JUMP AND VARIATIONAL INEQUALITIES FOR AVERAGING OPERATORS WITH VARIABLE KERNELS

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Abstract. In this paper, we prove that the jump function and variation of averaging operators with rough variable kernels are bounded on $L^2(\mathbb{R}^n)$ if $\Omega \in L^\infty(\mathbb{R}^n) \times L^q(S^{n-1})$ for $q > 2(n-1)/n$ and $n \geq 2$. Moreover, we obtain the boundedness on weighted $L^p(\mathbb{R}^n)$ spaces of the jump function and $\rho$-variations for averaging operators with smooth variable kernels. Finally, we extend the result to the Morrey spaces.

1. Introduction. As is well known, the jump and variational estimates, initially arising from probability theory, were introduced by Bourgain’s [2] well-known series of studies on a pointwise ergodic theorem regarding subsets of integers, and they have attracted substantial attention from harmonic analysis. The first $\rho$-variation for averaging operators was studied by Bourgain [2]. Subsequently, Jones and his collaborators made outstanding contributions to variational inequalities for ergodic averages ([9, 12]). The weighted case was investigated by Ma et al. ([18, 19]). Recently, several studies on the jump and variational inequalities for some averaging operators of the convolutional type have appeared in harmonic analysis (cf. e.g. [8, 10, 17, 20]).

The purpose of this paper is to study the jump and variational inequalities for averaging operators with variable kernels, a class of operators of the non-convolutional type. Let us first define some notations. Let $1 \leq \rho < \infty$. The $\rho$-variation norm of the family $a = \{a_t : t > 0\}$ of complex numbers is defined as

$$\|a\|_{V_\rho} = \sup \left( |a_{t_0}|^\rho + \sum_{k \geq 1} |a_{t_k} - a_{t_{k-1}}|^\rho \right)^{\frac{1}{\rho}},$$

where the supremum is taken over all increasing sequences $\{t_k : k \geq 0\}$ of positive numbers. The value of the strong $\rho$-variation function $V_\rho(\mathcal{F})$ of a sequence $\mathcal{F} = \{F_t\}_{t \geq 0}$ of Lebesgue measurable functions at $x$ is defined as

$$V_\rho(\mathcal{F})(x) = \||F_t(x)\|_{V_\rho}, \quad \text{for } 1 \leq \rho < \infty.$$
Suppose $\mathcal{A} = \{A_t\}_{t>0}$ is a family of operators on $L^p(\mathbb{R}^n)(1 \leq p \leq \infty)$. The strong $\rho$-variation operator can be defined as

$$V_\rho(\mathcal{A}f)(x) = \|\{A_t(f)(x)\}_{t>0}\|_{V_\rho}, \quad \forall f \in L^p(\mathbb{R}^n).$$

In 2008, Jones et al. [14] explained that the $\rho$-variation estimate is a crucial tool in studying pointwise convergence almost everywhere for a family of operators, owing to the fact that it immediately implies the pointwise convergence almost everywhere of the underlying family of operators without using the Banach principle via the corresponding maximal inequality. This is why the mapping property of the strong $\rho$-variation operator is so interesting in ergodic theory and harmonic analysis.

Moreover, for any $f \in L^p(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$,

$$\sup_{t>0} |A_t(f)(x)| \leq V_\rho(\mathcal{A}f)(x), \quad \text{for } 1 \leq \rho < \infty. \quad (1.1)$$

For $\lambda > 0$, the value of the $\lambda$-jump function $N_\lambda(\mathcal{F})$ of the family $\mathcal{F} = \{F_t\}_{t>0}$ of Lebesgue measurable functions at $x$ is defined as

$$N_\lambda(\mathcal{F})(x) = \sup\{J \in \mathbb{N} : \exists 0 < s_1 < t_1 \leq s_2 < t_2 \leq \cdots \leq s_J < t_J$$

such that $|F_{t_k}(x) - F_{s_k}(x)| > \lambda$ for each $k = 1, 2, 3, \cdots, J\}.$

In [14], Jones et al. found that for $\lambda > 0$ and $\rho \geq 1$, the $\lambda$-jump function $N_\lambda(\mathcal{F})$ is pointwisely controlled by the strong $\rho$-variation $V_\rho(\mathcal{F})$ in the following sense:

$$\lambda(N_\lambda(\mathcal{F})(x))^{1/\rho} \leq CV_\rho(\mathcal{F})(x). \quad (1.2)$$

Let $S^{n-1} = \{x \in \mathbb{R}^n : |x| = 1\}$ be the unit sphere with Lebesgue measure $d\sigma$ on $\mathbb{R}^n$ ($n \geq 2$). For a function $\Omega(x, z)$ defined on $\mathbb{R}^n \times \mathbb{R}^n$, for a fixed $q \geq 1$, we say that $\Omega(x, z') \in L^\infty(\mathbb{R}^n) \times L^q(S^{n-1})$, if it satisfies the following conditions:

$$\Omega(x, \lambda z) = \Omega(x, z), \quad \text{for every } x, z \in \mathbb{R}^n \text{ and } \lambda > 0, \quad (1.3)$$

$$||\Omega||_{L^\infty(\mathbb{R}^n) \times L^q(S^{n-1})} := \sup_{x \in \mathbb{R}^n} \left( \int_{S^{n-1}} |\Omega(x, z')|^q d\sigma(z') \right)^{1/q} < \infty, \quad (1.4)$$

where $z' = z/|z|$, for every $z \in \mathbb{R}^n \backslash \{0\}$.

Let $\mathcal{M}_\Omega = \{M_{\Omega, t}\}_{t>0}$ be a family of averaging operators with rough variable kernel, where $M_{\Omega, t}$ is defined as

$$M_{\Omega, t}f(x) = \frac{1}{t^n} \int_{|y| < t} \Omega(x, y')f(x-y)dy, \quad (1.5)$$

where $\Omega \in L^\infty(\mathbb{R}^n) \times L^q(S^{n-1})$.

The family $\mathcal{M}_\Omega = \{M_{\Omega, t}\}_{t>0}$ is of interest for two reasons. First, in the case in which $\Omega(x, z') = 1$, the jump and variational inequalities for the family $\mathcal{M} = \{M_t\}_{t>0}$ have been thoroughly studied in [13]. In 2017, Ding, Hong, and Liu [10] proved the $L^p(\mathbb{R}^n)$ boundedness of the jump function for the family of averaging operators $\mathcal{M}_\Omega$ with $\Omega(x, z') = \Omega(z') \in L(\log^+L)^{1/2}(S^{n-1})$ or $\Omega(z') \in H^1(S^{n-1})$.

In addition, the $L^p(\mathbb{R}^n)$ boundedness of the strong $\rho$-variation for the family of averaging operators $\mathcal{M}_\Omega$ with $\Omega(x, z') = \Omega(z') \in L^1(S^{n-1})$ was obtained in [10]. Second, the maximal operator associated with $\mathcal{M}_\Omega = \{M_{\Omega, t}\}_{t>0}$,

$$M^*_\Omega f(x) = \sup_{t>0} \frac{1}{t^n} \int_{|y| < t} |\Omega(x, y')f(x-y)|dy,$$

plays a very important role in studying singular integral operators with rough variable kernels (see [7]).

The first result in this paper can be stated as follows.
Theorem 1.1. Let $\mathcal{M}_\Omega = \{M_{\Omega,t}\}_{t>0}$ be the family of the averaging operators given in (1.3). If $\Omega \in L^\infty(\mathbb{R}^n) \times L^q(\mathbb{S}^{n-1})$ for $q > 2(n-1)/n$, then there exists a constant $C > 0$ such that
\[
\sup_{\lambda > 0} \|\lambda \sqrt{N_\lambda(M_{\Omega,f})}\|_{L^2(\mathbb{R}^n)} \leq C\|f\|_{L^2(\mathbb{R}^n)}.
\] (1.6)

Furthermore, there exists a constant $C > 0$ such that the strong $\rho$-variation inequality
\[
\|V_\rho(M_{\Omega,f})\|_{L^2(\mathbb{R}^n)} \leq C\|f\|_{L^2(\mathbb{R}^n)},
\] (1.7)
holds for all $\rho > 2$.

