Limiting distribution of extremal eigenvalues of d-dimensional random Schrödinger operator

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Abstract

We consider Schrödinger operator with random decaying potential on $\ell^2(\mathbb{Z}^d)$ and showed that, (i) IDS coincides with that of free Laplacian in general cases, (ii) the set of extremal eigenvalues, after rescaling, converges to a inhomogeneous Poisson process, under certain condition on the single-site distribution, and (iii) there are “border-line” cases, such that we have Poisson statistics in the sense of (ii) above if the potential does not decay, while we do not if the potential does decay.

1 Introduction

The study on Schrödinger operators with random decaying potential on $\ell^2(\mathbb{Z}^d)$ was initiated by [3], where they showed that if $d = 1$, it has various spectral properties depending on the decay rate of the potential. And a decade later, spectral properties related to Anderson localization were studied for general $d$-dimension [5]. Recently, topics related to eigenvalue/eigenfunction statistics are studied for $d = 1$: e.g., eigenvalue statistics for the bulk [8, 6, 9, 7], linear statistics [10, 2], eigenfunction statistics [12, 11], and limiting distribution of the maximal eigenvalue [4]. In this paper, we consider the Schrödinger operator with random decaying potential for general...
$d$-dimension, as is studied by [4]:

$$
H := H_0 + V, \\
(H_0 u)(n) = \sum_{|m-n|=1} u(m), \\
(V u)(n) := \frac{\omega_n}{\langle n \rangle^\alpha} u(n), \quad \langle n \rangle := (1 + |n|), \quad \alpha \geq 0, \quad u \in \ell^2(\mathbb{Z}^d).
$$

where $\{\omega_n\}_{n \in \mathbb{Z}^d}$ is i.i.d. with common distribution $\mu$. Let $L \in \mathbb{N}$ and we set the box $\Lambda_L$ of size $2L + 1$ and the finite-box Hamiltonian $H_L$ which is the restriction of $H$ on $\Lambda_L$:

$$
\Lambda_L := \{ n = (n_1, \ldots, n_d) \in \mathbb{Z}^d \mid |n_i| \leq L, \ i = 1, 2, \ldots, d \}, \\
H_L := 1_L H 1_L, \quad (1_L)(n) := 1(n \in \Lambda_L).
$$

Dolai [4] showed that, when $\mu$ has finite second moment : $E[\omega_0^2] < \infty$, then the IDS of $H$ is equal to that of the free Laplacian. Moreover, if the tail of $\mu$ satisfies $\mu[x, \infty) = x^{-p}$, $p > 0$, he obtained the limit distribution of the maximal eigenvalue of $H_L$. The purpose of this paper is to extend his result to show : (1) the same conclusion for IDS is valid without any restriction on $\mu$, (2) the joint distribution of the rescaled extremal eigenvalues converges to a Poisson process under more general condition on the tail of $\mu$, and (3) if the tail of $\mu$ is exponentially decaying, we have the same conclusion as in (2) if $\alpha = 0$, but do not if $\alpha > 0$.

To set up the problem, let $E_j(L)$, $j = 1, 2, \ldots, |\Lambda_L|$ be the eigenvalues of $H_L$ and let $\mu_L$ be the empirical measure for the eigenvalues of $H_L$ : a random probability measure on $\mathbb{R}$ defined by

$$
\mu_L := \frac{1}{|\Lambda_L|} \sum_j \delta_{E_j(L)}.
$$

Among many known results, we recall (i) let $\mu_L^0$ the empirical measure for the free Laplacian (that is, the Hamiltonian $H_0$). Then we have an ac probability measure $\mu^0$ with supp $\mu^0 = [-2d, 2d]$ such that $\mu_L^0 \Rightarrow \mu^0$. (ii) if $E[\omega_0^2] < \infty$, we have $\mu_L \Rightarrow \mu^0 ([4])$. We first remark that the second moment condition in [4] is not necessary:
Theorem 1  Let \( \alpha > 0 \). For any i.i.d. \( \{\omega_n\} \), we have
\[
\mu_L \xrightarrow{w} \mu^0, \quad \text{a.s.}
\]

We turn to study the extremal eigenvalues. We denote the tail of common distribution \( \mu \) by
\[
\mu(x, \infty) = \frac{1}{f(x)}, \quad x > 0,
\]
for a function \( f \). Let \( \{E^H_j(L)\}_{j \geq 1} : E^H_1(L) \geq E^H_2(L) \geq \cdots \) be positive eigenvalues of \( H_L \) in decreasing order, and let
\[
\tilde{E}^H_j := \frac{f(E^H_j(L))}{\Gamma_L}, \quad j = 1, 2, \cdots,
\]
be the scaling of those, where \( \Gamma_L \) will be chosen depending on \( f \) such that
\[
\lim_{L \to \infty} \Gamma_L = \infty.
\]

We set the following two assumptions on \( f \) and \( \Gamma_L \).

Assumption 1  \( f : (0, \infty) \to (0, \infty) \) and \( \Gamma_L \) satisfy the following conditions :
(1) \( f \) is strictly increasing on \( [R, \infty) \) for some \( R > 0 \), \( \lim_{x \to \infty} f(x) = \infty \), \( f \in C^1 \), and \( \lim_{L \to \infty} \Gamma_L = \infty \),
(2) \( f'(x) = o(f(x)) \), \( x \to \infty \)
(3) \( \sup_{|x-y| \leq 2d} |f(y)| \leq C |f(x)| \) for sufficiently large \( x \) and a positive constant \( C \).

The condition (1) is natural, since \( 1/f \) gives the tail of a measure. Conditions (2), (3) are satisfied if \( f \) is of regular variation. On the other hand, the following one is essential for our problem and non-trivial :

Assumption 2  Let
\[
p^{(L)}_n(x) := P \left( \frac{f(V(n))}{\Gamma_L} \geq x \right), \quad n \in \Lambda_L, \quad x > 0
\]
then we have
\[
\lim_{L \to \infty} \sum_{n \in \Lambda_L} p^{(L)}_n(x) = \frac{1}{x}, \quad x > 0.
\]
We note that, if $\alpha = 0$ and Assumption 1 is satisfied, Assumption 2 is always valid with $\Gamma_L = |\Lambda_L|$. In what follows, we omit the $L$-dependence of $p_n$ for simplicity. Under the two assumptions above, the rescaled extremal eigenvalues converge to a Poisson process:

**Theorem 2** Suppose $f, \Gamma_L$ satisfy Assumption 1, 2, and let $\nu$ be the measure on $(0, \infty)$ with $\nu(x, \infty) = \frac{1}{x}$. Then the point process $\xi_L := \sum_{j \geq 1} \delta_{\tilde{E}_j(L)}$ with atoms being composed of the rescaled eigenvalues converge in distribution to Poisson $(\nu)$, where Poisson $(\nu)$ is the Poisson process with intensity measure $\nu$.

