ON TRUNCATED $t$-FREE FOCK SPACES: SPECTRUM OF POSITION OPERATORS AND SHIFT-INVARIANT STATES

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Abstract. The ergodic properties of the shift on both full and $m$-truncated $t$-free $C^*$-algebras are analyzed. In particular, the shift is shown to be uniquely ergodic with respect to the fixed-point algebra. In addition, for every $m \geq 1$, the invariant states of the shift acting on the $m$-truncated $t$-free $C^*$-algebra are shown to yield a $m + 1$-dimensional Choquet simplex, which collapses to a segment in the full case. Finally, the spectrum of the position operators on the $m$-truncated $t$-free Fock space is also determined.

Mathematics Subject Classification: 46L55, 46L53, 60G20.
Key words: $t$-free Fock space, $C^*$-dynamical systems, Unique ergodicity

1. Introduction

A class of non-commutative $C^*$-dynamical systems arise from concrete models coming from Quantum Probability. The shift automorphism certainly represents an interesting example, not least because it is closely related to stationary stochastic processes. What is more relevant to non-commutative ergodic theory is that the shift displays different ergodic properties depending on the $C^*$-algebra it acts on. Little can be said when it acts on the CAR algebra since its shift-invariant states are probably too many to be characterized in any useful way, see [8, 9, 11]. On both Boolean and monotone $C^*$-algebras, however, the shift only features a segment of invariant states, whose extreme points are the vacuum and the state at infinity, [10]. Lastly, on the $C^*$-algebra generated by $q$-deformed commutation relations the shift is even better-behaved in that it acts as a uniquely ergodic automorphism whenever $|q| < 1$, [14].

In the present work we show that suitable choices of the $C^*$-algebra that we make the shift act on give rise to new ergodic properties, which are in a sense intermediate between those we recollected above. Since the third example includes in particular the free case, which corresponds
to $q = 0$, it is reasonable to analyze the ergodic properties of the shift when it acts on the $C^*$-algebra generated by the creators on truncated free Fock spaces. These spaces are not entirely new in the context of Quantum Probability. In fact, they have been studied by Franz and Lenczewski in [16], where they are referred to as the $m$-free Fock spaces, as a means to obtain what they call a hierarchy of freeness along with the corresponding central limit theorems. The spaces we are actually concerned with in this paper are slightly more general than those in [16] because they are obtained by truncating the $t$-free Fock space, which are at the same time a generalization of the full Fock space and a particular yet notable case of so-called one-mode type interacting Fock space, [3]. These were defined by Bożejko and Wysoczanski in [7] to provide a concrete model where the distribution in the vacuum of the position operators was exactly that of the central limit of the $t$-free convolution. This operation between measures had in turn been introduced in [6] by the same authors following the influential work of Bożejko, Leinert and Speicher [5] on conditionally free independence.

Let us now move on to present the results of our paper. After defining the truncated $t$-free Fock spaces for each natural $m \geq 1$, we first focus on the spectrum of the corresponding position operators. This turns out to be a finite set strictly larger than the support of the spectral measure associated with the vacuum vector, meaning that the latter fails to be separating for the $C^*$-algebra generated by a single position operator. In addition, in Proposition 3.1 we show that the spectral measure is a convex combination of $m + 1$ Dirac measures at the zeros of explicitly given polynomials whose weights are also given. We then analyze the ergodic properties of the action of the shift on the $C^*$-algebras generated by the creators on both the truncated $t$-free Fock spaces and on the full $t$-free Fock space. The corresponding dynamical systems are uniquely ergodic with respect to the fixed-point algebra, a notion introduced in [1] and further studied in [12, 13, 15]. Furthermore, for $m \geq 1$, the invariant states on the $C^*$-algebra relative to the $t$-free Fock space yield a $m + 1$-dimensional Choquet simplex which collapses to a segment when any number of particles is allowed, Propositions 4.6 and 4.7. Finally, in analogy with what happens in the Boolean and monotone cases, we show that the $C^*$-algebra generated by position operators agrees with the $C^*$-algebra generated by creation and annihilator operators and that it acts irreducibly on the truncated $t$-free Fock space.

1Actually a point when $t = 1$. 
2. Preliminaries

Let $\mathcal{H}$ be a fixed Hilbert space. For any positive real number $t$ the $t$-free Fock Hilbert space $\mathcal{F}_t(\mathcal{H})$, is the completion of the algebraic direct sum

$\mathbb{C}\Omega \oplus \mathcal{H} \oplus \mathcal{H}^{\otimes 2} \oplus \cdots \oplus \mathcal{H}^{\otimes n} \oplus \cdots$

with respect to the inner product

$\langle f_1 \otimes \cdots \otimes f_n, g_1 \otimes \cdots \otimes g_m \rangle = \delta_{n,m} t^{n-1} \prod_{i=1}^{n} \langle f_i, g_i \rangle$

for every $f_1, \ldots, f_n$ and $g_1, \ldots, g_m$ in $\mathcal{H}$ and $n, m \geq 1$, where $\Omega$ is the vacuum vector. We would like to mention that the choice $t = 1$ returns the free Fock space, and that this Hilbert space can also be viewed as a one-mode type interacting Fock space with $\lambda_n = t^n - 1$; for details the reader is referred to [2, 3]. In the sequel, we will always take $\mathcal{H} = \ell^2(\mathbb{Z})$ with canonical orthonormal basis $\{e_i : i \in \mathbb{Z}\}$. Creators and annihilators on the $t$-free Fock Hilbert space are defined as follows. For each $k \in \mathbb{N}$ and $i, j_1, \ldots, j_k \in \mathbb{Z}$

$$a_i^\dagger \Omega := e_i,$$

$$a_{i,j_1}^\dagger e_{j_1} \otimes \cdots \otimes e_{j_k} := e_i \otimes e_{j_1} \otimes \cdots \otimes e_{j_k}$$

$$a_i \Omega := 0,$$

(2.1)

$$a_{i,j_1} e_{j_1} \otimes \cdots \otimes e_{j_k} := \begin{cases} \delta_{i,j_1} \Omega & \text{if } k = 1 \\ t \delta_{i,j_1} e_{j_2} \otimes \cdots \otimes e_{j_k} & \text{if } k \geq 2 \end{cases}$$

