TOPOLOGICAL 2-DIMENSIONAL QUANTUM MECHANICS

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Abstract: We define a Chern-Simons Lagrangian for a system of planar particles topologically interacting at a distance. The anyon model appears as a particular case where all the particles are identical. We propose exact N-body eigenstates, set up a perturbative algorithm, discuss the case where some particles are fixed on a lattice, and also consider curved manifolds.

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1. INTRODUCTION

The anyon [1] (quantum mechanical, non relativistic) model is a fascinating system to study.

Firstly, it is a Aharonov-Bohm [2] system with no classical counterpart, in which planar particles interact at a distance topologically. Secondly, the particles being identical, the effect of the interaction is to make their statistics intermediate [1,3], nor Bose, neither Fermi.

Little is known about anyons despite huge efforts devoted to their study. In the N-anyon case with harmonic attraction to the origin, linear eigenstates have been constructed [4], but they are known to be only part of the spectrum. Perturbative approaches [5] have undercovered a very complex structure for the equation of state. Finally, several numerical analysis have been developped [6].

Here, one is going to drop the indistinguishibility of the particles and propose a general model of particles topogically interacting at a distance à la Aharonov-Bohm.

2. THE FORMALISM

Consider the density of Lagrangian for N particles moving in plane minimally coupled to vector gauge fields $A^i_\mu(\vec{r})$

$$L_N = \sum_{i=1}^{N} \left( \frac{1}{2} m \vec{v}_i^2 + \sum_{\alpha=1}^N e_{\alpha i} (\vec{A}_\alpha(\vec{r}_i) \cdot \vec{v}_i - A^\alpha_\alpha(\vec{r}_i)) \right) + \sum_{\alpha,\beta=1}^{N} \frac{\kappa_{\alpha\beta}}{2} \epsilon_{\mu\nu\rho} \int A^\mu_\alpha \partial^\nu A^\rho_\beta d\vec{r} \quad (1)$$

In (1), the index $i$ refers to the number affected to a given particle among $N$, the indices $\mu, \nu, \rho, \ldots$ correspond to the 3-dimensional Lorentz degrees freedom ($\mu = 0$ denotes the time direction, $\mu = 1, 2$ the space directions), and $\alpha, \beta, \ldots$ label some internal degrees of freedom carried by the vector fields ($1 < \alpha < N$). The $e_{\alpha i}$’s are the electromagnetic couplings (topological charges) between the matter particles $i$ and the gauge fields $\alpha$ at
position \( \vec{r}_i \). The \( \kappa_{\alpha\beta} \) are the Chern-Simons\(^3\) self-couplings of the gauge fields \( \alpha, \beta \).

One can easily see that under the gauge transformation

\[
\delta A_\alpha^\mu = \partial^\mu \Lambda_\alpha
\]  

(2)

the density of Lagrangian (1) changes by a time derivative, namely \( \delta \epsilon_{\mu
\nu\rho} A_\alpha^\mu \partial^\nu A_\beta^\rho = \partial^\mu (\epsilon_{\mu\nu\rho} \Lambda_\alpha \partial^\nu A_\beta^\rho) \). Also, by the very definition (1) one can restrict to \( \kappa_{\alpha\beta} = \kappa_{\beta\alpha} \), i.e to a symmetric matrix \([\kappa]\).

One proceeds by eliminating the time components of the gauge fields \( A_\beta^0 \) by varying the Lagrangian with respect to them (\( \delta L_N / \delta A_\beta^0 = 0 \)). One gets

\[
\sum_{i=1}^N \epsilon_{\beta i} \delta^2 (\vec{r} - \vec{r}_i) = \sum_{\alpha=1}^N \kappa_{\alpha\beta} \epsilon_{\alpha\nu} \partial^\nu A_\alpha^\nu
\]

(3)

where explicit use of the symmetry of \([\kappa]\) has been made.

The magnetic field \( B_\alpha = \epsilon_{\alpha\mu\nu} \partial^\mu A_\nu^\alpha \) appears in the right hand side of (3). Using the bidimensional identity \( \vec{\partial} \cdot \vec{r}^2 = 2\pi \delta^2 (\vec{r}) \) one finds in the Coulomb gauge \( \vec{\partial} \cdot \vec{A}_\alpha = 0 \)

\[
\sum_{\alpha} \kappa_{\alpha\beta} \vec{A}_\alpha (\vec{r}) = \sum_{i=1}^N \frac{\epsilon_{\beta i}}{2\pi} \vec{k} \times \frac{\vec{r} - \vec{r}_i}{(\vec{r} - \vec{r}_i)^2}
\]

(4)

(\( \vec{k} \) is the unit vector perpendicular to the plane). This equation can be symbolically rewritten as \([\kappa] [\vec{A}_\alpha (\vec{r})] = [e/2\pi][\vec{k} \times (\vec{r} - \vec{r}_i)/(\vec{r} - \vec{r}_i)^2] \) where \([\vec{A}_\alpha (\vec{r})]\) and \([\vec{k} \times (\vec{r} - \vec{r}_i)/(\vec{r} - \vec{r}_i)^2]\) are one column vectors on which the matrices \([\kappa]\) and \([e]\) act. If one assumes that \([\kappa]\) is invertible one gets

\[
[\vec{A}_\alpha (\vec{r})] = [\kappa]^{-1} \frac{e}{2\pi} [\vec{k} \times \frac{\vec{r} - \vec{r}_i}{(\vec{r} - \vec{r}_i)^2}]
\]

