Order and Chaos in Roto–Vibrational States of Atomic Nuclei

V.R. Manfredi
Dipartimento di Fisica “G. Galilei” dell’Università di Padova,
INFN, Sezione di Padova,
Via Marzolo 8, I 35131 Padova, Italy
Interdisciplinary Laboratory, SISSA,
Strada Costiera 11, 34014 Trieste, Italy

L. Salasnich
Dipartimento di Fisica ”G. Galilei” dell’Università di Padova,
INFN, Sezione di Padova,
Via Marzolo 8, I 35131 Padova, Italy
Departamento de Fisica Atomica, Molecular y Nuclear
Universidad ”Complutense” de Madrid,
Av. Complutense, E 28040 Madrid, Espana

1This work has been partially supported by the Ministero
dell’Università e della Ricerca Scientifica e Tecnologica (MURST).
2Author to whom all correspondence and reprint requests should be addressed.
E–Mail: VAXFPD::MANFREDI, MANFREDI@PADOVA.INFN.IT
Abstract

Using a classical analytical criterion (that of curvature) and numerical results (Poincarè sections and spectral statistics), a transition order–chaos–order in the roto–vibrational model of atomic nuclei has been shown. Numerical calculations were performed for some deformed nuclei.

PACS numbers: 21.10.-k, 05.45.+b, 03.65 Sq
1. Introduction

In the last few years many authors have shown great interest in the so-called "quantum chaos": the properties of quantal systems which are chaotic in the semiclassical limit $\hbar \to 0$ [1,2,3].

In atomic nuclei the coexistence of regular, chaotic and collective states makes the problem quite intricate. In order to disentangle it, many models have been used (see, for example [4,5,6] and references quoted therein). The aim of this paper is to study the transition from ordered to chaotic states in the roto–vibrational model of atomic nuclei. This model was introduced by Bohr and Mottelson [7] and discussed in great detail in [8,9,10].

2. The Model

Since the model has been amply described in [8], in this section we limit ourselves to reporting only a few basic formulae.

The hamiltonian is:

$$ H = E_{vib} + E_{rot}, \quad (1) $$

where:

$$ E_{vib} = \frac{1}{2} B (\dot{a}_0^2 + 2 \dot{a}_2^2) + V(a_0, a_2), \quad (2) $$

$$ E_{rot} = \frac{1}{2} \sum_{k=1}^{3} \omega_k^2 J_k(a_0, a_2), \quad (3) $$

with:

$$ V(a_0, a_2) = \sum_{m,n} c_{mn} (a_0^2 + 2a_2^2)^m a_0^n (6a_2^2 - a_0^2)^n. \quad (4) $$
The parameters $a_0$ and $a_2$ are connected to the deformation $\beta$ and asymmetry $\gamma$ by the standard relations [8]:

$$a_0 = \beta \cos \gamma, \quad a_2 = \frac{\beta}{\sqrt{2}} \sin \gamma.$$  \hspace{1cm} (5)

In terms of the new variables the components of the moment of inertia are [7]:

$$J_k = 4B\beta^2 \sin^2 \left(\gamma - \frac{2\pi}{3}k\right).$$  \hspace{1cm} (6)

If the nucleus has an axially symmetric deformation:

$$\omega_1 = \omega_2 = \frac{\omega}{\sqrt{2}}, \quad \omega_3 = 0,$$  \hspace{1cm} (7)

and if we take only the first terms of (4), the hamiltonian (1) can be written:

$$H = \frac{1}{2}B(\dot{a}_0^2 + 2\dot{a}_2^2) + V(a_0, a_2) + \frac{1}{2}B\omega^2(3a_0^2 + 2a_2^2),$$  \hspace{1cm} (8)

where:

$$V(a_0, a_2) = \frac{1}{2}C_2(a_0^2 + 2a_2^2) + \sqrt{\frac{2}{35}}C_3a_0(6a_2^2 - a_0^2) + \frac{1}{5}C_4(a_0^2 + 2a_2^2)^2 + V_0,$$  \hspace{1cm} (9)

with $V_0$ a parameter chosen to have the minimum of the potential equal to zero. As discussed in [8], the presence of bound states in atomic nuclei leads to a value of $C_4 > 0$, whereas for $C_3$ a positive value corresponds to a prolate shape, a negative value to an oblate shape. Similarly $C_2$ may also be either positive or negative.

