Quantum Games Entropy

Esteban Guevara Hidalgo
†Departamento de Física, Escuela Politécnica Nacional, Quito, Ecuador
‡SIÓN, Autopista General Ruminahui, Urbanización Edén del Valle, Sector 5, Calle 1 y Calle A # 79, Quito, Ecuador

We propose the study of quantum games from the point of view of quantum information theory and statistical mechanics. Every game can be described by a density operator, the von Neumann entropy and the quantum replicator dynamics. There exists a strong relationship between game theories, information theories and statistical physics. The density operator and entropy are the bonds between these theories. The analysis we propose is based on the properties of entropy, the amount of information that a player can obtain about his opponent and a maximum or minimum entropy criterion. The natural trend of a physical system is to its maximum entropy state. The minimum entropy state is a characteristic of a manipulated system i.e. externally controlled or imposed. There exist tacit rules inside a system that do not need to be specified or clarified and search the system equilibrium based on the collective welfare principle. The other rules are imposed over the system when one or many of its members violate this principle and maximize its individual welfare at the expense of the group.

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I. INTRODUCTION

In a recent work [1] we proposed certain quantization relationships based on the resemblances between quantum mechanics and game theory. Although both systems analyzed are described through two apparently different theories it was shown that both are analogous and thus exactly equivalents. So, we can take some concepts and definitions from quantum mechanics and physics for the best understanding of the behavior of economical and biological processes. The quantum analogues of the replicator dynamics and of certain relative frequencies matrix are the von Neumann equation and the density operator respectively. This would let us analyze the entropy of our system through the well known von Neumann (or Shannon) entropy. The properties that these entropies enjoy would let us analyze a “game” from a different point of view through statistical mechanics and quantum information theories.

There exists a strong relationship between game theories, statistical mechanics and information theory. The bonds between these theories are the density operator and entropy [2, 3]. The density operator is maybe the most important tool in quantum mechanics. It was introduced by von Neumann to describe a mixed ensemble in which each member has assigned a probability of being in a determined state. From the density operator we can construct and understand the statistical behavior about our system by using statistical mechanics and a criterion of maximum or minimum entropy. Also we can develop the system in function of its accessible information and analyze it through information theories.

Since Shannon [4, 5], information theory or the mathematical theory of communication changed from an engineering discipline that dealt with communication channels and codes [6] to a physical theory [7] in where the introduction of the concepts of entropy and information were indispensable to our understanding of the physics of measurement. Classical information theory has two primary goals [8]: The first is the development of the fundamental theoretical limits on the achievable performance when communicating a given information source over a given communication channel using coding schemes from within a prescribed class. The second goal is the development of coding schemes that provide performance that is reasonably good in comparison with the optimal performance given by the theory.

The von Neumann entropy [2, 3] is the quantum analogue of Shannon’s but it appeared 21 years before and it also generalizes Boltzmann’s expression. Entropy in quantum information theory plays prominent roles in many contexts, e.g., in studies of the classical capacity of a quantum channel [9, 10] and the compressibility of a quantum source [11, 12]. It may be defined [13] as the study of the achievable limits to information processing possible within quantum mechanics. The field of quantum information has two tasks: First, it aims to determine limits on the class of information processing tasks which are possible in quantum mechanics and provides constructive means for achieving information processing tasks. It is also the basis for a proper understanding of the emerging fields of quantum computation [14, 15], quantum communication [16, 17], and quantum cryptography [18, 19].

In the present work we will study the applicability of entropy and its properties to the study of games from two sources: Information theory and statistical mechanics and we will also propose possible applications to the solution of specific problems in economics.
II. ON QUANTUM REPLICATOR DYNAMICS & THE QUANTIZATION RELATIONSHIPS

Evolutionary game theory combines the principles of game theory, evolution and dynamical systems to explain the distribution of different phenotypes in biological populations. Instead of working out the optimal strategy, the different phenotypes in a population are associated with the basic strategies that are shaped by trial and error by a process of natural selection or learning. The natural selection process that determines how populations playing specific strategies evolve is known as the replicator dynamics [20, 23] whose stable fixed points are Nash equilibria [24].

