A global uniqueness for formally determined inverse electromagnetic obstacle scattering

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Abstract
It is proved that a perfect conducting obstacle in $\mathbb{R}^3$, consisting of finitely many solid polyhedra, is uniquely determined by the far-field pattern corresponding to a single incident electromagnetic plane wave.

1. Introduction and statement of the result
In this paper, we shall be mainly concerned with the inverse electromagnetic obstacle scattering, where one utilizes the time-harmonic electromagnetic far-field measurements to identify the inaccessible/unknown objects.

For a brief description of the forward scattering problem, we let a perfect conducting obstacle $D \subset \mathbb{R}^3$ be a compact set with connected Lipschitz complement $G := \mathbb{R}^3 \setminus D$, and

$$E'(x) := \frac{i}{k} \text{curl} \text{ curl} p e^{ikx \cdot d} = ik(d \times p) \times d e^{ikx \cdot d},$$

$$H'(x) := \text{curl} p e^{ikx \cdot d} = ikd \times p e^{ikx \cdot d},$$

be the incident electric and magnetic fields, where $p \in \mathbb{R}^3, k > 0$ and $d \in S^2 := \{x \in \mathbb{R}^3; |x| = 1\}$ represents respectively the polarization, wavenumber and direction of propagation. The incident wave propagating in the homogeneous background medium will be perturbed when it encounters an obstacle, and produces a scattered field. We denote by $E'$ and $H'$ the scattered electric and magnetic fields respectively, and define the total electric and magnetic fields to be

$$E(x) = E'(x) + E(x), \quad H(x) = H'(x) + H(x) \quad x \in \mathbb{R}^3.$$  \hspace{1cm} (1.3)

Then the direct scattering problem consists of finding a solution $(E, H) \in H^1_{\text{loc}} (\text{curl}; G) \times H^1_{\text{loc}} (\text{curl}; G)$ that satisfies the following time-harmonic Maxwell equations:

$$\text{curl} E - ikH = 0, \quad \text{curl} H + ikE = 0 \quad \text{in} \quad G := \mathbb{R}^3 \setminus D,$$

$$\nu \times E = 0 \quad \text{on} \quad \partial G. \quad \hspace{1cm} (1.4)$$

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\[
\lim_{|x|\to\infty} (\vec{H}^s \times x - |x|E^s) = 0, \tag{1.6}
\]

where the last limit corresponds to the so-called Silver–Müller radiation condition characterizing the fact that the scattered wave is radiating.

The forward scattering problem \((1.1)\)--\((1.6)\) is well understood (see e.g. [3]). Particularly, \(E\) and \(H\) are (real) analytic in \(G\) in the sense that both the real and imaginary parts of \(E\) and \(H\) are real analytic functions in \(G\); see [6] and we also refer to chapter 3 [10] for a formal definition of real analytic functions. The asymptotic behavior of the scattered field \((E^s, H^s)\) is governed by (see [6])

\[
E^s(x; \hat{D}, p, k, d) = \frac{e^{ik_d \cdot d}}{|x|} \left\{ E_\infty(\hat{x}; \hat{D}, p, k, d) + \mathcal{O}\left(\frac{1}{|x|}\right)\right\} \quad \text{as } |x| \to \infty, \tag{1.7}
\]

\[
H^s(x; \hat{D}, p, k, d) = \frac{e^{ik_d \cdot d}}{|x|} \left\{ H_\infty(\hat{x}; \hat{D}, p, k, d) + \mathcal{O}\left(\frac{1}{|x|}\right)\right\} \quad \text{as } |x| \to \infty, \tag{1.8}
\]

uniformly for all \(\hat{x} = x/|x| \in S^2\). The functions \(E_\infty(\hat{x})\) and \(H_\infty(\hat{x})\) in \((1.7)\) and \((1.8)\) are called, respectively, the electric and magnetic far-field patterns, and both are analytic on the unit sphere \(S^2\). As is noted above, \(E^s(x; \hat{D}, p, k, d), E_\infty(\hat{x}; \hat{D}, p, k, d), \) etc will be frequently used to specify their dependence on the observation direction \(\hat{x}\), the polarization \(p\), the wavenumber \(k\) and the incident direction \(d\).

The inverse scattering problem is the following. Assume that the obstacle \(D\) is unknown or inaccessible and we aim to image the object and thereby identify it by performing far-field measurements. That is, with the measurement of the electric far-field pattern (or, equivalently, the magnetic far-field pattern) of the wave which is scattered by \(D\) corresponding to a given incident wave, for one or more choices of its polarization \(p\) or of its wavenumber \(k\), or of its propagation direction \(d\), we would like to recover the obstacle whose scattered waves are compatible with the measurements performed. Let us define the map sending the obstacle to the corresponding far-field pattern as follows:

\[
\mathcal{F}_\epsilon(\partial G) = E_\infty(\hat{x}; \hat{D}, p, k, d) \quad \text{for } (\hat{x}, p, k, d) \in S^2 \times U \times K \times S^2, \tag{1.9}
\]

where \(S^2 \subset R^3, \cup \subset R^3, K \subset R^*_+ := \{x \in R; x > 0\}\) and the nonlinear operator \(\mathcal{F}_\epsilon\) is defined by the forward scattering system \((1.4)\)--\((1.6)\). Clearly, the inverse problem is to recover \(\partial G\) by knowledge of \(E_\infty\); namely, we need to invert the nonlinear operator equation \((1.9)\).

This problem, having its roots in the technology of radar and sonar, is also central to many other areas of science such as medical imaging, geophysical exploration and nondestructive testing, etc. We refer to [6] for a more detailed discussion and related literature. As usual in most of the inverse problems, the first question to ask in this context is the identifiability; i.e. whether an obstacle can be identified from knowledge of its far-field pattern. Mathematically, the identifiability is the uniqueness issue, which is the injectivity of the (nonlinear) operator \(\mathcal{F}_\epsilon\) in \((1.9)\). That is, if two obstacles \(D\) and \(\tilde{D}\) produce the same far-field data, i.e.

\[
E_\infty(\hat{x}; D, p, k, d) = E_\infty(\hat{x}; \tilde{D}, p, k, d) \quad \text{for } (\hat{x}, p, k, d) \in S^2 \times \cup \times K \times S^2, \tag{1.10}
\]

\(D\) have to be the same as \(\tilde{D}\)?