Here we consider the vague topology on the space of point processes on $\mathbb{R}$. As for the related results, the eigenvalue statistics on the bulk for $d = 1$ is well studied [8, 6, 9, 7] and the various limits such as clock, Sine $\beta$ and Poisson appear. However, there are not so many papers on the behavior of extremal eigenvalues even for $\alpha = 0$.

We have two classes of functions satisfying Assumption 1, 2. The first one is a family of power functions with some logarithmic corrections.

$$f(x) = f_{p,k}(x) := x^p (\log x)^{-k}, \quad p > 0, \quad k \in \mathbb{N} \cup \{0\}, \quad d \geq \alpha p.$$  

We remark that Dolai [4] obtained the limiting distribution of $\tilde{E}_1^H(L)$ when $p > 0, \; k = 0$.

**Theorem 3** $f_{p,k}, \Gamma_L$ satisfy Assumption 1, 2 if

(1) $\alpha p < d$

$$\Gamma_L := \gamma_{p,k} L^{d-\alpha p}, \quad \gamma_{p,k} := \frac{C_{d-1}}{d - \alpha p} \left(\frac{d}{d - \alpha p}\right)^k$$

(2) $\alpha p = d$

$$\Gamma_L = h_k^{-1} \left(\gamma_k (\log L)^{k+1}\right), \quad h_k(x) := x (\log x)^k, \quad \gamma_k := \frac{C_{d-1}}{k+1} \cdot p^k$$

where $C_{d-1} := |S^{d-1}|$ is the surface area of the $d$-dimensional unit ball.

We believe that for $\alpha p < d$ the result is true for any $k \in \mathbb{R}$. Note that Theorem 3(1) includes the case for the usual Anderson model where $\alpha = 0$. 

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It is natural to expect that the statement in Theorem 3 would be valid for general function \( f \) which is of regular variation of order \( p \):
\[
\lim_{x \to \infty} \frac{f(\lambda x)}{f(x)} = \lambda^p, \quad \lambda > 0.
\]
(2)

In fact, a formal computation indicates that \( f \) would satisfy Assumption 2 with \( \Gamma_L = L^{d-\alpha p}/(d-\alpha p) \) (say, for the case of \( \alpha p < d \)). However, the constant \( \gamma_{p,k} \) in Theorem 3 (1) implies that these observation is false in general and the quantity which vanishes in the limit in (2) has a non-zero contribution in the limiting behavior of \( \xi_L \).

We next consider a family of exponential functions :
\[
f(x) = f_\delta(x) := e^{x\delta}, \quad 0 < \delta \leq 1.
\]
In this case, the tail of \( \omega_n \) is smaller than the previous one, so that we expect the behavior of eigenvalues become more gentle. Here we modify the definition of the norm of \( n \in \mathbb{Z}^d \) for the sake of simplicity :
\[
\langle n \rangle := 1 + |n|_\infty, \quad |n|_\infty := \max_{i=1,2,\ldots,d} |n_i|.
\]

Since \( |n|_\infty \leq |n| \leq \sqrt{d}|n|_\infty \), this is not an essential modification. The following Theorem implies these exponential functions are on a borderline.

**Theorem 4**
(1) \( 0 < \delta < 1, \alpha = 0 : f_\delta \) satisfies Assumption 1,2 with \( \Gamma_L = |\Lambda_L| \).
(2) \( 0 < \delta \leq 1, \alpha > 0 : \) we can find positive constants \( C_j, j = 1,2 \) such that for sufficiently large \( x \), we have
\[
1 - C_1 e^{-x^\delta} \leq P \left( \bigcap_{L \geq 1} \left\{ E_1^H(L) \leq x \right\} \right) \leq \exp \left[ -C_2 x^{-d/\alpha} e^{-2D_{\alpha,\delta} x^\delta} \right]
\]
where \( D_{\alpha,\delta} = \max\{1,2^{\alpha\delta-1}\} \).

In Theorem 4(2), the first inequality implies that the sequence \( \{E_1^H(L)\}_{L \geq 1} \) is bounded for probability close to 1, while the second one implies that there is no constant \( M \) such that \( E_1^H(L) \leq M \), a.s., having completely different behavior from that in Theorem 4(1).
In later sections, we prove these theorems. The proof of Theorem 1 is simple: we cut off \( \omega_n \) suitably and let \( \tilde{H} \) be the corresponding Hamiltonian. Since its single site distribution has finite second moment, the IDS of \( \tilde{H} \) converges to \( \mu^0 \) by the result of [4]. It then suffices to compare IDS' of \( H \) and \( \tilde{H} \) by the rank inequality. For the proof of Theorem 2, we first consider the point process \( \xi_{\nu}^{V} \) composed of the rescaled eigenvalues of the multiplication operator \( V \) and show that it converges to Poisson (\( \nu \)), which is a non-identically distributed version of Poisson limit theorem. We then compare \( \xi_{\nu}^{L} \) and \( \xi_{\nu}^{V} \) by using Assumption 1 (2). For the proof of Theorem 3, we examine the condition in Assumption 2 explicitly for given \( f_{p,q} \) and \( \Gamma_{L} \). The proof of Theorem 4(2) is reduced to the study of \( E_{\nu}^{V}(L) \) and we can explicitly compute the quantity in question.

2 Proof of Theorem 1

We introduce a cut off parameter \( K \gg 1 \) and set
\[
\tilde{\omega}_n := \omega_n 1(\omega_n \leq K), \quad (\tilde{V}u)(n) := \frac{\tilde{\omega}_n}{n^{\alpha}} u(n),
\]
\[
\tilde{H} := H_0 + \tilde{V}.
\]
Let \( \tilde{\mu}_L \) be the empirical measure for \( \tilde{H} \). Since \( \{\tilde{\omega}_n\} \) is i.i.d. with \( E[\tilde{\omega}_0^2] \leq K^2 < \infty \), the result in [4] says that \( \tilde{\mu}_L \overset{w}{\to} \mu^0 \) which is equivalent to
\[
d_L(\tilde{\mu}_L, \mu^0) \overset{L \to \infty}{\to} 0, \quad a.s.
\]
where \( d_L(\mu, \nu) \) is the Lévy distance between the probability measures \( \mu \) and \( \nu \). For any \( \epsilon > 0 \), we can find \( K_{\epsilon} \gg 1 \) sufficient large such that
\[
P(|\omega_n| > K_{\epsilon}) < \epsilon.
\]
Set \( K = K_{\epsilon} \) in the definition of \( \tilde{H} \), and let \( d_{KS}(\mu, \nu) \) be the Kolmogorov–Smirnov distance between \( \mu \) and \( \nu \). By the rank inequality[1],
\[
d_{KS}(\mu_L, \tilde{\mu}_L) \leq \frac{1}{|\Lambda_L|} \text{rank}(H_L - \tilde{H}_L) \leq \frac{1}{|\Lambda_L|} \sum_{n \in \Lambda_L} 1(|\omega_n| > K_{\epsilon}).
\]
By the strong law of large numbers, for any \( \epsilon > 0 \), we can find \( \Omega_{\epsilon} \subset \Omega \) with \( P(\Omega_{\epsilon}) = 1 \) such that

