$N = 2$ KP and KdV hierarchies in extended superspace

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Abstract

We give the formulation in extended superspace of an $N = 2$ super-symmetric KP hierarchy using chirality preserving pseudo-differential operators. We obtain two quadratic hamiltonian structures, which lead to different reductions of the KP hierarchy. In particular we find two different hierarchies with the $N = 2$ classical super-$W_n$ algebra as a hamiltonian structure. The relation with the formulation in $N = 1$ superspace is carried out.

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Introduction

There has been recently an important activity in the study of $N = 2$ supersymmetric hierarchies (KP [1, 2, 3, 4, 5], generalizations of KdV [5, 6], Two Bosons [8, 9], NLS [10, 11], etc...). The most usual tools in this field are the algebra of $N = 1$ pseudo-differential operators and Gelfand-Dickey type Poisson brackets [12]. Although these systems have $N = 2$ supersymmetry, only for very few of them with very low number of fields is a formulation in extended superspace known. It is the purpose of this paper to partially fill this gap. The formalism which we shall present here partly originates from the article [13].

It turns out that in order to construct the Lax operators of $N = 2$ supersymmetric hierarchies, one should not use the whole algebra of $N = 2$ pseudo-differential operators, but rather the subalgebra of pseudo-differential operators preserving chirality. These operators were first considered in [14]. They will be defined in section 1, where we also study the KP Lax equations and the two associated Hamiltonian structures. It turns out that the first (linear) bracket is associated with a non-antisymmetric $r$ matrix [15]. Because of that, the second (quadratic) bracket is not of pure Gelfand-Dickey type. The main result of this paper is that we find two possibilities for this quadratic bracket. In fact, we show that there is an invertible map in the KP phase space which sends one of the quadratic Poisson structure into the other. However, this map does not preserve the Hamiltonians.

In section 2, we study the possible reductions of the KP hierarchy by looking for Poisson subspaces in the phase space. These are different depending on the quadratic bracket which is used. Among these reductions, there are two different hierarchies with the $N = 2$ classical super-$\mathcal{W}_n$ algebra [16] as a hamiltonian structure. In particular, two of the three known $N = 2$ supersymmetric extensions of the KdV hierarchy [17] are found. They correspond to $a = -2$ and $a = 4$ in the classification of Mathieu. These and some other examples are described in section 3. Notice that from the known cases with a low number of fields [17, 18, 19, 20, 21, 22], one expects for any $n$ three hierarchies with super-$\mathcal{W}_n$ as a hamiltonian structure. So our construction does not exhaust the possible cases.

We also found two hierarchies which Poisson structure is the classical “small” $N = 4$ superconformal algebra. In one case the evolution equations are $N = 4$ supersymmetric, while in the other they are only $N = 2$ super-
symmetric. Finally, in section 4 we give the relation of our formulation with
the usual formulation of the $N = 2$ supersymmetric KP Lax equations in
$N = 1$ superspace [20, 3, 3].

1 N=2 KP hierarchy

$N = 2$ supersymmetry We shall consider an $N = 2$ superspace with space
coordinate $x$ and two Grassmann coordinates $\theta, \bar{\theta}$. We shall use the notation $\underline{x}$ for the triple of coordinates $(x, \theta, \bar{\theta})$. The supersymmetric covariant
derivatives are defined by

$$\partial \equiv \frac{\partial}{\partial x}, \quad D = \frac{\partial}{\partial \theta} + \bar{\theta} \partial, \quad \bar{D} = \frac{\partial}{\partial \bar{\theta}} + \theta \partial, \quad D^2 = \bar{D}^2 = 0, \quad \{D, \bar{D}\} = \partial \quad (1.1)$$

Beside ordinary superfields $H(\underline{x})$ depending arbitrarily on Grassmann coor-
dinates, one can also define chiral superfields $\varphi(\underline{x})$ satisfying $D\varphi = 0$ and
antichiral superfields $\bar{\varphi}(\underline{x})$ satisfying $\bar{D}\bar{\varphi} = 0$. We define the integration over
the $N = 2$ superspace to be

$$\int d^3 \underline{x} H(x, \theta, \bar{\theta}) = \int dx \bar{D} D H(x, \theta, \bar{\theta})|_{\theta = \bar{\theta} = 0}. \quad (1.2)$$

The elements of the associative algebra of $N = 2$ pseudo-differential op-
erators ($\Psi$DOs) are the operators

$$P = \sum_{i < M} (a_i + b_i [D, \bar{D}] + \alpha_i D + \beta_i \bar{D}) \partial^i \quad (1.3)$$

where $a_i$, $b_i$ and $\alpha_i$, $\beta_i$ are respectively even and odd $N = 2$ superfields.
However, this algebra is not very manageable. In particular, the set of strictly
pseudo-differential operators ($M = 0$ in (1.3)) is not a proper subalgebra, but
only a Lie subalgebra. Also, there are too many fields in these operators.
We expect the phase space of the $N = 2$ KdV hierarchies to consist of the
supercurrents of the $N = 2$ $\mathcal{W}_n$ algebras. In extended superspace, these
supercurrents are bosonic superfields, and there is one such superfield for a
given integer dimension. But in (1.3), each power of $\partial$ corresponds to four
superfields, two even ones of integer dimension and two odd ones of half-
integer dimension. It is thus clear that one has to restrict suitably the form
of the $N = 2$ operators. It turns out that a possible restriction is to define...
the set $\mathcal{C}$ of pseudo-differential operators $L$ preserving chirality of the form

$$L = D\mathcal{L}\bar{D}, \quad \mathcal{L} = \sum_{i<M} u_i \partial^i.$$  

(1.4)

