Non-Gaussian dephasing in flux qubits due to $1/f$-noise

Y. M. Galperin,$^{1,2,3}$ B. L. Altshuler,$^{4,5}$ J. Bergli,$^6$ D. Shantsev,$^6$ and V. Vinokur$^2$

$^1$Department of Physics and Center of Advanced Materials and Nanotechnology, University of Oslo, PO Box 1048 Blindern, 0316 Oslo, Norway
$^2$Argonne National Laboratory, 9700 S. Cass Av., Argonne, IL 60439, USA
$^3$A. F. Ioffe Physico-Technical Institute of Russian Academy of Sciences, 194021 St. Petersburg, Russia
$^4$Department of Physics, Columbia University, 538 W. 120th St., New York, NY 10027, USA
$^5$NEC Research Institute, 4 Independence Way, Princeton, NJ 08540, USA
$^6$Department of Physics, University of Oslo, PO Box 1048 Blindern, 0316 Oslo, Norway

(Dated: September 12, 2018)

Recent experiments by F. Yoshihara et al. [Phys. Rev. Lett. 97, 167001 (2006)] and by K. Kakuyanagi et al. [cond-mat/0609564] provided information on decoherence of the echo signal in Josephson-junction flux qubits at various bias conditions. These results were interpreted assuming a Gaussian model for the decoherence due to $1/f$ noise. Here we revisit this problem on the basis of the exactly solvable spin-fluctuator model reproducing detailed properties of the $1/f$ noise interacting with a qubit. We consider the time dependence of the echo signal and conclude that the results based on the Gaussian assumption need essential reconsideration.

PACS numbers: 03.65.Yz, 85.25.Cp

Two remarkable works$^{1,2}$ have appeared recently, reporting on measurements of the two-pulse echo signal in Josephson junction flux qubits revealing its dependence of the time interval between the pulses. The authors interpreted their results in terms of the recent theory$^3$ of the decoherence in qubits caused by $1/f$ noise and obtained the qubit dephasing rate, $\Gamma_\phi$, based on the postulation of the Gaussian statistics of the noise. Although this looks like a natural starting point, the importance of the echo signal data for understanding the underlying mechanisms of the qubits decoherence, calls for careful examination of the assumptions built into the theoretical description.

In this paper we develop a theory of the time dependence of the echo signal making use of an exactly solvable but experimentally realistic model. We demonstrate that in many practical realizations of $1/f$ noise the results based on the Gaussian assumption need to be significantly corrected. We show that deviation of noise statistics from the Gaussian changes significantly the time dependence of the echo signal. Using the exactly solvable non-Gaussian spin-fluctuator model for the $1/f$ noise we analyze the dependence of the echo signal on both time and the qubit working point, and compare the obtained results with those derived within the Gaussian approximation.

Let us start with a brief review the procedure used in Refs.$^{1,2}$ In order to separate the relaxation due to direct transitions between the energy levels of the qubit, characterized by the time $T_1$, the echo signal is expressed as $e^{-t/2T_1} \rho(t)$, where $t$ is the arrival time of the echo signal. The factor $\rho(t)$ describes the pure dephasing.

If dephasing is induced by the noise of the magnetic flux in the SQUID loop, the echo signal is determined by the fluctuating part of the magnetic flux through the loop, $\Phi(\tau)$:

$$
\rho(t) = \exp \left[ -i v_\Phi \left( \int_{t/2}^{t} \Phi(\tau) d\tau \right) \right].
$$

(1)

The strength of the coupling between the qubit and the noise source, $v_\Phi$, depends on the working point of the qubit.$^{1,2}$

Assuming that the statistics of the fluctuations of $\Phi(t)$ is Gaussian one can express the decoherence rate through the magnetic noise spectrum, $S_\Phi(\omega) = (1/\pi) \int_0^\infty dt \cos \omega t \langle \Phi(t)\Phi(0) \rangle$:

$$
\rho(t) = \exp \left[ -\frac{v_\Phi^2 t^2}{2} \int_{-\infty}^{\infty} d\omega S_\Phi(\omega) \sin^4(\omega t/4) / (\omega t/4)^2 \right].
$$

(2)

This expression is common in the theory of spin resonance and allows one to find the decoherence rate from the noise spectrum once the coupling, $v_\Phi$, is known. For the $1/f$ noise spectrum

$$
S_\Phi(\omega) = A_\Phi / \omega,
$$

(3)

one obtains$^{1,4,5}$

$$
\rho(t) = e^{-(v_\Phi^2 t)^2}, \quad \Gamma_\Phi = |v_\Phi| \sqrt{A_\Phi \ln 2}.
$$

(4)

The experimental results of Ref.$^2$ were fitted to

$$
\rho(t) = \exp \left[ -t/2T_1 - \Gamma_\Phi^0 t - (\Gamma_\Phi^0 t)^2 \right],
$$

(5)

and $T_1$ and $\Gamma_\Phi^0$ were extracted from the analysis of the echo near the optimal point. The dephasing rate, $\Gamma_\Phi$, was in its turn found from the fitting and displayed as a function of the working point.

