Center Problem for the Group of Rectangular Paths

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Abstract
We solve the center problem for ODEs $\frac{dv}{dx} = \sum_{i=1}^{\infty} a_i(x) v^{i+1}$ such that the first integrals of vectors of their coefficients determine rectangular paths in finite dimensional complex vector spaces.

1. Introduction

1.1. The classical Poincaré Center-Focus problem for planar vector fields

$$\frac{dx}{dt} = -y + F(x, y), \quad \frac{dy}{dt} = x + G(x, y),$$

where $F$ and $G$ are real polynomials of a given degree without constant and linear terms asks about conditions on $F$ and $G$ under which all trajectories of (1.1) situated in a small neighbourhood of $0 \in \mathbb{R}^2$ are closed. It can be reduced passing to polar coordinates $(x, y) = (r \cos \phi, r \sin \phi)$ in (1.1) and expanding the right-hand side of the resulting equation as a series in $r$ (for $F$, $G$ with sufficiently small coefficients) to the center problem for the ordinary differential equation

$$\frac{dv}{dx} = \sum_{i=1}^{\infty} a_i(x) v^{i+1}, \quad x \in [0, 2\pi],$$

whose coefficients are trigonometric polynomials depending polynomially on the coefficients of $F$ and $G$.

More generally, consider equation (1.2) with coefficients $a_i$ from the Banach space $L^\infty(I_T)$ of bounded measurable complex-valued functions on $I_T := [0, T]$ equipped with the supremum norm. Condition

$$\sup_{x \in I_T, i \in \mathbb{N}} \sqrt{|a_i(x)|} < \infty$$

2000 Mathematics Subject Classification. Primary 37L10, Secondary 34C07.
Key words and phrases. Center problem, the group of rectangular paths, iterated integrals.
This can be solved by Picard iteration to obtain a solution $F \in \mathcal{L}_X$ whose coefficients in expansion in $W$ we set $\tilde{a}$ so that its end meets 0 and then taking it with the opposite orientation. We say that equation (1.2) determines a center guarantees that (1.2) has Lipschitz solutions on $X$ for all sufficiently small initial values. By $X$ we denote the complex Fréchet space of sequences $a = (a_1, a_2, \ldots)$ satisfying (1.3). We consider the set of paths with the standard operations of multiplication and taking the inverse. Then we introduce similar operations $*$ and $^{-1}$ on $X$ so that the correspondence $a \mapsto \tilde{a}$ is a monomorphism of semigroups.

Let $\mathbb{C}(X_1, X_2, \ldots)$ be the associative algebra with unit $I$ of complex noncommutative polynomials in $I$ and free noncommutative variables $X_1, X_2, \ldots$ (i.e., there are no nontrivial relations between these variables). By $\mathbb{C}(X_1, X_2, \ldots)[[t]]$ we denote the associative algebra of formal power series in $t$ with coefficients from $\mathbb{C}(X_1, X_2, \ldots)$. Also, by $\mathcal{A} \subset \mathbb{C}(X_1, X_2, \ldots)[[t]]$ we denote the subalgebra of series $f$ of the form

$$f = c_0 I + \sum_{n=1}^{\infty} \left( \sum_{i_1 + \cdots + i_k = n} c_{i_1, \ldots, i_k} X_{i_1} \cdots X_{i_k} \right) t^n$$

with $c_0, c_{i_1, \ldots, i_k} \in \mathbb{C}$ for all $i_1, \ldots, i_k, k \in \mathbb{N}$.

By $G \subset \mathcal{A}$ we denote the closed subset of elements $f$ of form (1.4) with $c_0 = 1$. We equip $\mathcal{A}$ with the adic topology determined by powers of the ideal $\mathcal{I} \subset \mathcal{A}$ of elements of form (1.4) with $c_0 = 0$. Then $(G, \cdot)$ is a topological group. Its Lie algebra $\mathcal{L}_G \subset \mathcal{A}$ is the vector space of elements of form (1.4) with $c_0 = 0$; here for $f, g \in \mathcal{L}_G$ their product is defined by the formula $[f, g] := f \cdot g - g \cdot f$. Also, the map $\exp : \mathcal{L}_G \to G$, $\exp(f) := e^f = \sum_{n=0}^{\infty} \frac{f^n}{n!}$, is a homeomorphism.

