q-STABILITY CONDITIONS VIA q-QUADRATIC DIFFERENTIALS FOR CALABI-YAU-X CATEGORIES

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ABSTRACT. Categorically, we introduce the Calabi-Yau-X categories $D_X$ of a graded marked surface $S^\lambda$, as a $q$-deformation of Haiden-Katzarkov-Kontsevich’s topological Fukaya category $D_\infty$ of $S^\lambda$. We show that $D_\infty$ can be identified with the cluster-X category associated to $D_X$.

Geometrically, we construct and identify the space of $q$-quadratic differentials on the logarithm surface $\log c S^\lambda \Delta$ with the space of induced $q$-stability conditions on $D_X$, for a complex parameter $s$ satisfying $\text{Re}(s) \gg 1$. When $s = N$ is an integer, the result gives an $N$-analogue of Bridgeland-Smith’s result for realizing stability conditions on the Calabi-Yau-N orbit category $D_{\infty} \sslash [X - N]$ via CY-N type quadratic differentials.

When the genus of $S$ is zero, the spaces of $q$-quadratic differentials can be also identified with framed Hurwitz spaces. As a byproduct, we confirms the conjectural almost Frobenius structure on the spaces of $q$-stability conditions for type $A$.

Key words: $q$-stability conditions, Calabi-Yau-X categories, cluster categories, $q$-quadratic differentials, twisted periods

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INTRODUCTION

In the prequel [IQ], we introduced \( q \)-stability conditions on a class of triangulated categories \( D_X \) with a distinguished auto-equivalence \( X \).

In this paper, we study the case when \( D_X \) are Calabi-Yau-\( X \) categories associated to graded decorated marked surfaces. We introduce \( q \)-quadratic differentials as counterpart of the \( q \)-stability conditions in prequel and prove they can be identified in the surface case.

0.1. Quadratic differentials as stability conditions. Motivated by Douglas’ II-stability of D-branes in string theory, Bridgeland [B2] introduced stability conditions on triangulated categories. Since then, the theory of stability conditions has played an important role in the study of mirror symmetry, Donaldson-Thomas invariants, cluster theory, etc. In particular, Kontsevich and Seidel observed that spaces of stability conditions and spaces of abelian/quadratic differentials share similar properties a couple of years ago (cf. [BS]). Recently, there are two seminal works in this direction:

• In [BS], Bridgeland-Smith (BS) showed that
  
  \[ \text{Stab}^\circ D_3(S^3)/\text{Aut} \cong \text{Quad}_3(S^3) \]
  which is upgraded in [KQ2] as
  
  \[ \text{Stab}^\circ D_3(S^3) \cong \text{FQuad}_3(S^3) \]  \hspace{1cm} (0.1)

  Here \( D_3(S^3) \cong D_3(S^3) \) is the Calabi-Yau-3 category associated to a (decorated) marked surface \( S^3 \) (or \( S^3 \)), \( \text{Stab}^\circ D_3 \) the space of stability conditions on a triangulated category \( D_3 \), \( \text{Quad}_3(S^3) \) the moduli space of (GMN/CY-3 type) quadratic differentials on \( S^3 \) and \( \text{FQuad}_3(S^3) \) the \( S^3 \)-framed version.

• In [HKK], Haiden-Katzarkov-Kontsevich (HKK) showed that
  
  \[ \text{Stab}^\circ D_\infty(S^\lambda) \cong \text{FQuad}_\infty(S^\lambda) \]  \hspace{1cm} (0.2)

  where \( D_\infty(S^\lambda) \) is the topological Fukaya category of a graded marked surface \( S^\lambda \), \( \text{Stab}^\circ D_\infty(S^\lambda) \) the space of stability conditions on \( D_\infty(S^\lambda) \) and \( \text{Quad}_\infty(S^\lambda) \) the moduli space of (exponential type) quadratic differentials on \( S^\lambda \).

These works share many similarities, which fit into the framework [DHKK] that relates dynamical systems and triangulated categories. For one thing, these categories are of topological Fukaya type. More precisely, the (spherical/indecomposable) objects in these categories correspond to (closed/graded) arcs on the surfaces (i.e., [Q2, Thm. 6.6], [HKK, Prop. 4.2]). Note that \( D_3(S^3) \) can be embedded into the derived Fukaya category of some symplectic manifold constructed from \( S^3 \) ([S], cf. the survey [Q4]). For another, the constructions of a stability condition \( \sigma = (Z, P) \) from a quadratic differential \( \xi \) are both roughly as follows:
- a saddle connection $\eta$ with phase $\varphi$ corresponds to (semi)stable objects $X_\eta$ in the slicing $\mathcal{P}(\phi)$;
- the integral of the square root of $\xi$ along (the double cover of) $\eta$ gives the central charge of the corresponding stable object $X_\eta$:

$$Z(X_\eta) = \int_{\tilde{\eta}} \sqrt{\xi}.$$ (0.3)

One of the main motivations of this paper is to give concrete links between these results [BS, HKK]. Categorically, we construct the (deformed) Calabi-Yau-$X$ completion $\mathcal{D}_X(S^\lambda) := \mathcal{D}_X(A)$ of $\mathcal{D}_\infty(S^\lambda)$ via a full formal arc system $A$ of the decorated version of $S^\lambda$, such that $\mathcal{D}_3(S^\lambda_{\Delta})$ is the (triangulated hull of the) orbit quotient:

$$\mathcal{D}_3(S^\lambda_{\Delta}) = \mathcal{D}_X(S^\lambda) \cap X = [X - 3].$$ (0.4)

Geometrically, we will $q$-deform HKK’s result to produce a Calabi-Yau-$s$ version of BS’ result, that realizes $q$-stability conditions via $q$-quadratic differentials, for a complex parameter $s$.

A by-product we obtained along the way is that HKK’s topological Fukaya category can be realized as the cluster category in the following sense:

$$\mathcal{D}_\infty(S^\lambda) \cong \text{per}(S^\lambda) \cong C_X(S^\lambda) : = \text{per}_X(S^\lambda)/\mathcal{D}_X(S^\lambda),$$ (0.5)

where $\text{per}(S^\lambda)$ is the Koszul dual of $\mathcal{D}_\infty(S^\lambda)$ and $C_X(S^\lambda)$ is the Calabi-Yau-$X$ version of the cluster category, in the sense of Amiot-Guo-Keller [A, K1, IY]. This explains the cluster-like structure of $\text{Stab}^o \mathcal{D}_\infty(S^\lambda)$ in [HKK, § 6], cf. [KQ1, Q1].

### 0.2. The $q$-deformations.

Given a triangulated category $\mathcal{D}_\infty$ with Grothendieck group $K(\mathcal{D}_\infty) \cong \mathbb{Z}^\oplus n$, Bridgeland shows that the set of all stability conditions (with support property) on $\mathcal{D}_\infty$ forms a complex manifold with dimension $n$. There is a local homeomorphism/coordinate

$$Z_{\infty}: \text{Stab} \mathcal{D}_\infty \rightarrow \text{Hom}_\mathbb{Z}(K(\mathcal{D}_\infty), \mathbb{C}),$$ (0.6)

provided by the central charge $Z$ of a stability condition $\sigma = (Z, \mathcal{P})$. To $q$-deform the notion of stability conditions and this result, we consider the Calabi-Yau-$X$ version of $\mathcal{D}_\infty$, which is a triangulated category $\mathcal{D}_X$ admitting a distinguish autoequivalence $X$, that makes the Grothendieck group as

$$K(\mathcal{D}_X) \cong R^\oplus n, \quad R_0 := \mathbb{Z}[q^{\pm 1}].$$

Recall that there are natural $\mathbb{C}$-action and Aut-action on the set of stability conditions.

A $q$-stability condition on $\mathcal{D}_X$ consists of a stability condition $\sigma$ and a complex number $s$ satisfying the Gepner equation

$$X(\sigma) = s \cdot \sigma,$$ (0.7)
(and a technical condition, the $q$-support property). In the prequel [IQ] we show that when fixing $s$, the $q$-stability conditions $((\sigma, s))$ also form a complex manifold with dimension $n$ with local homeomorphism/coordinate

$$Z_s : QStab_s D_X \rightarrow \text{Hom}_R(K(D_X), \mathbb{C}_s),$$

(0.8)

Here $\mathbb{C}_s$ is the complex plane equipped with the $R$-structure given by $q \cdot z := e^{i \pi s} z$.

On the quadratic differential side, we also need to perform $q$-deformation. The moduli space $FQuad_{\infty}(S^\lambda)$ that corresponds to $Stab D_{\infty}(S^\lambda)$ in (0.2) consists of quadratic differentials with exponential type singularities, locally of the form

$$e^{z - k z - l \log(z)} d^2 z,$$

(0.9)

for some $(k, l) \in (\mathbb{Z}_{>0} \times \mathbb{Z})$ and some locally holomorphic non-vanishing function $g(z)$. The local homeomorphism/coordinate is given by the period map

$$\Pi_{\infty} : FQuad_{\infty}(S^\lambda) \rightarrow \text{Hom}(H_1(S^\lambda, \partial S^\lambda; \mathbb{Z}_{sp}), \mathbb{C}), \phi \mapsto \int \sqrt{\phi},$$

(0.10)

which corresponds to (0.6).

We would like to $q$-deform the singularity (0.9) as an $s$-pole of the form

$$e^{-k(s-2)-l} d^2 z$$

(0.11)

together with $k$ $s$-zeroes of the form $e^{(s-2)} d^2 z$. Analytically, we introduce a multi-valued quadratic differential $\Theta$ on $S^\lambda$. Equivalently, we construct log-surface $\log c_S^\lambda \Delta$ and view $\Theta$ as a single-valued quadratic differentials on $\log c_S^\lambda \Delta$. The deck transformation, also denoted by $q$, on $\log c_S^\lambda \Delta$ corresponds to the distinguish autoequivalence $X$ and the equation (0.7) becomes

$$q^*(\Theta) = e^{i \pi s} \cdot \Theta$$

(0.12)

in this setting. Then the local homeomorphism/coordinate is given by the period map

$$\Pi_s : QQuad_s(\log c S^\lambda \Delta) \rightarrow \text{Hom}_{\mathbb{R}}(\hat{H}(\log c S^\lambda \Delta; \mathbb{Z}), \mathbb{C}_s), \xi \mapsto \int \sqrt{\xi},$$

(0.13)

that corresponds to (0.8), where $QQuad_s(\log c S^\lambda \Delta)$ is the moduli space of $\log c S^\lambda \Delta$-framed $q$-quadratic differentials.

When $\text{Re}(s) \gg 1$, we use HKK’s isomorphism (0.2) to prove the main result (Theorem 7.9) of the paper, which is a $q$-deformed generalization of BS’ isomorphism (0.1).

**Theorem 1.** Suppose $\text{Re}(s) \gg 1$. There is an isomorphism

$$QQuad_s(\log c S^\lambda \Delta) \xrightarrow{\cong} QStab_s D_X(A)$$

between complex manifolds, where $QStab_s D_X(A)$ consists of (possibly many) connected components in the space $QStab_s D_X(A)$ of $q$-stability conditions.
0.3. **Frobenius structures on genus zero Hurwitz spaces.** A Hurwitz space $\text{HS}$ is the moduli space of meromorphic functions on a Riemann surface $S$. In [D1, § 5], Dubrovin constructed Frobenius structures of Saito type (also called flat structure in [Sa1]) on Hurwitz spaces. The primitive form $[\text{Sa1, SaTa}]$ (primary differential [D1]) on a Hurwitz space plays an essential role in his construction.

In Section 9, we identify the space of $q$-quadratic differentials with the regular locus of the Hurwitz space when the surface $S$ is $\mathbb{P}^1$. As in Section 6, for $q$-quadratic differentials, we consider $s$-zeroes of the form $z^{(s-2)} d z^2$ and $s$-poles of the form $z^{-k(s-2)-l} d z^2$, which have the numerical data/polar type $(k,l) \in \mathbb{Z}_{\geq 1} \times \mathbb{Z}$. On the one hand, the positive integer $k$ determines the complex structure of Hurwitz spaces. On the other hand, the choice of the integer $l$ specifies the choice of the primitive form. Thus our introduction of the new parameter $s$ and the division of the pole order into $k(s-2) + l$ are essential for studying the moduli spaces of multi-valued quadratic differentials. The key numerical calculation is to match the winding numbers (in Section 6.5), which shows that a combination of $k$ $s$-zeroes and one $s$-pole is indeed the $q$-deformation of HKK’s exponential type singularity (0.9), with the same parameters $(k,l)$.

On the regular locus $\text{HS}_{\text{reg}}$ of a Hurwitz space $\text{HS}$, we can consider the almost Frobenius structure $[\text{D2}]$ as the almost dual of the original Frobenius structure. Under the identification between Hurwitz spaces and moduli spaces, the twisted periods of the almost Frobenius structure on $\text{HS}_{\text{reg}}$ can be identified with the periods of $q$-quadratic differentials. Combining with our main result that identifies the moduli space of $q$-quadratic differentials with the space of $q$-stability conditions, we obtain the correspondence between the Hurwitz space and the space of $q$-stability conditions in this case.

In particular, twisted periods of the almost Frobenius structure should be identified with the central charges of $q$-stability conditions. In fact, we present a conjectural description of the almost Frobenius structure on the space $\text{QStab}^0$ of $q$-stability conditions on a Calabi-Yau-X category in the case of type ADE through isomorphism between an $\text{QStab}^0$ and the universal covering of the regular locus of the Cartan subalgebra of the corresponding type. One application of our $q$-deformation of quadratic differentials/stability conditions is to prove this conjecture in the type $A_n$ case. The observation here is that the Cartan subalgebra of the type $A_n$ can be identified with the unfolding of a type $A_n$ singularity, which is a special case of a Hurwitz space.

In all, our new constructions provide an approach to understand almost Frobenius structure on the spaces of $q$-stability conditions.

0.4. **Content.** All the modules and categories will be over $k$, an algebraically closed field. Denote $\mathbb{Z}/m\mathbb{Z}$ by $\mathbb{Z}_m$ for a positive integer $m$.

The paper is organized as follows.

In Part 1, we establish categorical setups:

- In Section 1, we recall the notations and results from the prequel [IQ] on $q$-deformation of stability conditions.
In Section 2, we describe topological Fukaya categories from graded marked surfaces, partially following [HKK, LP, IQZ].

In Section 3, we recall Keller’s deformed Calabi-Yau completion from [K1] in the Calabi-Yau-X setting.

In Section 4, we introduce the Calabi-Yau-X categories associated to graded marked surfaces, as q-deformations of the topological Fukaya categories. We recall technical results for such categories from [IQZ].

In Part 2, we study the geometric side of the story:

- In Section 5, we recall the theory of quadratic differentials of GMN type and exponential type on Riemann surfaces from [BS] and [HKK], respectively.
- In Section 6, we introduce q-quadratic differentials as q-deformations of quadratic differentials of exponential type. The key result (Theorem 6.21) here is that the numerical data (mainly, the winding numbers) for these two type of differentials do match.
- In Section 7, we prove the main results of the paper (Theorem 7.4 and Theorem 7.9), that the induced q-stability conditions on Calabi-Yau-X categories from surfaces can be identified with q-quadratic differentials.

In Part 3, we give applications:

- In Section 8, we show (Theorem 8.7) that when specializing s = N to be a sufficiently large integer, the stability conditions on Calabi-Yau-N orbit categories can be realized as CY-N type quadratic differentials.
- In Section 9, we show (Theorem 9.9) that when the genus of the surface is zero, then framed q-quadratic differential can be identified with framed Hurwitz covers. In particular, this provides a solution of gluing q-stability conditions along the s-direction (cf. Corollary 9.12). Furthermore, when restricted to the disk case (type A_n), the moduli space of framed Hurwitz covers can be identified with the universal cover of the regular part of the space of polynomials of degree n + 1 (cf. Proposition 9.15).
- In Section 10, we describe a conjectural almost Frobenius structure on spaces of q-stability conditions for Dynkin quivers and confirm it (Theorem 10.15) for the type A case using the result above on Hurwitz covers.

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### Table 1. List of notations

| Symbol | Description |
|--------|-------------|
| $R$    | $\mathbb{Z}[q,q^{-1}]$ |
| $D_X(-)$ | a Calabi-Yau-$q$ category for $? \in \mathbb{Z} \cup \{X, \infty\}$ |
| $\sigma = (Z, \mathcal{P})$ | a stability conditions with central charge $Z$ and slicing $\mathcal{P}$ |
| $S = (S, M, Y)$ | a marked surface, i.e., surface with sets of open/closed marked points |
| $S^\lambda$ | a graded marked surface with grading $\lambda$ |
| $S_\Delta$ | a decorated marked surface (DMS), with decoration $\Delta$ |
| $c$ | a cut, i.e., a pairing between the set $Y$ and the set $\Delta$ |
| $S_\Delta = (S_\Delta, c, \Lambda)$ | a graded DMS with grading $\Lambda$ (compatible with $S^\lambda$) |
| $\log_c S^{\lambda}_\Delta$ | the log surface associated to $S_\Delta$, w.r.t. $c$ |
| $S^{\lambda(m)}_\Delta$ | $m^{th}$ sheet of $\log_c S^{\lambda}_\Delta$ |
| $C$ | a marked surface $S$ of genus 0 |
| $D$ | a marked disk |
| $\mathfrak{m}(?)$ | the numerical data of $?$ |
| $(k, l)$ | the polar type for exponential type/$q$-quadratic differentials |
| $A = \{\tilde{\gamma}_i\}$ | a full formal open arc system for $S^\lambda$ |
| $A^* = \{\tilde{\eta}_i\}$ | the dual full formal closed arc system |
| $\tilde{C}A(?)$ | the set of graded closed arcs on $?$ |
| $\hat{H}(?) = H_1(?)^-$ | the hat homology of $?$ |
| $D_\infty(A)$ | the topological Fukaya category of $S^\lambda$ |
| $D_X(A)$ | the Calabi-Yau-$X$ category of $S_\Delta$ |
| $(S, \phi, h)$ | a quadratic differential $\phi$ on a Riemann surface $S$ with framing $h$ |
| $\text{Core}(\phi)$ | the core of a quadratic differential |
| $\Theta = (S, \xi, h; s)$ | a $\log_c S^{\lambda}_\Delta$-framed $q$-quadratic differential |
| $\text{FQuad}_\infty(S^\lambda)$ | the moduli space of framed quadratic differential on $S^\lambda$ |
| $\text{QQuad}_\infty(\log_c S^{\lambda}_\Delta)$ | the moduli space of $\log_c S^{\lambda}_\Delta$-framed $q$-quadratic differential |
| $\text{QStab}(D_X(A))$ | the space of certain type $q$-stability conditions on $D_X(A)$ |
| $\text{HS}(g, k)$ | the moduli space of Hurwitz cover of genus $g$ and polar type $k$ |
| $\text{HS}(\log C_\Delta)$ | the moduli space of $\log C_\Delta$-framed Hurwitz covers |
| $\mathcal{M}_n$ | the space of polynomials of degree $n + 1$ |
| $\text{reg}$ | the regular part of space $?$ |
| $S_N^N, S_N^N$ | CY-$N$ marked surface and DMS |
| $\overline{\mathbb{C}} = \mathbb{P}^1$ | Riemann sphere |
Part 1. Categorical Part

1. q-Deformation of stability conditions

In this section, we recall the $q$-deformation of stability conditions from the prequel [IQ].

1.1. Bridgeland stability conditions. First we recall the definition of Bridgeland stability conditions on triangulated categories from [B2]. We assume that for a triangulated category $\mathcal{D}$, its Grothendieck group $K(\mathcal{D})$ is free of finite rank in this subsection, i.e., $K(\mathcal{D}) \cong \mathbb{Z}^n$ for some $n$.

**Definition 1.1.** Let $\mathcal{D}$ be a triangulated category. A stability condition $\sigma = (Z, \mathcal{P})$ on $\mathcal{D}$ consists of a group homomorphism $Z : K(\mathcal{D}) \to \mathbb{C}$ called the central charge and a family of full additive subcategories $\mathcal{P}(\varphi) \subset \mathcal{D}$ for $\varphi \in \mathbb{R}$ called the slicing satisfying the following conditions:

(a) if $0 \neq E \in \mathcal{P}(\varphi)$, then $Z(E) = m(E) \exp(i\pi \varphi)$ for some $m(E) \in \mathbb{R}_{>0}$,

(b) for all $\varphi \in \mathbb{R}$, $\mathcal{P}(\varphi + 1) = \mathcal{P}(\varphi)[1],$

(c) if $\varphi_1 > \varphi_2$ and $A_i \in \mathcal{P}(\varphi_i)$ ($i = 1, 2$), then $\text{Hom}_\mathcal{D}(A_1, A_2) = 0$,

(d) for $0 \neq E \in \mathcal{D}$, there is a finite sequence of real numbers

$$\varphi_1 > \varphi_2 > \cdots > \varphi_m$$

and a collection of exact triangles (Harder-Narasimhan filtration)

$$0 \to E_0 \to E_1 \to E_2 \to \cdots \to E_{m-1} \to E_m = E$$

with $A_i \in \mathcal{P}(\varphi_i)$ for all $i$.

For each non-zero object $0 \neq E \in \mathcal{D}$, we define two real numbers by $
 \varphi_+^\sigma(E) := \varphi_1$ and $\varphi_-^\sigma(E) := \varphi_m$ where $\varphi_1$ and $\varphi_m$ are determined by the axiom (d). Nonzero objects in $\mathcal{P}(\varphi)$ are called semistable of phase $\varphi$ and simple objects in $\mathcal{P}(\varphi)$ are called stable of phase $\varphi$. For a non-zero object $E \in \mathcal{D}$ with extension factors $A_1, \ldots, A_m$ given by axiom (d), define the mass of $E$ by

$$m_{\sigma}(E) := \sum_{i=1}^m |Z(A_i)|.$$ 

For a stability condition $\sigma = (Z, \mathcal{P})$, we introduce the set of semistable classes $\text{ss}(\sigma) \subset K(\mathcal{D})$ by

$$\text{ss}(\sigma) := \{ \alpha \in K(\mathcal{D}) \mid \text{there is a semistable object } E \in \mathcal{D} \text{ such that } [E] = \alpha \}. \quad (1.3)$$

Let $\| \cdot \|$ be some norm on $K(\mathcal{D}) \otimes \mathbb{R}$. A stability condition $\sigma = (Z, \mathcal{P})$ satisfies the support property if there is some constant $C > 0$ such that

$$C \cdot |Z(\alpha)| > \|\alpha\| \quad (1.4)$$
for all $\alpha \in \text{ss}(\sigma)$. We will assume that the stability conditions we considered always satisfy support property.

The key result by Bridgeland is the following.

**Theorem 1.2** ([B2], Theorem 1.2). *The projection map of taking central charges*

$$Z : \text{Stab}\mathcal{D} \to \text{Hom}(K(D), \mathbb{C}), \quad (Z, P) \mapsto Z$$

*is a local homeomorphism of topological spaces. In particular, $Z$ induces a complex structure on $\text{Stab}\mathcal{D}$."

There are two group actions on $\text{Stab}\mathcal{D}$ commuting with each other. The first one is the natural $\mathbb{C}$ action

$$s \cdot (Z, P) = (Z \cdot e^{-\text{ir} s}, P_{\text{Re}(s)}),$$

where $P_x(\phi) = P(\phi + x)$. There is also a natural action on $\text{Stab}\mathcal{D}$ induced by $\text{Aut}\mathcal{D}$, namely:

$$\Phi(Z, P) = (Z \circ \Phi^{-1}, \Phi(P)).$$

1.2. *q*-stability conditions. Let $\mathcal{D}_X$ be a triangulated category with a distinguished auto-equivalence

$$\mathcal{X} : \mathcal{D}_X \to \mathcal{D}_X.$$

We will write $E[[X]]$ instead of $X^l(E)$ for $l \in \mathbb{Z}$ and $E \in \mathcal{D}_X$. Set $R = \mathbb{Z}[q^{\pm 1}]$ and define the action of $R$ on $K(D_X)$ by

$$q^n \cdot [E] := [E[nX]].$$

Then $K(D_X)$ has an $R$-module structure.

**Definition 1.3.** A *q*-stability condition consists of a (Bridgeland) stability condition $\sigma = (Z, P)$ on $\mathcal{D}_X$ and a complex number $s$ satisfying

$$\mathcal{X}(\sigma) = s \cdot \sigma. \quad (1.5)$$

In the rest of this paper, we assume the following:

**Assumption 1.4.** The Grothendieck group $K(D_X)$ is free of finite rank over $R$, i.e.,

$$K(D_X) \cong R^{\oplus n}$$

for some $n$.

Similar to usual stability conditions, we also always require *q*-stability conditions satisfy the *q*-support property (see [IQ, Def. 3.8]).

Denote by $\text{QStab}_s \mathcal{D}_X$ the set of all *q*-stability conditions satisfying *q*-support property and with fixed $s$. Let

$$\text{QStab} \mathcal{D}_X = \bigcup_{s \in \mathbb{C}} \text{QStab}_s \mathcal{D}_X.$$

Let $\mathbb{C}_s$ denote the complex numbers with the $R$-module structure given by

$$q \cdot z := e^{\text{ir} s} z$$

for $z \in \mathbb{C}_s$, that corresponds to the specialization (1.8). The following is our *q*-deformed version of Bridgeland’s original result (Theorem 1.2).
Theorem 1.5. [IQ, Thm. 3.10] Let $\mathcal{D}_X$ be a triangulated category with $K(\mathcal{D}_X) \cong R^\oplus n$ and fix $s \in \mathbb{C}$. The projection map of taking central charges $$Z_s: \text{QStab}_s \mathcal{D}_X \to \text{Hom}_R(K(\mathcal{D}_X), \mathbb{C}_s), \quad (Z, P) \mapsto Z$$ is a local homeomorphism of topological spaces. In particular, $Z_s$ induces a complex structure on $\text{QStab}_s \mathcal{D}_X$.

1.3. $X$-baric heart and $q$-stability conditions.

Definition 1.6 ($X$-baric heart). An $X$-baric heart $\mathcal{D}_\infty \subset \mathcal{D}_X$ is a full triangulated subcategory of $\mathcal{D}_X$ satisfying the following conditions:

1. if $k_1 > k_2$ and $A_i \in \mathcal{D}_\infty[k_i X] (i = 1, 2)$, then $\text{Hom}_{\mathcal{D}_X}(A_1, A_2) = 0$,

2. for $0 \neq E \in \mathcal{D}_X$, there is a finite sequence of integers $k_1 > k_2 > \cdots > k_m$ and a collection of exact triangles

$$0 = E_0 \to E_1 \to E_2 \to \cdots \to E_{m-1} \to E_m = E$$

with $A_i \in \mathcal{D}_\infty[k_i X]$ for all $i$.

Note that, by definition, the classes of objects in $\mathcal{D}_\infty$ span $K(\mathcal{D}_X)$ over $R$ and there is a canonical isomorphism

$$K(\mathcal{D}_\infty) \otimes_\mathbb{Z} R \cong K(\mathcal{D}_X). \quad (1.6)$$

Recall that the triangulated category $\mathcal{D}_X$ is Calabi-Yau-$X$ if $X$ is the Serre functor (cf. (3.1)). For an $X$-baric heart $\mathcal{D}_\infty$ in a Calabi-Yau-$X$ category $\mathcal{D}_X$, Condition (1) in Definition 1.6 is equivalent to

$$\text{Hom}_{\mathcal{D}_X}(A_1, A_2) = 0 \quad (1.7)$$

for $A_i \in \mathcal{D}_\infty[k_i X]$ and $k_1 - k_2 \notin \{0, 1\}$.

For a fixed complex number $s \in \mathbb{C}$, consider the specialization

$$q_s: \mathbb{C}[q, q^{-1}] \to \mathbb{C}, \quad q \mapsto e^{i\pi s}. \quad (1.8)$$

Construction 1.7. Consider a triple $(\mathcal{D}_\infty, \hat{\sigma}, s)$ consisting of an $X$-baric heart $\mathcal{D}_\infty$, a (Bridgeland) stability condition $\hat{\sigma} = (\hat{Z}, \hat{P})$ on $\mathcal{D}_\infty$ and a complex number $s$. We construct

1°. the additive pre-stability condition $\sigma_{\oplus} = (Z, P_{\oplus})$ and

2°. the extension pre-stability condition $\sigma_* = (Z, P_*)$,

where
we first extend $\hat{Z}$ to

$$Z_q: = \hat{Z} \otimes 1: K(D_X) \rightarrow \mathbb{C}[q, q^{-1}]$$

via (1.6) and

$$Z = \hat{s} \circ (\hat{Z} \otimes 1): K(D_X) \rightarrow \mathbb{C}$$

gives a central charge function on $D_X$;

- the pre-slicing $P_{\oplus}$ is defined as

$$P_{\oplus}(\phi) = \text{add}^* \hat{P}[ZX] : = \text{add} \bigoplus_{k \in \mathbb{Z}} \hat{P}(\phi - k \text{Re}(s))[kX];$$

(1.9)

- the pre-slicing $P_*$ is defined as

$$P_*(\phi) = \langle \hat{P}[ZX]\rangle^*: = \langle \hat{P}(\phi - k \text{Re}(s))[kX]\rangle.$$

(1.10)

Note that $\sigma$ does not necessarily satisfy condition (d) in Definition 1.1 and hence may not be a stability condition.

Definition 1.8. [IQ, Q5] Given a slicing $P$ on a triangulated category $D$. Define the global dimension of $P$ by

$$\text{gldim } P = \sup \{\phi_2 - \phi_1 | \text{Hom}(P(\phi_1), P(\phi_2)) \neq 0\}. \quad (1.11)$$

For a stability condition $\sigma = (Z, P)$ on $D$, its global dimension $\text{gldim } \sigma$ is defined to be $\text{gldim } P$.

Definition 1.9. An open $q$-stability condition on $D_X$ is a pair $(\sigma, s)$ consisting of a stability condition $\sigma$ on $D_X$ and a complex parameter $s$ such that

- $\sigma = \sigma_{\oplus}$ is an additive pre-stability condition induced from some triple $(D_{\infty}, \hat{\sigma}, s)$ as in Construction 1.7.

A closed $q$-stability condition on $D_X$ is a pair $(\sigma, s)$ consisting of a stability condition $\sigma$ on $D_X$ and a complex parameter $s$ such that

- $\sigma = \sigma_*$ is an extension pre-stability condition induced from some triple $(D_{\infty}, \hat{\sigma}, s)$ as in Construction 1.7.

