The complex concept of quantity in the past and (possibly) the future of the International Vocabulary of Metrology

L Mari\textsuperscript{1}, A Chunovkina\textsuperscript{2} and C Ehrlich\textsuperscript{3}

\textsuperscript{1} Università Cattaneo – LIUC, Italy
\textsuperscript{2} Mendeleyev Institute for Metrology (VNIIM), Russian Federation
\textsuperscript{3} National Institute of Standards and Technology (NIST), United States of America

E-mail: lmari@liuc.it

Abstract. The \textit{International Vocabulary of Metrology} (VIM) is a foundational document of measurement science, with the ambition of providing a system of “basic and general concepts and associated terms” of metrology. Such a system has evolved with the evolution of measurement science, and this is particularly manifest in the case of the cluster of concepts around ‘quantity’. The present paper presents some aspects of this development, by first remarking on the importance of terminology for measurement science and then introducing the basic problem \textit{what is a quantity?} as a way to provide a well-grounded explanation to the usual claims that two quantities can be equal, or two values of quantities can be equal, or a quantity of an object can be equal to a value of a quantity. The analysis develops along three questions: (1) what is the relation between general properties and individual properties? (2) what is the relation between properties and quantities? (3) what is the relation between quantities of objects and values of quantities?

1. Why terminology is important for our subject

Our knowledge of the world around us is based on two kinds of entities: \textit{objects} (such as electrons, rods, galaxies, persons and companies, industrial processes and software systems) and \textit{properties} (physical quantities, such as mass, temperature, etc, but also non-physical properties, such as beauty, wealth, competence, and wellbeing – “object” and “property” are not standard terms: by “object” we generalize here what in [1] is called “phenomenon, body, or substance” (e.g., in def. 1.1); more or less equivalent to “property” are “attribute”, “observable”, “variable”, etc.; we use the term “entity” for “anything perceivable or conceivable”, as in [2], def. 3.1.1, where the term “object” is used instead; for a terminological analysis of this foundational topic, see [3]).

Objects and properties are so embedded in our knowledge systems that this basic epistemology becomes an ontology: we behave as if what is knowable is made of objects that have properties. Accordingly, a key knowledge construct is \textit{property of an object}. Indeed, properties of objects

- can be causally relevant: it is the mass of the rod that stretches the spring of the dynamometer,
• establish the behavior of objects in their comparison with other objects: the comparison of two rods by length and by mass generally produces different outcomes, and
• define the identity of objects: the rod can maintain its identity even if some of its properties change, provided that some other of its properties remain the same.

Some properties of some objects are empirical: we model them on the basis of the available knowledge, but the underlying hypothesis is that they exist independently of our models. For example, we accept that the sun emitted radiant power also before life appeared on the earth, and therefore also before the concept ‘power’ was formulated. As a consequence, such properties are not conceptual entities: we construct concepts to understand them, but they are not concepts. And of course they are not linguistic entities: we construct terms to designate them, but they are not terms. Indeed, the semiotic triangle applies both to objects and properties (in the diagram of figure 1).

![Figure 1. The semiotic triangle, and two relevant examples.](image)

This is an effective representation of the basic function of language as a tool of intersubjective communication. We understand each other when we use the same term (“rod”, “mass”) to refer to the same object (a given rod, the property mass). While sometimes this can be stipulated and learned by direct reference (“from now on “rod ABC123” is the name of this rod”, uttered while pointing to a given object, and thus providing a contextual meaning of “this rod”), in most cases such a strategy is impractical or just unfeasible. Concepts are then the critical connectors between terms and objects: we refer to the same object when uttering “rod” if we share the same concept of what a rod is. Terms are introduced and agreed for the common understanding of concepts and the common reference to objects. This highlights the importance of terminology, in scientific and technical contexts and not only: the definitions contained in vocabularies are linguistic expressions aimed to characterize concepts. Hence a good definition of, say, ‘measurement’ (the concept, with single quotes) is expected to provide a criterion to identify measurements (the objects), as designated as “measurement” (the term, with double quotes). In the case of property-related concepts this is a particularly delicate result to achieve, due to their elusive nature. On the other hand, properties, and specifically quantities, have a pivotal role in measurement – what is measured are properties of objects – and not surprisingly therefore the International Vocabulary of Metrology (VIM) since its first edition (VIM1, [4]; we refer to the subsequent editions as VIM2 [5] and VIM3 [1]) devoted great attention to them, as witnessed by the number of property-related definitions included in the Vocabulary.

