Entropic uncertainty relations for electromagnetic beams

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Abstract
The symplectic tomograms of two-dimensional (2D) Hermite–Gauss beams are found and expressed in terms of the Hermite polynomials squared. It is shown that measurements of optical-field intensities may be used to determine the tomograms of electromagnetic–radiation modes. Furthermore, entropic uncertainty relations associated with these tomograms are found and applied to establish the compatibility conditions of the field profile properties with Hermite–Gauss beam description. Numerical evaluations for some Hermite–Gauss modes illustrating the corresponding entropic uncertainty relations are finally given.

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1. Introduction

The paraxial optical beams propagating in media are known to obey the Schrödinger-like equation [1, 2]. The properties of the paraxial beams described by quantum-like equations were intensively discussed in both standard optics (see, for example, [3, 4]) and charged-particle-beam optics [5]. In particular, by means of such quantum-like equations, the paraxial propagation of the optical-field modes in both free space and in media, where the refractive index is a given function of the spatial coordinates, can be described in terms of the Hermite–Gauss (HG) or the Laguerre–Gauss modes [6, 7].

Recently, the tomographic analysis of electromagnetic beams received a great deal of attention, since it allows for an accurate characterization of the beam-mode profile required when direct observations are not able to determine the phase of the optical fields and, therefore, measurement of the beam intensity at all points is, in principle, needed [8].

In view of a tight connection between the paraxial beam propagation and quantum mechanics mentioned above, it seems quite natural to employ a quantum tomographic approach [9] to characterize optical-beam-mode profiles. Remarkably, in this regard, it was found that the quantum states, being the solutions to the Schrödinger equation, can be mapped by means of the Radon transform onto the standard probability distributions called symplectic [10] and optical [9] tomograms. The tomograms were shown [11] to obey the constraints that correspond to entropic uncertainty relations [12] for Shannon entropy [13] and Rényi entropy [14] of quantum states.

In the present paper, we construct symplectic tomograms of two-dimensional (2D) optical modes and discuss their properties. Furthermore, we introduce the concept of tomographic entropy of such modes and show for them the existence of entropic uncertainty relations in close analogy to the Heisenberg or Robertson–Schrödinger uncertainty relations. Numerical results showing the capability of entropic uncertainty relations to characterize the spatial content of 2D optical modes are presented.

2. Symplectic tomography of 2D beams

Let us consider a 2D (in Cartesian x and y transverse coordinates) optical mode field of wavelength λ that satisfies...
the following paraxial wave equation [1, 2]:

\[ i \frac{\partial \psi(x, y, z)}{\partial z} = -\frac{\lambda^2}{2} \nabla^2 \psi + U(x, y, z) \psi(x, y, z), \]  

(1)

where

\[ \nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}, \]

\[ \lambda = \lambda/2\pi, \]

\[ U \propto -\delta n(x, y, z) \]

(\(n\) being the refractive index close to the \(z\) propagation direction), and \(\psi(x, y, z)\) is the complex amplitude of electromagnetic field. In this equation, the longitudinal coordinate plays the role of time and \(\lambda\) plays the role of Planck's constant. The refractive-index profile provides the possibility to control the modes. In fact, the refractive index, being dependent on the transversal and longitudinal coordinates, determines 'the potential energy' as a function of the position and time in the corresponding Schrödinger-like equation.

Since we assume a real \(U\), the following normalization condition can be imposed:

\[ \int |\psi(x, y, z)|^2 \, dx \, dy = 1. \]  

(2)

The tomographic-probability distribution (called also tomogram) is defined by the Radon transform [15] generalized for two dimensions as symplectic tomogram [16] and given in terms of the squared modulus of the Fresnel integral [17]

\[ w(X_1, \mu_1, v_1, X_2, \mu_2, v_2, z) = \frac{1}{4\pi^2|v_1|v_2} \int \psi(x, y, z) \times \exp \left[ \frac{i}{2} \left( \frac{\mu_1}{v_1} x^2 + \frac{\mu_2}{v_2} y^2 - \frac{2X_1}{v_1} x - \frac{2X_2}{v_2} y \right) \right] \, dx \, dy \, |z|^2. \]  

(3)

Hereafter, to simplify the notation, we do not show explicitly the \(z\) dependence of the tomogram and related expressions. For instance, \(w(X_1, \mu_1, v_1, X_2, \mu_2, v_2, z)\) is replaced by \(w(X_1, \mu_1, v_1, X_2, \mu_2, v_2)\). The tomogram is a non-negative function of six real variables \(X_1, \mu_1, v_1, X_2, \mu_2, v_2\), and it satisfies the normalization condition

\[ \int w(X_1, \mu_1, v_1, X_2, \mu_2, v_2) \, dX_1 \, dX_2 = 1. \]  

