Poincaré-De Sitter Flow and Cosmological Meaning*

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We introduce the Poincaré-de Sitter flow with real numbers \(\{r, s\}\) to parameterize the relativistic quadruple \(\Omega_{PdR} = [P, P_2, D_+, D_-]_{M/M_\pm/D_\pm}\) for the triple of Poincaré/dS/AdS group \(P/D_+/D_-\) invariant special relativity. The dual Poincaré group \(P_2\)-invariant degenerated Einstein manifold \(M_\pm\) of \(\Lambda_\pm = \pm 3l^{-2}\) is for the space/time-like domain \(R_\pm\) of the compact lightcone \(\bar{C}_O\) associated to the common space/time-like region \(R_\pm\) of the lightcone \(C_O\) at common origin on Minkowski/dS/AdS spacetime \(M/D_+/D_-\). Based on the principle of relativity with two universal constants \((c, l)\), there are the law of inertia, coordinate time simultaneity and so on for the flow on a Poincaré-dS symmetric Einstein manifold of \(\Lambda_s = 3sl^{-2}\). Further, there is Robertson-Walker-like cosmos of the flow for the propertime simultaneity. The dS special relativity with double \([D_+, P_2]_{D_+/M_+}\) can provide a consistent kinematics for the cosmic scale physics with an upper entropy bound \(S_R = k_B\pi g^{-2}, g^2 := (\ell_P/R)^2 \sim 10^{-122}, \) for \(R \simeq (3/\Lambda)^{1/2} \sim 13.7\text{ Gly}\).

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I. INTRODUCTION

Precise cosmology shows that our universe is accelerated expanding with a tiny positive cosmological constant $\Lambda_+$ and quite possibly asymptotic to a Robertson-Walker-de Sitter ($dS$) spacetime. This opens up the era of the cosmic scale physics with new kinematics and dynamics characterized by the $\Lambda_+$. In Einstein’s theory of relativity [1, 2], special relativity ($SR$) is on the flat Minkowski ($\textit{Mink}$) spacetime, general relativity ($GR$) is on the curved spacetime with the $\textit{Mink}$-spacetime as a tangent space and the cosmological constant $\Lambda_+$ is put in by hand. For the kinematics at the cosmic scale with $\Lambda_+$, Einstein’s theory has to be re-examined from the very beginning: the principles. In particular, the principle of relativity ($PoR$) in Einstein’s $SR$.

As was well-known, the $PoR$ and the postulate on invariance of light velocity $c$ lead to Einstein’s $SR$ on the Poincaré invariant $\textit{Mink}$ spacetime $M$. Inherited from Newton’s theory, the space and time of the $\textit{Mink}$-spacetime are Euclidean and the projective infinity be excluded. The Euclidean assumption for rigid ruler and ideal clock are reasonable in conventional scales. It is priori, however,
for the cosmic scale kinematics. In order to avoid the Euclidean assumption and to allow the inertial motion of Newton’s first law may reach the (projective) infinity as a projective straightline, the PoR and the postulate should be extended and weakened to the PoR with two universal constants \((c, l)\) (denoted as \(\text{PoR}_{cl}\)). In fact, in addition to \(c\), the other universal constant \(l\) of length dimension should be introduced to characterize the dimension of space coordinates and to link the PoR with the cosmological constant \(\Lambda_+\).

If the inertial motion is allowed may reach the infinity, the most general transformations among inertial frames can be found first as the linear fractional transformations with common denominator (LFTs) [3–7], which may send finite events to the infinity. And it can be classified afterwards what kinds of kinematical symmetries of space and time may be included for real physics with metric \([8, 9]\). This is quite different from conventional understanding. Actually, these LFTs form the inertial motion group \(\text{IM}(1,3)\) homomorphic to 4d real projective group \(\text{PGL}(5,R)\) with the inertial motion algebra \(\text{im}(1,3)\) isomorphic to 4d real projective algebra \(\text{pgf}(5,R)\). The non-orientation of 4d real projective space can be avoided without taking antipodal identification of the inhomogeneous projective coordinates \(x^\mu\) [8, 9]. Hereafter, the term of projective geometry is still used in this sense.

If the \(\text{PoR}_{cl}\) is regarded as the fundamental principle for the physics at cosmic scale, it is needed to answer the questions: How to set up these inertial frames? And how to face Einstein’s “argument in a circle” for the PoR? [2]. With the help of the evolution of our universe, the inertial frames can definitely be set up via the extremely asymptotic behavior of the universe. Actually, the time arrow of the universe should coincide with the cosmic time of the Robertson-Walker-like \(dS\) cosmos that is related to the Beltrami inertial frames on the Beltrami-\(dS\) spacetime by changing the propertime simultaneity to the Beltrami coordinate time and vice versa. Thus, the evolution of our universe can definitely mark the time axis of Beltrami inertial frames so does for all kinds of inertial frames [11]. This is completely different from Einstein’s manner for determining the inertial frames with “an argument in a circle” [2]. As for Einstein’s question: “Are there at all any inertial systems for very extended portions of the space-time continuum, or, indeed, for the whole universe?” [2] Our answer in positive, otherwise it is impossible to set up kinematics at cosmic scale based on the \(\text{PoR}_{cl}\). We will explain this later.

As was shown recently, based on the \(\text{PoR}_{cl}\), with common Lorentz isotropy for the \(\text{Mink}\)-lightcone \(C_O\) with its space/time-like region \(R_\pm\) at the origin (see Eqs. (2.17) and (2.18)), there is an \(\text{SR}\) triple [8–10] for three kinds of \(\text{SR}\) of Poincaré/\(dS/\text{AdS}\) group \(\mathcal{P}/\mathcal{D}_\pm\) invariance on \(\text{Mink}/\text{Beltrami-}\(dS/\text{AdS}\) spacetime \(M/D_\pm\), respectively [11–19]. In addition, there also exists a dual Poincaré group \(\mathcal{P}_2\) of LFTs that include the projective infinity. So, they turn the lightcone (2.17) with its regions (2.18) to the compact one \(\tilde{C}_O\) with its space/time-like domains \(\tilde{R}_\pm\) and also preserve them as a compact origin lightcone structure [\(\tilde{C}_O\)] in (2.24) and (2.25). Although the formulae are almost the same, the meaning is quite different. For the domain \(\tilde{R}_\pm\), there is some (projective) infinity from the LFTs of the dual Poincaré group. And with boundary as the compact lightcone \(\tilde{C}_O\), \(\tilde{R}_\pm\) is associated
to region $R_\pm$ (2.18) at the common origin in $M\text{ink}/dS/AdS$-spacetime $M/D_+/D_-$, respectively [10]. Further, there is also a pair of $\mathcal{P}_2$-invariant degenerate Einstein manifolds $M_\pm := (M_{\pm}^{D_2}, g_\pm, g_\pm^{-1}, \nabla_{\Gamma_\pm})$ of $\Lambda_\pm = \pm 3l^{-2}$ for the domain $\hat{R}_\pm$ induced from $C_O = \partial \hat{R}_\pm$ as an absolute in the projective differential geometry approach [10]. Since for $M_\pm$ the compact lightcone $C_O$ with the origin is excluded, the dual Poincaré group $\mathcal{P}_2$ cannot offer independently a meaningful physical 4d kinematics, but as a kinematical symmetry it still is associated to three kinds of $SR$ as a triple and form a relativistic quadruple together with the Poincaré/$dS/AdS$ groups and their invariant spacetimes.

As subalgebras of $\mathfrak{im}(1,3) \cong \mathfrak{pgl}(5, R)$ with common $\mathfrak{so}(1,3)$, the $dS/AdS$ algebras $\mathfrak{d}_\pm := \mathfrak{so}(1, 4)/\mathfrak{so}(2, 3)$ and the Poincaré/dual Poincaré algebras $\mathfrak{p}/\mathfrak{p}_2$ are related by the combinatorial relation, which will be shown in (2.12), [8, 9] among their translations $(\mathbf{P}_\mu, \mathbf{P}_\mu', \mathbf{P}_\mu^\pm)$ including the pseudo ones $\mathbf{P}_\mu'$ as representations of Lorentz algebra, respectively. Thus, with common $\mathfrak{so}(1,3)$, each two of them construct an algebraic doublet, so there are six doublets in total. All of them form an algebraic quadruplet $\mathfrak{q} := (\mathfrak{p}, \mathfrak{p}_2, \mathfrak{d}_+, \mathfrak{d}_-)$. This is not only different from the contraction [20, 21] and deformation [22] approaches, but also very different from conventional concept on the inertial frames in certain given spacetime geometry. This relation, in fact, indicates that different kinds of inertial frames of $\mathcal{P}/\mathcal{P}_2/D_\pm$-invariance can be described uniformly by the same inhomogeneous projective coordinates and their differences can be marked by the corresponding translations.

Actually, corresponding to the algebraic relation (2.12), four groups are also related with common Lorentz isotropy $\mathcal{L} := SO(1, 3) = \mathcal{P}\cap\mathcal{P}_2\cap\mathcal{D}_+\cap\mathcal{D}_-$. And for the Poincaré/$dS/AdS$ group $\mathcal{P}/\mathcal{D}_+/\mathcal{D}_-$, their invariant spacetimes share the common origin lightcone structure $[C_O]$ shown in (2.19) with the same space/time-like region $R_\pm = M \cap D_+ \cap D_-$. While, for the lightcone $C_O$ and its region $R_\pm$, the LFTs of dual Poincaré group generate the $\mathcal{P}_2$-invariant compact lightcone $\hat{C}_O$ with infinity included and its space/time-like domain $\hat{R}_\pm$, $\partial \hat{R}_\pm = \hat{C}_O$. And induced from the compact origin lightcone $\hat{C}_O$ as the absolute, there is an Einstein’s manifold $M_\pm$ for the domain $\hat{R}_\pm$, respectively. Then, with the lightcone structure $[C_O]$ or its $\mathcal{P}_2$-invariant compact partner $[\hat{C}_O]$ (2.24), there are six doubles $[\mathcal{P}, \mathcal{P}_2]_{M/M_\pm}$, $[\mathcal{D}_+, \mathcal{P}]_{D_+/M}$, $[\mathcal{D}_-, \mathcal{P}_2]_{D_-/M_\pm}$ and $[\mathcal{D}_+\mathcal{D}_-]_{D_\pm}$. They form a relativistic quadruple $\mathcal{Q}_{\text{PoR}} = [\mathcal{P}, \mathcal{P}_2, \mathcal{D}_+, \mathcal{D}_-]_{M/M_\pm/D_\pm}$. It is associated to the triple of three kinds of $SR$ on $M\text{ink}/dS/AdS$-spacetimes [10].