For an element $a = (a_1, a_2, \ldots) \in X$ consider the equation

$$F'(x) = \left( \sum_{i=1}^{\infty} a_i(x) X_i t^i \right) F(x), \quad x \in I_T.$$  

This can be solved by Picard iteration to obtain a solution $F_a : I_T \to G$, $F_a(0) = I$, whose coefficients in expansion in $X_1, X_2, \ldots$ and $t$ are Lipschitz functions on $I_T$. We set

$$E(a) := F_a(T), \quad a \in X.$$  

The product of paths $\tilde{a} \circ \tilde{b}$ is the path obtained by translating $\tilde{a}$ so that its beginning meets the end of $\tilde{b}$ and then forming the composite path. Similarly, $\tilde{a}^{-1}$ is the path obtained by translating $\tilde{a}$ so that its end meets 0 and then taking it with the opposite orientation.
Then
\[ E(a \ast b) = E(a) \cdot E(b), \quad a, b \in X. \] (1.7)

An explicit calculation leads to the formula
\[ E(a) = I + \sum_{n=1}^{\infty} \left( \sum_{i_1 + \cdots + i_k = n} I_{i_1, \ldots, i_k}(a) X_{i_1} \cdots X_{i_k} \right) t^n \] (1.8)

where
\[ I_{i_1, \ldots, i_k}(a) := \int \cdots \int_{0 \leq s_1 \leq \cdots \leq s_k \leq T} a_{i_k}(s_k) \cdots a_{i_1}(s_1) \, ds_k \cdots ds_1 \] (1.9)

are basic iterated integrals on \( X \).

The kernel of the homomorphism \( E : X \to G \) is called the set of universal centers of equation (1.2) and is denoted by \( U \). The elements of \( U \) are of a topological nature, see [Br1] for their description. The set of equivalence classes \( G(X) := X/\sim \) with respect to the equivalence relation \( a \sim b \iff a \ast b^{-1} \in U \) has the structure of a group so that the factor-map \( \pi : X \to G(X) \) is an epimorphism of semigroups. Moreover, for each function \( I_{i_1, \ldots, i_k} \) on \( X \) there exists a function \( \hat{I}_{i_1, \ldots, i_k} \) on \( G(X) \) such that \( \hat{I}_{i_1, \ldots, i_k} \circ \pi = I_{i_1, \ldots, i_k} \). In particular, there exists a monomorphism of groups \( \hat{E} : G(X) \to G \) defined by \( E = \hat{E} \circ \pi \), i.e.,
\[ \hat{E}(g) = I + \sum_{n=1}^{\infty} \left( \sum_{i_1 + \cdots + i_k = n} \hat{I}_{i_1, \ldots, i_k}(g) X_{i_1} \cdots X_{i_k} \right) t^n, \quad g \in G(X). \] (1.10)

We equip \( G(X) \) with the weakest topology in which all functions \( \hat{I}_{i_1, \ldots, i_k} \) are continuous. Then \( G(X) \) is a topological group and \( \hat{E} \) is a continuous embedding. The completion of the image \( \hat{E}(G(X)) \subset A \) is called the group of formal paths in \( C^\infty \) and is denoted by \( G_f(X) \). The group \( G_f(X) \) is defined by the Reeh shuffle relations for the iterated integrals. Its Lie algebra \( \mathcal{L}_{Lie} \) consists of all Lie elements of \( A \), see [Br2] for details.

