On Groups Factorized by Mutually Permutable Subgroups

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Abstract. The aim of the paper is to provide a large extension of the recent results of de Giovanni and Ialenti (Commun Algebra 44:118–124, 2016), strengthening at the same time their conclusions. Our second main theorem is actually a complete generalization of a result obtained in the finite case by Beidleman and Heineken (Arch Math (Basel) 85:18–30, 2005) to periodic linear groups.

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1. Introduction

Let $G$ be a group. Two subgroups $A$ and $B$ of $G$ are said to be mutually permutable if $AY = YA$ and $XB = BX$ for all subgroups $X$ of $A$ and $Y$ of $B$. Of course any two normal subgroups are mutually permutable, on the other hand the example in [1] shows that there are non-supersoluble groups which are factorized by two mutually permutable proper (supersoluble) subgroups but not by two proper normal subgroups. Groups which are products of two mutually permutable subgroups have been recently investigated by several authors, and we refer to the monograph [4] for questions and results concerning this subject (see also [6,7] and their reference lists). In particular, it has been proved in [6] that if $G = AB$ is a finite group which is factorized by two mutually permutable subgroups, then $A \cap B$, $A'$ and $B'$ are subnormal subgroups of $G$. It follows...
that if any of those subgroups is locally nilpotent, then its normal closure in $G$ is locally nilpotent as well. It is precisely this latter corollary that has been generalized to infinite groups in the main result of [7] under some additional hypotheses: the whole group must be soluble-by-finite with finite abelian section rank, and the two subgroups $A$ and $B$ must be finite-by-nilpotent (that is, they must contain finite normal subgroups whose factor groups are nilpotent).

Our first main result shows that the assumptions on $A$ and $B$ can be omitted, while that on the rank can be weakened (see also the remark before Corollary 2.2). In the following, $\mathcal{M}$ denotes the class of all minimax groups containing a soluble subgroup of finite index, while, for any group $G$, $\rho_{\mathcal{L}\mathcal{N}}(G)$ denotes the Hirsch-Plotkin radical of $G$.

**Theorem A.** If $G = AB$ is a locally-$\mathcal{M}$ group which is factorized by two mutually permutable subgroups $A$ and $B$, then $\langle \rho_{\mathcal{L}\mathcal{N}}(A'), \rho_{\mathcal{L}\mathcal{N}}(B') \rangle^G \leq \rho_{\mathcal{L}\mathcal{N}}(G') \leq \rho_{\mathcal{L}\mathcal{N}}(G)$.

In Sect. 2, we prove the above theorem and give some corollaries; moreover, we also point out that the proof of Theorem A can be used to prove some more general theorems (see Theorems 2.6–2.8).

The second main result of [7] is a proper extension of the result of [6], and shows that if a Černikov group is factorized by two mutually permutable subgroups $A$ and $B$, then $A'$ and $B'$ are ascendant in $G$ provided that $A'$ and $B'$ are finite. Our second main result shows that the finiteness assumption can be omitted and gives the stronger conclusion that those commutator subgroups are subnormal. This is a complete generalization of the result of [6] to the case of periodic linear groups (so in particular to Černikov groups).

**Theorem B.** Let $G = AB$ be a linear group which is factorized by two mutually permutable periodic subgroups $A$ and $B$. Then $A \cap B$, $A'$ and $B'$ are subnormal subgroups of $G$.

The proof of Theorem B is carried out in Sect. 3, and is obtained as a composition of Theorems 3.4 and 3.6. A relevant step in the proof is our Lemma 3.2, which holds for arbitrary locally finite groups and shows that the subgroups $A \cap B$, $A'$ and $B'$ considered in the statement of Theorem B are always at least serial subgroups of the group.

Notice also that Theorem B is the key ingredient of [11], where a complete generalization (for periodic linear groups) of the results of [6] is provided.

Our notation is standard and can be found for instance in [17,19].

### 2. Proof of Theorem A

The following lemma is essentially due to Beidleman and Heineken [6] (although they only consider finite groups and do not deal with the subnormal case directly), but since its proof is very easy and we often need to use this result, we decided to reproduce it here.
**Lemma 2.1.** Let $G = AB$ be a group which is factorized by two mutually permutable subgroups $A$ and $B$. If $U$ is a subgroup (resp. subnormal subgroup of defect at most $c$) of $G$, then $(U \cap A)(U \cap B)$ is a subgroup (resp. subnormal subgroup of defect at most $c$) of $G$, which is factorized by mutually permutable subgroups $U \cap A$ and $U \cap B$.

**Proof.** Let $X \leq (U \cap A)$. Then $XB = BX$ and so

\[ X(U \cap B) = U \cap XB = U \cap BX = (U \cap B)X. \]

Similarly, one proves that if $Y$ is any subgroup of $U \cap B$, then $Y(U \cap A) = (U \cap A)Y$. Thus $U \cap A$ and $U \cap B$ are mutually permutable subgroups of $G$.

