PRESERVING POSITIVITY
FOR RANK-CONSTRAINED MATRICES

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Abstract. Entrywise functions preserving the cone of positive semidefinite matrices have been studied by many authors, most notably by Schoenberg [Duke Math. J. 9, 1942] and Rudin [Duke Math. J. 26, 1959]. Following their work, it is well-known that entrywise functions preserving Loewner positivity in all dimensions are precisely the absolutely monotonic functions. However, there are strong theoretical and practical motivations to study functions preserving positivity in a fixed dimension \( n \). Such characterizations for a fixed value of \( n \) are difficult to obtain, and in fact are only known in the \( 2 \times 2 \) case. In this paper, using a novel and intuitive approach, we study entrywise functions preserving positivity on distinguished submanifolds inside the cone obtained by imposing rank constraints. These rank constraints are prevalent in applications, and provide a natural way to relax the elusive original problem of preserving positivity in a fixed dimension. In our main result, we characterize entrywise functions mapping \( n \times n \) positive semidefinite matrices of rank at most \( l \) into positive semidefinite matrices of rank at most \( k \) for \( 1 \leq l \leq n \) and \( 1 \leq k < n \). We also demonstrate how an important necessary condition for preserving positivity by Horn and Loewner [Trans. Amer. Math. Soc. 136, 1969] can be significantly generalized by adding rank constraints. Finally, our techniques allow us to obtain an elementary proof of the classical characterization of functions preserving positivity in all dimensions obtained by Schoenberg and Rudin.

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1. Introduction and main results

This version contains a few edits to the published paper, which are minor in nature and mostly clarify the statements of results. The proofs remain virtually unchanged. A list of these edits is on the final page.

The study of entrywise functions mapping the space of positive semidefinite matrices into itself has been the focus of a concerted effort throughout the past century (see e.g. Schoenberg [36], Rudin [35], Herz [26], Horn [28], Christensen and Ressel [10], Vasudeva [38], FitzGerald and Horn [14], FitzGerald, Micchelli, and Pinkus [15], Hiai [27], Guillot and Rajaratnam [22], Guillot, Khare and Rajaratnam [19, 20] and others). Following the work of Schoenberg and Rudin, it is well-known that functions \( f : (-1, 1) \rightarrow \mathbb{R} \) such that \( f[A] := (f(a_{ij})) \) is positive semidefinite for all positive semidefinite matrices \( A = (a_{ij}) \) of all dimensions with entries in \((-1,1)\) are necessarily analytic with nonnegative Taylor coefficients; i.e., they are absolutely monotonic on the positive real axis. The converse follows easily from the Schur product theorem.

On the other hand, one is often interested in studying functions that preserve positivity for a fixed dimension \( n \). In this case, it is unnecessarily restrictive to characterize functions that preserve positivity in all dimensions. Results for fixed \( n \) are available only for \( n = 2 \), in which case entrywise functions mapping \( 2 \times 2 \) positive semidefinite matrices into themselves have been characterized by Vasudeva (see [38, Theorem 2]) in terms of multiplicatively mid-convex functions. Finding tractable descriptions of the functions that preserve positive semidefinite matrices in higher dimensions is far more involved; see [28, Theorem 1.2] for partial results for arbitrary but fixed \( n \geq 3 \). To the authors’ knowledge, no full characterization is known for \( n \geq 3 \).

The primary goal in this paper is to investigate entrywise functions mapping \( n \times n \) positive semidefinite matrices of a fixed dimension \( n \) and rank at most \( l \) into positive semidefinite matrices of rank at most \( k \) for given integers \( 1 \leq k, l \leq n \). Introducing such rank constraints is very natural from a modern applications perspective. Indeed, in high-dimensional probability and statistics, it is common to apply entrywise functions to regularize covariance/correlation matrices in order to improve their properties (e.g., condition number, Markov random field structure, etc.); see [7, 19–22, 24, 25, 30, 40]. In such settings, the rank of a covariance/correlation matrix corresponds to the sample size used to estimate it. Many modern-day applications require working with covariance/correlation matrices arising from small samples, and so these high-dimensional matrices are very often rank-deficient in practice. Applying functions entrywise is a popular way to increase the rank of these matrices. For many downstream applications, it is a requirement that the regularized covariance/correlation matrices be positive definite. It is thus very important and useful for applications to understand how the rank of a matrix is affected when a given function is applied to its entries, and whether Loewner positivity is preserved.

Our approach yields novel and explicit characterizations for functions preserving positivity for a fixed dimension, under various rank constraints. In particular, we show that a “special family” of matrices of rank at most 2 plays a fundamental role in preserving positivity. Furthermore, our techniques yield solutions to other characterization problems, in particular, those involving rank constraints without the positivity requirement. For instance, we provide characterizations of entrywise
functions mapping symmetric $n \times n$ matrices of rank at most $l$ into matrices of rank at most $k$ (for any $1 \leq k, l < n$).

Finally, our methods can also be used to solve the original problem of characterizing entrywise functions that preserve positivity for all dimensions, thereby providing a more direct and intuitive proof of the results by Schoenberg, Rudin, Vasilevski, and others.

**Notation.** To state our main results, some notation and definitions are needed. These are now collected together here, for the convenience of the reader.

**Definition 1.1.** Let $\mathbb{R} \supset \mathbb{Z} \supset \mathbb{N}$ denote the real numbers, the integers, and the positive integers, respectively. Let $I \subset \mathbb{R}$. Define $S_n(I)$ to be the set of $n \times n$ symmetric matrices with entries in $I$, and $P_n(I) \subset S_n(I)$ to be the subset of symmetric positive semidefinite matrices. Let rank $A$ denote the rank of a matrix $A$. Now define:

\begin{align}
S_n^k(I) &:= \{ A \in S_n(I) : \text{rank } A \leq k \}, \\
P_n^k(I) &:= \{ A \in P_n(I) : \text{rank } A \leq k \}.
\end{align}

Next, if $f : I \subset \mathbb{R} \to \mathbb{R}$ and $A = (a_{ij})$ is a matrix with entries in $I$, denote by $f[A]$ the matrix obtained by applying $f$ to every entry of $A$, i.e., $f[A] := (f(a_{ij}))$. Finally, denote by $f[-] : S_n(I) \to S_n$ the map sending $A \in S_n(I)$ to $f[A]$.

When $I = \mathbb{R}$, we denote $S_n^k(I)$ and $P_n^k(I)$ by $S_n^k$ and $P_n^k$, respectively. Note that

\begin{equation}
S_n(I) = S_n^n(I), \quad P_n(I) = P_n^n(I),
\end{equation}

and when $I \subset [0, \infty)$, $S_n^0(I) = P_n^1(I)$. Also, in what follows, given (possibly scalar) matrices $A_1, \ldots, A_n$ of arbitrary orders, denote by $A_1 \oplus \cdots \oplus A_n$ the corresponding block diagonal matrix $\text{diag}(A_1, \ldots, A_n).$ Note that this differs from the Kronecker sum of matrices.

Next, given a set of vectors or matrices $A_1, \ldots, A_m$ of equal orders, denote their entrywise product by $A_1 \circ \cdots \circ A_m$, which is a matrix of the same order with $(i, j)$th entry equal to $\prod_{k=1}^m (A_k)_{ij}$. Given a vector or matrix $A = (a_{ij})$ and $\alpha \in \mathbb{R}$ such that $\alpha_{ij}$ is defined for all $i, j$, define the $\alpha$th Hadamard power of $A$ to be $A^{\circ \alpha} := (\alpha_{ij}^{\alpha})_{i,j}$. Given $\alpha \in \mathbb{R}$, define the even and odd extensions of the power functions $f_\alpha(x) := x^\alpha$, to the entire real line, as follows:

\begin{equation}
\phi_\alpha(0) = \psi_\alpha(0) := 0, \quad \phi_\alpha(x) := |x|^\alpha, \quad \psi_\alpha(x) := \text{sgn}(x)|x|^\alpha, \quad \forall \alpha \in \mathbb{R}, \ x \in \mathbb{R} \setminus \{0\}.
\end{equation}

Finally, given a function $f : I \to \mathbb{R}$, and a vector or matrix $A = (a_{ij})$ with entries $a_{ij} \in I \subset \mathbb{R}$, define $f[A] := (f(a_{ij}))_{i,j}$.

**Remark 1.2.** Note that if the entrywise function $f[-]$ maps $n \times n$ positive semidefinite matrices of rank exactly $l$ into matrices of rank $k$ (for $1 \leq l, k \leq n$), and $f$ is continuous, then $f[-]$ necessarily maps $P_l$ into $P_k$. Moreover, we observe that there are no (continuous) entrywise maps $f[-]$ sending $P_n^l$ into matrices of rank bounded below by $k \geq 2$, since $f[1_{n \times n}]$ has rank at most 1. Here $1_{n \times n}$ denotes the matrix with all entries equal to 1. For these reasons we will study functions sending $P_n^l$ to $P_n^k$.

We now state the three main results of the paper. The first result characterizes entrywise functions mapping rank 1 matrices to rank at most $k$ matrices, under the
mild hypothesis that the function $f$ admits at least $k$ nonzero derivatives of some orders at the origin.

**Theorem A** (Rank 1, fixed dimension). Let $0 < R \leq \infty$, $I = [0, R)$ or $(-R, R)$, and $f : I \to \mathbb{R}$. Fix $1 \leq k < n$, and suppose $f$ admits at least $k$ nonzero derivatives at 0.

1. Then $f[\cdot] : \mathbb{P}^1_n(I) \to \mathbb{S}^k_n$ if and only if $f$ is a polynomial with exactly $k$ nonzero coefficients.
2. Similarly, $f[\cdot] : \mathbb{P}^1_n(I) \to \mathbb{P}^k_n$ if and only if $f$ is a polynomial with exactly $k$ nonzero coefficients which are all positive.
3. Suppose $f$ admits at least $n - 1$ nonzero derivatives at zero, of orders $0 \leq m_1 < \cdots < m_{n-1}$. If $f[\cdot] : \mathbb{P}^1_n(I) \to \mathbb{P}_n$, then writing $f$ in its Taylor expansion at 0:

$$f(x) = \sum_{i=1}^{n-1} \frac{f^{(m_i)}(0)}{m_i!} x^{m_i} + x^{m_{n-1}} h(x),$$

the Taylor coefficients $\frac{f^{(m_i)}(0)}{m_i!}$ are positive, and $h([0, R)) \subset [0, \infty)$.

Note that throughout this paper, we take the derivatives of $f$ to include the zeroth derivative function $f(x)$. For instance, according to our convention, the function $f(x) = 1 + x$ has two nonzero derivatives at the origin.

The intuitive approach adopted in this paper to prove Theorem A yields rich rewards in tackling more challenging problems. One of the primary goals of this paper is to classify all functions which take $n \times n$ matrices with rank at most $l$ to matrices of rank at most $k$. Our main theorem in this paper completely classifies the functions $f$ such that $f[\cdot] : \mathbb{P}^1_n(I) \to \mathbb{S}^k_n$ or $\mathbb{P}^k_n$ for $l \geq 2$, under the stronger assumption that $f$ is $C^k$ on $I$. Surprisingly, when $k \leq n - 3$ and $f[\cdot] : \mathbb{P}^1_n([0, R)) \to \mathbb{P}^k$, the $C^k$ assumption is not required.

**Theorem B** (Higher rank, fixed dimension). Let $0 < R \leq \infty$ and $I = [0, R)$ or $(-R, R)$. Fix integers $n \geq 2$, $0 \leq k < n - 1$, and $2 \leq l \leq n$. Suppose $f \in C^k(I)$. Then the following are equivalent:

1. $f[A] \in \mathbb{S}^k_n$ for all $A \in \mathbb{P}^l_n(I)$;
2. $f(x) = \sum_{i=1}^r a_i x^{i_1}$ for some $a_i \in \mathbb{R}$ and some distinct $i_t \in \mathbb{Z}_{\geq 0}$ such that

$$\sum_{i=1}^r \left( \frac{i_t + l - 1}{l - 1} \right) \leq k.$$ 

Similarly, $f[\cdot] : \mathbb{P}^l_n(I) \to \mathbb{P}^k_n$ if and only if $f$ satisfies (2) and $a_i \geq 0$ for all $i$. Moreover, if $I = [0, R)$ and $k = 0$ or $k \leq n - 3$, then the assumption that $f \in C^k(I)$ is not required.

Note that when $l = 1$, to obtain Theorem A it suffices to assume $f$ has $k$ nonzero derivatives of arbitrary orders at only the origin. For higher rank $l > 1$, we prove Theorem B under the stronger assumption of $f$ being $C^k$ on all of $I$.

Our last main result concerns entrywise functions preserving positivity for all dimensions, as in the classical case studied by Schoenberg, Rudin, Vasudeva, and many others. We demonstrate that functions preserving positivity over a small family of rank 2 matrices of all dimensions are automatically analytic with nonnegative Taylor coefficients. By contrast, classical results are generally proved under the far
stronger assumption that the function preserves positivity for all positive semidefinite matrices. Before we state the result, first recall the notion of an absolutely monotonic function.

**Definition 1.3.** Let $I \subset \mathbb{R}$ be an interval with interior $I^o$. A function $f \in C(I)$ is said to be **absolutely monotonic** on $I$ if $f \in C^\infty(I^o)$ and $f^{(k)}(x) \geq 0 \ \forall x \in I^o$ and every $k \geq 0$.

**Theorem C** (Rank 2, arbitrary dimension). Let $0 < R \leq \infty$, $I = (0,R)$, and $f : I \to \mathbb{R}$. Then the following are equivalent:

1. For all $n \geq 1$, $f[a1_{n \times n} + uu^T] \in \mathbb{P}_n$ for every $a \in [0, R]$ and $u \in \mathbb{R}^n$ such that $a + u_iu_j \in I$.
2. For all $n \geq 1$, $f[A] \in \mathbb{P}_n$ for every $A \in \mathbb{P}_n(I)$;
3. The function $f$ is absolutely monotonic on $I$.

Theorem C is a significant refinement of the original problem, in which one studies entrywise functions which preserve Loewner positivity among positive semidefinite matrices of all orders, and with no rank constraints. Moreover, condition (1) is a new and much simpler characterization of preserving positivity for all $\mathbb{P}_n$, in the sense that it simplifies condition (2) significantly. The equivalence $(2) \iff (3)$ has been known for some time in the literature in related settings. Vasudeva showed in [38] that $(2) \iff (3)$ for $I = (0, \infty)$, whereas Schoenberg and Rudin showed the same result for $I = (-1, 1)$. See Theorems 2.1 and 2.2.

**Remark 1.4.** Unlike Theorems A and B, Theorem C makes no continuity or differentiability assumptions on $f$. Also note that the family of rank 2 matrices in part (1) of Theorem C is a one-dimensional extension of the set of rank 1 matrices. This is in some sense the smallest family of matrices for which the result can hold. More precisely, the corresponding result to Theorem C for the smaller set of rank 1 matrices is false. For example, $f(x) := x^\alpha$ for $\alpha > 0$ sends $\mathbb{P}^1_1([0, \infty))$ to $\mathbb{P}^1_1$ for all $n$; however, $f$ is not absolutely monotonic unless $\alpha \in \mathbb{N}$. In this sense, Theorem C provides minimal assumptions under which entrywise functions preserving positivity are absolutely monotonic.

The rest of the paper is organized as follows: we begin by reviewing previous work on functions preserving positivity in Section 2 and show how several of these results can be extended to more general settings. In Section 3 we develop a general three-step approach for studying entrywise functions mapping $\mathbb{P}^1_1$ to $\mathbb{S}^k_n$, and use it to prove Theorem A. The results of Section 3 are then extended in Sections 4 and 5 to study the general case of entrywise functions mapping $\mathbb{P}^1_1$ into $\mathbb{S}^k_n$. Finally we demonstrate in Section 6 how our results and techniques can be used to obtain novel and intuitive proofs of the classical results of Schoenberg and Rudin. Along the way, we obtain several characterizations of entrywise functions preserving Loewner positivity in a variety of settings.

2. Previous results and extensions

We begin by reviewing known characterizations of functions preserving positivity, and extending them to other settings (other domains, Hermitian matrices). We first recall a fundamental result by Schoenberg and Rudin, characterizing entrywise...
functions preserving all positive semidefinite matrices. This celebrated characterization has been studied and generalized by many authors under various restrictions; only the most general version is presented here.

**Theorem 2.1** (see Schoenberg [36], Rudin [35], Herz [26], Christensen and Ressel [10]). Given $0 < R \leq \infty$, $I = (-R, R)$, and $f : I \to \mathbb{R}$, the following are equivalent:

1. $f[A] \in \mathbb{P}_n$ for all $A \in \mathbb{P}_n(I)$ and all $n \geq 1$.
2. $f$ is analytic on the disc $D(0, R) := \{z \in \mathbb{C} : |z| < R\}$ and absolutely monotonic on $(0, R)$; i.e., $f$ has a convergent Taylor series on $D(0, R)$ with nonnegative coefficients.

Similarly, the following result was shown by Vasudeva [38] for $I = (0, \infty)$ and also follows from Horn [28] Theorem 1.2.

**Theorem 2.2** (Vasudeva, [38] Theorem 6). Given $I = (0, \infty)$ and $f : I \to \mathbb{R}$, the following are equivalent:

1. $f[A] \in \mathbb{P}_n$ for all $A \in \mathbb{P}_n(I)$ and all $n \geq 1$.
2. $f$ can be extended analytically to $\mathbb{C}$ and is absolutely monotonic on $I$.

Note that the assertions in Theorems 2.1 and 2.2 are very similar, but for different domains of definition. We now extend Theorem 2.2 to general nonnegative intervals.

**Theorem 2.3.** Let $0 \leq a < b \leq \infty$. Assume $I = (a, b)$ or $I = [a, b)$ and let $f : I \to \mathbb{R}$. Then each of the following assertions implies the next one:

1. $f[A] \in \mathbb{P}_n$ for all $A \in \mathbb{P}_n(I)$ and all $n \geq 1$;
2. The function $f$ can be extended analytically to $D(0, b)$ and

$$f(z) = \sum_{n=0}^{\infty} c_n z^n, \quad z \in D(0, b)$$

for some $c_n \geq 0$;
3. $f$ is absolutely monotonic on $I$.

If furthermore, $I = [0, b)$, then (3) $\Rightarrow$ (2) and so all the assertions are equivalent.

Proof. Without loss of generality, we can assume $b < \infty$ (otherwise, the result follows by considering bounded intervals contained in $I$). That (2) $\Rightarrow$ (1) holds follows by the Schur product theorem and the continuity of the eigenvalues.

We now prove (1) $\Rightarrow$ (3). Consider the function $g : (-b, b) \to \mathbb{R}$ given by

$$g(x) := f \left( \frac{b - a}{2b} x + \frac{a + b}{2} \right).$$

Assume first that $I = (a, b)$. Consider the linear change of variables $T : (-b, b) \to (a, b)$, given by

$$T(x) := \left( \frac{b - a}{2b} \right) x + \frac{a + b}{2}.$$ 

Thus $T[B] \in \mathbb{P}_n((a, b))$ for all $B \in \mathbb{P}_n((-b, b))$. This implies that $g[-] = (f \circ T)[-]$ maps $\mathbb{P}_n((-b, b))$ into $\mathbb{P}_n$ for all $n \geq 1$. Thus, by Theorem 2.1 $g$ is analytic on $D(0, b)$ and absolutely monotonic on $(0, b)$. It follows that $f$ is analytic on $(a, b)$ and absolutely monotonic on $((a+b)/2, b)$. Repeating the above construction for all $I = (a, b_0)$ with $a < b_0 < b$, it follows that $f$ is absolutely monotonic on $I = (a, b)$ as desired.
We next assume that \( I = [0, b) \) and (1) holds. Then, clearly, (1) also holds for matrices with entries in \((0, b)\). Thus, from above, \( f \) is analytic and absolutely monotonic on \((0, b)\). To prove that \( f \) is continuous at 0, consider the matrix

\[
A = \begin{pmatrix}
2 & 1 & 1 \\
1 & 2 & 0 \\
1 & 0 & 2
\end{pmatrix} \in \mathbb{P}_3,
\]

and, for \( t > 0 \), let \( A_t = tA \). Since \( A_t \oplus 0_{(n-3)\times(n-3)} \in \mathbb{P}_n((0, b)) \) for \( t \) small enough, applying \( f \) entrywise, we conclude by (1) that \( f[A_t] \in \mathbb{P}_3 \). Also, since \( f \) is absolutely monotonic on \((0, b)\), it is nonnegative and increasing there, and so \( f^+(0) := \lim_{x \to 0^+} f(x) \) exists and is nonnegative. Moreover,

\[
\lim_{t \to 0^+} f[A_t] = \begin{pmatrix}
f^+(0) & f^+(0) & f^+(0) \\
f^+(0) & f^+(0) & f^+(0) \\
f^+(0) & f^+(0) & f^+(0)
\end{pmatrix} \in \mathbb{P}_3.
\]

This is possible only if the principal minors of this matrix are nonnegative. It follows that \( 0 \leq f(0) \leq f^+(0) \) and the determinant, which equals \( -f^+(0)(f^+(0) - f(0))^2 \), is nonnegative. But then \( f^+(0) = f(0) \), i.e., \( f \) is right-continuous at 0. This proves (1) \( \Rightarrow \) (3) if \( I = [0, b) \). The result for \( I = [a, b) \) follows from that for \([0, b-a)\) by using the translation \( g(x) = f(x+a) \), as above.

