GLUING $n$-TILTING AND $n$-COTILTING SUBCATEGORIES

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Abstract. Recently, Wang, Wei and Zhang define the recollement of extriangulated categories, which is a generalization of both recollement of abelian categories and recollement of triangulated categories. For a recollement $(\mathcal{A}, \mathcal{B}, \mathcal{C})$ of extriangulated categories, we show that $n$-tilting (resp. $n$-cotilting) subcategories in $\mathcal{A}$ and $\mathcal{C}$ can be glued to get $n$-tilting (resp. $n$-cotilting) subcategories in $\mathcal{B}$ under certain conditions.

1. Introduction

The recollement of triangulated categories was first introduced by Beilinson, Bernstein, and Deligne [BBD]. It is an important tool in algebraic geometry and representation theory. A fundamental example of a recollement of abelian categories appeared in the construction of perverse sheaves by MacPherson and Vilonen [MV], appearing as an inductive step in the construction.

The notion of extriangulated categories was introduced by Nakaoaka and Palu [NP] as a common generalization of exact and triangulated categories. Wang, Wei and Zhang [WWZ] gave a simultaneous generalization of recollements of abelian categories and triangulated categories, which is called recollements of extriangulated categories (see Definition 2.1 for details). A recollement of triangulated (or, abelian, extriangulated) categories is a diagram of functors between triangulated (or, abelian, extriangulated) categories of the following shape, which satisfies certain assumptions.

For a recollement $(\mathcal{A}, \mathcal{B}, \mathcal{C})$ of triangulated categories, Chen [C] has described how to glue together cotorsion pairs (which are essentially equal to torsion pairs in [IY]) in $\mathcal{A}$ and $\mathcal{C}$ to obtain a cotorsion pair in $\mathcal{B}$, which is a natural generalization of a similar result in [BBD] on gluing together $t$-structures of $\mathcal{A}$ and $\mathcal{C}$ to obtain a $t$-structure in $\mathcal{B}$.

The notion of cotorsion pair on extriangulated category was introduced in [NP], which is a generalization of cotorsion pair on triangulated and exact categories.

Definition 1.1. [NP, Definition 4.1] Let $\mathcal{U}$ and $\mathcal{V}$ be two subcategories of an extriangulated category $\mathcal{E}$. We call $(\mathcal{U}, \mathcal{V})$ a cotorsion pair if it satisfies the following conditions:

(a) $\mathcal{E}(\mathcal{U}, \mathcal{V}) = 0$.

(b) For any object $B \in \mathcal{E}$, there are two $\mathcal{E}$-triangles

$$V_B \to U_B \to B \to *, \quad B \to V^B \to U^B \to *$$

satisfying $U_B, U^B \in \mathcal{U}$ and $V_B, V^B \in \mathcal{V}$.

For a recollement $(\mathcal{A}, \mathcal{B}, \mathcal{C})$ of extriangulated categories, Wang, Wei and Zhang [WWZ] provided conditions such that the glued pair with respect to cotorsion pairs in $\mathcal{A}$ and $\mathcal{C}$ is still a cotorsion pair in $\mathcal{B}$. This result recovered a result given by Chen [C] for the recollement of triangulated categories.

We provide a slightly weaker assumption on the functors in (1.1) to get glued cotorsion pairs, which fits the recollement of abelian categories better (see Proposition 2.9 for details).

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Proposition 1.2. Let \((\mathcal{A}, \mathcal{B}, \mathcal{C})\) be a recollement of extriangulated categories. Assume that \(\mathcal{B}\) has enough projectives and \(i^*, j_*\) are exact. Let \((\mathcal{U}_1, \mathcal{V}_1)\) and \((\mathcal{U}_3, \mathcal{V}_3)\) be cotorsion pairs in \(\mathcal{A}\) and \(\mathcal{C}\), respectively. Define
\[
\mathcal{U}_2 = \{B \in \mathcal{B} \mid i^*B \in \mathcal{U}_1 \text{ and } j^*B \in \mathcal{U}_3\};
\]
\[
\mathcal{V}_2 = \{B \in \mathcal{B} \mid i^*B \in \mathcal{V}_1 \text{ and } j^*B \in \mathcal{V}_3\}.
\]
Then \((\mathcal{U}_2 := \perp \mathcal{V}_2, \mathcal{V}_2)\) is a cotorsion pair in \(\mathcal{B}\), where \(\mathcal{U}_2 \subseteq \text{add}\mathcal{U}_2\). We call cotorsion pair \((\mathcal{U}_2, \mathcal{V}_2)\) the glued cotorsion pair with respect to \((\mathcal{U}_1, \mathcal{V}_1)\) and \((\mathcal{U}_3, \mathcal{V}_3)\).

Tilting module was introduced by Brenner-Butler [BB] and Happel-Ringel [HR]. Ma and Zhao [MZ] gave a way of constructing a tilting module by gluing together two tilting modules in a recollement of module categories, this result also glued the correspondence torsion pairs. The notion of \(n\)-tilting module was first introduced by Miyashita [M], this coincides with the definition of tilting module when \(n = 1\). An analog concept of \(n\)-tilting module is introduced in [LZZZ], which is called \(n\)-tilting subcategory (see Definition 3.2 for details). Dually \(n\)-cotilting subcategory can be defined. Every \(n\)-tilting (resp. \(n\)-cotilting) subcategory admits a cotorsion pair, we call such cotorsion pair a tilting (resp. cotilting) cotorsion pair. We describe how to glue together two \(n\)-tilting (resp. \(n\)-cotilting) subcategories in \(\mathcal{A}\) and \(\mathcal{C}\) to obtain an \(n\)-tilting (resp. \(n\)-cotilting) subcategory in \(\mathcal{B}\) for a recollement of extriangulated categories, by glue the correspondent tilting (resp. cotilting) cotorsion pairs (see Theorem 4.5 and Proposition 4.3 for details).

Theorem 1.3. Assume that \((\mathcal{A}, \mathcal{B}, \mathcal{C})\) is a recollement of extriangulated categories, where \(\mathcal{B}\) has enough projectives and enough injectives, functors \(i^*, j_*\) are exact.

1. Let \((\mathcal{U}_1, \mathcal{V}_1)\) and \((\mathcal{U}_3, \mathcal{V}_3)\) be cotilting cotorsion pairs in \(\mathcal{A}\) and \(\mathcal{C}\) respectively. Then the glued cotorsion pair \((\mathcal{U}_2, \mathcal{V}_2)\) in \(\mathcal{B}\) is a cotilting cotorsion pair.
2. Let \((\mathcal{U}_1, \mathcal{V}_1)\) and \((\mathcal{U}_3, \mathcal{V}_3)\) be tilting cotorsion pairs in \(\mathcal{A}\) and \(\mathcal{C}\) respectively. Assume \(\mathcal{A}\) has finite projective global dimension, then the glued cotorsion pair \((\mathcal{U}_2, \mathcal{V}_2)\) in \(\mathcal{B}\) is a tilting cotorsion pair.

We also discuss how to glue \(n\)-tilting objects on abelian categories (see Theorem 4.4 for details).

