CM VALUES OF REGULARIZED THETA LIFTS AND HARMONIC WEAK MAASS FORMS OF WEIGHT ONE

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ABSTRACT. We study special values of regularized theta lifts at complex multiplication (CM) points. In particular, we show that CM values of Borcherds products can be expressed in terms of finitely many Fourier coefficients of certain harmonic weak Maass forms of weight one. As it turns out, these coefficients are logarithms of algebraic integers whose prime ideal factorization is determined by special cycles on an arithmetic curve. Our results imply a conjecture of Duke and Li [DL15] and give a new proof of the modularity of a certain arithmetic generating series of weight one studied by Kudla, Rapoport and Yang [KRY99].

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1. INTRODUCTION

Special values of automorphic forms often encode interesting arithmetic information. The type of automorphic forms we consider in this article are obtained as regularized theta lifts of harmonic weak Maass forms. We will study their values at CM points on orthogonal Shimura varieties. This includes the CM values of Borcherds products and as a special case (generalizations of) the singular moduli considered by Gross and Zagier in their seminal paper [GZ85]. The average value over a CM cycle was previously studied by Bruinier-Yang [BY09] and Schofer [Sch09] using a seesaw dual pair and the Siegel-Weil formula. We use the same seesaw identity without applying the Siegel-Weil formula to relate the values at individual points to coefficients of harmonic weak Maass forms of weight one. We then show that the coefficients of an appropriate normalization of these weak Maass forms are logarithms of algebraic numbers whose prime factorizations are determined by certain special cycles on the stack of elliptic curves with complex multiplication. We use this relation to prove a conjecture by Duke and Li [DL15].

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1.1. Setup and notation. Let $L$ be an even lattice with quadratic form $Q$ of type $(2, n)$ and write $V = L \otimes_{\mathbb{Z}} \mathbb{Q}$ for the corresponding rational quadratic space. We write $(\cdot, \cdot)$ for the associated bilinear form, such that $(x, x) = 2Q(x)$. Associated with such a lattice is a finite quadratic module given by the discriminant group $A_L = L' / L$, where $L'$ is the dual of $L$ with respect to $(\cdot, \cdot)$, and the reduction of $Q$ modulo $\mathbb{Z}$. For simplicity, we assume in the introduction that $n$ is even. The Weil representation [Wei64; Shi75; Bor98] $\rho_L$ of $\text{SL}_2(\mathbb{Z})$ associated with $L$ acts on the group ring $S_L = \mathbb{C}[A_L]$. We write $\phi_\mu$ with $\mu \in A_L$ for the standard basis elements of $S_L$. Moreover, we write $(\cdot, \cdot)$ for the $\mathbb{C}$-bilinear pairing such that $\langle \phi_\mu, \phi_\mu \rangle = 1$ and $\langle \phi_\mu, \phi_\lambda \rangle = 0$ for $\lambda \neq \mu$.

We denote the space of $S_L$-valued weakly holomorphic modular forms of weight $k$ transforming with representation $\rho_L$ by $M_{k,L}$ and the spaces of holomorphic modular forms and cusp forms by $M_{k,L}^!$ and $S_{k,L}$, respectively. We denote the hermitian symmetric domain associated with the orthogonal group $\text{SO}_V(\mathbb{R})$ by $\mathbb{D}$ and realize $\mathbb{D}$ as the Grassmannian of oriented 2-dimensional positive definite subspaces of $V(\mathbb{R}) = V \otimes_{\mathbb{Q}} \mathbb{R}$.

A function $f : \mathbb{H} \to S_L$ is called a harmonic weak Maaß form of weight $k$ and representation $\rho_L$ if it transforms like a modular form of weight $k$ with respect to $\rho_L$, is harmonic with respect to the weight $k$ Laplace operator and grows at most exponentially at the cusp $\infty$. We write $\mathcal{H}_{k,L}$ for the space of such functions. An element $f \in \mathcal{H}_{k,L}$ admits a unique decomposition $f = f^+ + f^-$ into a holomorphic part $f^+$ and a non-holomorphic part $f^-$. We refer to Section 2 for more details.

The antilinear differential operator $\xi_k$ defined by
\[
\xi_k(f)(\tau) := 2iv^k \frac{\partial}{\partial \tau} f(\tau)
\] (1.1)
plays an important role in the theory of harmonic weak Maaß forms. It was shown by Bruinier and Funke [BF04] that $\xi_k : \mathcal{H}_{k,L} \to M_{2-k,L}^!$ is surjective [BF04, Theorem 3.7] with kernel $M_{k,L}$.

Here, the lattice $L^-$ is given by the lattice $L$ together with the quadratic form $-Q$. The associated Weil representation can be identified with the dual of $\rho_L$. We denote by $H_{k,L} \subset \mathcal{H}_{k,L}$ the subspace of those harmonic weak Maaß forms that map to a cusp form under $\xi_k$.

Let $H = \text{GSpin}_V$ be the general spin group, which is a central extension of the special orthogonal group $\text{SO}_V$ and let $K \subset H(\mathbb{A}_f)$ be a compact open subgroup such that $K$ stabilizes $L$ and acts trivially on $A_L$; here, $\mathbb{A}_f$ are finite adeles of $\mathbb{Q}$. There is a Shimura variety $X_K$ with complex points
\[
X_K(\mathbb{C}) = H(\mathbb{Q}) \backslash (\mathbb{D} \times H(\mathbb{A}_f) / K).
\]
For an appropriate choice of $K$, the Siegel theta function $\Theta_L(\tau, z, h)$ attached to $L$ defines a non-holomorphic function on $X_K$ (here, $\tau \in \mathbb{H}$, $z \in \mathbb{D}$, and $h \in H(\mathbb{A}_f)$). In $\tau \in \mathbb{H}$, it transforms like a modular form of weight $(2-n)/2$ with representation $\rho_L$, but is also non-holomorphic.

The regularized theta lift of $f \in H_{k,L}$ with $k = (2-n)/2$ is defined as
\[
\Phi_L(z, h, f) = \int_{\text{SL}_2(\mathbb{Z}) \backslash \mathbb{H}} (f(\tau), \Theta_L(\tau, z, h)) v^k \frac{dudv}{v^2};
\] (1.2)
where $\tau = u + iv \in \mathbb{H}$ and the integral is regularized using Borcherds’ regularization (following Harvey-Moore) [Bor98]. We obtain a $\Gamma_L$-invariant function $\Phi_L(z, h, f)$ which is real analytic outside a divisor $Z(f)$ given by codimension one sub-Grassmannians of $\mathbb{D}$ depending only on the Fourier coefficients of $f^+$ with negative index, its principal part.

1.2. CM values. We will now describe the CM points we are considering. Let $U \subset V(\mathbb{Q})$ be a rational 2-dimensional positive definite subspace of $V$. Then $U$ defines two rational points $z_U^\pm$ in $\mathbb{D}$, given by $U(\mathbb{R})$ together with the two possible orientations. Let $T = \text{GSpin}_U$ and $K_T = K \cap T(\mathbb{A}_f)$. 

We obtain a CM cycle in $X_K$ by considering the Shimura variety with complex points

$$Z(U)(\mathbb{C}) = T(\mathbb{Q}) \backslash \{ \mathbb{Z}^{1/2} \} \times T(\mathbb{A}_f)/K_T \hookrightarrow X_K(\mathbb{C}).$$

For simplicity, we assume in the introduction that $L = P \oplus N$, where $P = L \cap U$ is positive definite and 2-dimensional and $N = L \cap U^\perp$ is a negative definite $n$-dimensional lattice in $V$. Under this assumption, we have $S_L \cong S_P \otimes_{\mathbb{C}} S_N$ and $\Theta_L(\tau, z_U^\perp, h) = \Theta_P(\tau, h) \otimes_{\mathbb{C}} \Theta_N(\tau)$ for $h \in T(\mathbb{A}_f)$.

Our first result is an analytic formula for the CM value $\Phi_L(z, h, f)$ for any $(z, h) \in Z(U)$.

**Theorem 1.1.** Let $f \in H_{1-n/2, L}$ and let $\mathring{\Theta}_P(\tau, h) \in \mathcal{H}_{1-P}$ be a harmonic weak Maaß form of weight 1 with the property that $\xi_1(\mathring{\Theta}_P(\tau, h)) = \Theta_P(\tau, h)$. Then for any $(z, h) \in Z(U)$ the value of $\Phi_L(z, h, f)$ is given by

$$\Phi_L(z, h, f) = CT \left( \langle f^+(\tau), \Theta_N^-(\tau) \otimes \mathring{\Theta}_P^+(\tau, h) \rangle \right)$$

$$- \int_{\text{SL}_2(\mathbb{Z}) \backslash \mathbb{H}} \xi_{1-n/2}(f) \langle \Theta_N^-(\tau) \otimes \mathring{\Theta}_P^+(\tau, h) \rangle \frac{v^{1+n/2} \, du dv}{v^2}. \tag{1.3}$$

Here, we write $CT(\cdot)$ for the constant term in the Fourier expansion. It is a finite sum of products of coefficients of $f^+$ and $\Theta_N^-(\tau) \otimes \mathring{\Theta}_P^+(\tau, h)$. Note that if $f$ is weakly holomorphic, then the regularized integral above vanishes, as $\xi_{1-n/2}(f) = 0$ in that case. Moreover, in this case the left-hand side of (1.3) is the logarithm of a Borcherds product $[\text{Bor98}]$, a meromorphic modular form for $\Gamma_L$ with divisor $Z(f)$. In particular, we obtain a formula for CM values of Borcherds products which involves only a finite number of coefficients of $\mathring{\Theta}_P^+(\tau, h)$ weighted by representation numbers of the lattice $N$ and the coefficients of $f^+$. Note that $\Theta_P(\tau, h)$ is not uniquely determined. It can be modified by adding any weakly holomorphic modular form and the theorem is still valid.

If we consider the weighted average value of $\Phi_L(z, h, f)$ over the CM-cycle $Z(U)$, we obtain a slight generalization of the results of Schofer [Sch09] (who considered weakly holomorphic $f$) and Bruinier-Yang [BY99]. Their formula involves the function $\mathcal{E}_P(\tau)$, a special value of the derivative of an incoherent Eisenstein series. The coefficients of $\mathcal{E}_P(\tau)$ have been calculated in many cases, also by Schofer [Sch09], Bruinier and Yang [BY99] and Kudla and Yang [KY10]. The Eisenstein series $\mathcal{E}_P(\tau)$ is a harmonic weak Maaß form and has the property that $\xi_1(\mathcal{E}_P(\tau)) = E_P(\tau)$ is the genus Eisenstein series attached to $P$. Using these explicit formulas, it was shown in [KRY99] and [BY99], that $\mathcal{E}_P(\tau)$ is in fact the generating series of the arithmetic degrees of certain special cycles on an arithmetic moduli stack.

### 1.3. Harmonic weak Maaß forms and arithmetic geometry

Motivated by these results and to make Theorem 1.1 more explicit, we study the coefficients of the holomorphic parts of appropriate choices for $\mathring{\Theta}_P(\tau, h)$ and their arithmetic meaning. We will show that they are also related to the special cycles studied by Kudla, Rapoport and Yang [KRY04] and give a new proof of the modularity of the degree generating series. We let $k$ be an imaginary quadratic number field of discriminant $D < 0$. We assume in the introduction that $D = -l$ for a prime $l \equiv 3$ mod 4 with $l > 3$ (so that the class number of $k$ is odd and there is only one genus). We will remove this restriction in the body of the paper and work with odd fundamental discriminants. We write $\text{Cl}_k$ for the class group of $k$ and $h_k$ denotes the class number of $k$. Let $P = \mathcal{O}_k$ be the ring of integers in $k$, which is a lattice of type $(2, 0)$ together with the quadratic form $Q(x) = N(x) = xx'$, where $x \mapsto x'$ denotes the non-trivial Galois automorphism of $k$. The dual lattice $P^* = \partial_k^{-1}$ is given by the inverse different in $k$ and the discriminant group $P^*/P$ is cyclic of order $|D|$. If we let $K = \hat{\mathbb{O}}_k^\times = \mathcal{O}_k^\times \otimes_{\mathbb{Z}} \hat{\mathbb{Z}}$ and denote $H = \text{GSpin}_V$ for $V = k$, then $H(\mathbb{A}_f)$ is given by the finite
ideles $\mathbb{A}_k^\times$ over $k$, $\mathbb{D} = \{z^\pm\}$ consists of two points (given by $k \otimes_{\mathbb{Q}} \mathbb{R}$ with the two possible choices of orientation), and $X_K(C)$ is given by two copies of the class group (see also Lemma 3.1)

$$X_K(C) = H(\mathbb{Q}) \backslash \{\{z^\pm\} \times H(\mathbb{A}_f)/K \cong \{z^\pm\} \times \text{Cl}_k.$$  

Moreover, if $a$ is the ideal determined by the idele $h$, we can identify the theta function $\Theta_P(\tau, h) = \Theta_P(\tau, z^\pm, h)$ with

$$\Theta_{a^2}(\tau) = \sum_{\mu \in \mathfrak{O}_k^{-1}a^2/a^2} \sum_{x \in a^2 + \mu} e \left( \frac{N(x)}{N(a^2)} \tau \right) \phi_\mu,$$

which only depends on the class $[a]^2 \in \text{Cl}_k$ of $a^2$. Note that since the class number $\text{Cl}_k$ is odd, every class $[b] \in \text{Cl}_k$ is of the form $[b] = [a]^2$ for some $a$ (i.e. there is only one genus).

Let $C_D$ be the (Deligne-Mumford) moduli stack of elliptic curves with complex multiplication by $\mathcal{O}_k$. The coarse moduli scheme of $C_D$ is isomorphic to $\text{Spec} \mathcal{O}_k$, where $\mathcal{O}_k$ is the Hilbert class field of $k$. Kudla, Rapoport and Yang define cycles $Z(m)$ on this arithmetic curve given by elliptic curves with certain “special endomorphisms”. These endomorphisms only occur in positive characteristic and the cycles $Z(m)$ are always supported in the fiber above a unique prime $p$, which is non-split in $k$. They showed that the degree generating series

$$\sum_{m \in \mathfrak{O}_k^\times, m > 0} \hat{\deg} Z(m)e(m \tau)(\phi_m + \phi_{-m}) + 2\Lambda'(\chi_D, 0)\phi_0,$$

is the holomorphic part of $-h_k\mathcal{E}_P(\tau)$. Here, we simply wrote $\phi_m$ for $\phi_\mu$ if $m \in \mathbb{Z}$ is an integer with $Q(\mu) + m \in \mathbb{Z}$ and $\Lambda'(\chi_D, s)$ denotes the derivative of the completed Dirichlet $L$-function $\Lambda(\chi_D, s) = |D|^{\frac{s}{2}}\pi^{-\frac{s+1}{2}}\Gamma\left(\frac{s+1}{2}\right) L(\chi_D, s)$ for the character $\chi_D = (\frac{D}{\cdot})$. If $m$ is not represented by $-Q$ modulo $\mathbb{Z}$, then the corresponding coefficient vanishes. The modularity of the generating series (1.4) is proven in [KRY99] by explicitly computing the Fourier coefficients of $\mathcal{E}_P(\tau)$ and the arithmetic degrees separately.

As no “explicit” construction of the forms $\widetilde{\Theta}_P(\tau, h)$ is known, we have to resort to a different method to show a relation to the cycles $Z(m)$. To state this second result in more detail, we write the pushforward (which has to be appropriately normalized) of the cycle $Z(m)$ to $\text{Spec} \mathcal{O}_k$ as an Arakelov divisor with vanishing archimedean contribution as

$$Z(m) = \sum_{\mathfrak{P} \subset \mathcal{O}_k} Z(m)_\mathfrak{P} \mathfrak{P},$$

where the sum runs over all nonzero prime ideals of $\mathcal{O}_k$. The arithmetic degree above is obtained as

$$\hat{\deg} Z(m) = \sum_{\mathfrak{P} \subset \mathcal{O}_k} Z(m)_\mathfrak{P} \log N_{\mathcal{O}_k/\mathbb{Q}}(\mathfrak{P}).$$

We obtain the following result which follows from Theorem 4.21 in Section 4.4.

**Theorem 1.2.** For every $[a] \in \text{Cl}_k$ there is a harmonic weak Maaß form $\widetilde{\Theta}_{a^2}(\tau) \in \mathcal{H}_{1,p}$ with holomorphic part

$$\widetilde{\Theta}_{a^2}^+(\tau) = \sum_{m > -\infty \atop m \in \mathfrak{O}_k^\times} c^+(a^2, m)e(m \tau)(\phi_m + \phi_{-m})$$

satisfying the following properties.
(i) We have \(\xi_1(\tilde{\Theta}_a(\tau)) = \Theta_a(\tau)\) for every \(a\) and the harmonic weak Maass form

\[
\tilde{E}_P(\tau) := \frac{1}{h_k} \sum_{[i] \in \mathcal{O}_k} \tilde{\Theta}_a(\tau)
\]

satisfies \(\xi_1(\tilde{E}_P(\tau)) = E_P(\tau)\), where \(E_P(\tau)\) is the Eisenstein series attached to \(P\) and the Fourier coefficients of negative index of \(\tilde{E}_P(\tau)\) vanish.

(ii) For all \(m \in \frac{1}{|D|} \mathbb{Z}\) with \(\chi_D(|D|m) = 1\), we have \(c^+(a^2, m) = 0\).

(iii) If \(\chi_D(|D|m) \neq 1\), then

\[
c^+(a, m) = -\frac{2}{r} \log |\alpha(a^2, m)|,
\]

where \(\alpha(a, m) \in \mathcal{O}_{H_k}\) and \(r \in \mathbb{Z}_{>0}\) depends only on \(D\).

(iv) Furthermore, we have for \(m > 0\) that

\[
\text{ord}_{\mathfrak{p}}(\alpha(a^2, m)) = 2r \mathbb{Z}(m)_{\mathfrak{p}^e}
\]

for all prime ideals \(\mathfrak{p} \subset \mathcal{O}_{H_k}\), where \(\sigma = \sigma(a^{-1}) \in \text{Gal}(H_k/k)\) corresponds to the image of the ideal class of \(a^{-1}\) under the Artin map of class field theory.

(v) If \(m < 0\) with \(\chi_D(-Dm) \neq 1\), then \(\alpha(a, m) \in \mathcal{O}_{H_k}^\times\).

Note that the theorem implies that we can write \(c^+(a, m) = \log(\beta(a, m))\), where \(\beta(a, m)\) is contained in a finite field extension of \(H_k\).

The idea of the proof of Theorem 1.2 (or Theorem 4.21 in the body of the paper) is to use certain seesaw dual reductive pairs \([\text{Kud84}]\). The following simple identity is central for our argument:

\[
\langle f(\tau) \otimes_C g(\tau), \Theta_P(\tau, h) \otimes_C \Theta_N(\tau) \rangle = \Phi_P(z_{Df}^+, h, f) \cdot \langle g(\tau), \Theta_N(\tau) \rangle,
\]

for \(f\) valued in \(S_P\) and \(g\) valued in \(S_N\) and where \(h \in T(\mathbb{A}_f)\). We use this together with Theorem 1.1 to relate the coefficients of \(\tilde{\Theta}_P(\tau, h)\) (occurring via Theorem 1.1 in the first factor on the right-hand side of (1.6)) to Borcherds products of weight zero on modular curves (the left-hand side of (1.6) in a special case). These Borcherds products are modular functions with special divisors. The second factor on the right-hand side can be shown to be constant for a special choice of \(g\) giving the desired relation. We then apply the main result of \([\text{Ehl15}]\) which relates the CM values of Borcherds products to the special cycles.

Duke and Li \([\text{DL15}]\) independently obtained related results on the coefficients of such harmonic weak Maass forms in the case of a prime discriminant using different methods. Based on numerical evidence, they formulated a conjecture on the prime factorization of the algebraic numbers \(\alpha(a, m)\). To state the conjectured formula, which we prove in this paper, we need a bit more notation. For \(m \in \mathbb{Q}_{>0}\), we define a set of rational primes by

\[
\text{Diff}(m) = \{ p < \infty \mid (-m, D)_p = -1 \},
\]

where \((\cdot, \cdot)_p\) denotes the \(p\)-adic Hilbert symbol \([\text{Ser73}]\). Moreover, let

\[
\nu_p(m) = \begin{cases} \frac{1}{2}(\text{ord}_p(m) + 1), & \text{if } p \text{ is inert in } k, \\ \text{ord}_p(m | D), & \text{if } p \text{ is ramified in } k, \end{cases}
\]

and, finally \(o(m) = 1\) if \(\text{ord}_p(m | D) > 0\) and \(o(m) = 0\), otherwise. The following formula has been conjectured in \([\text{DL15}, \text{p. 46}]\) (stated slightly differently). We fix the embedding of \(k\) into \(\mathbb{C}\), such that \(\text{Im}(\sqrt{D}) > 0\) and identify \(H_k = k(j)\), where \(j\) is equal to the \(j\)-invariant \((\frac{1 + \sqrt{D}}{2}) = j(\mathcal{O}_k)\).
Theorem 1.3. Assume that \( l > 3 \) and let \( m \in \frac{1}{2m} \mathbb{Z}_{>0} \). Then \( \alpha(a, m) \in \mathcal{O}_k^X \) unless \( |\text{Diff}(m)| = 1 \). Thus, assume that \( \text{Diff}(m) = \{ p \} \) and let \( \mathfrak{p}_0 \mid p \) be the unique prime above \( p \) fixed by complex conjugation. If \( \mathfrak{p} = \mathfrak{p}_0^{\sigma(b)} \) for any fractional ideal \( b \) of \( k \), then
\[
\text{ord}_{\mathfrak{p}}(\alpha(a^2, m)) = 2^{\nu(m)} \cdot r \cdot \sum_{n \geq 1} \rho \left( \frac{|D|}{p^n}, [a]^2[b]^2 \right) = 2^{\nu(m)} r \cdot \nu_p(m) \rho(m |D| / p, [a]^2[b]^2),
\]
where \( \rho(m, \mathcal{C}) \) is the number of integral ideals of \( \mathcal{O}_k \) of norm \( m \) in the class \( \mathcal{C} \in \text{Cl}_k \).

Theorem 1.3 follows from the new relation (iv) of the coefficients to the special cycles, together with Proposition 1.4 below. We note that Duke and Li also conjectured that \( r \mid 24 h_k h_k \), where \( h_k \) is the class number of \( k \), which does not follow from our proof (we do get a bound on \( r \) but it might be larger than what they conjecture). See Section 5.1 for details on the scalar-valued case as in [DL15] and Corollary 5.2 for a generalization to composite fundamental discriminants.

As a corollary of our results (see Theorem 4.29), we are also able to show that (1.4) is (up to a constant) the holomorphic part of a harmonic weak Maass form \( \tilde{E}_P(\tau) \) without using any explicit formulas and therefore give a completely new proof of the modularity of the (degree) generating series of these cycles. This follows essentially from (i) together with (iv) in Theorem 1.2 because taking the sum as in (i) in fact computes the degree of the cycles by (iv). See also Section 4.6.

In Section 7, we also consider a holomorphic analogue of the generating series (1.4), which can be seen as the generating series of the cycles \( Z(m) \) equipped with an automorphic Green function, and show that is modular, as well.

In [Ehl15], we gave formulas for the multiplicities of the cycles \( Z(m) \) which provide a prime ideal decomposition of the algebraic numbers \( \alpha(a, m) \) in Theorem 1.2. In the case of a prime discriminant the formulas are particularly simple and explicit.

Proposition 1.4 (Proposition 2 in [Ehl15]).

(i) We have \( Z(m)_{\mathfrak{p}} = 0 \) unless \( |\text{Diff}(m)| = 1 \).

(ii) Assume that \( \text{Diff}(m) = \{ p \} \). Let \( \mathfrak{p}_0 \mid p \) be the unique prime ideal that is fixed by complex conjugation, \( \mathfrak{p}_0 = \mathfrak{p}_0 \). For \( \mathfrak{p} = \mathfrak{p}_0^{\sigma} \), where \( \sigma = \sigma(a) \in \text{Gal}(H_k/k) \) corresponds to the ideal class of \( a \) under the Artin map, we have
\[
Z(m)_{\mathfrak{p}} = 2^{\nu(m)-1} \nu_p(m) \rho(m |D| / p, [a]^{-2}).
\]

In Section 6, we walk through the proof of Theorem 4.21 in a concrete example for \( D = -23 \) and give in fact a finite formula for the coefficients of the holomorphic part in this case. It should be possible to generalize the results obtained in the example and we will come back to this in a sequel to this paper.

Some remarks regarding the relation of our work to [Via12] and [DL15] are in order. We make use of the same seesaw identity as in [Via12], where it has been used to obtain formulas for regularized Petersson inner products of weakly holomorphic modular forms of weight one and the theta function \( \Theta_P(\tau, 1) \) for the lattice \( P = \mathcal{O}_k \) as above in the case of a prime discriminant. We use the same seesaw identity (essentially (1.6)) but focus on the coefficients of harmonic Maass forms and applications in arithmetic geometry, which do not appear in [Via12] and requires some extra work. Duke and Li [DL15] make use of the Rankin-Selberg method and the construction of (higher level) automorphic Green functions as in [GZ85] and [GZ86], whereas we entirely rely on the (regularized) theta lift machinery. In particular, Theorem 1.2 of [DL15] is a special case of Theorem 1.1 (or rather Theorem 3.5 in the body of the paper) in signature \((2, 1)\). We refer the reader to [Ehl12, Section 8] for this particular case.
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2. **Preliminaries**

We introduce some notation for the rest of the article. For a place $p$ of a field $k$ we let $k_p$ denote the completion of $k$ with respect to the valuation $v_p$ corresponding to $p$. For non-archimedean $p$, we denote by $O_p \subset k_p$ the corresponding valuation ring. We consider the adeles $A_k = \prod_p k_p$ over $k$ and the finite adeles are denoted by $A_k, f = \prod_{p \mid \infty} k_p$.

Moreover, we denote by $A_k^{\times}$ and $A_k, f^{\times}$ the groups of (finite) ideles over $k$. If $k = \mathbb{Q}$, we denote by $A$ the adeles over $\mathbb{Q}$ and by $A, f$ the finite adeles.

Throughout, we let $n$ be a non-negative integer and let $V$ be a quadratic space over $\mathbb{Q}$ of type $(2, n)$ with a non-degenerate quadratic form $Q$.

2.1. **Shimura varieties attached to** $G_{\text{Spin}} V$. As in the introduction we abbreviate $H := G_{\text{Spin}} V$. One has the following exact sequence of algebraic groups:

$$1 \rightarrow \mathbb{G}_m \rightarrow G_{\text{Spin}} V \rightarrow SO_V \rightarrow 1.$$  

Here, $\mathbb{G}_m$ denotes the multiplicative group.

