Packing Interval Graphs with Vertex-Disjoint Triangles

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Abstract. We show that there exists a polynomial algorithm to pack interval graphs with vertex-disjoint triangles.

1 Introduction
Finding the maximal number of vertex-disjoint triangles in a graph is a well-known NP-complete problem. Whether there exists a polynomial algorithm that solves this problem for interval graphs has been open for a long time.

The packing problem is NP-complete for chordal graphs [2,3]. Interestingly, the question whether a chordal graph can be partitioned into vertex-disjoint triangles can be answered in polynomial time [2].

For split graphs the packing problem can be solved via maximum matching. There exists a polynomial algorithm that solves the packing problem for unit interval graphs [5].

In this note we show that the packing problem can be solved in polynomial time for the class of interval graphs.

2 Packing interval graphs
A graph $G = (V, E)$ is an interval graph if and only if it has a consecutive clique arrangement [4]. That is a linear arrangement $\sigma = [C_1, \ldots, C_t]$ of the maximal cliques in $G$ such that for each vertex $x$, the maximal cliques that contain $x$ are consecutive in $\sigma$.

Theorem 1. There exists a polynomial algorithm that computes a triangle packing in interval graphs.

Proof. Let $G$ be an interval graph and let

$$\sigma = [C_1, \ldots, C_t]$$

be a consecutive clique arrangement for $G$. Write $n$ for the number of vertices in $G$. Notice that $t \leq n$ where $n$ is the number of vertices in $G$, since a chordal graph with $n$ vertices has at most $n$ maximal cliques.
Consider a triangle packing \( \mathcal{T} \). A vertex \( x \) is covered if \( x \in T \) for some \( T \in \mathcal{T} \). Notice that in any maximal packing, every maximal clique in \( G \) has at most two vertices that are not covered.

We describe an algorithm that computes a triangle packing via dynamic programming. For \( i \in \{1, \ldots, t\} \), let
\[
\sigma_i = [C_1, \ldots, C_i]
\]
and let \( G_i \) be the subgraph of \( G \) induced by the maximal cliques in \( \sigma_i \).

For every \( i \in \{1, \ldots, t\} \) we keep the following invariant. Let \( S \subseteq C_i \) with \( |S| \leq 2 \). If there is a triangle packing in \( G_i \) that covers all vertices of \( C_i \) except those in \( S \), then \( f_i(S) \) is the maximal number of triangles in such a triangle packing. If there is no such triangle packing in \( G_i \) then \( f_i(S) = 0 \).

First consider \( i = 1 \). Any maximal triangle packing for \( G_1 \) consists of
\[
\left\lfloor \frac{|C_1|}{3} \right\rfloor
\]
triangles. Let
\[
k = |C_1| - 3 \cdot \left\lfloor \frac{|C_1|}{3} \right\rfloor = |C_1| \mod 3
\]
be the number of vertices that are not covered by triangles.

By definition, we have that
\[
f_1(S) = \begin{cases} 
\frac{|C_1|}{3} & \text{if } |S| = k, \text{ and} \\
0 & \text{otherwise}, 
\end{cases}
\]
where \( k \) is given by Equation (4).

Consider the transition from \( i \) to \( i + 1 \). Let \( S \subseteq C_{i+1} \) with \( |S| \leq 2 \). We claim that
\[
f_{i+1}(S) = \max_{S'} f_i(S') + \kappa
\]
where
(a)
\[
S' \subseteq C_i \quad \text{and} \quad |S'| \leq 2 \quad \text{and} \quad S \cap C_i \subseteq S' \cap C_{i+1} \quad \text{and}
\]
(b) \( S'' = (S' \cap C_{i+1}) \setminus S \), and
(c)
\[
\kappa = \frac{|C_{i+1} \setminus (C_i \cup S) \cup S''|}{3} \text{ is integer.}
\]
Obviously, the right-hand side of Equation (6) is a lowerbound for $f_{t+1}(S)$. We show that it is also an upperbound.

Consider a maximum triangle packing $T_{t+1}$ for $G_{t+1}$. Let $S \subseteq C_{t+1}$ be the set of vertices that are not covered by $T_{t+1}$. Let $T_i \subseteq T_{t+1}$ be the set of triangles that have all vertices in $G_i$ and let $S'$ be the set of vertices in $C_i$ that are not covered by $T_i$.

We show that we may assume that $|S'| \leq 2$. Assume $|S'| \geq 3$. Let $\alpha$, $\beta$, and $\gamma$ be three vertices of $S'$ that are covered by triangles in $T_{t+1}$. First assume that $\{\alpha, \beta, p\}$ and $\{\gamma, q, r\}$ are triangles of $T_{t+1}$ with $p$, $q$ and $r$ in $C_{t+1} \setminus C_i$. Then replace these triangles with $\{\alpha, \beta, \gamma\}$ and $\{p, q, r\}$. Now assume that $\{\alpha, p, q\}$, $\{\beta, r, s\}$ and $\{\gamma, u, v\}$ are three triangles of $T_{t+1}$ with $p$, $q$, $r$, $s$, $u$ and $v$ in $C_{t+1} \setminus C_i$. Then replace the three triangles by $\{\alpha, \beta, \gamma\}$, $\{p, q, r\}$ and $\{s, u, v\}$.

Now assume that there are exactly two vertices $\alpha$ and $\beta$ in $S'$ that are covered by triangles $\{\alpha, p, q\}$ and $\{\beta, r, s\}$ in $T_{t+1}$ with $p$, $q$, $r$ and $s$ in $C_{t+1} \setminus C_i$. Then we may replace these triangles in $T_{t+1}$ with $\{\alpha, \beta, p\}$ and $\{q, r, s\}$. Assume that there exists a vertex $z \in S' \setminus \{\alpha, \beta\}$. Then replace $\{\alpha, \beta, p\}$ with $\{\alpha, \beta, z\}$.

Assume that there exists exactly one vertex $\alpha$ in $S'$ which is covered by a triangle $\{\alpha, p, q\} \in T_{t+1}$ with $p$ and $q$ in $C_{t+1} \setminus C_i$. Assume that there are two vertices $y$ and $z$ in $S' \setminus \{\alpha\}$. Then replace the triangle $\{\alpha, p, q\}$ in $T_{t+1}$ with $\{\alpha, y, z\}$.

The replacements above don’t change the number of triangles in $T_{t+1}$. In all cases, $|S'|$ decreases, and so this proves the claim.

The vertices of $C_{t+1} \setminus (C_i \cup S) \cup S''$ are in triangles that have at least one vertex in $C_{t+1} \setminus C_i$, where $S''$ is defined as in item (5). The number of these triangles is $\kappa$, as defined in item (5). Thus $\kappa$ must be an integer.

We show that this algorithm runs in polynomial time. For the computation of $f_{t+1}$, the algorithm considers $O(|C_{t+1}|^2)$ subsets $S$. To compute the maximum in Equation (6) the algorithm considers all subsets $S' \subseteq C_t$ with at most two vertices. The table look-up of $f_t(S')$ and the check if the choice of $S$ and $S'$ yields an integer $\kappa$ as in item (5) take constant time. The final answer is obtained by looking up the maximal value $f_t(S)$ over $S \subseteq C_t$ with $|S| \leq 2$. This shows that the algorithm can be implemented to run in time proportional to

$$
\sum_{i=1}^{t-1} |C_i|^2 \cdot |C_{t+1}|^2 + |C_t|^2 = O(n^5).
$$

This proves the theorem. \( \square \)

**References**

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