Theorem 1.10. [IQ] Suppose that $D_X$ is Calabi-Yau-$\mathbb{X}$ with an $\mathbb{X}$-baric heart $D_{\infty}$. A triple $(D_{\infty}, \hat{\sigma}, s)$ induces an open $q$-stability condition $(\sigma, s)$ for $\sigma = \sigma_{\oplus}$ if and only if

$$\text{gldim } \hat{\sigma} + 1 < \text{Re}(s). \quad (1.12)$$

A triple $(D_{\infty}, \hat{\sigma}, s)$ induces a closed $q$-stability condition $(\sigma, s)$ for $\sigma = \sigma_*$ if and only if

$$\text{gldim } \hat{\sigma} + 1 \leq \text{Re}(s) \quad (1.13)$$

We will still write

$$(\sigma, s) = (D_{\infty}, \hat{\sigma}, s) \otimes_{\oplus} R, \quad (\sigma, s) = (D_{\infty}, \hat{\sigma}, s) \otimes_* R$$

for the two induced $q$-stability conditions in the above theorem if there causes no confusion. We will study subspaces $Q\text{Stab}_{\oplus}^q D_X$ and $Q\text{Stab}_*^q D_X$ of $Q\text{Stab}_q D_X$ consisting of certain induced open and closed $q$-stability conditions.
2. Topological Fukaya categories

2.1. Graded marked surface. We recall the relative notions and notations on graded marked surfaces, cf. [IQZ, HKK, LP]. Note that we will use a single marked point to denote a marked boundary arc in their setting, where if one performs the real blow-up (with respect to some quadratic differential) at the corresponding singularity, such a marked point corresponds to a line (isomorphic to $\mathbb{R}$).

**Definition 2.1.** A graded marked surface $(S, M, Y, \lambda)$ consists of the following data:

- $S$ is a smooth oriented surface with $\partial S \neq \emptyset$,
- $M$ is a set of open marked points on $\partial S$, such that $M \cap \partial_1 \neq \emptyset$ for each boundary component $\partial_1$ of $\partial S$ (cf. the blue bullet points in Figure 2.1).
- $Y$ is a set of closed marked points on $\partial S$, such that points in $M$ and in $Y$ are alternating in $\partial S$ (cf. the red circle points in Figure 2.1). Let $\aleph := |Y|$.
- $\lambda$ is a grading (or line field) on $S$, that is, a section $\lambda : S \rightarrow \mathbb{PTS}$ of the projectivized tangent bundle $\mathbb{PTS}$.

For simplicity, we write $S$ for the triple $(S, M, Y)$, called marked surface, and $S^\lambda$ for the graded version. We will exclude the case when $S$ is a disk with $\aleph = 2$.

A morphism $S^\lambda \rightarrow S'^{\lambda'}$ of graded marked surfaces is a pair $(f, \tilde{f})$ consisting of an orientation preserving local diffeomorphism $f : S \rightarrow S'$, that preserves the sets of marked points setwise, and a (homotopy class of) path $\tilde{f}$ from $f^*(\lambda')$ to $\lambda$ in the space of sections of $\mathbb{PTS}$. Every graded marked surface $S$ admits an automorphism $\lambda_t$, known as the shift, given by the pair $(\text{id}_S, \overrightarrow{\lambda_t})$, where restricted to the generator of $\pi_1(\mathbb{PTS})$, $\overrightarrow{\lambda_t}$ is rotating clockwise by an angle of $\pi$, for any $p \in S$.

A curve in a graded marked surface $S^\lambda$ is an immersion $c : I \rightarrow S$ for a 1-manifold $I$. A grading $\tilde{c}$ on a curve $c$ is given by a family of (homotopy classes of) paths in $\mathbb{PTS}$ from $\lambda(c(t))$ to $c(t)$, varying continuously with $t \in (0, 1)$. The pair $(c, \tilde{c})$ is called a graded curve, and will be simply denoted by $\tilde{c}$ usually. The pushforward of a graded curve of a graded morphism $(f, \tilde{f})$ is given by $(I, f \circ c, (c^*\tilde{f}) \cdot \tilde{c})$.

A point of transverse intersection of a pair $((I_1, c_1, \tilde{c}_1), (I_2, c_2, \tilde{c}_2))$ of graded curves determines an integer as follows. Suppose $t_i \in I_i$ such that

$$c_1(t_1) = c_2(t_2) = p \in S \quad \text{and} \quad \dot{c}_1(t_1) \neq \dot{c}_2(t_2) \in \mathbb{PT}_p S. \quad (2.1)$$

We have the following homotopy classes of paths in $\mathbb{PT}_p S$:

1. $\tilde{c}_1(t_1)$ from $\lambda(p)$ to $\dot{c}_1(t_1)$,
2. $\tilde{c}_2(t_2)$ from $\lambda(p)$ to $\dot{c}_2(t_2)$,
3. $\kappa$ from $\dot{c}_1(t_1)$ to $\dot{c}_2(t_2)$ given by counterclockwise rotation in $T_p S$ by an angle less than $\pi$.

Define the intersection index of $c_1, c_2$ at $p$

$$\text{ind}_p(c_1, c_2) = \tilde{c}_1(t_1) \cdot \kappa \cdot \tilde{c}_2(t_2)^{-1} \in \pi_1(\mathbb{PT}_p S) \cong \mathbb{Z}. \quad (2.2)$$

It is well-defined if $c_1$ and $c_2$ are in general position.
2.2. Grading as cohomology/covering. The grading $\lambda : S \to \mathbb{PTS}$ can be interpreted as a cohomology class $[\lambda]$ in $H^1(\mathbb{PTS})$ in this case. The projection $\mathbb{PTS} \to S$ with $\mathbb{RP}^1 \cong S^1$-fiber gives a short exact sequence

$$0 \to \pi_1(S) \to \pi_1(\mathbb{PTS}) \to \pi_1(S) \to 0,$$

or (taking the abelization)

$$0 \to H_1(S) \to H_1(\mathbb{PTS}) \to H_1(S) \to 0.$$  \hspace{1cm} (2.3)

[LP2, Lemma 1.2] shows that $\lambda$ is determined by a class $[\lambda] \in H^1(\mathbb{PTS})$, induced from a split of $\pi_S$ (i.e., $\pi_S([\lambda]) = 1$ in $H^1(S)$). Such a data is equivalent to a split of $H_1(S) \to H_1(\mathbb{PTS})$ or a split of $\pi_1(S) \to \pi_1(\mathbb{PTS})$.

A more comprehensive way to think about $\lambda$ is via the Maslov (Z-)covering

$$\pi_\lambda : \mathbb{RTS}^\lambda \to \mathbb{PTS}$$
determined by $\lambda$, where $\mathbb{RTS}^\lambda$ is the $\mathbb{R}$-bundle of $S$ that can be constructed by gluing $Z$ copies of $\mathbb{PTS}$ cut out by $\lambda$. A morphism between two graded marked surfaces then consists of a map $f : S^\lambda \to S'^\lambda$ such that it preserves the marked points and $[\lambda'] = f^*[\lambda]$, regarding $[\lambda] \in H^1(\mathbb{PTS})$, together with a map $\tilde{f}$ between their Maslov covering that commutes with $f$. The automorphism/grading shift $[1]$ is then the deck transformation of the Maslov covering.

In this description, a graded curve $\tilde{c}$ is one of $\mathbb{Z}$ lifts in $\mathbb{RTS}^\lambda$, of the tangent $\dot{c}$ of an usual curve $c$ on $S$. The intersection index $i = \text{ind}_p([\tilde{c}_1, \tilde{c}_2])$ is the shift $[i]$ such that the lift $\tilde{c}_2[i] \mid_p$ of $c_2[i]$ at $p$ is in the length one interval

$$(\tilde{c}_1 \mid_p, \tilde{c}_1[1] \mid_p) \subset \mathbb{RT}_p \cong \mathbb{R}.$$  

2.3. Arc systems. We have the following notions.

- $M$ (resp. $Y$) divides $\partial S$ into $\mathbb{R}$ open (resp. closed) boundary arcs, each of which contains a closed (resp. open) marked point.
- An open (resp. closed) curve $\gamma$ in $S$ is (the isotopy class of) a (non-trivial) curve connecting points in $M$ (resp. in $Y$), such that $\gamma$ is in $S^\circ = S \setminus \partial S$ except for its endpoints.
- An (open/closed) arc is an (open/closed) curve without self-intersection in $S^\circ$.
- An open (resp. closed) arc system $A$ is a collection of pairwise disjoint graded open (resp. closed) arcs.
- An open (resp. closed) full formal arc system is an open (resp. closed) arc system $A$ such that it cuts $S$ into polygons (called $A$-polygons), each of which has exactly one open (resp. closed) boundary arc. See blue arcs in Figure 2.1.

The graded version of an open/closed curve $a$ will usually be denoted by $\tilde{a}$. Denote by $\text{OA}(S^\lambda)$ (resp. $\text{CA}(S^\lambda)$) the set of open (resp. closed) arcs on $S^\lambda$, and by $\tilde{\text{OA}}(S^\lambda)$ (resp. $\tilde{\text{CA}}(S^\lambda)$) the set of graded ones.
Choose (and fix) an initial full formal open arc system $A = \{\tilde{\gamma}_j\}$. The dual graph of $A$, denoted by $A^* = \{\tilde{\eta}_j\}$, is also a full formal closed arc system satisfying

$$\text{Int}^*(\tilde{\gamma}_i, \tilde{\eta}_j) = \text{Int}^0(\tilde{\gamma}_i, \tilde{\eta}_j) = \delta_{ij},$$

where $\text{Int}^d$ is the index-$d$ intersection number and $\text{Int}^* = \sum_{d \in \mathbb{Z}} \text{Int}^d$ is the total intersection number. As we only care about full formal arc systems, we will omit full formal for simplicity. The picture on the top of Figure 2.1 shows an example of a dual arc system $(A, A^*)$, where $A$ consists of the blue arcs and $A^*$ consists of red arcs.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure2_1.png}
\caption{The quiver with relation (in blue) associated to an open arc system and the Ext-quiver corresponding to the dual closed arc system (in red)}
\end{figure}

2.4. Numerical data.

**Definition 2.2.** The numerical data

$$\mathfrak{M} = \mathfrak{M}(S^\lambda) = (g, b; k, l; \text{LP}_g)$$

of $S^\lambda$ consists of

- the genus $g = g_S$;
- the number $b = |\partial S|$ of boundary components;

where $S^\lambda$ is an open arc system.
the ordered multi-set \( k = (k_1, \ldots, k_b) \in \mathbb{Z}_{\geq 0}^b \) of orders, where for each boundary component \( \partial_i \), \( k_i \) is the number of marked points on \( \partial_i \), satisfying
\[
\sum_{i=1}^{b} k_i = |M| = |Y|; \tag{2.4}
\]

the ordered multi-set \( l = (l_1, \ldots, l_b) \in \mathbb{Z}^b \) of indices, where for each boundary component \( \partial_i \), one can calculate \( l_i \) as follows: let \( \{Y_{j}^i \mid j \in \mathbb{Z}_{k_i}\} \) be the closed marked points on \( \partial_i \) (in the clockwise order of the induced orientation of \( S \)) and \( \tilde{n}_{j}^i = Y_{j}^i Y_{j+1}^i \) be the closed boundary arcs (with any grading), then
\[
\text{wind}_{\lambda}(\partial_i) = 2 - l_i = \sum_{j \in \mathbb{Z}_{k_i}} \left( \text{ind}_{Y_j^i} (\tilde{n}_{j-1}^i, \tilde{n}_{j}^i) - 1 \right). \tag{2.5}
\]

Here \( \text{wind}_{\lambda}(\partial_i) \) is the \((clockwise)\) winding number of \( \partial_i \) with respect to \( \lambda \) (cf. \([LP2, \S 1.2]\) and \([HKK, (3.21)]\)). More precisely, we have \( \text{wind}_{\lambda}(-) = (\lambda, -) \), cf. (2.3), for the natural paring:
\[
\langle -, - \rangle : H_1(\mathbb{P} S) \times H_1(\mathbb{P} S) \to \mathbb{Z}.
\]

Note that the indices satisfy \((LP2, (1.5))\)
\[
\sum_{i=1}^{b} l_i = 4 - 4g. \tag{2.6}
\]

We will use quadratic differentials to calculate the winding number \( \text{wind}_{\lambda}(\partial_i) \) and \( l_i \) in Section 6.5. Denote by \( k = (k_1, \ldots, k_b) \) and \( l = (l_1, \ldots, l_b) \) the data of \( (k_i, l_i) \).

the Lekili-Polishchuk (LP) data \( LP_g \), defined as follows:
- if \( g = 0 \), then \( LP_0 = \emptyset \);
- if \( g = 1 \), then \( LP_1 = \tilde{A} \), where \( \tilde{A} := \tilde{A}(\lambda) \) is defined to be the non-negative integer \( \gcd\{\text{wind}_{\lambda}(\gamma) \mid \text{non-separating} \ \gamma\} \);
- if \( g > 1 \), then \( LP_g = (A, \kappa) \) consisting of the Arf invariant \( A \) (see \([LP2, \S 1.2]\)) and the indicator \( \kappa \), where \( \kappa = 0 \) if \( \text{wind}_{\lambda}^{(2)} \equiv 0 \) and \( \kappa = 1 \) otherwise.

Here
\[
\text{wind}_{\lambda}^{(2)} : H_1(S; \mathbb{Z}_2) \to \mathbb{Z}_2 \tag{2.7}
\]
is the induced function \((LP2, \text{Definition 1.2.1})\).

Example 2.3. Consider two genus zero cases.

- When \( S \) is a disk, which corresponds to a type \( A_n \) quiver, then the numerical data is
  \[
  (g = 0, \ b = 1; \ k = (n + 1), \ l = (4); \ LP_0 = \emptyset). \]
- When \( S \) is an annulus that corresponds to the affine type \( \widetilde{A}_{p,q} \) quiver (with zero grading for all arrows), the numerical data is
  \[
  (g = 0, \ b = 2; \ k = (p, q), \ l = (2, 2); \ LP_0 = \emptyset). \]
In general, the indices can be chosen to be \( l = (2 + w, 2 - w) \) for some \( w \in \mathbb{Z} \) and the corresponding quiver \( \tilde{A}_{p,q} \) will be graded.

Note that in the case \( g = 0 \), \( A \) and \( \kappa \) are irrelevant.

2.5. **The topological Fukaya categories.** The graded quiver with relation \((Q_{A}^{(0)}, R_{A}^{(0)})\) associated to an open arc system \( A \) is defined as follows:

- its vertices \( \{i\} \) of \( Q_{A}^{(0)} \) correspond to the open arcs \( \{\tilde{\gamma}_i\} = A \);
- its arrows correspond to the (anticlockwise) angle between arcs of \( A \) in the polygons of \( S \) cutting out by \( A \) and the grading of the arrows is given by the intersection index at the intersection (vertex of the angle);
- the quadratic relations are non-composable arrows=angles.

Let \( k \) be an algebraic closed field and \( A_{0} = kQ_{A}^{(0)}/R_{A}^{(0)} \). Next, we produce a differential graded algebra (we will write dga for short) from \( A_{0} \), replacing the relation with differential.

**Construction 2.4.** [Op, Cons. 2.2] Consider the basic finite dimensional \( k \)-algebra \( A_{0} \). Let \( Q_{A}^{(1)} \) be the (graded) quiver obtained from \( Q_{A}^{(0)} \) by adding the arrows corresponding to the (minimal set of) relations of \( R_{A}^{(0)} \), whose degrees are the degrees of the corresponding relations minus one, and a differential (of degree 1), such that \( H^{0}(A_{0}) \) is a quotient of \( H^{0}(kQ_{A}^{(1)}) \). Now pick a generating set \( R_{A}^{(1)} \) for \( \ker \left( H^{1}(kQ_{A}^{(1)}) \to H^{0}(A_{0}) \right) \) to satisfy that \( kQ_{A}^{(1)}/R_{A}^{(1)} \) is quasi-isomorphic to \( A_{0} \). Iterating this process till we obtain a graded quiver \( Q_{A} \) (without relation) and a differential \( d \) such that \( A := kQ_{A} \) is quasi-isomorphic to \( A_{0} \).

**Remark 2.5.** By simple-projective duality

\[
\text{Irr}^{d}(P_{i}, P_{j}) \cong \text{Ext}^{1-d}(S_{j}, S_{i})^{*},
\]

\( Q_{A} \) can be also constructed from Ext-quiver ([KQ1, Def. 6.2]) of \( A_{0} \). More precisely, Let \( Q^{\text{Ext}}(A_{0}^{A}) \) be the graded Ext-quiver of \( A_{0}^{A} \), whose vertices are simple \( A_{0}^{A} \) module and whose graded arrows are given by graded morphisms between them. Then \( Q_{A} \) is obtained from \( Q^{\text{Ext}}(A_{0}^{A}) \) by the degree changing: \( d \mapsto 1 - d \).

Moreover, \( Q^{\text{Ext}}(A_{0}^{A}) \) is the intersection quiver of \( A^{*} \), in the sense that its vertices one-one correspond to graded closed arcs in \( A^{*} \) and its graded arrows one-one correspond to graded (by index) intersections between them. For instance, the blue solid quiver in Figure 2.1 is \( Q_{A}^{(0)} \) while the relation \( R_{A}^{(0)} \) is presented by blue dashed arrows; the red quiver is the Ext-quiver of \( Q_{A} \).

Furthermore, such a duality topologically corresponds to the graph duality between \( A \) and \( A^{*} \) (see [Q5] for the Calabi-Yau-3 case of this correspondence).
Denote by $\text{TFuk}(S^\lambda)$ the topological Fukaya category of $S^\lambda$, which in this case can be identified as

$$\text{TFuk}(S^\lambda) \cong \text{per } \mathcal{A}_A \cong D_{fd}(\mathcal{A}_A).$$

The above triangle equivalence follows from the fact that we require that any boundary component contains closed and open marked points (so that $\mathcal{A}_A$ is homological smooth and proper). Lekili-Polishchuk proves the following.

**Theorem 2.6. [LP2]** Let $S_i^{\lambda_i}, i = 1, 2,$ be a graded marked surface with the numerical data $\mathfrak{N}(S_i)$. Then $\mathfrak{N}(S_1^{\lambda_1}) = \mathfrak{N}(S_2^{\lambda_2})$ implies the triangle equivalence $\text{TFuk}(S_1^{\lambda_1}) \cong \text{TFuk}(S_2^{\lambda_2})$.

We primarily interest in

$$D_\infty(A) : = D_{fd}(\mathcal{A}_A). \quad (2.7)$$

Denote by $\text{Ind } \mathcal{C}$ the set of indecomposable objects in a category $\mathcal{C}$. The classification of objects in $D_\infty(A)$ is given in [HKK]. However, we will need the version in [IQZ, § 6], which is slightly weaker on objects but with an additional intersection formula.

**Theorem 2.7. [HKK, IQZ]** There is an injection

$$\tilde{X}_\infty : \tilde{C}A(S^\lambda) \longrightarrow \text{Ind } D_\infty(A) \quad (2.8)$$

such that

$$\text{Int}^d(\tilde{\eta}_1, \tilde{\eta}_2) = \dim \text{Hom}^d(\tilde{X}_\infty(\tilde{\eta}_1), \tilde{X}_\infty(\tilde{\eta}_2)), \quad (2.9)$$

for any $\tilde{X}_\infty(\tilde{\eta}_i) \in \tilde{C}A(S^\lambda)$.

Similarly, there is an injection

$$\tilde{X}_\infty^\vee : \tilde{O}A(S^\lambda) \longrightarrow \text{Ind } \text{per } \mathcal{A}_A,$$

but we will not use this map. Moreover, Theorem 2.7 implies that there is a canonical embedding (recall that we exclude the case when $S$ is a disk with two closed marked points)

$$\text{MCG}_{\ast}(S^\lambda) \rightarrow \text{Aut}^\circ D_\infty(A).$$

Here, $\text{MCG}_{\ast}(S^\lambda)$ is the (marked) mapping class group of $S^\lambda$, i.e., the group of isotopy classes of morphisms of $S^\lambda$, where all maps are required to fix $M$ and $Y$ setwise. And $\text{Aut}^\circ D_\infty(A)$ is the quotient group of the auto-equivalence group $\text{Aut } D_\infty(A)$ by those auto-equivalences of $D_\infty(A)$ which acts trivially on the corresponding space of stability conditions $\text{Stab } D_\infty(A)$. Note that in this case, the auto-equivalence that preserves $\tilde{X}_\infty(\tilde{\text{CA}}(S^\lambda))$ will preserve $\text{Stab } D_\infty(A)$ since $\tilde{X}_\infty(\tilde{\text{CA}}(S^\lambda))$ contains stable objects (see Theorem 5.13 for more details).

### 3. Deformed Calabi-Yau–K completion

In this section, we review Keller’s deformed Calabi-Yau completion [K1].
3.1. \(\mathbb{Z}^2\)-graded differential modules and categories. We will work with \(\mathbb{Z}^2\)-graded \(k\)-modules (\(\mathbb{Z}^2\)-graded mods). So such a \(\mathbb{Z}^2\)-graded mod \(M\) admits a decomposition \(M = \bigoplus_{(q,s) \in \mathbb{Z}^2} M^q_s\). Denote by \(\deg m = (q, s) \in \mathbb{Z}^2\) the degree of an element \(m \in M^q_s\).

A \(\mathbb{Z}^2\)-graded morphism \(f: M \to N\) between \(\mathbb{Z}^2\)-graded mods with degree \((p, t)\) is a \(k\)-linear morphism such that \(f(M^q_s) \subset N^{q+p}_{s+t}\). The first component of \(\mathbb{Z}^2\)-grading is the cohomological grading and denote the cohomological degree of \(m \in M^q_s\) by \(|m| = q\).

Similarly \(|f|\) denotes the cohomological degree of a \(\mathbb{Z}^2\)-graded morphism. The tensor product \(M \otimes L\) is \((M \otimes L)^q_s := \bigoplus M^q_{s_1} \otimes L^q_{s_2}\) for \((q_1, s_1) + (q_2, s_2) = (q, s)\) and \(f \otimes g: M \otimes L \to M' \otimes L'\) is

\[((f \otimes g)(m \otimes l)): = (-1)^{|m|\cdot|g|} f(m) \otimes g(l),\]

for \(f: M \to M', g: L \to L'\), where only the cohomological degree effects the sign.

A differential \(\mathbb{Z}^2\)-graded \(k\)-module (\(\mathbb{Z}^2\)-dg-mod) is a pair consisting of a \(\mathbb{Z}^2\)-graded mod \(M\) and a (\(k\)-linear) differential map \(d_M: M \to M\) with degree \((1, 0)\) such that \(d_M^2 = 0\). The cochain \(\mathbb{Z}^q(M)\) and cohomology \(H^q(M)\) are defined with respect to the differential \(d\) (or the cohomological degree), which admits a \(\mathbb{Z}\)-grading (that corresponds the second grading on \(M\))

\[\mathbb{Z}^q(M) = \bigoplus_s \mathbb{Z}_s^q(M), \quad H^q(M) = \bigoplus_s \mathbb{H}_s^q(M).\]

**Remark 3.1.** There are two natural shifts \(M\{1\}\) and \(M[1]\) on \(\mathbb{Z}^2\)-dg-mod \(M\):

\[\begin{align*}
(M\{1\})_q^s &= M_{q+1}^s, & d_{M\{1\}} &= d_M \\
(M[1])_q^s &= M_{q+1}^s, & d_{M[1]} &= -d_M.
\end{align*}\]

A \(\mathbb{Z}^2\)-dg-morphism \(f: M \to L\) between \(\mathbb{Z}^2\)-dg-mods is a \(k\)-linear map such that \(\deg f = (0, 0)\) and \(f \circ d_M = d_L \circ f\), which induces a morphism (of graded modules) on cohomologies \(H^\bullet\). The tensor product \(M \otimes L\) of two \(\mathbb{Z}^2\)-dg-mods is the tensor \(\mathbb{Z}^2\)-graded mod with \(d_{M \otimes L} = d_M \otimes \text{id}_L + \text{id}_M \otimes d_L\). The morphism space \(\text{Hom}(M, L)\) is also a \(\mathbb{Z}^2\)-dg-mod, where the component \(\text{Hom}^q_s(M, L)\) consists of all \(k\)-linear maps of degree \((q, s)\) and

\[d_f := d_M \circ f - (-1)^{|f|} f \circ d_L.\]

A \(\mathbb{Z}^2\)-graded differential category (\(\mathbb{Z}^2\)-dg-cat) \(\mathcal{A}\) is a \(\mathbb{Z}\)-category whose morphism spaces are \(\mathbb{Z}^2\)-dg-mods and whose compositions \(\mathcal{A}(Y, Z) \otimes \mathcal{A}(X, Y) \to \mathcal{A}(X, Z)\) are \(\mathbb{Z}^2\)-dg-morphisms. A \(\mathbb{Z}^2\)-dg-functor \(F: \mathcal{A} \to \mathcal{B}\) between \(\mathbb{Z}^2\)-dg-cats is given by a map between their objects and by morphisms of \(\mathbb{Z}^2\)-dg-mods \(F(X, Y): \mathcal{A}(X, Y) \to \mathcal{B}(FX, FY)\) for \(X, Y \in \text{Obj} \mathcal{A}\).

The opposite \(\mathbb{Z}^2\)-dg-cat \(\mathcal{A}^{op}\) of \(\mathcal{A}\) has the same objects with morphisms \(\mathcal{A}^{op}(Y, X) = \mathcal{A}(Y, X)\) and the composition of \(f \in \mathcal{A}^{op}(Y, X)\) and \(g \in \mathcal{A}^{op}(Z, Y)\) is given by \((-1)^{|f||g|} g \circ f\). The cochain and homology category \(\mathbb{Z}^0(\mathcal{A}), H^0(\mathcal{A})\) of \(\mathcal{A}\) have the same objects of \(\mathcal{A}\) with morphisms \((Z^0 \mathcal{A})(X, Y) = \mathbb{Z}^0(\mathcal{A}(X, Y))\) and \((H^0 \mathcal{A})(X, Y) = H^0(\mathcal{A}(X, Y))\) respectively.

For two \(\mathbb{Z}^2\)-dg-functors \(F, G: \mathcal{A} \to \mathcal{B}\), the complex of graded morphisms \(\text{Hom}(F, G)\) consists of a family of morphisms \(\phi_X: B(FX, FY)^n\) such that \((Gf)(\phi_X) = (\phi_Y)(Ff)\) with the induced differential from \(B(FX, FY)\). The set of morphisms between \(F\) and \(G\) is given by the set \(\mathbb{Z}^0(\text{Hom}(\mathcal{A}, \mathcal{B}))(F, G)\).
3.2. Derived categories of $\mathbb{Z}^2$-dg-cats. Denote by $C_{dg}(A)$ the $\mathbb{Z}^2$-dg-cat of a $\mathbb{Z}$-graded $k$-algebra $A = \bigoplus_{s \in \mathbb{Z}} A_s$. Let $\mathcal{A}$ be a small $\mathbb{Z}^2$-dg-cat. A (right) $\mathbb{Z}^2$-dg-$\mathcal{A}$-mod $M$ is a $\mathbb{Z}^2$-dg-functor

$$M: \mathcal{A}^{op} \to C_{dg}(k).$$

There are also two natural shifts $[1], \{1\}$ for a $\mathbb{Z}^2$-dg-$\mathcal{A}$-mod $M$, the cohomological shift $[1]$ and the extra grading shift $\{1\}$, such that

$$M[1]\{1\}(X) := M(X)[1]\{1\} \in C_{dg}(k).$$

For each object $X$ of $\mathcal{A}$, there is a right module $X^\wedge = \mathcal{A}(?, X)$ represented by $X$.

The $\mathbb{Z}^2$-dg-cat of $\mathcal{A}$ is defined to be

$$C_{dg}(\mathcal{A}) := \text{Hom}(\mathcal{A}^{op}, C_{dg}(k))$$

and we denote its morphism by $\text{Hom}_\mathcal{A}$. The category of $\mathbb{Z}^2$-dg-$\mathcal{A}$-mods $\mathcal{C}(\mathcal{A})$ has the $\mathbb{Z}^2$-dg-$\mathcal{A}$-mods as objects and the morphisms of $\mathbb{Z}^2$-dg-functors as morphisms and we have

$$\mathcal{C}(\mathcal{A}) = \mathbb{Z}^2 C_{dg}(\mathcal{A}).$$

The homotopy category of $\mathbb{Z}^2$-dg-$\mathcal{A}$-mods is $\mathcal{H}(\mathcal{A}) = H^0 C_{dg}(\mathcal{A})$ whose morphisms are given by $H^0 \text{Hom}_\mathcal{A}$. Note that there are canonical isomorphisms $\text{Hom}(X^\wedge, M) \xrightarrow{\sim} M(X)$ and

$$\mathcal{H}(\mathcal{A})(X^\wedge, M[l]) \xrightarrow{\sim} H^l M(X).$$

Definition 3.2. Denote by $\mathcal{D}(\mathcal{A})$ the derived category of $\mathcal{C}(\mathcal{A})$ (or $\mathcal{H}(\mathcal{A})$) with respect to the class of quasi-isomorphisms and its homomorphisms by $\text{Hom}_{\mathcal{D}(\mathcal{A})}$. An important fact is that $\mathcal{D}(\mathcal{A})$ also admits two equivalences, the triangulated shift $[1]$ and the extra grading shift $\{1\}$. We will write $[X]$ for $\{1\}$ and $[m + lX]$ for $[m] \circ [X]^l$, where $m, l \in \mathbb{Z}$.

Recall that a $\mathbb{Z}^2$-dg-mod $P$ is cofibrant if any morphism $P \to M$ factors through $L$ for a surjective quasi-iso-morphism $L \to M$: a $\mathbb{Z}^2$-dg-mod $I$ is fibrant if any morphism $L \to I$ extends to $M$ for an injective quasi-iso-morphism $L \to M$.

Lemma 3.3. [K5, Prop. 3.1] For each $\mathbb{Z}^2$-dg-mod $M$, there is a quasi-iso-morphism (known as cofibrant resolution) $pM \to M$ with cofibrant $pM$ and a quasi-iso-morphism (known as fibrant resolution) $M \to iM$ with fibrant $iM$. Moreover, the projection functor $\mathcal{H}(\mathcal{A}) \to \mathcal{D}(\mathcal{A})$ admits a fully faithful left/right adjoint given by $M \mapsto pM$ and $M \mapsto iM$ respectively. Thus,

$$\mathcal{H}(\mathcal{A})(pL, M) = \text{Hom}_{\mathcal{D}(\mathcal{A})}(L, M) = \mathcal{H}(\mathcal{A})(L, iM).$$

3.3. The inverse dualizing complex and Calabi-Yau categories. Suppose that $\mathcal{A}$ is homologically smooth, i.e., $\mathcal{A}$ is perfect as an $\mathcal{A}^{op}$-mod. Here, $\mathcal{A}^e$ is the $\mathbb{Z}^2$-dg-cat $\mathcal{A} \otimes \mathcal{A}^{op}$, which admits an involution $V^e$ such that

$$V^e(X, Y) = (Y, X) \quad \text{and} \quad V^e(f \otimes g) = (-1)^{|f| |g|} g \otimes f,$$

which is a preduality on $\mathcal{A}^e$.

A preduality $\mathbb{Z}^2$-dg-functor $V$ on a $\mathbb{Z}^2$-dg-cat $\mathcal{A}$ is a $\mathbb{Z}^2$-dg-functor $V: \mathcal{A} \to \mathcal{A}^{op}$, such that the opposite functor $V^{op}$ is a right adjoint of $V$. For a (right) $\mathbb{Z}^2$-dg-mod $M$, its
dual left $\mathbb{Z}^2$-dg-mod $M^*: \mathcal{A} \to \mathcal{C}_{dg}(k)$ sends $X$ to $\mathcal{H}om_{\mathcal{A}}(M, X^\wedge)$. The $V$-dual of $M$ is a $\mathbb{Z}^2$-dg-mod

$$M^V = M^* \circ V^{op} : \mathcal{A} \to \mathcal{C}_{dg}(k),$$

$$X \mapsto \mathcal{H}om_{\mathcal{A}}(M, (V^{op}X)^\wedge).$$

This induces a preduality functor, still denoted by $V$:

$$V : \mathcal{C}_{dg}(\mathcal{A}) \to \mathcal{C}_{dg}(\mathcal{A}^{op}).$$

Now let $\mathcal{A}$ be a small $\mathbb{Z}^2$-dg-cat. As $k$ is a field, $\mathcal{A}$ is cofibrant over $k$.