Let \( G[[r]] \) be the set of formal complex power series \( f(r) = r + \sum_{i=1}^{\infty} d_i r^{i+1} \). Let \( d_i : G[[r]] \to \mathbb{C} \) be such that \( d_i(f) \) is the \((i+1)\)st coefficient in the series expansion of \( f \). We equip \( G[[r]] \) with the weakest topology in which all \( d_i \) are continuous functions and consider the multiplication \( \circ \) on \( G[[r]] \) defined by the composition of series. Then \( G[[r]] \) is a separable topological group. By \( G_c[[r]] \subset G[[r]] \) we denote the subgroup of power series locally convergent near 0 equipped with the induced topology. Next, we define the map \( P : X \to G[[r]] \) by the formula
\[ P(a)(r) := r + \sum_{i=1}^{\infty} \left( \sum_{i_1 + \cdots + i_k = i} p_{i_1, \ldots, i_k}(i) \cdot I_{i_1, \ldots, i_k}(a) \right) r^{i+1} \] (1.11)

where
\[ p_{i_1, \ldots, i_k}(t) := (t - i_1 + 1)(t - i_1 - i_2 + 1) \cdots (t - i + 1). \]

Then \( P(a \ast b) = P(a) \circ P(b) \) and \( P(X) = G_c[[r]] \). Moreover, let \( v(x; r; a) \), \( x \in I_T \), be the Lipschitz solution of equation (1.2) with initial value \( v(0; r; a) = r \). Clearly for
every \( x \in I_T \) we have \( v(x; r; a) \in G_e[[r]] \). It is shown in [Br1] that \( P(a) = v(T; \cdot; a) \) (i.e., \( P(a) \) is the first return map of (1.2)). In particular, we have

\[
a \in \mathcal{C} \iff \sum_{i_1, \ldots, i_k = i} p_{i_1, \ldots, i_k}(i) \cdot I_{i_1, \ldots, i_k}(a) \equiv 0 \quad \text{for all} \quad i \in \mathbb{N}. \quad (1.12)
\]

Equation (1.11) implies that there exists a continuous homomorphism of groups \( \hat{P} : G(X) \rightarrow G[[r]] \) such that \( P = \hat{P} \circ \pi \). Identifying \( G(X) \) with its image under \( \hat{E} \) we extend \( \hat{P} \) by continuity to \( G_f(X) \) retaining the same symbol for the extension.

We set \( \hat{\mathcal{C}} := \pi(\mathcal{C}) \) and define \( \hat{\mathcal{C}}_f \) as the completion of \( \hat{E}(\hat{\mathcal{C}}) \). Then \( \hat{\mathcal{C}}_f \) coincides with the kernel of the homomorphism \( \hat{P} \). The groups \( \hat{\mathcal{C}} \) and \( \hat{\mathcal{C}}_f \) are called the groups of centers and formal centers of equation (1.2).

It was established in [Br1], [Br2] that

\[
\gamma_{i_1, \ldots, i_k} \cdot c_{i_1, \ldots, i_k} = 0 \quad \text{for all} \quad n \in \mathbb{N} \quad \text{where} \quad \gamma_n = 1 \quad \text{and} \quad \gamma_n = 1 \quad \text{and} \quad (1.14)
\]

such that

\[
\sum_{i_1, \ldots, i_k = n} c_{i_1, \ldots, i_k} \cdot \gamma_{i_1, \ldots, i_k} = 0 \quad \text{for all} \quad n \in \mathbb{N} \quad \text{where} \quad \gamma_n = 1 \quad \text{and} \quad \gamma_n = 1 \quad \text{and} \quad (1.14)
\]

In particular, the map \( \exp: \mathcal{L}_{\hat{\mathcal{C}}_f} \rightarrow \hat{\mathcal{C}}_f \) is a homeomorphism.

