Heat kernels for reflected diffusions with jumps
on inner uniform domains

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Abstract

In this paper, we study sharp two-sided heat kernel estimates for a large class of symmetric reflected diffusions with jumps on the closure of an inner uniform domain $D$ in a length metric space. The length metric is the intrinsic metric of a strongly local Dirichlet form. When $D$ is an inner uniform domain in the Euclidean space, a prototype for a special case of the processes under consideration are symmetric reflected diffusions with jumps on $\mathbb{R}^d$, whose infinitesimal generators are non-local (pseudo-differential) operators $L$ on $D$ of the form

$$\mathcal{L}u(x) = \frac{1}{2} \sum_{i,j=1}^{d} \frac{\partial}{\partial x_i} \left( a_{ij}(x) \frac{\partial u(x)}{\partial x_j} \right) + \lim_{\varepsilon \to 0} \int_{\{y \in D : \rho_D(y,x) > \varepsilon\}} (u(y) - u(x)) J(x,y) \, dy$$

satisfying “Neumann boundary condition”. Here, $\rho_D(x,y)$ is the length metric on $D$, $A(x) = (a_{ij}(x))_{1 \leq i,j \leq d}$ is a measurable $d \times d$ matrix-valued function on $D$ that is uniformly elliptic and bounded, and

$$J(x,y) := \frac{1}{\Phi(\rho_D(x,y))} \int_{[a_1,a_2]} \frac{c(\alpha,x,y)}{\rho_D(x,y)^{d+\alpha}} \nu(d\alpha),$$

where $\nu$ is a finite measure on $[a_1,a_2] \subset (0,2)$, $\Phi$ is an increasing function on $[0,\infty)$ with $c_1 e^{c_2 r^\beta} \leq \Phi(r) \leq c_3 e^{c_4 r^\beta}$ for some $\beta \in [0,\infty]$, and $c(\alpha,x,y)$ is a jointly measurable function that is bounded between two positive constants and is symmetric in $(x,y)$.

Keywords: reflected diffusions with jumps; symmetric Dirichlet form; inner uniform domain; heat kernel

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1 Introduction and setup

Let $E$ be a locally compact separable metric space, and $m$ a $\sigma$-finite Radon measure with full support on $E$. Let $(\mathcal{E}^0, \mathcal{F}^0)$ be a strongly local regular Dirichlet form on $L^2(E;m)$, and $\mu^0_{(u)}$ be the $(\mathcal{E}^0,\mathcal{F}^0)$-energy measure of $u \in \mathcal{F}^0$ so that $\mathcal{E}^0(u,u) = \frac{1}{2} \mu^0_{(u)}(E)$. We assume that the intrinsic metric $\rho$ of $(\mathcal{E}^0, \mathcal{F}^0)$ defined by $\rho(x,y) = \sup \{ f(x) - f(y) : f \in \mathcal{F}^0 \cap C_c(E) \text{ with } \mu^0_{(f)}(dz) \leq m(dz) \}$ is finite for any $x,y \in E$ and induces the original topology on $E$, and that $(E,\rho)$ is a complete metric space. In the above definition of $\rho(x,y)$, the condition $\mu^0_{(f)}(dz) \leq m(dz)$ means that $\mu^0_{(f)}(A) \leq m(A)$ for every $A \in \mathcal{B}(E)$. We refer the reader to [GS, Theorem 2.11] for a summary of basic properties of $(E,\rho)$. In particular, $(E,\rho)$ is a geodesic length space; that is, for any pair $x,y \in E$, there exists a continuous curve $\gamma : [0,1] \to E$ with $\gamma(0) = x$, $\gamma(1) = y$ and for every $s,t \in [0,1]$, $\rho(\gamma(s), \gamma(t)) = |t-s| \rho(x,y)$. Such a curve is called a minimal geodesic (parameterized by a multiple of arc length). In the following, we will always use the intrinsic metric $\rho$ for $E$, and use $B(x,r)$ for the ball centered at $x$ with radius $r$ under $\rho$-metric. Denote $m(B(x,r))$ by $V(x,r)$. We consider two conditions as follows.
(VD) **(Volume doubling property)** There is a constant $c_1 > 0$ so that for every $x \in E$ and $r > 0$, $V(x, 2r) \leq c_1 V(x, r)$.

(PI(2)) **(Poincaré inequality)** There is a constant $c_2 > 0$ so that for every $x \in E$, $r > 0$ and $f \in \mathcal{F}^0$,

$$\min_{a \in \mathbb{R}} \int_{B(x,r)} (f(y) - a)^2 m(dy) \leq c_2 r^2 \mu_{(f)}^0(B(x,r)). \quad (1.1)$$

The Dirichlet form $(\mathcal{E}^0, \mathcal{F}^0)$ satisfying the above properties is called a **Harnack-type Dirichlet space** in [GS Chapter 2]. The state space $E$ for Harnack-type Dirichlet space $(\mathcal{E}^0, \mathcal{F}^0)$ is connected and the diffusion process $Z^0$ associated with $(\mathcal{E}^0, \mathcal{F}^0)$ is conservative (see [GS Lemma 2.33]).

The following is Sturm’s generalization to the strongly local Dirichlet form setting of a celebrated result by Grigor’yan and Saloff-Coste on Riemannian manifolds. (See [St1, St2, St3].) In this paper, we use $:=$ as a way of definition.

**Theorem 1.1.** The following are equivalent for the strongly local Dirichlet form $(\mathcal{E}^0, \mathcal{F}^0)$ on $L^2(E; m)$.

(i) (VD) and (PI(2)) hold.

(ii) The uniform parabolic Harnack inequality holds. Namely, there exists a constant $C > 0$ such that for every $x_0 \in E$, $t_0 \geq 0$, $r > 0$ and for every non-negative function $u = u(t, x)$ on $[0, \infty) \times E$ that is caloric (or space-time harmonic) in cylinder $(t_0, t_0 + 4r^2) \times B(x_0, 2r)$,

$$\text{ess sup}_{Q_-} u \leq C \text{ ess inf}_{Q_+} u,$$

where $Q_- := (t_0 + r^2, t_0 + 2r^2) \times B(x_0, r)$ and $Q_+ := (t_0 + 3r^2, t_0 + 4r^2) \times B(x_0, r)$.

(iii) Aronson-type heat kernel estimates for the diffusion process $Z^0$ associated with $(\mathcal{E}^0, \mathcal{F}^0)$ hold, i.e., $Z^0$ admits a jointly continuous transition density $p^0(t, x, y)$ with respect to $m$ on $(0, \infty) \times E \times E$ and there are constants $c_1, c_2 \geq 1$ so that

$$\frac{1}{c_1 V(x, \sqrt{t})} \exp \left( -\frac{c_2 \rho(x, y)^2}{t} \right) \leq p^0(t, x, y) \leq \frac{c_1}{V(x, \sqrt{t})} \exp \left( -\frac{\rho(x, y)^2}{c_2 t} \right)$$

for every $x, y \in E$ and $t > 0$.

For a domain (i.e. connected open subset) $D$ of the length metric space $(E, \rho)$, define for $x, y \in D$,

$$\rho_D(x, y) = \inf \{ \text{length}(\gamma) : \text{a continuous curve } \gamma \text{ in } D \text{ with } \gamma(0) = x \text{ and } \gamma(1) = y \}. \quad (1.2)$$

Denote by $\bar{D}$ the completion of $D$ under the metric $\rho_D$. Note that $(\bar{D}, \rho_D)$ is a length metric space but may not be locally compact in general. For example, see [GS Remark 2.16]. We extend the definition of $m|_D$ to $D$ by setting $m|_{D(\bar{D} \setminus D)} = 0$.

For notational simplicity, by abusing the notation a little bit, we will use $m$ to denote this measure $m|_D$.

Following [GS Definition 3.6], we say that $D$ is **inner uniform** if there are constants $C_1, C_2 \in (0, \infty)$ such that, for any $x, y \in D$, there exists a continuous curve $\gamma_{x,y} : [0, 1] \to D$ with $\gamma_{x,y}(0) = x$, $\gamma_{x,y}(1) = y$ and satisfying the following two properties:

(i) The length of $\gamma_{x,y}$ is at most $C_1 \rho_D(x, y)$;

(ii) For any $z \in \gamma_{x,y}([0, 1])$,

$$\rho(z, \partial D) := \inf_{w \in \partial D} \rho(z, w) \geq C_2 \frac{\rho_D(z, x) \rho_D(z, y)}{\rho_D(x, y)}.$$
We call the above constants $(C_1, C_2)$ the characteristics of the inner uniform domain $D$. Note that, when $D$ is inner uniform, $(\bar{D}, \rho_D)$ is locally compact (see [GS Lemma 3.9]). Let $F_D^0 = \{ f \in F^0 : f = 0$ $\mathcal{E}$-q.e. on $D^c \}$. It is well known that $(\mathcal{E}^0, F_D^0)$ is the part Dirichlet form of $(\mathcal{E}^0, F^0)$ on $D$, or equivalently, it is the Dirichlet form on $L^2(D; m)$ of the subprocess of the diffusion process $X^0$ associated with $(\mathcal{E}^0, F^0)$ killed upon leaving $D$. A function $f$ is said to be locally in $F_D^0$, denoted as $f \in F_D^{0, \text{loc}}$, if for every relatively compact subset $U$ of $D$, there is a function $g \in F_D^0$ such that $g = m$ $\text{a.e.}$ on $U$. By [GS Proposition 2.13], we have that for $x, y \in D$,

$$
\rho_D(x, y) = \sup \left\{ f(x) - f(y) : f \in F_D^{0, \text{loc}} \cap C_c(D) \text{ with } \mu_{\{f\}}(dz) \leq m(dz) \right\}.
$$

Define $F_D^{0, \text{ref}} := \{ f \in F_D^{0, \text{loc}} : \mu_{\{f\}}(D) < \infty \}$ and

$$
\mathcal{E}^{0, \text{ref}}(f, f) := \frac{1}{2} \mu_{\{f\}}(D) \quad \text{for } f \in F_D^{0, \text{ref}}.
$$

The bilinear form $(\mathcal{E}^{0, \text{ref}}, F_D^{0, \text{ref}} \cap L^2(D; m))$ is the active reflected Dirichlet form of $(\mathcal{E}^0, F_D^0)$, which is known to be a Dirichlet form on $L^2(\bar{D}; m) = L^2(D; m)$, see [CR] Chapter 6]. Denote $B_D(x, r)$ by the ball centered at $x$ with radius $r$ under $\rho_D$-metric, i.e., $B_D(x, r) := \{ y \in \bar{D} : \rho_D(x, y) < r \}$. Denote $m(B_D(x, r))$ by $V_D(x, r)$. Let $\text{Lip}_c(\bar{D})$ be the space of Lipschitz functions with compact support in $\bar{D}$. The following is established in [GS Lemma 3.9, Theorem 3.10, Theorem 3.13 and Corollary 3.31].

**Theorem 1.2.** Suppose that $(\mathcal{E}^0, F^0)$ is a strongly local regular Dirichlet form on $L^2(E; m)$ that admits a carré du champ operator $\Gamma_0$ (that is, for every $u \in F$, $\mu_0^u(dx) = \Gamma_0(u, u) m(dx)$ and $\Gamma_0(u, u) \in L^1(E; m)$). Assume that (VD) and (PI(2)) hold for $(\mathcal{E}^0, F^0)$ on $(E, \rho, m)$. Suppose that $D$ is an inner uniform subdomain of $E$. Then $(\mathcal{E}^{0, \text{ref}}, F_D^{0, \text{ref}} \cap L^2(D; m))$ is a strongly local regular Dirichlet form on $L^2(D; m)$ with core $\text{Lip}_c(\bar{D})$ and that the volume doubling property and the Poincaré inequality hold for $(\mathcal{E}^{0, \text{ref}}, F_D^{0, \text{ref}} \cap L^2(D; m))$ on $(\bar{D}, \rho_D, m)$:

(VD) (Volume doubling property on $\bar{D}$) There is a constant $C_3 > 0$ so that for every $x \in \bar{D}$ and $r > 0$,

$$
V_D(x, 2r) \leq C_3 V_D(x, r).
$$

(PI(2)) (Poincaré inequality on $\bar{D}$) There is a constant $C_4 > 0$ so that for every $x \in \bar{D}$, $r > 0$ and $f \in F_D^{0, \text{ref}} \cap L^2(D; m)$,

$$
\min_{a \in \mathbb{R}} \int_{B_D(x, r)} (f(y) - a)^2 m(dy) \leq C_4 r^2 \mu_{\{f\}}(B_D(x, r)).
$$

Consequently, the diffusion process associated with $(\mathcal{E}^{0, \text{ref}}, F_D^{0, \text{ref}})$ admits a jointly continuous transition density function $p_{D}^N(t, x, y)$ on $(0, \infty) \times \bar{D} \times \bar{D}$, and there are constants $c_3, c_4 \geq 1$ depending on $C_3, C_4$ so that

$$
\frac{c_3^{-1}}{V_D(x, \sqrt{t})} \exp \left( -c_4 \rho_D(x, y)^2 \frac{t}{t} \right) \leq p_{D}^N(t, x, y) \leq \frac{c_3}{V_D(x, \sqrt{t})} \exp \left( -\frac{\rho_D(x, y)^2}{c_4 t} \right)
$$

for every $x, y \in \bar{D}$ and $t > 0$.

Throughout this paper, we assume that $(\mathcal{E}^0, F^0)$ and $D$ satisfy the assumptions of Theorem 1.2. Recall that $\rho_D$ is defined in [12], $\bar{D}$ is the completion of the metric space $(D, \rho_D)$, and (by abusing the notation) $m$ is the measure $m|_D$ extended to $\bar{D}$ by setting $m(\bar{D} \setminus D) = 0$. By Theorem 1.2 the volume doubling property (VD) holds for the inner uniform subdomain $D$ of $E$. Note that, (VD) condition on $V_D(x, r)$ is equivalent to the following: there exist constants $c_2, d_2 > 0$ depending on $C_3$ only such that for all $x \in \bar{D}$,

$$
\frac{V_D(x, R)}{V_D(x, r)} \leq c_2 \left( \frac{R}{r} \right)^{d_2} \quad \text{for } R \geq r > 0.
$$

(1.6)
Since $D$ is connected, the reverse volume doubling property (RVD) holds for $\bar{D}$; that is, there exist constants $c_1, d_1 > 0$, depending on $C_1, C_3, C_4$ only, such that for all $x \in \bar{D}$,
\[
\frac{V_D(x, R)}{V_D(x, r)} \geq c_1 \left( \frac{R}{r} \right)^{d_1} \quad \text{for } 0 < r \leq R \leq 2 \text{diam}(D).
\] (1.7)

See [YZ, Proposition 2.1 and a paragraph before Remark 2.1]. We further note that, by (1.6), for all $x, y \in \bar{D}$ and $0 < r \leq R$,
\[
\frac{V_D(x, R)}{V_D(y, r)} \leq \frac{V_D(y, \rho(x, y) + R)}{V_D(y, r)} \leq c_2 \left( \frac{\rho(x, y) + R}{r} \right)^{d_2}.
\] (1.8)

This in particular implies that, with $\tilde{c}_2 := 2^{d_2} c_2$,
\[
\tilde{c}_2^{-1} \left( \frac{\rho(x, y)}{r} \right)^{d_2} \leq \frac{V_D(x, r)}{V_D(y, r)} \leq \tilde{c}_2 \left( \frac{\rho(x, y)}{r} \right)^{d_2} \quad \text{for all } x, y \in \bar{D} \text{ and } r > 0.
\]

Here and in what follows, we use notations $a \land b := \min\{a, b\}$ and $a \lor b := \max\{a, b\}$ for $a, b \in \mathbb{R}$.

**Characteristic constants.** Many estimates in this paper will depend on the characteristics $(C_1, C_2)$ of the inner uniform domain $D$ and on the constants $C_3$ and $C_4$ in (VD) and (PI(2)) of Theorem 1.2. For convenience, we will call $(C_1, C_2, C_3, C_4)$ the characteristic constants of the domain $D$ in this paper.

In this paper, we are concerned with Markov processes on $\bar{D}$ associated with the following type of non-local symmetric Dirichlet forms $(\mathcal{E}, \mathcal{F})$ on $L^2(D; m)$:
\[
\mathcal{F} = \mathcal{F}_D^{0, \text{ref}} \cap L^2(D; m),
\] (1.9)

and, for $u \in \mathcal{F}$,
\[
\mathcal{E}(u, u) = \mathcal{E}_D^{0, \text{ref}}(u, u) + \frac{1}{2} \int_{D \times D} (u(x) - u(y))^2 J(x, y) m(dx) m(dy),
\] (1.10)

where $J(x, y)$ is a non-negative symmetric measurable function on $D \times D \setminus \text{diag}$ satisfying certain conditions to be specified below. Here and in what follows, diag is the diagonal of $D \times D$; that is, diag := \{(x, x) : x \in D\}.

Let $\phi$ be a strictly increasing function on $[0, \infty)$ such that $\phi(0) = 0$, $\phi(1) = 1$ and there exist constants $C_5 \geq 1$ and $0 < \alpha_* \leq \alpha^* < 2$ so that
\[
C_5^{-1} \left( \frac{R}{r} \right)^{\alpha_*} \leq \frac{\phi(R)}{\phi(r)} \leq C_5 \left( \frac{R}{r} \right)^{\alpha^*} \quad \text{for every } 0 < r < R < \infty.
\] (1.11)

Since the constant $\alpha^*$ in (1.11) is strictly less than 2, there exists $c_3 > 0$ depending on $C_5$ and $\alpha^*$ such that
\[
\int_0^r \frac{s}{\phi(s)} ds \leq c_3 r^2 \frac{\phi(r)}{\phi(r)} \quad \text{for every } r > 0.
\] (1.12)

**Definition 1.3.** Let $\beta \in [0, \infty]$ and $\phi$ be a strictly increasing function on $[0, \infty)$ with $\phi(0) = 0$ and $\phi(1) = 1$ that satisfies the condition (1.11) (with $0 < \alpha_* \leq \alpha^* < 2$). For a non-negative symmetric measurable function $J(x, y)$ on $D \times D \setminus \text{diag}$, we say

(i) condition $(J_{\phi, \beta, \leq})$ holds if there are positive constants $\kappa_1$ and $\kappa_2$ so that
\[
J(x, y) \leq \frac{\kappa_1}{V_D(x, \rho_D(x, y)) \phi(\rho_D(x, y)) \exp(\kappa_2 \rho_D(x, y)^\beta)} \quad \text{for } (x, y) \in D \times D \setminus \text{diag}; \quad (J_{\phi, \beta, \leq})
\]

(ii) condition $(J_{\phi, \beta, \geq})$ holds if there are positive constants $\kappa_3$ and $\kappa_4$ so that
\[
J(x, y) \geq \frac{\kappa_3}{V_D(x, \rho_D(x, y)) \phi(\rho_D(x, y)) \exp(\kappa_4 \rho_D(x, y)^\beta)} \quad \text{for } (x, y) \in D \times D \setminus \text{diag}; \quad (J_{\phi, \beta, \geq})
\]

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(iii) condition \((J_{\phi,0,\le})\) holds if both \((J_{\phi,\beta,\le})\) and \((J_{\phi,\beta,\ge})\) hold with possibly different constants \(\kappa_i\) in the upper and lower bounds;

(iv) condition \((J_{\phi,0,\le})\) holds if there are positive constants \(\kappa_5\) and \(\kappa_6\) so that

\[
\begin{cases}
J(x,y) \leq \frac{\kappa_5}{V_D(x,\rho_D(x,y))\phi(\rho_D(x,y))} & \text{for } (x,y) \in D \times D \setminus \text{diag}, \\
\sup_{x \in D} \int_{\{y \in D: \rho_D(x,y) > 1\}} \rho_D(x,y)^2 J(x,y) m(dy) \leq \kappa_6 < \infty,
\end{cases}
\]

where

\[
\phi_r := \phi(r) \mathbb{1}_{\{r \leq 1\}} + r^2 \mathbb{1}_{\{r > 1\}} \quad \text{for } r \geq 0;
\]

(v) condition \((J_{\phi,0,\le})\) holds if there are positive constants \(\kappa_7\) and \(\kappa_8\) so that

\[
\begin{cases}
J(x,y) \leq \frac{\kappa_7}{V_D(x,\rho_D(x,y))\phi(\rho_D(x,y))} & \text{for } x \neq y \text{ in } D \text{ with } \rho_D(x,y) \leq 1, \\
\sup_{x \in D} \int_{\{y \in D: \rho_D(x,y) > 1\}} \rho_D(x,y)^2 J(x,y) m(dy) \leq \kappa_8 < \infty;
\end{cases}
\]

(vi) condition \((J_{\phi,\le})\) holds if there are positive constants \(\kappa_9\) and \(\kappa_{10}\) so that

\[
\begin{cases}
J(x,y) \leq \frac{\kappa_9}{V_D(x,\rho_D(x,y))\phi(\rho_D(x,y))} & \text{for } x \neq y \text{ in } D \text{ with } \rho_D(x,y) \leq 1, \\
\sup_{x \in D} \int_{\{y \in D: \rho_D(x,y) > 1\}} J(x,y) m(dy) \leq \kappa_{10} < \infty.
\end{cases}
\]

Clearly, \((J_{\phi,0,\le}) \implies (J_{\phi,0,\le}) \implies (J_{\phi,\le})\) and

\[(J_{\phi,\beta,\le}) \implies (J_{\phi,0,\le}) \implies (J_{\phi,0,\le}) \implies (J_{\phi,\le}) \quad \text{for any } \beta \in (0, \infty].\]  

(1.14)

When \(\beta = 0\), we will simply write \((J_{\phi,\le})\), \((J_{\phi,\ge})\) and \((J_{\phi})\) for \((J_{\phi,0,\le})\), \((J_{\phi,0,\le})\) and \((J_{\phi,0})\), respectively. When \(\beta = \infty\), conditions \((J_{\phi,\le})\) and \((J_{\phi,\ge})\) are equivalent, respectively, to

\[
J(x,y) \leq \frac{\tilde{\kappa}_1}{V_D(x,\rho_D(x,y))\phi(\rho_D(x,y))} \mathbb{1}_{\{\rho(x,y) \leq 1\}} \quad \text{for } (x,y) \in D \times D \setminus \text{diag}
\]

(1.15)

and

\[
J(x,y) \geq \frac{\tilde{\kappa}_3}{V_D(x,\rho_D(x,y))\phi(\rho_D(x,y))} \mathbb{1}_{\{\rho(x,y) \leq 1\}} \quad \text{for } (x,y) \in D \times D \setminus \text{diag}.
\]

(1.16)

We will see from Proposition 2.1 below that, under condition \((J_{\phi,\le})\), \((E,\mathcal{F})\) is a regular Dirichlet form on \(L^2(D; m)\), and there is a symmetric Hunt process \(Y\) on \(D\) associated with it that can start from \(E\)-quasi-everywhere in \(D\). Moreover, the process \(Y\) is conservative; that is, \(Y\) has infinite lifetime almost surely. For notational convenience, we regard 0 as an “added” or “extended” number and declare that it is larger than 0 but smaller than any positive real number. With this notation, we can write, for instance, \((J_{\phi,\beta,\le})\) for \(\beta \in [0, \infty) \cup \{0\}\).