(2.2)

For each $i \in \mathbb{Z}$ the position operators $x_i$ are the self-adjoint part of the annihilators, namely $x_i = a_i + a_i^\dagger$. We denote by $\mathfrak{A}_t \subset B(\mathcal{F}_t(\mathcal{H}))$ the unital $C^*$-algebra generated by all creators.

For any fixed $m \geq 1$ we introduce what we call the $m$-truncated $t$-free Fock space as

$\mathcal{F}_t^{(m)}(\mathcal{H}) := \mathbb{C}\Omega \oplus \mathcal{H} \oplus \mathcal{H}^{\otimes 2} \oplus \cdots \oplus \mathcal{H}^{\otimes m} \subset \mathcal{F}_t(\mathcal{H}).$

We denote by $P_m$ the orthogonal projection of $\mathcal{F}_t(\mathcal{H})$ onto $\mathcal{F}_t^{(m)}(\mathcal{H})$. It is worth noting that, for each $m \geq 1$, the Hilbert space $\mathcal{F}_t^{(m)}(\mathcal{H})$ is again a one-mode interacting Fock space with $\lambda_0 = 1$, $\lambda_n = t^n - 1$ for $n = 1, 2, \ldots, m$ and $\lambda_n = 0$ for all $n \geq m + 1$.

The corresponding $m$-truncated annihilation and position operators are defined as $a_i^{(m)} := P_m a_i P_m$ and $x_i^{(m)} := P_m x_i P_m$. We also consider the $C^*$-algebras generated by the truncated operators defined above:

$\mathfrak{A}_t^{(m)} := C^*(\{a_i^{(m)} : i \in \mathbb{Z}\}) \subset B(\mathcal{F}_t^{(m)}(\mathcal{H})),$
\[ \mathfrak{A}_t^{(m)} := C^*\left(\{x_i^{(m)} : i \in \mathbb{Z}\}\right) \subset \mathcal{B}(\mathcal{F}_t^{(m)}(\mathcal{H})). \]

These \( C^* \)-algebras will be every so often referred to as \( m \)-truncated \( t \)-free \( C^* \)-algebras. Note both \( \mathfrak{A}_t^{(m)} \) and \( \mathfrak{R}_t^{(m)} \) are acted upon by \( * \)-automorphism \( \tau \) sending \( a_i^{(m)} \) in \( a_{i+1}^{(m)} \), and \( x_i^{(m)} \) in \( x_{i+1}^{(m)} \), respectively.

### 3. Spectrum of the position operators

Note that for each fixed \( m \geq 1 \) the operators \( x_i^{(m)} \) have the same spectrum. We denote by \( x^{(m)} \) any of them and by \( \mu_t^{(m)} \) the spectral measure of \( x^{(m)} \) associated with the vacuum vector, namely the unique Borel measure \( \mu_t^{(m)} \) on \( \sigma(x^{(m)}) \) such that

\[
\langle f(x^{(m)})\Omega, \Omega \rangle = \int_{\sigma(x^{(m)})} f \, d\mu_t^{(m)}
\]

for every \( f \in C(\sigma(x^{(m)})) \). Let \( \text{supp} \mu_t^{(m)} \) be the support of \( \mu_t^{(m)} \), and denote by \( r^{(m)} \) the position operator \( x^{(m)} \) corresponding to \( t = 1 \).

At this point, it is worth recalling that \( \mu_1^{(m)} \) is known explicitly for every \( m \geq 1 \). Indeed, from Theorem 5.3 in [16] one has

\[
\mu_1^{(m)} = \sum_{k=1}^{m+1} b_{m,k} \delta_{z_{m,k}},
\]

where \( z_{m,k} = 2 \cos\left(\frac{k\pi}{m+2}\right) \) and \( b_{m,k} = \frac{2\sin^2[k\pi/(m+2)]}{m+2} \) for every \( k = 1, \ldots, m+1 \).

To our knowledge, the general case has not been addressed elsewhere.

We need to recall a few basic facts. The Cauchy transform of a probability measure on \( \mathbb{R} \) is by definition the complex function

\[
G_\mu(z) = \int_{-\infty}^{\infty} \frac{d\mu(z)}{z-x}
\]

defined for \( z \) in the open upper half-plane \( \mathbb{C}^+ \). If \( \text{supp} \mu \) is compact, then it is known (see [4]) that \( G_\mu \) can be written as a convergent continuous fraction of the type

\[
G_\mu(z) = \frac{1}{z - \alpha_0 - \frac{\beta_0}{z - \alpha_1 - \frac{\beta_1}{z - \alpha_2 - \frac{\beta_2}{\ddots}}}},
\]

where \( \{\alpha_n : n \in \mathbb{N}\} \) and \( \{\beta_n : n \in \mathbb{N}\} \) are the sequences of the so-called Jacobi parameters, see [20]. The \( \alpha_n \)'s are all zero when \( \mu \) is
symmetric, and furthermore, by virtue of [2, Theorem 4.1] in one-mode type interacting Fock spaces the sequence \( \{\beta_n : n \in \mathbb{N}\} \) is given by \( \beta_0 = 1, \beta_n = \lambda_{n}^{\lambda_{n-1}} \) for \( n \geq 1 \).

