A proof algorithm associated with the dipole splitting algorithm

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We present a proof algorithm associated with the dipole splitting algorithm (DSA). The proof algorithm (PRA) is a straightforward algorithm to prove that the summation of all the subtraction terms created by the DSA vanishes. The execution of the PRA provides a strong consistency check including all the subtraction terms—the dipole, I, P, and K terms—in an analytical way. Thus we can obtain more reliable QCD NLO corrections. We clearly define the PRA with all the necessary formulae and demonstrate it in the hadron collider processes \( pp \rightarrow \mu^+\mu^-, 2 \text{ jets}, \) and \( n \text{ jets}. \)

Subject Index B00, B62, B65

1. Introduction

The present article follows Ref. [1]. We would like to start with a summary of Ref. [1]. At the CERN Large Hadron Collider (LHC), the standard-model Higgs boson was discovered in 2012 during run 1 with collision energies of 7 and 8 TeV. Run 2, with an energy of 13 TeV, is planned to start in 2015. In order to identify the discovery signals, we need both precise experimental results and precise theoretical predictions. For the precise theoretical predictions, at least the inclusion of the quantum chromodynamics (QCD) next-to-leading order (NLO) corrections is required. One of the most successful procedures to obtain the QCD NLO corrections for multiparton leg processes is the Catani–Seymour dipole subtraction procedure [2,3]. This procedure has been already applied to the huge number of processes happening at the LHC. A partial list of the achievements there is collected in the bibliography of Ref. [1].

Now that the dipole subtraction has been applied to so many processes, we can see some drawbacks about the use of the procedure. Among these drawbacks, we would like to point out three difficulties. The first difficulty is to confirm whether the subtraction terms created are necessary and sufficient ones, and whether the expression of each term includes no mistake. The second difficulty is for one person to reproduce the results in an article written by another person, in particular, the difficulty of specifying all the subtraction terms used. This difficulty is equally valid in the inverse case, namely, the difficulty for one person to tell the other person all the subtraction terms used in a reasonably short form without confusion. The third difficulty is about the use of computer packages in which the dipole subtraction procedure is automated. Publicly available packages are presented in Refs. [4–10]. Users sometimes have difficulties in understanding the algorithms implemented in the packages and the outputs of the run.
In order to solve some of these difficulties, it is required that a practical algorithm to use the dipole subtraction is clearly defined, and that the documentation of the algorithm includes the clear presentation of all the subjects in the wishlist as shown in Ref. [1]:

1. Input, output, creation order, and all formulae in the document,
2. Necessary information to specify each subtraction term,
3. Summary table of all subtraction terms created,
4. Associated proof algorithm.

We succeeded in constructing an algorithm that allows the clear presentation of all the entries in the wishlist. It is named the dipole splitting algorithm (DSA). The DSA has been already presented, focusing on entries 1–3 in the wishlist, in Ref. [1]. Thus, the purpose of this article is to present the last entry in the wishlist, “4. Associated proof algorithm”.

An associated proof algorithm means a straightforward algorithm to prove that the summation of all the subtraction terms created by the DSA vanishes. The proof algorithm is abbreviated as PRA hereafter. What the PRA proves is expressed in a formula, as follows. The QCD NLO corrections for an arbitrary process in a hadron collider are written as

$$\sigma_{\text{NLO}} = \sigma_R + \sigma_V + \sigma_C, \quad (1.1)$$

where the symbols $\sigma_R$, $\sigma_V$ and $\sigma_C$ represent the real correction, the virtual correction, and the collinear subtraction term, respectively. In the framework of dipole subtraction, the NLO corrections are reconstructed as

$$\sigma_{\text{NLO}} = (\sigma_R - \sigma_D) + (\sigma_V + \sigma_I) + \sigma_P + \sigma_K, \quad (1.2)$$

where the symbols $\sigma_D$, $\sigma_I$, $\sigma_P$, and $\sigma_K$ represent the dipole, I, P, and K terms, respectively. The quantities $(\sigma_R - \sigma_D)$, $(\sigma_V + \sigma_I)$, $\sigma_P$, and $\sigma_K$ are separately finite. When the NLO cross section in Eq. (1.1) is equated with the cross section in Eq. (1.2), we obtain the identity for the arbitrary process as

$$\sigma_{\text{subt}} = \sigma_D + \sigma_C - \sigma_I - \sigma_P - \sigma_K = 0. \quad (1.3)$$

We call this relation the consistency relation of the subtraction terms. The PRA is a straightforward algorithm to prove the consistency relation in Eq. (1.3) for any given process, if all the subtraction terms—the dipole, I, P, and K terms—are created by the DSA.

We clarify the main advantages of the PRA. Since the relation in Eq. (1.3) includes all the subtraction terms—the dipole, I, P, and K terms—created, the proof of the consistency relation gives the confirmation of all the terms. As mentioned above, the subtracted cross sections $(\sigma_R - \sigma_D)$ and $(\sigma_V + \sigma_I)$ are separately finite. When we use any wrong collection or wrong expression for the dipole or the I term, the subtracted cross sections would diverge. In this way, at least the divergent parts of the dipole and I terms are confirmed by the successful cancellation against the real and virtual corrections. Compared to the dipole and I terms, the P and K terms $\sigma_P$ and $\sigma_K$ are separately finite themselves, and confirmation by cancellation is impossible. The PRA can provide the precious confirmation including the P and K terms. This is the first advantage of the PRA. The PRA is executed in an analytical way and does not rely on any numerical evaluation. All the dipole terms are integrated over the soft and collinear regions of the phase space in an analytical way in $d$ dimensions. The expressions for the integrated dipole terms are available in Refs. [2,3]. The PRA utilizes the integrated dipole terms and all the steps of the PRA are executed in an analytical way. This is the second advantage of the PRA. In consequence, we can have a strong consistency check of all the subtraction terms by the execution of the PRA, and we can obtain more reliable QCD NLO corrections as a result.
In the dipole subtraction procedure, some algorithms to create the subtraction terms may be constructed. The construction of a straightforward algorithm to prove the consistency relation in Eq. (1.3), is not always possible for all of them. The cancellations in the consistency relation are realized between cross sections with the same initial states and the same reduced Born processes. In the DSA, the subtraction terms created are classified by real processes and the kind of the parton splitting. The classification of the subtraction terms is converted to the classification by the initial states and the reduced Born processes. Then we can easily identify the cross sections that cancel each other in Eq. (1.3). The systematical identification of the cancellations for an arbitrary process makes possible for us to construct a straightforward proof algorithm of the consistency relation. In the original article about Catani–Seymour dipole subtraction [2], the dipole terms are constructed to subtract the soft and collinear divergences from the real corrections, and then the dipole terms are analytically integrated over the soft and collinear regions in $d$-dimensional phase space. The integrated dipole terms are transformed to the I, P, and K terms. The method of transformation is explained for the proton–proton collider case in Sect. 10 in Ref. [2]. The PRA is just the transformation in a different order. The PRA is constructed in such a way that when the subtraction terms are created by the DSA, the consistency relation of the subtraction terms can be proved just by following well defined steps in a straightforward algorithm.

The present paper is organized as follows: The PRA is defined in Sect. 2. All the necessary formulae are collected in Appendix A. The PRA is demonstrated in the processes $pp \rightarrow \mu^+ \mu^-$, 2 jets, and $n$ jets in Sects. 3, 4, and 5, respectively. The results of the PRA in the dijet and $n$ jet processes are summarized in Appendixes B and C, respectively. Section 6 is devoted to the summary.

2. Proof algorithm

2.1. Definition

The prediction of the cross section including the QCD NLO corrections is generally written as

$$\sigma_{\text{prediction}} = \sigma_{\text{LO}} + \sigma_{\text{NLO}},$$

(2.1)

where the symbol $\sigma_{\text{LO}}$ represents the leading order (LO) cross section or a distribution, and the symbol $\sigma_{\text{NLO}}$ represents the QCD NLO corrections to the LO cross section. The LO cross section does not appear in the present paper hereafter. For a given collider process, the real emission processes that contribute to the collider process are written as $R_i$. The set consisting of all the real emission processes is denoted as

$$\{R_i\} = \{R_1, R_2, \ldots, R_{n_{\text{real}}}\},$$

(2.2)

where $n_{\text{real}}$ is the number of all the real processes. When the NLO corrections are treated within the framework of the DSA [1], they are expressed as

$$\sigma_{\text{NLO}} = \sum_{i=1}^{n_{\text{real}}} \sigma(R_i).$$

(2.3)

In the DSA, all the corrections are classified by the real processes $R_i$, and each contribution is denoted as $\sigma(R_i)$. The cross section $\sigma(R_i)$ is defined as

$$\sigma(R_i) = \left[ \sigma_{\text{R}}(R_i) - \sigma_{\text{D}}(R_i) \right] + \left[ \sigma_{\text{V}}(B1(R_i)) + \sigma_{\text{T}}(R_i) \right] + \sigma_{\text{P}}(R_i) + \sigma_{\text{K}}(R_i),$$

(2.4)

where the cross sections $\sigma_{\text{R}}(R_i)$ and $\sigma_{\text{V}}(B1(R_i))$ represent the real and virtual corrections, respectively. The process $B1(R_i)$ is the Born process reduced from $R_i$ by the rule $B1(R_i) = R_i-(a$ gluon
in the final state), as defined in Ref. [1]. The cross sections \( \sigma_\text{D}(R_i) \), \( \sigma_\text{I}(R_i) \), \( \sigma_\text{P}(R_i) \), and \( \sigma_\text{K}(R_i) \) represent the contributions of the dipole, I, P, and K terms, respectively. The cross sections are factorized into the parton distribution function (PDF) and the subpartonic cross sections as

\[
\sigma(R_i) = \int dx_1 \int dx_2 f_{F(x_a)}(x_1) f_{F(x_b)}(x_2) \times \left[ (\hat{\sigma}_\text{R}(R_i) - \hat{\sigma}_\text{D}(R_i)) + (\hat{\sigma}_\text{V}(B1(R_i)) + \hat{\sigma}_\text{I}(R_i)) + \hat{\sigma}_\text{P}(R_i) + \hat{\sigma}_\text{K}(R_i) \right],
\]

(2.5)

where \( f_{F(x_a)}(x_1) \) represents the PDF and the subscript \( F(x_a/b) \) denotes the field species of the initial state parton in the leg \( a/b \) as defined in Ref. [1]. The symbols \( \hat{\sigma}_\text{R}(R_i) \) and \( \hat{\sigma}_\text{V}(B1(R_i)) \) represent the partonic real and virtual corrections, respectively. The quantities \( \hat{\sigma}(R_i) \), with the subscripts, D, I, P, and K, represent the contributions of the dipole, I, P, and K terms to the partonic cross sections, respectively. The definitions of all the partonic cross sections are collected in Appendix A1. In order to specify the jet observables, the corresponding jet functions, \( F_j^{(n/n+1)} \), must be multiplied by all the cross sections. The use of the jet functions in the dipole subtraction is explained in Ref. [2]. For compact expression, we do not show the jet functions explicitly in the present article.

In the original calculation method for the QCD NLO corrections, the corrections can be constructed as

\[
\sigma_{\text{NLO}} = \sum_{i=1}^{n_{\text{real}}} \sigma_{\text{orig}}(R_i),
\]

(2.6)

where the symbol \( \sigma_{\text{orig}}(R_i) \) denotes each correction belonging to the real process \( R_i \). \( n_{\text{real}} \) is the same number as in Eq. (2.3). The cross section \( \sigma_{\text{orig}}(R_i) \) consists of three terms:

\[
\sigma_{\text{orig}}(R_i) = \sigma_\text{R}(R_i) + \sigma_\text{V}(B1(R_i)) + \sigma_\text{C}(R_i),
\]

(2.7)

where the symbols \( \sigma_\text{R}(R_i) \) and \( \sigma_\text{V}(B1(R_i)) \) are the same real and virtual corrections as appear in Eq. (2.4). \( \sigma_\text{C}(R_i) \) represents the collinear subtraction term. When the NLO cross section in Eq. (2.6) is equated with the cross section in Eq. (2.3), we obtain the identity for an arbitrary process as

\[
\sum_{i=1}^{n_{\text{real}}} \sigma_{\text{subt}}(R_i) = 0,
\]

(2.8)

where the cross section \( \sigma_{\text{subt}}(R_i) \) is defined as

\[
\sigma_{\text{subt}}(R_i) = \sigma_{\text{orig}}(R_i) - \sigma(R_i)
\]

\[
= \sigma_\text{D}(R_i) + \sigma_\text{C}(R_i) - \sigma_\text{I}(R_i) - \sigma_\text{P}(R_i) - \sigma_\text{K}(R_i).
\]

(2.9)

The cross section \( \sigma_{\text{subt}}(R_i) \) includes all the subtraction terms. We call the relation in Eq. (2.8) the consistency relation of the subtraction terms. The aim of the PRA is to prove the consistency relation for an arbitrary collider process. In order to construct the proof algorithm in separate steps, we reconstruct the dipole term \( \sigma_\text{D}(R_i) \) into four terms as

\[
\sigma_\text{D}(R_i) = \sigma_\text{D}(R_i, I) + \sigma_\text{D}(R_i, P) + \sigma_\text{D}(R_i, K) + \sigma_\text{D}(R_i, d\bar{d}p2).
\]

(2.10)

The definitions of the four terms will be given in Sects. 2.3–2.6, respectively. Using the four terms, we can rewrite \( \sigma_{\text{subt}}(R_i) \) in Eq. (2.9) as

\[
\sigma_{\text{subt}}(R_i) = [\sigma_\text{D}(R_i, I) - \sigma_\text{I}(R_i)] + [\sigma_\text{D}(R_i, P) + \sigma_\text{C}(R_i) - \sigma_\text{P}(R_i)]
\]

\[
+ [\sigma_\text{D}(R_i, K) - \sigma_\text{K}(R_i)] + \sigma_\text{D}(R_i, d\bar{d}p2).
\]

(2.11)
The execution of the proof algorithm proceeds according to the steps in such a way that the first three terms in square brackets are calculated in turn. At this stage, we can define all the six steps of the PRA as follows:

**Step 1.** Convert the dipole terms \( \sigma_D(R_i) \) to the integrated form,

2. \( \sigma_D(R_i, I) - \sigma_I(R_i) = -\sigma(I, (2)-1/2, N_f \mathcal{V}_{ff}) \),

3. \( \sigma_D(R_i, P) + \sigma_C(R_i) - \sigma_P(R_i) = 0 \),

4. \( \sigma_D(R_i, K) = \sigma_K(R_i, \bar{d} i p 1, (3)/(4)-1, N_f h) \),

5. \( \sigma_{\text{subt}}(R_i) = -\sigma_I(R_i, (2)-1/2, N_f \mathcal{V}_{ff}) - \sigma_K(R_i, \bar{d} i p 1, (3)/(4)-1, N_f h) + \sigma_D(R_i, \bar{d} i p 2) \),

6. \( \sum_{i=1}^{n_{\text{real}}} \sigma_{\text{subt}}(R_i) = 0 \).  

(2.12)

All the six steps will be separately explained in Sects. 2.2–2.7, respectively. The premise for the execution of the PRA for a given process is that all the dipole, I, P, and K terms are created by the DSA in Ref. [1].

### 2.2. Step 1: Integrated dipole terms \( \sigma_D \)

**Step 1** of the PRA is to convert all the dipole terms, which are created by the DSA, into the integrated form. The contribution of each dipole term to the partonic cross section is generally written in \( d \) dimensions,

\[
\hat{\sigma}_D(R_i, \bar{d} i p j) = \frac{1}{S_{R_i}} \Phi(R_i) d \cdot \frac{1}{n_s(a)n_s(b)} D(R_i, \bar{d} i p j)_{IJ,K}. 
\]

(2.13)

where \( R_i \) is a real correction process and \( \bar{d} i p j \) is the category to which the dipole term belongs. \( S_{R_i} \) is the symmetric factor of the process \( R_i \). The spin degree of freedom, \( n_s(a/b) \), is determined as \( n_s(\text{quark}) = 2 \) and \( n_s(\text{gluon}) = d - 2 = 2(1 - \epsilon) \). Each dipole term is specified with three legs \( I, J, \) and \( K \) of the real process \( R_i \). The dipole term is generally written as

\[
D(R_i, \bar{d} i p j)_{IJ,K} = \frac{1}{S_{JJ}} \frac{1}{x_{JJ}} \frac{1}{T^2_{\Phi(y_{\text{emi})}}} \left( B_j \right| T_{y_{\text{emi}}} \cdot T_{y_{\text{spe}}} V^y_{\text{emi}} \right| B_j \right), 
\]

(2.14)

where the details of the notation expressing the dipole terms are explained in Ref. [1]. The category Dipole \( j \) (in short, \( \bar{d} i p j \)) and the subcategory of the splittings are shown in Fig. A1 in Appendix A2. It is noted that in the DSA we introduce the field mapping, \( y = f(\bar{x}) \), and in Eq. (2.14) the legs of the reduced Born process \( B_j \), on which the color and helicity operators act, are specified with the elements \( (y_{\text{emi}}, y_{\text{spe}}) \) of set \{y\}.

The partonic cross section of the dipole term in Eq. (2.13) is converted to the integrated form as

\[
\hat{\sigma}_D(R_i, \bar{d} i p j, x_{a/b}) = -\frac{A_d}{S_{R_i}} \cdot \int_0^1 dx \frac{1}{T^2_{\Phi(y_{\text{emi})}}} \mathcal{V}(x; \epsilon) \cdot \Phi_{a/b}(B_j, x) \right| y_{\text{emi}}, y_{\text{spe}} \right|, 
\]

(2.15)

where the overall factor \( A_d \) is defined as

\[
A_d = \frac{\alpha_s (4\pi \mu^2)^\epsilon}{2\pi \Gamma(1 - \epsilon)}. 
\]

(2.16)
The phase space $\Phi_{a/b}(B_j, x_d)$ is defined in $d$ dimensions in Eq. (A35)/(A36). The explicit expressions of the integrated dipole terms depend on four types of dipole terms, final–final (FF), final–initial (FI), initial–final (IF), and initial–initial (II). The four types are defined in Refs. [2] and [1]. The types of dipole terms are denoted by subcategories:

$$\hat{\sigma}_D(R_i, c_i p j) \supset \hat{\sigma}_D(R_i, c_i p j, FF/FI/IF/II).$$

(2.17)

The expressions are separately shown in Eqs. (A20), (A28), (A39), and (A56), respectively, in Appendix A2. The expression of the the factor $\mathcal{V}(x; \epsilon)$ is determined in each type as

$$\mathcal{V}(x; \epsilon) = \begin{cases} 
\mathcal{V}_{F(x_i)}F(x_j)(\epsilon)\delta(1 - x) : FF \ (ij, k), \\
\mathcal{V}_{F(x_i)}F(x_j)(x; \epsilon) : FI \ (ij, a), \\
\mathcal{V}_{F(x_a/b)}F(ym_{\epsilon})(x; \epsilon) : IF \ (ai, k), \\
\mathcal{V}_{F(x_a/b)}F(ym_{\epsilon})(x; \epsilon) : II \ (ai, b).
\end{cases}$$

(2.18)

The concrete expressions of the universal singular functions, $\mathcal{V}_{F(x_i)}F(x_j)(\epsilon)$, $\mathcal{V}_{F(x_i)}F(x_j)(x; \epsilon)$, $\mathcal{V}_{F(x_a/b)}F(ym_{\epsilon})(x; \epsilon)$, and $\mathcal{V}_{F(x_a/b)}F(ym_{\epsilon})(x; \epsilon)$, further depend on the kinds of splittings, which are written with the factor $1/T^2_{ym_{\epsilon}}$ in Eqs. (A21), (A29), (A40), and (A57), respectively. The symbol $\{ym_{\epsilon}, y_{spe}\}$ is defined as

$$[ym_{\epsilon}, y_{spe}] = s^{-\epsilon} \cdot \{B j | T_{ym_{\epsilon}} \cdot T_{y_{spe}} | B j\}_d,$$

(2.19)

where the Lorentz scalar, $s$, also depends on the types FF, FI, IF, and II, which are defined in Eqs. (A27), (A38), (A55), and (A71), respectively. The quantity $\{B j | T_{ym_{\epsilon}} \cdot T_{y_{spe}} | B j\}$ is the color-correlated Born squared amplitude after the spin–color is summed and averaged in $d$ dimensions.

Thanks to the notation employed in the DSA [1], we have a simple and universal expression for the integrated dipole term in Eq. (2.15). The expression can be further abbreviated in a fixed form as

$$\hat{\sigma}_D(R_i, c_i p j, x_{a/b}) = \frac{A_d}{S_{R_i}} \cdot (\text{Factor } 1) \cdot (\text{Factor } 2),$$

(2.20)

where Factors 1 and 2 are denoted as

$$(\text{Factor } 1) = \int_0^1 dx \frac{1}{T^2_{ym_{\epsilon}}} \mathcal{V}(x; \epsilon),$$

(2.21)

$$(\text{Factor } 2) = \Phi_{a/b}(B_j, x_d)[ym_{\epsilon}, y_{spe}].$$

(2.22)

Once the concrete expressions of Factors 1 and 2 are determined, the integrated dipole term is also uniquely determined. Factor 1 is universal in the category with the same splitting and has a spectator in the same state among the initial and final states. In order to summarize all the integrated dipole terms in a short form, it is sufficient that the information on the reduced Born process, the kind of splitting, and Factors 1 and 2 is supplied in a table format. The summary tables for the dijet process are shown in Tables B1–B11 in Appendix B. These tables can be a template of the format for the summary tables for an arbitrary process.

We would like to show here two examples in the real process $R_1 = uu \rightarrow u\bar{u} g$, which contributes to the dijet process $pp \rightarrow 2 \text{ jets } + X$. The creation of the dipole terms by the DSA is explained in Ref. [1].

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Example 1: Table B1 in Appendix B1 in Ref. [1], 1.(13,2).

The reduced Born process is taken as

$$B_1 = u(y_a)\tilde{u}(y_b) \rightarrow u(y_1)\tilde{u}(y_2), \quad (2.23)$$

and the field mapping for this dipole term is made as

$$(y_a, y_b; y_1, y_2) = (a, b; \tilde{1}, \tilde{2}). \quad (2.24)$$

The contribution to the cross section is written as

$$\hat{\sigma}_D(R_1, \text{dip}1) = \frac{1}{S_{R_1}} \Phi(R_1)_d \cdot \frac{1}{n_s(u)n_s(\tilde{u})} D(R_1, \text{dip}1)_{13,2}. \quad (2.25)$$

and the dipole term is written as

$$D(R_1, \text{dip}1)_{13,2} = -\frac{1}{s_{13}} \frac{1}{C_F} V_{13,2} (B_1 \mid T_1 \cdot T_2 \mid B_1). \quad (2.26)$$

The dipole term is of final–final type and is converted to the integrated form in Eq. (A20) as

$$\hat{\sigma}_D(R_1, \text{dip}1, \text{FF}) = -\frac{A_d}{S_{R_1}} \frac{1}{C_F} V_{fg}(\epsilon) \cdot \Phi(B_1)_d[1, 2]. \quad (2.27)$$

where the color-correlated Born squared amplitude is denoted as

$$(1, 2) = (s_{y_1,y_2})^{-\epsilon} \cdot (B_1 \mid T_{y_1} \cdot T_{y_2} \mid B_1)_d. \quad (2.28)$$

with the Lorentz scalar $s_{y_1,y_2} = 2P(y_1) \cdot P(y_2)$. The expression of the dipole term is further abbreviated in the fixed form in Eq. (2.20) with Factors 1 and 2 as

$$(\text{Factor 1}) = \frac{1}{C_F} V_{fg}(\epsilon), \quad (2.29)$$

$$(\text{Factor 2}) = \Phi(B_1)_d[1, 2]. \quad (2.30)$$

In the summary table, in addition to Factors 1 and 2, we can explicitly specify the reduced Born process and the kind of splitting, which are $B_1$ and (1)-1 in the present case. Actually, the integrated dipole term is shown with the above information in the first one, $1. (a, b; \tilde{1}, \tilde{2}, \tilde{3})$, in Table B1 in Appendix B1.

Example 2: Table B1 in Ref. [1], 18. (a1,b).

The reduced Born process and the field mapping are determined as

$$B_3u = g(y_a)\tilde{u}(y_b) \rightarrow \tilde{u}(y_1)g(y_2) \quad \text{and} \quad (y_a, y_b; y_1, y_2) = (a\tilde{1}, \tilde{b}; 2, 3). \quad (2.31)$$

The cross section and the dipole term are written as

$$\hat{\sigma}_D(R_1, \text{dip}3u) = \frac{1}{S_{R_1}} \Phi(R_1)_d \cdot \frac{1}{n_s(u)n_s(\tilde{u})} D(R_1, \text{dip}3u)_{a1,b}. \quad (2.32)$$

$$D(R_1, \text{dip}3u)_{a1,b} = -\frac{1}{s_{a1}} \frac{1}{C_A} V_{a1,ab} (B_3u \mid T_{y_a} \cdot T_{y_b} \mid B_3u), \quad (2.33)$$

The dipole term is of initial–initial type and is converted to the integrated form in Eq. (A56) as

$$\hat{\sigma}_D(R_1, \text{dip}3u, \Pi, x_a) = -\frac{A_d}{S_{R_1}} \cdot \int_0^1 dx \frac{1}{C_A} \tilde{\gamma}_{fg}(x; \epsilon) \cdot \Phi_d(B_3u, x)_d[a, b], \quad (2.34)$$

where the symbol $[a, b]$ is denoted as

$$[a, b] = (s_{x_a,y_b})^{-\epsilon} \cdot (B_3u \mid T_{y_a} \cdot T_{y_b} \mid B_3u)_d. \quad (2.35)$$
with the Lorentz scalar \( s_{\alpha, \beta} = 2 p_\alpha \cdot p_\beta \). Factors 1 and 2 in Eq. (2.20) are determined as

\[
\text{(Factor 1)} = \int_0^1 dx \frac{1}{C_A} \hat{V}_{f,g}^\epsilon(x; \epsilon), \tag{2.36}
\]

\[
\text{(Factor 2)} = \Phi_d(3u, x)_d[a, b]. \tag{2.37}
\]

The integrated dipole term is shown in the entry 18, \((\vec{a} \vec{b}; 2, 3)\) in Table B1.

In this way, all the dipole terms can be converted into the integrated form. Then the summary tables of all the dipole terms created by the DSA are converted to summary tables of the integrated dipole terms with the necessary information. One original dipole term is converted to one integrated dipole term, and the total number of dipole terms is conserved through the conversion. As an example, for the dijet process, all the summary tables of the dipole terms created by the DSA are shown in Tables B1–B11 in Appendix B in Ref. [1]. All the summary tables are converted into summary tables of the integrated dipole terms as Tables B1–B11 in Appendix B in the present article.

2.3. Step 2: \( \sigma_D(I) - \sigma_I \)

**Step 2** of the PRA is to prove the relation

\[
\sigma_D(R_i, I) - \sigma_I(R_i) = -\sigma_I(R_i, (2)-1/2, N_f \mathcal{V}_{f,f}). \tag{2.38}
\]

The relation in Eq. (2.38) stands for an arbitrary process and is regarded as an identity. The left-hand side of Eq. (2.38) is the first term in square brackets in Eq. (2.11). We define the three cross sections \( \sigma_D(R_i, I), \sigma_I(R_i) \), and \( \sigma_I(R_i, (2)-1/2, N_f \mathcal{V}_{f,f}) \) in Eq. (2.38) as follows.

We first define the cross section \( \sigma_D(R_i, I) \). In **Step 1** all the dipole terms are converted into the integrated form. Among them, we take only the dipole terms in the Dipole 1 category. Dipole 1 includes splittings (1)–(4) with the cases of the spectator in the final/initial state denoted as subcategory -1/2, which are shown in Fig. A2 in Appendix A3. We extract the following parts from all the expressions, depending on splittings (1)–(4) and on the subcategory -1/2. For the dipole terms with splitting (1)-1, we take all of the singular function \( \mathcal{V}_{f,g}^\epsilon(x; \epsilon) \) in Eq. (A22). For the dipole terms with splitting (1)-2, we extract the part \( \delta(1 - x) \mathcal{V}_{f,g}^\epsilon(x; \epsilon) \) from the singular function \( \mathcal{V}_{f,g}^\epsilon(x; \epsilon) \) in Eq. (A30). Then, for both of the splittings (1)-1 and -2, the partonic cross sections \( \hat{\sigma}_D(R_i, I, (1)-1/2) \) are extracted as the same expression:

\[
\hat{\sigma}_D(R_i, I, (1)-1/2) = -\frac{A_d}{S_{R_i}} \cdot \frac{1}{C_A} \mathcal{V}_{f,g}^\epsilon(x; \epsilon) \cdot \Phi(B1)_d \left[ y_{emi}, y_{spe} \right]. \tag{2.39}
\]

The cross section is obtained by multiplying the PDFs by the partonic cross section in Eq. (2.39), as shown in Eq. (2.5). In the cases of the other splittings, the partonic cross sections are defined as follows, and the cross sections are similarly obtained by the multiplication of the PDFs. For splitting (2)-1, we take all of \( \mathcal{V}_{gg}^\epsilon(x; \epsilon) \) in Eq. (A23), and, for splitting (2)-2, we extract the part \( \delta(1 - x) \mathcal{V}_{gg}^\epsilon(x; \epsilon) \) from \( \mathcal{V}_{gg}^\epsilon(x; \epsilon) \) in Eq. (A31). Then, for both splittings (2)-1/2, the cross section \( \hat{\sigma}_D(R_i, I, (2)-1/2) \) is defined as

\[
\hat{\sigma}_D(R_i, I, (2)-1/2) = -\frac{A_d}{S_{R_i}} \cdot \frac{1}{C_A} \mathcal{V}_{gg}^\epsilon(x; \epsilon) \cdot \Phi(B1)_d \left[ y_{emi}, y_{spe} \right]. \tag{2.40}
\]

For splitting (3)-1, we extract the part \( \delta(1 - x) \mathcal{V}_{f,g}^\epsilon(x; \epsilon) \) from \( \mathcal{V}_{f,f}^\epsilon(x; \epsilon) \) in Eq. (A41), and, for splitting (3)-2, we extract the same part \( \delta(1 - x) \mathcal{V}_{f,g}^\epsilon(x; \epsilon) \) from \( \mathcal{V}_{f,f}^\epsilon(x; \epsilon) \) in Eq. (A59). Then, for both splittings (3)-1/2, the partonic cross sections \( \hat{\sigma}_D(R_i, I, (3)-1/2) \) are extracted as the identical expression in Eq. (2.39). For splitting (4)-1, we extract the part \( \delta(1 - x) \mathcal{V}_{g}^\epsilon(x; \epsilon) \) from \( \mathcal{V}_{g,g}^\epsilon(x; \epsilon) \) in
Eq. (A42), and, for splitting (4)-2, we extract the same quantity \( \delta (1 - x) V'_g (\epsilon) \) from \( \tilde{V}^{g,g} (x; \epsilon) \) in \( \tilde{V}^{g,g} (x; \epsilon) \) in Eq. (A60). Then, for both splittings (4)-1/2, the cross sections \( \hat{\sigma}_D (R_i, I, (4)-1/2) \) are defined as

\[
\hat{\sigma}_D (R_i, I, (4)-1/2) = - \frac{A_d}{S_{B1}} \cdot \frac{1}{C_A} \frac{1}{C_A} \frac{1}{C_A} V'_g (\epsilon) \cdot \Phi (B1)_d \left[ y_{emi}, y_{spe} \right].
\]

In Eqs. (2.39), (2.40), and (2.41), the color-correlated Born squared amplitude, \( [y_{emi}, y_{spe}] \), is denoted as

\[
[y_{emi}, y_{spe}] = \left( s_{y_{emi}, y_{spe}} \right)^{-\epsilon} \cdot \left( B1 \right| T_{y_{emi}} \cdot T_{y_{spe}} \left| B1 \right) _d,
\]

with the Lorentz scalar \( s_{y_{emi}, y_{spe}} = 2 \rho (y_{emi}) \cdot \rho (y_{spe}) \). The partonic cross section \( \hat{\sigma}_D (R_i, I) \) is the summation of all the existing partonic cross sections \( \hat{\sigma}_D (R_i, I, (1)-(4)-1/2) \) defined above. The formulae for \( \hat{\sigma}_D (R_i, I) \) are collected in Appendix A3.

Next we define the cross section \( \sigma_I (R_i) \). The partonic cross section \( \hat{\sigma}_I (R_i) \) is the contributions of the I terms that are created by the DSA [1]. The contribution of each I term is written as

\[
\hat{\sigma}_I (R_i)_{IK} = - \frac{A_d}{S_{B1}} \cdot \frac{1}{T_{F(I)}^2} V_{F(I)} \cdot \Phi (B1)_d [I, K],
\]

where again the notation is defined in Ref. [1]. \( S_{B1} \) is the symmetric factor of the reduced Born process \( B1 (R_i) \). In the DSA, the creation of the I terms is ordered by the species of the first leg \( I \), (1)-(4), which are shown in Fig. A7 in Appendix A8. There are four cases for the choices for the first leg \( I \), and each case has further choices for the second leg \( K \) in the final/initial state denoted as -1/2. The factor \( \frac{1}{T_{F(I)}^2} V_{F(I)} \) is determined in each case as

\[
\frac{1}{T_{F(I)}^2} V_{F(I)} = \begin{cases} 
\frac{1}{C_A} V'_f (\epsilon) & : (1), (3) - 1/2, \\
\frac{1}{C_A} V'_g (\epsilon) & : (2), (4) - 1/2, 
\end{cases}
\]

where the universal singular functions \( V'_f (\epsilon) \) and \( V'_g (\epsilon) \) are defined in Eqs. (A73) and (A45), respectively. The symbol \([I, K]\) is denoted as

\[
[I, K] = s_{IK}^{-\epsilon} \cdot \left( B1 \right| T_I \cdot T_K \left| B1 \right) _d,
\]

with the Lorentz scalar \( s_{IK} = 2 \rho_I \cdot \rho_K \). The partonic cross section \( \hat{\sigma}_I (R_i) \) is the summation of all the created I terms as

\[
\hat{\sigma}_I (R_i) = \sum_{I, K} \hat{\sigma}_I (R_i)_{IK}.
\]

Similar to the case \( \sigma_D (R_i, I) \), the cross section \( \sigma_I (R_i) \) is obtained by multiplying the PDFs by the partonic cross section \( \hat{\sigma}_I (R_i) \). The formulae for the I term \( \sigma_I (R_i) \) are collected in Appendix A8.