**Definition 3.4.** The inverse dualizing complex $\Theta_{\mathcal{A}}$ is any cofibrant replacement of the image of $\mathcal{A}$ (considered as a right $\mathbb{Z}^2$-dg-$\mathcal{A}^{op}$-mod) under the total derived functor of $V^{e}$.

Denote by $\mathcal{D}_{fd}(\mathcal{A})$ the finite dimensional derived category of $\mathcal{A}$, which is the full subcategory of $\mathcal{D}(\mathcal{A})$ formed by the $\mathbb{Z}^2$-dg-mod $M$ such that $\sum_i \dim H^i(M)$ is finite.

**Lemma 3.5.** [K1, Lemma 3.4] Then for any $L \in \mathcal{D}(\mathcal{A})$ and $M \in \mathcal{D}_{fd}(\mathcal{A})$, there is a canonical isomorphism

$$\mathcal{H}om_{\mathcal{D}(\mathcal{A})}(L \otimes_{\mathcal{A}} \Theta_{\mathcal{A}}, M) \xrightarrow{\sim} D \mathcal{H}om_{\mathcal{D}(\mathcal{A})}(M, L),$$

where $D = \mathcal{H}om_k(? \otimes \mathcal{A}, ?)$.

Let $\mathcal{N} = m + l\mathcal{X}$. A triangulated category $\mathcal{D}$ is called Calabi-Yau-$\mathcal{N}$ (CY-$\mathcal{N}$) if, for any objects $X, Y$ in $\mathcal{D}$ we have a natural isomorphism

$$\mathcal{G} : \mathcal{H}om(X, Y) \xrightarrow{\sim} D \mathcal{H}om(Y, X[\mathcal{N}]).$$

Further, an object $S$ is $\mathcal{N}$-spherical if $\mathcal{H}om^{\mathbb{Z}^2}(S, S) = k \oplus k[-\mathcal{N}]$ and (3.1) holds functorially for $X = S$ and $Y$ in $\mathcal{D}$, where

$$\mathcal{H}om^{\mathbb{Z}^2}(X, Y) := \bigoplus_{m,l \in \mathbb{Z}} \mathcal{H}om(X, Y[m + l\mathcal{X}]).$$

By Lemma 3.5, if $\Theta_{\mathcal{A}} \cong \mathcal{A}[−\mathcal{N}]$, then $\mathcal{D}_{fd}(\mathcal{A})$ is Calabi-Yau-$\mathcal{N}$. We are in particular interested in the case when $\mathcal{N} = \mathcal{X}$ or $\mathcal{N} = N \in \mathbb{Z}$. In these two cases, there is a twist functor $\Phi_S \in \text{Aut } \mathcal{D}$ for each $\mathcal{N}$-spherical object $S$, defined by

$$\Phi_S(X) = \text{Cone} \left( S \otimes \mathcal{H}om^{\mathbb{Z}^2}(S, X) \to X \right)$$

with inverse

$$\Phi_S^\ast(X) = \text{Cone} \left( X \to S \otimes \mathcal{H}om^{\mathbb{Z}^2}(X, S)^\vee \right)[-1].$$

Note that the graded dual of a graded $k$-vector space $V = \bigoplus_{m,l \in \mathbb{Z}} V_{m,l}[m + l\mathcal{X}]$ is

$$V^\vee = \bigoplus_{m,l \in \mathbb{Z}} D V_{m,l}[-m - l\mathcal{X}].$$
3.4. Deformed Calabi-Yau completion. Let $\Theta_{\mathcal{A}}$ be the inverse dualizing complex of $\mathcal{A}$ and $\theta = \Theta_{\mathcal{A}}[X - 1]$. The Calabi-Yau-$X$ completion of $\mathcal{A}$ is the tensor DG category $\Pi_X(\mathcal{A}) = T_A(\theta)$.

Moreover, let $c$ be an element of the Hochschild homology $\text{HH}_{X-2}(\mathcal{A}) = \text{Tor}_{\mathcal{A}}^e(X-2)(\mathcal{A}, \mathcal{A})$ (3.4) or a closed morphism of degree 1 form $\Theta$ to $\mathcal{A}$ (i.e., in $\text{Hom}_{\mathcal{D}(\mathcal{A})}(\Theta, \mathcal{A}[1])$ (cf. [K1, § 5.1]).

Definition 3.6. The deformation $\Pi_X(\mathcal{A}, c)$ obtained from $\Pi_X(\mathcal{A})$ by adding $c$ to the differential is called a deformed Calabi-Yau completion of $\mathcal{A}$ with respect to $c$.

Theorem 3.7. [Y, Thm. 3.17] Suppose $\mathcal{A}$ is finitely cellular. Assume that the element $c \in \text{HH}_{X-2}(\mathcal{A})$ can be lifted to an element $\tilde{c}$ of the negative cyclic homology $\text{HN}_{X-2}(\mathcal{A})$. Then the deformed Calabi-Yau completion $\Pi_X(\mathcal{A}, c)$ is homologically smooth and Calabi-Yau-$X$.

Remark 3.8. Recently, Keller [K4] corrected some error in [K1]. As remarked in [K4], the assumption in the above theorem can be satisfied in the case of Ginzburg dga which we will deal with in Section 4.

Moreover, we prove a lemma, which is a slightly weaker Calabi-Yau-$X$ version of [K1, Lemma 4.4 and Remark 5.3].

Definition 3.9. [KQ1, Def. 7.2] A functor $\mathcal{L} : \mathcal{D}(\mathcal{A}) \to \mathcal{D}(\Pi_X(\mathcal{A}, c))$ is a Lagrangian(-$X$) immersion if

- for $L, M \in \mathcal{D}_{fd}(\mathcal{A})$, we have
  $$\text{RHom}_{\Pi}(\mathcal{L}(L), \mathcal{L}(M)) = \text{RHom}_{\mathcal{A}}(L, M) \oplus D \text{RHom}_{\mathcal{A}}(M, L)[-X].$$

In particular, it is fully faithful restricted to the finite dimensional derived categories, i.e.

$$\text{Hom}_{\mathcal{D}_{fd}(\Pi_X(\mathcal{A}, c))}(\mathcal{L}(L), \mathcal{L}(M)) = \text{Hom}_{\mathcal{D}_{fd}(\mathcal{A})}(L, M)$$

for any $L, M \in \mathcal{D}_{fd}(\mathcal{A})$.

Lemma 3.10. The canonical projection (on the first component) $\Pi_X(\mathcal{A}, c) \to \mathcal{A}$ induces a Lagrangian immersion

$$\mathcal{L} : \mathcal{D}_{fd}(\mathcal{A}) \to \mathcal{D}_{fd}(\Pi_X(\mathcal{A}, c)).$$

Moreover, the image of $\mathcal{L}$ is an $X$-baric heart of $\Pi_X(\mathcal{A}, c)$.

Proof. By definition, (3.5) holds for any (double) shifts of simple $\mathcal{A}$-modules. Denote by $\text{Sim} \mathcal{D}_{fd}(\mathcal{A})$ the set of all shifts of simple $\mathcal{A}$-modules. Note that any object $M$ in $\mathcal{D}_{fd}(\mathcal{A})$ admits a simple filtration with factors in $\text{Sim} \mathcal{D}_{fd}(\mathcal{A})$ (which is a refinement of the canonical filtration with respect to the heart $\mathcal{A} - \text{mod}$-the category of $\mathcal{A}$-modules). Thus, (3.5) follows by induction (on the numbers of factors of simple filtration of $M$ and $L$).
For the second statement: note that any object in $\mathcal{D}_{fd}(\Pi_X(A,c))$ has a filtration with factors in
$$\{\text{Sim} \mathcal{D}_{fd}(A)[m + lX]\}_{m,l \in \mathbb{Z}}.$$ Moreover, $A$ has non-negative $X$-grading. Thus $(\text{Sim} \mathcal{D}_{fd}(A)[m])_{m \in \mathbb{Z}} = \mathcal{D}_{fd}(A)$ is an $X$-baric heart. \hfill $\square$

4. \textit{q-Deformation of topological Fukaya categories}

Recall that we have a graded marked surface $S^\lambda = (S, M, Y, \lambda)$ with dual arc systems $(A, A^*)$.

4.1. \textbf{Quivers with superpotential.}

\textbf{Definition 4.1.} The $\mathbb{Z}^2$-graded quiver with superpotential $(\tilde{Q}_A, W_A)$ is defined as follows:

- the vertices in $\tilde{Q}_A$ are arcs in $A$;
- for each arrow $a: \tilde{\gamma} \to \tilde{\gamma}'$ in $\tilde{Q}_A$, add a dual arrow $a^*: \tilde{\gamma}' \to \tilde{\gamma}$, where their degrees in $\tilde{Q}_A$ are
  $$\begin{aligned}
  \text{deg } a &= \text{deg } a, \\
  \text{deg } a^* &= (2 - X) - \text{deg } a;
  \end{aligned}$$
- for each $A$-polygon $D$, let its edges be $\tilde{\gamma}_1, \ldots, \tilde{\gamma}_m$ in clockwise order (and follows by the open boundary arc of $D$). Then there are graded arrows
  $$\begin{aligned}
  \tilde{\gamma}_i \xrightarrow{a_{ij}} \tilde{\gamma}_j \quad &\text{for } 1 \leq i < j \leq m - 1.
  \end{aligned}$$
  in $\tilde{Q}_A$ and let (composing from left to right)
  $$W_D = \sum_{1 \leq i < j < k \leq m} a_{ij}a_{jk}a_{ik}^*; \quad (4.1)$$
- there is also a loop $\tilde{\gamma}^*$ at each vertex $\tilde{\gamma}$ of degree $1 - X$ in $\tilde{Q}_A$;
- the superpotential $W_A$ is the sum of $W_D$’s for all $A$-polygons $D$.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{quiver_with_potential.png}
\caption{From the quiver with relation to the double graded quiver with potential}
\end{figure}
Lemma 4.2. The superpotential $W_A$ defined as above is homogeneous of degree $3 - X$.

Proof. It suffices to show that, for any $A$-polygon $D$, $W_D$ is homogeneous of degree $3 - X$. Since $A$ is full formal, there is a unique marked point $Y_D$ in $Y$ contained in $D$. Then the closed arcs in $A^*$ starting at $Y_D$, denoted by $\tilde{\eta}_1, \ldots, \tilde{\eta}_m$ (in clockwise order, cf. Figure 2.1 for $m = 4$), are the dual arcs of edges $\tilde{\gamma}_1, \ldots, \tilde{\gamma}_m$ of $D$. Note that an arc appears twice in $\{\tilde{\eta}_i\}$ if it is a loop based at $Y$. Suppose that the indices of intersection at $Y$ between graded arrows are $d_{ij} = \text{ind}_Y(\tilde{\eta}_i, \tilde{\eta}_j)$, for $1 \leq i < j \leq m - 1$.

By the composition rule, we have
\[
\deg d_{ij} = \sum_{k=i}^{j-1} d_{kk+1}
\] (4.2)
for any $1 \leq i < j \leq m$. Hence the corresponding arrow $a_{ij}$ in $Q_A$ has degree
\[
\deg a_{ij} = 1 - d_{ij},
\] (4.3)
and we have
\[
\begin{align*}
\tilde{\deg} a_{ij} + \tilde{\deg} a_{jk} + \tilde{\deg} a_{ik}^* &= (1 - d_{ij}) + (1 - d_{jk}) + ((2 - X) - (1 + d_{ik})) \\
&= 3 - X
\end{align*}
\]
for $1 \leq i < j < k \leq m$, as required. \hfill \square

4.2. Calabi-Yau-$X$ and cluster-$X$ categories of surfaces.

Definition 4.3. The Calabi-Yau-$X$ Ginzburg $\mathbb{Z}^2$-dg algebra $\Gamma_A^X$ is defined as follows:

- the underlying graded algebra of $\Gamma_A^X$ is the completion of the graded path algebra $k\tilde{Q}_A$;
- the differential $d = d_A$ of degree 1 is the unique continuous linear endomorphism satisfying $d^2 = 0$ and the Leibniz rule (with respect to degree 1) and it takes the following values:
  \[ d \quad \text{for} \quad a \in (\tilde{Q}_A)_1; \]
  \[ d \sum_{\gamma \in (\tilde{Q}_A)_0} \gamma^* = \sum_{a \in (\tilde{Q}_A)_1} [a, a^*]. \]

Note that we have the following convention:

- $\partial abc = (-1)^{\deg c} ab + (-1)^{\deg a bc} + (-1)^{\deg b ca}$ for any term $abc$ in $W_A$;
- $[a, a^*] = (-1)^{\deg a a^*} a + (-1)^{\deg a^* a} a$;

to ensure $d^2 = 0$.

Let $\mathcal{D}(\Gamma_A^X)$ be the derived category of $\mathbb{Z}^2$-dg $\Gamma_A^X$-modules, $\mathcal{D}_{fd}(\Gamma_A^X)$ the finite dimensional derived category and $\text{per} \Gamma_A^X$ the perfect derived category. They satisfy $\mathcal{D}_{fd}(\Gamma_A^X) \subset \text{per} \Gamma_A^X \subset \mathcal{D}(\Gamma_A^X)$. 
Equivalently, we have the following. Let $\mathcal{R}_A$ be the discrete $k$-category associated to the vertex set of $Q_A$ and $\mathcal{A}_A$ the path category of $Q_A$. Then $\Gamma_A^X$ is the tensor category over $\mathcal{R}_A$ of the bimodule

$$\tilde{Q}_A = Q_A \oplus (Q_A)^\vee [X-2] \oplus \mathcal{R}_A[X-1].$$

Denote by $c_A$ the image of $W_A$ in (3.4).

**Theorem 4.4.** [K3, Prop. 6.3] [Y, Thm. 3.17] The Ginzburg dga $\Gamma_A^X$ is quasi-isomorphic to the deformed Calabi-Yau completion $\Pi_X(A_A, c_A)$. Hence it is homologically smooth and Calabi-Yau-X.

Moreover, we obtain the following statement by applying Lemma 3.10.

**Corollary 4.5.** The projection $\Pi_X(A_A, c) \to A_A$ induces a Lagrangian immersion

$$\mathcal{L}_A : D_{fd}(A_A) \to D_{fd}(\Gamma_A^X)$$

and its image is an $X$-baric heart of

$$D_X(A) : = D_{fd}(\Gamma_A^X).$$

By abuse of notations, we will identify $D_{fd}(A_A) = D_\infty(A_A)$ sometimes.

In the case for $D_X = D_X(A)$, we will study the principal part of the spaces of open/closed $q$-stability conditions.

**Definition 4.6.** Denote by $QStab^\circ D_X(A)$ the subspace of $QStab D_X(A)$ consisting of closed $q$-stability conditions $\Psi(\sigma)$ for $\Psi \in ST D_X(A)$ and $\sigma$ is induced from some triple $(D_\infty(A), \hat{\sigma}, s)$ as in Theorem 1.10 for some $\hat{\sigma} \in \text{Stab}^\circ D_\infty(A)$ and $s \in \mathbb{C}$. Similarly, denote by $QStab^\circ D_X(A)$ the open version.

Applying the proof of [IQ, Thm. 6.7], we have the corresponding statement.

**Theorem 4.7.** There is a canonical triangle equivalence between the topological Fukaya category

$$\text{TFuk}(S^\lambda) : = \text{per } A_A$$

and the cluster-$X$ category

$$\mathcal{C}(\Gamma_A^X) : = \text{per } \Gamma_A^X / D_{fd}(\Gamma_A^X).$$

4.3. Graded decorated marked surfaces. We fix the dual arc systems $(A, A^*)$ on $S^\lambda$.

**Definition 4.8.** We introduce the following notions.

- The **decorated marked surface** (DMS) $S_\Delta$ of $S$ is obtained from $S$ by decorating a set $\Delta = \{Z_i\}_{i=1}^\infty$ of points in the interior of $S$, where $|\Delta| = |Y| = |M|$.
- A **(topological) cut** $c = \{c_i\}$ is a set of curves on $S$, pairing (connecting) points in $\Delta$ and $Y$ with no intersections or self-intersections. See green arcs in Figure 4.2.
- A **grading** $\Lambda$ on $S_\Delta$ (with respect to $c$) is a class in $H^1(F^T(S \setminus \Delta), \mathbb{Z})$, with values $- (1,0)$ on each (clockwise) loop $\{p\} \times \mathbb{R}P^1$ on $T_p(S \setminus \Delta)$ for $p \notin \Delta$. 


We have the following characterization of cuts.

Proof. First, suppose that there is an element \( b(c) = c \) such that \( b \in \text{SBr}(S_\Delta) \) and \( c \) is a cut on \( S_\Delta \). As \( \text{SBr}(S_\Delta) = \text{MCG}(S_\Delta) / \text{MCG}(S) \), we can assume that \( b \) acts as the identity on \( S_\Delta - c \) and hence \( b \) is the identity on \( S_\Delta \). Second, for any two cuts \( c_1 \) and \( c_2 \), one can construct an element \( b \in \text{SBr}(S_\Delta) \), satisfying \( b(c_1) = b(c_2) \), as follows:

- Take a homeomorphism/diffeomorphism \( b_1 : S_\Delta \to S_\Delta \) that acts as identity outside the neighbourhood of \( c_1 = \{ c_1^1 \} \) and pulls decorations \( \Delta = \{ Z_i \} \) along \( \{ c_1^1 \} \) to the neighbourhood of \( Y = \{ Y_i \} \);
- Then take another homeomorphism \( b_2 : S_\Delta \to S_\Delta \) that moves the decorations (around \( Y \)) on the arcs \( c_2 = \{ c_2^2 \} \);
- Finally, take a homeomorphism \( b_2 : S_\Delta \to S_\Delta \) that moves the decorations along \( \{ c_2^2 \} \) back to \( \Delta \).

The composition \( b_2 \circ b_2 \circ b_1 \) gives the required \( b \). This completes the proof. \( \square \)
4.4. Topological log surfaces. To understand the grading of arcs/curves on $S_{\Delta}$, we introduce log surface. In Section 6.4, we will discuss an alternative construction of log surface. Such a surface can be viewed as the $q$-deformation of $S_{\Delta}$.

**Definition 4.10.** The (topological) log surface $\log c S_{\Delta}^\lambda$ of $S_{\Delta}$ with respect to $c$ is constructed as follows:

- For every $m \in \mathbb{Z}$, take a copy of $S_{\Delta}$, denoted by $S_{\Delta}^{(m)}$, with the cut $c^{(m)}$. Cut it along $c_i^{(m)} \in c^{(m)}$ from the decorating point.
- Gluing $c_i^{(m)}$ with $c_i^{m+1}$ for $m \in \mathbb{Z}$ and $c_i \in c$, we obtain $\log c S_{\Delta}^\lambda$.
- $\log c S_{\Delta}^\lambda$ inherits a ($Z$-)grading $\Lambda_1 \in H^1(\mathcal{P}T(\log c S_{\Delta}^\lambda \setminus \Delta), \mathbb{Z})$ from $\Lambda$, which is the projection of $\Lambda$ to the first $\mathbb{Z}$-coordinate on each sheet.
- There is a deck transformation $q$ such that $q(S_{\Delta}^{(m)}) = S_{\Delta}^{(m+1)}$, which is also known as the Adams grading of $\log c S_{\Delta}^\lambda$.
- Denote by $\pi_\Delta : \log c S_{\Delta}^\lambda \to S_{\Delta}$ the covering map. For any (graded) curve on $S_{\Delta}$ (and is in a minimal position with respect to $c$), there are $\mathbb{Z}$ lifts on $\log c S_{\Delta}^\lambda$ via the deck transformation $q$. Denote by $\text{CA}(\log c S_{\Delta}^\lambda)$ and $\text{CA}(\log c S_{\Delta}^\lambda)$ the set of all lifts of $\text{CA}(S_{\Delta})$ and $\text{CA}(S_{\Delta})$ respectively. Similarly for $\text{AC}(\log c S_{\Delta}^\lambda)$ and $\text{AC}(\log c S_{\Delta}^\lambda)$.
- The grading of $\log c S_{\Delta}^\lambda$ inherits from the first grading $\Lambda_1$ of $\Lambda$ on $S_{\Delta}$, which will be denoted by $\log \lambda$.

The mapping class group $\text{MCG}_*(S_{\Delta})$ acts canonically on $\log c S_{\Delta}^\lambda$ via the action on each sheet. Denote by $\text{MCG}^c(\log c S_{\Delta}^\lambda)$ the mapping class group of $\log c S_{\Delta}^\lambda$, which is generated by $q$ and homeomorphisms induced from $\text{MCG}(S_{\Delta})$. Clearly, $q$ is in the center of $\text{MCG}^c(\log c S_{\Delta}^\lambda)$.

A cut $c$ is compatible with $\mathbf{A}$ if they do not intersect. From now on, we will fix the log surface $\log c S_{\Delta}^\lambda$ with respect to an initial cut $c$ that is compatible with the initial arc system $\mathbf{A}$ of $\tilde{S}^\lambda$. In fact, $c$ is uniquely determined by $\mathbf{A}$. Moreover, the (open full formal) arc system $\mathbf{A}$ is then also an arc system of $S_{\Delta}$.

The lifts of graded closed arcs in $\log c S_{\Delta}^\lambda$ provide a topological model for $\mathcal{D}_X(\mathbf{A})$. More precisely, we have the following.

**Theorem 4.11.** [IQZ, Thm. A] There is a map

$$\tilde{X}_X : \tilde{\text{AC}}(\log c S_{\Delta}^\lambda) \to \text{Ind} \mathcal{D}_X(\mathbf{A}).$$

(4.7)

Then restricted to $\tilde{\text{CA}}(\log c S_{\Delta}^\lambda)$, $\tilde{X}_X$ gives a bijection between the set of graded closed arcs and the set of reachable spherical objects (denoted by $\text{Sph} \mathcal{D}_X(\mathbf{A})$). Let $\text{ST} \mathcal{D}_X(\mathbf{A})$ be the spherical twists group, the subgroup of $\text{Aut} \mathcal{D}_X(\mathbf{A})$ generated by spherical twist $\Phi_S$ of $S \in \text{Sph} \mathcal{D}_X(\mathbf{A})$. Then $X$ induces an isomorphism

$$\iota_X : \text{BT}(S_{\Delta}) \cong \text{ST} \mathcal{D}_X(\mathbf{A}),$$

(4.8)

sending the braid twist $B_\eta$ of a closed arc $\eta$ to the spherical twist $\Phi_{\tilde{X}_X(\eta)}$. 


Furthermore, similar to [BS, Thm. 9.9] (cf. [KQ2, § 4.4]), we can consider a quotient group $\text{Aut}^e \mathcal{D}_X(A)$ of certain subgroup of $\text{Aut} \mathcal{D}_X(A)$, that fits in the short exact sequence

$$0 \rightarrow \text{ST} \mathcal{D}_X(A) \rightarrow \text{Aut}^e \mathcal{D}_X(A) \rightarrow \text{MCG}(S) \rightarrow 0.$$  

**Remark 4.12.** The numerical data $\mathcal{N}(\log_c S^\lambda_\Delta)$ of $\log_c S^\lambda_\Delta$ is given by $\mathcal{N}(S^\lambda)$ in Definition 2.2.

![Figure 4.2. Constructing DMS $S_\Delta$ via pulling closed marked points inside](image)

**4.5. Topological immersion.** The DMS $S_\Delta$ can be thought as obtained from $S$ by pulling the closed marked points along $c$ that become decorations, cf. Figure 4.2. This operation induces a map

$$\text{im}_c : \overline{\text{CA}}(S^\lambda) \rightarrow \overline{\text{CA}}(S^\lambda_\Delta) \subset \overline{\text{CA}}(\log_c S^\lambda_\Delta)$$

in [IQZ, (6.2)], cf. [IQZ, Fig. 24], that takes a graded closed arc in $S^\lambda$ to a graded closed arc in the 0-sheet $S^\lambda_\Delta$ of $\log_c S^\lambda_\Delta$.

Moreover, we have the following topological realization of Lagrangian immersion.

**Theorem 4.13.** [IQZ, Thm. 6.6] The following commutative diagram commutes

$$\begin{array}{ccc}
\overline{\text{CA}}(S^\lambda) & \xrightarrow{\text{im}_c} & \overline{\text{CA}}(\log_c S^\lambda_\Delta) \\
\bar{x}_\infty & \Downarrow & \bar{x}_\infty \\
\text{Ind} \mathcal{D}_\infty(A) & \xrightarrow{\mathcal{L}_A} & \text{Ind} \mathcal{D}_X(A).
\end{array}$$

Also notice that the full formal closed arc system $A^\ast$ on $S^\lambda$, dual to $A$, becomes the closed arc system

$$A^\Delta : = \text{im}_c(A^\ast)$$

on $S_\Delta$, still dual to $A$ (regarded on $S_\Delta$).
Remark 4.14. Similar to Remark 2.5, graph duality between $A_\Delta^*$ and $A$ corresponds to the simple-projective duality. More precisely, $\tilde{Q}_A$ can be obtained from the Ext-quiver $Q^{\text{Ext}}(\Gamma^X_A)$ by the degree changing: $d \mapsto 1 - d$. And $Q^{\text{Ext}}(\Gamma^X_A)$ can be identified with the intersection quiver of $A_\Delta^*$.

Furthermore, the Ext-quiver $Q^{\text{Ext}}(\Gamma^X_A)$ is the Calabi-Yau $X$-double, in the sense of $[KQ1, \text{Def. 6.2}]$, of the Ext-quiver $Q^{\text{Ext}}(A_0^\Delta)$.

Part 2. Geometric Part

5. Quadratic differentials on Riemann surfaces

In this section, we review the theory of Bridgeland-Smith [BS] (cf. [KQ2]) and Haiden-Katzarkov-Kontsevich [HKK], that the spaces of stability conditions on Fukaya type categories associated to Riemann surfaces can be realized by the moduli spaces of framed quadratic differentials (of certain type) on the Riemann surfaces.

5.1. Preliminaries. Let $S$ be a Riemann surface and $\omega_S$ its holomorphic cotangent bundle. A meromorphic quadratic differential $\phi$ on $S$ is a meromorphic section of the line bundle $\omega_S^2$. In terms of a local coordinate $z$ on $S$, such a $\phi$ can be written as $\phi(z) = g(z) \, d\, z \otimes \omega_S^2$, where $g(z)$ is a meromorphic function. Denote by $\text{Zero}_j(\phi)$ the set of zeroes of $\phi$ of order $j$ and $\text{Zero}(\phi) = \bigsqcup_{j>0} \text{Zero}_j(\phi)$. Similar for the set $\text{Pol}(\phi) = \bigsqcup_{j>0} \text{Pol}_j(\phi)$ of poles. Set $\text{Crit}(\phi) := \text{Zero}(\phi) \cup \text{Pol}(\phi)$.

At a point of $S^o = S \setminus \text{Crit}(\phi)$, there is a distinguished local coordinate $\omega$, uniquely defined up to transformations of the form $\omega \mapsto \pm \omega + \text{const}$, with respect to which one has $\phi(\omega) = d\, \omega \otimes \omega^2$. In terms of a local coordinate $z$, we have $w = \int \sqrt{g(z)} \, d\, z$. A quadratic differential $\phi$ on $S$ determines the $\phi$-metric on $S^o$, defined locally by pulling back the Euclidean metric on $\mathbb{C}$ via the distinguished coordinate $\omega$. Therefore, there are geodesics on $S^o$ and each geodesic has a constant phase, with respect to $\omega$.

Next, we summarize the global structure of the horizontal trajectories and horizontal strip decompositions (cf. [BS, § 3] and [HKK, § 2]).

Definition 5.1. A horizontal trajectory of $\phi$ on $S^o$ is a maximal horizontal geodesic $\gamma : (0,1) \to S^o$ with respect to the $\phi$-metric. The horizontal trajectories of a meromorphic quadratic differential $\phi$ provide the horizontal foliation on $S$.

The following are types of trajectories of a quadratic differential $\phi$ we are interested in:

- saddle trajectories with both ends in $\text{Zero}(\phi)$;
- separating trajectories with one end in $\text{Zero}(\phi)$ and the other in $\text{Pol}(\phi)$;
- generic trajectories with both ends in $\text{Pol}(\phi)$;
- closed trajectories that are simple closed curves;
- recurrent trajectories that are recurrent in at least one direction.

Definition 5.2. By removing all horizontal separating trajectories (which are finitely many) from $S^o$, the remaining open surface splits as a disjoint union of connected components. Each component is one of the following types:
• a half-plane, i.e., it is isomorphic to $\{ z \in \mathbb{C} : \text{Im}(z) > 0 \}$ equipped with the differential $dz^2$. It is swept out by generic trajectories which connect a fixed pole to itself.

• a horizontal strip, i.e., it is isomorphic to $\{ z \in \mathbb{C} : a < \text{Im}(z) < b \}$ equipped with the differential $dz^2$. It is swept out by generic trajectories which connect two (not necessarily distinct) poles.

• a ring domain, i.e., it is isomorphic to $\{ z \in \mathbb{C} : a < |z| < b \} \subset \mathbb{C}^\ast$ equipped with the differential $r \, dz^2/z^2$ for some $r \in \mathbb{R}_{<0}$. It is swept out by closed trajectories.

• a spiral domain, defined to be the interior of the closure of a recurrent trajectory.

We call this union the horizontal strip decomposition of $S$ with respect to $\phi$.

A quadratic differential $\phi$ on $S$ is called saddle-free, if it has no saddle trajectory. Note the following:

• In each horizontal strip, the trajectories are isotopy to each other.

• If $\phi$ is saddle-free, then $\text{Pol}(\phi)$ must be nonempty and, by [BS, Lemma 3.1], $\phi$ has no closed or recurrent trajectories. Thus, in the horizontal strip decomposition of $S$ with respect to $\phi$, there are only half-planes and horizontal strips.

• If $\phi$ is saddle-free, then the boundary of any strips consists of separating trajectories.

• If $\phi$ is saddle-free, then there is a unique geodesic in each horizontal strip, the saddle connection, connecting the two zeroes on its boundary.

5.2. Classical type of singularities in the Calabi-Yau-3 setting.

Definition 5.3. A classical type of singularity $p$ of a meromorphic quadratic differential has the local form

$$z^k g(z) \, dz^2,$$

where $g(z)$ is some non-zero holomorphic function and

- if $k > 0$, then $p$ is a zero of order $k$;
- if $k < 0$, then $p$ is a pole of order $k$.

The foliation near a singularity $p$ is shown in Figure 5.1, where $p$ is a zero of order 1 and 2 in the left two pictures and $p$ is a pole of order 3 and 4 in the right two pictures.

