Realistic definition of Grassmann numbers and fermionic states

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Abstract

In this paper we attempt to define Grassmann numbers as elements living in multi-dimensional space, that exist independently of path integral. Furthermore, we define path integral as literally a "limit of the sum" in such a way that it approximates the expected mathematical properties of Grassmann integration. We then proceed to show that "Harmonic oscillator" in Fermionic space produces a linear function. The latter has two degrees of freedom per variable (the slope and the height at the origin), just as expected from Fermi exclusion principle. At the same time, the linear function is living in a continuum space; in other words, only two out of uncountably many degrees of freedom are "utilized". However, we argue that we would be able to utilize the rest of the degrees of freedom if we were to modify the structure of our Lagrangian. Such modification involves the use of functions that can not be generated from the usual wedge products of Grassmann numbers. Thus, it requires an appeal to the fact that Grassmann numbers are well defined mathematical objects, independently of standardly-accepted formalism.

1. Introduction

It is well known that, in Feynmann path integral picture, Grassmann numbers are used in order to define fermionic field. At the same time, the integral of Grassmann numbers is not viewed as a limit of a sum. Instead, it is believed to be a "formal" operation of symbols. This is probably the case for two reasons. From the mathematical point of view, as will be outlined in the next section, naive attempts to view Grassmann integral as literally a "limit of a sum" would create some paradoxes that would need to be addressed. On the physical side, Fermi exclusion principle predicts only two states per variable. Thus, if we were to assign to it a continuum spectrum (as would be required in order to define integral in a "usual" way) we would have uncountably many degrees of freedom, out of which only two will be utilized. At the same time, however, it is common place to "pretend" that Grassmann
variables are somehow "physical". Examples of this range from the "rotations" of fermionic fields in standard model, to the physics beyond standard model, including supersymmetry.

Of course, there are ways to justify the "conventional" outlook on Grassmann numbers. For example, one can point out that in Fock space formulation the creation and annihilation operators are equally well defined for bosons and fermions, and neither require path integral. Furthermore, it is possible to redo the entire supersymmetry without reference to Grassmann numbers altogether ([2] and [1]). Now, it is logically possible to continue to use Grassmann numbers and path integrals we "don't believe to be real" and at the same time continue to hold true to realistic interpretation of quantum mechanics. After all, when we perform an addition by writing numbers in a column and drawing a line in the bottom, we don't provide physical meaning for either the position on the column or that line. This is merely a tool to compute the sum; yet, we do know physical meaning of the latter. In principle it is possible for a realist to understand, and use, Grassmann numbers and their "integers" in that same way. The Fock space formulation is "analogous" to the "sum", which we do know the meaning of, while path integral is "analogous" to writing numbers in column which has no meaning.

Nevertheless, while the above allows us to continue to be realist, it imposes some limitations on the possible "realistic" viewpoints we can consider. As we have just stated, we were forced to view Fock space as more fundamental than path integral. In curved space-time, however, this would imply "preferred" foliation of spacetime into hypersurfaces and "preferred" definition of particle number, contrary to the spirit of Bogolubov's transformations. We can then attempt to justify the preferred frame by claiming that we "happened" to believe in one anyway for the "independent" purposes related to quantum measurement problem. The other alternative would be to claim that the reality is to be computed through Feynmann diagrams whose in- and out- states are bosonic, while fermions only come into the loop terms. We can then re-interpret Grassmannian integral along "fermionic line" as a "two-point function" on bosonic line, the Grassmannian integral on fermionic loop as "one point function" on bosonic line, etc. which would be defined independently of fermionic fields. In the context of pilot wave models, this thinking was used by Struyve and Westmann [9]). While it is possible to argue that fermions don't exist and we "falsely infer" their existence by looking at bosons they emit, this imposes some limitations on the theories we can consider. For example, Valentini in his thesis attempted to use fermionic fields as beables which would now have to be deemed invalid. This begs the question: do we really want to continue to impose restrictions similar to above, or is it perhaps possible to realistically define Grassmann numbers which would make us free to consider broader spectrum of theories?

In fact, the argument that "everything involving Grassmann numbers can be explained away by other means" can be turned around. After all, we don't have "direct evidence" for anything else either. Instead, we gather a pattern of "coincidences" and, once that pattern is long enough, we conjecture that whatever this set of "coincidences" point to, indeed, exist. From this point of view, the "coincidence" that Grassmann numbers provide excellent "computational tools" for topics as diverse as path integral formulation of standard model as well as supersymmetry seem to suggest that perhaps they do, in fact, exist as literal
mathematical object. One thing that draws a big hole in this evidence is the fact that we would not expect the behavior of ”limit of the sum” to match the Grassmann integral as we know it. This is the main obstacle we overcame in this paper. Indeed, we have shown in Section 3 that if we assume two different kinds of products we will specify, and if we are to perform an integral over a contour that resembles ”random walk” in a space with very large number of anticommuting dimensions, and, finally, if we assume a metric as defined in Equation [15] we will, in fact, predict that ”limit of the sum” will obey the Grassmann integration properties!

As mentioned earlier, the idea of defining Grassmann numbers would probably meet a physics objection that their ”continuous” spectrum would contain a lot more degrees of freedom than could be directly observed. At the same time, however, it is possible to claim that from pure kinematical point of view we could observe these degrees of freedom, after all. We will claim, however, that the fermionic ”dynamics” predict that the system with infinitely many degrees of freedom will immediately ”collapse” into the linear combination of only two of them, and stay there. This will be shown to be a consequence of Grassmannian ”path integral” taken over the ”continuous” anticommuting space we have defined. At the same time, this will, in principle, allow for the possibility that under some modifications of the theory (in particular, the way in which we treat ”time derivative” term) such collapse would either be slowed down or temporarily reversed, thus leading to non-trivial physical predictions. This idea might particularly be fruitful in case of quantum measurement.

2. Key mathematical difficulties and how to avoid them

Before we proceed to define Grassmann numbers, let us address some of the mathematical arguments as to why the pursuit might seem ”impossible”. In other words, let us first convince the reader that what we will do does not imply any mathematical contradiction, and after that, in the subsequent sections, we will actually do it. Let us, therefore, make a list of apparent ”contradictions” we would face (a through f) and after that we will ”answer” each of the ”contradictions” in the corresponding items a’ through f’:

a) Anticommutativity implies that the product of any two co-linear elements is zero. Now, if a one-variable integral is over a line, then multiple integral is over all possible choices of \( n \) points on that line. Since these points are bound to be colinear, their wedge product will return zero, contrary to what we want.

b) Even though products of two Grassmann numbers commute, \((\theta_1\theta_2)\theta_3 = +\theta_3(\theta_1\theta_2)\), they can not be identified with real line. After all, \((\theta_1\theta_2)^2 = 0\). Thus, it is a bit puzzling why the integration of ”products” \( \theta d\theta \) would, in fact, return a ”real number” (namely 1).

c) In the definition of the integral we demand that the integral over odd function is 1 while integral over the constant is 0. On the other hand, if we were to view integral as a sum, we would be expecting just the opposite.
d) The fact that
\[ \int d\theta_1 d\theta_2 \theta_1 \theta_2 = +1 \] (1)
with plus rather than minus on the right hand side, implies that we assumed that \((d\theta_1 d\theta_2)(\theta_1 \theta_2) = +(d\theta_1)(d\theta_2)\), contrary to what anticommutativity would imply.

e) In order to define the integral as ”limit of the sum”, we need the notion of ”norm” since the latter is part of the definition of the limit. But, if we define norm in a ”naïve” way, \(|\theta|^2 = \theta \theta\), then anticommutativity will imply that \(|\theta| = 0\) for all \(\theta\). This would render the definition of limit and, therefore, integral, meaningless.

f) Scaling symmetry implies that the integral of \(e^{a\theta}\) is \(a^{-1} e^{a\theta}\); yet, Grassmannian integral is \(ae^{a\theta}\) instead.

Because of these difficulties, it is commonly assumed that Grassmann numbers are not mathematical objects at all; they are merely ”symbols”, and their integral is likewise viewed to be ”manipulation of symbols” as opposed to ”limit of the sum”. In this paper, however, we found a way to get around these difficulties and ultimately come up with ”realistic” view of both Grassmann numbers and their integrals. In particular, we address the above list of issues in the following way:

a’) Instead of assuming that one-variable integral is over a line, we assume that it is over a contour that is living in infinitely dimensional space, spanned by \(\{e_1, e_2, \cdots\}\). The contour has some specific afore-given, non-trivial shape, which is expressily different from the straight line. The \(n\)-variable integral will be the integration over \(n\) random points on the contour. In light of infinite dimensionality, combined with the curvature of the contour, the vectors pointing towards these points are linear independent. Thus, their product is no longer zero.

b’) We have two kinds of products: dot product (\(\cdot\)) and wedge product (\(\wedge\)), where we assume that only the latter anticommutes, but not the former. The typical expression under the integral takes the form \(d\theta_1 \cdot d\theta_2 \cdot (\theta_1 \wedge \theta_2)\). The presence of dot-product prevents us from saying that wedging it with itself will return zero. Thus it is conceivable that the above product is, indeed, a real number.

c’) We can turn the integral over even function into 0 and over odd function into 1 by using one of the two tricks:

(i) We can displace the center of the region of integration away from the origin. As a result, both even and odd functions will integrate to non-zero values. After that we can introduce a sequence of these regions and take the limit over the corresponding sequence of integrals. It is conceivable to select the sequence in such a way that the limit will return zero for even functions but not the odd ones.

(ii) We can, contrary to part (i), assume that the center is still at the origin. However, we can impose non-trivial measure that changes sign; in fact, the measure that we impose happens to be an odd function! As a result, the product of measure with the odd and even function will be even and odd respectively. This would imply the desired result: even functions would integrate to zero while odd ones would not.
d’) This is merely the issue of convention. We can go back between the two conventions by defining an extra product operation, \( \circ \), by \( \chi_1 \circ \chi_2 = \chi_2 \cdot \chi_1 \). Thus, in light of transitivity of \( \cdot \), we have
\[
d\theta_1 \cdot d\theta_2 \cdot (\theta_1 \wedge \theta_2) = (d\theta_2 \circ d\theta_1) \cdot (\theta_1 \wedge \theta_2).
\]
We will agree that
\[
1 = \int (d\theta_1 \circ d\theta_2) \cdot (\theta_1 \wedge \theta_2) = \int d\theta_2 \cdot d\theta_1 \cdot (\theta_1 \wedge \theta_2)
\]
(2)

e’) We will introduce a definition of norm independent of products.

f’) As stated in a’, we are integrating over a contour that has non-trivial shape. The shape of the contour violates scaling symmetry. Now, in order to prove that integral of \( e^{a\theta} \) is \( a^{-1}e^{a\theta} \), one needs to appeal to scaling symmetry. Since the scaling symmetry is no longer valid, neither is alleged property of the integral.

In light of the fact that we have gotten rid of ”no go” arguments, we will now attempt to go ahead and define Grassmann numbers as literal mathematical objects that exist independently of integration, and which might have a wide spectrum of continuous values. We will also define integral as a ”limit of the sum” and demonstrate that, if the dot product is defined in a certain way, the outcomes of the integrals over contours that make consecutive terms in different dimensions (inside almost-infinite dimensional space) can in fact be expected to return the typical properties of Grassmann integral. We will then work out the harmonic oscillator in this formulation. Finally, in Conclusion we will briefly mention possible physics applications of our ”new” definition of Grassmannian integral, that are to be explored in future research.