We refer to [11] for a general discussion of the critical role of uniqueness which plays in inverse problems theory theoretically as well as numerically. It is observed that the uniqueness results
also provide the practical information on how many measurement data one should use to identify the underlying object. As an important ingredient in the uniqueness study, we recall a conspicuous property of a real analytic function defined over an open connected set $\Omega \subset \mathbb{R}^3$; namely the unique continuation/analytic continuation which says that it is already determined uniquely everywhere in $\Omega$ by its value in an arbitrarily small open subset of $\Omega$ (cf [10]). Noting $E_\infty$ is analytic, one sees that if $S_0^2$ in (1.9) is only an open subset of the unit sphere, no matter how small the subset is, we can always recover such data on the whole unit sphere by analytic continuation. Hence, for our uniqueness study, without loss of generality, we can assume that the far-field data are given on the whole unit sphere, i.e. in every possible observation direction. Then it is easily seen that the inverse obstacle scattering is formally determined with fixed $p_0 \in \mathbb{R}^3$, $k_0 > 0$ and $d_0 \in S^2$, since the far-field data depend on the same number of variables as does the obstacle which is to be recovered. Due to such observation, there is a widespread belief that one can establish the uniqueness by using the far-field pattern corresponding to a single incident wave. However, this has remained to be a longstanding conjecture, though extensive study has been made in this aspect (see [2, 7]). The only previous result that we are aware of this kind is in [12], where it is shown that a simple ball can be uniquely determined by its far-field measurement corresponding to a single incident wave.

In the past few years, significant progress has been achieved on the unique determination of general polyhedral-type obstacles by several far-field measurements. The breakthrough is first made in the inverse acoustic obstacle scattering, where one utilizes the acoustic far-field measurement to identify the underlying scattering objects (see [1, 4, 9, 13, 14]). Among the arguments for the proofs of those results, the methodology developed in [13] which we call path argument is proved to be particularly suitable for attacking such problems. Based on suitably devised path arguments, together with some novel reflection principles for the solutions of Maxwell equations, various uniqueness results have been established in different settings with general polyhedral-type obstacles in [15, 16], but all with the far-field measurements corresponding to two different incident waves. In the current work, we are able to improve significantly on this result to the formally determined setting. It is shown that the measurement of the far-field pattern corresponding to a single incident wave uniquely determines a general polyhedral perfect conducting obstacle. For the proof, we follow the general strategy in [15, 16], but several technical new ingredients must be developed and the path argument in this work is refined significantly. We next state more precisely the main result.

It is first recalled that a compact polyhedron in $\mathbb{R}^3$ is a simply connected compact set whose boundary is composed of (open) faces, edges and vertices. In the following, we call $D$ a polyhedral obstacle if it is composed of finitely many (but unknown a priori) pairwise disjoint compact polyhedra. That is,

$$D = \bigcup_{l=1}^m D_l, \tag{1.10}$$

where $m$ is an unknown integer but must be finite and each $D_l$, $1 \leq l \leq m$ is a compact polyhedron such that

$$D_i \cap D_j = \emptyset \quad \text{if} \quad i \leq j \quad \text{and} \quad 1 \neq i, j \leq m.$$

Clearly, the forward scattering problem (1.4)–(1.6) with such a polyhedral obstacle $D$ is well-posed. Moreover, we know that the singular behaviors of the weak solution only attach to the edges and vertices, that is, $(E, H)$ satisfies (1.4) in the classical sense in any subdomain of $G$, which does not meet any corner or edge of $D$ (see [5]). By the regularity of the strong solution for the forward scattering problem, we know that both $E$ and $H$ are at least $C^{0,\alpha}$ continuous
(0 < α < 1) up to the regular points, namely, points lying in the interior of the open faces of $D$.

The main result of this paper is the following.

**Theorem 1.1.** Let $D$ and $\tilde{D}$ be two perfect polyhedral obstacles. For any fixed $k_0 > 0$, $d_0 \in S^2$ and $p_0 \in \mathbb{R}^3$ such that $d_0$ and $p_0$ are linearly independent, we have $D = \tilde{D}$ as long as

$$E_\infty(\hat{x}; D, p_0, k_0, d_0) = E_\infty(\hat{x}; \tilde{D}, p_0, k_0, d_0) \quad \text{for} \quad \hat{x} \in S^2. \quad (1.11)$$

**Remark 1.2.** As mentioned earlier, there are some uniqueness results established in [15, 16] in the unique determination of general polyhedral obstacles, but with the far-field data corresponding to two different incident waves. However, the polyhedral obstacles considered in [15] are more general than the present ones, and they admit the simultaneous presence of crack-type components (namely, screens). Whereas the uniqueness in [16] is established without knowing the a priori physical properties of the underlying obstacle. In section 4, we would make concluding remarks on that the uniqueness result in theorem 1.1 cannot cover completely those obtained in [15, 16].

The remainder of the paper is organized as follows. In section 2, we introduce the perfect set and perfect planes, and then show several crucial properties of them which shall play a key role in proving theorem 1.1. Section 3 is devoted to proof of theorem 1.1 and in section 4, we give some concluding remarks.

### 2. Perfect set and perfect planes

First, we fix some notations which shall be used throughout in the remainder of the paper.

We denote an open ball in $\mathbb{R}^3$ with center $x$ and radius $r$ by $B_r(x)$, the closure of $B_r(x)$ by $\bar{B}_r(x)$ and the boundary of $B_r(x)$ by $S_r(x)$. The notation $T_r(x)$ is defined to be an open cube of edge length $r$, centered at $x$, while $\bar{T}_r(x)$ is its corresponding closure. Unless specified otherwise, $\nu$ shall always denote the inward normal to a concerned domain, or the normal to a two-dimensional plane in $\mathbb{R}^3$. The distance between two sets $A$ and $B$ in $\mathbb{R}^3$ is understood as usual to be $d(A, B) = \inf_{x \in A, y \in B} |x - y|$. Finally, a curve $\gamma = \gamma(t), t \geq 0$ is said to be regular if it is $C^1$ smooth and $\nabla \gamma(t) \neq 0$.

Henceforth, we let $k_0 > 0, d_0 \in S^2$ and $p_0 \in \mathbb{R}^3$ be fixed such that $d_0$ and $p_0$ are linearly independent, and denote by $E(x) := E(x; D, p_0, k_0, d_0)$ the total electric field in (1.4)–(1.6) corresponding to a polyhedral perfect conducting obstacle $D$ as described in (1.10). The following definition of a perfect set is modified from that in [15] to fit the problem being under investigation.

**Definition 2.1.** $\mathcal{P}_E$ is called a perfect set of $E$ in $G := \mathbb{R}^3 \setminus D$ if

$$\mathcal{P}_E = \{x \in G; \nu \times E|_{\Pi \cap B_r(x)} = 0 \text{ for some } r > 0 \text{ and plane } \Pi \text{ passing through } x\},$$

where $\nu$ is the unit normal to the plane $\Pi$.