Remark 1. The condition $q > 2(n-1)/n$ in Theorem 1.1 is optimal by a proof similar to the one in [3].

The idea of the proof of Theorem 1.1 was taken from [14]. To prove (1.6), we reduce the $\lambda$-jump estimate to a dyadic $\lambda$-jump estimate and short 2-variation estimate (see Section 2 for related notions). However, the details are substantially different because we are working with rough variable kernels. The well-known techniques for treating “variable kernel” operators through spherical harmonics can be dated back to a paper of Calderón [5]. Then, we use the Fourier transform, square function estimates, and the spherical harmonic expansions of the kernel to obtain a dyadic $\lambda$-jump estimate and short 2-variation operators. Regarding the proof of (1.7), we cannot use Lemma 1.2 in [14] directly because it appears to be difficult to obtain the result that the jump function for averaging operators with rough variable kernel is bounded on $L^p(\mathbb{R}^n)$ for a particular range of $p$’s. However, we also show the desired estimate by proving the long and short variational estimates separately (see the end of Section 2).

Let $w$ be a non-negative locally integrable function defined on $\mathbb{R}^n$. For $1 < p < \infty$, we say that $w$ is an $A_p$ weight if there exists a constant $C > 0$ such that for any cube $Q$,
\[
\sup_Q \left( \frac{1}{|Q|} \int_Q w(x) \, dx \right) \left( \frac{1}{|Q|} \int_Q w(x)^{1-p'} \, dx \right)^{p'-1} \leq C,
\]
where $p'$ is the conjugate index of $p$.

If we assume that $\Omega(x,z')$ has very strong smoothness on $\mathbb{S}^{n-1}$ in its second variable, then we can prove that the weighted $L^p(\mathbb{R}^n)$ boundedness of the jump function implies a strong $\rho(p > 2)$-variation operator for averaging operators with smooth variable kernels.

Theorem 1.2. Let $\mathcal{M}_\Omega = \{M_{\Omega,t}\}_{t>0}$ be the family of the averaging operators given in (1.5). Suppose that $\Omega$ satisfies (1.3) and
\[
\max_{|\ell| \leq 2n} \|\partial^{\ell}/\partial y^{\ell}\Omega(x,y)\|_{L^\infty(\mathbb{R}^n \times \mathbb{S}^{n-1})} < \infty.
\] (1.8)

Then, the following $\lambda$-jump inequality holds. For $1 < p < \infty$ and $w \in A_p$
\[
\sup_{\lambda > 0} \|\lambda \sqrt{N_\lambda(M_{\Omega,f})}\|_{L^p(w)} \leq C\|f\|_{L^p(w)}.
\] (1.9)

Consequently, for all $\rho > 2$, there exists a constant $C > 0$ such that
\[
\|V_\rho(M_{\Omega,f})\|_{L^p(w)} \leq C\|f\|_{L^p(w)},
\]
for $1 < p < \infty$ and $w \in A_p$. 
We can extend the results of Theorem 1.2 to the Morrey spaces. For $1 < p < \infty$, let
\[ \|f\|_{L^p(\mathbb{R}^n)} = \left( \sup_{x \in \mathbb{R}^n, r > 0} \frac{1}{r^\gamma} \int_{Q(x,r)} |f(y)|^p \, dy \right)^{1/p}, \]
where $Q(x,r)$ is any cube of radius $r$ centered at $x$, $\gamma \in (0, n)$. Let
\[ L^p(\mathbb{R}^n) = \{ f \in L^p_{\text{loc}}(\mathbb{R}^n) : \|f\|_{L^p(\mathbb{R}^n)} < \infty \}. \]
If $\gamma = 0$, then $L^p(\mathbb{R}^n)$ coincides with the Lebesgue space $L^p(\mathbb{R}^n)$. Consequently, the boundedness of the jump function implies strong $\rho(\rho > 2)$-variation operator for averaging operators with smooth variable kernel on the Morrey spaces $L^p(\mathbb{R}^n)$.

**Theorem 1.3.** Let $M_{\Omega} = \{ M_{\Omega,r} \}_{r > 0}$ be the family of the averaging operators given in (1.5). If $\Omega$ satisfies (1.3) and (1.8), then for $1 < p < \infty$ and $\gamma \in (0, n)$,
\[ \sup_{\lambda > 0} \| \lambda^\gamma N_{\lambda}(M_{\Omega} f)\|_{L^p(\mathbb{R}^n)} \leq C \|f\|_{L^p(\mathbb{R}^n)}. \] (1.10)
Moreover, for all $\rho > 2$, there exists a constant $C > 0$ such that
\[ \| V^\rho_{\rho}(M_{\Omega} f)\|_{L^p(\mathbb{R}^n)} \leq C \|f\|_{L^p(\mathbb{R}^n)} \]
for $1 < p < \infty$ and $\gamma \in (0, n)$.

The remainder of this paper is organized as follows. Section 2 provides the proof of Theorem 1.1, Section 3 provides the proof of Theorem 1.2, and Section 4 provides the proof of Theorem 1.3. Throughout this paper, the letter “$C$” will denote a positive constant whose value can change at each appearance.

### 2. Proof of Theorem 1.1.

**Proof.** Before proving Theorem 1.1, let us recall some relevant notions. Set $\Delta$ to be a Laplacian operator. Any homogeneous polynomial $P$ of degree $m$, a solution of $\Delta P(x) = 0$, is referred to as a spherical harmonic of degree $m$ if it’s restricted to the unit sphere $S^{n-1}$. Denote by $\mathcal{H}_m$ the space of spherical harmonics of degree $m$, let $\mathcal{H}_m$ be a finite-dimensional vector space, and let $\dim \mathcal{H}_m = D_m$. Call $\{Y_{m,j}\}$, $j = 1, 2, \ldots, D_m$, $m = 0, 1, \ldots$, an orthonormal system of spherical harmonics complete in $L^2(S^{n-1})$. As was pointed out in [5], $D_m \approx m^{(n-2)/2}$, and for fixed $m \in \mathbb{N}$ and any $\xi' \in S^{n-1}$,
\[ \left( \sum_{j=1}^{D_m} |Y_{m,j}(\xi')|^2 \right)^{\frac{1}{2}} \sim m^{(n-2)/2}. \] (2.1)

Now, we return to proving Theorem 1.1. We first provide the proof of (1.6). Given
\[ \Omega(x, y') = \left[ \Omega(x, y') - \frac{1}{\sigma(S^{n-1})} \int_{S^{n-1}} \Omega(x, y')d\sigma(y') \right] + \frac{1}{\sigma(S^{n-1})} \int_{S^{n-1}} \Omega(x, y')d\sigma(y') \]
\[ =: \Omega_0(x, y') + \Upsilon(x). \]
It follows that $\Omega_0$ satisfies
\[ \int_{S^{n-1}} \Omega_0(x, y')d\sigma(y') = 0, \] (2.2)
and $\Omega_0 \in L^\infty(\mathbb{R}^n) \times L^q(S^{n-1})$ for $q > 2(n-1)/n$. Thus,
\[ M_{\Omega_0,f}(x) = \frac{1}{t^n} \int_{|y| \leq t} \Omega_0(x, y')f(x - y)dy + \Upsilon(x) \frac{1}{t^n} \int_{|y| \leq t} f(x - y)dy \]
\[ =: M_{\Omega_0,f}(x) + \Upsilon(x)M_1f(x). \] (2.3)
We reduce the desired estimate to
\[
\sup_{\lambda > 0} \| \lambda \sqrt{N_\lambda(M_{\Omega_0} f)} \|_{L^2(\mathbb{R}^n)} \leq C \| \Omega \|_{L^\infty(\mathbb{R}^n) \times L^q(S^{n-1})} \| f \|_{L^2(\mathbb{R}^n)} \tag{2.4}
\]
and
\[
\sup_{\lambda > 0} \| \lambda \sqrt{N_\lambda(\Upsilon(\cdot) M f)} \|_{L^2(\mathbb{R}^n)} \leq C \| \Omega \|_{L^\infty(\mathbb{R}^n) \times L^q(S^{n-1})} \| f \|_{L^2(\mathbb{R}^n)}, \tag{2.5}
\]
where \( M_{\Omega_0} = \{ M_{\Omega_0, t} \}_{t > 0} \) and \( M = \{ M_t \}_{t > 0} \).

