A Delayed Black and Scholes Formula I

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Abstract

In this article we develop an explicit formula for pricing European options when the underlying stock price follows a non-linear stochastic differential delay equation (sdde). We believe that the proposed model is sufficiently flexible to fit real market data, and is yet simple enough to allow for a closed-form representation of the option price. Furthermore, the model maintains the no-arbitrage property and the completeness of the market. The derivation of the option-pricing formula is based on an equivalent martingale measure.

1 Introduction

The Black and Scholes Formula has been one of the most important consequences of the study of continuous time models in finance ([Bac], [C], [Me1], [Me2]). However, the fitness of the model has been questioned on the basis of the assumption of constant volatility ([Sc], [R]), since empirical evidence shows that volatility actually depends on time in a way that is not predictable. This is sometimes pointed out as the reason for inaccurate predictions made by the Black and Scholes formula. On the other hand, the need for better ways of understanding the behavior of many natural processes has motivated the development of dynamic models of these processes that take into consideration the influence of past events on the current and future states of the system ([I.N], [Ku], [K.N], [Mo1], [Mo2], [M.T], [E.Ø.S]). This view is specially appropriate in the study of financial variables, since predictions about their evolution take strongly into account the knowledge of their past ([H.Ø], [S.K]).

In this paper we consider the effect of the past in the determination of the fair price of a call option. In particular, we assume that the stock price satisfies a stochastic functional differential equation (sfde). We consider call options that can be exercised only at the maturity date, viz. European call options. We derive an explicit formula for the valuation of a European call option on a given stock (Theorem 4) (cf. ([B.S], [Me1], [H.R])). Note that based on the

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Tests of the classical Black and Scholes model against real market data suggest the existence of significant levels of randomness in the volatility of the stock price, as manifested in the observed phenomenon of frowns and smiles ([Bat]). One of the motivations behind our model for the stock price is to account for such volatility in a natural manner, while at the same time maintain an explicit formula for the option price. It is hoped that the parameters of the proposed model will allow enough flexibility for a better fit than that of the Black and Scholes model when tested against real market data.

International markets for contingent claims have experienced remarkable growth in the last thirty years. This makes the study of option pricing of special interest in the present context, since this theory may lead to a general theory of pricing contingent claims or derivatives with hereditary structure.

2 Stochastic delay models for the stock price

In this section we propose a stochastic delay model for the evolution of the stock price. We prove that the proposed model is feasible. In Section 3, we formulate and solve the option pricing problem for the model.

Consider a stock whose price at time $t$ is given by a stochastic process $S(t)$ satisfying the following stochastic delay differential equation (sdde):

\[
\begin{align*}
    dS(t) &= \mu S(t-a)S(t) dt + g(S(t-b))dW(t), & t \in [0, T] \\
    S(0) &= \varphi(t), & t \in [-L, 0].
\end{align*}
\]  

(1)

on a probability space $(\Omega, \mathcal{F}, P)$ with a filtration $(\mathcal{F}_t)_{0 \leq t \leq T}$ satisfying the usual conditions. In the above sdde, $\mu, a, b$ are positive constants, $L := \max\{a, b\}$, and $g : \mathbb{R} \to \mathbb{R}$ is a continuous function. The initial process $\varphi : \Omega \to C([-L, 0], \mathbb{R})$ is $\mathcal{F}_0$-measurable with respect to the Borel $\sigma$-algebra of $C([-L, 0], \mathbb{R})$. The process $W$ is a one-dimensional standard Brownian motion adapted to the filtration $(\mathcal{F}_t)_{0 \leq t \leq T}$.

In our next result, we will show that the above model is feasible in the sense that it admits a pathwise unique solution $S$ such that $S(t) > 0$ almost surely for all $t \geq 0$ whenever $\varphi(0) > 0$.

**Theorem 1** The sdde (1) has a pathwise unique solution $S$ for a given $\mathcal{F}_0$-measurable initial process $\varphi : \Omega \to C([-L, 0], \mathbb{R})$. Furthermore, if $\varphi(0) \geq 0$ ($\varphi(0) > 0$) a.s., then $S(t) \geq 0$ ($S(t) > 0$) for all $t \geq 0$ a.s..

