COBOUNDARIES AND EIGENVALUES OF FINITARY S-ADIC SYSTEMS

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Abstract. An S-adic system is a symbolic dynamical system generated by iterating an infinite sequence of substitutions or morphisms, called a directive sequence. A finitary S-adic dynamical system is one where the directive sequence consists of morphisms selected from a finite set. We study eigenvalues and coboundaries for finitary recognizable S-adic dynamical systems, i.e., those where points can be uniquely desubstituted using the given sequence of morphisms. To do this we identify the notions of straightness and essential words, and use them to define a coboundary, inspired by Host’s formalism, which allows us to express necessary and sufficient conditions that a complex number must satisfy in order to be a continuous or measurable eigenvalue. We then apply our results to finitary directive sequences of substitutions of constant length, and show how to create constant-length S-adic shifts with non-trivial coboundaries. We show that in this case all continuous eigenvalues are rational and we give a complete description of the rationals that can be an eigenvalue, indicating how this leads to a Cobham-style result for these systems.

1. Introduction

The Rokhlin-Kakutani lemma in ergodic theory allows one to approximate, up to a set of small measure, a measure-preserving dynamical system by a partition of measurable sets, depicted as a tower, where the transformation consists of moving up the tower, see, e.g., [Pet83, Lemma 4.7]. Recognizable S-adic dynamical systems are a family of symbolic systems which can be approximated by well-controlled Rokhlin towers, where the instructions of how to construct finer Rokhlin towers from coarser ones are defined by a directive sequence \((\sigma_n)_{n \geq 0}\) of morphisms or substitutions. In a previous work [BSTY19], recognizability for sequences of morphisms was studied, and conditions which guaranteed it were identified. In this article, we apply these results by studying (dynamical) eigenvalues for recognizable S-adic dynamical systems, where the directive sequence consists of morphisms selected from a finite set. We will call such directive sequences finitary (see Section 2 for definitions of recognizability and substitutions).

There are various related notions in the measurable and topological dynamics literature that produce recognizable S-adic systems. In particular, cutting-and-stacking measurable systems on a Lebesgue space (see e.g. [KS67, Fer97]) are built with recursive instructions which can be interpreted as a directive sequence, and provided the latter is recognizable, the S-adic shift is a symbolic representation of the cutting-and-stacking system. Also, under the condition that the morphisms \(\sigma\) in the directive sequence are proper, i.e., \(\sigma(a)\) starts with the same letter and ends with the same letter for each letter \(a\), recognizable S-adic systems can be directly seen as topological Bratteli-Vershik systems, see [BSTY19], and [DHS99] for the stationary case.

Moreover, the model theorem from [HPS92, Theorem 4.7] guarantees us a proper Bratteli-Vershik representation for minimal systems, but we cannot usually obtain such a representation by simple manipulations of the data we are given in the form of the directive sequence: an S-adic system is not generally topologically conjugate to the natural Bratteli–Vershik system associated...
to the directive sequence, although they are almost–conjugate (see Theorem 2.8 and [BSTY19] for details). The manipulations required to obtain a proper Bratteli-Vershik representation are already complicated even in the stationary case, see [DHS99, BKM09]. More generally, even if finitary \( S \)-adic subshifts are known to be representable in as a finite rank Bratteli-Vershik system, and therefore represented in an \( S \)-adic and proper way, see [Esp23, GH22], the construction allowing one to go from a finitary \( S \)-adic representation to a proper \( S \)-adic one is not totally effective. The approach of this article is to work directly with the given \( S \)-adic description of the shift under study. As there are several important results concerning the eigenvalues of the latter families of systems, we describe throughout how our findings intersect or extend the existing literature.

A substitution shift \((X_\sigma, T)\) is an \( S \)-adic dynamical system where the directive sequence \((\sigma_n)_{n\geq 0}\) is constant, i.e., there exists a substitution \(\sigma\) such that \(\sigma_n = \sigma\) for each \(n\). Our departure point for this work is Host’s article [Hos86], where he identifies coboundaries as an important tool for describing eigenvalues of a primitive substitution shift \((X_\sigma, T)\), which allowed him moreover to prove the striking result that measurable eigenvalues are continuous eigenvalues. We use here the term coboundary following Host’s definition, which is not to be confused with the usual meaning of a coboundary, i.e., a function of the form \(f - f \circ T\). A coboundary is basically partial information that identifies the relationship between the values that a putative eigenfunction \(f\) can take on the orbit of some select set of points \(D\) called limit words, see Definition 3.4. In the substitution case, \(D\) is the finite set of \(\sigma\)-periodic points, and primitivity ensures that the \(T\)-orbit of any of these points is dense in \(X_\sigma\). For a primitive substitution \(\sigma\), Host shows that any coboundary defines a continuous eigenvalue, i.e., one with a continuous eigenfunction, and every measurable eigenvalue, i.e., one with a measurable eigenfunction, defines a coboundary, and therefore is a topological eigenvalue; see Theorem 3.19. In this proof, Host identifies that for \(\lambda\) to be an eigenvalue, \(\lambda^{(\sigma^n(\alpha))}\) must converge to a coboundary for each letter \(a\), but in fact he needs, and shows, the stronger fact that this convergence must be geometric.

In this paper, we extend Host’s notion of a coboundary to the finitary and straight \( S \)-adic setting. When moving from a substitution shift to an \( S \)-adic shift, we move from working with one language to working with a sequence of languages; this is reflected in Definition 4.1 for the notion of a coboundary. One other novelty that appears in our definition of a coboundary is the requirement that the directive sequence be straight (this notion is introduced in Section 3.1.) This is simply so that there is a well-defined set of points \(D\) of limit words as above, on which to partially define an eigenfunction. Straight directive sequences abound, for example, all stationary directive sequences can be uniformly telescoped to be straight; in fact in [Hos86], this is a property that is used. In Section 3.2 we show that many well-studied families of substitutions define straight directive sequences.

We investigate the distinction between continuous and measurable eigenvalues in the \( S \)-adic case, a program which was initiated in [Hos86] for the substitutive case. We show in Section 4 (devoted to the continuous case), and more particularly in Theorems 4.4 and 4.7 that, with the given conditions, coboundaries define continuous eigenvalues. Moreover, in Theorem 4.7, we show that if the convergence to the coboundary is fast, then the coboundary defines an eigenvalue, and in Section 4.3, we deduce from stronger convergence properties that eigenvalues associated to trivial coboundaries are easy to compute, in the sense that they are related to measures of letter cylinders. In Section 5, Theorem 5.10 gives a necessary condition for the existence of a measurable eigenvalue, to be compared to the condition in Theorem 4.7.

Sufficient conditions (both for the continuous and measurable case) for being an eigenvalue are expressed as summation-type conditions; such conditions are natural and classical, and appear in several places in the literature, for example in [Nad11, Corollary 15.57]. In particular, Theorem 5.10 can be seen as an extension of [CDHM03], where the authors study finitary, proper directive sequences which define linearly recurrent \( S \)-adic systems. Note that the condition that the morphisms are proper ensures that the only coboundary is the trivial one, which is why their result makes no mention of coboundaries whilst ours does; we comment on why this is the case in Proposition 4.6. More generally, let us refer to the important literature devoted to the study of dynamical eigenvalues of Cantor; see e.g. [BDM05, BDM10, CDHM03, DFM15, DFM19].
We then apply our results in Section 6 to the special family of $S$-adic shifts defined by directive sequences of constant-length substitutions. This is a natural family to consider in this context, as Host’s notion of a coboundary generalises the notion of height for constant-length substitutions, and indeed it is for this family that we most easily find non-trivial coboundaries. Here also, we show in Example 6.10 that it is not difficult to create constant-length $S$-adic shifts with non-trivial coboundaries. In the case where the directive sequence is also finitary and satisfies some mild assumption, we show in Theorem 6.5 that all continuous eigenvalues are rational, and we give a complete description of the rationals that can be an eigenvalue. This allows us to conclude with an $S$-adic version of Cobham’s theorem in Corollary 6.9.

Some results concerning sequences of constant-length substitutions exist in the finite rank literature, namely, work by Mentzen in [Men91], who shows that uniform exact finite rank systems have only rational measurable eigenfunctions; see Section 5 for the relevant definitions. Also, in the case where all the substitutions are proper, we can look at the literature concerning eigenvalues of Toeplitz shifts, given by Durand, Frank and Maass in [DFM15], with relevant results also in [CDHM03] and [BDM05]. The results in the latter works for constant length $S$-adic shifts intersect ours in the family of Toeplitz shifts; see Section 6.2 for definitions and a comparison of our results to these aforementioned works. We also give in Theorem 6.11 a generalisation of a result of Kamae, and Dekking [Kam72, Dek78], which gives broad conditions that ensure that a constant-length $S$-adic system is Toeplitz.

We illustrate our work with the running Example ??, which gives an example of a finitary strongly primitive and non-proper $S$-adic shift which is not linearly recurrent, and whose spectrum can be explicitly described using our methods. This example is from Durand’s work [Dur00, Dur03].

The study of eigenvalues for substitutions has given rise to an abundant literature. The approach developed in [FMN96] relies on the notion of return words which avoids the use of coboundaries. The series of papers [BDM05, BDM10, CDP16, CDHM03, DFM15, DFM19, ADE24] deals with proper substitutions, here again avoiding coboundaries. See also [DG19, BGmY23] for the constant-length case and [CFM08] for the Arnoldx-Rauzy shifts. The $S$-adic Pisot case is handled in [BST23] where continuous eigenvalues are considered. Observe that similar results are obtained for flows and $\mathbb{R}$-actions (instead of $\mathbb{Z}$-actions, which is the viewpoint developed here). See [Sol97, Sol07] for an extension of Host’s result for substitution tilings and see [FS14] for the constant-length case for multidimensional tilings of $S$-adic type, with an approach based on return words avoiding the use of coboundaries. Let us cite also [BS20] for the study of the modulus of continuity of spectral measures in the weakly mixing case via a spectral cocycle. Note that in the case of flows, towers have the same height, which is reminiscent of the constant-length case.

In conclusion, the papers we have cited above show situations where it is possible to characterize topological and measurable eigenvalues, and to distinguish them. The present work is a preliminary step towards the extension of the existing work in the proper case mentioned above, in order to get a more complete characterization of continuous vs. measurable eigenvalues in the $S$-adic setting, by highlighting the role played by coboundaries.

Let us sketch the contents of this paper. Preliminaries are recalled in Section 2. In Section 3 we introduce the combinatorial notions of essential words and of straightness; we also discuss eigenvalues and coboundaries for substitutions. In Section 4 we introduce $S$-adic coboundaries and discuss their connection to the existence of continuous eigenvalues of $S$-adic shifts; we also investigate the relation between continuous eigenvalues and measures of letters in the case of a trivial coboundary. Section 5 deals with measurable eigenvalues in the case of so-called finite exact rank. Lastly, in Section 6 we focus on the family of $S$-adic systems generated by constant-length directive sequences and in particular, on the Toeplitz $S$-adic shifts.

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2. Preliminaries

This section recalls the notions that will be needed hereafter. Section 2.1 first recalls basic definitions from symbolic dynamics. Section 2.2 is devoted to S-adic shifts, Section 2.3 to recognizability, and Section 2.4 discusses the partitions consisting of Rokhlin towers associated to recognizable S-adic shifts.

2.1. Shifts, substitutions and eigenvalues. Let $\mathcal{A}$ be a finite set of symbols, also called an alphabet, and let $\mathcal{A}^\mathbb{Z}$ denote the set of two-sided infinite sequences over $\mathcal{A}$. Endowing $\mathcal{A}$ with the discrete topology, we equip $\mathcal{A}^\mathbb{Z}$ with the metrizable product topology. In this work we consider shift dynamical systems, or shifts $(X,T)$, where $X$ is a closed $T$-invariant set of $\mathcal{A}^\mathbb{Z}$ and $T: \mathcal{A}^\mathbb{Z} \rightarrow \mathcal{A}^\mathbb{Z}$ is the left shift map $(x_n)_{n \in \mathbb{Z}} \mapsto (x_{n+1})_{n \in \mathbb{Z}}$. We call these invertible shifts two-sided. We use letters $x, y, z$ to denote points in two-sided shift spaces $\mathcal{A}^\mathbb{Z}$. In some cases we will discuss non-invertible, or one-sided, shifts $(\tilde{X}, \tilde{T})$, where $\tilde{X} \subset \mathcal{A}^\mathbb{N}$ stands for the set of one-sided, or right-infinite sequences over $\mathcal{A}$. A shift is minimal if it has no non-trivial closed shift-invariant subset. We say that $x \in \mathcal{A}^\mathbb{Z}$ is periodic if $\tilde{T}^k(x) = x$ for some $k \geq 1$, aperiodic otherwise. The shift $(X,T)$ is said to be aperiodic if each $x \in X$ is aperiodic. For basics on continuous and measurable dynamics, see e.g. [CFS82, Pet83, Wal82].

Given a finite alphabet $\mathcal{A}$, let $\mathcal{A}^*$ be the free monoid of all (finite) words over $\mathcal{A}$ under the operation of concatenation, and let $\mathcal{A}^+$ be the set of all non-empty words over $\mathcal{A}$. We let $|w|$ denote the length of a finite word $w$ and let $|\mathcal{A}|$ denote the cardinality of the set $\mathcal{A}$. A subword of a word or a sequence $x$ is a finite word $x_{[i,j]}$, $i \leq j$, with $x_{[i,j]} := x_i x_{i+1} \ldots x_{j-1}$. (We use here the term subword instead of factor, which we use only for topological or measurable factors.) For any word $w \in \mathcal{A}^+$ and any $0 \leq j < |w|$, the notation $T^j w$ refers to the word $w_{[j,|w|]}$. A language is a collection of words in $\mathcal{A}^*$. The language $\mathcal{L}_x$ of $x = (x_n)_{n \in \mathbb{Z}} \in \mathcal{A}^\mathbb{Z}$ is the set of all its subwords. The language $\mathcal{L}_X$ of a one- or two-sided shift $(X,T)$ is the union of the languages of all $x \in X$; it is closed under the taking of subwords and every word $w$ in $\mathcal{L}_X$ is left- and right-extendable to a word in $\mathcal{L}_X$, i.e., there exist letters $a, b$ such that $awb \in \mathcal{L}_X$. Conversely, a language $\mathcal{L}$ on $\mathcal{A}$ which is closed under the taking of subwords and such that each word is both left- and right-extendable defines a one- or two-sided shift $(X_\mathcal{L}, T)$, where $X_\mathcal{L}$ consists of the set of points all of whose subwords belong to $\mathcal{L}$, so that $\mathcal{L}_{X_\mathcal{L}} = \mathcal{L}$. We remark that the two-sided shift defined by $\mathcal{L}$ is the natural extension of the one-sided shift defined by $\mathcal{L}$; for a definition and details, see [CFS82].

Let $X$ be a one-sided (respectively two-sided) minimal shift. If $\mu$ is a $T$-invariant probability Borel measure on $X$, then $T$ induces an isometric (respectively unitary) Koopman operator on $L^2(X,\mu)$, namely $f \mapsto f \circ T$. The eigenvalues of $(X,T,\mu)$ are by definition the eigenvalues of this operator, and the eigenfunctions of $(X,T,\mu)$ are defined as being its eigenvectors. The set of eigenvalues is an invariant for conjugacy in measure and is called the discrete spectrum of $(X,T,\mu)$. If there exists an orthonormal basis of $L^2(X,\mu)$ consisting of eigenfunctions, then $(X,T,\mu)$ is said to have discrete spectrum. One can also consider the continuous eigenvalues of $(X,T)$: these are values $\lambda$ such that there exists a continuous function $f: X \rightarrow \mathbb{C}$ with $f \circ T = \lambda f$. Eigenvalues belong to the set $\mathbb{S}^1$ of complex numbers with modulus 1. If $t$ is such that $e^{2\pi i t}$ is a (continuous) eigenvalue, then $t$ is said to be an additive (continuous) eigenvalue. Abusing language, we say that $\lambda = e^{2\pi i t}$ is rational whenever $t$ is rational, i.e., whenever $\lambda$ is a root of unity. We denote by $E(X,T)$ the additive group of additive continuous eigenvalues of $(X,T)$. It is a topological invariant for $(X,T)$ for conjugacy. Continuous eigenvalues are measurable if the invariant measure is a Borel measure. An example of a family for which $E(X,T)$ equals the set of measurable eigenvalues is the family of primitive substitution shifts (see [Hos86] and also see below), but in general the set of measurable eigenvalues strictly contains the continuous ones. See for example [DL96] and the comments at the end of Section 6.2.
Let $v, w$ be two words. The cylinder $[v, w]$ is defined as the set $\{x \in X \mid x|_{[v, w]} = vw\}$ and the cylinder $[w]$ is the set $\{x \in X \mid x|_{[0, w]} = w\}$. Cylinders are clopen sets. If $w$ is a letter, the cylinder is called a one-letter cylinder, and the same holds for two letters. Let $\mathcal{M}(X, T)$ be the set of all $T$-invariant probability measures on $(X, T)$. Let $I$ be the additive group generated by the measures of cylinders, i.e.,

$$I(X, T) = \bigcap_{\mu \in \mathcal{M}(X, T)} \left\{ \int f \, d\mu : f \in C(X, \mathbb{Z}) \right\}.$$ 

One has $E(X, T) \subset I(X, T)$ by [CDP16, Proposition 11] and [GHH18, Corollary 3.7]. In particular, if $(X, T)$ is uniquely ergodic with unique $T$-invariant probability measure $\mu$, then $I(X, T) = \{ \int f \, d\mu : f \in C(X, \mathbb{Z}) \} = \{ \mu([w]) : w \in \mathcal{L}(X) \}$. Sufficient conditions for the additive group generated by measures of one-letter cylinders being included in $E$ under the assumption of unique ergodicity are given in Section 4.3.

The following result relating eigenvalues for one-sided and two-sided shifts is graciously provided by Karl Petersen. Note that a measure $\tilde{\mu}$ on a one-sided shift space defines $\mu$ on the natural extension. Both measures give the same mass to cylinders, the only difference is that they are defined on different spaces; see [CFS82, Chapter 10, Section 4] for details.

**Proposition 2.1.** Let $(\tilde{X}, T)$ be a one-sided minimal shift, and let $(X, T)$ be its natural extension. Then

- $E(\tilde{X}, T) = E(X, T)$.
- Let $\tilde{\mu}$ be a shift invariant measure on $(\tilde{X}, T)$ and $\mu$ the corresponding measure on $(X, T)$.

Then $\lambda$ is a measurable eigenvalue for $(\tilde{X}, T, \tilde{\mu})$ if and only if it is also one for $(X, T, \mu)$.

**Proof.** One easily checks that $E(\tilde{X}, T) \subset E(X, T)$. Conversely, suppose that $\lambda$ is a continuous eigenvalue of $(X, T)$ and let $f \in C(X)$ be a corresponding eigenfunction with $||f||_\infty = 1$. Given $\tilde{x} \in \tilde{X}$, let $x \in X$ agree with $\tilde{x}$ on the non-negative indices. Choose such an $x$ and fix it. Define $\tilde{f} : \{T^n(\tilde{x}) : n \in \mathbb{N}\} \to \mathbb{C}$ by $\tilde{f}(T^n(\tilde{x})) = \lambda^n(f(x))$. If we can show that $\tilde{f}$ is uniformly continuous on $\{T^n(\tilde{x}) : n \in \mathbb{N}\}$, then we are done as we can then extend $\tilde{f}$ to a continuous eigenfunction on $\tilde{X}$. (Note that we only need the existence of a point with a dense orbit.)

As $\tilde{f}$ is continuous on a compact space, so it is uniformly continuous. Thus for each $\varepsilon > 0$ there is a $k$ such that for all $y, z \in X$, if $y|_{-k, k} = z|_{-k, k}$, then $||f(y) - f(z)|| < \varepsilon$. Now if $T^n(\tilde{x})|_{0, 2k+1} = T^m(\tilde{x})|_{0, 2k+1}$, then by definition $T^{n+k}(x)|_{-k, k} = T^{m+k}(x)|_{-k, k}$, so that one has $||f(T^{n+k}(x)) - f(T^{m+k}(x))|| < \varepsilon$. Therefore

$$||\tilde{f}(T^n(\tilde{x})) - \tilde{f}(T^m(\tilde{x}))|| = ||\lambda^k \tilde{f}(T^n(\tilde{x})) - \lambda^k \tilde{f}(T^m(\tilde{x}))|| = ||\lambda^{k+n} f(x) - \lambda^{k+m} f(x)|| = ||f(T^{n+k}(x)) - f(T^{m+k}(x))|| < \varepsilon,$n

and $\tilde{f}$ is uniformly continuous on the orbit of $\tilde{x}$.

As $\tilde{\mu}$ and $\mu$ give the same mass to cylinders, the second statement then follows from [SNF58, Theorem 1], combined with the fact that the unitary operator $f \mapsto f \circ T$ on $L^2(X, T, \mu)$ is the minimal unitary dilation of the isometric operator on $L^2(\tilde{X}, T, \tilde{\mu})$ [Cam86, page 385].

A dynamical system $(Z, F)$ is called equicontinuous if the family $\{F^n : n \in \mathbb{Z}\}$ is equicontinuous. An equicontinuous system $(Z, F)$ which is minimal must be a minimal rotation, that is, there is a continuous abelian group structure on $Z$ and an element $g \in Z$ such that the homeomorphism $F$ is given by adding $g$, i.e., $F(z) = z + g$. Moreover, the orbit $\{ng : n \in \mathbb{Z}\}$ is dense in $Z$, i.e., $g$ is a topological generator of $Z$. Recall that $\pi : (X, T) \to (Z, F)$ is a factor map if it is continuous, onto, and $\pi \circ T = F \circ \pi$. If $(Z, +g)$ is equicontinuous and there is a factor map $\pi : (X, T) \to (Z, +g)$, we say that $(Z, +g)$ is an equicontinuous factor of $(X, T)$. Any $\mathbb{Z}$-action $(X, T)$ admits a maximal equicontinuous factor $\pi : (X, T) \to (Z, +g)$. This equicontinuous factor must be maximal in the sense that any equicontinuous factor of $(X, T)$ factors through it, and it encodes all continuous eigenvalues of $(X, T)$. 

□
Let $\sigma : A^* \to A^+$ be a morphism, also called a substitution. Note that the image of any letter is a non-empty word. We will abuse notation and write $\sigma : A \to A^+$. Using concatenation, we extend $\sigma$ to $A^{|}$ and $A^{|}$. The finiteness of $A$ guarantees that $\sigma$-periodic points, i.e., points $x$ such that $\sigma^k(x) = x$ for some positive $k$, exist. The incidence matrix of the substitution $\sigma$ is the $|A| \times |A|$ matrix $M_\sigma = (m_{ij})$ with $m_{ij}$ being the number of occurrences of $i$ in $\sigma(j)$. A substitution is primitive if its incidence matrix admits a power with positive entries. A substitution $\sigma$ is left-(right-) proper if for each letter $a$, $\sigma(a)$ starts (ends) with the same letter, and it is proper if it is both left- and right-proper. Given a substitution $\sigma : A \to A^+$, the language $L_\sigma$ defined by $\sigma$ is

$$L_\sigma = \{ w \in A^* : w \text{ is a subword of } \sigma^n(a) \text{ for some } a \in A \text{ and } n \in \mathbb{N} \}.$$ 

If $\sigma$ is primitive, then each word in $L_\sigma$ is left- and right-extendable, and $L_\sigma$ is closed under the taking of subwords, so we define $X_\sigma := X_{L_\sigma}$. We call $(X_\sigma, T)$ a substitution shift and the language of a substitution shift is called a substitution language. The substitution $\sigma$ is aperiodic if $(X_\sigma, T)$ is aperiodic.