Figure 5.1. Foliations near classical type of zeroes/poles
Next, we briefly review Bridgeland-Smith’s result (cf. [BS, KQ2]). For simplicity, we will not consider the case when the pole has order 1 or 2.

**Definition 5.4.** A *GMN differential* on $S$ is a meromorphic differential with classical type of singularities satisfying:

- all the zeroes are simple zeroes (i.e., of order 1);
- the order of any pole is at least 3.

The *real (oriented) blow-up* of $(S, \phi)$ is a differentiable surface $S^\phi$ obtained from the underlying differentiable surface by replacing a pole $P \in \text{Pol}(\phi)$ with order $\text{ord}_\phi(P) > 3$ by a boundary $\partial P$, where the points on the boundary correspond to the real tangent directions at $P$. We mark the points on $\partial P$ that correspond to the distinguished tangent directions. Thus there are $\text{ord}_\phi(P) - 2$ (open) marked points on $\partial P$. We denote by $S^\phi$ the (open) marked surface associated to $(S, \phi)$, that consists of the surface $S^\phi$ together with those marked points. The *decorated marked surface* $S^\phi_{\text{Zero}(\phi)}$ associated to $(S, \phi)$ is obtained from $S^\phi$ by decorating the set $\text{Zero}(\phi)$ of zeroes of $\phi$.

**Example 5.5.** Definition 5.4 fits perfectly with triangulated surfaces in the theory of cluster algebras [FST]. For instance, the foliation of a quadratic differential on $\mathbb{P}^1$ with type $(1, 1, 1, -7)$, i.e., it has three simple zeroes and one order 7 pole, is shown in Figure 5.2. The left picture is the real blow-up part and the right picture is the neighbourhood of the pole. Note that, in the pictures,

- the blue vertices are poles or marked points;
- the red vertices are simple zeroes;
- the green arcs are geodesics;
- the black arcs are separating trajectories;
- the red arcs are the saddle connections in the horizontal strips.

Observe that the isotopy classes of separating trajectories in the left picture of Figure 5.2 form a *WKB-triangulation* of the marked surface (a pentagon in this case). In general, such a WKB-triangulation will be formed provided the GMN differential is saddle-free.

Let $S^3_{\Delta}$ be a decorated marked surface with $M$ the set of open marked points and $\Delta$ the set of decorations in the interior. Its numerical data satisfies (cf. [Q2, Def. 3.1])

$$|\Delta| = 4g + 2|\partial S| + |M| - 4 \quad (\geq 1),$$

(5.2)

where $g$ is the genus of $S$.

**Definition 5.6.** A *$S^3_{\Delta}$-framed quadratic differential* $(S, \phi, \psi)$ consists of a GMN differential $\phi$ on a Riemann surface $S$ together with a diffeomorphism $\psi: S^3_{\Delta} \to S^\phi_{\text{Zero}(\phi)}$ preserving the marked/decorating points setwise. Two $S^3_{\Delta}$-framed quadratic differentials $(S_1, \phi_1, \psi_1)$ and $(S_2, \phi_2, \psi_2)$ are equivalent if

- there exists a biholomorphism $f: S_1 \to S_2$ such that $f^*(\phi_2) = \phi_1$ and,
- $\psi_2^{-1} \circ f_* \circ \psi_1 \in \text{Homeo}_0(S^3_{\Delta})$, where $f_*: (S_1)^{\phi_1}_{\text{Zero}(\phi_1)} \to (S_2)^{\phi_2}_{\text{Zero}(\phi_2)}$ is the induced homeomorphism. Recall that $\text{Homeo}_0$ consists of isotopy classes of homeomorphisms that are isotopy to identity.
We denote by $\text{FQuad}_3(S^3_{\Delta})$ the moduli space of $S^3_{\Delta}$-framed quadratic differentials. For $S^3_{\Delta}$, there is a canonical associated Calabi-Yau-3 category $\mathcal{D}_3(S^3_{\Delta})$, which can be constructed as follows.

- Take any triangulation $T$ of $S = S^9$ (or a WKB-triangulation mentioned above) and consider the corresponding quiver with potential $(Q_T, W_T)$, where
  - the vertices of $Q_T$ correspond to the open arcs in $T$;
  - the arrows of $Q_T$ correspond to the angles of triangles in $T$;
  - the terms/cycles of $W_T$ correspond to triangles in $T$.
- Let $\Gamma_T$ be the Ginzburg dg algebra of $(Q_T, W_T)$ and then $\mathcal{D}_3(S^3_{\Delta})$ is the finite dimensional derived category of $\Gamma_T$.

Recall that a triangulation is a maximal compatible collection of open arcs on $S$. Moreover, [BQZ, Thm. A] shows that $\mathcal{D}_3(S^3_{\Delta})$ is unique (up to natural transformation) in a suitable sense. Furthermore, $\mathcal{D}_3(S^3_{\Delta})$ can be embedded into the derived Fukaya category of a symplectic 3-fold, constructed from fibrations of $S$ via a quadratic differential [S].

Let $\tilde{S}^3_{\Delta}$ be the branched double cover (spectral cover) of $S^3_{\Delta}$, branching at $\Delta$. We finish this subsection with the following result, whose original version/idea is due to [BS], while the quoted upgraded version is taken from [KQ2].

**Theorem 5.7.** [BS, KQ2] There is an isomorphism

$$\chi_3: \hat{\mathcal{H}}(S^3_{\Delta}) \rightarrow K(\mathcal{D}_3(S^3_{\Delta})), \tag{5.3}$$

where the hat homology $\hat{\mathcal{H}}(S^3_{\Delta}) = H_1(\tilde{S}^3_{\Delta}; \mathbb{Z})$ is the anti-invariant part of $H_1(\tilde{S}^3_{\Delta}; \mathbb{Z})$ with respect to the covering involution (cf. Definition 6.11 for the precise definition in the general setting). Moreover, there is an isomorphism $\Xi_3$ of complex manifolds that
fits into the commutative diagram

\[
\begin{array}{ccc}
\text{FQuad}^\circ(S^3_\Lambda) & \xrightarrow{\Xi_3} & \text{Stab}^\circ D_3(S^3_\Lambda) \\
\Pi_3 & & \downarrow Z_3 \\
\text{Hom}(\hat{H}(S^3_\Lambda), C) & \xrightarrow{\chi_3} & \text{Hom}(K(D_3(S^3_\Lambda)), C),
\end{array}
\]

such that the period map \(\Pi_3: \phi \mapsto \int \sqrt{\phi}\) becomes the map \(Z_3\) (i.e., central charge). More precisely, (0.3) holds for

\[
X = X_3: \text{CA}(S^3_\Lambda) \to \text{Sph} D_3(S^3_\Lambda)/[1] (5.5)
\]

in [Q2, Thm. 6.6], where \(\text{CA}(S^3_\Lambda)\) the set of closed arcs in \(S^3_\Lambda\) and \(\text{Sph} D_3(S^3_\Lambda)\) the set of reachable spherical objects in \(D_3(S^3_\Lambda)\).

Furthermore, for \(\sigma = \Xi_3(\phi)\), the map \(X_3\) induces one-one correspondence between saddle connections of \(\phi\) on \(S^3_\Lambda\) and \(\sigma\)-semistable objects in \(D_3(S^3_\Lambda)\).

Here, \(^\circ\) in \(\text{FQuad}^\circ\) and \(\text{Stab}^\circ\) means taking the principal component (cf. [KQ2, § 4]).

We also have the moduli space of \(S^3\)-framed quadratic differentials such that

\[
\text{FQuad}(S^3) = \text{FQuad}^\circ(S^3_\Lambda)/\text{BT}(S^3_\Lambda) \cong \text{Stab} D_3(S^3_\Lambda)/\text{ST} D_3(S^3_\Lambda), (5.6)
\]

noticing that (5.5) induces the isomorphism between \(\text{BT}(S^3_\Lambda)\) and \(\text{ST} D_3(S^3_\Lambda)\), cf. [Q2, Thm.1], which is the Calabi-Yau-3 version of (4.8).

5.3. Exponential type singularities in the Calabi-Yau-∞ setting.

**Definition 5.8.** [HKK] The exponential type singularity \(p\) of index \((k, l) \in \mathbb{Z}_{>0} \times \mathbb{Z}\) of a quadratic differential \(\phi\) on a Riemann surface \(S\) is a singularity \(p\) with local coordinate as

\[
e^{-k z} z^{-l} g(z) \, d z^2 (5.7)
\]

for some non-zero holomorphic function \(g(z)\).

A flat surface \((S, \phi)\) is a Riemann surface \(S\) together with quadratic differential \(\phi\) whose singularities are all of exponential type as above.

By [HKK, Prop. 2.5], for a singularity \(p\) in \(S^\phi\) with local coordinate (5.7), there are exactly \(k\) additional points, with respect to the \(\phi\)-metric completion in the neighborhood of \(p\); all of which are \(\infty\)-angle singularities, also known as the conical singularity with infinity angle described as below.

**Definition 5.9.** Consider the spaces

\[
C_N: = \mathbb{Z}_N \times \mathbb{R} \times \mathbb{R}_{\geq 0}/ \sim, \quad (k, x, 0) \sim (k + 1, -x, 0) \text{ for } x \leq 0 (5.8)
\]

for \(N \in \mathbb{Z}_{>0}\) and

\[
C_\infty: = \mathbb{Z} \times \mathbb{R} \times \mathbb{R}_{\geq 0}/ \sim, \quad (k, x, 0) \sim (k + 1, -x, 0) \text{ for } x \leq 0. (5.9)
\]

The origin \(0 = (0, 0, 0) \sim (k, 0, 0)\) is a distinguished point in these spaces. The standard flat structure of \(\mathbb{R}^2\) makes \(C_\infty \setminus \{0\}\) a smooth flat surface with metric completion \(C_\infty\). The
origin is a conical singularity with $N\pi$ angle for $C_N$, $N \neq 2$ and is a conical singularity with infinity angle for $C_\infty$.

Note that the conical singularities with $N\pi$ angle are exactly classical type zeroes of order $N - 2$ mentioned above, for $N \geq 3$. For GMN differentials (i.e., in BS’ setting), we always have $N = 3$. We will discuss the $N \geq 3$ case in Section 8. Furthermore, the local behavior in the origin of $C_\infty$ is similar to zeroes instead of poles since a trajectory attending to such points has finite length.

Let $S_{sg}$ be the set of singularities of the flat surface $S^\phi$ (where the smooth part of $S^\phi$ is just $S$). We have the following notions.

- A saddle connection is a maximal geodesic converging towards closed points in $S_{sg}$ (i.e., zeroes or points in the metric completion) in both directions. Note that saddle trajectories are saddle connections which are horizontal. Also, the endpoints are not required to be distinct.
- The core $\text{Core}(\phi)$ is the convex hull of $S_{sg}$ by [HKK, Prop. 2.2], which contains all saddle connections. Moreover, $\text{Core}(\phi)$ is a deformation retract of $S^\phi$ provided that each point on $S^\phi$ has finite distance to $\text{Core}(\phi)$ (with respect to the $\phi$-metric), cf. [HKK, Prop. 2.3].

**Definition 5.10.** The graded marked surface $S^\phi$ associated to a flat surface $(S, \phi)$ is the real blow-up of $S$ at all the singularities of $\phi$. So

- Each boundary component of $S^\phi$ corresponds to the blow-up of a singularity $p$ of exponential type $(k, l)$.
- The closed marked points correspond to the additional $k$ points with respect to $\phi$-metric completion mentioned above.
- The open marked points correspond to the directions between closed marked points (hence are dual to closed marked points on the boundaries).
- The grading is given by the horizontal foliation.

**Remark 5.11.** In fact, any graded marked surface $(S, M, Y, \lambda)$ in Section 2 is the real blow-up of some flat surfaces with conical singularities of infinity angle.

**Definition 5.12.** Let $S^\lambda$ be a graded marked surface with the numerical data

$$\mathfrak{M}(S^\lambda) = (g, b; k; l; LP_g).$$

A $S^\lambda$-framed quadratic differential $(S, \phi, h)$ on $S^\lambda$ consists of a quadratic differential $\phi$ on some Riemann surface $S$ together with a diffeomorphism $h: S^\lambda \to S^\phi$ such that the numerical data of the grading $h^*(\phi)$ on $S^\lambda$ (the pull-back of the $\phi$-induced grading on $S$) equals $\mathfrak{M}(S^\lambda)$. Two $S^\lambda$-framed quadratic differentials $(S_1, \phi_1, h_1)$ are equivalent if

- there exists a biholomorphism $f: S_1 \to S_2$ such that $f^*(\phi_2) = \phi_1$;
- $h_2^{-1} \circ f_\ast \circ h_1 \in \text{Diff}_0(S^\lambda)$, where $f_\ast: S_1^{\phi_1} \to S_2^{\phi_2}$ is the induced diffeomorphism.

Denote by $\text{FQuad}_\infty(S^\lambda)$ the space of $S^\lambda$-framed quadratic differentials on $S^\lambda$. For simplicity, we will only mention a quadratic differential $\phi$ on $S^\lambda$ (omitting $S$ and $h$).
Now we recall the main result in [HKK], where the surjection of the $\Xi_\infty$ is shown by [Ta].

**Theorem 5.13.** [HKK, Ta] There is the following identification

$$\chi_\infty : H_1(S^\lambda, \partial S^\lambda; Z_{np}) \to K(D_\infty(A)),$$

(5.10)

where $Z_{np} = Z \otimes \mathbb{Z}_2$ for $\mathbb{Z}_2$ being the canonical double cover of $S^\lambda$. Moreover, there is an isomorphism $\Xi_\infty$ of complex manifolds that fits into the commutative diagram

$$\begin{array}{ccc}
F\text{Quad}_{\infty}(S^\lambda) & \xrightarrow{\Xi_\infty} & \text{Stab} D_\infty(A) \\
\Pi_\infty & & \downarrow \cong \\
\text{Hom}(H_1(S^\lambda, \partial S^\lambda; Z_{np}), \mathbb{C}) & \xrightarrow{\chi_\infty} & \text{Hom}(K(D_\infty(A)), \mathbb{C}),
\end{array}
$$

(5.11)

such that the period map $\Pi_\infty : \phi \mapsto \int \sqrt{\phi}$ becomes the map $\mathcal{Z}_\infty$ (i.e., central charge). More precisely, (0.3) holds for $X = \tilde{X}_\infty$ in (2.8) in Theorem 2.7. Furthermore, for $\tilde{\sigma} = \Xi_\infty(\phi)$, the map $\tilde{\phi}_\infty$ induces a one-one correspondence between saddle connections of $\phi$ on $S^\lambda$ and $\tilde{\sigma}$-semistable objects in $D_\infty(A)$. 

6. **q-Deformation of quadratic differentials**

In this section, we fix a complex number $s \in \mathbb{C}$ with $\text{Re}(s) > 2$ and set $q_s = e^{i\pi s}$.

6.1. **Multi-valued quadratic differentials.** Now we introduce the $q$-deformed generalization of GMN-differentials in Section 5.2. We first define $q_s$-quadratic differentials on a disk. Fix the following notations:

- $D := \{z \in \mathbb{C} \mid |z| < 1\}$ is the unit disk;
- $D^* := D \setminus \{0\}$ is the punctured unit disk;
- $\tilde{D}^*$ is the universal cover of $D^*$.

In fact, $\tilde{D}^*$ is isomorphic to the half plane $\{w \in \mathbb{C} \mid \text{Re}(w) < 0\}$ with covering map

$$\begin{align*}
\tilde{D}^* & \to D^* \\
w & \mapsto e^w.
\end{align*}
$$

Consider a holomorphic quadratic differential $e^{(ks+l+2)w} \, dw \otimes 2$ on $\tilde{D}^*$, where $k, l \in \mathbb{Z}$. Set $z = e^w$. Then $dz = e^w \, dw$ and one can view

$$\xi = z^{ks+l} \, dz \otimes 2$$

as a multi-valued quadratic differential on $D$ ramified at 0. Let $\gamma$ be a counter-clockwise loop around 0. Since the action of the deck transformation group $\pi_1(D^*) \cong \langle \gamma \rangle$ on $\tilde{D}^*$ is given by $\gamma(w) = w + 2\pi i$, the monodromy of $\xi$ is given by $\gamma^* \xi = q_s^{2k} \xi$. 

Now we turn to general case. Let $S$ be a compact Riemann surface and $\mathcal{R} \subset S$ a set of finite points. Denote by

$$\Pi: \widetilde{S^\phi} \rightarrow S^\phi$$

(6.1)

the universal cover of $S^\phi := S \setminus \mathcal{R}$. In our setup, a \textit{multi-valued quadratic differential on $S$ ramified at $\mathcal{R}$} refers to a holomorphic quadratic differential on $\widetilde{S^\phi}$. The fundamental group $\pi_1(S^\phi, \ast)$ acts on $\widetilde{S^\phi}$ as deck transformations. We denote $\gamma^*\xi$ the pull-back of a multi-valued quadratic differential $\xi$ via the action of a deck transformation $\gamma \in \pi_1(S^\phi, \ast)$.

**Definition 6.1.** A \textit{$q_s$-quadratic differential} $\xi$ on $S$ ramified at $\mathcal{R}$ refers to a holomorphic quadratic differential on $\widetilde{S^\phi}$. The fundamental group $\pi_1(S^\phi, \ast)$ acts on $\widetilde{S^\phi}$ as deck transformations. We denote $\gamma^*\xi$ the pull-back of a multi-valued quadratic differential $\xi$ via the action of a deck transformation $\gamma \in \pi_1(S^\phi, \ast)$.

$$\gamma^*\xi = q_s^{2m(\gamma)}\xi$$

(6.2)

for some $m(\gamma) \in \mathbb{Z}$.

For a point $p \in S$, let $D_p$ be a small disk (neighbourhood) centered at $p$ with coordinate $z$. Similarly as above, $D_p^\ast = D_p \setminus \{p\}$ and $\widetilde{D_p^\ast}$ its universal cover. Then $\widetilde{D_p^\ast} \subset S^\phi$ for any $p \in \mathcal{R}$ and the pull-back

$$\Pi^{-1}(D_p^\ast) = \bigsqcup_{\alpha} \widetilde{D_p^\ast}_\alpha$$

(6.3)

is the disjoint unions of countable copies of $\widetilde{D_p^\ast}$. We say a $q_s$-quadratic differential $\xi$ has the local form

$$\xi = cz^{k+lp} \, dz \otimes 2$$

(6.4)

on $D_p$ if

$$\xi = c e^{(k+lp+2)w} \, dw \otimes 2$$

on some connected component $\widetilde{D_p^\ast}_\alpha \subset \Pi^{-1}(D_p^\ast)$, for some $c \in \mathbb{C}$ and $k, l \in \mathbb{Z}$. Note that in such a case, by Definition 6.1, $\xi$ can be written as

$$\xi = c q_s^{2m_\alpha} e^{(s-2)(k+lp+2)w} \, dw \otimes 2,$$

for some $m_\alpha \in \mathbb{Z}$, on any connected component $\widetilde{D_p^\ast}_\alpha \subset \Pi^{-1}(D_p^\ast)$.

Now we introduce the notion of zeroes and poles for $q_s$-quadratic differentials.

**Definition 6.2.** Let $\xi$ be a $q_s$-quadratic differential and assume that it has the local form (6.4) on $D_p$ for some $p \in \mathcal{R}$, where $c \in \mathbb{C}$ and $k, l \in \mathbb{Z}$. We call $p$

- a \textit{zero of order $(k, l)$} if $\text{Re}(k+lp) > 0$;
- a \textit{pole of order $-(k, l)$} if $\text{Re}(k+lp) < 0$.

And a pole $p$ of $\xi$ is a \textit{higher order pole} if $\text{Re}(k+lp) < -2$. Moreover,

- a zero $p$ of $\xi$ is called \textit{$s$-simple} if $\xi$ has the local form
  $$\xi = cz^{s-2} \, dz \otimes 2$$
  (6.5)
- a pole $p$ of $\xi$ is called an \textit{$s$-pole of type $(k, l)$} if $\xi$ has the local form
  $$\xi = cz^{-k(s-2)-l} \, dz \otimes 2.$$
Associating to a $q_s$-quadratic differential $\xi$, there is a multi-set of pairs of integers
\[(k, l) : = \{(k_1, l_1), \ldots, (k_b, l_b)\}\]  
(6.7)
defined via gathering all types of $s$-poles of $\xi$, where $b$ is the number of poles of $\xi$. We call the multi-set $(k, l)$ the $s$-polar type of $\xi$.

**Remark 6.3.** If $s \in \mathbb{Q}$, the pair of integers $(k_i, l_i)$ is not uniquely determined by the local form $\xi$ around the corresponding $s$-pole. Therefore the $s$-polar type $(k, l)$ for $\xi$ carries the additional information.

Recall that a $q_s$-quadratic differential $\xi$ defines the monodromy representation $\rho_s : \pi_1(S^\emptyset, \ast) \to \mathbb{C}^*$
\[\gamma \mapsto q_s^{2m(\gamma)},\]  
(6.8)
where $m(\gamma)$ is given by (6.2).

**Assumption 6.4.** For a $q_s$-quadratic differential $\xi$, there is a simply connected open domain $U \subset S$ such that:
- $U$ contains all ramification points $\mathcal{R}$, i.e., $\mathcal{R} \subset U$;
- the induced monodromy representation $\rho_s \circ \iota_* : \pi_1(S \setminus U, \ast) \to \mathbb{C}^*$, via the inclusion $\iota : S \setminus U \to S^\emptyset$, is trivial.

In what follows, we first introduce a special class of $q_s$-quadratic differentials and then consider their moduli spaces.

**Definition 6.5.** Let $g$ be a non-negative integer and $(k, l)$, as in (6.7), a multi-set of pairs of integers satisfying $k_i > 0$ and (2.6). A CY-$s$-type $q_s$-quadratic differential $\xi$ on genus $g$ Riemann surfaces $S$ is a $q_s$-quadratic differential satisfying:
- all zeroes are $s$-simple;
- the number of $s$-poles is $b$ and the $s$-polar type is $(k, l)$;
- all poles are higher order $s$-poles, namely,
  \[\text{Re}(k_i(s - 2) + l_i) > 2, \quad \forall i = 1, \ldots, b;\]  
(6.9)
- Assumption 6.4.

We fix the following convention of notations.
- Zero($\xi$) = $\{Z_1, \ldots, Z_\aleph\}$ is the set of $\aleph$ $s$-simple zeroes.
- Pol($\xi$) = $\{p_1, \ldots, p_b\}$ is the set of $s$-poles with $p_i$ of type $(k_i, l_i)$.

In the setting above, the set of ramification points is $\mathcal{R} = \text{Zero}(\xi) \cup \text{Pol}(\xi)$.

The next statement clarifies the relationship between the number $\aleph$ of $s$-simple zeroes of a CY-$s$ type $q_s$-quadratic differential and its $s$-polar type.

**Lemma 6.6.** Let $\xi$ be a CY-$s$-type $q_s$-quadratic differential of $s$-polar type $(k, l)$ as in (6.7). Then we have
\[\aleph = \sum_{i=1}^b k_i.\]  
(6.10)
Proof. For the case $s \notin \mathbb{Q}$, denote by $\gamma_j$ an anticlockwise loop around $Z_j$ and by $\delta_i$ an anticlockwise loop around $p_i$. Then Assumption 6.4 implies that

$$\prod_{j=1}^{\mathbb{R}} \rho_s(\gamma_j) \cdot \prod_{i=1}^{b} \rho_s(\delta_i) = 1.$$ 

Since $\rho_s(\gamma_j) = q_s^2$ and $\rho(\delta_i) = q_s^{-2k_i}$, the above equality yields

$$\left(2\mathbb{R} - 2 \sum_{i=1}^{b} k_i\right) s \in 2\mathbb{Z}.$$ 

Therefore, by the fact that $s \notin \mathbb{Q}$, we have (6.10).

For the case $s \in \mathbb{Q}$, we can assume $s = j/m$ for some $j, m \in \mathbb{Z}$. Note that $\xi^m$ is a single-valued meromorphic section of $\omega_{S}^{\otimes 2m}$ with $\mathbb{R}$ zeroes of order $m(s-2)$ and $b$ poles of orders $m(k_i(s-2) + l_i)$, for $1 \leq i \leq b$. By the Riemann-Roch theorem, we obtain

$$m(4g - 4) = m\mathbb{R}(s - 2) - m \sum_{i=1}^{b} (k_i(s - 2) + l_i)$$

which implies (6.10) noticing (2.6). This completes the proof. \hfill \Box

Remark 6.7. When $s = 3$, we can recover the GMN differential on Riemann surfaces in Definition 5.4 ([BS]). For instance, when $S$ is $\mathbb{P}^1$ with numerical data (i.e., the normal $A_2$ case) (cf. Example 2.3) $g = 0, b = 1, (k, l) = (3, 4)$,

we will get the corresponding CY-3 $A_2$ example in Example 5.5 with singularity type $(1, 1, 1, -7)$, whose foliation as shown in Figure 5.2.

6.2. Log surfaces. Continuing Section 6.1, we consider a CY-$s$ type $q_s$-quadratic differential $\xi$ on a genus $g$ Riemann surface $S$ with an $s$-polar type $(k, l)$. As before, denote by $\gamma_j$ the loop around $Z_j$ and by $\delta_i$ the loop around $p_i$ in the simply connected domain $U \subset S$.

Define the representation of $\pi_1(S^\phi, *)$ on the infinite cyclic group $\langle q \rangle \cong \mathbb{Z}$ as follows. To start with, we have a representation

$$\rho: \pi_1(U \setminus \mathcal{R}, *) \rightarrow \langle q \rangle$$

defined by setting $\rho(\gamma_j) := q$ and $\rho(\delta_i) := q^{-k_i}$. Note that for a loop $\Gamma \subset U$ encircling all points in $\mathcal{R}$, we have $\rho(\Gamma) = 1$. Then we can extend $\rho$ to the representation

$$\rho: \pi_1(S^\phi, *) \rightarrow \langle q \rangle,$$

such that

- the restriction of $\rho$ on $U \setminus \mathcal{R}$ is $\rho$,
- $\rho(\gamma) = 1$ for any loop $\gamma \in \pi_1(S \setminus U)$.

Using the representation $\rho$, we introduce logarithmic surfaces.
Definition 6.8. The logarithmic (log) surface $\log S^\phi$ associated to $\xi$ is defined to be the covering space $\pi: \log S^\phi \to S^\phi$ corresponding to the representation $\rho$. Since $\xi$ is a single-valued quadratic differential on $\log S^\phi$, the latter has a natural grading in the sense of Definition 2.1 given by the horizontal foliation of $\xi$. The infinite cyclic group $\langle q \rangle$ acts on $\log S^\phi$ as deck transformations.

We end this subsection with one more remark.

Remark 6.9. For an $s$-simple zero $Z_j$ with loop $\gamma_j$ around it, we have $\rho(\gamma_j) = q$. Taking $p = Z_j$, we know that $\Pi^{-1}(D^*_p)$ in (6.3) consists of just one copy of $\tilde{D}^*_p$. Similarly, for an $s$-pole $p_i$ of type $(k_i, l_i)$ with loop $\delta_i$ around it, we have $\rho(\delta_i) = k_i$. Taking $p = p_i$, we know that $\Pi^{-1}(D^*_p)$ in (6.3) becomes $\Pi^{-1}(D^*_p) = \prod_{i=1}^{k_i} \tilde{D}^*_p$ (6.12) consisting of $k_i$ copy of $\tilde{D}^*_p$. The deck transformation $q$ maps $\tilde{D}^*_p$ to $\tilde{D}^*_{p+1}$, where the subscript $i, i+1$ is considered modulo $k_i$.

6.3. Hat homology groups. Let $\log S^\phi$ be the log surface associated to a $q_s$-quadratic differential $\xi$ on $S$ and $\psi_s := \sqrt{\xi}$ (6.13) the square root of $\xi$. Note that $\psi_s$ is a double-valued holomorphic 1-form on $\log S^\phi$ and it defines the monodromy representation $\rho^\psi: \pi_1(S^\phi, *) \to \langle q, -q \rangle$ $\gamma \mapsto \pm q^{m(\gamma)}$, (6.14) where $\gamma^* \psi_s = \pm q^{m(\gamma)} \psi_s$. The image of $\rho^\psi$ is given by $\text{Im} \rho^\psi = \langle q \rangle$ or $\langle q, -q \rangle$.

Now we consider the Riemann surface associated to $\psi_s$.

Definition 6.10. The spectral cover $\hat{s}_\rho: \log \hat{S}^\phi \to \log S^\phi$ is defined to be the covering space corresponding to the representation $\rho^\phi$.

The holomorphic 1-form $\psi_s$ is single-valued on $\log \hat{S}^\phi$. Note that

- $\log \hat{S}^\phi$ is a double cover of $\log S^\phi$ if $\text{Im} \rho^\phi = \langle q, -q \rangle$;
- $\log \hat{S}^\phi = \log S^\phi$ if $\text{Im} \rho^\phi = \langle q \rangle$.

Let $\hat{S}^\phi$ be the covering space $\hat{s}_\rho: \hat{S}^\phi \to S^\phi$ corresponding to the representation $\pi_1(S^\phi, *) \to \{ \pm 1 \}$,
which is obtained by setting \( q \to 1 \) in the definition of \( \hat{\rho} \). The above constructions give rise to the following commutative diagram:

\[
\begin{array}{ccc}
\log \hat{S}^o & \overset{\bar{\pi}}{\longrightarrow} & \hat{S}^o \\
\downarrow \breve{\phi} & & \downarrow sp \\
\log S^o & \overset{\bar{\pi}}{\longrightarrow} & S^\phi.
\end{array}
\]

Moreover, the double cover \( sp: \hat{S}^o \to S^\phi \) can be extended to a branched double cover \( Sp: \hat{S} \to S \), branching at the poles of \( \xi \) with odd winding numbers (equivalently, with odd \( l_i \)).

To introduce an appropriate integration of \( \psi_s \) on \( \log \hat{S}^o \), we define the following homology group.

**Definition 6.11.** The *hat homology group* of \( \xi \) is defined as follows.

- If \( \log \hat{S}^o = \log S^\phi \), then
  \[
  \hat{H}(\xi) := H_1(\log S^\phi; \mathbb{Z}).
  \]
- If \( \log \hat{S}^o \) is a double cover of \( \log S^\phi \), denote by \( \tau \) the covering involution and
  \[
  \hat{H}(\xi) := H_1(\log S^\phi; \mathbb{Z})^-,
  \]

where \( H_1(\log S^\phi; \mathbb{Z})^- \) is the \( \tau \)-anti-invariant part, i.e., consists of \( \gamma \in H_1(\log \hat{S}^o; \mathbb{Z}) \) satisfying \( \tau_* \gamma = -\gamma \). The group \( \hat{H}(\xi) \) has an \( R = \mathbb{Z}[q^{\pm 1}] \)-module structure induced from the action of the deck transformation group \( \langle q, -q \rangle \) (or \( \langle q \rangle \)) on \( \log \hat{S}^o \).

