INTEGRAL POINTS ON CONVEX CURVES

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Abstract. We estimate the maximal number of integral points which can be on a convex arc in $\mathbb{R}^2$ with given length, minimal radius of curvature and initial slope.

To the memory of Javier Cilleruelo

1. Introduction

Evaluating the number of integral points (points with integral coordinates) on finite continuous curves in $\mathbb{R}^2$ is a fairly general Diophantine question. Since the distance between two distinct elements in $\mathbb{Z}^2$ is at least 1, on a simple curve with length $\ell$ there cannot be more than $\ell + 1$ integral points, a bound which is only achieved for some linear curves.

Besides the study of specific curves, the first general result is due to Jarník [5] who proved in 1925 that the number of points on a strictly convex arc $y = f(x)$ of length $\ell$ is at most $3(4\pi)^{-1/3}\ell^{2/3} + O(\ell^{1/3})$, and that this bound is reached for some arc. From there Jarník deduced a similar result for strictly convex simple closed curves, giving the optimal bound $3(2\pi)^{-1/3}\ell^{2/3} + O(\ell^{1/3})$.

In 1963, Andrews [1] gave an upper bound for the number $N$ of integral points on the boundary of a strictly convex body in $\mathbb{R}^n$ in terms of the volume $V$ of that body, which is $N \ll V^{1/3}$ when $n = 2$.

Grekos [4], in 1988, revisited Jarník’s method in the case of strictly convex flat $C^2$ curves, i.e. curves $\Gamma$ for which the length $\ell = \ell(\Gamma)$ is smaller than the minimum of the radius of curvature along $\Gamma$. Denoting by $r = r(\Gamma)$ this minimal radius of curvature and by $N = N(\Gamma)$ the number of integral points on $\Gamma$, he first obtains the upper bound

\begin{equation}
N \leq 2\ell r^{-1/3}.
\end{equation}

With an unspecified constant, this result can be derived from [1].

The second result of [4] implies that, up to the constant, [1] is best possible, as long as $\Gamma$ is not too flat — i.e. $\log \ell / \log r > 2/3$ — and the lower bound he obtains for families of curves is uniform in terms of the slope $w = w(\Gamma)$ of the curve (i.e. the tangent of its angle with the $x$-axis).
The relevance of the slope is pointed out in [3]: Grekos and the first named author of the present paper showed that for any strictly convex $C^2$ curve with a tangent at the origin parallel to the $x$-axis, the number of its integral points satisfies

\begin{equation}
N \leq \ell^2/r + \ell/r + 1,
\end{equation}

a quantity which is essentially less than $\ell r^{-1/3}$ when $\log \ell / \log r < 2/3$.

On the other hand, for any $\alpha \in [1/3, 2/3]$ they also constructed curves $\Gamma$ for which $N > 0.79 \ell r^{-1/3}$ and $\log \ell / \log r = \alpha$.

Our main result expresses how the maximal number of integral points on a very flat strictly convex $C^2$ curve depends on the rational approximation of its slope. In particular, we show that (1.1) is essentially best possible for any fixed initial slope in the case $\ell \geq r^{2/3}$, which slightly improves on the result of Grekos ($\ell \geq C \epsilon r^{2/3 + \epsilon}$).

Let us first precise our notation. A strictly convex $C^2$ curve $\Gamma$ is (the image of) a $C^2$ map $\gamma = (x, y)$ from $[0, 1]$ to $\mathbb{R}^2$ such that $x'' y' - x' y''$ never vanishes. Up to an isometry which preserves $\mathbb{Z}^2$ (composition of symmetries with respect to the axes or the main bisectors) we may assume that $0 \leq y'(0) \leq x'(0)$; we then let $w = w(\Gamma) = y'(0)/x'(0)$ which belongs to $[0, 1]$. The radius of curvature of $\Gamma$ at the point $(x(t), y(t))$ is given by $r(t) = (x'^2 + y'^2)^{3/2}/|x'y'' - x'' y'|$ and we let $r = r(\Gamma) = \min_{t \in [0, 1]} r(t)$. We recall that $\ell = \ell(\Gamma)$ denotes the length of $\Gamma$ and $N = N(\Gamma)$ the number of its integral points. We consider curves satisfying $\ell(\Gamma) \leq r(\Gamma)$ and notice that they are really graphs (or arcs) $y = f(x)$.

A real number $x$ can be decomposed in a unique way as $x = \lfloor x \rfloor + \{x\}$, where $\lfloor x \rfloor$ is a rational integer called the integral part of $x$ and $\{x\}$ is a real number in $[0, 1)$. If $\{x\} \neq 1/2$, there exists a unique integer $[x]$ such that $\|x\| = |x - \lfloor x \rfloor| < 1/2$; if $\{x\} = 1/2$, we define $[x]$ to be $\lfloor x \rfloor$; in both cases, we call $[x]$ the nearest integer to $x$.

Finally, for functions $f$ and $g \geq 0$, we will also use either $f = O(g)$ or $f \ll g$ as shortcut for $|f| \leq C g$ for some positive constant $C$; $f \asymp g$ meaning both $f \ll g$ and $g \ll f$.

With those convention and notation, we have

**Theorem 1.1** (Main result). There exist two positive numbers $c_1$ and $c_2$ having the following property: for any $r \geq \ell \geq 1$ and $w \in [0, 1]$, the maximum $N_{w, \ell, r}$ of $N(\Gamma)$ where $\Gamma$ are curves with $\ell(\Gamma) = \ell$, $r(\Gamma) = r$ and $w(\Gamma) = w$ satisfies

\begin{equation}
N_{w, \ell, r} \leq c_2 \left(1 + \min(\ell r^{-1/3}, \ell \delta_{w, \ell r^{-1}})\right)
\end{equation}
and
\[
(1.4) \quad c_1 \left( 1 + \min(\ell r^{-1/3}, \ell \delta_{w,\ell r^{-1}}) \right) \leq N_{w,\ell,r},
\]
with \( \delta_{w,x} = \min_{q \in \mathbb{N}} (qx + \|qw\|) \) for \( x > 0 \).

**Remark 1.2.** In the excluded case \( \ell < 1 \) we trivially have \( N_{w,\ell,r} = 1 \). When \( \ell < r^{1/3} \), Theorem 1.1 says that \( N_{w,\ell,r} \asymp 1 \), so the result does not depend on the slope \( w \). The same happens when \( \ell > r^{2/3} \), since then the result simply claims that \( N_{w,\ell,r} \asymp \ell/r^{1/3} \) (due to the inequality \( \delta_{w,x} \geq x \)); this is a slight extension of Grekos work [4], since he proved it for \( \ell \gg r^{2/3+\epsilon} \). We shall actually show that the result for \( \ell = r^{2/3}/12 \) implies the case \( \ell \geq r^{2/3}/12 \). In that sense, we shall be able to assume that
\[
(1.5) \quad 1 \leq \ell \leq r^{2/3}/12.
\]

**Remark 1.3.** If \( w = 0 \), we have \( q = 1 \) and then (1.3) reads \( N_{0,\ell,r} \leq c_2(1 + \ell^2 r^{-1}) \) which is, up to a constant factor, the first result of [3].

**Remark 1.4.** The construction used in [3] for the above mentioned lower bound consists in considering, for \( \alpha \in [1/3, 2/3] \), curves (selected parts of parabolas) with given \( r, \ell = r^\alpha \) and \( w = r^{-1/3} \).

**Remark 1.5.** It may seem curious to restrict the consideration of the slope to one end point of the curve and one may ask what about the other end, or another point of the curve. Indeed, the consideration of the slope \( w \) is relevant only when \( \ell \) is less than \( r^{2/3} \), in which case the curve is extremely flat and the slope of the tangent can be considered as constant over the whole curve. We let the Reader make this point precise.

Theorem 1.1 is completely uniform, with \( \delta_{w,x} \) measuring how well \( w \) can be approximated by rationals with small denominator. From this result we can derive precise consequences for curves which begin with a fixed rational or irrational slope.