For any $x \in \mathcal{P}_E$, we let $\Pi$ be the plane involved in the definition of $\mathcal{P}_E$. Furthermore, we let $\bar{\Pi}$ be the connected component of $\Pi \setminus D$ containing $x$, then by the analyticity of $E$ in $G$, we see $\nu \times E = 0$ on $\bar{\Pi}$ by classical continuation. In the following, such $\bar{\Pi}$ will be referred to as a perfect plane. The introduction of the perfect set and perfect plane is motivated by the observation that, when proving theorem 1.1 by contradiction, if two different obstacles produce the same far-field pattern, then outside one obstacle there exists a perfect plane which
is extended from an open face of the other obstacle. Starting from now on, \( \tilde{\Pi}_l \) with an integer \( l \), shall always represent a perfect plane in \( G \) which lies on the plane \( \Pi_l \) in \( \mathbb{R}^3 \).

A very fine property of perfect planes is the so-called reflection principle, which constitutes an indispensable ingredient in the path arguments for proving the uniqueness results in [15, 16]. We formulate the principle in the following theorem. Subsequently, we use \( \mathcal{R}_l \) to denote the reflection in \( \mathbb{R}^3 \) with respect to a plane \( \Pi_l \).

**Theorem 2.2.** For a connected polyhedral domain \( \Omega \) in \( G := \mathbb{R}^3 \setminus D \), let \( \tilde{\Pi} \) be one of its faces that lies on some perfect plane. Furthermore, let \( \Pi \) be the plane in \( \mathbb{R}^3 \) containing \( \tilde{\Pi} \) and \( \Omega \cup \mathcal{R}_l \Omega \subset G \). We have two consequences:

(i) \( \nu_l \times E = 0 \) on \( \Pi \cap (\Omega \cup \mathcal{R}_l \Omega) \);

(ii) Suppose that \( \Sigma \subset \partial \Omega \) is a subset of one face of \( \Omega \) other than \( \tilde{\Pi} \), and the following condition holds

\[
\nu \times E = 0 \quad \text{on} \quad \Sigma,
\]

where \( \nu \) is the unit normal to \( \Sigma \) directed to the interior of \( \Omega \). Then we have

\[
\nu' \times E = 0 \quad \text{on} \quad \Sigma',
\]

where \( \Sigma' = \mathcal{R}_l \Sigma \) and \( \nu' \) is the unit normal to \( \Sigma' \) directed to the interior of \( \mathcal{R}_l \Omega \).

**Proof.** The verification for (i) can be found in proof of theorem 3.3 in [15], while for (ii), is given in theorem 2.3 in [15]. See also theorem 2.9 and proof of theorem 3.2 in [16]. \( \square \)

The reflection principle in item (i) of theorem 2.2 is particularly useful when \( (\Pi \cap (\Omega \cup \mathcal{R}_l \Omega)) \setminus \tilde{\Pi} \neq \emptyset \). Clearly, in such a case, we can find a perfect plane also lying on the plane \( \Pi \), but different from \( \tilde{\Pi} \).

Next, we would classify all those perfect planes in \( G \) into two sets in \( \mathbb{R}^3 \), one is bounded and the other is unbounded. In fact, it is verified directly that there might exist unbounded perfect planes. This constitutes one of the major differences from those perfect planes introduced in [15, 16]. All the perfect planes defined there are bounded due to the use of two different incident waves, see lemma 3.2 in [15]. In our subsequent path argument for proving theorem 1.1, the procedure of continuation of perfect planes along an exit path might be broken down with the presence of an unbounded perfect plane, since one may not be able to find another perfect plane with an unbounded perfect plane by using the reflection principle in theorem 2.2. In the remainder of this section, we shall show some critical properties on the unbounded perfect planes.

**Lemma 2.3.** All the unbounded perfect planes associated with \( E \) in \( G \) are conplane.

Obviously, lemma 2.3 is divided into the following two lemmata:

**Lemma 2.4.** There cannot exist two unbounded perfect planes \( \tilde{\Pi}_1 \) and \( \tilde{\Pi}_2 \) such that \( \Pi_1 \parallel \Pi_2 \).

**Lemma 2.5.** There cannot exist two different unbounded perfect planes \( \tilde{\Pi}_1 \) and \( \tilde{\Pi}_2 \) such that \( \Pi_1 \parallel \Pi_2 \).

**Proof of lemma 2.4.** Assume contrarily that \( \tilde{\Pi}_1 \) and \( \tilde{\Pi}_2 \) are two unbounded perfect planes in \( G \) such that \( \Pi_1 \parallel \Pi_2 \). Let \( \nu_1 \) and \( \nu_2 \), respectively, be the unit normals to \( \Pi_1 \) and \( \Pi_2 \). Noting that \( E(x) = O(1/|x|) \) as \( |x| \to \infty \), we have from

\[
\nu_l \times E(x) = 0 \quad \text{on} \quad \tilde{\Pi}_l \quad \text{for} \quad l = 1, 2,
\]
that
\[
\lim_{x \in \Pi_l; |x| \to \infty} |v_l \times E_l(x)| = 0 \quad \text{for} \quad l = 1, 2.
\]
Using (1.1), we further deduce
\[
v_l \times ((d_0 \times p_0) \times d_0) = 0 \quad \text{for} \quad l = 1, 2.
\]
That is, \( v_1 \parallel v_2 \) since they are both parallel to a fixed vector \((d_0 \times p_0) \times d_0\), contradicting to our assumption that \( \Pi_1 \parallel \Pi_2 \) and completing the proof. \( \square \)

