Two-dimensional QCD and instanton contribution *

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Abstract

In these notes Yang-Mills theories in $1 + 1$ dimensions are reviewed. Instantons on a sphere prove to be —in the decompactification limit— the key issue to clarify an old controversy between equal-time and light-front quantization.

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I. INTRODUCTION

In the past few years a large amount of efforts has been devoted to two-dimensional gauge theories (YM$_2$), mainly motivated by the possibility of exact solutions which are believed to share features with the real four-dimensional world.

Although YM$_2$ seems trivial when quantized in the light-cone gauge, still topological degrees of freedom occur if the theory is put on a (partially or totally) compact manifold, whereas the simpler behaviour enforced by the light-cone gauge condition on the plane entails a severe worsening in its infrared structure. One can say that, in light-cone gauge, dynamics gets hidden in the very singular nature of correlators at large distances (IR singularities).

The first quantity that comes to mind is the two-point correlator. If the theory is quantized in the gauge $A_- = 0$ at equal-times, the free propagator has the following causal expression (WML prescription) in two dimensions

$$D_{++}^{WML}(x) = \frac{1}{2\pi} \frac{x^-}{-x^+ + i\epsilon x^-}, \quad x^\pm = \frac{x^0 \pm x^1}{\sqrt{2}},$$

(1)

first proposed by T.T. Wu [1]. In turn this propagator is nothing but the restriction in two dimensions of the expression proposed by S. Mandelstam and G. Leibbrandt [2] in four dimensions and derived by means of a canonical quantization in ref. [3].

In dimensions higher than two, where “physical” degrees of freedom are switched on (transverse “gluons”), causality is mandatory in order to get correct analyticity properties, which in turn are the basis of any consistent renormalization program, and to obtain agreement with Feynman gauge results [4].

The situation is somewhat different in exactly two dimensions. Here the theory can be quantized on the light-front (at equal $x^+$); with such a choice, no dynamical degrees of freedom occur as the non-vanishing component of the vector field does not propagate

$$D_{++}^{P}(x) = -\frac{i}{2} |x^-| \delta(x^+),$$

(2)

but rather gives rise to an instantaneous (in $x^+$) Coulomb-like potential.
A formulation based essentially on the potential in Eq. (2) was originally proposed by G. ’t Hooft in 1974 [5], to derive beautiful solutions for the $q\bar{q}$-bound state problem under the form of rising Regge trajectories. On the other hand, when Wu’s prescription is adopted, the bound state equation at large-$N$ turns out to be very difficult, and Regge trajectories disappear. A suitable tool to clarify the origin of this discrepancy is the Wilson loop, owing to its gauge invariance and to its reasonable infrared properties.

II. WILSON LOOP ON THE PLANE

When inserted in perturbative Wilson loop calculations, expressions Eqs. (1) and (2) lead to completely different results. In two dimensions pure-area exponentiation is expected. This is due to the invariance of the theory under area-preserving diffeomorphisms, which suggests that the Wilson loop is a function of the dimensionless quantity $g^2A$, with $A$ the encircled area, and to unitary evolution.

If a rectangle with light-like sides is chosen as a contour, with ’t Hooft potential only planar diagrams can be built, for any value of $N$. Therefore the perturbative series is easily resummed leading to the expected result $^1$

$$W[A] = \exp \left( -\frac{1}{2} C_F g^2 A \right),$$

where $C_F$ is the quadratic Casimir operator for the fundamental representation, i.e. $\frac{N^2-1}{2N}$, $\frac{N^2}{2}$ for $SU(N)$, $U(N)$, respectively. Confinement holds and, in the ’t Hooft limit $N \rightarrow \infty$, $g^2N$ fixed, Eq. (3) exhibits a finite string tension.

