Lacunary Power Series and $U_m$-Numbers

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Abstract

In this work, the values of certain lacunar power series with rational coefficients for $U_m$-number arguments were determined to be either in a particular algebraic number field or in the set of transcendental numbers under specific circumstances in the complex numbers field. The result was also applied on some of the lacunar power series with coefficients in an algebraic number field. Roth's theorem which is the essential result in Diophantine approximation to algebraic numbers was used to reach the present results.

Keywords: Algebraic number field, power series, transcendental number, $U_m$-number.

1. Introduction

A classification of complex transcendental numbers in terms of polynomial approximation was introduced in 1932 by Mahler [1]. Pursuant to Mahler's classification, the set of complex numbers are separated into four disjoint subsets such as $A$, $S$, $T$ and $U$ numbers. Depending on approximation by algebraic numbers, another classification was suggested in 1939 by Koksma. Therefore, the set of complex numbers are separated into four subsets such as $A'$, $S'$, $T'$ and $U'$ numbers [2]. Wirsing [3] concluded that Mahler’s and Koksma’s classifications are equivalent, in other words, the classes $A, S, T$ and $U$ are identical to the classes $A', S', T'$ and $U'$, respectively. The class $A$ corresponds to the set of algebraic numbers. The class $U$ are divided into the subclasses $U_m$, where $m$ is a positive integer and $m > 1$. Furthermore, $U_m \cap U_n = \emptyset$ if $m \neq n$, and $U = \bigcup_{m=1}^{\infty} U_m$.

The theory of transcendental numbers has been studied from many different point of view such as exponents of diophantine approximation [4, 5], and the transcendence of value of some series or functions [6, 7, 8]. Notably, it was proved by Maillet [9] that the value of a rational function with rational coefficients for Liouville number arguments is a Liouville number. Inspired by this result, Mahler [10] asks the question of “Which analytic functions $f(x)$ have the property that if $X$ is any Liouville number, then so is $f(X)$? In particular, are there entire transcendental functions with this property?”. That question has interested a lot of mathematicians [11, 12, 13, 14].

In this manuscript some particular lacunar power series with rational coefficients for some transcendental number arguments were taken into account. First of all, some properties related with the coefficients of this series were given and it was shown that the convergence radius is infinite. Afterwards, using the Roth's theorem [15] it was proved in the field of complex numbers that the values of series for $U_m$-number arguments are associated with either a particular algebraic number field or the set of transcendental numbers. At last, these results were applied to some series with coefficients which are from a certain particular algebraic number field. Consequently, the results in [11] were generalized for the power series for Liouville number arguments to lacunar power series for $U_m$-number arguments.

The manuscript is organized in the three parts. In Section 2, some notations and some basic results are given. In Section 3, the main results of this manuscript are presented. In Section 3.1, some particular lacunar power series whose coefficients are rational numbers are focused on. It is proved that the values of these series for $U_m$-number arguments are either an element of a particular algebraic number field or a transcendental number. In Section 3.2, the values of some lacunar power series whose coefficients are in a particular algebraic number field are investigated.

2. Materials and Methods

The height $H(G)$ for a polynomial $G$ in $\mathbb{Z}[x]$ is the maximum of the absolute values of the coefficients of...
\( g(x) = \sum_{k_n=0}^{m} c_{k_n} x^{k_n} \) (3.1)

where \( c_{k_n} = \frac{b_{k_n}}{d_{k_n}} \) (\( d_{k_n} \) and \( b_{k_n} \) are integers; \( d_{k_n} > 1 \) for sufficiently large \( n \)) is a non-zero rational number and \( (k_n)_{n=0}^{m} \) is an increasing sequence of natural numbers.

**Lemma 3.1.** Assume the notations given above. If

\[
\sigma = \liminf_{n \to \infty} \frac{\log d_{k_n+1}}{\log d_{k_n}} > 1
\]

then we have

\[
\lim_{n \to \infty} \log d_{k_n} = +\infty \text{ and } \lim_{n \to \infty} d_{k_n} = +\infty. \quad (3.3)
\]

**Proof:** As a consequence of (3.2), we get that there is a sufficiently small \( \epsilon_1 > 0 \) such that \( \sigma_1 = \sigma - \epsilon_1 > 1 \).