The coefficient functions $u_i$ are bosonic $N = 2$ superfields. These operators satisfy $DL = \bar{L}D = 0$. The product of two chiral operators is again a chiral operator. The explicit product rule is easily worked out

$$LL' = D\left(\mathcal{L}\partial\mathcal{L}' + (D.\mathcal{L})(\bar{D}.\mathcal{L}')\right)\bar{D},$$

(1.5)

where we have used the notation

$$(D.\mathcal{L}) = \sum_{i<M} (Du_i) \partial^i.$$  

(1.6)

Notice that $I = D\partial^{-1}\bar{D}$ is the unit of the algebra $\mathcal{C}$. We could have used as well the algebra $\tilde{\mathcal{C}}$ of $\Psi$DOs satisfying $\bar{D}\bar{L} = \bar{L}D = 0$. Notice that the product of an element in $\mathcal{C}$ by an element in $\tilde{\mathcal{C}}$ vanishes. In fact $\mathcal{C}$ and $\tilde{\mathcal{C}}$ are related by transposition, $L^t = -\bar{D}\mathcal{L}^t\bar{D} \in \tilde{\mathcal{C}}$. Although the transposition leads from $\mathcal{C}$ to $\tilde{\mathcal{C}}$, there exists an anti-involution which acts inside $\mathcal{C}$. It is given by

$$\tau(L) = DL^t\partial^{-1}\bar{D}, \quad \tau(L_1L_2) = \tau(L_2)\tau(L_1).$$  

(1.7)

Notice that it does not make sense in the algebra $\mathcal{C}$ to multiply a $\Psi$DO by a function. However, it is possible to multiply on the left by a chiral function $\phi$, $D\phi = 0$

$$\phi L = D\phi \mathcal{L}\bar{D} = \lambda(\phi)L, \quad \lambda(\phi) \equiv D\phi\partial^{-1}\bar{D},$$

(1.8)

and on the right by an antichiral function $\bar{\phi}$, $\bar{D}\bar{\phi} = 0$

$$L\bar{\phi} = D\mathcal{L}\bar{\phi}\bar{D} = L\bar{\lambda}(\bar{\phi}), \quad \bar{\lambda}(\bar{\phi}) \equiv D\partial^{-1}\bar{\phi}\bar{D}.$$  

(1.9)

We define the residue of the pseudo-differential operator $L$ by $\text{res}L = u_{-1}$. The residue of a commutator is a total derivative, $\text{res}[L, L'] = D\bar{\omega} + \bar{D}\omega$. The trace of $L$ is the integral of the residue

$$\text{Tr} L = \int d^3 x \text{res}L, \quad \text{Tr}[L, L'] = 0.$$  

(1.10)

1) Operators of this type were first considered in [14].
\( \mathcal{C} \) can be divided into two proper subalgebras \( \mathcal{C} = \mathcal{C}_+ \oplus \mathcal{C}_- \), where \( L \) is in \( \mathcal{C}_+ \) if \( \mathcal{L} \) is a differential operator and \( L \) is in \( \mathcal{C}_- \) if \( \mathcal{L} \) is a strictly pseudo-differential operator \( (M = 0 \text{ in (1.4)}) \). We shall note
\[
L = L_+ + L_-; \quad L_+ = D\mathcal{L}_+ \bar{D} \in \mathcal{C}_+, \quad L_- = D\mathcal{L}_- \bar{D} \in \mathcal{C}_-.
\] (1.11)

Here an important difference with the usual bosonic and \( N = 1 \) cases occurs. For any two \( \Psi DOs \) \( L \) and \( L' \) in \( \mathcal{C} \) one has \( \text{Tr}(L_- L'_-) = \int d^3 \mathbf{x} \text{res}(L) \text{res}(L') \neq 0 \). While \( \mathcal{C}_+ \) is an isotropic subalgebra, \( \mathcal{C}_- \) is not. One important consequence of this fact is that if one defines the endomorphism \( R \) of \( \mathcal{C} \) by \( R(L) = \frac{1}{2}(L_+ - L_-) \), then \( R \) is a non-antisymmetric classical \( r \) matrix,
\[
\text{Tr}(R(L)L' + LR(L')) = -\int d^3 \mathbf{x} \text{res}L \text{res}L'.
\] (1.12)

Notice that a non-antisymmetric \( r \) matrix in the context of bosonic KP Lax equations first appeared in [23].

**KP equations** Let us now write the evolution equations of the \( N = 2 \) supersymmetric KP hierarchy. We consider operators \( L = D\mathcal{L} \bar{D} \) in \( \mathcal{C} \) of the form
\[
\mathcal{L} = \partial^{n-1} + \sum_{i=1}^{\infty} V_i \partial^{n-i-1}.
\] (1.13)

\( L \) has a unique \( n \)th root in \( \mathcal{C} \) of the form
\[
L^{\frac{1}{n}} = D(1 + \sum_{i=1}^{\infty} W_i \partial^{-i}) \bar{D},
\] (1.14)

and we are led to consider the commuting flows (see [14])
\[
\frac{\partial}{\partial t_k} L = [(L^{\frac{1}{n}})_+, L] = [R(L^{\frac{1}{n}}), L].
\] (1.15)

There are symmetries of these equations which may be described as follows. Let us first introduce a chiral, Grassmann even superfield \( \varphi \) which satisfies
\[
\frac{\partial}{\partial t_k} \varphi = (L^{\frac{1}{n}})_+ \varphi.
\] (1.16)
where the right-hand side is the chiral field obtained by acting with the differential operator \((L^\pm)\) on the field \(\varphi\). Then the transformed operator

\[ s(L) = \lambda(\varphi^{-1})L\lambda(\varphi) \quad (1.17) \]

satisfies an evolution equation of the same form \((1.13)\) as that of \(L\).