(5)

The Hamiltonian corresponding to \( L_N \) stems from

\[
\vec{p}_i \equiv \frac{\partial L_N}{\partial \dot{v}_i} = m \vec{v}_i + \sum_{\alpha} \epsilon_{\alpha i} \vec{A}_\alpha (\vec{r}_i)
\]

(6)

\(^3\)Here we concentrate on abelian Chern-Simons gauge fields. The generalization to the non-abelian case could be studied along the same lines.
One gets

$$H_N = \frac{1}{2m} \sum_i \left( \vec{p}_i - \sum_\alpha e_{\alpha i} \vec{A}_\alpha (\vec{r}_i) \right)^2$$

(7)

If one defines \( \vec{A}_i(\vec{r}) \equiv \sum_\alpha e_{\alpha i} \vec{A}_\alpha (\vec{r}) \) one sees that \( [\vec{A}_i(\vec{r})] = [e]^t [\kappa]^{-1} [e/2\pi] [\vec{k} \times (\vec{r} - \vec{r}_i)/(\vec{r} - \vec{r}_i)^2] \)

at position \( \vec{r}_i \) enters the definition of \( H_N \).

Let us consider the coupling matrix \( [\alpha] \equiv [e]^t [\kappa]^{-1} [e/2\pi] \). It is by definition a symmetric matrix. The usual anyon model is nothing but taking \( [\kappa] \) and \( [e] \) to be one-dimensional matrices (i.e. one single gauge field) with a single anyonic coupling constant \( \alpha = \frac{e^2}{2\pi \kappa} \)

between the "flux" \( \phi = \frac{\ell}{\kappa} \) and the "charge" \( e \) carried by each anyon. Singular self-interaction, which are present in \( \vec{A}(\vec{r}) \) at position \( \vec{r} = \vec{r}_i \), have to be left aside.

Here, we get a general model where the couplings \( \alpha_{ij} = \alpha_{ji} \) can depend on \( i \) and \( j \).

One notes that \( \vec{A}_i(\vec{r}) \) is correctly defined at \( \vec{r} = \vec{r}_i \) if and only if one asks for the matrix \( [\alpha] \) to have its diagonal elements equal to 0. Thus one ends up with \( N(N - 1)/2 \) independent anyonic coupling constant (the entries of \( [\alpha] \)) and a Hamiltonian

$$H_N = \frac{1}{2m} \sum_{i=1}^N \left( \vec{p}_i - \vec{A}_i \right)^2$$

(8)

where the gauge field \( \vec{A}_i = \sum_{j \neq i} \alpha_{ij} \frac{\vec{k} \times \vec{r}_{ij}}{\vec{r}_{ij}^2} \) with \( \vec{r}_{ij} = \vec{r}_i - \vec{r}_j \).

In this more general point of view, the anyon model can be recovered by taking all the \( \alpha_{ij} \) equal to \( \alpha \). But now, one gets as a bonus that \( \vec{A}_i(\vec{r}) |_{\vec{r} = \vec{r}_i} \) is defined at the position of the particles, which was not the case in the original formulation. A quantum mechanical model of flux tubes \( \phi_j \) interacting à la Aharonov-Bohm with electric charges \( e_i \) would correspond to taking \( e_{\alpha i} = e_i \) implying \( \phi_j = (\sum_{\alpha\beta} [\kappa]^{-1}_{\alpha\beta})e_j \) and \( \alpha_{ij} = \frac{1}{2\pi} e_i (\sum_{\alpha\beta} [\kappa]^{-1}_{\alpha\beta}) e_j \).

\( [\kappa] \) matrices have been already used [7] in order to reproduce fractional values for the Hall conductivity given as \( \sigma_H = \sum_{\alpha\beta} [\kappa]^{-1}_{\alpha\beta} \). In this sghightly different context (in particular there is no kinetic term for matter), one insists on a matrix \( [\kappa] \) with integer entries in order to reproduce the quantum numbers (statistics, charge) of the electron.
Here the entries are not constrained to be integers, in $[\kappa]$ as well as in $[\epsilon]$.

As in the anyon model, $\vec{A}_i(\vec{r}_i)$ is pure gauge, the singular gauge parameter being

$$\vec{A}_i(\vec{r}_i) = \vec{\nabla}_i \left( \sum_{k<l} \alpha_{kl} \theta_{kl} \right)$$  \hspace{1cm} (9)

where $\theta_{kl}$ is the relative angle of the particles $k$ and $l$. The strong analogy between the anyon model and the model proposed above will allow for the generalization of interesting results of the former to the latter.

3. SOME EXACT AND PERTURBATIVE RESULTS

i) Exact results

The structure of the Hamiltonian $H_N(\alpha_{ij})$ given in (8) allows for the following general comment. Suppose one has an eigenstate $\psi(\alpha_{ij})$ of energy $E(\alpha_{ij})$. By the virtue of the gauge transformation

$$\psi'(\alpha_{ij}) = \exp(i \sum_{k<l} m_{kl} \theta_{kl}) \psi(\alpha_{ij})$$  \hspace{1cm} (10)

one finds that $\psi'(\alpha_{ij})$ is a monovalued eigenstate of the Hamiltonian $H_N(\alpha_{ij} - m_{ij})$ with the same energy $E(\alpha_{ij})$. This implies that $\psi'(\alpha_{ij} + m_{ij})$ is a monovalued eigenstate of the original Hamiltonian $H_N(\alpha_{ij})$ with energy $E(\alpha_{ij} + m_{ij})$. Thus as soon one knows an exact eigenstate, one can associate to it a tower of orbital eigenstates indexed by the quantum numbers $m_{ij}$.

