A FINITENESS CONDITION ON QUASI-LOCAL OVERRINGS
OF A CLASS OF PINCHED DOMAINS

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Abstract. An integral domain is called Globalized multiplicatively pinched-Dedekind domain (GMPD domain) if every nonzero non-invertible ideal can be written as $J P_1 \cdots P_k$ with $J$ invertible ideal and $P_1, \ldots, P_k$ distinct ideals which are maximal among the nonzero non-invertible ideals, cf. [2]. The GMPD domains with only finitely many overrings have been recently studied in [15]. In this paper we find the exact number of quasi-local overrings of GMPD domains that only finitely many overrings. Also we study the effect of quasi-local overrings on the properties of GMPD domains. Moreover, we consider the structure of the partially ordered set of prime ideals (ordered under inclusion) in a GMPD domain.

1. Introduction

A. Jaballah gave necessary and sufficient conditions for the set of overrings of Prüfer domains to be finite [8, Corollary 2.1]. He asked for the exact number of overrings of Prüfer domains that have only finitely many overrings and also for the characterization of domains with finitely many overrings [8, Question 2.2]. R. Gilmer labeled such domains as FO-domains in [5]. Many related results about FO-domains can be found in [3], [6], and [7].

A class of domains, called Globalized multiplicatively pinched-Dedekind domains (GMPD domains), was introduced in [2] as an extension of the class of Dedekind domains and have been recently studied in [15] with finiteness condition on the set of its overrings. In this paper we continue to investigate the overring-theoretic properties of GMPD domains and determine the exact number of quasi-local overrings of GMPD domains that only finitely many overrings. Further we investigate whether the number of quasi-local overrings affects the properties of GMPD domains. More precisely, if a GMPD domain is given with finitely many quasi-local overrings, then what properties could be characterized by the number of quasi-local overrings?

A short introduction for the notions involved is given here for the reader’s convenience. An intermediate ring in the ring extension $A \subseteq B$ is a subring of $B$ that contains $A$. The set of all intermediate rings in $A \subseteq B$ is denoted by $[A, B]$. Let $D$ be an integral domain with quotient field $K$. Then $[D, K]$ denotes the set of all overrings of $D$. For simplicity we use $O(D)$ instead of $[D, K]$. For the set of quasi-local overrings of $D$, we we use the notation $O_{ql}(D)$. If every prime ideal $P$

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in $D$ is strongly prime i.e., whenever $xy \in P$ for some $x, y \in K$ then either $x \in P$ or $y \in P$, then $D$ is called pseudo-valuation domain (PVD). Two incomparable valuation domains with the same quotient field are said to be independent if they have no nonzero common prime ideal. A DVR is a valuation domain with value group $\mathbb{Z}$. A graph that can be drawn in the shape of the letter $Y$ is called a $Y$-graph and a graph that does not contain a $Y$-graph as a subgraph is called $Y$-free, cf. [6 Section 2]. If $\text{Spec}(D)$ endowed with natural partial ordering forms a tree then $D$ is called a treed domain, cf. [12]. If $\text{Spec}(D)$ has no $Y$-graph as a subgraph then $D$ is said to have $Y$-free spectrum. If each nonzero ideal of $D$ is contained in only finitely many maximal ideals and each nonzero prime ideal is contained in a unique maximal ideal then $D$ is called $h$-local. A treed domain is $h$-local if and only if it has $Y$-free spectrum. If $D$ has a unique MNI ideal $Q$ (by an MNI ideal of $D$ we mean an ideal of $D$ which is maximal among the non-invertible ideals of $D$) such that every nonzero non-invertible ideal of $D$ can be factored as $JQ$ for some invertible ideal $J$ then $D$ is called an MPD domain, cf. [1]. A Dedekind domain is an MPD domain with zero MNI ideal. The quasi-local MPD domains are: the DVRs, the two-generated pseudo-valuation domains, the (rank-one) valuation domains with value group $\mathbb{R}$ and the rank-two strongly discrete valuation domains [11 Propositions 2.3 and 2.5]. If $D$ is $h$-local and all localizations of $D$ in maximal ideals are MPD domains then $D$ is called a GMPD domain. Each nonzero non-invertible ideal of $D$ can be written as $JQ_1 \cdot Q_2 \cdot Q_3 \cdots Q_n$ for some invertible ideal $J$ and distinct MNI ideals $Q_1, Q_2, Q_3, \ldots, Q_n$ if and only if $D$ is a GMPD domain, cf. [2 Theorem 4] and [14 Theorem 2]. The following implications hold.