We may also consider an antichiral, Grassmann odd superfield \(\bar{\chi}\) which satisfies

\[ \frac{\partial}{\partial t_k} \bar{\chi} = -(L^\pm_l)_+ \bar{\chi} \quad (1.18) \]

Then the transformed operator

\[ \sigma(L) = (-1)^n\lambda((D\bar{\chi})^{-1})\tau(L)\lambda(D\bar{\chi}) \quad (1.19) \]

satisfies an evolution equation of the same form \((1.13)\) as that of \(L\), with the direction of time reversed.

**Poisson brackets** The Lax equations \((1.13)\) are bi-hamiltonian with respect to two compatible Poisson brackets which we now exhibit. Let \(X\) be some \(\Psi\)DO in \(\mathcal{C}\) with coefficients independent of the phase space fields \(\{V_i\}\), then define the linear functional \(l_X(L) = \text{Tr}(LX)\). The generalization of the first Gelfand-Dickey bracket is obvious and reads

\[ \{l_X, l_Y\}(1) = \text{Tr} \left( L[X_+, Y_+] - L[X_-, Y_-] \right). \quad (1.20) \]

This is nothing but the linear bracket associated with the matrix \(R\).

Now we turn to the construction of the second bracket. It will turn out more complicated than the standard Gelfand-Dickey bracket because of the non-antisymmetry of the \(r\) matrix. An analogous situation in the bosonic case is studied in [24]. We finally found two different possibilities. In order to write them down, we need to be able to separate the residue of a \(\Psi\)DO in \(\mathcal{C}\) into a chiral and an antichiral part. For an arbitrary superfield \(H(\underline{x})\), we define

\[ H = \Phi[H] + \bar{\Phi}[H], \quad D\Phi[H] = 0, \quad \bar{D}\bar{\Phi}[H] = 0. \quad (1.21) \]

This is not a local operation in \(\mathcal{C}\). An explicit form may be chosen as

\[ \Phi[H] = DD \int d^3\underline{x}' \Delta(\underline{x} - \underline{x}') H(\underline{x}'), \quad \bar{\Phi}[H] = \bar{D}\bar{D} \int d^3\underline{x}' \Delta(\underline{x} - \underline{x}') H(\underline{x}'), \quad (1.22) \]
where $\Delta$ is the distribution

$$
\Delta(x - x') = (\theta - \theta')(\bar{\theta} - \bar{\theta})\epsilon(x - x'),
$$
(1.23)

$$
\partial \epsilon(x - x') = \delta(x - x'), \quad \epsilon(x - x') = -\epsilon(x' - x).
$$

In the following, we shall use the short-hand notations $\Phi[\text{res}[L,X]] = \Phi_X$, $\bar{\Phi}[\text{res}[L,X]] = \bar{\Phi}_X$. In general, $\Phi_X$ will not satisfy the same boundary conditions as the phase space fields do. However, we noted earlier that in the case of a commutator, the residue is a total derivative, $\text{res}[L,X] = D\bar{\omega} + \bar{D}\omega$. Here $\omega$ and $\bar{\omega}$ are differential polynomials in the fields. Then one easily shows that $\Phi_X = D\bar{\omega} + \alpha$, $\bar{\Phi}_X = \bar{D}\omega - \alpha$, where $\alpha$ is a constant reflecting the arbitrariness in the definition of $\Phi$, $\bar{\Phi}$. Up to this constant, $\Phi_X$ will respect the boundary conditions.

We are now in a position to write the two possibilities for the second bracket as

$$
\{l_X, l_Y\}_a(L) = \text{Tr} \left( LX(Y) - XL(Y) + \Phi_Y LX + XL\Phi_Y \right),
$$
(1.26)

and

$$
\{l_X, l_Y\}_b(L) = \text{Tr} \left( LX(Y) - XL(Y) + \Phi_Y XL + LX\Phi_Y \right).
$$
(1.27)

These expressions do not depend on the arbitrary constant $\alpha$. Checking the antisymmetry of the Poisson brackets and the Jacobi identity can be done with a little effort. As usual, the first bracket is a linearization of the two quadratic ones, that is to say

$$
\{l_X, l_Y\}_{(2)}^{a,b}(L + zD\partial^- D) = \{l_X, l_Y\}_{(2)}^{a,b}(L) + z\{l_X, l_Y\}_{(1)}(L),
$$
(1.28)

and the linear bracket is compatible with each of the two quadratic brackets.