Another consequence is that if one considers the $m_{ij}$’s as gauge parameter coefficients (not as actual quantum numbers), and choose $m_{ij} = E[\alpha_{ij}]$, where $E[\alpha_{ij}]$ is the integer part of $\alpha_{ij}$, the gauge transformed Hamiltonian $H_N(\alpha_{ij} - m_{ij})$ is then defined in terms of the couplings $\alpha'_{ij} = \alpha_{ij} - E[\alpha_{ij}]$ with the constraint that $\alpha'_{ij} \in [0, 1]$. Clearly, the physics described by both models is the same, thus one will always assume in the sequel that $0 < \alpha_{ij} < 1$. In the anyon case with bosonic (fermionic) wavefunctions, the same
reasoning leads to $0 < \alpha < 2$ ($-1 < \alpha < 1$) since then the $m_{ij}$’s are constrained to be equal to a given even (odd) integer.

The question is of finding particular eigenstates. Let us confine the particles by a harmonic attraction to the origin. This procedure [1,8] is commonly used in the anyon context since it yields a discrete spectrum. The $N$-particle Hamiltonian with a harmonic interaction reads

$$H_N = \frac{1}{2m} \sum_{i=1}^{N} \left[ (\vec{p}_i - \sum_{j \neq i} \alpha_{ij} \frac{\vec{k} \times \vec{r}_{ij}}{r_{ij}^2})^2 + m^2 \omega_r^2 \vec{r}_i^2 \right]$$

(11)

One finds the relative eigenstates

$$\langle \vec{r}_i | n, m_{ij} \rangle = \mathcal{N} e^{\frac{i}{\hbar} \sum_{i<j} m_{ij}\theta\_ij - \beta r^2/2} \prod_{i<j} r_{ij}^{m_{ij} - \alpha_{ij}} L_n^{N-2+\sum_{i<j} |m_{ij} - \alpha_{ij}|} (\beta r^2)$$

(12)

($\mathcal{N}$ is a normalization factor, $\beta \equiv \frac{m\omega}{\hbar}$ and $r^2 \equiv \sum_{i<j} r_{ij}^2$) with eigenvalues

$$(2n + N - 1 + \sum_{i<j} |m_{ij} - \alpha_{ij}|) \omega$$

(13)

Note that since the states (12) are eigenstates of the total angular momentum operator, they are still eigenstates in the presence of a uniform magnetic field ($\omega \to \sqrt{\omega_r^2 + \omega_c^2}$). The integers $m_{ij}$ have to satisfy simultaneously either $m_{ij} > 0$ (case I) or $m_{ij} \leq 0$ (case II). For a given $N$, the relative states $\langle \vec{r}_i | n, m_{ij} \rangle$ have obviously too many quantum numbers. The $m_{ij}$’s can be choosen as independant orbital quantum numbers, the other being either 1 (case I) or 0 (case II). One gets $\langle n, m_{ij} | n', m'_{ij} \rangle = \delta_{n,n'}\delta_{m,m'}$ where $m = \sum_{i<j} m_{ij}$, and $m' = \sum_{i<j} m'_{ij}$, leading to sectors labelled by the quantum numbers $n, m$. In a given sector, the states can be separately orthonormalized.

The states (12) with linear dependance on the $\alpha_{ij}$’s narrow down to the usual linear anyonic eigenstates when one sets $\alpha_{ij} = \alpha$. In this particular case one has additionnal conditions on the $m_{ij}$’s depending on the statistics (Bose or Fermi) imposed when $\alpha = 0$. If the $\alpha_{ij}$’s are not constrained to be in the interval $[0, 1]$, one gets either $m_{ij} > E[\alpha_{ij}]$ or $m_{ij} \leq E[\alpha_{ij}]$. 

6
One way to reproduce (12) is to work [9] in the singular gauge (9)

\[ \psi' = \exp(-i \sum_{k<l} \alpha_{kl} \theta_{kl}) \psi \]  

(14)

In complex coordinates \( z_i = x_i + iy_i \) the free gauge transformed Hamiltonian is

\[ H'_N = \sum_{i=1}^{N} \left( -\frac{2}{m} \partial_{z_i} \partial_{\bar{z}_i} + \frac{1}{2} m \omega^2 z_i \bar{z}_i \right) \]  

(15)

The states

\[ \psi' = \exp(-\frac{N\beta}{2} \sum_{i} z_i \bar{z}_i) \phi \]  

(16)

are eigenstates of the Hamiltonian (15) if \( \phi \) is an homogeneous meromorphic function of degree \( d \) of \( z_1, \ldots, z_N \) (case I) or of \( \bar{z}_1, \ldots, \bar{z}_N \) (case II), with for eigenvalues \( (N+d)\omega \). Since the prefactor \( \exp(-\frac{N\beta}{2} \sum_{i} z_i \bar{z}_i) \) can be rewritten as \( \exp(-\frac{\beta}{2} \sum_{i<j} z_{ij} \bar{z}_{ij}) \exp(-\frac{\beta}{2} N^2 \bar{Z} \bar{Z}) \), the center of mass coordinate \( Z = \sum_i z_i / N \) factorizes out in (16) with energy \( \omega \).