**Remark 2.1.** Our setup is basically the same as in [Kud03; Sch09; BY09]. However, we warn the reader that we are working with a quadratic space of type $(2, n)$, whereas these references mostly use type $(n, 2)$ quadratic spaces.

Let $K \subset SO_V(\mathbb{R})$ be a maximal compact subgroup of $SO_V(\mathbb{R})$. Since $V$ is a quadratic space over $\mathbb{Q}$ of type $(2, n)$, the quotient $SO_V(\mathbb{R})/K$ is a symmetric space with a complex structure. There are several ways to realize $SO_V(\mathbb{R})/K$.

Consider the Grassmannian $\mathbb{D}$ of oriented two-dimensional positive definite subspaces of $V(\mathbb{R}) = V \otimes_{\mathbb{Q}} \mathbb{R}$. That is, we let

$$\mathbb{D} := \{ z^\pm \mid z \subset V(\mathbb{R}), \dim z = 2, \; Q|_z > 0 \}.$$ 

Here, for each 2-dimensional positive definite subspace $z \subset V(\mathbb{R})$, we write $z^+$ and $z^-$ for $z$ together with one of the two possible choices of orientation. The group $H(\mathbb{R})$ acts naturally on $\mathbb{D}$ and this action is transitive by Witt’s theorem. The Grassmannian has two connected components and each of them is isomorphic to the symmetric space $SO_V(\mathbb{R})/K$.

Let $K \subset H(A_f)$ be a compact open subgroup. We write $X_K$ for the associated Shimura variety with complex points

$$X_K(\mathbb{C}) = H(\mathbb{Q}) \backslash (\mathbb{D} \times H(A_f))/K.$$
2.2. Special divisors. There is a natural family of divisors on Shimura varieties of orthogonal type that will play an important role. We also refer to and [Kud97], [Kud03], and [BY09]. Let \( L \subset V(\mathbb{Q}) \) be an even lattice and let \( K \subset H(\mathbb{A}_f) \) be an open compact subgroup such that \( KL \subset L \) and \( K \) acts trivially on \( L'/L \). We will make these assumptions throughout this section. In this situation, we consider the group

\[ \Gamma_K = H(\mathbb{Q}) \cap K, \]

which is an arithmetic subgroup of \( H(\mathbb{Q}) \).

Let \( x \in V(\mathbb{Q}) \) be a vector of negative norm and denote the orthogonal complement \( x^\perp \subset V(\mathbb{Q}) \) by \( V_x \). We let \( H_x \) be the stabilizer of \( x \) in \( H \). Then \( H_x \cong \text{GSpin}(V_x) \) and the Grassmannian

\[ \mathbb{D}_x = \{ z \in \mathbb{D} \mid z \perp x \} \subset \mathbb{D} \]

defines an analytic set of codimension one in \( \mathbb{D} \).

Let \( h \in H(\mathbb{A}_f) \) and consider

\[ H_x(\mathbb{Q}) \setminus \mathbb{D}_x \times H_x(\mathbb{A}_f)/(H_x(\mathbb{A}_f) \cap hKh^{-1}) \rightarrow X_K \tag{2.1} \]
given by

\[ (z, h_1) \mapsto (z, h_1h). \]

The image of this map defines a divisor \( Z(x, h) \) on \( X_K \) which is rational over \( \mathbb{Q} \) [Kud97]. For \( m \in \mathbb{Q}_{<0} \) consider the quadric \( \Omega_m \subset V \) given by

\[ \Omega_m = \{ x \in V \mid Q(x) = m \}. \]

By Witt’s theorem, if \( \Omega_m(\mathbb{Q}) \neq \emptyset \), the orthogonal group acts transitively and thus for every \( x_0 \in \Omega_m(\mathbb{Q}) \) we have \( \Omega_m(\mathbb{Q}) = H(\mathbb{Q}):x_0 \) and \( \Omega_m(\mathbb{A}_f) = H(\mathbb{A}_f):x_0 \). Here,

\[ \Omega_m(\mathbb{A}_f) = \left( \prod_{p<\infty} \Omega_m(\mathbb{Q}_p) \right) \cap V(\mathbb{A}_f) \]

and \( \Omega_m(\mathbb{Q}_p) = \{ x \in V(\mathbb{Q}_p) \mid Q(x) = m \} \). Moreover, for any compact open subgroup \( K \subset H(\mathbb{A}_f) \), we have \( \Omega_m(\mathbb{A}_f) = K\Omega_m(\mathbb{Q}) \) (see Lemma 5.1 of [Kud97]).

We let \( S(V(\mathbb{A})) \) be the space of Schwartz(-Bruhat) functions on \( V(\mathbb{A}) \). That is, the space \( S(V(\mathbb{R})) \) is the usual space of Schwartz (rapidly decreasing) functions on \( V(\mathbb{R}) \) and \( S(V(\mathbb{Q}_p)) \) is the space of locally constant functions \( V(\mathbb{Q}_p) \rightarrow \mathbb{C} \) with compact support and we let

\[ S(V(\mathbb{A}_f)) = \bigotimes_{p<\infty} S(V(\mathbb{Q}_p)) \]

and \( S(V(\mathbb{A})) = S(V(\mathbb{A}_f)) \otimes S(V(\mathbb{R})) \).

Let \( L \) be an even lattice and \( \mu \in L'/L \simeq \hat{L}'/\hat{L} \), where \( \hat{L} = L \otimes \hat{\mathbb{Z}} \) with \( \hat{\mathbb{Z}} = \prod_{p<\infty} \mathbb{Z}_p \). We let \( \phi_\mu \in S(V(\mathbb{A}_f)) \) be the characteristic function of \( \mu \). We consider the finite dimensional subspace

\[ S_L = \bigoplus_{\mu \in L'/L} \mathbb{C}\phi_\mu \subset S(V(\mathbb{A}_f)). \]

**Definition 2.2.** For a Schwartz function \( \varphi \in S_L \) write

\[ \text{supp}(\varphi) \cap \Omega_m(\mathbb{A}_f) = \bigcup_j K\xi_j^{-1}x_0, \]
where $\xi_j \in H(\mathbb{A}_f)$. We define the special divisor
\[
Z(m, \varphi) := \sum_j \varphi(\xi_j^{-1}x_0)Z(x_0, \xi_j).
\]
For $\mu \in L'/L$, briefly write $Z(m, \mu) := Z(m, \phi_\mu)$.

In this context, we also let
\[
L_m := \Omega_m \cap L' \quad \text{and} \quad L_{m,\mu} := L_m \cap (L + \mu).
\]

2.3. **The Weil representation.** Let $L \subset V$ be an even lattice and let $\mu \in L'/L$.

We write $\langle \phi, \chi \rangle$ for the standard bilinear pairing between $S(V(\mathbb{A}_f))$ and its dual $S(V(\mathbb{A}_f))^\vee$. In particular
\[
\langle a \phi_\mu, b \phi_\nu \rangle = ab \delta_{\mu,\nu}
\]
for $a, b \in \mathbb{C}$ and $\mu, \nu \in L'/L$, where we identify $S_L$ with its dual.

**Remark 2.3.** We note that the space $S_L$ can be identified with the group ring $\mathbb{C}[L'/L]$ of the finite abelian group $L'/L$ via $\phi_\mu \mapsto \mathbf{c}_\mu$, if $\{\mathbf{c}_\mu \mid \mu \in L'/L\}$ is the standard basis for $\mathbb{C}[L'/L]$. In the latter space the corresponding scalar product is conjugate-linear in the second argument.

We write $\tilde{\Gamma} := \text{Mp}_2(\mathbb{Z})$ for the two-fold metaplectic cover of $\text{SL}_2(\mathbb{Z})$. The elements of $\tilde{\Gamma}$ are pairs $(A, \phi)$, where $A = \left( \begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right) \in \text{SL}_2(\mathbb{Z})$ and $\phi : \mathbb{H} \to \mathbb{C}$ is a holomorphic function with $\phi^2(\tau) = c\tau + d$.

There is a representation $\rho_L : \tilde{\Gamma} \to \text{Aut} S_L$, usually called the Weil representation associated with $L$. This representation can be described explicitly as follows.

The group $\tilde{\Gamma}$ is generated by
\[
S = \left( \begin{array}{cc} 0 & -1 \\ 1 & 0 \end{array} \right), \quad T = \left( \begin{array}{cc} 1 & 1 \\ 0 & 1 \end{array} \right).
\]

For these generators, we have
\[
\rho_L(T)\phi_\mu = e(Q(\mu))\phi_\mu, \\
\rho_L(S)\phi_\mu = \frac{e(-\text{sgn}(V)/2)}{\sqrt{|L'/L|}} \sum_{\nu \in L'/L} e(-\langle \mu, \nu \rangle)\phi_\nu. \quad (2.3)
\]

Here, $\text{sgn}(V)$ denotes the signature of $V$, which is equal to $2 - n$ in our case.

2.4. **Harmonic weak Maaß forms.** The main reference for this section is the fundamental article by Bruinier and Funke [BF04].

Let $(V, Q)$ be a rational quadratic space and let $L \subset V$ be an even lattice. Moreover, let $k \in \frac{1}{2} \mathbb{Z}$. For $(\gamma, \phi) \in \tilde{\Gamma}$, we define the Petersson slash operator on functions $f : \mathbb{H} \to S_L$ by
\[
(f \mid_{k, L} (\gamma, \phi))(\tau) = \phi(\tau)^{-2k}\rho_L((\gamma, \phi))^{-1}f(\gamma \tau).
\]

**Definition 2.4.** A twice continuously differentiable function $f : \mathbb{H} \to S_L$ is called a harmonic weak Maaß form (of weight $k$ with respect to $\tilde{\Gamma}$ and $\rho_L$) if it satisfies:

(i) $f \mid_{k, L} \gamma = f$ for all $\gamma \in \tilde{\Gamma}$,

(ii) there is a $C > 0$ such that $f(\tau) = O(e^{Cu})$ as $v \to \infty$ (uniformly in $u$, where $\tau = u + iv$),
Alternatively, we could define this to be the space of differential equation gives rise to a unique decomposition of the Fourier expansion of $H$ space was denoted by $\tau$ as $\Im(\tau)$.

We let $H_{k,L}$ be the subspace of cusp forms and holomorphic modular forms. As usual, elements of all three spaces $S_{k,L} \subset M_{k,L} \subset M_{k,L}^1$ are assumed to be holomorphic on the upper half-plane; weakly holomorphic modular forms are allowed to have a pole at the cusp, holomorphic modular forms are required to be holomorphic at the cusp and cuspidal forms are holomorphic modular forms that vanish at the cusp.

We write the Fourier expansion of $f \in H_{k,L}$ as

$$f(\tau) = \sum_{\mu \in \mathbb{L}/\mathbb{L}} \sum_{n \in \mathbb{Q}} c_f(n, \mu, v) q^n \phi_\mu. \tag{2.4}$$

Since $f$ is harmonic with respect to the weight $k$ Laplace operator, the coefficients $c_f(n, \mu, v)$ satisfy $\Delta_k c_f(n, \mu, v) \exp(2\pi i(n(u + iv))) = 0$. Computing a basis for the space of solutions to this differential equation gives rise to a unique decomposition of the Fourier expansion of $f$ into a holomorphic part $f^+$ and a non-holomorphic part $f^-$. If $n \neq 0$, we write accordingly $c_f(n, \mu, v) = c_f^+(n, \mu) + c_f^-(n, \mu)W_k(2\pi nv)$, where $W_k(a) = \int_{-2a}^\infty e^{-t^k} dt$.

In weight one, which is of particular interest for us, the expansion of the non-holomorphic part is of the form

$$f^-(\tau) = \sum_{\mu \in \mathbb{L}/\mathbb{L}} \left( c_f^-(0, \mu) \log(v) + \sum_{n \in \mathbb{Q}, n \neq 0} c_f^-(n, \mu)W_1(2\pi nv) q^n \right) \phi_\mu, \tag{2.5}$$

where $W_1(a) = \Gamma(0, -2a)$ for $a > 0$.

We recall a few more facts that can all be found in [BF04, Section 3]. We denote by $L^-$ the lattice given by the $\mathbb{Z}$-module $L$ together with the quadratic form $-Q$. There is an antilinear differential operator $\xi := \xi_k : H_{k,L} \to M_{k-L}^1$, defined by

$$f(\tau) \mapsto \xi(f)(\tau) := v^{k-2}L_k f(\tau) = R_{-k} v^k f(\tau). \tag{2.6}$$

Here $L_k$ and $R_k$ are the Maaß lowering and raising operators,

$$L_k = -2i v^2 \frac{\partial}{\partial \tau} \quad \text{and} \quad R_k = 2i \frac{\partial}{\partial \tau} + kv^{-1}.$$ 

We let $H_{k,L} \subset H_{k,L}$ be the subspace of forms with cuspidal “shadow”,

$$H_{k,L} := \{ f \in H_{k,L} \mid \xi(f) \in S_{k,L^-} \}.$$ 

Alternatively, we could define this to be the space of $f \in H_{k,L}$, such that there is a Fourier polynomial

$$P_f(\tau) = \sum_{\mu \in \mathbb{L}/\mathbb{L}} \sum_{m < 0} c_f^+(m, \mu) \exp(m \tau), \quad \text{with} \quad f - P_f(\tau) = O(1)$$

as $\Im(\tau) \to \infty$. The Fourier polynomial $P_f(\tau)$ is also called the principal part of $f$. Note that this space was denoted by $H_{k,L}^+$ in [BF04].
The kernel of $\xi_k$ is equal to $M^1_{k,L}$ and by [BF04, Corollary 3.8], the sequences

\begin{equation}
0 \longrightarrow M^1_{k,L} \longrightarrow H_{k,L} \xrightarrow{\xi_k} M^1_{2-k,L^-} \longrightarrow 0 \tag{2.7}
\end{equation}

\begin{equation}
0 \longrightarrow M^1_{k,L} \longrightarrow H_{k,L} \xrightarrow{\xi_k} S_{2-k,L^-} \longrightarrow 0 \tag{2.8}
\end{equation}

are exact.

For $f \in S_{k,L}$ and $g \in M_{k,L}$, we define the Petersson inner product of $f$ and $g$ as

\[(f,g) = \int_{S_{1,\mathbb{Z}}/\mathbb{H}} \langle f(\tau), g(\tau) \rangle v^k \, d\mu(\tau). \tag{2.9} \]

We denote by $\partial$ and $\overline{\partial}$ the usual Dolbeault operators, such that we have $d = \partial + \overline{\partial}$ for the exterior derivative on differential forms on $\mathbb{H}$.

**Lemma 2.5.** In terms of differential forms, we have

\[\overline{\partial}(f \, d\tau) = -v^{2-k} \xi_k(f) \, d\mu(\tau) = -L_k f \, d\mu(\tau). \]

Using the Petersson inner product and the operator $\xi_k$, we obtain a bilinear pairing between $g \in M_{2-k,L^-}$ and $f \in H_{k,L}$ via

\[\{g, f\} := \langle g, \xi_k(f) \rangle_{2-k} = \int_{S_{1,\mathbb{Z}}/\mathbb{H}} \langle g, \xi_k(f) \rangle v^{2-k} \, d\mu(\tau) = \int_{S_{1,\mathbb{Z}}/\mathbb{H}} \langle g, L_k f \rangle \, d\mu(\tau). \tag{2.9} \]

Using Lemma 2.5, the following result is essentially an application of Stokes’ theorem (see [BF04, Proposition 3.5]).

**Lemma 2.6.** Let $f \in H_{k,L}$ and $g \in M_{2-k,L^-}$. Then

\[\{g, f\} = \sum_{\mu \in L'/L} \sum_{n \leq 0} c^+(n, \mu) b(-n, \mu), \]

which implies that the pairing only depends on the principal part of $f$ (and on $g$). The exact sequence (2.7) implies that the pairing between $S_{2-k,L^-}$ and $H_{k,L}/M^1_{k,L}$ is non-degenerate.

### 2.5. Regularized theta lifts

We let $\Theta_L(\tau, z, h)$ be the Siegel theta function associated with $L$ as in [BY09], where $\tau \in \mathbb{H}$, $z \in \mathbb{D}$ and $h \in H(\mathbb{A}_f)$, where $H = \text{GSpin}_+$. If $V$ is positive definite and $\mathbb{D} = \{z^2\}$ consists of only two points, we frequently drop $z$ from the notation and just write $\Theta_L(\tau, h)$. The following theorem can be found (in a “more classical” language) in [Bor98]. A reference that uses our (adelic) setup is Kudla’s seminal paper [Kud03].

**Theorem 2.7.** If $K \subset H(\mathbb{A}_f)$ is an open compact subgroup preserving $L$ and acting trivially on $L'/L$, then the Siegel theta function $\Theta_L(\tau, z, h_f)$ defines a function on the Shimura variety $X_K$ (in $(z, h_f)$). Moreover, as a function in $\tau \in \mathbb{H}$, it is a non-holomorphic vector-valued modular form of weight $(2 - n)/2$, that is, for $\gamma = \left( \begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right)$, $\phi(\gamma) \in \tilde{\Gamma}$, we have

\[\Theta_L(\gamma \tau, z, h) = \phi(\tau)^{2-n} \rho_L(\gamma) \Theta_L(\tau, z, h).\]

Let $\mathcal{F} := \{\tau \in \mathbb{H}; |\tau| \geq 1, -1/2 \leq \text{Re}(\tau) \leq 1/2\}$ the standard fundamental domain for the action of $\text{SL}_2(\mathbb{Z})$ on $\mathbb{H}$ and let $\mathcal{F}_T := \{\tau \in \mathcal{F}; \text{Im}(\tau) \leq T\}$. For $f \in H_{1-n/2,L}$, we consider the regularized theta integral

\[\Phi_L(z, h, f) = \int_{\Gamma \backslash \mathbb{H}}^{\text{reg}} \langle f(\tau), \Theta_L(\tau, z, h) \rangle v^k \, d\mu(\tau) := \text{CT} \lim_{T \to \infty} \int_{\mathcal{F}_T} \langle f(\tau), \Theta_L(\tau, z, h) \rangle v^{k-s} \, d\mu(\tau).\]
Here, CT denotes the constant term in the Laurent expansion at \( s = 0 \) of the meromorphic continuation of the function enclosed in \([\cdot]\), which is initially defined for \( \text{Re}(s) \) large enough.

Associated with \( f \) is the divisor
\[
Z(f) = \sum_{\mu \in \mathbb{L}/\mathbb{L}} \sum_{m < 0} c^+(m, \mu)Z(m, \mu). \tag{2.10}
\]

**Theorem 2.8** ([Bor98], Theorem 13.3, cf. Theorem 1.3 in [Kud03]). Let \( f \in \mathcal{M}_{(2-n)/2, \mathbb{L}} \) with \( c(m, \mu) \in \mathbb{Z} \) for all \( m < 0 \) and \( c(m, \mu) \in \mathbb{Q} \) for all \( m \in \mathbb{Q} \). There is a function \( \Psi_L(z, h, f) \) on \( \mathbb{D} \times H(\mathbb{A}_f) \), such that:

(i) \( \Psi_L(z, h, f) \) is a meromorphic modular form for \( H(\mathbb{Q}) \) of weight \( c_f(0, 0)/2 \) and level \( K \), with some unitary multiplier system of finite order,

(ii) the divisor of \( \Psi_L(z, h, f)^2 \) on \( X_K \) is given by \( Z(f) \).

(iii) and we have
\[
\Phi_L(z, h, f) = -2 \log \| \Psi_L(z, h, f) \|^2 - c_f(0, 0)(\log(2\pi) + \Gamma'(1)),
\]
where \( \| \Psi_L(z, h, f) \|^2 = |\Psi_L(z, h, f)|^2 |y|^{c_f(0, 0)} \).

The modular form \( \Psi_L(z, h, f) \) admits an expansion as an infinite product, giving it the name Borchers product. We will only use the product expansion in a special case in Section 4. Therefore, we omit the general case and refer to Theorem 13.3 of [Bor98].

Bruinier extended the space of input functions of the theta lift to include harmonic weak Maaß forms and this extension will play an important role for us. Recall that a function \( G(z) \) on \( \mathbb{D} \) has a logarithmic singularity along a divisor \( D \), if every point in \( \mathbb{D} \) has a small neighborhood \( U \), such that for any meromorphic function \( g \) locally defining \( D \), we have that \( G - \log |g| \) extends to a smooth function on \( U \).

**Theorem 2.9** ([Bru02], Theorem 2.12, Theorem 4.7).

Let \( f \in H_{1-n/2, \mathbb{L}} \). Then the following holds.

(i) The function \( \Phi_L(z, h, f) \) is smooth on \( X_K \setminus Z(f) \) and has a logarithmic singularity along \( -2Z(f) \).

(ii) The differential \( dd^c\Phi_L(z, h, f) \) extends to a smooth \((1,1)\) form on \( X_K \). As a current on \( X_K \), we have
\[
\dd^c [\Phi_L(z, h, f)] + \delta_{Z(f)} = [\dd^c \Phi_L(z, h, f)].
\]

2.6. Operations on vector-valued modular forms. Let \( A_{k, \mathbb{L}} \) be the space of \( S_L \)-valued functions that are invariant under the weight \( k \) slash operator. Let \( M \subset \mathbb{L} \) be a sublattice of finite index. Then if \( f \in A_k, \mathbb{L} \), it can be naturally viewed as an element of \( A_k, \mathbb{M} \). Indeed, we have the inclusions \( M \subset \mathbb{L} \subset \mathbb{L}' \subset \mathbb{M} \) and therefore
\[
\mathbb{L}/\mathbb{M} \subset \mathbb{L}'/\mathbb{M} \subset \mathbb{M}'/\mathbb{M}.
\]
We have the natural map \( \mathbb{L}'/\mathbb{M} \rightarrow \mathbb{L}'/\mathbb{L}, \mu \mapsto \bar{\mu} \).

**Lemma 2.10.** There are two natural maps
\[
\text{res}_{\mathbb{L}/\mathbb{M}} : A_{k, \mathbb{L}} \rightarrow A_{k, \mathbb{M}}, \quad f \mapsto f_{\mathbb{M}}
\]
and
\[
\text{tr}_{\mathbb{L}/\mathbb{M}} : A_{k, \mathbb{M}} \rightarrow A_{k, \mathbb{L}}, \quad g \mapsto g^L
\]
such that for any \( f \in A_{k, \mathbb{L}} \) and \( g \in A_{k, \mathbb{M}} \)
\[
\langle f, g^L \rangle = \langle f_M, \bar{g} \rangle.
\]
They are given as follows. For \( \mu \in M'/M \) and \( f \in A_{k,L} \),

\[
(f_M)_\mu = \begin{cases} f_{\bar{\mu}}, & \text{if } \mu \in L'/M, \\ 0, & \text{if } \mu \notin L'/M. \end{cases}
\]

For any \( \bar{\mu} \in L'/L \), and \( g \in A_{k,M} \), let \( \mu \) be a fixed preimage of \( \bar{\mu} \) in \( L'/M \). Then

\[
(g_L)_\bar{\mu} = \sum_{\alpha \in L/M} g_{\alpha + \mu}. 
\]

Proof. See [Sch04, Proposition 6.9] for the map \( \text{res}_{L/M} \). The proof for \( \text{tr}_{L/M} \) is a similar calculation. \( \square \)

2.7. Jacobi forms and vector valued modular forms. In this section we recall the notion of a (weakly holomorphic) Jacobi form. We will use Jacobi forms of scalar index which have been intensively studied by Eichler and Zagier [EZ85] in Section 4.

**Definition 2.11.** Let \( k, m \in \mathbb{Z} \) and \( \varphi : \mathbb{H} \times \mathbb{C} \to \mathbb{C} \) be a holomorphic function. Then \( \varphi \) is called a **holomorphic Jacobi form** of weight \( k \) and index \( m \) if

(i) \( \varphi(\gamma \tau, \frac{z}{c \tau + d}) = (c \tau + d)^k e(mcz^2/(c \tau + d)) \varphi(\tau, z) \), for all \( \gamma = \left( \begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right) \in \text{SL}_2(\mathbb{Z}) \),

(ii) \( \varphi(\tau, z + r \tau + s) = e(-m(r^2 \tau + 2rz)) \varphi(\tau, z) \) for all \( r, s \in \mathbb{Z} \), and

(iii) \( \varphi(\tau, z) \) is holomorphic at the cusp \( \infty \).

Note that such a \( \varphi \) has a Fourier expansion of the form

\[
\varphi(\tau, z) = \sum_{n,r \in \mathbb{Z}} c(n, r) q^n \zeta^r,
\]

where \( q = e^{2\pi i \tau} \) and \( \zeta = e^{2\pi iz} \). The last condition in the definition means that \( c(n, r) = 0 \) if the discriminant \( 4nm - r^2 \) is negative.

A **weakly holomorphic Jacobi form** satisfies all the preceding conditions except that we only require it to be meromorphic at \( \infty \). This means in terms of the Fourier expansion that there are only finitely many non-vanishing Fourier coefficients with negative discriminant. We denote the space of holomorphic Jacobi forms of weight \( k \) and index \( m \) by \( J_{k,m} \) and by \( J^!_{k,m} \) the space of weakly holomorphic Jacobi forms of weight \( k \) and index \( m \).

Using the definitions, it is easy to check that

(i) If \( f \in J^!_{k_1,m_1} \) and \( g \in J^!_{k_2,m_2} \), then \( fg \in J^!_{k_1+k_2,m_1+m_2} \).

(ii) If \( f \in M^!_{k_1}(\text{SL}_2(\mathbb{Z})) \) and \( \varphi \in J^!_{k_2,m} \), then \( f \varphi \in J^!_{k_1+k_2,m} \).

(iii) If \( \varphi \in J^!_{k,m} \), then \( \varphi(\tau, 0) \in M^!_k \).

For \( r \in \mathbb{Z}/2m \mathbb{Z} \) we write

\[
\theta_r(\tau, z) = \sum_{n \in \mathbb{Z}}^{n \equiv r \bmod 2m} q^{\frac{n^2}{4m}} \zeta^n,
\]

for the corresponding theta function.

**Proposition 2.12.** Let \( \varphi \in J^!_{k,m} \) be a weakly holomorphic Jacobi form. Then \( \varphi(\tau, z) \) has a theta expansion of the form

\[
\varphi(\tau, z) = \sum_{r \bmod 2m} \varphi_r(\tau) \theta_r(\tau, z).
\]
Moreover, let $L = \mathbb{Z}$ be the lattice with quadratic form $Q(x) = -mx^2$. Then

$$\Phi(\tau) = \sum_{r \mod 2m} \varphi_r(\tau) \phi_r$$

is a vector valued modular form contained in $M^1_{k - 1/2, L}$. Here, $\phi_r$ is the characteristic function of the coset $r + 2m\mathbb{Z}$. This correspondence establishes an isomorphism

$$M^1_{k - 1/2, L} \cong J^1_{k, m}.$$

Proof. For holomorphic Jacobi forms, this is Theorem 5.1 of [EZ85]. It is straightforward to extend this to weakly holomorphic forms. See also [Zag02, Section 8].

