TRAVELING WAVE SOLUTIONS IN ADVECTION HYPERBOLIC-PARABOLIC SYSTEM WITH NONLOCAL DELAY

KUN LI
School of Mathematics and Computational Science, Hunan First Normal University
Changsha, 410205, China

JIANHUA HUANG *
College of Science, National University of Defense Technology
Changsha, 410073, China

XIONG LI
School of Mathematical Sciences, Beijing Normal University
Laboratory of Mathematics and Complex Systems, Ministry of Education
Beijing, 100875, China

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Abstract. This paper is concerned with a class of advection hyperbolic-parabolic systems with nonlocal delay. We prove that the wave profile is described by a hybrid system that consists of an integral transformation and an ordinary differential equation. By considering the same problem for a properly parameterized system and the continuous dependence of the wave speed on the parameter involved, we obtain the existence and uniqueness of traveling wave solutions in advection hyperbolic-parabolic system with nonlocal delay under bistable assumption. The influence of advection on the propagation speed is also considered.

1. Introduction. In the past two decades, great progress has been made on the existence of traveling wave solutions in reaction-diffusion equations with monostable nonlinearities, especially for nonlocal delays, see Al-Omari and Gourley [2], Ashwin et al. [4], Billingham [7], Gourley [21, 22], Gourley and Kuang [23], Gourley and Ruan [24], Ruan and Xiao [35], So et al. [41], Wang et al. [43] and Zou [47] and references cited therein. There are also well-known results with bistable nonlinearities. For example, Fife and McLeod [19] considered the following system without delay

$$\frac{\partial u}{\partial t} = d \frac{\partial^2 u}{\partial x^2} + g(u), \quad x \in \mathbb{R}, \ t > 0.$$ (1)

They studied the globally exponential stability of traveling wave solutions of (1), see also Volpert et. al. [42]. Schaaf [36] considered the following delayed system

$$\frac{\partial u(x, t)}{\partial t} = d \frac{\partial^2 u(x, t)}{\partial x^2} + g(u(x, t), u(x, t - \tau)), \quad x \in \mathbb{R}, \ t > 0, \ \tau > 0$$ (2)
for the so-called Huxley nonlinearity as well as Fisher nonlinearity. He proved the existence of traveling wave solutions of (2) by using the phase-plane technique, the maximum principle for parabolic functional differential equations and the general theory for ordinary functional differential equations. By first establishing the existence and comparison theorem of solutions for (2), where they applied the theory of abstract functional differential equations developed by Martin and Smith [32], and then using the elementary sub- and supersolutions comparison and the squeezing technique developed by Chen [13], Smith and Zhao [39] studied the global asymptotic stability, Lyapunov stability and uniqueness of traveling wave solutions of (2) under the bistable assumption.

The nonlocal system

\[
\frac{\partial u(x, t)}{\partial t} = d \frac{\partial^2 u(x, t)}{\partial x^2} + g\left(u(x, t), \int_{-\infty}^{\infty} J(x-y)b(u(y, t-\tau))dy\right), x \in \mathbb{R}, t > 0, \tau \geq 0
\]

was also studied by many researchers. When \( \tau = 0 \), Chen [13] considered the existence, uniqueness and global asymptotic stability of traveling wave solutions of (3) by the squeezing technique. For more similar results related to this technique, one can refer to Alikakos et al. [1], Berestycki and Nirenberg [6], Chen [12], Chen and Guo [14, 15], Ermentrout and McLeod [17], Evans et. al. [18], Fife and McLeod [19], Ma and Zou [27, 28] and Shen [37, 38]. When \( \tau > 0 \), \( g(u, v) = -\alpha u + v \), by using similar method in Smith and Zhao [39] to establish the existence and comparison theorem of solutions for (3), Ma and Wu [26] considered the uniqueness and global asymptotic stability of traveling wave solutions of (3) with under bistable assumption by means of the moving plane technique and the squeezing technique. In fact, Ma and Wu [26] studied the existence of traveling wave solutions with the help of a nonlocal system without time delay, see also Chen [13] for a similar technique, and then passing to (3). For more similar results related to this technique, one can refer to Chen [11] for a neural network model and Ou and Wu [33] for a delayed hyperbolic-parabolic model.

For some special cases of the following system

\[
\frac{\partial u(x, t)}{\partial t} = D \Delta u(x, t) - d(u(x, t)) + \int_0^\tau \int_{-\infty}^{\infty} \frac{1}{\sqrt{4\piDs}} e^{-\frac{(x-y)^2}{4Ds}} f(s)e^{-\gamma s}b(u(y, t-s))dyds,
\]

the existence of traveling wave solutions has been extensively studied by many researchers, one can refer to Gourley and Kuang [23] and Al-Omari and Gourley [3] for \( d(u(x, t)) = \alpha u^2(x, t) \) and \( b(u(x, t)) = \beta u(x, t) \), So et. al. [41] for \( d(u(x, t)) = \alpha u(x, t) \), \( b(u(x, t)) = \beta u(x, t)e^{-au(x,t)} \). It is easy to see that (4) with discrete delay is more general equation than (3).

In 2005, Ou and Wu [33] considered a delayed hyperbolic-parabolic model

\[
\frac{\partial u(x, t)}{\partial t} + r \frac{\partial^2 u(x, t)}{\partial t^2} = D \frac{\partial^2 u(x, t)}{\partial x^2} - du(x, t) + \varepsilon \int_{-\infty}^{\infty} f(x-y)b(u(y, t-\tau))dy + r \frac{\partial}{\partial t} \left[ \varepsilon \int_{-\infty}^{\infty} f(x-y)b(u(y, t-\tau))dy \right].
\]
where \( f(x) = \frac{1}{\sqrt{4\pi\alpha}} e^{-x^2/4\alpha} \), \( r > 0 \) was a time lag that the density of adult population \( u(x,t) \) moves randomly in space. For more about the usual structured population model including (5), one can refer to [8, 9, 34, 40, 45, 46]. When \( r = 0 \), So et al. [41] considered the existence of traveling wave front of (5) with monostable case by using the standard techniques involving sub- and supersolutions. The existence, uniqueness and asymptotic stability of a traveling wavefront to (5) were recently studied in [26] by using the comparison and squeezing techniques under bistable case. When \( r > 0 \), under bistable case, Ou and Wu [33] considered the existence and uniqueness of traveling wave fronts by considering the same problem for a properly parameterized parabolic system, and then by considering the continuous dependence of the wave speed on the parameter involved.

In [25], Liang and Wu derived a reaction advection diffusion equation with nonlocal delay

\[
\frac{\partial u(x,t)}{\partial t} = \frac{\partial}{\partial x} \left( D \frac{\partial u(x,t)}{\partial x} + Bu(x,t) \right) - du(x,t) + \varepsilon \int_{-\infty}^{\infty} J_\alpha(x + B\tau - y)b(u(y,t - \tau))dy,
\]

where \( J_\alpha(x) = \frac{1}{\sqrt{4\pi\alpha}} e^{-x^2/4\alpha}, \tau > 0 \) is the time delay, \( \varepsilon \) reflects the impact of the death rate of the immature, \( \alpha \) represents the effect of the dispersal rate of the immature on the growth rate of the matured population, and \( B \) is the velocity of the spatial transport field. By choosing three different birth functions \( b(u) \), they established the existence of traveling wave fronts of (6). We note that they only considered (6) with monostable nonlinearity.

In fact, some reaction-diffusion processes taking place in moving media such as fluids can be described by reaction advection diffusion equations, for example, combustion, atmospheric chemistry, and plankton distributions in the sea, see [5, 10, 20] and the references therein. We note that the advection terms on the propagation of traveling wave solutions play an important role, see [5, 20, 29, 30, 31]. However, the above results can not applied to the equations with time delay and nonlocal effect.

In [44], Wang, Li and Ruan considered the reaction advection diffusion equation with nonlocal delay

\[
\frac{\partial u(x,t)}{\partial t} = D \frac{\partial^2 u(x,t)}{\partial x^2} + Bu(x,t) + g(u(x,t), (h * b(u))(x,t)),
\]

where \( d > 0, B \in \mathbb{R}, h \) is a nonnegative kernel satisfying

\[
\int_0^\tau \int_{-\infty}^{\infty} h(y,s)dyds = 1, \quad \int_0^\tau \int_{-\infty}^{\infty} |y|h(y,s)dyds < \infty
\]

and the convolution is defined by

\[
(h * b(u))(x,t) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(x - y, -s)b(u(y,t + s))dyds.
\]

Under bistable assumption, they proved existence, uniqueness and globally asymptotic stability of traveling wave fronts. Their method is similar to Chen [13] and Smith and Zhao [39]. As applications, they also studied (4) and (6) under bistable case.

If we take into account the velocity of the spatial transport field in (5) or a time lag that the density of adult population \( u(x,t) \) moves randomly in space in (6), then...
the model (5) or (6) becomes the following form
\[
\begin{align*}
\frac{\partial u(x,t)}{\partial t} + r \frac{\partial^2 u(x,t)}{\partial t^2} &= \frac{\partial}{\partial x} \left( D_m \frac{\partial u(x,t)}{\partial x} + Bu(x,t) \right) - d_m u(x,t) \\
&\quad + \varepsilon \int_{-\infty}^{\infty} J_\alpha(x + B\tau - y)b(u(y,t - \tau))dy \\
&\quad + r \frac{\partial}{\partial t} \left[ \varepsilon \int_{-\infty}^{\infty} J_\alpha(x + B\tau - y)b(u(y,t - \tau))dy \right],
\end{align*}
\]
where \( J_\alpha(x) = \frac{1}{\sqrt{4\pi \alpha}} e^{-x^2/4\alpha}, \tau > 0, \alpha \) is the probability that a new born at time \( t - \tau \) and location 0 moves to the location \( x \) after maturation time \( \tau \), \( \varepsilon \in (0, 1] \) is the survival rate during the maturation period. \( r > 0 \) is a time lag that the density of adult population \( u(x,t) \) at time \( t \) and spatial location \( x \in \mathbb{R} \) of a given single species population with two age classes (the immature and mature with maturation time \( \tau > 0 \) being a constant) moves randomly in space, \( D_m \) and \( d_m > 0 \) are constant diffusion and death rates of the adult at time \( t \) and location \( x \), \( B \) is the velocity of the spatial transport field, \( b(u(x,t)) \) is the birth function.

In this paper we are interested in the existence and uniqueness of traveling wave solutions of (8). Motivated by [13, 25, 26, 33, 39, 44], we are concerned with more general advection hyperbolic-parabolic system with nonlocal delay
\[
\begin{align*}
\frac{\partial}{\partial t} u(x,t) + r \frac{\partial^2}{\partial t^2} u(x,t) &= D \frac{\partial^2}{\partial x^2} u(x,t) + B \frac{\partial}{\partial x} u(x,t) \\
&\quad + g(u(x,t), \int_{-\infty}^{\infty} h(x + B\tau - y)b(u(y,t - \tau))dy) \\
&\quad + r \frac{\partial}{\partial t} \left( \int_{-\infty}^{\infty} h(x + B\tau - y)b(u(y,t - \tau))dy \right),
\end{align*}
\]
where \( D > 0, B \in \mathbb{R}, r > 0 \) and \( h(x) = \frac{1}{\sqrt{4\pi \alpha}} e^{-x^2/4\alpha} \).

We make the following fundamental hypotheses:

(H1) \( g \in C^2(I \times I, \mathbb{R}) \) for some open interval \( I \subset \mathbb{R} \) with \([0, K] \subset I, K > 0 \) and \( \partial_2 g(u,v) > 0 \) for \((u,v) \in I \times I; b \in C^2(I, \mathbb{R})\) and \( b'(u) \geq 0 \) for \( u \in I; \)

(H2) \( g(0, b(0)) = g(K, b(K)) = 0, \partial_1 g(0, b(0)) + \partial_2 g(0, b(0))b'(0) < 0 \) and \( \partial_1 g(K, b(K)) + \partial_2 g(K, b(K))b'(K) < 0 \);

(H3) \( f \in C^4(I, \mathbb{R}) \) and \( f(u) > 0 \) for \( u \in I \).

The rest of this paper is organized as follows. In Section 2, we introduce associated parameterized advection parabolic system. In Section 3, by constructing a pair of super- and subsolutions and comparison principle, which are similar to [13, 39, 44], we consider the uniqueness of traveling wave solutions of associated system. In Section 4, by using the method in [13, 33, 44], the existence of traveling wave solutions for a class of reaction-diffusion equation without delay is established. Our key point is to investigate the continuity of wave speed \( C(c) \) of associated system and \( C(c) = c \) has a solution, then the existence of traveling wave solutions of (9) is obtained. In Section 5, we apply our results to (8).

