On Kerr-Schild spacetimes in higher dimensions

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Abstract. We summarize main properties of vacuum Kerr-Schild spacetimes in higher dimensions.

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INTRODUCTION

Kerr-Schild (KS) spacetimes [1] possess a rare property of being physically important and yet mathematically tractable. In \( n = 4 \) dimensions they contain important exact vacuum solutions such as the Kerr metric and pp-waves. Vacuum KS spacetimes are algebraically special and thus, because of the Goldberg-Sachs theorem, the KS null congruence is geodetic and shearfree. Thanks to this and to the Kerr theorem [2, 3, 4], the general \( n = 4 \) vacuum KS solution is in fact known \([1, 2, 3, 6]\). In arbitrary higher dimensions, the KS ansatz led to the discovery of rotating vacuum black holes [7]. Here we will study the general class of \( n > 4 \) KS spacetimes [8].

Note that in \( n > 4 \) gravity there is no unique generalization of the shearfree condition [9, 10, 11, 12, 13, 14] and, a fortiori, it is not obvious how to extend the Goldberg-Sachs theorem. In fact, it has been pointed out that this can not be done in the most direct way [7, 15, 16, 17, 18]. Our results below will suggest a possible weak generalization of the shearfree condition, and a partial extension of the Goldberg-Sachs theorem to \( n > 4 \) (limited to KS solutions). It is worth mentioning that an \( n > 4 \) extension of the Robinson theorem has been proven in even dimensions [10] assuming a generalization of the shearfree condition (see also [11, 12, 13, 14]) different from ours. The relation to our work will be discussed elsewhere. Let us also recall that properties of KS transformations in arbitrary dimensions have been studied in [19]. This does not overlap significantly with our contribution.

GEOMETRIC OPTICS AND ALGEBRAICAL PROPERTIES

By definition, Kerr-Schild spacetimes in \( n \geq 4 \) dimensions are metrics of the form

\[
g_{ab} = \eta_{ab} - 2\mathcal{H} k_a k_b, \tag{1}\]
where $\eta_{ab}=\text{diag}(-1,1,\ldots,1)$ is the Minkowski metric, $\mathcal{H}$ a scalar function and $k^a$ a 1-form that is assumed to be null with respect to $\eta_{ab}$, i.e. $\eta^{ab}k_ak_b=0$ ($\eta^{ab}$ is defined as the inverse of $\eta_{ab}$). Hence $k^a\equiv \eta^{ab}k_b=g^{ab}k_b$, so that $k^a$ is null also with respect to $g_{ab}$.

It can be shown that optical properties of $k^a$ in the full KS geometry $g_{ab}$ are inherited from the flat background spacetime $\eta_{ab}$. More specifically, the matrix $L_{ij}$ is the same in both spacetimes, i.e

$$L_{ij}\equiv k_{a;b}m^{(i)a}m^{(j)b} = k_{a;b}m^{(i)a}m^{(j)b},$$

(2)

and so are the optical scalars expansion $\theta=L_{ii}/(n-2)$, shear $\sigma^2 = L_{(ij)}L_{(ij)} - (n-2)\theta^2$ and twist $\omega^2 = L_{[ij]}L_{[ij]}$. Furthermore, $k^a$ is geodetic with respect to $g_{ab}$ iff it is geodetic in $\eta_{ab}$. This geometric condition on $k^a$ turns out to constraint the possible form of the energy-momentum tensor $T_{ab}$ compatible with $g_{ab}$, i.e.

**Proposition 1** The null vector $k^a$ in the KS metric (1) is geodetic iff $T_{ab}k^ak^b = 0$.

Note that this condition is satisfied, e.g., in the case of vacuum spacetimes, also with a possible cosmological constant, or in the presence of matter fields aligned with the KS vector $k^a$, such as an aligned Maxwell field or aligned pure radiation.

Further computation constrains also the algebraic type of the Weyl tensor

**Proposition 2** If $k^a$ is geodetic, KS spacetimes (1) are of type II (or more special).

From the results of [18], it follows that static and (a specific subclass of) stationary spacetimes belonging to the KS class are necessarily of type D, and $k^a$ is a multiple WAND. As a consequence, Myers-Perry black holes must be of type D (cf. also [20]). By contrast, black rings do not admit a KS representation, since they are of type $I$ [21].

