On large $q$ expansion in the Sachdev-Ye-Kitaev model

Grigory Tarnopolsky

1Department of Physics, Harvard University, Cambridge MA 02138, USA

We consider the Sachdev-Ye-Kitaev (SYK) model where interaction involves $q$ fermions at a time. We find the next order correction to the thermal two-point function in the large $q$ expansion. Using this result we find the next order correction to the SYK free energy.

INTRODUCTION

The Sachdev-Ye-Kitaev (SYK) model is a quantum mechanical model of $N$ interacting Majorana fermions $\chi_i, i = 1, \ldots, N$ with the Hamiltonian [1, 2]:

$$H_{\text{SYK}} = (i)^{\frac{N}{2}} \sum_{1 \leq i_1 < i_2 < \cdots < i_N \leq N} J_{i_1 \cdots i_N} \chi_{i_1} \chi_{i_2} \cdots \chi_{i_N},$$

(1)

where $\{\chi_i, \chi_i \}$ = $\delta_{ij}$ and $J_{i_1 \cdots i_N}$ are random couplings drawn from a Gaussian distribution with zero mean and a width $\langle J_{i_1 \cdots i_N} \rangle = (q - 1)!J^2/Nq^{-1}$. One is usually interested in computing correlation functions, and particularly two-point function at temperature $T = 1/\beta$:

$$G(\tau) = \langle T \chi(\tau) \chi(0) \rangle_\beta,$$

(2)

At the large $N$ limit only melonic Feynman diagrams contribute to the two-point function in the SYK model. These diagrams can be resummed and one obtains a non-perturbative Schwinger-Dyson equation:

$$G(i\omega_n)^{-1} = -i\omega_n - \Sigma(i\omega_n), \quad \Sigma(\tau) = J^2 G(\tau)^{q-1},$$

(3)

where $G(i\omega_n) = \int_0^\beta d\tau e^{i\omega_n \tau} G(\tau)$ and $\omega_n = 2\pi\beta^{-1}(n + 1/2)$. It is not possible to solve this equation analytically, but one can find solution in the infrared limit, where $\omega$ is small and the bare $-i\omega_n$-term in (3) can be neglected [1, 2].

$$G_c(\tau) = b\left(\frac{\pi}{\beta \sin \frac{\pi}{q}}\right)^{2/q} \text{sgn}(\tau),$$

(4)

where $J^2b^2\pi = (1/2 - 1/q) \tan(\pi/q)$. Nevertheless it is still interesting to obtain some analytic approximation for $G(\tau)$ which interpolates both UV and IR regions. One way to proceed is to use the large $q$ expansion. The first order in $1/q$ was found in [3]. In this note we compute the next $1/q^2$ correction and argue that it improves the approximation significantly, such that it agrees with numerical results quite well.

At the next section we compute $1/q^2$ correction to the two-point function. Next we compare the large $q$ results and numerics. At the end we compute the large $q$ free energy and the coefficient of the Schwarzian action.

LARGE $q$ TWO-POINT FUNCTION

We consider the large $q$ ansatz for the two-point function [3]:

$$G(\tau) = \frac{1}{2} \text{sgn}(\tau) \left(1 + \frac{1}{q} g(\tau) + \frac{1}{q^2} h(\tau) + \ldots\right),$$

(5)

For the self-energy [3] we find (we assume that $q$ is even)

$$\Sigma(\tau) = \frac{J^2}{q} \text{sgn}(\tau) e^g \left(1 + \frac{1}{q} (h - g - \frac{1}{2} g^2) + \ldots\right),$$

(6)

where a new coupling constant $J^2 = 2^{1-q}qJ^2$ is introduced. From now on we work on the interval $\tau \in [0, \beta]$ and we can omit $\text{sgn}(\tau)$ in all formulas. Expanding $G(i\omega_n)^{-1}$ in $1/q$ series up to $1/q^2$ term using (4) we obtain

$$G(i\omega_n)^{-1} = -i\omega_n + \frac{1}{2q} \omega_n^2 g(i\omega_n)$$

$$+ \frac{\omega_n^2}{2q^2} \left(h(i\omega_n) + \frac{\omega_n}{q} (g(i\omega_n))\right),$$