**Definition 2.2** (Transition and return words). Let $L$ be a language on $A$, and let $a, b \in A$. Suppose that the word $w \in L$ starts with $a$ and is such that $wb$ belongs to $L$. Then $w$ is called a transition word from $a$ to $b$. Furthermore if $wb$ contains exactly one occurrence of each of $a$ and $b$, then we say that $w$ is a strict transition word from $a$ to $b$. If $a = b$, $w$ is also called a return word or a strict return word to $a$.

For example, if $accd\in L$, then $accd$ is a strict transition word from $a$ to $b$. Also, if $aa \in L$, then $a$ is a strict return word to $a$.

A subshift $(X, T)$ is linearly recurrent if there exists a constant $L$ such that for any word $w$ belonging to $L_X$, the length of any strict return word to $w$ is of length at most $L|w|$. Examples of linearly recurrent shifts are primitive substitution shifts [Dur00].

2.2. $S$-adic shifts. We recall basic definitions concerning $S$-adic shifts. They are obtained by replacing the iteration of a single substitution by the iteration of a sequence of morphisms, which are defined like substitutions, except that letters in $A$ are mapped to non-empty words on a possibly different alphabet $B$. In this article we restrict to the case of an $S$-adic shift defined over one alphabet $A$, however in discussing other works, we refer to morphisms if that is their context.

Let $\sigma = (\sigma_n)_{n \geq 0}$ be a sequence of substitutions with $\sigma_n : A \to A^+$; we call $\sigma$ a directive sequence. For $0 \leq n < N$, let

$$\sigma_{[n, N)} = \sigma_n \circ \sigma_{n+1} \circ \cdots \circ \sigma_{N-1}.$$ 

For $n \geq 0$, define

$$\tilde{L}_\sigma^{(n)} = \{ w \in A^* : w \text{ is a subword of } \sigma_{[n, N)}(a) \text{ for some } a \in A, \text{ for some } N > n \}.$$ 

Define

$$X_{\sigma}^{(n)} : = \{ x \in A^| : \text{ for each } k \leq \ell, x[k, \ell] \in \tilde{L}_\sigma^{(n)} \}$$

and note that $\tilde{L}_\sigma^{(n)} := \{ w : w \text{ appears as some subword in some } x \in X_{\sigma}^{(n)} \}$ is a subset of $\tilde{L}_\sigma^{(n)}$, but it can be a proper subset. We call $(\tilde{L}_\sigma^{(n)})_{n \geq 0}$ the sequence of languages associated to $\sigma$.

We say that $\sigma$ is primitive if for each $n \geq 0$ there is an $N > n$ such that the incidence matrix $M_{[n, N)} := M_{\sigma_n} M_{\sigma_{n+1}} \cdots M_{\sigma_{N-1}}$ of $\sigma_{[n, N)}$ is a positive matrix. Under the assumption of primitivity for $\sigma$, each word in $\tilde{L}_\sigma^{(n)}$ is left- and right-extendable. Note that if $\sigma$ is primitive, then each letter appears in each language $\tilde{L}_\sigma^{(n)}$. If $\sigma$ is primitive, then $(X_{\sigma}^{(n)}, T)$ is minimal for all $n$, by [Dur00, Lemma 7], and $\tilde{L}_\sigma^{(n)} = L_{\sigma}^{(n)}$ for each $n$. As we only work with minimal shifts in this article, even if we do not always need to work with primitive directive sequences, we assume throughout that $\sigma$ is primitive, and hence that each letter in $A$ appears in each language $L_{\sigma}^{(n)}$. Nevertheless, some of our results hold with the weaker assumption of minimality of $(X_{\sigma}, T)$.

To abbreviate notation, we set $X_{\sigma} = X_{\sigma}^{(0)}$ and call $(X_{\sigma}, T)$ the $S$-adic shift generated by the directive sequence $\sigma$. The directive sequence $\sigma$ is everywhere growing if for each $a \in A$, $|\sigma_{[0, n)}(a)| \to \infty$ as $n$ tends to infinity. If a directive sequence is primitive, then it is everywhere growing.
COBOUNDARIES AND EIGENVALUES OF FINITARY S-ADIC SYSTEMS

1. Introduction

We consider two-sided dynamical recognizability. Notation two-sided recognizability does not in general imply one-sided recognizability. Also, we are concerned with dynamical recognizability, and not Mossé’s combinatorial version [Mos92, Mos96] (based on cutting points for finite words), and while dynamical recognizability implies combinatorial recognizability [BSTY19, Theorem 2.5], the converse is false. A two-sided aperiodic primitive substitution shift is recognizably in both senses, but not necessarily one-sided recognizable [Mos92, Mos96]. There even exist substitutions that are constant-length and injective on letters, that are combinatorially one-sided recognizable, but not dynamically one-sided recognizable as in the following example provided to us by Dominique Perrin (see also [BBPR22]).

Example 2.5. Consider

\[ a \xrightarrow{\sigma} ba \]

\[ b \xrightarrow{\sigma} aa. \]

Since \( \sigma \) is a primitive, aperiodic and constant-length substitution which is injective on the letters, it is one-sided recognizable in the combinatorial sense of Mossé [Mos92]. However, the sequence \( x = a\sigma(a)\sigma^2(a)\sigma^3(a)\cdots \in X_\sigma \) has two different centered \( \sigma \)-representations in \( X_\sigma \), namely \( x = T(\sigma(ax)) = T(\sigma(bx)) \), thus it is not dynamically one-sided recognizable.

In what follows, when we write that \( \sigma \) is recognizable, we mean that the corresponding two-sided shift is (dynamically) recognizable.

A morphism \( \sigma : A \to A^+ \) is elementary if it cannot be written as \( \sigma = \gamma \circ \delta \), with \( \delta : A \to B^+ \) and \( |B| < |A| \). A morphism \( \sigma : A \to B^+ \) is left- (right-) permutative if the first (last) letters of \( \sigma(a) \) and \( \sigma(b) \) are different for all distinct \( a,b \in A \). Two morphisms \( \sigma, \tilde{\sigma} : A \to B^+ \) are rotationally conjugate if there is a word \( w \in B^+ \) such that \( \sigma(a)w = \tilde{\sigma}(a)w \) for all \( a \in A \) or \( w\sigma(a) = \tilde{\sigma}(a)w \) for all \( a \in A \). We state here conditions on morphisms in a directive sequence \( (\sigma_n)_{n \geq 0} \) of morphisms, which guarantee recognizability. This is one of the results on recognizability from [BSTY19, Theorem 3.1], with the more general result concerning elementary morphisms obtained earlier by Karhumäki, Manušč and Plandowski in [KMP03], as discussed in [BPR23]. Note that there is a decision procedure to conclude whether a constant-length substitution generates an aperiodic fixed point [ARS09]. Since we work with morphisms on a fixed alphabet, the case that concerns us is \( \mathcal{A}_n = \mathcal{A} \) for each \( n \).

2. Recognizability

We start with the notion of recognizability which expresses the idea of performing a “desubstitution”.

Definition 2.4 (Dynamical recognizability, \( \sigma \)-representations and recognizable directive sequences). Let \( \sigma : A \to A^+ \) be a substitution and \( y \in A^\mathbb{Z} \). If \( y = T^k \sigma(x) \) with \( x = (x_n)_{n \in \mathbb{Z}} \in A^\mathbb{Z} \), and \( 0 \leq k < |\sigma(x_0)| \), then we say that \( (k,x) \) is a (centered) \( \sigma \)-representation of \( y \). For \( X \subseteq \mathbb{A}^\mathbb{Z} \), we say that the \( \sigma \)-representation \( (k,x) \) is in \( X \) if \( x \in X \).

Given a nonempty \( X \subseteq \mathbb{A}^\mathbb{Z} \) and \( \sigma : A \to A^+ \), we say that \( \sigma \) is recognizable in \( X \) if each \( y \in \mathbb{A}^\mathbb{Z} \) has at most one centered \( \sigma \)-representation in \( X \). A directive sequence \( \sigma \) is recognizable at level \( n \) if \( \sigma_n \) is recognizable in \( X^{\sigma(n+1)} \). The sequence \( \sigma \), or the \( S \)-adic shift \( X_\sigma \) if the sequence \( \sigma \) is given, is recognizable if each \( x \in X_{T^n(\sigma)} \) has exactly one \( \sigma_n \)-centred representation in \( X_{T^{n+1}(\sigma)} \).

3. Recognizability growing. We also say that \( \sigma \) is strongly primitive if there exists \( r \) such that, for each \( n \), \( \sigma_{[n,n+r]} \) has a positive incidence matrix.

We consider particular families of directive sequences.

Definition 2.3. Let \( \sigma = (\sigma_n)_{n \geq 0} \) be a directive sequence. We say that \( \sigma \) is

- finitary if there is a finite set \( S \) such that \( \sigma_n \in S \) for each \( n \);
- stationary if there exists a substitution \( \sigma \) such that \( \sigma_n = \sigma \) for all \( n \);
- constant-length, with length sequence \( (q_n)_{n \geq 0} \) if for each \( n \), \( \sigma_n \) has constant-length \( q_n \) (note that the sequence \( q_n \) is not necessarily constant);
- unimodular if for each \( n \), the incidence matrix \( M_n \) of \( \sigma_n \) is unimodular, i.e., \( |M_n| = \pm 1 \).

2.3. Recognizability. We first start with the notion of recognizability which expresses the idea of performing a “desubstitution”.
Theorem 2.6. Let $\sigma = (\sigma_n)_{n \geq 0}$ be a primitive sequence of morphisms with $\sigma_n : A_{n+1} \to A_n^+$. Suppose that $X_\sigma$ is aperiodic. If each morphism $\sigma_n$ is elementary, then $\sigma$ is recognizable. In particular, if each morphism $\sigma_n$ satisfies one of

- $\text{rk}(M_{\sigma_n}) = |A_{n+1}|$, or
- $|A_{n+1}| = 2$, or
- $\sigma_n$ is (rotationally conjugate to) a left- or right-permutative morphism,

then $\sigma$ is recognizable.

Example 2.7. We will use this running example to clarify definitions and results. Consider the substitutions $S = \{\sigma, \tau\}$ with

\[
\begin{align*}
  a & \mapsto abc & a & \mapsto abc \\
  b & \mapsto bab & b & \mapsto acb \\
  c & \mapsto cbc & c & \mapsto aac
\end{align*}
\]

We will work throughout this example with the directive sequence $\sigma$, where

$\sigma = \sigma, \tau, \sigma, \tau, \sigma, \sigma, \tau, \sigma, \sigma, \tau, \sigma, \tau, \sigma, \ldots$

This directive sequence is introduced by Durand in [Dur03] to produce a finitary strongly primitive constant-length directive sequence whose associated shift $(X_\sigma, T)$ is minimal, not linearly recurrent, and hence aperiodic, and has linear subword complexity\(^1\). This directive sequence is also recognizable, applying Theorem 2.6 to the aperiodic shift $(X_\sigma, T)$. To see this, note that $\sigma$ is left-permutative, and $\tau$ is rotationally conjugate to a left-permutative substitution: its middle column contains all letters, and its first column consists of one letter.

The relevance of recognizability is that it gives us a framework within which to approximate our dynamical system, in terms of generating partitions, see Section 2.4. We end this section with a statement of [BSTY19, Theorems 6.5 & 6.7], stated below as Theorem 2.8, which we will use when comparing our results to those in the literature, specifically those concerning Bratteli-Vershik systems and also finite-rank (cutting-and-stacking) systems. We assume that the reader is familiar with those systems, and refer them to [BSTY19, Section 6], or [BKMS13] for definitions and terminology concerning Bratteli-Vershik systems and [Fer97] concerning finite rank systems, recalling only a few definitions, used mainly for Theorem 2.8.

A Bratteli diagram is an infinite graph $B = (V, E)$ such that the vertex set $V = \bigcup_{n \geq 0} V_n$ and the edge set $E = \bigcup_{n \geq 0} E_n$ are partitioned into pairwise disjoint, non-empty subsets $V_n$ and $E_n$, where

(i) $V_0 = \{v_0\}$ is a single point;
(ii) $V_n$ and $E_n$ are finite sets;
(iii) there exists a range map $r : E \to V$ and a source map $s : E \to V$ such that $r(E_n) = V_{n+1}$ and $s(E_n) = V_n$ for each $n \geq 0$.

A finite or infinite sequence of edges $(e_n)$ with $e_n \in E_n$ such that $r(e_n) = s(e_{n+1})$ is called a finite or infinite path, respectively. An ordered Bratteli diagram is a Bratteli diagram together with a linear ordering on $r^{-1}(v)$ for each $v \in V \setminus V_0$. Given a directive sequence $\sigma$, we can define an associated natural ordered Bratteli diagram, as follows. For $n \geq 1$, $V_n$ is a copy of $A_{n-1}$. Since we work with a uniform alphabet, this means that $V_1$ is fixed for each $n$. Given a vertex $v \in V_{n+1}$ labelled by the letter $a \in A_n$, we order the edges with range $a$ as follows: if $b$ is the $j$-th letter in $\sigma_{n-1}(a)$, then we label an edge with source $b$ and range $a$ with $j-1$. Such an ordered Bratteli diagram allows the definition of a measurable Bratteli-Vershik dynamical system (see [BSTY19, Section 6]). Note that the ordered Bratteli diagram associated to $\sigma$ has much in common with prefix-suffix automata, with labels of edges in the $n$-level of the Bratteli diagram being replaced by prefixes of the words $\sigma_n(a)$; see for example [CS01a].

\(^1\)The subword complexity of a subshift $(X, T)$ is the function $n \mapsto p_X(n)$ that counts the number of words of length $n$ that belong to its language.
We say that a transformation \( \Phi : (X, T) \rightarrow (Y, S) \) is an almost-conjugacy (also called essential-conjugacy) if there is a \( T \)-invariant set \( D \subset X \) and an \( S \)-invariant set \( E \subset Y \), with \( \Phi : X \setminus D \rightarrow Y \setminus E \) a continuous bijection satisfying \( \Phi \circ T = S \circ \Phi \), and such that \( D \) (resp. \( E \)) has zero measure for every fully supported invariant probability measure on \( (X, T) \) (resp. \( (Y, S) \)). Thus, if \( (X, T) \) and \( (Y, S) \) are almost conjugate, \( \nu \) is any fully supported probability measure on \( X \) that is preserved by \( T \), \( \mu \) is any fully supported probability measure on \( Y \) that is preserved by \( S \), and \( \Phi \) maps \( \nu \) to \( \mu \), then \( (X, T, \nu) \) and \( (Y, S, \mu) \) are conjugate in measure. We can therefore apply [BSTY19, Lemma 6.4] to obtain the following, which is [BSTY19, Theorem 6.5, Corollary 6.7]. We refer to [KS67, Fer97] for details on cutting-and-stacking measurable transformations.

**Theorem 2.8.** Let \( \sigma \) be an everywhere growing recognizable directive sequence defined on \( A \) and \( (X_B, \varphi_\omega) \) the natural Bratteli-Vershik dynamical system associated to \( \sigma \). We assume that \((X_\sigma, T)\) aperiodic. Then

- the system \((X_\sigma, T)\) is almost-conjugate to \((X_B, \varphi_\omega)\), and
- if \(|A| = k\), and \( \nu \) is any fully supported probability measure such that \((X_\sigma, T, \nu)\) is measure preserving, then \((X_\sigma, T, \nu)\) is measurably conjugate to a cutting-and-stacking measurable transformation of a Lebesgue space which is of measurable rank at most \( k \).

The model theorem of [HPS92] tells us that any minimal \( S \)-adic shift has a proper Bratteli-Vershik representation, i.e., a Bratteli-Vershik representation where \( X_B \) has only one maximal and one minimal path, so that the successor map \( \varphi_\omega \) is a homeomorphism. If in addition, this proper representation defines a recognizable directive sequence, then it defines a proper \( S \)-adic representation. Note though that this proper \( S \)-adic representation may no longer be finitary, even if the original directive sequence is. By choosing to work with a non-proper representation we take advantage of the finitary nature of the directive sequence.

2.4. Generating partitions. Substitution dynamical systems have received much attention in the past decades, and this is in no small part due to the fact that aperiodic substitution shifts possess a natural sequence of partitions \((Q_n)\), consisting of Rokhlin towers, which generates in measure (see below for a definition). This is a consequence of the fact that \( \sigma \) is recognizable [Mos92, Mos96, BKM09]. We refer the reader to [Fog02, Que10] for expositions of the basic aspects of these systems and all the undefined terms we use below. This extends to the \( S \)-adic case and we present a sequence of partitions \((Q_n)\) relying here also on the recognizability property, such as described in Section 2.3.

Given a primitive directive sequence \( \sigma \) such that \( X_\sigma \) is aperiodic, and \( n \in \mathbb{N} \), for a letter \( a \in A \) define

\[
B_n(a) := \sigma_{[0,n)}([a])
\]

and for a word \( w \in A^* \), define

\[
h_n(w) := |\sigma_{[0,n)}(w)|.
\]

Let

\[
Q_n = \{T^k \sigma_{[0,n)}([a]) : 0 \leq k < h_n(a)\}.
\]

Note that for all \( n, Q_n \) is a cover of \( X_\sigma \). Indeed, for every \( \ell \in \mathbb{N} \)

\[
X^{(\ell)}_\sigma = \{T^k \sigma_{[0,n)}(x) : x \in X^{(\ell+1)}_\sigma, 0 \leq k < |\sigma_{\ell}(x_0)|\}
\]

(this equality is true without assuming recognizability), thus by iterating

\[
X_\sigma = \{T^k \sigma_{[0,n)}(x) : x \in X^{(n)}_\sigma, 0 \leq k < h_n(x_0)\}.
\]

Then, using the partition of \( X^{(n)}_\sigma \) by cylinders \([a], a \in A \), we get that \( Q_n \) is a cover.

We call \( T_n(a) := \cup_{0 \leq k < h_n(a) } T^k \sigma_{[0,n)}([a]) \) the \( n \)-tower defined by \( a \), we call \( T^k \sigma_{[0,n)}([a]) \) the \( k \)-th level of this tower, and \( B_n(a) \) its base. Thus the elements of \( Q_n \) can be arranged to form \(|A|\) towers. If \( \sigma \) is everywhere growing, then the height \( h_n(a) \) of each \( n \)-tower \( T_n(a) \) increases to \( \infty \) as \( n \) grows.

We say that a sequence \((k_n, a_n)_{n \in \mathbb{N}} \) is a \((Q_n)\)-address for \( x \in X_\sigma \) if \( x \in T^{k_n} \sigma_{[0,n)}([a_n]) \) for each \( n \in \mathbb{N} \), where \( 0 \leq k_n < |\sigma_{[0,n)}(b_n)| \). Each point \( x \in X_\sigma \) has at least one \((Q_n)\)-address.
Let $\mu$ be a shift invariant probability measure on $X_\sigma$. The covers $(Q_n)_{n=0}^\infty$ are generating in measure if $\mu$-almost every $x \in X_\sigma$ has a $(Q_n)$-address that uniquely determines $x$.

We next identify the importance of recognizability in ensuring that $(Q_n)$ is a sequence of partitions. Indeed it is straightforward to check that if $\sigma$ is recognizable, then each $Q_n$ is a partition of $X_\sigma$, i.e., each point has exactly one $(Q_n)$-address. Note also that since $\sigma_n(X_\sigma^{(n+1)}) \subseteq X_\sigma^{(n)}$, the sequence $(Q_n)_{n=0}^\infty$ is nested, i.e., for each $n \in \mathbb{N}$, every element of $Q_{n+1}$ is a subset of some element of $Q_n$. We summarise this as follows.

**Lemma 2.9.** Let $\sigma$ be a sequence of substitutions defined on $A$. If $\sigma$ is recognizable, then the sequence of covers $(Q_n)_{n=0}^\infty$ defined in (2.2) is a nested sequence of partitions of $X_\sigma$.

We will see in Lemma 5.1 that the sequence $(Q_n)$ generates in measure, and therefore we can use it when we investigate measurable eigenvalues in Section 5.

Incidence matrices of substitutions allow a partial description of towers. Indeed, the incidence matrix $M_{\sigma_{n-1}} = (m_{ij})$ for $\sigma_{n-1}$ gives us partial information about how elements from $Q_n$ are built from elements of $Q_{n-1}$. For, recalling that $m_{ab}$ equals the number of occurrences of the letter $a$ in the image of the letter $b$ under $\sigma_{n-1}$, $T_n(a)$ consists of $m_{ba}$ copies of a subtower of $T_{n-1}(b)$. What the incidence matrix does not tell us is in which order we stack these subtowers; the order of arrangement of the subtowers in $T_n(a)$ is given by $\sigma_{n-1}(a)$. Indeed, define $t_n(a,b) := \{0 \leq t < h_n(a) : T^n(B_n(a)) \subset B_{n-1}(b)\}$. One has $T^n(B_n(a)) \subset B_{n-1}(b)$ if and only if $b$ occurs as the $j$-th letter of $\sigma_{n-1}(a)$ and $t = h_{n-1}(\sigma_{n-1}(a)_{[0,j]})$. This implies that $t_n(a,b) \neq \emptyset$ if and only if $b$ appears in $\sigma_{n-1}(a)$, and that the cardinality of $t_n(a,b)$ equals the $(b,a)$ entry $m_{ba}$ of $M_{\sigma_{n-1}}$. See Figure 1 for an illustration, and [BR10, Chapter 6] for an exposition.

![Figure 1](image-url)  
In this example, we construct $T_n(a)$ with $\sigma_{n-1}(a) = cbd$, by concatenating portions of the towers of level $n - 1$ for $c$, $b$ and then $d$.

3. From combinatorics to spectral theory

We first introduce in Section 3.1 two key combinatorial notions here, namely the notions of essential words and of straightness for discussing limit words. We then provide some families of
examples in Section 3.2 and discuss in Section 3.3 eigenvalues and coboundaries for stationary directive sequences, i.e., for substitutions. Lastly, Section 3.4 deals with the notion of height for constant-length substitutions.

3.1. Limit words and straightness. We identify some distinguished points in \(X_\sigma\), namely limit words, which are analogues of substitution fixed points.

**Definition 3.1** (Essential and fully essential words). We say that a word \(w\) is essential (for \(\sigma\)) if it occurs in \(\mathcal{L}_\sigma^{(n)}\) for infinitely many \(n\). An essential word is fully essential if it occurs in \(\mathcal{L}_\sigma^{(n)}\) for each \(n\).

**Example 3.2.** Consider the substitutions

\[
\begin{align*}
a \mapsto aaba, & \quad a \mapsto bbbab \quad (\text{for } a) \\
b \mapsto ababa, & \quad b \mapsto babab. 
\end{align*}
\]

The words \(ab\) and \(ba\) are fully essential for any directive sequence. The word \(aa\) appears in \(\mathcal{L}_\sigma^{(n)}\) if and only if \(\sigma_n = \tau_1\), and \(bb\) appears in \(\mathcal{L}_\sigma^{(n)}\) if and only if \(\sigma_n = \tau_2\). Now take any directive sequence \(\sigma\) from the set \(\{\tau_1, \tau_2\}\). Then \(aa\) (respectively \(bb\)) is an essential word for \(\sigma\) if and only if we see \(\tau_1\) (respectively \(\tau_2\)) infinitely often in \(\sigma\). Note that \(aa\) (respectively \(bb\)) appears in \(\mathcal{L}_\sigma^{(0)}\) if and only if \(\sigma_0 = \tau_1\) (respectively \(\sigma_0 = \tau_2\)). In particular, essential words do not necessarily appear in \(\mathcal{L}_\sigma^{(0)}\).