\[ \text{Dedekind domain} \Rightarrow \text{MPD domain} \Rightarrow \text{GMPD domain} \]

After summarizing some basic properties of GMPD domains (Proposition 1 in section 2, we prove the following results. For a GMPD domain $D$, $\mathcal{O}(D)$ is finite if and only if each chain of overrings of $D$ is finite if and only if $\text{Max}(D)$ is finite (Proposition 2). For a quasi-local GMPD domain $D$, $\mathcal{O}(D) = \mathcal{O}_q(D)$ (Lemma 3). For a GMPD domain $D$, $\mathcal{O}_q(D) = \bigcup_{M_i \in \text{Max}(D)} \mathcal{O}(M_i)$ (Lemma 5). If $D$ is a GMPD domain with $|\mathcal{O}(D)| < \infty$, then $|\mathcal{O}_q(D)| = \sum_{M_i \in \text{Max}(D)} |\mathcal{O}(M_i)| - |\text{Max}(D)| + 1$ (Lemma 6). If $\mathcal{V} = \{V_1, V_2, \ldots, V_n\}$ is the collection of pairwise independent valuation domains all with quotient field $K$ and if $D = V_1 \cap V_2 \cap \cdots \cap V_n$ is a GMPD domain such that for each non-negative integer $n_i$, $1 \leq i \leq 3$, $|\{V \in \mathcal{V} \mid V \text{ is } V_i \}| = n_i$, where $V_i$ are valuation domains with value group $G_i, G_1 = \mathbb{R}, G_2 = \mathbb{Z} \times \mathbb{Z}$, and $G_3 = \mathbb{Z}$, then $|\mathcal{O}_q(D)| = n_1 + 2n_2 + n_3 + 1$ (Theorem 7). Also $|\mathcal{O}_q(D)| = |\text{Max}(D)| + 1$ if and only if no $V_i$ has value group $\mathbb{Z} \times \mathbb{Z}$ and $|\mathcal{O}_q(D)| = 2|\text{Max}(D)| + 1$ if and only if all $V_i$’s have value group $\mathbb{Z} \times \mathbb{Z}$ (Corollary 8). For any positive integer $n < \infty$, we can construct a GMPD domain having exactly $n$ quasi-local overrings (Example 9). If $D$ is GMPD domain with finite maximal spectrum such that for every non-negative integer $n_i$, $1 \leq i \leq 4$, $|\{M \in \text{Max}(D) \mid D_M \text{ is } K_i \}| = n_i$, where $K_1$ is a two generated PVD but not DVR, $K_j$ is valuation domain with value group $G_j, j = 2, 3, 4, G_2 = \mathbb{R}, G_3 = \mathbb{Z} \times \mathbb{Z}$ and $G_4 = \mathbb{Z}$, then $|\mathcal{O}_q(D)| = 2(n_1 + n_3) + (n_2 + n_4) + 1; D$ is Noetherian if and only if $|\mathcal{O}_q(D)| = 2n_1 + n_4 + 1; D$ is Prüfer if and only if $|\mathcal{O}_q(D)| = 2n_3 + (n_2 + n_4) + 1; D$ is Dedekind if and only if $|\mathcal{O}_q(D)| = n_3 + 1$ (Theorem 10). If $D$ is an MPD domain with $|\text{Max}(D)| = n < \infty$ then $|\mathcal{O}(D)| = 2^n$ or $3 \cdot 2^{n-1}$ and $|\mathcal{O}_q(D)| = n + 2$ or $n + 1$ (Theorem 11). A GMPD domain $D$ with $|\text{Spec}(D)| = n < \infty$ has exactly
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$\left\lceil \frac{n^2}{2} \right\rceil$ partially ordered sets of prime ideals (Theorem 12). For a GMPD domain $D$ with $|\mathcal{O}(D)| < \infty$, there exist a Prüfer domain $R$ such that $\text{Spec}(D) \cong \text{Spec}(R)$ as a partially ordered set (Remark 13).

Throughout this paper all rings are (commutative unitary) integral domains. Any unexplained material is standard as in [4] and [9].

2. Main Results

The first part of this paper deals with overring-theoretic properties of GMPD domains. Some basic facts related to GMPD domains are recalled from [2].

