Investigation of the generalization of Leith's model of the phenomenological theory of wave turbulence

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Abstract. A generalization of the Leith model of the phenomenological theory of the wave turbulence is studied. With the methods of group analysis, the basic models possessing nontrivial symmetries are obtained. The invariant solutions describing the invariant submodels of rank 0, are found in an explicit form. The physical meaning of these solutions is obtained. In particular, with the help of these solutions the turbulent processes for which there are "destructive waves" both with fixed wave numbers and with varying wave numbers are described. On the example of an invariant solution of rank 1 it was shown that the search of the invariant solutions of rank 1 that can not be found explicitly, can be reduced to solving of the integral equations. For this solution turbulent processes are researched for which at the initial instant of a time and for a fixed value of the wave number either the turbulence energy and rate of its change or the turbulence energy and its gradient are given. Under certain conditions, the existence and uniqueness of the solutions of the boundary value problems describing these processes are established.

1. Introduction
Leith’s model of turbulence without of the external forces is written by the following degenerate nonlinear partial differential equation [1]:

$$\frac{\partial E}{\partial t} = \frac{1}{8} \frac{\partial}{\partial k} \left( k^{1/2} E^{1/2} \frac{\partial}{\partial k} \left( k^{-2} E \right) \right) - \nu(t) k^2 E$$

where $E(k,t)$ is a turbulence energy spectrum, $t$ is a time, $k$ is the modulus of the wavenumber, $\nu(t)$ is a kinematic viscosity. The spectrum $E(k, t)$ is normalized so that the kinetic energy density equals $\int E \, dk$, where $k_0 > 0$ is an initial value of the wave number modulus. The Leith’s model is constructed such that in the case of vanishing viscosity there exist two steady-state solutions: the Kolmogorov spectrum that corresponds to the cascade state and the Rayleigh–Jeans distribution corresponding to thermodynamic equilibrium [1]. The most general steady state of this model is given by a ‘nonlinear blend’ of a constant energy flux with a thermodynamic component [2]. The non-
stationary spectrum arising from low-\( k \) initial conditions is characterized by a front propagating toward larger \( k \), with a power law spectrum steeper than a Kolmogorov one behind the front. It is described by a self-similar solution of the second kind [3], for which a spectrum at large wavenumbers is formed over a finite amount of time. This behaviour was demonstrated in [2, 4] via numerical experiments with vanishing viscosity and no external forces. An analytical justification of the existence of self-similar regime for the spectral energy density of the turbulence \( E(k, t) \) in the wavenumber space is given in [5]. The integral equations describing all essentially different invariant solutions of this equation in the absence of external forces impact in the inhomogeneous media, both in the absence and in the presence of viscosity are obtained in [6]. The invariant submodels of some generalization of the Leith’s model without viscosity and external forces were obtained and researched in [7, 8].

In this report we consider generalization of the Leith’s model of the phenomenological theory of turbulence in the absence of external forces, described by the equation

\[
\frac{\partial E}{\partial t} = \frac{1}{8} \frac{\partial}{\partial k} \left( k^a E^b \frac{\partial}{\partial k} \left( k^c E \right) \right) - \mu \epsilon k^\delta E
\]

(1)

where \( a, b, c, \epsilon, \delta, \mu (\mu>0) \) are real numbers. Parameter \( a \) depends on the type of waves: capillary, gravitational, other waves. Parameter \( b \) depends on the number of colliding waves.

Parameter \( c \) depends on the dimension of wave turbulence. Function \( \nu(t) = \mu \epsilon \) is a kinematic viscosity. Parameter \( \delta \) depends on the non-uniform viscosity distribution.

We are assume further that these parameters satisfy to the condition

\[
abc(c-1) \neq 0
\]

(2)

The model (1) under condition (2) will be studied by methods of group analysis [9–11].

2. Group classification

We fulfilled group classification of equation (1) under condition (2). We solved the problem of the group classification of this equation using the algorithm proposed in [10, 11]. This algorithm has been successfully used in [6 – 8, 10 – 18] for group classification of the various equations of mechanics and mathematical physics. The results of the group classification are as follows:

The kernel of the main groups consists only of the identity transformation of the space \( R^3 (t, k, E) \).