(4)

Tomogram (3) can be interpreted as the probability density of two random variables \(X_1 = \mu_1 x + v_1 p_x\) and \(X_2 = \mu_2 y + v_2 p_y\), where \(x\) and \(y\) are coordinates of an intersection point in the transversal plane of the light ray and \(p_x\) and \(p_y\) are small angles determining the unit direction vector parallel to the light ray propagating along the fiber axis.

The tomographic-probability density determines the modulus and phase factor of the mode function \(\psi(x, y)\) due to the inverse relation

\[ \psi(x, y)\psi^*(x', y') = \frac{1}{4\pi^2} \int w(X_1, \mu_1, v_1, X_2, \mu_2, v_2) \times \delta(v_1 - x + x') \delta(v_2 - y + y') \times \exp \left[ \frac{i}{2} (2X_1 - \mu_1 x + \mu_1 x') + 2X_2 (y + y') \right] \, dx_1 \, dX_2 \, d\mu_1 \, d\mu_2 \, dv_1 \, dv_2. \]  

(5)

Tomogram (3) has the homogeneity property

\[ w(\lambda_1 X_1, \lambda_1 \mu_1, \lambda_1 v_1, \lambda_2 X_2, \lambda_2 \mu_2, \lambda_2 v_2) = \frac{1}{|\lambda_1 \lambda_2|} w(X_1, \mu_1, v_1, X_2, \mu_2, v_2), \]  

(6)

and it can be expressed in terms of an optical tomogram depending on four real variables

\[ w_{\text{opt}}(X_1, \theta_1, X_2, \theta_2) = \frac{1}{4\pi^2} \sin \theta_1 \sin \theta_2 \left| \int \psi(x, y) \exp \left[ \frac{i}{2} \left( \cot \theta_1 (x^2 + X_1^2) + \cot \theta_2 (y^2 + X_2^2) \right) - \frac{2iX_1}{\sin \theta_1} - \frac{2iyX_2}{\sin \theta_2} \right] \, dx \, dy \right|^2. \]  

(7)

In fact, due to the definitions (3) and (7), one has

\[ w_{\text{opt}}(X_1, \theta_1, X_2, \theta_2) = w(X_1, \cos \theta_1, \sin \theta_1, X_2, \cos \theta_2, \sin \theta_2) \]  

(8)

and, due to the homogeneity property (6),

\[ w(X_1, \mu_1, v_1, X_2, \mu_2, v_2) = \frac{1}{\sqrt{(\mu_1^2 + v_1^2)(\mu_2^2 + v_2^2)}} \times w_{\text{opt}} \left( \frac{X_1}{\sqrt{\mu_1^2 + v_1^2}}, \frac{\mu_1}{\sqrt{\mu_1^2 + v_1^2}}, \frac{X_2}{\sqrt{\mu_2^2 + v_2^2}}, \frac{\mu_2}{\sqrt{\mu_2^2 + v_2^2}} \right). \]  

(9)

Another tomogram called Fresnel tomogram of the light mode in the optical fiber is given by the integral transform [18]

\[ w_F(X_1, v_1, X_2, v_2) = \frac{1}{4\pi^2} \left| \frac{v_1}{v_2} \right| \int \psi(x, y) \exp \left[ \frac{i(X_1 - x)^2}{2v_1} + \frac{i(X_2 - y)^2}{2v_2} \right] \, dx \, dy \right|^2. \]  

(10)

Fresnel tomogram \(w_F\) can be obtained from symplectic tomogram by putting \(\mu_1 = 1\) and \(\mu_2 = 1\), i.e.

\[ w_F(X_1, v_1, X_2, v_2) = w(X_1, 1, v_1, X_2, 1, v_2). \]  

Furthermore, the following relationship holds:

\[ w(X_1, \mu_1, v_1, X_2, \mu_2, v_2) = \frac{1}{|\mu_1 \mu_2|} w_F \left( \frac{X_1}{\mu_1}, \frac{v_1}{\mu_1}, \frac{X_2}{\mu_2}, \frac{v_2}{\mu_2} \right). \]  

(11)
In view of mutual tomogram relations (9) and (10), one can find the mode function $\psi(x, y)$ using (5) and express this function either in terms of optical tomogram or in terms of Fresnel tomogram.