The jumping intensity kernel \(J(x,y)\) determines a Lévy system of \(Y\), which describes the jumps of the process \(Y\): for any non-negative measurable function \(f\) on \(\mathbb{R}_+ \times \bar{D} \times \bar{D}\) with \(f(s,x) = 0\) for all \(s > 0\) and \(x \in \bar{D}\), and stopping time \(T\) (with respect to the filtration of \(Y\)),

\[
\mathbb{E}_x \left[ \sum_{s \leq T} f(s, Y_{s-}, Y_s) \right] = \mathbb{E}_x \left[ \int_0^T \left( \int_D f(s, Y_s, y) J(Y_s, y) m(dy) \right) ds \right].
\]

(1.17)
See, for example, [CK1 proof of Lemma 4.7] and [CK2 Appendix A].

The goal of this paper is to derive global two-sided sharp estimates on the heat kernel of Dirichlet form $(\mathcal{E}, \mathcal{F})$ (i.e., the transition density function of the process $Y$) under the assumption that $J(x,y)$ satisfies \((J_{\beta_i,\leq})\) and \((J_{\beta_i,\geq})\) for some strictly increasing functions $\phi_1$ and $\phi_2$ satisfying $\phi_i(0) = 0$, $\phi_i(1) = 1$ and \((1.11)\) (with $\phi_i$ in place of $\phi$) for $1 \leq i \leq 2$, and for $\beta_*$ and $\beta^*$ in $[0, \infty] \cup \{0_+\}$ in the following two cases:

- $\beta_* = \beta^* = 0$ and $\phi_1 = \phi_2$; and
- $0_+ \leq \beta_* \leq \beta^* \leq \infty$ excluding $\beta_* = \beta^* = 0_+$.

See Subsection 6.3 in the Appendix of this paper for brief discussions on the case under conditions \((J_{\phi_i,\leq})\) and \((J_{\phi_i,\geq})\) with $\beta^* \in (0, \infty)$.

We start with the $\beta_* = \beta^* = 0$ case. Define for $x \in \overline{D}$, $t > 0$ and $r \geq 0$,

$$p^{(c)}(t,x,r) := \frac{1}{V_D(x, \sqrt{t})} \exp(-r^2/t),$$

(1.18)

and for a strictly increasing function $\phi$ on $[0, \infty)$,

$$p^{(j)}_\phi(t,x,r) := \frac{1}{V_D(x, \phi^{-1}(t))} \wedge \frac{t}{V_D(x, r)\phi(r)},$$

(1.19)

where $\phi^{-1}(r)$ denotes the inverse function of $\phi$. Set

$$H_{\phi,0}(t,x,r) := \frac{1}{V_D(x, \phi^{-1}(t) \vee \sqrt{t})} \wedge \left( p^{(c)}(t,x,r) + p^{(j)}_\phi(t,x,r) \right).$$

(1.20)

**Theorem 1.4.** (The $\beta_* = \beta^* = 0$ case.) Assume that condition $(J_{\phi})$ holds for a strictly increasing function $\phi$ satisfying \((1.11)\). Then $(\mathcal{E}, \mathcal{F})$ defined by \((1.9)-(1.10)\) is a regular Dirichlet form on $L^2(D;m)$ and there is a conservative Feller process $Y$ associated with it that starts from every point in $\overline{D}$. Moreover, $Y$ has a jointly Hölder continuous transition density function $q(t,x,y)$ that enjoys the following two-sided estimates. There are positive constants $c_i$, $1 \leq i \leq 4$, depending only on the characteristic constants $(C_1, C_2, C_3, C_4)$ of $D$ and the constant parameters in \((1.11)\) and $(J_{\phi})$, such that for every $t > 0$ and $x, y \in \overline{D}$,

$$c_1 H_{\phi,0}(t,x,c_2\rho_D(x,y)) \leq q(t,x,y) \leq c_2 H_{\phi,0}(t,x,c_4\rho_D(x,y)).$$

(1.21)

Note that, in the present setting, the fact that the underlying state space $D$ has infinite diameter is equivalent to the fact that $D$ has infinite volume; see [CH Corollary 5.3]. So, when $\text{diam}(D) = \infty$, we can deduce Theorem 1.4 from [CKW3 Theorem 1.13]. Furthermore, as seen from [GGH] and [CKW2 Remark 1.19], all results of the paper [CKW3] continue to hold for bounded state space with obvious localized versions. Thus when $\text{diam}(D) < \infty$, according to [CKW3 Theorem 1.13] again, for any $T_0 > 0$, one can obtain estimates of $q(t,x,y)$ for $t \in (0,T_0]$ with constants $c_i$ further dependent on $T_0$. So for Theorem 1.4 it remains to prove the statement for $t \geq T_0$ and $\text{diam}(D) < \infty$. In fact, when $D$ is bounded, it holds that $V_D(x, \sqrt{t}) \approx 1$ for all $x \in D$ and $t \geq 1$, and the large time estimates \((1.21)\) (i.e., $t \in [1, \infty)$) of the heat kernel $q(t,x,y)$ are simply

$$q(t,x,y) \approx 1 \quad \text{for } x, y \in \overline{D} \text{ and } t \geq 1.$$  

(1.22)

Here and in what follows, $f(t,x,r) \approx g(t,x,r)$ means that there exist constants $c_1, c_2 > 0$ such that $c_1 g(t,x,r) \leq f(t,x,r) \leq c_2 g(t,x,r)$ for the specified range of the argument $(t,x,r)$. The full proof of Theorem 1.4 will be given in Section 2.

Note also that when $D$ is bounded, conditions $(J_{\beta,\leq})$ and $(J_{\beta,\geq})$ with $\beta \in \{0_+\} \cup (0, \infty)$ are reduced to $(J_{\beta,\leq})$ and $(J_{\beta,\geq})$, respectively. In view of the above theorem, the main task of the paper is to consider the case where $D$ is unbounded and the jumping density kernel $J(x,y)$ satisfying conditions $(J_{\beta_1,\leq})$ and $(J_{\beta_2,\geq})$ with $0_+ \leq \beta_* \leq \beta^* \leq \infty$, which is harder than the case $\beta_* = \beta^* = 0_+$ case.
In particular, $p^{(j)}_{\phi,\beta}(t, x, r)$ and $\phi$ are defined by (1.18) and (1.19), respectively. For $\beta \in [0, \infty)$ and a strictly increasing function $\phi$ on $[0, \infty)$ with $\phi(0) = 0$ and $\phi(1) = 1$, set for $x \in \tilde{D}$, $t > 0$ and $r \geq 0$,

$$p^{(j)}_{\phi,\beta}(t, x, r) := \frac{1}{V_D(x, \phi^{-1}(t))} \frac{t}{V(x, r) \phi(r) \exp(\beta r)},$$

(1.23)

In particular, $p^{(j)}_{\phi,0}(t, x, r) \simeq p^{(j)}_{\phi}(t, x, r)$. Define $\beta \in (0, 1]$,

$$H_{\phi,\beta}(t, x, r) := \begin{cases} \frac{1}{V_D(x, \sqrt{t})} \wedge \left( p^{(c)}(t, x, r) + p^{(j)}_{\phi,\beta}(t, x, r) \right) & \text{if } t \in (0, 1], \\ \frac{1}{V_D(x, \sqrt{t})} \exp \left( - (r^\beta \wedge (r^2/t)) \right) & \text{if } t \in (1, \infty); \end{cases}$$

for $\beta \in (1, \infty)$,

$$H_{\phi,\beta}(t, x, r) := \begin{cases} \frac{1}{V_D(x, \sqrt{t})} \wedge \left( p^{(c)}(t, x, r) + p^{(j)}_{\phi,\beta}(t, x, r) \right) & \text{if } t \in (0, 1] \text{ and } r \leq 1, \\ \frac{1}{V_D(x, \sqrt{t})} \exp \left( - \left( r \left( 1 + \log^+(r/t) \right) \right) \beta / \beta \right) \wedge r^\beta & \text{if } t \in (0, 1] \text{ and } r > 1, \\ \frac{1}{V_D(x, \sqrt{t})} \exp \left( - \left( r \left( 1 + \log^+(r/t) \right) \right) \beta / \beta \right) \wedge (r^2/t) & \text{if } t \in (1, \infty); \end{cases}$$

where $\log^+(x) := \log(x \vee 1)$, and $H_{\phi,\infty}(t, x, r) := \lim_{\beta \to \infty} H_{\phi,\beta}(t, x, r)$ for $\beta = \infty$, that is,

$$H_{\phi,\infty}(t, x, r) := \begin{cases} \frac{1}{V_D(x, \sqrt{t})} \wedge \left( p^{(c)}(t, x, r) + p^{(j)}_{\phi,\beta}(t, x, r) \right) & \text{if } t \in (0, 1] \text{ and } r \leq 1, \\ \frac{1}{V_D(x, \sqrt{t})} \exp \left( - r \left( 1 + \log^+(r/t) \right) \right) & \text{if } t \in (0, 1] \text{ and } r > 1, \\ \frac{1}{V_D(x, \sqrt{t})} \exp \left( - \left( r \left( 1 + \log^+(r/t) \right) \right) \wedge (r^2/t) \right) & \text{if } t \in (1, \infty). \end{cases}$$

We further define for $x \in \tilde{D}$, $t > 0$ and $r \geq 0$,

$$H_{\phi,\alpha_+}(t, x, r) := \frac{1}{V_D(x, \sqrt{t})} \wedge \left( p^{(c)}(t, x, r) + p^{(j)}_{\phi_+}(t, x, r) \right),$$

where $\phi_+$ is given by (1.13).

We can write the above functions in a more explicit way. For $\beta \in (0, 1]$,

$$H_{\phi,\beta}(t, x, r) := \begin{cases} \frac{1}{V_D(x, \sqrt{t})} \wedge \left( p^{(c)}(t, x, r) + p^{(j)}_{\phi,\beta}(t, x, r) \right) & \text{if } t \in (0, 1] \text{ and } r \leq 1, \\ \frac{t}{V_D(x, r) \phi(r)} \exp(-r^\beta) & \text{if } t \in (0, 1] \text{ and } r > 1, \\ \frac{1}{V_D(x, \sqrt{t})} \exp(-r^2/t) & \text{if } t \in (1, \infty) \text{ and } t \geq r^{2-\beta}, \\ \frac{1}{V_D(x, \sqrt{t})} \exp(-r^\beta) & \text{if } t \in (1, \infty) \text{ and } t \leq r^{2-\beta}; \end{cases}$$

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and for $\beta \in (1, \infty)$,

$$H_{\phi, \beta}(t, x, r) = \begin{cases} \frac{1}{V_{D}(x, \sqrt{t}}) \Lambda \left(\rho^{(c)}(t, x, r) + p^{(j)}_{\phi, \beta}(t, x, r)\right) & \text{if } t \in (0, 1] \text{ and } r \leq 1, \\
\frac{t}{V_{D}(x, r)\phi(r)} \exp\left(-r \left(1 + \log^{+}(r/t)\right)^{(\beta-1)/\beta}\right) & \text{if } t \in (0, 1], r \geq 1 \text{ and } t \geq r \exp(-r^{\beta}), \\
\frac{t}{V_{D}(x, r)\phi(r)} \exp(-r^{\beta}) & \text{if } t \in (0, 1], r \geq 1 \text{ and } t \leq r \exp(-r^{\beta}), \\
\frac{1}{V_{D}(x, \sqrt{t}}) \exp(-r^{2}/t) & \text{if } t \in (1, \infty) \text{ and } t \geq r, \\
\frac{1}{V_{D}(x, \sqrt{t}}) \exp\left(-r \left(1 + \log^{+}(r/t)\right)^{(\beta-1)/\beta}\right) & \text{if } t \in (1, \infty) \text{ and } t \leq r. 
\end{cases}$$

Here and in what follows, we write $f(t, x, r) \asymp g(t, x, r)$, if there exist constants $c_{k} > 0$, $k = 1, \ldots, 4$, such that $c_{1}g(t, x, c_{2}r) \leq f(t, x, r) \leq c_{3}g(t, x, c_{4}r)$ for the specified range of $(t, x, r)$.

The next theorem gives the existence and its upper bound for the transition density function of $Y$ associated with the Dirichlet form $(\mathcal{E}, \mathcal{F})$ of $[1.9] - [1.10]$ under condition $(J_{\phi, \alpha, \gamma, \leq})$. Its proof will be given in Section 3. Note that it holds on any inner uniform domain $D$, regardless whether it is bounded or not.

**Theorem 1.5.** Suppose that condition $(J_{\phi, \alpha, \gamma, \leq})$ holds for some strictly increasing function $\phi$ on $[0, \infty)$ satisfying the condition (1.11). Then the Dirichlet form $(\mathcal{E}, \mathcal{F})$ of $[1.9] - [1.10]$ is regular on $L^{2}(D; m)$ and there is a conservative Feller process $Y$ on $D$ associated with it that starts from point $x$ and $Y$ has a jointly Hölder continuous transition density function $q(t, x, y)$ on $(0, \infty) \times \overline{D}$ with respect to the measure $m$, and there are constants $c_{1}, c_{2} > 0$ such that

$$q(t, x, y) \leq c_{1}H_{\phi, \alpha, \gamma}(t, x, c_{2}\rho_{D}(x, y))$$

for all $x, y \in \overline{D}$ and $t > 0$. (1.24)

Furthermore, for every $c_{3} \in (0, 1/2)$, there are positive constants $c_{4}$ and $c_{5}$ so that for any $x \in \overline{D}$ and $r \in (0, c_{3}\text{diam}(D))$,

$$c_{4}r^{2} \leq E_{x} \left[\tau_{B_{D}(x, r)}\right] \leq c_{5}r^{2},$$

where $\tau_{B_{D}(x, r)} := \inf\{t > 0 : Y_{t} \notin B_{D}(x, r)\}$ is the first exit time from the ball $B_{D}(x, r)$ by the process $Y$. The positive constants $c_{1}, c_{2}$ and $c_{4}, c_{5}$ depend only on the characteristic constants $(C_{1}, C_{2}, C_{3}, C_{4})$ of $D$ and on the constant parameters in $(J_{\phi, \alpha, \gamma, \leq})$ and $(1.11)$ for the function $\phi$, with constants $c_{4}$ and $c_{5}$, in addition, on $c_{3}$ as well.

The following is the main result of this paper on the two-sided heat kernel estimates of $Y$.

**Theorem 1.6.** (The $\beta_{*} \leq \beta^{*} \leq \infty$ in $\{0_{+}\} \cup (0, \infty]$ case.) Suppose that $D$ is unbounded. Assume that $J(x, y)$ satisfies conditions $(J_{\phi, \beta_{*}, \leq})$ and $(J_{\phi, \beta^{*}, \geq})$ for some strictly increasing functions $\phi_{1}, \phi_{2}$ satisfying $\phi_{i}(0) = 0$, $\phi_{i}(1) = 1$ and $(1.11)$ (with $\phi_{i}$ in place of $\phi$) for $i = 1, 2$, and for $\beta_{*} \leq \beta^{*} \in \{0_{+}\} \cup (0, \infty]$ excluding $\beta_{*} = \beta^{*} = 0_{+}$, then the transition density function $q(t, x, y)$ of the conservative Feller process $Y$ associated with $(\mathcal{E}, \mathcal{F})$ has the following estimates: for every $t > 0$ and $x, y \in \overline{D}$,

$$c_{1}H_{\phi_{2}, \beta*}(t, x, c_{2}\rho_{D}(x, y)) \leq q(t, x, y) \leq c_{3}H_{\phi_{1}, \beta*}(t, x, c_{4}\rho_{D}(x, y)),$$

(1.26)

where $c_{1}, 1 \leq i \leq 4$, are positive constants that depend only on the characteristic constants $(C_{1}, C_{2}, C_{3}, C_{4})$ of $D$ and the constant parameters in $(J_{\phi_{1}, \beta_{*}, \leq})$ and $(J_{\phi_{2}, \beta^{*}, \geq})$ as well as in $(1.11)$ for $\phi_{1}$ and $\phi_{2}$, respectively.

**Remark 1.7.** (i) A prototype of the model considered in this paper is the following. Suppose that $D$ is a Lipschitz domain in $\mathbb{R}^{d}$, and that $m(dx) = dx$ is the Lebesgue measure on $\mathbb{R}^{d}$. Let

$$\mathcal{E}(f, f) = \frac{1}{2} \int_{D} \nabla u(x) \cdot A(x) \nabla v(x) dx + \frac{1}{2} \int_{D \times D} (f(x) - f(y))^{2} J(x, y) dx dy,$$

$$\mathcal{F} = W^{1,2}(D) := \{ f \in L^{2}(D; dx) : \nabla f \in L^{2}(D; dx) \}.$$
Here, $A(x) := (a_{ij}(x))_{1 \leq i,j \leq d}$ is a measurable $d \times d$ matrix-valued function on $D$ that is uniformly elliptic and bounded, and $J(x,y)$ is a symmetric function on $\mathbb{R}^d \times \mathbb{R}^d \setminus \text{diag}$ defined by

$$J(x,y) = \int_{[\alpha_1, \alpha_2]} \frac{c(\alpha, x, y)}{|x - y|^{d+\alpha} \Phi(|x - y|)} \nu(d\alpha),$$

where $\nu$ is a probability measure on $[\alpha_1, \alpha_2] \subset (0, 2)$, $\Phi$ is an increasing function on $[0, \infty)$ with

$$c_1 e^{c_2 r^\beta} \leq \Phi(r) \leq c_3 e^{c_4 r^\beta} \quad \text{for some } \beta \in [0, \infty],$$

and $c(\alpha, x, y)$ is a jointly measurable function that is symmetric in $(x, y)$ and is bounded between two positive constants. It is easy to check that $(\mathcal{E}, \mathcal{F})$ is a regular Dirichlet form on $L^2(D; dx)$. When $\beta = 0$ in (1.29) for $\Phi$ and $D = \mathbb{R}^d$, two-sided heat kernel estimates for the Dirichlet form (1.27) were obtained in [CK3].

