If I Only Had A Brane!

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Akademityck AB, Edsbruk.
To Nanna, who would be very proud
and Baji who would be really glad it’s finally over!
The Wizard Of Oz Theme

For some inexplicable reason, this thesis has taken on a life and a theme of its own. It started with the title I guess and then everything else just followed suit. As a result, there are references to 'The Wizard Of Oz' all over the place. For those of you who have not seen the classic 1931 MGM musical starring Judy Garland, here is a quick synopsis.

Dorothy is a young girl who lives with her Aunt and Uncle on a farm in Kansas. She is on her way home when a huge tornado sweeps the land but by the time she reaches the farm-house, everyone else is already safe in the storm shelter underground. The tornado blows the house away carrying Dorothy and her dog Toto inside it. When the house finally lands, Dorothy opens the door and steps out into a magical world where things are unfamiliar, but beautiful. The film which started out in sepia bursts into Technicolour at this point as Dorothy utters one of the most famous sentences of the film, “Toto, I have a feeling we’re not in Kansas anymore. We must be over the rainbow!” The song Over the Rainbow plays in the background.

Before Dorothy has had time to look around in this strange new place, a shimmering ball floats in, carrying the lovely Glinda, the Good Witch of the North. Glinda announces to Dorothy that her house has landed on, and killed, the Wicked Witch of the East who had ruled over Munchkinland – the wonderful country where Dorothy finds herself now. All that remains of the Wicked Witch are her feet which stick out from under the house with their sequin encrusted Ruby slippers. The jolly dwarf-like Munchkins, having quite suddenly and unexpectedly been liberated from a tyrannical reign burst into joyous celebration. Their cheer is brought to an abrupt end with the arrival of the Wicked Witch of the West who comes to swear vengeance on Dorothy for killing her sister – and to collect the Ruby slippers. When she turns to them however, they are magically transferred to Dorothy’s feet. The witch tries to convince Dorothy to hand the slippers over, saying that they will be no use to her .. but Glinda tells her to hang on to them. They must be powerful, she says, if the Wicked Witch wants them so badly. Thwarted at her efforts, the Wicked Witch renews her menacing warning and disappears.

Fearful now, Dorothy expresses her desire to return home for safety, ”Which is the way back to Kansas? I can’t go the way I came”. Glinda suggests that she travel to the far-off Emerald City in the Land of Oz, since ”The only person who might know would be the great and wonderful Wizard

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1 Though the movie is based on a book by L.Frank Baum, the images and quotes used here are from the film.
of Oz himself.” To get there, Dorothy is told ”It’s always best that you start at the beginning, and all you have to do is follow the Yellow Brick Road,” The Munchkins guide Dorothy to the border of Munchkinland to start her on her journey as they bid her farewell singing Follow the Yellow Brick Road and Glinda reminds her, Never let those ruby slippers off your feet for a moment, or you will be at the mercy of the Wicked Witch of the West.

Dorothy and Toto then begin the long walk to the Emerald City, along the Yellow Brick Road. Quite soon they come across a fork in the road and when Dorothy wonders out loud where to go, she gets her answer from a talking Scarecrow! The Scarecrow, upon finding out Dorothy’s mission asks if he can join her on her quest to ask the Wizard for a brain. He has no brain he says, which is why the crows are not scared of him at all. In order to explain his predicament, he sings the song If I only had a brain. Dorothy is only too glad to have him go along. On their journey down the Yellow Brick Road, Dorothy and the Scarecrow are joined by a Tin Woodman, who is in need of a heart and a Cowardly Lion who wants the Wizard to give him some courage.

After a few adventures and set-backs, the friends reach the glittering towers of the Emerald City. From the outside the Emerald City looks like a huge shimmering castle and the four friends feel sure they will obtain their hearts’ desires in such a wonderous place. Getting in to the City does not prove an easy task however, as there is only one door and that is bolted shut. After many attempts at knocking, the four friends finally manage to attract the attention of the gate-keeper who talks to them through a small window which he reluctantly opens. The following exchange then takes place.

Dorothy: We want to see the Wizard.
Gateman: The Wizard? But nobody can see the great Oz. Nobody’s ever seen the great Oz. Even I’ve never seen him.
Dorothy: Well then, how do you know there is one?
Gateman: Because he, uh..., you’re wasting my time.

Finally Dorothy hits upon the happy notion of saying that it was Glinda who sent them there. As if that name was the magic password he had been waiting for, the gate-keeper flings open the doors and lets Dorothy and her friends in to the charming and very unique Emerald City. After being feted and introduced to the wonders of the Emerald City, Dorothy tries to seek an audience with the Wizard. Once again she meets with incredulous faces. The Guard outside the Wizard’s room puts it best when he says “Orders
“are, nobody can see the Great Oz, not nobody, not no how...not nobody, not no how!” Once again, Dorothy has to explain who she is, and that the Wicked Witch of the West is after her, in order to be let in.

The meeting with the Wizard is brief to say the least. The little band of travellers is told that it is of course in the Wizard’s power to grant their wishes, but first they must prove to him that they are worthy by performing a ‘small task’. They must bring to him, the broomstick of the Wicked Witch of the West. High drama follows including a scary run in with the evil Winged Monkeys, before the friends succeed in capturing the sought after broomstick – by pouring cold water on the Witch and melting her to death!

With the broomstick in hand, the group returns victorious to the Emerald City, confident that their desires will at least be fulfilled. This time, they are allowed to meet the Wizard with no further ado. However, when the Wizard is at his most grand and mysterious, a little accident happens. Toto the dog, sees a curtain and runs towards it, pulling it playfully — and revealing a little man who sits there, like a puppeteer controlling the illusion that is ‘The Wizard Of Oz’. The true story now unfolds. It turns out that the man is a magician (also from Kansas) who landed in Oz one day when his hot air balloon flew astray. Since he came ‘from the sky’ he was heralded in Oz as a Wizard. Rather than fight this myth, he furthered it and established himself in the Emerald City keeping the legend alive by distancing himself from all people and not allowing his authority to be questioned.

He is however, a good hearted man, who only became reconciled to a life in Oz when he thought there was no way for him to return home. He points out to the friends how they already have what they thought they were lacking – all that they need now is material proof. The Lion has displayed great courage, and in recognition is presented a medal. The Tinman has shown great feeling and is gifted a real ‘beating’ heart as proof. The scarecrow has shown great intelligence and is awarded a ThD (Doctor of Thinkology) degree as a result. The only person who the Wizard can not help, is Dorothy.

At this point, just when Dorothy starts feeling rather deflated, Glinda shows up again. She proves that Dorothy too, already has what she needs to make her dreams come true: the Ruby Slippers. The magic, it seems, is that the person who wears the slippers can go where-ever they want to, as long as they Click the heels three times and tell the slippers where to go. Dorothy does this, repeating three times, there’s no place like home.. and she finds herself back at the farm.
Oh I could tell you why
The ocean’s near the shore
I could think of things
I’d never thunk before
And then I’d sit ...
And think some more.

- If I only had a Brain
The Wizard of Oz.
Abstract

This thesis starts with a review of supersymmetric solutions of 11-dimensional supergravity; in particular flat M-branes and BPS configurations which can be constructed from them. The harmonic function rule is discussed and it is shown why this cannot be expected to apply to intersecting brane configurations where the intersection is localised. A new class of spacetimes is needed to cover these situations and the Fayyazuddin-Smith metric ansatz is introduced as the answer to this problem. The ansatz is then used to find supergravity solutions for M-branes wrapped on holomorphic curves. This method is discussed in detail and its various components explained; for instance, the way holomorphicity dictates the projection conditions on the Killing spinor and the construction of a spinor in terms of Fock space states. The method is illustrated via various examples. We then move on and discuss a mathematical concept known as calibrations. Calibrations are forms which can be used to pick out the minimal surfaces in a given background. Since BPS configurations of M-branes are minimal, it turns out that they must wrap cycles ’calibrated’ by such p-forms. For the case when the background contains no flux, calibrations have been classified. However, for more general cases where there is a non-trivial field strength, such a classification does not yet exist. This would be desirable for various reasons, one of which is the following. Given a particular calibrated form, there is a simple and very elegant method of writing down the supergravity solution for a brane wrapped on the corresponding calibrated cycle. So far, this method was applied only to Kahler calibrations as these were the only ones known to exist in backgrounds with non-trivial flux. We extend this method to a wider class which contains Kahler calibrations. A constraint is used to classify possible calibrations; this constraint incorporates the geometry of the space transverse to the submanifold which contains the supersymmetric cycle. A rule is given which can generate the required constraint for any given M-brane wrapped on a holomorphic cycle. Ways in which this constraint can be satisfied are also discussed.
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Our all-girl group of students has to be a first – on this planet at least! I count on Cecilia Albertsson and Teresia Mansson for various things, from bailing me out of trouble with the printer, to a friendly chat in the middle of the day; when panic hits and reassurance is needed, Cecilia can be found at the desk next to mine. But before it was just us girls, Bjorn Brinne and Maxim Zabzine were around as well. I learnt a lot from both of them and still count on Bjorn for assorted help and advice.

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Most of all, my love and my deepest gratitude is for Abbu and Ammi, who gave me the roots which steady me and wings with which to fly. Between them, they have created such a haven that no matter where I go “there’s no place like home”.

"There’s no place like home".
Chapter 1

Over The Rainbow

Anyone stumbling onto M-Theory by chance would experience an overwhelming feeling that they are certainly 'not in Kansas anymore'. It takes some getting used to, the idea that we live in 11 dimensions; I suspect it has a lot to do with a slight resentment that all our lives, so much has been going on that we were unaware of, but I'll leave that analysis for my dissertation in Psychology! For now, the point is to realise that while it might seem a little overpowering initially, when Munchkins in colourful costumes jump at you from all angles, 'Over the Rainbow' is a wonderful place to be.
1.1 The Rainbow

String theories have been around long enough that we are now more or less used to them. They are beautiful, but to the despair of many a string theorist a decade ago, there are almost as many consistent theories as colours in the rainbow! The theories looked lovely and were enjoyed for their ‘decorative’ appeal but for a while it seemed that the promise of unification which had resulted in so much excitement was not a promise that would be kept.

Hope dawned anew in the mid-nineties when dualities were discovered and it was found that the string theories merely appear to be different but in fact are all linked! We can move around from one to the other in a continuous circle and no one theory is any more fundamental than the rest. So it was conjectured that perhaps there is something deeper, something which draws all the string theories into its folds and unifies them. This something was given the name M-Theory [1].

In the history of string theory, we stand now at the moment of the prism; all evidence points to the fact that the rainbow of string theories results from a single ray of light which is diffracted into the multi-coloured richness we see by the prism we unwittingly put in its path. However, for the time being, the white light of M-Theory is so bright that we are almost blinded. It will take a while before our eyes get accustomed to the light and we can see our way clearly in this new land over the rainbow.

1.2 um ... M?

Irreducible representations of a SUSY algebra dictate the possible particle content of the corresponding supersymmetric field theory. States in massless irreducible representations are labelled by helicity. Using the SUSY generators, we can define fermionic creation and annihilation operators which raise or lower helicity. Given the number of supersymmetry generators $N$ in the algebra and the maximal helicity $\Lambda$ we want in a particular theory, we can then work out its possible particle content.

Supergravity [2] is a theory of massless particles with helicity $\Lambda \leq 2$. Maximally extended supergravity in 4 dimensions has 8 supersymmetry generators. Since these generators are Majorana spinors, they have 4 components (supercharges) each which means that the maximum number of supercharges in a supergravity theory is 32. It takes all these supercharges to form a single 11-dimensional Majorana spinor. Consequently, $D = 11$ is the highest dimension in which a supergravity theory can exist.
11-d Supergravity

The particle content of 11-dimensional supergravity [3] can be obtained by studying irreducible representations of the algebra:

\[ \{Q_\alpha, Q_\beta\} = (\gamma^m C^{-1})_{\alpha\beta} P_m + (\Gamma^{mn} C^{-1})_{\alpha\beta} Z_{mn} + (\Gamma^{mnqp} C^{-1})_{\alpha\beta} Z_{mnqp} \]

(1.1)

where \( Q_\alpha \) is a Majorana spinor, \( P_m \) is the momentum operator and the \( Z \)'s are central charges. We find that the spectrum contains a graviton \( G_{MN} \), a rank three anti-symmetric tensor \( A_{MNP} \) and a gravitino \( \psi^M_\alpha \). The existence of all these fields can be intuitively justified: the graviton is needed for invariance under local coordinate transformations and the gravitino is needed for invariance under local supersymmetric transformations. However, the graviton has 128 fermionic degrees of freedom whereas the gravitino has only 44 bosonic degrees of freedom, so an additional 84 bosonic degrees of freedom are needed to make the counting come out right\(^1\) – these are provided by the 3-form \( A_{MNP} \).

Just as a point particle is an electric source for a gauge field (also known as a one-form) \( A_M \) through the coupling \( e \int A_M dX^M \), a \( p \)-brane is an electric source for a \((p + 1)\)-form \( A_{M_1...M_{p+1}} \) through the coupling:

\[ \mu_p \int A_{M_1...M_{p+1}} dX^{M_1} \wedge \ldots \wedge dX^{M_{p+1}} \]

(1.2)

where \( \mu_p \) is the charge of the \( p \)-brane under the \((p + 1)\)-form. Moreover, a \( p \)-brane is also a magnetic source for the \((D - p - 3)\)-form whose field strength is the Hodge dual in \( D \) dimensions of the \((p + 2)\)-form \( F = dA \).

We can hence see that the three-form \( A_{MNP} \) in 11 dimensions couples electrically to a two-brane and magnetically to a five-brane. Accordingly, 11-dimensional supergravity has two types of branes, which are known as the M2-brane (or membrane) and the M5-brane or fivebrane.

Dimensional Reduction

Now consider what happens when the \( X^{10} \) coordinate is a circle of radius \( R \). Since \( X^{10} \equiv y \) is compact, we can Fourier expand 11-dimensional fields as follows:

\[ \Phi_{11}(X) = \sum_n e^{in\pi} \Phi_{10}^n(X) \]

(1.3)

\(^1\)All the counting of the degrees of freedom is done on shell
Each field $\Phi_{11}(X)$ in 11 dimensions leads to an infinite tower of states $\Phi^n_{10}(X)$ in 10 dimensions, with masses that go like $n/R$. This can be seen by writing out the Klein-Gordon equation for the scalar field $\Phi$

$$\nabla^2_{11} \Phi_{11} = 0$$

$$\sum_n e^{inx} [\nabla^2_{10} - \partial_y^2] \Phi^n_{10}(X) = 0$$

$$\Rightarrow [\nabla^2_{10} - (n/R)^2] \Phi^n_{10}(X) = 0 \quad (1.4)$$

At low energies or large distance scales, we do not see the eleventh direction. From our ten-dimensional point of view, the momentum in this $y$ direction hence seems like a mass. In the $R \to 0$ limit where the circle shrinks, all the states $\Phi^n_{10}(X)$ for $n \neq 0$ become infinitely massive so we are left with a ten dimensional theory in which only the massless zero mode $\Phi^0_{10}(x)$ survives.

Concentrating on just the bosonic fields, we find:

$$G_{MN} \rightarrow A_{\mu} = G_{\mu 10} \quad A_{MNP} \rightarrow B_{\mu \nu} = A_{\mu \nu 10}$$

$$\Phi = G_{10,10} \quad A_{\mu \nu \lambda}$$

The 11-dimensional metric gives rise to a metric $G_{\mu \nu}$, a gauge field $A_{\mu}$ and a dilaton $\Phi$ in ten dimensions, whereas the 3-form in eleven dimensions reduces to a 2-form $B_{\mu \nu}$ and a 3-form $A_{\mu \nu \lambda}$ in ten dimensions.

**IIA Supergravity**

In ten dimensions, we can arrange 32 supercharges to form two Majorana-Weyl spinors. If these spinors have the same chirality we get IIB supergravity and if they have opposite chiralities, we end up with IIA supergravity. The particle content of IIA supergravity can be obtained by looking at irreducible representations of the following algebra:

$$\{Q_{\alpha}, Q_{\beta}\} = (\Gamma^m C^{-1})_{\alpha \beta} P_m + (\Gamma^{mn} C^{-1})_{\alpha \beta} Z_{mn} \quad (1.6)$$

$$+ (\Gamma^{mnpqr} C^{-1})_{\alpha \beta} Z_{mnpqr} + (\Gamma_{11} C^{-1})_{\alpha \beta} Z$$

$$+ (\Gamma^m \Gamma_{11} C^{-1})_{\alpha \beta} Z_m + (\Gamma^{mnpq} \Gamma_{11} C^{-1})_{\alpha \beta} Z_{mnpq}$$

The bosonic fields in this theory are a dilaton $\phi$, a metric $G_{\mu \nu}$ a NS-NS 2-form $B_{\mu \nu}$, and R-R one and three forms, $C_1$ and $C_3$. Notice that this is precisely the spectrum obtained in the $R \to 0$ limit of dimensional reduction of 11-dimensional supergravity on a circle.
1.3 aheM!