3. Grassmann Integral

As was mentioned in part a’, anticommutativity is accommodated through a very large dimensionality. In particular, our space will be spanned by unit vectors \( e_1, \ldots, e_N \). Furthermore, as was mentioned in part b’, we are going to introduce two kinds of products: dot-product and wedge-product. The two products coincide when it comes to orthogonal unit vectors:
\[
i \neq j \Rightarrow e_i \cdot e_j = e_i \wedge e_j
\]
(3)

On the other hand, if we are to multiply the unit vector by itself then the wedge product will give us zero, whereas the dot product will give us 1:
\[
e_k \wedge e_k = 0, \quad e_k \cdot e_k = 1
\]
(4)

It is easy to see that the same object has more than one dot-product representation (for example, \( 1 = e_5 \cdot e_5 = e_7 \cdot e_7 \)). At the same time, wedge product representation is unique as long as we make sure to write it down in an order of increasing of indexes (thus avoiding the confusion between \( e_1 \wedge e_2 \) and \( e_2 \wedge e_1 \)). From the above properties one can compute dot products of more complicated objects. For example,
\[
(e_1 \wedge e_5 \wedge e_7) \cdot (e_4 \wedge e_5) = (-e_1 \wedge e_7 \wedge e_5) \cdot (-e_5 \wedge e_4) = (e_1 \wedge e_7 \wedge e_5) \cdot (e_5 \wedge e_4) = \]
(5)
\( = (e_1 \cdot e_7) \cdot (e_5 \cdot e_4) = (e_1 \cdot e_7) \cdot 1 \cdot e_4 = e_1 \cdot e_7 \cdot e_4 = e_1 \wedge e_7 \wedge e_4 = -e_1 \wedge e_4 \wedge e_7 \)

It should be noted that the consistency of the procedure such as above needs to be formally proven. Strictly speaking, we have to define dot- and wedge- products of the "chains" of arbitrary length and then prove their associativity. Such proof, however, turns out to be surprisingly long and for that reason we have decided to leave it out. A Grassmann number \( \theta \) is given as a linear combination of unit vectors without any products:

\[
\theta = \sum \lambda_k e_k \tag{6}
\]

where \( \lambda_k \) are commuting reals while \( e_k \) anticommute as just described. When we are performing Grassmann integral, we are integrating over afore-given contour \( C \) as opposed to entire domain. That contour is given by some agreed upon set of differential functions \( \lambda_k(\sigma) \):

\[
\theta(\sigma) = \sum \lambda_k(\sigma) e_k \tag{7}
\]

Now we will assume that we have very large, but finite, dimensionality, \( N \). At any given time, only \( n \) out of \( N \) coordinates evolve, where \( n \) is some small integer (such as 2 or 5). For example, if \( n = 4 \) then our contour will start out parallel to \( x_1 x_2 x_3 \) hyperplane, then it would slowly "rotate" to the direction parallel to \( x_2 x_3 x_4 \) hyperplane, then it would move parallel to \( x_3 x_4 x_5 \) hyperplane, and so forth (this agrees with the "non-trivial shape" we mentioned in part a’). During the time of "shift" from \( x_1 x_2 x_3 \) hyperplane to \( x_2 x_3 x_4 \) all four coordinates evolve, which is why in this particular example we say \( n = 4 \) rather than \( n = 3 \). We will, therefore, assume that the total increment of each coordinate is reasonably small

\[
\lambda_k(\sigma = \infty) - \lambda_k(\sigma = -\infty) = \Delta \lambda \tag{8}
\]

Now, if we are perform the integral of \( d \theta \) over the above contour, we obtain

\[
\int_C d\theta = \Delta \lambda \sum_{k=1}^N e_k \tag{9}
\]

On the other hand, the integral of \( d\theta \cdot \theta \) is given by

\[
\int_C d\theta \cdot \theta = \sum_{i,j} (e_i \cdot e_j \int \frac{d\lambda_i}{d\sigma} \lambda_j(\sigma) d\sigma) \tag{10}
\]

Now, the above sum goes over both \( i \neq j \) and \( i = j \). The \( i \neq j \) terms will produce \( e_i \cdot e_j = e_i \wedge e_j \), while \( i = j \) terms will produce \( e_k \cdot e_k = 1 \). Thus, we obtain

\[
\int_C d\theta \cdot \theta = \sum_k \left( \int \frac{d\lambda_k}{d\sigma} \lambda_k(\sigma) d\sigma \right) + \sum_{i \neq j} \left( e_i \wedge e_j \int \frac{d\lambda_i}{d\sigma} \lambda_j(\sigma) d\sigma \right) \tag{11}
\]

By evaluating the first integral on the right hand side, we obtain

\[
\int_C d\theta \cdot \theta = \frac{N(\Delta \lambda)^2}{2} + \sum_{i \neq j} \left( e_i \wedge e_j \int \frac{d\lambda_i}{d\sigma} \lambda_j(\sigma) d\sigma \right) \tag{12}
\]
Now, if we take "Pythagorean" absolute value of Equation 9, we obtain

\[ \text{Pythagorean} \implies \left| \int d\theta \right|^2 = N(\Delta \lambda)^2 \] (13)

which happens to be twice larger than the integer part of Equation 12. In light of the fact that we "want" the left hand side of Equation 9 to be zero and the left hand side of Equation 12 to be 1, this doesn’t appear too optimistic. However, situation improves drastically if we replace the Pythagorean metric with "Special Manhattan" one (where "special Manhatten" differs from Manhatten in that we take maximum of coordinate values instead of summing them):

\[ \text{Pythagorean} \implies \left| \sum \lambda_k e_k \right| = \sqrt{\sum \lambda_k^2} \] (14)

\[ \text{Manhatten} \implies \left| \sum \lambda_k e_k \right| = \sum \lambda_k \] (15)

\[ \text{Special Manhatten} \implies \left| \sum \lambda_k e_k \right| = \max\{\lambda_k\} \] (16)

If we now adopt the "Special Manhatten" metric in our Grassmann space, and generalize it to tensors according to

\[ \left| \sum_k \sum_{a_1,\ldots,a_k} \lambda_{a_1,\ldots,a_k} e_{a_1} \cdots e_{a_k} \right| = \max\{\lambda_{a_1,\ldots,a_k} \mid k \in \mathbb{N}\} \] (17)

we will, indeed, conclude that the real part of right hand side of Equation 9 is "much smaller" than the real part of right hand side of Equation 12 as desired. The only thing that is left to do is to select \( \Delta \lambda \) in such a way that the latter is equal to 1. One can easily see that this is accomplished through

\[ \Delta \lambda = \sqrt{\frac{2}{N}} \] (18)

Let us now look at the integral of two Grassmann variables. We would like to introduce two versions of double integral,

\[ \int d\theta_1 \cdot d\theta_2 \cdot (\theta_1 \wedge \theta_2) = \sum_{abcd} (e_a \cdot e_b \cdot (e_c \wedge e_d) \int d\sigma_1 d\sigma_2 \lambda'_a(\sigma_1) \lambda'_b(\sigma_2) \lambda_c(\sigma_1) \lambda_d(\sigma_2)) \] (19)

and

\[ \int d\theta_1 \cdot d\theta_2 \cdot (\theta_1 \wedge \theta_2) = \sum_{abcd} (e_a \cdot e_b \cdot (e_c \wedge e_d) \int d\sigma_1 d\sigma_2 \lambda'_a(\sigma_1) \lambda'_b(\sigma_2) \lambda_c(\sigma_1) \lambda_d(\sigma_2)) \] (20)

Now we would like to represent both integrals as

\[ \left( \int d\theta_1 \cdot \theta_1 \right) \left( \int d\theta_2 \cdot \theta_2 \right) = \left( \sum_{a,c} (e_a \cdot e_c \int d\sigma_1 \lambda'_a(\sigma_1) \lambda_c(\sigma_1)) \right) \left( \sum_{b,d} (e_b \cdot e_d \int d\sigma_2 \lambda'_b(\sigma_2) \lambda_d(\sigma_2)) \right) \] (21)
This would require us to assume
\[ \Re(e_a \cdot e_b \cdot (e_c \wedge e_d)) = \text{WANT} \Re(e_a \cdot e_c) \Re(e_b \cdot e_d) \] (22)
\[ (e_a \wedge e_b) \cdot (e_c \wedge e_d) = \text{WANT} \Re(e_a \cdot e_c) \Re(e_b \cdot e_d) \] (23)
where \( \Re \) stands for "real part". The above, however, is not correct. For example, if we set \( a = 1, b = 2, c = 2 \) and \( d = 1 \), we obtain
\[ \Re(e_1 \cdot e_2 \cdot (e_2 \wedge e_1)) = \Re((e_1 \wedge e_2) \cdot (e_2 \wedge e_1)) = \Re(-1) = -1 \] (24)
and, at the same time,
\[ \Re(e_1 \cdot e_2) \Re(e_2 \cdot e_1) = 0 \times 0 = 0 \] (25)
Intuitively, what we are facing is "unwanted contraction". We would like to separately "contract" \( d\theta_1 \) with \( \theta_1 \) and \( d\theta_2 \) with \( \theta_2 \), leading to separation of variables. But, unfortunately, we obtain "unwanted" contractions of \( d\theta_1 \) with \( \theta_2 \) and \( d\theta_2 \) with \( \theta_1 \) in the process. These contractions are also real-valued and, therefore, special Manhattan metric is no longer of any use. Instead, we get unwanted large real component in our integral.

However, we still have one "trick" left! In particular, we can design the shape of the contour in such a clever way that the "unwanted contractions" occur only when \( \sigma_1 \) and \( \sigma_2 \) are relatively close to each other. As a result, the part of the integral "without" such contractions will simply "outweigh" the part of the integral "with" them. In particular, we will postulate
\[ \frac{d\lambda_k}{d\sigma} \neq 0 \implies (k - \frac{n}{2})\sigma_0 < \sigma < (k + \frac{n}{2})\sigma_0 \] (26)
Now, if we assume that the contour starts at the origin,
\[ \lambda(-\infty) = 0 \] (27)
this will imply that for \( i \ll j \) we will have \( d\theta_i \cdot \theta_j \) contraction, but we will not have \( d\theta_j \cdot \theta_i \) one. Now, in the example of Equations 24 and 25, we need both contractions present at the same time in order to produce a real number. On the other hand, if one is present while the other is not, it is easy to see that we will get some "flowing" components, whose total contribution will end up being "very small" thanks to special Manhattan metric.

What we have just stated about two-variable integral can be generalized to several variable case. However, that generalization relies on the fact that \( |\sigma_i - \sigma_j| \gg n\sigma_0 \) throughout the "dominant" part of \( \sigma_1 \times \cdots \times \sigma_D \) (after all, this is a prerequisite for the claim that, per Equation 26, there are no unwanted contractions). The latter, in turn, holds true only if
\[ D \ll N \] (28)
where \( D \) is the number of variables we are integrating over,
\[ \int d\theta_1 \ldots d\theta_D \wedge f(\theta_1, \ldots, \theta_D) \] (29)
while $N$ is the number of variables the contour lives in:

$$\theta = \lambda_1 e_1 + \cdots + \lambda_N e_N$$

Geometrically, we are integrating over arbitrary distribution of $D$ points on a contour that makes the order of $N$ ”turns”. If two of these points we integrate over accidentally falls within the same ”turn” then we would have unwanted contraction. If they fall within two neighboring turns, we might still have unwanted contraction due to the ”directions” of these ”turns” sharing some of the coordinates but not all; in fact, this will be a concern even if they are distant neighbors, as long as they are either $n$-th neighbors of each other or closer. If, on the other hand, the points are placed at ”far away” turns, the tangent lines towards the curve at these points do not share any of the coordinates and, therefore, there are no unwanted contractions. If $D \ll N$ then the latter scenario is almost guaranteed to happen. If, on the other hand, $N \ll D$, then we are forced to face the former scenario and deal with unwanted contractions.

