0. Explanation

What follows is a lightly edited version of the author’s unpublished master’s essay, submitted in partial fulfillment of the requirements of the degree of Master of Arts at the Pennsylvania State University, dated June 1994, written under the supervision of Professor George E. Andrews. It was retyped by the author on November 23, 2022. Obvious typographical errors in the original were corrected without comment; hopefully not too many new errors were introduced during the retyping. Explanatory text added by the author in 2022 is notated by Remark added in 2022. After the initial posting on the arXiv on November 29, 2022, the author received email from Wadim Zudilin and George Andrews, pointing out some typos and making some interesting comments. These comments have been incorporated in this revised submission to the arXiv. The bibliography in this version is more extensive than that of the original.

Lastly, in 1994, I neglected to mention that $q$ is being treated as a formal variable throughout.

1. Introduction

In Chapter 4 of his monograph q-series: Their Development and Application in Analysis, Number Theory, Combinatorics, Physics, and Computer Algebra [3], George Andrews showed that “[g]iven the Rogers–Ramanujan identities, . . . [certain] results become easy consequences of constant term arguments.” The results referred to are other series–product identities which are similar in form to the Rogers–Ramanujan identities.

The method of constant terms is executed as follows: one starts with a series involving powers of $q$ in the numerator and $q$-factorials in the denominator and possibly the numerator. Then the series is reexpressed as the constant term in a product of two series involving a new variable, $z$. The $q$-binomial theorem or one of its corollaries is invoked to convert the two series into products, which are then grouped in a new way, the $q$-binomial theorem and one or two of its corollaries are then invoked once again, and a new series appears. In the case of all of the identities addressed by Andrews in his monograph [3], the new series produced was always easily recognizable as some form of the Rogers–Ramanujan identities multiplied by an infinite product. As we shall demonstrate here, sometimes an unfamiliar series is derived via this method, and combining this information with established results, we generate new series–product identities.

The following standard notation will be used throughout:

$$
(a)_n = (a; q)_n = (1 - a)(1 - aq)(1 - aq^2) \cdots (1 - aq^{n-1})
$$

$$
(a)_\infty = (a; q)_\infty = \lim_{n \to \infty} (a; q)_n
$$

The following non-standard notation will also be used: $CT[X]$ will denote the constant term, that is, the coefficient of $z^0$ in the series or infinite product $X$.

The following identities will be assumed:

Date: Original: June 1994; annotated with some corrections on November 23, 2022.
Equations (1) and (2) are the Rogers–Ramanujan identities. Equation (3) is the \( q \)-analog of the binomial series [1, p. 17, Theorem 2.1]. Equations (4) [3, p. 115, Eq. (C.2)] and (5) [1, p. 19, Eq. (2.2.6)] can be easily deduced from (3). Equations (6) and (7) were originally proved by Rogers and reproved by Slater [9, p. 153–154, Eqs. (20) and (16) respectively. Equations (8), (9), and (10) were proved by Slater [9, pp. 154, 157; Eqs. (25), (52), and (26) respectively].

**Remark added in 2022.** I should have given more precise references for the various identities due to L. J. Rogers. The Rogers–Ramanujan identities (1), (2) first appeared in [7, p. 328 (2); p. 330 (2), resp.]. Eq. (6) first appeared in [7, p. 330], and (7) in [7, p. 331, above (7)]. Additionally, Eq. (8) first appeared in Ramanujan’s lost notebook [5, p. 85, Entry 4.2.7].

## 2. New proofs for old identities

As demonstrated in the proofs of Theorems 1 and 2 below, when the method of constant terms is applied to certain “mod 5” identities of the Rogers–Ramanujan type, we simply reprove known results, as in Andrews [3].