We first verify (2.5). By the Hölder inequality, there exists a constant \( C_1 > 0 \) such that \( |\Upsilon(x)| \leq C_1 \| \Omega \|_{L^\infty(\mathbb{R}^n) \times L^q(S^{n-1})} \). By the definition of the jump function, letting \( x \in \mathbb{R}^n \),
\[
N_\lambda(\Upsilon(\cdot) M f)(x) = \sup \{ J \in \mathbb{N} : \exists 0 < s_1 < t_1 \leq s_2 < t_2 \leq \cdots \leq s_f < t_f \\
\quad \text{ s.t. } |\Upsilon(x) M_{t_k} f(x) - \Upsilon(x) M_{s_k} f(x)| > \lambda \},
\]
and
\[
|\Upsilon(x) M_{t_k} f(x) - \Upsilon(x) M_{s_k} f(x)| \leq C_1 \| \Omega \|_{L^\infty(\mathbb{R}^n) \times L^q(S^{n-1})} |M_{t_k} f(x) - M_{s_k} f(x)|.
\]
Then,
\[
\{ J \in \mathbb{N} : \exists 0 < s_1 < t_1 \leq s_2 < t_2 \leq \cdots \leq s_f < t_f \\
\quad \text{ s.t. } |\Upsilon(x) M_{t_k} f(x) - \Upsilon(x) M_{s_k} f(x)| > \lambda \}
\subseteq \{ J \in \mathbb{N} : \exists 0 < s_1 < t_1 \leq s_2 < t_2 \leq \cdots \leq s_f < t_f \\
\quad \text{ s.t. } C_1 \| \Omega \|_{L^\infty(\mathbb{R}^n) \times L^q(S^{n-1})} |M_{t_k} f(x) - M_{s_k} f(x)| > \lambda \}.
\]
The above estimate yields the following pointwise estimate
\[
N_\lambda(\Upsilon(\cdot) M f)(x) \leq C_1 \| \Omega \|_{L^\infty(\mathbb{R}^n) \times L^q(S^{n-1})} (M f)(x).
\]
Then,
\[
\lambda \sqrt{N_\lambda(\Upsilon(\cdot) M f)(x)} \leq C_1 \| \Omega \|_{L^\infty(\mathbb{R}^n) \times L^q(S^{n-1})} \sqrt{N_\lambda(\Upsilon(\cdot) M f)(x)}.
\]
Let \( \lambda' = C_1 \| \Omega \|_{L^\infty(\mathbb{R}^n) \times L^q(S^{n-1})} \), it has been proved in [12] that
\[
\sup_{\lambda' > 0} \| \lambda' \sqrt{N_{\lambda'}(M f)} \|_{L^2(\mathbb{R}^n)} \leq C \| f \|_{L^2(\mathbb{R}^n)}.
\tag{2.7}
\]
Therefore, (2.6) and (2.7) yield (2.5).

Now, we verify (2.4) by proving the dyadic jump functions and short variational estimates separately (see [14, Lemma 1.3]). That is, we must show that
\[
\sup_{\lambda > 0} \left\| \lambda \sqrt{N_\lambda(M_{\Omega_0, 2^k} f)} \right\|_{L^2(\mathbb{R}^n)} \leq C \| \Omega \|_{L^\infty(\mathbb{R}^n) \times L^q(S^{n-1})} \| f \|_{L^2(\mathbb{R}^n)} \tag{2.8}
\]
and
\[
\| S_2(M_{\Omega_0} f) \|_{L^2(\mathbb{R}^n)} \leq C \| \Omega \|_{L^\infty(\mathbb{R}^n) \times L^q(S^{n-1})} \| f \|_{L^2(\mathbb{R}^n)}, \tag{2.9}
\]
where
\[
S_2(M_{\Omega_0} f)(x) = \left( \sum_{k \in \mathbb{Z}} |V_{2, k}(M_{\Omega_0} f)(x)|^2 \right)^{1/2}
\]
with
\[ V_{2,k}(\mathcal{M}_{\Omega_0}f)(x) = \left( \sup_{t_1,\ldots,t_{j_k+1} \in [2^k,2^{k+1}]} \left| \sum_{t=t_1}^{t_{j_k+1}} \mathcal{M}_{\Omega_0,t} f(x) - \mathcal{M}_{\Omega_0,t} f(x) \right| \right)^{1/2}. \]

For (2.8), as in [5], using a limit argument, we can reduce the proof of (2.8) to the case in which \( f \in C_0^\infty(\mathbb{R}^n) \) and \( \Omega_0(x,z') = \sum_{m \geq 0} D_m \sum_{j=1}^{a_m,j(x)Y_{m,j}(z')} \)

is a finite sum. Take note that \( \Omega_0(x,z') \) satisfies the cancellation condition (2.2), so \( a_{0,j} \equiv 0 \). Let
\[ a_m(x) = \left( \sum_{j=1}^{D_m} |a_{m,j}(x)|^2 \right)^{1/2} \]
and
\[ b_{m,j}(x) = \frac{a_{m,j}(x)}{a_m(x)}. \]
Then, 
\[ \sum_{j=1}^{D_m} b_{m,j}^2(x) = 1. \] (2.10)

Then,
\[ \mathcal{M}_{\Omega_0,2^k} f(x) = \sum_{m \geq 1} a_m(x) \sum_{j=1}^{D_m} b_{m,j}(x) \frac{1}{2kn} \int_{|y| < 2^k} Y_{m,j}(y') f(x-y) dy \]
\[ = \sum_{m \geq 1} a_m(x) \sum_{j=1}^{D_m} b_{m,j}(x) \tau_{k,m,j} * f(x), \]
where \( \tau_{k,m,j}(x) = 2^{-kn} Y_{m,j}(x') \chi_{\{|x'| < 2^k\}}(x) \), for \( k \in \mathbb{Z} \). Next, by (1.2), the Hölder inequality twice, and (2.10), for any fixed \( 0 < \theta < 1 \),
\[ \lambda \sqrt{N_\mathcal{A}(\{M_{\Omega_0,2^k} f\}_{k \in \mathbb{Z}})}(x) \]
\[ \leq C \left( \sum_{k \in \mathbb{Z}} \left| \sum_{m \geq 1} a_m(x) \sum_{j=1}^{D_m} b_{m,j}(x) \tau_{k,m,j} * f(x) \right|^2 \right)^{1/2} \]
\[ \leq C \left( \sum_{k \in \mathbb{Z}} \left\{ \sum_{m \geq 1} a_m^2(x) \right\} \left\{ \sum_{m \geq 1} m^{\theta} \left( \sum_{j=1}^{D_m} b_{m,j}(x) \tau_{k,m,j} * f(x) \right)^2 \right\} \right)^{1/2} \]
\[ \leq C \left( \sum_{m=1}^{\infty} a_m^2(x) m^{-\theta} \right)^{1/2} \left( \sum_{m=1}^{\infty} m^{\theta} \sum_{j=1}^{D_m} |\tau_{k,m,j} * f(x)|^2 \right)^{1/2}. \]
By [5], for \( q > 2(n-1)/n \), \( \theta \) is sufficiently close to 1, then
\[ \left( \sum_{m=1}^{\infty} a_m^2(x) m^{-\theta} \right)^{1/2} \leq C \| \Omega \|_{L^n(\mathbb{R}^n) \times L^q(\mathbb{R}^n)}. \] (2.11)
Then, we need to prove only that
\[ \left\| \sum_{m=1}^{\infty} m^\theta \sum_{j=1}^{D_m} \sum_{k \in \mathbb{Z}} |\tau_{k,m,j} \ast f|^2 \right\|_{L^2(\mathbb{R}^n)}^{1/2} \leq C \|f\|_{L^2(\mathbb{R}^n)}. \]  
(2.12)

Proving (2.12) requires the use of the following result, whose proof is similar to the proof of Lemma 2.2 in [6]. We omit the details.

