List covering of regular multigraphs

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Abstract. A graph covering projection, also known as a locally bijective homomorphism, is a mapping between vertices and edges of two graphs which preserves incidences and is a local bijection. This notion stems from topological graph theory, but has also found applications in combinatorics and theoretical computer science. It has been known that for every fixed simple regular graph $H$ of valency greater than 2, deciding if an input graph covers $H$ is NP-complete. In recent years, topological graph theory has developed into heavily relying on multiple edges, loops, and semi-edges, but only partial results on the complexity of covering multigraphs with semi-edges are known so far. In this paper we consider the list version of the problem, called LIST-$H$-COVER, where the vertices and edges of the input graph come with lists of admissible targets. Our main result reads that the LIST-$H$-COVER problem is NP-complete for every regular multigraph $H$ of valency greater than 2 which contains at least one semi-simple vertex (i.e., a vertex which is incident with no loops, with no multiple edges and with at most one semi-edge). Using this result we almost show the NP-co/polytime dichotomy for the computational complexity of LIST-$H$-COVER of cubic multigraphs, leaving just five open cases.

1 Introduction

Graph covering projections and related notions. For simple graphs $G$ and $H$, a covering projection from $G$ to $H$ is a mapping $f : V(G) \cup E(G) \to V(H) \cup E(H)$, such that (i) vertices are mapped to vertices and edges are mapped to edges, (ii) incidencies are retained, and (iii) $f$ is bijective in the neighborhood of each vertex. The last condition means that if for some $v \in V(G)$ and $x \in V(H)$ we have $f(v) = x$, then for each edge $e$ of $H$ containing $x$ there must be exactly one edge containing $v$ that is mapped to $e$. For a fixed graph $H$, in the $H$-COVER problem we ask if an instance graph $G$ admits a covering projection to $H$.

The notion of a graph covering projection, as a natural discretization of the covering projection used in topology, originates (not surprisingly) in topological graph theory. However, since then it has found numerous applications
elsewhere. Covering projections were used for constructing highly symmetrical graphs [4, 9, 22, 25], embedding complete graphs in surfaces of higher genus [39], and for analyzing a model of local computations [2].

Graph covering projections are also known as locally bijective homomorphisms and as such they fall into a family of locally constrained homomorphisms. Other problems from this family are locally surjective and locally injective graph homomorphisms, where we ask for the existence of a homomorphism that is, respectively, surjective or injective in the neighborhood of each vertex. Locally surjective homomorphisms play an important role in social sciences [20] (there this problem is called the Role Assignment Problem). On the other hand, a prominent special case of the locally injective homomorphism problem is the well-studied $L(2, 1)$-labeling problem [24] and, more generally, $H(p, q)$-coloring [15, 31].

Computational complexity. The complexity of finding locally constrained homomorphisms was studied by many authors. For locally surjective homomorphisms we know a complete dichotomy [20]. The problem is polynomial-time solvable if the target graph $H$ either (a) has no edge, or (b) has a component that consists of a single vertex with a loop, or (c) is simple and bipartite, with at least one component isomorphic to $K_2$. In all other cases the problem is NP-complete.

The dichotomy for locally injective homomorphisms is still unknown, despite some work [12, 18]. However, we understand the complexity of the list variant of the problem [13]: it is polynomial-time solvable if every component of the target graph has at most one cycle, and NP-complete otherwise.

To the best of our knowledge, Abello et al. [1] were the first to ask about the computational complexity of $H$-Cover. Note that in order to map a vertex of $G$ to a vertex of $H$, they must be of the same degree, a natural interesting special case is when $H$ is regular. It is known that for every $k \geq 3$, the $H$-Cover problem is NP-complete for every simple $k$-regular graph $H$ [28, 19]. Some other partial results are known, mostly focusing on small graphs $H$ [14, 29, 30]. Let us point out that in all the above results it was assumed that $H$ has no multiple edges.

Recall further that there is also some more work concerning the complexity of locally surjective and injective homomorphisms if $G$ is assumed to come from some special class [3, 5, 8, 10, 38]. We also refer the reader to the survey concerning various aspects of locally constrained homomorphisms [19].

(Multi)graphs with semi-edges. In the course of development of topological graph theory, it became standard to consider loops and multiedges, but recently also semi-edges are playing a more and more important role. Intuitively, a semi-edge (sometimes also called a half-edge or a fin) is an edge with just one end (this is in contrast with a loop, which has two ends, both in the same vertex). To name just a few most significant examples, Malnič et al. [34] considered semi-edges during their study of abelian covers to allow for a broader range of applications. Furthermore, the concept of graphs with semi-edges was introduced independently and naturally in mathematical physics [23]. It is also natural to consider semi-edges in the mentioned framework of local computations (we refer to the introductory section of [6] for more details). Finally, a well-known
Leighton’s theorem on finite common covers has been recently generalized to the semi-edge setting in. To highlight a few other contributions, the reader is invited to consult, the surveys, and finally for more recent results, the series of papers and the introductions therein. From now on, when we talk about graphs, we allow multiple edges, loops and semi-edges without explicitly stating it.

The complexity study of \( H \)-Cover for graphs \( H \) that allow semi-edges has been initiated only very recently in. We continue this line of research. In particular, our far-reaching goal is to prove the following conjecture.

**Strong Dichotomy Conjecture.** For every \( H \), the \( H \)-Cover problem is either polynomial-time solvable for general graphs, or NP-complete for simple graphs.

**Our results.** The goal of this paper is to push further the understanding of the complexity of \( H \)-Cover for regular graphs. Recall that the problem is known to be NP-complete for every fixed \( k \)-regular simple graph \( H \) of valency \( k \geq 3 \). Though it was known already from that in order to fully understand the complexity of covering general simple graphs, it is necessary (and sufficient) to prove a complete characterization for colored mixed multigraphs, the result of was formulated and proved only for simple graphs. In this paper we revisit the method developed in and we conclude that though it does not seem to work for multigraphs in general, it is possible to modify it and – under certain assumptions – prove hardness for the list variant of the problem, \( \text{List-} \) \( H \)-Cover, where the vertices and edges of the instance graph are given lists of admissible targets. Our main result is the following theorem (a vertex is semi-simple if it belongs to no loops nor multiple edges, and is incident to at most one semi-edge).

**Theorem 1.** Let \( k \geq 3 \) and let \( H \) be a \( k \)-regular graph. If \( H \) contains a semi-simple vertex, then \( \text{List-} \) \( H \)-Cover is NP-complete for simple input graphs.

We do believe that the Strong Dichotomy Conjecture holds true for \( \text{List-} \) \( H \)-Cover.

The second goal of the current paper is to show how Theorem could be used to prove the Strong Dichotomy Conjecture for cubic graphs. Recall that for the closely related locally injective homomorphism problem, introducing lists was helpful in obtaining the full complexity dichotomy. In Theorem we fully characterize the computational complexity of \( \text{List-} \) \( H \)-Cover for almost all cubic graphs, and identify just five exceptionally stubborn graphs \( H \) for which the complexity of the problem is still open.