A suitable basis for \( \phi \) is \( \{ \prod_{i<j} z_{ij}^{m_{ij}-\alpha_{ij}} \} \) or \( \{ \prod_{i<j} z_{ij}^{\alpha_{ij}-m_{ij}} \} \), where the \( m_{ij} \)'s are integers, thus \( d = \sum_{i<j} (m_{ij} - \alpha_{ij}) \) or \( d = \sum_{i<j} (\alpha_{ij} - m_{ij}) \) (the \( m_{ij} \)'s are easily seen to be not independent since \( z_{ij}^{m_{ij}} = (z_{1j} - z_{1i})^{m_{ij}} \) can be expanded in a power series of \( z_{1i} \)).

The space of eigenstates has to be a Hilbert space of square integrable functions where the Hamiltonian is self-adjoint. A simple requirement is to impose that the eigenstates have no divergence, implying that one has simultaneously \( m_{ij} > 0 \) (class I), or \( m_{ij} \leq 0 \) (class II). Thus one reproduces the eigenstates (12) with \( n = 0 \). An explicit calculation shows that non vanishing radial quantum number \( n > 0 \) correspond to the Laguerre polynomials appearing in (12). The energy \( (2n + N - 1 + \sum_{i<j} (m_{ij} - \alpha_{ij})) \omega \) (class I) or \( (2n + N - 1 + \sum_{i<j} (\alpha_{ij} - m_{ij})) \omega \) (class II) coincides with (13).

Imposing that the eigenstates vanish when any two particles come close together is an anyon-like requirement, amounting to the exclusion of the diagonal of the configuration space. Self-adjoint extensions [10] corresponding to diverging behavior at small distance
are possible. Indeed, allowing for short distance singularities implies that the \( m_{ij} \)’s have to satisfy simultaneously \( m_{ij} \geq 0 \) or \( m_{ij} \leq 1 \). These states diverge at the origin but are still square integrable. Asking for the Hamiltonian to be self adjoint implies additional constraints leading to the possible self adjoint extensions \( m_{ij} \geq 0 \) or \( m_{ij} \leq -1 \), and \( m_{ij} \geq 2 \) or \( m_{ij} \leq 1 \).

It is amusing to note that, in the situation where the \( \alpha_{ij} \)’s are either equal to \( \alpha \) or null, the eigenstates (12) can be directly deduced from the anyonic eigenstates

\[
e^{i \sum_{i<j} m_{ij} \theta_{ij}} e^{-\beta \sum_{i<j} |m_{ij}-\alpha| L_n^{-2} \sum_{i<j} |m_{ij}-\alpha| |(\beta r)^2| (17)}
\]

These states are monovalued eigenstates of the \( N \)-anyon Hamiltonian if the integers \( m_{ij} \) simultaneously satisfy \( m_{ij} > E[\alpha] \) or \( m_{ij} \leq E[\alpha] \) (here one cannot restrict \( \alpha \) to be in the interval \([0,2]\)). However, if one drops the monovaluedness criterion one finds that these states are still solutions of the eigenvalue equation if some of the \( m_{ij} \) are replaced by \( m_{ij} = m'_{ij} + \alpha \) where the integers \( m'_{ij} \) have to be simultaneously > 0 or \( \leq 0 \). But one can get rid of the multivaluedness of the wavefunction by means of the singular gauge transformation \( \exp(i \alpha \sum_{i<j} \theta_{ij}) \) where the last summation is performed only on those indices \( i, j \) for which \( m_{ij} = m'_{ij} + \alpha \). One then reproduces the eigenstates defined above where the indices \( i, j \) for which \( \alpha_{ij} \) has been set to 0 correspond to \( m_{ij} = m'_{ij} + \alpha \).

ii) Perturbative results

Leaving aside the exact eigenstates (12), very little is known about the model defined above. In the case where the \( \alpha_{ij} \)'s are assumed to be small, a perturbative analysis can give some information on the system. Here again the experience gained in the study of the anyon model is helpful. A naïve perturbative analysis might make no sense due to the very singular\(^5\) Aharonov-Bohm interaction \( \alpha^2_{ij}/r_{ij}^2 \). One can circumvent this difficulty by

\(^5\)here we assume that the Hilbert space of unperturbed wavefunctions does not contain any states with
noticing that the singular gauge parameter \( \Omega'' = \sum_{i<j} \alpha_{ij} \theta_{ij} \) is the imaginary part of the meromorphic function

\[
\Omega(z_1, z_2, \ldots, z_N) = \sum_{i<j} \alpha_{ij} \ln z_{ij}
\]  
(18)

Let us "gauge transform" the Hamiltonian \( H_N \) by taking as gauge parameter the real part of \( \Omega \), \( \Omega' = \sum_{i<j} \alpha_{ij} \ln r_{ij} \) i.e.