When the propagator is endowed with the causal prescription Eq. (1), instead, disagreement with the result Eq. (3) already shows up at order $O(g^4)$, and is due to graphs with crossed vector propagators which produce a contribution proportional to $C_F C_A$, $C_A$ being the Casimir of the adjoint representation. Although non-trivial, resummation of the perturbative series at all orders in the coupling constant $g$ is still possible. This was performed

$^1$in the Euclidean formulation
in [6]: once a circular contour is chosen, geometry factorizes out and the task of determining
the Wilson loop reduces to the purely combinatorial problem of finding the group factors
corresponding to the Wick contractions. Fortunately these factors are generated by a matrix
integral in the space of hermitian (traceless) $N \times N$ matrices for $U(N)$ ($SU(N)$). The result
for $U(N)$ reads

$$W = \frac{1}{N} \exp \left[ -\frac{g^2 A}{4} L_{N-1}^{(1)}(g^2 A/2) \right], \quad (4)$$

the function $L_{N-1}^{(1)}$ being a generalized Laguerre polynomial. In addition to the extra polyno-
mial appearing in Eq. (4), the string tension turns out to be different from the one in
Eq. (3). More dramatically, Eq. (4) in the 't Hooft limit becomes

$$\lim_{N \to \infty} W = \sqrt{\frac{2}{\hat{g}^2 A}} J_1(\sqrt{2\hat{g}^2 A}), \quad (5)$$

where $\hat{g}^2 = g^2 N$, and confinement is lost, thereby explaining the failure of Wu’s spectrum.

We are now standing before two different scenarios and there is apparently no motivation
to prefer one to the other, since both 't Hooft’s and Wu’s results are analytic in $g^2$. The
need for reconciliation becomes urgent.

### III. THE INSTANTON EXPANSION

In order to gain a deeper insight, it is worthwhile to study the problem on a compact
two-dimensional manifold [7]. Let it be the sphere for simplicity. Therein the procedure
is purely geometrical and group-theoretical, so that no gauge-fixing has to be adopted and
infra-red problems are absent.

The starting point are the well-known expressions [8] of the exact p-
artition function and
of a Wilson loop for a pure $U(N)$ Yang-Mills theory on a sphere $S^2$ with area $A$

$$Z(A) = \sum_R (d_R)^2 \exp \left[ -\frac{g^2 A}{4} C_2(R) \right], \quad (6)$$

$$W(A - A', A) = \frac{1}{ZN} \sum_{R,S} d_R d_S \exp \left[ -\frac{g^2 (A - A')}{4} C_2(R) - \frac{g^2 A}{4} C_2(S) \right]$$
$$\times \int dU \text{Tr} [U \chi_R(U) \chi_S^+(U)], \quad (7)$$
being the dimension of the irreducible representation $R(S)$ of $U(N)$; $C_2(R)$ ($C_2(S)$) is
the quadratic Casimir, $A - \mathcal{A}, \mathcal{A}$ are the areas singled out by the loop, the integral
in Eq. (7) is over the $U(N)$ group manifold and $\chi_{R(S)}$ is the character of the group
element $U$ in the $R(S)$ representation. In order to evaluate $\mathcal{W}(A - \mathcal{A}, \mathcal{A})$ in the decompactification limit, for
$N > 1$ Eqs. (6), (7) can be explicitly written in the form

$$Z(A) = \frac{1}{N!} \exp \left[ -\frac{g^2 A}{48} N (N^2 - 1) \right] \sum_{m_i = -\infty}^{+\infty} \Delta^2 (m_1, ..., m_N)$$

$$\times \exp \left[ -\frac{g^2 A}{4} \sum_{i=1}^{N} \left( m_i - \frac{N-1}{2} \right)^2 \right], \quad (8)$$

$$\mathcal{W}(A - \mathcal{A}, \mathcal{A}) = \frac{1}{Z(N)} \exp \left[ -\frac{g^2 A}{48} N (N^2 - 1) \right]$$

$$\times \sum_{k=1}^{N} \sum_{m_i = -\infty}^{+\infty} \Delta (m_1 + \delta_{1,k}, ..., m_N + \delta_{N,k}) \Delta (m_1, ..., m_N)$$

$$\times \exp \left[ -\frac{g^2 (A - \mathcal{A})}{4} \sum_{i=1}^{N} \left( m_i - \frac{N-1}{2} \right)^2 - \frac{g^2 \mathcal{A}}{4} \sum_{i=1}^{N} \left( m_i - \frac{N-1}{2} + \delta_{i,k} \right)^2 \right]. \quad (9)$$

