Fulfilling entanglement’s benefit via converting correlation to coherence

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Entanglement boosts performance limits in sensing and communication, and surprisingly even more in presence of entanglement-breaking noise. However, to fulfill such advantages requires a practical receiver design, a challenging task as information is encoded in the feeble quantum correlation after entanglement’s death. We propose a conversion module to capture and transform such correlation to coherent quadrature displacement, and therefore enables the optimal receiver design for a wide range of entanglement-enhanced protocols, including target detection (quantum illumination), phase estimation, classical communication, target ranging and arbitrary thermal-loss channel pattern classification. The conversion module maps the quantum detection problem to the semi-classical detection of noisy coherent states via heterodyne and passive linear optics. Our module is completely off-the-shelf and provides a paradigm for processing noisy quantum correlations.

Quantum entanglement not only refreshes our understanding of the world but also brings unprecedented power to boost our capabilities in sensing [9–12] and communication [1–8]. Entanglement is fragile—noise can easily destroy entanglement. Surprisingly, by evaluating information-theoretical limits of sensing and communication, people find that benefits from entanglement can even survive entanglement-breaking noise, for example in target detection [13], target ranging [14, 15] and classical communication [16]. However, despite entanglement’s surprisingly robust advantage, it is hard to actually design and build systems to fulfill such advantages, as it requires extracting information that is delicately hidden in quantum correlations. Indeed, till now the experimental demonstrations of these protocols are far from optimal [17, 18] and optimal receivers are either beyond near-term technology [19] or completely unknown.

We resolve this open problem and fulfill entanglement’s benefit with a conversion module from intermodal correlation to intramodal coherence. Via heterodyne detection on the return, the information encoded in the delicate remaining cross-correlation after entanglement’s death is revealed in the coherent quadrature displacement of the ancilla. We prove that such a correlation-to-displacement (C-D) conversion preserves all information and therefore enables the optimal performance in entanglement-assisted (EA) target detection (quantum illumination, QI), phase sensing, classical communication, target ranging and arbitrary thermal-loss channel pattern classification. Moreover, the conversion module enables exact performance analyses and extends quantum advantages to the non-asymptotic region, unexplored due to the limitations of asymptotic tools [19–23]. It also allows the proof of a folklore of a six-decibel error exponent advantage in an arbitrary thermal-loss channel pattern classification problem. Combining our conversion module with a standard optimal coherent-state receiver [24, 25], we enable optimal receiver designs, relying only on off-the-shelf components of linear optics and photon detection. Our C-D conversion module has broad application and brings new insights into how quantum correlations can be processed.

I. ENTANGLEMENT-ASSISTED PROTOCOL

As shown in Fig. 1(a), in a sensing protocol, a probe signal is sent out to interact with the physical process (such as reflection induced by a target), and then unavoidably encounters noise, before finally get detected. Similarly, in a communication protocol, the signal carrying a classical message goes through a noisy link to get detected. In both cases, the final detection requires a structured receiver to extract information. An EA protocol entangles the initial signal with an ancilla, which is jointly measured with the received signal to boost the information extraction performance.

In the above paradigm, the physical process and the noise can be modeled as an overall phase-shift thermal-loss channel $\Phi_{\kappa, \theta}$ [26], with $\kappa$ being the transmissivity and $\theta$ the phase-shifts.
\( \theta \) being the phase shift. For an input mode described by the annihilation operators \( \hat{a}_S \), the received mode is

\[
\hat{a}_R = e^{i\theta} \sqrt{\kappa} \hat{a}_S + \sqrt{1 - \kappa} \hat{a}_B,
\]  

where \( \hat{a}_B \) is in a thermal state with mean photon number \( N_B/(1 - \kappa) \) to model the noise [27].

In a target detection scenario [13, 14], when the target is present, it reflects part of the signals embedded in noise, and the channel is modeled as \( \Phi_{r,0} \); while when the target is absent, only noise can be received, and the channel is modeled as \( \Phi_{0,0} \). In a phase sensing [28] scenario that models bio-sensing, ranging [15] and gravitational-wave detection [29], we consider the estimation of the parameter \( \theta \) of channel \( \Phi_{r,\theta} \); In a communication scenario, phase-shift-keying (PSK) [30] encodes the message into a phase shift \( \theta \), therefore the signal goes through an overall channel \( \Phi_{r,\theta} \) before getting detected.