(ii) When a Dirichlet form $(\mathcal{E}, \mathcal{F})$ is of the form (1.27) with $D = \mathbb{R}^d$ and $A(x) \equiv 0$, and $J(x, y)$ a symmetric function defined on $\mathbb{R}^d \times \mathbb{R}^d \setminus \text{diag}$ that satisfies

$$\frac{1}{c|x - y|^{d} \phi(|x - y|)} \leq J(x,y) \leq \frac{1}{c|x - y|^{d} \phi(|x - y|)} \quad \text{for } (x, y) \in \mathbb{R}^d \times \mathbb{R}^d \setminus \text{diag},$$

where $c \geq 1$, and $\phi$ satisfies $\phi(0) = 0$, $\phi(1) = 1$ and condition (1.11), it is established in [CK2] that the transition density function $q(t, x, y)$ of the associated pure jump process is jointly Hölder continuous and has the following two-sided sharp estimates: there is a constant $C > 1$ so that

$$C^{-1} p_0^{(J)}(t, |x - y|) \leq q(t, x, y) \leq C p_0^{(J)}(t, |x - y|) \quad \text{for } (t, x, y) \in (0, \infty) \times \mathbb{R}^d \times \mathbb{R}^d,$$

where

$$p_0^{(J)}(t, r) := \frac{1}{\phi^{-1}(t)^d} \wedge \frac{1}{r d\phi(r)}.$$

When a Dirichlet form $(\mathcal{E}, \mathcal{F})$ is of the form (1.27) with $D = \mathbb{R}^d$, $A(x) \equiv 0$ and $J(x, y)$ a symmetric function defined on $\mathbb{R}^d \times \mathbb{R}^d \setminus \text{diag}$ that satisfies the condition (J$^\phi$, $\beta$) for some $\beta \in (0, \infty]$ as well as the UJS condition when $\beta \in (0, \infty)$ (see Subsection 6.2 below for its definition, and [BBK] [CKK2] [CKK3] [CKK4] for discussions on UJS), it is shown in [CKK3] that the transition density function $q(t, x, y)$ of the associated pure jump process is jointly continuous and has the two-sided estimates

$$q(t, x, y) \sim p_0^{(J)}(t, |x - y|) \quad \text{for } (t, x, y) \in (0, \infty) \times \mathbb{R}^d \times \mathbb{R}^d,$$

where for $\beta \in (0, 1]$,

$$p_0^{(J)}(t, r) := \begin{cases} \frac{1}{\phi^{-1}(t)^d} \wedge \frac{t}{r \phi(r) \Phi(r)} & \text{if } t \leq 1, \\ t^{-d/2} \exp \left( -r^\beta \wedge (r^2/t) \right) & \text{if } t > 1; \end{cases}$$

and for $\beta \in (1, \infty]$,

$$p_0^{(J)}(t, r) := \begin{cases} \frac{1}{\phi^{-1}(t)^d} \wedge \frac{r \phi(r) \Phi(r)}{t} & \text{if } t \leq 1, r < 1, \\ t \exp \left( -(r(1 + \log^+(r/t))^{(\beta-1)/\beta} \wedge r^\beta) \right) & \text{if } t \leq 1, r \geq 1, \\ t^{-d/2} \exp \left( -(r(1 + \log^+(r/t))^{(\beta-1)/\beta} \wedge (r^2/t)) \right) & \text{if } t > 1. \end{cases}$$

See [CKK3] Theorems 1.2 and 1.4 for more details. Such processes with tempered jumps at infinity arise in statistical physics to model turbulence as well as in mathematical finance to model stochastic volatility; see, for example, [MS] [Wu]. When a Dirichlet form $(\mathcal{E}, \mathcal{F})$ is of the form (1.27) with $D = \mathbb{R}^d$, $A(x)$ uniformly elliptic and bounded, and $J(x, y)$ a symmetric function defined on $\mathbb{R}^d \times \mathbb{R}^d \setminus \text{diag}$ that satisfies the condition (J$^\phi$, $\beta$) for some $\beta \in (0, \infty]$ as well as the UJS condition, one can easily verify
by checking the expressions of $p^{(J)}_\beta(t,r)$ in each case that the estimates in Theorem 1.6 can be stated as

$$c_0^{-1} \left( t^{-d/2} \wedge (p^{(c)}(t,c_1 \rho_D(x,y)) + p^{(J)}_\beta(t,c_1 \rho_D(x,y))) \right) \leq q(t,x,y) \leq c_0 \left( t^{-d/2} \wedge (p^{(c)}(t,c_2 \rho_D(x,y)) + p^{(J)}_\beta(t,c_2 \rho_D(x,y))) \right),$$

where

$$p^{(c)}(t,r) := t^{-d/2} \exp(-r^2/t).$$

See [CKW3] for the corresponding results for symmetric diffusions with jumps on metric measure spaces where $\beta = 0$ but the diffusion part can be sub-diffusive and the upper growth exponent $\alpha^*$ for $\phi$ in (1.11) can be possibly larger than 2. We further remark that the two-sided heat kernel estimates for diffusions with jumps of the form (1.20)–(1.21) was first given in [CK3].

We now make some comments on the approach of the main result Theorem 1.6. Two-sided heat kernel estimates for symmetric diffusions with stable-like jumps on metric measure spaces were obtained in the recent paper [CKW3] by three of the authors of this paper. In that paper, some powerful tools for the study of stability of heat kernel estimates and parabolic Harnack inequalities for symmetric jump processes developed in [CKW1, CKW2] are adapted; on the other hand, a new self-improvement argument for upper bounds via exit time probability estimates is proposed, see [CKW3, Section 4.3] for details. However, as for symmetric diffusions with (sub- or super-)exponential decay jumps in the present paper, the approach of [CKW3] (in particular, the self-improvement argument for upper bounds as mentioned above) does not work, due to light tails of jumps for the associated process. The novelty of the proof for Theorem 1.6 is twofold.

- Concerning upper bounds, we apply the stability result from [CKW3] to obtain on-diagonal heat kernel estimates, and obtain precise tail probability estimates for truncated processes, the processes with jumps of size larger than some suitable level removed, to achieve off-diagonal estimates. Our main technical tool to obtain tail probability estimates for truncated processes is a new generalization of Davies’ method from our recent work [CKKW] for Dirichlet heat kernels of the truncated process. The Meyer’s construction and the relations among the original process, the truncated process and the killed truncated processes are also fully used.

- While we rely heavily on techniques from [CKW1, CKW2, CKW3] to study lower bounds for the heat kernel $q(t,x,y)$, a new observation here is that parabolic Harnack inequalities for full ranges do not hold. In fact under conditions (J$\phi$1, $\beta_*, \leq$) and (J$\phi$2, $\beta^*, \geq$) with $\beta_* < \beta^*$ in $\{\alpha^* \} \cup (0, \infty)$, the jumping kernel $J(x,y)$ may not satisfy the UJS condition, see Subsection 6.2. Thus, it follows from (the proof of) [CKW2, Proposition 3.3] that parabolic Harnack inequalities for full ranges do not hold. Our results in Theorem 1.6 in particular give a family of Feller processes that satisfy global two-sided heat kernel estimates, but the associated parabolic Harnack inequalities for full ranges fail, which is of independent interest. We shall mention that, under condition (J$\phi$, $\alpha^*, \leq$), we always have the joint Hölder continuity for the heat kernel $q(t,x,y)$ so that we can establish two sided estimates for $q(t,x,y)$ for every $t > 0$ and $x, y \in \overline{D}$ without introducing any exceptional set.

We further emphasize that our approach to Theorem 1.6 is quite robust in the sense that, we can deal with heat kernel estimates for diffusions with jumps in the general (VD) setting, and most importantly the case that the jumping kernel $J(x,y)$ is bounded by different weighted functions (in particular, with possibly different increasing functions $\phi_1$ and $\phi_2$ and possibly different indexes $\beta_*$ and $\beta^*$ in the exponential terms). In fact, we have more. Roughly speaking, we have upper bound estimates $H_{\phi_1, \beta_*}$ for the heat kernel $q(t,x,y)$ under condition (J$\phi_1$, $\beta_*, \leq$) with $\beta_* \in \{0_+ \} \cup (0, \infty)$; and we have lower bound estimates $H_{\phi_2, \beta^*}$ under conditions (J$\phi_1$, $\alpha^*, \leq$) and (J$\phi_2$, $\beta^*, \geq$) with $\beta^* \in (0, \infty]$. See Theorems 4.4 and 5.4 below for more details. We further mention that, in Theorems 4.6 and 5.5 we do not require the underlying state space...
In the Appendix of this paper, we have that for $f$ has infinite diameter. This section is technically the most important section in this paper. Here, we start from every point in $D$ and is in fact a Feller process having the strong Feller property. By using the results from [CKW2, CKW3], we further present near-diagonal lower bounds for the heat kernel $q(t,x,y)$ and possibly different strictly increasing functions $\phi_1$ and $\phi_2$. Its proof uses the approaches from [CKW1, CKW2, CKW3]. The main result of this paper, Theorem 1.10 follows directly by combining Theorem 1.14 with Theorem 5.3 as well as Theorem 1.5. Some preliminary integral estimates are given in the Appendix (Subsection 6.1) of this paper, as well as a brief discussion in Subsection 6.2 on a property concerning the UJS condition.

Notations Throughout this paper, we use $c_1, c_2, \ldots$ to denote generic constants, whose exact values are not important and can change from one appearance to another. The labeling of the constants $c_1, c_2, \ldots$ starts anew in the statement of each result. For $p \in [1, \infty]$, we will use $\|f\|_p$ to denote the $L^p$-norm in $L^p(D;m)$. For any open subset $U$ of $D$, $C_c(U)$ is the space of continuous functions on $U$ having compact support with respect to the $\rho_D$-metric.

2 The case $\beta_* = \beta^* = 0$

We first give a general statement that holds under $(J_{\phi, \leq 1})$ given in Definition 1.3(vi).

Proposition 2.1. Suppose that condition $(J_{\phi, \leq 1})$ holds and $(\mathcal{E}, \mathcal{F})$ is the bilinear form defined by (1.9) – (1.10). Then there exists a constant $c > 0$, depending on constants in $(J_{\phi, \leq 1})$ and (1.11) for $\phi$ such that

$$\mathcal{E}_1(f,f) \leq c \mathcal{E}^{0, \text{ref}}_1(f,f) \quad \text{for every } f \in \mathcal{F},$$

where $\mathcal{E}_1(f,f) := \mathcal{E}(f,f) + \|f\|_2^2$ and $\mathcal{E}^{0, \text{ref}}_1(f,f) := \mathcal{E}^{0, \text{ref}}(f,f) + \|f\|_2^2$. Consequently, $(\mathcal{E}, \mathcal{F})$ is a regular Dirichlet form on $L^2(D;m)$ and so there is a symmetric Hunt process $Y$ on $D$ associated with it. Moreover, it is conservative in the sense that the Hunt process $Y$ has infinite lifetime.

Proof. Recall that $Z := (Z_t)_{t \geq 0}$ is the diffusion process associated with the regular Dirichlet form $(\mathcal{E}^{0, \text{ref}}, \mathcal{F})$ on $L^2(D;m)$, and that $\alpha^* \in (0,2)$ is the exponent in (1.11) for the strictly increasing function $\phi$. Let $S := (S_t)_{t \geq 0}$ be an $(\alpha^*/2)$-stable subordinator that is independent of $Z$. The subordinated diffusion $\{Z_{S_t}, t \geq 0\}$ is $m$-symmetric and we use $J_{\phi^*}(x,y)$ to denote its jumping kernel. By condition $(J_{\phi, \leq 1})$ and Lemma 6.2 from the Appendix of this paper, we have that for $f \in \mathcal{F}$,

$$\mathcal{E}_1(f,f) = \mathcal{E}^{0, \text{ref}}(f,f) + \frac{1}{2} \int_D \int_D (f(y) - f(x))^2 J(x,y)1_{\{\rho_D(x,y) \leq 1\}} m(dx) m(dy).$$
where \( \mathcal{C} \) for these definitions (see also Subsection 3.1 for some details). First, we write and cut-off Sobolev inequality for general Dirichlet forms. The readers are referred to \([\text{AB, CKW1, CKW2}]\), main result of \([\text{CKW3}]\), which we explain below. In the proof of this part, we use the Poincaré inequality \( Y \) the process \( E \) diameter is equivalent to \( \mathcal{D} \). Recall that, by \([\text{GH, Corollary 5.3}]\), the underlying state space \( \Gamma \) 0 is the carré du champ operator corresponding to \((\mathcal{E}, \mathcal{F})\). Let \( \text{MUW} \) Theorem 1, under (VD) and condition \((J_{\phi,1})\), the process \( Y \) is conservative.

**Proof of Theorem 1.4** Recall that, by \([\text{GH Corollary 5.3}]\), the underlying state space \( D \) has infinite diameter is equivalent to \( D \) has infinite volume. Then the proof is divided into two parts.

1. The case that \( \text{diam}(D) = \infty \). In this case, \( D \) has infinite volume and the desired result follows from the main result of \([\text{CKW3}]\), which we explain below. In the proof of this part, we use the Poincaré inequality and cut-off Sobolev inequality for general Dirichlet forms. The readers are referred to \([\text{AB, CKW1, CKW2}]\) for these definitions (see also Subsection 3.1 for some details). First, we write

\[
\mathcal{E}(u, u) = \mathcal{E}^{0,\text{ref}}(u, u) + \mathcal{E}^{(J)}(u, u),
\]

(2.1)

where

\[
\mathcal{E}^{(J)}(u, u) = \frac{1}{2} \int_{D \times D} (u(x) - u(y))^2 J(x, y) m(dx) m(dy).
\]

Recall that \( \Gamma_0 \) is the carré du champ operator corresponding to \((\mathcal{E}^0, \mathcal{F}^0)\). Let

\[
d \frac{\Gamma_J(u, u)}{dm}(x) = \int_D (u(y) - u(x))^2 J(x, y) m(dy), \quad u \in \mathcal{F}.
\]

Denote by \( \Gamma \) the carré du champ operator corresponding to \((\mathcal{E}, \mathcal{F})\). Then,

\[
\Gamma(u, u) = \Gamma_0(u, u) + \Gamma_J(u, u), \quad u \in \mathcal{F}.
\]

According to Theorem 1.2 and \([\text{AB Remark 1.11(4)}] \) (or \([\text{GH, p. 1942}]\)), the Poincaré inequality \( PI(2) \) and the cut-off Sobolev inequality \( CS(2) \) hold for \((\mathcal{E}^{0,\text{ref}}, \mathcal{F}); \) that is, there is a constant \( c_1 > 0 \) so that for every \( x \in \bar{D}, r > 0 \) and \( f \in \mathcal{F}, \)

\[
\min_{a \in \mathbb{R}} \int_{B_D(x, r)} (f(y) - a)^2 m(dy) \leq c_1 r^2 \int_{B_D(x, r)} \Gamma_0(0, f)(x) m(dx);
\]

and there is a constant \( c_2 > 0 \) such that for every \( 0 < r \leq R, x_0 \in \bar{D} \) and any \( f \in \mathcal{F}, \)

\[
\int_{B_D(x_0, R)} f^2 d\Gamma_0(\psi, \psi) \leq \frac{c_2}{r^2} \int_{B_D(x_0, R)} f^2 dm,
\]

(2.2)

where \( \psi(x) = h(\rho_D(x_0, x)) \) is a non-negative cut-off function on \( B(x_0, r) \subset B(x_0, R) \) with \( h \in C^1([0, \infty]) \) being such that \( 0 \leq h \leq 1, h(s) = 1 \) for all \( s \leq R, h(s) = 0 \) for \( s \geq R + r \) and \( |h'(s)| \leq 2/r \) for all \( s \geq 0. \)
Furthermore, by (J$_{\phi}$), with $\alpha^* < 2$ and Remark 1.7, the following cut-off Sobolev inequality $CS(\phi)$ holds for $(E^{(j)},\mathcal{F})$: for every $0 < r \leq R$, $C_0 \in (0,1]$, $x_0 \in \tilde{D}$ and any $f \in \mathcal{F}$,

$$\int_{B_D(x_0,R+(1+C_0)r)} f^2 \, d\Gamma_J(\psi,\psi) \leq \frac{c_2}{\phi(r)} \int_{B_D(x_0,R+(1+C_0)r)} f^2 \, dm,$$

where $\psi(x) = h(\rho_D(x_0,x))$ is a non-negative cut-off function on $B(x_0,r) \subset B(x_0, R)$ as in (2.4). (Note that $CS(2)$ that appeared before is a special case of $CS(\phi)$ when $\phi(r) = r^2$ for the corresponding Dirichlet form.)

On the other hand, according to Theorem 1.13 and Corollary 1.3 and Theorem 1.18, we know that the following Poincaré inequality $PI(\phi)$ is also satisfied for $(E^{(j)},\mathcal{F})$: there is a constant $c_4 > 0$ so that for every $x \in \tilde{D}$, $r > 0$ and $f \in \mathcal{F}$,

$$\min_{a \in \mathbb{R}} \int_{B_D(x,r)} (f(y) - a)^2 \, m(dy) \leq c_4 \phi(r) \left( \int_{B_D(x,r)} \Gamma_J(\psi,\psi)(x) \, m(dx) \right.$$

$$+ \int_{B_D(x,r)} \int_{B_D(x,r)} (f(x) - f(y))^2 J(x,y) \, m(dx) \, m(dy) \left. \right);$$

and for every $0 < r \leq R$, $C_0 \in (0,1]$, $x_0 \in \tilde{D}$ and any $f \in \mathcal{F}$,

$$\int_{B_D(x_0,R+(1+C_0)r)} f^2 \, d\Gamma(\psi,\psi) \leq \frac{c_6}{\phi(r)} \int_{B_D(x_0,R+(1+C_0)r)} f^2 \, dm,$$

where $\psi(x) = h(\rho_D(x_0,x))$ is a non-negative cut-off function on $B(x_0,r) \subset B(x_0, R)$ as in (2.4). This together with condition (J$_{\phi}$) and Theorem 1.13 and Theorem 1.18 yields the desired assertion.

(2) The case that $\text{diam}(D) < \infty$. In particular, $\tilde{D}$ is compact. (This is because, by (VD), bounded sets are totally bounded, and totally bounded closed sets on complete metric spaces are compact.) As mentioned in Remark 1.19, with minor adjustments of the proofs localized versions of the results in $\text{CKW1} \text{ CKW2} \text{ CKW3}$ should continue to hold. In particular, the heat kernel $q(t,x,y)$ of $(E,\mathcal{F})$ is jointly Hölder continuous on $(t,x,y)$, and (2.1) holds for all $t \in (0,t_0]$ and $x,y \in \tilde{D}$, where $t_0 = \phi(r_0)$ for some $r_0 \in (0,\text{diam}(D))]$. The readers are referred to $\text{GHH}$ for heat kernel estimates for stable-like Dirichlet forms with the Ahlfors $d$-set condition in both bounded and unbounded cases. Next, we will claim that for all $t \geq t_0$ and $x,y \in \tilde{D}$,

$$q(t,x,y) \simeq 1. \quad (2.3)$$

Since $\tilde{D}$ is compact, thanks to the joint Hölder continuity of $q(t,x,y)$ and the fact that holds for $t \in (0,t_0]$ and $x,y \in \tilde{D}$, the process $Y$ associated with $(E,\mathcal{F})$ is irreducible and has the strong Feller property. Hence, by a modification of Doeblin's celebrated result (see p. 365, Theorem 3.1]), the process $Y$ is strongly exponentially ergodic in the sense that there exist positive constants $\lambda_1$ and $c_0$ so that

$$\sup_{x \in \tilde{D}} |E_x f(Y_t) - \bar{m}(f)| \leq c_0 e^{-\lambda_1 t} \|f\|_{\infty} \quad \text{for every } t > 0 \text{ and } f \in B_b(\tilde{D}), \quad (2.4)$$

where $\bar{m} = m(D)^{-1}m$ is the unique invariant probability measure of the process $Y$. For any $y \in \tilde{D}$, applying $f(z) = q(t_0,z,y)$ into (2.4), we find that for any $t > 0$,

$$\left| q(t+t_0,x,y) - \frac{1}{m(D)} \right| \leq c_1 e^{-\lambda_1 t}, \quad (2.5)$$

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where \(c_1 = c_0 \sup_{x,u \in \mathcal{D}} q(t_0, z, u) < \infty\), due to the fact that \(\text{[1.2]}\) holds for \(t \in (0, t_0)\) and \(x, y \in \mathcal{D}\). Here, we used the semigroup property of \(q(t, x, y)\) and the conservativeness of the process \(Y\). According to \(\text{[2.5]}\), for all \(t > 0\) and \(x, y \in \mathcal{D}\),

\[
q(t + t_0, x, y) \leq \frac{1}{m(D)} + c_1 e^{-\lambda_1 t} \leq c_1 + \frac{1}{m(D)} =: c_2
\]

and

\[
q(t + t_0, x, y) \geq \frac{1}{m(D)} - c_1 e^{-\lambda_1 t}.
\]

In particular, we obtain that for all \(t \in (t_0, \infty)\) and \(x, y \in \mathcal{D}\), \(q(t, x, y) \leq c_2\), and that for all \(t \in [t_1, \infty)\) and \(x, y \in \mathcal{D}\), \(q(t, x, y) \geq \frac{1}{2m(D)}\), where \(t_1 > t_0\) satisfies that \(2m(D)c_1 \leq e^{\lambda_1(t_1-t_0)}\). Furthermore, for any \(t \in (t_0, t_1]\), we take \(n \geq 1\) such that \(t' := t/n \in [t_0/2, t_0]\). Then, by the semigroup property of \(q(t, x, y)\), for any \(x, y \in \mathcal{D}\),

\[
q(t, x, y) = \int_{\mathcal{D}} \cdots \int_{\mathcal{D}} q(t', x, z_1)q(t', z_1, z_2) \cdots q(t', z_{n-1}, y) m(dz_1)m(dz_2) \cdots m(dz_{n-1})
\]

\[
\geq \left( m(D) \left( \inf_{s \in [t_0/2, t_0]} \inf_{z,u \in \mathcal{D}} q(s, z, u) \right) \right)^{n-1}
\]

\[
\geq \left( 1 \wedge \left( m(D) \left( \inf_{s \in [t_0/2, t_0]} \inf_{z,u \in \mathcal{D}} q(s, z, u) \right) \right) \right)^{2t_1/t_0} =: c_3 > 0,
\]

where \(c_3\) may depend on \(m(D)\). Here in the last inequality, we used again the fact that \(\text{[1.2]}\) holds for all \(t \in (0, t_0)\) and \(x, y \in \mathcal{D}\). Putting all the estimates above together, we get \(\text{[2.6]}\), and, consequently, the desired assertion when \(\text{diam}(D) < \infty\).

\[\square\]

### 3 General framework under condition \((J_{\phi,0^+,\leq})\)

In this section, \((\mathcal{E}, \mathcal{F})\) is the bilinear form defined by \(\text{[1.9]}-\text{[1.10]}\) with \(J(x, y)\) satisfying condition \((J_{\phi,0^+,\leq})\) of Definition \(\text{[1.3]}\text{iv})\), where \(\phi\) is a strictly increasing function on \([0, \infty)\) satisfying \(\phi(0) = 0, \phi(1) = 1\) and \((\text{[1.1]}\). By \(\text{[1.14]}\) and Proposition \(\text{[2.4]}\) \((\mathcal{E}, \mathcal{F})\) is a regular Dirichlet form on \(L^2(D; m)\) and there is a conservative symmetric Markov process \(Y\) associated with it. We will show the existence of the heat kernel \(q(t, x, y)\) for the Hunt process \(Y\) and its joint Hölder regularity, and obtain on-diagonal upper bound estimates as stated in Theorem \(\text{[1.5]}\) as well as its near diagonal lower bound estimates for \(q(t, x, y)\).

