A NOTE ON TWISTED DIRAC OPERATORS ON CLOSED SURFACES

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Abstract. We derive an inequality that relates nodal set and eigenvalues of a class of twisted Dirac operators on closed surfaces. Moreover, we apply this technique to the twisted Dirac operator appearing in the context of Dirac-harmonic maps.

1. Introduction and results

Throughout this note we assume that \((M, g)\) is a closed Riemannian spin surface with fixed spin structure. On the spinor bundle \(\Sigma M\) we have a metric connection \(\nabla^{\Sigma M}\) induced from the Levi-Cevita connection and we fix a hermitian scalar product. Sections in the spinor bundle are called spinors. In addition, we have the Clifford multiplication of spinors with tangent vectors, denoted by \(X \cdot \psi\) for \(X \in TM\) and \(\psi \in \Gamma(\Sigma M)\). Clifford Multiplication is skew-symmetric

\[
\langle X \cdot \psi, \xi \rangle_{\Sigma M} = -\langle \psi, X \cdot \xi \rangle_{\Sigma M}
\]

and satisfies the Clifford relations

\[
X \cdot Y \cdot \psi + Y \cdot X \cdot \psi = -2g(X, Y)\psi
\]

for \(X,Y \in TM\) and \(\psi, \xi \in \Sigma M\).

The Dirac operator is a first order differential operator acting on spinors and is given by

\[
D := e_1 \cdot \nabla^{\Sigma M}_{e_1} + e_2 \cdot \nabla^{\Sigma M}_{e_2},
\]

where \(e_1, e_2\) is an orthonormal basis of \(TM\). It is an elliptic differential operator, which is self-adjoint with respect to the \(L^2\)-norm. On a compact Riemannian manifold the spectrum of the Dirac operator is discrete and consists of both positive and negative eigenvalues. In general, the spectrum cannot be computed explicitly. However, it is possible to estimate the spectrum.

On a \(n\)-dimensional manifold there is Friedrich’s inequality

\[
\lambda^2 \geq \frac{n}{4(n-1)} \inf_M R.
\]

Here, \(R\) denotes the scalar curvature of the manifold. On closed surfaces the following inequality was given by Bär in [1]

\[
\lambda^2 \geq \frac{2\pi \chi(M)}{\Vol(M, g)},
\]

where \(\chi(M)\) is the Euler characteristic of \(M\). For more details on the spectrum of the Dirac operator see the book [13].

Moreover, there exists the canonical splitting of the spinor bundle \(\Sigma M\) into the bundle of positive spinors \(\Sigma^+ M\) and the bundle of negative spinors \(\Sigma^- M\), that is \(\Sigma M = \Sigma^+ M \oplus \Sigma^- M\).

Using the complex-volume form \(\omega_C = ie_1 \cdot e_2\) we can decompose the spinor bundle into its positive and negative parts, that is

\[
\Sigma^{\pm} M = \frac{1}{2} (1 \pm \omega_C) \cdot \Sigma M.
\]

The Dirac type operators that appear in theoretical physics usually do not act on sections in \(\Sigma M\) but on sections in \(\Sigma M \otimes E\), where \(E\) is a given vector bundle. We call \(\Sigma M \otimes E\) twisted spinor bundle. The connections on \(\Sigma M\) and \(E\) induce a connection on \(\Sigma M \otimes E\) and the same holds true for the scalar product on \(\Sigma M \otimes E\).
The twisted Dirac operator $D^E$ maps sections of $\Sigma M \otimes E$ to sections of $\Sigma M \otimes E$ and is given by

$$D^E := e_1 \cdot \nabla_{e_1} \Sigma M \otimes E + e_2 \cdot \nabla_{e_2} \Sigma M \otimes E.$$  

The principal symbol of $D^E$ can be computed as

$$\sigma_1(D^E) = \sigma_1(D, \eta) \otimes id_E,$$

hence this operator is still elliptic. Moreover, $D^E$ is formally self-adjoint whenever we have a metric connection on $E$. The square of the twisted Dirac operator satisfies the following Weitzenböck formula

$$(D^E)^2 = \nabla^* \nabla \Sigma \otimes E + \frac{R}{4} + \frac{1}{2} \sum_{i,j=1}^{2} e_i \cdot e_j \cdot R^E(e_i, e_j),$$

where $R$ denotes the scalar curvature of $M$, $R^E$ the curvature endomorphism of the twist bundle $E$ and $E_i, i = 1, 2$ is an orthonormal frame field. For a derivation of this formula see [19], p.164. For more background material on spin geometry and the Dirac operator we refer the reader to the books [19] and [11].