Proposition 1. ([2, Theorems 6, 9]) Let $D$ be a GMPD domain. Then
(a) $\dim(D) \leq 2$.
(b) Every maximal ideal contains a unique height-one prime ideal.
(c) $D$ is a treed domain with $Y$-free spectrum.
(d) Every overring of $D$ is a GMPD domain.
(e) The integral closure $D'$ is a Prüfer GMPD domain.

A result analogous to [8, Corollary 2.1] is attained in the following Proposition under GMPD condition which is also an improvement of [15, Proposition 4].

Proposition 2. For a GMPD domain $D$, the following assertions are equivalent.
(a) $\mathcal{O}(D)$ is finite.
(b) Each chain of overrings of $D$ is finite.
(c) $\text{Spec}(D)$ is finite.
(d) $\text{Max}(D)$ is finite.

Proof. (a) $\Rightarrow$ (b) and (c) $\Rightarrow$ (d) are clear. (b) $\Rightarrow$ (c) follows from [3, Corollary 1.6].
(d) $\Rightarrow$ (a): As $|\mathcal{O}(D_M)| \leq 3$ for each $M \in \text{Max}(D)$, so by [3, Theorem 3.2], $\mathcal{O}(D)$ is finite. $\square$

Next we focus our attention to study the properties of GMPD domains based on quasi-local overrings.

We denote by $\mathcal{O}(D)$ the set of all overrings of a domain $D$ and by $\mathcal{O}_{ql}(D)$ the set of those overrings of a domain $D$ which are quasi local. Clearly, $\mathcal{O}_{ql}(D) \subseteq \mathcal{O}(D)$ but equality does not hold in general, even if $D$ is quasi-local. For example, if $K$ is a field and $X, Y$ are indeterminate over $K$ then the domain $K[X + YK(X)][Y]$ is quasi-local but its overring $K[X] + YK(X)[Y]$ is not quasi-local. Our first result shows that each overring of a quasi-local GMPD domain is quasi-local.

Lemma 3. Let $D$ be a quasi-local GMPD domain. Then $\mathcal{O}(D) = \mathcal{O}_{ql}(D)$.

Proof. Each overring of $D$ is either a valuation domain or a 2-generated PVD, cf. [2, Definition 2] and [5, Corollary 3.3]. $\square$

Remark 4. $|\mathcal{O}(D)| < \infty$ if and only if $|\mathcal{O}_{ql}(D)| < \infty$ for each integral domain $D$. Indeed, if $E \in \mathcal{O}(D)$ then $E = \bigcap \{E_M \mid M \in \text{Max}(E)\}$ where $E_M \in \mathcal{O}_{ql}(D)$ for each $M \in \text{Max}(E)$. Hence, if $|\mathcal{O}_{ql}(D)| < \infty$ then $|\mathcal{O}(D)| < \infty$.

Lemma 5. For a GMPD domain $D$, $\mathcal{O}_{ql}(D) = \bigcup \{\mathcal{O}(D_{M_i}) \mid M_i \in \text{Max}(D)\}$.
Further, suppose that \((E, M) \in \mathcal{O}(D_M) \subseteq \mathcal{O}(D)\). Further, suppose that \((E, M) \in \mathcal{O}_q(D)\). If \(Q = M \cap D\) then \(Q \subseteq M_i\) for some \(M_i \in \text{Max}(D)\) and so \(D_M \subseteq D_Q \subseteq E\). This implies \(E \in \mathcal{O}(D_M)\). \(\Box\)

Note that a domain \(D\) with finite maximal spectrum is h-local if and only if \(D_M D_N\) equals the quotient field of \(D\), for every two distinct maximal ideals \(M\) and \(N\) of \(D\).

**Lemma 6.** Let \(D\) be a GMPD domain with \(|\mathcal{O}(D)| < \infty\). Then

\[
|\mathcal{O}_q(D)| = \sum_{M_i \in \text{Max}(D)} |\mathcal{O}(D_{M_i})| - |\text{Max}(D)| + 1
\]

**Proof.** Since \(D\) is h-local, so \(|\mathcal{O}(D_M) \cap \mathcal{O}(D_N)| = 1\) for every two distinct maximal ideals \(M\) and \(N\) of \(D\). Now apply Lemma 5. \(\Box\)