2) The Poisson brackets (1.26,1.27) may be put in the general form introduced in

$$
\{l_X, l_Y\}_{(2)}^{a,b}(L) = \text{Tr} \left( LX(a(LY)) + XLb(LY) - LXc(YL) - XLd(YL) \right)
$$
(1.24)

However, the price to pay is that $a$, $b$, $c$, $d$ are non-local endomorphisms of $C$. As an example, for the first quadratic bracket one finds

$$
a(X) = \frac{1}{2}(X_+ + \lambda(\Phi[\text{res}X])) - \frac{1}{2}(X_- - \lambda(\Phi[\text{res}X])), \quad b(X) = \lambda(\Phi[\text{res}X]).
$$
(1.25)

One easily checks in particular that $a$ is a non-local antisymmetric $r$ matrix.
Introducing the hamiltonians $H_k = \frac{n}{k} \text{Tr}(L^k_n)$, the KP evolution equations (1.15) may be written as
\[
\partial_t (l_X(L)) = \{l_X, H_{k+n}\}_{(1)}(L) = \{l_X, H_k\}_{(2)}(L) \tag{1.29}
\]

**Poisson maps** Before turning to the study of the reductions of the KP hierarchies, let us exhibit some relations between the two quadratic brackets. We will use the invertible map in $C$
\[
p(L) = \partial^{-1} \tau(L) = D \partial^{-1} L' \partial^{-1} \bar{D}. \tag{1.30}
\]
Then a straightforward calculation leads to
\[
\{l_X \circ p, l_Y \circ p\}_{(2)}^a = -\{l_X, l_Y\}_{(2)}^b \circ p, \tag{1.31}
\]
which shows that (1.26) and (1.27) are equivalent Poisson brackets. However there is no relation between the hamiltonians $\text{Tr}(L^k_n)$ and $\text{Tr}(L^k \bar{n})$.

There is another relation between the two brackets, which involves the chiral superfield $\varphi$ satisfying the evolution equation (1.16). Let us introduce the linear functional $l_t = \int d^3 \varphi(t \varphi)$, where $t(\varphi)$ is a Grassmann even superfield. We consider an enlarged phase space including $\varphi$, and extend the Poisson bracket (1.26) to this phase space by
\[
\{l_t, l_Y\}_{(2)}^a (L, \varphi) = \int d^3 \varphi (\varphi_Y + \Phi_Y \varphi), \quad \{l_t, l_{t'}\}_{(2)}^a = 0. \tag{1.32}
\]
Then one finds
\[
\{l_X \circ s, l_Y \circ s\}_{(2)}^a = \{l_X, l_Y\}_{(2)}^b \circ s, \tag{1.33}
\]
where the transformation $s$ has been defined in (1.17). Notice that the hamiltonians are invariant functions for the transformation $s$, $\text{Tr}(L^k \bar{n}) = \text{Tr}(s(L)^k \bar{n})$.

A last relation uses the antichiral superfield $\bar{\chi}$ satisfying the evolution (1.18). Let us introduce the linear functional $l_t = \int d^3 \bar{\chi}(\bar{t} \bar{\chi})$, where $\bar{t}(\bar{\chi})$ is a Grassmann odd superfield. We consider an enlarged phase space including $\bar{\chi}$, and extend the Poisson bracket (1.26) to this phase space by
\[
\{l_t, l_Y\}_{(2)}^a (L, \bar{\chi}) = \int d^3 \bar{\chi}(\Phi_Y \bar{\chi}), \quad \{l_t, l_{t'}\}_{(2)}^a = -2 \int d^3 \bar{\chi} \bar{\chi} \Phi[\bar{t} \bar{\chi}], \tag{1.34}
\]
\[
\{l_t, l_{t'}\}_{(2)}^a = -2 \int d^3 \bar{\chi} \bar{\chi} \Phi[\bar{t} \bar{\chi}], \tag{1.35}
\]
\[
\]
where $\Phi$, $\bar{\Phi}$ are defined in equations (1.22, 1.23). Notice that this is a non-local Poisson bracket. One finds

$$\{l_X \circ \sigma, l_Y \circ \sigma\}_\sigma^{(a)} = -\{l_X, l_Y\}_\sigma^{(b)} \circ \sigma,$$

where the transformation $\sigma$ has been defined in (1.19).

## 2 Reductions of the KP hierarchy

In order to obtain consistent reductions of the KP hierarchy, we need to find Poisson submanifolds of the KP phase space. Considering first the quadratic bracket (1.26), we rewrite it as

$$\{l_X, l_Y\}_\sigma^{(a)} = \text{Tr} X \xi_{l_Y}^{a} = (LY)_+ L - L(YL)_+ + \Phi_Y L + L\bar{\Phi}_Y.$$

(2.1)

$\xi_{l_Y}^{a}$ is the Hamiltonian vector field associated with the function $l_Y$. One easily checks that if $L$ has the form (1.13), then for any $Y$, $\xi_{l_Y}^{a}$ has the form $D(\sum_{i<n-1} \xi_i \partial^i) \bar{D}$. It is obvious from (2.1) that for any $Y$, if $L$ is in $C_+$, then $\xi_{l_Y}^{a}$ is also in $C_+$. This means that the constraint

$$L = L_+$$

(2.2)

defines a Poisson submanifold. The hierarchies obtained in this way are the $N = 2$ supersymmetric KdV hierarchies studied by Inami and Kanno [26], and the Lax operators (2.2) already appeared in [14]. The lowest order cases will be presented in the next section.