Finally, by standard results from classical analysis (see Theorem 6.1), the implication (3) \( \Rightarrow \) (2) holds when \( 0 \in I \). \( \square \)

When the function \( f \) is analytic, Theorem 2.4 can easily be extended to complex-valued functions as follows:

**Theorem 2.5.** Suppose \( 0 < R \leq \infty \) and \( f : D(0, R) \to \mathbb{C} \) is analytic. The following are equivalent:

1. \( f[A] \) is Hermitian positive semidefinite for all Hermitian positive semidefinite matrices \( A \) with entries in \( D(0, R) \).
2. \( f \) is absolutely monotonic on \( D(0, R) \); i.e., \( f(z) = \sum_{n \geq 0} a_n z^n \) for real scalars \( a_n \geq 0 \).

The proof is an easy exercise using Theorem 2.4 and the uniqueness principle for analytic functions. Note that there also exist nonanalytic functions preserving positivity in arbitrary dimensions; see [15, Theorem 3.1] for the complete classification of these maps.

As mentioned earlier, very few results characterizing functions preserving positivity on \( \mathbb{P}_n \) for a fixed dimension \( n \) exist. To our knowledge, the only such known result is for \( n = 2 \) by Vasudeva [35, Theorem 2], and it characterizes functions mapping \( \mathbb{P}_2([0, \infty)) \) to \( \mathbb{P}_2 \). We now prove an extension of this result to more general intervals.

**Theorem 2.5.** Let \( 0 < b \leq \infty \), and \( I = (a, b) \) for \( |a| \leq b \), or \( I = [a, b) \) for \(-b \leq a \leq 0 \). Given \( f : I \to \mathbb{R} \), the following are equivalent:

1. \( f[A] \in \mathbb{P}_2 \) for every \( 2 \times 2 \) matrix \( A \in \mathbb{P}_2(I) \).
2. \( f \) satisfies

\[
f(\sqrt{xy})^2 \leq f(x)f(y) \quad \forall x, y \in I \cap [0, \infty)
\]

and

\[
|f(x)| \leq f(y) \quad \forall |x| \leq y \in I.
\]
In particular, if (1) holds, then either \( f \equiv 0 \) on \( I \setminus \{-b\} \) or \( f(x) > 0 \) for all \( x \in I \cap (0, \infty) \). Moreover \( f \) is continuous on \( I \cap (0, \infty) \).

Note that the condition \( |a| \leq b \) is assumed in Theorem 2.5 because no \( 2 \times 2 \) matrix in \( \mathbb{P}_2(I) \) can have any entry in \(( -\infty, -b) \). Also see [27] Lemma 2.1 for the special case where \( I = (0, R) \) for some \( 0 < R < \infty \).

Proof. Let \( A = \begin{pmatrix} p & q \\ q & r \end{pmatrix} \). Clearly, (1) holds if and only if \( f(q)^2 \leq f(p)f(r) \) whenever \( q^2 \leq pr \) for \( p, q, r \in I \), and \( f(p) \geq 0 \) for \( p \in I \cap (0, \infty) \). Thus, if (1) holds, then so does equation (2.5). Equation (2.6) follows easily by considering the matrix \( \begin{pmatrix} y & x \\ x & y \end{pmatrix} \) for \( |x| \leq y \in I \).

Conversely, assume (2) holds. Setting \( x = 0 \) in (2.6) shows that \( f(y) \geq |f(0)| \geq 0 \) whenever \( y \in I \cap (0, \infty) \). Now if \( q^2 \leq pr \) with \( p, r \geq 0 \) and \( q, r \in I \), then applying (2.4) with \( x = q \) and \( y = |q| \), we obtain \( f(q)^2 \leq f(|q|)^2 \). Therefore, by (2.6) and (2.4),

\[
f(q)^2 \leq f(|q|)^2 \leq f(\sqrt{pr})^2 \leq f(p)f(r).
\]

This proves (2) \( \Rightarrow \) (1).

Next, suppose (1) holds and \( f(x) = 0 \) for some \( x \in I \cap (0, \infty) \). We claim that \( f \equiv 0 \) on \( I \cap [0, b) \), which proves via (2.6) that \( f \equiv 0 \) on \( I \setminus \{-b\} \). To see the claim, first define \( x_0 := \sup\{x \in I \cap (0, \infty) : f(x) = 0\} \). Then \( f \) vanishes on \( I \cap [0, x_0) \) by (2.6). We now produce a contradiction if \( x_0 < b \), which proves the claim, and hence all of (2). Indeed if \( x_0 < y \in I \), then choose any \( x_1 \in I \cap (x_0^2/y, x_0) \). Thus, \( \sqrt{x_1y} \in (x_0, y) \subset I \), so by (2.5),

\[
f(\sqrt{x_1y})^2 \leq f(x_1)f(y) = 0.
\]

This contradicts the definition of \( x_0 \), and proves the claim.

Finally, define \( a' := \inf(I \cap (0, \infty)) \), and \( g(x) := \ln f(e^x) \) for \( x \in (\ln a', \ln b) \). It is clear that \( g \) is nondecreasing and mid(point)-convex on the interval \( (\ln a', \ln b) \subset \mathbb{R} \), whenever \( f \) satisfies (2.4). (By a midpoint convex function \( g : J \to \mathbb{R} \) we mean \( g((x + y)/2) \leq (g(x) + g(y))/2 \) for \( x, y \in J \).) Hence by [23] Theorem 71.C, \( g \) is necessarily continuous (and hence convex) on \( (\ln a', \ln b) \). We conclude that \( f \) is continuous on \( (a', b) = I \cap (0, \infty) \).

We note that functions preserving other forms of positivity have been studied by various authors in many settings, including by Ando and Hiai [1], Ando and Zhan [2], Bharali and Holtz [4], Bhatia and Karandikar [6], de Pillis [11], Hansen [23], Marcus and Katz [31], Marcus and Watkins [32], Michelli and Willoughby [33], Thompson [37], Zhang [41], and in previous work [17, 22] by the authors.

3. Preserving positivity under rank constraints I:

The rank 1 case

We begin by studying entrywise functions mapping \( \mathbb{P}_n^1 \) into \( \mathbb{S}_n^k \). In Subsection 3.3 we introduce a three-step approach for studying entrywise functions \( f[-] : \mathbb{P}_n^1 \to \mathbb{S}_n^k \) for \( 1 \leq k < n \) and use it to prove Theorem A. Next, in Subsection 3.2 we study entrywise functions \( f[-] : \mathbb{P}_n^1 \to \mathbb{P}_n \) and demonstrate important connections with the Laplace transform. Finally, in Subsection 3.3 we study generalizations of Theorem A involving the two-sided extensions of the power functions.
3.1. A three-step approach: functions preserving positivity of rank 1 matrices. This subsection is devoted to proving Theorem A. To do so, we adopt the following general and intuitive three-step strategy:

(S1) We begin by characterizing entrywise functions mapping $\mathbb{P}_n^1$ into $S_n^1$.

(S2) Assuming $f(0) = 0$, we show that the rank of $f[A]$ remains the same if $f$ is replaced by $f(x)/x^r$ for a suitable $r > 0$.

(S3) Assuming $f(0) \neq 0$, we prove that if $f$ maps $\mathbb{P}_n^1$ into $S_n^k$, then $f - f(0)$ maps $\mathbb{P}_n^1$ into $S_{n-1}^k$.

Several of our characterization results (including Theorem A) follow by repeatedly applying steps (S2) and (S3) until the result reduces to characterizing functions mapping $\mathbb{P}_n^1$ into $S_n^1$; and then we apply (S1) to obtain the desired result. We thus prove three propositions in this section, which pertain to the above three steps.

We start by recalling the following basic result from linear algebra, which we shall invoke frequently in this paper (see e.g. [12, Theorem IV.16]). Note that the result is seemingly presented in loc. cit. only over any subfield of $\mathbb{C}$, but in fact holds over any field.

**Lemma 3.1** ([12, Theorem IV.16]). Let $A$ be a symmetric $n \times n$ matrix over any field, and let $1 \leq r \leq n$. Then the following are equivalent:

1. rank $A \leq r$;
2. All $(r+1) \times (r+1)$ minors of $A$ (when defined) vanish;
3. All $(r+1) \times (r+1)$ and $(r+2) \times (r+2)$ principal minors of $A$ (when defined) vanish.

We now characterize functions mapping $\mathbb{P}_n^1$ into $S_n^1$, as in the first step (S1) outlined at the beginning of this subsection.

**Proposition 3.2.** Let $0 < b \leq \infty$, and $I = (a, b)$ for $|a| \leq b$, or $I = [a, b)$ for $|a| < b$. Let $f : I \to \mathbb{R}$. Then:

1. $f[A] \in S_n^1$ for every $A \in \mathbb{P}_n^1(I)$ if and only if $f(\pm \sqrt{xy})^2 = f(x)f(y)$ for all $x, y \in I \cap [0, \infty)$ such that $\pm \sqrt{xy} \in I$.
2. Suppose $0 \in I$, $f(p) = 0$ for some $p \in I \setminus \{0\}$, and $f[-] : \mathbb{P}_n^1(I) \to S_n^1$ for some $n \geq 2$. Then $f \equiv 0$ on $I$.

**Proof.**

(1) This follows immediately since $\det f[A] = 0$.

(2) Suppose $f(p) = 0$ for some $p \in I \setminus \{0\}$. We first claim that there exists a sequence of positive numbers $p_m \in I \cap (0, \infty)$ increasing to $b$ such that $f(p_m) = 0$. Define the numbers $p_m$ and the matrices $A_m$ inductively as follows:

$$p_1 := |p| \in (0, b), \quad p_{m+1} := \sqrt[p_m]{b} \quad \forall m \in \mathbb{N},$$

$$A_m := \begin{pmatrix} p_m & p_{m+1} \\ p_{m+1} & p^2_{m+1}/p_m \end{pmatrix} \oplus 0_{(n-2) \times (n-2)}.$$ 

It is easily verified that $p_m$ lies in $I$ and increases to $b$, and $A_m \in \mathbb{P}_n^1(I)$ for all $m$. Now applying $f$ entrywise to the matrix $\begin{pmatrix} |p| & p \\ p & |p| \end{pmatrix} \oplus 0_{(n-2) \times (n-2)} \in \mathbb{P}_n^1(I)$ shows by (1) that $f(p_1) = 0$. Next, using that $f[A_m] \in S_n^1$ implies inductively that $f(p_{m+1}) = 0$ for all $m$, which shows the claim.
Now let $q \in I$. Then $q \in I \cap (-p_m, p_m)$ for some $m$, in which case applying $f$ entrywise to the rank 1 matrix \((p_m q q^2/p_m) \oplus 0_{(n-2) \times (n-2)} \in \mathbb{P}_n^1(I)\) shows by (1) that $f(q) = 0$.

\[\square\]

Remark 3.3. Note that the functions that send $\mathbb{P}_n^1(I)$ to $\mathbb{P}_2$ instead of the larger set $\mathbb{S}_n^1$ can be characterized by the same conditions as in Proposition 3.2 (1), together with the fact that $f \geq 0$ on $I \cap [0, \infty)$. The proof is similar to the one above.

We now present an elegant characterization of continuous functions mapping $\mathbb{P}_n^1(I)$ into $\mathbb{S}_n^1$. To state the result, we first define the even and odd extensions of the power functions $f_\alpha(x) := x^\alpha$, to the entire real line, as follows:

\[(3.1) \quad \phi_\alpha(x) := |x|, \quad \psi_\alpha(x) := \text{sgn}(x)|x|^\alpha, \quad \forall \alpha > 0, \ x \in \mathbb{R}.

**Lemma 3.4.** Let $0 < b \leq \infty$, and $I = (a, b)$ for $|a| \leq b$, or $I = [a, b)$ for $|a| < b$. Let $n \geq 2$ and $f : I \to \mathbb{R}$ be continuous. Then the following are equivalent:

1. $f$ is continuous on $I$.
2. $f[A] \subseteq \mathbb{S}_n^1(I)$ for all $A \subseteq \mathbb{S}_n^1(I)$.
3. $f$ is continuous on $I \cap [0, \infty)$.

Moreover, $f[-] : \mathbb{P}_n^1(I) \to \mathbb{P}_n^1$ if and only if (3) holds with $c \geq 0$.

**Proof.** Clearly (3) $\Rightarrow$ (1) $\Rightarrow$ (2) by (a variant of) Proposition 3.2 (1). We now show that (2) $\Rightarrow$ (3). If (2) holds, then the function $f$ satisfies

$$f(\sqrt{xy}) = \pm \sqrt{|f(x)||f(y)|} \quad \forall x, y \in I \cap [0, \infty),$$

since $\mathbb{P}_n^1(I)$ embeds into $\mathbb{P}_n^1(I)$ via padding by zeros. Since $f$ is continuous on $I \setminus \{0\}$, it follows that for every $0 \leq \lambda \leq 1$,

$$f(x^\lambda y^{1-\lambda}) = \pm |f(x)|^\lambda |f(y)|^{1-\lambda} \quad \forall x, y \in I \cap (0, \infty).$$

Equivalently, the function $g(x) := \ln |f(e^x)|$ satisfies

$$g(\lambda x + (1-\lambda)y) = \lambda g(x) + (1-\lambda)g(y) \quad \forall x, y \in \ln(I \cap (0, \infty)).$$

Thus, $g(x) = \alpha x + \beta$ for some constants $\alpha, \beta \in \mathbb{R}$. As a consequence, $f(x) = cx^\alpha$ for all $x \in I \cap (0, \infty)$, where $|c| = e^\beta$.

It remains to compute $f$ on $I \cap (-\infty, 0)$. Suppose $x \in I \cap (-\infty, 0)$; then applying $f$ entrywise to the matrices

\[(3.4) \quad \begin{pmatrix} x \mid \phi_\alpha(x) \mid \psi_\alpha(x) \end{pmatrix} \oplus 0_{(n-2) \times (n-2)} \in \mathbb{P}_n^1(I)\]

shows that $f(x) = \pm f(|x|)$. There are now two cases: first if $f(x) = cx^\alpha$ on $I \cap (0, \infty)$, with $c = 0$ or $\alpha = 0$, then since $f$ is continuous on $I$, it is easy to check that $f$ is constant on $I$. The second case is if $c \neq 0$ and $\alpha \neq 0$. Then $\alpha > 0$ as $f$ is continuous on $I$ and $0 \in I$ by assumption. Moreover, on the interval $I \cap (-\infty, 0)$, the function $f(|x|)$ is continuous and has image in $[-c, c]$, by the above analysis. Hence $f(x)/|x|^\alpha$ is constant on $I \cap (-\infty, 0)$. Thus $f(x) \equiv c|x|^\alpha$ or $-c|x|^\alpha$ for all $0 > x \in I$, which shows (3). Given these equivalences, it is clear that $f[-] : \mathbb{P}_n^1(I) \to \mathbb{P}_n^1$ if and only if $c \geq 0$. \[\square\]

Lemma 3.3 addresses the first step (S1) outlined at the beginning of this subsection. The following proposition will play a central role later and addresses the second step (S2).
Proposition 3.5. Let $0 < R \leq \infty$, $I = [0, R)$ or $(-R, R)$, and $f, g : I \to \mathbb{R}$ such that $g(x)$ is nonzero whenever $x$ is nonzero. Assume $c := \lim_{x \to 0, x \in I} f(x)/g(x)$ exists and define

$$h_c(x) := \begin{cases} \frac{f(x)}{g(x)}, & x \neq 0 \\ c, & x = 0. \end{cases}$$

Fix integers $n \geq 2$ and $1 \leq k \leq n$.

1. Suppose $g[-] : \mathbb{P}_n^1(I) \to \mathbb{S}_n^k$. If $f[-] : \mathbb{P}_n^1(I) \to \mathbb{S}_n^k$, then $h_c[-] : \mathbb{P}_n^1(I) \to \mathbb{S}_n^k$.

The converse holds if $f(0) = cg(0)$.

2. Suppose $g[-] : \mathbb{P}_n^k(I) \to \mathbb{P}_n^k$, and $f[-] : \mathbb{P}_n^k(I) \to \mathbb{P}_n$. Then $c \geq 0$ and $h_c[A] \in \mathbb{P}_n$ for all $A \in \mathbb{P}_n^k(I)$.

Proof. We prove the result for $I = (-R, R)$; the proof is similar for $I = [0, R)$. Note that (1) is trivial if $k = n$; thus, we assume that $k < n$ and prove (1). First note that if $A, B \in \mathbb{S}_n$ and $B$ is a rank 1 matrix with nonzero entries, then $B^{c(-1)} := (b_{ij}^{(1)}/\sqrt{n})_{i,j=1}$ also has rank 1. Hence, since rank $A \circ B \leq$ (rank $A$)(rank $B$) (see [29], Theorem 5.1.7) we obtain

$$\text{rank } A = \text{rank } (A \circ B) \circ B^{c(-1)} \leq \text{rank } (A \circ B) \text{ rank } B^{c(-1)}$$

$$= \text{ rank } A \circ B \leq \text{ rank } A \cdot \text{ rank } B = \text{ rank } A.$$

We conclude that rank $A \circ B = \text{ rank } A$.

Now suppose $f, g$ satisfy the assumptions. Then by Proposition 3.2(2) applied to $g$. $g(x) \neq 0$ for all $0 \neq x \in I$ since $g \neq 0$. Thus given $c \in \mathbb{R}$, and nonzero $u_i$ such that $u_i u_j \in I \forall i, j$, we have

$$h_c[u_i u_j^T] = \left(\frac{f(u_i u_j)}{g(u_i u_j)}\right)_{i,j=1}^n = f[u_i u_j^T] \circ (1/g)[u_i u_j^T].$$

It follows from (3.6) that

$$\text{rank } h_c[u_i u_j^T] = \text{ rank } f[u_i u_j^T] \leq k.$$

Next, suppose $u \in \mathbb{R}^n$ such that $u_i u_j \in I \forall i, j$, and let $0 < \epsilon < \sqrt{R}$. Define $u^\epsilon \in \mathbb{R}^n$ to be the vector with coordinates $u_i + \epsilon \delta_{u_i, 0}$, where $\delta_{a,b}$ denotes the Kronecker delta. Hence by (3.6), rank $h_c[u^\epsilon u^\epsilon^T] = \text{rank } f[u^\epsilon u^\epsilon^T] \leq k$. Now, by Lemma 3.1, every $(k + 1) \times (k + 1)$ minor of $h_c[u^\epsilon u^\epsilon^T]$ is equal to zero for all $\epsilon > 0$. By the continuity of $h_c$ at 0 and continuity of the determinant function, it follows that the same is true for $h_c[u u^T]$. Thus, rank $h_c[u u^T] \leq k$ for all $u \in \mathbb{R}^n$ such that $u_i u_j \in I \forall i, j$, proving the first result. Conversely, if $f(0) = cg(0)$ and $h_c[-] : \mathbb{P}_n^1(I) \to \mathbb{S}_n^k$, then using a similar argument as above, we obtain $f[-] : \mathbb{P}_n^1(I) \to \mathbb{P}_n$.

We now prove the second part. Note first that $f(x), g(x) \geq 0$ for all $x \in I \cap [0, \infty)$ since $f[A], g[A] \in \mathbb{P}_n$ for all $A \in \mathbb{P}_n^k(I)$. Thus, for every $\epsilon \in I \cap [0, \infty)$, $(f/g)(\epsilon) \geq 0$ and so $c = \lim_{\epsilon \to 0+} (f/g)(\epsilon) \geq 0$. Now let $A = u u^T \in \mathbb{P}_n^k(I)$. If $u_i \neq 0$ for all $i$, then $h_c[A] \in \mathbb{P}_n$ by (3.7) and the Schur product theorem. The general case follows by a limiting argument, replacing $u$ by $u^\epsilon$ as above.

Our last proposition in this section addresses the third step (S3) outlined at the beginning of the present subsection.
Proposition 3.6. Let $0 < R \leq \infty$, $I = [0, R)$ or $(-R, R)$, and $h : I \to \mathbb{R}$ be such that $h(0) \neq 0$. Fix integers $n \geq 2$ and $1 \leq k, l \leq n$. Consider the following statements:

(1) $h[A] \in S_k^n$ for all $A \in P_n(I)$;
(2) $(h - h(0))[A] \in S_{k-1}^{n-1}$ for all $A \in P_n(I)$.