**Proof of Lemma 2.5.** By contradiction, we assume that there exist two different perfect planes \( \tilde{\Pi}_1 \) and \( \tilde{\Pi}_2 \) such that \( \Pi_1 \parallel \Pi_2 \). Let \( T := T_e(0) \) be a sufficiently large cube such that \( D \subset T \), and by suitable rotation, we may without loss of generality assume that both \( \Pi_1 \) and \( \Pi_2 \) are perpendicular to one face of \( T \). Next, with a little bit abuse of notations, we still denote by \( \tilde{\Pi}_1 \) and \( \tilde{\Pi}_2 \) those parts of \( \tilde{\Pi}_1 \) and \( \tilde{\Pi}_2 \) lying outside of \( T \), namely, \( \tilde{\Pi}_1 \setminus T \) and \( \tilde{\Pi}_2 \setminus T \), and the same rule applies to \( \Pi_1, l \in \mathbb{Z} \) appearing in the remainder of the proof. Now, in the (unbounded) polyhedral domain \( \mathbb{R}^3 \setminus T \), we can make use of the reflection as stated in (ii) of theorem 2.2, and from \( \tilde{\Pi}_1 \) and \( \tilde{\Pi}_2 \) to find that
\[
v \times E = 0 \quad \text{on} \quad \tilde{\Pi}_1 := \mathcal{R}_{\Pi_1}(\tilde{\Pi}_1).
\]
Continuing with such argument, from \( \tilde{\Pi}_2 \) and \( \tilde{\Pi}_3 \) we have
\[
v \times E = 0 \quad \text{on} \quad \tilde{\Pi}_4 := \mathcal{R}_{\Pi_1}(\tilde{\Pi}_2).
\]
By repeating this reflection, we eventually find a sequence of perfect planes \( \tilde{\Pi}_l, l = 1, 2, 3, \ldots \) such that all \( \tilde{\Pi}_l \)'s are parallel to each other. Clearly, \( d(\tilde{\Pi}_l, \tilde{\Pi}_{l+1}) = d(\tilde{\Pi}_1, \tilde{\Pi}_2) > 0 \) being fixed for \( l = 1, 2, 3, \ldots \) Hence, there must exist some \( l_0 < \infty \) such that \( T \) lies entirely at one side of \( \tilde{\Pi}_{l_0} \). That is, \( \tilde{\Pi}_{l_0} = \Pi_{l_0} \) is the whole plane in \( \mathbb{R}^3 \). Obviously, \( D \) also lies at one side of \( \Pi_{l_0} \). Using again the reflection principle in theorem 2.2 (ii), we see \( v \times E = 0 \) on \( \mathcal{R}_{\Pi_{l_0}}(\partial D) \). Finally, let \( \Sigma_1 \) and \( \Sigma_2 \) be two adjacent faces of \( \mathcal{R}_{\Pi_{l_0}}(\partial D) \) and we have from the extension of \( \Sigma_1 \) and \( \Sigma_2 \) two non-parallel unbounded perfect planes, which contradicts to lemma 2.4. The proof is completed. \( \square \)

We proceed to make an important observation of the reflection principle (i) in theorem 2.2, when \( \Omega \cup \mathcal{R}_{\Pi} \Omega \) is unbounded while \( \Pi \) is bounded. In this case, it is clear that the extension of some part of \( (\Pi \cap (\Omega \cup \mathcal{R}_{\Pi} \Omega)) \setminus \Pi \) gives at least one unbounded perfect plane. That is, some bounded perfect plane might imply the existence of some correspondingly unbounded perfect plane. Next, we study carefully such special bounded perfect plane \( \tilde{\Pi}_0 \), which can be regarded as ‘unbounded’. To localize our investigation, we fix an arbitrary point \( x_0 \in \Pi_0 \cap G \) and take a sufficiently small ball \( B_{l_0} := B_{l_0}(x_0) \) such that \( B_{l_0} \subset G \). \( B_{l_0} \) is divided by \( \tilde{\Pi}_0 \) into two half balls, which we respectively denote by \( B_{l_0}^+ \) and \( B_{l_0}^- \). Let \( G_{0}^+ \) be the connected component of \( G \setminus \tilde{\Pi}_0 \) containing \( B_{l_0}^+ \), and \( G_{0}^- \) be the connected component of \( G \setminus \tilde{\Pi}_0 \) containing \( B_{l_0}^- \). We remark that it may happen that \( G_{0}^+ = G_{0}^- \). Next, let \( \Lambda_{0}^+ \) be the connected component of \( G_{0}^+ \cap \mathcal{R}_{\Pi_0}(G_{0}) \) containing \( B_{l_0}^+ \), and \( \Lambda_{0}^- \) be the connected component of \( G_{0}^- \cap \mathcal{R}_{\Pi_0}(G_{0}) \) containing \( B_{l_0}^- \). Finally, set \( \Lambda_{0} = \Lambda_{0}^+ \cup \tilde{\Pi}_0 \cup \Lambda_{0}^- \) and we see that \( \Lambda_{0} \) is a polyhedral domain which symmetric with respect to \( \Pi_0 \), and moreover, \( B_{l_0} \subset \Lambda_{0} \). One can easily see that the construction of \( \Lambda_{0} \) is only dependent on the perfect plane \( \Pi_0 \). Since \( \partial \Lambda_{0} \) is composed of subsets lying either on \( \partial D \) or on \( \mathcal{R}_{\Pi_0}(\partial D) \), by the reflection principle in (ii) of theorem 2.2, we have \( v \times E = 0 \) on \( \partial \Lambda_{0} \).

From now on, we shall denote by \( \Lambda_{\tilde{\Pi}_0} \) the symmetric set constructed as above corresponding to a bounded perfect plane \( \tilde{\Pi}_1 \), namely, in the above, \( \Lambda_{\tilde{\Pi}_0} := \Lambda_{0} \). Clearly, in case \( \Lambda_{\tilde{\Pi}_0} \) is unbounded, we see from our earlier discussion that there must exist an unbounded...
perfect plane which is extended from some part of \((\Pi \cap \Lambda_{\tilde{\Pi}}) \setminus \tilde{\Pi})\). Such observation in combination with the result in lemma 2.3 gives the following.

**Lemma 2.6.** All the bounded perfect planes \(\tilde{\Pi}\) with unbounded \(\Lambda_{\tilde{\Pi}}\), and all the unbounded perfect planes are conplane.

Based on lemma 2.6, we introduce the following set consisting of all the ‘unbounded’ perfect planes
\[
Q_E := \{\tilde{\Pi}; \tilde{\Pi} \text{ is an unbounded perfect plane} \}
\]

or \(\tilde{\Pi}\) is a bounded perfect plane but with unbounded \(\Lambda_{\tilde{\Pi}}\).

(2.3)

Since all the members in \(Q_E\) are conplane, one verifies directly that \(Q_E\) consists of at most finitely many perfect planes by noting the fact that \(D\) is composed of finitely many pairwise disjoint compact polyhedra. We further define \(\tilde{Q}_E\) to be the subset of \(Q_E\) consisting of those bounded perfect planes in \(Q_E\). Next, we show some topological properties of the sets \(Q_E\) and \(\tilde{Q}_E\).

**Lemma 2.7.** Let \(G := \mathbb{R}^3 \setminus D\), then

(i) \(G \setminus \tilde{Q}_E\) is connected;

(ii) \(G \setminus Q_E\) has no bounded connected component.