The third cross section \( \sigma_I (R_i, (2)-1/2, N_f V_f) \) is defined as follows. We take only the I terms that belong to the splitting (2)-1/2 in Fig. A7. Among them we extract the part \( N_f V_f (\epsilon) \) in the function \( V'_g (\epsilon) \) in Eq. (A45). The extracted part is defined as the partonic cross section,

\[
\hat{\sigma}_I (R_i, (2)-1/2, N_f V_f)_{IK} = - \frac{A_d}{S_{B1}} \frac{N_f}{C_A} V_f (\epsilon) \cdot \Phi (B1)_d [I, K],
\]

where the leg \( I \) is in the final state and the field species is a gluon. By definition, the term exists only if the reduced Born process \( B1 (R_i) \) includes any gluon in the final state. This statement is equivalent to saying that the term exists only if the real process \( R_i \) includes two or more gluons in the final state. The partonic cross section \( \hat{\sigma}_I (R_i, (2)-1/2, N_f V_f) \) is the summation of all the existing cross sections.
in Eq. (2.47). The formula is also added in Appendix A8. Now that all three terms in Eq. (2.38) are defined, we can interpret the equation in such a way that the extracted part from the integrated dipole term, \( \hat{\sigma}_D(R_i, I) \), cancels the I term, \( \sigma_I(R_i) \), except for the part \( \sigma_I(R_i, (2)-1/2, N_f V_{ff} f \). The remaining part \( \sigma_I(R_i, (2)-1/2, N_f V_{ff} f) \) is canceled by a term created in a different real process. The mechanism of the cancellation will be clarified in Step 6 in Sect. 2.7.

Finally, we see one example. We take the same real emission process as used in Step 1, \( R_1 = u\bar{u} \to u\bar{u}g \). The reduced Born process, \( B_1(R_1) = u\bar{u} \to u\bar{u} \), does not include any gluon in the final state, and the right-hand side of Eq. (2.38) does not exist. Then the relation to be proved is written as

\[
\sigma_D(R_1, I) - \sigma_I(R_1) = 0. 
\]

(2.48)

First we construct the partonic cross section \( \hat{\sigma}_D(R_1, I) \). All the integrated dipole terms of \( \hat{\sigma}_D(R_1) \) are summarized in Table B1 in Appendix B1. Among all the 21 dipole terms, we take only the first twelve terms in the category Dipole 1. Following the definition of \( \sigma_D(R_i, I) \) given above, we extract the cross section as

\[
\hat{\sigma}_D(R_1, I) = -\frac{A_d}{S_{R_1}} \cdot \frac{1}{C_F} V_{fg}(\epsilon) \cdot \Phi(B_1)_d \left[ [1, 2] + [2, 1] + [1, a] + [1, b] + [2, a] + [2, b] + [a, 1] + [a, 2] + [b, 1] + [b, 2] + [a, b] + [b, a] \right],
\]

(2.49)

with the symmetric factor \( S_{R_1} = 1 \). Second, we construct the contribution of the I terms, \( \sigma_I(R_1) \). The I terms that are created by the DSA are summarized in Table B12 in Appendix B2 in Ref. [1]. The twelve I terms are written as

\[
\sigma_I(R_1) = -\frac{A_d}{S_{B_1}} \cdot \frac{1}{C_F} V_f(\epsilon) \cdot \Phi(B_1)_d \left[ [1, 2] + [2, 1] + [1, a] + [1, b] + [2, a] + [2, b] + [a, 1] + [a, 2] + [b, 1] + [b, 2] + [a, b] + [b, a] \right],
\]

(2.50)

with the symmetric factor \( S_{B_1} = 1 \). Using the relations \( S_{R_1} = S_{B_1} = 1 \) and \( V_f(\epsilon) = V_{fg}(\epsilon) \), we prove the relation in Eq. (2.48). Step 2 for the process \( R_1 \) is completed. The result is shown in Eq. (B1) in Appendix B1. As mentioned above, in this example, the right-hand side of Eq. (2.38) does not exist. Cases where the right-hand side exists will be seen in Sects. 4 and 5.

2.4. Step 3: \( \sigma_D(P) + \sigma_C - \sigma_P \)

Step 3 of the PRA is to prove the relation

\[
\sigma_D(R_i, P) + \sigma_C(R_i) - \sigma_P(R_i) = 0.
\]

(2.51)

The left-hand side of Eq. (2.51) is the second term in the square brackets in Eq. (2.11). We define the three cross sections \( \sigma_D(R_i, P), \sigma_C(R_i), \) and \( \sigma_P(R_i) \) in Eq. (2.51) as follows.

First we define the cross section \( \sigma_D(R_i, P) \). The quantity is extracted from the integrated dipole terms converted in Step 1 in the following way. We choose only the dipole terms with splittings (3), (4), (6), and (7), as shown in Fig. A3 in Appendix A4. For the initial–final dipole terms, namely, splittings (3)-, (4)-, (6)-, and (7)-1, we extract the factors \((-1/\epsilon + \ln x) \cdot P_{ff,gg,gg,gg}(x)\) in the functions \( \tilde{V}^f_f(x; \epsilon), \tilde{V}^g_f(x; \epsilon), \tilde{V}^g_f(x; \epsilon), \) and \( \tilde{V}^g_f(x; \epsilon) \) in Eqs. (A41)–(A44), respectively. For the initial–initial dipole terms, namely, splittings (3)-, (4)-, (6)-, and (7)-2, we extract the same factors in the functions \( \tilde{V}^f_f(x; \epsilon), \tilde{V}^g_f(x; \epsilon), \tilde{V}^f_f(x; \epsilon), \) and \( \tilde{V}^g_f(x; \epsilon) \) in Eqs. (A59)–(A62). For all the dipole
terms with splittings (3), (4), (6), and (7)-1/2, the partonic cross sections \( \hat{\sigma}_D(R_i, P) \) are defined as the universal expression

\[
\hat{\sigma}_D(R_i, P, \tilde{\Omega}; x_{a/b}) = \frac{A_d}{S_{R_i}} \int_0^1 dx \left( \frac{1}{\epsilon} - \ln x \right) \frac{1}{T^2_{F(\gamma_{emi})}(x)} p^{F(x_{a/b})(\gamma_{emi})}(x) \times \Phi_{a/b}(B_j, x)_{d} \{ y_{emi}, y_{spec} \}, \tag{2.52}
\]

where the factor \( \frac{p^{F(x_{a/b})(\gamma_{emi})}(x) / T^2_{F(\gamma_{emi})}}{T^2_{F(\gamma_{emi})}} \) depends on the splittings as written in Eq. (A77). The symbol \( \{ y_{emi}, y_{spec} \} \) is defined in Eqs. (A54) and (A70) for the initial–final and initial–initial dipole terms, respectively. The cross section \( \sigma_D(R_i, P) \) is obtained by multiplying the PDFs with the partonic cross section \( \hat{\sigma}_D(R_i, P) \). The formulae for \( \sigma_D(R_i, P) \) are collected in Appendix A4.

Second, we define the cross section \( \sigma_C(R_i) \). This quantity is the collinear subtraction term that is introduced in the QCD NLO corrections as in Eq. (2.7). Some algorithms to create collinear subtraction terms for an arbitrary process may be available. We introduce here an algorithm that is analogous with the algorithm to create the P term in the DSA [1]. The input is taken in a real process \( R_i \). As with the DSA, \( R_i \) defines the set \( \{ x \} = \{ x_a, x_b, x_1, \ldots, x_{n+1} \} \). The field species and the momenta are denoted as \( F(x) = \{ F(x_a), F(x_b), F(x_1), \ldots, F(x_{n+1}) \} \) and \( \{ p_a, p_b, p_1, \ldots, p_{n+1} \} \). Then we check whether the process \( R_i \) can have splittings (3), (4), (6), and (7), shown in Fig. A6 in Appendix A7. We start with splitting (3), including the leg-a \( (x_a) \). When the process \( R_i \) can have splitting (3), a pair \( (x_a, x_i) \) is chosen and the new element \( x_{a_i} \) is created with the field species \( F(x_{a_i}) \), which is the species of the root of the splitting: in the present splitting (3), a quark. The reduced Born process \( B_1 \) is taken in the same one as determined for the dipole terms in the DSA. The \( B_1 \) associates the set \( \{ y \} = \{ y_a, y_b, y_1, \ldots, y_n \} \), the field species \( F(\gamma) = \{ F(y_a), F(y_b), F(y_1), \ldots, F(y_n) \} \), and the momenta \( P(\gamma) = \{ P(y_a), P(y_b), P(y_1), \ldots, P(y_n) \} \). There are two possible cases, \( F(x_{a_i}) = F(y_a) \) or \( F(x_{a_i}) = F(y_b) \), which are denoted as \( y_{emi} = y_a \) or \( y_{emi} = y_b \), respectively. For both cases, the collinear subtraction terms with leg-a \( (x_a) \) are created as

\[
\hat{\sigma}_C(R_i, \tilde{\Omega}; x_a) = \frac{A_d}{S_{B1}} \int_0^1 dx \left( \frac{1}{\epsilon} - \ln \mu_F^2 \right) P_{ff}/(x) \cdot \Phi_{a}(B_1, x)_{d}(B_1), \tag{2.53}
\]

where \( S_{B1} \) is the symmetric factor of the Born process \( B_1 \) and the Altarelli–Parisi splitting function \( P_{ff}/(x) \) is defined in Eq. (A46). The symbol \( \langle B_1 \rangle \) represents the square of the matrix elements of the process \( B_1 \) after the spin–color is summed and averaged in \( d \) dimensions, which is written with the input momenta as

\[
\langle B_1 \rangle = |M_{B1}(P(y_a), P(y_b) \rightarrow P(y_1), \ldots, P(y_n))|^2. \tag{2.54}
\]

In the case in which \( y_{emi} = y_a \) or \( y_{emi} = y_b \), the input momenta in the initial state are determined as \( (P(y_a), P(y_b)) = (xp_a, p b) \) or \( (p b, xp_a) \), respectively. It is noted that, when the final state of \( R_i \) includes identical fields, one kind of splitting has as many possible pairs \( (x_a, x_i) \) as the number of identical fields. In this case, only one pair is taken and the other pairs must be discarded. For instance, the process \( R_i = u(x_a)\bar{u}(x_b) \rightarrow g(x_1)g(x_2)g(x_3) \) has three pairs \( (x_a, x_1), (x_a, x_2), \) and \( (x_a, x_3) \) for splitting (3). Among them, only one pair, for instance, \( (x_a, x_1) \), is taken and the others, \( (x_a, x_2) \) and \( (x_a, x_3) \), must be discarded. The discard rule is the same as the creation algorithm of the P term in the DSA. The creation algorithm shown above is similarly applied for leg-b \( (x_b) \) and for the other splittings (4), (6), and (7). We summarize the general formulae for the collinear subtraction term in an arbitrary process as follows. The collinear subtraction terms consist of terms with different
splittings, and, equivalently, different reduced Born processes as
\[ \hat{\sigma}_C(R_i) = \sum_{\text{dip}j} \hat{\sigma}_C(R_i, \text{dip}j), \]  
(2.55)
where each term \( \hat{\sigma}_C(R_i, \text{dip}j) \) can have contributions with leg-a \((x_a)\) and leg-b \((x_b)\) as
\[ \hat{\sigma}_C(R_i, \text{dip}j) = \hat{\sigma}_C(R_i, \text{dip}j, x_a) + \hat{\sigma}_C(R_i, \text{dip}j, x_b). \]  
(2.56)

The collinear subtraction term with the reduced Born process \( Bj \) and leg-a/b is universally written as
\[ \hat{\sigma}_C(R_i, \text{dip}j, x_{a/b}) = \frac{A_d}{S_{Bj}} \int_0^1 dx \left( \frac{1}{\epsilon} - \ln \frac{\mu^2_F}{x} \right) p^{F(x_{a/b})F(\gamma_{emi})}(x) \cdot \Phi_{a/b}(B_j, x)_{d(Bj)}, \]  
(2.57)
where the splitting function \( p^{F(x_{a/b})F(\gamma_{emi})}(x) \) is determined as shown in Eq. (A106). The formulae for the collinear subtraction term are collected in Appendix A7.

The third cross section \( \sigma_P(R_i) \) is the contribution of the P terms created by the DSA [1]. The input for the creation is taken in a real process \( R_i \) and the creation of the P terms is ordered by splittings (3), (4), (6), and (7), and also by the spectators in final/initial states denoted as -1/2, which are shown in Fig. A8 in Appendix A9. The partonic cross section \( \hat{\sigma}_P(R_i) \) is written in 4 dimensions as the universal form
\[ \hat{\sigma}_P(R_i, \text{dip}j, x_{a/b}) = \frac{A_4}{S_{Bj}} \int_0^1 dx \frac{1}{T^2_{F(\gamma_{emi})}} p^{F(x_{a/b})F(\gamma_{emi})}(x) \cdot \ln \frac{\mu^2_F}{x s_{x_{a/b}, \gamma_{spe}}} \times \Phi_{a/b}(B_j, x)_4 \{ \gamma_{emi}, \gamma_{spe} \}, \]  
(2.58)
where the factor \( A_4 \) is denoted as \( A_4 = \alpha_s/2\pi \) and \( S_{Bj} \) is the symmetric factor of the process \( Bj \). The definition of the factor \( p^{F(x_{a/b})F(\gamma_{emi})}(x)/T^2_{F(\gamma_{emi})} \) is the same as the factor for the cross section \( \hat{\sigma}_D(R_i, P) \) in Eq. (A77). The definition of the Lorentz scalar \( s_{x_{a/b}, \gamma_{spe}} \) represents the color-correlated Born squared amplitude in 4 dimensions as \( \{ \gamma_{emi}, \gamma_{spe} \} = \langle Bj| T_{\gamma_{emi}} \cdot T_{\gamma_{spe}} | Bj \rangle_4 \). The formulae for the P term \( \sigma_P(R_i) \) are collected in Appendix A9.

We see one example in the same process as used in Step 1 and 2, \( R_1 = u\bar{u} \to u\bar{u}g \). The relation to be proved for the process is written as
\[ \sigma_D(R_1, P) + \sigma_C(R_1) - \sigma_P(R_1) = 0. \]  
(2.59)
As shown in Table B1 in Appendix B1, the dipole terms include splittings (3), (6)\( u \), and (6)\( \bar{u} \), which have the following reduced Born processes, respectively:
\[ B1 = u\bar{u} \to u\bar{u}, \]  
(2.60)
\[ B3u = g\bar{u} \to \bar{u}g, \]  
(2.61)
\[ B3\bar{u} = ug \to ug. \]  
(2.62)

Then the relation in Eq. (2.59) is divided into three independent relations:
\[ \sigma_D(R_1, P, \text{dip}j) + \sigma_C(R_1, \text{dip}j) - \sigma_P(R_1, \text{dip}j) = 0. \]  
(2.63)
for \( \text{dip } j = \text{dip } 1, 3u, \) and \( 3\bar{u} \). First, we construct \( \sigma_D(R_1, P) \). We only show the expressions for the terms \( \sigma_D(R_1, P, \text{dip } 1, x_a) \) and \( \sigma_D(R_1, P, \text{dip } 3u) \) for convenience as

\[
\sigma_D(R_1, P, \text{dip } 1, x_a) = \frac{A_d}{S_{R_1}} \int_0^1 dx \left( \frac{1}{\epsilon} - \ln x \right) \frac{1}{C_F} P^{ff}(x) \Phi_a(B1, x)_d \\
\times ([a, 1] + [a, 2] + [a, b]), \tag{2.64}
\]

\[
\sigma_D(R_1, P, \text{dip } 3u) = \frac{A_d}{S_{R_1}} \int_0^1 dx \left( \frac{1}{\epsilon} - \ln x \right) \frac{1}{C_A} P^{fg}(x) \Phi_a(B3u, x)_d \\
\times ([a, 1] + [a, 2] + [a, b]), \tag{2.65}
\]

with the symmetric factor \( S_{R_1} = 1 \). In Eq. (2.64), the color-correlated Born squared amplitudes are denoted as

\[
[a, 1/2/b] = (s_{x_a, y_{1/2/b}})^{-\epsilon} \cdot \langle B1| T_{y_a} \cdot T_{y_{1/2/b}} |B1 \rangle_d. \tag{2.66}
\]

where the Lorentz factors are written as \( s_{x_a, y_{1/2}} = 2p_a \cdot P(y_{1/2}) \) and \( s_{x_a, y_b} = 2p_a \cdot p_b \). The corresponding quantities in Eq. (2.65) are written in such a way that \( B1 \) is replaced with \( B3u \) in Eq. (2.66).

Next we construct the collinear subtraction term \( \sigma_C(R_1) \). The algorithm to create the collinear subtraction term is executed as follows. The process \( R_1 \) associates set \{x\} as \( R_1 = u(x_a) \bar{u}(x_b) \rightarrow u(x_1) \bar{u}(x_2) g(x_3) \). The process \( R_1 \) can have splitting (3) with leg-a and the pair \((x_a, x_3)\) is chosen. The reduced Born process \( B1 \) associates set \{y\} as \( B1 = u(y_a) \bar{u}(y_b) \rightarrow u(y_1) \bar{u}(y_2) \), and the relation \( F(x_{a3}) = F(y_a) = u \) stands. The process \( R_1 \) can also have splitting (6) with leg-a and the pair \((x_a, x_1)\) is chosen. The reduced Born process \( B3u \) associates set \{y\} as \( B3u = g(y_a) \bar{u}(y_b) \rightarrow \bar{u}(y_1) g(y_2) \) and the relation \( F(x_{a1}) = F(y_a) = g \) stands. Then we write down the collinear subtraction terms as

\[
\hat{\sigma}_C(R_1, \text{dip } 1, x_a) = \frac{A_d}{S_{B1}} \int_0^1 dx \left( \frac{1}{\epsilon} - \ln \mu_F^2 \right) P^{ff}(x) \cdot \Phi_a(B1, x)_d \langle B1 \rangle, \tag{2.67}
\]

\[
\hat{\sigma}_C(R_1, \text{dip } 3u) = \frac{A_d}{S_{B3}} \int_0^1 dx \left( \frac{1}{\epsilon} - \ln \mu_F^2 \right) P^{fg}(x) \cdot \Phi_a(B3u, x)_d \langle B3u \rangle, \tag{2.68}
\]

with the symmetric factors \( S_{B1} = S_{B3} = 1 \). Then we construct the third term \( \sigma_P(R_1) \). The P terms created by the DSA are summarized in Table B17 in Appendix B3 in Ref. [1]. We write down the terms \( \sigma_P(R_1, \text{dip } 1, x_a) \) and \( \sigma_P(R_1, \text{dip } 3u) \) as

\[
\hat{\sigma}_P(R_1, \text{dip } 1, x_a) = \frac{A_4}{S_{B1}} \int_0^1 dx \frac{1}{C_F} P^{ff}(x) \cdot \Phi_a(B1, x)_4 \\
\times \left[ \ln \frac{\mu_F^2}{x_{s_{x_a, y_1}}} (a, 1) + \ln \frac{\mu_F^2}{x_{s_{x_a, y_2}}} (a, 2) + \ln \frac{\mu_F^2}{x_{s_{x_a, y_b}}} (a, b) \right]. \tag{2.69}
\]

\[
\hat{\sigma}_P(R_1, \text{dip } 3u) = \frac{A_4}{S_{B3}} \int_0^1 dx \frac{1}{C_A} P^{fg}(x) \cdot \Phi_a(B3u, x)_4 \\
\times \left[ \ln \frac{\mu_F^2}{x_{s_{x_a, y_1}}} (a, 1) + \ln \frac{\mu_F^2}{x_{s_{x_a, y_2}}} (a, 2) + \ln \frac{\mu_F^2}{x_{s_{x_a, y_b}}} (a, b) \right]. \tag{2.70}
\]
Now that all the terms are explicitly written down, we start the calculation from the summation, 
\[ \hat{\sigma}_D(R_1, P, \text{dip} 1, x_a) + \hat{\sigma}_C(R_1, \text{dip} 1, x_a), \]
as
\[
\hat{\sigma}_D(R_1, P, \text{dip} 1, x_a) + \hat{\sigma}_C(R_1, \text{dip} 1, x_a) = \frac{A_d}{S_{B1}} \int_0^1 dx \frac{1}{C_F} P_{ff}(x) \Phi_d(B1, x)_d 
\times \left[ \left( \frac{1}{\epsilon} - \ln x \right) \frac{1}{C_F} \left( s_{x_a,y_1}^{-\epsilon} \langle a, 1 \rangle + s_{x_a,y_2}^{-\epsilon} \langle a, 2 \rangle + s_{x_a,y_b}^{-\epsilon} \langle a, b \rangle \right) + \left( \frac{1}{\epsilon} - \ln \mu_F^2 \right) (B1) \right].
\tag{2.71}
\]
where the relation \( S_{R1} = S_{B1} = 1 \) is used. We expand the factor with the Lorentz scalar as \( s^{-\epsilon} = 1 - \epsilon \ln s \), and also expand the squared amplitude \( (B1) \) by using the color conservation law as
\[
\langle B1 \rangle = \frac{1}{C_F} \left| B1 \right| T_{y_a} \cdot T_{y_a} \left| B1 \right| = -\frac{1}{C_F} \left( \langle a, 1 \rangle + \langle a, 2 \rangle + \langle a, b \rangle \right).
\tag{2.72}
\]
The use of the color conservation law in dipole subtraction is explained in Ref. [2]. Then we obtain the expression as
\[
\hat{\sigma}_D(R_1, P, \text{dip} 1, x_a) + \hat{\sigma}_C(R_1, \text{dip} 1, x_a) = \frac{A_d}{S_{B1}} \int_0^1 dx \frac{1}{C_F} P_{ff}(x) \cdot \Phi_d(B1, x)_d 
\times \left[ \ln \frac{\mu_F^2}{x s_{x_a,y_1}} \langle a, 1 \rangle + \ln \frac{\mu_F^2}{x s_{x_a,y_2}} \langle a, 2 \rangle + \ln \frac{\mu_F^2}{x s_{x_a,y_b}} \langle a, b \rangle \right].
\tag{2.73}
\]
Since the expression is now finite in 4 dimensions, it is safely reduced back to 4 dimensions, which becomes nothing but the term \( \hat{\sigma}_P(R_1, \text{dip} 1, x_a) \) in Eq. (2.69). In this way, the relation
\[
\sigma_D(R_1, P, \text{dip} 1, x_a) + \sigma_C(R_1, \text{dip} 1, x_a) - \sigma_P(R_1, \text{dip} 1, x_a) = 0
\tag{2.74}
\]
is proved. Finally, we calculate the summation \( \hat{\sigma}_D(R_1, P, \text{dip} 3u) + \hat{\sigma}_C(R_1, \text{dip} 3u) \) as
\[
\hat{\sigma}_D(R_1, P, \text{dip} 3u) + \hat{\sigma}_C(R_1, \text{dip} 3u) = \frac{A_d}{S_{B3}} \int_0^1 dx \frac{1}{C_A} P_{ff}(x) \Phi_d(B3u, x)_d 
\times \left[ \left( \frac{1}{\epsilon} - \ln x \right) \frac{1}{C_A} \left( s_{x_a,y_1}^{-\epsilon} \langle a, 1 \rangle + s_{x_a,y_2}^{-\epsilon} \langle a, 2 \rangle + s_{x_a,y_b}^{-\epsilon} \langle a, b \rangle \right) + \left( \frac{1}{\epsilon} - \ln \mu_F^2 \right) \langle B3u \rangle \right].
\tag{2.75}
\]
Similar to the calculation in Eq. (2.71), we expand the factor with the Lorentz scalar and use the color conservation law as \( \langle B3u \rangle = -\left( \langle a, 1 \rangle + \langle a, 2 \rangle + \langle a, b \rangle \right) / C_A \). Then the quantity in Eq. (2.75) becomes finite in 4 dimensions, which is equal to the term \( \hat{\sigma}_P(R_1, \text{dip} 3u) \). The relation in Eq. (2.63) for \( \text{dip} 3u \) is proved. The relations in Eq. (2.63) for the remaining cases, Dipole 1 with leg-b and Dipole 3u, are similarly proved. Thus the relation in Eq. (2.59) is proved and Step 3 for the process \( R_1 \) is completed. The results are shown in Eqs. (B2) and (B3) in Appendix B1.

2.5. Step 4: \( \sigma_D(K) - \sigma_K \)

Step 4 of the PRA is to prove the relation
\[
\sigma_D(R_i, K) - \sigma_K(R_i) = -\sigma_K(R_i, \text{dip} 1, (3)/(4)-1, N_f h).
\tag{2.76}
\]
The left-hand side of Eq. (2.76) is the third term in the square brackets in Eq. (2.11). We define the three cross sections in Eq. (2.76) as follows.
We first define the cross section \(\sigma_D(R_i, K)\). The integrated dipole terms are separated into the four terms shown in Eq. (2.10). Among them, we have already defined the terms \(\sigma_D(R_i, I)\) and \(\sigma_D(R_i, P)\) in Sects. 2.3 and 2.4, respectively. The term \(\sigma_D(R_i, \text{dip}2)\) is defined as the dipole term with splitting (5) in the category Dipole 2, which is concretely shown in Sect. 2.6. Then the term \(\sigma_D(R_i, K)\) is defined as the remaining term in Eq. (2.10) as

\[
\sigma_D(R_i, K) = \sigma_D(R_i) - \sigma_D(R_i, I) - \sigma_D(R_i, P) - \sigma_D(R_i, \text{dip}2).
\]  

(2.77)

All the poles \(1/\epsilon^2\) and \(1/\epsilon\) in the integrated dipole term \(\sigma_D(R_i)\) are extracted by the three terms \(\sigma_D(R_i, I/P/\text{dip}2)\). Then the cross section \(\sigma_D(R_i, K)\) is finite and defined in 4 dimensions. The term \(\sigma_D(R_i, K)\) is classified by the splittings shown in Fig. A4 in Appendix A5. We start with the category Dipole 1, which includes splittings (1), (2), (3), and (4). For splittings (1)/(2)-2, the remaining factors \(C_F(g(x) - 3h(x)/2)\) and \(C_A(2g(x) - 11h(x)/3)\) in the functions \(V_{fg}(x; \epsilon)\) and \(V_{gg}(x; \epsilon)\) in Eqs. (A30) and (A31), respectively, are taken. We only show here the full expression for splitting (1)-2 with leg-a as

\[
\hat{\sigma}_D(R_i, K, \text{dip}1, (1)-2, x_a) = -\frac{A_4}{S_{R_i}} \int_0^{1} dx \left( g(x) - \frac{3}{2}h(x) \right) \cdot \Phi_a(B_1, x)_4 \langle y_{emi}, y_{spe} \rangle.
\]  

(2.78)

For splittings (3)/(4)-1, the remaining terms \(V_{\text{other}}^{f,f/g,g}(x; \epsilon)\) in the functions \(V_{\text{other}}^{f,f/g,g}(x; \epsilon)\) in Eqs. (A41)/(A42), respectively, are taken. The cross section for splitting (3)-1 with leg-a is written as

\[
\hat{\sigma}_D(R_i, K, \text{dip}1, (3)-1, x_a) = -\frac{A_4}{S_{R_i}} \int_0^{1} dx \frac{1}{C_F} \gamma_{\text{other}}^{f,f}(x; \epsilon) \cdot \Phi_a(B_1, x)_4 \langle y_{emi}, y_{spe} \rangle.
\]  

(2.79)

For splittings (3)/(4)-2, the terms \(\gamma_{\text{other}}^{f,f}(x; \epsilon) + C_F g(x) + \tilde{K}^{f,f}(x)\) in \(\tilde{V}_{\text{other}}^{f,f}(x; \epsilon)\) and \(\tilde{V}_{\text{other}}^{g,g}(x; \epsilon)\) in Eqs. (A59) and (A60), respectively, are extracted. The expression for splitting (3)-2 is written as

\[
\hat{\sigma}_D(R_i, K, \text{dip}1, (3)-2, x_a) = -\frac{A_4}{S_{R_i}} \int_0^{1} dx \frac{1}{C_F} \left( \gamma_{\text{other}}^{f,f}(x; \epsilon) + C_F g(x) + \tilde{K}^{f,f}(x) \right) \times \Phi_a(B_1, x)_4 \langle y_{emi}, y_{spe} \rangle.
\]  

(2.80)

As mentioned above, the cross section \(\sigma_D(R_i, K)\) is defined in 4 dimensions, and so in Eqs. (2.78)–(2.80) the overall factor \(A_4\), the phase space \(\Phi_a(B_1, x)_4\), and the reduced Born squared amplitude \(\langle y_{emi}, y_{spe} \rangle\) are all defined in 4 dimensions. The formulae for \(\hat{\sigma}_D(R_i, K, \text{dip}1)\) are collected in Eqs. (A80)–(A82) in Appendix A5. Then we proceed to the Dipoles 3 and 4 category, i.e., splittings (6) and (7), respectively. For splittings (6)/(7)-1, the terms \(\gamma_{\text{other}}^{f,g/g,f}(x; \epsilon)\) in Eqs. (A43)/(A44) are taken. For splittings (6)/(7)-2, the terms \(\gamma_{\text{other}}^{f,g/g,f}(x; \epsilon) + \tilde{K}^{f,g/g,f}(x)\) in Eqs. (A61)/(A62) are taken. The formulae for splittings (6)/(7)-1/2 are collected in Eqs. (A83) and (A84).

Next we define the cross section \(\sigma_K(R_i)\). This quantity is the contribution of the K terms created by the DSA [1]. The K terms are classified into categories Dipoles 1 and 3/4 with subcategories (3)/(4)-0/1/2 and (6)/(7)-0/2, respectively, as shown in Fig. A9 in Appendix A10. The contributions to the partonic cross sections are denoted as \(\sigma_K(R_i, \text{dip}1, (3)/(4)-0/1/2)\) and \(\sigma_K(R_i, \text{dip}3/4, (6)/(7)-0/2)\), respectively. The explicit expressions for the cross sections are collected in Eqs. (A115)–(A116) and Eqs. (A117) and (A118), respectively. We show here the expression
for the cross section in Dipole 1 with splitting (3)/(4)-1 and leg-a as

$$\hat{\sigma}_K(R_i, \text{dip}1, (3)/(4)-1, x_a) = \frac{A_4}{S_{B1}} \int_0^1 dx \frac{\gamma F(y_{spe})}{T_F^2(y_{spe})} h(x) \cdot \Phi_{a/B1, x_4} (y_{emi}, y_{spe}).$$  (2.81)

This term has a special feature that the factor \(\gamma F(y_{spe})/T_F^2(y_{spe})\) is determined by the field species of the spectator, \(F(y_{spe})\), unlike the other terms.

The third cross section, \(\sigma_K(R_i, \text{dip}1, (3)/(4)-1, N_f h)\), is defined as follows. When the cross section \(\hat{\sigma}_K(R_i, \text{dip}1, (3)/(4)-1)\) in Eq. (2.81), which is in the category Dipole 1 with splitting (3)/(4)-1, has the gluon in the final state of the reduced Born process as the spectator, i.e., \(F(y_{spe}) = \text{gluon}\), the factor \(\gamma g/T_g^2\) is determined as \(\gamma g/T_g^2 = 11/6 - T_R N_f/3C_A\) in Eq. (A121). From the factor we extract the second term, \(-2T_R N_f/3C_A\), and define \(\sigma_K(R_i, \text{dip}1, (3)/(4)-1, N_f h)\) as

$$\sigma_K(R_i, \text{dip}1, (3)/(4)-1, N_f h) = \frac{A_4}{S_{B1}} \int_0^1 dx \left( -2 \frac{T_R N_f}{3C_A} \right) h(x) \cdot \Phi_{a/B1, x_4} (y_{emi}, y_{spe}).$$  (2.82)

Now all three cross sections in Eq. (2.76) are defined.

The relation in Eq. (2.76) is separated into three independent ones,

$$\sigma_D(R_i, K, \text{dip}1) - \sigma_K(R_i, \text{dip}1) = -\sigma_K(R_i, \text{dip}1, (3)/(4)-1, N_f h),$$  (2.83)

$$\sigma_D(R_i, K, \text{dip}3) - \sigma_K(R_i, \text{dip}3) = 0,$$  (2.84)

$$\sigma_D(R_i, K, \text{dip}4) - \sigma_K(R_i, \text{dip}4) = 0,$$  (2.85)

because the cancellations are realized between the cross sections with the same reduced Born processes. In order to prove the relations in Eqs. (2.83)–(2.85) in a systematic way, we also divide Step 4 into substeps defined as follows.