Then (2) $\Rightarrow$ (1) $\Rightarrow$ (3). If $k < n - 1$ then (1) $\Rightarrow$ (2). The implications (1) $\Rightarrow$ (2) (when $k < n - 1$) and (1) $\Rightarrow$ (3) also hold upon replacing the sets $S_k^n, S_{k-1}^{n-1}$, $S_{k-1}^{n-1}$ by $P_k^n, P_{k-1}^{n-1}, P_{k-1}^{n-1}$ respectively.

Proof. That (2) $\Rightarrow$ (1) is clear. We now show that (1) $\Rightarrow$ (3). Note that the statement is trivial if $k = n$ so we assume $1 \leq k < n$. To do so, we fix $A \in P_{n-1}(I)$ and without loss of generality, consider any $k \times k$ minor $M$ of $A$. Then

$$
M = \begin{pmatrix}
A \\
0_{1 \times (n-1)}
\end{pmatrix}
$$

is a $(k+1) \times (k+1)$-minor of the matrix $B := \begin{pmatrix} A & 0_{(n-1) \times 1} \\
0_{1 \times (n-1)} & 0 \end{pmatrix} \in P_n(I)$. Consequently by applying (1) to $B$,

$$
\det \left( \begin{pmatrix} h[M] & h(0)1_{k \times 1} \\
0_{1 \times k} & h(0) \end{pmatrix} \right) = 0.
$$

Equivalently, subtracting the last column from every other column, we obtain

$$
\det \left( \begin{pmatrix} (h - h(0))[M] & h(0)1_{k \times 1} \\
0_{1 \times k} & h(0) \end{pmatrix} \right) = h(0) \cdot \det((h - h(0))[M] = 0.
$$

We conclude that $\det((h - h(0))[M] = 0$ for all $k \times k$ minors $M$ of $A$. In particular, $\rk(h - h(0))[M] < k$ by Lemma 3.1.2. This concludes the proof of (1) $\Rightarrow$ (3).

We now show that (1) $\Rightarrow$ (2) when $k < n - 1$. If (1) holds, then by Lemma 3.1.3, (2) holds if and only if every principal $(k+i) \times (k+i)$ minor of $(h - h(0))[A]$ vanishes for each $A \in P_n(I)$ and $i = 0, 1$. To show that this is indeed the case, fix $A \in P_n(I)$ and a subset $J \subset \{1, \ldots, n\}$ of $k + i$ indices. Without loss of generality, consider the principal submatrix $A_J$ formed by the rows and columns of $A$ corresponding to $J$. Let $A' := A_J \oplus 0_{(n-k-i) \times (n-k-i)} \in P_n(I)$. By (1) and Lemma 3.1.3, the leading principal $(k+i+1) \times (k+i+1)$ minor of $h[A']$ vanishes. In other words,

$$
\det h[A_J \oplus 0_{1 \times 1}] = \det \left( \begin{pmatrix} h[A_J] & h(0)1_{(k+i) \times 1} \\
0_{1 \times (k+i)} & h(0) \end{pmatrix} \right) = 0.
$$

As in the previous case, it follows that every principal $(k+i+1) \times (k+i+1)$ minor of $(h - h(0))[A]$ vanishes for every $A \in P_n(I)$ and $i = 0, 1$. Therefore, by Lemma 3.1.3, $(h - h(0))[A] \in S_{k-1}^{n-1}$ for every $A \in P_n(I)$, which proves (2).

Finally, we show that (1) $\Rightarrow$ (2) (when $k < n - 1$) and (1) $\Rightarrow$ (3) when the $S$-sets are replaced by the $P$-sets. We first claim that for all $n_1, n_2 \in \mathbb{N}$, $c \in \mathbb{R}$, and $B \in P_{n_1}(\mathbb{R})$

$$
B_c := \begin{pmatrix} B & c1_{n_1 \times n_2} \\
1_{n_2 \times n_1} & c1_{n_2 \times n_2} \end{pmatrix} \in P_{n_1+n_2}(\mathbb{R}) \iff c \geq 0, \quad B - c1_{n_1 \times n_1} \in P_{n_1}(\mathbb{R}).
$$

(3.9)
The claim \((3.9)\) is obvious for \(c = 0\); thus, we now assume that \(c \neq 0\). Let \(C := c I_{n_2 \times n_2} \) and \(D := c I_{n_2 \times n_1}\). Then \(C^T = \frac{1}{c} I_{n_2 \times n_2}\), and we have the decomposition

\[
B_c = \begin{pmatrix} B & DT^T \\ D & C \end{pmatrix} = \begin{pmatrix} Id_{n_1} & DT^T C^T \\ 0 & Id_{n_2} \end{pmatrix} \begin{pmatrix} B - DC^T D^T & 0 \\ 0 & C \end{pmatrix} \begin{pmatrix} Id_{n_1} & 0 \\ C^T D & Id_{n_2} \end{pmatrix},
\]

where \(Id_n\) denotes the \(n \times n\) identity matrix. By Sylvester’s law of inertia, \(B_c\) is positive semidefinite if and only if \(B - DC^T D^T = B - c I_{n_1 \times n_1}\) and \(C\) is positive semidefinite. This proves the claim.

Now suppose for the remainder of the proof that \(h[-] : \mathbb{P}^l_n(I) \rightarrow \mathbb{P}^k_n\). Observe that \(h(0) > 0\) because \(h[0_{n \times n}] \in \mathbb{P}^k_n\). We now assume \(1 \leq k < n - 1\) and show that the modified statement of \((3)\) holds, i.e., \((h - h(0))[A] \in \mathbb{P}^k_{n-1}\) for all \(A \in \mathbb{P}^l_{n-1}(I)\).

Indeed, given \(A \in \mathbb{P}^l_{n-1}(I)\), it follows from \((1) \Rightarrow (3)\) that \((h - h(0))[A] \in \mathbb{P}^k_{n-1}\). Applying the claim \((3.9)\) with \(B = h[A]\) and \(c = h(0)\), it follows that \((h - h(0))[A] \in \mathbb{P}^k_{n-1}\) as well, proving that \((1) \Rightarrow (3)\) for the \(\mathbb{P}\)-sets.

Finally, suppose \(h[-] : \mathbb{P}^l_n(I) \rightarrow \mathbb{P}^k_n\) and \(A \in \mathbb{P}^l_n(I)\). We now show that \((h - h(0))[A] \in \mathbb{P}^k_n\). Indeed, it follows from \((1) \Rightarrow (2)\) implication that \((h - h(0))[A] \in \mathbb{P}^k_{n-1}\). Since \((h - h(0))[A]\) is singular, it suffices to show that all its \(n_1 \times n_1\) principal minors are nonnegative for \(1 \leq n_1 \leq n - 1\). Let \(C\) be any \(n_1 \times n_1\) principal submatrix of \(A\). Applying the claim \((3.9)\) with \(B = h[C]\) and \(c = h(0)\), it follows that \((h - h(0))[C] \in \mathbb{P}^k_n\). This concludes the proof.

In the special case where \(l = 1\) and \(k = 2\), Proposition \((3.6)\) immediately characterizes the functions \(f\) mapping \(\mathbb{P}^1_n\) to \(\mathbb{P}^2_n\) under the assumption \(f(0) \neq 0\):

**Corollary 3.7.** Let \(n \geq 3\), \(I = [0, R)\) or \((-R, R]\) for some \(0 < R \leq \infty\), \(f : I \rightarrow \mathbb{R}\) be continuous and suppose \(f(0) \neq 0\). Then the following are equivalent:

1. \(f[A] \in \mathbb{P}^2_n\) for all \(A \in \mathbb{P}^1_n(I)\);
2. \(f(x) = a + b \phi_\alpha(x)\) or \(f(x) = a + b \psi_\alpha(x)\) for \(a \neq 0\), \(\alpha > 0\), and \(b \in \mathbb{R}\).

Recall that the power functions \(\phi_\alpha, \psi_\alpha\) were defined in equation \((3.1)\).

**Proof.** Clearly \((2) \Rightarrow (1)\) by Lemma \((3.4)\). To show the converse, apply the implication \((1) \Rightarrow (3)\) of Proposition \((3.6)\) with \(h\) replaced by \(f\) to obtain that \((f - f(0))[\cdot] : \mathbb{P}^1_{n-1}(I) \rightarrow \mathbb{P}^2_{n-1}\). The result now follows by Lemma \((3.4)\).

We now have all the ingredients needed to prove the first main result of the paper.

**Proof of Theorem A** We show the result for \(I = (-R, R)\); the proof is similar for \(I = [0, R)\). The proof proceeds by building up the polynomial function \(f(x)\) step by step, in a way that is similar to Horner’s algorithm. In order to do so, given \(r \in \mathbb{Z}_{\geq 0}\), define an operator \(T_r\) mapping any function \(h : I \rightarrow \mathbb{R}\) admitting at least \(r\) left and right derivatives at zero, via:

\[
T_r(h)(x) := \begin{cases} 
\frac{h(x) - h(r) x^r}{x^r} & \text{if } x \neq 0 \\
0 & \text{if } x = 0.
\end{cases}
\]

Denote by \(f^{(m_1)}, f^{(m_2)}, \ldots, f^{(m_k)}\) the first \(k\) derivatives of \(f\) that are nonzero at \(0\), with \(0 \leq m_1 < \cdots < m_k\) (where \(k\) is taken to be \(n\) for part \((3)\) of the statement). Here we define \(f^{(m)} = f\) when \(m = 0\). Also define \(m_0 := 0\) for notational convenience. Now inductively construct the function \(M_i(x)\) for \(0 \leq i \leq k - 1\) via
$M_0(x) := f(x)$ and $M_i(x) := T_{m_i - m_{i-1}} M_{i-1}(x)$ for $1 \leq i \leq k - 1$. We now claim that

$$M_i(x) = \sum_{j=i+1}^{k-1} \frac{f^{(m_j)}(0)}{m_j!} x^{m_j - m_i} + O(x^{1 + m_{k-1} - m_i}), \quad \forall x \in I, \ 0 \leq i \leq k - 1.$$  

(3.11)

For $i = 0$, the claim is easily verified using Taylor’s theorem. Now apply the operators $T_{m_i - m_{i-1}}$ inductively to verify the claim for each $0 \leq i \leq k - 1$.

It follows from (3.11) that $M_i$ is continuous at zero for all $0 \leq i \leq k - 1$. Next, we claim that for $i = 0, \ldots, k - 1$ and $A \in \mathbb{P}_{n-1}(I)$, we have $M_i[A] \in S^{k-i}_{n-i}$. We will prove the claim by induction on $i \geq 0$. Clearly the result holds if $i = 0$. Now assume it holds for some $i - 1 \geq 0$. By definition, for $x \neq 0$,

$$M_i(x) = x^{-(m_i - m_{i-1})} \left( M_{i-1}(x) - \frac{f^{(m_i)}(0)}{m_i!} x^{m_i - m_{i-1}} \right) = \frac{M_{i-1}(x) - f^{(m_i)}(0)}{x^{m_i - m_{i-1}}}. \quad (3.12)$$

By Propositions 3.5 and 3.6 and the induction hypothesis, it follows that $M_i[A] \in S^{k-(i-1)-1}_{n-(i-1)-1} = S^{k-i}_{n-i}$ for all $A \in \mathbb{P}_{n-i}^{1}(I)$. This completes the induction and proves the claim for all $0 \leq i \leq k - 1$. In particular, $M_{k-1}[A] \in S^{1}_{n-k+1}$ for all $A \in \mathbb{P}^{1}_{n-k+1}(I)$. Moreover, $M_{k-1}$ is continuous on $I$ by (3.11). We now complete the proofs of the three parts separately.

Proof of (1). By Lemma 3.5, $M_{k-1}(x) = a \phi_{\alpha}(x)$ or $a \psi_{\alpha}(x)$ for some $\alpha > 0$ and $a \in \mathbb{R}$. Working backwards, it follows that $f(x) = P(x) + c \phi_{\alpha}(x)$ or $f(x) = P(x) + c \psi_{\alpha}(x)$ where $P$ is a polynomial with exactly $k - 1$ nonzero coefficients, $c \in \mathbb{R}$, and $\gamma \geq m_{k-1}$. Let $m_k$ be the least positive integer such that $m_k > m_{k-1}$ and $f^{(m_k)}(0)$ exists and is nonzero. Then we obtain that $\gamma = m_k$, so that $f$ is a polynomial with exactly $k$ nonzero coefficients (and hence exactly $k$ nonzero derivatives at zero). This proves the first part of the theorem.

Proof of (2). To show the second part, we claim that $M_i[A] \in \mathbb{P}_{n-i}^{k-i}$ for all $A \in \mathbb{P}_{n-i}^{1}(I)$ and $0 \leq i \leq k - 1$. Indeed, the result clearly holds for $i = 0$. Now assume the result holds for $i - 1 \geq 0$. Note first that by applying Proposition 3.5(2) to $f(x) = M_{i-1}(x)$, $g(x) = x^{m_i - m_{i-1}}$ and $c = f^{(m_i)}(0)$, we obtain that $f^{(m_i)}(0) \geq 0$. Moreover, by (3.12) and Propositions 3.5 and 3.6 it follows that $M_i[A] \in \mathbb{P}_{n-i}^{k-i}$ for all $A \in \mathbb{P}_{n-i}^{1}(I)$. Using a similar argument as in part (1) together with Lemma 3.5 it follows that $f$ is a polynomial with nonnegative coefficients and exactly $k$ nonzero coefficients.

Proof of (3). For the third part, we obtain that $h(x) := M_{k-1}(x) = M_{n-1}(x)$ satisfies $h[-] : \mathbb{P}_{n}^{1}(I) \to \mathbb{P}_{n}$. Therefore, $h$ maps $I \cap [0, \infty)$ to $[0, \infty)$. Working backwards and reasoning as in the previous parts, it follows that $f(x) = P(x) + x^{m_n - 1} h(x)$ for a polynomial $P(x)$ with exactly $k - 1$ positive coefficients, and $h : I \to \mathbb{R}$ such that $h(I \cap [0, \infty)) \subset [0, \infty)$. □

Remark 3.8. Part (3) of Theorem A provides a necessary condition for a function to map every rank 1 $n \times n$ positive semidefinite matrix with positive entries to an $n \times n$ positive semidefinite matrix. Note that even when $f(x) = \sum_{i=0}^{N} c_{i} x^{\alpha_i}$ for $\alpha_i \geq 0$, condition (3) does not imply that all the coefficients $c_{i}$ are nonnegative. Indeed, the function $h$ (in the statement of the theorem) could be a sum of powers
containing some negative coefficients, as long as $h$ is nonnegative on $I$. It can however be shown that the first and last $n$ coefficients of $f$ have to be positive (see [13] for more details).

An important special case of interest in the literature is to study which analytic functions preserve positivity. The following result characterizes the analytic entrywise maps $f[-]$ sending $P^+_n(I)$ to $S^k_n, P^k_n$. Note that the third part of Theorem 3.9 generalizes Theorem A.3.

**Theorem 3.9** (Rank 1, fixed and arbitrary dimension). Let $0 < R \leq \infty$, $I = [0, R)$ or $(-R, R)$, and let $f : I \to \mathbb{R}$ be analytic on $I$. Also fix $1 \leq k < n \in \mathbb{N}$.

1. Then $f[A] \in S^k_n$ for all $A \in P^+_n(I)$ if and only if $f$ is a polynomial with at most $k$ nonzero coefficients.

2. Similarly, $f[A] \in P^k_n$ for all $A \in P^+_n(I)$ if and only if $f$ is a polynomial with at most $k$ nonzero coefficients, all of which are positive.

3. Furthermore, $f[A] \in P_n$ for all $A \in P^+_n(I)$ and all $n \in \mathbb{N}$, if and only if $f$ is absolutely monotonic on $I$.

**Proof.** First suppose $I = [0, R)$. The first two parts follow immediately from Theorem A since $f$ is analytic (considering the cases when $f$ has at least $k$ nonzero derivatives at the origin, and when it does not). Next, the sufficiency in the third part follows from the Schur product theorem. To show the necessity, it suffices to show by standard results from classical analysis (see Theorem 6.1) that $f(x) = \sum_{i=0}^{\infty} a_i x^i$ on $I$, with $a_i \geq 0$ for all $i$. Now applying Theorem A.3, it follows that $f^{(i)}(0) \geq 0$ for every $i \geq 0$; i.e., $f$ is absolutely monotonic on the positive real axis. Finally, if $I = (-R, R)$, then the result follows from the above analysis and the uniqueness principle for analytic functions. □

3.2. **Preserving positivity of rank 1 matrices and Laplace transforms.** We continue our study of rank constrained functions by exploring functions mapping $P^+_n$ into $P_n$ for all $n \geq 1$. Such functions can be characterized using the Laplace transform via the theory of positive definite kernels, which we recall for the reader’s convenience.

**Definition 3.10** ([39, Chapter VI]). Let $I \subset \mathbb{R}$. A function $k : I \times I \to \mathbb{R}$ is a **positive definite kernel** on $I$ if for every finite sequence $(x_i)_{i=1}^n \subset I$ of distinct numbers, the quadratic form

$$Q(\xi) = \sum_{i=1}^n \sum_{j=1}^n k(x_i, x_j) \xi_i \xi_j \quad (\xi \in \mathbb{R}^n)$$

(3.13)

is positive semidefinite. Equivalently, for every finite sequence $(x_i)_{i=1}^n \subset I$ of distinct numbers, the matrix $(k(x_i, x_j))_{ij}$ is positive semidefinite.

Recall from classical results in analysis that positive definite kernels can be characterized using the Laplace transform:

**Theorem 3.11** ([39, Chapter VI, Theorem 21]). A function $f : (0, \infty) \to \mathbb{R}$ can be represented as $f(x) = \int_{-\infty}^{\infty} e^{-\alpha x} d\mu(\alpha)$ for a positive measure $\mu$ on $\mathbb{R}$ if and only if $f$ is continuous and the kernel $k(x, y) := f(x + y)$ is positive definite on $(0, \infty)$. Moreover, if $f$ can be written in the above form, then $f$ is analytic on $(0, \infty)$.

Using Theorem 3.11, we now easily obtain the following characterization of entrywise functions defined on $(0, R)$, which map $P^+_n((0, R))$ into $P_n$ for every $n \geq 1$. 
Theorem 3.12. Given $0 < R \leq \infty$ and $f : (0, R) \to \mathbb{R}$, the following are equivalent:

1. $f$ is continuous on $(0, R)$ and $f[A] \in \mathbb{P}_n$ for all $A \in \mathbb{P}^1((0, R))$ and all $n$;
2. $f$ is continuous on $(0, R)$ and the kernel $k(x, y) = f(e^{-(x+y-ln R)})$ is positive definite on $(0, \infty)$;
3. $g(x) := f(e^{-x})$ is continuous on $(-\ln R, \infty)$ and the kernel $k(x, y) := g(-\ln R + x + y)$ is positive definite on $(0, \infty)$;
4. There exists a positive measure $\mu$ such that the function $g(x) := f(e^{-x})$ can be represented as

$$g(x) = \int_{-\infty}^{\infty} e^{-\alpha x} d\mu(\alpha) \quad (x > -\ln R);$$

5. There exists a positive measure $\nu$ such that

$$f(x) = \int_{-\infty}^{\infty} x^\alpha d\nu(\alpha) \quad (0 < x < R);$$

In particular, $g(x) := f(e^{-x})$ is analytic on $(-\ln R, \infty)$.

Proof. Clearly, (5) $\Leftrightarrow$ (4). Now suppose (4) is true and consider the function $h(x) := g(x - \ln R)$ for $x > 0$. Then

$$h(x) = \int_{-\infty}^{\infty} R^\alpha e^{-\alpha x} d\mu(\alpha) = \int_{-\infty}^{\infty} e^{-\alpha x} d\nu(\alpha) \quad (x > 0),$$

where $d\nu(\alpha) = R^\alpha d\mu(\alpha)$. By Theorem 3.11 the kernel $h(x + y) = g(x + y - \ln R)$ is positive definite. This proves (3). Conversely, if (3) is true, then there exists a positive measure $\nu$ such that

$$h(x) = \int_{-\infty}^{\infty} e^{-\alpha x} d\nu(\alpha) \quad (x > 0).$$

It follows that

$$g(x) = \int_{-\infty}^{\infty} R^{-\alpha} e^{-\alpha x} d\nu(\alpha) = \int_{-\infty}^{\infty} e^{-\alpha x} d\mu(\alpha) \quad (x > -\ln R),$$

where $d\mu(\alpha) = R^{-\alpha} d\nu(\alpha)$. This proves (4) $\Leftrightarrow$ (3). Clearly, (3) $\Leftrightarrow$ (2). Finally, (2) is true if and only if the matrix $(f(Re^{-x_i}e^{-x_j}))_{ij}$ is positive semidefinite for every $x_1, \ldots, x_n \geq 0$. Equivalently, $f[A]$ is positive semidefinite for every positive semidefinite matrix $A$ of rank 1 with entries on $(0, R)$. Therefore, (2) $\Leftrightarrow$ (1). Finally, that $g$ is analytic also follows from Theorem 3.11. \qed

Recall that part (3) of Theorem 3.9 provides a direct characterization from first principles of the analytic maps sending $\mathbb{P}^1_n(I)$ to $\mathbb{P}_n$. The same result can also be obtained using deep results about the representability of functions as the Laplace transforms of positive measures, and the uniqueness principle for the Laplace transform (see [39, Chapter VI, Theorem 6a]).

