TOROIDAL Z-ALGEBRAS

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Abstract

The toroidal Lie algebras an N variable generalizations of affine Kac-Moody Lie algebras. As in the affine Lie algebra there exists finite order automorphisms corresponding to Dynkin diagram automorphisms. The fixed point subalgebra are called twisted toroidal Lie algebras. In this paper we construct faithfull representations for toroidal Lie algebras (this includes the non-twisted case also) using methods developed by Lepowsky Wilson [LW]. This construction recovers the result by Eswara Rao - Moody [EM] in the homogeneous picture and by Yuly Billig [B1] in the principal picture. The proofs given in this paper are much shorter than above works. The results for the twisted case are completely new.

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Introduction

In recent times a great amount of research has been done on Extended Affine Lie Algebras (EALA), which are natural generalization of affine Kac-Moody Lie algebras. See [AABGP], [AG] and references there in. It is an accepted fact that the Lie algebras gains importance only when it admits a natural realization in other words a faithful representation. It is an open problem to find a realization for an EALA. An important class of EALA’s are the ones obtained from the so called toroidal Lie algebras. Toroidal Lie algebras are $N$ variable generalization of affine Kac-Moody Lie algebras. For the first time a large class of (integrable) modules are constructed for toroidal Lie algebras in [EM] and [MEY], the so called homogeneous picture. In [B1] similar construction has been made for the principal picture. These constructions are very important and have found applications in differential equations in the works of [B2], [ISW1] and [ISW2]. To obtain an EALA from toroidal Lie algebra one need to add infinite set of derivations. In 2006 Yuly Billig [B3] obtained a realization by making use of Vertex operator algebras. The next class examples of EALA’s are the one corresponding to the twisted toroidal Lie algebras. Thus the purpose of this paper is to construct faithful representation of twisted toroidal Lie algebras which arise as fixed points of certain automorphisms of toroidal Lie algebras of type ADE. The main idea is to use the Z-algebra theory developed by Lepowsky-Wilson [LW] in the study of Vertex operator representation for affine Kac-Moody Lie algebra. In the process of our construction of representations using Z-algebra theory, we recover the results of [EM], [MEY] in the homogeneous picture and the results of [B1] and [T] in the principle picture. Our proofs in these cases are much shorter than the existing proofs. This is the first time we have a faithful realization for the twisted toroidal Lie algebra.

Let $\mathcal{G}$ be the simple finite dimensional Lie algebra over the complex num-
bers. Let $A$ be a Laurent polynomial ring in $N + 1$ commuting variables. Consider the multiloop algebra $\mathcal{G} \otimes A$, its universal central extension $\tilde{\tau}$ the toroidal Lie algebra. Let $\theta$ be an automorphism of $\mathcal{G}$ of order $m$. Then $\theta$ can be extended to an automorphism of $\tilde{\tau}$ (Section 1). Then the subalgebra of $\theta$ fixed points inside $\tilde{\tau}$ is called twisted toroidal Lie algebra $\tilde{L}(\mathcal{G}, \theta)$. It is the universal central extension of the underlying multiloop algebra (See [BK]).

In Section 1, we define a category $\mathcal{C}_k$ of $\tilde{L}(\mathcal{G}, \theta)$ modules which satisfy a factorisation property first introduced in [BY]. The factorisation property is not satisfied for a general class of integrable modules. But there are enough of integrable modules which satisfy the factorisation property. For example the vertex representation defined in [EM] and the representation considered in [BY] satisfy factorisation property.

Next by following [LW] closely we define toroidal $\mathbb{Z}$-algebras (1.10) and define a category $\mathcal{D}_k$-of $\mathbb{Z}$-algebra modules. We then prove the important Proposition (2.6) which says that the categories $\mathcal{C}_k$ and $\mathcal{D}_k$ are equivalent. Thus by constructing a $\mathbb{Z}$- algebra module we get a module for $\tilde{L}(\mathcal{G}, \theta)$.

In section 3 we specialise to the homogeneous picture for the nontwisted case of type ADE. We construct a module for the $\mathbb{Z}$-toroidal Lie algebra closely following the results of [LP]. Thereby constructing a module for $\tilde{L}(\mathcal{G}, I_d) \cong \tilde{\tau}$ which is faithful. This recovers the main result of [EM]. Our calculations are certainly much shorter.

In Section 4 we specialise to the principal picture. This includes the twisted and nontwisted toroidal Lie algebras. We again construct a module for the $\mathbb{Z}$-toroidal Lie algebra by making use of the corresponding results for the affine Kac-Moody Lie algebra from [LW]. We have to consider the additional Fock space for this purpose. Thus we get a module for our $\tilde{L}(\mathcal{G}, \theta)$. This result recovers the main result of [B1] and [T]. Again our proof are much shorter. The twisted case is completely new.

In the process we have given the following realization of twisted toroidal
Lie-algebra. Let \( \pi \) be a Dynkin diagram automorphism of \( \mathcal{G} \). Define an automorphism \( \theta \) of \( \mathcal{G} \) as in the section 4. Then we prove that \( \tilde{\tau}(\mathcal{G}, \theta) \cong \tilde{L}(\mathcal{G}, \pi) \). This is what is called the principal realization in the affine case. The isomorphism is given explicitly in twisted case and it is completely new even in the affine case (Proposition 4.10).

Section 1

Let \( \mathcal{G} \) be a finite dimensional semisimple Lie-algebra over the complex numbers \( \mathbb{C} \). Let \( \langle, \rangle \) be a non-degenerate symmetric \( \mathcal{G} \)-invariant bilinear form on \( \mathcal{G} \). We fix a non-negative integer \( N \). Let \( A = \mathbb{C}[t^\pm 1, t_1^\pm 1, \ldots, t_N^\pm 1] \) be the ring of Laurent polynomials in \( N+1 \) commuting variables. Let \( r = (r_1, \ldots, r_N) \in \mathbb{Z}^N \). Let \( \theta = t_1^r t_2^r \cdots t_N^r \). Fix a positive integer \( m \). Let \( \Omega_A \) be a free \( A \)-module of rank \( N+1 \) with basis \( \{k_0, \ldots, k_N\} \). Let \( d_A \) be the subspace of \( \Omega_A \) spanned by elements of the form \( \frac{1}{m} r_0 t^r t^s k_0 + \cdots + r_N t^r t^s k_N \). Let \( x(r_0, r) = x \otimes t^r t^s \in \mathcal{G} \otimes A \). Then the toroidal Lie-algebra \( \tau = \mathcal{G} \otimes A \oplus \Omega_A / d_A \) is defined by the following bracket.

\[
[x(r_0, r), y(s_0, s)] = [x, y] (r_0 + s_0, r + s) + \frac{<x, y>}{m} r_0 t^{r_0 + s_0} t^{l^2} k_0 + \sum_{i=1}^{N} r_i t^{r_0 + s_0} t^{l^2} k_i
\]

for \( x, y \in \mathcal{G}, r, s \in \mathbb{Z}^N, r_0, s_0 \in \mathbb{Z} \).

\( \Omega_A / d_A \) is central.

It is known that \( \tau \) is the universal central extension of \( \mathcal{G} \otimes A \). (See [K], [MEY]). (First note that the toroidal Lie algebra defined by \( m = 1 \) is isomorphic to the above). Let \( h \) be a Cartan subalgebra of \( \mathcal{G} \). Let \( \theta \) be an automorphism of \( \mathcal{G} \) such that \( \theta(h) = h \) and of order \( m \). Let \( \mathbb{Z}_m = \mathbb{Z} / m\mathbb{Z} \) be the cyclic group of order \( m \). Let \( w \) be a primitive \( m \)th root of unity.
(1.2) Let $G_i = \{ x \in G \mid \theta x = w^i x \}$ for $i \in \mathbb{Z}$. Then $G = \oplus_{i \in \mathbb{Z}_m} G_i$. Note that $G_i, G_j = 0$ unless $i + j \equiv 0(\text{mod } m)$. For $x \in G$ write $x = \sum_{i \in \mathbb{Z}_m} x_i$ where $\theta x_i = w^i x_i$. Define $x_i = x_{\overline{i}}$ for $i \in \mathbb{Z}$ and $\overline{i} \in \mathbb{Z}_m$.

Extend the automorphism $\theta$ to $\tau$ by $\theta(x(r_0, \overline{r})) = w^{-\overline{r}} \theta(x(r_0, \overline{r})$ and $\theta(t_0^r t_k) = w^{-\overline{r}} t_k$, $0 \leq i \leq N$. Let $\tilde{\tau} = \tau \oplus D$ where $D$ is spanned by derivations $\{d_0, \cdots, d_N\}$ with bracket $[d_i, x(r_0, \overline{r})] = r_i x(r_0, \overline{r})$, for $0 \leq i \leq N$, $[d_i, t_k] = t_k$, and $[d_i, d_j] = 0$. Extend the automorphism $\theta$ to $\tilde{\tau}$ by $\theta(d_i) = d_i$. Let $(\Omega_A/d_A)_0$ be the linear span of $t^r t_k$ where $r_0 \equiv 0(\text{mod } m)$. Consider the $\theta$ fixed points of $\tilde{\tau}$ say $\tilde{L}(G, \theta)$.

(1.3) Let $L(G, \theta) = \oplus_{i \in \mathbb{Z}} G_i(i, \overline{r})$ and $\overline{L}(G, \theta) = L(G, \theta) \oplus (\Omega_A/d_A)_0$. Then clearly $\tilde{L}(G, \theta) = \overline{L}(G, \theta) \oplus D$.

Since $<,>$ is non-degenerate and $G$-invariant, its restriction to $\mathfrak{h}$ is also non-degenerate. We identify $\mathfrak{h}$ and $\mathfrak{h}^*$ via this form. Let $\Phi$ be the root system of $G$. For $\beta \in \Phi$, choose the corresponding non-zero root vectors $x_\beta$ such that $[x_\beta, x_{-\beta}] = <x_\beta, x_{-\beta}> \beta$. Let $\epsilon(\beta, \gamma)$ be a non-zero number such that $[x_\beta, x_\gamma] = \epsilon(\beta, \gamma)x_{\beta+\gamma}$. Clearly the set of roots $\Phi$ is $\theta$- stable. Then define $\eta(p, \beta)$ a non-zero scalar such that

(1.4) $\theta^p x_\beta = \eta(p, \beta) x_{\theta^p \beta}$.