**Proof.** We first observe that \(Q_E\) is bounded since \(Q_E \subset ch(D)\), where \(ch(D)\) is the convex hull of \(D\). By further noting that \(\partial G\) is bounded, we know that \(G \setminus Q_E\) has exactly one unbounded connected component. Hence, if \(G \setminus \tilde{Q}_E\) is not connected, it must have some bounded connected component, say \(C_0 \subset G\). Clearly, there must be one face of the polyhedral domain \(C_0\) that comes from exactly a perfect plane in \(\tilde{Q}_E\), say \(\tilde{\Pi}_0\). Now, one can verify directly that \(\Lambda_{\tilde{\Pi}_0} \subset C_0 \cup \partial_{\tilde{\Pi}_0} C_0\), which is bounded since \(C_0\) is bounded. But this contradicts to the assumption that \(\tilde{\Pi}_0 \in Q_E\), thus proving assertion (i). Next, assertion (ii) is readily seen from (i). In fact, if \(G \setminus \tilde{Q}_E\) has a bounded connected component, say \(D_0\), then one must have \(D_0 \subset G \setminus Q_E\), which is certainly not true. The proof is completed. \(\square\)

Correspondingly, we set

\[
S_E := \{\tilde{\Pi}; \tilde{\Pi} \text{ is a bounded perfect plane with bounded } \Lambda_{\tilde{\Pi}}\}.
\]

(2.4)

Finally, we give a lemma concerning the fundamental property of a connected set (see e.g., theorem 3.19.9 in [8]), which shall be needed in the following section on proving theorem 1.1.

**Lemma 2.8.** Let \(E\) be a metric space, \(\mathcal{A} \subset E\) be a subset and \(\mathcal{B} \subset E\) be a connected set such that \(\mathcal{A} \cap \mathcal{B} \neq \emptyset\) and \((E \setminus \mathcal{A}) \cap \mathcal{B} \neq \emptyset\), then \(\partial \mathcal{A} \cap \partial \mathcal{B} \neq \emptyset\).

**3. Proof of theorem 1.1**

The entire section is devoted to proof of theorem 1.1 by contradiction. Assume that \(D \neq \tilde{D}\) and

\[
E_\infty(\hat{x}; D, p_0, k_0, d_0) = E_\infty(\hat{x}; \tilde{D}, p_0, k_0, d_0) \quad \text{for} \quad \hat{x} \in S^2.
\]

(3.1)
Let $\Omega$ be the (unique) unbounded connected component of $\mathbb{R}^3 \setminus (D \cup \tilde{D})$. By Rellich’s theorem (see theorem 6.9, [6]), we infer from (3.1) that

$$E(x; D) = E(x; \tilde{D}) \quad \text{for} \quad x \in \Omega,$$

where $E(x; D)$ and $E(x; \tilde{D})$ are, respectively, abbreviations $E(x; D, p_0, k_0, d_0)$ and $E(x; \tilde{D}, p_0, k_0, d_0)$. Next, noting that $D \neq \tilde{D}$, we see that either $(\mathbb{R}^3 \setminus \tilde{\Omega}) \setminus D \neq \emptyset$ or $(\mathbb{R}^3 \setminus \Omega) \setminus \tilde{D} \neq \emptyset$. Without loss of generality, we assume the former case and let $D^* := (\mathbb{R}^3 \setminus \tilde{\Omega}) \setminus D \neq \emptyset$. It is easily seen that $D^* \subset \tilde{D}$, so $D^*$ is bounded. Moreover, by choosing connected component if necessary, we assume that $D^*$ is connected. Clearly, $D^*$ is a bounded polyhedral domain in $G = \mathbb{R}^3 \setminus D$ and $E(x; D)$ is defined over $D^*$.

Let $\partial D^* \subset \partial \Omega \cup \partial D \subset \partial D \cup \partial \tilde{D}$ and using (3.2), we have from the perfect boundary conditions of $E(x; D)$ and $E(x; \tilde{D})$ on $\partial D$ and $\partial \tilde{D}$ that

$$\nu \times E(x; D) = 0 \quad \text{on} \quad \partial D^*.$$  

In the following, in order to simplify notations, we use as those introduced in section 2, e.g., we write $E(x)$ to denote $E(x; D)$, etc. The remainder of the proof will be proceeded into three steps and a brief outline is as follows. In the first step, we will find a perfect plane $\bar{\Pi}_1 \subset \Sigma_0$, and this is the starting point of the subsequent path argument. In the second step, we would construct implicitly an exit path, which is a regular curve lying entirely in the exterior of $D$ and connecting to infinity. As we mentioned earlier that the path argument might be broken down with the presence of some ‘unbounded’ perfect planes (namely, perfect planes in $Q_k$), in order to avoid our subsequent argument being trapped at such ‘unbounded’ perfect planes, the curve is required to have at most one intersection with $Q_k$. Fortunately, this can be done by using lemma 2.7. Finally, using the reflection principle in theorem 2.2, we make continuation of (bounded) perfect planes along the exit path to find a sequence of perfect planes. Then a contradiction is constructed by showing that the continuation must follow the exit path to infinity since we always step a length larger than a fixed positive constant when making such continuation, but on the other hand, all the bounded perfect planes are contained in the convex hull of $D$ being bounded. In this final step, we must be carefully treating the possible presence of ‘unbounded’ perfect planes and this is the main difference of the present path argument from those implemented in [15, 16].

**Step I: existence of a bounded perfect plane $\bar{\Pi}_1$ with bounded $\Lambda_{\bar{\Pi}_1}$**

We first note that $\partial D^* \setminus \partial D \neq \emptyset$. Hence, there must be an open face say $\Sigma_0$ on $\partial D^*$ that can be extended in $G$ to form a perfect plane and it is denoted by $\bar{\Pi}_0$. Since $\bar{\Pi}_0$ is extended from a face of the bounded polyhedral domain $D^*$ in $G$, we infer from lemma 3.1 that $\bar{\Pi}_0$ is bounded. Now, if the symmetric set $\Lambda_{\bar{\Pi}_0}$ corresponding to $\bar{\Pi}_0$ is bounded, then we are done since we can take $\bar{\Pi}_0$ as $\bar{\Pi}_1$. So, without loss of generality, we assume that $\Lambda_{\bar{\Pi}_0}$ is unbounded. Next, based on $\Sigma_0$, we construct a bounded polyhedral domain in $G$ which is symmetric with respect to $\Pi_0$ but different from $\Lambda_{\bar{\Pi}_0}$. The construction procedure is similar to that for $\Lambda_{\bar{\Pi}_0}$, and we nonetheless present it here for clearness.