For any \( w \) irrational number, we are going to measure its good approximation by rationals by the exponent
\[
\beta = \beta(w) = \limsup_{j \to \infty} \beta_j
\]
where \((a_j/q_j)_{j \in \mathbb{N}}\) is the continued fraction of \( w \) and \( \beta_j \) is defined by the equation
\[
\left| w - \frac{a_j}{q_j} \right| = q_j^{-\beta_j}.
\]
This is the same as the irrationality exponent defined in [2, page 168]. If \( w \) is a rational number, we define \( \beta(w) = \infty \). It is known that
2 \leq \beta \leq \infty and that in that range the set of real numbers with exponent \beta has Hausdorff measure $2/\beta$ (Jarník-Besicovitch). We are going to show that this exponent determines also the number of integral points on curves with initial slope $w$.

**Theorem 1.6 (Curves with fixed initial slope).** Let $1/3 < \alpha < 2/3$. Let $w$ be an irrational number with $\beta(w) = \beta$. Then, we have

$$\limsup_{r \to \infty} \frac{\log N_{w, r^\alpha, r}}{\log r} = \alpha - \frac{1}{3}$$

and

$$\liminf_{r \to \infty} \frac{\log N_{w, r^\alpha, r}}{\log r} = \min \left( \alpha - \frac{1}{3}, 2\alpha - 1 + \frac{1 - \alpha}{\beta} \right).$$

In the previous result we excluded the ranges $\alpha \leq 1/3$ and $\alpha \geq 2/3$ because on them we trivially have

$$\lim_{r \to \infty} \frac{\log N_{w, r^\alpha, r}}{\log r} = \max \left( 0, \alpha - \frac{1}{3} \right),$$

and in particular the result does not depend on $w$.

2. **Upper bounds**

We use what is defined in the previous section. We begin by recalling an upper bound obtained in [4] which does not depend on the initial slope. This result directly follows from the understanding of the case $\ell = r^{1/3}$. We are going to give an arithmetic proof based on looking at the slopes between consecutive integral points.

**Proposition 2.1 (Local upper bound).** For any $r, \ell \geq 1$ we have

$$N_{w, \ell, r} \leq 2\frac{\ell}{r^{1/3}} + 2.$$

**Proof.** The result is a direct consequence of the fact that a curve with length $\ell = r^{1/3}$ cannot have more than two integral points. Suppose this were not true. We can assume that $0 \leq w = \tan(\theta_0) \leq 1$ and $x'(t), y'(t) > 0$ throughout the curve. The maximal slope that the curve can reach corresponds to the case of an arc of a circle of radius $r$, in which case that slope would be $w_1 = \tan(\theta_0 + \frac{\ell}{2\pi r})$. Since $\ell/r = 1/r^{2/3} \leq 1$ and $w \leq 1$, we have $w_1 \leq w + \frac{\ell}{r}$.

Then, if $(x_1, y_1), (x_2, y_2)$ and $(x_3, y_3)$ are three integral points on the curve with $x_1 < x_2 < x_3$, we have: $(a, b) = (x_2, y_2) - (x_1, y_1)$ and $(A, B) = (x_3, y_3) - (x_2, y_2)$ satisfy

$$a + A < \ell, \quad w \leq \frac{b}{a} < \frac{B}{A} \leq w + \frac{\ell}{r}.$$
Since \(a, A, b, B\) are natural numbers, this implies

\[
\frac{1}{\ell^2} < \frac{1}{Aa} \leq \frac{B}{A} - \frac{b}{a} < \frac{\ell}{r}
\]

which gives \(\ell > r^{1/3}\), a contradiction. \(\square\)

We now begin the study of an upper bound for slopes \(w\) that are near to a rational \(a/q\) with small \(q\). The tools will be geometric in nature. In the case \(w = 0\), in [3] it was shown that one can bound the number of integral points on the curve by the number of horizontal lines \(y = n, n \in \mathbb{Z}\) that touch the curve. We shall do the same for the case \(w = a/q\) with the “rational” lines \(y = \frac{a}{q}x + \frac{n}{q}\). The following lemma is essentially in [3].

**Lemma 2.2.** Let \(3 < \ell \leq r\) and \(\Gamma\) as in the previous section. Then \(\Gamma\) is included in a curvilinear triangle \(T(A, C, D)\) with: \(A = (x(0), y(0))\), \(AC\) a straight line with length \(\ell\) tangent to \(\Gamma\) at \(A\); \(\widehat{AD}\) is an arc of a circle with radius \(r\) and tangent to \(\Gamma\) at \(A\); \(CD\) is orthogonal to \(AC\).

By using that lemma we are going to prove the following result

**Lemma 2.3.** Let \(3 < \ell \leq r/3\). Then \(\Gamma\) is included in a parallelogram with two sides parallel to the \(y\) axis, two sides having slope \(w\) and such that the size of the sides parallel to the \(y\) axis is at most \(1.6\ell^2/r\) and its projection over the \(x\) axis has length \(1.02\ell\).

**Proof.** Throughout the proof of this lemma, we consider coordinates in the frame \((A, \vec{i}, \vec{j})\), with \(A = (x(0), y(0))\) and \(\vec{i}\) (resp. \(\vec{j}\)) is a unitary vector parallel to the \(x\)-axis (resp. \(y\)-axis). Notice that in the previous lemma we have \(|CD| = r - \sqrt{r^2 - \ell^2}\).

We begin by showing that \(|CD| \leq 0.6\ell^2/r\). To prove it is equivalent to show \(\sqrt{r^2 - \ell^2} \leq r - 0.6\ell^2/r\). Both \(r^2 - \ell^2\) and \(r - 0.6\ell^2/r\) are positive so it is equivalent to \(r^2 - \ell^2 \geq r^2 - 1.2\ell^2 + 0.36\ell^4/r^2\), i.e. \(0.2\ell^2 \geq 0.36\ell^4/r^2\), or \((\ell/r)^2 \leq 0.2/0.36\); but we have \((\ell/r)^2 \leq 1/9 < 0.2/0.36 = 5/9\).

Since \(\ell \leq r/3\), we further deduce that \(|CD| \leq 0.6\ell^2/r \leq 0.2\ell\) and

\[|AD| = \sqrt{|AC|^2 + |CD|^2} \leq \ell\sqrt{1.04} \leq 1.02\ell,
\]

so for any point \(k = (x_k, y_k)\) in the triangle \(ACD\), we have \(x_k \leq 1.02\ell\).

On the line \(x = 1.02\ell\), we consider the point \(P\) which is also on the line \(AC\) and \(Q\) which is also on the line \(AD\). We have the following properties:

(i) Any point in \(\Gamma\) is in the triangle \(APQ\).
(ii) \(|PQ| \leq 1.6\ell^2/r\).
The first property is clear. Let us prove the second: if \( \varphi \) is the angle between \( AC \) and \( AD \), since \( |CD| \leq 0.2\ell \), we have

\[
- \arctan(0.2) \leq \theta + \varphi \leq \frac{\pi}{4} + \arctan(0.2) \leq 0.983
\]

which implies \( \cos(\theta + \varphi) \geq 0.55 \) and since \( \theta + \varphi \) is the angle between \( \vec{i} \) and \( AD \), the horizontal component of the point \( D \) satisfies

\[
x_D \geq 0.55|AD| \geq 0.55\ell.
\]

Let \( C_1 \) be the intersection of the line \( AC \) with the line \( x = x_D \). Since \( \theta \in [0, \frac{\pi}{4}] \), we have

\[
|C_1D| = \frac{|CD|}{\cos \theta} \leq \sqrt{2}|CD| \leq 0.6\sqrt{2}\frac{\ell^2}{r}
\]

and so

\[
|PQ| = \frac{1.02}{x_D}|C_1D| \leq \frac{1.02}{0.55}0.6\sqrt{2}\frac{\ell^2}{r} \leq 1.6\frac{\ell^2}{r}.
\]

Now the triangle \( APQ \) is contained in the unique parallelogram \( APQK \) satisfying the properties in the statement of the lemma.