**Proof.**

Suppose $\varphi(0) \geq 0$. Define $l := \min\{a,b\} > 0$ and let $t \in [0, l]$. Then (1) gives

\[
\begin{align*}
    dS(t) &= S(t)[\mu \varphi(t-a) dt + g(\varphi(t-b)) dW(t)], & t \in [0, l] \\
    S(0) &= \varphi(0).
\end{align*}
\]  

(2)

Define the semimartingale

\[
N(t) := \mu \int_0^t \varphi(u-a) \, du + \int_0^t g(\varphi(u-b)) \, dW(u), \quad t \in [0, l],
\]

and denote by $[N, N](t) = \int_0^t g(\varphi(u-b))^2 \, du$, $t \in [0, l]$, its quadratic variation. Then (2) becomes

\[
dS(t) = S(t) dN(t), \quad t > 0, \quad S(0) = \varphi(0),
\]

with $N(t) > 0$. Hence

\[
dS(t) = \mu S(t-a)S(t) dt + g(\varphi(t-b))dW(t), \quad t \in [0, l],
\]

and we have a path of the process $S(t)$ up to time $l$. The proof is complete.
which has the unique solution
\[ S(t) = \varphi(0)\exp\{N(t) - \frac{1}{2}[N,N](t)\}, \]
\[ = \varphi(0)\exp\Big\{\mu \int_0^t \varphi(u - a)\,du \]
\[ + \int_0^t g(\varphi(u - b))\,dW(u) - \frac{1}{2} \int_0^t g(\varphi(u - b))^2\,du\Big\}, \]
for \( t \in [0, l] \). This clearly implies that \( S(t) > 0 \) for all \( t \in [0, l] \) almost surely, when \( \varphi(0) > 0 \) a.s. By a similar argument, it follows that \( S(t) > 0 \) for all \( t \in [l, 2l] \) a.s.. Therefore \( S(t) > 0 \) for all \( t \geq 0 \) a.s., by induction. Note that the above argument also gives existence and pathwise-uniqueness of the solution to (1).

## 3 A delayed option pricing formula

Consider a market consisting of a riskless asset (e.g., a bond or bank account) \( B(t) \) with rate of return \( r \geq 0 \) (i.e., \( B(t) = e^{rt} \)) and a single stock whose price \( S(t) \) at time \( t \) satisfies the sdde (1) where \( \varphi(0) > 0 \) a.s.. In the sdde (2), assume further that the delays \( a, b \) are positive and \( g \) is continuous. Consider an option, written on the stock, with maturity at some future time \( T > t \) and an exercise price \( K \). Assume also that there are no transaction costs and that the underlying stock pays no dividends. Our main objective is to derive the fair price of the option at time \( t \). In the following discussion, we will obtain an equivalent martingale measure with the help of Girsanov’s theorem.

Let
\[ \tilde{S}(t) := \frac{S(t)}{B(t)} = e^{-rt}S(t), \quad t \in [0, T], \]
be the discounted stock price process. Then by Itô’s formula (the product rule), we obtain
\[ d\tilde{S}(t) = e^{-rt}dS(t) + S(t)(-re^{-rt})\,dt \]
\[ = \tilde{S}(t)\{[\mu S(t - a) - r]\,dt + g(S(t - b))\,dW(t)\}. \]

Let
\[ \tilde{S}(t) := \int_0^t \{\mu S(u - a) - r\}\,du + \int_0^t g(S(u - b))\,dW(u), \quad t \in [0, T]. \]

Then
\[ d\tilde{S}(t) = \tilde{S}(t)d\tilde{S}(t), \quad 0 < t < T. \tag{3} \]

Taking into account that \( \tilde{S}(0) = \varphi(0) \), we have
\[ \tilde{S}(t) = \varphi(0) + \int_0^t \tilde{S}(u)\,d\tilde{S}(u), \quad t \in [0, T]. \tag{4} \]

We now recall Girsanov’s theorem (see, e.g., Theorem 5.5 in [K.K]).

