An Analytic Model for Blue Straggler Formation in Globular Clusters

Nathan Leigh$^1$, Alison Sills$^1$, Christian Knigge$^2$

$^1$Department of Physics and Astronomy, McMaster University, 1280 Main St. W., Hamilton, ON, L8S 4M1, Canada
$^2$School of Physics and Astronomy, University of Southampton, Highfield, Southampton, SO17 1BJ, United Kingdom

ABSTRACT

We present an analytic model for blue straggler formation in globular clusters. We assume that blue stragglers are formed only through stellar collisions and binary star evolution, and compare our predictions to observed blue straggler numbers taken from the catalogue of Leigh, Sills & Knigge (2011). We can summarize our key results as follows: (1) Binary star evolution consistently dominates blue straggler production in all our best-fitting models. (2) In order to account for the observed sub-linear dependence of blue straggler numbers on the core masses (Knigge, Leigh & Sills 2009), the core binary fraction must be inversely proportional to the total cluster luminosity and should always exceed at least a few percent. (3) In at least some clusters, blue straggler formation must be enhanced by dynamical encounters (either via direct collisions or by stimulating mass-transfer to occur by altering the distribution of binary orbital parameters) relative to what is expected by assuming a simple population of binaries evolving in isolation. (4) The agreement between the predictions of our model and the observations can be improved by including blue stragglers that form outside the core but later migrate in due to dynamical friction. (5) Longer blue straggler lifetimes are preferred in models that include blue stragglers formed outside the core since this increases the fraction that will have sufficient time to migrate in via dynamical friction.

Key words: stars: blue stragglers – globular clusters: general – stellar dynamics – stars: statistics.

1 INTRODUCTION

Commonly found in both open and globular clusters (GCs), blue stragglers (BSs) appear as an extension of the main-sequence (MS) in cluster colour-magnitude diagrams (CMDs), occupying the region that is just brighter and bluer than the main-sequence turn-off (MSTO) (Sandage 1953). BSs are thought to be produced via the addition of hydrogen to low-mass MS stars (e.g. Sills et al. 2001; Lombardi et al. 2002). This can occur via multiple channels, most of which involve the mergers of low-mass MS stars since a significant amount of mass is typically required to reproduce the observed locations of BSs in CMDs (e.g. Sills & Bailyn 1999). Stars in close binaries can merge if enough orbital angular momentum is lost, which can be mediated by dynamical interactions with other stars, magnetized stellar winds, tidal dissipation or even an outer triple companion (e.g. Leonard & Linde 1992; Li & Zhang 2006; Perets & Fabrycky 2009; Dervisoglu, Tout & Ibanoglu 2010). Alternatively, MS stars can collide directly, although this is also thought to usually be mediated by multiple star systems (e.g. Leonard 1989; Leonard & Livio 1995; Freged et al. 2004; Leigh & Sills 2011). First proposed by McCrea (1964), BSs have also been hypothesized to form by mass-transfer from an evolving primary onto a normal MS companion via Roche lobe overflow.

Despite numerous formation mechanisms having been proposed, a satisfactory explanation to account for the presence of BSs in star clusters eludes us still. Whatever the dominant BS formation mechanism(s) operating in dense clusters, it is now thought to somehow involve multiple star systems. This was shown to be the case in even the dense cores of GCs (Leigh, Sills & Knigge 2007, 2008; Knigge, Leigh & Sills 2009) where collisions between single stars are thought to occur frequently (Leonard 1989). In Knigge, Leigh & Sills (2009), we showed that the numbers of BSs in the cores of a large sample of GCs correlate with the core masses. We argued that our results are consistent with what is expected if BSs are descended from binary stars since this would imply a dependence of the form $N_{BS} \sim f_b M_{core}$, where $N_{BS}$ is the number of BSs in the core, $f_b$ is the binary fraction of the core, and $M_{core}$ is the core mass.
fraction in the core and $M_{\text{core}}$ is the total stellar mass contained within the core. Mathieu & Geller (2009) also showed that at least 76% of the BSs in the old open cluster NGC 188 have binary companions. Although the nature of these companions remains unknown, it is clear that binaries played a role in the formation of these BSs.

Blue stragglers are typically concentrated in the dense cores of globular clusters where the high stellar densities should result in a higher rate of stellar encounters (e.g. Leonard 1989). Whether or not this fact is directly related to BS formation remains unclear, since mass segregation also acts to migrate BSs (or their progenitors) into the core (e.g. Saviane et al. 1993; Guhathakurta et al. 1998). Additionally, numerous BSs have been observed in more sparsely populated open clusters (e.g. Andrievsky et al. 2000) and the fields of GCs where collisions are much less likely to occur and mass-transfer within binary systems is thought to be a more likely formation scenario (e.g. Mapelli et al. 2004).

Several studies have provided evidence that BSs show a bimodal spatial distribution in some GCs (Ferraro et al. 1997, 1999; Lanzoni et al. 2007). In these clusters, the BS numbers are the highest in the central cluster regions and decrease with increasing distance from the cluster centre until a second rise occurs in the cluster outskirts. This drop in BS numbers at intermediate cluster radii is often referred to as the “zone of avoidance”. Some authors have suggested that it is the result of two separate formation mechanisms occurring in the inner and outer regions of the cluster, with mass-transfer in primordial binaries dominating in the latter and stellar collisions dominating in the former (Ferraro et al. 2004; Mapelli et al. 2006). Conversely, mass segregation could also give rise to a “zone of avoidance” for BSs if the time-scale for dynamical friction exceeds the average BS lifetime in only the outskirts of GCs that exhibit this radial trend (e.g. Leigh, Sills & Knigge 2011).

Dynamical interactions occur frequently enough in dense clusters that they are expected to be at least partly responsible for the observed properties of BSs (e.g. Strickland 1993; Leigh & Sills 2011). It follows that the current properties of BS populations should reflect the dynamical histories of their host clusters. As a result, BSs could provide an indirect means of probing the physical processes that drive star cluster evolution (e.g. Heggie & Hut 2003; Hurley et al. 2002; Leigh & Sills 2011).

In this paper, our goal is to constrain the dominant BS formation mechanism(s) operating in the dense cores of GCs by analyzing the principal processes thought to influence their production. To this end, we use an analytic treatment to obtain predictions for the number of BSs expected to be found within one core radius of the cluster centre at the current cluster age. Predicted numbers for the core are calculated for a range of free parameters, and then compared to the observed numbers in order to find the best-fitting model parameters. In this way, we are able to quantify the degree to which each of the considered formation mechanisms should contribute to the total predicted numbers in order to best reproduce the observations.

