Quantum Hall Skyrmions with Higher Topological Charge

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We have investigated quantum Hall skyrmions at filling factor $\nu = 1$ carrying more than one unit of topological, and hence electric, charge. Using a combination of analytic and numerical methods we find the counterintuitive result that when the Zeeman energy is tuned to values much smaller than the interaction energy ($\mu B/(e^2/\ell) < 9 \times 10^{-5}$), the creation energy of a charge two skyrmion becomes less than twice the creation energy of a charge one skyrmion, i.e. skyrmions bind in pairs. The doubly charged skyrmions are stable to further accretion of charge and exhibit a 10 % larger spin per unit charge than charge one skyrmions which would, in principle, signal this pairing.

Ferromagnetic quantum Hall (QH) systems have quasiparticle excitations, "skyrmions", that involve a texturing of the spins \cite{1}. There is now strong experimental evidence that the lowest energy charged excitations at filling factor $\nu = 1$ are skyrmions \cite{2} \cite{3}.

A smooth texturing of the spins can be described by an effective non-linear sigma model (see \cite{4} below) where the spin is represented by a unit vector $\mathbf{n}(r)$. Skyrmions are topological excitations characterized by an integer topological charge $Z = \int d\mathbf{q}(\mathbf{r})$, where $q = \mathbf{n} \cdot (\partial_\mathbf{r} \times \partial_\mathbf{r} \mathbf{n})/4\pi$ is the topological (Pontryagin) density; $Z$ is the winding number of the mapping $\mathbf{n}(r)$ from the compactified plane ($S_2$) to the target space of the sigma model ($S_2$). A characteristic feature of QH skyrmions is that their topological density is proportional to the deviation of the electron density, $\rho$, from its uniform value, $\bar{\rho}$: $q = \nu(\rho - \bar{\rho})$. Thus, at $\nu = 1$, skyrmions with topological charge $Z$ carry electric charge $-Ze$.

Up till now only skyrmions with charge one have been considered for the natural reason that one would expect Coulombically repelling skyrmions to disfavor charge aggregation. In this note we study skyrmions at $\nu = 1$ with higher charge and find that this expectation is violated for extremely small values of the Zeeman energy where the skyrmions are large objects. More precisely, we find that a charge two skyrmion has lower energy than a pair of charge one skyrmions for $\tilde{g} < \tilde{g}_c = 9 \times 10^{-5}$, where $\tilde{g} \equiv \mu B/(e^2/\ell)$ is the dimensionless Zeeman energy. This implies that the interaction between two skyrmions, which must be repulsive at long distances on account of their electric charges, has an attractive core in a region where their identities are no longer distinct. We also find that skyrmions with charge three and higher are energetically disfavored at all values of $\tilde{g}$.

We begin with variational estimates of the skyrmion energies for various topological charges at small $\tilde{g}$ that were the motivation for this work. We then study the skyrmions numerically, both in the effective sigma model by integrating the equations of motion \cite{5}, and more microscopically using a Hartree-Fock scheme \cite{6}. These establish the result quoted above and determine $\tilde{g}_c$. Finally, we discuss the prospects for observing this effect in experiments.

Effective Sigma Model: The long wavelength physics of the spin degrees of freedom in ferromagnetic QH states is described by the effective Lagrangian density \cite{7}

$$L_{\text{eff}} = \frac{1}{2} \rho \mathbf{A} \cdot \partial \mathbf{n} - \frac{1}{2} \rho^s (\nabla \mathbf{n})^2 + \frac{1}{2} g^2 \mu_B \mathbf{n} \cdot \mathbf{B} - \frac{e^2}{2\epsilon} \int d^2 r \rho(r) \rho(r') \frac{|\mathbf{r} - \mathbf{r}'|}{|\mathbf{r} - \mathbf{r}'|}. \quad (1)$$

Here $\mathbf{A}$ is the vector potential of a unit magnetic monopole, $\rho^s$ is the spin stiffness ($\rho^s = e^2/(16\sqrt{2}\pi\ell$) for $\nu = 1$), $\epsilon$ is the dielectric constant and $\ell$ the magnetic length.