**Theorem 2 (Girsanov)** Let \( W(t), t \in [0, T], \) be a standard Wiener process on \((\Omega, \mathcal{F}, P)\). Let \( \Sigma \) be a predictable process such that \( \int_0^T |\Sigma(u)|^2\,du < \infty \) a.s., and let
\[ \varrho_t := \exp\left\{ \int_0^t \Sigma(u)\,dW(u) - \frac{1}{2} \int_0^t |\Sigma(u)|^2\,du \right\}, \quad t \in [0, T]. \]

Suppose that \( E_P(\varrho_T) = 1 \), where \( E_P \) denotes expectation with respect to the probability measure \( P \). Define the probability measure \( Q \) on \((\Omega, \mathcal{F})\) by \( dQ := \varrho_T\,dP \). Then the process
\[ \tilde{W}(t) := W(t) - \int_0^t \Sigma(u)\,du, \quad t \in [0, T], \]
is a standard Wiener process under the measure \( Q \).
From now on, we will assume that the function $g : \mathbb{R} \to \mathbb{R}$ in the sde (1) satisfies the following hypothesis:

**Hypothesis (B).** $g(v) \neq 0$ whenever $v \neq 0$.

We want to apply Girsanov’s theorem with the process

$$\Sigma(u) := -\frac{\{\mu S(u - a) - r\}}{g(S(u - b))}, \quad u \in [0, T].$$

Hypothesis (B) implies that $\Sigma$ is well-defined, since by Theorem 1, $S(t) > 0$ for all $t \in [0, T]$ a.s.. Clearly $\Sigma(t)$, $t \in [0, T]$, is a predictable process. Moreover, $\int_0^T |\Sigma(u)|^2 du < \infty$ a.s., since sample-path continuity of the process $S(t)$, $t \in [0, T]$, implies almost sure boundedness of $S(t)$, $t \in [0, T]$, and Hypothesis (B) implies that $1/g(v)$, $v \in (0, \infty)$, is bounded on bounded intervals. Now let $l := \min(a, b)$. Set $\mathcal{F}_t := \mathcal{F}_0$ for $t \leq 0$. Then $\Sigma(u)$, $u \in [0, T]$, is measurable with respect to the $\sigma$-algebra $\mathcal{F}_{T-l}$. Hence, the stochastic integral $\int_{T-l}^T \Sigma(u) dW(u)$ conditioned on $\mathcal{F}_{T-l}$ has a normal distribution with mean zero and variance $\int_{T-l}^T |\Sigma(u)|^2 du$. Consequently, by the formula for the moment generating function of a normal distribution, we obtain

$$E_P \left( \exp \left\{ \int_{T-l}^T \Sigma(u) dW(u) \right\} \mid \mathcal{F}_{T-l} \right) = \exp \left\{ \frac{1}{2} \int_{T-l}^T |\Sigma(u)|^2 du \right\} \text{ a.s.}$$

Hence

$$E_P \left( \exp \left\{ \int_{T-l}^T \Sigma(u) dW(u) - \frac{1}{2} \int_{T-l}^T |\Sigma(u)|^2 du \right\} \mid \mathcal{F}_{T-l} \right) = 1 \text{ a.s.}$$

Now the above relation easily implies that

$$E_P \left( \exp \left\{ \int_0^T \Sigma(u) dW(u) - \frac{1}{2} \int_0^T |\Sigma(u)|^2 du \right\} \mid \mathcal{F}_{T-l} \right)$$

$$= \exp \left\{ \int_0^{T-l} \Sigma(u) dW(u) - \frac{1}{2} \int_0^{T-l} |\Sigma(u)|^2 du \right\} \text{ a.s.}$$