**Definition 6.12.** The *period* of \( \xi \) is the integration of \( \psi_s \) on cycles in \( \hat{H}(\xi) \), which is an \( R \)-linear map:

\[
Z_\xi: \hat{H}(\xi) \to \mathbb{C}_s \\
\gamma \mapsto \int_\gamma \psi_s. \quad (6.15)
\]

Note that the construction of \( \rho \) implies that

\[
q^* \psi_s = q_s \psi_s \quad (6.16)
\]

and hence

\[
\int_{q^*\gamma} \psi_s = \int_\gamma q^* \psi_s = q_s \int_\gamma \psi_s
\]

for any cycle \( \gamma \) in \( \hat{H}(\xi) \).

The rest of the subsection is devoted to the calculation of the rank of \( \hat{H}(\xi) \).
Proposition 6.13. Let $\xi$ be a CY-$s$ type $q_s$-quadratic differential of genus $g$ and $s$-polar type $(k, l)$ as in (6.7). Then the associated hat homology group $\hat{H}(\xi)$ is a free $R$-module of rank

$$n = 2g - 2 + b + \sum_{i=1}^{b} k_i.$$ 

To prove the statement, we prepare several lemmas first. Let $B^{(m)}$ be a bouquet of $m$ circles joined at the point $\ast$, namely,

$$B^{(m)} = S^1 \lor \cdots \lor S^1.$$ 

The fundamental group of $B^{(m)}$ is a free group with $m$ generators, i.e.,

$$\pi_1(B^{(m)}, \ast) \cong \langle \gamma_1, \ldots, \gamma_m \rangle,$$

where, for $1 \leq i \leq m$, each $\gamma_i$ is a loop around the $i$-th circle of $B^{(m)}$.

Let $\rho: \pi_1(B^{(m)}, \ast) \to \langle q \rangle$ be any given surjective group homomorphism and take the covering space

$$\pi: \tilde{B}^{(m)} \to B^{(m)}$$

corresponding to $\rho$. Set $\tilde{\ast} = \pi^{-1}(\ast)$.

Lemma 6.14. The relative homology group $H_1(\tilde{B}^{(m)}, \tilde{\ast}; \mathbb{Z})$ is a free $R$-module of rank $m$ generated by the classes $[\tilde{\gamma}_1], \ldots, [\tilde{\gamma}_m]$, where $\tilde{\gamma}_i$ is a lift of the loop $\gamma_i$ around the $i$-th circle in $B^{(m)}$.

Proof. For each $1 \leq i \leq m$, take a lift $\tilde{\gamma}_i$ of $\gamma_i$ on $\tilde{B}^{(m)}$. Then all other lifts are of the forms $(q_*)^{n_i} \tilde{\gamma}_i$ for some $n_i \in \mathbb{Z}$. Due to the fact that any path which connects two points in $\tilde{\ast}$ is homotopy equivalent to the composition of lifts of $\gamma_i^{\pm 1}$, any relative homology class in $H_1(\tilde{B}^{(m)}, \tilde{\ast}; \mathbb{Z})$ is a linear combination of classes $[\tilde{\gamma}_i]$’s over $R$. It is clear that the set

$$\{ [(q_*)^{n_i} \tilde{\gamma}_i] \mid i = 1, \ldots, m, n_i \in \mathbb{Z} \}$$

is linearly independent. The claim follows. \qed

Lemma 6.15. The homology group $H_1(\tilde{B}^{(m)}; \mathbb{Z})$ is a free $R$-module of rank $m - 1$.

Proof. Consider the exact sequence

$$0 \to H_1(\tilde{B}^{(m)}; \mathbb{Z}) \to H_1(\tilde{B}^{(m)}, \tilde{\ast}; \mathbb{Z}) \to \tilde{H}_0(\tilde{\ast}; \mathbb{Z}) \to 0,$$

where $\tilde{H}_0(\tilde{\ast}; \mathbb{Z})$ is the 0-th reduced homology group of $\tilde{\ast}$. By definition and Lemma 6.14, we have $\tilde{H}_0(\tilde{\ast}; \mathbb{Z}) \cong R$ and $H_1(\tilde{B}^{(m)}, \tilde{\ast}; \mathbb{Z}) \cong R^m$. Therefore the proof completes. \qed
Proposition 6.16. Let \( \log S^\phi \) be the logarithmic surface associated to a CY-s type \( q_s \)-quadratic differential of genus \( g \) and \( s \)-polar type \( (k, l) \) as in (6.7). Then the homology group \( H_1(\log S^\phi; \mathbb{Z}) \) is a free \( R \)-module of rank

\[
 n = 2g - 2 + b + \sum_{i=1}^{b} k_i.
\]  

(6.17)

Proof. By Lemma 6.6, we have

\[
|\mathcal{R}| = b + \mathcal{N} = b + \sum_{i=1}^{b} k_i.
\]

Note that an oriented surface of genus \( g \) with \( |\mathcal{R}| \) points excluded is homotopy equivalent to a bouquet of \( 2g - 1 + |\mathcal{R}| \) circles. Therefore \( S^\phi = S \setminus \mathcal{R} \) is homotopy equivalent to the bouquet \( B^{(n+1)} \) and \( \log S^\phi \) is its covering that corresponds to \( \rho \) in (6.11). As \( \rho(\gamma_j)q \) for loop \( \gamma_j \) around any \( s \)-zero \( Z_j \), \( \rho \) is surjective. Hence the claim follows from Lemma 6.15.

Proof of Proposition 6.13. If \( \log \widehat{S}^\phi = \log S^\phi \), the statement follows from Proposition 6.16. Now consider the case when \( \log \widehat{S}^\phi \) is a double cover of \( \log S^\phi \). We first compute the rank of \( H_1(\log \widehat{S}^\phi; \mathbb{Z}) \) over \( R \). Recall that we have a underlying spectral/branched double cover \( \widehat{s}_p : \widehat{S}^\phi \to S^\phi \) for \( \widehat{S}^\phi = \widehat{S} \setminus \mathcal{R} \) and \( \mathcal{R} = \widehat{s}_p^{-1}(\mathcal{R}) \). The Riemann-Hurwitz formula implies that

\[
2\widehat{g} - 2 = 2(2g - 2) + b_1,
\]

where \( \widehat{g} \) is the genus of \( \widehat{S} \) and \( b_1 \) is the number of odd integers in \( \{l_1, \ldots, l_b\} \). In addition, we have

\[
|\mathcal{R}| = 2|\mathcal{R}| - b_1.
\]

Therefore the surface \( \widehat{S}^\phi \) is homotopy equivalent to the bouquet of \( m \) circles where

\[
m = 2\widehat{g} - 1 + |\mathcal{R}| = 4g - 3 + 2|\mathcal{R}| = 2n + 1.
\]

Then, by Proposition 6.16, the homology group \( H_1(\log \widehat{S}^\phi; \mathbb{Z}) \) is a free \( R \)-module of rank \( 2n \). Since the \( \tau \)-invariant part of \( H_1(\log \widehat{S}^\phi; \mathbb{Z}) \) can be identified with \( H_1(\log S^\phi; \mathbb{Z}) \), the rank of \( \tau \)-anti-invariant part is given by

\[
\text{rank}_R H_1(\log \widehat{S}^\phi; \mathbb{Z}) = n.
\]

6.4. Comparing log surfaces. Keep the notations as above, i.e. \( \xi \) is a CY-s type \( q_s \)-quadratic differential of genus \( g \) and \( s \)-polar type \( (k, l) \) on \( S \) ramified at

\[
\mathcal{R} = \text{Zero}(\xi) \cup \text{Pol}(\xi).
\]

As in Section 6.2, near each \( s \)-simple zero \( Z \in \text{Zero}(\xi) \), we can take a disk \( D_Z \subset S \) with center \( Z \) and the restriction of covering map \( s_p : \log S^\phi \to S^\phi \) on \( D_Z^* = D_Z \setminus \{Z\} \) is the universal covering \( \widetilde{D}_Z^* \subset \log S^\phi \). By adding a point \( \widetilde{Z} \) on \( \widetilde{D}_Z^* \) as the fiber of \( Z \), we can extend the covering \( \widetilde{D}_Z^* : \widetilde{D}_Z \to D_Z \) to a branched \( \mathbb{Z} \)-covering \( \widetilde{D}_Z \to D_Z \), branching at \( \widetilde{Z} \).
Lemma 6.17. The construction above, i.e., adding $\tilde{Z}$ can be viewed as the metric completion with respect to the $\xi$-metric on $\log S^\phi$.

Proof. Take coordinate $w \in \tilde{D}_Z^s$ and $z \in D_Z^s$ satisfying $z = e^w$. Then $\xi$ on $\tilde{D}_Z^s$ can be written as

$$\xi|_{\tilde{D}_Z^s} = (e^w)^{s-2}(d e^w)^{\otimes 2} = e^{sw} d w^{\otimes 2}.$$ 

We identify $\tilde{D}_Z^s \simeq \{w \in \mathbb{C} \mid \text{Re } w < 0\} \subset \mathbb{C} \cup \{\infty\} = \mathbb{CP}^1$.

Then due to [HKK, Prop. 2.1 and Lemma 2.5], the point $\infty$ is an exponential type singularity and we can obtain a single additional point $\tilde{Z}$ by the (partial) completion of $\tilde{D}_Z^s$ with respect to the $\xi$-metric. $\square$

Denote by $\log S^\xi_{\Delta}$ the topological surface obtained from $\log S^\phi$ by adding $\tilde{Z}_j$ for each $Z_j \in \text{Zero}(\xi)$. When there is no confusion, we will identify $\tilde{Z}_j$ with $Z_j$ and use the notation $\Delta = \text{Zero}(\xi)$.

Next we consider an $s$-pole $p \in \text{Pol}(\xi)$ of type $(k, l)$ of $\xi$, and take a disk $D_p \subset S$ with center $p$. Then the real blow-up of $D_p$ at $p$ is an annulus $A_p$ obtained by replacing $p$ with a boundary $\partial_p \simeq S^1$.

Now we consider the fiber of $D_p^s$ in $\log S^\phi$, as calculated in (6.12), consisting of $k$ connected component $\tilde{D}_p^s$. Corresponding to the real blow-up of $D_p$ at $p$, we add a real line $\mathbb{R}$ on each $\tilde{D}_p^s$ as the fiber of $\partial_p \subset A_p$ and extend the universal covering $\tilde{D}_p^s \to D_p^s$ to the following one.

Lemma 6.18. The construction above leads to an extended universal covering $\tilde{D}_p^s \cup \mathbb{R} =: \tilde{A}_p^s \to A_p$.

In the coordinate $\tilde{D}_p^s \simeq \{x + iy \mid x < 0, y \in \mathbb{R}\}$, the real line $\mathbb{R}$ corresponds to $\{-\infty + iy \mid y \in \mathbb{R}\}$.

We call this construction an real blow-up of $\log S^\phi$ at a pole $p$. Denote by $\log S^\phi_{\Delta}^{\xi}$ the surface obtained from $\log S^\phi$ by performing real blow-up at all poles of $\xi$.

Definition 6.19. The log surface $\log S^\xi_{\Delta}$ associated to a CY-s type $q_s$-quadratic differential $\xi$ on $S$ is the topological surface obtained from $\log S^\phi$ by adding back all zeroes in $\text{Zero}(\xi)$ as above and performing real blow-up at all poles in $\text{Pol}(\xi)$.

Remark 6.20. We define the numerical data $\mathfrak{N}(\log S^\phi_{\Delta}^{\xi}) = \mathfrak{N}(S, \xi) = (g, b, k, l; LP)$ of $\log S^\phi_{\Delta}^{\xi}$ as follows:

- The genus $g$ and the number $b$ of boundary components are inherited from $S$.
- $(k, l)$ is given by the polar type of $\xi$. 

• The Lekili-Polishchuk data $\text{LP}_q$ is defined the same way as in Definition 2.2, via the winding numbers on $S$ with respect to $\xi$. Note that, when choosing (e.g. non-separating) curves, one needs to avoid the simply connected domain $U$ in Assumption 6.4.

Then we have the following, which basically says that the winding number of an exponential pole of polar type $(k, l)$ with local coordinate $(5.7)$ matches the winding number of a CY-s pole of polar type $(k, l)$ with local coordinate $(6.6)$. Cf. Figure 7.1 for illustration for the case when $(k, l) = (3, 4)$ and $s = 3$.

**Theorem 6.21.** Let $S^\lambda$ be a graded marked surface and $(S, \xi)$ a Riemann surface with a CY-s type $q_s$-quadratic differential. If $\mathcal{N}(S^\lambda) = \mathcal{N}(S, \xi)$, then there is a homeomorphism $h: \text{log}_c S^\lambda \rightarrow \text{log}_c S^\xi$, which commutes with the action of the deck transformation group $\langle q \rangle$. Moreover, the numerical data $\mathcal{N}\text{log}_c S^\lambda$ matches $\mathcal{N}(\text{log}_c S^\lambda)$ under $h$.

**Proof.** The data $(g, b, k)$ is trivial to check while the tricky part is to show that winding numbers (i.e., index $l_i$) matches. This becomes the calculation of winding numbers in the following two situations:

• of a loop $\gamma$ around an exponential type singularity, where the local coordinate of the quadratic differential is in the form of $e^{z^k}z^{-l}g(z) \, dz \otimes 2$ as in $(5.7)$. This gives the grading change in $\text{log}_c S^\lambda$;

• of a loop $\gamma$ that contains an $s$-pole of type $(k, l)$ together with $k$ s-simple zeroes. This gives the corresponding grading change in $\text{log}_c S^\xi$, cf. left picture in Figure 7.1.

The two calculations are shown in Proposition 6.26 and Proposition 6.27, respectively, in the next subsection. As the Lekili-Polishchuk data $\text{LP}_q$ is determined by the winding function, the theorem follows. $\square$

### 6.5. Winding numbers.

In this subsection, we compute angle changes/winding numbers of certain cycles on a Riemann surface with a quadratic differential.

First we recall the definition of the curvature for smooth paths in $\mathbb{R}^2$ with the classical standard flat metric $dx^2 + dy^2$. Take a coordinate $(x, y) \in \mathbb{R}^2$ and consider a smooth path

$$\gamma: [0, 1] \rightarrow \mathbb{R}^2, \quad t \mapsto \gamma(t) = (x(t), y(t)).$$

The curvature $\kappa(t)$ of a path $\gamma(t)$ is defined by

$$\kappa(t) := \frac{\hat{x} \hat{y} - \hat{y} \hat{x}}{(\hat{x}^2 + \hat{y}^2)^{3/2}},$$

where $\hat{x} = \frac{d}{dt}x(t)$ and $\hat{y} = \frac{d}{dt}y(t)$. Then we define the angle change of a path $\gamma$ by

$$\text{Ang}(\gamma) := \int_0^1 \kappa(t)|\dot{\gamma}(t)| \, dt = \int_0^1 \frac{\dot{x} \dot{y} - \dot{y} \dot{x}}{\hat{x}^2 + \hat{y}^2} \, dt.$$
If we identify $\mathbb{R}^2$ with $\mathbb{C}$ by $w = x + iy$, then the angle change can be written as

$$\text{Ang}(\gamma) = \text{Im} \int_0^1 \frac{\dot{w} \ddot{w}}{|w|^2} \, dt = \text{Im} \int_0^1 \frac{\ddot{w}}{\dot{w}} \, dt.$$ 

We note that if two paths $\gamma$ and $\gamma'$ are regularly homotopic and have the same tangent vectors at boundaries $t = 0, 1$, then $\text{Ang}(\gamma) = \text{Ang}(\gamma')$. We also note that by definition, the angle change is invariant under the transformation $w \mapsto \alpha w + \beta$, where $\alpha \in \mathbb{C}^*$ and $\beta \in \mathbb{C}$ are some constants.

Now we extend the definition of the angle change on a Riemann surface $S$ with a quadratic differential $\xi$ (of various type). Take a small open subset $U \subset S \setminus \text{Crit}(\xi)$ with a coordinate $z \in U$ and consider the distinguished coordinate $w = \int \sqrt{\xi}$.

For a smooth path $\gamma: [0, 1] \to U$, $t \mapsto \gamma(t) = z(t)$, we can similarly define the angle change of $\gamma$ by

$$\text{Ang}(\gamma) := \text{Im} \int_0^1 \frac{\ddot{w}(z(t))}{\dot{w}(z(t))} \, dt,$$

where $\ddot{w}(z(t)) = \frac{d}{dt} w(z(t))$ and $\dot{w}(z(t)) = \frac{d^2}{dt^2} w(z(t))$. In the $q_s$-quadratic differential setting, since the coordinate $w$ is determined up to $w \mapsto \pm q^m_s w + c$, the angle change $\text{Ang}(\gamma)$ is also well-defined. For a general path $\gamma: [0, 1] \to S \setminus \text{Crit}(\xi)$, we can define the angle change by taking an open covering $\gamma \subset \bigcup \alpha \ U_\alpha$ and using the distinguished coordinate $w$ on each $U_\alpha$.

**Lemma 6.22.** Assume that on an open subset $U \subset S \setminus \text{Crit}(\xi)$ with a coordinate $z \in U$, the quadratic differential $\xi$ takes the form $\xi = f(z) \, dz^2$. Then for a path $\gamma: [0, 1] \to U$ with the $z$-coordinate expression $z(t)$, we have

$$\text{Ang}(\gamma) = \frac{1}{2} \text{Im} \int_0^1 \frac{f'(z(t))}{f(z(t))} \dot{z}(t) \, dt + \text{Im} \int_0^1 \frac{\ddot{z}(t)}{\dot{z}(t)} \, dt,$$

where $f'(z) = \frac{d}{dz} f(z)$.

**Proof.** First we note that by definition $\frac{d}{dt} w(z) = \sqrt{f(z)}$. So we have

$$\frac{d}{dt} w(z(t)) = \sqrt{f(z(t))} \dot{z}(t),$$

$$\frac{d^2}{dt^2} w(z(t)) = \frac{f'(z(t))}{2\sqrt{f(z(t))}} \dot{z}(t)^2 + \sqrt{f(z(t))} \ddot{z}(t).$$

Then by the definition of $\text{Ang}(\gamma)$, we have the following alternative definition of winding numbers. □
Definition 6.23. When $\gamma$ is a loop, the winding number is given by (cf. Definition 2.2)

$$\text{wind}_\xi(\gamma) := \frac{1}{\pi} \text{Ang}(\gamma).$$

(6.18)

We now compute the winding numbers around $s$-simple zeroes, $s$-poles (of type $(k,l)$) and exponential type singularities.

Lemma 6.24. Consider a $q_s$-quadratic differential $\xi = z^{s-2} d z \otimes 2$ on $\mathbb{C}$. Then for a small loop $\gamma$ around $s$-simple zero at $0 \in \mathbb{C}$, we have

$$\text{wind}_\xi(\gamma) = \text{Re}(s).$$

(6.19)

Proof. Set $z(t) = e^{it}$. Then by Lemma 6.22, we have

$$\text{Ang}(\gamma) = \frac{1}{2} \text{Im} \int_0^{2\pi} \frac{(s-2)z(t)^{s-3}}{z(t)^{s-2}} \overline{z}(t) dt + \text{Im} \int_0^{2\pi} \frac{\overline{z}(t)}{z(t)} dt$$

$$= \frac{1}{2} \text{Im} \int_0^{2\pi} \frac{(s-2)e^{i(s-3)}}{e^{i(s-2)}} e^{it} dt + \text{Im} \int_0^{2\pi} \frac{e^{it}}{1} dt$$

$$= \frac{1}{2} \text{Im} 2\pi i(s-2) + \text{Im} 2\pi i$$

$$= \pi \text{Re}(s).$$

We similarly have the following.

Lemma 6.25. Consider a $q_s$-quadratic differential $\xi = z^{-k(s-2)-l} d z \otimes 2$ on $\mathbb{C}$. Then for a small loop $\gamma$ around $s$-pole of type $(k,l)$ at $0 \in \mathbb{C}$, we have

$$\text{wind}_\xi(\gamma) = -k(\text{Re}(s) - 2) - l + 2.$$  

(6.20)

Proposition 6.26. Let $\xi$ be a $q_s$-quadratic differential on $S$ and $U \subset S$ be a simply connected domain which contains just $k$ $s$-simple zeroes and an $s$-pole of type $(k,l)$. Then for a loop $\gamma \subset U$ encircling these zeroes and a pole, we have

$$\text{wind}_\xi(\gamma) = 2 - l.$$  

(6.21)

Proof. This follows from combing (6.19), (6.20) and the following fact:

- if the loop $\gamma$ is obtained from a loop $\gamma'$ by including a single $s$-simple zero, then

$$\text{wind}_\xi(\gamma) = \text{wind}_\xi(\gamma') + \text{Re}(s - 2).$$

The proof of this claim is illustrated in Figure 6.1.

Set $D = \{ z \in \mathbb{C} \mid |z| < 2 \}$.

Proposition 6.27. Consider a quadratic differential $\phi = e^{z^{-k}} z^{-l} g(z) d z \otimes 2$ on $D$ with an exponential type singularity of index $(k,l)$ at $0 \in D$, where $g(z)$ is non-zero holomorphic function on $D$. Then, for a small loop $\gamma$ around $0$, we have

$$\text{wind}_\xi(\gamma) = 2 - l.$$  

(6.22)
Proof. Set \( z(t) = e^{lt} \). We note that the formula in Lemma 6.22 can be written as

\[
\text{Ang}(\gamma) = \frac{1}{2} \text{Im} \int_{\gamma} \frac{d}{dz} \log f(z) \, dz + \text{Im} \int_{0}^{1} \frac{\dot{z}(t)}{z(t)} \, dt.
\]

Then the first term is

\[
= \frac{1}{2} \text{Im} \int_{\gamma} \frac{d}{dz} \log e^{z-k} - l \log z + \log g(z) \, dz
\]

\[
= \frac{1}{2} \text{Im} \int_{\gamma} \left( -kz^{-k-1} + \frac{g'(z)}{g(z)} \right) \, dz
\]

\[
= \frac{1}{2} \text{Im}(2\pi i l) = -\pi l.
\]

Here we use the two facts:

- the residue of \( z^{-k-1} \) at \( z = 0 \) is zero since \( k \geq 1 \) and
- the residue of \( \frac{g'(z)}{g(z)} \) at \( z = 0 \) is zero since this function is holomorphic near zero by the condition that \( g(z) \) is non-vanishing on \( D \).

Next, we can easily compute that the contribution of the second term is \( 2\pi \). Thus we have the proposition follows.

6.6. Surface framing and period maps. Let \( S^\Delta \) be a graded marked surface with numerical data \( \mathcal{R}(S^\Delta) \) and an associated graded DMS \((S_\Delta, c, \Lambda)\). Let \( \log_c S^\Delta_\Delta \) be its topological log surface (cf. Definition 4.10). Fix \( s \in \mathbb{C} \).

Definition 6.28. A \( \log_c S^\Delta_\Delta \)-framed \( q \)-quadratic differential \( \Theta = (S, \xi, h; s) \) consists of a CY-\( s \) type \( q_s \)-quadratic differential \( \xi \) on a genus \( g \) Riemann surface \( S \) of \( s \)-polar type \((k, l)\) together with an isotopy class of a homeomorphism

\[
h: \log_c S^\Delta_\Delta \rightarrow \log S^\xi_\Delta,
\]

such that

- \( h \) identifies the decoration \( \Delta \) of \( S_\Delta \) with the set \( \text{Zero}(\xi) \) of zeroes of \( \xi \),
In the next section, we interpret Remark 6.31. Note that the monodromy representation \(\rho\) defined in Section 6.1, cf. (6.8), factors through the representation \(\rho\) above. More precisely, define a group homomorphism
\[
\text{ev}^\otimes_2: (q) \rightarrow \mathbb{C}^*,
q \mapsto q^2,
\]
Two \(q\)-quadratic differential \(\Theta_1 = (\xi_1, h_1; s)\) are log \(e\)-equivalent if
- there is a biholomorphism \(F: S_1 \rightarrow S_2\) satisfying \(F(\text{Pol}(\xi_1)) = \text{Pol}(\xi_2)\), together with the choice of a lift \(\tilde{F}: \log S_1 \rightarrow \log S_2\) such that \(\tilde{F}^*\xi_2 = \xi_1\) and
- \(h_2^{-1} \circ \log \tilde{F} \circ h_1 \in \text{Homeo}(\log e S^\lambda_\Delta),\)

is the induced homeomorphism.

Here, \(\text{Homeo}(\log e S^\lambda_\Delta)\) consists of isotopy classes of homeomorphisms of log \(e\)-connected \(q\)-quadratic differentials.

Denote by \(\text{QQuad}_s(\log e S^\lambda_\Delta)\) the moduli space of log \(e\)-framed \(q\)-quadratic differentials.

Consider the spectral cover \(\log \hat{S}^\phi \rightarrow \log S^\phi\). This can be naturally extended to the double cover \((\log \hat{S}^\phi_\Delta)^\xi \rightarrow (\log S^\phi_\Delta)^\xi\) with metric completions at zeroes and oriented blow-ups at poles. Correspondingly, we can also define the spectral cover of the topological log surface \(\log \hat{S}_\Delta \rightarrow \log e S^\lambda_\Delta\) with the covering involution \(\tau^\Delta\) such that \(h: \log e S^\lambda_\Delta \rightarrow \log S^\phi_\Delta\) lifts to the homeomorphism between spectral covers \(\hat{h}: \log \hat{S}_\Delta \rightarrow (\log S^\phi_\Delta)^\xi\). Finally, the framing \(h\) in (6.23) induces an isomorphism of the hat homology groups
\[
\hat{h}_*: \hat{H}(\log e S^\lambda_\Delta) \rightarrow \hat{H}(\xi),
\]
where \(\hat{H}(\log e S^\lambda_\Delta)\) is the corresponding hat homology group (i.e. defined by \(\tau^\lambda_s \gamma = -\gamma\)).

Recall that the period (6.15) of \(\xi\) is an \(R\)-linear map. Thus a log \(e\)-framed \(q_s\)-quadratic differential \((\xi, h)\) gives an \(R\)-linear map
\[
Z_\xi \circ \hat{h}_* \in \text{Hom}_R(\hat{H}(\log e S^\lambda_\Delta), \mathbb{C}_s).
\]

**Definition 6.29.** The period map
\[
\Pi_s: \text{QQuad}_s(\log e S^\lambda_\Delta) \rightarrow \text{Hom}_R(\hat{H}(\log e S^\lambda_\Delta), \mathbb{C}_s)
\]
is defined by sending \((\xi, h) \mapsto Z_\xi \circ \hat{h}_*\).

**Conjecture 6.30.** The period map \(\Pi_s\) is a local homeomorphism.

In the next section, we interpret \(q\)-quadratic differentials as \(q\)-stability conditions. Along the way, we will prove the conjecture above under certain assumptions.

### 6.7. Quadratic differentials on log surface.

**Remark 6.31.** Note that the monodromy representation \(\rho_s\) defined in Section 6.1, cf. (6.8), factors through the representation \(\rho\) above. More precisely, define a group homomorphism
\[
\text{ev}^\otimes_2: (q) \rightarrow \mathbb{C}^*,
q \mapsto q^2,
\]
then we have $\rho_s = ev_{\otimes^2} \circ \rho$. This implies that the CY-$s$ type $q_s$-quadratic differential $\xi$ is a single-valued non-zero holomorphic quadratic differential on $log S^\phi$ satisfying
\[ q^*\xi = q_s^2\xi. \tag{6.24} \]

Therefore, we have the following alternative definition of $\log cS^\lambda_{\Delta}$-framed $q$-quadratic differentials.

**Lemma 6.32.** A $\log cS^\lambda_{\Delta}$-framed $q$-quadratic differential $\Theta = (S, \xi, h; s)$ is equivalent to the data $\Theta = (\log S, \xi, h; s)$ as follows:

- a genus $g$ Riemann surface $log S$ with an automorphism $q$.
- a meromorphic quadratic differential $\xi$ on $log S$ such that
  - there are $|\Delta|$ zeroes of $\xi$, each of which is a conical singularity of infinity angle, as in Definition 5.9, with local coordinate of the form
    \[ \xi = ce^{\omega} d\omega^\otimes. \]
  - The automorphisms $q$ and $\xi$ satisfy (6.24). Thus, $q$ preserves zeroes and $q(p^i_j) = p^i_{j+1}$ in particular.
- a homeomorphism $h: log cS^\lambda_{\Delta} \to log S^\xi_{\Delta}$ such that
  - the numerical data of $log S^\xi_{\Delta}$ (similarly defined as in Remark 6.20) matches with the one of $log cS^\lambda_{\Delta}$ under $h$.
  - the deck transformation $q$ on $\log cS^\lambda_{\Delta}$ becomes the automorphism $q$ on $log S^\xi_{\Delta}$, where $log S^\xi_{\Delta}$ is the real blow-up of $log S$ at all the poles.

**Convention.** Let $\Theta = (log S, \xi, h; s)$ be a $\log cS^\lambda_{\Delta}$-framed $q$-quadratic differential. Using $h$, we will pull back various structures (i.e., foliation, metric, etc.) of $log S^\xi_{\Delta}$ to $\log cS^\lambda_{\Delta}$ and omit $(log S, h)$ in the following discussion when there is no confusion. Namely, we will simply write $\Theta = \xi$ and call it a $q$-quadratic differential on $\log cS^\lambda_{\Delta}$.

### 7. $q$-Stability Conditions via $q$-Quadratic Differentials

Recall that we have the following topological setting:

- $S^\lambda = (S, M, Y, \lambda)$ is a graded marked surface,
- $A$ is an (initial) full formal arc system of $S$,
- $c$ is the cut that is compatible with $A$,
- $(S_{\Delta}, c, \Lambda)$ is a graded DMS associated to $S^\lambda$.
- $\log cS^\lambda_{\Delta} = \bigcup_{m \in \mathbb{Z}} S^\lambda_{\Delta}$ is the topological log surface of $S_{\Delta}$ with grading $log \lambda$, with respect to $c$;

and categorical setting:
• $D_\infty(A)$ is the topological Fukaya category associated to $S$.
• $D_\infty(A)$ is the Calabi-Yau-$\mathcal{X}$ category of $D_\infty(A)$.
• we identify $D_\infty(A)$ with an $\mathcal{X}$-baric heart of $D_\infty(A)$ under (4.4).

Note that the cut $c$ induces an isomorphism
$$\hat{H}_1(\text{log}_c S^\lambda) \cong H_1(S, M; \mathbb{Z}_{sp}) \otimes R. \quad (7.1)$$

7.1. $q$-Quadratic differentials with cuts. Denote by $\text{Core}(\xi)$ the core of $\xi$ on $\text{log}_c S^\lambda$. Let $\pi_\Delta: \text{log}_c S^\lambda \to S^\Delta$ be the projection. Denote by
$$\text{Core}(\xi): = \pi_\Delta(\text{Core}(\xi)) \subset S^\Delta$$
the projection of $\text{Core}(\xi)$ on $S^\Delta$. Note that $\text{Core}(\xi)$ is well-defined topologically since (6.24) implies that $\text{Core}(\xi)$ and angles/direction of geodesics are invariants under the deck transformation (but the length of geodesics may be scaled). In order to give the metric information on $\text{Core}(\xi)$, we need to choose a cut. We will discuss the existence of cuts in the next subsection.

