SPACETIME INTEGRAL BOUNDS FOR THE ENERGY-CRITICAL NONLINEAR WAVE EQUATION

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Abstract. In this paper we prove a global spacetime bound for the quintic, nonlinear wave equation in three dimensions. This bound depends on the $L^\infty_t L^2_x$ and $L^\infty_t \dot{H}^2_x$ norms of the solution to the quintic problem.

1. Introduction

It has been known for a long time that the defocusing, quintic nonlinear wave equation,

\begin{equation}
    u_{tt} - \Delta u + u^5 = 0, \quad u(0, x) = u_0, \quad u_t(0, x) = u_1, \quad u : I \times \mathbb{R}^3 \to \mathbb{R},
\end{equation}

has a global solution that scatters for initial data lying in the critical Sobolev space $(u_0, u_1) \in \dot{H}^1 \cap L^6 \times L^2$. Equation (1.1) is called energy-critical because the solution conserves the energy

\begin{equation}
    E(u(t)) = \frac{1}{2} \int |\nabla u(t, x)|^2 dx + \frac{1}{2} \int |u_t(t, x)|^2 dx + \frac{1}{6} \int |u(t, x)|^6 dx,
\end{equation}

which is invariant under the scaling symmetry

\begin{equation}
    u(t, x) \mapsto \lambda^{1/2} u(\lambda t, \lambda x).
\end{equation}

Conservation of energy implies that for all $t \in I$, where $I$ is the maximal interval of existence for a solution to (1.1),

\begin{equation}
    \|(u(t), u_t(t))\|_{\dot{H}^1 \cap L^2} \lesssim \|u_0\|_{\dot{H}^1} \|u_1\|_{L^2}.
\end{equation}

The Sobolev embedding theorem guarantees that in $\mathbb{R}^3$, $\|u_0\|_{L^6} \lesssim \|u_0\|_{\dot{H}^1}$.

Equation (1.1) belongs to the large class of equations for which the local and small data theory is entirely determined by its scaling symmetry, (1.3). Indeed, the local well-posedness result of [Seg63] implies that $I$ is an open interval for any $(u_0, u_1) \in \dot{H}^1 \times L^2$. Scattering was proved in [Pec84] and [Rau81] for small energy data. Observe that $u(t, x)$ is a solution to (1.1) if and only if $v(t, x) = \lambda^{1/2} u(\lambda t, \lambda x)$ is a solution to (1.1) for different initial data.

For radial initial data, [Str88] proved global well-posedness of (1.1) for any initial data lying in the energy space. Later, [GSV92] combined the Morawetz estimate

\begin{equation}
    \int_I \int |x| u(t, x)^6 dx \lesssim E(u_0, u_1),
\end{equation}

with the radial Sobolev embedding theorem,

\begin{equation}
    \|u\|_{L^6_t(L^\infty_x)} \lesssim E(u_0, u_1),
\end{equation}

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to obtain the bound
\begin{equation}
\int \int \mathbb{R}^3 u(t, x)^8 \, dx \, dt \lesssim E(u_0, u_1)^2. \tag{1.7}
\end{equation}
This bound is enough to prove that (1.1) is globally well-posed and scattering for any radial initial data \((u_0, u_1) \in \dot{H}^1 \times L^2\). Scattering is defined as there exist \((u_0^+, u_1^+) \in \dot{H}^1 \times L^2\) and \((u_0^-, u_1^-) \in \dot{H}^1 \times L^2\) such that
\begin{equation}
\lim_{t \to +\infty} \| (u(t), u_t(t)) - S(t)(u_0^+, u_1^+) \|_{\dot{H}^1 \times L^2} = 0, \tag{1.8}
\end{equation}
and
\begin{equation}
\lim_{t \to -\infty} \| (u(t), u_t(t)) - S(t)(u_0^-, u_1^-) \|_{\dot{H}^1 \times L^2} = 0, \tag{1.9}
\end{equation}
where \(S(t)\) is the solution operator to the free wave equation \(u_{tt} - \Delta u = 0\).

For non-radial initial data, [Gri90] proved global well-posedness and persistence of regularity for (1.1) with smooth initial data. This result was extended to higher dimensions by [Gri92] and [Kap94]. Later, [SS94] proved global well-posedness for initial data in the energy space. Using the profile decomposition of [BC99], [Nak99a] proved global well-posedness and scattering for the quintic nonlinear wave equation (1.1) with initial data in the energy space. See also [Nak99b] for the Klein-Gordon equation.