We will always assume that we have a metric connection on $E$. A twisted spinor $\psi \in \Gamma(\Sigma M \otimes E)$ is called eigenspinor of the twisted Dirac operator $D^E$ with eigenvalue $\lambda$ if it satisfies

$$D^E \psi = \lambda \psi.$$  

The twisted Dirac operator $D^E$ interchanges positive and negative twisted spinors, more precisely $D^E: \Sigma^\pm M \otimes E \to \Sigma^\mp M \otimes E$, whereas $(D^E)^2: \Sigma^\pm M \otimes E \to \Sigma^\mp M \otimes E$ does not.

Of course, one cannot expect to compute the spectrum of the twisted Dirac operator explicitly. Nevertheless some eigenvalue estimates exist, see for example [17], [18], [16], [20] and [21]. By the main result of [2] we know that the nodal set of solutions to semilinear elliptic equations of first order is discrete, this result can be applied to all equations considered in this note.

For a complex line bundle $E$ we will prove the following

**Theorem 1.1.** Let $(M, g)$ be a closed spin surface with fixed spin structure and let $\Sigma M \otimes E$ be the spinor bundle twisted by a complex line bundle $E$ with metric connection. Suppose that $\lambda_k^2$ is the $k$-th eigenvalue of $(D^E)^2$ acting on sections in $\Sigma^\pm M \otimes E$. Then the following eigenvalue estimate holds

$$\lambda_k^2 \geq \frac{2\pi \chi(M)}{\text{Vol}(M, g)} \pm \frac{4\pi \deg(E)}{\text{Vol}(M, g)} + \frac{4\pi N_k}{\text{Vol}(M, g)},$$

where $\chi(M)$ is the Euler characteristic of $M$ and $\deg(E)$ the degree of the complex line bundle $E$. The sign in $(-2)$ depends on whether $\psi$ is a positive or negative spinor. Moreover $N_k$ denotes the sum of the order of the zero’s of an eigenspinor $\psi_k$ belonging to the $k$-th eigenvalue of the twisted Dirac operator, that is

$$N_k = \sum_{p \in M, \left|\psi_k(p)\right| = 0} n_p.$$  

**Remark 1.2.** There are several cases in which we have equality in (1.2). On the one hand twisted Killing spinors, which do not have any zeros, realize equality. On the other hand, we will see in some explicit examples that equality can also hold in the case of $\lambda = 0$.

**Corollary 1.3.** Of course, (1.2) can also be interpreted as an estimate on the nodal set of an eigenspinor $\psi_k$ belonging to the $k$-th eigenvalue of $(D^E)^2$, that is

$$N_k \leq \frac{\text{Vol}(M, g) \lambda_k^2}{4\pi} \mp \frac{\chi(M)}{2} \mp \deg(E).$$

**Remark 1.4.** A similar estimate for the classical Dirac operator acting on spinors was derived in [5], see also [15] for a more detailed analysis. Moreover, our main result generalizes some of the eigenvalue estimates from [18].
2. Proof of the main Theorem

By the main result of [2] we know that on a two-dimensional manifold the zero-set of twisted eigenspinors is discrete. In the following we will make use of the energy-momentum tensor $T^E(X,Y)$

\[ T^E(X,Y) := \langle X \cdot \nabla_Y^{\Sigma M \otimes E} \psi + Y \cdot \nabla_X^{\Sigma M \otimes E} \psi, \psi \rangle_{\Sigma M \otimes E}. \]

This tensor arises if one varies the functional $E(\psi) = \int_M \langle \psi, D^E \psi \rangle dM$ with respect to the metric. The following Lemma is a generalization of Lemma 5.1 from [12].

Lemma 2.1. For all $\psi \in \Gamma(\Sigma M \otimes E)$ the following inequality holds

\[ \frac{\langle \psi, (D^E)^2 \psi \rangle}{|\psi|^2} \geq \frac{R}{4} + \frac{1}{2} \sum_{i,j=1}^2 \frac{\langle e_i \cdot e_j \cdot R^E(E_i, E_j) \psi, \psi \rangle}{|\psi|^2} + \frac{|T^E|^2}{4|\psi|^4} - \Delta \log |\psi|. \]