For the relation with Dipole 1 in Eq. (2.83), Step 4 is divided into three substeps as

4-1. \(\hat{\sigma}_D(K, \text{dip}1, (1)-(4)-2, g) \hat{\sigma} + \hat{\sigma}_D(K, \text{dip}1, (3)/(4)-1/2, \gamma_{other}^{a,a})\)

$$\hat{\sigma}_K(\text{dip}1, (3)/(4)-0) = 0,$$  (2.86)

4-2. \(\hat{\sigma}_D(K, \text{dip}1, (1)/(2)-2, h) - \hat{\sigma}_K(\text{dip}1, (3)/(4)-1) = -\hat{\sigma}_K(\text{dip}1, (3)/(4)-1, N_f h),$$  (2.87)

4-3. \(\hat{\sigma}_D(K, \text{dip}1, (3)/(4)-2, \tilde{K}^{a,a}) - \hat{\sigma}_K(\text{dip}1, (3)/(4)-2) = 0,$$  (2.88)

where the argument \(R_i\) in the cross sections is omitted as \(\hat{\sigma}(R_i, X) \rightarrow \hat{\sigma}(X)\) for the compact notation. The cross section \(\sigma_D(R_i, K, \text{dip}1)\) is reconstructed into four terms as

$$\sigma_D(R_i, K, \text{dip}1) = \hat{\sigma}_D(K, \text{dip}1, (1)-(4)-2, g) \hat{\sigma} + \hat{\sigma}_D(K, \text{dip}1, (3)/(4)-1/2, \gamma_{other}^{a,a})$$

$$+ \hat{\sigma}_D(K, \text{dip}1, (1)/(2)-2, h) + \hat{\sigma}_D(K, \text{dip}1, (3)/(4)-2, \tilde{K}^{a,a}(x)),$$  (2.89)

which appear in Eqs. (2.86)–(2.88). We define the four terms as follows. The term \(\hat{\sigma}_D(K, \text{dip}1, (1)-(4)-2, g) \hat{\sigma}\) is defined in such a way that, in the case with splittings (1)–(4)-2 in Eqs. (A80) and (A82), the terms including the function \(g(x)\) are extracted. The expressions are shown in Eq. (A86). The symbol \(\hat{\sigma}\) in the term \(\hat{\sigma}_D(K, \text{dip}1, (1)-(4)-2, g) \hat{\sigma}\) means that, when all the existing terms of \(\hat{\sigma}_D(K, \text{dip}1, (1)-(4)-2, g)\) are summed, the expressions include the summation of the
color-correlated Born squared amplitudes such as

$$\langle a, 1 \rangle + \langle a, 2 \rangle + \cdots + \langle a, n \rangle + \langle a, b \rangle.$$  (2.90)

The color conservation law can be applied to the summation and the color correlations are factorized as

$$\sum_{i=1}^{n} \langle a, i \rangle + \langle a, b \rangle = -\langle a, a \rangle = -T_f^2(\gamma_a) \cdot (B1).$$  (2.91)

The next term $\hat{\sigma}_D(K, dip1, (3)/(4)-1/2, \gamma_{\text{other}}^{a,b})$ is defined in such a way that, in the case with splittings $(3)/(4)-1/2$ in Eqs. (A81) and (A82), the terms $\gamma_{\text{other}}^{a,b}(x; \epsilon)$ are extracted. The expressions are shown in Eqs. (A87) and (A88). The meaning of the symbol ‘$\xi$’ in the term $\hat{\sigma}_D(K, dip1, (3)/(4)-1/2, \gamma_{\text{other}}^{a,b})$ is the same as in the previous term $\hat{\sigma}_D(K, dip1, (1)-(4)-2, g)^{\xi}$. The third term $\hat{\sigma}_D(K, dip1, (1)/(2)-2, h)$ is defined in such a way that the terms including the function $h(x)$ in splittings $(1)/(2)-2$ in Eq. (A80) are extracted. The expressions are written in Eq. (A89). The last term $\hat{\sigma}_D(K, dip1, (3)/(4)-2, \tilde{K}^{a,b})$ is defined in such a way that the terms $\tilde{K}^{f/g}(x)$ in splittings $(3)/(4)-2$ in Eq. (A82) are extracted, which are written in Eq. (A90). The way of executing substeps 4-1–4-3 is demonstrated in the example later in the present subsection.

For the relations with Dipoles 3 and 4 in Eqs. (2.84) and (2.85), Step 4 is divided into two substeps as

**Step 4**

1. $\hat{\sigma}_D\left(K, dip3/4, (6)/(7)-1/2, \gamma_{\text{other}}^{a,b}\right) - \hat{\sigma}_K(\text{dip3/4, (6)/(7)-0}) = 0,$  (2.92)
2. $\hat{\sigma}_D\left(K, dip3/4, (6)/(7)-2, \tilde{K}^{ab}\right) - \hat{\sigma}_K(\text{dip3/4, (6)/(7)-2}) = 0,$  (2.93)

where again the argument $R_i$ is abbreviated. The cross section $\sigma_D(R_i, K, dip3/4)$ is reconstructed into two terms as

$$\sigma_D(R_i, K, dip3/4) = \hat{\sigma}_D\left(K, dip3/4, (6)/(7)-1/2, \gamma_{\text{other}}^{a,b}\right) + \hat{\sigma}_D\left(K, dip3/4, (6)/(7)-2, \tilde{K}^{ab}\right).$$  (2.94)

The terms $\gamma_{\text{other}}^{a,b}(x; \epsilon)$ in Eqs. (A83) and (A84) are extracted and the cross sections $\hat{\sigma}_D(K, dip3/4, (6)/(7)-1/2, \gamma_{\text{other}}^{a,b})$ are defined in Eqs. (A92) and (A93), respectively. The meaning of the symbol ‘$\xi$’ for this term is the same as in the case of Dipole 1. The terms $\tilde{K}^{ab}(x)$ in Eq. (A84) are extracted and the cross sections $\hat{\sigma}_D(K, dip3/4, (6)/(7)-2, \tilde{K}^{ab})$ are defined in Eq. (A94). The method of execution is demonstrated in the following example.

We demonstrate the execution of Step 4 in the same process as used in the previous sections, $R_1 = u\bar{u} \rightarrow u\bar{u}g$. The relation in Eq. (2.76) is written for the process $R_1$ as

$$\sigma_D(R_1, K) - \sigma_K(R_1) = 0,$$  (2.95)

where the right-hand side of Eq. (2.76) does not exist, because the reduced Born process $B1(R_1) = u\bar{u} \rightarrow u\bar{u}$ does not include any gluon in the final state. The integrated dipole terms $\sigma_D(R_1)$ are summarized in Table B1 in Appendix B1. The dipole terms include three categories, Dipoles 1, 3u,
and $\hat{\sigma}$, and the relation in Eq. (2.95) is separated into the three as

$$\sigma(D(R_1, K, dip1) - \sigma_K(R_1, dip1) = 0, \quad \sigma(D(R_1, K, dip3u) - \sigma_K(R_1, dip3u) = 0, \quad \sigma(D(R_1, K, dip3\bar{u}) - \sigma_K(R_1, dip3\bar{u}) = 0. \quad (2.96)$$

First, we prove the relation for Dipole 1 in Eq. (2.96). In order to prove it, we have the three steps as shown in Eqs. (2.86)–(2.88):

4-1. $\hat{\sigma}_D(K, dip1, (1)/(3)-2, g)^\circ + \hat{\sigma}_D(K, dip1, (3)-1/2, \nu_{\text{other}}^{ff})$

$$- \hat{\sigma}_K(dip1, (3)-0) = 0. \quad (2.99)$$

4-2. $\hat{\sigma}_D(K, dip1, (1)-2, h) - \hat{\sigma}_K(dip1, (3)-1) = 0. \quad (2.100)$

4-3. $\hat{\sigma}_D(K, dip1, (3)-2, K^{ff}) - \hat{\sigma}_K(dip1, (3)-2) = 0. \quad (2.101)$

$\hat{\sigma}_D(R_1, K, dip1)$ with leg-a is written as a summation of the possible splittings:

$$\hat{\sigma}_D(R_1, K, dip1, x_a) = \hat{\sigma}_D(R_1, K, dip1, (1)-2, x_a) + \hat{\sigma}_D(R_1, K, dip1, (3)-1, x_a)
+ \hat{\sigma}_D(R_1, K, dip1, (3)-2, x_a). \quad (2.102)$$

We explicitly prove the relations in Eqs. (2.99)–(2.101) with leg-a as follows. The relations with leg-b are also similarly proved. $\hat{\sigma}_D(R_1, K, dip1, x_a)$ is reconstructed into the four terms in Eq. (2.89) as

$$\sigma_D(R_1, K, dip1) = \hat{\sigma}_D(K, dip1, (1)/(3)-2, g)^\circ + \hat{\sigma}_D(K, dip1, (3)-1/2, \nu_{\text{other}}^{ff})^\circ
+ \hat{\sigma}_D(K, dip1, (1)-2, h) + \hat{\sigma}_D(K, dip1, (3)-2, K^{ff}). \quad (2.103)$$

where each term is written as

$$\hat{\sigma}_D(K, dip1, (1)/(3)-2, g)^\circ = -\frac{A_4}{S_{R_1}} \int_0^1 dx g(x) \Phi_a(B1)4((1, a) + (2, a) + (a, b)), \quad (2.104)$$

$$\hat{\sigma}_D(K, dip1, (3)-1/2, \nu_{\text{other}}^{ff})^\circ = -\frac{A_4}{S_{R_1}} \int_0^1 dx \frac{1}{C_F} \nu_{\text{other}}^{ff}(x; \epsilon) \cdot \Phi_a(B1)4 \times ((a, 1) + (a, 1) + (a, b)), \quad (2.105)$$

$$\hat{\sigma}_D(K, dip1, (1)-2, h) = -\frac{A_4}{S_{R_1}} \int_0^1 dx \left(-\frac{2}{3} h(x)\right) \Phi_a(B1)4((1, a) + (2, a)), \quad (2.106)$$

$$\hat{\sigma}_D(K, dip1, (3)-2, K^{ff}) = -\frac{A_4}{S_{R_1}} \int_0^1 dx \frac{1}{C_F} K^{ff}(x) \cdot \Phi_a(B1)4(a, b). \quad (2.107)$$

Here the argument $x_a$ is suppressed. Next we write down the cross sections of the K terms, $\hat{\sigma}_K(R_1, dip1, (3)-0/1/2).$ The K terms created by the DSA are summarized in Table B17 in Appendix B.3 in Ref. [1]. They are written down as

$$\hat{\sigma}_K(R_1, dip1, (3)-0, x_a) = \frac{A_4}{S_{B_1}} \int_0^1 dx \tilde{K}^{ff}(x) \cdot \Phi_a(B1)4(B1). \quad (2.108)$$

$$\hat{\sigma}_K(R_1, dip1, (3)-1, x_a) = \frac{A_4}{S_{B_1}} \int_0^1 dx \frac{3}{2} h(x) \cdot \Phi_a(B1)4((a, 1) + (a, 2)), \quad (2.109)$$

$$\hat{\sigma}_K(R_1, dip1, (3)-2, x_a) = \frac{A_4}{S_{B_1}} \int_0^1 dx \frac{1}{C_F} \tilde{K}^{ff}(x) \cdot \Phi_a(B1)4(a, b). \quad (2.110)$$
Then we execute from step 4-1. We calculate the summation \( \hat{\sigma}_D(K, \text{dip}3u, (1)-2, g) \) + \( \hat{\sigma}_D(K, \text{dip}1, (3)-1/2, \nu_{\text{other}}^f,g) \) as

\[
\hat{\sigma}_D(K, \text{dip}1, (1)/(3)-2, g) + \hat{\sigma}_D(K, \text{dip}1, (3)-1/2, \nu_{\text{other}}^f,g) = \frac{A_4}{S_{R_1}} \int_0^1 dx \Phi_a(B1)_4 \times (B1) \left( C_{FG}(x) + \nu_{\text{other}}^f,g(x; \epsilon) \right),
\]

(2.111)

where we use the color conservation law as \( \langle a, 1 \rangle + \langle a, 1 \rangle + \langle a, b \rangle = -C_F(B1) \). Noting the relations \( K_{ff} (x) = \nu_{\text{other}}^f,g(x; \epsilon) + C_{FG}(x) \) and \( S_{R_1} = S_{B_1} = 1 \), the relation in Eq. (2.99) is proved. Then we proceed to step 4-2. The left-hand side of the relation in Eq. (2.100) is calculated as

\[
\hat{\sigma}_D(K, \text{dip}1, (1)-2, h) - \hat{\sigma}_K(\text{dip}1, (3)-1) = \frac{A_4}{S_{B_1}} \int_0^1 dx \Phi_a(B1)_4 \frac{3}{2} h(x) \times \left[ \langle (1, a) + \langle 2, a \rangle - \langle (1, 1) + \langle 1, 2 \rangle \rangle \right] = 0,
\]

(2.112)

where the relation \( \langle a, 1/2 \rangle = \langle 1/2, a \rangle \) is used. In step 4-3 the relation in Eq. (2.101) is trivial with Eqs. (2.107) and (2.110). In this way, the relation with leg-a in Eq. (2.96) is proved.

Finally, we prove the relation for Dipole 3u in Eq. (2.97). To prove it, we have two substeps as

4-1. \( \hat{\sigma}_D(K, \text{dip}3u, (6)-1/2, \nu_{\text{other}}^f,g) = 0 \),

4-2. \( \hat{\sigma}_D(K, \text{dip}3u, (6)-2, \tilde{K}_{fg}) - \hat{\sigma}_K(\text{dip}3u, (6)-2) = 0 \),

(2.114)

\( \hat{\sigma}_D(R_1, K, \text{dip}3u) \) is written as the summation

\[
\hat{\sigma}_D(R_1, K, \text{dip}3u) = \hat{\sigma}_D(R_1, K, \text{dip}3u, (6)-1) + \hat{\sigma}_D(R_1, K, \text{dip}3u, (6)-2),
\]

(2.115)

which is reconstructed as

\[
\hat{\sigma}_D(R_1, K, \text{dip}3u) = \hat{\sigma}_D(K, \text{dip}3u, (6)-1/2, \nu_{\text{other}}^f,g) + \hat{\sigma}_D(K, \text{dip}3u, (6)-2, \tilde{K}_{fg}).
\]

(2.116)

The two terms are written as

\[
\hat{\sigma}_D(K, \text{dip}3u, (6)-1/2, \nu_{\text{other}}^f,g) = -\frac{A_4}{S_{R_1}} \int_0^1 dx \frac{1}{C_A} \nu_{\text{other}}^f,g(x; \epsilon) \cdot \Phi_a(B3u)_4 \times (\langle a, 1 \rangle + \langle a, 2 \rangle + \langle a, b \rangle),
\]

(2.117)

\[
\hat{\sigma}_D(K, \text{dip}3u, (6)-2, \tilde{K}_{fg}) = -\frac{A_4}{S_{R_1}} \int_0^1 dx \frac{1}{C_A} \tilde{K}_{fg}(x) \cdot \Phi_a(B3u)_4 (a, b).
\]

(2.118)

Referring to the K terms with splittings (6)u-0/2 in Table B17 in Ref. [1], the contributions of the K terms are written down as

\[
\hat{\sigma}_K(\text{dip}3u, (6)-0) = \frac{A_4}{S_{B_{3u}}} \int_0^1 dx \tilde{K}_{fg}(x) \cdot \Phi_a(B3u)_4 (B3u),
\]

(2.119)

\[
\hat{\sigma}_K(\text{dip}3u, (6)-2) = \frac{A_4}{S_{B_{3u}}} \int_0^1 dx \frac{1}{C_A} \tilde{K}_{fg}(x) \cdot \Phi_a(B3u)_4 (a, b).
\]

(2.120)
Then we start from step 4-1. The left-hand side of Eq. (2.113) is calculated as

\[
\hat{\sigma}_D \left( K, \text{dip}3u, (6)-1/2, \sigma_{\text{other}}^{f,g} \right) - \hat{\sigma}_K (\text{dip}3u, (6)-0) = \frac{A_4}{S_{B3u}} \int_0^1 dx \Phi_d (B3u)_A \times \langle B3u \rangle \left( \sigma_{\text{other}}^{f,g}(x; \epsilon) - \bar{K}^{f,g}(x) \right) = 0, \tag{2.121}
\]

where we use the relations \( S_{R1} = S_{B3u} = 1 \), the color conservation \( \langle a, 1 \rangle + \langle a, 1 \rangle + \langle a, b \rangle = -C_A (B3u) \), and \( \bar{K}^{f,g/r,g/r}(x) = \sigma_{\text{other}}^{f,g}(x; \epsilon) \). In step 4-2, the relation in Eq. (2.114) trivially stands with Eqs. (2.118) and (2.120). Then the relation in Eq. (2.97) is proved.

In a similar way, the relations for Dipole 1 with leg-\( b \) and for Dipole 3\( \bar{u} \) can be proved. Then the relation in Eq. (2.95) is proved and Step 4 for the process \( R_1 \) is completed. The results are shown in Eqs. (B4) and (B5) in Appendix B1. In this example, the right-hand side of Eq. (2.76), \( \sigma_K (R_i, \text{dip}1, (3)/(4)-1, N_f h) \), does not exist. The presence or absence of the right-hand side is the same as the right-hand side \( \sigma_1 (R_i, (2)-1/2, N_f \sigma_{\text{other}}^{f,g}) \) in Eq. (2.38) in Step 2. The cases where the right-hand side exists will be seen in Sects. 4 and 5.

2.6. Step 5: \( \sigma_{\text{subt}} \)

Step 5 of the PRA is to write down the cross section \( \sigma_{\text{subt}}(R_i) \) in Eq. (2.11). We substitute the first three terms in square brackets by the proved relations in Eqs. (2.38), (2.51), and (2.76) in Steps 2, 3, and 4, respectively. Then we obtain \( \sigma_{\text{subt}}(R_i) \) in the expression

\[
\sigma_{\text{subt}}(R_i) = -\sigma_1 (R_i, (2)-1/2, N_f \sigma_{\text{other}}^{f,g}) - \sigma_K (R_i, \text{dip}1, (3)/(4)-1, N_f h) + \sigma_D (R_i, \text{dip}2).
\tag{2.122}
\]

The cross sections \( \sigma_1 (R_i, (2)-1/2, N_f \sigma_{\text{other}}^{f,g}) \) and \( \sigma_K (R_i, \text{dip}1, (3)/(4)-1, N_f h) \) are defined in Steps 2 and 4 in Sects. 2.3 and 2.5, respectively. We here define the cross section \( \sigma_D (R_i, \text{dip}2) \) as follows.

The term \( \sigma_D (R_i, \text{dip}2) \) is nothing but the integrated dipole terms in the category Dipole 2 with splitting (5)-1/2 in Fig. A1 in Appendix A2. The term is written as the summation of the two terms with splittings (5)-1 and -2:

\[
\hat{\sigma}_D (R_i, \text{dip}2) = \hat{\sigma}_D (R_i, \text{dip}2, (5)-1) + \hat{\sigma}_D (R_i, \text{dip}2, (5)-2).
\tag{2.123}
\]

The expressions for the two terms are shown in Eqs. (A96) and (A97), respectively, in Appendix A6. For use in the last step, Step 6, we reconstruct the two terms into two different ones as

\[
\hat{\sigma}_D (R_i, \text{dip}2) = \hat{\sigma}_D (R_i, \text{dip}2, (5)-1/2, \sigma_{\text{other}}^{f,g}) + \hat{\sigma}_D (R_i, \text{dip}2, (5)-2, h).
\tag{2.124}
\]

The term \( \hat{\sigma}_D (R_i, \text{dip}2, (5)-1/2, \sigma_{\text{other}}^{f,g}) \) is defined in such a way that, for splitting (5)-1, all the terms are taken, and, for splitting (5)-2, the term including the function \( \sigma_{\text{other}}^{f,g}(\epsilon) \) in Eq. (A32) is extracted. The term \( \hat{\sigma}_D (R_i, \text{dip}2, (5)-2, h) \) is defined in such a way that, for splitting (5)-2, the term with the function \( h(x) \) in Eq. (A32) is extracted. The expressions are written as

\[
\hat{\sigma}_D (R_i, \text{dip}2, (5)-1/2, \sigma_{\text{other}}^{f,g}) = -\frac{A_d}{S_{R_i}} \cdot \frac{1}{C_A} \sigma_{\text{other}}^{f,g}(\epsilon) \cdot \Phi (B2)_d \left[ y_{\text{emi}}, y_{\text{spe}} \right],
\tag{2.125}
\]

\[
\hat{\sigma}_D (R_i, \text{dip}2, (5)-2, h) = -\frac{A_d}{S_{R_i}} \int_0^1 dx \frac{T_R}{C_A^2} h(x) \cdot \Phi (B2)_d (B2, x)_4 \left[ y_{\text{emi}}, y_{\text{spe}} \right].
\tag{2.126}
\]

The first term \( \hat{\sigma}_D (R_i, \text{dip}2, (5)-1/2, \sigma_{\text{other}}^{f,g}) \) is defined in \( d \) dimensions with the color-correlated Born squared amplitude \( \left[ y_{\text{emi}}, y_{\text{spe}} \right] = (s_{y_{\text{emi}}, y_{\text{spe}}})_{-\epsilon} \cdot \langle y_{\text{emi}}, y_{\text{spe}} \rangle_d \). The second term
\[ \hat{\sigma}_D(R_i, \text{dip}, (5)-2, h) \text{ is finite in 4 dimensions, and so the squared amplitude is defined in 4 dimensions as } \{y_{\text{em}}, y_{\text{spe}}\}. \]

The formulae for the term \( \sigma_D(R_i, \text{dip2}) \) are collected in Appendix A6. As mentioned in Steps 2 and 4, the terms \( \sigma_1 \left( R_i, (2)-1/2, N_f V_f \right) \) and \( \sigma_K(R_i, \text{dip1}, (3)/(4)-1, N_f h) \) on the right-hand side of Eq. (2.122) exist only if the process \( R_i \) includes two or more gluons in the final state. The term \( \hat{\sigma}_D(R_i, \text{dip2}) \) exists only if \( R_i \) includes a quark–antiquark pair (\( q\bar{q} \)) pair in the final state. For instance, if the final state of \( R_i \) includes neither a gluon nor a \( q\bar{q} \) pair, the cross section \( \sigma_{\text{subt}}(R_i) \) vanishes itself as \( \sigma_{\text{subt}}(R_i) = 0 \).

Finally, we show one example in the process used in the previous sections, \( R_1 = u\bar{u} \rightarrow u\bar{u}g \). We have seen the results for \( R_1 \) in Steps 2, 3, and 4 in Eqs. (2.48), (2.59), and (2.95), respectively. In Eq. (2.11), we substitute the three terms in square brackets by the three proved relations and obtain the expression for \( \sigma_{\text{subt}}(R_1) \) as

\[
\sigma_{\text{subt}}(R_1) = \sigma_D(R_1, \text{dip2}).
\]  

(2.127)

Since the process \( R_1 \) includes only one gluon in the final state, the first two terms on the right-hand side of Eq. (2.122) do not exist. The cross section \( \sigma_D(R_1, \text{dip2}) \) is written as the summation of the two terms as in Eq. (2.123). Referring to Table B1 in Appendix B1, the two terms in the category Dipole 2 are written as

\[
\hat{\sigma}_D(R_1, \text{dip2}, (5)-1) = -\frac{A_d}{S_{R_1}} \cdot \frac{1}{C_A} V_f(x; \epsilon) \cdot \Phi(B2)_d[1, 2].
\]  

(2.128)

\[
\hat{\sigma}_D(R_1, \text{dip2}, (5)-2) = -\frac{A_d}{S_{R_1}} \int_0^1 dx \frac{1}{C_A} V_f(x; \epsilon) \left[ \Phi_a(B2, x)_d[1, a] + \Phi_b(B2, x)_d[1, b] \right].
\]  

(2.129)

where the reduced Born process is determined as \( B2 = u(y_a)\bar{u}(y_b) \rightarrow g(y_1)g(y_2) \). \( \sigma_D(R_1, \text{dip2}) \) is reconstructed into the two terms in Eq. (2.124) as

\[
\hat{\sigma}_D(R_1, \text{dip2}, (5)-1/2, V_f) = -\frac{A_d}{S_{R_1}} \cdot \frac{1}{C_A} V_f(x; \epsilon) \cdot \Phi(B2)_d[1, 2] + [1, a] + [1, b],
\]  

(2.130)

\[
\hat{\sigma}_D(R_1, \text{dip2}, (5)-2, h) = -\frac{A_d}{S_{R_1}} \int_0^1 dx \frac{T_R}{C_A} \frac{2}{3} h(x)
\]

\[
\times \left( \Phi_a(B2, x)_{\bar{a}}(1, a) + \Phi_b(B2, x)_{\bar{a}}(1, b) \right).
\]  

(2.131)

The results are summarized in Eqs. (B6)–(B9) in Appendix B1. In this way, we write down the term \( \sigma_{\text{subt}}(R_1) \) and Step 5 for \( R_1 \) is completed. In Sects. 4 and 5, we will see the cases where the two terms \( \sigma_1(R_i, (2)-1/2, N_f V_f) \) and \( \sigma_K(R_i, \text{dip1}, (3)/(4)-1, N_f h) \) in Eq. (2.122) exist.

2.7 Step 6: \( \sum_i \sigma_{\text{subt}}(R_i) = 0 \)

For a given collider process, the set of all the real emission processes is denoted as \( \{R_i\} = \{R_1, R_2, \ldots, R_{n_{\text{real}}}\} \) in Eq. (2.2). Steps 1–5 are repeated over all the real processes, \( R_1, R_2, \ldots, R_{n_{\text{real}}} \), and we obtain the corresponding cross sections \( \sigma_{\text{subt}}(R_i) \) as

\[
\{\sigma_{\text{subt}}(R_i)\} = \{\sigma(R_1), \sigma(R_2), \ldots, \sigma(R_{n_{\text{real}}})\}.
\]  

(2.132)

Each cross section is obtained in the expression in Eq. (2.122). Step 6 of the PRA is to prove that the summation of all the cross sections \( \sigma_{\text{subt}}(R_i) \) vanishes as

\[
\sum_{i=1}^{n_{\text{real}}} \sigma_{\text{subt}}(R_i) = 0.
\]  

(2.133)
We can prove the relation in Eq. (2.133) in a systematic way as follows. For the process $R_i$, which has two or more gluons in the final state, we introduce a set, $\text{Con}(R_i)$, defined as $\text{Con}(R_i) = \{R_i, (R_i^C)\}$, where the subset $(R_i^C)$ is defined with the massless quark flavors as $(R_i^C) = \{R_i^{C, u\bar{u}}, R_i^{C, d\bar{d}}, R_i^{C, s\bar{s}}, \ldots\}$. The element of the process $R_i^{C,q\bar{q}}$ is defined in such a way that two gluons in the final state of $R_i$ are replaced with the $q\bar{q}$ pair as

$$R_i^{C,q\bar{q}} = R_i - (gg)_{\text{final}} + (q\bar{q})_{\text{final}}. \quad (2.134)$$

We call the process $R_i^{C,q\bar{q}}$ the $gg$-$q\bar{q}$ conjugation of the process $R_i$. The process $R_i^{C,q\bar{q}}$ exists for all the massless quark flavors, for instance, $q = u, d, s, c$, and $b$, at the energy scale of the LHC. Then we introduce the cross section $\sigma(\text{Con}(R_i))$ as

$$\sigma(\text{Con}(R_i)) = -\sigma_l \left( R_i, (2)-1/2, N_f \mathcal{V}_{ff} \right) - \sigma_K \left( R_i, \text{dip}1, (3)/(4)-1, N_f h \right) + \sigma_D \left( R_i^{C,q\bar{q}}, \text{dip}2 \right) \cdot N_f. \quad (2.135)$$

Here the cross sections with different quark flavors $\sigma_D(R_i^{C,q\bar{q}}, \text{dip}2)$ have the same contributions and the summation of the cross sections with all the massless quark flavors is equal to the multiplication of the number of massless quark flavors, $N_f$, by the cross section with the single flavor. The cross section $\sigma_D(R_i^{C,q\bar{q}}, \text{dip}2)$ is separated into two terms, $\hat{\sigma}_D(R_i^{C,q\bar{q}}, \text{dip}2, (5)-1/2, \mathcal{V}_{ff})$ and $\hat{\sigma}_D(R_i^{C,q\bar{q}}, \text{dip}2, (5)-2, h)$, as shown in Eq. (2.124). On the right-hand side of Eq. (2.135), we prove the two relations of the cancellation as

$$-\sigma_l \left( R_i, (2)-1/2, N_f \mathcal{V}_{ff} \right) + \hat{\sigma}_D \left( R_i^{C,q\bar{q}}, \text{dip}2, (5)-1/2, \mathcal{V}_{ff} \right) \cdot N_f = 0, \quad (2.136)$$

$$-\sigma_K \left( R_i, \text{dip}1, (3)/(4)-1, N_f h \right) + \hat{\sigma}_D \left( R_i^{C,q\bar{q}}, \text{dip}2, (5)-2, h \right) \cdot N_f = 0, \quad (2.137)$$

which leads to the cancellation of $\sigma(\text{Con}(R_i))$ as

$$\sigma(\text{Con}(R_i)) = 0. \quad (2.138)$$

We introduce the set $\{\text{Con}(R_i)\}$, which consists of all the possible sets $\text{Con}(R_i)$ for the processes $(R_i)$. We further introduce the set $\text{Self}$. The set $\text{Self}$ contains the process $R_i$, the final state of which includes one or no gluon, and the final state includes no $q\bar{q}$ pair. As mentioned in the previous sections, for any element $R_i$ of the set $\text{Self}$, the cross section $\sigma_{\text{subt}}(R_i)$ itself vanishes as

$$\sigma_{\text{subt}}(R_i) = 0. \quad (2.139)$$

Using the sets $\{\text{Con}(R_i)\}$ and $\text{Self}$, the left-hand side of Eq. (2.133) is always reconstructed as

$$\sum_{i=1}^{n_{\text{real}}} \sigma_{\text{subt}}(R_i) = \sum_{\{\text{Con}(R_i)\}} \sigma(\text{Con}(R_i)) + \sum_{\text{Self} \supset R_j} \sigma_{\text{subt}}(R_j), \quad (2.140)$$

where the symbol $\sum_{\{\text{Con}(R_i)\}}$ represents the summation over all the sets $\text{Con}(R_i)$ of the set $\{\text{Con}(R_i)\}$, and the symbol $\sum_{\text{Self} \supset R_j}$ represents the summation over all the processes $R_j$ of the set $\text{Self}$. Due to the cancellations, $\sigma(\text{Con}(R_i)) = 0$ and $\sigma_{\text{subt}}(R_j) = 0$, the relation in Eq. (2.133) is proved.

We briefly explain why the terms $\sigma_l(R_i, (2)-1/2, N_f \mathcal{V}_{ff})$ and $\sigma_K(R_i, \text{dip}1, (3)/(4)-1, N_f h)$ and the term $\sigma_D(R_i^{C,q\bar{q}}, \text{dip}2)$ cancel each other, as shown in Eq. (2.138). The cancellations in Eq. (2.133) are always realized between the cross sections with the same initial states and the same reduced Born processes. Among them, the cancellation inside $\sigma_{\text{subt}}(R_i)$ itself is calculated
in Steps 1–5, and the results are generally written in Eq. (2.122). When the process \( R_i \) includes two or more gluons in the final state, the reduced Born process \( B1(R_i) \) includes one or more gluon in the final state. The 1-loop virtual correction of the process \( B1(R_i) \) is denoted as \( (M_{LO}(B1(R_i)) - M_{1-loop}(B1(R_i))^* + c.c. ) \). In the original calculation of the QCD NLO correction without the dipole subtraction, the soft and collinear divergences of the 1-loop corrections are canceled by the real corrections of the processes \( R_i \) and \( R_i^{C,qq} \). The 1-loop corrections to the gluon legs in the final state include the contribution of the quark loops to the gluon propagators. The collinear divergence from the quark loop is canceled by the collinear divergence from the collinear limit of the \( qq \) splitting in the real correction \( R_i^{C,qq} \). In the dipole subtraction, the collinear divergence from the quark loop is subtracted by the dipole term \( \hat{\sigma}_D(R_i^{C,qq}, dip2, (5)-2, h) \), and the collinear divergence from the collinear limit of the \( qq \) splitting is subtracted by the dipole term \( \hat{\sigma}_D(R_i^{C,qq}, dip2, (5)-2, h) \). Then the cancellations of the collinear divergences from the quark loop and the \( qq \) splitting in the original calculation are represented in the dipole subtraction as in Eq. (2.136). It is supposed that the remaining finite term, \( \hat{\sigma}_D(R_i^{C,qq}, dip2, (5)-2, h) \), in \( \hat{\sigma}_D(R_i^{C,qq}, dip2) \) is transferred into the \( K \) term \( \sigma_K(R_i) \), and identified as the term \( \sigma_K(R_i, dip1, (3)/(4)-1, N_f h) \) in the construction of the dipole subtraction by the authors in Ref. [2]. The relation is represented in Eq. (2.137). In this way, the cancellation \( \sigma(\text{Con}(R_i)) = 0 \) is understood. The above explanation simultaneously also becomes the explanation of why \( \sigma_{\text{subt}}(R_f) \) for any \( R_f \subset \text{Self} \) completely cancels inside itself as \( \sigma_{\text{subt}}(R_f) = 0 \).