To benefit from entanglement, we consider \( M \) signal-idler pairs \( \{\hat{a}_{S,m}, \hat{a}_{I,m}\}_{m=1}^M \), where each pair is in a two-mode squeezed-vacuum (TMSV) state with mean photon number \( N_S \) [26], known to be optimal in these applications [30, 31]. While the signals are sent through the channel \( \Phi_{r,\theta} \), the idlers are stored or pre-shared to the receiver side, leading to \( M \) return-idler pairs \( \{\hat{a}_{R,m}, \hat{a}_{I,m}\}_{m=1}^M \), each maintaining a phase-sensitive cross-correlation \( \langle \hat{a}_{R,m} \hat{a}_{I,m} \rangle = e^{i\theta} C_p \) with the amplitude \( C_p \equiv \sqrt{\kappa N_S} (N_S + 1) \). When \( N_S \) is small, the amplitude of the correlation \( \propto \sqrt{N_S} \) in an EA protocol; while for a classical reference, the correlation \( \propto N_S \) and is therefore much smaller. In this regard, the crucial part of a receiver design to fulfill entanglement’s benefit is to detect the phase-sensitive cross-correlation.

II. CORRELATION-TO-DISPLACEMENT CONVERSION

Now we design a module to convert phase-sensitive cross-correlation between each signal-idler pair to the complex displacement amplitude of a coherent state, therefore mapping the quantum problem to a semi-classical one.

Given the return-idler pairs, \( \{\hat{a}_{R,m}, \hat{a}_{I,m}\}_{m=1}^M \), as shown in Fig. 1(b), the module first performs individual heterodyne measurement on each \( \hat{a}_{R,m} \), producing the complex measurement result \( \mathcal{M}_m \), which obeys a circularly-symmetric complex Gaussian distribution with variance \( \nu_m \equiv (N_B + \kappa N_S + 1)/2 \). Conditioned on the output \( \mathcal{M}_m \) [26, 32], each \( \hat{a}_{I,m} \) is in a displaced thermal state \( \hat{\rho}_{d,m,\xi} \), with mean \( d_m = (C_p/2\nu_m) e^{i\theta} \mathcal{M}_m^* \) and mean thermal photon number \( E = N_S (N_B + 1 - \kappa)/2\nu_m \leq N_S \) (see Appendix C). At this stage, various measurement strategies for the idler modes are possible for information extraction. For example, a passive linear optical transform, i.e., a beamsplitter array with proper weights, can combine all outputs into a single mode in state \( \hat{\rho}_{d,E} \), with mean \( d_T = |d_T| e^{i\theta} \) and thermal noise \( E \). Here the amplitude square \( |d_T|^2 = \sum_{m=1}^M |d_m|^2 \) satisfies the \( \chi^2 \) distribution of \( 2M \) degrees of freedom

\[
P_\kappa^{(M)}(x) \sim \frac{1}{\xi} \left( \frac{x}{\xi} \right)^{M-1} e^{-x/(2\xi)}
\]

with mean \( 2M\xi \) and variance \( 4M\xi^2 \), where \( \xi \equiv C_p^2/4\nu_m \).

III. PERFORMANCE LIMITS

We now analyze the performance limit of the C-D conversion module, while we defer the semi-classical coherent-state processing to later part.

A. Quantum illumination

QI for target detection considers the discrimination between two channels \( \Phi_{0,0} \) and \( \Phi_{r,0} \). In this case, the conversion module produces two displaced thermal state \( \hat{\rho}_{0,N_S} \) (target absent) and \( \hat{\rho}_{\sqrt{\kappa},E} \) (target present), where

\[\begin{align*}
\text{(a)} \quad & \quad \text{(b)} \quad \text{(c)} \quad \text{(d)}
\end{align*}\]
x \sim P\{M\}() obeys the distribution in Eq. (2). This leads to the error probability performance limit

P_{C,D} = \int dx P\{M\}(x) P_H(\hat{\rho}_{0,N_S}, \hat{\rho}_{\sqrt{F},E}),

(3)

where $P_H(\hat{\rho}, \hat{\sigma})$ is the Helstrom limit of error probability in discriminating states $\hat{\rho}$ and $\hat{\sigma}$ with equal prior probability [33–35] (see Appendix E).

