Solution of the Navier-Stokes problem

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Abstract

A new a priori estimate for solutions to Navier-Stokes equations is derived. Uniqueness and existence of these solutions in $\mathbb{R}^3$ for all $t > 0$ is proved in a class of solutions locally differentiable in time with values in $H^1(\mathbb{R}^3)$, where $H^1(\mathbb{R}^3)$ is the Sobolev space. By the solution a solution to an integral equation is understood. No smallness restrictions on the data are imposed.

1 Introduction

There is a large literature on the Navier-Stokes (NS) problem in $\mathbb{R}^3$ (see [2], [3] and references therein.) The global existence and uniqueness of a solution in $\mathbb{R}^3$ was not yet proved. It is mentioned as one of the millennium problems in [3]. This problem is solved in our paper. One of the new technical points is the usage of the Fourier transform of the unknown velocity and the reduction of the problem to one-dimensional Volterra integral inequalities. Large-time behavior of the solutions to evolution equations is studied in [4], [6]. Various notions of the solution were used, see [2], [3], [5], [7]. In this paper, which is based on [7] but is essentially self-contained, the solution to NS problem is defined as a solution to some integral equation. It is proved that this solution is unique, it exists globally, i.e. for all $t \geq 0$, and is uniformly bounded in the Sobolev space $H^1(\mathbb{R}^3)$ provided that the free term $f$ decays sufficiently fast as $|x| + t \to \infty$ and the initial data $v_0(x)$ decays sufficiently fast as $|x| \to \infty$.

The NS problem in $\mathbb{R}^3$ consists of solving the equations

$$v' + (v, \nabla)v = -\nabla p + \nu \Delta v + f, \quad x \in \mathbb{R}^3, \ t \geq 0, \ \nabla \cdot v = 0, \ v(x, 0) = v_0(x).$$

(1)

Vector-functions $v = v(x, t)$, $f = f(x, t)$ and the scalar function $p = p(x, t)$ decay as $|x| \to \infty$ uniformly with respect to $t \in \mathbb{R}_+: = [0, \infty)$, $v' := v_t$, $\nu = \text{const} > 0$, the velocity $v$ and the pressure $p$ are unknown, $v_0$ and $f$ are known, $\nabla \cdot v_0 = 0$. Equations (1) describe viscous incompressible fluid with density $\rho = 1$. By $\overline{v}$ complex conjugate of $v$ is denoted.

We assume that $|f| + |\nabla f|$ and $|v_0| + |\nabla v_0|$ decay as $O(|x|^{-a})$ as $|x| \to \infty$, where $a > 3$, and $\int_0^\infty N_0(f) dt < \infty$, $N_0(f) := \|f\|_{L^2(\mathbb{R}^3)}$. By $\nabla v$ any of the first derivatives of $v$ is understood.

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We use the integral equation (equation (17) from [7]):

\[ v(x, t) = F - \int_0^t ds \int_{\mathbb{R}^3} G(x - y, t - s)(v, \nabla)vdy. \]  \hspace{1cm} (2)

Equation (2) is equivalent to (1) (see [7]). Formula for the tensor \( G \) is derived in [7]. The term \( F = F(x, t) \) depends only on the data \( f \) and \( v_0 \) (see equation (18) in [7]):

\[ F := \int_{\mathbb{R}^3} g(x - y)v_0(y)dy + \int_0^t ds \int_{\mathbb{R}^3} G(x - y, t - s)f(y, s)dy. \]  \hspace{1cm} (3)

Let us formulate a priori estimates for \( v \). We assume throughout that \( f \) and \( v_0 \) are such that \( F \) is bounded in all of the norms we use.

Let \( X \) be the Banach space of continuous functions with respect to \( t \) with values in \( L^2(\mathbb{R}^3) \) with the norm \( \|v\| = \|v\|_X := \sup_{t \in [0, T]} \|v(x, t)\|_{L^2(\mathbb{R}^3)} \), where \( T > 0 \) is an arbitrary large fixed number. Denote \( N_0(v) := \|v\|_{L^2(\mathbb{R}^3)} \) and \( (v, v) := N_0^2(v) \).