Theorem 1.4. Assume \((\mathcal{A}, \mathcal{B}, \mathcal{C})\) is a recollement of abelian categories, where \(\mathcal{B}\) has enough projectives and enough injectives, functors \(i^*, j_*\) are exact. Let \((\mathcal{U}_1, \mathcal{V}_1)\) and \((\mathcal{U}_3, \mathcal{V}_3)\) be tilting cotorsion pairs in \(\mathcal{A}\) and \(\mathcal{C}\) respectively. Assume \(\mathcal{U}_i \cap \mathcal{V}_i = \text{add}\mathcal{T}_i, i = 1, 3\). Then the glued cotorsion pair \((\mathcal{U}_2, \mathcal{V}_2)\) in \(\mathcal{B}\) is a tilting cotorsion pair such that \(\mathcal{U}_2 \cap \mathcal{V}_2\) is the additive closure of an \(n\)-tilting object.

This article is organized as follows. In Section 2, we first recall the definition and some basic properties of recollements of extriangulated categories, then we show how to glue cotorsion pairs under certain assumptions. In Section 3, we recall the definition of \(n\)-tilting (resp. \(n\)-cotilting) subcategory in extriangulated categories and show some basic properties that we need. In Section 4, we glue \(n\)-tilting and \(n\)-cotilting subcategories in a recollement of extriangulated categories. In Section 5, we give some examples of our results.

2. Preliminaries

The definition and basic properties of extriangulated categories can be find in [NP, Section 2, 3]. In this article, let \(k\) be a field, \((\mathcal{E}, \mathcal{E}_E, \mathcal{E}_F)\) be an extriangulated category. Denote the subcategory of projective (resp. injective) objects by \(\mathcal{P}_E\) (resp. \(\mathcal{I}_E\)).

When we say that \(\mathcal{C}\) is a subcategory of \(\mathcal{E}\), we always assume that \(\mathcal{C}\) is full, closed under isomorphisms, direct sums and direct summands. Note that we do not assume any subcategory we construct has such property.

In this paper, we assume that \(\mathcal{E}\) satisfies Condition (WIC) ([NP, Condition 5.8]):

- If we have a deflation \(h : A \xrightarrow{f} B \xrightarrow{g} C\), then \(g\) is also a deflation.
- If we have an inflation \(h : A \xrightarrow{f} B \xrightarrow{g} C\), then \(f\) is also an inflation.

Note that any triangulated category and Krull-Schmidt exact category satisfies Condition (WIC).
2.1. Recollement of extriangulated categories. We recall the definition of recollement of extriangulated categories from [WWZ]. We only state the settings that we need, for details, one can see [WWZ, Section 3].

**Definition 2.1.** [WWZ, Definition 3.1] Let $A$, $B$ and $C$ be three extriangulated categories. A recollement of $B$ relative to $A$ and $C$, denoted by $(A, B, C)$, is a diagram

$$
\begin{array}{ccc}
A & \xrightarrow{i_*} & B \\
\downarrow{i^*} & & \downarrow{j_*} \\
C & \xrightarrow{j^*} & C
\end{array}
$$

given by two exact functors $i_*$, $j^*$, two right exact functors $i^*$, $j_*$ and two left exact functors $i^*$, $j_*$, which satisfies the following conditions:

1. $(i^*, i_*, i^*)$ and $(j_*, j^*, j_*)$ are adjoint triples.
2. $\text{Im} i_* = \text{Ker} j^*$.
3. $i_*$, $j_*$ and $j_*$ are fully faithful.
4. For each $X \in B$, there exists a left exact $\mathbb{E}_B$-triangle sequence
   \[ i_* i^* X \xrightarrow{\delta_X} X \xrightarrow{\theta_X} j_* j^* X \xrightarrow{i_* A} \]
   with $A \in A$, where $\theta_X$ and $\delta_X$ are given by the adjunction morphisms.
5. For each $X \in B$, there exists a right exact $\mathbb{E}_B$-triangle sequence
   \[ i_* A' \xrightarrow{i_* A'} \xrightarrow{\nu_X} X \xrightarrow{\nu_X} i_* i^* X \]
   with $A' \in A$, where $\nu_X$ and $\nu_X$ are given by the adjunction morphisms.

We omit the definitions of left, right exact functors and left, right exact $\mathbb{E}_C$-triangle, since they will not be used in the argument. The following remarks are useful.

**Remark 2.2.** [WWZ, Proposition 3.3] An additive covariant functor $F : A \rightarrow B$ is exact if and only if $F$ is both left exact and right exact. Recall that $F$ is exact if for any $\mathbb{E}_A$-triangle $A \xrightarrow{f} B \xrightarrow{g} C \rightarrow$, the sequence $F(A) \xrightarrow{F(f)} F(B) \xrightarrow{F(g)} F(C)$ is an $\mathbb{E}_B$-triangle.

**Remark 2.3.** (1) If $A$, $B$ and $C$ are abelian categories, then Definition 2.1 coincides with the definition of recollement of abelian categories (cf. [FP, P, MH]).

(2) If $A$, $B$ and $C$ are triangulated categories, then Definition 2.1 coincides with the definition of recollement of triangulated categories (cf. [BBD]).

(3) There exist examples of recollement of an extriangulated category in which one of the categories involved is neither abelian nor triangulated, see [WWZ].

We collect some properties of a recollement of extriangulated categories, which will be used in the sequel.

**Proposition 2.4.** [WWZ, Proposition 3.3] Let $(A, B, C)$ be a recollement of extriangulated categories.

1. All the natural transformations
   \[ i^*_! \Rightarrow \text{Id}_A, \  \text{Id}_A \Rightarrow i^*_! i_*, \  i^*_! \Rightarrow j^*_! j_*, \  j^*_! j_* \Rightarrow \text{Id}_C \]
   are natural isomorphisms.
2. $i^*_! 0 = 0$ and $i^*_! j_* = 0$.
3. $i^*_!$ preserves projective objects and $i^*_!$ preserves injective objects.
4. $j^*_!$ preserves projective objects and $j^*_!$ preserves injective objects.
5. If $B$ has enough projectives, then $A$ has enough projectives $\text{add}(i^*_! (P_B))$; if $B$ has enough injectives, then $A$ has enough injectives $\text{add}(i^*_! (I_B))$.
6. If $B$ has enough projectives and $j_*$ is exact, then $C$ has enough projectives $\text{add}(j^*_! (P_B))$; if $B$ has enough injectives and $j_*$ is exact, then $C$ has enough injectives $\text{add}(j^*_! (I_B))$.
7. If $B$ has enough projectives and $i^*_!$ is exact, then $\mathbb{E}_B(i_* X, Y) \cong \mathbb{E}_A(X, i^*_! Y)$ for any $X \in A$ and $Y \in B$.
8. If $C$ has enough projectives and $j_*$ is exact, then $\mathbb{E}_B(j_* Z, Y) \cong \mathbb{E}_C(Z, j^*_! Y)$ for any $Y \in B$ and $Z \in C$. 

(8) If \( i^* \) is exact, then \( j_! \) is exact.

\( (8') \) If \( i^* \) is exact, then \( j_* \) is exact.

**Proposition 2.5.** [WWZ, Proposition 3.4] Let \((A, B, C)\) be a recollement of extriangulated categories and \(X \in B\). Then the following statements hold.