The proof of scattering is equivalent to the proof that for any \((u_0, u_1) \in \dot{H}^1 \times L^2\), (1.1) has a global solution \(u\) that satisfies the bound
\begin{equation}
\| u \|_{L^4_t \dot{H}^1_x(\mathbb{R} \times \mathbb{R}^3)} \lesssim E(u_0, u_1)^{1/2}. \tag{1.10}
\end{equation}
(The argument proving this fact may be found in [Dod18], for example.) However, for non-radial data, the best spacetime bounds for the scattering size are much weaker than the bounds for the radial data, (1.7).

**Theorem 1.** Let \((u_0, u_1) \in \dot{H}^1_x(\mathbb{R}^3) \times L^2_x(\mathbb{R}^3)\) be initial data with the energy bound
\begin{equation}
\int_{\mathbb{R}^3} \frac{1}{2} |u_1|^2 + \frac{1}{2} |\nabla u_0|^2 + \frac{1}{6} |u_0|^6 \, dx \leq E. \tag{1.11}
\end{equation}
Then there exists a unique global solution \(u \in C^0_t \dot{H}^1_x(\mathbb{R} \times \mathbb{R}^3) \cap C^1_t L^2_x(\mathbb{R} \times \mathbb{R}^3) \cap L_t^4 L^1_x L^12(\mathbb{R} \times \mathbb{R}^3)\) with the spacetime bound
\begin{equation}
\| u \|_{L^4_t \dot{H}^1_x(\mathbb{R} \times \mathbb{R}^3)} \leq C(1 + E) E^{105/2}, \tag{1.12}
\end{equation}
for some absolute constant \(C > 0\). Interpolating (1.12) with \(\| u \|_{L^2_t L^2_x(\mathbb{R} \times \mathbb{R}^3)} \leq E^{1/6}\) gives
\begin{equation}
\| u \|_{L^4_t \dot{H}^1_x(\mathbb{R} \times \mathbb{R}^3)} \leq C(1 + E) E^{105/2}, \tag{1.13}
\end{equation}
This theorem was proved in [Tao00]. The proof used the induction on energy argument. This argument was previously used in [Ben99] to prove scattering for the radially symmetric, nonlinear Schrödinger equation in dimensions three and four. This argument was also used in [Tao05], proving scattering in dimensions five and higher for the radially symmetric nonlinear Schrödinger equation. Explicit scattering size bounds were also obtained in [Tao05].

In this note we obtain the following bounds for the size of the spacetime integral of a solution to (1.1).
Theorem 2. Let \((u_0, u_1) \in H^1_t \times L^2_x\) be initial data with the energy bound
\[
\int_{\mathbb{R}^3} \frac{1}{2} |u_1|^2 + \frac{1}{2} |\nabla u_0|^2 + \frac{1}{6} |u_0|^6 \, dx \leq E.
\]
Also suppose that the solution to (1.1) has the a priori bounds
\[
\|(u, u_t)\|_{L^\infty_t L^2_x, \dot{H}^{-1}_x} \leq A,
\]
for some \(A < \infty\). Then there exists a unique global solution \(u \in C^0_0 H^1_t (\mathbb{R} \times \mathbb{R}^3) \cap C^1_1 L^2_t (\mathbb{R} \times \mathbb{R}^3) \cap L^8_t, x(\mathbb{R} \times \mathbb{R}^3)\) with the spacetime bound
\[
\|u\|_{L^8_t, x(\mathbb{R} \times \mathbb{R}^3)} \leq CE^{4/7} A \exp(CE^{85/6} E^{13/14} A^{11}),
\]
for some absolute constant \(C > 0\).

Remark 1. One may use the scaling symmetry (1.3) to obtain the equality that arises in (1.15).

The proof of Theorem 2 follows the line of argument previously used for the nonlinear Schrödinger equation in [DM17], [ADM20], especially in [DM18]. We utilize an interaction Morawetz estimate to show that the energy of a solution must eventually spread out in \(\mathbb{R}^3\). More precisely, for any \(\varepsilon > 0\) and \(T < \infty\), there exists \(T(T, \varepsilon) < \infty\) such that for some \(t_0 \in [0, T(\varepsilon)]\),
\[
\int_{t_0}^{t_0 + T} \int |u(t, x)|^8 \, dx \, dt \leq \varepsilon.
\]
Combining this fact with some dispersive estimates for the linear wave equation implies Theorem 2.