\[
\psi = \exp(\pm \Omega') \tilde{\psi} = \prod_{i<j} r_{ij}^{\pm \alpha_{ij}} \tilde{\psi}(\vec{r}_1, \ldots, \vec{r}_N)
\]  
(19)

Because of the Cauchy-Riemann relations in 2 dimensions, \( \vec{\nabla}_i \Omega'' = \vec{k} \times \vec{\nabla}_i \Omega' \) (implying \( \partial_{z_k} \Omega = i \vec{A}_k \)), one finds that the singular terms are absent in the Hamiltonian \( \tilde{H}_N \) acting on \( \tilde{\psi} \)

\[
\tilde{H}_N = \sum_{i=1}^{N} \left( \frac{p_i^2}{2m} + \sum_{j \neq i} \frac{i \alpha_{ij} \vec{k} \times \vec{r}_{ij} \vec{\partial}_i \tilde{\psi} + \sum_{j \neq i} \frac{\alpha_{ij} \vec{r}_{ij} \vec{\partial}_i \tilde{\psi}}{m \vec{r}_{ij}^2} \right).
\]  
(20)

As in the anyon model, one gets 2-body interactions with short distance behavior adapted to a perturbative analysis. One notes the \( \pm \) sign freedom in the choice of the redefinition of \( \tilde{\psi} \). This sign freedom describes two possible short distance behaviors of the exact eigenstates as emphasized in the context of self adjoint extensions. The \( - \) sign in the redefinition (19) corresponds to the self adjoint extansion \( m_{ij} \geq 0 \) (class I) or \( m_{ij} \leq -1 \) (class II).

iii) Some of the particles are fixed.

So far the \( N \) particles are dynamical. It is however interesting to consider some of the particles fixed in the plane. To do so, take for the free Hamiltonian of the \( N \)-particle system in the singular gauge

\[
H'_N = \frac{1}{2m} \sum_{i=1}^{N} \epsilon_i \vec{p}_i^2
\]  
(21)

singular short distance behavior.
where $\epsilon_i = 1$ (moving particle) or $\epsilon_i = 0$ (fixed particle). In the Lagrange formulation (1), this amounts to affecting the speed $\vec{v}_i$ with the factor $\epsilon_i$. Going back to the regular gauge via the gauge transformation (14) yields the desired Hamiltonian describing fixed and moving particles interacting via the Aharonov-Bohm couplings $\alpha_{ij}$

$$H_N = \frac{1}{2m} \sum_{i=1}^{N} \epsilon_i (p_i - \vec{A}_i)^2$$

(22)

Most of the results presented above are still operative, in particular the perturbative considerations. However, the eigenstates (12) cannot be used anymore.

Let us consider the scattering of a single particle by a finite lattice of $N - 1$ identical flux tubes. This amounts to take $\epsilon_1 = 1$ and $\epsilon_i = 0$ for $i = 2 \ldots, N$. Only the scattering by a single flux line is solvable [2]. In the case of two flux lines one has in the singular gauge

$$\psi' = \exp(-i\alpha_{12}\theta_{12} - i\alpha_{13}\theta_{13})\psi$$

(23)

Let us assume that 2 and 3 are located at $z_2 = -h$ and $z_3 = h$. For simplicity we set $\alpha_{12} = \alpha_{13} = \alpha$. In the limit where the two flux lines are at the same point $h \to 0$, one should reproduce the scattering of one electron by a flux line, where polar coordinates are used to separate the eigenstates equation. There exists a single coordinate system where the eigenstate’s equation separates, and which contains the polar coordinates as a particular limiting case. This is the elliptic coordinates system [12]. Still working in the singular gauge but with the conformal mapping $z_1 = h \cosh(\mu + i\phi)$ where $(\mu, \phi)$ are the elliptic coordinates, one gets the free Hamiltonian

$$H' = -\frac{1}{2m \ h^2 (\cosh^2 \mu - \cos^2 \phi)} (\partial^2_\mu + \partial^2_\phi)$$

(24)

Notice that the gauge transformation (23) does not separate when one uses elliptic coor-
coordinates. Indeed one gets

\[ \exp(i\theta_{12} + i\theta_{13}) = \frac{\sinh^2(\mu + i\phi)}{\cosh^2 \mu - \cos^2 \phi} \]  

(25)

This set of coordinates has been used in [11], however we stress that it does not describe the scattering of a charged particle by two isolated fixed flux lines, but instead the scattering of a charged particle by an elliptic flux tube. Indeed the eigenvalue equation is now separable and one can factorize the eigenstates as \( M(\mu)\Phi(\phi) \), leading to Mathieu’s equations\footnote{note that if a central harmonic interaction is added, Hill’s equations have to be considered}. In the singular gauge, the angular function \( \Phi(\phi) \) is multivalued. With this choice of coordinates, the contour of the flux tube (which controls the multivaluedness of the singular free wavefunction) must necessarily coincide with a geodesic defined by a constant \( \mu \). This is an ellips or in the singular case the line \([-h, h]\] that connects \( z_2 = -h \) to \( z_3 = +h \). It follows that the singular gauge transformation implied by the choice of elliptic coordinates is \( \psi' = \exp(-i\alpha\phi)\psi \).