Recall [3] Section 22] that two incomparable valuation domains with the same quotient field are said to be independent if they have no non-zero common prime ideal. Equivalently, two valuation domains with the same quotient field are said to be independent if there exist no non-trivial valuation overring containing the both. More precisely, if \(V_1\) and \(V_2\) are valuation domains with the same quotient field \(K\), then \(V_1\) and \(V_2\) are independent if and only if \(V_1 V_2 = K\) if and only if no non-zero prime ideal of \(V_1 \cap V_2\) survives in both \(V_1\) and \(V_2\). Let \(V_1, V_2, ..., V_n\) be pairwise independent valuation domains all with quotient field \(K\). Then \(D = V_1 \cap V_2 \cap ... \cap V_n\) is GMPD if and only if each \(V_i\) has value group \(\mathbb{Z}, \mathbb{Z} \times \mathbb{Z}\) or \(\mathbb{R}\), cf. [15] Theorem 6. In the next result we count the quasi-local overrings of those GMPD domains which are obtained by intersection of valuation domains.

Let \(\mathcal{V} = \{V_1, V_2, ..., V_n\}\) be the collection of pairwise independent valuation domains all with quotient field \(K\) and let \(D = V_1 \cap V_2 \cap ... \cap V_n\) be a GMPD domain such that for each non-negative integer \(n_i, 1 \leq i \leq 3\), \(|\{V \in \mathcal{V} | V\ is \ V_i\}| = n_i\), where \(\mathcal{V}_i\) are valuation domains with value group \(G_i, G_1 = \mathbb{R}, G_2 = \mathbb{Z} \times \mathbb{Z}\), and \(G_3 = \mathbb{Z}\).

**Theorem 7.** With notation above \(|\mathcal{O}_q(D)| = n_1 + 2n_2 + n_3 + 1\).

**Proof.** As \(V_i\)’s are pairwise independent, so \(D\) is h-local and Bézout, cf. [11] Section 3. Let \(P_i\) be the center of \(V_i\) on \(D\). Then \(\text{Max}(D) = \{P_1, P_2, ..., P_n\}\) and \(D_{P_i} = V_i\), cf. [9] Theorem 107 or [13] Corollary 2]. Now apply Lemma 6 and the fact that a valuation domain of finite dimension \(d\) has \(d + 1\) overrings. \(\Box\)

**Corollary 8.** With notation above;

(a) \(|\mathcal{O}_q(D)| = n + 1\ if and only if no \(V_i\) has value group \(\mathbb{Z} \times \mathbb{Z}\).

(b) \(|\mathcal{O}_q(D)| = 2n + 1\ if and only if all \(V_i\)’s have value group \(\mathbb{Z} \times \mathbb{Z}\).

Now for any positive integer \(n < \infty\), we are able to construct a GMPD domain having exactly \(n\) quasi-local overrings, as illustrated in the next example. Recall [11] Section 43] that \(D\) is a Krull domain if \(D = \bigcap_{P \in X^1(D)} D_P\), this intersection has finite character and \(D_P\) is a DVR for each \(P \in X^1(D)\), where \(X^1(D)\) is the set of height-one prime ideals of \(D\). Clearly a UFD is a Krull domain, cf. [11] Proposition 43.2].
Example 9. Let $D$ be a Krull domain with quotient field $K$, $\{P_i\}_{i=1}^{n-1}$ be a finite collection of height-one prime ideals of $D$ and let $S = D - \cup_{i=1}^{n-1} P_i$. Then $D_S$ is a GMPD domain having exactly $n$ quasi-local overrings. Indeed, because $\{D_{P_i}\}_{i=1}^{n-1}$ are DVRs with the same quotient field $K$ and $D_S = \cap_{i=1}^{n-1} D_{P_i}$.

Recall [2 Corollary 19] that for every cardinal numbers $c_i, 1 \leq i \leq 4$, there exists a GMPD domain $D$ such that the set $\{M \in \text{Max}(D) \mid D_M \text{ is } K_i\}$ has cardinality $c_i$, where $K_1 = \text{two-generated PVD but not DVR, } K_j = \text{valuation domain with value group } G_j, j = 2, 3, 4, G_2 = \mathbb{R}, G_3 = \mathbb{Z} \times \mathbb{Z}$ and $G_4 = \mathbb{Z}$. Recall [5 Lemma 2] that a non-integrally closed two-generated PVD $D$ with quotient field $K$ has exactly three overrings $D, D'$, and $K$.