**Theorem 1.**

\[
\sum_{n=0}^{\infty} \frac{(-1)^n q^{3n^2}}{(q^4; q^4)_n (-q; q^2)_n} = \prod_{n=1}^{\infty} \frac{1}{(1 + q^n)(1 - q^{5n-4})(1 - q^{5n-1})}. \tag{11}
\]
Proof.

\[ \sum_{n=0}^{\infty} \frac{(-1)^n q^{3n^2}}{(q^4; q^4)_n(-q; q^2)_n} = CT \left[ \sum_{n=0}^{\infty} \frac{(-1)^n q^{n^2} z^n}{(q^4; q^4)_n} \times \sum_{m=0}^{\infty} \frac{z^m q^{2m^2}}{(q^4; q^4)_m} \right] \]

\[ = CT \left[ \frac{zq; q^2}{(zq^2; q^2)_{\infty}} \times \frac{(q^2; q^2)_{\infty}}{(-zq; q^2)_{\infty}} \times \frac{(-z^{-1}; q^4)_{\infty}}{(-z^{-1}q; q^2)_{\infty}} \times \frac{1}{(-z^{-1}q^2; q^4)_{\infty}} \right] \text{ (by (4) and (3))} \]

\[ = \frac{1}{(-q^2; q^2)_{\infty}} \sum_{n=0}^{\infty} \frac{q^{n^2}}{(q^4; q^4)_n} \times \frac{G(q)}{(-q^2; q^2)_{\infty}} \text{ (by (6))} \]

\[ = \frac{G(q)}{(-q)^{\infty}} \]

\[ = \prod_{n=1}^{\infty} \frac{1}{(1 + q^n)(1 - q^{5n-4})(1 - q^{5n-1})}. \]

\[ \square \]

Theorem 2.

\[ \sum_{n=0}^{\infty} \frac{(-1)^n q^{3n^2 - 2n}}{(q^4; q^4)_n(-q; q^2)_n} = \prod_{n=1}^{\infty} \frac{1}{(1 + q^n)(1 - q^{5n-3})(1 - q^{5n-2})}. \] (12)

Proof.

\[ \sum_{n=0}^{\infty} \frac{(-1)^n q^{3n^2 - 2n}}{(q^4; q^4)_n(-q; q^2)_n} = CT \left[ \sum_{n=0}^{\infty} \frac{(-1)^n q^{n^2} z^n}{(q^4; q^4)_n} \times \sum_{m=0}^{\infty} \frac{z^m q^{2m^2 - 2m}}{(q^4; q^4)_m} \right] \]

\[ = CT \left[ \frac{zq; q^2}{(zq^2; q^2)_{\infty}} \times \frac{(q^2; q^2)_{\infty}}{(-z^{-1}q; q^2)_{\infty}} \times \frac{1}{(-z^{-1}q^2; q^4)_{\infty}} \right] \text{ (by (4) and (3))} \]

\[ = \frac{1}{(-q^2; q^2)_{\infty}} \sum_{n=0}^{\infty} \frac{(-1)^n q^{n^2} z^n}{(q^4; q^4)_n} \times \frac{1}{(q^2; q^2)_{\infty}} \text{ (by (4) and (3))} \]

\[ = \frac{1}{(-q^2; q^2)_{\infty}} \sum_{n=0}^{\infty} \frac{q^{n^2 + 2n}}{(q^4; q^4)_n} \]

\[ \square \]
was not really new. It can be shown to be equivalent to Slater [9, pp. 155-156, Eqs. (19) and (15) respectively], so nothing new appeared as a result of employing the method of constant terms.

Remark added in 2022. Theorems 1 and 2 are due to Rogers [7, p. 339, Ex. 2; p. 330 (5), respectively].

3. A NEW SERIES–PRODUCT IDENTITY

Remark added in 2022. With the benefit of hindsight, I see now that the identity presented in Theorem 3 was not really new. It can be shown to be equivalent to Slater [9, p. 154, Eq. (48)] with q replaced by −q.

In light of the proofs of Theorems 1 and 2, and the eight identities of L. J. Rogers that Andrews proves via constant terms [3, pp. 33–36], one might wonder if all identities of the Rogers–Ramanujan type are provable by this method. However, when we look at “mod 6” identities, we see immediately that this is not always the case.