The first two terms in $L_{\text{eff}}$ are the leading terms in the effective action for any ferromagnet and on account of the scale invariance of the gradient term lead, prima facie, to charge $Z$ skyrmions of arbitrary size $\lambda$ and energy $E_Z = 4\pi Z|\rho^s| \tilde{g}$.

The Zeeman and Coulomb terms in $L_{\text{eff}}$ break the scale invariance and their competition sets the size and energy of the skyrmions and modifies the detailed form of their profiles; these now depend on the dimensionless ratio $\tilde{g} = (\mu B)/\rho^s$. The Zeeman energy to the Coulomb energy. A rotationally symmetric ansatz for a skyrmion with topological charge $Z$ is

$$\begin{align*}
  n_x &= \sqrt{1 - f^2(r) \cos(Z\theta)} , \\
  n_y &= \sqrt{1 - f^2(r) \sin(Z\theta)} , \\
  n_z &= f(r) ,
\end{align*} \quad (2)$$

where $f$ obeys the boundary conditions $f(0) = -1$, $f(\infty) = 1$. The topological density is $q = (Z/4\pi r) df/dr$. The skyrmions in the scale invariant sigma model are given by $f(r) = [(r/\lambda)^{2Z} - 4]/[(r/\lambda)^{2Z} + 4]$. For $Z > 1$, their topological density has a hollow core.

Substituting (2) in the action (1) and minimizing leads to a non-linear, non-local integro-differential equation for $f$. Below, we discuss results obtained by numerically integrating this equation using a relaxation technique \cite{3}. But first, it is instructive to consider approximate solutions at small $\tilde{g}$; the details are discussed elsewhere \cite{6}.

For $Z > 1$ we take the solution to be of the form of the
scale invariant sigma model solution with an optimized scale parameter \( \lambda \). This yields,

\[
\lambda = 0.780 \tilde{g}^{-1/3} \quad \text{and} \quad E = \frac{e^2}{\ell} \left[ 2 \sqrt{\frac{\pi}{32}} + 2.86 \tilde{g}^{1/3} \right] \quad (Z = 2)
\]
\[
\lambda = 1.20 \tilde{g}^{-1/3} \quad \text{and} \quad E = \frac{e^2}{\ell} \left[ 3 \sqrt{\frac{\pi}{32}} + 4.82 \tilde{g}^{1/3} \right] \quad (Z = 3)
\]
\[
\lambda = 1.75 \tilde{g}^{-1/3} \quad \text{and} \quad E = \frac{e^2}{\ell} \left[ 4 \sqrt{\frac{\pi}{32}} + 7.21 \tilde{g}^{1/3} \right] \quad (Z = 4)
\]

for the lowest three values of \( Z \). Note that the energy per unit charge is the same for the \( \tilde{g} \) independent piece as is appropriate for solutions of the scale invariant problem \( \tilde{g} \) while for the \( \tilde{g} \) dependent piece it increases monotonically with \( Z \). Hence, there is no binding between this set of skyrmions.

However, this procedure runs into trouble with the \( Z = 1 \) skyrmion where the scale invariant solution yields a Zeeman energy that diverges logarithmically with system size for any \( \lambda \). This can be fixed by matching the scale invariant solution onto the exact asymptotic solution in the outer region which decays exponentially with a length \( \propto \ell \tilde{g}^{-1/2} \). As a consequence we get logarithmically modified expressions \( \tilde{g}, \)

\[
\lambda = 0.558 (\tilde{g} | \ln \tilde{g} |)^{-1/3} \quad \text{and} \quad E = \frac{e^2}{\ell} \left[ \sqrt{\frac{\pi}{32}} + 0.622 (\tilde{g} | \ln \tilde{g} |)^{1/3} \right] \quad (Z = 1)
\]