#### 3.1 On-diagonal upper bound estimates

Recall that \(\Gamma_0\) is the carré du champ operator corresponding to the strongly local regular Dirichlet form \((\mathcal{E}^0, \mathcal{F}^0)\) on \(L^2(E; m)\). The inequality \(\text{[1.4]}\) in particular gives the following Poincaré inequality \(\text{PI}(2)\) of \(\text{[3.1]}\) for the regular Dirichlet form \((\mathcal{E}, \mathcal{F})\) on \(L^2(D; m)\), which in turn implies (see, e.g. [9]) the local Nash-type inequality \(\text{[3.2]}\) for \((\mathcal{E}, \mathcal{F})\).

**Lemma 3.1.** Suppose that \((J_{\phi,0^+,\leq})\) holds. Then there is a constant \(c_1 > 0\) so that for every \(x \in \mathcal{D}\), \(r > 0\) and \(f \in \mathcal{F}\),

\[
\min_{a \in \mathbb{R}} \int_{B_D(x, r)} (f(y) - a)^2 m(dy) \leq c_1 r^2 \left( \int_{B_D(x, r)} \Gamma_0(f)(x) m(dx) \right. \left. + \int_{B_D(x, r)} \int_{B_D(x, r)} (f(x) - f(y))^2 J(x, y) m(dx) m(dy) \right).
\]
Consequently, there is a constant $c_2 > 0$ such that for any $r > 0$ and $u \in \mathcal{F} \cap L^1(D;m)$,

$$\|u\|_2^2 \leq c_2 \left( \frac{\|u\|_1^2}{\inf_{z \in \text{supp}|u|} V_D(z,r)} + r^2 \mathcal{E}(u,u) \right).$$  \hspace{1cm} (3.2)

Note that under general (VD) condition on the metric measure space ($\bar{D}, \rho_D, m$), one can not expect the global Nash-type inequality such as that of [CK2, Theorem 2.5] to hold. In the present setting, we can only expect the local Nash-type inequality (3.2) as shown in Lemma 3.1.

We next introduce a localized version of Faber-Krahn inequality. For any open set $U \subset \bar{D}$, let $\mathcal{F}_U$ be the $\mathcal{E}_1$-closure in $\mathcal{F}$ of $\mathcal{F} \cap C_c(U)$. Define

$$\lambda_1(U) = \inf \{ \mathcal{E}(f,f) : f \in \mathcal{F}_U \text{ with } \|f\|_2 = 1 \}.$$

**Definition 3.2.** We say that the (localized version of) *Faber-Krahn inequality* $\text{FK}(2)$ holds for the regular Dirichlet form $(\mathcal{E}, \mathcal{F})$ on $L^2(D;m)$ if there exist positive constants $C$ and $\nu$ such that for any $r \in (0, \nu \text{diam}(D))$, any ball $B_D(x,r)$ and any open set $U \subset B_D(x,r)$,

$$\lambda_1(U) \geq \frac{C}{r^2} (V_D(x,r)/m(U))^\nu.$$

In view of (RVD) of [17] and the Poincaré inequality $\text{PI}(2)$ of [3.1] for $(\mathcal{E}, \mathcal{F})$, we conclude by the proof of [CKW1, Proposition 7.4] (see also the proof of [GHH, Lemma 3.5]) that $\text{FK}(2)$ holds for $(\mathcal{E}, \mathcal{F})$.

We next show that the cut-off Sobolev inequality $\text{CS}(2)$ holds for $(\mathcal{E}, \mathcal{F})$; see [CKW3, Definition 1.6] for its definition. Recall again that $\Gamma_0$ is the carré du champ operator corresponding to $(\mathcal{E}^0, \mathcal{F}^0)$. A function $f$ is said to be locally in $\mathcal{F}^0$, denoted as $f \in \mathcal{F}^0_{\text{loc}}$, if for every relatively compact subset $E$ of $U$, there is a function $g \in \mathcal{F}^0$ such that $f = g$ m-a.e. on $U$. It is well known that the following chain rule holds (see [CE, Theorem 4.3.7] and [FOTI, Theorem 3.2.2]): for every $C^1$ function $h : \mathbb{R} \to \mathbb{R}$ and bounded function $f \in \mathcal{F}^0_{\text{loc}}$,

$$\Gamma_0(h(f), h(f)) = (h'(f))^2 \Gamma_0(f,f).$$

We remark that the assumption $h(0) = 0$ in [CE, Theorem 4.3.7] and [FOTI, Theorem 3.2.2] is not needed since, by the strongly local property of $(\mathcal{E}^0, \mathcal{F}^0)$ and the fact that $1 \in \mathcal{F}^0_{\text{loc}}$,

$$\Gamma_0(h(f) - h(0), h(f) - h(0)) = \Gamma_0(h(f), h(f)).$$

In particular, for every bounded function $f \in \mathcal{F}^0_{\text{loc}}$, $\Gamma_0(e^f, e^f) = e^{2\int} \Gamma_0(f,f)$. Let

$$\frac{d\Gamma_J(u,u)}{d\nu}(x) = \int_D (u(y) - u(x))^2 J(x,y) m(dy), \quad u \in \mathcal{F}.$$ 

Denote by $\Gamma$ the carré du champ operator corresponding to $(\mathcal{E}, \mathcal{F})$. Then,

$$\Gamma(u,u) = \Gamma_0(u,u) + \Gamma_J(u,u), \quad u \in \mathcal{F}.$$ 

For any fixed $x_0 \in \bar{D}$ and $r, R > 0$, we choose a non-negative cut-off function $\psi(x) = h(\rho_D(x_0,x))$, where $h \in C^1([0, \infty))$ such that $0 \leq h \leq 1$, $h(s) = 1$ for all $s \leq R$, $h(s) = 0$ for $s \geq R + r$ and $|h'(s)| \leq 2/r$ for all $s \geq 0$. It holds that for all $x \in \bar{D}$,

$$\frac{d\Gamma_0(\psi, \psi)}{d\nu}(x) = h'(\rho_D(x_0,x))^2 \frac{d\Gamma_0(\rho_D(x_0, \cdot), \rho_D(x_0, \cdot))}{d\nu}(x) \leq 4/r^2.$$

On the other hand, under condition $(\mathcal{J}_{\phi, a, \phi, \alpha, \leq})$,

$$\frac{d\Gamma_J(\psi, \psi)}{d\nu}(x) = \int_D (\psi(x) - \psi(y))^2 J(x,y) m(dy) \leq \frac{4}{r^2} \int_D \rho_D(x, y)^2 J(x, y) m(dy) \leq \frac{c_1}{r^2}.$$
In this subsection, we present the sketch of the proof of near diagonal lower bound estimates for the heat kernel $q(t,x,y)$. As shown in Subsection 3.1, the symmetric Hunt process $Y$, or equivalently, its associated regular Dirichlet form $(\mathcal{E},\mathcal{F})$ on $L^2(\mathcal{D};m)$, satisfies PI(2), CS(2) and (J.φ,≤). Hence, the elliptic Hölder regularity (EHR) holds for the process $Y$; that is, there exist constants $c > 0$, $\theta \in (0,1]$ and $\varepsilon \in (0,1)$ such that for every $x_0 \in \mathcal{D}$, $r \in (0,\text{diam}(\mathcal{D})/3)$ and every bounded measurable function $u$ on $\mathcal{D}$ that is harmonic in $B_\mathcal{D}(x_0, r)$ with respect to the Hunt process $Y$,

$$|u(x) - u(y)| \leq c \left( \frac{\rho_{D}(x,y)}{r} \right)^{\theta} \text{ess sup}_{\mathcal{D}}|u|$$

for any $x, y \in B_\mathcal{D}(x_0, \varepsilon r)$. On the other hand, two-sided exit time estimates (1.25) hold for $Y$. With those two properties, we can follow the proof of [CKW2, Proposition 4.10] to get the following proposition. Recall
that \( q(t,x,y) \) is the transition density function of \( Y \). For any open set \( U \subset \bar{D} \), denote by \( q^U(t,x,y) \) the transition density function of the subprocess \( Y^U \) of \( Y \) killed up exiting \( U \), which is the (Dirichlet) heat kernel associated with the part Dirichlet form \((\mathcal{E}, \mathcal{F}_U)\).

**Proposition 3.3.** Under \((J_{0,0,+}, \leq)\), there exist \( \varepsilon \in (0,1) \) and \( c_1 > 0 \) such that for any \( x_0 \in \bar{D}, r \in (0, \text{diam}(D)/3) \) and \( 0 < t \leq r^2 \),

\[
q^{B_D(x_0,r)}(t,x,y) \geq \frac{c_1}{V_D(x_0, \sqrt{t})}, \quad x,y \in B_D(x_0, \varepsilon t^{1/2}).
\]

In particular, there exist \( c_2, c_3 > 0 \) such that for any \( t \in (0, \text{diam}(D)^2/9) \) and any \( x,y \in \bar{D} \) with \( \rho_D(x,y) \leq c_2 t^{1/2} \),

\[
q(t,x,y) \geq \frac{c_3}{V_D(x, \sqrt{t})}.
\]

**Remark 3.4.** Note that \( \text{Pl}(2) \) of [8] follows directly from [11] as \( \mathcal{E}(u,u) \geq \mathcal{E}^{\text{ref}}(u,u) \), so it holds under \((J_{0,0,+}, \leq)\). Under condition \((J_{0,0,+}, \leq)\) of Definition 1.3 v), we can easily see that CS(2) is satisfied. With these at hand, all assertions in this section still hold true under condition \((J_{0,0,+}, \leq)\) except that \[12\] holds for all \( t \in (0,1) \) and \( x,y \in \bar{D} \) with \( \rho_D(x,y) \leq 1 \), \[12\] holds for all \( x \in \bar{D} \) and \( r \in (0,1, \text{diam}(D)/3) \), and \[8\] holds for all \( t \in (0,1, \text{diam}(D)^2/9) \) and any \( x,y \in \bar{D} \) with \( \rho_D(x,y) \leq c_2 t^{1/2} \).

## 4 Upper bound estimates under \((J_{0,0,+}, \leq)\) with \( \beta_* \in (0, \infty) \)

In this section, the inner uniform domain \( D \) is assumed to be unbounded. We will derive full upper bounds of the heat kernel associated with the Dirichlet form \((\mathcal{E}, \mathcal{F})\) of [11] or (equivalently, of its associated Hunt process \( Y \)) under the assumption \((J_{0,0,+}, \leq)\) for some \( \beta_* \in (0, \infty) \). Recall that \( \phi_1 \) is a strictly increasing function on \([0, \infty)\) satisfying \( \phi_1(0) = 0, \phi_1(1) = 1 \) and [11]. In view of [11], all the assertions in Section 3 hold in this section under \((J_{0,0,+}, \leq)\).

To study off-diagonal upper bounds for the transition density function \( q(t,x,y) \) of \( Y \), we will use the following Meyer’s construction [Mey] which is to decompose the process by removing (large) jumps. For any \( \lambda > 0 \), set

\[
J^{(\lambda)}(\xi,\eta) := \mathbb{1}_{(\rho_D(\xi,\eta) \leq \lambda)} J(\xi,\eta) \quad \text{and} \quad J^{(\lambda)}(\xi,\eta) := \mathbb{1}_{(\rho_D(\xi,\eta) > \lambda)} J(\xi,\eta).
\]

One can remove the jumps of \( Y \) of size larger than \( \lambda \) to obtain a new process \( Y^{(\lambda)} \) as follows. For each \( x \in \bar{D} \), start a copy \( Y^1 \) of the process \( Y \) with initial position \( x \). Run it until the stopping time

\[
T_1 := \inf \{ t > 0 : \rho_D(Y^1_t, Y^1_t) > \lambda \},
\]

and define \( Y^{(\lambda)}_t := Y^1_t \) for \( t \in [0,T_1) \). If \( T_1 < \infty \), let \( Y^2 \) be a copy of \( Y \) that is independent of \( Y^1 \) and starts from the point \( Y^1_{T_1} \). Let

\[
T_2 := \inf \{ t > 0 : \rho_D(Y^2_t, Y^2_t) > \lambda \},
\]

and define \( Y^{(\lambda)}_{T_1+t} := Y^2_t \) for \( t \in [0,T_2) \). Repeat this procedure until one of \( T_n \)'s is infinity or countably many times if all \( T_n < \infty \). Let \( \zeta^{(\lambda)} = \infty \) in the first case and \( \zeta^{(\lambda)} = \sum_{k=1}^{\infty} T_k \) in the second case. Meyer [Mey] showed that the resulting process \( Y^{(\lambda)} \) is a Hunt process having jumping kernel \( J^{(\lambda)}(x,y) \) and lifetime \( \zeta^{(\lambda)} \) that can start from every point in \( D \). It is easy to see that the Dirichlet form of \( Y^{(\lambda)} \) on \( L^2(D;m) \) is \((\mathcal{E}^{(\lambda)}, \mathcal{F})\), where

\[
\mathcal{E}^{(\lambda)}(v,v) = \frac{1}{2} \int_D \Gamma(0,v,x) m(dx) + \frac{1}{2} \int_D \int_D (v(\xi) - v(\eta))^2 J^{(\lambda)}(\xi,\eta) m(d\eta) m(d\xi).
\]

Note that by \((J_{0,0,+}, \leq)\) and Lemma [11],

\[
\sup_{\eta \in \bar{D}} \int_{\bar{D}} J^{(\lambda)}(\eta,\xi) m(d\xi) \leq c_1(\lambda) < \infty.
\]

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Thus, for \( v \in \mathcal{F} \),
\[
0 \leq \mathcal{E}(v, v) - \mathcal{E}^{(\lambda)}(v, v) \leq 2 \int_D v(\xi)^2 \int_D J^{(\lambda)}(\xi, \eta) m(d\eta) m(d\xi) \leq 2c_1(\lambda) \int_D v(\xi)^2 m(d\xi)
\]
and so
\[
(1 + 2c_1(\lambda))^{-1} \mathcal{E}(v, v) \leq \mathcal{E}^{(\lambda)}(v, v) \leq \mathcal{E}(v, v) \quad \text{for every } v \in \mathcal{F}.
\]
Thus a set \( A \subset \hat{D} \) is \( \mathcal{E}^{(\lambda)} \)-polar if and only if it is \( \mathcal{E} \)-polar.

According to (1.24), (J_{\phi,\beta,\leq}) and the proof of [CKKW, Lemma 5.2], the process \( Y^{(\lambda)} \) admits the transition density function \( q^{(\lambda)}(t,x,y) \) defined on \((0, \infty) \times \hat{D} \times \hat{D} \), which satisfies that for all \( t > 0 \) and all \( x,y \in \hat{D} \),
\[
q^{(\lambda)}(t,x,y) \leq c_2 \left( \frac{1}{V_D(x,\sqrt{t})} + \frac{1}{V_D(y,\sqrt{t})} \right) \exp \left( \frac{ct^2}{\lambda^2} - \frac{c_4 \rho_D(x,y)}{\lambda} \right).
\]
Consequently, by (1.3), for all \( t > 0 \) and \( x,y \in \hat{D} \),
\[
q^{(\lambda)}(t,x,y) \leq \frac{c_5}{V_D(x,\sqrt{t})} \left( 1 + \frac{\rho_D(x,y)}{\sqrt{t}} \right)^{d_2} \exp \left( \frac{ct^2}{\lambda^2} - \frac{c_4 \rho_D(x,y)}{\lambda} \right). \tag{4.1}
\]
On the other hand, there is a close relation between \( q(t,x,y) \) and \( q^{(\lambda)}(t,x,y) \) in the viewpoint of Meyer’s decomposition. In particular, according to (J_{\phi,\beta,\leq}) and (CKKW, 4.34), Proposition 4.24 and its proof, for any \( t, \lambda > 0 \) and any \( x,y \in \hat{D} \),
\[
q(t,x,y) \leq q^{(\lambda)}(t,x,y) + \frac{c_6 t}{V_D(x,\phi_\lambda(\lambda))} \exp(c_7 \lambda^{d_2}). \tag{4.2}
\]

In order to obtain sharp upper bounds of \( q(t,x,y) \), we need to refine the estimate (4.1) for \( q^{(\lambda)}(t,x,y) \). For this, we will use the following bounds for the Dirichlet heat kernel \( q^{(\lambda),B_D(x_0,\lambda)}(t,x,y) \) of \( Y^{(\lambda)} \) in \( B_D(x_0,\lambda) \), which are based on a generalized Davies’ method recently developed in [CKKW]. For an open set \( U \subset \hat{D} \), let \( q^{(\lambda),U}(t,x,y) \) be the (Dirichlet) heat kernel of the subprocess \( Y^{(\lambda),U} \) of \( Y^{(\lambda)} \) killed up exiting \( U \); that is,
\[
\mathbb{E}_x \left[ f(Y^{(\lambda)}_t) : t < \tau^{(\lambda)}_U \right] = \int_U f(y) q^{(\lambda),U}(t,x,y) m(dy), \quad t > 0, \ x \in U, \ f \in L^2(U;m),
\]
where \( \tau^{(\lambda)}_U \) is the first exit time from \( U \) by the process \( Y^{(\lambda)} \). Recall that in this section, we assume that \( \text{diam}(\hat{D}) = \infty \). Then, by (1.6), (1.7), (1.24) and [CKKW, Theorem 5.1], for any \( \beta_* \in (0, \infty) \) and \( t \geq 2 \), there exists a constant \( c_0 > 0 \) such that for any \( x_0 \in \hat{D} \), \( \lambda > 0 \), \( f \in \text{Lip}_c(\hat{D}) \), \( t > 0 \) and any \( x,y \in B_D(x_0,\lambda) \),
\[
q^{(\lambda),B_D(x_0,\lambda)}(t,x,y) \leq \frac{c_0 t}{V_D(x_0,\lambda)} \left( \frac{\lambda}{\sqrt{t}} \right)^{d_1} \left( \frac{\lambda}{\sqrt{t}} \right)^{d_2} \exp \left( -|f(y) - f(x)| + 2\Lambda^{(\lambda)}(f)^2 t \right), \tag{4.3}
\]
where \( d_1, d_2 > 0 \) are the constants in (1.7) and (1.6) respectively, and
\[
\Lambda^{(\lambda)}(f) = \|e^{-2\Gamma^{(\lambda)}(e^f)}\|_\infty \vee \|e^{2\Gamma^{(\lambda)}(e^{-f})}\|_\infty,
\]
with
\[
\Gamma^{(\lambda)}(f)(\xi) = \Gamma(f,f)(\xi) + \int_D (f(\xi) - f(\eta))^2 f^{(\lambda)}(\xi, \eta) m(d\eta).
\]

**Proposition 4.1.** Suppose that (J_{\phi,\beta,\leq}) holds for some \( \beta_* \in (0, \infty) \). Then the following hold.