The deficiencies of this procedure become clear when we compare the approximate expression Eq. (4) with exact equations that follow from the spin-fluctuator model which we will now introduce.
Spin-fluctuator (SF) model for $1/f$-noise. One of the most common sources of the low-frequency noise is the rearrangement of dynamic two-state defects, fluctuators, see, e. g., book [3] and references therein. Random switching of a fluctuator between its two metastable states (1 and 2) produces a random telegraph noise. The process is characterized by the switching rates $\gamma_{12}$ and $\gamma_{21}$ for the transitions $1 \rightarrow 2$ and $2 \rightarrow 1$. Only the fluctuators with the energy splitting $E$ less than temperature, $T$, contribute to the dephasing since the fluctuators with large level splittings are frozen in their ground states. As long as $E < T$ the rates $\gamma_{12}$ and $\gamma_{21}$ are of the same order of magnitude, and without loss of generality one can assume that $\gamma_{12} \approx \gamma_{21} \equiv \gamma/2$. The random telegraph process, $\xi(t)$, is defined as switching between the values $\pm 1/2$ at random times, the probability to make $n$ transitions during the time $\tau$ being given by the Poisson distribution. The correlation function of the random telegraph processes is $\langle \xi(t)\xi(t+\tau) \rangle = e^{-|\gamma|\tau}/4$ and the corresponding contribution to the noise spectrum is a Lorentzian, $\gamma/4\pi(\omega^2 + \gamma^2)$. Accordingly, should there be many fluctuators coupled to the given qubit via constants $v_i$ and having the switching rates $\gamma_i$, the dephasing of the noise power spectrum is expressed through the sum $\sum_i v_i^2/\gamma_i(\omega^2 + \gamma_i^2)$. If the number of effective fluctuators is sufficiently large, the sum over the fluctuators transforms into an integral over $v$ and $\gamma$ weighted by the distribution function $P(v, \gamma)$. Upon the assumption that the coupling constants and switching rates are uncorrelated, the distribution density factorizes, $P(v, \gamma) = P_v(v)P_\gamma(\gamma)$, and the noise spectral function due to fluctuators reduces to $\langle v^2 \rangle S(\omega)$, where

$$\langle v^2 \rangle \equiv \int dv v^2 P_v(v), \quad S(\omega) = \frac{1}{4\pi} \int d\gamma \frac{P_\gamma(\gamma)}{\omega^2 + \gamma^2}. \quad (6)$$

The distribution, $P_\gamma(\gamma)$, is determined by the details of the interaction between the fluctuators and their environment, which causes their switchings. A fluctuator viewed as a two-level tunneling system, is characterized by two parameters – diagonal splitting, $\Delta$, and tunneling coupling, $\Lambda$, the distance between the energy levels being $E = \sqrt{\Delta^2 + \Lambda^2}$ (we follow notations of Ref. [3] where the model is described in detail). The environment is usually modeled as a boson bath, which can represent not only the phonon field, but, e. g., electron-hole pairs in the conducting part of the system. The external degrees of freedom are coupled with the fluctuator via modulation of $\Delta$ and $\Lambda$, modulation of the diagonal splitting $\Delta$ being most important. Under this assumption the fluctuatorenvironment interaction Hamiltonian acquires the form

$$H_{F-env} = \hat{c} \left( \frac{\Delta}{E} \sigma_z - \frac{\Lambda}{E} \sigma_x \right), \quad (7)$$

where $\hat{c}$ is an operator depending on the concrete interaction mechanism. Accordingly, the factor $(\Lambda/E)^2$ appears in the inter-level transition rate:

$$\gamma(E, \Lambda) = (\Lambda/E)^2 \gamma_0 (E). \quad (8)$$

Here the quantity $\gamma_0$ has a meaning of the maximal relaxation rate for fluctuators with the given energy splitting, $E$. The coupling, $\Lambda$, depends exponentially on the smoothly distributed tunneling action, leading to the $\Lambda^{-1}$-like distribution of $\Lambda$, and consequently $P_\gamma \propto \gamma^{-1}$.