(1) If \( i^* \) is exact, there exists an \( \mathbb{E}_G \)-triangle

\[
i_i^* X \to X \to j_* j^* X \to.\]

(2) If \( i^* \) is exact, there exists an \( \mathbb{E}_B \)-triangle

\[
j j^* X \to X \to i_* i^* X \to.\]

2.2. **Gluing cotorsion pairs.** From now on, we assume all extriangulated categories are Krull-Schmidt, Hom-finite, \( k \)-linear. We first introduce some notions.

**Definition 2.6.** Let \( \mathcal{C} \) and \( \mathcal{D} \) be two subcategories in \( \mathcal{E} \).

(a) Denote by \( \text{CoCone}(\mathcal{C}, \mathcal{D}) \) the subcategory

\[
\{X \in \mathcal{E} \mid \exists \text{\( \mathbb{E}_E \)-triangle} \ X \to C \to D \to. \text{ where} \ C \in \mathcal{C} \text{ and} \ D \in \mathcal{D} \}.\]

Let \( \Omega_2 \mathcal{C} = \text{CoCone}(\mathcal{P}_E, \mathcal{C}) \). We write an object \( X \) in the form \( \Omega_2 \mathcal{C} \) if it admits an \( \mathbb{E}_E \)-triangle \( X \to P \to C \to D \to \) with \( P \in \mathcal{P}_E \). Let \( \Omega_2 \mathcal{C} = \mathcal{C} \) and \( \Omega_2^1 \mathcal{C} = \Omega_2 \mathcal{C} \). Assume we have defined \( \Omega_2 \mathcal{C} \), \( i \geq 1 \), then we can denote \( \text{CoCone}(\mathcal{P}_E, \Omega_i \mathcal{C}) \) by \( \Omega_i^{i+1} \mathcal{C}. \)

(b) Denote by \( \text{Cone}(\mathcal{C}, \mathcal{D}) \) the subcategory

\[
\{Y \in \mathcal{E} \mid \exists \text{\( \mathbb{E}_E \)-triangle} \ C' \to D' \to Y \to. \text{ where} \ C' \in \mathcal{C} \text{ and} \ D' \in \mathcal{D} \}.\]

Let \( \Sigma_2 \mathcal{D} = \text{Cone}(\mathcal{D}, \mathcal{E}) \). We write an object \( Y \) in the form \( \Sigma_2 \mathcal{D} \) if it admits an \( \mathbb{E}_E \)-triangle \( D \to I \to Y \to \) with \( I \in \mathcal{I}_E \). Let \( \Sigma_2 \mathcal{D} = \mathcal{D} \) and \( \Sigma_2^1 \mathcal{D} = \Sigma_2 \mathcal{D} \). Assume we have defined \( \Sigma_2 \mathcal{D}, \)

\[
j \geq 1 \), then we can denote \( \text{Cone}(\Sigma_2^{i+1} \mathcal{D}) \) by \( \Sigma_2^{i+1} \mathcal{D}. \)

(c) Let \( C'_0 = C'_0 = C. \) We denote \( \text{Cone}(C'_0 \mathcal{C}) \) by \( C_\wedge \) and \( \text{CoCone}(\mathcal{C}, C'_1) \) by \( C'_1 \) for any \( i \geq 1. \)

We denote \( \bigcup_{i \geq 0} C'_i \) by \( C^\wedge \) and \( \bigcup_{i \geq 0} C'_i \) by \( C'^\wedge. \)

In the rest of this article, we assume that \( \mathcal{E} \) has enough projectives and enough injectives, then we can define higher extension groups as \( \mathbb{E}_E^{i+1}(X, Y) := \mathbb{E}_E(\Omega_i \mathcal{C} X, Y). \) Liu and Nakaoka [LN, Proposition 5.2] proved that

\[
\mathbb{E}_E(\Omega_2 \mathcal{C} X, Y) \simeq \mathbb{E}_E(X, \Sigma_2 \mathcal{D} Y). \]

For any subcategory \( \mathcal{C} \subseteq \mathcal{E} \), let

1. \( \mathcal{C}^\perp = \{X \in \mathcal{E} \mid \exists \mathbb{E}_E(X, C) = 0, \forall i > 0\};\)
2. \( \mathcal{C}^{\perp 1} = \{X \in \mathcal{E} \mid \mathbb{E}_E(X, C) = 0\};\)
3. \( \perp C = \{X \in \mathcal{E} \mid \mathbb{E}_E(X, C) = 0, \forall i > 0\};\)
4. \( \perp 1 C = \{X \in \mathcal{E} \mid \mathbb{E}_E(X, C) = 0\}.\)

**Definition 2.7.** [NP, Definition 4.1] Let \( \mathcal{U} \) and \( \mathcal{V} \) be two subcategories of \( \mathcal{E} \). We call \( (\mathcal{U}, \mathcal{V}) \) a cotorsion pair if it satisfies the following conditions:

(a) \( \mathbb{E}_E(\mathcal{U}, \mathcal{V}) = 0. \)

(b) For any object \( B \in \mathcal{E} \), there are two \( \mathbb{E}_E \)-triangles

\[
V_B \to U_B \to B \to U^B \to \]

satisfying \( U_B, U^B \in \mathcal{U} \) and \( V_B, V^B \in \mathcal{V} \).

A cotorsion pair \( (\mathcal{U}, \mathcal{V}) \) is said to be hereditary if \( \mathbb{E}_E(\mathcal{U}, \mathcal{V}) = 0. \)

By definition, we can conclude the following result.

**Lemma 2.8.** Let \( (\mathcal{U}, \mathcal{V}) \) be a cotorsion pair in \( \mathcal{B} \). Then

(a) \( \mathcal{V} = \mathcal{U}^{\perp 1}; \)

(b) \( \mathcal{U} = \perp 1 \mathcal{V}; \)

(c) \( \mathcal{U} \) and \( \mathcal{V} \) are closed under extensions;

(d) \( \mathcal{I}_E \subseteq \mathcal{V} \) and \( \mathcal{P}_E \subseteq \mathcal{U}. \)
The following are equivalent for \((\mathcal{U}, \mathcal{V})\).

1. \(\mathcal{E}^2_2(\mathcal{U}, \mathcal{V}) = 0\);
2. \(\mathcal{E}^i_1(\mathcal{U}, \mathcal{V}) = 0\) for any \(i \geq 1\);
3. \(\text{CoCone}(\mathcal{U}, \mathcal{U}) = \mathcal{U}\);
4. \(\text{Cone}(\mathcal{V}, \mathcal{V}) = \mathcal{V}\).

**Proposition 2.9.** Let \((\mathcal{A}, \mathcal{B}, \mathcal{C})\) be a recollement of extriangulated categories. Assume that \(\mathcal{B}\) has enough projectives and \(i^*, j^*\) are exact. Let \((\mathcal{U}_1, \mathcal{V}_1)\) and \((\mathcal{U}_3, \mathcal{V}_3)\) be cotorsion pairs in \(\mathcal{A}\) and \(\mathcal{C}\), respectively. Define

\[
\mathcal{U}_2 = \{ B \in \mathcal{B} \mid i^* B \in \mathcal{U}_1 \text{ and } j^* B \in \mathcal{U}_3 \};
\]

\[
\mathcal{V}_2 = \{ B \in \mathcal{B} \mid i^* B \in \mathcal{V}_1 \text{ and } j^* B \in \mathcal{V}_3 \}.
\]

Then \((\mathcal{U}_2 := \frac{1}{i^*} \mathcal{V}_2, \mathcal{V}_2)\) is a cotorsion pair in \(\mathcal{B}\), where \(\mathcal{U}_2 \subseteq \text{add}\mathcal{U}_2\).