(i) If \( \beta_* \in (0,1] \), then, for any \( t \geq 2 \), there exist \( c_1, c_2 \) and \( C_* > 0 \) such that for all \( x_0 \in \hat{D} \), \( \lambda > 0 \), \( t > 0 \) and \( x,y \in B_D(x_0,\lambda) \),
\[
q^{(\lambda),B(x_0,\lambda)}(t,x,y) \leq \frac{c_1}{V_D(x_0,\lambda)} \left( \frac{\lambda}{\sqrt{t}} \right)^{d_1} \left( \frac{\lambda}{\sqrt{t}} \right)^{d_2} \times \begin{cases} e^{-c_2 \rho_D(x,y)^2/t} & \text{if } C_* \lambda^{\beta_*} \rho_D(x,y) < t, \\
 e^{-c_2 \lambda^{\beta_* - 1} \rho_D(x,y)} & \text{if } C_* \lambda^{\beta_*} \rho_D(x,y) \geq t. 
\end{cases}
\]
(ii) If $\beta_\ast \in (1, \infty)$, then, for every $l \geq 2$ and $C_\ast \in (0, 1)$, there exist $c_1, c_2, c_3 > 0$ such that for all $x_0 \in \tilde{D}$, $
abla > 0$ and $x, y \in B_\tilde{D}(x_0, 1\nabla)$ with $\rho_\tilde{D}(x, y) \leq t/C_\ast$,

$$q^{(\nabla), B(x_0, 1\nabla)}(t, x, y) \leq \frac{c_1}{V_\tilde{D}(x_0, \nabla)} \left( \left( \frac{\nabla}{\sqrt{t}} \right)^{d_1} \vee \left( \frac{\nabla}{\sqrt{t}} \right)^{d_2} \right) \exp \left( -\frac{c_2 \rho_\tilde{D}(x, y)^2}{t} \right);$$

and that for all $x_0 \in \tilde{D}$, $\nabla > 0$, $t > 0$ and $x, y \in B_\tilde{D}(x_0, 1\nabla)$ with $\rho_\tilde{D}(x, y) \leq t/C_\ast$,

$$q^{(\nabla), B(x_0, 1\nabla)}(t, x, y) \leq \frac{c_1}{V_\tilde{D}(x_0, \nabla)} \left( \left( \frac{\nabla}{\sqrt{t}} \right)^{d_1} \vee \left( \frac{\nabla}{\sqrt{t}} \right)^{d_2} \right) \exp \left( -c_3 \rho_\tilde{D}(x, y) \left( \log \frac{\rho_\tilde{D}(x, y)}{t} \right)^{\frac{\beta_\ast - 1}{\nabla}} \right).$$

(iii) Suppose that the jumping kernel $J(x, y)$ satisfies (J\(\phi_\ast, \tilde{\nabla}, \leq\)), or equivalently, [1.13] with $\phi_\ast$ in place of $\phi$ there. Then, for every $l \geq 2$ and $C_\ast \in (0, 1)$, there exist $c_1, c_2, c_3 > 0$ such that for all $x_0 \in \tilde{D}$, $\nabla > 0$, $t > 0$ and $x, y \in B_\tilde{D}(x_0, 1\nabla)$ with $\rho_\tilde{D}(x, y) \leq t/C_\ast$,

$$q^{(\nabla), B(x_0, 1\nabla)}(t, x, y) \leq \frac{c_1}{V_\tilde{D}(x_0, \nabla)} \left( \left( \frac{\nabla}{\sqrt{t}} \right)^{d_1} \vee \left( \frac{\nabla}{\sqrt{t}} \right)^{d_2} \right) \exp \left( -\frac{c_2 \rho_\tilde{D}(x, y)^2}{t} \right);$$

and that for all $x_0 \in \tilde{D}$, $\nabla > 0$, $t > 0$ and $x, y \in B_\tilde{D}(x_0, 1\nabla)$ with $\rho_\tilde{D}(x, y) \leq t/C_\ast$,

$$q^{(\nabla), B(x_0, 1\nabla)}(t, x, y) \leq \frac{c_1}{V_\tilde{D}(x_0, \nabla)} \left( \left( \frac{\nabla}{\sqrt{t}} \right)^{d_1} \vee \left( \frac{\nabla}{\sqrt{t}} \right)^{d_2} \right) \exp \left( -c_3 \rho_\tilde{D}(x, y) \left( \log \frac{\rho_\tilde{D}(x, y)}{t} \right)^{\frac{\beta_\ast - 1}{\nabla}} \right).$$

**Proof.** Let $l \geq 2$, $x_0 \in \tilde{D}$ and $\nabla > 0$. For fixed $x, y \in B_\tilde{D}(x_0, 1\nabla)$, let $R := \rho_\tilde{D}(x, y)$ and $f(\tilde{\xi}) := s (\rho_\tilde{D}(x_0, \nabla) \wedge R$) for $\tilde{\xi} \in \tilde{D}$, where $s > 0$ is a constant to be chosen later for each case. Then, $f \in \text{Lip}_s(\tilde{D})$ and $|\tilde{f}(\tilde{\xi}) - f(\tilde{\eta})| \leq s \rho_\tilde{D}(\tilde{\xi}, \tilde{\eta})$ for every $\tilde{\xi}, \tilde{\eta} \in \tilde{D}$. Thus, by [CS, Proposition 2.21], $\|\tilde{f}(\tilde{\xi})\|_{\infty} \leq s^2$. This together with the elementary inequality that $|e^r - 1|^2 \leq r^2 e^{2|r|}$ for all $r \in \mathbb{R}$ yields that

$$e^{-2f(\tilde{\xi}) \Gamma(\nabla)}(e^f) (\tilde{\xi}) = e^{-2f(\tilde{\xi}) \Gamma_0(e^f, e^f)} (\tilde{\xi}) + \int_{\tilde{\eta} \in D \rho_\tilde{D}(\tilde{\xi}, \tilde{\eta}) \leq \nabla} (e^f(\tilde{\xi}) - f(\tilde{\eta})) (1 - 1)^2 J(\tilde{\xi}, \tilde{\eta}) m(d\eta)$$

$$\leq \Gamma_0(f, f) (\tilde{\xi}) + s^2 \int_{\tilde{\eta} \in D \rho_\tilde{D}(\tilde{\xi}, \tilde{\eta}) \leq \nabla} \rho_\tilde{D}(\tilde{\xi}, \tilde{\eta})^2 e^{2s \rho_\tilde{D}(\tilde{\xi}, \tilde{\eta})} J(\tilde{\xi}, \tilde{\eta}) m(d\eta)$$

$$\leq s^2 + c_0 \nabla s^2 \int_{\tilde{\eta} \in D \rho_\tilde{D}(\tilde{\xi}, \tilde{\eta}) \leq \nabla} \rho_\tilde{D}(\tilde{\xi}, \tilde{\eta})^2 e^{2s \rho_\tilde{D}(\tilde{\xi}, \tilde{\eta}) - \kappa_2 \rho_\tilde{D}(\tilde{\xi}, \tilde{\eta})^2} V_\tilde{D}(\tilde{\xi}, \rho_\tilde{D}(\tilde{\xi}, \tilde{\eta})) \phi_1(\rho_\tilde{D}(\tilde{\xi}, \tilde{\eta})) m(d\eta),$$

where the last inequality is due to (J\(\phi_\ast, \tilde{\nabla}, \leq\)).

(i) Suppose that the jumping kernel $J(x, y)$ satisfies (J\(\phi_\ast, \tilde{\nabla}, \leq\)) with $\beta_\ast \in (0, 1)$. Take $s = \kappa_2 \lambda^{\beta_\ast - 1}/4$. We have for $u \in [0, \lambda]$,

$$2su - \kappa_2 u^{2\beta_\ast} = \kappa_2 \lambda^{\beta_\ast - 1} u^2 / 2 - \kappa_2 u^{2\beta_\ast} = \kappa_2 u^{2\beta_\ast} ((u/\lambda)^{1-\beta_\ast} / 2 - 1) \leq -\kappa_2 u^{2\beta_\ast} / 2.$$
In particular, letting $C_\ast := (C \kappa_2)^{-1}$, it holds that
\[
- |f(y) - f(x)| + 2 \Lambda^{(\lambda)}(f)^2 t \leq - \frac{\kappa_2}{8} \lambda^{\beta_* - 1} R \quad \text{for } 0 < t \leq C_\ast \lambda^{1 - \beta_*} R. \tag{4.5}
\]

On the other hand, if $t > C_\ast \lambda^{1 - \beta_*} R$, then, with $s := \frac{R}{4 Ct} = \frac{\kappa_2 C_\ast R}{4 t} \leq \frac{\kappa_2}{4} \lambda^{\beta_* - 1}$, we have
\[
2 s u - \kappa_2 \lambda^{\beta_* - 1} u - \kappa_2 \lambda^{\beta_*} = - \kappa_2 \lambda^{\beta_*} \left( - \frac{1}{2} (u / \lambda)^{1 - \beta_*} + 1 \right) \leq - \kappa_2 \lambda^{\beta_*} / 2
\]
for all $u \leq \lambda$. Hence, for $t > C_\ast \lambda^{1 - \beta_*} R$,
\[
- |f(y) - f(x)| + 2 \Lambda^{(\lambda)}(f)^2 t \leq s (- R + 2 C t s) = \frac{R}{4 Ct} \left( - R + \frac{R}{2} \right) = - \frac{1}{8 C} R^2 t. \tag{4.6}
\]

Now, applying (4.3) and (4.6) to (4.3), we get the desired assertion.

(ii) Suppose next that the jumping kernel $J(x, y)$ satisfies $(J_{\phi_1, \beta_*}, \leq)$ with $\beta_* \in (1, \infty)$. When $0 < s \leq \kappa_2 / 4$, we have by (4.3)
\[
e^{-2f(\xi)} \Gamma(\lambda)(e^f)(\xi) \leq s^2 + c_0 \kappa_1 s^2 \int_{\{\eta \in D : \rho_D(\xi, \eta) \leq 1\}} \frac{\rho_D(\xi, \eta)^2}{V_D(\xi, \rho_D(\xi, \eta))} m(d\eta)
\]
\[
+ c_0 \kappa_1 s^2 \int_{\{\eta \in D : \rho_D(\xi, \eta) \geq 1\}} \frac{\rho_D(\xi, \eta)^2 e^{-\kappa_2 \rho_D(\xi, \eta)^{\beta_*}}}{V_D(\xi, \rho_D(\xi, \eta))} m(d\eta).
\]
Thus by (i) and (ii) of Lemma 6.1 and 1.12,
\[
e^{-2f(\xi)} \Gamma(\lambda)(e^f)(\xi) \leq s^2 + c_2 s^2 \left( \int_0^1 \frac{u}{\phi_1(u)} du + 1 \right) \leq c_3 s^2 / 2 \quad \text{for all } 0 < s \leq \kappa_2 / 4,
\]
which implies that
\[
- |f(y) - f(x)| + 2 \Lambda^{(\lambda)}(f)^2 t \leq - s R + c_3 t s^2 = s (- R + c_3 t s) \quad \text{for all } 0 < s \leq \kappa_2 / 4. \tag{4.7}
\]
Choose $c_3$ larger if necessary so that $c_3 \geq \frac{2}{C_\ast \kappa_2}$. For each $R \leq t / C_\ast$, take $s := \frac{R}{2 c_3 t} \leq \frac{1}{2 c_3 C_\ast} \leq \frac{\kappa_2}{4}$. Then it follows from (4.7) that
\[
- |f(y) - f(x)| + 2 \Lambda^{(\lambda)}(f)^2 t \leq - \frac{R^2}{4 c_3 t}.
\]
Putting this into (4.3), we obtain the assertion for $R \leq t / C_\ast$.

On the other hand, with $c_4 := (4 / \kappa_2)^{1/(\beta_* - 1)}$, we have by (4.4) and (ii) and (iii) of Lemma 6.1 that for all $s > 0$,
\[
e^{-2f(\xi)} \Gamma(\lambda)(e^f)(\xi) \leq s^2 + c_0 \kappa_1 s^2 \int_{\{\eta \in D : \rho_D(\xi, \eta) \leq c_4 s^{1/\beta_* - 1}\}} \frac{\rho_D(\xi, \eta)^2 e^{2 \kappa_2 \rho_D(\xi, \eta)}}{V_D(\xi, \rho_D(\xi, \eta))} m(d\eta)
\]
\[
+ c_0 \kappa_1 s^2 \int_{\{\eta \in D : \rho_D(\xi, \eta) \geq c_4 s^{1/\beta_* - 1}\}} \frac{\rho_D(\xi, \eta)^2 e^{-\kappa_2 \rho_D(\xi, \eta)^{\beta_*} / 2}}{V_D(\xi, \rho_D(\xi, \eta))} m(d\eta)
\]
\[
\leq s^2 + c_5 s^2 \int_0^R \frac{u e^{s u}}{\phi_1(u)} du + c_5 \kappa_1 s^2 \int_{D} \frac{\rho_D(\xi, \eta)^2 e^{-\kappa_2 \rho_D(\xi, \eta)^{\beta_*} / 2}}{V_D(\xi, \rho_D(\xi, \eta))} m(d\eta)
\]
\[
\leq c_6 \left( s^2 \exp(4 c_4 s^{3/\beta_* - 1}) \int_0^R \frac{u}{\phi_1(u)} du + s^2 \right).
\]
According to (1.11) and (4.12), the above is less than or equal to
\[ c_7 s^2 \left( \frac{e^{2/(\beta - 1)}}{\phi_1(s^{1/(\beta - 1)})} \exp(4c_4 s^{\beta_1/(\beta - 1)}) + 1 \right) \leq 2^{-1} c_8 s^2 \exp(c_8 s^{\beta_1/(\beta - 1)}), \quad s > 0. \]

So,
\[-|f(y) - f(x)| + 2\Lambda^{(a)}(f)^2 t \leq -sR + c_8 ts^2 \exp(c_8 s^{\beta_1/(\beta - 1)}) = sR \left( c_8 s(t/R) \exp(c_8 s^{\beta_1/(\beta - 1)}) - 1 \right).\]

Choose \(c_9 \leq (2c_8)^{-\beta_1/(\beta - 1)} \) small so that
\[ c_8 c_9 \sup_{a \geq C_*} \left( \log a \right)^{(\beta - 1)/\beta} a^{-1/2} < \frac{1}{2}, \]
and take \( s := c_9 \left( \log(\cdot/t) \right)^{(\beta - 1)/\beta} \). Then,
\[(t/R) \exp(c_8^{\beta_1/(\beta - 1)}) = (t/R)(R/t)^{c_9 \left( \log(\cdot/t) \right)^{(\beta - 1)/\beta}} \leq (t/R)(R/t)^{1/2} = (R/t)^{-1/2}, \]
and so, for \( R/t > C_*^{-1} > 1, \)
\[-|f(y) - f(x)| + 2\Lambda^{(a)}(f)^2 t \leq sR \left( c_8 c_9 \left( \log(\cdot/t) \right)^{(\beta - 1)/\beta} \left( R/t \right)^{-1/2} - 1 \right) \leq -\frac{1}{2} sR = -\frac{1}{2} c_9 R \left( \log(\cdot/t) \right)^{(\beta - 1)/\beta}. \]

Putting this into (4.3), we obtain the desired assertion for \( R > t/C_* \).

(iii) Now suppose that the jumping kernel \( J(x, y) \) satisfies (J\(_{\beta, \infty, \leq} \)), or equivalently, (1.14) with \( \phi_1 \) in place of \( \phi \) there. In this case, the argument is similar to that of (ii). Since \( J(\xi, \eta) = 0 \) on \( \rho_D(\xi, \eta) > 1 \), it follows from (4.4) and Lemma 6.1(iii) that
\[ e^{-2f}\Gamma(a)(e^f)(\xi) \leq s^2 + c_{10} s^2 \int_0^1 u e^{4su} \phi(u^2) du. \quad (4.8) \]

When \( 0 < s \leq 1 \), by (4.8),
\[-|f(y) - f(x)| + 2\Lambda^{(a)}(f)^2 t \leq -sR + c_{11} ts^2 = s(-R + c_{11} ts). \quad (4.9) \]

Let \( c_{12} = c_{11} \vee \frac{1}{2C_*} \). For each \( R \leq t/C_* \), take \( s := \frac{R}{2c_{12} t} \leq \frac{1}{2c_{12} C_*} \leq 1 \). We have from (4.9) that
\[-|f(y) - f(x)| + 2\Lambda^{(a)}(f)^2 t \leq s(-R + c_{12} ts) \leq -\frac{R^2}{4c_{12} t}. \]

Putting this into (4.3), we obtain the desired assertion for \( \rho_D(\xi, \eta) \leq t/C_* \).

On the other hand, by (4.8), for all \( \lambda > 0 \),
\[-|f(y) - f(x)| + 2\Lambda^{(a)}(f)^2 t \leq -sR + c_{13} ts^2 e^{4s} = sR \left( c_{13} s(t/R) e^{4s} - 1 \right). \quad (4.10) \]

Choose \( c_{14} \in (0, 1/2) \) small so that
\[ c_{13} c_{14} \sup_{a \geq 1/C_*} \log a \left/ \sqrt{a} \right. < \frac{1}{2}. \]

For any \( x, y \in B_D(x_0, l \lambda) \) and \( t > 0 \) with \( R/t > C_*^{-1} > 1 \), take \( s := c_{14} \left( \log(\cdot/t) \right) > 0 \) and we have
\[(t/R) e^{4s} = (t/R)(R/t)^{4c_{14}} \leq (t/R)(R/t)^{1/2} = (R/t)^{-1/2}. \]
Thus, from (4.10), we get
\[-|f(y) - f(x)| + 2\Lambda^{(\lambda)}(f)^2 t \leq sR \left( c_{13} c_{14} \left( \log(R/t) \right) (R/t)^{-1/2} - 1 \right) \leq -\frac{1}{2} sR = -\frac{1}{2} c_{14} R \log(R/t).\]
Putting this into (4.3), we obtain the desired assertion for $R > t/C_*$ as well. \(\square\)

The estimates for the following short time region require more sophisticated choices of test functions in order to obtain the right polynomial exponents.

**Proposition 4.2.** Suppose that $(J_{\phi, \beta_* \leq})$ holds for some $\beta_* \in (0, \infty]$. Then the following hold.

(i) If $\beta_* \in (0, 1]$, then, for every $l \geq 2$ and $a > 0$, there exist $c_1, c_2 > 0$ and $C^* \in (0, 1)$ such that for every $x_0 \in \bar{D}$, $\lambda > a/(4C^*(d_2 + 2))$, $x, y \in B_D(x_0, l\lambda)$ with $\rho_D(x, y) \geq t/C^*$,

\[ q^{(\lambda)}(t, x, y) \leq \frac{c_1}{V_D(x_0, \lambda)} \left( \frac{\lambda}{\sqrt{t}} \right)^{d_1} \vee \left( \frac{\lambda}{\sqrt{t}} \right)^{d_2} \left( \frac{\rho_D(x, y)}{t} \right) -\rho_D(x, y)/(8\lambda) e^{-c_2 \lambda^{\beta_* - 1} \rho_D(x, y)}. \]

(ii) If $\beta_* \in (1, \infty)$, then, for every $l \geq 2$ and $a > 0$, there exist $c_1, c_2 > 0$ and $C^* \in (0, 1)$ such that for every $x_0 \in \bar{D}$, $\lambda > a/(4C^*(d_2 + 2))$, $x, y \in B_D(x_0, l\lambda)$ with $\rho_D(x, y) \geq t/C^*$,

\[ q^{(\lambda)}(t, x, y) \leq \frac{c_1}{V_D(x_0, \lambda)} \left( \frac{\lambda}{\sqrt{t}} \right)^{d_1} \vee \left( \frac{\lambda}{\sqrt{t}} \right)^{d_2} \left( \frac{\rho_D(x, y)}{t} \right) -\rho_D(x, y)/(8\lambda) \times \exp \left( -c_2 \rho_D(x, y) (\log(\rho_D(x, y)/t))^{(\beta_* - 1)/\beta_*} \right). \]

(iii) Suppose that the jumping kernel $J(x, y)$ satisfies $(J_{\phi, \infty, \leq})$, or equivalently, $(1.14)$ with $\phi_1$ in place of $\phi$ there. Then, for every $l \geq 2$ and $a > 0$, there exist $c_1, c_2 > 0$ and $C^* \in (0, 1)$ such that for every $x_0 \in \bar{D}$, $\lambda > a/(4C^*(d_2 + 2))$, $x, y \in B_D(x_0, l\lambda)$ with $\rho_D(x, y) \geq t/C^*$,

\[ q^{(\lambda)}(t, x, y) \leq \frac{c_1}{V_D(x_0, \lambda)} \left( \frac{\lambda}{\sqrt{t}} \right)^{d_1} \vee \left( \frac{\lambda}{\sqrt{t}} \right)^{d_2} \left( \frac{\rho_D(x, y)}{t} \right) -\rho_D(x, y)/(8\lambda) \times \exp \left( -c_2 \rho_D(x, y) (\log(\rho_D(x, y)/t)) \right). \]

**Proof.** Let $l \geq 2$, $x_0 \in \bar{D}$ and $\lambda > 0$. Fix $x, y \in B_D(x_0, l\lambda)$ with $\rho_D(x, y) \geq a/C^*$, where $C^* > 0$ is a constant to be chosen later for each case. Set $R := \rho_D(x, y)$ and $f(\xi) := \frac{\xi}{\lambda} \left( \rho_D(\xi) \wedge R \right)$ for all $\xi \in \bar{D}$, where $s$ and $v$ are two positive constants to be chosen later. Since $|f(\eta) - f(\xi)| \leq \frac{2s}{\lambda} \rho_D(\xi, \eta)$ for all $\xi, \eta \in \bar{D}$, by the same argument as that for (4.3),

\[ e^{-2f(\xi)} \Gamma(\lambda)(e^f)(\xi) \leq c_0 (s + v)^2 \left( 1 + \int_{\eta \in D: \rho_D(\xi, \eta) \leq \lambda} \rho_D(\xi, \eta)^2 e^{2(s + v) \rho_D(\xi, \eta)/3 - \kappa_2 \rho_D(\xi, \eta)^{3/2}} V_D(\xi, \rho_D(\xi, \eta)) m(\eta) \right) \]

\[ \leq c_0 (s + v)^2 \left( 1 + e^{2s/3} \int_{\eta \in D: \rho_D(\xi, \eta) \leq \lambda} \rho_D(\xi, \eta)^2 \left( \frac{e^{2\rho_D(\xi, \eta)/3 - \kappa_2 \rho_D(\xi, \eta)^{3/2}}}{V_D(\xi, \rho_D(\xi, \eta))} m(\eta) \right) \right). \]

(i) Suppose that the jumping kernel $J(x, y)$ satisfies $(J_{\phi, \beta_* \leq})$ with $\beta_* \in (0, 1]$. Take $v = \kappa_2 \lambda^{\beta_* - 1}$. Since $2(vu - \kappa_2 u^3) \leq -(\kappa_2/3) u^3$ for $u \leq \lambda$, we have by Lemma (4.1) and (ii) that, for $\lambda > 0$,

\[ e^{-2f(\xi)} \Gamma(\lambda)(e^f)(\xi) \leq c_0 (s + v)^2 \left( 1 + e^{2s/3} \int_{\eta \in D: \rho_D(\xi, \eta) \leq \lambda} \rho_D(\xi, \eta)^2 \left( \frac{e^{2\rho_D(\xi, \eta)/3 - \kappa_2 \rho_D(\xi, \eta)^{3/2}}}{V_D(\xi, \rho_D(\xi, \eta))} m(\eta) \right) \right). \]
for every $\xi \in \bar{D}$. The same estimate holds for $\Gamma(\lambda)(-f)(\xi)$. Thus we have $\Lambda(\lambda)(f)^2 \leq \frac{e}{6} (s + \kappa_2 \lambda_{\beta_1} - 1)^2 e^{2k\lambda/3}$ and so

$$- |f(y) - f(x)| + 2\Lambda(\lambda)(f)^2 t \leq \frac{s + \kappa_2 \lambda_{\beta_1} - 1}{3} R \left( -1 + c_1 \left( s + \kappa_2 \lambda_{\beta_1} - 1 \right) \frac{t}{R} e^{2k\lambda/3} \right). \tag{4.11}$$

Now let $s = \frac{1}{4\lambda} \log \left( \frac{R}{t} \right)$, and then $(t/R)e^{2k\lambda/3} = (t/R)e^{1/2 \log \frac{R}{t}} = \sqrt{t/R}$. Choose $C^* \in (0, 1)$ such that

$$\frac{3C^*(d_2 + 2)}{a} \left( \sup_{0 < v \leq C^*} \sqrt{v} \log \left( \frac{1}{v} \right) \right) + \kappa_2 \left( \frac{4C^*(d_2 + 2)}{a} \right)^{1-\beta_1} \sqrt{C^*} < (2c_1)^{-1},$$

where $c_1 > 0$ is the constant in (4.11). Then for $R \geq t/C^*$ and $\lambda \geq \frac{\log}{4 \sqrt{c(d_2 + 2)}},$ we have by (4.11),

$$- |f(y) - f(x)| + 2\Lambda(\lambda)(f)^2 t \leq \frac{s + \kappa_2 \lambda_{\beta_1} - 1}{3} \frac{R C^*}{a} \left( -1 + c_1 \kappa_2 \lambda_{\beta_1} - 1 \frac{t}{R} \right) \sqrt{\frac{t}{R}} + \frac{c_1}{R} \left( \frac{4C^*(d_2 + 2)}{a} \right)^{1-\beta_1} \sqrt{\frac{t}{R}} \leq -R \frac{s}{3\lambda} \log \left( \frac{R}{t} \right) - \frac{\kappa_2}{6} \lambda_{\beta_1} - 1 R.$$

Applying the estimate above into (4.13), we get the desired assertion.