In addition, for each \( i \in \mathbb{Z} \) and \( n \in \mathbb{N} \), by (2.1) and (2.2) one has that \( \langle x_i^{2n+1}\Omega, \Omega \rangle = 0 \). Consequently, if we now denote by \( \mu_t \) the spectral measure of any of the self-adjoint operators \( x_i \) on the full \( t \)-free Fock space \( \mathcal{F}_t(\mathcal{H}) \) w.r.t. the vacuum, its Cauchy transform is nothing else than

\[
G_{\mu_t}(z) = \frac{1}{1 - \frac{t}{z}}.
\]

see also [7]. Since for our measures \( \mu_{t}^{(m)} \) the sequence \( \{\beta_n\}_{n \in \mathbb{N}} \) is given by \( \beta_0 = 1, \beta_n = t \) for \( n = 1, 2, \ldots, m \) and \( \beta_n = 0 \) as soon as \( n \geq m + 1 \), their Cauchy transforms \( G_{\mu_{t}^{(m)}} \) can be written as finite fractions as follows

\[
G_{\mu_{t}^{(0)}} = \frac{1}{z}, \quad G_{\mu_{t}^{(1)}} = \frac{1}{z - \frac{1}{z}}, \quad G_{\mu_{t}^{(2)}} = \frac{1}{z - \frac{t}{z}}, \quad G_{\mu_{t}^{(3)}} = \frac{1}{z - \frac{t}{z} - \frac{t}{z}}
\]

and so on.

In order to state our first result, we first need to establish some notation, for the atoms of the discrete measure \( \mu_{t}^{(m)} \) are in fact the zeros of polynomial expressions in which the Chebyshev polynomials show up. In the sequel, we will denote by \( U_m, m \geq 0 \), the Chebyshev polynomials of the second kind, see [20]. For completeness' sake, we recall that they are explicitly given by the formula

\[
U_m(x) = \frac{\sin[(m + 1) \arccos x]}{\sin[\arccos x]}, \quad x \in (-1, 1).
\]

**Proposition 3.1.** For each \( m \geq 1 \), the measure \( \mu_{t}^{(m)} \) is given by

\[
\mu_{t}^{(m)} = \sum_{i=1}^{m+1} b_i \delta_{z_i},
\]
where \( \{ z_i : i = 1, \ldots, m + 1 \} \) is the set of the \((m + 1)\) real zeros of the polynomial
\[
\sqrt{t_2} U_m \left( \frac{z}{2\sqrt{t}} \right) - U_{m-1} \left( \frac{z}{2\sqrt{t}} \right),
\]
and
\[
b_i = \text{Res}_{z = z_i} \frac{\sqrt{t} U_m \left( \frac{z}{\sqrt{t}} \right)}{U_m \left( \frac{z}{2\sqrt{t}} \right) - U_{m-1} \left( \frac{z}{2\sqrt{t}} \right)}
\]
for \( i = 1, \ldots, m + 1 \).

**Proof.** Throughout the proof by \( x^{(m)} \) we will always mean \( x_1^{(m)} \). By looking at the formula of \( G_{\mu_t}^{(m)} \) given above, one sees that \( G_{\mu_t}^{(m)} \) can also be written in terms of \( G_{\mu_1^{(m-1)}} \) as
\[
(3.1) \quad G_{\mu_1^{(m)}} = \frac{1}{z - \frac{1}{\sqrt{t}} G_{\mu_1^{(m-1)}} \left( \frac{z}{\sqrt{t}} \right)}.
\]
As is known, the atoms of a measure are the poles of its Cauchy transform. Now the poles of \( G_{\mu_1^{(m)}} \) are the zeros of \( z - \frac{1}{\sqrt{t}} G_{\mu_1^{(m-1)}} \left( \frac{z}{\sqrt{t}} \right) \). From [16, Lemma 5.2] we know that
\[
(3.2) \quad G_{\mu_1^{(m-1)}}(z) = \frac{U_{m-1}(z/2)}{U_m(z/2)}.
\]
Inserting (3.2) into (3.1) we arrive at
\[
G_{\mu_1^{(m)}} = \frac{\sqrt{t} U_m \left( \frac{z}{2\sqrt{t}} \right)}{\sqrt{t_2} U_m \left( \frac{z}{2\sqrt{t}} \right) - U_{m-1} \left( \frac{z}{2\sqrt{t}} \right)}.
\]
Therefore, the poles of \( G_{\mu_1^{(m)}} \) are the zeros of the polynomial \( \sqrt{t_2} U_m \left( \frac{z}{2\sqrt{t}} \right) - U_{m-1} \left( \frac{z}{2\sqrt{t}} \right) \). Define \( V^{(m)} := \text{span}\{ \Omega, e_1, e_1 \otimes e_1, \ldots, e_1^{\otimes m} \} \). Note that \( V^{(m)} \) is an invariant subspace for \( x^{(m)} \) and \( \Omega \) is cyclic for the restriction \( x^{(m)} \mid_{V^{(m)}} \). By cyclicity it follows that the eigenvalues of \( x^{(m)} \mid_{V^{(m)}} \) do not have multiplicity, that is \( |\sigma(x^{(m)} \mid_{V^{(m)}})| = m + 1 \). Since \( \Omega \) is also separating for \( x^{(m)} \mid_{V^{(m)}} \), we have \( \text{supp} \mu_t^{(m)} = \sigma(x^{(m)} \mid_{V^{(m)}}) \). As a result, the support of \( \mu_t^{(m)} \) is finite and thus it is the same as the set of the zeros of \( \sqrt{t_2} U_m \left( \frac{z}{2\sqrt{t}} \right) - U_{m-1} \left( \frac{z}{2\sqrt{t}} \right) \).
Finally, the formula of the weights of the atoms follows from applying e.g. Proposition 1.104 in [17].

\[ \mu_t^{(2)} = \frac{1}{2\sqrt{t}(1+t)^2} \delta_{t \geq t_0} + \frac{1}{t+1} \delta_0 + \frac{1}{2\sqrt{t}(1+t)} \delta_{t_0}. \]

The next result provides a formula for the spectrum of \( x^{(m)} \) in terms of the supports of \( \mu_t^{(m)} \) and all \( \mu_t^{(k)} \) with \( k = 1, 2, \ldots, m - 1 \).

