Strong (D)QBF Dependency Schemes via Tautology-free Resolution Paths

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Dependencies
Dependencies

\[ \begin{array}{ccc}
\text{E} & \text{E} & \text{A} \\
\text{A} & \text{E} & \text{E} \\
\text{E} & \text{A} & \text{E} \\
\end{array} \]
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We consider (closed, prenex) *dependency quantified Boolean formulas* of the following form (a.k.a. *S-form DQBF*):

\[
\Psi = \forall u_1 \cdots \forall u_m \exists x_1(S_{x_1}) \cdots \exists x_n(S_{x_n}) \cdot C_1 \land \cdots \land C_r
\]

A DQBF is *true* if there exist functions \( f_{x_i} : \{0,1\}^{S_{x_i}} \rightarrow \{0,1\} \) whose substitution for \( x_i \) yields a propositional tautology.
DQBF extends QBF:

\[ \Phi = \bigwedge \forall U_1 \exists X_1 \forall U_2 \exists X_2 \cdots \forall U_k \exists X_k \cdot C_1 \land \cdots \land C_r \]

If \( x_i \in X_i \), then \( S_{x_i} = \bigcup_{j<i} U_j \).

A DQBF is a QBF if and only if the support sets are linearly ordered under inclusion.
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We are interested in solving DQBFs as efficiently as possible.
Spurious Dependencies

Consider the formula $\forall u \exists x (\{u\}) \cdot (x \lor u) \land (x \lor \neg u)$.

It is obviously true by setting $x := 1$. But that does not need the dependency on $u$. Hence, the dependency of $x$ on $u$ is spurious.
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Dependency Schemes

- A *dependency scheme* as defined for QBF is a mapping:

\[ D : \Phi \mapsto D(\Phi) \subseteq D^{\text{trv}}(\Phi) = \{(x, y) \mid x < y\} \]
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- Because dependency schemes were created for QBF, dependencies are defined both ways. This turned out unnecessary in the analysis of refutational proof systems, and becomes meaningless in DQBF.
Proof Systems

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\[
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\vdots \\
\phi_3 \\
\phi_4 \\
\vdots \\
\phi_5 \\
\phi_6 \\
\perp
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- A Q-Res refutation is a sequence of clauses that are either existential resolvents or universal reducts.
- A $\forall$Exp+Res refutation is a resolution refutation of the universally expanded formula (a.k.a. Shannon expansion);
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**Theorem ([SS16])**

A QBF is false if, and only if, it has a $Q(\mathcal{D}^{\text{rrs}}, \mathcal{D}^{\text{std}})$-Res refutation.
Recap

We are trying to solve a DQBF; identify as many spurious dependencies as possible; while maintaining soundness of the proof system.
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Overview of Contributions

1. A clean DQBF-centric definition of dependency schemes along with a characterisation of when a dependency scheme can be used in any DQBF proof system;
2. A new, tautology-free dependency scheme D\textsubscript{tf} that generalizes the to-date strongest known resolution-path dependency scheme;
3. DQBF-genuine exponential separations of ∀\text{Exp}+\text{Res} with and without D\textsubscript{rrs} and D\textsubscript{tf};
4. QBF-genuine exponential separations of Q-Res with D\textsubscript{rrs} and with D\textsubscript{tf}.

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4. *QBF-genuine* exponential separations of $\text{Q-Res}$ with $\mathcal{D}^{\text{rrs}}$ and with $\mathcal{D}^{\text{tf}}$. 
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We say that a dependency scheme $\mathcal{D}$ is fully exhibited if $\mathcal{D}(\Psi) \equiv \Psi$ for every DQBF $\Psi$. 
Parameterising Proof Systems

**Definition (P(D))**

Let $P$ be a DQBF proof system and let $D$ be a dependency scheme. A $P(D)$ refutation of a DQBF $\Psi$ is a $P$ refutation of $D(\Psi)$.

**Proposition**

*Given a DQBF proof system $P$ and a dependency scheme $D$, $P(D)$ is sound and complete if, and only if, $D$ is fully exhibited.*
The Tautology-free Dependency Scheme
The reflexive resolution path dependency scheme ($\mathcal{D}^{\text{rrs}}$) is defined as the mapping $\Psi \mapsto \Psi'$, where

$$\Psi := \forall u_1 \cdots \forall u_m \exists x_1(S_{x_1}) \cdots \exists x_n(S_{x_n}) \cdot \psi,$$

$$\Psi' := \forall u_1 \cdots \forall u_m \exists x_1(S'_{x_1}) \cdots \exists x_n(S'_{x_n}) \cdot \psi,$$

and $S'_{x_i}$ is the set of universal variables $u \in S_{x_i}$ for which there exists a sequence $C_1, \ldots, C_k$ of clauses in $\psi$ and a sequence $p_1, \ldots, p_{k-1}$ of existential literals satisfying the following conditions:

(a) $u \in C_1$ and $\overline{u} \in C_k$;
(b) for some $j \in [k-1]$, $x_i = \text{var}(p_j)$;
(c) for each $j \in [k-1]$, $p_j \in C_j$, $\overline{p}_j \in C_{j+1}$, and $u \in S_{\text{var}(p_j)}$;
(d) for each $j \in [k-2]$, $\text{var}(p_j) \neq \text{var}(p_{j+1})$.