(ii) Suppose next that the jumping kernel $J(x, y)$ satisfies $(J_{\phi_1, \beta_1} \leq)$ with $\beta_1 \in (1, \infty).$ Let $c_\star := (4/3\kappa_2)^{1/(\beta_1 - 1)}.$ Then, since $2u_\star/3 - \kappa_2 u_{\beta_1} \leq -\kappa_2 u_{\beta_1}/2$ for $c_\star u^{1/(\beta_1 - 1)} \leq u,$ by Lemma 6.1 (ii) and (iii), for every $\xi \in \bar{D},$

$$e^{-2f(\xi)} \Gamma(\lambda)(f)(\xi) \leq c_0 (s + v)^2 \left[ 1 + e^{2k\lambda/3} \left( \int_{\{\eta \in D: \rho_D(\xi, \eta) \leq c_\star u^{1/(\beta_1 - 1)} \}} \rho_D(\xi, \eta)^2 e^{2\epsilon_D(D, \xi, \eta)} \frac{m(d\eta)}{\rho_D(D, \xi, \eta)} \right) + \int_{\{\eta \in D: \rho_D(D, \xi, \eta) \leq c_\star u^{1/(\beta_1 - 1)} \}} \rho_D(D, \xi, \eta)^2 e^{2\epsilon_D(D, \xi, \eta)} \frac{m(d\eta)}{\rho_D(D, \xi, \eta)} \right]$$

$$\leq c_0 (s + v)^2 \left[ 1 + e^{2k\lambda/3} \left( e^{c_\star u^{1/(\beta_1 - 1)}} \int_{\{\eta \in D: \rho_D(D, \xi, \eta) \leq c_\star u^{1/(\beta_1 - 1)} \}} \rho_D(D, \xi, \eta)^2 e^{-\epsilon_D(D, \xi, \eta)} \frac{m(d\eta)}{\rho_D(D, \xi, \eta)} \right) + \int_{\{\eta \in D: \rho_D(D, \xi, \eta) \leq c_\star u^{1/(\beta_1 - 1)} \}} \rho_D(D, \xi, \eta)^2 e^{-\epsilon_D(D, \xi, \eta)} \frac{m(d\eta)}{\rho_D(D, \xi, \eta)} \right]$$

$$\leq c_0 (s + v)^2 \left[ 1 + c_1 e^{2k\lambda/3} e^{c_\star u^{1/(\beta_1 - 1)}} \right] \leq \frac{c_0}{6} (s + v)^2 e^{2k\lambda/3} e^{c_\star u^{1/(\beta_1 - 1)}},
where \( c_3 := 1/(4C_*) \). The same estimate holds for \( \Gamma_\lambda ((-f)(\xi)) \). Hence we have

\[
\Lambda_\lambda(f)^2 \leq \frac{c_2}{6} (s + v)^2 e^{2s\lambda/3} e^{(4c_3)^{-1} e^{\beta_*/(\beta_* - 1)}}.
\]

Take \( v := (c_3 \log \frac{R}{t})^{(\beta_* - 1)/\beta_*} \). Then for \( t > 0 \),

\[
-|f(y) - f(x)| + 2\Lambda_\lambda(f)^2 t \leq \frac{s + v}{3} \left[ -R + c_2 \left( s + \left( c_3 \log \frac{R}{t}\right)^{(\beta_* - 1)/\beta_*} \right) e^{2s\lambda/3} \left( \frac{R}{t}\right)^{1/4} \right]
\]

Next we take \( s := \frac{3}{4} \log(\frac{R}{t}) \). Then,

\[
e^{2s\lambda/3} \left( \frac{t}{R}\right)^{3/4} = \exp \left( \frac{1}{2} \left( \log \frac{R}{t}\right) \right) \left( \frac{t}{R}\right)^{3/4} = \left( \frac{t}{R}\right)^{1/4}.
\]

Choose \( C^* \in (0,1) \) such that

\[
c_2 \sup_{0 < c < \infty} \left( \frac{3C^*}{a} (d_2 + 2) \log \frac{1}{v} + \left( c_3 \log \frac{1}{v}\right)^{(\beta_* - 1)/\beta_*} \right)^{s/4} < 1/2.
\]

Then, for \( R \geq t/C^* \), it holds that

\[
-|f(y) - f(x)| + 2\Lambda_\lambda(f)^2 t \\
\leq \frac{s + v}{3} \left[ -R + c_2 \left( \frac{3C^*}{a} (d_2 + 2) \log \left( \frac{R}{t}\right) + \left( c_3 \log \frac{R}{t}\right)^{(\beta_* - 1)/\beta_*} \right) \left( \frac{t}{R}\right)^{1/4} \right]
\]

\[
\leq \frac{-s + v}{6} R \leq \frac{R}{8\lambda} \log \frac{R}{t} - \frac{1}{6} \left( c_3 \log \frac{R}{t} \right)^{(\beta_* - 1)/\beta_*} R.
\]

Applying the estimate above into (4.3), we get the desired assertion.

(iii) Now suppose the jumping kernel \( J(x,y) \) satisfies \( (J_{\phi_1, 1, \infty}) \), or equivalently, (1.15) with \( \phi_1 \) in place of \( \phi \) there. In this case, for every \( \xi \in \bar{D} \) we have by (1.12) and Lemma 6.1 (iii),

\[
e^{-2f(\xi)} \Gamma_\lambda(f)(\xi) \leq c_0 (s + v)^2 \left[ 1 + e^{2s\lambda/3} \int_{\{\eta \in \bar{D}, \rho_\Delta(\xi, \eta) \leq 1\}} \rho_\Delta(\xi, \eta)^2 \frac{e^{2\rho_\Delta(\xi, \eta)/3}}{V_{\Delta}(\xi, \rho_\Delta(\xi, \eta)) \phi_1(\rho_\Delta(\xi, \eta))} m(d\eta) \right]
\]

\[
\leq c_0 (s + v)^2 \left( 1 + e^{2s\lambda/3} e^{2v/3} \left( \int_0^1 \frac{v}{\phi_1(u)} du + c_1 \right) \right)
\]

\[
\leq \frac{c_2}{6} (s + v)^2 e^{2s\lambda/3} e^{2v/3}.
\]

With this estimate at hand, one can follow the argument in (ii) with \( v = \frac{3}{8} \log \frac{R}{t} \) and \( s = \frac{3}{4\lambda} \log \frac{R}{t} \) to get the desired assertion. \( \square \)

Using Propositions (4.1) and (4.2) we have the following result.

**Proposition 4.3.** Suppose that \( (J_{\phi_1, \beta_*, \leq}) \) holds for some \( \beta_* \in (0, \infty) \). Then the following hold.
(i) If $\beta_* \in (0,1]$, then, for any $a > 0$ and any $l > 2$ there are constants $c_1, c_2, c_3, c_4, c_5, C^* > 0$ such that for all $\lambda \geq t > a$, $x_0 \in \bar{D}$ and $x \in B_{\bar{D}}(x_0, l\lambda)$,

$$\int_{\{y \in D_{PD}(x,y) \geq \lambda\}} q^{(\lambda),B(x_0,1\lambda)}(t,x,y) \, m(dy) \leq \begin{cases} c_1 e^{-c_2 \lambda^{2}/t} & \text{if } C_* \lambda^{2-\beta_*} \leq t \leq \lambda^{2}, \\ c_1 e^{-c_2 \lambda^{2} \beta_*} & \text{if } C_* \lambda^{2-\beta_*} > t; \end{cases}$$

and that for all $0 < t \leq a$, $4(d_2 + 2) \leq l_0 \leq l - 1$, $\lambda \geq a/(4C^*(d_2 + 2))$, $x_0 \in \bar{D}$ and $x \in B_{\bar{D}}(x_0, l\lambda)$,

$$\int_{\{y \in \bar{D}_{PD}(x,y) \geq l_0 \lambda\}} q^{(\lambda),B(x_0,1\lambda)}(t,x,y) \, m(dy) \leq c_3 \left( \frac{\lambda}{\sqrt{t}} \right)^{-l_0/8} e^{-c_4 \lambda^{2} \beta_*}.$$  

(4.13)

(ii) If $\beta_* \in (1, \infty]$, then, for every $C_* \in (0,1)$, $a > 0$ and every $l > 2$, there exist $c_1, c_2, c_3, c_4, c_5, C^* > 0$ such that for all $x_0 \in \bar{D}$, $t > 0$, $0 < \lambda \leq t/C_*$ and $x \in B_{\bar{D}}(x_0, l\lambda)$,

$$\int_{\{y \in D_{PD}(x,y) \geq \lambda\}} q^{(\lambda),B(x_0,1\lambda)}(t,x,y) \, m(dy) \leq c_1 \exp \left( -c_2 \lambda^{2} \beta_* \right),$$

and for all $x_0 \in \bar{D}$, $t > 0$, $\lambda > t/C_*$ and $x \in B_{\bar{D}}(x_0, l\lambda)$,

$$\int_{\{y \in D_{PD}(x,y) \geq \lambda\}} q^{(\lambda),B(x_0,1\lambda)}(t,x,y) \, m(dy) \leq c_3 \left( \frac{\lambda}{\sqrt{t}} \right)^{-l_0/8} e^{-c_4 \lambda^{2} \beta_*}.$$

Proof. (i) We first prove (4.12). Let $x_0 \in \bar{D}$ and $l \geq 2$. Let $C_*$ be the constant in Proposition 4.1(i). When $\lambda^2 \wedge (C_* \lambda^{2-\beta_*}) \geq t \geq a$, according to Proposition 4.1(i) and (3.18), we have that for $x \in B_{\bar{D}}(x_0, l\lambda)$,

$$\int_{\{y \in D_{PD}(x,y) \geq \lambda\}} q^{(\lambda),B(x_0,1\lambda)}(t,x,y) \, m(dy) \leq \frac{c_1}{V_{\bar{D}}(x_0, \lambda)} \left( \frac{\lambda}{\sqrt{t}} \right)^{d_2} \int_{\{y \in D_{PD}(x,y) \geq \lambda\}} e^{-c_2 \lambda^{2-1} \rho_{PD}(x,y)} \, m(dy)$$

$$\leq \frac{c_3}{V_{\bar{D}}(x, \lambda)} \left( \frac{\lambda}{\sqrt{t}} \right)^{d_2} \sum_{i=0}^\infty e^{-c_2 \lambda^{\beta_*-1} \beta^{2+1} \lambda} V_{\bar{D}}(x, 2^{i+1} \beta)$$

$$\leq c_4 \left( \frac{\lambda}{\sqrt{t}} \right)^{d_2} \sum_{i=0}^\infty e^{-c_2 \lambda^{2} \beta_* \lambda} \leq c_5 e^{-c_6 \lambda^{2} \beta_*},$$

where $c_5, c_6$ depend on $a$. On the other hand, when $\lambda^{2} \geq t \geq C_* \lambda^{2-\beta_*}$, also by Proposition 4.1(i),

$$\int_{\{y \in D_{PD}(x,y) \geq \lambda\}} q^{(\lambda),B(x_0,1\lambda)}(t,x,y) \, m(dy)$$

$$\leq \frac{c_7}{V_{\bar{D}}(x_0, \lambda)} \left( \frac{\lambda}{\sqrt{t}} \right)^{d_2} \int_{\{y \in D_{PD}(x,y) \leq C_* \lambda^{2-1}\}} e^{-c_8 \rho_{PD}(x,y)^{2}/t} \, m(dy)$$

$$+ \frac{c_7}{V_{\bar{D}}(x, \lambda)} \left( \frac{\lambda}{\sqrt{t}} \right)^{d_2} \int_{\{y \in D_{PD}(x,y) \geq C_* \lambda^{2-1}\}} e^{-c_8 \lambda^{\beta_*-1} \rho_{PD}(x,y)} \, m(dy)$$

$$\leq \frac{c_7}{V_{\bar{D}}(x, \lambda)} \left( \frac{\lambda}{\sqrt{t}} \right)^{d_2} \sum_{i=0}^\infty e^{-c_8 \lambda^{2} \beta_* \lambda} V_{\bar{D}}(x, 2^{i+1} \lambda) + \sum_{i=0}^\infty e^{-c_8 \lambda^{2} \beta_* \lambda} V_{\bar{D}}(x, 2^{i+1} C_* \lambda^{2-1})$$

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Furthermore, by the strong Markov property, for any \( x, \lambda \in \mathbb{R} \),

\[
\lambda / t \leq c_9 \left( \frac{\lambda}{\sqrt{t}} \right)^{d_2} \left( \sum_{i=0}^{\infty} e^{-c_9 \lambda^2 / 2t} + \sum_{i=0}^{\infty} e^{-c_9 \lambda^2 / 2t} \right)
\]

\[
\leq c_9 \left( \frac{\lambda}{\sqrt{t}} \right)^{d_2} \left( \sum_{i=0}^{\infty} e^{-c_9 \lambda^2 / 2t} + c_9 \left( \lambda^2 / 2t \right) \right)^{d_2 / 2} \sum_{i=0}^{\infty} e^{-c_9 \lambda^2 / 2t} \leq c_10 \left( e^{-c_11 \lambda^2 / 2t} + e^{-c_12 \lambda^2 / 2t} \right) \leq c_{13} e^{-c_14 \lambda^2 / t}.
\]

Next, we consider (4.15). According to Proposition 4.2 (ii), for every \( l > 4(d_2 + 2) \) and \( a > 0 \), there exists \( C^* \in (0, 1) \) such that for every \( x_0 \in \bar{D}, 4(d_2 + 2) \leq l_0 \leq l - 1, \lambda > a/(4C^*(d_2 + 2)), 0 < t \leq a \) and \( x \in B_D(x_0, l\lambda) \),

\[
\int_{\{y \in \bar{D}: \rho_D(x,y) \geq l_0 \lambda\}} q^{(x)} \cdot B(x_0,l\lambda)(t, x, y) m(dy)
\]

\[
\leq \frac{c_1}{\sqrt{d_2}} \left( \frac{\lambda}{\sqrt{t}} \right)^{d_2} \int_{\{y \in \bar{D}: \rho_D(x,y) \geq l_0 \lambda\}} \left( \frac{\rho_D(x,y)}{t} \right)^{-(\rho_D(x,y)/(8\lambda))} e^{-c_2 \rho_D(x,y) \lambda^{\beta_{\lambda}} - 1} m(dy)
\]

\[
\leq \frac{c_3}{\sqrt{d_2}} \left( \frac{\lambda}{\sqrt{t}} \right)^{d_2} \int_{\{y \in \bar{D}: \rho_D(x,y) \geq l_0 \lambda\}} e^{-c_2 \rho_D(x,y) \lambda^{\beta_{\lambda}} - 1} m(dy)
\]

\[
= \frac{c_3}{\sqrt{d_2}} \left( \frac{\lambda}{\sqrt{t}} \right)^{d_2} \int_{\{y \in \bar{D}: \rho_D(x,y) \geq l_0 \lambda\}} e^{-c_2 \rho_D(x,y) \lambda^{\beta_{\lambda}} - 1} m(dy).
\]

Noting \( \lambda \geq 1/(4C^*(d_2 + 2)) \) and \( l \geq l_0 + 1 \), we have

\[
\leq \frac{c_1}{\sqrt{d_2}} \left( \frac{\lambda}{\sqrt{t}} \right)^{d_2} \left( \frac{1}{4C^*(d_2 + 2)} \right)^{l_0/16 + d_2/2} \left( \frac{\lambda}{t} \right)^{l_0/16} \left( \frac{1}{4C^*(d_2 + 2)} \right)^{l_0/8} \int_{\{y \in \bar{D}: \rho_D(x,y) \geq l_0 \lambda\}} e^{-c_2 \rho_D(x,y) \lambda^{\beta_{\lambda}} - 1} m(dy)
\]

\[
\leq c_4 \lambda^{d_2/2 + l_0/16} \left( \frac{\lambda}{\sqrt{t}} \right)^{l_0/8} \frac{1}{\sqrt{d_2}} \int_{\{y \in \bar{D}: \rho_D(x,y) \geq l_0 \lambda\}} e^{-c_2 \rho_D(x,y) \lambda^{\beta_{\lambda}} - 1} m(dy)
\]

\[
\leq c_5 \lambda^{d_2/2 + l_0/16} \left( \frac{\lambda}{\sqrt{t}} \right)^{l_0/8} e^{-c_6 \lambda^{\beta_{\lambda}}} \leq c_7 \left( \frac{\lambda}{\sqrt{t}} \right)^{l_0/8} e^{-c_6 \lambda^{\beta_{\lambda}}}.
\]

Here, \( c_4 \) in the second inequality depends on \( l_0 \), and the third inequality follows from (4.10) by splitting the integral into pieces.