We have seen that IIA supergravity is a supersymmetric field theory in its own right. However, it can also be obtained as the low energy effective action of Type IIA string theory! The $\alpha' \to 0$ or low energy limit is a consistent truncation of string theory in which all massive states become infinitely massive (since $M^2 \sim 1/\alpha'$) and hence decouple, leaving behind only massless fields. The massless spectrum of IIA string theory coincides exactly with the field content of IIA supergravity.

Recall from the previous section that IIA supergravity can be obtained from dimensional reduction of 11-dimensional supergravity compactified on a circle. More precisely, we have the following expression:

$$ds_{11}^2 = e^{-\frac{2}{3}\phi} ds_{10}^2 + e^{\frac{4}{3}\phi} (dy^2 + A_\mu dX^\mu)$$ (1.7)

Once again $y$ denotes the direction which has been compactified and $\phi$ is the dilaton. It can immediately be seen from the metric that the radius $R$ of the circular direction $y$ is given by

$$R = e^{\frac{2}{3}\phi} = g_s$$ (1.8)

where we have used the fact that the dilaton is related to the string coupling constant as $e^\phi = g_s$. Hence, small $R$ corresponds to weak string coupling and large $R$ to strong coupling.

Putting all these facts together, we are lead to postulate the existence of a theory which would fill the missing corner in this rectangle of relations — we call this M-Theory!

Though we do not know yet what this theory is, we can deduce from the above discussion what it reduces to in certain limits. The low energy
limit of M-Theory is 11d supergravity and the small radius limit of M-Theory compactified on $S^1$ is IIA string theory! This is sometimes turned around and stated as follows: The strong coupling limit of IIA approaches a Lorentz invariant theory in 11 dimensions (M-Theory), whose low energy limit is 11d supergravity.

**Branes descending to IIA**

We can have two types of M-branes, those transverse to $S^1$ and those wrapped on it. An Mp-brane wrapped on the circle ‘loses’ a worldvolume direction when the circle is shrunk ($R \to 0$) and has only $p - 1$ spatial directions remaining when it arrives in IIA. On the other hand, the spatial extension of an M-brane transverse to the circle will not be affected by the $R \to 0$ limit; such a brane will continue to have a $p + 1$ dimensional world-volume in IIA.

An M5-brane wrapped on the M-theory circle thus appears as a D4-brane in Type IIA, whereas an M2-brane wrapped on the circle would become a fundamental string. M5 and M2-branes transverse to the $S^1$ reduce to NS5 and D2 branes respectively in IIA string theory.

**Supergravity as a detective.**

In what follows we will be dealing exclusively with BPS states as solutions of 11-dimensional supergravity. The term BPS refers to the fact that these states saturate a bound which relates their mass to their charge $M \geq Q$. Charge conservation prevents the decay of the least massive state with charge $Q$; it is obvious that the least massive state is the one which saturates the bound! It is also useful to note that supersymmetric states are automatically BPS.

The wonderful thing about BPS states is that since they are stable, they are expected to survive the perilous passage from supergravity to the full quantum theory of which the supergravity is just a low energy limit. In other words, supersymmetric M-brane configurations are present no matter how we slide along the energy scale in 11 dimensions, moving from the all encompassing M-Theory to a small sector therein. Since this small sector, supergravity, is currently the only handle we have on M-Theory, we study M-branes in this context and hope that the clues we find will help us build a picture of the elusive M-Theory.
1.4 Building Blocks

Flat M2-branes and M5-branes are supersymmetric objects in M-Theory and can in fact be thought of as building blocks for the BPS spectrum since a large number of supersymmetric states can be constructed from them [5].

BPS states arise for example, when flat M-branes are wrapped on supersymmetric cycles, so-called because branes wrapped on them preserve some spacetime supersymmetry. There are various other ways in which flat branes can be combined to create supersymmetric configurations; a little later on, we will discuss the rule which dictates how these branes must be placed such that the resulting set-up is BPS.

We start by reviewing a few basic facts about M-branes. In the expressions which follow, $X^\mu$ denotes coordinates tangent to the brane, $X^\alpha$ is used to denote transverse coordinates and $r = \sqrt{X^\alpha X^\alpha}$ is the radial coordinate in this transverse space.

The M5-brane:

A flat M5-brane with worldvolume $X^{\mu_0} \ldots X^{\mu_5}$ is a half-BPS object which preserves 16 real supersymmetries corresponding to the components of a spinor $\chi$ which satisfies the condition:

$$\hat{\Gamma}_{\mu_0 \mu_1 \mu_2 \mu_3 \mu_4 \mu_5} \chi = \chi. \quad (1.9)$$

When this brane is placed in a flat background, the geometry is modified and the resulting spacetime is described by the following metric

$$ds^2 = H^{-1/3} \eta_{\mu \nu} dX^\mu dX^\nu + H^{2/3} \delta_{\alpha \beta} dX^\alpha dX^\beta \quad (1.10)$$

where

$$H = 1 + \frac{a}{r^3} \quad (1.11)$$

and the four-form field strength

$$F_{\alpha \beta \gamma \delta} = \frac{1}{2} \epsilon_{\alpha \beta \gamma \delta \rho} \partial^\rho H. \quad (1.12)$$

of the supergravity three-form.

Together, these equations (1.11) and (1.12) specify the full bosonic content of the supergravity solution for the M5-brane.
The M2-brane:

A flat M2-brane spanning directions $X^{\mu_0}X^{\mu_1}X^{\mu_2}$ is also a half-BPS object. Preserved spacetime supersymmetries correspond to the 16 components of a spinor $\chi$ which survive the following projection

$$\hat{\Gamma}_{\mu_0\mu_1\mu_2}\chi = \chi.$$ (1.13)

Like the M5 brane discussed above, the M2 brane is also a charged massive object which warps the flat space-time in which it is placed. The bosonic fields in the M2-brane supergravity solution are given by

$$ds^2 = H^{-2/3}\eta_{\mu\nu}dX^\mu dX^\nu + H^{1/3}\delta_{\alpha\beta}dX^\alpha dX^\beta$$ (1.14)

$$F_{\mu_0\mu_1\mu_2\alpha} = \frac{\partial_\alpha H}{2H^2},$$ (1.15)

where $$H = 1 + \frac{a}{r^6}.$$ (1.16)

We now pause for a minute to discuss certain features of the landscape which will guide us later when we try to navigate the supergravity solutions of more complicated brane configurations.

Construction Site Rules

One way of generating BPS states from the flat M-branes described above is to construct configurations of intersecting branes. In order for two M-branes to have a dynamic intersection, there must exist a worldvolume field to which this intersection can couple, either electrically or magnetically.

Consider a $p$-brane which has a $q$-dimensional intersection with another $p$-brane. From the point of view of the worldvolume, this intersection must couple to a $(q + 1)$-form in order to be a dynamical object in the $(p + 1)$ dimensional theory.

All $p$-branes contain scalar fields $\phi$ which describe their transverse motion. The 1-form field strength of these scalars $F_1$ is the Hodge dual (on the worldvolume) of the $p$-form field strength $F_p$ of a $(p - 1)$ form gauge field $A_{p-1}$, i.e

$$d\phi = F_1 = *F_p = *dA_{p-1}.$$ Since the gauge field $A_{p-1}$ couples to an object with $(p - 2)$ spatial directions, we see that a $p$-brane can have a $(p - 2)$ dimensional dynamical self intersection [6].
This rule can be derived also using the BPS → no-force argument. Orienting branes of the same type so that they exert no force on each other, as must be the case for stable supersymmetric configurations (in the absence of world-volume fields) it is found that each pair of branes must share \((p - 2)\) spatial directions.

### Intersecting Branes

BPS states can thus be built from multiple M-branes if these are oriented such that each pair of Mp-branes has a \((p - 2)\)-dimensional spatial intersection. Killing spinors of the resulting intersecting brane system are those which survive the projection conditions imposed by each of its flat M-brane constituents.

Consider for example a system of multiple M2-branes. In order to obey the self intersection rule, these membranes must be oriented such that no two membranes share any spatial directions. This criterion is satisfied by, for example, a system of four M2-branes with worldvolumes 012, 034, 056 and 078. The Killing spinors of this configuration are proportional to a constant spinor \(\eta\) obeying the following constraints:

\[
\hat{\Gamma}_{012}\eta = \eta \\
\hat{\Gamma}_{034}\eta = \eta \\
\hat{\Gamma}_{056}\eta = \eta \\
\hat{\Gamma}_{078}\eta = \eta
\]

Since all the Gamma matrices\(^2\) here commute, they are simultaneously diagonalisable and we can proceed to search for eigenstates of the system.

To begin with, notice that every Gamma matrix above squares to one so all the eigenvalues must be either 1 or -1. Since the trace of a matrix is the sum of its eigenvalues, we know that each Gamma matrix has an equal number of ±1 eigenvalues, as all of the above matrices are traceless. Using these matrices we can construct projection operators\(^3\) for each Gamma matrix \(\Gamma_i\) as follows:

\[
P_i^+ = \frac{1}{2}[1 + \Gamma_i] \quad P_i^- = \frac{1}{2}[1 - \Gamma_i]
\]

\(^2\)By Gamma matrices I mean the \(\hat{\Gamma}_{012}\) etc appearing in the above expression.

\(^3\)It is obvious that \(P_i^+\) and \(P_i^-\) are projection operators as they obey

\[
(P_i^+)^2 = P_i^+, \quad (P_i^-)^2 = P_i^-, \quad P_i^+ + P_i^- = 1, \quad P_i^+ P_i^- = 0
\]
Acting $P_i^+$ on a spinor $\eta$, we see that components for which the $\Gamma_i$ eigenvalue is -1 are projected out and the ones which survive must have eigenvalue +1; thus the oft-quoted statement that Gamma matrices (which square to one and are traceless) project out exactly half the spinors. Acting now a second projection operator $P_j^+$, corresponding to another one of the Gamma matrices, we find the following:

$$P_j^+ P_j^+ \eta = \frac{1}{4}[1 + \Gamma_i + \Gamma_j + \Gamma_i \Gamma_j] \eta \quad (1.19)$$

Using the fact that the products of the Gamma matrices are also traceless and square to one, we know that the eigenvalues of $\Gamma_i \Gamma_j$ are also +1 and -1, in equal numbers. We now have the following 4 options corresponding to the possible eigenvalues of $\Gamma_i$ and $\Gamma_j$:

- Both eigenvalues are -1 $\Rightarrow$ The spinor is projected out.
- The eigenvalues are +1 and -1 (or -1 and +1) respectively $\Rightarrow$ Again, the spinor is projected out.
- Both eigenvalues are +1 $\Rightarrow$ The spinor survives.

Hence, only 1/4 of the spinors survive the combined projections due to two Gamma matrices.

Extending this construction, it is easy to see that for a set of $n$ independent gamma matrices the corresponding projection operators will project out an independent half of the supercharges, leaving behind $1/2^n$ supersymmetry.

Applying this to the intersecting membrane configuration described above we see that the four projections imposed together leave behind only two supercharges, or equally, the brane configuration preserves 1/16 supersymmetry.

When Wrappings Become Intersections..

A system of self-intersecting $M_p$-branes corresponds to the singular limit of a single $M_p$-brane wrapping a particular kind of smooth cycle. The cycle in question must be described by embedding functions that factorise. Each of the factors then gives the world-volume of a constituent brane.

In order to clarify this somewhat complicated statement, consider the following simple example of a membrane wrapped on a holomorphic curve.

---

4By which we mean that each matrix squares to one, as do products of the matrices; each matrix is traceless, as are the product matrices, and further all matrices in the set commute.
Take this curve to be \( f(u, v) = uv - c = 0 \) in \( \mathbb{C}^2 \) where \( c \) is a constant. In the limiting case when \( c = 0 \) the curve becomes singular and the function \( f \) obviously factorises to describe a system of two membranes spanning the \( u \) and \( v \)-planes respectively and intersecting only at a point.

In general, a complex structure can be defined on the relative transverse directions (those which are common to at least one but not all of the constituents) of a system of intersecting branes. The intersecting brane configuration then describes the singular limit of an M-brane wrapping a smooth cycle embedded in this complex subspace of spacetime. Due to the \((p - 2)\) self intersection rule, a system of \( n \) orthogonally intersecting membranes has a relative transverse space \( \mathbb{C}^n \).

Since preserved supersymmetries should be invariant under changes of the constant \( c \) in the holomorphic function \( f(u, v) \), the Killing spinors of the wrapped brane configuration \((c \neq 0)\) are the same as those for a system of \( n \) orthogonal membranes (the \( c = 0 \) limit) intersecting according to the \((p - 2)\) rule.

1.5 Landmarks

The Munchkins have now brought you to the end of their domain. To get to your destination though, you’ll just have to ‘Follow the Yellow Brick Road’. However, during your long walk, it might be useful to keep in mind some of the things you have learnt in this new land Over The Rainbow.

On the world-volume of a flat M-brane, we would expect to have Poincare invariance. Hence, a metric describing space-time in the presence of such a brane should not have any dependence on coordinates tangent to the M-brane. Also, we expect the effects of the brane on spacetime to decrease as we move away from it and furthermore, the configuration is invariant under rotations in the transverse directions. From these considerations of isometry alone, we can conclude that the metric depends only on the radial coordinate in the space transverse to the flat M-brane world-volume. Notice that this is true for both the M-brane solutions discussed previously.

Moreover, each solution can be specified completely in terms of not just an arbitrary but in fact a harmonic function of this radial coordinate. Tracing the origin of this condition, we are lead to the equation of motion for the field strength \( d*F = 0 \). Writing this out in component form, we find

\[
\partial_I(\sqrt{|\det g_{IJ}|}F^{IJKL}) = 0
\]

\( (1.20) \)
which implies that

\[ \nabla^2 H = \sum_{\alpha} \frac{\partial^2 H}{\partial X_\alpha^2} = 0 \quad (1.21) \]

so the function \( H \) must be a solution to the flat space Laplacian. Both these characteristics (i.e. the isometries and the role of the harmonic function) will have interesting generalisations when we consider supergravity solutions for non-trivial brane configurations.
Chapter 2
The Yellow Brick Road

In the paper [7] where the Harmonic Function rule was proposed, before even stating what the rule was, Tseytlin pointed out the assumptions that went into it. We should bear these in mind so that we only apply the rule to the configurations it was designed to describe in the first place. For starters,

- The harmonic function rule can be used only to construct supergravity solutions for intersecting $p$-brane systems which are smeared along the relative transverse directions (those which are tangent to at least one but not all of the constituent branes). As a result, the metric is then independent of these coordinates and is a function only of the overall transverse directions, which are not tangent to the world-volume of
any brane in the system. It is mentioned in passing that a class of more general solutions is expected to exist, such that each constituent brane has a different transverse space. As we will see later on, these are the cases covered by the Fayyazuddin-Smith metric ansatz!

- Secondly, since these supergravity solutions are governed by harmonic functions of the radial coordinate in the transverse space, this (overall) transverse space must be at least three dimensional if the harmonic functions are to decay at infinity.

This excludes from the present consideration certain intersecting M-brane systems which are allowed by the \((p - 2)\) self-intersection rule [6]. As will later be seen, these missing configurations are encompassed by the Fayyazuddin-Smith ansatz.

- Lastly, it is useful also to remember that the only configurations to which the rule is expected to be applicable are those for which the Chern-Simons contribution to the equation of motion for the four-form vanishes, i.e \(F \wedge F = 0\).

Bosonic backgrounds constructed by combining the individual supergravity solutions of constituent branes can still be solutions to the equations of motion of \(D=11\) supergravity. The basic observation is that for brane bound states with zero binding energy, it is possible to assign an independent harmonic function to each constituent intersecting brane. The argument for this is sketched below.