Assume $U$ is subnormal in $G$ of defect $c$ (which we may assume positive). Let $V = U^G$ and notice that

\[ V \cap (A(V \cap B)) = (V \cap A)(V \cap B) = V \cap ((V \cap A)B), \]

so $L = (V \cap A)(V \cap B)$ is normalized by both $A$ and $B$, and hence is normal in $G$. Now, $U \cap L$ is subnormal of defect $c - 1$ in $L$, and the subgroups $V \cap A$ and $V \cap B$ are mutually permutable by the first part of the proof, so induction yields that

\[ (U \cap L \cap V \cap A)(U \cap L \cap V \cap B) = (U \cap A)(U \cap B) \]

is subnormal in $L$ of defect at most $c - 1$. Combining the previous facts we see that $(U \cap A)(U \cap B)$ is subnormal in $G$ of defect at most $c$. \hfill \Box

Before going into the proof of Theorem A, recall that a group is minimax if it admits a series of finite length whose factors satisfy either the minimal or the maximal condition on subgroups. The structure of soluble-by-finite minimax groups is well known and can be found in Theorem 10.33 of [17]; here, we just recall that the finite residual $J$ of such a group $G$ is the direct product of finitely many infinite locally cyclic primary groups, and the factor group $G/J$ contains a nilpotent subgroup $N/J$ whose factor group $G/N$ is finitely generated and abelian-by-finite.

**Proof of Theorem A.** Put $H = \rho_{L\cap A}'(A')$ and $K = \rho_{L\cap B}'(B')$. Clearly, $\langle H, K \rangle^G = H^G K^G$, so we only need to prove that both subgroups $H^G = H^B$ and $K^G = K^A$ are locally nilpotent. Let $E$ be a finitely generated subgroup of $H^B$. Then there are finitely generated subgroups $E_1$ of $A$ and $E_2$ of $B$ such that $E \leq (E_1 \cap H)^{E_2}$.

Let $U = \langle E_1, E_2 \rangle$. Then $G_1 = (U \cap A)(U \cap B)$ is factorized by the two mutually permutable subgroups $U \cap A$ and $U \cap B$ (Lemma 2.1), and

\[ E \leq ((U \cap A)' \cap H)^{G_1} \leq \langle \rho_{L\cap A}'((U \cap A)'), \rho_{L\cap B}'((U \cap B)') \rangle^{G_1}. \]

Since $G_1 \leq U$ is soluble-by-finite and minimax, the symmetry of the argument shows that we only need to prove the statement for a group $G$ which is soluble-by-finite and minimax.

Now, assuming by contradiction that the statement is false, we choose a soluble-by-finite minimax counterexample $G$ with smallest possible torsion-
free rank \( r \). Put \( W = \langle H, K \rangle^G \) and let \( \overline{W} \) be the hypercentre of \( W \). Then \( \overline{W} \) is normal in \( G \) and we may assume \( \overline{W} = \{1\} \) by replacing \( G \) with \( G/\overline{W} \). We break the proof up into cases.

**Case 1.** There exists a periodic normal subgroup \( N \) of \( G \) such that \( \text{WO}_{p'}(N)/\text{O}_{p'}(N) \) is finite-by-(locally nilpotent) for any prime \( p \)

Let \( p \) be a prime. Then \( \text{WO}_{p'}(N)/\text{O}_{p'}(N) \) is finite-by-(locally nilpotent) and hence even locally nilpotent by Lemma 1 of [7] (and Theorem 1 of [6]). It follows from Theorem 6.38 of [17] that \( \text{O}_p(N) \cap W \) is contained in the hypercentre of \( W \), so that \( \text{O}_p(N) \cap W = \{1\} \). Since \( G \) is soluble-by-finite, it follows that \( N \cap W \) is finite, so again \( W \) is finite-by-(locally nilpotent) and hence even locally nilpotent by Lemma 1 of [7], a contradiction.

Let \( T \) be the largest periodic normal subgroup of \( G \), and let \( J \) be the finite residual of \( T \). By [17], Theorem 10.33, \( J \) is divisible abelian.

**Case 2.** \( WT/T \) is locally nilpotent, and \( J = \{1\} \)

Since \( T \) is minimax, its Sylow subgroups are Černikov and so even finite because \( T \) is residually finite. Let \( p \) be any prime number. Then Theorem 2.5.12 of [10] yields that \( T/\text{O}_p(T) \) is finite, so \( \text{WO}_{p'}(T)/\text{O}_{p'}(T) \) is finite-by-(locally nilpotent) and \( G/\text{O}_p(N) \cap W = \{1\} \). Since \( G \) is soluble-by-finite, it follows that \( J \) is periodic, and so \( J \) is divisible, a contradiction.

**Case 3.** \( G \) is locally (polycyclic-by-finite)

The same argument employed in the beginning of the proof shows that we can assume \( G \) is polycyclic-by-finite. Now, by Case 2 and Lemma 1 of [7], we may assume \( T = \{1\} \), so \( G \) has no non-trivial periodic normal subgroups. Let \( F \) be the Fitting subgroup of \( G \), so \( F \) is torsion-free and non-trivial. Let \( Z \) be the centre of \( F \). Then there is a positive integer \( n \) such that \( Y = Z^n \) is free abelian. Let \( p \) be any prime number and notice that \( |Y/Y^p| \leq p^r \) and that \( WY/Y^p \) is nilpotent by assumption on \( r \). Then \( WY/Y^p \) is nilpotent, so \( Y/Y^p \leq \zeta_r(WY/Y^p) \) and hence \( [Y_r, Y] \leq Y^p \). Since \( \bigcap_{q \in \mathbb{P}} Y^q = \{1\} \), it follows that \( [Y, W] = \{1\} \) and so \( Y \leq \zeta_r(WY) \), proving that \( W \) is locally nilpotent, a contradiction.

As a consequence of Case 3, we have that \( G \) cannot be periodic, and so \( r \) is positive.