**Example 3.3.** Recall the directive sequence

\[
\sigma, \tau, \sigma, \tau, \sigma, \sigma, \tau, \sigma, \sigma, \ldots
\]

from Example 2.7. It can be verified that \(\{ab, ac, ba, bc, ca, cb\}\) are all fully essential. For example, we show that \(ca\) is fully essential. We have \(ab \in \mathcal{L}_\sigma^{(k)}\) for each \(k\). Therefore, if \(\sigma_{k-1} = \tau\), we have \(ca \in \mathcal{L}_\sigma^{(k-1)}\), since \(\tau\) occurs as an isolated letter in the directive sequence \(\sigma\). Also, if \(ca \in \mathcal{L}_\sigma^{(k-1)}\) and \(\sigma_{k-2} = \sigma\), then \(ca \in \mathcal{L}_\sigma^{(k-2)}\). Hence \(ca \in \mathcal{L}_\sigma^{(j)}\) for all \(j \leq k - 1\). Since \(\tau\) appears infinitely often, the claim follows.

Otherwise, the word \(bb\) is essential, but does not appear at any level \(\mathcal{L}_\sigma^{(k)}\) where \(\sigma_k = \tau\), and \(aa\) is essential, only appearing in \(\mathcal{L}_\sigma^{(k)}\) when \(\sigma_k = \tau\). Finally \(cc\) does not belong to any \(\mathcal{L}_\sigma^{(k)}\).

Telecoping a directive sequence \((\sigma_n)_{n \geq 0}\) means taking a sequence \((n_k)_{k \geq 1}\) and considering instead the directive sequence \((\tilde{\sigma}_k)\) where \(\sigma_0 = \sigma_{[0,n_1]}\) and \(\tilde{\sigma}_k = \sigma_{[n_k,n_{k+1}]}\) for \(k \geq 1\). Telecoping a directive sequence does not change the dynamics, i.e., \(X_\sigma^{(0)} = X_{\tilde{\sigma}}^{(0)}\). Note that the telescoped sequence \(\tilde{\sigma}\) may not have the same set of essential words as the original directive sequence, in particular it may have fewer essential words.

Suppose that the word \(ab\) is essential. If there is a sequence \((n_k)_{k \geq 1}\) such that for each \(k\), \(ab \in \mathcal{L}_\sigma^{(n_k)}\), \(\sigma_{[0,n_k]}(a)\) shares a common prefix with \(\sigma_{[0,n_{k+1}]}(a)\) of increasing length, and \(\sigma_{[0,n_k]}(b)\) also shares a common suffix with \(\sigma_{[0,n_{k+1}]}(b)\) of increasing length, then the sequence of finite words \((\sigma_{[0,n_k]}(a) \cdot \sigma_{[0,n_k]}(b))_{k \geq 1}\) converges to a bi-infinite sequence in \(X_\sigma\). We denote it by \(\lim_k \sigma_{[0,n_k]}(a) : \lim_k \sigma_{[0,n_k]}(b)\). Here the indices to the right of the radix point \(\ldots\) start at 0, i.e., \(x \cdot y = \ldots x_{-1}y_0y_1\ldots\). The same convergence property holds similarly for one-sided words. Recall that we assume that each letter in \(\mathcal{A}\) appears in each language \(\mathcal{L}_\sigma^{(n)}\).

**Definition 3.4** (Limit word). We say that \(x\) in the two-sided shift \(X_\sigma\) is a limit word if \(x = \lim_k \sigma_{[0,n_k]}(a) : \lim_k \sigma_{[0,n_k]}(b)\) for some essential word \(ab\) that belongs to \(\mathcal{L}_\sigma^{(n_k)}\) for each \(k\). We say that an element \(x\) in the one-sided shift \(\hat{X}_\sigma\) is a limit word if \(x = \lim_k \sigma_{[0,n_k]}(a)\) for some sequence \((n_k)\).

For each \(n\), two-sided limit words of recognizable directive sequences belong to the base of some \(n\)-tower in the partition \(Q_n\) from (2.2). Note that if the word \(ab\) is essential, and \(\sigma\) is everywhere growing, then compactness implies that there is at least one sequence \((n_k)\) such that

\[
\lim_k \sigma_{[0,n_k]}(a) : \lim_k \sigma_{[0,n_k]}(b)
\]
belongs to $X_{\sigma}$. Different sequences $(n_k)$ may lead to different limit words; this motivates the definition of straightness below. We define

$$a \cdot b := \{ u \in \mathcal{A}^\mathbb{Z} : \exists (n_k) \text{ such that } ab \in \mathcal{L}_{\sigma}^{(n_k)} \text{ for each } k \text{ and } u = \lim_{k} \sigma_{[0,n_k]}(a) \cdot \lim_{k} \sigma_{[0,n_k]}(b) \}$$

and similarly

$$a := \{ u \in \mathcal{A}^\mathbb{N} : \exists (n_k) \text{ such that } u = \lim_{k} \sigma_{[0,n_k]}(a) \}.$$

If $a \cdot b$ consists of a unique limit word, we use $a \cdot b$ to denote this limit word. Similarly if $a$ is a singleton we use it to denote that limit word. We will work mainly with one-sided limit words in the following.

We note that there are alternative definitions of limit words, such as discussed e.g. in [BSTY19, Section 4]. Indeed a second and natural definition of a limit word consists in considering elements of $\bigcap_{n \in \mathbb{N}} \sigma_{[0,n]}(\mathcal{A}^\mathbb{Z})$ (this is case for instance in [AMS14, NPF20]). Observe that this definition of a limit word can yield a different shift $X_{\sigma}$ to ours, already in the non-minimal substitutive case such as stressed in [AS14]. Indeed take the constant sequence $\sigma$ taking the constant value $\sigma : 0 \mapsto 00, 1 \mapsto 11$ on the alphabet $\{0,1\}$. Then $\ldots 0011 \ldots$ does not belong to $X_{\sigma}$, but does belong to $\bigcap_{n \in \mathbb{N}} \sigma^{n}(\{0,1\}^\mathbb{Z})$. However, we do have the following lemma, which partly links these two notions of a limit word.

**Lemma 3.5.** Let $\sigma$ be an everywhere growing directive sequence defined on the alphabet $\mathcal{A}$. If $u^{(0)} \in \bigcap_{n \in \mathbb{N}} \sigma_{[0,n]} \left( X_{\sigma}^{(n)} \right)$, then there exist $a, b \in \mathcal{A}$ such that $u^{(0)} \in a \cdot b$. Conversely, if $u^{(0)} \in a \cdot b$ for $a, b \in \mathcal{A}$, then there is a sequence $(m_k)$ such that $u^{(0)} \in \bigcap_{k} \sigma_{[0,m_k]} \left( X_{\sigma}^{(m_k)} \right)$.

**Proof.** By hypothesis, for each $n$ there exists $u^{(n)} \in X_{\sigma}^{(n)}$ such that $u^{(0)} = \sigma_{[0,n]}(u^{(n)})$. We can pass to a subsequence $(n_k)$, so that there exist letters $a, b$ with $ab$ such that $u^{(n_k)}_{[-1,0]} = ab$ in $\mathcal{L}_{\sigma}^{(n_k)}$ for each $k$. In particular $ab$ is essential. The first part of the result follows using the assumption that $\sigma$ is everywhere growing.

Conversely, if $u^{(0)} \in a \cdot b$, then by definition, there is a sequence $(n_k)$ such that $ab$ belongs to $\mathcal{L}_{\sigma}^{(n_k)}$ for each $k$ and $u^{(0)} = \lim_{k} \sigma_{[0,n_k]}(a) \cdot \lim_{k} \sigma_{[0,n_k]}(b)$. Let $u^{(n)} = \lim_{k} \sigma_{[n,m_k]}(a) \cdot \lim_{k} \sigma_{[n,m_k]}(b)$. One has $u^{(n)} \in X_{\sigma}^{(n)}$. The result follows. □

The next lemma shows that there are finitely many limit words for the systems we consider.

**Lemma 3.6.** Let $\sigma$ be a sequence of substitutions defined on $\mathcal{A}$. Then for each letter, the set $a$ contains at most $|\mathcal{A}|$ points, and for each essential word $ab$, the set $a \cdot b$ contains at most $|\mathcal{A}|^2$ points.

**Proof.** Fix $a \in \mathcal{A}$. Consider an element $\lim_{k} \sigma_{[0,n_k]}(a)$ of $a$. For each $m$, if $n_k > m$, the word $\sigma_{[0,n_k]}(a)$ admits as a prefix one of the $|\mathcal{A}|$ words $\sigma_{[0,m]}(a)$, $a \in \mathcal{A}$. Thus there is a set of one-sided sequences of cardinality at most $|\mathcal{A}|$, such that whenever $n_k \to \infty$, $\lim_{n_k \to \infty} \sigma_{[0,n_k]}(a)$ exists and is a sequence, then it must belong to this set. Hence $a$ contains at most $|\mathcal{A}|$ points. Similarly, $a \cdot b$ contains at most $|\mathcal{A}|^2$ points. □

**Example 3.7.** Consider the Thue-Morse substitution

$$a \xrightarrow{\sigma} ab$$
$$b \xrightarrow{\sigma} ba.$$

Then $a \cdot b$ consists of two points, both of which equal the $\sigma$-fixed point starting with $b$ on the right; however on the left we can see the left infinite words fixed by $\sigma^2$ defined either by $a$, or by $b$, i.e.,

$$a \cdot b = \{ \lim_{k \to \infty} \sigma_{[0,2k]}(a) \cdot \lim_{n \to \infty} \sigma_{[0,2k]}(b), \lim_{k \to \infty} \sigma_{[0,2k+1]}(a) \cdot \lim_{n \to \infty} \sigma_{[0,2k+1]}(b) \}.$$
A necessary condition for most of our results is that the directive sequence be \textit{straight} (see Definition 3.10 below and Definition 5.6 for a stronger notion). The terminology is borrowed from Shi[20], where it is defined for directive sequences which are stationary. We thus first state it for substitutions by introducing the notion of strong straightness (see Definition 3.8). It corresponds to the case where the initial period equals 1 in [Hos86, Section 2.1], which allows us to work with \( p = 1 \) in the statement of Host’s theorem from [Hos86, Section 1.4].

**Definition 3.8 (Strongly straight substitution).** A substitution \( \sigma \) is \textit{strongly straight} if it is primitive and for every letter \( a \), whenever \( b \) is the first letter of \( \sigma(a) \), then \( \sigma(b) \) starts with \( b \).

**Example 3.9.** The substitution \( \tau_1 \) below is strongly straight. However \( \tau_2 \) is not, as \( \tau_2(c) \) starts with \( a \) but \( \tau_2(a) \) does not start with \( a \):

\[
\begin{align*}
a & \mapsto bca \quad a \mapsto bc \\
b & \mapsto bbc \quad b \mapsto bbc \\
c & \mapsto cac \quad c \mapsto abc.
\end{align*}
\]

Every primitive substitution has a power which is strongly straight. As a shift defined by a primitive substitution \( \sigma \) equals the shift defined by any of its powers, then we may assume, by taking a power of \( \sigma \) if needed, that it is strongly straight. If \( \sigma \) is strongly straight, then any one-sided \( \sigma \)-periodic point (i.e., any one-sided fixed point for some power \( \sigma^n \), with \( n \) positive) is a fixed point of the substitution \( \sigma \) and for every letter \( a \), \( \lim \sigma^n(a) \) exists and is a fixed point of \( \sigma \).

Definition 3.8 should be compared to the stronger Definition 5.6 that we introduce later for directive sequences. The next definition for directive sequences is a less restrictive notion than that of strong straightness in the case that the directive sequence is stationary. By this we mean that if \( \sigma \) is strongly straight, then it is straight, and that straight substitutions are not necessarily straight. For instance, the substitution \( \tau_2 \) of Example 3.9 is straight but not strongly straight.

**Definition 3.10 (Straightness).** A directive sequence \( (\sigma_n)_{n \geq 0} \) is \textit{straight} if it is primitive and for each letter \( a \in \mathcal{A} \), the set \( a \) is a singleton. In other words, \( (\sigma_n)_{n \geq 0} \) is straight if and only if \( \lim_{n \to \infty} \sigma(a) \) exists for each letter \( a \in \mathcal{A} \).

Consider the bases \( B_n(a) \) of the towers of the generating partition \( \mathcal{Q}_n \) from Section 2.4. If \( (\sigma_n)_{n \geq 0} \) is straight, then the right-infinte part of the two-sided word \( \cap_{n \geq 0} B_n(a) \) is equal to \( a \) for any increasing sequence \( (n_k)_k \), and this will be useful when we build eigenfunctions as limits of simple functions in later sections.

Note that we can always telescope a primitive sequence \( (\sigma_n)_{n \geq 0} \) to a sequence

\[
\sigma_0 := \sigma_{[0,n_1)}, \sigma_1 := \sigma_{[n_1,n_2]}, \sigma_2 := \sigma_{[n_2,n_3)}, \ldots
\]

which is straight. However, if \( \sigma \) is finitary, we may lose this property when we telescope to \( (\sigma_n) \), and some of our later results are sensitive to this condition. Take for example two substitutions \( \sigma, \tau \) such that \( \sigma(a) \) starts with \( b \), \( \sigma(b) \) starts with \( a \), \( \tau(a) \) starts with \( a \) and \( \tau(b) \) starts with \( b \), and take \( \sigma = (\sigma_n)_{n \geq 0} \) where \( \sigma_n = \tau \) except when \( n \in \{2^k : k \in \mathbb{N} \} \), in which case \( \sigma_n = \sigma \). Assume \( \sigma \) primitive and \( a \neq b \). In this case, to obtain straightness, we must telescope to levels \( n_k \) where \( n_{k+1} - n_k \to \infty \), and the resulting directive sequence is not finitary.

We say that \( (\sigma_n)_{n \geq 0} \) can be \textit{boundedly telescoped} to a straight sequence if we can telescope via a syndetic sequence \( (n_k)_k \) to a straight directive sequence, where a sequence is syndetic if its letters occur with bounded gaps.

**3.2. Examples.** We now illustrate the notions of straightness and limit words with examples of families of substitutions, namely the Cassaigne-Selmer, Arnoux-Rauzy, Jacobi-Perron and Brun substitutions, which are associated to multidimensional continued fractions and are used to construct shifts with prescribed eigenfunctions (see e.g. [BST23]). For all these families of substitutions, straightness usually holds.

**Example 3.11.** Consider the directive sequence

\[
\sigma = \sigma, \tau, \sigma, \sigma, \tau, \sigma, \sigma, \tau, \sigma, \sigma, \ldots
\]
from Example 3.3. We have
\[ a = b = c = \left\{ \lim_{n \to \infty} \sigma_{(0,n)}(a) \right\}, \]
so that \( \sigma \) is straight.

Similarly, any directive sequence from \( \{\sigma, \tau\} \) where there are infinitely many occurrences of the substitution \( \tau \) and bounded runs of consecutive occurrences of either the substitution \( \sigma \), or the substitution \( \tau \), can be boundedly telescoped to a straight sequence.

Note that
\[ u = \lim_{n \to \infty} \sigma_{(0,n)}(b) \cdot \lim_{n \to \infty} \sigma_{(0,n)}(a), \text{ and } v = \lim_{n \to \infty} \sigma_{(0,n)}(c) \cdot \lim_{n \to \infty} \sigma_{(0,n)}(a) \]
are distinct two-sided limit words, and this directive sequence cannot be telescoped to a directive sequence with a unique two-sided limit word. In other words, although \((X_\sigma, T)\) is guaranteed a proper Bratteli-Vershik representation, by the model theorem of (Thm 4.7, [HPS92]) for minimal systems, we do not know how to obtain such a representation by simple manipulations of the data we are given.

**Example 3.12.** Consider the Cassaigne-Selmer substitutions (discussed e.g. in [CLL17, CLL22, BST23])
\[
\begin{align*}
a &\overset{\gamma_1}{\rightarrow} a & a &\overset{\gamma_2}{\rightarrow} b \\
b &\overset{\gamma_1}{\rightarrow} ac & b &\overset{\gamma_2}{\rightarrow} ac \\
c &\overset{\gamma_1}{\rightarrow} b & c &\overset{\gamma_2}{\rightarrow} c.
\end{align*}
\]
A directive sequence from \( \{\gamma_1, \gamma_2\} \) generates an aperiodic shift as soon as it is primitive, by [CLL17, Proposition 6]. As \( \gamma_1 \) is right-permutative and \( \gamma_2 \) is left-permutative, any primitive directive sequence from \( \{\gamma_1, \gamma_2\} \) is recognizable. We consider now a primitive directive sequence \((\sigma_n)_n\). By [CLL17, Lemma 1], for all \( N \), there exists \( i \) such that \( \sigma_{N+i+1} \neq \sigma_{N+2i+1} \). In particular \( \gamma_1 \) and \( \gamma_2 \) appear infinitely often, and this implies that there is only one one-sided limit word; hence the directive sequence is straight, as every \( \alpha \) consists of that unique limit word. To see this, we will show that the unique right-infinite limit word is \( \lim_n \sigma_{(0,n)}(a) \). Every time \( \sigma_n = \gamma_1 \), the first letter of \( \sigma_{[n,N]}(a) \) and \( \sigma_{[n,N]}(b) \) equals \( a \), while \( \sigma_{[n,N]}(c) \) starts with \( a \) or \( b \). Furthermore if \( N \) is large enough so that \( \sigma_{[n,N]} \) contains two occurrences of \( \gamma_1 \), then \( \sigma_{[n,N]}(c) \) also starts with \( a \). Now let \( n \) grow, this means that each of \( \sigma_{[0,N]}(a) \) starts with \( \sigma_{[0,n]}(a) \), where \( n \to \infty \) as \( N \to \infty \). The claim follows.

Therefore in the case where \( \gamma_1 \) and \( \gamma_2 \) each appear infinitely often the above example gives directive sequences whose corresponding S-adic systems have one left-infinite limit word and one right-infinite limit word, i.e., the corresponding Bratteli-Vershik diagrams have one maximal and one minimal path, although the substitutions are not proper. Thus directive sequences with one unique left-infinite limit word and one unique right-infinite limit word are not in general proper. Furthermore, depending on the given directive sequence, one may have to telescope unboundedly in order to satisfy needed conditions in articles which consider eigenvalues of such systems via their Bratteli-Vershik representation, such as [CDHM03, BDM05, BDM10, DFM15], which require that every substitution is proper.

**Example 3.13.** Consider the Arnoux-Rauzy substitutions on the three-letter alphabet \( \{a, b, c\} \) (see [AR91]):
\[
\begin{align*}
a &\overset{\alpha_1}{\rightarrow} a & a &\overset{\alpha_2}{\rightarrow} ba & a &\overset{\alpha_3}{\rightarrow} ca \\
b &\overset{\alpha_1}{\rightarrow} ab & b &\overset{\alpha_2}{\rightarrow} b & b &\overset{\alpha_3}{\rightarrow} cb \\
c &\overset{\alpha_1}{\rightarrow} ac & c &\overset{\alpha_2}{\rightarrow} bc & c &\overset{\alpha_3}{\rightarrow} c.
\end{align*}
\]
Consider a directive sequence where each of the three substitutions occurs infinitely often; then the directive sequence is primitive, the shift is aperiodic (also for subword complexity reasons, as in the previous example), and as each substitution in \( \{\alpha_1, \alpha_2, \alpha_3\} \) is right-permutative, the directive sequence is recognizable. Furthermore any primitive directive sequence is straight since
any product of length 3 of these substitutions is proper. For more on eigenvalues of Arnoux-Rauzy shifts, see [CFM08].

Example 3.14. Similarly, the infinite family of Jacobi-Perron substitutions \( \{\sigma_{jk}\}_{0 \leq j \leq k \neq 0} \) [BST23, Equation (6.6)], where \( \sigma_{jk} \) is defined by

\[
\begin{align*}
  a &\mapsto \sigma_{jk} b \\
  b &\mapsto \sigma_{jk} c \\
  c &\mapsto \sigma_{jk} ab^k c^k,
\end{align*}
\]

is left-permutative and so the directive sequence generates a recognizable shift, provided the shift is aperiodic. A primitive directive sequence is straight and here we only need to telescope any directive sequence to every third level to achieve this. For, if we telescope in this way, then the composed substitutions \( \sigma_{j_1k_1} \circ \sigma_{j_2k_2} \circ \sigma_{j_3k_3} \) will be such that for each each \( \sigma \) and each letter \( a \), \( \sigma(a) \) starts with \( a \).

Example 3.15. Almost the same can be said of the family of (unordered) Brun substitutions (see [DHS13] and [BST23, Equation (6.7)]), defined on an arbitrary alphabet as

\[
j \mapsto \sigma_{ijk} = \begin{cases} i & \text{for } j \neq i \neq k \neq j, \\ j & \text{if } j = i. \end{cases}
\]

Each substitution is right-permutative, so any primitive directive sequence that generates an aperiodic shift is recognizable. And, provided that each substitution appears infinitely often, there is only one right-infinite limit word.

3.3. Eigenvalues and coboundaries for substitution shifts. We recall that \( \mathbb{S}^1 \) stands for the unit circle in the complex plane. We say that \( h : A^* \to \mathbb{S}^1 \) is a morphism if \( h(w_1w_2) = h(w_1)h(w_2) \) whenever \( w_1, w_2 \) and \( w_1w_2 \) belong to \( A^* \).

Host [Hos86] shows that for the class of primitive substitution shifts, every measurable eigenvalue is continuous; we summarise his approach below, which is based on the notion of coboundaries.

Definition 3.16 (Substitutive coboundary). Let \( \sigma \) be a substitution on a finite alphabet \( A \). A morphism \( h : A^* \to \mathbb{S}^1 \) is a coboundary for \( \sigma \) if, for any \( a \in A \), \( h(\sigma(a)) = 1 \) whenever \( a \) is a return word to \( a \).

The coboundary \( h \) is said to be trivial if \( h(a) = 1 \) for each \( a \in A \).

The next lemma revisits Definition 3.16 in terms of two-letter words.

Lemma 3.17. [Hos86] Let \((\bar{X}_\sigma, T)\) be a one-sided substitution shift, \( \bar{X}_\sigma \subset A^\mathbb{N} \), with \( \sigma \) primitive. A morphism \( h \) is a coboundary on \( \mathcal{L}_\sigma \) if and only if there exists a function \( \bar{f} : A \to \mathbb{S}^1 \) such that

\[
\bar{f}(a) = \bar{f}(ab)(a)
\]

for every two-letter word \( ab \in \mathcal{L}_\sigma \), one has \( \bar{f}(b) = \bar{f}(a)h(a) \).