Theorem 3.13. Let $0 < R \leq \infty$, $I = [0, R)$ or $(-R, R)$, and let $f : I \to \mathbb{R}$ be analytic on $I$. Then $f[-] : \mathbb{P}^1_n(I) \to \mathbb{P}_n$ for all $n$ if and only if $f$ is absolutely monotonic on $I$. 


Proof. Let \( f(z) = \sum_{n=0}^{\infty} a_n z^n \) for every \( z \) in an open set in \( \mathbb{C} \) containing \( I \). Then
\[
g(x) := f(e^{-x}) = \sum_{n=0}^{\infty} a_n e^{-nx} \quad (x > -\ln R).
\]
Using this power series representation, the function \( g \) can be extended analytically to every \( z \) in the half-plane \( \{ z \in \mathbb{C} : \Re z > -\ln R \} \), i.e., \( g(z) = \sum_{n=0}^{\infty} a_n e^{-nz} \) whenever \( \Re z > -\ln R \). Since \( f[A] \) is positive semidefinite for every positive semidefinite matrix \( A \) of rank 1 with coefficients in \( (0, R) \) then, by Theorem 3.12, there exists a positive measure \( \mu \) such that
\[
g(x) = \int_{-\infty}^{\infty} e^{-\alpha x} d\mu(\alpha) \quad (x > -\ln R).
\]
The function
\[
\bar{g}(z) = \int_{-\infty}^{\infty} e^{-\alpha z} d\mu(\alpha) \quad (\Re z > -\ln R)
\]
provides an analytic extension of \( g \) to \( \{ z \in \mathbb{C} : \Re z > -\ln R \} \). Since \( g \) and \( \bar{g} \) are both analytic and coincide on \( x > -\ln R \), by the uniqueness principle for analytic functions, \( g(z) = \bar{g}(z) \) for every \( z \in \{ w \in \mathbb{C} : \Re w > -\ln R \} \). Now, since \( g \) and \( \bar{g} \) are both bilateral Laplace transforms and coincide in a common strip of convergence, by \textit{[9]} Chapter VI, Theorem 6a, the two representing measures must coincide. In other words, \( \mu = \sum_{n=0}^{\infty} a_n \delta_n \), where \( \delta_n \) denotes the Dirac measure at the integer \( n \). Since \( \mu \) is positive, it follows that \( a_n \geq 0 \) for every \( n \geq 0 \) and so \( f \) is absolutely monotonic. The converse follows immediately from the Schur product theorem. \hfill \Box

3.3. Two-sided extensions of power functions. Thus far in this section, we have worked mostly with polynomials as the continuous functions sending \( \mathbb{P}_n^1(I) \) to \( \mathbb{P}_n^k \) for integers \( 1 \leq k < n \), and \( I = [0, R] \) or \((-R, R)\) for some \( 0 < R \leq \infty \). The strategy in proving all of the characterizations obtained above in this section was to subtract the “lowest degree monomial in \( f \)” and obtain a function that sends \( \mathbb{P}_{n-1}^1(I) \) to \( \mathbb{P}_{n-1}^k \). The final step classified the continuous maps sending \( \mathbb{P}_n^1(I) \) to \( \mathbb{S}_n^1 \), and these are precisely the constants and scalar multiples of the maps \( \phi_\alpha, \psi_\alpha \) for \( \alpha > 0 \).

In this subsection, we take a closer look at the above steps, but under less restrictive assumptions. Our main result in this subsection generalizes Theorem \textbf{A} under more relaxed differentiability hypotheses on \( f \). To prove this result, we adopt the three-step approach from Section 3.3.

Theorem 3.14. Let \( 0 < R \leq \infty \), \( I = (-R, R) \), and \( f : I \to \mathbb{R} \). Fix \( 1 \leq k < n \), and let \( 0 \leq m_1 < m_2 < \cdots < m_k \) denote the orders of the first nonzero left and right derivatives of \( f \) at 0. Assume moreover that \( |f^{(m_i)}(0^+)| = |f^{(m_i)}(0^-)| \) for \( 1 \leq i \leq k \), that \( f(0) = 0 \), and either \( f \) is continuous on \( I \) or \( k > 1 \).

(1) Then \( f[-] : \mathbb{P}_n^1(I) \to \mathbb{S}_n^k \) if and only if \( f(x) = \sum_{i=1}^{k} c_i g_{m_i}(x) \) where \( c_i \in \mathbb{R} \setminus \{0\} \), \( 0 \leq m_1 < m_2 < \cdots < m_k \) are integers, and \( g_{m_i}(x) = \phi_{m_i}(x) \) or \( \psi_{m_i}(x) \).

(2) Similarly, \( f[-] : \mathbb{P}_n^1(I) \to \mathbb{P}_n^k \) if and only if \( f \) is of the same form with all \( c_i > 0 \).

In particular, \( f \) has exactly \( k \) nonzero left and right derivatives at 0.
Proof. By Taylor’s theorem,

\begin{align}
  (3.16) & \quad f(x) = \sum_{i=1}^{k} \frac{f^{(m_i)}(0^+)}{m_i!} x^{m_i} + o(x^{m_k}) \quad (x \in (0, R)), \\
  (3.17) & \quad f(x) = \sum_{i=1}^{k} (-1)^m \frac{f^{(m_i)}(0^-)}{m_i!} x^{m_i} + o(x^{m_k}) \quad (x \in (-R, 0)).
\end{align}

For \( i = 1, \ldots, k \), define a function \( g_m \) as follows: if \( f^{(m_i)}(0^+) = f^{(m_i)}(0^-) \), then \( g_{m_i} = \phi_{m_i} \), if \( m_i \) is even and \( g_{m_i} = \psi_{m_i} \), if \( m_i \) is odd. If instead \( f^{(m_i)}(0^+) = -f^{(m_i)}(0^-) \), then \( g_{m_i} = \psi_{m_i} \), if \( m_i \) is even and \( g_{m_i} = \phi_{m_i} \), if \( m_i \) is odd. Using this notation, equations (3.16) and (3.17) can be rewritten as

\begin{equation}
  (3.18) \quad f(x) = \sum_{i=1}^{k} \frac{f^{(m_i)}(0^+)}{m_i!} g_{m_i}(x) + o(x^{m_k}) \quad (x \in I).
\end{equation}

Note that (3.18) also holds at \( x = 0 \) since \( f(0) = 0 \) by assumption.

For \( r \in \mathbb{Z}_{\geq 0} \), define an operator \( T_r \) mapping any function \( h : I \to \mathbb{R} \) admitting at least \( r \) left and right derivatives at zero, via:

\begin{equation}
  (3.19) \quad T_r(h)(x) := \begin{cases} 
    h(x) - \frac{k(r)(0^+)}{r!} g_{r}(x) & \text{if } x \neq 0, \\
    0 & \text{if } x = 0,
  \end{cases}
\end{equation}

where

\begin{equation}
  (3.20) \quad g_{r}(x) := \begin{cases} 
    \phi_{r}(x) & \text{if } f^{(m_i)}(0^+) = f^{(m_i)}(0^-) \text{ and } m_i \text{ is even}, \\
    \psi_{r}(x) & \text{if } f^{(m_i)}(0^+) = f^{(m_i)}(0^-) \text{ and } m_i \text{ is odd}, \\
    \psi_{r}(x) & \text{if } f^{(m_i)}(0^+) = -f^{(m_i)}(0^-) \text{ and } m_i \text{ is even}, \\
    \phi_{r}(x) & \text{if } f^{(m_i)}(0^+) = -f^{(m_i)}(0^-) \text{ and } m_i \text{ is odd}.
  \end{cases}
\end{equation}

Now, inductively construct the function \( M_i(x) \) for \( 0 \leq i \leq k - 1 \) via: \( M_0(x) := f(x) \) and \( M_i(x) := T_{m_i}M_{i-1}(x) \) for \( 1 \leq i \leq k - 1 \). We have that

\begin{equation}
  (3.21) \quad M_i(x) = \sum_{j=i+1}^{k-1} \frac{f^{(m_j)}(0^+)}{m_j!} h_{m_{j-i}}(x) + o(x^{1+m_{k-1}-m_i}), \quad \forall x \in I, \ 0 \leq i \leq k - 1,
\end{equation}

where \( h_{m_{j-i}}(x) = \phi_{m_{j-i}}(x) \) or \( h_{m_{j-i}}(x) = \psi_{m_{j-i}}(x) \). The rest of the proof is now similar to the proof of Theorem A. Note that if \( f \) is continuous or \( k > 1 \), then \( M_{k-1}(x) \) is continuous and sends \( \mathbb{P}_{n-k+1}(f) \) to \( \mathbb{P}_{n-k+1}(f) \). Now apply Lemma 3.3 to conclude the proof of (1). The second part of the theorem follows using an argument similar to the one used in the proof of Theorem A.

We now present an application that further illustrates the power of the three-step approach described in this section. In order to do so, we first extend the definition of the power functions \( \phi_{\alpha}, \psi_{\alpha} \) introduced in (3.11) to also cover negative powers as follows:

\begin{align}
  (3.22) & \quad \phi_{\alpha}(0) = \psi_{\alpha}(0) := 0, \quad \phi_{\alpha}(x) := |x|^\alpha, \\
  & \quad \psi_{\alpha}(x) := \text{sgn}(x)|x|^\alpha, \quad \forall \alpha \in \mathbb{R}, \ x \in \mathbb{R} \setminus \{0\}.
\end{align}

It is easy to verify that for all \( \alpha \in \mathbb{R} \), the power maps \( \phi_{\alpha}, \psi_{\alpha} \) are continuous except possibly at 0, as well as multiplicative. In fact, FitzGerald and Horn [14], Bhatia...
and Elsner [5], and Hiai [27] analyzed the set of such power maps which preserve
entrywise Loewner positivity (as well as other properties such as monotonicity and
convexity). Recently in [19], we have completed the classification of these maps,
which was initiated by FitzGerald and Horn in loc. cit.

The following result generalizes a part of Theorem 3.9 to sums of (possibly non-
integer) powers.

**Theorem 3.15.** Fix $0 < R \leq \infty$, $I = [0, R)$ or $(-R, R)$, integers $n \geq 3$ and
$1 \leq k < n$, and define the function

$$f(x) := c_0 + \sum_{j=1}^{\infty} c_j g_{\alpha_j}(x), \quad x \in I,$$

where $c_j, \alpha_j \in \mathbb{R}$ with $\alpha_j < \alpha_{j+1}$ for $j \geq 1$, and $g_{\alpha_j} \equiv \psi_{\alpha_j}$ or $\psi_{\alpha_j}$ for all $j$. Assume
that $f$ is continuous on $I \setminus \{0\}$. Then the following are equivalent:

1. $f[1] \in S_n^k$ for every $A \in P_n(I)$;
2. $c_j \neq 0$ for at most $k$ values of $j$.

In particular, if $f[1] \in S_n^k$ for every $A \in P_n(I)$, then $c_j \geq 0$ for all $j$.

In order to carry out the first step of the three-step approach for sums of two-
sided powers, we need the following preliminary result, which is analogous to Lemma
3.4 but without assuming continuity at the origin.

**Lemma 3.16.** Suppose $n \geq 3$, $0 < R \leq \infty$, $I = [0, R)$ or $(-R, R)$, and $f : I \to \mathbb{R}$
is continuous except possibly at $0$. Then the following are equivalent:

1. $f[\cdot] : P_n^1(I) \to S_n^1$;
2. There exists $c \in \mathbb{R}$ such that either $f \equiv c$ on $I$ or $f(x) \equiv c\psi_{\alpha}(x)$ or $cv_{\alpha}(x)$
   for some $\alpha \in \mathbb{R}$.

If instead $n = 2$, then $f[\cdot] : P_2^1(I) \to S_n^1$ if and only if there exists $c \in \mathbb{R}$ such that
either (2) holds, or $f \equiv c$ on $I \cap (0, \infty)$ and $f \equiv -c$ on $(\infty, 0)$.

**Proof.** That (2) $\Rightarrow$ (1) (and the corresponding implication for $n = 2$) is clear. We
now prove the converse implications. For ease of exposition, we show this result in
three steps. In Step 1, we examine the behavior of $f$ on $(0, R)$. In Step 2, we study
the possible values for $f$ on $(-R, 0)$ when $I = (-R, R)$. We conclude by showing
in Step 3 that $f(0) = 0$.

**Step 1.** First note by Proposition 3.2.2 that if $f(a) = 0$ for some $a \in I \setminus \{0\}$, then
$f \equiv 0$ on $I$ and we are done. Thus for the remainder of the proof, we will assume
that $f$ is nonzero on $(0, R)$ and $f[-] : P_n^1(I) \to S_n^1$. Our next claim is that there
exist $c \neq 0$ and $\alpha \in \mathbb{R}$ such that $f(x) = cx^\alpha$ on $(0, R)$.

To see why the claim holds, first note that $|f[\cdot]|$ maps $P_n^1(I)$ into $S_n^1$, which
implies that $|f[\cdot]| : P_n^1(I) \to S_n^1$. Now given $0 < a < b \in I \cap (0, \infty) = (0, R)$, one
shows using equation (3.2) that

$$|f(\sqrt{ab}(b/a)^y)| = \sqrt{|f(a)f(b)|} \cdot \left| \frac{f(b)}{f(a)} \right|^y, \quad \forall y \in (-1/2, 1/2).$$

Define $\alpha := \ln |f(b) - \ln|f(a)|| / \ln(b) - \ln(a)$. Then the previous equation yields:

$$|f(x) := c' x^\alpha \forall x \in (a, b) \subset I, \quad c' := \frac{\sqrt{|f(a)f(b)|}}{(ab)^{\alpha/2}} > 0.$$

We now claim that $|f(x)| = c'x^\alpha$ for all $x \in (0, R)$. To see this, first choose $k \in \mathbb{N}$ such that $\sqrt{abt_k} \in (a, b)$, where $t_k := \sqrt[2]{x/\sqrt{ab}}$. Then $x = t_k^k \sqrt{ab}$, and $\sqrt{abt_k^m} \in I$ for $0 \leq m \leq k - 2$. Now define
\[
B_m := \left( \begin{array}{cc}
\sqrt{abt_k^m} & \sqrt{abt_k^{m+1}} \\
\sqrt{abt_k^m} & \sqrt{abt_k^{m+2}} \end{array} \right) \oplus 0_{(n-2) \times (n-2)} \in \mathbb{P}_n^1(I), \quad 0 \leq m \leq k - 2.
\]
Since $f[B_0] \in S_n^1$, we conclude by the above analysis that
\[
|f|(\sqrt{abt_k^2}) = \left| f(\sqrt{abt_k^2}) \right|^2 = \left( c' \right)^2 (\sqrt{abt_k^2})^{2\alpha} = c'(\sqrt{abt_k^2})^\alpha.
\]
Similar reasoning shows that $|f|(\sqrt{abt_k^{m+2}}) = c'(\sqrt{abt_k^{m+2}})^\alpha$ whenever $0 \leq m \leq k - 2$. In particular by setting $m = k - 2$, we obtain: $|f(x)| = |f|(\sqrt{abt_k^2}) = c'(\sqrt{abt_k^2})^\alpha$ for all $x \in (0, R)$. Now $f$ is continuous on $(0, R)$, whence so is $f/f(0) \to \{ \pm 1 \}$. Therefore $f/f(0)$ is constant on $(0, R)$, and we conclude that $f(x) = cx^\alpha$ for $x \in (0, R)$ with $c \neq 0$.

**Step 2.** The previous step shows that $f(x) = x^\alpha$ for all $x \in (0, R)$. Now assume $I = (-R, R)$. We claim that there exists a constant $\varepsilon \in \{ \pm 1 \}$ such that $f(x) = \varepsilon c|x|\alpha$ for all $0 > x \in I$. Indeed, given $0 > x \in I$, applying $f$ entrywise to the matrix
\[
\left( \begin{array}{cc}
x & x \\
x & x \\
|x| & x \\
|x| & x \\
\end{array} \right) \oplus 0_{(n-2) \times (n-2)} \in \mathbb{P}_n^1(I)
\]
shows that $f(x) = \pm c|x|^\alpha$. Once again, $f/|f|$ is constant on $I \cap (-\infty, 0)$, from which the claim follows.

**Step 3.** It remains to determine the value of $f(0)$. First suppose $\alpha \neq 0$. Then choose $x > 0$ such that $f(x) \neq f(0)$. Applying $f$ entrywise to the matrix $x1_{1 \times 1} \oplus 0_{(n-1) \times (n-1)} \in \mathbb{P}_n^1(I)$ shows that $f(0) = 0$. Therefore $f \equiv c\phi_\alpha$ or $c\psi_\alpha$ on $I$ for all $n \geq 2$, if $\alpha \neq 0$.

Finally, suppose $\alpha = 0$. Applying $f$ entrywise to the matrix $(R/2)1_{1 \times 1} \oplus 0_{(n-1) \times (n-1)} \in \mathbb{P}_n^1(I)$ shows that $f(0) = 0$ or $f(0) = c$. This proves the assertion (2) in all cases except when $n > 2$, $I \cap (-\infty, 0)$ is nonempty, and $f \equiv c\psi_\alpha$ on $I \setminus \{0\}$ with $c \neq 0$. In this case, choose $x > 0$ such that $\pm x \in I$, and apply $f$ entrywise to the matrix
\[
A := \left( \begin{array}{ccc}
x & -x & 0 \\
-x & x & 0 \\
0 & 0 & 0 \\
\end{array} \right) \oplus 0_{(n-3) \times (n-3)} \in \mathbb{P}_n^1(I).
\]
It follows that $f[A]$ has rank one, whence its leading principal $3 \times 3$ minor must vanish. Now this minor equals $-4cf(0)^2 = 0$, whence it follows that $f(0) = 0$. This concludes the proof.

In order to prove Theorem 3.15, we also need to extend classical results about Vandermonde determinants to the odd and even extensions of the power functions.

**Proposition 3.17.** Fix $0 < R \leq \infty$ and $I = [0, R)$ or $(-R, R)$.

1. The functions $\{\phi_\alpha, \psi_\alpha : \alpha \in \mathbb{R}\} \cup \{f \equiv 1\}$ are linearly independent on $I = (-R, R)$, while on $I = [0, R)$ the functions $\phi_\alpha = \psi_\alpha = x^\alpha : \alpha \in \mathbb{R}\} \cup \{f \equiv 1\}$ are linearly independent.

2. The functions in the previous part are also “countably linearly independent”. More precisely, suppose $f(x) = c_0 + \sum_{i=1}^\infty (c_i \phi_\alpha(x) + d_i \psi_\alpha(x))$ with
Theorem 3.19 (§11, Example 1]). Given real numbers $\alpha_1 < \cdots < \alpha_n$ and $0 < x_1 < \cdots < x_n$ for some $n \geq 1$, the generalization of the Dedekind Independence Theorem to arbitrary semigroups; see [16, Chapter XIII, §8, Example 1]).

Proof of Proposition 3.18. Given real numbers $\alpha_1 < \cdots < \alpha_n$ and $0 < x_1 < \cdots < x_n$ for some $n \geq 1$, the generalization of the Dedekind Independence Theorem to arbitrary semigroups; see [16, Chapter XIII, §8, Example 1]).

Proof of Proposition 3.18. Given real numbers $\alpha_1 < \cdots < \alpha_n$ and $0 < x_1 < \cdots < x_n$ for some $n \geq 1$, the generalization of the Dedekind Independence Theorem to arbitrary semigroups; see [16, Chapter XIII, §8, Example 1]).

Proof of Proposition 3.18. Given real numbers $\alpha_1 < \cdots < \alpha_n$ and $0 < x_1 < \cdots < x_n$ for some $n \geq 1$, the generalization of the Dedekind Independence Theorem to arbitrary semigroups; see [16, Chapter XIII, §8, Example 1]).

Proof of Proposition 3.18. Given real numbers $\alpha_1 < \cdots < \alpha_n$ and $0 < x_1 < \cdots < x_n$ for some $n \geq 1$, the generalization of the Dedekind Independence Theorem to arbitrary semigroups; see [16, Chapter XIII, §8, Example 1]).
We now have all the ingredients to prove Theorem 3.15.