We can represent the replicator dynamics in matrix commutative form and realize that follows the same dynamic than the von Neumann equation [1]

\[
\frac{dX}{dt} = [\Lambda, X].
\]

The matrix \( X \) is a relative frequencies matrix. Its elements are \( x_{ij} = (x_i x_j)^{1/2} \) and \( x_i \) is the relative frequency of individuals using the strategy \( s_i \). The matrix \( \Lambda \) is equal to \( \Lambda = [Q, X] \), where \( Q \) and \( \Lambda \) have as elements \( q_{ii} = \frac{1}{2} \sum_{k=1}^{n} a_{ik} x_k \) and \( (\Lambda)_{ij} = \frac{1}{2} \left( (\sum_{k=1}^{n} a_{ik} x_k) x_{ij} - x_{ji} (\sum_{k=1}^{n} a_{jk} x_k) \right) \) respectively, \( a_{ij} \) are the elements of certain payoff matrix \( A \). Each component of \( X \) evolves following the replicator dynamics. If we take \( \Theta = [\Lambda, X] \) equation (1) becomes into \( \frac{dX}{dt} = \Theta \), where the elements of the matrix \( \Theta \) are given by \( (\Theta)_{ij} = \frac{1}{2} \sum_{k=1}^{n} a_{ik} x_k x_{ij} + \frac{1}{2} \sum_{k=1}^{n} a_{jk} x_k x_{ji} - \sum_{k,l=1}^{n} a_{ik} x_k x_{lj} x_{ij} \).

A physical or a socioeconomical system (described through quantum mechanics and game theory respectively) is composed by \( n \) members (particles, subsystems, players, etc.). Each member is described by a state or a strategy which has assigned a probability \( (x_1 \text{ or } \rho_{ij}) \). The quantum mechanical system is described by the density operator \( \rho \) whose elements represent the system average probability of being in a determined state. In evolutionary game theory the system is described through relative frequencies vector \( x \) whose elements represent the frequency of players playing a determined strategy and its evolution is described through the replicator dynamics (in matrix commutative form) which follows the same dynamic than the evolution of the density operator (von Neumann equation). The properties of the correspondent elements \( (\rho \text{ and } X) \) in both systems are similar, and as expected, the properties of our quantum system are more general than the classical system [1].

Although both systems are different, both are analogous and thus exactly equivalents. This let us define and propose the next quantization relationships

\[
x_i \rightarrow \sum_{k=1}^{n} \langle i | \Psi_k \rangle p_k \langle \Psi_k | i \rangle = \rho_{ii},
\]

\[
(x_i x_j)^{1/2} \rightarrow \sum_{k=1}^{n} \langle i | \Psi_k \rangle p_k \langle \Psi_k | j \rangle = \rho_{ij}.
\]

A population will be represented by a quantum system in which each subpopulation playing strategy \( s_i \) will be represented by a pure ensemble in the state \( |\Psi_i(t)\rangle \) and with probability \( p_k \). The probability \( x_i \) of playing strategy \( s_i \) or the relative frequency of the individuals using strategy \( s_i \) in that population will be represented as the probability \( \rho_{ii} \) of finding each pure ensemble in the state \( |i\rangle \).

Through these quantization relationships the quantum analogue of the replicator dynamics (in matrix commutative form) is the von Neumann equation \( (i\hbar \frac{d\rho}{dt} = [\hat{H}, \rho] \), where \( X \rightarrow \rho, \Lambda \rightarrow -\frac{i}{\hbar} \hat{H} \) and \( H(x(t)) \rightarrow S(\rho) \). These quantization relationships could let us describe not only classical, evolutionary and quantum games but also the biological systems that were described before through the replicator dynamics.

It is important to note that equation (11) is nonlinear while its quantum analogue is linear i.e. the quantization eliminates the classical system nonlinearities. The classical system that was described through the matrix \( X \) can be described now through a density operator in where its non diagonal elements could be different from zero (and represent a mixed state) due to the presence of coherence between quantum states which could not be observed when the system was analyzed classically.

