EnKF-levelset method for an acoustics inverse medium scattering problem

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Abstract. This paper demonstrates a numerical method for reconstructing penetrable obstacles in a homogeneous background medium. The ensemble Kalman method is proposed as an inversion solver, and the level set technique is discussed as a flexible way to tracking the boundary. Through some numerical examples, we show that the proposed method is effective and flexible even without the priori number knowledge of the scatterers.

1. Introduction

In this paper, we consider a special kind of inverse medium scattering problem for determining the shape of permeable obstacles in a homogeneous background. It is assumed that the permeable obstacle to be piecewise constant. This assumption can arises in many situations such as layered medium. Unlike the direct problem, the inverse problem admits great challenges due to its nonlinearity and ill-posedness. Over the past decades, many researchers have been devoted to this problem and put forward different numerical methods, including Gauss-Newton method, linear sampling method, extended sampling method, etc. For more details one can refer to [1-3].

In this paper, we consider the ensemble Kalman filter (EnKF) as the inversion method to reconstruct the shape of the scatterer. The EnKF is a widely used methodology [3-5], which can be deduced from the perspective of both Bayesian statistics and optimization. For the inverse problem, the EnKF uses an ensemble of particles, through a sequential process, the states of these particles can be improved by combing the dynamic model and the data. We will show that in the whole inversion process, we do not need to evaluate the Fréchet derivative of the forward operator and its adjoint [6]. Compared with the traditional Newton-type algorithm, it is very attractive in many cases when the Fréchet derivative of the forward operator and its adjoint are hard to achieve. To be flexibly deal with the topological changes of the scatterer boundary, we introduce the level set technique [7] into the EnKF algorithm. To the best of our knowledge, this combination is rarely reported in the field of inverse scattering [8]. Through some numerical experiments, we show that our algorithm is effective and flexible, and it can not only deal with one scatterer, but also can recover the shape of multiple scatterers.

The organization of the paper is as follows. In Section 2, we briefly discuss the inverse medium scattering problem and the level set parameterization. The EnKF algorithm is proposed in Section 3. In Section 4, we do some numerical experiments to show the effectiveness and flexibility of our algorithm.
2. Inverse problem and level-set parameterization

2.1 The inverse medium problem

Denote by \( S = \{ x \in \mathbb{R}^2 : |x| = 1 \} \) a unit disk. Given the incident plane wave \( u'(x) = e^{ikx}d, x \in \mathbb{R}^2, d \in S \), where \( k > 0 \) is the wave number and \( d \) is incident direction. The propagation of the time-harmonic plane wave \( u'(x) \) scattered by an inhomogeneous medium can be modeled by the following Helmholtz equation

\[
\Delta u + k^2(1 + b(x))u = 0, \quad \text{in } \mathbb{R}^2, \tag{1}
\]

\[
u = u' + u', \tag{2}
\]

\[
\lim_{r \to \infty} \sqrt{r} \left( \frac{\partial u'}{\partial r} - iku' \right) = 0, \quad r = |x|, \tag{3}
\]

where \( u(x) \) is the total field, which can be regarded as the superposition of the incident field \( u'(x) \) and the scattering field \( u'(x) \); \( b(x) > -1 \) is the refractive index; (3) is the Sommerfeld radiation condition. The properties of the inhomogeneous medium is described by \( b(x) \). In this paper we assume \( b(x) \in L^\infty(\mathbb{R}^2) \) to be piecewise constant and

\[
b(x) = \begin{cases} q, & x \in D, \\ 0, & x \in \mathbb{R}^2 \setminus \bar{D}, \end{cases} \tag{4}
\]

where \( q > 0 \) and \( D \subset \mathbb{R}^2 \) is a bounded domain. The inverse medium scattering problem (IMSP) considered in this paper is stated as follows:

**IMSP**: Determine \( \partial D \) from the near-field data \( \{ u'(x_i), x_i \in \Omega \} \), where \( x_i \) are measurement points that collected on a closed surface \( \Omega \) outside \( D \).