In fact, the geometry of $M/M_\pm/D_\pm$ with $\mathcal{P}/\mathcal{P}_2/\mathcal{D}_\pm$-invariance is the subset of 4d projective geometry invariant under $\text{PGL}(5, R) \sim IM(1, 3)$, respectively. In [10], the transformations of the groups $\mathcal{D}_+/\mathcal{D}_-/\mathcal{P}_2$ and their geometric properties on the Beltrami-$dS/AdS$ spacetime $D_\pm$ and the degenerated Einstein manifold $M_\pm$ for domain $\hat{R}_\pm$ have been studied by the projective geometry method, respectively. For instance, regarding the boundary of the domain for the Beltrami-$dS/AdS$ spacetime as an absolute shown in (2.35) and (2.36), respectively, the corresponding LFTs of $dS/AdS$ group can be obtained from the LFTs of group $IM(1, 3)$ to preserve them. Then the metric in the domain and other geometric properties invariant under the $dS/AdS$ LFTs can also be gotten from the cross ratio invariant.
However, the $dS$ and $AdS$ spacetimes may have different radii. And the realistic cosmological constant in the universe may contain two parts: an invariant part as a fundamental constant and a slightly variable part. In order to describe these properties, some variable numbers may be introduced. On the other hand, it is easy to find that the corresponding formulae in different cases are very similar to each other. Actually, as long as the domain $r_\pm(x)$ in (2.18) for $\dot{R}_\pm$ is replaced by the domain $\sigma_\pm(x)$ (2.35) for the Beltrami-$dS/AdS$ spacetime, the metric, Christoffel connection, Riemann curvature and so on of the degenerated Einstein manifold $M_\pm$ with the cosmological constant $\Lambda_\pm = \pm 3l^{-2}$ become that of the Beltrami-$dS/AdS$ spacetime as the positive/negative constant curvature Einstein manifold with $\Lambda_\pm = \pm 3l^{-2}$ and vice versa, respectively. These indicate that all these spacetimes and their invariant groups may be dealt with in a uniform manner at same time based on the PoR1, no matter what the $dS/AdS$ radius should be and whether the $\Lambda$ is variable.

We will show in this paper that these can all be reached by what is named the Poincaré-$dS$ flow $\mathcal{F}_{r,s}$. It is just a parameterized quadruple with a pair of real numbers $\{r, s\}$ for the domain, absolute, group transformations and other properties. For four different regions of these numbers, the flow corresponds to four cases, respectively. Thus, to deal with the flow is just to study all the members in the quadruple simultaneously. Actually, in the flow the universal constant $l$ is replaced by $l|s|^{-1/2}$ in relevant cases, so the $dS/AdS$ spacetimes of variable radius $l|s|^{-1/2}$ appear. Correspondingly, the value of cosmological constant $\Lambda_s = 3sl^{-2}$ is also changed for different $s$. It may also allow the other number $\{r\}$ to change slightly at above fixed values, too, especially for the case of dual Poincaré group. For usual Poincaré group, the first region shown in Eqs. (3.3) and (3.4) makes sense only for on the hyperplane at infinity in 4d real projective space. It is the absolute invariant under usual Poincaré group.

In terms of the projective geometry method, especially Hua’s matrix geometry (see, e.g. [6, 7, 23, 24]), from the invariance of Poincaré-$dS$ absolute (3.2), it follows the Poincaré-$dS$ LFTs of the flow reduced from LFTs of $IM(1, 3)$ (2.2). And from the cross ratio invariant, the Beltrami metric and other properties follow. Further, Newton’s law of inertia and the $1 + 3$ decomposition with respect to Beltrami coordinate simultaneity can be given uniformly. The latter shows that the coordinate simultaneous hyper-surfaces are differomorphic to 3d Euclidean space, sphere or pseudo-sphere for suitable values of $\{r, s\}$, respectively. In fact, for four regions of $\{r, s\}$ shown in (3.4), the Beltrami metric of the flow turns to the $Mink$-metric of $Mink$-spacetime $M$, the $\mathcal{P}_2$-invariant metric of degenerated Einstein manifold $M_\pm$ for $\dot{R}_\pm$ with $\Lambda_s = 3sl^{-2}$, whose coordinate time is just the degenerated direction, and the Beltrami metric on the Beltrami-$dS/AdS$-spacetime $D_\pm$ of radius $l|s|^{-1/2}$, respectively.

As in the case of the Beltrami-$dS/AdS$ spacetimes, there is also another simultaneity in the flow, i.e. the propertime simultaneity. The Robertson-Walker-like cosmos of the flow with different space foliations can also be studied and it coincides the cosmological principle with Poincaré-$dS$ invariance. Thus, the Poincaré-$dS$ flow is out of the puzzle between the PoR and the cosmological
principle in Einstein’s theory of relativity, which had been emphasized by Bondi [25], Bergmann [26] and Rosen [27] long ago and implied by Coleman, Glashow [28] and others recently. For the case of $r = 1, s \in R^+$, the flow turns to the $dS$ spacetime with radius $ls^{-1/2}$. It is clear that the $dS$ spacetime can fit kinematically at the accelerated expanding phase with an entropy upper bound (4.5) for the evolution of the universe. Moreover, it is also away from Einstein’s “argument in a circle” for the PoR [2], since the time-arrow of the universe can indicate the existence of the Beltrami-frame of inertia via its Robertson-Walker-like counterpart whose time axis should coincide with the cosmic time-arrow of the universe [11]. This also proves the existence of the inertial frames in the universe. In addition, as different $s$ may be chosen, it may also provide an inflationary phase near the Planck scale $\ell_P$ with an entropy bound from below for the beginning of the universe or the one at other scales like the GUT scale for the inflation. Although the latter is different from the inflation model, its role to the inflation should be studied further.

Since the Poincaré-$dS$ flow is based on the PoR$_{cl}$, it is away from the framework of general relativity. So, there is no gravity for the spacetime of the flow as an Einstein manifold of $\Lambda_s$, which contains all spacetimes $M/M_\pm/D_\pm$.

The paper is arranged as follows. In sec. II, we briefly recall the PoR$_{cl}$, the inertial motion group and the relativistic kinematics that act as the SR triple associated with a relativistic quadruple of four types of related groups and geometries. In sec. III, we introduce the Poincaré-$dS$ flow and deal with the symmetry and geometry aspects as well as the physical issues for inertial motions uniformly by means of the projective differential geometry method and corresponding embedding approach. In sec. IV, we study the cosmological meaning of the flow. Especially, we show that the $dS$ SR with the $dS$-dual Poincaré double $[D_-, P_2]_{D_+/M_+}$ can provide the consistent kinematics for physics at the cosmic scale. It may also work near the Planck length or at other scales. Finally, we end with some remarks.

II. THE PoR$_{cl}$, INERTIAL MOTION GROUP AND RELATIVISTIC QUADRUPLE

A. The PoR$_{cl}$ and Inertial Motion Group

Long ago, Lu [12] suggested that Einstein’s SR [1, 2] should be extended to $dS/AdS$ spacetime of constant curvature, and studied the $dS/AdS$ invariant SR [12, 13]$^1$. Recently, motivated by precise cosmology, much studies have been made [15, 17]. In fact, if the Euclidean assumption on space is avoided, the inertial motion of Newton’s first law can reach the infinity. In fact, the kinematic aspect of Newton’s first law in inertial coordinate systems can be considered first in general and it follows

$^1$ Similar consideration had also been made by Fanttappié et al earlier in view of Erlangen program rather than the PoR (see, e.g. [14]).
the inertial motion group $IM(1,3) \sim PGL(5, R)$ with algebra $\mathfrak{im}(1,3) \cong \mathfrak{pgl}(5, R)$ kinematically [8, 9].

Umov [3], Weyl [4], Fock [5] and Hua [6, 7] studied the important issue long ago: What are the most general transformations among the inertial frames $\mathcal{F} := \{S(x)\}$ that keep the inertial motions?

Based on the PoR$_{cl}$, the answer can be stated as: The most general transformations among $\mathcal{F}$ that preserve inertial motion

$$x^i = x^i_0 + v^i(t - t_0), \quad v^i = \frac{dx^i}{dt} = \text{consts.} \quad i = 1, 2, 3,$$

(2.1)

where the isotropy among space coordinates is required, are the LFTs of twenty-four parameters

$$T : \quad l^{-1}x^\mu = A^\mu_\nu l^{-1}x^\nu + b^\mu/c_l l^{-1}x^\lambda + d, \quad x^0 = ct,$$

(2.2)

$$\det T = \begin{vmatrix} A & b^t \\ c & d \end{vmatrix} \neq 0,$$

(2.3)

where $A = (A^\mu_\nu)$ a $4 \times 4$ matrix, $b, c 1 \times 4$ matrixes, $d \in R$, $c_l x^\lambda = \eta_{\lambda\sigma} c^\lambda x^\sigma$, $t$ for transpose and $J = (\eta_{\mu\nu}) = \text{diag}(1, -1, -1, -1)$. These LFTs form the inertial motion group $IM(1,3)$ homomorphic to the $4d$ real projective group $PGL(5, R)$ with the inertial motion algebra $\mathfrak{im}(1,3)$ isomorphic to $\mathfrak{pgl}(5, R)$. Further, the time reversal $T$ and space inversion $P$ preserve the inertial motion (2.1). So, all issues are modulo the $T$ and $P$ invariance.

Inertial motion (2.1) can be viewed as a 4d straightline with an affine parameter $\lambda$

$$x^\mu(\lambda) = x^\mu_0 + \lambda v^\mu, \quad v^\mu = \text{consts},$$

(2.4)

where $-\infty < \lambda < \infty$. If the inertial motion may reach the infinity so the infinity should be included, it becomes a projective straightline [6, 7]. Then from the fundamental theorem in projective geometry [24], it follows the LFTs in (2.2) that transform some events $\{X(x^\mu) : c_l l^{-1}x^\lambda + d = 0\}$ to the infinity. And LFTs (2.2) form the real projective group $PGL(5, R)$ with algebra $\mathfrak{pgl}(5, R)$. But, for orientation in physics the antipodal identification for $x^\mu$ as inhomogeneous projective coordinates of 4d real projective space should not be taken so that $IM(1,3) \sim PGL(5, R)$. As was mentioned, we still call it the projective geometry approach in this sense, hereafter.