**Case 4.** \( WT/T \) is locally nilpotent, and \( J \neq \{1\} \)

By Case 1 there is a prime \( p \) such that \( \text{WO}_{p'}(J)/\text{O}_{p'}(J) \) is not finite-by-(locally nilpotent). Let \( U/\text{O}_{p'}(J) \) be the hypercentre of the group \( \text{WO}_{p'}(J)/\text{O}_{p'}(J) \). Notice that, since \( r \) is smallest possible, then \( U/\text{O}_{p'}(J) \) is periodic. Let \( R/\text{U} \) be the finite residual of \( T/\text{U} \). If \( R/\text{U} \) were not a \( p \)-group, then \( R/\text{O}_{p'}(J) \) would contain a divisible \( p' \)-subgroup (recall that Sylow subgroups are Černikov, so if they are infinite, then they have a non-trivial finite residual) and so should \( J/\text{O}_{p'}(J) \), a contradiction. Thus, replacing \( G \) with \( G/\text{U} \) allows us to assume that \( J \) is a \( p \)-group of finite rank \( s \), which we require to be smallest possible; by Case 2, \( s > 0 \).
Let P be any minimal infinite G-invariant subgroup of J (so P is divisible abelian). It follows from Lemma 6.6.4 of [2] that \( G/C_G(P) \) is abelian-by-finite. By the minimality assumption on s, WP/P is locally nilpotent, thus P \( \cap W \) cannot be finite by Lemma 1 of [7] and hence P = P \( \cap W \leq W \). Thus, by Theorem 5.3.7 of [2], there exists a subgroup X of G such that G = X \( \times P \). Of course, C = C_X(P) is normal in G and if WC/C were locally nilpotent, then P would be contained in the hypercentre of W, a contradiction. Thus WC/C is not locally nilpotent and G/C is again a counterexample, so C is periodic by minimality of r and (T \( \cap X \))/C is residually finite. Moreover, if Z/C is the hypercentre of WC/C, then [Z, P] \leq C and so Z = C. Therefore we may replace G with G/C, assuming that C_X(P) = {1}, so C_G(P) = P, that P has no infinite proper G-invariant subgroups, and that X is an abelian-by-finite group. In this case every finitely generated subgroup of G is polycyclic-by-finite, so we obtain a contradiction by Case 3.

**Case 5.** \( T = \{1\} \)

Let F be the Fitting subgroup of G. Then G/F is polycyclic-by-finite, while F is nilpotent, torsion-free and non-trivial (see Theorem 10.33 of [17]); in particular, WF/F is locally nilpotent. Let z be a non-trivial element of the centre of F and put Z = \( \langle z \rangle ^G \). By Corollary 1 to Lemma 9.53 of [17], there is a non-trivial free abelian G-invariant subgroup Y of Z. Now, as in Case 3, we see that Y is contained in \( \zeta_r(WY) \) and so that W is locally nilpotent, the final contradiction.

The proof is complete. \( \square \)

Of course, the above result holds whenever the group is *locally finite*, or *locally polycyclic*. Moreover, a well known theorem of Kropholler [14] shows that a finitely generated soluble group with finite abelian section rank is minimax, so the class of all locally (soluble-by-finite and of finite abelian section rank) groups is precisely the class of all locally-\( \mathfrak{M} \) groups, and hence Theorem A provides a large generalization of the main result of [7].

**Corollary 2.2.** Let G = AB be a soluble-by-finite group which is factorized by two mutually permutable subgroups A and B such that both A’ and B’ are locally nilpotent. If G has finite abelian section rank, then the normal closure \( \langle A', B' \rangle ^G \) is locally nilpotent.

If G is a group, we denote by \( \rho_{0, L^\infty}(G) := \{1\} \) the trivial subgroup of G.
If we have defined the subgroup \( \rho_{i, L^\infty}(G) \) for some positive integer i, then we put
\[
\rho_{i+1, L^\infty}(G)/\rho_{i, L^\infty}(G) = \rho_{L^\infty}(G/\rho_{i, L^\infty}(G)).
\]
This defines the subgroups \( \rho_{n, L^\infty}(G) \) for any integer \( n \geq 0 \). Clearly, \( \rho_{L^\infty}(G) = \rho_{1, L^\infty}(G) \).
Corollary 2.3. Let \( G = AB \) be a locally-\( \mathcal{M} \) group which is factorized by two mutually permutable (non-trivial) soluble subgroups \( A \) and \( B \). If the derived length of \( A \) is \( c \) and the derived length of \( B \) is \( d \), then \( \rho_{k,\mathcal{L}\mathcal{M}}(G'') = G'' \), where \( k \) denotes the maximum of \( c - 1 \) and \( d - 1 \).

Proof. If \( c = d = 1 \), the result follows from Itô’s theorem on the product of two abelian subgroups. Assume \( k = \max\{c - 1, d - 1\} \geq 1 \). If \( c > 1 \), put \( C = A^{(c-1)} \), otherwise put \( C = \{1\} \); if \( d > 1 \), put \( D = B^{(d-1)} \), otherwise put \( D = \{1\} \). Let \( L = \langle C, D \rangle^G \) and notice that by Theorem A the normal subgroup \( L \) is locally nilpotent, so it is contained in \( \rho_{1,\mathcal{L}\mathcal{M}}(G) \). Now, induction on \( k \) gives that \( \rho_{n,\mathcal{L}\mathcal{M}}(G''L/L) = G''L/L \), where \( h = \max\{c - 2, d - 2\} \). Since \( h + 1 = k \) the statement is proved.