2. Introducing the problem
The cluster of concepts around ‘property’ that need to be clarified and defined can be introduced in a simple and operative way with an example as in the following diagram.

![Figure 2. The relations about properties.](image)
There are three kinds of relations here, and for each of them we use the “=” sign if appropriate:
A. mass of rod 1 = mass of rod 2
B. 1.23 kg = 2.71 lbs
C. mass of rod 1 = 1.23 kg
The key question here is: what is the meaning of those “=” signs in the three cases?

3. First key point: general properties and individual properties
Relations A, B, and C convey some information because they are related to meaningful comparisons. While the mass of rod 1 and the mass of rod 2 can be compared, and the outcome may be, in particular, that they are discovered to be equal, or at least indistinguishable, nothing similar could be said of the mass of rod 1 and the length of rod 2: they are neither equal nor different, because they cannot be compared. This condition of comparability justifies introducing the distinction between general properties, such as mass, and individual properties, such as any given mass (these terms are not standard; in the specific context of quantities, the three editions of the VIM define the concept ‘quantity’ as encompassing both entities such as mass and the mass of a given object, and their distinction is only noted: in the VIM1 in terms of “quantities in a general sense” and “specific quantities” (def. 1.01 note 1), in the VIM2 in terms of “quantities in a general sense” and “particular quantities” (def. 1.1 note 1), and in the VIM3 in terms of “kinds of quantities” (def. 1.2 note 3, even though the definition seems to suggest something else) and “individual quantities” (def. 1.1 note 1)).

Any individual property is the instance of a general property, and two individual properties are comparable if they are instances of the same general property or, equivalently then, if they are properties of the same kind. Hence the mass of rod 1 and the mass of rod 2 are comparable because they are individual properties that are both instances of the general property mass. A problem of foundational ontology is about how to interpret the statement of equality of two individual properties, e.g., relation A above, and two positions can be envisaged. In the example, does it mean that

- Position 1: there is one individual mass that rod 1 and rod 2 share? or that,
- Position 2: each rod has its own individual mass, and the two masses are indistinguishable?

While this problem cannot be solved by experimental means, we will see that these positions lead to distinct ways to conceive measurement results and the way we report them.

The VIM does not provide separate definitions for the two concepts, ‘general quantity’ and ‘individual quantity’, and uses the same term “quantity” for both, but the context helps solving this polysemy. For example, a measurand, i.e., a “quantity intended to be measured” (VIM3, def. 2.3), is an individual quantity (one can measure the mass of a given rod, not mass as such), whereas the distinction between base quantities and derived quantities (VIM3, def. 1.4 and 1.5) applies to general quantities (a general quantity of the SI is mass, not the mass of a given rod). Future editions of the VIM could provide better explanations, but introducing distinct terms and separate definitions does not seem to be a priority.

4. Second key point: properties and quantities
We have already mentioned that quantities are specific properties (so that ‘quantity’ is defined as “property such that...”). The VIM has assumed so far that only quantities are measurable, i.e., that non-quantitative properties are not measurable (“Measurement does not apply to nominal properties.”, VIM3, def. 2.1 note 1). Hence identifying what characterizes a quantity as a specific property is important. According to the VIM1 and the VIM2, quantities are properties (they use the equivalent term “attribute”) which “may be distinguished qualitatively and determined quantitatively” (VIM1, def. 1.01 and VIM2, def. 1.1). This is unfortunately not so useful, given that defining ‘quantity’ as something “determinable quantitatively” is basically tautological. Hence the VIM3 changed the definition of ‘quantity’: “property of a phenomenon, body, or substance, where the property has a magnitude that can be expressed as a number and a reference” (def. 1.1). Let us analyze it.