**Lemma 2.13.** With the same notation as in Proposition 2.12, let $\varphi \in J^1_{k, m}$ be a weakly holomorphic Jacobi form. We let $\Theta_{L^-}(\tau)$ be the theta function

$$\Theta_{L^-}(\tau) = \sum_{n \in \mathbb{Z}} e\left(\frac{n^2}{4m}\tau\right) \phi_n = \sum_{r \in \mathbb{Z}/2m\mathbb{Z}} \Theta_{L^-, r}(\tau) \phi_r \in M^1_{2, L^-}$$

associated with the lattice $L^-$. Then we have

$$\langle \Phi(\tau), \Theta_{L^-}(\tau) \rangle = \varphi(\tau, 0).$$

Proof. This follows directly from Proposition 2.12 using

$$\langle \Phi(\tau), \Theta_{L^-}(\tau) \rangle = \sum_{r \in \mathbb{Z}/2m\mathbb{Z}} \varphi_r(\tau) \Theta_{L^-, r}(\tau) = \sum_{r \in \mathbb{Z}/2m\mathbb{Z}} \varphi_r(\tau) \theta_r(\tau, 0).$$

From the properties stated above, it follows that the bi-graded ring

$$J^1_{ev, s} = \bigoplus_{k,m \in \mathbb{Z}} J^1_{2k, m}$$

of weakly holomorphic Jacobi forms of even weight is a module over the graded ring

$$M^1_s = M^1_s(\text{SL}_2(\mathbb{Z})) = \bigoplus_{k \in \mathbb{Z}} M^1_k(\text{SL}_2(\mathbb{Z})).$$

(Note that $M^1_k(\text{SL}_2(\mathbb{Z})) = \{0\}$ for $k$ odd.)

**Proposition 2.14** (See [Zag02]). The $M^1_s$-module $J^1_{ev, s}$ of weakly holomorphic Jacobi forms of even weight is free of rank two and generated by

$$F(\tau, z) = \phi_{-2, 1}(\tau, z) = (\zeta - 2 + \zeta^{-1}) + (-2\zeta^2 + 8\zeta - 12 + 8\zeta^{-1} - 2\zeta^{-2})q + \ldots \in J^1_{-2, 1},$$

$$G(\tau, z) = \phi_{0, 1}(\tau, z) = (\zeta + 10 + \zeta^{-1}) + (10\zeta^2 - 64\zeta + 108 - 64\zeta^{-1} + 10\zeta^{-2})q + \ldots \in J^1_{0, 1},$$

defined by Eichler and Zagier [EZ85].
2.8. **Special endomorphisms.** Let $D$ be a negative odd fundamental discriminant and let $k = \mathbb{Q}(\sqrt{D})$ be the imaginary quadratic field of discriminant $D$. We write $\mathcal{O}_k$ for the ring of integers in $k$, $h_k$ for the class number of $h_k$, and $w_k$ for the number of roots of unity in $k$. Moreover, we let $H_k$ be the Hilbert class field of $k$ and $\mathcal{O}_{H_k} \subset H_k$ be its ring of integers. By class field theory, we have an isomorphism via the Artin map $[\text{Sil94, II, Example 3.3}]
(\cdot, H_k/k) : \text{Cl}_k \to \text{Gal}(H_k/k).$

We will use the convention that we write $\sigma(a) = \sigma([a])$ for $([a], H_k/k)$ for the image of the class of the fractional ideal $a$ under this map.

For any scheme $S$ we consider pairs $(E, \iota)$, where $E$ is an elliptic curve over $S$ with complex multiplication $\iota : \mathcal{O}_k \to \text{End}(E)$. The coarse moduli scheme of the corresponding moduli problem is isomorphic to $\text{Spec} \mathcal{O}_{H_k}$ [KRY99]. If $(E, \iota)$ is an elliptic curve with complex multiplication over $S$, we write $\mathcal{O}_E = \text{End}_S(E)$ and consider the lattice $L(E, \iota)$ of *special endomorphisms*

$$L(E, \iota) = \{x \in \mathcal{O}_E \mid \iota(\alpha)x = x\iota(\alpha) \text{ for all } \alpha \in \mathcal{O}_k \text{ and } tr x = 0\}$$

as in Definition 5.7 of [KRY99]. It is equipped with the positive definite quadratic form given by $N(x) = \deg(x) = -x^2.$ For $S = \text{Spec } \mathbb{C}$ or $S = \text{Spec } \mathbb{F}_p$ for a prime $p$ that is split in $k$, we have that $L(E, \iota)$ is zero.

For non-split primes, $L(E, \iota)$ is a positive definite lattice of rank 2 in $\mathcal{O}_E$ and $(E, \iota)$ is supersingular. In this case $\mathcal{O}_E$ is a maximal order in the quaternion algebra $\mathbb{B}_p$ over $\mathbb{Q}$, which is ramified exactly at $p$ and $\infty$. The quadratic form $N(x)$ corresponds to the reduced norm on $\mathcal{O}_E$.

We fix the embedding of $k = \mathbb{Q}(\sqrt{D})$ into $\mathbb{C}$ such that $\sqrt{D}$ has positive imaginary part and let $\omega = (1 + \sqrt{D})/2$ so that $\mathcal{O}_k = \mathbb{Z} + \mathbb{Z}\omega$. We let $j_D = j(\omega) = j(\mathcal{O}_k)$. Let $p$ be a rational prime that is not split in $k$ and let $\mathfrak{p}$ be the unique prime ideal of $k$ above $p$. If $\mathfrak{p}$ is a prime of $H_k := k(j_D)$ above $\mathfrak{p}$, then the image of $j_D$ under the reduction map $\mathcal{O}_{H_k} \to \mathcal{O}_{H_k}/\mathfrak{p}$ is the $j$-invariant of an elliptic curve $(E_\mathfrak{p}, \iota_\mathfrak{p})$ with complex multiplication by $\mathcal{O}_k$ over $\mathbb{F}_p$, unique up to isomorphism.

Fix a fractional ideal $a \subset k$ and let $\mu \in \mathcal{O}_k^{-1}a/a$ and $m \in \mathbb{Q}_{>0}$. The following cycles arise from a moduli problem that has been studied in [KRY99] and [BY09] and generalized in [KY13]. For $m \in \mathbb{Q}$, let $L(E, \iota, m, a, \mu)$ be the set of all $x \in L(E, \iota)\mathcal{O}_k^{-1}a$, such that

$$N(x) = m N(a), \text{ and } x + \mu \in \mathcal{O}_E a.$$

For every positive rational number $m$, we define an Arakelov divisor

$$\mathcal{Z}(m, a, \mu) = \sum_{\mathfrak{p} \subset \mathcal{O}_{H_k}} \mathcal{Z}(m, a, \mu)_{\mathfrak{p}} \mathfrak{p}$$

on $\text{Spec } \mathcal{O}_{H_k}$. Here, we let $\mathcal{Z}(m, a, \mu)_{\mathfrak{p}} = 0$ if the rational prime $p$ below $\mathfrak{p}$ is split in $k$ and otherwise

$$\mathcal{Z}(m, a, \mu)_{\mathfrak{p}} = \nu_p(m) \frac{w_k}{w_k} |L(E_\mathfrak{p}, \iota_\mathfrak{p}, m, a, \mu)|$$

with

$$\nu_p(m) = \begin{cases} \frac{1}{2} (\text{ord}_p(m) + 1), & \text{if } p \text{ is inert in } k, \\ \text{ord}_p(m|D|), & \text{if } p \text{ is ramified in } k. \end{cases}$$

If $\mathcal{Z}(m, a, \mu)$ is non-empty, then we have $m + Q(\mu) = m + N(\mu)/N(a) \in \mathbb{Z}$. The representation numbers $|L(E_\mathfrak{p}, \iota_\mathfrak{p}, m, a, \mu)|$ can be determined following [KRY99] (completely explicit in the case of a prime discriminant and up to Galois conjugation in general). We refer to [Ehl15] for details.
3. CM cycles and CM values of regularized theta lifts

In this section we fix a rational quadratic space \((V,Q)\) of type \((2,n)\). We fix a compact open subgroup \(K\) of \(H(\mathbb{A}_f)\) and consider the Shimura variety \(X_K\) as in Section 2.1.

The type of CM cycles we consider are given as follows. Let \(U \subset V\) be a 2-dimensional, positive definite rational subspace. This determines a two-point subset \(\{z_U^\pm\} \subset \mathbb{D}\) given by \(U(\mathbb{R})\) with the two possible choices of orientation. Denote by \(V_- = U^\perp \subset V\) the \(n\)-dimensional negative definite orthogonal complement of \(U\) over \(\mathbb{Q}\). Then we have a rational splitting

\[
V = U \oplus V_-.
\]

We obtain a cycle \(Z(U)_K \subset X_K\), which is called the CM cycle in \(X_K\) corresponding to \(U\). It is obtained by embedding a Shimura variety associated with \(U\) into \(X_K\), which is given as follows. Put \(T = \text{GSpin}_U\), which we view as a subgroup of \(H\) acting trivially on \(V_-\). The group \(K_T = K \cap T(\mathbb{A}_f)\) is a compact open subgroup of \(T(\mathbb{A}_f)\). We obtain a generically injective map

\[
Z(U)_K = T(\mathbb{Q}) \setminus \{\{z_U^\pm\} \times T(\mathbb{A}_f)/K_T\} \hookrightarrow X_K.
\]

Here, each point is counted with multiplicity \(\frac{2}{w_{K,T}}\), where we let \(w_{K,T} = |(T(\mathbb{Q}) \cap K_T)|\).

\[\text{Lemma 3.1. Suppose that } U \text{ is isomorphic as a rational quadratic space to an imaginary quadratic field } k \text{ and let } \mathcal{O}_k \subset k \text{ be its ring of integers. If } K_T = \mathcal{O}_k^\times, \text{ then } Z(U) \text{ is isomorphic to two copies of the ideal class group } \text{Cl}_k \text{ of } k, \text{ that is, } Z(U) \cong \text{Cl}_k \times \{z_U^\pm\}.\]

\[\text{Proof. We have } T(\mathbb{A}_f) \cong \mathbb{A}_{K,f}^\times \text{ (see, for instance [Kit93] or [Ehl16, Section 2.2]) and then the claim follows from [Neu07, VI. Satz 1.3].}\]

3.1. The average value. Fix an even lattice \(L \subset V\) and we abbreviate \(\Phi(z,h,f) := \Phi_L(z,h,f)\). Schofer [Sch09], Bruinier and Yang [BY09] studied the CM value

\[
\Phi(Z(U), f) = \frac{2}{w_{K,T}} \sum_{(z,h) \in \text{supp} Z(U)_K} \Phi(z,h,f).
\]

We review their main results.

The splitting (3.1) yields two lattices, \(P\) and \(N\), defined by

\[P = L \cap U, \quad N = L \cap V_-.
\]

The direct sum \(P \oplus N\) is a sublattice of \(L\) of finite index.

For \(z = z_U^\pm\) and \(h \in T(\mathbb{A}_f)\), the Siegel theta function \(\Theta_{P \oplus N}(\tau, z, h)\) splits as a product

\[
\Theta_{P \oplus N}(\tau, z_U^\pm, h) = \Theta_P(\tau, z_U^\pm, h) \otimes_{\mathbb{C}} \Theta_N(\tau).
\]

Here, \(\Theta_N(\tau) = \Theta_N(\tau, 1)\) is the \(S_N\)-valued theta function of weight \(n/2\) associated to the negative definite lattice \(N\). Note that \(v^{-n/2} \Theta_N(\tau)\) is the holomorphic theta function corresponding to the positive definite lattice \(N^\perp\). Moreover, we identified \(S_P \otimes_N S_N\) with the tensor product \(S_P \otimes_{\mathbb{C}} S_N\).

Attached to \(P\) there is a so-called incoherent Eisenstein series \(\hat{E}_P(\tau, s)\) of weight 1 transforming with representation \(\rho_{\hat{P}} = \rho_{P^-}\) [KRY99; KRY04]. Here, the term “incoherent” refers to the fact that it is built from local data at each place which does not correspond to a quadratic space over \(\mathbb{Q}\).

Its central value at \(s = 0\) vanishes but it is the value of the derivative \(\frac{\partial}{\partial s} \hat{E}_P(\tau, s)\) at \(s = 0\) that carries the arithmetic data which contributes to the CM values. The function

\[
\hat{E}_P(\tau) = \bigg. \frac{\partial}{\partial s} \hat{E}_P(\tau, s) \bigg|_{s=0}
\]
is a harmonic weak Maaß form of weight 1 with respect to $pF$.

If $S(q) = \sum_{n \in \mathbb{Z}} a_n q^n$ is a Laurent series in $q$, we write $CT(S) = a_0$ for the constant term in the $q$-expansion.

**Theorem 3.2.** Let $f \in H_{k,L}$ with $k = 1 - n/2$. The value of the theta lift $\Phi(z,h,f)$ at the CM cycle $Z(U)_K$ is given by

$$\Phi(Z(U), f) = \text{deg}(Z(U)) \left( CT \left( (f_{|\mathbb{R}N})^+ (\tau), \Theta_{N^-} (\tau) \otimes E^+_P(\tau) \right) - L'(\xi_k(f), U, 0) \right).$$

Here, $L'(\xi_k(f), U, s)$ is the derivative with respect to $s$ of the $L$-function defined by the convolution integral

$$L(\xi_k(f), U, s) = \int_{\text{SL}_2(\mathbb{Z}) \backslash \mathbb{H}} \langle \xi_k(f)(\tau), \check{E}_P(\tau, s) \otimes \Theta_{N^-}(\tau) \rangle v^{1+n/2} \frac{du dv}{v^2}.$$ 

**Proof.** This is Theorem 4.7 in [BY09] with a corrected sign.

The proof involves the Siegel-Weil formula and the standard Eisenstein series associated with $P$, which is defined as

$$E_P(\tau, s) = \frac{1}{2} \sum_{\gamma \in \Gamma_{\infty}} \left( \text{Im}(\gamma) \phi_0 \right) |_{1, P} \gamma,$$

where $\phi_0 \in S(V(\mathbb{A}_f))$ is the characteristic function of $\check{P}$. The series converges for $\text{Re}(s) > 1$ and has a meromorphic continuation to the whole complex $s$-plane. Note that we normalized $E_P(\tau, s)$, such that the constant term is equal to one.

As in [BY09], we fix the Haar measure on $SO_U(\mathbb{R}) \cong SO(2, \mathbb{R})$ such that $\text{vol}(SO_U(\mathbb{R})) = 1$. This implies that we have $\text{vol}(SO_U(\mathbb{Q}) \setminus SO_U(\mathbb{A}_f)) = 2$. Moreover, we use the usual Haar measure on $\mathbb{A}_f^\times$. It satisfies $\text{vol}(\mathbb{Z}_p^\times) = \text{vol}(\hat{\mathbb{Z}}^\times) = 1$ and $\text{vol}(\mathbb{Q}^\times \setminus \mathbb{A}_f^\times) = 1/2$.

For later reference, we write the Fourier expansion of $E_P(\tau, 0)$ as

$$E_P(\tau) = E_P(\tau, 0) = \phi_0 + \sum_{\beta \in P'/P} \sum_{n \in \mathbb{Q}_{>0}} \sum_{n \in \mathbb{Q}(\beta) + \mathbb{Z}} \rho(n, \beta) e(n \tau) \phi_\beta.$$ 

A crucial fact is that $E_P(\tau)$ maps to $E_P(\tau)$ under the $\xi_1$-operator. This has been stated by Bruinier and Yang [BY09, Remark 2.4] and follows directly from equation (2.19) in [BY09].

The Fourier expansion of $E_P(\tau)$ can be determined using a very general result by Kudla and Yang [KY10] on the coefficients of Eisenstein series on $SL_2$. See Proposition 7.2 in [KY10] and Schofer [Sch09].

### 3.2. The value of $\Phi(z,f)$ at an individual CM point.

We are now interested in computing the value of the theta lift $\Phi(z,f)$ at a CM point (rather than averaging over the cycle). Let $K_P \subset K_T \subset T(\mathbb{A}_f)$ be a compact open subgroup such that $K_P$ preserves $\check{P}$ and acts trivially on $P'/P$. Consider the Shimura variety

$$Z(U)_{P,K} = T(\mathbb{Q}) \backslash \{ z_U^\pm \} \times T(\mathbb{A}_f)/K_P.$$ 

This is isomorphic to two identical copies of the “class group”

$$C_{P,K} = T(\mathbb{Q}) \backslash T(\mathbb{A}_f)/K_P$$

and defines a cover of the CM cycle $Z(U)_K$ with $[C_K : C_{P,K}]$ branches.

Since $K_P$ acts trivially on $P'/P$, the value $\Theta_P(\tau, h) = \Theta_P(\tau, z_U^\pm, h)$ is well defined for an element $h \in C_{P,K}$. As a function of $\tau$, we have $\Theta_P(\tau, h) \in M_{1,P}$.

We would like to apply Stokes’ theorem to compute $\Phi(z,h,f)$ for $(z,h) \in Z(U)_{P,K}$. We use the existence of a preimage $\check{\Theta}_P(\tau, h)$ under $\xi_1$ of each theta function $\Theta_P(\tau, h)$ for $h \in C_{P,K}$, which
is guaranteed the exact sequence (2.7). However, such a preimage is only unique up to weakly holomorphic modular forms. Later, we will make specific choices but for now we can work with an arbitrary preimage for every $h$.

Lemma 2.5 becomes the following statement in our situation.

**Lemma 3.3.** Let $\Theta_P(\tau, h) \in H_{1, P}$, such that $\xi_1(\Theta_P(\tau, h)) = \Theta_P(\tau, h)$. We have the equality of differential forms

$$
\tilde{\partial}(\Theta_P(\tau, h))d\tau = -v\Theta_P(\tau, z_U^\pm, h)d\mu(\tau).
$$

Note that we can always assume that $L$ splits as $L = P \oplus N$ without loss of generality because we can replace $\xi$ by $f_{P \oplus N}$, yielding

$$
\langle f, \Theta_L \rangle = \langle f_{P \oplus N}, \Theta_P \otimes \Theta_N \rangle,
$$

since $\Theta_{P \oplus N} = \Theta_P \otimes \Theta_N$, and $\Theta_L = (\Theta_{P \oplus N})^L$ by [BY09], identifying $S_P \otimes \mathbb{C} S_N$ with $S_L$.

We express the theta integral in a way that is convenient for the following calculations.

**Lemma 3.4.** We have

$$
\Phi(z_U^\pm, h, f) = \lim_{T \to \infty} \left( \int_{\mathcal{F}_T} \langle f_{P \oplus N}(\tau), \Theta_{N-}(\tau) \otimes \Theta_P(\tau, z_U^\pm, h) \rangle v d\mu(\tau) - A_0 \log(T) \right),
$$

where

$$
A_0 = CT \left( \langle f_{P \oplus N}^+(\tau), \Theta_{N-}(\tau) \otimes \phi_{0+} \rangle \right).
$$

We remark that this expression makes sense even if $(z_U^\pm, h)$ is contained in the support of $Z(f)$ and thus provides a (discontinuous) extension of $\Phi(z, h, f)$ to $Z(U)$ in case that $Z(f) \cap Z(U) \neq \emptyset$.

**Proof.** This is Lemma 4.5 of [BY09]. Note that a similar statement in the case of signature $(2, 0)$ can be found in Lemma 2.19 of [Sch09]. The proof along the lines of the proof Proposition 2.5 of [Kud03] is straightforward. 

Using the same techniques as Bruinier and Yang [BY09], we obtain the following theorem which is central to all of our applications.

**Theorem 3.5.** Let $f \in H_{k, L}$ with $k = 1 - n/2$. Then the value of $\Phi(z, h, f)$ for any $(z, h) \in Z(U)_{P, K}$ is given by

$$
\Phi(z, h, f) = CT \left( \langle f_{P \oplus N}^+(\tau), \Theta_{N-}(\tau) \otimes \Theta_P^+(\tau, h) \rangle \right)
$$

$$
- \int_{\text{SL}_2(\mathbb{Z}) \backslash \mathbb{H}} \langle \xi_k(f_{P \oplus N}(\tau), \Theta_{N-}(\tau) \otimes \Theta_P(\tau, h) \rangle v^{1+n/2} d\mu(\tau).
$$

Here, the integral is regularized by taking the limit

$$
\lim_{T \to \infty} \int_{\mathcal{F}_T} \langle \xi_k(f_{P \oplus N}(\tau), \Theta_{N-}(\tau) \otimes \Theta_P(\tau, h) \rangle v^{1+n/2} d\mu(\tau).
$$

**Remark 3.6.** Note that for $f \in M^!_{k, L}$ the second summand does not occur since $\xi_k(f) = 0$ in that case. Moreover, we should remark that the regularized integral can also be written as

$$
\int_{\text{SL}_2(\mathbb{Z}) \backslash \mathbb{H}} \langle L_{1-n/2}(f_{P \oplus N}), \Theta_{N-}(\tau) \otimes \Theta_P(\tau, h) \rangle d\mu(\tau)
$$

$$
= \int_{\text{SL}_2(\mathbb{Z}) \backslash \mathbb{H}} \langle \xi_k(f_{P \oplus N}), \Theta_{N-}(\tau) \otimes \Theta_P(\tau, h) \rangle v^{1+n/2} d\mu(\tau).
$$
Proof of Theorem 3.5. Assume again that $L = P \oplus N$. According to Lemma 3.4, we write

$$\Phi(z, h, f) = \lim_{T \to \infty} (I_T(z, h, f) - A_0 \log(T)), \quad (3.9)$$

where

$$I_T(z, h, f) = \int_{\mathcal{F}_T} \langle f(\tau), \Theta_N^-(\tau) \otimes \Theta_P(\tau, z_U^+, h) \rangle v d\mu(\tau)$$

$$= - \int_{\mathcal{F}_T} \langle f(\tau), \Theta_N^-(\tau) \otimes \overline{\Theta}_P(\tau, h) \rangle d\tau$$

$$= - \int_{\mathcal{F}_T} d \left( \langle f(\tau), \Theta_N^-(\tau) \otimes \overline{\Theta}_P(\tau, h) \rangle d\tau \right)$$

$$+ \int_{\mathcal{F}_T} \langle \overline{\partial} f(\tau), \Theta_N^-(\tau) \otimes \overline{\Theta}_P(\tau, h) \rangle d\tau.$$

Here, we have used Lemma 3.3.

For the first integral, we apply Stokes’ theorem and obtain

$$\int_{\mathcal{F}_T} d \left( \langle f(\tau), \Theta_N^-(\tau) \otimes \overline{\Theta}_P(\tau, h) \rangle \right) d\tau = \int_{\partial \mathcal{F}_T} \langle f(\tau), \Theta_N^-(\tau) \otimes \overline{\Theta}_P(\tau, h) \rangle d\tau$$

$$= - \int_{iT}^{iT+1} \langle f(\tau), \Theta_N^-(\tau) \otimes \overline{\Theta}_P(\tau, h) \rangle d\tau,$$

since the integrand is an $SL_2(\mathbb{Z})$-invariant differential form and thus the integral over the equivalent pieces of $\partial \mathcal{F}_T$ cancel. We split this into three pieces, insert this splitting into (3.9) and regroup to obtain

$$\Phi(z, h, f) = \lim_{T \to \infty} \int_{iT}^{iT+1} \langle f^+(\tau), \Theta_N^-(\tau) \otimes \overline{\Theta}_P(\tau, h) \rangle d\tau \quad (3.10)$$

$$+ \lim_{T \to \infty} \left( \int_{iT}^{iT+1} \langle f^+(\tau), \Theta_N^-(\tau) \otimes \overline{\Theta}_P(\tau, h) \rangle d\tau - A_0 \log(T) \right) \quad (3.11)$$

$$+ \lim_{T \to \infty} \int_{iT}^{iT+1} \langle f^-(\tau), \Theta_N^-(\tau) \otimes \overline{\Theta}_P(\tau, h) \rangle d\tau \quad (3.12)$$

$$+ \lim_{T \to \infty} \int_{\mathcal{F}_T} \langle \overline{\partial} f(\tau), \Theta_N^-(\tau) \otimes \overline{\Theta}_P(\tau, h) \rangle d\tau. \quad (3.13)$$

Each of the limits above exist.

The limit in (3.12) is equal to zero due to the exponential decay of $f^-(\tau)$. We write the Fourier expansion of the integrand as

$$\langle f^-(\tau), \Theta_N^-(\tau) \otimes \overline{\Theta}_P(\tau, h) \rangle = \sum_{n \in \mathbb{Z}} a(n, v) e(n\tau).$$

and insert this to obtain

$$\int_0^1 \langle f^-(u + iT), \Theta_N^-(u + iT) \otimes \overline{\Theta}_P(u + iT, h) \rangle du$$

$$= \sum_{n \in \mathbb{Z}} a(n, iT) e(inT) \int_0^1 e^{2\pi inu} du.$$
The integral above is equal to 0 for all \( n \in \mathbb{Z} \setminus \{0\} \) and is equal to 1 for \( n = 0 \). Consequently,

\[
\lim_{T \to \infty} \int_{iT}^{iT+1} \langle f^-(\tau), \Theta_{N^-}(\tau) \otimes \tilde{\Theta}_P(\tau, h) \rangle d\tau = \lim_{T \to \infty} a(0, iT) = \lim_{T \to \infty} \sum_{\mu \in \mathbb{L}'/\mathbb{L}} \sum_{m \in \mathbb{Q}_{>0}} c_{\bar{\tau}}(-m, \mu)W_k(-2\pi mT)c_g(m, \mu, T),
\]

where \( g(\tau) = \Theta_{N^-}(\tau) \otimes \tilde{\Theta}_P(\tau, h) \). Using the standard growth estimates for the Whittaker function and the Fourier coefficients of \( f \) and \( g \), we obtain that there is an \( N \in \mathbb{Z}_{>0} \) and a constant \( C > 0 \), such that for all \( m \geq N \), we have

\[
c_{\bar{\tau}}(-m, \mu)W_k(-2\pi mT)c_g(m, \mu, T) = O(e^{-mCT}).
\]

Thus, for every \( T > 0 \), the constant term in the Fourier expansion of the function

\[
\langle f^-(u + iT), \Theta_{N^-}(u + iT) \otimes \tilde{\Theta}_P(u + iT, h) \rangle
\]

can be bounded by

\[
|a(0, iT)| \leq c \frac{r(T)}{1 - r(T)} \text{ with } r(T) = e^{-CT},
\]

where \( c, C > 0 \) are constants. Therefore, in the limit \( T \to \infty \), we have

\[
\lim_{T \to \infty} |a(0, iT)| = 0,
\]

which finally shows that \((3.12)\) vanishes.