2. Main results

2.1. Consider elements \( g \in G_f(X) \) of the form

\[
g = e^h \quad \text{where} \quad h = \sum_{i=1}^{\infty} c_i X_i t^i, \quad c_i \in \mathbb{C}, \quad i \in \mathbb{N}. \quad (2.1)
\]

By \( PL \subset G_f(X) \) we denote the group generated by all such \( g \). It is called the group of piecewise linear paths in \( C^\infty \). It was shown in [Br2, Proposition 3.14] that the

\[\text{This reflects the fact that the first integrals of the vectors of coefficients of formal equations (1.2) corresponding to elements of } PL \text{ are piecewise linear paths in } C^\infty.\]
group $\hat{C}_{PL} := PL \cap \hat{C}_f$ of piecewise linear centers is dense in $\hat{C}_f$. It was also asked about the structure of the set of centers represented by piecewise linear paths in $\mathbb{C}^n$ (i.e., represented by products of elements of form (2.1) with all $c_i = 0$ for $i > n$). In this paper we solve this problem for the group of rectangular paths.

2.2. Let $X_{rect} \subset X$ be a semigroup generated by elements $a \in X$ whose first integrals $\tilde{a}$ are rectangular paths in $\mathbb{C}^\infty$ consisting of finitely many segments parallel to the coordinate axes. The image of $X_{rect}$ under homomorphism $E$ is called the group of rectangular paths and is denoted by $G(X_{rect})$. Clearly, $G(X_{rect}) \subset PL$. It is generated by elements $e^{(a_n T)}X_n t^n \in G_f(X)$, $a_n \in \mathbb{C}$, $n \in \mathbb{N}$. Since there are no nontrivial relations between these elements (with $a_n \neq 0$), the group $G(X_{rect})$ is isomorphic to the free product of countably many copies of $\mathbb{C}$. Also, the group $G(X_{rect}) \subset G(X)$ is dense in $G_f(X)$ and the elements $X_n t^n$, $n \in \mathbb{N}$, form a generating subset of the Lie algebra $L_{Lie}$, see [Br2]. Moreover, for each $g \in G(X_{rect})$ the first return map $\hat{P}(g) \in G_c[[r]]$ can be explicitly computed and represents an algebraic function. Specifically, for the equation

$$\frac{dv}{dx} = a_n v^{n+1} \quad (2.2)$$

corresponding to the element $g_n := e^{(a_n T)}X_n t^n$ an explicit calculation shows that its first return map is given by the formula

$$\hat{P}(g_n)(r) := \frac{r}{\sqrt{1 - na_n Tr^n}} = r + \sum_{j=1}^{\infty} \frac{(-1)^j(n(j-1)+1)(n(j-2)+1)\cdots1}{j!} (a_n T)^j r^{nj+1}. \quad (2.3)$$

Here $\sqrt[\cdot]{r} : \mathbb{C} \setminus (-\infty, 0] \to \mathbb{C}$ stands for the principal branch of the power function. Then for a generic $g \in G(X_{rect})$, the first return map $\hat{P}(g)$ is the composition of series of form (2.3).

Our main result is

**Theorem 2.1.** The restriction $\hat{P}|_{G(X_{rect})} : G(X_{rect}) \to G_c[[r]]$ is a monomorphism. In particular, $\hat{C} \cap G(X_{rect}) = \{1\}$ and $\mathcal{C} \cap X_{rect} \subset U$. Moreover, $\mathcal{C} \cap X_{rect}$ consists of elements $a \in X_{rect}$ such that their first integrals $\tilde{a} : [0, T] \to \mathbb{C}^\infty$ are rectangular paths modulo cancellations\footnote{i.e., forgetting sub-paths of a given path consisting of a segment and then immediately of the same segment going in the opposite direction.} representing the constant path $[0, T] \to (0, 0, \ldots) \in \mathbb{C}^\infty$.