Let us consider the singular flux \([-h, h]\) line case. Mathieu’s equations read

\[ \partial^2_\phi \Phi - h^2 mE \cos(2\phi)\Phi + (\lambda - h^2 mE)\Phi = 0 \]  

(26)

\[ \partial^2_\mu M + h^2 mE \cosh(2\mu) M - (\lambda - h^2 mE) M = 0 \]  

(27)

where \( \lambda \) is a constant introduced to separate the coordinates. The general solution [12] for a multivalued \( \Phi \) is

\[ \Phi = Ae^{i\phi\phi} \sum_{r=-\infty}^{+\infty} c_{2r} e^{2ir\phi} \]  

(28)

where \( iu_\phi \) is not an integer. A possible \( \exp(-u_\phi\phi) \sum_{r=-\infty}^{+\infty} c_{2r} e^{-2ir\phi} \) solution has been omitted since parity is broken anyway by the singular flux \([-h, h]\) line. In the regular gauge, the wavefunction has to be monovalued when a complete winding encircling the
flux \([-h, h]\) line is performed. This implies that \((u_{\phi} + i\alpha)/i\) has to be an integer \(n\). Obviously one can assume that \(n \in [0, 2]\). The quantification condition on \(\lambda\) comes from the compatibility of the homogeneous system of linear algebraic equations determining the \(c_{2r}\)'s in terms of the coefficients

\[
\frac{m h^2 E}{2} \frac{1}{(2r - i u_{\phi})^2 - \lambda + m h^2}
\]

Thus one has two quantum numbers \(\lambda\) and \(E\), as desired (remember that \(u_{\phi}\) has been fixed by the monovaluedness criterium). The solution of the equation on \(M\) introduces a parameter \(u_\mu\) which does not yield any additional quantum number since there is accordingly a compatibility condition to satisfy\(^7\).

In the limit where the flux \([-h, h]\) line shrinks to a point, \(h \to 0\), one should reproduce the isolated flux tube case. One has \(\mu \to \infty\), and \(z_1 \to re^{i\phi}\) with \(r = (h/2) \exp \mu\). The Mathieu's and compatibility equations become

\[
\partial^2_{\phi} \Phi + \lambda \Phi = 0 \tag{29}
\]

\[
- \frac{1}{r} \partial_r r \partial_r M + \frac{\lambda}{r^2} M = 2m E M \tag{30}
\]

\[
\sin^2 \pi \frac{\alpha - n}{2} = \sin^2 \pi \frac{\sqrt{\lambda}}{2} \tag{31}
\]

leading to the usual Bessel eigenfunctions with \(\lambda = (\alpha - \ell)^2\), where \(\ell = n + 2p\) is the usual angular quantum number (remember that \(n = 0, 1\)).

4. ON CURVED SPACE

i) The formalism

Let us consider a bidimensional manifold \(M_2\) defined by its metric \(g_{ab}(x)\), where the

\(^7\) one gets \(u_\mu = \pm i(n - \alpha)\).
indices $a, b$ label the space coordinates $x^1, x^2$. The covariant Lagrangian is given by

$$L_N = \sum_{i=1}^{N} \left( \frac{m g_{ab}(x_i) \dot{x}_i^a \dot{x}_i^b}{2} + \sum_{\alpha=1}^{N} e_{\alpha i} (g_{ab}(x_i) A_{\alpha}^a(x_i) \dot{x}_i^b - A_{\alpha}^b(x_i)) \right) + \sum_{\alpha, \beta=1}^{N} \frac{\kappa_{\alpha \beta}}{2} \epsilon^{\mu \nu \rho} \int A_{\alpha \mu} \partial_{\nu} A_{\beta \rho} d^2x$$

(32)

$$(g = | \det (g_{ab})|).$$ It is invariant under the gauge transformation $\delta A_\alpha^\mu = \partial_\mu \Lambda_\alpha$. By definition $[\kappa]$ is a symmetric matrix.

The Chern-Simons topological term having not explicit dependence on the metric one gets, by varying the Lagrangian with respect to $A_\beta^0$, an equation for the gauge potential identical to (3)

$$\sum_{i=1}^{N} e_{\beta i} \delta^2 (x - x_i) = \sum_{\alpha=1}^{N} \kappa_{\alpha \beta} \epsilon^{ab} \partial_a A_{\alpha b}$$

(33)

Defining $A_{ia} = \sum_\alpha e_{\alpha i} A_{\alpha a}$ the covariant Hamiltonian reads

$$H_N = \frac{1}{2m} \sum_{i=1}^{N} \frac{1}{\sqrt{g}} (p_{ia} - A_{ia}(x_i)) g^{ab} \sqrt{g} (p_{ib} - A_{ib}(x_i))$$

(34)

where $p_{ia} = \frac{1}{i} \partial_{ia}$ are the first quantized momenta (in the free case (34) is the Laplace-Beltrami operator). The gauge potentials $A_{ia}$ are determined by

$$\epsilon^{ab} \partial_a A_{ib} (x) = 2\pi \sum_j \alpha_{ij} \delta^2 (x - x_j)$$

(35)

with $[\alpha] = [e]^t [\kappa]^{-1} [e/2\pi]$. The question is of solving this equation on a non trivial manifold. As on the plane one should proceed by first solving the 2-body problem

$$\epsilon^{ab} \partial_{ia} A_{ib} (x_i, x_j) = 2\pi \delta^2 (x_i - x_j)$$

(36)

Then the vector potentials $A_{ia}(x_i)$ are given as a sum of 2-body terms

$$A_{ia}(x_i) = \sum_{j \neq i} \alpha_{ij} A_a (x_i, x_j)$$

(37)
plus possible irrotational one body terms describing the topology of the manifold (see below the cylinder case for an illustration). \( A_a(x_i, x_j) \) is the gradient of a potential symmetric under the exchange \( x_i \to x_j \).