III. CLASSICAL GAMES ENTROPY

We can define the entropy of our system by

\[
H = -Tr \{ X \ln X \}
\]

with the non diagonal elements of matrix \( X \) equal to zero i.e. the Shannon entropy over the elements of the relative frequency vector \( x \)

\[
H = -\sum_{i=1}^{n} x_i \ln x_i.
\]

We can describe the evolution of the entropy of our system by supposing that the vector of relative frequencies \( x(t) \) evolves in time following the replicator dynamics

\[
\frac{dx_i}{dt} = [f_i(x) - (f(x))] x_i = U_i x_i
\]
with $U_t = [f_t(x) - \langle f(x) \rangle]$, $f_t(x) = \sum_{j=1}^{n} a_{ij} x_j$ and $\langle f(x) \rangle = \sum_{k,t=1}^{n} a_{kt} f_k f_l$.

$$\frac{dH}{dt} = Tr \left\{ U(\hat{H} - X) \right\}, \quad (6)$$

where $\hat{H}$ is a diagonal matrix whose trace is equal to the Shannon entropy i.e. $H = Tr \hat{H}$.

IV. QUANTUM GAMES ENTROPY

Let's consider a system composed by $N$ members, players, strategies, etc. This system is described completely through the density operator $\rho(t) = \sum_{k=1}^{N} |\Psi_k(t)\rangle p_k \langle \Psi_k(t)\rangle$. Each state can “interact” with the remaining $N - 1$ states. There exists a total of $N^2$ states, $N$ “pure” and $N^2 - N$ that appear “by interaction” between pure states. All of these are grouped in the matrix which represents the density operator. The $N$ pure states are represented by the diagonal elements of the density operator $\rho_{ii} = \sum_{k=1}^{n} \langle i | \Psi_k \rangle p_k \langle \Psi_k | i \rangle$ and the remaining $N^2 - N$ states that can appear “by interaction” by the non diagonal elements $\rho_{ij} = \sum_{k=1}^{n} \langle i | \Psi_k \rangle p_k \langle \Psi_k | j \rangle$.

In general the elements of the density operator vary in time

$$i\hbar \frac{\rho_{ii}(t)}{dt} = \sum_{l=1}^{n} (\hat{H}_i \rho_{ii} - \rho_{ii} \hat{H}_i),$$

$$i\hbar \frac{\rho_{ij}(t)}{dt} = \sum_{l=1}^{n} (\hat{H}_i \rho_{ij} - \rho_{ij} \hat{H}_j). \quad (7)$$

When the system reaches the thermodynamic equilibrium the density operator non diagonal elements becomes zero i.e. the coherences between stationary states disappears while the populations of the stationary states are exponentially decreasing functions of the energy [23].

We can define the entropy of our system by the von Neumann entropy

$$S(t) = -Tr \{ \rho \ln \rho \} \quad (8)$$

which in a far from equilibrium system also vary in time until it reaches its maximum value. When the dynamics is chaotic the variation with time of the physical entropy goes through three successive, roughly separated stages [20]. In the first one, $S(t)$ is dependent on the details of the dynamical system and of the initial distribution, and no generic statement can be made. In the second stage, $S(t)$ is a linear increasing function of time ($\frac{dS}{dt} = const.$). In the third stage, $S(t)$ tends asymptotically towards the constant value which characterizes equilibrium ($\frac{dS}{dt} = 0$). With the purpose of calculating the time evolution of entropy we approximate the logarithm of $\rho$ by series $\ln(\rho) = (\rho - I) - \frac{1}{2}(\rho - I)^2 + \frac{1}{3}(\rho - I)^3 \ldots$ and

$$\frac{dS(t)}{dt} = \frac{11}{6} \sum_{i} \frac{d\rho_{ii}}{dt} - 6 \sum_{i,j} \rho_{ij} \frac{d\rho_{ij}}{dt} + \frac{9}{2} \sum_{i,j,k} \rho_{ij}\rho_{jk} \frac{d\rho_{ij}}{dt} - \frac{4}{3} \sum_{i,j,k,l} \rho_{ij}\rho_{jk}\rho_{kl} \frac{d\rho_{ij}}{dt} + \zeta. \quad (9)$$

V. GAMES ANALYSIS FROM STATISTICAL MECHANICS & QIT

There exists a strong relationship between game theories, statistical mechanics and information theories. The bonds between these theories are the density operator and entropy. From the density operator we can construct and understand the statistical behavior about our system by using statistical mechanics. Also we can develop the system in function of its accessible information and analyze it through information theories under a criterion of maximum or minimum entropy.