**Proof.** According to the proof of [WWZ, Lemma 4.5(2)], any object \(X \in \mathcal{B}\) admits a commutative diagram

\[
\begin{array}{ccc}
X & \rightarrow & H \\
\downarrow & & \downarrow \text{id}_H \\
X & \rightarrow & j_i U_3 \\
\downarrow & & \downarrow \text{id}_{j_i U_3} \\
i_i U_1 & \rightarrow & i_j U_1
\end{array}
\]

where \(V \in \mathcal{V}_2\), \(U_1 \in \mathcal{U}_1\) and \(U_3 \in \mathcal{U}_3\). By applying \(\text{Hom}_B(-, \mathcal{V}_2)\) to the third column, we can get an exact sequence

\[
\mathbb{E}_B(i_i U_1, \mathcal{V}_2) \rightarrow \mathbb{E}_B(U, \mathcal{V}_2) \rightarrow \mathbb{E}_B(j_i U_3, \mathcal{V}_2).
\]

By Proposition 2.4, we have \(\mathbb{E}_B(i_i U_1, \mathcal{V}_2) \simeq \mathbb{E}_A(U_1, i_i \mathcal{V}_2) = 0\) and \(\mathbb{E}_B(j_i U_3, \mathcal{V}_2) \simeq \mathbb{E}_C(U_3, j^* \mathcal{V}_2) = 0\). Hence \(\mathbb{E}_B(U, \mathcal{V}_2) = 0\), which implies \(U \in \mathcal{U}_2\).

Since \(\mathcal{B}\) has enough projectives, \(X\) admits an \(\mathbb{E}_B\)-triangle \(\mathbb{E}_B X \rightarrow P_X \rightarrow X \rightarrow \cdot\cdot\cdot\) with \(P_X \in \mathcal{P}_B\). Since \(\mathbb{E}_B X\) admits an \(\mathbb{E}_B\)-triangle \(\mathbb{E}_B X \rightarrow V_2 \rightarrow U_2 \rightarrow \cdot\cdot\cdot\) where \(V_2 \in \mathcal{V}_2\) and \(U_2 \in \mathcal{U}_2\), we can get the following commutative diagram.

\[
\begin{array}{ccc}
\mathbb{E}_B X & \rightarrow & P_X \\
\downarrow & & \downarrow \text{id}_{P_X} \\
V_2 & \rightarrow & U_2' \\
\downarrow & & \downarrow \text{id}_{U_2'} \\
\mathbb{E}_B(U_2') & \rightarrow & \mathbb{E}_B(U_2) \\
\end{array}
\]

Since \(P_X \in \mathcal{U}_2\), we have \(U_2' \in \mathcal{U}_2\). To get that \((\mathcal{U}_2, \mathcal{V}_2)\) is a cotorsion pair, we still need to show that \(\mathcal{V}_2\) is closed under direct sums and direct summands. It is enough to show that for any object \(Y\), if \(\mathbb{E}_B(X, Y) = 0\) with \(X \in \mathcal{U}_2\), then \(Y \in \mathcal{V}_2\).

Let \(U_1 \in \mathcal{U}_1\), then \(i_i U_1 \in \mathcal{U}_2\) since \(\mathbb{E}_B(i_i U_1, \mathcal{V}_2) \simeq \mathbb{E}_A(U_1, i_i \mathcal{V}_2) = 0\). We have \(\mathbb{E}_A(U_1, i_i Y) \simeq \mathbb{E}_B(i_i U_1, Y) = 0\). Hence \(i_i Y \in \mathcal{V}_1\). By the similar method, we can get that \(j^* Y \in \mathcal{V}_3\). Hence \(Y \in \mathcal{V}_2\).

Let \(U_2'\) be any object in \(\mathcal{U}_2\). By [WWZ, Lemma 4.5(1)], \(U_2'\) admits an \(\mathbb{E}_B\)-triangle \(V \rightarrow U \rightarrow U_2' \rightarrow \cdot\cdot\cdot\) where \(V \in \mathcal{V}_2\) and \(U \in \mathcal{U}_2\). Since \(\mathbb{E}_B(U_2', V) = 0\), this sequence splits, which implies that \(U_2'\) is a direct summand of \(U\). Hence \(\mathcal{U}_2 \subseteq \text{add}\mathcal{U}_2\). \(\square\)

**Remark 2.10.** Note that by Proposition 2.4, \(i^*\) is exact implies that \(j_i\) is exact. If we assume the exactness of \(i^*\) instead of \(j_i\), we can get that \(\mathcal{U}_2 = \mathcal{U}_2\).

Under the same settings as in Proposition 2.9, we show the following proposition.
Proposition 2.11. Assume $B$ also has enough injectives. If $(U_1, V_1)$ and $(U_3, V_3)$ are hereditary, then $(U_2, V_2)$ is also hereditary.

Proof. We only need to show that $\text{CoCone}(V_2, V_2) \subseteq V_2$. Let

$$V \to V' \to X \to 
$$

be an $E_5$-triangle with $V, V' \in V_2$. Since $i^!$ is exact, we get an $E_4$-triangle

$$i^!V \to i^!V' \to i^!X \to .
$$

By definition of $V_2$, we have $i^!V, i^!V' \in V_1$. Since $(U_1, V_1)$ is hereditary, we have $\text{CoCone}(V_1, V_1) = V_1$, hence $i^!X \in V_1$. By the similar argument we can show that $j^*X \in V_2$. Hence by definition $X \in V_2$. □

We call the cotorsion pair $(U_2, V_2)$ got in Proposition 2.9 the glued cotorsion pair with respect to $(U_1, V_1)$ and $(U_3, V_3)$. We have the following observation.

Proposition 2.12. Let $(A, B, C)$ be a recollement of abelian categories. Assume that $i^!, i^*$ are exact, $B$ has enough projectives and enough injectives. Let $(U_1, V_1)$ and $(U_3, V_3)$ be torsion pairs in $A$ and $C$ respectively. Let $(U_2, V_2)$ be the glued cotorsion pair and $T_i = U_i \cap V_i, i = 1, 2, 3$. Then any indecomposable object $T \in T_2$ satisfies one of the following conditions:

1. There is an indecomposable object $T' \in T_1$ such that $T \simeq i_*T'$.
2. There is an indecomposable object $T'' \in T_3$ such that $T \simeq j^*T''$.

Proof. Let $T \in T_2$ be any indecomposable object. It admits an $E_5$-triangle

$$j^* T \to T \to i^* T \to .
$$

We have $i^*T \in U_1$ and $i^*i^*T \in U_2$. We also have $j^*T \in T_3$. Since $i^*j^* = 0$ when $i^*$ is exact in the recollement of abelian categories, we have $j^* i^* T \in T_2$. Thus this sequence splits and we get $T \simeq j^* i^* T$ or $T \simeq i^* i^* T$.