**Theorem 3.3.** For every real \( t > 0 \) and integer \( m \geq 1 \), the spectrum of \( x^{(m)} \) is given by

\[
\sigma(x^{(m)}) = \text{supp} \mu_t^{(m)} \bigcup \sqrt{t} \langle \text{supp} \mu_t^{(k)} \rangle \cup \{0\}.
\]

**Proof.** As in the previous proof, by \( x^{(m)} \) we will always mean \( x_1^{(m)} \).

We first decompose \( \mathcal{F}_t^{(m)}(\mathcal{H}) \) into a direct sum of suitable cyclic subspaces. Let \( V^{(m)} \) be the cyclic subspace generated by \( \Omega \) we introduced in the proof above. As already pointed out, the spectrum of the restriction \( x^{(m)} \mid_{V^{(m)}} \) is equal to \( \text{supp} \mu_t^{(m)} \).

For every \( j \neq 1 \), let \( V_j^{(m)} \subset \mathcal{F}_t^{(m)}(\mathcal{H}) \) be the cyclic subspace generated by \( e_j \). Note that each \( V_j^{(m)} \) is the \( m \)-dimensional subspace \( \text{span}\{e_j, e_1 \otimes e_j, \ldots, e_1^{m-1} \otimes e_j\} \). For any integer \( 2 \leq k \leq m - 1 \), denote by \( V_{j_1,j_2,\ldots,j_k}^{(m)} \subset \mathcal{F}_t^{(m)}(\mathcal{H}) \) the cyclic subspace generated by the vector \( e_{j_1} \otimes \cdots \otimes e_{j_k} \otimes e_1 \), where \( j_1 \neq 1 \) and \( j_2, \ldots, j_k \in \mathbb{Z} \). Note that each \( V_{j_1,j_2,\ldots,j_k}^{(m)} \) is the \((m - k + 1)\)-dimensional subspace \( \text{span}\{e_{j_1} \otimes \cdots \otimes e_{j_k} \otimes e_1 \otimes e_1, \ldots, e_{j_1} \otimes \cdots \otimes e_{j_k} \otimes e_1^{m-k}\} \). It is easy to see that the following decomposition holds

\[
\mathcal{F}^{(m)}(\mathcal{H}) = V^{(m)} \oplus (\oplus_{j \neq 1} V_j^{(m)}) \oplus (\oplus_{j_1 \neq 1} V_{j_1,j_2,\ldots,j_k}^{(m)}).
\]

By direct computation one can verify the following unitary equivalences:

\[
x^{(m)} \mid_{V_j^{(m)}} \cong \sqrt{t} r^{(m-1)} \mid_{V_1^{(m-1)}}
\]

\[
x^{(m)} \mid_{V_{j_1,j_2,\ldots,j_k}^{(m)}} \cong \sqrt{t} r^{(m-k-1)} \mid_{V_1^{(m-k-1)}}.
\]

with \( r^{(m)} \) being the position operator \( x^{(m)} \) for \( t = 1 \). Therefore, we find

\[
\sigma(x^{(m)}) = \text{supp} \mu_t^{(m)} \bigcup \sqrt{t} \langle \text{supp} \mu_t^{(m-k-1)} \rangle \cup \{0\}.
\]

\( \square \)
An entirely explicit formula can be given for the spectrum of the position operator for \( t = 1 \).

**Corollary 3.4.** For each \( m \geq 1 \), the spectrum of \( r^{(m)} \) is

\[
\sigma(r^{(m)}) = \bigcup_{k=0}^{m} \left\{ 2 \cos \left( \frac{l \pi}{m+2} \right) : l = 0, 1, \ldots, k \right\}.
\]

Going back to the case of a general \( t > 0 \), we next show that the support of the spectral measure is always properly contained in the spectrum of the position operator.

**Corollary 3.5.** For every real \( t > 0 \) and integer \( m \geq 1 \), the support of \( \mu_t^{(m)} \) is strictly contained in \( \sigma(x^{(m)}) \).

**Proof.** Thanks to Equality (3.3), it is enough to show that

\[
| \bigcup_{k=1}^{m} \text{supp} \mu_1^{(k)} | > | \text{supp} \mu_t^{(m)} | = m + 1,
\]

where \( | \cdot | \) is the cardinality of a (finite) set. Now \( | \bigcup_{k=1}^{m-1} \text{supp} \mu_1^{(k)} | \geq | \text{supp} \mu_1^{(m-1)} \cup \text{supp} \mu_1^{(m-2)} | \). Because \( \text{supp} \mu_1^{(m-1)} \) and \( \text{supp} \mu_1^{(m-2)} \) are disjoint sets, see e.g. Section 5 of [16], we find \( | \bigcup_{k=1}^{m-1} \text{supp} \mu_1^{(k)} | \geq m + m - 1 = 2m - 1 \). Since \( 2m - 1 > m + 1 \) as soon as \( m > 2 \), the conclusion follows for any such \( m \). Finally, the cases \( m = 1 \) and \( m = 2 \) have already been handled in Example 3.2 by explicit computation.