Fix an arbitrary point $x^* \in \Sigma_0$ and let $B^* := B_\varepsilon(x^*)$ with $\varepsilon > 0$ sufficiently small such that $B^*$ is divided by $\Sigma_0$ into two (open) half balls $B^+_* \subset \tilde{D}$ and $B^-_* \subset G \setminus D^*$. Next, let $\bar{\Theta}^*$ be the connected component of $\bar{\Pi}_0(G \setminus D^*) \cap D^*$ containing $B^*_*$ and $\bar{\Theta}^*$ be the connected component of $\bar{\Pi}_0(D^*) \cap (G \setminus D^*)$ containing $B^-_*$. Set $\Theta^* = \bar{\Theta}^* \cup \Sigma_0 \cup \bar{\Theta}^-_*$. Clearly, $\Theta^*$ is a non-empty bounded polyhedral domain in $G$ since $B^* \subset \Theta^* \subset D^* \cup \bar{\Pi}_0 \setminus D^*$. We remark that $\Theta_0$ is in fact the connected component of $(D^* \cup \bar{\Pi}_0 \setminus D^*) \cap \Lambda_{\bar{\Pi}_0}$ containing $\Sigma_0$. By the reflection principle (ii) of theorem 2.2, $\nu \times E(x) = 0$ on $\partial \Theta^*$. It is obvious that
\partial \Theta^* \setminus D \neq \emptyset. \text{ Let } \Sigma_1 \subset \partial \Theta^* \setminus D \text{ be an open face. By analytic continuation, } \Sigma_1 \text{ is extended in } G \text{ to give a perfect plane } \tilde{\Pi}_1. \text{ Since } \Theta^* \text{ is symmetric with respect to } \Pi_0, \text{ we know } \Sigma_1 \not\subseteq \Pi_0 \text{ and therefore } \tilde{\Pi}_1 \not\subseteq Q_E \text{ by lemma 2.6, i.e. } \tilde{\Pi}_1 \text{ is bounded with bounded } \Lambda_{\tilde{\Pi}_1}.

**Step II: construction of the exit path } \gamma \**

Since both \( \partial G \) and \( \tilde{\Pi}_1 \) are bounded, we see that \( G \setminus \tilde{\Pi}_1 \) has a unique unbounded connected component, which is denoted by \( \mathcal{V} \). It readily has that \( \tilde{\Pi}_1 \subset \partial \mathcal{V} \) and \( \mathcal{V} \) contains the exterior of a sufficiently large ball containing \( D \). Next, we fix an arbitrarily point \( x_1 \in \tilde{\Pi}_1 \). Let \( \gamma := \gamma(t)(t \geq 0) \) be a regular curve such that \( \gamma(t_1) = x_1 \) with \( t_1 = 0 \) and \( \gamma(t)(t > 0) \) lies entirely in \( \mathcal{V} \). Furthermore, \( \gamma \) connects to infinity, i.e. \( \lim_{t \to \infty} |\gamma(t)| = \infty \). The exit path \( \gamma \) constructed in this way might have non-empty intersection with \( Q_E \). In this case, we require that \( \gamma(t)(t > 0) \) has only one intersection point with \( Q_E \). In fact, in case \( \gamma(t) \cap Q_E \neq \emptyset \), we would modify the curve \( \gamma \) as follows to satisfy such requirement. Let \( x_T := \gamma(T) \) be the ‘first’ intersection point of \( \gamma(t)(t > 0) \) and \( Q_E \); that is,

\[
T = \min\{t > 0; \gamma(t) \in Q_E\} < \infty.
\]

Then, set \( \mathcal{V}' \) be the connected component of \( G \setminus Q_E \) such that \( x_T \in \partial \mathcal{V}' \). Let \( \mathcal{V} := \mathcal{V}' \cap \gamma \). It can be verified that \( \mathcal{V} \) is an unbounded connected open set such that \( x_T \in \partial \mathcal{V} \). Indeed, the connectedness of \( \mathcal{V} \) is obvious by noting that both \( \mathcal{V}' \) and \( \gamma \) are connected. Whereas the unboundedness of \( \mathcal{V} \) is due to the facts that \( \gamma \) is unbounded by lemma 2.7 and \( \mathcal{V}' \) contains the exterior of a sufficiently large ball containing \( D \) as mentioned earlier. Next, let \( \eta(t)(t > T) \) be a regular curve such that \( \eta(T) = x_T, \eta(t)(t > T) \) lies entirely in \( \mathcal{V} \) and connects to infinity (i.e. \( \lim_{t \to \infty} |\eta(t)| = \infty \)). Furthermore, it is trivially required that \( \eta(t) \) has \( C^1 \) connection with \( \gamma(t)(0 \leq t \leq T) \) at \( x_T \). Now, set

\[
\tilde{\gamma}(t) = \begin{cases}
\gamma(t) & 0 \leq t \leq T, \\
\eta(t) & t > T,
\end{cases}
\]

then \( \tilde{\gamma}(t)(t \geq 0) \) satisfies all our requirements of an exit path.

**Step III: continuation of bounded perfect planes along } \gamma \**

Let \( d_0 = d(\gamma, D) > 0 \), which is attainable since \( D \) is compact, and \( r_0 = d_0/2 \). Clearly, \( B_{r_0}(\gamma(t)) \subset G \) for any \( t \geq 0 \). Let \( \tilde{x}_2 = \gamma(t_2) \in S_{r_0}(x_1) \cap \gamma \), where \( t_2 \) is taken to be \( t_2 = \max\{t > 0; \gamma(t) \in S_{r_0}(x_1)\} \), and let \( \tilde{x}_2 \) be the symmetric point of \( \tilde{x}_2 \) with respect to \( \Pi_1 \). Next, let \( G_1^* \) be the connected component of \( G \setminus \tilde{\Pi}_1 \) containing \( \tilde{x}_2 \), and \( G_2^* \) be the connected component of \( G \setminus \tilde{\Pi}_1 \) containing \( \tilde{x}_2 \). Then let \( \Lambda_1^* \) be the connected component of \( G_1^* \cap \Pi_1 \), \( G_2^* \) containing \( \tilde{x}_2 \), and \( \Lambda_2^* \) be the connected component of \( G_2^* \cap \Pi_1 \), \( G_1^* \) containing \( \tilde{x}_2 \). Set \( \Lambda_1 = \Lambda_1^* \cap \Pi_1 \cup \Lambda_2^* \). In fact, \( \Lambda_1 \) is the symmetric set \( \Lambda_{\tilde{\Pi}_2} \) corresponding to the perfect plane \( \tilde{\Pi}_1 \) and we present its construction again for convenience of the subsequent argument. Since \( \tilde{\Pi}_1 \in D, \Lambda_1 \) is bounded. By lemma 2.8, it is easy to deduce that \( \gamma \cap \partial \Lambda_1 \neq \emptyset \). We let \( x_2 = \gamma(t_2) \) be the ‘last’ intersection point of \( \gamma \) and \( \partial \Lambda_1 \); namely, \( t_2 = \max\{t > 0; \gamma(t) \in \partial \Lambda_1\} < \infty \). This then implies the existence of a perfect plane passing through \( x_2 \) which is extended from an open face of \( \partial \Lambda_1 \) whose closure contains \( x_2 \).