Proof. Suppose first that \( h : A \to \mathbb{S}^1 \) is a morphism and \( \bar{f} : A \to \mathbb{S}^1 \) is a function where (3.1) is satisfied. Then for any letter \( a \), if \( au_1 \ldots u_{n-1}u_n \) is a return word to \( a \), we have

\[
\begin{align*}
\bar{f}(a) &= \bar{f}(u_n)h(u_n) \\
&= \bar{f}(u_{n-1})h(u_{n-1})h(u_n) \\
&\vdots \\
&= \bar{f}(a)h(a)h(u_1) \ldots h(u_{n-1})h(u_n),
\end{align*}
\]

so that \( h(au_1 \ldots u_{n-1}u_n) = 1 \), i.e., \( h \) is a one-sided coboundary. Conversely, suppose that \( h \) is a one-sided coboundary. Since \((\bar{X}_\sigma, T)\) is minimal, we have \( \mathcal{L}_\sigma = \mathcal{L}_u \) for any \( u \in \bar{X}_\sigma \), so fix such a \( u \). By assumption, for any two indices \( m < n \) such that \( u_n = u_m \), we have \( h(u_m) \ldots h(u_{n-1}) = 1 \).
We define a function \( g: \mathbb{Z} \to \mathbb{S}^1 \) as
\[
g(k) := \begin{cases} 
  h(u_0) \cdots h(u_{k-1}) & \text{if } k > 0 \\
  1 & \text{if } k = 0 \\
  h(u_k)^{-1} \cdots h(u_{-1})^{-1} & \text{if } k < 0.
\end{cases}
\]
We then define \( \tilde{f} \) as \( \tilde{f}(a) := g(k) \) if \( k \) is such that \( u_k = a \). One checks that the map \( \tilde{f} \) is well defined and satisfies (3.1). Note that the map \( \tilde{f} \) depends on the choice of \( u_0 \), with \( \tilde{f}(u_0) = h(u_0) \).

**Remark 3.18.** Note that if, for some \( a \in A \), \( a \alpha \in \mathcal{L}_\sigma \), then \( h(a)\tilde{f}(a) = \tilde{f}(a) \), so \( h(a) = 1 \). In particular, for any non-trivial substitution shift on a two-letter alphabet, any coboundary \( h \) satisfies \( h \equiv 1 \).

We now recall the seminal result by Host originally stated in [Hos86] for primitive substitutions. It is phrased in terms of one-sided shifts and we state it only for strongly straight substitutions \( \sigma \), as any primitive substitution has a power which is strongly straight. Recognizability of \( X_\sigma \) is a key requirement. Note that Host assumed that \( \sigma \) is injective on letters, but this may be relaxed. For, if \( \sigma \) is not injective on letters, we can introduce an equivalence relation \( \sim \) on \( A \) where \( a \sim b \) if and only if \( \sigma(a) = \sigma(b) \). We then work with \( \bar{\sigma} \) defined on \( A/\sim \), and if \( \sigma \) is recognizable then so is \( \bar{\sigma} \). Finally \((X_\sigma, T)\) and \((X_{\bar{\sigma}}, T)\) are topologically conjugate. See [BDM04] for details. Recall that a primitive substitution shift is uniquely ergodic; we use \( \mu \) to denote the unique invariant measure of such a shift.

**Theorem 3.19 ([Hos86]).** Let \( \sigma \) be a primitive substitution on the alphabet \( A \) which is strongly straight. Suppose that the one-sided shift \((\dot{X}_\sigma, T, \mu)\) is recognizable. Let \( h_n(a) = |\sigma^n(a)| \) for all \( n \) and all \( a \in A \). If for each \( a \in A \) the limit
\[
(3.2) \quad h(a) := \lim_{n \to \infty} \lambda^{h_n(a)}
\]
exists and defines a coboundary \( h \), then \( \lambda \) is a continuous (and hence measurable) eigenvalue of \((\dot{X}_\sigma, T)\). Conversely, if \( \lambda \in \mathbb{S}^1 \) is a measurable eigenvalue of \((\dot{X}_\sigma, T, \mu)\), then it also satisfies (3.2) for some coboundary \( h \).

We give an intuition for the proof of Theorem 3.19 in the continuous case. If \( f \) is a continuous eigenfunction for the eigenvalue \( \lambda \), taking values in \( \mathbb{S}^1 \), then the assumption that \( \sigma \) is strongly straight allows us to define \( \tilde{f} : A \to \mathbb{S}^1 \) by \( \tilde{f}(a) := f(u) \), where \( u \) is the one-sided fixed point such that \( \sigma^n(u) \to u \). If \( ab \in \mathcal{L}_\sigma \) is such that \( u \in [ab] \) (where here it can happen that \( a = b \), then by continuity of the eigenfunction \( f \), one gets
\[
\tilde{f}(a)\lambda^{h_n(a)} \to \tilde{f}(b),
\]
which implies the existence of \( \lim_n \lambda^{h_n(a)} \); this yields a well-defined function \( h : A \to \mathbb{S}^1 \), \( a \mapsto \lim_n \lambda^{h_n(a)} \). Further, if \( ab_1 \cdots b_k \in \mathcal{L}_\sigma \) is a return word to \( a \), then since \( T^{h_n(w)}u \to u \), we have
\[
\lambda^{h_n(a)+h_n(b_1)+\cdots+h_n(b_k)} \to 1, \text{ i.e., } h(a) h(b_1) \cdots h(b_k) = 1.
\]
This kind of argument leads to the definition of a coboundary.

Conversely, given a coboundary \( h \) and a function \( f \) guaranteed by Lemma 3.17, we can use \( \tilde{f} \) to define a map \( f \) on the shift orbit of a fixed point \( u \), as \( f(T^nu) := \lambda^n f(u_0) \). Note that if \( u \) has a dense orbit, the definition of \( f \) by \( f(T^nu) = \lambda \) completely determines the continuous function, if it extends by continuity. From the fact that \( M_n/\lambda^n \) converges geometrically to the projection on the Perron-Frobenius eigenspace defined by \( \lambda \), we can deduce that the convergence of (3.2) is fast enough, so that we can extend \( f \) to a continuous eigenfunction on \( X_\sigma \). For details, see [Hos86, Lemma 5, Proposition 1].

To complete the proof of the theorem, Host shows that any measurable eigenvalue must also define a coboundary. Suppose that \( f \in L^2(X_\sigma, T, \mu) \) is a measurable eigenfunction for the eigenvalue \( \lambda \). Let \( B_n(a) = \sigma^n([a]) \) be the base of the \( \sigma^n \)-tower defined by the letter \( a \). The restriction \( f|_{T_n(a)} \) of \( f \) to this tower is completely determined by \( f|_{B_n(a)} \). Let \( \beta_n \equiv \mathbb{E}_n(f)|_{B_n(a)} \) be the
value that the conditional expectation of $f$, conditioned on the partition $Q_n$, assumes on $B_n(a)$. Then if $ab \in L_\sigma$, once again we should have
\[
\beta_{n,a}^{h_n(a)} \approx \beta_{n,b},
\]
and
\[
\lambda^{h_n(a)+h_n(b_1)+\cdots+h_n(b_k)} \approx 1
\]
for large $n$, whenever $ab_1\ldots b_k$ is a return word to $a$. The arguments are more delicate, but here again it can be shown that the heights of the towers end up defining a coboundary. We revisit this in more detail in Section 5.

The definition of a coboundary is purely combinatorial. Let us see that more is needed in order to get eigenvalues with the next example.

**Example 3.20.** There exists $h$ which satisfies the conditions of Definition 3.16 without defining an eigenfunction. In particular, for a coboundary to define an eigenvalue, Equation (3.2) must also be satisfied.

Consider the strongly straight substitution $\sigma$ defined over $\{a, b, c\}$ by
\[
\begin{align*}
  a & \mapsto abc \\
  b & \mapsto aabc \\
  c & \mapsto aaa.
\end{align*}
\]

For this substitution, the words of length two that belong to $L_\sigma$ are $\{aa, ab, bca, ca\}$. Setting $h(a) = 1$ and $h(b) = h(c) = -1$ and defines a coboundary, with the corresponding $f$ being $f(a) = f(b) = 1$ and $f(c) = -1$. However, since $h_{n+1}(c) = 3h_n(a)$, there is no value of $\lambda$ such that (3.2) is true for this function.

\[
-1 = h(c) = \lim_{n \to \infty} \lambda^{h_{n+1}(c)} = \lim_{n \to \infty} \lambda^{3h_n(a)} = (\lim_{n \to \infty} \lambda^{h_n(a)})^3 = h(a)^3 = 1,
\]
a contradiction.

**Remark 3.21.** The reason why $h$ does not define an eigenvalue in Example 3.20 is that $f : A \to S^1$ is non-constant whereas $\sigma$ admits only one fixed point. Indeed, the map $f$ encodes what values an eigenfunction $f$ can take on the fixed points for $\sigma$, namely, $f(a)$ should be the value that $f$ assigns to the fixed point defined by the letter $a$. Since $\sigma$ only has one fixed point, $f$ must be constant. This example shows that for a coboundary $h$ to define an eigenvalue, it must be accompanied by the appropriate $f$. This explains why, in Definition 4.1 below, we define a coboundary to be a pair $(h, f)$ such that $f$ takes the same values for letters that define the same limit words.

The following lemma tells us that $f$ must take the same value on letters that define the same limit words.

**Lemma 3.22.** Let $\sigma$ be a strongly straight substitution on the alphabet $A$. Suppose that the one-sided shift $(X_\sigma, T, \mu)$ is recognizable, and that $h$ is a coboundary which satisfies (3.2), with associated $f$. If $a = b$, then $f(a) = f(b)$.

**Proof.** Suppose that $a, b \in A$ satisfy $a = b$. Primitivity implies the existence of a word $w$ such that $awb \in L_\sigma$. Since $h$ is a coboundary, one has on the one hand $f(b) = f(a)h(aw)$. On the other hand, Host’s theorem tells us that the $\lambda$ from (3.2) is a continuous eigenvalue. Let $f$ be the continuous eigenfunction associated to $\lambda$. Now $\sigma^n(aw)\sigma^n(b) \in L_\sigma$, then by continuity of $f$ $\lim_{n \to \infty} f(T^{\sigma^n(aw)}a) = f(a)$, so that $\lambda^{\sigma^n(aw)} \to 1$. Since $h$ satisfies (3.2), we have $h(aw) = 1$ and thus $f(b) = f(a)$. \(\square\)

**3.4. Height.** Host’s definition of a coboundary is motivated by the combinatorial definition of height for a constant-length substitution, a definition due to Kamae [Kam72] and Dekking [Dek78]. Its strength is that it allows a complete description of the set of eigenvalues as $\{j/q^n h : j \in \mathbb{Z}, n \in \mathbb{N}\}$ when $h$ stands for the height (see Corollary 6.8 below). We assume below that the substitution $\sigma$ has a fixed point, as otherwise we consider a power of $\sigma$. 

**Definition 3.23 (Height).** Let $\sigma$ be a primitive constant-length substitution with length $q$, and let $u$ be any one-sided fixed point for $\sigma$. The *height* $\tilde{h}$ of $\sigma$ is defined as

$$
\tilde{h} := \max\{n \geq 1 : \gcd(n, q) = 1, n| \gcd\{k : u_k = u_0\}\}
$$

$$
= \gcd\{|w| : w \text{ is a return word to } u_0, \text{ and } |w| \text{ is coprime to } q\}.
$$

For the equivalence between the two formulations above in the definition, see, e.g., [Que10, Section 6.1.1].

Equation (3.2) tells us that for a constant-length substitution, a coboundary $h$ must be constant. The substitution $\sigma$ has non-trivial height $\tilde{h} \neq 1$ if and only if it defines a non-trivial coboundary $h \neq 1$. When the height $\tilde{h}$ is non-trivial, the constant function $h \equiv e^{2\pi i / \tilde{h}}$ is a non-trivial coboundary associated to an eigenvalue. See also Corollary 6.8, which recalls the explicit and classical above-mentioned relation with eigenvalues.

Note that while a primitive constant length substitution $\sigma$ admits its length $q$ as the dominant eigenvalue of its incidence matrix $M_\sigma$, and that this eigenvalue leads to a dynamical eigenvalue $e^{2\pi i/\tilde{h}}$ of the shift $(X_\sigma, T)$, there exist primitive non-constant length substitutions such that the dominant eigenvalue of $M_\sigma$ is an integer, but where $(X_\sigma, T)$ is weakly mixing, i.e., it has no dynamical eigenvalues; see e.g., [ASY22].

In this article we investigate extending Theorem 3.19 to $S$-adic shifts. In particular, in Section 4, we apply our results to constant-length $S$-adic shifts and we define the appropriate version of height in this case (see Definition 6.6).

**Remark 3.24.** There are many important situations for which coboundaries are always trivial. In particular any coboundary of a *Pisot irreducible* substitution is trivial [BK06]; see Section 4.3 for definitions. More generally, and as noticed by Clemens M"ullner in private communication, a coboundary of a primitive substitution for which the measures of the cylinders associated to letters are rationally independent is trivial. Also, left proper substitutions have only trivial coboundaries; see Proposition 4.6. These situations are a manifestation of a so-called property of coincidence (see [ABB+15] for more on the subject, see also Section 6.2).

### 4. Coboundaries for Directive Sequences and Continuous Eigenvalues

In this section, we investigate continuous eigenvalues. We first define $S$-adic coboundaries in Section 4.1; the main results for the continuous case are then stated and proved in Section 4.2. Lastly, in Section 4.3 we investigate the relation between continuous eigenvalues and measures of letters in the case of a trivial coboundary.

#### 4.1. $S$-adic coboundaries.

We define a coboundary as follows:

**Definition 4.1 ($S$-adic coboundary).** Let $\sigma = (\sigma_n)_{n \geq 0}$ be a straight directive sequence on $A$. A coboundary for $\sigma$ is a morphism $h : A^* \to S^1$, and a map $\tilde{f} : A \to S^1$ satisfying

- $\tilde{f}(b) = \tilde{f}(a)$ whenever $a = b$, and
- $\tilde{f}(b) = \tilde{f}(a) \lim_{n_k \to \infty} h(w_{n_k})$ for any $a, b \in A$, and for any sequence $(w_{n_k})_{n_k}$ of transition words from $a$ to $b$ such that $w_{n_k} \in L_{n_k}$ for all $k$ and $\lim_{n_k \to \infty} h(w_{n_k})$ exists.

If we replace the second requirement above by the weaker condition that $\tilde{f}(b) = \tilde{f}(a) \lim_{n_k \to \infty} h(w_{n_k})$ for any $a, b \in A$, and for any sequence $(w_{n_k})_{n_k}$ of transition words from $a$ to $b$ of bounded length such that $w_{n_k} \in L_{n_k}$ for each $k$ and $\lim_{n_k \to \infty} h(w_{n_k})$ exists, then we say that $(h, \tilde{f})$ is a weak coboundary.

**Remark 4.2.**

(i) Sometimes we will simply refer to a coboundary as $h$, dropping mention of $\tilde{f}$. In particular, when we write that $h$ defines a constant coboundary, we mean that there exists a function $\tilde{f}$ such that $(h, \tilde{f})$ is a coboundary, with $h$ taking constant values. Moreover, in the case where the map $h \equiv 1$, the coboundary is said to be trivial.

(ii) Let $\sigma$ be a substitution that occurs infinitely often in $\sigma$. Let $a$ be a letter, and let $wb$ be a prefix of $\sigma(a)$. Then existence of a coboundary implies that $\tilde{f}(b) = h(w)\tilde{f}(a)$. To see this,
suppose that \( w \) starts with \( w_0 \), then \( \bar{f}(b) = h(w)\bar{f}(w_0) \). Also, the supposed conditions imply that \( a = w_0 \). The claim follows.

(iii) Note that in the substitutive case, that is when \( \sigma \) is a stationary directive sequence, if \( (h, f) \) satisfies Definition 4.1, then \( h \) also satisfies Definition 3.16. Conversely, if \( h \) is a coboundary that satisfies Definition 3.16 and also defines a dynamical eigenvalue via (3.2), then the function \( \bar{f} \) guaranteed by Lemma 3.17 satisfies the conditions of Definition 4.1. Note that coboundaries, as defined in Definition 3.16, that do not lead to eigenvalues do not have to satisfy the first condition of Definition 4.1; see Example 3.20.

(iv) Coboundaries are supposed to reflect eigenvalues, and, in Definition 4.1, the function \( \bar{f} \) is supposed to define the putative eigenfunction on one-sided limit words. This is why we impose the first requirement in Definition 4.1. See also Remark 3.21 and Lemma 3.22.

(v) Although the coboundaries that we have defined seem to be suited to the study of one-sided shifts, because they are defined in terms of one-sided limit words, nevertheless, they are sufficient to work with two-sided shifts. This is mainly due to Proposition 2.1.

(vi) Observe that in the substitutive case, if \( h \) is a coboundary defined by (3.2), then \( h(\sigma(w)) = h(w) \) for any word. Similarly, let \( \sigma = (\sigma_n)_{n \geq 0} \) be a primitive directive sequence on \( \mathcal{A} \). Suppose that each \( \sigma_n \) that appears in \( \sigma \) appears infinitely often, and that for each \( a \in \mathcal{A} \), the limit

\[
h(a) := \lim_{n \to \infty} \lambda^{h_n(a)}
\]

exists. Then \( h(\sigma(a)) = h(a) \) for each \( \sigma \) that appears in \( \sigma \) and each \( a \in \mathcal{A} \). To see this, if \( \sigma \) appears in \( \sigma \) then for some \( (n_j) \), \( \sigma = \sigma_{n_j} \) for each \( j \). Recalling the notation \( h_n(a) := [\sigma(0, n_j)](a) \), one has, for any \( a \in \mathcal{A} \),

\[
h(\sigma(a)) = \lim_{j \to \infty} \lambda^{h_{n_j}(\sigma(a))} = \lim_{j \to \infty} \lambda^{[\sigma(0, n_j)](\sigma(a))} = \lim_{j \to \infty} \lambda^{[\sigma(0, n_j+1)](a)} = h(a).
\]

4.2. **S-adic coboundaries and continuous eigenvalues.** In this section we describe the connection between coboundaries and continuous eigenvalues. There are two types of statements, namely necessary conditions and sufficient conditions stated in terms of the existence of limits of the form \( \lim_{n \to \infty} \lambda^{h_n(a)} \), together with a coboundary condition. In Theorem 4.3 we show that the existence of a continuous eigenvalue implies the existence of a weak coboundary, under some combinatorial condition. For sufficient conditions, in Theorem 4.4 we show that the existence of a coboundary associated to a rational \( \lambda \) implies the existence of a continuous eigenvalue, and in Proposition 4.5 we provide the return word version of Theorem 4.4. Lastly, we consider the case of a finitary, straight, recognizable, directive sequence, where each substitution appears infinitely often. With these restrictions, we show in Theorem 4.7 that, provided that the terms \( \lambda^{h_n(a)} \) converge sufficiently fast, \( \lambda \) is a continuous eigenvalue.

In the following theorem, we require that there is a fully essential word of length two. It is easy to find examples where this is satisfied, for example, we can take a directive sequence where there exists at least one substitution which appears infinitely often, so that one can telescope to a directive sequence with a fully essential word. Also, this condition is satisfied if the substitutions in our directive sequence each have a word of length two in common in the images of letters by each substitution. However, note for instance with the case of an Arnoux-Rauzy directive sequence on a three-letter alphabet, Example 3.13, we would require that one of the substitutions occurs boundedly often, to apply Theorem 4.3. By Proposition 2.1 and Remark 4.2(v), the following result can be applied to either one- or two-sided shifts.

**Theorem 4.3.** Let \( \sigma = (\sigma_n)_{n \geq 0} \) be a straight directive sequence on \( \mathcal{A} \). Suppose that for each \( a \in \mathcal{A} \), there exists \( \ell \in \mathcal{A} \) such that \( a\ell \) is a fully essential word. If \( \lambda \in \mathbb{S}^1 \) is a continuous eigenvalue, then

\[
h(a) := \lim_{n \to \infty} \lambda^{h_n(a)}
\]

exists and defines a weak coboundary for \( \sigma \).

**Proof.** Suppose that \( \lambda \) is a continuous eigenvalue. By Proposition 2.1, it is a one-sided continuous eigenvalue; let \( f \) be a corresponding eigenfunction for the one-sided shift \( X_\sigma \); since \( \lambda \in \mathbb{S}^1 \), then
\[ f(x) \] is constant on any orbit. Primitivity of \( \sigma \) implies that \( X_\sigma \) is minimal, and now continuity implies that \( f \) is constant on \( X_\sigma \). Without loss of generality, we assume \( |f| = 1 \). Let \( \alpha \in A \) and let \( a \ell \) be a fully essential word. Since \( a \ell \in L_\sigma^{(n)} \) for each \( n \), the set \( \sigma_{[0, n]}([a \ell]) \subset X_\sigma \) is non-empty for each \( n \), and so \( \sigma_{[0, n]}([a \ell]) \) belongs to the language \( L_\sigma^{(0)} \). Let \( (x^{(n)})_{n \in \mathbb{N}} \) be a sequence of points in \( X_\sigma \) such that \( x^{(n)} \in \sigma_{[0, n]}([a \ell]) \) for each \( n \in \mathbb{N} \). Note that \( x^{(n)} \to a \) by straightness, and \( f(x^{(n)}) \to f(a) \) since \( f \) is continuous. Note also that \( T^{h_{\alpha}(a)}(x^{(n)}) \to \ell \) by straightness. Since \( f \circ T^{h_{\alpha}(a)}(x^{(n)}) = \lambda_{h_{\alpha}(a)} f(x^{(n)}) \), we obtain that
\[
\lim_{n \to \infty} \lambda_{h_{\alpha}(a)} f(x^{(n)}) = f(\ell),
\]
and since \( \lim_{n \to \infty} f(x^{(n)}) = f(a) \neq 0 \), we conclude that \( h(a) := \lim_{n \to \infty} \lambda_{h_{\alpha}(a)} \) exists. Since each \( a \in A \) admits a fully essential word of the form \( a \ell \), \( h \) is defined on all of \( A \), i.e., \( \lim_{n \to \infty} \lambda_{h_{\alpha}(a)} \) exists for all \( a \).

Define \( \bar{f} : A \to S^1 \) to be \( \bar{f}(a) = f(a) \). We now verify that \( h \) and \( \bar{f} \) satisfy the conditions of a weak coboundary in Definition 4.1. Clearly \( \bar{f}(b) = \bar{f}(a) \) whenever \( b = a \). Suppose that \( (w_{n_k}) \) is a sequence of transition words from \( a \) to \( b \) in \( L_\sigma^{(n_k)} \), of bounded length. As before, we have
\[
\bar{f}(b) = \bar{f}(a) \lim_{k \to \infty} \lambda_{h_{\alpha}(w_{n_k})}.
\]
Since by definition \( \bar{f}(a) = f(a) \) for each letter \( a \), one has
\[
\bar{f}(b) = \bar{f}(b) = \bar{f}(a) \lim_{k \to \infty} \lambda_{h_{\alpha}(w_{n_k})} = \bar{f}(a) \lim_{k \to \infty} h(w_{n_k}).
\]

As a sufficient condition in the rational case for the existence of a continuous eigenvalue, we have the following.

**Theorem 4.4.** Let \( \sigma = (\sigma_n)_{n \geq 0} \) be a straight and recognizable directive sequence on \( A \). Let \( \lambda \in S^1 \) be rational. If
\[
h(a) := \lim_{n \to \infty} \lambda_{h_{\alpha}(a)}
\]
eastern for each \( a \in A \) and if the map \( h : a \mapsto h(a) \) defines a coboundary \( (h, \bar{f}) \) for \( \sigma \), then \( \lambda \) is a continuous eigenvalue.