Proof. We define a modified connection on $\Sigma \otimes E$ by

\[ \tilde{\nabla}_X^{\Sigma M \otimes E} \psi := \nabla_X^{\Sigma M \otimes E} \psi - 2\alpha(X)\psi - \beta(X) \cdot \psi - X \cdot \alpha \cdot \psi \]

with a one-form $\alpha$ and a symmetric $(1,1)$-tensor $\beta$ given by

\[ \alpha := \frac{d|\psi|^2}{2|\psi|^2}, \quad \beta := -\frac{T^E(\cdot, \cdot)}{2|\psi|^2}. \]

By a direct computation we then find summing over repeated indices

\[ |\tilde{\nabla}^{\Sigma M \otimes E} \psi|^2 = \langle \psi, (D^E)^2 \psi \rangle - \frac{R}{4} |\psi|^2 - \frac{1}{2} \sum_{i,j=1}^2 \langle e_i \cdot e_j \cdot R^E(E_i, E_j) \psi, \psi \rangle + \frac{1}{2} \Delta |\psi|^2, \]

\[ \alpha_{ei} \langle \nabla_{ei}^{\Sigma M \otimes E} \psi, \psi \rangle = |\alpha|^2 |\psi|^2 = \frac{|d|\psi|^2|^2}{4|\psi|^2}, \]

\[ \langle \beta(e_i) \cdot \nabla_{ei}^{\Sigma M \otimes E} \psi, \psi \rangle = -\frac{|T^E|^2}{4|\psi|^2}. \]

Thus, we arrive at (summing over repeated indices)

\[ 0 \leq |\tilde{\nabla}^{\Sigma M \otimes E} \psi|^2 = \langle \psi, (D^E)^2 \psi \rangle - \frac{R}{4} |\psi|^2 + \frac{1}{2} \sum_{i,j=1}^2 \frac{\langle e_i \cdot e_j \cdot R^E(E_i, E_j) \psi, \psi \rangle}{|\psi|^2} + \frac{1}{2} \Delta |\psi|^2 - \frac{|d|\psi|^2|^2}{2|\psi|^2} + \frac{|T^E|^2}{4|\psi|^2} \]

yielding the result. \qed

Lemma 2.2. Suppose $M$ is a closed Riemannian surface. If the zero set of $|\psi|$ is discrete and $|\psi|$ does not vanish identically, then the following equality holds

\[ \int_M \Delta \log |\psi| dM = -2\pi \sum_{p \in M, |\psi|(p) = 0} n_p, \]

where $n_p$ is the order of $|\psi|$ at the point $p$.

Proof. A proof can be found in [23]. \qed

As a next step we analyze the term arising from the curvature of the twist bundle $E$. Using the skew symmetry of the curvature endomorphism and the Clifford multiplication we get

\[ \sum_{i,j=1}^2 e_i \cdot e_j \cdot R^E(E_i, E_j) \psi = 2e_1 \cdot e_2 \cdot R^E(E_1, E_2) \psi = -2i\omega_C \cdot R^E(E_1, E_2) \psi = 4\pi c_1(E)\omega_C \cdot \psi, \]
where $c_1(E)$ denotes the first Chern-form of the complex line bundle $E$, see [22], p.303 ff. for more details. Hence, we find

$$
\frac{1}{2} \sum_{i,j=1}^{2} \frac{\langle e_i \cdot e_j \cdot R^E(E_i, E_j) \psi, \psi \rangle}{|\psi|^2} = 2\pi c_1(E) \frac{\langle \omega_C \cdot \psi, \psi \rangle}{|\psi|^2} = \pm 2\pi c_1(E),
$$

where the sign depends on if $\psi$ is a positive or negative spinor. Now, we apply Lemma 2.1 in the case that $\psi$ is an eigenspinor of $(D^E)^2$. We can estimate the energy momentum tensor by

$$
|T|_2 \geq 2 \lambda^2 |\psi|^4.
$$

Thus, from (2.1) we obtain

$$
\lambda^2 \geq K \pm 4\pi c_1(E) - 2\Delta \log |\psi|,
$$

where $K$ denotes the Gaussian curvature of $M$. Remember that the degree of a vector bundle is given by

$$
\deg(E) = \int_M c_1(E) dM.
$$

Thus, by integrating over the surface $M$ and using (2.2) we obtain Theorem 1.1.