\[
\frac{1}{|\Lambda_L|} \sum_{n \in \Lambda_L} 1(|\omega_n| > K_\epsilon) \to P(|\omega_0| > K_\epsilon) < \epsilon, \quad \omega \in \Omega_{\epsilon}
\]  \hspace{1cm} (5)

On the other hand, by the triangle inequality for \( d_L \) and by \( d_L(\mu, \nu) \leq d_{KS}(\mu, \nu) \),

\[
d_L(\mu_L, \mu^0) \leq d_L(\mu_L, \tilde{\mu}_L) + d_L(\tilde{\mu}_L, \mu^0) \leq d_{KS}(\mu_L, \tilde{\mu}_L) + d_L(\tilde{\mu}_L, \mu^0).
\]

We take \( \limsup_{L \to \infty} \) on both sides and use (3, 4, 5).

\[
\limsup_{L \to \infty} d_L(\mu_L, \mu^0) \leq P(|\omega_0| > K_\epsilon) < \epsilon, \quad \omega \in \Omega_{\epsilon}.
\]

Letting \( \Omega_0 := \bigcap_{n \geq 1} \Omega_{1/n} \), we have \( P(\Omega_0) = 1 \) and

\[
\lim_{L \to \infty} d_L(\mu_L, \mu^0) = 0, \quad \omega \in \Omega_0.
\]

\hfill \Box

3 Proof of Theorem 2

We first prepare a lemma, which gives a sufficient condition to have a Poisson limit theorem, and follows easily from Assumption 1, 2.

Lemma 1 Following equations are valid.

\begin{align*}
(0) \quad & \lim_{L \to \infty} \max_{n \in \Lambda_L} p_n(x) = 0, \quad x > 0, \\
(1) \quad & \lim_{L \to \infty} \sum_{n \in \Lambda_L} \frac{p_n(x)}{1 - p_n(y)} = \frac{1}{x}, \quad x, y > 0, \\
(2) \quad & \lim_{L \to \infty} \sum_{n \in \Lambda_L} \left( \frac{p_n(x)}{1 - p_n(y)} \right)^k = 0, \quad k \geq 2.
\end{align*}
Proof. By assumption, \( f^{-1}(y) \) is well-defined for sufficiently large \( y \). We then compute
\[
p_n(x) = \mathbb{P} \left( \omega(n) \geq \langle n \rangle^\alpha f^{-1}(\Gamma_L x) \right) = \frac{1}{f(\langle n \rangle^\alpha f^{-1}(\Gamma_L x))}.
\]
The monotonicity of \( f \) yields
\[
\frac{1}{f(\langle L \rangle^\alpha f^{-1}(\Gamma_L x))} \leq p_n(x) \leq \frac{1}{\Gamma_L x}
\]
from which we have (0). (1) follows from (0) and Assumption 2. (2) follows from (1) and the fact that \( \lim_{L \to \infty} \max_{n \in \Lambda_L} p_n(x)^k = 0. \)

We next consider the multiplication operator \( V \) and let \( \{E^V_j\}_j \geq E^V_1 \geq \cdots \) be the eigenvalues of \( V \) in decreasing order. Moreover, let
\[
\tilde{E}^V_j = \tilde{E}^V_j := \frac{f(E^V_j)}{\Gamma_L},
\]
be the rescaling of those. Then we show

**Lemma 2** The point process \( \xi^V_L := \sum_{j \geq 1} \delta_{\tilde{E}^V_j} \) whose atoms are \( \{\tilde{E}^V_j\}_j \) converges to Poisson \( (\nu) \).

*Proof.* Fix \( k \in \mathbb{N} \) and let \( x_1, \cdots, x_k \in \mathbb{R} \) with \( 0 < x_k < x_{k-1} < \cdots < x_1 \) and consider the following intervals:
\[
I_k := [x_k, x_{k-1}), \cdots, I_1 := [x_1, x_{i-1}), \cdots, I_2 := [x_2, x_1), I_1 := [x_1, \infty),
\]
Take \( a_1, a_2, \cdots, a_k \in \mathbb{Z}_{\geq 0} \), let \( M := a_1 + a_2 + \cdots + a_k \) and consider the following event.
\[
P := \mathbb{P} \left( \bigcap_{i=1}^k \left\{ \sharp \{j \mid \tilde{E}^V_j \in I_i \} = a_i \right\} \right).
\]
It then suffices to show
\[
\mathbb{P} \left( \bigcap_{i=1}^k \left\{ \sharp \{j \mid \tilde{E}^V_j \in I_i \} = a_i \right\} \right) \xrightarrow{L \to \infty} \frac{1}{a_1!a_2! \cdots a_k!} \left\{ \prod_{i=2}^k \left( 1 - \frac{1}{x_i} \right)^{a_i} \right\} \left( \frac{1}{x_1!} \right)^{a_1} e^{-\frac{1}{x_k}}.
\]
We compute:

$$P = \begin{pmatrix}
\tilde{E}^V(n) & n = 1, 2, \ldots, a_1 \\
\tilde{E}^V(n) & n = a_1 + 1, \ldots, a_1 + a_2 \\
\vdots & \vdots \\
\tilde{E}^V(n) & n = \sum_{j=1}^{i-1} a_j + 1, \ldots, \sum_{j=1}^{i-1} a_j + a_i \\
\vdots & \vdots \\
\tilde{E}^V(n) & n = \sum_{j=1}^{k-1} a_j + 1, \ldots, \sum_{j=1}^{k-1} a_j + a_k \\
\end{pmatrix}$$