In the previous formulae the generic irreducible representation has been described by means
of the set of integers $m_i = (m_1, ..., m_N)$, related to the Young tableaux, in terms of which
one gets

$$C_2(R) = \frac{N}{12} (N^2 - 1) + \sum_{i=1}^{N} \left( m_i - \frac{N-1}{2} \right)^2,$$

$$d_R = \Delta (m_1, ..., m_N). \quad (10)$$

$\Delta$ is the Vandermonde determinant and the integration in Eq. (7) has been performed
explicitly, using the well-known formula for the characters in terms of the set $m_i$.

Now, as first noted by Witten [9], it is possible to represent $Z(A)$ —and consequently
$\mathcal{W}(A - \mathcal{A}, \mathcal{A})$— as a sum over instable instantons, each instanton contribution being associ-
ated to a finite, non-trivial, perturbative expansion. One can observe that in the sum over
$\{m_i\}$ the dependence on the area is through the dimensionless quantity $g^2 A$, whereas an
instanton action typically depends on $\frac{1}{g^2 A}$. Therefore a duality transformation is required to
turn the fraction upside down. The mathematical tool to carry out such a task is provided
by a Poisson resummation [10] over $\{m_i\}$. 

5
\[
\sum_{m_i = -\infty}^{+\infty} F(m_1, ..., m_N) = \sum_{n_i = -\infty}^{+\infty} \tilde{F}(n_1, ..., n_N),
\]  
\[
\tilde{F}(n_1, ..., n_N) = \int_{-\infty}^{+\infty} dz_1 ... dz_N \exp[2\pi i (z_1 n_1 + ... + z_N n_N)] F(z_1, ..., z_N).
\]

When performed in Eqs. (8), (9), it gives
\[
Z(A) = C(g^2 A, N) \sum_{n_i = -\infty}^{+\infty} \exp[-S_{\text{inst}}(n_i)] Z(n_1, ..., n_N),
\]
\[
\mathcal{W}(A - A, A) = \frac{1}{Z N} C(g^2 A, N) \exp\left[-g^2 A(A - A) \frac{A - A}{4A}\right] \sum_{n_i = -\infty}^{+\infty} \exp[-S_{\text{inst}}(n_i)]
\]
\[
\times \sum_{k=1}^{N} \exp\left[-2\pi i n_k \frac{A - A}{A}\right] W_k(n_1, ..., n_N),
\]
\[
(12)
\]

where
\[
S_{\text{inst}}(n_i) = \frac{4\pi^2}{g^2 A} \sum_{i=1}^{N} n_i^2.
\]
\[
(13)
\]

Finally, \(S_{\text{inst}}(n_i)\) is interpreted as an instanton action \(^2\), while \(Z(n_1, ..., n_N)\) and \(W_k(n_1, ..., n_N)\) are the quantum corrections around the classical solutions \([11]\).

From the above representations it is rather clear why the decompactification limit \(A \to \infty\) should not be performed too early. As a matter of fact, on the plane fluctuations around the instanton solutions are undistinguishable from Gaussian fluctuations around the trivial field configuration, since \(S_{\text{inst}}(n_i)\) goes to zero for any finite set \(n_i\) when \(A \to \infty\). For finite \(A\) and finite \(n_i\) instead, in the limit \(g \to 0\), only the zero-instanton sector can survive in the Wilson loop expression.

**IV. RELATION WITH PERTURBATION THEORY**

In principle the zero-instanton contribution should be obtainable by means of perturbative calculations. If in Eqs. (12), (13) only the zero-instanton sector, i.e. \(n_i = 0\), is retained, \(^2\)Indeed, on \(S^2\) there are non-trivial solutions of the Yang-Mills equation, labelled by the set of integers \(\{n_i\} = (n_1, \ldots, n_N)\) (see \([11]\)).
after some technicalities the following result is found [11]

\[ W^{(0)} = \frac{1}{N} \exp \left[ -\frac{g^2 A (A - A)}{4A} \right] L^{\dagger}_{N-1} \left( \frac{g^2 A^2 (A - A)}{2A} \right). \]  

(14)

At this point a remark is in order: in the decompactification limit \( A \to \infty, \) \( A \) fixed, the quantity in the equation above exactly coincides, for any value of \( N, \) with Eq. (11), which was derived from a matrix model. Hence, the zero-instanton contribution corresponds in a sense to “integrating over the group algebra”.

