THE MONSTER AND BLACK-BOX GROUPS

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Abstract. We discuss ways in which the black-box model for computation is or is not applicable to the Monster sporadic simple group. Conversely, we consider whether methods of computation in the Monster can be generalised to other situations, for example to groups of ‘cross-characteristic’ type.

1. Introduction

The concept of black-box group was introduced by Babai and Szemerédi [1] as an abstraction and generalisation of the ideas of ‘matrix group’ and ‘permutation group’, for the purposes of algorithm design and complexity analysis. In a black-box group, the elements are represented (not necessarily uniquely) by bit-strings of some fixed length $n$, and there are ‘black boxes’ that perform the three operations of group multiplication, inversion, and testing whether a given bit-string represents the identity element, each in a specified maximum amount of time.

The Monster is the largest of the 26 sporadic simple groups, and the only one for which no matrix or permutation representation is small enough for naive computation to be effective (yet). For practical computations we generally use instead the computer construction described in [4], in which the Monster is generated by a subgroup $G = C(z) = \langle a, b \rangle \cong 2^{1+24} : Co_1$, together with a ‘triality element’ $T$ of order 3, which centralizes a subgroup $2^{11} : M_{24}$ of $\langle a, b \rangle$.

Thus an element of the Monster is represented as a word in $T$ and elements of $G$. Multiplication is concatenation of words, combined with reduction by $T^3 = 1$ and multiplying together any contiguous elements of $G$. Inversion can be implemented as reversal of a word, followed by replacing each element of $G$ by its inverse, and each occurrence of $T$ by $T^{-1}$. And there is a quick and straightforward test for whether a given word represents the identity element.

It is a natural question to ask, to what extent this form of computation is covered by the black-box paradigm, or whether a different model is required. At first glance, the Monster appears to conform to the black-box paradigm, except that

- we have not been given an effective bound on the number of bits required to represent any element of the group;
- the time taken for both the inverse operation and the identity test is proportional to the length of the word, so is potentially unbounded.

So, various questions arise as to the extent to which the insights gained from the black-box approach are applicable to the very real problem of actually computing anything interesting in the Monster. Issues of complexity, of course, do not arise, but issues of efficiency are of paramount importance. Moreover, since the Monster is so large, efficiency questions do almost look like complexity questions.

In this paper we attempt to analyse computational questions about the Monster from this point of view. Conversely, we ask to what extent insights gained from computing in the Monster can be applied more generally, to problems which are usually considered in the black-box model.

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2. Can we bound the length of the words?

Unless we have effective methods of shortening words, the Monster fails the black-box principle that there should be an effective upper bound on the number of bits needed to represent an element of the group. For these purposes, a word is taken to be an alternating sequence of elements \( g_i \in G \), and \( T_i = T^{±1} \), and we define the length of a word to be the total number of \( g_i \) plus the total number of \( T_i \). Note that this is a different definition from \([5]\), where only the number of \( T_i \) is counted. Multiplication of elements is, in the worst case, simply concatenation of words, and therefore normal paradigms of computation lead to exponential growth in the length of the words, and consequently exponential growth in the number of bits used in the representation of elements.

Some extremely useful methods for shortening certain words in certain circumstances are described in \([5, 6, 11]\) and other papers cited below. Combining these in ways suggested by Ryba’s constructive membership testing algorithm \([8]\) leads to an effective method for shortening any word of length greater than some reasonable bound (which can be taken to be as small as 17 if we wish). However, this is a randomised method, which has a small but non-zero probability of failure. Moreover, from a theoretical point of view, it is not clear whether repeated attempts are sufficiently independent to ensure eventual success in practice, or whether there actually exist elements of the Monster for which the method is guaranteed to fail. (This latter possibility, however, seems very unlikely, and even if it happens, small changes to the method should eliminate it, at the cost of an increase in the bound of 17.)

The first basic building block was introduced in \([5]\), and is a method of taking a word which represents an element of \( C(z) \), and writing it in canonical form as a word of length 1. We may then make a recursive application of Ryba’s algorithm (or some other constructive membership test) in \( 2^{1+24-C_01} \) to find a word in \( a \) and \( b \) which is equal to the given Monster element. In fact, Ryba’s algorithm does not work very well in groups with large normal 2-subgroups, so we make some modifications, as described below.