\( \Box \)

**Remark 3.6.** In particular, from Corollary 3.5 we see that for every \( m \geq 1 \) the vacuum vector \( \Omega \) is not separating for the \( C^* \)-algebra generated by \( x^{(m)} \).

In a similar fashion we can determine the spectrum of the position operators \( x_i \) on the full \( t \)-free Fock space. As usual, we will drop the subscripts and just write \( x \) to refer to any of the position operators.

**Proposition 3.7.** For every real \( t > 0 \), the spectrum of \( x \) is given by

\[
\sigma(x) = \text{supp} \mu_t,
\]

with \( \text{supp} \mu_t \) being \([-2\sqrt{t}, 2\sqrt{t}]\) for \( t \geq \frac{1}{2} \) and \( \text{supp} \mu_t = [-2\sqrt{t}, 2\sqrt{t}] \cup \{ \pm 1/\sqrt{1-t} \} \) for \( t < \frac{1}{2} \). In particular, for every \( t > 0 \) the vacuum vector is separating for \( x \).

**Proof.** Arguing as in the proof of Theorem 3.3, one finds that \( \sigma(x) = \text{supp} \mu_t \cup \sqrt{t} \text{supp} \mu_1 \). Now \( \sqrt{t} \text{supp} \mu_1 \) is \([-2\sqrt{t}, 2\sqrt{t}] \) since \( \mu_1 \) is the standard Wigner distribution, whereas \( \text{supp} \mu_t \) is \([-2\sqrt{t}, 2\sqrt{t}] \cup \{ \pm \frac{1}{\sqrt{1-t}} \} \) for \( t < \frac{1}{2} \) or \([-2\sqrt{t}, 2\sqrt{t}] \) for \( t \geq \frac{1}{2} \), as shown in [7, 19]. In either case, \( \sqrt{t} \text{supp} \mu_1 \) is contained in \( \text{supp} \mu_t \), hence \( \sigma(x) = \text{supp} \mu_t \). \( \Box \)
4. Ergodic properties of the shift

We denote by $\mathcal{A}_t^{(m)}$ the dense $*$-subalgebra of $\mathfrak{F}_t^{(m)}$ generated by the set $\{a_i^{(m)} : i \in \mathbb{Z}\}$. In the following, to ease the notation we will write $a_i$ instead of $a_i^{(m)}$ when $m$ is fixed. We will denote by $P_{\mathcal{H}^\otimes h}$, $h = 0, 1, \ldots, m$, the projection of $\mathcal{F}_t^{(m)}(\mathcal{H})$ onto $\mathcal{H}^\otimes h$, where $\mathcal{H}^\otimes 0$ is understood as $\mathbb{C}\Omega$.

Lemma 4.1. For every $t > 0$ and every integer $m \geq 1$, the projections $P_{\mathcal{H}^\otimes h}$ belong to $\mathcal{A}_t^{(m)}$ for $h = 0, 1, \ldots, m$. Moreover, the $*$-algebra $\mathcal{A}_t^{(m)}$ is contained in the linear span of all monomials of the type

$$a_i^\dagger \cdots a_i a_j \cdots a_j P_{\mathcal{H}^\otimes h},$$

with $i_1, \ldots, i_r, j_1, \ldots, j_l \in \mathbb{Z}$, $r, l = 0, 1, \ldots, m$ and $h = 0, 1, 2, \ldots, m - 1$.

Proof. We start by noting that for all $i \in \mathbb{Z}$ and $h = 0, 1, \ldots, m - 1$ the following equalities hold as a straightforward consequence of (2.1)–(2.2):

\[
\begin{align*}
a_i a_j^\dagger &= \delta_{i,j} \left( P_{\Omega} + t \sum_{h=1}^{m-1} P_{\mathcal{H}^\otimes h} \right), \\
a_i P_{\Omega} &= 0 = P_{\Omega} a_i^\dagger, \\
P_{\mathcal{H}^\otimes h} a_i &= a_i P_{\mathcal{H}^\otimes h+1}, \\
P_{\mathcal{H}^\otimes h+1} a_i^\dagger &= a_i^\dagger P_{\mathcal{H}^\otimes h}.
\end{align*}
\]

The first part of the statement can be proved by induction on $m$. The basis, that is $m = 1$, follows trivially from the first of Relations (4.1). For the inductive step, the first thing to observe is that $\sum_{h=0}^{m} P_{\mathcal{H}^\otimes h}$ is in $\mathcal{A}_t^{(m+1)}$ due to the first of Relations (4.1). Suppose that $P_{\Omega}, P_{\mathcal{H}}, \ldots, P_{\mathcal{H}^\otimes m}$ lie in $\mathcal{A}_t^{(m)}$. Thanks to the equality $a_i^{(m)} = (I - P_{\mathcal{H}^\otimes m+1})a_i^{(m+1)}(I - P_{\mathcal{H}^\otimes m+1})$, which is easily verified, the above projections are in $\mathcal{A}_t^{(m+1)}$ as well and we are done.

For the second part of the statement, from (4.1) it is easy to see that any monomial in the generators of $\mathcal{A}_t^{(m)}$ can be rewritten as a finite linear combination of words as in the statement. Indeed, if a word $a_{i_1}^k a_{i_2}^\dagger \cdots a_{i_r}^\dagger$, where $^k$ is either 1 or $\dagger$, is not in the desired order, then it must display a factor of the type $a_i a_j^\dagger$. But this is reduced by the first quality above. Applying the remaining equalities as many times as necessary, the sought form is finally arrived at.

We denote by $\tau$ the shift automorphism on $\mathfrak{F}_t^{(m)}$, that is the automorphism completely determined by $\tau(a_i) = a_{i+1}$, for every $i \in \mathbb{Z}$. 