2.2 Level-set parameterization

For the sake of simplicity, in this section we first transform the problem into the form of operator equations. According to (1)-(3), the forward problem can be transformed into \( u'(x) = F(b), x \in \mathbb{R}^2 \), where \( F: L^\infty(\mathbb{R}^2) \to H^1(\mathbb{R}^2) \) is the forward response operator. Denote by \( O: H^1(\mathbb{R}^2) \to \mathbb{C}^N \),

\[
(O \circ F)(b) = (u'(x_1); \cdots; u'(x_N)), \quad x_1, \cdots x_N \in \Omega,
\]

the observation operator and then taken the measurement noise into consideration, the problem IMSP can be written as

\[
z = (O \circ F)(b) + \delta, \tag{5}
\]

where \( z \) is the measurement data and \( \delta \) is the noise. In this paper we always assume \( \delta \) to be Gaussian with mean zero and covariance \( C \). Next, we consider the level-set parameterization of the problem (5). The scatterer \( b(x) \) can be characterized by

\[
b(x) = \sum_{i=1}^{I} q_i 1_{B_i}(x),
\]

where \( q_i \) are known positive constants and \( 1_{B_i}(x) \) denotes the indicator function of \( B_i \), here \( B_i \) are subsets of \( D \) and satisfying \( \bigcup_{i=1}^{I} \overline{B_i} = \overline{D} \) and \( B_i \cap B_j = \emptyset \), for \( i \neq j \). The purpose of the inverse problem is to determine the geometry of the interfaces described by \( B_i \). To this end, we consider the following parameterization \( B_i = \{ x \in D : \alpha_{i-1} \leq \xi(x) < \alpha_i \}, i = 1, \cdots, I \), where \( \xi: D \to \mathbb{R} \) are known as
the level set function and $\alpha_i$ are constants that satisfies $-\infty = \alpha_0 < \alpha_1 < \cdots < \alpha_I = +\infty$. Denote $\Xi = C(D; \mathbb{R})$, for given $q_1, \cdots, q_I$, we define the level set map $S: \Xi \to L^\infty(\mathbb{R}^2)$ by

$$
(S_\xi)(x) = \sum_{i=1}^{I} q_i I_{\eta_i}(x),
$$

(6)

From (6) we can see that the geometric field of interest can be obtained from the level set field through the relationship of the map $S: \Xi \to L^\infty(\mathbb{R}^2)$. Together with (5) and (6), we can finally transform the problem IMSP as: for given $z$ find $\xi$ such that

$$
z = G(\xi) + \delta,
$$

(7)

where $G = O \circ F \circ S$.

3. Ensemble Kalman filter

Based on the previous discussion of the level set parameterization of the inverse problem, in this section we propose the ensemble Kalman method to reconstruct the geometry of the scatterer. Let $Y: \Xi \times \mathbb{C}^N$ and $y = (\xi, \omega)^*$, we define the map $\psi: Y \to Y$ as

$$
\psi: \begin{bmatrix} \xi \\ \omega \end{bmatrix} \to \begin{bmatrix} \xi \\ G(\xi) \end{bmatrix},
$$

Then the artificial dynamics is given by

$$
y_{n+1} = \psi(y_n),
$$

(8)

From the artificial dynamics (8), the observation data can be obtained by the following form

$$
z_{n+1} = H y_{n+1} + \delta_{n+1},
$$

(9)

where $H: Y \to \mathbb{C}^N$ is defined by $H = (0, I)$ and $\{\delta_n\}_{n \in \mathbb{N}}$ are i.i.d. Gaussian noise, i.e. $\delta_n \sim N(0, C)$. The basic idea of the EnKF method is to use an ensemble of particles, through an update process by combining model (8) and (9) to get an improved state from the current one. The improved state can be obtained through the 3DVar technique by minimizing

$$
y_{n+1}^* = \arg\min_{y \in \mathbb{C}^N} \frac{1}{2} |H y - z_{n+1}|_C^2 + \frac{1}{2} |y - y_{n+1}^*|_C^2,
$$

(10)

where $y_{n+1}^* = \psi(y_n^{im})$ and $\Gamma$ is the covariance of the current state. The minimizer of (10) can be given by [9]

$$
y_{n+1}^{im} = (I - KH) \psi(y_n^{im}) + K z_{n+1},
$$

(11)

where $K = \Gamma H^* (H \Gamma H^* + C)^{-1}$ is the so called Kalman gain matrix, here $H^*$ is the adjoint of $H$. In the numerical implementation, if $H$ has low rank and $C^{-1}$ can be easy to obtained, the following Sherman-Morrison-Woodbury formula

$$(H \Gamma H^* + C)^{-1} = C^{-1} - C^{-1} H \Gamma^{1/2} (I + \Gamma H^* C^{-1} H \Gamma^{1/2})^{-1} \Gamma^{1/2} H^* C^{-1},$$

can be used as a cheap way to calculate the inverse term.

