The Least Singular Value of the General Deformed Ginibre Ensemble

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Abstract
We study the least singular value of the $n \times n$ matrix $H - z$ with $H = A_0 + H_0$, where $H_0$ is drawn from the complex Ginibre ensemble of matrices with iid Gaussian entries, and $A_0$ is some general $n \times n$ matrix with complex entries (it can be random and in this case it is independent of $H_0$). Assuming some rather general assumptions on $A_0$, we prove an optimal tail estimate on the least singular value in the regime where $z$ is around the spectral edge of $H$ thus generalize the recent result of Cipolloni et al. (Probab Math Phys 1(1):101–146, 2020) to the case $A_0 \neq 0$. The result improves the classical bound by Sankar et al. (SIAM J Matrix Anal Appl 28:446–476, 2006).

1 Introduction

Consider random $n \times n$ matrices

$$H = A + H_0,$$

where $A$ is some general $n \times n$ matrix with complex entries, and $H_0$ is drawn from the complex Ginibre ensemble, i.e. $H_0$ has i.i.d. complex Gaussian entries $\{h_{ij}^{(0)}\}_{i,j=1}^n$ such that

$$\mathbb{E}[h_{ij}^{(0)}] = 0, \quad \mathbb{E}(|h_{ij}^{(0)}|^2] = 1/n, \quad \mathbb{E}[(h_{ij}^{(0)})^2] = 0.$$

Deformation $A$ can be deterministic or random (but in this case it is independent of $H_0$).

Such matrices are important in communication theory, where $A$ is considered as a signal, and $H_0$ as a noise matrix. In particular, one is interested in effective numerical solvability of a large system of linear equations $Hx = b$ which is determined by the behaviour of the smallest singular value $\sigma_1(H)$ of $H$. 

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The classical bound of Sankar, Spielman and Teng [23] states that the smallest singular value \( \sigma_1(H) \) is of order not smaller than \( n^{-1} \) (equivalently, the smallest eigenvalue \( \lambda_1(HH^*) \) of \( HH^* \) is of order not smaller that \( n^{-2} \)), i.e.

\[
\mathbb{P}\left( \lambda_1(HH^*) = (\sigma_1(H))^2 \leq x/n^2 \right) \lesssim x, \quad x > 0.
\]  

(1.2)

up to logarithmic corrections, uniformly in \( A \). If \( H_0 \) is a real Ginibre matrix, then the bound is \( \sqrt{x} \) instead of \( x \).

The bound is proved to be optimal for the case of pure complex Ginibre ensemble (i.e. \( A = 0 \)), see [17]. Similar results for matrices with iid entries (i.e. \( A = 0 \), but \( H_0 \) is not necessary Gaussian) are also known (see, e.g., [22, 31] and references therein). The case \( H = H_0 - z \) with a real Ginibre \( H_0 \) was studied in [11] and [12].

The matrix \( HH^* \) can be considered as the so-called deformed Laguerre ensemble, and its limiting eigenvalue distribution is well-known, see [14]. Moreover, according to [2], almost surely no eigenvalues lie outside any finite support neighbourhood of the limiting measure. Therefore, we can have three possible situations: 0 is away of the support of limiting spectral measure, and then \( \lambda_1(HH^*) \) has a positive constant lower bound; 0 lies in the bulk of the spectrum, and then we get (1.2); the intermediate regime when 0 is near the edge of the spectrum.

In the bulk regime the lower bounds on \( \lambda_1(HH^*) \) with quite general \( A \) and even with the non-Gaussian \( H_0 \) (with iid elements) have been obtained in [32, 33] (but not uniformly in \( A \)); see also [13, 36] beyond the i.i.d. case.

The edge regime is much less studied. However, as it was shown in [8] for the case of the constant diagonal shift of the Ginibre ensemble, i.e. \( A = -zI \), the bound (1.2) can be improved in the edge regime \( |z| \sim 1 \). The aim of the current paper is to extend the result of [8] to the case \( A = A_0 - zI \) with a rather general complex deformation \( A_0 \).

Another important source of motivation is that an effective lower tail bound on the least singular value of \( H - z \) is an essential ingredient for the study of eigenvalues distribution of large non-Hermitian matrices. In particular, the results of Cipolloni, Erdős, Schröder [8] was used in their subsequent work [9] to remove the four moment matching condition in the classical edge universality result for non-Hermitian random matrices with iid entries by Tao, Vu [34]. Better understanding of higher order correlation functions of the shifted Ginibre ensemble in the bulk is expected to help to do the same thing for the universality in the bulk. In addition, [8] is also an important ingredient for the recent CLT result [10].

To obtain the lower bound on the least singular value of \( H - z = A_0 + H_0 - z \), we are going to study the resolvent of

\[
Y(z) = (H - z)(H - z)^*.
\]

(1.3)

More precisely, we are going to obtain the integral representation and the precise asymptotic behaviour of

\[
T(z, \varepsilon) = \mathbb{E}\left\{ n^{-1} \text{Tr} \left( Y(z) + \varepsilon^2 \right)^{-1} \right\},
\]

(1.4)

which is related to the average density of states (or one-point correlation function) of \( Y(z) \) at 0. For the small \( \varepsilon \) the main contribution to (1.4) comes from the lowest eigenvalue \( \lambda_1(Y(z)) \), hence an upper estimate on (1.4) implies a lower tail bound on \( \lambda_1(Y(z)) \).

The ensemble \( Y(z) \) was extensively studied in [4] by using so-called Brézin-Hikami formulas (see [7]). Theorem 7.1 of [4] allows to represent even higher order correlation function of \( Y(z) \) as the determinants of a certain kernel which can be computed as a double integral involving the Bessel kernel. Paper [4] did not analyse the resulting one point function.
(and, generally, used this integral representation to study the case when \( H = A_0 + H_0 \) is Gaussian divisible ensemble only), but one can use this integral representation to analyze \( T(z, \varepsilon) \).

In this paper, however, we use a different approach because of two main reasons. First, there is no analogue of Brézin-Hikami formulas for the real symmetric case, and, second, the local study of eigenvalues distribution of \( H \) requires the analysis of the joint distribution the smallest singular values of \( H - z_1, H - z_2, \ldots, H - z_k \) for different \( z_i \) which is not covered by the determinantal formulas of [4].

Our approach is based on supersymmetry (SUSY) techniques and is expected to be more robust, in particular, it is available in the case of real symmetry (see [8, 11, 12] for the case \( A_0 = 0 \)). It is based on the fact that SUSY techniques allows to rewrite the main spectral characteristics of random matrices (such as density of states, correlation functions, etc.) as an integral containing both complex and Grassmann (anticommuting) variables. The method is widely used in the physics literature (see, e.g., [18, 21]), but the rigorous analysis of such integral representations usually is quite difficult. However, the method was successfully applied to the rigorous study of some random matrix ensembles, including the most successful applications to the Gaussian random band matrices (see [3, 15, 16, 24–30]), as well as to the study of overlaps of non-Hermitian Ginibre eigenvectors [19].

The method of [8] for the case of \( A_0 = 0 \) also utilizes SUSY. Their technique is based on the superbosonization formula by Littelmann, Sommers and Zirnbauer [20] which significantly reduces the number integration variables: instead of \( 2n \) real and \( 2n \) Grassmann variables of integration one can get the integral over the two complex variables only. However, this formula is not applicable in the case of general \( A_0 \), so we need to use another approach based on the Hubbard-Stratonovich transformation (see (2.9)–(2.10) below) and Fourier transform (see Sect. 2 for the discussion).

Now let us formulate the main results of the paper. Consider the matrix

\[
H(z) = A_0 + H_0 - z,
\]

where \( H_0 \) is a complex Ginibre matrix defined in (1.1), and \( A_0 \) is some general \( n \times n \) matrix with complex entries (which can be random, and in this case is independent of \( H_0 \)) satisfying the following conditions

**Assumptions (A1)-(A4):**

(A1) There are some \( M, \lambda > 0 \) such that

\[
\text{Prob}\{n^{-1} \sum_{i,j=1}^{n} |A_{0,ij}|^2 < M\} \geq 1 - n^{-1-d}.
\]

(A2) For almost all \( z \) normalized counting measure of eigenvalues of the matrix \( Y_0(z) := (A_0 - z)(A_0 - z)^* \) converges, as \( n \to \infty \), to some limiting measure \( \nu_z \);

(A3) Denote \( \sigma_0 = \{z : 0 \in \text{supp } \nu_z\} \), \( \sigma_\varepsilon \) - \( \varepsilon \)-neighbourhood of \( \sigma_0 \). Set

\[
\Omega^{(1)}_{\varepsilon, \kappa} = \{\omega : \inf_{z \in \sigma_\varepsilon} \text{dist}\{\text{supp } \nu_{z,n}; 0\} \geq \kappa(\varepsilon)\},
\]

\[
\Omega^{(2)}_\varepsilon = \{\omega : \sup_{z \notin \sigma_\varepsilon} \left| n^{-1} \text{Tr } Y_0^{-1}(z) - \int \lambda^{-1} d\nu_z(\lambda)\right| \leq C_\varepsilon n^{-1/2-d_0}\},
\]

where \( d_0 > 0 \) is some fixed number.

Then for any \( \varepsilon > 0 \) there is \( \kappa(\varepsilon) > 0 \) and \( d > 0 \) such that

\[
\text{Prob}\{\Omega^{(1)}_{\varepsilon, \kappa(\varepsilon)} \cap \Omega^{(2)}_\varepsilon\} \geq 1 - Cn^{-1-d};
\]
(A4) There is $d_1 > 0$ such that for some sufficiently small $\epsilon_0$, $\epsilon$ if we set

$$\Omega^{(3)} = \left\{ \omega : \inf_{z\in\sigma_0} n^{-1} \text{Tr} \left( Y_0(z) + \epsilon^2 \right)^{-1} > 1 + d_1 \right\},$$

then

$$\text{Prob}\{\Omega^{(3)}\} > 1 - C'_\epsilon n^{-1-d}$$

Notice that in the case when $A_0$ is non-random, assumptions (A1)-(A4) mean that starting from some $n$ the inequalities in (A1)-(A4) (which in the random case we want to have with probability higher than $1 - n^{-1-d}$) are valid.

Below we give a few examples of $A_0$ satisfying the assumptions (A1)-(A4):

1. $A^{(n)}$ is a sequence of matrices whose limiting spectrum consists of a finite number of points $\sigma_0 = \{\xi_1, \ldots, \xi_k\}$ and we have some large deviation type bound for distribution of $\{\xi_j^{(n)}\}$.
2. $A_0 = A_0^\ast$ - any classical hermitian model like Wigner matrices, sample covariance matrices, sparse matrices, etc.
3. $A_0$ is a diagonal matrix $A_0 = \text{diag}\{\xi^{(n)}_j\}$ with $\{\xi^{(n)}_j\}$ having limiting distribution with a compact finitely connected support $\sigma_0$ with a smooth boundary and a non zero limiting density and such that we have some large deviation type bound (like (A3), (A4)).
4. $A_0$ is a Ginibre matrix with i.i.d. entries having all moments. Then, according to [35], its normalized counting eigenvalue distribution converges to a circular low and we have a large deviation type bound (A3), (A4) due to the result of [1].

According to the result of [35], under assumption A1-A2 there exists a non-random probability measure $\mu$ on the complex plain which is a limit of the normalized counting measure of eigenvalues of $H$. In addition, for almost all $z \in \mathbb{C}$ there exists a non-random probability measure $\eta_z$ on $\mathbb{R}$ which is a limit of the normalized counting measure of eigenvalues of $Y(z)$ defined in (1.3) (see [14]). The support $D$ of the limiting measure $\mu$ is not so easy to describe, however, according to [6], under an additional assumption

$$(A') \quad D = \text{supp } \mu = \{z : 0 \in \text{supp } \eta_z\},$$

it takes the nice form:

$$D = \{z : \limsup_{\epsilon \to 0} n^{-1} \text{Tr} \left( Y_0(z) + \epsilon^2 \right)^{-1} \geq 1\}. \quad (1.5)$$

Notice that the result of [6] also includes $\sigma_0$ of (A3) to $\text{supp } \mu$, but the assumptions (A3) – (A4) guarantee that $\sigma_0 \in D$ with probability 1, so $\text{supp } \mu$ coincides with (1.5). Remark also that the authors of [6] mentioned that they are not aware of any examples where (A') fails to hold.

Assumption (A4) guarantees that there is $\epsilon > 0$ such that the boundary $\partial D \cap \sigma_\epsilon = \emptyset$. Thus, $\partial D$ is a level line of the smooth function

$$\mathcal{F}(z) := \int \lambda^{-1} d\nu_z(\lambda), \quad (1.6)$$

and hence $\partial D$ is a set of piece-wise smooth closed curves enclosing $\sigma_0$ of (A3).