**Proof.** Write \( \lambda = e^{2\pi i p/q} \) where \( p \) and \( q \) are non-zero coprime integers (with \( q \geq 1 \)). Fix a limit word \( u \), and, for \( 0 \leq j \leq q - 1 \), define
\[
A_j := \{ T^i(u) : i \in \mathbb{Z}, \ i \equiv j \mod q \}.
\]
Minimality tells us that the union of the sets \( A_j \) is \( X_\sigma \). If we show that the sets \( A_j \) are pairwise disjoint, then this means that the closed sets \( A_j \) are also open, i.e., \( \{ A_0, \ldots, A_{q-1} \} \) forms a clopen partition with \( T(A_i) = T(A_{i+1} \mod q) \). We will see below that this implies that this partition defines a continuous dynamical eigenvalue.

Let us first prove that the sets \( \{ A_j : 0 \leq j \leq q - 1 \} \) are pairwise disjoint. Since \( \lambda \) is a \( q \)-th root of unity and \( \lim \lambda_{h_{\alpha}(a)} \) exists for all \( a \), \( \lambda_{h_{\alpha}(a)} = h(a) \) for all \( n \) large enough and each \( a \in A \). The map \( a \mapsto \lambda_{h_{\alpha}(a)} \) is a morphism for all \( n \). Since \( h \) is also a morphism, we get that for all \( w \in A^+ \), \( h(w) = \lambda_{h_{\alpha}(w)} \) for all \( n \) large enough.

The assumption of recognizability implies the following: for each \( \varepsilon > 0 \) and \( n \in \mathbb{N} \), there exists \( N_n \) such that if \( w = w_{-N_n} \ldots w_{-1} w_n \in L_\sigma \), then there exists \( M_n \geq N_n(1-\varepsilon) \) such that \( w_{-M_n} \ldots w_{M_n} \in L_\sigma^{(n)} \). By a centred representation of a word at level \( n \), we mean that there is a (finite) word \( v^{(n)} = v^{(n)}_k \ldots v^{(n)}_0 \in L_\sigma^{(n)} \), there exist a proper suffix \( s \) of \( \sigma_{[0, n]}(v^{(n)}_k) \), a proper prefix \( p \) of \( \sigma_{[0, n]}(v^{(n)}_0) \), and a proper prefix \( s' \) of \( \sigma_{[0, n]}(v^{(n)}_0) \), such that
\[
w_{[-M_n, -1]} = s \sigma_{[0, n]}(v^{(n)}_{k+1} \ldots v^{(n)}_1)p \quad \text{and} \quad w_{[0, M_n]} = s' \sigma_{[0, n]}(v^{(n)}_1 \ldots v^{(n)}_{k-1})p.
\]
By uniqueness we mean that there is a unique such pair \((v^{(n)}, |p'|)\). Otherwise, by compactness, we would find a point in \(X_\sigma\) with two desubstitutions at some level \(n\), and this contradicts recognizability.

Now suppose that \(x \in A_j \cap A_{j'}\), \(j \neq j'\). This means that there exist sequences \((m_k)\) and \((m_k')\) such that \(m_k \equiv j \mod q\), \(m_k' \equiv j' \mod q\), \(T^{m_k} u \to x\) and \(T^{m_k'} u \to x\). Without loss of generality, we can assume that \(m_k < m_k'\) for each \(k\). Fix \(\varepsilon > 0\). For each \(n\), choose \(N_n, M_n\), \((v^{(n)}, |p'|)\) as above (here \(p' = (p')^{(n)}\)). By dropping to a subsequence if necessary, we may assume that each \(v^{(n)}_0\) equals a fixed letter \(b\) and \(v^{(n)}_1\) equals a fixed letter \(q\) to the unique right-infinite limit word \(x\).

By uniqueness we mean that there is a unique such pair \((v^{(n)}, |p'|)\). Without loss of generality, we can assume that \(m_k < m_k'\) for each \(k\). Fix \(\varepsilon > 0\). For each \(n\), choose \(N_n, M_n\), \((v^{(n)}, |p'|)\) as above (here \(p' = (p')^{(n)}\)). By dropping to a subsequence if necessary, we may assume that each \(v^{(n)}_0\) equals a fixed letter \(b\) and \(v^{(n)}_1\) equals a fixed letter \(q\) to the unique right-infinite limit word \(x\).

Let \(u^{(n)} \in X_\sigma^{(n)}\) be such that \(u = \sigma_{[0,n]}(u^{(n)})\). In \(u^{(n)}\), we see two occurrences of \(v^{(n)}\), where the image under \(\sigma_{[0,n]}\) of the first occurrence of \(v^{(n)}\) in \(u\) starts at an index congruent to \(m_k - N_n - \ell\), and the image of the second one starts at an index congruent to \(m_k' - N_n - \ell\); they are separated by a concatenation \(R^{(n)}\) of return words to the letter \(b\) in \(u^{(n)}\). As \(h\) is assumed to be a cocycle, we have \(h_n(v^{(n)}R^{(n)}) = [\sigma_{[0,n]}(v^{(n)}R^{(n)})] \to 0 \mod q\) as \(n \to \infty\), and so \(|\sigma_{[0,n]}(v^{(n)}R^{(n)})| \equiv 0 \mod q\) for \(n\) large. This contradicts the fact that \(|\sigma_{[0,n]}(v^{(n)}R^{(n)})| = m_k' - m_k \neq 0 \mod q\) for all \(n\). We thus have proved that the sets \(A_j\) are disjoint.

Let \(a\) be such that \(u = a\), and set \(f(u) := \tilde{f}(a)\), where \(\tilde{f} : A \to S^1\) is a function associated to \(h\). We define \(f(x) = f(u)\lambda^j\) if \(x \in A_j\). Since \(f\) is constant on each \(A_j\), and since the sets \(\{A_j : 0 \leq j \leq q - 1\}\) are open, \(f\) is continuous. Since \(f(T(T^m(u))) = \lambda f(T^m(u))\) and the orbit of \(u\) is dense by minimality, \(f\) is an eigenfunction for \(\lambda\).

We will need in Section 6.3 a slightly modified version of Theorem 4.4, stated in terms of return words, which avoids the use of coboundaries (in the flavour of [FMN96]). We use below the notation from (2.1) for \(h_{n_k}\). The proof of the following proposition is contained in that of Theorem 4.4.

**Proposition 4.5.** Let \(\sigma = (\sigma_n)_{n \geq 0}\) be a straight recognizable sequence of substitutions on \(A\). If \(\lambda\) is rational and

\[
\lim_{k \to \infty} \lambda^{h_{n_k}(w_{n_k})} = 1
\]

wherever \((w_{n_k})\) is a sequence of return words to some \(a \in A\), with \(w_{n_k} \in L_{n_k}\), then \(\lambda\) is a continuous eigenvalue.

We next consider the analogue of [CDHM03, Proposition 7], namely Theorem 4.7. Note that [CDHM03, Proposition 7] makes no mentions of coboundaries whilst Theorem 4.7 does; this is because the assumption that an \(S\)-adic representation is left-proper simplifies some issues, as the next proposition shows, namely, that all coboundaries are trivial for left-proper substitutions. We state and prove this theorem for left-proper substitutions, but we note that there is an analogous statement and proof, for right-proper substitutions.

**Proposition 4.6.** Suppose that the primitive directive sequence \(\sigma\) consists of left-proper substitutions. If \(\lambda\) is a continuous eigenvalue of \((X_\sigma, T)\), then it must satisfy

\[
\lim_{n} \lambda^{h_n(a)} = 1
\]

for each letter \(a\).

**Proof.** By Proposition 2.1, we need only work with the one-sided shift \((\tilde{X}_\sigma, T)\). The assumption that each \(\sigma_n\) is left-proper means that for each \(n\) there is a letter \(a_n\) such that \(\sigma_n(b)\) starts with \(a_n\) for each letter \(b\). This implies that the sequence of words \((\sigma_{[0,n]}(a_n))\) is nested and converges to the unique right-infinite limit word \(u\) for the one-sided shift \(\tilde{X}_\sigma\). Furthermore, for any letter \(b\) and any \(n > 1\), \(\sigma_{[0,n]}(b)\) starts with \(\sigma_{[0,n-1]}(a_{n-1})\). For each \(b \in A\), let \((x^{(n)})_{n \in N}\) be a sequence of points in \(\tilde{X}_\sigma\) such that \(x^{(n)} \in \sigma_{[0,n]}((b))\) for all \(n\), and hence \(x^{(n)}\) starts with \(\sigma_{[0,n]}(b)\sigma_{[0,n-1]}(a_{n-1})\) for all \(n\). Then \(x^{(n)} \to u\) and \(T^{h_n(b)}(x^{(n)}) \to u\). Let \(f\) be a continuous eigenfunction associated to
Also, since that partitions (such that the first letter of \( \omega \) occurs infinitely often). This implies in particular that \( \lambda (X) \neq 0 \). Because of Proposition 4.6, the statements of [CDHM03, Propositions 7, 8] are stated in terms of convergence of the terms \( \lambda^h(b) \) to 1; we discuss in Section 4.3 consequences of such a convergence property in terms of the relation between eigenvalues and measures of one-letter cylinders; see Theorem 4.11 and Lemma 4.9. In general \( \lambda^h(b) \) converges to a term \( h(b) \), for some coboundary \( h \).

**Theorem 4.7.** Let \( \sigma = (\sigma_n)_{n \geq 0} \) be a finitary, straight, recognizable directive sequence on \( A \), where each substitution occurs infinitely often. Suppose that \( (\sigma_n(T)) \) is aperiodic. Let \( (h,f) \) be a weak coboundary for \( (X_\sigma,T) \). Let \( \lambda \in \mathbb{S}^1 \). If

\[
\sum_{n=1}^{\infty} |\lambda^{h_n(a)} - h(a)| < \infty
\]

for each \( a \in A \), then \( \lambda \) is a continuous eigenvalue of \( (X_\sigma,T) \).

**Proof.** Since all the substitutions appear infinitely often, all the factors of images of letters are essential. Now fix a substitution \( \sigma \) that occurs in the directive sequence and fix a pair of letters \( a,b \) such that \( b \) occurs in \( \sigma(a) \). Write \( \sigma(a) = wbu' \). The word \( w \) is an essential transition word from the first letter of \( w \) to \( b \). This means we can apply (ii) of Remark 4.6, to write \( h(w)f(u) = f(b) \).

Pick and fix \( x \in X_\sigma \). We use the notation of Section 2.4 and work with the sequence of partitions \( (Q_n) \). For each \( n \), there exist a letter \( a_n = a_n(x) \) and \( j_n = j_n(x) \), with \( 0 \leq j_n < h_n(a_n) \) such that \( x \in T^{j_n}B_n(a_n) \). In other words, \( (j_n,a_n)_{n \geq 0} \) is a \( (Q_n) \)-address for \( x \). Note that this forces \( a_n \) to appear in \( \sigma_n(a_n+1) \): if \( \sigma_n(a_n+1) = b_n^{(1)} \ldots b_n^{(L_n)} \), then there exists \( 0 \leq k_n < L_n \) such that \( b_n^{(k_n+1)} = a_n \). If \( k_n = 0 \), then \( j_{n+1} = j_n \). If \( k_n \geq 1 \), then \( j_{n+1} = \sum_{i=1}^{k_n} h(b_n^{(i)}) + j_n \). See Figure 1.

Note also that \( b_n^{(1)} \ldots b_n^{(k_n)} \) is an essential transition word from \( b_n^{(1)} \) to \( a_n \), from the remark above. Moreover, since \( \sigma \) is straight, one has \( a_{n+1} = b_n^{(1)} \) (we use the fact that the substitutions \( \sigma_n \) occur infinitely often). This implies in particular that \( f(b_n^{(1)}) = f(a_{n+1}) \). Since \( h \) is a weak coboundary, for each fixed morphism \( \sigma \), by taking the limit on the subset of indices \( n \) where \( \sigma \) occurs in the directive sequence, i.e., the \( n \) such that \( \sigma_n = \sigma \), we have that

\[
|h(b_n^{(1)} \ldots b_n^{(k_n)})f(b_n^{(1)}) - f(a_n)| \rightarrow 0,
\]

and so equals 0 since the expression is constant. We have shown that \( |h(b_n^{(1)} \ldots b_n^{(k_n)})f(a_{n+1}) - f(a_n)| = 0 \) for all \( n \).

For each \( n \), define \( f_n(x) = \lambda^{j_n} f(a_n) \).

**Claim:** the sequence \( (f_n) \) converges uniformly to a continuous function \( f \). We have, as \( \lambda \) has modulus 1,

\[
|f_{n+1}(x) - f_n(x)| = |\lambda^{j_n+1} f(a_{n+1}) - \lambda^{j_n} f(a_n)| = |\lambda^{\sum_{i=1}^{k_n} h_n(b_n^{(i)})} f(a_{n+1}) - f(a_n)| + |\lambda^{\sum_{i=1}^{k_n-1} h_n(b_n^{(i)})} f(a_{n+1}) - f(a_n)|
\]

Also, since \( |f| = 1 \),

\[
|\lambda^{\sum_{i=1}^{k_n} h_n(b_n^{(i)})} f(a_{n+1}) - f(a_n)| \leq |\lambda^{\sum_{i=1}^{k_n} h_n(b_n^{(i)})} f(a_{n+1}) - f(a_{n+1})| + |\lambda^{\sum_{i=1}^{k_n-1} h_n(b_n^{(i)})} f(a_{n+1}) - f(a_n)|
\]

\[
= |\lambda^{h_n(b_n^{(k_n)})} - h(b_n^{(k_n)})| + |\lambda^{\sum_{i=1}^{k_n-1} h_n(b_n^{(i)})} f(a_{n+1}) - f(a_n)|,
\]
and recursively,
\[
|f_{n+1}(x) - f_n(x)| = |\sum_{i=1}^{k_n} h_n(b^{(i)}_n) \bar{f}(a_{n+1}) - \bar{f}(a_n)|
\leq \sum_{i=1}^{k_n} |\lambda^{h_n(b^{(i)}_n)} - h(b^{(i)}_n)| + |h(b^{(i)}_1 \ldots b^{(i)}_{k_n}) \bar{f}(a_{n+1}) - \bar{f}(a_n)|
= \sum_{i=1}^{k_n} |\lambda^{h_n(b^{(i)}_n)} - h(b^{(i)}_n)|.
\]

Note that since \( \sigma \) takes values from a finite set, \( k_n \) is uniformly bounded on \( n \), and this implies that the sum on the right hand is uniformly bounded for all \( x \). Thus the series \( \sum_{n \geq 1} |f_n - f_{n-1}|_\infty \) converges, and the sequence \( (f_n) \) converges uniformly to a continuous function \( \bar{f} \).

**Claim:** \( f \) is an eigenfunction for \( \lambda \). Note first that the maps \( x \mapsto j_n(x) \) and \( x \mapsto a_n(x) \) are continuous. For all \( n \) and for all \( x \) such that \( 0 \leq j_n(x) < h_n(a_n(x)) - 1 \), we have that \( j_n(Tx) = j_n(x) + 1 \) and \( a_n(Tx) = a_n(x) \). Hence \( f_n(Tx) = \lambda f_n(x) \). One has either \( 0 \leq j_n(x) = h_n(a_n(x)) - 1 \) for all \( n \) or \( 0 \leq j_n(x) < h_n(a_n(x)) - 1 \) for \( n \) large enough. By the same argument as in the proof of Lemma 3.6 there are finitely many points \( x \in X_\sigma \) such that \( j_n(x) = h_n(a_n(x)) - 1 \) for all \( n \), and by aperiodicity, they are well approximated by points \( y \) such that \( j_n(y) < h_n(a_n(y)) - 1 \). This proves that \( f \) is an eigenfunction on a dense subset of \( X_\sigma \), from which the result follows.

\[
\square
\]

### 4.3. Convergence, eigenvalues and measures

We discuss in this section the convergence properties involved in Theorem 4.7. We focus on the case where the incidence matrices are invertible (i.e., they have non-zero determinant). With the assumption of strong convergence discussed below, the S-adic shifts are uniquely ergodic; see e.g. [BD14, Theorem 5.7]. Here, we relate eigenvalues to the measures of cylinders in the uniquely ergodic case, i.e., continuous eigenvalues form a subset of the group generated by the measures of cylinders in the full set \( X, T \) stand for the incidence matrix of \( M_{(0,n)} \) summing to \( h_n(a) \), the existence of the limit \( h(a) := \lim_{n \to \infty} \lambda^{h_n(a)} \), with \( \lambda = e^{2\pi i} \), is equivalent to

\[
\lim_{n \to \infty} tM_{(0,n)} \equiv \arg h \mod Z^d,
\]

where the \( a \)-indexed entry of \( h \) equals \( h(a) \) and \( \arg h \) denotes the column vector whose \( a \)-th index equals the argument of \( h(a) \) (where the argument is considered modulo \( 2\pi \)). We focus here on the condition \( \lim_{n \to \infty} tM_{(0,n)} \equiv 0 \mod Z^d \), which is the case when the coboundary is trivial, i.e., \( h \equiv 1 \) and \( f \equiv 1 \), and we relate it with the notion of convergence for the products of matrices \( M_{(0,n)} \). Let \( e_a \) denote the vector of the canonical basis associated to the letter \( a \).

**Definition 4.8** (Strong and exponential convergence). Let \( u \in \mathbb{R}_{\geq 0}^d \) have entries which sum to 1. We say that the directive sequence \( \sigma \) on the alphabet \( \{1, \ldots, d\} \) is strongly convergent with respect to \( u \in \mathbb{R}_{\geq 0}^d \) if

\[
\lim_{n \to +\infty} \text{dist}(M_{(0,n)}e_a, \mathbb{R}u) = 0 \quad \text{for all } a \in \{1, \ldots, d\},
\]

where the distance dist refers to the usual distance of a point to a line, i.e., dist(\( x, \mathbb{R}u \)) is the infimum of the set \( \{\|x - \lambda u\|_2 \mid \lambda \in \mathbb{R}\} \). In this case we call \( u \) a generalised right eigenvector for \( \sigma \). The vector \( u \) is normalised if the sum of its entries equals 1.
We say that the directive sequence $\sigma$ is exponentially convergent with respect to $u \in \mathbb{R}^d_+$ if there exist $C, \alpha > 0$ such that
\[
\text{dist}(M_{[0,n]} e_a, \mathbb{R}u) < Ce^{-\alpha n} \quad \text{for all } n \text{ and all } a \in \{1, \ldots, d\}.
\]

Note that strong convergence implies that
\[
\bigcap_n M_{[0,n]} \mathbb{R}^d_+ = \mathbb{R}^d_+ u,
\]
with the latter property being called cone convergence, see for example [NPF20]. Note also that exponential convergence implies the convergence of $\sum_n \text{dist}(M_{[0,n]} e_a, \mathbb{R}u)$ for each $a$; the latter is called sum convergence.

A generalised right eigenvector for $\sigma$ can be seen as the generalisation of the Perron–Frobenius eigenvector of a primitive matrix.

Note that the property that $\sigma$ is exponentially convergent with respect to a generalised eigenvector has been identified and studied in the literature. For example, see [BD14, Theorem 6.3], and also [BST19, BST23]. This property has emerged in the Pisot case. Let us first recall a few definitions. A Pisot number is an algebraic integer greater than 1 whose Galois conjugates are all contained in the open unit disk. A Pisot irreducible matrix is such that its characteristic polynomial is the minimal polynomial of a Pisot number; such a matrix is primitive by [CS01b]. A Pisot irreducible substitution is one whose incidence matrix is Pisot irreducible. A stationary directive sequence given by a Pisot irreducible substitution is exponentially convergent to the Perron eigenvector [Hos86]. We recall that the only coboundary $h$ of a Pisot irreducible substitution is the trivial one, by Remark 3.24, that is, $h = 1$. Also, if the directive sequence is made of substitutions with the same Pisot irreducible incidence matrix, then the directive sequence will converge exponentially to the Perron eigenvector. Sets of substitutions with the same incidence matrix have been studied in [AMS14], [BSS18] (where it is called semi-compatibility), and [Rus20] (where it is called compatibility).

In particular, if the incidence matrix $M_\sigma$ of a Pisot irreducible substitution $\sigma$ has unit determinant, then the eigenvalues of $(X_\sigma, T)$ lie in the linear span of the Perron eigenvector for $M_\sigma$ [Hos86, Example (6.2)]. In this section we show that a similar result holds if our directive sequence is finitary, recognizable, straight, under the condition of strong convergence, with invertible matrices.

We start with Lemma 4.9, which extends Assertion 1 of Exercice 7.3.30 from [Fog02] and [Hos86, Lemma 1]. Similar extensions can be found e.g. with [BDM05, Lemma 15].

**Lemma 4.9.** Let $(M_n)_n \in \mathcal{M}^d$ where $\mathcal{M}$ is a finite set of $d \times d$ invertible matrices with integer entries. Assume that $(M_n)_n$ is strongly convergent with respect to a normalized vector $u \in \mathbb{R}^d_+$. Let $t \in \mathbb{R}$. Suppose that
\[
\lim_{n \to \infty} tM_{[0,n)} = 0 \mod \mathbb{Z}^d.
\]
Then the following holds.

(i) The vector $t1$ can be decomposed as $t1 = t_1 + t_2$ where $t_1 M_{[0,n)} \to 0$ and $t_2 M_{[0,n)} \in \mathbb{Z}^d$ for all $n$ large.

(ii) The vector $t_1$ is orthogonal to $u$.

(iii) The number $t$ is a rational combination of the entries of $u$. If the matrices $M_n$ are all unimodular, then $t$ is an integer combination of the entries of $u$.

(iv) For each $a$, let $h_n(a) = (1\ldots1)M_{[0,n)} e_a$. There exists $C > 0$ such that for each letter $a$ and each $n$, then
\[
|\lambda^{h_n(a)} - 1| \leq C \text{dist}(M_{[0,n]} e_a, \mathbb{R}u).
\]

**Remark 4.10.** Note that the condition $\lim_{n \to \infty} tM_{[0,n)} = 0 \mod \mathbb{Z}^d$ is equivalent to $\lim_n \lambda^{h_n(a)} = 1$ for all $a$ where $\lambda = \exp(2\pi it)$. Note that Assertion (iv) above also implies that if sum convergence holds, i.e., if the series $\sum_n \text{dist}(M_{[0,n]} e_a, \mathbb{R}u)$ converges, then the series $\sum_n |\lambda^{h_n(a)} - 1|$ converges.

**Proof.** **Proof of (i).** By assumption, we can write
\[
t1 M_{[0,n)} = u_n + v_n.
\]
where \( u_n \in \mathbb{Z}^d \) and \( v_n \to 0 \) as \( n \to \infty \). Note that for each \( n \)
\[
\begin{align*}
  u_{n+1} + v_{n+1} &= t1M_{(0,n+1)} = (u_n + v_n)M_n \\
  &= u_nM_n + v_nM_n,
\end{align*}
\]
so
\[
  u_{n+1} - u_nM_n = v_nM_n - v_{n+1},
\]
and since \( M_n \) is one of finitely many matrices, the right hand side of this last expression converges to \( 0 \), so therefore \( u_{n+1} - u_nM_n \to 0 \) as \( n \to \infty \). But \( u_{n+1} - u_nM_n \) is an integer valued row vector. We conclude that there exists \( N \) such that
\[
u_{N+1} - u_NM_N = 0 \text{ and } u_{N+m} - u_NM_{N,N+m} = 0 \text{ for all } m \geq 1.
\]

As each of the matrices \( M_n \) is invertible, we can find a vector \( t_2 \) such that \( t_2M_{(0,N)} = u_N \), and so
\[
u_n = t_2M_{(0,n)} \text{ for } n \geq N.
\]
Write \( t_1 := t1 - t_2 \). Then, for every \( n \geq N \), one has
\[
t_1M_{(0,n)} = t1M_{(0,n)} - t_2M_{(0,n)} = t1M_{(0,n)} - u_n = v_n.
\]
But \( v_n \to 0 \) as \( n \to \infty \), so \( t_1M_{(0,n)} \to 0 \) as \( n \to \infty \).