The second basic building block of the method was introduced in \([6]\), where it was called ‘changing post’. What this means is, given any \( 2B \)-element \( t \) which centralizes \( z \), finding a word (of length at most 4) which conjugates \( t \) into \( z \). This in itself does not shorten any words: indeed, it lengthens them. The original word of length 1 for \( t \) is turned into a word of length (typically) 9.

The technical details of these two processes are described in the next section. Now we explain how they are combined into a word-shortening algorithm.

Given any word \( W \) representing an element of the Monster,

1. take (random) elements \( g \in G \) until the word \( W′ := Wg \) is an element which powers up to an involution \( t \) in class \( 2B \);
2. conjugate \( t \) by (random) \( c_0 \in G \) until \( t^{c_0}z \) has even order and powers up to a \( 2B \)-element \( y \);
3. find the word of length 1 for \( y \);
4. find a word \( c_1T^{c_{21}}T_{c2}^{c_2} \) of length 4 which conjugates \( y \) to \( z \);
5. find the word of length 1 for \( t^c \), where \( c = c_0c_1T^{-c_{21}}T_{c2}^{c_2} \);
6. find a word \( d = d_1T^{b_{21}}d_2T_{b2}^{b_2} \) of length 4 which conjugates \( t^c \) to \( z \);
7. find the word of length 1 for \( W′′ := W^{cd} \).

Finally we have

\[
W = W′g^{-1} = cdW′′d^{-1}c^{-1}g^{-1} = c_0c_1T^{c_{21}}c_2T_{c2}^{c_2}d_1T^{b_{21}}d_2T_{b2}^{b_2}W′′T^{−b_{21}}d_1^{−1}T^{−b_{21}}d_1^{−1}T^{−γ_{21}}c_2^{−1}T^{−γ_{21}}c_1^{−1}c_0^{−1}g^{-1}
\]
Since we can compute products within $G$, the total length of the word for $W$ is 17. In certain (unlikely) circumstances, the words for $c$ and/or $d$ may be shorter, in which case the word for $W$ is correspondingly shorter than 17.

3. The technical details

This section can be skipped by those readers who only want an overview, or an understanding of the general principles.

3.1. The underlying module. The generators, $T$ and elements of $G$, for the Monster, are stored in a format whereby their actions on the underlying module, of dimension 196883 over $F_3$, can be computed. This is the foundation for both the identity test and the order oracle. Two vectors have been pre-computed, whose joint stabilizer is proved to be trivial. Hence a word represents the identity element if and only if it fixes these two vectors. (If one is prepared to make do with a Monte Carlo algorithm, one can halve the cost by taking one random vector instead.)

3.2. Changing post. There are just five classes of involutions in $C(z)$ which lie in Monster class 2B. For a representative $x$ of each class we pre-compute a word which conjugates $x_i$ to $z$, as follows.

1. In the case $x_1 = z$, there is nothing to do.
2. In the case $x_2 \in 2^{1+24}$, we may choose $x_2$ such that $x_T^2 = z$.
3. Otherwise, $x_i$ maps to an involution in the quotient $Co_1$, and we may choose $x_i$ such that $x_T^i \in 2^{1+24}$. In each of the three cases we search for an element $y_i$ of $G$ which conjugates $x_T^i$ to the canonical representative $x_2$ of this class (see below for details).

If now $x$ is any 2B-element in $C(z)$, conjugate to $x_i$, say, we search for an element which does this conjugation. For $i = 3, 4, 5$, we perform the conjugation in $Co_1$ first, and then conjugate by suitable elements of $2^{1+24}$ as necessary afterwards. Since any involution centralizes at least $2^{12}$ in $2^{1+24}$, even an exhaustive search is not impossible. For $i = 2$, we adopt a randomised approach, and fingerprint around 1000 conjugates of each of $x$ and $x_2$. Sorting and merging the two lists of fingerprints we easily find a match, and read off an element of $G$ which conjugates $x$ to $x_2$.

In all cases we now have a word $c_1T y_i T$, or $c_1 T$, or the empty word, which conjugates $x$ to $z$. In fact for technical reasons it is easier to allow also the possibility of using $T^{-1}$ rather than $T$.