The shape of the nuclear potential $V(a_0, a_2)$ is a function of $C_2$ and $\chi = C_3^2/(C_2C_4)$. For $C_2 > 0$, and $0 < \chi < 56/9$ the nucleus is spherical; for $56/9 < \chi < 7$ the nucleus is spherical in the ground state (g.s.) and deformed.
in the excited states (e.s.); for $\chi > 7$ it is deformed in the g.s. and spherical in the e.s.; for $C_2 < 0$ it is deformed in the g.s. and $\gamma$–unstable in the e.s.

Near the equilibrium ($\bar{a}_0 = \beta_0$, $\bar{a}_2 = 0$), $V(a_0, a_2)$ can be written:

$$\tilde{V}(a_0, a_2) = \frac{\tilde{C}_0}{2}(a_0 - \beta_0)^2 + \tilde{C}_2 a_2^2. \quad (10)$$

In order to calculate the equilibrium configuration the following system must be solved:

$$\frac{\partial V}{\partial a_0|_{\beta_0, 0}} = \beta_0(C_2 - 3\sqrt{\frac{2}{35}}C_3\beta_0 + \frac{4}{5}C_4\beta_0^2) = 0, \quad (11)$$

$$\frac{\partial^2 V}{\partial a_0^2|_{\beta_0, 0}} = C_2 - 6\sqrt{\frac{2}{35}}C_3\beta_0 + \frac{12}{5}C_4\beta_0^2 = \frac{\partial^2 \tilde{V}}{\partial a_0^2|_{\beta_0, 0}} = \tilde{C}_0, \quad (12)$$

$$\frac{1}{2}\frac{\partial^2 V}{\partial a_2^2|_{\beta_0, 0}} = C_2 + 6\sqrt{\frac{2}{35}}C_3\beta_0 + \frac{12}{5}C_4\beta_0^2 = \frac{\partial^2 \tilde{V}}{\partial a_2^2|_{\beta_0, 0}} = \tilde{C}_2. \quad (13)$$

In roto–vibrational nuclei the parameters $\tilde{C}_0$ and $\tilde{C}_2$ are simply connected to some experimental quantities:

$$E_\beta = \hbar\omega_\beta = \hbar\sqrt{\frac{C_0}{B}}, \quad E_\gamma = \hbar\omega_\gamma = \hbar\sqrt{\frac{C_2}{B}}, \quad (14)$$

and:

$$\tilde{C}_0 = \frac{E_\beta^2}{3\epsilon\beta_0^2}, \quad \tilde{C}_2 = \frac{E_\gamma^2}{3\epsilon\beta_0^2}, \quad (15)$$

where $\epsilon = \hbar^2/J_0$ is taken from the experimental energy of the first rotational state and $\beta_0$ is the equilibrium deformation of the nucleus, which may be obtained from the $B(E_2, 2^+ \rightarrow 0^+)$ value [8]. Finally:

$$C_2 = -\frac{1}{18\epsilon\beta_0}(3E_\beta^2 - E_\gamma^2),$$
As a simple application of the above procedure, the nuclear potentials have been calculated for $^{164}Dy$ and $^{166}Er$ (see Figures 1,2). The numerical values of the parameters $\beta_0$, $\epsilon$, $E_\beta$, $E_\gamma$, taken from the experimental data, are shown in Table 1.

3. The onset of chaos by curvature criterion

As is well known, the transition order–chaos in systems with two degrees of freedom may be studied by the curvature criterion [11,12]. It is however important to point out that in general the curvature criterion guarantees only a local instability and should therefore be combined with the Poincaré sections [14]. For a fuller discussion of this point see [13]. As mentioned in the previous section, the total hamiltonian can be written:

$$H = \frac{1}{2} B (\dot{a}_0^2 + 2\dot{a}_2^2) + W(a_0, a_2), \quad (17)$$

where:

$$W(a_0, a_2) = V(a_0, a_2) + \frac{1}{2} B \omega^2 (3a_0^2 + 2a_2^2) \quad (18)$$

is the effective potential. As a function of $C_2$, $C_3$, $C_4$, (18) becomes:

$$W(a_0, a_2) = \frac{1}{2} C_2 (a_0^2 + 2a_2^2) + \sqrt{\frac{2}{35}} C_3 a_0 (6a_2^2 - a_0^2) + \frac{1}{5} C_4 (a_0^2 + 2a_2^2)^2 + V_0 + \frac{1}{2} B \omega^2 (3a_0^2 + 2a_2^2). \quad (19)$$