Entropy [4, 5, 27] is the central concept of information theories. The Shannon entropy expresses the average information we expect to gain on performing a probabilistic experiment of a random variable $A$ which takes the values $a_i$ with the respective probabilities $p_i$. It also can be seen as a measure of uncertainty before we learn the value of $A$. We define the Shannon entropy of a random variable $A$ by

$$H(A) \equiv H(p_1, ..., p_n) = - \sum_{i=1}^{n} p_i \log_2 p_i. \quad (10)$$

The Shannon entropy of the probability distribution associated with the source gives the minimal number of bits that are needed in order to store the information produced by a source, in the sense that the produced string can later be recovered.

If we define an entropy over a random variable $S^A$ (player’s $A$ strategic space) which can take the values $s_i^A$ with the respective probabilities $x_i^A$ i.e. $H(A) \equiv - \sum_{i=1}^{n} x_i \log_2 x_i$, we could interpret the entropy of our game as a measure of uncertainty before we learn what strategy player $A$ is going to use. If we do not know what strategy a player is going to use every strategy becomes equally probable and our uncertainty becomes maximum and it is greater while greater is the number of strategies. If we would know the relative frequency with which player $A$ uses any strategy we can prepare our reply in function of the most probable player $A$ strategy. That
would be our actual best reply which in that moment would let us maximize our payoff due to our uncertainty. Obviously our uncertainty vanish if we are sure about the strategy our opponent is going to use. The complete knowledge of the rules of a game and the reserve in our strategies becomes an advantage over an opponent who does not know the game rules or who always plays in a same predictive way. To become a game fair, an external referee should make the players to know completely the game rules and the strategies that the players can use.

If the player $B$ decides to play strategy $s^B_j$ against player $A$ (who plays strategy $s^A_i$) our total uncertainty about the pair $(A, B)$ can be measured by an external “referee” through the joint entropy of the system $H(A, B) \equiv - \sum_{i,j} x_{ij} \log_2 x_{ij}$ is the joint probability to find $A$ in state $s_i$ and $B$ in state $s_j$. This is smaller or at least equal than the sum of the uncertainty about $A$ and the uncertainty about $B$, $H(A, B) \leq H(A) + H(B)$. The interaction and the correlation between $A$ and $B$ reduces the uncertainty due to the sharing of information. There can be more predictability in the whole than in the sum of the parts. The uncertainty decreases while more systems interact jointly creating a new only system.

We can measure how much information $A$ and $B$ share and have an idea of how their strategies or states are correlated by their mutual or correlation entropy $H(A : B) \equiv - \sum_{i,j} x_{ij} \log_2 x_{ij}$, with $x_{ij} = \sum_i x_{ij}$. It can be seen easily as $H(A : B) \equiv H(A) + H(B) - H(A, B)$. The joint entropy would equal the sum of each of $A$’s and $B$’s entropies only in the case that there are no correlations between $A$’s and $B$’s states. In that case, the mutual entropy vanishes and we could not make any predictions about $A$ just from knowing something about $B$.

If we know that $B$ decides to play strategy $s^B_j$ we can determine the uncertainty about $A$ through the conditional entropy $H(A \mid B) \equiv H(A, B) - H(B) = - \sum_{i,j} x_{ij} \log_2 x_{ij}$ with $x_{ij} = \sum_i x_{ij}$. If this uncertainty is bigger or equal to zero then the uncertainty about the whole is smaller or at least equal than the uncertainty about $A$, i.e. $H(A : B) \leq H(A)$. Our uncertainty about the decisions of player $A$ knowing how $B$ and $C$ plays is smaller or at least equal than our uncertainty about the decisions of $A$ knowing only how $B$ plays $H(A \mid B, C) \leq H(A \mid B)$ i.e. conditioning reduces entropy.

If the behavior of the players of a game follows a Markov chain i.e. $A \to B \to C$ then $H(A) \geq H(A \mid B) \geq H(A \mid C)$ i.e. the information can only reduces in time. Also any information $C$ shares with $A$ must be information which $C$ also shares with $B$, $H(C : B) \geq H(C : A)$.

