Probing quantum entanglement, quantum discord, classical correlation and quantum state without disturbing them

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In this paper, we present schemes for a type of one-parameter bipartite quantum states to probe the quantum entanglement, the quantum discord, the classical correlation and the quantum state based on the cavity QED. It is shown that our detection does not influence all these measured quantities. We also discuss how the spontaneous emission introduced by our probe atom influences our detection.

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I. INTRODUCTION

In quantum dynamics, it is important to study whether a given bipartite quantum state is entangled, separable, quantum correlated, or classically correlated. Until now many efforts has been taken in this region. As we know, quantum entanglement is a very useful physical resource in quantum information processing. In recent years, the quantification of entanglement has attracted much more attention [1-8], in particular, entanglement of bipartite pure states and two-qubit mixed states has been obtained good understanding [1-5,8]. However, quantum entanglement is not the only quantum correlation in quantum dynamics. Quantum discord [9,10], first introduced by Ollivier and Zurek [10], can effectively capture the quantum correlation of quantum system. It has been shown that quantum discord can lead to a speedup in some quantum information tasks [11-14]. It is especially worth noting that quantum discord is not consistent with quantum entanglement in general cases [15]. Quantum discord captures quantum correlation even more general than entanglement. Specially separable mixed states, which have no entanglement, have proven to include nonzero quantum discord. Quantum discord is not always larger than quantum entanglement.

However, most interests of quantifying entanglement and quantum discord are focused in a pure mathematical frame [16,17,18]. For example, since quantum entanglement is not an observable in the strict quantum mechanical frame, no directly measurable observable has been found until now, to describe the entanglement of a given arbitrary quantum state, owing to the unphysical quantum operations in the usual entanglement measure, such as the complex conjugation of concurrence [2] and the partial transpose of negativity [1]. In recent years, some interesting methods have been proposed to construct direct observables related to entanglement [19-22], which can be used to precisely measure the entanglement in contrast to entanglement witness [23] and have less observables compared with quantum state tomography [24,25]. However, they require the simultaneous multiple copies of given quantum states, which brings new difficulties to the experiment. Quantum discord as a measure of quantum correlation can only be analytically calculated for some special states [15,16], unless the invariational definition of quantum discord is considered [26]. The key difficulty lies in the analytical calculation of the classical correlation, since quantum discord is defined as the difference between the total correlation (i.e. quantum mutual information) and the classical correlation [9,10]. Although the dynamical behavior and some operational understanding of quantum discord in quantum state evolution has attracted increasing interests recently [27-29], there still exist an open question how one can construct several directly measurable observables related to quantum discord.

In this paper, we propose a scheme to probe the entanglement, discord, and the classical correlation of a type of one-parameter bipartite quantum states based on the cavity QED [30-33]. Although the entanglement of one-parameter quantum state such as Werner state [34], isotropic state and so on [35,36] can be measured based on simple von Neumann measurements without the requirement of simultaneous copies of the state, these measurements usually cover projections on two qubits. However, our measurement is only performed on the probe qubit. In addition, the distinguished advantage of our scheme is that one can probe quantum entanglement, quantum discord and the classical correlation by introducing a probe qubit to interact with the measured systems, but the entanglement, the discord and the classical correlation of the system are not disturbed. In particular, in some cases, one can even realize the non-demolition measurement of the quantum state [37,38]. That is to say, we can probe the quantum information of the quantum state, but the state is not disturbed. The paper is organized as follows. In Sec. II, we present a scheme to probe the quantum entanglement, the quantum discord and the classical correlation, but they are not disturbed in the probe procedure. In Sec. III, we give our another model to demonstrate the non-demolition measurement of our given quantum state. The conclusion is drawn finally.
The Hamiltonian in the interaction picture as,
\[ H_I = \frac{g}{\sqrt{2}} [\sigma^+ a_A + \sigma^- a_A^\dagger + \sigma^+ a_B + \sigma^- a_B^\dagger]. \tag{3} \]

If the initial state of atom C is prepared in \( |g\rangle_C \), the evolution of the joint system of the two cavities and the atom can be given by
\[ \rho(t) = \exp(-iH_I t) (\rho_0 \otimes |g\rangle_C \langle g|) \exp(iH_I t). \tag{4} \]

After a simple calculation, one can find that
\[ \rho_{AB}(t) = \begin{pmatrix} (1-x)\cos^2(gt) & 0 & 0 & 0 \\ 0 & \frac{\sqrt{2}}{2} & \frac{2-3x}{2} & 0 \\ 0 & \frac{2-3x}{2} & \frac{2}{2} & 0 \\ 0 & 0 & 0 & (1-x)\sin^2(gt) \end{pmatrix}, \tag{5} \]
and
\[ \rho_C(t) = \begin{pmatrix} 2(1-x)\sin^2(gt) & 0 \\ 0 & 2(1-x)\cos^2(gt) + 2x - 1 \end{pmatrix}. \tag{6} \]