To show that the limit in \((3.11)\) is equal to zero is analogous.

The contribution from \((3.10)\) is given by the constant term in the Fourier expansion of

\[
\langle f^+_{P \oplus N}(\tau), \Theta_{N^-}(\tau) \otimes \tilde{\Theta}_P(\tau, h) \rangle.
\]

Finally, by Lemma 2.5, we see that \((3.13)\) is equal to

\[
\lim_{T \to \infty} \int_{F_T} \langle \tilde{\Theta} f(\tau), \Theta_{N^-}(\tau) \otimes \tilde{\Theta}_P(\tau, h) \rangle d\tau
= -\lim_{T \to \infty} \int_{F_T} \langle L_{1-n/2}f(\tau), \Theta_{N^-}(\tau) \otimes \tilde{\Theta}_P(\tau, h) \rangle d\mu(\tau).
\]

This is exactly the definition of the regularized integral in the statement of the theorem. We still have to justify that this limit exists. However, this now follows from the vanishing of \((3.11)\) and \((3.12)\) and the fact that \( \Phi(z, h, f) \) is defined at \((z, h)\). That is, we have shown that

\[
\lim_{T \to \infty} \int_{F_T} \langle \xi_k(f), \Theta_{N^-}(\tau) \otimes \tilde{\Theta}_P(\tau, h) \rangle v^{1+n/2} d\mu(\tau)
= -\Phi(z, h, f) + CT \left( (f^+_{P \oplus N}(\tau), \Theta_{N^-}(\tau) \otimes \tilde{\Theta}_P(\tau, h)) \right)
\]

and therefore, the limit exists.

\[\square\]

Remark 3.7. Note that the formula holds for any preimage of \( \Theta_P(\tau, h) \) under \( \xi_1 \).
4. The holomorphic part of $\tilde{\Theta}_P$

An important ingredient for Theorem 3.5 was the existence of a harmonic weak Maaß form $\tilde{\Theta}_P^+(\tau, h)$, such that $\xi(\tilde{\Theta}_P^+(\tau, h)) = \Theta_P(\tau, h)$ for a two-dimensional positive definite lattice $P$. In this section, we will prove our main result, Theorem 4.21, which gives detailed information about the Fourier coefficients of the holomorphic part of appropriately normalized $\tilde{\Theta}_P(\tau, h)$ that appear on the right-hand side of the formula for the CM value in Theorem 3.5. Theorem 1.2 in the introduction is a special case of this result.

The proof exploits a certain seesaw identity that will allow us to express the coefficients of the holomorphic part of $\tilde{\Theta}_P^+(\tau, h)$ essentially as special values of Borcherds products on modular curves. We will relate each of these coefficients to CM values of meromorphic modular forms of weight zero given by a Borcherds product. Then, we apply the results of [Ehl15] to determine their prime ideal factorization.

4.1. Embedding into a modular curve. In this section, we basically use the same setup as in Section 7.1 of [BY09], except that Bruinier and Yang work in signature $(1, 2)$.

Let $N$ be a positive integer and consider the congruence subgroup $\Gamma_0(N) \subset \text{SL}_2(\mathbb{Z})$, defined by

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{Z}) \mid c \equiv 0 \mod N \right\}.$$ 

The modular curve $Y_0(N) := \Gamma_0(N) \backslash \mathbb{H}$ can be obtained as a Shimura variety as follows.

Consider the vector space $V := \{ x \in M_2(\mathbb{Q}) \mid \text{tr}(x) = 0 \}$ and define the quadratic form by $Q(x) = -N \det(x)$. The corresponding bilinear form is $(x, y) = N \text{tr}(xy)$. The space $(V, Q)$ has signature $(2, 1)$.

The even part of the Clifford algebra is $C^0(V) = M_2(\mathbb{Q})$ and $H = \text{GSpin}_V \cong \text{GL}_2$. The action of $\gamma \in H$ on $x \in V$ is given by

$$\gamma \cdot x = \gamma x \gamma^{-1}.$$ 

We have an isomorphism $\mathbb{H} \cup \overline{\mathbb{H}} \rightarrow \mathbb{D}$ via

$$z = x + iy \mapsto \mathbb{R} \text{Re} \left( \begin{pmatrix} z & -z^2 \\ 1 & -z \end{pmatrix} \right) \oplus \mathbb{R} \text{Im} \left( \begin{pmatrix} z & -z^2 \\ 1 & -z \end{pmatrix} \right).$$

The action of $\gamma \in \text{GL}_2 = \text{GSpin}_V$ is explicitly given by

$$\gamma \cdot \begin{pmatrix} z & -z^2 \\ 1 & -z \end{pmatrix} = \frac{(cz + d)^2}{\text{det}(\gamma)} \begin{pmatrix} \gamma z & -(\gamma z)^2 \\ 1 & -\gamma z \end{pmatrix},$$

where $\gamma z$ is the action via linear fractional transformations on $\mathbb{H} \cup \overline{\mathbb{H}}$.

For a prime $p$, let

$$K_p = \{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{GL}_2(\mathbb{Z}_p) \mid c \in N\mathbb{Z}_p \}$$

and

$$K = \prod_p K_p.$$ 

Then $K$ is a compact open subgroup of the adelic group $H(\mathbb{A}_f)$ and by strong approximation [Bum97, Theorem 3.3.1], we have $H(\mathbb{A}_f) = H(\mathbb{Q})K$ and $H(\mathbb{A}) = H(\mathbb{Q})H(\mathbb{R})^+K$. This implies that $X_K \cong \Gamma \backslash \mathbb{H}$ where $\Gamma$ is given by $\Gamma = H(\mathbb{Q}) \cap H(\mathbb{R})^+K \cong \Gamma_0(N)$. The isomorphism is explicitly given by

$$Y_0(N) \rightarrow X_K, \Gamma_0(N)z \mapsto H(\mathbb{Q})(z, 1)K.$$ (4.1)
In $V$ we have the even lattice

$$L = \left\{ \begin{pmatrix} b & -a \\ c & -b \end{pmatrix} \bigg| a, b, c \in \mathbb{Z} \right\}.$$ 

The dual lattice of $L$ is given by

$$L' = \left\{ \begin{pmatrix} b \\ 2Nc \end{pmatrix} \bigg| a, b, c \in \mathbb{Z} \right\}.$$ 

The discriminant group $L'/L$ is cyclic of order $2N$ and we can identify the corresponding finite quadratic module with the group $\mathbb{Z}/2N\mathbb{Z}$ together with the quadratic form $x^2/4N$, valued in $\mathbb{Z}/2N\mathbb{Z} \subset \mathbb{Q}/\mathbb{Z}$. The isomorphism of finite quadratic modules is explicitly given by

$$\mathbb{Z}/2N\mathbb{Z} \rightarrow L'/L, \quad r \mapsto \mu_r = \begin{pmatrix} r/2N \\ -r/2N \end{pmatrix} =: \text{diag}(r/2N, -r/2N).$$

It is easy to check that the group $K$ preserves the lattice $L$ and acts trivially on $L'/L$.

As in section 7.1 of [BY09], we obtain a positive definite, two-dimensional lattice

$$\mathcal{P} := L \cap U = \mathbb{Z} \begin{pmatrix} 1 & 0 \\ -r & -1 \end{pmatrix} \oplus \mathbb{Z} \begin{pmatrix} 0 & -1 \\ \frac{N}{D-r^2} & 0 \end{pmatrix}$$

and a negative definite, one-dimensional lattice

$$\mathcal{N} := L \cap \mathbb{Q} \lambda_r = \mathbb{Z} \frac{2N}{t} \lambda_r \text{ with dual } \mathcal{N}' = \mathbb{Z} \frac{t}{D} \lambda_r.$$ 

Here, $t = (r, 2N)$.

From now on, assume that $D < 0$ is a fundamental discriminant and let $k = k_D = \mathbb{Q}(\sqrt{D})$. The following lemma is easy to prove and identifies the lattice $\mathcal{P}$ as an integral ideal (see Lemma 7.1 in [BY09]).

**Lemma 4.1.** With the same notation as above, we have an isometry of lattices

$$(\mathcal{P}, Q) \cong \left( n, \frac{N(x)}{N(n)} \right) \text{ with } n = \begin{pmatrix} N, r + \sqrt{D} \\ 2 \end{pmatrix} \subset \mathcal{O}_k.$$ 

It is explicitly given by

$$\begin{pmatrix} 1 & 0 \\ -r & 1 \end{pmatrix} \mapsto N \quad \text{and} \quad \begin{pmatrix} 0 & -1 \\ \frac{N}{D-r^2} & 0 \end{pmatrix} \mapsto \frac{r + \sqrt{D}}{2}.$$
We note that the two points $z_U^±$ corresponding to $\lambda_r$ satisfy the quadratic equation
\[
\frac{r^2 - D}{4} \tau + r\tau + 1 = 0
\]
in terms of coordinates of $\mathbb{H} \cup \overline{\mathbb{H}}$. In the following, we will often simply write $z_U$ for any of the two points $z_U^±$.

Recall that we write $T = \text{GSpin}_U$ as in Section 3 and we consider $T$ as a subgroup of $H = \text{GSpin}_V$, acting trivially on $U^⊥$. An explicit calculation shows the following lemma.

**Lemma 4.2.** We have that $T = \text{GSpin}_U \cong k_D^\times$ via

\[
1 \mapsto \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sqrt{D} \mapsto \left( \frac{r}{r^2 - D}, \frac{-2}{r^2 - D} \right).
\]

and for $K$ as above, we have

\[
K_T := K \cap T(\mathcal{A}_f) \cong \hat{\mathcal{O}}_k^\times.
\]

Recall the setup from Section 3.2. Note that $K_T$ acts trivially on $\mathcal{P}'/\mathcal{P}$ and thus we can take $K_P = K_T$ and we have $C_{P,K} = T(\mathcal{A}_f)/K_T$, which is isomorphic to $I_k/\hat{\mathcal{O}}_k^\times \cong \text{Cl}_k$ for $k = k_D$.

Now we specialize the setup for our application. From now on and for the rest of this section we fix an integral ideal $\mathfrak{a} \subset \mathcal{O}_k$ of the imaginary quadratic field $k = k_D$ of discriminant $D$. We assume throughout that $D$ is an odd fundamental discriminant. Consider the positive definite lattice $(P,Q) := \left( \mathfrak{a}, \frac{N(\mathfrak{a})}{\sqrt{D}} \right)$ and write $\mathfrak{a} = \left( A, \frac{B+\sqrt{D}}{2} \right)$ with $A, B \in \mathbb{Z}$, $A > 0$. That is, the ideal $\mathfrak{a}$ is generated by $A$ and $\frac{B+\sqrt{D}}{2}$. This is equivalent to saying that $P$ (or $\mathfrak{a}$) corresponds to the positive definite integral binary quadratic form $[A,B,C]$ of discriminant $D = B^2 - 4AC$, with $C \in \mathbb{Z}$ determined by $A, B$ and $D$. We will use the construction above to embed the lattice $P$ into the lattice $L$ for $N = A|D|$.

**Assumption 4.3.** Without loss of generality, we will assume that $(A,D) = 1$. If $A$ is not coprime to $D$, we can replace $[A,B,C]$ with an equivalent form. An integral binary quadratic form of discriminant $D$ represents infinitely many primes (cf. Theorem 9.12 of [Cox89]). Thus, we may in fact choose $A$ to be a prime not dividing $D$.

Under these assumptions, there are $E, F \in \mathbb{Z}$ with $2AE + BF = 1$. (Note that $(A,D) = 1$ implies $(2A,B) = 1$ because $D$ is odd.) Using this, we have for $R := F\mathcal{D}$ that

\[
R^2 \equiv D \mod 4A|D|.
\]

Indeed, we have $R^2 \equiv 0 \mod D$ and $F^2D^2 = F^2D \cdot D = F^2(B^2 - 4AC)D \equiv D \mod 4A$.

**Warning.** Note that the definition of $R$ depends of course on the ideal $\mathfrak{a}$ we started with.

With this setup, we put $M := \frac{1}{4A} = \frac{B}{4A|D|}$ and let $\lambda_R$ as in (4.2). Note that we obtain in this special case

\[
\mathcal{N} = L \cap \mathbb{Q}\lambda_R = \mathbb{Z}2A\lambda_R \quad \text{with dual} \quad \mathcal{N}' = \mathbb{Z}\lambda_R.
\]

In contrast to the general case, in our situation the lattice $L$ splits as $L = \mathcal{P} \oplus \mathcal{N}$. This follows from the fact that the discriminant group $\mathcal{N}'/\mathcal{N}$ is isomorphic to $\mathbb{Z}/2AZ$ and the following lemma.

**Lemma 4.4.** With the same notation as above, we have an isometry of lattices

\[
(\mathcal{P}, Q) \cong \left( b, \frac{N(x)}{N(b)} \right).
\]
with
\[ b = \left( A|D|, \frac{R + \sqrt{D}}{2} \right) = \partial_k a. \]

In particular, \([b] = [a] \in \text{Cl}_k\). The isometry is given explicitly in Lemma 4.1.

Throughout, we fix the isometry \(a \to b\) given by \(x \mapsto \sqrt{D}x\) and \(P = a \to b \to P\) given by applying \(x \mapsto \sqrt{D}x\) and then the isometry of Lemma 4.4. We also tacitly identify the theta functions \(\Theta_P(\tau, h)\) using this isometry and write \(P\) instead of \(P\) for simplicity whenever this does not make a difference.

We will now see that the special divisor \(Z(M, \mu_R)\) (Definition 2.2) is given by the CM cycle \(Z(U)\). For \(D\) coprime to \(N\), this was stated as Proposition 7.2 in [BY09] but it also holds in our special situation.

**Proposition 4.5.** With the same notation as above we have
\[ Z(U) = Z(M, \mu_R). \]

It is important for us to understand the action of \(T(\mathbb{A}_f) \cong \mathbb{A}_{k,f}^\times\) on modular functions precisely. It is given by the following Lemma which can be easily shown using Theorem 6.31 of [Shi94].

**Lemma 4.6.** Let \(h \in \mathbb{A}_{k,f}^\times\) and let \(f \in \mathbb{Q}(X_0(N))\) be a rational modular function. Let \(t \in T(\mathbb{A}_f)\) be the image of \(h\) and write \(t = \gamma k\) for \(\gamma \in H(\mathbb{Q})\) and \(k \in K\). Let \(z \in Z(M, \mu_R)\) be a CM point. Then we have
\[ f(z)^{\sigma(h)} = f(\gamma^{-1}z). \]

4.2. A seesaw identity. The motivation for this and the next section is the following. Let us write the Fourier expansion of \(\tilde{\Theta}_P^+(\tau, h)\) as
\[ \tilde{\Theta}_P^+(\tau, h) = \sum_{\beta \in P^*/P} \sum_{m \gg \infty} c_P^+(h, m, \beta)e(m\tau)\phi_\beta. \]

Suppose that there is a weakly holomorphic modular form \(f \in M_{1,p}^!\), with principal part
\[ P_f = q^{-m}(\phi_\beta + \phi_{-\beta}) \]
for \(m > 0\). (Note that such a form often does not exist but we will deal with this problem in the next section.) On the one hand, we obtain by Theorem 3.5 that
\[ \Phi_P(h, f) = \int_{\Gamma \backslash \mathbb{H}} \langle f(\tau), \tilde{\Theta}_P(\tau, h) \rangle v d\mu(\tau) \]
\[ = \text{CT}(f(\tau), \tilde{\Theta}_P^+(\tau, h)) = c_P^+(h, m, \beta) + c_P^+(h, m, -\beta) + \text{“error term”}, \]
where the “error term” is a contribution of the pairing of the principal part of \(\tilde{\Theta}_P\) with the coefficients of positive index of \(f\). Let us ignore this term for the moment.

On the other hand, there is a different expression for the theta lift \(\Phi_P(h, f)\) in terms of a CM value of the theta lift for the lattice \(L\). The basic principle goes back to Kudla [Kud84] who realized that many previously mysterious identities between theta lifts can be understood in the context of “seesaw dual reductive pairs”.

Explicitly, we will use the embedding defined above to obtain an expression for the regularized theta lift in the case of signature \((2,0)\) as a CM value of a modular function on \(X_0(N)\) for \(N = A|D|\).
Consider the theta lift $\Phi_L$ corresponding to $L$ defined as

$$
\Phi_L(z, h, g) = \int_{\Gamma \backslash \mathbb{H}} (g, \Theta_L(\tau, z)) v^{\frac{1}{2}} d\mu(\tau),
$$

for $(z, h) \in X_K$. Since the lattice $L$ splits as $L = P \oplus N$, the Weil representation $\rho_L$ is isomorphic to the tensor product $\rho_P \otimes \rho_N$. In particular, we have that $M^1_{1, P} \otimes M^1_{-\frac{3}{2}, N}$ is a subspace of $M^1_{\frac{3}{2}, L}$.

At a point $(z_U^+, h) \in Z(U)$ corresponding to the splitting $P \oplus N$ we obtain for an element $f \otimes_C \varphi$ in $M^1_{1, P} \otimes_C M^1_{-\frac{3}{2}, N}$ that

$$
\Phi_L(z_U, h, f \otimes \varphi) = \int_{\Gamma \backslash \mathbb{H}} \langle (f \otimes_C \varphi)(\tau), (\Theta_P \otimes_C \Theta_N)(\tau, h) \rangle v^{\frac{1}{2}} d\mu(\tau)
= \int_{\Gamma \backslash \mathbb{H}} \langle f(\tau), \Theta_P(\tau, h) \rangle \langle \varphi(\tau), \Theta_N(-\tau) \rangle v d\mu(\tau).
$$

Now we choose a specific function $\varphi$. Since $N'/N = \mathbb{Z}/2\mathbb{Z}$ with quadratic form $-x^2/4A$, the space $M^1_{k,N}$ is isomorphic to the space of weakly holomorphic Jacobi forms $J^1_{k+\frac{1}{2},A}$ of weight $k + \frac{1}{2}$ and index $A$. See also Section 2.7. In particular, $M^1_{-\frac{3}{2}, N}$ is isomorphic to $J^0_{0,A}$. It follows from Proposition 2.14, that $J^1_{0,A}$ is generated as a $\mathbb{C}$-vector space by elements of the form

$$
\sum_{j=0}^A \psi_j \mathcal{F}^j \mathcal{G}^{A-j},
$$

where $\psi_j \in M^1_{2j}({\text{SL}_2}(\mathbb{Z}))$ and $\mathcal{F}, \mathcal{G}$ are the generators of the ring of weak Jacobi forms of even weight as in Proposition 2.14. We will frequently identify these forms as vector valued modular forms in $M^1_{-5/2,N}$ and $M^1_{-1/2,N}$ via the theta development of Jacobi forms (see Section 2.7). Under this identification, the following relation is easy to obtain.

**Lemma 4.7.** We have

$$
\langle \mathcal{F}^j(\tau) \mathcal{G}^{A-j}(\tau), \Theta_N(-\tau) \rangle = \begin{cases} 12^A, & \text{if } j = 0, \\ 0, & \text{otherwise.} \end{cases}
$$

Using the lemma, we obtain that

$$
\sum_{j=0}^A \langle \psi_j \mathcal{F}^j \mathcal{G}^{A-j}(\tau), \Theta_N(-\tau) \rangle = 12^A \psi_0(\tau).
$$

Consequently,

$$
\Phi_L(z_U, h, f \otimes \sum_{j=0}^A \psi_j \mathcal{F}^j \mathcal{G}^{A-j}) = 12^A \int_{\Gamma \backslash \mathbb{H}} \psi_0(\tau) \langle f(\tau), \overline{\Theta_P(\tau, h)} \rangle v d\mu(\tau)
= 12^A \Phi_P(h, f \cdot \psi_0).
$$

This allows us to relate $c^+_P(h, m, \beta)$ to the special value $\Phi_L(z_U, h, g)$ for a weakly holomorphic modular form $g$, which is equal to $\log |\Psi((z_U, h, g)|$ by Borcherds’ theorem (Theorem 2.8) if the constant term of the input function $g$ vanishes. In this case, $\Psi(z, g)$ is a meromorphic modular form of weight zero. By choosing $g$ carefully, we can assure that those special values of $\Psi(z, g)$ lie in the Hilbert class field $H_k$ and determine their prime ideal factorization using the results of [Ehl15].
Remark 4.8. For $D$ a prime discriminant, a similar setup is used in [Via12]. The main result of [Via12] expresses $\Phi_P(1, f)$ essentially as the logarithm of an algebraic integer given by a special value of a Borcherds product on a modular curve of level $|D|$ and gives a formula for the prime ideal decomposition using work of Gross. We make use of Viazovska’s idea but our setup is slightly different and combines some of the ideas of [Via12] with those in Section 7 of [BY09]. This allows for square-free discriminants and by working adelically, we can keep track of all theta functions $\Theta_P(\tau, h)$ at once and determine the action of $\text{Gal}(\mathbb{H}_k/k)$ on the preimages $\tilde{\Theta}_P(\tau, h)$. Moreover, harmonic Maaß forms are not considered in [Via12] and to obtain a result that applies to the coefficients $c^p_{\tilde{\Phi}}(h, m, \beta)$ requires extra work as we shall see below. Finally, to establish the relation to the special cycles (without the use of explicit formulas), and to apply our results to arithmetic geometry, we use the results of [Ehl15] which build on the arithmetic pullback formula in [BY09] (and for the explicit formulas, of course on Gross’ result, as well).

4.3. Weakly holomorphic modular forms of weight 1. We will now define a set of weakly holomorphic modular forms $f_{m, \beta}$ in $M_{1, P}$ serving as a replacement for a form with principal part $q^{-m}(\phi_{\beta} + \phi_{-\beta})$ in case such a form does not exist.

Recall that, according to the exact sequence

$$0 \rightarrow M_{1, P}^1 \rightarrow H_{1, P} \rightarrow S_{1, P} \rightarrow 0,$$

the space $S_{1, P}$ is the space of obstructions for the existence of a weakly holomorphic modular form with prescribed principal part. Moreover, note that if $M_{1, P} \neq \{0\}$ only fixing the principal part is not enough to uniquely determine $f_{m, \beta}$, which is why we have to take this space into account as well.

We write $S_{k, L}(\mathbb{Q})$ for the space of cusp forms of weight $k$, representation $\rho_L$ with only rational Fourier coefficients. By [McG03], we have $S_{k, L}(\mathbb{Q}) \otimes \mathbb{Q} \subset S_{k, L}$. The following lemma gives a basis for $S_{k, L}(\mathbb{Q})$ (for any $L$ and $k$) that has a “simple” structure, similar to a $q$-expansion basis starting with increasing powers of $q$ for scalar valued forms. The proof is straightforward.

Lemma 4.9. Let $L$ be an even lattice. Then there is a basis $\{g_1(L), \ldots, g_{d(L)}(L)\}$ of $S_{k, L}(\mathbb{Q})$, where $d(L) = \dim S_{k, L}$ with the following property. There are rational numbers $n_1(L) \leq \ldots \leq n_{d(L)}(L)$ and elements $\mu_1(L), \ldots, \mu_{d(L)}(L) \in L'/L$, such that we have for their Fourier coefficients

$$c_{g_j}(n_1, \mu_1) = \delta_{j, 1}.$$

Definition 4.10. From now on, we let $p_j := n_j(P)$ and $\pi_j = \mu_j(P)$ be a set of indexes as in Lemma 4.9 for the space $S_{1, P}(\mathbb{Q})$. We also write $d^+ = d(P)$.

Moreover, we fix $n_j := n_j(P^-)$, $\beta_j := \mu_j(P^-)$, and $g_j := g_j(P^-)$ as in Lemma 4.9 for $S_{1, P^-}(\mathbb{Q})$. We write

$$g_j(\tau) = \sum_{\beta \in P'/P} \sum_{m \in \mathbb{Q}_{>0}} a_j(m, \beta)e(m\tau)\phi_{\beta}$$

for the Fourier expansion of $g_j$ and we write $d^- = d(P^-)$.

Proposition 4.11. For $m \in \mathbb{Q}_{>0}$ and $\beta \in P'/P$ with $m + Q(\beta) \in \mathbb{Z}$ and $(m, \beta) \neq (n_j, \beta_j)$ for all $j \in \{1, \ldots, d^-\}$, there is a weakly holomorphic modular form $f_{m, \beta} \in M_{1, P}$ with only rational Fourier coefficients, having a Fourier expansion of the form

$$f_{m, \beta}(\tau) = q^{-m}(\phi_{\beta} + \phi_{-\beta}) - \sum_{j=1}^{d^-} a_j(m, \beta)q^{-n_j}(\phi_{\beta_j} + \phi_{-\beta_j}) + O(1)$$

(4.10)

and with $c_{f_{m, \beta}}(0, 0) = 0$ and $c_{f_{m, \beta}}(p_j, \pi_j) = 0$ for all $j \in \{1, \ldots, d^+\}$. 
Proof. The existence of $\tilde{f}_{m,\beta} \in H_{1,P}$ with the principal part as above is clear by Proposition 3.11 of [BF04]. However, we have

$$\{ \tilde{f}_{m,\beta}, g \} = 0$$

for all $g \in S_{1,P}$ by construction and Lemma 2.6, where $\{ , \}$ is the pairing in (2.9). Thus, $\xi_1(\tilde{f}_{m,\beta}) = 0$ and by (2.7) we have $\tilde{f}_{m,\beta} \in M_{1,P}^1$.

If we multiply any such form $\tilde{f}_{m,\beta}$ with $\Delta(\tau)^{m_0}$, where $m_0 = \max\{m, n_1, \ldots, n_d\}$, to obtain an element of $M_{1+12m_0,P}$. Since this space has a basis of forms with integral Fourier coefficients [McG03], there is an element $g \in M_{1+12m_0,P}(\mathbb{Q})$ with only rational Fourier coefficients, such that $\tilde{f}_{m,\beta} \Delta^{m_0} - g = O(q^{m_0})$. Consequently, we have $\tilde{g} = g/\Delta^{m_0} \in M_{1,P}^1$ and $\tilde{g}$ has the correct principal part and only rational Fourier coefficients.