2. Preliminaries. A traveling wave solution of (9) is a translation invariant solution of the special form \( u(x, t) = U(x + ct) \), where \( c \) is a given positive constant, \( U \in C^2(\mathbb{R}, \mathbb{R}) \) is the profiles of the wave that propagates through the one-dimensional spatial domain at a constant velocity \( c > 0 \). Substituting \( u(x, t) = U(x + ct) \) into (9), denoting \( x + ct \) by \( t \), and denoting \( x + B\tau - y \) by \( z \), then the corresponding wave equation is
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\[ cU'(t) + rc^2U''(t) = DU''(t) + BU'(t) \]
\[ + g\left(U(t), \int_{-\infty}^{\infty} h(t + B\tau - z)b(U(z))dz\right) \]
\[ + cr \left[f\left(\int_{-\infty}^{\infty} h(t + B\tau - z)b(U(z))dz\right)\right]' \]

or equivalently,
\[ c\left[U(t) - rf\left(\int_{-\infty}^{\infty} h(t + B\tau - z)b(U(z))dz\right)\right]' = (D - rc^2)U''(t) + BU'(t) + g\left(U(t), \int_{-\infty}^{\infty} h(t + B\tau - z)b(U(z))dz\right), \tag{10} \]

Let
\[ V(t) = U(t) - rf\left(\int_{-\infty}^{\infty} h(t + B\tau - z)b(U(z))dz\right). \tag{11} \]

Therefore, we obtain
\[ cV'(t) = D(c)V''(t) + BV'(t) \]
\[ + g\left(V(t) + rf\left(\int_{-\infty}^{\infty} h(t + B\tau - z)b(U(z))dz\right)\right), \]
\[ + B \left(\int_{-\infty}^{\infty} h(t + B\tau - z)b(U(z))dz\right) \]
\[ + rf\left(\int_{-\infty}^{\infty} h(t + B\tau - z)b(U(z))dz\right) \]
\[ \times \left(\int_{-\infty}^{\infty} h(t + B\tau - z)b(U(z))dz\right)^2 \]
\[ + f'\left(\int_{-\infty}^{\infty} h(t + B\tau - z)b(U(z))dz\right) \]
\[ \times \int_{-\infty}^{\infty} h'(t + B\tau - z)b(U(z))dz \]
\[ + rf'\left(\int_{-\infty}^{\infty} h(t + B\tau - z)b(U(z))dz\right) \]
\[ \times \int_{-\infty}^{\infty} h'(t + B\tau - z)b(U(z))dz, \tag{12} \]

where \( D(c) = D - rc^2 \).

We will consider the existence of traveling wave solutions for (2.2) and (2.3). It is equivalent to consider the following associated parabolic system
\[ \frac{\partial}{\partial t}w(x,t) = D(c)\frac{\partial^2}{\partial x^2}w(x,t) + B\frac{\partial}{\partial x}w(x,t) \]
\[ + g\left(w(x,t) + rf\left(\int_{-\infty}^{\infty} h(x + B\tau - y)b(\varphi(y,t))dy\right)\right), \]
\[ \int_{-\infty}^{\infty} h(x + B\tau - y)b(\varphi(y,t))dy \]
where $c \in \mathbb{R}$ satisfies
\begin{equation}
D(c) = D - rc^2 > 0. 
\end{equation}

We shall find the existence of traveling wave solution $w(x, t) = V(x+C(c)t), \varphi(x, t) = U(x+C(c)t)$ to (13) and (14) satisfying
\begin{equation}
V(-\infty) = 0, V(\infty) = V_{\text{max}} = K - r f(b(K)), \quad U(-\infty) = 0, U(\infty) = K, 
\end{equation}
where $K > 0$ is the maximal positive solution of equation $g(u, b(u)) = 0$, see Lemma 3.1 and Remark 1 for $V_{\text{max}}$. For any $c, r$ and $D$ satisfying (15) and another technical conditions (24) and (33) given in the next section, there exists a wave speed $C(c)$, denoting $x + C(c)t$ by $t$, such that $U(t)$ and $V(t)$ satisfying
\begin{equation}
C(c)V(t) = D(c)V''(t) + BV'(t) 
\end{equation}
\begin{equation}
+ f'\left(\int_{-\infty}^{\infty} h(t + B\tau - c\tau - z)b(U(z))dz\right) 
\end{equation}
\begin{equation}
+ f''\left(\int_{-\infty}^{\infty} h(t + B\tau - c\tau - z)b(U(z))dz\right)^2 
\end{equation}
\begin{equation}
+ rD(c)\left[f''\left(\int_{-\infty}^{\infty} h(t + B\tau - c\tau - z)b(U(z))dz\right) 
\end{equation}
\begin{equation}
\times \left(\int_{-\infty}^{\infty} h'(t + B\tau - c\tau - z)b(U(z))dz\right)^2 
\end{equation}
\begin{equation}
+ f\left(\int_{-\infty}^{\infty} h(t + B\tau - c\tau - z)b(U(z))dz\right) 
\end{equation}
\begin{equation}
\times \int_{-\infty}^{\infty} h''(t + B\tau - c\tau - z)b(U(z))dz] 
\end{equation}
\begin{equation}
+ rBf'\left(\int_{-\infty}^{\infty} h(t + B\tau - c\tau - z)b(U(z))dz\right) 
\end{equation}
\begin{equation}
\times \int_{-\infty}^{\infty} h'(t + B\tau - c\tau - z)b(U(z))dz 
\end{equation}
and
\begin{equation}
U(t) = V(t) + rf\left(\int_{-\infty}^{\infty} h(t + B\tau - c\tau - y)b(U(y))dy\right) 
\end{equation}
as well as the asymptotic boundary condition (16). If there exists \( c = C(c) \), then for such \( c > 0 \), we find a solution for (11) and (12).

3. Uniqueness of traveling wave solutions for associated system. For \( x \in \mathbb{R} \), we begin with the following more general system

\[
\begin{align*}
\frac{\partial}{\partial t} w(x,t) &= D(c) \frac{\partial^2}{\partial x^2} w(x,t) + B \frac{\partial}{\partial x} w(x,t) + I_\varphi(x,t), \\
\varphi(x,t) &= w(x,t) + rf \left( \int_{-\infty}^{\infty} h(x + B\tau - ct - y) b(\varphi(y,t))dy \right)
\end{align*}
\]

with the following initial data

\[
\begin{align*}
\begin{cases}
  w(x,s) = \phi(x,s), s \in [-\tau_1,0], \\
  \varphi(x,s) = w(x,s) + rf \left( \int_{-\infty}^{\infty} h(x + B\tau - ct - y) b(\varphi(y,s))dy \right), s \in [-\tau_1,0],
\end{cases}
\end{align*}
\]

where

\[
I_\varphi(x,t) = g \left( w(x,t) + rf \left( \int_{-\infty}^{\infty} h(\xi)b(\varphi(y,t-\tau_1))dy \right) \int_{-\infty}^{\infty} h(\xi)b(\varphi(y,t-\tau_1))dy \right)
\]

\[
+ rD(c) \left[ f'' \left( \int_{-\infty}^{\infty} h(\xi)b(\varphi(y,t-\tau_1))dy \right) \left( \int_{-\infty}^{\infty} h'(\xi)b(\varphi(y,t-\tau_1))dy \right)^2 \right]
\]

\[
+ f' \left( \int_{-\infty}^{\infty} h(\xi)b(\varphi(y,t-\tau_1))dy \right) \int_{-\infty}^{\infty} h''(\xi)b(\varphi(y,t-\tau_1))dy 
\]

\[+ rf' \left( \int_{-\infty}^{\infty} h(\xi)b(\varphi(y,t-\tau_1))dy \right) \int_{-\infty}^{\infty} h'(\xi)b(\varphi(y,t-\tau_1))dy, \]

\( \xi = x + B\tau - ct + B\tau_1 - y \) and \( \tau_1 > 0 \).

In this section, we shall prove that (19) has at most one traveling wave solution (up to translation) \( w(x,t) = V(x + C(c)t) \), \( \varphi(x,t) = U(x + C(c)t) \) with the wave speed \( C(c) \) depending on \( c \). In particular, when \( \tau_1 = 0 \) system (19) reduces to (13) and (14).

Let \( X = BUC(\mathbb{R}, \mathbb{R}) \) be the Banach space of bounded and uniformly continuous functions from \( \mathbb{R} \) into \( \mathbb{R} \) with the usual supremum norm. Let

\[ X^+ = \{ \phi \in X : \phi(x) \geq 0, x \in \mathbb{R} \}. \]

It is easy to see that \( X^+ \) is a closed cone of \( X \) and \( X \) is a Banach lattice under the partial ordering induced by \( X^+ \). By Theorem 1.5 in [16], it then follows that the \( X \)-realization \( D\Delta X \) of \( D\Delta \) generates a strongly continuous analytic semigroup \( T(t) \) on \( X \) and \( T(t)X^+ \subset X^+ \), \( t \geq 0 \). Moreover, by the explicit expression of solutions of the heat equation

\[
\begin{align*}
\begin{cases}
\frac{\partial}{\partial t} v(x,t) &= D \frac{\partial^2}{\partial x^2} v(x,t), \quad x \in \mathbb{R}, \quad t > 0, \\
v(x,0) &= \phi(x), \quad x \in \mathbb{R},
\end{cases}
\end{align*}
\]

we have

\[ T(t)\phi(x) = \frac{1}{\sqrt{4\pi Dt}} \int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4Dt}} \phi(y)dy, \quad x \in \mathbb{R}, \quad t > 0, \quad \phi(\cdot) \in X. \]
Consider the following equation
\[
\begin{align*}
\frac{\partial}{\partial t}w(x, t) &= D \frac{\partial^2}{\partial x^2}w(x, t) + B \frac{\partial}{\partial x}w(x, t), \quad x \in \mathbb{R}, \ t > 0, \\
w(x, 0) &= \phi(x), \quad x \in \mathbb{R}.
\end{align*}
\] (22)

The relation between (21) and (22) is as follows. If \(v(x, t)\) is the solution of (21), then \(w(x, t) = v(x + Bt, t)\) is a solution of (22). Inversely, if \(w(x, t)\) is a solution of (22), then \(v(x, t) = w(x - Bt, t)\) is a solution of (21). So the existence and uniqueness of solutions of (22) follows from the existence and uniqueness of solutions of (21). In particular,
\[
w(x, t) = v(x + Bt, t) = \frac{1}{\sqrt{4\pi D t}} \int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4Dt}} \phi(y)dy.
\]

Define bounded linear operator \(S(t) : X \to X, t \geq 0,\) by
\[
S(t)\phi(x) = \phi(x),
\]
\[
S(t)\phi(x) = \frac{1}{\sqrt{4\pi D t}} \int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4Dt}} \phi(y)dy, \quad x \in \mathbb{R}, \ t > 0, \phi(\cdot) \in X.
\] (23)

It is easy to show that \(S(t)\) is a strongly continuous semigroup on \(X.\) Obviously, \(S(t)X^+ \subset X^+\), \(t \geq 0.\) In particular, when \(B = 0, S(t) = T(t).\)

Let \(f_0(\cdot) : I \to \mathbb{R}\) be defined by \(f_0(u) = g(u, b(u)), u \in I.\) By the continuity of \(f_0\) and (H2), it easily follows that there exist \(\delta_0, u^\pm, u^+ \in (0, K)\) with \([-2\delta_0, K + 2\delta_0] \subset I\) and \(u^- \leq u^+\) such that \(f_0(\cdot) : [-2\delta_0, K + 2\delta_0] \to \mathbb{R}\) satisfies
\[
\begin{align*}
&\begin{cases} 
\delta_0 = f_0(u^-) = f_0(u^+) = f_0(K) = 0, \\
f_0(u) > 0 \text{ for } u \in [-\delta_0, 0) \cup (u^+, K) \text{ and } f_0(u) < 0 \text{ for } u \in (0, u^-) \cup (K, K + 2\delta_0). 
\end{cases}
\end{align*}
\]

Let \(L_1 = \max\{|\partial_t g(u, v)| : -2\delta_0 \leq u \leq K + 2\delta_0, b(-2\delta_0) \leq v \leq b(K + 2\delta_0)\}\) and define
\[
\Theta(J, t) = \frac{1}{\sqrt{4\pi D t}} e^{-\frac{|J|^2}{4Dt}}
\]
\[
\left| \frac{J}{\sqrt{4Dt}} \right| - \left( L_1 + \frac{|J|}{\sqrt{4Dt}} \right) 1 - \frac{|J|}{\sqrt{4Dt}}^2, \quad J \geq 0, \ t > 0.
\]

Clearly, \(\Theta \in C([0, \infty) \times (0, \infty), \mathbb{R}).\)

Let \(C = C([-\tau_1, 0], X)\) be the Banach space of continuous functions from \([-\tau_1, 0]\) into \(X\) with the supremum norm, and let \(C^+ = \{\phi \in C : \phi(s) \in X^+, s \in [-\tau_1, 0]\}\). Then \(C^+\) is a positive cone of \(C.\) As usual, we identify an element \(\phi \in C\) as a function from \(\mathbb{R} \times [-\tau_1, 0]\) into \(\mathbb{R}\) defined by \(\phi(x, s) = \phi(s)(x).\) For any continuous function \(w : [-\tau_1, a] \to X, a > 0,\) we define \(w_t \in C, t \in [0, a],\) by \(w_t(s) = w(t + s), s \in [-\tau_1, 0].\) Then \(t \mapsto w_t\) is a continuous function from \([0, a]\) to \(C.\)

To show the existence and positivity of solutions to (19) and (20) when \(\phi \in C^+,\) we need some order preserving property for the second equation in (19). For any \(v \in X,\) we consider an operator \(R : X \to X\) defined by
\[
(Rv)(x) = v(x) + rf \left( \int_{-\infty}^{\infty} h(x + B\tau - c\tau - y)b(u(y))dy \right).
\]

Let
\[
X_{\delta_0} = \{\phi \in X : -\delta_0 \leq \phi(x) \leq K + \delta_0, \ x \in \mathbb{R}\}.
\]

We also assume that
\[
f'_{\text{max}}k'_{\text{max}} = \sup\{|f'(b(u))b'(u)| : -\infty \leq u < \infty\} > 0.
\]
The following lemma is similar to Lemma 3.1 in [33] and we omit the proof here.