Now, one arrive at

\[
\log d_{k_{n+1}} > \sigma_1 \log d_{k_n}, \quad (3.4)
\]

since \( d_{k_n} > 1 \). From (3.4) we get (3.3).

**Lemma 3.2.** Assume the notations given above. Let

\[
D_{k_n} = \left[ d_{k_0}, d_{k_1}, \ldots, d_{k_n} \right]
\]

and \( \epsilon_1 > 0 \) be a sufficiently small number such that \( \sigma_1 = \sigma - \epsilon_1 > 1 \).

Then we have

\[
d_{k_n} \leq D_{k_n} < C_0 \frac{d_{1}}{x_{-1}^{\epsilon_1}}, \quad (3.5)
\]

where \( C_0 > 0 \) is a suitable number.

**Proof:** From (3.4) we get (3.5).

**Lemma 3.3.** Assume the notations given above. If

\[
\theta = \liminf_{n \to \infty} \frac{\log |b_{k_n}|}{\log d_{k_n}} < 1, \quad (3.6)
\]

then the convergence radius of the series \( g(x) \) in (3.1) is infinite.

**Proof:** Using (3.6) we arrive at

\[
|b_{k_n}| < d_{k_n}^{\theta + \epsilon_2}, \quad (3.7)
\]

where \( \epsilon_2 > 0 \) is a sufficiently small number such that \( \theta + \epsilon_2 < 1 \). From (3.3), (3.4) and (3.7), the convergence radius of the series \( g(x) \) is infinite.

**Theorem 3.1** Let \( g(x) \) be the lacunary power series in (3.1) satisfying the conditions (3.2), (3.6) and

\[
\lim_{n \to \infty} \frac{\log d_{k_n}}{k_n} < 1. \quad (3.8)
\]
Let $\mathcal{S} = \mathbb{Q}(\zeta)$ be an algebraic number field such that $[\mathcal{S} : \mathbb{Q}] = m$ and $\alpha_{k_n}$ be algebraic number of degree $m$ over $\mathcal{S}$. Then let $\zeta$ be a $U_m$-number for which the following properties hold: $\zeta$ has an approximation with algebraic numbers $\alpha_{k_n}$ so that the following inequalities are true

$$|\zeta - \alpha_{k_n}| < \frac{1}{H(\alpha_{k_n})^2w(k_n)}, \quad (3.9)$$

$$\lim_{n \to \infty} w(k_n) = \infty$$

and

$$d_{k_n}^2 \leq H(\alpha_{k_n})^2 \leq d_{k_n}^2, \quad (3.10)$$

where $s_1$ and $s_2$ are real numbers such that $1 < s_1 \leq s_2$. If $\sigma(\sigma - 1)(1 - \theta) > 2m(\sigma + (\sigma - 1)(1 + \theta + s_2))$, then $g(\zeta)$ is either in $\mathcal{S}$ or in the set of transcendental numbers.

**Proof:** Let us consider the polynomial

$$g_n(x) = \sum_{p=0}^{n} c_{k_p} x^k_p \quad (n = 1, 2, ...).$$

Therefore we get $g_n(\alpha_{k_n}) \in K$ and then

$$\deg(g_n(\alpha_{k_n})) \leq m \quad (n = 1, 2, ...).$$

Taking

$$G(z, x) = D_{k_n} z - \sum_{p=0}^{n} D_{k_p} c_{k_p} x^k_p,$$

we arrive at

$$H(G) = \max_{0 \leq v \leq n} \{D_{k_n} D_{k_n} |c_{k_v}|\} = D_{k_n} B_{k_n},$$

where

$$B_{k_n} := \max_{0 \leq v \leq n} \{1, |b_{k_v}|\}.$$  

In view of (3.3) and (3.7), we obtain

$$B_{k_n} < d_{k_n}^{\theta + \varepsilon_2}. \quad (3.11)$$

Thus we have

$$H(G) = D_{k_n} d_{k_n}^{\theta + \varepsilon_2}. \quad (3.12)$$

Since

$$G(g_n(\alpha_{k_n}), \alpha_{k_n}) = 0,$$

if we consider Lemma 2.1, then from (3.5), (3.8) and (3.10) we get

$$H(g_n(\alpha_{k_n})) \leq 3^{2m + mk_n} H(G)^m H(\alpha_{k_n})^{mk_n} \leq d_{k_n}^{(\sigma - \varepsilon_1)(\theta + \varepsilon_2 + s_2 + 1)m} \quad (3.13)$$

where $\varepsilon_3 > 0$ is a suitable constant. Now, using (3.12) and (3.13) we can infer

$$|g_n(\zeta) - g_n(\alpha_{k_n})| \leq \frac{1}{\varepsilon_3} \frac{s_1 w(k_n) - \theta - \varepsilon_2 - \varepsilon_3}{s_1 w(k_n) - \theta - \varepsilon_2 - \varepsilon_3} \quad (3.14)$$