$$= \begin{pmatrix}
\tilde{V}(n) & n = n_1^{(1)}, \ldots, n_{a_1}^{(1)} \\
\tilde{V}(n) & n = n_1^{(2)}, \ldots, n_{a_2}^{(2)} \\
\vdots & \vdots \\
\tilde{V}(n) & n = n_1^{(i)}, \ldots, n_{a_i}^{(i)} \\
\vdots & \vdots \\
\tilde{V}(n) & n = n_1^{(k)}, \ldots, n_{a_k}^{(k)} \\
\tilde{V}(n) \notin \bigcup_{i=1}^{k} I_i, n \notin \{n_j^{(i)}\}_{j=1, \ldots, a_i}^{i=1, \ldots, k} (\subset \Lambda_L) : distinct \\
\end{pmatrix}$$

$$= \frac{1}{a_1! a_2! \cdots a_k!} \sum_{\{n_j^{(i)}\}_{j=1, \ldots, a_i}^{i=1, \ldots, k}} \prod_{i=1}^{k} \prod_{j=1}^{a_i} \frac{p_{n_j^{(i)}}(x_i) - p_{n_j^{(i-1)}}(x_{i-1})}{1 - p_{n_j^{(i)}}(x_k)} \cdot (1 - p_{n_j^{(i)}}(x_k))$$

$$\times \prod_{n \in \Lambda_L \setminus \{n_j^{(i)}\}_{j=1, \ldots, a_i}^{i=1, \ldots, k}} (1 - p_n(x_k)) \cdot 1 \left(\{n_j^{(i)}\}_{j=1, \ldots, a_i}^{i=1, \ldots, k} : distinct\right)$$

$$= \frac{1}{a_1! a_2! \cdots a_k!} \sum_{\{n_j^{(i)}\}_{j=1, \ldots, a_i}^{i=1, \ldots, k}} \prod_{i=1}^{k} \prod_{j=1}^{a_i} \frac{p_{n_j^{(i)}}(x_i) - p_{n_j^{(i-1)}}(x_{i-1})}{1 - p_{n_j^{(i)}}(x_k)}$$

$$\times \prod_{n \in \Lambda_L} (1 - p_n(x_k)) \cdot 1 \left(\{n_j^{(i)}\}_{j=1, \ldots, a_i}^{i=1, \ldots, k} : distinct\right).$$

Here we substitute

$$1 \left(\{n_j^{(i)}\}_{j=1, \ldots, a_i}^{i=1, \ldots, k} : distinct\right) = 1 - 1 \left(\exists (i, j), (i', j') s.t. n_j^{(i)} = n_j^{(i')}\right)$$
and decompose RHS into two terms,

$$P = P_1 - P_2.$$ 

We show that $P_1$ converges to the right one, while $P_2 \to 0$. For $P_1$,

$$P_1 = \frac{1}{a_1!a_2!\cdots a_k!} \sum_{\{n_j(i)\}_{i=1,\ldots,k}} \prod_{j=1}^a \prod_{j=1}^{a_i} \frac{p_{n_j(i)}(x_i) - p_{n_j(i)}(x_{i-1})}{1 - p_{n_j(i)}(x_k)} \times \prod_{n \in \Lambda_L} (1 - p_n(x_k))$$

$$= \frac{1}{a_1!a_2!\cdots a_k!} \prod_{j=1}^a \sum_{n \in \Lambda_L} \frac{p_{n_j(i)}(x_i) - p_{n_j(i)}(x_{i-1})}{1 - p_{n_j(i)}(x_k)} \times \prod_{n \in \Lambda_L} (1 - p_n(x_k))$$

$$\xrightarrow{L \to \infty} \frac{1}{a_1!a_2!\cdots a_k!} \prod_{j=1}^a \left( \frac{1}{x_i} - \frac{1}{x_{i-1}} \right) \frac{a_j}{x_k}$$

where we used Lemma 1 and the following fact.

$$\log \prod_{n \in \Lambda_L} (1 - p_n(x)) = \sum_{n \in \Lambda_L} \log(1 - p_n(x))$$

$$= - \sum_{n \in \Lambda_L} p_n(x) + \mathcal{O} \left( \sum_{n \in \Lambda_L} p_n(x)^2 \right) \xrightarrow{L \to \infty} - \frac{1}{x}.$$ 

For $P_2$,

$$P_2 = \frac{1}{a_1!a_2!\cdots a_k!} \sum_{\{n_j(i)\}_{i=1,\ldots,k}} \prod_{j=1}^a \prod_{j=1}^{a_i} \frac{p_{n_j(i)}(x_i) - p_{n_j(i)}(x_{i-1})}{1 - p_{n_j(i)}(x_k)}$$

$$\times \prod_{n \in \Lambda_L} (1 - p_n(x_k)) \cdot 1 \left( \exists (i, j), (i', j') \text{ s.t. } n_j(i) = n_{j'}(i') \right),$$

we consider the number $p_M$ of all cases to decompose the set $[M] = \{1, 2, \ldots, M\}$ of $M$ elements into subsets. Explicit value of $p_M$ is given by the Stirling number of second kind: $p_M = \sum_{k=1}^M S(M, k)$, but we do not need here. Let $P'$ be the one of $p_M$ terms which constitute $P_2$ and let $n_q^{(p)} = n_s^{(r)}$ be the one of those pairings corresponding to $P'$. We can then estimate

$$P' \leq \frac{1}{a_1!a_2!\cdots a_k!} \sum_{\{n_j(i)\}} 1 \left( n_q^{(p)} = n_s^{(r)} \right) \prod_{i=1}^a \prod_{j=1}^{a_i} p(i, j)$$

(6)
where we set

\[ p(i, j) := \frac{p_{n(i)}(x_i) - p_{n(i)}(x_{i-1})}{1 - p_{n(j)}(x_k)} \]

Then we estimate

\[
RHS \text{ of (6)} = \frac{1}{a_1!a_2! \cdots a_k!} \sum_{\{n_j^{(i)}\} \{n_q^{(p)}, n_s^{(r)}\} n_q^{(p)}, n_s^{(r)}} \left\{ \prod_{i=1}^{k} \prod_{j=1}^{a_i} p(i, j) 1 \left( n_j^{(i)} \neq n_q^{(p)}, n_s^{(r)} \right) \right\} \cdot \prod_{p, q, r, s} p(p, q) \cdot p(r, s) \\
\times \sum_{n \in \Lambda_L} \frac{p_n(x_p) - p_n(x_{p-1})}{1 - p_n(x_k)} \cdot p_n(x_r) - p_n(x_{r-1}) \\
\leq \frac{1}{a_1!a_2! \cdots a_k!} \prod_{i=1}^{k} \prod_{j=1}^{a_i} \left( 1 ((i, j) \neq (p, q), (r, s)) \right) \times \\
\times \left( \sum_{n_j^{(i)} \in \Lambda_L} p(i, j) \right) \sum_{n \in \Lambda_L} \left( \frac{p_n(x_p) - p_n(x_{p-1})}{1 - p_n(x_k)} \right)^2 \\
= o(1).
\]

Finally, we compare \( \tilde{E}^H \) and \( \tilde{E}^V \) to finish the proof of Theorem 2. This is elementary, but we mention it for completeness.