□
Clearly, \( \tau \) restricts to an automorphism of \( \mathfrak{H}_m \), which we continue to denote by \( \tau \). Note that \( \tau(x_i^{(m)}) = x_{i+1}^{(m)} \), for every \( i \in \mathbb{Z} \).

In the following lemma we prove an analogue of both estimates in [14, Proposition 3.2] and in [10, Proposition 4.2] adapted to the \( t \)-free result below. To avoid any ambiguity, we denote by \( l^r \) the creators on full the \( t \)-free Fock space. Furthermore, we denote by \( \xi \) either 1 or \( \dagger \).

**Lemma 4.2.** For every \( t > 0 \), one has

\[
\left\| \sum_{k=0}^{n-1} \tau^k (l_{i_1}^r l_{i_2}^s \cdots l_{i_r}^r l_{j_1}^s \cdots l_{j_s}^s) \right\| \leq \sqrt{nt} \max\{1, \sqrt{t}\}^{(s+r-1)}
\]

for all \( r, s \in \mathbb{N} \) and \( i_1, i_2, \ldots, i_r, j_1, j_2, \ldots, j_s \in \mathbb{Z} \).

**Proof.** Let \( \xi \) be a unit vector in \( \mathfrak{H}^{\otimes m} \). For \( m \geq s \), one has

\[
\left\| \sum_{k=0}^{n-1} \tau^k (l_{i_1}^r l_{i_2}^s \cdots l_{i_r}^r l_{j_1}^s \cdots l_{j_s}^s) \xi \right\|^2 = \left\| \sum_{k=0}^{n-1} l_{i_1+k}^r l_{i_2+k}^s \cdots l_{i_r+k}^r l_{j_1+k}^s \cdots l_{j_s+k}^s \xi \right\|^2
\]

with \( \xi_k := l_{i_2+k}^s \cdots l_{i_r+k}^s l_{j_1+k}^s \cdots l_{j_s+k}^s \xi \) for \( k = 0, 1, \ldots, n - 1 \). Now note that \( \xi_k \in \mathfrak{H}^{\otimes m+r+s-1} \) and \( \|\xi_k\|^2 \leq \max\{1, \sqrt{t}\}^{2(s+r-1)} \) due to \( \|l_i\| \leq \max\{1, \sqrt{t}\} \) for every \( i \in \mathbb{Z} \). But then

\[
\left\| \sum_{k=0}^{n-1} l_{i_1+k}^r \xi_k \right\|^2 = \left\langle \sum_{k=0}^{n-1} l_{i_1+k}^r \xi_k, \sum_{k'=0}^{n-1} l_{i_1+k'}^r \xi_{k'} \right\rangle
\]

\[
= \left\langle \sum_{k=0}^{n-1} e_{i_1+k} \otimes \xi_k, \sum_{k'=0}^{n-1} e_{i_1+k'} \otimes \xi_{k'} \right\rangle
\]

\[
= \sum_{k=0}^{n-1} \|e_{i_1+k} \otimes \xi_k\|^2 = \sum_{k=0}^{n-1} t \|\xi_k\|^2
\]

\[
\leq nt \max\{1, \sqrt{t}\}^{2(s+r-1)}.
\]

\[\square\]

A version of the lemma above tailored to the case of the truncated \( t \)-free spaces can be proved as well. For convenience, we single out the relative result below. The proof, though, is left out as it is a very minor variation of the proof above.
Corollary 4.3. For every $t > 0$ and every integer $m \geq 1$, one has
\[
\left\| \frac{1}{n} \sum_{k=0}^{n-1} \tau^k(a_{i_1}^\dagger \cdots a_{i_r}^\dagger a_{j_1} \cdots a_{j_s}) \right\| \leq \sqrt{nt} \max\{1, \sqrt{t}\}^{2m-1}
\]
for all $r, s \in \mathbb{N}$ and $i_1, i_2, \ldots, i_r, j_1, j_2, \ldots, j_s \in \mathbb{Z}$.

With the above lemmas at hand, we are ready to prove that the shift is uniquely ergodic with respect to its fixed-point subalgebra. For completeness, we recall that unique ergodicity w.r.t. the fixed-point subalgebra is a generalization of unique ergodicity due to Abadie and Dykema [1]. Before going on, let us recall a few definitions. By a $C^*$-dynamical system we mean a pair $(\mathfrak{A}, \Phi)$, where $\mathfrak{A}$ is a $C^*$-algebra and $\Phi$ a $*$-automorphism. The fixed-point subalgebra of $(\mathfrak{A}, \Phi)$ is the $C^*$-subalgebra $\mathfrak{A}^\Phi := \{ a \in \mathfrak{A} : \Phi(a) = a \}$. A $C^*$-dynamical system is said to be uniquely ergodic w.r.t. the fixed-point subalgebra when for every $a \in \mathfrak{A}$ the Cesàro averages $\frac{1}{n} \sum_{k=0}^{n-1} \Phi^k(a)$ converge in norm to some (fixed) element of $\mathfrak{A}$. Clearly, this notion reduces to unique ergodicity when the fixed-point subalgebra is trivial. Furthermore, in the aforementioned paper several equivalent conditions are provided for a dynamical system to be uniquely ergodic w.r.t. the fixed-point subalgebra. One particularly relevant to the present work is that a dynamical system enjoys this property if and only if every state on the fixed-point subalgebra has precisely one invariant extension to the whole $C^*$-algebra, see Theorem 3.2 in [1].

Proposition 4.4. For every $t > 0$ and integer $m \geq 1$, the $C^*$-dynamical system $(\mathfrak{A}_t^{(m)}, \tau)$ is uniquely ergodic w.r.t the fixed-point subalgebra.