**Definition 7.1.** Let $\xi$ be a $q$-quadratic differential on $\text{log}_c S^\lambda$. A cut $t = \{t_i\}$ of $S^\Delta$ is compatible with $\xi$ if any of its arc $t_i$ does not intersect the interior of $\text{Core}(\xi)$. Denote by $Q_{\text{Quad}}(\text{log}_c S^\lambda)$ the $q$-quadratic differentials on $\text{log}_c S^\lambda$ that admit some compatible cut.

Now take a $q$-quadratic differential $\xi$ with compatible cut $t$. Then by Lemma 4.9 there is a unique element $b_\ast \in S_{\text{Br}}(S^\Delta)$ such that $b_\ast(t) = c$. Consider the induced action of $b_\ast$ on $\text{log}_c S^\lambda$, which is also denoted by $b$. Denote by
$$\text{Core}(\xi): = \text{Core}(b_\ast(\xi)) \cap S^{(0)} \quad (7.2)$$
the induced core on $S^\Delta$, which is the intersection of the core $\text{Core}(b_\ast(\xi))$ with the zero sheet $S^{(0)}$ of $\text{log}_c S^\lambda$. Note that $\xi$ also provides foliation/metric on $\text{Core}(\xi)$. By pulling the set of vertices $\Delta$ of $\text{Core}(\xi)$ on $S^\Delta$ along the arcs in $c$ to the set $Y$ of closed marked points of $S^\Delta$, we obtain some convex hull $\text{Core}(\xi)$ (see Figure 7.1). More precisely, this is the inverse operation of $\text{im}_{\text{in}}$ in (4.9):
$$\text{Core}(\xi): = \text{im}_{\text{in}}^{-1}(\text{Core}(\xi)) \subset S^\lambda. \quad (7.3)$$

The foliation of $\text{Core}(\xi)$ gives a partial foliation of $S$, which can be extended to the rest of the surface by gluing upper half planes as the core determines a flat surface. Thus we obtain a flat surface, or equivalently, a quadratic differential $\phi$ on some graded marked surface $S^\lambda_0$ determined by
$$\text{Core}(\phi) = \text{Core}(\xi). \quad (7.3)$$

**Lemma 7.2.** $\phi$ is in $F_{\text{Quad}}(S^\lambda)$.

**Proof.** By definition, we only need to show that the numerical data of $\phi$ (or of $S^\lambda_0$) coincide with $\mathcal{R}(S^\lambda)$. This is exactly what we have shown in Theorem 6.21. \qed
By (5.11) in Theorem 5.13, $\phi$ corresponds to a stability condition
\[ \hat{\sigma}(\xi) = \Xi_\infty(\phi) \in \text{Stab}^o D_\infty(A). \] (7.4)
Therefore, we obtain a triple $(D_\infty(A), \hat{\sigma}(\xi), s)$. We proceed to show that such a triple induces a $q$-stability condition.

**Proposition 7.3.** Suppose that $\text{Re}(s) \geq 2$. Let $\xi \in \text{QQuad}_c(\log cS^\lambda_\Delta)$. Then the triple $(D_\infty(A), \hat{\sigma}(\xi), s)$, constructed as above, induces a closed $q$-stability condition
\[ (\sigma, s) = (D_\infty(A), \hat{\sigma}(\xi), s) \otimes_s R \]
on $D_X(A)$. Then we obtain a map
\[ \Xi_s: \text{QQuad}^c(\log cS^\lambda_\Delta) / \text{SBr} (S_\Delta) \to \text{QStab}^* D_X(A) / \text{ST}, \] (7.5)
which is in fact an isomorphism.

**Proof.** Without loss of generality, assume $b$ in (7.2) is trivial, i.e., $\xi$ and $c$ are compatible.

**Step 1.** We only need to prove (1.13), i.e., $\text{gldim} \hat{\sigma} + 1 \leq \text{Re}(s)$. Then Theorem 1.10 implies that the construction above gives a $q$-stability condition and we obtain a map $\Xi_s$. Suppose not, then there is a pair of semistable objects $\hat{M}_1$ and $\hat{M}_2$ in $D_\infty(A)$ such that
\[ \text{Hom}_{D_\infty}(\hat{M}_1, \hat{M}_2) \neq 0, \]
\[ \varphi_{\hat{\sigma}}(\hat{M}_2) - \varphi_{\hat{\sigma}}(\hat{M}_1) > \text{Re}(s) - 1(\geq 1), \] (7.6)
where $\varphi_{\hat{\sigma}}$ denotes the phase function with respect to $\hat{\sigma}$.

By Theorem 2.7, $\hat{M}_i$ correspond to graded closed arcs $\hat{\gamma}_i$ on $S$, $i = 1, 2$, and the non-zero morphism in (7.6) corresponds to an intersection
\[ p \in \hat{\gamma}_1 \cap \hat{\gamma}_2 \]

**Figure 7.1.** Horizontal strip decompositions on (a sheet of) $\log cS^\lambda_\Delta$ and $S$
with \( \text{ind}_p(\tilde{\gamma}_1, \tilde{\gamma}_2) = 0 \). We claim that \( p \) cannot be in the interior of \( S \) as shown below.

\[
\begin{array}{c}
\gamma_1 \\
\gamma_2 \\
\hline
p
\end{array}
\]

Otherwise, such an intersection in \( S^c \) between \( \tilde{\gamma}_i \) also contributes as

\[ p \in \tilde{\gamma}_2 \cap \tilde{\gamma}_1 \]

with \( \text{ind}_p(\tilde{\gamma}_2, \tilde{\gamma}_1) = 1 \) by [IQZ, Lemma 2.3]. By Theorem 2.7 again, we also have a (non-zero) induced morphism in

\[ \text{Hom}_{D_c}(\hat{M}_2, \hat{M}_1)[1]. \]

As both \( \hat{M}_i \) (and their shifts) are (semi-)stable, we have

\[ \varphi_\beta(\hat{M}_1[1]) \geq \varphi_\beta(\hat{M}_2) \]

which contradicts to the second inequality in (7.6).

Now let \( Y \in \tilde{\gamma}_1 \cap \tilde{\gamma}_2 \) be the intersection, which is in \( Y \), corresponds to the non-zero morphism in (7.6). More precisely, such a morphism corresponds to the angle \( \alpha \) from \( \tilde{\gamma}_1 \) to \( \tilde{\gamma}_2 \) at \( Y \), that equals to (with respect to the quadratic differential \( \phi \))

\[
(\varphi_\beta(M_2) - \varphi_\beta(M_1)) \cdot \pi.
\]

Let \( \tilde{\eta}_i = \text{im}_c(\tilde{\gamma}_i) \), where \( \text{im}_c \) is defined in (4.9). Since \( \hat{M}_i \) are semistable, \( \tilde{\gamma}_i \) are in \( \text{Core}(\phi) = \text{Core}^0_\phi(\xi) \) and \( \eta_i \) are in \( \text{Core}^0(\xi) \). Denote by \( Z \) the corresponding intersection between \( \eta_i \), which is the decoration connecting to \( Y \) by \( c \). By construction, the angle \( \beta \)
from $\eta_1$ to $\eta_2$ at $Z$ (with respect to the quadratic differential $\xi$) equals the angle $\alpha$ from $\tilde{\gamma}_1$ to $\tilde{\gamma}_2$ at $Y$, cf. Figure 7.2.

Let $\tilde{\eta}_i$ be some lifts of $\eta_i$ in $\log c S^1_{\Delta}$ so that $\beta$ is also lifted to an angle $\tilde{\beta}$ from $\tilde{\eta}_1$ to $\tilde{\eta}_2$. As the deck transformation $q$ acts as rotation by $e^{i\pi s}$ on $q$-quadratic differentials, the angle $\theta$ from $\eta_2$ to $q(\eta_1)$ at $Z$ is

$$\theta = \text{Re}(s) \cdot \pi - \tilde{\beta} = \text{Re}(s) \cdot \pi - \beta = \text{Re}(s) \cdot \pi - \alpha = \text{Re}(s) \cdot \pi - (\varphi(M_2) - \varphi(M_1)) \cdot \pi.$$

By (7.6), $\theta < \pi$. Since both $\eta_2$ and $q(\eta_1)$ are in the core $\text{Core}(\xi)$, which is a convex hull, there is a geodesic $\tilde{\eta}_0$ in $\text{Core}(\xi)$ obtained from $q(\eta_1) \cup \eta_2$ by smoothing the intersection at $Z$ (on the side of the angle $\theta$, cf. the violet dashed arc in Figure 7.2).

Then the projection $\pi_\Delta(\tilde{\eta}_0)$ of $\tilde{\eta}_0$ on $S_\Delta$, which is in $\text{Core}^0(\xi)$, will intersect the segment $ZY$ in the cut $c$. This contradicts to the compatibility between $\xi$ and $c$. Thus (1.13) holds and $\sigma$ is a closed $q$-stability condition and we obtain a map $\Xi_s$ in (7.5). The injectivity of $\Xi_s$ follows from the facts that: $\Xi_\infty$ in (5.11) is injective and the core determines quadratic differentials.

**Step 2.** Reversing the process above, we have the following.

Given a point in $Q\text{Stab}^* D^*_X(A)/ST$, it can be represented by a $q$-stability condition $(\sigma, s)$ induced from some triple $(D^*_\infty(A), \tilde{\sigma}, s)$ for $\tilde{\sigma} \in \text{Stab}^o D^*_\infty(A)$. Let $\phi = \Xi^{-1}_\infty(\tilde{\sigma})$ be the exponential type quadratic differential on $S^1$ that corresponds to $\tilde{\sigma}$. Pushing the vertices of $Y$ along the cut $c$ to points in $\Delta$, one gets a convex hull $\text{Core}^0 = \text{im}_c(\text{Core}(\phi))$ on $S_\Delta$ from the core $\text{Core}(\phi)$. The convexity of $\text{Core}^0$ on $\log c S^1_{\Delta}$ follows from (1.13):

- Take two saddle connections, say $\tilde{\eta}_1$ in Figure 7.2, that intersect at a decoration $Z$. Then the angle between them that does not intersect the cut, say from $\tilde{\eta}_1$ to $\tilde{\eta}_2$, is at most $\text{glidim}(\tilde{\sigma}) \cdot \pi$. This implies the other angle, e.g. $\theta$ from $\tilde{\eta}_2$ to $q(\tilde{\eta}_1)$ in Figure 7.2, is at least

$$\text{Re}(s) \cdot \pi - \text{glidim}(\tilde{\sigma}) \cdot \pi \geq \pi.$$

Now we shall use this convex hull to build a $q$-quadratic differential on $\log c S^1_{\Delta}$. Consider the horizontal strip decomposition of $S$ with respect to $\phi$. Delete all the upper half planes that do not intersect the core $\text{Core}(\phi)$. When pushing closed marked point in $Y$ along the cut $c$ to decorations in $\Delta$, the foliation of the remaining of the horizontal strip decomposition provides a partial foliation/grading of the zero sheet $S^1_{\Delta(0)} \subset \log c S^1_{\Delta}$. Note that this partial foliation contains $\text{Core}^0$ mentioned above.

The partial foliation for the $m$-th sheet $S^1_{\Delta(m)}$ is given in the same way from the horizontal strip decomposition on $S$ with respect to the rotated quadratic differential $e^{im\pi s} \cdot \phi$. Then it contains the corresponding convex hull $\text{Core}^m$ obtained from $\text{Core}(e^{im\pi s} \cdot \phi)$.

The rest of the foliation on $\log c S^1_{\Delta}$ will be given/filled-in by part of the log-surface $C^s = s \log z$ around each decoration $Z$ and by (non-essential) upper half planes. Here,
$C^*_\infty = s \log z$ is topologically homeomorphic to $C_\infty$ in (5.9). This can be done, i.e., the partial foliations do match because of the condition (1.13):

- Each angle of Core$^m$ between saddle connections at a decoration $Z$ is at most $\text{gldim} \hat{\sigma} \cdot \pi$. Therefore (the neighbourhood of) the cores Core$^m$ around $Z$, do not overlap in $C^*_\infty$.
- These foliations are compatible, i.e. they can be regarded as part of $C^*_\infty$ simultaneously, because by construction the deck transformation $q$ becomes the rotation by $e^{m \pi s}$.
- Thus we can fill in the gaps to get a foliation on $\log c S^\lambda_\Delta$.

Finally, it is straightforward to check that the foliation we just constructed is indeed a $q$-quadratic differential on $\log c S^\lambda_\Delta$ in Lemma 6.32. This completes the proof that $\Xi_s$ is surjective. \hfill $\square$

Now take a subspace $\text{QQuad}^*_s(\log c S^\lambda_\Delta)$ in $\text{QQuad}^c_s(\log c S^\lambda_\Delta)$ which consists of those $q$-quadratic differentials, whose compatible cuts are in $\{ b(c) \mid b \in BT(S^\Delta_\Delta) \}$. This is the analogue of taking a connected component $FQuad^3_\Delta(S^\Delta)$ in $FQuad(S^\Delta)$ in [KQ2, (4.13)], where they fit in the commutative diagram

$$
\begin{array}{ccc}
FQuad^3_\Delta(S^\Delta) & \rightarrow & FQuad_3(S) \\
\downarrow & & \downarrow \\
FQuad_3(S) & \rightarrow & \text{SB}(S)
\end{array}
$$

(7.7)

in that case (Calabi-Yau-3). As the subgroup $\text{ST}_{D\chi(A)}$ of $\text{SB}(S^\Delta)$ is isomorphic to $BT(S^\Delta)$ by Theorem 4.11, we have the following result, which is the Calabi-Yau-s version of Theorem 5.7, or the $q$-deformed version of Theorem 5.13.

**Theorem 7.4.** There is the following identification

$$
\chi_s : \hat{H}(\log c S^\lambda_\Delta) \rightarrow K(D\chi(A)).
$$

(7.8)

Moreover, there is an isomorphism $\Xi_s$ of complex manifolds that fits into the commutative diagram

$$
\begin{array}{ccc}
\text{QQuad}^*_s(\log c S^\lambda_\Delta) & \rightarrow & \text{QStab}^*_s D\chi(A) \\
\downarrow & & \downarrow \\
\text{Hom}_R(\hat{H}(\log c S^\lambda_\Delta), C_s) & \rightarrow & \text{Hom}_R(K(D\chi(A)), C_s),
\end{array}
$$

(7.9)

such that the period map $\Pi_s$ is given by

$$
\Pi_s = q_s \circ (\Pi_\infty \otimes 1) : \xi \mapsto \int \sqrt{\xi}
$$

via (7.1) and becomes the map $Z_s = q_s \circ (Z_\infty \otimes 1)$ (i.e., central charge). More precisely, (0.3) holds for $X = \tilde{X}_\chi$ in (4.7) of Theorem 4.11. Here $\Pi_\infty$ and $Z_\infty$ are the ones in (5.11).
Furthermore, for $(\sigma, s) = \Xi_s(\xi)$, the map $\tilde{X}$ induces one-one correspondence between saddle connections of $\xi$ on $\log S^\lambda_\Delta$ and $\sigma$-semistable objects in $\mathcal{D}_X(A)$.

**Proof.** Firstly, (7.8) follows from (5.11) by tensoring $R$, noticing (1.6) and (7.1).

Secondly, (7.9) is the lifted version of (7.5) in Proposition 7.3, noticing that we have (4.8).

Finally, suppose that $(\sigma, s)$ is induced from the triple $(\mathcal{D}_\infty(A), \hat{\sigma}, s)$ up to the action of $\text{ST} \mathcal{D}_X(A)$ and $\phi = \Xi_{-1}^{-1}(\hat{\sigma})$ as in the proof of Proposition 7.3. By Theorem 4.13, we deduce that the correspondence between $\sigma$-semistable objects and saddle connections of $\xi$ also inherits from the correspondence between the $\hat{\sigma}$-semistable objects and saddle connections of $\phi$ in Theorem 5.13. □

In particular, Conjecture 6.30 holds restricted to $\text{QQuad}^*_s(\log S^\lambda_\Delta)$.

### 7.2. Existence of cuts.

One technical issue is that if the compatible cuts always exist for a $q$-quadratic differential. In other words, such an existence is equivalent to the assumption below.

**Assumption 7.5.** $\text{QQuad}^*_s(\log S^\lambda_\Delta) = \text{QQuad}^*_s(\log S^\lambda_\Delta)$.

We present some sufficient conditions for the existence of cuts in this subsection. First we have the following fact.

**Lemma 7.6.** The projection $\pi_\Delta(Z)$ of any zero/decoration $Z$ of $\xi$ on $S_\Delta$ is in the boundary of $\text{Core}(\xi)$.

**Proof.** Up to rotation, we only need to consider the saddle-free case. Suppose that a zero $Z$ is in the interior of $\text{Core}(\xi)$.

Let $K \gg 1$ be a positive integer. Consider all saddle connections starting from $Z$ that are in sheets $\bigcup_{i=1}^{K} S^i_\Delta$. Let $l_i$ be the number of saddle connections in the sheet $S^i_\Delta$. As mentioned above, $\xi$ satisfies (6.24) and hence the deck transformation $q$ preserves geodesics/saddle connections. Thus $l_i$ are the same, denoted by $l$.

On the one hand, each of these saddle connections is a horizontal strip $H$, such that $H$ is incident at $Z$ with angle $\pi$. Thus the sum of angles at $Z$ of horizontal strips that contains these saddle connections satisfies

$$K \cdot l \cdot \pi < K \cdot \text{Re}(s) \cdot \pi + 2\pi,$$

where $\text{Re}(s) \cdot \pi$ is the angle at $Z$ for each sheet. Dividing by $K$ of the inequality above and taking $\lim_{K \to +\infty}$ we see that $l \leq \text{Re}(s)$.

On the other hand, the angle between any two adjacent saddle connections at $Z$ is less than $\pi$, since $\text{Core}(\xi)$ is convex and $Z$ is in the interior of $\text{Core}(\xi)$. So the sum of angles at $Z$ of horizontal strips that contains these saddle connections satisfies

$$2\pi + (K \cdot l - 1) \cdot \pi > K \cdot \text{Re}(s) \cdot \pi.$$
Similarly, we deduce that \( l \geq \text{Re}(s) \), which forces \( l = \text{Re}(s) \). However, this will further force that the angle between any two adjacent saddle connection at \( Z \) equals \( \pi \), which contradicts to the fact that \( Z \) is in the interior of \( \text{Core}(\xi) \). \( \Box \)

Now we show that Assumption 7.5 always holds in the disk case.

**Corollary 7.7.** If the genus \( g \) of \( S \) is zero, then Assumption 7.5 holds.

**Proof.** By Lemma 7.6, any decoration \( Z \in \Delta \) is in the boundary of \( \text{Core}(\xi) \), so that there is no obstruction to pair points in \( \Delta \) and \( Y \in \partial S \Delta \). Hence the corollary holds. \( \Box \)

Also, when \( \text{Re}(s) \gg 1 \), then Assumption 7.5 holds. More precisely, we have the following.

**Proposition 7.8.** Recall that \( (k, l) := \{(k_1, l_1), \ldots, (k_b, l_b)\} \). If

\[
\text{Re}(s) \geq \max_{1 \leq i \leq b} \{3 - k_i - l_i\},
\]

then Assumption 7.5 holds.

**Proof.** We use the winding number and a combinatorial fact to prove the proposition.

**Step 1.** Since the boundaries of \( \log_{c} S_{\Delta} \) contract to the core \( \text{Core}(\xi) \), the boundaries of \( S_{\Delta} \) contract to \( \text{Core}(\xi) \). If we cut \( S_{\Delta} \) along \( \text{Core}(\xi) \), we will get exactly \( b \) annulus \( \mathcal{A}_i \), each of which contains a boundary component \( \partial_i \) of \( S_{\Delta} \). Equivalently, we have

\[
S_{\Delta} \setminus \text{Core}(\xi) \cong \bigcup_{i=1}^{b} \mathcal{A}_i, \quad \partial_i \subset \partial \mathcal{A}_i,
\]

where \( \mathcal{A}_i \) can intersect only at boundaries:

\[
\mathcal{A}_i \cap \mathcal{A}_j = \emptyset, \quad i \neq j.
\]

Suppose that \( \partial_i \) is the real blow-up of a pole \( p_i \) of type \( (k_i, l_i) \) and \( Z_1, \ldots, Z_k \) are the \( s \)-simple zeroes on the other boundary \( C_i(\neq \partial_i) \) of \( \mathcal{A}_i \).

Let \( L \) be the unique simple closed curve in \( \mathcal{A}_i \) (up to isotopy), clockwise around \( C_i \). The winding number of \( L \), as an anticlockwise loop around \( \partial_i \) is

\[
\text{wind}(L) = -\text{wind}_\xi(\partial_i) = k_i(\text{Re}(s) - 2) + l_i - 2.
\]

On the other hand, such a winding number can be calculated as a clockwise loop around \( C_i \):

- boundary arcs (connecting zeroes) of \( C_i \) are (images of) saddle connections (with zero winding) and;
- around each \( s \)-simple zero \( Z_j \), the winding number is \( \text{Re}(s) \), which implies that \( Z_j \) contributes the winding of \( L \) is

\[
\text{wind}(L; Z_j) = \alpha - 1 < \text{Re}(s) - 1,
\]

for \( 0 < \alpha < \text{Re}(s) \) is the angle at the corner \( Z_j \) of \( C_j \) shown in Figure 7.3.
Thus, we have the estimation that
\[ \text{wind}(L) = \sum_{j=1}^{k} \text{wind}(L; Z_j) < k \cdot (\text{Re}(s) - 1) \]
or
\[ k > k_i + \frac{k_i + l_i - 2}{\text{Re}(s) - 1}. \tag{7.11} \]

By (7.10), we deduce that \( k > k_i - 1 \) or \( k \geq k_i \).

**Step 2.** Consider a bipartite (unoriented) graph \( G \), consisting of white vertices \( W_Z \) labelled by \( s \)-simple zeroes \( Z \) in \( \Delta \) and black vertices \( B_{\partial_i} \) labelled by boundary components \( \partial_i \) of \( S_{\Delta} \). There is an edge in \( G \) connecting the white vertex \( W_Z \) and the black vertex \( B_{\partial_i} \) if and only if \( Z \in \Delta \) are in the boundary of \( A_i \). We associate a number \( k_i \) to \( B_{\partial_i} \), where \((k_i, l_i)\) is the type of the corresponding \( s \)-pole of \( B_{\partial_i} \). We have the following conditions on \( G \):

- **Step 1** says that the valence of \( B_{\partial_i} \) is at least \( k_i \);
- Lemma 7.6 says that the valence of any \( W_Z \) is at least 1.

We claim that for any bipartite graph \( G \) with the conditions above, there is a matching \( E \) of \( G \), consisting of \( \mathbb{N} = \sum_{i=1}^{b} k_i \) edges of \( G \), such that each white vertex \( W_Z \) is incident in exactly one edge of \( E \) and each black vertex \( B_{\partial_i} \) is incident in exactly \( k_i \) edges of \( E \).

Use induction on the number of edges in \( G \), where the start step is trivial. For the inductive step, there are two cases:

- **Step 1** says that the valence of \( B_{\partial_i} \) is at least \( k_i \);
- Lemma 7.6 says that the valence of any \( W_Z \) is at least 1.

We claim that for any bipartite graph \( G \) with the conditions above, there is a matching \( E \) of \( G \), consisting of \( \mathbb{N} = \sum_{i=1}^{b} k_i \) edges of \( G \), such that each white vertex \( W_Z \) is incident in exactly one edge of \( E \) and each black vertex \( B_{\partial_i} \) is incident in exactly \( k_i \) edges of \( E \).

Use induction on the number of edges in \( G \), where the start step is trivial. For the inductive step, there are two cases:

- If there is a white vertex \( Z \) with valence 1, let \( B_{\partial_i} \) be the only black vertex connected directly to \( Z \) (by an edge \( e \)). Then delete \( Z \) and reduce the associated number \( k_i \) of \( B_{\partial_i} \) to \( k_i - 1 \) to get a graph \( G' \) which also satisfies the conditions above. By inductive assumption, there is a matching \( E' \) for \( G' \) and then \( E' \cup \{e\} \) is a matching for \( G \).
- Otherwise, the valence of each white vertex \( Z \) is at least 2. The sum of valences of all white vertices is at least
\[ 2\mathbb{N} = 2 \sum_{i=1}^{b} k_i, \]
which is also the sum of valences of all black vertices. Then we deduce that the
valence $v_j$ of some black vertex $B_j$ is strictly bigger than the associated number
$k_j$. Delete any edge incidents at $B_j$ from $G$ and we can apply the inductive
assumption again.

**Step 3.** Combining the two facts above, the matching tells how to pair $k_i$ $s$-simple
zeroes to the boundary $\partial_i$, which then induces a cut (by connecting them in each of the
annulus $A_i$). In fact, the cut can be then chosen to be images of geodesics ('separating
connections'). □

7.3. **Conclusion.** Up to the technical issue, i.e., Assumption 7.5 above, the spaces of
induced open and closed $q$-stability conditions coincide, which provide the ‘principal’
components of the space $QStab_s D_X(A)$ of $q$-stability conditions.

**Theorem 7.9.** Suppose that Assumption 7.5 holds (e.g. when $\Re(s) \gg 1$) and $\Re(s) >
2$. Then we have the following

1°. $QStab^s_s D_X(A) = QStab^*_s D_X(A)$. Thus,

$$QQuad_s(\log_s S^X_\lambda) \cong QStab^s_s D_X(A).$$ (7.12)

2°. The image of the inclusion

$$QStab^s_s D_X(A) \hookrightarrow QStab_s D_X(A)$$

is open and closed and hence consists of connected components.

We will write $QStab_s D_X(A)$ for $QStab^s_s D_X(A)$ in this case.

**Proof.** Take a closed $q$-stability condition $(\sigma, s)$ in $QStab^*_s D_X(A)$, which is induced from
some triple, i.e.

$$(\sigma, s) = (D_\infty(A), \hat{\sigma}, s) \otimes_s R$$

with $\text{gldim} \hat{\sigma} = \Re(s) - 1$ (here we take the canonical $X$-baric heart without loss of
generality). We will keep all the notations in Proposition 7.3, i.e.

- $\xi = \Xi^{-1}(\sigma, s)$ with a compatible cut $c$ and the corresponding (exponential type)
  quadratic differential
    $\phi = \Xi^{-1}(\hat{\sigma}) \in FQuad_\infty(S^X_\lambda)$,
  cf. (7.3) and (7.4).

For the first statement, we need to show that there exists another compatible cut $c_0$ such
that the corresponding quadratic differential $\phi_0 \in FQuad_\infty(S^X_\lambda)$ and stability condition
$\hat{\sigma}_0 = \Xi_\infty(\phi_0)$ satisfying

$$(\sigma, s) = (L_0(D_\infty(A)), \hat{\sigma}_0, s) \otimes_s R \quad \text{with} \quad \text{gldim} \hat{\sigma}_0 < \Re(s) - 1.$$
Step 1: Consider a pair of semistable objects \( \hat{M}_1 \) and \( \hat{M}_2 \) in \( D_\infty(A) \) such that

\[
\begin{align*}
\text{Hom}_{D_\infty}(\hat{M}_1, \hat{M}_2) & \neq 0, \\
\varphi_{\hat{\sigma}}(\hat{M}_2) - \varphi_{\hat{\sigma}}(\hat{M}_1) & = \text{gldim} \hat{\sigma}.
\end{align*}
\]

(7.13)

Let \( \tilde{\gamma}_i \) be the closed arcs on \( S \) that correspond to \( \hat{M}_i \) in \( D_\infty(A) \). As in the proof of Proposition 7.3, they intersect at some \( Y \in Y \), such that the angle \( \alpha = \text{gldim} \hat{\sigma} \cdot \pi \) at \( Y \) provides the non-zero morphism in (7.13), cf. Figure 7.2. By \( [Q6, \text{Prop.} 5.7] \), there are only finitely many such angles \( \alpha \).

Step 2: Pulling along the cut \( c \), the arcs \( \tilde{\gamma}_i \) become \( \tilde{\eta}_i = \text{im}_c(\tilde{\gamma}_i) \) on \( S_\Delta \) as before. Denote by \( Z \in \Delta \) the decoration corresponding to \( Y \) and \( \beta \) the angle corresponding to \( \alpha \). So we have

\[
\beta = \alpha = \text{gldim} \hat{\sigma} \cdot \pi = (\text{Re}(s) - 1) \cdot \pi,
\]

which implies the other angle \( \theta \) at \( Z \) between \( \tilde{\eta}_2 \) and \( q(\tilde{\eta}_1) \) is exactly \( \pi \). This also means that \( \tilde{\eta}_i \) are boundary arcs of (the zero sheet of) the core \( \text{Core}_0(\xi) \).

Step 3: Now we can perturb the quadratic differential \( \xi \) a little bit to get a new one \( \xi_0 \), such that, the only changes happen at all those angles \( (\beta, \theta) \) as in Step 1, namely, we require for each of such angles,

\[
\beta_0 = \beta - \epsilon, \quad \theta_0 = \theta + \epsilon = \pi + \epsilon
\]

for some small positive real number \( \epsilon \). By Assumption 7.5, \( \xi_0 \) admits a compatible cut \( c_0 \), which is also compatible with \( \xi \) then.

Use the cut \( c_0 \) to produce another quadratic differential \( \varphi_0 \in F\text{Quad}_\infty(S_\lambda) \) that corresponds to another stability condition \( \hat{\sigma}_0 \) in \( \text{Stab} D_\infty(A) \). Note that \( c_0 \) will cut the angle \( \beta \) in Step 2. By \( [Q6, \text{Prop.} 5.7] \),

\[
\text{gldim} \hat{\sigma}_0 < \text{gldim} \hat{\sigma} = \text{Re}(s) - 1.
\]

With the Lagrangian immersion \( L_0 \) that corresponds to \( c_0 \), we obtain another triple to induce \( (\sigma, s) \):

\[
(\sigma, s) = (L_0(D_\infty(A)), \hat{\sigma}_0, s) \otimes \oplus R,
\]

as required.

For the second statement, we note that \( \text{QStab}^B D_X(A) \) is open \( \text{QStab} s D_X(A) \) since condition (1.12) is an open one (cf. [IQ, Thm. 5.11]). For the closedness in \( \text{QStab} s D_X(A) \), one can use the analogue map \( K \) in [BS, § 11.5], as we have shown that these stability conditions are indeed coming from quadratic differentials. \( \square \)

Part 3. Applications

8. Bridgeland-Smith theory on Calabi-Yau-N categories

We keep all the settings at the beginning of Section 7.
8.1. $N$-reductions for categories. Suppose that $N \geq \text{gldim } \mathcal{A} + 1$ and $\Gamma^N_{\mathcal{A}}$ is the Calabi-Yau-N Ginzburg algebra obtained from $\Gamma^X_{\mathcal{A}}$ by collapsing the double degree $a + bX$ into a single degree $a + bN$. Then there is a projection between dg algebras

$$\pi_N: \Gamma^X_{\mathcal{A}} \rightarrow \Gamma^N_{\mathcal{A}},$$

which induces a fully faithful exact functor

$$\pi_N: \mathcal{D}_X(\mathcal{A})/[[X - N]] \rightarrow \mathcal{D}_N(\mathcal{A}) (: = \mathcal{D}_{fd}(\Gamma^N_{\mathcal{A}})),$$

that makes $\mathcal{D}_N(\mathcal{A})$ the $N$-reduction of $\mathcal{D}_X(\mathcal{A})/[[X - N]]$, in the sense that

- $\mathcal{D}_N(\mathcal{A})$ can be identified with a triangulated hull of $\mathcal{D}_X(\mathcal{A})/[[X - N]]$,
- $K(\mathcal{D}_N) \cong \mathbb{Z}^{\oplus n}$ and the induced $R$-linear map $[\pi_N]: K(\mathcal{D}_X(\mathcal{A})) \rightarrow K(\mathcal{D}_N(\mathcal{A}))$ is a surjection given by sending $q \mapsto (-1)^N$.