We see one example of the cancellation \( \sigma(\text{Con}(R_i)) = 0 \). We take the process \( R_{8u} = u\bar{u} \rightarrow ggg \) in Table B8 in Appendix B8 in the dijet process. After the execution of Steps 1–5, we obtain the cross section \( \sigma_{\text{subt}}(R_{8u}) \) in Eq. (B58) as

\[
\hat{\sigma}_{\text{subt}}(R_{8u}) = -\hat{\sigma}_I(R_{8u}, (2)-1/2, N_f V_{jj}) - \hat{\sigma}_K(R_{8u}, dip1, (3)-1, N_f h),
\]

where the two terms are written in Eqs. (B54) and (B57) as

\[
\hat{\sigma}_I(R_{8u}, (2)-1/2, N_f V_{jj}) = -\frac{A_d}{S_{B1}} \cdot \frac{N_f}{C_A} V_{jj}(\epsilon) \Phi(B1)_d \\
\quad \cdot \left( [1, 2] + [2, 1] + [1, a] + [1, b] + [2, a] + [2, b] \right),
\]

\[
\hat{\sigma}_K(R_{8u}, dip1, (3)-1, N_f h) = -\frac{A_4}{S_{B1}} \int_0^1 dx \cdot \frac{T_R N_f}{3} \cdot h(x) \\
\quad \times \left[ \Phi_d(B1, x)_a(\langle a, 1 \rangle + \langle a, 2 \rangle) + \Phi_d(B1, x)_b(\langle b, 1 \rangle + \langle b, 2 \rangle) \right].
\]

For the process \( R_{8u} \), the set \( \text{Con}(R_{8u}) \) is determined as \( \text{Con}(R_{8u}) = \{ R_{8u}, \{ R_{8u}^C \} \} \) with the subset \( \{ R_{8u}^C \} = \{ R_{8u}^{C, u\bar{u}}, R_{8u}^{C, dd} \ldots \} \). The \( gg - u\bar{u} \) conjugation of the process \( R_{8u} \) is written as \( R_{8u}^{C, u\bar{u}} = u\bar{u} \rightarrow u\bar{u}g \), which is nothing but the process \( R_1 \) used in the examples in the previous sections. Then the cross section \( \sigma(\text{Con}(R_{8u})) \) is written down as

\[
\sigma(\text{Con}(R_{8u})) = -\sigma_I(R_{8u}, (2)-1/2, N_f V_{jj}) - \sigma_K(R_{8u}, dip1, (3)/(4)-1, N_f h) \\
\quad + \hat{\sigma}_D(R_1, dip2, (5)-1/2, V_{jj}) \cdot N_f + \hat{\sigma}_D(R_1, dip2, (5)-2, h) \cdot N_f,
\]

where the last two terms are obtained in Eqs. (2.130) and (2.131). The color-correlated Born squared amplitudes in Eqs. (2.130) and (2.131) are extracted as \( \{1, 2\} + [1, a] + [1, b] \) and \( \langle 1, a/b \rangle \). The reduced Born process of \( R_1 \) in the category Dipole 2 is fixed as \( B2(R_1) = u(y_{\alpha})\bar{u}(y_{\alpha}) \rightarrow g(y_1)g(y_2) \). The reduced Born process \( B2(R_1) \) has two gluons in the final state and, in that sense, the legs of two...
 gluons, $y_1$ and $y_2$, are symmetric. When we require in Eq. (2.135) that the contributions of the I and K terms, $\sigma_I(R_1, (2)-1/2, N_fV_f)$ and $\sigma_K(R_1, 2i, 1, (3)/(4)-1, N_fh)$, are canceled by the integrated dipole term $\hat{\sigma}_D(R_i^{C,\bar{u}u}, 2i, 1, 2)$ at the level of the square of the matrix elements, $|M|^2$, on each phase-space point, the summation of the color-correlated Born squared amplitudes in $\hat{\sigma}_D(R_i^{C,\bar{u}u}, 2i, 1, 2)$ must be symmetrized about the color factor insertion operators on the identical fields. In the present case, the summation in the term $\hat{\sigma}_D(R_1^{C,\bar{u}u}, 2i, 1, 2)$ must be symmetrized about the color factor operators, $T_{y_1}$ and $T_{y_2}$, on the two gluon legs, $y_1$ and $y_2$, as
\[
[1, 2] + [1, a] + [1, b] = \frac{1}{2} \left( [1, 2] + [2, 1] + [1, a] + [2, a] + [1, b] + [2, b] \right),
\]
\[
\langle 1, a/b \rangle = \frac{1}{2} \left( \langle 1, a/b \rangle + \langle 2, a/b \rangle \right).
\] (2.146)

Using the symmetrized expressions in Eqs. (2.145) and (2.146), and the relation of the symmetric factors, $S_{B_1(R_{u\bar{u}})} = 2S_{R_1} = 2$, we prove the two relations of the cancellation in Eqs. (2.136) and (2.137) for $R_{u\bar{u}}$. Then we prove the cancellation as $\sigma(\text{Con}(R_{u\bar{u}})) = 0$. We will see the full treatment of Step 6, $\sum_i \sigma_{\text{subt}}(R_i) = 0$, in the dijet process in Sect. 4.

**3. Drell–Yan: $pp \to \mu^+\mu^- + X$**

**3.1. Results of the DSA**

The subtraction terms—the dipole, I, P and K terms—for the Drell–Yan process have been created by the DSA in Sect. 3 in Ref. [1]. Here we summarize the results. There are three real emission processes as follows:

\begin{align*}
R_1 &= u\bar{u} \to \mu^-\mu^+g, \\
R_2 &= ug \to \mu^-\mu^+u, \\
R_3 &= \bar{u}g \to \mu^-\mu^+\bar{u},
\end{align*}

and the set $\{R_i\}$ is denoted as $\{R_i\} = \{R_1, R_2, R_3\}$. The dipole, I, and P/K terms that belong to the process $R_1$ are summarized in Tables 1, 2, and 3, respectively, in Ref. [1]. The dipole and P/K terms that belong to the process $R_2$ are summarized in Tables 1 and 3, respectively, in Ref. [1]. Those for $R_3$ are summarized in Tables 1 and 3 in Ref. [1].

**3.2. Execution of the PRA**

For the subtraction terms created by the DSA, we execute the PRA as follows. We start with the execution for the process $R_1$. Since the final state of $R_1$ does not include two or more gluons, nor any $q\bar{q}$ pair, the relation to be proved is written as
\[
\sigma_{\text{subt}}(R_1) = 0.
\] (3.2)

To prove the relation, we start from Step 1. The dipole terms created by the DSA in Table 1 in Ref. [1] are converted to the integrated dipole terms shown in Table 1 in the present article.

Then we proceed to Step 2, where we prove the relation
\[
\sigma_D(R_1, I) - \sigma_I(R_1) = 0.
\] (3.3)

Referring to Table 1 and Appendix A3, the integrated dipole term $\hat{\sigma}_D(R_i, I)$ is written as
\[
\hat{\sigma}_D(R_1, I) = -\frac{A_d}{S_{R_1}} \cdot \frac{1}{C_F} V_f(\langle e \rangle \cdot \Phi(B1)_d ([a, b] + [b, a])).
\] (3.4)

Referring to Table 1 in Ref. [1] and Appendix A8, the contribution of the I terms $\sigma_I(R_1)$ is written as the same expression in Eq. (3.4) with the relation $S_{R_1} = S_B = 1$. Then the relation in Eq. (3.3) is
Table 1. Integrated dipole terms: $\sigma_D(R_1)$.

| Dip | BJ | Splitting | $\gamma_a, \gamma_b : \gamma_1, \gamma_2$ | Factor 1 | $\Phi(B_j)[\gamma_{emi}, \gamma_{esp}]$ |
|-----|----|-----------|----------------|----------|----------------------------------|
| Dip1 | $u\bar{u} \rightarrow \mu^+ \mu^-$ | (3) - 2 | 1.$(\bar{a}, \bar{b}; 1, 2)$ | $\int dx^f f(x)/C_F$ | $\Phi_a(B_1)[a, b]$ |
|     |     |           |                     |          | $\Phi_b(B_1)[b, a]$ |

In this way, the relation in Eq. (3.5) is proved. In Step 3, we prove the relation

$$\sigma_D(R_1, P) = \sigma_C(R_1) - \sigma_P(R_1) = 0.$$  (3.5)

Referring to Table 1 and Appendix A4, the integrated dipole term $\sigma_D(R_1, P)$ is written as

$$\hat{\sigma}_D(R_1, P) = \frac{A_d}{S_{R_1}} \int_0^1 dx \left( \frac{1}{\epsilon} - \ln x \right) \frac{1}{C_F} P_{ll}(x) \left[ \Phi_a(B_1, x)[a, b] + \Phi_b(B_1, x)[b, a] \right].$$  (3.6)

With Appendix A7, the collinear subtraction term is created as

$$\hat{\sigma}_C(R_1) = \frac{A_d}{S_{B_1}} \int_0^1 dx \left( \frac{1}{\epsilon} - \ln \mu_F^2 \right) P_{ll}(x) \left[ \Phi_a(B_1, x)[a, b] + \Phi_b(B_1, x)[b, a] \right].$$  (3.7)

Then we calculate the summation as

$$\hat{\sigma}_D(R_1, P) + \hat{\sigma}_C(R_1) = \frac{A_d}{S_{R_1}} \int_0^1 dx \frac{1}{C_F} P_{ll}(x) \ln \frac{\mu_F^2}{x_{ab}} \left[ \Phi_a(B_1, x)[a, b] + \Phi_b(B_1, x)[b, a] \right],$$  (3.8)

where we use the color conservation as $\langle B_1 \rangle = -\langle a, b \rangle/C_F = -\langle b, a \rangle/C_F$. The summation in Eq. (3.8) is finite and reduced to 4 dimensions. The summation in 4 dimensions is shown to be equal to the P term $\sigma_P(R_1)$, which is created in Table 3 in Ref. [1] and is written down in Appendix A9. In this way, the relation in Eq. (3.5) is proved. In Step 4, we prove the relation

$$\sigma_D(R_1, K) - \sigma_K(R_1) = 0.$$  (3.9)

With Table 1 and Appendix A5, the term $\hat{\sigma}_D(R_1, K)$ is written as

$$\hat{\sigma}_D(R_1, K) = -\frac{A_d}{S_{R_1}} \int_0^1 dx \frac{1}{C_F} \left( \frac{1}{\epsilon} - \ln \mu_F^2 \right) \left[ \Phi_a(B_1, x)[a, b] + \Phi_b(B_1, x)[b, a] \right].$$  (3.10)

With Table 3 in Ref. [1] and Appendix A10, the term $\hat{\sigma}_K(R_1)$ is written as

$$\hat{\sigma}_K(R_1) = \frac{A_d}{S_{B_1}} \int_0^1 dx \left[ \Phi_a(B_1, x)[a, b] \right] + (a \leftrightarrow b).$$  (3.11)

The present process, $R_1 = u\bar{u} \rightarrow \mu^- \mu^+ g$, is so simple that we do not have to divide Step 4 into substeps in Eqs. (2.86)-(2.88). Using the relations $K_{ll}^{ffg}(x) = \gamma_{other}^{ffg}(x; \epsilon) + C_F g(x)$, $\langle a, b \rangle = -C_F \langle B_1 \rangle$, and $S_{R_1} = S_{B_1} = 1$, the relation in Eq. (3.9) is proved. Then we obtain the relation in Eq. (3.2) in Step 5.

Next we apply the PRA to the process $R_2 = u g \rightarrow \mu^- \mu^+ u$. The relation to be proved is written as

$$\sigma_{sub}(R_2) = 0.$$  (3.12)

In Step 1, the dipole term in Table 1 in Ref. [1] is converted to the integrated dipole term shown in Table 2. For the process $R_2$, Step 2 does not exist because the final state does not include any gluon,
and the reduced Born process $B1(R2)$ does not exist. Then we proceed to Step 3, where we prove the relation

$$\sigma_D(R2, P) + \sigma_C(R2) - \sigma_P(R2) = 0. \tag{3.13}$$

Referring to Table 1 and Appendix A4, the term $\hat{\sigma}_D(R2, P)$ is written as

$$\hat{\sigma}_D(R2, P) = \frac{A_d}{S_{R2}} \int_0^1 dx \left( \frac{1}{\epsilon - \ln x} \right) \frac{1}{C_F} p^{gf}(x) \Phi_b(B4u, x)_d[b, a]. \tag{3.14}$$

With Appendix A7, the collinear subtraction term $\sigma_C(R2)$ is created, and, with Table 3 in Ref. [1] and Appendix A9, the P term $\sigma_P(R2)$ is written down. It is shown that the three terms $\hat{\sigma}_D(R2, P)$, $\hat{\sigma}_C(R2)$, and $\hat{\sigma}_P(R2)$ satisfy the relation in Eq. (3.13). In Step 4, we prove the relation

$$\sigma_D(R2, K) - \sigma_K(R2) = 0. \tag{3.15}$$

With Table 1 and Appendix A5, the term $\hat{\sigma}_D(R2, K)$ is written as

$$\hat{\sigma}_D(R2, K) = - \frac{A_4}{S_{R1}} \int_0^1 dx \frac{1}{C_F} \left( \sigma^{gf}_{other}(x; \epsilon) + \tilde{K}^{gf}(x) \right) \Phi_b(B4u, x)_4(b, a). \tag{3.16}$$

With Table 3 in Ref. [1] and Appendix A10, the term $\hat{\sigma}_K(R2)$ is written down, and is shown to be equal to $\hat{\sigma}_D(R2, K)$ in Eq. (3.16). In Step 5, we obtain the relation in Eq. (3.12). In a similar way, we can prove the relation for the process $R3$ as

$$\sigma_{subt}(R3) = 0. \tag{3.17}$$

Finally, we come to the last step, Step 6. All the real processes in the set $\{R_i\}$ belong to the set Self as $\text{Self} = \{R1, R2, R3\}$, and any set $\text{Con}(R_i)$ does not exist. Then we calculate the summation of the cross sections $\sigma_{subt}(R_i)$ as

$$\sum_{i=1}^3 \sigma_{subt}(R_i) = \sum_{\text{Self} \subset R_j} \sigma_{subt}(R_j) = 0, \tag{3.18}$$

where we use the cancellations in Eqs. (3.2), (3.12), and (3.17) for $R1$, $R2$, and $R3$, respectively. The contributions of the different flavors, $d$, $s$, $c$, and $b$, are identical to the process with the up quark, $u$, and the executions of Steps 1–6 are also identical. Thus the execution of the PRA for the Drell–Yan process is completed.

4. Dijet: $pp \to 2 \text{ jets } + X$

4.1. Results of the DSA

The DSA for the dijet process has been executed in Sect. 4 in Ref. [1]. We here summarize the results. The real processes are denoted as

$$R_{1u} = u\bar{u} \to u\bar{u}g, \quad (R_{1d})$$

$$R_{2u} = uu \to uu\bar{g}, \quad (R_{2d}, R_{2d}, R_{2d})$$

| Dip4u | $u\bar{u} \to \mu^+\mu^-$ | $\beta$ | Splitting $(\gamma_u, \gamma_d; \gamma_1, \gamma_2)$ | Factor 1 | $\Phi(B_j)[\gamma_{emi}, \gamma_{spe}]$ |
|-------|-----------------------------|--------|-----------------------------------------------|---------|-------------------------------------------|
|       |                             |        |                                               |         |                                           |

---

**Table 2.** Integrated dipole terms: $\sigma_D(R2)$.

$$\hat{\sigma}_D(R2) = ug \to \mu^+\mu^- u, \quad S_{R2} = 1,$$
\[
\begin{align*}
R_{3u} &= ug \rightarrow uu\bar{u}, \quad (R_{3u}, R_{3d}, R_{3d}) \\
R_{4u} &= u\bar{u} \rightarrow d\bar{d}g, \quad (R_{4d}) \\
R_{5ud} &= ud \rightarrow udg, \quad (R_{5gd}) \\
R_{6ud} &= u\bar{d} \rightarrow u\bar{d}g, \quad (R_{6ud}) \\
R_{7u} &= ug \rightarrow udd, \quad (R_{7u}, R_{7d}, R_{7d}) \\
R_{8u} &= u\bar{u} \rightarrow ggg, \quad (R_{8d}) \\
R_{9u} &= ug \rightarrow ugg, \quad (R_{9u}, R_{9d}, R_{9d}) \\
R_{10u} &= gg \rightarrow u\bar{u}g, \quad (R_{10d}) \\
R_{11} &= gg \rightarrow ggg, \quad (4.1)
\end{align*}
\]

where the processes that have identical expressions for the cross sections with different quark flavors are written in round brackets, e.g., \((R_{1d})\) for the process \(R_{1u}\). The dipole terms are summarized in Tables B1–B11 in Appendix B1 in Ref. [1], the I terms in Tables B12–B16 in Appendix B2 in Ref. [1], and the P and K terms in Tables B17–B27 in Appendix B3 in Ref. [1]. The details of the creation are explained in Ref. [1].

### 4.2. Execution of the PRA

We start from **Step 1**. The dipole terms \(\hat{\sigma}_D(R_i)\) for the processes \(R_1, \ldots, R_{11}\) in Tables B1–B11 in Appendix B.1 in Ref. [1] are converted into integrated ones in Tables B1–B11 in Appendixes B1–B11 in the present article, respectively. The results of **Steps 2–5** for \(R_1, \ldots, R_{11}\) are shown after Tables B1–B11 in Appendixes B1–B11. Here we write down the results of **Step 5** as follows:

\[
\hat{\sigma}_{\text{subt}}(R_{1u}) = \hat{\sigma}_D(R_{1u}, \text{dip}2), \\
\hat{\sigma}_{\text{subt}}(R_{2u}) = 0, \\
\hat{\sigma}_{\text{subt}}(R_{3u}) = \hat{\sigma}_D(R_{3u}, \text{dip}2), \\
\hat{\sigma}_{\text{subt}}(R_{4u}) = \hat{\sigma}_D(R_{4u}, \text{dip}2), \\
\hat{\sigma}_{\text{subt}}(R_{5ud}) = 0, \\
\hat{\sigma}_{\text{subt}}(R_{6ud}) = 0, \\
\hat{\sigma}_{\text{subt}}(R_{7u}) = \hat{\sigma}_D(R_{7u}, \text{dip}2), \\
\hat{\sigma}_{\text{subt}}(R_{8u}) = -\hat{\sigma}_I(R_{8u}, (2)-1/2, N_f V_{ff}) - \hat{\sigma}_K(R_{8u}, \text{dip}1, (3)-1, N_f h), \\
\hat{\sigma}_{\text{subt}}(R_{9u}) = -\hat{\sigma}_I(R_{9u}, (2)-1/2, N_f V_{ff}) - \hat{\sigma}_K(R_{9u}, \text{dip}1, (3)/(4)-1, N_f h), \\
\hat{\sigma}_{\text{subt}}(R_{10u}) = \hat{\sigma}_D(R_{10u}, \text{dip}2), \\
\hat{\sigma}_{\text{subt}}(R_{11}) = -\hat{\sigma}_I(R_{11}, (2)-1/2, N_f V_{ff}) - \hat{\sigma}_K(R_{11}, \text{dip}1, (4)-1, N_f h), \quad (4.2)
\]

where the expressions for the cross sections on the right-hand sides are all written in Appendix B. Then we proceed to **Step 6**. We construct the sets \(\text{Con}(R_i)\) and Self as

\[
\begin{align*}
\text{Con}(R_{8u}) &= \{R_{8u}, \{R_{1u}, R_{4u}\}\}, \quad (4.3) \\
\text{Con}(R_{9u}) &= \{R_{9u}, \{R_{3u}, R_{7u}\}\}, \quad (4.4)
\end{align*}
\]
In this section, we deal with the three real processes among all those that contribute to the collider process \( pp \rightarrow n \) jets +X as

\[
\begin{align*}
R_1 &= u\bar{u} \rightarrow (n + 1)\cdot g, \\
R_2 &= u\bar{u} \rightarrow u\bar{u} + (n - 1)\cdot g, \\
R_3 &= u\bar{u} \rightarrow d\bar{d} + (n - 1)\cdot g.
\end{align*}
\]

and prove the cancellation of the cross section as \( \sigma(\text{Con}(R_i)) = 0 \) in Step 6 of the PRA. We first apply the DSA to the three processes. Since the expressions of the created subtraction terms are too long, we do not show the expressions explicitly. Then we apply the PRA to the subtraction terms created by the DSA. The results of the PRA are collected in Appendix C. The results of Step 1 for \( R_1, R_2, \) and \( R_3 \) are shown in Tables C1, C2, and C3, respectively. The results of Steps 2–5 for \( R_1, R_2, \) and \( R_3 \) are shown after Tables C1, C2, and C3, respectively. We would like to clarify two points to notice in obtaining the results of the PRA. Both of the points concern the expressions of the integrated dipole term \( \hat{\sigma}_D(R_i) \). We explain them in the following two paragraphs, respectively.

The first point is about the expressions of the integrated dipole terms converted in Step 1. When the reduced Born process includes identical fields, the field mapping, which is determined for each dipole term, has the freedom to choose the emitter–spectator pair \((y_{\text{emi}}, y_{\text{spe}})\) among the identical fields in set \( \{y\} \). Using the freedom of the field mapping, we can transform the integrated dipole terms into an identical expression. We call this operation the unification of the integrated dipole terms. We show one example in the process \( R_1 = (x_u)\bar{u}(x_b) \rightarrow g(x_1)g(x_2) \cdots g(x_{n+1}) \). The creation of the dipole terms starts from splitting (2)-1, where the reduced Born process is fixed with set \( \{y\} \) as

\[
B1(R_1) = u(y_u)\bar{u}(y_b) \rightarrow g(y_1)g(y_2) \cdots g(y_n).
\]

Then we can choose the three legs \((x_1, x_j, x_k)\) in the final state in set \( \{x\} \) as

\[
(x_1, x_2, x_3), (x_1, x_2, x_4), \ldots, (x_1, x_2, x_{n+1}), \\
(x_1, x_3, x_2), (x_1, x_3, x_4), \ldots, (x_1, x_3, x_{n+1}), \\
\vdots \quad \vdots \\
(x_{n-1}, x_n, x_1), (x_{n-1}, x_n, x_{n-1}), \ldots, (x_{n-1}, x_n, x_{n-2}).
\]
The total number of pairs is \( n+1 C_2 \cdot (n - 1) \). One field mapping is fixed for each pair \((x_i, x_j, x_k)\), and specifies the two legs of the emitter and the spectator in set \( \{y\} \) as \((y_{emi}, y_{spe})\). All the specified pairs \((y_{emi}, y_{spe})\) for the pairs \((x_i, x_j, x_k)\) in Eq. (5.3) are written in the expression \((y_{a}, y_{b})\), where the indices \( \alpha \) and \( \beta \) take any value among \( 1, \ldots, n \) with the condition \( \alpha \neq \beta \). Using the freedom of the field mapping, we can reconstruct all the field mappings in such a way that they have an identical pair \((y_{emi}, y_{spe})\); for instance, \((y_1, y_2)\). We call this operation unification. After the operation of unification, the summation of the integrated dipole terms in splitting (2)-1 is rewritten as

\[
\hat{\sigma}_D(R_1, (2)-1) = -\frac{A_d}{S_{R_1}} \cdot \frac{1}{C_A} \cdot \frac{V_{gg}(\epsilon)}{\Phi(B1)_{d}[1, 2]} \times n_{deg}, \tag{5.4}
\]

where the degeneracy factor, \( n_{deg} \), is determined as \( n_{deg} = n+1 C_2 \cdot (n - 1) \). We can apply the operation of unification to all the other integrated dipole terms as well. The unification of all the dipole terms may be described as follows: “The results of Step 1 are transformed into the unification expression.” Actually, in Tables C1, C2, and C3, the results are represented in the unification expression. Compared to the format of Tables B1–B11 in Appendix B, the entry \((y_{a}, y_{b} : y_1, \ldots, y_n)\) is replaced with the entry \((y_{emi}, y_{spe})\) and the new entry for the degeneracy factor \( n_{deg} \) is added. The dipole term \( \hat{\sigma}_D(R_1, (2)-1) \) in Eq. (5.4) is represented by the first one, \( 1(y_1, y_2) \), with the degeneracy factor \( n_{deg} \) in Table C1. Referring to Table C1, for instance, we can read out the integrated dipole term \( \hat{\sigma}_D(R_1, I) \) used in Step 2 as

\[
\hat{\sigma}_D(R_1, I) = -\frac{A_d}{S_{R_1}} \cdot \Phi(B1)_{d} \left[ \frac{V_{gg}(\epsilon)}{C_A} \left[ [1, 2] \cdot n+1 C_2(n - 1) + ([1, a] + [1, b]) \cdot n+1 C_2 \right] \right. \\
+ \left. \frac{V_{fg}(\epsilon)}{C_F} \left[ ([a, 1] + [b, 1]) \cdot (n + 1)n + ([a, b] + [b, a]) \cdot (n + 1) \right] \right]. \tag{5.5}
\]

The advantage of the table format in the unification expression is that the length of the table is shorter than the original format. The disadvantage is that the one-to-one correspondences between the original dipole terms and the integrated dipole terms are lost. The format of Tables C1, C2, and C3, can be a template to show the results in the unification expression.

The second point is about the symmetric expression of the integrated dipole terms used for Steps 2, 3, and 4. In order that the integrated dipole terms \( \hat{\sigma}_D(R_i, I/P/K) \) cancel the I, P, and K terms \( \hat{\sigma}_I/P/K(R_i) \) at the level of the squared amplitude \([y_{emi}, y_{spe}]\) on each phase-space point, the summation of the color-correlated Born squared amplitudes \([y_{emi}, y_{spe}]\) in the integrated dipole terms \( \hat{\sigma}_D(R_i, I/P/K) \) must be symmetrized over the legs of the identical fields. We call this operation symmetrization of the integrated dipole terms. To demonstrate the symmetrization, we take the same process \( R_1 \). The reduced Born process \( B1(R_1) \) in Eq. (5.2) has \( n \) identical fields in the legs \( y_1, \ldots, y_n \). Then the color-correlated Born squared amplitude \([1, 2]\), for instance, in Eq. (5.5) can be symmetrized over the legs \( y_1, \ldots, y_n \) as

\[
[1, 2] = \frac{1}{n(n - 1)} \sum_{i,k=1}^{n} [i, k], \tag{5.6}
\]

with the condition \( i \neq k \). We call this operation symmetrization. The symmetrization is allowed by the freedom of the field mapping over the identical fields. The freedom is the same one by which the operation of unification is allowed, as explained above. When the symmetrization is applied to all the
terms in the integrated dipole term $\hat{\sigma}_D(R_1, I)$ in Eq. (5.5), the expression is transformed as

$$
\hat{\sigma}_D(R_1, I) = -A_d \frac{S_B}{S_B} \Phi(B1)_d \left[ \frac{1}{C_A} \cdot \frac{1}{2} V_{gg}(\epsilon) \sum_{i,k=1}^n [i, k] + \sum_{i=1}^n ([i, a] + [i, b]) \right] 
+ \frac{1}{C_F} V_{ff}(\epsilon) \left[ \sum_{k=1}^n ([a, k] + [b, k]) + [a, b] + [b, a] \right],
$$
(5.7)

where we use the relation of the symmetric factors $S_{R_1} = (n + 1) \cdot S_{B_1} = (n + 1)$. The I term $\hat{\sigma}_I(R_1)$ is created by the DSA in such an expression that the factor $V_{gg}(\epsilon)/2$ in Eq. (5.7) is replaced with the factor $V_{gg}(\epsilon)/2 + N_f V_{ff}(\epsilon)$. Then the integrated dipole term $\hat{\sigma}_D(R_1, I)$ in Eq. (5.7) is able to cancel the I term $\hat{\sigma}_I(R_1)$ at the integrand level, which means that, on each phase-space point of $\Phi(B1)_d$, the color-correlated Born squared amplitudes $[y_{emi}, y_{spe}]$ in the two terms $\hat{\sigma}_D(R_1, I)$ and $\hat{\sigma}_I(R_1)$ cancel each other. In this way, we prove the relation in Step 2 as

$$
\hat{\sigma}_D(R_1, I) - \hat{\sigma}_I(R_1) = -\hat{\sigma}_I(R_1, (2)-1/2, N_f V_{ff}).
$$
(5.8)

where the term $\hat{\sigma}_I(R_1, (2)-1/2, N_f V_{ff})$ is written in Eq. (C2). In a similar way, in Steps 3 and 4, the integrated dipole terms $\hat{\sigma}_D(R_1, P/K)$ are symmetrized and can be canceled against the P/K terms $\hat{\sigma}_{P/K}(R_1)$, as shown in Eqs. (C3)/(C4), respectively.

Finally, we show the relation of cancellation in Step 6 as

$$
\sigma(Con(R_1)) = 0.
$$
(5.9)

The set $Con(R_1)$ is denoted as $Con(R_1) = \{R_1, \{R_2, R_3\}\}$. After the execution of Steps 1–5 for $R_1, R_2$, and $R_3$, the cross section $\sigma(Con(R_1))$ is written down as

$$
\sigma(Con(R_1)) = -\hat{\sigma}_I(R_1, (2)-1/2, N_f V_{ff}) - \hat{\sigma}_K(R_1, \{\text{dip}1\}, (3)-1, N_f h) + \hat{\sigma}_D(R_2, \{\text{dip}2\}) \cdot N_f,
$$
(5.10)

where the three terms on the right-hand side are defined in Eqs. (C2), (C5), and (C17), respectively. The integrated dipole term $\hat{\sigma}_D(R_2, \{\text{dip}2\})$ is separated into two terms, $\hat{\sigma}_D(R_2, \{\text{dip}2\}, (5)-1/2, V_{ff})$ and $\hat{\sigma}_D(R_2, \{\text{dip}2\}, (5)-2, h)$, which are defined in Eqs. (C18) and (C19), respectively. After the symmetrization of the integrated dipole term $\hat{\sigma}_D(R_2, \{\text{dip}2\})$, the three terms on the right-hand side of Eq. (5.10) cancel each other in the way shown in Eqs. (2.136) and (2.137). Then the relation in Eq. (5.9) is proved. In the present section, we only prove the cancellation $\sigma(Con(R_1)) = 0$, and do not prove the whole relation of the cancellation in Step 6 as $\sum_{i=1}^{n_{real}} \sigma_{\text{subt}}(R_i) = 0$. The execution of the DSA and the PRA for all the processes $\{R_i\}$ contributing to the process $pp \to n$ jets can be regarded as the execution for an almost general process with an arbitrary large number $n$. The dipole terms created by the DSA have very long and general expressions. The proofs of the relations in the PRA also become long and general. For these reasons, we do not present the execution of the DSA and PRA for all the processes $\{R_i\}$ in the present article.

6. Summary

In the dipole subtraction procedure, we create subtraction terms and write down the expressions for the phase-space integration. While creating the subtraction terms and writing down the expressions, we sometimes have the chance to make mistakes. The main reason for this is that many subtraction terms exist for the multiparton processes and each term is not so simple. Among the subtraction terms, the singular parts of the dipole and I terms are confirmed by cancellation against the real and
virtual corrections during the calculation of the NLO corrections. The P and K terms are finite and confirmation by cancellation is impossible. The summation of all the subtraction terms created must vanish as

\[ \sigma_{\text{subt}} = \sigma_D + \sigma_C - \sigma_I - \sigma_P - \sigma_K = 0. \]  

(6.1)

We call this relation the consistency relation of the subtraction terms. The proof of the consistency relation provides one confirmation of the P and K terms as well as all the other subtraction terms. The cancellations in the consistency relation are realized between subtraction terms with the same initial states and the same reduced Born processes. In Ref. [1], we presented the dipole splitting algorithm (DSA), which is an algorithm to create the subtraction terms. In the DSA, the subtraction terms are classified by the real processes \( \{ R_i \} \) and the kinds of splittings. The classification can be translated as the classification by the initial states and the reduced Born processes. Thanks to such a classification in the DSA, we can construct a straightforward algorithm to prove the consistency relation. In this article, we have presented the proof algorithm (PRA) with the necessary formulae and demonstrated the PRA in the example processes. The PRA is defined in Sect. 2 and all the formulae are collected in Appendix A. The PRA is demonstrated in the Drell–Yan, dijet, and \( n \) jet processes in Sects. 3, 4, and 5, respectively. The results of the PRA for the dijet and the \( n \) jets are summarized in Appendixes B and C, respectively. In Step 1 of the PRA, the dipole terms are converted to the integrated dipole terms. We showed two templates for the tables representing the integrated dipole terms in Appendixes B and C. In the tables in Appendix B, one integrated dipole term corresponds to one original dipole term. In the tables in Appendix C, the integrated dipole terms are transformed to as many identical expressions as possible. The transformed expression is called the unification expression.

The executions of the PRA in the Drell–Yan and dijet processes can be easily completed by hand manipulation. For more complicated multiparton processes, execution by hand manipulation will be too long and may not be realistic. Thus, automation of the PRA as a computer code is desirable and may be realized in the future. The extension of the DSA and PRA to include massive quark cases is also left for future work. The execution should be straightforward because the algorithm structure of the dipole subtraction procedure with massive quarks in Ref. [3] is almost identical to the massless case in Ref. [2]. Beginners who start to use dipole subtraction may have some difficulties creating and confirming the subtraction terms. Thus, the DSA in Ref. [1] and the PRA in the present article may serve as supplementary materials by which beginners can learn how to create and confirm the subtraction terms. The reason for this is that the DSA and PRA are well defined with all the formulae in the documents and what the user needs to do is just to follow the steps in the straightforward algorithm. We hope that the DSA and PRA will help users to obtain reliable predictions at QCD NLO accuracy.

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Appendix A. Formulae for PRA

A.1. Cross sections: \( \hat{\sigma}(R_i) \)

The partonic cross sections in Eq. (2.5) are defined as

\[
\hat{\sigma}_R(R_i) = \frac{1}{S_{R_i}} \Phi(R_i) \cdot |M(R_i)|_a^2,
\]

(A1)

\[
\hat{\sigma}_D(R_i) = \frac{1}{S_{R_i}} \Phi(R_i) \cdot \frac{1}{n_s(a)n_s(b)} D(R_i),
\]

(A2)

\[
\hat{\sigma}_V(B1) = \frac{1}{S_{B1}} \Phi(B1) \cdot |M_{\text{vir}}(B1)|_a^2,
\]

(A3)

\[
\hat{\sigma}_I(R_i) = \frac{1}{S_{B1}} \Phi(B1) \cdot I(R_i),
\]

(A4)

\[
\hat{\sigma}_P(R_i) = \int_0^1 dx \sum_{B_j} \frac{1}{S_{B_j}} \Phi_d(R_i : B_j, x) \cdot P(R_i, x_a : B_j, x_p) + (a \leftrightarrow b),
\]

(A5)

\[
\hat{\sigma}_K(R_i) = \int_0^1 dx \sum_{B_j} \frac{1}{S_{B_j}} \Phi_d(R_i : B_j, x) \cdot K(R_i, x_a : B_j, x_p) + (a \leftrightarrow b).
\]

(A6)

The phase spaces including the flux factors are defined as

\[
\Phi(R_i)_d = \frac{1}{F(p_a, p_b)} \prod_{i=1}^{n+1} \int \frac{d^{d-1}p_i}{(2\pi)^{d-1}} \cdot \frac{1}{2E_i} \cdot (2\pi)^d \delta^{(d)}(p_a + p_b - \sum_{i=1}^{n+1} p_i),
\]

(A7)

\[
\Phi(B1)_d = \frac{1}{F(p_a, p_b)} \prod_{i=1}^{n} \int \frac{d^{d-1}p_i}{(2\pi)^{d-1}} \cdot \frac{1}{2E_i} \cdot (2\pi)^d \delta^{(d)}(p_a + p_b - \sum_{i=1}^{n} p_i).
\]

(A8)

\[
\Phi_d(R_i : B_j, x) = \frac{1}{F(x_{p_a}, p_b)} \prod_{i=1}^{n} \int \frac{d^3p_i}{(2\pi)^3} \cdot \frac{1}{2E_i} \cdot (2\pi)^4 \delta^{(4)}(x_{p_a} + p_b - \sum_{i=1}^{n} p_i).
\]

(A9)

The exact definitions of the factors and the symbols are given in the DSA in Ref. [1]. The jet functions \( F_j^{(n/n+1)}(p_1, \ldots, p_{n/n+1}) \) must be multiplied by the partonic cross sections in Eqs. (A1)–(A6).