Since only the fluctuators with $E \lesssim T$ are important and temperatures we are interested in are low as compared to the relevant energy scale, it is natural to assume the distribution of $E$ to be almost constant. Denoting the corresponding density of states in the energy space as $P_0$ we arrive at the distribution of the relaxation rates as

$$P_\gamma(\gamma) = \frac{P_0 T}{\gamma} \Theta(\gamma_0 - \gamma). \quad (9)$$

The product $P_0 T$ determines the amplitude of the $1/f$ noise. Indeed, the integral $\int P_\gamma(\gamma) d\gamma$ is nothing but the total number of thermally excited fluctuators, $N_T$. Consequently,

$$P_0 T = N_T/L, \quad L \equiv \ln(\gamma_0/\gamma_{\min}) \quad (10)$$

where $\gamma_{\min}$ is the minimal relaxation rate of the fluctuators. In the following we will assume that the number of fluctuators is large, so that $P_0 T \gg 1$ or $N_T \gg 2\pi L$. After that the Eq. (9) yields:

$$S(\omega) = \frac{A}{\omega} \times \left\{ \begin{array}{ll} 1 & , \omega \ll \gamma_0 \\ 2\gamma_0/\pi\omega & , \omega \gg \gamma_0 \end{array} \right. \quad A = \frac{1}{8} P_0 T. \quad (11)$$

We see that the SF model reproduces the $1/f$ noise power spectrum [3] for $\omega \ll \gamma_0$. The crossover from $\omega^{-1}$ to $\omega^{-2}$ behavior at $\omega \sim \gamma_0$ follows from the existence of a maximal switching rate $\gamma_0$. Below we will see that this crossover modifies the time dependence of the echo signal at times $t \ll \gamma_0^{-1}$.

The SF model has previously been used for description of effects of noise in various systems [9,10,11,12,13,14] and was recently applied to analysis of decoherence in charge qubits [7,15,16,17,18,19,20,21,22,23]. Quantum aspects of the model were addressed in Ref. [23]. These studies, demonstrated, in particular, that the SF model is suitable for the study of non-Gaussian effects and that these may be essential in certain situations.

Now we are ready to analyze consequences of the upper cutoff and the effect of non-Gaussian noise, and through this identify the validity region for the prediction of Eq. (4).

In the following we will express the fluctuation of the magnetic flux as a sum of the contributions of the statistically independent fluctuators, $\theta_i(t) = \sum_b b_i \xi_i(t)$, where $b_i$ are partial amplitudes while $\xi_i(t)$ are random telegraph processes. Consequently, we express the product $v_i^2 A \Phi$ as $\tilde{v}^2 A$ where $\tilde{v} \equiv \sqrt{\langle v_i^2 \rangle}$.

$a. \ \text{Echo signal in the Gaussian approximation.}$ Substitution of the distribution [9] into Eq. (2) yields for $K(t) \equiv -\ln \rho(t)$$

$$K_g(t) = \tilde{v}^2 A t^2 \times \left\{ \begin{array}{ll} \gamma_0 t/6 & , \gamma_0 t \ll 1 \\ \ln 2 & , \gamma_0 t \gg 1. \end{array} \right. \quad (12)$$



The subscript $g$ means that this result is obtained in the Gaussian approximation. One sees immediately that the crossover between the $\omega^{-1}$ and $\omega^{-2}$ behaviors in the noise spectrum does not affect the echo signal at long times $t \gg \gamma_0^{-1}$. However, at small times the decay decrement acquires an extra factor $\gamma_0 t$, which is nothing but the probability for a typical fluctuator to change its state during the time $t$. As a result, even in the the Gaussian approximation at small times $K_{g}(t) \propto t^3$ (replacing $K_{g}(t) \propto t^2$ behavior).

b. Non-Gaussian theory: The echo signal given by Eq. (11) can be calculated exactly using the method of stochastic differential equations, if the fluctuating quantity $\Phi$ is a single random telegraph process, see Ref. 7 and references therein. The method was developed in the contexts of spin resonance, spectral diffusion in glasses, and single molecular spectroscopy in disordered media. Averaging in Eq. (1) is performed over random realizations of $\Phi$ and its initial states and reflects the conventional experimental procedure, where the observable signal is accumulated over numerous repetitions of the same sequence of inputs.

