Harmonic maps in unfashionable geometries

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Summary. We describe some general constructions on a real smooth projective 4-quadric which provide analogues of the Willmore functional and conformal Gauss map in Lie sphere and projective differential geometry. Extrema of these functionals are characterized by harmonicity of this Gauss map.

1. Introduction

Many topics in integrable surface geometry\(^1\) may be unified by application of the highly developed theory of harmonic maps of surfaces into (pseudo-)Riemannian symmetric spaces. On the one hand, such harmonic maps comprise an integrable system with spectral deformations, algebro-geometric solutions and dressing actions of loop groups generated by Bäcklund transforms [5], [6], [14], [21], [24]. On the other hand, several integrable classes of surface are characterized by harmonicity of a suitable Gauss map. Thus, a surface \(\mathcal{f}: M^2 \to \mathbb{R}^3\) has constant mean curvature \(H\) if and only if its Gauss map \(M^2 \to S^2\) is harmonic. Again, such a surface has constant Gauss curvature \(K\) if and only if its Gauss map is harmonic with respect to the metric on \(M\) provided by the second fundamental form of \(\mathcal{f}\). The theory of harmonic maps now provides a conceptual explanation of the classical integrable aspects of such surfaces such as associated families, Lie and Bäcklund transformations.

These ideas gain wider applicability if we extend the notion of Gauss map. Consider, for example, the case of Willmore surfaces \([1]\); these are surfaces

\(^{1}\)But not all: isothermic surfaces, for example, do not fit into this picture.

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\( f : M^2 \to \mathbb{R}^3 \) which extremize the Willmore functional

\[
W(f) = \int_{M^2} (H^2 - K) \, dA.
\]

The functional \( W \) and so its critical points are preserved by the Möbius group of conformal diffeomorphisms of \( S^3 = \mathbb{R}^3 \cup \{ \infty \} \). A conformally immersed surface in \( S^3 \) also has a Gauss map which can be defined as follows: to each point \( x \in M^2 \), attach the oriented 2-sphere \( S(x) \) in \( S^3 \) which has first order contact with \( f \) at \( f(x) \) and the same mean curvature vector there. The map \( x \mapsto S(x) \) is variously known as the \textit{central sphere congruence} \([1]\) or the \textit{conformal Gauss map} \([3]\) and is a Möbius invariant of \( f \). The space of oriented 2-spheres in \( S^3 \) is naturally identified with the Lorentz 4-sphere which is a pseudo-Riemannian symmetric space. One shows that the harmonic map energy \( E(S) \) of \( S \) coincides with \( W(f) \) and further \([1],[3]\) that \( f \) is Willmore if and only if \( S \) is harmonic.

In this paper we will study two other classes of surfaces, Lie minimal and projectively minimal surfaces, and show that an exactly analogous theory applies. These surfaces are the analogues of Willmore surfaces in projective and Lie sphere geometry and were introduced and intensively studied around the turn of the last century (see, for example, \([22],[23],[1],[2]\)). More recently, Ferapontov \([10],[11]\) has demonstrated that these surfaces have integrable structure and we shall show how this structure is explained by the harmonicity of a Gauss map.

Let us begin by explaining what Lie and projectively minimal surfaces are. First contemplate an immersed surface \( f : M^2 \to \mathbb{R}^3 \) with curvature line coordinates \( u, v \) and corresponding principal curvatures \( \kappa_1, \kappa_2 \). We define a functional \( L_{\text{Lie}} \) using a formulation we learned from Ferapontov \([10]\):

\[
L_{\text{Lie}}(f) = \int_{M^2} \frac{\partial_u \kappa_1 \partial_v \kappa_2}{(\kappa_1 - \kappa_2)^2} \, du \wedge dv.
\]

The critical points of \( L_{\text{Lie}} \) are called \textit{Lie minimal} surfaces. One can show (and we will!) that the Lagrangian density (and so \( L_{\text{Lie}} \) and its critical points) is invariant under both the Möbius group and normal shifts (the passage to a parallel surface \( f + t \mathbf{n} \)). Otherwise said, the density is preserved by the group of Lie sphere transformations (see Cecil \([7]\) for a modern account of Lie sphere geometry).

Secondly, let \( f : M^2 \to \mathbb{R}P^3 \) be an immersed surface in real projective 3-space with (possibly complex conjugate) asymptotic coordinates\(^2\) \( u, v \). Thus we have\(^3\)

\[
\begin{align*}
\hat{f}_{uu} &= \ast \hat{f}_u + p \hat{f}_v + \ast \hat{f}, \\
\hat{f}_{uv} &= q \hat{f}_u + \ast \hat{f}_v + \ast \hat{f},
\end{align*}
\]

\(^2\) Note that the notion of asymptotic coordinates is projectively invariant as the conformal class of the second fundamental form is.

\(^3\) Note that here, and elsewhere, we do not distinguish between a map \( f : M^2 \to \mathbb{R}P^3 \) and any lift (expression in homogeneous coordinates) \( \tilde{f} : M^2 \to \mathbb{H}^4 \).
for some functions \(p, q\) (here and elsewhere, we use \(\ast\) to represent unknown functions that are irrelevant to our analysis). It is not difficult to check that the density \(pq \, du \wedge dv\) is independent of choices (both of asymptotic coordinates and lift) so that we have a well-defined functional

\[
L_{\text{proj}}(f) = \int_{M^2} pq \, du \wedge dv.
\]

The critical points are the projectively minimal surfaces \(^{[23]}\). In this case the density is invariant under the projective action of \(\text{SL}(4, \mathbb{R})\) on \(\mathbb{RP}^3\).

Our contention is that Lie minimality and projective minimality are characterized by harmonicity of an appropriate Gauss map and that, moreover, this Gauss map has a geometric interpretation as a congruence of “model surfaces” — either Dupin cyclides or quadrics — having second order contact with the immersion \(f\). Both Dupin cyclides and quadrics of fixed signature are parametrized by pseudo-Riemannian symmetric spaces (in fact, Grassmannians) and, in classical language, our main results are:

1. **Theorem.** \(f : M^2 \to \mathbb{R}^3\) is Lie minimal if and only if its congruence of Lie cyclides is harmonic.

2. **Theorem.** \(f : M^2 \to \mathbb{RP}^3\) is projectively minimal if and only if its congruence of Lie quadrics is harmonic.

We find a uniform treatment of these assertions in the following considerations: first we treat the contact lifts of immersions rather than the immersions themselves — indeed, for Lie sphere geometry, this is compulsory since the symmetry group of the situation does not act by point-transformations: for example, a circle and a torus of revolution are Lie sphere equivalent via a normal shift. Second, we exploit the fact that the space of contact elements in \(S^3 = \mathbb{R}^3 \cup \{\infty\}\) and in \(\mathbb{RP}^3\) share a common description as the space \(Z\) of lines in a 4-dimensional quadric \(Q \subset \mathbb{RP}^5\). For Lie sphere geometry, this comes from the fact that oriented 2-spheres in \(S^3\) (including points) are parametrized by the Lie quadric: the projective light cone of \(\mathbb{RP}^{3,2}\) and lines in this quadric correspond to parabolic pencils of spheres or, equivalently, contact elements in \(S^3\) (see \(^{[7]}\)). In projective geometry, the double cover \(\text{SL}(4, \mathbb{R}) \to \text{O}(3, 3)\) gives rise to the Klein correspondence between the space of lines in \(\mathbb{RP}^3\) and the Plücker quadric: the projective light cone of \(\mathbb{RP}^{3,3}\). Then lines in the Plücker quadric parametrize contact elements in \(\mathbb{RP}^3\).

Thus we arrive at a uniform approach by considering Legendre immersions into the space \(Z\) of lines in a 4-dimensional quadric \(Q\) as described in Section 2. In this setting, with the aid of the focal surfaces and conjugate parameters attached to such an immersion, we shall, in Section 3, equip each Legendre immersion with a Grassmannian-valued Gauss map. We shall use this in Section 4 to define a functional on Legendre immersions whose critical points (with respect to Legendre variations) are characterized by harmonicity of this Gauss map.
These constructions proceed independently of the signature of the metric on $\mathbb{R}^6$ defining $Q$ but specialize, as we shall see in Section 5, to give our main results once that signature is declared. Thus our methods may be viewed as a practical implementation of the famous line-sphere correspondence of Lie$^4$.

We conclude our study by indicating in Section 6 some applications of harmonic map theory to Lie and projectively minimal surfaces.