Corollary 2.4 (see also [6], Corollary 2). Let \( G = AB \) be a finite group which is factorized by two mutually permutable soluble subgroups \( A \) and \( B \). Then \( G \) is soluble.

Corollary 2.5. Let \( G = AB \) be a locally (soluble-by-finite) group of finite rank which is factorized by two mutually permutable soluble subgroups \( A \) and \( B \). Then \( G \) is hyperabelian.

Proof. By Corollary 2.3 there is \( n \) such that \( \rho_{n,\mathcal{L}\mathcal{M}}(G) = G \). The statement is proved observing that every locally nilpotent group of finite rank is hyper-central.

We remark that the proof of Theorem A shows something more than what we have summarized in the statement of the same theorem. Recall first that if \( \Theta \) is any set of words (on the alphabet \( x_1, x_2, \ldots, x_n, x_{n+1}, \ldots \)) and \( H \) is any group, we denote by \( \Theta(H) \) the verbal subgroup of \( H \) determined by \( \Theta \), that is, the subgroup generated by all the evaluations \( \theta(h_1, \ldots, h_n) \) where \( h_i \in H \) and \( \theta \in \Theta \); if, for instance, \( \Theta = \{[x_1, x_2]\} \), then \( \Theta(H) = H' \). Now, the proof of Theorem A gives the following result.

Theorem 2.6. Let \( \Theta_1, \Theta_2 \) be sets of words and let \( G \) be a group which is locally-\( \mathcal{M} \). Suppose the following property holds:

\[(\ast)\text{ if } H = UV \text{ is a finite group which is factorized by two mutually permutable subgroups } U \text{ and } V \text{. Then } \langle \rho_{\mathcal{L}\mathcal{M}}(\Theta_1(U)), \rho_{\mathcal{L}\mathcal{M}}(\Theta_2(V)) \rangle^G \leq \rho_{\mathcal{L}\mathcal{M}}(H).\]

Then \( \langle \rho_{\mathcal{L}\mathcal{M}}(\Theta_1(A)), \rho_{\mathcal{L}\mathcal{M}}(\Theta_2(B)) \rangle^G \leq \rho_{\mathcal{L}\mathcal{M}}(\Theta_1 \cup \Theta_2(G)) \leq \rho_{\mathcal{L}\mathcal{M}}(G) \), whenever \( G = AB \) is factorized by two mutually permutable subgroups \( A \) and \( B \).

Theorem 1 of [6] shows that property \((\ast)\) holds in particular whenever \( \Theta_1 \) and \( \Theta_2 \) are sets of commutator words, that is, \( \Theta_1(H) \leq H' \) for any group \( H \) and \( i = 1, 2 \). It follows that Theorem A is actually a corollary of Theorem 2.6.

A relevant type of commutator words is that of outer commutator words (of weight strictly larger than 1). These words are recursively defined as follows: all the letters of the alphabet are outer commutator words of weight 1, and...
if $\theta_1, \theta_2$ are outer commutator words of weights $r$ and $s$, then $[\theta_1, \theta_2]$ is an outer commutator word of weight $r + s$. If $G$ is a group and $\Theta$ is a set of outer commutator words, it is easy to see (by induction on the smallest weight of a word in $\Theta$) that $G/\Theta(G)$ is soluble. If we restrict our attention to this kind of word, we can move the rank assumption from the whole group to the factors.

Recall that two subgroups $A$ and $B$ of a group $G$ are said to be totally permutable if $HK = KH$ whenever $H \leq A$ and $K \leq B$. Of course, totally permutable subgroups are mutually permutable.

**Theorem 2.7.** Let $\Theta_1, \Theta_2$ be sets of outer commutator words of weight strictly larger than 1, and let $G = AB$ be a locally (soluble-by-finite) group which is factorized by two mutually (resp., totally) permutable subgroups $A$ and $B$ (resp., locally) of finite abelian section rank. Then $\langle \rho_{\text{ln}}(\Theta_1(A)), \rho_{\text{ln}}(\Theta_2(B)) \rangle^G \leq \rho_{\text{ln}}(\Theta_1 \cup \Theta_2(G)) \leq \rho_{\text{ln}}(G)$.

**Proof.** As in the beginning of the proof of Theorem A, we may assume $G$ is soluble-by-finite. Let $S$ be the soluble radical of $G$. By Lemma 2.1, $X = (S \cap A)(S \cap B)$ is a soluble normal subgroup of $G$ which is factorized by the two mutually permutable subgroups $S \cap A$ and $S \cap B$. Moreover, Lemma 1.2.5 of [2] yields that $X$ has finite index in $G$.

Now, if $A$ and $B$ have finite abelian section rank, it follows from Theorem 4.6.11 of [2] that $X$ has finite abelian section rank. Thus $G$ is locally-$\mathfrak{M}$ by the above mentioned result of Krogholler and so Theorem A completes the proof in the case of mutually permutable subgroups $A$ and $B$.

If $A$ and $B$ are totally permutable, then every finitely generated subgroup of $X$ is contained in a subgroup which is factorized by two finitely generated totally permutable subgroups. Then again Theorem 4.6.11 of [2] (together with the above mentioned result of Krogholler) shows that $G$ is locally-$\mathfrak{M}$, so we can apply Theorem A and we are done. \qed

Using Lemma 2.1 one easily see that in the above statement it is possible to replace “totally permutable subgroups $A$ and $B$ locally of finite abelian section rank” by “mutually permutable subgroups $A$ and $B$ such that $A \cap B$ is polycyclic-by-finite”.