We can put the parallelogram from Lemma 2.3 inside one with two sides having rational slope \( a/q \) and the other two sides being vertical. This gives the following result.

**Proposition 2.4** (Curve inside rational parallelogram). Let \( q \geq 1 \) and \( 0 \leq a \leq q \) with \((a, q) = 1\). If \( 3 < \ell < r/3 \) then \( \Gamma \) is included in a parallelogram with two sides parallel to \( \vec{j} \), two sides having slope \( a/q \) and the size of the sides parallel to \( \vec{j} \) is at most \( 1.02\ell|w-a/q|+1.6\ell^2/r \).

Now, it is possible to control the integral points inside such a parallelogram by grouping them onto lines of slope \( a/q \), the number of those lines being easy to understand.

**Lemma 2.5** (Integral points in a rational parallelogram). Let \( q \) and \( a \) be coprime integers with \( q \geq 1 \) and \( a \geq 0 \), let \( u, v, h, k \) be real numbers with \( h > 0 \) and \( k > 0 \) and let \( P \) be the parallelogram with vertices \((u, v), (u, v+h), (u+k, v+ak/q), (u+k, v+h+ak/q)\). The number of straight lines with slope \( a/q \) which contain at least one integral point from \( P \) is at most equal to \( qh+1 \).

**Proof.** Let \( y = ax/q + m \) be the equation of such a straight line. Since it contains at least one integral point, \( mq \) is an integer \( j \). Since it contains one point in \( P \), we have \( v \leq au/q + j/q \leq v + h \). Thus the number of straight lines we are counting is at most the number of integers \( j \) in the interval \([vq - au, (v+h)q - au]\), whence the result. \( \square \)
With the two previous results we can finally prove our upper bound for the number of integral points on \( \Gamma \).

**Theorem 2.6** (Upper bound for “rational” slopes). Let \( 3 < \ell \leq r/3 \) and \( \Gamma \) with \( \ell(\Gamma) = \ell \) be such that for any \( M \in \Gamma \), \( r(M) \geq r \). Then for any \( q \geq 1 \), \( a \geq 0 \) and \( (a,q) = 1 \), we have

\[
N(\Gamma) \leq 2.04q \ell \left| w - \frac{a}{q} \right| + 3.2q \frac{\ell^2}{r} + 2.
\]

**Proof.** By Proposition 2.4, \( \Gamma \) is contained in a parallelogram \( \mathcal{P} \), and by Lemma 2.5 the number of lines with slope \( a/q \) inside that parallelogram which contain at least one integral point is at most

\[
q \left( 1.02 \ell \left| w - \frac{a}{q} \right| + 1.6 \frac{\ell^2}{r} \right) + 1.
\]

Now, each integral point on \( \Gamma \) is contained on one of those lines. Moreover, since \( \Gamma \) is strictly convex, each line cannot contain more than two points, so the result follows. \( \square \)

Notice that by choosing the best \( a/q \) possible in Theorem 2.6 we get the bound \( N_{w,\ell,r} \ll 1 + \ell \delta_{w,\ell,r}^{-1} \) from Theorem 1.1, and considering also Proposition 2.1 we have (1.3). In the following section we shall show that those bounds are the only restrictions for \( N_{w,\ell,r} \).

### 3. Lower bound for “irrational” slopes

The proof of Proposition 2.1 shows a relationship between integral points on the curve and rational slopes. For obtaining lower bounds, both Jarník \[5\] and Grekos \[4\] used Farey fractions as slopes in order to build curves with many integral points. We begin by writing a general result capturing those ideas.

**Lemma 3.1** (Curve with Farey tangents). Let \( I \) be an interval contained in \([0,1]\) with \( |I| \leq 1/30 \). Let \( M \in \mathbb{N} \) and \( F_M \) be the family of Farey fractions with denominators up to \( M \). If \( |F_M \cap I| \geq 3 \), then there exists a twice differentiable curve \( \Gamma \subset \mathbb{R}^2 \) such that

(i) its length is at most \( 32M^3|I| \),

(ii) its radius of curvature at each point is in the interval \( \left[ \frac{1}{16}M^3, 16M^3 \right] \),

(iii) it has at least \( |F_M \cap I| - 1 \) points with integer coordinates,

(iv) the slope at its initial point is \( \frac{h_1}{k_1} + \frac{h_2}{k_2} \),

with \( h_1/k_1 \) and \( h_2/k_2 \) being the first two terms in \( F_M \cap I \).
Proof. Let \( I = [s, s + \Delta s] \); let us write the elements of \( F_M \) in increasing order

\[
\frac{h_0}{k_0} < s \leq \frac{h_1}{k_1} < \frac{h_2}{k_2} < \ldots < \frac{h_N}{k_N} \leq s + \Delta s < \frac{h_{N+1}}{k_{N+1}}.
\]

We are going to use those elements to build our curve. We first list \( N - 1 \) points with integer coordinates, which will be on the curve:

\[
(x_1, y_1) = (0, 0) \quad \text{and} \quad \forall j \in [2, N - 1]: (x_j, y_j) = (x_{j-1}, y_{j-1}) + (\lambda_j k_j, \lambda_j h_j)
\]

with \( \lambda_j = \lfloor M^2/k_j^2 \rfloor \); we recall that \( \lfloor x \rfloor \) denotes the nearest integer to \( x \).

Next, we fix the slope of the curve at the point \((x_j, y_j)\) to be \( \tan \theta_j = \frac{h_j + h_{j+1}}{k_j + k_{j+1}} \) for \( 1 \leq j \leq N \). We are going to use Proposition 6.2 to make sure that a curve satisfying those requirements and the ones in the statement of the lemma does exist. In fact, in order to build the curve between \( A = (x_{j-1}, y_{j-1}) \) and \( B = (x_j, y_j) \), since the line between them has slope \( \tan \theta = h_j/k_j \), we check that

\[
\tan \theta_{j-1} - \tan \theta = \frac{h_{j-1} + h_j}{k_{j-1} + k_j} - \frac{h_j}{k_j} = -\frac{1}{k_j(k_{j-1} + k_j)}.
\]

And

\[
\tan \theta_j - \tan \theta = \frac{h_{j+1} + h_j}{k_{j+1} + k_j} - \frac{h_j}{k_j} = \frac{1}{k_j(k_{j+1} + k_j)}.
\]

Now, \( k_{j+1} + k_j > M \) because otherwise \( (h_{j+1} + h_j)/(k_{j+1} + k_j) \) would be a Farey fraction in \( F_M \) between \( h_j/k_j \) and \( h_{j+1}/k_{j+1} \). Thus, \( k_{j+1} + k_j \in [M, 2M] \) for every \( j \), so that

\[
\frac{\tan \theta_{j-1} - \tan \theta}{\tan \theta_j - \tan \theta} = -\frac{k_{j+1} + k_j}{k_{j-1} + k_j} \in [-2, -1/2],
\]

and then by applying Lemma 6.1 we have

\[
\frac{\alpha}{\beta} = \frac{\tan(\theta_{j-1} - \theta)}{\tan(\theta_j - \theta)} \in [-3, -1/3].
\]

Also

\[
\frac{|AB|}{\tan \theta_j - \tan \theta} = \frac{\lambda_j \sqrt{h_j^2 + k_j^2}}{1/(k_j(k_{j+1} + k_j))} = \lambda_j k_j^2 (k_{j+1} + k_j) \sqrt{1 + (\tan \theta)^2}
\]

so that

\[
\frac{|AB|}{\tan \theta_j - \tan \theta} \in \left[ \frac{1}{2} M^3, 3\sqrt{2} M^3 \right],
\]
and then by Lemma 6.1 we have
\[ \frac{|AB|}{\beta} = \frac{|AB|}{\tan(\theta_j - \theta)} \in \left[ \frac{1}{3} M^3, 9M^3 \right]. \]

By Proposition 6.2 we can build a curve between \(A\) and \(B\) with radius of curvature between \(M^3/16\) and \(16M^3\), and joining those pieces the same is true between \((x_1, y_1)\) and \((x_{N-1}, y_{N-1})\).