V. THE WILSON LOOP WITH WINDING NUMBER \( n \)

Disagreement of Eq. (14) with the pure-area exponentiation is at this stage no longer surprising since \( W^{(0)} \) does not contain any genuine non-perturbative contribution, \( \text{viz} \) instantons. For any value of \( N \) the pure-area exponentiation follows, after decompactification, from resummation of all instanton sectors, changing completely the zero-sector behaviour and, in particular, the value of the string tension.

What might instead be unexpected in this context is the fact that, using the instantaneous ’t Hooft potential and just resumming at all orders the related perturbative series, one still ends up with the correct pure-area exponentiation.

This is true also in more general instances, such as Wilson loops winding \( n \) times on a closed contour [12]. For a pure \( U(N) \) Yang-Mills theory on a sphere \( S^2 \) with area \( A \) it holds

\[ W_n(A - A, A) = \frac{1}{Z_N} \sum_{R,S} d_R d_S \exp \left[ -\frac{g^2 (A - A)}{4} C_2(R) - \frac{g^2 A}{4} C_2(S) \right] \times \int dU Tr[U^n] \chi_R(U) \chi^\dagger_S(U), \]  

(15)

the notation being as in Eq. (7). It can be shown that in the decompactification limit \( A \to \infty, \) \( A \) fixed, the following expression is recovered

\[ W_n(A; N) = \frac{1}{nN} \exp \left[ -\frac{g^2 A}{4} n(N + n - 1) \right] \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \frac{\Gamma(N + n - k)}{\Gamma(N - k) \Gamma(n - k)} \times \exp \left[ \frac{g^2 A}{2} n k \right]. \]  

(16)
The series is actually a finite sum, stopping at $k = n - 1$ or $k = N - 1$, depending on the smaller one. As happened for $n = 1$, Eq. (16) is expected to come out from the resummation of ’t Hooft perturbative series, that corresponds to a light-front quantization of the theory. Some comments are now in order. First of all notice that when $n > 1$ the simple abelian-like exponentiation is lost. In other words the theory starts to feel its non-abelian nature and the combinatorial coefficients in Eq. (16) signal that the light-front vacuum, though simpler than its analog in the equal-time quantization, cannot be considered trivial. Actually, from the sphere point of view, Eq. (16) can be understood as coming from an instantons’ resummation. The procedure to see this is exactly the one outlined above for the case $n = 1$. On the other hand to neglect instantons, and then to send the area of the sphere to infinity, is likely to reproduce the WML computation. In fact the perturbative analysis have confirmed these claims. Furthermore, Eq. (16) exhibits an intriguing duality

$$W_n(A; N) = W_N \left( \frac{n}{N} A; n \right), \quad (17)$$

a relation that is far from being trivial, involving an unexpected interplay between the geometrical and the algebraic structure of the theory. Looking at Eq. (17), the abelian-like exponentiation for $U(N)$ when $n = 1$ appears to be connected to the $U(1)$ loop with $N$ windings, the “genuine” triviality of Maxwell theory providing the expected behaviour for the string tension. Finally, one should observe that the large-$N$ limit (with $n$ fixed) is equivalent to the limit in which an infinite number of windings is considered with vanishing rescaled loop area. Alternatively, this rescaling could be thought to affect the coupling constant $g^2 \rightarrow \frac{n}{N} g^2$. In detail, the former limit ($N \rightarrow \infty$, $\hat{g}^2 = g^2 N$ fixed) coincides with the Kazakov-Kostov result [13]

$$W_n(A; \infty) = \frac{1}{n} L_{n-1}^{(1)} \left( g^2 A n/2 \right) \exp \left[ -\frac{\hat{g}^2 A n}{4} \right]. \quad (18)$$