**Lemma 3.1.** Assume that \( r \) is sufficiently small so that

\[
r < \frac{\delta_0}{(K + 2\delta_0)f'_{\text{max}}'v_{\text{max}}}'.
\]

Then

(i) for every \( v \in X \), the equation

\[
u(x) = v(x) + rf\left(\int_{-\infty}^{\infty} h(x + Br - cr - y)b(u(y))dy\right)
\]

has one and only one solution \( u = F(v) \in X \). In particular, if \( v \in X^+ \), then

\( u = F(v) \in X^+ \);

(ii) for every \( v \in X_{\delta_0}, F(v) \in X_{2\delta_0} \);

(iii) \( F(v) - F(\bar{v}) \in X^+ \), if \( v, \bar{v} \in X_{\delta_0} \);

(iv) if \( v \in X_{\delta_0} \) is nondecreasing on \( \mathbb{R} \), then so is \( F(v) \);

(v) for \( v, \bar{v} \in X_{\delta_0} \), we have

\[
\| F(v) - F(\bar{v}) \| \leq 2 \| v - \bar{v} \| ;
\]

(vi) if \( v \) is a constant function of value \( \bar{v} \), then (3.6) reduces to the algebraic equation

\[
\ddot{u} = \bar{v} + rf(b(\bar{u})),
\]

which has a unique solution \( \bar{u} \) such that \( \bar{u} = F(\bar{v}) \). Moreover,

\[
\frac{d\bar{u}}{d\bar{v}} = F'(\bar{v}) = \frac{1}{1 - rf'(b(\bar{u}))b'(\bar{u})}.
\]

**Remark 1.** We can see from Lemma 3.1 that \( 0 \leq v < V_{\text{max}} = K - rf(b(K)) \) when \( 0 \leq u < K, V_{\text{max}} = K - rf(b(K)) \) and \( g(K, b(K)) = 0 \).

**Remark 2.** For any fixed \( t \) in system (19), we can solve the second equation by Lemma 3.1 to obtain

\[
\varphi(x, t) = F(w(x, t)).
\]

Thus system (19) with initial data (20) can be transformed into

\[
\begin{aligned}
&\frac{\partial}{\partial t}w(x, t) = D(c)\frac{\partial^2}{\partial x^2}w(x, t) + B\frac{\partial}{\partial x}w(x, t) + IF(w(x, t)), \\
&w(x, s) = \varphi(x, s).
\end{aligned}
\]

Now we return to system (19). For any \( \phi \in [-\delta_0, K + \delta_0] \), \( \{ \phi \in C : \phi(x, s) \in [-\delta_0, K + \delta_0], x \in \mathbb{R}, s \in [-\tau_1, 0] \} \), define \( F_1(\phi)(x) = I_{F(\phi)(x, 0)} \). The definition of \( I_{F(\phi)(x, 0)} \) is similar to \( I_{\varphi(x, t)} \), in what follows, all the similar definitions will not be given. By the global Lipschitz continuity of \( g(\cdot, \cdot) \) on \([{-2\delta_0, K + 2\delta_0}] \times [b(-2\delta_0), b(K + 2\delta_0)]\), we can verify that \( F_1(\phi) \in X \) and \( F_1 : [-\delta_0, K + \delta_0] \rightarrow X \) is globally Lipschitz continuous.

**Definition 3.2.** A continuous function pair \((w, \varphi) = (w, F(w)) \in C([-\tau_1, a], X) \times C([-\tau_1, a], X) \) is called a supersolution (subsolution) of (19) if

\[
\begin{aligned}
w(t) &\geq (\leq)S(t - t_0)w(t_0) + \int_{t_0}^{t}(\leq)S(t - s)F_1(w_s)ds, \\
\varphi(t) &= F(w)(t), \quad t \geq t_0 - \tau_1
\end{aligned}
\]

for all \( 0 \leq t_0 < t < a \). We call \((w, F(w))\) as a mild solution of (3.1) if it is both a supersolution and a subsolution on \([0, a]\).
Remark 3. Assume that there is a bounded and continuous function pair \((w, \varphi)\) defined on \(\mathbb{R} \times [-\tau_1, 1]\) that are \(C^2\) in \(x \in \mathbb{R}\) and \(C^1\) in \(t \in (0, a)\) and
\[
\begin{align*}
\frac{\partial}{\partial t} w(x, t) &\geq (\leq) D(c) \frac{\partial^2}{\partial x^2} w(x, t) + B \frac{\partial}{\partial x} w(x, t) + I_{\varphi(x, t)}, \ t \geq 0, \\
\varphi(x, t) &= w(x, t) + rf \left( \int_{-\infty}^{\infty} h(x + Br - ct - y)b(\varphi(y, t))dy \right), \ t \geq -\tau_1
\end{align*}
\]
for \(x \in \mathbb{R}\). Then (30) holds by \(S(t)X^+ \subset X^+\), so \((w, \varphi)\) is a supersolution (subsolution) of (19) on \([0, a]\).

Now we give the following existence of solution and comparison theorem.

Lemma 3.3. For any initial value \(\phi \in [-\delta_0, \delta_0]_C\), system (19) with (20) have a mild solution \((w(x, t, \phi), F(w(x, t, \phi)))\) for \(t \in [0, \infty)\) with \((-\delta_0, F(-\delta_0)) \leq (w(x, t, \phi), F(w(x, t, \phi))) \leq (K + \delta_0, F(K + \delta_0))\) in the sense
\[-\delta_0 \leq w(x, t, \phi) \leq K + \delta_0, \ F(-\delta_0) \leq F(w(x, t, \phi)) \leq F(K + \delta_0),\]
and \((w(x, t, \phi), F(w(x, t, \phi)))\) is a classical solution to (19) and (20) for \((x, t) \in \mathbb{R} \times (-\tau_1, \infty)\). Moreover, for any pair of supersolution \((w^+, F(w^+))\) and subsolution \((w^-, F(w^-))\) of (19) and (20) with \(-\delta_0 \leq w^+(x, t), w^-(x, t) \leq K + \delta_0\) for \(t \in (-\tau_1, \infty)\) and \(x \in \mathbb{R}\), and \(w^+(x, s) \geq w^-(x, s)\) for \(x \in \mathbb{R}\) and \(s \in [-\tau_1, 0]\), \(w^+(x, t) \geq w^-(x, t)\) holds for all \(x \in \mathbb{R}\) and \(t \geq 0\), and
\[
w^+(x, t) - w^-(x, t) \geq \Theta(\|x - z\|, t - t_0) \int_{z}^{z+1} (w^+(y, t_0) - w^-(y, t_0))dy
\]
for every \(z \in \mathbb{R}\) and \(t \geq t_0 \geq 0\).

Proof. We know that \(\varphi(x, t) = F(w(x, t))\). It follows from [32] that a mild solution \((w, F(w))\) of (19) and (20) is a solution of the following integral equation
\[
\begin{align*}
\begin{cases}
\text{w}(t) = S(t - t_0)w(t_0) + \int_{t_0}^{t} S(t - s)F_1(w_s)ds, \\
\text{w}_0 = \phi \in [-\delta_0, K + \delta_0]_C.
\end{cases}
\end{align*}
\]
By the choice of \(\delta_0\), we have \(f_0(K + 2\delta_0) < 0\) and \(f_0(-2\delta_0) > 0\). Clearly, \(v^+ = (K + \delta_0, F(K + \delta_0))\) and \(v^- = (-\delta_0, F(-\delta_0))\) are supersolution and subsolution of (19) and (20), respectively. Notice that \(F_1 : [-\delta_0, K + \delta_0]_C\) is globally Lipschitz continuous. It also satisfies the quasi-monotone condition in the sense that
\[
\lim_{\rho \to 0^+} \frac{1}{\rho} \text{dist}(\psi(0) - \varphi(0) + \rho[F_1(\psi_s) - F_1(\varphi_s)], X^+) = 0
\]
for all \(\psi, \varphi \in [-\delta_0, K + \delta_0]_C\) with \(\psi \geq \varphi\). Let
\[
\begin{align*}
I_\psi &= \int_{-\infty}^{\infty} h(\xi)b(F(\psi)(y, -\tau_1))dy, \ I_\varphi = \int_{-\infty}^{\infty} h(\xi)b(F(\varphi)(y, -\tau_1))dy, \\
I'_\psi &= \int_{-\infty}^{\infty} h'(\xi)b(F(\psi)(y, -\tau_1))dy, \ I'_\varphi = \int_{-\infty}^{\infty} h'(\xi)b(F(\varphi)(y, -\tau_1))dy, \\
I''_\psi &= \int_{-\infty}^{\infty} h''(\xi)b(F(\psi)(y, -\tau_1))dy, \ I''_\varphi = \int_{-\infty}^{\infty} h''(\xi)b(F(\varphi)(y, -\tau_1))dy,\end{align*}
\]
hence,
\[ g(\psi(x, 0) + rf(I_\psi), I_\psi) - g(\varphi(x, 0) + rf(I_\varphi), I_\varphi) \]
\[ = \partial_1 g(\xi_1(x, 0), I_\psi)[\psi(x, 0) - \varphi(x, 0)] \]
\[ + \int_{-\infty}^x \{ r f'(\xi_2(x, 0)) + \partial_2 g(\varphi(x, 0) + rf(I_\varphi), \xi_3(x, 0)) + r B f''(\xi_4(x, 0)) I_\psi' \]
\[ + r D(c) f'''(\xi_5(x, 0)) (I_\psi')^2 + r D(c) f'''(\xi_6(x, 0)) I_\psi'' - \frac{1}{2\alpha} r D(c) f'(I_\varphi) \]
\[ - \frac{\xi}{2\alpha} \{ r B f'(I_\psi) + r D(c) f'''(I_\psi') (I_\psi' + I_\varphi') + \frac{\xi^2}{4\alpha^2} r D(c) f'(I_\varphi) \} \]
\[ \times h(\xi) [b(F(\psi)(y, -\tau_1)) - b(F(\varphi)(y, -\tau_1))] dy \]
\[ \geq -L_1 [\psi(x, 0) - \varphi(x, 0)] + \int_{-\infty}^x (L_2 - r M_1 - r M_2 |\xi| + r M_3 \xi^2) \]
\[ \times h(\xi) [b(F(\psi)(y, -\tau_1)) - b(F(\varphi)(y, -\tau_1))] dy, \]
where
\[ L_1 = \max \{ \partial_1 g(u, v) : (u, v) \in [-2\delta_0, K + 2\delta_0] \times [b(-2\delta_0), b(K + 2\delta_0)] \}, \]
\[ L_2 = \min \{ \partial_2 g(u, v) : (u, v) \in [F(-\delta_0), F(K + \delta_0)] \} \times [b(F(-\delta_0)), b(F(K + \delta_0))] > 0, \]
\[ L_2 = \max \{ \partial_2 g(u, v) : (u, v) \in [F(-\delta_0), F(K + \delta_0)] \} \times [b(F(-\delta_0)), b(F(K + \delta_0))] > 0, \]
\[ M_1 = \max \{ |f'(\xi_2(x, 0)) + B f''(\xi_4(x, 0)) I_\psi' + D(c) f'''(\xi_5(x, 0)) (I_\psi')^2 \]
\[ + D(c) f'''(\xi_6(x, 0)) I_\psi'' - \frac{1}{2\alpha} D(c) f'(I_\varphi) | : I_\psi, I_\varphi \]
\[ \in [b(F(-\delta_0)), b(F(K + \delta_0))], \]
\[ M_2 = \max \{ |B f'(I_\psi) + D(c) f'''(I_\psi') (I_\psi' + I_\varphi') | : I_\psi, I_\varphi \}
\[ \in [b(F(-\delta_0)), b(F(K + \delta_0))], \]
\[ M_3 = \min \left\{ \frac{1}{4\alpha^2} D(c) f'(u) : u \in [b(F(-\delta_0)), b(F(K + \delta_0))] \right\} > 0, \]
\[ M_3 = \max \left\{ \frac{1}{4\alpha^2} D(c) f'(u) : u \in [b(F(-\delta_0)), b(F(K + \delta_0))] \right\} > 0. \]
Now we choose $r$ sufficiently small such that
\[ L_2 - rM_1 - rM_2 |\xi| + rM_3 \xi^2 \geq 0 \]  
for all $\xi \in \mathbb{R}$. Therefore
\[ F_1(\psi_x) - F_1(\varphi_x) \geq -L_1 [\psi(x,0) - \varphi(x,0)], \]
and hence, for any $h > 0$, with $L_1 h < 1$,
\[ \psi(0) - \varphi(0) + h[F_1(\psi_x) - F_1(\varphi_x)] \geq (1 - L_1 h) [\psi(x,0) - \varphi(x,0)] \geq 0, \]
which implies that (32) holds. Hence, it follows that the existence and uniqueness of $w(x,t,\phi)$ by Corollary 5 in [32] with $S(t,s) = T(t,s) = T(t-s)$ for $t \geq s \geq 0$ and $B(t,\phi) = F_1(\phi)$. Moreover, by using a similar argument to Theorem 1 in [32], it follows that $(w,F(w))$ is a classical solution for $t \geq \tau_1$.