2. Line congruences in quadrics and Legendre surfaces

We consider a nonsingular 4-dimensional quadric $Q \subset \mathbb{P}^5$ that contains (real) lines, that is, $Q = \mathbb{P}(\mathcal{L})$ is the projectivized light cone of some $\mathbb{R}^{m,n}$ where $m + n = 6$ and $m \geq n \geq 2$ (this last inequality is the condition that $Q$ contains real lines). Further, let $Z^{m,n} := G_2(\mathbb{R}^{m,n})$ denote the space of null 2-planes in $\mathbb{R}^{m,n}$, that is, the space of lines in $Q$. In the sequel, we will refer to the space of lines in $Q$ as $Z$ unless the signature of the underlying quadric $Q$ has to be emphasized.

The orthogonal group $O(m, n)$ acts transitively on these spaces (and others we shall consider below) and this gives convenient algebraic models for their tangent spaces which we shall use repeatedly. This being the case, let us briefly recall the basic setting of homogeneous geometry: so let $G$ be a Lie group with Lie algebra $\mathfrak{g}$ and $N$ a homogeneous $G$-space. Each $\xi \in \mathfrak{g}$ gives rise to a vector field $\tilde{\xi}$ on $N$ via $\tilde{\xi}_x = \frac{d}{dt}igr|_{t=0} \exp(t \xi) \cdot x$ and then

$$[\tilde{\xi}, \tilde{\eta}] = -[\xi, \eta] .$$

(1)

Since $G$ acts transitively on $N$, we have a surjection $\mathfrak{g} \ni \xi \mapsto \tilde{\xi}_x \in T_xN$ whose kernel is the (infinitesimal) stabiliser $\mathfrak{g}^x$ of $x$. Thus, $T_xN \cong \mathfrak{g}/\mathfrak{g}^x$ and, more globally, we have identified $TN$ with the quotient of the trivial bundle $N \times \mathfrak{g}$ by the bundle of stabilisers.

Let us consider the example of $Z$ in more detail: $\pi \in Z$ is a null 2-plane in $\mathbb{R}^{n+m}$ and so $\mathfrak{o}(m, n)^\pi = \{\xi \in \mathfrak{o}(m, n) : \xi \pi \subset \pi\}$. Thus $T_{\pi}Z \cong \mathfrak{o}(m, n)/\mathfrak{o}(m, n)^\pi$. However, it is a simple matter to see that restriction gives a surjection

$$\mathfrak{o}(m, n) \ni \xi \mapsto \xi|_\pi \mod \pi \in \{A \in \text{Hom}(\pi, \mathbb{R}^{m,n}/\pi) \mid \langle A s_1, s_2 \rangle + \langle s_1, A s_2 \rangle = 0\}$$

with kernel $\mathfrak{o}(m, n)^\pi$ so that we have an isomorphism

$$T_{\pi}Z \cong \{A \in \text{Hom}(\pi, \mathbb{R}^{m,n}/\pi) \mid \langle A s_1, s_2 \rangle + \langle s_1, A s_2 \rangle = 0\}.$$

(2)

Explicitly, this isomorphism is given by $X \mapsto A_X$ where $A_X s = d_X s \mod \pi$ for any local section $s$ of the tautological bundle over $Z$ (note that $A_X$ so defined is algebraic).

$^4$Note that Lie’s line-sphere correspondence does not provide an isomorphism between the Lie and Plücker quadrics (these two spaces are topologically different). However, it can certainly be considered as a correspondence between concepts in projective line geometry and Lie sphere geometry (cf. [17], [15]).
Two structures on $Z$ will be important to us in the sequel: first $Z$ is a contact manifold. For this, define a line bundle $L$ by $L_{\pi} = \Lambda^2\pi^*$ and note that we have a surjective bundle map $\vartheta : TZ \to L$ via $\vartheta(A) = (A_{..})|_{\pi \times \pi}$, for $A \in T_{\pi}Z$. Denote the kernel of $\vartheta$ by $D$. Clearly

$$D_{\pi} = \text{Hom}(\pi, \pi^\perp/\pi).$$

We claim that $\vartheta$ provides a contact structure, that is, the (algebraic) map $D \times D \to L$ given by $X, Y \mapsto \vartheta([X, Y])$ is non-degenerate. In fact, using (1), one sees that, for $A, B \in D_{\pi}$,

$$\vartheta([A, B])(s_1, s_2) = \langle Bs_1, As_2 \rangle - \langle As_1, Bs_2 \rangle$$

which is readily checked to be non-degenerate.

Secondly, both $\pi$ and $\pi^\perp/\pi$ are 2-dimensional so that we can equip $D$ with a $(2, 2)$-conformal structure by setting $(A,A) = \det A$. Of course, this requires a choice of bases on $\pi$ and $\pi^\perp/\pi$ but changing this choice merely rescales the result.

**Definition.** A map $f : M^2 \to Z$ is called Legendre if it is tangent to the contact distribution $D$, that is, if $\langle ds_1, s_2 \rangle \equiv 0$ for $s_1, s_2 : M^2 \to L$ with $f = s_1 \wedge s_2$.

A Legendre immersion pulls back the conformal structure on $D$ to one on $M$ that will be useful to us. Let us consider the possibilities for its signature: clearly any tangent plane to $f$ is Lagrangian for the symplectic form (3) and, when $(m,n) = (4,2)$, it is easily seen that this forces the conformal structure on any $T_{\pi}M$ to vanish or have signature $(1,1)$. When $(m,n) = (3,3)$ there is no such restriction and all signature are possible.

**Assumption.** From this point on, we assume that $f : M^2 \to Z$ is a Legendre map such that the induced conformal structure (df takes values in $D$!) is non-degenerate. In particular, wherever applicable, $f$ will be assumed to be an immersion. Further, let $(u,v)$ be (possibly complex conjugate) coordinates along the null-directions of the induced conformal structure.

As $f_u, f_v \in \text{Hom}(f, f^\perp/f) = D_f$ are null directions of the conformal structure on $D$ we have $\det f_u = \det f_v = 0$ so that there are $l, s : M^2 \to L$ with $\ker f_u = \text{span}\{l\}$ and $\ker f_v = \text{span}\{s\}$. Thus, $l_u, s_v \equiv 0 \mod f$. In classical language, $l, s$ are the focal surfaces of the line congruence $f$ with $(u,v)$ forming their common conjugate net.

Since $f$ is an immersion, we have $l_v, s_u \notin f$ as, otherwise, $f_u = 0$ or $f_v = 0$. In particular, $l$ and $s$ are linearly independent so that $f = l \wedge s$. Similarly, $l_v$ and $s_u$ are linearly independent mod $f$ as, otherwise, the induced conformal structure would have a third null direction.

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5) More invariantly, one can view the conformal structure as the inner product $A, B \mapsto \frac{1}{2}(A \wedge B + B \wedge A)$ taking values in the line $\Lambda^2\pi^* \otimes \Lambda^2\pi^\perp/\pi$. 

As we shall see, these constructions have a direct geometric interpretation: in case \((m, n) = (4, 2)\), \(f\) is the contact lift of a surface in \(S^3\) for which \((u, v)\) are curvature line coordinates\(^6\) and \(l, s\) are the curvature spheres (cf. \([7]\)); similarly, in case \((m, n) = (3, 3)\), \(f\) is the contact lift of a surface in \(\mathbb{R}^3\) for which \((u, v)\) are asymptotic coordinates and \(l, s\) are the corresponding asymptotic line congruences (cf. \([17]\)).

### 3. The conformal Gauss map

We first prove that \(s, s_u, s_{uu}\) and \(l, l_v, l_{vv}\) define two orthogonal 3-dimensional bundles with non-degenerate induced metrics.

Differentiating \((l, s) = 0\) with respect to \(v\) twice yields \(l, l_v, l_{uv} \perp s\) since \(s_v \in l \wedge s\); similarly, \(s, s_u, s_{uu} \perp l\). As \(s_{uv} \in l \wedge s \wedge s_u \perp l\), we find \(\langle l_v, s_u \rangle = \langle l, s_u \rangle_v = 0\).

Thus, we also have \(s_{uv} \in l \wedge s \wedge s_u \perp l_v\), so that we obtain \(\langle l_{vv}, s_u \rangle = \langle l_v, s_u \rangle_v = 0\); similarly, \(\langle l_v, s_{uu} \rangle = 0\). Finally, we have \(s_{uv} \in l \wedge s \wedge s_u \wedge s_{uu} \perp l_v\) which yields \(\langle l_{vv}, s_{uu} \rangle = \langle l_v, s_{uu} \rangle_v = 0\). This gives the first assertion \(s, s_u, s_{uu} \perp l, l_v, l_{vv}\).

