Voting power and Qualified Majority Voting with a “no vote” option

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Summary. In recent years, enlargement of the European Union has led to increased interest in the allocation of voting weights to member states with hugely differing population numbers. While the eventually agreed voting scheme lacks any strict mathematical basis, the Polish government suggested a voting scheme based on the Penrose definition of voting power, leading to an allocation of voting weights proportional to the square root of the population (the "Jagielonian Compromise"). The Penrose definition of voting power is derived from the citizens’ freedom to vote either “yes” or “no”. This paper defines a corresponding voting power based on “yes”, “no” and “abstain” options, and it is found that this definition also leads to a square root law, and to the same optimal vote allocation as the Penrose scheme.

Keywords: Penrose voting; Qualified Majority Voting; Square root voting; Voting power; Voting systems

1. Introduction

Following the failure of the draft EU constitution in referenda in France and the Netherlands, and the subsequent negotiations on a Reform Treaty, the voting arrangements in the Council of the EU have received a significant amount of public attention, not least through the Polish proposal of a voting system that gives every member state a voting weight proportional to the square root of its population. The idea of this voting scheme is to give every EU citizen the same influence on decisions in the Council, based on an analysis of their voting power. The concept of voting power used in this context was first introduced by Penrose (1946), and adapted to the EU framework by Zyczkowski and Slomczynski (2004). In this scheme for Qualified Majority Voting, the threshold for a motion to pass in the Council of the EU is then set according to an optimality condition, see Zyczkowski and Slomczynski (2004), and the details.

The Penrose definition of voting power assumes that all citizens have the freedom to vote either “yes” or “no” in elections or referenda. However, this definition of voting power ignores the freedom not to vote at all, which is a freedom very frequently used by the citizens of EU member states, as voter turnout in elections is generally much less than 100%.

The purpose of this paper is to provide an analogous definition of voting power which includes this “no vote” option. In section 2.1 a brief introduction to the Penrose voting power scheme is given, while in section 2.2 it is generalised to the case where citizens have the possibility to abstain from voting. A short conclusion summarises the result.

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2 Voting power

Wherever constituencies with different numbers of voters elect representatives for a council where the representatives for each constituency have to cast a joint vote, the question of allocation of voting weights to the constituencies arises. This is, for example, the case in the European Union, where population numbers vary between about 400,000 (Malta) and 82,300,000 (Germany), see EuroStat (2007). While it may be intuitive to allocate voting weights proportional to the population numbers, there is actually no sound mathematical foundation for such a scheme. What is needed is a rigorous definition of citizens’ voting power.

2.1 Penrose voting power

Penrose defined voting power as the probability that the vote of a single voter is decisive, provided all other voters vote randomly, in an election where voters choose between “yes” and “no”.

More precisely, let us assume there are \( N + 1 \) voters in the country, and a motion is successful if it receives more “yes” than “no” votes. Whether or not the vote of a single voter is decisive depends on how the remaining \( N \) voters have voted. For a single voter to be decisive, the motion must be successful if they vote “yes”, and fail if they vote “no”. If \( N \) is even, this is the case when the \( N \) votes are evenly split between “yes” and “no”, if \( N \) is odd, the individual voter will be decisive if there is one “yes” vote more than there are “no” votes.

Penrose’s fundamental assumption is that these \( N \) voters vote randomly, i.e. all \( 2^N \) possible voting outcomes are equally likely.

The number of voting outcomes where the remaining voter is decisive is given by the binomial coefficients

\[
\binom{N}{N/2} \quad (N \text{ even}), \quad \binom{N}{(N-1)/2} \quad (N \text{ odd}).
\]

(1)

This means that the probability for the remaining voter to be decisive, i.e. their voting power \( P_N \), is

\[
P_N = \frac{1}{2^N} \left( \frac{N}{\lfloor N/2 \rfloor} \right),
\]

where \( \lfloor N/2 \rfloor \) denotes the floor function, i.e. the largest integer not exceeding \( N/2 \). Using the Stirling approximation for the factorials we obtain

\[
P_N \approx \sqrt{\frac{2}{\pi}} \frac{1}{\sqrt{N}} \approx \sqrt{\frac{2}{\pi}} \frac{1}{\sqrt{N + 1}} \quad (N \gg 1),
\]

(3)

ie the voting power decreases like one divided by the square root of the population.