3. Tomographic entropies of optical beams

In this section, we consider in detail tomographic entropies associated with optical beams following the general formalism developed in [18]. There exists the Shannon construction [13] of entropy associated with the probability distribution function $P(n)$ of a discrete variable $n$

$$H = - \sum_n P(n) \ln P(n),$$  \hspace{1cm} (12)

which can be easily extended to the classical probability distribution function of a continuous variable. In particular, we can use the tomographic-probability distribution (3) to introduce the following tomographic entropy for an optical beam:

$$H(\mu_1, v_1, \mu_2, v_2) = - \int w(x_1, \mu_1, v_1, x_2, \mu_2, v_2) \times \ln w(x_1, \mu_1, v_1, x_2, \mu_2, v_2) \, dX_1 \, dX_2.$$  \hspace{1cm} (13)

The above entropy is a new information characteristic of the optical-beam profile in media, which yields also the optical tomographic entropy of the optical beam

$$H_{\text{opt}}(\theta_1, \theta_2) = - \int w_{\text{opt}}(x_1, \theta_1, x_2, \theta_2) \times \ln w_{\text{opt}}(x_1, \theta_1, x_2, \theta_2) \, dX_1 \, dX_2,$$  \hspace{1cm} (14)

and the Fresnel tomographic entropy

$$H_F(v_1, v_2) = - \int w_{\text{F}}(x_1, v_1, x_2, v_2) \times \ln w_{\text{F}}(x_1, v_1, x_2, v_2) \, dX_1 \, dX_2.$$  \hspace{1cm} (15)

All three entropies (13)–(15) are mutually related.

An interesting property of the tomographic entropies is connected with the uncertainty relation associated with the light-beam intensities related to the mode function $\psi(x, y)$ and its Fourier transform

$$\tilde{\psi}(p_x, p_y) = (2\pi)^{-1} \int \psi(x, y) \times \exp^{-i(p_x x + p_y y)} \, dx \, dy.$$  \hspace{1cm} (16)

This entropic uncertainty relation reads (see e.g. [12])

$$- \int |\psi(x, y)|^2 \ln |\psi(x, y)|^2 \, dx \, dy - \int |\tilde{\psi}(x, y)|^2 \ln |\tilde{\psi}(x, y)|^2 \, dx \, dy \geq 2 \ln(\pi e).$$  \hspace{1cm} (17)

The entropic uncertainty relation was generalized for the symplectic tomographic entropy as well as for the optical and Fresnel tomographic entropies [18].

To consider, for example, the optical tomographic entropy, we introduce the function

$$R(\theta_1, \theta_2) = H_{\text{opt}}(\theta_1, \theta_2) + H_{\text{opt}}(\theta_1 + \pi/2, \theta_2 + \pi/2) - 2 \ln(\pi e),$$  \hspace{1cm} (18)

where $H_{\text{opt}}(\theta_1, \theta_2)$ is given by (14).

According to the new entropic uncertainty relations [18], the function $R(\theta_1, \theta_2)$ must be non-negative for all values of the angles $\theta_1$ and $\theta_2$, i.e.

$$R(\theta_1, \theta_2) \geq 0.$$  \hspace{1cm} (19)

This means that, if one measures the modulus and phase of the mode function $\psi(x, y)$ by any method, the results of the measurement yield also the function (18) which must be non-negative. Non-negativity of this function for all the angles $\theta_1$ and $\theta_2$ can serve as an extra control of accuracy of the measurements.

4. Tomography of 2D HG beams

We want to illustrate here the tomographic properties of a 2D HG beam of order $(n, m)$ described by a wave function $\psi_{nm}(x_1, x_2)$. These beams are of particular interest since they form a complete set of solutions of the paraxial equation (1) in the case of a linear medium whose refractive index is a quadratic function of $x_1$ and $x_2$ or in vacuo (note that in (1) we have replaced $x$ and $y$ by $x_1$ and $x_2$, respectively). Any other solution can be written, in principle, as an expansion in terms of HG beams over all the indices $n$ and $m$. The HG beams play an important role in the design of spherical resonators since they represent the resonators’ modes having the same wavefronts as a Gaussian beam but different amplitude distributions.