In Section 2 we describe the BS catalogue used for comparison to our model predictions. In Section 3 we present our analytic model for BS formation as well as the statistical technique we have developed to compare its predictions to the observations. These predictions are then compared to the observations in Section 4 for a range of model parameters. In Section 5 we discuss the implications of our results for BS formation, as well as the role played by the cluster dynamics in shaping the current properties of BS populations.

2 THE DATA

The data used in this study was taken from Leigh, Sills & Knigge (2011). In that paper, we presented a catalogue for blue straggler, red giant branch (RGB), horizontal branch (HB) and main-sequence turn-off stars obtained from the colour-magnitude diagrams of 35 Milky Way GCs taken from the ACS Survey for Globular Clusters (Sarajedini et al. 2007). The ACS Survey provides unprecedented deep photometry in the F606W ($\sim V$) and F814W ($\sim I$) filters that extends reliably from the HB all the way down to about 7 magnitudes below the MSTO. The clusters in our sample span a range of total masses (by nearly 3 orders of magnitude) and central concentrations (Harris et al. 1996). We have confirmed that the photometry is nearly complete in the BS region of the CMD for every cluster in our sample. This was done using the results of artificial star tests taken from Anderson et al. (2008).

Each cluster was centred in the ACS field, which extends out to several core radii from the cluster centre in most of the clusters in our sample. Only the core populations provided in Leigh, Sills & Knigge (2011) are used in this paper. We have taken estimates for the core radii and central luminosity densities for the clusters in our sample from Harris et al. (1996), whereas central velocity dispersions were taken from Webbin (1985). Estimates for the total stellar mass contained within the core were obtained from single-mass King models, as described in Leigh, Sills & Knigge (2011). All of the clusters in our sample were chosen to be non-post-core collapse, and have surface brightness profiles that provide good fits to our King models.

3 METHOD

In this section, we present our model and outline our assumptions. We also present the statistical technique used to compare the observed number counts to our model predictions in order to identify the best-fitting model parameters.

3.1 Model

Consider a GC core that is home to $N_{\text{BS,0}}$ BSs at some time $t = t_0$. At a specified time in the future, the number of BSs in the core can be approximated by:

$$N_{\text{BS}} = N_{\text{BS,0}} + N_{\text{coll}} + N_{\text{bin}} + N_{\text{in}} - N_{\text{out}} - N_{\text{ev}},$$  

(1)

where $N_{\text{coll}}$ is the number of BSs formed from collisions during single-single (1+1), single-binary (1+2) and binary-binary (2+2) encounters, $N_{\text{bin}}$ is the number formed from binary evolution (either partial mass-transfer between the binary components or their complete coalescence), $N_{\text{in}}$ is the number of BSs that migrate into the core due to dynamical friction, $N_{\text{out}}$ is the number that migrate out of the...
core via kicks experienced during dynamical encounters, and 
\(N_{ev}\) is the number of BSs that have evolved away from being  
brighter and bluer than the MSTO in the cluster CMD due to  
stellar evolution.

We adopt an average stellar mass of \(m = 0.65 M_\odot\)  
and an average BS mass of \(m_{BS} = 2m = 1.3 M_\odot\). The mass of a  
BS can provide a rough guide to its lifetime, although a range  
of lifetimes is still possible for any given mass. For  
instance, Sandquist, Bolte & Hernquist (1997) showed that  
a 1.3 M_\odot blue straggler will have a lifetime of around 0.78  
Gyrs in unmixed models, or 1.57 Gyrs in completely mixed  
models. Combined with the results of Sills et al. (2001),  
Lombardi et al. (2002) and Glebbeek & Pols (2008), we  
expect a lifetime in the range 1-5 Gyrs for a 1.3 M_\odot BS. As  
a first approximation, we choose a likely value of \(\tau_{BS} = 1.5\)  
Gyrs for the average BS lifetime (e.g. Sills et al. 2001). The  
effects had on our results by changing our assumption for  
the average BS lifetime will be explored in Section 4 and  
discussed in Section 5.

We consider only the last \(\tau_{BS}\) years. This is because  
we are comparing our model predictions to current observa-  
tions of BS populations, so that we are only concerned with  
those BSs formed within the last few Gyrs. Any BSs formed  
before this would have evolved away from being brighter  
and bluer than the MSTO by the current cluster age.  
Consequently, we set \(N_{BS,0} = N_{ev}\) in Equation 1. We further  
assume that all central cluster parameters have not changed  
in the last \(\tau_{BS}\) years, including the central velocity disper-  
sion, the central luminosity density, the core radius and the  
core binary fraction. It follows that the rate of BS  
formation is constant for the time-scale of interest. This time-scale  
is comparable to the half-mass relaxation time but much  
longer than the central relaxation time for the majority of  
the clusters in our sample (Harris et al. 1996). This  
suggests that core parameters such as the central density and  
the core radius will typically change in a time \(\tau_{BS}\) since the  
time-scale on which these parameters vary is the central  
relaxation time (Heggie & Hut 2003). Therefore, our  
assumption of constant rates and cluster parameters is not strictly  
correct, however it provides a suitable starting point for our  
model. We will discuss the implications of our assumption  
of time-independent cluster properties and rates in Section 2.

In the following sections, we discuss each of the remaining  
terms in Equation 1.

### 3.1.1 Stellar Collisions

We can approximate the number of BSs formed in the last  
\(\tau_{BS}\) years from collisions during dynamical encounters as:

\[
N_{\text{coll}} = f_{1+1} N_{1+1} + f_{1+2} N_{1+2} + f_{2+2} N_{2+2},
\]

where \(N_{1+1}, N_{1+2}\) and \(N_{2+2}\) are the number of single- 
single, single-binary and binary-binary encounters, respectiv- 
ely. The terms \(f_{1+1}, f_{1+2}\) and \(f_{2+2}\) are the fraction of  
1+1, 1+2 and 2+2 encounters, respectively, that will produce a  
BS in the last \(\tau_{BS}\) years. We treat these three variables as  
free parameters since we do not know what fraction of collis- 
sion products will produce BSs (i.e. stars with an appro- 
appropriate combination of colour and brightness to end up in the BS  
region of the CMD), nor do we know what fraction of 1+2 and 2+2  
encounters will result in a stellar collision. Numerical  
scattering experiments have been performed to study the  
outcomes of 1+2 and 2+2 encounters (e.g. Hut & Bahcall  
1983, McMillan 1986, Fregeau et al. 2004), however a large  
fraction of the relevant parameter space has yet to be  
explored.

