“PEIERLES SUBSTITUTION” AND
CHERN–SIMONS QUANTUM MECHANICS*

G. Dunne

Department of Mathematics
and
Center for Theoretical Physics
Massachusetts Institute of Technology
Cambridge, Massachusetts 02139  U.S.A.

and

R. Jackiw

Center for Theoretical Physics
Laboratory for Nuclear Science
and Department of Physics
Massachusetts Institute of Technology
Cambridge, Massachusetts 02139  U.S.A.

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ABSTRACT

An elementary derivation is given for the “Peierls substitution” used in projecting dynamics in a strong magnetic field onto the lowest Landau level. The projection of wavefunctions and the ordering prescription for the projected Hamiltonian is explained.
The “Peierles substitution”\textsuperscript{1} advances calculations for the following problem. Consider a charged particle with mass $m$ moving on the plane in a constant magnetic field $B$ perpendicular to the plane, and also subject to planar forces arising from the potential $V(x, y)$. Motion along the $z$-axis (along $B$) is ignored and it is assumed that $B$ is sufficiently strong and $V$ sufficiently weak so that the lowest Landau level retains its identity in the presence of $V$. The question one wants to answer is how the energy-degenerate states in the lowest Landau level are modified by the interaction with $V$. The result in the approximation of strong $B$ and weak $V$ is that the energy eigenvalues become

$$ E = \frac{B}{2m} + \epsilon_n $$

where $B/2m$ is the lowest Landau level energy [we set $\hbar$, $c$ and particle charge to unity] while $\epsilon_n$ are eigenvalues of the operator obtained from $V(x, y)$ by the Peierles substitution,

$$ V(p, q) |n\rangle = \epsilon_n |n\rangle $$

where $p$ and $q$ are canonically conjugate (with $1/B$ playing the role of Planck’s constant).

$$ i[p, q] = \frac{1}{B} $$

The Peierles substitution has been used for over half a century; a recent application is to the Azbel\textsuperscript{2}–Hofstadter\textsuperscript{3} problem where $V(x, y) = V_0 \cos x + V_0 \cos y$. But its justification\textsuperscript{4} remains a “most difficult step.”\textsuperscript{3}

In this Letter we draw on our experience with Chern–Simons quantum mechanics\textsuperscript{5} to give an immediate derivation of the Peierles substitution, and to answer two immediate questions: first, how do the wavefunctions $\psi(x, y)$ which depend on two commuting coordinates $x$ and
reduce to wavefunctions depending only on the single coordinate $q$? Second, how does one resolve the operator ordering ambiguity inherent in (2) when the potential $V(x, y)$ becomes a function of the non-commuting operators $p$ and $q$? We answer these questions and explicitly verify our prescription when $V$ is rotationally symmetric: $V = V(x^2 + y^2)$.

Our derivation relies on the following fact about quantizing a system governed by a Lagrangian that is \textit{linear} in time derivatives. Let $\xi^i \ (i = 1, 2)$ be a two-component quantity with Lagrangian

$$L = \frac{g}{2} \sum_{ij} \dot{\xi}^i \epsilon_{ij} \xi^j - V(\xi) \quad (4)$$

Here $g$ is a constant, $\epsilon_{ij}$ the anti-symmetric tensor $\epsilon_{ij} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, and $V$ a function depending on $\xi^i$ but not on its time-derivative $\dot{\xi}^i$. Although lacking the usual kinetic term, quadratic in time derivatives, (4) gives rise to non-singular equations of motion, which may be quantized in a Hamiltonian framework, provided the fundamental quantum commutator is taken as

$$[\xi^i, \xi^j] = \frac{i}{g} \epsilon^{ij} \quad (5)$$

While (5) is a standard result\textsuperscript{6} about systems like (4), it is not well-known, hence in the Appendix we sketch a derivation.

Consider now the magnetic problem described by the Lagrangian

$$L_m = \frac{1}{2}m \dot{\mathbf{r}}^2 + \frac{B}{2} \mathbf{r} \times \dot{\mathbf{r}} - V(\mathbf{r}) \quad (6)$$

Dynamics is confined to the plane, $\mathbf{r} = (x, y)$, the $z$-coordinate is suppressed and the vector potential $\mathbf{A}$ giving rise to the constant magnetic field $B = \nabla \times \mathbf{A}$ is taken in the rotationally
symmetric gauge $A^i(x) = -\epsilon^{ij}r^jB\frac{r}{2}$. The Hamiltonian corresponding to (6) is the familiar expression

$$H_m(p, r) = \frac{1}{2m}(p - A(r))^2 + V(r)$$

(7)

In the absence of $V$, the spectrum is well-known: the infinitely degenerate lowest Landau level has energy $B/2m$ and the higher levels are separated by gaps of magnitude proportional to $B/m$. When $V$ enters weakly and $B$ is strong, the degeneracy is lifted, but the pattern remains unchanged, and the lowest level may be isolated by setting $m$ to zero, provided the energy eigenvalue $B/2m$ is subtracted (“renormalized”). Thus the structure of the lowest level is governed by the $m = 0$ limit.