Proof of Theorem 3.15. Clearly (2) ⇒ (1). Now assume (1) holds. If $c_j$ is nonzero for at most $k - 1$ values of $j \geq 0$ then we are done; thus suppose that $c_j$ is nonzero for at least $k$ values of $j$. To simplify the proof, we will assume $c_0, \ldots, c_{k-1} \neq 0$; the proof of the general case is similar. Using the three-step approach as in Section 3.5 (1) of $f(x) = P_n^{-1}(I) \rightarrow S_{n-1}^k$ by Proposition 3.6. Now applying Proposition 3.6 (1) with $g(x) := g_{a_1}(x)$, $g := c_1$, we conclude that $f_1(x) := c_1 + \sum_{j=2}^{\infty} c_j h_{a_j - a_1}(x)$ satisfies $f_1[-] : P_n^1(I) \rightarrow S_{n-1}^k$, where $h_{a_j - a_1}(x) := g_{a_j}(x)/g_{a_1}(x)$ is of the form $\phi_{a_j - a_1}(x)$ or $\psi_{a_j - a_1}(x)$. Continuing inductively in this manner, we arrive at $f_{k-1} : I \rightarrow \mathbb{R}$, of the form $f_{k-1}(x) = c_{k-1} + \sum_{j=k}^{\infty} c_j h_{a_j - a_{k-1}}(x)$ where $h_{a_j - a_{k-1}}(x) = \phi_{a_j - a_{k-1}}(x)$ or $\psi_{a_j - a_{k-1}}(x)$, $c_{k-1} \neq 0$, and $f_{k-1}[-] : P_n^{k-1}(I) \rightarrow S_{n-k+1}^1$. Moreover, $f_{k-1}$ is continuous on $I \setminus \{0\} = (0, R)$ by construction. There are now two cases:

(1) The first case is when $k < n - 1$. Then $n - k + 1 \geq 3$, so by Lemma 3.16, $f_{k-1}$ is either a constant or a scalar multiple of $\phi_{a}$ or $\psi_{a}$ for some $a \in \mathbb{R}$. Evaluating at the origin shows that $f_{k-1} \equiv c_{k-1}$. Now applying Proposition 3.17 shows that $c_j = 0$ for all $j \geq k$. This concludes the proof of the first equivalence.

(2) The other case is when $k = n - 1$, i.e., $n - k + 1 = 2$. Then by Lemma 3.16 either $f_{k-1}$ is a constant or a scalar multiple of $\phi_{a}$ or $\psi_{a}$ for some $a \in \mathbb{R}$ (in which case the same reasoning as in the previous case yields the result), or else $I = (-R, R)$ and $f_{k-1} \equiv K_c$ on $I$, where $K_c(x) \equiv c \neq 0$ on $[0, R)$ and $K_c \equiv -c$ on $(-R, 0)$. We now show that this latter possibility cannot occur. Indeed, suppose by contradiction that

$$f_{k-1}(x) = c_{k-1} + \sum_{j=k}^{\infty} c_j h_{a_j - a_{k-1}}(x) \equiv K_c(x) (c \neq 0).$$

Evaluating both sides at zero yields: $c_{k-1} = c$, so that when restricted to $[0, R)$, we obtain

$$\sum_{j=k}^{\infty} c_j h_{a_j - a_{k-1}}(x) \equiv 0, \quad x \in [0, R).$$

Using Proposition 3.17 (2) on $[0, R)$, we conclude that $c_j = 0$ for all $j \geq k$, so that $f_{k-1} \equiv c_{k-1}$ on $I$, which contradicts our assumption that $f_{k-1} \equiv K_c$ on $I$.

The final assertion is shown similarly using Proposition 3.6 (2) instead of Proposition 3.5 (1). \hfill \square

4. Preserving positivity under rank constraints II: The special rank 2 case

Recall that in Section 3.5 we had studied functions mapping rank 1 matrices into $\mathbb{P}_n^k$. We now study the entrywise functions mapping $\mathbb{P}_n^2$ to $\mathbb{P}_n^k$. More precisely, we study functions which preserve positivity on a class of special rank 2 matrices, i.e., matrices of the form

$$a_1 + au^T, \quad a \in \mathbb{R}, \ u \in \mathbb{R}^n.$$
(We abuse notation slightly here, as the matrix $a1_{n \times n} + uu^T$ is of rank at most 1 if $a = 0$.) As we demonstrate in this section, preserving positivity on these special rank 2 matrices greatly constrains the possible entrywise functions. We begin by generalizing a previous result by Horn [28, Theorem 1.2] (attributed to Loewner), which provides a necessary condition for an entrywise function to preserve positivity on special rank 2 matrices. To our knowledge Horn’s result is the only known result in the literature involving entrywise functions preserving positivity for matrices of a fixed dimension.

**Theorem 4.1** (Necessary conditions, fixed dimension). Suppose $0 < R \leq \infty$, $I = (0, R)$, and $f : I \to \mathbb{R}$. Fix $2 \leq n \in \mathbb{N}$ and suppose that $f[A] \in \mathbb{P}_n$ for all $A \in \mathbb{P}_n^2(I)$ of the form $A = a1_{n \times n} + uu^T$, with $a \in [0, R), u \in [0, \sqrt{R} - a]^n$. Then $f \in C^{n-3}(I)$,

$$f^{(k)}(x) \geq 0, \quad \forall x \in I, \quad 0 \leq k \leq n - 3,$$

and $f^{(n-3)}$ is a convex nondecreasing function on $I$. In particular, if $f \in C^{n-1}(I)$, then $f^{(k)}(x) \geq 0$ for all $x \in I, 0 \leq k \leq n - 1$.

**Remark 4.2.** Note that Theorem 4.1 generalizes [28, Theorem 1.2] by weakening the hypotheses in the following three ways: (1) $f$ is no longer assumed to be continuous; (2) $f$ is assumed to preserve Loewner positivity on a far smaller subset of matrices in $\mathbb{P}_n^2(I)$; (3) the entries of the matrices can come from $(0, R)$ instead of $(0, \infty)$, for any $0 < R \leq \infty$.

**Proof of Theorem 4.1.** For the sake of exposition, we carry out the proof in three steps.

**Step 1: Smooth case.** First suppose that $f \in C^\infty(I)$ is smooth on $I = (0, R)$. The result is then true for all $0 < R \leq \infty$, by repeating the argument in the proof of [28, Theorem 1.2] on $I$, but using $0 < a < R$ now.

**Step 2: Continuous case.** Next, suppose $f$ is continuous but not necessarily smooth on $I = (0, R)$. Given any probability distribution $\theta \in C^\infty(-1, 0)$ with compact support in $(-1, 0)$, let $\theta_{\varepsilon}(x) := \theta(x^{-1})$ for $\varepsilon > 0$. Consider the function $f_\varepsilon : (0, R - \varepsilon) \to \mathbb{R}$, given by

$$f_\varepsilon(x) := \frac{1}{\varepsilon} \int_{-\varepsilon}^{\varepsilon} f(x - t)\theta_{\varepsilon}(t) \, dt \in C^\infty(0, R - \varepsilon). \quad (4.2)$$

Fix $0 < \varepsilon_0 < R$, and choose $A = a1_{n \times n} + uu^T \in \mathbb{P}_n(0, R - \varepsilon_0)$ with $a \in [0, R - \varepsilon_0)$. Then equation (4.2) shows that for $0 < \varepsilon \leq \varepsilon_0$,

$$f_\varepsilon[A] = \int_{-\varepsilon}^\varepsilon \theta_{\varepsilon}(t)f[A - t1_{n \times n}] \, dt \in \mathbb{P}_n$$

by assumption on $f$. Then $f_\varepsilon$ is smooth and satisfies the other assumptions of the theorem on $I = (0, R - \varepsilon_0)$. Therefore, by the previous step, all of the derivatives of $f_\varepsilon$ are nonnegative on $(0, R - \varepsilon_0)$. In particular, all the finite differences of $f_\varepsilon$ are nonnegative. Since the finite differences of $f_\varepsilon$ converge to the finite differences of $f$ as $\varepsilon \to 0^+$, it follows that the finite differences of $f$ are also nonnegative on $(0, R - \varepsilon_0)$. Hence by [3, Theorem, p. 497], $f \in C^{n-3}(0, R - \varepsilon_0)$. The result now follows for $I = (0, R - \varepsilon_0)$ by carrying out the steps at the end of the proof of [28, Theorem 1.2]. (We remark that the continuity of $f$ is needed in loc. cit.) Finally, the result holds on all of $I = (0, R)$ because $\varepsilon_0$ was arbitrary.
Step 3: General case. It remains to show that every function $f$ satisfying the hypotheses is necessarily continuous on $I = (0, R)$. Consider any $a, b, c \in I$ such that

$$B := \begin{pmatrix} a & b \\ b & c \end{pmatrix} \in \mathbb{P}_2(I).$$

We first claim that there exists $\tilde{B} \in \mathbb{P}_n(I)$ of the form $a'\mathbf{1}_{n \times n} + uu^T$, whose principal $2 \times 2$ submatrix is $B$. To show the claim, define

$$M_1 := \begin{pmatrix} a\mathbf{1}_{(n-1)\times(n-1)} & b\mathbf{1}_{(n-1)\times1} \\ b\mathbf{1}_{1\times(n-1)} & b^2a^{-1} \end{pmatrix}, \quad M_2 := (c-b^2a^{-1})E_{n,n},$$

where $E_{n,n}$ is the elementary matrix, with $(i, j)$th entry equal to 1 if $i = j = n$ and 0 otherwise. Now let $\tilde{B} := M_1 + M_2$. Note that $\tilde{B} \in \mathbb{P}_n^2(I)$ since $M_1, M_2 \in \mathbb{P}_n^1$. Moreover, $\tilde{B}$ contains $B$ as a principal submatrix. We now show that $\tilde{B}$ is indeed of the form $a'\mathbf{1}_{n \times n} + uu^T$. There are two sub-cases:

1. If $a + c \leq 2b$, then $ac \geq b^2 \geq (a + c)^2/4$, whence $a = c$ by the arithmetic mean-geometric mean inequality. It follows that $b = (a + c)/2 = a$, so that $\tilde{B} = a\mathbf{1}_{n \times n}$ is of the desired form.

2. If $a + c > 2b$, set $a' := \frac{ac-b^2}{a+c-2b}$. It is easy to verify that $0 \leq a' < \min(a, c)$. Therefore $\tilde{B} = a'\mathbf{1}_{n \times n} + uu^T$ is of the desired form, where

$$u := \left(\sqrt{a-a'}, \ldots, \sqrt{a-a'}, \sqrt{c-a'}\right)^T.$$

Finally, suppose $f[A] \in \mathbb{P}_n$ for all $A \in \mathbb{P}_n^2(I)$ of the form $a'\mathbf{1}_{n \times n} + uu^T$. Setting $A = \tilde{B}$ for $B \in \mathbb{P}_2(I)$, we conclude that $f[B] \in \mathbb{P}_2$ for all $B \in \mathbb{P}_2(I)$. By Theorem 2.5, $f$ is continuous on $I$, and the proof is now complete. \hfill \Box

Remark 4.3. An immediate consequence of Theorem 4.1 is that for all non-integer values $t \in (0, n-2)$, there exists $A \in \mathbb{P}_n^2(I)$ of the form $a'\mathbf{1}_{n \times n} + uu^T$, such that

$$A^{\otimes l} := (a_{ij}^l) \text{ is not in } \mathbb{P}_n.$$  

This strengthens [28 Corollary 1.3]. A specific example of such a matrix $A$ was constructed in [14, Theorem 2.2]. More generally for any $2 \leq l \leq n$, one can produce examples of matrices $A \in \mathbb{P}_n(I)$ of rank exactly $l$ such that $A^{\otimes l} \notin \mathbb{P}_n$; see [19, Section 6] for more details.

Remark 4.4. Note that applying Theorem 4.1 for all $n \in \mathbb{N}$ easily yields a generalization of Theorem 2.2 for any interval $I = (0, R)$. Thus, Theorem 4.1 immediately implies Theorem C. Later in Section 6, we will provide an alternate, elementary proof of Theorem C.

Note that Theorem C follows immediately from Theorem 4.1. In Section 6, we also provide an intuitive proof of Theorem C that uses the rank techniques developed in this paper to prove Theorems A and B.

Recall that Theorem A shows that functions mapping $\mathbb{P}_n^1(I)$ to $\mathbb{P}_n^k$ under some differentiability assumptions were polynomials of arbitrary degree. We now show that the rank 2 situation is far more restrictive than the rank 1 case, and it requires no assumptions on $f$ if $k \leq n - 3$.

**Theorem 4.5** (Special rank 2, fixed dimension). Suppose $0 < R \leq \infty$, $I = [0, R)$ or $(-R, R)$, and $f \in C^k(I)$ for some $1 \leq k < n$.

1. Then the following are equivalent:
   (a) $f[a\mathbf{1}_{n \times n} + uu^T] \in \mathbb{P}_n^k$ for all $a \in I$ and all $u \in \mathbb{R}^n$ with $a + uu^T \in I$;
   (b) $f$ is a polynomial of degree at most $k - 1$. 


Similarly, when $I = [0, R)$, we have $f[a\mathbf{1}_{n \times n} + uu^T] \in \mathbb{P}^k_n$ for all $a \in [0, R)$ and all $u \in \mathbb{R}^n$ with $a + u_i u_j \in I$, if and only if $f$ is a polynomial of degree at most $k - 1$ with nonnegative coefficients. Moreover if $k \leq n - 3$, the assumption that $f \in C^k(I)$ is not required.

Remark 4.6. Note that part (2) of Theorem 4.5 is stated only for $I = [0, R)$ since $a\mathbf{1}_{n \times n} + uu^T \not\in \mathbb{P}^2_n$ in general if $a < 0$. When $I = (-R, R)$ and $f$ is analytic on $I$, Theorem 4.5(2) also holds for $I = (-R, R)$ and follows immediately by the uniqueness principle from the $I = [0, R)$ case. The result also holds if $I = (-R, R)$ and $f$ admits at least $k$ nonzero derivatives at the origin, since in that case $f$ is a polynomial by Theorem A.

The following result is crucially used in the proof of Theorem 4.5 as well as in later sections.

Proposition 4.7. Let $a \in \mathbb{R}$, $n \geq 2$, $1 \leq k \leq n$, $0 < R \leq \infty$, $I = (a - R, a + R)$, and $f : I \to \mathbb{R}$. Suppose $f$ admits at least $k$ nonzero derivatives at $a$. Then there exists $u \in \mathbb{R}^n$ such that $a + u_i u_j \in I$ and $f[a\mathbf{1}_{n \times n} + uu^T]$ has rank at least $k$.

Proof. Suppose to the contrary that $f[a\mathbf{1}_{n \times n} + uu^T]$ has rank less than $k$ for all $u \in \mathbb{R}^n$ such that $a + u_i u_j \in I$. Define $g : (-R, R) \to \mathbb{R}$ by $g(x) := f(a + x)$. By hypothesis, $g[-] : \mathbb{P}^k_n((-R, R)) \to \mathbb{S}^{k-1}_n$. Moreover, $g$ admits at least $k$ nonzero derivatives at 0. Thus, by Theorem A(1), the function $g$ is a polynomial with exactly $k - 1$ nonzero coefficients, which is impossible. Therefore, there exists $u \in (-R, R)^n$ such that $g[uu^T] = f[a\mathbf{1}_{n \times n} + uu^T]$ has rank at least $k$. □

We now have all the ingredients to prove Theorem 4.5.

Proof of Theorem 4.5. We begin by proving the first set of equivalences.

(a) $\Rightarrow$ (b) Clearly, (b) holds if $f^{(k)} = 0$ on $I$. Thus, assume there is a point $a_{k} \in I$ such that $f^{(k)}(a_{k}) \neq 0$. By continuity, there is an open interval $I_k \subset I$ such that $f^{(k)}$ has no roots in $I_k$. It follows by repeatedly applying Rolle’s Theorem that $f^{(i)}$ has at most $k - i$ roots in $I_k$ for all $0 \leq i < k$. Now pick any point $a_0 \in I_k$ which is not one of these finitely many roots of $f^{(i)}$ for any $0 \leq i \leq k$, i.e., $f^{(0)}(a_0), f^{(1)}(a_0), \ldots, f^{(k)}(a_0) \neq 0$. Therefore, by Proposition 4.7 there exists $A = a\mathbf{1}_{n \times n} + uu^T \in \mathbb{P}^k_n(I)$ such that $f[A]$ has rank at least $k + 1$. This is impossible by assumption. Thus $f^{(k)} \equiv 0$ and $f$ is a polynomial of degree at most $k - 1$, proving (b).

(b) $\Rightarrow$ (a) Conversely, suppose $f(x) = \sum_{m=0}^{k-1} c_m x^m$. Then we compute for $a \in I$ and $u \in \mathbb{R}^n$ such that $a + u_i u_j \in I$:

$$f[a\mathbf{1}_{n \times n} + uu^T]_{ij} = \sum_{m=0}^{k-1} c_m (a + u_i u_j)^m = \sum_{m=0}^{k-1} \sum_{l=0}^{m} c_m \left( \frac{m}{l} \right) a^{m-l} u_i^l u_j^l$$

$$= \sum_{l=0}^{k-1} u_i^l u_j^l \sum_{m=l}^{k-1} c_m \left( \frac{m}{l} \right) a^{m-l} = \sum_{l=0}^{k-1} u_i^l u_j^l d_l,$$

say. Therefore $f[a\mathbf{1}_{n \times n} + uu^T] = \sum_{l=0}^{k-1} d_l u_i^l (u^l)^T$, where $u^l := (u_1^l, u_2^l, \ldots, u_n^l)^T$.

In particular, $f[a\mathbf{1}_{n \times n} + uu^T]$ has rank at most $k$. 

We now prove the second set of equivalences. Clearly if \( f \) is a polynomial of degree \( \leq k - 1 \) with nonnegative coefficients, then \( f[a1_{n \times n} + uu^T] \in R^n_k \) for all \( a \geq 0 \) and \( u \in \mathbb{R}^n \) such that \( a + u_iu_j \in [0, R] \), by the calculation in equation 4.3. Conversely if (1) holds, the first set of equivalences already shows that \( f \) is a polynomial of degree \( \leq k - 1 \). That the coefficients of \( f \) are nonnegative follows by Theorem 4.9. Finally, if \( k \leq n - 3 \) then the condition that \( f \in C^k(I) \) actually follows by Theorem 4.1, and hence does not need to be assumed.

Remark 4.8. Note that the implication \( (a) \Rightarrow (b) \) in Theorem 4.5 also holds under the weaker assumption that \( f[a1_{n \times n} + uu^T] \in S^n_k \) for all \( a \in I \) and \( u \in (-\epsilon(a), \epsilon(a))^n \) where \( 0 < \epsilon(a) < \sqrt{R - |a|} \). This observation will be important later in proving Theorem 4.13.

Recall by Theorem 4.5 that polynomials of degree at most \( k - 1 \) take special rank 2 matrices to \( S^n_k \). We now show that this behavior is not shared by arbitrary linear combinations of powers - for instance, if there is even one noninteger power involved.

**Proposition 4.9.** Fix \( 0 < R \leq \infty \), integers \( n \geq 2 \) and \( m \geq 1 \), and suppose \( \alpha_1 < \cdots < \alpha_m \in \mathbb{R} \) with \( \alpha_i \notin \{0, 1, \ldots, n - 2\} \) for at least one \( i \). Define \( f(x) = \sum_{i=1}^m c_i x^{\alpha_i} \), with \( c_i \neq 0 \). Then there exist \( a \in (0, R) \) and \( u \in (-\epsilon, \epsilon)^n \) where \( \epsilon := \min(\sqrt{a}, \sqrt{R - a}) \), such that \( f[a1_{n \times n} + uu^T] \) has full rank.

**Proof.** We first claim that there exists an open interval \((p, q) \subset (0, R)\) such that \( f, f', \ldots, f^{(n - 1)} \) are all nonzero on \((p, q)\). Indeed, let \( I_0 := (0, R) \), and note that for any \( x_1 < \cdots < x_m \) in \( I_0 \), the matrix \((x_j^{\alpha_i})_{i,j=1}^m \) is nonsingular by Proposition 3.18. Hence there exists \( j \) such that \( f(x_j) = \sum_i c_i x_j^{\alpha_i} \neq 0 \). We conclude by continuity of \( f \) that \( f = f^{(0)} \) is nonzero on a nonempty open interval \( I_1 \subset I_0 \). Repeatedly applying the above arguments, we obtain a nested sequence of nonempty open intervals on which all sufficiently low-degree derivatives of \( f \) are nonzero. This shows the existence of the interval \( I_{n - 1} = (p, q) \). (We need at least one \( \alpha_i \) to not lie in \( \{0, \ldots, n - 2\} \), otherwise \( f^{(n - 1)} \equiv 0 \).) Finally, fix \( a := (p + q)/2 \) and \( \epsilon = (q - p)/2 \). The result then follows by Proposition 4.7.

Proposition 4.7 also has the following important consequence, which will be useful later. Recall that \( f_\alpha(x) := x^\alpha \) for \( \alpha > 0 \).

**Corollary 4.10.** Let \( \alpha \in (0, \infty) \) and \( n \geq 2 \). For \( u \in \mathbb{R}^n \), define \( A_u := 1_{n \times n} + uu^T \). Then the following are equivalent:

1. There exists \( u \in \mathbb{R}^n \) (in fact in \((-1, 1)^n\)) such that the matrix \( f_\alpha[A_u] \) is nonsingular.
2. Either \( \alpha \in \mathbb{N} \cap [n - 1, \infty) \) or \( \alpha \not\in \mathbb{N} \).