**Proof.** The proof of Theorem 2.1 is based on the deep result of S. Cohen [C].

Consider an irreducible word $g = g_k \cdots g_k \in G(X_{rect})$ where $g_k := e^{(a_k T)}X_k t^{k_k}$ and $a_k \in \mathbb{C}^* := \mathbb{C} \setminus \{0\}$. We must show that $\hat{P}(g) = \hat{P}(g_k) \circ \cdots \circ \hat{P}(g_k) \neq 1$; here 1 is the unit of the group $G_c[[r]]$. Assume, on the contrary, that $\hat{P}(g) = 1$. Then
from equation (2.3) for all \( r \in \mathbb{C} \) sufficiently close to 0 we obtain

\[
\hat{P}(g)(r) = \left( \prod \left( 1 + b_{k_i} r^{k_i} \right)^{\frac{k_{i-1}}{k_i}} + b_{k_{i-1}} r^{k_{i-1}} + \cdots \right)^{\frac{k_1}{k_2}} \frac{1}{r^{\frac{1}{k_1}}}.
\] (2.4)

Here we set \( b_{k_i} := -k_i a_k T \).

Making the substitution \( t = \frac{1}{r} \) from equations \( \hat{P}(g) = 1 \) and (2.4) we get for all sufficiently large positive \( t \)

\[
\left( \prod \left( t^{k_i} + b_{k_i} \right)^{\frac{k_{i-1}}{k_i}} + b_{k_{i-1}} t^{k_{i-1}} + \cdots \right)^{\frac{k_1}{k_2}} + b_{k_1} \right) \frac{1}{t^{\frac{1}{k_1}}} = t
\] (2.5)

Here from the irreducibility of \( g \) we obtain that \( k_i \neq k_{i+1} \) for all \( 1 \leq i \leq l - 1 \).

Consider the multi-valued algebraic function over \( \mathbb{C} \) defined by the left-hand side of equation (2.5). Then there exist a connected Riemann surface \( S \), a finite surjective holomorphic map \( \pi : S \to \mathbb{C} \) and a (single-valued) holomorphic function \( f \) defined on \( S \) such that the pullback by \( \pi^{-1} \) of the restriction of \( f \) to a suitable open subset of \( S \) corresponds to the branch of the original function satisfying (2.5) defined on an open subset of \( \mathbb{C} \) containing a ray \([R, \infty)\) for \( R \) sufficiently large. Equation (2.5) implies that \( f \) coincides with the pullback \( \pi^* z \) of the function \( z \) on \( \mathbb{C} \).

Let \( P \) be the abelian group of maps \( \mathbb{C} \to \mathbb{C} \) generated by \( \{ x \mapsto x^p ; p \in \mathbb{N} \} \). Then there exist a connected Riemann surface \( S \), a finite surjective holomorphic map \( \pi : S \to \mathbb{C} \) and a (single-valued) holomorphic function \( f \) defined on \( S \) such that the pullback by \( \pi^{-1} \) of the restriction of \( f \) to a suitable open subset of \( S \) corresponds to the branch of the original function satisfying (2.5) defined on an open subset of \( \mathbb{C} \) containing a ray \([R, \infty)\) for \( R \) sufficiently large. Equation (2.5) implies that \( f \) coincides with the pullback \( \pi^* z \) of the function \( z \) on \( \mathbb{C} \).

Let \( T_{\mathbb{C}} \) be the abelian group of the maps \( \{ x \mapsto x + a ; a \in \mathbb{C} \} \). Then Theorem 1.5 of [C] states that the group of complex maps of \( \mathbb{C} \) generated by \( P \) and \( T_{\mathbb{C}} \) is their free product \( P \ast T_{\mathbb{C}} \).

Next, the function \( h : \mathbb{C} \to \mathbb{C} \) defined by the left-hand side of (2.5) belongs to the group generated by \( P \) and \( T_{\mathbb{C}} \) (in the definition of \( h \) we define the fractional powers as in the above cited theorem). In turn, by the definition of \( S \) there exists a subset \( U \) of \( S \) such that \( \pi : U \to \mathbb{C} \) is a bijection and \( f \circ (\pi|_U)^{-1} = h \). Since \( f = \pi^* z \), the latter implies that \( h(t) = t \) for all \( t \in \mathbb{C} \). But according to our assumptions the word in \( P \ast T_{\mathbb{C}} \) representing \( h \) is irreducible. Thus it cannot be equal to the unit element of this group.