**Lemma 2.1.** For $0 < \beta < 1$, $k \in \mathbb{Z}$, $m \in \mathbb{N}$. Let
\[ \tau_{k,m,j}(x) = 2^{-kn}Y_{m,j}(x)\chi_{\{|x|<2^k\}}(x). \]

Then, for $m \geq 1$,
\[ |\tau_{k,m,j}(\xi)| \leq C m^{-\theta+1+\beta/2} \min \{2^k |\xi|, |2^k \xi|^{-\beta/2}\} |Y_{m,j}(\xi')|, \]
(2.13)

where $\varrho = (n-2)/2$ and $\xi' = \xi/|\xi|$.

Applying Lemma 2.1, the Plancherel theorem, and $\sum_{j=1}^{D_m} |Y_{m,j}(\xi')|^2 \sim m^{2\varrho}$ and taking $0 < \beta < 1 - \theta$ yield
\[ \left\| \sum_{m=1}^{\infty} m^\theta \sum_{j=1}^{D_m} \sum_{k \in \mathbb{Z}} |\tau_{k,m,j} \ast f|^2 \right\|_{L^2(\mathbb{R}^n)}^{1/2} \leq C \int_{\mathbb{R}^n} m^\theta \sum_{m=1}^{\infty} m^\theta \sum_{j=1}^{D_m} \sum_{k \in \mathbb{Z}} |\tau_{k,m,j}(\xi)|^2 |\hat{f}(\xi)|^2 d\xi \]
\[ \leq C \int_{\mathbb{R}^n} m^\theta \sum_{m=1}^{\infty} m^{-\varrho-2+\beta} \left( \sum_{2^k \leq |\xi|^{-1}} |2^k \xi|^2 + \sum_{2^k > |\xi|^{-1}} |2^k \xi|^{-\beta} \right) \sum_{j=1}^{D_m} |Y_{m,j}(x')|^2 |\hat{f}(\xi)|^2 d\xi \]
\[ \leq C \int_{\mathbb{R}^n} m^\theta \sum_{m=1}^{\infty} m^{-\varrho-2+\beta+2\beta} |\hat{f}(\xi)|^2 d\xi \leq C \|f\|_{L^2(\mathbb{R}^n)}. \]

This completes the proof of (2.8).

For (2.9). Let $\psi(\xi) = \psi(|\xi|) \in C_c^\infty(\mathbb{R}^n)$ such that $0 \leq \psi \leq 1$, supp$\psi \subset \{ \xi : 1/2 \leq |\xi| \leq 2 \}$ and $\int_{\mathbb{R}^n} \psi^2 (2^{-t} \xi) = 1$ for $|\xi| \neq 0$. Define the multiplier $\Delta_l$ by $\Delta_l f(\xi) = \psi(2^{-t} \xi) \hat{f}(\xi)$. Then, by the Minkowski inequality,
\[ S_2(\mathcal{M}_{\Omega_0} f)(x) = \left( \sum_{k \in \mathbb{Z}} |V_{2,k}(\mathcal{M}_{\Omega_0} f)(x)|^2 \right)^{1/2} \]
\[ = \left( \sum_{k \in \mathbb{Z}} \left\| \left\{ \sum_{m \geq 1} a_m(x) \sum_{j=1}^{D_m} b_{m,j}(x) u_{k,t,m,j} \ast f(x) \right\}_{t \in [1,2]} \right\|_{V_2}^2 \right)^{1/2} \]
\[ \leq \sum_{t \in \mathbb{Z}} \left( \sum_{k \in \mathbb{Z}} \left\| \left\{ \sum_{m \geq 1} a_m(x) \sum_{j=1}^{D_m} b_{m,j}(x) u_{k,t,m,j} \ast \Delta_{l-k}^t f(x) \right\}_{t \in [1,2]} \right\|_{V_2}^2 \right)^{1/2} \]
\[ =: S_{2,l}(\mathcal{M}_{\Omega_0} f)(x), \]
where $u_{k,t,m,j}(x) = (2^k t)^{-n} Y_{m,j}(x') \chi_{\{|x|<2^k t\}}(x)$, for $k \in \mathbb{Z}$ and $t \in [1,2]$. Then, it is necessary only to provide the estimate of $\|S_{2,l}(\mathcal{M}_{\Omega_0} f)\|_{L^2(\mathbb{R}^n)}$ for $l \in \mathbb{Z}$.
Since $\|a_t\|_{L^2} \leq C\|a_t\|^{1/2}_X \frac{d}{dt}a_t^{1/2}_X$, where $X = L^2([1, 2], \mathbb{R})$ (see [8]). Then $a_t = \{\sum_{m \geq 1} a_m(x) \sum_{j=1}^{D_m} b_{m,j}(x)u_{k,t,m,j} \ast \Delta^2_{t-k} f(x)\}_{t \in [1, 2]}$ means that

$$\|S_{2,t}(\mathcal{M}_\Omega f)(x)\|^2 \leq C \left\| \sum_{k \in \mathbb{Z}} \left( \sum_{m \geq 1} a_m(\cdot) \sum_{j=1}^{D_m} b_{m,j}(\cdot)u_{k,t,m,j} \ast \Delta^2_{t-k} f \right) \right\|_{L^2([1, 2], \mathbb{R}^n)}^2 \cdot \left\| \frac{d}{dt} \left( \sum_{m \geq 1} a_m(\cdot) \sum_{j=1}^{D_m} b_{m,j}(\cdot)u_{k,t,m,j} \ast \Delta^2_{t-k} f \right) \right\|_{L^2([1, 2], \mathbb{R}^n)}^2.$$  

By the Cauchy-Schwarz inequality,

$$\|S_{2,t}(\mathcal{M}_\Omega f)\|_{L^2(\mathbb{R}^n)}^2 \leq C \left( \left\| \sum_{k \in \mathbb{Z}} \left( \sum_{m \geq 1} a_m(\cdot) \sum_{j=1}^{D_m} b_{m,j}(\cdot)u_{k,t,m,j} \ast \Delta^2_{t-k} f \right) \right\|_{L^2([1, 2], \mathbb{R}^n)}^2 \right)^{1/2} \cdot \left\| \frac{d}{dt} \left( \sum_{m \geq 1} a_m(\cdot) \sum_{j=1}^{D_m} b_{m,j}(\cdot)u_{k,t,m,j} \ast \Delta^2_{t-k} f \right) \right\|_{L^2([1, 2], \mathbb{R}^n)}^{1/2}. $$

We now estimate the first term on the right-hand side of (2.14). Using the Hölder inequality twice, (2.10) and (2.11), yields

$$\left\| \left( \sum_{k \in \mathbb{Z}} \left( \sum_{m \geq 1} a_m(\cdot) \sum_{j=1}^{D_m} b_{m,j}(\cdot)u_{k,t,m,j} \ast \Delta^2_{t-k} f \right) \right)^{1/2} \right\|_{L^2(\mathbb{R}^n)}^2 \leq C \left\| \Omega \right\|_{L^\infty(\mathbb{R}^n) \times L^1(\mathbb{Z}^{n-1})}^2 \left( \left\| \sum_{k \in \mathbb{Z}} \sum_{m=1}^{\infty} m^\theta \sum_{j=1}^{D_m} \left\| u_{k,t,m,j} \ast \Delta^2_{t-k} f \right\|_{L^2([1, 2], \mathbb{R}^n)}^2 \right)^{1/2} \right\|_{L^2(\mathbb{R}^n)}^2.$$ 