Next, we will employ Wootters’ concurrence [2] as an entanglement measure, the quantum discord [9,10] as the measure of quantum correlation to discuss their invariability after our detection. Concurrence of a bipartite quantum state \( \rho_{AB} \) is defined as
\[ C(\rho_{AB}) = \max\{0, \lambda_1 - \lambda_2 - \lambda_3 - \lambda_4\}, \tag{7} \]
where the \( \lambda_i \) are the square roots of the eigenvalues of the non-Hermitian matrix \( \rho_{AB} \rho_{AB}^\dagger \) with \( \rho_{AB} = (\sigma_y \otimes \sigma_y)\rho_{AB}^\dagger (\sigma_y \otimes \sigma_y) \) in decreasing order. Substitute \( \rho_{AB}(t) \) and \( \rho_0 \) into eq. (7), one can easily obtain
\[ C(\rho_{AB}(t)) = \max\{0, |2 - 3x| - (1-x)\sin(2gt)| \} \tag{8} \]
and
\[ C(\rho_0) = |2 - 3x|. \tag{9} \]

It is obvious that \( C(\rho_{AB}(t_n)) = C(\rho_0) \) when \( t_n = \frac{n\pi}{2g}, n = 0, 1, 2, \cdots \). At the same time, if one measures \( \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \) on \( \rho_C(t_n) \), one will get
\[ C(\rho_0) = \frac{|3(\sigma_z(t_n)) - 1|}{4}, \tag{10} \]
where \( \langle \sigma_z(t_n) \rangle = \text{Tr}[\rho_C(t_n)\sigma_z] \). That is to say, in the current ideal case (there is no noise), so long as we measure \( \sigma_z \) on \( \rho_C(t_n) \) at \( t_n = \frac{n\pi}{2g} \), we can probe the concurrence of \( \rho_0 \) without disturbing the concurrence of \( \rho_{AB} \). However, when in the realistic scenario, one has to select \( t_1 = \frac{\pi}{2g} \), by which one can reduce the time of interaction of the whole system with the unavoidable noise.

Next, we will show that the quantum and the classical correlations can not be disturbed by our detection.
Quantum discord is used to measure the quantum correlation between two subsystems. For a bipartite quantum system $\rho^{ab}$ with $\rho^a$ ($\rho^b$) denoting the reduced density matrix of subsystem $a$ ($b$), then the quantum discord between subsystems $a$ and $b$ can be defined as follows

$$Q(\rho^{ab}) = I(\rho^{ab}) - C(\rho^{ab}),$$

(11)

where

$$I(\rho^{ab}) = S(\rho^a) + S(\rho^b) - S(\rho^{ab})$$

(12)

is the quantum mutual information and $C(\rho^{ab})$ is the classical correlation between the two subsystems. In particular, the classical correlation is given by

$$C(\rho) = \max_{\{B_k\}}[S(\rho^a) - S(\rho|\{B_k\})],$$

(13)

where $\{B_k\}$ is a set of von Neumann measurements performed on subsystem $b$ locally, $S(\rho|\{B_k\}) = \sum_k p_k S(\rho_k)$ is the quantum conditional entropy, $\rho_k = (I \otimes B_k)\rho(I \otimes B_k)/\text{Tr}(I \otimes B_k)\rho(I \otimes B_k)$ is the conditional density operator corresponding to the outcome labeled by $k$, and $p_k = \text{Tr}(I \otimes B_k)\rho(I \otimes B_k)$. Here $I$ is the identity operator performed on subsystem $a$.

For the bipartite quantum state $\rho_0$ and $\rho_{AB}(t_n)$, one can analytically calculate the quantum and the classical correlations according to Ref. [15]. In order to explicitly show the invariability of quantum discord and classical correlation after our detection, we would like to first consider the density matrix

$$\rho = \begin{pmatrix}
\rho_{11} & 0 & 0 & 0 \\
0 & \rho_{22} & \rho_{23} & 0 \\
0 & \rho_{32} & \rho_{33} & 0 \\
0 & 0 & 0 & \rho_{44}
\end{pmatrix},$$

(14)

where all the entries are real. Based on eq. (12), one can get

$$I(\rho_{AB}(t)) = S(\rho_{AB}^A) + S(\rho_{AB}^B) + \sum_{j=1}^{4} \lambda_j \log_2 \lambda_j,$$