In terms of the core radius \(r_c\) (in parsecs), the central  
number density \(n_0\) (in pc\(^{-3}\)), the root-mean-square velocity  
\(v_m\) (in km s\(^{-1}\)), the average stellar mass \(m\) (in M_\odot)  
and the average stellar radius \(R\) (in R_\odot), the mean time-scale be- 
tween single-single collisions in the core of a GC is (Leonard  
1983):

\[
\tau_{1+1} = 1.1 \times 10^{10} (1 - f_b)^{-2} \left(\frac{1pc}{r_c}\right)^3 \left(\frac{10^3 pc^{-3}}{n_0}\right)^2 \left(\frac{v_m}{5 km/s}\right) \left(\frac{0.5 M_\odot}{m}\right) \left(\frac{0.5 R_\odot}{R}\right) \text{ years}
\]

The additional factor \((1-f_b)^{-2}\) comes from the fact that we  
are only considering interactions between single stars and  
the central number density of single stars is given by  
\(1-f_b n_0\), where \(f_b\) is the binary fraction in the core (i.e. the  
fraction of objects that are binaries). For our chosen mass,  
we assume a corresponding average stellar radius using the  
relation \(M/M_\odot = R/R_\odot\) (Iben 1991). The number of 1+1  
collisions expected to have occurred in the last \(\tau_{BS}\) years is  
then approximated by:

\[
N_{1+1} = \frac{\tau_{BS}}{\tau_{1+1}}
\]

The rate of collisions between single stars and binaries,  
as well as between two binary pairs, can be roughly approxi- 
imated in the same way as for single-single encounters  
(Leonard 1983, Sigurdsson & Phinney 1993, Bacon et al.  
1996, Fregeau et al. 2004). We adopt the time-scales derived  
in Leigh & Sills (2011) for the average times between 1+2  
and 2+2 encounters. These are:

\[
\tau_{1+2} = 3.4 \times 10^7 f_b^{-1} (1 - f_b)^{-1} \left(\frac{1pc}{r_c}\right)^3 \left(\frac{10^3 pc^{-3}}{n_0}\right)^2 \left(\frac{v_m}{5 km/s}\right) \left(\frac{0.5 M_\odot}{m}\right) \left(\frac{1AU}{a}\right) \text{ years}
\]

and

\[
\tau_{2+2} = 1.3 \times 10^7 f_b^{-2} \left(\frac{1pc}{r_c}\right)^3 \left(\frac{10^3 pc^{-3}}{n_0}\right)^2 \left(\frac{v_m}{5 km/s}\right) \left(\frac{0.5 M_\odot}{m}\right) \left(\frac{1AU}{a}\right) \text{ years},
\]

where \(a\) is the average binary semi-major axis in the core in AU  
and we have assumed that the average binary mass is  
equal to twice the average single star mass. The numbers of  
1+2 and 2+2 encounters expected to have occurred in the last  
\(\tau_{BS}\) years are given by, respectively:

\[
N_{1+2} = \frac{\tau_{BS}}{\tau_{1+2}}
\]

and

\[
N_{2+2} = \frac{\tau_{BS}}{\tau_{2+2}}.
\]

The outcomes of 1+2 and 2+2 encounters will ultimate-  
ly contribute to the evolution of the binary fraction in  
the core. How and with what frequency binary hardening/  
softening as well as capture, exchange and ionization  
interactions affect the binary fraction in the dense cores  
of GCs is currently a subject of debate (e.g. Ivanova et al.  
2007).
3.1.2 Binary Star Evolution

Although we do not know the rate of BS formation from binary star evolution, we expect a general relation of the form $N_{\text{bin}} = \tau_{BS}/\tau_{\text{tot}}$ for the number of BSs produced from binary mergers in the last $\tau_{BS}$ years, where $\tau_{\text{tot}}$ is the average time between BS formation events due to binary star evolution. We can express the number of BSs formed from binary star evolution in the last $\tau_{BS}$ years as:

$$N_{\text{bin}} = f_{\text{tot}} t_{\text{tot}} N_{\text{core}},$$

where $N_{\text{core}}$ is the total number of objects (i.e. single and binary stars) in the core and $f_{\text{tot}}$ is the fraction of binary stars that evolved internally to form BSs within the last $\tau_{BS}$ years. We treat $t_{\text{tot}}$ as a free parameter since it is likely to depend on the mass-ratio, period and eccentricity distributions characteristic of the binary populations of evolved GC cores, for which data is scarce at best.

3.1.3 Migration Into and Out of the Core

Due to the relatively small sizes of the BS populations considered, the migration of BSs into or out of the core is an important consideration when calculating the predicted numbers. In other words, we are dealing with relatively small number statistics and every blue straggler counts. In order to approximate the number of stars in the core as a function of time, two competing effects need to be taken into account: (1) mass stratification/segregation (or, equivalently, dynamical friction) and (2) kicks experienced during dynamical interactions. These effects are accounted for with the variables $N_{\text{in}}$ and $N_{\text{out}}$ in Equation (1), respectively.

Blue stragglers are among the most massive stars in clusters (e.g. Shara et al. 1993; van den Berg et al. 2001; Mathieu & Geller 2009), so they should typically be heavily mass segregated relative to other stellar populations (e.g. Spitzer 1969; Shara et al. 1993; King, Sosin & Cool 1992). The time-scale for this process to occur can be approximated using the dynamical friction time-scale ($\tau_{\text{dyn}}$) (Binney & Tremaine 1987):

$$\tau_{\text{dyn}} = \frac{3}{4\ln A G^2 (2\pi)^{1/2}} \frac{\sigma(r)^3}{m_{BS} \rho(r)},$$

where $\sigma(r)$ and $\rho(r)$ are, respectively, the velocity dispersion and stellar mass density at the given distance from the cluster centre $r$. Both $\sigma(r)$ and $\rho(r)$ are found from single-mass King models (Sigurdsson & Phinney 1993), which are fit to each cluster using the concentration parameters provided in McLaughlin & van den Marel (2003). The Coulomb logarithm is denoted by $\Lambda$, and we adopt a value of $\ln \Lambda \sim 10$ throughout this paper (e.g. Spitzer 1969; Heggie & Hut 2003). If $\tau_{\text{dyn}} > \tau_{BS}$ at a given distance from the cluster centre, then any BSs formed at this radius in the last $\tau_{BS}$ years will not have had sufficient time to migrate into the core by the current cluster age. The maximum radius $r_{\text{max}}$ at which BSs can have formed in the last $\tau_{BS}$ years and still have time to migrate into the core via dynamical friction is given by setting $\tau_{\text{dyn}} = \tau_{BS}$. Therefore, $N_{\text{in}}$ depends only on the number of BSs formed in the last $\tau_{BS}$ years at a distance from the cluster centre smaller than $r_{\text{max}}$.