**Lemma 2** \( \xi_L^V \xrightarrow{d} \text{Poisson(} \nu \text{)} \) implies \( \xi_L^\nu \xrightarrow{d} \text{Poisson(} \nu \text{)} \).

**Proof.** By Assumption 1(1), for any \( \epsilon > 0 \) we can find \( M_\epsilon > 0 \) such that \( |f'(x)| \leq \epsilon f(x) \) for \( x > M_\epsilon \). And by Assumption 1(2), we can find positive constant \( M' \) such sup \( |x-y| \leq 2d \ |f(y)| \leq C|f(x)| \) for \( x > M' \). Set \( M'_\epsilon := \)
max\{M_{\epsilon}, M^{'}\}. Take any \( j \in \mathbb{N} \). By mean value theorem,

\[
|\widetilde{E}_{j}^{H} - \widetilde{E}_{j}^{V}| = \left| \frac{f(E_{j}^{H})}{\Gamma_{L}} - \frac{f(E_{j}^{V})}{\Gamma_{L}} \right| = \left| \frac{f'(E_{j}^{i})}{\Gamma_{L}} (E_{j}^{H} - E_{j}^{V}) \right|, \quad E_{j}^{'} \in [E_{j}^{H}, E_{j}^{V}]
\]  

(7)

Since \( \xi_{L}^{V} \rightarrow \text{Poisson}(\nu) \), we can suppose \( \lim_{L \rightarrow \infty} E_{j}^{V} = \infty \), and since \( |E_{j}^{H} - E_{j}^{V}| \leq 2d \), we can also suppose \( E_{j}^{H}, E_{j}^{'} \rightarrow \infty \), under those events to be considered below. Hence by taking \( L \) sufficiently large such that \( E_{j}^{V}, E_{j}^{H} > M_{\epsilon}^{'} \), we have

\[
|f'(E_{j}^{i})| \leq \epsilon |f(E_{j}^{i})| \leq \epsilon \sup_{|x - E_{j}^{i}| \leq 2d} |f(x)| \leq C \epsilon |f(E_{j}^{i})|, \quad \sharp = H, V
\]

Plugging this equation into (7) yields

\[
|\widetilde{E}_{j}^{H} - \widetilde{E}_{j}^{V}| \leq C \epsilon \bar{E}_{j}^{4} \cdot (2d).
\]  

(8)

Let \( I_{j} = (a_{j}, b_{j}), j = 1, 2, \cdots, K \), \( 0 < a_{j} < b_{j} < \infty \) be the disjoint intervals, and let \( k_{j} \in \mathbb{N} \cup \{0\} \). For \( \delta > 0 \) sufficiently small, let \( I_{j}^{\pm \delta} := (a_{j} \pm \delta, b_{j} \pm \delta) \) be the intervals inflated (resp. shrinked) by \( \delta \) from \( I_{j} \). Then by (8), we have for \( L \) sufficiently large,

\[
\begin{align*}
\{ \sharp \{ \widetilde{E}_{i}^{H} \in I \} = k \} & \subset \{ \sharp \{ \widetilde{E}_{i}^{V} \in I^{+\delta} \} \geq k \} \cap \{ \sharp \{ \widetilde{E}_{i}^{H} \in I \} = k \} \\
& \subset \{ \{ \sharp \{ \widetilde{E}_{i}^{V} \in I^{+\delta} \} = k \} \cup \{ \{ \sharp \{ \widetilde{E}_{i}^{V} \in I^{+\delta} \} \geq k + 1 \} \cap \{ \sharp \{ \widetilde{E}_{i}^{H} \in I \} = k \} \} \\
& \subset \{ \{ \sharp \{ \widetilde{E}_{i}^{V} \in I^{+\delta} \} = k \} \cup \{ \{ \sharp \{ \widetilde{E}_{i}^{H} \in I^{+2\delta} \} \geq k + 1 \} \cap \{ \sharp \{ \widetilde{E}_{i}^{H} \in I \} = k \} \} \\
& \subset \{ \{ \sharp \{ \widetilde{E}_{i}^{V} \in I^{+\delta} \} = k \} \cup \{ \{ \sharp \{ \widetilde{E}_{i}^{H} \in I^{+3\delta} \} \geq 1 \} \} \quad (9)
\end{align*}
\]

which implies

\[
\bigcap_{j=1}^{K} \{ \sharp \{ \widetilde{E}_{i}^{H} \in I_{j} \} = k_{j} \} \subset \bigcap_{j=1}^{K} \{ \sharp \{ \widetilde{E}_{i}^{V} \in I_{j}^{+\delta} \} = k_{j} \} \cup \bigcup_{j=1}^{K} \{ \{ \sharp \{ \widetilde{E}_{i}^{H} \in I_{j}^{+3\delta} \} \geq 1 \} \}
\]  

(10)

(11)
By (10),
\[
\left\{ \# \{ \tilde{E}_i^H \in I \} = k \right\} \subset \left\{ \# \{ \tilde{E}_i^V \in I^{+\delta} \} = k \right\} \cup \left\{ \# \{ \tilde{E}_i^H \in I^{+2\delta} \setminus I \} \geq 1 \right\}.
\]

Switching \(H\) and \(V\),
\[
\left\{ \# \{ \tilde{E}_i^V \in I \} = k \right\} \subset \left\{ \# \{ \tilde{E}_i^H \in I^{+\delta} \} = k \right\} \cup \left\{ \# \{ \tilde{E}_i^V \in I^{+2\delta} \setminus I \} \geq 1 \right\}.
\]

Replacing \(I\) by \(I^{-\delta}\),
\[
\left\{ \# \{ \tilde{E}_i^V \in I^{-\delta} \} = k \right\} \subset \left\{ \# \{ \tilde{E}_i^H \in I \} = k \right\} \cup \left\{ \# \{ \tilde{E}_i^V \in I^{+\delta} \setminus I^{-\delta} \} \geq 1 \right\}.
\]

Therefore, we have
\[
\bigcap_{j=1}^{K} \left\{ \# \{ \tilde{E}_i^V \in I_j^{-\delta} \} = k_j \right\} \subset \bigcap_{j=1}^{K} \left\{ \# \{ \tilde{E}_i^H \in I_j \} = k_j \right\} \cup \bigcup_{j=1}^{K} \left\{ \# \{ \tilde{E}_i^V \in I_j^{+\delta} \setminus I_j^{-\delta} \} \geq 1 \right\}.
\]