The solution to (2) is a continuously differentiable function of \( t \) with values in the Sobolev space \( H^{1}(\mathbb{R}^3) \) as follows from Theorem 1.

**Theorem 1.** A solution to problem (1) satisfies the following a priori estimates:

\[ N_0(v) \leq c_0, \quad N_0(\nabla v) \leq c(t), \]  \hspace{1cm} (4)

where the constant \( c_0 > 0 \) does not depend on \( t \) and \( c(t) \) is a continuous function defined for all \( t \geq 0 \).

The first estimate (4) is known, see [2] and [7], while the second estimate (4) is new.

**Theorem 2.** Problem (1) has a solution in \( X \) and this solution is unique in \( X \).

**Remark 1.** If \( N_0(\|\cdot\|\tilde{F}) = O\left(\frac{1}{(t+1)^\gamma}\right) \) as \( t \to \infty, \gamma > 1/4 \), then \( \sup_{t \geq 0} c(t) < \infty \). This follows, for example, from estimate (13), see below.

### 2 Proof of Theorem 1

a) Multiply (1) by \( \nabla \) and integrate over \( \mathbb{R}^3 \). Using the standard transformations, one gets

\[ 2N_0(v)\frac{dN_0(v)}{dt} + 2vN_0^2(\nabla v) = 2Re(f, v). \]

Thus, \( \frac{dN_0(v)}{dt} + \nu N_0^2(\nabla v)/N_0(v) \leq N_0(f) \), and \( \frac{dN_0(v)}{dt} \leq N_0(f) \), so \( N_0(v) \leq \int_0^t N_0(f)ds + N_0(v_0) \). If \( \int_{0}^{\infty} N_0(f)ds < \infty \), then the first inequality (4) holds.

b) Let us prove that \( \psi(t) := N_0(\nabla v) \leq c(t) \).

Take the Fourier transform of (2), denote \( \tilde{v}(\xi, t) := (2\pi)^{-\frac{3}{2}} \int_{\mathbb{R}^3} e^{-i\xi \cdot x} v(x, t)dx \) and let \( \tilde{G} \) denote the Fourier transform of \( G \). Then

\[ \tilde{v} = \tilde{F} - \int_0^t \tilde{G}(\xi, t - s)\tilde{v} \star (-i\xi \tilde{v})ds := B(\tilde{v}). \]  \hspace{1cm} (5)

Here \( \star \) denotes the convolution in \( \mathbb{R}^3 \) and for brevity we omitted the tensorial indices: instead of \( \tilde{G}_{mp}\tilde{v}_j \star (-i\xi_j)\tilde{v}_p \), where one sums up over the repeated indices, we wrote \( \tilde{G}(\xi, t - s)\tilde{v} \star (-i\xi \tilde{v}) \).

Let \( \psi(t) := N_0(\nabla v) = N_0(\|\widetilde{v}\|), \quad N_0(v) = N_0(\widetilde{v}) \), where the Parseval’s identity is used. From (5) one gets

\[ \psi(t) \leq N_0(\|\cdot\|\tilde{F}) + c \int_0^t N_0(\|\tilde{G}(\xi, t - s)\|)\psi(s)ds. \]  \hspace{1cm} (6)
Here and below $c$ stands for various constants independent of $t$; we have used the a priori estimate $\sup_{t \geq 0} N_0(\tilde{v}) \leq c_0$ and the standard estimate $|\tilde{v} * \tilde{w}| \leq N_0(\tilde{v}) N_0(\tilde{w})$. One can check that
\[ N_0(|\tilde{G}(\xi, t - s)|) \leq c [\nu(t - s)]^{-\frac{1}{2}}, \quad N_0(|\xi||\tilde{G}(\xi, t - s)|) \leq c [\nu(t - s)]^{-\frac{1}{2}}. \tag{7} \]
In the derivation of (7) we use the estimate $|\tilde{G}(\xi, t)| \leq c e^{-\nu t \xi^2}$, where $|G|$ is a norm of the matrix, see formula (9) in [7]. We need some lemmas.