Owing to the symmetry properties of the effective potential $W$, our study may be restricted to the case $W(a_0, a_2 = 0)$ [8]. To apply the above criterion
to our system, the sign of the curvature $K$ can be studied by solving the equation:

$$K(a_0) = \frac{\partial^2 W}{\partial a_0^2}(a_0, a_2 = 0) = \frac{12}{5} C_4 a_0^2 - 6 \sqrt{\frac{2}{35}} C_3 a_0 + (C_2 + 3 B \omega^2) = 0, \quad (20)$$

whose discriminant $\Delta$ is given by:

$$\Delta = \frac{72}{35} C_2 C_4 (\chi - \frac{14}{3} - \frac{14 B \omega^2}{C_2}). \quad (21)$$

If $\Delta \leq 0$ the curvature $K$ is always positive and the motion is regular; if $\Delta > 0$ there is a region of negative curvature and the motion may be chaotic.

For $C_2 > 0$, $0 < \chi < 14/3$ (spherical nuclei) the curvature is positive and therefore the motion is regular for all $\omega$. For $\chi \geq 14/3$ (spherical and deformed nuclei) and:

$$0 \leq \omega < \sqrt{\frac{C_2}{14 B}(\chi - \frac{14}{3})} \quad (22)$$

the curvature is negative and chaotic motion may appear.

For $C_2 < 0$ ($\gamma$–unstable nuclei) there is a region with negative curvature for $0 \leq \omega < \omega_c$, where:

$$\omega_c = \sqrt{\frac{C_2}{14 B}(\chi - \frac{14}{3})} \quad (23)$$

is the critical frequency of the system. As a function of $\epsilon$ and $\beta_0$ the critical energy can be written:

$$\hbar \omega_c = \sqrt{(\chi - \frac{14}{3}) \frac{3 \epsilon \beta_0^2 C_2}{14}} \quad (24)$$

and the critical angular momentum $I_c$ is:

$$I_c = \frac{\hbar}{\epsilon} \sqrt{(\chi - \frac{14}{3}) \frac{3 \epsilon \beta_0^2 C_2}{14}}. \quad (25)$$
It is perhaps noteworthy that the shape of the effective potential \( W(a_0,0) \) changes drastically as a function of \( \omega \) (see Fig. 3). If \( \omega \) increases there is a transition from chaos to order: the region of chaotic motion decreases and becomes zero for \( \omega > \omega_c \). For a fixed value of the parameter \( \chi \), the region of chaotic motion is limited, on the line \( K(a_0) = 0 \), by the two branches defined by:

\[
a_0^\pm = \frac{5}{4C_4} \left[ \sqrt{\frac{2}{35} C_3} \pm \sqrt{\frac{2}{35} C_2 C_4 (\chi - \frac{14}{3} - \frac{14B\omega^2}{C_2})} \right].
\]  

(26)

Table 3 shows, for some nuclei, the energies \( E_{-\text{chaos}}^\pm \) and \( E_{+\text{chaos}}^\pm \), which limit the chaotic region for \( \omega = 0 \), and also the critical angular momentum \( I_c \).

4. Numerical study of the order–chaos transition

As mentioned in the previous section, the curvature criterion is able to characterize the local behaviour of the system (for example the local instability) and may give only a rough signature of the global properties (e.g. the global instability) [13]. As is well known, a very useful tool for the study of global properties is provided by the Poincarè sections [14]. With this aim the classical trajectories have been calculated by a fourth order Runge–Kutta method. In order to avoid numerical errors connected to the use of finite temporal intervals, a first–order interpolation has been used [15].

The Hamilton equations of the systems are:

\[
\dot{a}_0 = Bp_0,
\]

\[
\dot{a}_2 = 2Bp_2,
\]

\[
\dot{p}_0 = -C_2 a_0 - 2 \sqrt{\frac{2}{35} C_3 (3a_2^2 - 3a_0^2)} - \frac{4}{5} C_4 a_0 (a_0^2 + 2a_2^2) - 3B\omega^2 a_0.
\]  

(27)
\[ \dot{p}_2 = -2C_2a_2 - 12\sqrt{\frac{2}{35}}C_3a_0a_2 - \frac{8}{5}C_4a_2(a_0^2 + 2a_2^2) - 2B\omega^2a_2, \]

where \( p_0 \) and \( p_2 \) are the conjugate momenta:

\[ p_0 = B\dot{a}_0, \quad p_2 = 2B\dot{a}_2. \]  \hspace{1cm} (28)

Figure 4 shows the Poincarè sections for \(^{160}\text{Gd}\) at the energy 5.5 MeV and for different values of rotational frequency. The figure clearly shows a chaos–order transition as the frequency \( \omega \) increases. In Figure 5 for \(^{166}\text{Er}\) the Poincarè sections are shown for different values of the energy and rotational frequency \( \omega = 0 \). As can be seen, there is a chaos–order transition, albeit not so sharp as in the previous case.