**Proof.** Clearly, if \( \alpha \in \mathbb{N} \), then for any \( u \in \mathbb{R}^n \) such that \( 1 + u_iu_j \in (0, \infty) \), the matrix \( f_\alpha[A_u] = \sum_{k=0}^\alpha \binom{\alpha}{k} f_k[u]^k f_k[u]^T \) has rank at most \( 1 + \alpha \). Therefore if (1) holds, then \( \alpha \notin \mathbb{N} \cap (0, n - 1) \) and so (1) \( \Rightarrow \) (2). Conversely, suppose \( \alpha \in \mathbb{N} \cap [n - 1, \infty) \) or \( \alpha \not\in \mathbb{N} \). Then the function \( f(x) = x^\alpha \) admits at least \( n \) nonzero derivatives at \( x = 1 \). Thus, by Proposition 4.7, there exists \( u \in \mathbb{R}^n \) such that \( 1 + u_iu_j \in (0, 2) \) for all \( i, j \) and \( f_\alpha[A_u] \) has full rank. This shows that (2) \( \Rightarrow \) (1) and concludes the proof.
5. Preserving positivity under rank constraints III: The higher rank case

The goal of this section is to study entrywise functions mapping $\mathbb{P}_k^n$ to $\mathbb{P}_l^n$ for general $1 \leq k, l \leq n$. The $l = 1$ case has been explored in Section 3 so we assume throughout this section that $l > 1$. Note by the results shown in Section 4 that $C^k$ functions sending special rank 2 matrices to $S_k^n$ automatically have to be polynomials. Using the aforementioned results, in Subsection 5.1 we prove Theorem 5.1 which classifies the entrywise maps $f$ which are $C^k$ and send $\mathbb{P}_l^n$ to $S_k^n$. We then show in Subsections 5.2 and 5.3 that the assumptions on $f$ can be relaxed even further if the rank “does not double”. Namely, we obtain a complete classification of the entrywise maps sending $\mathbb{P}_l^n$ to $S_k^n$ for the special regime where $0 \leq k < \min(n, 2l)$, under either continuity assumptions on $f$ or no assumption at all.

5.1. Proof of the main Theorem 5.1. Before proceeding to the proof of Theorem 5.1 we need some preliminary results.

Proposition 5.1. Fix a field $\mathbb{F}$ of characteristic zero, as well as $N, l \in \mathbb{N}$. Let $m_i = (m_{ij}) \in \mathbb{F}^l$ be distinct vectors for $1 \leq i \leq N$. Then:

1. For any $r_1, \ldots, r_{l-1} \in \mathbb{N}$, there exists $\alpha = (\alpha_1, \ldots, \alpha_l) \in \mathbb{N}^l \subset \mathbb{F}^l$, such that $\alpha_{i+1} > r_i \alpha_i$ for all $0 < i < l$, and $\alpha^T m_i$ are pairwise distinct.
2. Suppose $m_i \in \mathbb{Z}_{\geq 0}^l$ are distinct for all $i$, and $0 < v_1 < \cdots < v_N \in \mathbb{Q} \subset \mathbb{F}$.

Then there exists $\alpha \in \mathbb{N}^l$, such that defining $u_j := v_i^{\alpha_i}$ for $1 \leq j \leq l$, the vectors

$$w_i := u_1^{m_{i1}} \circ \cdots \circ u_l^{m_{il}}$$

are $\mathbb{F}$-linearly independent for $1 \leq i \leq N$. If $\mathbb{F} = \mathbb{R}$ then the result holds even if we assume that $m_i \in [0, \infty)^l$ for all $i$.

Remark 5.2. Note that the second part generalizes the nonsingularity of generalized Vandermonde determinants, but in $\mathbb{Q}$ (and hence, every field of characteristic zero). This is because in the special case of $l = 1$, we can choose $\alpha = \alpha_1 = 1$. Moreover, we will show below that Proposition 5.1(2) is in fact equivalent to the nonsingularity of generalized Vandermonde determinants.

Proof of Proposition 5.1. We first claim that if $V$ is a vector space over a field $\mathbb{F}$ of characteristic zero, and $C$ is any $\mathbb{Q}$-convex subset of $V$, then the following are equivalent:

1. $C$ is contained in a proper subspace of $V$.
2. $C$ is contained in a finite union of proper subspaces of $V$.

Clearly (1) implies (2). Conversely, suppose $C$ is not contained in any proper subspace of $V$. We show that (2) also fails to hold, by induction on the number $n$ of proper subspaces of $V$. This is clearly true for $n = 1$. Next, suppose $C$ is not contained in a finite union of $n - 1$ proper subspaces of $V$ and let $V_1, \ldots, V_n$ be proper subspaces of $V$. Fix elements $v_1 \in V_1 \setminus \bigcup_{i \geq 1} V_i$ and $v_2 \in V_2 \setminus (V_1 \cup \bigcup_{i \geq 2} V_i)$, and consider the infinite set $\{(1/n)v_1 + ((n - 1)/n)v_2 : n \in \mathbb{N}\} \subset C$. If $C \subset \bigcup_{i=1}^n V_i$, then at least two elements of this infinite set lie in some $V_i$, in which case we obtain that $v_1, v_2 \in V_i$. This is false by assumption. It therefore follows by induction that $C$ is not contained in a finite union of proper subspaces of $V$, and so (2) implies (1).
We now show the first part. Given \( r_i \) as above, define
\[
r_0 := 0, \quad N := \prod_{j=0}^{l-1} (1 + r_j),
\]
\[
C := \mathbb{Q}^l \cap \times_{i=1}^{l} \left( N^{-1} r_i \prod_{j=0}^{l-1} (1 + r_j), N^{-1} \prod_{j=0}^{l} (1 + r_j) \right) \subset \mathbb{F}^l,
\]
where \( \times_{i=1}^{n} \) denotes an \( n \)-fold Cartesian product of intervals. Now clearly, \( C \) is \( \mathbb{Q} \)-convex. Moreover, it is easy to check that if \( C \) is contained in any \( \mathbb{F} \)-vector subspace \( V_0 \subset \mathbb{F}^l \), then the standard basis \( \{ e_i : 1 \leq i \leq l \} \) is contained in \( V_0 \), whence \( V_0 = \mathbb{F}^l \). Hence by the previous part, \( C \) is not contained in a finite union of proper subspaces of \( \mathbb{F}^l \). In particular, \( C \) is not contained in the orthogonal complements to the vectors \( m_i - m_j \) (for all \( i \neq j \) in \( \mathbb{Q}^l \)). Take any point in \( C \) that is not contained in the union of these orthogonal complements, and rescale it by a sufficiently large integer \( M \in \mathbb{N} \). This provides the desired vector \( \alpha \in \mathbb{N}^l \).

Now fix \( 0 < v_1 < \cdots < v_N \in \mathbb{Q} \subset \mathbb{F} \) and let \( \alpha \in \mathbb{N}^l \) be as in the above analysis. Since \( \alpha^T m_i \) are distinct, the generalized Vandermonde matrix \( B := (\alpha_i^T m_i)_{i=1}^N \) is nonsingular by Proposition 3.13. Now define \( u_j := (v_1^{a_j}, \ldots, v_N^{a_j})^T \) for \( 1 \leq j \leq l \). Then the linearly independent columns of \( B \) are precisely \( w_i \) as defined in the statement, which concludes the proof. The assertion for \( \mathbb{F} = \mathbb{R} \) is similarly proved. \( \square \)

The next two results are technical tools which will be useful in proving Theorem \( \square \) when \( I = (-R, R) \).

**Lemma 5.3.** Let \( 0 < R \leq \infty \), \( I = [0, R) \) or \((−R, R)\), and \( f : I \to \mathbb{R} \). Fix \( n \geq 2 \), \( 0 \leq k < n - 1 \), and \( 1 \leq l \leq n \). Then the following are equivalent:

1. \( f[-] : \mathbb{P}_n^k(I) \to \mathbb{S}_n^k \).
2. \( f[-] : \mathbb{P}_n^{l'}(I) \to \mathbb{S}_n^{k'} \) for all \( n' \geq \max(k + 2, l) \).

**Proof.** Clearly (2) \( \Rightarrow \) (1). Suppose (1) holds. If \( n' \leq n \) then clearly \( f[-] : \mathbb{P}_n^{l'}(I) \to \mathbb{S}_n^{k'} \). Now suppose \( n' \geq \max(n + 1, k + 2, l) \) and let \( A' \in \mathbb{P}_n^{l'}(I) \). Note that every \( n \times n \) principal submatrix of \( A' \) belongs to \( \mathbb{P}_n^{l'}(I) \). It follows by Lemma 3.1 that all \( (k+1) \times (k+1) \) and \( (k+2) \times (k+2) \) principal minors of \( f[A'] \) vanish. Again using Lemma 3.1 we conclude that \( f[A'] \in \mathbb{S}_n^k \). \( \square \)

**Corollary 5.4.** Let \( 0 < R \leq \infty \), \( I = (-R, R) \), and \( f : I \to \mathbb{R} \). Fix \( n \geq 2 \) and \( 0 \leq k < n - 1 \). Suppose \( f[-] : \mathbb{P}_n^k(I) \to \mathbb{S}_n^k \). Then \( f[a_1 1_{n \times n} + u_i u_j] \in \mathbb{S}_n^k \) for all \( a < 0 \) and all \( u \in \mathbb{R}^n \) such that \( \pm a + u_i u_j \in I \forall i, j \).

**Proof.** Let \( u \in \mathbb{R}^n \) such that \( \pm a + u_i u_j \in (-R, R) \forall i, j \). Define
\[
x := \left( -\sqrt{|a|} 1_{n \times 1} \right) \in \mathbb{R}^{2n}, \quad y := \left( \frac{u}{u} \right) \in \mathbb{R}^{2n},
\]
and consider the matrix
\[
A := xx^T + yy^T = \begin{pmatrix} |a| 1_{n \times n} + uu^T & a 1_{n \times n} + uu^T \\ a 1_{n \times n} + uu^T & |a| 1_{n \times n} + uu^T \end{pmatrix} \in \mathbb{P}_n^k(I).
\]
By Lemma 5.3 we have \( f[-] : \mathbb{P}_n^k(I) \to \mathbb{S}_n^k \), and so \( f[A] \in \mathbb{S}_n^k \). Thus, by Lemma 3.1, all \((k + 1) \times (k + 1) \) minors of \( f[A] \) vanish. In particular, all \((k + 1) \times (k + 1) \) minors of the upper right block of \( f[A] \) vanish. Thus, \( f[a 1_{n \times n} + uu^T] \in \mathbb{S}_n^k \). \( \square \)
Proof of Theorem 3.17. If \( k = 0 \) then the result is immediate to prove, so we suppose henceforth that \( k \geq 1 \).

We first prove that (2) \( \Rightarrow \) (1). Let \( i \geq 0 \) and \( A = \sum_{j=1}^{l} u_j u_j^T \in P_n(I) \). Then
\[
A^{\circ i} = \sum_{m_1+\cdots+m_i \leq i} \binom{i}{m_1,\ldots,m_i} w_m w_m^T,
\]
where \( \binom{i}{m_1,\ldots,m_i} \) denotes the multinomial coefficient. Note that there are exactly \( i+1 \) terms in the previous summation. Therefore \( \text{rank } A^{\circ i} \leq \binom{i+1}{i-1} \), and so (1) easily follows from (2).

Conversely, suppose (1) holds. If \( I = [0,R) \), then \( f \) is a polynomial of degree at most \( k-1 \) by Theorem 4.5. Similarly, if \( I = (-R,R) \), then an application of Corollary 4.4 shows that \( f[a 1_{n \times n} + uu^T] \in S_n^k \) for all \( a \in (-R,R) \) and all \( u \in \mathbb{R}^n \) such that \( \pm a + uu^T \in (-R,R) \). Thus \( f \) is also a polynomial of degree at most \( k-1 \) by Theorems 3.5 and Remark 4.8.

Now denote by \( m_1,\ldots,m_l \) the collection of vectors \( \{\mathbb{Z}_{\geq 0} \cap [0,k-1]\}^l \). By Proposition 5.1 with all \( r_i = 1 \), there exists \( \alpha \in \mathbb{N}^l \) with distinct coordinates such that \( \alpha^T m_i \) are pairwise distinct for \( 1 \leq i \leq N \). Let \( g_\alpha : [0,\infty) \to \mathbb{R} \) be defined by
\[
g_\alpha(x) := R \cdot \frac{x^{m_1} + \cdots + x^{m_l}}{R^{m_1} + \cdots + R^{m_l}}.
\]
Note that \( g_\alpha[-] : P_n^1(I) \to P_n^1(I) \) and so \( f \circ g_\alpha[-] : P_n^1(I) \to S_n^k \). Thus, by Theorem 4.9, the polynomial \( f \circ g_\alpha \) is a linear combination of at most \( k \) integer powers. On the other hand, writing \( f(x) = \sum_{t=1}^r a_t x^{i_t} \) for some distinct integers \( i_t \in [0,k-1] \) with all \( a_t \) nonzero, by the choice of \( \alpha \), the function \( f \circ g_\alpha \) is a linear combination of exactly \( \sum_{t=1}^r (i_t+1) \) distinct integer powers. Therefore (2) follows since the power functions \( \{x^n : n \in \mathbb{Z}_{\geq 0}\} \cup \{f \equiv 1\} \) are linearly independent on \([0,\infty)\) by Proposition 3.17.

We now prove the second set of equivalences. Clearly if \( f \) is a polynomial with nonnegative coefficients which satisfies assertion (2) in the theorem, then \( f[-] : P_n^1(I) \to P_n^k \) by the calculation in equation (4.3) (and the Schur product theorem). Conversely if (1) holds, the first set of equivalences already shows that \( f \) is a polynomial of degree \( \leq k-1 \) satisfying (1.6). That the coefficients of \( f \) are nonnegative follows by Theorem 3.9. Finally, if \( k \leq n-3 \) then the condition that \( f \in C^k(I) \) actually follows by Theorem 4.1, and hence does not need to be assumed.

Remark 5.5. Note that if \( l > 1 \), Theorem 3.4 immediately provides a constraint on the degree of a polynomial \( p(x) \) mapping \( P_n^1 \to P_n^k \). Indeed, the degree must satisfy \( \binom{\deg(p)+l-1}{l-1} \leq k \). On the other hand, the degree can be arbitrary when \( l = 1 \), by Theorem A.

Recall that Theorem A shows that under appropriate differentiability hypotheses, entrywise functions mapping \( P_n^1(I) \) into \( S_n^k \) are precisely the set of polynomials with \( k \) nonzero coefficients. Similarly, Theorem B shows that an analytic function maps \( P_n^1(I) \to S_n^k \) if and only if it satisfies equation (1.6). We now prove that the conclusion of the theorems are optimal in the following precise sense.
Proposition 5.6. Let \( 0 < R \leq \infty, I = [0, R) \) or \((-R, R)\), and \( f : I \to \mathbb{R} \). Fix \( 1 \leq k < n \). Then:

1. If \( f \) is a polynomial with \( k \) nonzero coefficients, then there exists a matrix \( A \in \mathbb{P}_n(I) \) such that \( f[A] \) has rank exactly \( k \).
2. If \( f(x) = \sum_{i=1}^{k} a_i x^{m_i} \) with \( a_i \neq 0 \) and \( m_i \in \mathbb{Z}_{\geq 0} \) satisfying equation (1.6), then there exists \( A \in \mathbb{P}_n(I) \) such that \( f[A] \) has rank exactly \( \sum_{i=1}^{k} (i+1-l-1) \).

Proof. It suffices to show the result for \( I = [0, R) \). To prove (1), let \( f(x) = \sum_{i=1}^{k} c_i x^{m_i} \), with \( c_i \neq 0 \) and \( m_i \in \mathbb{N} \). Let \( v \in I^n \) be a vector with distinct components and let \( A = vv^T \in \mathbb{P}_n^1(I) \). Clearly \( f[A] \) has rank at most \( k \) since \( f[A] \) is a sum of \( k \) rank 1 matrices. Now, by Proposition 3.15 the vectors \( v^{c_{m_1}}, \ldots, v^{c_{m_k}} \) are linearly independent. Denote by \( U \) the \( k \times n \) matrix whose columns are \( v^{c_{m_1}}, \ldots, v^{c_{m_k}} \), and let \( C \) be the \( k \times k \) diagonal matrix with diagonal entries \( c_1, \ldots, c_k \). Note that \( f[A] = U^T C U \). Clearly \( U^T C U \) and \( U \) have rank \( k \). Thus, by Sylvester’s rank inequality, \( \text{rank } f[A] = \text{rank } U^T C U \geq \text{rank } U^T C + \text{rank } U = k \). It follows that \( \text{rank } f[A] = k \). This proves (1).

To prove (2), first note that by Proposition 5.1(2), there exist \( u_1, \ldots, u_l \) in \( I^n \) such that the vectors

\[
\{ u_{a_1}^{\circ} \cdots u_{a_l}^{\circ} : a_1, \ldots, a_l \in \mathbb{Z}_{\geq 0}, a_1 + \cdots + a_l = i_t, t = 1, \ldots, r \}
\]

are linearly independent. Note that there are \( \sum_{t=1}^{r} \binom{i_t+l-1}{l-1} \) such vectors. Define

\[
A := \sum_{t=1}^{r} u_{i_t}^{\circ} \in \mathbb{P}_n(I). \]

Expanding \( f[A] \) using the multinomial theorem, we obtain a linear combination of the vectors in \( \{ u_{a_1}^{\circ} \cdots u_{a_l}^{\circ} \} \) with nonzero coefficients. Using the same argument as in the first part, it now follows that \( f[A] \) has rank \( \sum_{t=1}^{r} \binom{i_t+l-1}{l-1} \), as desired. \( \square \)

5.2. The regime \( 1 \leq k < l \). Recall that the characterization obtained in Theorem 13 was obtained under the assumption that \( f \in C^k \). Surprisingly, this assumption can be relaxed significantly if additional constraints are known on \((l, k)\). In this Subsection and Subsection 5.3 we study the cases where \( 1 \leq k < l \) and \( l \leq 2k \) respectively. We now demonstrate that when \( k < l \), no assumption on \( f \) is required in order to obtain the conclusion of Theorem 13.

Theorem 5.7. Suppose \( 0 < R \leq \infty, I = [0, R) \) or \((-R, R)\), and \( f : I \to \mathbb{R} \) with \( f \neq 0 \). Fix integers \( n \geq 3 \) and \( 1 \leq k < l \leq n \). Suppose \( 1 \leq k < n-1 \) when \( I = (-R, R) \). Then the following are equivalent:

1. \( f[A] \in \mathbb{S}_n^k \) for every \( A \in \mathbb{P}_n(I) \);
2. \( f[A] \in \mathbb{S}_n^k \) for every \( A \in \mathbb{P}_n^l(I) \);
3. \( f \equiv c \) on \( I \) for some \( c \neq 0 \).

Moreover, \( f[-] : \mathbb{P}_n^l(I) \to \mathbb{P}_n^k \) if and only if \( f \equiv c \) for some \( c > 0 \).

Proof. Clearly, (3) \( \Rightarrow \) (1) \( \Rightarrow \) (2). We first show that (2) \( \Rightarrow \) (3) if \( I = [0, R) \). Suppose first \( f(0) = 0 \). Observe that \( f[-] : \mathbb{P}_n^{k+1}(I) \to \mathbb{S}_n^k \), so for all \( a \in I \cap [0, \infty) \), we have \( f[a \mathbf{Id}_{k+1} \oplus 0_{(n-k-1) \times (n-k-1)}] \in \mathbb{S}_n^k \) if (2) holds. Thus its leading principal \((k+1) \times (k+1)\) minor vanishes, which shows that \( f \equiv 0 \) on \( I \in [0, \infty) \), which contradicts (2) if \( I = [0, R) \) and \( f(0) = 0 \). Therefore \( f(0) \neq 0 \). Now the (1) \( \Rightarrow \) (3) implication of Proposition 3.6 shows that \( f[-] : \mathbb{P}_n^{l-1}(I) \to \mathbb{S}_n^{l-1} \). It follows from the argument above that \( f - f(0) \equiv 0 \) on \( I = [0, R) \). This proves (2) \( \Rightarrow \) (3) if \( I = [0, R) \).
Now suppose \( I = (-R, R) \) and \( k < n - 1 \). Given \( a \in [0, R) \), define the following matrices:
\[
A_a := a \Id_{k+1} \oplus 0_{(n-k-1) \times (n-k-1)} \in \mathbb{P}_n(I), \quad \tilde{A}_a := \begin{pmatrix} A_a & -A_a \\ -A_a & A_a \end{pmatrix}.
\]

Note that \( \tilde{A}_a \) is the Kronecker product of \( B := \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \) and \( A_a \), so its eigenvalues are the products of the eigenvalues of \( B \) and \( A_a \). It follows that \( \tilde{A}_a \in \mathbb{P}_n^d(I) \). Now assume (2) holds. By Lemma 5.3 we have \( f[\cdot] : \mathbb{P}_n^d(I) \to \mathbb{S}^k_{2n} \). Thus, \( f[\tilde{A}_a] \in \mathbb{S}^k_{2n} \). Therefore by Lemma 5.1 the \((k+1) \times (k+1)\) principal minors of both \( f[A_a] \) and \( f[-A_a] \) vanish. It follows that \( f \equiv 0 \) on \( I = (-R, R) \), proving (3) if \( I = (-R, R) \) and \( f(0) = 0 \). Now suppose \( f(0) \neq 0 \). Then Proposition 5.6 shows that \( (f - f(0))[\cdot] : \mathbb{P}_{n-1}^d(I) \to \mathbb{S}_{n-1}^{k-1} \) and the result follows.