for the core scale parameter and energy. It is evident that the presence of the logarithm by itself implies that charge one skyrmions will pair bind for small enough \( \tilde{g} \) and that the binding energy will vanish as \( \tilde{g} \) vanishes. Using the above expressions one finds a critical value of \( \tilde{g}_c \approx 5.3 \times 10^{-6} \) for this to happen. That such a critical value has to exist can be inferred by a separate microscopic computation of the pair-binding energy at large \( \tilde{g} \), where the skyrmion size is \( O(\ell) \) and they reduce to polarized quasiparticles, which shows the absence of binding. However, the quantitative issue is rather delicate. Even at small \( \tilde{g} \) the result \( \tilde{g}_c \) is expected to have logarithmically subdominant corrections and is known numerically to be not terribly accurate in estimating the difference \( E(\tilde{g}) - E(0) \), even for small \( \tilde{g} \) as small as \( 10^{-6} \). Also, the asymptotic validity of our small \( \tilde{g} \) expressions is not on completely rigorous footing at present. So it follows that while the pair binding should be present only at small \( \tilde{g} \), a reliable estimate of the critical \( \tilde{g}_c \) requires the more accurate computations of the skyrmion energies reported below.

**Hartree-Fock:** We complement the effective action calculations by the Hartree-Fock method introduced by Fertig et. al. [8]. For skyrmions with topological charge \( Z \) the Hartree-Fock wave function is fixed by requiring it to be in the lowest Landau level and by the symmetries of the classical solution \( \tilde{g} \) to be

\[
|\Psi\rangle = \prod_{m=0}^{\infty} (u_m c_m^\dagger + v_m c_{m+1}^\dagger) \prod_{p=0}^{Z-1} c_p^\dagger |0\rangle \quad \text{if} \quad Z > 0
\]

\[
|\Psi\rangle = \prod_{m=-Z}^{\infty} (u_m c_m^\dagger + v_m c_{m-1}^\dagger) |0\rangle \quad \text{if} \quad Z < 0 \quad (5)
\]

where \( |u_m|^2 + |v_m|^2 = 1 \), \( c_m |0\rangle = 0 \) and \( c_m^\dagger \) creates electrons with spin \( \sigma \) in the angular momentum \( m \)-state in the lowest Landau level. We choose \( u_m, v_m \) real and require \( u_m \to 1 \) as \( m \to \infty \). The coefficients \( u_m, v_m \) are determined by numerically iterating the Hartree-Fock equations until a self-consistent solution is found. Obviously, \( |\Psi\rangle \) has electric charge \(-Z|c|\) relative to the polarized groundstate \( |\Psi_0\rangle = \prod_{m=0}^{\infty} c_m^\dagger |0\rangle \). Skyrmions with charge \( Z < 0 \) are often referred to as antiskyrmions. In the actual numerical calculation we have a finite number of particles, \( m \leq N \), and impose \( u_N = 1 \) (or \( u_N = u_{N-1} = 1 \) see below).

**Results:** Our results are shown in Fig. 1-3. In Fig. 1 we give the energies to create one \( Z = 2 \) skyrmion and to create two \( Z = 1 \) skyrmions, both at fixed particle number and magnetic field, as functions of \( \tilde{g} \). These energies are equal for the skyrmion \((Z > 0)\) and the antiskyrmion \((-Z < 0)\). This means that we have shifted the Hartree-Fock result for the state \( \tilde{g} \) by a constant. For large \( \tilde{g} \), the \( Z = 2 \) skyrmion has higher energy than two \( Z = 1 \) skyrmions. In this region the skyrmions are small, much smaller than the system size, and the Hartree-Fock calculation is reliable.