In particular, \(f^1 = l \wedge s \wedge s_u \wedge l_v\) so that the non-degeneracy of \(\langle \cdot, \cdot \rangle\) shows that we must have \(\langle s, s_u \rangle, \langle l_v, l_u \rangle \neq 0\). The pairwise scalar products of \(l, l_v, l_{vv}\) and \(s, s_u, s_{uu}\), respectively, are given by

\[
\begin{pmatrix}
0 & 0 & -\langle l_v, l_v \rangle \\
0 & \langle l_v, l_v \rangle & * \\
-\langle l_v, l_v \rangle & * & *
\end{pmatrix}
\quad \text{and} \quad
\begin{pmatrix}
0 & 0 & -\langle s_u, s_u \rangle \\
0 & \langle s_u, s_u \rangle & * \\
-\langle s_u, s_u \rangle & * & *
\end{pmatrix}
\]

showing that \(l, l_v, l_{vv}\) and \(s, s_u, s_{uu}\) span 3-dimensional subspaces with non-degenerate metrics. Note that both spaces contain null lines and, in case \((m, n) = (4, 2)\), these are real (since \(u, v\) are real) so that their signatures are both \((2, 1)\). When \((m, n) = (3, 3)\) there are two cases to consider: if \((u, v)\) are real, both spaces are real and contain null lines so that, as before, the signatures are \((2, 1)\) and \((1, 2)\). If, however, \((u, v)\) are complex conjugate then these spaces are complex and conjugate to each other.

In all cases, we have to do with a splitting \((\mathbb{R}^m, n)^{\mathbb{C}} = \mathbb{C}^6 = S \oplus S^\perp\) where \(S, S^\perp\) are 3-dimensional orthogonal complex spaces satisfying a reality condition

\[
\bar{S} = S \quad \text{for} \ (u, v) \text{ real, and} \\
\bar{S} = S^\perp \quad \text{for} \ (u, v) \text{ complex conjugate.}
\]

Such a splitting corresponds to a real symmetric endomorphism \(*_S : \mathbb{C}^6 \rightarrow \mathbb{C}^6\) with \(*_S^2 = \varepsilon^2, \varepsilon = 1\ or = i\), by setting \(S, \text{ resp. } S^\perp,\) to be the \(+\varepsilon, \text{ resp. } -\varepsilon,\) eigenspaces of \(*_S\). Now, \(*_S\) gives rise to a Legendre submanifold \(Z_S\) of \(Z\) by

\[
Z_S := \{ \pi \in Z | *_S \pi = \pi \}.
\]

\(^6\) Curvature lines are real: this explains the restriction on the signature of the induced conformal structure in the \((4, 2)\)-case.
Indeed, \( T_xZ_S = \{ A \in T_xZ \mid \ast_S A = A\ast_S \} \) and, with \( \varepsilon_{\pm}\)-eigenvectors \( s_{\pm} \) of \( \ast_S \) in \( \pi^C \), we have \( \langle As_+, s_- \rangle = -\frac{1}{\varepsilon}(As_+, \ast_S s_-) = -\frac{1}{\varepsilon}(A \ast_S s_+ , s_-) = -\langle As_+, s_- \rangle = 0 \) so that \( T_xZ_S \subset D_\pi \). Thus
\[
T_xZ_S = \{ A \in \text{Hom}(\pi, \pi^\bot) \mid \ast_S A = A\ast_S \}.
\]

Moreover, the null vectors in \( T_xZ_S \) have \( \ast_S \)-stable kernels and images and so are real or complex conjugate according to whether \( \varepsilon = 1 \) or \( = i \) (the latter case only being possible when \( (m, n) = (3, 3) \)). Thus \( Z_S \) has a conformal structure of signature \( (1, 1) \) for \( \varepsilon = 1 \) or \( (2, 0) \) for \( \varepsilon = i \). We label the Grassmannian of all such \( S \) by the signature \( (m, n) \) of the real structure and that of the conformal structure on \( Z_S \): thus we set
\[
G_{i,j}^{m,n} := \{ \ast_S : \mathbb{R}^{m,n} \to \mathbb{R}^{m,n} \mid \ast_S \text{ is symmetric, } s_S^2 = \pm 1, \text{ and } Z_S \text{ has signature } (i, j) \}.
\]

To be absolutely explicit, this means
\[
G_{1,1}^{1,1} = \{ S = (S_R)^C \subset (\mathbb{R}^{m,n})^C \mid S_R \subset \mathbb{R}^{m,n} \text{ has signature } (2, 1) \},
\]
\[
G_{3,2}^{3,3} = \{ S \subset (\mathbb{R}^{3,3})^C \mid S \cap S^\bot = \{ 0 \}, \text{ and } S = S^\bot \}.
\]

We will see later that \( G_{1,1}^{4,2} \) parametrizes contact lifts of \( Z_S \) of Dupin cyclides while \( G_{3,2}^{3,3} \) parametrizes (contact lifts of) quadrics in \( \mathbb{R}^{3,3} \) with conformal structure of signature \( (i, j) \).

In the sequel, we shall drop the decorations and simply refer to any of these Grassmannians as \( \mathcal{G} \) unless the signatures need emphasis.

**Definition.** Given a Legendre surface \( f : M^2 \to Z \) with induced conformal structure of signature \( (i, j) \), focal surfaces \( l, s : M^2 \to \mathcal{L}^C \), and conjugate parameters \( (u, v) \), we define the conformal Gauss map \( S : M^2 \to \mathcal{G}_{i,j}^{m,n} \) of \( f \) by \( S := \text{span}_C \{ l, l_v, l_{uv} \} \).

It is clear that this definition is indeed independent of any choices (besides swapping the roles of the subspaces in the Lie sphere case).

Here is the geometry of the situation: by construction, \( l, s \) are eigenvectors of \( \ast_S \) so that \( f = l \wedge s \) is \( \ast_S \)-stable. That is, for each \( x \in M \), we have \( f(x) \in Z_S(x) \). Additionally, the images \( s_u, l_v \) mod \( f_u, f_v \) are eigenvectors of \( \ast_S \) so that \( f_u \) and \( f_v \) both commute with \( \ast_S \). Thus, each \( df_x(M) = T_{f(x)}Z_S(x) \) so that \( f(M) \) and \( Z_S(x) \) have first order contact at \( f(x) \). Otherwise said, the congruence \( Z_S \) envelops \( f \).

We now justify our terminology by showing that our conformal Gauss map \( S \) is indeed conformal. For this, first contemplate the Grassmannians \( \mathcal{G} \): these are pseudo-Riemannian symmetric spaces. In fact, \( O(m, n) \) acts isometrically on each and the identification \( T_S \mathcal{G} \) with \( o(m, n) / o(m, n)^S \) together with the natural isomorphism \( \mathcal{C}^6 / S \cong S^\bot \) gives an identification of \( T_S \mathcal{G} \) with \( \text{Hom}(S, S^\bot) \) via \( X \mapsto [\sigma \mapsto \pi^\bot(d_X\sigma)] \), where \( \pi^\bot : \mathcal{C}^6 \to S^\bot \) denotes orthogonal projection. We
define a real indefinite\(^7\) metric on \(\text{Hom}(S, S^\perp)\) by \(\langle A, B \rangle = -\text{tr}B^* A = -\text{tr}AB^*\) where \(B^*: S^\perp \to S\) is the adjoint of \(B: S \to S^\perp\). Moreover, \(*_S\) induces an isometric involution on \(\mathcal{G}\) which is the symmetric involution at \(S\).

With this in hand, consider a conformal Gauss map \(S: M \to \mathcal{G}\). We identify \(S^* T^2 \mathcal{G}\) with \(\text{Hom}(S, S^\perp)\) and then \(dS = [\sigma \mapsto dS \cdot \sigma = \pi^\perp d\sigma]\) for \(\sigma\) a local section of the bundle \(S \to M\). Note that, for \(\sigma \in \Gamma(S)\) and \(\varrho \in \Gamma(S^\perp)\), we have \(\langle dS \cdot \sigma, \varrho \rangle = \langle d\sigma, \varrho \rangle = -\langle \sigma, d\varrho \rangle\) so that \(dS^*\) is given by \(dS^* \varrho = -\pi d\varrho\) where \(\pi\) is the orthogonal projection \(\mathcal{C}^0 \to S\).

By construction, \(S_v l = S_u l_v = 0\), that is, \(\ker S_u \supset l^\perp \cap S\), hence \(\text{im} S^*_u \subset \text{span}\{l\}\). Similarly, \(s^\perp \cap S^\perp \subset \ker S^*_v\), or \(\text{im} S^*_v \subset \text{span}\{s\}\). Consequently, \(S^*_u \circ S_u = 0\) and \(S_v \circ S^*_v = 0\) and taking traces gives \(\langle S_u, S_u \rangle = \langle S_v, S_v \rangle = 0\). Since \((u, v)\) are null coordinates for the conformal structure induced by \(f\) we conclude that \(S\) is indeed conformal.