Finally, if \( f[\cdot] : \mathbb{P}_n^d(I) \to \mathbb{S}^k_n \), then in particular \( f[\cdot] : \mathbb{P}_n^d(I) \to \mathbb{S}^k_n \) and so \( f \equiv c \). It follows easily that \( c > 0 \).

\[ \square \]

5.3. The regime \( l \leq k < 2l \). We now study the regime \( l \leq k < 2l \). Theorem 5.8 classifies all continuous functions \( f \) which send \( \mathbb{P}_n^l \) into \( \mathbb{S}^k_n \) with \( l \leq k < 2l \) and yields the same classification as in Theorem 1.

**Theorem 5.8.** Fix \( 0 < R \leq \infty \) and integers \( 2 < l < k < \min(2l, n - 1) \). Suppose \( I = [0, R) \) or \((-R, R)\), and \( f : I \to \mathbb{R} \) is continuous. The following are equivalent:

1. \( f[A] \in \mathbb{S}^k_n \) for every \( A \in \mathbb{P}_n^l(I) \);
2. \( f[A] \in \mathbb{S}^k_n \) for every \( A \in \mathbb{P}_n^l(I) \);
3. There exists \( c_1 \in \mathbb{R} \) such that \( f(x) = f(0) + c_1 x \) for all \( x \in I \).

Moreover, \( f[\cdot] : \mathbb{P}_n^l(I) \to \mathbb{S}^k_n \) if and only if \( f(0) \geq 0 \) and \( f(x) = f(0) + c_1 x \) for some \( c_1 \geq 0 \) and all \( x \in I \). Also, if \( f[\cdot] : \mathbb{P}_n^l(I) \to \mathbb{S}^k_n, I = [0, R), \) and \( k \leq n - 3, \) the continuity assumption is not required.

Theorem 5.8 addresses the general case \( 2 < l < k < \min(2l, n - 1) \). Whether or not the theorem holds for the remaining possible values for \( l, k, \) and \( n \) is discussed in Remark 5.13.

**Remark 5.9.** Note that when \( 1 \leq k < 2l \), the combinatorial condition (1.6) implies that \( f \) is constant or linear. Theorems 5.7 and 5.8 thus show that the \( C^k \) assumption in Theorem 1 can be replaced by continuity when \( 1 \leq k < 2l \), without changing the characterization.

In order to prove Theorem 5.8 we need the following two preliminary results.

**Lemma 5.10.** Let \( A, B, C \) be three positive semidefinite matrices of dimension \( n_1, n_2, \) and \( n_3 \) respectively. Then the \( 2(n_1 + n_2 + n_3) \times 2(n_1 + n_2 + n_3) \) matrix
\[
M := \begin{pmatrix} A \oplus B \oplus C & A \oplus -B \oplus C \\ A \oplus -B \oplus C & A \oplus B \oplus C \end{pmatrix}
\]
is positive semidefinite and \( \text{rank } M = \text{rank } A + \text{rank } B + \text{rank } C \). (Recall here that \( A \oplus B \oplus C = \text{diag}(A, B, C) \) denotes a block diagonal matrix.)

**Proof.** To compute \( \text{rank } M \), note that the first half and the second half of the columns of \( M \) are linearly dependent. It follows easily that \( \text{rank } M = \text{rank } A + \text{rank } B + \text{rank } C \). To prove that \( M \) is positive semidefinite, suppose first that \( A, B, \)
and $C$ are invertible. Clearly, $A \oplus B \oplus C$ is positive definite. The Schur complement of the (2,2) block $A \oplus B \oplus C$ in $M$ is

$$S = (A \oplus B \oplus C) - (A \oplus -B \oplus C)(A^{-1} \oplus B^{-1} \oplus C^{-1})(A \oplus -B \oplus C)$$

$$= (A \oplus B \oplus C) - (\text{Id}_{n_1} \oplus -\text{Id}_{n_2} \oplus \text{Id}_{n_3})(A \oplus -B \oplus C)$$

$$= 0_{(n_1+n_2+n_3) \times (n_1+n_2+n_3)}.$$  

It follows that $M$ is positive semidefinite (see [9, Appendix A.5.5]). Finally, if $A$, $B$, or $C$ is not invertible, then the result follows by replacing $(A, B, C)$ by $(A + \epsilon \text{Id}_{n_1}, B + \epsilon \text{Id}_{n_2}, C + \epsilon \text{Id}_{n_3})$ for $\epsilon > 0$, applying the above argument to the resulting block matrix $M$, and letting $\epsilon \to 0^+$. \hfill \Box

The next preliminary result demonstrates that applying $\phi_1$ entrywise to rank $l$ matrices can double the rank.

**Proposition 5.11.** Let $0 < R \leq \infty$ and let $I = (-R, R)$. Fix integers $n > l \geq 2$ and $1 \leq k < \max(2l, n)$. Then there exists a matrix $A \in \mathbb{P}_n^l(I)$ such that $\phi_1[A] \notin \mathbb{S}_{n}^k$.

**Proof.** It suffices to prove the result for $I = \mathbb{R}$ since $\phi_\alpha(ax) = a^\alpha \phi_\alpha(x)$ for all $a > 0$. Suppose first that $l > k$. Let $A \in \mathbb{P}_n^l([0, R))$ have rank exactly $l$. Then $\phi_1[A] = A \notin \mathbb{S}_n^k$ and so $\phi_1$ does not map $\mathbb{P}_n^l(I)$ to $\mathbb{S}_n^k$ if $l > k$. Now suppose $2 \leq l \leq k < \max(2l, n)$. Let

$$A_4 := \begin{pmatrix}
 8 & 4 & -2 & 6 \\
 4 & 4 & 2 & 4 \\
 -2 & 2 & 5 & 0 \\
 6 & 4 & 0 & 5
\end{pmatrix}, \quad A_6 := \begin{pmatrix}
 9 & 7 & 7 & 1 & 1 & -3 \\
 7 & 6 & 5 & 3 & 2 & -2 \\
 7 & 5 & 6 & -1 & 0 & 2 \\
 1 & 3 & -1 & 9 & 5 & 1 \\
 1 & 2 & 0 & 5 & 3 & 1 \\
 -3 & -2 & -2 & 1 & 1 & 3
\end{pmatrix}. $$

It is not difficult to verify that $A_4 \in \mathbb{P}_2^2(\mathbb{R})$, $A_6 \in \mathbb{P}_6^3(\mathbb{R})$, and all the leading principal minors of $\phi_1[A_4]$ and $\phi_1[A_6]$ are nonzero.

Suppose first $l$ is even, say $l = 2a$ for some integer $a > 0$. If $n \geq 4a = 2l$, then the matrix

$$M := B_a \oplus 0_{(n-4a) \times (n-4a)}; \quad B_a := A_4 \oplus \cdots \oplus A_4, \quad \text{a times}$$

satisfies $M \in \mathbb{P}_n^l(\mathbb{R})$, and $\phi_1[M] \in \mathbb{S}_{2l}^n \setminus \mathbb{S}_{2l-1}^n$. This proves that $\phi_1[-]$ does not send $\mathbb{P}_n^l(I)$ to $\mathbb{S}_n^k$ if $n \geq 2l$ and $l$ is even. Now suppose $l < n < 2l$. Let $M$ be the leading $n \times n$ principal submatrix of $B_a$, i.e., the submatrix formed by its first $n$ rows and columns. Then $M \in \mathbb{P}_n^l(\mathbb{R})$ since $B_a$ has rank $l$. Moreover, since every leading principal submatrix of $\phi_1[A_4]$ is nonsingular, it follows that $\phi_1[M]$ is also nonsingular, i.e., $\phi_1[M] \in \mathbb{S}_n^k \setminus \mathbb{S}_{2l-1}^n$. This proves the result for $n < 2l$ if $l$ is even.

Now suppose $l$ is odd, say $l = 2a + 1$ for some integer $a \geq 0$. For $n \geq 4a + 2 = 2l$, consider the matrix $M := B_{a-1} \oplus A_6 \oplus 0_{(n-4a-2) \times (n-4a-2)}$. Then rank $M = 2(a-1) + 3 = 2a + 1 = l$. Thus $M \in \mathbb{P}_n^l(\mathbb{R})$. However, rank $\phi_1[M] = 4(a-1) + 6 = 4a + 2 = 2l$. This shows that $\phi_1[-]$ does not send $\mathbb{P}_n^l(I)$ to $\mathbb{S}_n^k$ if $n \geq 4a + 2 = 2l$ and $l$ is odd. If $l < n < 2l$, then the result follows by considering a leading principal submatrix of $B_{a-1} \oplus A_6$ as in the case of even $l$. \hfill \Box

With the above results in hand, we can now prove Theorem 5.8.
Proof of Theorem 5.8. Clearly, (3) $\Rightarrow$ (1) $\Rightarrow$ (2). We first show that (2) $\Rightarrow$ (3) if $I = [0, R)$, $f(0) = 0$, and $2 < l \leq k < \min(2l, n - 1)$. This assertion clearly holds if $f$ is constant on $I$. Now suppose $f$ is not constant on $I$. Fix $c \in I$ such that $f(c) \neq 0$ and choose arbitrary $a, b \in \mathbb{R}$ such that $a^2, ab, b^2 \in I$. Let $u := (a, b)^T$ and define

$$C_j := uu^T \oplus \cdots \oplus uu^T, \quad j \text{ times}$$

(5.7)

$$A_{k,l}(a, b, c) := C_{k-l+1} \oplus cI_{2l-k-1} \oplus 0_{(n-k-1) \times (n-k-1)} \in \mathbb{P}_n^k(I).$$

Then $f[A_{k,l}(a, b, c)] \in S^k_n$ by hypothesis, whence its leading principal $(k+1) \times (k+1)$ submatrix is singular, i.e.,

(5.8)

$$f(c)^{2l-k-1}(f(a^2) - f(ab)^2)^{k-l+1} = 0$$

(assuming that $f(0) = 0$). Since $a, b$ are arbitrary and $f(c) \neq 0$, it follows that $f[-] : \mathbb{P}^l_n(I) \rightarrow S^2_n$. Since $f$ is not constant, it follows by Lemma 5.3 that $f(x) = c_1x^\alpha$ for some $c_1 \in \mathbb{R}$, $\alpha > 0$, and all $x \in I = [0, R)$.

We now show that $\alpha = 1$. First, if $\alpha \notin \{1, \ldots, n-1\}$, then Corollary 4.10 implies that there exists $u \in \mathbb{R}^n$ such that $A_u := 1_{n \times n} + uu^T \in \mathbb{P}_n((0, \infty))$ and $f_u[A_u]$ is nonsingular (which contradicts the assumptions). We conclude that $\alpha \in \{1, \ldots, n-1\}$. By Theorem \[\text{applied on } I = [0, R), \text{ it follows that } \binom{n+l-1}{l-1} \leq k.

If $\alpha \geq 2$ we verify that $\binom{n+l-1}{l-1} \leq 2l$ since $l > 2$. Thus, $f \equiv c_1x$ on $I$ and (3) follows.

Now suppose that $I = [0, R)$ and $f(0) \neq 0$. Then Proposition 4.6 shows that $f$ is constant on $I$. If $k = l$, then it follows that $f(0)$ is constant by Theorem 5.4 and so $f \equiv f(0)$. If $k > l$, then the above reasoning shows that $f \equiv c_1x$ for some $c_1 \in \mathbb{R}$. Thus $f(x) = f(0) + c_1x$ for some $c_1 \in \mathbb{R}$ and all $x \in I$.

Next, we prove that (2) $\Rightarrow$ (3) when $I = (-R, R)$. The result clearly holds if $f$ is constant. Thus, assume $f$ is nonconstant. Suppose first $f(0) = 0$. We use a technique similar to the one used in the proof of Theorem 5.7 for $I = (-R, R)$. Let $c \in I$ be such that either $f(c) \neq 0$ or $f(-c) \neq 0$ and let $a, b \in \mathbb{R}$ such that $a^2, b^2, ab \in I$. Consider the matrix

(5.9)

$$\tilde{A} := \begin{pmatrix} A_{k,l}(a, b, |c|) & A_{k,l}(a, b, -|c|) \\ A_{k,l}(a, b, -|c|) & A_{k,l}(a, b, |c|) \end{pmatrix}$$

with $A_{k,l}(x, y, z)$ as in (5.7). By Lemma 5.10 we have $\tilde{A} \in \mathbb{P}_n^k(I)$. Also, by Lemma 5.3 we have $f[-] : \mathbb{P}_n^k(I) \rightarrow S^2_n$. Thus, $f[\tilde{A}] \in S^2_n$. By Lemma 5.10, the $(k+1) \times (k+1)$ principal minors of $A_{k,l}(a, b, |c|)$ and $A_{k,l}(a, b, -|c|)$ both vanish. Computing the determinant of these two matrices as in (5.8), we conclude that $f(a^2)(b^2) - f(ab)^2 = 0$ for all $a, b \in \mathbb{R}$ such that $a^2, b^2, ab \in I$. It follows by Proposition 5.2 that $f[-] : \mathbb{P}_n^k(I) \rightarrow S^2_n$. Since $f$ is not constant, by Lemma 5.4, we have that either $f(x) = c_1\phi_1(x)$ or $f(x) = c_1\phi_2(x)$ for some $c_1 \in \mathbb{R}$, $\alpha > 0$, and all $x \in I = (-R, R)$. Now since $f[-] : \mathbb{P}_n^k(I) \rightarrow S^2_n$, then in particular $f[-] : \mathbb{P}_n^k((0, R)) \rightarrow S^2_n$. Therefore, by the previous part $f \equiv c_1x$ on $[0, R)$. It follows that $\alpha = 1$, i.e., $f \equiv c_1\psi_1$ or $f \equiv c_1\psi_1$ on $I = (-R, R)$. By Proposition 5.11, the function $\psi_1$ does not map $\mathbb{P}_n^k(I)$ into $S^2_n$. Thus, $f(x) = c_1\psi_1(x) = c_1x$ for all $x \in I$. This proves the result for $I = (-R, R)$ if $f(0) = 0$. If $f(0) \neq 0$, then
applying Proposition 3.6 shows that \((f - f(0))[\cdot] : \mathbb{P}_{n-1}^l(I) \to S_{n-1}^{k-1}\) and the result easily follows.

Now suppose \(f[\cdot] : \mathbb{P}_n^k(I) \to \mathbb{P}_n^k\). Then in particular \(f[\cdot] : \mathbb{P}_n^k(I) \to S_n^k\) and so \(f(x) = f(0) + c_1x\) for all \(x \in I\). It follows easily that \(f(0), c_1 \geq 0\). The converse is obvious. Finally, if \(k \leq n - 3\) and \(f[\cdot] : \mathbb{P}_n^k([0, R)) \to \mathbb{P}_n^k\), then by Theorem 5.8 we do not need any continuity assumption on \(f\).

**Remark 5.12.** When \(I = (-R, R)\), the proof of Theorem 5.8 depends on Lemmas 5.3 and 5.10. We now provide a direct argument that avoids using these lemmas. Let \(a, b \in \mathbb{R}\) such that \(a^2, b^2, ab \in I\) and let \(u := (a, b)^T\). Let \(A := C_{(k+1)/2} \oplus O_{(n-k-1) \times (n-k-1)}\) if \(k\) is odd, and \(A := C_{(k+2)/2} \oplus O_{(n-k-2) \times (n-k-2)}\) if \(k\) is even, where \(C_J\) was defined in (5.7). Note that \(n \geq k + 2\) since \(k < n - 1\) by hypothesis and so \(A\) is well-defined. Also, since \(k < 2l\), then \(l \geq (k + 1)/2\) if \(k\) is odd, and \(l \geq (k + 2)/2\) if \(k\) is even. Therefore, \(A \in \mathbb{P}_n^l(I)\), which implies that \(f[A] \in S_n^k\) if \(k\) is odd, and \(f[A] \in S_{n+1}^{k+1}\) if \(k\) is even. It follows that \(f[\cdot] : \mathbb{P}_n^k(I) \to S_n^k\). The proof can now be concluded by using the rest of the argument in the proof of Theorem 5.8.

Note that when \(k = 0\) and \(n = k - 1\), the proof given above is also valid.

**Remark 5.13.** Various cases were left unresolved in Theorem 5.8 in order to simplify the statement of the theorem. We now address each one of them separately.

**Case 1:** \(l = k\). When \(l = k = 0\), the arguments used in the proof of Theorem 5.8 show that \(f(x) = c_1x\) for some \(c_1 \in \mathbb{R}\). The converse also clearly holds. If \(f(0) \neq 0\), Proposition 3.6 shows that \((f - f(0))[\cdot] : \mathbb{P}_{n-1}^l \to S_{n-1}^{k-1} = S_{n-1}^{l-1}\), and Theorem 5.7 implies that \(f = f(0)\).

**Case 2:** \(k = 2l\) and \(f(0) \neq 0\). In this case, Proposition 3.6 shows that \((f - f(0))[\cdot] : \mathbb{P}_{n-1}^l \to S_{n-1}^{k-1} = S_{2l-1}^{l-1}\). We conclude by Theorem 5.8 that \(f(x) = f(0) + c_1x\) for some \(c_1 \in \mathbb{R}\) and all \(x \in I\).

**Case 3:** \((l, k) = (2, 3)\). If \(l = 2\), \(k = 3\), and \(n \geq 4\), the result becomes slightly different. First, if \(f(0) \neq 0\), then \(f - f(0) = c_1x\) by Proposition 3.6 and the \(l = k\) case. Suppose \(f(0) = 0\). The proof of Theorem 5.8 shows that either \(f \equiv c_1 \phi_\alpha\) or \(f \equiv c_1 \psi_\alpha\) on \(I\) for some \(\alpha \in \{1, \ldots, n - 1\}\). If \(\alpha > 2\), then \((\alpha + 1)/2 = (\alpha + 1) = \alpha \geq 4\), and Theorem 5.8 applied on \([0, R)\) implies that \(c_1 \phi_\alpha\) or \(c_1 \psi_\alpha\) cannot send \(\mathbb{P}_n^2(I)\) to \(S_n^3\) if \(c_1 \neq 0\). Thus \(\alpha = 1\) or 2. By Theorem 5.8 the functions \(f \equiv c_1x, c_1x^2\) do map \(\mathbb{P}_n^2(I)\) to \(S_n^3\). We now claim that \(\phi_1\) and \(\psi_2\) don’t. This is clear for \(\phi_1\) by Proposition 5.11. To prove that \(\psi_2[\cdot]\) does not send \(\mathbb{P}_n^2\) to \(S_n^3\), let \(B_4 := (\cos (\frac{(k-1)\pi}{4})^i_j\) be the matrix constructed in [5], Section 1]. Then one easily verifies that \(\psi_2[B_4]\) is nonsingular. Therefore, if \(l = 2, k = 3\), and \(f(0) = 0\), then \(f \equiv c_1x\) or \(f \equiv c_1x^2\) on \(I\).

**Case 4:** \(k = n - 1\). In Theorem 5.8, we assume \(k < \min(2l, n - 1)\). It is natural to ask if the assumption can be relaxed to \(k < \min(2l, n)\). Note that when \(I = (0, R)\), the proof of Theorem 5.8 goes through for \(k < \min(2l, n)\). The result also holds when \(I = (-R, R)\) and \(k\) is odd; see Remark 5.12. However, the result fails in general when \(I = (-R, R)\) and \(k\) is even. For instance, the \((2) \Rightarrow (3)\) implication in Theorem 5.8 does not hold if \(k = l = 2, n = 3,\) and \(I = (-R, R)\). In fact, we claim that every function \(f\) such that \(f \equiv 0\) on \([-R/2, R]\) automatically satisfies (2) when \(n = 3\). To show the claim, first suppose that at least one of \(r, s, t\) lies in
[-R/2, R). Now given any matrix
\[ A = \begin{pmatrix} a & r & s \\ r & b & t \\ s & t & c \end{pmatrix} \in \mathbb{P}_3^2(I), \]
it is clear that \( \det f[A] = 2f(r)f(s)f(t) = 0 \), so \( f[A] \in \mathbb{S}_3^2 \) and (b) holds.