2. Local well-posedness and small data arguments

As was mentioned in the introduction, [Gri90] proved that (1.1) is globally well-posed for initial data in the energy space. Additionally, the a priori bound
\[
\|(u, u_t)\|_{L^\infty_t \dot{H}^2_x \times \dot{H}^1_x} \leq A,
\]
implies that

Theorem 3. For any \(t_0 \in \mathbb{R}\) and \(T > 0\),
\[
\|u\|_{L^8_t, x([0, T] \times \mathbb{R}^3)} \lesssim A^8(T).
\]

Proof. This follows from the Sobolev embedding theorem. Indeed, since \(S(t)\) is a unitary operator on \(\dot{H}^s \times \dot{H}^{s-1}\), for any \(s \in \mathbb{R}\), the bounds (2.1) and (1.15) imply
\[
\|S(t-t_0)(u(t_0), u_t(t_0))\|_{L^8} \lesssim \|u(t_0), u_t(t_0)\|_{\dot{H}^{9/8} \times \dot{H}^{1/8}} \lesssim A.
\]
Taking \(|I|\) sufficiently small depending on \(E\) and \(A\), small data arguments imply
\[
\|u\|_{L^8_t, x(\mathbb{R} \times \mathbb{R}^3)} \lesssim 1.
\]
Since the bound (2.1) is uniform on \(\mathbb{R}\), we can partition \([t_0, t_0 + T]\) into \(\lesssim A^8(T)\) such intervals for which (2.4), which proves (2.2).

Remark 2. In fact, for any \(\dot{H}^1\)-admissible pair \((p, q)\) in \(\mathbb{R}^3\), we have proved
\[
\|u\|_{L^p_t L^q_x([t_0, t_0 + T] \times \mathbb{R}^3)} \lesssim A(T)^{1/p}.
\]

We can prove a small data well-posedness result.
Theorem 4. There exists $\epsilon(E) > 0$ such that if $I = [t_0, t_0 + T]$ is an interval for which
\begin{equation}
\|S(t - t_0)(u(t_0), u_t(t_0))\|_{L^8_{t,x}(I \times \mathbb{R}^3)} \leq \epsilon,
\end{equation}
and
\begin{equation}
\|u\|_{L^\infty_t H^1(I \times \mathbb{R}^3)} \leq E^{1/2},
\end{equation}
then
\begin{equation}
\|u\|_{L^8_{t,x}(I \times \mathbb{R}^3)} \leq 2\epsilon.
\end{equation}

Proof. By Duhamel’s principle, for any $t \in I$,
\begin{equation}
u(t) = S(t - t_0)(u(t_0), u_t(t_0)) - \int_{t_0}^t S(t - \tau)(0, u^5)\,d\tau.
\end{equation}
By interpolation,
\begin{equation}
\|\nabla|^{1/2} u^5\|_{L^{16/9}(L^{18/13}(I \times \mathbb{R}^3)} \lesssim \|\nabla u\|_{L^2_t L^2_x} \|u\|_{L^8_{t,x}}^{9/2}.\end{equation}
Then interpolating the estimates,
\begin{equation}
\|S(t)(0,F)\|_{L^\infty_t \lesssim \frac{1}{t} \|F\|_{L^1}} \quad \|S(t)(0,F)\|_{L^2_t \|\nabla|^{-1} F\|_{L^1}} \quad \|S(t)(0,F)\|_{L^2_t \|F\|_{L^1}},
\end{equation}
the Hardy-Littlewood-Sobolev inequality implies
\begin{equation}
\|\nabla|^{1/4}, \partial_t|^{1/4} \int_{t_0}^t S(t - \tau)(0, u^5)\,d\tau\|_{L^{16/3}(I \times \mathbb{R}^3)} \lesssim \|\nabla u\|_{L^2_t L^2_x} \|u\|_{L^8_{t,x}}^{9/2}.\end{equation}
Then by the Sobolev embedding theorem,
\begin{equation}
\|\int_{t_0}^t S(t - \tau)(0, u^5)\,d\tau\|_{L^8_{t,x}(I \times \mathbb{R}^3)} \lesssim \|\nabla u\|_{L^2_t L^2_x} \|u\|_{L^8_{t,x}(I \times \mathbb{R}^3)}^{9/2}.
\end{equation}
Choosing $\epsilon(E)$ sufficiently small, $\epsilon(E) \sim E^{-1/14}$ will do, the proof is complete. \qed