**VACUUM SOLUTIONS**

In the rest of the paper we will focus on vacuum solutions and, by Proposition 1, $k^a$ will thus be geodetic. Some of the vacuum equations are remarkably simple thanks to the metric ansatz (1). In particular, imposing $R_{ij}=0$ we obtain

$$(D\ln \mathcal{H})S_{ij} = L_{ik}L_{jk} - (n-2)\theta S_{ij},$$

(3)

(where $D\equiv k^a\nabla_a$) and its contraction with $\delta^{ij}$ gives

$$(n-2)\theta (D\ln \mathcal{H}) = \sigma^2 + \omega^2 - (n-2)(n-3)\theta^2.$$  

(4)

The latter involves $\mathcal{H}$ only when $\theta \neq 0$, so that KS spacetimes naturally split into two families with either $\theta = 0$ (non-expanding) and $\theta \neq 0$ (expanding).

**Non-expanding solutions**

It turns out that the vacuum KS subfamily $\theta = 0$ can be integrated in full generality. Our analysis, combined with the results of [22] (where all vacuum Kundt type N
solutions have been given) shows that in arbitrary \( n \geq 4 \) dimensions

**Proposition 3** The subfamily of Kerr-Schild vacuum spacetimes with a non-expanding KS congruence \( k^a \) coincides with the class of vacuum Kundt solutions of type N.

A simple explicit example of non-expanding KS solutions is given by \( pp \)-waves of type N (cf. [22] and references therein). But note that, as opposed to the case \( n = 4 \), for \( n > 4 \) not all \( pp \)-waves fall into the KS class, and in fact they can also be of Weyl types different from N (see [8] for details).

**Expanding solutions**

The subfamily of expanding solutions is more complex and contains, in particular, Myers-Perry black holes [7]. When \( \theta \neq 0 \), from eqs. (3) and (4) one gets

\[
L_{ik}L_{jk} = \frac{L_{ik}L_{jk}}{(n-2)}\theta S_{ij}.
\]

Remarkably, this equation is independent of the function \( \mathcal{H} \). It is thus a purely geometric condition on the KS null congruence \( k^a \) in the Minkowskian “background” \( \eta_{ab} \) (an optical constraint). It is important in proving further properties of expanding solutions.

**Optics**

First, the optical constraint implies \( LL^T - L^TL = 0 \), i.e. \( L \) is a normal matrix. Combining this with the Ricci identities [17], one can prove that there exists a “canonical” frame in which the matrix \( L_{ij} \) takes a specific block-diagonal form, with a number \( p \) of \( 2 \times 2 \) blocks \( \mathcal{L}_\mu \), and a single diagonal block \( \mathcal{L} \) of dimension \((n-2-2p) \times (n-2-2p)\). They are given by

\[
\mathcal{L}_\mu = \begin{pmatrix}
s(2\mu) & A_{2\mu,2\mu+1} \\
-A_{2\mu,2\mu+1} & s(2\mu)
\end{pmatrix} \quad (\mu = 1, \ldots, p),
\]

\[
s(2\mu) = \frac{r}{r^2 + (d_0^2(2\mu))^2}, \quad A_{2\mu,2\mu+1} = \frac{d_0^2(2\mu)}{r^2 + (d_0^2(2\mu))^2},
\]

\[
\mathcal{L} = \frac{1}{r} \text{diag}(1, \ldots, 1, 0, \ldots, 0),
\]

with \( 0 \leq 2p \leq m \leq n-2 \). (The integer \( m \geq 2 \) is the rank of \( L_{ij} \). From now on, a superscript (or subscript) index 0 denotes quantities independent of \( r \), which is an affine parameter along \( k^a \).) The above special properties of the matrix \( L_{ij} \) can be viewed as a “generalization” of the shearfree condition and considered in a weak formulation of the Goldberg-Sachs theorem in \( n > 4 \) dimensions, restricted to KS solutions [8].
Singularity

Together with the Einstein equation (4), eqs. (6)–(8) in turn enable one to fix also the $r$-dependence of $\mathcal{H}$, i.e.

$$\mathcal{H} = \mathcal{H}_0 \frac{1}{r^{m-2p-1} \prod_{\mu=1}^{p} \frac{1}{r^2 + (a^0_{(2\mu)})^2}}.$$  \hspace{1cm} (9)

The above functional dependence suggests there may be singularities at $r = 0$, at least for $2p \neq m$ ($m$ even) and $2p \neq m - 1$ ($m$ odd). This singular behaviour can indeed be confirmed by examining the Kretschmann scalar. Singularities may also be present in the special cases $2p = m$ and $2p = m - 1$ at “special points” with $r = 0$ and where some of the $a^0_{(2\mu)}$ vanish. See [8] for more details and [7] for a thorough discussion of singularities in the special case of rotating black hole spacetimes.

Weyl type

Along with the Bianchi identities [16], the optical constraint also imply that expanding vacuum KS solutions can not be of the type III or N, so that in arbitrary dimension $n \geq 4$

**Proposition 4** Kerr-Schild vacuum spacetimes with an expanding KS congruence $k^a$ are of algebraic type II or D.

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