2 Preliminaries

In the sequel, a graph is allowed to have loops, multiple edges and semi-edges, and all these objects are referred to as edges. Edges are thus distinguished to be of three types: ordinary edges that are incident with two distinct vertices, loops that have two ends, both in the same vertex, and semi-edges that have
only one end. By saying that we allow multiedges we mean that our graph may have more edges with the same set of endpoints (so we may have multiple loops at the same vertex, multiple semi-edges at the same vertex, or multiple ordinary edges incident with the same pair of vertices). Given a graph \(G\) and a vertex \(u \in V(G)\), the set of edges of \(G\) incident with \(u\) will be denoted by \(E_G(u)\).

The degree (or valency) of a vertex \(u\) is the number of edge endpoints equal to \(u\). In particular, each ordinary edge and each semi-edge contribute 1 to the degree of each of its vertices, and each loop contributes 2. A graph is regular if all of its vertices have the same degree. We further say that:

– a vertex is semi-simple if it belongs to no loops, no multiple edges and at most one semi-edge,
– a graph is semi-simple if each of its vertices is semi-simple,
– a vertex is simple if it is semi-simple and is incident with no semi-edges,
– a graph is simple if each of its vertices is simple,

– a graph is bipartite if it has no loops, no semi-edges and no odd cycles.

Given graphs \(G\) and \(H\), a mapping \(f : V(G) \cup E(G) \rightarrow V(H) \cup E(H)\) is a graph covering projection if vertices of \(G\) are mapped onto vertices of \(H\), edges of \(G\) are mapped onto edges of \(H\) so that incidences are retained, and in such a way that the preimage of a loop is a disjoint union of cycles spanning the preimage of the vertex incident with the loop, the preimage of a semi-edge is a disjoint union of semi-edges and ordinary edges spanning the preimage of the vertex incident with this semi-edge, and the preimage of an ordinary edge is a matching spanning the preimage of the two vertices incident with this edge.

The computational problem of deciding whether an input graph \(G\) covers a fixed graph \(H\) is denoted by \(H\)-Cover.

The mapping \(f : V(G) \cup E(G) \rightarrow V(H) \cup E(H)\) is a partial covering projection when the preimages are not required to be spanning subgraphs, but all other properties are fulfilled, i.e., the vertex- and edge-mappings are both surjective and the incidences are retained, the preimage of an ordinary edge connecting vertices say \(u\) and \(v\) is a matching consisting of edges each connecting a vertex from \(f^{-1}(u)\) to a vertex from \(f^{-1}(v)\), the preimage of a semi-edge incident with vertex \(u\) is a disjoint union of semi-edges and ordinary edges all incident only with vertices from \(f^{-1}(u)\), and the preimage of a loop incident with a vertex \(u\) is a disjoint union of cycles (including loops) and paths whose all edges are incident only with vertices from \(f^{-1}(u)\).

In the List-\(H\)-Cover problem the input graph \(G\) is given with lists \(\mathcal{L} = \{L_u, L_e : u \in V(G), e \in E(G)\}\), such that \(L_u \subseteq V(H)\) for every \(u \in V(G)\) and \(L_e \subseteq E(H)\) for every \(e \in E(G)\). A covering projection \(f : G \rightarrow H\) respects the lists of \(\mathcal{L}\) if \(f(u) \in L_u\) for every \(u \in V(G)\) and \(f(e) \in L_e\) for every \(e \in E(G)\).

3 Proof of Theorem 1

In the first two subsections we will prove the theorem for the case when \(H\) is bipartite (and hence does not contain loops) and has no semi-edges. By the celebrated König-Hall theorem, such a graph is \(k\)-edge-colorable.
3.1 Multicovers

The following construction will be used to build gadgets for the hardness proof.\footnote{The proofs of statements marked with (♠) will appear in the journal version of the paper.}

**Proposition 1 (♠).** Let $H$ be a connected $k$-regular $k$-edge-colorable graph with no loops or semi-edges. Let $x, y$ be two adjacent vertices of $H$. Then there exists a connected simple $k$-regular $k$-edge-colorable graph $G$ and $u \in V(G)$, such that

(a) for any bijection from $E_G(u)$ onto $E_H(x)$, there exists a covering projection from $G$ to $H$ which extends this bijection and maps $u$ to $x$, and

(b) for any bijection from $E_G(u)$ onto $E_H(y)$ there exists a covering projection from $G$ to $H$ which extends this bijection and maps $u$ to $y$.

The main building block of our reduction is the graph $G_u$ obtained from $G$ by splitting vertex $u$ into $k$ pendant vertices of degree 1. For each edge $e$ of $G$ incident with $u$, we formally keep this edge with the same name in $G_u$, denote its pendant vertex of degree 1 by $u_e$ and denote by $w_e$ the other endpoint of $e$. (Thus, with this slight abuse of notation, $E_G(u) = \bigcup_{e \in E_G(u)} E_{G_u}(u_e)$.) Then we have the following proposition.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig1.png}
\caption{An illustration to the construction of $G_u$.}
\end{figure}

**Proposition 2 (♠).** The graph $G_u$ constructed from the multicover $G$ of $H$ as above satisfies the following:

(a) for every bijection $\sigma_x : E_G(u) \rightarrow E_H(x)$, there exists a partial covering projection of $G_u$ onto $H$ that extends $\sigma_x$ and maps each $u_e, e \in E_G(u)$ to $x$;

(b) for every bijection $\sigma_y : E_G(u) \rightarrow E_H(y)$, there exists a partial covering projection of $G_u$ onto $H$ that extends $\sigma_y$;

(c) in every partial covering projection from $G_u$ onto $H$, the pendant vertices $u_e, e \in E_G(u)$ are mapped onto the same vertex of $H$;

(d) in every partial covering projection from $G_u$ onto $H$, the pendant edges are mapped onto different edges (incident with the image of the pendant vertices).

3.2 Reduction from hypergraph coloring

The reduction is exactly the same as in [28], but the proof for the case when multiple edges are allowed needs some extra analysis. Hence we need to describe
the reduction in full detail in here. We reduce from $k$-edge-colorability of $(k-1)$-uniform $k$-regular hypergraphs. In the wording of the incidence graph of the hypergraph, suppose we are given a simple bi-regular bipartite graph $K = (A \cup B, E)$ such that all vertices in $A$ (which represent the edges of the hypergraph) have degree $k - 1$ and all vertices in $B$ (which represent the vertices of the hypergraph) have degree $k$. The question is if the vertices of $A$ can be colored by $k$ colors so that the neighborhood of each vertex from $B$ is rainbow colored (i.e., each vertex from $B$ sees all $k$ colors on its neighbors, each color exactly once). This problem is NP-complete for every fixed $k \geq 3$ [28].

Given such a graph $K$, we build an input graph $G_K$ by local replacements. Recall that we are working with a $k$-edge-colorable $k$-regular graph $H$ with a simple vertex $x$, and with respect to this vertex (and one of its neighbors $y$) we are guaranteed the existence of a graph $G_u$ which satisfies the properties stated in Proposition 2. This $G_u$ will be a key building block in our construction.