**Proof of (ii).** By the assumption of strong convergence, we have \( \lim_{n \to +\infty} \text{dist}(M_{(0,n)}e_a, \mathbb{R}u) = 0 \) for each \( a \). In particular, cone convergence holds. By continuity of the scalar product, one has \( \lim_{n \to +\infty} \frac{(t_1, M_{(0,n)}u)}{|(t_1, M_{(0,n)}u)|} = \frac{(t_1, u)}{|(t_1, u)|} \). Moreover, \( \lim_{n \to +\infty} (t_1, M_{(0,n)}u) = 0 \) from above. Therefore \( (t_1, u) = 0 \).

**Proof of (iii).** One has \( t1 = t1 + t2 \) and \( t = (t1, u) = (t1 + t2, u) = (t2, u) \) and \( t2 = u_{N+1}^{-1}(M_{(0,N+2)})^{-1} \). Here \( u_{N+1}^{-1} \) has integer entries, as well as the invertible matrices \( M_n \), so the entries of \( t2 \) are also rational numbers. Hence \( t \) is a rational combination of the entries of \( u \), and if the matrices are unimodular, then it is even a linear combination of the entries of \( u \).

**Proof of (iv).** By Part (i), \( t1M_{(0,n)} \equiv t1M_{(0,n)} \mod \mathbb{Z}^d \) for all \( n \). Write \( \lambda^{h_n(a)} = \exp(2i\pi t_n(a)) \).

For each \( a \) and each \( n \), one has
\[
|\lambda^{h_n(a)} - 1| \leq 2\pi \|t1, M_{(0,n)}e_a\| \quad \text{and} \quad |\langle t1, M_{(0,n)}e_a\rangle| \leq \|t1\|_2 \text{dist}(M_{(0,n)}e_a, \mathbb{R}u),
\]
by the Cauchy-Schwartz inequality together with the fact that \( t1 \) is orthogonal to \( u \) for the second inequality. \( \square \)

The following result relates eigenvalues and measures of letter cylinders.

**Theorem 4.11.** Let \( \sigma = (\sigma_n)_{n \geq 0} \) be a finitary, straight, recognizable directive sequence on \( \mathcal{A} \), where each substitution occurs infinitely often, and where each incidence matrix is invertible. Suppose that \( (X_\sigma, T) \) is aperiodic, and that \( \sigma \) is strongly convergent. Let \( \mu \) be the (unique) shift-invariant measure on \( (X_\sigma, T) \). Let \( \lambda = e^{2\pi i} \) in \( S^1 \) such that \( \lim_n \lambda^{h_n(a)} = 1 \) for every \( a \in \mathcal{A} \). If
\[
\sum_n \max_a \text{dist}(M_{(0,n)}e_a, \mathbb{R}u) \leq \infty,
\]
then \( \lambda \) is a continuous eigenvalue, and \( t \) is a \( \mathbb{Q} \)-linear combination of the measures of the letter cylinders \( \mu[a] \), and even a \( \mathbb{Z} \)-linear combination if \( \sigma \) is unimodular. Finally, \( \mu[a] \) is a continuous eigenvalue for every letter \( a \).

**Proof.** Strong convergence, together with the assumption of having invertible matrices, implies unique ergodicity [BD14, Theorem 5.7], and similarly as in the substitutive case, the entries of the generalised normalised right eigenvector \( u \) are given by the measures \( \mu[a] \) of one-letter cylinders. The fact that \( \lambda \) is a continuous eigenvalue comes from Theorem 4.7, with the constant trivial coboundary \( (h, f) \), with \( h \equiv 1 \) and \( f \equiv 1 \), and Condition (iv) of Lemma 4.9. Lemma 4.9 then implies that \( t \) is a \( \mathbb{Q} \)-linear combination of the measures of the letter cylinders \( \mu[a] \), and if \( \sigma \) is unimodular, \( t \) is a \( \mathbb{Z} \)-linear combination.
It remains to prove that for each letter $a$, $t = \mu[a]$ satisfies $\lim_n \lambda^{h_n(a)} = 1$, where $\lambda = e^{2\pi it}$. The projection of $M_{[0,n]}e_a$ along the hyperplane orthogonal to the vector 1 onto $\mathbb{R}^+u$ is $h_n(a)u$, whose $a$th entry is $\mu[a]h_n(a)$. Since the matrix $M_{[0,n]}$ has integer entries, the convergence to 0 of $\text{dist}(M_{[0,n]}e_a, \mathbb{R}^+u)$ implies the convergence of $\mu[a]h_n(a)$ to 0 modulo 1. We again conclude with Theorem 4.7, together with Condition (iv) of Lemma 4.9 that $\mu[a]$ is a continuous eigenvalue. \hfill \Box

Remark 4.12. We can also use the fact that $\sum_n \max_\sigma \text{dist}(M_{[0,n]}e_a, \mathbb{R}^+u) < +\infty$ implies the so-called balance property under the finitary assumption (see [BD14, Theorem 5.8]). We then again deduce that $\mu[a]$ is a continuous eigenvalue, via [BCB19, Theorem 1], which relies on the Gottshalk-Hedlund theorem.

Remark 4.13. The convergence condition of Theorem 4.11 holds in particular if one has exponential convergence for the directive sequence but we can go beyond here: the convergence just needs to be sufficiently fast for the series to converge. This has to be compared with the measurable case where the convergence is slower (see Theorem 5.10). This is a classical phenomenon observed for some proper directive sequences; see [DFM19] or [CFM08] for Arnoux-Rauzy shifts; see also [Nad11, Corollary 15.57].

5. Measurable eigenvalues

In this section we equip an $S$-adic shift with an invariant measure and study the existence of measurable eigenvalues. For this, Theorem 2.8 is particularly useful, as, with the conditions that the directive sequence is recognizable and everywhere growing, it allows us to use and compare results concerning Bratteli-Vershik systems. In particular we state results from [BKMS13], [Men91] and [CDHM03] which are couched in terms of Bratteli diagrams, or cutting-and-stacking transformations, but which we rephrase in our language. Our main result in this section is Theorem 5.10. It requires a strengthening of the notion of straightness to that of strong straightness, in Definition 5.6. The combination of strong primitivity and finitary ensures that our systems are of exact finite rank, a condition which has been extensively studied in the literature [Men91, Bos92], and one which is useful in the proof of Theorem 5.10. We recall the partitions $(Q_n)$ that were defined in Section 2.4.

Lemma 5.1. [BSTY19, Lemma 6.3] Let $\sigma = (\sigma_n)_{n \geq 0}$ be a recognizable and everywhere growing directive sequence on $A$. If $\mu$ is a shift invariant probability measure on $X_{\sigma}$, then one has $\mu(\cap_{n=0}^{\infty} \bigcup_{a \in A} B_n(a)) = 0$, and $(Q_n)$ is generating in measure.

We give an indication of the proof of Lemma 5.1. In fact, the partitions $Q_n$ determine the past of any point which is not in the $T$-orbit of a limit word. This is because for such a point $x$, we have that infinitely often $x$ belongs neither to the base $\sigma_{[0,n]}([a])$ of any $T_n(a)$, nor to the top level $T^{[h_n(b)-1]1_{\sigma_{[0,n]}([b])}}$ of any $T_n(b)$. Therefore, if $n$ is such that $x \not\in \bigcup_{a \in A} \sigma_{[0,n]}([a]) \cup \bigcup_{b \in A} T^{[h_n(b)-1]1_{\sigma_{[0,n]}([b])}}$, then we have complete knowledge of $x_{[-k_n,k_n]}$, where $k_n = \min_{a \in A} h_n(a)$. We then let $n$ become large and use the hypothesis that $\sigma$ is everywhere growing.

We now introduce the following notion, which is key to the proof of Theorem 5.10.

Definition 5.2 ($S$-adic exact finite rank). We say that the directive sequence $\sigma$ is of exact finite rank if it is primitive and recognizable, and if there is an invariant probability measure $\mu$ on $X_{\sigma}$ and a constant $\delta > 0$ such that $\mu(T_n(a)) \geq \delta$ for all $n \geq 0$ and all $a \in A$.

We keep in mind that $\mu(T_n(a)) = h_n(a)\mu(B_n(a))$. Thus if the heights of the towers $T_n(a)$ and $T_n(b)$ are comparable in an exact finite rank system, then so is the $\mu$-mass of their bases. Note also that the vector with entries $(\mu(B_n(a))_a)$ is equal to the generalised normalised right eigenvector of the sequence of matrices $(M_k)_{k \geq n}$, i.e., $\mu([a])_a = M_{[0,n]}(\mu(B_n(a))_a)$ for a finitary, primitive and strongly convergent directive sequence made of invertible matrices (see e.g. [BR10, Section 6.8.1] and [BD14, Theorem 3.10] for the existence of frequencies under these assumptions). Indeed one has $\mu(B_n(a)) = (M_n\mu(B_{n+1}(b))_a)$.

Even though the notion of exact finite rank is essentially one that describes the given dynamical system, nevertheless whether it holds is a function of the given $S$-adic representation of the
system, for, as discussed in [BKMS13, Remark 6.9], a dynamical system may have two S-adic representations, only one of which is of exact finite rank. Boshernitzan showed that dynamical systems which have an exact finite rank representation are uniquely ergodic by [Bos92, Theorem 1.2], so we will denote the unique invariant measure by $\mu$.

Finally we repeat here a condition, [BKMS13, Proposition 5.7], which identifies a large family of exact finite rank directive sequences. Note that most of the results in [BKMS13] concerning exactness do not depend on the ordering of the Bratteli diagram, i.e., there is no requirement that the substitutions in the directive sequence be proper.

**Proposition 5.3.** Let $\sigma$ be a recognizable directive sequence on $A$, where the incidence matrix $M_{\sigma_n} = (m_{ab}^{(n)})_{a,b}$ of $\sigma_n$ is positive for each $n$. If there is a constant $c > 0$ such that

$$\frac{\min_{a,b} m_{ab}^{(n)}}{\max_{a,b} m_{ab}^{(n)}} \geq c$$

for each $n$, then $\sigma$ is of exact finite rank, and $h_n(a)/h_n(b) \geq c$ for each $n$ and pair of letters $a,b$.

As a direct consequence of Proposition 5.3, we get that if the system is finitary and strongly primitive, then $\sigma$ is of exact finite rank, and thus uniquely ergodic as recalled above by [Bos92, Theorem 1.2]. Exact finite rank systems and linearly recurrent shifts enjoy similar properties, though they are different. Linear recurrence requires specific additional combinatorial properties.

**Theorem 5.4.** [Dur03, Proposition 1.1] The subshift $(X,T)$ is linearly recurrent if and only if there exists a strongly primitive, finitary and proper directive sequence $\sigma$ such that $(X,T) = (X_\sigma, \sigma)$.

In particular, linearly recurrent systems have an exact finite rank representation. However, the next example shows that exact finite rank directive sequences are not necessarily linearly recurrent.

**Example 5.5.** We have shown that the directive sequence

$$\sigma = \sigma \circ \tau \circ \sigma \circ \tau \circ \sigma \circ \tau \circ \sigma \circ \tau \circ \sigma \circ \tau \circ \sigma \circ \tau \circ \sigma \circ \tau \circ \sigma \circ \tau \circ \sigma \circ \tau \circ \sigma \ldots$$

from Example 2.7 is recognizable. Note that $\tau \circ \sigma$ has a positive incidence matrix, as do $\sigma \circ \sigma$ and $\sigma \circ \tau \circ \sigma$. Hence we can boundedly telescope the directive sequence $\sigma$ as

$$\sigma \circ \tau \circ \sigma \circ \tau \circ \sigma \circ \tau \circ \sigma \circ \tau \circ \sigma \circ \tau \circ \sigma \circ \tau \circ \sigma \circ \tau \circ \sigma \circ \tau \circ \sigma \circ \tau \circ \sigma \circ \tau \circ \sigma \ldots$$

where this directive sequence takes values in $\{\sigma \circ \tau \circ \sigma, \sigma \circ \sigma, \tau \circ \sigma, \sigma \circ \tau \circ \sigma, \sigma \circ \sigma, \sigma \circ \sigma, \tau \circ \sigma, \ldots\}$. We conclude by Proposition 5.3 that $(X_\sigma, T)$ is of exact finite rank, although it is not linearly recurrent such as stressed in [Dur03].

We now introduce the S-adic counterpart of the notion of strong straightness from Definition 3.8 (stated in the stationary case). This notion allows one to relax the condition of properness, which we often do not have with a given directive sequence, even if it defines a linearly recurrent shift. For example, the conditions of Theorem 5.10 imply that the shift is linearly recurrent, by [Dur03, Lemma 3.1], without assuming properness.

**Definition 5.6 (Strong straightness).** Let $\sigma$ be a straight directive sequence on $A$. For each letter $a \in A$ and for each $n$, let $a_n$ be the first letter of $\sigma_n(a)$. We call $\sigma$ **strongly straight** if

(i) for each letter $a \in A$ and each non-negative $n$, $\sigma_n(a_{n+1})$ starts with $a_n$,

(ii) if $a = b$, then $(a_n) = (b_n)$, and

(iii) there exists $r$ such that for each $n$, for each two-letter word $a\beta \in L_{\sigma}^{(n)}$, there exists a letter $\gamma$ such that $a\beta$ is a subword of $\sigma(a_{n+r})\gamma$.

Note that if $\sigma$ is stationary and if its constant term $\sigma$ is strongly straight in the sense of Definition 3.8, then it satisfies the conditions of Definition 5.6. Moreover a strongly straight directive sequence is a straight directive sequence, i.e., each letter $a$ defines a unique right-infinite limit word $a$. Indeed Condition (i) guarantees that the prefix $\sigma_{[0,n]}(a_n)$ of $\sigma_{[0,n+1]}(a)$ is also a prefix of $\sigma_{[0,n+k]}(a)$ for each $a$, $n \geq 1$ and $k \geq 0$, and it is also a prefix of the unique right-infinite limit word $a$ defined by $a$, since the sequence of words $(\sigma_{[0,n]}(a_n))$ is a nested sequence of words
that increases in length. See Figure 2 for an illustration. Note also that Condition (ii) holds if \( \sigma \) is one-sided recognizable. We stress the fact that strongly straight directive sequences are distinct from proper ones. For instance, a directive sequence made of substitutions that are such that the image of each letter \( a \) starts with \( a \) is an example of a strongly straight sequence. See also Condition (i) in Example 6.10.

**Example 5.7.** The directive sequence \( \sigma \) from our running example 2.7 is not strongly straight since \( a = b \), \( a_n = a \) for all \( n \), and \( b_n = b \) for all \( n \) such that \( \sigma_n = \sigma \). Moreover one checks that Condition (iii) does not hold.

The following example shows that there exist \( S \)-adic systems that are finitary, recognizable, strongly prefix-straight and not linearly recurrent. However note that the combination of the conditions finitary, strongly straight and strongly primitive implies linear recurrence, since factors of length 2 occur with uniform bounded gaps, by [Dur03, Lemma 3.1].

**Example 5.8.** Consider the substitutions \( S = \{ \sigma, \tau \} \) with

\[
\begin{align*}
    a & \mapsto aaaa & a & \mapsto bbbb \\
    b & \mapsto abaa & b & \mapsto babb,
\end{align*}
\]

and let

\[
\sigma = \tau, \sigma, \tau, \sigma, \sigma, \tau, \sigma, \sigma, \tau, \sigma, \sigma, \tau, \sigma, \sigma, \ldots.
\]

We follow here the same scheme of proof as in [Dur03, Section 2]. One verifies that the difference between two successive occurrences of the word \( abb \) in \( \tau \circ \sigma^n(z) \) is greater than \( 4^n \) for any sequence \( z \). For \( n \geq 1 \), let \( \rho_n = \tau \circ \sigma \circ \tau \circ \sigma^2 \circ \cdots \circ \tau \circ \sigma^{n-1} \). One checks that if \( w \) is a strict return word of \( abb \) in \( \tau \circ \sigma^n(z) \), then \( \rho_n(w) \) is a strict return word of \( \rho_n(abb) \) in \( X_\sigma \). Thus there exists \( C > 0 \) such that \( \frac{1}{\rho_n(abb)} \geq C4^n \). This implies that \( X_\sigma \) is not linearly recurrent.

We remark that the notion of a strongly straight directive sequence is restrictive. However the conditions are not difficult to verify. We have gathered all notions required to state the following generalisation of Proposition [CDHM03, Proposition 8]. Its topological counterpart is given by Theorems 4.7. We start with a preliminary lemma. For all \( n \), let \( B_n \) denote the \( \sigma \)-algebra generated by \( Q_n \) and let \( \mathbb{E}_n : L^1(X_\sigma, B, \mu) \to L^1(X_\sigma, B_n, \mu) \) be the conditional expectation operator; write

![Figure 2. Part of the natural Bratteli diagram for a strongly straight directive sequence, where the letters \( a \) and \( b \) determine the same right-infinite limit word. Dashed edges correspond to minimal edges from \( a \) or \( b \) which lead directly to the limit word \( a = b \). The solid edges depict the limit word \( a \).](image-url)
\( f_n = \mathbb{E}_n(f) \). The function \( f_n \) is constant on elements of \( Q_n \). The sequence \( (f_n) \) is a martingale and converges to \( f \) in \( L^2(\mu) \).

**Lemma 5.9.** Suppose that \((X_\sigma, T, \mu)\) is of exact finite rank, with eigenfunction \( f \) such that \(|f| = 1\).

For each \( a \in \mathcal{A} \), let \( v_n(a) := \mathbb{E}_n(f) \) on \( B_n(a) \). Then \( |v_n(a)| \to 1 \).

**Proof.** As \( f \) is an eigenfunction, for the eigenvalue \( \lambda \), then \( \mathbb{E}_n(f) = f = \lambda^j v_n(a) \) on \( T^j(B_n(a)) \) for \( 0 \leq j < \sigma_{(0,n)}(a) \), and so for all \( x \in X_\sigma \)

\[
  f_n(x) = \sum_{a \in \mathcal{A}} \sum_{j=0}^{\sigma_{(0,n)}(a) - 1} v_n(a) \lambda^j \chi_{T^j(B_n(a))}(x),
\]

where \( \chi \) refers to the indicator function. Therefore \( \|f_n\|_2^2 = \sum_{a \in \mathcal{A}} |v_n(a)|^2 \mu(T_n(a)) \). As \( |v_n(a)| = \left| \frac{1}{\mu(B_n(a))} \int_{B_n(a)} f(x) \, d\mu(x) \right| \leq 1 \), then for any \( a \in \mathcal{A} \), we have

\[
  \|f_n\|_2^2 \leq |v_n(a)|^2 \mu(T_n(a)) + \mu(\cup_{b \neq a} \mu(T_n(b)))
  \leq |v_n(a)|^2 \mu(T_n(a)) + 1 - \mu(T_n(a)) = (|v_n(a)|^2 - 1) \mu(T_n(a)) + 1.
\]

Now suppose that \( |v_n(a)| \to \rho < 1 \) for some subsequence \( (n_k)_k \). The hypothesis of exact finite rank tells us that there is a \( \delta > 0 \) such that \( \mu(T_n(a)) \geq \delta \) for each \( n \). Thus for \( \varepsilon \) sufficiently small

\[
  \lim_k \|f_{n_k}\|_2^2 < (\rho + \varepsilon - 1)\delta + 1 < 1
\]

for \( k \) large enough, contradicting the fact that \( \|f_n\|_2^2 \to \|f\|_2^2 = 1 \). \( \Box \)

Recall the comment after Proposition 5.3 that finitary, strongly primitive directive sequences define uniquely ergodic shifts \((X_\sigma, T, \mu)\).

**Theorem 5.10.** Let \( \sigma \) be a finitary, strongly straight, recognizable and strongly primitive directive sequence on \( \mathcal{A} \). If \( \lambda \in \mathbb{S}^1 \) is a measurable eigenvalue of \((X_\sigma, T, \mu)\), then

\[
  \sum_{n=1}^{\infty} |\chi^{b_n}(w_n(a,b)) - 1|^2 < \infty
\]

for any \( a, b \) such that \( a = b \) and any \( w_n(a,b) \in \{ w \in \mathcal{L}_\sigma^{(n)} : w \) is a strict transition word from \( a \) to \( b \} \).

**Proof.** We follow [CDHM03]. We recall from Section 2.4 that given \( a, b \in \mathcal{A} \) and \( n \),

\[
  t_n(a, b) = \{ 0 \leq t < h_n(a) : T^t(B_n(a)) \subset B_{n-1}(b) \},
\]

and the cardinality of \( t_n(a, b) \) equals the \((b,a)\) entry of \( M_{\sigma_{n-1}} \). In particular \( t_n(a, b) \neq \emptyset \) if and only if \( b \) appears in \( \sigma_{n-1}(a) \), and \( 0 \in t_n(a, b) \) means that \( b \) is the first letter of \( \sigma_{n-1}(a) \). See Figure 1 for an illustration.

Let \( f \) be an eigenfunction for \( \lambda \). As \( |\lambda| = 1 \), and \((X_\sigma, \sigma)\) is minimal, we can take \(|f| = 1\) almost everywhere. Recall that \( f_n = \mathbb{E}_n(f) \) where \( \mathbb{E}_n \) is the conditional expectation defined by the \( \sigma \)-algebra generated by \( Q_n \). The functions \( f_n - f_{n-1} \) are mutually orthogonal (see for example [Doo53] for an exposition on martingales), so that \( \sum_{n=1}^{\infty} \|f_n - f_{n-1}\|_2^2 < \infty \).

