The fractional derivative Kelvin–Voigt model of viscoelasticity with and without volumetric relaxation

Yu A Rossikhin and M V Shitikova
Research Center on Dynamics of Solids and Structures, Voronezh State Technical University, Voronezh, Russia
E-mail: mvs@vgasu.vrn.ru

Abstract. The fractional derivative Kelvin–Voigt model of viscoelasticity involving the time-
dependent Poisson’s operator has been studied not only for the case of a time-independent bulk
modulus, but also when the volumetric relaxation is taken into account. It has been shown that
such a model could describe the features of auxetic materials.

1. Introduction
It is well known that each isotropic elastic material possesses only two independent constants,
and all others are expressed in terms of two constants which should be preassigned [1]. Thus,
if Young’s modulus $E$ and Poisson’s ratio $\nu$ are known, then Lame constants $\lambda$ and $\mu$ and bulk
modulus $K$ are expressed as

$$\mu = \frac{E}{2(1 + \nu)}, \quad \lambda = \frac{E\nu}{(1 - 2\nu)(1 + \nu)}, \quad K = \frac{E}{3(1 - 2\nu)},$$

(1)
or in the case, when shear modulus $\mu$ and bulk modulus $K$ are preassigned, other constants, $E$,
$\nu$, and $\lambda$, are defined as

$$E = \frac{9K\mu}{3K + \mu}, \quad \nu = \frac{3K - 2\mu}{2(3K + \mu)}, \quad \lambda = \frac{3K - 2\mu}{9}.$$  

(2)

In the case of an isotropic viscoelastic material, material properties are time-dependent, and
once again only two time-dependent viscoelastic operators should be known, while others could
be expressed in terms of two preassigned operators utilizing the correspondence principle and
relationships (1) or (2).

In the present paper, viscoelastic operators will be constructed for the fractional derivative
Kelvin–Voigt model with and without volume relaxation, since this model is of frequent use in
engineering applications [2].

2. Application of the simplest viscoelastic models for the description of dynamic
response of current materials
Simplified approach of many researchers for solving dynamic viscoelastic problems is connected
first of all with the ineptitude to decode intricate operator expressions involving fractional
operators which appear during the solution of dynamic problems. The desire to solve a formulated problem by any means often leads to the fact that some simplifying assumptions are introduced into operator expressions, resulting in incorrectness of these expressions, and finally to the loss of their physical meaning.

That is why, despite that in different applications the utilization of fractional derivatives in problems of mechanics and physics is growing significantly during last two decays, the variety of rheological models used is rather restricted. During description of such viscoelastic bodies as beams, plates and shells, the most frequently used models for the Young operator are the fractional derivative Kelvin–Voigt relation [3–8], the fractional derivative Maxwell model [9, 10], and standard linear solid model [11, 12], in so doing the Poisson ratio of viscoelastic material is assumed as a constant.

However as experimental data have shown [13, 14], Poisson’s ratio is always a time-dependent operator $\tilde{\nu}$ [15–17], and only the bulk extension-compression operator $\tilde{K}$ may be expressed as the time-independent value, which for the most viscoelastic materials weakly varies during deformation.

However, the fractional derivative Kelvin–Voigt model with time-independent Poisson’s ratio is only acceptable for the description of the dynamic behavior of elastic bodies in a viscoelastic medium [2, 18–20] or on a viscoelastic foundation [21, 22].

The review of ‘traditional’ fractional calculus models (‘traditional’ in the sense that such models consider time-independent Poisson’s ratios) could be found in [2, 23–25].

2.1. Modelling the Young operator $\tilde{E}$ using the fractional derivative Kelvin–Voigt model without volumetric relaxation

Let us first choose Young’s operator as

$$\tilde{E} = E_0[1 + (\tau^E_\sigma)^\gamma D^\gamma], \quad (3)$$

where $\tau^E_\sigma$ is the retardation, or creep, time during longitudinal deformations, $D^\gamma$ is the Riemann-Liouville fractional derivative

$$D^\gamma x(t) = \frac{d}{dt} \int_0^t (t-t')^{-\gamma} \frac{x(t')}{\Gamma(1-\gamma)} dt',$$  \quad (4)

$\Gamma(1-\gamma)$ is the Gamma-function, and $x(t)$ is a certain function.

It should be noted that the choice of the Riemann-Liouville derivative of the fractional order is not random. Since during the solution of dynamic problems of viscoelasticity it is often needed to find resolvent operators and to decode intricate operator relationships [25, 26], then the utilization of other definitions of fractional derivatives tunes out to be inconvenient. Details could be found in [27].