For any vector space $V$ and for indeterminates $\zeta_1, \cdots, \zeta_\ell$, denote $V\{\zeta_1, \cdots, \zeta_\ell\}$ the space of formal Laurent series. Further $V[\zeta_1^{\pm 1}, \cdots, \zeta_\ell^{\pm 1}]$ denote finite formal Laurent series. We recall the following Proposition from [LW]. Define $\delta(\zeta) = \sum_{i \in \mathbb{Z}} \zeta^i \in \mathbb{C}\{\zeta\}$.

Proposition (1.5) (a) (Proposition (2.2) of [LW]). Let $f(\zeta) \in V[\zeta, \zeta^{-1}]$. Then

$$f(\zeta)\delta(\zeta^m) = m^{-1} \sum_{p \in \mathbb{Z}_m} f(w^p)\delta(w^{-p}\zeta).$$
For \( x \in G, \underline{r} \in \mathbb{Z}^N \) let

\[
x(\underline{r}, \zeta) = \sum_i x_i \otimes t^i \xi^i, \quad k_i(\underline{r}, \zeta^m) = \sum_{p \in \mathbb{Z}} t^m \eta^i r \xi^m.
\]

For any infinite series \( f(\zeta) = \sum b_i \zeta^i \), let \( Df(\zeta) = \sum i b_i \zeta^i \).

**Proposition (1.5)** (b) The following relations hold for \( x(\underline{r}, \zeta) \) and \( k_i(\underline{r}, \zeta^m) \).

In fact they define a Lie-algebra \( \tilde{L}(\mathcal{G}, \theta) \). For \( \beta_1, \beta_2 \in \Phi, \underline{r}, \underline{s} \in \mathbb{Z}^N \).

(1)

\[
[x_{\beta_1}(\underline{r}, \zeta_1), x_{\beta_2}(\underline{s}, \zeta_2)] = \frac{1}{m} \sum_{\theta^p \beta_1 + \beta_2 = 0} \eta(p, \beta_1) \epsilon(\theta^p \beta_1, \beta_2) x_{\theta^p \beta_1 + \beta_2}(\underline{r} + \underline{s}, \zeta_2) \delta(w^{-p} \zeta_1 / \zeta_2)
\]

\[
- \frac{1}{m} \langle x_{\beta_2}, x_{-\beta_2} \rangle \sum_{\theta^p \beta_1 + \beta_2 = 0} \eta(p, \beta_1) \beta_2(\underline{r} + \underline{s}, \zeta_2) \delta(w^{-p} \zeta_1 / \zeta_2)
\]

\[
+ \frac{1}{m} \langle x_{\beta_2}, x_{-\beta_2} \rangle \sum_{\theta^p \beta_1 + \beta_2 = 0} \eta(p, \beta_1) \left( \sum_{i=1}^{N} r_i k_i(\underline{r} + \underline{s}, \zeta_2^m) \delta(w^{-p} \zeta_1 / \zeta_2) \right)
\]

\[
+ \frac{k_0(\underline{r} + \underline{s}, \zeta_2^m)}{m} D \delta(w^{-p} \zeta_1 / \zeta_2))
\]

(2)

\[
[\beta_1(\underline{r}, \zeta_1), \beta_2(\underline{s}, \zeta_2)] = \frac{1}{m} \sum_{p \in \mathbb{Z}_m} \langle \theta^p \beta_1, \beta_2 \rangle \left( \sum_{i=1}^{N} r_i k_i(\underline{r} + \underline{s}, \zeta_2^m) \delta(w^{-p} \zeta_1 / \zeta_2) \right)
\]

\[
+ \frac{k_0(\underline{r} + \underline{s}, \zeta_2^m)}{m} D \delta(w^{-p} \zeta_1 / \zeta_2))
\]

(3)

\[
[\beta_1(\underline{r}, \zeta_1), x_{\beta_2}(\underline{s}, \zeta_2)] = \frac{1}{m} \sum_{p \in \mathbb{Z}_m} \langle \theta^p \beta_1, \beta_2 \rangle x_{\beta_2}(\underline{r} + \underline{s}, \zeta_2) \delta(w^{-p} \zeta_1 / \zeta_2)
\]

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(4) \[ \frac{1}{m} Dk_0(\mathbf{r}, \zeta^m) + \sum_{i=1}^{N} r_i k_i(\mathbf{r}, \zeta^m) = 0 \]

(5) \[ [d_i, k_j(\mathbf{r}, \zeta^m)] = r_i k_j(\mathbf{r}, \zeta^m) \text{ for } 1 \leq i \leq N \text{ and } 0 \leq j \leq N \]

(6) \[ [d_0, k_j(\mathbf{r}, \zeta^m)] = Dk_j(\mathbf{r}, \zeta^m), \quad 0 \leq j \leq N \]

(7) \[ x_\beta(\mathbf{r}, \mu^p \zeta) = \eta(p, \beta) x_{\theta(p, \beta)}(\mathbf{r}, \zeta) \]

(8) \[ k_i(\mathbf{r}, \zeta^m) \text{ is central in } \mathcal{L}(\mathcal{G}, \theta) \text{ for } 0 \leq i \leq N \]

(9) \[ [d_i, x_\beta(\mathbf{r}, \zeta)] = r_i x_\beta(\mathbf{r}, \zeta), \quad 1 \leq i \leq N \]

(10) \[ [d_0, x_\beta(\mathbf{r}, \zeta)] = D x_\beta(\mathbf{r}, \zeta), \quad [d_i, d_j] = 0 \]

**Proof** (4) follows from the definition of $(\Omega_A/d_A)_0$. (5), (6), (7) and (8) are easy to see. First consider the following:

\[ [x(\mathbf{r}, \zeta_1), y(\mathbf{s}, \zeta_2)] = F + G_1 + G_2. \]

Where

\[ F = \sum_{i,j} [x_i, y_j] t^{i+j} \mathcal{L}^{p+2} \zeta_1^i \zeta_2^j, \]

\[ G_1 = \frac{1}{m} \sum_{i,j} i < x_i, y_j > t^{i+j} \mathcal{L}^{p+2} k_0 \zeta_1^i \zeta_2^j, \]

\[ G_2 = \sum_{\ell=1}^{N} \sum_{i,j} r_\ell < x_i, y_j > t^{i+j} \mathcal{L}^{p+2} k_\ell \zeta_1^i \zeta_2^j. \]

From the proof of Theorem (2.3) of [LW] it follows that

\[ F = \frac{1}{m} \sum_{p \in \mathbb{Z}_m} [\theta^p x, y](\mathbf{r} + \mathbf{s}, \zeta_2) \delta(w^{-p} \zeta_1 / \zeta_2). \]
Now consider $[\theta^p x, \beta_1, x, \beta_2] = \eta(p, \beta_1)[x_{\theta^p \beta_1}, x, \beta_2]$.

$$
= \begin{cases}
\eta(p, \beta_1) < x, \beta_2, x, \beta_2 > \beta_2 & \text{if } \theta^p \beta_1 + \beta_2 = 0 \\
0 & \text{if } \theta^p \beta_1 + \beta_2 \neq 0 \text{ and not a root.}
\end{cases}
$$

Thus for $x = x, \beta_1$ and $y = x, \beta_2$, $F$ equals to the first and second term of right hand side in (1). For $G_1$ and $G_2$, first note that $< x, y, > = 0$ if $i + j \neq 0(m)$ by (1.2). Thus

$$
mG_1 = \sum_{i,p} i < x, y, > < \theta^p t^{\ell^p + \ell^q k_0(\zeta_1/\zeta_2)^i}^m.
$$

$$
= \sum_{i,p} i < x, y, > < \theta^p t^{\ell^p + \ell^q k_0(\zeta_1/\zeta_2)^i}^m
$$

$$
= \sum_{i,p} i < x, y, > < \theta^p t^{\ell^p + \ell^q k_0(\zeta_1/\zeta_2)^i}^m
$$

$$
= D(\sum_{i} < x, y, > (\zeta_1/\zeta_2)^i) k_0(r + s, \zeta_2^m)
$$

$$
= D(\sum_{i=0}^{m-1} < x, y, > (\zeta_1/\zeta_2)^i) k_0(r + s, \zeta_2^m)
$$

$$
= \frac{1}{m} \sum_{\ell} < \theta^p x, y, > D\delta(w^{-p}\zeta_1/\zeta_2) k_0(r + s, \zeta_2^m).
$$

1.5(1)

Similarly

$$
G_2 = m^{-1} \sum_{\ell=1}^{N} r_{\ell} < \theta^p x, y, > \delta(w^{-p}\zeta_1/\zeta_2) k_\ell(r + s, \zeta_2^m).
$$

1.5(2)

Again for $x = x, \beta_1$ and $y = x, \beta_2$, $< \theta^p x, y, > = \eta(p, \beta_1) < x_{\theta^p \beta_1}, x, \beta_2 >= 0$ if $\theta^p \beta_1 + \beta_2 \neq 0$

$$
= \eta(p, \beta_1) < x, \beta_2, x, \beta_2 > \text{ if } \theta^p \beta_1 + \beta_2 = 0.
$$

This completes the proof (1). To see (2), take $x = \beta_1$ and $y = \beta_2$. Then $F = 0$. From 1.5 (1) and 1.5(2), (2) will follow. To see (3) take $x = \beta_1$ and
\[ y = x_{\beta_2} \] and note that \( G_1 = 0 \) and \( G_2 = 0 \) and \( F \) is equal to right hand side of 3. This completes the proof of the Proposition (1.5) (b).

Now we define a category \( C_k \) of \( \tilde{L}(G, \theta) \)-modules.

**Definition (1.6)** A \( \tilde{L}(G, \theta) \)-module \( V \) is in \( C_k \) if

1. \( k_o \) acts as \( k \).
2. \( V = \oplus_{z \in \mathbb{C}} V_z, V_z = \{ v \in V \mid d_0 v = z v \} \). Assume for any \( z \) there exists \( \ell_0 \) such that \( V_{z+\ell} = 0 \) for \( \ell > \ell_0 \)
3. \( \frac{1}{k} x(r, \zeta) k_0 (s, \zeta^m) = x(r + s, \zeta) \)
   \( \frac{1}{k} k_i (s, \zeta^m) k_0 (s, \zeta^m) = k_i (r + s, \zeta^m) \) for \( 0 \leq i \leq N \).