To compare with the ultimate performance, we evaluate the Nair-Gu (NG) lower bound on the error probability applicable to any source of illumination [31] $P_{NG}$ (see Appendix A). To benchmark for entanglement advantage, we also consider the Helstrom limit of the optimal classical scheme based on coherent states [13], $P_H,CS = P_H(\hat{\rho}_{0,N_B}, \hat{\rho}_{\sqrt{F},E})$.

We begin with the asymptotic limit of low brightness $N_S \ll 1$ and low reflectivity $\kappa \ll 1$, where $M$ is large to guarantee a decent signal-to-noise ratio. At this limit, we can approximate $\hat{\rho}_{\sqrt{F},E}$ as a coherent state and $\hat{\rho}_{0,N_S}$ as vacuum; Therefore, the Helstrom limit $P_H(\hat{\rho}_{0,N_S}, \hat{\rho}_{\sqrt{F},E}) \simeq e^{-2}/4$ and Eq. (3) leads to

$$P_{C,D} \simeq \frac{1}{4}(1 + 2\xi)^{-M} \simeq \frac{1}{4}\exp\{-Mr_{C,D}\},$$

(4)

which saturates the lower bound $P_{NG}$ in Eq. (A1) with the error exponent $r_{C,D} = 2\xi$. In fact, one can easily check that the optimality holds as long as $N_S \ll 1$ and $\kappa \ll 1 + N_B$. We verify this optimality in Fig. 2(a), where a close agreement is seen between $P_{C,D}$ (red) and $P_{NG}$ (green). At the same time, huge advantage over the classical limit $P_H,CS$ (black) can be observed.

Now we examine the exponent more closely. In general, when $\xi \ll 1$ (e.g., due to $\kappa \ll 1$) we can obtain a lower bound on the error exponent, $r_{C,D} \geq 2((\sqrt{N_S} + 1) - \sqrt{N_S})^2$, where the coherent state error exponent $r_{CS} = \kappa N_S((\sqrt{N_B} + 1) - \sqrt{N_B})^2$ [13] (see Appendix E). We can show that the entanglement advantage exists as long as the signal brightness is smaller than the noise brightness, i.e., $N_S \leq N_B$, as also confirmed in Fig. 2(b) via plotting $r_{C,D}/r_{CS}$.

Finally, we emphasize that Eq. (3) provides an efficiently calculable and achievable error-probability lower bound for QI, in contrast to the asymptotically tight quantum Chernoff (upper) bound (QCB) [13, 20, 21]. As a consequence, Eq. (3) allows the exploration of QI’s advantage in the non-asymptotic region. As shown in Fig. 3(a), when the classical Helstrom limit is fixed at $P_H,CS = 0.05$, the ratio $P_{C,D}/P_{H,CS} \leq 1$ when $N_S \leq N_B$ (above the red dashed line). However, QCB can only show quantum advantage in a strictly smaller region above the orange dashed curve. We also pick a set of parameters $N_S, N_B$ to explicitly plot the error probability versus the number of copies $M$ in Fig. 3(b)—when $M$ is small, QCB fails to identify quantum advantage, while our Eq. (3) shows advantage.

### B. Quantum phase estimation

For quantum phase estimation, we aim at estimating the phase shift $\theta$ of the channel $\Phi_{\kappa,0}$ described in Eq. (1). After the conversion module, we obtain a displaced thermal state $\hat{\rho}_{\sqrt{F},E}$, where $x \sim P\{M\}(\cdot)$ in Eq. (2). The variance of unbiased estimators has an asymptotically tight lower bound $\delta \theta^2 = 1/\mathcal{F}$, with $\mathcal{F}$ being the quantum Fisher information (QFI). The overall QFI enabled by the conversion module can be obtained as (see Appendix F)

$$\mathcal{F}_{C,D} = \frac{4M\kappa N_S(N_S + 1)}{1 + N_B + N_S(2N_B + 2 - \kappa)}.$$  

(5)

We compare $\mathcal{F}_{C,D}$ with the ultimate upper bound of Fisher information $\mathcal{F}_{UB}$ derived in Ref. [36] (see Appendix A). At low brightness $N_S \ll 1$, we find $\mathcal{F}_{C,D} \simeq \frac{1 - \kappa}{(1 + N_B)} \mathcal{F}_{UB}$ and if reflectivity $\kappa$ is low, we can further show that $\mathcal{F}_{C,D} \simeq \mathcal{F}_{UB}$ achieves the optimum.