Moreover, by our definition of the curve, in order to finish the proof we just have to show that its length satisfies the condition in the statement of the lemma. But by convexity and considering the slope of the curve we have
\[
\text{Length}(\Gamma) \leq \Delta x + \Delta y \leq 2\Delta x.
\]
On the other hand, by the mean value theorem and our control over the curvature of the curve we have
\[
\Delta s \geq \Delta \left( \frac{dy}{dx} \right) \geq \frac{1}{16M^3} \Delta x
\]
so finally
\[
\text{Length}(\Gamma) \leq 32M^3 \Delta s
\]
and the result follows.

In order to take advantage of the previous result we need to control the distribution of Farey fractions in certain intervals. The question is that for \(N_{w, \ell, r}\) we are interested in the Farey fractions near \(w\), and that depends on whether \(w\) is near a rational with small denominator or not. In the first case, that rational repels other rationals, so we would not have other Farey fractions. In the second one, we should have the amount of Farey fractions that would be expected from probabilistic reasoning.

The problem with the analysis in [4, Lemme 3] and [6, Corollary 1] is that their counting of Farey fractions on an interval \(I\) only takes into account its length \(|I|\) and does not capture the subtlety described in the previous paragraph. We solve that problem with the following result.

**Lemma 3.2** (Farey fractions in an interval). There exists a constant \(C > 1\) such that for any \(a, q\) coprime natural numbers with \(a/q + 1/q^2 < 1\) and \(M\) with \(\frac{M}{q} > C\), \(z > C\), the number of Farey fractions with denominators up to \(M\) in the interval
\[
\left[ \frac{a}{q}, \frac{a}{q} + \frac{z}{Mq} \right]
\]
is at least \(\pi^{-2}z(M/q)\).
**Remark 3.3.** Notice that for $z < 1$ the only possible Farey fraction is $a/q$, so there is a sudden change in behaviour when $z$ increases (especially if $q$ is much smaller than $M$).

**Proof.** We need to count the coprime $h, k$ with $1 \leq k \leq M$ such that

$$0 < \frac{h}{k} - \frac{a}{q} < \frac{z}{Mq}$$

which is equivalent to

$$0 < qh - ak < \frac{zk}{M}.$$ 

By restricting to $M/2 < k < M$ we see that the number of Farey fractions we want to control is at least

$$J = \sum_{M/2 < k < M} \sum_{m < \frac{z}{2}} \sum_{\substack{qh - ak = m \ (h,k) = 1}} 1.$$ 

We can parametrize the integer solutions of $qh - ak = m$ as

$$k = qj - \overline{a}m \quad h = aj - sm$$

with $1 \leq \overline{a} \leq q$ the inverse of $a$ modulo $q$ and $s = \frac{a\overline{a} - 1}{q}$. Also, $(h, k) = 1$ is equivalent to $(j, m) = 1$, and thus

$$J = \sum_{m < \frac{z}{2}} \sum_{\substack{x < \frac{j}{2} - \delta m < x \ (j,m) = 1}} 1$$

with $x = M/q$ and $\delta = \overline{a}/q$. Since $\sum_{d|m} \mu(d) = 0$ for $l > 1$ and equals 1 for $l = 1$ we have

$$J = \sum_{m < \frac{z}{2}} \sum_{\substack{x < j - \delta m < x \ (j,m) = 1}} \sum_{d|m} \mu(d).$$

By writing $j = j_d d$ and $m = m_d d$ and rearranging the sums we have

$$J = \sum_{d < \frac{z}{2}} \mu(d) F(d)$$

with

$$F(d) = \sum_{m_d < \frac{z}{2d}} \sum_{\substack{x \leq j_d - \delta m_d < \frac{x}{d}}} 1.$$ 

We split the sum as

$$J = \sum_{d < \frac{z}{2}} \mu(d) F(d) + \sum_{\frac{z}{2} \leq d < \frac{z}{2}} \mu(d) F(d)$$
with \( y = \min(z, x) \). We can estimate

\[
F(d) = \sum_{m_* < \frac{x}{d}} \left( \frac{x}{2d} + O(1) \right) = \frac{zx}{4d^2} + O \left( \frac{z+x}{d} \right)
\]

so

\[
\sum_{d < \frac{y}{2}} \mu(d) F(d) = \sum_{d < \frac{y}{2}} \mu(d) \left[ \frac{zx}{4d^2} + O \left( \frac{z+x}{d} \right) \right]
\]

so by using \( \sum_{d=1}^{\infty} \mu(d)d^{-2} = \zeta(2)^{-1} = 6/\pi^2 \) we have

\[
\sum_{d < \frac{y}{2}} \mu(d) F(d) = \frac{zx}{4} \frac{6}{\pi^2} + O \left( \frac{zx}{y} \right) + O((z+x) \log y)
\]

so for \( z, x > C \) with \( C \) large enough we have

\[
\sum_{d < \frac{y}{2}} \mu(d) F(d) \geq \frac{zx}{4} \frac{5}{\pi^2}.
\]

To control the other sum let us look at \( S_{D,D'} = \sum_{D \leq d < D'} \mu(d) F(d) \) for any \( x \leq D \leq D' \leq 2D \). Rearranging the sums we have

\[
|S_{D,D'}| \leq \sum_{m_* < \frac{D}{\log D}} \sum_{\frac{D}{\log D} < j_* - \delta m_* \frac{1}{D}} \left| \sum_{a_{j_*}, m_* < d < b_{j_*}, m_*} \mu(d) \right|
\]

for some \( a_{j_*}, m_* \), \( b_{j_*}, m_* \) in the interval \([D, 2D]\). Thus, applying the prime number theorem we have

\[
S_{D,D'} \ll \sum_{m_* < \frac{D}{\log D}} \sum_{\frac{D}{\log D} < j_* - \delta m_* \frac{1}{D}} \frac{D}{(\log D)^2} \ll \sum_{m_* < \frac{D}{\log D}} \frac{D}{(\log D)^2} \ll \frac{z}{(\log D)^2}.
\]

Then, by splitting the sum into dyadic intervals we have

\[
\sum_{\frac{y}{2} \leq d < \frac{y}{2}} \mu(d) F(d) \ll \sum_{\frac{y}{2} \leq 2^n < \frac{y}{2}} \frac{z}{(\log 2^n)^2} \ll \sum_{\frac{y}{2} \leq 2^n < \frac{y}{2}} \frac{z}{(\log 2^n)^2} \ll \frac{z}{(\log 2^n)^2}.
\]

so this sum is at most \( zx/4\pi^2 \) for \( x > C \) for \( C \) large enough, so that we finally get

\[
J \geq \frac{zx}{4 \pi^2} - \frac{zx}{4 \pi^2} = \frac{zx}{\pi^2}.
\]

Now we are going to build curves with many integral points by using Farey fractions. But we are going to do it just in the “irrational” case, namely when \( w \) is not near to a rational with small denominator. Afterwards we shall see that in the other case, the “rational” case, we shall need other tools.
Theorem 3.4 (Lower bound for “irrational” slopes). There exists a constant $C > 1$ such that: for every $r, \ell > 1$ with $800C^2 r^{1/3} < \ell < r^{2/3}$ and $w \in (0, 1)$, if there is no rational $a/q$ with $q \leq 800C^4 r^{2/3}/\ell$ and $|w - a/q| \leq 1/qr^{1/3}$ then we can build a curve $\Gamma$ in $\mathbb{R}^2$ satisfying the following properties:

(i) $\Gamma$ is twice differentiable and its radius of curvature is always in the range $[C^3 r/32, 32C^3 r]$, 
(ii) the length of $\Gamma$ is less than $\ell$, 
(iii) the initial slope of $\Gamma$ is $w$, 
(iv) $|\Gamma \cap \mathbb{Z}^2| \geq \frac{1}{3200\pi C^2 \ell r^{1/3}}$.