Let $k$ to be a positive integer such that $0 \leq T - kl \leq l$. Then by successive conditioning using backward steps of length $l$, an inductive argument gives

$$E_P \left( \exp \left\{ \int_0^T \Sigma(u) dW(u) - \frac{1}{2} \int_0^T |\Sigma(u)|^2 du \right\} \mid \mathcal{F}_{T-kl} \right)$$

$$= \exp \left\{ \int_0^{T-kl} \Sigma(u) dW(u) - \frac{1}{2} \int_0^{T-kl} |\Sigma(u)|^2 du \right\} \text{ a.s.}$$

Taking conditional expectation with respect to $\mathcal{F}_0$ on both sides of the above equation, we obtain

$$E_P \left( \exp \left\{ \int_0^T \Sigma(u) dW(u) - \frac{1}{2} \int_0^T |\Sigma(u)|^2 du \right\} \mid \mathcal{F}_0 \right)$$

$$= E_P \left( \exp \left\{ \int_0^{T-kl} \Sigma(u) dW(u) - \frac{1}{2} \int_0^{T-kl} |\Sigma(u)|^2 du \right\} \mid \mathcal{F}_0 \right) = 1 \text{ a.s.}$$

Taking the expectation of the above equation, we immediately obtain

$$E_P(g_T) = 1.$$

Therefore, the Girsanov’s theorem (Theorem 2) applies and the process

$$\tilde{W}(t) := W(t) + \int_0^t \frac{\{\mu S(u - a) - r\}}{g(S(u - b))} du, \quad t \in [0, T],$$
is a standard Wiener process under the measure $Q$ defined by $dQ := \varrho_t \, dP$ with

$$
\varrho_t := \exp \left\{ - \int_0^T \frac{\mu S(u-a) - r}{g(S(u-b))} \, dW(u) - \frac{1}{2} \int_0^T \left[ \frac{\mu S(u-a) - r}{g(S(u-b))} \right]^2 \, du \right\}
$$
a.s.. Since the process $\tilde{S}(t), \ t \in [0, T],$ can be written in the form

$$
\tilde{S}(t) = \int_0^t g(S(u-b)) \, d\tilde{W}(u), \quad t \in [0, T],
$$

we conclude that $\tilde{S}(t), \ t \in [0, T],$ is a continuous $Q$-martingale (i.e., a continuous martingale under the measure $Q$). Furthermore, by the representation (3), the discounted stock price process $\hat{S}(t), \ t \in [0, T],$ is also a continuous $Q$-martingale. In other words, $Q$ is an equivalent martingale measure. By the well-known theorem on trading strategies (e.g., Theorem 7.1 in [K.K]), it follows that the market consisting of $\{B(t), S(t) : t \in [0, T]\}$ satisfies the no-arbitrage property: There is no admissible self-financing strategy which gives an arbitrage opportunity.

We now establish the completeness of the market $\{B(t), S(t) : t \in [0, T]\}.$

From the proof of Theorem 1, it follows that the solution of the sde (1) satisfies the relation

$$
S(t) = \varphi(0) \exp \left\{ \int_0^t g(S(u-b)) \, dW(u) + \mu \int_0^t (u-a) \, du - \frac{1}{2} \int_0^t g(S(u-b))^2 \, du \right\}
$$
a.s. for $t \in [0, T].$ Hence we have

$$
\tilde{S}(t) = \varphi(0) \exp \left\{ \int_0^t g(S(u-b)) \, d\tilde{W}(u) - \frac{1}{2} \int_0^t g(S(u-b))^2 \, du \right\}
$$

for $t \in [0, T].$ By examining the definitions of $\tilde{S}, \tilde{W}, \hat{S}$ and equation (6), it is not hard to see that for $t \geq 0,$ $F_t^S = F_t^{\tilde{S}} = F_t^{\tilde{W}} = F_t^W,$ the $\sigma$-algebras generated by $\{S(u) : u \leq t\},$ $\{\tilde{S}(u) : u \leq t\}, \{\tilde{W}(u) : u \leq t\}, \{W(u) : u \leq t\},$ respectively. (Clearly, $F_t^W \subseteq F_t.$) Now, let $X$ be a contingent claim, viz. an integrable non-negative $F_t^W$-measurable random variable. Consider the $Q$-martingale

$$
M(t) := E_Q(e^{-rT}X \mid F_t^S) = E_Q(e^{-rT}X \mid F_t^{\tilde{W}}), \quad t \in [0, T].
$$