**Remark (1.7)** Condition (3) is not satisfied for most of the modules. But there are enough of them which are sufficient for a realization of \( L(G, \theta) \). For examples vertex operator representation of \([EM]\) satisfy the condition (3) as well as the representations considered in \([BY]\).

**Remark (1.8)** Consider the Lie subalgebra of \( \tilde{L}(G, \theta), \tilde{\mathfrak{h}} = \oplus_{i \in \mathbb{Z}} h_i \otimes t^i \oplus C k_0 \oplus C d_0 (h \in \mathfrak{h}) \). The bracket is given by

\[
[h_i \otimes t^i, h_j \otimes t^j] = \frac{ik_0}{m} < h_i, h_j > \delta_{i+j,0}.
\]
Clearly \( \tilde{\mathfrak{h}} \) is \( \mathbb{Z} \)-graded. Let \( M(k) \) be a Verma module of level \( k \) for \( \tilde{\mathfrak{h}} \). Then it is a standard fact that \( M(k) \) is irreducible whenever \( k \) is non-zero.

**Proposition (1.9)** Any module \( V \) in \( C_k (k \neq 0) \) has the following decomposition as \( \tilde{\mathfrak{h}} \)-modules. \( V \cong M(k) \otimes \Omega_V \) where

\[
\Omega_V = \{ v \in V \mid h_i \otimes t^i v = 0 \text{ for } i > 0 \}
\]
see Proposition 5.4 of \([LW]\).
We now define toroidal \( Z \) algebras. Notation as earlier. For \( \alpha \in \Phi, r \in \mathbb{Z}^N \) let \( Z(\alpha, r, \zeta) \) be a series in \( \zeta \) with integral powers. For \( r \in \mathbb{Z}^N \) let \( k_i(r, \zeta^m) \) be a series in \( \zeta^m \). The toroidal \( Z_k \)-algebra or simply \( Z_k \)-algebra is an algebra generated by the components of \( Z(\alpha, r, \zeta), k_i(r, \zeta^m), h_0 \) and \( d_0, d_1, \ldots, d_N \) by the following relations.

(1.10) **Relations** \( \alpha, \beta, \beta_1, \beta_2 \in \Phi, r, s \in \mathbb{Z}^N \).

1. \( \frac{1}{k} Z(\alpha, r, \zeta) k_0(s, \zeta^m) = Z(\alpha, r + s, \zeta) \)

2. \( \frac{1}{k} k_0(r, \zeta^m) k_i(s, \zeta^m) = k_i(s + r, \zeta^m), \quad 0 \leq i \leq N \).

3. \( \sum_{i=1}^{N} r_i k_i(r, \zeta^m) + \frac{1}{m} Dk_0(r, \zeta^m) = 0 \).

4. \( [d_0, Z(\beta, r, \zeta)] = DZ(\beta, r, \zeta) \)

5. \( [d_0, k_i(r, \zeta^m)] = Dk_i(r, \zeta^m), \quad 0 \leq i \leq N \)

6. \( [d_i, k_j(r, \zeta^m)] = r_i k_j(r, \zeta^m), \quad 1 \leq i \leq N, 0 \leq j \leq N \)

7. \( \prod_{p \in \mathbb{Z}_m} (1 - w^{-p} \zeta_1/\zeta_2)^{<\theta^p \beta_1, \beta_2> / k} Z(\beta_1, r, \zeta_1) Z(\beta_2, s, \zeta_2) \)

8. \( \prod_{p \in \mathbb{Z}_m} (1 - w^{-p} \zeta_2/\zeta_1)^{<\theta^p \beta_2, \beta_1> / k} Z(\beta_2, s, \zeta_2) Z(\beta_1, r, \zeta_1) \)

\[ \begin{align*}
\frac{1}{m} \sum_{\theta^p \beta_1 + \beta_2 = 0} \eta(p, \beta_1) \epsilon(\theta^p \beta_1, \beta_2) Z(\theta^p \beta_1 + \beta_2, r + s, \zeta_2) \delta(w^{-p} \zeta_1/\zeta_2) \\
- \frac{1}{mk} < x_{\beta_2}, x_{-\beta_2} > \sum_{\theta^p \beta_1 + \beta_2 = 0} \eta(p, \beta_1)(\beta_2) k_0(r + s, \zeta^m) \delta(w^{-p} \zeta_1/\zeta_2) \\
+ \frac{1}{m} < x_{\beta_2}, x_{-\beta_2} > \sum_{\theta^p \beta_1 + \beta_2 = 0} \eta(p, \beta_1)(\sum_{i=1}^{N} r_i k_i(r + s, \zeta^m) \delta(w^{-p} \zeta_1/\zeta_2) + \frac{1}{m} k_0(r + s, \zeta^m) D \delta(w^{-p} \zeta_1/\zeta_2)) \end{align*} \]

(8) \( [\alpha, Z(\beta, r, \zeta)] = <\alpha, \beta > Z(\beta, r, \zeta), \quad \alpha \in h_0 \)
\[(9)\] \[Z(\beta, r, \omega^p \zeta) = \eta(\rho, \beta)Z(\theta^p \beta, r, \zeta)\]

\[(10)\] \[k_i(r, \zeta^m)\text{ commutes with } Z(\alpha, r, \zeta)\text{ and } h_0, \quad 0 \leq i \leq N.\]

As it is we do not know whether a \(Z_k\) algebra is non-zero or not but certainly it is well defined.

(1.11) **Definition** A \(Z_k\) module \(V\) is said to be in the category \(\mathcal{D}_k\) if
\begin{enumerate}
  \item \(k_0\) acts by scalar \(k\).
  \item \(V = \oplus_{z \in \mathbb{C}} V_z, V_z = \{v \in V \mid d_0 v = zv\}\). Assume for any given \(z\) there exists \(\ell_0\) such that \(V_{z+\ell} = 0\) for \(\ell > \ell_0\).
\end{enumerate}

**Section 2**

In this section we establish equivalence between the categories \(\mathcal{C}_k\) and \(\mathcal{D}_k\). The proof are very similar to [LW]. In fact most of the results go through.

Let \(V \in \mathcal{C}_k\). Define for \(\beta \in \Phi\).

\[E^\pm(\beta, \zeta) = \exp(\pm m \sum_{j>0} \beta \pm j \otimes t^{\pm j} \zeta^{\pm j}/j k)\]

and \(Z(\beta, r, \zeta) = E^-(\beta, \zeta)x_\beta(r, \zeta)E^+(\beta, \zeta)\). We first prove that these \(Z\)-operators satisfy relations in (1.10).

We first recall the following from section 3 of [LW]. We are taking \(\underline{a} = h\) and \(\underline{m} = 0\) (in decomposition (3.5)) in [LW].

**Proposition (2.1)** \(\alpha, \beta, \gamma \in \Phi\). Let
\begin{enumerate}
  \item \(i > 0\)
  \item \(\alpha_i \otimes t^{i}, E^+(\beta, \zeta) = 0\)
\end{enumerate}
(b) \( [\alpha_{-i} \otimes t^{-i}, E^-(\beta, \zeta)] = 0 \)

(c) \( [\alpha_i \otimes t^i, E^-(\beta, \zeta)] = -<\alpha_i, \beta > \zeta^{-i} E^-(\beta, \zeta) \).

(d) \( [\alpha_{-i} \otimes t^{-i}, E^+(\beta, \zeta)] = -<\alpha_{-i}, \beta > \zeta^i E^+(\beta, \zeta) \)

\[(2) \quad (a) \quad E^{\pm}(\beta + \gamma, \zeta) = E^{\pm}(\beta, \zeta) E^{\pm}(\gamma, \zeta) \]

(b) \( E^{\pm}(\theta^p \beta, \zeta) = E^{\pm}(\beta, w^p \zeta) \)

(c) \( D E^{\pm}(\beta, \zeta) = \frac{m}{k} \beta(\zeta)^{\pm} E^{\pm}(\beta, \zeta) \)

where \( \beta(\zeta)^{\pm} = \sum_{i>0} \beta_{\pm i} \otimes t^{\pm i} \zeta^{\pm i} \)

\[(3) \quad (a) \quad x_\beta(\underline{r}, \zeta) = E^-(\gamma, \zeta) Z(\beta, \underline{r}, \zeta) E^+(\beta, \zeta) \]

(b) \( Z(\beta, \underline{r}, w^p \zeta) = \eta(p, \beta) Z(\theta^p \beta, \underline{r}, \zeta) \)

\[(4) \quad E^+(\beta, \zeta_1) E^-(\gamma, \zeta_2) = E^-(\gamma, \zeta_2) E^+(\beta, \zeta_1) \prod_{p \in Z_m} (1 - w^{-p} \zeta_1 / \zeta_2)^{-<\theta^p \beta, \gamma>/k} \]

\[(5) \quad E^+(\beta, \zeta_1) x_\gamma(\underline{r}, \zeta_2) = x_\gamma(\underline{r}, \zeta_2) E^+(\beta, \zeta_1) \prod_{p \in Z_m} (1 - w^{-p} \zeta_1 / \zeta_2)^{-<\theta^p \beta, \gamma>/k} \]

\[(6) \quad x_\beta(\underline{r}, \zeta_1) E^-(\gamma, \zeta_2) = E^-(\gamma, \zeta_2) x_\beta(\underline{r}, \zeta_1) \prod_{p \in Z_m} (1 - w^p \zeta_1 / \zeta_2)^{-<\theta^p \beta, \gamma>/k}. \]
Proof (1) a and b follows from the definition. To see 1(c) consider
\[
[\alpha_i \otimes t^i, - \sum_{j < 0} \beta_j \otimes t^j/j_k \zeta^j]
\]
\[
= \frac{1}{m} < \alpha_i, \beta_{-i} > \zeta^{-i}.
\]
Now
\[
[\alpha_i \otimes t^i, (- \sum_{j < 0} \beta_j \otimes t^j/j_k \zeta^j)^{\ell}/\ell!]
\]
\[
= - \frac{< \alpha_i, \beta_{-i} >}{m(\ell - 1)!} \zeta^{-i}(\sum_{j < 0} \beta_j \otimes t^j/j_k \zeta^j)^{\ell-1}).
\]
First note that $< \alpha_i, \beta_{-i} > = < \alpha_i, \beta >$ by (1.2). Now 1(c) follows from the definition.
(1)(d) follows from similar argument.
(2) and (3) follows from definition. See also Proposition 3.2 and 3.3 of [LW].
(4) follows from Proposition 3.4 of [LW].
(5) and (6) follows from Proposition 3.5 and 3.6 of [LW].

