Relaxed ISS Small-Gain Theorems for Discrete-Time Systems

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Abstract

In this paper ISS small-gain theorems for discrete-time systems are stated, which do not require input-to-state stability (ISS) of each subsystem. This approach weakens conservatism in ISS small-gain theory, and for the class of exponentially ISS systems we are able to prove that the proposed relaxed small-gain theorems are non-conservative in a sense to be made precise. The proofs of the small-gain theorems rely on the construction of a dissipative finite-step ISS Lyapunov function which is introduced in this work. Furthermore, dissipative finite-step ISS Lyapunov functions, as relaxations of ISS Lyapunov functions, are shown to be sufficient and necessary to conclude ISS of the overall system.

Keywords: input-to-state stability, Lyapunov methods, small-gain conditions, discrete-time nonlinear systems, large-scale interconnections

1 Introduction

Large-scale systems form an important class of systems with various applications such as formation control, logistics, consensus dynamics, networked control systems and further applications. While stability conditions for such large-scale systems have already been studied in the 1970s and early 1980s cf. [29, 32, 35] based on linear gains and Lyapunov techniques, nonlinear approaches are more recent. An efficient tool in the analysis of large-scale nonlinear control systems is the concept of input-to-state stability (ISS) as introduced in [33], and the introduction of ISS Lyapunov functions in [33, 34]. This concept of ISS was originally formulated for continuous-time systems, but has also been established for discrete-time control systems ([21, 19]) of the form $x(k+1) = G(x(k), u(k))$, as considered in this work.

As ISS Lyapunov functions are assumed to decrease at each step (while neglecting the input) the search for ISS Lyapunov functions is a hard task, in general. To relax this assumption we introduce the concept of dissipative finite-step ISS Lyapunov functions, where the function is assumed to decrease after a finite time, rather than at each time step. This approach originates from [1] and was recently used in [27, 8, 7] for the stability analysis of discrete-time systems without inputs.

We provide, in a first step, an equivalent characterization of input-to-state stability in terms of the existence of a dissipative finite-step ISS Lyapunov function. Thereby, the sufficiency part follows the lines of [19, Lemma 3.5], which shows that the existence of a continuous

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(dissipative) ISS Lyapunov function implies ISS of the system. Necessity is shown using a converse ISS Lyapunov theorem \[19, 25\]. Moreover, for the case of exponential ISS systems, we can show that any norm is a dissipative finite-step ISS Lyapunov function.

For large-scale nonlinear systems it may be difficult to prove ISS directly. But if a large-scale system is defined through the interconnection of a number of smaller components then there exist small-gain type conditions guaranteeing the ISS property for the interconnected system. Whereas small-gain theorems have a long history, the first ISS versions in a trajectory-based formulation and in a Lyapunov-based formulation are given in \[18\] and \[17\], respectively. In both cases the results are stated for systems consisting of two subsystems. These results have been extended to large-scale interconnections in \[5\] and \[6\].

The first ISS small-gain theorems for discrete-time systems were presented in \[19\], which parallel the results of \[18\] and \[17\] for continuous-time systems. For interconnections consisting of more than two subsystems, small-gain theorems are presented in \[16\] and in \[5\], whereas in \[16\] ISS was defined in a maximum formulation and in \[5\] the results are given in a summation formulation. Further extensions to the formulation via maximization or summation are ISS formulations via \textit{monotone aggregation functions}. In this formulation, the ISS small-gain results are shown to hold in a more general form, see \[30\]. In \[6\] the authors present an ISS small-gain theorem in a Lyapunov-based formulation that allows to construct an overall ISS Lyapunov function. This idea is picked up in \[28\], where the authors present a discrete-time version in a maximum formulation and construct an ISS Lyapunov function for the overall system.

The classical idea of ISS small-gain theory is that the interconnection of ISS subsystems remains ISS if the influence of the subsystems on each other is small enough. This is a sufficient criterion, but the requirement on all systems being ISS is not necessary, even for linear systems as we recall in Section \[5\]. Hence, classical small-gain theorems come with a certain conservatism. The main purpose of this work to reduce conservatism in ISS small-gain theory. This is achieved as the results presented in this work do not require each subsystem to be ISS. Indeed, each subsystem may be unstable when decoupled from the other subsystems. This is a crucial difference to classical ISS small-gain results, where it is implicitly assumed that the other subsystems act as perturbations. Here the subsystems may have a stabilizing effect on each other.

The requirement imposed is that Lyapunov-type functions for the subsystems have to decrease after a finite time. This relaxation also includes previous ISS small-gain theorems, but nevertheless applies to a larger class of interconnected systems. Furthermore, if the overall system is expISS, i.e., solutions of the unperturbed system are decaying exponentially, the ISS small-gain theorems are indeed non-conservative, i.e., they are also necessary.

The proof of the ISS small-gain theorems presented give further insight in the systems behavior. For the sufficiency part a dissipative finite-step ISS Lyapunov function is constructed from the Lyapunov-type functions and the gain functions involved. On the other hand, for expISS systems suitable Lyapunov-type and gain functions are derived. This in particular implies a constructive methodology for applications. However, if the overall system is ISS but not expISS the application of the results is challenging to implement.

To illustrate this methodology we consider a nonlinear discrete-time system consisting of two subsystems that are not both ISS. Thus, to the best of the authors’ knowledge, previous ISS small-gain theorems do not apply. Following the methodology we derive a dissipative finite-step ISS Lyapunov function to show ISS of the overall system.

The outline of this work is as follows. The preliminaries are given in Section \[2\] followed
by the problem statement including the definition of a dissipative finite-step ISS Lyapunov function, in Section 3. The sufficiency of the existence of dissipative finite-step ISS Lyapunov functions to conclude ISS is stated in Section 4. In this section we also state a particular converse finite-step ISS Lyapunov theorem that shows that for any expISS system any norm is a dissipative finite-step ISS Lyapunov function. Section 5 contains the main results of this work. Firstly, in Section 5.1 ISS small-gain theorems are presented that do not require each system to admit an ISS Lyapunov function. And secondly, in Section 5.2 we show the non-conservativeness of the relaxed ISS small-gain theorems for the class of expISS systems, by stating a converse of the presented ISS small-gain theorems. We close the paper in Section 6 with an illustrative example.

2 Preliminaries

2.1 Notation and conventions

By \( \mathbb{N} \) we denote the natural numbers and we assume \( 0 \in \mathbb{N} \). Let \( \mathbb{R} \) denote the field of real numbers, \( \mathbb{R}_+ \) the set of nonnegative real numbers and \( \mathbb{R}^n \) the vector space of real column vectors of length \( n \); further \( \mathbb{R}^+_n \) denotes the positive orthant. For any vector \( v \in \mathbb{R}^n \) we denote by \( [v]_i \) its \( i \)th component. Then \( \mathbb{R}^n \) induces a partial order for vectors \( v, w \in \mathbb{R}^n \) as follows: We denote \( v \geq w : \iff [v]_i \geq [w]_i \) and \( v > w : \iff [v]_i > [w]_i \), each for all \( i \in \{1, \ldots, n\} \). Further; \( v \not\geq w : \iff \) there exists an index \( i \in \{1, \ldots, n\} \) such that \( [v]_i < [w]_i \).

For \( x_i \in \mathbb{R}^m, i \in \{1, \ldots, N\} \) let \( (x_1, \ldots, x_N) := (x_1^\top, \ldots, x_N^\top)^\top \). For a sequence \( \{u(k)\}_{k \in \mathbb{N}} \) with \( u(k) \in \mathbb{R}^m \), (or for short \( u(\cdot) \subset \mathbb{R}^m \)) we define \( \|u\|_\infty := \sup_{k \in \mathbb{N}}\{\|u(k)\|\} \in \mathbb{R}_+ \cup \{\infty\} \).

If \( u(\cdot) \) is bounded, i.e., \( \|u\|_\infty < \infty \), then \( u(\cdot) \in L^\infty \). By \( \cdot \) we denote an arbitrary fixed monotonic norm on \( \mathbb{R}^n \), i.e., if \( v, w \in \mathbb{R}^n_+ \) with \( w \geq v \) then \( |w| \geq |v| \).

We will make use of the following consequence of the equivalence of norms in \( \mathbb{R}^n \): For any norm \( | \cdot | \) on \( \mathbb{R}^n \) there exists a constant \( \kappa \geq 1 \) such that for all \( x = (x_1, \ldots, x_N) \in \mathbb{R}^n \) with \( x_i \in \mathbb{R}^n_i \) and \( n = \sum_{i=1}^N n_i \), it holds

\[
|x| \leq \kappa \max_{i \in \{1, \ldots, N\}} |x_i|, \tag{1}
\]

where \( |x_i| := |(0, \ldots, 0, x_i, 0, \ldots, 0)| \). In particular, if \( | \cdot | \) is a \( p \)-norm then \( \kappa = N^{1/p} \) is the smallest constant satisfying (1).

2.2 Comparison functions and induced monotone maps

It has become standard to use comparison functions to state stability properties of nonlinear systems. Here we use functions of class \( \mathcal{K}, \mathcal{K}_\infty, \mathcal{L}, \mathcal{KL} \). For a definition see 22.

A \( \mathcal{K}_\infty \)-function \( \alpha \) is called sub-additive, if for any \( s_1, s_2 \in \mathbb{R}_+ \) it holds

\[
\alpha(s_1 + s_2) \leq \alpha(s_1) + \alpha(s_2).
\]

In the following lemma we collect some facts about \( \mathcal{K}_\infty \)-functions, which are useful not only in the particular proofs of this work. Note that the symbol \( \circ \) denotes the composition of two functions.

**Lemma 2.1** (i) [2 Prop. 3] The pair \( (\mathcal{K}_\infty, \circ) \) is a non-commutative group. In particular, for \( \alpha, \alpha_1, \alpha_2 \in \mathcal{K}_\infty \) the inverse \( \alpha^{-1} \in \mathcal{K}_\infty \) exists, and \( \alpha_1 \circ \alpha_2 \in \mathcal{K}_\infty \).
Consider the map $\Gamma$.

2.3 Small-gain conditions

For all $\Gamma$ (resp. diagonal operator $D$) it holds $\Gamma(s) < s$ for all $s > 0$. A continuous function $\eta : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is called positive definite, if $\eta(0) = 0$ and $\eta(s) > 0$ for all $s > 0$. By $\id$ we denote the identity function $\id(s) = s$ for all $s \in \mathbb{R}_+$, and by $0 : \mathbb{R}_+ \mapsto 0$ we denote the zero function.

Given $\gamma_{ij} \in \mathcal{K}_\infty \cup \{0\}$ for $i, j \in \{1, \ldots, n\}$, we define the map $\Gamma : \mathbb{R}^n_+ \rightarrow \mathbb{R}^n_+$ by

$$
\Gamma(s) := \begin{pmatrix}
\max \{\gamma_{11}([s]_1), \ldots, \gamma_{1n}([s]_n)\} \\
\vdots \\
\max \{\gamma_{n1}([s]_1), \ldots, \gamma_{nn}([s]_n)\}
\end{pmatrix}.
$$

(2)

For the $k$ times composition of this map we write $\Gamma^k$. Note that $\Gamma$ is monotone, i.e., $\Gamma(s^1) \leq \Gamma(s^2)$ for all $s^1, s^2 \in \mathbb{R}^n_+$ with $s^1 \leq s^2$, also $\Gamma(0) = 0$.

2.3 Small-gain conditions

Consider the map $\Gamma$ from (2), let $\delta_i \in \mathcal{K}_\infty$, $D_i = (\id + \delta_i)$, $i \in \{1, \ldots, n\}$, and define the diagonal operator $D : \mathbb{R}^n_+ \rightarrow \mathbb{R}^n_+$ by

$$
D(s) := (D_1([s]_1), \ldots, D_n([s]_n))^\top.
$$

(3)

Definition 2.2 The map $\Gamma$ from (2) satisfies the small-gain condition if

$$
\Gamma_i(s) \not> s \quad \text{for all } s \in \mathbb{R}^n_+ \setminus \{0\}.
$$

(4)

The map $\Gamma$ satisfies the strong small-gain condition if there exists a diagonal operator $D$ as in (3) such that

$$
(D \circ \Gamma)(s) \not> s \quad \text{for all } s \in \mathbb{R}^n_+ \setminus \{0\}.
$$

(5)

The condition $\Gamma_i(s) \not> s$ for all $s \in \mathbb{R}^n_+ \setminus \{0\}$, or for short $\Gamma \not> \id$, means that for any $s > 0$ the image $\Gamma_i(s)$ is decreasing in at least one component $i^* \in \{1, \ldots, n\}$, i.e., $\Gamma(s)[i^*] < s[i^*]$. Furthermore, we can assume that all functions $\delta_i \in \mathcal{K}_\infty$ of the diagonal operator $D$ are identical, by setting $\delta(s) := \min_i \delta_i(s)$. We will then write $D = \diag(\id + \delta)$. For any splitting $D = D_{II} \circ D_I$ with diagonal operators $D_I, D_{II} : \mathbb{R}^n_+ \rightarrow \mathbb{R}^n_+$ as defined above, it holds $D \circ \Gamma \not> \id \iff D_I \circ \Gamma \circ D_{II} \not> \id$, and, in particular, $D \circ \Gamma \not> \id \iff \Gamma \circ D \not> \id$.