The bulk extension-compression operator $\tilde{K}$ is assumed to be time-independent, i.e., volumetric relaxation is neglected (this assumption is due to the fact that for many viscoelastic materials volumetric relaxation is much smaller than the shear relaxation)

$$\tilde{K} = K_0, \quad (5)$$

resulting in

$$\frac{\tilde{E}}{1-2\tilde{\nu}} = 3K_0, \quad (6)$$

where $K_0$ is a certain constant.

Substituting (3) in (6), we could find Poisson’s operator

$$\tilde{\nu} = \nu_0 - \frac{E_0}{6K_0} (\tau^E_\sigma)^\gamma D^\gamma, \quad (7)$$
and finally

\[ \nu(t) = \tilde{\nu}H(t) = \nu_0 - \frac{E_0}{6K_0} (\tau_\nu^\mu)^\gamma \frac{t^{-\gamma}}{\Gamma(1-\gamma)}, \]

whence it follows that

\[ \lim_{t \to 0} \tilde{\nu}H(t) = -\infty, \quad \lim_{t \to \infty} \tilde{\nu}H(t) = \nu_0, \]

where \( \nu_0 = (3K_0 - E_0)/(6K_0) \).

From (9) it is seen that this model lacks the physical meaning, since for real materials the low limiting value of Poisson’s ratio \( \nu(0) \) could not take on extremely negative value \(-\infty\). The similar result for this model was reported in [28] but with reverse sign: \( \nu(0) = +\infty \), and \( \nu(\infty) = \nu_0 \). But nevertheless this model is inappropriate for dealing with real viscoelastic materials.

2.2. Modelling the shear operator \( \tilde{\mu} \) using the fractional derivative Kelvin–Voigt model without volumetric relaxation

The shear operator \( \tilde{\mu} \) most frequently is preassigned using the fractional derivative Kelvin–Voigt model

\[ \tilde{\mu} = \mu_0[1 + (\tau_\mu^\mu)^\gamma D^\gamma], \]

where \( \mu_0 \) is the relaxed shear modulus, \( \tau_\mu^\mu \) is the retardation time during shear deformations, while the bulk operator is assumed to be constant according to (5).

In order to evaluate the dynamic response of viscoelastic bodies, it is necessary to calculate Young’s operator. For this purpose using the Volterra correspondence principle, the following formula could be utilized:

\[ \tilde{E} = \frac{9K_0 \tilde{\mu}}{3K_0 + \tilde{\mu}} \]

First we write the operator

\[ 3K_0 + \tilde{\mu} = (3K_0 + \mu_0)(1 + t_\sigma^\sigma D^\gamma), \]

where \( t_\sigma^\sigma = \mu_0(\tau_\sigma^\mu)^\gamma(3K_0 + \mu_0)^{-1} \).

Then we find the operator reverse to (12), i.e.,

\[ (3K_0 + \tilde{\mu})^{-1} = \frac{1}{3K_0 + \mu_0} \frac{1}{1 + t_\sigma^\sigma D^\gamma} = (3K_0 + \mu_0)^{-1} \exists_\gamma^*(t_\sigma^\sigma), \]

where

\[ \exists_\gamma^*(t_\sigma^\sigma) = \frac{1}{1 + t_\sigma^\sigma D^\gamma} \]

is the dimensionless Rabotnov fractional operator [23, 25, 26].

Before calculating operator \( \tilde{E} \), let us remind one useful formula [29]

\[ t_\sigma^\sigma D^\gamma. \exists_\gamma^*(t_\sigma^\sigma) = \frac{t_\sigma^\sigma D^\gamma}{1 + t_\sigma^\sigma D^\gamma} = 1 - \exists_\gamma^*(t_\sigma^\sigma). \]

Substituting (10) and (13) in (11) and considering (15) yield

\[ \tilde{E} = 9K_0 \left[ 1 - \frac{E_0}{3\mu_0} \exists_\gamma^*(t_\sigma^\sigma) \right]. \]

Now we could calculate Poisson’s operator \( \tilde{\nu} \) via the formula

\[ \frac{\tilde{E}}{1 - 2\tilde{\nu}} = 3K_0. \]
Substituting (16) in (17), we have

$$\tilde{\nu} = -1 + \frac{E_0}{2\mu_0} \gamma^*\left(t_\gamma^\sigma\right). \tag{18}$$