Another possible reduction is to take $L$ of the form

$$L = L_+ + D \varphi \partial^{-1} \bar{\varphi} \bar{D}, \quad D\varphi = \bar{D}\bar{\varphi} = 0.$$

(2.3)

where $\varphi$ and $\bar{\varphi}$ are Grassmann even or odd chiral superfields. With $L$ of the form (2.3) and $Y$ arbitrary, one finds

$$\left(\xi_{l_Y}^{a}\right)_- = D((LY)_+ \varphi + \Phi_Y \varphi) \partial^{-1} \bar{\varphi} + \varphi \partial^{-1} (- (YL)_+^t \bar{\varphi} + \Phi_Y \bar{\varphi}) \bar{D},$$

(2.4)

Noticing that $(LY)_+ \varphi$ is a chiral superfield and $(YL)_+^t \bar{\varphi}$ an antichiral superfield, it is easily checked that $\xi_{l_Y}^{a}$ is indeed tangent to the submanifold.
defined by the constraints (2.3). It is possible to consider an enlarged phase space which coordinates are the fields in \( L \) and \( \varphi, \bar{\varphi} \). Let us introduce the linear functionals

\[
l_t = \int d^3x (\varphi t), \quad l_i = \int d^3x (\bar{\varphi}^-),
\]

(2.5)

where \( t \) and \( \bar{t} \) are general superfields, of the same Grassmann parity as \( \varphi \) and \( \bar{\varphi} \). In this enlarged phase space, the second Poisson bracket, in the case when \( \varphi \) and \( \bar{\varphi} \) are Grassmann even, is defined by (1.26) and

\[
\{l_t, l_Y\}^{a(2)}_t(L, \varphi, \bar{\varphi}) = \int d^3x ((LY)_+ \varphi + \Phi_Y \varphi)t,
\]

\[
\{l_t, l_Y\}^{a(2)}_t(L, \varphi, \bar{\varphi}) = \int d^3x (t (YL)_+ t \bar{\varphi} + \bar{\Phi}_Y \varphi),
\]

(2.6)

and

\[
\{l_t, l_i\}^{a(2)} (L, \varphi, \bar{\varphi}) = \int d^3x (L_+ \bar{t}), \quad \{l_t, l_i\}^{a(2)}_t = 0, \quad \{l_t, l_i\}^{a(2)}_t = 0.
\]

(2.7)

In the case when \( \varphi \) and \( \bar{\varphi} \) are Grassmann odd, the last two lines should be modified to

\[
\{l_t, l_i\}^{a(2)} (L, \varphi, \bar{\varphi}) = \int d^3x ((L_+ \bar{t}) t - 2 \varphi t \Phi[\bar{\varphi}]t),
\]

\[
\{l_t, l_i\}^{a(2)}_t = 2 \int d^3x \varphi t \Phi[\varphi t], \quad \{l_t, l_i\}^{a(2)}_t = -2 \int d^3x \bar{t} \varphi \Phi[\bar{\varphi} t],
\]

(2.8)

where the applications \( \Phi \) and \( \bar{\Phi} \) have been defined in (1.22). The lowest order case is \( L = D(1 + \varphi \partial^{-1} \bar{\varphi}) \bar{D} \). Then if \( \varphi \) and \( \bar{\varphi} \) are odd, the equation \( \frac{d}{dt} L = [L^2_+, L] \) is the \( N = 2 \) supersymmetric extension of the NLS equation [27]. The next-to-lowest order case is \( L = D(\partial + H + \varphi \partial^{-1} \bar{\varphi}) \bar{D} \). If \( \varphi \) and \( \bar{\varphi} \) are even, the hamiltonian structure (1.26) reduces in this case to the classical version of the “small” \( N = 4 \) superconformal algebra. Although the Poisson algebra contains 4 supersymmetry generators, the evolution equations (1.13) have only \( N = 2 \) supersymmetry. This case was first obtained by another method which will be given, as part of a detailed study, in [28].

We now turn to the second quadratic bracket (1.27). We rewrite it as

\[
\{l_X, l_Y\}^{b(2)}_t(L) = \text{Tr} X \xi^b l_Y,
\]

\[
\xi^b l_y = (LY)_+ L - L(YL)_+ + L \lambda(\Phi_Y) + \bar{\lambda}(\bar{\Phi}_Y) L.
\]

(2.9)

It is easily seen that neither the condition (2.2), nor the more complicated condition (2.3) are admissible reductions in this case. The easiest way to
find Poisson subspaces for the bracket (1.27) is to apply the map (1.30) to the Poisson subspaces of the first quadratic bracket. From (2.2), we are then lead to the restriction:

\[ L = L_+ + D \bar{D} \partial^{-1} H \partial^{-1} D \bar{D} \]  \hspace{1cm} (2.10)

With \( L \) of the form (2.10) and \( Y \) arbitrary, one finds

\[ (\xi_Y^I) = D \bar{D} \partial^{-1} ((LY)_+ H - (YL)^I_+ H + \text{res}[L,Y]H) \partial^{-1} D \bar{D}, \]  \hspace{1cm} (2.11)

which directly shows that condition (2.10) defines a Poisson submanifold for the Poisson bracket (1.27). It turns out that (2.10) also defines a Poisson submanifold for the linear Poisson bracket (1.20). To show this we rewrite the linear bracket as

\[ \{l_X, l_Y\} = \text{Tr} X \eta_Y, \quad \eta_Y = [L,Y]_+ - [L,Y]_+ + \lambda(\Phi_Y) + \bar{\lambda}(\bar{\Phi}_Y). \]  \hspace{1cm} (2.12)

With \( L \) of the form (2.10) and \( Y \) arbitrary, one finds

\[ (\eta_Y)_+ = D \bar{D} \partial^{-1} ((Y_+ - Y^I_+) H + \text{res}[L,Y]H) \partial^{-1} D \bar{D}. \]  \hspace{1cm} (2.13)

Thus the reduced hierarchies defined by condition (2.10) are bi-hamiltonian. The lowest order cases will be studied in the next section.