Since \( \sigma \) is strongly primitive, there exists \( r \) such that for each \( n \), \( \sigma^{(n,r+n)} \) has a positive incidence matrix. As \( \sigma \) takes values from a finite set, we can telescope \( \sigma \) so that it takes values from a finite set and all incidence matrices are positive. This implies that there exists \( c \) such that

\[
  \frac{\min_{a,b} m_{ab}^{(n)}}{\max_{a,b} m_{ab}^{(n)}} \geq c,
\]

which allows us to conclude, by Proposition 5.3, that \( \sigma \) is of exact finite rank, so that there is \( \delta > 0 \) with \( h_n(a) \mu(B_n(a)) \geq \delta \) for each \( a \).
Let \( j \in t_n(a, b) \). Recall \( v_n(a) := E_n(f) \) on \( B_n(a) \). Then, for each \( 0 \leq p < h_{n-1}(b) \), we have \( T^{j+p}B_n(a) \subset T^pB_n(b) \), and if \( x \in T^{j+p}B_n(a) \), then \( f_n(x) = \lambda^{j+p}v_n(a) \), \( f_{n-1}(x) = \lambda^p v_n(b) \). So

\[
||f_n - f_{n-1}||_2^2 \geq h_{n-1}(b) \mu(B_n(a)) |\lambda^j v_n(a) - v_{n-1}(b)|^2 \geq \delta \frac{h_{n-1}(b)}{h_n(a)} |\lambda^j v_n(a) - v_{n-1}(b)|^2.
\]

Let \( L \) be the maximum sum of the entries of a column of any of the incidence matrices, and let \( b^* \) be such that \( h_{n-1}(b^*) = \min_{a \in A} h_{n-1}(b) \). One has

\[
\max_{a, b} \frac{h_n(a)}{h_{n-1}(b)} = \max_{a, b} \frac{h_n(a)}{\min_{b} h_{n-1}(b)} = \frac{\max_{a} h_n(a)}{h_{n-1}(b^*)} \leq \frac{L}{e},
\]

where the last statement follows from Proposition 5.3. This implies that \( |\lambda^j v_n(a) - v_{n-1}(b)|^2 \leq \frac{1}{e^2} ||f_n - f_{n-1}||_2^2 \). Taking the maximum, over all \( j \in t_n(a, b) \) and \( a, b \in A \), we have

\[
\sum_{n=1}^{\infty} \max_{a, b} \max_{j \in t_n(a, b)} |\lambda^j v_n(a) - v_{n-1}(b)|^2 < \infty.
\]

We use the notation of Definition 5.6, writing that \( \sigma_{n-1}(a) \) starts with \( a_{n-1} \). This gives \( 0 \in t_n(a, a_{n-1}) \), and letting \( j = 0 \) in (5.2), we get

\[
\sum_{n=1}^{\infty} |v_n(a) - v_{n-1}(a_{n-1})|^2 < \infty.
\]

Now assume that \( a = b \). Since \( \sigma \) is strongly straight, then in addition to \( 0 \in t_n(a, a_{n-1}) \) for each \( n \) (i.e., \( \sigma_{n-1}(a) \) starts with \( a_{n-1} \)), we also have \( 0 \in t_n(b, a_{n-1}) \) for each \( n \) (i.e., \( \sigma_{n-1}(b) \) starts with \( a_{n-1} \)), since \( b_{n-1} = a_{n-1} \). See also Figure 2.

Using (5.2) two more times with \( j = 0 \), we get

\[
\sum_{n=2}^{\infty} |v_{n-1}(a_{n-1}) - v_{n-2}(a_{n-2})|^2 < \infty \text{ and } \sum_{n=2}^{\infty} |v_{n-2}(a_{n-2}) - v_{n-1}(b)|^2 < \infty,
\]

and we obtain

\[
\sum_{n=1}^{\infty} \max_{a, b} |v_n(a) - v_{n-1}(b)|^2 < \infty.
\]

As \( \sigma \) is of exact finite rank, then by Lemma 5.9, \( |v_n(a)| \to 1 \) for each \( a \in A \). Therefore from (5.2) and (5.3), we obtain

\[
\sum_{n=1}^{\infty} \max_{a, b} \max_{j \in t_n(a, b)} |\lambda^j - 1|^2 < \infty.
\]

Since all incidence matrices are positive, any strict transition word \( aw_n \in L(n) \) from \( a \) to \( b \) must appear in \( \sigma_n(\alpha \beta) \) for some word \( \alpha \beta \in L_\sigma^{n+1} \) of length 2. Also, by the assumption (iii) of the definition of strong straightness, there exists \( r' \) such that the word \( \alpha \beta \) appears in \( \sigma_{[n, n+r']}(\gamma) \) for some \( \gamma \). Since all incidence matrices are positive, \( \alpha \beta \) appears in \( \sigma_{[n, n+r'+1)}(a) \). Thus by telescoping boundedly, we can assume that \( aw_n b \) appears in \( \sigma_n(a) \). We write \( \sigma_n(a) = p_n aw_n b s_n \). Then \( j_n := |h_n(p_n)| \in t_{n+1}(a, a) \), \( J_n := |h_n(p_n a w_n)| \in t_{n+1}(a, b) \), and we have

\[
\sum_{n=1}^{\infty} |\lambda^{j_n} - 1|^2 < \infty \text{ and } \sum_{n=1}^{\infty} |\lambda^{J_n} - 1|^2 < \infty.
\]

Thus

\[
\sum_{n=1}^{\infty} |\lambda^{h_n(a w_n)} - 1|^2 = \sum_{n=1}^{\infty} |\lambda^{j_n} - 1|^2 = \sum_{n=1}^{\infty} |\lambda^{J_n} - \lambda^{j_n}|^2 \leq \sum_{n=1}^{\infty} |\lambda^{j_n} - 1|^2 + \sum_{n=1}^{\infty} |\lambda^{j_n} - 1|^2 < \infty.
\]
6. THE CONSTANT-LENGTH $S$-ADIC CASE

We apply the results of the previous sections to the family of $S$-adic systems generated by constant-length directive sequences. This is a natural starting point; after all Host’s work on coboundaries generalises the notion of height in Dekking’s and Kamae’s works on constant-length substitutions; see [Dek78, Kam72] and also [Que10]. We first discuss eigenvalues and constant coboundaries in the continuous case in Section 6.1, and then focus on the Toeplitz $S$-adic shifts in Section 6.2, and we lastly consider measurable eigenvalues in Section 6.3.

6.1. Eigenvalues and constant coboundaries. We first start with the following remark, easily providing continuous eigenvalues.

Remark 6.1. Let $(\sigma_n)$ be a recognizable and constant-length directive sequence with sequence of lengths $(q_n)$. Assume that $X_{\sigma}$ is aperiodic. Note that for any $n$, $\lambda = e^{2\pi i q_n}$ is an eigenvalue of the recognizable $S$-adic system of constant length $(X_{\sigma}, T)$. For, as $h_{m+1}(a) = q_0 q_1 \ldots q_m$, for each letter $a$, we can use the partition $Q_{m+1} = \{ T^k [\sigma]_m (+1) : a \in A, 0 \leq k < q_0 q_1 \ldots q_m \}$ to build an eigenfunction. Namely, let $f(x) = \lambda^k$ if $x \in T^k [\sigma]_m (+1)$. Then $f$ is a continuous eigenfunction for the eigenvalue $\lambda$.

We can translate the previous remark to information about equicontinuous factors of $(X_{\sigma}, T)$. Indeed, we first recall that if $\exp(2\pi i /r)$ is an eigenvalue of $(X, T)$, then the addition of 1 on the finite group $\mathbb{Z}/r\mathbb{Z}$ is a factor of $(X, T)$. We now extend this fact to the case where $\exp(2\pi i /q_n)$ is an eigenvalue of $(X, T)$ for each $n$. We first recall the definition of an odometer associated to a sequence $(q_n)_n$ of positive integers, where we assume $q_n \geq 2$. Given a sequence $(q_n)_n \geq 1$ of positive integers, define the group

$$\mathbb{Z}_{(q_n)} := \prod_{n \geq 1} \mathbb{Z}/q_n \mathbb{Z},$$

where the group operation of addition is performed with carry. For a detailed exposition of equivalent formal definitions, we refer the reader to [Dow05]. Endowed with the product topology over the discrete topology on each $\mathbb{Z}/q_n \mathbb{Z}$, the group $\mathbb{Z}_{(q_n)}$ is a compact metrizable topological group, where the multiples of the unit $z = \ldots 0 1$, which we simply write as $z = 1$, are dense.

We write elements $(z_n)_{n \geq 1}$ of $\mathbb{Z}_{(q_n)}$ as left-infinite sequences $\ldots z_2 z_1$ where $z_n \in \mathbb{Z}/q_n \mathbb{Z}$, so that addition in $\mathbb{Z}_{(q_n)}$ has the carries propagating to the left in the usual way as in $\mathbb{Z}$. If $q_n = p$ is constant, then $\mathbb{Z}_{(q_n)} = \mathbb{Z}_{(p)}$ is the classical ring of $p$-adic integers, and the addition of 1 on the $p$-adic group $\mathbb{Z}_{(p)}$ is a factor if and only if $\exp(2\pi i /p^n)$ is an eigenvalue for every $n \geq 1$ (see e.g. [Fog02]). Given $(q_n)$ and $h \in \mathbb{N}$, we use $\mathbb{Z}_{h(q_n)}$ to denote the group defined by the sequence $\hat{h}, q_1, q_2, \ldots$. Finally recall that an odometer is a dynamical system $(Z, +1)$ where $Z = \mathbb{Z}_{(q_n)}$ for some sequence $(q_n)$ and $+1$ denotes the homeomorphism of adding the unit 1.

The following arithmetic lemma will be essential when finding either measurable or topological eigenvalues of a constant-length $S$-adic shift. It can be considered as an analogue of the discussion in Section 2.2 in [Hos86] (see also Lemma 4.9). We stress the fact that in the constant-length case, coboundaries $h$ are constant they do not depend on letters), since they are given by limits of the form $h = \lim_n \lambda^0 \ldots q_n$. The following statement is motivated by the fact that taking the quotient of two successives terms yields $\lim_n \lambda^0 \ldots q_n (q_{n+1} - 1) = 1$, from which we can deduce crucial information in the finitary case.

Lemma 6.2. Let $\lambda \in S^1$, let the sequence $(q_n)$ of integers take on finitely many values, and suppose that $q_n \geq 2$ for all $n$. If

$$\lim_n \lambda^0 \ldots q_n (q_{n+1} - 1) = 1,$$

then $\lambda$ is rational, and more precisely, for each positive integer $m$ sufficiently large, there exists $k$ such that

$$\lambda = e^{2\pi i /q_0 \ldots q_m (q_{m+1} - 1)}.$$

Proof. Let $\lambda = e^{2\pi i t}$. By assumption

$$q_0 \ldots q_n (q_{n+1} - 1)t = k_n + \varepsilon_n$$

for all $n$. Since the sequence $(q_n)$ is aperiodic, we can replace it by a sequence of integers $n_k$ which has bounded partial quotients (e.g. $n_k = q_1 \ldots q_m$ for all $k$). Then

$$\lambda^{k_n + \varepsilon_n} = e^{2\pi i t (k_n + \varepsilon_n)} = e^{2\pi i t k_n} e^{2\pi i t \varepsilon_n} = \lambda^{k_n} e^{2\pi i t \varepsilon_n}.$$

Since $e^{2\pi i t \varepsilon_n}$ is a bounded function, it follows that $\lambda^{k_n}$ is a bounded function of $\lambda$. Therefore, $\lambda$ is rational.
where \( k_n \in \mathbb{Z} \) and \( \varepsilon_n \to 0 \). Let

\[
    r_n := q_{n+1} \frac{q_{n+2} - 1}{q_{n+1} - 1}.
\]

Note that the sequence \((r_n)\) takes on finitely many values. We have that

\[
    k_{n+1} - r_n k_n = q_0 \cdots q_{n+1}(q_{n+2} - 1)t - \varepsilon_{n+1} - r_n (q_0 \cdots q_n(q_{n+1} - 1)t - \varepsilon_n)
    = q_0 \cdots q_{n+1}(q_{n+2} - 1)t - r_n (q_0 \cdots q_n(q_{n+1} - 1)t) - \varepsilon_{n+1} + r_n \varepsilon_n
    = -\varepsilon_{n+1} + r_n \varepsilon_n.
\]

Since the sequence \((r_n)\) takes on finitely many values and \( \varepsilon_n \to 0 \), we get \( k_{n+1} - r_n k_n \to 0 \). Since in addition \((k_n)\) is integer valued, we have \( k_{n+1} - r_n k_n = -\varepsilon_{n+1} + r_n \varepsilon_n = 0 \) for \( n \) large enough. But then, recursively, we get \( \varepsilon_{n+\ell} = r_{n+\ell-1} \cdots r_{n+1} r_n \varepsilon_n \) for all \( \ell \geq 1 \). Since \( r_j > 1 \) for each \( j \), we conclude that \( \varepsilon_n = 0 \) for all \( n \) large. The statement of the lemma follows from (6.2).

With the notation of the previous lemma, then writing \( k \) and \( q_0 \cdots q_m(q_{m+1} - 1) \) in their unique prime factorization and simplifying the common factors, we even get an irreducible quotient \( \frac{q}{q'} \), where the prime factors of \( q \) are in those of \( \{q_0, \ldots, q_m, q_{m+1} - 1\} \).

Example 6.3. We give an example to show that the finitary assumption (i.e., of finitely many values \( q_n \)) in Lemma 6.2 cannot be relaxed without additional qualification. Take \( \lambda = e^{2\pi i \sum_{k=1}^{\infty} \frac{1}{10^k}} \). The exponent \( t = \sum_{k=1}^{\infty} \frac{1}{10^k} \) is the Liouville number, known to be transcendental. Take \((q_n)\) such that \( q_0 q_1 \cdots q_n = 10^{n!} \) for \( n \geq 1 \). Then \( q_0 \cdots q_n \sum_{k=1}^{\infty} \frac{1}{10^k} \mod \mathbb{Z} \equiv 10^{n!} \sum_{k=n+1}^{\infty} \frac{1}{10^k} \to n \to 0 \), so (6.1) is satisfied.

Remark 6.4. If \( \lambda \) is rational, i.e., if \( \lambda = e^{2\pi i t} \) with \( t \in \mathbb{Q} \), and \( h(a) = \lim_{n \to \infty} \lambda^{h_n(a)} \) for some \( a \), then \( h(a) \) is rational. Note though that \( h(a) \) rational does not necessarily imply that \( \lambda \) is rational: take e.g. the case of a trivial coboundary \( h \equiv 1 \) with \( \sigma \) being the Fibonacci substitution \( \sigma : a \mapsto ab, b \mapsto a \). The topological eigenvalues of this Pisot irreducible substitution belong to \( \mathbb{Z}[\phi] + \mathbb{Z} \), where \( \phi = \frac{1 + \sqrt{5}}{2} \) (see Section 4.3). However, if we are in the finitary constant-length case, Lemma 6.2 implies that \( \lambda \) has to be rational.

In the case where the directive sequence consists of constant-length substitutions, inspection of the proof of Theorem 4.3 yields that a weak coboundary \((h, f)\) associated to a continuous function is constant, as in the constant-length substitution case, and also that we can lighten the hypothesis concerning fully essential words. Indeed, in Theorem 4.3 we require that for each \( a \in \mathcal{A} \), there exists \( \ell \in \mathcal{A} \) such that \( a \ell \) is a fully essential word. Below, with Theorem 6.5, we just need that there exists a fully essential word of length 2.

We can now state the following theorem.

Theorem 6.5. Let \( \sigma = (\sigma_n)_{n \geq 0} \) be a straight constant-length directive sequence on \( \mathcal{A} \), with length sequence \((q_n)_{n \geq 0} \), where each \( q_n \geq 2 \). Suppose that \( X_{\sigma} \) is aperiodic and that \( \sigma \) has a fully essential word of length 2.

If \( \lambda \in \mathbb{S}^1 \) is a continuous eigenvalue of \((X_{\sigma}, T)\), then

\[
    h := \lim_{n \to \infty} \lambda^{q_0 \cdots q_n}
\]

exists and defines a constant weak coboundary.

Assume in addition that \( \sigma = (\sigma_n)_{n \geq 0} \) is finitary and recognizable. Suppose that \( h := \lim_{n \to \infty} \lambda^{q_0 \cdots q_n} \) exists for some \( \lambda \in \mathbb{S}^1 \), and that \( h \) is a coboundary. Then \( \lambda \) is a continuous eigenvalue, and \( e^{2\pi i t_m} \) is also a continuous eigenvalue, for any \( m \). Moreover, each continuous eigenvalue is rational, and there exists \( h \in \mathbb{N} \)

- which is coprime to each \( q_n \), and
- which divides \( p := \prod_{q \in \{q_n : n \geq 0\}} (q - 1) \), where the product is over the set of distinct values of elements in \( \{q_n : n \geq 0\} \),

such that the maximal equicontinuous factor of \((X_{\sigma}, T)\) is the odometer \((\mathbb{Z}_{h, (q_n)}, +1)\).
Proof. To prove the first statement, we repeat the proof of Theorem 4.3, except working only with one fully essential word \( ab \). We obtain

\[
\lim_{n \to \infty} \lambda^{q_0q_1 \ldots q_n} f(a) = \lim_{n \to \infty} \lambda^{h_n(a)} f(a) = f(b),
\]

so that (6.3) holds. The rest of the proof is as in Theorem 4.3.

If \( \sigma \) is finitary and recognizable, and \( h = \lim_{n \to \infty} \lambda^{q_0 \ldots q_n} \) exists, then by Lemma 6.2, \( \lambda = e^{2\pi it} \) where \( t \) is rational; write \( t = p/q \) with \( q \geq 1 \) and where \( p \) and \( q \) are non-zero coprime integers. The claim that \( \lambda \) is a continuous eigenvalue follows by Theorem 4.4.

Remark 6.1 tells us that \( (\mathbb{Z}/(q_n), +1) \) is an equicontinuous factor of \( (X_{\sigma}, T) \). Finally, Lemma 6.2 tells us that \( \frac{p}{q} = \frac{a_0}{q_0 + a_1(q_0 + 1)} \) for some \( m \). Therefore, if \( \frac{p}{q} \) is an additional continuous eigenvalue, we need only add some factor of \( (\mathbb{Z}/(q_{n+1})\mathbb{Z}, +1) \), to \( (\mathbb{Z}/(q_n), +1) \) to obtain possibly a larger equicontinuous factor of \( (X_{\sigma}, T) \). The result follows.

The following definition of height in the \( S \)-adic case is a generalisation of Definition 3.23. Recall the discussion there, of the relationship between the height of a constant length substitution, and the existence of a nontrivial coboundary. The following is motivated by the close relation between eigenvalues and height.

**Definition 6.6 (\( S \)-adic height).** Let \( \sigma = (\sigma_n)_{n \geq 0} \) be a finitary, straight, aperiodic, recognizable, constant-length directive sequence on \( A \), with length sequence \( (q_n)_{n \geq 0} \), where \( q_n \geq 2 \) for all \( n \geq 2 \). Suppose that \( \sigma \) has a fully essential word of length two. Then, by Theorem 6.5,

\[
\tilde{h} := \text{lcm}\{q \in \mathbb{N} : \lambda = e^{2\pi i/q} \text{ is an eigenvalue of } (X_{\sigma}, T), \text{ with } q \text{ coprime to each } q_n \}
\]

is finite. We call \( \tilde{h} \) the height of \( \sigma \).

As in the case of constant-length substitutions, the factors of \( \tilde{h} \) define the set of eigenvalues of \( (X_{\sigma}, T) \) not determined by the sequence of lengths \( (q_n) \) (as expressed in Theorem 6.5).

**Example 6.7.** Here we consider the constant-length directive sequence in Example 5.5 which is recognizable, aperiodic and straight as we previously discussed. Since \( ba \) is a fully essential word, Theorem 6.5 tells us that if \( \lambda \) is a continuous eigenvalue, then \( h := \lim_n \lambda^{3^n} \) exists. As we computed earlier in Example 3.11, both \( a \) and \( b \) consist of the single limit word \( u = \lim_{n \to \infty} \sigma_{(0,0)}(a) \). Therefore if \( f \) is an eigenfunction associated to \( \lambda \), this gives, according to the proof of Theorem 6.5,

\[
f(b)h = f(a),
\]

so that we must have \( h = 1 \). This implies that \( h \) is a strong coboundary. By Theorem 6.5, all continuous eigenvalues are rational. Also, any eigenvalue must be of the form \( \lambda = e^{2\pi it} \) where \( \tilde{h} \in \{1, 2\} \). Since \( \lim_k \lambda^{3^n} = 1 \), one has \( h = 1 \). This \( S \)-adic system therefore has trivial height, and so its maximal equicontinuous factor is \( (\mathbb{Z}/(3), +1) \).

Recall the definition of height in Definition 3.23. From Theorem 6.5 we recover its role in the case of a constant-length substitution, as seen in [Kam72, Dek78], [Que10, Theorem 6.1].

**Corollary 6.8.** Let \( \sigma \) be a straight aperiodic substitution of constant-length \( q \). Then the continuous eigenvalues of \( (X_{\sigma}, T) \) are generated by \( \{e^{2\pi i/n} : n \in \mathbb{N}\} \cup \{e^{2\pi i/\tilde{h}}\} \), where \( \tilde{h} \) is the height of \( \sigma \). Furthermore \( \tilde{h} \) divides \( q - 1 \).

Similarly, if a finitary directive sequence satisfies the conditions of Theorem 6.5, and the shift is aperiodic, then the continuous eigenvalues of \( (X_{\sigma}, T) \) are generated by \( \{e^{2\pi i/n} : n \in \mathbb{N}\} \cup \{e^{2\pi i/\tilde{h}}\} \), where \( \tilde{h} \) is the height of \( \sigma \).

From Theorem 6.5 we also obtain a version of Cobham’s theorem. For more on its classical substitutive version, see [Dur11].

**Corollary 6.9.** Let \( \sigma = (\sigma_n)_{n \geq 0} \) and \( \bar{\sigma} = (\bar{\sigma}_n)_{n \geq 0} \) be two finitary, straight, aperiodic, recognizable, constant-length directive sequences on \( A \), with length sequences \( (q_n)_{n \geq 0} \) and \( (\bar{q}_n)_{n \geq 0} \), where each \( q_n \geq 2 \) and \( \bar{q}_n \geq 2 \). Suppose that each directive sequence possesses a fully essential word of length 2. If there is a prime factor \( p \) of infinitely many of the lengths \( q_n \) that is a prime factor of only finitely many of the lengths \( \bar{q}_n \), then \( (X_{\sigma}, T) \) and \( (X_{\bar{\sigma}}, T) \) cannot be topologically conjugate.
Proof. The conditions we have imposed allow us to deduce from Theorem 6.5 that the maximal equicontinuous factor of \((X, T)\) is \((\mathbb{Z}_{h_1}, (\tilde{q}_n), +1)\), and that of \(\tilde{\sigma} = (\tilde{q}_n)_{n \geq 0}\) is \((\mathbb{Z}_{h_2}, (\tilde{q}_n), +1)\), for some finite natural numbers \(h_1\) and \(h_2\). The fact that the prime factor \(p\) appears infinitely often in the lengths \(q_n\) implies that \((\mathbb{Z}_p, +1)\) is an equicontinuous factor of \((X, T)\), while the fact that \(p\) divides only finitely often in the lengths \(\tilde{q}_n\) implies that \((\mathbb{Z}_p, +1)\) is not an equicontinuous factor of \((X, T)\). Now equicontinuous factors encode continuous eigenvalues. In particular, for each \(n\), \(e^{2\pi i/p^n}\) is an eigenvalue of \((X, T)\), but only finitely many of the \(e^{2\pi i/p^n}\) can be eigenvalues of \((X, T)\). The statement follows since the set of continuous eigenvalues is a topological invariant. \(\square\)

To see why the requirement that \(p\) has to divide \(q_n\) infinitely often is necessary, recall that the height of the second system can in principle absorb a \(p\) which divides \(q_n\) only finitely many times. For example, consider the situation where \(q_n = 3\) for each \(n\), and \((\tilde{q}_n) = 2, 3, 3, 3, \ldots\); then it can happen that both \((X, T)\) and \((X, \tilde{T})\) have \(\mathbb{Z}_{2,3}(3)\) as maximal equicontinuous factor, in particular if \((X, T)\) has height 2. Take for example the stationary \(\sigma\) generated by the substitution \(a \mapsto aba, b \mapsto bac, c \mapsto cab\). It can be verified that this substitution has height 2, so that its maximal equicontinuous factor is \(\mathbb{Z}_{2,3}^{(3)}\).