If we study the relaxation process, i.e., consider the longitudinal deformation in a rod as constant, then operator $\tilde{E}$ will act on the unit Heaviside function $\tilde{E}H(t)$. With due account for

$$\int_0^\infty \gamma\left(t - \frac{t'}{\tau_i}\right)H(t')\,dt' = 1, \tag{19}$$

where

$$\gamma\left(t - \frac{t}{\tau_i}\right) = \frac{t^{\gamma-1}}{\tau_i^\gamma} \sum_{n=0}^\infty \frac{(-1)^n (t/\tau_i)^n}{\Gamma[\gamma(n+1)]} \tag{20}$$

is Rabotnov fractional exponential function [30] which at $\gamma = 1$ goes over into the traditional exponent, we obtain

$$\lim_{t\to 0} \tilde{E}H(t) = 9K_0, \quad \lim_{t\to \infty} \tilde{E}H(t) = \frac{9K_0\mu_0}{3K_0 + \mu_0} = E_0. \tag{21}$$

For operator $\tilde{\nu}H(t)$ we have [29]

$$\lim_{t\to 0} \tilde{\nu}H(t) = -1, \quad \lim_{t\to \infty} \tilde{\nu}H(t) = \frac{3K_0 - 2\mu_0}{2(3K_0 + \mu_0)} = \nu_0. \tag{22}$$

According to the classical theory of elasticity it has been considered that for the conventional materials the Poisson’s ratio $\nu$ varies within the interval $0 < \nu \leq 0.5$. However, nowadays a wide variety of so-called auxetic materials has been fabricated, including polymeric and metallic foams, microporous polymers, carbonfibre laminates and honeycomb structures [31], which possess such unusual mechanical property as a negative Poisson’s ratio, i.e., the Poisson’s ratio $\nu$ could vary within the interval $-1 < \nu \leq 0.5$. Therefore, the proposed model (10) with $\tilde{K} = K_0 = \text{const}$ could describe the behavior of viscoelastic auxetic materials, in so doing Poisson’s ratio could vary from $-1$ to its relaxed magnitude $\nu_0$.

In [29], substituting (16) and (18) in formula

$$\tilde{d} = \frac{\tilde{E}}{1 - \tilde{\nu}^2} \tag{23}$$

the rigidity operator was calculated, which is needed when solving different dynamic problems of thin plates and shells.

In conclusion of this subsection note that Zhu et al. [32, 33] for studying the dynamic response of a Timoshenko beam considered shear operator $\tilde{\mu}$ (10) and time-independent volumetric modulus (5) with further interpretation of the Young operator in the form of (11). The same model was adopted in [34] for quasi-static analysis of a Bernoulli-Euler beam.

However, the authors were not able to decode operator (11), although in order to it they would have to know only one formula from the algebra of fractional operators, namely: (15), where $\gamma^*\left(t_\gamma^\sigma\right)$ is defined in (14), and this formula even does not need demonstration, since it is evident.

So Poisson’s operator in [32] is given in the form

$$\tilde{\nu} = \frac{3K_0 - 2\mu_0[1 + (\tau_\sigma^\mu)^\gamma D^\gamma]}{6K_0 + 2\mu_0[1 + (\tau_\sigma^\mu)^\gamma D^\gamma]}, \tag{24}$$
Further the formulated problem was solved without expanding the expression (24), and deflection of the beam was found. However, expanding formula (24) and considering (14) yield
\[ \tilde{\nu} = \nu_0 (1 - \tau^\gamma D^\gamma) \mathcal{E}^\gamma (t_\sigma^\gamma), \]
where \( \nu_0 \) and \( t_\sigma^\gamma \) have been defined above, and
\[ \tau^\gamma = \frac{2\mu_0 (\tau^\mu) \gamma}{3K_0 - 2\mu_0}. \]

Now removing parenthesis in (25) and considering (15), we have
\[ \tilde{\nu} = \nu_0 \left\{ \mathcal{E}^\gamma (t_\sigma^\gamma) - \frac{\tau^\gamma}{t_\sigma^\gamma} [1 - \mathcal{E}^\gamma (t_\sigma^\gamma)] \right\} = \nu_0 \left[ -\frac{1}{\nu_0} + \left( 1 + \frac{1}{\nu_0} \right) \mathcal{E}^\gamma (t_\sigma^\gamma) \right], \]
that is
\[ \tilde{\nu} = -1 + (1 + \nu_0) \mathcal{E}^\gamma (t_\sigma^\gamma). \]

Since \( 1 + \nu_0 = E_0 (2\mu_0)^{-1} \), then (26) coincides with (18). Thus, the fact that for this model Poisson’s ratio could increase from \(-1\) to \(0\) (what is characteristic for viscoelastic auxetic materials) and then it could increase further from \(0\) to \(\nu_0\) (what is valid for ordinary viscoelastic materials with positive magnitudes of Poisson’s ratio) has not been revealed in [32–34].