The set $\{T\}^{im} := (H^\pm, P^\pm_i, J_i, K_i, N_i, R_{ij}, M_\mu)$ of generators for LFTs (2.2) spans the $\mathfrak{im}(1,3)$ [9], where

$$N_i := t \partial_i + c^{-2}x_i \partial_t,$$

$$R_{ij} := x_i \partial_j + x_j \partial_i, \quad (i < j), \quad M_\mu := x^{(\mu)} \partial_{(\mu)},$$

(2.5)

where no summation for repeated indexes in brackets. The Lorentz isotropy algebra $\mathfrak{so}(1,3)$ of Lorentz group $\mathcal{L}$ generated by space rotation $J_i$ defined as

$$J_i = \frac{1}{2} \epsilon_{ijk} L_{jk}, \quad L_{jk} := x_j \partial_k - x_k \partial_j,$$

(2.6)
and Lorentz boosts $K_i$ defined as

$$K_i := t \partial_i - c^2 x_i \partial_t.$$  \hspace{1cm} (2.7)

Among four time and space translations $\{H\} := \{H, H', H^\pm\}$ of dimension $[\nu]$, $\{P\} := \{P_i, P_i', P_i^\pm\}$ of dimension $[l^{-1}]$ including the pseudo-ones $(H', P_i')$, and four boosts $\{K\} := \{K_i, N_i, K_i^0, K_i^t\}$ of Lorentz, geometry, Galilei and Carroll boosts of dimension $[c^{-1}]$, there are two independences, respectively

$$H := \partial_t, \quad H' := -v^2 t x^\nu \partial_\nu, \quad H^\pm := \partial_t \mp v^2 t x^\nu \partial_\nu;$$  \hspace{1cm} (2.8)

$$P_i := \partial_i, \quad P_i' := -l^{-2} x_i x^\nu \partial_\nu, \quad P_i^\pm := \partial_i \mp l^{-2} x_i x^\nu \partial_\nu;$$  \hspace{1cm} (2.9)

$$K_i := t \partial_i - c^2 x_i \partial_t, \quad N_i := t \partial_i + c^2 x_i \partial_t,$$

$$K_i^0 := t \partial_i, \quad K_i^t := -c^2 x_i \partial_t,$$  \hspace{1cm} (2.10)

where $\nu := c/l$ is called the Newton-Hooke constant. These generators are scalar and vector representation of $\mathfrak{so}(3)$ generated by space rotation $J_i$ without dimension as follows [8, 9]

$$[J, J] = J, \quad [J, H] = 0, \quad [J, P] = P, \quad [J, K] = K,$$  \hspace{1cm} (2.11)

where with $\epsilon_{123} = -\epsilon_{12}^3 = 1$ $[J, P] = P$ is, e.g. a shorthand of $[J_i, P_j^\pm] = -\epsilon_{ij}^k P_k^\pm$ etc. All generators and commutators have right dimensions expressed by the constants $c, l$ or $\nu$.

It is important that there are combinatorial relations among those translations and boosts

$$H \pm H' = H^\pm, \quad P_i \pm P_i' = P_i^\pm,$$  \hspace{1cm} (2.12)

$$K_j/N_j = K_i^0 \pm K_i^t.$$  \hspace{1cm} (2.13)

These relations mean that for different kinematics with corresponding translations and boost for different spacetimes as well as spaces and times the inhomogeneous projective coordinates $x^\mu$ do make sense as inertial coordinates. This is completely different from usual understanding in conventional approach.

In Table I, all relativistic kinematics are listed symbolically. For the geometrical and non-relativistic cases [9], we shall study them in detail elsewhere.

| TABLE I: All relativistic kinematics |
|--------------------------------------|
| Group Algebra | Generator Set | $[H, P]$ | $[H, K]$ | $[P, P]$ | $[K, K]$ | $[P, K]$ |
| $\mathcal{D}_+$ | $\partial_+$ | $(H^+, P_i^+, K_i^0, J_i)$ | $\nu^2 K$ | $P$ | $l^{-2} J$ | $-c^{-2} J c^{-2} H$ |
| $\mathcal{D}_-$ | $\partial_-$ | $(H^-, P_i^-, K_i^0, J_i)$ | $-\nu^2 K$ | $P$ | $l^{-2} J$ | $-c^{-2} J c^{-2} H$ |
| $\mathcal{P}$ | $p$ | $(H, P_i, K_i, J_i)$ | $0$ | $P$ | $0$ | $-c^{-2} J c^{-2} H$ |
| $\mathcal{P}_2$ | $p_2$ | $(H', P_i', K_i, J_i)$ | $0$ | $P$ | $0$ | $-c^{-2} J c^{-2} H$ |
It is clear that with common $\mathfrak{so}(1, 3)$ isotropy, the Poincaré algebraic doublet $(p, p_2)$ leads to the $dS/AdS$ algebraic doublet $(\partial_+, \partial_-)$ with the Beltrami time and space translations $(H^\pm, P^\pm_j)$ in the Beltrami-$dS/AdS$-spacetime, respectively, and vice versa [8, 9]. In addition, from the algebraic relations of $\text{im}(1, 3)$ [9] both the Poincaré doublet $(p, p_2)$ and the $dS/AdS$ doublet $(\partial_+, \partial_-)$ are closed in the $\text{im}(1, 3)$, while the generators $(R_{ij}, M_\mu)$, which generate $A^\mu$ in LFTs (2.2) together with $N_\mu$ in (2.10) and $(K_i, J_i)$, exchange the translations in the $dS/AdS$ doublet $(\partial_+, \partial_-)$ and keep that in the Poincaré doublet $(p, p_2)$ [8, 9]. Since all issues are in $IM(1, 3)$ based on the $PoR_d$, there should be three kinds of $SR$ that form the $SR$ triple of four related groups $L = \mathcal{P} \cap \mathcal{P}_2 \cap \mathcal{D}_+ \cap \mathcal{D}_-$ as a relativistic quadruple $\mathcal{O}_{PoR} := [\mathcal{P}, \mathcal{P}_2, \mathcal{D}_+, \mathcal{D}_-]_{M/M \pm /D \pm}$ [10] with the algebraic quadruplet $q = (p, p_2, \partial_+, \partial_-)$ [9].

**B. The Poincaré Double, Special Relativity Triple and Relativistic Quadruple**

It is clear that for $A = L \in \mathcal{L}$, $c = 0$ and $d = 1$, the LFTs in (2.2) reduce to the transformations of usual Poincaré group $\mathcal{P} := ISO(1, 3) = R(1, 3) \rtimes \mathcal{L}$

$$P : \ x'^\mu = (x^\nu - a^\mu)L^\mu_\nu, \ detP = det \left( \begin{array}{cc} L & b^t \\ 0 & 1 \end{array} \right) \neq 0,$$

(2.14)

where $R(1, 3)$ is 4d translation group with respect to parameters $a^\mu$ and $L = (L^\mu_\nu) \in SO(1, 3)$, $b^t = -l^{-1}(aL)^t$. And $\forall P \in \mathcal{P}$, it follows the Mink-spacetime $M := \mathcal{P}/\mathcal{L}$ with the Mink-metric and the Mink-lightcone at event $(a^\mu)$

$$ds^2 = \eta_{\mu\nu}dx^\mu dx^\nu = dxJdx^t,$$

$$C_A : \ (x-a)J(x-a)^t = 0,$$

(2.15)

with the lightcone structure $[C_A]$:

$$[C_A] : \ (x-a)J(x-a)^t \gtrless 0, \ ds^2|_A = dxJdx^t|_A = 0.$$

(2.16)

For the origin lightcone $C_O$ and its space/time-like region $R_\pm$, we have

$$C_O : \ \eta_{\mu\nu}x^\mu x^\nu = xJx^t = 0,$$

(2.17)

$$R_\pm : \ xJx^t \lesssim 0;$$

(2.18)

$$[C_O] : \ xJx^t \gtrless 0, \ ds^2|_O = dxJdx^t|_O = 0.$$

(2.19)

The set $\{T\}^p := (H, P_i, K_i, J_i)$ spans $p := \mathfrak{iso}(1, 3)$ listed in Table I symbolically. As was just mentioned, in view of projective differential geometry, Poincaré group keeps a hyperplane at infinity as its absolute in 4d real projective space $RP^4$. In terms of homogeneous projective coordinates $\xi^\mu, \xi^4$, if $\xi^4 \neq 0$, the Mink-coordinates $x^\mu$ may be regarded as

$$x^\mu = l^{\xi^4-1}\xi^\mu, \ \xi^4 \neq 0,$$

(2.20)
while the hyperplane at infinity corresponds to
\[ \xi^4 = 0. \] (2.21)

It is important that for \( A = L \in \mathcal{L} \) and \( b = 0 \) and \( d = 1 \), the LFTs in (2.2) reduce to the transformations
\[ l^{-1} x^\mu \to l^{-1} x'^\mu = \frac{L^\mu_\nu l^{-1} x^\nu}{c_l l^{-1} x^\lambda + d}, \] (2.22)
in which for \( d = 1 \) form a group called the dual Poincaré group, \( \forall P_2 \in \mathcal{P}_2 \cong ISO(1, 3) \),
\[ \det P_2 = \det \begin{pmatrix} L & 0 \\ c & d \end{pmatrix} \neq 0, \quad \text{for } d = 1. \] (2.23)

It is important that LFTs (2.22) send those events \( X(x^\mu) \) satisfying \( c_l l^{-1} x^\lambda + d = 0 \) to infinity. Therefore, corresponding infinity exists with the lightcone \( C_O \) (2.17) and its space/time-like region \( R_\pm \) (2.18) and the dual Poincaré group turn them to the compact one \( \bar{C}_O \) and its space/time-like domain \( \bar{R}_\pm \), and the former is the boundary of the latter, \( \bar{C}_O = \partial \bar{R}_\pm \), as follows
\[
\bar{C}_O : \quad x J x^t / p_2 = 0, \\
\bar{R}_\pm : \quad r_\pm(x) := \mp x J x^t / p_2 > 0, \quad \partial \bar{R}_\pm = \bar{C}_O. \] (2.24)

Clearly, LFTs (2.22) do preserve the compact lightcone structure consists of (2.24) and (2.25). Hence, the symmetry of them is not just the Lorentz group, but the semi-product of the dual Poincaré group \( \mathcal{P}_2 \) and a dilation.

In fact, in terms of the projective differential geometry, from \( \bar{C}_O \) (2.24) and \( \bar{R}_\pm \) (2.25) as absolute and domain, it follows that LFTs (2.2) are reduced to their subset (2.22). And the set \( \{ T \}^{p_2} = (H^\prime, P^\prime_i, K_i, J_i) \) spans \( p_2 \cong \mathfrak{iso}(1, 3) \) and the generator of dilation \( d \) in (2.22) is \( D = x^\mu \partial_\mu = \sum M_\mu \in \text{im}(1, 3) \).