As shown in [6, Theorem 5.2] the (strong) small-gain condition (4) (resp. (5)) implies the existence of a so-called $\Omega$-path $\tilde{\sigma}$ with respect to $\Gamma$ (resp. $D \circ \Gamma$) (6, Definition 5.1). The essential property of $\Omega$-paths used in this work is that $\tilde{\sigma} = (\tilde{\sigma}_1, \ldots, \tilde{\sigma}_n) \in \mathcal{K}^\infty$, and satisfies $\Gamma(\tilde{\sigma}(r)) < \tilde{\sigma}(r)$ (resp. $(D \circ \Gamma)(\tilde{\sigma}(r)) < \tilde{\sigma}(r)$) for all $r > 0$. The numerical construction of $\Omega$-paths can be performed using the algorithm proposed in [10]. A simple calculation shows that if $\tilde{\sigma}$ is an $\Omega$-path with respect to $D \circ \Gamma$ if and only if $D_{II}^{-1} \circ \tilde{\sigma}$ is an $\Omega$-path with respect to $D_I \circ \Gamma \circ D_{II}$, where $D = D_{II} \circ D_I$ is split as above.

Remark 2.3 Condition (4) originates from [2] and is in fact equivalent to the equilibrium $s^* = 0$ of the system $s(k + 1) = \Gamma(s(k))$ being globally asymptotically stable (GA), see [6, Theorem 5.6]. The idea comes from the linear case with $\Gamma \in \mathbb{R}^{nxn}_+$, where the following is equivalent (see [37, Lemma 1.1] and [2, Section 4.5]):

\[\text{see Remark 3.5 for a definition}\]
(i) for the spectral radius it holds $\rho(\Gamma) < 1$;

(ii) $\Gamma s \preceq s$ for all $s \in \mathbb{R}_+^n \setminus \{0\}$;

(iii) $\Gamma^k \to 0$ for $k \to \infty$;

(iv) the system $s(k + 1) = \Gamma(s(k))$ is GAS.

For the map $\Gamma_\oplus$ defined in (2) we have the following equivalent condition, which gives the possibility to check the small-gain condition (see [31, Theorem 6.4]).

**Proposition 2.4** The map $\Gamma_\oplus : \mathbb{R}_+^n \to \mathbb{R}_+^n$ defined in (2) satisfies the small-gain condition (4) if and only if all cycles in the corresponding graph of $\Gamma_\oplus$ are weakly contracting, i.e.,

$$\gamma_{i_0 i_1} \circ \gamma_{i_1 i_2} \circ \ldots \circ \gamma_{i_k i_0} < \text{id}$$

for $k \in \mathbb{N}, i_j \neq i_l$ for $j \neq l$.

Note that it is sufficient that all minimal cycles of the graph of $\Gamma_\oplus$ are weakly contracting, which means that $i_j \neq i_l$ for all $j, l \in \{0, \ldots, k\}$. Thus, $k < n$.

## 3 Problem statement

We consider discrete-time systems of the form

$$x(k + 1) = G(x(k), u(k)), \quad k \in \mathbb{N}. \quad (6)$$

Here $u(k) \in \mathbb{R}^m$ denotes the input at time $k \in \mathbb{N}$. Note that an input is a function $u : \mathbb{N} \to \mathbb{R}^m$. By $x(k, \xi, u(\cdot)) \in \mathbb{R}^n$ we denote the state at time $k \in \mathbb{N}$, starting in $x(0) = \xi \in \mathbb{R}^n$ with input $u(\cdot) \subset \mathbb{R}^m$.

Throughout this work the map $G : \mathbb{R}_+^n \times \mathbb{R}_+^m \to \mathbb{R}_+^n$ satisfies the following standing assumption.

**Assumption 3.1** The function $G$ in (6) is globally $K$-bounded, i.e., there exist functions $\omega_1, \omega_2 \in K$ such that for all $\xi \in \mathbb{R}^n$ and $\mu \in \mathbb{R}^m$ we have

$$|G(\xi, \mu)| \leq \omega_1(|\xi|) + \omega_2(|\mu|) \quad (7)$$

This in particular implies $G(0, 0) = 0$, but it does not require the map $G$ to be continuous away from the fixed point $(0, 0)$ (as assumed e.g. in [19, 20, 28]) or (locally) Lipschitz (as assumed e.g. in [2, 11]). For further remarks on Assumption 3.1 see Remark 3.3 and Appendix A.2.

**Definition 3.2** We call system (6) input-to-state stable if there exist $\beta \in KL$ and $\gamma \in K$ such that for all initial states $\xi \in \mathbb{R}^n$ and all inputs $u(\cdot) \subset \mathbb{R}^m$ and all $k \in \mathbb{N}$

$$|x(k, \xi, u(\cdot))| \leq \beta(|\xi|, k) + \gamma(\|u\|_\infty). \quad (8)$$

If the $KL$-function $\beta$ in (8) can be chosen as

$$\beta(r, t) = C \kappa^t r \quad (9)$$

with $C \geq 1$ and $\kappa \in [0, 1)$, then system (6) is called exponentially input-to-state stable (expISS).
An alternative definition of ISS replaces the sum in (8) by the maximum. Note that both definitions are equivalent, and the equivalence even holds for more general definitions of ISS using monotone aggregation functions, see [11] Proposition 2.5.

**Remark 3.3** Since we are interested in checking the ISS property of system (6), it is clear that the existence of functions \( \omega_1, \omega_2 \in \mathcal{K} \) satisfying (7) in Assumption 3.1 is no restriction, since every ISS system necessarily satisfies (7). In particular, by (8), we have

\[
|G(\xi, \mu)| = |x(1, \xi, \mu)| \leq \beta(|\xi|, 1) + \gamma(|\mu|)
\]

and we may choose \( \omega_1(\cdot) = \beta(\cdot, 1) \) and \( \omega_2(\cdot) = \gamma(\cdot) \) to obtain (7). Moreover, for expISS systems we can take \( \omega_1(s) = Crs \), so that there exists a linear \( \omega_1 \in \mathcal{K} \) satisfying (7).

The following lemma shows that by a suitable change of coordinates, i.e., a homeomorphism \( T: \mathbb{R}^n \to \mathbb{R}^n \) with \( T(0) = 0 \) (see e.g. [23]), we can always assume that \( \omega_1 \in \mathcal{K} \) in (7) is linear.

**Lemma 3.4** Consider system (6). Then there exists a change of coordinates \( T \) such that for \( z(k) := T(x(k)) \) the system

\[
z(k + 1) = \tilde{G}(z(k), u(k)), \quad \forall k \in \mathbb{N}
\]

satisfies (7) with linear \( \omega_1 \in \mathcal{K} \).

**Proof.** Consider a change of coordinates \( T: \mathbb{R}^n \to \mathbb{R}^n \), and define \( z(k) := T(x(k)) \), where \( x(k) \) comes from (6). Then \( z \) satisfies (10) with

\[
\tilde{G}(z, u) = T(G(T^{-1}(z), u)).
\]

Note that \( \tilde{G}(0, 0) = 0 \) since \( T \), and its inverse, fixes the origin. Furthermore, let \( \omega_1, \omega_2 \in \mathcal{K} \) satisfy (7) for the map \( G \). Without loss of generality, we assume that \( (2\omega_1 - \text{id}) \in \mathcal{K}_\infty \), else increase \( \omega_1 \). Take any \( \lambda > 1 \). By [24] Lemma 19 there exists a \( \mathcal{K}_\infty \)-function \( \varphi \) satisfying

\[
\varphi(2\omega_1(s)) = \lambda \varphi(s), \quad \forall s \geq 0.
\]

Define \( T(x) := \varphi(|x|) \frac{x}{|x|} \) for \( x \neq 0 \), and \( T(0) = 0 \). Clearly, \( T \) is a homeomorphism. With \( z = T(x) \) a direct computation yields \( T^{-1}(z) := \varphi^{-1}(|z|) \frac{z}{|z|} \) for \( z \neq 0 \) and \( T^{-1}(0) = 0 \). With this, we obtain the following estimate

\[
|\tilde{G}(\xi, \mu)| = \varphi \left( G \left( \varphi^{-1}(|\xi|) \frac{\xi}{|\xi|}, \mu \right) \right) \leq \varphi \left( \omega_1(\varphi^{-1}(|\xi|)) + \omega_2(|\mu|) \right)
\]

\[
\leq \varphi \left( 2\omega_1(\varphi^{-1}(|\xi|)) \right) + \varphi \left( 2\omega_2(|\mu|) \right) = \lambda|\xi| + \varphi \left( 2\omega_2(|\mu|) \right).
\]

So, \( \tilde{G} \) satisfies (7) for a linear \( \omega_1 \in \mathcal{K}_\infty \), which concludes the proof. \( \square \)

**Remark 3.5** We further note that ISS implies global asymptotic stability of the origin with 0 input (0-GAS), i.e., the existence of a class-\( \mathcal{KL} \) function \( \beta \) such that for all \( \xi \in \mathbb{R}^n \) and all \( k \in \mathbb{N} \),

\[
x(k, \xi, 0) \leq \beta(|\xi|, k).
\]

In [23] the author shows that in the discrete-time setting integral input-to-state stability (iISS) is equivalent to 0-GAS. Note that this is not true in continuous time.
To prove ISS of system (6) the concept of ISS Lyapunov functions is widely used (see e.g. [19]). Note that the definition of an ISS Lyapunov function stated here does not require continuity. To shorten notation, we call a function $W : \mathbb{R}^n \to \mathbb{R}_+$ proper and positive definite if there exist $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ such that for all $\bar{\xi} \in \mathbb{R}^n$

$$\alpha_1(|\xi|) \leq W(\xi) \leq \alpha_2(|\xi|).$$

**Definition 3.6** A proper and positive function $W : \mathbb{R}^n \to \mathbb{R}_+$ is called a dissipative ISS Lyapunov function for system (6) if there exist $\sigma \in \mathcal{K}$ and a positive definite function $\rho$ with $(\text{id} - \rho) \in \mathcal{K}_\infty$ such that for any $\xi \in \mathbb{R}^n, \mu \in \mathbb{R}^m$

$$W(G(\xi, \mu)) \leq \rho(W(\xi)) + \sigma(|\mu|).$$

**Remark 3.7** (i) In many prior works (e.g. [19, 23]) the definition of a dissipative ISS Lyapunov function requires the existence of a function $\alpha_3 \in \mathcal{K}_\infty$ and a function $\sigma \in \mathcal{K}$ such that

$$W(G(\xi, \mu)) - W(\xi) \leq -\alpha_3(|\xi|) + \sigma(|\mu|)$$

holds for all $\xi \in \mathbb{R}^n, \mu \in \mathbb{R}^m$. Let us briefly explain, that this requirement is equivalent to Definition 3.6. Note that $0 \leq W(G(\xi, \mu)) \leq W(\xi) - \alpha_3(|\xi|) + \sigma(|\mu|) \leq (\text{id} - \alpha_3 \circ \alpha_2^{-1})(W(\xi)) + \sigma(|\mu|) = \rho(W(\xi)) + \sigma(|\mu|)$ with $\rho := (\text{id} - \alpha_3 \circ \alpha_2^{-1})$ positive definite, and $\text{id} - \rho = \alpha_3 \circ \alpha_2^{-1} \in \mathcal{K}_\infty$. Note that since $0 \leq W(G(\xi, \mu)) \leq (\alpha_2 - \alpha_3)(|\xi|) + \sigma(|\mu|)$ holds it follows $\alpha_2 \geq \alpha_3$ by taking $\mu = 0$. On the other hand for given $\rho \in \mathcal{K}$ with $(\text{id} - \rho) \in \mathcal{K}_\infty$, we get $W(G(\xi, \mu)) - W(\xi) \leq -\alpha_3(|\xi|) + \sigma(|\mu|)$ for $\alpha_3 := (\text{id} - \rho) \circ \alpha_1 \in \mathcal{K}_\infty$.

(ii) For systems with external inputs $u$ there are usually two forms of ISS Lyapunov function. The first one is the dissipative ISS Lyapunov functions as in Definition 3.6, and the other one is the implication-form ISS Lyapunov function, i.e., a proper and positive definite function $W : \mathbb{R}^n \to \mathbb{R}_+$ satisfying

$$|\xi| \geq \chi(|\mu|) \Rightarrow W(G(\xi, \mu)) \leq \tilde{\rho}(W(\xi)).$$

for all $\xi \in \mathbb{R}^n, \mu \in \mathbb{R}^m$, and some positive definite function $\tilde{\rho} < \text{id}$ and $\chi \in \mathcal{K}$.