Let us briefly discuss one equality case, which occurs if $\psi$ is a twisted Killing spinor, that is a solution of

$$
\nabla_{X}^\Sigma M \otimes \phi^*TN \psi = -\lambda X \cdot \psi.
$$

It can easily be checked by differentiating $|\psi|^2$ and using the equation (2.3) that a twisted Killing spinor has constant norm and thus does not have any zeros. Hence, we obtain equality in (2.1). Moreover, the energy momentum tensor satisfies

$$
|T|_2 = 2\lambda^2 |\psi|^4,
$$

leading to

$$
\lambda^2 = K + 2 \sum_{i,j=1}^{2} \frac{\langle e_i \cdot e_j \cdot R^E(E_i, E_j) \psi, \psi \rangle}{|\psi|^2}.
$$

Performing the same analysis of the curvature of the line bundle as before we then obtain

$$
\lambda^2 = \frac{2\pi \chi(M)}{Vol(M, g)} \pm \frac{4\pi \deg(E)}{Vol(M, g)}.
$$

3. Applications

3.1. Dirac-harmonic maps from surfaces. In this section we focus on Dirac-harmonic maps [9] from surfaces. Thus, let $N$ be another Riemannian manifold, let $\phi: M \to N$ be a map and $E$ be the pull-back bundle $\phi^*TN$ over $M$. We are considering so-called vector spinors, that is $\psi \in \Gamma(\Sigma M \otimes \phi^*TN)$. Dirac-harmonic maps are critical points of the energy functional (with the first term being the usual Dirichlet energy)

$$
E(\phi, \psi) = \frac{1}{2} \int_M (|d\phi|^2 + \langle \psi, D^{\phi^*TN} \psi \rangle) dM
$$

and satisfy the Euler-Lagrange equations

$$
\tau(\phi) = \frac{1}{2} \sum_{i=1}^{2} R^N(e_i \cdot \psi, \psi) d\phi(e_i), \quad D^{\phi^*TN} \psi = 0.
$$

Here, $\tau(\phi)$ is the tension field of the map $\phi$, $R^N$ the curvature tensor on $N$ and $e_i$ is an orthonormal basis of $TM$.

**Proposition 3.1.** Let $M$ be a closed Riemannian spin surface with fixed spin structure and $N$ another Riemannian manifold. Suppose that $(\phi, \psi)$ is a smooth non-trivial Dirac-harmonic from $M$ to $N$. Then the following inequality holds

$$
|R^N|_{L^\infty} \int_M |d\phi|^2 dM \geq \pi \chi(M) + 2\pi N_0,
$$

where $N_0$ denotes the sum of the order of the zero’s of $\psi$. 

Proof. First of all we note that
\[ \sum_{i,j=1}^{2} \frac{1}{2} \langle e_i \cdot e_j \cdot R^N(d\phi(e_i),d\phi(e_j)\psi,\psi) \rangle \geq -|R^N|_{L^\infty} |d\phi|^2. \]
Using (2.1) with \( E = \phi^*TN \) and the assumption \( D^\phi^*TN \psi = 0 \) we get
\[ |R^N|_{L^\infty} |d\phi|^2 \geq \frac{R}{4} - \Delta \log |\psi|. \]
The result then follows by integration. \( \Box \)

Making use of Proposition 3.1 we obtain the following

**Corollary 3.2.** Let \( M \) be a closed Riemannian spin surface and \( N \) another Riemannian manifold. Suppose that \((\phi,\psi)\) is a smooth non-trivial Dirac-harmonic from \( M \) to \( N \). If \( \chi(M) > 0 \) and the energy of \( \phi \) is sufficiently small, that is
\[ \int_M |d\phi|^2 dM < \epsilon \]
for some small \( \epsilon > 0 \) then the pair \((\phi,\psi)\) is trivial.

**Proof.** The triviality of \( \psi \) follows from (3.1), for the vanishing of the map \( \phi \) see [6], Lemma 4.9. \( \Box \)

If both \( M = N = S^2 \) it was proven in [26] that the Euler-Lagrange equations for Dirac-harmonic maps decouple, that is
\[ \tau(\phi) = 0 = 1 \sum_{i=1}^{2} R^N(e_i \cdot \psi,\psi)d\phi(e_i), \quad D^\phi^*TN \psi = 0 \]
and making use of this fact all non-trivial Dirac-harmonic maps between spheres could be classified. Moreover, it is well-known that harmonic maps between two-spheres are holomorphic or anti-holomorphic maps with energy
\[ \frac{1}{2} \int_M |d\phi|^2 dM = E(\phi) = 4\pi |\deg(\phi)|. \]
Hence, we obtain the following inequality for uncoupled Dirac-harmonic maps between spheres
\[ 4|R^N|_{L^\infty} |\deg(\phi)| \geq 1 + N_0. \]
(3.2)

### 3.2. Dirac-harmonic maps between surfaces.

In this section we apply Theorem 1.1 in the case of Dirac-harmonic maps between surfaces. More precisely, we assume that \( N \) is a closed, connected and oriented Riemannian surface. Thus, we can think of \( TN \) as a complex line bundle. In particular, it is interesting to analyze if the eigenvalue 0 is in the spectrum of \( D^\phi^*TN \).