Corollary (2.2) Let $\tilde{h}^1 = \oplus_{i \neq 0} h_i \otimes t^i$. Then as operators the following hold. (1) $[\tilde{h}^1, Z(\alpha_r, \zeta)] = 0$
(2) $[a_0, Z(\alpha_r, \zeta)] = < a_0, \alpha > Z(\alpha_r, \zeta), a \in \tilde{h}.$

Proof (1) Follows from above. (2) is easy to see.

Proposition (2.3) (Proposition (3.9) of [LW]).

Let $W$ be a vector space and let $f(\zeta_1, \zeta_2) = \sum w_{ij}\zeta_1^i\zeta_2^j$ where each $w_{ij} \in W$ and suppose for some $n \in \mathbb{Z}$ either $w_{ij} = 0$ where one of $i$ or $j > n$ or $w_{ij} = 0$ whenever $i$ or $j < n$.

Set
\[
D_i f(\zeta_1, \zeta_2) = \zeta_i \frac{df}{d\zeta_i}(\zeta_1, \zeta_2)
\]
Then for \( a \neq 0 \)

1. \[
\delta(a\zeta_1/\zeta_2)f(\zeta_1, \zeta_2) = \delta(a\zeta_1/\zeta_2)f(\zeta_1, a\zeta_1)
\]

2. \[
D\delta(a\zeta_1/\zeta_2)f(\zeta_1, \zeta_2) = (D\delta)(a\zeta_1/\zeta_2)f(\zeta_1, a\zeta_1) + \delta(a\zeta_1/\zeta_2)Df(\zeta_1, a\zeta_1)
\]

Proposition (2.4)

\[
\prod_{p \in \mathbb{Z}_m} (1 - w^{-p}\zeta_1/\zeta_2)^{<\theta^p\beta_1, \beta_2>/k} Z(\beta_1, \underline{r}, \zeta_1)Z(\beta_2, \underline{s}, \zeta_2)
\]

\[
- \prod_{p \in \mathbb{Z}_m} (1 - w^{-p}\zeta_2/\zeta_1)^{<\theta^p\beta_2, \beta_1>/k} Z(\beta_2, \underline{s}, \zeta_2)Z(\beta_1, \underline{r}, \zeta_1)
\]

\[
= E^- (\beta_1, \zeta_1)E^- (\beta_2, \zeta_2)[x_{\beta_1}(\underline{r}, \zeta_1), x_{\beta_2}(\underline{s}, \zeta_2)]
\]

\[
E^+(\beta_1, \zeta_1)E^+(\beta_2, \zeta_2)
\]

Proof Consider

\[
Z(\beta_1, \underline{r}, \zeta_1)Z(\beta_2, \underline{s}, \zeta_2)
\]

\[
= E^- (\beta_1, \underline{r}, \zeta_1)x_{\beta_1}(\underline{r}, \zeta_1)E^+(\beta_1, \underline{r}, \zeta_1).
\]

\[
E^- (\beta_2, \underline{s}, \zeta_2)x_{\beta_2}(\underline{s}, \zeta_2)E^+(\beta_2, \underline{s}, \zeta_2)
\]

\[
= \prod_{p \in \mathbb{Z}_m} (1 - w^{-p}\zeta_1/\zeta_2)^{<\theta^p\beta_1, \beta_2>/k}.
\]

\[
E^- (\beta_1, \underline{r}, \zeta_1)x_{\beta_1}(\underline{r}, \zeta_1)E^- (\beta_2, \underline{s}, \zeta_2).
\]

\[
E^+(\beta_1, \underline{r}, \zeta_1)x_{\beta_2}(\underline{s}, \zeta_2)E^+(\beta_2, \underline{s}, \zeta_2)
\]

from 4 of Proposition (2.1).

\[
= \prod_{p \in \mathbb{Z}_m} (1 - w^{-p}\zeta_1/\zeta_2)^{<\theta^p\beta_1, \beta_2>/k}.
\]

\[
E^- (\beta_1, \underline{r}, \zeta_1)E^- (\beta_2, \underline{s}, \zeta_2)x_{\beta_1}(\underline{r}, \zeta_1).
\]

\[
x_{\beta_2}(\underline{s}, \zeta_2)E^+(\beta_1, \underline{r}, \zeta_1)E^+(\beta_2, \underline{s}, \zeta_2)
\]

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by 5 of Proposition (2.1).

Multiplying both sides by the inverse of the first factor on the right, and subtracting the expression obtained by interchanging the roles of the subscripts 1 and 2 we have Proposition (2.4).

**Proposition (2.5)** For these $Z$ operators the relation at (1.10) hold.

**Proof** (2) to (6), holds from definition of $d_i$. Since $V \in \mathbb{C}_k$ we have

$$
\frac{1}{k} x_\beta(r, \zeta)k_0(s, \zeta^m) = x_\beta(r + s, \zeta).
$$

Thus (1) holds from definition of $Z$ operator. (9) holds from Proposition 2.1(3). (6), (10) and (8) are easy to see. We only need to prove (7). The right hand side of the Proposition (2.4) and by using Proposition 1.5 (b) (1) is equal to

$$
E_1 + E_2 + E_3 + E_4
$$

where

$$
E_1 = E^-(\beta_1, \zeta_1)E^-(\beta_2, \zeta_2).
$$

$$
E_2 = E^-(\beta_1, \zeta_1)E^-(\beta_2, \zeta_2).
$$

$$
E_3 = E^+(\beta_1, \zeta_1)E^-(\beta_2, \zeta_2).
$$

$$
E_4 = E^-(\beta_1, \zeta_1)E^+(\beta_2, \zeta_2).
$$
Thus we get

\[ E = \frac{1}{m} \sum_{\theta^p \beta_1 + \beta_2 \in \Phi} \left( \eta(p, \beta_1) E_1(\theta^p \beta_1 + \beta_2, \zeta_2) \right). \]

\[ \epsilon(\theta^p \beta_1 + \beta_2)(x^{\theta^p \beta_1 + \beta_2}(x + s, \zeta_2)\delta(w^{-p}\zeta_1/\zeta_2). \]

\[ E^+(\theta^p \beta_1 + \beta_2, \zeta_2). \]

Now

\[ E_1 = \frac{1}{m} \sum_{\theta^p \beta_1 + \beta_2 \in \Phi} \left( \eta(p, \beta_1) E_1(\theta^p \beta_1 + \beta_2, \zeta_2) \right). \]

\[ \epsilon(\theta^p \beta_1 + \beta_2)(x^{\theta^p \beta_1 + \beta_2}(x + s, \zeta_2)\delta(w^{-p}\zeta_1/\zeta_2). \]

\[ E_2 = -\frac{1}{m} < x_{\beta_2}, x_{-\beta_2} > \sum_{\theta^p \beta_1 + \beta_2 = 0} \eta(p, \beta_1) \beta_2(x + s, \zeta_2)\delta(w^{-p}\zeta_1/\zeta_2) \]

\[ E_3 = \frac{1}{m} < x_{\beta_2}, x_{-\beta_2} > \sum_{\theta^p \beta_1 + \beta_2 = 0} \eta(p, \beta_1) \sum_{i=1}^{N} \eta_i k_1(x + s, \zeta_2)\delta(w^{-p}\zeta_1/\zeta_2). \]

For \( E_4 \) we use Proposition 2.3 (2) (\( a = w^{-p} \)) and Proposition 2.1 (2) (b).
Thus we get

\[ E_4 = \frac{1}{m^2} < x_{\beta_2}, x_{-\beta_2} > \sum_{\theta^p \beta_1 + \beta_2 = 0} \eta(p, \beta_1) k_0(x + s, \zeta_2) \theta^p \delta(w^{-p}\zeta_1/\zeta_2). \]

\[ \frac{-1}{m^2} < x_{\beta_2}, x_{-\beta_2} > \sum_{\theta^p \beta_1 + \beta_2 = 0} \eta(p, \beta_1) \frac{m}{k} \sum_{i \neq 0} \beta_i \overset{\theta^p \zeta_2}{\underset{w^i}Y} t^i (w^p \zeta_2)^i k_0(x + s, \zeta_2) \delta(w^{-p}\zeta_1/\zeta_2) \]

We will use the fact that \( \sum_{i \neq 0} \beta_1_i \overset{\theta^p \zeta_2}{\underset{w}Y} t^i (w^p \zeta_2)^i = \beta_2(\zeta_2) - (\beta_2)_0 \) and the fact that

\[ (2.6) \]

\[ \frac{1}{k} \beta_2(\zeta_2) k_0(x + s, \zeta_2) = \beta_2(x + s, \zeta_2) \]

So we get

\[ E_4 = \frac{1}{m^2} < x_{\beta_2}, x_{-\beta_2} > \sum_{\theta^p \beta_1 + \beta_2 = 0} \eta(p, \beta_1) k_0(x + s, \zeta_2) \theta^p \delta(w^{-p}\zeta_1/\zeta_2). \]

\[ -\frac{1}{mk} < x_{\beta_2}, x_{-\beta_2} > \sum_{\theta^p \beta_1 + \beta_2 = 0} \eta(p, \beta_1) \sum_{i \neq 0} \eta_i \beta_i \overset{\theta^p \zeta_2}{\underset{w^i}Y} t^i (w^p \zeta_2)^i k_0(x + s, \zeta_2) \delta(w^{-p}\zeta_1/\zeta_2) \]

\[ = \frac{1}{m^2} < x_{\beta_2}, x_{-\beta_2} > \sum_{\theta^p \beta_1 + \beta_2 = 0} \eta(p, \beta_1) k_0(x + s, \zeta_2) \delta(w^{-p}\zeta_1/\zeta_2) \]
\[ \frac{-1}{mk} < x_{\beta_2}, x_{-\beta_2} > \sum_{\theta_1, \theta_2 = 0} \eta(p, \beta) \theta^p \beta_1(\zeta_2) k_0(\zeta + \zeta_2^m) \delta(w^{-p}\zeta_1/\zeta_2) \]
\[ + \frac{1}{mk} < x_{\beta_2}, x_{-\beta_2} > \sum_{\theta_1, \theta_2 = 0} \eta(p, \beta_1) \theta^p \beta_2(\zeta_2) k_0(\zeta + \zeta_2^m) \delta(w^{-p}\zeta_1/\zeta_2). \]

Note that the second term is \(-E_2\) by (2.6). Since \(\theta^p \beta_1 + \beta_2 = 0\) we have \((\beta_1)_0 + (\beta_2)_0 = 0\). Thus \(E_2 + E_4\)
\[ = \frac{1}{m^2} < x_{\beta_2}, x_{-\beta_2} > \sum_{\theta_1, \theta_2 = 0} \eta(p, \beta_1) \theta^p \beta_1(\zeta_2) k_0(\zeta + \zeta_2^m) D\delta(w^{-p}\zeta_1/\zeta_2) \]
\[ - \frac{1}{mk} < x_{\beta_2}, x_{-\beta_2} > \sum_{\theta_1, \theta_2 = 0} \eta(p, \beta_2) \theta^p \beta_2(\zeta_2) k_0(\zeta + \zeta_2^m) \delta(w^{-p}\zeta_1/\zeta_2). \]

Now adding \(E_1, E_2, E_3\) and \(E_4\) we get the desired result. Thus we proved that \(\Omega(V) \in D_k\).