It follows that \( A_a(x_i) \) is the gradient with respect to \( x_i \) of some function \( \Omega''(x_1, \ldots, x_N) \) which is multivalued for any loop enclosing the singular points \( x = x_j \). In the singular gauge

\[
\psi' = \exp (-i\Omega'') \psi
\]

the Hamiltonian is free but \( \psi' \) is multivalued.

In the Coulomb gauge \( \partial_a \sqrt{g} A^a = 0 \), one can rewrite \( A^a_i(x_i) \) as the dual tensor \(-1/\sqrt{g}\epsilon^{ab}\partial_b \Omega'(x_1, \ldots, x_N)\). If one redefines

\[
\psi = \exp (\pm \Omega') \tilde{\psi}
\]

the Hamiltonian acting on \( \tilde{\psi} \) is

\[
\tilde{H}_N = \sum_{i=1}^{N} \left( \frac{1}{2m} \sqrt{g} g^{ab} \partial_b \Omega'(x_i) \partial_a + \frac{1}{m} \sqrt{g} \epsilon_{ab} A^a_i(x_i) \partial_b \right)
\]

\( \Omega' \) and \( \Omega'' \) are by definition harmonic for the Laplace-Beltrami operator. In the simple case where \( g_{ab} = \pm \sqrt{g} \delta_{ab} \) (plane, cylinder [13], sphere [14],...), \( \Omega' \) and \( \Omega'' \) satisfy the Cauchy-Riemann relation, and \( \Omega = \Omega' + i\Omega'' \) is meromorphic.

ii) An example: the cylinder

Let us consider the particular case of the cylinder \( \mathbb{R} \times S_1 \). One has that \( x^2 \) and \( x^2 + 2\pi \) have to be identified. We define \( z = x^1 + ix^2 \). Let us first construct \( \Omega \) in the 2-body case (36). On the one hand it must behave as \( \alpha_{ij} \ln z_{ij} \) (see [18]) when \( z_{ij} \to 0 \), since locally the cylinder is equivalent to a plane. On the other hand \( \partial_z \Omega = i(A_{i1} - iA_{i2}) \) has to be periodic in the variable \( x^2_{ij} \). One arrives at \( \Omega = \alpha_{ij} \ln(e^{2i\pi} - 1) \) which, however, has yet to
be symmetrized under the exchange of $i$ and $j$, yielding

$$
\Omega = \alpha_{ij} \ln 2 \sinh \frac{z_{ij}}{2}
$$

(41)

One way to reproduce this result consists in considering the planar
$$
\Omega' = \alpha_{ij} \ln \sqrt{(x_{ij}^1)^2 + (x_{ij}^2)^2}
$$

and making it periodic by introducing the infinite series

$$
\alpha_{ij} \sum_{n=-\infty}^{\infty} \ln \sqrt{(x_{ij}^1)^2 + (x_{ij}^2 + 2\pi n)^2}
$$

(42)

This series is formally divergent, but it can be given a non ambiguous meaning by a usual procedure (derive with respect to $x_{ij}^1$, perform the summation, and then integrate). One then obtains [11].

The essential difference between the plane and the cylinder consists in the non vanishing contour integral of a gauge field on a non contractible loop around the cylinder. Consequently one has to introduce a line of flux inside the hole and, in the Coulomb gauge, consider a 1-body multivalued term $\phi_i z_i$. In the 2-body case this amounts to add

$$
\phi_i z_i + \phi_j z_j
$$

to (41)

$$
\Omega = \alpha_{ij} \ln 2 \sinh \frac{z_i - z_j}{2} + \phi_i z_i + \phi_j z_j
$$

(43)

Thus, in the $N$-body case, one has

$$
\Omega = \sum_{0 \leq i < j \leq N} \alpha_{ij} \ln 2 \sinh \frac{z_i - z_j}{2} + \sum_{i=1}^{N} \phi_i z_i
$$

(44)

We now specialize to the 2-anyon case $\alpha_{12} = \alpha, \phi_1 = \phi_2 = \phi$. It follows that the relative motion is controlled by the anyonic $\alpha$ term, whereas the $\phi$ term concerns the center of mass motion.

Let us define the center of mass $Z = (z_1 + z_2)/2$ and relative $z = z_1 - z_2$ coordinates. In the singular gauge [18] the Hamiltonian is free. Since the configuration space of two identical particles is defined by the identification $z \rightarrow -z$, the conformal mapping $w =
$2 \sinh(z/2)$ maps the cylinder on the plane and is thus well adapted. In polar coordinates $w = re^{i\theta}$, the relative Hamiltonian reads

$$H_2 = -\frac{1}{m} \sqrt{(1 - \frac{1}{4}r^2)^2 + r^2 \cos^2 \theta \left( \frac{1}{r} \partial_r r \partial_r + \frac{1}{r^2} \partial_\theta^2 \right)}$$

(45)

Going back to the regular gauge amounts to the shift $\partial_\theta \rightarrow \partial_\theta - i\alpha$.