By Lemma 2.1, for fixed $0 < \beta < 1$ and $\theta = \frac{n-2}{2}$,

$$|u_{k,t,m,j}(x)| \leq m^{-\varepsilon+1/2} \min \{|2^k \xi|, |2^k \xi|^{-\beta/2}\} |Y_{m,j}(\xi')|,$$

uniformly in $t \in [1, 2]$. Therefore, using the Plancherel theorem, Littlewood-Paley theory, and $\sum_{j=1}^{D_m} |Y_{m,j}(\xi')|^2 \sim m^{2\varepsilon}$ and taking $0 < \beta < 1 - \theta$, for some $0 < \mu < 1$,

$$\left\| \left( \sum_{k \in \mathbb{Z}} \sum_{m=1}^{\infty} m^\theta \sum_{j=1}^{D_m} \left\| u_{k,t,m,j} \ast \Delta^2_{t-k} f \right\|_{L^2([1, 2], \mathbb{R}^n)}^2 \right)^{1/2} \right\|_{L^2(\mathbb{R}^n)}^2 \leq C \left\| \Omega \right\|_{L^\infty(\mathbb{R}^n) \times L^1(\mathbb{Z}^{n-1})} \left( \left\| \sum_{k \in \mathbb{Z}} \sum_{m=1}^{\infty} m^\theta \sum_{j=1}^{D_m} \left\| u_{k,t,m,j} \ast \Delta^2_{t-k} f \right\|_{L^2([1, 2], \mathbb{R}^n)}^2 \right)^{1/2} \right\|_{L^2(\mathbb{R}^n)}^2. $$(2.15)

$$= \sum_{m=1}^{\infty} m^\theta \sum_{k \in \mathbb{Z}} \int_1^2 \int_{\mathbb{R}^n} |\psi(2^k \xi)|^2 |u_{k,t,m,j}(\xi')|^2 |\Delta_{t-k} f(\xi')|^2 d\xi dt,$$

$$\leq C \sum_{m=1}^{\infty} m^{-2\varepsilon+2+\beta+\theta - \mu} \int_{\mathbb{R}^n} |\Delta_{t-k} f(\xi')|^2 \sum_{j=1}^{D_m} |Y_{m,j}(\xi')|^2 d\xi \leq C 2^{-\mu} \|f\|_{L^2(\mathbb{R}^n)}^2. $$

Now, we turn to estimating the second term on the right-hand side of (2.14). By the spherical coordinate transformation, a short calculation shows

$$\frac{d}{dt} \left( \sum_{m \geq 1} a_m(x) \sum_{j=1}^{D_m} b_{m,j}(x)u_{k,t,m,j} \ast h(x) \right)$$

$$= \frac{d}{dt} \left( \int_{|y| < 2^t} \Omega(x, y') h(x - y) dy \right) = \frac{d}{dt} \left( \int_{\mathbb{S}^{n-1}} \Omega(x, y') \int_0^{2^t} \frac{1}{r} h(x - ry') dr \sigma(y') \right).$$
Note that \( t \in [1, 2] \), by the Minkowski inequality,

\[
\left( \int_{\mathbb{R}^n} \left| \frac{d}{dt} \left( \sum_{m \geq 1} a_m(x) \sum_{j=1}^{D_m} b_{m,j}(x) u_{k,t,m,j} \ast h(x) \right) \right|^2 dx \right)^{1/2} = \left( \int_{\mathbb{R}^n} \frac{1}{t} \int_{S^{n-1}} \Omega(x, y') h(x - 2^k t y') d\sigma(y') \right)^{1/2}
\]

then, by Littlewood-Paley theory,

\[
\left\| \left( \sum_{k \in \mathbb{Z}} \left\| \frac{d}{dt} \left( \sum_{m \geq 1} a_m(x) \sum_{j=1}^{D_m} b_{m,j}(x) u_{k,t,m,j} \ast \Delta^2_{-k} f \right) \right\|_{L^2([1, 2])}^2 \right\|_{L^2(\mathbb{R}^n)}^{1/2} \leq C \left\| \Omega \right\|^2_{L^\infty(\mathbb{R}^n) \times L^1(S^{n-1})} \int_{1}^{2} \int_{\mathbb{R}^n} \left| \Delta^2_{-k} f(x) \right|^2 dx \frac{dt}{t} \leq C \left\| \Omega \right\|^2_{L^\infty(\mathbb{R}^n) \times L^1(S^{n-1})} \left\| f \right\|_{L^2(\mathbb{R}^n)}. \tag{2.16}\]

Combining (2.15) with (2.16) yields that

\[
\left\| S_{2,1}(\mathcal{M}_\Omega f) \right\|_{L^2(\mathbb{R}^n)} \leq C 2^{-\mu(|l|/2)} \left\| \Omega \right\|_{L^\infty(\mathbb{R}^n) \times L^1(S^{n-1})} \left\| f \right\|_{L^2(\mathbb{R}^n)}. \tag{2.17}\]

Then,

\[
\left\| S_{2}(\mathcal{M}_\Omega f) \right\|_{L^2(\mathbb{R}^n)} \leq C \sum_{l \in \mathbb{Z}} 2^{-\mu(|l|/2)} \left\| \Omega \right\|_{L^\infty(\mathbb{R}^n) \times L^1(S^{n-1})} \left\| f \right\|_{L^2(\mathbb{R}^n)} \leq C \left\| \Omega \right\|_{L^\infty(\mathbb{R}^n) \times L^1(S^{n-1})} \left\| f \right\|_{L^2(\mathbb{R}^n)}.
\]

This completes the proof of (1.6).

Finally, we turn to the proof of (1.7). Equality (2.3) reduces the desired estimate (1.7) to

\[
\left\| V_\rho(\mathcal{M}_\Omega f) \right\|_{L^2(\mathbb{R}^n)} \leq C \left\| \Omega \right\|_{L^\infty(\mathbb{R}^n) \times L^1(S^{n-1})} \left\| f \right\|_{L^2(\mathbb{R}^n)} \tag{2.18}\]

and

\[
\left\| V_\rho(\mathcal{M}(\cdot) f) \right\|_{L^2(\mathbb{R}^n)} \leq C \left\| \Omega \right\|_{L^\infty(\mathbb{R}^n) \times L^1(S^{n-1})} \left\| f \right\|_{L^2(\mathbb{R}^n)}. \tag{2.19}\]

The definition of the strong \( \rho \)-variation operator and the Hölder inequality together imply that

\[
\left\| V_\rho(\mathcal{M}(\cdot) f) \right\|_{L^2(\mathbb{R}^n)} \leq C \left\| \Omega \right\|_{L^\infty(\mathbb{R}^n) \times L^1(S^{n-1})} \left\| V_\rho(\mathcal{M} f) \right\|_{L^2(\mathbb{R}^n)} \leq C \left\| \Omega \right\|_{L^\infty(\mathbb{R}^n) \times L^1(S^{n-1})} \left\| f \right\|_{L^2(\mathbb{R}^n)},
\]

where the second inequality has been proved in [12]. To finish the proof of (2.18), by the Hölder inequality twice, (2.10), (2.11), and (2.12),

\[
\left\| V_\rho(\mathcal{M}_{\Omega_{0,2k}} f) \right\|_{L^2(\mathbb{R}^n)} \leq C \left\| \left( \sum_{k \in \mathbb{Z}} \left( \sum_{m \geq 1} a_m(x) \sum_{j=1}^{D_m} b_{m,j}(x) \tau_{k,m,j} \ast f(x) \right) \right)^2 \right\|_{L^2(\mathbb{R}^n)}^{1/2}
\]
Then, combining (2.20) with (2.9) yields inequality (2.18). This completes the proof of Theorem 1.1.