If the function $G$ in (6) is continuous then conditions (12) and (13) are equivalent, see [19, Remark 3.3] and [13, Proposition 3.3 and 3.6]. On the other hand, if $G$ is discontinuous then any dissipative ISS Lyapunov function is in fact an implication-form ISS Lyapunov function, but the converse does not hold in general, see [13]. In particular, an implication-form ISS Lyapunov function is not sufficient to conclude ISS, see also [26, Remark 2.1] and [13, Example 3.7].Moreover, for any ISS system (6) there exists a dissipative ISS Lyapunov function $W$ with linear decrease function $\rho$, see [13, Theorem 2.6].

(iii) In Definition 3.6 the assumption $(\text{id} - \rho) \in \mathcal{K}_\infty$ can be weakened to the condition $(\text{id} - \rho) \in \mathcal{K}$ and sup$(\text{id} - \rho) > \sup \sigma$, still proving ISS of system (6), see [13, Proposition 2.4].

We relax the assumption in Definition 3.6(ii) by allowing the function $W$ to decrease after a finite time. This relaxation was recently introduced in [27, 8, 7] for systems without inputs. In the context of ISS, i.e., for systems with inputs, this concept appears to be new.
**Definition 3.8** A proper and positive definite function $V : \mathbb{R}^n \to \mathbb{R}_+$ is called a dissipative finite-step ISS Lyapunov function for system (6) if there exist an $M \in \mathbb{N}$, $\sigma \in \mathcal{K}$, a positive definite function $\rho$ with $(\text{id} - \rho) \in \mathcal{K}_\infty$ such that for any $\xi \in \mathbb{R}^n$, $u(\cdot) \subset \mathbb{R}^m$

$$V(x(M, \xi, u(\cdot))) \leq \rho(V(\xi)) + \sigma(\|u\|_\infty). \quad (14)$$

We point out however that in applications it is not sufficient to know a finite-step ISS Lyapunov function, but we also require to know the constant $M$, which may be hard to characterize.

In the remainder of this work we show that the existence of dissipative finite-step ISS Lyapunov functions, as relaxations of dissipative ISS Lyapunov functions, is necessary (Proposition 4.5) and sufficient (Theorem 4.1) to conclude ISS of systems of the form (6). The hope, that finding dissipative finite-step ISS Lyapunov functions is easier than finding ISS Lyapunov functions is motivated by Theorem 4.6, where we show that for expISS systems any norm is a dissipative finite-step ISS Lyapunov function.

As an application we drop the common requirement in small-gain theory that all subsystems are ISS, still proving ISS of the overall system by constructing a dissipative finite-step ISS Lyapunov function (Theorems 5.1 and 5.4). The improvement of the ISS small-gain theorems stated here compared to previous ISS small-gain theorems becomes clear in the case of expISS systems, where the presented ISS small-gain theorems are shown to be non-conservative, i.e., necessary and sufficient to conclude expISS, see Theorem 5.8.

## 4 Dissipative Finite-Step ISS Lyapunov Theorems

We start this section by proving that the existence of a dissipative finite-step ISS Lyapunov function is sufficient to conclude ISS of the system (6). As any dissipative ISS Lyapunov function is a particular dissipative finite-step ISS Lyapunov function, this result includes [19, Lemma 3.5]. Furthermore, the class of suitable dissipative ISS Lyapunov functions is a strict subset of the class of suitable dissipative finite-step ISS Lyapunov functions. Hence, this result is more general than showing that the existence of a dissipative ISS Lyapunov function implies ISS of the underlying system. The proof includes a comparison lemma and an additional lemma, which are given in the appendix.

**Theorem 4.1** If there exists a dissipative finite-step ISS Lyapunov function for system (6), then system (6) is ISS.

**Proof.** The proof follows the lines of [19, Lemma 3.5], which establishes that the existence of a continuous dissipative ISS Lyapunov function implies ISS of the system. Note that in [19] the authors assume continuity, whereas in this work global $\mathcal{K}$-boundedness is considered. Moreover, [19, Lemma 3.5] only considers the case $M = 1$.

Let $V$ be a dissipative finite-step ISS Lyapunov function satisfying Definition 3.8 for system (6) with suitable $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$, $M \in \mathbb{N}$, $\sigma \in \mathcal{K}$, and a positive definite function $\rho$ with $(\text{id} - \rho) \in \mathcal{K}_\infty$. Fix any $\xi \in \mathbb{R}^n$ and any input $u(\cdot) \in l^\infty$. We will abbreviate the state $x(k, \xi, u(\cdot))$ by $x(k)$. Let $\nu \in \mathcal{K}_\infty$ be such that $\text{id} - \nu \in \mathcal{K}_\infty$ and consider the set

$$\Delta := \{ \xi \in \mathbb{R}^n : V(\xi) \leq \delta := (\text{id} - \rho)^{-1} \circ \nu^{-1} \circ \sigma(\|u\|_\infty) \}.$$
We will now show that for any \( k_0 \in \mathbb{N} \) with \( x(k_0) \in \Delta \) we have \( x(k_0 + lM) \in \Delta \) for all \( l \in \mathbb{N} \). Using \( \{14\} \), a direct computation yields
\[
V(x(k_0 + M)) \leq \rho(V(x(k_0))) + \sigma(\|u\|_\infty) \\
\leq \rho(\delta) + \sigma(\|u\|_\infty) \\
= -(\text{id} - \nu) \circ (\text{id} - \rho)(\delta) + \delta - \nu \circ (\text{id} - \rho)(\delta) + \sigma(\|u\|_\infty) \\
= -(\text{id} - \nu) \circ (\text{id} - \rho)(\delta) + \delta \leq \delta.
\]
Hence, \( x(k_0 + M) \in \Delta \) and by induction we get \( x(k_0 + lM) \in \Delta \) for all \( l \in \mathbb{N} \).

Now assume that there exists a \( j_0 \in \mathbb{N} \) satisfying \( j_0 := \min \{ k \in \mathbb{N} : x(k), \ldots, x(k + M - 1) \in \Delta \} \). Else, per definition, \( j_0 = \infty \). Then for all \( k \geq j_0 \) we get
\[
V(x(k)) \leq (\text{id} - \rho)^{-1} \circ \nu^{-1} \circ \sigma(\|u\|_\infty) =: \tilde{\gamma}(\|u\|_\infty).
\]

If \( k < j_0 \), we have to consider two cases. Firstly, if \( x(k) \in \Delta \), then \( V(x(k)) \leq \tilde{\gamma}(\|u\|_\infty) \). On the other hand, if \( x(k) \not\in \Delta \) then \( V(x(k)) > \tilde{\gamma}(\|u\|_\infty) \) implying
\[
V(x(k + M)) \leq \rho(V(x(k))) + \sigma(\|u\|_\infty) \\
< \rho(V(x(k))) + \nu \circ (\text{id} - \rho) \circ V(x(k)) \\
= (\rho + \nu \circ (\text{id} - \rho)) \circ V(x(k)).
\]

Note that \( (\rho + \nu \circ (\text{id} - \rho)) = (\text{id} - (\text{id} - \nu)) \circ (\text{id} - \rho) < \text{id} \). So we can use the comparison Lemma \([A.1]\) which implies the existence of some \( \tilde{\beta} \in K\mathcal{L} \) such that for all \( 0 \leq k \leq j_0 \)
\[
V(x(k)) \leq \tilde{\beta}(V_M^{\max}(\xi, u(\cdot)), k),
\]
where \( V_M^{\max}(\xi, u(\cdot)) := \max_{j \in \{0, \ldots, M-1\}} V(x(j, \xi, u(\cdot))) \). Using Lemma \([A.3]\) it is easy to see that with \( \tilde{\vartheta} := \max_{j \in \{0, \ldots, M-1\}} \alpha_2(2\tilde{\vartheta}_j) \) and \( \tilde{\zeta} := \max_{j \in \{0, \ldots, M-1\}} \alpha_2(2\tilde{\zeta}_j) \), where \( \vartheta_j, \zeta_j \in \mathcal{K} \) come from Lemma \([A.3]\) we get
\[
V_M^{\max}(\xi, u(\cdot)) \leq \max_{j \in \{0, \ldots, M-1\}} \alpha_2(|x(j)|) \leq \tilde{\vartheta}(|\xi|) + \tilde{\zeta}(\|u\|_\infty).
\]

So all in all we have for all \( k \in \mathbb{N} \)
\[
V(x(k)) \leq \max \left\{ \tilde{\beta}(\tilde{\vartheta}(|\xi|) + \tilde{\zeta}(\|u\|_\infty), k), \tilde{\gamma}(\|u\|_\infty) \right\} \\
\leq \max \left\{ \tilde{\beta}(2\tilde{\vartheta}(|\xi|), k) + \tilde{\beta}(2\tilde{\zeta}(\|u\|_\infty), 0), \tilde{\gamma}(\|u\|_\infty) \right\} \\
\leq \tilde{\beta}(2\tilde{\vartheta}(|\xi|), k) + \left( \tilde{\beta}(2\tilde{\zeta}(\|u\|_\infty), 0) + \tilde{\gamma}(\|u\|_\infty) \right).
\]

Then we get \([8]\) by defining \( \beta(s, r) := \alpha_{1}^{-1}(2\tilde{\beta}(2\tilde{\vartheta}(s, r))) \) and \( \gamma(s) := \alpha_{1}^{-1}(2\tilde{\beta}(2\tilde{\vartheta}(s, 0) + \tilde{\gamma}(\|u\|_\infty))) \). Note that for fixed \( r \geq 0 \), \( \beta(\cdot, r) \) is a \( \mathcal{K} \)-function as the composition of \( \mathcal{K} \)-functions, and for fixed \( s > 0 \), \( \beta(s, \cdot) \in \mathcal{L} \), since the composition of \( \mathcal{K} \)- and \( \mathcal{L} \)-functions is of class \( \mathcal{L} \) (see \([14\) Section 24]), \([22\) Section 2]), so really \( \beta \in \mathcal{K}\mathcal{L} \). Further note that the summation of class-\( \mathcal{K} \) functions yields a class-\( \mathcal{K} \) function, so \( \gamma \in \mathcal{K} \). \( \square \)
Remark 4.2 To better understand the concept of dissipative finite-step ISS Lyapunov functions we will now give the connection to higher order iterates of system (16).

Let $G : \mathbb{R}^n \times \mathbb{R}^m$ from (6) be given. Then, for any $i \in \mathbb{N}$ with $i \geq 1$, we define the $i$th iterate of $G$ as follows:

$$
\xi \in \mathbb{R}^n, \ w_1 = u_1 \in \mathbb{R}^m \mapsto G^1(\xi, w_1) := G(\xi, u_1),
\xi \in \mathbb{R}^n, \ (u_1, \ldots, u_i) \in (\mathbb{R}^m)^i \mapsto G^i(\xi, w_i) := G(G^{i-1}(\xi, w_{i-1}), u_i)
$$

with $i \in \mathbb{N}$, $i \geq 2$. Now fix any $M \in \mathbb{N}$, and consider the system

$$
\bar{x}(k+1) = G^M(\bar{x}(k), w_M(k))
$$

(15)

with state $\bar{x} \in \mathbb{R}^n$ and input function $w_M(\cdot) = (u_1(\cdot), \ldots, u_M(\cdot)) \subset (\mathbb{R}^m)^M$. Firstly, for any $k \in \mathbb{N}$ there exist unique $l \in \mathbb{N}$ and $i \in \{1, \ldots, M\}$ such that $k = lM + i$. Let $u(\cdot) \in l^\infty$ be given. If we define the input sequences $u_1, \ldots, u_M$ by

$$
u_i(l) := u(lM + i), \quad l \in \mathbb{N}, \ i \in \{1, \ldots, M\}
$$

we call (15) the corresponding $M$th iterate system of system (6). Note that $\|w_M\|_\infty := \max\{\|u_1\|_\infty, \ldots, \|u_M\|_\infty\} = \|u\|_\infty$. It is not hard to see that for all $j \in \mathbb{N}$ and all $\xi \in \mathbb{R}^n$ we have

$$
x(jM, \xi, u(\cdot)) = \bar{x}(j, \xi, w_M(\cdot)).
$$

(16)

Thus, if system (6) is ISS, i.e., it satisfies (5), then also the corresponding $M$th iterated system (15) is ISS and satisfies

$$
|\bar{x}(j, \xi, w_M(\cdot))| \leq \beta(|\xi|, jM) + \gamma(\|u\|_\infty) =: \beta(|\xi|, j) + \gamma(\|w_M\|_\infty).
$$

Moreover, a dissipative finite-step ISS Lyapunov function for system (6) is also a dissipative ISS Lyapunov function for system (15).