Further, for a pair of events \( A(a^\mu), B(b^\mu) \in \bar{R}_\pm \), a line between them
\[ (1 - \tau)a + \tau b \] (2.26)
crosses the absolute \( \bar{C}_O = \partial(\bar{R}_\pm) \) at \( \tau_1, \tau_2 \). For four events with \( \tau = (\tau_1, 1, \tau_2, 0) \), a cross ratio can be given. From the power-2 cross ratio invariant
\[ \Delta^2_{R_\pm}(A, X) = \pm l^2 \{ r_\pm^{-1}(a)r_\pm^{-1}(x)r_\pm^2(a, x) - 1 \}, \] (2.27)
\[ r_\pm(a) := r_\pm(a, a) > 0, \quad r_\pm(a, x) := \mp a J x^t, \]
for two closely nearby events \( X(x^\mu), X + dX(x^\mu + dx^\mu) \in \bar{R}_\pm \), it follows a metric
\[ g_{\pm \mu \nu} = (\eta_{\mu \nu} r_\pm(x) + l^{-2} x^\mu x^\nu r_\pm^2(x)) / r_\pm(x), \quad x_\mu = \eta_{\mu \lambda} x^\lambda, \]
\[ g_\pm := (g_\pm)_{\mu \nu} = r_\pm^{-1}(x)(J + l^{-2} x x J / r_\pm(x)). \] (2.28)
Although it is degenerated, i.e. \( \text{det} \ g_\pm = 0 \), formally there is still a contra-variant metric as its inverse, respectively:

\[
g_\pm^{-1}(r_\pm) = r_\pm(x)(J \pm l^{-2} \frac{Jx^t xJ}{r_\pm(x)})^{-1},
\]

(2.29)

which is divergent with a finite part

\[
g_\pm^{-1}(r_\pm) \bigg|_{\text{finite}} = l^{-2} r_\pm J x^t xJ.
\]

(2.30)

The Christoffel symbol for dual Poincaré symmetry can still be formally obtained

\[
\Gamma_\pm^{\lambda \mu \nu}(x) = \pm r_\pm^{-1}(x) (\delta_\mu^\lambda x_\nu + \delta_\nu^\lambda x_\mu).
\]

(2.31)

It is obviously metric compatible, i.e. \( \nabla \Gamma_\pm = 0 \), \( \nabla_{\Gamma_\pm} = 0 \), \( \nabla_{\Gamma_\pm} \big|_{\text{finite}} = 0 \). Further, the Riemann, Ricci and scale curvature reads, respectively:

\[
R_\pm^\mu_{\nu \lambda \sigma}(x) = \pm l^{-2} (g^\pm_{\nu \lambda} \delta_\sigma^\mu - g^\pm_{\nu \sigma} \delta_\lambda^\mu),
\]

\[
R_\pm_{\mu \nu}(x) = \pm 3 l^{-2} g^\pm_{\mu \nu}, \quad R_\pm(x) = \pm 12 l^{-2}.
\]

(2.32)

So, the \( M_\pm := (M_\pm^P, g_\pm, g^{-1}_\pm, \nabla_{\Gamma_\pm}) \) is an Einstein manifold with \( \Lambda_\pm = \pm 3/l^2 \) for \( \hat{R}_\pm \) (2.25), respectively. In [29], a different \( \mathcal{P}_2 \)-invariant kinematics with degenerated geometry for \( \hat{R}_\pm \) is given in another manner.

Since all these objects are given originally from the cross ratio invariance of \( \bar{C}_O \) (2.24) under \( LFTs \) (2.2), such a \( \mathcal{P}_2 \)-invariant kinematics is invariant under (2.23) and is based on the \( PoR_{cl} \). It easy to check that their Lie derivatives vanish with respect to the \( \mathfrak{p}_2 \)-generators as Killing vectors on \( M_\pm \), respectively, and that Eq. (2.1) is indeed the equation of motion for a free particle, if any, on such a pair of Einstein manifolds \( M_\pm \).

It is clear that in all the formulae the origin and the compact lightcone \( \bar{C}_O \) are automatically excluded. For the origin excluded, the name of dual Poincaré group makes sense dual to usual Poincaré group, for which the infinity is excluded. Due to the origin is excluded, the \( \mathcal{P}_2 \)-invariant kinematics is not independently a meaningful physical one, rather as a kinematical symmetry, which is always associated to three kinds of \( SR \) as a triple.

In fact, the above \( \mathcal{P}_2 \)-structure also exists with respect to the lightcone structure \( [C_A] \) (2.16), while the \( \mathcal{P}_2 \)-structure with respect to the \( [C_O] \) is just a representative. This can be easily seen from all above relevant formulae if all \( x^\mu \) are replaced by \( x^\mu - a^\mu \). There are intersections for \( \mathcal{P} \) and \( \mathcal{P}_2 \), i.e. \( \mathcal{L} = \mathcal{P} \cap \mathcal{P}_2 \). While for \( M \) and \( M_\pm \), \( R_\pm \in M \) and \( \hat{R}_\pm = \cap M_\pm \) are very closely related. Thus, relevant to the \( Mink \)-spacetime there exists a Poincaré double \( [\mathcal{P}, \mathcal{P}_2]_{M/M_\pm} \) with a pair of Einstein manifolds \( M_\pm \) induced from \( C_O \) for \( \hat{R}_\pm \), respectively. Further, under usual Poincaré transformations (2.14), the Poincaré algebraic doublet \( (p, p_2) \) at the origin can be transformed to the event \( A(a^\mu) \) and vice versa. In fact, under transformations of usual Poincaré group \( \mathcal{P} \), the generators \( P'_\mu := (H', P'_i) \) as a
4-vector on $M$ are transferred and the action is closed in the algebra $\text{im}(1, 3)$ with $\mathcal{d}_\pm$ for $dS/AdS$ SR as the subsymmetry:

$$\mathcal{L}_{P_\mu} P'_\nu = [P_\mu, P'_\nu] \in \text{im}(1, 3) \cong \mathfrak{pgl}(5, R). \quad (2.33)$$

Thus, as long as the inertial motions for free particles and light signals are naturally allowed to reach the infinity, the symmetry and geometry related to the Mink-spacetime are dramatically changed. In order to eliminate the Poincaré double and restore Einstein’s SR, $l = \infty$ must be taken. This is equivalent to the Euclidean assumption that the inertial motions cannot reach the infinity. However, the existence of the cosmological constant $\Lambda_+$ leads to a maximum value of $l$ and the Poincaré double $[\mathcal{P}, \mathcal{P}_2]_{M/M^+}$ would appear with following bound for modern relativistic physics

$$\nu^2 := (c/l)^2 \simeq c^2 \Lambda_+/3 \sim 10^{-35} \text{sec}^{-2}. \quad (2.34)$$

This value is so tiny that their effects can still be ignored up to now locally except for the cosmic scale physics.

It is also shown \cite{10} that the $dS/AdS$ double $[\mathcal{D}_+, \mathcal{D}_-]_{D_+}$ of the algebraic doublet $(\mathcal{d}_+, \mathcal{d}_-)$ can be reached via the Mink-lightcone $C_0$ in (2.17) by relaxing the flatness of Mink-spacetime. In fact, shifting the lightcone by $\mp l^2$ leads to the expression

$$\mathcal{D}_\pm : \quad \sigma_\pm(x) := \sigma_\pm(x, x) = 1 \mp l^{-2} x J x^t > 0, \quad (2.35)$$

which denote a pair of related regions on $M$ with boundary as a ‘pseudosphere’

$$\mathcal{B}_\pm = \partial(\mathcal{D}_\pm) : \quad \sigma_\pm(x) = 1 \mp l^{-2} x J x^t = 0, \quad (2.36)$$

respectively (see, e.g. \cite{30}). However, if the flatness of spacetime is relaxed, regions (2.35) and their boundaries (2.36) without flatness are just the the domain conditions and their absolutes in the projective differential geometry approach to the Beltrami model of $dS/AdS$-spacetime $D_\pm$ with radius $l$, respectively \cite{7, 10}. Actually, if the coordinates $x^\mu$ are regarded as the Beltrami coordinates in a chart $U_4$, say,

$$x^\mu = l \xi^{4-1}_\mu, \quad \xi^4 > 0, \quad (2.37)$$

regions (2.35) and boundaries (2.36) become the $dS/AdS$-hyperboloid $\mathcal{H}_\pm$ and their boundaries, respectively

$$\mathcal{H}_\pm : \quad \eta_{\mu\nu} \xi^\mu \xi^\nu \mp (\xi^4)^2 \leq 0, \quad (2.38)$$

$$\mathcal{B}_\pm = \partial \mathcal{H}_\pm : \quad \eta_{\mu\nu} \xi^\mu \xi^\nu \mp (\xi^4)^2 = 0. \quad (2.39)$$

Clearly, the expressions of $\eta_{\mu\nu} \xi^\mu \xi^\nu$ and of $(\xi^4)^2$ are in intersections of the $\mathcal{H}_\pm$ and their boundaries $\mathcal{B}_\pm = \partial \mathcal{H}_\pm$ and their vanishing expressions just correspond to the (compact) origin lightcone and
hyperplane at infinity as the absolutes of the dual Poincaré group and of the usual Poincaré group, respectively. Then, by means of the projective differential geometry method, the intersected Beltrami-
\( dS/AdS \)-spacetimes \( D_\pm \) invariant under the \( dS/AdS \) group \( \mathcal{D}_\pm \) can be set up and form the \( dS/AdS \) double \([\mathcal{D}_+,\mathcal{D}_-]_{D_\pm}\) with common Lorentz isotropy, i.e. \( \mathcal{L} = \mathcal{D}_+ \cap \mathcal{D}_- \). And they are closely related to both the Poincaé group and the dual Poincaré group.