The first theorem gives an asymptotic behavior of $T(z, \epsilon)$ in the edge regime where $z$ lies inside the spectrum of $A_0 + H_0$, but the distance to the boundary of the spectrum is of order $n^{-1/2}$. 
Theorem 1.1 Given \( z \) such that \( \text{dist}(z, \partial D) \leq \delta^2 n^{-1/2} \) and \( \partial D \) is a smooth curve in some neighbourhood of \( z \), and given \( \varepsilon^2 = \tilde{\varepsilon}^2 n^{-3/2} \) with \( |\tilde{\delta}| \leq C_0 \) (where \( C_0 \) is any fixed \( n \)-independent constant) and \( \tilde{\varepsilon} \sim 1 \), under assumptions (A1)–(A4) and (A') we have:

\[
T(z, \varepsilon) = n^{1/2} I(\tilde{\delta}, \tilde{\varepsilon})(1 + O(n^{-1/2})),
\]

where \( I(\tilde{\delta}, \tilde{\varepsilon}) \sim 1 \) is represented by some integral (see (3.26)), depending on \( \tilde{\delta}, \tilde{\varepsilon} \), and the integral is bounded uniformly in \( \tilde{\delta} \leq C_0 \).

The next theorem gives an asymptotic behavior of \( T(z, \varepsilon) \) in the bulk regime where \( z \) is well inside the spectrum of \( A_0 + H_0 \). In this regime the bound (1.2) becomes optimal.

Theorem 1.2 Under assumptions (A1)–(A4) and (A'), if we choose \( z \in D \), \( \text{dist}(z, \partial D) = \delta^2 \sim 1 \) and \( \varepsilon = (\delta n)^{-1} \tilde{\varepsilon} \) with \( \tilde{\varepsilon} \sim 1 \), then

\[
T(z, \varepsilon) = \frac{n\delta}{\tilde{\varepsilon}} u_s \tilde{I}(z, \tilde{\varepsilon}, (u_s/\delta))(1 + O(n^{-1})),
\]

where \( \tilde{I} \sim 1 \) is a function represented by (4.28), and \( u_s \) is a solution of (4.1).

The transition regime between Theorem 1.1 and Theorem 1.2 is given by

Theorem 1.3 Given \( z \in D \) such that \( \text{dist}(z, \partial D) = \delta^2 \) with \( 1 \gg \delta^2 \gg n^{-1/2} \) and \( \partial D \) is a smooth curve in some neighbourhood of \( z \), and given \( \varepsilon = (\delta n)^{-1} \tilde{\varepsilon} \) with \( \tilde{\varepsilon} \sim 1 \), under assumptions (A1)–(A4) and (A') we have:

\[
T(z, \varepsilon) = \frac{n\delta^2}{\tilde{\varepsilon}} (k/c_2)^{1/2} \tilde{I}(z, \tilde{\varepsilon}, (k/c_2)^{1/2})(1 + O((\delta^4 n)^{-1/2}) + O(\delta^2)),
\]

where \( \tilde{I} \sim 1 \) is a function represented by (4.28), and parameters \( k \) and \( c_2 \) are defined in (3.17) and (3.19).

Since the main contribution to (1.4) comes from the lowest eigenvalue \( \lambda_1(Y(z)) \), by a straightforward Markov inequality Theorems 1.1–1.3 give the following corollary

Theorem 1.4 Under assumptions (A1)–(A4) and (A'), we have

\[
P\left( \lambda_1(Y(z)) \leq c(n, z) x \right) \lesssim x
\]

uniformly in \( z \in D \cup \{ z : \text{dist}(z, \partial D) \leq C n^{-1/2} \} \) and in \( x \in [c_1, C_1], c_1, C_1 > 0 \). Here

\[
c(n, z) = \min\left\{ \frac{1}{n^{3/2}}, \frac{1}{n^2 \cdot \text{dist}(z, \partial D)} \right\}.
\]

Remark 1.1 Note that the analogous result of [8] for the case \( A_0 = 0 \) is uniform in \( x \in [0, C] \). Considering \( x < c_1 \) requires a better control of \( \tilde{I}, \tilde{I} \) above, as \( \tilde{\varepsilon} \ll 1 \), which is not performed here.

Notice that conditions (A3) and (A4) allow us to prove Theorems 1.1–1.3 only for \( \omega \in \Omega^{(1)}_{\varepsilon_0/2, \varepsilon_0/2} \cap \Omega^{(2)}_{\varepsilon_0/2} \cap \Omega^{(3)}_{\varepsilon_0/2} \) since on the complement we can use the trivial bound

\[
n^{-1} \text{Tr} \ (Y(z) + \varepsilon^2)^{-1} \leq \varepsilon^{-2}
\]

combined with the inequality

\[
\text{Prob}(\Omega^{(1)}_{\varepsilon_0/2, \varepsilon_0/2} \cap \Omega^{(2)}_{\varepsilon_0/2} \cap \Omega^{(3)}_{\varepsilon_0/2}) \lesssim C n^{-1-d}.
\]
These bounds give us that the contributions of \((\Omega_{e0/2,\epsilon(e0/2)}^{(1)} \cap \Omega_{e0/2}^{(2)} \cap \Omega_{e0/2}^{(3)})^c\) are small in comparison with the main terms of the r.h.s. of (1.7) – (1.9).

The paper is organized as follows. In Sect. 2 we give the brief outline of SUSY techniques and obtain the SUSY integral representation of \(T(\epsilon, z)\) of (1.4). Sections 3 and 4 deal with the proof of Theorems 1.1 – 1.3.

2 Integral Representation of \(T(z, \epsilon)\)

The aim of this section is to derive an integral representation for \(T(z, \epsilon)\) using SUSY. For the reader convenience, we start with a very brief outline of the basic formulas of SUSY techniques. More detailed information about the techniques and its applications to random matrix theory can be found, e.g., in [18] or [21].

2.1 SUSY Techniques: Basic Formulas

Let us consider two sets of formal variables \(\{\psi_j\}^n_{j=1}\) and \(\{\overline{\psi}_j\}^n_{j=1}\), which satisfy the anticommutation conditions

\[
\psi_j \psi_k + \psi_k \psi_j = \overline{\psi}_j \overline{\psi}_k + \overline{\psi}_k \overline{\psi}_j = 0, \quad j, k = 1, \ldots, n.
\]

(2.1)

Note that this definition implies \(\psi_j^2 = \overline{\psi}_j^2 = 0\). These two sets of variables \(\{\psi_j\}^n_{j=1}\) and \(\{\overline{\psi}_j\}^n_{j=1}\) generate the Grassmann algebra \(\mathfrak{A}\). Taking into account that \(\psi_j^2 = 0\), we have that all elements of \(\mathfrak{A}\) are polynomials of \(\{\psi_j\}^n_{j=1}\) and \(\{\overline{\psi}_j\}^n_{j=1}\) of degree at most one in each variable. We can also define functions of the Grassmann variables. Let \(\chi\) be an element of \(\mathfrak{A}\), i.e.

\[
\chi = a + \sum_{j=1}^n (a_j \psi_j + b_j \overline{\psi}_j) + \sum_{j \neq k} (a_{j,k} \psi_j \psi_k + b_{j,k} \psi_j \overline{\psi}_k + c_{j,k} \overline{\psi}_j \overline{\psi}_k) + \cdots
\]

(2.2)

For any sufficiently smooth function \(f\) we define by \(f(\chi)\) the element of \(\mathfrak{A}\) obtained by substituting \(\chi - a\) in the Taylor series of \(f\) at the point \(a\):

\[
f(\chi) = a + f'(a)(\chi - a) + \frac{f''(a)}{2!}(\chi - a)^2 + \cdots
\]

Since \(\chi\) is a polynomial of \(\{\psi_j\}^n_{j=1}\), \(\{\overline{\psi}_j\}^n_{j=1}\) of the form (2.2), according to (2.1) there exists such \(l\) that \((\chi - a)^l = 0\), and hence the series terminates after a finite number of terms and so \(f(\chi) \in \mathfrak{A}\).

Following Berezin [5], we define the operation of integration with respect to the anticommuting variables in a formal way:

\[
\int d \psi_j = \int d \overline{\psi}_j = 0, \quad \int \psi_j d \psi_j = \int \overline{\psi}_j d \overline{\psi}_j = 1,
\]

and then extend the definition to the general element of \(\mathfrak{A}\) by the linearity. A multiple integral is defined to be a repeated integral. Assume also that the “differentials” \(d \psi_j\) and \(d \overline{\psi}_k\) anticommute with each other and with the variables \(\psi_j\) and \(\overline{\psi}_k\). Thus, according to the definition, if
\[ f(\psi_1, \ldots, \psi_k) = p_0 + \sum_{j_1=1}^{k} p_{j_1} \psi_{j_1} + \sum_{j_1 < j_2} p_{j_1,j_2} \psi_{j_1} \psi_{j_2} + \cdots + p_{1,2,\ldots,k} \psi_1 \ldots \psi_k, \]

then

\[ \int f(\psi_1, \ldots, \psi_k) d\psi_k \ldots d\psi_1 = p_{1,2,\ldots,k}. \]

Let \( A \) be an ordinary Hermitian matrix with a positive real part. The following Gaussian integral is well-known

\[ \int \exp \left\{ -\sum_{j,k=1}^{n} A_{jk} z_j \bar{z}_k \right\} \prod_{j=1}^{n} \frac{d\Re z_j d\Im z_j}{\pi} = \frac{1}{\det A}. \] (2.3)

One of the important formulas of the Grassmann variables theory is the analog of this formula for the Grassmann algebra (see \([5]\)):

\[ \int \exp \left\{ -\sum_{j,k=1}^{n} A_{jk} \bar{\psi}_j \psi_k \right\} \prod_{j=1}^{n} d\bar{\psi}_j d\psi_j = \det A, \] (2.4)

where \( A \) now is any \( n \times n \) matrix.

Let

\[ F = \begin{pmatrix} A & \chi \\ \eta & B \end{pmatrix}, \]

where \( A \) and \( B \) are Hermitian complex \( k \times k \) matrices such that \( \Re B > 0 \) and \( \chi, \eta \) are \( k \times k \) matrices of independent anticommuting Grassmann variables, and let

\[ \theta = (\psi_1, \ldots, \psi_k, x_1, \ldots, x_k)^t, \]

where \( \{\psi_j\}_{j=1}^{k} \) are independent Grassmann variables and \( \{x_j\}_{j=1}^{k} \) are complex variables. Combining (2.3) – (2.4) we obtain (see \([5]\))

\[ \int \exp\{-\theta^+ F \theta\} \prod_{j=1}^{k} d\bar{\psi}_j d\psi_j \prod_{j=1}^{k} \frac{\Re x_j \Im x_j}{\pi} = \text{Sdet}^{-1} F, \] (2.5)

where

\[ \text{Sdet} F = \frac{\det (B - \eta A^{-1} \chi)}{\det A}. \] (2.6)

Notice also that if we define

\[ \text{Str} F = \text{Tr} B - \text{Tr} A, \] (2.7)

then

\[ \log \left( \text{Sdet} F \right) = \text{Str} \left( \log F \right). \] (2.8)

We will need also the following Hubbard-Stratonovich transform formulas based on Gaussian integration.

\[ e^{ab} = \pi^{-1} \int e^{a \bar{u} + b u - \bar{u} u} d\bar{u} du, \] (2.9)

\[ e^{-\rho \tau} = \int e^{\rho \chi + \tau \eta + \chi \eta} d\eta d\chi. \] (2.10)
Here \( a, b \) can be complex numbers or sums of the products of even numbers of Grassmann variables (i.e. commuting elements of Grassmann algebra), and \( \rho, \tau \) are sums of the products of odd numbers of Grassmann variables (i.e. anticommuting elements of Grassmann algebra).

## 2.2 Derivation of the Main Formula for \( T(z, \varepsilon) \)

**Proposition 2.1** Given (1.4), we have

\[
T(z, \varepsilon) = -\frac{n^3}{\pi^3 \varepsilon} \int_{-\infty}^{\infty} (u_1 + \varepsilon) du_1 du_2 \int_L dt_1 dt_2 \int_{0}^{\infty} r_1 r_2 dr_1 dr_2
\]

\[
\varphi(u_1, u_2, t_1, t_2) \exp[n F_1(u_1, u_2)] \exp[n F_2(t_1, t_2, r_1, r_2)]
\]

where \( L := \mathbb{R} + i \varepsilon_0 \)

\[
F_1(u_1, u_2) = L_n(u_1^2 + u_2^2) - (u_1 + \varepsilon)^2 - u_2^2
\]

\[
F_2(t_1, t_2, r_1, r_2) = -L_n(-t_1 t_2) - (r_1 r_2)^2 - i r_1 r_2 (t_1 + t_2) - \varepsilon(r_1^2 + r_2^2)
\]

\[
\varphi(u_1, u_2, t_1, t_2) = \left( 1 - \frac{1}{n} \text{Tr} G(-t_1 t_2) + \frac{|u_1|^2}{n} \text{Tr} G(u\bar{u}) G(-t_1 t_2) \right)^2
\]

\[
+ t_1 t_2 |u_1|^2 \left( \frac{1}{n} \text{Tr} G(u\bar{u}) G(-t_1 t_2) \right)^2
\]

\[
- \frac{|u_1|^2 + t_1 t_2}{n^2} \left( \text{Tr} G(u\bar{u}) G^2(-t_1 t_2) - |u_1|^2 \text{Tr} G^2(u\bar{u}) G^2(-t_1 t_2) \right)
\]

with

\[
L_n(x) := n^{-1} \log \det(Y_0(z) + x), \quad Y_0(z) := (A_0 - z)(A_0 - z)^* \]

\[
G(x) := (Y_0(z) + x)^{-1}.
\]