Theorem 10. Let $D$ be a GMPD domain with $|\text{Max}(D)| < \infty$ such that for every non-negative integer $n_i, 1 \leq i \leq 4$, $\{|M \in \text{Max}(D) \mid D_M \text{ is } K_i\} = n_i$, where $K_1 = \text{two-generated PVD but not DVR, } K_j = \text{valuation domain with value group } G_j, j = 2, 3, 4, G_2 = \mathbb{R}, G_3 = \mathbb{Z} \times \mathbb{Z}$ and $G_4 = \mathbb{Z}$. Then

(a) $|O_{q\ell}(D)| = 2(n_1 + n_3) + (n_2 + n_4) + 1$.
(b) $D$ is Noetherian if and only if $|O_{q\ell}(D)| = 2n_1 + n_4 + 1$.
(c) $D$ is Prüfer if and only if $|O_{q\ell}(D)| = 2n_3 + (n_2 + n_4) + 1$.
(d) $D$ is Dedekind if and only if $|O_{q\ell}(D)| = n_4 + 1$.

Proof. Apply Lemma 5 and the facts that a non-integrally closed two-generated PVD has exactly three overrings and a valuation domain of finite dimension $d$ has $d + 1$ overrings.

Next we find the exact number of overrings and quasi-local overrings of an MPD domain. Recall that an MPD domain is a GMPD domain with unique MNI ideal, cf. [1].

Theorem 11. Let $D$ be an MPD domain with $|\text{Max}(D)| = n < \infty$. Then

(a) $|O(D)| = 2^n$ or $3 \cdot 2^{n-1}$.
(b) $|O_{q\ell}(D)| = n + 2$ or $n + 1$.

Proof. (a): Let $\text{Max}(D) = \{M_1, M_2, M_3, ..., M_n\}$. From [1 Proposition 2.7], there exist a maximal ideal $M_i$ such that $D_{M_i}$ is MPD and $D_{M_j}$ is DVR for each $j \neq i$. If $D_{M_i}$ is not a DVR, then $D_{M_i}$ is either a 2-generated PVD or a valuation domain with value group $\mathbb{R}$ or $\mathbb{Z} \times \mathbb{Z}$. Therefore, $|O(D_{M_i})| = 2$ or $3$ and $|O(D_{M_j})| = 2$ for each $i \neq j$. Hence by [13 Theorem 10], we get that $|O(D)| = 3 \cdot 2^{n-1}$. If $D_{M_i}$ is a DVR, then again by [13 Theorem 10], we get that $|O(D)| = 2^n$.

(b): Apply [1 Proposition 2.7] and Lemma 6.

At the end we consider the structure of the partially ordered set of prime ideals (ordered under inclusion) in a GMPD domain. For a GMPD domain $D$ with $|\text{Spec}(D)| < \infty$, we consider the question that what partially ordered sets could be arise as $\text{Spec}(D)$? Keeping in view the basic properties of GMPD domain, given in Proposition 11 we make first the following observations:
After these observations a natural question arises that how many structurally distinct partially ordered sets of prime ideals a GMPD domain can have? The answer to this question is provided in the following proposition.

**Proposition 12.** Let $D$ be a GMPD domain with $|\text{Spec}(D)| = n < \infty$. Then $D$ can have at most $\left\lceil \frac{n^2}{2} \right\rceil$ partially ordered sets of prime ideals.

**Proof.** We can assume that $n > 1$. Since $D$ is a treed domain with $Y$-free spectrum, so $n - 1 \leq 2|\text{Max}(D)|$ and hence $\frac{n - 1}{2} \leq |\text{Max}(D)| \leq n - 1$. If $n$ is odd, the possibilities for the number of maximal ideals in each spectrum is $n - 1, n - 2, \ldots, \frac{n - 1}{2}$ respectively. If $n$ is even, the possibilities for the number of maximal ideals in each spectrum is $n - 1, n - 2, \ldots, \frac{n}{2}$ respectively. Also each prime spectrum has distinct cardinality of maximal ideals. Hence the total possibilities for the number of distinct partially ordered sets of prime ideals is $\left\lceil \frac{n^2}{2} \right\rceil$. □

**Remark 13.** Any two partially ordered sets $U$ and $V$ are said to be isomorphic if there is an order preserving bijection $f : U \to V$ such that $f^{-1}$ is also order preserving. By [10, Theorem 3.1] and Proposition we can easily deduce that for a GMPD domain $D$ with $|O(D)| < \infty$ there exist a Prüfer domain $R$ such that $\text{Spec}(D) \cong \text{Spec}(R)$ (as a partially ordered set).

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