If $T \simeq j^* i^* T$, since $j_!$ is faithful, we can get that $j^*T$ is indecomposable. Hence condition (2) is satisfied.

Assume $T \simeq i^* i^* T$. $i_* T$ admits an $E_4$-triangle

$$i^* T \to T \to U_1 \to .$$

By applying $i_*$, we can get an $E_5$-triangle $T \to i_* T \to i_* U_1 \to$. Since $i_* U_1 \in U_2$, this sequence splits. Hence $T$ is a direct summand of $i_* T_1$, which implies that $i^* T$ is a direct summand of $T_1$. Since $i_*$ is faithful, we can get that $i^* T$ is indecomposable. Hence the condition (1) is satisfied. □

We can get the following corollary immediately.

Corollary 2.13. Under the settings of the previous proposition, assume $T_1 = \text{add} T_1$ and $T_3 = \text{add} T_3$, then $T_2 = \text{add} T_2$ where $T_2 = i_* T_1 \oplus j^* T_3$.

3. $n$-tilting and $n$-cotilting subcategories

Definition 3.1. A subcategory $D \subseteq \mathcal{E}$ is said to have finite projective dimension if there is a natural number $n$ such that $D \subseteq (P_{E})_{n}$. The minimal $n$ that satisfies this condition is called the projective dimension of $D$. In this case, we write $\text{pd}_{E} D = n$. The projective dimension of an object $D$ is just the projective dimension of $\text{add} D$.

Dually, we can define the injective dimension $\text{id}_{E} D$ (resp. $\text{id}_{E} D$) of a subcategory $D$ (resp. an object $D$).

Definition 3.2. [LZZZ, Definition 3.2] A subcategory $T \subseteq \mathcal{E}$ is called an $n$-tilting subcategory ($n \geq 1$) if

- $(P_1)$ $\text{pd}_{E} T \leq n$.
- $(P_2)$ $E_{n}(T, T) = 0, i = 1, 2, ..., n$.
- $(P_3)$ Any projective object $P$ admits $E_{n}$-triangles

$$P \to T_0 \to R_1 \to ..., R_1 \to T_1 \to R_2 \to ..., R_{n-1} \to T_{n-1} \to T_n \to
$$

where $T_i \in T, i = 0, 1, ..., n$. For convenience, any $n$-tilting subcategory can be simply called a generalized tilting subcategory. An object $T$ is called a generalized tilting object if $\text{add} T$ is a generalized tilting subcategory. Dually we can define $n$-cotilting subcategory and $n$-cotilting object.
Remark 3.3. According to [ZhZ, Remark 4], the tilting subcategory of projective dimension $n$ defined in [ZhZ, Definition 7] is a special case of $n$-tilting subcategory in Definition 3.2.

The following results are useful.

Lemma 3.4. (see [AT, Section 3] for details) Let $S$ be a subcategory in $E$ such that $E_i^E(S, S) = 0, \forall i > 0$. Then

1. $E_i^E(S^V, S^\wedge) = 0, \forall i > 0$ and $S^V \cap S^\wedge = S$.
2. $S^\wedge$ is closed under direct sums, direct summands and extensions. $\text{CoCone}(S^V, S^\wedge) = S^V$.
3. $S^\wedge$ is closed under direct sums, direct summands and extensions. $\text{Cone}(S^\wedge, S^V) = S^\wedge$.
4. $S^\perp = (S^V)^\perp$ and $^\perp S = (^\perp S)^\perp$.

Proposition 3.5. [LZZZ, Proposition 3.7] Let $T \subseteq B$ such that $E_i^B(T, T) = 0, \forall i > 0$ and $pd_T T \leq n$. Consider the following conditions:

(a) $T$ is contravariantly finite and $n$-tilting;
(b) $(T^V, T^\perp)$ is a cotorsion pair;
(c) $T$ is an $n$-tilting subcategory.

We have (a)$\Rightarrow$(b)$\Rightarrow$(c).

Definition 3.6. We call a hereditary cotorsion pair $(U, V)$ a tilting cotorsion pair if the following conditions are satisfied:

1. $T = U \cap V$ is a generalized tilting subcategory.
2. $U = T^V$ and $V = T^\perp$.

Dually we can define cotilting cotorsion pair.

Proposition 3.7. Let $(U, V)$ be a hereditary cotorsion pair. Then $(U, V)$ is a tilting cotorsion pair if and only if $pd_U U < \infty$.

Proof. Let $T = U \cap V$. Assume that $pd_U U \leq n$, then $E_i^E(U, U) = 0, \forall i > n$. For any object $U \in U$, we have $E_i^E$-triangles $U_{i-1} \rightarrow T_i \rightarrow U_i, i = 1, ..., T_i \in T, U_i \in U, U_0 = U$. By applying Hom$_E(U, -)$ to these $E_i$-triangles, we can get the following exact sequences

$$0 = E_i^E(U, T_{n-i+2}) \rightarrow E_i^E(U, U_{n-i+2}) \rightarrow E_i^{i+1}(U, U_{n-i+1}) \rightarrow E_i^{i+1}(U, T_{n-i+2}) = 0$$

with $i = 1, ..., n$. Hence we have $E_i^E(U, U_{n-i+2}) \cong E_i^{i+1}(U, U_{n-i+1}), i = 1, ..., n$. But $E_i^{i+1}(U, U_1) = 0$, we get that $E_i^E(U, U_{n-i+1}) = 0$. Hence $U_{n+1} \in T$. This implies $U \subseteq T^n$. Since $(U, V)$ is a hereditary cotorsion pair, we have $V \leq T^n$. This shows that $U = T^n$. We have $(T^V)^\perp \supseteq (T^\wedge)^\perp = T^\perp$. On the other hand, $\Omega_E T \subseteq T^\vee, \forall i > 0$, then $X \in (T^\vee)^\perp$ implies that $E_i^E(\Omega_E T, X) = E_i^E(T, X) = 0, \forall i > 0$. Hence $V = T^\perp$.

Now let $(U, V)$ be a tilting cotorsion pair. Then $U = T^V$. Let $pd_T T \leq n$. We can easily get that for each object $U \in T^V$, $pd_U U \leq n$. Hence $pd_U U \leq n$. From the proof this proposition, we can easily get the following corollary.

Corollary 3.8. Let $(U, V)$ be a hereditary cotorsion pair and $T = U \cap V$. If $pd_U U \leq n$, then $U = T^n$.

4. Gluing $n$-tilting and $n$-cotilting subcategories

Definition 4.1. $E$ is said to have finite projective global dimension if there is a natural number $n$ such that $E = (P^n)^\perp$. The minimal $n$ that satisfies this condition is called the global dimension of $E$. In this case, we write $\text{p.gl. dim} E = n$.

Dually we can define the injective global dimension $\text{i.gl. dim} E$ of $E$.

In this section, we always assume the following:

(a) $(A, B, C)$ is a recollement of extriangulated categories.
(b) $B$ has enough projectives and enough injectives.
(c) $i_j, j_i$ are exact.
(d) $(U_1, V_1)$ and $(U_3, V_3)$ are hereditary pairs in $A$ and $C$ respectively. $(U_2, V_2)$ is the glued cotorsion pair in $B$. Denote $U_i \cap V_i$ by $T_i, i = 1, 2, 3$.