3. Theorem. The conformal Gauss map \(S\) of a Legendre surface \(f\) is conformal.

For later use, we note that \(S^*_u \circ S_u (l) = -\langle S_u, S_u \rangle l\) since \(S^*_u \subset \text{span}\{l\}\). Thus, defining functions \(p, q\) on \(M\) by

\[
\begin{align*}
    l_u &= *l + ps \\
    s_v &= ql + *s
\end{align*}
\]

yields \(\langle S_u, S_v \rangle = pq\) (4) since \(S^*_u \circ S_v l = pS^*_u s = -pq l\). As a consequence, \(S\) induces a non-degenerate metric if and only if the focal surfaces of \(f\) are non-degenerate, that is, are immersions into \(\mathcal{Q}\).

Finally, we address the question: when is a given map \(S: M^2 \to \mathcal{G}\) the conformal Gauss map of some Legendre surface \(f: M^2 \to Z\)?

Clearly, we have some necessary conditions: first, the metric induced on \(M\) by \(S\) must have signature \((i, j)\) away from critical points of \(S\). Secondly, with \((u, v)\) null coordinates for that metric, we have

\(S^*_u \circ S_u = 0\) and \(S_v \circ S^*_v = 0\) (5)

(modulo interchanging the roles of \(u\) and \(v\)). Consider therefore \(S: M \to \mathcal{G}\) satisfying these conditions, and additionally assume that the induced metric is non-degenerate. In particular \(S\) is an immersion. From (5), we deduce \(\text{im} S_u \subset \ker S^*_u = (\text{im} S_u)^\perp\) so that \(\text{im} S_u\) is light-like and 1-dimensional (since \(S_u \neq 0\)); similarly, \(\text{im} S^*_v\) is light-like and 1-dimensional. Thus, we obtain a candidate \(f = l \wedge s\), where \(l, s: M \to \mathcal{L}^2\) satisfy \(\text{span}\{l\} = \text{im} S^*_u\) and \(\text{span}\{s\} = \text{im} S_u\). Note that \(f = \text{im} S_u \wedge \text{im} S^*_u\) and so is real\(^8\) since \(dS\) is.

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\(^7\) It may amuse the reader to compute the signature.

\(^8\) If we are in the situation of complex conjugate parameters \((u, v)\), that is, \(S = S^\perp\) then we have \(S^*_v = -S_u\) so that \(\text{span}\{s\} = \text{span}\{l\}\).
Since $l$ takes values in $\text{im} S_u^* \subset \ker S_v = l^\perp \cap S$, we have $S_v l = 0$, that is, $l_v \in \Gamma(S)$; consequently, $l_v \in l^\perp \cap S$ which implies $S_v l_v = 0$, that is, $l_v \in \Gamma(S)$. We have therefore established that $l, l_v, l_{vv} \in S$ and, similarly, $s, s_u, s_{uu} \in S^\perp$. We would like to show that $f$ is a Legendre immersion which induces the same conformal structure on $M$ as $S$. Sadly, there are counterexamples to this assertion: one can construct $S$ with constant $f$. However, as soon as $f$ is an immersion the Legendre and conformality conditions amount to $l_v, s_v \in l \cap s = f$ and this is always true. Indeed, $S_v l = \pi^l l_v \in \text{im} S_u = \text{span}\{s\}$ which gives us half of the first assertion and it remains to prove $\pi l_u \in \text{span}\{l\}$. At this point, we wheel out our non-degeneracy assumption: since $\text{im} S_u = \text{span}\{s\}$ we get $S_u \circ S_u^*(s) = -(S_u, S_v)s \neq 0$ to find that $S_u^*s$ is a nonzero multiple of $l$. Hence, $\pi l_u \in \text{span}\{l\}$ if and only if $\pi(S_u^*s) \in \text{span}\{l\}$. The flat differentiation $d$ decomposes according to the bundle decomposition $C^6 = S \oplus S^\perp$,
\[
 d\sigma = \nabla \sigma + dS \cdot \sigma \quad \text{for} \quad \sigma \in \Gamma(S)
\]
\[
 d\sigma^\perp = -dS^* \cdot \sigma^\perp + \nabla^\perp \sigma^\perp \quad \text{for} \quad \sigma^\perp \in \Gamma(S^\perp),
\]
and we obtain a Codazzi equation $\nabla_v S_u^* + S_u^* \nabla^\perp_v = \nabla_u S_v^* + S_v^* \nabla^\perp_u$ from the $S^\perp \to S$ part of the (vanishing) curvature of $d$. Apply this to our distinguished section $s \in \Gamma(S^\perp)$: since $s, \nabla_u S_u^*s \in s^\perp \cap S^\perp = (\text{im} S_u^*)^\perp = \ker S_v^*$, we have $\nabla_v S_u^*s - S_u^* \nabla^\perp_v s = 0$ so that the Codazzi equation yields $\nabla_u S_u^*(s) = S_u^* \nabla^\perp_u s \in \text{span}\{l\}$. This proves that $l_u \in l \cap s$; similarly, we find $s_v \in l \cap s$. To summarize:

4. Theorem (Blaschke [1], §93). Let $S : M^2 \to G_{i,j}^{m,n}$ be an immersion which induces on $M$ a metric of signature $(i, j)$ for which $(u, v)$ are null coordinates. Then $S$ is the conformal Gauss map of a possibly degenerate Legendre map $f : M^2 \to Z$ if and only if $S_u^* \circ S_u = 0$ and $S_v \circ S_v^* = 0$. In this case $f = \text{im} S_u \wedge \text{im} S_v^*$.

4. The variational problem

We now come to the main point of our considerations. Let $f : M^2 \to Z$ be a Legendre map with non-degenerate induced conformal structure and conformal Gauss map $S : M^2 \to G$.

**Definition.** We define the Willmore energy of $f$ to be the harmonic map energy of its conformal Gauss map $S$. Thus,
\[
 W(f) := E(S) = \frac{1}{2} \int_M \langle dS, dS \rangle \text{dvol} = \frac{1}{2} \int_M \langle dS \wedge \star dS \rangle
\]
where $\star$ is the Hodge $\star$-operator on $M$ provided by the conformal structure induced by $f$.

We say that $f$ is $W$-minimal if it extremizes $W$ with respect to variations through Legendre maps.

---

9) It may happen that the same equations additionally hold with the roles of $u$ and $v$ interchanged: in this case, $f$ is the contact lift of a surface of Demoulin [8] (or its Lie sphere geometry equivalent) where the congruence of Lie quadrics only possesses two envelopes; if only one of these additional conditions holds, we are in the case of a Godaux-Rozet surface [12] [20] (resp. its sphere geometry equivalent). See also §122 in [3] and [9].
We shall see in Section 5 that the contact lift of an immersion $f$ is $W$-minimal if and only if $f$ is Lie or projectively minimal.

It is clear from the definition that $f$ extremizes $W$ as soon as $S$ is harmonic. Our mission is to prove the converse:

5. Proposition. If $\frac{\partial}{\partial t}|_{t=0} W(f_t) = 0$ for every variation $f_t$ of $f_0 = f$ through Legendre maps, then the conformal Gauss map $S$ of $f$ is harmonic.

This will require a little preparation.

First, a standard computation in harmonic map theory (see for example [4]) gives

$$\frac{\partial}{\partial t}|_{t=0} W(f_t) = -\int_M \langle \hat{S}, d^D \ast dS \rangle + \frac{1}{2} \int_M \langle dS \wedge dS \circ \hat{J} \rangle,$$

where $D$ is the pull-back of the Levi-Civita connection on $G$ and $J_t$ is the adjoint of $\ast_t$, that is, $\ast_t df_t = df_t \circ J_t$. We now show that the second integrand vanishes: $\langle dS \wedge dS \circ \hat{J} \rangle = 0$. Since $J_t^2 = \pm 1$, we have $JJ + JJ = 0$. Hence, $\hat{J}$ intertwines the eigenspaces of $J$: if $X_\pm$ denote eigenvectors of $J$, $JX_\pm = \pm \varepsilon X_\pm$ with $\varepsilon = 1$ in case $J^2 = 1$ and $\varepsilon = i$ in case $J^2 = -1$, then $JX_\pm \parallel X_\mp$. Moreover, the eigendirections of $J$ are isotropic for the conformal structure showing that $\langle dS(X_\pm), dS(JX_\mp) \rangle = 0$ since $S$ is conformal, that is, $X_\pm$ are isotropic for the metric induced by $S$.