Theorem 3.

\[
\sum_{n=0}^{\infty} \frac{(-1; q^2)_n q^{n^2+n}}{(q^2; q^2)_n (-q; q^2)_n} = \prod_{n=1}^{\infty} \frac{(1 - q^{6n-3})^2(1 - q^{6n})}{(1 - q^{4n-2})(1 - q^{2n})}.
\]

Proof.

\[
\frac{(q^3; q^3)^2_\infty (q^6; q^6)_\infty (-q; q^2)_\infty}{(q^2; q^2)_\infty} = \sum_{n=0}^{\infty} \frac{q^{n^2} (-q; q^2)_n}{(q^4; q^4)_n} (by \ (8))
\]

\[
= \sum_{n=0}^{\infty} \frac{q^{n^2} (-q; q^2)_n}{(-q^2; q^2)_n (q^2; q^2)_n}
\]

\[
= CT \left[ \sum_{n=\infty}^{\infty} \frac{q^{n^2} z^n}{(-q^2; q^2)_n} \sum_{m=0}^{\infty} \frac{z^{-m} (-q; q^2)_m}{(q^2; q^2)_m} \right]
\]

\[
= CT \left[ \frac{(-z q; q^2)_\infty (-z^{-1} q; q^2)_\infty (q^2; q^2)_\infty}{(z^{-1} q; q^2)_\infty} \times \frac{(-z^{-1} q; q^2)_\infty}{(z^{-1} q; q^2)_\infty} \right] (by \ (4) \ and \ (3))
\]

\[
= \frac{1}{(-q^2; q^2)_\infty} CT \left[ \sum_{n=\infty}^{\infty} q^{n^2} z^n \sum_{m=0}^{\infty} \frac{(-1; q^2)_m z^{-m} q^m}{(q^2; q^2)_m} \sum_{r=0}^{\infty} \frac{z^{-r}}{(q^2; q^2)_r} \right] (by \ (4) \ and \ (3))
\]

□

Equations (11) and (12) were published previously with different proofs [9, pp. 155–156, Eqs. (19) and (15) respectively], so nothing new appeared as a result of employing the method of constant terms.
\[ \frac{1}{(-q^2; q^2)_{\infty}} \sum_{m, r \geq 0} q^{(m+r)^2 - (1; q^2)_m} = \frac{1}{(-q^2; q^2)_{\infty}} \sum_{m=0}^{\infty} q^{m^2 + m - (1; q^2)_m} \sum_{r=0}^{\infty} q^{r^2 + 2mr} = \frac{1}{(-q^2; q^2)_{\infty}} \sum_{m=0}^{\infty} q^{m^2 + m - (1; q^2)_m} (-q^{2m+1}; q^2)_{\infty} \] (by (5))

\[ \frac{(-q^2; q^2)_{\infty}}{(-q^2; q^2)_{\infty}} \sum_{m=0}^{\infty} q^{m^2 + m - (1; q^2)_m} = \frac{(-q^2; q^2)_{\infty}}{(-q^2; q^2)_{\infty}} \sum_{m=0}^{\infty} (q^2; q^2)_m (-q; q^2)_m. \] (16)

Here, the method of constant terms leads us from the series (14) to another series (16), and (16) is not a straightforward restatement of either of the Rogers–Ramanujan identities. However, we could have proceeded from (14) to (16) using Heine’s transformation [1, p. 19, Cor. 2.3], which would have been easier than using constant terms.

4. new double series–product identities

What then, is the value of the method of constant terms? The key lies in the step justifying equation (15) above. We were able to collapse the double series into a single series above because of the particular exponents on \( q \). Many times this will not be the case, and we will be left with an “irreducible” double series. Theorems 4 and 5 are examples of this situation.

Theorem 4.

\[ \sum_{m, r \geq 0} q^{4m^2 + 4mr + 2r^2 - r} (q^4; q^4)_{m} (q^2; q^2)_r = \prod_{n=1}^{\infty} (1 + q^{2n-1}). \] (17)

Proof.

\[ (-q)_{\infty} = \sum_{n=0}^{\infty} q^{2n^2 - n} (-q; q^2)_n (q^2; q^2)_n \] (by (9))

\[ = CT \left[ \sum_{n=-\infty}^{\infty} \frac{q^{2n^2 - n} z^n}{(q^2; q^4)_n} \sum_{m=0}^{\infty} z^m (-q; q^2)_m \right] (by (4) and (3))

\[ = \frac{1}{(q^2; q^4)_{\infty}} CT \left[ (-zq; q^4)_{\infty} (-z^{-1} q^3; q^2)_{\infty} (q^4; q^4)_{\infty} \times \frac{(-z^{-1} q; q^2)_{\infty}}{(z^{-1}; q^2)_{\infty}} \right] \]

\[ = \frac{1}{(q^2; q^4)_{\infty}} \sum_{m, r \geq 0} q^{2(m+r)^2 - (m+r) + 2m^2 + m} \frac{(-1)^n q^{2n^2 - n} z^n}{(q^4; q^4)_m} \sum_{r=0}^{\infty} z^{-r} \] (by (4), (5), and (3))

\[ = \frac{1}{(q^2; q^4)_{\infty}} \sum_{m, r \geq 0} q^{4m^2 + 4mr + 2r^2 - r} (q^2; q^2)_m (q^2; q^2)_r \]
Theorem 5.  