For

\[
\delta(e+1) \neq 0
\]

(3)

the main group is generated by the operators

\[
Y_1 = -\delta bt \partial_t + b \left( 1+\epsilon \right) k \partial_k + \left( 2-a-c \right) \left( 1+\epsilon \right) + \delta \right) E \partial_E
\]

For

\[
a = 2 - 3c, \quad b = -2, \quad \delta = 1 - c, \quad \epsilon = -1
\]

(4)

the main group is generated by the operator \( Y_1 \) and an operator

\[
Y_2 = 2 \mu t \ln t \partial_t - \frac{2 \mu k + k^c}{1-c} \partial_k + \left( \mu \left( 1+\ln t \right) + \frac{2c-1}{1-c} \right) E \partial_E
\]

For
\[ \delta = 0, \ \varepsilon \neq -1 \]  

the main group is generated by the operator \( Y_1 \) and the operators

\[ Y_3 = \exp \left( \frac{\mu b}{\varepsilon + 1} t^{\varepsilon + 1} \right) \left( \partial_t - \mu t^\varepsilon E \partial_E \right), \quad Y_4 = b f(t) \exp \left( \frac{\mu b}{\varepsilon + 1} t^{\varepsilon + 1} \right) \left( \partial_t - \mu t^\varepsilon E \partial_E \right) - E \partial_E, \]

\[ f(t) = \int_0^t \exp \left( \frac{-\mu b \tau^{\varepsilon + 1}}{\varepsilon + 1} \right) d\tau \]

in particular, for \( \varepsilon = 1, \ \mu b = 1 \) function \( f(t) \) with accuracy up to a constant multiplier coincides with the integral for the density of the normal distribution a random variable.

For

\[ \delta = 0, \ \varepsilon = -1, \ \mu b = 1 \]  

the main group is generated by the operators

\[ Y_5 = b b_k \partial_k + \left( c (b-1) - a + 2 \right) E \partial_E, \quad Y_6 = b t \partial_t - E \partial_E, \quad Y_7 = b t \ln t \partial_t - \left( 1 + \ln t \right) E \partial_E \]

For

\[ \delta = 0, \ \varepsilon = -1, \ \mu b \neq 1 \]  

the main group is generated by the operators \( X_6, X_7 \) and an operator

\[ Y_8 = t^{2b-1} \left( \partial_t - \mu E \partial_E \right) \]

Thus, the set of the models (1), having essentially different group properties consists of the models (3) – (7). We give some invariant solutions for these models.

3. Exact solutions

Exact solutions for each mathematical model are important. They allow us to assess, the degree of the adequacy of the mathematical model of real physical processes, after carrying out experiments appropriate to these solutions, and an evaluation of the arising deviations. Exact solutions can be used to describe of some physical processes. Exact solutions are good tests to check the approximate numerical solutions.

The set of physically meaningful nonzero invariant solutions of rank 0 of equation (1) consists of the following solutions.

\begin{itemize}
  \item The submodel (5) has only two essentially distinct nonzero invariant \( H \)-solutions of rank 0, where \( H \in \left\{ \left( Y_1 + \sigma Y_4, Y_3 \right), \left( Y_1, Y_4 \right) \right\} \). Parameter \( \sigma \) is any real number.
    
  1. A nonzero invariant \( \left( Y_1 + \sigma Y_4, Y_3 \right) \) - solution exists only in the following three cases: 1) under \( \sigma = 2 - a + c (b-1), \ b \neq -1, \) 2) under \( b \neq -1, \ \sigma = \frac{2 + b - a - c}{b + 1}, \) 3) under \( a = 1 - c, \ b = -1. \) This solution is determined by the formula
    
    \[ E = c_1 k^{2+c(b-2)-a-\sigma} \exp \left( \frac{\mu t^{\varepsilon + 1}}{\varepsilon + 1} \right) \]  

\end{itemize}

where \( c_1 \) is any positive constant. It describes a turbulent process with a kinematic viscosity \( \nu(t) = \mu t^\varepsilon \ (\varepsilon \neq -1) \) that depends on the time by the power law with exponent equal to \( \varepsilon \). At each
fixed point in time when \( \sigma \leq 4 + c(b-2) - a \), the solution (8) describes a “destructive wave”. This concept was introduced in [7]. "Destructive wave" has an infinitely large kinetic energy. Using this concept, a qualitative description of the so-called "rogue waves" or "freak waves" was proposed in [7].

The spectrum \( E(k, t) \) at each modulus of the wavenumber varies with time according to the exponential law. In particular for \( \varepsilon = 1 \), the dependence of the spectrum on the time at each the modulus of the wavenumber is determined by the density of the normal distribution of the random variable. If \( \varepsilon > -1 \), then at each modulus of the wavenumber the spectrum tends to zero when \( t \to \infty \). If \( \varepsilon < -1 \), then at each modulus of the wavenumber the spectrum tends to finite (non-zero) value when \( t \to \infty \).