The following corollary is a direct conclusion of Proposition 3.7.
Corollary 4.2. If \( \text{p.gl. dim } B < \infty \) (resp. \( \text{i.gl. dim } B < \infty \)), then \((U_2, V_2)\) is a tilting (resp. cotilting) cotorsion pair.

4.1. Gluing \( n \)-tilting subcategories.

Proposition 4.3. Let \((U_1, V_1)\) and \((U_3, V_3)\) be tilting cotorsion pairs. Assume \( \text{p.gl. dim } A < \infty \), then \((U_2, V_2)\) is a tilting cotorsion pair.

Proof. By Proposition 3.7 and Proposition 2.11, it is enough to check that \( \text{pd}_B U_2 < \infty \). Assume \( \text{p.gl. dim } A \leq n_1 \) and \( \text{pd}_C U_3 \leq n_3 \).

Let \( U \in U_2 \) be any object. By Proposition 2.9, there exists an object \( U' \in \text{add} \tilde{U}_2 \) such that \( U' = U' \oplus U \) satisfies \( i^* U \in U_1 \) and \( j^* U \in U_3 \). There exists a right exact \( E_B \)-triangle sequence

\[
i_* A \twoheadrightarrow j_* j^* \tilde{U} \twoheadrightarrow \tilde{U} \twoheadrightarrow i_* i^* \tilde{U}
\]

with \( A \in A \), which is, in fact, a combination of two \( E_B \)-triangles

\[
i_* A \twoheadrightarrow j_* j^* \tilde{U} \twoheadrightarrow X \to X \to i_* i^* \tilde{U}
\]

Since \( \text{p.gl. dim } A \leq n_1 \), we have \( \text{pd}_A A \leq n_1 \). Since \( i_* \) preserves projectives, we have \( \text{pd}_B i_* A \leq n_1 \).

Since \( j^* \tilde{U} \in U_3 \), we have \( \text{pd}_C j^* \tilde{U} \leq n_3 \). Since \( j_* \) preserves projectives, we have \( \text{pd}_B (j_* j^* \tilde{U}) \leq n_3 \). Hence \( \text{pd}_B X \leq \max\{n_1 + 1, n_3\} \). Since \( i^* U \in U_1 \), we have \( \text{pd}_A i^* \tilde{U} \leq n_1 \), then \( \text{pd}_B (i_* i^* \tilde{U}) \leq n_1 \). Hence \( \text{pd}_B \tilde{U} \leq \max\{n_1 + 1, n_3\} \). This implies \( \text{pd}_B U \leq \max\{n_1 + 1, n_3\} \).

Thus \( \text{pd}_B U_2 \leq \max\{n_1 + 1, n_3\} \) and by Proposition 3.7, \((U_2, V_2)\) is a tilting cotorsion pair.

\( \square \)

When we glue tilting objects on abelian categories, we can drop the assumption of projective dimension finiteness. We need some preparation.

First note that \( i_* T_1 \subseteq T_2 \).

For any object \( T'' \in T_3 \), we have \( j_1 T'' \in U_2 \). It admits an \( E_B \)-triangle

\[
i_* j_1 T'' \rightarrow j_* T'' \rightarrow j_* T'' \to X
\]

Since \( i_* j_1 T'' \in A \), it admits an \( E_A \)-triangle

\[
i_1 j_1 T'' \rightarrow V_1 \rightarrow U_1 \to X
\]

where \( U_1 \in U_1 \) and \( V_1 \in V_1 \). Then we have an \( E_B \)-triangle

\[
i_* j_1 T'' \rightarrow i_* V_1 \rightarrow i_* U_1 \to X
\]

where \( i_* U_1 \in U_2 \) and \( i_* V_1 \in V_2 \). Now we have the following commutative diagram of \( E_B \)-triangles.

\[
\begin{array}{ccc}
i_* j_1 T'' & \rightarrow & j_* T'' \\
i_1 V_1 & \leftarrow & K_{T''} \\
i_* U_1 & \downarrow & j_* T'' \to X
\end{array}
\]

Since \( j_* T'' \in U_2 \), \( j_* T'' \in V_2 \), we have \( K_{T''} \in T_2 \).

Assume \( T_3 = \text{add} T_3 \) such that \( T_3 = \bigoplus_{i=1}^n T_3^i \), where \( T_3^i \) are indecomposable objects. Let \( K_{T_3} \) be the object got in the above diagram with respect to \( T_3^i \). Denote \( \bigoplus_{i=1}^n K_{T_3} \) by \( K_{T_3} \).

Theorem 4.4. Let \((A, B, C)\) be a recollement of abelian categories. Let \((U_1, V_1)\) and \((U_3, V_3)\) be tilting cotorsion pairs. Assume \( T_i = \text{add} T_i, i = 1, 3 \), then \((U_2, V_2)\) is a tilting cotorsion pair such that \( T_2 = \text{add}(i_* T_1 \oplus K_{T_3}) \).
Proof. Denote $i_* T_1 \oplus K_{T_1} = T_2'$ and $\text{add} T_2'$ by $T_2''$. Assume $pd_A U_1 \leq n_1$ and $pd_A U_3 \leq n_3$.

Since $T_2' \in \mathcal{T}_2$, we have $\text{Ext}_2 (T_2', T_2) = 0$. Definition 3.2(P2) is satisfied.

Since $T_1 \in \mathcal{U}_1$ and $pd_A U_1 \leq n_1$, we have $pd_A i_* T_1 \leq n_1$. Since $T_3 \in \mathcal{U}_3$ and $pd_A U_3 \leq n_3$, according to diagram (★), we have $pd_B K_{T_3} \leq \max \{n_1, n_3\}$. Hence $pd_B T_2' \leq \max \{n_1, n_3\}$. Definition 3.2(P1) is satisfied.

Let $P \in \mathcal{P}_B$ be an indecomposable object. It admits two short exact sequences

$$i_* A \rightarrow j_! j^* P \rightarrow X, \quad X \rightarrow P \rightarrow i_* i^* P$$

with $A \in \mathcal{A}$. Since $i^*$ and $i_*$ preserve projectives, we have $i_* i^* P \in \mathcal{P}_B$. Hence the second sequence splits. Since $P$ is indecomposable, we can get that $P \cong i_* i^* P$ or $P \cong X$. The second case implies that $P$ is a direct summand of $j_! j^* P$. Since $j^*$ preserves projectives, we get that any indecomposable object $P \in \mathcal{P}_B$ satisfies one of the following conditions:

1. There is an object $P_1 \in \mathcal{P}_A$ such that $P \cong i_* P_1$.
2. There is an object $P_3 \in \mathcal{P}_C$ such that $P \cong j_! P_3$.

If $P \cong i_* P_1$, since $P_1 \in \mathcal{U}_1 = (T_1)_n^\vee$, we have $P \cong (i_* T_1)_n^\vee \subseteq (T_2)_n^\vee$.