A single brane supergravity solution can be expressed completely in terms of one harmonic function. For an extremal (no binding energy) BPS configuration of \(N\) branes, there is no force between the constituents and hence no obstruction to moving one of the branes far apart from the others. When a brane is moved sufficiently far apart from the rest, their effects on it are negligible and it is to all intents and purposes free. Fields near it should thus approximate the fields in the supergravity solution of a single brane. Since any or all \(N\) of the branes can arbitrarily be moved back and forth at no cost to the energy, we would expect the solution describing a configuration of \(N\) branes to be parameterized by \(N\) independent harmonic functions.

### 2.1 The Rule of the Harmonic Functions

In the presence of an intersecting M-brane configuration, the \((10 + 1)\) dimensional spacetime naturally 'splits up' into three separate parts. The
directions common to all M-branes are referred to as the common tangent directions. Tangent space indices here will be denoted by \( a, b \), and curved space indices by \( \mu \nu \). In the wrapped brane picture, these are the worldvolume directions which are left flat. The supersymmetric cycle which the M-brane wraps is embedded in to the subspace spanned by the relative transverse directions, so called as they are tangent to at least one but not all of the constituent branes. We define complex coordinates on this space where flat and curved indices are denoted by \( m, \overline{m} \), and \( M, \overline{N} \) respectively. Finally there are the overall transverse directions which span the space transverse to all constituent branes in the intersecting brane system; in the wrapped brane picture, these are the directions which are transverse to both the brane worldvolume and the embedding space.

**It is Decreed · · ·**

As mentioned above, the harmonic function rule gives a recipe for ‘superposing’ the individual bosonic fields in the supergravity solutions of each of the component branes in an intersecting brane system.

- **The Metric**
  Assigning a harmonic function \( H_a \) to each constituent M-brane, we proceed to construct the metric for a multi-brane configuration by taking our cue from the metric for a single brane. In analogy to that case, directions tangent to the \( i^{th} \) M5-brane are multiplied by a factor of \( H_i^{-1/3} \) whereas directions transverse to it are multiplied by \( H_i^{2/3} \).

  Similarly, a metric describing the background created by a system of intersecting membranes, can be constructed by ensuring that the coordinates along the worldvolume of the \( j^{th} \) M2-brane carry a factor of \( H_j^{-2/3} \) while transverse coordinates are multiplied by \( H_j^{1/3} \).

- **The Field Strength**

  Since the field strength components due to each constituent M-brane carry different indices, the field strength of the intersecting brane configuration can be obtained merely by adding the individual field strengths corresponding to each M-brane.

This rule is made clear by its application to the following systems.
Two Membranes

|   | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|----|
| M2| × | × | × |   |   |   |   |   |   |   |    |
| M2| × | × | × |   |   |   |   |   |   |   |    |

Assigning a factor of $H_1$ to the first membrane and $H_2$ to the second, we use the harmonic function rule to write down the metric and field strength. The metric is given by

$$ds^2 = H_1^{1/3} H_2^{1/3} \left[ -H_1^{-1} H_2^{-1} dX_0^2 + H_1^{-1} (dX_1^2 + dX_2^2) + H_2^{-1} (dX_3^2 + dX_4^2) + (dX_5^2 + dX_6^2 + dX_7^2 + dX_8^2 + dX_9^2 + dX_{10}^2) \right]$$

and the non-vanishing components of the field strength are

$$F_{012\alpha} = \frac{\partial_\alpha H_1}{H_1^2} \quad F_{034\alpha} = \frac{\partial_\alpha H_2}{H_2^2}$$

where $H_1$ and $H_2$ are functions only of $X^\alpha$ for $\alpha = 5, 6 \ldots 10$.

Three Membranes

Introducing now a third membrane, characterized by the harmonic function $H_3$, we have the following configuration

|   | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|----|
| M2| × | × | × |   |   |   |   |   |   |   |    |
| M2| × | × | × |   |   |   |   |   |   |   |    |
| M2| × | × | × |   |   |   |   |   |   |   |    |

with metric

$$ds^2 = H_1^{1/3} H_2^{1/3} H_3^{1/3} \left[ -H_1^{-1} H_2^{-1} H_3^{-1} dX_0^2 + H_1^{-1} (dX_1^2 + dX_2^2) + H_2^{-1} (dX_3^2 + dX_4^2 + dX_5^2 + dX_6^2) + dX_7^2 + dX_8^2 + dX_9^2 + dX_{10}^2 \right]$$

and field strength,

$$F_{012\alpha} = \frac{\partial_\alpha H_1}{H_1^2} \quad F_{034\alpha} = \frac{\partial_\alpha H_2}{H_2^2} \quad F_{056\alpha} = \frac{\partial_\alpha H_3}{H_3^2}$$

Here, $H_1$, $H_2$ and $H_3$ are functions of the transverse directions $X^\alpha$ where $\alpha = 7 \ldots 10$. 
2.1 The Rule of the Harmonic Functions

Two Fivebranes

\[
\begin{array}{cccccccccc}
0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\
M5 & \times & \times & \times & \times & \times & \times & \times & \times & \times & \times \\
M5 & \times & \times & \times & \times & \times & \times & \times & \times & \times & \times \\
\end{array}
\]

The harmonic function rule dictates the following metric

\[
ds^2 = H_1^{2/3} H_2^{2/3} [H_1^{-1} H_2^{-1} (-dX_0^2 + dX_1^2 + dX_2^2 + dX_3^2) + H_1^{-1} (dX_4^2 + dX_5^2) + H_2^{-1} (dX_6^2 + dX_7^2) + (dX_8^2 + dX_9^2 + dX_{10}^2)]
\]

and field strength components

\[
F_{67\alpha\beta} = \epsilon_{\alpha\beta\gamma} H_1 \\
F_{45\alpha\beta} = \epsilon_{\alpha\beta\gamma} H_2 \\
\]

The functions $H_1$ and $H_2$ depend only on the overall transverse directions labelled by $\alpha$ which takes values $8, 9, 10$.

Three Fivebranes

Adding now a third M5-brane in the following manner:

\[
\begin{array}{cccccccccc}
0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\
M5 & \times & \times & \times & \times & \times & \times & \times & \times & \times & \times \\
M5 & \times & \times & \times & \times & \times & \times & \times & \times & \times & \times \\
M5 & \times & \times & \times & \times & \times & \times & \times & \times & \times & \times \\
\end{array}
\]

we find that spacetime is described by the metric

\[
ds^2 = H_1^{2/3} H_2^{2/3} H_3^{2/3} [H_1^{-1} H_2^{-1} H_3^{-1} (-dX_0^2 + dX_1^2) + H_1^{-1} H_2^{-1} (dX_4^2 + dX_5^2) + H_1^{-1} H_3^{-1} (dX_6^2 + dX_7^2) + H_2^{-1} H_3^{-1} (dX_8^2 + dX_9^2 + dX_{10}^2)]
\]

and field strength components are

\[
F_{67\alpha\beta} = \epsilon_{\alpha\beta\gamma} H_1 \\
F_{45\alpha\beta} = \epsilon_{\alpha\beta\gamma} H_2 \\
F_{23\alpha\beta} = \epsilon_{\alpha\beta\gamma} H_3 \\
\]

The harmonic functions $H_1$, $H_2$ and $H_3$ depend on $X^\alpha$ where $\alpha = 8, 9, 10$. 
Exiled!

Under the Rule of the Harmonic Functions, the configurations described below are banished on the charge that they do not have a sufficient number of overall transverse directions. The minimum number of transverse directions required for a law abiding brane configuration living under the Harmonic Function rule is three. At least three overall transverse directions are needed in order for the functions $H_i$ to be solutions of the flat Laplacian in this subspace with the right behaviour at infinity and thus be deserving of the title 'Harmonic' functions.

Two Transverse Directions

|   | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|----|
| M2 | × | × | × |   |   |   |   |   |   |   |    |
| M2 | × |   | × | × |   |   |   |   |   |   |    |
| M2 | × |   |   | × | × |   |   |   |   |   |    |
| M2 | × |   |   |   |   |   |   | × | × |   |    |

(2.12)

One Transverse Direction

|   | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|----|
| M5 | × | × | × | × |   | × |   |   |   |   |    |
| M5 | × | × | × | × |   |   | × | × |   |   |    |
| M5 | × | × | × | × |   |   |   | × | × |   |    |

(2.13)

No Transverse Directions!

|   | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|----|
| M2 | × | × | × |   |   |   |   |   |   |   |    |
| M2 | × |   | × | × |   |   |   |   |   |   |    |
| M2 | × |   |   | × | × |   |   |   |   |   |    |
| M2 | × |   |   |   |   |   |   | × | × |   |    |
| M2 | × |   |   |   |   |   |   |   |   | × | × |

(2.14)

2.2 The Cycles we Turn To

Supersymmetric cycles have the defining property that branes wrapping them preserve some supersymmetry. Holomorphic cycles are known to be supersymmetric; in fact being the simplest examples of supersymmetric cycles they are the ones considered most often. All the configurations studied in this thesis describe M-branes wrapping holomorphic cycles, so we now
2.2 The Cycles we Turn To

take a small detour and see how holomorphicity leads to supersymmetry and study the conditions it imposes on the surviving supercharges.

A wrapped $p$-brane whose embedding into spacetime is described by $X^\mu(\sigma^1)$ is said to be supersymmetric only if the background it gives rise to admits at least one Killing spinor $\chi$ such that

$$\chi = \frac{1}{p!} \sqrt{h} \varepsilon_{\alpha_1 \ldots \alpha_p} \Gamma_{M_1 \ldots M_p} \partial_{\alpha_1} X^{M_1} \ldots \partial_{\alpha_p} X^{M_p} \chi. \quad (2.15)$$

where $\Gamma_{M_1 \ldots M_p}$ is the completely anti-symmetrized product of $p$ eleven dimensional $\Gamma$ matrices, $\varepsilon$ is the $p$-dimensional Levi-Civita symbol and $h$ denotes the determinant of the induced metric $h_{ij}$ on the brane.

For a brane embedded in a complex space with Hermitean metric $G_{\bar{U}\bar{V}}$, the induced metric on the worldvolume is

$$h_{ij} = [\partial_i X^U \partial_j X^{\bar{V}} + \partial_j X^U \partial_i X^{\bar{V}}] G_{U\bar{V}} \quad (2.16)$$

If the brane is holomorphically embedded, the supersymmetric cycle it wraps must be even-dimensional; we denote its dimension by $2m$. Defining now a complex structure on this cycle which is compatible with the complex structure in spacetime, the induced metric can be written in the form:

$$h_{u\bar{v}} = \partial_u X^U \partial_{\bar{v}} X^{\bar{V}} G_{U\bar{V}} \quad (2.17)$$

Hermiticity of the embedding space metric $G_{U\bar{V}}$, will imply that the induced metric on the worldvolume is Hermitian as well; as such $\det h_{u\bar{v}} = \sqrt{h}$ is given by

$$\sqrt{h} = \frac{1}{m!} \varepsilon^{u_1 \ldots u_m} \varepsilon^{\bar{v}_1 \ldots \bar{v}_m} h_{u_1 \bar{v}_1} \ldots h_{u_m \bar{v}_m} \quad (2.18)$$

The Killing spinor equation (2.15) now takes the form

$$\sqrt{h} \chi = \left( \frac{1}{m!} \right)^2 \varepsilon^{u_1 \ldots u_m} \varepsilon^{\bar{v}_1 \ldots \bar{v}_m} \Gamma_{U_1 \bar{V}_1 \ldots U_m \bar{V}_m} \partial_{u_1} X^{U_1} \ldots \partial_{\bar{v}_m} X^{\bar{V}_m} \chi$$

$$= \left( \frac{1}{m!} \right)^2 \varepsilon^{u_1 \ldots u_m} \varepsilon^{\bar{v}_1 \ldots \bar{v}_m} \Gamma_{u_1 \bar{v}_1 \ldots u_m \bar{v}_m} \quad (2.19)$$

Inserting the expression for $h$ from (2.18), we can read off the projection conditions on the Killing spinor associated with an M-brane wrapping the given $2m$-dimensional holomorphic cycle.

In particular, for a two-cycle, we find

$$\Gamma_{u\bar{v}} \chi = h_{u\bar{v}} \chi \quad (2.20)$$

and for a four-cycle,

$$\Gamma_{u\bar{v}s\bar{s}} \chi = [h_{u\bar{v}} h_{s\bar{s}} - h_{u\bar{s}} h_{v\bar{v}}] \chi \quad (2.21)$$
Both these conditions will be used later on when we discuss holomorphic embeddings of M-branes.

### The Determinant of a Hermitean Metric.

Assume that the metric $h_{MN}$ in an $m$ dimensional complex space is Hermitean. Hermiticity implies that $h_{ij} = h_{\bar{i}\bar{j}} = 0$. Thus, in the $2m \times 2m$ dimensional matrix which would conventionally be used to represent the metric, there are only $4m^2 - m^2 - m^2$ entries which are non-zero and none of these are along the diagonal. Using then the fact that a metric must be symmetric, we find that the degrees of freedom are reduced by a further half, leaving only $m^2$ entries to determine the metric completely. These can be arranged into an $m \times m$ matrix which specifies the Hermitean metric.

Consider a simple example to illustrate this point: Let $h_{MN}$ be the metric in a $2m$ real dimensional space.

$$\det h_{MN} = \frac{1}{(2m)!} \varepsilon^{I_1 \ldots I_{2m}} \varepsilon_{J_1 \ldots J_{2m}} h_{I_1 J_1} \cdots h_{I_{2m} J_{2m}}$$

If a complex structure is defined on this space, the anti-symmetric tensor $\varepsilon^{I_1 \ldots I_{2m}}$ splits up into a product of two tensors, with holomorphic and anti-holomorphic indices as follows:

$$\frac{1}{(2m)!} \varepsilon^{I_1 \ldots I_{2m}} = \frac{1}{m!} \varepsilon^{i_1 \ldots i_m} \frac{1}{m!} \varepsilon^{\bar{j}_1 \ldots \bar{j}_m}$$

If this complex structure is such that the resulting metric is hermitean, the determinant can be written as

$$h \equiv \det h_{MN} = [\det h_{ij}]^2$$

where

$$\det h_{ij} = \frac{1}{m!} \varepsilon^{i_1 \ldots i_m} \varepsilon^{\bar{j}_1 \ldots \bar{j}_m} h_{i_1 \bar{j}_1} h_{i_2 \bar{j}_2} \cdots h_{i_m \bar{j}_m}$$

### 2.3 Killing (some) Spinors

The amount of supersymmetry preserved by a p-brane with worldvolume $X^{M_1} \cdots X^{M_p}$ is given by the number of spinors which satisfy (2.15).
Because we are dealing with holomorphic cycles, and a complex structure has been defined on the embedding space, it makes sense to re-write the Clifford algebra in complex coordinates as well. This is in fact a very useful thing to do, as it turns out that the flat space Clifford algebra when expressed in complex coordinates resembles the algebra of fermionic creation and annihilation operators! We can thus express spinors on a complex manifold as states in a Fock space \[\mathbb{F}\]. Defining \(\Gamma\) matrices for a complex coordinate \(z_j = x_j + iy_j\) as follows:

\[
\Gamma_{z_j} = \frac{1}{2}(\Gamma_{x_j} + i\Gamma_{y_j})
\]

\[
\Gamma_{\bar{z}_j} = \frac{1}{2}(\Gamma_{x_j} - i\Gamma_{y_j})
\]

the Clifford algebra in \(C^n\) takes the form

\[
\{\Gamma_{z_i}, \Gamma_{z_j}\} = 2\eta_{ij}. \tag{2.23}
\]

Declaring \(\Gamma_{z_i}\) to be creation operators and \(\Gamma_{\bar{z}_j}\) to be annihilation operators, it is now clear that a Fock space can be generated by acting the creation operators on a vacuum. Because there are \(n\) creation operators, each state in the Fock space is labelled by \(n\) integers taking values 0 or 1 which correspond to its fermionic occupation numbers.