![FIG. 1. Energies for one \( Z = 2 \) skyrmion and for two \( Z = 1 \) skyrmions. The lines are Hartree-Fock results for \( N = 1000 \) and the points are sigma model results. The spin of one \( Z = 2 \) and two \( Z = 1 \) skyrmions is indicated along the lines. The inset shows that the energies cross at \( \tilde{g}_c \sim 6 \times 10^{-6} \) (sigma model results only). \( E(0) \) is the \( \tilde{g} = 0 \) energy.](image)
\( Z = 1 \) skyrmions decreases. We see that, except possibly for very small \( \tilde{g} \), the \( Z = 2 \) skyrmion has larger energy. As noted earlier, in the limit \( \tilde{g} \to 0 \) the skyrmions become infinitely large and only the scale invariant term in (1) contributes to the energy. Thus both energies must eventually approach the pure scale invariant result \( E(0) = 8\pi\rho^* = \sqrt{\pi/2} e^2/\ell \) as \( \tilde{g} \to 0 \). To investigate the region of small \( \tilde{g} \) we proceed in two different ways.

The numerical sigma model method can be used for small \( \tilde{g} \). The inset in Fig. 1 shows the difference in energy as a function of \( \tilde{g} \) obtained using the sigma model. This predicts a crossing at \( \tilde{g}_c = (9 \pm 1) \times 10^{-5} \). At this point the energy is \( E_{\tilde{g}_c} = 0.751 e^2/\ell \) and there is a 10% change in the spin per charge \((S/Z)\) of the lowest energy excitation: \((S/Z)_{Z=1} = 221\) and \((S/Z)_{Z=2} = 243\). The size of the excitation also changes: The charge radii (per unit charge) are \( r_{Z=1} = 21\ell \) and \( r_{Z=2} = 31\ell \). We can also see that when \( \tilde{g} \) decreases below \( \tilde{g}_c \), the difference in energy first increases but then eventually starts to decrease as it must, since it should vanish at \( \tilde{g} = 0 \).

Using Hartree-Fock for smaller \( \tilde{g} \) than shown in Fig. 1, we find apparent crossing points, \( \tilde{g}_c(N) \). However, the \( \tilde{g}_c(N) \) are so small that the skyrmions are large and there are large finite size effects: \( \tilde{g}_c \) depends on \( N \) and on whether one imposes as boundary condition that one or several of the spins are up at the edge of the system. It is difficult to increase \( N \) very much beyond \( N = 1000 \) since the convergence is very slow for the small values of \( \tilde{g} \) we are interested in. We conclude that within Hartree-Fock we are not able to directly probe a region of \( \tilde{g} \) where an \( N \)-independent crossing might take place. However, we can reach smaller \( \tilde{g} \) by finite size scaling. In Fig. 2 we plot the crossing point, \( \tilde{g}_c(N) \), as a function of \( 1/N \) for three different sets of boundary conditions (as indicated in the figure) [10]. To each set we fit a quadratic polynomial in \( 1/N \) and read off \( \tilde{g}_c \equiv \tilde{g}_c(\infty) = (8.9 \pm 1.4) \times 10^{-5} \). The error is the standard deviation in the three extrapolated values. The errors in the fit for each curve are much smaller. Also, adding or subtracting a point at small \( N \) does not affect the results. That the three different sets, which give very different \( \tilde{g}_c(N) \), predict virtually the same nonzero \( \tilde{g}_c \) indicates strongly that there is a crossing. This value of \( \tilde{g}_c \) agrees with the one obtained in the effective theory. Thus we conclude that the doubly charged skyrmions have lower energy for \( \tilde{g} < \tilde{g}_c = 9 \times 10^{-5} \).

It is straightforward to calculate not only the energy but also the total spin as well as spin density and charge density. The total spin, which is indicated along the line in Fig. 1, is much larger for the doubly charged skyrmion than for the charge one skyrmion. In Fig. 3 we give examples of densities for the doubly charged skyrmion and compare to one charge one skyrmion. Note that the charge density has its maximum a finite distance away from the origin for the doubly charged skyrmion (as the scale invariant skyrmion has).