If $P \cong j_! P_3$, since $P_3 \in \mathcal{U}_3 = (T_3)_n^\vee$, we have the following short exact sequences

$$P \rightarrow j_! T_3 \rightarrow j_! U_3 \rightarrow j_! T_3 \rightarrow j_! U_3 \rightarrow j_! T_3 \rightarrow \cdots, \quad j_! U_3^{n-1} \rightarrow j_! T_3^{n-1} \rightarrow j_! T_3^n$$

where $T_3 \in \mathcal{T}_3$ and $U_3 \in \mathcal{U}_3$. Since $j_! T_3^n$ admits a short exact sequence $j_! T_3^n \rightarrow K_{T_3^n} \rightarrow i_* U_1^n$ where $K_{T_3^n} \in \text{add}(K_{T_3^n})$ and $U_1^n \in \mathcal{U}_1$, we get the following commutative diagram.

$$
\begin{array}{ccc}
P & \rightarrow & j_! T_3^n \\
\downarrow & \downarrow & \downarrow \\
K_{T_3^n} & \rightarrow & U_1^n \\
\downarrow & \downarrow & \downarrow \\
i_* U_1^n & = & i_* U_1^n
\end{array}
$$

Since $j_! U_1^n, i_* U_1^n \in \mathcal{U}_2$. $j_! U_1^n$ admits the following commutative diagram

$$
\begin{array}{ccc}
j_! U_1^n & \rightarrow & j_! T_1^n \\
\downarrow & \downarrow & \downarrow \\
j_! U_1^n & \rightarrow & K_{T_1^n} \\
\downarrow & \downarrow & \downarrow \\
i_* U_1^n & = & i_* U_1^n
\end{array}
$$

where $K_{T_1^n} \in \text{add}(K_{T_1^n})$ and $U_1^n \in \mathcal{U}_1$. This implies $U_2^n \in \mathcal{U}_2$. Since $i_* U_1^n$ admits a short exact sequence $i_* U_1^n \rightarrow i_* T_1^n \rightarrow i_* (U_1^n)'$ where $T_1^n \in \mathcal{T}_1$ and $(U_1^n)' \in \mathcal{U}_1$, we have the following commutative diagram

$$
\begin{array}{ccc}
j_! U_1^n & \rightarrow & K_{T_1^n} \\
\downarrow & \downarrow & \downarrow \\
U_2^n & \rightarrow & K_{T_1^n} \oplus i_* T_1^n \\
\downarrow & \downarrow & \downarrow \\
i_* U_1^n & = & i_* (U_1^n)'
\end{array}
$$

where $K_{T_1^n} \oplus i_* T_1^n \in \text{add} T_2'$. Now we only need to focus on $U_2^n$. Since it admits a short exact sequence $j_! U_2^n \rightarrow U_2^n \rightarrow i_* U_1^n$, we can continue this process and get the following exact sequences:

$$
P \rightarrow T_0^n \rightarrow U_2^n \rightarrow U_2^n \rightarrow U_2^n \oplus i_* (U_1^n)' \rightarrow \cdots \rightarrow U_2^{n-1} \rightarrow U_2^n \rightarrow U_2^n \oplus i_* (U_1^{n-1})'$$

where $U_2^n \oplus i_* (U_1^n)' \in \text{add} T_2'$. Since it admits a short exact sequence $j_! U_2^n \rightarrow U_2^n \rightarrow i_* U_1^n$, we can continue this process and get the following exact sequences:
where \( T'_2 \in T'_2, \quad U'_2 \in U_2 \) and \( (U_1^k)' \in U_1 \). Moreover, \( U'_2 \) admits a short exact sequence
\[
jT'_2^{m_2} \rightarrow U'_2 \rightarrow i_*U_1^{n_3-1}.
\]

Then we have the following commutative diagram.
\[
\begin{array}{ccc}
jT'_2^{m_2} & \rightarrow & K_{T'_2}^{m_2} \\
\downarrow & & \downarrow \\
U'_2 & \rightarrow & K_{T'_2}^{m_2} \oplus i_*T'_1 \\
\downarrow & & \downarrow \\
i_*U_1^{n_3-1} & \rightarrow & i_*T'_1 \\
\end{array}
\]

Hence we get that \( P \in (T'_2)^{\vee} \). Definition 3.2(P3) is satisfied.

Note that the argument above also shows that \( j_!U_1 \subseteq (T'_2)^{\vee} \).

By Proposition 3.5, \( ((T'_2)^{\vee}, (T'_2)^{\perp}) \) is a cotorsion pair. Since \( T'_2 \in T_2 \), we have \( (T'_2)^{\perp} \supseteq V_2 \). Let
\( X \in (T'_2)^{\perp} \). We show that \( X \in V_2 \).

\[ \text{Ext}^j_1(j_!U_1, X) \cong \text{Ext}^j_2(j_!U_1, X) \]
\( \text{Ext}^j_2(j_!U_1, X) \) is exact, we can get that \( \text{id}_{V_2} \), hence \( \text{id}_C \).

We have \( j'_*X \in V_3 \).

\[ \text{Ext}^j_2(j_!U_1, i'_!X) \cong \text{Ext}^j_2(i'_!U_1, X) \]
\( \text{id}_B \), \( i'_!X \in V_1 \).

This \( X \in V_2 \).

Thus \((U_2, V_2) = ((T'_2)^{\vee}, (T'_2)^{\perp}) \) is a tilting cotorsion pair and \( T'_2 = T_2 \).

\[ \square \]

4.2. Gluing \( n \)-cotilting subcategories.

**Theorem 4.5.** Let \((U_1, V_1)\) and \((U_2, V_2)\) be cotilting cotorsion pairs. Then \((U_2, V_2)\) is a cotilting cotorsion pair.

**Proof.** By Proposition 2.11 and the dual of Proposition 3.7, it is enough to show that \( \text{id}_B V_2 \leq \infty \). By the dual of Proposition 3.7, we can assume that \( \text{id}_A V_1 \leq n_1 \) and \( \text{id}_C V_3 \leq n_3 \).

For any \( I \in I_B \), since \( i'_! \) is exact, \( I \) admits an \( E_B \)-triangle
\[ i_*i'_!I \rightarrow I \rightarrow j_*j^*I \rightarrow \]

Since \( j_*j^* \) preserves injectives, we have \( j_*j^*I \in I_B \). This implies \( \text{id}_B(i_*i'_!I) \leq 1 \).

For any \( V \in V_2 \), we have an \( E_B \)-triangle
\[ i_*i'_!V \rightarrow V \rightarrow j_*j^*V \rightarrow \]

We have \( j_*V \in V_3 \), hence \( \text{id}_B(j_*j^*V) \leq n_3 \). We also have \( i'_!V \in V_1 \), then \( \text{id}_A i'_!V \leq n_1 \). By Proposition 2.4(5), \( A \) has enough injectives add \((i'_!I_B) \), we have the following \( E_A \)-triangles.
\[ i'_!V \rightarrow I_1 \rightarrow R_1 \rightarrow \cdots, \quad R_1 \rightarrow I_2' \rightarrow R_2 \rightarrow \cdots, \quad R_{n_1} \rightarrow I_{n_1}' \rightarrow I_{n_1+1}' \rightarrow \]

where \( I_j' \in \text{add}(i'_!I_B) \), \( j = 1, 2, ..., n_1 + 1 \). Then \( i_*I_j' \) is a direct summand of some object in \( i_*i'_!I_B \), which implies that \( \text{id}_B(i_*I_j') \leq j_1, j = 1, 2, ..., n_1 + 1 \). Since \( i_* \) is exact, we can get that \( \text{id}_B(i_*i'_!V) \leq n_1 + 1 \).