3.3. Computing words of length 1 for elements which centralize $z$. We first work in the quotient $Co_1$ of $C(z)$, so that Ryba’s algorithm (or some other constructive membership test) can be applied directly. It is straightforward, if somewhat technical, to obtain elements of $2 Co_1$ as $24 \times 24$ matrices over $F_3$, corresponding in pairs (modulo sign) to elements of the quotient of $2^{1+24} Co_1$ by the normal 2-subgroup.

This process can be carried out for any element of the Monster which commutes with the central involution $z$ of $2^{1+24} Co_1$, even if it is only given as a word in the generators of the Monster. We just have to compute the images of a carefully selected set of 24 coordinate vectors, and extract another (not necessarily the same) carefully selected set of 24 coordinates from the answer.

Now we need to lift to $2^{1+24} Co_1$. Suppose that $w$ is the (long) word, and $x$ is the element given as a word in $a$ and $b$. Then we know that $wx^{-1}$ is an element of the group $2^{1+24}$. It so happens that there is an easy constructive membership test in $2^{1+24}$. Thus we have words in $a$ and $b$ for both $x$ and $wx^{-1}$, and we combine them to get a word in $a$ and $b$ for $w$. 

4. **Is the Monster a black-box group?**

Of course, this is a meaningless question. Or at best, it is a philosophical question, not a mathematical one. (A black-box group, after all, is one which in principle one knows nothing about.) One can perhaps best interpret the question at a purely phenomenological level, and ask whether black-box algorithms (a) work at all, or (b) are effective, in the given computational environment for the Monster.

With the word-shortening method described in the previous sections, we have some bounds on the number of bits required to represent any element of the Monster, and the time required for each operation. In the current implementation, $n$ is around $6 \times 10^8$. (This compares with about $7 \times 10^{10}$ for the underlying matrices, or $4 \times 10^{10}$ for the minimal representation.) The time required for the identity test is around 5 seconds, and that for inversion is about 1 minute.

The multiplication algorithm envisaged here has not been implemented, but, assuming that typical elements will require close to the maximum word length of 17, all the work is in shortening the resulting word of length around 33. Currently every shortened word we require is made by hand, and takes a day or two to make. If the method were automated and efficiently implemented I guess it would take an hour or two. (For comparison, the time taken to multiply two $196882 \times 196882$ matrices over $\mathbb{F}_3$ on the same system would be more than a week.)

So, on the face of it, this would make the Monster into a black-box group according to the official definition. (Actually, this is not quite true, because the black box for element multiplication is now only a Las Vegas algorithm: it may report failure instead.) Black-box algorithms can be used, although the very high cost of multiplication is a barrier. Indeed, the computations that are required for practical problems like determining the maximal subgroups, require hundreds of thousands of multiplications, and therefore would take years, as opposed to the small number of days taken by the calculations we have actually done.

For these reasons, then, we conclude that the black-box model is still not a very useful model for the Monster.

5. **Compare and contrast**

It is clear by now that, at least in some respects, we are in a better position than in the black-box situation. This must be so, for black-box algorithms alone could not achieve the results that are described in the papers we cite below. To compensate for the lack of a generic multiplication algorithm, various other techniques are available.

Most importantly, a fantastically efficient order oracle is available. Since every element has order at most 119, computing the order of a word can be done in at most 119 times the time taken to test whether it is the identity element, so around 10 minutes for a word of length 17.

Philosophically, the black-box model is a socialist model: all elements of the group are treated equally. The basic operation is multiplication of (arbitrary) group elements, and complexity is measured in terms of the (worst case) time taken for this basic operation. An order oracle is often assumed, and is generally taken to be at least as expensive as multiplication of elements.

But the model of computation in the Monster is an elitist one: the elements of the subgroup $G$ are highly favoured, because they can be multiplied together in about 5 seconds with no increase in word length. Multiplication of arbitrary group elements, on the other hand, is too expensive for indiscriminate use. Practical computations tend to be dominated either by computations in $G$, or by the order oracle, depending on the context.

We should also consider to what extent matrix invariants are available in the Monster. Each word which represents an element of the Monster can in principle be converted into a $196882 \times 196882$ matrix over $\mathbb{F}_3$. The time taken for this operation...
is of the order of a few hours per letter of the word. Thus a trace can be computed in around a couple of days. This is unlikely ever to be cost-effective, however. Other matrix operations are in general not available in an effective manner. We know of no efficient way of calculating the characteristic polynomial, or the Jordan block structure, for example.