Now, in the real physical situation we have separate fermionic coordinate associated with every point in space. Thus, in order to say that $D \ll N$ we have to first assume that the spacetime is discretized and then further assume that the number of discrete points is much smaller than $N$. These two assumptions are quite separate. As far as discretization itself, it has to be assumed in bosonic case as well since integral with infinitely many dimensions is simply not defined. But the second assumption that the number of lattice points is much smaller than $N$ is strictly a consequence of Grassmannian nature of the integral. Of course, since we have not specified $N$, any ”finite” number of lattice points will have to be ”much smaller” than something. Thus, the statement that we have made might be interpreted as simply saying that ”$N$ is bounded below” as opposed to ”$D$ is bounded above”. The fermion doubling issue, of course, needs to be addressed; this, however, is beyond the scope of this paper.

4. Complex extension and linear combinations

As the reader probably noticed, we have defined Grassmann integral as the integral over the contour rather than the entire domain. This raises an important question: how can we ”rotate” Grassmann numbers? Yet, rotation of Grassmann numbers seems to be a very important property. First of all, it is common place to perform integral over $\theta = \theta_1 + i\theta_2$ and $\bar{\theta} = \theta_1 - i\theta_2$, instead of directly integrating over $\theta_1$ and $\theta_2$. Even more importantly, in gauge theory, the symmetry transformations are commonly applied to fermions; and they would not be defined if we couldn’t rotate Grassmann numbers. We propose to address the above by making clear distinction between Grassmann numbers and Grassmann variables as follows:

a) Grassmann numbers are defined in the entire $N$-dimensional space, and are not constrained to any contour nor any other subspace of that space. Likewise, all of our products and sums are defined on the space of all Grassmann numbers.

b) The term variable is used only if a Grassmann number under discussion is constrained
c) We can, in principle constrain different Grassmann variables to different contours from each other; or we can choose to constrain them to the same contour. It is up to us!

d) A given ”variable” (constrained to afore-specified contour)’ is said to be Grassmann variable if the contour it is constrained to leads to the integration properties described in the previous sections, up to the agreed-upon approximation.

e) A variable that violates part ”d” by living in the ”wrong contour” can still be referred to as ”variable”, as long as we are not calling it ”Grassmann variable”. Likewise, we can still integrate over that ”wrong” contour, as long as we don’t expect the results to match with Grassmann integration.

f) Since Grassmann numbers are defined independently of the presence of any contours, the ”variables” that violate part ”d” by living on the ”wrong contour” are still Grassmann numbers even though they are no longer Grassmann variables.

g) To summarize d, e and f, a variable living in the wrong contour is both a ”Grassmann number” and a ”variable”, but it is not a ”Grassmann variable”.

We claim that, particularly due to item c, the desired ”rotation” properties can be accomodated both for the case of Grassmann numbers and Grassmann variables, although we need a lot less care in the former case than in the latter.

When we are discussing symmetry transformations, we are discussing transformations of Grassmann numbers rather than Grassmann variables. Since the set of Grassmann numbers is a continuum, we can envision a ”classical” fermionic field $\psi$ which varies continuously and differentiably in space. However, we should not let the ”continuity in $x$” confuse us. When we talk about ”variable” we are claiming that, for any fixed $x$, the value of $\psi(x)$ can still ”vary”. This, of course, would be true in the context of path integral (and in this case we would be integrating $\psi(x_1)$ over $C_1$, $\psi(x_2)$ over $C_2$, and so forth; or we could instead decide that they are all being integrated around the same contour $C$; but in either case, the statement about ”contours” would turn them into ”variables”). But for ”classical” fermionic field, the value of $\psi(x = x_0)$ is not assumed to ”vary”, which means it is not a variable. Since in case of Grassmann numbers we don’t have to worry about ”staying on a contour”, we are free to rotate them. This, naturally, applies to ”classical” fermionic field and allows us to do symmatry transformations in ”classical” context.

Now, of course, we would also like to be able to rotate Grassmann ”variables” as well, and also we would like to think of an end-product as a ”Grassmann variable”. For example, it is common place to replace the integral over $\theta_1$ and $\theta_2$ with an integral over $\theta$ and $\bar{\theta}$, where

$$\begin{align*}
\theta &= \frac{\theta_1 + i\theta_2}{\sqrt{2}},
\bar{\theta} &= \frac{\theta_1 - i\theta_2}{\sqrt{2}},
\end{align*}$$

(31)

and we would like to be able to think of the latter as something well defined in a literal sence, as opposed to mere convention. Let us start with ”real valued” case,

$$\begin{align*}
\theta'_1 &= \frac{\theta_1 + \theta_2}{\sqrt{2}},
\theta'_2 &= \frac{\theta_1 - \theta_2}{\sqrt{2}}.
\end{align*}$$

(32)
In this case, we can simply agree that whenever different physical parameters are related by certain relations, the same applies to the contours they reside on. For example suppose we have a single spin 1/2 particle. In this case, we agree upon that the Grassmann numbers related to spin-up and spin-down are "constrained" to the contours $\theta = \theta_{+z}(\sigma)$ and $\theta_{-z}(\sigma)$. Then the corresponding spins in +x and -x directions are constrained to contours given by

$$\theta_{+x}(\sigma) = \frac{1}{\sqrt{2}}(\theta_{+z}(\sigma) + \theta_{-z}(\sigma)), \quad \theta_{-x}(\sigma) = \frac{1}{\sqrt{2}}(\theta_{+z}(\sigma) - \theta_{-z}(\sigma)) \quad (33)$$

Now, due to the $1/\sqrt{2}$ coefficient, the "new" contours obey the properties discussed in the previous section, which is why the "variables" associated with +x and -x are, in fact, Grassmann variables. The coefficients, in principle, could have been different. In this case, we would still have new contour, prescribed by the "wrong" coefficients, and we would still have new variables, but they would no longer be called "Grassmann variables". For example, if we assume that

a) $\eta$ is a Grassmann variable

b) $\chi = 2\eta$

Then we would obtain

$$\int d\chi \cdot \chi = 4 \int d\eta \cdot \eta = 4 \quad (34)$$

which would imply that $\chi$ is not a Grassmann variable. It is also related to the issue of integrating of exponents. In particular, it is true that

$$\int d\eta \cdot e^{2\eta} = \frac{1}{2} \int d\chi \cdot e^\chi, \quad \text{where } e^A = 1 + A + \frac{A \wedge A}{2} + \cdots \quad (35)$$

but, at the same time,

$$\int d\chi \cdot e^\chi = 4 \quad (36)$$

which is why

$$\int d\eta \cdot e^{2\eta} = \frac{1}{2} \times 4 = 2 \quad (37)$$

Again, the number 4 is a consequence of the fact that $\chi$ is not a Grassmann variable. At the same time, $\chi$ is still a "variable", and we are free to continue to use $\chi$ as long as we remember the "new" integration properties. Furthermore, since $\chi$ shares the same $N$-dimensional space with $\eta$, it is still a Grassmann number. Thus, it continues to have all of the "algebraic" properties of dot and wedge products outside the integration; but the rules of integration are now different.

Of course, what we have just said about $\eta$ and $\chi$ is also true in the "conventional" presentation of Grassmann numbers. But in the "conventional" case the agreement as to when the integration properties hold and when they don’t is merely formal. In our case, on the other hand, they are a consequence of the geometry of the specific contour. Thus, even if $\eta$ has never been mentioned, one can still know that the integral over $\chi$ returns 4 rather than 1 by simple inspection of the contour. This, in particular, means that the contour should
change the shape under linear transformations such as above. Since we can "stretch" straight line without changing its shape, this immediately tells us that the contour in question either has to have limited length, or it has to have non-trivial curvature, or both.

From what we have seen in the previous section, we already know that we need the contour to "bend around" for other purposes. In particular we wanted the properties of multiple integral to continue to hold true even if the same contour is selected to "constrain" all of the variables under discussion. In order for their wedge product not to return zero, we had to assume that the contour makes arbitrary many "turns". This, of course, has a side-benefit for the issue we are currently discussing. In particular, it means that we don't have to do "anything more" to assure that the "doubling" of a contour is distinct from the original contour. After all, the presence of different "turns" guarantees this to be the case.

Finally, let us turn to complex case. As expected, this would require us to extend the space of Grassmann numbers from real valued to complex valued, which amounts to doubling the number of "real" dimensions:

$$S = \{ r_{11}e_{11} + ir_{12}e_{11} + r_{21}e_{21} + ir_{22}e_{21} + \cdots \}$$

(38)

This creates a problem. As we have just mentioned, if we multiply a Grassmann variable by a "wrong" factor, it would no longer be a Grassmann variable. Now, $i$ is an example of such a "wrong factor". For example, if $\chi = i\eta$ then

$$\int d\chi \cdot \chi = - \int d\eta \cdot \eta$$

(39)

But, due to the difference in dimensionality, avoiding the factor of $i$ would have imposed much more severe limitations on us than avoiding the factor of 2. Instead, we will deal with the issue by taking "complex conjugate" of the "differential" part, without taking corresponding conjugate on the "finite" part. Thus, if we assume that

$$\theta = \frac{\theta_1 + i\theta_2}{\sqrt{2}}$$

(40)

and if we also assume that $\theta_1$ and $\theta_2$ are Grassmann variables, we obtain

$$\int d\bar{\theta} \cdot \theta = \frac{1}{2} \int d\theta_1 \cdot \theta_1 + \frac{i \times i}{2} \int d\theta_2 \cdot \theta_2 = 1$$

(41)

On the other hand,

$$\int d\theta_1 \cdot \theta = \frac{1}{2} \int d\theta_1 \cdot \theta_1 + \frac{i \times i}{2} \int d\theta_2 \cdot \theta_2 = 0$$

(42)

The reader should notice that the above two equations are just the opposite to what we would expect to get. The way we "fix" it is by introducing a third kind of product, $\circ$, defined as

$$A \circ B = \overline{A} \cdot B$$

(43)
Thus, we obtain
\[ \int d\theta \circ \theta = \int d\bar{\theta} \cdot \theta = 1 \] (44)
\[ \int d\bar{\theta} \circ \theta = \int d\theta \cdot \theta = 0 \] (45)

Since we are defining another kind of product for the purposes of "convention", we might as well kill two birds with one stone and deal with one other "convention" that needs to be addressed. Going back to "real valued" case, it is usually assumed that
\[ \int d\theta d\theta_2 \theta_1 \theta_2 = 1 \] (46)
while in our case we have, by using quasi-anticommutativity of the dot-product between $d\theta_2$ and $\theta_1$ in the "typical" scenario (that is, "typical" in a sense of special Manhattan metric discussed earlier), we obtain
\[ \int d\theta_1 \cdot d\theta_2 \cdot (\theta_1 \wedge \theta_2) = - \int d\theta_1 \cdot (\int d\theta_2 \cdot \theta_2) = - \int d\theta_1 \cdot \theta_1 = -1 \] (47)

We address the sign issue by claiming that
\[ A \wedge B = B \wedge A \] (48)

Now, by substituting this into the definition of $\circ$, we obtain
\[ \int (d\theta_1 \cdot d\theta_2) \circ (\theta_1 \wedge \theta_2) = \int d\theta_1 d\theta_2 \cdot (\theta_1 \wedge \theta_2) = \int d\bar{\theta}_2 \cdot d\bar{\theta}_1 \cdot (\theta_1 \wedge \theta_2) = 1 \] (49)

In general,
\[ \int (d\theta_1 \cdot \ldots \cdot \theta_n) \circ (\theta_1 \wedge \ldots \wedge \theta_n) = \int d\bar{\theta}_n \cdot \ldots \cdot d\bar{\theta}_1 \cdot (\theta_1 \wedge \ldots \wedge \theta_n) = 1 \] (50)

The left hand side of the above equation, indeed, agrees with commonly accepted convention. The price to pay for this, however, is the use of $\circ$ which is not associative.