Thus, we have the following by [IQ, Thm. 4.5].

**Corollary 8.1.** There is an embedding

$$\text{QStab}_N \mathcal{D}_X(\mathcal{A}) \hookrightarrow \text{Stab}_N \mathcal{D}(\mathcal{A})$$

between complex manifolds. Denote by $\text{Stab}^\circ \mathcal{D}_N(\mathcal{A})$ the image of $\text{QStab}_N \mathcal{D}_X(\mathcal{A})$.

We also give the following conjecture, which is the surface version of [IQ, Cor. 6.8].

**Conjecture 8.2.** The cluster-$X$ category $\mathcal{C}(\Gamma^X_{\mathcal{A}})$ in (4.6) admits the Serre functor $\tau \circ [1]$ (which should correspond to $[X - 1]$), for $\tau$ the Auslander-Reiten functor. Moreover, the Calabi-Yau-$(N - 1)$ cluster category $\mathcal{C}_{N-1}(\mathcal{A})$ of $\Gamma^N_{\mathcal{A}}$ is the (unique) triangulated hull of $\mathcal{C}(\Gamma^X_{\mathcal{A}})/\tau[2 - N]$ that fits into the commutative diagram:

$$0 \rightarrow \mathcal{D}_{fd}(\Gamma^X_{\mathcal{A}}) \rightarrow \text{per } \Gamma^X_{\mathcal{A}} \rightarrow \mathcal{C}(\Gamma^X_{\mathcal{A}}) \rightarrow 0$$

$$0 \rightarrow \mathcal{D}_{fd}(\Gamma^N_{\mathcal{A}}) \rightarrow \text{per } \Gamma^N_{\mathcal{A}} \rightarrow \mathcal{C}(\Gamma^N_{\mathcal{A}}) \rightarrow 0.$$

**Remark 8.3.** One interesting question is what is the condition for $\mathcal{D}_X(\mathcal{A})/[[X - N]]$ to be $N$-reductive. The condition $N \geq \text{gldim } \mathcal{A} + 1$ here is actually for $\Gamma^N_{\mathcal{A}}$ being non-positive, so that it is a silting object in its perfect derived category and hence ensures $\text{Stab}_N \mathcal{D}(\mathcal{A})$ is nonempty.

8.2. $N$-reductions for $q$-quadratic differentials. When $s = N \geq 3$ is a positive integer, then the $q_s$-quadratic differential $\xi$, locally of the form (6.4), becomes a classical type quadratic differential on some Riemann surface in Definition 5.3. We have the definition of generalized GMN differentials as follows.

**Definition 8.4.** A CY-$N$ type differential on a Riemann surface $S$ is a meromorphic differential with classical type of singularities satisfying:

- all the zeroes have order $N - 2$;
- the order of any pole is at least 3.
The (decorated) marked surface associated to \((S, \phi)\) is defined as in Definition 5.4.

Next, from a graded marked surface \(S^\lambda\) with associated graded DMS \(S^\lambda_\Delta\), we construct the marked surface \(S^N_\Delta\) for \(q_N\)-quadratic differentials.

**Definition 8.5.** The \((CY-N)\) marked surface \(S^N\) is a marked surface modified from \(S^\lambda\) by

- forgetting the grading and closed marked points and
- changing the number of open marked points on the boundary \(\partial_i\) to be
  \[ k_i \cdot (N - 2) + l_i - 2, \]
  \[ (8.2) \]
  where \((k_i, l_i)\) is the associated numerical data.

The \((CY-N)\) DMS \(S^N_\Delta\) is modified from \(S_\Delta\) same as above.

Note that the condition (6.9) in Definition 6.5 that \(s\)-poles are higher \(s\)-poles ensures the number in (8.2) is a positive integer.

Similar to Definition 5.6, we have moduli spaces \(F_{\text{Quad}}^N(S^N_\Delta)\) of \(S^N_\Delta\)-framed quadratic differentials. So we have the following straightforward observation, as the counterpart to Corollary 8.1.

**Lemma 8.6.** There is an embedding

\[ \text{QQuad}_N(\log c S_\Delta^\lambda) \rightarrow F_{\text{Quad}}(S^N_\Delta). \]

Denote its image by \(F_{\text{Quad}}^N(S^N_\Delta)\) and the corresponding quotient space in \(F_{\text{Quad}}^N(S^N)\) by \(F_{\text{Quad}}^N(S^N_\Delta)\).

Combining Corollary 8.1 and Lemma 8.6 we have the following version of Theorem 7.9 as an \(N\)-analogue of Bridgeland-Smith’s result (or King-Qiu’s upgraded version). The condition \(N \gg 1\) ensures that Assumption 7.5 holds. But one expects that this is not necessary: condition (??) is more reasonable.

**Theorem 8.7.** If \(N \gg 1\), there is an isomorphism between complex manifolds

\[ F_{\text{Quad}}^N(S^N_\Delta) \cong \text{Stab}^0 D_{fd}(\Gamma^N_\lambda). \]  

(8.3)

When \(S\) is the disk (i.e., type A case), the theorem above is the main theorem in [I]. However, notice that the method in [I] is a directly generalization of [BS] while our approach is different.

**Remark 8.8.** Let us explain that in the unpunctured case of [BS], i.e., \(N = 3\) and when the marked surface has no punctures (equivalent to the higher order pole condition), how the result above should reproduce (0.1). To see this, let \(K = (K_1, \cdots, K_b)\) be the numerical data of numbers of marked points on \(\partial S\) in the setting of [BS]. Recall that \(g\) and \(b\) are the genus and number of boundaries components of \(S^g\). One chooses a graded marked surface \(S^\lambda\) of genus \(g\), with integers \(l = (l_1 \cdots, l_b)\) that sums to \(4 - 4g\) and a set of integer \(k = (k_1 \cdots, k_b)\) for \(k_i = K_i - l_i + 2\) as in (8.2). Here we need \(k_i \geq 1\), which can be done since
• if \( g \geq 1 \) or \( b \geq 2 \), then
\[
\sum_{i=1}^{b} k_i = \sum_{i=1}^{b} K_i + (4g - 4) + 2b \geq b;
\]

• if \( g = 0 \) and \( b = 1 \), then \( K_1 \geq 3 \) (so that \( \Delta \geq 1 \) in (5.2)) and the estimation above also holds.

Then (8.3) in Theorem 8.7 indeed becomes (0.1) as required.

We hope to cover the punctured case of [BS, Theorem 1.2] in future works.

**Example 8.9.** Let \( S = D \) be a disk with 4 marked points on its boundary. So the numerical data is \( (g = 0, b = 1; k = (4), l = (4); LP_0 = \emptyset) \).

Choose the full formal open arc system \( A \) with dual closed arc system \( A^* \) of \( S^\lambda \) as in the first picture in Figure 8.1. Note that the open arcs are drawn in blue while the closed ones are in red. Suppose that the intersection indexes between open arcs in \( A \) are all zero. Then the corresponding dual closed arc systems \( A^*_\Delta \) in \( S_\Delta \) and \( S^3_\Delta \) are shown in the second and third pictures in Figure 8.1. Then we could have the corresponding graded quivers as:

\[
\text{Figure 8.1. The } A_3 \text{ case: } S^\lambda, S_\Delta \text{ and } S^3_\Delta \text{ with arc systems}
\]

where the potentials associated to the second and third quivers are just the inner three cycles of degree \( 3 - X \) and degree 0 respectively.
Example 8.10. Keep the surface $S = D$ as above. Choose another full formal open arc system $A$ with dual closed arc system $A^*$ of $S^\lambda$ as in the left picture in Figure 8.2. Then the corresponding dual full formal closed arc systems $A^*_\Delta$ in $S_\Delta$ is shown in the right pictures in Figure 8.2. The associated quivers $(Q_A, \tilde{Q}_A)$ are just the standard $A_3$ quiver $1 \to 2 \to 3$ and its Calabi-Yau-X double respectively, whose gradings are controlled by grading of curves. We list three cases of the gradings of $Q_A$ with the corresponding 4-reduction pictures as follows:

- If $\deg a = \deg b = 0$, then the corresponding dual full formal closed arc system $A^*_\Delta$ in $S^4_\Delta$ is shown in the left picture of Figure 8.3.
- If $\deg a = \deg b = 1$, then the corresponding dual full formal closed arc system $A^*_\Delta$ in $S^4_\Delta$ is shown in the middle picture of Figure 8.3.
- If $\deg a = 1$ and $\deg b = 2$, then the corresponding dual full formal closed arc system $A^*_\Delta$ in $S^4_\Delta$ is shown in the right picture of Figure 8.3.

Note that the grading of quivers in the Calabi-Yau-$N$ case is controlled by the numbers of marked points on the boundaries of $S^N_\Delta$. In the final subsection of this section, we explain the cluster theory in the Calabi-Yau-$N$ case.
8.3. Cluster combinatorics. The skeletons (cf. [KQ2, Remark 2.17, Lemma 4.8 and Lemma 4.9], see also [Q1, BS]) for manifolds in (8.3) should be

$$\text{EG}(S^N) \cong \text{CEG}(C_{N-1}(A)),$$

where on the left hand side, it is the exchange graph of $N$-angulation of $S_N$ and on the right hand side, it is the cluster exchange graph of $C_{N-1}(A)$. The vertices of $\text{EG}(S^N)$ are $N$-angulation of $S_N$ and the edges are forward flips. The vertices of $\text{CEG}(A)$ are $(N-1)$-cluster tilting sets (cf. e.g. [KQ1, Def. 4.1]) in the cluster categories $C_{N-1}(A)$ mentioned in Conjecture 8.2 with edges correspond to forward mutations.

For instance, Figure 8.4 shows a sequence of forward flips for $N = 4$-angulations of an 12-gon, which corresponds to meromorphic quadratic differentials of singularity type

$$(2^5,-14) = (2, 2, 2, 2, 2, -14)$$
on $\mathbb{C} = \mathbb{P}^1$. More precisely, these 4-angulations consist of isotopy classes of separated saddle connections of the corresponding meromorphic quadratic differentials. Moreover, an example of a decorated forward flip for 4-angulation is from the middle picture of Figure 8.3 to the right picture there. For more details, cf. [KQ2, § 3].

![Figure 8.4. A sequence of flips of 4-angulations](image)

One can also associate Calabi-Yau-$N$ quivers with superpotential to $N$-angulations directly, parallel to Definition 4.1, where one replaces degree $X$ with $N$. For instance, in figure 8.5 which is the case when $N = 5$, the superpotential should contains all those 3-cycle of degree $-2$ (two of them are shown in the figure).

![Figure 8.5. Some 3-cycles in the CY-5 case.](image)
Remark 8.11. Finally, we remark that one may want to get (8.3) by generalizing the approach in [BS], as [I] did for the type A case. However, this is not a short cut by all means (nor easy) due to lack of the following results:

- the topological cluster theory for surfaces, e.g. (8.5), has not been established.
- the categorical cluster theory for surfaces, e.g. [KQ1, Thm. 8.6] for $\Gamma^X_A$, has not been established.

9. Hurwitz spaces via quadratic differentials in genus zero

The aim of this section is to identify the moduli space of CY-$s$ type quadratic differentials with the Hurwitz space in the case $g = 0$.

9.1. Hurwitz spaces.

Definition 9.1. Let $k = (k_1, \ldots, k_b)$ be a multi-set of positive integers $k_i \in \mathbb{Z}_{\geq 1}$. A Hurwitz cover of genus $g$ and polar type $k$ is a pair of a compact Riemann surface $S$ of genus $g$ and a meromorphic function $f : S \to \mathbb{P}^1$ such that poles of $f$ consists of $b$ points, i.e., $f^{-1}(\infty) = \{p_i\}_{i=1}^b$ with orders $\text{ord}_f(p_i) = -k_i$.

Two Hurwitz covers $(S, f)$ and $(S', f')$ are equivalent if there is a biholomorphism $h : S \to S'$ such that $f = h^* f'$. The Hurwitz space $\text{HS}(g, k)$ is the moduli space of Hurwitz covers of genus $g$ and polar type $k$.

For a Hurwitz cover $(S, f)$, denote by Zero($f$), Pol($f$) $\subset S$ the set of zeroes and poles of $f$ respectively. Write $\text{Crit}(f) = \text{Zero}(f) \cup \text{Pol}(f)$. It is known that the dimension of the Hurwitz space is given by

$$\dim \text{HS}(g, k) = 2g - 2 + b + \sum_{i=1}^{b} k_i,$$

which will be denoted by $n$.

The regular locus $\text{HS}(g, k)_{\text{reg}} \subset \text{HS}(g, k)$ is defined to be the subset consisting of Hurwitz covers with simple zeroes, which will be called regular Hurwitz covers. For $(S, f) \in \text{HS}(g, k)_{\text{reg}}$, denote $\Delta = \text{Zero}(f)$. Then since

$$0 = \sum_{p \in \text{Crit}(f)} \text{ord}_f(p) = |\Delta| - \sum_{i=1}^{b} k_i,$$

we have $|\Delta| = \sum_{i=1}^{b} k_i$ in this case.

9.2. Marked surfaces and log surfaces from Hurwitz covers. In this subsection, we construct (ungraded) marked surfaces and (ungraded) log surfaces from Hurwitz covers.

Construction 9.2. For a Hurwitz cover $(S, f)$, we can associate a marked surface $S^f = (S^f, M, Y)$ as follows:
• A surface $S^f$ is defined as the real blow up of the underlying smooth surface of $S$ at the poles $f^{-1}(\infty) = \{z_1, \ldots, z_b\}$.
• Define marked arcs $\overline{M} \subset \partial S^f$ as open intervals corresponding to asymptotic directions, where $e^f$ decays exponentially as $z \to z_i$.
• By shrinking each marked arc of $\overline{M}$ to a single point in $S^f$, we obtain a set of closed marked points $M$.
• The set of open marked points in $Y \subset \partial S^f$ are the dual to $M$, i.e., they are alternative in $\partial S^f$.

Denote by $\partial_i$ the $S^1$-boundary component obtained from the real blow up at $z_i$ for $i = 1, \ldots, b$. Then by definition, there are $k_i = -\text{ord}_f(z_i)$ points on the boundary component $\partial_i$.

Remark 9.3. Though we can construct the marked surface $S^f = (S^f, M, Y)$ from the Hurwitz cover $(S, f)$, this information is not enough to construct a grading $\lambda$ on $S^f$. In Section 9.5, we introduce a primary quadratic differential in the case when the genus of $S^f$ is zero, which allows us construct a natural grading.

Next we consider the construction of log surfaces from Hurwitz covers. Let $(S, f)$ be a regular Hurwitz cover. Consider a multi-valued holomorphic function $\log f$ on $S^\phi := S \setminus \text{Crit}(f)$. Take a base point $* \in S^\phi$ and choose a value $\log f(*)$. First we note that the analytic continuation of $\log f(*)$ along with a cycle $\gamma \in \pi_1(S^\phi, *)$ is given as the form $\log f(*) + 2\pi i m$ for some $m \in \mathbb{Z}$. Then we obtain a representation

$$\rho: \pi_1(S^\phi, *) \to \langle q \rangle, \quad \gamma \mapsto \rho(\gamma) := q^m.$$  

Definition 9.4. The logarithmic (log) surface $\log S^\phi$ associated to a Hurwitz cover $(S, f)$ is defined to be the covering $\pi: \log S^\phi \to S^\phi$ corresponding to the representation $\rho: \pi_1(S^\phi, *) \to \langle q \rangle$. The infinite cyclic group $\langle q \rangle$ acts on $\log S^\phi$ as the group of deck transformations.

By definition, the multi-valued function $\log f$ can be lifted on the log surface $\log S^\phi$ as a single-valued function through the analytic continuation. In other words, the surface $\log S^\phi$ is the Riemann surface of the function $\log f$. We also note that for any complex number $s \in \mathbb{C}$, the function

$$f^{s-2} = e^{(s-2)\log f}$$

is also a single-valued function on $\log S^\phi$.

Finally we note that as in Section 6.4, for a log surface $\log S^\phi$ we can construct the topological log surface $\log S^f_\Delta$ by adding zeroes $\Delta$ of $f$ and doing the real blow-ups at poles of $f$.

9.3. Primary quadratic differentials. In the following, we assume $g = 0$, namely $S \cong \mathbb{P}^1$, and fix the coordinate $z \in \mathbb{C} \cup \{\infty\} = \mathbb{P}^1$.

Definition 9.5. Let $l = (l_1, \ldots, l_b) \in \mathbb{Z}^b$ be an ordered multi-set of $b$ integers satisfying the condition

$$\sum_{i=1}^{b} l_i = 4.$$
For a Hurwitz cover \((\mathbb{C}, f)\) with poles \(f^{-1}(\infty) = \{z_1, \ldots, z_b\}\), we define a meromorphic quadratic differential \(\Omega_1\) by
\[
\Omega_1(z) := \prod_{i=1}^{b} \frac{dz^2}{(z - z_i)^{l_i}}.
\]  
(9.1)

We call \(\Omega_1\) the primary quadratic differential compatible with \((\mathbb{C}, f)\).

For a Hurwitz cover \(f: \mathbb{C} \to \mathbb{C}\) with poles of order \(k_1, \ldots, k_b\) at \(z_1, \ldots, z_b\) respectively, consider the primary quadratic differential \(\Omega_1\). Then the quadratic differential
\[
\phi = \phi(f, 1) := e^{f} \Omega_1
\]
(9.2)
on \(\mathbb{C}\) gives a quadratic differential with exponential type singularities of indexes
\[(k, l) = \{(k_1, l_1), \ldots, (k_b, l_b)\}\]
at \(z_1, \ldots, z_b\) respectively. The following statement gives the other direction.

**Proposition 9.6.** Let \(\phi\) be a quadratic differential on \(\mathbb{C}\) with exponential type singularities of indexes \((k, l)\) at \(z_1, \ldots, z_b\) respectively. Then there is a unique Hurwitz cover \(f: \mathbb{C} \to \mathbb{C}\) with poles of order \(k_1, \ldots, k_b\) at \(z_1, \ldots, z_b\), up to \(f \mapsto f + 2\pi i m\) for \(m \in \mathbb{Z}\), such that \(\phi\) is of the form in (9.2).

**Proof.** First we consider the holomorphic function \(g(z) := \phi(z)/\Omega_1(z)\) on \(\mathbb{C} \setminus \{z_1, \ldots, z_b\}\). Then by definition, \(g(z)\) is non-zero on \(\mathbb{C} \setminus \{z_1, \ldots, z_b\}\) and points \(z_1, \ldots, z_b\) are exponential type singularities. Since \(g(z)\) is non-zero away from \(z_i\), it is sufficient to show that the function \(\log g(z)\) has a pole of order \(k_i\) at \(z_i\) and is single-valued around \(z_i\). By definition of exponential type singularity, there is a coordinate change \(v(z)\) satisfying \(v(z_i) = 0\) and \(v'(z_i) \neq 0\) near \(z_i\) such that
\[
\phi = e^{v - k_i} v^{-l_i} \cdot h_1(v) \, dv^2,
\]
where \(h_1(v)\) is some non-zero holomorphic function near \(v = 0\). We also note that under the above coordinate change, the primary quadratic differential \(\Omega_1\) can be written as
\[
\Omega_1 = v^{-l_i} \cdot h_2(v) \, dv^2,
\]
where \(h_2(v)\) is also some non-zero holomorphic function near \(v = 0\). Thus \(g(z(v)) = \exp(v^{-k_i}) \cdot h_1(v)/h_2(v)\) near \(v = 0\). Thus \(\log g\) has a pole of order \(k_i\) at \(z_i\) as required. \(\square\)

Note that the primary quadratic differential \(\phi = \phi(f, 1)\) in (9.2) provides a canonical grading \(\lambda = \lambda(f, 1)\) on the associated marked surface \(S^f\). Define the numerical data for the Hurwitz cover \((\mathbb{C}, f)\) by
\[
\mathcal{M}(\mathbb{C}, f) = \mathcal{M}\left((S^f)^{\lambda}\right).
\]
(9.3)
9.4. $q$-Quadratic differentials. Similar to Proposition 9.6, we consider the analogue statement for CY-s type quadratic differentials. Fix a complex number $s$ with $\text{Re}(s) > 2$. For a Hurwitz cover $f: \mathbb{C} \to \mathbb{C}$ with simple zeroes and poles of order $k_1, \ldots, k_b$ at $z_1, \ldots, z_b$ respectively, define a multi-valued quadratic differential on $\mathbb{C}$ ramified at $\text{Crit}(f)$ by

$$\xi = \xi(f, s, 1) := f^{s-2} \Omega_1 = e^{(s-2) \log f} \Omega_1$$

(9.4)

Then $\xi$ is a CY-s type $q_s$-quadratic differential with $s$-poles of type $(k, l)$ at $z_1, \ldots, z_b$.

The following statement gives the other direction in this case.

**Proposition 9.7.** Let $\xi$ be a CY-s type $q_s$-quadratic differential on $\mathbb{C}$ with $s$-poles of type $(k_1, l_1), \ldots, (k_b, l_b)$ at $z_1, \ldots, z_b$ together with a fixed value $\xi(*)$ on some point $* \in \log(\mathbb{C})^\circ$. Then there is a unique regular Hurwitz cover $f: \mathbb{C} \to \mathbb{C}$ with poles of order $k_1, \ldots, k_b$ at $z_1, \ldots, z_b$ and a single-valued function $\log f$ on $\log \mathbb{C}^\circ$, up to multiplying by $\omega_{s-2}^m$, for $\omega_{s-2} = e^{2\pi i/(s-2)}$ and $m \in \mathbb{Z}$:

$$f \mapsto \omega_{s-2}^m f \quad \text{or} \quad \log f \mapsto \log f + \frac{2\pi im}{s-2}.$$  \hspace{2cm} (9.5)

such that

$$\xi = f^{s-2} \Omega_1,$$

where $l = (l_1, \ldots, l_b)$ and the value $\log f(*)$ is determined by $\xi(*) = e^{(s-2) \log f(*)} \Omega_1(*)$.

**Proof.** First we note that the ambiguity $f \mapsto \omega_{s-2}^m f$ comes from $f^{s-2} = (\omega_{s-2}^m f)^{s-2}$ since $\omega_{s-2}^{s-2} = 1$. Consider a multi-valued function $g(z) := \xi(z)/\Omega_1(z)$ and let

$$f(z) := g(z)^{1/(s-2)} = e^{\frac{1}{s-2} \log g(z)}$$

up to multiplying $\omega_{s-2}^m$. We only need to show that $f(z)$ has a pole of order $k_i$ at $z_i$ and is single-valued near $z_i$.

Similar to the proof of Proposition 9.6, there is a coordinate change $v(z)$ satisfying

$$v(z_i) = 0 \quad \text{and} \quad v'(z_i) \neq 0 \quad \text{near} \quad z_i$$

such that

$$\xi = v^{-k_i(s-2)-l_i} \, dv^2$$

and

$$\Omega_1 = v^{-l_i} \cdot h(v) \, dv^2,$$

where $h(v)$ is also some non-zero holomorphic function near $v = 0$. Then $g(z(v)) = v^{-k_i(s-2)} \cdot h(v)^{-1}$ near $v = 0$. Thus $f(z(v)) = g(z(v))^{1/(s-2)}$ has a pole of order $k_i$ at $v = 0$ and is single-valued near $v = 0$ since $h(v)^{-1}$ is non-zero. \hfill \square

9.5. Framed regular Hurwitz spaces. Let $C^\lambda = (C, M, Y, \lambda)$ be a graded marked surface of genus zero. Then the numerical data of $C$ is just

$$\mathfrak{G}(C^\lambda) = (b; k, l)$$

since $g = 0$ and LP$_0$ is empty. We also choose an arc system $A$ and a cut $c$ and thus have the log surface $\log_c C^\lambda$, which will be simply written as $\log C^\lambda$.
Let \((\overline{C}, f)\) be a regular Hurwitz cover of polar type \(k\) and \(\log \overline{C}_\Delta\) be the associated topological log surface constructed in Section 9.2 with numerical data \(\mathcal{N}(\overline{C}, f)\) in (9.3). Then Theorem 6.21 implies that there is a homeomorphism
\[ h: \log \overline{C}_\Delta \to \log \overline{C}_\Delta^{f} \]
which commutes with the action of \(\langle q \rangle\).

**Definition 9.8.** A \(\log \overline{C}_\Delta\)-framed Hurwitz cover \((\overline{C}, f, \log f, h)\) consists of a regular Hurwitz cover \((\overline{C}, f)\), a fixed choice of the (single-valued) function \(\log f\) on \(\log \overline{C}\) and a homeomorphism
\[ h: \log \overline{C}_\Delta \to (\log \overline{C}_\Delta)^{f}, \]
which commutes with the action of \(\langle q \rangle\) and matches the numerical data \(\mathcal{N}(\overline{C}, f)\) with \(\mathcal{N}(\log \overline{C}_\Delta)\).

Two \(\log \overline{C}_\Delta\)-framed Hurwitz cover \((\overline{C}, f_i, \log f_i, h_i)\) are \(\log \overline{C}_\Delta\)-equivalent if

- there is a biholomorphism \(F: \overline{C} \to \overline{C}\) satisfying \(F(f_1^{-1}(\infty)) = f_2^{-1}(\infty)\), together with the choice of a lift \(\tilde{F}: \log \overline{C}^{f_1} \to \log \overline{C}^{f_2}\) such that \(\tilde{F}^{*}(\log f_2) = \log f_1\) and
- \(h_2^{-1} \circ \tilde{F} \circ h_1 \in \text{Homeo}_0(\log \overline{C}_\Delta)\), where
\[ \tilde{F}: \log \overline{C}^{f_1} \to \log \overline{C}^{f_2} \]
is the induced homeomorphism.

Denote by \(\text{HS}(\log \overline{C}_\Delta)\) the moduli space of \(\log \overline{C}_\Delta\)-framed Hurwitz covers.

Recall that \(\text{MCG}^{\circ}(\log \overline{C}_\Delta)\) is generated by the deck transformation \(q\) and those homeomorphism induced from \(\text{MCG}(\overline{C}_\Delta)\). The deck transformation \(q \in \text{MCG}^{\circ}(\log \overline{C}_\Delta)\) acts on \(\text{HS}(\log \overline{C}_\Delta)\) as
\[ (\overline{C}, f, \log f, h) \cdot q = (\overline{C}, f, \log f + 2\pi i, h). \]
The right action of \(g \in \text{MCG}^{\circ}(\log \overline{C}_\Delta)\) on \(\text{HS}(\log \overline{C}_\Delta)\) is given by
\[ (\overline{C}, f, \log f, h) \cdot g = (\overline{C}, f, \log f, h \circ g). \]

Choose a boundary component \(\partial_1 \subset \overline{C}_\Delta\) with \(k_1\) marked points and set \(\overline{\partial_1} := \pi^{-1}(\partial_1)\) for the covering map \(\pi_\Delta: \log \overline{C}_\Delta \to \overline{C}_\Delta\). For a \(\log \overline{C}_\Delta\)-framed Hurwitz cover \((\overline{C}, f, \log f, h)\), assume that a pole \(z_1\) of \(f\) corresponds to \(\overline{\partial_1}\) through the framing \(h\). Then by the automorphism of \(\overline{C}\), we can always take the representative \((\overline{C}, f, \log f, h)\) with \(z_1 = \infty\) and the coefficient of the pole of \(f\) at \(z_1 = \infty\) is one, namely
\[ f = z^{k_1} + O(z^{k_1+1}). \] (9.6)

Recall from Definition 6.28 that \(\text{QQuad}_q(\log \overline{C}_\Delta)\) is the moduli space of \(q\)-quadratic differentials with the numerical data \(\mathcal{N}(\log \overline{C}_\Delta) = (b, k, l)\).

**Theorem 9.9.** There is an isomorphism of complex manifolds
\[ \Psi_q: \text{HS}(\log \overline{C}_\Delta) \to \text{QQuad}_q(\log \overline{C}_\Delta) \]
for \(\xi = \xi(f, x, l)\) as in (9.4). Here we use the representative \(f\) with the form (9.6).
Proof. The existence and injectivity of the map $\Psi_s$ are explicit. We only need to show the surjectivity of $\Psi_s$. For a $q$-quadratic differential, by the automorphism of $\mathbb{C}$, we can take its representative as $(\mathbb{C}, \xi, h, s)$ with $z_1 = \infty$, where $z_1$ is an $s$-pole corresponding to $\bar{\partial}l$. In addition, we can normalize as

$$\xi = e(z_{k_1}^{(s-2)} + \cdots)\Omega_l,$$

where $c = q_s^{2m}$ for some $m \in \mathbb{Z}$. Take some point $\ast$ from the interior of $\log \mathbb{C}$. Then by Proposition 9.7, there is a regular Hurwitz cover $f: \mathbb{C} \to \mathbb{C}$ and a single-valued function $\log f$ on $\log \mathbb{C}$, up to multiplying by $\omega^{n-2}$ as in (9.5), such that

$$\xi = f^{s-2}\Omega_l \quad \text{and} \quad \xi(h(\ast)) = e((s-2)\log f(h(\ast)))\Omega_l(h(\ast)).$$

By the construction of $f$ in the proof of Proposition 9.7, we can take $f$ with the form

$$f = z_{k_1} + \cdots.$$ 

This implies the surjectivity of $\Psi_s$. □

Remark 9.10. As a complex manifold, the moduli space $Q_{\text{Quad}}(\log \mathbb{C})$ is independent of the date $l$ of $N(\log \mathbb{C})$. However, the period map $\Pi_s = \int \sqrt{\xi}$ strongly depends on $l$ since $\xi = f^{s-2}\Omega_l$. The choice $l$ corresponds to the choice of the primitive form [Sa1, SaTa] (the primary differential [D1, Section 5]) and controls the Frobenius structure on $\text{HS}(\log \mathbb{C})$.

Corollary 9.11. Consider the union

$$\text{QQuad}_{>2}(\log \mathbb{C}) := \bigcup_{s \in \mathbb{C}^2} Q_{\text{Quad}}(\log \mathbb{C}),$$

where $\mathbb{C}^2 = \{s \in \mathbb{C} \mid \text{Re}(s) > 2\}$. Then the space $\text{QQuad}(\log \mathbb{C})$ has the structure of a complex manifold of dimension $(n+1)$ induced from the bijection

$$\text{QQuad}_{>2}(\log \mathbb{C}) \cong \text{HS}(\log \mathbb{C}) \times \mathbb{C}^2$$

given by Theorem 9.9.

Passing to $q$-stability conditions, we have the following result on gluing $q$-stability conditions along $s$-direction for

$$\text{QStab}^q_{>2} \mathcal{D}_X(A) = \bigcup_{s \in \mathbb{C}^2} \text{QStab}^q \mathcal{D}_X(A),$$

which provides an answer for [IQ, Conj. 3.16].