Since $(w^+(x,t), F(w^+)(x,t)) \geq (w^-(x,t), F(w^-)(x,t))$, it follows from Corollary 5 in [32] that
\[ -\delta_0 \leq w(x,t,w^-) \leq w(x,t,w^+) \leq K + \delta_0, \quad t \geq 0, \quad x \in \mathbb{R}. \]

By applying Corollary 5 in [32] with $v^+(x,t) = K + \delta_0$ and $v^-(x,t) = w^-(x,t)$, $v^+(x,t) = w^+(x,t)$ and $v^-(x,t) = -\delta_0$, respectively, we get
\[ w^+(x,t) \leq w(x,t,w^-) \leq K + \delta_0, \quad t \geq 0, \quad x \in \mathbb{R} \]
and
\[ -\delta_0 \leq w(x,t,w^+) \leq w^+(x,t), \quad t \geq 0, \quad x \in \mathbb{R}. \]

Combining (34)-(36), we have $w^-(x,t) \leq w^+(x,t)$ for all $t \geq 0$ and $x \in \mathbb{R}$, when $B = 0$.

Now we consider the case $B \neq 0$. Define $\bar{h}(y) = h(y - B\tau)$ for $y \in \mathbb{R}$. Consider the initial value problem
\[
\begin{aligned}
\frac{\partial}{\partial t} w(x,t) &= D(c) \frac{\partial^2}{\partial x^2} w(x,t) + \tilde{I}_\varphi(x,t), \\
w(x,0) &= \varphi \in [-\delta_0, K + \delta_0],
\end{aligned}
\]
where
\[
\tilde{I}_\varphi(x,t) = g \left( \int_{-\infty}^{\infty} \bar{h}(\xi) \delta_b(\varphi(y,t - \tau_1))dy \right) \int_{-\infty}^{\infty} \bar{h}(\xi) \delta_b(\varphi(y,t - \tau_1))dy
\]
\[
+ r\bar{D}(c) \left[ f'' \left( \int_{-\infty}^{\infty} \bar{h}(\xi) \delta_b(\varphi(y,t - \tau_1))dy \right) \int_{-\infty}^{\infty} \bar{h}(\xi) \delta_b(\varphi(y,t - \tau_1))dy \right]^2
\]
\[
+ f' \left( \int_{-\infty}^{\infty} \bar{h}(\xi) \delta_b(\varphi(y,t - \tau_1))dy \right) \int_{-\infty}^{\infty} \bar{h}(\xi) \delta_b(\varphi(y,t - \tau_1))dy
\]
\[
+ rBf' \left( \int_{-\infty}^{\infty} \bar{h}(\xi) \delta_b(\varphi(y,t - \tau_1))dy \right) \int_{-\infty}^{\infty} \bar{h}(\xi) \delta_b(\varphi(y,t - \tau_1))dy
\]
with $\bar{x} = x + B\tau - ct - y$.

By the relation between (19) and (37), we have the following fact. If $w(x,t)$ is a (mild) solution of (19) with initial value $\phi \in [-\delta_0, K + \delta_0]$, then $w(x-Bt,t)$ is a (mild) solution of (37). Conversely, if $w(x,t)$ is a (mild) solution of (37), then $w(x+Bt,t)$ is a solution of (19) with initial value $\phi \in [-\delta_0, K + \delta_0]$. Moreover, if $w(x,t)$ is a supersolution (subsolution) of (19), then $w(x-Bt,t)$ is a supersolution (subsolution) of (37). Conversely, if $w(x,t)$ is a supersolution (subsolution) of (37), then $w(x+Bt,t)$ is a supersolution (subsolution) of (19). Applying the conclusions
for the case $B = 0$ to (37), the results hold for $B \neq 0$ except the last inequality in this lemma.

Now we prove the last inequality in this lemma. Let $v(x, t) = w^+(x, t) - w^-(x, t), x \in \mathbb{R}, t \in [-\pi, \infty)$. Clearly, $w^+_t, w^-_t \in [-\delta_0, K + \delta_0]C$ with $w^+_t \geq w^-_t$ in $C$ for all $t \geq 0$. For any given $t_0 \geq 0$, it follows from [32] that

$$v(t) \geq S(t - t_0)v(t_0) + \int_{t_0}^{t} S(t - s)[F_1(w^+_s) - F_1(w^-_s)] ds$$

$$\geq S(t - t_0)v(t_0) - L_1 \int_{t_0}^{t} S(t - s)v(s) ds. \tag{38}$$

Since $v(t) = \exp(-L_1(t - t_0))S(t - t_0)v(t_0)$ satisfies the following integral equation

$$v(t) = S(t - t_0)v(t_0) - L_1 \int_{t_0}^{t} S(t - s)v(s) ds,$$

it follows from proposition 3 [32] that

$$w^+(t) - w^-(t) \geq e^{-L_1(t - t_0)}S(t - t_0)(w^+(t_0) - w^-(t_0)), \quad t \geq t_0. \tag{39}$$

Combining (23), (39) with the definition of $\Theta(J, t)$, it follows that for $t \geq t_0 \geq 0$ and $x \in \mathbb{R},$

$$w^+(x, t) - w^-(x, t) \geq \Theta(|x - z|, t - t_0) \int_{z}^{z+1} [w^+(y, t_0) - w^-(y, t_0)] dy.$$

This completes the proof. \hfill \Box

**Remark 4.** It is easy to see from Lemma 3.3 that if $w^+(x, 0) \neq w^-(x, 0)$, then for any $t > 0,$

$$w^+(x, t) - w^-(x, t) \geq \Theta(|x - z|, t) \int_{z}^{z+1} [w^+(y, 0) - w^-(y, 0)] dy > 0.$$

In particular, if $(w(x, t, \phi), F(w)(x, t, \phi))$ is a solution of (3.1) with the initial data $\phi \in [-\delta_0, K + \delta_0]C$ and $\phi(\neq$ constant) is nondecreasing on $\mathbb{R}$, then for any fixed $t > 0, w(x, t)$ is strictly increasing in $x \in \mathbb{R}.$

Now we estimate the derivative for the traveling wave solutions.

**Lemma 3.4.** Let $(V(x + C(c)t), F(V)(x + C(c)t))$ be a nondecreasing traveling wave solution of (19). Then

$$0 < V'(\xi) \leq \frac{G}{2\sqrt{D(c)L_1}} \text{ and } \lim_{|\xi| \to \infty} V'(\xi) = 0,$$

where $G = L_1 + \max\{|g(u, v)| : (u, v) \in [F(-\delta_0), F(K + \delta_0)] \times [b(F(-\delta_0)), b(F(K + \delta_0))]\}$.

**Proof.** By Lemma 3.3, We have that for $\xi = x + C(c)t$ and every $h > 0,$

$$V(\xi + h) - V(\xi) \geq \max_{z \in \mathbb{R}} \Theta(|x - z|, t) \int_{z}^{z+1} [V(y + h) - V(y)] dy > 0,$$

which implies that

$$V'(\xi) \geq \max_{z \in \mathbb{R}} \Theta(|x - z|, t) [V(z + 1) - V(z)] > 0.$$
Let
\[ \lambda_1 = \frac{C(c) - B - \sqrt{(C(c) - B)^2 + 4D(c)L_1}}{2D(c)} < 0, \]
\[ \lambda_2 = \frac{C(c) - B + \sqrt{(C(c) - B)^2 + 4D(c)L_1}}{2D(c)} > 0. \]
Then
\[ V(\xi) = \frac{1}{D(c)(\lambda_2 - \lambda_1)} \left( \int_{-\infty}^{\xi} e^{\lambda_1(\xi-s)}H(V)(s)ds + \int_{\xi}^{\infty} e^{\lambda_2(\xi-s)}H(V)(s)ds \right) \]
and
\[ V'(\xi) = \frac{1}{D(c)(\lambda_2 - \lambda_1)} \left( \lambda_1 \int_{-\infty}^{\xi} e^{\lambda_1(\xi-s)}H(V)(s)ds + \lambda_2 \int_{\xi}^{\infty} e^{\lambda_2(\xi-s)}H(V)(s)ds \right) \leq \ \frac{\lambda_2}{D(c)(\lambda_2 - \lambda_1)} \int_{\xi}^{\infty} e^{\lambda_2(\xi-s)}H(V)(s)ds, \]
where
\[ H(V)(\xi) = L_1V(\xi) + g(V(\xi) + rf \left( \int_{-\infty}^{\infty} h(\eta)b(F(V)(\eta))d\eta \right), \int_{-\infty}^{\infty} h(\eta)b(F(V)(\eta))d\eta \right) \]
\[ + rD(c) \left( \int_{-\infty}^{\infty} h(\eta)b(F(V)(\eta))d\eta \right)^2 \int_{-\infty}^{\infty} h(\eta)b(F(V)(\eta))d\eta \right) \]
\[ + rBf \left( \int_{-\infty}^{\infty} h(\eta)b(F(V)(\eta))d\eta \right) \int_{-\infty}^{\infty} h(\eta)b(F(V)(\eta))d\eta \right), \]
where \( \eta = \xi + Br - cr + B\tau_1 - C(c)\tau_1 - z. \)

Since \( \lambda_2 - \lambda_1 = \sqrt{(C(c) - B)^2 + 4D(c)L_1}/D(c) \geq 2\sqrt{L_1/D(c)}, \) it is easy to prove that
\[ |V'(\xi)| \leq \frac{G}{2\sqrt{D(c)L_1}}. \]

It follows from the dominant convergence theorem that \( \lim_{|\xi| \to \infty} V'(\xi) = 0. \) The proof is completed.

**Lemma 3.5.** Let \( (V(x + C(c)t), F(V)(x + C(c)t)) \) be a nondecreasing traveling wave solution of (19). Then there exist \( \beta_0 > 0 \) (which is independent of \( V \)), \( \sigma_0 > 0 \) and \( \delta > 0 \) such that for every \( \delta \in (0, \delta] \) and every \( \xi_0 \in \mathbb{R} \), the functions \( (w^+, F(w^+)) \) and \( (w^-, F(w^-)) \) defined by
\[ w^\pm(x, t) := V(x + C(c)t + \xi_0 + \sigma_0\delta (1 - e^{-\beta_0t})) \pm \delta e^{-\beta_0t} \]
are a supersolution and a subsolution of (19) and (20) on \( t \in [0, +\infty) \), respectively.