(15)

where

$$S(\rho_{AB}^A) = -[(\rho_{11} + \rho_{22}) \log_2 (\rho_{11} + \rho_{22}) + (\rho_{33} + \rho_{44}) \log_2 (\rho_{33} + \rho_{44})],$$

(16)

$$S(\rho_{AB}^B) = -[(\rho_{11} + \rho_{33}) \log_2 (\rho_{11} + \rho_{33}) + (\rho_{22} + \rho_{44}) \log_2 (\rho_{22} + \rho_{44})],$$

(17)

and

$$\lambda_1 = \rho_{11}, \lambda_2 = \rho_{22} - \rho_{23}, \lambda_3 = \rho_{22} + \rho_{23}, \lambda_4 = \rho_{44}. $$

(18)

The classical correlation $C(\rho_{AB}(t))$ can be given by

$$C(\rho_{AB}(t)) = S(\rho_{AB}^A) - \min_{B_i}[S(\rho_{AB}(t)|\{B_i\})],$$

(19)

where

$$\min_{B_i}[S(\rho_{AB}(t)|\{B_i\})] = \left\{ \begin{array}{ll}
(\rho_{22} + \rho_{33}) \log_2 (\rho_{22} + \rho_{33}) - \rho_{22} \log_2 \rho_{22} - \frac{\theta}{2}(1 - \theta) \log_2 (1 - \theta) + (1 + \theta) \log_2 (1 + \theta)
\end{array} \right.$$

(20)

with $\theta = \sqrt{(\rho_{11} - \rho_{44})^2 + 4\rho_{23}^2}$.

From the above calculation, one can find that $\rho_{11}$ and $\rho_{44}$ are symmetric in eqs. (15-18,20) if $\rho_{22} = \rho_{33}$. In particular, one can also find that the difference between $\rho_0$ and $\rho_{AB}(t_n)$ is the exchange of $\rho_{11}$ and $\rho_{44}$. That is to say, $\rho_0$ and $\rho_{AB}(t_n)$ have the same quantum discord and the same classical correlation. What’s more, since $\langle \sigma_z(t_n) \rangle = 3 - 4x$ from eq. (10) and all the entries of $\rho_0$ given in eq. (1) are the function of $x$, one can conclude that the quantum discord $Q(\rho_0)$ and the classical correlation $C(\rho_0)$ can be obtained by measuring $\sigma_z$ on $\rho_C(t_n)$. Namely, we can detect the quantum and the classical correlations of $\rho_0$ without disturbing them.

Finally, we would like to discuss the influence of the noise. As we know, the decoherence will inevitably happen in $\rho_0$ if the two cavity modes interact with noise no matter whether we introduce the third qubit to probe the system. So we want to emphasize here that our detection has no influence on the concurrence, the quantum discord and the classical correlation of the system. From the viewpoint of noise, we say that the noise due to our detection will have slight influence on the system. In this sense, we only need to consider the noise relevant to the third qubit—the spontaneous emission of atom C. In the interaction picture, the master equation governing the evolution of the “cavity+atom” system can be given by

$$\dot{\rho} = -i[H_I, \rho] + \gamma(2\sigma^- \rho \sigma^+ - \sigma^+ \sigma^- \rho - \rho \sigma^+ \sigma^-),$$

(21)}
where $\gamma$ is the atomic spontaneous emission rate and $H_I$ is defined as eq. (3). The initial state of the cavities is given by eq. (1) and the atom is prepared in the ground state initially. We have numerically solved the master equation. The concurrence $C(\rho_{AB}(t_n))$, the quantum discord $Q(\rho_{AB}(t_n))$ and the classical correlation $C(\rho_{AB}(t_n))$ with and without noise vs. $x$ are plotted in Fig. 2 and $\langle \sigma_z(t_n) \rangle$ with and without noise vs. $x$ are plotted in Fig. 3. We set $\gamma = 0.1g$. From the figure, one can find that the spontaneous emission has slight influence on the entanglement and correlation of $\rho_0$ as well as our detection. In particular, the quantum discord of the original state is influenced more slightly by the small $\gamma$. In addition, we also find a very interesting phenomenon that the existence of $\gamma$ might lead to the increment of quantum discord for some large $x$, such as $x > 0.8$.