Proof. Dirichlet lemma tells us that there is always an irreducible rational $a/q$ with $q < r^{1/3}$ such that $|w - a/q| \leq \frac{1}{qr^{1/3}}$, and by our hypotheses we can assume that $q > 800C^4 r^{2/3}/\ell$.

We shall be able to choose $C$ as the maximum of $8\pi^2$ and the constant in the statement of Lemma 3.2. Pick $M = \left\lfloor C r^{1/3} \right\rfloor$. By applying Lemma 3.1 with the interval $I = [a/q, a/q + \ell/400C^3 r]$ and by Lemma 3.2, since $M > Cq$ and $z = \frac{zM/q}{400\pi C^3 r} > C$ we can build a curve $\tilde{\Gamma}$ with at least

$$\frac{zM/q}{\pi^2} - 1 \geq \frac{zM/q}{2\pi^2} = \frac{\ell}{2\pi^2 400C^3 r} M^2 \geq \frac{1}{1600\pi^2 C r^{1/3}} \ell$$

points with integer coordinates, with radius of curvature always in the range $[M^3/16, 16M^2]$, length at most $32M^3 \ell/400C^3 r \leq \frac{3}{4} \ell$ and with initial slope

$$\tan \theta = \frac{a + h_2}{q + k_2}.$$

In order to finish building our curve $\Gamma$, we only need to fix the problem that the initial slope should be $w = \tan \theta_w$ instead of $\tan \theta$, but

$$|\tan \theta - \tan \theta_w| \leq \left| \frac{a + h_2}{q + k_2} - \frac{a}{q} \right| + \left| \frac{a}{q} - w \right| \leq \frac{1}{qM} + \frac{1}{q r^{1/3}} \leq \frac{1}{400C^4 r} \ell.$$

Then, if $w \leq \frac{a + h_2}{q + k_2}$, by prolonging $\tilde{\Gamma}$ to the left maintaining the curvature constant we can make sure that the initial slope is $w$ and the length of $\Gamma$ will be the length of $\tilde{\Gamma}$ plus at most the length of an arc of a circle of radius $16M^3$ and angle $\theta - \theta_w$, namely

$$\text{Length}(\Gamma) \leq \frac{3}{4} \ell + (\theta - \theta_w)(16M^3) \leq \frac{3}{4} \ell + \frac{1}{400C^4 r} 16M^3 \leq \ell,$$

by the inequality $\theta - \theta_w \leq \tan \theta - \tan \theta_w$. On the other hand, in the case $w > \frac{a + h_2}{q + k_2}$, we shall delete the first part of $\tilde{\Gamma}$ to get $\Gamma$, precisely until the slope is $w$. In doing so, we delete some of the points with integer coordinates belonging to $\tilde{\Gamma}$. By construction, the number of points which we counted previously and we are deleting now is at most
1 plus the number of Farey fractions with denominator up to $M$ in the interval

$\left[ \frac{a}{q}, w \right]$.

Since the distance between two consecutive Farey fractions is at least $M^{-2}$, the number of them in that interval is at most

$$1 + M^2 |w - \frac{a}{q}| \leq 1 + M^2 \frac{1}{qr^{1/3}} \leq \frac{1}{400C^2} \ell r^{-1/3}.$$ 

Thus, the number of “surviving” integral points is at least

$$\frac{1}{1600\pi^2 C} \frac{l}{r^{1/3}} - \frac{1}{400C^2} \ell r^{-1/3} \geq \frac{1}{3200\pi^2 C} \frac{\ell}{r^{1/3}}.$$ 

□

4. Lower bound for “rational” slopes

Now we are going to handle the “rational” case, that is when $w$ is near to a rational $a/q$ with small $q$. In this case, the method of Farey fractions stops working when $\ell < r^{2/3 - \epsilon}$, since then there will not be any rational with denominator up to $r^{1/3}$ in the interval $\left( \frac{a}{q}, \frac{a}{q} + \frac{\ell}{r} \right)$.

We know from the proof of Theorem 2.6 that in this case we can essentially control the amount of points in the curve by the number of lines $y = \frac{a}{q}x + \frac{n}{q}, n \in \mathbb{Z}$ that touch the curve (since by convexity the curve cannot have more than two points on a line).

Then, in order to build a curve with many integral points, we shall explicitly choose a sequence of integral points $((x_n, y_n))_{0 \leq n \leq N}$, with $(x_n, y_n)$ belonging to the the line $y = \frac{a}{q}x + \frac{n}{q}$ such that they can be put on a curve with the requested curvature, slope and length.

From the point of view of slopes, we can say that the fractions relevant to the problem can be parametrized in terms $a, q$. We shall split the proof into two parts: the first one will work for $w$ very near $a/q$ (with $q$ small), and the second one for $w$ near to $a/q$, but not that much.

**Theorem 4.1** (First lower bound for “rational” slopes). Let $C > 1$, $r, \ell > 1$ with $(800C^4)^2 r^{1/3} < \ell < r^{2/3}$, $w \in (0, 1)$ and $|w - a/q| \leq \ell/25r$ for some irreducible rational $a/q$ with $q < 800C^4 r^{2/3}/\ell$, we can build a curve $\Gamma$ in $\mathbb{R}^2$ satisfying the following properties:

(i) $\Gamma$ is twice differentiable and its radius of curvature is always in the range $[r/16, 16r]$,

(ii) the length of $\Gamma$ is less than $\ell$,

(iii) the initial slope of $\Gamma$ is $w$. 

Proof. We can assume that $q\ell^2/r \geq (800C^4)^6$ since otherwise we just need to build a curve with at least one integral point, which is trivial. Let us suppose that $w \geq a/q$, the other case being similar. Define $\Omega = q[kr/q^2\ell]$ with $k = (800C^4)^2$ and consider the sequence $(x_j, y_j)_{0 \leq j \leq N}$ with $x_0 = 0, y_0 = 0$,

$$\Delta x_j = \Omega - \pi - (j - 1)q[\Omega^3/r]; \\ \Delta y_j = \frac{a}{q} \Delta x_j + \frac{1}{q},$$

with $\Delta b_j = b_j - b_{j-1}$, $\pi$ the number between 1 and $q$ which is the inverse of $a$ modulo $q$ and $N = [(1/k)r/q\Omega^2]$. The definition implies that $x_j, y_j \in \mathbb{Z}$. Also, our hypotheses imply that $\Omega > 800Cq$, $\Omega^3/r \geq 1$ so that

$$(4.1) \Omega \left(1 - \frac{1}{400C}\right) < \Omega - q - Nq\Omega^3/r < \Delta x_j < \Omega$$

and in particular $\Delta x_j > 0$.