First, the expression “phenomenon, body, or substance” refers to what we are generically calling here an object. Second, the condition “that can be expressed as a number and a reference” seems to be a clarification more than an actual specification, given that everything can be expressed as a number and
a reference if the expression is not somehow constrained, and therefore can be dropped as inessential (the VIM3 considers ordinal properties as quantities, so that in the case of ratio quantities the reference is a measurement unit (def. 1.19 note 1)). With these simplifications, we obtain that a quantity is a property of an object, where the property has a magnitude. The VIM does not define what a magnitude is, but a plausible hypothesis is that “magnitude” can be intended here as “amount”, so that for example mass is a quantity because it is a property that objects have in amounts (as contrasted, e.g., to shape, that is a property that objects do not have in amounts). Accordingly, the phrase “the magnitude of the mass of this rod” means “the amount of mass of this rod”. On the other hand, we commonly refer to the mass of objects, not to the magnitude of the mass of objects: why? The distinction between Position 1 and Position 2 provides an explanation.

According to Position 1, “mass of rod 1 = mass of rod 2” means that there is one individual mass that rod 1 and rod 2 share. Hence for two objects having the same (individual) mass and having the same amount of mass are identical conditions: a magnitude of mass is just an individual mass.

According to Position 2, “mass of rod 1 = mass of rod 2” means that each rod has its own individual mass and the two masses are indistinguishable. Since two objects having the same (individual) mass is not possible, they may only have the same amount of mass: a magnitude of mass is what indistinguishable masses have in common.

This explains why measurement results are customarily reported by referring to quantities of objects, such as the (individual) mass of rod 1, instead of to magnitudes of general quantities as considered of objects, such as the magnitude of (general) mass as considered of rod 1.

According to Position 2, “mass of rod 1 = mass of rod 2” means that each rod has its own individual mass and the two masses are indistinguishable. Since two objects having the same (individual) mass is not possible, they may only have the same amount of mass: a magnitude of mass is what indistinguishable masses have in common.

In this sense, the expression “mass of rod 1 = mass of rod 2” cannot be literally true, and must be intended as an elliptical reference to the condition that the two masses have the same magnitude, i.e., magnitude of mass of rod 1 = magnitude of mass of rod 2.

Under this interpretation, the definition of ‘quantity’ given by the VIM3 is then compatible with both Position 1 and Position 2. Future editions of the VIM could provide even better explanations, making this concept of magnitude/amount more explicit by characterizing it in terms of invariance conditions among comparable individual properties. For example, ratio quantities could be defined as general properties whose instances can be compared by ratio (where then double amount corresponds to ratio = 2, and so on). Furthermore, while the VIM3 definition of ‘nominal property’ (def. 1.30) is phrased in negative terms, “property of a phenomenon, body, or substance, where the property has no magnitude”,

![Figure 3](image-url)  
Figure 3. The interpretation of relation mass of rod 1 = mass of rod 2 according to Position 1.

![Figure 4](image-url)  
Figure 4. The interpretation of relation mass of rod 1 = mass of rod 2 according to Position 2.
such a strategy would lead to characterize nominal properties as general properties whose instances can be compared by equality: a simple, operational option.

A further comparative analysis of Position 1 and Position 2 requires taking also values of quantities into account.

5. Third key point: quantities of objects and values of quantities

We have provided some explanation about relation A in the diagram of figure 2, but relations B (1.23 kg = 2.71 lbs) and C (mass of rod 1 = 1.23 kg) still require our analysis: again, what is the meaning of those “=” signs? Both relations involve values of quantities, and therefore let us consider how the concept ‘value of a quantity’ – the VIM3 has “quantity value” as the preferred term, but the difference is only lexical – is defined in the three editions of the VIM (we leave to other contexts the more general analysis about values of properties):

VIM1: “expression of a quantity in terms of a number and an appropriate unit of measurement” (def. 1.17);
VIM2: “magnitude of a particular quantity generally expressed as a unit of measurement multiplied by a number” (def. 1.18);
VIM3: “number and reference together expressing magnitude of a quantity” (def. 1.19).