III. NON-DEMOLITION MEASUREMENT OF THE QUANTUM STATE

In the previous section, we have presented a model to probe the concurrence and correlation of a type one-parameter quantum state without disturbing them. In fact, in a different framework, one can realize the non-demolition measurement of the quantum state. In this case, we consider two identical two-level atoms A and B which are trapped in an optical cavity, respectively. In particular, the two cavities are arranged to be crossed as is sketched in Fig. 1 (b). We will introduce a third atom C as a probe qubit. One can find that the measurement on the qubit C can reveal the information of the quantum state of the joint system 'A+B' without disturbing it.

The Hamiltonian of our whole system is given by (setting $\hbar = 1$)

$$H = \sum_{i=A,B,C} \omega_i \sigma_i^+ \sigma_i^- + \sum_j v \sigma_j^a \sigma_j^b + \frac{g}{\sqrt{2}} \sum_{j=1,2} (\sigma_j^+ \sigma_j^- + \sigma_j^+ \sigma_j^-),$$

where $\omega_i, i = A, B, C$ and $\omega_c$ are the frequencies of the atomic transition and the cavity modes respectively; $\frac{g}{\sqrt{2}}$ is the coupling constant of the atom and cavity mode; $\sigma_i, i = A, B, C$ denotes the lowering operators of the atom $i$; $a_j$ is the annihilation operator of the $j$th cavity. In addition, we let $\omega_A = \omega_B$ and $\omega_C - \omega_A = g^2/2\delta$ with $\delta = \omega_C - \nu$. If there is not any photons initially in both the cavities, under the large detuning condition, i.e. $\delta \gg \frac{g^2}{\sqrt{2}}$, one can obtain the effective Hamiltonian of the joint system in a proper rotation frame as

$$H_{eff} = \frac{g^2}{2\delta} (\sigma_A^+ \sigma_A^- + \sigma_B^+ \sigma_B^- + \sigma_C^+ \sigma_C^- + \sigma_0^+ \sigma_0^-).$$

Thus, the evolution of the system can be given by the time evolution operator

$$U(t) = \exp(-iH_{eff}t).$$

Now we turn to the original state $\rho_0$ given in eq. (1). After the evolution governed by $U(t)$, one can find that

$$\rho_0 = \sum_{n=0}^{\infty} \rho_{AB}(t_n) \rho_{AB}(t_m) (\sigma_x \otimes \sigma_x) \rho_0 (\sigma_x \otimes \sigma_x),$$

since one can find that $H_{eff}$ have the consistent form with the Hamiltonian of eq. (3). Here $\langle c | c \rangle$ means that the initial state of the probe atom is $| c \rangle$, and $t_m = \frac{5\pi}{g} (m = 1, 2, 3, ...)$ The quantum state of atom C at this time is consistent with that in eq. (6). It is very interesting that one can find

$$\rho_{AB}(t_m) \frac{U(t_n)}{\rho_{AB}(t_n)} = \rho_0,$$

which means the initial state $\rho_{AB}(t_m)$ evolves to $\rho_{AB}(t_n)$ with the initial atom C in $| g \rangle_C$ and the evolution time $t_n = \frac{5\pi}{g} (n = 1, 2, 3, ...)$.

Besides the non-demolition measurement of the quantum state, we would like to compare this model with that in Sec. II. One can find that the initial state of the probe
qubit must be $|g\rangle_C$ in the model of Sec. II, otherwise the original state of $\rho_0$ will have to be disturbed. That is to say, in Sec. II, the quantum state can only be probed once. The residual quantum state has the same entanglement, discord and classical correlation as $\rho_0$. However, in the model of this section, one can alternately select $|g\rangle_C$ and $|e\rangle_C$ as the initial quantum state of the probe qubit. One divides the measurement outcomes into two groups according to the different choice of $|g\rangle_C$ and $|e\rangle_C$. For each group (or any one of the two groups), one can complete the measurement statistics. In this way, we can implement the measurement of entanglement and correlation without disturbing them.

IV. CONCLUSION AND DISCUSSION

We have presented a scheme for a type of one-parameter quantum state to probe quantum entanglement, quantum discord and classical correlation without disturbing them. We also analyze how the spontaneous emission of our probe qubit influences our detection, which shows that small spontaneous emission rate of the probe atom has slight influence on the detection. In particular, one can find that the influence on the quantum discord and classical correlations is much weaker than the concurrence. It is very interesting that we have found that spontaneous emission might benefit to the quantum discord for some quantum states. In addition, we also present a scheme to implement the non-demolition measurement of the quantum state. That is to say, based on our scheme, one can extract the information of the quantum state, but the quantum state per se is not disturbed. However, it seems that our scheme is not universal for a general quantum state. How to develop a scheme for multiple-parameter quantum state to probe the concurrence, quantum discord or classical correlation, even the quantum state without disturbing them deserves our forthcoming efforts.

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