

where the hermitian operator $iF(\phi)/2\epsilon_0$ can be thought of as its “Hamiltonian”. It is well-known from quantum mechanics that any solution $\chi(\phi)$ to this equation can be expressed in terms of a constant spinor $\chi(0)$ and an unitary evolution operator $U(\phi)$ that is also a solution to Eq. (15).
Figure 4. Left: density of the degenerate ground state $\eta_{\mu+0}(\phi; q)$ and first excited state $\eta_{\mu+1}(\phi; q)$ for a ring of radius 200 Å, a conduction band effective mass of 0.1$m_0$, with $m_0$ the bare electron mass, and the SO coupling constants $\alpha_R = 8$ nm ps$^{-1}$ and $\beta_D = 14$ nm ps$^{-1}$ ($q \approx 0.033$). Right: the spectrum for the ground and excited states as a function of $\alpha_R$ and $\beta_D$. Notice that, when $\alpha_R = \beta_D = 0$, the energies of the ground and the excited states tend to zero and $\epsilon_0 \approx 0.95$ meV, respectively. These results are consistent with the exact solution (see Sec. IV A).

\[ \chi(\phi) = U(\phi)\chi(0). \]  

(A2)

With recourse to Floquet theory, it can also be shown that, in the cases we are considering, the 2$\pi$-periodicity of $F(\phi)$ endows the evolution operator $U(\phi)$ with the pseudo-periodic property

\[ U(\phi + 2\pi) = U(\phi)U(2\pi), \]  

(A3)

As $U(2\pi)$ is also unitary, its eigenvectors form an orthonormal set. Therefore, a pair of orthonormal pseudo-periodic solutions to Eq. (A1) can be readily constructed by applying the evolution operator $U(\phi)$ to the (constant) eigenvectors of $U(2\pi)$. More specifically, if $\chi_\rho(0)$ is an eigenvector of $U(2\pi)$ with eigenvalue $\rho$, then there exists a solution $\chi_\rho(\phi)$ of the form (A2) that satisfies

\[ \chi_\rho(\phi + 2\pi) = U(\phi + 2\pi)\chi_\rho(0) = U(\phi)U(2\pi)\chi_\rho(0) = \rho U(\phi)\chi_\rho(0) = \rho \chi_\rho(\phi). \]  

(A4)

The unitarity of $U(2\pi)$ implies that all of its eigenvalues satisfy $|\rho| = 1$ and therefore are of the form $e^{i\gamma}$, with $\gamma \in \mathbb{R}$. It can also be shown, by computing the determinant of $U(2\pi)$ through
Jacobi’s formula, that its (two) eigenvalues $\rho_{\pm}$ are related by

$$\rho_+ \rho_- = \det U(2\pi) = \exp \left( \frac{1}{2\epsilon_0} \int_0^{2\pi} \text{Tr} F(\phi') d\phi' \right) = 1,$$

(A5)
since $\text{Tr} S^\pm = 0$ and therefore $\text{Tr} F(\phi) = 0$. This relation, together with $|\rho_{\pm}| = 1$, shows that these eigenvalues are of the form $\rho_{\pm} = e^{\pm 2\pi i(\mu + n_{\pm})}$, with $0 \leq \mu \leq 1/2$ and $n_{\pm} \in \mathbb{Z}$. The quantities $\rho_{\pm}$ are the characteristic Floquet multipliers of the system and $i(\pm \mu + n_{\pm})$ their Floquet exponents.

Finally, it stems from the unitarity of $U(2\pi)$ and the orthonormality of its eigenvectors $\chi_{\rho}(0)$ that the solutions $\chi_\rho(\phi)$ are orthonormal for all $\phi$, since

$$\chi_\rho(\phi)^\dagger \chi_{\rho'}(\phi) = [U(\phi)\chi_{\rho}(0)]^\dagger [U(\phi)\chi_{\rho'}(0)] = \chi_{\rho'}(0)^\dagger U(\phi)\chi_{\rho}(0) = \chi_{\rho}(0)^\dagger \chi_{\rho'}(0) = \delta_{\rho\rho'}. \quad (A6)$$

Appendix B

In what follows, we mention some of the properties of Mathieu functions of both integer and non-integer order that relate directly to their orthonormality in the interval $0 \leq \phi \leq 2\pi$. For proofs and detailed derivations, we refer the reader to Refs. [16]–[18].

It can be shown that, when $q, \phi \in \mathbb{R}$, the even $ce_m(\phi; q)$ ($m \geq 0$) and odd $se_m(\phi; q)$ ($m > 0$) Mathieu functions of integer order are real and satisfy

\[
\begin{align*}
\int_0^{2\pi} ce_m(\phi; q) ce_n(\phi; q) d\phi &= \pi \delta_{mn}, \\
\int_0^{2\pi} se_m(\phi; q) se_n(\phi; q) d\phi &= \pi \delta_{mn}; \\
\int_0^{2\pi} ce_m(\phi; q) se_n(\phi; q) d\phi &= 0; 
\end{align*}
\]

(B1)

where $\delta_{mn}$ is the Kronecker delta.

In turn, it can be shown that, if $\nu, \phi, q \in \mathbb{R}$, Mathieu functions $me_\nu(\phi; q)$ satisfy

$$me_\nu(\phi; q)^* = me_\nu(-\phi; q),$$

(B2)

$$me_\nu(\phi + \pi; q) = e^{i\nu \pi} me_\nu(\phi; q),$$

(B3)

$$\int_0^{\pi} me_{\nu+2n}(-\phi; q) me_{\nu+2m}(\phi; q) d\phi = \pi \delta_{nm};$$

(B4)

where $n, m \in \mathbb{Z}$. These three properties can be applied to write any well-behaved pseudo-periodic function $f_\nu(\phi + \pi) = e^{i\nu \pi} f_\nu(\phi)$ as a series in terms of the $me_\nu(\phi; q)$ [17].

As to how properties (B3) and (B4) change when the interval is extended to $0 \leq \phi \leq 2\pi$, it can be readily seen that the former implies $me_\nu(\phi + 2\pi; q) = e^{2\pi i\nu} me_\nu(\phi; q)$, whereas for the latter one
can show, using the Fourier series of the functions $m_{\nu,\eta}(\phi; q)$, that it takes the form

$$
\int_0^{2\pi} m_{-(\nu+n)}(\phi; q) m_{\nu+m}(\phi; q) d\phi = 2\pi \delta_{mn}.
$$

This relation, together with property (B2), allows the product $m_{-(\nu+n)}(\phi; q) m_{\nu+m}(\phi; q)$ to be interpreted as a density.

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