We are going to see that it is possible to build a curve $\Gamma$ which satisfies the conditions of the statement and contains the previous sequence of points. First we fix the slope of $\Gamma$ at the point $(x_j, y_j)$, with $1 \leq j \leq N - 1$, to be

$$\tan \theta_j = \frac{\Delta y_{j+1} + \Delta y_j}{\Delta x_{j+1} + \Delta x_j} = \frac{a}{q} + \frac{1}{q} \frac{2}{\Delta x_{j+1} + \Delta x_j}.$$  

Now, to see that a curve satisfying the curvature condition in the statement does exist, it is enough to see that we can apply Proposition 6.2 with $A = (x_{j-1}, y_{j-1}), B = (x_j, y_j), T_A$ and $T_B$ lines with slopes $\tan \theta_{j-1}$ and $\tan \theta_j$ respectively, and $\rho = r$. Since the line between $A$ and $B$ has slope

$$\tan \theta = \frac{\Delta y_j}{\Delta x_j} = \frac{a}{q} + \frac{1}{q} \frac{1}{\Delta x_j};$$

by using (4.1) we have

$$\tan \theta_{j-1} - \tan \theta = \frac{1}{q} \frac{\Delta(\Delta x_j)}{\Delta x_j(\Delta x_{j-1} + \Delta x_j)} = -\frac{[\Omega^3/r]}{2\Omega^2(1 - \epsilon)^2};$$

$$\tan \theta_j - \tan \theta = -\frac{1}{q} \frac{\Delta(\Delta x_{j+1})}{\Delta x_j(\Delta x_{j+1} + \Delta x_j)} = -\frac{[\Omega^3/r]}{2\Omega^2(1 - \epsilon')^2},$$

with $0 < \epsilon, \epsilon' < 1/400C$, so

$$\frac{\tan \theta_{j-1} - \tan \theta}{\tan \theta_j - \tan \theta} \in \left[-1/(1 - 1/400C)^2, -(1 - 1/400C)^2\right]$$
and since $C \geq 1$, by Lemma 6.1, we have
\[ \frac{\alpha}{\beta} = \frac{\tan(\theta_{j-1} - \theta)}{\tan(\theta_j - \theta)} \in [-3, -1/3]. \]
Moreover $|AB| = s \Delta x_j$ with $1 < s < 2$ and $\beta = \tan(\theta_j - \theta) = t(\theta_j - \tan \theta)$ with $1/2 < t < 3$, hence
\[ \frac{|AB|}{\beta \rho} = \frac{s \Delta x_j}{rt(\tan \theta_j - \tan \theta)} = \frac{s}{t r(\Omega^2 / r)} \frac{\Omega(1 - \epsilon)}{2\Omega^2 (1 - \epsilon')^2} = \frac{2s}{t} (1 - \epsilon)(1 - \epsilon')^2 \frac{\Omega^3 / r}{[\Omega^3 / r]} \in [1/3, 9], \]
for some $0 < \epsilon, \epsilon' < 1/400C$, so indeed we can build a suitable curve between $A$ and $B$. By convexity, the length of $\Gamma$ is at most
\[ \sum_{j \leq N} (\Delta x_j + \Delta y_j) \leq \sum_{j \leq N} 3\Delta x_j \leq 3\left(\frac{1}{k \Omega \sqrt[3]{r}}\right) \Omega \leq \frac{6}{k^2} \leq \frac{\ell}{800}. \]
Then, the curve $\Gamma$ we have just built, beginning at the point $(x_1, y_1)$, satisfies all the requirements but not necessarily the one about the initial slope. By (4.1) we have
\[ \tan \theta_1 - \frac{a}{q} = \frac{1}{q} \frac{2}{\Delta x_2 + \Delta x_1} \leq \frac{1}{q} \frac{1}{(1 - 1/400C') \Omega} \leq \frac{4}{k^2} \leq \frac{\ell}{r} \frac{\ell}{800}. \]
so the condition $w - a/q < \ell/25r$ implies that
\[ |\tan \theta_1 - w| < \frac{\ell}{20r}. \]
Thus, as in the proof of Theorem 3.4, it is enough to enlarge the curve $\Gamma$ by an arc of a circle of radius at most $16r$ and angle at most $\ell/20r$. Then, the expanded curve will satisfy the initial condition and have length at most
\[ \frac{\ell}{800} + 16r \left(\frac{1}{20r}\right) < \ell. \]

\[ \square \]

**Theorem 4.2** (Second lower bound for “rational” slopes). Let $C > 1$, $r, \ell > 1$ with $(800C^4)^{2r^{1/3}} < \ell < r^{2/3}$ and $w = \tan \theta_w$ with $0 < w < 1$ and $\ell/25r < |w - a/q| \leq 1/qr^{1/3}$ for some irreducible rational $a/q$ with $q < 800C^4r^{2/3}/\ell$. We can build a curve $\Gamma$ in $\mathbb{R}^2$ satisfying the first three properties in Theorem 4.1 and
\[ |\Gamma \cap \mathbb{Z}^2| \geq \frac{1}{4} + \frac{1}{4(800C^4)^{2r^{1/3}}} \left|w - \frac{a}{q}\right|. \]
Proof. We can assume that $q^6 | w - a/q | \geq (800C^4)^6$. We also suppose that $w \geq a/q$. We proceed to build the sequence and the curve as in the proof of Theorem 4.1, but define

$$\Omega = q \left\lfloor \frac{1}{q(qw - a)} \right\rfloor.$$ 

By our hypotheses we again have $\Omega > 800Cq$ and $\Omega^3/r \geq 1$, so that the curvature condition is satisfied (since those inequalities are the only thing about $\Omega$ that we used to get the curvature condition in the proof of Theorem 4.1). Now, we are going to keep just the part of the curve which has the first $M = \left\lfloor \frac{1}{(800C^4)^6 q^6 | w - a/q |} \right\rfloor$ points from the sequence. This is possible since $M \leq N = [(1/k)r/q]\Omega^2$ occurs whenever

$$\left| w - \frac{a}{q} \right| \geq \frac{16 \ell}{k^2 r},$$

which is true due to our hypothesis $|w - a/q| \geq \ell/25r$. Thus, this curve also has the desired number of integral points. Moreover, its length is at most

$$\sum_{j \leq M} (\Delta x_j + \Delta y_j) \leq \sum_{j \leq M} 3\Delta x_j \leq 3M\Omega \leq \frac{\ell}{800}.$$ 

It remains to see what happens with the initial slope. We have

$$\Delta x_1 = \Omega - \pi = \frac{1}{qw - a} - \delta q$$

for some $0 < \delta < 2$, hence the slope of the line between $(x_0, y_0)$ and $(x_1, y_1)$ is

$$\tan \theta = \frac{a}{q} + \frac{1}{q} \frac{w - a/q}{1 - \delta q(qw - a)} = \frac{a}{q} + \frac{w - a/q}{1 - \delta q(qw - a)} = \frac{w - a/q}{1 - \delta q(qw - a)} + \frac{\delta(qw - a)^2}{1 - \delta q(qw - a)}$$

and since $|q(qw - a)| \leq 1/4$ we have

$$0 < \tan \theta - w < 4(qw - a)^2 \leq \frac{4}{r^{2/3}} \leq \frac{1}{1600r}.$$ 

Moreover

$$0 < \tan \theta_1 - \tan \theta \leq \frac{\Omega}{r} \leq \frac{1}{r(qw - a)} \leq \frac{1}{1600r} \leq \frac{\ell}{r}$$

since we assumed $q^6 | w - a/q | \geq (800C^4)^6$. Thus, finally we have

$$0 < \tan \theta_1 - w < \frac{\ell}{800r}.$$
so we can finish as in the proof of Theorem 4.1 by enlarging the curve to the left of \((x_1,y_1)\) with an arc of a circle.

\[\square\]

5. Proofs of the main results

We begin by proving Theorem 1.1. First we are going to show that, as mentioned in Remark 1.2, we can restrict ourselves to the case \(\ell \leq r^{2/3}/12\).