**Lemma 1.** The operator $Af := \int_0^t (t - s)^p f(s) ds$ in the Banach space $X_0 := C([0, T])$ has spectral radius $r(A) = 0$ for $p > 0$ and any fixed $T > 0$, $0 \leq t \leq T$.

**Proof.** The spectral radius of a bounded linear operator $A$ can be calculated by the formula $r(A) = \lim_{n \to \infty} \|A^n\|^{1/n}$. One checks by induction that

\[ |A^nf| \leq t^{n(p+1)} \frac{\Gamma(n+1)}{\Gamma(n(p+1)+1)} \|f\|_{X_0}, \quad n \geq 1, \tag{8} \]

where $\|f\|_{X_0}$ is the norm in $X_0$ and $\Gamma(z)$ is the Gamma function. The conclusion of Lemma 1 follows immediately from (8) and from the asymptotic of $\Gamma(p(n + 1) + 1)$ as $n \to +\infty$.

Lemma 1 is proved. \qed

**Lemma 2.** Let $A$ be a linear operator in a Banach space $X$. If $f = Af + f_0$ and $r(A) = 0$, then $f = \sum_{j=n}^{\infty} A^j f_0$ for any element $f_0 \in X$. If $f_0 = 0$ then $f = 0$.

The conclusion of this Lemma follows from a well known variant of Lemma 2 with the assumption $\|A\| < 1$. \qed

Denote $\Phi_\lambda(t) := \frac{t^{\lambda-1}}{\Gamma(\mu)}$, $\lambda \neq 0, -1, -2, \ldots$, where $t^{\lambda-1}_+ := 0$ for $t < 0$ and $t^{\lambda-1}_+ := t^{\lambda-1}$ for $t > 0$. This $\Phi_\lambda$ is defined as a distribution, $\Phi_\lambda * \Phi_\mu = \Phi_{\lambda+\mu}$, $\Phi_\lambda * \Phi_{-\lambda} = I$, where $*$ here denotes the convolution in $[0, \infty)$ and $I$ is the identity operator whose kernel is the $\delta$-function. The convolution of the distribution $\Phi_\lambda(t)$ with any distribution vanishing for $t < 0$ is well defined, see [1]. The following lemma allows one to claim that a solution of an integral equation (inequality) with a strongly singular kernel solves an integral equation (inequality) with an integrable kernel, see [8].

**Lemma 3.** If $h = h_0 + \Phi_{-\lambda} * h$, then $h = \Phi_\lambda * h - \Phi_{-\lambda} * h_0$, $\lambda > 0$.

**Proof.** Apply the operator $\Phi_\lambda *$ to the equation $h = h_0 + \Phi_{-\lambda} * h$ and use the property $\Phi_\lambda * \Phi_{-\lambda} = I$.

One has $\int_0^t (t - s)^{-\frac{1}{4}} \psi ds = \Gamma(-\frac{1}{4}) \Phi_{-\frac{1}{4}} * \psi$. Inequality (9) can be written as
\[ \psi \leq \psi_0 + c \Gamma(-\frac{1}{4}) \Phi_{-\frac{1}{4}} * \psi, \quad \psi_0 := N_0(|\xi| \tilde{F}), \tag{9} \]
and $\Gamma(-\frac{1}{4}) = -4 \Gamma(3/4) := -b^{-1}, b > 0$. Applying to (9) the operator $\Phi_{1/4} *$ and multiplying by $c^{-1}b$ one gets
\[ \psi \leq c^{-1}b \Phi_{1/4} * \psi_0 - c^{-1}b \Phi_{1/4} * \psi. \tag{10} \]

Using Lemmas 1 and 2 one derives from (10) by iterations that
\[ \psi(t) \leq h(t), \tag{11} \]
where $h(t)$ is the unique solution of the equation
\[ h = c^{-1}b \Phi_{1/4} * \psi_0 - c^{-1}b \Phi_{1/4} * h. \tag{12} \]
Equation (12) is solvable by iterations and the iterations converge by Lemma 2. The solution $h$ is bounded by a constant depending only on the data, that is, on $f, v_0$ and $T$, $0 \leq t \leq T$. By (11) the $\psi$ is bounded by $h$. Since $T > 0$ is arbitrary, the second estimate (4) is proved.