(7)

where $g \ast g(i\omega_n) = \int_0^\beta d\tau e^{i\omega_n \tau} g^2(\tau)$. Then using the equations (3) and (4) and going back to the coordinate space we find differential equations for each order of $1/q$:

$$\partial^2_{\tau} g = 2\mathcal{J}^2 e^g,$$

$$\partial^2_{\tau} h = 2\mathcal{J}^2 e^h + \frac{1}{2} \partial^2_{\tau} (g \ast g) - 2\mathcal{J}^2 e^g (g + \frac{1}{2} g^2),$$

(8)

and the functions $g(\tau)$ and $h(\tau)$ satisfy the boundary conditions $g(0) = g(\beta) = 0$ and $h(0) = h(\beta) = 0$. Now we introduce a convenient variable $x = \frac{\pi \tau}{\beta} - \frac{\pi v}{\beta}$. Then the first equation has the solution

$$g(x) = \log \left(\frac{\cos \frac{\pi v}{\cos x}}{\cos \frac{\pi x}{\cos x}}\right)^2, \quad \beta \mathcal{J} = \frac{\pi v}{\cos \frac{\pi x}{2}}.$$

(9)

Using this solution the second equation can be represented as

$$\left(\partial^2_{x} - \frac{2}{\cos^2 x}\right) h(x) =$$

$$= -\frac{\pi v}{\beta} \frac{1}{2} \partial^2_{x} (g \ast g) - \partial^2_{x} g(x) (g(x) + \frac{1}{2} g^2(x)).$$

(10)

The solution to this equation can be written as

$$h(x) = -\int_{-\frac{\pi v}{\beta}}^{\frac{\pi v}{\beta}} dy G(x, y) \left(\frac{\pi v}{\beta} \frac{1}{2} \partial^2_{y} (g \ast g) \right.$$}

$$\left. + \partial^2_{y} g(y) (g(y) + \frac{1}{2} g^2(y))\right),$$

(11)
where the Green's function $\mathcal{G}(x, y)$ obeys the equation
\[
\left(\frac{\partial^2}{\partial x^2} - \frac{2}{\cos^2 x}\right)\mathcal{G}(x, y) = \delta(x - y)
\]
with the boundary conditions $\mathcal{G}(-\frac{\pi v}{2}, y) = \mathcal{G}(\frac{\pi v}{2}, y) = 0$. One can solve this equation and obtain an explicit formula for the Green's function
\[
\mathcal{G}(x, y) = \frac{1}{2V}(\tan x < (V + x < + 1)(\tan x > (V - x >) - 1),
\]
where $V \equiv \frac{\pi v}{2} + \cot \frac{\pi v}{2}$ and $x > \equiv \max(x, y)$ and $x < \equiv \min(x, y)$. Computing the convolution
\[
\frac{\pi v}{\beta} \frac{1}{2} \partial_x^3 (g \ast g) = 2 \partial_x \left( g(x) \left( \cot \left( \frac{\pi v}{2} + x \right) - \cot \left( \frac{\pi v}{2} - x \right) \right) \right) - 4,
\]
and using the explicit formula for the Green's function we obtain from (11)
\[
\mathcal{h}(x) = \frac{1}{2} g^2(x) - 2 \ell(x) - 4 \left( \tan x \int_0^\infty dy \ell(y) + 1 \right)
+ 4 \frac{1 + x \tan x}{1 + \frac{\pi v}{2} \tan \frac{\pi v}{2}} \left( \tan \frac{\pi v}{2} \int_0^{\pi v/2} dy \ell(y) + 1 \right),
\]
where $\ell(x) \equiv g(x) - e^{-g(x)} \text{Li}_2(1 - e^{g(x)})$ and $g(x)$ is given in [3]. One can compute explicitly the integral (formulas from [3] are useful)
\[
\int_0^{\pi v/2} dy \ell(y) = -\frac{\pi^2 v^2}{24 \cos^2 \frac{\pi v}{2}} \left( \pi v + 3 \sin \pi v \right).
\]