We illustrate the utility of this construction by considering in detail an M5-brane wrapping a two-cycle in \(\mathbb{C}^3\). Let the holomorphic coordinates in \(\mathbb{C}^3\) be \(u, v, w\). The Clifford algebra can then be explicitly written out as

\[
\{\Gamma_u, \Gamma_v\} = \{\Gamma_v, \Gamma_w\} = \{\Gamma_w, \Gamma_u\} = 1; \tag{2.24}
\]

all other anti-commutators are zero. Since the Fock vacuum is the state where none of the oscillators are excited it is denoted by \(|000\rangle\) and the highest weight state where all oscillators are excited is denoted by \(|111\rangle\). Hence we have

\[
\Gamma_z|000\rangle = 0 \tag{2.25}
\]

\[
\Gamma_z|111\rangle = 0.
\]

where \(z\) can take values \(u, v, w\). A generic spinor \(\psi\) in \(\mathbb{C}^3\) can then be decomposed in terms of Fock space states as follows:

\[
\psi = a|000\rangle + b|100\rangle + c|010\rangle + d|001\rangle + e|110\rangle + f|101\rangle + g|011\rangle + h|111\rangle \tag{2.26}
\]
An eleven dimensional spinor $\chi$ can be written as a sum of terms of the form $\alpha \otimes \psi \equiv \alpha \otimes |n_u, n_v, n_w>$ where $\alpha$ is a four-dimensional spinor and $n_z$, for $z = u, v, w$ are the fermionic occupation numbers of the state. Expressing spinors in this way and writing down the supersymmetry preservation conditions in terms of $\Gamma_z$, it becomes very easy to figure out the Killing spinors.

Consider an M5-brane wrapping a two-cycle in $\mathbb{C}^3$. From \eqref{2.15} we see that supersymmetry is preserved only if solutions can be found to the equation \eqref{2.27}

\begin{equation}
    i \Gamma_{0123} \Gamma_{m\bar{n}} \chi = \eta_{m\bar{n}} \chi
\end{equation}

For the time being, we restrict ourselves to the implications of the above condition on $\psi$ alone. We then find the following constraints:

\begin{align}
    \Gamma_{uv} \psi &= \Gamma_{uv} \bar{\psi} = 0 \\
    \Gamma_{vw} \psi &= \Gamma_{vw} \bar{\psi} = 0
\end{align}

(2.28)

Those which were Killed, Live!

We pause for a minute to see what these constraints say about the spinor. Take a simple constraint, say $\Gamma_u \chi = 0$. Since $\Gamma_u$ is a creation operator, it acts on $\psi$ such that all states with $n_u = 0$ are taken to $n_u = 1$ but states which already had $n_u = 1$ are killed. Writing this out we find

\begin{equation}
    \Gamma_u \psi = a|100> + b|110> + c|101> + d|111> = 0
\end{equation}

Since all Fock space states are independent, this implies that $a = b = c = d = 0$. The spinor $\psi$ can now only have the components $|000>, |010>, |001>$ and $|011>$. Hence, in effect what a constraint of the type $\Gamma \psi = 0$ does is to kill the spinors which survive the action of $\Gamma$ and keep the spinors which were eliminated!

Hence \eqref{2.28} eliminates all states which have the following occupation numbers.

\begin{align}
    n_u = 0 &\text{ and } n_v = 1, \text{ or } n_u = 0 & n_w = 1, \\
    n_v = 0 &\text{ and } n_u = 1, \text{ or } n_v = 0 & n_w = 1, \\
    n_w = 0 &\text{ and } n_u = 1, \text{ or } n_w = 0 & n_v = 1.
\end{align}

(2.29)
The only components of $\psi$ which survive this treatment are $|000>$ and $|111>$. Recall that (2.27) also gives rise to constraints

$$i\Gamma_{0123}\Gamma_{zz}\chi = \frac{1}{2}\chi$$ (2.30)

Since

$$\Gamma_{zz}|000> = |000>$$
$$\Gamma_{zz}|111> = -|111>$$ (2.31)

these constraints can be used to determine the four-dimensional chirality of the spinors $\alpha$ and $\beta$ when $\chi$ is expressed as

$$\chi = \alpha \otimes |000> + \beta \otimes |111>$$ (2.32)

Imposing (2.30) we find that

$$i\Gamma_{0123}\alpha = \alpha$$
$$i\Gamma_{0123}\beta = -\beta$$ (2.33)

Counting

The last step is to count the number of supercharges preserved by this configuration. As a generic spinor in 11 dimensions, $\chi$ starts out life with 32 complex components: 4 complex components come from the Dirac spinor $\alpha$ and, as we have seen explicitly, 8 components come from $\psi$. After the constraints (2.28) are imposed, only 2 of these 8 components survive, so the spinor $\chi$ is left with $4 \times 2$ complex degrees of freedom. Determining the chirality of $\alpha$ and $\beta$ cuts the degrees of freedom down by a further half. Finally we impose the Majorana condition, which essentially states that $\chi$ can be completely determined by $\alpha|000>$ and thus has only the 4 real degrees of freedom of this state.

An M5-brane wrapping a holomorphic two-cycle in $C^3$ hence preserves 4 of the 32 possible supercharges, or 1/8 of the total spacetime supersymmetry.

2.4 Pushing the Boundaries

Now that we have a rule which allows us to construct supergravity solutions for a certain class of BPS bound states of branes, we can try to extend the scope of this rule to a wider class of supersymmetric configurations.
A natural step in this direction would be to make the harmonic functions depend on coordinates in the relative transverse space so that they may describe branes which are not smeared in this subspace. However, even if we start from this assumption it turns out that at least one of the branes becomes delocalized because of the translational invariance which results from smearing.

To illustrate this point consider the simplest possible example, that of 2 M2-branes, which span worldvolumes $X^0X^1X^2$ and $X^0X^3X^4$ intersecting at a point, and are characterized by functions $H_1$ and $H_2$ respectively. $H_1$ and $H_2$ are now allowed to depend on all spatial coordinates, in contrast to what was acceptable previously for the harmonic function rule.

If the membrane spanning $X^0X^3X^4$ is localized then we should have translational invariance on its world-volume implying that the membrane with worldvolume $X^0X^1X^2$ cannot be localized in $X^3$ or $X^4$ and must instead be smeared. So, as far as this M2-brane is concerned, life is just as it was previously under the harmonic function rule; $H_1$ still depends only on the overall transverse directions $X^5 \ldots X^{10}$: Moreover, since it also obeys the flat space Laplace equation in these directions, $H_1$ is a Harmonic function of the radial coordinate in this space.

However, since the M2-brane with worldvolume $X^0X^3X^4$ is localised, the function $H_2$ characterising it obeys a different equation. Recall that the origin of the Harmonic function condition was the equation of motion for the four-form $d \ast F = 0$. Applying this to the case at hand, we now find that $H_2$ obeys the curved space Laplace equation [17]

$$H_2(\partial_3^2 + \partial_4^2)H_1 + (\partial_5^2 \ldots \partial_{10}^2)H_1 = 0$$

and is hence called a generalised harmonic function

2.5 The Kingdom is larger than it seems..

The Harmonic Functions rule very successfully in their own domain, but they never claimed to exercise absolute control over the entire kingdom of supergravity solutions. Branes smeared over the relative transverse directions are loyal subjects, (as long as they have three or more overall transverse directions) but, as we have seen in the previous section, localised intersections defy the Rule and consequently must be described by a wider class of spacetimes.

In an attempt to address this problem, Fayyazuddin and Smith [13] came up with a metric ansatz whose form is dictated by the isometries of
the background in the presence of a wrapped brane configuration, (which could have an intersecting brane system as its singular limit).

The spacetimes they considered describe M-branes wrapping holomorphic cycles so a complex structure has been described on the embedding space in order to make holomorphicity transparent. If the supersymmetric cycle is embedded into an $2n$ dimensional subspace, then we would expect the remaining directions of spacetime $X^\alpha$, (i.e those transverse to the flat worldvolume directions $X^\mu$ and also to the embedding space), to be rotationally invariant. Rotational invariance in $X^\alpha$ implies that the metric is diagonal in this subspace and that the undetermined functions in the metric ansatz depend only on the radial coordinate $\rho = \sqrt{X_\alpha X^\alpha}$.

Isometries, however, fail to guide us when it comes to dictating the form of the Hermitean metric $G_{MN}$ in the complex space where the holomorphic cycle is embedded. All we can say is about the Hermitean metric is that it too, along with $H_1$ and $H_2$, must be independent of $X^\mu$ since we would Lorentz symmetry to be preserved along the un-wrapped directions of the M-brane worldvolume. This Lorentz symmetry also implies that the metric is diagonal in $X^\mu$.

A metric incorporating the above symmetries takes the form:

$$ds^2 = H_1^2 \eta_{\mu\nu} dX^\mu dX^\nu + 2G_{MN} dz^M dz^N + H_2^2 \delta_{\alpha\beta} dX^\alpha dX^\beta$$  \hspace{1cm} (2.35)

We now describe some of its key features.

- Comparing this metric to that of a flat M-brane, we see one major difference. The metric for a wrapped brane too, depends only on the radial coordinate $r$ in the space transverse to the brane but because of the non-trivial worldvolume of this brane the transverse space is not as simple to define as it was earlier. We know that the transverse space now is some combination of what we call the relative transverse (or embedding space) directions and the overall transverse directions, but a more exact statement is hard to make, unless we know explicitly the geometry of the worldvolume.

Depending on how the supersymmetric cycle lies in the relative transverse/embedding manifold, it could of course happen that $r$ does not depend on all the coordinates in this space. But in order to cover all possible cases and for the purposes of making a general ansatz, we assume $r$ is a function of all but the overall common directions.

- The Hermitean metric in the above ansatz will have, in general, off diagonal components as well. As will be shown later in an explicit
example, it is these components of the metric which allow us to move from a supergravity solution with a smeared intersection to one which has a localised intersection. It also allows us to incorporate configurations of branes intersecting at angles – a class of systems not encompassed by the harmonic function rule.

- Allowing $r$ to depend on the embedding space coordinates has another implication also. For supergravity solutions constructed via the harmonic function rule, the field strength $F_{p+2} = dA_{p+1}$ consisted only of components obtained from the gauge potential by acting $d$ in the overall transverse directions, as these were the only ones on which the metric was allowed to depend.

Using the Fayyazuddin-Smith ansatz instead, where the fields are allowed to depend on the relative transverse directions as well, we can now apply the exterior derivative in these transverse directions also, leading to previously unknown components for the field strength!

- Moreover, by allowing $r$ to depend on the embedding space coordinates, we get rid of the objection which the harmonic function rule levelled at brane configurations with less than three overall transverse directions. Such configurations too, are welcomed into the fold of this new ansatz.

New lands however, can not be added to the kingdom of supergravity solutions without paying a price. The battle we must wage now is against non-linear differential equations. Earlier, while discussing the flat M-brane solutions, we pointed out that the functions $H$ were harmonic due to the fact that equations of motion for the four-form field strength $d^* F = 0$ reduced to the flat space Laplace equation. With the more complicated metric we have now, this is no longer the case. The undetermined functions in the metric ansatz are thus harmonic no longer, and obey a more complicated differential equation, which will in practise be very difficult to solve.

**New Frontiers**

The fact that a brane configuration preserves supersymmetry implies that the supersymmetric variation of the gravitino $\delta_\chi \Psi$ vanishes in this background if the variation parameter is a Killing spinor. If we require this to be true for a given metric, we find a set of conditions relating the metric to components of the field strength of the supergravity three-form. If the resulting
metric and four-form also obey the constraints $dF = 0$ and $d \star F = 0$ then Einstein’s equations are guaranteed to be satisfied and we have determined the bosonic components of a BPS solution to 11-dimensional supergravity.

Denoting flat (tangent space) indices in 11-dimensional spacetime by $i, j$ and curved indices by $I, J$, the bosonic part of the action for 11d supergravity can be written as

$$S = \int \sqrt{-G} \{ R - \frac{1}{12} F^2 - \frac{1}{432} \epsilon^{I_1 \ldots I_{11}} F_{I_1 \ldots I_4} F_{I_5 \ldots I_8} A_{I_9 \ldots I_{11}} \} \quad (2.36)$$

and the supersymmetric variation of the gravitino is given by

$$\delta \Psi_I = (\partial_I + \frac{1}{4} \omega^{ij}_I \Gamma_{ij} + \frac{1}{144} \Gamma^{JKLM}_{IJKL} F_{JKLM} - \frac{1}{18} \Gamma^{JKLM}_{IJ} F_{IJ}) \chi. \quad (2.37)$$

Following the logic outlined in the previous section, we begin our search for solutions of this theory by writing down an ansatz for the space-time metric following Fayyazuddin and Smith.

We then enforce supersymmetry preservation by setting $\delta \Psi_I = 0$. This expression involves components of the spin connection which can be calculated from the metric ansatz using the formula

$$2 \omega^{ij}_I = e^{ij}(\partial_I e^j - \partial_J e^i) - e^{ij}(\partial_I e^j - \partial_J e^i) - e^{ik} e^{jL}(\partial_K e^l - \partial_L e^k) e_{iI} \quad (2.38)$$

Since $\chi$ can be expressed as a sum of Fock space states, the condition $\delta \Psi_I = 0$ amounts to a sum of linearly independent constraints (one arising from every Fock state), each of which must be put to zero separately. This leads to a set of relations between various components of the field strength and the metric.

The supergravity solution is obtained \[15\] when these conditions are supplemented by the equations of motion and Bianchi identity for the four-form field strength $dF = 0$ and $d \star F = 0$. This is the method followed in the preceding section, to find the bosonic solutions of 11-dimensional supergravity for all M-branes wrapping holomorphic curves such that $F \wedge F = 0$.

\section*{2.6 Charging forth.}

\subsection*{Membranes}

The Fayyazuddin-Smith ansatz for a metric describing the supergravity background created by an M2-brane wrapping a holomorphic curve in $\mathbb{C}^n$ is \[15\]:

$$ds^2 = -H_1^2 dt^2 + 2H_1^{-1} g_{MN} dZ^M dZ^N + H_2^2 \delta_{\alpha\beta} dX^\alpha dX^\beta. \quad (2.39)$$
Here $Z^M$ are used to denote the $n$ complex coordinates, $X^\alpha$ span the remaining $(10 - 2n)$ transverse directions and a factor of $H_1^{-1}$ is pulled out of the Hermitian metric for later convenience.

A Hermitian two-form $\omega_g$ associated with the metric is defined such that 

$$\omega_g = ig_{MN}dz^M \wedge d\bar{z}^N.$$ 

Supersymmetry preservation imposes a constraint on the Hermitian metric $g_{MN}$ and in addition states that

$$H^{-1} = \sqrt{\det g_{MN}}$$

where $H \equiv H_1^{-3} = H_2^6$. Further, a set of relations between components of the metric and field strength is obtained. These can be solved for the non-zero components of the four-form.

Killing spinors of this configuration being Majorana can be expressed as 

$$\chi = \alpha + C\alpha^*$$

if $C$ is the charge conjugation matrix. Hence, the spinor $\alpha$ is all that is needed to completely specify $\chi$.

The Bianchi identity $dF = 0$ is automatically satisfied, but the equation of motion $d * F = 0$ must be imposed by hand. This leads to a complicated non-linear differential equation involving $g_{MN}$ and $H$ which must be satisfied by any supergravity solution.

Having thus discussed the structure of the supergravity solutions for membranes wrapping holomorphic two-cycles in manifolds of various dimension, we proceed to present the results.

In some cases, when the holomorphic functions describing the embedding of the manifold are factorizable, the wrapped brane configuration has an intersecting brane interpretation when the curve becomes singular. Along with the components of the solution described above, we will also present in our analysis of each configuration [15], [19], the intersecting brane system which would arise in the singular limit of this curve, if indeed such a singular limit exists.

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1As pointed out in [15], this determines $H$ only up to a rescaling by an arbitrary holomorphic function
2.6 Charging forth.

**M2 wrapping a 2-cycle in $\mathbb{C}^2$**

**Metric Constraint**

\[ \partial(H \omega_g) = 0 \]  \hspace{1cm} (2.40)

**Four-Form Field Strength**

The non-zero components of the four-form field strength are given by:

\[ F_{0M\bar{N}\alpha} = -\frac{i}{2} \partial_\alpha g_{M\bar{N}}, \]  \hspace{1cm} (2.41)

\[ F_{0M\bar{N}\bar{P}} = -\frac{3i}{4}[\partial_{\bar{P}}g_{M\bar{N}} - \partial_{\bar{N}}g_{M\bar{P}}] - \frac{i}{2}(\partial_{\bar{P}}\ln H)g_{M\bar{N}} - (\partial_{\bar{N}}\ln H)g_{M\bar{P}} \]

and their complex conjugates.