Now let system (15) be ISS then there exists a dissipative ISS Lyapunov function $V$ for system (15) (see e.g. [19, Theorem 1] for continuous $G^M$ or [13, Lemma 2.3] for discontinuous $G^M$). From (16) we see that $V$ is also a dissipative finite-step ISS Lyapunov function for system (6), and by Theorem 4.4 we conclude that system (6) is ISS.

Summarizing, we obtain the following corollary.

Corollary 4.3 System (6) is ISS if and only if the corresponding $M$th iterated system (15) is ISS. In particular, a function $V : \mathbb{R}^n \to \mathbb{R}_+$ is a dissipative Lyapunov function for system (6) if and only if it is a dissipative finite-step Lyapunov function for system (6).

As finding a (dissipative) ISS Lyapunov function is a hard task, we will see in the remainder of this work that finding a dissipative finite-step ISS Lyapunov function (or equivalently a dissipative Lyapunov function of a corresponding iterated system) is sometimes easier. Furthermore, if we impose stronger conditions on the dissipative finite-step ISS Lyapunov function and the dynamics, then we can conclude an exponential decay of the bound on the system’s state.

Theorem 4.4 If there exists a dissipative finite-step ISS Lyapunov function for system (6) satisfying Definition 3.8 with $\alpha_1(s) = as^\lambda$, $\alpha_2(s) = bs^\lambda$, $\rho(s) = cs$, and $\sigma(s) = ds$ with $b \geq a > 0, c \in [0, 1)$ and $d, \lambda > 0$, then system (6) is expISS.
**Proof.** The proof follows the lines of the proof of Theorem 4.1. Hence, we will omit the detailed proof, and only give a sketch.

As discussed in Remark 3.3 and Lemma 3.4 we assume that Assumption 3.1 is satisfied for a linear $\omega_1 \in K_\infty$. Furthermore, we assume that $\omega_2(s) = w_2 s$ is also a linear function. This second assumption is only for simplifying the proof, but does not change the result. First note that in the proof of Theorem 4.1 we can choose $\nu(s) = h s$ with $h \in (0, 1)$, and since $\rho$ and $\sigma$ are linear $K_\infty$-functions, we obtain that $\sigma(s) := (\text{id} - \rho)^{-1} \circ \nu^{-1} \circ \sigma(s) = \frac{d}{\rho(1-c)} s$ is a linear function. Furthermore, in the case that $x(k) \notin \Delta$, we see that

$$V(x(k, M, \xi, u(\cdot))) \leq (c + h(1 - c)) V(x(k, \xi, u(\cdot))).$$

Define $\bar{\kappa} := (c + h(1 - c)) < 1$. In this case using the comparison Lemma A.2 we obtain the estimate

$$V(x(k, \xi, u(\cdot))) \leq \left(\frac{\bar{\kappa}^{1/M}}{\kappa}\right)^k \max_{M} V^\infty(x, u(\cdot)),$$

where $V^\infty(x, u(\cdot)) := \max_{j \in \{0, \ldots, M-1\}} V(x(j, \xi, u(\cdot)))$. Using Lemma A.4 (32) is satisfied for $\tilde{v}_j(s) = w_j^t s$ and $\tilde{\zeta}_j(s) = w_2 \sum_{i=0}^{j-1} w_i^t s$. Thus,

$$V(x(j, \xi, u(\cdot))) \leq b \left(\sum_{i=0}^{j-1} w_i^t \|u\|_\infty\right)^\lambda$$

with $\tilde{w}_1 := \max_{j \in \{0, \ldots, M-1\}} w_j^t$, and $\tilde{w}_2 := \max_{j \in \{0, \ldots, M-1\}} w_2 \sum_{i=0}^{j-1} w_i^t$, and hence,

$$V(x(k, \xi, u(\cdot))) \leq \max_{j \in \{0, \ldots, M-1\}} \left(\frac{\bar{\kappa}^{k/M}}{\kappa}\right) b \left(\sum_{i=0}^{j-1} w_i^t |\xi| + w_2 \sum_{i=0}^{j-1} w_i^t \|u\|_\infty\right)^\lambda$$

This implies

$$|x(k, \xi, u(\cdot))| \leq \left(a^{-1} V(x(k, \xi, u(\cdot)))\right)^{1/\lambda}$$

$$\leq \left(\frac{b \bar{\kappa}^{k/M}}{a \kappa}\right)^{1/\lambda} (\tilde{w}_1 |\xi| + \tilde{w}_2 \|u\|_\infty)$$

$$\leq \left(\frac{b \bar{\kappa}^{1/\lambda}}{a \kappa}\right)^{1/\lambda} (\tilde{w}_1 |\xi| + \tilde{w}_2 \|u\|_\infty)$$

with $\kappa := \bar{\kappa}^{1/\lambda} < 1$. So, system (6) satisfies (5) with $\beta$ as in (9) with $C = \left(\frac{b \bar{\kappa}^{1/\lambda}}{a \kappa}\right)^{1/\lambda}$ and $\kappa < 1$ as defined above. Hence, system (6) is expISS. 

While Theorem 4.1 shows the sufficiency of dissipative finite-step ISS Lyapunov functions to conclude ISS of system (6), we are now interested in the necessity. At this stage we can exploit the fact that any dissipative ISS Lyapunov function as defined in Definition 3.6 is a particular dissipative finite-step ISS Lyapunov function satisfying Definition 3.8 with $M = 1$.

**Proposition 4.5** If system (6) is ISS then there exists a dissipative finite-step ISS Lyapunov function for system (6).
Proof. If the right-hand site function $G : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ of system (6) is continuous, then [19, Theorem 1] implies the existence of a smooth function $V : \mathbb{R}^n \to \mathbb{R}_+$ satisfying

$$
\alpha_1(|\xi|) \leq V(\xi) \leq \alpha_2(|\xi|) \quad \text{and} \quad V(G(\xi, \mu)) - V(\xi) \leq -\alpha_3(|\xi|) + \sigma(|\mu|)
$$

for all $\xi \in \mathbb{R}^n$, $\mu \in \mathbb{R}^m$, and suitable $\alpha_1, \alpha_2, \alpha_3 \in \mathcal{K}_\infty$, $\sigma \in \mathcal{K}$. Then with Remark 3.7 and $M = 1$ it is easy to see that $V$ is a dissipative finite-step ISS Lyapunov function. Furthermore, this result also applies to discontinuous dynamics, see [13, Lemma 2.3]. □

Proposition 4.5 makes use of the converse ISS Lyapunov theorem in [19, 13] to guarantee the existence of a dissipative (finite-step) ISS Lyapunov function. Converse Lyapunov theorems have, in general, the disadvantage that they are theoretical results and no (ISS) Lyapunov can be explicitly constructed (see also the explanation in [7]). Hence finding a suitable (finite-step) (ISS) Lyapunov function is, in general, a hard task.

However, for the case of expISS systems of the form (6), we can show that norms are dissipative finite-step ISS Lyapunov functions.

Theorem 4.6 If system (6) is expISS then the function $V : \mathbb{R}^n \to \mathbb{R}_+$ defined by

$$
V(\xi) := |\xi|
$$

for all $\xi \in \mathbb{R}^n$ is a dissipative finite-step ISS Lyapunov function for system (6).

Proof. System (6) is expISS if it satisfies (8) with (9) for constants $C \geq 1$ and $\kappa \in [0, 1)$. Take $M \in \mathbb{N}$ such that $C\kappa^M < 1$, and $V$ as defined in (17). Clearly, $V$ is proper and positive definite with $\alpha_1 = \alpha_2 = \text{id} \in \mathcal{K}_\infty$. On the other hand, for any $\xi \in \mathbb{R}^n$, we have

$$
V(x(M, \xi, u(\cdot))) = |x(M, \xi, u(\cdot))| \leq C\kappa^M |\xi| + \gamma(\|u\|_\infty)
$$

$$
= C\kappa^M V(\xi) + \gamma(\|u\|_\infty) =: \rho(V(\xi)) + \sigma(\|u\|_\infty)
$$

where $\rho(s) := C\kappa^M s < s$ for all $s > 0$, since $C\kappa^M < 1$. Note that $(\text{id} - \rho)(s) = (1 - C\kappa^M)s \in \mathcal{K}_\infty$ and $\sigma := \gamma \in \mathcal{K}_\infty$, which shows (14). So $V$ defined in (17) is a dissipative finite-step ISS Lyapunov function for system (6). □

We emphasize that the hard task in Theorem 4.6 is finding a suitable, large enough $M \in \mathbb{N}$. However, Theorem 4.6 gives a procedure to check the ISS property as follows.

Procedure 4.7 Consider system (6) and its iterated systems (15).

[1] Check the $\mathcal{K}$-boundedness of $G$, Assumption 3.1.

[2] Set $k = 1$.

[3] Check $|G^k(\xi, w_k)| \leq c|\xi| + \sigma(\|w_k\|_\infty)$ for all $\xi \in \mathbb{R}^n$, $w_k \in (\mathbb{R}^m)^k$ with suitable $c \in [0, 1)$ and $\sigma \in \mathcal{K}_\infty$. If the inequality holds set $M = k$; else set $k = k + 1$ and repeat.

If this procedure is successful, then $V(\xi) := |\xi|$ is a dissipative finite-step ISS Lyapunov function for system (6) implying exponential ISS by Theorem 4.4.
5 Relaxed ISS Small-Gain Theorems

In this section we consider system (6) split into $N$ subsystems of the form

$$x_i(k+1) = g_i(x_1(k), \ldots, x_N(k), u(k)), \quad k \in \mathbb{N}, \quad (18)$$

with $x_i(0) \in \mathbb{R}^{n_i}$ and $g_i : \mathbb{R}^{n_1} \times \cdots \times \mathbb{R}^{n_N} \times \mathbb{R}^m \rightarrow \mathbb{R}^{n_i}$ for $i \in \{1, \ldots, N\}$. We further let $n = \sum_{i=1}^{N} n_i$, $x = (x_1, \ldots, x_N) \in \mathbb{R}^n$, then with $G = (g_1, \ldots, g_N)$ we call (6) the overall system of the subsystems (18).

A typical assumption of classical small-gain theorems is the requirement that each subsystem (18) has to admit an ISS Lyapunov function (see e.g. [15, 6, 28]). This assumption comes from the fact that in small-gain theory the influence of the other subsystems is considered as disturbance. Loosely speaking, the small-gain theorem (without external inputs) says that if all subsystems are considered GAS when decoupled from the other subsystems, and the influences of the other subsystems are small enough (i.e., a small-gain condition as (4) or (5) has to be satisfied), then GAS of the overall system can still be guaranteed. This is quite conservative as it can be seen from the following linear system

$$x(k+1) = \begin{pmatrix} 1.5 & 1 \\ -2 & -1 \end{pmatrix} x(k), \quad k \in \mathbb{N}.$$ 

As the right-hand site matrix is Schur stable the linear discrete-time system is GAS. But the first decoupled subsystem $x_1$ with dynamic $x_1(k+1) = 1.5x_1(k)$ is unstable. So classical small-gain theorems cannot be applied to this linear example.

The aim of this section is to derive a small-gain theorem which relaxes the assumption of classical small-gain theorems to admit (ISS) Lyapunov functions for each subsystem, by assuming that each system has to admit a Lyapunov-type function that decreases after a finite time rather than at each step. So the here proposed Lyapunov-type functions do not require 0-GAS of the subsystems as decoupled from the others.

The results in this section build upon the small-gain theorems presented in [12, 8] for systems without inputs, and the construction of (finite-step) ISS Lyapunov functions presented in [6]. The section is divided in two parts. In the first one, Section 5.1 we prove ISS of system (6) by constructing an overall dissipative finite-step ISS Lyapunov function. In Section 5.2 we show that the relaxed small-gain theorems derived are necessary for expISS systems.

5.1 Dissipative ISS small-gain theorems

In this section we derive small-gain theorems by constructing dissipative finite-step ISS Lyapunov functions for the overall system (6). We highlight that we do not require the subsystems (18) to be ISS, see also Remark 5.2.

We start with the case that the effect of the external input $u$ can be captured via maximization.

**Theorem 5.1** Assume there exist an $M \in \mathbb{N}$, $M \geq 1$, functions $V_i : \mathbb{R}^{n_i} \rightarrow \mathbb{R}_+$, $\gamma_{ij} \in \mathcal{K}_\infty \cup \{0\}$, $\gamma_{iu} \in \mathcal{K} \cup \{0\}$, and positive definite functions $\delta_i$, with $d_i := (\text{id} + \delta_i) \in \mathcal{K}_\infty$, for $i,j \in \{1, \ldots, N\}$ such that with $\Gamma_D$ defined in (2), and the diagonal operator $D$ defined by $D = \text{diag}(d_i)$ the following conditions hold.
(i) For all $i \in \{1, \ldots, N\}$, the functions $V_i$ are proper and positive definite.