For the real correction in Eq. (A1), the jet function with \((n+1)\) fields is multiplied as

\[
\hat{\sigma}_R(R_i) = \frac{1}{S_{R_i}} \Phi(R_i) \cdot |M(R_i)|_a^2 \cdot F_j^{(n+1)}(p_1, \ldots, p_{n+1}).
\]

(A10)

For the cross sections in Eqs. (A3)–(A6), the jet function with \(n\) fields, \( F_j^{(n)}(p_1, \ldots, p_n) \), is multiplied. For the dipole term in Eq. (A2), the jet function \( F_j^{(n)} \) is multiplied and the \( n \) reduced momenta of the arguments are identified with the \( n \) reduced momenta \((P(y_1), \ldots, P(y_n))\). The use of the jet functions in the dipole subtraction is explained in Ref. [2]. For compact notation, we do not show the jet functions explicitly in the present article.

We summarize the PRA as follows. We clarify the subsections where the various cross sections appearing in the PRA are defined in Appendix A. The cross section \( \sigma_{\text{subt}}(R_i) \) is defined as

\[
\sigma_{\text{subt}}(R_i) = \sigma_D(R_i) + \sigma_C(R_i) - \sigma_I(R_i) - \sigma_P(R_i) - \sigma_K(R_i).
\]

(A11)

The term \( \sigma_D(R_i) \) is the integrated dipole term that is converted in \textbf{Step 1}, and the formulae are collected in Appendix A2. The integrated dipole term \( \sigma_D(R_i) \) is separated into four terms as

\[
\sigma_D(R_i) = \sigma_D(R_i, I) + \sigma_D(R_i, P) + \sigma_D(R_i, K) + \sigma_D(R_i, d\text{i}p2).
\]

(A12)
Fig. A1. The classification of the dipole term $\sigma_D$.

The formulae for the four terms, $\sigma_D(R_i, I/P/K)$ and $\sigma_D(R_i, \text{dip}2)$, are collected in Appendixes A3, A4, A5, and A6, respectively. Then $\sigma_{\text{subt}}(R_i)$ is reconstructed as

$$\sigma_{\text{subt}}(R_i) = [\sigma_D(R_i, I) - \sigma_I(R_i)] + [\sigma_D(R_i, P) + \sigma_C(R_i) - \sigma_P(R_i)]$$

$$+ [\sigma_D(R_i, K) - \sigma_K(R_i)] + \sigma_D(R_i, \text{dip}2).$$

(A13)

The formulae for the terms $\sigma_C(R_i)$ and $\sigma_{I/P/K}(R_i)$ are collected in Appendix A7, A8, A9, and A10, respectively. In Steps 2, 3, and 4, the following relations are proved:

**Step 2.** $\sigma_D(R_i, I) - \sigma_I(R_i) = -\sigma_I(R_i, (2)-1/2, N_f \mathcal{V}_{fj}),$

3. $\sigma_D(R_i, P) + \sigma_C(R_i) - \sigma_P(R_i) = 0,$

4. $\sigma_D(R_i, K) - \sigma_K(R_i) = -\sigma_K(R_i, \text{dip}1, (3)/(4)-1, N_f h)$,

where the terms $\sigma_I(R_i, (2)-1/2, N_f \mathcal{V}_{fj})$ and $\sigma_K(R_i, \text{dip}1, (3)/(4)-1, N_f h)$ are defined in Appendixes A8 and A10, respectively. We substitute the first three terms in square brackets in Eq. (A13) by the three relations in Eq. (A14), and obtain $\sigma_{\text{subt}}(R_i)$ in the expression

**Step 5.**

$$\sigma_{\text{subt}}(R_i) = -\sigma_I(R_i, (2)-1/2, N_f \mathcal{V}_{fj}) - \sigma_K(R_i, \text{dip}1, (3)/(4)-1, N_f h)$$

$$+ \sigma_D(R_i, \text{dip}2).$$

(A15)

In the last step, we prove that the summation of all the terms $\sigma_{\text{subt}}(R_i)$ vanishes as

**Step 6.** $\sum_{i=1}^{n_{\text{real}}} \sigma_{\text{subt}}(R_i) = 0.$

(A16)

A.2. **Integrated dipole term: $\sigma_D$**

The integrated dipole term is universally written as

$$\hat{\sigma}_D(R_i) = -\frac{A_d}{S_{R_i}} \cdot \int_0^1 dx \frac{1}{F(\gamma_{\text{emi}})} \mathcal{V}(x; \epsilon) \cdot \Phi_d(B_j, x) d [y_{\text{emi}}, y_{\text{spe}}],$$

(A17)
where the overall factor $A_d$ is defined as

$$A_d = \frac{\alpha_s (4\pi \mu^2)^\epsilon}{2\pi \Gamma(1-\epsilon)}.$$  \hspace{1cm} (A18)

The integrated dipole term is classified into four types as

$$\hat{\sigma}_D(R_i, \text{dip} j) \supset \hat{\sigma}_D(R_i, \text{dip} j, \text{FF/FI/IF/II}),$$  \hspace{1cm} (A19)

which are defined as follows.

\[ D_{ij,a}: \text{Final–initial dipole} \]

Dipole 1 (1)-2, (2)-2,
Dipole 2 (5)-2:

$$\hat{\sigma}_D(R_i, \text{dip} j, \text{FL}, x_{a/b}) = -\frac{A_d}{S_{R_i}} \int_0^1 dx \frac{1}{T_F(y_{emi})} \mathcal{V}_{F(x_i)F(x_j)}(x; \epsilon) \cdot \Phi_{a/b}(B_j, x) \cdot \Phi_{b/a}(B_j, x) \cdot \Phi_{b/a}(B_j, x) \cdot \Phi_{b/a}(B_j, x).$$  \hspace{1cm} (A28)

\[ D_{ij,k}: \text{Final–final dipole} \]

Dipole 1 (1)-1, (2)-1,
Dipole 2 (5)-1:

$$\hat{\sigma}_D(R_i, \text{dip} j, \text{FF}) = -\frac{A_d}{S_{R_i}} \frac{1}{T_F(y_{emi})} \mathcal{V}_{F(x_i)F(x_j)}(\epsilon) \cdot \Phi(B) \cdot \Phi(B) \cdot \Phi(B) \cdot \Phi(B) \cdot \Phi(B).$$  \hspace{1cm} (A20)

\[ \text{Definition of the symbols} \]

Universal singular functions:

$$\mathcal{V}_{F(x_i)F(x_j)}(\epsilon) = \begin{cases} \frac{1}{C_F} & : \text{Dipole 1 (1)-1,} \\ \frac{1}{CA} & : \text{Dipole 1 (2)-1,} \\ \frac{1}{CA} & : \text{Dipole 2 (5)-1.} \end{cases}$$  \hspace{1cm} (A21)

$$\mathcal{V}_{fg}(\epsilon) = C_F \left[ \frac{1}{\epsilon^2} + \frac{3}{2\epsilon} + 5 - \frac{\pi^2}{2} \right],$$  \hspace{1cm} (A22)

$$\mathcal{V}_{gg}(\epsilon) = 2C_A \left[ \frac{1}{\epsilon^2} + \frac{11}{6\epsilon} + \frac{50}{9} - \frac{\pi^2}{2} \right],$$  \hspace{1cm} (A23)

$$\mathcal{V}_{ff}(\epsilon) = T_F \left[ -\frac{2}{3\epsilon} - \frac{16}{9} \right].$$  \hspace{1cm} (A24)

Phase space:

$$\Phi(B) = \frac{1}{\mathcal{F}(p_a, p_b)} \prod_{i=1}^n \int \frac{d^{d-1}p_t}{(2\pi)^{d-1}} \frac{1}{2E_i} \cdot (2\pi)^d \delta^{(d)} \left( p_a + p_b - \sum_{i=1}^n p_t \right).$$  \hspace{1cm} (A25)

Color-correlated Born squared amplitude:

$$[\epsilon_{emi}, \epsilon_{spe}] = (s_{emi}, s_{spe})^{-\epsilon} \cdot (B_j | T_{y_{emi}} \cdot T_{y_{spe}} | B_j) \cdot \Phi(B) \cdot \Phi(B) \cdot \Phi(B) \cdot \Phi(B) \cdot \Phi(B).$$  \hspace{1cm} (A26)

Lorentz scalar:

$$s_{emi, spe} = 2P(\epsilon_{emi}) \cdot P(\epsilon_{spe}).$$  \hspace{1cm} (A27)
\textit{Definition of the symbols}

Universal singular functions:

\[
\frac{1}{T_{\text{F}(\text{yemi})}} \mathcal{V}_{\text{F}(x_i)\text{F}(x_j)}(x; \epsilon) = \begin{cases} 
\frac{1}{C_{\text{F}}} \mathcal{V}_{fg}(x; \epsilon) & \text{: Dipole 1 (1)-2,} \\
\frac{1}{C_{\text{A}}} \mathcal{V}_{g\bar{g}}(x; \epsilon) & \text{: Dipole 1 (2)-2,} \\
\frac{1}{C_{\text{A}}} \mathcal{V}_{f\bar{f}}(x; \epsilon) & \text{: Dipole 2 (5)-2.} 
\end{cases} \tag{A29}
\]

\[
\mathcal{V}_{fg}(x; \epsilon) = \delta(1-x)\mathcal{V}_{fg}(\epsilon) + C_{\text{F}} \left[ g(x) - \frac{3}{2}h(x) \right], \tag{A30}
\]

\[
\mathcal{V}_{g\bar{g}}(x; \epsilon) = \delta(1-x)\mathcal{V}_{g\bar{g}}(\epsilon) + C_{\text{A}} \left[ 2g(x) - \frac{11}{3}h(x) \right], \tag{A31}
\]

\[
\mathcal{V}_{f\bar{f}}(x; \epsilon) = \delta(1-x)\mathcal{V}_{f\bar{f}}(\epsilon) + T_{\text{R}} \frac{3}{2}h(x). \tag{A32}
\]

\[
g(x) = \left( \frac{2}{1-x} \ln \frac{1}{1-x} \right) + \frac{2}{1-x} \ln(2-x), \tag{A33}
\]

\[
h(x) = \left( \frac{1}{1-x} \right) + \delta(1-x). \tag{A34}
\]

\textbf{Phase spaces:}

\[
\Phi_{a}(\text{B}_j, x)_d = \frac{1}{\mathcal{F}(xp_a, p_b)} \prod_{i=1}^{n} \int \frac{d^{d-1}p_i}{(2\pi)^{d-1}} \frac{1}{2E_i} \cdot (2\pi)^d \delta^{(d)} \left( xp_a + p_b - \sum_{i=1}^{n} p_i \right), \tag{A35}
\]

\[
\Phi_{b}(\text{B}_j, x)_d = \frac{1}{\mathcal{F}(pa, xp_b)} \prod_{i=1}^{n} \int \frac{d^{d-1}p_i}{(2\pi)^{d-1}} \frac{1}{2E_i} \cdot (2\pi)^d \delta^{(d)} \left( p_a + xp_b - \sum_{i=1}^{n} p_i \right). \tag{A36}
\]

\textbf{Color-correlated Born squared amplitude:}

\[
\left[ \text{yemi, yspe} \right] = (s_{\text{yemi}, xp_a/b})^{-e} \cdot \langle \text{B}_j | T_{\text{yemi}} \cdot T_{\text{yspe}} | \text{B}_j \rangle_d. \tag{A37}
\]

\textbf{Lorentz scalar:}

\[
s_{\text{yemi}, xp_a/b} = 2p(y_{\text{emi}}) \cdot p_{a/b}. \tag{A38}
\]

\textbf{D}_{\text{ai,k}}: \text{Initial–final dipole}

Dipole 1 (3)-1, (4)-1,
Dipole 3 (6)-1,
Dipole 4 (7)-1:

\[
\delta_D(\text{R}_i, \text{dip} j, \text{IF}, x_{a/b}) = -\frac{A_d}{S_{\text{R}_i}} \int_0^1 dx \frac{1}{T_{\text{F}(\text{yemi})}^2} \mathcal{V}_{\text{F}(x_{a/b})\text{F}(\text{yemi})}(x; \epsilon) \cdot \Phi_{a/b}(\text{B}_j, x)_d \left[ \text{yemi, yspe} \right]. \tag{A39}
\]
Definition of the symbols
Universal singular functions:

\[
\frac{1}{T_F^{2}}\varphi_{F(x_{0/1}),F_{Y_{e/mi}}}(x;\epsilon) = \begin{cases} 
\frac{1}{C_F} \varphi_{f-f}(x;\epsilon) & : \text{Dipole 1 (3)-1}, \\
\frac{1}{C_A} \varphi_{g-g}(x;\epsilon) & : \text{Dipole 4 (7)-1}, \\
\frac{1}{C_A} \varphi_{g-f}(x;\epsilon) & : \text{Dipole 3 (6)-1}, \\
\frac{1}{C_F} \varphi_{g-f}(x;\epsilon) & : \text{Dipole 4 (7)-1}.
\end{cases}
\] (A40)

\[
\varphi_{f-f}(x;\epsilon) = \delta(1-x) \varphi_{f-f}(\epsilon) + \left(\frac{1}{\epsilon} + \ln \epsilon\right) P_{f-f}(x) + \varphi_{f-f}(x;\epsilon),
\] (A41)

\[
\varphi_{g-g}(x;\epsilon) = \delta(1-x) \varphi_{g-g}(\epsilon) + \left(\frac{1}{\epsilon} + \ln \epsilon\right) P_{g-g}(x) + \varphi_{g-g}(x;\epsilon),
\] (A42)

\[
\varphi_{f-g}(x;\epsilon) = \left(\frac{1}{\epsilon} + \ln \epsilon\right) P_{f-g}(x) + \varphi_{f-g}(x;\epsilon),
\] (A43)

\[
\varphi_{g-f}(x;\epsilon) = \left(\frac{1}{\epsilon} + \ln \epsilon\right) P_{g-f}(x) + \varphi_{g-f}(x;\epsilon).
\] (A44)

\[
\varphi_{g}(\epsilon) = \frac{1}{2} \varphi_{g-g}(\epsilon) + N_f \varphi_{f-f}(\epsilon).
\] (A45)

\[
P_{f-f}(x) = C_F \left(\frac{1+x^2}{1-x}\right)_+, C_F \left[\frac{3}{2} \delta(1-x) + \frac{1+x^2}{(1-x)_+}\right],
\] (A46)

\[
P_{g-g}(x) = \delta(1-x) \left(\frac{11}{6} C_A - \frac{2}{3} N_f T_R\right)
+ 2CA \left[\left(\frac{1}{1-x}\right)_+ + \frac{1-x}{x} - 1 + x(1-x)\right],
\] (A47)

\[
P_{f-g}(x) = C_F \frac{1}{x} \left(1-(1-x)^2\right),
\] (A48)

\[
P_{g-f}(x) = T_R \left[x^2 + (1-x)^2\right].
\] (A49)

\[
\varphi_{f-f}(x;\epsilon) = \delta(1-x) C_F \left(\frac{2}{3} \pi^2 - 5\right) - \ln x \cdot P_{f-f}(x)
+ C_F \left[- \left(\frac{4}{1-x} \ln \frac{1}{1-x}\right)_+ + \frac{2}{1-x} \ln(2-x) + 1 - x - (1+x) \ln(1-x)\right],
\] (A50)

\[
\varphi_{g-g}(x;\epsilon) = \delta(1-x) \left[C_A \left(\frac{2}{3} \pi^2 - 50\right) + \frac{16}{9} N_f T_R\right] - \ln x \cdot P_{g-g}(x)
+ C_A \left[- \left(\frac{4}{1-x} \ln \frac{1}{1-x}\right)_+ - \frac{2}{1-x} \ln(2-x)
+ 2 \left(-1 + x(1-x) + \frac{1-x}{x}\right) \ln(1-x)\right],
\] (A51)

\[
\varphi_{g-f}(x;\epsilon) = \ln \frac{1-x}{x} P_{f-g}(x) + C_F x,
\] (A52)
\[ y_{\text{other}}^g(x; \epsilon) = \ln \frac{1 - x}{x} p_{gf}(x) + T_R 2x(1 - x). \]  
\hspace{1cm} (A53)

**Color-correlated Born squared amplitude:**

\[ [y_{\text{emi}}, y_{\text{spe}}] = (s_{x/a, y_{\text{spe}}})^{-\epsilon} \cdot \langle B j | T_{y_{\text{emi}}} \cdot T_{y_{\text{spe}}} | B j \rangle_d. \]  
\hspace{1cm} (A54)

**Lorentz scalar:**

\[ s_{x/a, y_{\text{spe}}} = 2p_{a/b} \cdot P(y_{\text{spe}}). \]  
\hspace{1cm} (A55)

**D\(_{ai,b\text{: Initial–initial dipole}}**

Dipole 1 (3)-2, (4)-2,
Dipole 3 (6)-2,
Dipole 4 (7)-2:

\[ \sigma_D(R_i, \text{dip} j, II, x_{a/b}) = - \frac{A_d}{S_{R_i}} \int_0^1 dx \frac{1}{T_{F(y_{\text{emi})}}} \tilde{\nu}^{F(x_{a/b}),F(y_{\text{emi})}}(x; \epsilon) \cdot \Phi_{a/b}(B_j, x)_d [y_{\text{emi}}, y_{\text{spe}}]. \]  
\hspace{1cm} (A56)

**Definition of the symbols**

Universal singular functions:

\[ \frac{1}{T_{F(y_{\text{emi})}}} \tilde{\nu}^{F(x_{a/b}),F(y_{\text{emi})}}(x; \epsilon) = \begin{cases} 
\frac{1}{C_F} \tilde{\nu}^{f,f}(x; \epsilon) & \text{: Dipole 1 (3)-2,} \\
\frac{1}{C_F} \tilde{\nu}^{g,g}(x; \epsilon) & \text{: Dipole 1 (4)-2,} \\
\frac{1}{C_A} \tilde{\nu}^{f,g}(x; \epsilon) & \text{: Dipole 3 (6)-2,} \\
\frac{1}{C_F} \tilde{\nu}^{g,f}(x; \epsilon) & \text{: Dipole 4 (7)-2.} 
\end{cases} \]  
\hspace{1cm} (A57)

\[ \tilde{\nu}^{a,b}(x; \epsilon) = \nu^{a,b}(x; \epsilon) + \delta^{ab} T_a^2 g(x) + \tilde{K}^{ab}(x), \]  
\hspace{1cm} (A58)

\[ \tilde{\nu}^{f,f}(x; \epsilon) = \nu^{f,f}(x; \epsilon) + C_{fg}(x) + \tilde{K}^{ff}(x), \]  
\hspace{1cm} (A59)

\[ \tilde{\nu}^{g,g}(x; \epsilon) = \nu^{g,g}(x; \epsilon) + C_{Ag}(x) + \tilde{K}^{gg}(x), \]  
\hspace{1cm} (A60)

\[ \tilde{\nu}^{f,g}(x; \epsilon) = \nu^{f,g}(x; \epsilon) + \tilde{K}^{fg}(x), \]  
\hspace{1cm} (A61)

\[ \tilde{\nu}^{g,f}(x; \epsilon) = \nu^{g,f}(x; \epsilon) + \tilde{K}^{gf}(x). \]  
\hspace{1cm} (A62)

\[ \tilde{K}^{ab}(x) = P_{reg}^{ab}(x) \ln(1 - x) + \delta^{ab} T_a^2 \left[ \frac{2}{x} \ln(1 - x) + \frac{\pi^2}{3} \delta(1 - x) \right]. \]  
\hspace{1cm} (A63)

\[ \tilde{K}^{ff}(x) = P_{reg}^{ff}(x) \ln(1 - x) + C_F \left[ \frac{2}{x} \ln(1 - x) + \frac{\pi^2}{3} \delta(1 - x) \right]. \]  
\hspace{1cm} (A64)
Fig. A2. The classification of the integrated dipole term $\sigma_D(I)$.

\[
\tilde{K}^{gs}(x) = P_{reg}^{gs}(x) \ln(1-x) + C_A \left[ \left( \frac{2}{1-x} \ln(1-x) \right)_+ - \frac{\pi^2}{3} \delta(1-x) \right],
\]
(A65)

\[
\tilde{K}^{fg}(x) = P^{fg}(x) \ln(1-x),
\]
(A66)

\[
\tilde{K}^{gf}(x) = P^{gf}(x) \ln(1-x).
\]
(A67)

\[
P_{reg}^{ff}(x) = -C_F(1+x),
\]
(A68)

\[
P_{reg}^{gg}(x) = 2C_A \left\{ \frac{1-x}{x} - 1 + x(1-x) \right\}.
\]
(A69)

Color-correlated Born squared amplitude:

\[
[y_{emi}, y_{spe}] = (s_{xa/b, y_{spe}})^{-\epsilon} \cdot \langle B_j | T_{y_{emi}} \cdot T_{y_{spe}} | B_j \rangle d.
\]
(A70)

Lorentz scalar:

\[
s_{xa/b, y_{spe}} = 2p_a \cdot p_b.
\]
(A71)

A.3. Integrated dipole term: $\sigma_D(I)$

Dipole 1 (1)-1/2, (2)-1/2, (3)-1/2, (4)-1/2:

\[
(1)-1/2 : \hat{\sigma}_D(R_i, I, (1)-1/2) = -\frac{A_d}{\delta R_i} \cdot \frac{1}{C_F} V_f(\epsilon) \cdot \Phi(B1)_d \left[ y_{emi, y_{spe}} \right],
\]

\[
(2)-1/2 : \hat{\sigma}_D(R_i, I, (2)-1/2) = -\frac{A_d}{\delta R_i} \cdot \frac{1}{C_A} V_{gg}(\epsilon) \cdot \Phi(B1)_d \left[ y_{emi, y_{spe}} \right],
\]

\[
(3)-1/2 : \hat{\sigma}_D(R_i, I, (3)-1/2) = -\frac{A_d}{\delta R_i} \cdot \frac{1}{C_F} V_f(\epsilon) \cdot \Phi(B1)_d \left[ y_{emi, y_{spe}} \right],
\]

\[
(4)-1/2 : \hat{\sigma}_D(R_i, I, (4)-1/2).
\]
Fig. A3. The classification of the integrated dipole term $\sigma_D(P)$.

\[(4)\cdot\frac{1}{2} : \hat{\sigma}_D(R_i, I, (4)\cdot\frac{1}{2}) = -\frac{A_d}{S_{R_i}} \cdot \frac{1}{C_A} \nu_g(\epsilon) \cdot \Phi(B1)_d \left[ y_{emi}, y_{spe} \right]. \quad (A72)\]

**Definition of the symbols**

Universal singular functions:

\[\nu_f(\epsilon) = \nu_{fg}(\epsilon), \quad (A73)\]
\[\nu_g(\epsilon) = \frac{1}{2} \nu_{gg}(\epsilon) + N_f \nu_{ff}(\epsilon). \quad (A74)\]

Color-correlated Born squared amplitude:

\[\left[ y_{emi}, y_{spe} \right] = (s_{y_{emi}, y_{spe}})^{-\epsilon} \cdot (B1|T_{y_{emi}} \cdot T_{y_{spe}}|B1)_d. \quad (A75)\]

Lorentz scalar: $s_{y_{emi}, y_{spe}} = 2P(y_{emi}) \cdot P(y_{spe})$.

**A.4. Integrated dipole term: $\sigma_D(P)$**

Dipole 1 (3)\(-\frac{1}{2}, (4)\cdot\frac{1}{2}, \)

Dipole 3 (6)\(-\frac{1}{2}, \)

Dipole 4 (7)\(-\frac{1}{2}: \)

\[\hat{\sigma}_D(R_i, P, dip_j, x_{a/b}) = \frac{A_d}{S_{R_i}} \int_0^1 dx \left( \frac{1}{\epsilon} - \ln x \right) \frac{1}{T^2_{F(y_{emi})}} P^{F(x_{a/b})} F(y_{emi})(x) \]
\[\cdot \Phi_{a/b}(B_j, x)_d \left[ y_{emi}, y_{spe} \right]. \quad (A76)\]
The classification of the integrated dipole term $\sigma_D(K)$.

**Definition of the symbols**

Splitting functions:

$$\frac{1}{T^2_{F(Y_{emi})}} p^{F(x_a/b) F(Y_{emi})}(x) = \begin{cases} \frac{1}{C_F} p^{ff}(x) : \text{Dipole 1 (3)-1/2,} \\ \frac{1}{C_A} p^{gg}(x) : \text{Dipole 1 (4)-1/2,} \\ \frac{1}{C_A} p^{fg}(x) : \text{Dipole 3 (6)-1/2,} \\ \frac{1}{C_F} p^{gf}(x) : \text{Dipole 4 (7)-1/2.} \end{cases}$$

(A77)

Color-correlated Born squared amplitude:

$$\left[ y_{emi}, y_{spe} \right] = (s_{x_a/b, y_{spe}})^{-\epsilon} : (B_j | T_{y_{emi}} \cdot T_{y_{spe}} | B_j)_d.$$  

(A78)

Lorentz scalars:

$$s_{x_a/b, y_{spe}} = \begin{cases} 2 p_{a/b} \cdot P(y_{spe}) : (3, (4), (6), (7)-1, \text{initial–final,} \\ 2 p_a \cdot p_b : (3, (4), (6), (7)-2, \text{initial–initial.} \end{cases}$$

(A79)

**A.5. Integrated dipole term: $\sigma_D(K)$**

Dipole 1 (1)-2, (2)-2, (3)-1/2, (4)-1/2,

Dipole 3 (6)-1/2,

Dipole 4 (7)-1/2:
Dipole 1

\[ (1)-2 : \hat{\sigma}_D(R_i, K, \text{dip}1, (1)-2, x_d) = -\frac{A_4}{5R_i} \int_0^1 dx \left( g(x) - \frac{3}{2} h(x) \right) \cdot \Phi_a(B1, x)_4 \langle \gamma_{\text{emi}}, \gamma_{\text{spe}} \rangle, \]

\[ (2)-2 : \hat{\sigma}_D(R_i, K, \text{dip}1, (2)-2, x_d) = -\frac{A_4}{5R_i} \int_0^1 dx \left( 2g(x) - \frac{11}{3} h(x) \right) \cdot \Phi_a(B1, x)_4 \langle \gamma_{\text{emi}}, \gamma_{\text{spe}} \rangle, \]

\[ (3)-1 : \hat{\sigma}_D(R_i, K, \text{dip}1, (3)-1, x_d) = -\frac{A_4}{5R_i} \int_0^1 dx \frac{1}{C_F} V_{\text{other}}^f \langle \epsilon \rangle \cdot \Phi_a(B1, x)_4 \langle \gamma_{\text{emi}}, \gamma_{\text{spe}} \rangle, \]

(3) \[ (3)-2 : \hat{\sigma}_D(R_i, K, \text{dip}1, (3)-2, x_d) = -\frac{A_4}{5R_i} \int_0^1 dx \frac{1}{C_F} \left( V_{\text{other}}^f \langle \epsilon \rangle + C_F g(x) + \tilde{K}^f \langle \epsilon \rangle \right) \cdot \Phi_a(B1, x)_4 \langle \gamma_{\text{emi}}, \gamma_{\text{spe}} \rangle, \]

(4) \[ (3)-2 : \hat{\sigma}_D(R_i, K, \text{dip}1, (4)-2, x_d) = -\frac{A_4}{5R_i} \int_0^1 dx \frac{1}{C_A} \left( V_{\text{other}}^g \langle \epsilon \rangle + C_A g(x) + \tilde{K}^g \langle \epsilon \rangle \right) \cdot \Phi_a(B1, x)_4 \langle \gamma_{\text{emi}}, \gamma_{\text{spe}} \rangle. \]

Dipole 3/4

\[ (6)-1 : \hat{\sigma}_D(R_i, K, \text{dip}3, (6)-1, x_d) = -\frac{A_4}{5R_i} \int_0^1 dx \frac{1}{C_A} \left( V_{\text{other}}^f \langle \epsilon \rangle + C_A g(x) + \tilde{K}^f \langle \epsilon \rangle \right) \cdot \Phi_a(B3, x)_4 \langle \gamma_{\text{emi}}, \gamma_{\text{spe}} \rangle, \]

\[ (7)-1 : \hat{\sigma}_D(R_i, K, \text{dip}4, (7)-1, x_d) = -\frac{A_4}{5R_i} \int_0^1 dx \frac{1}{C_A} \left( V_{\text{other}}^g \langle \epsilon \rangle + C_A g(x) + \tilde{K}^g \langle \epsilon \rangle \right) \cdot \Phi_a(B4, x)_4 \langle \gamma_{\text{emi}}, \gamma_{\text{spe}} \rangle, \]

\[ (6)-2 : \hat{\sigma}_D(R_i, K, \text{dip}3, (6)-2, x_d) = -\frac{A_4}{5R_i} \int_0^1 dx \frac{1}{C_A} \left( V_{\text{other}}^f \langle \epsilon \rangle + C_A g(x) + \tilde{K}^f \langle \epsilon \rangle \right) \cdot \Phi_a(B3, x)_4 \langle \gamma_{\text{emi}}, \gamma_{\text{spe}} \rangle, \]

\[ (7)-2 : \hat{\sigma}_D(R_i, K, \text{dip}4, (7)-2, x_d) = -\frac{A_4}{5R_i} \int_0^1 dx \frac{1}{C_A} \left( V_{\text{other}}^g \langle \epsilon \rangle + C_A g(x) + \tilde{K}^g \langle \epsilon \rangle \right) \cdot \Phi_a(B4, x)_4 \langle \gamma_{\text{emi}}, \gamma_{\text{spe}} \rangle. \]

Reconstruction of Dipole 1

\[ \sigma_D(R_i, K, \text{dip}1) = \hat{\sigma}_D(K, \text{dip}1, (1)-(4)-2, g)^\oplus \]

\[ + \hat{\sigma}_D(K, \text{dip}1, (3)/(4)-1/2, V_{\text{other}}^a)^\oplus \]

\[ + \hat{\sigma}_D(K, \text{dip}1, (1)/(2)-2, h) \]

\[ + \hat{\sigma}_D(K, \text{dip}1, (3)/(4)-2, \tilde{K}^{aa}), \]

\[ \text{(A85)} \]
\[
\hat{\sigma}_D(K, \text{ dip}1, (1)-(4)-2, g) = \frac{A_4}{S_{R_s}} \int_0^1 dx \Phi_a(B1, x) 4 \langle y_{emi}, y_{spe} \rangle g(x) \\
\times \begin{cases} 
1: (1)-, (3)-, (4)-2, \\
2: (2)-2. 
\end{cases} 
\]  
(A86)

\[
\hat{\sigma}_D \left( K, \text{ dip}1, (3)-1/2, \nu_{other}^{ff, f} \right) = -\frac{A_4}{S_{R_s}} \int_0^1 dx \frac{1}{C_F} \nu_{other}^{ff, f}(x; \epsilon) \cdot \Phi_a(B1, x) 4 \langle y_{emi}, y_{spe} \rangle, 
\]  
(A87)

\[
\hat{\sigma}_D \left( K, \text{ dip}1, (4)-1/2, \nu_{other}^{gg, g} \right) = -\frac{A_4}{S_{R_s}} \int_0^1 dx \frac{1}{C_A} \nu_{other}^{gg, g}(x; \epsilon) \cdot \Phi_a(B1, x) 4 \langle y_{emi}, y_{spe} \rangle, 
\]  
(A88)

\[
\hat{\sigma}_D \left( K, \text{ dip}1, (1)/(2)-2, h \right) = -\frac{A_4}{S_{R_s}} \int_0^1 dx \Phi_a(B1, x) 4 \langle y_{emi}, y_{spe} \rangle h(x) \\
\times \begin{cases} 
-3/2 : (1)-2, \\
-11/3 : (2)-2, 
\end{cases} 
\]  
(A89)

\[
\hat{\sigma}_D \left( K, \text{ dip}1, (3)/(4)-2, \tilde{K}^{ff/ff} \right) = -\frac{A_4}{S_{R_s}} \int_0^1 dx \frac{1}{C_{F/A}} \tilde{K}^{ff/ff}(x) \cdot \Phi_a(B1, x) 4 \langle y_{emi}, y_{spe} \rangle. 
\]  
(A90)

Reconstruction of Dipole 3/4

\[
\sigma_D(R_s, K, \text{ dip}3/4) = \hat{\sigma}_D \left( K, \text{ dip}3/4, (6)/(7)-1/2, \nu_{other}^{a,b} \right) \\
+ \hat{\sigma}_D \left( K, \text{ dip}3/4, (6)/(7)-2, \tilde{K}^{a,b} \right), 
\]  
(A91)

\[
\hat{\sigma}_D \left( K, \text{ dip}3, (6)-1/2, \nu_{other}^{f/g} \right) = -\frac{A_4}{S_{R_s}} \int_0^1 dx \frac{1}{C_A} \nu_{other}^{f/g}(x; \epsilon) \cdot \Phi_a(B3, x) 4 \langle y_{emi}, y_{spe} \rangle, 
\]  
(A92)

\[
\hat{\sigma}_D \left( K, \text{ dip}4, (7)-1/2, \nu_{other}^{g/f} \right) = -\frac{A_4}{S_{R_s}} \int_0^1 dx \frac{1}{C_F} \nu_{other}^{g/f}(x; \epsilon) \cdot \Phi_a(B4, x) 4 \langle y_{emi}, y_{spe} \rangle, 
\]  
(A93)

\[
\hat{\sigma}_D \left( K, \text{ dip}3/4, (6)/(7)-2, \tilde{K}^{f/g/f} \right) = -\frac{A_4}{S_{R_s}} \int_0^1 dx \frac{1}{C_{A/F}} \tilde{K}^{f/g/f}(x) \cdot \Phi_a(B3/4, x) 4 \langle y_{emi}, y_{spe} \rangle. 
\]  
(A94)

The cross sections with leg-b \( (x_b) \), \( \hat{\sigma}_D(R_s, K, \text{ dip}j, x_b) \), are obtained by the replacements \( \Phi_a(Bj, x)_4 \rightarrow \Phi_b(Bj, x)_4 \) in the above formulae for leg-a \( (x_a) \).