Let $\Theta$ be a set of words (on the alphabet $x_1, x_2, \ldots, x_n, x_{n+1}, \ldots$). If $H$ is any group, we recall that the marginal subgroup $\Theta^*(H)$ of $H$ determined by $\Theta$ is the largest normal subgroup $N$ of $H$ such that $\theta(a_1, \ldots, a_n) = \theta(b_1, \ldots, b_n)$ for any $\theta \in \Theta$ and $a_i \equiv b_i \text{ mod } N$. The proof of Theorem A also yields the following result.

**Theorem 2.8.** Let $\Theta_1, \Theta_2$ be sets of words and let $G$ be a group which is locally-$\mathfrak{M}$. Suppose the following property holds:

\((*)\) if $H = UV$ is a finite group which is factorized by two mutually permutable subgroups $U$ and $V$. Then $\langle \rho_{\text{ln}}(\Theta_1^*(U)), \rho_{\text{ln}}(\Theta_2^*(V)) \rangle^G \leq \rho_{\text{ln}}(H)$. 


Then \( \langle \rho_{LN}(\Theta^*_1(A)), \rho_{LN}(\Theta^*_2(B)) \rangle^G \leq \rho_{LN}(G) \), whenever \( G = AB \) is factorized by two mutually permutable subgroups \( A \) and \( B \).

As a final remark, we observe that we do not have an analogue of Theorem B for groups which are factorized but not mutually factorized. In fact, let \( G = H \wr K \), where \( H \simeq K \simeq \text{Sym}(3) \), and let \( B \) be the base group of \( G \). Clearly, \( G = KB \) and both \( K' \) and \( B' \) are abelian; on the other hand, \( (K')^G \) is not even locally nilpotent, because it contains a copy of \( \text{Sym}(3) \).

3. Proof of Theorem B

Let \( G = AB \) be a finite group which is factorized by two subgroups \( A \) and \( B \), and let \( H \) be a subgroup of \( A \cap B \) which is subnormal in both \( A \) and \( B \). In such circumstances, it has been proved by Maier and Wielandt that the subgroup \( H \) is actually subnormal in \( G \) (see for instance Theorem 7.7.1 of [15]). Although the corresponding problem for an arbitrary infinite group seems to be still open, there are some classes of groups for which the result holds: this is for instance the case of the class of nilpotent-by-abelian-by-finite groups (see [12], Theorem A) and so in particular of the class of soluble-by-finite linear groups (see [12], Corollary to Theorem A). The first part of our Theorem B is thus a variation of the Maier-Wielandt theorem for periodic linear groups which are factorized by two mutually permutable subgroups, and it is a fundamental preliminary piece of the proof of the whole statement of Theorem B.

First we need to recall some known facts about linear groups. If \( G \) is any group, we denote by \( S = S(G) \) the soluble radical of \( G \), i.e. the product of all its normal soluble subgroups. It is well known that if \( G \) is linear, then \( S \) is a soluble subgroup of \( G \) which is closed in the Zariski topology and contains all soluble ascendant subgroups of \( G \) (see for instance Lemma 2.11 of [9]); moreover, if \( G \) is periodic linear, then \( G/S \) contains a normal subgroup \( B/S \) which is the direct product of finitely many simple non-abelian groups, and the index \(|G : B|\) is finite (see 5.1.5 and Theorem 5.1.6 of [18], or [9], Lemma 2.15). We denote by \( u(G) \) the unipotent radical of a linear group \( G \), i.e. its largest unipotent normal subgroup. If \( G \) is periodic linear of characteristic \( p \), then \( u(G) = \{1\} \) if \( p = 0 \), and \( u(G) = O_p(G) \) if \( p > 0 \). It has been proved in [16] that if \( G \) is a periodic group with trivial unipotent radical, then all its homomorphic images are linear groups; observe that, if \( G \) is linear, then also \( G/u(G) \) is linear (see for instance Lemma 2.13 of [9]).

Finally, we will frequently make use of some results that have recently been proved: 1) every serial subgroup of a periodic linear group is ascendant (see Theorem 2.14 of [9]); 2) every descendant subgroup of a periodic linear group is subnormal (see Theorem 3.5 of [9]; 3) every serial subgroup of a periodic linear group with trivial soluble radical is subnormal (see Lemma 3.4 of [9], but this also follows not difficulty from the structure we have described above); 4) every permutable subgroup of a periodic linear group is subnormal.
(see [8]); 5) if $X$ is a serial subgroup of a periodic linear group, then $X$ is subnormal in $u(G) X$ (see Corollary 2.5 of [9]).

We refer to the first chapter of [17] for the exact definitions of descendant, ascendant, and serial subgroup.

We also recall that a subgroup $X$ of a group $G$ is *permutable* in $G$ if $HX = XH$ for every subgroup $X$ of $G$. It is well known that permutable subgroups are ascendant, and permutable subgroups naturally arise when we deal with groups which are factorized by mutually permutable subgroups.

**Lemma 3.1.** Let $G = AB$ be a group which is factorized by two mutually permutable subgroups $A$ and $B$. Then $A \cap B$ is permutable in both $A$ and $B$.

**Proof.** Let $X$ be any subgroup of $A$ and notice that

$$X(A \cap B) = A \cap XB = A \cap BX = (A \cap B)X,$$

so $A \cap B$ is a permutable subgroup of $A$. Similarly, $A \cap B$ is permutable in $B$. \hfill \Box

**Lemma 3.2.** Let $G = AB$ be a locally finite group which is factorized by two mutually permutable subgroups $A$ and $B$. Then $A \cap B$, $A'$ and $B'$ are serial subgroups of $G$.