3. A Reduction of the Solution

Strichartz estimates imply that
\begin{equation}
\|S(t)(u_0, u_1)\|_{L^8_{t,x}(\mathbb{R} \times \mathbb{R}^3)} \lesssim E^{1/2}.
\end{equation}
Next, partition $\mathbb{R}$ into $\frac{E^{1/2}}{\epsilon}$ subintervals $I_j$ such that
\begin{equation}
\|S(t)(u_0, u_1)\|_{L^8_{t,x}(I_j \times \mathbb{R}^3)} \leq \frac{\epsilon}{4},
\end{equation}
where $\epsilon \sim E^{-1/14}$. Then by the triangle inequality,
\begin{equation}
\|u\|_{L^8_{t,x}(\mathbb{R} \times \mathbb{R}^3)} \lesssim E^{4/7} \sup_j \|u\|_{L^8_{t,x}(I_j \times \mathbb{R}^3)}.
\end{equation}
Therefore, to obtain a bound on the scattering size of $u$, it is enough to obtain a bound on
\begin{equation}
\|u\|_{L^8_{t,x}(I_j \times \mathbb{R}^3)}
\end{equation}
that is uniform in $j$.

Fix $I_j = [a_j, b_j]$. If there exists some $t_j \in [a_j, b_j]$ such that
\begin{equation}
\|S(t - t_j)(u(t_j), u_t(t_j))\|_{L^8_{t,x}([t_j, b_j] \times \mathbb{R}^3)} \leq \epsilon,
\end{equation}

then Theorems 3 and 4 imply
\(\|u\|_{L^4_{t,x}(\mathbb{T}_j \times \mathbb{R}^3)} \lesssim A(t_j - t)^{1/8} + \epsilon.\)

For any \(t \in \mathbb{R},\) Duhamel’s principle implies that the solution to \((1.1)\) has the form
\[
S(t)(u_0, u_1) = \int_0^t S(t - \tau)(0, u^5) d\tau.
\]
Therefore, for any \(t > t_j,\)
\[
S(t - t_j)(u(t_j), u_t(t_j)) = S(t)(u_0, u_1) - \int_0^{t_j} S(t - \tau)(0, u^5) d\tau.
\]
By definition of \(I_j, (3.2)\) implies
\[
\|S(t)(u_0, u_1)\|_{L^4_{t,x}([t_j, b_j] \times \mathbb{R}^3)} \leq \frac{\epsilon}{4}.
\]
Therefore, to obtain \((3.4)\) it only remains to prove
\[
\|\int_0^{t_j} S(t - \tau)(0, u^5) d\tau\|_{L^4_{t,x}([t_j, b_j] \times \mathbb{R}^3)} \leq \frac{3\epsilon}{4}.
\]
Suppose without loss of generality that \(a_j \geq 0\) and decompose
\[
\int_0^{t_j} S(t - \tau)(0, u^5) d\tau = \int_0^{t_{j} - T} S(t - \tau)(0, u^5) d\tau + \int_{t_{j} - T}^{t_j} S(t - \tau)(0, u^5) d\tau.
\]

**Lemma 1.** There exists \(T \sim A^{2/k^{13/6}}\) such that
\[
\|\int_0^{t_{j} - T} S(t - \tau)(0, u^5) d\tau\|_{L^4_{t,x}([t_j, b_j] \times \mathbb{R}^3)} \leq \frac{\epsilon}{4},
\]
for any \(t_j \in \mathbb{R}.
\]

**Proof.** For any \((t, x) \in [t_j, b_j] \times \mathbb{R}^3,
\]
\[
\langle \int_0^{t_{j} - T} S(t - \tau)(0, u^5) d\tau(t, x) \rangle = \int_0^{t_{j} - T} \frac{1}{4\pi(t - \tau)} \int_{|x - x'| = |t - \tau|} u^5(\tau, x') dS(x'),
\]
where \(dS\) denotes the surface measure of a sphere in \(\mathbb{R}^3.\) See for example [Sog95]. Also, by computing the energy flux, for any \(R > 0, x_0 \in \mathbb{R}^3,
\]
\[
\frac{d}{dt} \int_{|x - x_0| \leq R} \frac{1}{2} |\nabla u(t, x)|^2 + \frac{1}{2} |u_t(t, x)|^2 + \frac{1}{6} |u(t, x)|^6 dx \leq -\frac{1}{2} \int_{|x - x_0| = R} |u(t, x)|^6 d\sigma(x).
\]
Therefore, by conservation of energy, \((3.13),\) and Hölder’s inequality,
\[
\int_0^{t_{j} - T} \frac{1}{4\pi(t - \tau)} \int_{|x - x'| = |t - \tau|} u^5(\tau, x') dS(x') \lesssim E^{5/6}(\int_0^{t_{j} - T} \frac{1}{|t - \tau|^4})^{1/6} \lesssim E^{5/6} T^{1/2}.
\]
Strichartz estimates, the energy identity, and Duhamel’s principle also imply that
\[
\|\int_0^{t_{j} - T} S(t - \tau)(0, u^5) d\tau\|_{L^4_{t,x}([t_j, b_j] \times \mathbb{R}^3)} \lesssim \|(u(t_j - T), u_t(t_j - T))\|_{\dot{H}^{1/2} \times \dot{H}^{-1/2}} + \|(u_0, u_1)\|_{\dot{H}^{1/2} \times \dot{H}^{-1/2}} \lesssim E^{1/4} A^{1/2}.\]
Interpolating (3.14) and (3.15),