Proof. We will prove the statement by showing that, for every $a \in \mathfrak{A}_t^{(m)}$, the Cesàro averages $\frac{1}{n} \sum_{k=0}^{n-1} \tau^k(a)$ converge in norm. Since $\left\| \frac{1}{n} \sum_{k=0}^{n-1} \tau^k \right\| \leq 1$ for every $n \in \mathbb{N}$, thanks to Lemma 4.1 it is enough to prove convergence only for elements of the form
\[
a_{i_1}^\dagger \cdots a_{i_r}^\dagger a_{j_1} \cdots a_{j_l} P_{\mathfrak{H}^\otimes h}
\]
for $r, l \in \mathbb{N}, i_1, \ldots, i_r, j_1, \ldots, j_l \in \mathbb{Z}$, and $h = 0, 1, \ldots, m$. Furthermore, only words with $r + l \geq 1$ need to be taken care of. For words of this type, the sequence $\frac{1}{n} \sum_{k=0}^{n-1} \tau^k(a_{i_1}^\dagger \cdots a_{i_r}^\dagger a_{j_1} \cdots a_{j_l} P_{\mathfrak{H}^\otimes h})$ tends to 0 in norm by Corollary 4.3.

Unique ergodicity can now be taken advantage of to determine what the fixed-point subalgebra is like.
Proposition 4.5. For every $t > 0$ and every integer $m \geq 1$, the subalgebra of $\tau$-invariant elements of $\mathfrak{A}_t^{(m)}$ is span$\{P_\Omega, P_{3_1^h}, \ldots, P_{3_1^{(m)}}\}$.

Proof. Thanks to Proposition 4.4, the formula $E := \lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} \tau^k$ defines a conditional expectation onto the fixed-point subalgebra. Now the argument employed in the proof of the proposition above shows that the latter is linearly spanned by the projections $P_{3_1^{(h)}}^h$, $h = 0, 1, \ldots, m$. \hfill $\square$

We are now in a position to describe the structure of the $*$-weakly compact convex set of all shift-invariant states on $\mathfrak{A}_t^{(m)}$, which we denote by $S_Z(\mathfrak{A}_t^{(m)})$.

Proposition 4.6. For every $t > 0$ and every $m \geq 1$, $S_Z(\mathfrak{A}_t^{(m)})$ is a finite-dimensional (Choquet) simplex with $m + 1$ extreme states $\omega_i$, $i = 0, 1, \ldots, m$, uniquely determined by $\omega_i(P_{3_1^{(m)}}) = \delta_{i,j}$, for $i, j = 0, 1, \ldots, m$.

Proof. By unique ergodicity w.r.t. the fixed-point subalgebra, $(\mathfrak{A}_t^{(m)})^\tau$, the restriction map $S_Z(\mathfrak{A}_t^{(m)}) \ni \omega \to \omega|_{(\mathfrak{A}_t^{(m)})^\tau} \in S((\mathfrak{A}_t^{(m)})^\tau)$ is an affine homeomorphism between compact convex sets. The conclusion now follows from Proposition 4.5 because $(\mathfrak{A}_t^{(m)})^\tau \cong \mathbb{C}^{m+1}$ (as $C^*$-algebras). \hfill $\square$

The case of the full $t$-free Fock space looks different insofar as the fixed-point subalgebra is two-dimensional for $t \neq 1$.

Proposition 4.7. For every $t > 0$, $(\mathfrak{A}_t, \tau)$ is uniquely ergodic w.r.t. the fixed-point subalgebra. Moreover, for $t = 1$ the fixed-point subalgebra is trivial, whereas it is span$\{P_\Omega, I - P_\Omega\}$ for $t \neq 1$.

Proof. We need only deal with $t \neq 1$ since the free case, i.e. $t = 1$, is known, see [14, Theorem 3.3]. From the relations $l_i P_\Omega = 0$ (hence $l_i(I - P_\Omega) = l_i$) and $l_i l_j^\dagger = \delta_{i,j}(P_\Omega + t(I - P_\Omega))$, $i, j \in \mathbb{Z}$, one sees that any word in the $t$-free creators and annihilators can be written as a linear combinations of $P_\Omega$, $(I - P_\Omega)$, $w$, $wP_\Omega$ and their adjoints, where $w$ is a word (possibly of length 1) either starting with a creator or ending in an annihilator. Applying Lemma 4.2 to $w$, one finally finds that the range of the conditional expectation onto the fixed-point subalgebra is spanned by $P_\Omega$ and $I - P_\Omega$. \hfill $\square$

Remark 4.8. As an immediate consequence of Proposition 4.7, we have that $S_Z(\mathfrak{A}_t)$ has exactly two extreme states when $t \neq 1$. 

The rest of the section is devoted to the $C^*$-subalgebra generated by the position operators, that is $\mathfrak{A}_t^{(m)}$. In particular, we are going to show that $\mathfrak{A}_t^{(m)}$ is actually the same as $\mathfrak{A}_t^{(m)}$. To this end, we start by ascertaining that all projections onto the $k$-particle spaces lie in $\mathfrak{A}_t^{(m)}$.

**Lemma 4.9.** For every $t > 0$ and for every integer $m \geq 1$, the set of projections $\{ P_{\mathfrak{H}^k} : h = 0, 1, \ldots, m \}$ sits in $\mathfrak{A}_t^{(m)}$.