In order to estimate the contribution to $N_{BS}$ in Equation (1) from BSs formed outside the core, we calculate the number of BSs formed in radial shells between the cluster centre and $r_{\text{max}}$. Each shell is taken to be one core radius thick, and we calculate the contribution from each formation mechanism in every shell. This is done by assuming a constant (average) density and velocity dispersion in each shell. Specifically, we estimated the density and velocity dispersion at the half-way point in each shell using our single-mass King models, and used these to set average values. The number of BSs expected to have migrated into the core within the last $\tau_{BS}$ years can be written:

$$N_{\text{in}} = \sum_{i=2}^{N} \left( f_{i+1} N_{(1+1),i} + f_{1+2} N_{(1+2),i} + f_{2+2} N_{(2+2),i} \right) + f_{\text{tot}} N_{(\text{bin}),i} \times \left( 1 - \frac{\tau_{\text{dyn}}}{\tau_{BS}} \right),$$

where $i = 1$ refers to the core, $i = 2$ refers to the shell immediately outside the core, etc. and $N$ is taken to be the integer nearest to $r_{\text{max}}/r_c$. We let the terms with $N_{(1+1),i}$, $N_{(1+2),i}$, $N_{(2+2),i}$ and $N_{(\text{bin}),i}$ represent the number of BSs formed in shell $i$ from single-single collisions, single-binary collisions, binary-binary collisions and binary star evolution, respectively. The time-scale for dynamical friction in shell $i$ is denoted by $\tau_{\text{dyn},i}$, and the factor $(1 - \tau_{\text{dyn},i}/\tau_{BS})$ is included to account for the fact that we are assuming a constant formation rate for BSs, so that not every BS formed in shells outside the core will have sufficient time to fall in by the current cluster age.

It is typically the least massive stars that are ejected from 1+2 and 2+2 interactions as single stars (e.g. Sigurdsson & Phinney 1993). Combined with conservation of momentum, this suggests that BSs are the least likely to be ejected from dynamical encounters with velocities higher than the central velocity dispersion due to their large masses. This has been confirmed by several studies of numerical scattering experiments (e.g. Hut & Bahcall 1983; Fregeau et al. 2004). Based on this, we expect that $N_{\text{out}}$
3.2 Statistical Comparison with Observations

Our model contains 4 free parameters, which correspond to the fraction of outcomes that produce a blue straggler for each formation mechanism (1+1 collisions, 1+2 collisions, 2+2 collisions, and binary star evolution). These are the \( f \) values described in the previous section: \( f_{1+1}, f_{1+2}, f_{2+2}, f_{\text{max}} \). We assume that these values are constant throughout each cluster, and are also constant between clusters. By fitting the predictions of our model to the observational data, we can determine best-fit values for each of these \( f \) parameters, and therefore make predictions about which blue straggler formation processes are more important.

In order to determine the best values for these \( f \) parameters, we need an appropriate statistical test. For this, we follow the approach of Verbunt, Poole & Bassel (2008). The number of BSs seen in the core of a globular cluster can be described by Poisson statistics. In particular, the probability of observing \( N \) sources when \( \mu \) are expected is:

\[
P(N, \mu) = \frac{\mu^N}{N!} e^{-\mu}
\]

We can calculate a probability for each cluster, and then calculate an overall probability \( P' \) for the model by multiplying the individual \( P \) values. We can then vary the \( f \) values to maximize this value.

In practice, these \( P \) values are typically of order ten percent per cluster, and with 35 clusters, the value of \( P' \) quickly becomes extremely small. Therefore we chose to work with a modified version of this value: the deviance of our model to the saturated model. A saturated model is one in which the observed number of sources is exactly equal to the expected number in each cluster. In other words, this is the best that we can possibly do. However, because of the nature of Poisson statistics, the probability \( P \) of such a model (calculated by setting \( N = \mu \) in Equation 12) is not equal to 1, but in fact has some smaller value. For the numbers of blue stragglers in our clusters, the \( P \) values for the saturated model run from 0.044 to 0.149, and the value of \( P' \) is \( 2.08 \times 10^{-42} \).

The deviance of any model from the saturated model is given by

\[
D = 2.0(\ln(P'_{\text{saturated}}) - \ln(P'_{\text{model}}))
\]

The model which minimizes this quantity will be our best-fit model. Ideally, the scaled deviance \( (D/(N_{\text{data}} - N_{\text{parameters}})) \) should be equal to 1 for a best fit. Given the simplicity of our model, we expect that our values will not provide this kind of agreement, and we simply look for the model which provides the minimum of the scaled deviance.

4 RESULTS

In this section, we present the results of comparing our model predictions to the observations. After presenting the results for a constant core binary fraction for all clusters, we explore the implications of adopting a core binary fraction that depends on the cluster luminosity, as reported in Sollima et al. (2007) and Milone et al. (2008).

4.1 Initial Assumptions

The predictions of our model for our initial choice of assumptions are shown in Figure 1. These numbers are plotted against the total stellar mass in the core along with the number of BSs observed in the core (filled triangles). We plot both the total number of BSs predicted to have formed within \( r_{\text{max}} \) in the last \( \tau_{BS} \) years (\( N_{BS} \) in Equation 14, open circles), as well as the total number formed only in the core (\( N_{\text{core}} + N_{\text{bin}} \); small filled circles). Upon comparing \( N_{BS} \) to the observed number of BSs in the core, the best-fitting model parameters predict that most BSs are formed from binary star evolution, with a small contribution from 2+2 collisions being needed in order to obtain the best possible match to the observations. The ideal contribution from 2+2 collisions constitutes at most a few percent of the predicted total for most of the clusters in our sample. Even for our best-fitting model parameters, our initial choice of assumptions predicts too few BSs in clusters with small core masses.