\[
\sum_{m,r \geq 0} \frac{q^{2m^2+2mr+r^2}}{(q^2;q^2)_m(q^2)_r} = \prod_{n=1}^{\infty} \frac{(1 - q^{6n-3})^2(1 - q^{6n})}{1 - q^n}.
\]  

Proof.

\[
\frac{(q^3;q^6)_\infty^2(q^6;q^6)_\infty}{(q)_\infty^2} = \sum_{n=0}^{\infty} \frac{q^n(-q)_n}{(q;q^2)_{n+1}(q)_n} \quad \text{(by (10))}
\]

\[
= CT \left[ \frac{1}{1 - q} \sum_{n=-\infty}^{\infty} \frac{q^n z^n}{(q^3;q^2)_n} \sum_{m=0}^{\infty} \frac{z^{-m}(-q)_m}{(q)_m} \right]
\]

\[
= CT \left[ \frac{(-zq;q^2)_\infty(-z^{-1}q;q^2)_\infty(q^2;q^2)_\infty}{(-z^{-1}q^2;q^2)_\infty(q^2;q^2)_\infty} \times \frac{(-z^{-1}q)_\infty}{(z-1)_\infty} \right] \quad \text{(by (4) and (3))}
\]

\[
= \frac{1}{(q;q^2)_\infty} \frac{1}{(q^3;q^6)_\infty} \left[ (-zq;q^2)_\infty(-z^{-1}q;q^2)_\infty(q^2;q^2)_\infty \times (-z^{-1}q;q^2)_\infty \times \frac{1}{(z-1)_\infty} \right]
\]

\[
= \frac{1}{(q;q^2)_\infty} \frac{1}{(q^3;q^6)_\infty} \left[ \sum_{n=-\infty}^{\infty} q^n z^n \sum_{m=0}^{\infty} \frac{q^m z^{-m}}{(q^2;q^2)_m} \sum_{r=0}^{\infty} \frac{z^{-r}}{(q)_r} \right] \quad \text{(by (4), (5), and (3))}
\]

\[
= \frac{1}{(q;q^2)_\infty} \sum_{m,r \geq 0} \frac{q^{(m+r)^2+m^2}}{(q^2;q^2)_m(q)_r}
\]

\[
= \frac{1}{(q;q^2)_\infty} \sum_{m,r \geq 0} \frac{q^{2m^2+2mr+r^2}}{(q^2;q^2)_m(q)_r}.
\]

Comment from Wadim Zudilin, 29 November 2022. [T]his is a particular case of Bressoud’s identities [6]; when you replace 2s by 3s on the left, you get a Kanade–Russell mod 9 (see a discussion at the end of my paper with Ali Uncu [10]). The very same identity is recently treated again as the constant term, but in a more sophisticated way! See [11, Theorem 1.1].... The techniques of using the integrals for the complex terms and combining those with some theorems from the q-Bible are nicely given by Hjalmar [Rosengren] in [8].

One might be suspicious that the double series in (17) and (19) are actually collapsible into single series. Just because (5) is not applicable to the double series in (17) and (19) as it was in (15), does not mean that there is not some other way to simplify the double series. However, currently there is no known general method for simplifying arbitrary multiple q-series. Andrews [2] has dealt with several special cases, but these apparently do not apply here. So, by referring to these double series as “irreducible,” I mean that they cannot be transformed into a single-fold sum by any currently known method.
5. Conclusion

This study suggests several directions for further research. One could, of course, apply the method of constant terms to any identity of the Rogers–Ramanujan type (there are 130 such identities in Slater [9]), to see what series are generated by the method. In a more advanced study, one could attempt to identify which partitions are enumerated by these series.

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