Corollary 9.12. For any $s$ with $\text{Re}(s) > 2$, we have

$$\Xi_s \circ \Psi_s: \text{HS}(\log \mathbb{C}) \cong \text{QStab}^q \mathcal{D}_X(A),$$

where $\Xi_s$ and $\Psi_s$ are as in (7.9) and (9.7). Hence $\text{QStab}^q_{>2} \mathcal{D}_X$ admits the structure of a complex manifold of dimension $n+1$ and the projection map

$$\pi: \text{QStab}^q \mathcal{D}_X \to \mathbb{C}, \quad (\sigma, s) \mapsto s$$

is holomorphic.
Proof. In our case, $S = \mathbb{C}$ has genus zero and thus Corollary 7.7 applies. By Theorem 7.9, we have (7.12), i.e., $\text{QStab}_X(\mathbf{A}) \cong \text{QQuad}_X(\log C_\Delta)$ in this case. Thus the claim follows from Theorem 7.4 and Corollary 9.11 respectively. 

9.6. Disk case: type A example. Now consider the type A case, where $S = \mathbb{C}$, $b = 1$, $k = (n + 1)$ and $I = (4)$. In this case, the marked surface $C$ is a disk with $n + 1$ points on the boundary, which will be denoted by $D$. Consider the space of polynomials of degree $n + 1$

$$
\mathcal{M}_n := \{ f(z) \mid f(z) = z^{n+1} + a_1 z^{n-1} + a_2 z^{n-2} + \cdots + a_n, a_i \in \mathbb{C} \} \cong \mathbb{C}^n \quad (9.10)
$$

and its regular subset

$$
\mathcal{M}_{n,\text{reg}} := \{ f(z) \in \mathcal{M}_n \mid \text{all zeroes of } f(z) \text{ are simple} \}.
$$

Then we can regard $f(z) \in \mathcal{M}_{n,\text{reg}}$ as a genus 0 regular Hurwitz cover

$$
f : \mathbb{C} \to \mathbb{C}
$$

with a unique pole of order $n + 1$ at $z = \infty$.

Let $\mathfrak{h}$ be the Cartan subalgebra of type $A_n$ and $\mathfrak{h}_{\text{reg}} \subset \mathfrak{h}$ be the regular subset. Then there is an identification

$$
\mathcal{M}_{n,\text{reg}} \cong \mathfrak{h}_{\text{reg}}/W,
$$

where $W$ is the Weyl group of type $A_n$, namely the symmetric group of degree $n + 1$.

More details on $\mathfrak{h}$ in the Dynkin case will be discussed in Section 10.3. Denote by $\tilde{\mathcal{M}}_{n,\text{reg}}$ the universal cover of $\mathcal{M}_{n,\text{reg}}$.

Let $\omega_{n+1} = e^{2\pi i/(n+1)}$ be the $(n + 1)$-th root of unity and denote by $C_{n+1} := \langle \omega_{n+1} \rangle$ the cyclic group of order $n + 1$ generated by $\omega_{n+1}$. We define the action of $C_{n+1}$ on $\mathcal{M}_{n,\text{reg}}$ by

$$
(\omega_{n+1} \cdot f)(z) := f(\omega_{n+1}^{-1} z).
$$

Alternatively, we can write $\omega_{n+1} \cdot (a_i) = (\omega_{n+1}^{i+1} a_i)_i$ for coefficients $(a_i)_i$ of $f(z)$.

Lemma 9.13. There is an isomorphism

$$
\mathcal{M}_{n,\text{reg}}/C_{n+1} \cong \text{HS}(0, (n + 1))_{\text{reg}} \quad (9.11)
$$

by regarding a regular polynomial as a regular Hurwitz cover on $\mathbb{C}$.

Proof. It is straightforward to check that the only automorphisms of $\mathbb{C}$ which preserve $\mathcal{M}_{n,\text{reg}}$ are $z \mapsto \omega z$ for $\omega \in C_{n+1}$. Hence the claim follows. 

We note that the cyclic group $C_{n+1}$ can be identified with the marked mapping class group $\text{MCG}_\bullet(D)$, whose generator is the rotation $\tau$ that sends each marked point on the boundary to the next (clockwise) point. Recall that $\text{MCG}_\bullet$ here is only required to preserve $M$ setwise. Let $D_\Delta$ be the decorated version of $D$ (with $|\Delta| = n + 1$). It is well-known that

$$
\text{MCG}_\bullet(D_\Delta) = (\text{Br}_{n+1} \times \langle \tau \rangle) / \langle \tau^{n+1} \rangle,
$$

which follows from the short exact sequence

$$
0 \to \text{SBr}(D_\Delta) \to \text{MCG}_\bullet(D_\Delta) \to \text{MCG}_\bullet(D) \to 0
$$

and its regular subset

$$
\mathcal{M}_{n,\text{reg}} := \{ f(z) \in \mathcal{M}_n \mid \text{all zeroes of } f(z) \text{ are simple} \}.
$$
for

$$SBr(D_\Delta) = MCG(D_\Delta, \partial D_\Delta) = Br_{n+1}.$$ 

Here, $MCG(D_\Delta, \partial D_\Delta)$ is the classical mapping class group that preserves the boundary pointwise. More precisely, the $\tau^{n+1}$ is the Dehn twist along $\partial D_\Delta$, which is also the generator of the center of $Br_{n+1}$.

Let $\log D_\Delta$ be the topological (ungraded) log surface of type A. Then first we note that the lift of the action $\tau \in MCG^\circ(D)$ on $\log D_\Delta$ can be identified with the deck transformation $q \in MCG^\circ(\log D_\Delta)$. Then we have the following description of the mapping group $MCG^\circ(\log D_\Delta)$.

**Lemma 9.14.** As

$$MCG^\circ(\log D_\Delta) = (Br_{n+1} \times \langle q \rangle) / \langle q^{n+1} \rangle,$$

where $q^{n+1}$ acts on $Br_{n+1}$ as the lift of $D_\partial D_\Delta$, there is an isomorphism of the groups

$$MCG^\circ(\log D_\Delta) \cong Br_{n+1} \rtimes C_{n+1}.$$ 

**Proposition 9.15.** There is an isomorphism of complex manifolds

$$\widetilde{\Phi}_{n+1}: \widetilde{\mathcal{M}}_{n,\text{reg}} \to HS(\log D_\Delta),$$

as universal covering of

$$\Phi_{n+1}: HS(0, (n+1))_{\text{reg}} \to HS(\log D_\Delta) / MCG^\circ(\log D_\Delta) \to (C, f, \log f, h).$$

**Proof.** By Definition, we have (9.13).

Let

$$conf_m(X) := \{\{x_1, \ldots, x_n\} \in X^n \mid x_i \neq x_j, \forall i \neq j\}$$

be the $m$th (unordered) configuration space of $X$. Then there is a natural identification

$$\mathcal{M}_{n,\text{reg}} = \{f(z) = \prod_{k=1}^{n+1} (z - z_k) \mid z_k \in C, z_i \neq z_j, \forall i \neq j\}$$

$$= conf_{n+1}(C).$$

It is well-known that $\pi_1(\mathcal{M}_{n,\text{reg}}) = Br_{n+1}$. By Lemma 9.13, we have

$$\pi_1(HS(0, (n+1))_{\text{reg}}) = \pi_1(\mathcal{M}_{n,\text{reg}}/C_{n+1}) = Br_{n+1} \rtimes C_{n+1}.$$ 

Combining Lemma 9.14, we obtain the required covering version (9.12).

10. **Almost Frobenius structures for stability conditions: ADE case**

The aim of this section is to give a conjectural description of the spaces of $q$-stability conditions on the Calabi-Yau-X completions of type ADE quivers.
10.1. Almost Frobenius structures. We recall the definition of an almost Frobenius structure from [D1, § 3].

**Definition 10.1.** A Frobenius algebra \((A, *, (, ,))\) is a pair of a commutative associative \(\mathbb{C}\)-algebra \((A, \ast)\) with the unit \(E \in A\) and a non-degenerate symmetric bilinear form 
\[(,): A \otimes A \to \mathbb{C}\]
satisfying the condition 
\[(X \ast Y, Z) = (X, Y \ast Z)\]
for \(X, Y, Z \in A\).

For a complex manifold \(M\), denote by \(T_M\) the holomorphic tangent sheaf of \(M\).

**Definition 10.2.** An almost Frobenius manifold \((M, *, (, ), E, e)\) of the charge \(d \in \mathbb{C} (d \neq 1)\) consists of the following data:

- a complex manifold \(M\) of dimension \(n\),
- a commutative associative \(\mathcal{O}_M\)-linear product 
\[\ast: T_M \otimes T_M \to T_M\]
- a non-degenerate symmetric \(\mathcal{O}_M\)-bilinear form 
\[(,): T_M \otimes T_M \to \mathcal{O}_M\]
satisfying the condition 
\[(X \ast Y, Z) = (X, Y \ast Z)\]
for (local) holomorphic vector fields \(X, Y, Z\),
- global holomorphic vector fields \(E\) (called the Euler vector field) and \(e\) (called the primitive vector field).

They satisfy the following axioms:

1. The metric \((, )\) is flat.
2. Let \(p^1, \ldots, p^n\) be flat coordinates of the flat metric \((, )\) on some open subset \(U \subset M\) and set \(\partial_i := \frac{\partial}{\partial p^i}\). Then there is some function \(F(p) \in \mathcal{O}_M(U)\) such that the multiplication \(\ast\) can be written as
\[\partial_i \ast \partial_j = \sum_{k=1}^n C^k_{ij}(p) \partial_k,\]
where
\[C^k_{ij}(p) = \sum_{l=1}^n G^{kl} \partial_i \partial_j \partial_l F(p)\]
and \((G^{kl})\) is the inverse matrix of the metric \((G_{kl} = (\partial_k, \partial_l))\).

3. The function \(F(p)\) satisfies the homogeneity equation
\[\sum_{i=1}^n p_i \partial_i F(p) = 2F(p) + \frac{1}{1 - d} \sum_{i,j=1}^n G_{ij} p_i p_j.\]
(4) The Euler vector field takes the form

\[ E = \frac{1 - d}{2} \sum_{i=1}^{n} p^i \partial_i \]

and is the unit of the multiplication \(*\).

(5) The primitive vector field \( e \) is invertible with respect to the multiplication \(*\) and the derivation \( P_\nu(p) \mapsto e P_\nu(p) \) on the space of solutions of the following integrable equation with the parameter \( \nu \in \mathbb{C} \)

\[ \partial_i \partial_j P_\nu(p) = \nu \sum_{k=1}^{n} C_{ij}^k(p) \partial_k P_\nu(p) \tag{10.1} \]

acts as the shift of the parameter \( \nu \mapsto \nu - 1 \), i.e., if \( P_\nu(p) \) is the solution of the equation (10.1) with the parameter \( \nu \), then \( e P_\nu(p) \) is the solution with the parameter \( \nu - 1 \).

We note that the restriction of the multiplication and the metric on the tangent space \( T_p M \) for each point \( p \in M \) gives a family of Frobenius algebras \( (T_p M, *_{p}, (\cdot, \cdot)_p) \).

The flat coordinates \( p^1, \ldots, p^n \) of the flat metric \((\cdot, \cdot)\) are called periods and independent solutions \( P^1_\nu, \ldots, P^n_\nu \) of the equation in Axiom (5) are called twisted periods. The twisted periods satisfy the following quasi-homogeneity condition.

**Proposition 10.3.** The derivative of the twisted period \( P_\nu \) along with the Euler field \( E \) satisfies

\[ EP_\nu = \left( \frac{1 - d}{2} + \nu \right) P_\nu + \text{const.} \]

**Proof.** We show

\[ \partial_j EP_\nu = \left( \frac{1 - d}{2} + \nu \right) \partial_j P_\nu. \]

Since \([\partial_j, E] = \frac{1-d}{2} \partial_j\), the above equality is equivalent to \( E \partial_j P_\nu = \nu \partial_j P_\nu \). Recall that the Euler field is the unit of the Frobenius algebra, i.e. \( E * \partial_j = \partial_j \). This is equivalent to

\[ \frac{1 - d}{2} \sum_{i,k=1}^{n} p^i C^k_{ij} \partial_k = \partial_j. \]

Together with the equation (10.1) for the twisted period \( \partial_i \partial_j P_\nu(p) = \nu \sum_k C^k_{ij} \partial_k P_\nu \), we have

\[ E \partial_j P_\nu = \frac{1 - d}{2} \sum_{i=1}^{n} p^i \partial_i \partial_j P_\nu = \nu \frac{1 - d}{2} \sum_{i,k=1}^{n} p^i C^k_{ij} \partial_k P_\nu = \nu \partial_j P_\nu. \]

\[ \square \]

In the following, we always normalize the twisted period \( P_\nu \) to satisfy the quasi-homogeneity condition

\[ EP_\nu = \left( \frac{1 - d}{2} + \nu \right) P_\nu \tag{10.2} \]
by shifting some constant.

10.2. **Flat torsion-free connections and twisted periods.** In this section, we summarize the relation between flat torsion-free connections and affine structures on manifolds. In the following we consider in the category of complex manifolds and all vector fields, differential forms and connections are holomorphic. Let $M$ be a complex manifold of dimension $n$ and $\nabla$ be a connection of the holomorphic tangent bundle $TM$. We recall that $\nabla$ is called *torsion-free* if

$$\nabla_X Y - \nabla_Y X - [X,Y] = 0$$

for vector fields $X,Y$ on $M$. Recall that a connection $\nabla$ on the tangent bundle $TM$ induces the connection $\nabla$ on the cotangent bundle $T^*M$ by the formula

$$\langle d \langle X, \omega \rangle, \eta \rangle = \langle \nabla_X \omega, \eta \rangle + \langle X, \nabla_\eta \omega \rangle$$

for a vector field $X$ and a 1-form $\omega$ on $M$. The following statement is well-known.

**Proposition 10.4.** Let $M$ be a simply connected complex manifold of dimension $n$ and $\nabla$ be a flat torsion-free connection on $TM$. Then there are $n$ holomorphic functions $P^1, \ldots, P^n : M \to \mathbb{C}$ such that

$$d P^1 \wedge \cdots \wedge d P^n \neq 0$$

everywhere on $M$ and

$$\nabla d P^i = 0.$$

These functions are unique up to affine transformation

$$P^i \mapsto \sum_{j=1}^n A^i_j P^j + B^i,$$

where $(A^i_j) \in \text{GL}(n, \mathbb{C})$ and $(B^i) \in \mathbb{C}^n$.

Thus functions $P^1, \ldots, P^n$ give local coordinates on $M$. This coordinate system is called a *flat coordinate system*. Consider the vector space

$$V_M := \bigoplus_{i=1}^n \mathbb{C} d P^i.$$

Then since $d P^1, \ldots, d P^n$ are independent global sections of $T^*M$, we have a trivialization of the cotangent bundle

$$T^*M \cong V_M \times M$$

and the flat torsion-free connection $\nabla$ is just equal to the exterior differential $d$ on the trivial bundle.

Consider an almost Frobenius manifold $(M, \ast, (\cdot, \cdot), E, e)$. Let $\nabla$ be the Levi-Civita connection with respect to the flat metric $(\cdot, \cdot)$ on $M$. Fix a complex number $\nu \in \mathbb{C}$. Then we can define a flat torsion-free connection $\tilde{\nabla}$ on $TM$

$$\tilde{\nabla}_X Y := \nabla_X Y + \nu X \ast Y$$
for vector fields $X,Y$. Then the equation (10.1) for twisted periods $P_\nu$ is equivalent to the equation

$$\nabla dP_\nu = 0.$$  

**Remark 10.5.** The connection $\nabla$ is called the Dubrovin connection or the second structure connection of a Frobenius manifold. For details, we refer to [D1, § 3] and [M, § 2.1].

10.3. ADE type. We recall the almost Frobenius structure on the universal covering of the regular locus of the Cartan subalgebra in the case of type ADE following [D1, § 5.2]. Let $Q$ be a quiver of type ADE and $(L_Q, (\cdot, \cdot))$ be the root lattice associated with $Q$. Let $\Delta_+ \subset L_Q$ be the set of positive roots. The Cartan subalgebra is

$$\mathfrak{h} := \text{Hom}_\mathbb{Z}(L_Q, \mathbb{C})$$

and let $e_1, \ldots, e_n \in \mathfrak{h}$ be the dual basis of $\alpha_1, \ldots, \alpha_n$. Then

$$\mathfrak{h} \cong \bigoplus_{i=1}^n \mathbb{C} e_i$$

with metric $(e_i, e_j) := (A^{-1})_{ij}$, where $A^{-1}$ is the inverse of the Cartan matrix. We regard a root $\alpha \in \Delta_+$ as a function on $\mathfrak{h}$ via the pairing between $\mathfrak{h}$ and $L_Q$. Consider the regular subset

$$\mathfrak{h}_\text{reg} := \mathfrak{h} \setminus \bigcup_{\alpha \in \Delta_+} \ker \alpha$$

(10.3)

and its universal covering space

$$\tilde{M}_Q := \tilde{\mathfrak{h}}_{\text{reg}}.$$  

When $Q$ is an $A_n$ quiver, these definitions/notations match the ones in Section 9.6. In the following, we construct the almost Frobenius structure on $\tilde{M}_Q$.

Let $\alpha_1, \ldots, \alpha_n$ be simple roots and introduce the coordinate on $\mathfrak{h}$ given by $p^i := \alpha_i(p)$ for $p \in \mathfrak{h}$, namely $p = \sum_{i=1}^n p^i e_i$. Then the coordinates $p^1, \ldots, p^n$ also form the local coordinates on $\tilde{M}_Q$ through the projection $\tilde{M}_Q \to \mathfrak{h}_{\text{reg}}$. Since $T\tilde{M}_Q \cong \mathfrak{h} \times \tilde{M}_Q$, the metric $(\cdot, \cdot)$ on $\mathfrak{h}$ induces the flat metric on $\tilde{M}_Q$ and the coordinates $p^1, \ldots, p^n$ form the flat coordinates.

Define the potential function $F: \tilde{M}_Q \to \mathbb{C}$ by

$$F(p) := \frac{1}{2} \sum_{\alpha \in \Delta_+} \alpha(p)^2 \log \alpha(p).$$

Since $\tilde{M}_Q$ is the universal covering of $\mathfrak{h}_{\text{reg}}$, $F$ is a single-valued holomorphic function on $\tilde{M}_Q$. On an open subset $U \subset \tilde{M}_Q$ with the flat coordinate $p^1, \ldots, p^n$, the third derivative of $F$ is given by

$$C_{ijk}(p) := \partial_i \partial_j \partial_k F(p) = \sum_{\alpha \in \Delta_+} \frac{\alpha(e_i)\alpha(e_j)\alpha(e_k)}{\alpha(p)}.$$  

(10.4)
This defines the multiplication * together with the inverse of the metric, which is just the Cartan matrix \((dp^i, dp^j) = A_{ij}\). The charge \(d\) is given by
\[
d = 1 - \frac{2}{h},
\]
where \(h\) is the Coxeter number. In particular, the Euler field takes the form
\[
E = \frac{1}{h} \sum_{i=1}^{n} p^i \partial_i.
\]

For the construction of the primitive vector field \(e\), we refer to [D1, § 5.2].

### 10.4. Coxeter-KZ connections
Following [C] (and the survey [TL, § 4]), we consider a certain integrable connection on \(h_{\text{reg}}\), called the Coxeter-KZ connection.

As in the previous section, let \((L_Q, (, ))\) be the root lattice associated with a quiver \(Q\) of type ADE. For a root \(\alpha \in \Delta\), consider reflections
\[
r_\alpha: L_Q \to L_Q, \quad \lambda \mapsto \lambda - (\lambda, \alpha)\alpha,
\]
for \(\alpha \in L_Q\). These reflections form the Weyl group
\[
W_Q := \langle r_\alpha | \alpha \in \Delta \rangle,
\]
which naturally acts on the Cartan subalgebra \(h\).

Consider a representation of \(W_Q\) on a finite dimensional vector space \(V\):
\[
\rho: W_Q \to \text{GL}(V).
\]

On the trivial bundle \(V \times h_{\text{reg}} \to h_{\text{reg}}\), we define the Coxeter-KZ (CKZ) connection \(\nabla_{\nu}^{\text{CKZ}}\) by
\[
\nabla_{\nu}^{\text{CKZ}} := d - \nu \sum_{\alpha \in \Delta_+} (\rho(r_\alpha) - \text{id}_V) \frac{d\alpha}{\alpha},
\]
where \(\nu \in \mathbb{C}\) is some complex number.

**Theorem 10.6.** The connection \(\nabla_{\nu}^{\text{CKZ}}\) is a \(W\)-invariant flat connection on \(V \times h_{\text{reg}}\).

The monodromy representation
\[
\rho_\nu: \pi_1(h_{\text{reg}}/W) \cong \text{Br}_Q \to \text{GL}(V)
\]
factors through the Hecke algebra \(H_Q\):
\[
\begin{array}{c}
\text{R}[\text{Br}_Q] \quad \text{R}[\text{GL}(V)] \\
\xrightarrow{H_Q} \quad \xrightarrow{\rho_\nu, (\psi_i)} \\
\text{psi} \quad \text{psi} \quad \text{psi}
\end{array}
\]
\[
\begin{array}{c}
\xleftarrow{T_i} \quad \xleftarrow{T_i} \quad \xleftarrow{T_i}
\end{array}
\]
where we specialize \(q = e^{2i\pi \nu}\) for the Hecke algebra \(H_Q\).

**Proof.** See [C] and also the survey [TL, § 4]. \(\square\)
Remark 10.7. In [TL], the CKZ connection is given by
\[ \nabla^\nu_{\text{CKZ}} = d - \nu \sum_{\alpha \in \Delta_+} \rho(r_\alpha) \frac{d\alpha}{\alpha}. \]

Two connections \( \nabla^\nu_{\text{CKZ}} \) and \( \nabla'^\nu_{\text{CKZ}} \) are related as the following gauge transformation. Consider the function
\[ f : = \prod_{\alpha \in \Delta_+} \alpha \]
on \( \mathfrak{h} \). Then the equality
\[ \nabla'^\nu_{\text{CKZ}} = f^\nu \circ \nabla^\nu_{\text{CKZ}} \circ f^{-\nu} \]
holds since
\[ d f^{-\nu} = -\nu \sum_{\alpha \in \Delta_+} \text{id}_{\nu} \frac{d\alpha}{\alpha}. \]

The connection \( \nabla^\nu_{\text{CKZ}} \) can be naturally extended on \( \tilde{M}_Q = \tilde{\mathfrak{h}}_{\text{reg}} \). As the special case, we consider the CKZ connection on the tangent bundle of \( \tilde{M}_Q \). We note that the tangent bundle of \( M \) is given by
\[ T\tilde{M}_Q \cong \mathfrak{h} \times \tilde{M}_Q. \]

Since the Weyl group \( W \) acts on \( \mathfrak{h} \) as the reflection group, we have the CKZ connection on \( T\tilde{M}_Q \).

Proposition 10.8. The CKZ connection \( \nabla^\nu_{\text{CKZ}} \) on \( T\tilde{M}_Q \) coincides with the Dubrovin connection \( \tilde{\nabla} \). In particular, \( \nabla^\nu_{\text{CKZ}} \) on \( T\tilde{M}_Q \) is torsion-free and the twisted period \( P^\nu \) satisfies
\[ \nabla^\nu_{\text{CKZ}} d P^\nu = 0. \]

Proof. First we note that under the identification \( T\tilde{M}_Q \cong \mathfrak{h} \times \tilde{M}_Q \), the vector field \( \partial_i = \frac{\partial}{\partial p_i} \) is identified with \( e_i \). By using equation (10.4) we have
\[ \tilde{\nabla}_{\partial_i} \partial_j = \nu \partial_i * \partial_j = \nu \sum_{\alpha \in \Delta_+} \sum_{k,l} \frac{\alpha(e_i)\alpha(e_j)\alpha(e_k)}{\alpha(p)} G^{kl} \partial_l. \]

On the other hand,
\[ \nabla^\nu_{\psi,e_i} e_j = -\nu \sum_{\alpha \in \Delta_+} (r_\alpha(e_j) - e_j) \frac{\langle d\alpha, e_i \rangle}{\alpha(p)} = \nu \sum_{\alpha \in \Delta_+} (e_j - r_\alpha(e_j)) \frac{\alpha(e_i)}{\alpha(p)}. \]

Since
\[ e_j - r_\alpha(e_j) = e_j - (e_j - \alpha(e_j) \alpha') = \alpha(e_j) \sum_{k,l} \alpha(e_k) G^{kl} e_l, \]
the result follows. \( \square \)

Thus the twisted periods for Frobenius manifolds of type ADE can be characterized as the flat coordinates of the CKZ connections. Together with Theorem 10.6, we have the following.
Corollary 10.9. The monodromy representation of $\text{Br}_Q$ for the twisted period map $P_\nu$ factors through the Hecke algebra $H_Q$.

Remark 10.10. In $[\text{Sa}1]$, Saito defined the period map associated with the primitive form on the unfolding of an isolated singularity. He also defined the holonomic D-module (with the complex parameter $s \in \mathbb{C}$) whose solutions characterize his period map. The above twisted period maps $P_\nu$ on almost Frobenius manifolds of type ADE correspond to Saito’s period maps in the case that isolated singularities are of type ADE. For the details, we refer to $[\text{Sa}1, \S 5]$, $[\text{Sa}2, \S 4]$ and $[\text{D}1, \S 5.1]$.

10.5. Cotangent bundles of the spaces of stability conditions. Let $\mathcal{D}_X$ be a triangulated category satisfying Assumption 1.4 and take a basis $[E_1], \ldots, [E_n]$ of $K(\mathcal{D}_X)$. Then

$$K(\mathcal{D}_X) \cong \bigoplus_{i=1}^n R[E_i].$$

Consider holomorphic functions

$$Z^i: \text{QStab}_s \mathcal{D}_X \to \mathbb{C} \quad (i = 1, \ldots, n)$$

given as the components of the central charge defined by $Z^i(\sigma) := Z(E_i)$ for $\sigma = (Z, P) \in \text{QStab}_s \mathcal{D}_X$. Consider holomorphic 1-forms $dZ^1, \ldots, dZ^n$ on $\text{QStab}_s \mathcal{D}_X$ and define a vector space

$$V_S = \bigoplus_{i=1}^n \mathbb{C} dZ^i.$$

The following is a direct consequence of Theorem 1.5.

Proposition 10.11. The cotangent bundle of the space $\text{QStab}_s \mathcal{D}_X$ is the trivial bundle given by

$$T^* \text{QStab}_s \mathcal{D}_X \cong V_S \times \text{QStab}_s \mathcal{D}_X.$$

Proof. By Theorem 1.5, for any small open subset $U \subset \text{QStab}_s \mathcal{D}_X$, functions $Z^1, \ldots, Z^n$ form local coordinates on $U$. This implies the result. □

If we consider the trivial connection $d$ on $V_S \times \text{QStab}_s \mathcal{D}_X$, the central charges $Z^i$ satisfies the trivial equation

$$d dZ^i = 0.$$

Together with the content of Section 10.2, this suggests the following fundamental problem on the spaces of stability conditions.

Problem 10.12. To find a simply connected complex manifold $M$ and a torsion-free flat connection $\nabla$ such that there is an isomorphism of complex manifolds

$$\phi: M \to \text{QStab}_s \mathcal{D}_X$$

satisfying $\phi^* d = \nabla$. 
In the next subsection, we shall give a conjectural answer for $D_X(Q)$ with type ADE-quiver $Q$.

If we find an isomorphism $\phi$ in the above problem, then the flat coordinates $P^i$ of $\nabla$ on $M$ can be identified with the central charges $Z^i$ up to affine transformation. In Section 10.1, we fix the ambiguity of constant shift of twisted periods by the equation (10.2) via the Euler field. Correspondingly, we introduce the Euler field in the side of stability conditions as follows. Let $\frac{\partial}{\partial Z^i}$ be the dual basis of $dZ^i$. Then the tangent bundle of $Q_{\text{Stab}} D_X(Q)$ is given by

$$T Q_{\text{Stab}} D_X(Q) \cong \left( \bigoplus_{i=1}^n \mathbb{C} \frac{\partial}{\partial Z^i} \right) \times Q_{\text{Stab}} D_X(Q).$$

The Euler vector field $\mathcal{E}$ of charge $d$ on $Q_{\text{Stab}} D_X(Q)$ is defined by

$$\mathcal{E} := \left( \frac{1 - d}{2} + \frac{s - 2}{2} \right) \sum_{i=1}^n Z^i \frac{\partial}{\partial Z^i}. $$

Then central charges $Z^i$ clearly satisfy the quasi-homogeneity condition

$$\mathcal{E} Z^i = \left( \frac{1 - d}{2} + \frac{s - 2}{2} \right) Z^i.$$

10.6. Conjectural descriptions. Similar to Definition 4.3, we have the Calabi-Yau-$X$ version of a path algebra.

**Definition 10.13.** [IQ, § 2.4] Let $Q$ be a quiver of type ADE. Denote by $\Gamma_X Q$ the associated Calabi-Yau-$X$ Ginzburg $\mathbb{Z}^2$-graded differential algebra, $D_X(Q)$ its finite dimensional derived category and $Q_{\text{Stab}} D_X(Q)$ the space of $q$-stability conditions on $D_X(Q)$.

**Conjecture 10.14.** Assume that $\text{Re}(s) \geq 2$ and set $\nu = (s - 2)/2$. There is a $\text{Br}_Q$-equivariant isomorphism $\hat{\varphi}_s$ of complex manifolds such that the diagram

$$\begin{array}{ccc}
\tilde{M}_Q & \xrightarrow{\hat{\varphi}_s} & Q_{\text{Stab}} D_X(Q) \\
P_v \downarrow & & \downarrow Z_s \\
\mathbb{C}^n & & 2_s
\end{array}$$

commutes. In particular, $\hat{\varphi}_s$ satisfies that

$$\begin{cases}
\hat{\varphi}_s^* E = E, \\
\hat{\varphi}_s^* d = \nabla_{\nu}^{\text{CKZ}}
\end{cases}$$

for Euler fields and torsion-free flat connections on each side.

10.7. Type A case. When $Q$ is of type A, i.e., $S = D$ is a (graded) disk with $n + 1$ marked points on its boundary as in Section 9.6. Take a full formal arc system $A$ of $D$ and we identify $D_X(Q) = D_X(A)$.

Then putting the results of Corollary 9.12 and Proposition 9.15 together, we have the identification

$$Q_{\text{Stab}} D_X(A_n) \cong \tilde{M}_{A_n} = \mathfrak{h}_{\text{reg}}/W$$  (10.6)
via moduli spaces of (framed) $q$-quadratic differentials and (framed) Hurwitz spaces. Thus, this implies the following.

**Theorem 10.15.** Assume that $\text{Re}(s) > 2$. Let $Q$ be the $A_n$-quiver. Then Conjecture 10.14 holds.

**Remark 10.16.** For integer valued $s$, there are following previous works:

- $s = 2$ is essentially shown in [B5];
- $s = 3$ is essentially shown in [BS];
- $s \in \mathbb{Z}_{\geq 2}$ is shown in [I] based on the theory of [BS].

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