Next, using Eq. (17) one is able to perform the latter limit, namely $n \rightarrow \infty$ with fixed $n^2 A$

$$\lim_{n \rightarrow \infty} W_n(A; N) = \frac{1}{N} L_{N-1}^{(1)} \left( g^2 A n^2/2 \right) \exp \left[ -\frac{g^2 A n^2}{4} \right]. \quad (19)$$
Eq. (19) turns out to be exactly the zero-instanton contribution $W_n^{(0)}$ in $W_n(A - \mathcal{A}, \mathcal{A})$ after decompactification \[12\]. Such a coincidence can be explained by observing that, having the instantons a finite size, small loops are essentially blind to them. Again, this is not the end of the story: as expected, Eq. (19) can be derived through resummation of the perturbative series defined via WML prescription (matrix model).

The conclusion to be drawn is that the interpretation of both the equal-time and the light-front vacua in the case of a simple Wilson loop can be extended to the more general case of a loop with winding number $n$. In the light of the considerations above, WML, even resummed, appears to be truly perturbative since it provides only the expression of the zero-instanton contribution to Wilson loops $W^{(0)}$. As opposed to this, ’t Hooft is non-perturbative, in the sense that a “perturbative” series in a “static” potential reproduces a complex instanton expansion. This seems to be the case also for loops in the adjoint representation with non trivial $\theta$-vacua (when the theory is based on the group $SU(N)/\mathbb{Z}_N$). Again $W^{(0)}$ corresponds to a matrix model after decompactification and is insensitive to the choice of the vacuum sector. This will be the subject of a forthcoming publication.
REFERENCES

[1] T.T. Wu, Phys. Lett. **71B**, 142 (1977).

[2] S. Mandelstam, Nucl. Phys. **B213**, 149 (1983); G. Leibbrandt, Phys. Rev. **D29**, 1699 (1984).

[3] A. Bassetto, M. Dalbosco, I. Lazzizzera and R. Soldati, Phys. Rev. **D31**, 2012 (1985).

[4] A. Bassetto, G. Nardelli and R. Soldati, *Yang-Mills Theories in Algebraic Non-Covariant Gauges*, World Scientific, Singapore 1991; A. Bassetto, I.A. Korchemskaya, G.P. Korchemsky and G. Nardelli, Nucl. Phys. **B408**, 62 (1993); A. Bassetto and M. Ryskin, Phys. Lett. **B316**, (1993); C. Acerbi and A. Bassetto, Phys. Rev. **D49**, 1067 (1994); A. Bassetto, Nucl. Phys. Proc. Suppl. **51C**, 281 (1996); A. Bassetto, G. Heinrich, Z. Kunszt and W. Vogelsang, Phys. Rev. **D58**, 094020 (1998).

[5] G. ’t Hooft, Nucl. Phys. **B75**, 461 (1974).

[6] M. Staudacher and W. Krauth, Phys. Rev. **D57**, 2456 (1998).

[7] D. J. Gross and A. Matytsin, Nucl. Phys. **B437**, 541 (1997).

[8] A.A. Migdal, Sov. Phys. JETP **42**, 413 (1975); B.E. Rusakov, Mod. Phys. Lett. **A5**, 693 (1990).

[9] E. Witten, Commun. Math. Phys. **141**, 153 (1991) and J. Geom. Phys. **9**, 303 (1992).

[10] M. Caselle, A. D’Adda, L. Magnea and S. Panzeri, *Two dimensional QCD on the sphere and on the cylinder*, in Proceedings of Workshop on High Energy Physics and Cosmology, (Trieste 1993) eds. E.Gava, A.Masiero, K.S.Narain, S.Randjbar-Daemi and Q.Shafi, World Scientific, Singapore, 1994.

[11] A. Bassetto and L. Griguolo, Phys. Lett. **B443**, 325 (1998).

[12] A. Bassetto, L. Griguolo and F. Vian, Nucl. Phys. **B559**, 563 (1999).
[13] V.A. Kazakov and I.K. Kostov, Nucl. Phys. B176, 199 (1980); V.A. Kazakov, *ibid.* 179, 283 (1981).