**Killing Spinors**

The eight Killing spinors of this configuration are specified by

\[ \alpha = H^{-1/6} \eta |00> \]  \hspace{1cm} (2.42)

such that $\eta$ is a constant spinor obeying the projection condition

\[ \Gamma_{012} \eta = \Gamma_{034} \eta = \eta \]  \hspace{1cm} (2.43)

\[ d*F = 0: \]

\[ \partial_\alpha (H g_{MN}) + 2 \partial_M \partial_N H = 0 \]  \hspace{1cm} (2.44)

**Intersecting Brane Limit**

|      | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|------|---|---|---|---|---|---|---|---|---|---|----|
| M2   | × | × | × |   |   |   |   |   |   |   |    |
| M2   | × |   | × | × |   |   |   |   |   |   |    |

(2.45)
The Yellow Brick Road

**M2 wrapping a 2-cycle in $\mathbb{C}^3$**

**Metric Constraint**

$$\partial(H \omega_g \wedge \omega_g) = 0 \quad (2.46)$$

**Four-Form Field Strength**

The following expressions, together with their complex conjugates give the non-zero components of the field strength.

$$F_{0M\bar{N}\alpha} = -\frac{i}{2} \partial_{\alpha} g_{M\bar{N}}, \quad (2.47)$$

$$F_{0M\bar{N}\bar{P}} = -\frac{i}{2} [\partial_{\bar{P}} g_{M\bar{N}} - \partial_{\bar{N}} g_{M\bar{P}}]$$

**Killing Spinors**

The four Killing spinors of this configuration are specified by

$$\alpha = H^{-1/6} \eta |000\rangle \quad (2.48)$$

where $\eta$ is a constant spinor such that

$$\Gamma_{012} \eta = \Gamma_{034} \eta = \Gamma_{056} \eta = \eta \quad (2.49)$$

$$d * F = 0:$$

$$\partial^2_{\alpha} [H \omega_g \wedge \omega_g] = 0. \quad (2.50)$$

**Intersecting Brane Limit**

|     | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|-----|---|---|---|---|---|---|---|---|---|---|----|
| M2  |   | × | × | × |   |   |   |   |   |   |    |
| M2  | × |   |   | × |   |   |   |   |   |   |    |
| M2  | × |   | × | × |   |   |   |   |   |   |    |

(2.51)
2.6 Charging forth.

**M2 wrapping a 2-cycle in $\mathbb{C}^4$**

Since this is a configuration which has only two overall transverse directions, note that its supergravity solution could not be obtained using the harmonic function rule.

**Metric Constraint**

Supersymmetry requires the metric to obey

$$\partial(H \omega_g \wedge \omega_g \wedge \omega_g) = 0$$

(2.52)

**Four-Form Field Strength**

The field strength components are the same as the previous case. They are given by the following expressions and their complex conjugates.

$$F_{0M\bar{N}\alpha} = -\frac{i}{2} \partial_\alpha g_{M\bar{N}},$$

(2.53)

$$F_{0M\bar{N}\bar{P}} = -\frac{i}{2} [\partial_{\bar{P}} g_{M\bar{N}} - \partial_{\bar{N}} g_{M\bar{P}}]$$

**Killing Spinors**

The two Killing spinors are specified by

$$\alpha = H^{-1/6} \eta |0000 >$$

(2.54)

where the constant spinor $\eta$ obeys the projection conditions

$$\Gamma_{012} \eta = \Gamma_{034} \eta = \Gamma_{056} \eta = \Gamma_{078} \eta = \eta$$

(2.55)

$$d \ast F = 0.$$  

dr^2 [H \omega_g \wedge \omega_g \wedge \omega_g] + 2 \delta [H \omega_g \wedge \omega_g] = 0.$$  

(2.56)

**Intersecting Brane Limit**

|   | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|----|
| M2 | × | × |   |   |   |   |   |   |   |   |    |
| M2 | × | × |   | × |   |   |   |   |   |   |    |
| M2 | × |   | × | × |   |   |   |   |   |   |    |
| M2 | × |   | × | × |   |   |   |   |   |   |    |
M2 wrapping a 2-cycle in $\mathbb{C}^5$

This is the maximal dimensional complex manifold we can have, since it now spans all ten spatial directions. There is hence no $X^\alpha$ and consequently no $F_{0\bar{M}\bar{N}\alpha}$ component in the field strength. Moreover, there is no $H_2$, so $H$ is defined simply as $H = H_1^{-3}$.

The absence of any overall transverse directions also means that the supergravity solution of this configuration has no counterpart which can be obtained via the harmonic function rule.

**Metric Constraint**

$$\partial (H \omega_g \wedge \omega_g \wedge \omega_g \wedge \omega_g) = 0$$ (2.58)

**Four-Form Field Strength**

The only non-zero contributions come from:

$$F_{0\bar{M}\bar{N}\bar{P}} = -\frac{i}{2} \left[ \partial_P (g_{MN}) - \partial_N (g_{MP}) \right],$$ (2.59)

and its complex conjugate, $F_{0\bar{M}\bar{N}\bar{P}}$.

**Killing Spinors**

The single Killing spinor of this configuration is

$$\chi = H^{-1/6} \eta (|00000 > + |11111 >)$$ (2.60)

where the constant spinor $\eta$ is subject to

$$\Gamma_{012} \eta = \Gamma_{034} \eta = \Gamma_{056} \eta = \Gamma_{078} \eta = \Gamma_{09(10)} \eta = \eta$$ (2.61)

$d * F = 0$:

$$g^{MN} \partial_M H (\partial_C g_{B\bar{N}} - \partial_B g_{C\bar{N}}) = 0$$ (2.62)

$$g^{MC} \partial_M [H^{-\frac{3}{2}} (\partial_B g_{A\bar{C}} - \partial_A g_{B\bar{C}})] = 0$$

$$g^{NP} \partial_P (\partial_B g_{N\bar{C}} - \partial_N g_{B\bar{C}}) = 0$$

$$g^{NP} g^{AM} \partial_P [g_{AC} \partial_B g_{N\bar{M}} + g_{BM} \partial_A g_{N\bar{C}} - \partial_N (g_{BM} g_{A\bar{C}})]$$ (2.63)
2.6 Charging forth.

Intersecting Brane Limit

\[
\begin{array}{cccccccccc}
0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\
\text{M2} & X & X & X & & & & & & & \\
\text{M2} & X & & X & X & & & & & & \\
\text{M2} & X & & & X & X & & & & & \\
\text{M2} & X & & & & X & X & & & & \\
\text{M2} & X & & & & & X & X & & & \\
\end{array}
\]  

(2.64)

Fivebranes

We now turn to supergravity solutions describing fivebranes wrapped on holomorphic curves. The relevant ansatz for the spacetime metric is:

\[
ds^2 = H^{-1/3} \eta_{\mu\nu} dX^\mu dX^\nu + 2G_{MN} dz^M dz^N + H^{2/3}\delta_{\alpha\beta} dX^\alpha dX^\beta. \tag{2.65}
\]

where \(\mu\) labels the flat coordinates of the fivebrane worldvolume, \(z^M\) are the holomorphic coordinates on the embedding space \(\mathbb{C}^n\) and \(\alpha\) takes values in the overall transverse space.

Supersymmetry preservation has already been used here to fix the relative coefficients \(H^{-1/3}\) and \(H^{2/3}\). It further dictates that, (upto rescaling by an arbitrary holomorphic function), the function \(H\) is related to the determinant \(G\) of the Hermitian metric in a way that depends on the dimensions of the holomorphic curve and the complex space into which it is embedded. The Hermitian two-form \(\omega_G\) associated with the metric in the complex subspace is defined such that

\[
\omega_G = iG_{MN} dz^M \wedge dz^N.
\]

Components of the four-form field strength can be expressed in terms of the functions in the metric ansatz by solving a set of constraints which arise from \(\delta \Psi = 0\). Once again, the Killing spinors are Majorana and hence of the form \(\chi = \alpha + C\alpha^*\) if \(C\) is the charge conjugation matrix. The spinor \(\alpha\) depends on the particular configuration under study.

The non-linear differential equation involving \(g_{MN}\) and \(H\) now follows from imposing the Bianchi Identity \(dF = 0\), since the four-form \(F\) couples magnetically to the fivebranes and the roles of the equations of motion and Bianchi Identity are consequently interchanged.

All the supergravity solutions for fivebranes wrapping holomorphic cycles in manifolds of various dimension will be of the form outlined above. We present the results [13, 14, 15, 20] including in addition, the intersecting brane system which arises when the holomorphic cycle becomes singular. Note that such a limit does not always exist.
M5 wrapping a 2-cycle in \( \mathbb{C}^2 \)

**Metric Constraint**

\[
\partial[\frac{1}{3}G H^{1/3}] = 0. \tag{2.66}
\]

where \( H^{2/3} = \sqrt{G} \)

**Four-Form Field Strength**

The non vanishing components of the supergravity four-form are:

\[
F_{M89(10)} = -\frac{i}{2} \partial_M H \tag{2.67}
\]

\[
F_{NM\beta\gamma} = \frac{i}{2} \epsilon_{\alpha\beta\gamma} \partial_\alpha \left[ \frac{1}{3} G H^{1/3} G_{NM} \right] \tag{2.68}
\]

and complex conjugates.

**Killing Spinors**

The eight Killing spinors of this configuration are specified by

\[
\alpha = H^{-1/12} \eta |00> \tag{2.69}
\]

where the constant spinor \( \eta \) is subject to

\[
i \Gamma_{0123} \eta = \eta \tag{2.70}
\]

\( df = 0; \)

\[
\partial_a^2 g_{MN} + 2 \partial_M \partial_N H = 0 \tag{2.71}
\]

**Intersecting Brane Limit**

\[
\begin{array}{cccccccccc}
0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\
M5 & \times & \times & \times & \times & \times & \times & \times & \times & \times & \times \\
M5 & \times & \times & \times & \times & \times & \times & \times & \times & \times & \times \\
\end{array} \tag{2.72}
\]
M5 wrapping a 2-cycle in $\mathbb{C}^3$

Metric Constraint

$$\partial[H^{-1/3}\omega_G \wedge \omega_G] = 0. \quad (2.73)$$

and $H = \sqrt{G}$ must hold.

Four-Form Field Strength

The four-form field strength is given by the expressions below, and their complex conjugates:

$$F_{NP\bar{M}y} = \frac{1}{2}[\partial_P(H^{1/3}G_{\bar{M}N}) - \partial_N(H^{1/3}G_{M\bar{P}})], \quad (2.74)$$

$$F_{MNP\bar{Q}} = \frac{i}{2}\partial_y[H^{-1/3}(G_{\bar{M}\bar{Q}}G_{N\bar{P}} - G_{MP}G_{N\bar{Q}})] \quad (2.75)$$

Killing Spinors

The four Killing spinors of this configuration are specified by

$$\alpha = H^{-1/12}\eta|000> \quad (2.76)$$

where the constant spinor $\eta$ obeys the projection condition

$$i\Gamma_{0123}\eta = \eta \quad (2.77)$$

$$dF = 0:$$

$$\partial_y^2(H^{-1/3}\omega_G \wedge \omega_G) \wedge dy + 2i\partial(H^{-1/3}\omega_G) = 0 \quad (2.78)$$

Intersecting Brane Limit

|     | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|-----|---|---|---|---|---|---|---|---|---|---|----|
| M5  | x | x | x |   | x | x |   |   |   |   |    |
| M5  | x | x | x |   | x | x |   |   |   |   |    |
| M5  | x | x | x |   | x | x |   |   |   |   |    |
Non-Kahler Metrics.

A flat M5-brane can be thought of as being wrapped on a trivial supersymmetric cycle embedded into a subspace of 11-dimensional spacetime. Take this subspace to be $\mathbb{C}^3$, spanned by holomorphic coordinates $u,v,w$. Two equations $f = f(u,v,w) = 0$ and $g = g(u,v,w) = 0$ are then needed to define a holomorphic two-cycle. If these equations are $v = 0$ and $w = 0$, the two-cycle in question is simply the complex $u$ plane.

In the presence of a flat M5-brane with worldvolume $012\bar{3}u\bar{u}$, the spacetime metric is given by:

$$ds^2 = H^{-1/3}(-dt^2 + dx_1^2 + dx_2^2 + dx_3^2 + dud\bar{u}) + H^{2/3}(dvd\bar{v} + dwd\bar{w} + dy^2)$$

where

$$H = \text{constant} \quad \frac{1}{(|v|^2 + |w|^2 + y^2)^{3/2}}$$

In general, an M5-brane wrapping a Riemann surface embedded in $\mathbb{C}^3$ is expected to give rise to a metric of the form [14]:

$$ds^2 = H^{-1/3}(-dt^2 + dx_1^2 + dx_2^2 + dx_3^2) + 2G_{M\bar{N}}dz^Mdz^\bar{N} + H^{2/3}dy^2$$

Comparing the two expressions and defining a re-scaled Hermitian metric $g_{M\bar{N}} = H^{-1/6}G_{M\bar{N}}$, we find

$$2g_{u\bar{u}} = H^{-1/2}, \quad 2g_{v\bar{v}} = H^{1/2}, \quad 2g_{w\bar{w}} = H^{1/2}.$$ 

It can trivially be seen that this metric is not Kahler; moreover since a Kahler metric cannot be obtained even by rescaling, $g_{M\bar{N}}$ is not warped Kahler either. However, the components of this blatantly non-Kahler metric satisfy the following curious relations:

$$\partial_u(g_{v\bar{v}}g_{w\bar{w}}) = \partial_v(g_{u\bar{u}}g_{w\bar{w}}) = \partial_w(g_{u\bar{u}}g_{v\bar{v}}) = 0. \quad (2.80)$$

In terms of the Hermitian form $\omega = ig_{M\bar{N}}dz^Mdz^\bar{N}$ associated with the metric, this can be re-expressed as follows:

$$\partial[\omega \wedge \omega] = 0 \quad \text{but} \quad \partial\omega \neq 0$$
M5 wrapping a 4-cycle in $\mathbb{C}^3$

It turns out to be convenient to define a rescaled metric in this case

$$g_{M\bar{N}} \equiv H^{1/3} G_{M\bar{N}}$$  \hspace{1cm} (2.81)

**Metric Constraint**

$$\partial \omega_g = 0$$  \hspace{1cm} (2.82)

and $g = \det g_{M\bar{N}} = H$

**Four-Form Field Strength**

$$F_{89(10)M} = \frac{i}{2} \partial_M H$$  \hspace{1cm} (2.83)

$$F_{N\bar{M}\beta\gamma} = \frac{i}{2} \epsilon_{\alpha\beta\gamma} \partial_{\alpha} g_{N\bar{M}}$$  \hspace{1cm} (2.84)

**Killing Spinors**

This configuration has four Killing spinors specified by

$$\alpha = H^{-1/12} \eta |000>$$  \hspace{1cm} (2.85)

where the constant spinor $\eta$ obeys

$$\Gamma_{89(10)} \eta = -i \eta$$  \hspace{1cm} (2.86)

$dF = 0$:

$$\partial_{\alpha}^2 g_{M\bar{N}} + 2 \partial_M \partial_{\bar{N}} H = 0$$  \hspace{1cm} (2.87)

**Intersecting Brane Limit**

In the subcases for which an intersecting brane picture exists, this would take the form:

|   | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|----|
| M5 | × | × | × | × | × | × | × | × | 8 | 9 | 10 |
| M5 | × | × | × | × | × | × | × | 8 | 9 | 10 | 11 |
| M5 | × | × | × | × | × | × | 8 | 9 | 10 | 11 | 12 |
The Harmonic Function Rule is Safe!