5. Derivatives with respect to Grassmann variables

Let us now turn to a much simpler issue and attempt to define the derivatives with respect to Grassmann coordinates. The only obstacle to overcome is the fact that we have to "divide" by "vectors". From conventional point of view, whenever we compute $\partial f / \partial \theta$, $f$ is either constant in $\theta$ or proportional to $\theta$ (as always); thus, derivative is well defined in both cases. In the context of our model, however, we should be allowed to define some non-trivial function that can not be generated through wedge products; after all this is a simple consequence of the fact that we now have a set with well defined mathematical objects, rather than mere formal manipulation of symbols. Thus, it is possible for example, to define
\[ \theta(\lambda) = \lambda e_5 \ , \ f(\lambda e_5) = e_7 \sin \lambda, \] (51)
which means that the derivative will force us to evaluate $e_7/e_5$. The way to do it is to define the division in terms of dot-product rather than wedge-product:

$$\frac{A}{\theta} = B \iff \theta \cdot B = A$$

(52)

in this case, we immediately obtain

$$e_5 \cdot (e_5 \wedge e_7) = e_7 \Rightarrow \frac{e_7}{e_5} = e_5 \wedge e_7$$

(53)

In fact, the above definition of ratio is equivalent to

$$\frac{A}{\theta} = \frac{\theta \cdot A}{\theta \cdot \theta}$$

(54)

where the right hand side is well defined due to $\theta \cdot \theta$ being a scalar. After all, by using the associativity of scalar multiplication on the first step, and associativity of the dot-product on the second step we obtain

$$\theta \cdot \theta \cdot A = \frac{1}{\theta \cdot \theta} (\theta \cdot \theta) \cdot A = A$$

(55)

where on the last step we were using the assumption that

$$k \in \mathbb{C} \Rightarrow \forall A (k \wedge A = A \wedge k = k \cdot A = A \cdot k = kA)$$

(56)

Thus, if we have $n$ Grassmann variables, $\theta_1, \cdots, \theta_n$ living in the same contour $\theta(\sigma)$,

$$\theta_k = \theta(\sigma_k)$$

(57)

we can define

$$\tilde{F}(\sigma_1, \cdots, \sigma_n) = F(\sigma(\tau_1), \cdots, \sigma(\tau_n))$$

(58)

we can formally define the partial derivative as

$$\frac{\partial F}{\partial \theta_k} = \frac{\dot{\theta}(\sigma_k) \cdot (\partial \tilde{F}/\partial \sigma_k)}{\dot{\theta}(\sigma_k) \cdot \dot{\theta}(\sigma_k)}$$

(59)

On the other hand, if the Grassmann variables live in the different contours (namely $\theta_k$ lives on its own contour $\theta_k(\sigma)$ instead of the common contour $\theta(\sigma)$), one can simply rewrite the above expression while adding extra index $k$ to all $\theta$-s:

$$\frac{\partial F}{\partial \theta_k} = \frac{\dot{\theta}_k(\sigma_k) \cdot (\partial \tilde{F}/\partial \sigma_k)}{\dot{\theta}_k(\sigma_k) \cdot \dot{\theta}_k(\sigma_k)}$$

(60)

where one should remember that $k$ is fixed, and therefore the ”repetition” of index $k$ does not imply summation.

The above definitions would occasionally imply that the derivative has more components than we would expect. For example, we would normally expect the derivative of one vector
with respect to the other to return a scalar; yet, in some situations, including the example of Equation 51, we obtain a tensor instead:

$$\frac{df}{d\theta} = e_5 \wedge e_7 \cos \lambda$$  \hspace{1cm} (61)

If, on the other hand, we were to define \(g\) as

$$\theta(\lambda) = \lambda e_5, \quad g(\lambda e_5) = e_t \sin \lambda,$$  \hspace{1cm} (62)

then we would, in fact, obtain a scalar:

$$\frac{df}{d\theta} = \cos \lambda$$  \hspace{1cm} (63)

Since in the actual physical situation we typically consider the functions that can be generated through wedge products, we would not encounter the "surprising" scenario. But, in principle, it is possible to explore some non-algebraic modifications of the theory in which case we will begin to encounter them.

Another thing that is important to mention is that, contrary to the situation with the integral, in case of derivative we do not need a contour in order for it to be well defined (although of course we were using the contour in the version of the derivative we described above). We can, instead, define the derivative in terms of a "gradient" with respect to entire space:

$$(\nabla F)(\theta = \theta_0) = \sum_k \left( e_k \cdot \frac{dF(\theta_0 + te_k)}{dt} \right)$$  \hspace{1cm} (64)

where we have omitted the division by \(e_k \cdot e_k\) since we know that the latter is equal to 1. It should be noted that the "gradient" can be applied both to scalar, to vector and to higher rank tensor. If it is applied to scalar, we would get a vector as one might expect. Applying to vector will return a linear combination of a scalar and rank 2 tensor. Applying gradient to rank \(n\) tensor will return a linear combination of tensors of rank \(n - 1\) and \(n + 1\).

In case of "taking the derivative" of path integral with respect to "source terms" in order to compute a correlation function, we can think of it as either "derivative with respect to a path" or "gradient with respect to entire space"; in either case we would get the same answer. From physics point of view, however, we understand that there is no such thing as "source term". Instead, we are dealing with one of the fermionic fields which, for the purposes of our own human convenience, we are separating from everything else. Now, since we postulate that physical fields are constrained to a contour, the same is true for the sources and sinks. Thus, from this perspective, we are taking the contour-based derivatives rather than gradients. But, again, this does not affect the actual answer we will arrive at.

6. Harmonic oscillator in two Grassmann coordinates

So far we have found a way to model grassmann numbers as mathematical objects. The next logical question is to do the physics in the Grassmann space we have just defined. Doing the
lattice quantum field theory on fermions is beyond the scope of this paper. Instead, we will look at the ”first quantization” of the single particle living in two Grassmann coordinates. Since we have modeled Grassmann numbers as elements of a continuum, one would expect that the first quantization of physical system will look like a wave function,

$$\psi = \psi(\theta_1, \theta_2; t)$$  \hspace{1cm} (65)

We will derive its evolution by means of path integral. Just like in bosonic case, in order to define path integral properly, one would need to discretize time. However, due to the anticommuting nature of Grassmann variables, the effects of the discretization will end up being much more significant than they were in commuting case. After all, \(\theta_1(t_0) \text{”commutes” with itself and, at the same time, it ”anticommutes” with } \theta(t_0 + \delta t)\). In fact, the product \(\theta(t_0) \wedge \theta(t_0 + \delta t)\) is ”large” despite the fact that \(\theta_1(t_0) \wedge \theta_1(t_0) = 0\). This opens some room for discontinuities and might possibly imply that our result will depend on the discretization of our choice. The study of different kinds of discretizations is beyond the scope of the paper. For our purposes, we will simply split time into even intervals of length \(\delta t\). Being inspired by bosonic path integral, we will postulate the ”inductive” evolution of \(\psi\) according to

$$\psi(\theta_1, \theta_2; t = t_0 + \delta t) = \int (d\theta'_1 \cdot d\theta'_2) \circ \left(e^{-i\mathcal{L}(\theta'_1, \theta'_2, \theta_1, \theta_2)} \wedge \psi(\theta'_1, \theta'_2; t = t_0)\right)$$  \hspace{1cm} (66)

where the exponentiation is defined in terms of wedge-products:

$$e^A = 1 + A + \frac{A \wedge A}{2} + \frac{A \wedge A \wedge A}{6} + \cdots$$  \hspace{1cm} (67)

Let us now start at \(t = 0\) and see how it evolves. By substituting \(t = 0\) into Equation (66) we obtain

$$\psi(\theta_1, \theta_2; t = \delta t) = \int (d\theta'_1 \cdot d\theta'_2) \circ \left(e^{-i\mathcal{L}(\theta'_1, \theta'_2, \theta_1, \theta_2)} \wedge \psi(\theta'_1, \theta'_2; t = 0)\right)$$  \hspace{1cm} (68)

If we now substitute \(t = \delta t\) into Equation (66) we obtain

$$\psi(\theta_1, \theta_2; t = 2\delta t) = \int (d\theta'_1 \cdot d\theta'_2) \circ \left(e^{-i\mathcal{L}(\theta'_1, \theta'_2, \theta_1, \theta_2)} \wedge \psi(\theta'_1, \theta'_2; t = \delta t)\right)$$  \hspace{1cm} (69)

By substituting Equation (68) into Equation (69) we obtain

$$\psi(\theta_1, \theta_2; t = 2\delta t) = \int (d\theta'_1 \cdot d\theta'_2) \circ \left(e^{-i\mathcal{L}(\theta'_1, \theta'_2, \theta_1, \theta_2)} \wedge \left(\int (d\theta''_1 \cdot d\theta''_2) \circ \left(e^{-i\mathcal{L}(\theta''_1, \theta''_2, \theta'_1, \theta'_2)} \wedge \psi(\theta''_1, \theta''_2; t = 0)\right)\right)\right)$$  \hspace{1cm} (70)

Now, as we recall from our previous discussion, due to the fact that we have three different products, we can not simply pull \(e^{i\mathcal{L}(\theta'_1, \theta'_2, \theta_1, \theta_2)}\) inside the integral over \(\theta''_1 \circ \theta''_2\) and expect exactly the same result. At the same time, we have also previously argued that, in light of the special Manhattan metric, we will get approximately the same result, where the level of
approximation gets better and better as dimensionality of Grassmann space gets higher and higher. Thus, assuming that approximation, we rewrite Equation 70 as

$$\psi(\theta_1, \theta_2; t = 2\delta t) = \int (d\theta_1' \cdot d\theta_2' \cdot d\theta_1' \cdot d\theta_2') \circ$$

$$\circ \left( e^{-iL(\theta_1', \theta_2', \theta_1, \theta_2)} \wedge e^{-iL(\theta_1', \theta_2', \theta_1, \theta_2)} \wedge \psi(\theta_1', \theta_2'; t = 0) \right)$$

The other subtle point is combining the exponentials. In general non-commuting case, the product of exponentials is given by Baker-Campbell-Hausdorff formula

$$e^A e^B = e^{A + B + \frac{1}{2}[A, B] + \cdots}$$

However, in light of the fact that a typical Lagrangian contains only the products of even number of Grassmann variables, we can assume that it commutes with everything,

$$\mathcal{L} \wedge A = A \wedge \mathcal{L}$$

If we make that assumption, Equation 72 implies that the product of exponents is just the usual exponent of the sum! Thus,

$$e^{-iL(\theta_1', \theta_2', \theta_1, \theta_2)} \wedge e^{-iL(\theta_1', \theta_2', \theta_1, \theta_2)} = e^{-iL(\theta_1', \theta_2', \theta_1, \theta_2) - iL(\theta_1', \theta_2', \theta_1, \theta_2)}$$

By substituting Equation 74 into Equation 71 we obtain

$$\psi(\theta_1, \theta_2; t = 2\delta t) = \int (d\theta_1' \cdot d\theta_2' \cdot d\theta_1' \cdot d\theta_2') \circ \left( e^{-iL(\theta_1', \theta_2', \theta_1, \theta_2) - iL(\theta_1', \theta_2', \theta_1, \theta_2)} \wedge \psi(\theta_1', \theta_2'; t = 0) \right)$$

By simple induction argument, it is easy to see that, for arbitrary $k$, we will have

$$\psi(\theta_1, \theta_2; t = k\delta t) = \int (d\theta_{1k} \cdot d\theta_{2k} \cdot \ldots \cdot d\theta_{11} \cdot d\theta_{21}) \circ$$

$$\circ \left( e^{-iL(\theta_{11}, \theta_{21}, \theta_1, \theta_2) - \cdots - iL(\theta_{1k}, \theta_{2k}, \theta_{1k-1}, \theta_{2k-1})} \wedge \psi(\theta_{1k}, \theta_{2k}; t = 0) \right)$$

This, indeed, shows that we obtain the path integral, as expected.