**Proposition 3.3.** Let \( E = \phi^*TN \) and suppose that \( \psi \in \Gamma(\Sigma \pm M \otimes \phi^*TN) \). Then all eigenvalues of \((D^\phi^*TN)^2\) satisfy the following inequality
\[ \lambda_k^2 \geq \frac{2\pi \chi(M)}{\Vol(M,g)} \pm \frac{4\pi \deg(\phi) \chi(N)}{\Vol(M,g)} + \frac{4\pi N_k}{\Vol(M,g)}, \]
(3.3)
where \( \deg(\phi) \) denotes the degree of the map \( \phi \) and \( \chi(N) \) is the Euler-characteristic of \( N \). The sign in front of the second term depends on if \( \psi \) is a positive or negative vector spinor.

**Proof.** We apply Theorem 1.1 in the case that \( E = \phi^*TN \). By the properties of the first Chern-form we find
\[ \deg(\phi^*TN) = \int_M c_1(\phi^*TN) dM = \int_M \phi^* c_1(TN) dM = \frac{1}{2\pi} \int_M \phi^* K^N dM. \]
Using the degree theorem we get
\[ \int_M \phi^* K^N dM = \deg(\phi) \int_N K^N dN = 2\pi \deg(\phi) \chi(N), \]
which proves the result. \( \Box \)
Remark 3.4. Note that for \( \lambda_0 = 0 \), which corresponds to \( D^{\phi^*TN}\psi = 0 \), we get
\[
0 \leq N_0 \leq g_M - 1 + \deg(\phi)(2g_N - 2).
\]
Consequently, we get restrictions for the existence of non-trivial solutions to \( D^{\phi^*TN} = 0 \). This was already proven in [26] giving rise to a structure theorem for Dirac-harmonic maps between closed surfaces. Moreover, it was also shown that in this case there holds equality in (3.4). Note that (3.4) improves the estimate (3.2) in the case that \( M = N = S^2 \).

Remark 3.5. The estimate (3.3) still holds if we consider Dirac-harmonic maps with torsion between surfaces, which were introduced in [4]. In this case one considers a metric connection with torsion on \( TN \). Moreover, it was also shown that in this case there holds equality in (3.4). Note that for \( \epsilon > 0 \) forcing \( \psi \) to be trivial. This has already been proven in [6], Lemma 4.9 with the help of the Sobolev embedding theorem. However, in (3.6) all the constants are explicit.

3.3. Dirac-harmonic maps with curvature term. Dirac-harmonic maps with curvature term arise as critical points of the energy functional for Dirac-harmonic maps taking into account an additional curvature term, see \([8\) and \([6\) for more details. We again consider Proposition 3.6.

Proposition 3.6. Let \((\phi, \psi)\) be a smooth Dirac-harmonic map with curvature term with energy \( E(\phi, \psi) = \int_M |d\phi|^2 + |\psi|^4 |dM\). Then the following inequality holds
\[
C_N E(\phi, \psi) \geq \pi \chi(M) + 2\pi N \tag{3.6}
\]
with the constant \( C_N := \max\{|R^N|_{L^{\infty}}, |R^N|_{L^\infty}\} \) and \( N \) is defined as in \([13\).