Conversely assume that \(W \in D_k\). Let \(V = M(k) \otimes W\). Define \(X_\alpha(r, \zeta) = E^-(\alpha, \zeta) E^+(\alpha, \zeta) \otimes Z(\alpha, r, \zeta)\).
\[ \beta(r, \zeta) = \frac{1}{k} \beta(\zeta) k_0(r, \zeta^m). \]

The central elements to be same. Since \(W \in D_k\). The operators \(X_\alpha(r, \zeta)\) and \(k_0(r, \zeta)\) satisfy
\[ \frac{1}{k} X_\alpha(r, \zeta) k_0(s, \zeta^m) = X_\alpha(r + s, \zeta) \text{ for all } X_\alpha \in G. \]

Conditions 4 to 7 of Proposition (1.6) are easily satisfied as the corresponding conditions are satisfied for \(Z\)-operators.

Condition (3) can be proved exactly as in the proof of Proposition 5.3 of \([LW]\). Condition (1) is satisfied as the same relation holds for \(Z\)-operators and \(Z\) operator commutes with \(E^\pm\)-operators.

Consider \([\beta_1(\zeta_1), \beta_2(\zeta_2)] = \frac{1}{m^2} k \sum_{p \in \mathbb{Z}_m} < \theta^p \beta_1, \beta_2 > D\delta(w^{-p}\zeta_1/\zeta_2).\) See Theorem 2.4 of \([LW]\).
Now by Proposition 2.3 (2) we have

\[
\frac{1}{m^2} \sum_{p \in \mathbb{Z}_m} < \theta^p \beta_1, \beta_2 > k_0(\zeta + \zeta^m_1) D \delta(w^{-p} \zeta_1/\zeta_2)
\]

\[
- \frac{1}{m^2 k} \sum_{p \in \mathbb{Z}_m} < \theta^p \beta_1, \beta_2 > D_1 k_0(\zeta - \zeta^m_1) |_{\zeta_1 = w^p \zeta_2} k_0(\zeta^m_2) \delta(w^{-p} \zeta_1/\zeta_2)
\]

\[
= \frac{1}{m^2} \sum_{p \in \mathbb{Z}_m} < \theta^p \beta_1, \beta_2 > k_0(r + s, \zeta^m_2) D \delta(w^{-p} \zeta_1/\zeta_2)
\]

Thus we have proved the following:

**Proposition 2.6** The category \( C_k \) of \( \tilde{L}(G, \theta) \)-modules are equivalent to the category \( D_k \) of \( Z_k \)-modules.

### Section 3 (Homogeneous picture)

In this section our aim is to construct a faithful representation for the untwisted toroidal Lie algebra \( \tilde{\tau} \) coming from simple, simply connected Lie-algebra \( G \). (First note that on any representation \( \tau \) where centre \( \Omega_A/d_A \) acts faithfully, then \( \tilde{\tau} \) acts faithfully). That is we are giving a realization. This recovers the main result of [EM]. For this we give a representation for the \( Z \)-algebra such that the centre acts faithfully. Thus we have a faithful representation for the toroidal Lie algebra \( \tau \).

We take the automorphism \( \theta = id \). We first give a presentation for the Lie-algebra \( G \). Let \( \tilde{Q} \) be the root lattice spanned by simple roots. The nondegenerate form is chosen so that \( (\alpha, \alpha) = 2 \) for a highest root \( \alpha \). Then it is known that

\[
\Phi = \{ \alpha \in \tilde{Q} | (\alpha, \alpha) = 2 \}
\]
The following cocycle on $\hat{Q} \times \hat{Q}$ is known to exist:

$$
\epsilon : \hat{Q} \times \hat{Q} \rightarrow \{\pm 1\}
$$

3.1

(1) $\epsilon(\alpha, \alpha) = (-1)^{(\alpha, \alpha)}$

(2) $\epsilon(\alpha, \beta)\epsilon(\beta, \alpha) = (-1)^{\langle \alpha, \beta \rangle}$

(3) $\epsilon(\alpha + \beta, \gamma) = \epsilon(\alpha, \gamma)\epsilon(\beta, \gamma)$

(4) $\epsilon(\alpha, \beta + \gamma) = \epsilon(\alpha, \beta)\epsilon(\alpha, \gamma)$

Note that $\epsilon(\alpha, \alpha) = \epsilon(\alpha, -\alpha) = -1$ for $\alpha \in \Phi$. Then there exist vectors $x_\alpha, h_\alpha$ in $G$, $\alpha \in \Phi$ satisfying the following:

3.2

(1) $[x_\alpha, x_\beta] = \epsilon(\alpha, \beta)x_{\alpha+\beta}$, if $\alpha + \beta \in \Phi$

(2) $[x_\alpha, x_\beta] = 0$ if $\alpha + \beta \notin \Phi \cup \{0\}$

(3) $h_\alpha, h_\beta = 0$ if $\alpha + \beta = 0$

(4) $[h, x_\alpha] = \alpha(h)x_\alpha \forall h \in h$

Let $\Gamma$ be a $\mathbb{Z}$-lattice spanned by $\alpha_1, \ldots, \alpha_\ell, \delta_1, \ldots, \delta_N, d_1, \ldots, d_N$. Define a non-degenerate bilinear form on $\Gamma$ extending the one on $\hat{Q}$ by

$$(\hat{Q}, \delta_i) = (\hat{Q}, d_i) = 0$$

$$(d_i, d_j) = (\delta_i, \delta_j) = 0$$

$$(\delta_i, d_j) = \delta_{ij}$$

Any vector which is integral linear combination of $\delta_i$ is called a null root. For $\underline{r} \in \mathbb{Z}^N$, define $\delta_{\underline{r}} = \sum r_i\delta_i$. Note that $\langle \delta_{\underline{r}}, \delta_{\underline{s}} \rangle = 0$. Let $Q$ be the sub lattice spanned by $\hat{Q}$ and $\delta_1, \ldots, \delta_N$. Extend the co-cycle $\epsilon$ to $Q$ by $\epsilon(\alpha, \delta_{\underline{r}}) = 1$.
for $\alpha \in Q$. Now extend $\epsilon$ to $Q \times \Gamma$ to be bimultiplicative in any convenient way. Consider the group algebra $\mathbb{C}[\Gamma]$ and make $\mathbb{C}[\Gamma]$ a $\mathbb{C}[Q]$ module by the following multiplication.

$$e^\alpha \cdot e^\gamma = \epsilon(\alpha, \gamma)e^{\alpha+\gamma}.$$ 

Let $\mathcal{H} = Q \otimes \mathbb{C}$. Let $\mathcal{H}_\pm = \bigoplus_{n>0} \mathcal{H} \otimes t^n$. Consider the Fock space

$$V(\Gamma) = S(\mathcal{H}_-) \otimes \mathbb{C}[\Gamma].$$

Define operators $\alpha(0)$ on $V(\Gamma)$ by

$$\alpha(0) \cdot u \otimes e^\gamma = (\alpha, \gamma)u \otimes e^\gamma, \alpha \in Q.$$ 

For $\delta$ nullroot

$$\delta(n)u \otimes e^\gamma = \delta(n)u \otimes e^\gamma, n \neq 0$$

$\delta(n)u$ is multiplication if $n < 0$ and differentiation if $n > 0$. This is the standard Fock space representation of $\mathcal{H} \otimes \mathbb{C}[t, t^{-1}]$ on $V[\Gamma]$.

For a null root $\delta$ define $E^\pm(\delta, \zeta) = \exp \sum_{n>0} \delta_{\pm n} \zeta^{\pm n}$. Define operators

$$\zeta^{\alpha(0)} u \otimes e^\gamma = \zeta^{(\alpha, \gamma)}u \otimes e^\gamma, \alpha \in Q.$$ 

Consider the vertex operator

$$X(\delta, \zeta) = E^-(\delta, \zeta)\zeta^{\delta(0)}E^+(\delta, \zeta).$$

Let $k_i(\delta, \zeta) = \delta_i(\zeta)X(\delta, z)$ for $1 \leq i \leq N$ and $k_0(\delta, z) = X(\delta, \zeta)$. From [EM] it is known that each $k_i(\delta, z)$ acts non trivially and $Dk_0(\delta, \zeta) + \sum_{i=1}^N r_i k_i(\delta, \zeta) = 0$. Further any relation among $k_i(\delta, \zeta)$ is the one given above. (see Lemma $C$ of [EM]).

We will now define $Z$ operators. Define $Z(\alpha, 0, \zeta) = \zeta^{(\alpha, \zeta)}\zeta^{-\alpha(0)}e^\alpha, \alpha \in \Phi$. Then define $Z(\alpha, \ell, \zeta) = Z(\alpha, 0, \zeta)k_0(\ell, \zeta)$. $d_0, d_1, \cdots, d_N$ are defined naturally as grading on $Z(\alpha, \ell, \zeta)$. 20
We will now check the relation at (1.10) for the above Z-operator. (1) to (6) are clearly satisfied from definition. We will rewrite the relation (7) using the fact that $\theta = Id$ and $m = 1$. Notice also $X(0, \zeta) = 1$ and hence $k_0$ acts as 1 so that $k = 1$.