Let us now attempt to tackle the problem of harmonic oscillator in fermionic coordinates. The first step is to write down a "classical" action. Consider a particle living in two-dimensional Grassmann space, and moving along a trajectory $((\theta_1(t), \theta_2(t))$. In a continuum setting, we would like the action to be

$$S = \int dt (m\theta_1(t) \wedge \dot{\theta}_2(t) - m\omega \theta_1(t) \wedge \theta_2(t))$$

Now, the definition of path integral we have provided is based on continuum space and discrete time. Thus, while leaving $\theta_1$ and $\theta_2$ continuous, we will discretize $t$. Thus, we will assume that

$$t = k\delta t, \ k \in \{0, \ldots, n\}$$
We will replace "time derivative" at $t$ according to

$$\dot{\theta}_2(t) \rightarrow \frac{\theta_2((k+1)\delta t) - \theta_2(k\delta t)}{\delta t}$$

(79)

In light of Grassmannian nature of parameters, it is conceivable that we would get different results depending on whether we chose "right derivative" or "left derivative". In order to be as general as possible, we will "split" constant terms between slice $t$ and $t + \delta t$. Thus, if we put "most" of the constant terms to $t$ we would obtain "almost-right" derivative, while putting "most" of the constant terms to $t + \delta t$ would give us "almost-left" derivative; splitting constant terms "evenly" would give us "middle derivative". Thus, we will replace "same-time" terms according to

$$m_1\theta_1(t) \rightarrow m_1\theta_1(k\delta t) + m_2\theta_1((k+1)\delta t)$$

(80)

$$m_2\theta_1(t) \wedge \theta_2(t) \rightarrow m_3\omega\theta_1(n\delta t) \wedge \theta_2(n\delta t)$$

(81)

Where we have replaced $m$ with three parameters $m_1$, $m_2$ and $m_3$, given by

$$m_3 = m_1 + m_2$$

(82)

Intuitively, $m_3$ is the replacement of $m$, while $m_1$ and $m_2$ are the two "parts" of $m$ we "split" it as. Thus, if we want to have "left derivative" we would set $0 = m_2 < m_1 = m_3$, for right derivative we would set $0 = m_1 < m_2 = m_3$, and for middle derivative we set $0 < m_1 = m_2 = m_3/2$. By making these substitutions, discretized action (for general $m_1$, $m_2$ and $m_3$) becomes

$$S = \sum_{k=0}^{n-1} (m_1\theta_1(k\delta t) \wedge \theta_2((k+1)\delta t) - m_1\theta_1(k\delta t) \wedge \theta_2(k\delta t) +$$

$$+ m_2\theta_1((k+1)\delta t) \wedge \theta_2((k+1)\delta t) - m_2\theta_1((k+1)\delta t) \wedge \theta_2(k\delta t) - m_3\omega\delta t \theta_1(k\delta t) \wedge \theta_2(k\delta t))$$

(83)

We notice that the of the form $\theta_1((k+1)\delta t) \wedge \theta_2((k+1)\delta t)$ for $k \leq n - 1$ match the terms of the form $\theta_1(k) \wedge \theta_2(k)$ for $k \rightarrow k + 1$. By combining such terms for $1 \leq k \leq n - 1$ while taking separate care regarding the "surface terms" ($k = 0$ and $k = n$), we obtain

$$S = m_2\theta_1(n\delta t) \wedge \theta_2(n\delta t) - m_1\theta_1(0) \wedge \theta_2(0) - m_3\omega\delta t \theta_1(0) \wedge \theta_2(0) -$$

$$- m_3\omega\theta_1(n\delta t) \wedge \theta_2(n\delta t) + \sum_{k=1}^{n-1} (m_1\theta_1(k\delta t) \wedge \theta_2((k+1)\delta t) -$$

$$- m_2\theta_1((k+1)\delta t) \wedge \theta_2(k\delta t) + (m_2 - m_1 - m_3\omega\delta t)\theta_1(k\delta t) \wedge \theta_2(k\delta t))$$

(84)

The above expression can be rewritten as

$$S = \sum_{k=0}^{n} \mathcal{L}_k$$

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where we define \( \mathcal{L}_k \) according to

\[
\mathcal{L}_0 = -(m_1 + m_3 \omega \delta t) \theta_1(0) \land \theta_2(0)
\]

\[
\mathcal{L}_k = (m_1 \theta_1(k \delta t) \land \theta_2((k + 1) \delta t)) - m_2 \theta_1((k + 1) \delta t) \land \theta_2(k \delta t) +
\]

\[
1 \leq k \leq n - 1 \tag{85}
\]

\[
\mathcal{L}_n = (m_2 - m_3 \omega) \theta_1(n \delta t) \land \theta_2(n \delta t)
\]

Let us now rewrite it in a notation of Equation \([66]\). Thus, we will denote \( \theta_1(k \delta t) \) and \( \theta_2(k \delta t) \) by \( \theta'_1 \) and \( \theta'_2 \), while denoting \( \theta_1((k + 1) \delta t) \) and \( \theta_2((k + 1) \delta t) \) by \( \theta_1 \) and \( \theta_2 \). Thus, we rewrite \( \mathcal{L} \) as

\[
\mathcal{L}(\theta'_1, \theta'_2, \theta_1, \theta_2) = -(m_1 + m_3 \omega \delta t) \theta_1 \land \theta_2 , \ t = 0
\]

\[
\mathcal{L}(\theta'_1, \theta'_2, \theta_1, \theta_2) = (m_1 \theta'_1 \land \theta_2 - m_2 \theta_1 \land \theta'_2) +
\]

\[
+ (m_2 - m_1 - m_3 \omega \delta t) \theta'_1 \land \theta'_2 , \ \delta t \leq t \leq (n - 1) \delta t
\]

\[
\mathcal{L}(\theta'_1, \theta'_2, \theta_1, \theta_2) = (m_2 - m_3 \omega) \theta'_1 \land \theta'_2 , \ t = t_n
\]

Let us now take the exponent of the above. As before, we will define the exponent in terms of a series of wedge products. This means that the expression terminates after linear terms. From straightforward calculation, we obtain

\[
e^{-i\mathcal{L}_0} = 1 - i(m_1 + m_3 \omega \delta t) \theta_1 \land \theta_2 , \ t = 0
\]

\[
e^{-i\mathcal{L}_k} = 1 - im_1 \theta'_1 \land \theta_2 + im_2 \theta_1 \land \theta'_2 -
\]

\[
- i(m_2 - m_1 - m_3 \omega) \theta'_1 \land \theta'_2 + m_1 m_2 \theta'_1 \land \theta_2 \land \theta_1 \land \theta'_2
\]

\[
e^{-i\mathcal{L}_n} = 1 - i(m_2 - m_3 \omega) \theta_1(n \delta t) \land \theta_2(n \delta t)
\]

Let us now compute the evolution of \( \psi \). Suppose \( \psi \) was arbitrary at \( t = 0 \) (that is, it can be something completely non-linear). Now, any \( \theta \)-dependence of \( \psi \) at \( t = 0 \) will become \( \theta' \)-dependence as far as \( t = \delta t \) is concerned. That dependence will disappear via the integration over \( \theta' \). On the other hand, new dependence will be created; namely, the dependence on un-primed \( \theta \)-s. That dependence, however, will be strictly linear, since \( \theta \)-s will enter linearly into Equation \([87] \). Thus, even if \( \psi(\theta) \) "starts out" non-linear at \( t = 0 \), it will "collapse" into linear function by \( t = \delta t \). Therefore, if we are concerned about \( t \geq \delta t \), we can write down \( \psi(\theta_1, \theta_2) \) in a form of a general linear function:

\[
\psi(\theta_1, \theta_2; t) = \theta_1 f_1(t) + \theta_2 f_2(t) + \theta_1 \land \theta_2 f_3(t) + f_4(t)
\]

\[
\psi(\theta_1, \theta_2; t) = \int (d\theta'_1 \cdot d\theta'_2) \circ ((1 - im_1 \theta'_1 \land \theta_2 + im_2 \theta_1 \land \theta'_2 -
\]

\[
- i(m_2 - m_1 - m_3 \omega) \theta'_1 \land \theta'_2 + m_1 m_2 \theta'_1 \land \theta_2 \land \theta_1 \land \theta'_2) \land
\]

\[
\tag{89}
\]

The time evolution of \( \psi \), therefore, is completely described through the time evolution of \( f_1, f_2, f_3 \) and \( f_4 \). Now, if we substitute Equations \([87] \) and \([88] \) into Equation \([66] \) we obtain that, for \( \delta t \leq t \leq (n - 1) \delta t \), the dynamics of \( \psi \) is given by

\[
\psi(\theta_1, \theta_2; t) = \int (d\theta'_1 \cdot d\theta'_2) \circ ((1 - im_1 \theta'_1 \land \theta_2 + im_2 \theta_1 \land \theta'_2 -
\]

\[
- i(m_2 - m_1 - m_3 \omega) \theta'_1 \land \theta'_2 + m_1 m_2 \theta'_1 \land \theta_2 \land \theta_1 \land \theta'_2) \land
\]

\[
\tag{89}
\]
\[ \psi(\theta_1, \theta_2, t) = -im_2 \theta_1 f_1(t - \delta t) + im_1 \theta_2 f_2(t - \delta t) - m_1 m_2 \theta_1 \wedge \theta_2 f_1(t - \delta t) + f_3(t - \delta t) + i(m_3 \omega + m_1 - m_2) f_4(t - \delta t) \]

By comparing Equation 88 to Equation 90, we obtain

\[
\begin{pmatrix}
 f_1(t) \\
f_2(t) \\
f_3(t) \\
f_4(t)
\end{pmatrix} =
\begin{pmatrix}
 0 & -im_2 & 0 & 0 \\
im_1 & 0 & 0 & 0 \\
0 & 0 & 0 & -m_1 m_2 \\
0 & 0 & 1 & i(m_1 - m_2 + m_3 \omega \delta t)
\end{pmatrix}
\begin{pmatrix}
 f_1(t - \delta t) \\
f_2(t - \delta t) \\
f_3(t - \delta t) \\
f_4(t - \delta t)
\end{pmatrix}
\]

Thus, naively we would expect that \( e^{-iE \delta t} \) coincides with the eigenvalues of the above tensor. Now, the eigenvalues associated with the top left block are

\[ c_1 = -\sqrt{m_1 m_2}, \quad c_2 = \sqrt{m_1 m_2} \]

while the eigenvalues associated with bottom right block are

\[ c_3 = \frac{i(m_1 - m_2 + m_3 \omega \delta t) - \sqrt{-(m_1 - m_2 + m_3 \omega \delta t)^2 - 4m_1 m_2}}{2} \]

\[ c_4 = \frac{i(m_1 - m_2 + m_3 \omega \delta t) + \sqrt{-(m_1 - m_2 + m_3 \omega \delta t)^2 - 4m_1 m_2}}{2} \]

Now, if we assume that \( m \)-s as well as \( \omega \) are real, it is easy to see that the expression under square root for \( c_3 \) and \( c_4 \) is negative, which makes \( c_3 \) and \( c_4 \) purely imaginary. At the same time, \( c_1 \) and \( c_2 \) are purely real. Thus, if we stick to the view that

\[ c_k = |c_k| e^{-iE_k \delta t} \]

we would be forced to conclude that

\[ E_{1,2} = \frac{\pi}{\omega \delta t}, \quad E_{3,4} = \frac{\pi}{2\omega \delta t} \]

This of course has several major problems. Just to name a few, we have four eigenvalues instead of two, in contradiction to Fermi exclusion principle. In fact, two of the eigenstates would have twice larger energy level than the other two, while in a simple fermion system we would expect the two energies ("vacuum" and "one particle") to have the same absolute value and opposite sign. Finally, the energies are simply too high. The fact that the "periods" of the "oscillation" processes match either \( \delta t \) or \( 2\delta t \) would not account for any "more obviously" sinusoidal processes seen in the lab. We avoid the above issues by proposing more clever way of defining energies:

a) Assume that \( \omega \) is imaginary. In fact, set \( \omega = iE \) (for now, \( E \) will simply be a letter in the alphabet but we will soon show this in fact represents energy).
b) Factor out the overall complex phase in the eigenvalues and look at the deviations in phase due to $\omega = iE$.

c) Try to see whether or not the overall factors we have "ignored" in part b might lead to some problems, such as interference between states and so forth.