If \(\ell \gg r^{2/3}\), Theorem 1.1 says that \(N_{w,\ell,r} \simeq \ell r^{-1/3}\). Thus, by cutting a curve of length \(\ell > r^{2/3}/12\) into pieces of length between \(r^{2/3}/24\) and \(r^{2/3}/12\) we see that applying the bound \(N_{w,\ell,r} \ll \ell r^{-1/3}\) for \(\ell \leq r^{2/3}/12\) implies the same bound for \(\ell > r^{2/3}/12\). Regarding the lower bound, we can assume first that \(r\) is sufficiently large, since in the case \(\ell \leq r \ll 1\) trivially \(N_{w,\ell,r} \asymp 1\). We begin by building a curve \(\Gamma_1\) with initial slope \(w \leq 1\), length \((500r)^{2/3}/24\) and radius of curvature larger than \(500r\), with \(\gg r^{1/3}\) integral points. If this curve ends at a point \(A\) with slope \(\tan \theta_1\), we build another curve \(\Gamma_2\) with the same conditions but initial slope \(\tan(\theta_1 + 2r^{-1/3})\). Consider the point \(\tilde{B}\) such that the line passing through \(A\) and \(\tilde{B}\) has slope \(\tan(\theta_1 + r^{-1/3})\) and the distance from \(A\) to \(\tilde{B}\) equals \(r^{2/3}\). By an integral translation, we can assume that the initial point \(B\) of \(\Gamma_2\) is at distance at most 1 from \(\tilde{B}\). This implies that \(|AB| = r^{2/3} + O(1) \sim r^{2/3}\), \(\tan(AB, T_A) = -r^{-1/3} + O(1/r^{2/3}+O(1)) \sim -r^{-1/3}\) with \(T_A\) the line tangent to \(\Gamma_1\) at \(A\), and \(\tan(AB, T_B) \sim r^{-1/3}\) with \(T_B\) the line tangent to \(\Gamma_2\) at \(B\). Then, we can apply Proposition 2.1 with \(\rho = 250r\) to join \(\Gamma_1\) to \(\Gamma_2\) so that the full curve is \(C^2\) and of length \(O(r^{2/3})\). We continue this procedure with curves \(\Gamma_1, \Gamma_2, \Gamma_3, \ldots\), joining \(\Gamma_i\) to \(\Gamma_{i+1}\) until we get a \(C^2\) curve \(\Gamma\) with length at most \(\ell\), radius of curvature at least \(r\) and \(\gg (\ell/r^{2/3}) r^{1/3} = \ell r^{-1/3}\) integral points.

After the last paragraph, we can assume \(\ell \leq r^{2/3}/12\). Let \(q_0\) be a natural number for which the minimum \(\delta_{w,\ell r^{-1}} = \min_{q \in \mathbb{N}}(q\ell r^{-1} + \|qw\|)\) is reached, and let \(a_0 = [q_0w]\). This implies \((a_0, q_0) = 1\), and applying Proposition 2.1 and Theorem 2.6 with \(a_0\) and \(q_0\) we have

\[(5.1) \quad N_{w,\ell,r} \ll 1 + \min(\ell/r^{1/3}, \ell \delta_{w,\ell r^{-1}}).\]

For \(l \ll r^{1/3}\) this implies \(N_{w,\ell,r} \ll 1\). In this range it is trivial to build a curve satisfying the curvature condition and with at least one integral point, so that \(N_{w,\ell,r} \asymp 1\).

Then we can assume \(Kr^{1/3} < \ell \leq r^{2/3}/12\) for any fixed constant \(K\). Then we can apply either Theorem 3.4 or Theorem 4.1 or Theorem 4.2.

\[\square\]
in order to get the bound
\[ N_{w, \ell, r} \gg 1 + \min(\ell/r^{1/3}, \ell(\|qw\| + q\ell/r)). \]

for some \( q \in \mathbb{N} \), which clearly implies
\[ N_{w, \ell, r} \gg 1 + \min(\ell r^{-1/3}, \ell \delta_{w, \ell r^{-1}}) \]

so from this and (5.1) we deduce Theorem 1.1.

Now we are going to derive Theorem 1.6 from Theorem 1.1. For \( w \) rational this deduction is trivial. For \( w \) irrational, let us pick any \( r \) sufficiently large, and choose the pair of convergents \( q_j, q_{j+1} \) of \( w \) such that
\[ (5.2) \quad q_j \leq r^{1/3} < q_{j+1}. \]

In this range
\[ \frac{1}{2q_j q_{j+1}} \leq \left| w - \frac{a_j}{q_j} \right| \leq \frac{1}{q_j q_{j+1}} \leq \frac{1}{q_j r^{1/3}} \]

and we can show that
\[ \min(r^{-1/3}, \delta_{w, \ell r^{-1}}) \leq \min(r^{-1/3}, \|q_j w\| + q_j \ell r^{-1}) \leq 10 \min(r^{-1/3}, \delta_{w, \ell r^{-1}}) \]

with \( \ell = r^a \). The first inequality comes from the definition of \( \delta_{w, \ell r^{-1}} \); if the second were not true we would have \( \delta_{w, \ell r^{-1}} < \frac{r^{-1/3}}{10} \) and then \( q < \frac{r^{1/3}}{10}, \|qw\| < \frac{r^{1/3}}{10} \) for the \( q \) such that \( \delta_{w, \ell r^{-1}} = \|qw\| + q\ell r^{-1} \), and this would contradict the inequality \( (q_j q_{j+1})^{-1} \leq |w - a_j| + |w - a_{j+1}| \), \( a = [qw] \).

Then, by Theorem 1.1 we have
\[ N = N_{w, r^a, r} \ll \min \left( r^{a-1/3}, q_j r^{2a-1} + r^a q_j^{\beta_j-1} \right) \]

since \( q_{j+1} \asymp q_j^{\beta_j-1} \). This also gives the inequality \( q_j \leq r^{1/3} \ll q_j^{\beta_j-1} \). By choosing \( r^{1/3} = q_j \) we have \( q_j r^{2a-1} = (r^{a-1/3})^2 \geq r^{a-1/3} \) so \( N \asymp r^{a-1/3} \) and the result for the upper limit follows. On the other hand, any \( r \) in the range (5.2) can be written as \( r \asymp q_j^{\theta} \) with \( 1 \leq \theta \leq \beta_j - 1 \) or, in other terms, \( q_j \asymp r^\epsilon \) with \( 1/3 (\beta_j - 1) \leq \epsilon \leq 1/3 \). Then
\[ N \asymp \min(r^{a-1/3}, r^{\epsilon+2a-1} + r^{a-\epsilon(\beta_j-1)}). \]

Now, we want to compute the minimum of this function \( N = N(\epsilon) \) in the interval \( 1/3 (\beta_j - 1) \leq \epsilon \leq 1/3 \). Since \( r^{a-1/3} \) is constant in \( \epsilon \), we only need to look at the minimum of the second term. Moreover, both at \( \epsilon = 1/3 (\beta_j - 1) \) and \( \epsilon = 1/3 \) we have \( N(\epsilon) \asymp r^{a-1/3} \), so we only
need to check the case in which the second term has a minimum in the interior of the interval, and this happens at $\epsilon = (1 - \alpha)/\beta_j$ whenever
\[
\frac{1}{3(\beta_j - 1)} < \frac{1 - \alpha}{\beta_j} < \frac{1}{3}.
\]
The second inequality is always true, but the first amounts to
\[
\beta_j > 1 + \frac{1}{2 - 3\alpha}.
\]
In this case, the minimum for the second term is \(r_2 \propto 1 - \alpha\beta_j\). This implies the result for the lower limit.

6. Appendix

Lemma 6.1 (Trigonometric lemma). Let $\tan \theta$ and $\tan(\theta + \tilde{\beta})$ be in the interval $[s, s + \Delta s] \subset [0, 1]$, with $0 \leq \Delta s < 1/2$. Then we have
\[
1 - \Delta s \leq \frac{\tan(\theta + \tilde{\beta}) - \tan \theta}{(1 + (\tan \theta)^2) \tan \beta} \leq \frac{1 - \Delta s}{1 - 2\Delta s}.
\]

Proof. The formula for the tangent of the sum of two angles gives
\[
\tan(\theta + \tilde{\beta}) - \tan \theta = \tan \tilde{\beta} \frac{1 + (\tan \theta)^2}{1 - \tan \theta \tan \tilde{\beta}},
\]
which implies
\[
|\tan \tilde{\beta}| \leq |\tan(\theta + \tilde{\beta}) - \tan \theta| (1 + |\tan \tilde{\beta}|) \leq \Delta s (1 + |\tan \tilde{\beta}|)
\]
so that $|\tan \tilde{\beta}| \leq \Delta s/(1 - \Delta s)$. Substituting this bound into the previous identity ends the proof. \(\square\)

The following result is essentially a variation on construction in [4].