Notice that the transformation (1.17) maps the systems satisfying the condition (2.3) with Grassmann even fields \( \varphi \) and \( \bar{\varphi} \) into systems satisfying condition (2.10) with

\[ H = \varphi \bar{\varphi} + \varphi^{-1} L_+ \varphi. \]  \hspace{1cm} (2.14)

Analogously, the transformation (1.19) maps the systems satisfying the condition (2.3) with Grassmann odd fields \( \varphi \) and \( \bar{\varphi} \) into systems satisfying condition (2.10) with

\[ H = (-1)^n \left( \bar{\varphi} \varphi + (D \bar{\varphi})^{-1} D(L^I_+ \varphi) \right). \]  \hspace{1cm} (2.15)

Such transformations may be found in [9, 5].

Finally we may consider the image of the Poisson subspace defined by (2.3) under the map \( p \). One finds the condition

\[ L = L_+ + D \bar{D} \partial^{-1} (H + \varphi \partial^{-1} \varphi) \partial^{-1} D \bar{D}. \]  \hspace{1cm} (2.16)
The lowest order case is when $L_+ = DD$. The hamiltonian structure (1.27) reduces in this case to the classical version of the “small” $N = 4$ superconformal algebra. The equation $\frac{d}{dt}L = [(L^3)_+, L]$ becomes, after suitable redefinitions, the $N = 4$ supersymmetric extension of the KdV equation derived in [29] and written in $N = 2$ superspace in [30].

One can again consider an enlarged phase space which coordinates are the fields in $L$ and $\phi, \bar{\phi}$. The second quadratic bracket in this phase space is easily obtained from the first one by applying the map $p$ to the first quadratic bracket. $p$ acts as the identity on $\phi$ and $\bar{\phi}$. As a consequence the Poisson brackets (2.7) and (2.8) keep the same form, whereas (2.6) should be modified to

$$\{l_t, l_Y\}_{(2)} = \int d^3x \left( \text{res} \left( \tau((YL)_+\lambda(\phi)) + \Phi_Y \phi \right) \right).$$

3 Examples and comparison with other works

This paragraph is devoted to the presentation of the simplest integrable equations obtained using our formalism.

Considering first the condition (2.2), the simplest example is the lax operator $L = D(\partial + W)\bar{D}$. Then the evolution equation

$$\frac{d}{dt}L = [L^3_+, L],$$

leads to the equation

$$8\partial_t W = 2W_{xxx} + 6 \left( (DW)(\bar{D}W) \right)_x - \left( W^3 \right)_x,$$

which coincide after the redefinition $W = 2i\Phi$ with the $a = -2$ $N = 2$ extension of the KdV equation in the classification of Mathieu [17, 18]. The Lax operator given in [17] may be obtained from $L$ in the following way. Let us consider the operator

$$L_{-2} = L + L^t = \partial^2 + W[D, \bar{D}] + (DW)\bar{D} - (\bar{D}W)D.$$
\( L \) is in \( \mathcal{C} \) and \( L^t \) is in \( \bar{\mathcal{C}} \). If we remember that the product of an element in \( \mathcal{C} \) and an element in \( \bar{\mathcal{C}} \) always vanishes, we immediately get that a square root of \( L_2 \) with highest derivative term equal to \( \partial \) is \((L_{-2})^{\frac{1}{2}} = L^{\frac{1}{2}} - (L^{\frac{1}{2}})^t\). From this we deduce the relation \((L_{-2})^{\frac{1}{2}} = L^{\frac{1}{2}} - (L^{\frac{1}{2}})^t\). As a consequence \( L_{-2} \) satisfies the evolution equation

\[
\frac{d}{dt} L_{-2} = [L^{\frac{3}{2}}_+, L] + ([L^{\frac{3}{2}}_+, L])^t = [(L_{-2})^{\frac{3}{2}}, L_{-2}],
\]

which is thus an equivalent Lax representation for equation (3.3).

As the next example, we consider the Lax operator

\[
L = D(\partial^2 + V \partial + W) \bar{D}
\]

Then the evolution equation \( \frac{d}{dt} L = [L^{2/3}_+, L] \) should coincide, after suitable redefinitions, with one of the three \( N = 2 \) supersymmetric extensions of the Boussinesq equations derived in [21]. Indeed one can check that the Lax operator they give for the \( \alpha = -1/2 \) equation may be written as \( L^{(1)} = L + \bar{D} \partial^2 D \). Then one easily obtains \((L^{(1)})^{\frac{3}{2}} = L^{\frac{3}{2}} + \bar{D} \partial D\), and the evolution equation for \( L^{(1)} \) is easily deduced from that of \( L \)

\[
\frac{d}{dt} L^{(1)} = \frac{d}{dt} L = [(L^{(1)})^{\frac{3}{2}}, L^{(1)}].
\]

Turning now to condition (3.9), the lowest order case corresponds to the Lax operator \( L = D\bar{D} + D\bar{D} \partial^{-1} W \partial^{-1} D\bar{D} \). Then the equation \( \frac{d}{dt} L = [(L^{3})_+, L] \) becomes, after suitable redefinitions, the \( N = 2 \) supersymmetric extension of the KdV equation with parameter \( a = 4 \),

\[
\partial_t W = W_{xxx} + \frac{3}{2} ( [D, \bar{D}] W^2 )_x - 3 \left( (DW)(\bar{D}W) \right)_x + (W^3)_x
\]

Notice that, all integer powers of \( L \) define conserved charges in this case (an alternative Lax operator with the same property was derived in [10]).