**Proof.** Obviously, \( 0 < V(\xi) < V_{\text{max}} \). So \( 0 < V(x + C(c)t) < V_{\text{max}} \) for \( x \in \mathbb{R}, t \in \mathbb{R} \). It follows from Lemma 3.3 and the monotonicity of \( V(\cdot) \) that \( V(\cdot) \in C^1(\mathbb{R}) \) and \( V'(\xi) > 0, \xi \in \mathbb{R} \). Since
\[ \lim_{(u,v,l,s,w,\beta) \to (0,0,0,0,0,0,0,0)} \left[ \partial_1 g(u,v) + e^{\beta\tau_1} e^{\beta^2} \partial_2 g(l,s) + \beta \right] \]
\[ = \partial_1 g(0,0) + e^{\beta\tau_1} e^{\beta^2} \partial_2 g(0,0) < 0 \]
and
\[
\lim_{(u,v,l,s,\varpi,\beta)\to(K,b(K),K,b(K),b'(K),0)} [\partial_t g(u, v) + e^{\beta \tau_1} \varpi \partial_2 g(l, s) + \beta] \\
= \partial_t g(K, b(K)) + e^{\beta \tau_1} b'(K) \partial_2 g(K, b(K)) < 0,
\]
for fixed \(\beta_0 > 0\) and \(0 < \delta^* < 2\delta_0\) such that
\[
\partial_t g(u, b(u)) + e^{\beta \tau_1} b'(u) \partial_2 g(u, b(u)) < -\beta_0
\]
(40)
for all
\[
(u, v, l, s, \varpi) \in [-\delta^*, \delta^*] \times [b(0) - \delta^*, b(0) + \delta^*] \times [-\delta^*, \delta^*] \\
\times [b(0) - \delta^*, b(0) + \delta^*] \times [b'(0) - \delta^*, b'(0) + \delta^*]
\]
and
\[
(u, v, l, s, \varpi) \in [K - \delta^*, K + \delta^*] \times [b(K) - \delta^*, b(K) + \delta^*] \times [K - \delta^*, K + \delta^*] \\
\times [b(K) - \delta^*, b(K) + \delta^*] \times [b'(K) - \delta^*, b'(K) + \delta^*].
\]
By
\[
\lim_{(\xi, \delta)\to(\infty, 0)} \int_{-\infty}^{\infty} h(y) b(F(V(\xi + B \tau - c \tau + B \tau_1 - C(c) \tau_1 - y) + \delta)) dy = b(K),
\]
\[
\lim_{(\xi, \delta)\to(-\infty, 0)} \int_{-\infty}^{\infty} h(y) b(F(V(\xi + B \tau - c \tau + B \tau_1 - C(c) \tau_1 - y) + \delta)) dy = b(0),
\]
\[
\lim_{(\xi, \delta)\to(\infty, 0)} \int_{-\infty}^{\infty} h(y) b'(F(V(\xi + B \tau - c \tau + B \tau_1 - C(c) \tau_1 - y) + \delta)) dy = b'(K),
\]
\[
\lim_{(\xi, \delta)\to(-\infty, 0)} \int_{-\infty}^{\infty} h(y) b'(F(V(\xi + B \tau - c \tau + B \tau_1 - C(c) \tau_1 - y) + \delta)) dy = b'(0),
\]
there exist \(M_0 = M_0(V, \beta_0, \delta^*) > 0\) and \(\hat{\delta} = \hat{\delta}(V, \beta_0, \delta^*) \in (0, \delta^*)\) such that for all \(\xi \geq M_0\) and \(\delta \in [0, \hat{\delta}],\)
\[
F(V(\xi)) \geq K - \delta^*,
\]
\[
b(K) + \delta^* \geq \lim_{(\xi, \delta)\to(\infty, 0)} \int_{-\infty}^{\infty} h(y) b(F(V(\xi + B \tau - c \tau + B \tau_1 - C(c) \tau_1 - y) + \delta)) dy \\
\geq b(K) - \delta^*,
\]
\[
b'(K) + \delta^* \geq \lim_{(\xi, \delta)\to(\infty, 0)} \int_{-\infty}^{\infty} h(y) b'(F(V(\xi + B \tau - c \tau + B \tau_1 - C(c) \tau_1 - y) + \delta)) dy \\
\geq b'(K) - \delta^*.
\]
(41)
and for all \(\xi \leq -M_0\) and \(\delta \in [0, \hat{\delta}],\)
\[
F(V(\xi)) \leq \delta^*,
\]
\[
b(0) - \delta^* \leq \lim_{(\xi, \delta)\to(\infty, 0)} \int_{-\infty}^{\infty} h(y) b(F(V(\xi + B \tau - c \tau + B \tau_1 - C(c) \tau_1 - y) + \delta)) dy \\
\leq b(0) + \delta^*,
\]
\[
b'(0) - \delta^* \leq \lim_{(\xi, \delta)\to(\infty, 0)} \int_{-\infty}^{\infty} h(y) b'(F(V(\xi + B \tau - c \tau + B \tau_1 - C(c) \tau_1 - y) + \delta)) dy \\
\leq b'(0) + \delta^*.
\]
(42)
Let
\[ c_1 = c_1(\beta_0, \delta^*) = \max \left\{ |\partial_t g(u, v)| + \kappa e^{\beta_0 \tau_1} \int_{-\infty}^{\infty} |\partial_2 g(l, s)| + rM_1 + rM_2|y| + r\tilde{M}_3 y^2 h(y)dy : u, l \in [0, K + \delta^*], v, s \in [b(0), b(K + \delta^*)] \right\} \]
and
\[ m_0 = m_0(V, \beta_0, \delta^*) = \min \{ V'(\xi) : |\xi| \leq M_0 \} > 0, \]
where \( \kappa = \max \{ b'(u) : u \in [0, K + \delta^*] \} > 0 \) and define
\[ \sigma_0 = \sigma_0(V, \beta_0, \delta^*) = \frac{\beta_0 + c_1}{m_0 \beta_0}, \quad \delta = \min \left\{ \delta e^{-\beta_0 \tau_1}, (F^{-1}(K + \delta^*) - V_{max})e^{-\beta_0 \tau_1} \right\}. \]

We only need to show that \( w^+(x, t) \) is a supersolution of (19) and \( w^-(x, t) \) is similar.

In the following, we show that the following inequality holds for \( (x, t) \in \mathbb{R} \times [0, \infty) \),
\[
\frac{\partial w^+(x, t)}{\partial t} - D(c) \frac{\partial^2 w^+(x, t)}{\partial x^2} - B \frac{\partial w^+(x, t)}{\partial x} - I_{F(w^+)(x, t)} \geq 0. \tag{43}
\]
For any given \( \delta \in (0, \bar{\delta}), t \geq 0 \), let \( \xi(x, t) = x + C(c)t + \xi_0 + \sigma_0 \delta(1 - e^{-\beta_0 t}) \),
\[
\frac{\partial w^+(x, t)}{\partial t} - D(c) \frac{\partial^2 w^+(x, t)}{\partial x^2} - B \frac{\partial w^+(x, t)}{\partial x} - I_{F(w^+)(x, t)}
= V'(\xi(x, t))(C(c) + \sigma_0 \delta \beta_0 e^{-\beta_0 t} - \beta_0 \delta e^{-\beta_0 t})
- D(c)V''(\xi(x, t)) - BV'(\xi(x, t)) - I_{F(w^+)(x, t)}
=(\sigma_0 V'(\xi(x, t)) - 1) \beta_0 \delta e^{-\beta_0 t} + I_V - I_{F(w^+)(x, t)} \tag{44}
\geq \delta e^{-\beta_0 t} \left[ \sigma_0 V'(\xi(x, t)) - \beta_0 - \left| \partial_t g(\xi(x, t), \int_{-\infty}^{\infty} h(y)b(F(V)(\eta_1))dy) \right| \right]
- e^{\beta_0 \tau_1} \left[ \int_{-\infty}^{\infty} (|\partial_2 g(l, s)| + rM_1 + rM_2|y| + r\tilde{M}_3 y^2 h(y)b'(\eta))dy \right],
\]
where the definitions of \( M_1, M_2 \) and \( \tilde{M}_3 \) are similar to that of Lemma 3.3 with \( l \in [0, K + \delta^*], s \in [b(0), b(K + \delta^*)] \),
\[
I_V = g(F(V)(\xi(x, t))), \int_{-\infty}^{\infty} h(y)b(F(V)(\eta_1))dy
+ rD(c) \left[ f'' \left( \int_{-\infty}^{\infty} h(y)b(F(V)(\eta_1))dy \right) \left( \int_{-\infty}^{\infty} h(y)b(F(V)(\eta_1))dy \right)^2 + f' \left( \int_{-\infty}^{\infty} h(y)b(F(V)(\eta_1))dy \right) \int_{-\infty}^{\infty} h'(y)b(F(V)(\eta_1))dy \right]
+ rBf' \left( \int_{-\infty}^{\infty} h(y)b(F(V)(\eta_1))dy \right) \int_{-\infty}^{\infty} h'(y)b(F(V)(\eta_1))dy
\]
with \( \eta_1 = \xi(x, t) + B\tau - c\tau + B\tau_1 - C(c)\tau_1 - y, \) and
\[
|I_V - I_{F(w^+)(x, t)}|
\leq \left| \partial_t g(\xi(x, t), \int_{-\infty}^{\infty} h(y)b(F(V)(\eta_1))dy) \right| |V(\xi(x, t)) - w^+(x, t)|.
\]
\[ + \int_{-\infty}^{\infty} |\partial_2 g(l, s)| + rM_1 + rM_2|y| + r\bar{M}_3y^2|b(y)b(F(V)(\eta_1)) - b(F(V)(\eta_2))|dy \leq \partial_1 g\left(\xi^*(x, t), \int_{-\infty}^{\infty} h(y)b(F(V)(\eta_1))dy\right) \delta e^{-\beta_0 t} \]

\[ + \int_{-\infty}^{\infty} (|\partial_2 g(l, s)| + rM_1 + rM_2|y| + r\bar{M}_3y^2)h(y)b'(\bar{\eta})\delta e^{-\beta_0(t-\tau_1)}dy, \]

\[ \xi^*(x, t) = \theta F(V)(\xi(x, t)) + (1 - \theta)F(w^+(x, t), V(\eta_2) = V(\xi(x, t) + B_\tau - c\tau + B\tau_1 - \bar{C}(\epsilon)\tau_1 - y - \sigma_0(\delta e^\beta_0 - 1)e^{-\beta_0 t}) + M\epsilon^{-\beta_0(t-\tau_1)}, \bar{\eta} = \theta F(V(\eta_1)) + (1 - \theta)F(V(\eta_2)). \]

We need to consider three cases.

**Case (i)** \(|\xi(x, t)| \leq M_0\). By the definition of \(\delta\), we have

\[ 0 \leq F(V(\eta_2)) \leq K + \delta^*, \]

\[ S(0) \leq \int_{-\infty}^{\infty} h(y)b(F(V)(\eta_2))dy \leq b(K + \delta^*) \]

and

\[ \int_{-\infty}^{\infty} (|\partial_2 g(l, s)| + rM_1 + rM_2|y| + r\bar{M}_3y^2)h(y)b'(\bar{\eta})dy \leq c_1. \]

Therefore, by the choice of \(c_1\), we have

\[ \left|\partial_1 g\left(\xi^*(x, t), \int_{-\infty}^{\infty} h(y)b(F(V)(\eta_1))dy\right)\right| + e^{\beta_0\tau_1} \int_{-\infty}^{\infty} (|\partial_2 g(l, s)| + rM_1 + rM_2|y| + r\bar{M}_3y^2)h(y)b'(\bar{\eta})dy \leq c_1. \]

**Case (ii)** \(\xi(x, t) \geq M_0\). By (41) we have

\[ b(K) - \delta^* \leq \int_{-\infty}^{\infty} h(y)b(F(V)(\eta_1))dy \leq b(K), \]

\[ b(K) - \delta^* \leq \int_{-\infty}^{\infty} h(y)b(F(V)(\eta_2))dy \leq b(K) + \delta^* \]

and

\[ b'(K) - \delta^* \leq \int_{-\infty}^{\infty} h(y)b'(F(V)(\eta_2))dy \leq b'(K) + \delta^*. \]

Therefore, by (40) and (44), it follows that for sufficiently small \(r > 0\) satisfying (15), (24) and (33)

\[ \frac{\partial w^+(x, t)}{\partial t} - D(c)\frac{\partial^2 w^+(x, t)}{\partial x^2} - B\frac{\partial w^+(x, t)}{\partial x} - I_{F(w^+)(x, t)} \geq \delta e^{-\beta_0}(\sigma_0\beta_0 V'(\xi(x, t))) - \beta_0 + \beta_0 \geq 0. \]

**Case (iii)** \(\xi(x, t) \leq -M_0\). The proof is similar to that in case (ii) and hence is omitted.
Now we prove that (30) holds for \(w^+\). Let \(\tilde{w}^+(x,t) = w^+(x - Bt, t)\). We only need to prove that \(\tilde{w}^+(x,t)\) is a supersolution of (3.20), namely,

\[
\tilde{w}^+(t) \geq T(t - t_0)\tilde{w}^+(t_0) + \int_{t_0}^{t} T(t-s)\tilde{F}_1(\tilde{w}^+)ds
\]

holds, where \(\tilde{F}_1(\tilde{\phi})(x) = \tilde{I}_{F(\phi)}(x,0)\).