By use of (3.4), (3.7) and (3.8) we have

$$|g(\zeta) - g_n(\zeta)| \leq \frac{n}{\varepsilon_3} \left| \sum_{p=1}^{n} c_{k_p} |\zeta|^{k_p n + v} \right| \leq \frac{2}{\varepsilon_3} \left| \zeta \right|^{k_{n+1}} d_{k_{n+1}}^{-\theta - \varepsilon_2} \leq \frac{2}{\varepsilon_3} d_{k_n}^{(\sigma - \varepsilon_1)(1 - \theta - \varepsilon_2 - \varepsilon_4)}, \quad (3.15)$$

where $\varepsilon_4 > 0$ is a suitable constant. Hence from (3.12) and (3.15) we find

$$|g(\zeta) - g_n(\zeta)| \leq \frac{1}{\varepsilon_3} d_{k_n}^{(\sigma - \varepsilon_1)(1 - \theta - \varepsilon_2 - \varepsilon_4)} \quad (3.16)$$

Since $\lim_{n \to \infty} w(k_n) = \infty$ and $s_1 > 0$, the inequality

$$s_1 w(k_n) - (\theta + \varepsilon_2 + \varepsilon_3) > (\sigma - \varepsilon_1)(1 - \theta - \varepsilon_2 - \varepsilon_4)$$

hold. From (3.14) and (3.16) we get

$$|g(\zeta) - g_n(\alpha_{k_n})| \leq \frac{1}{\varepsilon_3} d_{k_n}^{(\sigma - \varepsilon_1)(1 - \theta - \varepsilon_2 - \varepsilon_4)} \quad (3.17)$$

For the appropriate numbers $\varepsilon_1, \varepsilon_2$ and $\varepsilon_4$, a suitable positive number $\varepsilon$ exists such that

$$\frac{(\sigma - \varepsilon_1)(1 - \theta - \varepsilon_2 - \varepsilon_4)}{(\sigma - \varepsilon_1 - 1 + \theta + \varepsilon_2 + s_2 + 1)} m > \frac{\sigma(\sigma - 1)(1 - \theta)}{(\sigma + (\sigma - 1)(1 + \theta + s_2)) m} - \varepsilon.$$

Hence it can be found a positive number $\varepsilon$ satisfies the inequality

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\[
\sigma(\sigma - 1)(1 - \theta) \geq 2 + \epsilon
\]

Since
\[
\sigma(\sigma - 1)(1 - \theta) > 2m(\sigma + (\sigma - 1)(1 + \theta + s_2)).
\]

Using (3.17), we get
\[
\left| g(\zeta) - g_n(\alpha_{k_n}) \right| \leq \frac{1}{H(g_n(\alpha_{k_n}))^{2+\varepsilon}}, \quad (3.18)
\]

where \( \varepsilon > 0 \) is a suitable number. If the sequence \( (g_n(\alpha_{k_n})) \) is constant, then \( g(\zeta) \) is in \( \mathbb{S} \). Otherwise, pursuant to Roth's theorem [15], \( g(\zeta) \) is a transcendental number.

### 3.2. Lacunary Power Series with Algebraic Coefficients

In this subsection we concentrate our study on lacunary power series with algebraic coefficients. We extend the results in the Section 3.1 considering lacunary power series with algebraic coefficients. Suppose that \( \mathcal{S} \) is an algebraic number field of degree \( q \) and

\[
g(x) = \sum_{k_n=n}^{n} \tau_{k_n} x^{k_n} \quad (3.19)
\]

is a lacunary power series with non-zero algebraic coefficients, where \( \tau_{k_n} \) is in \( \mathcal{S} \), \( d_{k_n} \) is a rational integer such that \( d_{k_n} > 1 \) for sufficiently large \( n \) and \( (k_n)_{n=0}^{+} \) is an increasing sequence of natural numbers.