We now show how this new ansatz encompasses and generalises the harmonic function rule, using the above example to illustrate the point. The Fayyazuddin-Smith metric ansatz for this system can be written as

\[ ds^2 = H^{2/3} \left[ H^{-1} \eta_{\mu\nu} dX^\mu dX^\nu + 2H^{-1} g_{MN} dz^M dz^N + \delta_{\alpha\beta} dX^\alpha dX^\beta \right] \] (2.89)

For ease of comparison, the metric dictated by the harmonic function rule can also be written using complex coordinates on the embedding (relative transverse) space. Defining

\[ u = X^2 + iX^3, \quad v = X^4 + iX^5 \quad \text{and} \quad w = X^6 + iX^7, \]

this metric takes the form

\[ ds^2 = H^{2/3} H_1^{2/3} H_2^{2/3} H_3^{2/3} \left[ H_1^{-1} H_2^{-1} H_3^{-1} (-dX_8^2 + dX_9^2) + H_1^{-1} H_2^{-1} dud\bar{u} + H_1^{-1} H_3^{-1} dvd\bar{v} + H_2^{-1} H_3^{-1} dwd\bar{w} + (dX_8^2 + dX_9^2 + dX_{10}^2) \right] \]

Comparing the two expressions, one can immediately see that

\[ H = H_1 H_2 H_3, \quad 2g_{u\bar{u}} = H_3, \quad 2g_{v\bar{v}} = H_2, \quad 2g_{w\bar{w}} = H_1 \] (2.90)

and all other components of \( g_{MN} \) are zero. Since \( H_1, H_2 \) and \( H_3 \) are only functions of the overall transverse coordinates, it follows that \( H \) must be too, so \( \partial_M H = 0 \). Substituting in (2.84), we find that this implies \( F_{u\bar{u}{}^{(10)}} \) and \( F_{N\bar{M}\gamma} \propto \eta_{N\bar{M}} \) such that

\[ F_{u\bar{u}}{}_{\beta\gamma} = \frac{1}{2} \epsilon_{\beta\gamma\alpha} \partial_\alpha H_3, \quad F_{u\bar{u}}{}_{\beta\gamma} = \frac{1}{2} \epsilon_{\beta\gamma\alpha} \partial_\alpha H_2 \]

\[ F_{u\bar{u}}{}_{\beta\gamma} = \frac{1}{2} \epsilon_{\beta\gamma\alpha} \partial_\alpha H_1 \]

in perfect agreement with the results obtained in (2.75).

It should by now be clear, through this explicit discussion, that the Harmonic Function Rule is the restriction of the Fayyazuddin-Smith ansatz to cases where \( H \) is not allowed to depend on the embedding space!
Chapter 3

The Emerald City

You might have arrived at the doors of the Emerald City, but in order to get through the gate-keeper and set foot inside the hallowed portals, you must know the password: **calibrations**.

Even if this is just another word to you right now, the reason for it being so revered will become clear when you enter the shimmering jewelled towers of the Emerald City; in fact once you become acquainted with calibrations, you will wonder how you ever got along without them!
3.1 The Gate-Keeper

A calibration is a mathematical construction which enables us to track down preferred (minimal) submanifolds in a given spacetime. The minimization in question does not necessarily refer to the volume of the submanifold; in fact it is only in purely geometric backgrounds that this is so. As we shall soon see, when the background contains a non-vanishing flux, calibrations pick out minimum energy submanifolds. In general, the quantity being minimized tells us about the structure of the ambient space-time, so essentially calibrations provide us with a way of describing a particular background.

A useful way to discover the calibrations in a particular background is to use 'test branes'. The idea is similar to a test particle in electrodynamics which is placed in an already existing electric field merely to measure its value at a given point. In order to focus our attention on the properties of the existing background, all possible distractions are set to zero, i.e it is assumed that the test particle is truly just a probe in that it gives rise to no fields of its own. Similarly, test branes are branes which do not themselves cause any deformations of the background in which they are placed, (i.e all such deformations are neglected) but merely act as probes for the existing geometry.

If a test brane is BPS, it takes on a stable shape which must by definition, be a supersymmetric cycle in the given background. When this background is flat space, we know what these cycles are. When the background is a manifold of special holonomy, again we know what the cycles are, Berger classified them for us. The problem though is to figure out what happens when the background is one created by a charged gravitating p-brane source! Such a background has a non-trivial flux, and the supersymmetric cycles in these geometries have not yet been classified. These are the cases we will attempt to study as we tour the Emerald City.

Requiring the world-volume of a wrapped brane to be supersymmetric determines or defines the supersymmetric cycles. In the same way, requiring the energy of a wrapped brane to be minimal determines the calibrations! This is like the age old chicken and egg conundrum ⋯ who knows what came first? The logic can flow either way.

Reduced Holonomy

In the absence of the four-form field strength, a supersymmetric compactification of M-Theory is only possible on a manifold $\mathcal{M}$ of reduced holonomy. The reason for this is as follows. Supersymmetry is only preserved if the
supersymmetric variation of the gravitino vanishes. In a background with no field strength, we can see from (2.37) that setting the gravitino variation to zero is equivalent to the statement that Killing spinors \( \chi \) on the manifold \( \mathcal{M} \) are covariantly constant\(^1\), since

\[
\delta \chi \psi_I = (\partial_I + \frac{1}{4} \omega_I^{ab} \hat{\Gamma}_{ab}) \chi = D_I \chi = 0,
\]

where \( a = 1, \ldots, \dim \mathcal{M} \). From \( D_I \chi = 0 \), it follows that \( \mathcal{M} \) is Ricci flat. Furthermore, since

\[
R_{ablJ} \gamma^{ab} \eta = D_{[J} D_{I]} \eta = 0
\]

some generators of the holonomy group \( R_{ablJ} \gamma^{ab} \) annihilate the vacuum, so the holonomy group of the manifold must be a subgroup of the maximal \( \text{SO}(\dim \mathcal{M}) \).

In other words, for backgrounds where there is no field strength flux, supersymmetry preservation implies the existence of covariantly constant spinors on the compactification manifold. These exist only if \( \mathcal{M} \) has reduced holonomy and is Ricci flat, ensuring that Einstein’s field equations are automatically satisfied by the internal manifold\(^2\).

For compactifications of string theory down to four dimensions, these conditions imply that the manifold \( \mathcal{M} \) should be a Calabi-Yau, ie, a 3 complex dimensional Ricci flat manifolds with SU(3), rather than SO(6), holonomy; for 4-dimensional compactifications of M-theory, we find instead that \( \mathcal{M} \) is a seven dimensional manifold with \( G_2 \) holonomy.

### Introducing Calibrations

Calibrations \( \phi \) are \( p \)-forms, which enable us to classify minimal \( p \)-dimensional submanifolds in a particular background space-time \([10]\). The (standard) calibration \( \phi \) satisfies

\[
\int_{\mathcal{M}_p} \phi \leq \text{Vol} (\mathcal{M}_p) \quad \text{and} \quad d\phi = 0
\]

for any \( p \)-dimensional manifold \( \mathcal{M}_p \). In every homology class there is a manifold \( \Sigma_p \) which minimizes the volume, saturating the above inequality. Since any other manifold \( \Sigma'_p \) in the same homology class can be written as

\[
\Sigma'_p = \Sigma_p + \partial \mathcal{M}_{p+1}
\]

\(^1\)with respect to a torsion-free metric

\(^2\)A covariantly constant spinor is always a signal for reduced holonomy, but the converse is true for Ricci flat manifolds only
Using Stokes’ Theorem, we find
\[ \oint_{\Sigma'} \phi = \int_{\Sigma} \phi + \int_{\partial M} \phi = \int_{\Sigma} \phi + \int_{M} d\phi = \int_{\Sigma} \phi = Vol(\Sigma). \tag{3.5} \]
This can be used to prove that
\[ \int_{\Sigma'_p} \phi = Vol(\Sigma_p) \leq Vol(\Sigma'_p). \tag{3.6} \]
Hence, integrating the calibration \( \phi \) over any \( p \)-dimensional submanifold gives the volume of the minimal manifold in that particular homology class; this minimal manifold is known as a calibrated manifold.

Since we will mostly be dealing with branes of infinite spatial extent, their volumes will obviously be infinite as well. It thus makes more sense to express the definition of a calibrating form as follows:
\[ P_{M_p}(\phi_p) \leq dV_{M_p} \tag{3.7} \]
i.e, the pullback of the calibrating form \( \phi_p \) onto any \( p \)-dimensional manifold \( M_p \) is less than or equal to the volume form of this manifold.

**Berger’s Classification.**

For space-times with no background field strength, we have seen that the compactification manifold must have special holonomy. There is a classification, due to Berger, of the calibrations which can exist on such manifolds.

There are several different kinds of calibrations; Kahler calibrations and special Lagrangian calibrations which exist in any Calabi-Yau space and exceptional calibrations which exist only in exceptional holonomy spaces.

- In a Calabi-Yau \( n \)-fold, the \( 2p \)-forms \( \phi = \frac{1}{p!} \omega^p \) which are constructed from the Kahler form \( \omega \) are closed and are known as Kahler calibrations.
- A Calabi-Yau \( n \)-fold admits a unique nowhere vanishing \((n,0)\)-form, \( \phi_n \), called the Special Lagrangian calibration.
- In seven dimensional manifolds, there is a 3-form \( \psi \) (and its dual 4-form \( *\psi \)) which is invariant under the exceptional group \( G_2 \); these give rise to three and four dimensional calibrations known as the Associative and Co-associative calibrations respectively.
- Similarly, in eight dimensional manifolds, there is a self dual 4-form \( \Phi \) which is invariant under \( Spin(7) \) and gives rise to a four dimensional calibration called the Cayley calibration.
Branes and Calibrations

In the absence of flux, stable branes are those whose world-volumes are minimized. Given what we have learnt about calibrations, it is obvious that the volume form on such a brane must be a calibrated form in the ambient spacetime!

For a membrane wrapped on a holomorphic cycle in $\mathbb{C}^n$, the volume form decomposes into a trivial one form contribution from the time direction and the two form associated with the Hermitian metric in the complex embedding space of the two-cycle. In order for the volume form to be a calibration, it turns out that the metric on $\mathbb{C}^n$ should in fact be Kahler.

It is hence easy to see why a membrane wrapping a holomorphic two-cycle in a Calabi-Yau space gives rise to a supersymmetric configuration.

3.2 An Audience with the Wizard

In the presence of a charged brane, the bosonic fields needed to specify the background are the metric and the flux of the gauge field which couples to the brane. Killing spinors of the brane configuration are determined by the metric as well as the field strength and are hence no longer covariantly constant. Since the coupling of the gauge potential to the brane must now be taken into account, it is only natural that the criterion for brane stability also should change. In fact it turns out that stability requires now that the energy of the brane be minimized, where the energy is a measure of not only the volume but also the charge $[12]$.

**Generalised calibrations** $\phi_p$ are defined such that

$$d(A_p + \phi_p) = 0$$

$$\mathcal{P}_{\Sigma_p}(\phi_p) \leq \tilde{d}V_{\Sigma_p}$$

So, a generalised calibration $\phi_p$ is a $p$-form whose pullback on to any $p$-dimensional manifold $\Sigma_p$ is less than or equal to the (curved space) volume form of this manifold. Note that $\tilde{d}V$ is used to denote the volume form in curved space, as opposed to the flat space volume form $dV$ used in the definition of a standard calibration.

It is clear from the above that a generalised calibration is not closed, but rather is gauge equivalent to the potential $A_p$ under which the $(p-1)$brane is charged. In the trivial case where the field strength flux vanishes, generalised calibrations reduce to the standard calibrations $d\phi = 0$ discussed earlier.

Recall from our earlier discussion that a calibrated manifold i.e, one which saturates the bound, must have minimal volume. We now proceed
The Emerald City

to show that an analogous statement holds for generalised calibrations as well, except that calibrated manifolds now have minimum energy. As we will show later [12], the energy $E$ is given by

$$E(\Sigma_p) = \int_{\Sigma_p} [d\tilde{V} + A_p] \quad (3.10)$$

We start by assuming that $\Sigma_p$ is a calibrated manifold and thus saturates the calibration bound

$$\int_{\Sigma_p} \phi_p = \int_{\Sigma_p} d\tilde{V} = Vol(\Sigma_p)$$

Since $\Sigma_p$ is minimal, any other generic manifold $\Sigma_p'$ in the same homology class can be expressed as $\Sigma_p + \partial M_{p+1}$. This implies the following

$$\int_{\Sigma_p'} \phi_p = \int_{\Sigma_p} \phi_p + \int_{\partial M_{p+1}} \phi_p = \int_{\Sigma_p} \phi_p + \int_{M_{p+1}} d\phi_p \quad (3.11)$$

From the definition of generalised calibrations, we have $dA_p = -d\phi_p$,

$$\int_{\Sigma_p} \phi_p = \int_{\Sigma_p} \phi_p - \int_{\partial M_{p+1}} dA_p = \int_{\Sigma_p} \phi_p - \int_{\partial M_{p+1}} A_p$$

$$= \int_{\Sigma_p} \phi_p + \int_{\Sigma_p} A_p - \int_{\Sigma_p'} A_p \quad (3.12)$$

Hence,

$$\int_{\Sigma_p'} (\phi_p + A_p) = \int_{\Sigma_p} (\phi_p + A_p) \quad (3.13)$$

For the manifold $\Sigma_p'$,

$$\int_{\Sigma_p'} \phi_p \leq \int_{\Sigma_p'} d\tilde{V} \quad \Rightarrow \quad \int_{\Sigma_p'} [\phi_p + A_p] \leq E(\Sigma_p')$$

However, for the calibrated manifold the inequality is saturated, and

$$\int_{\Sigma_p} [\phi_p + A_p] = E(\Sigma_p)$$

Hence (3.13) reduces simply to the statement

$$E(\Sigma_p) \leq E(\Sigma_p')$$

which says that the calibrated manifold is the one which minimizes its energy in a given holonomy class.

Having seen the way things are in the Emerald City, let us now try and understand, as far as we can, the reasons why they are so.
### Charged Branes

The Lagrangian density for a $p$-brane charged under a potential $A_{p+1}$ is

$$\mathcal{L} = -\sqrt{-\det h_{ab}} - \mathcal{P}(A)$$

(3.14)

where $h_{ab}$ is the induced metric on the world-volume, and

$$\mathcal{P}(A) = \epsilon^{a_0 a_1 \ldots a_p} \partial_{a_0} X^{\mu_0} \partial_{a_1} X^{\mu_1} \ldots \partial_{a_p} X^{\mu_p} A_{\mu_0 \mu_1 \ldots \mu_p}$$

is the pullback of the spacetime $(p+1)$-form gauge potential onto the brane. In what follows, we will consider only those spacetimes for which the component $G_{0I}$ of the metric vanishes, if $I$ is a spatial index. Separating the purely spatial part of the induced metric by defining

$$g_{ij} \equiv \partial_i X^\mu \partial_j X^\nu G_{\mu\nu}$$

(3.15)

we can express $\det h_{ab}$ as:

$$\det h_{ab} = (\partial_t X^\nu \partial_j X^\rho G_{\mu\nu}) \det g_{ij}$$

(3.16)

Choosing now the gauge $X^0 = t$, the Lagrangian density takes the form

$$\mathcal{L} = -\sqrt{(G_{tt} + \partial_t X^I \partial_I X^J G_{IJ}) \det g_{ij}} - \epsilon^{i_1 \ldots i_p} \partial_{i_1} X^{I_1} \ldots \partial_{i_p} X^{I_p} A_{I_1 \ldots I_p} - \epsilon^{i_1 \ldots i_p} \partial_t X^M \partial_{i_1} X^{I_1} \ldots \partial_{i_p} X^{I_p} A_{M I_1 \ldots I_p}$$

(3.17)

Since the canonical momentum $p_I$ is given by the expression

$$p_K = \frac{\partial \mathcal{L}}{\partial(\partial_t X^K)} = -\frac{\partial_t X^J G_{JK} - \epsilon^{i_1 \ldots i_p} \partial_{i_1} X^{I_1} \ldots \partial_{i_p} X^{I_p} A_{K I_1 \ldots I_p}}{\sqrt{-\det h_{ab}}}$$

(3.18)

it follows that the Hamiltonian density $\mathcal{H} = p_I \partial_I X^J - \mathcal{L}$ is

$$\mathcal{H} = \sqrt{-\det h_{ab}} - \frac{\partial_t X^J G_{JK} + \epsilon^{i_1 \ldots i_p} \partial_{i_1} X^{I_1} \ldots \partial_{i_p} X^{I_p} A_{I_1 \ldots I_p}}{\sqrt{-\det h_{ab}}}$$

(3.19)

Restricting ourselves to static configurations, we can set $\partial_t X^J = 0$ to get

$$\mathcal{H} = \sqrt{-\det h_{ab}} + \epsilon^{i_1 \ldots i_p} \partial_{i_1} X^{I_1} \ldots \partial_{i_p} X^{I_p} A_{I_1 \ldots I_p}$$

(3.20)
For notational convenience we now introduce the $p$-form $A_p$ such that

$$A_{I_1 \ldots I_p} \equiv A_{t I_1 \ldots I_p}$$  \hspace{1cm} (3.21)

This allows us to define $\Phi$, the worldvolume pull-back of the space-time gauge potential, as follows

$$\Phi \equiv * P(A) = \varepsilon^{i_1 \ldots i_p} \partial_{i_1} X^{I_1} \ldots \partial_{i_p} X^{I_p} A_{I_1 \ldots I_p}$$  \hspace{1cm} (3.22)

Finally, we are in a position to write down the energy $\mathcal{E}$, associated with the $p$-brane as follows

$$\mathcal{E} = \int d^p \sigma \left[ \sqrt{-G_{tt}} \sqrt{\det g_{ij}} + \Phi \right]$$

$$= \int d^p \sigma [\tilde{dV} + A]$$  \hspace{1cm} (3.23)

**Spatial Isometries**

Typically, only those directions along the brane worldvolume which are wrapped on a supersymmetric curve have a non-trivial space-time embedding; the remaining directions are flat\(^3\). Had its entire world-volume been flat, the brane would be a 1/2 BPS object; wrapped branes however, generically break more than half the supersymmetry. Since the preserved supercharges depend on the geometry of the supersymmetric cycle, it is the wrapped directions of the brane world-volume which play a essential role in this analysis whereas flat directions contribute trivially.