First, every vertex $v \in B$ will be replaced by a copy of the so-called vertex gadget, which is a disjoint union of a copy $G_v^u$ of $G_u$ and a single vertex $B_v$. For every neighbor $a \in A$ of $v$, one of the pendant vertices of $G_v^u$ will be denoted by $u^{va}$, and its neighbor within $G_v^u$ will be denoted by $w^{va}$.

Fig. 2. An illustration to the construction of $G_K$ for $k = 4$. 
The hyperedge gadgets used to replace the vertices of \( A \) are more complicated. This gadget consists of \( 2(k-1) \) copies of \( G_u \) linked together in the following way. Let \( a \in A \). We take \( 2(k-1) \) vertices \( \ell_i^a, r_i^a, i = 1, 2, \ldots, k-1 \), and for every neighbor \( v \) of \( a \), we take two copies \( G_L^a \) and \( G_R^a \) of \( G_u \). The pendant vertices of \( G_L^a \) will be unified with \( B_v \) and \( \ell_1^a, \ell_2^a, \ldots, \ell_{k-1}^a \), while the pendant vertices of \( G_R^a \) will be unified with \( w^a \) and \( r_1^a, r_2^a, \ldots, r_{k-1}^a \). The neighbor of \( B_v \) in \( G_L^a \) will be denoted by \( z^a \). Lastly, the matching \( \ell_i^a r_i^a, i = 1, 2, \ldots, k-1 \) is added.

The resulting graph \( G_K \) is \( k \)-regular. To make it an instance of the \textsc{List-H-Cover} problem, we prescribe that the vertices \( B_v, v \in B \) and \( \ell_i^a, a \in A, i = 1, 2, \ldots, k-1 \) are all mapped onto \( x \) (this means, that for these vertices, their lists of admissible target vertices are one-element and all the same, while for the remaining vertices, their lists are full, as well as for all the edges).

The fact that \( x \) is simple implies the following observation: For every partial covering projection from \( G_u \) to \( H \) which maps all the pendant vertices onto \( x \), their neighbors in \( G_u \) are mapped onto distinct vertices of \( H \) (this immediately follows from property (d) of Proposition 2). Similarly, if any vertex of \( G_K \) is mapped onto \( x \) by a covering projection, then its neighbors are mapped onto distinct vertices of \( H \) (the neighborhood \( N_H(x) \) of \( x \) in \( H \)). We will exploit these observations in the following argumentation.

Suppose \( f : G_K \to H \) is a covering projection such that all vertices \( B_v, v \in B \) and \( \ell_i^a, a \in A, i = 1, 2, \ldots, k-1 \) are mapped onto \( x \). Consider an \( a \in A \), and let \( f(r_i^a) = y \), whence \( y \in V(H) \) is a neighbor of \( x \) in \( H \). Property c) applied to any \( G_L^a \), for \( v \) being a neighbor of \( a \) in \( K \), implies that \( f(r_i^a) = f(w^a) = y \) for all \( i = 2, \ldots, k-1 \). Since each \( \ell_i^a \) has a neighbor \( r_i^a \) mapped onto \( y \), none of their neighbors in \( G_L^a \) is mapped onto \( y \). Property (d) then implies that \( f(z^a) = y \).

Define a coloring \( \phi \) of \( A \) by colors \( N_H(x) \) as \( \phi(a) = f(r_i^a) \). Consider a vertex \( v \in B \). The neighbors of \( B_v \) in \( G_K \) are \( z^a, a \in N_K(v) \). Since \( f(B_v) = x \) and \( x \) is simple, the vertices \( z^a, a \in N_K(v) \) are mapped onto different neighbors of \( x \) by \( f \), and hence the colors \( \phi(a), a \in N_K(v) \) are all distinct. Thus \( \phi \) is a \( k \)-coloring of \( A \) of the required property.

Suppose for the opposite direction that \( A \) allows a \( k \)-coloring \( \phi \) such that each vertex \( v \in B \) sees all \( k \) colors on its neighbors, and identify the colors with the names of the neighbors of \( x \) in \( H \). Set

\[
\begin{align*}
f(B_v) &= \ell_i^a = x \text{ for all } v \in B, a \in A, i = 1, 2, \ldots, k-1 \text{ (as required by the lists)}, \\
f(u^a) &= x \text{ for all } v \in B \text{ and } a \in N_K(v), \text{ and} \\
f(r_i^a) &= f(w^a) = f(z^a) = \phi(a) \text{ for all } a \in A \text{ and } v \in N_K(a).
\end{align*}
\]

and define \( f \) on the edges incident to \( \ell_i^a \) (\( r_i^a \), respectively) so that for every \( i \), these edges are mapped onto different edges incident to \( x \) (to \( \phi(a) \), respectively), and on the other hand for every \( v \in N_K(a) \), the pendant edges of \( G_L^a \) (of \( G_R^a \), respectively) are mapped onto distinct edges incident to \( x \) (to \( \phi(a) \), respectively). (This is a simple exercise.) The properties (a) and (b) of Proposition 2 imply that this mapping can be extended to partial covering projections within each copy of \( G_u \) used in the construction of \( G_K \). To see that they altogether provide a covering projection from \( G_K \) to \( H \), note that for each \( v \in B \), the edges incident
with the vertex $B_v$ are mapped onto different edges because their other endpoints are $\phi(a), a \in N_K(v)$, and hence all different by the assumption on the coloring $\phi$, and also each copy $G_v^u$ has its pendant edges mapped onto different edges incident to $x$, since the pendant vertices $u^{va}, a \in N_K(v)$ are all mapped onto $x$ and their neighbors $w^{va}$ in $G_v^u$ are mapped onto distinct vertices $\phi(a), a \in N_K(v)$. This concludes the proof of the case of bipartite $H$.

3.3 The non-bipartite case

Suppose the graph $H$ is not $k$-edge-colorable (this includes the case when $H$ contains loops and/or semi-edges). Consider $H' = H \times K_2$. This $H'$ may still contain multiple edges (the product of a multiple ordinary edge with $K_2$ is again a multiple edge, but also the product of a loop with $K_2$ is a double ordinary edge, and the product of a multiple semi-edge with $K_2$ results in a multiple ordinary edge as well), but it is bipartite (and thus has neither semi-edges nor loops) and therefore is $k$-edge-colorable. In the product with $K_2$, every semi-simple vertex of $H$ results in two simple vertices of $H'$. Hence, by the result of the preceding subsection, List-$H'$-Cover is NP-complete.

It is proved in [19] that for simple graphs, $G$ covers $H \times K_2$ if and only if $G$ is bipartite and covers $H$. This proof readily extends to graphs that allow loops, semi-edges and multiple edges. The proof for the list version of the problem may get more complicated in general. However, the list version that we have proven NP-complete in the preceding subsection is very special: the lists of all edges are full, and so are the lists of all the vertices except for those which are prescribed to be mapped onto the same simple vertex, say $x'$. If we take such an instance of List-$H'$-Cover, this $x'$ is a copy of a semi-simple vertex $x \in V(H)$, and all vertices of the input graph $G$ that are prescribed to be mapped onto $x'$ are from the same class of its bipartition. We just prescribe them to be mapped onto $x$ as an instance of List-$H$-Cover. It is easy to see that this mapping can be extended to a covering projection to $H$ if and only if $G$ allows a covering projection to $H'$ in which all these prescribed vertices are mapped onto $x'$. This concludes the proof of Theorem 1.