2. A nonzero invariant \( \{Y_1, Y_4\} \)-solution exists only for \( (2 - a + c(b-1))(a + c - b - 2) \neq 0 \) and has the form:

\[
E = \left(\frac{8b^2}{(2 - a + c(b-1))(a + c - b - 2)f(t)}\right)^{\frac{1}{b}} \left(\frac{(1-c)(a-2-c(b-1))}{b}\right)^{-c} \exp\left(-\frac{\mu \varepsilon + 1}{\varepsilon + 1}\right)
\]

The solution (9) has a physical meaning if and only if \( (2 - a + c(b-1))(a + c - b - 2) > 0 \). This solution, also like the solution (8) describes a turbulent process with the same kinematic viscosity. The spectrum at each modulus of the wavenumber varies with time by a more complex law. At each fixed point in time when \( (1-c)(a-2-c(b-1)) \geq c - 2 \), this solution describes a “destructive wave”.

- The submodel (6) has also only two essentially distinct nonzero invariant \( H \)-solutions of rank 0 where \( H \in \{\{Y_5 + \sigma Y_7, Y_6\}, \{Y_5, Y_7 + \rho Y_6\}\} \). Parameters \( \sigma \) and \( \rho \) are any real numbers.

1. A nonzero invariant \( \{Y_5 + \sigma Y_7, Y_6\} \)-solution exists only in the following three cases: 1) under \( \sigma = 2 - a + c(b-1) \), \( b \neq -1 \), 2) under \( b \neq -1 \), \( \sigma = \frac{2+b-a-c}{b+1} \), 3) under \( a = 1-c \), \( b = -1 \).

This solution is determined by the formula

\[
E = c_2 t^{\frac{1}{b}} k^{\frac{1}{b}(2-c-a-\sigma)}
\]

where \( c_2 \) is any positive constant. It describes a turbulent process with a singular kinematic viscosity \( \nu(t) = \mu t^{-1} \). At each fixed point in time when \( \frac{2-c-a-\sigma}{b} \geq -2 \), the solution (10) describes a “destructive wave”. For \( b > 0 \) at each modulus of the wavenumber the spectrum tends to zero when \( t \to \infty \). For \( b < 0 \) at each modulus of the wavenumber the spectrum increases indefinitely when \( t \to \infty \).

2. A nonzero invariant \( \{Y_5, Y_7 + \rho Y_6\} \)-solution exists only for \( a = 1-c \), \( b = -1 \), \( \rho = \ln\left(\frac{(1-c)^2}{8}\right) \), and is given by the formula

\[
E = c_3 k^{-1}
\]

where \( c_3 \) is any positive constant. This solution describes a turbulent process with a singular
The kinematic viscosity \( \nu(t) = \mu t^{-1} \). The solution (11) linearly depends on time and at each fixed point of the time describes a “destructive wave”.

- The submodel (7) has only one nonzero invariant \( H \)-solutions of rank 0, where

\[
H = \left\{ Y_5 + \sigma Y_6, Y_8 \right\}.
\]

Parameter \( \sigma \) is any real number. This solution is determined by the formula

\[
E = c_4 \xi \mu^{-\frac{2c-a+\sigma(b\mu-1)}{b}},
\]

where \( c_4 \) is any positive constant. This solution describes a turbulent process with a singular kinematic viscosity \( \nu(t) = \mu t^{-1} \). At each fixed point in time when \( \frac{2c-a+\sigma(b\mu-1)}{b} \geq -2 \), the solution (12) describes a “destructive wave”. For \( \mu > 0 \) at each modulus of the wavenumber the spectrum tends to zero when \( t \to \infty \). For \( \mu < 0 \) at each modulus of the wavenumber the spectrum increases indefinitely when \( t \to \infty \).

As an example of an invariant solution of rank 1, we give an invariant \( \left\{ Y_2 + \sigma Y_1 \right\} \)-solution for the submodel (4). For \( \sigma = \frac{\mu}{1-c} \ln \left( \frac{(1-c)^2}{8} \right) \) this solution has a form:

\[
E = k^{1-2c} \xi \frac{1}{2} \ln^2 \left( \frac{(1-c)^2}{8} - t \right) U^{-1}(\xi), \quad \xi = \left(1 + 2 \mu k^{1-c}\right) \ln \left( \frac{(1-c)^2}{8} - t_0 \right)
\]

The factor-equation is reduced to the following integral equations

\[
U(\xi) = c_5 + c_6 \xi + \frac{2}{(\mu(1-c))^2} \int_0^\xi \eta(\eta - 1) - \xi(\xi + 1) U^{-1}(\eta) d\eta
\]

where \( \xi_0 = \left(1 + 2 \mu k^{1-c}_0\right) \ln \left( \frac{(1-c)^2}{8} - t_0 \right) \).