\[(3.16)\] \[\| \int_{t_j-T}^{t_j} S(t-\tau)(0, u^5) d\tau \|_{L^8_t x \langle [t_j, b_j] \times \mathbb{R}^3 \rangle} \lesssim A^{1/2} E^{13/24} \]

Taking \( T \sim A^{2.5^{13/6}} \) gives the estimate

\[(3.17)\] \[\| \int_{t_j-T}^{t_j} S(t-\tau)(0, u^5) d\tau \|_{L^8_t x \langle [t_j, b_j] \times \mathbb{R}^3 \rangle} \leq \frac{\epsilon}{4}.\]

Therefore, to obtain uniform spacetime integral bounds on a solution to (1.1), it only remains to prove that there exists \( T(T) \) such that for any \( j \), if \( b_j - a_j \geq T(T) \), there exists \( t_j \in I_j \), \( 0 < t_j - a_j \leq T(T) \) that satisfies

\[(3.18)\] \[\| \int_{t_j-T}^{t_j} S(t-\tau)(0, u^5) d\tau \|_{L^8_t x \langle [t_j, b_j] \times \mathbb{R}^3 \rangle} \leq \frac{\epsilon}{2}.
\]

This will be the topic of the next section. The proof will utilize an interaction Morawetz estimate.

4. Interaction Morawetz estimate

Theorem 5. For \( T \sim A^{2.5^{13/6}} \) there exists \( T(T) \) such that for any \( j \), if \( b_j - a_j \geq T(T) \), there exists \( t_j \in I_j \), \( 0 < t_j - a_j \leq T(T) \) that satisfies

\[(4.1)\] \[\| \int_{t_j-T}^{t_j} S(t-\tau)(0, u^5) d\tau \|_{L^8_t x \langle [t_j, b_j] \times \mathbb{R}^3 \rangle} \leq \frac{\epsilon}{2}.
\]

**Proof.** The proof uses an interaction Morawetz estimate. Choose \( \phi \in C_0^\infty(\mathbb{R}^3) \) that is supported on \(|x| \leq 2 \), \( \phi(x) = 1 \) on \(|x| \leq 1 \), \( \phi(x) \geq 0 \). Let \( M_R(t) \) denote the Morawetz potential,

\[(4.2)\] \[M_R(t) = \int e(t, y)\phi\left(\frac{x-y}{R}\right)(x-y) \cdot (u_t, \nabla u) dxdy + \int e(t, y)\phi\left(\frac{x-y}{R}\right)\langle u_t, u \rangle dxdy,
\]

where \( R > 0 \) is a fixed constant and \( e(t, y) \) is the energy density

\[(4.3)\] \[e(t, y) = \frac{1}{2} u_t(t, y)^2 + \frac{1}{2} \| \nabla u(t, y) \|^2 + \frac{1}{6} |u(t, y)|^6.
\]