For a single fluctuator with switching rate $\gamma/2$ and coupling $\nu$ we find

$$\rho_1(t) = \frac{e^{-\gamma t/2}}{2\mu^2} \left[ (1 + \mu) e^{\nu t/2} + (1 - \mu) e^{-\nu t/2} - \frac{2\nu^2}{\gamma^2} \right],$$

where $\mu = \sqrt{1 - (v/\gamma)^2}$. In the appropriate limits this can be expanded to give

$$-\ln \rho_1(t|v, \gamma) \approx \begin{cases} v^2 \gamma t^3/48, & t^{-1} \gg \gamma, v \\ \gamma t/2, & \gamma, v \ll t^{-1} \gg \gamma \gg v, t^{-1}. \end{cases}$$

Note that the last limiting case here is similar to the motional narrowing of spectral lines well known in physics of spin resonance.

In the case of many fluctuators producing $1/f$-like noise, $N_T \gg 1$, the sum of the contributions from individual fluctuators gives the echo decay decrement

$$K_{sf}(t) = -\int dv P_v(v) \int d\gamma P_{\gamma}(\gamma) \ln \rho_1(t|v, \gamma).$$

Let us assume that the distribution of $v$ is a sharp function centered at some value $\bar{v}$. This reduces integration over $v$ to mere replacing $v \to \bar{v}$ in the expressions for $\rho(v, t|\gamma)$. This approximation is valid as long as $\langle v^2 \rangle$ is finite. This is seemingly the case, e. g., for magnetic noise induced by tunneling of vortices between different pinning centers within the SQUID loop. The case of divergent $\langle v^2 \rangle$ is considered in Refs. 3, 21, 33. Using then Eq. (9) for the distribution function $P_{\gamma}$ and using the appropriate terms from Eq. (13) in Eq. (11) we find the time dependence of the logarithm of the echo signal, $K_{sf}(t)$. For $\bar{v} \ll \gamma_0$

$$K_{sf}(t) \approx \begin{cases} \frac{\gamma_0 A\bar{v}^2 t^3/6}{4\gamma_0 A t}, & t \ll \bar{v}^{-1} \\ \ln 2\bar{v} A t^2, & \gamma_0^{-1} \ll t \ll \bar{v}^{-1}, \\ \frac{\alpha v A t}{2}, & v^{-1} \ll t. \end{cases}$$

where $\alpha \approx 6$.

At small times $t \ll \bar{v}$ we arrive at the same result as in the properly treated Gaussian approach, Eq (12). However, at large times, $t \gg \bar{v}^{-1}$, the exact result dramatically differs from the prediction of the Gaussian approximation. To understand the reason, notice that for $P_{\gamma}(\gamma) \propto 1/\gamma$, the decoherence is dominated by fluctuators with $\gamma \approx v$. The physical reason for that is clear: very "slow" fluctuators produce slow varying fields, which are effectively refocused in course of the echo experiment, while the influence of too "fast" fluctuators is reduced due to the effect of motional narrowing. As shown in Ref. 24, only the fluctuators with $v \ll \gamma$ produce Gaussian noise. Consequently, the noise in this case is essentially non-Gaussian. Only at short times $t \ll \bar{v}^{-1}$ when these most important fluctuators did not yet have time to switch, and only the faster fluctuators contribute, is the Gaussian approximation valid.

For $\bar{v} \gg \gamma_0$ we find

$$K_{sf}(t) \approx \begin{cases} \frac{\gamma_0 A\bar{v}^2 t^3/6}{4\gamma_0 A t}, & t \ll \bar{v}^{-1} \\ \ln 2\bar{v} A t^2, & \gamma_0^{-1} \ll t \ll \bar{v}^{-1}, \\ \frac{\alpha v A t}{2}, & v^{-1} \ll t. \end{cases}$$

In this case all fluctuators have $v \gg \gamma_0$, hence the result at long times again differs significantly from the Gaussian result Eq. (12).

Discussion: Here we apply the results obtained above to analyze quantitatively the decoherence of the flux qubit. Fig. 1 shows the time dependence of the echo signal measured in Ref. 1 and a fit based on the SF model, Eqs. (15) and (16), and to the $\rho = e^{-t^2/\bar{v}^2}$ law. The fitting took into account all data points including those that fall outside the range of the plot (e. g. those with $\rho > 1$).