3. Proof of Theorem 1.2.

Proof. Let us first recall some properties \{Y_{m,j}(x')\}_{j=1}^{D_m}. For any multi-index \(\alpha\) (see [4])

\[
\sup_{x' \in \mathbb{S}^{n-1}} \left| \frac{\partial}{\partial x'}^{\alpha} Y_{m,j}(x') \right| \leq C \|x^{\alpha+(n-2)/2}\| \quad \text{for any positive integer } l.
\]

If \(\varphi \in C^\infty(\mathbb{S}^{n-1})\), then \(\sum_m \sum_j a_{m,j} Y_{m,j}(x')\) is the Fourier series expansion of \(\varphi(x')\) with respect to \(\{Y_{m,j}(x')\}_{m,j}\), where

\[ a_{m,j} = \int_{\mathbb{S}^{n-1}} \varphi(y') Y_{m,j}(y') \, d\sigma(y') \]

and (see [4])

\[ a_{m,j} = (-1)^l m^{-l} (m+n-2)^{-l} \int_{\mathbb{S}^{n-1}} L^l(\varphi) Y_{m,j}(x') \, d\sigma(x') \quad m \geq 1, \quad (3.1) \]

where \(L(\varphi) = |x|^2 \Delta \varphi(x)\) and \(L^{l+1}(\varphi) = L(L^l(\varphi))\) for any positive integer \(l\). In particular, the expansion of \(\varphi\) into spherical harmonics converges uniformly to \(\varphi\). As was pointed out in [5],

\[ D_m \leq C m^{n-2}, \quad m \geq 1. \quad (3.2) \]

We perform the following decomposition:

\[
\Omega(x,y') = \left[ \Omega(x,y') - \frac{1}{\sigma(\mathbb{S}^{n-1})} \int_{\mathbb{S}^{n-1}} \Omega(x,y') \, d\sigma(y') \right] + \frac{1}{\sigma(\mathbb{S}^{n-1})} \int_{\mathbb{S}^{n-1}} \Omega(x,y') \, d\sigma(y')
\]

\[ =: \Omega_1(x,y') + \Gamma(x). \]

Then, \(\Omega_1(x,y')\) satisfies

\[
\int_{\mathbb{S}^{n-1}} \Omega_1(x,y') \, d\sigma(y') = 0, \quad (3.3)
\]

and (1.8). Thus,

\[
M_{\Omega_1} f(x) = \frac{1}{t^n} \int_{|y'|<t} \Omega_1(x,y') f(x-y) \, dy + \Gamma(x) \frac{1}{t^n} \int_{|y'|<t} f(x-y) \, dy
\]

\[ =: M_{\Omega_1} f(x) + \Gamma(x) M_t f(x). \quad (3.4) \]

Denote the operator family \(\{M_{\Omega_1}(t)\}_{t>0}\) by \(M_{\Omega_1}\), and \(\{M_t\}_{t>0}\) by \(M\). It is sufficient to show that

\[
\sup_{\lambda>0} \|\lambda^{\alpha}(M_{\Omega_1} f)\|_{L^p(w)} \leq C \|f\|_{L^p(w)} \quad (3.5)
\]

and

\[
\sup_{\lambda>0} \|\lambda^{\alpha}(\Gamma(\cdot) M f)\|_{L^p(w)} \leq C \|f\|_{L^p(w)} \quad (3.6)
\]

for \(1 < p < \infty\) and \(w \in A_p\).
We can see that $|\Gamma(x)| \leq \|\Omega\|_{L^\infty(\mathbb{R}^n) \times L^\infty(\mathbb{R}^{n-1})}$. Similarly to the proof of the inequality of (2.5), we also obtain the following pointwise estimate:

$$\lambda \sqrt{N_A(\Gamma f)}(x) \leq \frac{\lambda \|\Omega\|_{L^\infty(\mathbb{R}^n) \times L^\infty(\mathbb{R}^{n-1})}}{\|\Omega\|_{L^\infty(\mathbb{R}^n) \times L^\infty(\mathbb{R}^{n-1})}} \left( \int_{\mathbb{R}^n} \lambda (Mf)(x) \right).$$

Let $\lambda_1 = \frac{\lambda}{\|\Omega\|_{L^\infty(\mathbb{R}^n) \times L^\infty(\mathbb{R}^{n-1})}}$. To obtain (3.6), it is sufficient to prove

$$\sup_{\lambda > 0} \|\lambda_1 \sqrt{N_A(\lambda Mf)}\|_{L^p(w)} \leq C \|f\|_{L^p(w)}$$

for $1 < p < \infty$ and $w \in A_p$. The inequality of (3.7) is proved in [16].

For (3.5). Similarly to the proof of (2.4), we prove that for $1 < p < \infty$ and $w \in A_p$,

$$\sup_{\lambda > 0} \|\lambda_1 \sqrt{N_A(\lambda M_{\Omega_1} f)}\|_{L^p(w)} \leq C \|f\|_{L^p(w)}$$

and

$$\|S_2(\lambda M_{\Omega_1} f)\|_{L^p(w)} \leq C \|f\|_{L^p(w)}.$$  

To prove (3.8), we define

$$M_{\Omega_1,2^k}f(x) = \frac{1}{2^{kn}} \int_{|y| < 2^k} \Omega_1(x,y')f(x-y)dy.$$  

Similarly to the decomposition of $\Omega_0(x,z')$ in the proof of Theorem 1.2,

$$\Omega_1(x,z') = \sum_{m \geq 1} \sum_{j=1}^{D_m} a_{m,j}(x) Y_{m,j}(z').$$

Then,

$$M_{\Omega_1,2^k}f(x) = \sum_{m \geq 1} \sum_{j=1}^{D_m} a_{m,j}(x) \frac{1}{2^{kn}} \int_{|y| < 2^k} Y_{m,j}(y')f(x-y)dy$$

$$= \sum_{m \geq 1} \sum_{j=1}^{D_m} a_{m,j}(x) \tau_{k,m,j} * f(x),$$

where $\tau_{k,m,j}(x) = 2^{-kn}Y_{m,j}(x')\lambda_{\{|x| < 2^k\}}(x)$, for $k \in \mathbb{Z}$. By (1.8) and (3.1),

$$\|a_{m,j}\|_{L^\infty} \leq C m^{-2n}.$$  

By the Minkowski inequality and (3.11),

$$\lambda \sqrt{N_A(\lambda M_{\Omega_1,2^k} f)}(x) \leq C \left( \sum_{k \in \mathbb{Z}} \left| \sum_{m \geq 1} \sum_{j=1}^{D_m} a_{m,j}(x) \tau_{k,m,j} * f(x) \right|^2 \right)^{\frac{1}{2}}$$

$$\leq C \sum_{m \geq 1} \sum_{j=1}^{D_m} m^{-2n} \left( \sum_{k \in \mathbb{Z}} \left| \tau_{k,m,j} * f(x) \right|^2 \right)^{\frac{1}{2}}$$

$$= C \sum_{m \geq 1} \sum_{j=1}^{D_m} m^{-2n} R_{m,j} f(x).$$

where

$$R_{m,j} f(x) := \left( \sum_{k \in \mathbb{Z}} \left| \tau_{k,m,j} * f(x) \right|^2 \right)^{\frac{1}{2}}.$$
It is sufficient to show that
\[ \| R_{m,j} f \|_{L^p(w)} \leq C m^{n/2} \| f \|_{L^p(w)}. \]  
(3.13)

Then, (3.12), (3.13), and \( D_m \leq C m^{n-2} \) yield that for \( 1 < p < \infty \) and \( w \in A_p \),
\[ \sup_{\lambda > 0} \| \lambda \sqrt{N_{\lambda}(M_{\Omega_{2,\lambda}} f)} \|_{L^p(w)} \leq C \sum_{m \geq 1} m^{-2n+n-2+n/2} \sum_{l \in \mathbb{Z}} 2^{-|l|} \| f \|_{L^p(w)} \leq C \| f \|_{L^p(w)}. \]