A less precise estimate

$$\psi(t) \leq c^{-1}b\Phi_{1/4} \ast \psi_0,$$

than estimate (11), can be obtained from equation (10) if one takes into account that $c^{-1}b\Phi_{1/4} \ast \psi \geq 0$.

Using formulas (2), (5) and (4) one can estimate $|v(x, t)|$ and $|\tilde{v}(\xi, t)|$.

### 3 Proof of Theorem 2

Since problem (1) is equivalent to (2) and to (5), it is sufficient to prove that (5) has a solution and this solution is unique in the space $X$.

**Proof of the uniqueness of the solution to (5) in the space $X$.**

Let $\tilde{v}_1$ and $\tilde{v}_2$ solve equation (5). Define

$$w := \tilde{v}_1 - \tilde{v}_2, \quad N_0(w) := q, \quad N_0(|\xi|w) := Q,$$

(14)

Subtracting from equation (5) for $\tilde{v}_1$ equation (5) for $\tilde{v}_2$, one gets the equation

$$w = - \int_0^t \tilde{G}[w \ast (-i\xi \tilde{v}_2) + \tilde{v}_2 \ast (-i\xi w)]ds,$$

(15)

where the $\ast$ is the convolution in $\mathbb{R}^3$. Taking the norm $N_0$ of both sides of (15), denoting $z := q + Q$, using estimates (4) and formulas (7) and (14) one derives the following relation:

$$z \leq c\left(\Gamma(1/4)\Phi_{1/4} \ast z + \Gamma(-1/4)\Phi_{-1/4} \ast z\right),$$

(16)

where $\ast$ here denotes the convolution in $[0, \infty)$. The operator $\Phi_{\lambda} \ast$ with $\lambda > 0$ maps the set of non-negative functions (or distributions) into itself, it preserves inequality sign. Applying the operator $\Phi_{1/4} \ast$ to (16), taking into account that $\Gamma(-1/4) = -4\Gamma(3/4)$, $\Phi_{\lambda} \ast \Phi_{\mu} = \Phi_{\lambda+\mu}$, and $\Phi_{\lambda} \ast \Phi_{-\lambda} = I$, one gets:

$$4c\Gamma(3/4)z \leq c\Gamma(1/4)\Phi_{1/2} \ast z - \Phi_{1/4} \ast z.$$

(17)

It follows from (17) and (11) that $z = 0$ (because $f_0 = 0$ in our case, see Lemma 2). This proves the uniqueness of the solution to problem (1) in the space $X$ for any $T > 0, 0 \leq t \leq T$. $\square$

**Proof of the existence of the solution to (5).**

Let us show that the operator $B$ in (5) is a contraction in $X$ for a sufficiently small $\tau$, $0 \leq t \leq \tau$. With $w = \tilde{v}_1 - \tilde{v}_2$ one obtains from (5) the following equation:

$$B(\tilde{v}_1) - B(\tilde{v}_2) = - \int_0^t \tilde{G}[w \ast (-i\xi \tilde{v}_1) + \tilde{v}_2 \ast (-i\xi w)]ds,$$

(18)

Here the $\ast$ denotes the convolution in $\mathbb{R}^3$. Using the second estimate (4) one derives:

$$|\tilde{v}_2 \ast (-i\xi w)| \leq N_0(w)N_0(\xi + |\xi'|)\tilde{v}_2(\xi') \leq cN_0(w)(1 + |\xi|^2)^{1/2}.$$  

(19)
Using estimates (4), (7) and (19), one obtains from (18) the following inequality:

\[ N_0(B(\tilde{v}_1) - B(\tilde{v}_2)) \leq c\left( \int_0^t (t-s)^{-\frac{3}{4}} qds + \int_0^t (t-s)^{-\frac{5}{4}} qds \right), \quad q = N_0(w). \tag{20} \]