**Proof of Proposition 2.1** The first step is to rewrite \( T(z, \varepsilon) \) in the following way:

\[
T(z, \varepsilon) = \frac{1}{2\pi n} \frac{d}{d \varepsilon} Z(\varepsilon, \varepsilon_1) \bigg|_{\varepsilon_1 = \varepsilon}, \quad Z(\varepsilon, \varepsilon_1) = \mathbb{E} \left\{ \frac{\det(Y(z) + \varepsilon_1^2)}{\det(Y(z) + \varepsilon_2^2)} \right\}
\]

Introduce now Grassmann variables and complex variables

\[
\Psi_l = (\psi_{l1}, \ldots, \psi_{ln})^t, \quad \bar{\Psi}_l = (\bar{\psi}_{l1}, \ldots, \bar{\psi}_{ln})^t, \quad l = 1, 2, \quad \text{Grassmann};
\]

\[
X_l = (x_{l1}, \ldots, x_{ln})^t, \quad \bar{X}_l = (\bar{x}_{l1}, \ldots, \bar{x}_{ln})^t, \quad l = 1, 2, \quad \text{complex}.
\]

We are going to use the standard linearisation formula:

\[
\det(Y(z) + \varepsilon^2) = \det \tilde{Y}(z, \varepsilon), \quad \tilde{Y}(z, \varepsilon) = \begin{pmatrix} -\varepsilon & i(H - z) \\ i(H - z)^* & -\varepsilon \end{pmatrix}.
\]

Since all eigenvalues of \( \tilde{Y}(z, \varepsilon) \) have real part \(-\varepsilon\), we can apply (2.3), (2.4) to write

\[
(\det(Y(z) + \varepsilon^2))^{-1} = \int d\bar{X} dX \exp\{ \tilde{Y}(z, \varepsilon) X, \bar{X} \},
\]

\[
\det(Y(z) + \varepsilon^2) = \int d\bar{\Psi} d\Psi \exp\{ \tilde{Y}(z, \varepsilon_1) \Psi, \bar{\Psi} \},
\]
where we denoted $X = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix}$, $\Psi = \begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix}$, 

$$d\tilde{X}dX = \prod_{j=1}^{n} \frac{d\tilde{x}_1 dx_1 d\tilde{x}_2 dx_2}{\pi^2}, \quad d\tilde{\Psi}d\Psi = \prod_{j=1}^{n} \frac{d\tilde{\psi}_1 d\psi_1 d\tilde{\psi}_2 d\psi_2}{\pi^2},$$

and for $m$-dimension vectors $V_1 = \{v_{1i}\}_{i=1}^{m}$, $V_2 = \{v_{2j}\}_{j=1}^{m}$ (coordinates can be both complex or Grassmann) we denote

$$(V_1, V_2) = \sum_{i=1}^{m} v_{1i} v_{2i}.$$ 

Hence, we obtain

$$Z(\varepsilon, \varepsilon_1) = \int d\tilde{X}dX d\tilde{\Psi}d\Psi Z_{\varepsilon}(X) Z_{\varepsilon_1}(\Psi) E(\Psi, X), \quad (2.15)$$

where

$$Z_{\varepsilon}(X) = \exp \left\{ -\varepsilon ((X_1, \tilde{X}_1) + (X_2, \tilde{X}_2)) + i((A_0 - z) X_1, \tilde{X}_2) + i((A_0 - z)^{\ast} X_2, \tilde{X}_1) \right\},$$

$$Z_{\varepsilon_1}(\Psi) = \exp \left\{ -\varepsilon_1 ((\Psi_1, \tilde{\Psi}_1) + (\Psi_2, \tilde{\Psi}_2)) + i((A_0 - z) \Psi_1, \tilde{\Psi}_2) + i((A_0 - z)^{\ast} \Psi_2, \tilde{\Psi}_1) \right\},$$

$$E(\Psi, X) = \mathbb{E} \left\{ \exp \left\{ i \sum_{i,j} (H_{ij}(x_{ij}\tilde{x}_i + \psi_{ij}\tilde{\psi}_i) + \bar{H}_{ij}(x_{ij}\bar{\tilde{x}}_i + \psi_{ij}\bar{\tilde{\psi}}_i)) \right\} \right\}. \quad (2.16)$$

Taking the expectation with respect to $H_{ij}$ we get

$$E(\Psi, X) = \exp \left\{ -\frac{1}{n} \sum_{i,j} (x_{ij}\tilde{x}_i + \psi_{ij}\tilde{\psi}_i) (x_{ij}\tilde{x}_j + \psi_{ij}\tilde{\psi}_j) \right\}$$

$$= \exp \left\{ -\frac{1}{n} \sum_{i,j} \left( \tilde{x}_i x_i \cdot \tilde{x}_j x_j - \tilde{\psi}_i \psi_i \cdot \tilde{\psi}_j \psi_j + \tilde{x}_i \psi_i \cdot \tilde{\psi}_j x_j \\
+ \psi_j \tilde{x}_j \cdot \tilde{\psi}_i x_i \right) \right\}$$

$$= \exp \left\{ -\frac{1}{n} \left( (X_1, \tilde{X}_1)(X_2, \tilde{X}_2) - (\Psi_1, \tilde{\Psi}_1)(\Psi_2, \tilde{\Psi}_2) + (\tilde{X}_1, \Psi_1)(X_2, \tilde{\Psi}_2) \right) \right\}.$$ 

Notice that, in contrast to the case $A_0 = 0$ considered in [8], for general $A_0$ the functions $Z_{\varepsilon}(X), Z_{\varepsilon_1}(\Psi)$ of (2.16) are not the functions of variables $p_{i,j} = (X_i, \tilde{X}_j), q_{i,j} = (\Psi_i, \tilde{\Psi}_j)$ only, so we cannot apply the superbosonization formula of [20]. Instead, we use the Hubbard-Stratonovich transformation (2.9) to get

$$e^{(\Psi_1, \tilde{\Psi}_1)(\Psi_2, \tilde{\Psi}_2)/n} = \frac{n}{\pi} \int \exp[u'(\Psi_1, \tilde{\Psi}_1) + \bar{u}'(\Psi_2, \tilde{\Psi}_2) - nu'\bar{u}'] du'd\bar{u}', \quad u' = u_1' + iu_2'.$$ 

(2.17)

Unfortunately, one cannot use the same transformation for $e^{-(X_1, \tilde{X}_1)(X_2, \tilde{X}_2)/n}$ since the integral with respect to $X_1, X_2$ becomes divergent. Instead, we can apply the Fourier transform formula.
More precisely, set

\[
\begin{align*}
    r_1 r_2 \Phi(r_1^2, r_2^2) &= \int_{(X_1, X_1) = r_1^2} dX_1 d\tilde{X}_1 \int_{(X_2, X_2) = r_2^2} dX_2 d\tilde{X}_2 \exp\{i((A_0 - z)^*X_1, \tilde{X}_2) + i((A_0 - z)X_2, X_1) - \varepsilon(X_1, X_1) - \varepsilon(X_2, \tilde{X}_2) - \frac{1}{n}(\tilde{X}_1, \Psi_1)(X_2, \tilde{\Psi}_2) - \frac{1}{n}(X_2, \tilde{\Psi}_2)(X_1, \tilde{\Psi}_1)\}.
\end{align*}
\]

We remark that \( \Phi \) evidently depends also on \( \Psi_1, \tilde{\Psi}_1, \Psi_2, \tilde{\Psi}_2 \), but we omit these arguments here in order to simplify formulas. Changing \( r_1 \to \sqrt{n} r_1, r_2 \to \sqrt{n} r_2 \), we obtain from (2.15) and (2.17)

\[
Z(\varepsilon, \varepsilon_1) = \frac{n^3}{\pi} \int_0^\infty \int r_1 r_2 \Phi(nr_1^2, nr_2^2) \cdot \exp\{-nr_1^2 r_2^2\} \cdot Z_{\varepsilon_1}(\Psi) \times \exp[u'(\Psi_1, \tilde{\Psi}_1) + \tilde{u}'(\Psi_2, \tilde{\Psi}_2) - nu' \tilde{u}'] du' d\tilde{u}' d\tilde{\Psi} d\Psi dr_1 dr_2. \tag{2.18}
\]

Using the inverse Fourier transform formula we get

\[
\Phi(nr_1^2, nr_2^2) = \frac{1}{(2\pi)^2} \int \hat{\Phi}(t_1, t_2) e^{-int_1 r_1^2 - int_2 r_2^2} dt_1 dt_2,
\]

where

\[
\hat{\Phi}(t_1, t_2) = 4 \int e^{it_1 r_1^2 + it_2 r_2^2} r_1 r_2 \Phi(r_1^2, r_2^2) dr_1 dr_2
\]

\[
= 4 \int e^{it_1 r_1^2 + it_2 r_2^2} r_1 r_2 \int d\tilde{X} dX \exp\{(it_1 - \varepsilon)(X_1, \tilde{X}_1) + (it_2 - \varepsilon)(X_2, \tilde{X}_1) + i((A_0 - z)^*X_1, \tilde{X}_2) + i((A_0 - z)X_2, \tilde{X}_1) - \frac{1}{n}(\tilde{X}_1, \Psi_1)(X_2, \tilde{\Psi}_2) - \frac{1}{n}(\tilde{X}_2, \tilde{\Psi}_2)(X_1, \tilde{\Psi}_1)\}.
\]

Thus, we obtain

\[
\Phi(nr_1^2, nr_2^2) = \frac{1}{\pi^2} \int dt_1 dt_2 \int d\tilde{X} dX \exp\{i((A_0 - z)^*X_1, \tilde{X}_2) + i((A_0 - z)X_2, \tilde{X}_1) + it_1((X_1, \tilde{X}_1) - nr_1^2) + it_2((X_2, \tilde{X}_2) - nr_2^2) - \varepsilon(X_1, X_1) - \varepsilon(X_2, \tilde{X}_2) - \frac{1}{n}(\tilde{X}_1, \Psi_1)(X_2, \tilde{\Psi}_2) - \frac{1}{n}(\tilde{X}_2, \tilde{\Psi}_2)(X_1, \tilde{\Psi}_1)\}.
\]

Let us make the change of variables

\[
X_1 = r_1 X_1', \quad X_2 = r_2 X_2', \quad t_1 = \frac{r_2}{r_1} t_1', \quad t_2 = \frac{r_1}{r_2} t_2'.
\]

and denote

\[
R = r_1 r_2.
\]
Then we get
\[ \Phi(nr_1^2, nr_2^2) = \frac{R^{2n}}{\pi^2} \int dt_1 dt_2 \int dX' dX' \exp(i R((A_0 - z)^*X_1' + X_2') + i R((A_0 - z)X_2', X_1') \\
+ it_1^R R((X_1', X_1') - n) + it_2^R R((X_2', X_2') - n) - \frac{R}{r_1}(X_1', X_1') \\
+ \frac{i r_2}{r_1}(X_2', X_2')) \\
- \frac{R}{n}(X_1', \bar{\Psi}_1)(X_2', \bar{\Psi}_2) - \frac{R}{n}(X_1', \bar{\Psi}_2)(X_1', \bar{\Psi}_1). \tag{2.19} \]

Now we use the Hubbard-Stratonovich transformation (2.10) for the Grassmann variables:
\[ e^{-\frac{R}{n}(\bar{\Psi}_1, \Psi_1)(X_1', \bar{\Psi}_2)} = \int d\eta_1 d\chi_1 \exp(\chi_1(\bar{\chi}_1', \bar{\Psi}_1)(\frac{R}{n})^{1/2} + \eta_1(X_2', \bar{\Psi}_2)(\frac{R}{n})^{1/2} + \chi_1 \eta_1), \]
\[ e^{-\frac{R}{n}(\bar{\Psi}_2, \Psi_2)(X_1', \bar{\Psi}_1)} = \int d\eta_2 d\chi_2 \exp(\chi_2(X_2', \bar{\Psi}_2)(\frac{R}{n})^{1/2} + \eta_2(X_1', \bar{\Psi}_1)(\frac{R}{n})^{1/2} + \chi_2 \eta_2). \tag{2.20} \]