Finally, we can subtract suitable multiple of $E_P$ to obtain the vanishing of the Fourier coefficient of index $(0,0)$ and multiples of the special basis elements of $S_{1,P}$ to ensure that $c_{f_{m,\beta}}(p_j, \pi_j) = 0$ for all $j \in \{1, \ldots, d^+\}$. \hfill \Box

Lemma 4.12. Let $f \in M_{1,P}^1$. There exists a weakly holomorphic modular form $f \in M_{1/2,L}^1$, such that

$$\langle f(\tau), \Theta_N(\tau) \rangle = 12^A f(\tau)$$

(4.11)

and the constant term $c_f(0,0)$ of $f$ vanishes. If $f$ has rational Fourier coefficients, the same is true for $f$ and if $f$ has only integral Fourier coefficients, then the Fourier coefficients of $f$ are rational with denominators bounded by the maximum of

$$h_k \cdot \left( \frac{2A}{A} \right) \text{ and } 12h_k + 2w_k.$$

Proof. For the proof, recall the definition of $F$ and $G$ from Proposition 2.14. Let us take $f = f \otimes G^A$, the simplest choice. Note that $G^A$ has integral Fourier coefficients. Then we consider two cases.

The first case is $A = 1$. There is a $g \in M_2^1(\text{SL}_2(\mathbb{Z}))$ with only integral Fourier coefficients such that we have $g(\tau) = q^{-1} + O(q)$. The weakly holomorphic modular form $E_P \otimes Fg \in M_{1/2,L}^1$ has a non-vanishing constant term

$$c_{E_P}(1,0)c_F(0,0) + c_F(1,0).$$

In fact, $c_{E_P}(1,0) = w_k/h_k$ and we have $c_F(0,0) = -2$ and $c_F(1,0) = -12$.

Therefore, replacing $f$ by

$$f + \frac{c_f(0,0)h_k}{12h_k + 2w_k} E_P \otimes Fg.$$ 

eliminates the constant term and does not change (4.11).

In the other case, when $A > 1$, we can take $g \in M_{2A}(\text{SL}_2(\mathbb{Z}))$ with $g(\tau) = 1 + O(q)$ and subtract a multiple of $E_P \otimes F^Ag$. Note that the constant term of $F^A$ is given by the constant term of $(\zeta - 2 + \zeta^{-1})^A$, which can be easily seen to be equal to

$$(-1)^A \left( \frac{2A}{A} \right)$$

and is in particular nonzero. Thus, we can replace $f$ by

$$f - \frac{c_f(0,0)}{(-1)^A 2^A} E_P \otimes F^Ag$$

to obtain a vanishing constant term without changing (4.11). \hfill \Box
We denote by $F_{m,\beta} := f_{m,\beta}$ the weakly holomorphic modular form corresponding to $f_{m,\beta}$ via Lemma 4.12.

It is easy to see that $f_{m,\beta}$ has Fourier coefficients with bounded denominators using the result of McGraw [McG03] that the space of vector valued modular forms has a basis of modular forms with only integral Fourier coefficients.

**Definition 4.13.** For each $\beta \in P'/P$ and $m \in \mathbb{Q}_{>0}$ with $m + Q(\beta) \in \mathbb{Z}$ we let $c_{m,\beta} \in \mathbb{Z}$, such that $c_{m,\beta}F_{m,\beta}$ has only integral Fourier coefficients.

We have that
\[ 12^A c_{m,\beta} \int_{\Gamma \backslash \mathbb{H}} \langle f_{m,\beta}(\tau), \Theta_P(\tau, h) \rangle v \, d\mu(\tau) = \Phi_L(z_U, h, c_{m,\beta}F_{m,\beta}). \quad (4.12) \]

By the theorem of Borcherds (Theorem 2.8), the value on the right hand side is essentially the logarithm of a special value of a rational function on $Y_0(N)$ (or rather on its compactification $X_0(N)$) which is defined over $\mathbb{Q}$, as long as the coefficients of the input function (in our case given by $c_{m,\beta}f_{m,\beta}$) are all integers. Moreover,
\[ \int_{\Gamma \backslash \mathbb{H}} \langle f_{m,\beta}(\tau), \Theta_P(\tau, h) \rangle v \, d\mu(\tau) = - \frac{2}{12^A c_{m,\beta}} \log |\Psi_L(z_U, h, c_{m,\beta}F_{m,\beta})|^2. \quad (4.13) \]

By CM theory (Lemma 4.16), an integral power of the value $\Psi_L(z_U, h, c_{m,\beta}F_{m,\beta})$ on the right is contained in the Hilbert class field $H_k$ of $k$. On the left hand side, we essentially obtain the coefficient $c_P^L(h, m, \beta)$ we are interested in and some "error terms".

4.4. **The arithmetic pullback and the coefficients of the holomorphic part.** In this section we will determine the prime ideal factorization of the algebraic number given by the special value $\Psi_L(z_U, h, c_{m,\beta}F_{m,\beta})$.

Our basic setup is the following. Given $P \cong \mathfrak{a}$, we let $\mathfrak{b} = \delta_k \mathfrak{a}$ as in Lemma 4.4 and put $N = A|D|$, where $(P, Q)$ corresponds to the integral binary quadratic form $[A, B, C]$. As before, we assume that $(A, D) = 1$. Moreover, we let $M, R$ be as defined on page 23.

**Definition 4.14.** For a harmonic weak Maass form $f \in H_{1/2,L}(\mathbb{Z})$, we write
\[ Z(f) = \sum_{\mu \in L'/L} \sum_{m < 0} c_f^+(m, \mu) Z(m, \mu) \]
for the divisor associated with $f$ on $Y_0(N)$. Here, $Z(m, \mu)$ is the extension of the Heegner divisor $Z(m, \mu)$ to the stack $X_0(N)$ as in [BY09] and [Ehl15]. It is equal to the flat closure of $Z(f)$, defined analogously.

For $f \in H_{1/2,L}$ with integral principal part the pair $\hat{Z}^c(f) = (Z^c(f), \Phi_L(\cdot, g))$ defines an arithmetic divisor on $X_0(N)$. Here, $Z^c(f) = Z(f) + C(f)$ is a suitable extension of $Z(f)$ to $X_0(N)$ where $C(f)$ is supported at the cusps.

**Lemma 4.15.** Let $f \in M_{1/2,L}$ with constant coefficient $c_f(0, 0) = 0$ and $c_f(m, \mu) \in \mathbb{Q}$ for all $m \in \mathbb{Q}$ and $\mu \in L'/L$. Then there exists an integer $M_f$, such that the Borcherds product $\Psi_L(z, h, M_f \cdot f)$ defines a meromorphic modular function contained in $\mathbb{Q}(j, j_N)$.

**Proof.** See Lemma 8.1 of [Ehl15].

The lemma implies that the arithmetic divisor $(\text{div}(\Psi_L(\cdot, M_f f)),- \log |\Psi(\cdot, M_f f)|^2)$ associated with the Borcherds lift of $f$ is principal.
Lemma 4.16. The CM value $\Psi_L(z_U^\pm, h, M_f \cdot f)$ is contained in the Hilbert class field $H_k$ of $k$ for every $(z_U^\pm, h) \in Z(U)$ and we have

$$\Psi_L(z_U^\pm, h, M_f \cdot f) = \Psi_L^{(h)}(z_U^\pm, 1, M_f \cdot f).$$

Proof. If we write $h = \gamma k \in H(\mathbb{Q})K$, then we have according to Lemma 4.6 that

$$\Psi_L(z_U^\pm, h, f) = \Psi_L(\gamma^{-1} z_U^\pm, 1, f) = \Psi_L^{(h)}(z_U^\pm, 1, f).$$

The following Theorem shows that $\text{div}(\Psi_L(z, h, f))$ is a horizontal divisor.

Theorem 4.17. Let $f \in M^1_{1/2, L}$ be a weakly holomorphic modular form with only integral Fourier coefficients and assume that $N$ is square-free. Suppose that the multiplier system of $\Psi_L(z, h, f)$ is trivial. Then the divisor of the rational function defined by $\Psi_L(z, h, f)$ on $\mathcal{Y}_0(N)$ is equal to $Z(f)$, the flat closure of $Z(f)$.

Proof. See Theorem 8.2 of [Ehl15].

The following Proposition is crucial for the proof of the main theorem in this section. It shows that we do not have to deal with bad intersection in order to obtain arithmetic information about the coefficients of $\Theta^+_P(\tau, h)$.

Proposition 4.18. Let $f \in M^1_{1, P}$ with $c_f(0, 0) = 0$ and $H \in M^1_{1/2, L}$, both with integral principal parts, such that

$$\langle H(\tau), \phi_{0+P} \otimes \Theta^{N-}_{N}(\tau) \rangle = \langle f(\tau), \phi_{0+P} \rangle.$$

Then $Z(H)$ and $Z(M, \mu_R)$ (see page 23) intersect properly.

Proof. The divisors $Z(M, \mu_R)$ and $Z(n, \nu)$ do not intersect properly if they have complex points in common, which can only occur if $Dd$ is a perfect square for $D = 4NM$ and $d = 4Nn$.

This means that improper intersection might occur here if there is a coefficient $c_{H}(m, \mu)$ with $4Nn = Dn^2$ for some $n \in \mathbb{Z}$ and $\mu = n \cdot \mu_R$ and $Z(m, \mu)(\mathbb{C}) \cap Z(M, \mu_R)(\mathbb{C})$ is given by points coming from non-primitive elements $n \lambda \in L_{m, \mu}$ with $\lambda \in L_{M, \mu_R}$. Thus, we obtain

$$Z(H)(\mathbb{C}) \cap Z(M, \mu_R)(\mathbb{C}) = \sum_{n \in \mathbb{Z}} c_{H} \left( -\frac{n^2}{4A}, n \cdot \mu_R \right) Z(M, \mu_R)(\mathbb{C})$$

and we need to show that the sum on the right-hand side is zero. On the one hand, we have by assumption $\text{CT}(\langle H(\tau), \phi_0 \otimes \Theta^{N-}_{N}(\tau) \rangle) = c_f(0, 0) = 0$. On the other hand, recalling that $N^\prime$ is spanned by $\lambda_R \equiv \mu_R \mod L$, we obtain

$$\text{CT}(\langle H(\tau), \phi_{0+P} \otimes \Theta^{N-}_{N}(\tau) \rangle) = \sum_{n \in \mathbb{Z}} c_{H} \left( -\frac{n^2}{4A}, n \cdot \mu_R \right),$$

by the definition of the theta function.

Therefore, the total multiplicity of improper intersection is zero.

Remark 4.19. An alternative approach to avoid improper intersection would be to show a moving lemma, i.e. that it is always possible to subtract a weakly holomorphic modular form $H \in M^1_{1/2, L}$, such that $\tilde{F}_{m, \beta} = F_{m, \beta} - H$ satisfies

$$c_{\tilde{F}_{m, \beta}} \left( -\frac{n^2}{4A}, n \cdot \mu_R \right) = 0$$

for all $n \in \mathbb{Z}$. This is in fact possible but tedious and, as the proposition shows, completely unnecessary for our applications.
The following proposition expresses the prime ideal factorization of CM values of Borcherds products that appear in the seesaw identity \((4.9)\) in terms of the special cycles \(\text{Spec} \mathcal{O}_{H_k}\) defined in Section 2.8.

**Proposition 4.20.** Let \(f \in M^1_{1,P} \) and \(g \in M_{-\frac{1}{2}N}^-\) with

\[
g = \sum_{j=0}^{A} \psi_j f^j g^{A-j},
\]

where \(\psi_j \in M^1_{2j}(\text{SL}_2(\mathbb{Z}))\) for every \(j\). Suppose that \(f \otimes g\) has only integral Fourier coefficients, \(c_{f \otimes g}(0,0) = 0\), and the multiplier system of \(\Psi_L((z,h,f \otimes g))\) is trivial. Then we have

\[
\text{ord}_\mathfrak{P}(\Psi_L(zU, h, f \otimes g)) = 12A w_k \sum_{\beta \in P'/P} \sum_{m>0} c_{f \otimes g}(-m,\beta) Z(m, a, h, \beta) \mathfrak{P} \quad (4.15)
\]

for every prime ideal \(\mathfrak{P}\) of the Hilbert class field \(H_k\).

**Proof.** By Lemma 4.7, we have

\[
\sum_{\beta \in P'/P} \langle f \otimes g, \phi_\beta \otimes \Theta_{X^-} \rangle \phi_\beta = 12Af \psi_0.
\]

Therefore,

\[
12A c_{f \psi_0}(-m,\beta) = \sum_{n \in \mathbb{Z}} c_{f \otimes g} \left( -m - \frac{n^2}{4A}, \beta + \mu_{\tilde{n}} \right).
\]

Here \(\tilde{n} = -Dn\) so that \(\mu_{\tilde{n}}\) corresponds to \(n \mod 2A\).

By Theorem 7.4 of [Ehl15], we have

\[
\text{ord}_\mathfrak{P}(\Psi_L(zU, 1, f \otimes g)) = \sum_{\beta \in P'/P} \sum_{\nu \in N'/N} \sum_{m_1 < 0} c_{f \otimes g}(m_1, \beta + \nu)
\]

\[
\times \sum_{n \equiv F_1 \mod 2A \atop n^2 \leq |D_1/D|} Z \left( |m_1| - \frac{n^2}{4A}, a, r_1 \left( 1 + \frac{F_\sqrt{D}}{2\sqrt{D}} \right) \right) \mathfrak{P}.
\]

Here, \(r_1\) corresponds to \(\mu_{r_1} = \beta + \nu\). Note that

\[
r_1 \left( \frac{1 + F\sqrt{D}}{2\sqrt{D}} \right) \in \partial_k^{-1} a/a
\]

does not depend on \(\nu\). Moreover, the generator \(A/\sqrt{D}\) of \(\partial_k^{-1} a\) maps to \(2A \mod 2N\) using the isometry in Lemma 4.4 and our identification of \(r \mod 2N\) with \(\mu_r\). Consequently,

\[
2A \frac{1 + F\sqrt{D}}{2\sqrt{D}} = \frac{A}{\sqrt{D}} + AF \equiv \frac{A}{\sqrt{D}} \mod a.
\]

Thus, we obtain

\[
\text{ord}_\mathfrak{P}(\Psi_L(zU, 1, f \otimes g)) = \sum_{\beta \in P'/P} \sum_{m>0} \sum_{n \in \mathbb{Z}} c_{f \otimes g} \left( -m - \frac{n^2}{4A}, \beta + \mu_{\tilde{n}} \right) Z(m, a, \beta)
\]

\[
= 12A \sum_{m>0} \sum_{\beta \in P'/P} c_{f \psi_0}(-m,\beta) Z(m, a, \beta) \mathfrak{P}.
\]
The result for $h \neq 1$ follows from the action of the Galois group $\text{Gal}(H_k/k)$ described in Lemma 4.6 above for the left-hand side and Proposition 3.5 of [Ehl15] for the cycles (note that the numbers in the published version have been accidentally shifted by the publisher and Proposition 3.5 is contained in Section 5 op. cit.)

We are now in a position to state our main result.

**Theorem 4.21.** Assume that the lattice $P$ is given by an integral ideal $a \subset \mathcal{O}_k$ with quadratic form $Q(x) = N(x)/N(a)$.

For every $h \in C_{P,K} \cong \text{Cl}_k$ there is a harmonic weak Maaß form $\tilde{\Theta}_P(\tau, h) \in \mathcal{H}_{1, P}$ with holomorphic part

$$\tilde{\Theta}^+_P(\tau, h) = \sum_{\beta \in P^*/P} \sum_{m \gg -\infty} c^+_P(h, m, \beta)e(m\tau)\phi_\beta$$

satisfying the following properties:

(i) We have $\xi(\tilde{\Theta}_P(\tau, h)) = \Theta_P(\tau, h)$ and

$$E_P(\tau) := \frac{1}{h_k} \sum_{h \in C_{P,K}} \tilde{\Theta}_P(\tau, h)$$

satisfies $\xi_1(E_P(\tau)) = E_P(\tau)$ and the principal part of $E_P(\tau)$ vanishes.

(ii) For all $\beta \in P^*/P$ and all $m \in \mathbb{Q}$, $m \neq 0$ with $m \equiv -Q(\beta) \mod \mathbb{Z}$, we have

$$c^+_P(h, m, \beta) = -\frac{2}{r} \log |\alpha(h, m, \beta)|,$$

where $\alpha(h, m, \beta) \in \mathcal{O}_{H_k}$ and $\tau \in \mathbb{Z}_{>0}$ only depends on $D$.

(iii) Moreover, if $m > 0$, then

$$\text{ord}_P(\alpha(h, m, \beta)) = r \cdot w_k \cdot Z(m, h, a, h, \beta),$$

for all prime ideals $\mathfrak{q} \subset \mathcal{O}_{H_k}$.

(iv) For $m < 0$, we have $\alpha(h, m, \beta) \in \mathcal{O}_{H_k}^\times$ and $c^+_P(h, m, \beta) = 0$ if $m < -p_{d^2}$

(see Definition 4.10).

4.5. **Proof of Theorem 4.21.** The proof is structured as follows. In the next subsection, we will prove (i), (ii) for $m < 0$ and (iv). After that, in Section 4.5.2 we will turn to the coefficients of index $m > 0$ and show (iii). We will first show that (iii) holds with integers $r(m, \beta)$ in place of $r$ and finally show in Section 4.5.3, that we can choose $r(m, \beta)$ to not depend on $m$ and $\beta$.

4.5.1. **The principal part and compatibility with the Siegel-Weil formula.** Since $P$ is positive definite and $P'/P$ is anisotropic, we have the decomposition

$$\Theta_P(\tau, h) = E_P(\tau, 0) + g_P(\tau, h),$$

where for each $h \in C_{P,K} \cong \text{Cl}_k$ the form $g_P(\tau, h) \in S_{1, P}$ is a cusp form of weight 1 which has rational Fourier coefficients.

Let $h_1, \ldots, h_{d^2} \in S_{1, P}$ be a basis of $S_{1, P}$ as constructed in Section 4.3. There are harmonic weak Maaß forms $H_1, \ldots, H_{d^2} \in H_{1, P}$ with $\xi_1(H_i) = h_i$ and such that the principal part of $H_i$ is of the form

$$P_{H_i}(\tau) = \frac{1}{2} \sum_{j=1}^{d^2} q^{-p_j}(\epsilon_{\pi_j} + \epsilon_{-\pi_j})(h_i, h_j).$$

Indeed, if we define $H_i$ to have this principal part, then the pairing defined in Lemma 2.6 yields $(\xi_1(H_i), h_j) = \{H_i, h_j\} = (h_i, h_j)$ for all $j$ and thus $\xi_1(H_i) = h_i$ by the non-degeneracy of the
Petersson inner product on cusp forms. We define coefficients $a_i(h) \in \mathbb{Q}$ by $\sum a_i(h) h_i = g_P(\tau, h)$. Using the same set of coefficients, we put

$$\tilde{g}_P(\tau, h) := \sum_{i=1}^{d^+} a_i(h) H_i$$

(4.19)

**Remark 4.22.** After our normalization and fixing a basis of $S_{1,P}$ as in Section 4.3, the forms $\tilde{g}_P(\tau, h)$ are uniquely determined up to holomorphic modular forms in $M_{1,P-} = S_{1,P-}$.

Moreover, it is possible to show that the orthogonal complement of the space of theta functions in $M_{1,P}$ has a basis with rational Fourier coefficients. This implies that the Petersson inner products in the principal part will in fact only involve inner products of cusp forms coming from theta functions. In [Ehl16], we give an explicit formula for these Petersson inner products in terms of CM values of $\log |v^{1/2}\eta^2(\tau)|$.

In order to ensure (4.16), we will first just pick a preimage $\tilde{E}_P(\tau)$ of $E_P(\tau)$ to define a preimage of $\Theta_P(\tau, h)$ by $\tilde{\Theta}_P(\tau, h) := \tilde{E}_P(\tau) + \tilde{g}_P(\tau, h)$. First note that such a preimage exists by the surjectivity of $\xi_1$, see (2.7). It is convenient to pick a preimage that has a vanishing principal part. To see that this is possible, note that $\{\tilde{E}_P, g\} = (E_P, g) = 0$ for all $g \in S_{1,P}$. Thus, using the fact that there is a harmonic Maass form in $f \in H_{1,P-}$, such that $\tilde{E}_P - f$ has vanishing principal part (Proposition 3.11 of [BF04]), we see that $0 = \{\tilde{E}_P, g\} = \{f, g\}$. By the exactness of the sequence (2.8) we obtain $f \in M_{1,P}$, and thus $\xi_1(\tilde{E}_P - f) = \tilde{E}_P$. Later, in Lemma 4.26, we will pick a specific $\tilde{E}_P$. The following proposition summarizes our normalization.

**Proposition 4.23.** Let $\tilde{E}_P \in H_{1,P-}$ with $\xi_1(\tilde{E}_P) = E_P$ and such that the principal part of $\tilde{E}_P(\tau)$ vanishes. For $h \in C_{P,K}$ define

$$\tilde{\Theta}_P(\tau, h) := \tilde{E}_P(\tau) + \tilde{g}_P(\tau, h),$$

where $\tilde{g}_P(\tau, h)$ has been defined in (4.19). Then $\xi_1(\tilde{\Theta}_P)(\tau, h) = \Theta_P(\tau, h)$ and we have

$$\frac{1}{h_k} \sum_{h \in C_{P,K}} \tilde{\Theta}_P(\tau, h) = \tilde{E}_P(\tau).$$

For the proof, we quote the following Lemma of Schofer [Sch09, Lemma 2.13].

**Lemma 4.24.** Let $B(h)$ be a function on $T(\mathbb{A}_f)$ depending only on the image of $h$ in $\text{SO}_U(\mathbb{A}_f)$. Assume that $B$ is invariant under $K_T$ and $T(\mathbb{Q})$. Then

$$2\frac{\text{vol}(K_P)}{w_T} \sum_{h \in C_{P,K}} B(h) = \int_{\text{SO}_U(\mathbb{Q}) \backslash \text{SO}_U(\mathbb{A}_f)} B(h) dh,$$

where $w_T = |T(\mathbb{Q}) \cap K_T|$.

**Proof of Proposition 4.23.** Setting $B(h) = \Theta_P(\tau, (z_U, h))$ in Lemma 4.24 we get

$$\sum_{h \in C_{P,K}} \Theta_P(\tau, (z_U, h)) = \frac{w_P}{2\text{vol}(K_P)} \int_{\text{SO}_U(\mathbb{Q}) \backslash \text{SO}_U(\mathbb{A}_f)} \Theta_P(\tau, (z_U, h)) dh.$$

Note that the factor $2/w_P$ is missing in [Sch09].
The latter integral is equal to $2E_P(\tau, 0)$ by the Siegel-Weil formula (Theorem 2.1 of [BY09]). Therefore, since $\Theta_P(\tau, z_U^\pm, h) = E_P(\tau, 0) + g_P(\tau, h)$, we have indeed
\[
\sum_{h \in C_{P,K}} g_P(\tau, h) = 0.
\]
Consequently,
\[
\sum_{h \in C_{P,K}} a_i(h) = 0
\]
for all $i$, since the $h_i$ are linearly independent. It follows that
\[
\xi_1(\tilde{g}_P(\tau, h)) = g_P(\tau, h) \text{ from the definition of } \tilde{g}_P(\tau, h) \text{ and }
\sum_{h \in C_{P,K}} \tilde{g}_P(\tau, h) = 0.
\]
Thus, we obtain
\[
\sum_{h \in C_{P,K}} \tilde{\Theta}_P(\tau, h) = |C_{P,K}| \cdot \tilde{E}_P(\tau).
\]
Using Lemma 4.24 with $B(h) = 1$ shows that the factor $\frac{\text{vol}(K_T)}{w_T} = |C_{P,K}| = h_k$ in our case. \hfill \Box

**Lemma 4.25.** For $\tilde{\Theta}_P(\tau, h)$ as in Proposition 4.23 and $m < 0$, we have
\[
c_P^+(h, m, \beta) = -2r(m, \beta)^{-1} \log |\alpha(h, m, \beta)|
\]
with $\alpha(m, \beta) \in \mathcal{O}_{H_k}^\times$ and $r(m, \beta) \in \mathbb{Z}_{>0}$.

**Proof.** By construction every coefficient in the principal part of $\Theta_P(\tau, h)$ is of the form
\[
(g, g_P(\tau, h)) = (g, \Theta_P(\tau, h)),
\]
for some $g \in S_{1,P}(\mathbb{Q})$, a cusp form with rational Fourier coefficients. Therefore, fixing $h \in C_{P,K}$, it is enough to show that
\[
(g, \Theta_P(\tau, h)) = \frac{2}{c} \log |\alpha|,
\]
with $c \in \mathbb{Z}_{>0}$ and $\alpha \in \mathcal{O}_{H_k}^\times$. This follows easily from the fact that
\[
(g, \Theta_P(\tau, h)) = \Phi_P(h, g)
\]
and that this quantity is equal to
\[
12^A \Phi_P(h, g) = -2 \log |\Psi_L(z_U, h, g)|^2
\]
by (4.9). Thus, replacing $g$ by $s \cdot g$ for an appropriate positive integer $s$ if necessary, we have by (4.15) that
\[
\alpha := \Psi_L(z_U, h, s \cdot g)^2 \in \mathcal{O}_{H_k}^\times.
\]
Thus, we obtain (4.20) with $c = 12^A s$ and this finishes the proof. \hfill \Box
4.5.2. The coefficients of positive index.

**Lemma 4.26.** We can choose the functions \( \tilde{\Theta}_P(\tau, h) \), such that:

(i) Proposition 4.23 holds,

(ii) for \( j \in \{1, \ldots, d^-\} \), we have

\[
\sigma_P^+(h, n_j, \beta_j) = -\frac{2}{r_0} \log |\alpha(h, n_j, \beta_j)|
\]

with

\[
\text{ord}_P(\alpha(h, n_j, \beta_j)) = r_0 \cdot w_k \cdot Z(n_j, h, a, h, \beta_j)_\mathfrak{p}.
\]

Here, the indices \( n_j, \beta_j \) for \( j = 1, \ldots, d^- \) belong to our special basis of \( S_{1,P}^- \) as in Lemma 4.9 and \( r_0 \) does not depend on \( j \).

**Proof.** We choose \( r_0 \in \mathbb{Z}_{>0} \) minimal such that for all primes \( \mathfrak{p} \) of \( \mathcal{O}_{H_k} \) with \( \mathfrak{p} \mid p \) for any \( p \in \bigcup_{j=1}^{d^-} \text{Diff}(n_j) \) there is an element \( \alpha(1, n_j, \beta_j) \in \mathcal{O}_{H_k} \) with

\[
\text{ord}_P(\alpha(1, n_j, \beta_j)) = r_0 \cdot w_k \cdot Z(n_j, a, \beta_j)_\mathfrak{p}
\]

for all \( \mathfrak{p} \). Note that \( r_0 \) is a divisor of \( h_{H_k} \), the class number of \( H_k \). Using this, we let

\[
\alpha(h, n_j, \beta_j) = \alpha(1, n_j, \beta_j)^{\sigma(h)},
\]

where \( \sigma(h) \) corresponds to the class of the idele \( h \) under the Artin map.