By direct calculation,

\[(4.4)\] \[\frac{d}{dt} M_R(t) = \int \nabla \cdot (u_t, \nabla u)\phi\left(\frac{x-y}{R}\right)(x-y) \cdot (u_t, \nabla u)
\]

\[(4.5)\] \[+ \int \nabla \cdot (u_t, \nabla u)\phi\left(\frac{x-y}{R}\right)\langle u_t, u \rangle dxdy
\]

\[(4.6)\] \[+ \int e(t, y)\phi\left(\frac{x-y}{R}\right)(x-y) \cdot (\nabla u, \nabla u) - \langle u^5, \nabla u \rangle dxdy
\]

\[(4.7)\] \[+ \int e(t, y)\phi\left(\frac{x-y}{R}\right)\langle u_t, \nabla u \rangle dxdy
\]

\[(4.8)\] \[+ \int e(t, y)\phi\left(\frac{x-y}{R}\right)\langle u, \nabla u \rangle - \langle u, u^5 \rangle dxdy
\]
\[ (4.9) \quad + \int e(t, y) \phi(\frac{x-y}{R}) \langle u_t, u_t \rangle dxdy. \]

Integrating by parts,

\[ (4.10) \quad \int e(t, y) \phi(\frac{x-y}{R}) u_t^2 dxdy - \frac{1}{2} \int e(t, y) \phi'(\frac{x-y}{R}) \frac{|x-y|}{R} u_t^2 dxdy, \]

so

\[ (4.11) \quad + \frac{1}{2} \int e(t, y) \phi'(\frac{x-y}{R}) \frac{|x-y|}{R} u_t^2 dxdy. \]

Also integrating by parts,

\[ (4.12) \quad - \int e(t, y) \phi(\frac{x-y}{R}) |\nabla u|^2 dxdy - \int e(t, y) \phi(\frac{x-y}{R}) u^6 dxdy + \frac{1}{2} \int e(t, y) \phi''(\frac{x-y}{R}) \frac{1}{R^2} u^6 dxdy. \]

Rewriting,

\[ (4.13) \quad \langle \Delta u, \nabla u \rangle = \partial_k \langle \partial_j u, \partial_k u \rangle - \frac{1}{2} \partial_j \langle \partial_k u, \partial_k u \rangle, \]

and integrating by parts,

\[ (4.14) \quad = \frac{1}{2} \int e(t, y) \phi(\frac{x-y}{R}) |\nabla u|^2 dxdy + \frac{1}{6} \int e(t, y) \phi(\frac{x-y}{R}) u^6 dxdy \]
\[ + \frac{1}{2} \int e(t, y) \phi'(\frac{x-y}{R}) \frac{(x-y)_j (x-y)_k}{|x-y| R^2} \langle \partial_j u, \partial_k u \rangle dxdy \]
\[ - \frac{1}{2} \int e(t, y) \phi'(\frac{x-y}{R}) \frac{|x-y|}{R} |\nabla u|^2 dxdy \]
\[ + \frac{1}{6} \int e(t, y) \phi'(\frac{x-y}{R}) \frac{|x-y|}{R^2} u^6 dxdy. \]

Also, integrating by parts,

\[ (4.15) \quad = \int \langle \partial_k u, u_t \rangle \phi(\frac{x-y}{R}) \delta_{jk} \langle \partial_j u, u_t \rangle dxdy \]
\[ + \int \langle \partial_k u, u_t \rangle \phi'(\frac{x-y}{R}) \frac{(x-y)_j (x-y)_k}{|x-y| R^2} \langle \partial_j u, u_t \rangle dxdy. \]

Finally, integrating by parts,

\[ (4.16) \quad = \int \langle \partial_k u, u_t \rangle \phi'(\frac{x-y}{R}) \frac{(x-y)_k}{|x-y| R} \langle u, u_t \rangle dxdy. \]
Therefore,

\[
\begin{align*}
&= - \int \left[ \frac{1}{2} u_t^2 + \frac{1}{2} |\nabla u|^2 + \frac{1}{6} u^6 \phi \left( \frac{x - y}{R} \right) \right] \frac{1}{2} u_t^2 + \frac{1}{2} |\nabla u|^2 + \frac{2}{3} u^6 dxdy \\
&\quad + \int \langle \partial_k u, u_t \rangle \phi \left( \frac{x - y}{R} \right) \delta_{jk} \langle \partial_j u, u_t \rangle dxdy \\
&\quad + \int \langle \partial_k u, u_t \rangle \phi' \left( \frac{x - y}{R} \right) (x - y)_j (x - y)_k \langle \partial_j u, u_t \rangle dxdy \\
&\quad - \frac{1}{2} \int e(t, y) \phi' \left( \frac{x - y}{R} \right) \frac{|x - y|}{R} u_t^2 dxdy \\
&\quad + \frac{1}{2} \int e(t, y) \phi' \left( \frac{x - y}{R} \right) \frac{|x - y|}{R} u_t^2 dxdy \\
&\quad + \int \langle \partial_k u, u_t \rangle \phi' \left( \frac{x - y}{R} \right) (x - y)_k (u_t, u_t) dxdy.
\end{align*}
\]