This contradiction shows that \( \hat{P}(g) \neq 1 \) and proves the first statement of the theorem.

The other statements follow straightforwardly from the corresponding definitions. We leave the details to the reader.

**Remark 2.2.** Theorem 2.1 implies that \( G[[r]] \) contains a dense subgroup isomorphic to the free product of countably many copies of \( \mathbb{C} \) generated by series of form (2.3).
In fact, the subgroup of \( G[[r]] \) generated by series of form (2.3) with \( n = 1 \) and \( n = 2 \) only is already dense in \( G[[r]] \). It follows, e.g., from [Br2, Proposition 3.11].

2.3. Let \( a \in X_{rect} \) be such that \( \hat{E}(\pi(a)) = e^{a_{k_1}X_{k_1}t^{k_1}} \cdots e^{a_{k_l}X_{k_l}t^{k_l}} \in G_f(X) \) for some \( a_{k_1}, \ldots, a_{k_l} \in \mathbb{C} \), i.e., the path \( \tilde{a} : [0, T] \to \mathbb{C}^\infty \) consists of \( l \) segments parallel to the coordinate axes \( z_{k_1}, \ldots, z_{k_l} \) of \( \mathbb{C}^\infty \). Considering \( a_{k_1}, \ldots, a_{k_l} \) as complex variables in \( \mathbb{C}^l \) we obtain a family \( \mathcal{F} \) of rectangular paths. The first return maps \( \hat{P}(a) \) of elements of \( \mathcal{F} \) can be computed by expanding the functions \( e^{a_{k_1}X_{k_1}t^{k_1}} \cdots e^{a_{k_l}X_{k_l}t^{k_l}} \) in infinite series in variables \( X_{k_i}t^{k_s} \), \( 1 \leq s \leq l \), then replacing each \( X_{k_i} \) by \( DL^{k_i-1} \) where \( D \) and \( L \) are the differentiation and the left translation in the algebra of formal power series \( \mathbb{C}[[z]] \), and then evaluating the resulting series in \( D, L, t \) at elements \( z^p \), see [Br1] for similar arguments. As a result we obtain (with \( t \) substituted for \( r \))

\[
\hat{P}(a)(r) = r + \sum_{i=1}^{\infty} \left( \sum_{k_1s_1 + \cdots + k_is_i = i} q_{k_1; s_1, \ldots, k_is_i}(i) \frac{a_{k_1}^{s_1}}{s_1!} \cdots \frac{a_{k_i}^{s_i}}{s_i!} \right) r^{i+1}
\]

(2.6)

(here we set for convenience \( s_0k_0 := 0 \)).

By \( c_i(a_{k_1}, \ldots, a_{k_l}) \) we denote the coefficient at \( r^{i+1} \) of \( \hat{P}(a) \). It is a holomorphic polynomial on \( \mathbb{C}^l \). The center set \( C \) of equations (1.2) corresponding to the family \( \mathcal{F} \) is the intersection of sets of zeros \( \{ (a_{k_1}, \ldots, a_{k_l}) \in \mathbb{C}^l; c_i(a_{k_1}, \ldots, a_{k_l}) = 0 \} \) of all polynomials \( c_i \). According to Theorem 2.1 \((a_{k_1}, \ldots, a_{k_l}) \in C \) if and only if the word \( e^{a_{k_1}t^{k_1}X_{k_1}} \cdots e^{a_{k_l}t^{k_l}X_{k_l}} = I \) in \( \mathcal{A} \). Since the groups generated by elements \( e^{a_{k_{p_1}}t^{p_1}X_{k_{p_1}}} \cdots e^{a_{k_{p_m}}t^{p_m}X_{k_{p_m}}} \) with mutually distinct \( X_{k_{p_j}} \) and nonzero numbers \( a_{k_{p_j}} \) are free, the last equation implies that \( C \) is the union of finitely many complex subspaces of \( \mathbb{C}^l \). (For instance, if all \( k_j \) are mutually distinct, then \( C = \{ 0 \} \subset \mathbb{C}^l \).