Definition of the symbols

Color-correlated Born squared amplitude:

\[
\langle y_{emi}, y_{spe} \rangle = \langle Bj | T_{y_{emi}} \cdot T_{y_{spe}} | Bj \rangle 4. 
\]  
(A95)
Fig. A5. The classification of the integrated dipole term $\sigma_D(\text{dip}2)$.

A.6. Integrated dipole term: $\sigma_D(\text{dip}2)$

Dipole 2 (5)-1/2:

\begin{equation}
\hat{\sigma}_D(R_i, \text{dip}2, (5)-1) = -\frac{A_d}{S_{R_i}} \cdot \frac{1}{C_A} \Phi(B2) \cdot \left( y_{\text{emi}}, y_{\text{spe}} \right),
\end{equation}

\begin{equation}
\hat{\sigma}_D(R_i, \text{dip}2, (5)-2, x_{a/b}) = -\frac{A_d}{S_{R_i}} \int_0^1 dx \frac{1}{C_A} \Phi(B2) \cdot \left( y_{\text{emi}}, y_{\text{spe}} \right).
\end{equation}

Reconstruction of Dipole 2

\begin{equation}
\hat{\sigma}_D(R_i, \text{dip}2, (5)-1/2, \Phi_{ij}) = -\frac{A_d}{S_{R_i}} \cdot \frac{1}{C_A} \Phi(B2) \cdot \left( y_{\text{emi}}, y_{\text{spe}} \right),
\end{equation}

\begin{equation}
\hat{\sigma}_D(R_i, \text{dip}2, (5)-2, h, x_{a/b}) = -\frac{A_4}{S_{R_i}} \int_0^1 dx \frac{1}{C_A} \Phi(B2) \cdot \left( y_{\text{emi}}, y_{\text{spe}} \right).
\end{equation}

Definition of the symbols

Color-correlated Born squared amplitude:

\begin{equation}
\left[ y_{\text{emi}}, y_{\text{spe}} \right] = \left( s_{\text{emi}}, s_{\text{spe}} \right) e \cdot \left( y_{\text{emi}}, y_{\text{spe}} \right).
\end{equation}

\begin{equation}
\left( y_{\text{emi}}, y_{\text{spe}} \right)_{d/4} = (B2|T_{\text{emi}} \cdot T_{\text{spe}} |B2)_{d/4}.
\end{equation}

Lorentz scalar: $s_{\text{emi}}, s_{\text{spe}} = 2P(y_{\text{emi}}) \cdot P(y_{\text{spe}})$.

A.7. Collinear subtraction term: $\sigma_C$

\begin{align*}
\text{Input:} & \ R_i \\
\text{Dipole 1 (3), (4),} & \\
\text{Dipole 3 (6),} & \\
\text{Dipole 4 (7):} & \\
\end{align*}
\( \hat{\sigma}_C(R_i) = \sum_{\text{dip}} \hat{\sigma}_C(R_i, \text{dip}). \)  
\( \hat{\sigma}_C(R_i, \text{dip}) = \hat{\sigma}_C(R_i, \text{dip}, x_a) + \hat{\sigma}_C(R_i, \text{dip}, x_b). \)  
\( \hat{\sigma}_C(R_i, \text{dip}, x_{a/b}) = \frac{\alpha_s}{2\pi} \frac{1}{\Gamma(1-\epsilon)} \int_0^1 dx \left[ \frac{1}{\epsilon} \left( \frac{4\pi \mu^2}{\mu_F^2} \right)^\epsilon p^{F(x_{a/b})F(y_{\text{emi})})(x)} \right] 
\times \frac{1}{S_{Bj}} \Phi_{a/b}(B_j, x) d(B_j), \)  
\( = \frac{A_d}{S_{Bj}} \int_0^1 dx \left( \frac{1}{\epsilon} - \ln \mu_F^2 \right) p^{F(x_{a/b})F(y_{\text{emi})})(x)} : \Phi_{a/b}(B_j, x) d(B_j). \)  

**Definition of the symbols**

Born squared amplitude: \( \langle B j \rangle = \langle B j | B j \rangle_d. \)

Splitting functions:

\[
P^{F(x_{a/b})F(y_{\text{emi})})(x)} = \begin{cases} 
P^{ff}(x) & : \text{Dipole 1 (3),} 
P^{gg}(x) & : \text{Dipole 1 (4),} 
P^{sf}(x) & : \text{Dipole 3 (6),} 
P^{gf}(x) & : \text{Dipole 4 (7).} \end{cases}
\]

**A.8. I term: \( \sigma_I \)**

**Input:** \( B1(R_i) \)

**Legs:** \( (1)-1/2, (2)-1/2, (3)-1/2, (4)-1/2. \)
Fig. A7. The classification of the I term $\sigma_1$.

$$\hat{\sigma}_1(R_i)_{IK} = -\frac{A_d}{B_1} \cdot \frac{1}{T^2_{F(I)}} \cdot \mathcal{V}_F(I) \cdot \Phi(B1)_{d}[I, K].$$ \hspace{1cm} (A107)

**Definition of the symbols**

Common factor: $A_d$ is the same as in Eq. (2.16).

Universal singular functions:

$$\frac{1}{T^2_{F(I)}} \mathcal{V}_F(I) = \begin{cases} \frac{1}{C_F} \mathcal{V}_f(\varepsilon) = \frac{1}{C_F} \mathcal{V}_{fg}(\varepsilon) & : (1), (3)-1/2, \\ \frac{1}{C_A} \mathcal{V}_g(\varepsilon) = \frac{1}{C_A} (\frac{1}{2} \mathcal{V}_{gg}(\varepsilon) + N_f \mathcal{V}_{ff}(\varepsilon)) & : (2), (4)-1/2. \end{cases}$$ \hspace{1cm} (A108)

Color-correlated Born squared amplitude:

$$[I, K] = s^{-\varepsilon}_{IK} \cdot \langle B1 | T_I \cdot T_K | B1 \rangle_d.$$ \hspace{1cm} (A109)

$$\hat{\sigma}_1(R_i, (2)-1/2, N_f \mathcal{V}_{ff})_{IK}$$

$$\hat{\sigma}_1(R_i, (2)-1/2, N_f \mathcal{V}_{ff})_{IK} = -\frac{A_d}{S_{B1}} \cdot \frac{N_f}{C_A} \mathcal{V}_{ff}(\varepsilon) \cdot \Phi(B1)_{d}[I, K].$$ \hspace{1cm} (A110)

**A.9. P term: $\sigma_P$**

**Input:** $R_i$

- Dipole 1 (3)-1/2, (4)-1/2,
- Dipole 3 (6)-1/2,
- Dipole 4 (7)-1/2:

$$\hat{\sigma}_P(R_i, dipj, x_{a/b}) = \frac{A_4}{S_{B_j}} \int_0^1 dx \frac{1}{T^2_{F(y_{emi})}} p^{F(x_{a/b})F(y_{emi})}(x)$$

$$\cdot \ln \frac{\mu_r^2}{x_{s_{x_{a/b}, y_{spe}}}} \cdot \Phi_{a/b}(B_j, x)_{4} \{y_{emi}, y_{spe}\}.$$ \hspace{1cm} (A111)
Fig. A8. The classification of the P term $\sigma_P$.

**Definition of the symbols**

**Common factor:**

\[ A_4 = \frac{\alpha_s}{2\pi}. \]  \hfill (A112)

**Splitting functions:** $P_{\text{F}(x_a/b)\text{F}(\gamma_{\text{emi}})}(x)/T_{\text{F}(\gamma_{\text{emi}})}^2$ is the same as in Eq. (A77).

**Lorentz scalar:** $s_{x_a/b, y_{\text{spe}}}$ is the same as in Eq. (A79).

**Color-correlated Born squared amplitude:**

\[ \langle y_{\text{emi}}, y_{\text{spe}} \rangle = \langle B_{\gamma} | T_{\gamma_{\text{emi}}} \cdot T_{y_{\text{spe}}} | B_{\gamma} \rangle. \]  \hfill (A113)

**A.10. K term:** $\sigma_K$

**Input:** $R_i$

Dipole 1: (3) -0/1/2, (4) -0/1/2,

Dipole 3: (6) -0/2,

Dipole 4: (7) -0/2:

Dipole 1 (3)/(4)

\[ \hat{\sigma}_K(R_i, \text{dip1}, (3)/(4)-0, x_a) = \frac{A_4}{\mathcal{B}_1} \int_0^1 dx \tilde{K}_{ff/gg}^{\gamma y_{\text{spe}}} (x) \cdot \Phi_a(B_1, x)_{4(B1)}. \]  \hfill (A114)

\[ \hat{\sigma}_K(R_i, \text{dip1}, (3)/(4)-1, x_a) = \frac{A_4}{\mathcal{B}_1} \int_0^1 dx \frac{\gamma_{\text{F}(y_{\text{spe}})} h(x) \cdot \Phi_a(B_1, x)_{4(y_{\text{emi}}, y_{\text{spe}})}}{T_{\gamma y_{\text{spe}}}^2}. \]  \hfill (A115)

\[ \hat{\sigma}_K(R_i, \text{dip1}, (3)/(4)-2, x_a) = \frac{A_4}{\mathcal{B}_1} \int_0^1 dx \frac{1}{C_{\gamma y_{\text{spe}}}} \tilde{K}_{ff/gg}^{\gamma y_{\text{spe}}} (x) \cdot \Phi_a(B_1, x)_{4(y_{\text{emi}}, y_{\text{spe}})}. \]  \hfill (A116)
Fig. A9. The classification of the K term $\sigma_K$.

Dipole 3/4 (6)/(7)

$$\hat{\sigma}_K(R_i, \text{dip}3/4, (6)/(7)-0, x_a) = \frac{A_4}{S_{B3/4}} \int_0^1 dx \tilde{K}^{f/g/g}(x) \cdot \Phi_4(B_{3/4}, x)_4(B_{3/4}), \quad (A117)$$

$$\hat{\sigma}_K(R_i, \text{dip}3/4, (6)/(7)-2, x_a) = \frac{A_4}{S_{B3/4}} \int_0^1 dx \frac{-1}{C_A/F} \tilde{K}^{f/g/g}(x) \cdot \Phi_4(B_{3/4}, x)_4 \langle y_{\text{emi}}, y_{\text{spe}} \rangle. \quad (A118)$$

The cross sections with leg-b ($x_b$), $\hat{\sigma}_K(R_i, \text{dip}j, x_b)$, are obtained by the replacements $\Phi_4(B_j, x)_4 \rightarrow \Phi_4(B_j, x)_4$ in the formulae for leg-a ($x_a$).

**Definition of the symbols**

Color-correlated Born squared amplitude: $\langle y_{\text{emi}}, y_{\text{spe}} \rangle$ is the same as in Eq. (A113).

Relations of the functions, $K^{f/g/g}(x)$, $V^{f,f/g,g/\text{other}}_\text{other}(x; \epsilon)$, and $g(x)$:

$$K^{f/g/g}(x) = V^{f,f/g,g/\text{other}}_\text{other}(x; \epsilon) + C_F/A g(x), \quad (A119)$$

$$K^{f/g/g/\text{other}}(x) = V^{f,g/g,f}_\text{other}(x; \epsilon). \quad (A120)$$

The factor $T^2_{F(y_{\text{spe}})}/T^2_{F(y_{\text{spe}})}$:

$$\frac{T^2_{F(y_{\text{spe}})}/T^2_{F(y_{\text{spe}})}}{\gamma_{F(y_{\text{spe}})}} = \begin{cases} \frac{3}{2} : F(y_{\text{spe}}) = \text{quark}, \\ \frac{11}{6} - \frac{2}{3} \frac{T_R N_f}{C_A} : F(y_{\text{spe}}) = \text{gluon}. \end{cases} \quad (A121)$$

$$\sigma_K(R_i, \text{dip}1, (3)/(4)-1, N_f h, x_{a/b}) = \frac{A_4}{S_{B1}} \int_0^1 dx \left( -2 \frac{T_R N_f}{3 C_A} \right) h(x) \cdot \Phi_{a/b}(B_1, x)_4 \langle y_{\text{emi}}, y_{\text{spe}} \rangle. \quad (A122)$$
Appendix B. Summary for the dijet process

B.1. \( \hat{\sigma}_{\text{subt}}(R_{1u}) \)

Table B1. Summary table of \( \hat{\sigma}_D(R_{1u}) \). The universal singular functions are abbreviated as \( \mathcal{V}_{ij}(e) = \mathcal{V}_{ij} \), \( \mathcal{V}_{ij}(x; e) = \mathcal{V}_{ij}(x) \), \( \mathcal{V}_{ij}^{a,a'}(x; e) = \mathcal{V}_{ij}^{a,a'}(x) \), and \( \mathcal{V}_{ij}^{a,a'}(x; e) = \mathcal{V}_{ij}^{a,a'}(x) \).

| Dip \( j \) | Bj | Splitting | \((y_u, y_i, y_1, y_2)\) | Factor 1 | \( \Phi(B_j) \) | \( y_{\text{emi}}, y_{\text{spe}} \) |
|------------|----|-----------|-----------------|---------|------------|----------------|
| Dip 1 \( u\bar{u} \rightarrow u\bar{u} \) | (1) - 1 | 1. \((a, b; 3\bar{2}, \bar{2}3)\) | \( \mathcal{V}_{fg}/C_F \) | \( \Phi(B1)[1, 2] \) | |
| | (1) - 2 | 3. \((a\bar{b}; 3\bar{2}, \bar{2}3)\) | \( \int dx \mathcal{V}_{fg}(x)/C_F \) | \( \Phi_\nu(B1)[1, a] \) | |
| | (3) - 1 | 7. \((a\bar{3}, b; \bar{2}1, 2\bar{3})\) | \( \int dx \mathcal{V}_{fg}(x)/C_F \) | \( \Phi_\nu(B1)[a, 1] \) | |
| | (3) - 2 | 11. \((a\bar{3}, b; 12, \bar{2}3)\) | \( \int dx \tilde{\mathcal{V}}_{fg}(x)/C_F \) | \( \Phi_\nu(B1)[b, a] \) | |
| Dip 2u \( u\bar{u} \rightarrow gg \) | (5) - 1 | 13. \((a, b; 2\bar{3}, \bar{2}3)\) | \( \mathcal{V}_{fg}/C_A \) | \( \Phi(B2u)[1, 2] \) | |
| | (5) - 2 | 14. \((a\bar{b}; 2\bar{3}, \bar{2}3)\) | \( \int dx \mathcal{V}_{fg}(x)/C_A \) | \( \Phi_\nu(B2u)[1, a] \) | |
| | (6) - 1 | 16. \((a\bar{1}, b; 3\bar{2}, 2\bar{3})\) | \( \int dx \mathcal{V}_{fg}(x)/C_A \) | \( \Phi_\nu(B3u)[a, 1] \) | |
| | (6) - 2 | 17. \((a\bar{1}, b; 2\bar{3}, 3\bar{2})\) | \( \int dx \tilde{\mathcal{V}}_{fg}(x)/C_A \) | \( \Phi_\nu(B3u)[a, 2] \) | |
| Dip 3u \( g\bar{u} \rightarrow u\bar{g} \) | (6) - 1 | 19. \((a\bar{b}; 2\bar{1}, \bar{2}3)\) | \( \int dx \mathcal{V}_{fg}(x)/C_A \) | \( \Phi_\nu(B3\bar{u})[b, a] \) | |
| | (6) - 2 | 20. \((a\bar{b}; 1\bar{3}, \bar{2}3)\) | \( \int dx \tilde{\mathcal{V}}_{fg}(x)/C_A \) | \( \Phi_\nu(B3\bar{u})[b, 2] \) | |
| Dip 3\( \bar{u} \) \( ug \rightarrow ug \) | (6) - 1 | 21. \((a\bar{b}; 1\bar{3}, \bar{2}3)\) | \( \int dx \mathcal{V}_{fg}(x)/C_A \) | \( \Phi_\nu(B3\bar{u})[b, a] \) | |

Step 1

\[
\hat{\sigma}_D(R_1) = \frac{A_d}{S_{R_1}} \cdot \text{(Factor 1)} \cdot \Phi_d(R_1 : Bj, x) \delta \left[ y_{\text{emi}}, y_{\text{spe}} \right].
\]

\[
\hat{\sigma}_D(R_{1u} = u\bar{u} \rightarrow u\bar{u}g) : \quad S_{R_1} = 1,
\]

Step 2

\[
\hat{\sigma}_D(R_1, I) - \hat{\sigma}_I(R_1) = 0. \tag{B1}
\]

Step 3

\[
\hat{\sigma}_D(R_1, P) + \hat{\sigma}_C(R_1) - \hat{\sigma}_P(R_1) = 0, \tag{B2}
\]

which is separated into three relations for Dipoles 1, 3\( u \), and 3\( \bar{u} \) as

\[
\hat{\sigma}_D(R_1, P, \text{dip} \, j) + \hat{\sigma}_C(R_1, \text{dip} \, j) - \hat{\sigma}_P(R_1, \text{dip} \, j) = 0. \tag{B3}
\]
Step 4

\[ \hat{\sigma}_D (R_1, K) - \hat{\sigma}_K (R_1) = 0, \]  
which is separated into three relations for Dipoles 1, 3u, and 3\( \bar{u} \) as

\[ \hat{\sigma}_D (R_1, K, \text{dip} j) - \hat{\sigma}_K (R_1, \text{dip} j) = 0. \]  

Step 5

\[ \hat{\sigma}_{\text{sub}} (R_1) = \hat{\sigma}_D (R_1, \text{dip} 2). \]  
\[ \hat{\sigma}_D (R_1, \text{dip} 2) = \hat{\sigma}_D \left( R_1, \text{dip} 2, (5)-1/2, \mathcal{V}_{f \bar{f}} \right) + \hat{\sigma}_D \left( R_1, \text{dip} 2, (5)-2, h \right). \]

\[ \hat{\sigma}_D \left( R_1, \text{dip} 2, (5)-1/2, \mathcal{V}_{f \bar{f}} \right) = -\frac{A_d}{S_{R_1}} \cdot \frac{1}{C_A} \mathcal{V}_{f \bar{f}}(\epsilon) \]
\[ \cdot \Phi(B2)_d \left( [1, 2] + [1, a] + [1, b] \right). \]

\[ \hat{\sigma}_D (R_1, \text{dip} 2, (5)-2, h) = -\frac{A_4}{S_{R_1}} \int_0^1 dx \frac{T_R}{C_A^2} h(x) \]
\[ \times \left( \Phi_a(B2, x)_4(1, a) + \Phi_b(B2, x)_4(1, b) \right). \]

B.2. \( \hat{\sigma}_{\text{sub}} (R_{2u}) \)

| Dip | Bj | Splitting | \((y_u, y_b : y_1, y_2)\) | Factor 1 | \( \Phi(B_j) \) |
|-----|----|-----------|-----------------|----------|----------------|
| Dip 1 uu \( \rightarrow \) uu | (1) - 1 | 1. \( (a, b; \tilde{1}, \tilde{2}) \) | \( \mathcal{V}_{f \bar{f}} / C_F \) | \( \Phi(B1)[1, 2] \) |
|     |     | 2. \( (a, b; \tilde{1}, \tilde{2}) \) | \( \Phi(B1)[2, 1] \) |
|     |     | 3. \( (\tilde{a}, b; \tilde{1}, \tilde{2}) \) | \( \int dx \mathcal{V}_{f \bar{f}}(x) / C_F \) | \( \Phi_a(B1)[1, a] \) |
|     |     | 4. \( (a, \tilde{b}; \tilde{1}, \tilde{2}) \) | \( \Phi_b(B1)[1, b] \) |
|     |     | 5. \( (\tilde{a}, b; 1, \tilde{2}) \) | \( \Phi_a(B1)[2, a] \) |
|     |     | 6. \( (\tilde{a}, b; 1, \tilde{2}) \) | \( \Phi_b(B1)[2, b] \) |
|     | (3) - 1 | 7. \( (\tilde{a}, \tilde{b}; \tilde{1}, \tilde{2}) \) | \( \int dx \mathcal{V}_{f \bar{f}}(x) / C_F \) | \( \Phi_a(B1)[a, 1] \) |
|     |     | 8. \( (\tilde{a}, b; 1, \tilde{2}) \) | \( \Phi_a(B1)[a, 2] \) |
|     |     | 9. \( (a, \tilde{b}; \tilde{1}, \tilde{2}) \) | \( \Phi_b(B1)[b, 1] \) |
|     |     | 10. \( (a, \tilde{b}; 1, \tilde{2}) \) | \( \Phi_b(B1)[b, 2] \) |
|     | (3) - 2 | 11. \( (\tilde{a}, \tilde{b}; 1, \tilde{2}) \) | \( \int dx \mathcal{V}_{f \bar{f}}(x) / C_F \) | \( \Phi_a(B1)[a, b] \) |
|     |     | 12. \( (\tilde{a}, \tilde{b}; 1, \tilde{2}) \) | \( \Phi_b(B1)[b, a] \) |
| Dip 3u gu \( \rightarrow \) ug | (6) - 1 | 13. \( (\tilde{a}, 1, b; \tilde{2}, 3) \) | \( \int dx \mathcal{V}_{f \bar{f}}(x) / C_A \) | \( \Phi_a(B3u)[a, 1] \) |
|     |     | 14. \( (\tilde{a}, b; 2, \tilde{3}) \) | \( \Phi_a(B3u)[a, 2] \) |
|     |     | 15. \( (\tilde{a}, b; 1, 3) \) | \( \Phi_a(B3u)[a, 1] \) |
|     |     | 16. \( (\tilde{a}, 1, b; \tilde{3}) \) | \( \Phi_a(B3u)[a, 2] \) |
|     |     | 17. \( (\tilde{b}, 1, a; \tilde{2}, 3) \) | \( \Phi_b(B3u)[a, 1] \) |

Continued
Table B2. Continued

| Dip j | Bj | Splitting | \( \langle y_a, y_b : y_1, y_2 \rangle \) | Factor 1 | \( \Phi(B_j) \) |
|-------|----|-----------|-----------------|----------|----------------|
| Dip 2u | ug → gu | (5) - 1 | \( 1. (a, b; \tilde{3}, 2) \) | \( \mathcal{V}_{ij}/C_A \) | \( \Phi(B2u)[1, 2] \) |
| | | | \( 2. (a, b; \tilde{3}, 1) \) | | \( \Phi(B2u)[1, 2] \) |
| | | | \( 3. (\tilde{a}, b; \tilde{3}, 2) \) | \( \int d x \tilde{V}_{ij}(x)/C_A \) | \( \Phi(B2u)[1, a] \) |
| | | | \( 4. (a, \tilde{b}; \tilde{3}, 2) \) | | \( \Phi(B2u)[1, b] \) |
| | | | \( 5. (\tilde{a}, b; \tilde{2}, 1) \) | | \( \Phi(B2u)[1, a] \) |
| | | | \( 6. (a, \tilde{b}; \tilde{2}, 1) \) | | \( \Phi(B2u)[1, b] \) |

Step 1

\[
\hat{\sigma}_D(R_{2u} = uu \rightarrow uug) : \quad S_{R_2} = 2,
\]

\[
\hat{\sigma}_{\text{subt}}(R_{2u} = uu \rightarrow uug)
\]

Step 2

\[
\hat{\sigma}_D(R_{2u}, 1) - \hat{\sigma}_l(R_{2u}) = 0. \tag{B10}
\]

Step 3

\[
\hat{\sigma}_D(R_{2u}, P) + \hat{\sigma}_C(R_{2u}) - \hat{\sigma}_p(R_{2u}) = 0, \tag{B11}
\]

which is separated into two relations for Dipoles 1 and 3u as

\[
\hat{\sigma}_D(R_{2u}, P, \text{dip} j) + \hat{\sigma}_C(R_{2u}, \text{dip} j) - \hat{\sigma}_p(R_{2u}, \text{dip} j) = 0. \tag{B12}
\]

Step 4

\[
\hat{\sigma}_D(R_{2u}, K) - \hat{\sigma}_K(R_{2u}) = 0, \tag{B13}
\]

which is separated into two relations for Dipoles 1 and 3u as

\[
\hat{\sigma}_D(R_{2u}, K, \text{dip} j) - \hat{\sigma}_K(R_{2u}, \text{dip} j) = 0. \tag{B14}
\]

Step 5

\[
\hat{\sigma}_{\text{subt}}(R_{2u}) = 0. \tag{B15}
\]

B.3. \( \hat{\sigma}_\text{subt}(R_{3u}) \)

Table B3. Summary table of \( \hat{\sigma}_D(R_{3u}) \).

| Dip j | Bj | Splitting | \( \langle y_a, y_b : y_1, y_2 \rangle \) | Factor 1 | \( \Phi(B_j) \) |
|-------|----|-----------|-----------------|----------|----------------|
| Dip 3u | ug → gu | (5) - 1 | \( 1. (a, b; \tilde{3}, 2) \) | \( \mathcal{V}_{ij}/C_A \) | \( \Phi(B3u)[1, 2] \) |
| | | | \( 2. (a, b; \tilde{3}, 1) \) | | \( \Phi(B3u)[1, 2] \) |
| | | | \( 3. (\tilde{a}, b; \tilde{3}, 2) \) | \( \int d x \tilde{V}_{ij}(x)/C_A \) | \( \Phi(B3u)[1, a] \) |
| | | | \( 4. (a, \tilde{b}; \tilde{3}, 2) \) | | \( \Phi(B3u)[1, b] \) |
| | | | \( 5. (\tilde{a}, b; \tilde{2}, 1) \) | | \( \Phi(B3u)[1, a] \) |
| | | | \( 6. (a, \tilde{b}; \tilde{2}, 1) \) | | \( \Phi(B3u)[1, b] \) |
Table B3. Continued

| Dip 3u     | B  | Splitting | ($\gamma_u, \gamma_b : \gamma_1, \gamma_2$) | Factor 1 | $\Phi(B_j) | \gamma_{emi}, \gamma_{spec} |
|------------|----|-----------|------------------------------------------|----------|-----------------------------------------|
| Dip 3u $gg \rightarrow uu$ (6) - 1 | 7.($a'1, b'; 2, 3$) | $\int dx V^{f,s}(x)/C_A$ | $\Phi_{a'}(B3u)[a, 1]$ | $\Phi_{a'}(B3u)[a, 2]$ |
|            | 8.($a'1, b; 2, 3$) | $\int dx V^{f,s}(x)/C_A$ | $\Phi_{a'}(B3u)[a, 1]$ | $\Phi_{a'}(B3u)[a, 2]$ |
|            | 9.$(a'2, b; \bar{i}, 3$) | $\int dx V^{f,s}(x)/C_A$ | $\Phi_{a'}(B3u)[a, 1]$ | $\Phi_{a'}(B3u)[a, 2]$ |
|            | 10.$(a'2, b; 1, \bar{3}$) | $\int dx V^{f,s}(x)/C_A$ | $\Phi_{a'}(B3u)[a, 1]$ | $\Phi_{a'}(B3u)[a, 2]$ |
| Dip 4u $uu \rightarrow uu$ (7) - 1 | 11.$(a', b; 2, 3$) | $\int dx V^{f,s}(x)/C_A$ | $\Phi_{a'}(B4u)[a, 1]$ | $\Phi_{a'}(B4u)[b, 2]$ |
|            | 12.$(a', \bar{b}; 1, 3$) | $\int dx V^{f,s}(x)/C_A$ | $\Phi_{a'}(B4u)[a, 1]$ | $\Phi_{a'}(B4u)[b, 2]$ |
| Dip 4u $uu \rightarrow uu$ (7) - 1 | 13.$(a, b; 2, 3$) | $\int dx V^{f,s}(x)/C_A$ | $\Phi_{b}(B4u)[a, 1]$ | $\Phi_{b}(B4u)[b, a]$ |
|            | 14.$(a, \bar{b}; 2, 3$) | $\int dx V^{f,s}(x)/C_A$ | $\Phi_{b}(B4u)[a, 1]$ | $\Phi_{b}(B4u)[b, a]$ |
|            | 15.$(a, \bar{b}; 1, 3$) | $\int dx V^{f,s}(x)/C_A$ | $\Phi_{b}(B4u)[a, 1]$ | $\Phi_{b}(B4u)[b, a]$ |
|            | 16.$(a, \bar{b}; 1, 3$) | $\int dx V^{f,s}(x)/C_A$ | $\Phi_{b}(B4u)[a, 1]$ | $\Phi_{b}(B4u)[b, a]$ |

**Step 1**

$$\hat{\sigma}_D(R_{3u} = ug \rightarrow uuuu) : \ S_{R_3} = 2.$$  
$$\hat{\sigma}_{sub}(R_{3u} = ug \rightarrow uuuu)$$

**Step 2**

$$\hat{\sigma}_D(R_{3u}, 1) = 0 \quad \text{and} \quad \hat{\sigma}_I(R_{3u}) = 0.$$  
(B16)

**Step 3**

$$\hat{\sigma}_D(R_{3u}, P) + \hat{\sigma}_C(R_{3u}) - \hat{\sigma}_p(R_{3u}) = 0,$$  
which is separated into three relations for Dipoles 3u, 4u, and 4$\bar{u}$ as

$$\hat{\sigma}_D(R_{3u}, P, \text{ dip } j) + \hat{\sigma}_C(R_{3u}, \text{ dip } j) - \hat{\sigma}_p(R_{3u}, \text{ dip } j) = 0.$$  
(B17)

**Step 4**

$$\hat{\sigma}_D(R_{3u}, K) - \hat{\sigma}_K(R_{3u}) = 0,$$  
(B19)

which is separated into three relations for Dipoles 3u, 4u, and 4$\bar{u}$ as

$$\hat{\sigma}_D(R_{3u}, K, \text{ dip } j) - \hat{\sigma}_K(R_{3u}, \text{ dip } j) = 0.$$  
(B20)

**Step 5**

$$\hat{\sigma}_{sub}(R_{3u}) = \hat{\sigma}_D(R_{3u}, \text{ dip } 2).$$  
(B21)

$$\hat{\sigma}_D(R_{3u}, \text{ dip } 2) = \hat{\sigma}_D \left( R_{3u}, \text{ dip } 2, (5)-1/2, V_{fj} \right)$$  
$$+ \hat{\sigma}_D \left( R_{3u}, \text{ dip } 2, (5)-2, h \right).$$  
(B22)

$$\hat{\sigma}_D \left( R_{3u}, \text{ dip } 2, (5)-1/2, V_{fj} \right) = \frac{A_d}{S_{R_{3u}}} \cdot \frac{1}{C_A} V_{fj}(\epsilon)$$  
$$\cdot \Phi(B2)^d \left( [1, 2] + [1, a] + [1, b] \right) \cdot 2.$$  
(B23)
\[ \hat{\sigma}_D(R_{3u}, \text{dip}2, (5)-2, h) = -\frac{A_4}{S_{R_{3u}}} \int_0^1 dx \frac{T_R}{C_A} \frac{2}{3} h(x) \]
\[ \times (\Phi_a(B_2, x)_A(1, a) + \Phi_b(B_2, x)_A(1, b)) \cdot 2. \]

(B24)

**B.4. \(\hat{\sigma}_{\text{subt}}(R_{4u})\)**

| Dip\textsubscript{j} | B \textsubscript{j} | Splitting | \((y_a, y_b : y_1, y_2)\) | Factor \(I\) | \(\Phi(B)\) | \(y_{\text{emi}}, y_{\text{spe}}\) |
|----------------------|----------------|------------|--------------------------|-------------|-----------------|---------------------|
| Dip \textsubscript{1} | \(u\bar{u} \to d\bar{d}\) | (1) - 1 | \((a, b; \bar{1}, 2)\) | \(V_{j\bar{a}}/C_F\) | \(\Phi(B_1)[1, 2]\) |                      |
| | | | \((a, b; \bar{1}, \bar{2})\) | | \(\Phi(B_1)[2, 1]\) | |                      |
| | | (1) - 2 | \((\tilde{a}, b; \bar{1}, 2)\) | \(\int dx V_{j\bar{a}}(x)/C_F\) | \(\Phi_a(B_1)[1, a]\) |                      |
| | | | \((a, \bar{b}; \bar{1}, 2)\) | | \(\Phi(b_1)[1, b]\) | |                      |
| | | | \((a, \bar{b}; \bar{1}, \bar{2})\) | | \(\Phi_a(B_1)[2, a]\) | |                      |
| | | (3) - 1 | \((\tilde{a}, b; \bar{1}, 2)\) | \(\int x V_{j\tilde{a}}(x)/C_F\) | \(\Phi_a(B_1)[a, 1]\) |                      |
| | | | \((a, \bar{b}; 1, 2)\) | | \(\Phi_a(B_1)[1, a]\) | |                      |
| | | | \((a, 1, \bar{2})\) | | \(\Phi_a(B_1)[a, 2]\) | |                      |
| | | | \((a, \tilde{b}; 1, \bar{2})\) | | \(\Phi_a(B_1)[b, 1]\) | |                      |
| | | | \((a, \tilde{b}; 1, 2)\) | | \(\Phi_a(B_1)[b, 2]\) | |                      |
| | | (3) - 2 | \((\tilde{a}, \bar{b}; 1, 2)\) | \(\int dx V_{j\tilde{a}}(x)/C_F\) | \(\Phi_a(B_1)[a, b]\) |                      |
| Dip \textsubscript{2d} | \(u\bar{u} \to gg\) | (5) - 1 | \((a, b; \bar{1}, \bar{3})\) | \(V_{j\bar{a}}/C_A\) | \(\Phi(B_2d)[1, 2]\) |                      |
| | | | \((a, b; 1, \bar{3})\) | | \(\Phi(B_2d)[2, 1]\) | |                      |
| | | (5) - 2 | \((\tilde{a}, \bar{b}; 1, \bar{3})\) | \(\int dx V_{j\tilde{a}}(x)/C_A\) | \(\Phi_a(B_2d)[1, a]\) |                      |
| | | | \((a, \bar{b}; 1, \bar{3})\) | | \(\Phi_a(B_2d)[1, b]\) | |                      |

**Step 1**

\(\hat{\sigma}_D(R_{4u} = u\bar{u} \to d\bar{d}g) : \quad S_{R_4} = 1,\)

\(\hat{\sigma}_{\text{subt}}(R_{4u} = u\bar{u} \to d\bar{d}g)\)

**Step 2**

\(\hat{\sigma}_D(R_{4u}, L) - \hat{\sigma}_1(R_{4u}) = 0.\)

(B25)

**Step 3**

\(\hat{\sigma}_D(R_{4u}, P) + \hat{\sigma}_C(R_{4u}) - \hat{\sigma}_P(R_{4u}) = 0.\)

(B26)

which includes only Dipole 1.