6. The Monster as an infinite group

The ideas in this section were expressed to me by Sasha Borovik. It has often been remarked that the Monster is so large that it is ‘morally’ an infinite group, or even ‘bigger’ than many infinite groups. As he put it to me, the Monster is *de jure* finite but *de facto* infinite. The same may be said of black-box groups, if in a somewhat different way.

The difference, as Borovik explained it, is that black-box groups have invariant probability measures, whereas the Monster does not. Therefore black-box groups behave like compact Lie groups, while the Monster behaves more like an HNN-extension or free amalgamated product.

Indeed, the form of the Monster construction we are using even looks like an HNN-extension. It is not actually *free*, of course, in the sense that an HNN-extension is free, but for the purposes of many computations, it might as well be. Computing in the Monster is very like computing in an HNN-extension, in that very little can be done except in conjugates of the base subgroup.

7. The Pacific Island model

To use a geographical analogy, the group $G$ is a small island where productive work can be done, in a Monstrously vast ocean of other group elements most of which are apparently useless. Occasionally one has to fish in this vast ocean for elements outside $G$ that will perform useful functions. (Perhaps $T$ is the boat that enables us to travel on these fishing trips?—$T$ stands for ‘travel’ as well as ‘triality’.) And when even that fails, one has to navigate to distant islands and trade for the elements one requires ($T$ also stands for ‘trade’). Typically, this may involve scouring an entire unfamiliar island in search of the elusive prize. (Actually, the Pacific Ocean is too small, and the islands too large, for this analogy. The number of elements in the Monster is about 100 million times the number of water molecules in the Pacific Ocean, whereas the number of elements in $G$, if converted into silica molecules, will give you not an island, but a bucket of sand.)

What are the essential ingredients of this model of computation? We seem to need the following:

- black boxes to perform group operations in $G = C(z)$;
- an order oracle for (short) words;
- an oracle to solve constructively the conjugacy problem in the conjugacy class of $z$;
- an oracle for constructive membership testing in $G$.

These are listed in approximate order of cost in the Monster, from cheapest to most expensive.

The first and last are not really computations in the whole group, but only in the subgroup $G$, so should perhaps be taken as read. The third we have already shown how to do, given the other three. This reduces the essential requirements to one thing only: namely, an order oracle, which is much cheaper than multiplication of elements.

To summarize, in the Pacific Island model of group computation, we are given

- an involution centralizer $G = C(z)$, in which all problems can be solved,
- one extra generator $T$ (which probably should satisfy $zz^T = 1$), and
- an order oracle for words of bounded length,
8. Pacific Island, or Pacific Ocean?

As currently practiced, computation in the Monster is largely restricted to working on one particular island, that is the subgroup $G$, with the occasional fishing or trading trip to collect additional elements to perform particular functions. This is not the only possible way to work, however. Often we want to work on other islands, i.e. compute in different subgroups.

Occasionally this has been done in the Monster. A collection of words is found for elements generating a (usually maximal, or close to maximal) subgroup. The representation of this subgroup on the underlying 196882-dimensional module is investigated via a specially designed form of ‘condensation’, in order to construct explicitly the action on a suitably small invariant subspace. Then this subgroup can be explored in the usual way as a matrix group.

9. Is the Pacific Island model useful in other contexts?

It is of course well-known that the black-box model, while extremely useful, does not capture every important aspect of computation in finite groups. It is commonplace to use other information if it is available, for example traces of matrices, numbers of fixed points of permutations, order oracles, and so on. But most of this extra information is still used in a socialist paradigm: all group elements are treated equally.

In practice, however, many modern algorithms, for matrix groups in particular, recurse to a subgroup, often an involution centralizer, as quickly as possible. Efficient implementations will generally convert elements of the subgroup into a form where computations proceed much more quickly. In this scenario, the black-box model seems less applicable than the Pacific Island model alluded to above.

The black-box approach is most useful in the beginning stages of an investigation, when we have essentially no knowledge about the group under discussion. But in the later stages, we generally have a lot of knowledge, and often even know the isomorphism type of the group (up to a certain probability of error). This again favours the Pacific Island model.