By (43), for any \((x,t) \in \mathbb{R} \times [0, \infty)\), we have

\[
\frac{\partial \tilde{w}^+(x,t)}{\partial t} - D(c) \frac{\partial^2 \tilde{w}^+(x,t)}{\partial x^2} - \tilde{I}_{F(\tilde{w}^+)}(x,t) \geq 0.
\]

Define

\[
\Phi(\tilde{w}^+)(x,t,s) = \frac{1}{\sqrt{4\pi D(c)(t-s)}} \int_{-\infty}^{\infty} e^{\frac{-(x-y)^2}{4D(c)(t-s)}} \tilde{w}^+(y,s)dy, \quad t > s \geq 0
\]

and

\[
H(\tilde{w}^+)(x,t) = -\frac{\partial \tilde{w}^+(x,t)}{\partial t} + D(c) \frac{\partial^2 \tilde{w}^+(x,t)}{\partial x^2} + \tilde{I}_{F(\tilde{w}^+)}(x,t) \leq 0.
\]

Set \(\tilde{F}_1(\tilde{w}^+)(x) = \tilde{I}_{F(\tilde{w}^+)}(x,t)\), then a direct calculation implies

\[
\frac{\partial}{\partial s} \Phi(\tilde{w}^+)(x,t,s) = \frac{1}{2(t-s)\sqrt{4\pi D(c)(t-s)}} \int_{-\infty}^{\infty} e^{\frac{-(x-y)^2}{4D(c)(t-s)}} \tilde{w}^+(y,s)dy
\]

Furthermore, integration by parts gives

\[
\frac{\partial}{\partial s} \Phi(\tilde{w}^+)(x,t,s) = \frac{1}{\sqrt{4\pi D(c)(t-s)}} \int_{-\infty}^{\infty} e^{\frac{-(x-y)^2}{4D(c)(t-s)}} \frac{\partial^2 \tilde{w}^+(y,s)}{\partial y^2}dy
\]

Hence, it follows that

\[
\frac{\partial}{\partial s} \Phi(\tilde{w}^+)(x,t,s) = \frac{1}{\sqrt{4\pi D(c)(t-s)}} \int_{-\infty}^{\infty} e^{\frac{-(x-y)^2}{4D(c)(t-s)}} \left[\tilde{F}_1(\tilde{w}^+)_s(y) - H(\tilde{w}^+)(y,s)\right]dy.
\]

Since \(\frac{\partial}{\partial s} \Phi(\tilde{w}^+)(x,t,s)\) is continuous in \(s \in [0,t]\), and

\[
\lim_{s \to t-0} \frac{1}{\sqrt{4\pi D(c)(t-s)}} \int_{-\infty}^{\infty} e^{\frac{-(x-y)^2}{4D(c)(t-s)}} \tilde{w}^+(y,s)dy = \tilde{w}^+(x,t),
\]
it follows that for $0 \leq t_0 < t$,
\[
\hat{w}^+(x, t) = \lim_{\eta \to 0+} \Phi(\hat{w}^+)(x, t, t - \eta)
\]
\[
\begin{align*}
&= \Phi(\hat{w}^+)(x, t, t_0) + \lim_{\eta \to 0+} \int_{t_0}^{t-\eta} \frac{\partial}{\partial s} \Phi(\hat{w}^+)(x, t, s) ds \\
&= \frac{1}{\sqrt{4\pi D(c)(t - t_0)}} \int_{-\infty}^{\infty} e^{-\frac{(\xi - y)^2}{4D(c)(t - t_0)}} \hat{w}^+(y, t_0) dy \\
&\quad + \int_{t_0}^{t} \frac{1}{\sqrt{4\pi D(c)(t - s)}} \int_{-\infty}^{\infty} e^{-\frac{(\xi - y)^2}{4D(c)(t - s)}} [\bar{F}(\bar{w}^+_s)(y) - H(\bar{w}^+)(y, s)] dy ds.
\end{align*}
\]

Then (45) holds since $H(\bar{w}^+)(y, s) \leq 0$, which implies that $w^+(x, t)$ is a supersolution of (19). This completes the proof. \hfill \Box

**Theorem 3.6.** Assume that (19) has a nondecreasing traveling wave solution $(V(x + C(c)t), F(V)(x + C(c)t))$. Then for any traveling wave solution $\bar{V}(x + \bar{C}(c)t), F(\bar{V})(x + \bar{C}(c)t)$ with $0 \leq \bar{V} \leq V_{\text{max}}$, we have $\bar{C}(c) = C(c)$ and $\bar{V}(\cdot) = V(\cdot + \xi_0)$ for some $\xi_0 \in \mathbb{R}$.

**Proof.** We extend the standard proof in [13, 26, 39] to our case. Since $V$ and $\bar{V}$ have the same limit as $\xi \to \pm \infty$, there exist $\xi \in \mathbb{R}$ and large enough $h > 0$ such that for every $s \in [-\tau_1, 0]$ and $x \in \mathbb{R}$,
\[
V(x + C(c)s + \bar{\xi}) - \bar{\delta} < \bar{V}(x + C(c)s) < V(x + C(c)s + \bar{\xi} + h) + \bar{\delta}
\]
and
\[
V(x + C(c)s + \bar{\xi} - \sigma_0 \bar{\delta} (e^{\beta_0 \tau_1} - e^{-\beta_0 s})) - \bar{\delta} e^{-\beta_0 s} < V(x + \bar{C}(c)s) < V(x + C(c)s + \bar{\xi} + h + \sigma_0 \bar{\delta} (e^{\beta_0 \tau_1} - e^{-\beta_0 s})) + \bar{\delta} e^{-\beta_0 s},
\]
where $\beta_0, \sigma_0$ and $\bar{\delta}$ are constants given in Lemma 3.5. Noting that the operator $F(v)(\cdot)$ defined in Lemma 3.1 is non-decreasing if $v$ is nondecreasing, we can still use the comparison result to obtain that for all $t \geq 0$ and $x \in \mathbb{R}$,
\[
V(x + C(c)t + \bar{\xi} - \sigma_0 \bar{\delta} (e^{\beta_0 \tau_1} - e^{-\beta_0 t})) - \bar{\delta} e^{-\beta_0 t} < \bar{V}(x + \bar{C}(c)t) < V(x + C(c)t + \bar{\xi} + h + \sigma_0 \bar{\delta} (e^{\beta_0 \tau_1} - e^{-\beta_0 t})) + \bar{\delta} e^{-\beta_0 t}.
\]
Keeping $\xi = x + \bar{C}(c)t$ fixed and letting $t \to \infty$, we have from the first inequality that $C(c) \leq \bar{C}(c)$ and from the second inequality that $C(c) \geq \bar{C}(c)$. This yields $C(c) = \bar{C}(c)$. Moreover, we get
\[
V(\xi + \bar{\xi} - \sigma_0 \bar{\delta} e^{\beta_0 \tau_1}) < V(\xi) < V(\xi + \bar{\xi} + h + \sigma_0 \bar{\delta} e^{\beta_0 \tau_1}), \quad \text{for} \quad \xi \in \mathbb{R}.
\]

Define
\[
\xi^* = \inf\{\xi : \bar{V}(\cdot) \leq V(\cdot + \xi)\}, \quad \xi_* = \sup\{\xi : \bar{V}(\cdot) \geq V(\cdot + \xi)\}.
\]
By (46), both $\xi^*$ and $\xi_*$ are well defined. In particular, since $V(\cdot + \xi^*) \leq \bar{V}(\cdot) \leq V(\cdot + \xi_*)$, we have $\xi_* \leq \xi^*$.

We only need to show $\xi_* \geq \xi^*$. If $\xi_* < \xi^*$ and $\bar{V}(\cdot) \neq V(\cdot + \xi^*)$. It follows from \[
\lim_{\xi \to \infty} V'(\xi) = 0 \quad \text{that there exists large enough} \quad \bar{M} = \bar{M}(V) > 0 \quad \text{such that}
\]
\[
2\sigma_0 e^{\beta_0 \tau_1} V'(\xi) \leq 1, \quad \text{if} \quad |\xi| \geq \bar{M}.
\]
By the fact that \( \bar{V}(\cdot) \leq V(\cdot + \xi^*) \) and \( \bar{V}(\cdot) \not= V(\cdot + \xi^*) \), we can conclude from Remark 4 that \( \bar{V}(\cdot) < V(\cdot + \xi^*) \) on \( \mathbb{R} \). Therefore, by the continuity of \( V \) and \( \bar{V} \), there exists a small \( \bar{h} \in (0, \delta] \) with \( 2\sigma_0 e^{\beta_0 \tau_1} \bar{h} \leq 1 \) such that
\[
\bar{V}(\xi) < V(\xi + \xi^* - 2\sigma_0 e^{\beta_0 \tau_1} \bar{h}) \tag{47}
\]
provided that \( \xi \in [-\bar{M} - 1 - \xi^*, \bar{M} + 1 - \xi^*] \). When \( |\xi + \xi^*| \geq \bar{M} + 1 \), we have
\[
V(\xi + \xi^* - 2\sigma_0 e^{\beta_0 \tau_1} \bar{h}) - \bar{V}(\xi) > V(\xi + \xi^* - 2\sigma_0 e^{\beta_0 \tau_1} \bar{h}) - V(\xi + \xi^*) = -2\sigma_0 e^{\beta_0 \tau_1} \bar{h} V'(\xi + \xi^* - 2\sigma_0 e^{\beta_0 \tau_1} \bar{h}) \geq -\bar{h},
\]
which, together with (47), implies that for any \( s \in [-\tau_1, 0] \) and \( x \in \mathbb{R} \),
\[
V(x + C(c)s + \xi^* - 2\sigma_0 e^{\beta_0 \tau_1} \bar{h} + \sigma_0 \bar{h}(e^{\beta_0 \tau_1} - e^{-\beta_0 s})) + \bar{h} e^{-\beta_0 s} \geq \bar{V}(x + C(c)s).
\]
Therefore, we have by the comparison principle that
\[
V(x + C(c)t + \xi^* - 2\sigma_0 e^{\beta_0 \tau_1} \bar{h} + \sigma_0 \bar{h}(e^{\beta_0 \tau_1} - e^{-\beta_0 t})) + \bar{h} e^{-\beta_0 t} \geq \bar{V}(x + C(c)t). \tag{48}
\]
In (48), as before we keep \( \xi = x + C(c)t \) fixed and let \( t \to \infty \) to obtain
\[
V(\xi + \xi^* - \sigma_0 e^{\beta_0 \tau_1} \bar{h}) \geq \bar{V}(\xi).
\]
This contradicts the definition of \( \xi^* \), since \( \bar{h} > 0 \). Hence \( \xi_* = \xi^* \). The proof is completed. \( \square \)

**Remark 5.** If \( r = 0 \) and \( \tau = 0 \), system (19) is independent of \( c \). Then for \( \tau_1 = \tau \), we can see from Theorem 3.6 that the traveling wave solution of (19) (or (9)) is unique (up to translation), see also [26, 44].

### 4. Existence of traveling wave solutions.

As discussed in in Section 2, in order to the existence of traveling wave solution for (9), we consider system (19) with \( \tau_1 = 0 \), that is,
\[
\begin{aligned}
\frac{\partial}{\partial t} w(x, t) &= D(c) \frac{\partial^2}{\partial x^2} w(x, t) + B \frac{\partial}{\partial x} w(x, t) + g(w(x, t) + rf \left( \int_{-\infty}^{\infty} h(\xi) b(\varphi(y, t)) dy \right) \int_{-\infty}^{\infty} h(\xi) b(\varphi(y, t)) dy) \\
&\quad + rD(c) \left( f'' \left( \int_{-\infty}^{\infty} h(\xi) b(\varphi(y, t)) dy \right) \left( \int_{-\infty}^{\infty} h'(\xi) b(\varphi(y, t)) dy \right)^2 \\
&\quad + f' \left( \int_{-\infty}^{\infty} h(\xi) b(\varphi(y, t)) dy \right) \int_{-\infty}^{\infty} h''(\xi) b(\varphi(y, t)) dy \\
&\quad + rBf' \left( \int_{-\infty}^{\infty} h(\xi) b(\varphi(y, t)) dy \right) \int_{-\infty}^{\infty} h'(\xi) b(\varphi(y, t)) dy, \ t \geq 0,
\end{aligned}
\]
\[
\varphi(x, t) = w(x, t) + rf \left( \int_{-\infty}^{\infty} h(x + B\tau - ct - y) b(\varphi(y, t)) dy \right), \ t \geq 0,
\]
where \( \xi = x + B\tau - ct - y \) and \( c \in \mathbb{R} \) is a parameter.

In this section, we consider the case \( u^+ = u^- \), namely, \( f_0(u) = g(u, b(u)) \) has only three zeros. If \( f_0(u) \) has more than three zeros, one can refer to [19, 42]. Denote \( \bar{u} = u^+ = u^- \). In addition, we assume the following condition hold.

(H4) \( f_0(u) < 0 \) for \( u \in (0, \bar{u}) \), \( f_0(u) > 0 \) for \( u \in (\bar{u}, K) \) and \( f'_0(\bar{u}) > 0 \).
By in Lemma 3.3, we have the following fact. If $V(x + ct)$ is a traveling wave solution of (19) with wave speed $c$, then $V(x + (B + c)t)$ is a traveling wave solution of (37) with wave speed $-B + c$. Inversely, if $V(x + ct)$ is a traveling wave solution of (37), then $V(x + (B + c)t)$ is a traveling wave solution of (19) with wave speed $B + c$. Hence we only need to consider the existence of traveling wave solutions of (37). We first prove the existence of traveling wave solutions of the following equation

\[
\begin{cases}
\frac{\partial}{\partial t} w(x,t) = D(c) \frac{\partial^2}{\partial x^2} w(x,t) + \bar{I}_\varphi(x,t), & t \geq 0, \\
\varphi(x,t) = w(x,t) + rf \left( \int_{-\infty}^{\infty} h(x + B\tau - ct - y)b(\varphi(y,t))dy \right), & t \geq 0,
\end{cases}
\]  

where

\[
\bar{I}_\varphi(x,t) = g\left(w(x,t) + rf \left( \int_{-\infty}^{\infty} h(\xi)b(\varphi(y,t))dy \right), \int_{-\infty}^{\infty} h(\xi)b(\varphi(y,t))dy \right) + rD(c) \left[f'' \left( \int_{-\infty}^{\infty} h(\xi)b(\varphi(y,t))dy \right) \left( \int_{-\infty}^{\infty} h'(\xi)b(\varphi(y,t))dy \right) \right]^2 + f' \left( \int_{-\infty}^{\infty} h(\xi)b(\varphi(y,t))dy \right) \int_{-\infty}^{\infty} h''(\xi)b(\varphi(y,t))dy + rBf' \left( \int_{-\infty}^{\infty} h(\xi)b(\varphi(y,t))dy \right) \int_{-\infty}^{\infty} h'(\xi)b(\varphi(y,t))dy,
\]

\[\xi = x + B\tau - ct - y \text{ and } c \in \mathbb{R}\] is a parameter.