**Step 4**

\(\hat{\sigma}_D(R_{4u}, K) - \hat{\sigma}_K(R_{4u}) = 0.\)

(B27)

which includes only Dipole 1.
Step 5

\[
\hat{\sigma}_{\text{subt}}(R_{4u}) = \hat{\sigma}_D(R_{4u}, \text{dip}2d).
\]

(B28)

\[
\hat{\sigma}_D(R_{4u}, \text{dip}2d) = \hat{\sigma}_D \left( R_{4u}, \text{dip}2d, (5)-1/2, \mathcal{V}_{ff} \right)
+ \hat{\sigma}_D(R_{4u}, \text{dip}2d, (5)-2, h). \tag{B29}
\]

\[
\hat{\sigma}_D \left( R_{4u}, \text{dip}2d, (5)-1/2, \mathcal{V}_{ff} \right) = -\frac{A_d}{S_{R_{2d}}} \cdot \frac{1}{\mathcal{C}_A} \mathcal{V}_{ff}(\epsilon)
\]

\[
\cdot \Phi(B2_d) \left( [1, 2] + [1, a] + [1, b] \right). \tag{B30}
\]

\[
\hat{\sigma}_D(R_{4u}, \text{dip}2d, (5)-2, h) = -\frac{A_d}{S_{R_{2d}}} \int_0^1 dx \frac{\mathcal{T}_R}{\mathcal{C}_A} h(x)
\]

\[
\times (\Phi_d(B2, x)_4(1, a) + \Phi_d(B2, x)_4(1, b)). \tag{B31}
\]

B.5. \( \hat{\sigma}_{\text{subt}}(R_{5ud}) \)

Table B5. Summary table of \( \hat{\sigma}_D(R_{5ud}) \).

| Dip | Bj  | Splitting | \( (y_1, y_2 : y_1, y_2) \) | Factor 1 | \( \Phi(B_j) \) | \( y_{\text{emis}}, y_{\text{spe}} \) |
|-----|-----|-----------|-----------------|----------|-----------------|-----------------|
| Dip 1 ud \( \rightarrow \) ud | (1) | 1. \( (a, b; 13, 2) \) | \( \mathcal{V}_{f g}/\mathcal{C}_F \) | \( \Phi(B1)[1, 2] \) |          |
|     |     | 2. \( (a, b; 1, 23) \) |                      | \( \Phi(B1)[2, 1] \) |          |
|     |     | (1) | 2. \( (\bar{a}, b; \bar{13}, 2) \) | \( \int dx \mathcal{V}_{f g}(x)/\mathcal{C}_F \) | \( \Phi_d(B1)[1, a] \) |          |
|     |     | (4) | \( (a, \bar{b}; \bar{13}, 2) \) |                      | \( \Phi_d(B1)[1, b] \) |          |
|     |     | (5) | \( (\bar{a}, b; 1, \bar{23}) \) |                      | \( \Phi_d(B1)[2, a] \) |          |
|     |     | (6) | \( (a, \bar{b}; 1, \bar{23}) \) |                      | \( \Phi_d(B1)[2, b] \) |          |
|     |     | (3) | 1. \( (\bar{a}3, b; \bar{1}, 2) \) | \( \int dx \mathcal{V}^{f-f}(x)/\mathcal{C}_F \) | \( \Phi_d(B1)[a, 1] \) |          |
|     |     |     | 2. \( (\bar{a}3, b; 1, \bar{2}) \) |                      | \( \Phi_d(B1)[a, 2] \) |          |
|     |     |     | 3. \( (a, \bar{b}3; \bar{1}, \bar{2}) \) |                      | \( \Phi_d(B1)[b, 1] \) |          |
|     |     |     | 4. \( (a, \bar{b}3; 1, \bar{2}) \) |                      | \( \Phi_d(B1)[b, 2] \) |          |
|     |     | (3) | 1. \( (\bar{a}3, \bar{b}; 1, 2) \) | \( \int dx \mathcal{V}^{f-f}(x)/\mathcal{C}_F \) | \( \Phi_d(B1)[a, b] \) |          |
|     |     |     | 2. \( (\bar{a}3, \bar{b}; 1, 2) \) |                      | \( \Phi_d(B1)[b, a] \) |          |
| Dip 3u gd \( \rightarrow \) dg | (6) | 1. \( (\bar{a}1, b; \bar{2}, 3) \) | \( \int dx \mathcal{V}^{f-g}(x)/\mathcal{C}_A \) | \( \Phi_a(B3_u)[a, 1] \) |          |
|     |     | 2. \( (\bar{a}1, b; 2, \bar{3}) \) |                      | \( \Phi_a(B3_u)[a, 2] \) |          |
|     |     | (6) | 1. \( (\bar{a}1, \bar{b}; 2, 3) \) | \( \int dx \mathcal{V}^{f-g}(x)/\mathcal{C}_A \) | \( \Phi_a(B3_u)[a, b] \) |          |
| Dip 3d ug \( \rightarrow \) ug | (6) | 1. \( (a, \bar{b}2; \bar{1}, 3) \) | \( \int dx \mathcal{V}^{f-g}(x)/\mathcal{C}_A \) | \( \Phi_b(B3_d)[b, 1] \) |          |
|     |     | 2. \( (a, \bar{b}2; 1, \bar{3}) \) |                      | \( \Phi_b(B3_d)[b, 2] \) |          |
|     |     | (6) | 1. \( (\bar{a}, \bar{b}2; 1, 3) \) | \( \int dx \mathcal{V}^{f-g}(x)/\mathcal{C}_A \) | \( \Phi_b(B3_d)[b, a] \) |          |

Step 1

\[
\hat{\sigma}_D(R_{5ud} = ud \rightarrow udg) : \quad S_{R_5} = 1,
\]

\[
\hat{\sigma}_{\text{subt}}(R_{5ud} = ud \rightarrow udg)
\]

Step 2

\[
\hat{\sigma}_D(R_{5ud}, I) - \hat{\sigma}_I(R_{5ud}) = 0. \tag{B32}
\]
Step 3
\[ \hat{\sigma}_D (R_{5ud}, P) + \hat{\sigma}_C (R_{5ud}) - \hat{\sigma}_P (R_{5ud}) = 0. \quad (B33) \]
which is separated into three relations for Dipoles 1, 3u, and 3d as
\[ \hat{\sigma}_D (R_{5ud}, P, \text{dip}_{j}) + \hat{\sigma}_C (R_{5ud}, \text{dip}_{j}) - \hat{\sigma}_P (R_{5ud}, \text{dip}_{j}) = 0. \quad (B34) \]

Step 4
\[ \hat{\sigma}_D (R_{5ud}, K) - \hat{\sigma}_K (R_{5ud}) = 0, \quad (B35) \]
which is separated into three relations for Dipoles 1, 3u, and 3d as
\[ \hat{\sigma}_D (R_{5ud}, K, \text{dip}_{j}) - \hat{\sigma}_K (R_{5ud}, \text{dip}_{j}) = 0. \quad (B36) \]

Step 5
\[ \hat{\sigma}_{\text{subt}} (R_{5ud}) = 0. \quad (B37) \]

B.6. \( \hat{\sigma}_{\text{subt}} (R_{6ud}) \)

Table B6. Summary table of \( \hat{\sigma}_D (R_{6ud}) \).

| Dip \( j \) | \( B j \) | Splitting | \( (y_a, y_b; y_1, y_2) \) | Factor 1 | \( \Phi (B_j) [y_{\text{emi}}, y_{\text{spe}}] \) |
|---|---|---|---|---|---|
| Dip 1 \( u\bar{d} \to u\bar{d} \) (1) – 1 | \( 1. (a, b; 13, 2) \) | \( \nu_{fg}/C_F \) | \( \Phi (B1)[1, 2] \) |
| | \( 2. (a, b; \bar{1}, \bar{2}) \) | | \( \Phi (B1)[2, 1] \) |
| | (1) – 2 | \( 3. (\bar{a}, b; \bar{13}, 2) \) | \( \int dx \nu_{fg}(x)/C_F \) | \( \Phi_a (B1)[1, a] \) |
| | | \( 4. (a, \bar{b}; 13, 2) \) | | \( \Phi_a (B1)[1, b] \) |
| | | \( 5. (\bar{a}, b; 1, \bar{23}) \) | | \( \Phi_a (B1)[2, a] \) |
| | | \( 6. (a, \bar{b}; 1, \bar{23}) \) | | \( \Phi_a (B1)[2, b] \) |
| | (3) – 1 | \( 7. (\bar{a}3, b; \bar{1}, 2) \) | \( \int dx \nu^{f-f}(x)/C_F \) | \( \Phi_a (B1)[a, 1] \) |
| | | \( 8. (\bar{a}3, b; 1, \bar{2}) \) | | \( \Phi_a (B1)[a, 2] \) |
| | | \( 9. (a, \bar{b}3; \bar{1}, 2) \) | | \( \Phi_b (B1)[b, 1] \) |
| | | \( 10. (a, \bar{b}3; 1, \bar{2}) \) | | \( \Phi_b (B1)[b, 2] \) |
| | (3) – 2 | \( 11. (\bar{a}3, \bar{b}; 1, 2) \) | \( \int dx \nu^{f-f}(x)/C_F \) | \( \Phi_a (B1)[a, b] \) |
| | | \( 12. (\bar{a}3, \bar{b}; 1, 2) \) | | \( \Phi_a (B1)[b, a] \) |
| Dip 3u \( g\bar{d} \to \bar{d}g \) (6) – 1 | \( 13. (\bar{a}1, b; \bar{2}, 3) \) | \( \int dx \nu^{f-g}(x)/C_A \) | \( \Phi_a (B3u)[a, 1] \) |
| | | \( 14. (\bar{a}1, b; 2, \bar{3}) \) | | \( \Phi_a (B3u)[a, 2] \) |
| | (6) – 2 | \( 15. (\bar{a}1, \bar{b}; 2, 3) \) | \( \int dx \nu^{f-g}(x)/C_A \) | \( \Phi_a (B3u)[a, b] \) |
| Dip 3\( \bar{d} \) \( u\bar{g} \to u\bar{g} \) (6) – 1 | \( 16. (a, \bar{b}2; \bar{1}, 3) \) | \( \int dx \nu^{f-g}(x)/C_A \) | \( \Phi_b (B3\bar{d})[b, 1] \) |
| | | \( 17. (a, \bar{b}2; 1, \bar{3}) \) | | \( \Phi_b (B3\bar{d})[b, 2] \) |
| | (6) – 2 | \( 18. (\bar{a}, \bar{b}2; 1, 3) \) | \( \int dx \nu^{f-g}(x)/C_A \) | \( \Phi_b (B3\bar{d})[b, a] \) |

Step 1
\[ \hat{\sigma}_D (R_{6ud} = u\bar{d} \to u\bar{d}g) : \quad S_{R_6} = 1, \]
\[ \hat{\sigma}_{\text{subt}} (R_{6ud} = u\bar{d} \to u\bar{d}g) \]

Step 2
\[ \hat{\sigma}_D (R_{6ud}, l) - \hat{\sigma}_l (R_{6ud}) = 0. \quad (B38) \]
Step 3

\[ \delta_D(R_{6ud}, P) + \delta_C(R_{6ud}) - \delta_P(R_{6ud}) = 0, \]  
\[ \text{(B39)} \]

which is separated into three relations for Dipoles 1, 3u, and 3d as

\[ \delta_D(R_{6ud}, P, dipj) + \delta_C(R_{6ud}, dipj) - \delta_P(R_{6ud}, dipj) = 0. \]  
\[ \text{(B40)} \]

Step 4

\[ \delta_D(R_{6ud}, K) - \delta_K(R_{6ud}) = 0, \]  
\[ \text{(B41)} \]

which is separated into three relations for Dipoles 1, 3u, and 3d as

\[ \delta_D(R_{6ud}, K, dipj) - \delta_K(R_{6ud}, dipj) = 0. \]  
\[ \text{(B42)} \]

Step 5

\[ \hat{\delta}_{\text{subt}}(R_{6ud}) = 0. \]  
\[ \text{(B43)} \]

B.7. \( \hat{\delta}_{\text{subt}}(R_{7u}) \)

Table B7. Summary table of \( \hat{\delta}_D(R_{7u}) \).

| Dipj | Bj | Splitting | \((y_u, y_b : y_1, y_2)\) | Factor 1 | \(\Phi(B_j)\) | \(\gamma_{\text{emi}}, \gamma_{\text{spe}}\) |
|------|----|-----------|----------------|--------|-------------|----------------|
| Dip 2u | ug → ug | (5) - 1 | \((a, b; \overline{1}, 2\overline{3})\) | \(V_{fj}/C_A\) | \(\Phi(B2u)[2, 1] \) | |
| | | (5) - 2 | \((\overline{a}, b; 1, 2\overline{3})\) | | \(\Phi(B2u)[2, a]\) | |
| Dip 3u | gg → dd | (6) - 1 | \((a, \overline{\tilde{b}}; 1, 2\overline{3})\) | \(f dx V_{fj}(x)/C_A\) | \(\Phi(b)(B3u)[a, 1]\) | |
| | | (6) - 2 | \((a, \overline{\tilde{b}}, \overline{\tilde{b}}; 2, 3)\) | \(\Phi_b(B3u)[a, 2]\) | | |
| Dip 4u | uu → dd | (7) - 1 | \((a, \overline{\tilde{b}}; 2, 3)\) | \(f dx V_{fj}(x)/C_A\) | \(\Phi_b(B4u)[a, b]\) | |
| | | (7) - 2 | \((\overline{a}, \overline{\tilde{b}}; 1, 2)\) | \(\Phi_b(B4u)[b, 2]\) | | |
| Dip 4d | uu → uu | (7) - 1 | \((a, \overline{\tilde{b}}; 1, 3)\) | \(f dx V_{fj}(x)/C_F\) | \(\Phi_b(B4d)[a, b]\) | |
| | | (7) - 2 | \((\overline{a}, \overline{\tilde{b}}; 2, 1)\) | \(\Phi_b(B4d)[b, 2]\) | | |
| Dip 4d | dd → ud | (7) - 1 | \((a, \overline{\tilde{b}}; 1, 2)\) | \(f dx V_{fj}(x)/C_F\) | \(\Phi_b(B4d)[b, a]\) | |
| | | (7) - 2 | \((\overline{a}, \overline{\tilde{b}}; 1, 2)\) | \(\Phi_b(B4d)[b, a]\) | | |

Step 1

\[ \hat{\delta}_D(R_{7u} = ug → udd) : S_{R7} = 1, \]

\[ \hat{\delta}_{\text{subt}}(R_{7u} = ug → udd) \]

Step 2

\[ \hat{\delta}_D(R_{7u}, I) = 0 \text{ and } \hat{\delta}_I(R_{7u}) = 0. \]  
\[ \text{(B44)} \]
\[ \hat{\sigma}_D(R_{7u}, P) + \hat{\sigma}_C(R_{7u}) - \hat{\sigma}_P(R_{7u}) = 0, \quad (B45) \]

which is separated into four relations for Dipoles 3\(u\), 4\(u\), 4\(d\), and 4\(\bar{d}\) as

\[ \hat{\sigma}_D(R_{7u}, P, dipj) + \hat{\sigma}_C(R_{7u}, dipj) - \hat{\sigma}_P(R_{7u}, dipj) = 0. \quad (B46) \]

**Step 4**

\[ \hat{\sigma}_D(R_{7u}, K) - \hat{\sigma}_K(R_{7u}) = 0, \quad (B47) \]

which is separated into four relations for Dipoles 3\(u\), 4\(u\), 4\(d\), and 4\(\bar{d}\) as

\[ \hat{\sigma}_D(R_{7u}, K, dipj) - \hat{\sigma}_K(R_{7u}, dipj) = 0. \quad (B48) \]

**Step 5**

\[ \hat{\sigma}_{subt}(R_{8u}) = \hat{\sigma}_D(R_{7u}, dip2). \quad (B49) \]

\[ \hat{\sigma}_D(R_{7u}, dip2) = \hat{\sigma}_D \left( R_{7u}, dip2, (5)-1/2, V_{fj} \right) + \hat{\sigma}_D(R_{7u}, dip2, (5)-2, h). \quad (B50) \]

\[ \hat{\sigma}_D \left( R_{7u}, dip2, (5)-1/2, V_{fj} \right) = -\frac{A_d}{S_{R_{7u}}} \cdot \frac{1}{C_A} V_{fj}(\epsilon) \cdot \Phi(B2)_d \]
\[ \times \left( [2, 1] + [2, a] + [2, b] \right). \quad (B51) \]

\[ \hat{\sigma}_D \left( R_{7u}, dip2, (5)-2, h \right) = -\frac{A_4}{S_{R_{7u}}} \int_0^1 dx \frac{Tr}{C_A} \frac{2}{3} h(x) \]
\[ \times \left( \Phi_4(B2, x)_4(2, a) + \Phi_6(B2, x)_4(2, b) \right). \quad (B52) \]

**B.8.** \( \hat{\sigma}_{subt}(R_{8u}) \)

| Dipj | Bj | Splitting | \((y_a, y_b : y_1, y_2)\) | Factor 1 | \( \Phi(Bj) \) | \( y_{emi}, y_{spe} \) |
|------|----|-----------|------------------|-----------|------------|------------------|
| Dip 1 | \( u\bar{u} \rightarrow gg \) | (2) - 1 | 1. \((a, b; \tilde{I}_2, \tilde{3})\) | \( V_{gg}/C_A \) | \( \Phi(B1)_4[1, 2] \) |                       |
|       |     |          | 2. \((a, b; \tilde{13}, \tilde{2})\) |          | \( \Phi(B1)_4[1, 2] \) |                       |
|       |     |          | 3. \((a, b; \tilde{23}, \tilde{1})\) |          | \( \Phi(B1)_4[1, 2] \) |                       |
|       |     |          | (2) - 2 | 4. \((\tilde{a}, b; \tilde{I}_2, 3)\) | \( \int dx V_{gg}(x)/C_A \) | \( \Phi_4(B1)_4[1, a] \) |                       |
|       |     |          | 5. \((a, \tilde{b}; \tilde{I}_2, 3)\) |          | \( \Phi_6(B1)_4[1, b] \) |                       |
|       |     |          | 6. \((\tilde{a}, b; \tilde{13}, 2)\) |          | \( \Phi_6(B1)_4[1, a] \) |                       |
|       |     |          | 7. \((a, \tilde{b}; \tilde{13}, 2)\) |          | \( \Phi_6(B1)_4[1, b] \) |                       |
|       |     |          | 8. \((\tilde{a}, b; \tilde{23}, 1)\) |          | \( \Phi_4(B1)_4[1, a] \) |                       |
|       |     |          | 9. \((a, \tilde{b}; \tilde{23}, 1)\) |          | \( \Phi_6(B1)_4[1, b] \) |                       |
|       | (3) - 1 |          | 10. \((\tilde{a}^1, b; \tilde{2}, 3)\) | \( \int dx V_{fj}(x)/C_F \) | \( \Phi_4(B1)_4[a, 1] \) |                       |
|       |     |          | 11. \((a^1, b; 2, \tilde{3})\) |          | \( \Phi_4(B1)_4[a, 2] \) |                       |
|       |     |          | 12. \((\tilde{a}^2, b; 1, 3)\) |          | \( \Phi_4(B1)_4[a, 1] \) |                       |
|       |     |          | 13. \((\tilde{a}^2, b; 1, \tilde{3})\) |          | \( \Phi_4(B1)_4[a, 2] \) |                       |

Continued
Table B8. Continued

| Dip | Bj | Splitting | \( (y_a, y_b : y_1, y_2) \) | Factor 1 | \( \Phi(B_j) \) |
|-----|----|-----------|----------------|---------|---------------|
| 14. | \( (a \bar{3}, b; \bar{1}, 2) \) | \( \Phi_a(B1)[a, 1] \) |
| 15. | \( (a \bar{3}, b; 1, \bar{2}) \) | \( \Phi_a(B1)[a, 2] \) |
| 16. | \( (a, \tilde{b}1; \bar{2}, 3) \) | \( \Phi_b(B1)[b, 1] \) |
| 17. | \( (a, \tilde{b}1; 2, \bar{3}) \) | \( \Phi_{b}(B1)[b, 2] \) |
| 18. | \( (a, \tilde{b}2; \bar{1}, 3) \) | \( \Phi_{b}(B1)[b, 1] \) |
| 19. | \( (a, \tilde{b}2; 1, \bar{3}) \) | \( \Phi_{b}(B1)[b, 2] \) |
| 20. | \( (a, b \bar{3}; \bar{1}, 2) \) | \( \Phi_{b}(B1)[b, 1] \) |
| 21. | \( (a, \tilde{b}3; 1, \bar{2}) \) | \( \Phi_{b}(B1)[b, 2] \) |
| 22. | \( (\tilde{a}1, \tilde{b}; 2, 3) \) | \( \int dx \tilde{V}/j(x)/C_F \) | \( \Phi_{b}(B1)[a, b] \) |
| 23. | \( (\tilde{a}2, \tilde{b}; 1, 3) \) | \( \Phi_{b}(B1)[a, b] \) |
| 24. | \( (\tilde{a}3, \tilde{b}; 1, 2) \) | \( \Phi_{b}(B1)[a, b] \) |
| 25. | \( (\tilde{a}, \tilde{b}1; 2, 3) \) | \( \Phi_{b}(B1)[b, a] \) |
| 26. | \( (\tilde{a}, \tilde{b}2; 1, 3) \) | \( \Phi_{b}(B1)[b, a] \) |
| 27. | \( (\tilde{a}, b \bar{3}; 1, 2) \) | \( \Phi_{b}(B1)[b, a] \) |

**Step 1**

\[
\hat{\sigma}_D(R_{8u} = u\bar{u} \to ggg) : \quad S_{R_8} = 6, \quad \hat{\sigma}_{subt}(R_{8u} = u\bar{u} \to ggg)
\]

**Step 2**

\[
\hat{\sigma}_D(R_{8u}, I) - \hat{\sigma}_I(R_{8u}) = -\hat{\sigma}_I(R_{8u}, (2)-1/2, N_f V_{f\bar{f}}).
\]

\[
\hat{\sigma}_I(R_{8u}, (2)-1/2, N_f V_{f\bar{f}}) = -\frac{A_d}{S_{B_1}} \cdot \frac{N_f}{C_A} V_{f\bar{f}}(\epsilon) \Phi(B1)_d \cdot [(1, 2) + [2, 1] + [1, a] + [1, b] + [2, a] + [2, b]).
\]

**Step 3**

\[
\hat{\sigma}_D(R_{8u}, P) + \hat{\sigma}_C(R_{8u}) - \hat{\sigma}_P(R_{8u}) = 0,
\]

which includes only Dipole 1.

**Step 4**

\[
\hat{\sigma}_D(R_{8u}, K) - \hat{\sigma}_K(R_{8u}) = -\hat{\sigma}_K(R_{8u}, dip1, (3)-1, N_f h),
\]

which includes only Dipole 1.

\[
\hat{\sigma}_K(R_{8u}, dip1, (3)-1, N_f h) = -\frac{A_d}{S_{B_1}} \int_0^1 dx \frac{T_R N_f}{C_A} \frac{2}{3} h(x) \times [\Phi_d(B1, x)_4 ((a, 1) + (a, 2)) + \Phi_{b}(B1, x)_4 ((b, 1) + (b, 2))].
\]

**Step 5**

\[
\hat{\sigma}_{subt}(R_{8u}) = -\hat{\sigma}_I(R_{8u}, (2)-1/2, N_f V_{f\bar{f}}) - \hat{\sigma}_K(R_{8u}, dip1, (3)-1, N_f h).
\]
B.9. $\hat{\sigma}_{\text{subt}}(R_{9u})$

Table B9. Summary table of $\hat{\sigma}_D(R_{9u})$.

| Dipj | Bj | Splitting | $(y_a, y_b; \gamma_1, \gamma_2)$ | Factor 1 $\Phi(B_j)$ $|y_{\text{emi}}, y_{\text{spe}}|$ |
|------|----|-----------|-------------------------------|----------------|
| Dip 1 | $ug \rightarrow ug$ (1) | $1.(a, b; \tilde{1}, \tilde{2})$ | $\mathcal{V}_{68}/C_F$ | $\Phi(B1)[1, 2]$ |
|      |     | $2.(a, b; \tilde{1}, \tilde{3})$ | $\mathcal{V}_{69}/C_A$ | $\Phi(B1)[1, 2]$ |
|      | (2) | $7.(a, b; \tilde{1}, \tilde{2})$ | $\mathcal{V}_{68}/C_A$ | $\Phi(B1)[1, 2]$ |
|      | (2) | $8.(a, b; 1, \tilde{2})$ | $\mathcal{V}_{69}/C_A$ | $\Phi(B1)[1, 2]$ |
|      | (3) | $10.(a, b; 1, \tilde{3})$ | $\mathcal{V}_{69}/C_A$ | $\Phi(B1)[1, 2]$ |
|      | (3) | $11.(a, b; 1, 3)$ | $\mathcal{V}_{69}/C_A$ | $\Phi(B1)[1, 2]$ |
| Dip 3u | $gg \rightarrow gg$ (6) | $16.(a, b; \tilde{1}, \tilde{2})$ | $\mathcal{V}_{69}/C_A$ | $\Phi(B1)[1, 2]$ |
|      |     | $17.(a, b; 1, \tilde{2})$ | $\mathcal{V}_{69}/C_A$ | $\Phi(B1)[1, 2]$ |
|      |     | $19.(a, b; 1, \tilde{2})$ | $\mathcal{V}_{69}/C_A$ | $\Phi(B1)[1, 2]$ |
| Dip 4u | $uu \rightarrow gg$ (7) | $25.(a, b; 1, \tilde{2})$ | $\mathcal{V}_{69}/C_A$ | $\Phi(B1)[1, 2]$ |
|      |     | $26.(a, b; 1, \tilde{2})$ | $\mathcal{V}_{69}/C_A$ | $\Phi(B1)[1, 2]$ |
|      |     | $27.(a, b; 1, \tilde{2})$ | $\mathcal{V}_{69}/C_A$ | $\Phi(B1)[1, 2]$ |

**Step 1**

$$\hat{\sigma}_D(R_{9u} = ug \rightarrow ugg) : S_{R_9} = 2,$$

$$\hat{\sigma}_{\text{subt}}(R_{9u} = ug \rightarrow ugg)$$

**Step 2**

$$\hat{\sigma}_D(R_{9u}, I) - \hat{\sigma}_I(R_{9u}) = -\frac{\hat{\sigma}_I(R_{9u}, (2)-1/2, N_f \mathcal{V}_{ff})}{(B59)}.$$

**Step 3**

$$\hat{\sigma}_D(R_{9u}, P) + \hat{\sigma}_C(R_{9u}) - \hat{\sigma}_P(R_{9u}) = 0.$$
which is separated into three relations for Dipoles 1, 3u, and 4u as

\[ \hat{\sigma}_D(R_{9u}, P, \text{dip} j) + \hat{\sigma}_C(R_{9u}, \text{dip} j) - \hat{\sigma}_P(R_{9u}, \text{dip} j) = 0. \] (B62)

**Step 4**

\[ \hat{\sigma}_D(R_{9u}, \text{K}) - \hat{\sigma}_K(R_{9u}) = -\hat{\sigma}_K(R_{9u}, \text{dip} 1, (3)/(4)-1, N_f h), \] (B63)

which is separated into three relations for Dipoles 1, 3u, and 4u as

\[ \hat{\sigma}_D(R_{9u}, \text{K}, \text{dip} 1) - \hat{\sigma}_K(R_{9u}, \text{dip} 1) = -\hat{\sigma}_K(R_{9u}, \text{dip} 1, (3)/(4)-1, N_f h), \] (B64)
\[ \hat{\sigma}_D(R_{9u}, \text{K}, \text{dip} 3u) - \hat{\sigma}_K(R_{9u}, \text{dip} 3u) = 0, \] (B65)
\[ \hat{\sigma}_D(R_{9u}, \text{K}, \text{dip} 4u) - \hat{\sigma}_K(R_{9u}, \text{dip} 4u) = 0. \] (B66)

\[ \hat{\sigma}_K(R_{9u}, \text{dip} 1, (3)/(4)-1, N_f h) = -\frac{A_4}{S_{B1}} \int_{0}^{1} dx \frac{T_R N_f}{C_A} \frac{2}{3} h(x) \times [\Phi_a(B1, x)_{4}(a, 2) + \Phi_b(B1, x)_{4}(b, 2)] \] (B67)

**Step 5**

\[ \hat{\sigma}_{\text{subt}}(R_{9u}) = -\hat{\sigma}_1(R_{9u}, (2)-1/2, N_f V_{fj}) - \hat{\sigma}_K(R_{9u}, \text{dip} 1, (3)/(4)-1, N_f h). \] (B68)

### B.10. \( \hat{\sigma}_{\text{subt}}(R_{10u}) \)

**Table B10. Summary table of \( \hat{\sigma}_D(R_{10u}) \).**

| Dipj/  | Bj     | Splitting | \((y_a, y_b : y_1, y_2)\) | Factor 1 | \(\Phi(B_j)\) \(\{y_{\text{emi}}, y_{\text{spec}}\}\) |
|--------|--------|-----------|--------------------------|-----------|-------------------------------------------------|
| Dip 1  | \(gg \rightarrow uu\) | (1) – 1  | \(1. (a, b; \bar{1}, \bar{2})\) | \(V_{fj}/C_F\) | \(\Phi(B1)[1, 2]\) |
|        |        |           | 2.\((a, b; \bar{1}, \bar{2})\) | \(\Phi(B1)[2, 1]\) |
|        |        | (1) – 2  | 3.\((\bar{a}, b; \bar{1}, \bar{2})\) | \(\int dx V_{fj}(x)/C_F\) | \(\Phi_{a}(B1)[1, a]\) |
|        |        |           | 4.\((a, \bar{b}; \bar{1}, \bar{2})\) | \(\Phi_{a}(B1)[1, b]\) |
|        |        |           | 5.\((\bar{a}, b; 1, 23)\) | \(\Phi_{a}(B1)[2, a]\) |
|        |        | (4) – 1  | 6.\((a, \bar{b}; 1, 23)\) | \(\Phi_{a}(B1)[2, b]\) |
|        |        |           | 7.\((a\bar{3}, b; \bar{1}, 2)\) | \(\int dx \sqrt{\sigma_{\text{eff}}}(x)/C_A\) | \(\Phi_{a}(B1)[a, 1]\) |
|        |        |           | 8.\((a\bar{3}, b; 1, \bar{2})\) | \(\Phi_{a}(B1)[a, 2]\) |
|        |        |           | 9.\((a, \bar{b}3; \bar{1}, 2)\) | \(\Phi_{b}(B1)[b, 1]\) |
|        |        |           | 10.\((a, \bar{b}3; 1, \bar{2})\) | \(\Phi_{b}(B1)[b, 2]\) |
|        |        | (4) – 2  | 11.\((a\bar{3}, \bar{b}; 1, \bar{2})\) | \(\int dx \sqrt{\sigma_{\text{eff}}}(x)/C_A\) | \(\Phi_{a}(B1)[b, a]\) |
|        |        |           | 12.\((a\bar{3}, \bar{b}; 1, \bar{2})\) | \(\Phi_{b}(B1)[b, a]\) |
| Dip 2u | \(gg \rightarrow gg\) | (5) – 1  | \(13. (a, b; \bar{1}, \bar{3})\) | \(V_{fj}/C_A\) | \(\Phi(B2u)[1, 2]\) |
|        |        |           | 14.\((a, b; \bar{1}, \bar{3})\) | \(\Phi(B2u)[1, a]\) |
|        |        | (5) – 2  | 15.\((a, \bar{1}, \bar{2}, 3)\) | \(\Phi_{a}(B2u)[1, b]\) |
| Dip 4u | \(\bar{u}g \rightarrow \bar{u}g\) | (7) – 1  | \(16. (a, b; 2, 3)\) | \(\Phi(B4u)[a, 1]\) |
|        |        |           | 17.\((\bar{a}, b; 2, \bar{3})\) | \(\Phi_{a}(B4u)[a, 2]\) |
|        |        |           | 18.\((b, a; 2, \bar{3})\) | \(\Phi_{b}(B4u)[a, 1]\) |
|        |        |           | 19.\((b, a; 2, \bar{3})\) | \(\Phi_{b}(B4u)[a, 2]\) |