The following example gives a recipe to find constant-length directive sequences which have a non-trivial height. We formulate it to generate straight, recognizable directive sequences, by modifying the first two ingredients appropriately.

Example 6.10. Let \(S\) be any finite set of aperiodic, primitive constant-length substitutions on \(A = \{a, b, c, d\}\) where

(i) for all \(\sigma \in S\) for each \(\alpha \in \{a, b\}\), \(\sigma(\alpha)\) begins with the same letter in \(\{a, b\}\) for all \(\sigma \in S\), and for each \(\alpha \in \{c, d\}\), \(\sigma(\alpha)\) begins with the same letter in \(\{c, d\}\) for all \(\sigma \in S\),

(ii) each substitution in \(S\) is rotationally conjugate to a left- or right-permutative substitution,

(iii) there is a word \(w\) of length two such that for each \(\sigma \in S\) there is \(\alpha \in A\) where \(w\) occurs in \(\sigma(\alpha)\),

(iv) for each \(\sigma \in S\), any occurrence of a letter in \(\{a, b\}\) in the image of a letter by a substitution is always followed by a letter in \(\{c, d\}\), and any occurrence of a letter in \(\{c, d\}\) is always followed by a letter \(\{a, b\}\), and

(v) each substitution in \(S\) has odd length at least three.

Then we claim that \(-1\) is an eigenvalue, provided that the resulting shift is aperiodic. Indeed, Condition (i) ensures that any primitive directive sequence from \(S\) is straight. Condition (ii) ensures that any directive sequence is recognizable, using Theorem 2.6. Condition (iii) ensures that there is a fully essential word of length two. Therefore Theorem 6.5 applies. Finally, Conditions (iv) and (v) ensure that in all words of the language \(L_\sigma\), a letter from \(\{a, b\}\) is always followed by a letter from \(\{c, d\}\) and conversely. Thus \(-1\) is an eigenvalue, as

\[P = \{[a] \cup [b], [c] \cup [d]\}\]

is then a clopen partition which forms a Rokhlin tower of height 2, that is \(X_\sigma = A \cup B\) with \(T(A) = B, T(B) = A\), here with \(A = [a] \cup [b], B = [c] \cup [d]\).

Take for example \(S = \{\sigma, \tau\}\) with

\[
\begin{align*}
a &\mapsto bda, & a &\mapsto bcada \\
b &\mapsto bcb, & b &\mapsto bdadb \\
c &\mapsto cac, & c &\mapsto cacbc \\
d &\mapsto cbd, & d &\mapsto cdbad.
\end{align*}
\]

The above general remarks tell us that \(-1\) is an eigenvalue. The conditions of Theorem 6.5 are satisfied, so we could in principle have an eigenvalue \(\lambda\) which is a fourth root of unity. We show that this does not occur. If \(\lambda\) is a continuous eigenvalue, then by Theorem 6.5, \(\lambda\) defines a weak coboundary \((h, f)\). Assume first that \(\tau\) appears infinitely often in the directive sequence. Now the
fact that $\tau(b)$ appears infinitely often and contains the subword $d a d$ tells us that $f(d)h^2 = f(d)$, and this means that $h$ cannot be a fourth root of unity, and hence the eigenvalue -1 is the only eigenvalue that appears in addition to those in Remark 6.1. Finally if $\tau$ appears only finitely often, then, since $a = b$, we can repeat the argument above, but with the essential word $b d a$, to arrive at the same conclusion.

For the above family, if $X_\sigma$ is aperiodic, then we can use Theorem 6.5 to deduce the explicit form of the maximal equicontinuous factor space of the given $S$-adic shift. For example, if $\sigma$ and $\tau$ above each appear infinitely often in a directive sequence from $S$, then the appropriate group is isomorphic to $\mathbb{Z}_2(15)$.

Note that any directive sequence chosen from the specific $S = \{\sigma, \tau\}$ will give a strongly straight directive sequence (see Definition 5.6); in general though examples satisfying (i) above are not necessarily strongly straight.

Finally we remark that this kind of construction is essentially the only kind that yields height in the constant length $S$-adic case, see [BGMnY23, Corollary 5.5]. Also, this technique can be extended to obtain nontrivial coboundaries for non-constant length directive sequences.

6.2. Toeplitz $S$-adic shifts and discrete spectrum. In this section we focus on a special family of constant-length $S$-adic shifts which has been extensively studied, namely those that are almost-automorphic.

We apply our previous results to this family and then compare our results to existing results in the literature.

We first start with a few definitions. A factor $(Z, S)$ of $(X, T)$ via a map $\pi : (X, T) \to (Z, S)$ is almost one-to-one if the set $\{x \in X : \pi^{-1}(\{\pi(x)\}) = \{x\}\}$ is dense in $X$. In this case, we call the system $(X, T)$ an almost one-to-one extension of $(Z, S)$. A system $(X, T)$ is almost automorphic if it is an almost one-to-one extension of a minimal equicontinuous system. Almost automorphic systems are necessarily minimal, and for minimal systems $\{x \in X : \pi^{-1}(\{\pi(x)\}) = \{x\}\}$ is a dense $G_\delta$ if it is non-empty. Thus if there is some $x \in X$ such that $|\pi^{-1}(\pi(x))| = 1$, then $(X, T)$ an almost one-to-one extension of $(Z, S)$. A Toeplitz shift is a symbolic shift $(X, \sigma)$, $X \subset A^\mathbb{Z}$ with $A$ finite, which is an almost automorphic extension of an odometer $(Z, +1)$, via some factor $\pi : (X, \sigma) \to (Z, +1)$. Note that Gjerde and Johansen [GJ02] give a nice characterisation of Toeplitz shifts in terms of the equal path number property for their Bratteli-Vershik representations.

Given a substitution $\sigma_n : \mathcal{A} \to \mathcal{A}^q_n$, we write it as $\sigma_n = \sigma_{n,0}\sigma_{n,1} \ldots \sigma_{n,q_n-1}$, a concatenation of $q_n$ maps $\sigma_{n,i} : \mathcal{A} \to \mathcal{A}$. We say that $\sigma_n$ has a coincidence at $i$ if $|\sigma_{n,i}(\mathcal{A})| = 1$. We say that the directive sequence $\sigma$ has a coincidence if each of the substitutions $\sigma_n$ has a coincidence at some $i$.

We have the following result, a straightforward generalisation of a result due to Kamae and Dekking [Kam72, Theorem 7]; see also [Dek78, Theorem 7], which states that a measurable constant length substitution shift has discrete spectrum if and only if the pure base of the substitution has a coincidence; see the aforementioned papers for definitions and details.

Theorem 6.11. Let $\sigma = (\sigma_n)_{n \geq 0}$ be a finitary, straight, recognizable, constant-length directive sequence on $A$, with length sequence $(q_n)_{n \geq 0}$, where each $q_n \geq 2$. Suppose that $\sigma = (\sigma_n)_{n \geq 0}$ has a fully essential word of length $2$, that infinitely many of the substitutions $\sigma_n$ have a coincidence, and that $X_\sigma$ is aperiodic. Then $(X_\sigma, T)$ is Toeplitz and uniquely ergodic. If $\mu$ is the unique invariant measure, then the measure preserving system $(X_\sigma, T, \mu)$ has discrete spectrum, $\sigma$ has trivial height, and the maximal equicontinuous factor of $(X_\sigma, T)$ is $(\mathbb{Z}_{(q_n)}, +1)$.

Proof. We assume that each $\sigma_n$ has a coincidence; this does not change the generality of our arguments (otherwise we just need to add a layer in the indexing of subsequences). By Theorem 6.5, the maximal equicontinuous factor of $(X_\sigma, T)$ is $(\mathbb{Z}_{(q_n)}, +1)$ for some $\tilde{h}$ which divides $\prod_{j=1}^N (q_j - 1)$ for some $N$.

Note that $(\mathbb{Z}_{(q_n)}, +1)$ is an equicontinuous factor of the system. Recognizability means that for each $n$ and each $x \in X_\sigma$, there are a unique $m_i$ with $0 \leq m_i < q_i$ and a unique $x^{(n)} \in X_\sigma^{(n)}$ such that $x = T^{m_0}\sigma_0 T^{m_1}\sigma_1 \ldots T^{m_{n-1}}\sigma_{n-1}(x^{(n)})$, and furthermore the data $m_0, \ldots, m_{n-1}$, which let us desubstitute $x$ to $X_\sigma^{(n+1)}$, agree with the data $m_0, \ldots, m_n$, which let us desubstitute $x$ to $X_\sigma^{(n)}$. This allows us to define a maximal equicontinuous factor map $\pi : X_\sigma \to \mathbb{Z}_{(q_n)}$, where $\pi(x) = \ldots m_2 m_1 m_0$. Let $\lambda$ be the Haar probability measure on $\mathbb{Z}_{(q_n)}$, and let $Z$ be the set of points
in \( \mathbb{Z}_{(q_n)} \) which are not invertible under \( \pi \). Denote by \( C_j \) the set of indices \( 0 < i < q_j - 1 \) such that \( \sigma_j \) has a coincidence, i.e., \( |\sigma_{j,i}(A)| = 1 \). If needed to ensure that this set is nonempty, we can telescope two substitutions at a time, to ensure that the set of coincidences indices is not “extremal”. We have that \( z \notin \mathcal{Z} \) if \( z_j \in C_j \) infinitely often. In other words, if \( [C_j]^c := \{ z \in \mathbb{Z}_{(q_n)} : z_j \notin [C_j] \} \), then

\[
\mathcal{Z} \subset \bigcup_{k=1}^{\infty} \bigcap_{j \geq k} [C_j]^c,
\]

so that

\[
\lambda(\mathcal{Z}) \leq \lim_{k \to \infty} \lambda \left( \bigcap_{j \geq k} [C_j]^c \right) \leq \lim_{k \to \infty} \prod_{j \geq k} \frac{q_j - 1}{q_j} = 0,
\]

with the second inequality following because we assumed that each \( \sigma_n \) has a coincidence, and the last equality following since the sequence \( (q_n) \) takes on a finite number of values. It follows that \( (X_\sigma, T) \) is uniquely ergodic, see for example \([ABKL15, \text{Theorem 4.12}]\). Let \( \mu \) be the unique invariant measure.

To show that \( (X_\sigma, T, \mu) \) has discrete spectrum, we must show that \( \mu(\pi^{-1}(\mathcal{Z})) = 0 \). Define \( D_j := \{ x : (\pi(x))_j \notin [C_j] \} \). Then as above, \( \pi^{-1}(\mathcal{Z}) \subset \bigcup_{k=1}^{\infty} \bigcap_{j \geq k} D_j \), and as for each \( j \), \( \sigma_j \) defines a \( q_j \)-cyclic partition of \( X_{\sigma_j}^{(j+1)} \), hence each element of this partition has measure \( 1/q_j \) and we have that \( \mu(\bigcap_{j \geq k} D_j) \leq \prod_{j \geq k} \frac{q_j - 1}{q_j} = 0 \). Therefore \( \mu(\pi^{-1}(\mathcal{Z})) = 0 \).

Any point \( z \in \mathbb{Z}_{(q_n)} \) such that \( z_j \in C_j \) infinitely often satisfies \( |\pi^{-1}(z)| = 1 \). For minimal systems \( \{ x \in X : \pi^{-1}(\pi(x)) = \{ x \} \} \) is a dense \( G_\delta \) set if it is non-empty (which holds from the above) Thus \( (X_\sigma, T, \mu) \) is almost automorphic and the fact that \( \pi \) is somewhere injective implies that it is a maximal equicontinuous factor map, by \([\text{Wil84, Proposition 1.1}]\). Thus its maximal equicontinuous factor is \( (\mathbb{Z}_{(q_n)}, +1) \); this forces \( \tilde{h} = 1 \).

Example 6.12. We have shown that the directive sequence

\[
\sigma, \tau, \sigma, \sigma, \tau, \sigma, \sigma, \tau, \sigma, \sigma, \ldots
\]

is recognizable in Example 2.7, and also straighten Example 3.11. The \( S \)-adic shift \( (X_\sigma, T) \) has a unique ergodic measure \( \mu \), see for example Proposition 5.3 and the comment below it. The substitution \( \tau \) has a coincidence at index 0. Since \( \tau \) appears infinitely often in the directive sequence, Theorem 6.11 tells us that \( (X_\sigma, T) \) is Toeplitz and that \( (X_\sigma, T, \mu) \) has discrete spectrum.

We compare our results on constant-length \( S \)-adic shifts to those in the literature for Toeplitz shifts, of whose spectrum there is an extensive study. The maximal equicontinuous factor of a Toeplitz shift is always an odometer \([\text{Wil84}]\). In other words, each continuous eigenvalue of a Toeplitz shift is rational. Furthermore, it was known that there could exist rational continuous eigenvalues that are not given by the period sequence \( (p_n)_{n \geq 0} \) \([\text{Dow05}]\). Our contribution in the case of Toeplitz shifts is to quantify the eigenvalues if the Toeplitz shift is given as a constant-length \( S \)-adic shift, as given by Theorem 6.5, and to describe them as heights, as in the constant-length substitution case, and finitely extending Cobham’s theorem to this setting.

In the measurable setting, Downarowicz and Lacroix showed that if \( K \subset \mathbb{S}^1 \) is a countable subgroup containing infinitely many rationals, then there exists a minimal, uniquely ergodic Toeplitz shift whose measurable eigenvalues equals \( K \) \([\text{DL96}]\). Therefore we see that the set of measurable eigenvalues of a Toeplitz shift can be much larger than the set of continuous eigenvalues. This is further studied by Durand, Frank and Maass \([\text{DFM15}]\), who use the characterisation of Toeplitz shifts as expansive symbolic systems which have proper Bratteli-Vershik representations defined by a sequence of constant-length morphisms \([\text{GJ02}]\). In this setting recognizability is built into the representation, but the morphisms need to be assumed \( proper \), which is a restriction we do not impose, and automatically implies that the \( S \)-adic shifts considered in \([\text{DFM15}]\) are Toeplitz, whereas constant-length \( S \)-adic shifts are generally not; see also \([\text{ADE24}]\) where the proper condition is extended in terms of coincidences. Finally, in moving to the setting of proper Bratteli-Vershik
representations, the notion of height disappears. If one considers a Toeplitz shift with a proper Bratteli-Vershik representation in which the morphisms are taken from a finite set and all of whose incidence matrices are positive, then these systems are linearly recurrent [Dur03], and all their measurable eigenvalues are continuous [BDM05].

Mentzen studies a family of measure preserving systems on the unit interval, those of exact uniform rank [Men91]; these are the closest dynamical systems to constant-length $S$-adic shifts, as they can be seen as a geometric realisation of such symbolic systems. We study this next.

6.3. **Measurable spectrum.** The notion of uniform exact rank is investigated in [Men91] - these are measure preserving finite rank dynamical systems on the unit interval which have a sequence of generating partitions $(R_n)$, where $R_n$ consists of a fixed number of Rokhlin towers, all of which are of the same height, and where this representation is of exact finite rank. By Theorem 2.8, if an $S$-adic shift $(X_\sigma, T)$ is defined using a recognizable, directive sequence of constant length, such that $(X_\sigma, T, \mu)$ is of exact finite rank, then it is measurably conjugate to a uniform exact rank system as defined by Mentzen. Mentzen shows that such a system must necessarily have rational measurable spectrum, i.e., translating his result to the $S$-adic setting, we have the following which should be compared with Theorem 6.5; his proof techniques are different.

**Theorem 6.13.** [Men91, Corollary 1] Let $\sigma = (\sigma_n)_{n \geq 0}$ be a recognizable, constant-length directive sequence on $A$, with length sequence $(q_n)_{n \geq 0}$. If $\sigma$ is of exact finite rank, then every measurable eigenvalue is rational.

If we specialise to constant-length directive sequences the statement of Theorem 5.10, we may relax the condition that the directive sequence be strongly straight in a variety of ways. We illustrate with one that we will use. Note that the condition that the directive sequence admits a unique right-infinite limit word, say $a$, and moreover, that $\sigma_n(a)$ starts with $a$ for each $n$, is weaker than the condition that there is a letter $a$ such that for each $n$ and each $\alpha$, $\sigma_n(\alpha)$ starts with $a$, which is the required condition of properness in [CDHM03]. For example, recall the substitutions $\sigma$ and $\tau$ from Example ??, and note that any directive sequence $\sigma$ selected from $\{\sigma, \tau\}$ satisfies this condition, provided that $\tau$ appears infinitely often.

Recall that strongly primitive finitary directive sequences generate uniquely ergodic shifts. We denote the unique invariant measure by $\mu$ in the following.

**Theorem 6.14.** Let $\sigma = (\sigma_n)_{n \geq 0}$ be a finitary, strongly primitive constant-length directive sequence on $A$, with length sequence $(q_n)_{n \geq 0}$, where each $q_n \geq 2$. Suppose that the directive sequence admits a unique right-infinite limit word, say $a$, and moreover that $\sigma_n(a)$ starts with $a$ for each $n$. If $\lambda \in S^1$ is a measurable eigenvalue of $(X_\sigma, T, \mu)$, then

$$\sum_{n=1}^{\infty} |\lambda^{q_0\ldots q_n} - 1|^2 < \infty.$$ 

**Proof.** We follow the proof of Theorem 5.10, to the step which yields (5.2), i.e.,

$$\sum_{n=1}^{\infty} \max_{a,b} \max_{t_n(a,b)} |\lambda^j v_n(a) - v_{n-1}(b)|^2 < \infty.$$ 

Since $\sigma_{n-1}(a)$ starts with $a$ for each $n$, then $0 \in t_n(a,a)$, and letting $j = 0$ in (5.2), we get

$$\sum_{n=1}^{\infty} |v_n(a) - v_{n-1}(a)|^2 < \infty. \tag{6.4}$$

By Lemma 5.9, $\min_{a \in A} |v_n(a)| \to 1$. Therefore from (6.4), we obtain, since $q_0 \ldots q_n$ belongs to $t_{n+1}(a,b)$ for some $b$, and any such $b$ satisfies $a = b$,

$$\sum_{n=1}^{\infty} |\lambda^{q_0\ldots q_n} - 1|^2 < \infty.$$ 

$\square$
Corollary 6.15. Let \( \sigma = (\sigma_n)_{n \geq 0} \) be a finitary, recognizable and strongly primitive constant-length directive sequence on \( A \), with length sequence \((q_n)_{n \geq 0}\), where each \( q_n \geq 2 \). Suppose that either

- \( \sigma \) is strongly straight, or
- there is a unique right-infinite limit word \( a \), and such that \( \sigma_n(a) \) starts with \( a \) for each \( n \).

Then every measurable eigenvalue for the uniquely ergodic \((X_\sigma, T, \mu)\) is continuous.

Proof. By Proposition 5.3 and Theorem 6.13, every measurable eigenvalue \( \lambda \) is rational. With the first hypothesis, we have that the sum

\[
\sum_{n=1}^{\infty} \max_{a, b: a = b} |\lambda^{h_n(w_n(a,b))} - 1|^2
\]

is finite for strict transition words \( w_n(a,b) \) by Theorem 5.10. We conclude in particular that \( \lambda^{h_n(w_n(a,a))} = 1 \) for each \( n \geq n_0 \) and each \( a \), since \( \lambda \) is rational.

Furthermore, the third condition of the definition of strongly straight tells us that any strict return word to \( a \) can be contained in \( \sigma_n(\alpha) \) for some \( \alpha \), by telescoping boundedly if needed. This means that there is a finite set of words to which a strict return word at any level belongs. Thus \( n_0 \) can be taken to be independent of the given sequence of strict return words. This also implies that \( \lambda^{h_n(w_n(a,a))} - 1 = 0 \) when \( w_n(a,a) \) is a (not necessarily strict) return word to \( a \). Now, Proposition 4.5 gives the result.

With the second hypothesis, Theorem 6.14 implies

\[
\sum_{n=1}^{\infty} |\lambda^{q_0 \ldots q_n} - 1|^2 < \infty,
\]

or \( \lambda^{q_0 \ldots q_n} - 1 = 0 \) for large enough \( n \). Now, Theorem 4.4 implies that \( \lambda \) is a continuous eigenvalue.

Example 6.16. We conclude by recapping the eigenvalues, measurable and topological of the \( S \)-adic shift of our running Example 2.7. We have shown that the directive sequence

\[ \sigma, \tau, \sigma, \sigma, \tau, \sigma, \sigma, \sigma, \sigma, \sigma, \ldots \]

is recognizable, straight, and if we re-write it as

\[ \sigma, \tau \circ \sigma, \sigma, \tau \circ \sigma, \sigma, \tau \circ \sigma, \sigma, \tau \circ \sigma, \sigma, \sigma, \ldots, \]

it is strongly primitive. We have also seen that \((X_\sigma, T)\) is of exact finite rank, although it is not linearly recurrent [Dur03], and that its maximal equicontinuous factor is \((\mathbb{Z}^3, +1)\). Finally, while it is not strongly straight, it satisfies the second condition of Corollary 6.15, so \((\mathbb{Z}^3, +1)\) is also its Kronecker factor, which we define as the maximal measure theoretic factor of the system that is isomorphic to a rotation on a compact abelian group.

References

[ABB+15] S. Akiyama, M. Barge, V. Berthé, J.-Y. Lee, and A. Siegel, On the Pisot substitution conjecture, Mathematics of aperiodic order, Progress in mathematics, vol. 309, Birkhäuser, Basel, 2015, pp. 33–72.

[ABKL15] J.-B. Aujogue, M. Barge, J. Kellendonk, and D. Lenz, Equicontinuous factors, proximality and Ellis semigroup for Delone sets, Mathematics of aperiodic order, Progr. Math., vol. 309, Birkhäuser/Springer, Basel, 2015, pp. 137–194.

[ADE24] Felipe Arbulú, Fabien Durand, and Bastián Espinoza, The Jacobs-Keane theorem from the \( S \)-adic viewpoint, Discrete Contin. Dyn. Syst. 44 (2024), no. 10, 3077–3108. MR 4770766

[AMS14] P. Arnoux, M. Muzitani, and T. Sellami, Random product of substitutions with the same incidence matrix, Theoret. Comput. Sci. 543 (2014), 68–78.

[AR91] P. Arnoux and G. Rauzy, Représentation géométrique de suites de complexité \( 2n+1 \), Bull. Soc. Math. France 119 (1991), no. 2, 199–215.

[ARS09] J.-P. Allouche, N. Rampersad, and J. Shallit, Periodicity, repetitions, and orbits of an automatic sequence, Theoret. Comput. Sci. 410 (2009), no. 30-32, 2795–2803. MR 2543333

[AS14] N. Aubrun and M. Sablik, Multidimensional effective \( S \)-adic subshifts are sofic, Unif. Distrib. Theory 9 (2014), no. 2, 7–29.
[Sol07] Solomyak, Eigenfunctions for substitution tiling systems, Probability and number theory—Kanazawa 2005, Adv. Stud. Pure Math., vol. 49, Math. Soc. Japan, Tokyo, 2007, pp. 433–454.

[Wal82] P. Walters, An introduction to ergodic theory, Graduate Texts in Mathematics, vol. 79, Springer-Verlag, New York-Berlin, 1982.

[Wil84] S. Williams, Toeplitz minimal flows which are not uniquely ergodic, Z. Wahrsch. Verw. Gebiete 67 (1984), no. 1, 95–107.