*Proof.* An easy adapatation of Lemma 2.2 in [21] shows that the sequence

$$\frac{1}{2n+1} \sum_{k=-n}^{n} (x_i^{(m)})^2$$

converges strongly to $P_\Omega + t(P_{\kappa} + \cdots + P_{\kappa^m-1})$. By virtue of Proposition 4.4 the convergence of the sequence is actually in norm, which means the projection $P_\Omega + t(P_{\kappa} + \cdots + P_{\kappa^m-1})$ belongs to $\mathfrak{A}_t^{(m)}$. In particular, $P_\Omega$ and $P_{\kappa} + \cdots + P_{\kappa^m-1}$ lie in $\mathfrak{A}_t^{(m)}$. The conclusion is then reached by induction on $m$ as was done in the proof of Lemma 4.1. □

**Proposition 4.10.** For each $m \geq 1$, one has $\mathfrak{A}_t^{(m)} = \mathfrak{A}_t^{(m)}$.

*Proof.* All we have to prove is that, for every $i \in \mathbb{Z}$, the operator $a_i^{(m)}$ sits in $\mathfrak{A}_t^{(m)}$. This is a straightforward consequence of Lemma 4.9 and the equality

$$a_i^{(m)} = \sum_{k=0}^{m-1} P_{\mathfrak{H}^k} x_i^{(m)} P_{\mathfrak{H}^k+1},$$

which can be checked by direct computation. □

We end the section by showing that $\mathfrak{A}_t^{(m)}$ is an irreducible $C^*$-subalgebra of $\mathcal{B}(\mathcal{F}_t^{(m)}(\mathfrak{H}))$. This was already known in the full case, [21].

**Proposition 4.11.** For every $t > 0$ and for every integer $m \geq 1$, the $C^*$-algebra $\mathfrak{A}_t^{(m)} = \mathfrak{A}_t^{(m)}$ acts irreducibly on $\mathcal{F}_t^{(m)}(\mathfrak{H})$.

*Proof.* We will show that the commutant $(\mathfrak{A}_t^{(m)})'$ is trivial. If $T \in \mathcal{B}(\mathcal{F}_t^{(m)}(\mathfrak{H}))$ is in $(\mathfrak{A}_t^{(m)})'$, then in particular $TP_\Omega = P_\Omega T$, hence $T \Omega = \lambda \Omega$, for some $\lambda \in \mathbb{C}$. Since $\Omega$ is cyclic for $\mathfrak{A}_t^{(m)}$, $\Omega$ is separating for $(\mathfrak{A}_t^{(m)})'$, which means $T = \lambda I$. □

**Acknowledgments**

We acknowledge the support of Italian INDAM-GNAMPA.
References

[1] Abadie B., Dykema K. Unique ergodicity of free shifts and some other automorphisms of $C^*$-algebras, J. Operator Theory 61 (2009), 279–294.
[2] Accardi L., Bożejko M. Interacting Fock spaces and Gaussianization of probability measures, Infin. Dimens. Anal. Quantum Probab. Relat. Top. 1 (1998), 663–670.
[3] Accardi L., Crismale V., Lu Y.G. Constructive universal central limit theorems based on interacting Fock spaces, Infin. Dimens. Anal. Quantum Probab. Relat. Top. 8 (2005), 631–650.
[4] Akhiezer N. I. The classical moment problem, Oliver and Boyd, London, 1965.
[5] Bożejko M., Leinert M., Speicher R. Convolution and limit theorems for conditionally free random variables, Pacific J. Math. 175 (1996), 357–388.
[6] Bożejko M., Wysoczanski J. New examples of convolutions and non-commutative central limit theorems, Banach Center Publ. 43 (1998), 95–103.
[7] Bożejko M., Wysoczanski J. Remarks on t-transformations of measures and convolutions, Ann. Inst. H. Poincaré Probab. Statist. 37 (2001), 737–761.
[8] Crismale V., Fidaleo F. De Finetti theorem on the CAR algebra, Commun. Math. Phys. 315 (2012), 135–152.
[9] Crismale V., Fidaleo F. Exchangeable stochastic processes and symmetric states in quantum probability, Ann. Mat. Pura Appl., 194 (2015), 969–993.
[10] Crismale V., Fidaleo F., Lu Y.G. Ergodic theorems in quantum probability: an application to monotone stochastic processes, Ann. Sc. Norm. Super. Pisa Cl. Sci. (5), XVII (2017), 113–141.
[11] Crismale V., Rossi S. Failure of the Ryll-Nardzewski theorem on the CAR algebra, submitted.
[12] Del Vecchio S., Fidaleo F., Rossi S. Skew-product dynamical systems for crossed product $C^*$-algebras and their ergodic properties, J. Math. Anal. Appl. 503 (2021), 125302.
[13] Del Vecchio S., Fidaleo F., Rossi S. Invariant conditional expectations and unique ergodicity for Anzai skew-products, Arxiv preprint 2108.11938.
[14] Dykema K., Fidaleo F. Unique mixing of the shift on the $C^*$-algebras generated by the $q$–canonical commutation relations, Houston J. Math. 36 (2010), 275–281.
[15] Fidaleo F. On the Uniform Convergence of Ergodic Averages for $C^*$-Dynamical Systems, Mediterr. J. Math. 17 (2020), 135.
[16] Franz U., Lenczewski R. Limit theorems for the hierarchy of freeness, Probab. Math. Statist. 19 (1999), 23–41.
[17] Hora A., Obata N. Quantum probability and spectral analysis of graphs. Theoretical and Mathematical Physics. Springer, Berlin 2007.
[18] Ricard E. The von Neumann algebras generated by t-Gaussians, Ann. Inst. Fourier (Grenoble) 56 (2006), 475–498.
[19] Ryll-Nardzewski V. On stationary sequences of random variables and the de Finetti’s equivalence, Colloq. Math. 4 (1957), 149–156.
[20] Szego G. Orthogonal polynomials, Volume XXIII, 4th edn. American Math. Society Colloquium Publication, Providence 1975.
[21] Wysoczanski J. The von Neumann algebra associated with an infinite number of t-free noncommutative Gaussian random variables, Quantum probability,
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