4 Sausages and rings

In this section we consider two special classes of cubic graphs. These graphs play a special role in the classification in Theorem 4. The $k$-ring is the cubic graph obtained from the cycle of length $2k$ by doubling every second edge. We call a $k$-sausage every cubic graph that is obtained from a path on $k$ vertices by doubling every other edge and adding loops or semi-edges to the end-vertices of the path to make the graph 3-regular. Note that while for every $k$, the $k$-ring is defined uniquely, there are several types of $k$-sausages, as depicted in Fig. 3.

**Proposition 3.** For every $k \geq 2$, let $S_k$ be a $k$-sausage. Then $S_k \times K_2$ is isomorphic to the $k$-ring.
Proof. The product $H \times K_2$ is a bipartite graph with no loops or semi-edges, in which every ordinary edge in $H$ gives rise to a pair of ordinary edges of the same multiplicity, a loop, or a pair of semi-edges incident to the same vertex of $H$ gives rise to a double ordinary edge, and a single semi-edge in $H$ gives rise to a simple ordinary edge in $H \times K_2$. Thus $S_k \times K_2$ has a cyclic structure and the number of double edges is equal to the number of vertices of $S_k$, see Fig. 4.

In the following two theorems we show that for every $k \neq 4$, the List-$k$-ring-Cover problem is NP-complete in simple graphs.

**Theorem 2 (♠).** The $k$-ring-Cover problem is NP-complete for simple input graphs for every $k = 2^\alpha (2\beta + 3)$ such that $\alpha$ and $\beta$ are non-negative integers.

**Theorem 3 (♠).** The List-$k$-ring-Cover problem is NP-complete for simple input graphs for every $k = 2^\alpha$ such that $\alpha \geq 3$ is an integer.

The following observation shows that hardness for $k$-rings implies the hardness for $k$-sausages.

**Proposition 4 (♠).** For every $k \geq 2$ and every $k$-sausage $S_k$, $k$-ring-Cover $\propto S_k$-Cover and List-$k$-ring-Cover $\propto$ List-$S_k$-Cover.
Proof. A graph $G$ covers $H \times K_2$ if and only if it is bipartite and covers $H$. Since bipartiteness can be tested in polynomial time, testing if $G$ covers the $k$-ring polynomially reduces to testing if $G$ covers $S_k$.

The proof for the list version is a bit more complicated and we defer it to the journal version of the paper.

5 Towards Strong Dichotomy for cubic graphs

In this section we are getting close to proving the Strong Dichotomy Conjecture for cubic graphs.

Theorem 4. Let $H$ be a connected cubic graph which is neither the 4-ring nor a 4-sausage. Then List-$H$-Cover is polynomial-time solvable for general graphs when $H$ has only one vertex and at most one semi-edge, and it is NP-complete even for simple input graphs otherwise.

Proof. The proof is divided into several cases, depending on the structure of $H$.

Case 1: $|V(H)| = 1$. We distinguish two subcases.

Case 1A - $H$ has one semi-edge and one loop. The preimage of the semi-edge should be a disjoint union of the semi-edges of the input graph $G$ and of a perfect matching on the vertices not incident to a semi-edge. Then the remaining edges of $G$ form a spanning collection of cycles (including loops) which form the preimage of the loop. The existence of a spanning subgraph of $G$ that is a preimage of the semi-edge can be tested in polynomial time.

If lists are present as part of the input, the situation gets a little more tricky. We start with a preprocessing phase. We check the below conditions:

(a) $G$ has a vertex or an edge with an empty list.
(b) $G$ has a vertex incident to two or more semi-edges,
(c) $G$ has a semi-edge whose list does not contain the semi-edge of $H$,
(d) $G$ has a vertex incident to a semi-edge and an edge, whose list does not contain the loop of $H$,
(e) $G$ has a vertex incident to two ordinary edges, whose lists do not contain the loop of $H$,
(f) $G$ has a loop whose list does not contain the loop of $H$.

It is clear that if any of the above conditions is satisfied, then $(G, \mathcal{L})$ is a no-instance. Thus we reject and quit.

Now we shall construct an auxiliary graph $G'$. We start our construction with $G$ and perform the following steps.

1. If some vertex $v$ is incident to a semi-edge, then delete $v$ with all its edges.
2. If some edge $e$ does not have the semi-edge of $H$ in its list, remove $e$ from the graph.
3. If some edge $e$ does not have the loop of $H$ in the list, leave $e$, but remove all edges incident to $e$. 
Let $G'$ be the graph after the exhaustive application of steps 1, 2, and 3. It is straightforward to verify that steps 1 and 2 ensure that the union of a perfect matching in $G'$ and the semi-edges removed in step 1 can be a preimage of the semi-edge of $H$. Furthermore, by step 3 we ensure that if some edge has to be mapped to the semi-edge, then it will be so.

We can verify in polynomial time if $G'$ has a perfect matching. If not, we reject and quit. So let $M$ be a perfect matching in $G'$, and let $M'$ be the union of $M$ and the set of semi-edges removed in step 1. Observe that the graph $G-M$ is 2-regular, i.e., is a disjoint union of cycles (including loops). Furthermore, every edge of $G - M$ has the loop of $H$ in its list, this is guaranteed by step 3 and the preprocessing phase. Thus in this case we report a yes-instance.

**Case 1B - $H$ has three semi-edges.** In this case already $H$-Cover is NP-complete, as it is equivalent to 3-edge-colorability of cubic graphs.

**Case 2:** $|V(H)| = 2$. If $H$ has neither loops nor semi-edges, then $H$ is a bipartite graph formed by a triple edge between two vertices. Only bipartite graphs can cover a bipartite one. Hence a covering projection corresponds to a 3-edge-coloring of the input graph. Thus $H$-Cover is polynomial-time solvable (every cubic bipartite graph is 3-edge-colorable), but List-$H$-Cover is NP-complete, because List 3-Coloring is NP-complete for line graphs of cubic bipartite graphs [11]. If $H$ has a loop or a semi-edge, then it is one of the four graphs in Fig. 5, and for each of these already the $H$-Cover problem is NP-complete [6].

![Fig. 5. The non-bipartite 2-vertex graphs.](image)

**Case 3:** $|V(H)| \geq 3$. Here we split into several subcases.

**Case 3A - $H$ is acyclic.** If we shave all semi-edges off from $H$, we get a tree with at least three vertices. At least one of them has degree greater than 1, and such vertex is semi-simple in $H$. Thus List-$H$-Cover is NP-complete by Theorem 1.

**Case 3B - $H$ has a cycle of length greater than 2 which does not span all of its vertices.** Then $H$ has a vertex outside of this cycle, and thus $H$ has a semi-simple vertex and List-$H$-Cover is NP-complete.

**Case 3C - $H$ has a cycle of length greater than 2 with a diagonal.** Then again $H$ has a semi-simple vertex and List-$H$-Cover is NP-complete.