![FIG. 1: Dephasing component of the echo measurements replotted from Fig 4a of Ref. 1 away from the optimal point. The curves show fits to the SF model, Eqs. (15) and (16), and to the $\rho = e^{-t^2/\bar{v}^2}$ law. The fitting took into account all data points including those that fall outside the range of the plot (e. g. those with $\rho > 1$).](image-url)
also shows the commonly used fit \( \rho = e^{-T_\gamma t^2} \), which is represented by a straight line with slope 2. The fit to the SF model seems slightly better, or at least equally good. From this fit we can extract the average change in the qubit energy splitting \( E_0 \) due to a flip of one fluctuator, \( \bar{v} \approx 4 \mu s^{-1} \). It allows us to evaluate the change of flux in the qubit loop induced by a fluctuator flip, \( \bar{v} = \bar{v}/\Phi_0 \), since \( \Phi_0 = (1/\hbar)\partial E_0/\partial \Phi \) was measured in Ref. [1]. Using the experimental values for all parameters we get \( \bar{v} \approx 4.2 \times 10^{-4} \Phi_0 \), where \( \Phi_0 \) is the flux quantum while the deviation from the optimal working point was \( \approx 10^{-3} \Phi_0 \). Hence the flip of one fluctuator changes the qubit energy splitting by 0.4%. The fit also determines the value of the product \( \gamma \rho \) within our assumption \( \gamma_0 < v \) we find a lower estimate for the flux noise amplitude \( \sqrt{A_\phi/\Phi_0} > 1.3 \times 10^{-6} \).

Similarly, if \( \bar{v} \ll \gamma_0 \) we can fit to Eq. [14], using the interpolation formula \( K_{\text{sf}}(t) = \bar{v}^2 A^3/(6\gamma_0 + t/\ln 2) \). We do not include here the \( K_{\text{sf}} \propto t \) behavior at large \( t \) since it corresponds to \( K_{\text{sf}} > A \gg 1 \). The fitting gives \( \gamma_0 \approx 30 \mu s^{-1} \), which is then also an upper estimate for \( \bar{v} \). We also get \( \sqrt{A_\phi/\Phi_0} = 1.05 \times 10^{-6} \).

The quality of the data does not allow us to determine which of the cases \( \bar{v} \gg \gamma_0 \) or \( \bar{v} \ll \gamma_0 \) is realized in the experiments. However, in both cases the value of \( A_\phi \) is similar and close to the value obtained in the Gaussian approach. Besides, even without determining which case is realized we obtain that both \( \bar{v} \) and \( \gamma_0 \) cannot be larger than \( 3 \times 10^{-5} \Phi_0 \). This gives an upper estimate for the change of flux in the qubit loop as one fluctuator flips, \( b < 3 \times 10^{-5} \Phi_0 \). Since \( \gamma_0 \) is expected to grow with temperature, we believe that it would be instructive to perform similar analysis of the echo decay at different temperatures.

**Conclusions:** By introducing the spin fluctuator model for \( 1/f \) noise in the qubit level splitting we have determined the time dependence of the echo signal. We show that the standard quadratic time dependence in the Gaussian approximation Eq. [14] has a limited range of applicability, and \( t^3 \) or \( t^4 \) dependencies are found beyond this range. Fitting to the SF model also allows us to determine the strength of individual fluctuators, and for the flux qubits reported in Ref. [1] the change of flux in the qubit loop due to the flip of one fluctuator was found to be \( b < 3 \times 10^{-5} \Phi_0 \).

**Acknowledgments**

This work was partly supported by the Norwegian Research Council, Fimmat@UiO, and by the U. S. Department of Energy Office of Science through contract No. DE-AC02-06CH11357. We are thankful to Y. Nakamura for helpful discussions and for providing experimental data.