Let us write the HG beam of the order $(n, m)$ in the following normalized form:

$$\psi_{nm}(x_1, x_2) = N_{nm} H_n \left( \frac{\sqrt{2} x_1}{\sigma_0} \right) H_m \left( \frac{\sqrt{2} x_2}{\sigma_0} \right) \times \exp \left[ - \frac{x_1^2 + x_2^2}{2 \sigma_0^2} \right].$$  \hspace{1cm} (20)

where $\sigma_0$ is the width assumed to be the same along the two transverse directions $x_1$ and $x_2$ and

$$N_{nm} = \sqrt{\frac{2}{\pi \sigma_0^2 n! m! 2^{n+m}}}$$

is the normalization factor.

The 2D tomogram $w_{nm}$ of the HG beam of the order $(n, m)$ is given by the following 6D expression:

$$w_{nm}(X_1, X_2, \mu_1, \mu_2, v_1, v_2) = \frac{1}{(2\pi)^2} \frac{1}{|v_1 v_2|} \int \psi_{nm}(\xi_1, \xi_2) \times \exp \left[ i \left( \frac{\mu_1}{2 v_1} \xi_1 + \frac{\mu_2}{2 v_2} \xi_2 - \frac{X_1}{v_1} \xi_1 - \frac{X_2}{v_2} \xi_2 \right) \right] \, d\xi_1 \, d\xi_2.$$  \hspace{1cm} (21)
Substituting the HG field given by equation (20) into equation (21) and taking into account the integral relation

\[ \int_{-\infty}^{\infty} H_n(\alpha y) \exp \left[ -\frac{(y-\beta)^2}{2} \right] dy = \sqrt{\pi} \left( 1 - \alpha^2 \right)^{\nu/2} H_n \left( \frac{\alpha \beta}{\sqrt{1 - \alpha^2}} \right), \tag{22} \]

we obtain for the tomogram the following expression:

\[ w_{nm}(X_1, X_2, \mu_1, \mu_2, v_1, v_2) = \frac{1}{(2\pi)^2} \frac{1}{|v_2v_1|} N_{nm}^2 |I(X_1, \mu_1, v_1)|^2 |I(X_2, \mu_2, v_2)|^2, \tag{23} \]

where

\[ I(X_k, \mu_k, v_k) = \sqrt{\frac{2}{\pi}} \left( 1 - \alpha_k^2 \right)^{\nu/2} H_n \left( \frac{\alpha_k \beta_k}{\sqrt{1 - \alpha_k^2}} \right) \]

\[ \times \exp \left[ -\frac{X_k^2/4v_k^2q_k}{\sqrt{q_k}} \right], \quad k = 1, 2. \tag{24} \]

and the parameters \( \alpha_k, \beta_k \) and \( q_k \) are given by

\[ \alpha_k = \frac{\sqrt{2}}{\sigma_0 q_k}, \]

\[ \beta_k = \frac{X_k}{2v_k \sqrt{q_k}}, \tag{25} \]

\[ q_k = \frac{1}{\sigma_0^2} - \frac{1}{2v_k}. \]

In view of the definition given by equation (21) inserted into equation (24) for \( k = 1, 2 \), we can write the tomogram

\[ w_{nm}(X_1, X_2, \mu_1, \mu_2, v_1, v_2) = \frac{1}{|\mu_1 \mu_2|} N_{nm}^2 \left( \frac{\sigma_k^2}{\sigma_1(\sigma_1 z_1) \sigma_2(\sigma_2 z_2)} \right) \]

\[ \times H_n \left( \sqrt{\frac{2}{\mu_1 \sigma_1(z_1)}} \right)^2 H_m \left( \sqrt{\frac{2}{\mu_2 \sigma_2(z_2)}} \right)^2 \]

\[ \times \exp \left[ -2 \left( \frac{X_1^2}{\mu_1^2 \sigma_1(z_1) \sigma_2(\sigma_2 z_2)^2} + \frac{X_2^2}{\mu_2^2 \sigma_2(z_2) \sigma_1(z_1)^2} \right) \right], \tag{26} \]

where the following quantities are defined:

\[ \sigma_k(z) = \sigma_0 \sqrt{1 + \left( \frac{z_k}{z_0} \right)^2}, \]

\[ \frac{\mu_k}{v_k} = \frac{2\pi}{\lambda z_k}, \tag{27} \]

\[ z_0 = \frac{\pi \sigma_0^2}{\lambda}. \]

Now we consider a special case \( \mu_1/v_1 = \mu_2/v_2 = \rho \).