**Case 3D - $H$ has a cycle of length greater than 2, but none of the previous cases apply.** Then $H$ is the $k$-ring for some $k \geq 2$. If $k = 2$, 2-Ring-Cover is NP-complete by [6]. If $k \neq 2^\alpha$ for some $\alpha \geq 2$, $k$-ring-Cover is NP-complete by
In the case of \( k = 2^\alpha \) with \( \alpha \geq 3 \), the List-\( k \)-RING-COVER problem is NP-complete by Theorem 3. The case of \( k = 4 \) remains open.

**Case 3E** - \( H \) has a cycle, but all cycles are of length one or two. If, in addition, \( H \) has no semi-simple vertex, then \( H \) is a \( k \)-sausage for some \( k \geq 2 \). If \( k \neq 4 \), the NP-completeness of List-\( H \)-COVER follows from Case 3D via Proposition 4. The case of \( k = 4 \) remains open.

6 Concluding remarks

We have studied the complexity of the List-\( H \)-COVER problem in the setting of graphs with multiple edges, loops, and semi-edges for regular target graphs. We have shown in Theorem 1 a general hardness result under the assumption that the target graph contains at least one semi-simple vertex. It is worthwhile to note that in fact we have proved the NP-hardness for the more specific \( H \)-PRECOVERING EXTENSION problem, when all the lists are either one-element, or full. Actually, we proved hardness for the even more specific VERTEX \( H \)-PRECOVERING EXTENSION version, when only vertices may come with prescribed covering projections, but all edges have the lists full.

On the contrary, the nature of the NP-hard cases that appear in the almost complete characterization of the complexity of List-\( H \)-COVER of cubic graphs given by Theorem 4 is more varied. Some of them are NP-hard already for \( H \)-COVER, some of them are NP-hard for \( H \)-PRECOVERING EXTENSION, but apart the VERTEX \( H \)-PRECOVERING EXTENSION version in applications of Theorem 4 this time we also utilize the EDGE \( H \)-PRECOVERING EXTENSION version for the case of the bipartite 2-vertex graph formed by a triple edge between two vertices. Finally, for the cases of sausages and rings of length power of two, nontrivial lists are required to make our proof technique work.

Needless to say, we are leaving the following problem open. An affirmative answer would imply the hardness of List \( H \)-COVER for 4-sausages \( H \), and thus prove the list variant of the Strong Dichotomy Conjecture for cubic graphs.

**Conjecture.** The 4-RING-COVER problem is NP-complete for simple input graphs.

Note added in proof

After submitting the paper to IWOCA 2022, we have proved the above stated Conjecture. The proof will appear in the journal version of the paper.

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Appendix A: omitted proofs from Section 3.

The following operation will be an important tool used in our hardness proof.

**Definition 1 (Colored product).**

1. Let $M_1, M_2, \ldots, M_m$ be $m$ perfect matchings, possibly on different vertex sets. Their product is the graph

$$
\prod_{i=1}^{m} M_i = (\prod_{i=1}^{m} V(M_i), \{uv : u_i v_i \in M_i \text{ for each } i = 1, 2, \ldots, m\})
$$

where it is assumed that the notation of the vertices of the product is such that $u = (u_1, u_2, \ldots, u_m)$ with $u_i \in V(M_i)$ for all $i = 1, 2, \ldots, m$. Then $\prod_{i=1}^{m} M_i$ is a perfect matching as well.

2. Let $G_1, G_2, \ldots, G_m$ be $k$-regular $k$-edge-colorable graphs without loops or semi-edges. For each $i = 1, 2, \ldots, m$, let $\phi_i : E(G_i) \to \{1, 2, \ldots, k\}$ be a proper edge-coloring of $G_i$. The colored product of $G_i$’s is the graph $\prod_{i=1}^{m} G_i$ with vertex set $\prod_{i=1}^{m} V(G_i)$ and edge set being the union of $E(\prod_{i=1}^{m} M^j_i)$, $j = 1, 2, \ldots, k$, where for each $i$ and $j$, $M^j_i = \phi^{-1}_i(j)$ is the perfect matching in $G_i$ formed by edges colored by color $j$ in the coloring $\phi_i$. If we define $\phi : E(\prod_{i=1}^{m} G_i) \to \{1, 2, \ldots, k\}$ by setting $\phi(e) = j$ iff $e \in E(\prod_{i=1}^{m} M^j_i)$, we see that $\phi$ is a proper $k$-edge-coloring of $\prod_{i=1}^{m} G_i$.

3. Let $G_1, G_2, \ldots, G_m$ and $\phi : E(G_i) \to \{1, 2, \ldots, k\}$ be as in 2) above. For each $i = 1, 2, \ldots, m$, define the projection $\pi_i$ from the colored product $\prod_{i=1}^{m} G_i$ to its $i$-th coordinate by setting $\pi_i(u) = u_i$ and $\pi_i(e) = e_i$ for such an edge $e_i \in E(G_i)$ that satisfies $\phi_i(e_i) = \phi(e)$ and whose end-vertices are $u_i$ and $v_i$, provided the end-vertices of $e$ are $u$ and $v$.

![Fig. 6. An example of the colored product is on the left. Projections of vertex $u$ and its neighborhood are visualized on the right – $u, r, s, t$ map in this order onto $a, b, c, c$ in the vertical projection, and onto $a, c, c, b$ in the horizontal one.](image-url)
Lemma 1. Let $G_1, G_2, \ldots, G_m$ be $k$-regular graphs without loops or semi-edges, and let for each $i = 1, \ldots, m$, $\phi_i : E(G_i) \to \{1, 2, \ldots, k\}$ be a proper edge-coloring of $G_i$. Then each projection $\pi_i$, $i = 1, 2, \ldots, m$, is a covering projection from $\prod_{i=1}^m G_i$ onto $G_i$.

Proposition 1 (♠). Let $H$ be a connected $k$-regular $k$-edge-colorable graph with no loops or semi-edges. Let $x, y$ be two adjacent vertices of $H$. Then there exists a connected simple $k$-regular $k$-edge-colorable graph $G$ and $u \in V(G)$, such that
(a) for any bijection from $E_G(u)$ onto $E_H(x)$, there exists a covering projection from $G$ to $H$ which extends this bijection and maps $u$ to $x$, and
(b) for any bijection from $E_G(u)$ onto $E_H(y)$ there exists a covering projection from $G$ to $H$ which extends this bijection and maps $u$ to $y$.