Replacing \( \tilde{\Theta}_P(\tau, h) \) by

\[
\tilde{\Theta}_P(\tau, h) - \sum_{j=1}^{d^-} \left( \frac{2}{r_0} \log |\alpha(h, n_j, \beta_j)| + \sigma_P^+(h, n_j, \beta_j) \right) g_j(\tau)
\]

and accordingly \( \tilde{E}_P(\tau) \) by

\[
\tilde{E}_P(\tau) - \sum_{h \in C_{p,K}} \sum_{j=1}^{d^-} \left( \frac{2}{r_0} \log |\alpha(h, n_j, \beta_j)| + \sigma_P^+(h, n_j, \beta_j) \right) g_j(\tau),
\]

we obtain that the coefficients \( \sigma_P^+(h, n_j, \beta_j) \) satisfy the assertion for all \( j \) and all \( h \), Proposition 4.23 is still satisfied and the principal part is unaltered. \( \square \)

We can now finish the proof of Theorem 4.21. So far, we have shown (ii) for \( m < 0 \), (ii) and (iii) for \( (m, \beta) = (n_j, \beta_j) \) and also (iv). We now show (ii) and (iii) for all \( m > 0 \) and \( h = 1 \). The case \( h \neq 1 \) then follows by the reciprocity laws in Lemma 4.6 and the action of \( h \) on the special cycles as described in Sections 4 and 5 of [Ehl15].

Let \( F_{m,\beta} \in M_{-1/2,L} \) as in Lemma 4.12 and recall Equation (4.13):

\[
\int_{\Gamma \setminus H}^{\text{reg}} (f_{m,\beta}(\tau), \Theta_P(\tau, h)) \nu d\mu(\tau) = -\frac{4}{12^d c_{m,\beta}} \log |\Psi_L(z_U, h, c_{m,\beta} F_{m,\beta})| .
\]  

By Theorem 3.5, the left hand side is equal to

\[
\{ \tilde{\Theta}_P(\tau, h), f_{m,\beta} \} = 2\sigma_P^+(h, m, \beta) + \sum_{\gamma \in \mathcal{P}' / P} \sum_{n > 0} \sigma_P^+(h, -n, \gamma) c_{f_{m,\beta}}(n, \gamma) - 2 \sum_{j=1}^{d^-} \sigma_P^+(h, \beta_j, n_j) a_j(m, \beta).
\]
Note that the first sum in the second line vanishes because $c_P^+(h, -n, \gamma) = 0$ for $(n, \gamma) \notin \{(p_1, \pi_1), \ldots, (p_{d^+}, \pi_{d^+})\}$ and $c_{f_{m,\beta}}(p_j, \pi_j) = 0$ for all $j$.

We let $M_{m,\beta} \in \mathbb{Z}_{>0}$ with $c_{m,\beta} | M_{m,\beta}$, such that $\Psi_L(z, h, M_{m,\beta} \cdot r_0 \cdot F_{m,\beta}) \in \mathbb{Q}(j\gamma_N)$, where $r_0$ has been defined in Lemma 4.26.

We replace $c_{m,\beta}$ by $r_0 \cdot M_{m,\beta}$ and the right-hand side of (4.21) now becomes

$$-4r(m, \beta)^{-1} \log |\tilde{\alpha}(1, m, \beta)|,$$

with $\tilde{\alpha}(1, m, \beta) := \Psi(z_U, 1, M_{m,\beta} \cdot r_0 \cdot F_{m,\beta}) \in \mathbb{H}_k$ and $r(m, \beta) := 12^A \cdot r_0 \cdot M_{m,\beta}$.

Using (4.21), (4.22), (4.23) and Lemma 4.26, we obtain

$$c_P^+(1, m, \beta) = -\frac{2}{r(m, \beta)} \log |\tilde{\alpha}(1, m, \beta)|$$

$$-\frac{2 \cdot 12^A M_{m,\beta}}{r(m, \beta)} \sum_{j=1}^{d^-} a_j(m, \beta) \log |\alpha(1, n_j, \beta_j)|.$$}

This implies that

$$c_P^+(1, m, \beta) = -\frac{2}{r(m, \beta)} \log \left|\frac{\tilde{\alpha}(1, m, \beta)}{\prod_{j=1}^{d^-} \alpha(1, n_j, \beta_j)^{12^A M_{m,\beta} a_j(m, \beta)}}\right|.$$}

Note that the number in the absolute value is algebraic and contained in $\mathbb{H}_k$. To see this, note that the denominator of $a_j(m, \beta)$ is bounded by $c_{m,\beta}$. This implies that

$$\alpha(1, n_j, \pm \beta_j)^{12^A M_{m,\beta} a_j(m, \beta)} \in \mathcal{O}_{\mathbb{H}_k}$$}

for all $j$. Finally, let

$$\alpha(1, m, \beta) = \tilde{\alpha}(1, m, \beta) \prod_{j=1}^{d^-} \alpha(1, n_j, \beta_j)^{12^A M_{m,\beta} a_j(m, \beta)} \in \mathbb{H}_k.$$}

This shows (4.17) for $m > 0$ and $\alpha(1, m, \beta)$ satisfies

$$\text{ord}_\mathbb{Q}(\alpha(1, m, \beta)) = \text{ord}_\mathbb{Q}(\tilde{\alpha}(1, m, \beta))$$

$$+ 12^A w_k M_{m,\beta} r_0 \sum_{j=1}^{d^-} a_j(m, \beta) Z(n_j, a, \beta_j),$$}

by Lemma 4.26.

By Proposition 4.18, we know that $Z(F_{m,\beta})$ and $Z(M, \mu_R)$ intersect properly and we have by (4.15) that

$$\text{ord}_\mathbb{Q}(\tilde{\alpha}(1, m, \beta)) = r(m, \beta) \frac{w_k}{2} \sum_{\gamma \in \mathbb{P}^+/\mathbb{F} n > 0} \sum_{\gamma \in \mathbb{P}^+/\mathbb{F} n > 0} c_{m,\beta}(-n, \gamma) Z(n, a, \gamma)\mathfrak{p}$$}

which we can further expand to

$$\text{ord}_\mathbb{Q}(\tilde{\alpha}(1, m, \beta)) = r(m, \beta) w_k \sum_{\gamma \in \mathbb{P}^+/\mathbb{F} n > 0} \sum_{\gamma \in \mathbb{P}^+/\mathbb{F} n > 0} c_{m,\beta}(-n, \gamma) Z(n, a, \gamma)\mathfrak{p}$$

$$- r(m, \beta) w_k \sum_{j=1}^{d^-} a_j(m, \beta) Z(n_j, a, \beta_j),$$}

where we used that $Z(n, a, -\gamma)\mathfrak{p} = Z(n, a, \gamma)\mathfrak{p}$ and $c_{f_{m,\beta}}(n, \gamma) = c_{f_{m,\beta}}(n, -\gamma)$.
Plugging (4.25) into (4.24) and using and \( r(m, \beta) = 12^4 M_{m,\beta} n_j \beta_j \) we conclude
\[
\text{ord}_{P}(\alpha(1,m,\beta)) = r(m, \beta) w_k Z(m, \beta)_{P},
\]
which proves (iii), but with \( r(m, \beta) \) possibly depending on \( m \) and \( \beta \).

4.5.3. A bound for \( r(m, \beta) \). We will now finish the proof by showing that the integers \( r(m, \beta) \) in the theorem can be bounded so that we can choose an integer \( r \) only depending on the isomorphism class of \( P \) (and thus can be chosen to only depend on \( D \)).

**Proposition 4.27.** There is an \( r \in \mathbb{Z}_{>0} \), such that we can take \( r(m, \beta) = r \) for all \( m \in \mathbb{Q} \), \( \beta \in P'/P \) and \( h \in C_{P,K} \).

**Proof.** Recall that for square-free \( N \) each cusp of \( \Gamma_0(N) \) can be represented a fraction \( 1/c \) with \( c \mid N \).

For \( f \in M_{1/2,L}^1(\mathbb{Z}) \), the Weyl vector associated with \( f \) at the cusp \( 1/c \), is defined by
\[
\rho_{f,c} = \frac{\sqrt{N}}{8\pi} \int_{\mathcal{F}} \langle f(\tau), \Theta_{K_c}(\tau) \rangle v^{1/2} d\mu(\tau),
\]
where \( K_c \) is a certain one-dimensional positive definite lattice attached to the cusp \( 1/c \) satisfying \( K'/K_c \cong L'/L \).

The significance of the Weyl vectors is that the divisor of \( \Psi_L(z,f) \) on \( X_0(N) \) is given by \( Z(f) + C(f) \), where \( C(f) = \sum_{c|N} \rho_{f,c} \cdot (1/c) \).

We assume that the constant term of \( f \) vanishes, \( c_f(m,\mu) \in \mathbb{Q} \) for all \( m \in \mathbb{Q} \) and \( \mu \in L'/L \) and \( c_f(m,\mu) \in \mathbb{Z} \) for \( m \leq 0 \). Since the multiplier system \( \sigma \) of \( \Psi_L(z,f) \) is of finite order \( M \), we have that \( M \cdot (Z(f) + C(f)) \) is the divisor of a rational function on \( X_0(N) \). Note that this also implies that \( \text{deg}(Z(f) + C(f)) = 0 \). Conversely, as in the proof of Theorem 6.2 in [BO10], we have that if \( M \cdot (Z(f) + C(f)) \) is the divisor of a rational function on \( X_0(N) \) then \( \sigma^M \) is trivial. If \( \rho_{f,c} \in \mathbb{Z} \) for all cusps, \( \sqrt{N} + C(f) \) defines a rational point in the Jacobian \( J \) of \( X_0(N) \) because the Heegner divisors are defined over \( \mathbb{Q} \) and so are the cusps of \( \Gamma_0(N) \) for \( N \) square-free. The group of rational points \( J(\mathbb{Q}) \) is a finitely generated abelian group by the Mordell-Weil theorem. Therefore, \( M \mid M(N) \), where \( M(N) \) is the order of the torsion subgroup of \( J(\mathbb{Q}) \).

The idea is now, similar to the general theme of this paper, to construct a harmonic Maaß form \( \bar{\Theta}_K(\tau) \in \mathcal{H}_{3/2,K} \), such that \( \xi_{3/2}(\bar{\Theta}_{K_c}(\tau)) = \Theta_{K_c}(\tau) \) to compute the Weyl vectors and ensure that they are integral so that the above reasoning applies.

In the case of a prime discriminant, the theta function \( \Theta_{K_c}(\tau) \) is in fact an Eisenstein series. A preimage is given by a generalization of Zagier’s Eisenstein series, constructed as the Kudla-Millson lift of the constant functions \( 1 \) as in Theorem 4.5 of [BF06] (see also Theorem 5.5 of [AE13] for the specialization to \( \Gamma_0(N) \)).

For the general case, the construction is a bit more complicated and involves showing the existence of weakly holomorphic modular forms of weight \( 0 \) for \( \Gamma_0(N) \) with certain prescribed principal parts. The details can be found in [BS16]; in particular, by Theorem 4.5 together with Theorem 5.1 (1) op. cit. we obtain that there is a constant \( \kappa \) that only depends on \( A \mid D = N \), such that \( \kappa \cdot \rho_{f,c} \in \mathbb{Z} \) for all \( c \mid N \) and all \( f \in M_{1/2,L}^1 \) with only integral Fourier coefficients.

Without loss of generality we can assume that \( A \) is a prime and does not divide \( D \). Recall the numbers \( n_1, \ldots, n_d \) that belong to our special basis of \( S_{1,P} \) constructed in Section 4.3. Let \( n > \max\{n_1, \ldots, n_d\} \) be a fixed integer and \( S = \{n + Q(\beta) \mid \beta \in P'/P \} \). For \( x \in S \) and \( \beta \in P'/P \), such that \( x \equiv Q(\beta) \mod \mathbb{Z} \), we let \( F_{x,\beta} := f_{x,\beta} \) as in Lemma 4.12. From our discussion above, it follows that there is an \( r \in \mathbb{Z}_{>0} \), such that the following properties are satisfied:
(i) The functions $r \cdot f_{x, \beta}$ have integral Fourier coefficients for all $x \in S$;

(ii) if $r \cdot f \in M^1_{1, P}$ has integral Fourier coefficients, then $r' \cdot f \in M^1_{1/2, L}$ constructed as in

Lemma 4.12 also has integral Fourier coefficients with $r' = r/12^A$;

(iii) and we have that $\rho_{r \cdot f, c} \in \mathbb{Z}$ for all $c \mid N$ and $\Psi_L((z, h), r' \cdot f) \in \mathbb{Q}(j, j_N)$.

(iv) Moreover, for all $m \leq n$, we have

$$c^+_P(h, m, \beta) = \frac{2}{r} \log |\alpha(h, m, \beta)|$$

with $\alpha(h, m, \beta) \in \mathcal{O}_{\mathbf{H}}$ and $\text{ord}_\mathfrak{p}(\alpha(h, m, \beta)) = w_k \cdot r \cdot \mathcal{Z}(m, h, a, h, \beta)\mathfrak{p}$.

Now let $m \in \mathbb{Q}$ and $\beta \in P'/P$, such that $m > n$ and $m \equiv Q(\beta) \mod \mathbb{Z}$. Then $m = x + n'$ for

some $x \in S$ and $n' \in \mathbb{Z}_{>0}$. We let $f(\tau) = j''(\tau)f_{x, \beta}(\tau)$, where $j(\tau)$ is the $j$-invariant. Moreover,

we let $f(\tau) \in M^1_{1/2, L}$ given by

$$f = f \otimes \mathcal{Q}^A - CE_P \otimes \mathcal{F}g,$$

for a suitable constant $C \in \mathbb{Q}$ and $g \in M^1_{2A}(\text{SL}_2(\mathbb{Z}))$ as in the proof of Lemma 4.12, such that

$c_f(0, 0) = 0$. Note that $\mathcal{Q}$ and $j''$ have integral Fourier coefficients. By (i), Lemma 4.12 and (ii) we have that $r' \cdot f$ has integral Fourier coefficients and by (iii) the Weyl vectors of $r' \cdot f$ are integral and $\Psi_L((z, h), r' \cdot f)$ is contained in $\mathbb{Q}(j, j_N)$.

As before, we have on the one hand

$$\Phi_P(h, f) = 2c^+_P(h, m, \beta) + \sum_{\gamma \in P'/P} \sum_{m' < m} c^+_P(h, m', \gamma)c_f(-m', \gamma)$$

and on the other hand

$$\Phi_P(h, f) = -\frac{2}{r} \log |\Psi_L(z_U, h, r' \cdot f)|^2.$$  

By our assumptions we have that $\Psi_L(z_U, h, r' \cdot f) \in \mathcal{O}_{\mathbf{H}}$. The statement of the proposition now

easily follows by induction on $n' \in \mathbb{Z}_{>0}$.

\begin{flushright} \Box \end{flushright}

\noindent **Remark 4.28.** Let $l \geq 5$ be a prime, let $n$ be the numerator of $(l - 1)/12$ and consider the Jacobian $J$ of $X_0(l)$. It follows from a Theorem of Mazur [Maz77] that the torsion subgroup $J(\mathbb{Q})_{\text{tors}}$ is cyclic of order $n$. Together with this explicit result, Proposition 4.27 describes an algorithm to determine the number $r$ for a given prime discriminant $D = -l$ explicitly.

\subsection{4.6. Consequences.}

The following theorem shows the modularity of the generating series of the degrees of the special cycles, as mentioned in the introduction. This result was proven by Kudla, Rapoport and Yang using an explicit comparison of Fourier coefficients.

\begin{theorem}
There exists a harmonic weak Maass form $\tilde{E}_P(\tau) \in M_{1, P}$ with vanishing principal part, such that

$$-\frac{h_k}{w_k} \tilde{E}_P^+(\tau) = \sum_{\mu \in P'/P} \sum_{m > 0} \deg \mathcal{Z}(m, a, \mu)e(m\tau)\phi_\mu + c\phi_0,$$

where $c \in \mathbb{C}$ is a constant.

\end{theorem}

\begin{proof}
Consider the sum

$$\tilde{E}_P(\tau) = \frac{1}{h_k} \sum_{h \in C_K/K} \tilde{\Theta}_P(\tau, h).$$

\end{proof}
We obtain by Theorem 4.21 that the holomorphic part of $\tilde{E}_P(\tau)$ has a vanishing principal part and we have

$$\frac{h_k}{w_k}E^+_P(\tau) = \frac{1}{w_k} \sum_{\beta \in P'/P} \sum_{m \geq 0} \sum_{h \in \mathcal{C}_{P',K}} c^+_P(h, m, \beta) e(m\tau) \phi_{\beta}$$

$$= -\frac{1}{w_k} \sum_{\beta \in P'/P} \sum_{m \geq 0} r^{-1} \sum_{h \in \mathcal{C}_{P',K}} \log |\alpha(h, m, \beta)|^2 e(m\tau) \phi_{\beta},$$

where $\alpha(h, m, \beta) \in \mathcal{O}_{H_k}$.

Moreover, for $m > 0$ and for every prime $\mathfrak{P}$ of the Hilbert class field $H_k$ of $k$, we have

$$\text{ord}_{\mathfrak{P}} \left( \prod_{h} \alpha(h, m, \beta) \right) = rw_k \sum_{h} \mathcal{Z}(m, h.a, h.\beta)_{\mathfrak{P}}.$$

By Proposition 5.6 in [Ehl15], we have for $\sigma = \sigma(h)$ that

$$\mathcal{Z}(m, a, \beta)_{\mathfrak{P}^\sigma} = \mathcal{Z}(m, h^{-1}.a, h^{-1}.\beta)$$

and thus

$$\sum_{h \in \mathcal{C}_{k}/K} \mathcal{Z}(m, h.a, h.\beta)_{\mathfrak{P}} = \sum_{\sigma \in \text{Gal}(H_k/k)} \mathcal{Z}(m, a, \beta)_{\mathfrak{P}^\sigma}.$$

This shows in fact that

$$\sum_{h} \log |\alpha(h, m, \beta)|^2 = \log N_{H_k/Q}(|\alpha(1, m, \beta)|).$$

But since $\mathcal{Z}(m, a, \beta)$ is supported at a unique prime $p$, we also have that

$$\hat{\text{deg}} \mathcal{Z}(m, a, \beta) = \sum_{\sigma \in \text{Gal}(H_k/k)} \mathcal{Z}(m, a, \beta)_{\mathfrak{P}^\sigma} \log N_{H_k/Q}(|\mathfrak{P}|).$$

Here, we used that $p$ is non-split in $k$. Then we find

$$\sum_{h \in \mathcal{C}_{k}/K} \log |\alpha(h, m, \beta)|^2 = w_k \cdot r \cdot \sum_{\sigma \in \text{Gal}(H_k/k)} \mathcal{Z}(m, a, \beta)_{\mathfrak{P}^\sigma} \log N_{H_k/Q}(|\mathfrak{P}|)$$

$$= w_k \cdot r \cdot \hat{\text{deg}} \mathcal{Z}(m, a, \beta),$$

which implies the statement of the proposition. \qed

We collect some consequences of Theorem 4.21 in connection with the explicit formulas given in [Ehl15].

For $m \in \mathbb{Q}_{>0}$, we define a set of rational primes by

$$\text{Diff}(m) = \{ p < \infty \mid (-mN(a), D)_p = -1 \}.$$ 

A starting point is the following Corollary.

**Corollary 4.30.** The algebraic numbers in Theorem 4.21 satisfy the following properties.

(i) If $|\text{Diff}(m)| \neq 1$, then $c^+_P(h, m, \beta) = -\frac{2}{3} \log |\epsilon|$ for $\epsilon \in \mathcal{O}_{H_k}^\times$.

(ii) If $\text{Diff}(m) = \{ p \}$, we have $\text{ord}_{\mathfrak{P}}(\alpha(h, m, \beta)) = 0$ for all primes $\mathfrak{P} \nmid p$.

(iii) The Shimura reciprocity

$$\alpha(h, m, \beta) = \alpha(1, m, \beta)^{\sigma(h)}$$

holds, where $\sigma(h)$ corresponds to $h$ under the Artin map.
Therefore, we may assume from now on that $\text{Diff}(m) = \{p\}$. In particular, $p$ is nonsplit in $k$. We recall some notation needed to describe the results – for details we refer the reader to [Ehl15]. We let $p_0 \in \mathbb{Z}$ be a prime with $p_0 \mid 2pD$ such that if $p$ is inert in $k$, we have

$$(D, -pp_0)_v = \begin{cases} -1, & v = p, \infty, \\ 1, & \text{otherwise}, \end{cases}$$

and if $p$ is ramified in $k$, we have

$$(D, -p)_v = \begin{cases} -1, & v = p, \infty, \\ 1, & \text{otherwise}. \end{cases}$$

With this choice, we let $p_0$ be any fixed prime ideal of $\mathcal{O}_k$ lying above $p_0$. Here, $(\cdot, \cdot)_v$ denotes the $v$-adic Hilbert symbol. If $p$ is inert in $k$, let $\mathfrak{c}_0 = p_0 \mathfrak{o}_k$. If $p$ is ramified and $p \subset \mathcal{O}_k$ is the prime above $p$, let $\mathfrak{c}_0 = p_0 \mathfrak{o}_k^{-1} \mathfrak{c}_k$.

Moreover, we define for $m \in \mathbb{Q}$:

$$\nu_p(m) = \begin{cases} \frac{1}{2}(\text{ord}_p(m) + 1), & \text{if } p \text{ is inert in } k, \\ \text{ord}_p(m | D)!, & \text{if } p \text{ is ramified in } k. \end{cases}$$

We let $\mathfrak{p}_0$ be the prime ideal corresponding to the elliptic curve with complex multiplication $(E_0, \iota_0)$ over $\mathbb{F}_p$ such that the class $[L(E_0, \iota_0)]$ of the rank one $\mathcal{O}_k$-module $L(E_0, \iota_0)$ in $\text{Pic}(\mathcal{O}_k)$ is equal to $[\mathfrak{p}_0]^{-1}$. Finally, for $m \in \mathbb{Q}$, we let $o(m)$ denote the number of primes $q \mid D$ such that $\text{ord}_q(m | D) > 0$.

In the following corollary, we restrict our attention to the simpler cases.

**Corollary 4.31.**

(i) Let $L$ be the subfield of $H_k$ fixed by all elements of order less or equal than two in $\text{Gal}(H_k / k)$ and let $\mathfrak{f} \subset \mathcal{O}_L$ be the prime ideal below the fixed prime $\mathfrak{p}_0$. We have

$$\frac{\text{ord}_l(N_{H_k / L}(\alpha(h, m, \beta)))}{r \cdot w_k} = \mathcal{Z}(m, h, \iota_0),$$

where

$$2^{o(m) - 1} \nu_p(m) \rho(m | D) \rho([h]^{-2}[\mathfrak{c}_0 \mathfrak{o}_k]).$$

(ii) If $D = -l$ is prime, then there is a unique prime $\mathfrak{p} \mid p$ that is fixed by complex conjugation. We have

$$\frac{\text{ord}_l(\alpha(h, m, \beta))}{r \cdot w_k} = 2^{o(m) - 1} \nu_p(m) \rho(m | D) \rho([h]^{-2}[\mathfrak{c}_0 \mathfrak{o}_k]).$$

Note that we did not use the known explicit formulas for $\mathcal{E}_P(\tau)$ so far. Employing these formulas [KY10], however, we obtain another corollary.

**Corollary 4.32.** With the same notation as in Theorem 4.21, we have

$$\tilde{E}_P(\tau) = E_P(\tau),$$

where $E_P(\tau) = \frac{\partial}{\partial s} E_P(\tau, s)|_{s=0}$ is defined in (3.5).

Using the pairing (2.9), the following Corollary is immediate.

**Corollary 4.33.** The constant term of $\tilde{\Theta}_P(\tau, h)$ is given by

$$c_P^+(h, 0, 0) = - \sum_{\beta \in L / L} \sum_{m > 0} c_P(h, -m, \beta) \hat{p}(m, \beta) - 2 \frac{\Lambda'(\chi D, 0)}{\Lambda(\chi D, 0)},$$

where $\hat{p}(m, \beta)$ is the coefficient of index $(m, \beta)$ of $E_P(\tau)$. 

5. The scalar valued case

To complete the picture and give a slightly more classical description of our results, we briefly describe the scalar valued case in this section. Throughout, we let $D < 0$ with $D \equiv 1 \mod 4$ be a fundamental discriminant and let $\mathfrak{A} \in \text{Cl}_k/\text{Cl}_k^+$ be a genus. For every class $[a] \in \mathfrak{A}$, there is a theta function $\theta_a(\tau)$ contained in $M_1(|D|, \chi_D)$. A special case of the Siegel-Weil formula states that

$$\sum_{b \in \text{Cl}_k} \theta_{ab^2}(\tau) = h_k E_{\mathfrak{A}}(\tau),$$

where $E_{\mathfrak{A}}(\tau)$ is the normalized genus Eisenstein series attached to $\mathfrak{A}$. It is well known that

$$E_{\mathfrak{A}}(\tau) = 1 + \frac{w_k}{h_k} \sum_{n=1}^{\infty} \rho_\mathfrak{A}(n) e(n\tau),$$

where $\rho_\mathfrak{A}(n)$ is equal to the number of integral ideals of norm $n$ in the genus $\mathfrak{A}$. Note that $E_{\mathfrak{A}}(\tau)$ is the $\phi_0$-component of $E_P(\tau)$.

**Corollary 5.1.** For every $[a] \in \mathfrak{A}$, there is a $\tilde{\theta}_a \in \mathcal{H}_1(|D|, \chi_D)$, where $\chi_D$ is given by the Kronecker symbol, with the following properties. We write $c_a^+(n)$ for the coefficients of the holomorphic part of $\tilde{\theta}_a$.

(i) We have $\xi(\tilde{\theta}_a) = \theta_a$ and

$$\sum_{[b] \in \text{Cl}_k} \tilde{\theta}_{b^2a} = h_k E_{\mathfrak{A}},$$

where $\xi(E_{\mathfrak{A}}) = E_{\mathfrak{A}}$ and the principal part of $E_{\mathfrak{A}}$ vanishes.

(ii) For all $n > 0$ we have

$$c_a^+(n) = -\frac{2}{r} \log |\alpha(a, n)|,$$

where $\alpha(a, n) \in \mathcal{O}_{H_k}$ and $r \in \mathbb{Z}_{>0}$ is independent of $n$.

(iii) Finally, for all $n < 0$, we have that

$$c_a^+(n) = -\frac{2}{r} \log |\alpha(a, n)|,$$

where $\alpha(a, n) \in \mathcal{O}_{H_k}^\times$.