Solution (13) can be used for the submodel (4), to describe the turbulent process for which at an initial time \( t = t_0 > 0 \) for waves with a fixed wave number \( k = k_0 \) the spectrum and its rate of change or the spectrum and its gradient are given respectively by the formulas

\[
E(k_0, t_0) = E_0 > 0, \quad \frac{\partial E}{\partial t}(k_0, t_0) = E_1
\]

or

\[
E(k_0, t_0) = E_0 > 0, \quad \frac{\partial E}{\partial k}(k_0, t_0) = E_2
\]

It is established that in the neighborhood of the point \((k_0, t_0)\) there is a unique solution of the equation (1) satisfying to the conditions (15) or (16), for which the value

\[
k^{2c-1} \frac{1}{2} \ln \left( \frac{3}{2} \left( \frac{(1-c)^2}{8} - t \right) E(k, t) \right)
\]

is constant along each trajectory.
\[ t = \frac{8}{(1-c)^2} \exp \left( \frac{A}{1+2\mu k^{1-c}} \right) (A = \text{const}) \]. These solutions are given by the formulas (13), (14)

where for the conditions (15) the constants \( c_5 \) and \( c_6 \) have the values given by the formulas:

\[
c_5 = k_0^{1-2c} t_0^{\frac{1}{2}} \ln^2 \left( \frac{(1-c)^2}{8} \right) E_0^{-1} - c_6 \left( 2\mu k_0^{1-c} + 1 \right) \ln \left( \frac{(1-c)^2}{8} \right) + \]

\[
c_6 = \frac{2}{\mu^2 \left( 2\mu k_0^{1-c} + 1 \right)} \left( 2\mu k_0^{1-c} + 1 \right)^2 k_0^{2c-1} E_0 + \]

\[
\left( \frac{(1-c)^2}{2} k_0^{-1-2c} \right) t_0 \ln \left( \frac{(1-c)^2}{8} t_0 \right) E_0^{-2} \left( t_0^{-1} + 3 \ln^{-1} \left( \frac{(1-c)^2}{8} t_0 \right) \right) \left( E_0 - E_1 \right) + \]

For the conditions (16) the constant \( c_5 \) takes the same values as for the conditions (15), and the constant \( c_6 \) has the value given by the formula

\[
c_6 = \frac{2}{\mu^2 \left( 2\mu k_0^{1-c} + 1 \right)} \left( 2\mu k_0^{1-c} + 1 \right) E_0 + \]

\[
\left( \frac{(1-c)^2}{4} k_0^{1-2c} t_0 \ln \left( \frac{(1-c)^2}{8} t_0 \right) E_0^{-2} \left( \left( 1-2c \right) E_0 - k_0 E_2 \right) \right) + \]

**4. Conclusion and discussion**
We investigated a generalization of the Leith’s model of the phenomenological theory of the wave turbulence. With the methods of group analysis, we obtained basic models possessing nontrivial symmetries. We found in an explicit form all invariant solutions describing invariant submodels of rank 0. We found the physical meaning of these solutions. In particular, with the help of these solutions, we described turbulent processes for which there are "destructive waves". Perhaps, "destructive waves" is connected with the known "rogue waves" or "freak waves" as follows:

A strong non-stationary undercurrent during the flow around of the irregularities on the ocean bottom creates turbulent zones. With certain combinations of the characteristics of this non-stationary undercurrent and irregularities on the ocean bottom, an underwater "destructive wave" may arise. This "destructive wave" has a very high energy. This wave generates a "rogue wave" or a "freak wave" on the surface of the ocean. The validity of this hypothesis can be established only by experimental studies.

On the example of an invariant solution of rank 1 we showed that the search of the invariant solutions of rank 1, which can not be found explicitly, can be reduced to solving integral equations. For this solution we investigated the turbulent processes for which at the initial instant of time for a fixed value of the wave number either the turbulence energy and rate of its change or the turbulence energy and its gradient are given. Under certain conditions, we established the existence and uniqueness of solutions of boundary value problems describing these processes.

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