Therefore,

\[
\begin{align*}
&= - \frac{1}{J} \int_{R_0}^{e^f R_0} \frac{1}{R} \int \left[ \frac{1}{2} u_t^2 + \frac{1}{2} |\nabla u|^2 + \frac{1}{6} u^6 \phi \left( \frac{x - y}{R} \right) \right] \frac{1}{2} u_t^2 + \frac{1}{2} |\nabla u|^2 + \frac{2}{3} u^6 dxdy dR \\
&\quad + \frac{1}{J} \int_{R_0}^{e^f R_0} \frac{1}{R} \int \langle \partial_k u, u_t \rangle \phi \left( \frac{x - y}{R} \right) \langle \partial_j u, u_t \rangle dxdydR \\
&\quad + O \left( \frac{1}{J} \int_{e^f R_0}^{2 e^f R_0} e(t, y) e(t, x) dxdy \right) \\
&\quad + O \left( \frac{1}{J} \int_{e^f R_0}^{2 e^f R_0} e(t, y) \frac{1}{|x - y|^2} u(t, x)^2 dxdy \right).
\end{align*}
\]

Therefore, by the fundamental theorem of calculus,

\[
\begin{align*}
&= - \frac{1}{J} \int_{e^f R_0}^{a_j + T} \frac{1}{R} \int \left[ \frac{1}{2} u_t^2 + \frac{1}{2} |\nabla u|^2 + \frac{1}{6} u^6 \phi \left( \frac{x - y}{R} \right) \right] \frac{1}{2} u_t^2 + \frac{1}{2} |\nabla u|^2 + \frac{2}{3} u^6 dxdy dRdt \\
&\quad - \frac{1}{J} \int_{a_j}^{T} \frac{1}{R} \int \langle \partial_k u, u_t \rangle \phi \left( \frac{x - y}{R} \right) dxdydRdt \lesssim \frac{T}{J} E^2 + \frac{e^f R_0}{J} E^2.
\end{align*}
\]
Now by positive definiteness argument, if we choose \( T = e^j R_0 \),
\[
\int_{a_j + T}^{a_j + T + R} \frac{1}{R} \int_{R_0}^{e^j R_0} \frac{1}{J} \int \left[ \frac{1}{2} (|u_t| - |\nabla u|^2 + \frac{1}{6} u_t^2) \phi \left( \frac{x-y}{R} \right) \right] dx dy dt \lesssim \frac{T}{J} E^2.
\]

Inequality (4.20) implies that there exists some \( t_j \in [a_j + T, a_j + T + T] \) and some \( R_0 \leq R \leq e^j R_0 \) such that
\[
\int_{t_j - T}^{t_j} \int \left[ \frac{1}{2} (|u_t| - |\nabla u|^2 + \frac{1}{6} u_t^2) \phi \left( \frac{x-y}{R} \right) \right] dx dy dt \lesssim \frac{T}{J} E^2.
\]

Rewriting (4.21) as a sum,
\[
\int_{t_j - T}^{t_j} \sum_{k \in \mathbb{Z}^3} \left( \int \chi \left( \frac{Rk-x}{R} \right) \left( \frac{1}{2} (|\nabla u| - |u_t|)^2 + \frac{1}{6} u_t^2 \right) dx \right) \left( \int \chi \left( \frac{Rk-x}{R} \right) \left( \frac{1}{2} u_t^2 + \frac{1}{2} |\nabla u|^2 + \frac{2}{3} u^6 \right) dx \right) dt \lesssim \frac{T}{J} E^2,
\]
where \( \chi \) is a smooth, compactly supported function, \( \chi(x) \geq 0 \) on the ball of radius 1.