**COMPARISON WITH NUMERICAL RESULTS**

In this section we compare the large $q$ result with the numerical solution of the Schwinger-Dyson equation [3]. In general we expect the large $q$ formula to work well when $|g(\tau)| \ll q$ and $|h(\tau)| \ll q^2$. These inequalities are fulfilled when $\beta J \ll \pi e^{q/2}$.

Looking at the explicit formula it is tempting to exponentiate the result and to introduce an exponentiated large $q$ two-point function
\[
G(\tau) = \frac{1}{2} \text{sgn}(\tau) \exp \left( \frac{1}{q} g + \frac{1}{q^2} \left( h - \frac{1}{2} g^2 \right) \right),
\]
which is equivalent to [3] up to order $1/q^2$. We plot numerical and the large $q$ results for $q = 4$ and different values of $\beta J$ in figure [4]. We can see that the exponentiated result works very precisely even for large $\beta J$, whereas the large $q$ answer [3] deviates significantly from numerics at large $\beta J$.
**LARGE q FREE ENERGY**

The leading large $N$ approximation to the free energy in the SYK model is \[ [2, 3] \]

\[- \frac{\beta F}{N} = \log \mathcal{P}(\partial_\tau - \Sigma) \]

\[- \frac{1}{2} \int_0^\beta d\tau_1 d\tau_2 \left( \Sigma(\tau_{12}) G(\tau_{12}) - \frac{J^2}{q} G(\tau_{12})^2 \right). \quad (18) \]

To avoid evaluating the Pfaffian it is convenient to differentiate the free energy by $J \partial_J$ \[ [3] \]

\[ J \partial_J(-\beta F/N) = -\frac{\beta}{q} \partial_\tau G|_{\tau \to +0} \]

\[ = \frac{\pi v}{2} \left( \frac{1}{q^2} \partial_\tau g + \frac{1}{q^2} \partial_\tau h \right)|_{\tau \to +0}, \quad (19) \]

where from \[ [19] \] and \[ [15] \] we find

\[ \partial_\tau g|_{\tau \to +0} = 2 \tan \frac{\pi v}{2}, \]

\[ \partial_\tau h|_{\tau \to +0} = \frac{4}{1 + \frac{\pi v}{2} \tan \frac{\pi v}{2}} \left( \frac{\pi v}{2} - \tan \frac{\pi v}{2} (1 + \frac{\pi v}{2} \tan \frac{\pi v}{2}) - \int_0^{\pi v/2} dy \frac{\pi v}{2} \right). \quad (20) \]

Next, using \[ [16] \] and

\[ J \partial_J = \frac{v \partial_\tau}{1 + \frac{\pi v}{2} \tan \frac{\pi v}{2}}, \quad (21) \]

we can integrate back and obtain $-\beta F/N = \frac{1}{2} \log 2 + \frac{1}{q^2} F_1/q^2 + \frac{1}{q^4} F_1/q^4 + \ldots$, where

\[ F_1/q^2(v) = \pi v \left( \tan \frac{\pi v}{2} - \frac{\pi v}{2} \right), \]

\[ F_1/q^4(v) = \pi v \left( \tan \frac{\pi v}{2} - 2 \tan \frac{\pi v}{2} \left( 1 - \frac{\pi^2 v^2}{12} \right) \right). \quad (22) \]