Our reasoning is supported by [35], where it is noted that “\( \nu(0) = -1 \) indicates that such a material experiences lateral expansion under axial tension at the very beginning of the elongation process.”

### 2.3. Models considering volumetric relaxation

In the state-of-the-art papers [2, 26] it has been emphasized that in the majority of articles utilizing the fractional calculus models the second, or bulk, viscosity is not taken into account, in spite the fact that there are numerous evidence of its existence in scientific literature [36–38]. It is more important for modern composite materials [37].

One of the first papers in the field was published by Meshkov and Pachevskaya [36], wherein the attempt for considering the influence of the bulk relaxation on the internal friction phenomenon was carried out on the example of longitudinal harmonic vibrations of a three-dimensional hereditary elastic rod under the conditions of homogeneous deformation. A fractional exponential function has been used as a hereditary kernel. Investigation of the frequency dependence of the tangent of the phase shift between the stress and deformation, i.e., mechanical loss tangent, has revealed two peaks, namely, shear and bulk, in so doing the peak due to the shear deformation is five times larger than that resulting from the bulk deformation.

However, the account for the volumetric relaxation results in more cumbersome calculations. Thus, if operators \( \tilde{E} \) and \( \tilde{\mu} \), which are defined, respectively, by (3) and (10), are known, then first it is necessary to find operator \( \tilde{K}^{-1} \) utilizing the relationship
\[ \tilde{J} = \tilde{E}^{-1} = \frac{3\tilde{K} + \tilde{\mu}}{9\tilde{K} \tilde{\mu}} = \frac{1}{3} \tilde{\mu}^{-1} + \frac{1}{9} \tilde{K}^{-1}, \]
whence it follows that
\[ \tilde{K}^{-1} = 9\tilde{E}^{-1} - 3\tilde{\mu}^{-1} = a_1 \mathcal{E}^\gamma (\tau^\mu) \gamma - a_2 \mathcal{E}^\gamma (\tau^\sigma) \gamma, \]
where \( a_1 = 9E_0^{-1} \) and \( a_2 = 3\mu_0^{-1} \).
Now we could calculate the ratio of operators considering formulas (3) and (28)

\[
\frac{\tilde{E}}{K} = E_0 [1 + (\tau_{\sigma}^E)^\gamma D^\gamma] [a_1 \varphi_1^r (\tau_{\sigma}^E)^\gamma - a_2 \varphi_2^r (\tau_{\sigma}^E)^\gamma],
\]

or

\[
\frac{\tilde{E}}{K} = E_0 \left[ a_1 - a_2 \left( \frac{\tau_{\sigma}^E}{\tau_{\sigma}^\mu} \right)^\gamma \right] - E_0 a_2 \left[ 1 - \left( \frac{\tau_{\sigma}^E}{\tau_{\sigma}^\mu} \right)^\gamma \right] \varphi_1^r (\tau_{\sigma}^\mu)^\gamma. \tag{29}
\]

Computing operator \( \tilde{\nu} \) via formula

\[
\frac{\tilde{E}}{K} = 3(1 - 2\tilde{\nu}) \tag{30}
\]

and considering (29) yield

\[
\tilde{\nu} = \frac{1}{2} - \frac{E_0}{6} \left[ a_1 - a_2 \left( \frac{\tau_{\sigma}^E}{\tau_{\sigma}^\mu} \right)^\gamma \right] + \frac{E_0}{6} a_2 \left[ 1 - \left( \frac{\tau_{\sigma}^E}{\tau_{\sigma}^\mu} \right)^\gamma \right] \varphi_1^r (\tau_{\sigma}^\mu)^\gamma. \tag{31}
\]

Considering the product \( \nu(t) = \tilde{\nu} \cdot H(t) \), we find

\[
\nu(0) = \frac{1}{2} - \frac{E_0}{6} \left[ a_1 - a_2 \left( \frac{\tau_{\sigma}^E}{\tau_{\sigma}^\mu} \right)^\gamma \right] = -1 + \frac{E_0}{2\mu_0} \left( \frac{\tau_{\sigma}^E}{\tau_{\sigma}^\mu} \right)^\gamma, \tag{32}
\]

\[
\nu(\infty) = \frac{1}{2} + \frac{E_0}{6} (a_2 - a_1) = -1 + \frac{E_0}{2\mu_0} = \nu_0. \tag{33}
\]

This model with two preassigned operators \( \tilde{E} (3) \) and \( \tilde{\mu} (10) \) was used in [39] for the analysis of dynamic response of viscoelastic Timoshenko beam. Since the Timoshenko beam equation does not involve the Poisson’s ratio in the explicit form, then there was no need for the authors of [39] to find operators \( K \) and \( \tilde{\nu} \).