Proposition 6.2. Let $A$ and $B$ be two points in the euclidean plane $E = \mathbb{R}^2$, $T_A$ (resp. $T_B$) be a straight line containing $A$ (resp. $B$) and $\alpha, \beta, \rho_1, \rho_2, \rho$ be real numbers such that
\[
(i) \quad \tan(AB, T_A) = \alpha \text{ and } \tan(AB, T_B) = \beta,
\]
\[
(ii) \quad \beta \in (0, 1/3], \alpha \in [-3\beta, -\beta/3] \text{ and } 0 < \rho \leq \min(\rho_1, \rho_2),
\]
\[
(iii) \quad |AB| \in [(1/3)\beta \rho , 9\beta \rho].
\]

There exists a two times differentiable curve with end points $A$ and $B$, which admits for tangent at the point $A$ (resp. $B$) the line $T_A$ (resp. $T_B$), such that its radius of curvature is always between $\rho/250$ and $250 \max(\rho_1, \rho_2)$ and which takes the value $(1 + \alpha^2)^{3/2}\rho_1$ (resp. $(1 + \beta^2)^{3/2}\rho_2$) at the point $A$ (resp. $B$).

We first prove a technical Lemma, in the spirit of Lemma 1 in [4].
Lemma 6.3. Let $\rho > 0$, $a < b$, $\beta > 0$, $\alpha = -\lambda \beta$ with $\lambda \in [1/3, 3]$ be real numbers such that

\[(1/3)\beta \rho \leq b - a \leq 9\beta \rho.\]

There exists a differentiable real function $f$ defined on $[a,b]$ such that

\[\begin{array}{ll}
(i) & f(a) = \alpha \text{ and } f(b) = \beta, \\
(ii) & f'(a) = 1/\rho_1, \ f'(b) = 1/\rho_2, \\
(iii) & \forall x \in [a,b] : 0.01/\max(\rho_1, \rho_2) \leq f'(x) \leq 100/\rho, \\
(iv) & \int_{a}^{b} f(t)dt = 0.
\end{array}\]

Proof. We consider the points $M_A = (a, \alpha)$ and $M_B = (b, \beta)$. The slope of the segment $M_A M_B$ lies in the interval $[0.3/\rho, 12/\rho]$: we have indeed

\[0.3 \leq \frac{4}{9\rho} \leq \frac{(4/3)\beta}{3\beta \rho} \leq \frac{\beta(1+\lambda)}{b-a} = \frac{\beta - \alpha}{b-a} = \frac{\beta(1+\lambda)}{b-a} \leq \frac{4\beta}{\beta \rho/3} = \frac{12}{\rho}.
\]

This implies that the straight lines $D_A$ passing through $M_A$ and having the slope $0.01/\rho < 0.3/\rho$ and the lines $D_B$ passing through $M_B$ and having the slope $100/\rho > 12/\rho$ will meet at a point $M_1 = (x_1, y_1)$ with $a < x_1 < b$ and $\alpha < y_1 < \beta$.

We now consider the function $h_1$ defined on $[a,b]$, linear on $[a,x_1]$ and on $[x_1, b]$ and which takes the values: $h_1(a) = \alpha$, $h_1(x_1) = y_1$ and $h_1(b) = \beta$. Let us show that

\[I_1 = \int_{a}^{b} h_1(x)dx < 0.
\]

We have

\[2I_1 = (x_1 - a)(\alpha + y_1) + (b - x_1)(\beta + y_1);
\]

The coordinates of the point $M_1$ are defined by

\[\begin{align*}
\frac{y_1 - \alpha}{x_1 - a} &= 0.01/\rho \quad \text{and} \quad \frac{\beta - y_1}{b - x_1} = 100/\rho,
\end{align*}\]

from which we get

\[\begin{align*}
2I_1 &= \rho \left\{100(y_1^2 - \alpha^2) + 0.01(\beta^2 - y_1^2)\right\} \\
&= \rho \beta^2 \left\{99.99 \left(\frac{y_1}{\beta}\right)^2 + (0.01 - 100\lambda^2)\right\}.
\end{align*}\]

From (6.2) we can compute $y_1$ and get

\[\frac{1}{\rho} = 100(y_1 - \alpha) + 0.01(\beta y_1)\]
which leads to
\[
\frac{b - a}{\rho \beta} = 100 \left( \frac{y_1}{\beta} - \frac{\alpha}{\beta} \right) + 0.01 \left( 1 - \frac{y_1}{\beta} \right) \\
= 99.99 \left( \frac{y_1}{\beta} \right) + (100\lambda + 0.01).
\]

Since \((b - a)/(\rho \beta) \in [1/3, 9]\) and \(\lambda \in [1/3, 3]\), we have
\[-99.893\lambda \leq -100\lambda + 1/3 - 0.01 \leq 99.99(y_1/\beta) \leq 9 - (100\lambda + 0.01) < 0,
\]
and so
\[\left( \frac{y_1}{\beta} \right)^2 \leq 0.9981\lambda^2.
\]

We incorporate this last relation in (6.3) and get
\[2I_1/(\rho \beta^2) \leq 99.99 \times 0.9981\lambda^2 + 0.01 - 100\lambda^2 < 0.01 - 0.19\lambda^2 < -0.011,
\]
which proves that \(I_1\) is negative.

For \(\delta\) positive and sufficiently small, we also have \(\int_a^b (h_1(x) + 2\delta)dx < 0\). We can then slightly modify the function \(h_1 + \delta\) to get a function \(f_1(x)\) which satisfies the conditions (i), (ii) and (iii) of Lemma 6.3 and such that \(J_1 = \int_a^b f_1(x)dx < 0\).

In a similar way, considering first the straight lines \(\Delta_A\) (resp. \(\Delta_B\)) passing through \(M_A\) (resp. \(M_B\)) and having the slope \(100/\rho\) (resp. \(0.01/\rho\)), one can construct a function \(f_2(x)\) which satisfies the conditions (i), (ii) and (iii) of Lemma 6.3 and such that \(J_2 = \int_a^b f_2(x)dx > 0\).

The function \(f\) defined by \(f(x) = (J_2f_1(x) - J_1f_2(x))/(J_2 - J_1)\) satisfies the conditions (i), (ii), (iii) and (iv) of the Lemma 6.3.

For a two times differentiable function \(g\), we denote by \(\text{rad}_g(x)\) its radius of curvature at the point \(x\), given by
\[
\text{rad}_g(x) = \frac{(1 + g'(x)^2)^{3/2}}{|g''(x)|}.
\]

**Corollary 6.4.** Let \(\min(\rho_1, \rho_2) \geq \rho > 0, a < b, \beta \in (0, 1/3], \alpha = -\lambda\beta\) with \(\lambda \in [1/3, 3]\) be real numbers such that
\[
\frac{1}{3}\beta \rho \leq b - a \leq 9\beta \rho.
\]
There exists a two times differentiable real function $F$ defined on $[a, b]$ such that

(i) $F(a) = F(b) = 0$,

(ii) $F'(a) = \alpha$, $F'(b) = \beta$,

(iii) We have $\text{rad}_F(a) = \rho_1(1 + \alpha^2)^{3/2}$ and $\text{rad}_F(b) = \rho_2(1 + \beta^2)^{3/2}$,

(iv) $\forall x \in [a, b] : \text{rad}_F(x) \in [0.01 \rho, 300 \max(\rho_1, \rho_2)]$.

Proof. For the given parameters $\rho, \rho_1, \rho_2, a, b, \alpha, \beta, \lambda$, we construct a function $f$ satisfying the conditions of Lemma 6.3. We define

$$\forall x \in [a, b] : F(x) = \int_a^x f(t) dt.$$ 

Relations (i), (ii) and (iii) come directly from Lemma 6.3 and (6.5). Relation (ii) of Lemma 6.3 implies that for all $x$ one has $0.01 \rho \leq 1/F''(x) \leq 100 \max(\rho_1, \rho_2)$; so $F'$ is increasing and then $|F'(x)| \leq \max(|\alpha|, \beta) \leq 1$; relation (iv) easily follows from those relations and (6.5).

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