**Lemma 3.4.** Assume the notations given above. If
\[
\sigma := \liminf_{n \to \infty} \frac{\log d_{k_{n+1}}}{\log d_{k_n}} > 1 \quad (3.20)
\]

then we have
\[
\lim_{n \to \infty} \log d_{k_n} = +\infty \quad \text{and} \quad \lim_{n \to \infty} d_{k_n} = +\infty. \quad (3.21)
\]

**Proof:** The proof of Lemma 3.4 can be obtained as in the proof of Lemma 3.1.

**Lemma 3.5.** Assume the notations given above. Let
\[
D_{k_n} := \left[ d_{k_n}, d_{k_4}, \ldots, d_{k_{s_2}} \right].
\]

Then we have
\[
d_{k_n} \leq D_{k_n} < C_0 d_{k_n}^{\left( \frac{\sigma_1}{\varepsilon_2} - 1 \right)}, \quad (3.22)
\]

where \( C_0 > 0 \) is a suitable number and \( \varepsilon_2 > 0 \) is a sufficiently small number such that \( \sigma_1 = \sigma - \varepsilon_2 > 1 \).

**Proof:** Using (3.20) we have (3.22).

**Lemma 3.6.** Assume the notations given above. If
\[
\theta := \liminf_{n \to \infty} \frac{\log H(\tau_{k_n})}{\log d_{k_n}} < 1, \quad (3.23)
\]

then the convergence radius of the series \( g(x) \) in (3.19) is infinite.

**Proof:** From (3.23) we get
\[
H(\tau_{k_n}) < d_{k_n}^{\theta+\varepsilon_2}. \quad (3.24)
\]

where \( \varepsilon_2 > 0 \) is a sufficiently small number such that \( \theta + \varepsilon_2 < 1 \). Using Lemma 2.2, we have
\[
|\tau_{k_n}| < 2d_{k_n}^{\theta+\varepsilon_2}. \quad (3.25)
\]

In view of (3.21) and (3.25), the convergence radius of \( g(x) \) is infinite.

**Theorem 4.1.** Let \( g(x) \) be the lacunary power series in (3.19) satisfying the conditions (3.20), (3.23) and
\[
\lim_{n \to \infty} \frac{\log d_{k_n}}{w(k_n)} = +\infty. \quad (3.26)
\]

Let \( \mathcal{L} = \mathbb{Q}(\zeta) \) be an algebraic number field such that \( [\mathcal{L} : \mathbb{Q}] = m \) and let \( \alpha_{k_n} \) be algebraic number of degree \( m \) over \( \mathcal{L} \). Then let \( \zeta \) be a \( \mathcal{U}_m \)-number for which the following properties hold true: \( \zeta \) has an approximation with algebraic numbers \( \alpha_{k_n} \) so that the following inequalities are true
\[
|\zeta - \alpha_{k_n}| < \frac{1}{H(\alpha_{k_n})^{w(k_n)}}. \quad (3.27)
\]

\[
\lim_{n \to \infty} w(k_n) = \infty \quad \text{and} \quad \frac{d_{k_n}^{\alpha_1}}{H(\alpha_{k_n})^{\varepsilon_2}} \leq d_{k_n}^{s_1} \quad (3.28)
\]

for sufficiently large \( n \), where \( s_1 \) and \( s_2 \) are two real numbers such that \( 1 < s_1 < s_2 \). If \( \sigma(\sigma - 1)(1 - \theta) > 2t(\sigma(1 + \theta) + (1 + s)(\sigma - 1)) \), then \( g(\zeta) \) is either in \( \mathcal{M} \) or a transcendental number, where \( \mathcal{S} \subseteq \mathcal{M}, \mathcal{L} \subseteq \mathcal{M} \) and \( [\mathcal{M} : \mathbb{Q}] = t \).