Hence \( \text{id}_B V_2 \leq \max\{n_1 + 1, n_3\} \).

\[ \square \]

5. Examples

In this section we give some examples of our results.

Let \( \Lambda', \Lambda'' \) be artin algebras and \( A, N_{\Lambda'} \) an \( (\Lambda', \Lambda'') \)-bimodule, and let \( \left( \begin{array}{c} \Lambda' \\ N_{\Lambda'} \end{array} \right) \) be a triangular matrix algebra. Then any module in \( \text{mod} \Lambda \) can be uniquely written as a triple \( \left( \begin{array}{c} X \\ Y \end{array} \right) \) with \( X \in \text{mod} \Lambda' \), \( Y \in \text{mod} \Lambda' \) and \( f \in \text{Hom}_{\Lambda'} (N \otimes_{\Lambda''} Y, X) \), see [ARS, page 76].

**Example 5.1.** Let \( \Lambda' \) be the finite dimensional algebra given by the quiver
\[
\begin{array}{c}
1 \rightarrow 2
\end{array}
\]
and \( \Lambda'' \) be the finite dimensional algebra given by the quiver
\[
\begin{array}{c}
3 \rightarrow 4 \rightarrow 5
\end{array}
\]
with the relation \( \beta \alpha = 0 \). Define a triangular matrix algebra \( \Lambda = \left( \begin{array}{c} \Lambda' \\ N_{\Lambda'} \end{array} \right) \), where the right \( \Lambda' \)-module structure on \( \Lambda' \) is induced by the unique algebra
surjective homomorphism $\Lambda'' \xrightarrow{\varphi} \Lambda'$ satisfying $\varphi(e_3) = e_1, \varphi(e_4) = e_2, \varphi(e_5) = 0$. Then $\Lambda$ is a finite dimensional algebra given by the quiver

$$
\begin{array}{c}
\delta \\
\downarrow \gamma \\
\epsilon \\
\downarrow \alpha
\end{array}
$$

with the relation $\gamma \alpha = \delta \epsilon$ and $\beta \alpha = 0$. The Auslander-Reiten quiver of $\Lambda$ is

$$
\begin{array}{c}
\begin{array}{c}
(P(5)) \\
\downarrow (S(2)(S(4)) \\
(P(4)) \\
\downarrow (S(2)) \\
0
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
(S(2)(P(4))) \\
\downarrow (P(1)(S(4)) \\
0 \\
\downarrow (P(1)(P(4))) \\
0
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
(S(1)(P(3))) \\
\downarrow (P(1)(S(3))) \\
0 \\
\downarrow 0
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
0
\end{array}
\end{array}
\end{array}
$$

By [P, Example 2.12], we have that

$$
\begin{array}{c}
\text{mod} \Lambda' \underbrace{\longrightarrow}_{\text{j}} \text{mod} \Lambda \underbrace{\longleftarrow}_{\text{i}} \text{mod} \Lambda''
\end{array}
$$

is a recollement of module categories, where

$$
i^*(\begin{pmatrix} X \\ Y \end{pmatrix}_f) = \text{Coker} f,
\quad i_*(X) = \begin{pmatrix} X \\ 0 \end{pmatrix},
\quad i^*(\begin{pmatrix} X \\ Y \end{pmatrix}_f) = X,
\quad j^*(\begin{pmatrix} X \\ Y \end{pmatrix}_f) = Y,
\quad j_*(Y) = \begin{pmatrix} 0 \\ Y \end{pmatrix}.
$$

(1) Let $T_1 = P(1) \oplus S(1) \in \text{mod} \Lambda'$ and $T_3 = P(3) \oplus P(4) \oplus P(5) \in \text{mod} \Lambda''$. Then $T_1$ is a cotilting $\Lambda'$-module and $T_3$ is a 2-cotilting $\Lambda''$-module. We have two cotorsion pairs:

$$
(U_1, V_1) = (\text{mod} \Lambda', \text{add} T_1),
(U_3, V_3) = (\text{add} T_3, \text{mod} \Lambda'').
$$

Note that $T_1, T_3$ are just $T'$ and $T''$ given in [MZ, Example 4.1(1)] respectively. They are also tilting modules, and the tilting cotorsion pairs they induce are still $(U_1, V_1)$ and $(U_3, V_3)$ respectively. By Theorem 4.5 we have a cotilting $\Lambda$-module

$$
T = \begin{pmatrix} 0 \\ P(5) \end{pmatrix} \oplus \begin{pmatrix} S(1) \\ 0 \end{pmatrix} \oplus \begin{pmatrix} P(1) \\ 0 \end{pmatrix} \oplus \begin{pmatrix} 0 \\ P(3) \end{pmatrix} \oplus \begin{pmatrix} P(1) \\ P(4) \end{pmatrix} \oplus \begin{pmatrix} 0 \\ 0 \end{pmatrix},
$$

which is different from the tilting $\Lambda$-module

$$
\begin{pmatrix} 0 \\ P(5) \end{pmatrix} \oplus \begin{pmatrix} S(2) \\ P(4) \end{pmatrix} \oplus \begin{pmatrix} P(1) \\ 0 \end{pmatrix} \oplus \begin{pmatrix} P(1) \\ P(3) \end{pmatrix} \oplus \begin{pmatrix} 0 \\ 0 \end{pmatrix},
$$

got in [MZ, Example 4.1(1)].

(2) Let $T_1 = P(1) \oplus S(2) \in \text{mod} \Lambda'$ and $T_3 = P(3) \oplus P(4) \oplus S(3) \in \text{mod} \Lambda''$. Then $T_1$ is a tilting $\Lambda'$-module and $T_3$ is a 2-tilting $\Lambda''$-module. We have two cotorsion pairs:

$$
(U_1, V_1) = (\text{add} T_1, \text{mod} \Lambda'),
(U_3, V_3) = (\text{mod} \Lambda'', \text{add} T_3).
$$
Then we have a 2-tilting \( \Lambda \)-module
\[
T = \begin{pmatrix} S(2) \\ 0 \end{pmatrix} \oplus \begin{pmatrix} S(2) \\ P(4) \end{pmatrix} \oplus \begin{pmatrix} P(1) \\ 0 \end{pmatrix} \oplus \begin{pmatrix} P(1) \\ P(3) \end{pmatrix} \oplus \begin{pmatrix} S(1) \\ S(3) \end{pmatrix}
\]
by gluing \( T_1 \) and \( T_3 \). We also have
\[
i_*(P(1)) = \begin{pmatrix} P(1) \\ 0 \end{pmatrix}, \quad i_*(S(2)) = \begin{pmatrix} S(2) \\ 0 \end{pmatrix}, \quad j_!(P(3)) = \begin{pmatrix} P(1) \\ P(3) \end{pmatrix}, \quad j_!(P(4)) = \begin{pmatrix} S(2) \\ P(4) \end{pmatrix}, \quad j_!(S(3)) = \begin{pmatrix} S(1) \\ S(3) \end{pmatrix}.
\]

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