**Proof.** All statements follow from Theorem 4.21 by considering

$$\tilde{\theta}_{ab^2}(\tau) = \tilde{\Theta}_{P,0}(\tau, h) \in \mathcal{H}_1(|D|, \chi_D)$$

where $P = a$, $(h) = b$ and

$$\tilde{\Theta}_{P}(\tau, h) = \sum_{\beta \in P'/P} \tilde{\Theta}_{P,\beta}(\tau, h) \phi_{\beta}.$$

Since we have $\theta_a(\tau) = \Theta_{P,0}(\tau)$, we conclude $\xi(\tilde{\theta}_a) = \theta_a(\tau)$. \qed

**Corollary 5.2.** We use the same notation as in Corollary 5.1.

(i) We have $\text{ord}_p(\alpha(a, n)) = 0$, unless $|\text{Diff}(n)| = 1$.

(ii) If $\text{Diff}(n) = \{p\}$, then

$$\text{ord}_p(\alpha(a, n)) = r \cdot w_k \cdot \nu_p(n) \rho(n |D|/p, [ac_0]),$$

where $\mathfrak{P}_0 | p$ is the distinguished prime as above.
(iii) The Shimura reciprocity
\[ \text{ord}_P(\alpha(a, n)) = \alpha(b^{-2}a, n), \]
holds, where \( \sigma = \sigma(b) \).
(iv) For the coefficients in the principal part, we have that \( \alpha(a, n) \in O_{H_k} \times H_k \).

Proof. This follows from Theorem 4.21 and the proof of Propositions 3.8 and 3.9 in [Ehl15] by setting \( \mu = 0 \) and noting that the condition \( \lambda_a x \in a \) is superfluous if \( m \in \mathbb{Z} \) (using the notation of loc. cit.). \( \square \)

Remark 5.3. Note that we are able to give a relatively simple formula for the \( P \)-valuations of the coefficients in comparison to the vector valued case, where this is only possible after passing to the fixed field of elements of order two in the Galois group \( \text{Gal}(H_k/k) \). In fact, considering scalar valued theta series corresponds to “forgetting” the additional congruence conditions that the vector valued forms keep track of.

5.1. The conjecture of Duke and Li. In [DL15], W. Duke and Y. Li also study the scalar valued case. Motivated by numerical experiments, the authors were led to formulate a conjecture, which we state now, adapted to our notation.

Let \( k = \mathbb{Q}(\sqrt{-l}) \) for a prime \( l \equiv 3 \mod 4, l > 3 \). We consider the lattice \( P = \mathcal{O}_k \) given by the ring of integers in \( k \), which is sufficient because there is only one genus in this case. We note that the preimages are normalized differently in op. cit. than in the previous section, leading to a slightly different result if we compare the conjecture with Corollary 5.2.

Now let \( p \) a prime that is non-split in \( k \), let \( \mathfrak{p}_0 \mid p \) be the unique prime above \( p \) fixed by complex conjugation, and \( \mathfrak{P}^{\sigma} = \mathfrak{p}_0 \) for a fractional ideal \( \mathfrak{b} \).

Conjecture 5.4 ([DL15]). The functions \( \tilde{\theta}_a(\tau) \) can be chosen such that for \( n \in \mathbb{Z}_{>0} \) with \( \chi_p(n) \neq 1 \), we have
\[ c_{a^2}^+(n) = -\frac{2}{r} \log |u(a^2, n)| \]
with \( u(a^2, n) \in \mathcal{O}_{H_k} \) and, if \( \text{Diff} \left( \frac{n}{D} \right) = \{ p \} \), then
\[ \text{ord}_\mathfrak{p}(u(a^2, n)) = 2r \sum_{m \geq 1} \rho \left( \frac{n}{p_m}, [ab]^2 \right), \]
with \( r \in \mathbb{Z} \) independent of \( n \) and divides \( 24h_kh_{H_k} \).

Remark 5.5. 1) Compare with the conjecture as stated in op. cit., note that the normalization of the theta functions differs by a factor of \( 1/2 \) from our normalization; this leads to the factor 2 on the right-hand side above. 2) We added the condition \( \text{Diff} \left( \frac{n}{D} \right) = \{ p \} \) because it is needed for the conjecture to hold and be compatible with the Siegel-Weil formula (which essentially corresponds to (iv) in Theorem 1.1 of op. cit.).

We will see below that our results imply the conjecture regarding the valuations, but we were not able to show the explicit bound \( 24h_kh_{H_k} \) for \( r \).

The next lemma gives a simpler expression for the right-hand side of (5.1), relating it to our formula.

Lemma 5.6. If \( n \in \mathbb{Z}_{>0} \) with \( \text{Diff} \left( \frac{n}{D} \right) = \{ p \} \), then
\[ \sum_{m \geq 1} \rho \left( \frac{n}{p_m}, [c] \right) = \nu_p \left( \frac{n}{D} \right) \rho(n/p, [c]) \]
for any class $[c] \in \text{Cl}_k$.

Proof. Suppose that $p \neq l$. Then $\chi_l(-n) = -1$, $l \nmid n$ and we only have a non-zero contribution if $\text{ord}_p(n) > 0$ and $\text{ord}_p(n)$ is necessarily odd. More precisely, we have

$$\rho\left(\frac{n}{p^m}, [c]\right) = \begin{cases} 0, & \text{if } m \text{ is even}, \\ \rho(n/p, [c]), & \text{if } m \text{ is odd}. \end{cases}$$

Since $p$ is inert, there are no ideals of norm $p$. Therefore, the map $c \mapsto pc$ establishes a bijection between ideals of norm $N(c)$ and ideals of norm $p^2N(c)$, leaving the class invariant. Thus, we obtain the contribution $\rho(n/p, [c])$ times $\frac{1}{2}(\text{ord}_p(n) + 1)$, as in our formula.

The case $p = l$ is similar, but this time we obtain $\text{ord}_p(n) \cdot \rho(n/p, [c])$ because there is a unique ideal of norm $p$. \hfill $\square$

**Theorem 5.7.** Conjecture 5.4 is true.

Proof. To prove the theorem, we take $\tilde{\theta}_a^2(\tau) := \sum_{\beta \in P'/P} \tilde{\Theta}_{P,\beta}(|D|, \tau, h)$, where $(h) = a$ instead of the zeroth component as in the proof of Corollary 5.1. First, note that by Equation (2) in [BB03] $\tilde{\theta}_a^2(\tau) \in \mathcal{H}_1(|D|, \chi_D)$. Moreover, it is easy to see that in fact

$$\sum_{\beta \in P'/P} \Theta_{P,\beta}(|D|, \tau, h) = \theta_a^2(\tau).$$

Indeed, multiplication by $\sqrt{D}$ gives a bijection between elements of norm $m$ in $\hat{\mathcal{O}}^{-1}a^2$ and elements of norm $m|D|$ in $a^2$. Hence, $\xi_1(\tilde{\theta}_a^2(\tau)) = \theta_a^2(\tau)$.

By Theorem 4.21 and Corollary 4.31, we have that the $n$-th Fourier coefficient of $\tilde{\theta}_a^2(\tau)$ is equal to

$$\frac{-2}{r} \sum_{\beta \in P'/P} \log \left| \alpha(h, \frac{n}{|D|}, \beta) \right|$$

with (setting $m := n/|D|$)

$$\text{ord}_{\mathbb{P}_0}(\alpha(h, m, \beta)) = 2^{\rho(m)} r \cdot \nu_p(m) \rho(n/p, [a]^2)$$

if $m + Q(\beta) \in \mathbb{Z}$. Now, if $l \nmid n$, then $o(m) = 0$ and if $\rho(n/p, [a]^2) \neq 0$, then there are exactly two elements in $\beta \in P'/P$ with $m + Q(\beta) \in \mathbb{Z}$. If $l \mid n$, then $\beta = 0$ is the only element in $P'/P$ satisfying $m + Q(\beta) \in \mathbb{Z}$ and also $o(m) = 1$. In any case

$$\sum_{\beta \in P'/P} \text{ord}_{\mathbb{P}_0}(\alpha(h, \frac{n}{|D|}, \beta)) = 2r \cdot \nu_p\left(\frac{n}{|D|}\right) \rho(n/p, [a]^2),$$

which agrees with the conjecture by Lemma 5.6. \hfill $\square$

Note that our theorem also covers the case $l = 3$ and we provide a generalization to composite discriminants.

**Proof of Theorem 1.3.** Theorem 1.3 is a reformulation of the conjecture in the vector-valued case and follows directly from Theorem 4.21, and Corollary 4.31 (ii) together with Lemma 5.6. \hfill $\square$
6. A COMPREHENSIVE EXAMPLE FOR $D = -23$

In this section, we will give an example where we can give an explicit, finite formula for all the coefficients of the forms $\Theta_P(\tau, h)$ and follow the proof of Theorem 4.21.

The method of this section can be used to obtain such explicit formulas for all primes $p \equiv 3 \mod 4$ such that the modular curve $X_0^+(l)$ has genus zero. Here, $X_0^+(l)$ is the compactification of $\Gamma_0(l)^* \backslash \mathbb{H}$, where $\Gamma_0(l)^*$ is the extension of $\Gamma_0(p)$ by the Fricke involution. It is known that there are only finitely many such primes. In order to keep the example as explicit as possible, we specialize to $l = 23$. It is the first prime congruent to 3 modulo 4 such that $k = \mathbb{Q}(\sqrt{-l})$ has class number $h_k > 1$. In this case, we have $h_k = 3$. The three ideal classes can be represented by the ideals $\mathcal{O} = (1), \mathfrak{a} = (2, (2 + \sqrt{-23})/2)$ and $\bar{\mathfrak{a}}$.

Note that $M_{1, \mathfrak{p}}$, for both $P = \mathcal{O}$ and $P = \mathfrak{a}$, is isomorphic [BB03] to $M_{1+}^+(\Gamma_0(23), \chi_{-23})$, the subspace of modular forms whose Fourier coefficients of index $n$ vanish whenever $\left(\frac{-23}{n}\right) = -1$. Thus, for simplicity we will work with scalar valued modular forms in this example.

The two corresponding scalar valued theta functions have Fourier expansions starting with

\[
\theta_\mathcal{O}(\tau) = 1 + 2q + 2q^4 + 4q^6 + 4q^8 + 2q^9 + 4q^{12} + O(q^{15})
\]

\[
\theta_\mathfrak{a}(\tau) = \theta_\mathfrak{a}(\tau) = 1 + 2q^2 + 2q^3 + 2q^4 + 2q^6 + 2q^8 + 2q^9 + 4q^{12} + 2q^{13} + O(q^{15}).
\]

The difference

\[
g(\tau) = \frac{1}{2}(\theta_\mathcal{O}(\tau) - \theta_\mathfrak{a}(\tau)) = q - q^2 - q^3 + q^6 + q^8 - q^{13} + O(q^{15})
\]

can easily be identified as $\eta(\tau)\eta(23\tau)$. We will write $c_g(n)$ for the coefficient of index $n$ of $g$ and $c_\mathcal{O}(n)$ and $c_\mathfrak{a}(n)$ for the coefficients of the two theta series above. The cusp form $g$ is a normalized newform and does not have any zeroes on $\mathbb{H}$.

In fact, we have in our case $M_{1+}^+(\Gamma_0(23), \chi_{-23}) = M_1(\Gamma_0(23), \chi_{-23})$ and this space is spanned by $g$ and the genus Eisenstein series

\[
E(\tau) = \frac{1}{3}(\theta_\mathcal{O}(\tau) + 2\theta_\mathfrak{a}(\tau)) = 1 + \frac{2}{3}q + \frac{4}{3}q^2 + \frac{4}{3}q^3 + 2q^4 + \frac{8}{3}q^6 + \frac{8}{3}q^8 + 2q^9 + 4q^{12} + \frac{4}{3}q^{13} + O(q^{15}).
\]

We will write $c_E(n)$ for the coefficient of index $n$ of $E(\tau)$. The vanishing of $M_{1-}^-(\Gamma_0(23), \chi_{-23})$ corresponds to the fact that $M_{1, \mathfrak{p}} = \{0\}$ for either choices of $\mathfrak{p}$.

We would like to determine the coefficients of $\tilde{\theta}_\mathcal{O}(\tau)$ and $\tilde{\theta}_\mathfrak{a}(\tau)$. Part of our normalization is that $\tilde{\theta}_\mathcal{O}(\tau) = \bar{E}(\tau) + \frac{2}{3}g(\tau)$ and $\tilde{\theta}_\mathfrak{a}(\tau) = \bar{E}(\tau) - \frac{2}{3}g(\tau)$, where $\xi_1(g(\tau)) = g(\tau)$ and $\xi_1(E(\tau)) = E(\tau)$ with vanishing principal part.

An important consequence of the vanishing of $M_{1, \mathfrak{p}}$ is that there are no obstructions to finding a weakly holomorphic modular form in $M_{1, \mathfrak{p}} \cong M_{1+}^+(\Gamma_0(23), \chi_{-23})$. Therefore, the technical part concerning the cusp forms in $S_{1, \mathfrak{p}}$ does not concern us in this simple example. We can choose $\bar{g}(\tau)$ to have principal part $(g, g)_{\Gamma_0(23)}g^{-1}$, where $(g, g)_{\Gamma_0(23)} = \int_{\Gamma_0(23)\backslash \mathbb{H}} g(\tau)\bar{g}(\tau)\,d\tau/n$ is the Petersson norm of $g$. This can be identified as the theta lift of $3g/4$ using the technique of [Ehl16]. But it is also the example that Stark gives on page 91 of [Sta75] and it turns out that $(g, g)_{\Gamma_0(23)} = 3\log|\alpha|$, where $\alpha \in H$ is the unit which is the unique real root of the polynomial $X^3 - X - 1$ with complex embedding equal to $1.324717\ldots$.

To determine the coefficients $c_{\bar{g}}^+(m)$ of $\bar{g}$ with positive index we first show that for every $m > 0$ with $(\frac{m}{23}) \neq 1$ (so that $(\frac{m}{23}) \neq -1$), there is a weakly holomorphic modular form with all integral
Fourier coefficients
\[ f_m(\tau) = q^{-m} + O(q^2) \in M_{1,1}^+(\Gamma_0(23), \chi_{-23}). \]

We can show this by constructing such a form for \( 1 \leq m \leq 23 \) with \( \left( \frac{m}{23} \right) \neq -1 \) and then use the same strategy as in the proof of Proposition 4.27 to obtain all such forms. Using Sage, we see that the space \( M_{13}^+(\Gamma_0(23), \chi_{-23}) \) has dimension 13 and computed an integral basis. Dividing each form in the integral basis by \( \Delta(23\tau) \) gives us an element of \( M_{1,1}^+(\Gamma_0(23), \chi_{-23}) \). This way, we obtain the 12 weakly holomorphic modular forms \( f_m \) for \( m \in \{5, 7, 10, 11, 14, 15, 17, 19, 20, 21, 22, 23\} \). As an example, the Fourier expansions of the first three forms start with

\[
\begin{align*}
  f_5(\tau) &= q^{-5} - 6q^2 + q^3 - 7q^4 - 8q^6 + 19q^8 + 20q^9 + O(q^{11}), \\
  f_7(\tau) &= q^{-7} - 4q^2 - 10q^3 - 5q^4 + 8q^6 - 31q^8 + 35q^9 + O(q^{11}), \\
  f_{10}(\tau) &= q^{-10} - 13q^2 - 14q^3 + 13q^4 + 13q^8 - 78q^9 + O(q^{11}).
\end{align*}
\]

Now we obtain \( f_m \) for \( m > 23 \) as follows. Write \( m = 23a + b \) with \( 0 < b \leq 23 \). We construct \( f_m \) for all \( b \) by induction on \( a \). Then \( j(23\tau)^a f_b \) lies in the plus-space and its Fourier expansion has only integral Fourier coefficients and starts with \( q^{-m} + O(q^{-23a+2}) \). Therefore, we can subtract off integral multiples of \( f_n \) with \( n < 23a \) to obtain the desired principal part. We will write \( c_m(n) \) for the Fourier coefficient of index \( n \) of \( f_m \).

According to Theorem 3.5, we have
\[
c^+_m(m) = 3(\Phi_\mathcal{O}(1, f_m) - \Phi_\mathcal{O}(h_\mathcal{O}, f_m)) = 3(\Phi_\mathcal{O}(1, f_m) - \Phi_\mathcal{O}(1, f_m)),
\]
where \( h_\mathcal{O} \) is the idele corresponding to \( \mathcal{O} \). To determine the theta lifts on the right hand side, we use the see-saw identity (4.9). We will need to know the principal part of \( F_m = f_m \otimes \mathcal{O} \) for every \( m \), where we identify \( f_m \) with a vector valued modular form in \( M_{1,1}^+ \). We will simply write \( C_m(n) = C_m(n, \mu) \) with \( \mu = x + L_{23} \) and \( x^2 \equiv n \mod (46) \) for the coefficients of \( F_m \). This is independent of the choice of \( x \) as every square modulo 92 has at most two roots modulo 46. The nonzero coefficients in the principal part are then given in Table 1. Note that the constant term of \( F_m \) vanishes for every \( m \).

| \( n \) | \( \frac{-4m-23}{92} \) | \( \frac{-4m}{92} \) | \( \frac{-15}{92} \) | \( \frac{-11}{92} \) | \( \frac{-7}{92} \) |
|---|---|---|---|---|---|
| \( C_m(n) \) | 1 | 10 | \( c_m(2) \) | \( c_m(3) \) | \( c_m(4) \) |

Table 1. The principal part of \( F_m \)

In order to determine the coefficients of \( f_m \) of index 2, 3 and 4 for any \( m \), we will use the forms \( f_2, f_3 \) and \( f_4 \) which are contained in \( M_{1,1}^-(\Gamma_0(23), \chi_{-23}) \) and are constructed in a similar fashion as \( f_m \) above. We have
\[
\begin{align*}
  f_2 &= q^{-2} + q^{-1} - 1 + 6q^2 + 4q^7 + 13q^{10} + O(q^{11}), \\
  f_3 &= q^{-3} + q^{-1} - 1 + 5q^7 + 14q^{10} + O(q^{11}), \\
  f_4 &= q^{-4} - 1 + 7q^5 + 5q^7 - 13q^{10} + O(q^{11}).
\end{align*}
\]

We can easily determine the Fourier expansion of these 3 forms to an arbitrary precision. We note that these are also the first few in a sequence of weakly holomorphic modular forms spanning the space \( M_{1,1}^-(\Gamma_0(23), \chi_{-23}) \), having all integral Fourier coefficients.

Lemma 6.1. We have \( c_m(2) = -c_2(m) \), \( c_m(3) = c_3(m) \) and \( c_m(4) = -c_4(m) \).
Using the Lemma and Theorem 2.8, we see that \( \Phi_{L_{23}}(F_m, (z, 1)) \) for \( L_{23} \) as in Section 4.1 for \( N = 23 \) is equal to
\[
\Phi_{L_{23}}((z, 1), F_m) = -2 \log |\Psi_{L_{23}}((z, 1), F_m)|^2
\]
and \( \Psi_{L_{23}}((z, 1), F_m)^2 \) is a meromorphic modular form for \( \Gamma_0(23) \), invariant under the Fricke involution and with divisor
\[
Z \left( \frac{4m + 23}{92} \right) + 10Z \left( \frac{4m}{92} \right) + c_2(m)Z \left( \frac{15}{92} \right) + c_3(m)Z \left( \frac{11}{92} \right) + c_4(m)Z \left( \frac{7}{92} \right).
\]
Here, we briefly wrote \( Z(n/92) \) for the divisor \( Z(n/92, \mu) + Z(n/92, -\mu) \) with \( \mu = x + L_{23} \) and \( x^2 \equiv -n \mod (46) \). The modular curve \( X = X_0^+(23) \) has genus 0. A generator \( H_{23} \) of the function field of \( X \) can be obtained as the Borcherds product \( H_{23} = \Psi_{L_{23}}((z, 1), F) \), where \( F(\tau) \in M_{1/2, L_{23}}^1 \) is the unique form with principal part equal to \( q^{-7/92}(\phi_{19} + \phi_{-19}) \) and constant term 0. It turns out that \( H_{23}(\tau) = \frac{2g(\tau)}{g(\tau)} - 2 \) and we have
\[
H_{23}(\tau) = q^{-1} + 4q + 7q^2 + 13q^3 + 19q^4 + 33q^5 + 47q^6 + 74q^7 + 106q^8 + 154q^9 + O(q^{10}).
\]
Using this information it is clear that
\[
\Psi_{L_{23}}((z, 1), F_m)^2 = R_m(H_{23}(z)),
\]
where \( R \) is the rational function given by
\[
R_m(x) = \prod_{z \in \text{div}(\Psi(F_m))^2} (x - H_{23}(z))^{2 \text{ord}_x(\Psi(F_m))} = \prod_{d > 0} P_d(x)^{c_m(-d)},
\]
where
\[
P_d(x) = \prod_{z \in Z(d/92)} (x - H_{23}(z)) \in Z[x].
\]
We see that the coefficients of the homomorphic part of \( \tilde{g} \) can all be obtained by rational functions in \( H_{23}(z_{23}) \), where \( z_{23} = \frac{-23 + \sqrt{-23}}{64} \). Explicitly, we obtain
\[
c_0^\dagger(m) = -\frac{1}{12} \log |R_m(H_{23}(z_{23}))|, \quad c_a^\dagger(m) = c_b^\dagger(m) = -\frac{1}{12} \log |R_m(H_{23}(z_{23}))^{\sigma(a)}|,
\]
\[
c_{\tilde{a}}^\dagger(m) = -\frac{1}{24} \log \left| \frac{R_m(H_{23}(z_{23}))}{R_m(H_{23}(z_{23}))^{\sigma(\tilde{a})}} \right|.
\]
We remark that the algebraic numbers defining \( c_a^\dagger(m) \) and \( c_{\tilde{a}}^\dagger(m) \) in Corollary 5.1 are different. However, \( |R_m(H_{23}(z_{23}))| = |R_m(H_{23}(z_{23}))^{\sigma(a)}| \) because \( H_{23}(z_{23})^{\sigma(a)} = \overline{H_{23}(z_{23})^{\sigma(a)}} \) and the coefficients of \( R_m \) are rational integers, which implies the equality \( c_a^\dagger(m) = c_{\tilde{a}}^\dagger(m) \).

A table with the polynomials \( P_d \) for some small values of \( d \) can be found in [Via12]. Note that we need to substitute \( x - 1 \) for \( x \) in the table because our normalization of the Hauptmodul \( H_{23} \) is different to match the Borcherds product. Using these explicit formulas it is also easy to verify the factorization formula we gave and the numerical values obtained in [DL15].

We conclude this section with an interesting observation: the values \( R_m(H_{23}(z_{23})) \) can also be interpreted as follows. Consider the modular function \( \tilde{H}_{23}(\tau) = H_{23}(\tau) - H_{23}(z_{23}) \) and let
\( \tilde{F}_m \) be the unique weakly holomorphic modular form in \( M_{1/2,L_{23}} \) with a principal part equal to \( q^{-m/92}(\phi_\mu + \phi_{-\mu}) \) and \( \mu = x + L_{23} \) with \( x^2 \equiv -m \text{ mod } (92) \), as above. Then we obtain

\[
\Phi((z_{23},1), \tilde{F}_m) = -2 \log \left| \prod_{z \in \mathbb{Z}(m)} (H_{23}(z_{23}) - H_{23}(z)) \right| = -2 \log \left| \tilde{H}_{23}(Z(m)) \right|
\]

and by Theorem 3.5, this implies that

\[
\Theta(\tau) \otimes \tilde{\Theta}^+(\tau) = \frac{3}{2} \log |\alpha| q^{-1/23}(\phi_{2/92} + \phi_{-2/92}) + c^+_{\mathcal{O}}(0)
- 2 \sum_{\mu \mod (46)} \sum_{m > 0, -m \equiv \mu^2 \text{ mod } (92)} \log \left| \tilde{H}_{23}(Z(m)) \right| q_{\frac{m}{92}}(\phi_\mu + \phi_{-\mu}),
\]

where

\[
\Theta(\tau) = \sum_{n \in \mathbb{Z}} e \left( \frac{n^2}{4} \right) \phi_{n/2+\mathbb{Z}}
\]

is the vector valued theta function for the lattice \( \mathbb{Z} \) with quadratic form \( x^2 \). Moreover, note that \( \Theta(\tau) \otimes \tilde{\Theta}_\mathcal{O}(\tau) \) is modular of weight 3/2 and transforms with representation \( \rho_{L_{23}} \). We can obtain the coefficients \( c^+_{\mathcal{O}}(m) \) as linear combinations of coefficients of \( \Theta(\tau) \otimes \tilde{\Theta}^+_\mathcal{O}(\tau) \).

This suggests that the harmonic weak Maaß forms of weight one that we described in this paper appear as building blocks for generating series as above – note the similarity to the result on the modularity of the generating series of traces of singular moduli [Zag02; BF06]! It should be possible to obtain these generating series explicitly using the Kudla-Millson theta lift as in [BF06; Fun07; AE13]. This would also yield an explicit construction of the harmonic Maaß forms we considered using a theta lift. However, note that the fact that the obstruction space \( S_{3/2,L_{23}} \) is zero plays a crucial role in our simple description. The situation is more complicated in general. We will come back to this description in a sequel to this paper.