Also, by the Sobolev embedding theorem,
\[
\| \chi \left( \frac{Rk-x}{R} \right) u \|_{L^\infty (\mathbb{R}^3)} \lesssim \| \chi \left( \frac{Rk-x}{R} \right) u \|_{L^6}^{1/2} \| \nabla (\chi \left( \frac{Rk-x}{R} \right) u) \|_{L^6}^{1/2}
\]
\[
\lesssim \frac{1}{R^{1/2}} \| \chi \left( \frac{Rk-x}{R} \right) u \|_{L^6}^{1/2} \| \chi \left( \frac{Rk-x}{R} \right) u \|_{L^6}^{1/2} + \| \chi \left( \frac{Rk-x}{R} \right) u \|_{L^6}^{1/2} \| \nabla (\chi \left( \frac{Rk-x}{R} \right) u) \|_{L^6}^{1/2}.
\]

Therefore, by the support properties of \( \chi \) and \( \chi' \), (4.20) implies
\[
\left( \sum_{k \in \mathbb{Z}^3} \| \chi \left( \frac{Rk-x}{R} \right) u \|_{L^6}^6 \right)^{1/6} \lesssim \frac{1}{R^{1/2}} \left( \| u \|_{L^6(\mathbb{R}^3)} + \| \nabla u \|_{L^6(\mathbb{R}^3)} \right) \| u \|_{L^6(\mathbb{R}^3)}^{1/2} \lesssim \frac{1}{R^{1/2}} E^{1/6} + E^{1/12} A^{1/2}.
\]

Taking \( R \geq e^{5/3} A \),
\[
\left( \sum_{k \in \mathbb{Z}^3} \| \chi \left( \frac{Rk-x}{R} \right) u \|_{L^6(\mathbb{R}^3)}^6 \right)^{1/6} \lesssim E^{1/12} A^{1/2}.
\]

Meanwhile, by (4.21),
\[
\left( \int_{t_j - T}^{t_j} \sum_{k \in \mathbb{Z}^3} \| \chi \left( \frac{Rk-x}{R} \right) u \|_{L^6(\mathbb{R}^3)}^{12} dt \right)^{1/12} \lesssim \frac{T^{1/12}}{J^{1/12}} E^{1/6}.
\]

Interpolating (4.25) and (4.26),
\[
\| u \|_{L^2_t L^6_x ([t_j - T, t_j] \times \mathbb{R}^3)} \lesssim \frac{T^{1/24}}{J^{1/24}} E^{1/8} A^{1/4}.
\]

Also by (4.24),
\[
\| u \|_{L^\infty_t L^2_x ([t_j - T, t_j] \times \mathbb{R}^3)} \lesssim A.
\]

Interpolating (4.27) and (4.28),
\[
\| u \|_{L^7_t L^\infty_x ([t_j - T, t_j] \times \mathbb{R}^3)} \lesssim \frac{T^{1/27}}{J^{1/27}} E^{1/9} A^{1/3}.
\]
Choosing $\delta > 0$ sufficiently small such that $\delta^3 E \lesssim \epsilon$, $\|u\|_{L^2_t L^{16}_x(\{t_j - T, t_j\} \times \mathbb{R}^3)} \lesssim \delta$ implies

$$\| \int_{t_j - T}^{t_j} S(t - \tau)(0, u^5) d\tau \|_{L^8_{t,x}(\mathbb{R} \times \mathbb{R}^3)} \lesssim \epsilon.$$  \hfill (4.30)

Indeed, by Strichartz estimates and conservation of energy, if $(p, q)$ is the $\dot{H}^1$-admissible pair $(p, q) = (\frac{9}{4}, 5)$,

$$\|u\|_{L^p_t L^q_x} \lesssim \frac{E^{1/2}}{\delta} + \delta^3 \|u\|_{L^p_t L^q_x},$$

which implies $\|u\|_{L^p_t L^q_x(\{t_j - T, t_j\} \times \mathbb{R}^3)} \lesssim E^{1/2}$. Plugging this fact into (4.30) gives the appropriate estimate.

Doing some algebra with (4.29), since $\epsilon \sim E^{-1/14}$ and $T \sim \frac{A^{11/27}E^{85/162}}{\epsilon^{13/27}}$, and therefore,

$$\frac{A^{11/27}E^{85/162}}{\epsilon^{13/27}} \sim A^{11/27}E^{85/162}E^{31/378} \lesssim T^{1/27},$$

implies that

$$\| \int_{t_j - T}^{t_j} S(t - \tau)(0, u^5) \|_{L^8_{t,x}(\mathbb{R} \times \mathbb{R}^3)} \lesssim \frac{\epsilon}{2}.$$  \hfill (4.34)

This proves the theorem. \hfill \blacksquare

Setting $T = \exp(C E^{85/6} E^{13/14} A^{11})$ for some constant $C$, (4.33) and (4.34) imply

$$\|u\|_{L^8_{t,x}(\mathbb{R} \times \mathbb{R}^3)} \lesssim E^{4/7} A \exp(C E^{85/6} E^{13/14} A^{11}),$$

which proves Theorem 2.

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