Thus one has a situation similar to the relative scattering of two anyons on a plane. However, the Jacobian of the conformal mapping makes the relative motion non separable, implying seemingly out of reach exact eigenstates.

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References

[1] J.M. Leinaas and J. Myrheim, Nuovo Cimento B 37, 1 (1977); J.M. Leinaas, Nuovo Cimento A 47, 1 (1978); M.G.G. Laidlaw and C.M. de Witt, Phys. Rev. D3, 1375 (1971)

[2] Y. Aharonov and D. Bohm, Phys. Rev. 115 (1959) 485; in the case of two flux tubes see for example P. Štovíček, Phys. Lett. A 142, 1 (1989) 5

[3] F. Wilczek, Phys. Rev. Lett. 49, 957 (1982); Y.S.Wu, Proc. 2nd Int. Symp. Foundations of Quantum Mechanics, Tokyo, 171 (1986) and Phys. Rev. Lett. 53 (1984); for a general review on the subject see F. Wilczek, "Fractional Statistics and Anyon Superconductivity", World Pub. (1990); Y.-H. Chen, F. Wilczek, E. Witten, et B. I. Halperin, Int. J. Mod. Phys. B 3 (1989) 1001

[4] Y. S. Wu, Phys. Rev. Lett. 53, 111 (1984); G.V. Dunne, A. Lerda and C.A. Trugenberger, Mod. Phys. Lett. A 6, 2891 (1991); ibid, Int. Jour. Mod. Phys. B 5, 1675 (1991); see also G.V. Dunne, A. Lerda , S. Sciuto and C.A. Trugenberger, “Exact multi-anyon wavefunctions in a magnetic field ”, preprint CTP#1978-91; J. Grundberg, T.H. Hansson, A. Karlhede and E. Westerberg, “Landau levels for anyons”, preprint USITP-91-2; A.P. Polychronakos, Phys. Lett. B 264, 362 (1991); C. Chou, Phys. Lett. A 155, 245 (1991); K. H. Cho and C. Rim, “ Many anyon wavefunctions in a constant magnetic field”, preprint SNUTP-91-21; A. Khare, J. McCabe and S. Ouvry, Phys. Rev. D46 (1992) 2714; G.V. Dunne, A. Lerda , S. Sciuto and C.A. Trugenberger, “ Magnetic Moment and Third virial of non-relativistic anyons ”, preprint MIT-CTP 2032, CERN-TH 6338; E. Verlinde, “ A Note on Braid Statistics and the Non-Abelian Aharonov-Bohm Effect”, preprint IASSNS-HEP-90/60

[5] J. McCabe and S. Ouvry , Phys. Lett. B 260 (1991) 113; A. Comtet, J. McCabe and S. Ouvry , Phys. Lett. B 260 (1991) 372; A. Dasnieres de Veigy and S. Ouvry, Phys. Lett. 291B (1992) 130; Nucl. Phys. B[FS] 388 (1992) 715; see also D. Sen, Nucl. Phys.
[6] M.V.N. Murthy, J. Law, M. Brack and R.K. Bhaduri, Phys. Rev. Lett. 67, 1817 (1991); M. Sporre, J.J.M. Verbaarschot and I. Zahed, Phys. Rev. Lett. 67, 1813 (1991); ibid, “Four anyons in an harmonic well”, preprint SUNY-NTG-91/40; “Anyon Spectra and the Third Virial Coefficient”, preprint SUNY-NTG-91/47; J. Law, M. Suzuki and R.K. Bhaduri, Phys. Rev. A46 (1992) 4693; J. Myrheim and K. Olaussen, ”The Third Virial Coefficient of Free Anyons” Unit Trondheim report (1992)

[7] J. Fröhlich and A. Zee, Nucl. Phys. B[FS] 364 (1991) 517; X. G. Wen, A. Zee, Phys. Rev. B 46, 4 (1992) 2290

[8] A. Comtet, Y. Georgelin and S. Ouvry, J. Phys. A : Math. Gen. 22, 3917-3925 (1989)

[9] See for example G. V. Dunne et al in [2]

[10] R. Jackiw, MIT preprint CTP1937 (1991); C. Manuel and R. Tarrach, Phys. Lett. B268 (1991) 22; J. Grundberg, T.H. Hansson, A. Karlhede and J. M. Leinaas, Mod. Phys. Lett. B 5 (1991) 539

[11] Z. Y. Gu and S. W. Qian, J. Phys. A : Math. Gen. 21 (1988) 2573

[12] I. S. Gradshteyn and I. M. Ryzhik, ”Table of integrals, series and products”, Academic Press, New York and London (1965) p. 991

[13] for alternative studies of the anyon model on a cylinder see Y. S. Wu, Int. J. Mod. Phys. B 5, 10 (1991) 1649; Y. Hatsugai, M. Kohmoto and Y. S. Wu, Phys. Rev. B 43 (1991) 2661; S. Chakravarty and Y. Hosotani, Phys. Rev. D44 (1991) 441

[14] A. Comtet, J. McCabe and S. Ouvry, Phys. Rev. D45 (1992) 709; D. Li, ”Anyons and Quantum Hall Effect on the Sphere” Sissa Report 129/91/EP