By the martingale representation theorem (e.g., Theorem 9.4 in [K.K]), there exists an $(F_t^{\tilde{W}})$-predictable process $h_0(t), \ t \in [0, T],$ such that

$$
\int_0^T h_0(u)^2 \, du < \infty \quad a.s.,
$$

and

$$
M(t) = E_Q(e^{-rT}X) + \int_0^t h_0(u) \, d\tilde{W}(u), \quad t \in [0, T].
$$

By (3) and (5) we obtain $d\tilde{S}(t) = \tilde{S}(t)g(S(t-b)) \, d\tilde{W}(u), \ t \in [0, T].$ Define

$$
\pi_S(t) := \frac{h_0(t)}{S(t)g(S(t-b))}, \quad \pi_B(t) := M(t) - \pi_S(t)\tilde{S}(t), \quad t \in [0, T].
$$

Consider the strategy $\{\pi_B(t), \pi_S(t) : t \in [0, T]\}$ which consists of holding $\pi_S(t)$ units of the stock and $\pi_B(t)$ units of the bond at time $t.$ The value of the portfolio at any time $t \in [0, T]$ is given by

$$
V(t) := \pi_B(t)e^{rt} + \pi_S(t)S(t) = e^{rt}M(t).
$$
Therefore, by the product rule and the definition of the strategy \(\{(\pi_B(t), \pi_S(t)) : t \in [0, T]\}\), it follows that

\[
dV(t) = e^{rt}dM(t) + M(t)de^{rt} = \pi_B(t)de^{rt} + \pi_S(t)dS(t), \quad t \in [0, T].
\]

Consequently, \(\{(\pi_B(t), \pi_S(t)) : t \in [0, T]\}\) is a self-financing strategy. Moreover, \(V(T) = e^{rT}M(T) = X\) a.s.. Hence the contingent claim \(X\) is attainable. This shows that the market \(\{B(t), S(t) : t \in [0, T]\}\) is complete, since every contingent claim is attainable. Moreover, in order for the augmented market \(\{B(t), S(t), X : t \in [0, T]\}\) to satisfy the no-arbitrage property, the price of the claim \(X\) must be

\[
V(t) = e^{-r(T-t)}E_Q(X | \mathcal{F}_t^S)
\]

at each \(t \in [0, T]\) a.s.. See, e.g., [B.R] or Theorem 9.2 in [K.K].

The above discussion may be summarized in the following formula for the fair price \(V(t)\) of an option on the stock whose evolution is described by the sdde (1).

**Theorem 3** Suppose that the stock price \(S\) is given by the sdde (1), where \(\varphi(0) > 0\) and \(g\) satisfies Hypothesis (B). Let \(T\) be the maturity time of an option (contingent claim) on the stock with payoff function \(X\), i.e., \(X\) is an \(\mathcal{F}_T^S\)-measurable non-negative integrable random variable. Then at any time \(t \in [0, T]\), the fair price \(V(t)\) of the option is given by the formula

\[
V(t) = e^{-r(T-t)}E_Q(X | \mathcal{F}_t^S),
\]

where \(Q\) denotes the probability measure on \((\Omega, \mathcal{F})\) defined by \(dQ := q_t \, dP\) with

\[
q_t := \exp \left\{ -\int_0^t \left\{ \frac{\mu S(u-a)}{g(S(u-b))} - r \right\} dW(u) - \frac{1}{2} \int_0^t \left\{ \frac{\mu S(u-a)}{g(S(u-b))} \right\}^2 du \right\}
\]

for \(t \in [0, T]\). The measure \(Q\) is a martingale measure and the market is complete.