The tautology-free dependency scheme ($\mathcal{D}^{\text{tf}}$) adds to $\mathcal{D}^{\text{rrs}}$ the condition

(e) for each $j \in [k-1]$, $(C_j \cup C_{j+1}) \upharpoonright \exists(\psi)$ is non-tautological.
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- \(D^\text{rrs}\) identifies potential information flows between variables as resolution paths;
- A resolution path is a sequence of clauses which can trigger unit propagation under a suitable assignment;
- If a resolution path connects \(u\) and \(x\), then assigning \(u\) may affect the choices for \(x\);
- However, certain resolution paths are blocked: they contain tautologies on variables that are “already assigned at the time” \(u\) is assigned, such as the independent existential variables \(I_\exists(\Psi)\).
Example ($\mathcal{D}^{\text{rrs}}$ vs. $\mathcal{D}^{\text{tf}}$)

\[
\forall u \exists x (\emptyset) \exists z (\{u\}) \cdot (x \lor u \lor z) \land (\neg x \lor \neg u \lor \neg z)
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Example ($\mathcal{D}^{\text{rrs}}$ vs. $\mathcal{D}^{\text{tf}}$)

\[ \forall u \exists x(\emptyset) \exists z(\{u\}) \cdot (x \lor u \lor z) \land (\neg x \lor \neg u \lor \neg z) \]

- The two clauses $(x \lor u \lor z)$ and $(\neg x \lor \neg u \lor \neg z)$ constitute a resolution path that connects $u$ and $z$. Indeed, if $x$ is set to false, the first clause simplifies to the implication $\neg u \implies z$, and if $x$ is set to true, the second clause simplifies to $u \implies \neg z$. The value of $u$ may potentially force either value of $z$.

Accordingly, $\mathcal{D}^{\text{rrs}}$ identifies $z$ as truly dependent on $u$. But $x$ has to be set “before” $z$, because it does not depend on anything. Hence one of the implications is always killed. In other words, the union of the clauses, restricted to independent existential variables, is a tautology. $\mathcal{D}^{\text{tf}}$ detects the tautology and concludes that $z$ is independent of $u$. Indeed $x \mapsto 0$ and $z \mapsto 1$ is a model that exhibits this.
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Example ($D^{rrs}$ vs. $D^{tf}$)

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- Accordingly, $D^{rrs}$ identifies $z$ as truly dependent on $u$.
- But $x$ has to be set “before” $z$, because it does not depend on anything. Hence one of the implications is always killed. In other words, the union of the clauses, restricted to independent existential variables, is a tautology.
- $D^{tf}$ detects the tautology and concludes that $z$ is independent of $u$. Indeed $x \mapsto 0$ and $z \mapsto 1$ is a model that exhibits this.
Properties of $\mathcal{D}^{tf}$

Proposition

$\mathcal{D}^{tf}$ is a monotone dependency scheme, i.e. $\Psi \leq \Psi' \implies \mathcal{D}^{tf}(\Psi) \leq \mathcal{D}^{tf}(\Psi')$. 
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$\mathcal{D}^{tf}$ is fully exhibited.
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**Proof.**

By reduction to full exhibition of $\mathcal{D}^{\text{rrs}}$ established for DQBF by Wimmer et al. [WSWB16]. If $\Psi$ is true, pick a satisfying assignment $\alpha$ to $I_\exists(\Psi)$, and restrict with it. Because $\Psi[\alpha]$ has no independent existential variables, $\mathcal{D}^{\text{tf}}$ reduces to $\mathcal{D}^{\text{rrs}}$ and the theorem follows by full exhibition of $\mathcal{D}^{\text{rrs}}$. 

\[\square\]
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**Corollary**

$\mathcal{D}^{\text{tf}}$ can be plugged in into any proof system, in particular $\forall\text{Exp+Res}$.
Separations
Genuine DQBF Separations

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Definition

Let $P$ and $Q$ be DQBF proof systems. We write $Q \not\leq^*_p P$ when there exists a DQBF family $\{\Psi_n\}_{n \in \mathbb{N}}$ such that:

(a) $\{\Psi_n\}_{n \in \mathbb{N}}$ has polynomial-size $Q$ refutations;

(b) $\{\Psi_n\}_{n \in \mathbb{N}}$ requires superpolynomial-size $P$ refutations;

(c) every QBF family $\{\Phi_n\}_{n \in \mathbb{N}}$ with $\Phi_n \leq \Psi_n$ has polynomial-size $P$ refutations.