**Proof.** Let $E$ be any finite subgroup of $G$. Since $G = AB$ we can find a finite subgroup $F = \langle g_1, \ldots, g_n \rangle$ of $G$ such that $E \leq F$ and $\{g_1, \ldots, g_n\} \subseteq A \cup B$. By Lemma 2.1, $(F \cap A)(F \cap B)$ is a subgroup of $F$, but since it contains $\{g_1, \ldots, g_n\}$, it is actually equal to $F$; moreover, $F \cap A$ and $F \cap B$ are mutually permutable. Thus Theorem 1 of [6] shows that $F \cap A \cap B$, $(F \cap A)'$ and $(F \cap B)'$ are subnormal subgroups of $F$. In particular, $E \cap A \cap B$ is subnormal in $E$. Now, a well known result of Hartley (see [13], or [9], Lemma 2.1, for a simpler proof) yields that $A \cap B$ is a serial subgroup of $G$.

It remains to prove that $A'$ and $B'$ are serial subgroups of $G$; we focus on $B'$, being symmetrically true for the commutator subgroup $A'$. Let $\mathcal{F}$ be the set of all finite subgroups $W$ of $G$ such that $W = (W \cap A)(W \cap B)$, and take $F \in \mathcal{F}$; as we noticed above, $(F \cap B)'$ is subnormal in $F$. Let $S_F$ be the set of all series between $Y$ and $F$, where $Y$ is a subgroup such that $(F \cap B)' \leq Y \leq F \cap B'$; clearly, $S_F$ is finite and non-empty. If $E \in \mathcal{F}$ is such that $F \leq E$, and $\Sigma \in S_E$, then $\tau_{E,F}(\Sigma) = \{F \cap K : K \in \Sigma\}$ belongs to $S_F$. Thus the set of all $S_F$, where $F \in \mathcal{F}$, and all maps $\tau_{E,F}$, where $E \geq F$ are elements of $\mathcal{F}$, is an inverse system, whose inverse limit is not empty (by a well known theorem of Kurosh). Let $\{\Xi_F\}_{F \in \mathcal{F}}$ be an element of this inverse limit, so

$$\Xi_F : F_0 \leq F_1 \leq \ldots \leq F_{\tau(F)}$$

is an element of $S_F$ and $\Xi_F = \{E_i \cap F : E_i \in \Xi_E\}$, whenever $E \geq F$ are elements of $\mathcal{F}$. Clearly we may assume $B' < G$. For any $g \in G \setminus B'$, let $d_g$ be the set of
all pairs \((F, i) \in \mathcal{F} \times \mathbb{N}\) such that \(g \in F_i \cap F_{i-1}\). Since \(\mathcal{F}\) is a local system for \(G\), it follows that \(\mathcal{I}_g\) is not empty, so we may put

\[
\Lambda_g = \bigcup_{(F, i) \in \mathcal{I}_g} F_i \quad \text{and} \quad V_g = \bigcup_{(F, i) \in \mathcal{I}_g} F_{i-1}.
\]

Then \(V_g\) is normal in \(\Lambda_g\), and \(g \in \Lambda_g \setminus V_g\) so

\[
\bigcup_{g \in G \setminus B'} \Lambda_g \supseteq G \setminus B' \quad \text{and} \quad \bigcap_{g \in G \setminus B'} V_g \subseteq B'.
\]

Moreover, if \(g \in G \setminus B'\) and \(b \in B'\), then there is \(E \in \mathcal{F}\) with \(b \in (E \cap B') \leq E_0 \subseteq V_g\); thus \(B' \subseteq V_g\). It follows that \(B' = \bigcap_{g \in G \setminus B'} V_g\) and \(G = \bigcup_{g \in G \setminus B'} \Lambda_g\), so the set \(\{(\Lambda_g, V_g) : g \in G \setminus B'\}\) is a series between \(B'\) and \(G\), and hence \(B'\) is a serial subgroup of \(G\).

\qed

In the statement of the above lemma, commutator subgroups can be replaced by verbal subgroups provided that an analogue result holds for finite groups.

**Corollary 3.3.** Let \(G = AB\) be a periodic linear group which is factorized by two mutually permutable subgroups \(A\) and \(B\). Then \(A \cap B\) is ascendant in \(G\). Moreover, if \(G\) has no non-trivial abelian normal subgroups, then \(A \cap B\) is subnormal in \(G\).

**Proof.** By Lemma 3.2, \(A \cap B\) is serial in \(G\), so it is ascendant in \(G\) (by Theorem 2.14 of [9]), and even subnormal in \(G\) (by Lemma 3.4 of [9]) if \(S(G) = \{1\}\).

**Theorem 3.4.** Let \(G = AB\) be a linear group (over a field \(\mathcal{K}\)) which is factorized by two mutually permutable periodic subgroups \(A\) and \(B\). Then \(A \cap B\) is subnormal in \(G\).

**Proof.** Of course, \(\mathcal{K}\) can be assumed algebraically closed. Let \(g \in G\). There exists \(a \in A\) and \(b \in B\) such that \(g = ab\), so \(g \in K = \langle a \rangle B\). On the other hand, \(B\) is a subgroup of finite index of \(K\), so \(K\) is periodic and hence \(g\) is periodic too. The arbitrariness of \(g\) in \(G\) shows that \(G\) is periodic.