In fact, \( LFTs \) (2.2) with (2.36) as the absolute reduce to the \( dS/AdS-LFTs \) with common \( L^\mu_\kappa \in \mathcal{L} \)

\[
\mathcal{D}_\pm : \ x^\mu \rightarrow x'^\mu = \pm \sigma_\pm^{1/2}(a) \sigma_\pm^{-1}(a,x)(x^\nu - a^\nu)D_\pm^\mu_\nu, \\
D_\pm^\mu_\nu = L^\mu_\nu \pm l^{-2}a_\nu a^\kappa(\sigma_\pm(a) + \sigma_\pm^{1/2}(a))^{-1}L^\mu_\kappa,
\]

which preserve (2.35) for the Beltrami-\( dS/AdS \) spacetime \( D_\pm \), respectively. Further, from a power-2 cross ratio invariant, it follows the interval between a pair of events \( A(a^\mu) \) and \( X(x^\mu) \) and the lightcone with the vertex at \( A(a^\mu) \) as follows, respectively

\[
\Delta_\pm^2(A, X) = \pm l^{-2}\{\sigma_\pm^{-1}(a)\sigma_\pm^{-1}(x)\sigma_\pm^2(a, x) - 1\} \gtrless 0, \quad (2.41) \\
F_\pm : \ \sigma_\pm^2(a, x) = \sigma_\pm(a)\sigma_\pm(x) = 0. \quad (2.42)
\]

For the closely nearby two events \( X(x^\mu), X + dX(x^\mu + dx^\mu) \in \mathfrak{D}_\pm \), the Beltrami metric \( g^B_\pm \) and its inverse of \( dS/AdS \) spacetime follows from (2.41), respectively

\[
ds^2_\pm = (\frac{n_{\mu\nu}}{\sigma_\pm(x)} \pm l^{-2}\frac{x^\mu x^\nu}{\sigma_\pm(x)})dx^\mu dx^\nu, \quad \sigma_\pm(x) > 0. \\
g^B_\pm := (g^B_\pm)^{\mu\nu} = \sigma_\pm^{-1}(x)(J \pm l^{-2}\frac{J x^t x^i J}{\sigma_\pm(x)}), \\
g^{B-1}_\pm = \sigma_\pm(J \pm l^{-2}\frac{J x^t x^i J}{\sigma_\pm(x)})^{-1} = \sigma_\pm(J - l^{-2}x^t x).
\]

Due to transitivity of (2.40), the Beltrami-\( dS/AdS \) spacetime \( D_\pm \cong \mathcal{D}_\pm/\mathcal{L} \) with \([C_O] = D_+ \cap D_- \) is homogeneous, respectively. It is also true for the entire Beltrami-\( dS/AdS \) spacetime globally. And the set \( \{T^O_\pm\} = (H^\pm, P^\pm_i, K_i, J_i) \) of (2.40) spans the \( dS/AdS \) algebra, respectively, and they form a doublet \( (\mathfrak{o}_+, \mathfrak{o}_-) \) with common Lorentz isotropy.

In addition, the generators in \( \{T^O_\pm\} = (H^\pm, P^\pm_i, K_i, J_i) \) of the \( dS/AdS \) algebra \( \mathfrak{d}_\pm \) can be regarded as Killing vectors of Beltrami-\( dS/AdS \) metric (2.43). With respect to these Killing vectors the Lie derivatives vanish for the metric, connection and curvature. Further, the geodesic motion of (2.43) is indeed the inertial motion (2.1).

In short, domains \( \mathfrak{D}_\pm \) (2.35) shifted from \( C_O \) (2.17) indicate that the Beltrami-\( dS/AdS \) spacetimes \( D_\pm \) share the same origin lightcone structure \([C_O] \) from (2.42) and (2.43). So, the \( dS/AdS \) SR on \( D_\pm \) form a double \([\mathcal{D}_+,\mathcal{D}_-]_{D_\pm}\). Since they also share the \([C_O] \) with \( Mink \)-spacetime \( M \), together with Poincaré, \( dS/AdS-P \), and \( dS/AdS \) doubles, there are also \( dS/AdS-P_2 \) doubles \([\mathcal{D}_+,\mathcal{P}_2]_{D_\pm/M_\pm}\) with Einstein manifold \( M_\pm \) for domain \( \hat{R}_\pm \) (2.25) generated by the \( \mathcal{P}_2 \)-invariance. All six doubles form the quadruple \( \mathfrak{Q}_{P\hat{R}} \) with common Lorentz isotropy. But, as far as the cosmological constants \( \Lambda_\pm \) are concerned, only two \( dS/AdS-P_2 \) doubles \([\mathcal{D}_+,\mathcal{P}_2]_{D_\pm/M_\pm}\) are consistent kinematically in principle.
III. THE POINCARÉ-DS FLOW AS A PARAMETERIZED QUADRUPLE

As was mentioned, in order to allow that the $dS/AdS$ spacetimes may have different radii and the realistic cosmological constant may contain a variable part in addition to the fundamental one, some variable numbers may be introduced together with the universal constant $l$. On the other hand, as long as domain $r_\pm(x) \ (2.18)$ is replaced by domain $\sigma_\pm(x) \ (2.35)$ the degenerate Einstein manifold $M_\pm$ of $\Lambda_\pm = \pm 3l^{-2}$ becomes the Beltrami-$dS/AdS$ spacetime with the same cosmological constant and vice versa, respectively. These indicate that all these spacetimes and relevant Poincaré, dual Poincaré, $dS$ and $AdS$ groups can be dealt with simultaneously in the same manner. In fact, these can all be done by introducing such a domain and its absolute with a pair of numbers $\{r, s\}$ that for their four regions shown in (3.4), it gives rise to four corresponding domains, their absolutes of the spacetimes and their invariant groups, respectively. Such a parameterized domain with absolute is called that of the Poincaré-$dS$ flow $F_{r,s}$. Then the flow is just the parameterized relativistic quadruple $Q_{r,s}$.

A. Domain, Absolute and Transformations of Poincaré-$dS$ Flow

Let us introduce the domain and the absolute of the Poincaré-$dS$ flow as follows

\[
\mathcal{D}_{r,s} : \quad \sigma_{r,s}(x) := \sigma_{r,s}(x, x) = r - sl^{-2}xJx^t > 0, \quad (3.1)
\]

\[
\mathfrak{A}_{r,s} = \partial(\mathcal{D}_{r,s}) : \quad \sigma_{r,s}(x) = r - sl^{-2}xJx^t = 0. \quad (3.2)
\]

It is clear that first for the fixed values of the numbers $\{r, s\}$

\[
\{r, s\}_{\text{fixed}} = \{1, 0\}, \{0, 1\}, \{1, 1\}, \{1, -1\}, \quad (3.3)
\]

absolute (3.2) and its domain (3.1) become that of Poincaré, dual Poincaré and $dS/AdS$ case, respectively. If one of the pair of numbers, i.e. $\{s\}$, is allowed to change its value in four regions as follows

\[
\{r, s\} \in \{\{1\}, \{0\}\}, \{\{0\}, (0, +\infty)\}, \{\{1\}, (0, +\infty)\}, \{\{1\}, (-\infty, 0)\}, \quad (3.4)
\]

the four types of domains and absolutes can also be reached for invariant group $\mathcal{P}/\mathcal{P}_2/\mathcal{D}_+/\mathcal{D}_-$, respectively. It is clear that not only (3.2) contains the absolutes of $\mathcal{P}, \mathcal{P}_2, \mathcal{D}_\pm$ for four pairs of fixed value of $\{r, s\}$ in (3.3), but also four types of the absolutes of $\mathcal{P}, \mathcal{P}_2, \mathcal{D}_\pm$ with four different regions (3.4) of $\{r, s\}$, respectively. Namely,

\[
\sigma_{r,s}(x) = r - sl^{-2}xJx^t = 0 : \quad \begin{cases} 
\xi^4 = 0, & r = 1, \quad s = 0, \\
\ r_\pm(x) = 0, & r = 0, \quad s > 0, \\
\sigma_\pm(x) = 0, & r = 1, \quad s \geq 0.
\end{cases} \quad (3.5)
\]
In order to preserve absolute (3.2) or (3.5), LFTs (2.2) must reduce to their subset with the following conditions

\[ sAJA^t - rc^t c = sJ, \quad sAJb^t - rc^t d = 0, \quad rd^2 - sbJb^t = r. \] (3.6)

Setting \( b = -rl^{-1} aA \) with respect to the point \( A(a^\mu) \in \mathcal{D}_{r,s} \) in domain (3.1) of the flow \( \mathcal{F}_{r,s} \), it follows

\[ sAJA^t - rc^t c = sJ, \quad srl^{-1} AJA^t a^t + rc^td = 0, \quad rd^2 - sr^2 l^{-2} aAJA^t a^t = r, \] (3.7)

In terms of the standard method of projective geometry, especially Hua’s matrix analysis (see, e.g. [6, 7, 23, 24]), Eq. (3.7) can be solved to get the transitive form of the parameterized LFTs of the flow

\[ T_{r,s} : \quad x \rightarrow \tilde{x} = (1 - srl^{-2} aJa^t)^{\frac{1}{2}} \frac{x - ra}{1 - sl^{-2} xJa^t} A \]
\[ A = L + srl^{-2}[(1 - srl^{-2} aJa^t)^{\frac{1}{2}} + (1 - srl^{-2} aJa^t)]^{-1} Ja^t aL, \] (3.8)

and all these parameterized LFTs form an invariant group. For four regions of \( \{r, s\} \) in Eqs. (3.3) and (3.4), it follows Poincaré transformations (2.14) as \( s = 0 \), dual Poincaré transformations in (2.22) as \( r = 0 \) with length parameter \( l|s|^{-1/2} \), and \( dS/AdS \) transformations (2.40) as \( s \gtrless 0 \) of radius \( l|s|^{-1/2} \), respectively. The infinitesimal generators of LFTs (3.8) read

\[ H_{r,s}^i = r \partial_i - s\nu^2 t x^\alpha \partial_\alpha, \quad P_{r,s}^i = r \partial_i + sl^{-2} x^i x^\alpha \partial_\alpha, \quad K_{r,s}^i = K_i, \quad J_{r,s}^i = J_i. \] (3.9)

They satisfy

\[ [P_{r,s}^i, P_{r,s}^j] = -rs l^{-2} \epsilon_{ij}^k J_k, \quad [H_{r,s}^i, P_{r,s}^j] = rs \nu^2 K_i, \quad [P_{r,s}^i, K_j] = \delta_{ij} c^{-2} H_{r,s}^i, \quad [H_{r,s}^i, K_j] = P_{r,s}^i, \quad [K_i, K_j] = rsc^{-2} \epsilon_{ij}^k J_k, \]
\[ [J_i, H_{r,s}^i] = 0, \quad [J_i, P_{r,s}^i] = -\epsilon_{ij}^k P_{r,s}^k, \quad [J_i, K_j] = -\epsilon_{ij}^k K_k. \] (3.10)

For four regions of \( \{t, s\} \) in Eqs. (3.3) and (3.4), it follows Poincaré, dual Poincaré and \( dS/AdS \) algebra, respectively.