Rewrite (20) as

\[ N_0(B(\tilde{v}_1) - B(\tilde{v}_2)) \leq c\left( \Gamma(1/4)\Phi_{1/4} \ast q - 4\Gamma(3/4)\Phi_{-1/4} \ast q \right), \tag{21} \]

where the \( \ast \) denotes the convolution in \([0, \infty)\). Applying \( \lambda_1 \Phi_{1/4} \ast \) to (21), one gets

\[ \lambda_1 \Phi_{1/4} \ast N_0(B(\tilde{v}_1) - B(\tilde{v}_2)) \leq \lambda_1 c\Gamma(1/4)\Phi_{1/2} \ast q - q, \quad \lambda_1^{-1} := 4c\Gamma(3/4) > 0. \tag{22} \]

Since \( q \geq 0 \) one has

\[ \lambda_1 \Phi_{1/4} \ast N_0(B(\tilde{v}_1) - B(\tilde{v}_2)) \leq \lambda_1 c\Gamma(1/4)\Phi_{1/2} \ast \sup_{t \in [0,\tau]} q(t). \tag{23} \]

From (23) one derives

\[ N_0(B(\tilde{v}_1) - B(\tilde{v}_2)) \leq c\Gamma(1/4)\Phi_{1/4} \ast 1 \sup_{t \in [0,\tau]} q(t). \tag{24} \]

Clearly, \( \lim_{t \to 0} \Phi_{1/4} \ast 1 = 0 \). Thus, \( B \) is a contraction on a ball \( B_R, \) \( B_R := \{ q : q \leq R \}, \) if \( \tau \) is sufficiently small, \( 0 \leq t \leq \tau, \) and \( R > 0 \) is an arbitrary large fixed number. Consequently, the operator \( \tilde{B} \) in (3) is a contraction on \( B_R \) for \( t \leq \tau. \) Therefore the solution to (5) exists in \( X \) for \( t \leq \tau. \) The a priori estimates (1) do not depend on \( \tau \leq T. \) Therefore one can repeat the argument for \( \tau \leq t \leq 2\tau \) considering the initial value to be \( \tilde{v}(\xi, \tau) \) and the free term to be \( \tilde{F}(\xi, t), t \in [\tau, 2\tau]. \) The solution \( \tilde{v} \) is unique in \( X, \) as we have proved. So, one gets the existence of the solution on \( 0 \leq t \leq 2\tau. \) Continue this process and in finitely many steps get the existence of the unique in \( X \) solution in \([0, T]. \) Since \( T > 0 \) is arbitrary, the solution exists for all \( T > 0. \)

It follows from Theorem 2 that there cannot be turbulent motions of fluid in the NS problem in the whole space \( \mathbb{R}^3 \) if the data \( f \) and \( v_0 \) are smooth and rapidly decaying.

References

[1] I. Gel’fand and G. Shilov, Generalized functions, Vol.1, AMS Chelsea Publ., 1964.

[2] O. A. Ladyzhenskaya, The mathematical theory of viscous incompressible flow, Gordon and Breach, New York, 1969.

[3] P.Lemarie-Rieusset, The Navier-Stokes problem in the 21-st century, CRC Press, Boca Raton, 2016.

[4] A.G.Ramm, Large-time behavior of the weak solution to 3D Navier-Stokes equations, Appl. Math. Lett., 26 (2013), 252-257.

[5] A.G.Ramm, Existence and uniqueness of the global solution to the Navier-Stokes equations, Applied Math. Letters, 49, (2015), 7-11.

[6] A.G.Ramm, Large-time behavior of solutions to evolution equations, Handbook of Applications of Chaos Theory, Chapman and Hall/CRC, 2016, pp. 183-200 (ed. C.Skiadas).

[7] A.G.Ramm, Global existence, uniqueness and estimates of the solution to the Navier-Stokes equations, Applied Math. Letters, 74, (2017), 154-160.

[8] A.G.Ramm, Existence of the solutions to convolution equations with distributional kernels, Global Journal of Math. Analysis, 6(1), (2018), 1-2.