(ii) For \( \beta_{\lambda} \in (1, \infty) \), the proof is similar to that for (i) by applying Proposition 4.1 (ii) (resp. Proposition 4.2 (ii)) instead of Proposition 4.1 (i) (resp. Proposition 4.2 (i)). The details are omitted here. From Proposition 4.1 (iii) and Proposition 4.2 (iii), we can also establish the desired assertion for the case \( \beta_{\lambda} = \infty \).
Theorem 4.4. Suppose that \((J_{\phi_1, \beta_*} \leq)\) holds for some \(\beta_* \in (0, \infty)\). Then there are positive constants \(c_1\) and \(c_2\) that depend only on characteristic constants \((C_1, C_2, C_3, C_4)\) of \(D\) and the constant parameters in \((J_{\phi_1, \beta_*} \leq)\) and \(\mathbb{1}_{1.23}\) for \(\phi_1\) so that
\[
q(t, x, y) \leq c_1 H_{\phi_1, \beta_*}(t, x, c_2 \rho_D(x, y)) \quad \text{for every } t > 0 \text{ and } x, y \in \bar{D}.
\]

Proof. We will give the proof for the case when \(\beta_* \in (0, 1]\). The other cases can be proved similarly. Let \(C > 1\). When \(t \in (0, 1]\) and \(\rho_D(x, y) \leq C\), the desired estimate follows from \(\mathbb{1}_{1.23}\) with \(\phi_1 = \phi_1\), since there is a constant \(c_0 := c_0(C)\) such that \(c_0 p_D^{(1/2)}(t, x, \rho_D(x, y)) \geq p_D(t, x, \rho_D(x, y))\) for all \(t > 0\) and \(x, y \in \bar{D}\) with \(\rho_D(x, y) \leq C\). Next, set \(k = 32(2 + d_2)\) and \(C := k/(4C^*(2 + d_2)) = 8/C^*\), where \(C^* > 0\) is the constant in Proposition \(\mathbb{1}_{1.24}\). For \(t \in (0, 1]\) and \(x, y \in \bar{D}\) with \(\rho_D(x, y) \geq C\), let \(\lambda = \rho_D(x, y)/k\). According to \(\mathbb{1}_{1.24}\), it suffices to verify that for any \(t \in (0, 1]\) and \(\rho_D(x, y) \geq C\),
\[
q^{(\lambda)}(t, x, y) \leq c_1 t \rho_D(x, y) e^{-c_2 \rho_D(x, y)^{\beta_*}}.
\]
Indeed, letting \(R = \rho_D(x, y) > 0\), by \(\mathbb{1}_{1.24}\) Lemma 7.2(2), it holds that for any \(t \in (0, 1]\) and \(\rho_D(x, y) \geq C\),
\[
q^{(\lambda)}(t, x, y) \leq q(t, x, y)e^{t/\epsilon^{(\lambda)}} \leq c_1 q(t, x, y) \leq \frac{c_2}{V_D(x, \sqrt{t})},
\]
where \(c(\lambda)\) is a positive increasing function of \(\lambda\). Here the second inequality is due to the facts that \(\lambda = \rho_D(x, y)/k \geq C/k = 1/(4C^*(2 + d_2))\) and \(t \in (0, 1]\), and the third inequality is by \(\mathbb{1}_{1.24}\). Using this, we have for any \(t > 0\) and \(x, y \in \bar{D}\) with \(\rho_D(x, y) \geq C\),
\[
q^{(\lambda)}(t, x, y) = \int_{D} q^{(\lambda)}(t/2, x, z)q^{(\lambda)}(t/2, z, y) m(dy) \\
\leq \left( \int_{B_D(x, R/2)} + \int_{B_D(y, R/2)} \right) \left( \int_{B_D(x, R/2)} + \int_{B_D(y, R/2)} \right) q^{(\lambda)}(t/2, x, z) q^{(\lambda)}(t/2, z, y) m(dy) \\
\leq \frac{c_2}{V_D(y, \sqrt{t})} \int_{B_D(x, R/2)} q^{(\lambda)}(t/2, x, z) m(dz) + \frac{c_2}{V_D(x, \sqrt{t})} \int_{B_D(y, R/2)} q^{(\lambda)}(t/2, y, z) m(dz) \\
\leq \frac{c_3}{V_D(x, \rho_D(x, y))} \left( \frac{\rho_D(x, y)}{\sqrt{t}} \right)^{d_2} \sup_{x \in D} \sup_{z \in \bar{D}} \sup_{t \leq s \leq 2t} \mathbb{P}_z(\rho_D(Y^{(\lambda)}(s), B(x, R/2)) \geq R/4) \\
\leq \frac{c_3}{V_D(x, \rho_D(x, y))} \left( \frac{\rho_D(x, y)}{\sqrt{t}} \right)^{d_2} \left( \frac{\lambda}{\sqrt{t}} \right)^{-k/32} e^{-c_4 \rho_D(x, y)^{\beta_*}} \\
\leq \frac{c_5}{V_D(x, \rho_D(x, y))} \frac{t}{\rho_D(x, y)} e^{-c_6 \rho_D(x, y)^{\beta_*}},
\]
where the third inequality follows from \(\mathbb{1}_{1.8}\) and \(\mathbb{1}_{1.16}\) and the fourth inequality follows from \(\mathbb{1}_{1.8}\) with \(l = 1 + (k/2)\) and \(l_0 = k/4\). Thus we have
\[
q^{(\lambda)}(t, x, y) \leq \frac{c_7 t}{V_D(x, \rho_D(x, y))} e^{-c_8 \rho_D(x, y)^{\beta_*}},
\]
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which proves (4.16).

Finally, suppose \( t > 1 \). For any \( c > 0 \), if \( \rho_D(x, y) \leq c\sqrt{t} \), then the estimate follows from (1.24); if \( \rho_D(x, y) \geq c\sqrt{t} \), we can follow the argument above and use (4.15) in place of (4.13) to get the desired result.

\( \square \)

The upper bounds of \( q(t, x, y) \) in Theorem 1.6 follow directly from Theorems 1.5 and 4.4 for the case that \( \beta_* = 0_+ \) and \( \beta_* \in (0, \infty) \), respectively.

## 5 Lower bound estimates under \((J_{\phi_1,0_+\leq})\) and \((J_{\phi_2,\beta_*\geq})\) with \( \beta_* \in (0, \infty) \)

Throughout this section, we assume that \( \text{diam}(D) = \infty \) and that \((J_{\phi_1,0_+\leq})\) holds. Recall from Subsections 3.1 and 3.2 that \( Y \) is the Feller process associated with the regular Dirichlet form \((\mathcal{E}, \mathcal{F})\) of (1.9)–(1.10) on \( L^2(D; \mu) \), which has a jointly Hölder continuous transition density function \( q(t, x, y) \) that enjoys the properties stated in Theorems 1.5 and Proposition 3.3.

In this section, we will assume the jumping kernel \( J \) satisfies condition \((J_{\phi_2,\beta_*\geq})\) for some \( \beta_* \in (0, \infty) \), where the strictly increasing function \( \phi_2 \) can be different from \( \phi_1 \) in the upper condition \((J_{\phi_1,0_+\leq})\) for \( J(x, y) \).

**Proposition 5.1.** Assume that \((J_{\phi_1,0_+\leq})\) and \((J_{\phi_2,\beta_*\geq})\) hold for some \( \beta_* \in (0, \infty) \). Then there are positive constants \( c_k, k = 1, \cdots, 4 \), and \( c_6 \), depending only on the characteristic constants \((C_1, C_2, C_3, C_4)\) of \( D \) and constant parameters in \((J_{\phi_1,0_+\leq})\) and \((J_{\phi_2,\beta_*\geq})\) as well as in (1.11) for \( \phi_1 \) and \( \phi_2 \), and on the constant \( C_* \) below (for \( c_1, \cdots, c_4 \) and on both \( C_* \) and \( c_5 \) (for \( c_6 \)), so that the following hold.

(i) Suppose that \( \beta_* \in (0, \infty) \). Then for every \( C_* > 0 \), there exist \( c_1, c_2 > 0 \) such that for every \( t > 0 \) and \( x, y \in \overline{D} \) with \( \rho_D(x, y) \geq C_*^{1/2} \),

\[
q(t, x, y) \geq \frac{c_1}{V_D(x, \sqrt{t})} \exp\left(-\frac{c_2 \rho_D(x, y)^2}{t}\right).
\]

(ii) Suppose that \( \beta_* \in (0, \infty) \). Then for every \( C_* > 0 \), there exist \( c_3, c_4 > 0 \) such that for every \( t > 0 \) and \( x, y \in \overline{D} \) with \( \rho_D(x, y) \geq C_*^{1/2} \),

\[
q(t, x, y) \geq \frac{c_3 t}{V_D(x, \rho_D(x, y))} \frac{c_4 t}{\phi_2(\rho_D(x, y))} e^{-c_4 \rho_D(x, y)^{\beta_*}}. \quad (5.1)
\]

(iii) Suppose that the jumping kernel \( J(x, y) \) satisfies \((J_{\phi_2,\beta_*\geq})\), or equivalently, (1.16) with \( \phi_2 \) in place of \( \phi \) there. Then for any \( c_5 \in (0, 1) \) and \( C_* > 0 \), there is a constant \( c_6 > 0 \) such that for every \( t > 0 \) and \( x, y \in \overline{D} \)

\[
q(t, x, y) \geq \frac{c_6 t}{V_D(x, \rho_D(x, y))} \phi_2(\rho_D(x, y)) \quad \text{whenever } C_*^{1/2} \leq \rho_D(x, y) \leq c_5.
\]

**Proof.** (i) We mainly follow the proof of [CKW3, Proposition 5.5]. Without loss of generality, we may and do assume that \( C_* \) is less than the constant \( c_2 \) in Proposition 3.3. Let \( r = \rho_D(x, y) \). Let \( N \geq 2 \) be an integer such that

\[
9r^2/(C_2^2 t) \leq N < 9r^2/(C_2^2 t) + 1 \quad \text{so that } C_* \sqrt{2t/N} \leq 3r/N \leq C_* \sqrt{t/N}.
\]

Let \( x_0 = x, x_1, \cdots, x_N = y \) such that \( \rho_D(x_i, x_{i+1}) \leq r/N \), and set \( B_i = B_D(x_i, r/N) \) for all \( 1 \leq i \leq N \). Then, by Proposition 3.3 and (1.6),

\[
q(t, x, y) \geq \int_{B_1} q(t/N, x, y_1)q(t/N, y_1, y_2) m(dy_1) \cdots \int_{B_{N-1}} q(t/N, y_{N-1}, y) m(dy_{N-1})
\]

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Proposition 5.3. Applying Propositions 3.3 and 5.1, and following the proof of \cite[Proposition 5.5]{CKW3}, we can obtain Corollary 5.2.

Recall that \( p^{(c)}(t, x, r) \) and \( p^{(j)}_{\phi_2}(t, x, r) \) are defined by \(1.13 \), \(1.19 \) and \(1.23 \), respectively. Applying Propositions 3.3 and 5.1 and following the proof of \cite[Proposition 5.5]{CKW3}, we can obtain

Corollary 5.2. Suppose that \((J_{\phi_1, \alpha}, \leq)\) and \((J_{\phi_2, \alpha}, \geq)\) hold with \(\beta^* \in (0, \infty)\). Then there are positive constants \(c_1, 1 \leq i \leq 8\), that depend on the characteristic constants \((C_1, C_2, C_3, C_4)\) of \(D\) and the constant parameters in \((J_{\phi_1, \alpha}, \leq)\) and \((J_{\phi_2, \alpha}, \geq)\) as well as in \(1.11\) for \(\phi_1\) and \(\phi_2\), so that the following hold.

(i) When \(\beta^* \in (0, \infty)\),
\[
q(t, x, y) \geq \frac{c_1}{V_D(x, \sqrt{t})} \wedge \left( p^{(c)}(t, x, c_2 \rho_D(x, y)) + p^{(j)}_{\phi_2, \alpha^*}(t, x, c_3 \rho_D(x, y)) \right) \quad \text{for } t > 0 \text{ and } x, y \in D.
\]

(ii) When \(\beta^* = \infty\),
\[
q(t, x, y) \geq \frac{c_1}{V_D(x, \sqrt{t})} \wedge \left( p^{(c)}(t, x, c_5 \rho_D(x, y)) + p^{(j)}_{\phi_2}(t, x, c_6 \rho_D(x, y)) \right)
\]
holds for every \(t \in (0, c_7] \) and \(x, y \in \bar{D} \) with \(\rho_D(x, y) \leq c_8\).

To further refine lower bound estimates of the heat kernel \(q(t, x, y)\) for the case that \(\beta^* \in (1, \infty)\), we need the following estimates.

Proposition 5.3. Assume that \((J_{\phi_1, \alpha}, \leq)\) and \((J_{\phi_2, \alpha}, \geq)\) hold for some \(\beta^* \in (0, \infty)\). Then for any \(C^* \in (0, 1)\) and \(c \in (0, 1)\), there are positive constants \(c_1, 1 \leq i \leq 4\), that depend only on the characteristic constants \((C_1, C_2, C_3, C_4)\) of \(D\), the constant parameters in \((J_{\phi_1, \alpha}, \leq)\) and \((J_{\phi_2, \alpha}, \geq)\) and in \(1.11\) for \(\phi_1\) and \(\phi_2\) as well as on the constants \(C^*\) for \(c_1\) and \(c_2\) and on both \(C^*\) and \(c\) for \(c_3\) and \(c_4\), so that the following hold.

(i) When \(\beta^* \in (1, \infty)\), for all \(t > 0 \) and \(x, y \in \bar{D} \) with \(\rho_D(x, y) \geq t/C^*\),
\[
q(t, x, y) \geq \frac{c_1 t}{V_D(x, \rho_D(x, y))} \exp \left( -c_2 \rho_D(x, y) \left( \log \frac{\rho_D(x, y)}{t} \right)^{\beta^* - 1} \right). \tag{5.2}
\]

(ii) When \(\beta^* = \infty\), for all \(t > 0 \) and \(x, y \in \bar{D} \) with \(\rho_D(x, y) \geq C^* \wedge (t/C^*)\),
\[
q(t, x, y) \geq \frac{c_3 t}{V_D(x, \rho_D(x, y))} \left( \frac{t}{\rho_D(x, y)} \right)^{c_4 \rho_D(x, y)}.
\]

Proof. (i) Let \(r := \rho_D(x, y)\). Note that \(\exp \left( -c r (\log(r/t))^{(\beta^* - 1)/\beta^*} \right) \geq \exp(-c r^{\beta^*})\) is equivalent to \(t \geq r \exp(-r^{\beta^*})\). Then, according to Propositions 3.3 and 6.1, it suffices to consider the case \(C^*r \geq t \geq r \exp(-r^{\beta^*})\). In this case we have \(r (\log(r/t))^{-1/\beta^*} \geq 1\). Let \(l \geq 2\) be an integer such that
\[
r (\log(r/t))^{-1/\beta^*} < l \leq r (\log(r/t))^{-1/\beta^*} + 1 \leq 2r (\log(r/t))^{-1/\beta^*},
\]
and in particular \(l \geq 2\). Then
\[
q(t, x, y) \geq \frac{c_1 t}{V_D(x, \rho_D(x, y))} \exp \left( -c_2 \rho_D(x, y) \left( \log \frac{\rho_D(x, y)}{t} \right)^{\beta^* - 1} \right).
\]
and let $x = x_0, x_1, \cdots, x_t = y \in \bar{D}$ be such that $r/(2l) \leq \rho_D(x_i, x_{i+1}) \leq 2r/l$ for $i = 0, \cdots, l-1$. We observe that

$$
\frac{t}{2r} (\log(r/t))^{1/\beta^*} \leq \frac{t}{r} (\log(r/t))^{1/\beta^*} \leq \sup_{s \geq 1/C^*} s^{-1} (\log s)^{1/\beta^*} =: t_0 < \infty
$$

and

$$\frac{r}{2l} \geq \frac{1}{4} (-\log C^*)^{1/\beta^*} =: r_0 > 0.
$$

Thus for all $(y_i, y_{i+1}) \in B_D(x_i, r/(8l)) \times B_D(x_{i+1}, r/(8l))$, $3r/l \geq \rho_D(y_i, y_{i+1}) \geq r/(4l)$ and so $\rho_D(y_i, y_{i+1}) \geq r/(4l) \geq r_0/2 \geq (r_0/(2t_0^{1/2})) \cdot (t/l)^{1/2}$. According to Corollary 5.2(i), for all $(y_i, y_{i+1}) \in B_D(x_i, r/(8l)) \times B_D(x_{i+1}, r/(8l)),$

$$q(t/l, y_i, y_{i+1}) \geq \frac{c_1 t/l}{V_D(y_i, r/l)} \exp(-c_2(r/l)^{\beta^*}) \geq \frac{c_3 t/l}{V_D(y_i, r/l)} \exp(-2c_2(r/l)^{\beta^*})
$$

$$\geq \frac{(c_3/2)(t/r)}{V_D(y_i, r/l)} (\log(r/t))^{1/\beta^*} (t/r)^{2c_2} \geq \frac{(c_3/2)(t/r)^{1+2c_2}}{V_D(y_i, r/l)} (\log(C^{-1})^{1/\beta^*} \geq \frac{c_4 (t/r)^{c_5}}{V_D(y_i, r/l)}. \tag{5.3}
$$

where $c_5 > 1$. Let $B_i = B_D(x_i, r/(8l))$ for all $1 \leq i \leq l$. Using (5.3) and (1.0), we have

$$q(t, x, y) \geq \int_{B_1} \cdots \int_{B_{l-1}} q(t/l, x, y_i) \cdots q(t/l, y_{l-1}, y) m(dy_1) \cdots m(dy_{l-1})
$$

$$\geq \frac{c_4 (t/r)^{c_5}}{V_D(x, r/l)} \prod_{i=1}^{l-1} c_4(t/r)^{c_5} \frac{V_D(x_i, r/l)}{V_D(x_i, 9r/(8l))} \geq \frac{c_6}{V_D(x, r/l)} (c_7 (t/r)^{c_5})^{l-1}
$$

$$\geq \frac{c_6}{V_D(x, r/l)} \exp \left( -c_8 r (\log(r/t))^{(-1/\beta^*)} (1 + \ln(r/t)) \right)
$$

$$\geq \frac{c_6}{V_D(x, r/l)} \exp \left( -c_9 r (\log(r/t))^{(\beta^*-1)/\beta^*} \right).
$$

This yields (5.2) by considering the cases that $t \in (0, 1]$ and $t > 1$ respectively.

(ii) Let $c, C^* \in (0, 1)$. For any $t > 0$ and $r := \rho_D(x, y) \geq c \vee (t/C^*)$, we take $c_0 = 10/c$, and let $l \geq 2$ be an integer such that $c_0 r \leq l \leq c_0 r + 1 \leq 2c_0 r$. Choose $x = x_0, x_1, \cdots, x_t = y \in \bar{D}$ such that $1/(4c_0) \leq r/(2l) \leq \rho_D(x_i, x_{i+1}) \leq 2r/l \leq 2/c_0$ for $i = 0, \cdots, l-1$. It is clear that

$$\frac{t}{l} \leq \frac{C^* r}{c_0} \leq \frac{C^*}{c_0} \quad \text{and} \quad \frac{1}{8c_0} \leq \frac{r}{4l} \leq \frac{1}{c_0}.
$$

Then, since $\rho_D(y_i, y_{i+1}) \leq (9/4)r/l \leq 9/c_0 = 9c/10 < 1$ for $(y_i, y_{i+1}) \in B_D(x_i, r/(8l)) \times B_D(x_{i+1}, r/(8l))$, according to Proposition 5.1(iii), we have for all $(y_i, y_{i+1}) \in B_D(x_i, r/(8l)) \times B_D(x_{i+1}, r/(8l))$

$$q(t/l, y_i, y_{i+1}) \geq \frac{c_1 t/l}{V_D(y_i, r/l)} \exp(-c_2(r/l)^{\beta^*}) \geq \frac{c_3 t/l}{V_D(y_i, r/l)}.
$$

Let $B_i = B_D(x_i, r/(8l))$ for all $1 \leq i \leq l$. Using (5.3) and (1.0) we have

$$q(t, x, y) \geq \int_{B_1} \cdots \int_{B_{l-1}} q(t/l, x, y_i) \cdots q(t/l, y_{l-1}, y) m(dy_1) \cdots m(dy_{l-1})
$$

$$\geq \frac{c_2 (t/l)}{V_D(x, r/l)} (c_3 (t/l))^{l-1} \geq \frac{c_4}{V_D(x, r)} (c_5 t/l)^{2c_0 r} \geq \frac{c_6}{V_D(x, r)} (t/r)^{c_7 r}.
$$

This completes the proof. 

Now, we can state the main result in this section, which provides lower bounds of $q(t, x, y)$. Recall that $H_{\phi, \beta}(t, x, r)$ is defined right after (1.23).
Theorem 5.4. Suppose that diam(D) = ∞, and that (I_{φ_{1,0},<}) and (I_{φ_{2,β^*},<}) hold for some β* ∈ (0, ∞). Then there are positive constants c_1 and c_2, which depend only on the characteristic constants (C_1, C_2, C_3, C_4) of D and the constant parameters (I_{φ_{1,0},<}) and (I_{φ_{2,β^*},<}) as well as in (5.1) for φ_1 and φ_2, so that

\[ q(t, x, y) ≥ c_1 H_{φ_{2,β^*}}(t, x, c_2 ρ_D(x, y)) \]  
for all t > 0 and x, y ∈ D. \hspace{1cm} (5.5)

Proof. (i) When β* ∈ (0, 1], the assertion for the case that t ∈ (0, 1] follows from Corollary 5.2 and Proposition 5.1. When t > 1, near diagonal lower bounds of q(t, x, y) is proved by Proposition 5.3. From (5.1) we see that for every C_* > 0, t > 0 and x, y ∈ D with ρ_D(x, y) ≥ C_* t^{1/2},

\[ q(t, x, y) ≥ \frac{c_1 t}{V_D(x, ρ_D(x, y))} e^{-c_2 ρ_D(x, y)^{β^*}} ≥ \frac{c_3 t}{V_D(x, \sqrt{t})} \left( \frac{t}{ρ_D(x, y)^2} \right)^{d_2/2} e^{-c_4 ρ_D(x, y)^{β^*}}, \]  
((5.6)

where in the second inequality we used (1.6). Thus, off diagonal lower bounds of q(t, x, y) for t ≥ 1 are a consequence of Proposition 5.1(i) and (5.6).

(ii) When β* ∈ (1, ∞), we can get from (5.2) that for any C_* ∈ (0, 1), t ∈ (0, 1) and x, y ∈ D with ρ_D(x, y) ≥ t/C_*,

\[ q(t, x, y) ≥ \frac{c_1 t}{V_D(x, ρ_D(x, y))} φ_2(ρ_D(x, y)) \exp \left( -c_2 ρ_D(x, y) \left( 1 + \log \frac{ρ_D(x, y)}{t} \right) (β^* - 1)/β^* \right), \]  

where the constants c_1, c_2 may be different from those in (5.2). Moreover, we can also obtain form (5.2) and (1.6) that for any t ∈ [1, ∞) and x, y ∈ D with ρ_D(x, y) ≥ t/C_* ≥ t^{1/2}/C_* with any C_* ∈ (0, 1),

\[ q(t, x, y) ≥ \frac{c_3 t}{V_D(x, \sqrt{t})} \left( \frac{t^{1/2}}{ρ_D(x, y)} \right)^{d_2} \exp \left( -c_4 ρ_D(x, y) \left( 1 + \log \frac{ρ_D(x, y)}{t} \right) (β^* - 1)/β^* \right) \]  

With these two estimates above at hand, we can obtain lower bounds of q(t, x, y) for the case that β* ∈ (1, ∞) from Proposition 5.3, Proposition 5.4, Corollary 5.2 and Proposition 5.3.