**(3.3)**

$$(1 - \zeta_1/\zeta_2)^{<\beta_1, \beta_2>} Z(\beta_1, \underline{r}, \zeta_1) Z(\beta_2, \underline{s}, \zeta_2) - (1 - \zeta_2/\zeta_1)^{<\beta_1, \beta_2>} Z(\beta_2, \underline{s}, \zeta_2) Z(\beta_1, \underline{r}, \zeta_1)$$

$$= \begin{cases} 
\epsilon(\beta_1, \beta_2) Z(\beta_1 + \beta_2, \underline{r} + \underline{s}, \zeta_2) \delta(\zeta_1/\zeta_2) & \text{if } \beta_1 + \beta_2 \in \Phi \\
- \langle x_{\beta_2}, x_{-\beta_2} \rangle \beta_2 k_0(\underline{r} + \underline{s}, \zeta_2) \delta(\zeta_1/\zeta_2) \\
+ \langle x_{\beta_2}, x_{-\beta_2} \rangle \left( \sum_{i=1}^{N} r_i k_i(\underline{r} + \underline{s}, \zeta_2) \delta(\zeta_1/\zeta_2) + k_0(\underline{r} + \underline{s}, \zeta_2) D\delta(\zeta_1/\zeta_2) \right) & \text{if } \beta_1 + \beta_2 = 0 \\
= 0 & \text{if } \beta_1 + \beta_2 \notin \Phi \cup \{0\}. 
\end{cases}$$

Suppose $\underline{r} = 0$ and $\underline{s} = 0$ then (3.3) follows from Theorem 5.3 of [LM]. For general $\underline{r}$ and $\underline{s}$ consider left hand side of (3.3) which is equal to

$$(1 - \zeta_1/\zeta_2)^{<\beta_1, \beta_2>} Z(\beta_1, 0, \zeta_1) k_0(\underline{r}, \zeta_1) Z(\beta_2, 0, \zeta_2) k_0(\underline{s}, \zeta_2)$$

$$- (1 - \zeta_2/\zeta_1)^{<\beta_1, \beta_2>} Z(\beta_2, 0, \zeta_2) k_0(\underline{s}, \zeta_2) Z(\beta_1, 0, \zeta_1) k_0(\underline{r}, \zeta_1)$$

$$= k_0(\underline{r}, \zeta_1) k_0(\underline{s}, \zeta_2) Z(\beta_1, \beta_2)$$

where

$$Z(\beta_1, \beta_2) = \begin{cases} 
\epsilon(\beta_1, \beta_2) Z(\beta_1 + \beta_2, 0, \zeta_1) \delta(\zeta_1/\zeta_2) & \text{if } \beta_1 + \beta_2 \in \Phi \\
- \langle x_{\beta_2}, x_{-\beta_2} \rangle (-\beta_2 \delta(\zeta_1/\zeta_2) + D\delta(\zeta_1/\zeta_2)) & \text{if } \beta_1 + \beta_2 = 0 \\
0 & \text{if } \beta_1 + \beta_2 \notin \Phi \cup \{0\}. 
\end{cases}$$

This follows from case $\underline{r} = 0 = s$. Now the case $\beta_1 + \beta_2 \in \Phi$, (3.3) follows from Proposition 2.3 (1). The case $\beta_1 + \beta_2 \notin \Phi \cup \{0\}$ is very standard as $< \beta_1, \beta_2 > \geq 0$. For the case $\beta_1 + \beta_2 = 0, - \langle x_{\beta_2}, x_{-\beta_2} \rangle > k_0(\underline{r}, \zeta_1) k_0(\underline{s}, \zeta_2) \beta_2 \delta(\zeta_1/\zeta_2)$ is equal to the first term of 3.3 which follows from Proposition 2.3(1).

Now
\[ <x_{\beta_2}, x_{-\beta_2} > k_0(\zeta_1)k_0(\zeta_2).D\delta(\zeta_1/\zeta_2) \]
\[ = <x_{\beta_2}, x_{-\beta_2} > (k_0(\zeta_1 + \zeta_2)D\delta(\zeta_1/\zeta_2) + \sum r_i k_i(\zeta_1 + \zeta_2)\delta(\zeta_1/\zeta_2)). \]

By proposition 2.3(2). This completes the proof of (3.3).

### Section 4  Principal realization

Recall that \( \mathcal{G} \) is simple finite dimensional Lie algebra and \( <,> \) a non-degenerate bilinear form an \( \mathcal{G} \). Let \( \eta \) be a finite order automorphism of order \( p \). Consider the affine Lie algebra \( \mathcal{G} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C} \mathcal{C} \) with Lie bracket \([x \otimes t^m, y \otimes t^n] = [x, y] \otimes t^{m+n} + \frac{x \otimes m \delta_{m+n,0}}{p} \mathcal{C} \). Let \( \mathcal{G}(\eta) \) be the corresponding twisted affine Lie algebra. See [Ka] for details.

Let \( \pi \) be an automorphism of order \( K (= 1, 2 \text{ or } 3) \) induced by an automorphism of the Dynkin diagram of \( \mathcal{G} \) with respect to some Cartan subalgebra \( h \) of \( \mathcal{G} \). Let \( \epsilon \) be \( K \)-th primitive root. We will now extend the automorphism \( \pi \) to \( \mathcal{G} \otimes \mathcal{A} \oplus \Omega_\mathcal{A}/d_\mathcal{A} = \tau \) by

(4.1)

\[
\pi(x t^{r_0} t^\mathcal{L}) = \epsilon^{-r_0} \pi(x) t^{r_0} t^\mathcal{L} \\
\pi(t^{r_0} t^\mathcal{L} k_i) = \epsilon^{-r_0} t^{r_0} t^\mathcal{L} k_i, \quad 0 \leq i \leq N.
\]

The aim of this section is to prove that \( L(\mathcal{G}, \pi) \cong L(\mathcal{G}, \theta) \) where \( \theta \) is a special automorphism depending on \( \pi \). This is a generalisation of the standard principal realization of affine Lie-algebras given in [KKLW].

To do this we first have to define the automorphism \( \theta \). For \( i \in \mathbb{Z} \), let \( \mathcal{G}[i] \) be the \( \epsilon^i \) eigenspace of \( \mathcal{G} \). Then the fixed point space \( \mathcal{G}[0] \) is a simple Lie-subalgebra of \( \mathcal{G} \) and \( \mathcal{G}[0] \) module \( \mathcal{G}[1] \) and \( \mathcal{G}[-1] \) are irreducible and contra-gradient.

Fix a Cartan subalgebra \( \mathcal{L} \) of \( \mathcal{G}[0] \) inside \( \mathcal{h} \). Let \( H_j, E_j, F_j \ (1 \leq j \leq \ell) \) be a corresponding set of canonical generators of \( \mathcal{G}[0] \). Let \( E_0 \) be the lowest
weight vector of $G_{[0]}$ module $G_{[1]}$, and let $F_0$ be the highest weight vector $G_{[0]}$-module $G_{[-1]}$, normalised so that $[H_0, F_0] = 2F_0$ where $H_0 = [E_0, F_0]$. Let $\psi_1, \cdots, \psi_\ell \in t^*$ be simple roots of $G_{[0]}$, and let $\psi_0 \in t^*$ be the lowest weight of the $G_{[0]}$ module $G_{[1]}$. For $i, j = 0, 1, \cdots, \ell$ set $A_{ij} = \psi_j(H_i)$. Then it is known that $A = (A_{ij})$ is an indecomposable affine Cartan matrix (see [LW] and [KKLW]). Let $a_0, \cdots, a_\ell, a_0^1, \cdots, a_\ell^1$ be positive integers such that

\begin{equation}
(4.2) \quad A(a_0, \cdots a_\ell)^T = 0 \quad (a_0^1, \cdots, a_\ell^1)A = 0 \quad \text{and} \quad g.c.d(a_0, \cdots, a_\ell) = 1 \quad g.c.d(a_0^1, \cdots, a_\ell^1) = 1.
\end{equation}

Then $a_0, a_1, \cdots, a_\ell$ are precisely the indices of the Dynkin diagram of $A$. (see Table $K$ of [KKLW]).

\begin{equation}
(4.3) \quad \text{Note that from above tables we see that $a_0 = 1$ always. Recall from [LW] that}
\end{equation}

\begin{equation}
(4.4) \quad \sum_{j=0}^{\ell} a_j \psi_j = 0 \quad \text{and} \quad \sum_{j=0}^{\ell} a_j^1 H_j = 0
\end{equation}

\begin{equation}
(4.5) \quad \text{Proposition (KKLW) \quad The Lie subalgebra $G(\pi)$ of $G \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}C$ generated by $E_i, F_i, H_i (1 \leq i \leq \ell), E_0 \otimes t, F_0 \otimes t^{-1}, H_0$ and $C$ is isomorphic to the affine Lie algebra corresponding to $A$.}
\end{equation}
Let $\mathbf{s} = (s_0, \cdots, s_\ell)$ be a sequence of non-negative integers, not all 0.

Take $m = K\sum_{j=0}^\ell s_j a_j$.

Define an automorphism $\theta$ of $G$ by the condition

$$\theta H_i = H_i, \theta E_j = w^{s_j} E_j.$$ 

where $w$ is $m$th root of unity.

Then $\theta$ defines an automorphism of $G$ of order $m$. See [LW]. Then from section 1 we can define $\mathcal{T}(G, \theta)$. Our aim in this section is to prove $\mathcal{T}(G, \pi) \cong \mathcal{T}(G, \theta)$ which is what we call principal realization of toroidal Lie algebras.

(4.6) Note that the $G$-invariant bilinear form $\langle \cdot, \cdot \rangle$ on $G$ is necessarily $\theta$ and $\pi$ invariant. This form $\langle \cdot, \cdot \rangle$ remains non-singular on the Cartan subalgebra $\mathfrak{t}$ of $G_{[0]}$. See section 8 of [LW].

Using the restricted form we identify $\mathfrak{t}$ and $\mathfrak{t}^*$. We normalise the form $\langle \cdot, \cdot \rangle$ such that

\begin{equation}
\langle \psi_0, \psi_0 \rangle = \frac{2a_0^1}{K}.
\end{equation}

Then we have $a_j^1 = K \langle \psi_j, \psi_j \rangle a_j/2$ for $j = 0, \cdots, \ell$. (see [LW]).

Now we have the following $s$-realization of the affine Lie algebra $G(\pi)$ from [KKLW].