This immediately tells us that the states associated with eigenvalues $c_1$ and $c_2$ have zero energy, since $\omega$ has no impact on their values:

$$E_1 = \gamma E_2 = \gamma 0 \quad (97)$$

At first, this might appear troubling since we know that fermionic vacuum has negative energy rather than 0. Upon further look, however, one realizes that due to the block-diagonal form of the matrix, these two states are "isolated" from the states associated with $c_3$ and $c_4$. Furthermore, since $\omega$-term originally came from coupling, the fact that states 1 and 2 lack that term shows that they are neither coupled to themselves nor to anything else. Thus, even if we were to perturb our system in some way, that perturbation will amount to the "additional interaction" between states 3 and 4, while states 1 and 2 will continue to stay "isolated". This simultaneously explains why we "observe" only two states and why we "don't observe" zero-energy that is apparently attributable to states 1 and 2.

As far as states 3 and 4 are concerned, the evaluation of Equations 93 and 94 up to 0(\tau) tells us that

$$c_3 = -im_2 + \frac{im_3 \omega \delta t}{2} - \frac{i(m_1 - m_2)m_3 \omega \delta t}{2(m_1 + m_2)} \quad (98)$$
$$c_4 = im_1 + \frac{im_3 \omega \delta t}{2} + \frac{i(m_1 - m_2)m_3 \omega \delta t}{2(m_1 + m_2)} \quad (99)$$

Without loss of generality, we can substitute

$$m_3 = m_1 + m_2 \quad (100)$$

After all, we still assume that $m$-s are real, while any mismatch in amplitude of $m_3$ can be absorbed into the amplitude of $\omega$. Thus, we go ahead and substitute the Equation 100 into the Equations 98 and 99 to obtain

$$c_3 = -im_2(1 - \omega \delta t) \ , \ c_4 = im_1(1 + \omega \delta t) \quad (101)$$

which, upon substitution $\omega = iE$ becomes

$$c_3 = -im_2(1 - iE \delta t) \ , \ c_4 = im_1(1 + iE \delta t) \quad (102)$$

This looks a lot more promising. If we were to ignore the overall coefficients and just look at the terms inside the bracket, we find that, indeed, we have a "very slow" time evolution that would produce sinusoidal process. Furthermore, we have $E$ comming with the same magnitude and opposite signs, just as expected from vacuum and one-particle states. Now, if the overall factors outside of the bracket were the same for $c_3$ and $c_4$, we would have been able to just normalize them away. Due to the fact that they are not the same, however, poses a possibility of interference between $c_3$ and $c_4$ that would "disclose" the unwanted "rapid
evolution”. However, in case of $m_1 = m_2$ the interference will simply amount to relative sign between $c_3$ and $c_4$ alternating between 1 and $-1$ in aforegiven order; its effect can be reasonably expected to average out to something trivial. If we assume that $m_1 \neq m_2$ things might get a little more interesting since the state with larger $m$ will dominate over the state with smaller $m$; in fact, the effect of the latter will quickly approach zero in comparison to the former. We will, therefore, have to assume that

\[ m_1 = m_2 \]  \hspace{1cm} (103)

in order to avoid the above situation. This setting is most natural anyway since it respects time reversal symmetry. If we now substitute Equation 103 along with

\[ m = m_3 \text{, } \omega = iE \]  \hspace{1cm} (104)

into the Equations 98 and 99, we obtain

\[ c_3 = -\frac{Em\delta t - \sqrt{m^2E^2(\delta t)^2 - m^2}}{2} \]  \hspace{1cm} (105)

\[ c_4 = -\frac{Em\delta t + \sqrt{m^2E^2(\delta t)^2 - m^2}}{2} \]  \hspace{1cm} (106)

Now, since $\delta t$ is “very small”, the expression under square root is negative, which means that square root itself is imaginary. Now, since the square root contains $m$, which is finite, while the term outside of square root has $\delta t$ in it, the square root dominates. Therefore, due to imaginary nature of square root, it makes sense to factor out $im$ rather than $m$. This implies

\[ c_3 = -\frac{im}{2} \left( \sqrt{1 - E^2(\delta t)^2} - iE\delta t \right) \]  \hspace{1cm} (107)

\[ c_4 = \frac{im}{2} \left( \sqrt{1 - E^2(\delta t)^2} + iE\delta t \right) \]  \hspace{1cm} (108)

By re-expressing this as

\[ c_3 = -\frac{im\sqrt{1 - E^2(\delta t)^2}}{2} \left( 1 - \frac{iE\delta t}{\sqrt{1 - E^2(\delta t)^2}} \right) \]  \hspace{1cm} (109)

\[ c_4 = \frac{im\sqrt{1 - E^2(\delta t)^2}}{2} \left( 1 + \frac{iE\delta t}{\sqrt{1 - E^2(\delta t)^2}} \right) \]  \hspace{1cm} (110)

which shows us that the higher order corrections due to square root do not imply the difference between the absolute values of positive and negative energy states. However, apart from this curiosity, we are only computing the subsequent exponential evolution up to finite order. After all, due to the higher order deviations between $1 - iE\delta t$ and $e^{-iE\delta t}$, doing higher order calculations would be increasingly more complicated; in this respect the situation is no different from bosons in discrete time. Up to finite terms, we see that the two eigenstates evolve as

\[ v_3 = \left( -\frac{im}{2} \right)^{t/\delta t} e^{-iEt} v_3(0) \]  \hspace{1cm} (111)
\[ v_4 = \left( \frac{im}{2} \right)^{t/\delta t} e^{iEt} v_4(0) \]  

(112)

It is easy to see that the corresponding eigenstates can be expressed as

\[
\begin{align*}
  v_3(0) &= C \begin{pmatrix} 0 \\ 0 \\ -i \sqrt{\frac{m^2}{2}} \end{pmatrix} \\
  v_4(0) &= D \begin{pmatrix} 0 \\ 0 \\ i \sqrt{\frac{m^2}{2}} \end{pmatrix}
\end{align*}
\]

(113)

The above is not to be confused with spinor. As was stated earlier, the only "physical" components are the third and fourth component of the above column, whereas the first two components are spurious. What we are describing corresponds to one spin-up particle state and vacuum state. The "spinor" would include four copies of the above (corresponding to spin-up particle, spin-down particle, spin-up antiparticle and spin-down antiparticle); thus, it would be having 16 components, 8 of which physical and the other 8 spurious. In this paper, for the sake of simplicity, we restrict ourselves only to one spinor component, which is why we get only 4 states (2 physical, 2 spurious) instead. Any similarity between what we have and spinor notation is purely coincidental.

Now, if we substitute Equations 111 and 112 into Equation 113, we obtain

\[
\begin{align*}
  v_3 &= C \left( -\frac{im}{2} \right)^{t/\delta t} e^{-iEt} \begin{pmatrix} 0 \\ 0 \\ -i \sqrt{\frac{m^2}{2}} \end{pmatrix} \\
  v_4 &= D \left( \frac{im}{2} \right)^{t/\delta t} e^{iEt} \begin{pmatrix} 0 \\ 0 \\ i \sqrt{\frac{m^2}{2}} \end{pmatrix}
\end{align*}
\]

(114)

This means that corresponding wave functions evolve as

\[
\begin{align*}
  \psi_3(\theta_1, \theta_2; t) &= C \left( -\frac{im}{2} \right)^{t/\delta t} e^{-iEt} \left( \sqrt{\frac{2}{m}} - i \sqrt{\frac{m}{2}} \theta_1 \wedge \theta_2 \right) \\
  \psi_4(\theta_1, \theta_2; t) &= D \left( \frac{im}{2} \right)^{t/\delta t} e^{iEt} \left( \sqrt{\frac{2}{m}} + i \sqrt{\frac{m}{2}} \theta_1 \wedge \theta_2 \right)
\end{align*}
\]

(115)

(116)

For the completeness purposes, let us now compute "spurious states" $\psi_1$ and $\psi_2$. It is easy to see that their eigenvalues are $c_1 = -m$ and $c_2 = m$, which correspond to

\[
\begin{align*}
  v_1(0) &= A \begin{pmatrix} i \\ 1 \\ 0 \end{pmatrix} \\
  v_2(0) &= B \begin{pmatrix} 1 \\ i \\ 0 \end{pmatrix}
\end{align*}
\]

(117)

This means that these states evolve as

\[
\begin{align*}
  v_1(0) &= A(-m)^{t/\delta t} \begin{pmatrix} i \\ 1 \\ 0 \end{pmatrix} \\
  v_2(0) &= Bm^{t/\delta t} \begin{pmatrix} 1 \\ i \\ 0 \end{pmatrix}
\end{align*}
\]

(118)
The lack of any exponential term in the coefficient is closely related to the fact that these states are spurious. Finally, they correspond to the wave functions $\psi_1$ and $\psi_2$ evolving as

$$
\psi_1(\theta_1, \theta_2; t) = A(-m)^{t/\delta t}(i\theta_1 + \theta_2), \quad \psi_2(\theta_1, \theta_2; t) = Bm^{t/\delta t}(\theta_1 + i\theta_2)
$$

(119)

The general wave function, therefore, evolves according to

$$\psi(\theta_1, \theta_2; t) = A(-m)^{t/\delta t}(i\theta_1 + \theta_2) + Bm^{t/\delta t}(\theta_1 + i\theta_2) + C\left(-\frac{im}{2}\right)^{t/\delta t}e^{-it\delta t}\left(\sqrt{\frac{2}{m}} - i\sqrt{\frac{m}{2}}\theta_1 \land \theta_2\right) + D\left(\frac{im}{2}\right)^{t/\delta t}e^{it\delta t}\left(\sqrt{\frac{2}{m}} + i\sqrt{\frac{m}{2}}\theta_1 \land \theta_2\right)
$$

(120)

The terms with coefficients $A$ and $B$ are spurious, the term with coefficient $C$ represents one-particle state and the term with coefficient $D$ represents vacuum state. The spurious states have zero energies, the one-particle state has energy $E$ and the vacuum state has energy $-E$.