Proof. In a way similar to the proof in [28] we first construct a colored product of many many copies of $H$ that covers $H$ in many ways, in particular, that satisfies (a) and (b). To do so, we take $k!$ copies of $H$ with edge colorings obtained by all permutations of colors, and in their colored product consider the vertex $a$ whose all projections are $x$. The edges incident with $a$ are projected onto $E_H(x)$ in all possible ways from this colored product. Then we take the same product and consider its vertex $b = (y, y, \ldots, y)$. The edges incident with $b$ are projected in all possible ways onto $E_H(y)$. Finally, in the product of these two graphs, the vertex $(a, b)$ plays the role of $u$. The difference to the approach in [28] is that it is not sufficient to require that all bijections of the vertex neighborhoods of $u$ and $x$ ($y$, respectively) can be extended to covering projections, but we must aim at extending bijections of the sets of incident edges. The product that we construct may still contain multiple edges. If this is the case, we further take the product with a simple graph, say $K_{k,k}$. This product is already a simple graph and still possesses all the requested covering projections. It may still be disconnected, though, and we denote by $G$ the component that contains the vitally important vertex $u$ in such a case. \hfill \Box

Proposition 2 (♠). The graph $G_u$ constructed from the multicovery $G$ of $H$ as above satisfies the following:
(a) for every bijection $\sigma_x : E_G(u) \to E_H(x)$, there exists a partial covering projection of $G_u$ onto $H$ that extends $\sigma_x$ and maps each $u_e, e \in E_G(u)$ to $x$;
(b) for every bijection $\sigma_y : E_G(u) \to E_H(y)$, there exists a partial covering projection of $G_u$ onto $H$ that extends $\sigma_y$;
(c) in every partial covering projection from $G_u$ onto $H$, the pendant vertices $u_e, e \in E_G(u)$ are mapped onto the same vertex of $H$;
(d) in every partial covering projection from $G_u$ onto $H$, the pendant edges are mapped onto different edges (incident with the image of the pendant vertices).

Proof. Items (a) and (b) follow directly from Proposition [1]. To prove (c) and (d), we first show that in any partial covering projection $f$ from $G_u$ onto $H$, the pendant edges are mapped onto edges of all colors.

Denote by $E^*$ the pendant edges of $G_u$, and set $V^* = V(G) \setminus \{u\} = V(G_u) \setminus \bigcup_{e \in E_G(u)} \{u_e\}$. Observe first that since $H$ has a perfect matching, the number
of its vertices is even, and since \( G \) covers \( H \), so is the number of vertices of \( G \). Hence \(|V^*| \equiv 1 \mod 2\).

Suppose \( f' : G_u \to H \) is a partial covering projection. Fix a proper \( k \)-coloring \( \phi \) of the edges of \( H \) and define a \( k \)-coloring \( \tilde{\phi} \) of the edges of \( G_u \) by setting \( \tilde{\phi}(e) = \phi(f(e)) \). Since \( f \) is a partial covering projection, \( \tilde{\phi} \) is a proper edge-coloring of \( G_u \). If some color is missing on all of the edges from \( E^* \), edges of this color would form a perfect matching in \( G_u[V^*] \) and \(|V^*| \) would be even, a contradiction. Hence every color appears on exactly one edge of \( E^* \).

For every \( a \in V(H) \), denote by \( h_a \) the number of vertices of \( V^* \) that are mapped onto \( a \) by \( f \). Consider an edge \( e' \) connecting vertices \( a \) and \( b \) of \( H \). If \( e' = f(e) \) for some edge \( e \in E^* \), we have \( f(u_a) = a \) and \( f(w_b) = b \), or vice versa.

Every vertex in \( f^{-1}(a) \cap V^* \) is adjacent to exactly one vertex in \( f^{-1}(b) \cap V^* \) via an edge of color \( \phi(e') \), whilst every vertex of \( f^{-1}(b) \cap V^* \) except for \( w_e \) is adjacent to exactly one vertex in \( f^{-1}(a) \cap V^* \) via an edge of the same color. Hence \( h_b = h_a + 1 \). Orient the edge \( e' \) from \( a \) to \( b \) in such a case. If, on the other hand, the edge \( e' \) is not the image of any edge from \( E^* \), the edges of color \( \phi(e') \) form a matching between the vertices of \( f^{-1}(a) \cap V^* \) and the vertices of \( f^{-1}(b) \cap V^* \), and \( h_a = h_b \). Leave the edge \( e' \) undirected in such a case.

After processing all edges of \( H \) in this way, we have constructed a mixed graph \( \overrightarrow{H} \) which has exactly one edge of each color directed. From the meaning of the orientations and non-orientations of edges of \( \overrightarrow{H} \) for the values of \( h_a, a \in V(H) \), it follows that the vertex set of \( H \) falls into levels, say \( L_r, L_{r+1}, \ldots, L_s \) such that undirected edges live inside the levels, while directed edges connect vertices of consecutive levels and are directed from \( L_i \) to \( L_{i+1} \) for suitable \( i \). Every two consecutive levels are connected by at least one directed edge in this way. Since \( H \) is connected, every vertex \( a \in L_i \) will satisfy \( h_a = i \) (the indices \( r \) and \( s \) are chosen so that \( r \) is the smallest value of \( h_a \) and \( s \) is the largest one). Directed edges connecting two consecutive levels form a cut in \( H \), and since edges of each color form a perfect matching, the parities of the numbers of edges of each color in this cut are the same. Since the cut contains at least one edge, but at most one edge of each color (exactly one edge of each color is directed), it follows that the cut contains all \( k \) directed edges and that \( H \) has only two levels, i.e., \( s = r + 1 \).

Thus \( H \) has \(|L_r| \) vertices \( a \) with \( h_a = r \) and \(|L_{r+1}| = |V(H)| - |L_r| \) vertices \( a \) with \( h_a = r + 1 \). It follows that

\[
|V^*| = r|L_r| + (r + 1)(|V(H)| - |L_r|) = (r + 1)|V(H)| - |L_r|.
\]

We know that \( G \) covers \( H \), and so \(|V(G)| = \ell|V(H)|\) for some \( \ell \). Thus \(|V^*| = \ell|V(H)| - 1\). Thus we obtain

\[
(r + 1)|V(H)| - |L_r| = \ell|V(H)| - 1,
\]

which implies

\[
(r + 1 - \ell)|V(H)| = |L_r| - 1,
\]

and since \( 1 \leq |L_r| \leq |V(H)| - 1 \), the only possible way for \(|L_r| - 1 \) to be divisible by \(|V(H)| \) is \(|L_r| = 1 \). But this implies that all directed edges of \( \overrightarrow{H} \) start in the
same vertex, say $z$, and from the construction of $\vec{H}$ it follows that $f(u_e) = z$ for all $e \in E^*$. This proves (c).

Now (d) follows from the two observations above. The $k$ pendant edges of $E^*$ have mutually distinct colors in $\vec{\phi}$, and thus they must be mapped to distinct edges of $E_H(z)$ by $f$. \qed
Appendix B: omitted proofs from Section 4

Theorem 2 (♠). The k-ring-Cover problem is NP-complete for simple input graphs for every $k = 2^\alpha (2^\beta + 3)$ such that $\alpha$ and $\beta$ are non-negative integers.

Proof. We reduce from $C_{(2^\beta+3)}$-Hom for $\beta$ being a non-negative integer which is known to be NP-complete by the dichotomy theorem of Hell and Nešetřil [26]. We call $k$-ring occasionally $H$. Furthermore, the vertices of $H$, i.e. $k$-ring, will have its vertices consecutively denoted by $1, 1', \ldots, k, k'$ with precisely edges $jj'$ being double for $j \in \{1, \ldots, k\}$.