(ii) For all $\xi \in \mathbb{R}^n$ and $u(\cdot) \subseteq \mathbb{R}^m$ it holds
\[
\begin{bmatrix}
V_1(x_1(M, \xi, u(\cdot))) \\
\vdots \\
V_N(x_N(M, \xi, u(\cdot)))
\end{bmatrix} \leq \max \left\{ \Gamma_\oplus \left( \begin{bmatrix}
V_1(\xi_1) \\
\vdots \\
V_N(\xi_N)
\end{bmatrix} \right), \begin{bmatrix}
\gamma_{1u}(\|u\|_\infty) \\
\vdots \\
\gamma_{Nu}(\|u\|_\infty)
\end{bmatrix} \right\}.
\]

(iii) The small-gain condition\footnote{Note that the strong small-gain condition in Definition \ref{def:small-gain-condition} requires the functions $\delta_i$ in the diagonal operator to be of class $\mathcal{K}_\infty$, where here we only require $\delta_i$ to be positive definite.} $\Gamma_\oplus \circ D \not\geq \text{id}$ holds.

Then there exists $\tilde{\sigma} \in \mathcal{K}_\infty^N$ with $(\Gamma_\oplus \circ D)(\tilde{\sigma}) < \tilde{\sigma}$. If for any $i \in \{1, \ldots, N\}$ there exists a $\mathcal{K}_\infty$-functions $\tilde{\alpha}_i$ satisfying
\[
\tilde{\sigma}_i^{-1} \circ d_i^{-1} \circ \tilde{\sigma}_i = \tilde{\sigma}_i^{-1} \circ (\text{id} + \delta_i)^{-1} \circ \tilde{\sigma}_i = \text{id} - \tilde{\alpha}_i
\]
then the function
\[
V(\xi) := \max_i(\tilde{\sigma}_i^{-1} \circ d_i^{-1})(V_i(\xi_i)).
\]
is a dissipative finite-step ISS Lyapunov function for system (6). In particular, system (6) is ISS.

**Proof.** Assume that $V_i$ and $\gamma_{ij}, \gamma_{iu}$ satisfy the hypothesis of the theorem. Denote $\gamma_u(\cdot) := (\gamma_{1u}(\cdot), \ldots, \gamma_{Nu}(\cdot))^T$. Then from condition (iii) and\footnote{see also Section 2.3} [6, Corollary 5.7] it follows that there exists a $\mathcal{K}_\infty$-function $\varphi$ and a function $\tilde{\sigma} \in \mathcal{K}_\infty^N$ such that $\max\{(\Gamma_\oplus \circ D)(\tilde{\sigma}(s)), \gamma_u(\varphi(s))\} < \tilde{\sigma}(s)$ holds for all $s > 0$. In particular,
\[
\max_{i,j \in \{1, \ldots, N\}} \tilde{\sigma}_i^{-1} \circ \gamma_{ij} \circ d_j \circ \tilde{\sigma}_j, \max_{i \in \{1, \ldots, N\}} \tilde{\sigma}_i^{-1} \circ \gamma_{iu} \circ \varphi < \text{id}.
\]

In the following let $i, j \in \{1, \ldots, N\}$. Let $V : \mathbb{R}^n \rightarrow \mathbb{R}_+$ be defined as in (20). We will show that $V$ is a dissipative finite-step ISS Lyapunov function for the overall system (6). To show this note that condition (i) implies the existence of $\alpha_{1i}, \alpha_{2i} \in \mathcal{K}_\infty$ such that for all $\xi_i \in \mathbb{R}^{n_i}$ we have $\alpha_{1i}(\|\xi_i\|) \leq V_i(\xi_i) \leq \alpha_{2i}(\|\xi_i\|)$. Thus,
\[
V(\xi) \geq \max_i(\tilde{\sigma}_i^{-1} \circ d_i^{-1})(\alpha_{1i}(\|\xi_i\|)) \geq \alpha_1(\|\xi\|)
\]
with $\alpha_1 := \min_j \tilde{\sigma}_j^{-1} \circ d_j^{-1} \circ \alpha_{1j} \circ \frac{1}{\kappa} \text{id} \in \mathcal{K}_\infty$, where $\kappa \geq 1$ comes from (11). On the other hand we get
\[
V(\xi) \leq \max_i(\tilde{\sigma}_i^{-1} \circ d_i^{-1})(\alpha_{2i}(\|\xi_i\|)) \leq \alpha_2(\|\xi\|)
\]
with $\alpha_2 := \max_i \tilde{\sigma}_i^{-1} \circ d_i^{-1} \circ \alpha_{2i} \in \mathcal{K}_\infty$, which shows $V$ defined in (20) is proper and positive.
Remark 5.2

(i) To understand the assumptions imposed in Theorem 5.1 consider the case that the maximum can be upper bounded by summation, this shows that $V$ is a dissipative finite-step ISS Lyapunov function as defined in Definition 3.8. Then from Theorem 4.1 we conclude that system (13) is ISS.

Define $\rho := \max_i (\text{id} - \hat{\alpha}_i)$, then $\rho \in K_\infty$ by (19), and satisfies $\text{id} - \rho = \max_i \hat{\alpha}_i \in K_\infty$. Noticing that the maximum can be upper bounded by summation, this shows that $V$ is a dissipative finite-step ISS Lyapunov function as defined in Definition 3.8. Then from Theorem 4.1 we conclude that system (13) is ISS.

Remark 5.2 (i) To understand the assumptions imposed in Theorem 5.1 consider the case that $M = 1$ and $\delta_i \in K_\infty$, $i \in \{1, \ldots, N\}$.

First, by condition (ii) we have for any $i \in \{1, \ldots, N\}$

$$V_i(x_i(1, \xi, u(\cdot))) \leq \max \left\{ \max_{j \in \{1, \ldots, N\}} \gamma_{ij}(V_j(\xi_j)), \gamma_{iu}(\|u\|) \right\}.$$  (22)

From the small-gain condition (iii) we conclude, since $\delta_i \in K_\infty$, that $\gamma_{ii} \circ (\text{id} + \delta_i) < \text{id}$ by applying the $i$th unit vector. Hence, $\gamma_{ii} < (\text{id} + \delta_i)^{-1} = \text{id} - \delta_i$ with $\delta_i \in K_\infty$ by (31), Lemma 2.4]. Thus, we can write (22) as

$$V_i(x_i(1, \xi, u(\cdot))) \leq \left[ \text{id} - \delta_i \right](V_i(\xi_i)) + \sum_{j=1}^{N} \gamma_{ij}(V_j(\xi_j)) + \gamma_{iu}(\|u\|).$$

Together with condition (i) this implies that the functions $V_i$ are dissipative ISS Lyapunov functions for the subsystem (13).

Therefore, if $M = 1$ and $\delta_i \in K_\infty$ then Theorem 5.1 is a dissipative small-gain theorem for discrete-time systems in the classical sense, see also Remark 5.5.

(ii) If $M = 1$ and $\delta_i$ are only positive definite, then the functions $V_i$ are not necessarily dissipative ISS Lyapunov functions, as we cannot ensure that the decay of $V_i$ in terms of the function $\rho_i$ satisfies $\text{id} - \rho_i \in K_\infty$. Hence, even for $M = 1$ we do not necessarily assume that the subsystems are ISS.

(iii) Now consider the case $M > 1$. Typically in classical ISS small-gain theorems the systems are required to be ISS with respect to internal and external inputs, and the small-gain condition ensures that the (internal and external) inputs cannot destabilize the subsystem. In particular, the subsystems are 0-GAS.

In Theorem 5.1 the internal inputs $x_j$ may have an stabilizing effect on system $x_i$ in the first
Theorem 5.4

In particular, the subsystem (18) can be 0-input unstable, i.e., the system \( x_i(k + 1) = g_i(0, \ldots, 0, x_i(k), 0, \ldots, 0) \) can be unstable. See also Section 6 devoted to the discussion of an example.

Example 5.3 In this example we will show that condition (19) is not trivially satisfied, even if \( \delta_i \in \mathcal{K}_\infty \). To this end consider the \( \mathcal{K}_\infty \)-functions

\[
\tilde{\sigma}(s) := e^s - 1, \quad \tilde{\sigma}^{-1}(s) = \log(s + 1), \quad \hat{\delta}(s) = (s + 1)(1 - \left(\frac{1}{s+1}\right)^{\frac{1}{s+1}}).
\]

First note that \( \hat{\delta} \in \mathcal{K}_\infty \), since the derivative \( \hat{\delta}'(s) > 0 \) for all \( s \geq 0 \), and \( \lim_{s \to \infty} \hat{\delta}(s) = \infty \).

Furthermore, \( (\text{id} - \hat{\delta})(s) = (s + 1)^{1 - \frac{1}{s+1}} - 1 \in \mathcal{K}_\infty \). Then, similarly as in [31, Lemma 2.4], there exists a function \( \delta \in \mathcal{K}_\infty \) such that \( (\text{id} + \delta)^{-1} = \text{id} - \delta \). But then we have

\[
\tilde{\sigma}^{-1} \circ (\text{id} + \delta)^{-1} \circ \tilde{\sigma}(s) = \tilde{\sigma}^{-1} \circ (\text{id} - \hat{\delta}) \circ \tilde{\sigma}(s) = s(1 - e^{-s}).
\]

But this \( \mathcal{K}_\infty \)-function approaches the identity function. So there cannot exist a \( \mathcal{K}_\infty \)-function \( \tilde{\alpha} \) satisfying (19).

If the inputs effect can be captured in an additive way rather than in a maximal way, the condition (ii) in Theorem 5.1 has to be changed. Note that in the case of summation the small-gain condition invoked in Theorem 5.1 is not strong enough to ensure that \( V \) defined in (20) is a dissipative finite-step ISS Lyapunov function (see [6]), so we also have to change condition Theorem 5.1 (iii). In particular, we have to assume that the functions \( \delta_i \) are of class \( \mathcal{K}_\infty \), and not only positive definite. We recall from Section 2.3 that if the diagonal operator \( D = \text{diag}(\text{id} + \delta_i) \) is split into

\[
D = D_\delta \circ D_1, \quad D_j = \text{diag}(\text{id} + \delta_{ij}), \quad \delta_{ij} \in \mathcal{K}_\infty, \ i \in \{1, \ldots, N\}, \ j \in \{1, 2\}
\]

then \( D \circ \Gamma_{\oplus} \not= \text{id} \) is equivalent to \( D_1 \circ \Gamma_{\oplus} \circ D_2 \not= \text{id} \).

Theorem 5.4 Assume there exist an \( M \in \mathbb{N} \), \( M \geq 1 \), functions \( V_i : \mathbb{R}^n_i \to \mathbb{R}_+ \), \( \gamma_{ij} \in \mathcal{K}_\infty \cup \{0\} \), \( \gamma_{iu} \in \mathcal{K}_\cup \{0\} \), and \( \delta_i \in \mathcal{K}_\infty \), for \( i, j \in \{1, \ldots, N\} \) such that with \( \Gamma_{\oplus} \) defined in (2), and the diagonal operator \( D \) defined by \( D = \text{diag}(d_i) := \text{diag}(\text{id} + \delta_i) \) the following conditions hold.

(i) For all \( i \in \{1, \ldots, N\} \), the functions \( V_i \) are proper and positive definite.

(ii) For all \( \xi \in \mathbb{R}^n \) and \( u(\cdot) \subset \mathbb{R}^m \) it holds

\[
\begin{bmatrix}
V_1(x_1(M, \xi, u(\cdot))) \\
\vdots \\
V_N(x_N(M, \xi, u(\cdot)))
\end{bmatrix} \leq \Gamma_{\oplus} \begin{bmatrix}
V_1(\xi_1) \\
\vdots \\
V_N(\xi_N)
\end{bmatrix} + \begin{bmatrix}
\gamma_{1u}(\|u\|_\infty) \\
\vdots \\
\gamma_{Nu}(\|u\|_\infty)
\end{bmatrix}.
\]

Note that [31] Lemma 2.4] argues that if \( \delta \in \mathcal{K}_\infty \) is given, there exists a suitable \( \hat{\delta} \). But the other direction follows by defining \( \delta = \delta \circ (\text{id} - \hat{\delta})^{-1} \in \mathcal{K}_\infty \).
(iii) The strong small-gain condition $D \circ \Gamma_\oplus \not\sim \text{id}$ holds.

Let $D = D_2 \circ D_1$ be split as in (23) then there exists a function $\tilde{\sigma} \in \mathcal{K}_\infty^N$ satisfying $(D_1 \circ \Gamma_\oplus \circ D_2)(\tilde{\sigma}) < \tilde{\sigma}$. If for any $i \in \{1, \ldots, N\}$ there exists a $\mathcal{K}_\infty$-functions $\tilde{\alpha}_i$ satisfying
\begin{equation}
\tilde{\alpha}_i^{-1} \circ d_i^{-1} \circ \tilde{\sigma}_i = \text{id} - \tilde{\alpha}_i \tag{24}
\end{equation}
then the function $V : \mathbb{R}^n \to \mathbb{R}_+$ defined by
\begin{equation}
V(\xi) := \max_i (\tilde{\alpha}_i^{-1} \circ d_i^{-1})(\mathcal{V}_i(\xi_i)) \tag{25}
\end{equation}
is a dissipative finite-step ISS Lyapunov function for system (6). In particular, system (6) is ISS.