Substituting this and (2.19) to (2.18) we obtain finally
\[ Z(\varepsilon, \varepsilon_1) = \frac{n^3}{\pi^3} \int_0^\infty d\tau_1 d\tau_2 R^{2n+1} \int dt_1 dt_2 du_1' du_2' \int d\bar{X}' dX' d\bar{\Psi} d\Psi \\
\times \exp \left\{ i R((A_0 - z)^*X_1' + X_2') + i R((A_0 - z)X_2', X_1') - n R^2 \\
+ it_1^R R((X_1', \bar{\chi}_1') - n) + it_2^R R((X_2', \bar{\chi}_2') - n) \\
- \frac{R}{r_1}(X_1', \bar{\psi}_1') + \frac{i r_2}{r_1}(X_2', \bar{\psi}_2') \right\} \\
\times P(X', \Psi, R) \cdot \exp \left\{ u'(\bar{\Psi}_1, \bar{\chi}_1') + \bar{u}'(\bar{\Psi}_2, \bar{\chi}_2') - n |u'|^2 \right\} : Z_{\varepsilon_1}(\Psi) \\
= \frac{n^3}{\pi^3} \int_0^\infty d\tau_1 d\tau_2 R^{2n+1} e^{- n R^2} e^{- n |u'|^2} \\
\int dt_1 dt_2 du_1' du_2' d\eta d\chi \ e^{\chi_1 \eta_1 + \chi_2 \eta_2 - i n R t_1 - i n R t_2} \\
\times \int d\bar{X}' dX' d\bar{\Psi} d\Psi \exp((Q \bar{\Psi}, \bar{\Psi}^*)), \tag{2.21} \]

where
\[ P(X', \Psi, R) = \int \exp \left\{ \chi_1(\bar{X}_1', \bar{\psi}_1)(R/n)^{1/2} + \eta_1(X_2', \bar{\psi}_2)(R/n)^{1/2} \\
+ \chi_2(X_2', \bar{\psi}_2)(R/n)^{1/2} + \eta_2(X_1', \bar{\psi}_1)(R/n)^{1/2} + \chi_1 \eta_1 + \chi_2 \eta_2 \right\} d\eta d\chi. \tag{2.22} \]

with \( d\eta d\chi = d\eta_1 d\chi_1 d\eta_2 d\chi_2 \). We also denoted \( \bar{\Psi}, \bar{\Psi}^* \) super-vectors of the form
\[ \bar{\Psi} = \begin{pmatrix} \Psi_1 \\ \Psi_2 \\ X_1' \\ X_2' \end{pmatrix}, \quad \bar{\Psi}^* = \begin{pmatrix} \bar{\psi}_1 \\ \bar{\psi}_2 \\ \bar{X}_1' \\ \bar{X}_2' \end{pmatrix}. \]
with Grassmann vectors $\Psi_\alpha, \bar{\Psi}_\alpha$ and complex vectors $X'_\alpha, \bar{X}'_\alpha, \alpha = 1, 2$. $Q$ is $4n \times 4n$ super-matrix of the form

$$Q = \left( A(u' - \varepsilon_1) \hat{\chi} (R/n)^{1/2} \right), \quad t_{1\varepsilon} = t'_1 + i \varepsilon \frac{r_1}{r_2}, \quad t_{2\varepsilon} = t'_2 + i \varepsilon \frac{r_2}{r_1}$$

and $2n \times 2n$ block matrices $A, B$ have complex coefficients, while $\hat{\chi}, \hat{\eta}$ are diagonal matrix with Grassmann coefficients $\chi_1, \chi_2, \eta_1, \eta_2$:

$$A(u) = \left( \begin{array}{cc} u I_n & i(A_0 - z) \\ i(A_0 - z)^* & \bar{u} I_n \end{array} \right), \quad B(t_1, t_2) = \left( \begin{array}{cc} i t_1 I_n & i(A_0 - z) \\ i(A_0 - z)^* & i t_2 I_n \end{array} \right).$$

It is easy to see that

$$\det A = \det((A_0 - z)(A_0 - z)^* + |u|^2),$$
$$\det B = \det((A_0 - z)(A_0 - z)^* - t_1 t_2).$$

(2.23)

Notice that to simplify formulas here and below we drop the variables dependence in the notations $A(u), B(t_1, t_2)$.

Now we can integrate with respect to $d\tilde{X}'dX'd\tilde{\Psi}d\Psi$ using (2.5):

$$\int \int dX'd\tilde{X}'d\Psi d\tilde{\Psi} \exp[(Q\tilde{\Psi}, \tilde{\Psi}^*)] = \text{Sdet}^{-1} Q = \frac{\det A}{\det RB} \text{Sdet}^{-1} (1 + \hat{Q})$$
$$= \frac{\det A}{\det(RB)} \exp[ - \text{Str} \log(1 + \tilde{Q}) ] = \frac{\det A}{R^{2n} \det B} \exp \left\{ \frac{1}{2} \text{Str} \tilde{Q}^2 + \frac{1}{4} \text{Str} \tilde{Q}^4 \right\},$$

(2.24)

where

$$\tilde{Q} := \left( \begin{array}{cc} 0 & A^{-1} \hat{\chi} (R/n)^{1/2} \\ -R^{-1} B^{-1} \hat{\eta} (R/n)^{1/2} & 0 \end{array} \right),$$
$$\tilde{Q}^2 = \frac{1}{n} \left( \begin{array}{cc} -A^{-1} \hat{\chi} B^{-1} \hat{\eta} & 0 \\ 0 & -B^{-1} \hat{\eta} A^{-1} \hat{\chi} \end{array} \right),$$

and $\text{Sdet}, \text{Str}$ are defined in (2.6) – (2.7). Here we used that

$$\text{Str} \tilde{Q} = \text{Str} \tilde{Q}^3 = 0, \quad \tilde{Q}^p = 0, \quad p \geq 5.$$

Define

$$I = \int d\eta d\chi \, e^{\chi_1 \eta_1 + \chi_2 \eta_2} \exp \left\{ \frac{1}{2} \text{Str} \tilde{Q}^2 + \frac{1}{4} \text{Str} \tilde{Q}^4 \right\}$$
$$= \int d\eta d\chi \, e^{\chi_1 \eta_1 + \chi_2 \eta_2} \exp \left\{ \frac{1}{n} \text{Tr} A^{-1} \hat{\chi} B^{-1} \hat{\eta} - \frac{1}{2n^2} \text{Tr} (A^{-1} \hat{\chi} B^{-1} \hat{\eta})^2 \right\}$$
$$= \int d\eta d\chi \, e^{\chi_1 \eta_1 + \chi_2 \eta_2} \left( 1 + \frac{1}{n} \text{Tr} A^{-1} \hat{\chi} B^{-1} \hat{\eta} + \frac{1}{2n^2} \text{Tr} (A^{-1} \hat{\chi} B^{-1} \hat{\eta})^2 \right) \right.$$  
$$- \frac{1}{2n^2} \text{Tr} (A^{-1} \hat{\chi} B^{-1} \hat{\eta})^2$$

Observe that

$$A^{-1} \hat{\chi} B^{-1} \hat{\eta} = \left( \begin{array}{cc} A_{111}^{-1} B_{11}^{-1} \chi_1 \eta_1 + A_{121}^{-1} B_{21}^{-1} \chi_2 \eta_2 & A_{111}^{-1} B_{12}^{-1} \chi_1 \eta_2 + A_{121}^{-1} B_{22}^{-1} \chi_2 \eta_2 \\ A_{211}^{-1} B_{11}^{-1} \chi_1 \eta_1 + A_{221}^{-1} B_{21}^{-1} \chi_2 \eta_2 & A_{211}^{-1} B_{12}^{-1} \chi_1 \eta_2 + A_{221}^{-1} B_{22}^{-1} \chi_2 \eta_2 \end{array} \right).$$
Hence
\[ I = \int d\eta d\chi e^{\chi_1 \eta_1 + \chi_2 \eta_2} (1 + \frac{1}{n} \chi_2 \eta_2 \text{Tr} A_{12}^{-1} B_{21}^{-1}) (1 + \frac{1}{n} \chi_1 \eta_1 \text{Tr} A_{21}^{-1} B_{12}^{-1}) \]
\[ - \frac{1}{n^2} \chi_1 \eta_1 \chi_2 \eta_2 (\text{Tr} A_{11}^{-1} B_{11}^{-1} \text{Tr} A_{22}^{-1} B_{22}^{-1} + \text{Tr} A_{12}^{-1} B_{22}^{-1} A_{21}^{-1} B_{11}^{-1}) \]
\[ - \text{Tr} A_{11}^{-1} B_{22}^{-1} A_{22}^{-1} B_{21}^{-1}) \]
\[ = (1 + \frac{1}{n} \text{Tr} A_{12}^{-1} B_{21}^{-1}) (1 + \frac{1}{n} \text{Tr} A_{21}^{-1} B_{12}^{-1}) \]
\[ - \frac{1}{n^2} \text{Tr} A_{11}^{-1} B_{11}^{-1} \text{Tr} A_{22}^{-1} B_{22}^{-1} - \frac{1}{n^2} (\text{Tr} A_{12}^{-1} B_{22}^{-1} A_{21}^{-1} B_{11}^{-1} - \text{Tr} A_{11}^{-1} B_{12}^{-1} A_{22}^{-1} B_{21}^{-1}) \].

(2.25)

Using the standard Schur inversion formula for block matrices, we get
\[
A^{-1} = \begin{pmatrix}
\tilde{u} G(u\tilde{u}) & -i G(u\tilde{u})(A_0 - z) \\
-i(A_0 - z)^* G(u\tilde{u}) & u\tilde{G}(u\tilde{u})
\end{pmatrix},
\]
\[
B^{-1} = \begin{pmatrix}
 it_2 G(-t_1 t_2) & -i G(-t_1 t_2)(A_0 - z) \\
-i(A_0 - z)^* G(-t_1 t_2) & it_1 \tilde{G}(-t_1 t_2)
\end{pmatrix},
\]

(2.26)

where $G$ is defined in (2.13), and
\[
\tilde{G}(x) := ((A_0 - z)^* (A_0 - z) + x)^{-1}.
\]

(2.27)

By (2.26) we have
\[
\text{Tr} A_{21}^{-1} B_{12}^{-1} = \text{Tr} A_{12}^{-1} B_{21}^{-1} = -\text{Tr} G(u\tilde{u})(A_0 - z)(A_0 - z)^* G(-t_1 t_2)
\]
\[ = - \text{Tr} G(-t_1 t_2) + |u|^2 \text{Tr} G(u\tilde{u}) G(-t_1 t_2);
\]
\[
\text{Tr} A_{11}^{-1} B_{11}^{-1} \text{Tr} A_{22}^{-1} B_{22}^{-1} = -t_1 t_2 |u|^2 (\text{Tr} G(u\tilde{u}) G(-t_1 t_2))^2;
\]
\[
\text{Tr} A_{12}^{-1} B_{22}^{-1} A_{21}^{-1} B_{11}^{-1} = t_1 t_2 \text{Tr} G(u\tilde{u})(A_0 - z) \tilde{G}(-t_1 t_2)(A_0 - z)^* G(u\tilde{u}) G(-t_1 t_2)
\]
\[ = t_1 t_2 \text{Tr} G^2(-t_1 t_2)(A_0 - z)(A_0 - z)^* G^2(u\tilde{u})
\]
\[ = t_1 t_2 \text{Tr} G^2(-t_1 t_2) G(u\tilde{u}) - t_1 t_2 |u|^2 \text{Tr} G^2(-t_1 t_2) G^2(u\tilde{u});
\]
\[
\text{Tr} A_{11}^{-1} B_{12}^{-1} A_{22}^{-1} B_{21}^{-1} = - |u|^2 \text{Tr} G(u\tilde{u}) G(-t_1 t_2)(A_0 - z) \tilde{G}(u\tilde{u})(A_0 - z)^* G(-t_1 t_2)
\]
\[ = - |u|^2 \text{Tr} G(u\tilde{u}) G^2(-t_1 t_2) + |u|^4 \text{Tr} G^2(u\tilde{u}) G^2(-t_1 t_2).
\]

Here we used the relation
\[
\tilde{G}(x)(A_0 - z)^* = (A_0 - z)^* G(x).
\]

Substituting this to (2.25), we obtain
\[
I = \left(1 - \frac{1}{n} \text{Tr} G(-t_1 t_2) + \frac{|u|^2}{n} \text{Tr} G(u\tilde{u}) G(-t_1 t_2)\right)^2 + t_1 t_2 |u|^2 \left(\frac{1}{n} \text{Tr} G(u\tilde{u}) G(-t_1 t_2)\right)^2
\]
\[ - \frac{|u|^2 + t_1 t_2}{n^2} \left(\text{Tr} G(u\tilde{u}) G^2(-t_1 t_2) - |u|^2 \text{Tr} G^2(u\tilde{u}) G^2(-t_1 t_2)\right).
\]

(2.28)

Finally, let us change the variables in the integral with respect to $u_1', u_2', t_1', t_2'$ of (2.21) as
\[ u = u' - \varepsilon_1, \quad t_1 = t_1' + i \varepsilon \frac{r_1}{r_2}, \quad t_2 = t_2' + i \varepsilon \frac{r_2}{r_1}.
\]
Notice, that in the case of \( t_1, t_2 \) this means that we move integration from the lines \( t_1' \in \mathbb{R} \) to the lines \( t_1 \in L_1 = \mathbb{R} + i \varepsilon_1 f_1' t_2, t_2 \in L_- = \mathbb{R} + i \varepsilon_1 f_1' \). This can be done since the function \( \mathcal{L}_n(-t_1 t_2) \) is analytic in \( t_1, t_2 \), if \( \Im t_1 > 0, \Im t_2 > 0 \). Moving the integration with respect to \( t_1, t_2 \) to \( L_\) gathering together (2.21), (2.23), (2.24), (2.28), and differentiating with respect to \( \varepsilon \), we obtain (2.11). \( \square \)

### 3 Proof of Theorem 1.1

Now we perform an asymptotic analysis of (2.11) in the regime \( \text{dist}[z, \partial D] \sim n^{-1/2}, \varepsilon \sim n^{-3/4} \) (recall that we consider only \( \omega \in \Omega^{(1)}_{\varepsilon_0/2, \kappa(\varepsilon_0/2)} \cap \Omega^{(2)}_{\varepsilon_0/2} \cap \Omega^{(3)} \) (see conditions (A3) – (A4)).