Thus the conformality of $S$ allows us to ignore the variation in conformal structure on $M$.

In the case at hand, under the identification $S^*T^*G \cong \text{Hom}(S, S^\perp)$, the connection $D$ is induced by the connections $\nabla, \nabla^\perp$ on $S$, $S^\perp$: $D\tau = \nabla^\perp \circ \tau - \tau \circ \nabla$. Fixing our null coordinates $(u, v)$ so that $J \frac{\partial}{\partial u} = \varepsilon \frac{\partial}{\partial u}$ and $J \frac{\partial}{\partial v} = -\varepsilon \frac{\partial}{\partial v}$ we compute

$$d^D \ast dS(\frac{\partial}{\partial u}, \frac{\partial}{\partial v}) = \varepsilon [-\nabla^\perp \circ S_u + S_v \circ \nabla_u - \nabla^\perp \circ S_u + S_u \circ \nabla_v] = -2\varepsilon [\nabla^\perp \circ S_v - S_v \circ \nabla_u] = -2\varepsilon \tau_S$$

where the second equality follows from the Codazzi equation. Thus,

$$\frac{\partial}{\partial t}|_{t=0} W(f_t) = 2\varepsilon \int_M \langle \hat{S}, \tau_S \rangle \, du \wedge dv.$$

The key point now is that the mean curvature vector $\tau_S : S \to S^\perp$ of a conformal Gauss map has very restricted image — it takes values in (the pullback of) $L^\ast$, the dual of the contact line bundle of $Z$:

**Lemma.** $\text{im} \tau_S \subset \text{span}\{s\}$, and $\ker \tau_S \supset l^\perp \cap S$.

**Proof.** From $\tau_S = \nabla^\perp \circ S_v - S_v \circ \nabla_u$ we learn that $\text{im} \tau_S \subset \text{span}\{s\}$ since $\text{im} S_u \subset \text{span}\{s\}$ and $\nabla^\perp s \in \text{span}\{s\}$. On the other hand, the Codazzi equations yield $\tau_S = \nabla^\perp \circ S_v - S_v \circ \nabla_u$ so that the second claim follows since $l^\perp \cap S \subset \ker S_v$ and $l^\perp \cap S$ is $\nabla_u$-stable, since $\text{span}\{l\}$ is.

$\diamond$
Thus, \( \tau_S \in \text{Hom}(S/S \cap l^\perp, \text{span}\{s\}) \cong \text{span}\{l\} \otimes \text{span}\{s\} \) where we use the metric to identify \( \text{span}\{l\} \) with \( (S/S \cap l^\perp)^* \).

Using this, we have

\[
(\tau_S^* \circ \hat{\mathcal{S}})l = \text{tr}(\tau_S^* \circ \hat{\mathcal{S}}) \cdot l = -\langle \hat{\mathcal{S}}, \tau_S \rangle \cdot l
\]

since \( \tau_S^* \in \text{Hom}(S^\perp/(S^\perp \cap S^\perp), \text{span}\{l\}) \), by the lemma. Moreover, fixing \( \sigma \in \Gamma(S^\perp) \) with \( \langle s, \sigma \rangle \equiv 1 \), we have

\[
\hat{\mathcal{S}}l = \langle s, \hat{\mathcal{S}}l \rangle \mod s^\perp \cap S^\perp
\]

whence \( (\tau_S^* \circ \hat{\mathcal{S}})l = \langle s, \hat{\mathcal{S}}l \rangle \tau_S^* \sigma \). Finally \( \langle s, \hat{\mathcal{S}}l \rangle = \langle s, \pi^\perp l \rangle = \langle s, \hat{l} \rangle \) since \( \hat{l} = l \mod f \). Thus

\[
\langle s, \hat{\mathcal{S}}l \rangle = -\theta(\hat{f})(s, l)
\]

where \( \theta \) is (the pullback of) our contact form so that \( (\tau_S^* \circ \hat{\mathcal{S}})l = -\theta(\hat{f})(s, l) \tau_S^* \sigma \).

Otherwise said \( (\tau_S^* \circ \hat{\mathcal{S}})l = -\theta(\hat{f})(s, \tau_S^* \sigma)l \) and taking a trace gives

\[
\langle \hat{\mathcal{S}}, \tau_S \rangle = \theta(\hat{f})(s, \tau_S^* \sigma).
\]

Now let \( \gamma \) be an arbitrary compactly supported section of \( L \). It is well known\(^{10}\) that \( \gamma \) generates an infinitesimal contactomorphism \( X_\gamma \) on \( Z \) with \( \vartheta(X_\gamma) = \gamma \). Let \( \Phi_t : Z \to Z \) be the flow by contactomorphisms of \( X_\gamma \) and set \( f_t = \Phi_t \circ f \). Then \( f_t \) is a variation of \( f \) through Legendre maps with \( \hat{f} = X_\gamma \circ f \) and we have

\[
\langle \hat{\mathcal{S}}, \tau_S \rangle = \gamma(f(s, \tau_S^* \sigma)).
\]

Thus, if \( \frac{\partial}{\partial t}|_{t=0}W(f_t) = 0 \), we see that \( \tau_S^* \sigma \) vanishes whence \( \tau_S^* \sigma \) and so \( \tau_S \) vanish by the lemma. We conclude that \( S \) is harmonic. To summarize:

6. **Theorem.** A Legendre surface \( f : M \to Z \) extremizes \( W \) with respect to Legendre variations if and only if its conformal Gauss map \( S : M \to \mathcal{G} \) is harmonic.

5. **Implications**

We now explore the implications of our analysis and consider each of the signatures \((m, n)\) in turn. In this way, we obtain results on Lie and projectively minimal surfaces as promised in the introduction.

\(^{10}\) Indeed, take any vector field \( X \) with \( \vartheta(X) = \gamma \) and observe that \( \Gamma(D) \ni Y \mapsto \vartheta([X, Y]) \in L \) is tensorial. Now use the nondegeneracy of \( \vartheta([., .]) \) on \( D \) to find \( Y \in \Gamma(D) \) with \( \vartheta([X, .]) = \vartheta([Y, .]) \) on \( \Gamma(D) \) and set \( X_\gamma = X - Y \).
5.1 Projectively minimal surfaces

Consider first the case \((m, n) = (3, 3)\). Here we are dealing with projective differential geometry. For this, view \(\mathbb{RP}^3\) as an \(\text{SL}(4, \mathbb{R})\)-space: thus we give \(\mathbb{R}^4\) a fixed volume form \(\text{vol} \in \Lambda^4(\mathbb{R}^4)^*\) and set \(\mathbb{RP}^3 = \{x \in \mathbb{R}^3 \setminus \{0\}\}\). The 6-dimensional space \(\Lambda^2(\mathbb{R}^4)\) gets a metric of signature \((3, 3)\) by \(\langle v, w \rangle = \text{vol}(v \wedge w)\) for which the action of \(\text{SL}(4, \mathbb{R})\) is clearly isometric. This gives a double covering \(\text{SL}(4, \mathbb{R}) \to O(3, 3)\). Henceforth, we write \(\Lambda^2(\mathbb{R}^4) = \mathbb{R}^{3, 3}\). Moreover, \(l \in \mathbb{R}^{3, 3}\) satisfies the Plücker relation \(\langle l, l \rangle = 0\) if and only if \(l\) is decomposable: \(l = x \wedge y\) for some \(x, y \in \mathbb{R}^4\). This yields a diffeomorphism, the *Klein correspondence*, between lines in \(\mathbb{RP}^3\) and the Plücker quadric \(Q\): span\{\(x, y\)\} \(\mapsto\) span\{\(x \wedge y\)\}. Further, if \(l_1, l_2 \in Q\) satisfy \(\langle l_1, l_2 \rangle = 0\) then there is an \(x \in \mathbb{R}^4\) such that \(l_1 = x \wedge y_1\) and \(l_2 = x \wedge y_2\). That is, orthogonal points in \(Q\) correspond to intersecting lines in \(\mathbb{RP}^3\). Moreover, span\{\(l_1, l_2\)\} = \{\(x \wedge (ay_1 + by_2)\) | \(a, b \in \mathbb{R}\)\} determines a plane span\{\(x, y_1, y_2\)\} in \(\mathbb{RP}^3\). Hence, we can identify the space \(Z\) of all \(x\)-null lines in \(Q\) with the manifold \(\{(p, P) \in \mathbb{RP}^3 \times (\mathbb{RP}^3)^* \mid p \in P\}\) of contact elements to \(\mathbb{RP}^3\).