### 7. An Arithmetic Theta Function of Weight One

In the spirit of the Kudla program, we are led to form another generating series related to the cycles \( Z(m, \alpha, \beta) \). The non-holomorphic Eisenstein series of weight one should be seen as the generating series of the arithmetic cycles equipped with Kudla’s Green functions. We will complete them instead with the automorphic Green functions obtained by the regularized theta lift and form the generating series of the completed cycles \( \hat{Z}(m, \alpha, \beta) \) with values in \( \widehat{CH}^1(C_D) \otimes_{\mathbb{Z}} \mathbb{Z}[P'/P] \):

\[
\hat{\Phi}(\tau) = \sum_{\beta \in P'/P} \sum_{m > 0} \hat{Z}(m, \alpha, \beta)e(m\tau)\phi_\beta.
\] (7.1)

Let us first define the completed cycles. To keep the setup as simple as possible, we define them as Arakelov divisors [Neu07, Kapitel III] on \( \text{Spec } \mathcal{O}_H \). We let

\[
\hat{Z}(m, \alpha, \beta) = \sum_{\mathfrak{P} \subseteq \mathcal{O}_H} \mathcal{Z}(m, \alpha, \beta)_{\mathfrak{P}} \mathfrak{P} + \sum_{\sigma} \lambda(m, \beta, \sigma)\sigma.
\]

Here, \( \sigma \) runs over all complex embeddings of \( H \) and we define \( \lambda(m, \beta, \sigma) \in \mathbb{R} \) as follows. For \( m \in \mathbb{Z}_{>0} \) and \( \beta \in P'/P \), we let \( g_{m, \beta} \in H_{1, P} \) with principal part

\[
P_g(\tau) = \frac{1}{2} q^{-m}(\phi_\beta + \phi_{-\beta})
\]
and $c^+_{g_{m,\beta}}(0,0) = 0$. We start from a fixed embedding $\sigma_0$ of $\mathcal{O}_{\mathbb{H}_k}$ into $\mathbb{C}$ and define

$$\lambda(m, \beta, \sigma_0) := \frac{1}{w_k} \Phi_P(g_{m,\beta}(\tau), 1)$$

Then, for any other embedding $\sigma$, if $\sigma = \sigma_0 \circ \sigma(h)$ or $\sigma = \sigma_0 \circ \sigma(h)$, we let $h(\sigma) := h(\sigma_0) := h$ and define

$$\lambda(m, \beta, \sigma) := \frac{1}{w_k} \Phi_P(g_{m,\beta}(\tau), h(\sigma)).$$

However, all of this only makes sense if $\Phi_P(g_{m,\beta}(\tau), h) \in \mathbb{R}$, which we did not prove here. It should be possible to show this using e.g. the embedding used in Section 4.1 and by working out the Fourier expansion as in Section 7 of [Bor98]. To be safe, we let

$$\tilde{\Phi}_P(g_{m,\beta}(\tau), h) := \frac{1}{2} \left( \Phi_P(g_{m,\beta}(\tau), h) + \Phi_P(g_{m,\beta}(\tau), h) \right) \in \mathbb{R}$$

and let $\lambda(m, \beta, \sigma) := \frac{1}{w_k} \tilde{\Phi}_P(g_{m,\beta}(\tau), h(\sigma))$. Then we define the formal power series $\tilde{\Phi}_P(\tau)$ with coefficients in $\hat{CH}^1(\mathcal{O}_{\mathbb{H}_k}) \otimes_{\mathbb{Z}} \mathbb{Z}[P'/P]$ via (7.1). To make this precise, we have to impose further conditions on $g_{m,\beta}$; we demand that $c^+_{g_{m,\beta}}(p_j, \pi_j) = 0$ for all $j$ and where $p_j$ and $\pi_j$ have been defined in Section 4.3.

Note that it is a consequence of Theorems 3.5 and 4.21 that $\deg \hat{Z}(m, a, \mu)$ vanishes. (However, this also follows directly from Theorem 6.5 of [BY09].) Consequently, the coefficients of $\tilde{\Phi}(\tau)$ vanish when viewed as a generating series valued in $\hat{CH}^1(\mathcal{O}_{\mathbb{H}_k})$ and the degree generating series is thus “trivially” modular.

However, a stronger statement is true. In fact, what we have shown amounts to:

**Corollary 7.1.** If $f \in M^1_{1,P}$ is a weakly holomorphic modular form with only integral Fourier coefficients in its principal part, then

$$\sum_{\beta \in P'/P} \sum_{n>0} c_f(-n, \beta) \hat{Z}(n, a, \beta) = 0 \in \hat{CH}^1(\mathcal{O}_{\mathbb{H}_k}) \otimes_{\mathbb{Z}} \mathbb{Q}.$$ 

**Proof.** First, we claim that

$$\sum_{\beta \in P'/P} \sum_{n>0} c_f(-n, \beta) g_{m,\beta}(\tau) = \frac{1}{2} \sum_{\beta \in P'/P} \sum_{n>0} c_f(-n, \beta) f_{m,\beta}(\tau) := \tilde{f}, \quad (7.2)$$

where $f_{m,\beta}$ has been defined in Proposition 4.11. In fact, the difference of the left-hand side and the right-hand side is a weakly holomorphic modular form with principal part equal to

$$\frac{1}{2} \sum_{\beta \in P'/P} \sum_{n>0} c_f(-n, \beta) a_j(n, \beta) q^{-n_j} (\phi_{\beta_j} + \phi_{-\beta_j}).$$

Since $f$ is weakly holomorphic, the double sum

$$\sum_{\beta \in P'/P} \sum_{n>0} c_f(-n, \beta) a_j(n, \beta) = \{f, g_j\}$$

vanishes for all $j$. Thus, the difference of the left-hand side and right-hand side in (7.2) is a holomorphic modular form with vanishing constant term in $M^1_{1,P}$ and thus a cusp form since we
assume that $P'/P$ has squarefree order $|D|$. However, since the normalization of $g_{m,\beta}$ for the positive index Fourier coefficients agrees with the normalization of $f_{m,\beta}$, (7.2) follows. Write
\[
\sum_{\beta \in P'/P} \sum_{nP > 0} c_f(-n, \beta) \hat{Z}(n, a, \beta) = \sum_{\beta \in P'/P} \sum_{nP > 0} c_f(-n, \beta) Z(n, a, \beta) + \frac{1}{w_k} \sum_{\sigma} \Phi_P(f, h(\sigma)) \sigma.
\]

Then Theorem 3.5 shows that
\[
\Phi_P(f, h(\sigma)) = \frac{1}{2} \sum_{\beta \in P'/P} \sum_{n > 0} c_f(-n, \beta) c_P^+(h(\sigma), n, \beta)
\]

and by Theorem 4.21, we have that $c_P^+(1, n, \beta) = -2^2 \log |\alpha(1, n, \beta)| \in \mathbb{R}$ with
\[
\text{ord}_\mathfrak{P}(\alpha(1, n, \beta)) = r \cdot w_k \cdot \mathcal{Z}(n, a, \beta)_{\mathfrak{P}}
\]

and $\alpha(h(\sigma), n, \beta) = \alpha(1, n, \beta)^{\sigma}$, which shows that the class defined by
\[
\sum_{\beta \in P'/P} \sum_{n > 0} c_f(-n, \beta) \hat{Z}(n, a, \beta)
\]
is torsion and thus vanishes in $\widehat{\text{CH}}^1(\mathcal{O}_k) \otimes \mathbb{Z} \mathbb{Q}$. □

**Lemma 7.2.** The classes $\hat{Z}(m, a, \beta)$ lie in a finite dimensional subspace of $\widehat{\text{CH}}^1(\mathcal{O}_k) \otimes \mathbb{Z} \mathbb{Q}$.

**Proof.** Clearly, taking $f = f_{m,\beta}$ in the corollary gives
\[
\frac{1}{2} \sum_{j=1}^{d^-} a_j(m, \beta) \hat{Z}(n_j, a, \beta_j) = \hat{Z}(m, a, \beta) \in \widehat{\text{CH}}^1(\mathcal{O}_k) \otimes \mathbb{Z} \mathbb{Q},
\]
i.e. the special divisors lie in the subspace spanned by the $\hat{Z}(n_j, a, \beta_j), j = 1, \ldots, d^-$. □

**Corollary 7.3.** The formal power series $\hat{\Phi}(\tau)$ is the $q$-expansion of a modular form of weight one and representation $\rho_{P^-}$.

**Proof.** This follows by applying the modularity criterion as in [Bor99, Lemma 4.3] and the existence of a basis of $M_{1, P^-}$ with integral Fourier coefficients [McG03]. □

**Remark 7.4.** It is a bit unsatisfactory that, in contrast to the non-holomorphic arithmetic generating series $\mathcal{E}_P(\tau)$ we do not know that $\hat{\Phi}(\tau) \neq 0$ and this seems to be a subtle question.

**List of Symbols**

- $(f, g)$: The Petersson inner product of $f$ and $g$ (vector valued modular forms), 11
- $(a, b)$: $\text{gcd}(a, b)$, the greatest common divisor of $a$ and $b$
- $\langle \cdot, \cdot \rangle$: The Petersson slash operator on vector valued functions, 9
- $|_{k, L}$: The Petersson slash operator on vector valued functions, 9
- $A$: Starting from Section 4.1: a prime which does not divide $D$, such that $[A, B, C]$ corresponds to $P = a$ with $D = B^2 - 4AC$, 23
- $a, b, c$: Fractional (most of the time integral) ideals
- $\mathbb{A}_{\mathbb{Q}}, \mathbb{A}_k$: The adeles over $\mathbb{Q}$ and $k$, respectively, 7
- $\mathbb{A}_{f, k, f}, \mathbb{A}^\times_{f, k}$: The finite adeles over $\mathbb{Q}$ and $k$, respectively, 7
- $\mathbb{A}^\times_k$: The finite ideles over $\mathbb{Q}$, 7
- $\mathbb{A}_{k}^\times$: The ideles over $\mathbb{Q}$ and $k$, 7
- $B$: Starting from Section 4.1: a fixed integer, see $A$, 23
\[ \beta_j = \mu_j(P^-), \]
\[ \mathbb{C} \]  
the field of complex numbers
\[ C \]  
starting from section 4.1: a fixed integer, see A, 23
\[ c_{m,\beta} \]  
an integer defined in definition 4.13, 28
\[ c^+_p(h, m, \beta) \]  
the coefficient of index \((m, \beta)\) of \(\tilde{\Theta}_p^+(\tau, h)\), 24
\[ C_{p,K} \]  
a certain cover of the CM cycle, 17
\[ CT_{s=0}[A(s)] \]  
the constant term in the Laurent series expansion of \(A(s)\) at \(s = 0\), 12
\[ CT \]  
also the constant term in a Laurent series in \(q\), 17
\[ D \]  
usually an odd fundamental discriminant, 15
\[ \mathbb{D} \]  
the symmetric domain attached to \(\text{SO}_V(\mathbb{R})\), realized as Grassmannian, 7
\[ \text{diag}(a_1, \ldots, a_n) \]  
the \(n \times n\) diagonal matrix with diagonal \((a_1, \ldots, a_n)\), 22
\[ d^- = d(P^-) = \dim(S_{1,P^-}), \]
\[ d^+ = d(P) = \dim(S_{1,P}), \]
\[ e(x) = e^{2\pi ix} \]
\[ E_P(\tau) \]  
the holomorphic Eisenstein series in \(M_{1,P}\), equal to \(E_P(\tau, 0)\), 17
\[ \tilde{E}_P(\tau) \]  
a preimage of \(E_P(\tau)\) under the \(\xi_1\)-operator, 31
\[ \hat{E}_P(\tau,s) \]  
an incoherent Eisenstein series attached to \(P\), 16
\[ f^+ \]  
the holomorphic part of \(f\), 10
\[ f^- \]  
the non-holomorphic part of \(f\), 10
\[ \mathcal{F} \]  
the standard fundamental domain for \(\text{SL}_2(\mathbb{Z})\), 11
\[ \mathcal{F}_T \]  
a truncated fundamental domain, 11
\[ \mathfrak{F} \]  
a weak Jacobi form, 14
\[ f^L \]  
the image of \(f\) under the trace map \(\text{tr}\), 13
\[ f^M \]  
the image of \(f\) under the restriction map \(\text{res}\), 13
\[ \tilde{f}_{m,\beta} \]  
the element in \(f_{m,\beta} \in M_{1/2,L}^1\) corresponding to \(f_{m,\beta}\) via lemma 4.12, 28
\[ f_{m,\beta} \]  
an element of \(M^1_{1,P}\), 27
\[ \mathbb{F}_q \]  
an algebraic closure of a finite field with \(q\) elements
\[ \bar{\mathbb{F}}_q \]  
an algebraic closure of a finite field with \(q\) elements
\[ \Gamma_0(N) \]  
the congruence subgroup \(\Gamma_0(N) \subset \text{SL}_2(\mathbb{Z})\), 21
\[ G \]  
a weak Jacobi form, 14
\[ \mathbb{H} \]  
the complex upper half-plane, \(\mathbb{H} = \{z \in \mathbb{C} \mid \text{Im}(z) > 0\}\)
\[ H \]  
frequently, \(H = \text{GSpin}_V\), the general spin group, 7
\[ H_k \]  
the Hilbert class field of the imaginary quadratic field \(k\), 15
\[ \mathcal{H}_{k,L} \]  
the space of harmonic weak Maass forms of weight \(k\) and representation \(\rho_L\), 10
\[ H_{k,L} \]  
harmonic weak Maass forms \(f \in \mathcal{H}_{k,L}\) such that \(\xi_k(f) \in S_{2-k,L^-}\), 10
\[ \text{Im}(z) \]  
the imaginary part of \(z\)
\[ k \]  
usually a half-integer or integer (a weight), often \(k = 1 - n/2\)
\[ k \]  
a field, usually an imaginary quadratic field
\[ k_D = \mathbb{Q}(\sqrt{D}) \]
\[ L^- \]  
the quadratic module given by \(L\) together with the quadratic form \(-Q\), 10
\[ \hat{L} = L \otimes_{\mathbb{Z}} \hat{\mathbb{Z}}, \]
\[ L_{m,\mu} \]  
norm \(m\) elements in \(L + \mu\), 9
\[ \lambda_r \]  
an element in \(L_{m,\mu}\), 22
\[ M_{k,L} \]  
the space of holomorphic modular forms of weight \(k\) and representation \(\rho_L\), 10
The space of weakly holomorphic modular forms of weight $k$ and representation $\rho_L$, 10

$\mu_r = \text{diag}(r/2N,-r/2N)$, 22

$N$ A one-dimensional negative definite lattice given by $N = L \cap U^\perp$, 22

$n$ a non-negative integer, s.t. $V$ has signature $(2,n)$, sometimes also used as index in sums, 7

$N$ A positive integer, $N = A|D|$ in Section 4, 23

Also a negative definite lattice, 16

$n_j = n_j(P^-)$, 26

$P$ Usually a two-dimensional positive definite even lattice, 21

From Section 4.1 on, $P = a \subset \mathcal{O}_k$ with quadratic form $N(x)/N(a)$, corresponding to $[A,B,C]$, s.t. $D = B^2 - 4AC$, 23

Often obtained as $P = L \cap U$, 16

$P_f(\tau)$ The principal part of $f$ (not including the constant term), 10

$\phi_\mu$ The characteristic function of $\mu + L$, 8

$\pi_j = \mu_j(P)$, 26

$p_j = n_j(P)$, 26

$\mathcal{P}$ A two-dimensional positive definite lattice given by $\mathcal{P} = L \cap U$, 22

$\mathbb{Q}$ The field of rational numbers

$q$ Usually $q = e^{2\pi i \tau}$

$\mathbb{Q}_p$ The field of $p$-adic numbers

$R$ A fixed integer, $R = FD$ with $R^2 \equiv D \mod 4A|D|$, also see $A$, 23

$\text{Re}(z)$ The real part of $z$

$\rho_L$ The Weil representation associated with $L$, 9

$\hat{\beta}(n,\beta)$ The coefficient of index $(n,\beta)$ of $E_P(\tau)$, 17

$S$ The matrix $\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ of $SL_2(\mathbb{Z})$ or the element $\left(\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right), \sqrt{\tau}\right) \in Mp_2(\mathbb{Z})$, 9

$\sigma(a)$ The image of $a$ under the Artin map $(\cdot, \mathbb{H}_k/k)$, 15

$S_{k,L}$ The space of cusp forms of weight $k$ and representation $\rho_L$, 10

$S_L$ The span of the characteristic functions $\phi_\mu$ in $S(V(\mathcal{A}f))$, 8

$\tau \tau = u + iv \in \mathbb{H}$

$T$ The matrix $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ of $SL_2(\mathbb{Z})$ or the element $\left(\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right), 1\right) \in Mp_2(\mathbb{Z})$, 9

Also $T = \text{GSpin}_U$, 16

$\bar{\Theta}_P(\tau,h)$ A preimage of $\Theta_P(\tau,h)$ under the $\xi_1$ operator, 18

$U$ A two-dimensional positive definite subspace of $V(\mathbb{Q})$, 16

$V$ A rational quadratic space of signature $(2,n)$, 7

$w_{K,T} = |(T(\mathbb{Q}) \cap K_T)|$, 16

$X_0(N)$ The compactification of $Y_0(N) = \Gamma_0(N)\backslash \mathbb{H}$, 21

$X_0(N)$ A certain integral model of $X_0(N)$, 28

$\xi$ Usually, $\xi = \xi_k$, a differential operator, 10

$X_K$ A Shimura variety, 7

$\mathbb{Z}$ The ring of integers

$\mathbb{Z}_{>0}$ The set of positive integers

$Z(f)$ A special divisor attached to $f$, 12

$Z(f)$ A special divisor on $X_0(N)$, 28

$\hat{\mathbb{Z}} = \prod_{p<\infty} \mathbb{Z}_p$, 8

$Z(m,\mu)$ A special divisor, 9

$Z(m,a,\mu)$ A certain arithmetic divisor on Spec $\mathcal{O}_{H_k}$, 15
REFERENCES

\[Z(m, a, \mu)_p\] see \(Z(m, a, \mu)\)

\[Z(m, \mu)\] A special divisor on \(X_0(N)\), 28

\(Z_p\) The \(p\)-adic integers

\(z_U^\pm = z_U\) The (two) point(s) corresponding to \(U \subset V(\mathbb{R})\), 16

\(Z(U)_K = Z(U)\) A CM cycle, see equation (3.2), 16

References

[AE13] C. Alfes and S. Ehlen. “Twisted traces of CM values of weak Maass forms”. In: \textit{J. Number Theory} 133.6 (2013), pp. 1827–1845. issn: 0022-314X. url: \texttt{http://dx.doi.org/10.1016/j.jnt.2012.10.008}.

[Bor98] R. E. Borcherds. “Automorphic forms with singularities on Grassmannians”. In: \textit{Invent. Math.} 132.3 (1998), pp. 491–562. issn: 0020-9910.

[Bor99] R. E. Borcherds. “The Gross-Kohnen-Zagier theorem in higher dimensions”. In: \textit{Duke Math. J.} 97.2 (1999), pp. 219–233. issn: 0012-7094.

[Bru02] J. H. Bruinier. \textit{Borcherds products on O(2, l) and Chern classes of Heegner divisors}. Vol. 1780. Lecture Notes in Mathematics. Berlin: Springer-Verlag, 2002, pp. viii+152. isbn: 3-540-43320-1.

[BB03] J. H. Bruinier and M. Bundschuh. “On Borcherds products associated with lattices of prime discriminant”. In: \textit{Ramanujan J.} 7.1-3 (2003). Rankin memorial issues, pp. 49–61. issn: 1382-4090. url: \texttt{http://dx.doi.org/10.1023/A:1026222507219}.

[BF04] J. H. Bruinier and J. Funke. “On two geometric theta lifts”. In: \textit{Duke Math. J.} 125.1 (2004), pp. 45–90. issn: 0012-7094.

[BF06] J. H. Bruinier and J. Funke. “Traces of CM values of modular functions”. In: \textit{J. Reine Angew. Math.} 594 (2006), pp. 1–33. issn: 0075-4102. url: \texttt{http://dx.doi.org/10.1515/CRELLE.2006.034}.

[BO10] J. H. Bruinier and K. Ono. “Heegner divisors, \(L\)-functions and harmonic weak Maass forms”. In: \textit{Ann. of Math. (2)} 172.3 (2010), pp. 631–681. issn: 0003-486X. url: \texttt{http://dx.doi.org/10.4007/annals.2010.172.2135}.

[BS16] J. H. Bruinier and M. Schwagenscheidt. “Algebraic formulas for the coefficients of mock theta functions and Weyl vectors of Borcherds products”. In: \textit{ArXiv e-prints} (July 2016).

[BY09] J. H. Bruinier and T. Yang. “Faltings heights of CM cycles and derivatives of \(L\)-functions”. In: \textit{Invent. Math.} 177.3 (2009), pp. 631–681. issn: 0020-9910. url: \texttt{http://dx.doi.org/10.1007/s00222-009-0192-8}.

[Bum97] D. Bump. \textit{Automorphic forms and representations}. Vol. 55. Cambridge Studies in Advanced Mathematics. Cambridge: Cambridge University Press, 1997, pp. xiv+574. isbn: 0-521-55098-X. url: \texttt{http://dx.doi.org/10.1017/CBO9780511609672}.

[Cox89] D. A. Cox. \textit{Primes of the form \(x^2 + ny^2\)}. A Wiley-Interscience Publication. Fermat, class field theory and complex multiplication. New York: John Wiley & Sons Inc., 1989, pp. xiv+351. isbn: 0-471-50654-0; 0-471-19079-9.

[DL15] W. Duke and Y. Li. “Harmonic Maass forms of weight 1”. In: \textit{Duke Math. J.} 164.1 (2015), pp. 39–113. issn: 0012-7094. url: \texttt{http://dx.doi.org/10.1215/00127094-2838436}.

[Ehl12] S. Ehlen. “CM values of Borcherds products and harmonic weak Maass forms of weight one”. In: \textit{ArXiv e-prints} (Aug. 2012).

[Ehl13] S. Ehlen. \textit{CM values of regularized theta lifts}. TU Darmstadt, 2013. url: \texttt{http://tuprints.ulb.tu-darmstadt.de/3731/}.

[Ehl15] S. Ehlen. “Singular moduli of higher level and special cycles”. In: \textit{Res. Math. Sci.} 2 (2015), Art. 16, 27. issn: 2197-9847. url: \texttt{http://dx.doi.org/10.1186/s40687-015-0030-0}.

[Ehl16] S. Ehlen. “Vector valued theta functions associated with binary quadratic forms”. In: \textit{Forum Mathematicum} 28.5 (2016).

[EZ85] M. Eichler and D. B. Zagier. \textit{The theory of Jacobi forms}. Vol. 55. Progress in Mathematics. Boston, MA: Birkhäuser Boston Inc., 1985, pp. v+148. isbn: 0-8176-3180-1.
[Fun07] J. Funke. “CM points and weight 3/2 modular forms”. In: Analytic number theory, Clay Math. Proc. 7. Providence, RI: Amer. Math. Soc., 2007, pp. 107–127.

[GZ85] B. H. Gross and D. B. Zagier. “On singular moduli”. In: J. Reine Angew. Math. 355 (1985), pp. 191–220. ISBN: 0075-4102. URL: http://dx.doi.org/10.1016/j.jnt.2006.04.009.

[GZ86] B. H. Gross and D. B. Zagier. “Heegner points and derivatives of $L$-series”. In: Invent. Math. 84.2 (1986), pp. 225–320. ISBN: 0020-9910. URL: http://dx.doi.org/10.1007/BF01388809.

[Kit93] Y. Kitaoka. Arithmetic of quadratic forms. Vol. 106. Cambridge Tracts in Mathematics. Cambridge: Cambridge University Press, 1993, pp. x+268. ISBN: 0-521-40475-4.

[Kud84] S. S. Kudla. “Seesaw dual reductive pairs”. In: Automorphic forms of several variables (Katata, 1983). Vol. 46. Progr. Math. Boston, MA: Birkhäuser Boston, 1984, pp. 244–268.

[Kud97] S. S. Kudla. “Algebraic cycles on Shimura varieties of orthogonal type”. In: Duke Math. J. 86.1 (1997), pp. 39–78. ISBN: 0012-7094. URL: http://dx.doi.org/10.1215/S0012-7094-97-08602-6.

[Kud03] S. S. Kudla. “Integrals of Borcherds forms”. In: Compositio Math. 140.3-4 (2013), pp. 393–440. ISSN: 0010-437X. URL: http://dx.doi.org/10.1007/s00229-012-0569-7.

[KRY99] S. S. Kudla, M. Rapoport, and T. Yang. “On the derivative of an Eisenstein series of weight one”. In: Internat. Math. Res. Notices 7 (1999), pp. 347–385. ISSN: 1073-7928. URL: http://dx.doi.org/10.1155/S1073792899000185.

[KRY04] S. S. Kudla, M. Rapoport, and T. Yang. “Derivatives of Eisenstein series and Faltings heights”. In: Compos. Math. 140.4 (2004), pp. 887–951. ISSN: 0010-437X. URL: http://dx.doi.org/10.1112/S0010437X03000459.

[KY10] S. S. Kudla and T. Yang. “Eisenstein series for SL(2)”. In: Sci. China Math. 53.9 (2010), pp. 2275–2316. ISSN: 1674-7283. URL: http://dx.doi.org/10.1007/s11425-010-4097-1.

[KY13] S. S. Kudla and T. Yang. “On the pullback of an arithmetic theta function”. In: Manuscripta Math. 140.3-4 (2013), pp. 393–440. ISSN: 0025-2611. URL: http://dx.doi.org/10.1007/s00229-012-0569-7.

[Maz77] B. Mazur. “Modular curves and the Eisenstein ideal”. In: Inst. Hautes Études Sci. Publ. Math. 47 (1977), 33–186 (1978). ISSN: 0073-8301. URL: http://www.numdam.org/item?id=PMIHES_1977__47__33_0.

[McG03] W. J. McGraw. “The rationality of vector valued modular forms associated with the Weil representation”. In: Math. Ann. 326.1 (2003), pp. 105–122. ISSN: 0025-5831. URL: http://dx.doi.org/10.1007/s00208-003-0413-1.

[Neu07] J. Neukirch. Algebraische Zahlentheorie. Berlin: Springer, xiv, 595 p., 2007.

[Sch04] N. Scheithauer. “Moonshine for Conway’s group”. In: Habilitation, University of Heidelberg (2004).

[Sch09] J. Schofer. “Borcherds forms and generalizations of singular moduli”. In: J. Reine Angew. Math. 629 (2009), pp. 1–36. ISSN: 0075-4102. URL: http://dx.doi.org/10.1515/CRELLE.2009.025.

[Ser73] J.-P. Serre. A course in arithmetic. Translated from the French, Graduate Texts in Mathematics, No. 7. New York: Springer-Verlag, 1973, pp. viii+115.

[Shi94] G. Shimura. Introduction to the arithmetic theory of automorphic functions. Vol. 11. Publications of the Mathematical Society of Japan. Reprint of the 1971 original, Kanô Memorial Lectures, 1. Princeton, NJ: Princeton University Press, 1994, pp. xiv+271. ISBN: 0-691-08092-5.

[Shi75] T. Shintani. “On construction of holomorphic cusp forms of half integral weight”. In: Nagoya Math. J. 58 (1975), pp. 83–126. ISSN: 0027-7630.

[Sil94] J. H. Silverman. Advanced topics in the arithmetic of elliptic curves. Vol. 151. Graduate Texts in Mathematics. New York: Springer-Verlag, 1994, pp. xiv+525. ISBN: 0-387-94328-5. URL: http://dx.doi.org/10.1007/978-1-4612-0851-8.
[Sta75] H. M. Stark. “L-functions at $s = 1$. II. Artin L-functions with rational characters”. In: Advances in Math. 17.1 (1975), pp. 60–92. issn: 0001-8708.

[Via12] M. Viazovska. “Petersson inner products of weight one modular forms”. In: ArXiv e-prints (2012).

[Wei64] A. Weil. “Sur certains groupes d’opérateurs unitaires”. In: Acta Math. 111 (1964), pp. 143–211. issn: 0001-5962.

[Zag02] D. B. Zagier. “Traces of singular moduli”. In: Motives, polylogarithms and Hodge theory, Part I (Irvine, CA, 1998). Vol. 3. Int. Press Lect. Ser. Int. Press, Somerville, MA, 2002, pp. 211–244.

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