4.2 Binary Fraction

We tried changing our assumption of a constant \( f_b \) for all clusters to one for which the core binary fraction varies with the total cluster magnitude. First, we adopted a dependence of the form:

\[
f_b = 0.13M_V + 1.07,
\]

where \( M_V \) is the total cluster V magnitude. This relation comes from fitting a line of best-fit to the observations of Sollima et al. (2007), who studied the binary fractions in a sample of 13 low-density GCs (we calculated an average of columns 3 and 4 in their Table 3 and used these binary fractions to obtain Equation 14). In order to prevent negative binary fractions, we impose a minimum binary fraction of \( f_{\text{min}}^{\text{BS}} = 0.01 \). In other words, we set \( f_{b} = f_{\text{min}}^{\text{BS}} \) if Equation 14 gives a binary fraction less than \( f_{\text{min}}^{\text{BS}} \). As before, we adopt an average semi-major axis of 2 AU. The results of this comparison are presented in Figure 2. As in Figure 1, both the numbers of observed (filled triangles) and predicted (open circles) BSs in the core are plotted versus the total stellar mass in the core. Once again, the predicted numbers include all BSs formed within \( r_{\text{max}} \) in the last \( \tau_{BS} \) years. The best-fitting model parameters for this comparison suggest that both single-single collisions and binary star evolution are significant contributors to BS formation. Single-single collisions contribute up to several tens of a percent of the predicted total in several clusters. We obtain a deviance in this case that is significantly larger than that obtained for the best-fitting model parameters assuming a constant \( f_b \) of 10% for all clusters. Equation 14 gives higher binary fractions in low-mass clusters relative to our initial assumption of a constant \( f_b \). This increases the number of BSs formed from binary star
evolution and improves the agreement between our model predictions and the observations in low-mass clusters. This is consistent with the results of Sollima et al. (2008). However, adopting Equation (14) for f_{min} also causes our model to under-predict the number of BSs in several high-mass clusters.

The best fit to the observations is found by adopting the relation for f_{b} provided in Equation (14) and setting f_{min} = 0.1 (however we note that a comparably good agreement is found with a slightly lower f_{min} = 0.05). This improves the agreement between our model predictions and the observations by increasing the number of BSs formed from binary star evolution in massive clusters. The result is a good agreement between our model predictions and the observations in both low- and high-mass cores, as shown in Figure 3. In this case, the best-fitting model parameters yield the lowest deviance of any of the assumptions so far considered. These best-fitting values suggest that most BSs are formed from binary star evolution, with a small contribution from 2+2 collisions being needed in order to obtain the best possible match to the observations. Similarly to what was found for our initial assumptions, the ideal contribution from 2+2 collisions constitutes at most a few percent of the predicted total for most of the clusters in our sample. On the other hand, if we change our imposed minimum binary fraction to f_{min} = 0.05 we find that a non-negligible (i.e. up to a few tens of a percent) contribution from single-single collisions is needed in several clusters to obtain the best possible agreement with the observations (which is very nearly as good as was found using f_{min} = 0.1). All of this shows that, although binary star evolution consistently dominates BS formation in our best-fitting models, at least some contribution from collisions (whether it be 1+1 or 2+2 collisions, or some combination of 1+1, 1+2 and 2+2 collisions) also consistently improves the agreement with the observations. Moreover, it is interesting to note that an improved agreement with the observations could alternatively be obtained if we keep f_{min} = 0.01 but all or some of f_{1+1}, f_{1+2} and f_{2+2} increase with increasing cluster mass. This would also serve to improve the agreement at the high-mass end. We will return to this in Section 5.

Finally, we also tried adopting the observed dependence of f_{b} on M_{V} reported in Milone et al. (2008), who also found evidence for an anti-correlation between the core binary fraction and the total cluster mass. Despite this change, we consistently find that our results are the same as found when using the empirical binary fraction relation provided in Equation (14).

4.3 Average BS Lifetime

We also tried changing our assumption for the average BS lifetime. We explored a range of plausible lifetimes based on values found throughout the literature. Specifically, we explored the range 0.5-5 Gyrs. We find that at the low end of this range, our model fits become increasingly poor. This is because lower values for \( \tau_{BS} \) correspond to smaller values for...
Figure 3. The number of BSs predicted in the cluster core using the binary fractions of Sollima et al. (2007) with $f_{b}^{min} = 0.1$ plotted against the total stellar mass contained within the core. The symbols used to indicate the observed and predicted numbers are the same as in Figure 1. The predicted numbers correspond to the best-fitting model parameters, which are $f_{1+1} = 0$, $f_{1+2} = 0$, $f_{2+2} = 3.6 \times 10^{-3}$, and $f_{mt} = 1.4 \times 10^{-3}$.

$r_{max}$ and decrease the term $(1 - t_{(dyn)},i/\tau_{BS})$ in Equation 11. This reduces the contribution to the total predicted numbers from BSs formed outside the core that fall in via dynamical friction. Conversely, our model fits improve for $r_{max}$ outside the core since this would also serve to increase $N_{in}$ from $N_{bs}$ (e.g. Sollima et al. 2007). It is important to note, however, that this same effect can be had by increasing the number of BSs formed outside the core, since this would also serve to increase $N_{in}$ in Equation 11. This can be accomplished by, for instance, increasing the binary fraction outside the core (which would increase the number of BSs formed from binary star evolution outside the core that migrate in due to dynamical friction) relative to inside the core. This seems unlikely, however, given that observations of low-density globular clusters and open clusters suggest that their binary fractions tend to drop off rapidly outside the core (e.g. Sollima et al. 2007). We will return to this issue in Section 5.

The best possible match to the observations is found by adopting an average BS lifetime of 5 Gyrs along with the relation for $f_{b}$ provided in Equation 11 with $f_{b}^{min} = 0.1$. The predictions of our model are shown in Figure 4 for these best-fitting model parameters. As shown, the agreement between our model predictions and the observed numbers is excellent. The best agreement is found by adopting an average BS lifetime of 5 Gyrs and the relation for the cluster binary fraction provided in Equation 14 with $f_{b}^{min} = 0.1$. The symbols used to indicate the observed and predicted numbers are the same as in Figure 1. The best-fitting model parameters used to calculate the predicted numbers are $f_{1+1} = 0$, $f_{1+2} = 0$, $f_{2+2} = 1.6 \times 10^{-3}$, and $f_{mt} = 9.9 \times 10^{-4}$. The agreement with the observations is excellent for these best-fitting values.

Figure 4. The predicted number of BSs plotted versus the total stellar mass in the core for the best-fitting model parameters found using an average BS lifetime of 5 Gyrs and the relation for the cluster binary fraction provided in Equation 11 with $f_{b}^{min} = 0.1$. The symbols used to indicate the observed and predicted numbers are the same as in Figure 1. The best-fitting model parameters used to calculate the predicted numbers are $f_{1+1} = 0$, $f_{1+2} = 0$, $f_{2+2} = 1.6 \times 10^{-3}$, and $f_{mt} = 9.9 \times 10^{-4}$. The agreement with the observations is excellent for these best-fitting values.

4.4 Migration

In order to explore the sensitivity of our results to our assumption for $r_{max}$, we also tried setting $N_{in}$ equal to the total number of BSs expected to form within 10 $r_{c}$ for all clusters. For comparison, for an average BS lifetime of $\tau_{BS} = 1.5$ Gyrs, $r_{max}$ ranges from 2 - 15 $r_{c}$ for the clusters in our sample. Despite implementing this change, our results remained the same. This is because the largest contribution to the total number of BSs comes from those BSs formed in the first few shells immediately outside the core that migrate in due to dynamical friction.