Our next result gives an effective bound on the number of coefficients in (2.6) determining the center set \( C \).

**Theorem 2.3.** The set \( C \subset \mathbb{C}^l \) is determined by equations \( c_1 = 0, \ldots, c_{d+1} = 0 \) where

\[
d := \prod_{i=1}^{l-1} \frac{k_i}{\gcd(k_i, k_{i+1})}
\]

(here \( \gcd(n, m) \) is the greatest common divisor of natural numbers \( n \) and \( m \)).

**Proof.** Since \( c_i(z^{k_1a_{k_1}}, \ldots, z^{k_la_{k_l}}) = z^i c_i(a_{k_1}, \ldots, a_{k_l}) \), \( i \in \mathbb{N} \), it suffices to prove that \( C \cap B \) where \( B \subset \mathbb{C}^l \) is the open unit Euclidean ball is determined by equations \( c_1 = 0, \ldots, c_{d+1} = 0 \).

Next, there exists a positive number \( R \) such that for each \( (a_{k_1}, \ldots, a_{k_l}) \in B \) the first return map \( \hat{P}(a) \) given by (2.6) determines a holomorphic function on \( \mathbb{D}_R := \{ z \in \mathbb{C}; |z| < R \} \). On the other hand, according to (2.3), \( \hat{P}(a) \) is the composite of algebraic functions. Now, from formula (2.4) we obtain straightforwardly that
equation $f(a)(r) = c$, $f(a)(r) := \frac{R(a)(r) - r}{r}$, has at most $d$ complex roots in $\mathbb{D}_R$ (counted with their multiplicities), i.e., the valency of $f(a)$ on $\mathbb{D}_R$ is at most $d$. From here and the result of Hayman [H, Theorem 2.3] we obtain

$$\sup_{r \in \mathbb{D}_R^2} |f(a)(r)| \leq e^{\frac{1+\pi^2}{2}} \cdot \frac{R^{d+1} - 1}{R - 1} \cdot \max_{1 \leq k \leq d+1} |c_k(a_k, \ldots, a_k)|. \quad (2.7)$$

This implies that if $c_1(a_k, \ldots, a_k) = \cdots = c_{d+1}(a_k, \ldots, a_k) = 0$, then $f(a) \equiv 0$, i.e., $(a_k, \ldots, a_k) \in \mathcal{C}$.

The proof is complete. \qed

**Remark 2.4.** It follows from equations $c_i(z^k a_k, \ldots, z^l a_l) = z^i c_i(a_k, \ldots, a_k)$, $i \in \mathbb{N}$, inequality (2.7) and the Cauchy integral formula for derivatives of $f(a)$ that for any $k \in \mathbb{N}$ and all $\lambda \in \mathbb{C}^l$

$$|c_{d+1+k}(\lambda)| \leq e^{\frac{1+\pi^2}{2}} \cdot \frac{R^{d+1} - 1}{R - 1} \cdot \left(\frac{2\sqrt{l}}{R}\right)^{d+k} \cdot (1 + \|\lambda\|_2)^{d+k} \cdot \max_{1 \leq k \leq d+1} |c_k(\lambda)|$$

where $\| \cdot \|_2$ is the Euclidean norm on $\mathbb{C}^l$.

This and Proposition 1.1 of [HRT] imply that each polynomial $c_{d+1+k}$ belongs to the integral closure $\mathcal{T}$ of the polynomial ideal $\mathcal{I}$ generated by $c_1, \ldots, c_{d+1}$.

An interesting question is: in which cases $\mathcal{T} = \mathcal{I}$? (If this is true, then the (Bautin) polynomial ideal generated by all polynomials $c_i$ is, in fact, generated by $c_1, \ldots, c_{d+1}$.)

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