Expanding the free energy at strong coupling by using that

\[ v = 1 - \frac{2}{\beta J} + \frac{4}{(\beta J)^2} - \frac{(24 + \pi^2)}{3(\beta J)^3} + \ldots, \quad (23) \]

we find

\[- \frac{\beta F}{N} = \beta J \left( \frac{1}{q^2} - \frac{12 - \pi^2}{6q^4} \right) + \frac{1}{2} \log 2 - \frac{\pi^2}{4q^2} + \frac{\pi^2}{3q^4} \]

\[ + \frac{1}{\beta J \left( \frac{\pi^2}{2q^2} + \frac{\pi^2 (\pi^2 + 12)}{12q^4} \right)} \quad + \ldots, \quad (24) \]

where the first three terms are the ground state energy, the zero-temperature entropy and the temperature dependent correction to the entropy. The zero temperature entropy coincides with the large $q$ expansion of the formula \[ [2, 3] \]

\[ S_0 = \frac{1}{2} \log 2 - \int_0^{1/q} dx \pi (\frac{1}{2} - x) \tan \pi x. \quad (25) \]

The last term in \[ (24) \] agrees with the formula reported in \[ [3, 10] \].

Using the result \[ (24) \] one can find the coefficient of the Schwarzian action. The Schwarzian action, which governs the low energy dynamics of the SYK model is given by the formula \[ [3, 11, 14] \]

\[ S = -N \frac{\alpha_S}{J} \int d\tau \{ f, \tau \} = \frac{f''}{f} - \frac{3}{2} \left( \frac{f''}{f} \right)^2, \quad (26) \]

where the coefficient $\alpha_S$ depends on $q$. This coefficient is related to the finite temperature correction to the free energy, so at large $q$ using \[ (24) \] we find

\[- \frac{\beta F}{N} \geq \frac{2\pi^2 \alpha_S}{\beta J} \geq \frac{1}{\beta J} \left( \frac{\pi^2}{2q^2} - \frac{\pi^2 (\pi^2 + 12)}{12q^4} + \ldots \right). \quad (27) \]

At $q = 2$ one has $\alpha_S = \frac{1}{24\pi}$. Using asymptotics for $\alpha_S$ at $q = \infty$ and $q = 2$

\[ \alpha_S(q) = \begin{cases} \frac{1}{24\pi} + \ldots, & q \to 2 \\ \frac{1}{24\pi} - \frac{\pi^2 + 12}{24q^2} + \ldots, & q \to \infty \end{cases} \quad (28) \]

we obtain two-sided Pade approximant:

\[ \text{Pade}_{[3,1]}: \quad \alpha_S(q) = \frac{\pi^2 - 18\pi + 24 + 3(3\pi - 2)q}{6q^2 (\pi^3 + 8 + 2(3\pi - 2)q)}. \quad (29) \]

We note that one can improve approximation by using more terms near $q = 2$ \[ [8] \]. We plotted Pade approximation and numerical results adapted from \[ [3] \] in figure \[ 2 \]. We see that the Pade approximation is very close to numerics.

![FIG. 2. (Color online) Plot of $\alpha_S$ as a function of $q$. The black circles correspond to numerical results adapted from \[ [3] \]. The blue solid line corresponds to the two-sided Pade approximation \[ [29] \).](image-url)
CONCLUSIONS

It would be interesting to generalize the result of this article to other SYK-type models, discussed in [15–18]. Especially it would be interesting to compute the thermalization time using large $q$ solution for the SYK models discussed in [19].

It is also interesting to develop $1/q$ expansion for the higher dimensional SYK models [20–24] where the stability of the large $N$ limit is unclear.

The large $q$ approximation to the two-point function can be used as well in studying tensor models [25–27]. Even though the general $q$ melonic tensor interaction have some ambiguities [27, 28], one can just formally consider large $q$ generalization of the Schwinger-Dyson equation.

G.T. would like to thank Yingfei Gu, Igor Klebanov, Subir Sachdev and Douglas Stanford for useful comments. Also G.T. thanks Douglas Stanford for providing numerical results for $\alpha_S$ from [2]. This research was supported by the MURI grant W911NF-14-1-0003 from ARO and by DOE grant de-sc0007870.