We denote the perfect plane by \( \tilde{\Pi}_2 \). Without loss of generality, we further assume that \( x_2 \) is the ‘last’ intersection point of \( \gamma \) with \( \tilde{\Pi}_2 \). By the following result, we know \( \tilde{\Pi}_2 \) is bounded. We shall prove at the end of this section.

**Lemma 3.1.** Suppose that \( \Lambda \subset G \) is a bounded polyhedral domain such that

\[
\nu \times E = 0 \quad \text{on} \quad \partial \Lambda.
\]

\[9\]
Then every open face lying on $\partial \Lambda \setminus D$ cannot be connectedly extended to an unbounded planar domain in $G$.

Now, we still need to distinguish between two cases of $\bar{\Pi}_2 \in \bar{Q}_E$ and $\bar{\Pi}_2 \in S_E$. But for the end of a more general discussion, we next give the induction procedure for the above reflection argument of finding a different perfect plane with a known one. Suppose that $\bar{\Pi}_n \in S_E$, $n \in \mathbb{N}$ and $x_n := \gamma(t_n) \in \gamma \cap \bar{\Pi}_n$ is the ‘last’ intersection point between $\gamma$ and $\bar{\Pi}_n$.

Let $\bar{x}_{n+1} = \gamma(t_{n+1}) \in S_n(x_n) \cap \gamma$, where $t_{n+1}$ is taken to be $t_{n+1} = \max \{t > 0; \gamma(t) \in S_n(x_n)\}$, and let $\bar{x}_{n+1}$ be the symmetric point of $\bar{x}_{n+1}$ with respect to $\Pi_1$. Next, let $G_n^*$ be the connected component of $G \setminus \bar{\Pi}_n$ containing $\bar{x}_{n+1}$, and $G_n^*$ be the connected component of $G \setminus \bar{x}_{n+1}$ containing $\bar{x}_{n+1}$. Then let $A_n^*$ be the connected component of $G_n^* \cap R_{\Pi_n}(G_n^*)$ containing $\bar{x}_{n+1}$, and $A_n^*$ be the connected component of $G_n \cap R_{\Pi_n}(G_n^*)$ containing $\bar{x}_{n+1}$. Set $A_n = A_n^* \cup \bar{x}_{n+1} \cup A_n^*$. By our earlier discussion, $A_n = A_n^*$, and it is bounded since $\bar{\Pi}_n \in S_E$. By Lemma 2.8, $\gamma \cap \partial A_n \neq \emptyset$. We let $x_{n+1} := \gamma(t_{n+1})$ with $t_{n+1} = \max \{t > 0; \gamma(t) \in \partial A_n\} < \infty$. This again implies the existence of a perfect plane $\Pi_{n+1}$ passing through $x_{n+1}$, and $\Pi_{n+1}$ is bounded by Lemma 3.1. Furthermore, we can also assume that $x_{n+1}$ is the ‘last’ intersection point of $\gamma$ with $\Pi_{n+1}$. In the following, we list several important results that have been achieved:

(i) $x_n, x_{n+1} \in \mathcal{P} := \{x \in \mathcal{P}_E; x \in \bar{\Pi} \in S_E \cup \bar{Q}_E\}$;

(ii) $\bar{\Pi}_{n+1}$ is different from $\bar{\Pi}_{n}$, since $t_n$ and $t_{n+1}$ with $t_{n+1} > t_n$ are respectively the ‘last’ intersection points between $\gamma$ and $\bar{\Pi}_n$ and $\bar{\Pi}_{n+1}$;

(iii) Both $\bar{\Pi}_n$ and $\bar{\Pi}_{n+1}$ are bounded;

(iv) Since $B_{r_0}(x_n) \subset A_n$, the length of $\gamma(t)$ from $t_n$ to $t_{n+1}$ is not less than $r_0$, i.e.

$$|\gamma(t_n \leq t \leq t_{n+1})| \geq |\gamma(t_n \leq t \leq t_{n+1})| \geq r_0.$$  (3.5)

If $\bar{\Pi}_{n+1} \in S_E$, by repeating the above reflection argument, we can find another bounded perfect plane $\bar{\Pi}_{n+2}$, and also $x_{n+2} := \gamma(t_{n+2})$, the ‘last’ intersection point between $\gamma$ and $\bar{\Pi}_{n+2}$, such that

$$|\gamma(t_n \leq t \leq t_{n+2})| \geq r_0.$$  (3.6)

In case $\bar{\Pi}_{n+1} \in \bar{Q}_E$, we can no longer guarantee that $\gamma \cap \partial A_{n+1} \neq \emptyset$ since $A_{n+1} = A_{\bar{\Pi}_{n+1}}$ is unbounded. Let $B_0 := B_{r_0}(x_{n+1})$ with $\epsilon_0 > 0$ sufficiently small such that $B_0 \subset B_{r_0}(x_{n+1})$ and one of the half ball of $B_0$ divided by $\bar{\Pi}_{n+1}$ is contained entirely in $A_n$. Here, we recall that $\bar{\Pi}_{n+1}$ is extended from an open face of $A_n$. Then, let $A_{n+1}^*$ be the connected component of $(A_n \cup \bar{\Pi}_{n+1}, A_n) \cap A_{n+1}$ containing $B_0$. It is remarked that this is similar to the construction of $\Theta^*$ from $D^*$ in step I. Since $A_n$ is bounded, we know $A_{n+1}^*$ is bounded. Moreover, by the reflection principle in (ii) of theorem 2.2, $\nu \times E(x) = 0$ on $\partial A_{n+1}^*$. Now, by Lemma 2.8, it is verified directly that $\gamma \cap A_{n+1}^* \neq \emptyset$. Also, we let $x_{n+2} := \gamma(t_{n+2})$ be the ‘last’ intersection point between $\gamma$ and $\partial A_{n+2}^*$. By analytic continuation, this implies the existence of a perfect plane $\bar{\Pi}_{n+2}$ passing through $x_{n+2}$, which must be bounded by Lemma 3.1. More importantly, noting that $\bar{\Pi}_{n+2}$ is not conplane to $\bar{\Pi}_{n+1}$, we know by Lemma 2.6 that $\bar{\Pi}_{n+2} \in S_E$. As what has been frequently done before, we can further assume that $x_{n+2}$ is the ‘last’ intersection point of $\gamma$ with $\bar{\Pi}_{n+2}$. Finally, it is easy to show

$$|\gamma(t_n \leq t \leq t_{n+2})| \geq \epsilon_0.$$
By induction and also by noting that \( \gamma(t)(t > 0) \) has at most one intersection point with \( Q_k \), we have constructed a sequence of different perfect planes \( \tilde{\Pi}_n \), \( n = 1, 2, 3, \ldots \), all belonging to \( S_k \) except possibly only one belonging to \( \tilde{Q}_k \). Moreover, there is a strictly increasing sequence \( \{t_n\}_{n=1}^\infty \) together with a sequence of points \( x_n = \gamma(t_n) \in \gamma \cap \tilde{\Pi}_n \), \( n = 1, 2, 3, \ldots \), such that

\[
|\gamma(t_n) - \gamma(t_{n+1})| \geq r_0 \quad \text{when} \quad n > n_0,
\]

where \( n_0 \) is the index such that \( \tilde{\Pi}_{n_0} \notin \tilde{Q}_k \) and it might be 0.