Moreover, there is an adapted and square integrable process \(h_0(u), u \in [0, T]\) such that

\[
E_Q(e^{-rT}X | \mathcal{F}_t^S) = E_Q(e^{-rT}X) + \int_0^t h_0(u) d\hat{W}(u), \quad t \in [0, T],
\]

where \(\hat{W}\) is a standard \(Q\)-Wiener process given by

\[
\hat{W}(t) := W(t) + \int_0^t \frac{\mu S(u-a)}{g(S(u-b))} du, \quad t \in [0, T],
\]

The hedging strategy is given by

\[
\pi_S(t) := \frac{h_0(t)}{S(t)g(S(t-b))}, \quad \pi_B(t) := M(t) - \pi_S(t)\hat{S}(t), \quad t \in [0, T].
\]

The following result is a consequence of Theorem 3. It gives a Black-Scholes-type formula for the value of a European option on the stock at any time prior to maturity.

**Theorem 4** Assume the conditions of Theorem 3. Let \(V(t)\) be the fair price of a European call option written on the stock \(S\) with exercise price \(K\) and maturity time \(T\). Let \(\Phi\) denote the distribution function of the standard normal law, i.e.,

\[
\Phi(x) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^x e^{-u^2/2} du, \quad x \in \mathbb{R}.
\]
Then for all $t \in [T - l, T]$ (where $l := \min\{a, b\}$), $V(t)$ is given by

$$V(t) = S(t)\Phi(\beta_+(t)) - Ke^{-r(T-t)}\Phi(\beta_-(t)), \quad (9)$$

where

$$\beta_{\pm}(t) := \frac{\log \frac{S(t)}{K} + \frac{1}{2} \int_t^T (r \pm \frac{1}{2} g(u)^2) du}{\sqrt{\frac{1}{2} \int_t^T g(u)^2 du}}.$$

If $T > l$ and $t < T - l$, then

$$V(t) = e^{rt}E_Q \left( H \left( \tilde{S}(T - l), -\frac{1}{2} \int_{T-l}^T g(S(u) - b)^2 du, \int_{T-l}^T g(S(u) - b)^2 du \right) \bigg| \mathcal{F}_t \right) \quad (10)$$

where $H$ is given by

$$H(x, m, \sigma^2) := xe^{m+\sigma^2/2}\Phi(\alpha_1(x, m, \sigma)) - Ke^{-rT}\Phi(\alpha_2(x, m, \sigma)),$$

and

$$\alpha_1(x, m, \sigma) := \frac{1}{\sigma} \left[ \log \left( \frac{x}{K} \right) + rt + m + \sigma^2 \right],$$

$$\alpha_2(x, m, \sigma) := \frac{1}{\sigma} \left[ \log \left( \frac{x}{K} \right) + rt + m \right],$$

for $\sigma, x \in \mathbb{R}^+$, $m \in \mathbb{R}$.

The hedging strategy is given by

$$\pi_S(t) = \Phi(\beta_+(t)), \quad \pi_B(t) = -Ke^{-rT}\Phi(\beta_-(t)), \quad t \in [T - l, T].$$

Remark 2.

If $g(x) = 1$ for all $x \in \mathbb{R}^+$ then equation (9) reduces to the classical Black and Scholes formula. Note that, in contrast with the classical (non-delayed) Black and Scholes formula, the fair price $V(t)$ in a general delayed model considered in Theorem 4 depends not only on the stock price $S(t)$ at the present time $t$, but also on the whole segment $\{S(v) : v \in [t - b, T - b]\}$. (Of course $[t - b, T - b] \subset [0, l]$ since $t \geq T - l$ and $l \leq b$.)

Proof of Theorem 4.

Consider a European call option in the above market with exercise price $K$ and maturity time $T$. Taking $X = (S(T) - K)^+$ in Theorem 3, the fair price $V(t)$ of the option is given by

$$V(t) = e^{-r(T-t)}E_Q((S(T) - K)^+ \big| \mathcal{F}_t) = e^{rt}E_Q((\tilde{S}(T) - Ke^{-rT})^+ \big| \mathcal{F}_t),$$

at any time $t \in [0, T]$.