Let us change the variables in (2.11):

\[
(t_1, t_2) \to (s, t), \quad s = \frac{t_1 - t_2}{2}, \quad t = \frac{t_1 + t_2}{2}, \quad s \in \mathbb{R}, \quad t \in L, \quad (3.1)
\]

and

\[
(r_1, r_2) \to (v, R), \quad v = r_1 - r_2, \quad R = r_1 r_2 \quad (3.2)
\]

Jacobian of the first change is \( J_1 = 2 \), and for the second change it is

\[
J_2 = \frac{1}{r_1 + r_2} = \frac{1}{(v^2 + 4R)^{1/2}},
\]

So (2.11) in new variables takes form

\[
T(z, \varepsilon) = -\frac{2n^3}{\pi^3 \varepsilon} \int_{-\infty}^{\infty} (u_1 + \varepsilon) du_1 du_2 \int_{-\infty}^{\infty} \frac{ds dv}{(v^2 + 4R)^{1/2}} \int_{L} dt \int_{0}^{\infty} R dR \\
\times \phi(u_1, u_2, t + s, t - s) \exp[n F_1(u_1, u_2)] \exp[n \tilde{F}_2(s, t, v, R)] \quad (3.3)
\]

where

\[
\tilde{F}_2(s, t, v, R) = -\mathcal{L}_n(s^2 - t^2) - (R + it + \varepsilon)^2 - (t - it - \varepsilon)^2 - \varepsilon v^2. \quad (3.4)
\]

Let us move the integration with respect to \( t \) to the contour

\[
L = L_+ \cup L'_+ \cup L_- \cup L'_-, \\
L_\pm = \{ t = iu_* \pm e^{\pm i \pi/4} \tau, \tau \in [0, C_0] \}, \\
L'_\pm = \{ t = iu_* \pm e^{\pm i \pi/4} C_0 \pm \tau, \tau > 0 \}. \quad (3.5)
\]

where \( C_0 \) is sufficiently big to provide the inequality

\[
\log C_0^2 > \mathcal{L}(0) + 2, \quad (3.6)
\]

and

\[
u_u = n^{-1/4} \tilde{u}_u, \quad (3.7)
\]

where \( \tilde{u}_u > 0 \) will be chosen later (see (3.25)).

To show the possibility of such contour shift, we notice that for any fixed \( s \) the function \( \mathcal{L}_n(s^2 - t^2) \) is analytic in \( t \) in the domain \( \{ t : \Im t > 0 \} \); moreover, for any fixed \( s \)

\[
-9 \Re \mathcal{L}_n(s^2 - t^2) < -\frac{1}{2} \log(|t|^2 + 1), \quad |t| \to \infty.
\]
Continuing the contours deformation, we deform the $R$-contour for any fixed $t \in L$ as follows:

\[ \mathcal{R}(t) = \mathcal{R}_1(t) \cup \mathcal{R}_2(t), \]

\[ \mathcal{R}_1(t) = \{ R : R = e^{i\theta(t)} \rho, \ 0 \leq \rho \leq |it + \varepsilon| \}, \quad \theta(t) = \arg(-it - \varepsilon) \]

\[ \mathcal{R}_2(t) = \{ R : R = -it - \varepsilon + \rho, \ \rho > 0 \}. \quad (3.8) \]

This deformation is possible since $\tilde{F}_2(s, t, v, R)$ is an analytic function and by (3.4) $e^{n\tilde{F}_2(s, t, v, R)} \to 0$ fast enough, as $R \to \infty$.

Below we will need the following straightforward inequalities:

\[ -\Re((R + it + \varepsilon)^2) \leq \begin{cases} 
-\cos(2\theta(t)) \cdot (|it + \varepsilon| - \rho)^2 & 0 \leq t \in L_+ \cup L_-, \ R \in \mathcal{R}_1(t), \\
-\rho^2 & t \in L_+ \cup L_-, \ R \in \mathcal{R}_2(t), \\
\Re((t - i\varepsilon)^2) + O(u_\ast) & t \in L'_+ \cup L'_-, \ R \in \mathcal{R}_1(t), \\
-\rho^2 & t \in L'_+ \cup L'_-, \ R \in \mathcal{R}_2(t).
\end{cases} \quad (3.9) \]

Notice that the term $O(u_\ast)$ appears only for $0 \leq \tau \leq u_\ast$ in (3.5). For $t \in L'_+ \cup L'_-$ we have

\[ -\Re(L_n(s^2 - t^2)) = -\Re(L_n(s^2 - (\pm C_0 e^{\pm i\pi/4} \pm \tau)^2) + O(u_\ast) \]

\[ = -\frac{1}{2} \int \log \left( (\lambda + s^2 - \sqrt{2}C_0 \tau - \tau^2)^2 + (C_0^2 + \sqrt{2}C_0 \tau)^2 \right) d\nu_{n, z}(\lambda) + O(u_\ast) \]

\[ \leq -\log C_0^2 + O(u_\ast) \leq -(\Re(L_n(0) + 1)). \quad (3.10) \]

Here and below we denote by $\nu_{n, z}(\lambda)$ the empirical spectral measure of $(A_0 - z)(A_0 - z)^\ast$.

We are going to integrate first with respect to $s$, then with respect to $R$, and then with respect to $u_1, u_2, t, v$.

To integrate with respect to $s$, observe that for $t \in L_+ \cup L_-$

\[ \frac{d}{ds} \tilde{F}_2(s, t, v, R) \bigg|_{s=0} = 0, \]

and

\[ \frac{d}{d(s^2)} \Re(\tilde{F}_2(s, t, v, R)) \]

\[ = -\frac{1}{2} \frac{d}{d(s^2)} \int \log \left( (\lambda + s^2 + \sqrt{2}u_\ast \tau + u_\ast^2)^2 + (\tau^2 + \sqrt{2}u_\ast \tau)^2 \right) d\nu_{n, z}(\lambda) < 0. \]

For $t \in L'_+ \cup L'_-$, under the condition (3.6) for $C_0$, we have by (3.9) and (3.10)

\[ n\Re(\tilde{F}_2(s, t, v, R)) = -(n - 2)(\Re(L_n(0) + 1) \]

\[ -\int \log \left( (\lambda + s^2 - \sqrt{2}C_0 \tau - \tau^2)^2 + C_0^4 \right) d\nu_{n, z}(\lambda). \]

The last two relations imply that

\[ \max_{(-n^{-1/2} \log n \geq s) \forall (s > n^{-1/2} \log n)} n\Re(\tilde{F}_2(s, t, v, R)) \leq n\Re(\tilde{F}_2(0, t, v, R) - \Re(F_s(t) \log^2 n/2, \]

\[ F_s = -\frac{d}{d(s^2)} \Re(\tilde{F}_2(s, t, v, R)) \bigg|_{s=0} = n^{-1} \text{Tr} G(-t^2). \]
In addition,
\[ \int e^{2\Re\tilde{F}_2(s,t,v,R)}ds < C < \infty \]

Hence we can apply a saddle-point method with respect to \( s \) expanding \( \Re \tilde{F}_2(s,t,v,R) \) at \( s = 0 \). We obtain
\[ \varphi(u_1,u_2,t+s,t-s)e^{n\tilde{F}_2(s,t,v,R)} \to \sqrt{\pi/(nF)}\varphi(u_1,u_2,t,t)e^{n\tilde{F}_2(0,t,v,R)}(1 + O(n^{-1})), \]
(3.11)

Notice \( F_\ast \to 1 \) as \( t \to 0 \) (see (2.13), (1.5) and recall \( \text{dist}[z, \partial D] = \tilde{d}^2n^{-1/2} \))

To integrate with respect to \( R \), we would like to restrict the integration domain to
\[ |t| \leq n^{-1/4} \log n. \]
(3.12)

To this end, observe that for any \( \tilde{F}_2(s,t,v,R) \)
\[ (it + \varepsilon)^2 = -(t - i\varepsilon)^2 = -(\pm \tau e^{\pm i\tau/4} + i(u_* - \varepsilon))^2 \]
\[ = \pm i(\tau^2 + \sqrt{2}(u_* - \varepsilon)) + (u_* - \varepsilon)^2 - \sqrt{2}(u_* - \varepsilon)\tau. \]
(3.13)

Hence, using also (3.9), for \( t \in L_+ \cup L_- \) we get
\[ \Re \tilde{F}_2(0,t,v,R) < \Re\left(-L_n(-t^2) - (t - i\varepsilon)^2\right) - \varepsilon v^2 \]
\[ = -\frac{1}{2} \int \log \left((\lambda + \sqrt{2}u_\ast \tau + u_\ast^2)^2 + (\tau^2 + \sqrt{2}u_\ast \tau)^2\right)dv_{n,z}(\lambda) \]
\[ + ((u_* - \varepsilon)^2 + (u_* - \varepsilon)\sqrt{2}\tau) - \varepsilon v^2. \]
(3.14)

Set
\[ \tilde{F}(t) := -L_n(-t^2) - (t - i\varepsilon)^2. \]
(3.15)

Using (3.14) it is straightforward to show that
\[ \Re \tilde{F}'(t) = 0 \Rightarrow |t| < C_1 n^{-1/12}. \]
(3.16)

Let us expand \( \tilde{F}(t) \) for \( |t| < C_1 n^{-1/12} \) as follows:
\[ \tilde{F}(t) = -L(0) + \frac{t^2}{n} \text{Tr} G(0) + \frac{t^4}{2n} \text{Tr} G^2(0) - (t - i\varepsilon)^2 + O(t^6). \]

Denote by \( z_* \) the point of \( \partial D \) such that \( |z - z_*| = \text{dist}[z, \partial D] \). Since we assume that \( \partial D \) is smooth near \( z \) we have
\[ z - z_* = \pm |z - z_*| |\nabla \mathcal{F}(z_*)|^{-1} \nabla \mathcal{F}(z_*) + O(\delta^4), \quad |\nabla \mathcal{F}(z_*)| \neq 0. \]
where \( \mathcal{F} \) was defined in (1.6) and the sign “+” corresponds to \( z \in D \), while “−” corresponds to \( z \notin D \). Then, taking into account that we consider \( \omega \in \Omega_{e_0/2}^{(2)} \) (see assumption (A3)), we obtain
\[ n^{-1} \text{Tr} G(0) = n^{-1} \text{Tr} Y_0^{-1}(z_*) + n^{-1} \left( \text{Tr} Y_0^{-1}(z) - \text{Tr} Y_0^{-1}(z_*) \right) \]
\[ = 1 + O(n^{-1/2-\delta_0}) + n^{-1/2} \delta^2 k + O(n^{-1}), \]
(3.17)
where the sign “+” corresponds to $z \in D$, while “−” corresponds to $z \notin D$, and we used also the assumption of Theorem 1.1 that $|z - z_0| = n^{-1/2} \delta^2$.

Hence

$$
\tilde{F}(t) = -L(0) + t^2 (1 + n^{-1/2} k \delta^2) - (t - i \epsilon)^2 + \frac{c_2 t^4}{2} + O(t^2 n^{-1}) + O(t^6)
$$

$$
= -L(0) + \frac{c_2 t^4}{2} + t^2 k \delta^2 n^{-1/2} (1 + o(1)) + 2it \epsilon + \epsilon^2 + O(t^6) + O(n^{-1} t^2),
$$

(3.18)

where

$$
c_2 = n^{-1} \text{Tr} G^2(0).
$$

(3.19)

Thus for $t \in L_+ \cup L_-$ and $|t| \leq C_1 n^{-1/12}$

$$
\Re \tilde{F}(t) + L(0) \leq \frac{c_2}{2} (u_3^2 + 2\sqrt{2} u_3 \tau - 2\sqrt{2} u_3 \tau^3 - \tau^4) - k \delta^2 n^{-1/2} (u_3^2 + \sqrt{2} u_3 \tau)
$$

$$
+ O(n^{-1/6}) \tau^4 + O(n^{-3/4}) \tau.
$$

Replacing $\tau \to n^{-1/4} \tilde{\tau}$, we get

$$
\Re \tilde{F}(\tau) + L(0) \leq n^{-1} \left( -\frac{c_2}{3} \tilde{\tau}^4 + q_3 \tilde{\tau}^3 + q_2 \tilde{\tau}^3 + q_1 \tau + q_0 \right)
$$

(3.20)

with some bounded $q_0, q_1, q_2,$ and $q_3$. Since $\Re \tilde{F}'(t)$ does not have zeros for $|\tau| > C_1 n^{-1/12}$, we conclude that $\Re \tilde{F}(t)$ is a decreasing function for $|\tau| > C_1 n^{-1/12}$. For $n^{-1/4} \log n \leq |\tau| \leq C_1 n^{-1/12}$ (3.20) implies

$$
\Re \tilde{F}(t) \leq -L(0) - \frac{c_2}{4} \tilde{\tau}^4 < -L(0) - Cn^{-1} \log^4 n, \quad t \in L_+ \cup L_-.
$$

For $t \in L'_+ \cup L'_-$ with $C_0$ satisfying (3.6) we have by (3.9) and (3.10)

$$
\Re \tilde{F}_2(0, t, v, R) < -(L_0(0) + 1).
$$

Thus finally we obtain the inequality (see (3.10))

$$
n \Re \tilde{F}_2(0, \tau, v, R) - n \tilde{F}_2(0, 0, 0, 0)
$$

$$
\leq -C \log^4 n - \log(C_0^2 + \tau C_0)^2 - R^2 - 2it R - \epsilon(2R + v^2).
$$

Thus we conclude that we can restrict the integration with respect to $t$ to the domain (3.12).