**FIG. 2.** Finite size scaling of \( \tilde{g}_c(N) \) using Hartree-Fock results for antiskyrmions with various boundary conditions (one or two spins up at the edge, as indicated in the figure). \( \tilde{g}_c(N) \) is where the energy of one \( Z = -2 \) skyrmion is equal to the energy of two \( Z = -1 \) skyrmions in a Hartree-Fock calculation with \( N \) particles. Note that \( \tilde{g}_c(N) \) depends strongly on \( N \) and on the boundary condition. Fitting to quadratic polynomials in \( 1/N \) gives \( \tilde{g}_c \equiv \tilde{g}_c(\infty) = 8.9 \times 10^{-5} \).

**FIG. 3.** Charge and spin densities at \( \tilde{g} = 0.02, 0.01, 0.005 \) for one \( Z = 2 \) skyrmion (using Hartree-Fock). Densities for one \( Z = 1 \) skyrmion at \( \tilde{g} = 0.02 \) is included for comparison.

The numerical sigma model gives \( \lim_{\tilde{g} \to 0} |E - \sqrt{\pi/2} e^2/\ell|/\tilde{g}^{1/3} = 2.86 e^2/\ell \) for \( Z = 2 \) in agreement with (3). Thus there is no logarithm in the \( Z = 2 \) energy and hence no reason to expect that \( Z > 2 \) skyrmions would ever have the lowest energy per charge. We have also checked that \( Z = 3 \) skyrmions have higher energy (per charge) than \( Z = 1,2 \) skyrmions for \( \tilde{g} > 10^{-7} \).

**Discussion and Experimental Implications:** Our analysis has established that two skyrmions are unstable to pair-binding at very small Zeeman energies. At distances much larger than their size, the inter-skyrmion potential must be repulsive as it will be dominated by their Coulomb interaction. It follows that the attraction
that we have found sets in only when the skyrmions overlap and their identities begin to merge. In computing the skyrmion energies, we have ignored the effects of Landau level mixing and of the softening of the Coulomb interaction at small distances by the finite extent of the subband wavefunctions transverse to the plane of the electron gas. It is likely that neither of these makes a huge difference to the energetics. In the sigma model description, the dominant effect of Landau level mixing is a renormalization of the spin stiffness which does not affect the inter-skyrmion energetics. The softening of the Coulomb interactions is significant at distances which are $O(\ell)$ and for much bigger skyrmions (such as in the region of interest where they exceed $20\ell$) most of the self-interaction is unsoftened. (We also note that the long wavelength arguments regarding the peculiarities of the $Z = 1$ solution that underlie the pairing depend only on the form of the sigma model action; they will apply mutatis mutandis to other ferromagnetic fillings as well.)

This is encouraging in that it suggests that it might be possible to see the binding of the skyrmions as a sharp decrease in the spin polarization with decreasing $\tilde{g}$ in the vicinity of $\nu = 1$, where the ground state will exhibit a dilute density of skyrmions. Unfortunately, this is unlikely on two grounds. First, it would require an extremely fine tuning of $\tilde{g}$ which seems hard to accomplish. Second, there is the more fundamental problem that when the skyrmions get to be very big, they become sensitive to the details of the disorder potential which will tend to limit their size. Effectively, we expect the disorder to provide a lower limit to $\tilde{g}$ and while we have not computed this for realistic disorder realizations, we are not greatly encouraged by the data in [5] which found a maximum skyrmion size, even for $\tilde{g}$ reduced under pressure, considerably smaller than the critical one calculated in this paper.

**Note Added:** While finishing this work we became aware of a recent paper by Nazarov and Khaetskii [11] that studies higher topological charge skyrmions in the effective sigma model in the spirit of our variational calculations. These authors also argue that the interaction between two charge one skyrmions is attractive at intermediate and short distances and that the binding distance is zero, at which they form a charge two skyrmion. However, their quoted results are independent of $\tilde{g}$, in contradiction with ours which hold only at small $\tilde{g}$, and they appear not to have noticed the existence or the extreme smallness of $\tilde{g}_c$.

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