It is well known that the fluctuation properties of quantal systems with underlying classical chaotic behaviour and time–reversal symmetry are in agreement with the predictions of the Gaussian Orthogonal Ensemble (GOE), and that quantum analogs of classically integrable systems display the characteristics of Poisson statistics [4,6].

For deformed nuclei, like those of the rare–earth region, it is not easy to obtain the correct energy levels, because the potential energy is an asymmetric triple well. To avoid the problems related to the tunneling effects, a numerical method based on the formulation of quantum mechanics using the euclidean path integral [16] should be used. The energy spectrum is then mapped into one with quasi–uniform level density by means of the local unfolding procedure described in detail in reference [17].

In Figure 6 the spectral statistics \( P(s) \) and \( \Delta_3 \) [18,19] are plotted for \(^{166}\text{Er}\). These statistics confirm the classical results: for energies above the
saddle energy, about 4 MeV, there is prevalently chaotic behaviour; for higher energies, about 15 MeV, there is mixed behaviour with a predominance of regular classical trajectories. The non-universal behaviour of $\Delta_3(L)$ for large values of $L$, not predicted by GOE, has been explained by Berry [20] using the semiclassical quantization.

5. Conclusions

We have shown by combining analytical results (the curvature criterion) with numerical ones (the Poincaré sections and spectral statistics) that, in the roto-vibrational model of atomic nuclei, an order-chaos-order transition occurs as a function of the energy. Our results are in good agreement with those of [12], but, unlike the authors of [12], we have also calculated the energy ranges of the chaotic regions (see Table 3) and the chaos-order transition which occurs as a function of the rotational frequency $\omega$.

The authors are greatly indebted to M. Rosa–Clot and S. Taddei for the provided numerical data of energy levels.
REFERENCES

[1] M.C. Gutzwiller: *Chaos in Classical and Quantum Mechanics* (Springer–Verlag, Berlin, 1990)

[2] M. Berry: in *Quantum Chaos*, Ed. H.A. Cerdeira, R. Ramaswamy, M.C. Gutzwiller, G. Casati (World Scientific, New York, 1991)

[3] *From Classical to Quantum Chaos*, Ed. G.F. Dell’Antonio, S. Fantoni, V.R. Manfredi, Conference Proceedings SIF, vol. 41 (Editrice Compositori, Bologna, 1993)

[4] O. Bohigas, H.A. Weidenmüller: *Ann. Rev. Nucl. Part. Sci.* 38, 421 (1988)

[5] B.R. Mottelson: *Nucl. Phys.* A 557, 717c (1993)

[6] M.T. Lopez–Arias, V.R. Manfredi, L. Salasnich: *La Rivista del Nuovo Cimento* 17, n. 5 (1994)

[7] A. Bohr, B. Mottelson: *Nuclear Structure*, vol. 2 (Benjamin, London, 1975)

[8] J.M. Eisenberg, W. Greiner: *Nuclear Models*, vol. 1 (North Holland, Amsterdam, 1970)

[9] A. Faessler, W. Greiner: *Zeit. Phys.* 168, 425 (1962); A. Faessler, W. Greiner R.K. Sheline: *Nucl. Phys.* 70, 33 (1965)

[10] U. Mosel, W. Greiner: *Zeit. Phys.* 217, 256 (1968)

[11] M. Toda: *Phys. Lett.* A 48, 335 (1974)

[12] Yu. Bolotin, V.Yu. Gonchar, E.V. Inopin, V.V. Levenko, V.N. Tarasov, N.A. Chekanov: *Sov. I. Part. Nucl.* 20, 372 (1989)

[13] G. Benettin, R. Brambilla, L. Galgani: *Physica A* 87, 381 (1977)
[14] H. Poincarè: *New Methods of Celestial Mechanics*, vol. 3, ch. 27
(Transl. NASA Washington DC 1967)

[15] M. Henon: *Physica D*, 5, 412 (1982)

[16] M. Rosa-Clot, S. Taddei: *Phys. Rev. C* 50, 627 (1994); M. Rosa-Clot, S. Taddei: *Phys. Lett. A* 197, 1 (1995)

[17] V.R. Manfredi: *Lett. Nuovo Cim.* 40, 135 (1984)

[18] F.J. Dyson, M.L. Metha: *J. Math, Phys.* 4, 701 (1963)

[19] O. Bohigas, M.J. Giannoni: Ann. Phys. (N.Y.) 89, 393 (1975)

[20] M. Berry: *Proc. Roy. Soc. Lond. A* 400, 229 (1985)
TABLE CAPTIONS

Table 1: Numerical values of the parameters $\beta_0, \epsilon, E_\beta, E_\gamma$, taken from the experimental data for some even–even nuclei. The parameters are defined in the text.