Several GCs have been reported to show evidence for a decrease in their binary fractions with increasing distance from the cluster centre (e.g. Sollima et al. 2007; Davis et al. 2008). This effect is often significant, with binary fractions decreasing by up to a factor of a few within only a few core radii from the cluster centre. Based on this, our assumption that $f_{b}$ is independent of the distance from the cluster centre likely results in an over-estimate of the true binary fraction at large cluster radii. In order to quantify the possible implications of this for our results, we tried setting $f_{b} = 0$ for all shells outside the core. Although this assumption is certainly an under-estimate for the true binary fraction outside the core, our results remain the same (albeit the agreement with the observations is considerably worse than for most of our previous model assumptions). Once again, the best-fitting model parameters suggest that most BSs are
formed from binary star evolution, with a non-negligible (i.e. up to a few tens of a percent in some clusters) contribution from binary-binary collisions. Our results indicate that, if the binary fraction is negligible outside the core, then the contribution from BSs that migrate into the core due to dynamical friction is also negligible. This is because the time between 1+1 collisions increases rapidly outside the core, and every other BS formation mechanism requires binary stars to operate.

We also explored the effects of assuming a non-zero value for \( N_{out} \) in Equation 1 by imparting a constant kick velocity to all BSs at birth. Using conservation of energy, we calculated the cluster radius to which BSs should be kicked upon formation, and used the time-scale for dynamical friction at the kick radius to calculate the fraction of BSs expected to return back into the core within a time \( t_{BS} \).

Regardless of our assumption for the kick velocity, this did not improve the deviance for any of our best-fitting model parameters.

### 4.5 Average Binary Semi-Major Axis

We investigated the dependence of our results on our assumption for the average binary semi-major axis. However, this had a very small effect on our results. This is because only \( N_{1+2} \) and \( N_{2+2} \) depend on the average semi-major axis, and neither of these terms dominated BS production regardless of our model assumptions. Only \( f_{2+2} \) is non-zero for our best-fitting models however, as before, it consistently suggests that far fewer BSs should be formed from 2+2 collisions than from binary star evolution.

### 5 SUMMARY & DISCUSSION

In this paper, we have presented an analytic model to investigate BS formation in globular clusters. Our model predicts the number of BSs in the cluster core at the current cluster age by estimating the number that should have either formed there from stellar collisions and binary star evolution, or migrated in via dynamical friction after forming outside the core. We have compared the results of our model predictions for a variety of input parameters to observed BS numbers in 35 GCs taken from the catalogue of \cite{Leigh,Sills,Knigge} (2011).

What has our model told us about BS formation in dense cluster environments? The agreement between the predictions of our model and the observations is excellent if we assume that:

- Binary star evolution dominates BS formation, however at least some contribution from 2+2 collisions (most of which occur in the core) must also be included in the total predicted numbers. Although it is clear that including a contribution from dynamical encounters gives the best possible match to the observations, it is not clear how exactly this is accomplished in real star clusters. Does the cluster dynamics increase BS numbers via direct collisions? Do dynamical interactions somehow modify primordial binaries to initiate more mass-transfer events? We will return to this point below.

- The binary fraction in the core is inversely correlated with the total cluster luminosity, similar to the empirical relations found by \cite{Sollima} and \cite{Milone} (2008). We also require a minimum core binary fraction of 5% – 10%. The inverse dependence of \( f_1 \) on the total cluster mass contributes to a better agreement with the observations at the low-mass end of the distribution of cluster masses, whereas the imposed condition that \( f_{1+2} = 0.05-0.1 \) contributes to improving the agreement at the high-mass end.

- BSs formed outside the core that migrate in by the current cluster age contribute to the total predicted numbers.

- The average BS lifetime is roughly a few (\( \sim 3-5 \)) Gyr, since this increases the fraction of BSs formed outside the core that will have sufficient time to migrate in due to dynamical friction.

Our model can only provide a reasonable fit to the observations for all cluster masses if we assume that the cluster binary fraction is inversely proportional to the total cluster mass. It is interesting to consider the possibility that such an inverse proportionality could arise as a result of the fact that the rate of two-body relaxation is also inversely proportional to the cluster mass \cite{Spitzer}. Consequently, since binaries tend to be the most massive objects in GCs, they should quickly migrate into the core in low-mass clusters, contributing to an increase in the core binary fraction over time \cite{Freguea}. This process should operate on a considerably longer time-scale in very massive GCs since the time-scale for two-body relaxation is very long. Mass segregation could then contribute to the observed sub-linear dependence of BS numbers on the core masses by acting to preferentially migrate the binary star progenitors of BSs into the cores of low-mass clusters. This is one example of how a direct link could arise between the observed properties of BS populations and the dynamical histories of their host clusters. Although this scenario is interesting to consider, we cannot rule out the possibility that an anti-correlation between the core binary fraction and the total cluster mass could be a primordial property characteristic of GCs at birth.

When interpreting our results, it is important to bear in mind that binary star evolution and dynamical interactions involving binaries may not always contribute to BS formation independently. For example, dynamics could play an important role in changing the distribution of binary orbital parameters so that mass-transfer occurs more commonly in some clusters. One way to perhaps compensate for this effect would be to include a factor of \( 1/a \) (where \( a \) is the average binary semi-major axis) in Equation 9. This would serve to account for the fact that we might naively expect clusters populated by more close binaries to be more likely to have a larger fraction of their binary populations undergo mass-transfer. This does not, however, guarantee that more BSs will form since our poor understanding of binary star evolution prevents us from being able to predict the outcomes of these mass-transfer events, and whether or not they will form BSs. Moreover, little is known about the distribution of orbital parameters characteristic of the binary populations in globular clusters, and how they are typically modified by the cluster dynamics. For these reasons, the interpretation
of our results must be done with care in order to ensure that reliable conclusions can be drawn.

In general, our results suggest that binary stars play a crucial role in BS formation in dense GCs. In order to obtain the best possible agreement with the observations, an enhancement in BS formation from dynamical encounters is required in at least some clusters relative to what is expected by assuming a simple population of binaries evolving in isolation. It is not clear from our results, however, how exactly this occurs in real star clusters. Dynamics could enhance BS formation directly by causing stellar collisions, or this could also occur indirectly if the cluster dynamics somehow induces episodes of mass-transfer by reducing the orbital separations of binaries. But in which clusters is BS formation the most strongly influenced by the cluster dynamics? Unfortunately, no clear trends have emerged from our analysis that provide a straightforward answer to this question. However, our results are consistent with dynamical interactions playing a more significant role in more massive clusters (although this does not imply that the cluster dynamics does not also contribute in low-mass clusters). This could be due to the fact that more massive clusters also tend to have higher central densities (e.g. Djorgovski & Meylan 1994), and therefore higher collision rates. This picture is, broadly speaking, roughly consistent with the results of Davies, Piotto & De Angeli (2004). These authors considered the observed dependence (or lack thereof) of BS numbers on the total cluster masses presented in Piotto et al. (2004), and suggested that primordial binary evolution and stellar collisions dominate BS production in low- and high-mass clusters, respectively.