Reference to relationship (32) shows that the low limiting magnitude of Poisson’s ratio \( \nu(0) \) depends on the ratio of two retardation times and fractional parameter \( \gamma \), as distinct from the Kelvin–Voigt model without volumetric relaxation for which \( \nu(0) = -1 \) according to (22).

However note that when \( \left( \frac{\tau_{\sigma}^E}{\tau_{\sigma}^\mu} \right)^\gamma \to 0 \), then from (32) it follows that \( \nu(0) \to -1 \).

It means that the model under consideration could be used for modelling the features of viscoelastic auxetic materials until \( \left( \frac{\tau_{\sigma}^E}{\tau_{\sigma}^\mu} \right)^\gamma \to 2\mu_0/E_0 \) resulting in \( \nu(0) \to 0 \). For magnitudes \( \left( \frac{\tau_{\sigma}^E}{\tau_{\sigma}^\mu} \right)^\gamma \geq 2\mu_0/E_0 \), \( \nu(0) \geq 0 \) what corresponds to traditional viscoelastic materials.

Note that the standard linear solid model with two preassigned Lame operators \( \hat{\mu} \) and \( \hat{\lambda} \)

\[
\hat{\mu} = \mu_0 \frac{1 + (\tau_{\sigma}^\mu)^\gamma D^\gamma}{1 + (\tau_{\sigma}^\mu)^\gamma D^\gamma}, \quad \hat{\lambda} = \lambda_0 \frac{1 + (\tau_{\sigma}^\lambda)^\gamma D^\gamma}{1 + (\tau_{\sigma}^\lambda)^\gamma D^\gamma}, \tag{34}
\]

has been studied in [29].

In this case, the account for bulk relaxation results in the viscoelastic Poisson’s operator \( \tilde{\nu} \) which could be computed via formula

\[
\tilde{\nu} = \frac{\hat{\lambda}}{2(\hat{\lambda} + \hat{\mu})} = \nu_\infty [1 + M_1 \varphi_1^r (T_1^\gamma) + M_2 \varphi_2^r (T_2^\gamma)], \tag{35}
\]

where \( M_i \) and \( T_i \) \((i = 1, 2)\) are material’s parameters, whence it follows that viscoelastic Poisson’s ratio varies from \( \nu(t)|_{t \to 0} = \nu_\infty \) to \( \nu(t)|_{t \to \infty} = \nu_0 \).
It is also very important to note that if there exists some ‘hypothetical’ viscoelastic material with retardation times equal in magnitude $\tau_E^E = \tau^\mu$, then from (32) and (33) it is seen that such a material should possess the time-independent Poisson’s ratio $\bar{\nu} = \nu = \text{const}$. As a result, all existing viscoelastic operators are proportional each other, what, generally speaking, does not refer with all real viscoelastic materials. Some authors based their research on such a ‘hypothetical’ viscoelastic material [7, 40] and tried to solve different intricate problems of Mechanics of Solids with such a so-called viscoelastic material, what, broadly speaking, is inappropriate from the point of view of Mechanics.

The critical review of the role of Poisson’s ratio in linear viscoelasticity is presented by Tschoegl et al. [17] and Hilton [13, 16], wherein it has been shown that “no dynamic effects and no body forces, as well as no moving boundaries, i.e., no penetration or ablation problems, can be included” in the consideration adopting the assumption of time-independent Poisson’s ratios.

Conclusions
For the fractional derivative Kelvin–Voigt model involving the time-dependent Poisson’s operator it has been shown that such a model could describe the features of auxetic materials. For the model without bulk relaxation the Poisson’s ratio varies from $-1$ to a positive magnitude $\nu_0$, while for the model considering the volumetric relaxation the Poisson’s ratio increases from a negative magnitude larger than -1 (which depends on the retardation times) to a positive magnitude $\nu_0$.

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