Let $\zeta \in C^\infty(\mathbb{R}, \mathbb{R})$ be a fixed function with the following properties

\[\zeta(s) = 0, \quad \text{if} \quad s \leq 0; \quad \zeta(s) = 1, \quad \text{if} \quad s \geq 4; \quad 0 < \zeta'(s) < 1, \quad |\zeta''(s)| \leq 1, \quad \text{if} \quad s \in (0, 4).
\]

**Lemma 4.1.** Assume that the parameters $c$ and $r$ satisfy (15), (24) and (33). Then there exist two small constants $\delta > 0, \varepsilon_0 > 0$ and a large constant $C_0 > 0$, which are independent of $c$ and $\tau$ such that

(i) the functions $(v_0^+(x,t), F(v_0^+)(x,t))$ and $(v_0^-(x,t), F(v_0^-)(x,t))$ defined by

\[v_0^+(x,t) = V_{\max} + \delta - V_{\max} \zeta(-\varepsilon_0(x + C_0t)) \]

and

\[v_0^-(x,t) = -\delta + V_{\max} \zeta(\varepsilon_0(x - C_0t)) \]

are a supersolution of (49) for $c \geq 0$ and a subsolution of (49) for $c \leq 0$, respectively;

(ii) the functions $(v_c^+(x,t), F(v_c^+)(x,t))$ and $(v_c^-(x,t), F(v_c^-)(x,t))$ defined by

\[v_c^+(x,t) = V_{\max} + \delta - V_{\max} \zeta(-\varepsilon_c(x + C_c t)) \]

and

\[v_c^-(x,t) = -\delta + V_{\max} \zeta(\varepsilon_c(x - C_c t)) \]

are a supersolution and a subsolution of (49) for all $c$ satisfying (15), respectively, where $\varepsilon_c = \varepsilon_0/(1 + |c|\tau)$ and $C_c = (1 + |c|\tau)C_0$.

**Proof.** We only prove for $v_0^-(x,t)$. From (28), we have

\[0 < F'(0) = \frac{1}{1 - rf'(b(0))b'(0)} \quad \text{and} \quad 0 < F'(V_{\max}) = \frac{1}{1 - rf'(b(K))b'(K)}.
\]
By (H2), for small enough $r > 0$ satisfying (15), (24) and (33), there exist $\rho \in [1/2, 1), \ell > 0$ and

$$
\delta < \delta_0 = \min \left\{ \frac{\tilde{u}}{2}, \frac{K - \tilde{u}}{2}, \frac{2F^{-1}(\tilde{u})}{3}, \frac{2(V_{\text{max}} - F^{-1}(\tilde{u}))}{3}, \delta_0 \right\}
$$

such that

$$
\rho \partial_t g(K, b(K)) + \frac{F(V_{\text{max}} + \delta) - F(V_{\text{max}})}{\delta} 
\times (b'(K) + \ell) \int_{-\infty}^{\infty} \left( \partial_2 g(K, b(K)) + rM_1 + rM_2 |y| + r\tilde{M}_3 y^2 \right) h(y) dy < 0,
$$

$$
0 \leq \left( \frac{1}{\rho} - \rho \right) \delta < K,
$$

and

$$
0 \leq b'(\eta) < b'(0) + \ell, \text{ for } \eta \in [-2\delta, 2\delta].
$$

It is sufficient to prove that for $(x, t) \in \mathbb{R} \times [0, \infty)$,

$$
\frac{\partial v_0^+(x, t)}{\partial t} - D(c) \frac{\partial^2 v_0^+(x, t)}{\partial x^2} - \bar{I}_{F(v_0^+)}(x, t) \geq 0. \tag{50}
$$

For fixed $\delta \in (0, \delta_0]$, let

$$
\varrho_1 = \varrho_1(\delta) = \int_{-\infty}^{\infty} (\tilde{L}_2 + rM_1 + rM_2 |y| + r\tilde{M}_3 y^2) h(y) dy,
$$

$$
\varrho_2 = \varrho_2(\delta) = \max \{ b'(u) : u \in [F(-\delta), F(K + \delta)] \},
$$

$$
\varrho_0 = \varrho_0(\delta) = \varrho_1 \varrho_2,
$$

$$
m_1 = m_1(\delta) = \min \left\{ -f_0(u) : u \in \left[ F(\delta), F \left( \frac{3\delta}{2} \right) \right] \right\} > 0,
$$

where the definitions of $\tilde{L}_2, M_1, M_2$ and $\tilde{M}_3$ are similar to those of Lemma 3.3.

Then there exist $\epsilon^* = \epsilon^*(\delta) > 0$ and $M_0 = M_0(\delta) > 0$, with $\epsilon^* < \delta$ sufficiently small and $M_0$ sufficiently large, such that

$$
V_{\text{max}} \epsilon^* < 2(1 - \rho) \delta
$$

and

$$
m_1 - \varrho_0 \epsilon^* - 2\varrho_2 F(V_{\text{max}} + \delta) \left( \int_{-\infty}^{-M_0} - \int_{M_0}^{\infty} \right) (\tilde{L}_2 + rM_1 + rM_2 |y| + r\tilde{M}_3 y^2) h(y) dy > 0.
$$

Take $\mu = \mu(\epsilon^*) \in (0, 1)$ sufficiently small such that

$$
0 \leq \zeta(x) < \frac{\epsilon^*}{2} \text{ if } x < \mu, \quad 1 - \frac{\epsilon^*}{2} < \zeta(x) \leq 1 \text{ if } x > 4 - \mu,
$$

$$
|F(V_{\text{max}} + \delta - V_{\text{max}}(x_1)) - F(V_{\text{max}} + \delta - V_{\text{max}}(x_2))| < \epsilon^* \text{ if } x_1 < \mu(\text{or } > 4 - \mu), i = 1, 2,
$$

and $\varpi = \varpi(\mu) > 0$ sufficiently small such that

$$
(1 - \varpi) \left( 4 - \frac{\mu}{2} \right) > 4 - \mu,
$$

and $\epsilon_0 = \epsilon_0(\delta) > 0$ sufficiently small such that

$$
\epsilon_0 M_0 \leq \varpi(4 - \mu), \quad \epsilon_0 \tau < \varpi \left( 4 - \frac{\mu}{2} \right),
$$

where $\varpi, \varrho, \varrho_0, \varrho_1, \varrho_2, \varpi, \varrho_1, \varrho_2, \varrho_3$, and $\varrho_4$ are positive constants.
We note that $\eta$ Case (i) where $\delta$ and $g$ follows that $V = \max\{\rho \partial_t g(K, b(K)) + \frac{F(V_{\max} + \delta)}{\delta} - F(V_{\max})\} > 0$. 

Set
$$m_0 = \min\left\{\zeta'(x) : \frac{\epsilon}{2} \leq x \leq 4 - \frac{\mu}{2}\right\} > 0.$$ 

Take $C_0 = C_0(\delta)$ such that $\epsilon_0 C_0 V_{\max} m_0 - D(c) \epsilon_0^2 V_{\max}^2 - \max\{|g(u, b(v))| : (u, v) \in [F(\delta), F(K + \delta)]\} > 0$. We note that $\epsilon_0$ and $C_0$ are independent of $c$.

Let $\eta = \epsilon_0 (x + C_0 t)$, we consider three cases.

**Case (i)** $\eta = \epsilon_0 (x + C_0 t) > -\frac{\epsilon}{2}$. Since $\zeta(-\epsilon_0 (x + C_0 t)) < \frac{\epsilon}{2}$,

$$V_{\max} + \delta \geq v_0^+(x, t) \geq V_{\max} + \delta - \frac{\epsilon}{2} \geq V_{\max} + \delta - (1 - \rho) \delta = V_{\max} + \rho \delta$$

and

$$F(V_{\max} + \delta) \geq F(v_0^+(x, t)),$$

by $g(F(V_{\max}), b(F(V_{\max}))) = g(K, b(K)) = 0$ and similar results in Lemma 20, it follows that

$$\frac{\partial v_0^+(x, t)}{\partial t} - D(c) \frac{\partial^2 v_0^+(x, t)}{\partial x^2} - I_{F(v_0^+)}(x, t) = C_0 \epsilon_0 V_{\max} \zeta'(-\epsilon_0 (x + C_0 t)) - D(c) \epsilon_0^2 V_{\max} \zeta''(-\epsilon_0 (x + C_0 t)) - I_{F(v_0^+)}(x, t)$$

$$\geq -D(c) \epsilon_0^2 V_{\max} - \rho \partial_t g\left(\theta v_0^+(x, t) \int_{-\infty}^{\infty} h(y) b(F(v_0^+))(x + B\tau - c\tau - y, t) dy\right)(v_0^+(x, t) - V_{\max})$$

$$- \int_{-\infty}^{\infty} [\partial_2 g(F(V_{\max}), b(v_1(y)) + rM_1 + rM_2|y| + r\bar{M}y^2]h(y) b'(v_2(y))$$

$$\times |F(v_0^+)(x + B\tau - c\tau - y, t) - F(V_{\max})|dy$$

$$\geq -D(c) \epsilon_0^2 V_{\max} - \rho \partial g(K, b(K)) - [F(V_{\max} + \delta) - F(V_{\max})](b'(K) + l)$$

$$\times \int_{-\infty}^{\infty} [\partial_2 g(K, b(K)) + rM_1 + rM_2|y| + r\bar{M}y^2]h(y) dy$$

$$= -D(c) \epsilon_0^2 V_{\max} - \delta \left\{\rho \partial g(K, b(K)) + \frac{F(V_{\max} + \delta) - F(V_{\max})}{\delta} \times (b'(K) + l)\right\} > 0,$$

where $v_i(y) = \theta_i F(v_0^+)(x + B\tau - c\tau - y, t) + (1 - \theta_i) F(V_{\max}) \in [F(V_{\max}), F(V_{\max} + \delta)] \subset [K, K + 2\delta], \theta \in (0, 1), \theta_i \in (0, 1), i = 1, 2.$
Case (ii) \( \eta = \epsilon_0(x + C_0 t) < -4 + \frac{\mu}{2} \). Then
\[
\zeta(-\epsilon_0(x + C_0 t)) > \zeta\left(-4 + \frac{\mu}{2}\right) > 1 - \frac{\epsilon^*}{2},
\]
\[
\delta \leq v^+_0(x, t) \leq V_{\max} + \delta - V_{\max}\left(1 - \frac{\epsilon^*}{2}\right) \leq \delta + (1 - \rho)\delta \leq \frac{3\delta}{2}.
\]
By the choice of \( \epsilon_0 \) and \( \varpi \), we have
\[
\frac{\eta \varpi}{\epsilon_0} = \varpi(x + C_0 t) \leq \frac{\varpi(-4 + \frac{\mu}{2})}{\epsilon_0} < \frac{\varpi(-4 + \mu)}{\epsilon_0} \leq -M_0.
\]
Let \( y - B \tau \in [\varpi(x + C_0 t), -\varpi(x + C_0 t)] \). Then for \( c \geq 0 \),
\[
\epsilon_0(x + B \tau - c \tau - y + C_0 t) \leq \epsilon_0(1 - \varpi)(x + C_0 t) - \epsilon_0 c \tau \leq (1 - \varpi)(-4 + \frac{\mu}{2}) < -4 + \mu.
\]
By \( \eta = \epsilon_0(x + C_0 t) < -4 + \frac{\mu}{2} < -4 + \mu \), we have
\[
|F(V_{\max} + \delta - V_{\max}\zeta(-\epsilon_0(x + B \tau - y - c \tau + C_0 t)))| - F(V_{\max} + \delta - V_{\max}\zeta(-\epsilon_0(x + C_0 t)))| \leq \epsilon^*.
\]
For all \( t \geq 0 \), we have
\[
\dot{I}_{F(v^+_0)(x, t)} = g(F(v^+_0)(x, t), b(F(v^+_0)(x, t)))
\]
\[
+ \int_{-\infty}^{\infty} \left( \tilde{L}_2 + rM_1 + rM_2|y| + rM_3 y^2 \right) h(y)
\]
\[
\times \left( b(v^+_0(y)) |F(v^+_0)(x + B \tau - c \tau - y, t) - F(v^+_0)(x, t)| dy, \right.
\]
where \( v^+_0(y) \) is between \( F(v^+_0)(x + B \tau - c \tau - y, t) \) and \( F(v^+_0)(x, t) \).