Our model neglects the dynamical evolution of GCs and the resulting changes to their global properties, including the central density, velocity dispersion, core radius and binary fraction. As a young cluster evolves, dynamical processes like mass segregation and stellar evaporation tend to result in a smaller, denser core. Within a matter of a few half-mass relaxation times, a gravothermal instability has set in and the collapse ensues on a time-scale determined by the rate of heat flow out of the core (e.g. Spitzer 1987). We are focussing on the last \( \tau_{BS} \) years of cluster evolution, a sufficiently late period in the lives of most GCs that gravothermal collapse will have long since taken over as the primary driving force affecting the stellar concentration in the core. Most of the GCs in our sample should currently be in a phase of core contraction (Fregeau, Ivanova & Rasio 2008; Gieles, Heggie & Zhao 2011), and their central densities and core radii should have been steadily decreasing over the last \( \tau_{BS} \) years. Therefore, by using the currently observed central cluster parameters and assuming that they remained constant over the last \( \tau_{BS} \) years, we have effectively calculated upper limits for the encounter rates. This could suggest that we have over-estimated the importance of dynamical interactions for BS formation. On the other hand, some theoretical models of GC evolution suggest that the hard binary fraction in the core of a dense stellar system will generally increase with time (e.g. Hurley et al. 2002; Fregeau, Ivanova & Rasio 2004). This can be understood as an imbalance between the migration of binaries into the core via mass segregation and the destruction of binaries in the core via both dynamical encounters and their internal evolution. This could suggest that our estimate for the number of BSs formed from binary star evolution should also be taken as an upper limit. The key point is that GC evolution can act to increase the number of BSs in the cluster core via several different channels. The effects we have discussed should typically be small, however, since \( \tau_{BS} \) is much shorter than the cluster age (De Angeli et al. 2003) and any changes to global cluster properties that occur during this time will often be small.

Our model adopts the same values for all free parameters in all clusters. In particular, this is the case for several global cluster properties, including the average stellar mass, the average BS mass, the average BS lifetime, and the average binary semi-major axis. With the exception of the average stellar mass, there is no conclusive observational or theoretical evidence to indicate that these parameters should differ from cluster-to-cluster, although we cannot rule out this possibility. For instance, the distribution of binary orbital parameters could depend on cluster properties like the total mass, density or velocity dispersion (e.g. Sigurdsson & Phinney 1993). In particular, the central velocity dispersion should be higher in more massive GCs (e.g. Djorgovski & Meylan 1994), which should correspond to a smaller binary orbital separation for the hard-soft boundary. This could contribute to massive GCs tending to have smaller average binary orbital separations since soft binaries should not survive for long in the dense cores of GCs (e.g. Heggie & Hut 2003). In turn, this could affect the occurrence of mass-transfer events, or of mergers during 1+2 and 2+2 encounters. This last point follows from the fact that numerical scattering experiments have shown that the probability of mergers occurring during 1+2 and 2+2 interactions increases with decreasing binary orbital separation (e.g. Fregeau et al. 2004). Both the average stellar mass and the average BS mass (and hence lifetime) could also depend on the total cluster mass, as discussed in Leigh, Sills & Knigge (2004) and Leigh, Sills & Knigge (2011).

We have also neglected to consider the importance of triples for BS formation throughout our analysis (e.g. Perets & Fabrycky 2009) since we are unaware of any observations of triples in GCs in the literature. Interestingly, however, our results for binary star evolution can be generalized to include the internal evolution of triples since they should have the same functional dependence on the core mass (i.e. \( N_{bc} \propto f_{1} M_{core} \)), where \( N_{bc} \) is the number of BSs formed from triple star evolution and \( f_{1} \) is the fraction of objects that are triples).

Finally, our model assumes that several parameters remain constant as a function of the distance from the cluster centre, including the binary fraction and the average semi-major axis. However, observations of GCs suggest that their binary fractions could fall off rapidly outside the core (e.g. Sollima et al. 2007; Davis et al. 2008). Our results suggest that, if the binary fraction is negligible outside the core, then the contribution from BSs that migrate into the core due to dynamical friction is also negligible. This is because the time between 1+1 collisions increases rapidly outside the core, and every other BS formation mechanism requires binary stars to operate. On the other hand, the presently observed binary fraction outside the core could be low as a result of binaries having previously migrated into the core due to dynamical friction (e.g. Fregeau, Ivanova & Rasio 2004). If
these are the binary star progenitors of the BSs currently populating the core, then dynamical friction remains an important effect in determining the number of BSs currently populating the core.

Despite all of these simplifying assumptions, we have shown that our model can reproduce the observations with remarkable accuracy. Notwithstanding, the effects we have discussed could be contributing to cluster-to-cluster differences in the observed BS numbers. Our model provides a well-suited resource for addressing the role played by these effects, however future observations will be needed in order to obtain the desired constraints (e.g. binary fractions, distributions of binary orbital parameters, etc.).

ACKNOWLEDGMENTS

We would like to thank Ata Sarajedini, Aaron Dotter and Roger Cohen for providing the observations to which we compared our model predictions, as well as for providing a great deal of guidance in analyzing the data. We would also like to thank Evert Glebbeek, Bob Mathieu and Aaron Geller for useful discussions. This research has been supported by NSERC and OGS.