Table 2: Numerical values of the parameters $C_2, C_3, C_4, V_0$ defining the effective potential $W$.

Table 3: Numerical values of the energies which delimit the chaotic region for $\omega = 0$ and the critical angular momentum.
FIGURE CAPTIONS

Figure 1: Nuclear potential for $^{164}\text{Dy}$.  
Figure 2: Nuclear potential for $^{166}\text{Er}$.  
Figure 3: Effective nuclear potential for $^{164}\text{Dy}$ as a function of the rotational frequency $\omega$; from left to right: $\hbar\omega = 0 \text{ MeV}$, $\hbar\omega = 0.5 \text{ MeV}$, $\hbar\omega = 1 \text{ MeV}$.  
Figure 4: The Poincarè sections for $^{160}\text{Gd}$ at the energy 5.5 MeV and for different values of rotational frequency; from the top: $\hbar\omega = 0 \text{ MeV}$, $\hbar\omega = 0.5 \text{ MeV}$, $\hbar\omega = 1 \text{ MeV}$.  
Figure 5: The Poincarè sections for $^{166}\text{Er}$ at the rotational frequency $\omega = 0$ and for different values of the energy: (a) $E = 1 \text{ MeV}$, (b) $E = 6 \text{ MeV}$, (c) $E = 9 \text{ MeV}$, (d) $E = 12 \text{ MeV}$.  
Figure 6: Spectral statistics $P(s)$ and $\Delta_3(L)$ for $^{166}\text{Er}$ at the rotational frequency $\omega = 0$ for different energy regions: $2 \leq E \leq 6 \text{ MeV}$ (below) and for $13 \leq E \leq 17 \text{ MeV}$ (above). The solid line is the GOE statistic curve and the dashed line is the Poisson one.
| Nucleus | $\beta_0$ | $\epsilon$ (MeV) | $E_\beta$ (MeV) | $E_\gamma$ (MeV) |
|--------|-----------|------------------|-----------------|-----------------|
| $^{160}$Gd | 0.47 | 0.022 | 1.80 | 0.96 |
| $^{164}$Dy | 0.41 | 0.021 | 1.76 | 0.73 |
| $^{166}$Er | 0.33 | 0.023 | 1.14 | 0.75 |
| $^{230}$Th | 0.23 | 0.015 | 0.63 | 0.76 |
| $^{238}$U | 0.28 | 0.014 | 0.99 | 1.05 |
| $^{240}$Pu | 0.28 | 0.013 | 0.86 | 0.92 |

Table 1
| Nucleus | $C_2$ (MeV) | $C_3$ (MeV) | $C_4$ (MeV) | $V_0$ (MeV) |
|---------|------------|------------|------------|------------|
| $^{160}$Gd | -100.64 | 37.57 | 668.01 | 5.56 |
| $^{164}$Dy | -138.41 | 34.97 | 1155.67 | 5.69 |
| $^{166}$Er | -73.77 | 64.96 | 1138.52 | 1.89 |
| $^{230}$Th | -43.56 | 229.67 | 2260.62 | 0.44 |
| $^{238}$Ur | -94.54 | 334.20 | 3276.70 | 1.56 |
| $^{240}$Pu | -73.64 | 281.53 | 2664.58 | 1.20 |

Table 2
| Nucleus   | $E^-_{\text{chaos}}$ (MeV) | $E^+_{\text{chaos}}$ (MeV) | $I_c (\hbar)$ |
|-----------|-----------------------------|----------------------------|---------------|
| $^{160}\text{Gd}$ | 2.36                        | 5.36                       | 31.86         |
| $^{164}\text{Dy}$  | 2.51                        | 5.59                       | 33.31         |
| $^{166}\text{Er}$  | 0.83                        | 1.89                       | 18.79         |
| $^{230}\text{Th}$  | 0.15                        | 0.44                       | 13.28         |
| $^{238}\text{U}$   | 0.61                        | 1.55                       | 23.88         |
| $^{240}\text{Pu}$  | 0.46                        | 1.19                       | 21.97         |

Table 3