7. Coupling between bosonic and fermionic coordinates

Let us now discuss the situation when we have three coordinates: two Grassmannian and one real, and see what happens. The wave function will evolve according to

$$\psi(\theta_1, \theta_2, x, t) = \int \left((dx' d\theta'_1 \cdot d\theta'_2) \circ e^{-i\delta t\left(\frac{mB(x - x')^2}{2(\delta t)^2} - U(x')\theta'_1 \land \theta'_2 - V(x')\right)} \land \psi(\theta'_1, \theta'_2, x', t - \delta t)\right)$$

(121)

Now, let $\{\psi_n(x)\}$ be an eigenfunction of Schrodinger’s operator; that is, 

$$-\frac{1}{2m} \frac{d^2\psi_n}{dx^2} + V\psi_n = E_n\psi_n$$

(122)

The above step is pure mathematics: since we have extra anticommuting coordinates, we can not claim that Schrodinger’s equation holds, unless we have demonstrated it through path integral. But, from the point of view of differential equations, the above set of eigenfunctions continues to exist. Since that set of eigenfunctions ”happen” to be complete, the general analytical function $\psi(\theta_1, \theta_2, x, t)$ can be written as

$$\psi(\theta_1, \theta_2, x, t) = \sum_n \theta_1 \psi_n(x)f_1(t) + \sum_n \theta_2 \psi_n(x)f_2(t) + \sum_n \theta_1 \land \theta_2 f_3(t) + \sum_n \psi_n(x)f_4(t)
$$

(123)

Let us now evaluate Equation [121] by substituting Equation [123] for $\psi(\theta_1, \theta_2, x, t)$ and Equation [87] for $e^{-i\delta t(\theta'_1, \theta'_2, \theta_1, \theta_2)}$. By evaluating the integral over $\theta'_1 \cdot \theta'_2$ we obtain

$$\psi(\theta_1, \theta_2, x, t) = \sum_n \int dx' e^{-i\delta t\left(\frac{mB(x - x')^2}{2(\delta t)^2} - V(x')\right)}(\psi_n(x')f_3(t - \delta t) +$$

(124)


\[ + im_1 \theta_2 \psi_n(x')(f_{2n}(t-\delta t)) - im_2 \theta_1 \psi_n(x')f_{1n}(t-\delta t) - \]

\[ - i(m_2 - m_1 - m_3 \omega) \psi_n(x')f_{4n}(t-\delta t) - m_1 m_2 \theta_1 \wedge \theta_2 \psi_n f_{4n}(t-\delta t) + iU(x') \psi_n(x')f_{4n}(t-\delta t) \]

Now, in light of the fact that \( \{ \psi_k \} \) is complete, we can express \( U(x)\psi_n(x) \) as

\[ U(x)\psi_n(x) = \sum_k C_{kn} \psi_k(x) \]  

(125)

where

\[ C_{kn} = \int U(x)\psi^*(x)\psi(x)dx \]  

(126)

Furthermore, from our knowledge of correlation between bosonic path integral and Schrödinger’s equation, we know that

\[ \int dx' e^{-i\delta t \left( \frac{m_1^2 (x-x')^2}{2m} - V(x') \right)} \phi_n(x') = \sqrt{\frac{2\pi\delta t}{im}} e^{-iE_n\delta t} \phi_n(x) \]  

(127)

By applying Equation [125] and [127] to Equation [124], we obtain

\[ \psi(\theta_1, \theta_2, x, t) = \sqrt{\frac{2\pi\delta t}{im}} \sum_n \left( e^{-iE_n\delta t} \right) \psi_n(x') f_{3n}(t-\delta t) + \]

\[ + im_1 \theta_2 \psi_n(x') f_{2n}(t-\delta t) - im_2 \theta_1 \psi_n(x') f_{1n}(t-\delta t) + \]

\[ + i(m_3 \omega + m_1 - m_2) \psi_n(x') f_{4n}(t-\delta t) - m_1 m_2 \theta_1 \wedge \theta_2 \psi_n f_{4n}(t-\delta t) + \]

\[ + \sqrt{\frac{2\pi\delta t}{im}} \sum_{k, n} iC_{kn} e^{-iE_k\delta t} \psi_k(x') f_{4n}(t-\delta t) \]  

(128)

Let us first consider the case where bosons and fermions are decoupled; that is, we will assume that \( C_{kn} = 0 \). Now, by comparing the Equation [128] to Equation [123], we see that the step-by-step evolution of terms with "fixed" index \( n \) in this section is similar to the step-by-step process of "everything" in the previous section (this similarity is "broken" via \( C_{nk} \) terms, which is why we are first considering the case of these terms being zero). Therefore we can "copy" the Equation [120] and insert the indeces \( n \) as well as a summation over them. Apart from that, we also have to be careful with coefficients that we now have which we did not have before. As far as the \( (2\pi\delta t/im)^{1/2} \) coefficient goes, that "extra multiplication" occurs every time we pass the interval of \( \delta t \). Thus, we have to take that coefficient to the power of \( t/\delta t \), which would produce \( (2\pi\delta t/im)^{t/2\delta t} \). As far as \( e^{-iE_n\delta t} \) in the exponent we would produce a sum of "very large" \( t/\delta t \) number of copies of "very small" numbers \( (-iE_n\delta t) \), which would produce a "finite" number \( -iE_n t \) in the exponential; that is, we get an extra coefficient of \( e^{-iE_n t} \). Thus, the "non-interacting" fermionic field (with \( C_{kn} = 0 \) will produce a copy of Equation [120] with the insertions of summation over \( n \) as well as the coefficients just discussed:

\[ \psi(\theta_1, \theta_2; t) = \left( \frac{2\pi\delta t}{im} \right)^{t/2\delta t} \sum_n \left[ e^{-iE_n t \left( A(-m)^{t/\delta t} (i\theta_1 + \theta_2) + Bm^{t/\delta t} (\theta_1 + i\theta_2) \right)} \right] \]  

(129)
+ C \left( -\frac{im}{2} \right)^{t/\delta t} e^{-iE \delta t} \left( \sqrt{\frac{2}{m} - i \sqrt{\frac{m}{2}} \hat{\theta}_1 \wedge \hat{\theta}_2} + D \left( \frac{im}{2} \right)^{t/\delta t} e^{iE \delta t} \left( \sqrt{\frac{2}{m} + i \sqrt{\frac{m}{2}} \hat{\theta}_1 \wedge \hat{\theta}_2} \right) \right) \right]

By combining the $e^{-iE \delta t}$ on the outside with the $e^{-iE \delta t}$ and $e^{iE \delta t}$ on the inside, we obtain

$$\psi(\hat{\theta}_1, \hat{\theta}_2; t) = \left( \frac{2\pi \delta t}{im} \right)^{t/\delta t} \sum_n \left( A(-m)^{t/\delta t}(i\hat{\theta}_1 + \hat{\theta}_2) + Bm^{t/\delta t}(\hat{\theta}_1 + i\hat{\theta}_2) + \right)$$

$$+ C \left( -\frac{im}{2} \right)^{t/\delta t} e^{-i(E + E_n) \delta t} \left( \sqrt{\frac{2}{m} - i \sqrt{\frac{m}{2}} \hat{\theta}_1 \wedge \hat{\theta}_2} + D \left( \frac{im}{2} \right)^{t/\delta t} e^{-i(E - E_n) \delta t} \left( \sqrt{\frac{2}{m} + i \sqrt{\frac{m}{2}} \hat{\theta}_1 \wedge \hat{\theta}_2} \right) \right)$$

This tells us that we obtain eigenstates with energies $E_n + E$ and $E_n - E$. The energy $E_n > 0$ comes from bosonic part, whereas subtraction or addition of $E$ comes from negative energy ”fermionic vacuum” and positive energy ”fermionic one particle state” respectively (the bosonic vacuum energy, on the other hand, is absorbed into $E_n$).

8. Multidimensional harmonic oscillator and multiparticle states

So far we have worked out the ”harmonic oscillator” in two Grassmann coordinates (which is the lowest number of coordinates where we could include ”mass term” into). Now, the simple harmonic oscillator generalizes to multidimensional one which serves as a model of second quantization of bosonic fields. Multidimensional bosonic system continues to have wave function representation, except that it is now a wave function in $N$ coordinates, where $N$ is the number of lattice points. Suppose, for example, we have one-dimensional chain of points. Then their potential is given by

$$V = \frac{1}{2(\delta x)^2} \sum_{k=1}^{D-1} (\phi_{k+1} - \phi_k)^2 + \frac{m^2}{2} \sum_{k=1}^{D} \phi_k^2$$

which, upon simple algebra becomes

$$V = \frac{\phi_1^2}{2(\delta x)^2} + \frac{\phi_D^2}{2(\delta x)^2} + \left( \frac{1}{(\delta x)^2} + \frac{m^2}{2} \right) \sum_{k=2}^{D-1} \phi_k^2 + \sum_{k=1}^{N-1} \phi_k \phi_{k+1}$$

By diagonalizing the coefficients, we see that in ”rotated” coordinates $V$ is given by

$$V = \sum_{k=1}^{D} \frac{\omega_k^2 \phi_k^2}{2} , \text{where } \phi_k = \sum A_{kl} \phi_l$$

(133)

This means that the wave function is given by

$$\psi(\phi_1', \cdots, \phi_D') = \sum_{n_1, \cdots, n_D}^D \left( C_{n_1, \cdots, n_D} \prod_{k=1}^{D} \psi_{n_k} \phi_k' \right)$$

(134)
where $\psi_{\omega,n}(x)$ is the $n$-th eigenfunction of simple harmonic oscillator of frequency $\omega$. If we like, we can rotate the wave function back into un-prime frame, thus obtaining

$$
\psi(\phi_1, \cdots, \phi_D) = \sum_{n_1, \cdots, n_D} \left( C_{n_1, \cdots, n_D} \prod_{k=1}^D \psi_{\omega_k:n_k} \left( \sum_l B_{kl}\phi_l \right) \right)
$$

where

$$
A_{ik}B_{kj} = \delta_{ij} \tag{136}
$$

Now, we "independently know" that the eigenstates of non-interacting field are fixed momentum states. Thus, we can guess that $\phi'_k$ represents a particle moving with a "fixed momentum" $p_k$, whereas $\phi_k$ represents the particle of fixed position $x_k$. This means that

$$
A_{kl} \approx e^{ip_k x_l}, \quad B_{kl} \approx e^{-ip_k x_l} \tag{137}
$$

up to some trivial power of $2\pi$. This essentially tells us that, even if we are "believers" in position, we can still "take seriously" wave function in a momentum-based Fock space. After all, Fourier transformation is merely a "rotation" inside the multidimensional space where our wave function lives. Since we agree that in "few" dimensions we are free to "rotate" wave function, the same is true for many dimensions. Thus, a general bosonic state

$$
|\psi\rangle = \sum_m \sum_{p_1, \cdots, p_m} \sum_{n_1, \cdots, n_m} C_{p_1, \cdots, p_m; n_1, \cdots, n_m} (a_{p_1}^\dagger)^{n_1} \cdots (a_{p_m}^\dagger)^{n_m} |0\rangle \tag{138}
$$

has a wave function representation

$$
\psi(\phi_1', \cdots, \phi_D') = \sum_m \sum_{p_1, \cdots, p_m} \sum_{n_1, \cdots, n_m} C_{p_1, \cdots, p_m; n_1, \cdots, n_m} \times
$$

$$
\times \left( \prod_{p \in \{p_1, \cdots, p_m\}} \psi_2^{p,n_k}(p_k) \right) \left( \prod_{p \notin \{p_1, \cdots, p_m\}} \psi_2^{p,0}(p_k) \right) \tag{139}
$$