Now, we turn to proving (3.13). By \( \sum_{l \in \mathbb{Z}} \Delta_l^2 f = f \) and the Minkowski inequality,
\[ R_{m,j} f(x) = \left( \sum_{k \in \mathbb{Z}} | \tau_{k,m,j} * \sum_{l \in \mathbb{Z}} \Delta_l^2 f(x) |^2 \right)^{1/2} \leq \sum_{l \in \mathbb{Z}} \left( \sum_{k \in \mathbb{Z}} | \tau_{k,m,j} * \Delta_l^2 f(x) |^2 \right)^{1/2} =: \sum_{l \in \mathbb{Z}} R_{m,j}^l f(x). \]
(3.14)

We first provide the \( L^2(\mathbb{R}^n) \) estimate of \( R_{m,j}^l f \) for \( l \in \mathbb{Z} \). Since \( \text{supp} \psi \subset \{ \xi : 1/2 \leq |\xi| \leq 2 \} \), Lemma 2.1 yields
\[ | \psi(2^{k-l} \xi) | | \tau_{k,m,j}(\xi) | \leq C m^{-1} \varphi + \beta/2 \min\{2^{-1/2}, 2^l\} | Y_{m,j}(\xi') |. \]
(3.15)

The Plancherel theorem, Littlewood-Paley theory, (3.15), and \( | Y_{m,j}(\xi') | \leq C m^\varphi, \varphi = \frac{n-2}{2} \) yield that for some \( 0 < \epsilon < 1 \),
\[ \| R_{m,j}^l f \|_{L^2(\mathbb{R}^n)} = \left( \int_{\mathbb{R}^n} \sum_{k \in \mathbb{Z}} \psi(2^{k-l} \xi) | \tau_{k,m,j}(\xi) |^2 | \Delta_l^2 f(\xi) |^2 d\xi \right)^{1/2} \leq C m^{-1} \varphi + \beta/2 \sum_{k \in \mathbb{Z}} | Y_{m,j}(\xi') |^2 \left( \int_{\mathbb{R}^n} | \Delta_l^2 f(\xi) |^2 d\xi \right)^{1/2} \leq C m^{n/2} 2^{-\epsilon |l|} \| f \|_{L^2(\mathbb{R}^n)}. \]
(3.16)

Next, we estimate \( \| R_{m,j}^l f \|_{L^p(w)} \) for \( l \in \mathbb{Z} \). By \( | Y_{m,j}(\xi') | \leq C m^{n/2} \),
\[ \left\| \sup_{k \in \mathbb{Z}} | \tau_{k,m,j} * f_k | \right\|_{L^p(w)} \leq C m^{n/2} \left\| M(\sup_{k \in \mathbb{Z}} | f_k |) \right\|_{L^p(w)} \quad 1 < p < \infty, \]
(3.17)

where \( M \) is the Hardy-Littlewood maximal operator. It follows from the \( L^p(w) \) \( (1 < p < \infty) \) boundedness of \( M \) and (3.17) that
\[ \left\| \sup_{k \in \mathbb{Z}} | \tau_{k,m,j} * f_k | \right\|_{L^p(w)} \leq C m^{n/2} \left\| \sup_{k \in \mathbb{Z}} | f_k | \right\|_{L^p(w)}. \]
(3.18)

Using (3.18) and duality yields that for \( 1 < p < \infty \) and \( w \in A_p \),
\[ \left\| \sum_{k \in \mathbb{Z}} | \tau_{k,m,j} * f_k | \right\|_{L^p(w)} \leq C m^{n/2} \left\| \sum_{k \in \mathbb{Z}} | f_k | \right\|_{L^p(w)} \].
(3.19)

Interpolating between (3.18) and (3.19) yields that for \( 1 < p < \infty \) and \( w \in A_p \),
\[ \left\| \left( \sum_{k \in \mathbb{Z}} | \tau_{k,m,j} * f_k |^2 \right)^{1/2} \right\|_{L^p(w)} \leq C m^{n/2} \left\| \left( \sum_{k \in \mathbb{Z}} | f_k |^2 \right)^{1/2} \right\|_{L^p(w)}. \]
(3.20)
Applying (3.20) and the weighted Littlewood-Paley theory (see [15]) yields that
\[ \|R_{m,j}^l f\|_{L^p(w)} \leq C m^{n/2} \left( \sum_{k \in \mathbb{Z}} |\Delta_{l-k}^2 f|^2 \right)^{1/2} \|f\|_{L^p(w)}. \tag{3.21} \]

Using Stein and Weiss’s interpolation theorem [21] with a change of measure to (3.16) and (3.21) yields that for some \( \delta \in (0,1) \)
\[ \|R_{m,j}^l f\|_{L^p(w)} \leq C 2^{-\delta |l|} m^{n/2} \|f\|_{L^p(w)}. \tag{3.22} \]

Finally, we consider (3.9). By the Minkowski inequality, \( \sum_{l \in \mathbb{Z}} \Delta_{l} f = f \), and \( \|a_{m,j}\|_{L^\infty} \leq C m^{-2n} \)
\[ S_2(\mathcal{M}_1 f)(x) \leq C \sum_{m \geq 1} \sum_{j=1}^{D_m} m^{-2n} \left( \sum_{k \in \mathbb{Z}} \left\| \{ u_{k,t,m,j} \ast f(x) \}_{t \in [1,2]} \right\|_{L^2(V_2)}^2 \right)^{1/2} \]
\[ \leq \sum_{m \geq 1} \sum_{j=1}^{D_m} m^{-2n} \sum_{t \in \mathbb{Z}} \left( \sum_{k \in \mathbb{Z}} \left\| \{ u_{k,t,m,j} \ast \Delta_{l-k}^2 f(x) \}_{t \in [1,2]} \right\|_{L^2(V_2)}^2 \right)^{1/2} \]
\[ =: \sum_{m \geq 1} \sum_{j=1}^{D_m} m^{-2n} \sum_{t \in \mathbb{Z}} S_{2,l}(\mathcal{M}_{m,j} f)(x), \]
where \( u_{k,t,m,j}(x) = (2^l t)^{-n} Y_{m,j}(x) \chi_{\{|x| < 2^l t\}}(x) \), for \( k \in \mathbb{Z} \) and \( t \in [1,2] \). Obtaining (3.9) requires showing that for all \( l \in \mathbb{Z} \), there exists some \( 0 < \theta' < 1 \) such that
\[ \|S_{2,l}(\mathcal{M}_{m,j} f)\|_{L^p(w)} \leq C 2^{-\theta' |l|} m^{n/2} \|f\|_{L^p(w)} \tag{3.23} \]
for \( 1 < p < \infty \) and \( w \in A_p \). Then, by \( D_m \leq C m^{n-2} \)
\[ \|S_2(\mathcal{M}_1 f)\|_{L^p(w)} \leq C \sum_{m \geq 1} \sum_{j=1}^{D_m} m^{-2n+n/2} \sum_{t \in \mathbb{Z}} 2^{-\theta' |l|} \|f\|_{L^p(w)} \leq C \|f\|_{L^p(w)} \]
for \( 1 < p < \infty \) and \( w \in A_p \).

Now, we turn to proving (3.23). First, for some \( 0 < \epsilon < 1 \)
\[ \|S_{2,l}(\mathcal{M}_{m,j} f)\|_{L^2(\mathbb{R}^n)} \leq C 2^{-\epsilon |l|} m^{n/2} \|f\|_{L^2(\mathbb{R}^n)}, \tag{3.24} \]
which can be proved in a manner similar to the proof of the estimate of (2.17). We omit the details here.