Suppose instead that \( r, s, t \in (-R, -R/2) \). We show that \( A \) (of the above form) cannot lie in \( \mathbb{P}_3^2(I) \) - in fact, not even in \( \mathbb{P}_3(I) \). (This shows more generally that if \( B \in \mathbb{P}_n(I) \), then for every principal \( 3 \times 3 \) submatrix \( A \) of \( B \), at least one off-diagonal entry lies in \([-R/2, R)\).) First compute:
\[
4 \det A = 4abc - 4at^2 - 4bs^2 - 4cr^2 + 8rst \leq 4abc - (a + b + c)R^2 - R^3
\]
by the arithmetic mean-geometric mean inequality. Let \( u := (a + b + c)/3 \) and define \( g(x) := 4x^3 - 3xR^2 - R^3 \). Note that if \( A \in \mathbb{P}_3^2(I) \) then \( u \in [0, R) \). The above computations thus show that if \( r, s, t \in (-R, -R/2) \), then \( 4 \det A < g(u) \), with \( u \in [0, R) \). Note that \( g'(x) = 12(x^2 - (R/2)^2) \), which is nonpositive on \([0, R/2]\) and positive on \((R/2, R]\). Thus \( g(x) \) is decreasing on \([0, R/2]\) and increasing on \([R/2, R]\); moreover, \( g(0) < 0 = g(R) \). Hence we get
\[
4 \det A < g(u) \leq 0,
\]
which shows that if \( r, s, t \in (-R, -R/2) \) then \( A \notin \mathbb{P}_3 \).

In Case 4 of Remark 5.13 we demonstrated that if \( k = l = 2, n = 3, \) and \( I = (-R, R) \), any function \( f \) such that \( f \equiv 0 \) on \([-R/2, R)\) maps \( \mathbb{P}_3^2(I) \) into \( \mathbb{S}_3^2 \). We now prove that the conclusion of Theorem 5.8 holds if \( f \neq 0 \) on \([-R/2, R)\) and \( k < \min(2l, n) \). Note that all the cases where \( k < n - 1 \) have already been considered in Theorem 5.8 under more general hypotheses.

**Theorem 5.14.** Fix \( 0 < R \leq \infty \) and integers \( 2 < l < k < \min(2l, n) \). Suppose \( I = (-R, R), f : I \rightarrow \mathbb{R} \) is continuous, and \( f \neq 0 \) on \([-R/2, R)\). Then the following are equivalent:

1. \( f[A] \in \mathbb{S}_n^k \) for every \( A \in \mathbb{S}_n^l(I) \);
2. \( f[A] \in \mathbb{S}_n^k \) for every \( A \in \mathbb{P}_n^l(I) \);
3. There exists \( c_1 \in \mathbb{R} \) such that \( f(x) = f(0) + c_1x \) for all \( x \in I \).

Moreover, \( f[-] : \mathbb{P}_n^l(I) \rightarrow \mathbb{P}_n^k \) if and only if \( f(0) \geq 0 \) and \( f(x) = f(0) + c_1x \) for some \( c_1 \geq 0 \) and all \( x \in I \). Also, if \( f[-] : \mathbb{P}_n^l(I) \rightarrow \mathbb{P}_n^k, I = [0, R), \) and \( k \leq n - 3 \), the continuity assumption is not required.

**Proof.** Clearly (3) \( \Rightarrow \) (1) \( \Rightarrow \) (2). The implication (2) \( \Rightarrow \) (3) was already proved in greater generality when \( k < n - 1 \) or \( k = n - 1 \) and \( k \) is odd (see Theorem 5.8 and Remark 5.13). Thus, it suffices to prove (2) \( \Rightarrow \) (3) under the assumption that \( k = n - 1 \) and \( k \) is even.

Let \( 2 < l < n - 1 < 2l \), let \( k := n - 1 \) be even, and suppose (2) holds. Also, suppose first that \( f(0) = 0 \). We consider two cases:

**Case 1:** \( f[-] : \mathbb{P}_n^2(I) \rightarrow \mathbb{S}_n^1 \). In this case, following the argument in the proof of Theorem 5.8, we conclude that \( f(x) = c_1x \) for all \( x \in I \) and some \( c_1 \in \mathbb{R} \).

**Case 2:** \( f[-] : \mathbb{P}_n^2(I) \not\rightarrow \mathbb{S}_n^1 \). In this case, there exists \( \hat{A}_2 \in \mathbb{P}_n^2(I) \) such that \( f[\hat{A}_2] \) has rank 2. Note that \( l \geq (n + 1)/2 \) and so \( f[-] : \mathbb{P}_n^{l+1}(I) \rightarrow \mathbb{S}_n^{l+1} \). Also note that
n \geq 5 \text{ and } n \text{ is odd, say } n = 3 + 2a \text{ for some } a \geq 1. \text{ Now given } A_3 \in \mathbb{P}_2^3(I), \text{ consider} \\
A = A(\hat{A}_2, A_3) := A_3 \oplus \hat{A}_2 \oplus \cdots \oplus \hat{A}_2 \in \mathbb{P}^{2+a}_n(I) = \mathbb{P}^{n+1}_n(I).

Since \( f[A(\hat{A}_2, A_3)] \in \mathbb{S}^{-1}_n \) for all \( A_3 \in \mathbb{P}^3_2(I), \) it follows that \( f[-] : \mathbb{P}^3_2(I) \to \mathbb{S}^2_2. \) Since \( f \not\equiv 0 \) on \([{-R}/2, R], \) we now claim that there exists \( c \in (0, R) \) such that \( f(c) \neq 0. \) The claim is clear if \( f \neq 0 \) on \((0, R); \) otherwise let \( x_0 \in (-R/2, 0) \) such that \( f(x_0) \neq 0. \) Define the block diagonal matrix
\[
(5.10)\quad B(x_0) := x_0 I_{3 \times 3} - 3x_0 Id_3 \in \mathbb{P}^2_3(I),
\]
Since \( f[-] : \mathbb{P}^2_3(I) \to \mathbb{S}^2_2, \) the matrix \( f[B(x_0)] \) has determinant zero, i.e.,
\[
0 = \det f[B(x_0)] = (f(-2x_0) + 2f(x_0))(f(-2x_0) - f(x_0))^2.
\]
Hence either \( f(-2x_0) = -2f(x_0) \neq 0 \) or \( f(-2x_0) = f(x_0) \neq 0 \) This proves the claim with \( c := -2x_0, \) Now considering the matrix \( A_{2,2}(a,b,c) \) as in \((5.7),\) we conclude that \( f[-] : \mathbb{P}^2_3(I) \to \mathbb{S}^2_2 \) and so \( f(x) = cx \) for some \( c_1 \in \mathbb{R} \) as in Case 1. This proves the result if \( f(0) = 0. \) Note that the proof goes through for \( f(0) = 0 \) even if \( l = k. \)

If \( f(0) \neq 0, \) then applying Proposition 6.6 shows that \( (f - f)(0)[-] : \mathbb{S}^k_{n-1}(I) \to \mathbb{S}^k_{n-1} \) and the result easily follows using the above analysis.

Now suppose \( f[-] : \mathbb{P}^k_n(I) \to \mathbb{P}^k_n. \) Then in particular \( f[-] : \mathbb{P}^l_n(I) \to \mathbb{S}^k_n \) and so \( f(x) = f(0) + cx \) for all \( x \in I. \) It follows easily that \( f(0), c_1 \geq 0. \) The converse is obvious. Finally, if \( k \leq n - 3 \) and \( f[-] : \mathbb{P}^k_n([0, R]) \to \mathbb{P}^k_n, \) then by Theorem 6.6 we do not need any continuity assumption on \( f. \) \( \square \)

6. Preserving positivity and absolutely monotonic functions

In the final section of this paper, we return to the original problem studied by Schoenberg, Rudin, Horn, Vasudeva, Hiai and others, i.e., the characterization of entrywise functions mapping \( \mathbb{P}_n(I) \) to \( \mathbb{P}_n \) for all \( n. \) We demonstrate a stronger result, namely, that preserving positivity on \( \mathbb{P}_2 \) (in fact on all special rank 2 matrices) for all \( n \geq 1 \) is equivalent to preserving positivity on \( \mathbb{P}_n, \) for all \( n \geq 1. \) In particular, we provide a new proof of a generalization of the result by Vasudeva [35].

First recall some classical results about absolutely monotonic functions. Define the \( m \)-th forward difference of a function \( f, \) with step \( h > 0 \) at the point \( x, \) to be
\[
(6.1)\quad \Delta^m f(x) := \sum_{i=0}^{m} (-1)^i \binom{m}{i} f(x + (m - i)h).
\]

We now state two important results about absolutely monotonic functions which will be needed to prove Theorem C.

**Theorem 6.1** (see [39], Chapter IV, Theorem 7). Let \( 0 < R \leq \infty \) and let \( f : [0, R] \to \mathbb{R}. \) Then the following are equivalent:

1. The function \( f \) is absolutely monotonic on \([0, R].\)
2. The function \( f \) can be extended analytically to the disc \( D(0, R) \subset \mathbb{C}, \) and \( f(z) = \sum_{i=0}^{\infty} a_i z^i \) for some \( a_i \geq 0. \)
3. For every \( m \geq 1, \) \( \Delta^m f(x) \geq 0 \) for all \( x \) and \( h \) such that
\[
0 \leq x < x + h < \cdots < x + mh < R.
\]
Theorem 6.1 can be used to show the following useful result.

**Lemma 6.2.** Let \( 0 < R \leq \infty \) and let \( f_n : [0, R) \to \mathbb{R} \), \( n \geq 1 \), be a sequence of absolutely monotonic functions and assume \( f_n(x) \to f(x) \) for every \( x \in [0, R) \). Then \( f \) is absolutely monotonic on \( [0, R) \).

**Proof.** By Theorem 6.1, the forward differences \( \Delta^k_n[f_n](x) \) of \( f_n \) are nonnegative for all integers \( n \geq 0 \) and for all \( x \) and \( h \) such that \( 0 \leq x < x + h < \cdots < x + nh < R \). Since \( f \) is the pointwise limit of the sequence \( f_n \), the same is true for \( \Delta^k_n[f](x) \). As a consequence, by Theorem 6.1, the function \( f \) is absolutely monotonic. \( \square \)

Recall that Vasudeva [58] proved that functions mapping all positive semidefinite matrices with positive entries into themselves are absolutely monotonic (see Theorem 2.2). Theorem \( \mathcal{C} \) strengthens Vasudeva’s result by working only with special matrices with positive entries into themselves are absolutely monotonic (see Theorem 6.1 (see Remark 4.4)). Using the techniques developed above, we now provide a more transparent and elementary proof of Theorem \( \mathcal{C} \).

**Proof of Theorem \( \mathcal{C} \).** That (3) \( \Rightarrow \) (2) follows from the Schur product theorem, and clearly (2) \( \Rightarrow \) (1). We now show that (1) \( \Rightarrow \) (3). Assume that \( f \in C^\infty(I) \) and let \( 0 < a < R \). Define \( f_a : [0, R-a) \to \mathbb{R} \) by \( f_a(x) := f(a + x) \). Then

\[
(6.2) \quad f_a[A] \in \mathbb{P}_n \quad \text{for every} \quad A \in \mathbb{P}_n^1([0, R-a)).
\]

Denote by \( \{ m_1, m_2, \ldots, m_k(a) \} \subset \{0, 1, \ldots, n-1 \} \) the (possibly empty) set of indices between 0 and \( n-1 \) such that \( f_a^{(i)}(0) \neq 0 \) if and only if \( i = m_j \) for some \( j \). By Theorem \( \mathcal{A}(3) \), we have \( f_a^{(m_i)}(0) > 0 \) for all \( 1 \leq j \leq k(a) \). Consequently, \( f^{(i)}(a) = f_a^{(i)}(0) \geq 0 \) for all \( 0 \leq i \leq n-1 \), for all \( n \in \mathbb{N} \), and all \( a \in I \). Since \( f \) is smooth, it follows from [52], Chapter IV that \( f \) has a power series representation with nonnegative coefficients, which proves the result when \( f \in C^\infty(I) \).

Now assume \( f \) is not necessarily smooth and let \( 0 < b < R \). First note by Step 3 of the proof of Theorem 6.1 that \( f \) is continuous on \( I \). Now given any probability distribution \( \theta \in C^\infty(b/R, \infty) \) with compact support in \( (b/R, \infty) \), let

\[
(6.3) \quad f_\theta(x) := \int_{b/R}^\infty f(xy^{-1})\theta(y)\frac{dy}{y}, \quad 0 < x < b.
\]

Then \( f_\theta \in C^\infty(0, b) \). Suppose \( A \in \mathbb{P}_n^1(0, b) \). Then, for every \( \beta \in \mathbb{R}^n \),

\[
\beta^T(f_\theta)[A] = \sum_{i,j=1}^n \beta_i \beta_j \int_{b/R}^\infty f(a_{ij}y^{-1})\theta(y)\frac{dy}{y} = \int_{b/R}^\infty \sum_{i,j=1}^n \beta_i \beta_j f(a_{ij}y^{-1})\theta(y)\frac{dy}{y}.
\]

Note that the integrand is nonnegative for every \( y > 0 \). It thus follows that \( \beta^T(f_\theta)[A] \geq 0 \). Since \( \beta \) is arbitrary, \( f_\theta[A] \in \mathbb{P}_n \).

Now consider a sequence \( \theta_m \in C^\infty(\mathbb{R}) \) of probability distributions with compact support in \( (b/R, \infty) \) such that \( \theta_m \) converges weakly to \( \delta_1 \), the Dirac measure at 1. Note that such a sequence can be constructed since \( b/R < 1 \). By the first part of the proof, \( f_{\theta_m} \) is absolutely monotonic on \( (0, b) \) for every \( m \geq 1 \). Therefore by Theorem 6.1, the forward differences \( \Delta^k_n[f_{\theta_m}](x) \) of \( f_{\theta_m} \) are nonnegative for \( l \geq 0 \) and all \( x \) and \( h \) such that \( 0 \leq x < x + h < \cdots < x + lh < R \). Since \( f \) is continuous, \( f_{\theta_m}(x) \to f(x) \) for every \( x \in (0, b) \). Therefore \( \Delta^k_n[f](x) \geq 0 \) for all such \( x \) and \( h \)
as well. As a consequence, by Theorem 6.1 the function \( f \) is absolutely monotonic on \((0, b)\). Since this is true for every \( 0 < b < R \), it follows that \( f \) is absolutely monotonic on \( I \).

It is natural to ask if results similar to Theorem C hold when \( I = [0, R) \). In other words, can one characterize functions that preserve positivity for all positive semidefinite matrices, or for positive semidefinite matrices of rank at most 2? Before answering this question, we first point out a subtle difference between functions that are absolutely monotonic on \((0, R)\) and \([0, R)\).

Remark 6.3. Recall that \( f \) is absolutely monotonic on \([0, R)\) if and only if its derivatives are all nonnegative on \([0, R)\) and \( f \) is continuous at 0. If instead \( f : [0, R) \to \mathbb{R} \) satisfies \( f[A] \in \mathbb{P}_n \) for all \( A \in \mathbb{P}_n([0, R)) \), then \( f \) is absolutely monotonic, nonnegative, and nondecreasing on \([0, R)\). Therefore \( f \) has (at most) a removable discontinuity at 0. Redefining \( f \) at 0 to be \( \lim_{x \to 0^+} f(x) \), we get that \( f \) is absolutely monotonic on \([0, R)\).

We now prove two characterization results analogous to Theorem C, but for \( I = [0, R)\).

**Theorem 6.4.** Suppose \( 0 < R \leq \infty \), \( I = [0, R)\), and \( f : I \to \mathbb{R} \). Then the following are equivalent:

(a) For all \( n \geq 1 \), \( f[a1_{n \times n} + uu^T] \in \mathbb{P}_n \) for every \( a \in I \) and \( u \in [0, \sqrt{R - a}]^n \).

(b) The function \( f \) is absolutely monotonic on \((0, R)\) and \( 0 \leq f(0) \leq f^+(0) := \lim_{x \to 0^+} f(x) \).

Similarly, the following are equivalent.

(1) \( f \) is right-continuous at 0, and for all \( n \geq 1 \), \( f[a1_{n \times n} + uu^T] \in \mathbb{P}_n \) for every \( a \in I \) and \( u \in [0, \sqrt{R - a}]^n \).

(2) For all \( n \geq 1 \), \( f[A] \in \mathbb{P}_n \) for every \( A \in \mathbb{P}_n(I) \).

(3) The function \( f \) is absolutely monotonic on \( I \).

In particular, the result is more involved for \( I = [0, R) \) than for \( I = (0, R) \), since functions preserving the set of matrices of the form \( a1_{n \times n} + uu^T \) need not preserve all matrices in \( \mathbb{P}_n \) for all \( n \).

**Proof.** If (a) holds, then \( f \) is absolutely monotonic on \((0, R)\) by Theorem C; hence nonnegative and nondecreasing. Thus the right-hand limit of \( f \) at zero, \( f^+(0) \), exists and is nonnegative. Now define the matrix \( A_t := \begin{pmatrix} t & 0 \\ 0 & 0 \end{pmatrix} \in \mathbb{P}_2 \). Then \( \lim_{t \to 0^+} f[A_t] \in \mathbb{P}_2 \), which implies: \( 0 \leq f(0) \leq f^+(0) \), proving (b).

Conversely, suppose (b) holds. If \( a > 0 \), then (a) follows from the (3) \( \Rightarrow \) (1) part of Theorem C. Now, assume \( a = 0 \). Suppose \( 0 \leq f(0) < f^+(0) \) and let \( A = uu^T \) for some \( u \in [0, \sqrt{R}]^n \). By permuting rows and columns of \( A \), we can assume it is of the form

\[
A = \begin{pmatrix} A' & 0_{n_1 \times n_2} \\ 0_{n_2 \times n_1} & 0_{n_2 \times n_2} \end{pmatrix},
\]

where \( 0 \leq n_1, n_2 \leq n \), and \( A' \) has no zero entry. By equation (3.9), the matrix \( f[A] \) is positive semidefinite if and only if \( f(0) \geq 0 \) (which holds by hypothesis) and \( (f - f(0))[A'] \in \mathbb{P}_n \). Now note by Remark 6.3 that the function \( \tilde{f} : [0, R) \to \mathbb{R} \) obtained by redefining \( f \) at 0 to be \( f^+(0) \) is absolutely monotonic on \([0, R)\), and
so (1) holds for \( \tilde{f} \) by the Schur product theorem. Hence the function \( \tilde{f} - f^+(0) \) is absolutely monotonic on \((0, R)\). Now since \( 0 \leq f(0) < f^+(0) \) by assumption,

\[
(f - f(0))[A'] = (\tilde{f} - f^+(0))[A'] + (f^+(0) - f(0))1_{n \times n} \in \mathbb{P}_n.
\]

This implies \( f[A] \in \mathbb{P}_n \) and concludes the proof of the first equivalence.

We now show the second set of equivalences. That \( (3) \Rightarrow (1),(2) \) follows from the right continuity of \( f \) at 0 and the Schur product theorem as in the \( I = (0, R) \) case. If (1) holds, then \( f \) is continuous at 0, as well as absolutely monotonic on \((0, R)\) by Theorem C which shows (3). We finally show that \( (2) \Rightarrow (1) \). As in the proof of the first equivalence, \( 0 \leq f(0) \leq f^+(0) \). Now proceed as in the proof of Theorem 2.3 to conclude that \( f^+(0) = f(0) \). □

Remark 6.5. Vasudeva’s proof of Theorem 2.2 can be adapted to functions with possibly finite domains \( f : (0, R) \to \infty \) for \( 0 < R \leq \infty \). However, Vasudeva’s methods do not extend to studying the problem of preserving positivity with rank constraints. In contrast, we solve the harder rank-constrained problem for fixed dimension by using a novel approach as described in Section 3.1. As a consequence, we prove Theorem 2.2 as a special case of Theorem C and moreover, by a more intuitive proof than that in [38].

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List of edits to the published version:

1. Definition 1.1: Added the first sentence.
2. Statement of Theorem 3.13: There are three changes: (i) $k$ can be 0, including in the final sentence; (ii) the polynomial $f$ in part (2) can have a constant term; and (iii) the $i_i$ are distinct (and non-negative).
3. The proof of Proposition 3.17 was earlier termed “Proof of Theorem 3.17.” This is now corrected.
4. Statement of Proposition 4.9: The integer $m$ is now specified to be positive.
5. Proof of Proposition 5.1, line 3: The definition of the set $C$ uses $Q^n$ – this should be $Q^l$. Similarly, the Cartesian product is over $l$ factors, not $n$.
6. Statements of Lemma 5.3 and Corollary 5.4: The value of $k$ is now allowed to be zero.
7. Proof of Theorem 13:
   (a) The first two sentences are new.
   (b) The vectors $m_j$ now comprise the set of all vectors in $(\mathbb{Z}_{\geq 0} \cap [0, k-1])^l$.
   (c) The third and fourth sentences after Equation (5.2) have been somewhat modified.
8. (For completeness:) While the authors’ affiliations have since changed, they are retained below as they are in the published paper.

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