The last example that we shall study is the Lax operator

\[
L = D(\partial + V) \bar{D} + D\bar{D} \partial^{-1} W \partial^{-1} D\bar{D}.
\]

Then the equation

\[
\partial_2 L = [L_+, L]
\]
explicitely reads
\[
\begin{align*}
\partial_2 V &= 2W_x, \\
\partial_2 W &= [D, \overline{D}]W_x + VW_x + (DV)(\overline{D}W) + (\overline{D}V)(DW).
\end{align*}
\] (3.10) (3.11)

This equation is identical, up to a rescaling of time, to the \(N = 2\) supersymmetric extension of the Boussinesq equation with parameter \(\alpha = -2\) derived in [21].

4 From \(N = 2\) to \(N = 1\) superspace

\(N = 2\) extensions of the KP and KdV hierarchies have been studied in several articles [20, 4, 3, 6] using an \(N = 1\) superspace formalism. In this section we wish to relate the KP hierarchies that we described in section 1 to those given in the literature. The first step will be to relate our \(N = 2\) algebra \(\mathcal{C}\) of pseudo-differential operators to the \(N = 1\) algebra of pseudo-differential operators. An operator \(L = D\mathcal{L}\overline{D}\) in \(\mathcal{C}\) should be considered as acting on a chiral object \(\Psi\), \(D\Psi = 0\), and this action writes
\[
L.\Psi = D\mathcal{L}\overline{D}.\Psi = \mathcal{L}\partial.\Psi + (D.\mathcal{L})\overline{D}.\Psi.
\] (4.1)

We shall use the following combinations of the chiral derivatives
\[
D_1 = D + \overline{D}, \quad D_2 = -D + \overline{D}, \quad D_1^2 = -D_2^2 = \partial, \quad \{D_1, D_2\} = 0.
\] (4.2)

Then the action of \(L\) on \(\Psi\) is
\[
L.\Psi = (\mathcal{L}\partial + (D.\mathcal{L})D_1).\Psi.
\] (4.3)

We then choose to associate to the \(N = 2\) pseudo-differential operator \(L\) the \(N = 1\) pseudo-differential operator \(\mathcal{L}\) given by
\[
\mathcal{L} = \mathcal{L}|_{\theta_2=0}\partial + (D.\mathcal{L})|_{\theta_2=0}D_1.
\] (4.4)

It is easily checked that this correspondence respects the product, \(LL' = \mathcal{L}\mathcal{L}'\). It also has the property
\[
L_{\pm} = L_{>0}.
\] (4.5)

13
That is to say that the image of an $N = 2$ differential operator is a strictly differential $N = 1$ operator, without the non-derivative term. Notice also the useful relations

$$\text{res}(L) = (D, \text{res}(L))|_{\theta_2=0}, \quad (4.6)$$

$$\text{Tr}(L) = \text{Tr}(L) \equiv \int d^2 \theta \text{res}(L), \quad \int d^2 \theta \equiv \int dx d\theta_1 \quad (4.7)$$

where the residue of the operator $L$ is the coefficient of $D_1^{-1} \equiv D_1 \partial^{-1}$. From now on, all expressions will be written in $N = 1$ superspace, and we drop the index of $D_1$ and $\theta_1$. The KP hierarchy described in section 1 may be described in $N = 1$ superspace as follows. We consider an operator $L$ of the form

$$L = D^{2n} + \sum_{p=1}^{\infty} w_p D^{2n-p-1} \quad (4.8)$$

and consider evolution equations

$$\frac{\partial}{\partial t_k} L = [L_{>0}^k, L] \quad (4.9)$$

This is nothing but the non-standard supersymmetric KP hierarchy described in [4, 3]. The evolution equations (4.9) admit the conserved quantities $H_p = \text{Tr}(L^p)$, and they are bi-hamiltonian. The first Poisson bracket is easily deduced from its $N = 2$ counterpart (1.20). With $l_X = \text{Tr}(L X)$, we have

$$\{l_X, l_Y\}_1 = \text{Tr}(L ([X_{>0}, Y_{>0}] - [X_{<0}, Y_{<0}])) \quad (4.10)$$

As in the $N = 2$ formalism, this is a standard bracket associated with a non-antisymmetric $r$ matrix. As a consequence, the two quadratic brackets deduced from (1.26) and (1.27) are quite complicated. They involve the quantity $\psi_X$ defined up to a constant by $D\psi_X = \text{res}[L, X]$. The first one is

$$\{l_X, l_Y\}_2^a(L) = \text{Tr}(L X(L X)_+ - X L(X L)_+) + \int d^2 \theta (-\psi_X \text{res}[L, X]$$

$$+ \text{res}[L, X] \text{res}(X L D^{-1}) - \text{res}[L, X] \text{res}(L X D^{-1})). \quad (4.11)$$

The Poisson bracket (1.27) becomes

$$\{l_X, l_Y\}_2^b(L) = \text{Tr}(L X(L Y)_+ - X L(Y L)_+) + \int d^2 \theta (\psi_X \text{res}[L, X]$$

$$+ \text{res}[L, X] \text{res}(L X D^{-1}) - \text{res}[L, X] \text{res}(L Y D^{-1})). \quad (4.12)$$

14
and already appeared in [3]. It is not a difficult task to obtain the $N = 1$ restrictions which correspond to the $N = 2$ conditions (2.2, 2.3, 2.10, 2.16). Some of the lax operators obtained in this way are already known, in particular those satisfying (2.2) from [26] and the lowest order operator coming from (2.3) with odd $\varphi$ and $\bar{\varphi}$, which is the super-NLS Lax operator obtained in [3].