Let \(X = A \cap B\), \(S = S(G)\) and \(U = u(G)\); in particular, \(G/U\) is linear. Then \(X\) is permutable in \(A\), so \(X\) is subnormal in \(A\) (see [8]), and hence \(XU/U\) is subnormal in \((AU \cap B)U/U\). By Corollary 3.3, \(X\) is ascendant in \(G\), so \(X\) is subnormal in \(XU\) by Corollary 2.5 of [9]. Thus, replacing \(G\) by \(G/U\), we may assume \(U = \{1\}\), so the connected component \(S^0\) of \(S\) is a diagonalizable (normal) subgroup of \(G\). Now, another application of Corollary 3.3 yields that \(X \cdot S^0/S^0\) is subnormal in \(G/S^0\), so there is a finite series of subgroups

\[
X \cdot S^0 =: L_0 \subseteq L_1 \subseteq \ldots \subseteq L_i \subseteq L_{i+1} \subseteq \ldots \subseteq L_n =: G
\]

where \(n\) is the subnormal defect of \(X \cdot S^0\) in \(G\). We aim to further reduce ourselves to the case in which \(G = XD\) for some diagonalizable normal subgroup \(D\) of \(G\). Suppose in our group \(G\), the (non-negative) integer \(n\) is positive. By Lemma 2.1, the subgroup \(H = (L_{n-1} \cap A)(L_{n-1} \cap B)\) is normal in \(G\) and factorized
by the mutually permutable subgroups $H_A = L_{n-1} \cap A$ and $H_B = L_{n-1} \cap B$; moreover,
\[ H_A \cap H_B = X \quad \text{and} \quad (X \cdot S^0) \cap H = X(S^0 \cap H). \]
Thus $X(S^0 \cap H)$ is subnormal in $H$ of defect strictly smaller than $n$. Continuing in this way, we see that it is possible to assume $G = XD$, for some diagonalizable normal subgroup $D$. By Lemma 1.12 of [19], $C_G(D)$ has finite index in $G$, so we may factor $C_X(D)$ out (see [16]) assuming therefore that $G$ is abelian-by-finite. In this case the statement is a consequence of [12], Theorem A.

Now we are able to complete the proof of Theorem B.

**Lemma 3.5.** Let $G = AB$ be a group which is factorized by two mutually permutable subgroups $A$ and $B$. If $H \leq A$ and $K \leq B$ are such that $A \cap B \leq H \cup K$, then $HK = KH$.

**Proof.** $HK = H(A \cap B)K = AK \cap HB = KA \cap BH = K(A \cap B)H = KH$. □

**Theorem 3.6.** Let $G = AB$ be a linear group (over a field $\mathbb{K}$) which is factorized by two mutually permutable periodic subgroups $A$ and $B$. Then $A'$ and $B'$ are subnormal subgroups of $G$.

**Proof.** Of course, $G$ is periodic (as in the beginning of Theorem 3.4) and $\mathbb{K}$ can be assumed algebraically closed. Theorem 3.4 yields that $A \cap B$ is subnormal in $G$. Moreover, $A$ is periodic and linear, so an application of the Lie-Kolchin theorem to the connected component $D$ of the soluble radical $S(A)$ of $A$ (plus our remarks at the beginning of the section) yields that:

1. $u(D)$ is a nilpotent group of finite exponent;
2. $D/u(D)$ is an abelian group of finite rank whose Sylow subgroups are Černikov groups (here $D/u(D)$ is a diagonalizable subgroup of $G/u(D)$);
3. $G/D$ admits a finite composition series.

Thus, there is a finite series
\[ A \cap B = X_0 \leq X_1 \leq \ldots \leq X_i \leq X_{i+1} \leq \ldots \leq X_n = A \]
whose factors are groups of one of the following types: simple, elementary abelian, Prüfer or direct sums of countably infinitely many, locally cyclic primary groups.

Choose $0 \leq i < n$ and put
\[ K = (X_i B)X_{i+1}B = (X_i B)X_{i+1}. \]
Then $A \cap B \leq X_i \leq K$, $K = K \cap X_i B = X_i(K \cap B)$ and $X_{i+1} \cap B \leq X_0$, so $X_{i+1}K/K \simeq X_{i+1}/X_i$. Now, put $U = BX_{i+1}$ and notice that by Lemma 2.1 the subgroups $B = U \cap B$ and $X_{i+1} = U \cap A$ are mutually permutable, so we may apply for instance Lemma 4 of [7] to deduce that the subgroups $X_{i+1}K/K$ and $BK/K$ are totally permutable. Since $BK/K$ admits no non-trivial normal
subgroups of $X_{i+1}B/K$, Theorem 1 of [5] yields that $BK/K$ is nilpotent, and we also observe that
$$X_{i+1}K \cap BK = (X_{i+1}(K \cap B)) \cap (BX_i) = K.$$ 
By Lemma 3.5, $X_{i+1}B' = B'X_{i+1}$ and $X_iB' = B'X_i$ are subgroups, and now, in order to show that $X_iB'$ is subnormal in $X_{i+1}B'$, we deal with each possibility of the factor group $X_{i+1}/X_i$ separately.

**Case 1.** $X_{i+1}/X_i$ is either cyclic of prime order or of type $p^\infty$ for some prime $p$. We prove that $B'$ is contained in $K$, so
$$K \cap X_{i+1}B' = (X_i(K \cap B)) \cap X_{i+1}B' = X_i((K \cap B) \cap X_{i+1})B' = X_iB',$$
and hence $X_iB'$ is normal in $X_{i+1}B'$, as we wanted.