The reader is advised to consult Figure 7 to better understand the gadgets and the reduction.

Let us describe the vertex gadget. Suppose we have a vertex $v$ of degree $\text{deg}(v)$. Then let us take two disjoint copies of $C_l$ where $l = 2k \times \text{deg}(v)$. Let us denote them $C^V_1$ and $C^V_2$ and their vertices $v_1, 1, v_2, 1, \ldots, v_l, l$. The construction of the gadget proceeds in the following way:

- Add edges $v_1, 2i - 1v_2, 2i$ for every $i \in \{1, \ldots, l/2\}$.
- Add edges $v_2, 2i - 1v_1, 2i$ for every $i \in \{1, \ldots, l/2\}$.
- Delete edge $v_2, 2kj - 1v_2, 2kj$ for every $j \in \{1, \ldots, \text{deg}(v)\}$ and to each of the endpoints of the deleted edge, add a pendant vertex.

We call these pendant vertices leafs of the vertex gadget and we speak about pairs of leafs when we refer to the two pendant vertices created after the deletion of the same edge. Further, since the gadget is bipartite, we can say that leafs are either black or white depending on the part of the bipartition they belong to. Observe that every pair of leafs in the above sense has one black and one white vertex.

Before we describe the edge gadget, let us introduce enforcing gadget which is simply the same as the vertex gadget for vertex of degree 1, i.e. it is created as was described in the preceding with $l = 2k$.

In the following, we number the vertices of every cycle consecutively starting with 1. We can thus speak about e.g. the $i$-th even vertex. Also, we automatically take the indices of cycles modulo $2k$.

For the actual edge gadget, take $k$ disjoint copies of cycles $C_{2k}$. Let us denote these cycles $C^E_1, \ldots, C^E_k$. We now insert enforcing gadgets in between the cycles as follows. For all $j$ being odd and $j < k$ and for all even vertices of $C^E_j$, we identify one of the leafs of the enforcing gadget with the $i$-th even vertex of $C^E_j$, and we identify the other leaf of the enforcing gadget with the $i$-th even vertex of $C^E_{j+1}$.

The similar connection is done in case of $j$ even and $j < k$, except that the $i$-th odd vertex of $C^E_j$ is connected by a copy of enforcing gadget to the $i$-th odd vertex of $C^E_{j+1}$.

Now, except for $C^E_1$ and $C^E_k$, all vertices are of degree 3. For every vertex of degree 2 in $C^E_1$ except for the first and the $(1 + 2^\alpha + 1)$-th vertex, let us say the $i$-th one, we place the enforcing gadget between the $i$-th vertex of $C^E_1$ and
the \((i + k)\)-th vertex of \(C_k^E\) again by identifying each of the leaves with one of the mentioned vertices. This completes the construction of the edge gadget. The only vertices of degree 2 are now the first and the \((1 + 2^{a+1})\)-th vertex in \(C_k^E\) and the \((1 + k)\)-th and the \((1 + 2^{a+1} + k)\)-th vertex in \(C_k^E\).

Let us have an instance \(G\) of \(C_{(2^b+3)}\)-HOM. We shall construct a new graph \(G'\). For each vertex in \(G\), we take a copy of the vertex gadget of corresponding size and insert it into \(G'\). For each edge \(uv\) in \(G\), we obtain a new copy of the edge gadget. We connect it with the vertex gadget corresponding to \(u\) as follows. We take one of the so-far unused pair of leaves coming from the vertex gadget corresponding to \(u\) and identify the black leaf of the pair with the first vertex in \(C_k^E\) of the edge gadget and the white leaf of the pair with the \((1 + k)\)-th vertex of \(C_k^E\). For the vertex gadget of \(v\), we again take one of the so-far unused pair of leaves coming from the vertex gadget of \(v\) and identify the black leaf of the pair with the \((1 + 2^{a+1})\)-th vertex in \(C_k^E\) of the edge gadget and the white leaf of the pair with the \((1 + 2^{a+1} + k)\)-th vertex of \(C_k^E\).

We shall now describe possible images of vertex and edge gadget under a covering projection to \(k\)-ring.

We claim that under every covering projection to \(H\), all black leaves of a given vertex gadget will be mapped to the same vertex of \(k\)-ring and white vertices to its prime version (or vice versa). The crucial observation is that \(v_{1,1}, v_{2,2}, v_{2,1}, v_{1,2}\) form a 4-cycle in vertex gadget. By a simple analysis then, \(v_{1,1}\) and \(v_{2,1}\) must be mapped to some \(\ell\) of \(H\) and \(v_{1,2}\) and \(v_{2,2}\) to \(\ell'\) (or vice versa, but let us further assume the first possibility). Furthermore this enforces the images of vertices \(v_{1,3}, v_{2,4}, v_{2,3}, v_{1,4}\) as well (and they form again a 4-cycle). A repeated use of this propagation ensures that images of \(v_{2,2k-1}v_{2,2k}\) are \((\ell - 1)\) and \((\ell - 1)'\), respectively (and possibly modulo \(2k\), which will be assumed from now on). By the construction, the pendant vertices have then images \((\ell - 1)'\) and \((\ell - 1)\). The argument then can be repeated further and further, until we arrive on the conclusion that all black leaves have inevitably the same image \((\ell - 1)'\) and the white leaves \((\ell - 1)\).

Specially, for the enforcing gadget, we get that its leaves must be mapped to the different endpoints of a specific double edge in \(k\)-ring. In other words, whenever we have a vertex which is being identified with one of the leaves of the enforcing gadget, then given its image \(i\) under a covering projection to the \(k\)-ring, the other vertex identified with the other leaf of the enforcing gadget has to be mapped to \(i'\) in \(k\)-ring or vice versa.

We claim that under every covering projection, edge gadget will be mapped to \(k\)-ring in the following way. Without loss of generality, the first vertex of \(C_1^E\), let us call it \(a\), will be mapped to 1. Clearly, as the edge gadget is connected here to a vertex gadget through \(a\), one of the neighbors of \(a\) in the edge gadget has to be mapped to \(1'\) and the other to \(k'\), or vice versa. In both cases these neighbors are connected through enforcing gadget to \(C_k^F\) and thus this enforces not only the images of vertices at distance 2 from \(a\) on \(C_1^E\) but also the images of the vertices at distance two from \(a\) on \(C_k^F\). Proceeding inductively, we arrive on the conclusion that the vertices of \(C_1^E\) are either consecutively 1, 1', 2, 2', ..., \(k, k'\).
or in the counter-clockwise fashion $1, k', k, \ldots, 1'$. Let us, again without loss of
generality, describe what follows in the first situation. The other one is in fact
just a mirrored situation and the argumentation is thus almost the same. In
the first situation, the images of $C^E_2$ are shifted clockwise by one, so the images
of vertices of $C^E_2$ are consecutively $k', 1, 1', \ldots, k$. This propagates the shifting
further to $C^E_3$ and inductively up to $C^E_k$, also than to the enforcing gadgets
between the layers.

Up to now, everything was enforced, all vertices of the edge gadget are
mapped. However, it remains to check two things.