By (11), (12),
\[
P \left( \bigcap_{j=1}^{K} \left\{ \# \{ \tilde{E}_i^V \in I_j^{-\delta} \} = k_j \right\} \right) - \sum_{j=1}^{K} P \left( \left\{ \# \{ \tilde{E}_i^V \in I_j^{+\delta} \setminus I_j^{-\delta} \} \geq 1 \right\} \right) \leq P \left( \bigcap_{j=1}^{K} \left\{ \# \{ \tilde{E}_i^H \in I_j \} = k_j \right\} \right) \leq \limsup_{L \to \infty} P \left( \bigcap_{j=1}^{K} \left\{ \# \{ \tilde{E}_i^H \in I_j \} = k_j \right\} \right) \leq P \left( \bigcap_{j=1}^{K} \left\{ \# \{ \tilde{E}_i^V \in I_j^{+\delta} \setminus I_j^{-\delta} \} \geq 1 \right\} \right).
\]

Taking \(\lim \inf_L, \lim \sup_L\) on both sides yields
\[
P \left( \bigcap_{j=1}^{K} \left\{ \# \{ X_i \in I_j^{-\delta} \} = k_j \right\} \right) - \sum_{j=1}^{K} P \left( \left\{ \# \{ X_i \in I_j^{+\delta} \setminus I_j^{-\delta} \} \geq 1 \right\} \right) \leq \liminf_{L \to \infty} \left( \bigcap_{j=1}^{K} \left\{ \# \{ \tilde{E}_i^H \in I_j \} = k_j \right\} \right) \leq \limsup_{L \to \infty} \left( \bigcap_{j=1}^{K} \left\{ \# \{ \tilde{E}_i^H \in I_j \} = k_j \right\} \right) \leq P \left( \bigcap_{j=1}^{K} \left\{ \# \{ X_i \in I_j^{+\delta} \setminus I_j^{-\delta} \} \geq 1 \right\} \right),
\]

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where \( \{X_j\}_{j=1}^{\infty} \sim \text{Poisson}(\nu) \). Since \( P(\#\{X_i \in I\} \geq 1) \xrightarrow{\nu(I) \to 0} O(\nu(I)) \), we have

\[
P \left( \bigcap_{j=1}^{K} \{ \#\{X_i \in I^{-\delta}_j\} = k_j \} \right) - (\text{Const.}) \sum_{j=1}^{K} \nu (I_{j}^{+2\delta} \setminus I_{j}^{-2\delta})
\]

\[
\leq \liminf_{L \to \infty} P \left( \bigcap_{j=1}^{K} \{ \#\{E_i^H \in I_j\} = k_j \} \right) \leq \limsup_{L \to \infty} P \left( \bigcap_{j=1}^{K} \{ \#\{E_i^H \in I_j\} = k_j \} \right)
\]

\[
\leq P \left( \bigcap_{j=1}^{K} \{ \#\{X_i \in I_j^{+\delta}\} = k_j \} \right) + (\text{Const.}) \sum_{j=1}^{K} \nu (I_{j}^{+3\delta} \setminus I_{j}^{-\delta}).
\]

Taking \( \delta \to 0 \), we get the desired conclusion. \( \square \)

4 Proof of Theorem 3

It is easy to see that \( f_{p,k} \) satisfies Assumption 1. For Assumption 2, we discuss the cases \( \alpha p < d \) and \( \alpha p = d \) separately. In this section we simply write \( f = f_{p,k} \).

4.1 \( \alpha p < d \)

Lemma 3 Let

\[
\Gamma_L := \gamma_{p,k} L^{d-\alpha p}, \quad \gamma_{p,k} := \frac{C_{d-1}}{d-\alpha p} \left( \frac{d}{d-\alpha p} \right)^k
\]

Then \( f \) and \( \Gamma_L \) satisfy Assumption 2:

\[
\sum_{n \in \Lambda_L} p_n(x) = \frac{1}{x} + o(1), \quad L \to \infty.
\]
Proof. It is straightforward to see

\[
\sum_{n \in \Lambda_L} p_n(x) = \frac{1}{\Gamma_L x} \sum_{n \in \Lambda_L} \frac{1}{\langle n \rangle^{\alpha p}} \left( 1 + \frac{\log \langle n \rangle^\alpha}{\log f^{-1}(\Gamma_L x)} \right)^k
\]

\[
= \frac{1}{\Gamma_L x} \sum_{n \in \Lambda_L} \frac{1}{\langle n \rangle^{\alpha p}} \sum_{l=0}^k \binom{k}{l} \left( \frac{\log \langle n \rangle^\alpha}{\log f^{-1}(\Gamma_L x)} \right)^l
\]

\[
= : \sum_{l=0}^k A_l
\]

\[
A_l := \frac{1}{\Gamma_L x} \binom{k}{l} \sum_{n \in \Lambda_L} \frac{(\log \langle n \rangle^\alpha)^l}{\langle n \rangle^{\alpha p}} \frac{1}{(\log f^{-1}(\Gamma_L x))^l}
\]

We here note that

\[
y = \log f^{-1}(z) = \frac{1}{p} \log z + \frac{k}{p} \log \log z(1 + o(1)), \quad z \to \infty. \quad (13)
\]

To see (13), we first set \( y := \log x \), and \( g(y) := y^{-k}e^y \). Then \( z = f(x) \) if and only if \( y = p^{-1}g^{-1}(p^{-k}z) \) so that we study the asymptotic behavior of \( g^{-1}(z) \) as \( z \to \infty \). It then suffices to integrate the following formula:

\[
\frac{dy}{dz} = \frac{1}{z} + \frac{k}{z} \cdot \frac{1}{\log z} + O \left( \frac{\log \log z}{z(\log z)^2} \right).
\]

By (13), we have

\[
\log f^{-1}(\Gamma_L x) = \frac{1}{p} \log \Gamma_L (1 + o(1)).
\]

On the other hand, since the leading term of the diverging series is given by the corresponding integral, we have

\[
\sum_{n \in \Lambda_L} \frac{(\log \langle n \rangle^\alpha)^l}{\langle n \rangle^{\alpha p}} = \alpha^l C_{d-1} \int_1^L dx x^{d-1-\alpha p}(\log x)^l(1 + o(1))
\]

\[
= \frac{\alpha^l C_{d-1}}{d - \alpha p} L^{d-\alpha p}(\log L)^l(1 + o(1)).
\]

Plugging them into the definition of \( A_l \) yields

\[
A_l = \frac{1}{\Gamma_L x} \binom{k}{l} \frac{(\alpha p)^l}{d - \alpha p} C_{d-1} \frac{L^{d-\alpha p}(\log L)^l}{(\log \Gamma_L)^l}(1 + o(1)), \quad l = 0, 1, \ldots, k.
\]
Since $A_l \sim L^{d-\alpha_p} \frac{(\log L)^l}{\Gamma_L (\log \Gamma_L)^l}$, we would like to choose $\Gamma_L$ such that $L^{d-\alpha_p} (\log L)^l \sim \Gamma_L (\log \Gamma_L)^l$. Thus we take a constant $\gamma$ to be fixed later and set $\Gamma_L := \gamma L^{d-\alpha_p}$.