**Proof:** Let us start by considering the polynomials
\[
g_n(x) = \sum_{n=0}^{n} \tau_{k_n} x^{k_n} \quad (n = 1, 2, \ldots).
\]

Since \( \gamma_n = g_n(\alpha_{k_n}) \) is an algebraic number in \( \mathcal{M} \) we have
\[
\deg(\gamma_n) \leq t \quad (n = 1, 2, \ldots).
\]

By using Lemma 2.1 with the polynomial
\[ G(y, x_0, \ldots, x_n, x_{n+1}) = D_{k_n} y - \sum_{\nu=0}^{n} D_{k_{\nu}} X_{\nu} x_{n+1} \]

from (3.22), (3.24), (3.26) and (3.28) one arrive at

\[ H(y_n) \leq d_{k_{n+1}} \left( \frac{\sigma(1 - \sigma - \epsilon_2 - \epsilon_4)}{\sigma(1 - \epsilon_2)} (1 + \theta + \epsilon_2 + s_2 + 1) \right) t, \quad (3.29) \]

where \( \epsilon_3 > 0 \) is a sufficiently small number and

\[ G(y_n, \tau_{k_0}, \ldots, \tau_{k_n}, \tau_{k_{n+1}}) = 0. \]

In view of (3.20), (3.25), (3.26) and (3.29) we get

\[ |g(\zeta) - g_n(\zeta)| \leq \frac{2}{d_{k_{n+1}}} \left( 1 + \frac{4}{d_{k_{n+1}}} - \epsilon_2 \right) \left( \sum_{\nu=0}^{n} |\tau_{k_{\nu}}| k_{\nu} (|\zeta| + 1)^{k_{\nu} - 1} \right) \]

\[ \leq \frac{1}{2} \frac{\sigma_1(1 - \sigma - \epsilon_2 - \epsilon_4)}{d_{k_{n+1}}} \]

\[ \leq \frac{1}{2} \frac{1}{d_{k_{n+1}}} \left( \frac{\sigma_1}{d_{k_{n+1}}} (1 + \theta + \epsilon_2 + 1)^{s_2 + 1} \right) t \]

\[ (3.30) \]

where \( \epsilon_4 > 0 \) is a sufficiently small number. As a consequence of (3.21), (3.25)-(3.29) it follows that

\[ |g_n(\zeta) - y_n| \leq |\zeta - \alpha_{k_n}| \sum_{\nu=0}^{n} \left( \frac{1}{d_{s_1(k_{\nu} + 3 + \theta + \epsilon_2)} \right) \left( \frac{1}{d_{k_n}} \left( \frac{\sigma_1}{d_{k_{n+1}}} (1 + \theta + \epsilon_2 + 1)^{s_2 + 1} \right) t \right) \]

\[ \leq \frac{1}{2} \frac{1}{d_{s_1(k_n + 3 + \theta + \epsilon_2)} \left( 1 + \theta + \epsilon_2 \right)} \]

\[ \leq H(y_n) \left( \frac{\sigma_1}{d_{k_{n+1}}} (1 + \theta + \epsilon_2 + 1)^{s_2 + 1} \right) t \]

\[ (3.31) \]

From (3.27), (3.30) and (3.31) we get

\[ |g_n(\zeta) - y_n| \leq \frac{1}{2} \frac{\sigma_1(1 - \sigma - \epsilon_2 - \epsilon_4)}{d_{k_{n+1}}} \left( \frac{1}{d_{k_{n+1}}} \left( 1 + \theta + \epsilon_2 + 1 \right) \right) t \]

\[ (3.32) \]

Since

\[ \sigma(\sigma - 1)(1 - \theta) > 2\epsilon(\sigma(1 + \theta) + (1 + s_2)(\sigma - 1)), \]

there is a suitable number \( \epsilon > 0 \) such that

\[ |g(\zeta) - y_n| \leq \frac{1}{2} \frac{\sigma_1(1 - \sigma - \epsilon_2 - \epsilon_4)}{d_{k_{n+1}}} \left( \frac{1}{d_{k_{n+1}}} \left( 1 + \theta + \epsilon_2 + 1 \right) \right) t \]

\[ (3.33) \]

for the suitable numbers \( \epsilon_1, \epsilon_2, \epsilon_3 \) and \( \epsilon_4 \). Now, it can be shown that

\[ |g(\zeta) - y_n| \leq \frac{1}{H(y_n)^{2 + \epsilon}}, \quad (3.34) \]

where \( \epsilon \) is a suitable positive number. If the sequence \( y_n \) is constant, then \( g(\zeta) \) is in \( M \). Otherwise, as stated in Roth's theorem [15], \( g(\zeta) \) is a transcendental number.

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Ethics

There are no ethical issues after the publication of this manuscript.

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