(4.8) Proposition ([LW], [KKLW])

Let $e_j = E_j \otimes t^{s_j}$, $f_j = F_j \otimes t^{-s_j}$, $h_j = H_j \otimes 1 + 2s_j \langle \psi_j, \psi_j \rangle^{-1} m^{-1} C$ inside $G(\theta)$. Then there is an isomorphism of affine Lie algebras $\varphi : G(\pi) \rightarrow$
$G(\theta)$ defined by

\[
\begin{align*}
\varphi(E_i \otimes 1) &= e_i \quad 1 \leq i \leq \ell \\
\varphi(F_i \otimes 1) &= f_i \quad 1 \leq i \leq \ell \\
\varphi(E_0 \otimes t) &= e_0 \\
\varphi(F_0 \otimes t^{-1}) &= f_0 \\
\varphi(H_i) &= h_i \quad 1 \leq i \leq \ell \\
\varphi(H_0 + \frac{<E_0,F_0>}{K}C) &= h_0.
\end{align*}
\]

Further $e_i, f_i, h_i \ (0 \leq i \leq \ell)$ forms a set of canonical generators for the affine Lie-algebra $G(\theta)$. Here the Lie bracket $G(\pi)$ is defined by the bilinear form $\frac{1}{K}<>$ and the Lie bracket in $G(\theta)$ is defined by $\frac{1}{m} <,>$. As we are interested in the principal realization we take $\underline{s} = (1, 1, \cdots, 1)$.

**Remark.** The $\pi$-invariants of $h$ equal to $t$. In particular they are spanned by $H_i, 1 \leq i \leq \ell$.

**Proposition** Let $w$ be the Chevalley involution automorphism of $G$. Let $\varphi : G(\pi) \rightarrow G(\theta)$ be the isomorphism of Lie-algebras given earlier. Then the following hold.

1. $\varphi(C) = C$

2. $\varphi(x_\alpha \otimes t^{r_0}) = x_\alpha \otimes t^{N(\alpha) + \frac{m}{K}r_0}$ where $x_\alpha \otimes t^{r_0}$ is a real root vector of $G(\pi)$. $N(\alpha)$ is an integer independent of $r_0$ but depends on $\alpha$.

3. Let $h_\alpha = [x_\alpha, w(x_\alpha)]$ then

   \[
   \varphi(h_\alpha \otimes t^{r_0}) = h_\alpha \otimes t^{\frac{m}{K}r_0} + <x_\alpha, w(x_\alpha)> \frac{N(\alpha)}{m} \delta_{r_0,0}
   \]

4. $<x_\alpha, x_\beta> \neq 0$ implies $N(\alpha) + N(\beta) = 0$

5. $<x_\alpha, x_\beta> = 0$ then
\[ \varphi([x_\alpha, x_\beta] \otimes \mathcal{t}^{r_0 s_0}) = [x_\alpha, x_\beta] \otimes \mathcal{t}^{(r_0 s_0) \frac{r}{2} + N(\alpha) + N(\beta)}. \]

Here \([x_\alpha, x_\beta]\) could be part of real or imaginary root.

**Proof**  Let \(x_\alpha \otimes \mathcal{t}^{r_0}\) be a real root vector of \(G(\theta)\). Then \(G(\theta)\) is spanned by \(x_\alpha \otimes \mathcal{r}^{r_0}, [x_\alpha, x_\beta] \otimes \mathcal{t}^{r_0}\) and \(C\). We have from (4.4)

\[ H_0 = -\sum_{j=1}^{\ell} \frac{a_1^1}{a_0^j} H_j. \]

From Proposition (4.8) we have

\[ H_0 \otimes 1+2 \langle \psi_0, \psi_0 \rangle^{-1} m^{-1} C = h_0 = [E_0 \mathcal{t}, F_0 \mathcal{t}^{-1}] = [E_0, F_0] + \langle E_0, F_0 \rangle > \frac{1}{m} C. \]

(4.11) This implies \(\langle E_0, F_0 \rangle = \frac{2}{\langle \psi_0, \psi_0 \rangle}.\) Consider

\[ \varphi(H_0) = -\sum_{j=1}^{\ell} \frac{a_1^1}{a_0^j} \varphi(H_j) \]
\[ = -\sum_{j=1}^{\ell} \frac{a_1^1}{a_0^j} (H_j + \frac{2}{\langle \psi_j, \psi_j \rangle} \frac{1}{m} C) \]
\[ = H_0 - \sum_{j=1}^{\ell} \frac{a_1^1}{a_0^j} \frac{1}{m} a_j^j KC \]
\[ = H_0 - \sum_{j=1}^{\ell} \frac{a_j^j KC}{a_0^j m} \]
\[ = H_0 - \frac{m - a_0^j K}{a_0^j m} C \]

But \(\varphi(H_0 + \frac{\langle E_0, F_0 \rangle}{K} C) = H_0 + \frac{2}{\langle \psi_0, \psi_0 \rangle} \frac{1}{m} C\)

(by Proposition 4.8)

\[ \frac{\varphi(C)}{a_0^j} = \frac{2}{K \langle \psi_0, \psi_0 \rangle} \frac{1}{m} C - \varphi(H_0) \]
\[ = \frac{2}{\langle \psi_0, \psi_0 \rangle} \frac{1}{m} C + \frac{1}{a_0^j} C^j - \frac{a_0^j K}{a_0^j m} C \]

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(We are using (4.11), (4.7)). This implies \( \varphi(C) = C \) by (4.3).

(2) Clearly \( \varphi(x_a \otimes t^0) = x_a \otimes t^N \) for some integer. Write \( N = N(\alpha) + r_0 \frac{m}{K} \) for some integer \( N(\alpha) \) which may depend on \( r_0 \). Clearly \( \varphi(w(x_a) \otimes t^{-r_0}) = w(x_a) \otimes t^{-r_0} \frac{m}{K} - N(\alpha) \). Let \( h_\alpha = [x_a, w(x_a)] \). Consider the following in \( G(\pi) \).

\[
[x_a \otimes t^0, w(x_a) \otimes t^{-r_0}] = h_\alpha + \frac{<x_a, w(x_a) > r_0 C}{K}
\]

Now apply \( \varphi \) both sides and the bracket takes place in \( G(\theta) \)

\[
\varphi(h_\alpha + \frac{r_0}{K} < x_a, w(x_a) > C) = h_\alpha + \frac{<x_a, w(x_a) >}{m} (N(\alpha) + \frac{m}{K} r_0) C.
\]

Since \( \varphi(C) = C \) we have

\[
\varphi(h_\alpha) = h_\alpha + < x_a, w(x_a) > \frac{N(\alpha)}{m} C.
\]

As \( h_\alpha \) is independent of \( r_0 \) it follows that \( N(\alpha) \) does not depend on \( r_0 \). Now consider the following in \( G(\pi) \).

\[
[x_a \otimes t^{r_0}, w(x_a) \otimes t^{\bar{s}_0}] = h_\alpha \otimes t^{r_0 + s_0} + \frac{1}{K}(x_a, w(x_a) r_0 \delta r_0 + s_0, 0) C.
\]

As earlier apply \( \varphi \) both sides

\[
\varphi(h_\alpha \otimes t^{r_0 + s_0}) = h_\alpha \otimes t^{r_0 + s_0} + < x_a, w(x_a) > \frac{N(\alpha)}{m} \delta r_0 + s_0, 0) C.
\]

This proves (3). For (4) suppose \( < x_a, x_\beta > \neq 0 \). Since \( <,> \) is \( t \)-invariant, it follows that \( \alpha + \beta \) is zero root. (root with respect to \( t \)). Thus \([x_a, x_\beta]\) is a part of imaginary root and so \([x_a, x_\beta] \in \mathfrak{h} \). Since \( \pi \) is an automorphism we have

\[
< \pi(x_a), \pi(x_\beta) > = < x_a, x_\beta > \neq 0.
\]

Let \( e \) be the \( K \) th root of unity.

Let \( \pi(x_a) = e^i x_a \) and \( \pi(x_\beta) = e^j x_\beta \). Since \( < x_a, x_\beta > \neq 0 \) it follows that \( i + j \equiv 0(K) \). Thus \([x_a, x_\beta] \) is \( \pi \)-invariant. Consider

\[
[x_a t^{r_0}, x_\beta t^{\bar{s}_0}] = [x_a, x_\beta] t^{r_0 + s_0} + \frac{1}{K} < x_a, x_\beta > r_0 \delta r_0 + s_0, 0) C
\]

\[
\varphi([x_a, x_\beta] t^{r_0 + s_0}) = [x_a, x_\beta] t^{r_0 + s_0} + \frac{1}{K} < x_a, x_\beta > r_0 \delta r_0 + s_0, 0) C + < x_a, x_\beta > NC \text{ for some } N.
\]

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Now from (3) and (4.9) it follows that

$$\varphi([x_\alpha, x_\beta]t^{r_0+s_0}) = [x_\alpha, x_\beta]t^{(r_0+s_0)m}.$$  

This forces $N(\alpha) + N(\beta) = 0$.

(5) is clear.

Now we define an isomorphism between $\overline{L}(\mathcal{G}, \pi)$ and $\overline{L}(\mathcal{G}, \theta)$.

(4.12) Proposition  The following map $\varphi$ define an isomorphism from $\overline{L}(\mathcal{G}, \pi)$ to $\overline{L}(\mathcal{G}, \theta)$.