The initial ensemble of particles in the EnKF approach can be generated from the prior distribution. In this paper, we assume the prior is Gaussian of the Whittle-Martérn type. The covariance function of the prior is given by

$$
C(x, y) = \frac{2^{|v|}}{\Gamma(v)} \left( \frac{|x - y|}{l} \right)^v K_{\nu} \left( \frac{|x - y|}{l} \right),
$$

(12)

where $\Gamma(\cdot)$ is the Gamma function, $v$ is the smooth parameter, $l$ is the length scale and $K_{\nu}$ is the $\nu$-th order second kind modified Bessel function. After the initial ensemble is generated, the update procedure of the EnKF method is stated as
(1) Generate a pre-estimated state according to
\[ \bar{\mathbf{y}}_{n+1} = \mathbf{y}(\mathbf{y}_{n+1}), \quad n = 0, 1, \ldots, \]
and calculate
\[ \mathbf{y}_{n+1} = \frac{1}{J} \sum_{j=1}^{J} \bar{\mathbf{y}}_{n+1}^{(j)} \mathbf{T} \mathbf{y}_{n+1}^{(j)} \mathbf{T}^{T} - \mathbf{y}_{n+1} \mathbf{y}_{n+1}^{T}, \]
where \( J \) is the number of particles.

(2) According to (11), calculate the Kalman gain matrix
\[ \mathbf{K}_{n+1} = \Gamma_{n+1} \mathbf{H}^{T} (\mathbf{H} \Gamma_{n+1} + \mathbf{R})^{-1}, \]
and update each particle as
\[ \mathbf{y}_{n+1}^{(j)} = (I - \mathbf{K}_{n+1} \mathbf{H}) \mathbf{y}_{n+1}^{(j)} + \mathbf{K}_{n+1} \mathbf{z}_{n+1}^{(j)}. \]

(3) Obtain the components \( \mathbf{z}_{n+1}^{(j)} \) and calculate their mean
\[ \bar{\mathbf{z}}_{n+1} = \frac{1}{J} \sum_{j=1}^{J} \mathbf{H}^{T} \mathbf{y}_{n+1}^{(j)} = \frac{1}{J} \sum_{j=1}^{J} \mathbf{z}_{n+1}^{(j)}, \]
where \( \mathbf{H}^{T} : \mathbf{Y} \to \mathbf{Z} \) is defined by \( \mathbf{H}^{T} = (I, 0) \).

4. Numerical examples
In this section, we test the performance of the EnKF-levelset method on the following examples. In all the numerical tests, we fix the wave number \( k = 1 \) and assume \( b(x) = 1 \) in \( \Omega \). The direct problem is solved by the DtN-FEM algorithm (see [10]). The measurement points are collected on a circle \( \Omega = \mathcal{R}(\cos \theta, \sin \theta) \) with \( \mathcal{R} = 5 \) and \( \theta = 2\pi j / N, \quad j = 0, 1, \ldots, N - 1, \quad N = 49. \) The clean data is contaminated by
\[ u_{\delta} = u + \gamma(\delta_{1} + i\delta_{2}) \max |u|, \]
where \( \delta_{1}, \delta_{2} \sim N(0, 1) \) and \( \gamma = 1\% \) is the noise level.

We choose two incident directions \( d = [\pm 1, 0] \) and consider the following three geometries:

(1) A kite given by
\[ (x_{1}, x_{2}) = (\cos \theta + 0.65 \cos 2\theta - 0.65, 1.5 \sin \theta). \]

(2) A rounded-triangle-hole given by
\[ (x_{1}, x_{2}) = \left( \frac{5 + \sin 3\theta}{3} \cos \theta, \frac{5 + \sin 3\theta}{3} \sin \theta \right) / (0.5 \cos \theta, 0.5 \sin \theta). \]

(3) A circle-ellipse given by
\[ (x_{1}, x_{2}) = (0.5 \cos \theta - 1.5, 0.5 \sin \theta - 1.5) \cup (1.2 \cos \theta + 1, 0.6 \sin \theta + 1). \]

The exact geometries are displayed in figure 1.

Figure 1. The true geometries of the scatterer.

Figure 2. The reconstructed geometries of the scatterer.
In the inversion process, we choose $J = 500$ particles, the smooth and length scale parameters in (12) are taken as $\nu = 2$ and $l = 0.1$. The inversion result are displayed in figure 2. From figure 2, we can see that our algorithm is effective, it can not only suitable for one scatterer, but also for multiple scatterers.

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