1. Whether all the vertices identified with leafs of vertex gadgets are mapped
   in the right way.
2. Whether the edges between $C^E_1$ and $C^E_k$ and their endpoints do have the
   right images.

Regarding (1), we assumed $a$ is mapped to 1. Thus the other, white leaf from
the pair containing $a$ has to be mapped to $1'$. This is indeed all right considering
the shifting of images between the cycles. The same can be argued for the pair
of leafs corresponding to the second vertex gadget attached. The argumentation
here is based on the fact that the black leaf of the other vertex gadget is identified
with vertex at distance $2^\alpha + 1$ on $C^E_1$.

The case (2) depends on analogous argument, again based on shifting of the
images of $C^E_1$ versus the images of $C^E_k$.

Finally, let us observe that $G'$ is again bipartite and thus it remains valid
to speak about black and white leafs of vertex gadgets (or about vertices corre-
spanding to such leafs). Furthermore, we can say that $G'$ has black and white
vertices.

Suppose that $G'$ covers the $H$. Clearly, as was described and without loss
of generality, all vertices corresponding originally to the black leafs of all vertex
gadgets in $G$ must be mapped to the black vertices of $H$ and white leafs to
the white vertices of $H$. We can further assume without loss of generality that
black vertices of vertex gadgets map to vertices $1 + (j - 1)(2^\alpha + 1)$ where $j \in
\{1, \ldots, 2\beta - 3\}$ of $H$.

Edge gadgets enforce that whenever there is an edge in $G$, then the images
of the respective black leafs of vertex gadgets connected to it are exactly at
distance $2^\alpha + 1$ from each other. This implies that there is a homomorphism of $G$
to $C^{(2\beta+3)}$ and we can say that the image of $v$ is the $j$-th vertex of $C^{(2\beta+3)}$ if
black leafs of vertex gadget of $v$ map to $1 + (j - 1)(2^\alpha + 1)$.

For the other direction, suppose there exists a homomorphism of $G$ to $C^{(2\beta+3)}$.
If a vertex $v$ of $G$ is mapped to the $j$-th vertex of $C^{(2\beta+3)}$, we map the ver-
tices corresponding to the black leafs of the vertex gadget of $v$ to the vertex
$1 + (j - 1)(2^\alpha + 1)$ of $H$. We already know that this ensures that there is only
one possible mapping of vertices of the vertex gadget of $v$ to $H$. The same can
be done and said for all the other vertex gadgets of $G'$. From the analysis of
possible mappings edge gadgets, we know than we can complete the mapping
now so that the result is a covering projection of $G'$ to $H$. 
We showed that there is a homomorphism of $G$ to $C_{(2\beta+3)}$ if and only if $G'$ covers $H$. This completes the reduction and thus the proof of the theorem. □

**Theorem 3 (♠).** The List-$k$-ring-Cover problem is NP-complete for simple input graphs for every $k = 2^\alpha$ such that $\alpha \geq 3$ is an integer.

**Proof.** By a seminal result of Feder et al. [10], List-$H'$-Hom is NP-complete if $H'$ is a so called bi-arc graph. By Corollary 3.1. therein, it follows that all cycles of size at least five are bi-arc graphs. Thus, we shall reduce from List-$C_k$-Hom.

The construction of $(G', L')$ for a given $(G, L)$ being an instance of List-$C_k$-Hom is almost the same as in Theorem 2. We shall only describe differences here. Suppose that $C_k$ has its vertices named consecutively $1, 2, \ldots, k$. Again as before, the $k$-ring will have its vertices denoted consecutively by $1, 1', \ldots, k, k'$ with precisely edges $jj'$ being double for $j \in \{1, \ldots, k\}$.

- Suppose we have a vertex $v$ in $G$. Then for every $\ell \in L(v)$, we add $\ell$ to the lists of all black leafs of its vertex gadget and $\ell'$ to all lists of all white leafs. All of the other vertices in $G'$ will have full lists, i.e. all vertices of $k$-ring. Also the edges will have full lists.
- Vertex gadgets will be connected to the respective edge gadget in a slightly different way. For a vertex $uv$ of $G$, we shall identify the black leaf of a pair of a vertex gadget for $u$ with first vertex of $C_{1E}$ and the white leaf of the same pair with the $(1+k)$-th vertex of $C_{kE}$ of the edge gadget for $uv$. We shall.
identify the black leaf of a pair of a vertex gadget for \( v \) with third vertex of \( C_E^k \) and the white leaf of the same pair with the \((3 + k)\)-th vertex of \( C_E^k \) of the edge gadget for \( uv \).

Clearly, under any possible list covering projection, all black leaves of a given vertex gadget choose simultaneously one vertex \( \ell \) from their lists and subsequently \( \ell' \) for all its white leaves (or vice versa).

Now the same analysis can be done as in Theorem 2 to show that there exists a list homomorphism of \((G, L)\) to \( C_k \) if and only if there exists a list covering projection of \((G', L')\) to \( k \)-ring. This completes the proof.

\[ \text{Proposition 4 (♠). For every } k \geq 2 \text{ and every } k \text{-sausage } S_k, k \text{-ring-Cover } \propto S_k \text{-Cover and List-} k \text{-ring-Cover } \propto \text{List-} S_k \text{-Cover.} \]

\textit{Proof}. A graph \( G \) covers \( H \times K_2 \) if and only if it is bipartite and covers \( H \). Since bipartiteness can be tested in polynomial time, testing if \( G \) covers the \( k \)-ring polynomially reduces to testing if \( G \) covers \( S_k \).

With the list version, one has to be a bit more explicit. Let \( S_k = (V, E), V(K_2) = \{b, w\} \) and let the \( k \)-ring be denoted by \( R_k \), with \( V(R_k) = \{ (u, \alpha) : u \in V, \alpha \in \{b, w\} \} \). After checking that the input graph \( G \) of the List-\( R_k \)-Cover problem is bipartite, let \( V(G) = A \cup B \) be the bipartition of \( G \). In a feasible covering projection, either the vertices of \( A \) are mapped onto the vertices of \( V \times \{w\} \) and the vertices of \( B \) onto \( V \times \{b\} \), or vice-versa. We try these two possibilities separately. For trying the former one, reduce first the lists to \( L_u' = L_u \cap (V \times \{w\}) \) for \( u \in A \) and \( L_u' = L_u \cap (V \times \{b\}) \) for \( u \in B \), and adjust the lists for edges accordingly. Then regard \( G \) as an instance of List-\( S_k \)-Cover with the lists \( L_u = \{ x : (x, w) \in L_u' \} \) or \( L_u = \{ x : (x, b) \in L_u' \} \), and lists for edges being adjusted accordingly. It is not difficult to see that \( G \) allows a covering projection onto \( R_k \) that respects \( L' \) if and only if it allows a covering projection onto \( S_k \) that respects \( \tilde{L} \). Check the latter possibility in a similar way and conclude that \( G, \tilde{L} \) is an feasible instance of List-\( R_k \)-Cover if and only if at least one of the \( G, \tilde{L} \) instances is feasible for the corresponding List-\( S_k \)-Cover problem. \( \Box \)