**B. Spacetime, Inertial Motion and Simultaneity of Poincaré-\( dS \) Flow**

Let us consider a line like (2.26) connecting two points \( (x_1, x_2) \) in domain (3.1) with \( x = x_1, x_2 \) for \( \tau = 0, 1 \), respectively. The line intersects absolute (3.2) at two points \( (\tau_+, \tau_-) \)

\[ \tau_+ + \tau_- = -\frac{2x_1 J(x_2 - x_1)^t}{(x_2 - x_1)J(x_2 - x_1)^t} \quad \tau_+ \tau_- = \frac{s x_1 J x_1^t - r l^2}{s (x_2 - x_1)J(x_2 - x_1)^t}. \] (3.11)
Thus, for four events with $\tau = (\tau_+, 1, \tau_-, 0)$, a cross ratio can be given. From the power-2 cross ratio invariant, it follows the interval between a pair of events $A(\mu)$ and $X(\nu)$ and the lightcone with the vertex at $A(\mu)$ as follows, respectively

$$\Delta^2_{r,s}(A, X) = s l^{-2} \{ \sigma_{r,s}^{-1}(a) \sigma_{r,s}^{-1}(x) \sigma_{r,s}^2(a, x) - 1 \},$$  \hspace{1cm} (3.12)

$$\mathcal{C}_{r,s} : \quad \sigma_{r,s}^2(a, x) - \sigma_{r,s}(a) \sigma_{r,s}(x) = 0.$$  \hspace{1cm} (3.13)

For two close events $X(\mu), X + dX(\mu + d\mu) \in \mathcal{O}_{r,s}$ (3.1), the Beltrami metric of the flow follows from (3.12)

$$ds^2_{r,s} = \sigma_{r,s}^{-1}(x) dx^J dx^I + s l^{-2} \sigma_{r,s}^{-2}(x) (x^J dx^I)^2,$$

$$g_{r,s} := (g_{r,s})_{\mu\nu} = \sigma_{r,s}^{-1}(x) \left( J \pm l^{-2} \frac{J^t x^J}{\sigma_{r,s}(x)} \right).$$  \hspace{1cm} (3.14)

The contra-variant metric is

$$g^{-1}_{r,s} = \sigma_{r,s}(x) \left( J \pm l^{-2} \frac{J^t x^J}{\sigma_{r,s}(x)} \right)^{-1} = \sigma_{r,s}(x) \left( J - \frac{r}{s} l^{-2} x^J x^J \right).$$  \hspace{1cm} (3.15)

The last step of above equation can be obtained only when $r \neq 0$. The metric (3.14) is degenerated for the region of dual Poincaré case in (3.5), i.e. $r = 0, s > 0$. The corresponding contra-variant metric (3.15) is divergent and can be given formally with the help of the flow when $r = 0$. In terms of projective geometry, $\frac{1}{r} = \infty (r = 0)$ is well-defined. It is straightforward to get the Christoffel connection, Riemann curvature, Ricci curvature and scale curvature of the flow, respectively, as follows

$$\Gamma^\lambda_{r,s \mu\nu} = s l^{-2} \sigma_{r,s}^{-1} (\delta^\lambda_\mu x^\nu + \delta^\lambda_\nu x^\mu),$$

$$R^\mu_{r,s\nu\lambda}(x) = s l^{-2} (g_{r,s\lambda\sigma} \delta^\mu_\sigma - g_{r,s\nu\sigma} \delta^\mu_\lambda),$$

$$R_{r,s\mu\nu}(x) = 3 s l^{-2} g_{r,s\mu\nu}, \quad R_{r,s}(x) = 12 s l^{-2}.$$  \hspace{1cm} (3.16)

Thus, the spacetime of the flow is an Einstein manifold with a parameterized cosmological constant $\Lambda_s = 3 s l^{-2}$.

It is also straightforward to check that the geodesic motion of the metric (3.14) is indeed inertial motion (2.1) of Newton’s first law. In fact, the geodesic of the flow leads to the conservation law of the (pseudo) 4-momentum of a particle with mass $m_{r,s}$ in the flow

$$\frac{d}{ds_{r,s}} p_{r,s}^\mu = 0, \quad p_{r,s}^\mu = m_{r,s} \sigma_{r,s}^{-1}(x) \frac{dx^\mu}{ds_{r,s}}.$$  \hspace{1cm} (3.17)

Then, it follows inertial motion (2.1)

$$p_{r,s}^\mu = \text{consts}, \quad \frac{p_{r,s}^i}{p_{r,s}^0} = \frac{dx^i}{cdt} = \text{consts}.$$  \hspace{1cm} (3.18)
It is easy to show that the corresponding 4d angular momentum is also preserved for the motion

$$L_{r,s}^{\mu\nu} = x^\mu p_{r,s}^\nu - x^\nu p_{r,s}^\mu. \quad (3.19)$$

And there is generalized Einstein’s formula for the particle

$$rE^2 - r\mathbf{p}_{r,s}^2 c^2 - \frac{sc^2}{l^2} j_{r,s}^2 + \frac{sc^4}{l^2} k_{r,s}^2 = m_{r,s}^2 c^4. \quad (3.20)$$

with energy $E = p_{r,s}^0$, momentum $p_{r,s}^i$, $p_{r,s} = \delta_{ij} p_{r,s}^j$, ‘boost’ $k_{r,s}^i$, $k_{r,s} = \delta_{ij} k_{r,s}^j$ and 3-angular momentum $j_{r,s}^i$, $j_{r,s} = \delta_{ij} j_{r,s}^j$. Similarly, it is also true for the light signals in the flow.

In order to make sense for inertial motion (2.1) in measurements, the coordinate simultaneity should be defined as two events $A(a^\mu)$ and $B(b^\nu)$ are simultaneous if and only if

$$a^0 := x^0(A) = x^0(B) =: b^0. \quad (3.21)$$

This simultaneity leads to the following 1 + 3 split of metric (3.14)

$$ds_{r,s}^2 = N_{r,s}^2(dx^0)^2 - h_{r,sij}(dx^i + N_{r,s}^i dx^0)(dx^j + N_{r,s}^j dx^0). \quad (3.22)$$

Here the lapse function $N_{r,s}$, shift vector $N_{r,s}^i$, and induced 3-geometry $h_{r,sij}$ on 3-hypersurface $\Sigma_{r,s}$ read

$$h_{r,sij} = -g_{r,sij} = \sigma_{r,s}^{-1}(x)\delta_{ij} - s\sigma_{r,s}^{-2}(x)l^{-2}x_ix_j$$

$$-h_{r,sij}N_{r,s}^j = g_{r,s0i} = s\sigma_{r,s}^{-2}(x)l^{-2}x_0x_i,$$

$$N_{r,s}^2 - h_{r,sij}N_{r,s}^iN_{r,s}^j = g_{r,s00} = \sigma_{r,s}^{-1}(x) + \sigma_{r,s}^{-2}(x)l^{-2}x_0x_0. \quad (3.23)$$

The inverse of $h_{r,sij}$ reads in the 1 + 3 split of contra-variant metric

$$h_{ij} = \sigma_{r,s}(x)(\delta_{ij} + s l^{-2} x_i x_j (r - s l^{-2} (x^0)^2)^{-1}). \quad (3.24)$$

And the shift vector $N_{r,s}^i$ and the lapse function $N_{r,s}$ reads, respectively

$$N_{r,s}^i = \frac{s x^0 x^i}{rl^2 - s(x^0)^2}, \quad N_{r,s}^2 = \frac{r l^2}{\sigma_{r,s}(x)(rl^2 - s(x^0)^2)}. \quad (3.25)$$

It is true that for four regions (3.3) and (3.4) of the numbers $\{r, s\}$, the flow reaches that of Poincaré, dual Poincaré, $dS$ and $AdS$ with different radius, respectively.

In particular, for the dual Poincaré case, degenerated metric (2.28) and its formal inverse (2.29) can also be defined as of the numbers $r = 0, s > 0$. And from 1 + 3 split (3.22), it follows that the non-degenerate part of degenerated metric (2.28) is a Beltrami metric of a 3d sphere/hyperboloid of radius $l|s|^{-1/2}$ for space/time-like domain $\hat{R}_\pm$ at the origin with the compact one $\hat{C}_O = \partial\hat{R}_\pm$ in Eq. (2.24), and the non-degenerate part of formal inverse (2.29) is a time axis, respectively.
C. Embedding Picture of Poincaré-dS Flow

Since the above 4d spacetime $S_{r,s}^4$ of the flow is constant curvature formally, it can be embedded in a parameterized 5d flat spacetime $M_{r,s}^5$ for $r \neq 0$, $s \geq 0$ of $dS/AdS$ case, respectively, as follows

\[ S_{r,s}^4 : \quad |s|\eta_{\mu \nu} \xi^\mu \xi^\nu - \frac{|s|}{s} r(\xi^4)^2 = -\frac{l^2}{s}, \]  

(3.26)

which can be written as

\[ |s|\eta_{rsAB} \xi^A \xi^B = -\frac{l^2}{s}, \quad J_{r,s} := \eta_{rsAB} = diag(J, -\frac{r}{s}). \]  

(3.27)

On $S_{r,s}^4 \subset M_{r,s}^5$, there is a parameterized metric

\[ ds^2_5 = |s|\eta_{\mu \nu} d\xi^\mu d\xi^\nu - \frac{r}{s}|\xi^4|^2 = |s|\eta_{rsAB} d\xi^A d\xi^B, \]  

(3.28)

which satisfies

\[ ds^2_5 = ds^2_{r,s}. \]  

(3.29)

It is clear that they are invariant under a subgroup $S_{rs}$ of LFTs (2.2) consists of all $T_{r,s}$ satisfying

\[ T_{r,s} J_{r,s} T_{r,s} = J_{r,s}, \quad \forall T_{r,s} = S_{rs}. \]  

(3.30)

These conditions are equivalent to Eqs. (3.6). In terms of the Beltrami coordinates like (2.37), Eqs. (3.26) or (3.27) and (3.28) become domain condition (3.1) and Beltrami metric (3.14) of the flow, respectively. And transformations $T_{r,s}$ in (3.30) are just that of LFTs in (3.8). It should be noted that all matrices $T_{r,s}$ satisfying (3.30) do form a Lie group with two numbers \{r, s\} (see, e.g. [31]).