Two external observers of the same game can measure the difference in their perceptions about certain strategy space of the same player $A$ by its relative entropy. Each of them could define a relative frequency vector, $x$ and $y$, and the relative entropy over these two probability distributions is a measure of its closeness $H(x \parallel y) \equiv \sum_i x_i \log_2 x_i - \sum_i x_i \log_2 y_i$. We could also suppose that $A$ could be in two possible states i.e. we know that $A$ can play of two specific but different ways and each way has its probability distribution (again $x$ and $y$ that also is known). Suppose that this situation is repeated exactly $N$ times or by $N$ people. We can made certain “measure”, experiment or “trick” to determine which the state of the player is. The probability that these two states can be confused is given by the classical or the quantum Sanov’s theorem [6, 28–30].

By analogy with the Shannon entropies it is possible to define conditional, mutual and relative quantum entropies which also satisfy many other interesting properties that do not satisfy their classical analogues. For example, the conditional entropy $S(A \mid B)$ can be negative and its negativity always indicates that two systems (in this case players) are entangled and indeed, how negative the conditional entropy is provides a lower bound on how entangled the two systems are [13]. If $\lambda_i$ are the eigenvalues of $\rho$ then von Neumann’s definition can be expressed as $S(\lambda) = - \sum_i \lambda_i \log \lambda_i$ and it reduces to a Shannon entropy if $\rho$ is a mixed state composed of orthogonal quantum states [31]. Our uncertainty about the mixture of states $S(\sum_i p_i \rho_i)$ should be higher than the average uncertainty of the states $\Sigma_i p_i S(\rho_i)$.

By other hand, in statistical mechanics entropy can be regarded as a quantitative measure of disorder. It takes its maximum possible value $\ln n$ in a completely random ensemble in which all quantum mechanical states are equally likely and is equal to zero if $\rho$ is pure i.e. when all its members are characterized by the same quantum mechanical state ket. Entropy can be maximized subject to different constraints. Generally, the result is a probability distribution function. We will maximize $S(\rho)$ subject to the constraints $\delta Tr(\rho) = 0$ and $\delta \langle E \rangle = 0$ and the result is

$$\rho_{i\alpha} = \frac{e^{-\beta E_i}}{\sum_k e^{-\beta E_k}}$$

(11)

which is the condition that the density operator must satisfy to our system tends to maximize its entropy $S$. Without the internal energy constraint $\delta \langle E \rangle = 0$ we obtain $\rho_{i\alpha} = \frac{1}{N}$ which is the $\beta \to 0$ limit (“high - temperature limit”) in equation (11) in which a canonical ensemble becomes a completely random ensemble in which all energy eigenstates are equally populated. In the opposite low - temperature limit $\beta \to \infty$ tell us that a canonical ensemble becomes a pure ensemble where only the ground state is populated [32]. The parameter $\beta$ is related inversely to the “temperature” $\tau$, $\beta = \frac{1}{\tau}$. We can rewrite entropy in function of the partition function $Z = \sum_k e^{-\beta E_k}$, $\beta$
and $\langle E \rangle$ via $S = \ln Z + \beta \langle E \rangle$. From the partition function we can know some parameters that define the system like $\langle E \rangle$ and $\langle \Delta E^2 \rangle$. We can also analyze the variation of entropy with respect to the average energy of the system

$$\frac{\partial S}{\partial \langle E \rangle} = \frac{1}{\tau},$$

$$\frac{\partial^2 S}{\partial \langle E \rangle^2} = -\frac{1}{\tau^2} \frac{\partial \tau}{\partial \langle E \rangle}$$

and with respect to the parameter $\beta$

$$\frac{\partial S}{\partial \beta} = -\beta \langle \Delta E^2 \rangle,$$

$$\frac{\partial^2 S}{\partial \beta^2} = \frac{\partial \langle E \rangle}{\partial \beta} + \beta \frac{\partial^2 \langle E \rangle}{\partial \beta^2}.$$

VI. DISCUSSION

In this point, it is important to remember that we are dealing with very general and unspecific terms, definitions and concepts like state, game and system. Due to this, the theories that have been developed around these terms like quantum mechanics, statistical physics, information theory and game theory enjoy of this generality quality and could be applicable to model any system depending on what we want to mean for game, state or system. Objectively these words can be and represent anything. Once we have defined what system is in our model, we could try to understand what kind of “game” is developing between its members and how they accommodate their “states” in order to get their objectives. This would let us visualize what temperature, energy and entropy would represent in our specific system through the relationships, properties and laws that were defined before when we described a physical system.