*Continued*
Table B10. Continued

| Dipj | Bj | Splitting | (\(y_u, y_d : y_1, y_2\)) | Factor 1 | \(\Phi(B_j) [y_{emi}, y_{spe}]\) |
|------|----|-----------|--------------------------|----------|-------------------------------|
| Dip 4\(\bar{u}\) ug \(\rightarrow\) ug | (7) - 2 | 20.\((a\bar{1}, \bar{b}; 2, 3)\) | \(\int dx \tilde{Y}^{x.f}(x)/C_F\) | \(\Phi_{a}(B4u)[a, b]\) |
| | | 21.\((\bar{b}1, a\bar{1}; 2, 3)\) | | \(\Phi_{b}(B4u)[a, b]\) |
| Dip 4\(\bar{u}\) ug \(\rightarrow\) ug | (7) - 1 | 22.\((\bar{a}\bar{2}, b; 1, 3)\) | \(\int dx \tilde{Y}^{x.f}(x)/C_F\) | \(\Phi_{a}(B4\bar{u})[a, 1]\) |
| | | 23.\((\bar{a}\bar{2}, b; 1, \bar{3})\) | | \(\Phi_{b}(B4\bar{u})[a, 2]\) |
| | | 24.\((\bar{b}\bar{2}, a; \bar{1}, 3)\) | | \(\Phi_{b}(B4\bar{u})[a, 1]\) |
| | | 25.\((\bar{b}\bar{2}, a; 1, \bar{3})\) | | \(\Phi_{b}(B4\bar{u})[a, 2]\) |
| | (7) - 2 | 26.\((\bar{a}\bar{2}, \bar{b}; 1, 3)\) | \(\int dx \tilde{Y}^{x.f}(x)/C_F\) | \(\Phi_{a}(B4\bar{u})[a, b]\) |
| | | 27.\((\bar{b}\bar{2}, \bar{a}; 1, 3)\) | | \(\Phi_{b}(B4\bar{u})[a, b]\) |

Step 1

\[\hat{\sigma}_D(R_{10u} = gg \rightarrow u\bar{u}g) : S_{R10} = 1,\]

\[\hat{\sigma}_{\text{subt}}(R_{10u} = gg \rightarrow u\bar{u}g)\]

Step 2

\[\hat{\sigma}_D(R_{10u}, l) - \hat{\sigma}_1(R_{10u}) = 0.\] (B69)

Step 3

\[\hat{\sigma}_D(R_{10u}, P) + \hat{\sigma}_C(R_{10u}) - \hat{\sigma}_P(R_{10u}) = 0,\] (B70)

which is separated into three relations for Dipoles 1, 4\(u\), and 4\(\bar{u}\) as

\[\hat{\sigma}_D(R_{10u}, P, \text{dip}j) + \hat{\sigma}_C(R_{10u}, \text{dip}j) - \hat{\sigma}_P(R_{10u}, \text{dip}j) = 0.\] (B71)

Step 4

\[\hat{\sigma}_D(R_{10u}, K) - \hat{\sigma}_K(R_{10u}) = 0,\] (B72)

which is separated into three relations for Dipoles 1, 4\(u\), and 4\(\bar{u}\) as

\[\hat{\sigma}_D(R_{10u}, K, \text{dip}j) - \hat{\sigma}_K(R_{10u}, \text{dip}j) = 0.\] (B73)

Step 5

\[\hat{\sigma}_{\text{subt}}(R_{10u}) = \hat{\sigma}_D(R_{10u}, \text{dip}2).\] (B74)

\[\hat{\sigma}_D(R_{10u}, \text{dip}2) = \hat{\sigma}_D\left(R_{10u}, \text{dip}2, (5)-1/2, \mathcal{V}_{jj}\right) + \hat{\sigma}_D\left(R_{10u}, \text{dip}2, (5)-2, h\right).\] (B75)

\[\hat{\sigma}_D\left(R_{10u}, \text{dip}2, (5)-1/2, \mathcal{V}_{jj}\right) = \frac{A_d}{S_{R10u}} \cdot \frac{1}{C_A} \mathcal{V}_{jj}(e) \cdot \Phi(B2)_{d}\left([1, 2] + [1, a] + [1, b]\right).\] (B76)

\[\hat{\sigma}_D(R_{10u}, \text{dip}2, (5)-2, h) = \frac{A_4}{S_{R10u}} \int_0^1 dx \frac{Tr R_2}{C_A^3} h(x)\]

\[\times (\Phi_{a}(B2, x)_{4}(1, a) + \Phi_{b}(B2, x)_{4}(1, b)).\] (B77)
### B.11. $\hat{\sigma}_{\text{subt}}(R_{11})$

**Table B11. Summary table of $\hat{\sigma}_D(R_{11})$.**

| Dipf | Bj  | Splitting | $(y_a, y_b : y_1, y_2)$ | Factor 1 | $\Phi(B_j)$ [yemi, yspe] |
|------|-----|-----------|--------------------------|----------|-------------------------|
| Dip 1 | gg → gg | (2) – 1 | 1. $(a, b; \tilde{1}2, \tilde{3})$ | $\mathcal{V}_{gg}/C_A$ | $\Phi(B1)[1, 2]$ |
|      |      |          | 2. $(a, b; \tilde{1}3, \tilde{2})$ |           | $\Phi(B1)[1, 2]$ |
|      |      |          | 3. $(a, b; \tilde{2}3, \tilde{1})$ |           | $\Phi(B1)[1, 2]$ |
| (2) – 2 |      |          | 4. $(\tilde{a}, b; \tilde{1}2, 3)$ | $\int dx\mathcal{V}_{gg}(x)/C_A$ | $\Phi_a(B1)[1, a]$ |
|      |      |          | 5. $(a, \tilde{b}; \tilde{1}2, 3)$ |           | $\Phi_a(B1)[1, b]$ |
|      |      |          | 6. $(\tilde{a}, b; \tilde{1}3, 2)$ |           | $\Phi_a(B1)[1, a]$ |
|      |      |          | 7. $(a, \tilde{b}; \tilde{1}3, 2)$ |           | $\Phi_a(B1)[1, b]$ |
|      |      |          | 8. $(\tilde{a}, b; \tilde{2}3, 1)$ |           | $\Phi_a(B1)[1, a]$ |
|      |      |          | 9. $(a, \tilde{b}; \tilde{2}3, 1)$ |           | $\Phi_a(B1)[1, b]$ |
| (4) – 1 |      |          | 10. $(\tilde{a}1, b; \tilde{2}2, 3)$ | $\int dx\mathcal{V}^{R,s}(x)/C_A$ | $\Phi_a(B1)[a, 1]$ |
|      |      |          | 11. $(\tilde{a}1, b; 2, \tilde{3})$ |           | $\Phi_a(B1)[a, 2]$ |
|      |      |          | 12. $(\tilde{a}2, b; \tilde{1}, 3)$ |           | $\Phi_a(B1)[a, 1]$ |
|      |      |          | 13. $(\tilde{a}2, b; 1, \tilde{3})$ |           | $\Phi_a(B1)[a, 2]$ |
|      |      |          | 14. $(\tilde{a}3, b; \tilde{1}, \tilde{2})$ |           | $\Phi_a(B1)[a, 1]$ |
|      |      |          | 15. $(\tilde{a}3, b; 1, \tilde{2})$ |           | $\Phi_a(B1)[a, 2]$ |
|      |      |          | 16. $(a, \tilde{b}1; \tilde{2}, 3)$ |           | $\Phi_b(B1)[b, 1]$ |
|      |      |          | 17. $(a, \tilde{b}1; 2, \tilde{3})$ |           | $\Phi_b(B1)[b, 2]$ |
|      |      |          | 18. $(a, \tilde{b}2; \tilde{1}, 3)$ |           | $\Phi_b(B1)[b, 1]$ |
|      |      |          | 19. $(a, \tilde{b}2; 1, \tilde{3})$ |           | $\Phi_b(B1)[b, 2]$ |
|      |      |          | 20. $(a, \tilde{b}3; \tilde{1}, 2)$ |           | $\Phi_b(B1)[b, 1]$ |
|      |      |          | 21. $(a, \tilde{b}3; 1, \tilde{2})$ |           | $\Phi_b(B1)[b, 2]$ |
| (4) – 2 |      |          | 22. $(\tilde{a}1, b; \tilde{2}2, 3)$ | $\int dx\tilde{\mathcal{V}}^{R,s}(x)/C_A$ | $\Phi_b(B1)[a, b]$ |
|      |      |          | 23. $(\tilde{a}2, b; 1, 3)$ |           | $\Phi_b(B1)[a, b]$ |
|      |      |          | 24. $(\tilde{a}3, b; 1, 2)$ |           | $\Phi_b(B1)[a, b]$ |
|      |      |          | 25. $(\tilde{a}, \tilde{b}1; 2, 3)$ |           | $\Phi_b(B1)[b, a]$ |
|      |      |          | 26. $(\tilde{a}, \tilde{b}2; 1, 3)$ |           | $\Phi_b(B1)[b, a]$ |
|      |      |          | 27. $(\tilde{a}, \tilde{b}3; 1, 2)$ |           | $\Phi_b(B1)[b, a]$ |

**Step 1**

$$\hat{\sigma}_D(R_{11} = gg \rightarrow ggg) : \quad S_{R_{11}} = 6$$

**Step 2**

$$\hat{\sigma}_D(R_{11}, \text{I}) - \hat{\sigma}_I(R_{11}) = -\hat{\sigma}_I(R_{11}, (2) - 1/2, N_f \mathcal{V}_{f\bar{f}}). \quad (B78)$$

$$\hat{\sigma}_I(R_{11}, (2) - 1/2, N_f \mathcal{V}_{f\bar{f}}) = -\frac{A_d}{S_{B_1}} \frac{N_f}{C_A} \mathcal{V}_{f\bar{f}}(\epsilon) \Phi(B_1)_d \times ([1, 2] + [2, 1] + [1, a] + [1, b] + [2, a] + [2, b]). \quad (B79)$$
Step 3
\[ \hat{\sigma}_D(R_{11}, P) + \hat{\sigma}_C(R_{11}) - \hat{\sigma}_P(R_{11}) = 0, \] (B80)
which includes only Dipole 1.

Step 4
\[ \hat{\sigma}_D(R_{11}, K) - \hat{\sigma}_K(R_{11}) = -\hat{\sigma}_K(R_{11}, \text{dip} 1, (4)-1, N_f h), \] (B81)
which includes only Dipole 1.
\[ \hat{\sigma}_K(R_{11}, \text{dip} 1, (4)-1, N_f h) = -\frac{A_d}{S_{B_1}} \int_0^1 dx \frac{T_{R N_f}}{C_A} \frac{2}{3} h(x) \left[ \Phi_a(B_1, x) (\langle a, 1 \rangle + \langle a, 2 \rangle) \right. \\
\left. + \Phi_b(B_1, x) (\langle b, 1 \rangle + \langle b, 2 \rangle) \right]. \] (B82)

Step 5
\[ \hat{\sigma}_{\text{subt}}(R_{11}) = -\hat{\tau}(R_{11}, \text{ (2)-1/2}, N_f V_{f f}) - \hat{\sigma}_K(R_{11}, \text{dip} 1, (4)-1, N_f h). \] (B83)

Appendix C. Summary for the \( n \) jet process

C.1 \[ \hat{\sigma}_{\text{subt}}(R_1) \]

| Dip | Bj | Splitting | \( (y_{\text{emi}}, y_{\text{spe}}) \) | Factor1 | \( \Phi(B_j) \) | \( (y_{\text{emi}}, y_{\text{spe}}) \) | \( n_{\text{deg}} \) |
|-----|----|-----------|-----------------|---------|-------------|-----------------|------|
| Dip 1 | u\( \bar{u} \) → \( n \)-g | (2) - 1 | \( 1(y_1, y_2) \) | \( V_{gg}/C_A \) | \( \Phi(B_1)[1, 2] \) | \( n+1C_2 \cdot (n-1) \) |
| | | (2) - 2 | \( 2(y_1, y_2) \) | \( \int dx V_{gg}(x)/C_A \) | \( \Phi_4(B_1)[1, a] \) | (n+1) \cdot 1 |
| | | 3.(y_1, y_2) | \( \Phi_5(B_1)[1, b] \) | (n+1) \cdot 1 |
| | | (3) - 1 | \( 4(y_1, y_2) \) | \( \int dx V^{f-f}(x)/C_F \) | \( \Phi_1(B_1)[a, 1] \) | (n+1) \cdot 1 |
| | | 5.(y_1, y_2) | \( \Phi_2(B_1)[b, 1] \) | (n+1) \cdot 1 |
| | | (3) - 2 | \( 6(y_1, y_2) \) | \( \int dx V^{f-f}(x)/C_F \) | \( \Phi_1(B_1)[a, b] \) | (n+1) \cdot 1 |
| | | 7.(y_1, y_2) | \( \Phi_6(B_1)[b, a] \) | (n+1) \cdot 1 |

Step 1
\[ \hat{\sigma}_D(R_i) = -\frac{A_d}{S_{R_i}} \cdot (\text{Factor1}) \cdot \Phi_{a/b}(R_i; B_j, x)_d[y_{\text{emi}}, y_{\text{spe}}] \times n_{\text{deg}}. \]
\[ \hat{\sigma}_D(R_1 = u\( \bar{u} \) → (n + 1)-g) : \ S_{R_1} = (n + 1)!. \]
\[ \hat{\sigma}_{\text{subt}}(R_1 = u\( \bar{u} \) → (n + 1)-g) \]

Step 2
\[ \hat{\sigma}_D(R_1, 1) - \hat{\tau}(R_1) = -\hat{\tau}(R_1, \text{ (2)-1/2}, N_f V_{f f}). \] (C1)
\[ \hat{\tau}(R_1, \text{ (2)-1/2}, N_f V_{f f}) = -\frac{A_d}{S_{B_1}} \cdot \frac{N_f}{C_A} V^{f,f}(\epsilon) \cdot \Phi(B_1)_d \sum_i K, \] (C2)
where the indices of the summation take the values \( i = 1, \ldots, n \) and \( K = 1, \ldots, n, a, b \) with the condition \( i \neq K \).

Step 3
\[ \hat{\sigma}_D(R_1, P) + \hat{\sigma}_C(R_1) - \hat{\sigma}_P(R_1) = 0, \] (C3)
which includes only Dipole 1.
Step 4

\[ \hat{\sigma}_D(R_1, K) - \hat{\sigma}_K(R_1) = -\hat{\sigma}_K(R_1, \text{dip} 1, (3)-1, N_f h), \]  
which includes only Dipole 1.

\[ \hat{\sigma}_K(R_1, \text{dip} 1, (3)-1, N_f h) = -\frac{A_4}{S_{B1}} \int_0^1 dx \left( \frac{2 T_R N_f}{3 C_A} \right) h(x) \]
\[ \times \left[ \Phi_a(B1, x) \sum_{k=1}^n \langle a, k \rangle + \Phi_b(B1, x) \sum_{k=1}^n \langle b, k \rangle \right]. \]

Step 5

\[ \hat{\sigma}_{\text{subt}}(R_1) = \hat{\sigma}_1(R_1, (2)-1/2, N_f v_f f) - \hat{\sigma}_K(R_1, \text{dip} 1, (3)-1, N_f h). \]

C.2. \( \hat{\sigma}_{\text{subt}}(R_2) \)

Table C.2. Summary table of \( \hat{\sigma}_D(R_{2u}) \).

| Dip / | Bj | Splitting | \( y_{\text{emi}} \), \( y_{\text{spe}} \) | Factor 1 | \( \Phi(B_j) [y_{\text{emi}}, y_{\text{spe}}] \) | \( n_{\text{deg}} \) |
|-------|----|-----------|-----------------|----------|-------------------------------|---------|
| Dip 1 | u\( \bar{u} \) | \( + (n - 2) \cdot g \) | \( 1. (y_1, y_2) \) | \( V_{f/g}/C_F \) | \( \Phi(B1)[1, 2] \) | \( (n - 1) \cdot 1 \) |
|       |     |           | \( 2. (y_1, y_3) \) |          | \( \Phi(B1)[1, 3] \) | \( (n - 1) \cdot (n - 2) \) |
|       |     |           | \( 3. (y_2, y_1) \) |          | \( \Phi(B1)[2, 1] \) | \( (n - 1) \cdot 1 \) |
|       |     |           | \( 4. (y_2, y_3) \) |          | \( \Phi(B1)[2, 2] \) | \( (n - 1) \cdot (n - 2) \) |
|       |     |           | \( 5. (y_1, y_2) \) | \( f dx V_{g/g}(x)/C_F \) | \( \Phi_a(B1)[1, a] \) | \( (n - 1) \cdot 1 \) |
|       |     |           | \( 6. (y_1, y_3) \) |          | \( \Phi_b(B1)[1, b] \) | \( (n - 1) \cdot 1 \) |
|       |     |           | \( 7. (y_2, y_1) \) |          | \( \Phi_a(B1)[2, a] \) | \( (n - 1) \cdot 1 \) |
|       |     |           | \( 8. (y_2, y_3) \) |          | \( \Phi_b(B1)[2, b] \) | \( (n - 1) \cdot 1 \) |
|       |     |           | \( 9. (y_3, y_1) \) | \( V_{g/g}/C_A \) | \( \Phi(B1)[3, 1] \) | \( (n - 1) \cdot C_2 \cdot 1 \) |
|       |     |           | \( 10. (y_3, y_2) \) |          | \( \Phi(B1)[3, 2] \) | \( (n - 1) \cdot C_2 \cdot 1 \) |
|       |     |           | \( 11. (y_3, y_4) \) |          | \( \Phi(B1)[3, 3] \) | \( (n - 1) \cdot C_2 \cdot 1 \) |
|       |     |           | \( 12. (y_3, y_4) \) | \( f dx V_{g/g}(x)/C_A \) | \( \Phi_a(B1)[3, a] \) | \( (n - 1) \cdot C_2 \cdot 1 \) |
|       |     |           | \( 13. (y_3, y_4) \) |          | \( \Phi_b(B1)[3, b] \) | \( (n - 1) \cdot C_2 \cdot 1 \) |
|       |     |           | \( 14. (y_4, y_1) \) | \( f dx V_{f/f}(x)/C_F \) | \( \Phi_a(B1)[a, 1] \) | \( (n - 1) \cdot 1 \) |
|       |     |           | \( 15. (y_4, y_2) \) |          | \( \Phi_a(B1)[a, 2] \) | \( (n - 1) \cdot 1 \) |
|       |     |           | \( 16. (y_4, y_3) \) |          | \( \Phi_a(B1)[a, 3] \) | \( (n - 1) \cdot (n - 2) \) |
|       |     |           | \( 17. (y_4, y_1) \) |          | \( \Phi_b(B1)[b, 1] \) | \( (n - 1) \cdot 1 \) |
|       |     |           | \( 18. (y_4, y_2) \) |          | \( \Phi_b(B1)[b, 2] \) | \( (n - 1) \cdot 1 \) |
|       |     |           | \( 19. (y_4, y_3) \) |          | \( \Phi_b(B1)[b, 3] \) | \( (n - 1) \cdot (n - 2) \) |
|       |     |           | \( 20. (y_4, y_3) \) | \( f dx V_{f/f}(x)/C_F \) | \( \Phi_a(B1)[a, b] \) | \( (n - 1) \cdot 1 \) |
|       |     |           | \( 21. (y_4, y_3) \) |          | \( \Phi_b(B1)[b, a] \) | \( (n - 1) \cdot 1 \) |
| Dip 2u | u\( \bar{u} \) | \( n \cdot g \) | \( 22. (y_1, y_2) \) | \( V_{f/f}/C_A \) | \( \Phi_a(B2u)[1, 2] \) | \( 1 \cdot (n - 1) \) |
|       |     |           | \( 23. (y_1, y_3) \) |          | \( \Phi_a(B2u)[1, a] \) | \( 1 \cdot 1 \) |
|       |     |           | \( 24. (y_1, y_4) \) |          | \( \Phi_b(B2u)[1, b] \) | \( 1 \cdot 1 \) |
| Dip 3u | g\( \bar{u} \) | \( + (n - 1) \cdot g \) | \( 25. (y_1, y_1) \) | \( f dx V_{f/f}(x)/C_A \) | \( \Phi_a(B3u)[a, 1] \) | \( 1 \cdot 1 \) |
|       |     |           | \( 26. (y_1, y_2) \) |          | \( \Phi_a(B3u)[a, 2] \) | \( 1 \cdot (n - 1) \) |
|       |     |           | \( 27. (y_1, y_3) \) | \( f dx V_{f/f}(x)/C_A \) | \( \Phi_a(B3u)[a, b] \) | \( 1 \cdot 1 \) |
|       |     |           | \( 28. (y_1, y_4) \) |          | \( \Phi_b(B3u)[b, 1] \) | \( 1 \cdot 1 \) |
|       |     |           | \( 29. (y_1, y_4) \) |          | \( \Phi_b(B3u)[b, 2] \) | \( 1 \cdot (n - 1) \) |
| Dip 3\( \bar{u} \) | u\( \bar{u} \) | \( n \cdot g \) | \( 30. (y_1, y_4) \) | \( f dx V_{f/f}(x)/C_A \) | \( \Phi_a(B3\bar{u})[b, a] \) | \( 1 \cdot 1 \) |

Step 1

\[ \hat{\sigma}_D(R_2 = u\bar{u} \rightarrow u\bar{u}(n - 1) \cdot g) : \quad S_{R_2} = (n - 1)!. \]

\[ \hat{\sigma}_{\text{subt}}(R_2 = u\bar{u} \rightarrow u\bar{u}(n - 1) \cdot g) \]
Step 2

\[ \hat{\sigma}_D(R_2, 1) - \hat{\sigma}_I(R_2) = -\hat{\sigma}_I(R_2, (2)-1/2, N_f \nu_{ff}). \]  
(C7)

\[ \hat{\sigma}_I(R_2, (2)-1/2, N_f \nu_{ff}) = -\frac{A_d}{S_{B_1}} \cdot \frac{N_f}{C_A} \nu_{ff}(\epsilon) \cdot \Phi(B1)_{d} \sum_{i, K} [i, K], \]  
(C8)

where the indices of the summation take the values \( i = 3, \ldots, n \) and \( K = 1, \ldots, n, a, b \) with the condition \( i \neq K \).

Step 3

\[ \hat{\sigma}_D(R_2, P) + \hat{\sigma}_C(R_2) - \hat{\sigma}_P(R_2) = 0, \]  
(C9)

which is separated into three relations for Dipoles 1, 3\( \bar{u} \), and \( 3\bar{u} \) as

\[ \hat{\sigma}_D(R_2, P, \text{dip}) + \hat{\sigma}_C(R_2, \text{dip}) - \hat{\sigma}_P(R_2, \text{dip}) = 0. \]  
(C10)

Step 4

\[ \hat{\sigma}_D(R_2, K) - \hat{\sigma}_K(R_2) = -\hat{\sigma}_K(R_2, \text{dip}1, (3)-1, N_f h), \]  
(C11)

which is separated into three relations for Dipoles 1, 3\( u \), and \( 3\bar{u} \) as

\[ \hat{\sigma}_D(R_2, K, \text{dip}1) - \hat{\sigma}_K(R_2, \text{dip}1) = -\hat{\sigma}_K(R_2, \text{dip}1, (3)-1, N_f h), \]  
(C12)

\[ \hat{\sigma}_D(R_2, K, \text{dip}3\bar{u}) - \hat{\sigma}_K(R_2, \text{dip}3\bar{u}) = 0, \]  
(C13)

\[ \hat{\sigma}_D(R_2, K, \text{dip}3\bar{u}) - \hat{\sigma}_K(R_2, \text{dip}3\bar{u}) = 0. \]  
(C14)

\[ \hat{\sigma}_K(R_2, \text{dip}1, (3)-1, N_f h) \]
\[ = -\frac{A_d}{S_{B_1}} \int_0^1 dx \left( \frac{2 T_R N_f}{3 C_A} \right) h(x) \left[ \Phi_a(B1, x)_{4} \sum_{k=3}^{n} (a, k) + \Phi_b(B1, x)_{4} \sum_{k=3}^{n} (b, k) \right]. \]  
(C15)

Step 5

\[ \hat{\sigma}_{\text{subt}}(R_2) = -\hat{\sigma}_I(R_2, (2)-1/2, N_f \nu_{ff}) \]

\[ - \hat{\sigma}_K(R_2, \text{dip}1, (3)-1, N_f h) + \hat{\sigma}_D(R_2, \text{dip}2). \]  
(C16)

\[ \hat{\sigma}_D(R_2, \text{dip}2) = \hat{\sigma}_D(R_2, \text{dip}2, (5)-1/2, \nu_{ff}) \]

\[ + \hat{\sigma}_D(R_2, \text{dip}2, (5)-2, h). \]  
(C17)

\[ \hat{\sigma}_D(R_2, \text{dip}2, (5)-1/2, \nu_{ff}) = -\frac{A_d}{S_{R_2}} \cdot \frac{1}{C_A} \nu_{ff}(\epsilon) \cdot \Phi(B2)_{d} \]

\[ \times \left( [1, 2] \cdot (n - 1) + [1, a] + [1, b] \right). \]  
(C18)

\[ \hat{\sigma}_D(R_2, \text{dip}2, (5)-2, h) = -\frac{A_d}{S_{R_2}} \int_0^1 dx \left( \frac{2 T_R}{3 C_A} \right) h(x) \]

\[ \times \left( \Phi_a(B2, x)_{4}(1, a) + \Phi_b(B2, x)_{4}(1, b) \right). \]  
(C19)
C.3. $\hat{\sigma}_{\text{subt}}(R_3)$

Table C.3. Summary table of $\hat{\sigma}_D(R_{3u})$.

| Dip | $B_j$ | Splitting | $(y_{emi}, y_{spe})$ | Factor1 | $\Phi(B_j) \left[ y_{emi}, y_{spe} \right]$ | $n_{\text{deg}}$ |
|-----|-------|-----------|-----------------|---------|---------------------------------|-------------|
| Dip 1 | $u\bar{u} \to d\bar{d}$ | (1) - 1 | $\mathcal{V}_{fg}/C_F$ | $\Phi(B1)[1, 2]$ | $(n - 1) \cdot 1$ |
| | | | | $\Phi(B1)[1, 3]$ | $(n - 1) \cdot (n - 2)$ |
| | | | | $\Phi(B1)[2, 1]$ | $(n - 1) \cdot 1$ |
| | | | | $\Phi(B1)[2, 3]$ | $(n - 1) \cdot (n - 2)$ |
| | | $(n - 2)\cdot g$ | | | |
| | | | | $\int d\mathcal{V}_{fg}(\epsilon)/C_F$ | $\Phi_d(B1)[1, a]$ | $(n - 1) \cdot 1$ |
| | | | | | $\Phi_b(B1)[1, b]$ | $(n - 1) \cdot 1$ |
| | | | | | $\Phi_d(B1)[2, a]$ | $(n - 1) \cdot 1$ |
| | | | | | $\Phi_b(B1)[2, b]$ | $(n - 1) \cdot 1$ |
| | | | | | $\Phi(B1)[3, 1]$ | $n_{\text{deg}} C_2 \cdot 1$ |
| | | | | | $\Phi(B1)[3, 2]$ | $n_{\text{deg}} C_2 \cdot 1$ |
| | | | | | $\Phi(B1)[3, 4]$ | $n_{\text{deg}} C_2 \cdot (n - 3)$ |
| | | | | | $\Phi(B1)[3, a]$ | $n_{\text{deg}} C_2 \cdot 1$ |
| | | | | | $\Phi(B1)[3, b]$ | $n_{\text{deg}} C_2 \cdot 1$ |
| | | | | | $\Phi_d(B1)[a, 1]$ | $(n - 1) \cdot 1$ |
| | | | | | $\Phi_d(B1)[a, 2]$ | $(n - 1) \cdot 1$ |
| | | | | | $\Phi_d(B1)[a, 3]$ | $(n - 1) \cdot (n - 2)$ |
| | | | | | $\Phi_b(B1)[b, 1]$ | $(n - 1) \cdot 1$ |
| | | | | | $\Phi_b(B1)[b, 2]$ | $(n - 1) \cdot 1$ |
| | | | | | $\Phi_b(B1)[b, 3]$ | $(n - 1) \cdot (n - 2)$ |
| | | | | | $\Phi_d(B2u)[a, b]$ | $(n - 1) \cdot 1$ |
| | | | | | $\Phi_b(B2u)[b, a]$ | $(n - 1) \cdot 1$ |
| Dip 2u | $u\bar{u} \to n\cdot g$ | (5) - 1 | $\mathcal{V}_{fj}/C_A$ | $\Phi(B2u)[1, 2]$ | $1 \cdot (n - 1)$ |
| | | | | | $\Phi_d(B2u)[a, 1]$ | $1 \cdot 1$ |
| | | | | | | $\Phi_b(B2u)[b, 1]$ | $1 \cdot 1$ |

Step 1

$\hat{\sigma}_D(R_3 = u\bar{u} \to d\bar{d} + (n - 1)\cdot g)$: $S_{R_3} = (n - 1)!$.

$\hat{\sigma}_{\text{subt}}(R_3 = u\bar{u} \to d\bar{d}(n - 1)\cdot g)$

Step 2

$\hat{\sigma}_D(R_3, 1) - \hat{\sigma}(R_3) = -\hat{\sigma}_1(R_3, (2)\cdot 1/2, N_f \mathcal{V}_{fj})$. \hspace{1cm} (C20)

$\hat{\sigma}_1(R_3, (2)\cdot 1/2, N_f \mathcal{V}_{fj}) = -\frac{A_d}{S_{B1}} \cdot \frac{N_f}{C_A} \mathcal{V}_{fj}(\epsilon) \cdot \Phi(B1)_{d} \sum_{i, K}[i, K]$. \hspace{1cm} (C21)

where the indices of the summation take the values $i = 3, \ldots, n$ and $K = 1, \ldots, n, a, b$ with the condition $i \neq K$.

Step 3

$\hat{\sigma}_D(R_3, P) + \hat{\sigma}_C(R_3) - \hat{\sigma}_P(R_3) = 0$. \hspace{1cm} (C22)

which includes Dipole 1.
Step 4

\[ \hat{\sigma}_D(R_3, K) - \hat{\sigma}_K(R_3) = -\hat{\sigma}_K(R_3, \text{dip}1, (3)-1, N_f h), \]  

which includes Dipole 1.

\[ \hat{\sigma}_K(R_3, \text{dip}1, (3)-1, N_f h) = -\frac{A_4}{S_{B_1}} \int_0^1 dx \left( \frac{2 T_R N_f}{3 C_A} \right) h(x) \times \left[ \Phi_a(B_1, x) \sum_{k=3}^n \langle a, k \rangle + \Phi_b(B_1, x) \sum_{k=3}^n \langle b, k \rangle \right]. \]  

(C24)

Step 5

\[ \hat{\sigma}_{\text{subt}}(R_3) = -\hat{\sigma}_1(R_3, (2)-1/2, N_f V_{f f}) \]

\[ - \hat{\sigma}_K(R_3, \text{dip}1, (3)-1, N_f h) + \hat{\sigma}_D(R_3, \text{dip}2). \]  

(C25)

\[ \hat{\sigma}_D(R_3, \text{dip}2) = \hat{\sigma}_D(R_3, \text{dip}2, (5)-1/2, V_{f f}) \]

\[ + \hat{\sigma}_D(R_3, \text{dip}2, (5)-2, h). \]  

(C26)

\[ \hat{\sigma}_D(R_3, \text{dip}2, (5)-1/2, V_{f f}) = -\frac{A_4}{S_{R_3}} \cdot \frac{1}{C_A} V_{f f}(e) \cdot \Phi(B_2)_d \times \left[ [1, 2] \cdot (n-1) + [1, a] + [1, b] \right]. \]  

(C27)

\[ \hat{\sigma}_D(R_3, \text{dip}2, (5)-2, h) = -\frac{A_4}{S_{R_3}} \int_0^1 dx \left( \frac{2 T_R}{3 C_A} \right) h(x) \times \left( \Phi_a(B_2, x) \langle 1, a \rangle + \Phi_b(B_2, x) \langle 1, b \rangle \right). \]  

(C28)

References

[1] K. Hasegawa, Prog. Theor. Exp. Phys. 2015, 113B07 (2015).
[2] S. Catani and M. H. Seymour, Nucl. Phys. B 485, 291 (1997) [arXiv:hep-ph/9605323] [Search inSPIRE].
[3] S. Catani, S. Dittmaier, M. H. Seymour, and Z. Trocsanyi, Nucl. Phys. B 627, 189 (2002) [arXiv:hep-ph/0201036] [Search inSPIRE].
[4] M. H. Seymour and C. Tevlin, [arXiv:0803.2231 [hep-ph]] [Search inSPIRE].
[5] K. Hasegawa, S. Moch, and P. Uwer, Nucl. Phys. Proc. Suppl. 183, 268 (2008) [arXiv:0807.3701 [hep-ph]] [Search inSPIRE].
[6] K. Hasegawa, S. Moch, and P. Uwer, Comput. Phys. Commun. 181, 1802 (2010) [arXiv:0911.4371 [hep-ph]] [Search inSPIRE].
[7] K. Hasegawa, Eur. Phys. J. C 70, 285 (2010) [arXiv:1007.1585 [hep-ph]] [Search inSPIRE].
[8] R. Frederix, T. Gehrmann, and N. Greiner, J. High Energy Phys. 0809, 122 (2008) [arXiv:0808.2128 [hep-ph]] [Search inSPIRE].
[9] R. Frederix, T. Gehrmann, and N. Greiner, J. High Energy Phys. 1006, 086 (2010) [arXiv:1004.2905 [hep-ph]] [Search inSPIRE].
[10] M. Czakon, C. G. Papadopoulos, and M. Worek, J. High Energy Phys. 0908, 085 (2009) [arXiv:0905.0883 [hep-ph]] [Search inSPIRE].

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