Let \(f : M^2 \to \mathbb{RP}^3\) be an immersion and contemplate its contact lift \(\tilde{f} : M^2 \to Z\) defined by \(\tilde{f} = \text{span}\{f \wedge df(x) \mid X \in TM\}\).

\(f\) is an immersion: indeed, if not, there is some \(X \in T_xM\) with \(dxl \in f(x)\) for any local section \(l\) of \(f\). In particular, \(\tilde{f}(x) \wedge df(X) = 0\). However, any such \(l\) is of the form \(f \wedge dy\) for some local vector field \(Y \in \Gamma(TM)\) so that \(0 = f \wedge dx(f \wedge dy) = f \wedge dxf \wedge dy\) for all \(Y \in T_xM\). Consequently, \(dxf \in \gamma Y\), span\{\(f(x), dyf\)\} = span\(\{\tilde{f}(x)\}\), a contradiction to the assumption that \(f\) is an immersion.

\(f\) is Legendre: fixing a local basis \((X, Y)\) in \(TM\), set \(l := f \wedge dxf\) and \(s := f \wedge dyf\), so that \(f = \text{span}\{l, s\}\). Then, \(|df\)vol = \(|dxf \wedge dfy \wedge df\) = 0 since \(df\) certainly takes values in span\{\(f, dxf, dyf\)\}.

Now, we identify the conformal structure induced by \(f\): first note that the conformal class of the second fundamental form of \(f\) is a projective invariant. Indeed, let \(\tilde{f} : M^2 \to (\mathbb{RP}^3)^*\) denote the dual surface of \(f\) and contemplate \(TM \ni X, Y \mapsto \tilde{f}(dxdf)\): this is tensorial and scales with \(f\) and \(\tilde{f}\). We fix a basis \((X, Y)\) in \(TM\) and choose \(n\) such that vol\(\{f, dxf, dyf, n\} = 1\); we adjust the scaling of \(\tilde{f}\) so that \(\tilde{f}\)\(\{n\}\) = 1. Then, \(f \wedge dxf\) and \(f \wedge dyf\) form a basis of \(f\) while \(f \wedge n\) and \(dxf \wedge dyf\) mod \(f\) form a basis of \(f^\perp/f\). Now, for \(Z = aX + bY \in TM\) we find

\[
\begin{align*}
    df(Z) \cdot f \wedge dxf &= d_2f \wedge dxf + f \wedge d_2dxf \mod f \\
    &= -bdxf \wedge dyf + f^*(d_2dxf) \wedge f \wedge n \mod f \\
    df(Z) \cdot f \wedge dyf &= adxf \wedge dyf + f^*(d_2dyf) \wedge f \wedge n \mod f
\end{align*}
\]

so that the determinant of \(df(Z)\) — that gives, by definition, the conformal structure — is \(-af^*(d_2dxf) - bf^*(d_2dyf) = -f^*(d_2f)\). Consequently, the conformal class on \(M\) induced by \(f\) is that of the second fundamental form of \(f\).

It follows that the null directions of the conformal structure are precisely the asymptotic directions of \(f\).
Finally, let $u,v$ be (possibly complex conjugate) asymptotic coordinates for $\mathfrak{f}$, and define $l := \mathfrak{f} \wedge \mathfrak{f}_u$ and $s := \mathfrak{f} \wedge \mathfrak{f}_v$. Then, $f = \text{span}\{l,s\}$, and $l,s$ are by construction the line congruences tangent to the asymptotic directions of $\mathfrak{f}$. We have $l, s$ in $\text{span}\{\mathfrak{f}, \mathfrak{f}_u, \mathfrak{f}_v\}$ so that

\[
\begin{align*}
\mathfrak{f}_{uu} &= *\mathfrak{f}_u + p\mathfrak{f}_v + *\mathfrak{f}, \\
\mathfrak{f}_{vv} &= q\mathfrak{f}_u + *\mathfrak{f}_v + *\mathfrak{f},
\end{align*}
\]

with suitable functions $p,q$. Now,

\[
\begin{align*}
l_u &= (\mathfrak{f} \wedge \mathfrak{f}_u)_u = \mathfrak{f}_u \wedge \mathfrak{f}_u + \mathfrak{f} \wedge \mathfrak{f}_{uu} = *l + ps, \\
s_v &= (\mathfrak{f} \wedge \mathfrak{f}_v)_v = \mathfrak{f}_v \wedge \mathfrak{f}_v + \mathfrak{f} \wedge \mathfrak{f}_{vv} = ql + *s.
\end{align*}
\]

From this we see that $l$ and $s$ are the focal surfaces of the line congruence $f$ in $\mathcal{Q}$, with $u$ and $v$ the corresponding conjugate parameters, and so we are in the situation of our main analysis. Moreover, from [22], we learn that the Willmore energy of $f$ is given by

\[
W(f) = \int_M \langle S_u, S_v \rangle du \wedge dv = \int_M pq du \wedge dv.
\]

This coincides with the projectively minimal Lagrangian $L_{\text{proj}}(\mathfrak{f})$ for the immersion $\mathfrak{f}$ described in the introduction. All our constructions are palpably $\text{SL}(4,\mathbb{R})$-invariant and we conclude, with [22], that $L_{\text{proj}}(\mathfrak{f})$ is a projectively invariant functional. Moreover, as $\mathfrak{f}$ varies through immersions, its contact lift varies through Legendre immersions and conversely so that $\mathfrak{f}$ is projectively minimal if and only if its contact lift $f$ is $W$-minimal.

What is the geometry of the conformal Gauss map $S$ of $f$? We have already seen that any $S(x), x \in M$, gives rise to a Legendre submanifold $Z_{S(x)}$ having first order contact with $f$ at $f(x)$. We claim that $Z_{S(x)}$ is the contact lift of a quadric $Q\subset \mathbb{RP}^3$ which therefore has second order contact with the underlying immersion $\mathfrak{f} : M \to \mathbb{RP}^3$ at $f(x)$.

A quadric $Q$ in $\mathbb{RP}^3$ is the null cone of an inner product, also called $Q$, on $\mathbb{R}^4$ which is unique up to homothety. The quadric has a conformal structure (given by the second fundamental form) of signature $(i,j)$ which is $(1,1)$ when the metric $Q$ has signature $(2,2)$ and is $(2,0)$ when the metric has Lorentz signature. Without loss of generality, we can take $\text{vol}_Q = \text{vol}$ so that the Hodge operator $*Q : \Lambda^2 \mathbb{R}^4 \to \Lambda^2 \mathbb{R}^4$ is given by

\[
\text{vol}(v \wedge *Qw) = Q(v, w).
\]

Now $*Q$ is clearly a symmetric endomorphism of $\Lambda^2 \mathbb{R}^4 = \mathbb{R}^{3,3}$ (remember that the $(3,3)$ metric on $\Lambda^2 \mathbb{R}^4$ is provided by $\text{vol}$) and $*Q^2 = \pm 1$ according to whether the metric $Q$ has signature $(2,2)$ or Lorentz signature. Otherwise said, we have defined a map $Q \mapsto *Q$ from the space of quadrics with $(i,j)$ conformal structure to $G_{i,j}$. Both domain and co-domain of this map are homogeneous $\text{SL}(4,\mathbb{R})$-spaces while this map is plainly $\text{SL}(4,\mathbb{R})$-equivariant and so is a surjection. In fact, it injects also so that a quadric is determined by its $*$-operator on 2-vectors. Indeed, as is well known, we have
Lemma. Let \( l = x \wedge y \in \Lambda^2(\mathbb{R}^4) \) be decomposable. Then \( l \) is an eigenvector of \( \star_Q \), \( \star_Q l = \pm \varepsilon l \), if and only if \( l \) is a line on the quadric given by \( Q \), that is, \( Q \) vanishes on the \( 2 \)-plane in \( \mathbb{R}^4 \) spanned by \( x \) and \( y \).

With this in hand, we see that the two families of generators of a quadric \( Q \) are given by the null (for the \((3,3)\) metric) eigenvectors of \( \star_Q \). From this, it is clear that \( \star_Q \) determines the quadric \( Q \) and, moreover, for \( S \in \mathcal{G}_{1,3} \),

\[
Z_S = \{ \text{span}\{l_+, l_-\} | \star_S l_\pm = \pm \varepsilon l_\pm \}
\]

comprises the span of pairs of generators, one from each family or, equivalently, the contact element given by the intersection of these lines, together with their span in \( T \mathbb{R}^3 \). Otherwise said, \( Z_S \) is the contact lift of the corresponding quadric \( Q_S \). To summarize, the conformal Gauss map \( S \) of \( f \) is (the contact lift of) a congruence of quadrics in \( \mathbb{RP}^3 \) having second order contact with the surface \( f \).