We now derive an explicit formula for the option price $V(t)$ at any time $t \in [T - l, T]$. The representation (6) of $\tilde{S}(t)$ implies that

$$\tilde{S}(T) = \tilde{S}(t) \exp \left\{ \int_t^T g(S(u) - b) d\tilde{W}(u) - \frac{1}{2} \int_t^T g(S(u) - b)^2 du \right\}$$

for all $t \in [0, T]$. Clearly $\tilde{S}(t)$ is $\mathcal{F}_t$-measurable. If $t \in [T - l, T]$, then $-\frac{1}{2} \int_t^T g(S(u) - b)^2 du$ is also $\mathcal{F}_t$-measurable. Furthermore, when conditioned on $\mathcal{F}_t$, the distribution of $\int_t^T g(S(u) -
b) $d\hat{W}(u)$ under $Q$ is the same as that of $\sigma \xi$, where $\xi$ is a Gaussian $N(0, 1)$-distributed random variable, and $\sigma^2 = \int_t^T g(S(u - b))^2 du$. Consequently, the fair price at time $t$ is given by

$$V(t) = e^{rt}H\left(\tilde{S}(t), -\frac{1}{2} \int_t^T g(S(u - b))^2 du, \int_t^T g(S(u - b))^2 du\right),$$

where

$$H(x, m, \sigma^2) := E_Q(xe^{m+\sigma \xi} - Ke^{-rT})^+, \quad \sigma, x \in \mathbb{R}^+, m \in \mathbb{R}.$$  

Now, an elementary computation yields the following:

$$H(x, m, \sigma^2) = xe^{m+\sigma^2/2} \Phi(\alpha_1(x, m, \sigma)) - Ke^{-rT} \Phi(\alpha_2(x, m, \sigma)).$$

Therefore, $V(t)$ takes the form:

$$V(t) = S(t)\Phi(\beta_+) - Ke^{-r(T-t)}\Phi(\beta_-),$$

where

$$\beta_\pm = \frac{\log \frac{S(t)}{K} + \int_t^T \left(r \pm \frac{1}{2} g(S(u - b))^2\right) du}{\sqrt{\int_t^T g(S(u - b))^2 du}}.$$  

For $T > l$ and $t < T - l$, from the representation (8) of $\tilde{S}(t)$, we have

$$\tilde{S}(T) = \tilde{S}(T - l) \exp \left\{ \int_{T-l}^T g(S(u - b)) d\hat{W}(u) - \frac{1}{2} \int_{T-l}^T g(S(u - b))^2 du \right\}.$$  

Consequently, the option price at time $t$ with $t < T - l$ is given by

$$V(t) = e^{rt}E_Q\left( H\left(\tilde{S}(T - l), -\frac{1}{2} \int_{T-l}^T g(S(u - b))^2 du, \int_{T-l}^T g(S(u - b))^2 du\right) \bigg| \mathcal{F}_t \right).$$

To calculate the hedging strategy for $t \in [T - l, T]$, it suffices to use an idea from [B.R], pages 95–96. This completes the proof of the theorem. \hfill \Box

**Remark 4.**

In the last delay period $[T - l, T]$, one can rewrite the option price $V(t), t \in [T - l, T]$ in terms of the solution of a random Black-Scholes pde of the form

$$\begin{align*}
\frac{\partial F(t, x)}{\partial t} &= -\frac{1}{2} g(S(t - b))^2 x^2 \frac{\partial^2 F(t, x)}{\partial x^2} - rx \frac{\partial F(t, x)}{\partial x} + rF(t, x), \quad 0 < t < T \\
F(T, x) &= (x - K)^+, \quad x > 0.
\end{align*}$$

(11)

The above time-dependent random final-value problem admits a unique $(\mathcal{F}_t)_{t \geq 0}$-adapted random field $F(t, x)$. Using the classical Itô-Ventzell formula ([Kun]) and (7) of Theorem 3, it can be shown that

$$V(t) = e^{-r(T-t)} F(t, S(t)), \quad t \in [T - b, T].$$

Note that the above representation is no longer valid if $t \leq T - b$, because in this range, the solution $F$ of the final-value problem (11) is **anticipating** with respect to the filtration $(\mathcal{F}_t)_{t \geq 0}$.

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