In their marked differences, each of them grasps an aspect of what values are. The VIM1 emphasizes the function of values to convey information, hence considering them “expressions”. On the other hand, the concept ‘expression’ is generic, and seems to be related to linguistic entities, so that in relation B, “1.23 kg” and “2.71 lbs” are plainly different expressions (say, only the former contains the character “3”). Indeed, what the relation states to be equal are not the expressions but what they express, i.e., the entities that are individual quantities. Analogously, in relation C, an empirical entity, such as the mass of rod 1, cannot be equated to an expression, so that what is stated to be equal is an individual quantity and what an expression such as “1.23 kg” expresses. With its operative flavor, the VIM1 does not maintain the distinction between expressions and what expressions express, and values are what expressions such as “1.23 kg” and “2.71 lbs” express. Not surprisingly, then, the VIM2 takes a different track, by emphasizing the ontological status of values: according to the analysis in the previous section, Position 2 is endorsed here and values are considered magnitudes/amounts of individual (“particular”) quantities, even though the characterization of such magnitudes is provided in terms of an expression like in the VIM1. Finally, the VIM3 definition does not actually specify what a value is (‘value and reference’ is not a single concept), and rather specifies what values are made of: numbers and references, and therefore numbers and units in the case of ratio quantities.

Even this short analysis shows that future editions of the VIM could bring significant improvements to our understanding of what values are. Let us then better explore this subject.

6. What are values of quantities?

Instead of dealing with the general case of values of properties, we focus here on values of ratio quantities, like 1.23 kg. As the VIM3 explicitly states (def. 1.19 note 1), such a value is the product of a number and a unit: Position 1 and Position 2 provide related but distinct interpretations of this, both based on the acknowledgment that 1.23 kg is an entity identified by the mathematical technique of somehow selecting a mass unit (the task of those who define what is the kilogram) and then identifying all other related entities as multiples or submultiples of the selected unit.

According to Position 1, a mass unit is unproblematically an individual mass, so that 1.23 kg is the individual mass identified as 1.23 times the kilogram, where the identification, which is in principle mathematical (in fact for assessing that, e.g., 1.23 kg < 2.34 kg no empirical processes are required), may be also realized by constructing an object – thus a measurement standard – that materializes a mass that is 1.23 times the kilogram. This provides a simple explanation of the meaning of the relations between individual quantities identified as quantities of objects and individual quantities identified as multiples or submultiples of units. Being entities of the same nature, they can be actually compared, so
that for example mass of rod 1 < 1.23 kg means exactly what it is written: it is the comparison of two masses, such that the mass of rod 1 is less than the mass that is 1.23 times the kilogram. Analogously, relation C in the diagram of figure 2, mass of rod 1 = 1.23 kg, means that there is one individual mass, that has been identified as both the mass of rod 1 and 1.23 times the kilogram (as then in figure 3). In summary, according to Position 1 values are individual quantities identified as multiples or submultiples of units, which are themselves individual quantities.

Position 2 introduces the further entity, magnitude, that is what indistinguishable individual quantities have in common, so that 1.23 kg is a magnitude of mass, identified as 1.23 times the kilogram, which now has to be considered a magnitude in turn. Exactly as for relation A between masses of objects, also in this case relation C needs to be interpreted as elliptical: it is written “mass of rod 1 = 1.23 kg” but it must be meant as magnitude of mass of rod 1 = 1.23 kg (as in figure 4). In summary, according to Position 2 values are magnitudes identified as multiples or submultiples of units, which are themselves magnitudes.

Interestingly, this analysis leads us also to interpret relation B, i.e., the equality of values having different units, like 1.23 kg = 2.71 lbs: it is the claim that there is one individual mass (in Position 1) or a magnitude of mass (in Position 2) that is identified by reference to different units. Back to the semiotic triangle, in both cases the equality of 1.23 kg and 2.71 lbs states that there is one entity (an individual mass, in Position 1; a magnitude of mass, in Position 2) identified by means of two different concepts, the concept of being 1.23 times the kilogram and the concept of being 2.71 times the pound.

7. Conclusion
Quantities have a pivotal role in measurement, and the three editions published so far of the International Vocabulary of Metrology witness the evolution of the understanding around the three basic questions: (1) what is the relation between general properties and individual properties? (2) what is the relation between properties and quantities? (3) what is the relation between quantities of objects and values of quantities? At least two general positions can be consistently maintained about them, and along these paths the future editions of the VIM will plausibly move forward in providing an always better foundation of measurement science.

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