Classically, these quadrics are known as the “Lie quadrics” (see [1], [2], [22], compare [16]) so that the conformal Gauss map of our Legendre surface \( f \) is the congruence of Lie quadrics of \( f \). Putting this all together, we have:

7. Theorem. A surface \( f : M^2 \to \mathbb{RP}^3 \) is projectively minimal if and only if its congruence of Lie quadrics\(^{11}\) \( S : M^2 \to \mathcal{G}_{1,3} \) is a harmonic map.

We complete this circle of ideas by briefly discussing how to recover an immersion \( f \) from a Legendre surface \( f \). Clearly, any surface \( f = \text{span}\{l, s\} \) gives a map \( f : M^2 \to \mathbb{RP}^3 \) as \((l, s) = 0 \) implies \( l = f \wedge g_1 \) and \( s = f \wedge g_2 \) and then the contact condition gives us \( 0 = df \wedge g_1 \wedge (f \wedge g_1) = df \wedge g_1 \wedge f \wedge g_2 \), that is, \( df \) takes values in \( \text{span}\{f, g_1, g_2\} \). The main issue is whether \( f \) is an immersion: this is the case as soon as the conformal structure induced by \( f \) is non-degenerate. To see this, introduce \( n \) and \( f^* \) as above so that \( \text{vol}(f, g_1, g_2, n) = 1 \), and \( f^* \) annihilates \( f, g_1, g_2 \), and has \( f^*(n) \equiv 1 \). Now suppose \( f \) is not immersed at some point \( x \in M \). Then, there are two cases to consider: if \( df \equiv 0 \mod f \) at \( x \), then \( df(Z) \cdot f \wedge g_i = f^*(dZ g_i) f \wedge n \mod f \) for any \( Z \in T_x M \), and the induced conformal structure degenerates completely. If, on the other hand, \( df \not\equiv 0 \mod f \) at \( x \), we can assume without loss of generality that there is a basis \((X, Y)\) in \( T_x M \) such that \( dX f \parallel f \) and \( dY f = g_2 \) — implying that \( dX (f \wedge g_2) \equiv 0 \mod f \). For \( Z = aX + bY \) we then find

\[
\begin{align*}
\text{df}(Z) \cdot f \wedge g_1 &= -b g_1 \wedge g_2 + f^*(dZ g_1) f \wedge n \mod f, \\
\text{df}(Z) \cdot f \wedge g_2 &= b f^*(dY g_2) f \wedge n \mod f.
\end{align*}
\]

Hence, the induced conformal structure is given by \( aX + bY = Z \mapsto -b^2 f^*(dY g_2) \) giving \( X \perp T_x M \) so that, in this case also, the conformal structure degenerates. As a consequence, any Legendre surface \( f : M^2 \to Z \) with a non-degenerate conformal structure gives rise to an immersion \( f : M^2 \to \mathbb{RP}^3 \) into projective 3-space.

\(^{11}\) Here, \((i, j)\) is the signature of the second fundamental form of \( f \) and so is \((2,0)\) for convex surfaces and \((1,1)\) for negatively curved \( f \) — our non-degeneracy assumption on \( f \) is precisely the non-degeneracy of this second fundamental form.
5.2 Lie minimal surfaces

Here, we skip most of the setup — all the background material can be found in the modern and readable introduction by Cecil [7]. The key points, however, are that the points of the quadric $Q \subset \mathbb{P}^3 \times \mathbb{R}^2$ parametrize oriented spheres in $S^3 \cong \mathbb{R}^3 \cup \{\infty\}$ and that two such points are orthogonal if and only if the corresponding spheres are in oriented contact. Thus we may identify $Z$ with a space of contact elements, this time the contact elements of $S^3$.

Let $\mathfrak{f} : M^2 \to \mathbb{R}^3$ be an immersion with Gauss map $\mathfrak{n} : M^2 \to S^2$. Let $(v_{-1}, v_0, v_1, v_2, v_3, v_{\infty})$ denote a basis of $\mathbb{R}^{4+2}$ such that $(v_1, v_2, v_3)$ form an orthonormal basis for an $\mathbb{R}^3$, $v_{-1}$ is an orthogonal time-like basis vector, and $(v_0, v_{\infty})$ are isotropic, spanning the remaining orthogonal $\mathbb{R}^{1+1}$, with scalar product $\langle v_0, v_{\infty} \rangle = -\frac{1}{3}$. Then, define $\varphi := v_0 + f + \mathfrak{f}^2 v_{\infty}$ to be the stereographic projection of $\mathfrak{f}$ in $S^3$, $\nu := v_{-1} + \mathfrak{n} + 2\mathfrak{n} \cdot f v_{\infty}$ its tangent plane map (here, $\nu$ denotes the scalar product in $\mathbb{R}^3$) and let $f = \varphi \wedge \nu$ be the corresponding contact lift. Choose (local) curvature line coordinates $(u, v)$ so that $0 = \mathfrak{n} + \kappa_1 f_u$ and $0 = \mathfrak{n} + \kappa_2 f_v$, where $\kappa_i$ are the principal curvatures of $\mathfrak{f}$. Further, define the curvature spheres $l, s : M^2 \to \mathcal{Q}$ by $l = \nu + \kappa_1 \varphi$ and $s = \nu + \kappa_2 \varphi$. Clearly, $f = l \wedge s$, while

$$l_u = (\partial_u \kappa_1) \varphi \in f \quad \text{and} \quad s_v = (\partial_v \kappa_2) \varphi \in f.$$  

Thus, the curvature spheres are the focal surfaces of $f$ and the curvature line coordinates $(u, v)$ the corresponding conjugate coordinates so that, once again, our analysis applies. Finally, rearranging (6), we have

$$l_u = \frac{\partial_u \kappa_1}{\kappa_1 - \kappa_2} (l - s) \quad \text{and} \quad s_v = \frac{\partial_v \kappa_2}{\kappa_1 - \kappa_2} (l - s),$$

and then [1] gives

$$\langle S_u, S_v \rangle \ du \wedge dv = -\frac{\partial_u \kappa_1 \partial_v \kappa_2}{(\kappa_1 - \kappa_2)^2} du \wedge dv.$$ 

This coincides with the Lie minimal Lagrangian $L_{\text{Lie}}(\mathfrak{f})$ for the immersion $\mathfrak{f}$ as described in the introduction. As all our constructions were Lie-invariant we conclude that the functional $L_{\text{Lie}}(\mathfrak{f})$ is invariant under Lie sphere transformations. Moreover, as $\mathfrak{f}$ varies through immersions, its contact lift varies through Legendre immersions and conversely [12]) so that $\mathfrak{f}$ is Lie minimal if and only if its contact lift $f$ is $W$-minimal.

Just as in the projective case, the conformal Gauss map $S$ of a Legendre surface $f$ provides a congruence of “simple” surfaces having second order contact with the underlying surface: in the case at hand, the conformal Gauss map defines a congruence of Dupin cyclides. This becomes clear when realizing that the spheres in each $S(x)$ and $S^\perp(x)$ are the principal spheres of $Z_{S(x)}$. Namely, the principal spheres of $Z_{S(x)}$ are constant along the corresponding curvature lines so that the surface has two families of circular curvature lines (is a channel surface in two ways). As in the projective case, the Dupin cyclides of the conformal Gauss map are the Lie cyclides [16] of the Legendre surface $f$. Thus:

[12]) Note that the condition on $\mathfrak{f}$ to be an immersion is an open condition.
8. Theorem. A surface \( f : M^2 \to \mathbb{R}^3 \subset S^3 \) is Lie minimal if and only if its congruence of Lie cyclides \( S : M^2 \to G^{1,2}_{1,1} \) is a harmonic map.

We want to conclude this section with a remark on the recovery of a surface in \( \mathbb{R}^3 \) or \( S^3 \) from a Legendre surface \( f : M^2 \to \mathbb{Z}^{4,2} \): in contrast to the projective picture, not every Legendre surface gives rise to an immersion \( f : M^2 \to \mathbb{R}^3 \). An easy example is given by the horn and spindle cyclides\(^{13}\) where the Legendre surface is immersed but an induced map \( f : M^2 \to \mathbb{R}^3 \) becomes singular at certain points. However, as all Dupin cyclides are Lie equivalent, one can find an immersed map \( f : M^2 \to \mathbb{R}^3 \) with contact lift \( f \), parametrizing a ring cyclide (torus) in this example. For a more comprehensive discussion see \([7]\).

6. Applications

We conclude our discussion by recalling, in abbreviated form, the main points of the integrable systems theory of harmonic maps and indicating how these apply to \( W \)-minimal surfaces.