Then
\[ A_l = \frac{1}{x} \binom{k}{l} \frac{C_{d-1}}{d-\alpha_p} \gamma \left( \frac{d}{d-\alpha_p} \right)^l \cdot \frac{1}{\left( 1 + \frac{\log \gamma}{(d-\alpha_p) \log L} \right)} \cdot (1 + o(1)) \]

\[ \sum_{n \in \Lambda_L} p_n(x) = \sum_{l=0}^{k} A_l = \frac{1}{x} \frac{C_{d-1}}{d-\alpha_p} \gamma \left( \frac{d}{d-\alpha_p} \right)^k \cdot (1 + o(1)). \]

Therefore choosing $\gamma = \gamma_{p,k}$, we have Assumption 2.

**4.2 $\alpha_p = d, \ q = -k, \ k \in \mathbb{N} \cup \{0\}$**

**Lemma 4** Let $\alpha_p = d, \ q = -k, \ k \in \mathbb{N} \cup \{0\}$. Let $\Gamma_L$ satify the following equation.

\[ \Gamma (\log \Gamma)^k = \gamma (\log L)^{k+1}, \ \gamma := \frac{C_{d-1}}{k+1} \cdot p^k. \]

Then $f$ and $\Gamma_L$ satisfy Assumption 2.

**Proof.** As in the proof of Lemma 3, we have

\[ \sum_{n \in \Lambda_L} p_n(x) = \frac{1}{\Gamma_L x} \sum_{n \in \Lambda_L} \frac{1}{\langle n \rangle^{\alpha_p}} \left( 1 + \frac{\log \langle n \rangle^{\alpha}}{\log f^{-1}(\Gamma_L x)} \right)^k = \sum_{l=0}^{k} A_l \]

\[ A_l := \frac{1}{\Gamma_L x} \binom{k}{l} \sum_{n \in \Lambda_L} \frac{\langle n \rangle^{\alpha_p}}{\langle n \rangle^{\alpha_p}} \frac{1}{\left( \log f^{-1}(\Gamma_L x) \right)^l}. \]

Plugging

\[ \sum_{n \in \Lambda_L} \frac{\langle n \rangle^{\alpha_p}}{\langle n \rangle^{\alpha_p}} = \frac{C_{d-1}}{l+1} (\log L)^{l+1} (1 + o(1)), \]

\[ (\log f^{-1}(\Gamma_L x))^l = \frac{1}{p^l} (\log \Gamma_L)^l (1 + o(1)) \]
into $A_l$ yields

$$A_l = \frac{1}{\Gamma_L x} \cdot \binom{k}{l} \cdot \frac{C_d - 1}{l + 1} (\log L)^{l+1} \cdot \frac{1}{p^l (\log \Gamma_L)^l} (1 + o(1)) \sim \frac{(\log L)^{l+1}}{\Gamma_L (\log \Gamma_L)^l}.$$ 

Since $A_k$ has major contribution, we will choose $\Gamma_L$ such that

$$(\log L)^{k+1} \sim \Gamma (\log \Gamma)^k$$

which would imply

$$\frac{(\log L)^{l+1}}{\Gamma_L (\log \Gamma_L)^l} = \left( \frac{\log \Gamma}{\log L} \right)^{k-l} \leq \left( \frac{\log \log L}{\log L} \right)^{k-l} = o(1), \quad l = 0, 1, \cdots, k-1$$

and thus,

$$A_k = O(1), \quad A_0, A_1, \cdots, A_{k-1} = o(1).$$

Therefore let $\Gamma_L$ satisfy

$$\Gamma (\log \Gamma)^k = \gamma (\log L)^{k+1}.$$ 

Then

$$\sum_{n \in \Lambda_L} p_n(x) = A_k + o(1) = \frac{1}{x} \cdot \frac{C_d - 1}{k + 1} \cdot p^k \cdot \frac{1}{\gamma}$$

which leads us to the conclusion. \qed

## 5 Proof of Theorem 4

### 5.1 Reduction to the study of $E_1^V$

Let

$$A_L(x) := \mathbb{P} \left( E_1^V(L) \leq x \right).$$

Then the following the lemma reduces our problem to a simpler one.

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Lemma 5
Suppose that we can find positive constants $C_j$, $j = 1, 2$ such that for large $x > 0$,

$$1 - C_1 e^{-x^\delta} \leq \lim_{L \to \infty} A_L(x) \leq \exp \left[ -C_2 x^{-d/\alpha} e^{-2D_{\alpha,\delta} x^\delta} \right].$$

Then we have Theorem 4(2).

Proof. Let $\Omega_L(x) := \{ \omega_n \leq (n)^\alpha x, \forall n \in \Lambda_L \}$. Then we have $\{ E_1^L \leq x \} = \Omega_L(x)$ and since the sequence of events $\{ \Omega_L(x) \}_{L \geq 1}$ is monotonically decreasing, we have

$$\lim_{L \to \infty} A_L(x) = \mathbb{P} \left( \bigcap_{L \geq 1} \Omega_L \right) = \mathbb{P} \left( \bigcap_{L \geq 1} \{ E_1^L \leq x \} \right).$$

Suppose we have (14). Then the inclusion $\{ E_1^L \leq x - 2d \} \subset \{ E_1^L \leq x \} \subset \{ E_1^L \leq x + 2d \}$ and some elementary manipulations of inequalities yield the statement of Theorem 4(2). \qed

5.2 Proof of Theorem 4(2) : lower bound

In this subsection we prove the first inequality in (14). By definition we have $A_L(x) = \prod_{n \in \Lambda_L} \mathbb{P} (\omega_n \leq (n)^\alpha x) = \prod_{n \in \Lambda_L} \left( 1 - e^{-(n)^\alpha x^\delta} \right)$, and thus

$$B(x) := \lim_{L \to \infty} \log A_L(x) = \sum_{n \in \mathbb{Z}^d} \log \left( 1 - e^{-(n)^\alpha x^\delta} \right) = -\sum_{k \geq 1} \frac{1}{k} \sum_{n \in \mathbb{Z}^d} e^{-k(n)^\alpha x^\delta}.$$

(15)