Similarly, if we change $u_1 = |u| \cos \phi, u_2 = |u| \sin \phi, du_1 du_2 = |u| |d| |u| |d| \phi$, then we can restrict the integration with respect to $|u|$ by $|u| \leq n^{-1/4} \log n$.

Since the $R$-dependent part of $\tilde{F}_2$ of (3.4) for any $|t| < n^{-1/4} \log n$ has the form

$$
\tilde{F}_R = -(R + it + \epsilon)^2,
$$

we can integrate over $R$. Notice that for $|t| < n^{-1/4} \log n$ and $R \in R_1(t)$

$$
\tilde{F}_R = -\cos(2\theta(t)) \cdot (\rho - |it + \epsilon|)^2
$$
(see (3.8) for the parametrization of $R_1(t)$). In addition, since $-it - \epsilon = u_{\ast} - \epsilon + i\tau e^{\pm i\pi/4}$, we have

$$\cos(2\theta(t)) = 1 - 2\sin^2 \theta(t) = 1 - \frac{\tau^2}{\tau^2 + (u_{\ast} - \epsilon)^2 + \sqrt{2}(u_{\ast} - \epsilon)\tau} \geq \frac{(u_{\ast} - \epsilon)}{(\tau^2 + (u_{\ast} - \epsilon)^2 + \sqrt{2}(u_{\ast} - \epsilon)\tau)^{1/2}} = \frac{(u_{\ast} - \epsilon)}{|it + \epsilon|} > \frac{C}{\log n}.$$

Here we have (3.7) and (3.12). Moreover, for $R \in R_2(t)$

$$\tilde{F}_R = -\rho^2.$$

Hence, the main contribution to the $R$-integral is given by the the saddle-point $R = -it - \epsilon$, and we obtain after integration with respect to $R$:

$$\int R e^{n\tilde{F}(t) - n(R + it + \epsilon)^2 - n\epsilon v^2 + n\epsilon^2} (v^2 + 4R)^{1/2} \to \sqrt{\frac{\pi}{n}} \frac{-(it + \epsilon)}{(v^2 - 4(it + \epsilon))^{1/2}} e^{n\tilde{F}(t) - n\epsilon v^2} (1 + O(n^{-1}))$$

(3.21)

with $\tilde{F}(t)$ of (3.15).

Relations (3.17), (3.18), and (3.12) yield

$$\tilde{F}(t) - \epsilon v^2 = -\mathcal{L}(0) + \frac{c_2 t^4}{2} + t^2 n^{-1/2} \kappa^2 + 2it\epsilon - \epsilon v^2 + O(n^{-3/2} \log^6 n)$$

Since condition (A4) guarantees that $\partial D \nsubseteq \sigma_{\epsilon_0}$, and then $z \notin \sigma_{\epsilon_0/2}$, we have by the first condition in (A3) that $(A_0 - z)(A_0 - z)^{\ast} \geq \kappa(\epsilon_0/2)$, hence $\|G(0)\| \leq \kappa^{-1}(\epsilon_0/2)$, and so $\|G(x)\| \leq 2\kappa^{-1}(\epsilon_0/2)$ for $|x| \leq \epsilon$ with some $n$-independent $\epsilon$. Thus we conclude that $n^{-1}\text{Tr} G(x)$ is an analytic function of $x$ for $|x| \leq \epsilon$, and so we can use the Taylor expansion for it:

$$n^{-1}\text{Tr} G(x) = n^{-1}\text{Tr} G(0) - xn^{-1}\text{Tr} G^2(0) + O(x^2),$$

$$n^{-1}\text{Tr} G^2(x) = n^{-1}\text{Tr} G^2(0) + O(x) =: c_2 + O(x).$$

(3.22)

which combined with (3.17) and (3.12) implies for $\varphi$ of (2.12)

$$\varphi(|u| \cos \phi, |u| \sin \phi, t, t) = (-n^{-1/2}k\delta^2 + (-t^2 + |u|^2)c_2)^2 + t^2 |u|^2 c_2^2 + O(n^{-3/2})$$

(3.23)

with $c_2$ of (3.19).

Now let us make the change of variables

$$|u| = n^{-1/4}(\bar{u} + \bar{u}_{\ast}),$$

$$t = n^{-1/4}(\bar{t} + i\bar{u}_{\ast}), \quad \bar{t} \in \bar{L}_+ \cup \bar{L}_-, \quad \bar{L}_\pm = \{z : z = \pm \tilde{\tau} e^{\pm i\pi/4}, \, \tilde{\tau} > 0\},$$

$$v^2 = n^{-1/4}\tilde{v}^2.$$
where the sign ± corresponds to \( \tilde{r} \in \widetilde{L}_+ \) and \( \tilde{r} \in \widetilde{L}_- \). Then we have

\[
\varphi(u_1, u_2, t_1, t_2) \rightarrow n^{-1/2} \tilde{\varphi}(\tilde{u}, \tilde{r}) + n^{-3/2} \tilde{r}(\tilde{u}, \tilde{r})
\]

\[
\tilde{\varphi}(\tilde{u}, \tilde{r}) = (-k \delta^2 + (\tilde{u} + \tilde{u}_*)^2 - (\tilde{r} + i \tilde{u}_*)^2) c_1^2 + (\tilde{r} + i \tilde{u}_*)^2 (\tilde{u} + \tilde{u}_*)^2 c_2^2
\]

\[
n F_1(u_1, u_2) \rightarrow n \mathcal{L}(0) + a_0(\tilde{u}_*) + a_1(\tilde{u}_*) \tilde{r} - a_2(\tilde{u}_*) \tilde{r}^2 - a_3(\tilde{u}_*) \tilde{r}^3 - \frac{c_2}{2} \tilde{u}^4
\]

\[-2 \delta (\tilde{u} + \tilde{u}_*) (\cos \theta + 1) + O(n^{-1/2}),
\]

\[
n \tilde{F}(t) - \varepsilon v^2 \rightarrow -n \mathcal{L}(0) - a_0(\tilde{u}_*) + a_1(\tilde{u}_*) i \tilde{r} - a_2(\tilde{u}_*) \tilde{r}^2 + ia_3(\tilde{u}_*) \tilde{r}^3 + \frac{c_2}{2} \tilde{r}^4
\]

\[-\varepsilon \tilde{v}^2 + O(n^{-1/2}),
\]

(3.24)

where \( \tilde{r}(\tilde{u}, \tilde{r}) \sim 1 \) is the remainder function and

\[
a_0(\tilde{u}_*) = -\frac{c_2}{2} \tilde{u}_*^4 + k \delta^2 \tilde{u}_*^2 + 2 \varepsilon \tilde{u}_*,
\]

\[
a_1(\tilde{u}_*) = -2c_2^2 \tilde{u}_*^3 + 2k \delta^2 \tilde{u}_* + 2 \varepsilon,
\]

\[
a_2(\tilde{u}_*) = 3c_2^2 \tilde{u}_*^2 - k \delta^2, \quad a_3(\tilde{u}_*) = 2c_2 \tilde{u}_*.
\]

Choose \( \tilde{u}_* \) as a smallest positive solution (evidently, it exists) of the equation

\[
a_1(\tilde{u}_*) = 0.
\]

(3.25)

Finally we obtain

\[
T(z, \varepsilon) = n^{1/2} I(\tilde{\delta}, \tilde{\varepsilon})(1 + O(n^{-1/2}))
\]

\[
I(\tilde{\delta}, \tilde{\varepsilon}) = -\frac{4}{\pi^2 \varepsilon} \Re \left[ e^{\pi \varepsilon / 4} \int_0^\infty (\tilde{u} + \tilde{u}_*)^2 \cos \theta e^{-c_2 \tilde{u}_*^4 / 2 - a_2 \tilde{u}_*^2 - a_3 \tilde{u}_*^3} d\tilde{u}
\]

\[
\times \int_0^\infty (\tilde{u}_* + i e^{\pi \varepsilon / 4} \tilde{r} e^{-c_2 \tilde{r}_*^4 / 2 - a_2 \tilde{r}_*^2 - a_3 e^{\pi \varepsilon / 4} \tilde{r}_*^3} d\tilde{r}
\]

\[
\times \int_0^{2\pi} e^{-2 \varepsilon (\tilde{u}_* + \tilde{u}_*) (\cos \theta + 1)} d\theta \int_{-\infty}^\infty \tilde{\varphi}(\tilde{u}, e^{\pi \varepsilon / 4} \tilde{r} e^{-\varepsilon \tilde{v}^2} d\tilde{v})
\]

(3.26)

with \( \tilde{\varphi} \) defined in (3.24). For simplicity, here we omitted the argument \( \tilde{u}_* \) of \( a_2 \) and \( a_3 \).

It is evident that \( |I(\tilde{\delta}, \tilde{\varepsilon})| \) is bounded uniformly from above, if the parameters \( c_2, \varepsilon \) satisfy the conditions \( \Re c_2, \Re \varepsilon > \sigma \) with some fixed \( \sigma > 0 \) and \( |\tilde{\delta}|, |\tilde{r}| \leq C \). In addition, \( I(\tilde{\delta}, \tilde{\varepsilon}) \) is an analytic function of \( \tilde{\varepsilon} : \forall \tilde{\varepsilon} > 0 \). Moreover, since \( T(z, \varepsilon) \) for fixed \( n \) is a positive and decreasing function of \( \varepsilon \) (see (1.4)), and we have proved that

\[
I(\tilde{\delta}, \tilde{\varepsilon}) = \lim_{n \to \infty} n^{-1/2} T(z, \varepsilon),
\]

one can conclude that \( I(\tilde{\delta}, \tilde{\varepsilon}) \) for \( \tilde{\varepsilon} \in \mathbb{R}_+ \) is a non-negative and decreasing function of \( \tilde{\varepsilon} \). It implies that if \( I(\tilde{\delta}, \tilde{\varepsilon}) \neq 0 \), then \( I(\tilde{\delta}, \tilde{\varepsilon}) \), as a function of \( \tilde{\varepsilon} \), cannot have zeros for real \( \tilde{\varepsilon} \) (because of analyticity of \( I(\tilde{\delta}, \tilde{\varepsilon}) \)). Hence, in order to prove that \( I \sim 1 \), it is sufficient to check that \( I(\tilde{\delta}, \tilde{\varepsilon}) \neq 0 \). To this aim, we check that \( I(\tilde{\delta}, \tilde{\varepsilon}) \), as \( \tilde{\varepsilon} \to \infty \), is not zero. More precisely, we have

**Proposition 3.1**

\[
I(\tilde{\delta}, \tilde{\varepsilon}) = \frac{\tilde{u}_*}{\tilde{\varepsilon}} (1 + o(1)), \quad \tilde{\varepsilon} \to \infty.
\]

(3.27)

The proof of Proposition 3.1 can be found in Appendix. □
4 Proof of Theorems 1.2, 1.3

Since the proofs of Theorems 1.2, 1.3 are very similar we will prove them together and make some remarks at the points where there is some difference in the proofs. Thus in this section we perform an asymptotic analysis of $T(z, \varepsilon)$ of (2.11) in the regime $n^{-1/4} \ll \delta \leq M$ (recall that $\delta^2 = \text{dist}(z, \partial D)$), where $M$ is an arbitrary positive constant and $n \varepsilon = \bar{\varepsilon}/\delta$ with $\bar{\varepsilon} \sim 1$.

As in the proof of Theorem 1.1, we consider only $\omega \in \Omega_{\varepsilon_0/2}^{(1)} \cap \Omega_{\varepsilon_0/2}^{(2)} \cap \Omega^{(3)}$ (see conditions (A3) – (A4)).

As in the proof of Theorem 1.1, we again start with the change the variables (3.1)–(3.2) to obtain (3.3).

Denote by $u_*$ a positive solution of the equation

$$n^{-1} \text{Tr} G(u_*^2) = 1$$

(4.1)

with $G(x)$ defined in (2.13). Since by (3.22) and (3.17)

$$n^{-1} \text{Tr} G(u^2) = n^{-1} \text{Tr} G(0) - c_2 u^2 + O(u^4) = 1 + k \delta^2 - c_2 u^2 + O(u^4) + O(\delta^4),$$

we obtain that the solution $u_*$ of (4.1) has the form

$$u_* = \left( \frac{k}{c_2} \right)^{1/2} \delta (1 + O(\delta^2)) \Rightarrow \frac{u_*}{\delta} = \left( \frac{k}{c_2} \right)^{1/2} (1 + O(\delta^2)).$$

Set

$$\tilde{F}(t) = -\mathcal{L}_n(-t^2) - t^2.$$  

(4.3)

Lemma 4.1 One can chose $0 < \kappa < u_* \text{ and } C_0 > 0$ satisfying (3.6), such that there exists a contour $L_+ \subset \{z : \Re z > 0 \wedge \arg z \geq \pi/4 \}$ and a constant $\sigma > 0$ depending on $L_+$ with the following conditions:

$$iu_0 + \kappa \in L_+, \quad iC_0 + C_0 \in L_+,$$

(4.4)

$$\Re \tilde{F}(t) \leq \Re \tilde{F}(iu_*) - \sigma, \quad t \in L_+.$$  

(4.5)

Proof Since it is easy to check that $x = 0$ is a minimum point of $\Re \tilde{F}(iu_0 + x) (x \in \mathbb{R})$, we conclude that $\Re \tilde{F}''(iu_*) < 0$ and we can choose $\kappa$ sufficiently small to provide that $\Re \tilde{F}$ decreases on the segment $[iu_*, iu_* + \kappa]$ and hence the segment is situated between two level lines of $\Re \tilde{F}$: $\ell_1$ and $\ell_2$ which intersect in $iu_* \ (\ell_1 \text{ is an upper level line})$. Here and below we denote $[a, b]$ the segment with edge points $a, b \in \mathbb{C}$. Notice that since $t = u_*$ is a maximum point of $\Re \tilde{F}(it)$, $\ell_1$ and $\ell_2$ cannot intersect the imaginary axis at $it \neq iu_*$. Let us also mention that $\ell_1$ and $\ell_2$ cannot form loops or intersect with each other in the upper half-plane since $\Re \tilde{F}$ is a harmonic function.