We can now generalize it to fermionic case. In Section 6 we have established that the fermionic harmonic oscillator can be viewed as a wave function in two Grassmann coordinates. This, in conjunction with what we have just said in the present section, suggests that multiparticle fermionic system can be viewed as multidimensional Harmonic oscillator which can likewise be modeled in terms of wave function. Furthermore, in Section 4 we have established that we are "allowed" to rotate Grassmann variables and, provided that the rotation is unitary, we can continue to think of end product as Grassmann variables, as well. Therefore, we can interpret Fourier transformation as that very rotation (after all, we have already agree that we can think of it as "rotation" in bosonic case and all of the obstacles that prevent us from generalizing it to fermions have been removed in previous sections). Now, a general fermionic wave function in two Grassmann coordinates has two physical states takes the form

$$
\psi(\theta_1, \theta_2; t) = f_1(t)\theta_1 + f_2(t)\theta_2 + f_3(t)\theta_1 \wedge \theta_2 + f_4(t) \tag{140}
$$
where $f_1(t)$ and $f_2(t)$ are spurious parameters, while $f_3(t)$ and $f_4(t)$ are physical. Now, $(\theta_1, \theta_2)$ represents a single spinor component in a single-point lattice. Upon "rotation" into momentum space, we can say that $(\theta_1, \theta_2)$ represents a spin-up particle with fixed momentum. In order to account for spin-down particle as well as antiparticles of both spins of that given momentum, we need 8 coordinates, $(\theta_1, \cdots, \theta_8)$. Finally, since we have $D$ values of momentum; after all, we have $D$ vaues of position coming from the lattice, so their "rotation" in a form of momentum should also have $D$ values. Thus, the total number of $\theta$-s is $8D$. Thus, the general wave function is represented by

$$\psi(\theta_1, \cdots, \theta_{8D}; t) = \sum_{n=0}^{8D} \sum_{1 \leq k_1 \leq \cdots \leq k_n \leq 8D} C_{k_1 \cdots k_n} \theta_{k_1} \wedge \cdots \wedge \theta_{k_n} \quad (141)$$

Now, if $\{2j + 1, 2j + 2\} \subset \{k_1, \cdots, k_n\}$, this means that the "slot" number $j$ is occupied by something "isomorphic" to $\theta_1 \wedge \theta_2$ which happens to be a linear combination of two "physical" states (a physical particle and physical vacuum). Likewise, if $\{2j + 1, 2j + 2\} \cap \{k_1, \cdots, k_n\} = \emptyset$, then the $j$-th "slot" is occupied by a state isomorphic to 1, which another linear combination of the above two physical states, and therefore also physical. On the other hand, if $2j + 1$ is an element of $\{k_1, \cdots, k_n\}$ but not $2j + 2$, the $j$-th slot will be filled by something isomorphic to $\theta_1$, which is spurious. If the opposite of these two statements is true then the $j$-th slot will be filled by something isomorphic to $\theta_2$ which is, likewise, spurious.

We can physically interpret the above situation in the following way. As usual, the vacuum energy is $-1/2$, while the energy of each particle is $+1$. But, on top of this, we also have "spurious particles" whose energy is $+1/2$. Thus, if the vacuum is occupied by the ordinary particle, the energy will be $-1/2 + 1 = +1/2$. If, on the other hand, it is occupied by spurious particle, the energy will be $-1/2 + 1/2 = 0$. Spurious particle is also a subject to Fermi exclusion principle. Thus, it prevents any other particle from filling a given slot; but at the same time, it does not interact with anything else either. This has several effects. First of all, it is possible that a given lattice point is occupied by spurious particle, but not spurious antiparticle. As a result, an antiparticle will be able to pass by that point, put a particle won’t. Or, if a given lattice point is occupied by both spurious particle and spurious antiparticle, of both spins, then a fermion can not pass that lattice point. However, that lattice point can still be occupied by a boson.

In addition to this, there is no interaction between different states whose spurious particles fill different spots. As a result, these states might be macroscopically distinct from each other, which raises a question why don’t they interact gravitationally. This question, however, can be asked in the context of any other quantum mechanical model involving decoherence. The ultimate answer to such question is some form of collapse mechanism. It is quite possible that we might take advantage of the fact that Grassmann numbers are defined outside path integral in order to go beyond the realm of analytic functions in proposing such mechanism.
9. Conclusion and outlook

In this paper we have defined Grassmann numbers in such a way that they are well defined outside path integral, which in principle allows for the realistic interpretations of fermionic fields. It turns out that if we properly define dot and wedge product, as well as "special manhattan" metric, then we would be able to expect the "limit of the sum" to approximate Grassmann integral up to some factor, provided that the path we are integrating over has some very long series of mostly non-overlapping "turns" in very large number of non-commuting dimensions. Furthermore, we have worked out the behavior of a toy system in two fermionic dimensions. The discrete nature of time ends up playing a crucial role in the dynamics. We assume the discreteness scale to be $\delta t$; this means that any given point of time is an integer factor of the latter. At slice $t = 0$, wave function can be arbitrary. Then, by the time $t = \delta t$ it "collapses" to something linear. After that, the linear function undergoes some periodic finite changes every time interval $\delta t$. These changes, however, fit a trivial pattern and, therefore, deemed to be unphysical (although we would have been facing extreme discontinuity of the theory were we to take them seriously). However, this "quickly changing" function is being multiplied by "slowly changing" harmonic, and that harmonic represents the physical evolution of the state. It turns out, however, that only the constant term and $\theta_1 \wedge \theta_2$ term are multiplied by corresponding harmonics, while the $\theta_1$ and $\theta_2$ terms are not. Thus, the former two terms are interpreted as physical while the latter two terms are interpreted as spurious. It turns out that both vacuum state and one particle state are linear combination of the two "physical" components we just mentioned, with coefficients specified in this paper.

One surprising result of our model is the prediction of "spurious states". The root of the issue is the fact that we need two variables ($\theta_1$ and $\theta_2$) in order to obtain two states (vacuum state and one particle state). However, once we have these two variables we have four possible linear functions rather than two: $\theta_1$, $\theta_2$, $\theta_1 \wedge \theta_2$ and 1. Thus, only two of these four degrees of freedom are physical, while the other two are spurious. It turns out that the two physical degrees of freedom are 1 and $\theta_1 \wedge \theta_2$, both of which correspond to linear combination of vacuum and one particle states with the coefficients given in the paper. The other two states are "spurious" and they don’t interact with any of the physical states. As a result, it is possible for "spurious particle" to permanently occupy one of the lattice points, thus not allowing the corresponding fermions to be created there, while the bosonic fields would continue to pass by that lattice point. This would potentially produce some non-trivial deviations from lattice field theory that are worth investigating. On the other hand, the "exact match" with conventional theory is still possible: namely, in a scenario devoid of spurious particles. Since spurious particles do not interact, their absence "at the beginning" implies that they will never appear at all. Thus, if we are desperate to match the results of conventional physics we can simply postulate the absence of spurious particles as initial conditions. Such initial conditions, however, are quite unnatural and in my view it would be interesting to allow the spurious particles to continue to exist and explore the various possible outcomes that might be produced. In fact, it might even be interesting to ask whether or not doubling problem will persist in such a radically different discretization of fermions (currently I am of no opinion on this issue, so it is just a blind guess).
One of the originally intended benefits of our framework is that we gain an ability to view Grassmann variables in a literal physical sense, which might potentially provide some extra insight into interpretation of quantum mechanics. However, our theory inherits some of the limitations that we expect from ordinary fermionic theory. In "conventional" case, the limitation on the number of fermionic states corresponds to the lack of continuous spectrum of Grassmann variables. In our case, we do have continuous spectrum but, instead, our dynamics is limited to producing only linear functions on that spectrum. In conventional case, we can’t be speaking of "smoothly varying" fermionic field because we would require to have continuous spectrum in order to define the latter. In our case, we do have continuous spectrum but the multitude of "turns" on Grassmann space create their own problem. As soon as the two Grassmann variables on a given contour are separated by more than one "turn", their distance would be expected to be in the same range regardless of whether they are separated by only one turn or a million of turns; this is particularly due to the Manhattan metric that we have defined. Thus, in order to truly speak of Grassmann variables being "very close" to each other, we have to assume that they are within the same "very small" part of the "turn" of the contour. In this case, however, most of the properties of Grassmann integration would disappear and it is questionable whether or not the physical implications of this would be of any interest.

On a good side, however, it is still possible to use the extra degrees of freedom offered by the continuous spectrum to come up with some physical applications. While it is true that on a scale of several different turns of the contour we can not speak of "continuous functions" in a classical sense, we can still find other creative ways of introducing extra degrees of freedom. For example, we can consider a possibility of function deviating from linearity in some way that correlates either with the shape of the contour or with the deviation of some other function from linearity. Now, the specific dynamics that we have proposed in this paper (which was derived from the "naive" use of path integral) would kill any non-linearity during the very first interval $\delta t$. It is, however, possible to explore some modifications of the definition of "time derivative", particularly the ones that can not be generated by the products, and see whether or not it is possible to retain some of the non-linearities. Of course, if these modifications are assumed to be small, then the non-linearities should be "almost killed" within "very short but not as short" period of time. This would, however, imply the possibility that during the big bang these non-linearities were still present in a non-negligible form. This would imply that Fermi exclusion principle did not work back at the time of big bang, which might lead to some novel cosmological theories.

Apart from that, it is possible to attempt to apply this paper to the following problems (although I personally feel it might be highly questionable due to the issue with "turns" mentioned earlier):

1. **Causal Set Theory** It has been recently attempted to define quantum field theory on a causal set \( \mathbb{S} \) (see [6]). In order to define Lagrangian density, we needed to consider Lorentzian neighborhood of any given point, which, in discrete case, has to be finite. Lorentz covariance implies that any finite Lorentzian neighborhood fills up the vicinity of light cone

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1A causal set is a model of Lorentz covariant discretization of spacetime manifold which was originally proposed by Rafael Sorkin (see [5])
and, therefore, is highly non-local. Bringing back locality would require "cutting off" the Lorentzian neighborhood and, therefore, violating relativity. In order to stay loyal to Lorentz covariance, we decided to come up with "Lorentz covariant" criteria of "selecting preferred frame". Namely, for any field configuration, the frame where field varies the least is "preferred". This requires us to look at a "fixed" field configuration outside of integral. We then repeat this "analysis" for every single field configuration and then "sum up" our results at the end. In order to generalize the above from bosons to fermions, we have to be able to ask ourselves "in which frame does fermionic field varies the least". This question can be asked and answered only if the values of fermionic field (that is, Grassmann numbers) are well defined outside of path integral.

II. Pilot Wave Models

There were recent attempts by Struyve and Westmann (see [9]) to come up with pilot wave model for bosonic fields, where the fields, themselves, are viewed as "beables". However, the fact that fermionic fields are Grassmannian makes it very difficult to come up with similar model for fermions (although an attempt to do so was made by Valentini in his thesis). Instead, fermions are typically modeled to be particles (see, for example, [7]). Having defined Grassmann numbers might now allow us to explore the possibility of field beables for fermions.

III. Lattice theories

As one would gather from the body of the paper, one of the underlying assumptions is that the dimensionality of Grassmann space is "much larger" than the number of variables we are integrating over. However, in a real physical situation, we are integrating over fermionic fields at every single point in space. Thus, in order to stay consistent with this paper, we have to insist that the space is discrete and, furthermore, claim that the number of discrete points is much smaller than the dimensinality of Grassmann space, which is still assumed to be finite! For future research, it might be interesting to attempt to relax the last assumption. In this case, we might possibly predict some non-trivial differences from the "typical" lattice models of fermions. It might, for example, be interesting to explore whether or not the "new" version of lattice model will still have doubling problem. At the moment, the author of this paper has no opinion one way or the other.

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