Proof. First of all we note, using that \( \psi \) is a solution of (3.5), that
\[
\langle \psi, (D^{\phi^*TN})^2 \psi \rangle = \frac{1}{3} \langle \psi, D^{\phi^*TN}(R^N(\psi, \psi)) \rangle = \frac{1}{3} \langle \psi, (\nabla (R^N(\psi, \psi))) \cdot \psi \rangle + \frac{1}{3} \langle \psi, R^N(\psi, \psi)D^{\phi^*TN}\psi \rangle = \frac{1}{9} |R^N(\psi, \psi)|^2,
\]
where we used the skew-symmetry of the Clifford multiplication. Moreover, we can estimate the energy momentum tensor as follows
\[
|T^{\phi^*TN}|^2 \geq 2 \langle \psi, D^{\phi^*TN}\psi \rangle^2 = \frac{2}{9} |\langle \psi, R^N(\psi, \psi)\psi \rangle|^2.
\]
Hence from (2.1) with \( E = \phi^*TN \) we get
\[
\frac{1}{9} |R^N(\psi, \psi)|^2 \geq K \frac{1}{2} + \frac{1}{2} \sum_{i,j=1}^2 \langle e_i \cdot e_j \cdot R^N(d\phi(e_i), d\phi(e_j)) \psi, \psi \rangle + \frac{|\langle \psi, R^N(\psi, \psi)\psi \rangle|^2}{18|\psi|^4} - \Delta \log |\psi|.
\]
Proceeding as in the proof of Proposition 3.1 we obtain
\[
C_N (|d\phi|^2 + |\psi|^4) \geq K \frac{1}{2} - \Delta \log |\psi|,
\]
where \( C_N := \max\{|R^N|_{L^{\infty}}, |R^N|_{L^\infty}\} \). The claim then follows by integration. \(\square\)

The above Proposition allows us to draw the following conclusions:

Corollary 3.7. (1) Suppose \((\phi, \psi)\) is a smooth Dirac-harmonic map with curvature term. If \( \chi(M) > 0 \) and if the energy is sufficiently small, that is \( E(\phi, \psi) < \epsilon \) for some small \( \epsilon > 0 \), then we get a contradiction from (3.6) forcing \( \psi \) to be trivial. This has already been proven in [6], Lemma 4.9 with the help of the Sobolev embedding theorem. However, in (3.6) all the constants are explicit.
(2) Moreover, \(\text{(3.10)}\) also gives an upper bound on the nodal set of solutions of \((3.9)\).

3.4. Spinor-valued one-forms. For \(E = T^* M\) we call sections in \(\Sigma M \otimes T^* M\) spinor-valued one-forms. These appear in the context of the Rarita-Schwinger operator, however one needs to introduce a certain projection in order to relate the twisted Dirac operator on \(\Sigma M \otimes T^* M\) to the Rarita-Schwinger operator, see \([3]\), section 2.6, \([25]\) and \([7]\) for more details. Nevertheless, we apply our main result in the case that \(E = T^* M\).

**Proposition 3.8.** Let \(E = T^* M\) and suppose that \(\psi \in \Gamma(\Sigma^{\pm}M \otimes T^* M)\). Then all eigenvalues of \((D^{T^*M})^2\) satisfy the following inequality
\[
\lambda_k^2 \geq \frac{2\pi \chi(M)}{\text{Vol}(M,g)} (1 + 2) + \frac{4\pi N_k}{\text{Vol}(M,g)}.
\]

The sign depends on if \(\psi\) is a positive or negative spinor-valued one-form.

**Proof.** This follows directly from (1.2) by using \(\deg(T^* M) = -\deg(TM) = -\chi(M)\). \(\square\)

3.5. Surfaces with Spin\(^c\) structure. Every Riemannian spin surface also possesses a Spin\(^c\) structure. We can think of the Dirac operator associated to a Spin\(^c\) structure as the classical Dirac operator acting on spinors which are twisted by a line bundle. For more details on Spin\(^c\) structures see \([19]\), Appendix D and \([14]\), \([23]\) for eigenvalue estimates of the Spin\(^c\) Dirac operator.

Making use of our main result (1.1) we find:

**Proposition 3.9.** Let \((M,g)\) be a closed spin surface with Spin\(^c\) structure. Suppose that \(\lambda_k^2 \) is the \(k\)-th eigenvalue of the Spin\(^c\) Dirac operator acting on \(\psi \in \Gamma(\Sigma^{\pm} M)\). Then the following eigenvalue estimate holds
\[
\lambda_k^2 \geq \frac{2\pi \chi(M)}{\text{Vol}(M,g)} \pm 4\pi \deg(L) + \frac{4\pi N_k}{\text{Vol}(M,g)},
\]
where \(\chi(M)\) is the Euler characteristic of \(M\) and \(\deg(L)\) the degree of the complex line bundle \(L\) associated to the Spin\(^c\) structure. The sign in (3.8) depends on whether \(\psi\) is a positive or negative spinor and \(N_k\) is defined as before.

Our result is similar to Theorem 3.4 in \([17]\), however in that reference the authors do not decompose the spinor bundle into its positive and negative parts. In addition, the authors characterize the equality case: It is given by a Spin\(^c\)-Killing spinor (which does not have any zeros) being an eigenspinor for the action of the curvature endomorphism of the line bundle at the same time.

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