Therefore,
\[
\frac{\partial v^+_0(x, t)}{\partial t} - D \frac{\partial^2 v^+_0(x, t)}{\partial x^2} - \dot{I}_{F(v^+_0)(x, t)} \geq -D(c)\epsilon^*_0 V_{\max} - g(F(v^+_0)(x, t), b(F(v^+_0)(x, t))) - \varrho_2 \int_{-\infty}^{\infty} \left( \tilde{L}_2 + rM_1 
\right.
\]
\[
+ rM_2|y| + rM_3 y^2 \right) h(y) \left\{ F(V_{\max} + \delta - V_{\max}\zeta(-\epsilon_0(x + B \tau - y + c \tau + C_0 t))) 
\right.
\]
\[
\left. - F(V_{\max} + \delta - V_{\max}\zeta(-\epsilon_0(x + C_0 t))) \right| dy
\]
\[
\geq -D(c)\epsilon^*_0 V_{\max} + m_1 - 2\varrho_2 F(V_{\max} + \delta)
\]
\[
\times \left( \int_{-\infty}^{\infty} - \int_{\frac{\varpi y}{\epsilon_0}}^{\frac{\varpi y}{\epsilon_0}} \right) \left( \tilde{L}_2 + rM_1 + rM_2|y| + rM_3 y^2 \right) h(y) dy - \varrho_0 \epsilon^*
\]
\[
\geq 0.
\]
Case (iii) \(-\frac{\mu}{2} \geq \epsilon_0(x - \xi + C_0 t) \geq -4 + \frac{\mu}{2} \). Then
\[
\frac{\partial v^+_0(x, t)}{\partial t} - D(c) \frac{\partial^2 v^+_0(x, t)}{\partial x^2} - \dot{I}_{F(v^+_0)(x, t)} \geq \epsilon_0 C_0 V_{\max}\zeta'(-\epsilon_0(x - \xi + C_0 t)) - D(c)\epsilon^*_0 V_{\max}
\]
\[
- \max\{ |g(u, b(v))| : (u, v) \in [F(\delta), F(K + \delta)] \}
\]
\[
\geq 0.
\]
Noting that \( \epsilon_c \leq \epsilon_0, \epsilon_c C_c = \epsilon_0 C_0 \), we can prove that (50) holds for \( v^+_0(x, t) \) in a similar way. The proof is completed. \( \square \)
Lemma 4.2. For every $c$ and $r$ satisfying (15), (24) and (23), there exists a unique strictly monotonic traveling wave solution $(V(x + C(c)t), F(V)(x + C(c)t))$ for system (49) with speed $C(c)$ being a continuous function of $c$.

Proof. The existence result is very similar to [13, 26, 44]. From Lemma 3.3 (or Remark 4), we can get the monotonicity result be obtained. And the uniqueness result can be obtained from Theorem 3.6.

Now we prove that $C(c)$ is continuous in $c$. Assume that $(V_c(x + C(c)t), F(V_c)(x + C(c)t))$ is a traveling wave solution with wave speed $C(c)$. Without loss of generality, we assume that $0 < V_c(0) = F^{-1}(\bar{u}) < V_{\text{max}}$. Then $V_c(x + C(c)t)$ satisfies

$$
\begin{align*}
C(c)V''_c(\xi) &= D(c)V''_c(\xi) - L_1 V_c(\xi) + H_c(V_c)(\xi), \\
U_c(\xi) &= V_c(\xi) + rf\left(\int_{-\infty}^{\infty} h(\xi + B\tau - c\tau - y)b(U_c(\xi))dy\right),
\end{align*}
$$

where $\xi = x + C(c)t,$

$$
H_c(V_c)(\xi) = L_1 V_c(\xi) + g\left(V_c(\xi) + rf\left(\int_{-\infty}^{\infty} h(\eta)b(F(V_c)(\eta))d\eta\right)\int_{-\infty}^{\infty} h(\eta)b(F(V_c)(\eta))d\eta\right)
$$

$$
+ rD(c)f\left(h''(\eta)b(F(V_c)(\eta))d\eta\right)^2
$$

$$
+ rf\left(\int_{-\infty}^{\infty} h(\eta)b(F(V_c)(\eta))d\eta\right)\int_{-\infty}^{\infty} h''(\eta)b(F(V_c)(\eta))d\eta
$$

$$
+ rBf\left(\int_{-\infty}^{\infty} h(\eta)b(F(V_c)(\eta))d\eta\right)\int_{-\infty}^{\infty} h'(\eta)b(F(V_c)(\eta))d\eta
$$

with $\eta = \xi + B\tau - c\tau - z$. Hence,

$$
V_c(\xi) = \frac{1}{D(c)\left(\lambda_2(C(c)) - \lambda_1(C(c))\right)} \times \left[\int_{-\infty}^{\xi} e^{\lambda_1(C(c))(\xi-s)}H_c(V_c)(s)ds + \int_{\xi}^{\infty} e^{\lambda_2(C(c))(\xi-s)}H_c(V_c)(s)ds\right],
$$

where

$$
\lambda_1(C(c)) = \frac{C(c) - \sqrt{C^2(c) + 4D(c)L_1}}{2D(c)} < 0,
$$

$$
\lambda_2(C(c)) = \frac{C(c) + \sqrt{C^2(c) + 4D(c)L_1}}{2D(c)} > 0.
$$

Since $0 \leq V_c \leq V_{\text{max}}, 0 \leq F(V_c) \leq K$ and

$$
\lambda_2(C(c)) - \lambda_1(C(c)) = \frac{\sqrt{C^2(c) + 4D(c)L_1}}{D(c)} \geq 2\sqrt{\frac{L_1}{D(c)}},
$$

By a similar argument as that in Lemma 3.4, we have

$$
|V'_c(\xi)| \leq \frac{G}{2\sqrt{D(c)L_1}}.
$$

Now we first prove that $C(c)$ is bounded for any bounded $c \in \mathbb{R}$ satisfying (15). For $V_c(x)$ and $v_c^-(x, 0)$ defined in Lemma 4.1, there exists $x_0 \in \mathbb{R}$ such that
Theorem 4.3. Assume that $v_c^-(x - x_0, 0) < V_c(x)$ for all $x \in \mathbb{R}$. It follows by comparison that $v_c^-(x - x_0, t) < V_c(x + C(c)t)$ for all $x \in \mathbb{R}$ and $t \in [0, \infty)$, that is,

$$-\delta + V_{\max}(\xi, x - x_0 - C_c t)) < U_c(x + C(c)t). \quad (52)$$

We claim that $-C(c) \leq C_c$. Otherwise, if $-C(c) > C_c$, then we fix $x - x_0 - C_c t = \xi^*$ with $V_{\max}(\xi, \xi^*) = 2\delta$, hence, $V_c(x + C(c)t) = V_c(\xi^* + x_0 + (C_c + C(c))t)$. Letting $t \to \infty$ in (52), then we have $\delta \leq V_c(-\infty)$, which is a contradiction to $V_c(-\infty) = 0$.

Thus, we have $-C(c) \leq C_c = (1 + |c\tau|)C_0$. Similarly, we obtain $-C(c) \geq -C_c = -(1 + |c\tau|)C_0$ by comparing $V_c(x)$ and $v_c^+(x, 0)$.

Assume that $c_n$ satisfies (15) and $c_n \to c$, but $C(c_n)$ does not converge to $C(c)$, then there exists a subsequence $c_{nk} \to c$ such that $C(c_{nk}) \to \tilde{C} \neq C(c)$. Let $H^* = \sup \{c_{nk}\}$ Since $V_{c_n}(\cdot)$ is nondecreasing, $V_{c_n}(0) = F^{-1}(\bar{u})$, and by (A.3) and (A.4) in Appendix of [26], see also [44], $V_{c_n}(\cdot)$ also satisfies, for sufficiently small $\delta > 0$,

$$V_{c_n}(x) \leq F^{-1}(\bar{u}) - \delta, \quad \text{if } x \leq -M^* - L^*H^*\tau \leq -M^* - L^*|c_{nk}|\tau,$$

$$V_{c_n}(x) \geq F^{-1}(\bar{u}) + \delta, \quad \text{if } x \geq M^* + L^*H^*\tau \geq M^* + L^*|c_{nk}|\tau.$$  

By the Arzela–Ascoli Theorem, we can choose a subsequence of $\{c_{nk}\}$, also denoting it by $\{c_{nk}\}$ such that $V_{c_{nk}}(\cdot)$ converges to a continuous function $\bar{V}(\cdot)$ in $\mathbb{R}$. It follows that $\bar{V}(\cdot)$ is non-decreasing, $0 \leq \bar{V}(\cdot) \leq V_{\max}$ and

$$\limsup_{x \to -\infty} \bar{V}(x) \leq F^{-1}(\bar{u}) - \delta \quad \text{and} \quad \liminf_{x \to -\infty} \bar{V}(x) \geq F^{-1}(\bar{u}) + \delta.$$  

In (51) with $c$ being replaced by $c_{nk}$, let $k \to \infty$, by the dominant convergence theorem, we have

$$\bar{V}(\xi) = \frac{1}{D(\tau)} \int_{-\infty}^{\infty} e^{\lambda_1(b)(\xi - s)}H_c(\bar{V})(s)ds + \int_{\xi}^{\infty} e^{\lambda_2(b)(\xi - s)}H_c(\bar{V})(s)ds$$

and hence $\bar{V}(x + \tilde{C}t)$ is a solution of (49). But for the given parameter $c$, it follows from Theorem 3.6 $\tilde{C} = C(c)$, which is a contradiction. This completes the proof. \qed

Theorem 4.3. Assume that $r > 0$ satisfies

$$D - r^2C_0 > 0. \quad (53)$$

Then (19) admits a strictly monotonic traveling wave solution $(V(x + (B + c^*)t), F(V)(x + (B + c^*)t))$ with $|c^*| \leq C_0$, where $C_0$ is described by Lemma 4.1. Furthermore, (9) admits a strictly monotonic traveling wave solution $U(x + c^*t) = F(V)(x + C(c)t))$.

Proof. By (53), we can see that (15) holds for any $c$ satisfying $|c| \leq C_0$. Therefore, by Lemma 50, (49) has a strictly monotonic traveling wave solution $(V(x + C(c)t), F(V)(x + C(c)t))$.

Motivated by the methods in [33, 44], now we prove that there exists at least one $c^*$ such that $C(c^*) = c^*$ and $|c^*| \leq C_0$. It only need to show that the curves $y = c$ and $y = C(c)$ have at least one common point in region $|y| \leq C_0$. For $c \leq 0$, let $v_0^-(x, t)$ be the subsolution of (49) described in Lemma 4.1. Then there exists a large constant $X > 0$ such that $V(\cdot) \geq v_0^-(\cdot - X, 0)$. So we have by the comparison that $V(x + C(c)t) \geq v_0^-(x - X, t)$ for all $t \geq 0$ and $x \in \mathbb{R}$. Hence by the choice of $\delta$ (letting $\delta \to 0$), it follows that $C(c) \geq -C_0$ for $c \leq 0$. Similarly, $C(c) \leq C_0$ for $c \geq 0$. By Lemma 4.2, we can see that $C(c)$ is continuous for any
|c| < C_0 < \sqrt{D/r}. Therefore, it follows from (53) that there is at least one common point c^* such that C(c^*) = c^* in the region |c| < C_0 < \sqrt{D/r} and |y| \leq C_0. Hence, when $\tau_1 = 0$, $(V(x + c^*t), F(V)(x + c^*t))$ is also a strictly monotonic traveling wave solution of (37) and $(V(x + (B + c^*)t), F(V)(x + (B + c^*)t))$ is also a strictly monotonic traveling wave solution of (19) with wave speed $B + c^*$, furthermore, $U(x + c^*t) = F(V)(x + c^*t)$ is a strictly monotonic traveling wave solution of (9) with wave speed $c^*$. This completes the proof. □

5. Applications. As mentioned in the introduction, in this section, we apply our results in section 3 and 4 to establish the existence and uniqueness of traveling wave solutions for (8).

Example 1. Assume that equation (8) satisfies the following conditions

(P1) There exists $0 < u^* < K$ such that $\varepsilon b(u) - d_mu = 0$ for $u = 0, u^*, K; \varepsilon b(u) - d_mu < 0$ for $u \in (0, u^*); \varepsilon b(u) - d_mu > 0$ for $u \in (u^*, K)$.

(P2) $b(\cdot) \in C^2(I)$ for some open interval $I \supset [0, K]$, $b(\cdot) \geq 0, \varepsilon b(0) < d_m, \varepsilon b(u^*) > d_m, \varepsilon b'(K) < d_m$.

Obviously, (H1)-(H4) hold for (8). By Theorems 3.6 and 4.3, (8) admits a unique monotonic traveling wave solution which satisfies $\lim_{\xi \to -\infty} U(\xi) = 0$ and $\lim_{\xi \to \infty} U(\xi) = K$.

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E-mail address: kli@mail.bnu.edu.cn
E-mail address: jhuang32@nudt.edu.cn
E-mail address: xli@bnu.edu.cn