1. $\varphi(x_\alpha \otimes t^{r_0}t^0) = x_\alpha \otimes t^{r_0+\frac{m\alpha}{m}N(\alpha)}t^0$  
2. $\varphi(h_\alpha t^{r_0}t^0) = h_\alpha t^{r_0+\frac{m\alpha}{m}N(\alpha)}t^0 + <x_\alpha, x_\beta> w(x_\alpha) > N(\alpha)t^0 \frac{m\alpha}{m}t^0k_0$  
3. $\varphi(t^{r_0}t^0k_i) = t^{r_0+\frac{m\alpha}{m}N(\alpha)+N(\beta)+t^0k_i}$, $0 \leq i \leq N$  
4. if $<x_\alpha, x_\beta> = 0$
5. $\varphi([x_\alpha, x_\beta]t^{r_0}t^0) = [x_\alpha, x_\beta]t^{r_0+\frac{m\alpha}{m}N(\alpha)+N(\beta)+t^0}t^0.$

Proof  In view of earlier Proposition the right hand side belongs to $\overline{L}(\mathcal{G}, \theta)$ except possible for (3). For (3) $t^{r_0}t^0k_i \in \overline{L}(\mathcal{G}, \pi)$ which means $r_0 \equiv (K)$. Thus $r_0 \frac{m\alpha}{m} \equiv 0(m)$ which means $t^{r_0+\frac{m\alpha}{m}N(\alpha)}t^0k_i$ belongs $\overline{L}(\mathcal{G}, \theta)$. The fact that $\varphi$ defines an isomorphism follows by the corresponding isomorphism of the earlier proposition. We will verify one bracket. Consider

$$4.13 \quad [x_\alpha t^{r_0}t^0, x_\beta t^{s_0}t^0] = [x_\alpha, x_\beta]t^{r_0+s_0}t^0 + \frac{<x_\alpha, x_\beta>}{K}t^{r_0+s_0}t^0k_0 + <x_\alpha, x_\beta> \sum r_it^{r_0+s_0}t^{r+s}k_i$$

Suppose $<x_\alpha, x_\beta> = 0$. Then the $\varphi$ of both sides are equal. Suppose $<x_\alpha, x_\beta> \neq 0$, then by previous proposition it follows that $[x_\alpha, x_\beta]$ is in $\mathcal{h}$
and \( \pi \)-invariant. Further \( N(\alpha) + N(\beta) = 0 \).

\[
\left[ \varphi(x_\alpha t^{r_0} t^\xi), \varphi(x_\beta t^{s_0} t^\xi) \right] = \left[ x_\alpha t^{r_0 \frac{m}{K} + N(\alpha) t^\xi}, x_\beta t^{s_0 \frac{m}{K} + N(\beta) t^\xi} \right] = \\
= \left[ x_\alpha, x_\beta \right] t^{(r_0 + s_0) \frac{m}{K} t^{r+s} k_0} + <x_\alpha, x_\beta> (N(\alpha) + \frac{rm}{K}) t^{(r_0 + s_0) \frac{m}{K} t^{r+s} k_0} + <x_\alpha, x_\beta> \sum r_i t^{(r_0 + s_0) \frac{m}{K} t^{r+s} k_i},
\]

which is exactly equal to \( \varphi \) of the right hand side of (4.13).

(4.14) Proposition \( \overline{L}(G, \pi) \) is the universal central extension of \( L(G, \pi) \).

Follows from Remark (2.4) of \([BK]\).

In the next section we give a faithful realisation to \( \overline{L}(G, \theta) \) thereby giving a realization to \( L(G, \pi) \) where the infinite dimensional centre acts faithfully.

Section 5 Principal picture.

In this section we construct level one module for the toroidal Lie-algebra of type \( A^K_n, D^K_n \) and \( E^K_n \). What we do is to construct representation for the \( Z \) algebras where the centre acts faithfully. That in turn constructs module for toroidal algebras of type ADE. This also covers the twisted case which is new result.

Notation as in section 4. Consider the cyclic element \( E = \sum_{i=1}^\ell E_i \in \mathcal{G}(1) \). We make the assumption that the \( \theta \)-stable Cartan subalgebra \( t \) is the centraliser of \( E \). (See [LW] for details).

Let \( t_0 = t \oplus CC \oplus Cd. \) Let \( L(\lambda) \) be a basic module for \( \mathcal{G}(\theta) \) where \( \lambda \in t_0^* \). We renormalize the root vector \( x_\beta, \beta \in \Phi \) such that \( [x_\beta, x_{-\beta}] = -2/ <\beta, \beta> \) and \( \eta(p, \beta) = 1 \) for all \( p \in \mathbb{Z}_m \) and for all \( \beta \in \Phi \). As in Theorem 8.7 of [LW] choose coset representatives \( \beta_1, \cdots, \beta_\ell \) for the action \( \theta \) on \( \Phi \) such that \( (\beta_1)_0, \cdots, (\beta_\ell)_0 \) is a basis for \( \ell \). Let \( C_j = \lambda((x_\beta)_0) \).

Since \( L(\lambda) \) is a basic module, we have \( \dim \Omega_{L(\lambda)} = 1 \). From section 8 of [LW] we have operators \( Z(\beta, \zeta) \) acting on \( \Omega_{L(\lambda)} \).
Proposition (5.1) We have the following from Section (8) of [LW].

(1) \( \dim \Omega_{L(\lambda)} = 1 \)

(2) \( Z(\beta_j, \zeta) = C_j \) on \( \Omega_{L(\lambda)} \)

(3) \( Z(\theta^p \beta_j, \zeta) = Z(\beta_j, w^p \zeta) \)

(4) \[
\prod_{p \in \mathbb{Z}_m} (1 - w^{-p} \zeta_1 / \zeta_2)^{<\theta^p \beta_1, \beta_2>} \frac{Z(\beta_1, \zeta_1) Z(\beta_2, \zeta_2)}{Z(\beta_1, \zeta_1) Z(\beta_2, \zeta_2)}
\]

\[
= \prod_{p \in \mathbb{Z}_m} (1 - w^{-p} \zeta_2 / \zeta_1)^{<\theta^p \beta_2, \beta_1>} \frac{Z(\beta_2, \zeta_2) Z(\beta_1, \zeta_1)}{Z(\beta_2, \zeta_2) Z(\beta_1, \zeta_1)}
\]

\[
= \frac{1}{m} \sum_{\theta^p \beta_1 + \beta_2 \in \Phi} \epsilon(\theta^p \beta_1, \beta_2) Z(\theta^p \beta_1 + \beta_2, \zeta_2) \delta(w^{-p} \zeta_1 / \zeta_2)
\]

\[
= -2m^{-2} \langle \beta_1, \beta_1 \rangle^{-1} \sum_{\theta^p \beta_1 + \beta_2 = 0} D \delta(w^{-p} \zeta_1 / \zeta_2).
\]

Proof (1), (2) and (3) follows from Section 8 of [LW]. For that just note that \( \eta(p, \beta) = 1 \). (4) follows from Theorem 8.7 of [LW] as \( Z \) operator defined in (2) satisfy (8.21) of [LW]. We are also using the fact that \( a_0 = 1 \).

Let \( \Gamma \) be \( \mathbb{Z} \)-lattice spanned by \( \delta_1, \ldots, \delta_N, d_1, \ldots, d_N \) with bilinear form \( (\delta_i, d_j) = \delta_{ij} \) and \( (\delta_i, \delta_j) = (d_i, d_j) = 0 \). Let \( H = \bigoplus \mathbb{C} \delta_i \) and \( H_+ = \bigoplus_{n \in \mathbb{Z}_+, i} \mathbb{C} \delta_i(n) \) Consider the symmetric algebra \( S(H_+) \).

Consider the space \( V(\Gamma) = S(H_+) \otimes e^\Gamma \) where \( e^\Gamma \) group algebra.

Let
\[
E^+(\delta, \zeta^m) = \exp \sum_{n > 0} \frac{\zeta(n)}{n} \zeta^{mn}
\]

\[
E^-(\delta, \zeta^m) = \exp \sum_{n > 0} \frac{\zeta(-n)}{-n} \zeta^{-mn}
\]

which act on \( S(H_+) \) and on \( V(\Gamma) \).

Let \( \delta(0) \) act on \( V(\Gamma) \) by

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\[
\delta(0)u \otimes e^r = (\delta, r)u \otimes e^r \\
\zeta^{\delta(0)}u \otimes e^r = \zeta^{(\delta, r)}u \otimes e^r \\
Cu \otimes e^r = 1u \otimes e^r
\]

Consider \(X(\delta, \zeta^m) = E^{-}(\delta, \zeta^m)\zeta^{m\delta(0)}E^{+}(\delta, \zeta^m)\)

Let \(r \in \mathbb{Z}^N\) and let \(\delta_r = \sum r_i \delta_i\)

Let \(k_0(\underline{u}, \zeta^m) = X(\delta_r, \zeta^m)\)

Let \(\delta(\zeta^m) = \sum \zeta^{mm}\)

Let \(k_1(\underline{u}, \zeta^m) = \delta_i(\zeta^m)X(\delta_r, \zeta^m)\). Then by standard argument one can prove that

\[(5.2) \quad Dk_0(\underline{u}, \zeta^m) = -m \sum_{i=1}^{N} r_ik_1(\underline{u}, \zeta^m).\]

Now define \(Z(\alpha, \underline{u}, \zeta) = Z(\alpha, \zeta)k_0(\underline{u}, \zeta)\). Now we will check all the relation define at (1.10). Remember \(k = 1\). (1) is true by definition. (2) is true by the fact that \(X(\delta_1, \zeta^m)X(\delta_2, \zeta^m) = X(\delta_1 + \delta_2, \zeta^m)\) (3) is just (5.2) (4), (5), (6) can be easily checked. (8) is clear, (9) is true as \(\eta(p, \beta) = 1\). (10) is true by definition. Thus it remains to prove (7). To see this multiply 4 of Prop 5.1 by \(k_0(\underline{u}, \zeta_1^m)k_0(\underline{u}, \zeta_2^m)\). Consider \(k_0(\underline{u}, \zeta_1^m)k_0(\underline{u}, \zeta_2^m)D\delta(w^{-p}\zeta_1/\zeta_2)\) which is equal to (from Proposition 2.3(2))

\[
D\delta(w^{-p}\zeta_1/\zeta_2)k_0(\underline{u}, \zeta_1^m)k_0(\underline{u}, \zeta_2^m) \\
-\delta(w^{-p}\zeta_1/\zeta_2) \left( \frac{d}{d\zeta'}(k_0(\underline{u}, \zeta_1^m)k_0(\underline{u}, \zeta_2^m) |_{\zeta = w^{-p}\zeta_2}) \right) \\
= D\delta(w^{-p}\zeta_1/\zeta_2)k_0(\underline{u} + \underline{\zeta}, \zeta_2^m) \\
+ \delta(w^{-p}\zeta_1/\zeta_2)m \sum r_ik_1(\underline{u}, \zeta_2^m)k_0(\underline{u}, \zeta_2^m) \\
= D\delta(w^{-p}\zeta_1/\zeta_2)k_0(\underline{u} + \underline{\zeta}, \zeta_2^m) \\
+ m\delta(w^{-p}\zeta_1/\zeta_2) \sum r_ik_1(\underline{u} + \underline{\zeta}, \zeta_2^m)
\]

Now (7) of (1.10) follows from the earlier arguments.
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