[1] S. Sachdev and J. Ye, Phys. Rev. Lett. 70, 3339 (1993), arXiv:cond-mat/9212030 [cond-mat]
[2] A. Kitaev, Talks at KITP, April 7, 2015 and May 27, 2015.
[3] J. Maldacena and D. Stanford, Phys. Rev. D94, 106002 (2016), arXiv:1604.07818 [hep-th]
[4] O. Parcollet and A. Georges, Phys. Rev. B 59, 5341 (1999)
[5] A. I. Davydychev and M. Yu. Kalmykov, Nucl. Phys. B699, 3 (2004) arXiv:hep-th/0303162 [hep-th]
[6] S. Sachdev, Phys. Rev. X5, 041025 (2015), arXiv:1506.05111 [hep-th]
[7] A. Georges, O. Parcollet, and S. Sachdev, Phys. Rev. B 63, 134406 (2001)
[8] J. S. Cotler, G. Gur-Ari, M. Hanada, J. Polchinski, P. Saud, S. H. Shenker, D. Stanford, A. Streicher, and M. Tezuka, JHEP 05, 118 (2017), arXiv:1611.04650 [hep-th]
[9] A. Jevicki and K. Suzuki, JHEP 11, 046 (2016)
[10] A. Kitaev and S. J. Suh, (2017), arXiv:1711.08467 [hep-th]
[11] J. Maldacena, D. Stanford, and Z. Yang, PTEP 2016, 12C104 (2016), arXiv:1606.01857 [hep-th]
[12] J. Engelsy, T. G. Mertens, and H. Verlinde, JHEP 07, 139 (2016) arXiv:1606.03438 [hep-th]
[13] K. Jensen, Phys. Rev. Lett. 117, 111601 (2016) arXiv:1605.06098 [hep-th]
[14] A. Jevicki, K. Suzuki, and J. Yoon, JHEP 07, 007 (2016), arXiv:1603.06246 [hep-th]
[15] D. J. Gross and V. Rosenhaus, JHEP 02, 093 (2017), arXiv:1610.01569 [hep-th]
[16] R. A. Davison, W. Fu, A. Georges, Y. Gu, K. Jensen, and S. Sachdev, Phys. Rev. B95, 155131 (2017), arXiv:1612.00849 [cond-mat.str-el]
[17] Y. Gu, X.-L. Qi, and D. Stanford, JHEP 05, 125 (2017), arXiv:1609.07832 [hep-th]
[18] W. Fu, D. Gaiotto, J. Maldacena, and S. Sachdev, Phys. Rev. D95, 026009 (2017), arXiv:1610.08917 [hep-th]
[19] A. Eberlein, V. Kasper, S. Sachdev, and J. Steinberg, Phys. Rev. B96, 205123 (2017), arXiv:1706.07803 [cond-mat.str-el]
[20] M. Berkoov, P. Narayan, M. Rozali, and J. Simm, JHEP 01, 138 (2017), arXiv:1610.02422 [hep-th]
[21] G. Turiaci and H. Verlinde, (2017), arXiv:1701.00528 [hep-th]
[22] J. Murugan, D. Stanford, and E. Witten, (2017), arXiv:1706.05362 [hep-th]
[23] S. Giombi, I. R. Klebanov, and G. Tarnopolsky, Phys. Rev. D96, 106014 (2017), arXiv:1707.03866 [hep-th]
[24] S. Prakash and R. Sinha, (2017), arXiv:1710.03935 [hep-th]
[25] E. Witten, (2016), arXiv:1610.09758 [hep-th]
[26] R. Gurau, Nucl. Phys. B916, 386 (2017), arXiv:1611.04032 [hep-th]
[27] I. R. Klebanov and G. Tarnopolsky, Phys. Rev. D95, 046004 (2017), arXiv:1611.08915 [hep-th]
[28] C. Jepsen, Private communications.