However, there remains the important question as to which of the additional operations that are practical in the Monster remain practical and efficient in these other contexts. The crucial issue is whether or not there is a fast order oracle.

On the face of it, it seems hard to imagine many contexts in which an order oracle is much faster than a single multiplication! What makes it work in the Monster is the fact that the degree of the representation is much bigger than the largest element order. This is a phenomenon associated with sporadic groups, or cross-characteristic representations.

It is probably not a coincidence, that these are situations in which the black-box model does not have a great deal to tell us. To make a sweeping generalisation, in this situation the input data is so large compared to the order of the group, that almost all algorithms are more-or-less linear. But that does not necessarily mean that all problems are soluble in practice: just as in the Monster. So maybe the Pacific Island model can tell us something useful about how to perform such calculations?

10. Pacific island algorithms

The word-shortening algorithm described in Section 2 is in effect a Monte Carlo ‘constructive membership testing’ algorithm. It takes an element of the Monster, and writes it as a word in the standard generators. If the input is not in fact an
element of the Monster, then the algorithm probably fails, but may just give the wrong answer.

Another problem one might wish to solve is the constructive recognition problem: given another group which is claimed to be the Monster, can we produce an isomorphism with the standard copy? This problem comes in various flavours, depending on how much information is compatible between the two copies.

Suppose first that we are given copies of $G$ in both groups. Then, by assumption, we can find isomorphisms between these copies of $G$. If $T$ is also given in both groups, then we want to adjust these isomorphisms so that they map $T$ to the right place. In each copy of $G$ we can find the centralizer of the relevant copy of $T$, and then inside $G$ we can conjugate one of these centralizers to the other. It remains only to identify which is which of the 8 elements of order 3 in $\langle z, T \rangle \cong A_4$, which can be achieved by computing orders of a few random elements.

A more likely scenario, which has actually occurred in practice, is that $G$ is not immediately available in one of the copies of the Monster. For example, we might have the ‘mod 2’ construction, in which, instead of $G$, a subgroup $H \cong 3^{1+12} \cdot 2 \cdot S\text{suz}:2$ is the island in which one can work. In [12] we have given a method of obtaining generators for $H$ from those of $G$, and then constructive recognition within $H$ allows us to find the standard generators. Moreover, the extra generator for the $H$-type Monster can be found within $G$, although this has not actually been done yet.

The isomorphism in the other direction may be obtained by a similar process. First, there is an involution whose centralizer in $H$ is $6 \cdot S\text{suz}:2$. The full centralizer of this involution in the Monster is $2^{1+24} \cdot C_{01}$, and involution centralizers can be found in the Monster either by Bray’s algorithm, or by some more subtle technique. On the other hand, $T$ does not lie in $H$, so cannot be found quite so easily.

In a more general situation, one might imagine that $G = C(z)$ and $H = C(y)$ are both involution centralizers, in some large ‘cross-characteristic’ group. In this case we would probably want to choose the isomorphisms so that $z$ and $y$ commute with each other. If it is possible also to choose $T$ commuting with $y$, then the above method will construct a suitable isomorphism.

On the other hand, these are not really the type of calculations which are most often going to arise in the Pacific Island situation. More likely, we already know what the group is, and we are interested in computing particular subgroups, such as centralizers and normalizers, Sylow subgroups, and the like. The references below give many examples of calculations of this type in the Monster. In this context, it would be normal to assume that we are working in a ‘standard copy’ of the group. This allows us the luxury of pre-computing a great deal of the structure. But it also rules out the cross-characteristic groups, and leaves us only with sporadic groups.

11. Conclusion

By comparing and contrasting (a) the black-box model of computation in finite groups, and (b) the Pacific Island model of practical computation in the Monster, we have seen that, in fact, there is surprisingly little overlap between the two, and surprisingly little application of either method in the other context.

There have been one or two major influences from the sporadic context to the black-box context: especially, the emphasis on computing involution centralizers; and arising from that, Ryba’s constructive membership algorithm. Moreover, black-box methods often involve a reduction to simple groups, in which case the sporadic groups have to be dealt with somehow. In the other direction, subgroups of sporadic groups can be efficiently investigated using general-purpose black-box algorithms.

But beyond these influences, the two inhabit rather different worlds. They complement each other, and both are necessary for a fully functional toolkit for computation in finite groups.
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