2.2 Voting power with a “no vote” option

Let us now consider a voting scheme where all voters have the choice between “yes”, “no” and “abstain”. Let us again assume that there are \( N + 1 \) voters, and a motion is successful if it receives more “yes” than “no” votes. As in the Penrose case, we define the voting power of a single voter as the probability for this voter to be decisive, provided the remaining \( N \) voters vote randomly.
There are $3^N$ possible voting outcomes, all of them equally likely according to the random voting assumption. For any integer $0 \leq K \leq N$, there are
\[
\binom{N}{K}
\]
ways to choose $K$ voters participating, i.e. voting either “yes” or “no”. If $K$ is even, the single remaining voter will be decisive if the numbers of “yes” and “no” votes are equal, either by voting “yes” and rendering the motion successful, or by voting “no” or abstaining, in which case the motion will fail. If $K$ is odd, the single voter will be decisive if there is one “yes” vote more than there are “no” votes; in this case a “yes” vote or an abstention will cause the motion to be successful, while a “no” vote will lead to its rejection. This means that for any choice of $K$ voters who have voted either “yes” or “no”, there are
\[
\binom{K}{\lfloor K/2 \rfloor}
\]
voting outcomes rendering the single remaining voter decisive. The probability for this one voter to be decisive, i.e. is their voting power, is thus
\[
P_N = \frac{1}{3^N} \sum_{K=0}^{N} \binom{N}{K} \binom{K}{\lfloor K/2 \rfloor}.
\]
(4)

There does not seem to be any simple expression for $P_N$. What we are interested in here is its behaviour for large $N$, as in the Penrose case. For $N > 0$, the sum can be written as
\[
\sum_{K=0}^{N} \binom{N}{K} \binom{K}{\lfloor K/2 \rfloor} = \binom{N}{0}_2 + \binom{N}{1}_2
\]
(5)
(Callan, 2004), where $(\cdot)_2$ denotes the trinomial coefficient defined by
\[
(x^2 + x + 1)^n = \sum_{k=-n}^{n} \binom{n}{k}_2 x^{n+k}.
\]
(6)
The trinomial coefficients have asymptotic expansions
\[
\binom{N}{0}_2 = \sqrt{\frac{3}{2}} \left( \frac{2N}{N} \right)^N \left( \frac{3N}{4} \right) \left( 1 + O \left( \frac{1}{N} \right) \right)
\]
(7)
and
\[
\binom{N}{1}_2 = \sqrt{\frac{3}{6}} \left( \frac{2(N+1)}{N+1} \right)^{N+1} \left( \frac{3}{4} \right)^{N+1} \left( 1 + O \left( \frac{1}{N} \right) \right)
\]
(Merlini et al., 2002). Using Stirling’s formula for the binomial coefficients, we get
\[
P_N = \frac{1}{2} \sqrt{\frac{3}{\pi}} \left( \frac{1}{\sqrt{N}} + \frac{1}{\sqrt{N+1}} \right) \left( 1 + O \left( \frac{1}{N} \right) \right)
\]
(9)
\[
\approx \sqrt{\frac{3}{\pi}} \frac{1}{\sqrt{N+1}}
\]
(10)
for large $N$, which means that even in the case of a “no vote” option, the voting power of each citizen decreases with the square root of the population number, as in the case of pure yes-no voting.

For some reasonable population numbers (ranging from 100,000 to 100 million), figure 1 shows the voting power calculated from equation 4 and compares it with the square root approximation from equation 10. It can be seen that for these population numbers, the error from truncating the asymptotic expansions is negligible, and the square root approximation provides a good measure for voting power.

3. Conclusion

In the previous section, it has been shown that in the case of voters having three voting options, “yes”, “no” and “abstain”, the voting power defined as the probability for a single voter to be decisive is proportional to one divided by the square root of the population, in the limit of large population numbers. This analysis is based on the random voting assumption, i.e. the assumption that all possible voting outcomes are equally likely. While this assumption has been criticised in the literature, e.g. in Garret and Tsebelis (2001), Albert (2003), Bafumi et al. (2004), as not being realistic, other authors, e.g. Zyczkowski and Slomczynski (2004), Zyczkowski and Slomczynski (2006), Leech (2003), Shapley and Shubik (1954), Banzhaf (1968), Coleman (1971), point out that this is an \textit{a priori} voting power based on voters’ possible options rather than their actual decisions in previous elections.

The square root behaviour of the voting power is similar to voting power in pure yes-no votes, the only difference being the constant factor. As the basis for voting weight allocation in the Jagiellonian Compromise scheme for the EU is relative rather than absolute voting power, it is not affected by the constant factor. This means that even the inclusion of a “no
vote” option in the analysis of citizens’ voting power leads to the allocation of exactly the same voting weights under the Jagiellonian Compromise scheme.

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