We can choose static gauge along the flat directions of the brane’s worldvolume. Assume there are $l$ such directions, we then set $X^i = \sigma^i$ for $i = 1, \ldots, l$ and find that as a result, the determinant of the induced metric on the worldspace factorises such that

$$\det h_{ab} = -G_{tt} \det G|_{l \times l} \det g_{rs} \equiv v_l^2 \det g_{rs}$$  \hspace{1cm} (3.24)

where $\det G|_{l \times l}$ denotes the restriction of the determinant of the bulk space metric to the $l$ spatial directions which are isometries of the system, and $\det g_{rs}$ is the determinant of the induced metric on the Mp-brane in directions $\sigma^{l+1} \ldots \sigma^p$. Also, the pullback of the gauge potential $P(A)$ can be expressed in static gauge as

$$P(A) = \varepsilon^{i_1 \ldots i_p} \partial_{i_1} X^{I_1} \ldots \partial_{i_p} X^{I_p} A_{I_1 \ldots I_p}$$

$$= \varepsilon^{i_{l+1} \ldots i_p} \partial_{i_{l+1}} X^{I_{l+1}} \ldots \partial_{i_p} X^{I_p} A_{I_{l+1} \ldots I_p}$$  \hspace{1cm} (3.25)\(^3\)

\(^3\)In the notation adopted here, these are the $X^\mu$
3.2 An Audience with the Wizard

Hence the energy of this configuration is given by

$$ E = \int d^p \sigma \sqrt{g_{rs}} + \epsilon^{i_1 \ldots i_p} \partial_{i_{l+1}} X^{I_{l+1}} \ldots \partial_{i_p} X^{I_p} A_{12 \ldots I_{l+1} \ldots I_p} $$

(3.26)

Due to the infinite extent of the brane, this energy too will be infinite. It thus makes sense to define the energy per unit \( l \)-volume

$$ E_l = \int d^{p-l} \sigma \sqrt{g_{rs}} + \epsilon^{i_1 \ldots i_p} \partial_{i_1} X^{I_1} \ldots \partial_{i_p} X^{I_p} A_{12 \ldots I_{l+1} \ldots I_p} $$

(3.27)

**Supersymmetry Preservation**

In order to be a Killing spinor, \( \xi \) must satisfy the projection condition [12]

$$ \frac{1}{\sqrt{\det h}} \epsilon^{a_0 \ldots a_p} \partial_{a_0} X^{\mu_0} \partial_{a_1} X^{\mu_1} \ldots \partial_{a_p} X^{\mu_p} \Gamma_{\mu_0 \mu_1 \ldots \mu_p} \xi = \xi $$

(3.28)

Assuming that the configuration is static, this condition takes the form

$$ \frac{1}{\sqrt{-G_{tt} \sqrt{\det h}}} e^{a_1 \ldots a_p} \partial_{a_1} X^{\mu_1} \ldots \partial_{a_p} X^{\mu_p} \Gamma_0 \Gamma_{\mu_1 \ldots \mu_p} \xi = \xi $$

(3.29)

Defining

$$ \gamma \equiv \epsilon^{a_1 \ldots a_p} \partial_{a_1} X^{\mu_1} \ldots \partial_{a_p} X^{\mu_p} \Gamma_{\mu_1 \ldots \mu_p} $$

(3.30)

we can re-write the above condition as

$$ [1 - \frac{1}{\sqrt{-G_{tt} \sqrt{\det h}}} \Gamma_0 \gamma] \xi \equiv [1 - \hat{\Gamma}] \xi = 0 $$

(3.31)

where \((\hat{\Gamma})^2 = 1\) since \( \gamma^2 = 1 \) and \( \Gamma_0^2 = -G_{tt} \).

**Establishing a bound**

The obvious statement that the square of a quantity is positive definite can be used to establish a general bound on \( \phi|\xi \).

Consider \([1 - \hat{\Gamma}] \xi \), acting on a generic configuration. Even if the system is not supersymmetric, the following will obviously hold

$$ \xi^{\dagger} [1 - \hat{\Gamma}] [1 - \hat{\Gamma}] \xi \geq 0 $$

(3.32)

Since \( \hat{\Gamma} \) is Hermitian and the spinor \( \xi \) has been normalised such that \( \xi^{\dagger} \xi = \sqrt{-G_{tt} \sqrt{\det h}} \), we find

$$ \sqrt{-G_{tt} \sqrt{\det h}} [1 - \xi^{\dagger} \hat{\Gamma} \xi] \geq 0 $$

(3.33)
Inserting the expression for $\hat{\Gamma}$ from (3.31), we can re-write the above as
\[
\sqrt{-G_{tt}} \sqrt{\det h} - \xi \Gamma_0 \gamma \xi \geq 0 \quad (3.34)
\]
Since $\sqrt{-G_{tt}} \sqrt{\det h} = \text{Vol}_\xi$ is the (world-space dual of the) curved space volume form, this implies that in terms of the $p$-form
\[
\phi \equiv * \xi \Gamma_0 \gamma \xi \quad (3.35)
\]
which is the world-space dual of $\xi \Gamma_0 \gamma \xi$, the inequality (3.34) reduces to
\[
\text{Vol}(M) \geq P_M(\phi) \quad (3.36)
\]
But this is merely the definition of a generalised calibration, (3.9)! So (3.35) is actually a way to explicitly construct generalised calibrations from Killing spinors!

**Supersymmetry Algebra**

The supersymmetry algebra is motivated in [12] from considerations of $\kappa$-symmetry. It turns out that,
\[
\xi \{ \bar{Q}, Q \} \xi = \int d^p \sigma [\sqrt{-G_{tt}} \sqrt{\det h} - \xi \Gamma_0 \gamma \xi] = \int d^p \sigma [\mathcal{H} - \Phi - *\phi] = H - \int d^p \sigma [A + \phi] \quad (3.37)
\]
So, $\int d^p \sigma [A - \phi]$ constitutes a central extension to the superalgebra; as such, it must be a topological term. This implies that
\[
d(A + \phi) = 0 \quad (3.38)
\]
which was one of the defining statements about generalised calibrations.

### 3.3 The Great and Powerful Oz has Spoken!!

The Great Oz now convinces us of his greatness by showing us a simple and elegant way to reproduce the results we worked so hard for, in our trek down the Yellow Brick Road. Had we never walked that road, he argues, we could still have reached the same conclusions, if we only had the knowledge he has. All the supergravity solutions which we found after a long and
arduous trek, the Wizard now reproduces in the comfort of his Emerald City quarters. The trick about to be performed will consist of a few short steps, carried out in quick succession, which result in the construction of a supergravity solution for any M-brane wrapped on a holomorphic curve and will replicate the expressions obtained from the Fayyazuddin-Smith analysis employed by us during our Yellow Brick Road travels.

“A supergravity solution”, says the Wizard, “consists of a metric, which may be expressed in terms of undetermined functions, as long as we also present the equations which these functions must satisfy and in addition the four-form field strength of the supergravity three-form”. He will now prove to us his unquestionable superiority, he says, through a point by point comparison of our earlier slower approach to his fast and elegant analysis.

Recall that we started out with the Fayyazuddin-Smith ansatz for a metric, and then obtained relations between the undetermined functions $H_1, H_2$ and $\det G_{\bar{M}\bar{N}}$ in the ansatz by appealing to the constraints which arose from setting $\delta \Psi = 0$. The Wizard replicates these relations as follows:

Given the Fayyazuddin-Smith ansatz for the metric in a particular supergravity background, the functions $H_1$ and $H_2$ can be eliminated in favour of a single function $H$:

- For a membrane, $H_1 = H^{-1/3}$ and $H_2 = H^{1/6}$,
- And for a fivebrane $H_1 = H^{-1/6}$ and $H_2 = H^{1/3}$.

Note in both cases, the analogy with the flat brane solutions.

The relation between the determinant of the Hermitean metric and $H$ can be obtained using the following simple relations which also follow from comparison with the flat brane case:

- For a membrane on a holomorphic cycle, the determinant of the full 11-dimensional metric is always $H^{2/3}$. The determinant $G$ of the Hermitean metric in a manifold of complex dimension $n$ must be given by $G = H^{(2n-6)/3}$ in order for this to hold$^4$.

- For a fivebrane wrapped on a holomorphic two-cycle, the determinant of the full 11-dimensional metric is $H^{4/3}$. The determinant $G$ of the $n$ dimensional Hermitean metric must therefore be $G = H^{(2n-6)/3}$.

$^4$Bear in mind that all such relations hold only upto multiplication by an arbitrary holomorphic function.
- For a fivebrane wrapped on a holomorphic four-cycle, the determinant of the full 11-dimensional metric is still $H^{4/3}$. The determinant $G$ of the $n$ dimensional Hermitean metric however is now $G = H^{(2n-12)/3}$.

“So that”, says the Wizard, “takes care of that. True,” he adds, “the non-linear differential equation involving the Hermitean metric and $H$ will have to be written out, but this follows simply from requiring $d*F = 0$ for membranes, and $dF = 0$ for fivebranes. So, once I present you with the expressions for components of $F$, my work is done.”

Once again, he reminds us, where we had to wade through many different constraints, one for each Fock state of the Killing spinor, and solve them simultaneously to arrive at expressions for $F$, the Wizard will reproduce these in the wink of an eye. “Pay attention,” he says, “because this is where the magic really happens”. The Wizard now claims that components of the field strength can be calculated simply by acting the exterior derivative on the volume form of the wrapped M-brane in question! Noticing the awe-struck expression on our faces, the gleam in his eye deepens and he elaborates as follows:

“It is the BPS bound,” says he, “which steps in to make matters so simple. Since this bound is saturated by a supersymmetric brane, the mass of such a brane must be equal to its charge. Equally, one could say that the pull-back of the space-time gauge potential on to a supersymmetric brane is equal to the volume form of the brane, by virtue of the BPS condition. As you doubtless remarked during your tour of the Emerald City, the generalised calibration corresponding to a particular stable brane is (gauge) equivalent to the spacetime gauge potential under which the brane is charged.

“Hence, the generalised calibration for a a stable (and hence BPS) brane is given simply by its volume form! Components of the field strength can thus be determined by using the fact that the calibrated form corresponding to a wrapped brane is equivalent to the gauge potential to which the brane couples electrically”.

For our edification, and to convince us beyond a shadow of a doubt, this procedure will now be illustrated in detail for each of the M-brane configurations we came across earlier, at a time when we were ignorant of calibrations.
3.3 The Great and Powerful Oz has Spoken!!

M2 wrapping a 2-cycle in $C^n$

Start with writing down the Fayyazuddin-Smith metric in the background of M2-brane wrapping a holomorphic curve in $C^n$:

$$ds^2 = -H^{-2/3} dt^2 + 2H^{1/3} g_{MN} dz^M dz^\bar{N} + H^{1/3} \delta_{\alpha\beta} dx^\alpha dx^\beta.$$ (3.39)

The $n$ holomorphic coordinates are denoted by $z^M$ and $\alpha = 2n + 1, \ldots, 10$. A factor of $H^{-1}$ has been pulled out of the Hermitean metric to facilitate comparison with the expressions found earlier.

We know from the BPS condition that the calibrating form $\Phi$ of the M2-brane must be identical to its volume form and can hence be read off directly from the metric

$$\Phi = i H^{-1/3} g_{MN} dt \wedge dz^M \wedge dz^\bar{N}$$ (3.40)

$$= dV_0 \wedge \phi_{MN}$$ (3.41)

Since $F_4$ is the electric field strength for the M2-brane, it can be calculated using $F_4 = d\Phi$ to yield the expressions in (2.42) through (2.42).

M5 wrapping a 2-cycle in $C^2$

When an M5-brane wraps a holomorphic 2-cycle in $C^2$, the relevant ansatz for the spacetime metric is:

$$ds^2 = H^{-1/3} \eta_{\mu\nu} dX^\mu dX^\nu + 2G_{MN} dz^M dz^\bar{N} + H^{2/3} \delta_{\alpha\beta} dx^\alpha dx^\beta$$ (3.42)

where $z^M$ are coordinates on $C^2$, $\alpha$ takes values 8, 9, 10 and $\mu$ runs over 0, 1, 2, 3. The harmonic function $H$ is related to the determinant $G$ of the Hermitian metric by $\sqrt{G} = H^{2/3}$. Since the calibrating form $\Phi$ of the BPS M5-brane is identical to its volume form, we can read it off directly from the metric to obtain

$$\Phi = i H^{-2/3} G_{MN} dt \wedge dX^1 \wedge dX^2 \wedge dX^3 \wedge dz^M \wedge dz^\bar{N}$$ (3.43)

$$= dV_{0123} \wedge \phi_{MN}$$ (3.44)

We can now calculate $F_4 = *dF_7 = *d\Phi$ and find the same expressions as in (2.68).

M5 wrapping a 2-cycle in $C^3$

When the M5-brane is wrapped on a holomorphic curve embedded in $C^3$, the metric takes the form:

$$ds^2 = H^{-1/3} \eta_{\mu\nu} dX^\mu dX^\nu + G_{MN} dz^M dz^\bar{N} + H^{2/3} dy^2.$$ (3.45)
where $z^M$ now span $\mathbb{C}^3$, $y$ is the single overall transverse direction, and the harmonic function $H$ is related to the determinant of the Hermitian metric by $H = \sqrt{G}$.

In this background, the wrapped M5-brane is calibrated by the volume form

$$
\Phi = H^{-2/3} G_{M\bar{N}} dt \wedge dX^1 \wedge dX^2 \wedge dX^3 \wedge dz^M \wedge dz^{\bar{N}} \quad (3.46)
$$

$$
= dV_{0123} \wedge \phi_{M\bar{N}} \quad (3.47)
$$

The field strength can be calculated using $F_4 = *dF_7 = *d\Phi$. This yields the same expressions as in (2.75).

**M5 wrapping a 4-cycle in $\mathbb{C}^3$**

For an M5-brane wrapped on a 4-cycle $\Sigma_4$ in $\mathbb{C}^3$, the metric takes the form:

$$
ds^2 = H_1^{1/2} \eta_{\mu\nu} dX^\mu dX^\nu + 2G_{MN} dz^M dz^{\bar{N}} + H_2^2 \delta_{\alpha\beta} dX^\alpha dX^\beta \quad (3.48)
$$

where $\mu = 0, 1$ labels the unwrapped directions, $z^M$ are holomorphic coordinates in $\mathbb{C}^3$ and $\alpha$ takes values 8, 9, and 10. The determinant of the Hermitian metric is given by $G = H^{-4/3}$

The generalised calibration $\Phi$ for this wrapped M-brane is the same as its volume form, hence the only non-vanishing component of $\Phi$ is the following:

$$
\Phi_{01MN\bar{P}\bar{Q}} = H^{-1/3}(G_{MP}G_{N\bar{Q}} - G_{M\bar{Q}}G_{NP}) \quad (3.49)
$$

$$
= dV_{01} \times \phi_{MN\bar{P}\bar{Q}}
$$

Using the fact that $F_4 = *d\Phi$ we can once again calculate the components of the four-form, obtaining the same results as in (??).

**3.4 Lifting the curtain.**

In this awe-inspiring show where supergravity solutions of wrapped M-branes make their appearance with such grace and speed, it is easy to get side-tracked and ignore the one major sleight of hand; all the information obtained via the Fayyazuddin-Smith method has been duplicated here · · · with the conspicuous omission of the metric constraint!