[The analogy to gauge theories with Chern–Simons terms is now evident: the kinetic and potential terms in (6) are point-particle analogues of the conventional Maxwell/Yang–Mills Lagrangian; the second, magnetic term models the field theoretic Chern–Simons interaction. Setting $m$ to zero in the quantum mechanics problem parallels the passage from a topologically massive gauge field theory (with the Maxwell/Yang–Mills and Chern–Simons terms) to a pure Chern–Simons field theory. Indeed Ref. [5] is devoted to an illustration of these field-theoretic issues in a quantum mechanical context. But here now we pursue the actual physical relevance of the quantum mechanical model.]

The $m \to 0$ limit may be discussed in the Lagrangian (6) or Hamiltonian (7) framework. In view of the previously stated result (4) and (5), the Lagrangian discussion directly leads to the conclusion that the symplectic structure of the reduced Lagrangian

$$L_0 = \frac{B}{2}r \times \dot{r} - V(r)$$

(8)

enforces the commutation relation

$$i [r^i, r^j] = \frac{1}{B} \epsilon^{ij}$$

(9)
and the reduced Hamiltonian $H_0$, which is obtained from $L_0$ by the usual Legendre transform

$$
H_0 = \frac{\partial L_0}{\partial \dot{r}} \cdot \dot{r} - L_0 = V(r),
$$

becomes evaluated with non-commuting arguments in view of (9). An alternative, more complicated, derivation starts with $H_m$ of (7): the limit $m = 0$ is achieved by imposing quantum mechanically the constraint

$$
0 = p - A \equiv m \dot{r}
$$

$H$ reduces to $H_0$ and (9) emerges as a consequence of constrained quantization.$^5,^6$

We have thus arrived at a derivation of the Peierls substitution: the perturbed structure of the lowest Landau level is governed by the effective Hamiltonian

$$
H_0 = V(p, q)
$$

where the non-commutativity of the arguments follows from (9) (expressed as in (3)).

When $V = V(r^2) = V(p^2 + q^2)$ it is tempting to write the eigenvalues of $H_0$ as

$$
\epsilon_n \sim V\left(\frac{2}{B}\left(n + \frac{1}{2}\right)\right)
$$

where $\frac{2}{B}\left(n + \frac{1}{2}\right)$ are the harmonic oscillator eigenvalues. However, this naive substitution ignores operator ordering issues, which produce "corrections" to (13).

To study the operator ordering we first describe the behavior of wavefunctions when the limit $m \to 0$ is taken. Note that in the limit, phase space is reduced from four dimensions $(p_x, p_y, x, y)$ to two $(p, q)$; this is seen also as a consequence of the constraint (3). Wave functions depend on half the phase-space variables: two before the reduction and one after,
but if normalization is maintained, one argument of $\psi_m(x, y)$ cannot simply disappear when $m$ vanishes. Detailed analysis\(^5\) gives the following story. In quantizing the reduced theory, the holomorphic polarization is chosen for wavefunctions;\(^7\) the single combination of the two phase space variables, on which wavefunctions are taken to depend, is the non-Hermitian combination $\sqrt{B/2}(x + iy)$. Thus we form the operators

$$a \equiv \sqrt{\frac{B}{2}}(x - iy), \quad a^\dagger \equiv \sqrt{\frac{B}{2}}(x + iy), \quad [a, a^\dagger] = 1 \quad (14)$$

Coherent states provide a basis

$$\langle \alpha | a^\dagger = \langle \alpha | \alpha \quad (15)$$

and states $|\psi\rangle$ are described by wavefunctions that depend on $\alpha$.

$$\langle \alpha | \psi \rangle = \psi(\alpha) \quad (16)$$

The operator $a^\dagger$ acts on these functions by multiplication, a by differentiation; the adjoint relationship between the two is maintained by virtue of a non-trivial measure.

$$\langle \alpha | a^\dagger | \psi \rangle = \alpha \psi(\alpha)$$

$$\langle \alpha | a | \psi \rangle = \frac{d}{d\alpha} \psi(\alpha) \quad (17)$$

$$\frac{1}{2\pi i} \int d\alpha^* d\alpha e^{-|\alpha|^2} |\alpha\rangle \langle \alpha | = I$$

$$\frac{1}{2\pi i} d\alpha^* d\alpha \equiv \frac{B}{2\pi} dx \, dy \quad . \quad (18)$$