Assume first $X_{i+1}/X_i$ is cyclic of prime order. By Theorem 3 of [5] there is a normal subgroup of $X_{i+1}B/K$ which is contained either in $X_{i+1}K/K$ or in $BK/K$. Since $BK/K$ admits no non-trivial normal subgroups of $X_{i+1}B/K$, it is safe to assume that $X_{i+1}K/K$ is normal by $BK/K$. But then the centralizer $C_{BK/K}(X_{i+1}K/K)$ is trivial, so again $B' \leq K$.

Assume now $X_{i+1}/X_i$ is of type $p^\infty$ for some prime $p$. Let $H/K$ be any cyclic subgroup of $BK/K$. Then $X_{i+1}H/K$ is Černikov, and $X_{i+1}K/K$ is its finite residual; in particular, $H/K$ normalizes $X_{i+1}K/K$. The arbitrariness of $H/K$ shows that $X_{i+1}K/K$ is normal in $X_{i+1}B/K$. Since the automorphism group of a group of type $p^\infty$ is abelian, it follows that $B'K/K$ is normal in $X_{i+1}B/K$ and hence even trivial, i.e. $B' \leq K$.

**Case 2.** $X_{i+1}/X_i$ is a direct sum of countably infinitely many, locally cyclic primary groups. There is a descending series running from $X_{i+1}$ to $X_i$
$$X_i = \bigcap_{j<\infty} Y_j \ldots \leq Y_{j+1} \leq Y_j \leq \ldots \leq Y_1 \leq Y_0 = X_{i+1}$$
whose factors are either cyclic of prime order or of type $p^\infty$ for some prime $p$. The argument of Case 1 yields that $Y_jB'$ is normal in $Y_{j+1}B'$ for any $j$. Let $U = \bigcap_{j<\infty} (Y_jB')$. Then $U$ is a descendant subgroup of $X_{i+1}B'$, so it is even subnormal in $X_{i+1}B'$ by Theorem 3.5 of [9]. It follows from Lemma 2.1 that $X_{i+1}B'$ is factorized by the two permutable subgroups $X_{i+1}$ and $(A \cap B)B'$, so (again by Lemma 2.1)
$$\left(\bigcap_{j<\infty} Y_j\right) B' = X_iB'$$
is subnormal in $X_{i+1}B'$ and we are done.

**Case 3.** $X_{i+1}/X_i$ is an elementary abelian $p$-group.
It follows from the main result of [3] that $X_{i+1}B/K$ is locally supersoluble, so, if $\sigma$ is the set of all primes $q > p$, then the set of all $\sigma$-elements of $X_{i+1}B/K$ is a subgroup and it must necessarily be a subgroup of $BK/K$ (this depends on the fact that $X_{i+1}/X_i$ is elementary abelian and the subgroups $X_{i+1}K/K$ and $BK/K$ are totally permutable). Since $BK/K$ has no non-trivial normal subgroups of $X_{i+1}B/K$, we have that $q \leq p$ for any $q \in \pi(BK/K)$.

Let $W_p/K$ be the unique Sylow $p$-subgroup of $BK/K$. Then $X_{i+1}W_p/K$ is locally nilpotent, so its maximal subgroups are normal and hence $W_pK/K$ is normalized by the elementary abelian $p$-group $X_{i+1}K/K$; it follows that $W_pK/K = \{1\}$. Now, Proposition 2 of [5] yields that $B'K/K$ centralizes $X_{i+1}K/K$, so $B' \leq K$.

As in Case 1 we see that $X_iB'$ is normal in $X_{i+1}B'$.

Case 4. $X_{i+1}/X_i$ is a non-abelian simple group.

Theorem 1 of [5] yields that $X_{i+1}K/K$ is centralized by $BK/K$, so $B' \leq B \leq K$ and we are done again as in Case 1.

So far, we have proved that $B'X_i$ is subnormal in $B'X_{i+1}$ for any choice of $i$, and this implies that $B'(A \cap B) = B'X_0$ is subnormal in $B'X_i = B'A$. Of course, $B'$ is normal in $B'(A \cap B)$, and hence $B'$ is subnormal in $B'A$, so there exists a positive integer $\ell$ such that $[A,_{\ell}B'] \leq B'$. Notice now that $[AB, B'] \leq [A, B']B'$, and if $C$ is any subgroup of $G$ with $CB' = B'C$, then $[CB', B'] \leq [C, B']B'$.

This shows that $[G,_{\ell}B'] = [AB,_{\ell}B'] \leq B'$, so $B'$ is subnormal in $G$ and the statement is proved.

Clearly, the above result applies in particular to any Černikov group and so we have the following consequence of our second main result which not only generalizes Theorem B of [7] but strengthens its conclusions.

Corollary 3.7. Let $G = AB$ be a Černikov group which is factorized by two mutually permutable subgroups $A$ and $B$. Then $A'$ and $B'$ are subnormal subgroups of $G$.

Corollary 3.8. Let $G = AB$ be a linear group which is factorized by the two mutually permutable subgroups $A$ and $B$. If $A$ and $B$ are both periodic and soluble-by-finite, then also $G$ is periodic and soluble-by-finite.

Proof. It is clearly possible to assume $S(G) = \{1\}$. Thus $A$ and $B$ are finite-by-abelian by Theorem 3.6 and hence even finite over their centres (being linear groups). Now, Theorem 2.2.5 of [2] shows that $G$ is soluble-by-finite.

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