Recall that $v_{n,z}$ is the normalized counting measure of $(A_0 - z)(A_0 - z)^{*}$. Taking into account that

$$\Re \tilde{F}(\tau e^{i\pi/4}) = -\frac{1}{2} \int \log(\lambda^2 + \tau^4) d\nu_{n,z}(\lambda)$$

(4.6)

decreases, as $\tau$ grows, it suffices to prove that $\ell_2$ intersects the ray $\ell_3 = \{z = t e^{i\pi/4}\}$ at some point $t_s e^{i\pi/4}$. Indeed, in this case, since $\ell_1$ cannot intersect $\ell_3$ and $\ell_2$ cannot intersect $\ell_3$ twice, we can choose any curve, satisfying (4.4), lying between $\ell_1$ and $\ell_2$, and above or on $\ell_3$, and such that the distances between $L_+$ and $\ell_1$ and $L_+$ and $\ell_2$ are not zero.
Hence it suffices to prove that $\ell_2$ intersects $\ell_3$. Choose $\lambda_*$ such that
\[ \lambda_*^2 \geq \sup(\lambda \in \text{supp } \nu_{n,z}). \]
Consider $\ell_4 = \{i\lambda_* + \tau, \tau > 0\}$. Then since
\[ \frac{d}{d(\tau^2)} \Im(\tilde{F}(i\lambda_* + \tau)) = -\int \frac{\lambda_*^2 + \tau^2 - \lambda_*}{(\lambda_*^2 + \lambda - \tau^2)^2 + 2\lambda_*^2 \tau^2} d\nu_{n,z}(\lambda) - 1 \leq -1, \]
we conclude that $\Im \tilde{F}$ decreases at $\ell_4$. Since $\ell_1$ cannot intersect $\ell_3$, it must intersect $\ell_4$. But then $\ell_2$ cannot intersect $\ell_4$, and therefore it must intersect $\ell_3$. \hfill \Box

Now we take $L_+$ given by Lemma 4.1, take $L_-$ to be symmetric to $L_+$ with respect to the imaginary axis, and consider the contour
\[ L = L_0 \cup L_+ \cup L_- \cup L'_+ \cup L'_-, \quad L_0 = \{(iu_* - \kappa), (iu_* + \kappa)\}, \quad L'_+ = \{(iC_0 \pm C_0) \pm \tau, \tau > 0\}, \]
As in the case of Theorem 1.1, we move the integration with respect to $t$ from the real line to $L$.

Then for any fixed $t \in L$ we move the contour of integration with respect to $R$ to $R(t)$ of (3.8) with $\epsilon = 0$. Notice that inequalities (3.9) with $\epsilon = 0$ are still valid for $L_+ \cup L_-$ and for $t \in L'_+ \cup L'_-$ (without $O(u_*)$ term), and the inequality (3.10) should be replaced with
\[ -\Re L_n(s^2 - t^2) < - (\Re n(u_*^2) + 1), \quad t \in L'_+ \cup L'_-. \]
To restrict the integration with respect to $s$, notice that for $t \in L_0$
\[ -\Re L_n(s^2 - (iu_* + \tau)^2) = -\frac{1}{2} \int \log((\lambda + s^2 + u_*^2 - \tau^2)^2 + 4u_*^2 \tau^2) d\nu_{n,z}(\lambda), \]
and thus, since $|\tau| \leq \kappa < u_*$, for $\tilde{F}_2$ from (3.4) we get
\[ \frac{d}{d(s^2)} \Re \tilde{F}_2 = -\int \frac{\lambda + s^2 + u_*^2 - \tau^2}{(\lambda + s^2 + u_*^2 - \tau^2)^2 + 4u_*^2 \tau^2} d\nu_{n,z}(\lambda) < 0, \]
Moreover, for $t \in L_+ \cup L_-$ we have
\[ -\Re L_n(s^2 - \Re(t^2) - i\Im(t^2)) = -\frac{1}{2} \int \log((\lambda + s^2 - \Re(t^2))^2 + \Im(t^2)^2) d\nu_{n,z}(\lambda), \]
and since $\Re(t^2) \leq 0$ (recall that by Lemma 4.1 $L_+$ is above $\ell_3$), one can see that $\Re \tilde{F}$ is decreasing function of $s^2$, when $t \in L_+ \cup L_-$. Combining the above argument with (4.8), we conclude that there is $C > 0$ such that
\[ -n\Re L_n(s^2 - t^2) \leq -n\Re L_n(-t^2) - C \log^2 n, \quad |s| > n^{-1/2} \log n, \]
and thus we can restrict the integration with respect to $s$ by
\[ |s| \leq n^{-1/2} \log n. \]
Now let us show that we can also restrict the integration with respect to $t$ by
\[ |t - iu_*| \leq (nu_*^2)^{-1/2} \log n. \]
We would like to notice that if $\delta \sim 1$ and $u_* \sim 1$, then $(nu_*^2)^{-1/2}$ differs from $n^{-1/2}$ only with a constant. But in the setting of Theorem 1.3, when $u_* \sim \delta \ll 1$ the factor $(nu_*^2)^{-1/2}$ gives the correct scaling, so this normalization is good for both theorems.
It follows from (3.9) and Lemma 4.1 that for $t \in L_0 \cup L_+ \cup L_-$, $|t - i u_s| > (u_s^2 n)^{-1/2} \log n$ and $s$ satisfying (4.10) we have

$$\Re \widetilde{F}_2(s, t, R, v) \leq \Re \widetilde{F}(t) - C s^2 \leq \Re \widetilde{F}(i u_s) - C n^{-1} \log^2 n.$$  

Thus, we prove that the integration with respect to $t$ can be restricted to (4.11). Similarly, if we change $u_1 = |u| \cos \theta, u_2 = |u| \sin \theta, du_1 du_2 = |u| d|u| d\theta$, then we can restrict the integration with respect to $|u|$ to

$$||u| - u_s| \leq (u_s^2 n)^{-1/2} \log n. \quad (4.12)$$

In addition, using that $\varphi$ of (2.12) does not depend on $R$, we can repeat the argument of Sect. 3 in order to integrate with respect to $R$ (recall $\varepsilon$ is of order $1/(n \delta)$):

$$\int \frac{R e^{-n(R+i t+\varepsilon)^2+n s^2}}{(v^2 + 4 R)^{1/2}} dR = \frac{\sqrt{\pi/n}(-it-\varepsilon)}{(v^2 - 4(it + \varepsilon))^{1/2}} (1 + O(n^{-1})) = \frac{\sqrt{\pi/n}(-it)}{(v^2 - 4it)^{1/2}} \left(1 + \frac{n|\varphi(t, v)|}{nu_s^2}\right), \quad (4.13)$$

where the remainder function $\varphi(t, v)$ is bounded uniformly in $t$ satisfying condition (4.11) and in $v \in \mathbb{R}$.

Now, denoting $\Omega_n$ the set of $|u|, s \in \mathbb{R}$ and $t \in i u_s + \mathbb{R}$ satisfying (4.10), (4.11), and (4.12), and changing the variable $v^2 = (-it) \vec{v}^2$, we can write (2.11) in the form

$$\varepsilon T(z, \varepsilon) = \left\{ \begin{array}{ll}
\int_{-\pi}^{\pi} \left( -\cos \theta \exp \left\{ -\frac{2\varepsilon |u|}{\delta} \cos \theta \right\} d\theta 
\int_{-\pi}^{\pi} \exp \left\{ -\frac{2\varepsilon |u|}{\delta} \cos \theta \right\} d\theta 
\end{array} \right\}, \quad (4.14)$$

where we introduce the averaging

$$\langle f(t, u, s) \rangle = \frac{2n^{5/2}}{\pi^{5/2}} \int_{-\pi}^{\pi} d\theta \int_{-\infty}^{\infty} \frac{d\vec{v}}{(v^2 + 4)^{1/2}} \int_{t, s, |u| \in \Omega_n} dudtds |u|(-it) f(t, u, s)$$

$$\times \vec{\varphi}(|u|, t, s) \exp^{nh(|u|)-nh(-it)} e^{-ns^2 F_s(t) - O(n^{-1/2})} e^{2\varepsilon |u|/\delta} e^{-\varepsilon ((-it)/\delta)(\vec{v}^2 + 2)}$$

$$\times (1 + (nu_s^2)^{-1/2} \vec{r}(t, \vec{v})) \quad (4.15)$$

with

$$h(x) = \mathcal{L}_n(x^2) - x^2, \quad F_s(t) = n^{-1} \text{Tr} \ G(-t^2),$$

$$\vec{\varphi}(|u|, t, s) = \varphi(|u| \cos \theta, |u| \sin \theta, t + s, t - s),$$

and use that $n \varepsilon = \tilde{\varepsilon}/\delta$.

Now let us make the change of variables in (4.15)

$$t = i u_s + (nu_s^2)^{-1/2} \tilde{v}, \quad s = n^{-1/2} \tilde{x}, \quad |u| = u_s + (nu_s^2)^{-1/2} \tilde{u}. \quad (4.16)$$

Since some functions under consideration depend on $t^2$ and $|u|^2$ we will use also representations

$$u^2 = u_s^2 + x_u, \quad t^2 = -u_s^2 + x_t$$

$$x_u = \frac{2\tilde{u}}{n^{1/2}} + \frac{\tilde{u}^2}{u_s^2 n}, \quad x_t = \frac{2\tilde{t}}{n^{1/2}} + \frac{\tilde{t}^2}{u_s^2 n} \quad (4.17)$$
We remark that since $n^{-1/2} \gg (u_2^2 n)^{-1} \sim (n\delta^2)^{-1}$,
\[ x_u = O(n^{-1/2}), \quad x_t = O(n^{-1/2}), \quad i \tilde{u} = O(1), \quad \tilde{t} = O(1). \quad (4.18) \]

Moreover,
\[ \tilde{\varphi}(u_\ast, iu_\ast, 0) = 0 \]
\[ \Rightarrow \tilde{\varphi}(|u|, t, s) = \Phi_{1u}x_u + \Phi_{1t}x_t + \Phi_{2u}x_u^2 + \Phi_{2t}x_t^2 + \Phi_{2ut}x_ux_t + n^{-1}\Phi_{2s}\tilde{\delta}^2 + O(n^{-3/2}) \]
\[ = \Phi(x_u, x_t, \tilde{s}) + O(n^{-3/2}), \quad (4.19) \]
where $\Phi_{1u}$, $\Phi_{1t}$, $\Phi_{2u}$, $\Phi_{1t}$, $\Phi_{2ut}$, $\Phi_{2s}$ are some constants. It will be important below that
\[ \Phi_{1u} = u_\ast^2 c_2^2, \quad \Phi_{1t} = -u_\ast^2 c_2^2, \quad \Phi(x_u, x_t, \tilde{s}) = u_\ast^2 c_2^2(x_u - x_v) + O(n^{-1}) \quad (4.20) \]

Notice that because of the change (4.16) we obtain an additional factor $n^{-3/2}u_\ast^{-2}$ in front of the integral in (4.15). Taking into account that in $\Omega_n$
\[ |u| \cdot (-it) = u_\ast^2(1 + o(1)), \]
we conclude that it is sufficient to control the terms up to the order $O(n^{-1})$ of $\tilde{\varphi}(u_\ast, iu_\ast, 0)$, hence one could replace $\tilde{\varphi}(u_\ast, iu_\ast, 0)$ by $\tilde{\Phi}$ in (4.15).

It is straightforward to see that
\[
\begin{align*}

nh(|u|) &= nh(u_\ast) - 2c_2\tilde{u}^2 + O(\tilde{u}^3(nu_\ast^4)^{-1/2}), \\
nh(-it) &= - nh(u_\ast) - 2c_2\tilde{t}^2 + O(\tilde{t}^3(nu_\ast^4)^{-1/2}). \\
\end{align*}
\]
(4.21)

Here and below
\[ c_k = n^{-1}\mathrm{Tr} G^k(u_\ast^2), \quad k = 2, 3 \quad (4.22) \]
with $G(x)$ defined in (2.13).