One is also interested in number states

$$a^\dagger a |n\rangle = n |n\rangle \quad (19a)$$

which are described within the holomorphc representation by

$$\langle \alpha | n \rangle = \frac{\alpha^n}{\sqrt{n!}} \quad . \quad (19b)$$
The phase-space reductive $m \to 0$ limit onto the lowest Landau level, when applied to wavefunctions, works in the following manner. The $(x, y)$-dependence in the finite-$m$ eigenfunctions of $H_m$ for the lowest Landau level, are presented in terms of complex variables

$$z = \sqrt{B/2} (x + iy), \quad z^* = \sqrt{B/2} (x - iy)$$

$$\psi_m = \psi_m(z, z^*) \quad (20a)$$

When $m$ is set to zero, the limiting form becomes

$$\lim_{m \to 0} \psi_m(z, z^*) = \left( \frac{B}{2\pi} \right)^{1/2} e^{-\frac{1}{2} |z|^2} \psi(z) \quad (20b)$$

with $\psi$ being a holomorphic eigenfunction of $H_0$. Note that the probability densities transform properly so that the norm is maintained.

$$d^2 r \ |\psi_m(r)|^2 \xrightarrow{m \to 0} \frac{B}{2\pi} dx \ dy \ e^{-|z|^2} |\psi(z)|^2 = \frac{d\alpha \ d\alpha^*}{2\pi i} e^{-|\alpha|^2} |\psi(\alpha)|^2 \quad (21)$$

(All this may be easily and explicitly checked for the solvable case when $V$ is a harmonic oscillator potential.\textsuperscript{5})

The ordering prescription for $H_0 = V$ must be such that the matrix elements $\epsilon = \langle n | V | n \rangle$ of $V$, when computed before the phase-space reduction as integrals over the function $V(x, y)$, coincide with the evaluation after reduction, when $V$ becomes an operator between the number states $|n\rangle$ in (19a). This prescribes that the operator $H_0$ be expressed as the anti-normal ordered (i.e. all $a$’s to the left, $a^\dagger$’s to the right) form of $V$ when $V(x, y)$ is written as $V(z, z^*)$ and $z^*$ is replaced by $a$ and $z$ by $a^\dagger$. (To see the need for anti-normal ordering, note that the coherent state resolution of the identity in (18) implies that $a^k \ (a^\dagger)^\ell = \int \frac{dz \ dz^*}{2\pi i} e^{-|z|^2} |z\rangle \ (z^*)^k \ z^\ell |z\rangle$.)

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To verify this explicitly, we consider a potential $V = V(r^2)$ as a function of $r^2 = p^2 + q^2 \sim \frac{2}{B} a a^\dagger$, a potential where the tilde indicates that the ordering is yet to be performed. Then $mH_m$ may be written as

$$mH_m = \frac{1}{2} (p - A)^2 + mV$$

(22)

and the energies $\epsilon_n$ of (1) can be evaluated by lowest order perturbation theory in $m$. The ground state of the unperturbed Hamiltonian $(p - A)^2/2$ carries energy $B/2$; it is infinitely degenerate with wavefunctions

$$u_n(r) = \frac{1}{\sqrt{\pi}} \left( \frac{B}{2} \right)^{(1+n)/2} \frac{z^n}{\sqrt{n!}} e^{-\frac{B}{2} |z|^2}, \quad n = 0, 1, \ldots$$

(23)

Degenerate perturbation theory must be employed, but fortunately a rotationally symmetric $V$ is already diagonal in the representation (23). Thus we find

$$\epsilon_n = \left( \frac{B}{2} \right)^{1+n} \int_0^\infty dr^2 r^{2n} V(r^2) e^{-\frac{B}{2} r^2}$$

(24)

If $V$ is represented as

$$V(r^2) = \int d\sigma e^{-\sigma r^2} v(\sigma)$$

(25)

it follows that

$$\epsilon_n = \int d\sigma \left( 1 + \frac{2}{B} \sigma \right)^{-1-n} v(\sigma)$$

(26)

On the other hand, in the reduced problem we have the operator Hamiltonian

$$V \sim \int d\sigma e^{-\frac{2}{B} \sigma a a^\dagger} v(\sigma)$$

(28a)