In this case, the tomogram is a function of four phase space variables only, namely, \( X_1, X_2, \mu \) and \( v \), and equation (26) can be written as follows:

\[ w_{nm}(X_1, X_2, \mu, v) = \frac{1}{|\mu \nu|} N_{nm}^2 \left( \frac{\sigma_0^2}{\sigma(\sigma z)} \right) H_n \left( \sqrt{\frac{2}{\mu \sigma(z)}} \right)^2 \]

\[ \times H_m \left( \sqrt{\frac{2}{\mu \sigma(z)}} \right)^2 \exp \left[ -2 \left( \frac{X_1^2}{\mu_1^2 \sigma(z)} + \frac{X_2^2}{\mu_2^2 \sigma(z)} \right) \right], \tag{28} \]

where, according to equation (27), we have \( \sigma(z) = \sigma_0 \sqrt{1 + (\sigma z/z_0)^2} \) and \( z = 2\pi/\lambda \rho \), with \( z_0 \) given in equation (27).

Apart from the scaling factor \((|\mu \nu|)^{-1}\), equation (28) represents the intensity distribution of the HG beam of order \((n, m)\) at a distance \( z \) in the transverse plane of rescaled transverse distances \( X_1/\mu \) and \( X_2/\mu \). The first part of equation (28) gives the well-known dependence of the beam radius \( \sigma(z) \) on the propagation distance \( z = 2\pi/\lambda \rho \), which is defined through the second part of equation (28) in terms of the ratio \( \rho = \mu/\nu \) of the phase space variables. At \( z = 0 \), the beam radius is given by \( \sigma(0) = \sigma_0 \).

The wavelength \( \lambda \) plays the role of a spatial scaling parameter in equations (27) and (28). We point out the formal analogy between the tomographic representation of HG beams and the paraxial propagation law governing the evolution of the intensity distribution in the rescaled transverse plane of an optical HG beam of the order \((n, m)\) and wavelength \( \lambda \). Indeed \( z_0 \) given by equation (27) is the well-known confocal parameter or the depth of the focus of the propagating beam. When the relation \( \mu_1/v_1 = \mu_2/v_2 = \rho \) is not satisfied and the phase-space variables assume arbitrary values, the simple interpretation of tomogram in terms of the intensity distribution of 2D HG beam at a fixed distance \( z = 2\pi/\lambda \rho \) does not hold more. In general, we have two different distances which, according to equation (27), are given by \( z_1 = 2\pi v_1/\lambda \mu_1 \) and \( z_2 = 2\pi v_2/\lambda \mu_2 \), respectively.

However, it is easy to see that equation (27) is separable along the two transverse directions \( X_1/\mu \) and \( X_2/\mu \) and that the corresponding factors represent the intensity distribution of two independent 1D HG beams of radii \( \sigma_1(z_1) \) and \( \sigma_2(z_2) \) at the distances \( z_1 \) and \( z_2 \).

Therefore, the tomogram of 2D HG comprises, in general, the features of propagation laws corresponding to the projections of the beam along two orthogonal directions depending on the values of the phase space variables.

In figures 1–3, we present plots of the function \( R(\theta_1, \theta_2) \) calculated for three different cases of 2D HG modes. One can see similar behavior of these functions for different mode indices and widths. Nevertheless, all the functions are non-negative according to the found uncertainty relations.
5. Conclusions and discussions

In the present paper, we have shown that the symplectic tomography of 2D optical beams can be employed to characterize the spatial distribution of the beam mode along the propagation direction. With a proper identification of the phase parameters, the tomographic map reduces to measuring the optical intensity distribution of the optical beam in a given transverse plane. However, a full tomographic characterization of the beam requires, in principle, an infinite set of measurements of the spatial distributions of the light intensity at different points in the cross section of the optical beam along the propagation direction, which, in practice, is impossible due to the bandwidth limitations of the optical detectors. Nevertheless, we have shown that the tomographic approach can be employed to derive a set of entropic inequalities in terms of the logarithmic measurement of the beam intensity. An advantage of this formulation is that it allows one to catch the relevant features of the propagation beam characteristics from a tomographic map. Here, the concept of entropy, well established by statistics and later extended by Shannon to information theory, plays the role of an integral descriptor of the tomogram of the optical mode in terms of two basic optical-beam parameters, the width and Rayleigh range. According to the tomographic-based entropic inequality, a strictly positive function of two space variables $R(\theta_1, \theta_2)$ can be associated with the optical mode. The numerical results obtained for the basic HG modes show clearly that this entropic-based optical descriptor is sensitive to the beam width and, in general, tends to increase for higher order modes. It is also expected that this formulation can be extended to optical vortex field where the presence of several defects or vortices is a manifest signature of high-order optical modes.

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