---

1. F. Yoshihara, K. Harabi, A. O. Niskanen, Y. Nakamura, and J. S. Tsai, Phys. Rev. Lett. 97 (2006).
2. K. Kakuyanagi, T. Men, S. Saito, H. Nakano, K. Semba, H. Takayanagi, F. Deppe, and A. Shnirman, cond-mat/0609564 (2006).
3. A. Cottet, Ph.D. thesis, Université Paris VI (2002).
4. A. Shnirman, Y. Makhlin, and G. Schönh, Phys. Scr. T102 (2002).
5. Y. Makhlin and A. Shnirman, Phys. Rev. Lett. 92, 178301 (2004).
6. S. Kogan, *Electronic Noise and Fluctuations in Solids* (Cambridge University Press, Cambridge, UK, 1996).
7. Y. M. Galperin, B. L. Altshuler, and D. V. Shantsev, in *Fundamental Problems of Mesoscopic Physics* edited by I. V. Lerner et al. (Kluwer Academic Publishers, The Netherlands, 2004), pp. 141–165.
8. There are other choices of \( P_\gamma \), based on the standard for the glassy systems assumption about the smooth distribution of relevant quantities. In particular, assuming wide distribution for \( \Delta \) one arrives at \( P_\gamma \) reads: \( P_\gamma(\gamma) = \left( P_0/2\gamma^2\sqrt{1-\gamma/\gamma_0} \right) \Theta(\gamma_0 - \gamma) \). This distribution is extensively used in physics of low-temperature properties of glasses where two-level tunneling systems are responsible for low-temperature thermal and transport properties. It leads to the same physical conclusion as the distribution since the most important ingredients – 1/\( \gamma \) behavior at small \( \gamma \) and cut-off at \( \gamma = \gamma_0 \) – are present. In particular, both distributions reproduce 1/\( \omega \) behavior of the noise spectra.
9. A. Ludviksson, R. Kree, and A. Schmid, Phys. Rev. Lett. 52, 950 (1984).
10. S. M. Kogan and K. E. Nagaev, Solid State Commun. 49, 387 (1984).
11. V. I. Kozub, Sov. Phys. Jetp 59, 1303 (1984).
12. Y. M. Galperin and V. L. Gurevich, Phys. Rev. B 43, 1290 (1991).
13. Y. M. Galperin, N. Zou, and K. A. Chao, Phys. Rev. B 49, 13728 (1994).
14. J. P. Hessling and Y. Galperin, Phys. Rev. B 52, 5082 (1995).
15. E. Paladino, L. Faoro, G. Falci, and R. Fazio, Phys. Rev. Lett. 88, 228304 (2001).
16. E. Paladino, L. Faoro, A. D’Arrigo, A. Mastelone, and E. Paladino, Physica E (Amsterdam) 18, 29 (2003).
17. G. Falci, E. Paladino, and R. Fazio, in *Quantum Phenomena in Mesoscopic Systems* edited by B. L. Altshuler and V. Tognetti (IOS Press, Amsterdam, 2003).
18. G. Falci, A. D’Arrigo, A. Mastelloni, and E. Paladino, Phys. Rev. A 70, 040101 (2004).
19. G. Falci, A. D’Arrigo, A. Mastelloni, and E. Paladino, Phys. Rev. Lett. 94, 167002 (2005).
20. Y. M. Galperin, B. L. Altshuler, J. Bergli, and D. V. Shantsev, Phys. Rev. Lett. 96, 097009 (2006).
21. I. Martin and Y. M. Galperin, Phys. Rev. B 73, 18020 (2006).
22. J. Bergli, Y. M. Galperin, and B. L. Altshuler, Phys. Rev.
Lett. 74, 024509 (2006).

23. D. P. DiVincenzo and D. Loss, Phys. Rev. B 71, 035318 (2005).

24. R. Klauder and P. W. Anderson, 125, 912 (1962).

25. J. L. Black and B. I. Halperin, Phys. Rev. B 16, 2879 (1977).

26. P. Hu and L. Walker, Solid State Commun. 24 (1977).

27. R. Maynard, R. Rammal, and R. Suchail, J. Phys. (Paris) Lett. 41, L291 (1980).

28. B. D. Laikhtman, Phys. Rev. B 31, 490 (1985).

29. W. E. Moerner, Science 265, 46 (1994).

30. E. Geva, P. D. Reily, and J. L. Skinner, Acc. Chem. Res. 29 (1996).

31. W. E. Moerner and M. Orrit, Science 283, 1670 (1999).

32. E. Barkai, Y. Jung, and R. Silbey, Phys. Rev. Lett. 87, 207403 (2001).

33. J. Schriefl, Y. Makhlin, A. Shnirman, and G. Schön, New J. Phys. 8, 1 (2006).

34. P. W. Anderson, B. I. Halperin, and C. M. Varma, Philos. Mag. 25, 1 (1972).

35. W. A. Phillips, J. Low Temp. Phys. 7, 351 (1972).

36. J. Jäckle, Z. Phys. 257, 212 (1972).

37. J. L. Black, in Glassy Metals I edited by H.-J. Gunterodt and H. Beck (Springer-Verlag, Berlin, 1981), p. 167.