**Proof.** Let $D = D_2 \circ D_1$ be split as in (23). Now assume that $\mathcal{V}_i$ and $\gamma_{ij}, \gamma_{iu}$ satisfy the hypothesis of the theorem. Then from condition (iii), [3] Lemma 2.6 and [3] Corollary 5.6 it follows that there exists a $\mathcal{K}_\infty$-function $\varphi$ and a function $\tilde{\sigma} \in \mathcal{K}_\infty^N$ such that
\begin{equation}
(\Gamma_\oplus \circ D_2)(\tilde{\sigma}(r)) + \gamma_u(\varphi(r)) < \tilde{\sigma}(r) \tag{26}
\end{equation}
holds for all $r > 0$. Note that here (in contrast to Theorem 5.1) we need that the functions $\delta_i$ in the diagonal operator $D$ are of class $\mathcal{K}_\infty$. In the following let $i, j \in \{1, \ldots, N\}$. Take $V$ from (20). First note that $V$ is proper and positive definite, which follows directly from the proof of Theorem 5.1.

To show the decay of $V$, i.e., (14), note that from (20) it follows
\begin{equation}
\max_{i,j} \tilde{\alpha}_i^{-1} \circ (\gamma_{ij} \circ d_j \circ \tilde{\sigma}_j(r) + \gamma_{iu} \circ \varphi(r)) < r \tag{27}
\end{equation}
for all $r > 0$. Then we have
\begin{align*}
V(x(M, \xi, u(\cdot))) &= \max_i (\tilde{\alpha}_i^{-1} \circ d_i^{-1})(\mathcal{V}_i(x_i(M, \xi, u(\cdot)))) \\
&\leq \max_i (\tilde{\alpha}_i^{-1} \circ d_i^{-1}) \left( \max_j \gamma_{ij}(\mathcal{V}_j(\xi_j)) + \gamma_{iu}(\|u\|_\infty) \right) \\
&\leq \max_{i,j} (\text{id} - \tilde{\alpha}_i) \circ \tilde{\alpha}_i^{-1} (\gamma_{ij}(\mathcal{V}_j(\xi_j)) + \gamma_{iu}(\|u\|_\infty)) \\
&= \max_{i,j} (\text{id} - \tilde{\alpha}_i) \circ \tilde{\alpha}_i^{-1} \left( \gamma_{ij} \circ d_j \circ \tilde{\sigma}_j \circ (\tilde{\alpha}_j^{-1} \circ d_j^{-1} \circ \mathcal{V}_j(\xi_j)) + \gamma_{iu} \circ \varphi^{-1}(\|u\|_\infty) \right) \\
&\leq \max_{i} (\text{id} - \tilde{\alpha}_i)(V(\xi) + \gamma_{iu}(\|u\|_\infty)) + \max_{i} (\text{id} - \tilde{\alpha}_i)(\varphi^{-1}(\|u\|_\infty)).
\end{align*}
Again, as in the proof of Theorem 5.1 this shows that $V$ is a dissipative finite-step ISS Lyapunov function as defined in Definition 5.8 with $\rho := \max_i (\text{id} - \tilde{\alpha}_i)$. From Theorem 4.1 we conclude that system (6) is ISS. \hfill \Box
Remark 5.5 If Theorem 5.1 (resp. Theorem 5.4) is satisfied with $M = 1$, then the constructed dissipative finite-step Lyapunov function $V$ in (20) (resp. (25)) is a dissipative ISS Lyapunov function. In particular, we obtain the following special cases: If Theorem 5.4 is satisfied for $M = 1$, then this gives a dissipative-form discrete-time version of [6, Corollary 5.6]. On the other hand, for $M = 1$, Theorem 5.1 includes the ISS variant of [16, Theorem 3] as a special case. We also note that the small-gain result [28, Theorem 1] considers the classical case that all subsystems are endowed with a continuous (implication-form) ISS Lyapunov function, which we do not require.

In Theorem 5.1 we introduced the diagonal operator $D$. Note that $D \circ \Gamma \not\geq \text{id}$ implies $\Gamma \not\geq \text{id}$ since $d_i = \text{id} + \delta_i > \text{id}$, while the converse implication does not hold in general. In addition, (19) is assumed to hold. In the following corollary we impose further assumptions such that we do not need the diagonal operator $D$ anymore. Under these stronger assumptions, system (6) is shown to be expISS.

Corollary 5.6 Assume there exist an $M \in \mathbb{N}$, $M \geq 1$, linear functions $\gamma_{ij} \in K_\infty$, and functions $V_i : \mathbb{R}^{n_i} \rightarrow \mathbb{R}_+$ satisfying Theorem 5.1-(i) with linear functions $\alpha_{1i}, \alpha_{2i}$. Let Theorem 5.1-(ii) hold, and instead of Theorem 5.1-(iii) let the small-gain condition (4) hold. Furthermore, assume that $\omega_1$ in Assumption 3.1 is linear. Then system (6) is expISS.

Proof. We follow the proof of Theorem 5.1. By the small-gain (4) there exists an $\Omega$-path $\tilde{\sigma} \in K_N^{\infty}$ satisfying $\Gamma_\oplus(\tilde{\sigma})(r) < \tilde{\sigma}(r)$ for all $r > 0$, see [6]. Moreover, as the functions $\gamma_{ij} \in K_\infty$ are linear, we can also assume the $\Omega$-path functions $\tilde{\sigma}_i \in K_\infty$ to be linear, see [10]. Thus, the function

$$V(\xi) := \max_i \tilde{\sigma}_i^{-1}(V_i(\xi_i))$$

has linear bounds $\alpha_1$ and $\alpha_2$. Furthermore, since $\tilde{\sigma}_i$ and $\gamma_{ij}$ are linear functions, we obtain (14) with linear function $\rho := \max_{i,j} \tilde{\sigma}_i^{-1} \circ \gamma_{ij} \circ \tilde{\sigma}_j < \text{id}$, and $\sigma := \max_i \tilde{\sigma}_i^{-1} \circ \gamma_{iu}$. Clearly, $(\text{id} - \rho) \in K_\infty$ by linearity of $\rho$. Thus, $V$ is a dissipative finite-step ISS Lyapunov function for system (6). Since $\omega_1$ in Assumption 3.1 is linear, we can apply Theorem 5.1 to show that system (6) is expISS. Note that the requirement that $\omega_1$ is linear is necessary for the system being expISS by Remark 3.3. □

The same reasoning applies in the case, where the external input enters additively.

Corollary 5.7 Assume there exist an $M \in \mathbb{N}$, $M \geq 1$, linear functions $\gamma_{ij} \in K_\infty$, and functions $V_i : \mathbb{R}^{n_i} \rightarrow \mathbb{R}_+$ satisfying Theorem 5.4-(i) with linear functions $\alpha_{1i}, \alpha_{2i}$. Let Theorem 5.4-(ii) and the small-gain condition (4) hold. Furthermore, assume that $\omega_1$ in Assumption 3.1 is linear. Then system (6) is expISS.

Proof. We omit the details as the proof follows the lines of the proof of Theorem 5.4 combined with the argumentation of the proof of Corollary 5.6. First, the small-gain condition implies the existence of an $\Omega$-path $\tilde{\sigma} \in K_N^{\infty}$, which is linear as the functions $\gamma_{ij}$ are linear. In particular, $\Gamma_\oplus(\tilde{\sigma}(r)) < \sigma(r)$ for all $r > 0$.

Next, note that the function $V$ defined in (28) has linear bounds as shown in the proof of Corollary 5.6, and satisfies

$$V(x(M, \xi, u(\cdot))) \leq \rho(V(\xi)) + \sigma(\|u\|_\infty)$$

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with \( \rho := \max_{i,j} \tilde{\sigma}_i^{-1} \circ \gamma_{ij} \circ \tilde{\sigma}_j \) and \( \sigma := \max_{i} \tilde{\sigma}_i^{-1} \circ \gamma_{iu} \), which can be seen by a straightforward calculation, invoking Theorem 5.4(ii) and the linearity of the \( \Omega \)-path \( \tilde{\sigma} \). Again as in the proof of Corollary 5.6 \((id - \rho) \in \mathcal{K}_{\infty}\) by linearity of \( \rho \). Thus, \( V \) defined in (28) is a dissipative finite-step ISS Lyapunov function for system (6). Since \( \omega_1 \) in Assumption 3.1 is linear, we can apply Theorem 4.4, and the result follows. \( \square \)

### 5.2 Non-conservative expISS Small-Gain Theorems

General small-gain theorems are conservative, as explained at the beginning of this section. In the remainder of this section we show that the relaxation of classical small-gain theorems given in Theorems 5.1 and 5.4 is non-conservative at least for expISS systems, i.e., Corollary 5.6 and 5.7 provide conditions that are not only sufficient but also necessary.

**Theorem 5.8** Let system (6) be the overall system of the subsystems (18). Then system (6) is expISS if and only if

(i) Assumption 3.1 holds with linear \( \omega_1 \), and

(ii) there exist an \( M \in \mathbb{N} \), \( M \geq 1 \), linear functions \( \gamma_{ij} \in \mathcal{K}_{\infty} \), proper and positive definite functions \( V_i : \mathbb{R}^{n_i} \to \mathbb{R}^+ \) with linear bounds \( \alpha_{1i}, \alpha_{2i} \in \mathcal{K}_{\infty} \) such that the following holds:

(a) Condition (ii) of Theorem 5.1 (and thus also condition (ii) of Theorem 5.4);

(b) The small-gain condition (4).

**Proof.** The if part is shown by Corollary 5.6 and Corollary 5.7 so we only have to prove the only if part. Clearly, since system (6) is expISS, Assumption 3.1 holds with linear \( \omega_1 \), see Remark 3.3. Furthermore, the function \( V(\xi) := |\xi| \) for all \( \xi \in \mathbb{R}^n \) is a dissipative finite-step ISS Lyapunov function for system (6) by Theorem 4.6. Hence, there exist \( \bar{M} \in \mathbb{N} \), \( \sigma \in \mathcal{K} \) and a positive definite function \( \rho \), where we can assume \( \rho(s) = cs \) with \( c < 1 \) by Theorem 4.6 such that for all \( \xi \in \mathbb{R}^n \) and all \( u(\cdot) \subset \mathbb{R}^m \)

\[ |x(\bar{M}, \xi, u(\cdot))| \leq c|\xi| + \sigma(\|u\|\infty). \]  

(29)

Define \( V_i(\xi_i) := |\xi_i| \) for \( i \in \{1, \ldots, N\} \), where the norm for \( \xi_i \in \mathbb{R}^{n_i} \) is defined in the preliminaries. Then \( V_i \) is proper and positive definite with \( \alpha_{1i} = \alpha_{2i} = \text{id} \) for all \( i \in \{1, \ldots, N\} \). Take \( \kappa \geq 1 \) from (11), and define

\[ l := \min \{ \ell \in \mathbb{N} : c^\ell \kappa < \frac{1}{2} \}. \]
Then we have

\[ V_i(x_i(l\hat{M}, \xi, u(\cdot))) = |x_i(l\hat{M}, \xi, u(\cdot))| \leq |x(l\hat{M}, \xi, u(\cdot))| \]

\[ \leq c|x((l-1)\hat{M}, \xi, u(\cdot))| + \sigma(\|u\|_\infty) \]

\[ \leq c|\xi| + \sum_{j'=1}^{l} c^{j'-1}\sigma(\|u\|_\infty) \]

\[ \leq \max_j c'\kappa_j|\xi_j| + \sum_{j'=1}^{l} c^{j'-1}\sigma(\|u\|_\infty) \]

\[ \leq \max_j \left( \max_j \gamma_{ij}(\xi_j), \gamma_{iu}(\|u\|_\infty) \right) \]

\[ \leq \max_j \gamma_{ij}(\xi_j) + \gamma_{iu}(\|u\|_\infty) \]

with \( \gamma_{iu}(\cdot) := 2\sum_{j=1}^{l} c^{j-1}\sigma(\cdot) \), and \( \gamma_{ij} := 2c'\kappa \text{id} \). The last inequality shows condition (ii) of Theorem 5.4-(ii), while the second last inequality shows condition (ii) of Theorem 5.1 for \( M = l\hat{M} \). Finally, by definition of \( l \in \mathbb{N} \), we have \( \gamma_{ij} < \text{id} \) for all \( i, j \in \{1, \ldots, N\} \). By Proposition 2.4, this shows the small-gain condition (4). This proves the theorem. ☐

The converse expISS small-gain Theorem 5.8 is proved in a constructive way, i.e., it is shown that under the assumption that the origin of system (6) is expISS, we can choose the Lyapunov-type functions \( V_i : \mathbb{R}^{n_i} \rightarrow \mathbb{R}_+ \) as norms, i.e., \( V_i(\cdot) = |\cdot| \). Then there exist an \( M \in \mathbb{N} \) and linear gains \( \gamma_{ij} \in \mathcal{K}_\infty \) satisfying Theorem 5.1-(ii), and thus also Theorem 5.4-(ii), as well as the small-gain condition (4). This suggests the following procedure.