6.1 Harmonic maps and moving frames

Let \( N \) be a pseudo-Riemannian symmetric \( G \)-space for some Lie group \( G \). Fix a base-point \( o \in N \) with stabilizer \( K \) so that \( N \cong G/K \) and the involution at \( o \) induces a symmetric decomposition \( g = k \oplus p \). The coset projection \( p : G \to N, p(g) = g \cdot o \) is a principal \( K \)-bundle.

Now let \( M \) be a surface equipped with a conformal structure of signature \( (i, j) \) and contemplate maps \( \varphi : M \to N \). A frame of \( \varphi \) is a map \( F : M \to G \) such that \( p \circ F = \varphi \). Let \( F \) be such a frame and consider its Maurer-Cartan form \( \alpha = F^{-1}dF \): a 1-form on \( M \) with values in \( g \). Write \( \alpha = \alpha_t + \alpha_p \) according to the symmetric decomposition and further write

\[
\alpha_p = \alpha'_p + \alpha''_p
\]

where \( \alpha'_p \) and \( \alpha''_p \) are the components along the null directions of the conformal structure on \( M \). Thus, if \( (u, v) \) are an oriented choice of null coordinates, \( \alpha'_p = \alpha_p(u^2)du \). Note that \( \alpha'_p, \alpha''_p \) are real, respectively complex conjugate according as \( (i, j) \) is \((1, 1)\) or \((2, 0)\).

Now introduce a spectral parameter \( \lambda \in \mathbb{C}^\times \) and a family of \( g^{C\times}\)-valued 1-forms by

\[
\alpha_\lambda := \alpha_t + \lambda \alpha'_p + \lambda^{-1} \alpha''_p
\]

The key observation (see \([19]\), \([25]\), \([24]\)) is that \( \varphi \) is harmonic if and only if each \( \alpha_\lambda \) is flat:

\[
d\alpha_\lambda + \frac{1}{2} [\alpha_\lambda \wedge \alpha_\lambda] = 0.
\]

\(^{13}\) These are Möbius transformations of circular cylinders and circular cones, respectively.
Conversely, given $\alpha_\lambda$ of the form (7) satisfying (8) for all $\lambda \in \mathbb{C}^\times$, we may (locally) integrate to find $F_\lambda : M \to G^r$ with $F_\lambda^*dF_\lambda = \alpha_\lambda$. In particular, taking $\lambda \in \mathbb{R}^\times$ when $(i, j) = (1, 1)$ and $\lambda \in S^1$ when $(i, j) = (2, 0)$, we see that $\alpha_\lambda$ is $\mathfrak{g}$-valued so that we may take $F_\lambda : M \to G$ and so obtain a 1-parameter family of maps $\varphi_\lambda = p \circ F_\lambda : M \to N$ with $\varphi_1 = \varphi$.

In the case at hand, we take $N = G_{i,j}^{m,n}$ with base-point $S_\circ$. Then $\mathfrak{g} = \mathfrak{o}(m, n)$ and

\[
f^C = \{X : \mathcal{C}^6 \to \mathcal{C}^6 | X^* = -X, XS_\circ \subset S_\circ\},
\]

\[
p^C = \{X : \mathcal{C}^6 \to \mathcal{C}^6 | X^* = -X, XS_\circ \subset S_\circ^\perp\}.
\]

Given $S : M^2 \to G_{i,j}^{m,n}$ with frame $F$, we have

\[
F\alpha'_{p}(\frac{\partial}{\partial u})F^{-1} = S_u - S_u^* \quad \text{and} \quad F\alpha''_{p}(\frac{\partial}{\partial v})F^{-1} = S_v - S_v^*.
\]

With this in hand, we offer some sample applications.

### 6.2 Spectral deformation

It is easy to see that $\varphi_\lambda : M \to N$ introduced above are all harmonic [24] so that (locally) we obtain a 1-parameter family of harmonic maps from a given one. In our setting, if $S$ is the conformal Gauss map of a $W$-minimal Legendre surface, then $S^\perp \circ S_u = S_v \circ S_v^\perp = 0$. From (9), we see that this condition amounts to

\[
o'_{p}(\frac{\partial}{\partial u}) \circ \alpha'_{p}(\frac{\partial}{\partial u}) S_o = 0 \quad \text{and} \quad \alpha''(\frac{\partial}{\partial u}) \circ \alpha''(\frac{\partial}{\partial v}) S_o^\perp = 0.
\]

The lift $F_\lambda$ of $S_\lambda$ has $(\alpha\lambda)'_p = \lambda \alpha'_p (\alpha\lambda)'_p = \lambda^{-1} \alpha''_p$ which clearly satisfies (10) so that, by Theorem 3, we see that the spectral deformation $S \to S_\lambda$ preserves the property of being a conformal Gauss map. We therefore conclude (as has Ferapontov [11] and [10] by different methods) that

**9. Theorem.** A $W$-minimal surface $f : M^2 \to Z$ gives (locally) rise to a 1-parameter family of $W$-minimal surfaces $f_\lambda : M^2 \to Z$ with $\lambda \in \mathbb{R}^\times$ or $S^1$ according to the signature of the conformal structure induced by $f$.

### 6.3 Dressing transformations and Bäcklund transforms

The lifts $F_\lambda, \lambda \in \mathbb{C}^\times$, patch together to give a map of $M$ into a loop group. There is a well developed theory of dressing actions where a point-wise action of a complementary loop group induces an action on such maps and so, eventually, on the underlying harmonic maps (see, for example, [6]). In general, such an action requires solution of a Riemann-Hilbert problem but for certain special elements of the complementary loop group, the simple factors, the action is explicitly computable and gives rise to Bäcklund transformations. For a careful account of these ideas see [21].
All we want to say here is that the dressing action preserves the class of conformal Gauss maps. For this, the only fact we need is that if \( \hat{\phi} \) arises from \( \phi \) by a dressing transformation then there are frames \( F, \hat{F} \) with

\[
\hat{\alpha}_p' = k_+ \alpha'_p k_+^{-1} \quad \text{and} \quad \hat{\alpha}_p'' = k_- \alpha''_p k_-^{-1}
\]

where \( k_\pm : M \to K^C \). Clearly the requirement (10) is invariant under conjugation by elements of \( K^C \) and we conclude:

**10. Theorem.** The dressing action on harmonic maps \( M \to G_{m,n}^{i,j} \) preserves the class of conformal Gauss maps and so there is an induced action of a loop group on \( W \)-minimal surfaces.

In particular, dressing by simple factors provides Bäcklund transforms of Lie and projectively minimal surfaces. We will return to this topic elsewhere.

### 6.4 A duality between harmonic maps

The celebrated duality between Riemannian symmetric spaces of compact and non-compact type [13] extends to arbitrary symmetric spaces: if \( g = \mathfrak{k} \oplus \mathfrak{p} \) is a symmetric decomposition then \( \hat{g} = \mathfrak{k} \oplus \sqrt{-1} \mathfrak{p} \) is also a symmetric decomposition of a second Lie algebra giving rise to a second symmetric space \( \hat{N} \).

Now suppose that \( \varphi : M^2 \to N \) is a harmonic map of a surface with \((1,1)\) conformal structure. Then the spectral deformation arises by integrating \( \alpha_\lambda \) for \( \lambda \in \mathbb{C} \). However, for \( \lambda \in \sqrt{-1} \mathbb{R}^* \), \( \alpha_\lambda \) is clearly \( \hat{g} \)-valued so that \( F_\lambda \) may be taken\(^{14}\) to be \( \hat{G} \)-valued and then \( \varphi_\lambda = p \circ F_\lambda \) is a harmonic map \( M \to N \). We therefore have:

**11. Proposition.** Let \( M \) be a surface with \((1,1)\) conformal structure. Then there is a (local) bijective correspondence between harmonic maps \( \varphi : M \to N \) and \( \hat{\varphi} : M \to \hat{N} \) modulo isometries.

In the case at hand, we have a duality between \( G_{3,1}^{3,3} \) and \( G_{4,2}^{4,2} \); the duality mechanism is implemented by taking \( \mathbb{R}^{3,3} = S_0 \oplus S_0^\perp \) and then setting \( \mathbb{R}^{4,2} = S_0 \oplus \sqrt{-1} S_0^\perp \). Just as in Section 6.2, the duality preserves conformal Gauss maps and we arrive at a conceptual explanation of an observation of Ferapontov [10].

**12. Theorem.** There is a local bijective correspondence between Lie minimal surfaces and negatively curved projectively minimal surfaces modulo congruence.

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\(^{14}\) By adjusting constants of integration.
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