It is with this slight over-sight that the Great Oz betrays himself as a mere magician rather than a real Wizard. His failure to constrain the metric in any way, leaves us with a very rich structure, but no clue where
3.4 Lifting the curtain.

to apply it. Having now pointed out the Wizard’s short-coming, which
doubtless he was hoping we would not catch on to, we are almost as far
away from an answer as we were before.

Without any further knowledge of the Hermitean metrics we can allow,
it seems that a journey home, or for that matter anywhere else, is currently
out of the question. As a last resort, we turn to the truly magical Ruby
Slippers to show us the way.
The Emerald City
We have seen, in our journey down the Yellow Brick Road, how to construct the supergravity solution for a wrapped M-brane by looking for bosonic backgrounds which admit Killing spinors. Having set the gravitino to zero, we have made sure that the supersymmetric variations of the bosonic fields vanish identically and it is left only to impose that the supersymmetry variation of the gravitino vanish as well. We require this to be true for our metric ansatz, when the variation parameter in the supersymmetry transformation is a Killing spinor. If in addition, the Bianchi identity and equations of motion for the field strength are also satisfied, the metric and four-form we have obtained are guaranteed to satisfy Einstein’s equations, furnishing a bosonic solution to 11-dimensional supergravity.

Once we arrived in the Emerald City, we learnt that there was an alternate way in which this problem could be solved. Since the generalised calibration corresponding to a wrapped M-brane is gauge equivalent to the gauge potential to which the brane couples electrically, the field strength
$F = dA = d\phi$, then immediately follows once we are given a suitable calibration. For membranes, this field strength is the four-form we are used to seeing in our supergravity solutions; for five-branes, the field strength is a seven-form and we have to dualise it in order to obtain the familiar four-form. Supersymmetry requirements fix the undetermined functions in the metric ansatz in terms of the Hermitian metric $G_{M\bar{N}}$ which obeys a non-linear differential equation that follows from $d*F = 0$.

This procedure is by far simpler than the previous one... however there is a catch! Since generalised calibrations which can exist in a background with non-zero flux have not yet been classified, there is no comprehensive (or even partial) list from which we can pick a suitable calibration $\phi$, from which to construct a supergravity solution.

### The Constraint

A persistent feature of the wrapped brane supergravity solutions we saw along the Yellow Brick Road is a constraint on the metric in the subspace where the supersymmetric cycle is embedded. This constraint in turn restricts the $(p+1)$-form potential to which the brane couples. Since the potential is gauge equivalent to the generalised calibration, we find that in fact the metric constraint can alternately be viewed as a condition which determines generalised calibrations in the given background.

This condition can be expressed as a constraint on the Hodge dual, with respect to the embedding space, of the generalised calibration in that space. When written this way, it shows clearly that calibrated (supersymmetric) cycles in a particular submanifold of space-time, are not specified completely by the submanifold and in fact have a non-trivial dependence on the surrounding spacetime as well. This is to be expected because unlike the case for $F=0$ when supersymmetric cycles depended only on geometry, we do expect now that the field strength flux (and through it, the remaining non-complex directions of spacetime) will make their presence felt and play their part in determining the calibrated cycles which can exist in a given subspace.

Expressing the results in this form also enables us to unify the M2 and M5 brane analysis and to show that the constraint can be expressed perfectly generally and arises for all M-branes embedded holomorphically into a subspace of 11 dimensional spacetime, such that $F \wedge F = 0$. 
In 11-dimensional backgrounds with non-zero four-form flux, a class of generalised calibrations in the embedding space $M$ is given by the $2m$-forms $\phi_{2m}$, for

$$\phi_{2m} = \left(\omega \wedge \omega \wedge \ldots \wedge \omega \wedge \omega\right)_{m}$$

if the following constraint holds:

$$\partial \ast_M [\phi_{2m} | G' |^{1/2m}] = 0.$$  \hspace{1cm} (4.2)

Here, $G'$ denotes the determinant of the metric restricted to directions transverse to the embedding space, and the Hodge dual is taken within the embedding space.

The Hodge Dual of the Hermitean Form.

If $\omega$ is the two-form associated with the Hermitean metric on a manifold $M$ of complex dimension $n$, and $\ast$ denotes the Hodge dual on this space, then we have the following:

$$\ast \omega_{\tilde{P}_{2} \ldots \tilde{P}_{n}Q_{2} \ldots Q_{n}} = \sqrt{\det g} \epsilon_{M_{1}P_{2} \ldots P_{n}}^{} \epsilon_{N_{1}}^{} \omega_{M_{1}N_{1}}^{}$$

$$= \sqrt{\det g} g_{M_{1}S_{1}}^{R_{1}} \epsilon_{S_{1}P_{2} \ldots P_{n}}^{R_{1}Q_{2} \ldots Q_{n}} i g_{M_{1}N_{1}}^{R_{1}}$$

$$= i \sqrt{\det g} g_{R_{1}S_{1}}^{R_{1}} \epsilon_{S_{1}P_{2} \ldots P_{n}}^{R_{1}Q_{2} \ldots Q_{n}}$$ \hspace{1cm} (4.3)

Substituting now the expression for the inverse metric

$$g_{R_{1}S_{1}}^{R_{1}} = \frac{1}{(n-1)!} \sqrt{\det g} g_{S_{1}S_{2} \ldots S_{n}}^{R_{1}R_{2} \ldots R_{n}} g_{R_{2}S_{2}} \ldots g_{R_{n}S_{n}}$$ \hspace{1cm} (4.4)

we find that

$$i^{(n-2)} \ast \omega = \left(\omega \wedge \omega \wedge \ldots \wedge \omega \wedge \omega\right)_{(n-1)}$$  \hspace{1cm} (4.5)

As a 'check' of the above, note that wedging both sides with $\omega$ gives the identity

$$\omega \wedge \ast \omega = \left(\omega \wedge \omega \wedge \ldots \wedge \omega \wedge \omega\right)_{n} = \text{Vol}(M)$$  \hspace{1cm} (4.6)
4.1 Click the Heels ...

M2-branes

Supergravity solutions for a class of BPS states corresponding to wrapped membranes were discussed in [19]. In keeping with the logic that holomorphicity implies supersymmetry, the M2-branes were wrapped on holomorphic cycles in complex subspaces of varying dimension $n$.

We will work with the following (standard) ansatz for the spacetime metric:

$$ ds^2 = -H^{-2/3}dt^2 + 2G_{MN}dz^Mdz^N + H^{1/3}\delta_{\alpha\beta}dX^\alpha dX^\beta, $$ \hspace{1cm} (4.7)

where $z^M$ are $n$ holomorphic coordinates and $X^\alpha$ span the $(10-2n)$ transverse directions.

In order to express the constraint on the generalised calibrations in each case, we resort to (4.2). Since the M2-branes wrap only two-cycle, the calibrating form in the complex space is simply the Hermitian form associated with the metric, $\omega = iG_{MN}dz^M \wedge dz^N$. The restricted determinant $|G'|$ is also simple to calculate and we find that

$$ |G'| = H^{-2/3}(H^{1/3})^{10-2n} = (H^{1/3})^{8-2n} $$ \hspace{1cm} (4.8)

The constraints now follow immediately. For a membrane wrapping a holomorphic curve in $\mathbb{C}^n$, (4.2) dictates that

$$ \partial *_{\mathbb{C}^n} [H^{(4-n)/3}\omega_G] = 0 $$ \hspace{1cm} (4.9)

This can be explicitly written out as follows:

$$ \partial[H^{2/3}\omega_G] = 0 \hspace{1cm} \text{for} \hspace{0.2cm} n = 2 $$

$$ \partial[H^{1/3}\omega_G \wedge \omega_G] = 0 \hspace{1cm} \text{for} \hspace{0.2cm} n = 3 $$

$$ \partial[\omega_G \wedge \omega_G \wedge \omega_G] = 0 \hspace{1cm} \text{for} \hspace{0.2cm} n = 4 $$

$$ \partial[H^{-1/3}\omega_G \wedge \omega_G \wedge \omega_G \wedge \omega_G] = 0 \hspace{1cm} \text{for} \hspace{0.2cm} n = 5 $$ \hspace{1cm} (4.10)

Note that these constraints reproduce (2.40) - (2.58)

M5-branes

The five-brane configurations do not fit this easily into a pattern, and will just have to be considered one by one.

We start with the two M5-branes which wrap a holomorphic two-cycle in $\mathbb{C}^n$. For both these systems, the calibrating form is the Hermitian two-form
associated with the metric, as long as it is subject to the proper constraint. In order to work out the constraint, we need to first calculate $\sqrt{|G|}$, where $G'$ is the determinant of the metric in the directions transverse to $\mathbb{C}^n$.

**M5 wrapping a 2-cycle in $\mathbb{C}^2$**

From the metric (3.42) it is clear that

$$|G'| = (H^{-1/3})^4(H^{2/3})^3 = H^{2/3}$$

so, the constraint (4.11) for this system is

$$\partial *_{\mathbb{C}^2}[H^{1/3}\omega_G] = \partial[H^{1/3}\omega_G] = 0$$

which agrees with (2.66).

**M5 wrapping a 2-cycle in $\mathbb{C}^3$**

The metric (3.45) allows us to read off

$$|G'| = (H^{-1/3})^4H^{2/3} = H^{-2/3}$$

so the constraint in this case takes the form

$$\partial *_{\mathbb{C}^3}[H^{-1/3}\omega_G] = \partial[H^{-1/3}\omega_G \wedge \omega_G] = 0$$

which reproduces (2.73).

We now turn to the last configuration on our list, and the only one which involves a brane wrapping a four-cycle. The calibrating form is now the square of the Hermitian two-form, $\omega_G \wedge \omega_G$ and in order to impose the relevant constraint, we need to compute $|G'|^{1/4}$.

**M5 wrapping a 4-cycle in $\mathbb{C}^3$**

In this case, we can see from the metric (3.48) that

$$G' = (H^{-1/3})^2(H^{2/3})^3 = H^{4/3}$$

and the constraint on the calibration is thus

$$\partial *_{\mathbb{C}^3}[H^{1/3}\omega_G \wedge \omega_G] = \partial[H^{1/3}\omega_G] = 0.$$
4.2 Count to Three....

Satisfying $\partial (\omega \wedge \omega) = 0$

Since $\omega_{\bar{M}P} = iG_{\bar{M}P}$, the above constraint can be written in component form as

$$\partial_R (G_{\bar{M}P} G_{N\bar{Q}}) = 0$$  \hspace{1cm} (4.14)

Upon contraction with the inverse metric $G^{N\bar{Q}}$ this gives

$$2(n - 3)[\partial_R G_{\bar{M}P} - \partial_M G_{R\bar{P}}] + G_{\bar{M}P} \partial_R \ln G =
2G^{N\bar{Q}}[G_{\bar{M}P} \partial_N G_{R\bar{Q}} - G_{R\bar{P}} \partial_N G_{M\bar{Q}}] + G_{R\bar{P}} \partial_M \ln G$$  \hspace{1cm} (4.15)

Here, $n$ denotes the complex dimension of the manifold which has $G_{M\bar{N}}$ as its Hermitean metric. Contracting again with $G^{M\bar{P}}$ we find the relation\(^1\)

$$\partial_R \ln G = 2G^{N\bar{Q}} \partial_N G_{R\bar{Q}}$$  \hspace{1cm} (4.16)

which can be substituted into (4.15) to give

$$(n - 3)[\partial_R G_{\bar{M}P} - \partial_M G_{R\bar{P}}] = 0$$  \hspace{1cm} (4.17)

This equation can be satisfied in two ways:

- Either $n = 3$, in which case $\partial (\omega \wedge \omega) = 0$ is a non-trivial requirement,
- Or we must have a Kahler metric, so that $\partial (\omega \wedge \omega)$ vanishes as a result of $\partial \omega = 0$.

Satisfying $\partial (\omega \wedge \omega \wedge \omega) = 0$

Employing the same procedure as in the previous case, we write the constraint out in component form

$$\partial_U (G_{MQ} G_{NR} G_{SP}) = 0$$  \hspace{1cm} (4.18)

and contract it with $G^{PS} G^{NR}$ to obtain

$$2(n - 4)[\partial_U G_{MQ} - \partial_M G_{U\bar{Q}}] + G_{MQ} \partial_U \ln G =
2G^{NR}[G_{MQ} \partial_N G_{UR} - G_{U\bar{Q}} \partial_N G_{M\bar{R}}] + G_{U\bar{Q}} \partial_M \ln G$$  \hspace{1cm} (4.19)

\(^1\)There is an overall factor of $(n-2)$ multiplying this equation, but that can be cancelled once we note that on manifolds of complex dimension $n \leq 2$, the five-form $\partial (\omega \wedge \omega)$ would vanish identically! Similar, obviously non-zero factors also occur in the analysis of the remaining constraints.
Contracting once more with the inverse metric $G^{M\bar{Q}}$ we arrive at the relation
\[ \partial_U \ln G = 2G^{M\bar{Q}} \partial_M G_{UQ} \] (4.20)
which can be substituted into (4.19) to give
\[ (n - 4)[\partial_U G_{MQ} - \partial_M G_{UQ}] = 0 \] (4.21)
Hence, it is only in a four-complex dimensional manifold that the constraint $\partial(\omega \land \omega \land \omega \land \omega) = 0$ can be satisfied \textit{without} having $\partial \omega = 0$.

In particular this implies that $\partial(\omega \land \omega \land \omega) = 0$ cannot be satisfied by a Hermitean two-form (of a non-Kahler metric) which obeys $\partial(\omega \land \omega) = 0$. This is in agreement with the analysis of the previous constraint, where it was found that $\partial(\omega \land \omega) = 0$ is a non-trivial constraint only in three complex dimensions.

\textbf{Satisfying $\partial(\omega \land \omega \land \omega \land \omega) = 0$}

Writing the constraint out in component form,
\[ \partial_A (G_{B\bar{C}} G_{DE} G_{F\bar{H}} G_{I\bar{J}}) = 0 \] (4.22)
and contracting with $G^{B\bar{C}} G^{D\bar{E}} G^{F\bar{H}}$, leads to the expression
\begin{align*}
2(n - 5)[\partial_A G_{I\bar{J}} \partial_I G_{A\bar{J}}] + 3G_{I\bar{J}} \partial_A \ln G =
& 6G^{B\bar{C}} [G_{I\bar{J}} \partial_B G_{A\bar{C}} - G_{A\bar{J}} \partial_B G_{I\bar{C}}] + 3G_{A\bar{J}} \partial_I \ln G
\end{align*}
(4.23)
This, when contracted further with $G^{I\bar{J}}$, yields the relation
\[ 2G^{BC} \partial_B G_{A\bar{C}} = \partial_A \ln G \] (4.24)
Substituting the above into (4.23), we find that
\[ (n - 5)[\partial_A G_{I\bar{J}} - \partial_I G_{A\bar{J}}] = 0 \] (4.25)
This statement implies that in order to satisfy
\[ \partial(\omega \land \omega \land \omega \land \omega) = 0 \]
non-trivially, we must have a manifold with complex dimension $n = 5$; in all other dimensions, this constraint can only be satisfied by a Kahler metric.
4.3 ... and You’re Home!

So, to summarize, what we learnt in the Emerald City was how to construct a supergravity solution for a wrapped M-brane by looking simply at the isometries of the configuration and using the technology of generalised calibrations. This analysis leads to an almost complete picture – the one missing piece in the puzzle is a constraint on generalised calibration. This constraint as we have now seen, can be generated by applying a remarkably simple rule \(^{(1.2)}\) to the spacetime in question. We have also seen that it is solved only by two possible classes of calibrations; those corresponding to metrics which are either Kahler, \(\partial \omega = 0\), or co-Kahler, \(\partial \ast \omega = 0\).

Now that you can construct the supergravity solution for an M-brane wrapped on any holomorphic cycle with a click of your heels, perhaps it is time to venture further into lands as yet unknown. Countless wrapped M-brane configurations exist which have not been considered here. The most obvious extension would be to look at M-branes on holomorphic cycles such that \(F \wedge F \neq 0\); an example is provided by an M5 wrapping a holomorphic four-cycle embedded in a four-complex dimensional manifold. Branching out further, one could consider M-branes wrapping Special Lagrangian cycles. It would be interesting to see if an analogous constraint arises on the embedding space metric in those cases and to explore its implications for the corresponding calibrations.

Whether or not the Ruby Slippers will be of any use to us in these new journeys remains to be seen. Certainly they have served us well in our little adventure here, and have provided us with plenty of material about which we can now “sit, and think some more”.
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