**Procedure 5.9** Consider (6) as the overall system of the subsystems (18).

(i) Check that Assumption 3.1 is satisfied with a linear \( \omega_1 \) (else the origin of system (6) cannot be expISS, see Remark 3.3);

(ii) Define \( V_i(\xi_i) := |\xi_i| \) for \( \xi_i \in \mathbb{R}^{n_i} \), and set \( k = 1 \);

(iii) Compute \( \gamma_{iu} \in \mathcal{K}_\infty \) and linear \( \gamma_{ij} \in \mathcal{K}_\infty \cup \{0\} \) satisfying

\[ V_i(x_i(k, \xi, u(\cdot))) = |x_i(k, \xi, u(\cdot))| \leq \max_{j \in \{1, \ldots, N\}} \gamma_{ij}|\xi_j| + \gamma_{iu}(\|u\|_\infty); \]

(iv) Check the small-gain condition (4) with \( \Gamma_{\Xi} \) defined in 2. If (4) is violated set \( k = k + 1 \) and repeat with (iii).

If this procedure is successful, then expISS of the origin of the overall system (6) is shown by Theorem 5.8.

**Remark 5.10** Although this methodology is straightforward, finding a suitable \( M \in \mathbb{N} \) may be computationally intractable (NP-hard), as it was shown in [4], even for simple classes of systems. Nevertheless, a systematic way to find a suitable number \( M \in \mathbb{N} \) for certain classes of systems is discussed in [7].

In the next section we consider a nonlinear system and show how Procedure 5.9 can be applied.
6 Example

In Section 5 the conservativeness of classical small-gain theorems was illustrated by a linear example without external inputs, which is GAS, but the decoupled subsystems are even unstable. In this section we consider a nonlinear example with external inputs and show how the relaxed small-gain theorem from Section 5 can be applied.

Consider the nonlinear system

\[ \begin{align*}
    x_1(k+1) &= x_1(k) - 0.3x_2(k) + u(k) \\
    x_2(k+1) &= x_1(k) + 0.3 \frac{x_2^2(k)}{1 + x_2^2(k)}.
\end{align*} \tag{30} \]

We will show that the origin of this system is ISS. Since in practice finding a suitable ISS Lyapunov function is a hard task, we are trying to find a suitable dissipative finite-step ISS Lyapunov function. Therefore, we split the system into two subsystems. Note that the origin of the first subsystem decoupled from the second subsystem with zero input is only globally stable and not 0-GAS, hence not ISS. So we cannot find an ISS Lyapunov function for this subsystem. At this point standard small-gain theorems would fail.

The converse small-gain results in Section 5.2 suggest to prove ISS of the origin by a search for suitable functions \( V_i \) and \( \gamma_{ij} \in \mathcal{K}_\infty \). Hence, we will follow Procedure 5.9. First, the right-hand side \( G \) of (30) is \( \mathcal{K} \)-bounded, since

\[ \|G(\xi, \mu)\|_\infty \leq \max\{|\xi_1| + 0.3|\xi_2| + |\mu|, |\xi_1| + 0.3 \frac{\xi_2^2}{1 + \xi_2^2}\} \leq 1.3 \|\xi\|_\infty + |\mu|, \]

where we used that for all \( x \in \mathbb{R} \) we have

\[ \frac{x^2}{1+x^2} \leq \frac{|x|^2}{2}. \tag{31} \]

Let \( V_i(\xi_i) := |\xi_i| \), \( i = 1, 2 \). Then we compute for all \( \xi \in \mathbb{R}^2 \),

\[ \begin{align*}
    V_1(x_1(1, \xi, u(\cdot))) &= |\xi_1 - 0.3\xi_2 + u(0)| \leq \max\{2V_1(\xi_1), 0.6V_2(\xi_2)\} + \|u\|_\infty, \\
    V_2(x_2(1, \xi, u(\cdot))) &= |\xi_1 + 0.3 \frac{\xi_2^2}{1 + \xi_2^2}| \leq \max\{2V_1(\xi_1), 0.6 \frac{V_2(\xi_2)}{1 + V_2(\xi_2)}\}.
\end{align*} \]

Since \( \gamma_{11}(s) = 2s \), the small-gain condition is violated and we cannot conclude stability. This is clear from the above observation that the origin of the first subsystem is not ISS.

Computing solutions \( x(k, \xi, u(\cdot)) \) with initial condition \( \xi \in \mathbb{R}^2 \) and input \( u(\cdot) \subset \mathbb{R} \) we see that for \( k = 3 \) we have

\[ x(3, \xi, u(\cdot)) = \begin{pmatrix}
    0.4\xi_1 - 0.2\xi_2 - 0.09 \frac{\xi_2^2}{1 + \xi_2^2} - 0.09 \frac{(\xi_1 + 0.3 \frac{\xi_2^2}{1 + \xi_2^2})^2}{1 + (\xi_1 + 0.3 \frac{\xi_2^2}{1 + \xi_2^2})^2} + 0.7u(0) + u(1) + u(2) \\
    0.7\xi_1 - 0.3\xi_2 - 0.09 \frac{\xi_2^2}{1 + \xi_2^2} + 0.3 \frac{(\xi_1 + 0.3 \frac{\xi_2^2}{1 + \xi_2^2})^2}{1 + (\xi_1 + 0.3 \frac{\xi_2^2}{1 + \xi_2^2})^2} + u(0) + u(1)
\end{pmatrix}. \]

\[ \begin{align*}
    &\xi_1 - 0.3\xi_2 + 0.3 \frac{\xi_2^2}{1 + \xi_2^2} \frac{2}{1 + (\xi_1 + 0.3 \frac{\xi_2^2}{1 + \xi_2^2})^2} + u(0) + u(1) \\
    &1 + (\xi_1 + 0.3 \frac{\xi_2^2}{1 + \xi_2^2})^2
\end{align*} \]

\[ \begin{align*}
    &\xi_1 - 0.3\xi_2 + 0.3 \frac{\xi_2^2}{1 + \xi_2^2} \frac{2}{1 + (\xi_1 + 0.3 \frac{\xi_2^2}{1 + \xi_2^2})^2} + u(0) + u(1) \\
    &1 + (\xi_1 + 0.3 \frac{\xi_2^2}{1 + \xi_2^2})^2
\end{align*} \]

We could also make the first system 0-input unstable by letting \( x_1(k+1) = (1 + \epsilon)x_1(k) - 0.3x_2(k) + u(k) \) and \( \epsilon > 0 \) small enough, see also §. But here we let \( \epsilon = 0 \) to simplify computations.
Using (31) we compute
\begin{align*}
V_1(x_1(3, \xi, u(\cdot))) &\leq 0.4|\xi_1| + 0.21|\xi_2| + \frac{0.09}{2}|\xi_2| + \frac{0.09}{2}(|\xi_1| + \frac{0.3}{2}|\xi_2|) \\
&\quad + 0.7|u(0)| + |u(1)| + |u(2)| \\
&= \max\{0.89V_1(\xi_1), 0.5235V_2(\xi_2)\} + 2.7\|u\|\infty,
\end{align*}
\begin{align*}
V_2(x_2(3, \xi, u(\cdot))) &\leq 0.7|\xi_1| + 0.3|\xi_2| + \frac{0.09}{2}|\xi_2| + \frac{0.3}{2}(|\xi_1| + 0.3|\xi_2| + \frac{0.3}{2}(|\xi_1| + \frac{0.3}{2}|\xi_2|)) \\
&\quad + |u(0)| + |u(1)| \\
&= \max\{1.745V_1(\xi_1), 0.78675V_2(\xi_2)\} + 2\|u\|\infty.
\end{align*}

From this we derive the linear functions
\begin{align*}
\gamma_{11}(s) &= 0.89s, & \gamma_{12}(s) &= 0.5235s, & \gamma_{1u}(s) &= 2.7s, \\
\gamma_{21}(s) &= 1.745s, & \gamma_{22}(s) &= 0.78675s, & \gamma_{2u}(s) &= 2s,
\end{align*}
yielding the map \( \Gamma_\oplus : \mathbb{R}_+^2 \to \mathbb{R}_+^2 \) from (2) as
\[
\Gamma_\oplus((s_1, s_2)) = \left( \begin{array}{c} \max\{0.89s_1, 0.5235s_2\} \\ \max\{1.745s_1, 0.78675s_2\} \end{array} \right).
\]

Since \( \gamma_{11} < \text{id}, \gamma_{22} < \text{id} \) and \( \gamma_{12} \circ \gamma_{21} < \text{id} \), we conclude from the cycle condition Proposition 2.4 that the small-gain condition (4) is satisfied. Hence, from Corollary 5.7 we can now conclude that the origin of system (30) is \( \text{expISS} \).

**Remark 6.1** The small-gain results in Section 5, and in particular Corollary 5.7, prove the ISS property of the origin of the interconnected system (6) by constructing a dissipative finite-step ISS Lyapunov function. The following shows that this procedure is straightforward. To do this we use the method proposed in [10] to compute an \( \Omega \)-path \( \tilde{\sigma}(r) := (\frac{0.5r}{0.9r}) \) that satisfies
\[
\Gamma_\oplus(\tilde{\sigma}(r)) = \left( \begin{array}{c} 0.47115r \\ 0.8725r \end{array} \right) < \left( \begin{array}{c} 0.5r \\ 0.9r \end{array} \right) = \tilde{\sigma}(r)
\]
for all \( r > 0 \). From the proof of Corollary 5.7 we can now conclude that
\[
V(\xi) := \max_i \tilde{\sigma}_i^{-1}(\mathbb{V}_i(\xi_i)) = \max\{2|\xi_1|, \frac{10}{9}|\xi_2|\}
\]
is a dissipative finite-step ISS Lyapunov function for the overall system (30). In particular, as shown in the proof of Corollary 5.7 we can compute
\[
\rho(s) := \max_{i,j \in \{1, 2\}} \tilde{\sigma}_i^{-1} \circ \gamma_{ij} \circ \tilde{\sigma}_j(s) = 0.9695s,
\]
and,
\[
\sigma(s) := \max_{i \in \{1, 2\}} \tilde{\sigma}_i^{-1} \circ \gamma_{iu} = 5.4s
\]
for which \( V \) satisfies \( V(x(3, \xi, u(\cdot))) \leq \rho(V(\xi)) + \sigma(\|u\|\infty) \) for all \( \xi \in \mathbb{R}^2 \).
7 Conclusion

In this work we introduced the notion of dissipative finite-step ISS Lyapunov functions as a relaxation of ISS Lyapunov functions. These finite-step ISS Lyapunov functions were shown to be necessary and sufficient to conclude ISS of the underlying discrete-time system. In particular, for expISS system, norms are always dissipative finite-step ISS Lyapunov functions. Furthermore, we stated relaxed ISS small-gain theorems that drop the common assumption of small-gain theorems that the subsystems are ISS. ISS of the overall systems was then proven by constructing a dissipative finite-step ISS Lyapunov function. For the class of expISS systems, these small-gain theorems are shown to be non-conservative, i.e., necessary and sufficient to conclude ISS of the overall system. An example showed how the results can be applied.

A Appendix

The proofs of the main results in Section 4 require some lemmata, which will be given in this section.

A.1 A comparison lemma

The following lemma is a particular comparison lemma for finite-step dynamics.