We can also consider a kind of uniform great ‘circular’ motions on (3.26) for a free massive particle with mass $m_{r,s}$ defined by a conserved 5d angular momentum, \textit{i.e.}

\[ \frac{dL^{AB}_{r,s}}{ds_5} = 0, \]  

(3.31)

\[ L^{AB}_{r,s} = m_{r,s} |s|(\xi^A \frac{d\xi^B}{ds_5} - \xi^B \frac{d\xi^A}{ds_5}). \]  

(3.32)

And there is an Einstein-like formula for the particle:

\[ -\frac{s}{2l^2} \eta_{rsAC} \eta_{rsBD} L^{AB}_{r,s} L^{CD}_{r,s} = m_{r,s}^2, \]  

(3.33)

It is invariant under linear transformations $T_{rs}$ (3.30) of $S_{rs}$ group of the flow.

In terms of the Beltrami coordinates like (2.37), the parameterized Beltrami metric (3.14) follows from (3.28) and the uniform ‘great circular’ motion (3.31) turns to the geodesic motion in Eq. (3.17) and the conserved 5d angular momentum in (3.32) becomes the conserved 4d pseudo-momentum in Eq. (3.17) and 4d angular momentum

\[ p^\mu_{r,s} = m_{r,s} \sigma_{r,s}^{-1}(x) \frac{dx^\mu}{ds_{r,s}} = \frac{1}{l} L^{4\mu}_{r,s}, \quad L^{\mu \nu}_{r,s} = L^{\mu \nu}_{r,s}. \]  

(3.34)
And Einstein-like formula (3.33) becomes the generalized Einstein’s formula (3.20).

In order to make sense for the uniform great circular motion (3.31), simultaneity should be defined on the spacetime $S^4_{r,s}$ (3.26) of the flow. For a pair of simultaneous events $(P(\xi_P), Q(\xi_Q))$, their time-like coordinates $\xi^0$, which is corresponding to the common inhomogeneous time coordinate $x^0$, should be equal to each other. Namely,

$$\xi^0_P = \xi^0_Q. \quad (3.35)$$

This simultaneity is the same with respect to the proper-time simultaneity on $S^4_{r,s} \subset M^5_{r,s}$. Then a simultaneous 3-space of $\xi^0 = \text{const}$ reads:

$$|s|\delta_{r,sIJ}\xi^I \xi^J = l^2 s^{-1} + |s|(\xi^0)^2, \quad I, J = 1, \ldots, 4; \quad (3.36)$$

$$dl^2|\xi^0=\text{const}| = |s|\delta_{r,sIJ}d\xi^I d\xi^J, \quad \delta_{r,sIJ} = \text{diag}(I^3, \frac{T}{S}).$$

For a kind of “observers” $O_{S^4_{r,s}}$ at the spacial origin $O|\xi^\alpha = 0$, where $\alpha$ takes three among $1, \ldots, 4$, it is a 3d hypersurface of variable radius $(l^2/s + |s|(\xi^0)^2)^{-1/2}$.

Along with $|\xi^0|$ enlarging, for the case of $r = 1, s > 0$, i.e. the $dS$ case, it is an expanding 3-sphere. For the case of $r = 1, s < 0$, i.e. the $AdS$ case, it is a 3-hyperboloid. Other two cases of Poincaré and dual Poincaré in Eqs. (3.3) and (3.4) correspond to $r = 1, s = 0$ and $r = 0$, respectively.

IV. THE COSMOLOGICAL MEANING OF POINCARÉ–dS FLOW

Let us now consider the cosmological meaning of the Poincaré–dS flow. As was mentioned, As in the case of the Beltrami–dS/AdS spacetimes, there are two kinds of simultaneity, i.e. the coordinate simultaneity and the propertime simultaneity, in the Poincaré–dS flow. Then it can also be studied the Robertson-Walker-like cosmos of the flow, which coincides the cosmological principle.

A. Robertson-Walker-like Metric of Poincaré–dS Flow

For the Robertson-Walker-like metric of the Poincaré–dS flow, we chose the simplest and most natural foliation. Namely, from the parameterized Beltrami metric (3.14) of the flow to this cosmic metric is just by changing the parameterized Beltrami coordinate time to the proper-time as a co-moving or the cosmic time and vise versa. Namely, As in the Beltrami model of $dS$ spacetime [15], for the flow we can also introduce the propertime $\tau$ as cosmic time that is related to Beltrami-time coordinate $x^0$ by

$$x^0\sigma^{-1/2}_{rs}(x) = (\xi^0 =) \frac{l}{\sqrt{s|s|}} \sinh \left(\sqrt{s}\tau/l\right). \quad (4.1)$$
The metric (3.14) of the flow becomes
\[ ds^2_{r,s} = d\tau^2 - \cosh^2 \left( \sqrt{s\tau/l} \right) dl_0^2, \]
\[ dl_0^2 = \sigma_0^{-1}(x)\delta_{ij}dx^idx^j - sl^{-2}\sigma_0^{-2}(x)\delta_{ik}\delta_{jl}x^ix^jdx^idx^j, \]
\[ \sigma_0(x) = r + sl^{-2}\delta_{ij}x^ix^j. \] (4.2)

This is just a Robertson-Walker-like metric with the cosmic time. The Beltrami metric (3.14) of the flow can also be obtained from the Robertson-Walker-like metric (4.2) by replacing \( \tau \) with Beltrami time via Eq. (4.1).

It is clear that here the space foliation is a 3d sphere and other flat or open space foliations may also be taken for other physical requirements.

The metric of 3d simultaneous hyper-surface with respect to the same cosmic time reads
\[ dl_{r=\text{const}}^2 = R(\sqrt{s\tau/l}) dl_0^2 = \cosh^2 \left( \sqrt{s\tau/l} \right) dl_0^2. \] (4.3)

It is clear that for \( s > 0 \), metric (4.2) describes a Robertson-Walker-like \( dS \)-spacetime as an accelerated expanding 3d spherical cosmos with \( dl_0^2 \) as metric of a 3d sphere with radius \( ls^{-1/2} \). For the models with other space foliations, there are corresponding formulae, but the scalar factor \( R(\sqrt{s\tau/l}) \) is the same.

**B. Kinematics for the Universe at Cosmic Scale**

From the above analysis, it follows that the Poincaré-\( dS \) flow links the \( PoR_{cl} \) and the cosmological principle by changing the simultaneity. As was emphasized by Bondi [25], Bergmann [26] and Rosen [27] long ago and implied by Coleman, Grashow [28] and others recently, there is the puzzle between the \( PoR \) and the cosmological principle in Einstein’s theory of relativity. However, due to the extreme asymptotical behaviors of our universe, the flow is out of the puzzle.

More importantly, for the numbers \( \{r, s\} \in \{(1), (0, +\infty)\} \), the flow turns to the \( dS \) case with variable curvature radius \( ls^{-1/2} \). In fact, for the Robertson-Walker-like \( dS \) cosmos with a variable radius \( ls^{-1/2} \) for different \( s \), there is a horizon with Hawking temperature and non-gravitational entropy due to non-inertial effect [16]
\[ S = \pi l_0^2 s^{-1} c^3 k_B G \hbar = k_B \pi l_0^2 s^{-1} \ell_P^{-2}, \] (4.4)
where \( k_B \) is the Boltzmann constant and \( \ell_P \) the Planck length. Obviously, for different value of \( s \), it works at cosmic scale with an accelerated expanding spherical cosmos of radius \( R \) if \( ls^{-1/2} = R \simeq \sqrt{3/\Lambda} \).

If \( \Lambda \) is directly taken from observation without slightly variation, it gives rise to an accelerated expanding \( S^3 \) model of radius \( R \) and entropy \( S_R \)
\[ R \simeq (3/\Lambda)^{1/2} \sim 13.7 \text{ Gly}, \quad S_R = k_B G g^{-2}, \quad g^2 := (\ell_P/R)^2 \simeq 10^{-122}, \] (4.5)
where \( S_R \) may provide an upper entropy bound for our universe. It is also important that the entropy bound is independent of the space foliation [16]. Here an important dimensionless constant \( g \) appears. It should characterize the dimensionless gauge coupling in the gauge theory model of \( dS \) gravity [11, 17, 19, 32, 33].

Since the time-arrow of the universe should coincide with the arrow of cosmic time axis of the Robertson-Walker-like \( dS \) cosmos, which is a counterpart of the Beltrami-\( dS \) spacetime with Beltrami time axis related to the cosmic time axis by changing the Beltrami-time to the proper-time as the cosmic time, the evolution of our universe definitely indicate the existence of the Beltrami-frame of inertia[11, 19]. This manner for determination of the Beltrami-frame of inertia is completely different from that described by Einstein and is also away from Einstein's "an argument in a circle" for the \( PoR \) [2].

In addition, with different \( s \) the Robertson-Walker-like \( dS \) cosmos may also provide an inflationary phase near the Planck length \( \ell_P \) if

\[
ls^{-1/2} \simeq \ell_P, \quad S_{\ell_P} = k_B\pi,
\]

or at other scales as the GUT scale for the inflation model. For the Planck scale, it may provide kinematically a lowest entropy bound for the universe. Of course, other inflationary phase with different scale may also work with corresponding entropy bound. But, (4.6) is the lowest one. The relation between this inflationary phase and the inflation model should be studied further.

What is real kinematics for the universe? Since the cosmological constant \( \Lambda \) is positive, it must be the \( dS \) \( SR \).

In fact, our theoretical analysis should also conclude that this is the case even if the cosmological constant could be unknown. It is important that our universe is always with entropy that is increasing and roughly described symmetrically by the cosmological principle. In addition, its kinematics should be based on the \( PoR \) as well. Therefore, in the sense of the \( PoR_d \), the cosmological principle and the principle of increasing entropy, the \( dS \) \( SR \) with double \([\mathcal{D}_+, \mathcal{P}_2]_{D_+/M_+} \) should definitely provide new kinematics for our universe. While two other kinds of \( SR \) of Poincaré/AdS invariance cannot. Thus, the conventional relativistic physics works up to the bound (2.34) locally except for the both cosmic scale and near Planck scale.

In conclusion, the \( dS \) \( SR \) should at least describe physics kinematically at the cosmic scale with the entropy upper bound (4.5) for the accelerated expanding universe. It and may also work for the beginning of our universe near the Planck scale with the entropy lowest bound (4.6) of an inflationary phase or at other scale for an inflation.

Since the Poincaré-\( dS \) flow may allow the \( dS \) spacetime with different radius \( ls^{-1/2} \), it may also allow correspondingly to divide the observed cosmological constant \( \Lambda \) into a slightly variable part \( \Lambda_v \) and the fundamental one \( \Lambda_0 \). The latter should be related to the value of the universal invariant constant \( l \). Then one may simply take that \( \Lambda \) is \( \Lambda_v \) and \( \Lambda_0 \) relates to \( l \) only, so the (slightly) variable
part reads

$$\Lambda_v = \Lambda_s - \Lambda_0 = 3(s - 1)l^{-2}, \quad s \sim 1. \quad (4.7)$$

In order to determine the concrete values of two parts or equivalently the value of \( l \), one should refer to the detailed analysis on the observation data at least.