(iii) The β* = ∞ case be verified by a similar argument as that for (ii). \hspace{1cm} □

The lower bounds of q(t, x, y) in Theorem 1.0 follows directly from (5.14) and Theorem 5.4.

6 Appendix

6.1 Preliminary estimates

Lemma 6.1. Assume that φ is a strictly increasing function on [0, ∞) such that φ(0) = 0, φ(1) = 1 and

\[ c_1 \left( \frac{R}{r} \right)^{g_1} ≤ \frac{φ(R)}{φ(r)} ≤ c_2 \left( \frac{R}{r} \right)^{g_2} \]  
for every 0 < r < R < ∞,  

(6.1)

where 0 < g_1 ≤ g_2 < 2 and c_1, c_2 > 0. Then there exist constants c_i ∈ (0, ∞), i = 3, 4, 5, depending on g_1, g_2, c_1 and c_2, such that the following are true.

(i) Let γ > 0 and β ∈ [0, ∞). Then, for all r > 0,

\[ \sup_{η∈D} \int_{D \setminus B_D(η,r)} \frac{e^{-γρ_D(η,ξ)}}{V_D(η, ρ_D(η,ξ)) φ(ρ_D(η,ξ))} m(dξ) ≤ c_3 e^{-γr^β}/φ(r). \]  

(6.2)
(ii) Let $\gamma > 0$ and $\beta \in [0, \infty)$. Then, for all $r > 0$,

$$
\sup_{\eta \in D} \int_{B_D(\eta, r)} \frac{e^{-\gamma \rho_D(\xi, \eta)^2}}{V_D(\eta, \rho_D(\xi, \eta)) \varphi(\rho_D(\eta, \xi))} \rho(\eta, \xi)^2 m(d\xi) \leq c_4 \int_0^r \frac{s}{\varphi(s)} e^{-\gamma s/2} ds.
$$

(6.2)

In particular, when $\beta \in (0, \infty)$,

$$
\sup_{\eta \in D} \int_D \frac{e^{-\gamma \rho_D(\xi, \eta)^2}}{V_D(\eta, \rho_D(\xi, \eta)) \varphi(\rho_D(\eta, \xi))} \rho(\eta, \xi)^2 m(d\xi) \leq c_4 \int_0^\infty \frac{s}{\varphi(s)} e^{-\gamma s/2} ds < \infty.
$$

(iii) Let $\gamma > 0$ and $\beta \in [0, \infty)$. Then, for all $r > 0$,

$$
\sup_{\eta \in D} \int_{B_D(\eta, r)} \frac{e^{\gamma \rho_D(\xi, \eta)^2}}{V_D(\eta, \rho_D(\xi, \eta)) \varphi(\rho_D(\eta, \xi))} \rho(\eta, \xi)^2 m(d\xi) \leq c_5 \int_0^r \frac{s}{\varphi(s)} e^{\gamma s/2} ds.
$$

Proof. (i) By (1.0), we have for any $\eta \in D$,

$$
\int_{D \setminus B_D(\eta, r)} \frac{e^{-\gamma \rho_D(\xi, \eta)^2}}{V_D(\eta, \rho_D(\xi, \eta)) \varphi(\rho_D(\eta, \xi))} \rho(\eta, \xi)^2 m(d\xi) = \sum_{n=0}^\infty \int_{\{\xi \in D: 2^n r \leq \rho_D(\xi, \eta) < 2^{n+1} r\}} \frac{e^{-\gamma \rho_D(\xi, \eta)^2}}{V_D(\eta, \rho_D(\xi, \eta)) \varphi(\rho_D(\eta, \xi))} \rho(\eta, \xi)^2 m(d\xi)
$$

$$
\leq c_1 \sum_{n=0}^\infty \frac{1}{\varphi(2^n r)} V_D(\eta, 2^n r) \left( V_D(\eta, 2^{n+1} r) - V_D(\eta, 2^n r) \right) 
\leq c_1 e^{-\gamma r^2} \sum_{n=0}^\infty \frac{\varphi(r)}{\varphi(2^n r)} \sum_{i=0}^\infty 2^{-i} \leq c_5 e^{-\gamma r^2},
$$

where the lower bound in (6.1) was used in the second to the last inequality.

(ii) By (1.0) and (6.1), for any $\eta \in D$, we have

$$
\int_{B_D(\eta, r)} \rho(\eta, \xi)^2 \frac{e^{-\gamma \rho_D(\xi, \eta)^2}}{V_D(\eta, \rho_D(\xi, \eta)) \varphi(\rho_D(\eta, \xi))} m(d\xi) \leq c_1 \sum_{i=0}^\infty (2^{-i} r)^2 e^{-\gamma (2^{-i-1} r)^2} \frac{V_D(\eta, 2^{-i} r) - V_D(\eta, 2^{-i-1} r)}{V_D(\eta, 2^{-i-1} r) \varphi(2^{-i-1} r)} 
\leq c_3 \sum_{i=0}^\infty (2^{-i} r)^2 e^{-\gamma (2^{-i-1} r)^2} \frac{c_2 - 1}{V_D(\eta, 2^{-i-1} r) \varphi(2^{-i-1} r)} 
= c_3 \sum_{i=0}^\infty \frac{(2^{-i} r)^2}{\varphi(2^{-i} r)} e^{-\gamma (2^{-i-1} r)^2} \leq c_4 \int_0^r \frac{s}{\varphi(s)} e^{-\gamma s/2} ds.
$$

This establishes (6.2). Taking $r \to \infty$ in (6.2) yields the second assertion in (ii).

(iii) Its proof is similar to that of (ii) and is thus omitted. \qed

Recall that $(E^{0, ref}, \mathcal{F})$ defined in (1.3) is a strongly local regular Dirichlet form on $L^2(D; m)$, and that $(VD)$ and (PI(2)) hold for $(E^{0, ref}, \mathcal{F})$ on $(D, \rho_D, m)$. The diffusion process $Z$ associated with $(E^{0, ref}, \mathcal{F}_D^{0, ref})$
admits a jointly continuous transition density function \( p_{D}^{N}(t, x, y) \) on \((0, \infty) \times \mathbb{D} \times \mathbb{D} \), which enjoys the estimate in (1.5). Let \( \alpha \in (0, 2) \). We consider \( \alpha/2 \)-stable subordinator \( S := (S_{t})_{t \geq 0} \) independent of \( Z \). Let \((Z_{S_{t}})_{t \geq 0} \) be the subordinated diffusion. It is easy to see that (e.g. see [Ok]) \((Z_{S_{t}})_{t \geq 0} \) has a transition density \( p_{\alpha}(t, x, y) \) given by

\[
p_{\alpha}(t, x, y) = \int_{0}^{\infty} p_{D}^{N}(s, x, y) \mathbb{P}(S_{t} \in ds), \quad x, y \in \mathbb{D}, t > 0,
\]

and the jumping function of \((Z_{S_{t}})_{t \geq 0} \) is given by

\[
\int_{0}^{\infty} p_{D}^{N}(s, x, y) \mu(s) ds, \quad x, y \in \mathbb{D},
\]

where \( \mu(t) = c(\alpha)t^{-1-\alpha/2} \) is the Lévy density of \( S \). Further, we have the following estimate for \( j_{\alpha}(x, y) \).

**Lemma 6.2.** There is a constant \( c_{0} \geq 1 \), depending on \( \alpha \in (0, 2) \) and the constants in (1.5), such that for every \( x, y \in \mathbb{D} \),

\[
\frac{c_{0}}{V_{D}(x, \rho_{D}(x, y))} \rho_{D}(x, y)^{\alpha} \leq j_{\alpha}(x, y) \leq \frac{c_{0}}{V_{D}(x, \rho_{D}(x, y))} \rho_{D}(x, y)^{\alpha}.
\]

**Proof.** By (1.5) and (1.6), \( j_{\alpha}(x, y) \leq c_{1} k_{D}^{\alpha}(x, c_{2} \rho_{D}(x, y)) \), where

\[
k_{D}^{\alpha}(x, r) := \int_{0}^{\infty} \frac{1}{V_{D}(x, \sqrt{s})} \exp \left( -\frac{r^{2}}{s} \right) s^{-1-\alpha/2} ds
\]

\[
= \int_{0}^{r^{2}} \frac{1}{V_{D}(x, \sqrt{s})} \exp \left( -\frac{r^{2}}{s} \right) s^{-1-\alpha/2} ds + \int_{r^{2}}^{\infty} \frac{1}{V_{D}(x, \sqrt{s})} \exp \left( -\frac{r^{2}}{s} \right) s^{-1-\alpha/2} ds.
\]

With the change of variable \( t = r^{2}/s \), we have

\[
\int_{r^{2}}^{\infty} \frac{1}{V_{D}(x, \sqrt{s})} \exp \left( -\frac{r^{2}}{s} \right) s^{-1-\alpha/2} ds = r^{-\alpha} \int_{0}^{1} \frac{1}{V_{D}(x, \sqrt{t})} \exp (-t) t^{-1+\alpha/2} dt.
\]

Since \( V_{D}(x, r) \leq V_{D}(x, r/\sqrt{t}) \) for \( x \in \mathbb{D}, r > 0 \) and \( t \in (0, 1] \), the above is less than or equal to

\[
r^{-\alpha} \int_{0}^{1} \exp (-t) t^{-1+\alpha/2} dt \leq \frac{c_{3}}{r^{\alpha} V_{D}(x, r)}.
\]

On the other hand, by (1.6), for \( s \leq r^{2} \),

\[
\frac{1}{V_{D}(x, \sqrt{s})} = \frac{1}{V_{D}(x, r)} \frac{V_{D}(x, r)}{V_{D}(x, \sqrt{s})} \leq \frac{c_{4}}{V_{D}(x, r)} r^{d_{2}/2}.
\]

Thus,

\[
\int_{0}^{r^{2}} \frac{1}{V_{D}(x, \sqrt{s})} \exp \left( -\frac{r^{2}}{s} \right) s^{-1-\alpha/2} ds \leq \frac{c_{4}}{V_{D}(x, r)} \int_{0}^{r^{2}} \exp \left( -\frac{r^{2}}{s} \right) r^{d_{2}s-1-\alpha/2-\alpha/2} ds \leq \frac{c_{4}}{V_{D}(x, r)} \left( \sup_{a \geq 1} e^{-a} a^{1+\alpha/2+d_{2}/2} \right) \int_{0}^{r^{2}} r^{-2-\alpha} ds \leq \frac{c_{5}}{r^{\alpha} V_{D}(x, r)}.
\]

Combining both estimates above and using (1.6), we obtain the upper bound.

Next, we consider the lower bound. By (6.3) and (1.5), \( j_{\alpha}(x, y) \geq c_{6} k_{D}^{\alpha}(x, c_{7} \rho_{D}(x, y)) \). According to (1.6), for all \( x \in \mathbb{D}, r > 0 \) and \( t \in (0, 1] \),

\[
\frac{1}{V_{D}(x, r/\sqrt{t})} = \frac{1}{V_{D}(x, r)} \frac{V_{D}(x, r)}{V_{D}(x, r/\sqrt{t})} \geq \frac{c_{8}}{V_{D}(x, r)} r^{d_{2}/2}.
\]

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Thus, using \(6.4\), we have
\[
k_0(x, r) \geq \frac{c_0}{r^\alpha V_D(x, r)} \int_0^1 \exp \left( -t \left( t^{-1/2} + d_2 / t \right) \right) dt \geq \frac{c_0}{r^\alpha V_D(x, r)}.
\]
Hence, the lemma follows from \(1.6\).\)

In the following, let \((E^{(\alpha)}, F^{(\alpha)})\) be the Dirichlet form associated with the subordinated diffusion \((Z_{S_t})_{t \geq 0}\), where \(S := (S_t)_{t \geq 0}\) is an \(\alpha/2\)-stable subordinator independent of \(Z\) with \(\alpha \in (0, 2)\). Then,
\[
E^{(\alpha)}(f, f) = \frac{1}{2} \int_D \int_D \left( f(y) - f(x) \right)^2 \alpha(x, y) m(dx)m(dy), \quad f \in F^{(\alpha)}.
\]
We have by \([Ok, Theorem 2.1(i)]\]

**Lemma 6.3.** \(F^{0,\text{ref}}_D \subset F^{(\alpha)}\), and, there is a constant \(c_1 > 0\) such that
\[
E_1^{(\alpha)}(f, f) := E^{(\alpha)}(f, f) + \|f\|^2 \leq c_1 E_1^{0,\text{ref}}(f, f) \quad \text{for every } f \in F^{0,\text{ref}}_D.
\]

### 6.2 UJS

Following \([BBK, CKK2, CKK3]\), for \(R \in (0, \infty]\), we say the jumping kernel \(J(x, y)\) satisfies the \(UJS_R\) condition if there is a constant \(c > 0\) such that for \(m\)-a.e. \(x, y \in D,\)
\[
J(x, y) \leq \frac{c}{V_D(x, r)} \int_{B_D(x, r)} J(z, r) m(dz) \quad \text{whenever } r \leq \rho_D(x, y) / 2 < R. \quad (UJS_R)
\]

When \(R = \infty\), the \(UJS_\infty\) condition will simply be called the \(UJS\) condition.

The \(UJS\) condition plays an important role in the study of parabolic Harnack inequalities for non-local Dirichlet forms; see \([CKK2, CKW2]\). If \((x, y)\) satisfies both \((J_{\phi, \beta, \leq})\) and \((J_{\phi, \beta', \geq})\) with \(\beta \leq \beta'\) in \([0, \infty] \cup \{0_+\}\), then it is easy to check that \(UJS_1\) condition holds. It follows from the proofs of \([CKK2, Theorem 5.2], [CKW2, Theorem 3.8]\) and \([CKW3, Theorem 1.18]\) that parabolic Harnack inequalities for finite ranges hold as well. See \([CKK2, page 1071]\) for the definition of parabolic Harnack inequalities for finite ranges. We note that the \(UJS\) condition holds if the jumping kernel \(J(x, y)\) has two-sided bounds \((J_{\phi, \infty, \leq})\) and \((J_{\phi, \infty, \geq})\). In this case, the scale-invariant parabolic Harnack inequality holds for full ranges.

Suppose there are positive constants \(c_1, c_2, \beta \in (0, \infty), \theta_1 \geq \theta_2 \geq 0\) and a strictly increasing function \(\phi\) on \([0, \infty)\) satisfying \(\phi(0) = 0, \phi(1) = 1\) and \([1.11]\) so that
\[
\frac{c_1}{V_D(x, \rho_D(x, y))} \exp \left( -\theta_1 \rho_D(x, y)^\beta \right) \leq J(x, y) \leq \frac{c_2}{V_D(x, \rho_D(x, y))} \exp \left( -\theta_2 \rho_D(x, y)^\beta \right)
\]
for every \((x, y) \in D \times D \setminus \text{diag}\). The next lemma shows that, for the jumping kernel \(J(x, y)\) satisfying \((6.5)\) with \(\theta_1 = \theta_2\), then \(UJS\) holds. However, when \(\theta_1 \neq \theta_2\) in \((6.5)\), then \(UJS\) does not hold in general. (See \([CKK3, Example 2.4]\).)

**Lemma 6.4.** Suppose that the jumping kernel \(J(x, y)\) satisfies \((6.5)\) with \(\theta_1 = \theta_2\). Then \(UJS\) holds.

**Proof.** The proof is a simple modification of that of \([CKK3, Lemma 2.1]\). Let \(x, y \in D\). Suppose \(2r \leq \rho_D(x, y)\). Let
\[
A_{x,y,r} := \{ z \in B_D(x, r) : \rho_D(z, y) \leq \rho_D(x, y) \},
\]
and define \(\tilde{\phi}(r) := \phi(r)e^{\theta_1 r^\beta}\). Note that
\[
\int_{B_D(x, r)} \frac{m(dz)}{V_D(z, \rho_D(z, y))} \tilde{\phi}(\rho_D(z, y)) \geq \int_{A_{x,y,r}} \frac{m(dz)}{V_D(z, \rho_D(z, y))} \tilde{\phi}(\rho_D(z, y)) \geq \frac{c_2 m(A_{x,y,r})}{V_D(x, \rho_D(x, y)) \tilde{\phi}(\rho_D(x, y))}.
\]
Let \( \gamma \) be a continuous curve with \( \gamma(0) = x \) and \( \gamma(1) = y \) such that length(\( \gamma \)) \( \leq \rho_D(x, y) + r/4 \). Choose \( w \in \gamma \) such that \( \rho_D(x, w) = r/2 \). Then

\[
\rho_D(x, y) + \frac{r}{4} \geq \text{length}(\gamma) = \text{length}(\gamma_{x,w}) + \text{length}(\gamma_{w,y}) \geq \frac{r}{2} + \rho_D(w, y),
\]

so that \( \rho_D(w, y) \leq \rho_D(x, y) - 4^{-1}r \). Thus, for every \( z \in B_D(w, 4^{-1}r) \), we have \( \rho_D(z, y) \leq \rho_D(z, w) + \rho_D(w, y) \leq \rho_D(x, y) \), and so \( B_D(w, r/4) \subset A_{x,y,r} \). Therefore, by (1.8),

\[
m(A_{x,y,r}) \geq m(B_D(w, r/4)) = V_D(w, r/4) \geq c_3 V_D(x, r).
\]

Using (6.5) with \( \theta_1 = \theta_2 \), we conclude that there exists a constant \( c_5 > 0 \) such that for every \( r > 0 \) and \( 2r \leq \rho_D(x, y) \), we have

\[
\int_{B_D(x,r)} J(z, y) \, m(dz) \geq c_3 \rho_1^{-1} V_D(x, r) \frac{c_3 V_D(x, r) \phi(\rho_D(x, y))}{V_D(x, \rho_D(x, y))} \geq c_5 V_D(x, r) J(x, y).
\]

The proof is complete. \( \square \)

### 6.3 Heat kernel estimates: the case \( 0 = \beta_* < \beta^* \leq \infty \)

Note that the lower bound heat kernel estimates on \( q(t, x, y) \) in Theorem 5.3 are obtained under the condition \((J_{\phi_1,0,\leq})\) in addition to \((J_{\phi_2,\beta^*,\geq})\) with \( \beta^* \in (0, \infty] \). In this subsection, we comment that it seems hard to replace condition \((J_{\phi_1,0,\leq})\) by a weaker condition condition \((J_{\phi_1,\leq})\).

Suppose that \( D \) is unbounded. We first observe, under the condition \((J_{\phi_1,0,\leq})\) which is stronger than \((J_{\phi_1,\leq})\), what estimates one can obtain. Let \((\mathcal{E}, \mathcal{F})\) be the non-local Dirichlet form given by (1.9 - 1.10) with jumping kernel \( J(x, y) \) satisfying \((J_{\phi_1,0,\leq})\) and \((J_{\phi_2,\beta^*,\geq})\) for some \( \beta^* \in (0, \infty] \). By Proposition 2.1, \((\mathcal{E}, \mathcal{F})\) is a regular Dirichlet form on \( L^2(D; m) \). By Remark 3.3 and Proposition 5.3, there is a conservative Feller process \( Y \) on \( D \) associated with the regular Dirichlet form \((\mathcal{E}, \mathcal{F})\) on \( L^2(D; m) \) that starts from every point in \( D \). Moreover, \( Y \) has a jointly Hölder continuous transition density function \( q(t, x, y) \) on \((0, \infty) \times \bar{D} \times \bar{D} \) with respect to the measure \( m \), and there are constants \( c_1, c_2 > 0 \) so that for all \( x, y \in \bar{D} \) with \( \rho_D(x, y) \leq 1 \) and \( t \in (0, 1) \),

\[
q(t, x, y) \leq c_1 \left( \frac{1}{V_D(x, \sqrt{t})} \wedge \left( p^{(c)}(t, x, c_2 \rho_D(x, y)) + p^{(j)}_{\phi_1}(t, x, \rho_D(x, y)) \right) \right),
\]

where \( p^{(c)}(t, x, r) \) and \( p^{(j)}_{\phi_1}(t, x, r) \) are defined by (1.8) and (1.12), respectively, and there are constants \( c_3, c_4 > 0 \) such that for any \( t \in (0, 1] \) and any \( x, y \in \bar{D} \) with \( \rho_D(x, y) \leq c_2 t^{1/2} \),

\[
q(t, x, y) \geq \frac{c_3}{V_D(x, \sqrt{t})}.
\]

Following the proof of [CKW3] Theorem 1.13 and the arguments in Section 5, we can obtain that for all \( x, y \in \bar{D} \) with \( \rho_D(x, y) \leq 1 \) and \( t \in (0, 1) \),

\[
c_3 H_{\phi_2,\beta^*}(t, x, c_4 \rho_D(x, y)) \leq q(t, x, y) \leq c_5 H_{\phi_1,0}(t, x, c_5 \rho_D(x, y)).
\]

However, it seems to be hard to obtain good explicit estimates for \( q(t, x, y) \) especially for large times when the jumping kernel \( J(x, y) \) satisfies the lower bound condition \((J_{\phi_2,\beta^*,\geq})\) for some \( \beta^* \in (0, \infty] \) and merely the upper bound condition \((J_{\phi_1,\leq})\). One of the reasons is that the behavior of the on-diagonal estimates for \( q(t, x, y) \) is completely different between the case \( \beta_* = \beta^* = 0 \) and \( 0 < \beta_* \leq \beta^* \leq \infty \) as shown in Theorems 1.4 and 1.6.
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