Now, we can conclude our proof of theorem 1.1 by a contradiction as follows. Since \( \gamma(t_n) \in \tilde{\mathcal{R}} \subset \partial h(D) \) being bounded and \( \lim_{t \to \infty} |\gamma(t)| = \infty \), we know there must exist some \( T_0 < \infty \) such that \( \lim_{n \to \infty} t_n = T_0 \). Then,

\[
\lim_{n \to \infty} |\gamma(t_n) - \gamma(t_{n+1})| = \lim_{n \to \infty} \int_{t_n}^{t_{n+1}} |\gamma'(t)| \, dt = 0.
\]

A contradiction to (3.6).

**Proof of lemma 3.1.** Assume contrarily that there is an open face \( \Gamma_0 \) on \( \partial A \setminus D \) which can be connectedly extended in \( G \) to give an unbounded planar domain. By analytic continuation, this gives an unbounded perfect plane \( \tilde{\Pi}_0 \). Since \( A \) is a bounded polyhedron in \( G \), \( \tilde{\Pi}_0 \) must be separated from \( A \) at some of its edge. Hence, there is another open face \( \Gamma_1 \) on \( \partial A \setminus D \), such that \( \Gamma_0 \) and \( \Gamma_1 \) have a common edge in \( G \). Again by analytic continuation, we have a perfect plane \( \tilde{\Pi}_1 \) from the connected extension of \( \Gamma_1 \) in \( G \). Noting \( \tilde{\Pi}_0 \notin \tilde{Q}_k \), we see \( \tilde{\Pi}_1 \in S_k \) by lemma 2.6. Next, the argument follows a similar manner as that of step III in proof of theorem 1.1.

Fix an arbitrary point \( x_1 \in \Gamma_0 \cap \Gamma_1 \). Let \( \gamma := \gamma(t)(t \geq 0) \) be a regular curve such that \( \gamma(t_1) = x_1 \) with \( t_1 = 0 \) and \( \gamma(t)(t > 0) \) lies entirely in the unbounded connected component of \( \tilde{\Pi}_0 \setminus A \) and \( \lim_{t \to \infty} |\gamma(t)| = \infty \). Set \( t_0 = d(\gamma, D) > 0 \). From our earlier discussion in step III of proof of theorem 1.1, we know \( \gamma \cap \partial A_{\tilde{\Pi}_1} \neq \emptyset \). Furthermore, letting \( x_2 := \gamma(t_2) \) be the ‘last’ intersection point of \( \gamma \) with \( \partial A_{\tilde{\Pi}_1} \), there is another perfect plane \( \tilde{\Pi}_2 \) extended from an open face on \( \partial A_{\tilde{\Pi}_1} \) such that \( \tilde{\Pi}_2 \) passes through \( x_2 \) and

\[
|\gamma(t_1) - \gamma(t_2)| \geq t_0.
\]

A crucial observation is that \( \gamma \subset \tilde{\Pi}_0 \), we can without loss of generality assume that \( \tilde{\Pi}_2 \) is non-parallel to \( \tilde{\Pi}_0 \); therefore \( \tilde{\Pi}_2 \in S_k \) by lemma 2.6. By repeating the above procedure, we can construct countably many different perfect planes \( \tilde{\Pi}_n \in S_k \), \( n = 1, 2, 3, \ldots \), together with a sequence of points \( x_n := \gamma(t_n) \in \gamma \cap \tilde{\Pi}_n \) satisfying

\[
|\gamma(t_n) - \gamma(t_{n+1})| \geq t_0.
\]

Finally, a similar contradiction is established as that in (3.7), thus completing the proof. \( \Box \)

**4. Concluding remarks**

In this paper, we have established a global uniqueness for the formally determined inverse electromagnetic obstacle scattering. That is, the far-field pattern \( E_\infty(\hat{x} ; D, p_0, k_0, d_0) \) for fixed \( p_0 \in \mathbb{R}^3, k_0 > 0, d_0 \in \mathbb{S}^2 \) and all \( \hat{x} \in \mathbb{S}^2 \), uniquely determine a general polyhedral scatterer \( D \). As mentioned in the introduction, some uniqueness results on the unique determination of general polyhedral obstacles have been established, but all with the far-field patterns corresponding to two different incident waves.

In [15], the underlying obstacle admits the simultaneous presence of finitely many cracks, where a *crack* is defined to be the closure of some bounded open subset of a plane in \( \mathbb{R}^3 \).
That is, in addition to finitely many solid polyhedra, the polyhedral obstacle $D$ in [15] may also contain finitely many cracks. In the case with the additional presence of a crack to the polyhedral obstacle $D$ considered in theorem 1.1, one verifies straightforwardly that the argument in step I of its proof might not hold any longer. In fact, one may not be able to find a bounded polyhedral domain $D^*$ in $G$, and it might be a sole crack instead. In turn, one may not be able to construct the bounded polyhedral domain $\Theta^*$, which is essential to find the starting perfect plane $\tilde{\Pi}_1 \in \tilde{S}_E$ for the subsequent path argument.

Since knowing $E_\infty(\hat{x}; D)$ and $H_\infty(\hat{x}; D)$ are equivalent, one can see that theorem 1.1 is still valid with the polyhedral obstacle $D$ associated with the following perfect boundary condition corresponding to $H$:

$$\nu \times H = 0 \quad \text{on} \quad \partial G.$$  \hspace{1cm} (4.1)

In [16], a more general situation is considered that we need not to know the a priori physical properties of the underlying obstacle. That is, the underlying obstacle $D$ may be either associated with boundary condition (1.5), or (4.1), or even with mixed type of (1.5) and (4.1). In such setting, we need to consider perfect planes corresponding to both the electric field $E$ and magnetic field $H$ (see [16]). By using a single incident wave, one can show that lemma 2.3 may not hold any longer. In fact, by direct calculations, two non-parallel unbounded perfect planes, one corresponding to $E$ and the other corresponding to $H$, may not give a contradiction as that in lemma 2.4. Consequently, theorem 1.1 might not be valid with the underlying polyhedral obstacle $D$ associated with mixed boundary conditions.

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