**V. CONCLUDING REMARKS**

The accelerated expanding universe strongly indicates that there are no longer Euclidean rigid ruler and ideal clock for physics at the cosmic scale characterized by the cosmological constant. As long as the Euclidean assumption for space and time is given up, Newton’s inertial motion is allowed may reach the projective infinity. Then, the inertial motion group \( IM(1, 3) \sim PGL(5, R) \) follows based on the \( PoR_{cl} \) and there are more candidates of metric geometry as space and time for kinematics.

With common Lorentz isotropy, there are three kinds of \( SR \) as a triple associated with the relativistic quadruple \( \mathcal{Q}_{PoR} = \{ \mathcal{P}_r, , \mathcal{D}_+, \mathcal{D}_- \}_{M/M_\pm/D_\pm} \). The quadruple can be parameterized as the Poincaré-dS flow \( \mathcal{F}_{r,s} \). Thus, three kinds of \( SR \) of Poincaré/dS/AdS-invariance associated with the dual Poincaré invariant degenerated spacetimes can be reached by the Poincaré-dS flow \( \mathcal{F}_{r,s} \) with a pair of numbers \( \{ r, s \} \) of different values. In fact, the radii of the \( dS/AdS \) spacetimes are allowed to be different as \( l|s|^{-1/2} \) and correspondingly the cosmological constant can also be changed as \( \Lambda_s \) like in Eq. (4.7).

For the flow, there are the parameterized Beltrami-time simultaneity and the parameterized inertial motions for a free particle with a parameterized Einstein’s formula. For different values of \( \{ r, s \} \), they give rise to the ones of the Poincaré/dual Poincaré/dS/AdS-invariance on \( M/M_\pm/D_\pm \), respectively. On the other hand, with respect to the parameterized propertime simultaneity of the flow, the Beltrami metric of the flow should turns to the Robertson-Walker-like one and vice versa. Thus, for the extremely asymptotical behavior of the universe, the flow gets ride of the puzzle between the \( PoR \) and the cosmological principle [25–28] and the evolution of our universe does fix all kinds of the inertial coordinate frames in the flow just like the case of the Beltrami-dS inertial frames [11].

The dual Poincaré kinematics can be studied more clearly as a member of the Poincaré-dS flow. Although there is no an independent meaningful 4d \( \mathcal{P}_2 \)-invariant kinematics, rather on a pair of degenerated Einstein manifolds \( M_\pm \) of \( \Lambda_\pm \), it always appears as an associated partner at the common origin on Minkowski/dS/AdS spacetime \( M/D_\pm \) for the space/time-like domain \( \hat{R}_\pm \) of the compact lightcone \( \hat{C}_O \) generated by the \( LFTs \) of dual Poincaré group \( \mathcal{P}_2 \) from the space/time-like region \( R_\pm \) of the lightcone \( C_O \) at the common origin, respectively.

In the sense that for the whole universe it holds the \( PoR_{cl} \), the cosmological principle and the principle of increasing entropy, the \( dS \) \( SR \) with double \( [\mathcal{D}_+, \mathcal{P}_2]_{D_+/M_+} \) should provide consistent
kinematics for the cosmic scale physics with the entropy upper bound (4.5). In addition, it may also work for physics near the Planck length with the entropy lowest bound (4.6) of an inflationary phase or at other scale. It is worthy while to study its relation to the inflation model further. Since the cosmological constant is very tiny, the conventional relativistic physics can still work up to the bound (2.34) locally except for the cosmic scale. It is clear that in order to determine the concrete values of a slightly variable part and the fundamental one of the cosmological constant or equivalently the value of $l$, the detailed analysis on the observation data are needed at least.

From the viewpoint of the PoR$_{cl}$, dynamics should coincide with kinematics. Actually, for Newton’s second law with an integral curve diffeomorphic to the projective straight line for the inertial motion of Newton’s first law, the symmetry of such kind of Newton’s second law is still the inertial motion group $IM(1, 3) \sim PGL(5, R)$. And gravity should be based on the local-globalization of kinematics. Namely, to localize the kinematic symmetry and its spacetime in patches first and to transit these patches together globally to form a kind of $1 + 3d$ manifolds, and to describe gravity with some gauge-like field equation characterized by the dimensionless coupling constant $g$ in Eq. (4.5) (see, e.g. [11, 17, 19, 32, 33]).

Although there are important results for the Poincaré-$dS$ flow and as was mentioned, the roles of the flow are still mainly in mathematics. In particular, as the number $s$ varies, it may realize the concrete procedures of different contractions of the universal constant $l$, although the constant $l$ is always invariant here. Thus, one may also introduce one more real number to realize the concrete procedures of the different contractions of $c$, i.e. $c \rightarrow \infty$ and $c \rightarrow 0$. Then the method employed and basic considerations here can be applied to other geometrical and non-relativistic kinematic symmetries.

Of course, more physical meaning of the flow with these numbers and above issues should be explored further.

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[1] A. Einstein, *Ann. Phys.* 17 (1905) 891.
[2] A. Einstein, *The Meaning of Relativity*, Princeton Univ. Press, 1966.
[3] N.A. Umow, *Physikalische Zeitschrift*, 11 (1910) 905.
[4] H. Weyl, *Mathemathische Analyse des Raumproblems*, Berlin, Springer, 1923.
[5] V. Fock, *The Theory of Space-Time and Gravitation*, Pergamon Press, 1964. And references therein.

[6] L.K. Hua (L.G. Hua) (1962), *Starting with the Unit Circle*, (A lecture notes), Scien. Press (1977); Springer 1981.

[7] L.K. Hua (1974), *Uniform velocity straight-line motions and projective geometry*. An unpublished manuscript.

[8] H.-Y. Guo, H.-T. Wu and B. Zhou, *Phys. Lett.* B670 (2009) 437-441.

[9] H.-Y. Guo, C.-G. Huang, H.-T. Wu and B. Zhou, *The Principle of Relativity, Kinematics and Algebraic Relations*, Sci. China Phys. Mech. Astr. (2010) 53 591-597, ArXiv: 0812.0871v3.

[10] H.-Y. Guo and H.-T. Wu, *The Principle of Relativity, Dual Poincaré Group and Relativistic Quadruple*, ArXiv: 0909.5497v3.

[11] H.-Y. Guo, *On principle of inertia in closed universe*, Phys. Lett. B653 (2007) 88, hep-th/0611341.

[12] K.H. Look (Q.K. Lu), *Why the Minkowski metric must be used?* (1970) An unpublished note.

[13] K.H. Look, C.L. Tsou (Z.L. Zou) and H.Y. Kuo (H.-Y. Guo), *Act. Phys. Sin.* 23 (1974) 225; *Nature* (Shanghai, Suppl.), Mod. Phys. 1 (1980) 97; H.Y. Kuo, *Kexue Tongbao* (Chin. Sci. Bull.) 22 (1977) 487; H.Y. Kuo, *Proc. of 2nd M. Grossmann Meeting on General Relativity*, ed. R. Ruffini, (North-Holland, 1982) 801; H.Y. Kuo, *Nucl. Phys. B* (Proc. Suppl.) 6 (1989) 381.

[14] See, e.g., G. Arcidiacono (1986), *Projective Relativity, Cosmology, and Gravitation*. Hadronic Press. And references therein.

[15] H.-Y. Guo, C.-G. Huang, Z. Xu and B. Zhou, *Mod. Phys. Lett.* A19 (2004) 1701-1710; *Phys. Lett.* A331 (2004) 1-7.

[16] H.-Y. Guo, C.-G. Huang, and B. Zhou, *Europhys. Lett.* 72 (2005) 1-7.

[17] H.-Y. Guo, C.-G. Huang, Y. Tian, H.T. Wu, Z. Xu and B. Zhou, *Snyder’s model – de Sitter special relativity duality and de Sitter gravity*, CQG 24 (2007) 4009, gr-qc/0703078.

[18] M.L. Yan, N.C. Xiao, W. Huang, S. Li, *Hamiltonian formalism of the de Sitter invariant special relativity*, Comm. Theor. Phys. (2007) 48 27, hep-th/0512319.

[19] H.-Y. Guo, *Special relativity and theory of gravity via maximum symmetry and localization — In honor of the 80th birthday of Professor Qikeng Lu*, Scien. China A51 (2008) 588-603; ArXiv: 0707.3855(gr-qc).

[20] E. Inönü and E.P. Wigner, *Proc. Nat. Acad. Sci.* 39 (1953) 510-524.

[21] H. Bacry and J.M. Lévy-Leblond, *J. Math. Phys.* 9 (1968) 1605.

[22] M. Gerstenhaber, *Ann. Math.* 79 (1964) 59.

[23] L.K. Hua, *Harmonic Analysis of Functions of Several Complex Variables in the Classical Domains*, Am. Math. Soc. (1963).

[24] L.K. Hua and Z.X. Wan, *Classic Groups*, Shanghai Scien. Tech. Press, 1963.

[25] H. Bondi, *Physics and cosmology*, Observatory (London), 82(1962) 133.

[26] P.G. Bergmann, *Cosmology as a Science*, Found. Phys. 1 (1970) 17.
[27] N. Rosen, *Theory of gravitation*, *Phys. Rev.* **D3** (1971) 2317.

[28] S.R. Coleman and S.L. Glashow, *Phys. Rev.* **D59**, 116008 (1999), hep-ph/9812418.

[29] C.-G. Huang, Y. Tian, X.N. Wu, Z.Xu and B. Zhou, arXiv:0909.2773v1 [gr-qc].

[30] H. Minkowski (1909), *Physikalische Zeitschrift* **10** 104-111. See also J.L. Synge, *Relativity: The Special Theory*, New York, Interscience (1956), p67.

[31] H.-Y. Guo, X.A. Ren, H.T. Wu and Y. Yu, in preparation.

[32] C.-G. Huang, Y. Tian, X.N. Wu and H.-Y. Guo, *On torsion-free vacuum solutions of the model of de Sitter gauge theory of gravity*, *Pront. Phys. China* **3** (2008) 191.

[33] C.-G. Huang, H.-Q. Zhang, H.-Y. Guo, *Cosmological solutions with torsion in a model of de Sitter gauge theory of gravity*, *JCAP* **10** (2008) 010, ArXiv:0801.0905.