The parameter “$\beta$” related with the “temperature” of a statistical system has been used like a measure of the rationality of the players [32] and in other cases like the average amount of money per economic agent [34]. Entropy is defined over a random variable that objectively can be anything. And depending on what the variable over which we want determinate its grade of order or disorder is we can resolve if the best for the system is its state of maximum or minimum entropy. If we measure the order or disorder of our system over a resources distribution variable the best state for that system is those in where its resources are fairly distributed over its members which would represent a state of maximum entropy. By the other hand, if we define an entropy over the acception of a presidential candidate in a democratic process the best would represent a minimum entropy state i.e. the acception of a candidate by the vast majority of the population.

Fundamentally, we could distinguish three states in every system: minimum entropy, maximum entropy, and when the system is tending to whatever of these two states. The natural trend of a physical system is to the maximum entropy state. The minimum entropy state is a characteristic of a “manipulated” system i.e. externally controlled or imposed. A system can be internally or externally manipulated or controlled with the purpose of guide it to a state of maximum or minimum entropy depending of the ambitions of the members that compose it or the “people” who control it.

If a physical system is not in an equilibrium state, the whole system will vary and rearrange its state and the states of its ensembles with the purpose of maximize its entropy which could be seen as the purpose and maximum payoff of a physical system. The system and its members will vary and rearrange themselves to reach the best possible state for each of them which is also the best possible state for the whole system. This can be seen as a microscopical cooperation between quantum objects to improve their states in order to reach or maintain the equilibrium of the system. All the members of our quantum system will play a game in which its maximum payoff is the welfare of the collective. We can resume the last analysis through what we have called as Collective Welfare Principle: “A system is stable only if it maximizes the welfare of the collective above the welfare of the individual. If it is maximized the welfare of the individual above the welfare of the collective the system gets unstable and eventually it collapses” [1].

There exist tacit rules inside a system. These rules do not need to be specified or clarified and search the system equilibrium under a collective welfare principle. The other “prohibitive” and “repressive” rules are imposed over the system when one or many of its members violate the collective welfare principle and search to maximize its individual welfare at the expense of the group. Then it is necessary to impose regulations on the system to try to reestablish the broken natural order. But, this “order” can not be reestablished due to the constant violation of the collective welfare principle making this world a far from equilibrium system which oscillate in a tendency state struggling desperately against chaos, anarchy and its collapse.

Is our system the best possible? Maybe we must imitate the behavior of the most perfect system and its equilibrium concept. In a next work we will analyze the entropy of the world by specifying distinct random variables which let us describe the social, political and economic behavior of our world. Are we near to a state of maximum entropy?
VII. CONCLUSIONS

Every game can be described by a density operator, the von Neumann entropy and the quantum replicator dynamics. There exists a strong relationship between game theories, statistical mechanics and information theory. The bonds between these theories are the density operator and entropy. From the density operator we can construct and understand the statistical behavior about our system by using statistical mechanics. Also we can develop the system in function of its accessible information and analyze it through information theories under a criterion of maximum or minimum entropy depending on what variable we have defined entropy.

The words system, state and game can be and represent anything. Once we have defined what system is in our model, we could try to understand what kind of “game” is developing between its members and how they arrange their “states” in order to get their objectives and we could realize if would be possible to model that system through the relationships, properties and laws that were defined before in the study of the physical system.

The natural trend of a physical system is to the maximum entropy state. The minimum entropy state is a characteristic of a “manipulated” system i.e. externally controlled or imposed. A system can be internally or externally manipulated or controlled with the purpose of guide it to a state of maximum or minimum entropy depending of the ambitions of the members that compose it or the “people” who control it.

There exist tacit rules inside a system that do not need to be specified or clarified and search the system equilibrium under the collective welfare principle. The other rules are imposed over the system when one or many of its members violate this principle and maximize its individual welfare at the expense of the group.

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