In what follows, we shall denote by $C$ the universal constants in those estimates below which may differ from line to line. Since the maximum point $y_{max}$ of the function $g(y) = y^{d-1}e^{-ky^\alpha x^\delta}$ is in the order of $x^{-1/\alpha}$, we can bound the series by the integral:

$$\sum_{n \in \mathbb{Z}^d} e^{-k(n)^\alpha x^\delta} = \sum_{n=0}^{L} \sum_{|m| = n} e^{-k(n)^\alpha x^\delta} \leq C \int_1^\infty y^{d-1}e^{-ky^\alpha x^\delta} dy + C e^{-kx^\delta}$$

$$= C \frac{1}{\alpha \delta} \frac{1}{k^\alpha x^\delta} \frac{1}{x} \int_{kx^\delta}^\infty z^{\frac{d-1}{\alpha}} e^{-z} dz + C e^{-kx^\delta}.$$
Plugging it into (15) yields
\[
B(x) \geq -C \frac{1}{\alpha \delta} \frac{1}{x^\delta} \sum_{k \geq 1} \int_{k\alpha \delta}^\infty z^{\frac{d}{\alpha \delta} - 1} e^{-z} \frac{1}{k^{1 + \frac{d}{\alpha \delta}}} + C \log \left( 1 - e^{-x^\delta} \right). \tag{16}
\]

To estimate the integral in (16), we set \( M := \left\lfloor \frac{d}{\alpha \delta} \right\rfloor \). Then we have
\[
\int_{k\alpha \delta}^\infty z^{\frac{d}{\alpha \delta} - 1} e^{-z} dz \leq \int_{k\alpha \delta}^\infty z^M e^{-z} dz = e^{-kx^\delta} \sum_{n=0}^M M! (kx^\delta)^n
\leq (M + 1)! e^{-kx^\delta} x^{\delta M} k^M. \tag{17}
\]

In what follows we assume \( M \geq 1 \); the argument for \( M = 0 \) is similar. Plugging (17) into (16) and using \( M \leq \frac{d}{\alpha \delta} \), we have
\[
B(x) := \lim_{L \to \infty} \log A_L(x)
\geq -C \frac{1}{\alpha \delta} \frac{1}{x^\delta} \sum_{k \geq 1} \sum_{k \geq 1} \frac{1}{k^{1 + \frac{d}{\alpha \delta}}} (M + 1)! e^{-kx^\delta} x^{\delta M} k^M + C \log \left( 1 - e^{-x^\delta} \right)
\geq -C(M + 1)! \frac{1}{x^{\frac{d}{\alpha \delta} - \delta M}} \sum_{k \geq 1} \frac{e^{-kx^\delta}}{k^{1 + \frac{d}{\alpha \delta} + 1 - M}} + C \log \left( 1 - e^{-x^\delta} \right)
\geq -C(M + 1)! \frac{1}{x^{\frac{d}{\alpha \delta} - \delta M}} \sum_{k \geq 1} \frac{e^{-kx^\delta}}{k^{1 + \frac{d}{\alpha \delta} + 1 - M}} + C \log \left( 1 - e^{-x^\delta} \right)
\geq C_0 \log \left( 1 - e^{-x^\delta} \right)
\]
where \( C_0 := \frac{CM}{\alpha \delta} + C \). Therefore
\[
B(x) = \lim_{L \to \infty} A_L(x) \geq \left( 1 - e^{-x^\delta} \right)^{C_0} \geq 1 - C_0 e^{-x^\delta}.
\]
Here we use an inequality \((1 - \theta)^\rho \geq 1 - \rho \theta\) for \(0 \leq \theta \ll 1, \rho \geq 1\) and obtain the lower bound in Lemma 5.
5.3 Proof of Theorem 4(2) : upper bound

In this subsection we prove the second inequality in (14). Starting from (15), we aim to have a lower bound of the following series

\[ S_L(k) := \sum_{n \in \Lambda_L} e^{-k(n)_{\alpha \delta} x^\delta}. \]

Then as in the previous subsection we have

\[ S_L(k) \geq C \int_1^L y^{d-1} e^{-kx^\delta(y)^{\alpha \delta}} dy. \]

We note \((y)^{\alpha \delta} \leq D_{\alpha,\delta}(1 + y^{\alpha \delta})\) so that

\[ S_L(k) \geq Ce^{D_{\alpha,\delta}kx^\delta} \int_1^L y^{d-1} e^{-D_{\alpha,\delta}kx^\delta y^{\alpha \delta}} dy \]

\[ J_L(k) := \int_{D_{\alpha,\delta}kx^\delta}^{D_{\alpha,\delta}kx^\delta L_{\alpha \delta}} z^{d-1} e^{-z} dz. \]

Here we recall \(M := \left\lfloor \frac{d}{\alpha \delta} \right\rfloor\) and assume \(M \geq 1\); for \(M = 0\), the argument becomes simpler. For large \(x\), we have \(D_{\alpha,\delta}kx^\delta \geq D_{\alpha,\delta}x^\delta > 1\) so that \(z^{d \alpha / \alpha \delta} \geq z^{M-1}\). Thus

\[ J_L(k) \geq \int_{D_{\alpha,\delta}kx^\delta}^{D_{\alpha,\delta}kx^\delta L_{\alpha \delta}} z^{M-1} e^{-z} dz \xrightarrow{L \to \infty} e^{-D_{\alpha,\delta}kx^\delta} \sum_{n=0}^{M-1} \left( \frac{M - 1}{n!} \right) (D_{\alpha,\delta}kx^\delta)^n. \]

Plugging them into (15) yields

\[
\left| \lim_{L \to \infty} \log A_L(x) \right| \\
\geq \sum_{k \geq 1} \frac{1}{k} \lim_{L \to \infty} S_L(k) \\
\geq \frac{C}{\alpha \delta \left( D_{\alpha,\delta} \right)^{d/\alpha \delta}} \frac{1}{x^{d/\alpha}} \sum_{k \geq 1} \frac{1}{k^{1 + d/\alpha \delta}} e^{-2D_{\alpha,\delta}kx^\delta} \sum_{n=0}^{M-1} \frac{(M - 1)!}{n!} (D_{\alpha,\delta}kx^\delta)^n.
\]

To have a bound of simple form, we pick up \(k = 1, n = 0\) term only and

\[
\left| \lim_{L \to \infty} \log A_L(x) \right| \geq \frac{C}{\alpha \delta \left( D_{\alpha,\delta} \right)^{d/\alpha \delta}} \frac{1}{x^{d/\alpha}} e^{-2D_{\alpha,\delta}x^\delta}
\]
yielding the second inequality in Lemma 5.

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