To anti-normal order, we expand the exponential

$$V = \int d\sigma v(\sigma) \sum_{N+0}^{\infty} \frac{1}{N!} \left( -\frac{2\sigma}{B} \right)^N (a)^N (a^\dagger)^N$$

$$= \int d\sigma v(\sigma) \left( 1 + \sum_{N=1}^{\infty} \frac{1}{N!} \left( -\frac{2\sigma}{B} \right)^N (a^\dagger a + N) (a^\dagger a + N - 1) \ldots (a^\dagger a + 2) (a^\dagger a + 1) \right)$$

(28b)
The number states $|n\rangle$ are eigenstates with eigenvalue

$$
\epsilon_n = \int d\sigma v(\sigma) \sum_{N=0}^{\infty} \frac{1}{N!} \left( \frac{-2\sigma}{B} \right)^N \frac{(N+n)!}{n!}
$$

(29)

Summing the series reproduces (26). These expressions, (26) and (29), encapsulate all the “corrections” to the naive replacement in (13).

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Dimensional reduction of phase space in the presence of strong magnetic fields has recently also been studied by Iso, Karabali, Sakita, Sheng and Su,\(^8\) as well as by Levit and Siran.\(^9\) We thank some of these authors for discussion. Levit and Sivan give a semi-classical (WKB) analysis of ordering corrections to $H_0$; their prescription in general disagrees with ours, agreement exists only for the first two terms of a large-$n$ expansion. We are grateful to C. Callan and D. Freed for acquainting us with the Peierles substitution and M. Berry for suggesting the perturbative calculation.

NOTE ADDED

After completing this work, we searched the literature and discovered that most of our points were made almost a decade ago by S. Girvin and T. Jach, \textit{Phys. Rev. D} 29, 5617 (1984); see also S. Kivelson, C. Kallin, D. Arovas and J. Schrieffer, \textit{Phys. Rev. B} 36, 1620 (1987).
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APPENDIX

We consider the following Lagrangian that governs dynamics for a multiplet of variables $\xi^i$ and is linear in the velocities $\dot{\xi}^i$.

\[ L = a_i(\xi)\dot{\xi}^i - V(\xi) \quad (A.1) \]

Here $a_i$ and $V$ are functions of $\xi^i$, but not of $\dot{\xi}^i$, and the summation convention is used. $L$ of (A.1) is a generalization of (4), where $a_i(\xi) = g^{ij} \xi_j \dot{\xi}^j$. Although (A.1) lacks the usual kinetic term, quadratic in velocities, it is in fact the generic form for a Lagrangian. The point is that familiar Lagrangians like

\[ L = \frac{1}{2} m \dot{\varphi}^2 - V(q) \quad (A.2a) \]

may be expressed as

\[ L(p, q) = p\dot{q} - H(p, q) \quad , \quad H(p, q) = \frac{p^2}{2m} + V(q) \quad (A.2b) \]

and the Euler-Lagrange equations for $L(p, q)$, with $p$ and $q$ taken as Lagrangian variables, reproduce the equations of motion of (A.2a). But (A.2b), is of the form (A.1) with $\xi^i = \binom{p}{q}$.

The equations of motion that follow from (A.1) read

\[ f_{ij}(\xi)\dot{\xi}^j = \frac{\partial V(\xi)}{\partial \xi^i} \quad (A.3) \]

\[ f_{ij}(\xi) = \frac{\partial}{\partial \xi^i} a_j(\xi) - \frac{\partial}{\partial \xi^i} a_i(\xi) \quad (A.4) \]

Next, we assume that the matrix $f_{ij}$ possesses an inverse $f^{ij}$ — if it does not, the subsequent development is more complicated, but unnecessary in our case (4). Then (A.3) implies

\[ \dot{\xi}^i = f^{ij}(\xi) \frac{\partial V(\xi)}{\partial \xi^j} \quad (A.5) \]
The Hamiltonian for (A.1) is constructed by the usual Legendre transform

\[ H = \frac{\partial L}{\partial \dot{\xi}^i} \dot{\xi}^i - L = V(\xi) \quad (A.6) \]

Hence if the aim is to reproduce (A.5) by bracketing \( \xi^i \) with the Hamiltonian, it must be that the \( [\xi^i, \xi^j] \) bracket is non-vanishing since \( H = V(\xi) \) depends only on \( \xi^i \). Evidently,

\[ i \left[ V(\xi), \xi^i \right] = \frac{\partial V(\xi_i)}{\partial \xi^j} i \left[ \xi^j, \xi^i \right] = \dot{\xi}^i \quad (A.7) \]

and (A.5) is regained provided

\[ [\xi^i, \xi^j] = i f^{ij}(\xi) \quad (A.8) \]

This gives the derivation of (5).