**Lemma A.1** Let $M \in \mathbb{N}$ and $y : \mathbb{N} \rightarrow \mathbb{R}_+$ be a function satisfying

$$y(k + M) \leq \chi(y(k))$$

for all $0 \leq k < K$ for some $K \in \mathbb{N}$ and $\chi \in \mathcal{K}$ with $\chi < \text{id}$. Then there exists a $\mathcal{KL}$-function $\beta$ such that

$$y(k) \leq \beta(y^\text{max}_M, k)$$

for all $0 \leq k < K$ and $y^\text{max}_M := \max\{y(0), \ldots, y(M - 1)\}$

**Proof.** Let $M \in \mathbb{N}, M \geq 1$. Then for any $k \in \mathbb{N}$ there exist unique $k_0 \in \{0, \ldots, M - 1\}$ and $l \in \mathbb{N}$ such that $k = k_0 + lM$. Assume $k < K$, then we have

$$y(k) \leq \chi(y(k_0 + (l - 1)M)) \leq \ldots \leq \chi^l(y(k_0)).$$

Note that since $\chi < \text{id}$ we have $\chi^l > \chi^{l+1}$, and $\chi^l(s) \rightarrow 0$ as $l \rightarrow \infty$ for any $s \in \mathbb{R}_+$. For any $j \in \{0, \ldots, M - 1\}$ define $t_{j,l} := lM + j$, $t_{j,l}^+ := (l + 1)M + j$ for $l \in \mathbb{N}$, and

$$\beta_j(s, r) := \begin{cases} \frac{1}{M} \left((t_{j,0} - r)\chi^{-1}(s) + (r + M - j)\text{id}(s)\right) & r \in [0, t_{j,0}), s \geq 0 \\ \frac{1}{M} \left((t_{j,l}^+ - r)\chi^l(s) + (r - t_{j,0})\chi^{l+1}(s)\right) & r \in [t_{j,l}, t_{j,l}^+), s \geq 0 \end{cases}.$$ 

Note that this construction is similar to the one proposed in [20, Lemma 4.3]. Clearly, for any $j \in \{0, \ldots, M - 1\}$, $\beta_j(\cdot, r)$ is a $\mathcal{K}$-function for any fixed $r \geq 0$. On the other hand, for any fixed $s \geq 0$, $\beta_j(s, \cdot)$ is a $\mathcal{L}$-function, as it is linear affine on any interval $[t_{j,l}, t_{j,l}^+]$ and strictly decreasing by

$$\beta_j(s, t_{j,l}) = \chi^l(s) > \chi^{l+1}(s) = \beta_j(s, t_{j,l}^+).$$
Hence $\beta_j \in KL$ for any $j \in \{0, \ldots, M - 1\}$. Define
\[ \beta(s, r) := \max_{j \in \{0, \ldots, M - 1\}} \beta_j(s, r). \]
Then $\beta \in KL$, and for any $k \in \mathbb{N}$, $k < K$ with $k = j + lM$ and $j \in \{0, \ldots, M - 1\}$, $l \in \mathbb{N}$, we have
\[ y(k) \leq \chi^l(y(j)) = \beta_j(y(j), k) \leq \beta_j(y_M^{\max}, k) \leq \beta(y_M^{\max}, k). \]
This concludes the proof. \( \square \)

If the function $\chi$ in Lemma A.1 is linear, then the function $\beta$ has a simple form.

**Lemma A.2** Let the assumptions of Lemma A.1 be satisfied with $\chi(s) = \theta s$ and $\theta \in (0, 1)$. Then we obtain the estimate
\[ y(k) \leq \frac{y_M^{\max}}{\theta} \theta^{k/M}. \]
for all $k < K$, and $y_M^{\max} := \max\{y(0, \ldots, y(M - 1))\}$.

**Proof.** A direct computation yields that for any $j \in \{0, \ldots, M - 1\}$, and any $l \in \mathbb{N}$ with $lM + j < K$ we have
\[ y(lM + j) \leq \chi^j(y((l - 1)M + j)) \leq \chi^l(y(j)) = \theta^j y(j). \]
Hence, for any $k = lM + j < K$, with $l \in \mathbb{N}$ and $j \in \{0, \ldots, M - 1\}$ we have
\[ y(k) \leq \max_{j \in \{0, \ldots, M - 1\}} \{ y(j) \theta^l \} \leq y_M^{\max} \theta^{k/M - 1}. \]
This proves the lemma. \( \square \)

**A.2 Bounds on trajectories**

As noted in Remark A.3 the requirement on the existence of $K$-functions $\omega_1, \omega_2$ satisfying (7) in Assumption A.1 is a necessary condition for system (6) to be ISS. The following lemma states that any system (6) satisfying Assumption A.1 induces $K$-bounds for finite-step trajectories. This result is needed in Theorem A.1 to show that the existence of a finite-step ISS Lyapunov function implies ISS of system (6).

**Lemma A.3** Let system (6) satisfy the $K$-boundedness condition of Assumption A.1. Then for any $j \in \mathbb{N}$, there exist $K$-functions $\vartheta_j, \zeta_j$ such that for any $\xi \in \mathbb{R}^n$, $u(\cdot) \subset \mathbb{R}^m$
\[ |x(j, \xi, u(\cdot))| \leq \vartheta_j(|\xi|) + \zeta_j(\|u\|_\infty). \] (32)

**Proof.** We prove this result by induction. Take any $\xi \in \mathbb{R}^n$ and any input $u(\cdot) \subset \mathbb{R}^m$. For $j = 0$ we have $|x(0, \xi, u(\cdot))| = |\xi|$ and there is nothing to show. For $j = 1$ it follows by Assumption A.1 that $|x(1, \xi, u(\cdot))| \leq \omega_1(|\xi|) + \omega_2(\|u\|_\infty)$. So take $\vartheta_1 := \omega_1$ and $\zeta_1 := \omega_2$.

Let us assume, that there exist $\vartheta_j, \zeta_j \in K$ satisfying (32) for some $j \in \mathbb{N}$. Then there exist functions $\vartheta_{j+1}, \zeta_{j+1} \in K$ such that (32) holds for $j + 1$, which follows by
\[
|\vartheta_{j+1}(|\xi|) + \zeta_{j+1}(\|u\|_\infty)| \\
\leq \omega_1(2\vartheta_j(|\xi|)) + \omega_2(2\zeta_j(\|u\|_\infty)) \\
\leq \omega_1(2\vartheta_j(|\xi|)) + \omega_2(\|u\|_\infty) =: \vartheta_{j+1}(|\xi|) + \zeta_{j+1}(\|u\|_\infty).
\]
By induction, the lemma holds for all $j \in \mathbb{N}$. \( \square \)

As in Section A.1 the use of linear functions simplifies the result a lot.
Lemma A.4 Let system (6) satisfy the $K$-boundedness condition of Assumption 3.1 with linear functions $\omega_1(s) := w_1 s$ and $\omega_2(s) = w_2 s$. Then (32) is satisfied with $\vartheta_j(s) = w_j 1_j s$ and $\zeta_j = w_2 \sum_{i=0}^{j-1} w_i s$.

Proof. Following the proof of Lemma A.3 we inductively obtain for any $j \in \mathbb{N}$, $\vartheta_{j+1}(s) = \omega_1(\vartheta_j(s)) = \omega_1^{j+1}(s) = w_1^{j+1} s$, and $\zeta_{j+1}(s) = \omega_1(\zeta_j(s)) + \omega_2(s) = w_2 \sum_{i=0}^{j} w_i s$. □

References

[1] D. Aeyels and J. Peuteman. A new asymptotic stability criterion for nonlinear time-variant differential equations. IEEE Trans. Autom. Control, 43(7):968–971, 1998.

[2] R. P. Agarwal. Difference equations and inequalities, volume 155 of Monographs and Textbooks in Pure and Applied Mathematics. Marcel Dekker Inc., New York, 1992. Theory, methods, and applications.

[3] D. Angeli. Intrinsic robustness of global asymptotic stability. Systems Control Lett., 38(4-5):297–307, 1999.

[4] V. D. Blondel and J. N. Tsitsiklis. Complexity of stability and controllability of elementary hybrid systems. Automatica J. IFAC, 35(3):479–489, 1999.

[5] S. N. Dashkovskiy, B. S. Rüffer, and F. R. Wirth. An ISS small gain theorem for general networks. Math. Control Signals Systems, 19:93–122, 2007.

[6] S. N. Dashkovskiy, B. S. Rüffer, and F. R. Wirth. Small gain theorems for large scale systems and construction of ISS Lyapunov functions. SIAM J. Control Optim., 48:4089–4118, 2010.

[7] R. Geiselhart, R. H. Gielen, M. Lazar, and F. R. Wirth. An alternative converse Lyapunov theorem for discrete-time systems. Systems & Control Letters (2014), http://dx.doi.org/10.1016/j.sysconle.2014.05.007 in press.

[8] R. Geiselhart, M. Lazar, and F. R. Wirth. A Relaxed Small-Gain Theorem for Interconnected Discrete-Time Systems. to appear in IEEE Trans. Autom. Control, 2015.

[9] R. Geiselhart and F. Wirth. Solving iterative functional equations for a class of piecewise linear $\mathcal{K}_\infty$-functions. J. Math. Anal. Appl., 411(2):652–664, 2014.

[10] R. Geiselhart and F. R. Wirth. Numerical construction of LISS Lyapunov functions under a small-gain condition. Math. Control Signals Systems, 24:3–32, 2012.

[11] R. Geiselhart and F. R. Wirth. On maximal gains guaranteeing a small gain condition. submitted to SIAM J. Control Optim., June 2013.

[12] R. Gielen and M. Lazar. Non-conservative dissipativity and small-gain conditions for stability analysis of interconnected systems. In Proc. 51th IEEE Conf. Decis. Control, pages 4187–4192, Maui, HI, Dec.10-13 2012.

[13] L. Grüne and C. M. Kellett. ISS-Lyapunov functions for discontinuous discrete-time systems. IEEE Trans. Automat. Control, PP(99), May 2014.

[14] W. Hahn. Stability of motion., volume 138 of Springer-Verlag. Die Grundlehren der mathematischen Wissenschaften in Einzeldarstellungen, Berlin, xii edition, 1967.

[15] Z.-P. Jiang, Y. Lin, and Y. Wang. Nonlinear small-gain theorems for discrete-time feedback systems and applications. Automatica J. IFAC, 40(12):2129–2136, 2004.
[16] Z.-P. Jiang, Y. Lin, and Y. Wang. Nonlinear small-gain theorems for discrete-time large-scale systems. In Proc. 27th Chinese Control Conf., pages 704–708, Kumming, Yunnan, China, July 16-18 2008.

[17] Z.-P. Jiang, I. M. Y. Mareels, and Y. Wang. A Lyapunov formulation of the nonlinear small-gain theorem for interconnected ISS systems. Automatica J. IFAC, 32(8):1211–1215, 1996.

[18] Z.-P. Jiang, A. R. Teel, and L. Praly. Small-gain theorem for ISS systems and applications. Math. Control Signals Systems, 7(2):95–120, 1994.

[19] Z.-P. Jiang and Y. Wang. Input-to-state stability for discrete-time nonlinear systems. Automatica J. IFAC, 37(6):857–869, 2001.

[20] Z.-P. Jiang and Y. Wang. A converse Lyapunov theorem for discrete-time systems with disturbances. Systems Control Lett., 45(1):49–58, 2002.

[21] D. Kazakos and J. Tsinias. The input to state stability condition and global stabilization of discrete-time systems. IEEE Trans. Automat. Control, 39(10):2111–2113, 1994.

[22] C. M. Kellett. A compendium of comparison function results. Math. Control Signals Syst., pages 1–36, 2014.

[23] C. M. Kellett and P. M. Dower. Stability of (integral) Input-to-State Stable Interconnected Nonlinear Systems via Qualitative Equivalences. In Control Conference (AUCC), 2013 3rd Australian, pages 41–46, November 2013.

[24] C. M. Kellett and A. R. Teel. Discrete-time asymptotic controllability implies smooth control-Lyapunov function. Systems Control Lett., 52(5):349–359, 2004.

[25] D. S. Laila and A. Astolfi. Input-to-state stability for discrete-time time-varying systems with applications to robust stabilization of systems in power form. Automatica J. IFAC, 41(11):1891–1903, 2005.

[26] D. S. Laila and D. Nešić. Discrete-time Lyapunov-based small-gain theorem for parameterized interconnected ISS systems. IEEE Trans. Automat. Control, 48(10):1783–1788, 2003. New directions on nonlinear control.

[27] M. Lazar, A. I. Doban, and N. Athanasopoulos. On stability analysis of discrete–time homogeneous dynamics. In Proc. of the 17th IEEE International Conference on Systems theory, Control and Computing, pages 297 – 305, Sinaia, Romania, October 11-13 2013.

[28] T. Liu, D. J. Hill, and Z.-P. Jiang. Lyapunov formulation of the large-scale, ISS cyclic-small-gain theorem: The discrete-time case. Systems Control Lett., 61(1):266–272, 2012.

[29] P. Moylan and D. Hill. Stability criteria for large-scale systems. IEEE Trans. Autom. Control, 23(2):143–149, 1978.

[30] B. S. Rüffer. Monotone dynamical systems, graphs, and stability of large-scale interconnected systems. PhD thesis, Universität Bremen, Germany, August 2007.

[31] B. S. Rüffer. Monotone inequalities, dynamical systems, and paths in the positive orthant of Euclidean n-space. Positivity, 14(2):257–283, 2010.

[32] D. D. ˇSiljak. Large-scale dynamic systems, volume 3 of North-Holland Series in System Science and Engineering. North-Holland Publishing Co., New York, 1979.

[33] E. D. Sontag. Smooth stabilization implies coprime factorization. IEEE Trans. Automat. Control, 34(4):435–443, 1989.

[34] E. D. Sontag and Y. Wang. On characterizations of the input-to-state stability property. Systems Control Lett., 24(5):351–359, 1995.

[35] M. Vidyasagar. Input-output analysis of large-scale interconnected systems, volume 29 of Lecture Notes in Control and Information Sciences. Springer-Verlag, Berlin, 1981.