REFERENCES

Anderson J., Sarajedini A., Bedin L. R., King I. R., Piotto G., Reid I. N., Siegel M., Majewski S. R., Paust N. E. Q., Aparicio A., Milone A. P., Chaboyer B., Rosenberg A. 2008, AJ, 135, 2055

Andrievsky S. M., Schönberner D., Drilling J. S. 2000, A&A, 356, 517

Bacon D., Sigurdsson S., Davies M. B. 1996, MNRAS, 281, 830

Binney J., Tremaine S. 1987, Galactic Dynamics (Princeton: Princeton University Press)

Cool A. M., Bolton A. S. 2002, in ASP Conference Series 263, Stellar Collisions, Mergers and their Consequences, ed. M. M. Shara (San Francisco: ASP), 163

Davies M. B., Piotto G., De Angeli F. 2004, MNRAS, 348, 129

Davis D. S., Richer H. B., Anderson J., Brewer J., Hurley J., Kalirai J. S., Rich R. M., Stetson P. B. 2008, AJ, 135, 2155

De Angeli F., Piotto G., Cassisi S., Busso G., Recio-Blanco A., Salaris M., Aparicio A., Rosenberg A. 2005, AJ, 130, 116

Derivisoghoh, Tout C. A., Ibanoglu C. 2010, MNRAS, 406, 1071

Djorgovski S., Meylan G. 1994, ApJ, 108, 1292

Ferraro F. R., Paltrinieri B., Fusi Pecci F., Cacciari C., Dorman B., Rood R. T., Buonanno R., Corsi C. E., Burgarella D., Laget M. 1997, A&A, 324, 915

Ferraro F. R., Paltrinieri B., Rood R. T., Dorman B. 1999, ApJ, 522, 983

Ferraro F. R., Beccari G., Rood, R. T., Bellazzini M., Sills A., Sabbas E. 2004, ApJ, 603, 127

Fregeau J. M., Cheung P., Portegies Zwart S. F., Rasio F. A. 2004, MNRAS, 352, 1

Fregeau J. M., Ivanova N., Rasio F. A. 2009, ApJ, 707, 1533

Geller A. M., Mathieu R. D., Harris H. C., McClure R. D., 2009, AJ, 137, 3743

Gieles M., Heggie D., Zhao H. 2011, MNRAS, accepted

Glebbeek E., Pols O. R. 2008, A&A, 488, 1017

Guethakurta P., Webster Z. T., Yanny B., Schneider D. P., Bahcall J. N. 1998, AJ, 116, 1757

Harris W. E. 1996, AJ, 112, 1487; 2010 update

Heggie D. C., Hut P. 2003, The Gravitational Million-Body Problem: A Multidisciplinary Approach to Sar Cluster Dynamics (Cambridge: Cambridge University Press)

Hurley J. R., Pols O. R., Aarseth S. J., Tout C. A. 2005, MNRAS, 363, 293

Hurley J. R., Aarseth S. J., Shara M. M. 2007, ApJ, 665, 707

Hut P., Bahcall J. N. 1983, ApJ, 268, 319

Iben I. Jr. 1991, ApJS, 76, 55

Ivanova N., Belczynski K., Fregeau J. M., Rasio F. A. 2005, MNRAS, 358, 572

King I. R., Cusin C., Cool A. M. 1995, ApJ, 452, L33

Kneige C., Leigh N., Sills A. 2009, Nature, 457, 288

Latham D. W., 2005, Highlights of Astronomy, 14, 444

Lanzi B., Dallemand E., Petosa S., Ferraro F. R., Rood R. T., Sollima A. 2007, ApJ, 670, 1065

Leigh N. W., Sills A., Knigge C. 2007, ApJ, 661, 210

Leigh N. W., Sills A., Knigge C. 2008, ApJ, 678, 564

Leigh N. W., Sills A., Knigge C. 2009, MNRAS, 399, L179

Leigh N. W., Sills A. 2011, MNRAS, 410, 2370

Leigh N. W., Sills A., Knigge C. 2011, MNRAS, accepted

Leonard P. J. T. 1989, AJ, 98, 217

Leonard P. J. T., Linnell A. P. 1992, AJ, 103, 1928

Leonard P. J. T., Livio M. 1995, ApJ, 447, L121

Li L., Zhang F. 2006, MNRAS, 369, 2001

Lombardi J. C., Warren J. S., Rasio F. A., Sills A., Warren R. A. 2002, ApJ, 568, 939

Mapelli M., Sigurdsson S., Colpi M., Ferraro F. R., Bosanti A., Rood R. T., Sills A., Beccari G. 2004, ApJ, 605, L29

Mapelli M., Sigurdsson S., Ferraro F. R., Colpi M., Bosanti A., Lanzoni B. 2006, MNRAS, 373, 361

Mathieu R. D., Geller A. R. 2009, Nature, 462, 1032

McCrea W. H. 1964, MNRAS, 128, 147

McLaughlin D. E., van der Marel R. P. 2005, ApJS, 161, 304

McMillan S. L. W. 1986, ApJ, 306, 552

Milone A. P., Piotto G., Bedin L. R., Sarajedini A. 2008, MmSAI, 79, 623

Piotto G., De Angeli F., King I. R., Djorgovski S. G., Bonn G., Cassisi S., Meylan G., Recio-Blanco A., Rich R. M., Davies M. B. 2004, ApJ, 604, L109

Perets H. B., Fabrycky D. C. 2009, ApJ, 697, 1048

Rubenstein E. P., Bailyn C. D. 1995, MmSAI, 79, 623

Sandage A. 1953, AJ, 58, 61

Sandquist E. L., Bolte M., Hernquist L. 1997, ApJ, 477, 335

Sarajedini A., Bedin L. R., Chaboyer B., Dotter A., Siegel M., Anderson J., Aparicio A., King I., Majewski S., Marin-Franch A., Piotto G., Reid I. N., Rosenberg A., Steven M. 2007, AJ, 133, 1658

Sigurdsson S., Phinney E. S. 1993, ApJ, 415, 631

Saviane I., Piotto G., Fagotto F., Zaggia S., Capaccioli M., Aparicio A. 1998, A&A, 333, 479
Shara M. M., Drissen L., Bergeron L. E., Paresce F. 1995, ApJ, 441, 617
Shara M. M., Saffer R. A., Livio M. 1997, ApJ, 489, L59
Sills A., Bailyn C. D. 1999, ApJ, 513, 428
Sills A. R., Faber J. A., Lombardi J. C., Rasio F. A., Warren A. R. 2001, ApJ, 548, 323
Sollima A., Beccari G., Ferraro F. R., Fusi Pecci F., Sarajedini A. 2007, MNRAS, 380, 781
Sollima A., Beccari G., Ferraro F. R., Fusi Pecci F., Sarajedini A. 2008, A&A, 481, 701
Spitzer L. Jr. 1969, ApJ, 168, L139
Spitzer L. 1987, Dynamical Evolution of Globular Clusters (Princeton: Princeton University Press)
Stryker L. L. 1993, PASP, 105, 1081
van den Berg M., Orosz J., Verbunt F., Stassun K. 2001, A&A, 375, 375
Verbunt F., Pooley D., Bassa C. 2008, IAU Symposium 246, 301
Webbink R. F. 1985, in Dynamics of Star Clusters, IAU Symp. 113, ed. J. Goodman & P. Hut (Dordrecht: Reidel), 541