We write $P \prec^*_p Q$ when both $P \leq^*_p Q$ and $Q \not\leq^*_p P$ hold.
Main Theorem

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\[ \forall \text{Exp+Res} \prec_p \forall \text{Exp+Res}(D^{rrs}) \prec_p \forall \text{Exp+Res}(D^{tf}). \]
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\[ \forall \text{Exp} + \text{Res} \preceq^* \forall \text{Exp} + \text{Res}(D_{rs}) \preceq^* \forall \text{Exp} + \text{Res}(D_{tf}). \]

Definition (EQ\(^0_n\) (adapted from [BBH19]))

EQ\(^0_n\) := \Pi_{EQ}^n \cdot \psi_{EQ}^n, where

\[ \Pi_{EQ}^n := \forall u_1 \cdots \forall u_n \exists x_1(\emptyset) \cdots \exists x_n(\emptyset) \exists z_1(u_1) \cdots \exists z_n(u_n), \]

\[ \psi_{EQ}^n := (\overline{z_1} \lor \cdots \lor \overline{z_n}) \land \bigwedge_{i=1}^{n} \left( (\overline{x_i} \lor \overline{u_i} \lor z_i) \land (x_i \lor u_i \lor z_i) \right). \]

Human readably:

- there are \(x_i\) and \(z_i\) depending on \(u_i\) such that for all values of the \(u_i\)
- if \(u_i = x_i\), then \(z_i\), but not all \(z_i\).
Theorem

\{EQ_0^n\}_{n \in \mathbb{N}} \text{ requires exponential-size } \forall \text{Exp+Res refutations.}
First Separation

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**Proposition ([BB19])**

*For all* \( n \), the dependency sets of \( D^{rrs}(\text{EQ}^0_n) \) are empty.*
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**Proposition** ([BB19])

*For all* \( n \), *the dependency sets of* \( D^{\text{rrs}}(\text{EQ}_n^0) \) *are empty.*

**Theorem** ([BB19])

\[ \{ \text{EQ}_n^0 \}_{n \in \mathbb{N}} \text{ has linear-size } \forall \text{Exp+Res}(D^{\text{rrs}}) \text{ refutations.} \]
**Second Separation**

**Definition (EQ\(^1\)_n (adapted from [BB17]))**

For each natural number \( n \),

\[
\text{EQ}^1_n := \Pi_n^{\text{EQ}} \forall \emptyset \exists \{u_1, \ldots, u_n\} \cdot \left( \psi_n^{\text{EQ}} \otimes (r \lor s) \right) \land \left( \psi_n^{\text{EQ}} \otimes (\overline{r} \lor \overline{s}) \right) \land (r \lor \overline{s}) \land (\overline{r} \lor s).
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\begin{align*}
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**Proposition**

*For each \(n\), \(D_{rrs}(EQ_1^n) = EQ_1^n\) and the dependency sets of \(D_{tf}(EQ_1^n)\) are all empty.*
Second Separation

**Definition (EQ\_1^n (adapted from [BB17]))**

For each natural number $n$,

$$EQ_n^1 := \Pi_n^{EQ} \exists r(\emptyset) \exists s\{u_1, \ldots, u_n\} \cdot \\
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**Proposition**

*For each $n$, $D^{rrs}(EQ_n^1) = EQ_n^1$ and the dependency sets of $D^{tf}(EQ_n^1)$ are all empty.*

**Theorem**

*Hence, $\{EQ_n^1\}_{n \in \mathbb{N}}$ requires exponential-size $\forall \text{Exp} + \text{Res}(D^{rrs})$ refutations, but has linear-size $\forall \text{Exp} + \text{Res}(D^{tf})$ refutations.*
Summary

- We proposed a clean framework for DQBF dependency schemes and their proof complexity centered around the notion of full exhibition;
- We defined a novel dependency scheme $D_{tf}$ based on the intuition about how resolution paths do and do not transfer information between variables;
- We showed that $D_{tf}$ is fully exhibited;
- We showed that the use of $D_{tf}$ in both $\forall\text{Exp}+\text{Res}$ and $\text{Q-Res}$ results in exponentially shorter proofs compared to $D_{rrs}$.

A short remark on the Equality formulas: the QBF template is hard for both proof system, but our DQBF version is only hard for $\forall\text{Exp}+\text{Res}$ and becomes easy in $\text{Q-Res}$. Why?
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A short remark on the Equality formulas: the QBF template is hard for both proof system, but our DQBF version is only hard for $\forall\text{Exp}+\text{Res}$ and becomes easy in $\text{Q-Res}$. Why?
Summary

- We proposed a clean framework for DQBF dependency schemes and their proof complexity centered around the notion of full exhibition;
- We defined a novel dependency scheme $D^\text{tf}$ based on the intuition about how resolution paths do and do not transfer information between variables;
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