Finally, averaging (4.15) takes the form
\[
\langle f(\tilde{t}, \tilde{u}, \tilde{s}) \rangle = 4n e^{-2(u_\ast/8)^2} \int d\tilde{t} d\tilde{u} d\tilde{s} f(\tilde{t}, \tilde{u}, \tilde{s}) \cdot e^{-2c_2(\tilde{t}^2 + \tilde{u}^2)/\delta^2} / (\sqrt{\pi})^3 \Phi(x_u, x_t, \tilde{s}) \cdot \Pi(\tilde{t}, \tilde{u}, \tilde{s})
\]
\[
\Pi(\tilde{t}, \tilde{u}, \tilde{s}) := J_0\left(-2i\tilde{t}\left(\frac{u_\ast}{\delta} + \frac{\tilde{u}}{\delta(nu_\ast^2)^{1/2}}\right)\right) \cdot \mathcal{V}\left(2\tilde{t}\left(\frac{u_\ast}{\delta} - \frac{i\tilde{t}}{\delta(nu_\ast^2)^{1/2}}\right)\right) \\
\times \left(1 + \frac{\tilde{u}}{u_\ast^{1/2}}\right)\left(1 - \frac{i\tilde{t}}{u_\ast^{1/2}}\right) \left(1 + O\left(\left(\frac{\tilde{u}^3}{(nu_\ast^2)^{1/2}}\right) + O\left(\frac{\tilde{t}^3}{(nu_\ast^2)^{1/2}}\right)\right)\right) + O(n^{-3/2}) \quad (4.23)
\]
Here the first two multipliers at the last line correspond to $(-it)$ and $|u|$ in (4.15) and the last multiplier comes from (4.21). We also set
\[ \mathcal{V}(x) = \int e^{-x\tilde{v}^2/2} \frac{d\tilde{v}}{\sqrt{\tilde{v}^2 + 4}}, \]
and used that
\[ \frac{1}{2\pi} \int e^{-2(u_\ast/\delta) + \frac{a}{\delta(nu_\ast^2)^{1/2}}\tilde{\phi}} \cos \tilde{\phi} \ d\tilde{\phi} = J_0\left(-2i\tilde{\phi}\left(u_\ast/\delta\right) + \frac{\tilde{u}}{\delta(nu_\ast^2)^{1/2}}\right), \]
where $J_0$ is a zero-order Bessel function.
Integrating by parts with respect to $\tilde{u}$ in (4.23), we get
\[
\left\langle \frac{\tilde{u}}{(nu_*^4)^{1/2}} \right\rangle = \frac{ne^{-2(u_*/\delta)\tilde{\theta}}}{c_2(nu_*^4)^{1/2}} \int \tilde{d}u \tilde{d}s \left( \frac{\sqrt{2}}{\pi} \right)^{3/2} \left( \frac{\partial \Phi}{\partial \tilde{u}} + \frac{\partial \Pi}{\partial \tilde{u}} \right) = e^{-2(u_*/\delta)\tilde{\theta}} J_0(-2i\tilde{\varepsilon}u_*/\delta) \cdot \mathcal{V}(2\tilde{\varepsilon}u_*/\delta) + O((nu_*^4)^{-1/2}).
\]

(4.24)

Here we used that (4.23) and (4.20) yield
\[
\frac{1}{(nu_*^4)^{1/2}} \frac{\partial \Pi}{\partial \tilde{u}} = \mathcal{O} \left( \frac{1}{nu_*^2} \right) \cdot \mathcal{O} \left( \frac{1}{n^{1/2}} \right) = \frac{1}{n} \mathcal{O} \left( \frac{1}{u_*^2 n^{1/2}} \right),
\]
\[
\frac{1}{(nu_*^4)^{1/2}} \frac{\partial \Phi}{\partial \tilde{u}} = \frac{2u_*^2c_2^2}{nu_*^4} \left( 1 + \mathcal{O} \left( \frac{1}{u_*^2 n^{1/2}} \right) \right).
\]

Similarly
\[
\left\langle \frac{\tilde{u}^2}{nu_*^4} \right\rangle = \frac{1}{4c_2nu_*^4} + \frac{ne^{-2(u_*/\delta)\tilde{\theta}}}{c_2 \cdot nu_*^4} \int \tilde{d}u \tilde{d}s \left( \frac{\sqrt{2}}{\pi} \right)^{3/2} \left( \frac{\partial \Phi}{\partial \tilde{u}} + \frac{\partial \Pi}{\partial \tilde{u}} \right) = \mathcal{O} ((nu_*^4)^{-1/2}).
\]

(4.25)

Here we used also that by construction of the averaging (4.15) we have
\[
1 = \mathcal{Z}(\varepsilon, \varepsilon) = \langle 1 \rangle.
\]

(4.26)

Observe that for small $\tilde{x}$ and any fixed $x$
\[
\int_{-\pi}^{\pi} (-\cos \theta)e^{-(x + \tilde{x})\cos \theta} d\theta \quad \int_{-\pi}^{\pi} e^{-(x + \tilde{x})\cos \theta} d\theta = (\log J_0(-i(x + \tilde{x})))^\prime
\]
\[
= (\log J_0(-ix))^\prime \left( 1 + \frac{(\log J_0(-ix))^\prime}{\log J_0(-ix)} \right) \cdot \tilde{x} + O(\tilde{x}^2).
\]

Thus, denoting
\[
\tilde{J}_1 = (\log J_0(-ix))^\prime \bigg|_{x=2\tilde{\varepsilon}u_*/\delta}, \quad \tilde{J}_2 = \frac{(\log J_0(-ix))^\prime}{\log J_0(-ix)} \bigg|_{x=2\tilde{\varepsilon}u_*/\delta},
\]

(4.27)

we get from (4.14), (4.24), (4.25), and (4.26)
\[
\varepsilon T(z, \varepsilon) = u_* \tilde{I}(z, \tilde{\varepsilon}, (u_*/\delta)),
\]
\[
\tilde{I}(z, \tilde{\varepsilon}, (u_*/\delta)) = \tilde{J}_1 \left( \left( 1 + \frac{\tilde{u}}{(nu_*^4)^{1/2}} \right) \left( 1 + \tilde{J}_2 \frac{\tilde{u}}{n^{1/2}u_*\delta} \right) \right)
\]
\[
= \tilde{J}_1 \left( 1 + (\tilde{J}_2 + 1) \left( \frac{\tilde{u}}{(nu_*^4)^{1/2}} \right) \right) + \mathcal{O} ((nu_*^4)^{-1})
\]
\[
= \tilde{J}_1 \left( 1 + (\tilde{J}_2 + 1) e^{-2(u_*/\delta)\tilde{\theta}} J_0(-2i\tilde{\varepsilon}u_*/\delta) \cdot \mathcal{V}(2\tilde{\varepsilon}u_*/\delta) \right) + \mathcal{O} ((nu_*^4)^{-1}).
\]

(4.28)
Notice, that for $x > 0$

$$J_0(-ix) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-x \cos \theta} d\theta > 0,$$

$$(J_0(-ix))' = \frac{1}{2\pi} \int_{-\pi}^{\pi} (-\cos \theta) e^{-x \cos \theta} d\theta = \frac{1}{2\pi} \int_{-\pi}^{\pi} x \sin^2 \theta e^{-x \cos \theta} d\theta > 0,$$

$$(\log J_0(-ix))'' = \frac{\pi}{2} e^{-x} \sin^2 x - \pi x \sin^2 x e^{-x} > 0.$$

Hence $\tilde{I} > \tilde{J}_1 \neq 0$. Remark also that to obtain (1.9) it is sufficient to use (4.2) in (4.28).

**Data Availability** Data sharing is not applicable to this article as no datasets were generated or analysed during the current study.

**Declarations**

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## 5 Appendix

### 5.1 Proof of Proposition 3.1

Let us introduce the averaging (see (3.26))

$$f = \frac{4}{\pi^2} \int_0^{\infty} f(\bar{u}, \theta) p(\bar{u}, \bar{\tau}) e^{\bar{F}_1(\bar{u})} d\bar{u} \int_0^{\infty} e^{\bar{F}_2(\bar{\tau})} d\bar{\tau} \int_{-\infty}^{\infty} -\bar{\epsilon} \bar{u}^2 d\bar{\epsilon},$$

$$\bar{F}_1(\bar{u}) = -c_2 \bar{u}^4/2 - a_2 \bar{u}^2 - a_3 \bar{u}^3,$$

$$\bar{F}_2(\bar{\tau}) = -c_2 \bar{\tau}^4/2 - i a_2 \bar{\tau}^2 - i a_3 e^{i\pi/4} \bar{\tau}^3,$$

$$p(\bar{u}, \bar{\tau}) := (\bar{u} + \bar{u}_*) (\bar{u}_* - i e^{i\pi/4} \bar{\tau}) \frac{\bar{\varphi}(\bar{u}, e^{i\pi/4} \bar{\tau})}{(\bar{u}_*^2 + 4\bar{u}_* - 4i e^{i\pi/4} \bar{\tau})^{1/2}}.$$  

where $\bar{\varphi}$ and $a_2$ and $a_3$ are defined in (3.24). Then by construction

$$1 = Z(\epsilon, \epsilon) = \langle 1 \rangle + O(n^{-1/2}).$$

Moreover, evidently,

$$I(\bar{\epsilon}, \bar{\epsilon}) = -\bar{\epsilon}^{-1} (\bar{u}_* + \bar{u}_*) \cos \theta = -\frac{\bar{u}_*}{\bar{\epsilon}} \langle \cos \theta \rangle - \frac{1}{\bar{\epsilon}} \langle \bar{u} \cos \theta \rangle.$$  

Equation (3.25), as $\epsilon \to \infty$, yields:

$$\bar{u}_*(\bar{\epsilon}, \bar{\epsilon}) = (\bar{\epsilon}/c_2)^{1/3} (1 + o(1)) \to \infty, \quad a_2 = 3c_2 \bar{u}_*^2 (1 + o(1)) \to \infty, \quad a_3 = 2c_2 \bar{u}_* \to \infty.$$  

Since $a_2 \to \infty$ and

$$\max \bar{F}_1(\bar{u}) = \bar{F}_1(0) = 0, \quad \max(-\cos \theta + 1) = 0, \quad \max(-\bar{u}^2) = 0,$$
one can use a standard saddle point method for integration with respect to \( \tilde{\mu} \), \( \theta \) and \( \tilde{v} \). To integrate with respect to \( \tilde{\tau} \) we move the contour of integration from \( \mathbb{R}_+ \) to \( e^{-i\pi/8} \mathbb{R}_+ \). Then

\[
\max \Re \tilde{F}_2(e^{-i\pi/8} \tilde{\tau}) = \max \Re(-a_2 e^{i\pi/4} \tilde{\tau}^2 - a_3 e^{3i\pi/8} \tilde{\tau}^3) = \tilde{F}_2(0) = 0,
\]

and we can apply the saddle point method for the integral with respect to \( \tilde{\tau} \) also. We restrict the integration domain to

\[
|\tilde{u}| \leq \tilde{u}_*^{-1} \log \tilde{u}_*, \quad |\tilde{\tau}| \leq \tilde{u}_*^{-1} \log \tilde{u}_*, \quad |v| \leq \tilde{\epsilon}^{-1/2} \log \tilde{\epsilon} \quad |\theta - \pi| \leq (\tilde{\epsilon} \tilde{u}_*)^{-1/2} \log(\tilde{\epsilon} \tilde{u}_*)
\]

and change the variables

\[
\tilde{u} = u'/\tilde{u}_*, \quad \tilde{\tau} = e^{-i\pi/8} \tau'/\tilde{u}_*, \quad \tilde{v} = v'/\tilde{\epsilon}^{1/2}, \quad \theta = -\pi + \theta'(\tilde{\epsilon} \tilde{u}_*)^{-1/2}
\]

Then

\[
\begin{align*}
\tilde{F}_1(\tilde{u}) &\rightarrow -3c_2(u')^2 + O(\tilde{u}_*^{-1}); \\
\tilde{F}_1(\tilde{\tau}) &\rightarrow -3c_2 e^{i\pi/4} (\tau')^2 + O(\tilde{u}_*^{-1}); \\
&-2\tilde{\epsilon} \tilde{u}_*(\cos \theta + 1) \rightarrow -(\theta')^2 + O((\tilde{\epsilon} \tilde{u}_*)^{-1/2}); \\
p(\tilde{u}, \tilde{\tau}) &\rightarrow 3(\tilde{u}_*)^6 c_2^2/2(\tilde{u}_*)^{1/2}(1 + O(\tilde{u}_*^{-1/2})),
\end{align*}
\]

and substituting this to (5.2) we get

\[
I(\tilde{\delta}, \tilde{\epsilon}) = \frac{\tilde{u}_*}{\tilde{\epsilon}} (1 + O(\tilde{u}_*^{-1/2})) + O(\frac{1}{\tilde{\epsilon} \tilde{u}_*})
\]

(5.4)

where we set

\[
\langle f \rangle_0 = \frac{c_2^2(\tilde{u}_*)^3}{\pi^2 \tilde{\epsilon}} \Re \left[ e^{i\pi/8} \int du' d\tau' d\theta' dv' f(u', \theta') e^{-3c_2(u')^2 - 3c_2 e^{i\pi/4} (\tau')^2 - (\theta')^2 - (v')^2} \right]
\]

By (5.1) and (5.3) we have

\[
\langle 1 \rangle_0 = 1 + O(\tilde{u}_*^{-1/2})
\]

and combining the above relation with (5.4), we obtain (3.27).

\[\square\]

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