However, we offer the following observation:
\[ S_{2,l}(\mathcal{M}_{m,j} f)(x) \]
\[ = \left( \sum_{k \in \mathbb{Z}} \sup_{t_1, \ldots, t_j} \left| \int_{t_{s+1} \leq |y| < 2^{t_s}} \frac{Y_{m,j}(y')}{|y|^{n}} \Delta_{l-k}^2 f(x-y) dy \right| \right)^{1/2} \]
\[ \leq \left( \sum_{k \in \mathbb{Z}} \sup_{t_1, \ldots, t_j} \left( \int_{t_{s+1} \leq |y| < 2^{t_s}} |Y_{m,j}(y')| \frac{|\Delta_{l-k}^2 f(x-y)| dy}{|y|^{n}} \right)^2 \right)^{1/2} \]
Proof of Theorem 1.3.

4. Proof of Theorem 1.3. We need to prove only (1.10). The strategy of the proof is taken from [11]. By (3.4), the proof of (1.10) is reduced to proving that for 1 < p < ∞ and w ∈ A_p,

\[ \left\| \sum_{k \in \mathbb{Z}} |M(f_k)|^2 \right\|_{L^p(w)} \leq C \left\| \sum_{k \in \mathbb{Z}} |\Delta_{-k}^{2}f|^2 \right\|_{L^p(w)} \]

which was established in [1]. Then, by the weighted Littlewood-Paley theory (see [15]),

\[ \|S_{2,l}(M_{m,j}f)\|_{L^p(w)} \leq C m^{n/2} \left\| \sum_{k \in \mathbb{Z}} |M(\Delta_{-k}^{2}f)|^2 \right\|_{L^p(w)} \leq C m^{n/2} \|f\|_{L^p(w)} \quad (3.25) \]

for 1 < p < ∞ and w ∈ A_p. Thus, using Stein and Weiss's interpolation theorem [21] with a change of measure to (3.24) and (3.25) completes the proof of (3.23). This completes the proof of Theorem 1.2. \[\square\]

4. Proof of Theorem 1.3. We need to prove only (1.10). The strategy of the proof is taken from [11]. By (3.4), the proof of (1.10) is reduced to proving that for 1 < p < ∞ and γ ∈ (0, n),

\[ \sup_{\lambda > 0} \|\lambda \sqrt{N_{\lambda}(M_{\Omega_1}f)}\|_{L^p(\mathbb{R}^n)} \leq C \|f\|_{L^p(\mathbb{R}^n)} \quad (4.1) \]

and

\[ \sup_{\lambda > 0} \|\lambda_1 \sqrt{N_{\lambda_1}(Mf)}\|_{L^p(\mathbb{R}^n)} \leq C \|f\|_{L^p(\mathbb{R}^n)}, \quad (4.2) \]

where \( M_{\Omega_1} \), \( M \) and \( \lambda_1 \) are as defined in Section 3.

For (4.1), In a manner similar to that used in the proof of (2.4), we prove that for 1 < p < ∞ and γ ∈ (0, n),

\[ \sup_{\lambda > 0} \|\lambda \sqrt{N_{\lambda}(M_{\Omega_1,2\kappa}f)}\|_{L^p(\mathbb{R}^n)} \leq C \|f\|_{L^p(\mathbb{R}^n)} \quad (4.3) \]

and

\[ \|S_2(M_{\Omega_1}f)\|_{L^p(\mathbb{R}^n)} \leq C \|f\|_{L^p(\mathbb{R}^n)}. \quad (4.4) \]

For (4.3). By the Minkowski inequality and \( \|a_{m,j}\|_{L^\infty} \leq C m^{-2n}, \)

\[ \sup_{\lambda > 0} \lambda \sqrt{N_{\lambda}(M_{\Omega_1,2\kappa}f)}(x) \leq C \sum_{m \geq 1} \sum_{j=1}^{D_m} m^{-2n} \left( \sum_{k \in \mathbb{Z}} |\tau_{k,m,j} * f(x)|^2 \right)^{\frac{1}{2}} \]

\[ = C \sum_{m \geq 1} \sum_{j=1}^{D_m} m^{-2n} R_{m,j}f(x), \quad (4.5) \]

where

\[ R_{m,j}f(x) := \left( \sum_{k \in \mathbb{Z}} |\tau_{k,m,j} * f(x)|^2 \right)^{\frac{1}{2}} \]
and \( \tau_{k,m,j}(x) = 2^{-kn} Y_{m,j}(x') \chi_{\{|x'|<2^k\}}(x) \), for \( k \in \mathbb{Z} \). Then, we need only to obtain the proof of the boundedness of \( R_{m,j} f \) on the Morrey space. For fixed \( t \in \mathbb{R}^n \) and \( r > 0 \), we abbreviate \( B = B(t,r) \). For \( f \in L^{p,\gamma}(\mathbb{R}^n) \), we write

\[
 f(y) = f(y) \chi_{2B}(y) + \sum_{\ell=1}^{\infty} f(y) \chi_{2^{\ell+1}B \setminus 2^\ell B}(y) =: \sum_{\ell=0}^{\infty} f_{\ell}(y).
\]

For \( k = 0 \), by (3.13) with \( w(x) = 1 \),

\[
 \int_B |R_{m,j} f_0(x)|^p \, dx \leq ||R_{m,j} f_0||_{L^p(\mathbb{R}^n)}^p \leq m^{n/2} ||f_0||_{L^p(\mathbb{R}^n)}^p
\]

\[
 = C m^{n/2} \int_{2B} |f(y)|^p \, dx \leq C m^{n/2}(2r)^{\gamma} ||f||_{L^{p,\gamma}(\mathbb{R}^n)}^p.
\]

For \( \ell > 0 \), \( M(\chi_B)(x) \sim 2^{-\ell n} \) when \( x \in 2^{\ell+1}B \setminus 2^\ell B \) \((\text{see [11]}\)). Fixing any \( \eta \in (\gamma/n, 1) \), \((M_XB)^{\eta} \in A_1 \subset A_2 \) with \( A_1 \) constant depending only on \( \gamma \) and \( n \). Now, for \( \ell > 0 \), by (3.13), for \( 1 < p < \infty \) and \( \gamma \in (0,n) \),

\[
 \int_B |R_{m,j} f_\ell(x)|^p \, dx
\]

\[
 = \int_{\mathbb{R}^n} |R_{m,j} f_\ell(x)|^p (\chi_B(x))^n \, dx \leq \int_{\mathbb{R}^n} |R_{m,j} f_\ell(x)|^p (M_XB(x))^\eta \, dx
\]

\[
 \leq C m^{n/2} \int_{\mathbb{R}^n} |f_\ell(x)|^p (M_XB(x))^\eta \, dx \leq C m^{n/2} 2^{-\ell n} \int_{2^{\ell+1}B} |f_\ell(x)|^p \, dx
\]

\[
 \leq C m^{n/2} 2^{-\ell n(\eta-\gamma/n)} 2^{-\ell t} \int_{2^{\ell+1}B} |f_\ell(x)|^p \, dx \leq C m^{n/2} 2^{-\ell n(\eta-\gamma/n)} r^{\gamma} ||f||_{L^{p,\gamma}(\mathbb{R}^n)}^p.
\]

Applying the above estimate, \( D_m \leq C m^{n-2} \), and the Minkowski inequality yields that

\[
 \sup_{\lambda > 0} \|\lambda \sqrt{N_\lambda(M_{\Omega_1,2^k} f)(x)}\|_{L^{p,\gamma}(\mathbb{R}^n)} \leq C \sum_{m \geq 1} \sum_{j=1}^{D_m} m^{n/2} m^{-2n} ||f||_{L^{p,\gamma}(\mathbb{R}^n)} \]

\[
 \leq C ||f||_{L^{p,\gamma}(\mathbb{R}^n)}.
\]

In a manner similar to that used in the proof of (4.3), we can also obtain the estimations of (4.2) and (4.4). We omit the details here. \( \square \)

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