5 Conclusion

An easy generalization of the hierarchies presented in this article would be to consider multi-components KP hierarchies, that is to say replace the fields $\varphi$ and $\bar{\varphi}$ in (2.3) and (2.16) by a set of $n + m$ fields $\varphi_i$ and $\bar{\varphi}_i$, $n$ of them being Grassmann even and the other $m$ being Grassmann odd. For the lowest order case of equation (2.3), such a generalization has been considered in [4]. The Lax representation that we propose for such hierarchies has the advantage that one does not need to modify the definition of the residue. For the next to lowest order case of equation (2.3), and the lowest order case of equation (2.16), it should be possible to obtain in this way hierarchies based on $W$-superalgebras with an arbitrary number of supersymmetry charges.

Little is known about the matrix Lax formulation of the hierarchies presented here. In the case of operators satisfying condition (2.2), such a matrix Lax formulation was constructed in $N = 1$ superspace by Inami and Kanno [24, 31]. It involves the loop superalgebra based on $sl(n|n)$. What we know about the matrix Lax formulation in $N = 2$ superspace for hierarchies based on Lax operators satisfying conditions (2.2) or (2.3) will be reported elsewhere. Notice that we obtained the form (2.2) of the scalar Lax operators from a matrix Lax representation, and only later became aware of reference [14] where these operators also appear.

References

[1] Z. Popowicz, J. Phys. A29 (1996) 1281.

[2] H. Aratyn and C. Rasinariu, preprint UICHEP-TH/96-15 and hep-th/9608107.
[3] J.C. Brunelli and A. Das, Rev. Math. Phys. 7 (1995) 1181.

[4] S. Ghosh and S. Paul, Phys. Lett. B 341 (1995) 293.

[5] L. Bonora, S. Krivonos and A. Sorin, preprint SISSA-56-96-EP and hep-th/9604163.

[6] A. Das and S. Panda, Mod. Phys. Lett. A 11 (1996) 723.

[7] E. Ivanov, S. Krivonos and R.P. Malik, Int. J. Mod. Phys. A 10 (1995) 253.

[8] A. Das and J.C. Brunelli, Phys. Lett. B 337 (1994) 303; Phys. Lett. B 354 (1995) 307; Int. J. Mod. Phys. A 10 (1995) 4563; preprint hep-th/9506096.

[9] S. Krivonos and A. Sorin, Phys. Lett. B 357 (1995) 94.

[10] S. Krivonos, A. Sorin and F. Toppan, Phys. Lett. A 206 (1995) 146.

[11] A. Das and J.C. Brunelli, Mod. Phys. Lett. A 10 (1995) 2019; J. Math. Phys. 36 (1995) 268.

[12] I.M. Gelfand and L.A. Dikii, Funct. Anal. Appl. 10 (1976) 259.

[13] F. Delduc and M. Magro, J. Phys. A: Math. Gen. 29 (1996) 4987.

[14] Z. Popowicz, Phys. Lett. B 319 (1993) 478.

[15] M.A. Semenov-Tian-Shansky, Funct. Anal. Appl. 17 (1983) 259.

[16] H. Lu, C.N. Pope, L.J. Romans, X. Shen and X.J. Wang, Phys. Lett. B 264 (1991) 91.

[17] C. A. Laberge, P. Mathieu, Phys. Lett. B 215 (1988) 718.

[18] P. Labelle and P. Mathieu, J. Math Phys. 32 (1991) 923.

[19] Z. Popowicz, Phys. Lett. A 174 (1993) 411.

[20] C.M. Yung, Phys. Lett. B 309 (1993) 175.
[21] S. Bellucci, E. Ivanov, S. Krivonos and A. Pichugin, Phys. Lett. B 312 (1993) 463.

[22] C.M. Yung and R.C. Warner, J. Math. Phys. 34 (1993) 4050.

[23] B. A. Kupershmidt, Commun. Math. Phy. 99, (1985) 51.

[24] W. Oevel and W. Straampp, Commun. Math. Phys. 157 (1993) 51.

[25] L. Freidel, J. M. Maillet, Phys. Lett. B 262 (1991) 278.

[26] T. Inami and H. Kanno, Int. J. Mod. Phys. A 7, Suppl. 1A (1992) 419.

[27] G. Roelofs and P. Kersten, J. Math Phys. 33 (1992) 2185.

[28] F. Delduc, E. Ivanov and L. Gallot, in preparation.

[29] F. Delduc and E. Ivanov, Phys. Lett. B 309 (1993) 312.

[30] F. Delduc, E. Ivanov and S. Krivonos, J. Math. Phys. 37 (1996) 1356.

[31] T. Inami and H. Kanno, J. Phys. A 25 (1992) 3729.