Now we show the quantum advantage by comparing with the classical limit using coherent-state sources $\mathcal{F}_{CS} = 4M\kappa N_S/(1 + 2N_B)$. When the noise $N_B \gg 1$, the
optimal performance of $\mathcal{F}_{\text{C-D}}$ has a factor of two advantage over $\mathcal{F}_{\text{CS}}$, as verified in Fig. 4 (a). Comparing $\mathcal{F}_{\text{C-D}}$ and $\mathcal{F}_{\text{CS}}$, we can show quantum advantage ($\mathcal{F}_{\text{C-D}} \geq \mathcal{F}_{\text{CS}}$) as long as $N_S \leq N_B/(1 - \kappa)$, as verified in Fig. 4(b).

### C. Entanglement assisted communication

Consider PSK with repetitions, where $M$ signal modes are modulated by the same phase $\theta$ uniformly randomly chosen from $[0, 2\pi)$. Below we present the results, while details can be found in Appendix G. The output of the conversion module is in state $\hat{\rho}_{\varphi,\pi,\theta, E}$, where $x \sim P_{\kappa}^{(M)}(\cdot)$ in Eq. (2). At this point, the achievable information rate per symbol from the output state is

$$\chi_{\text{C-D}} = \frac{1}{M} \int dx P_{\kappa}^{(M)}(x) \chi(\{\hat{\rho}_{\varphi,\pi,\theta, E}\}), \quad (6)$$

where $\chi(\{\hat{\rho}_{\varphi,\pi,\theta, E}\})$ is the Holevo information [11, 37] of the corresponding state ensemble. Due to the uniform phase modulation and the Gaussian nature of each state $\hat{\rho}_{\varphi,\pi,\theta, E}$, Eq. (6) can be efficiently evaluated.

To compare with the ultimate performance, we consider the EA classical capacity $C_E$ [16]. At the same time, to understand the advantage over the classical schemes, we also compare with the classical capacity without assistance $C$ [38]. In Fig. 5(a), we see that $\chi_{\text{C-D}}$ approaches $C_E$, therefore verifying the optimality of the conversion module to fulfill the EA advantage in communication. Indeed, at the limit of low brightness, $N_S \to 0$, we obtain $\chi_{\text{C-D}} \sim \kappa N_S \log(1/N_S)/(N_B + 1)$, which achieves the scaling of the EA capacity. The same optimality result also holds for the binary PSK modulation.

To fully understand the advantage enabled by the conversion module, we plot the ratio $\chi_{\text{C-D}}/C$ in Fig. 5(b) versus $N_S, N_B$. When $N_S \ll 1, N_B \gg 1$, we indeed see the huge advantage; Moreover, we find that entanglement’s benefit can be identified in a large region when $N_S \leq N_B$, similar to the previous cases.

### D. Channel Pattern classification

So far, we considered the sensing of a single phase-shift thermal-loss channel $\Phi_{\kappa,\theta}$. In general, complex sensing problems often involve composite channels with $M$ different sub-channels $\Phi_{\kappa,\theta} = \otimes_{m=1}^{M} \Phi_{\kappa_m,\theta_m}$, where the vector notation $\kappa = \{\kappa_m\}_{m=1}^{M}, \theta = \{\theta_m\}_{m=1}^{M}$ and we have assumed that the noise background $N_B$ is identical across all sub-channels. Previous works on quantum channel position-finding [39], barcode recognition [40], quantum ranging [15, 39], and absorption spectrum recognition [41] can all be considered as special cases of this composite channel.

For hypothesis testing between general composite channels, we are able to prove a universal error exponent advantage from entanglement.

**Theorem 1** In the high noise $N_B \gg 1$ and low signal brightness $N_S \ll 1$ limit, entanglement from two-mode squeezed vacuum enables a factor of four (six decibel) advantage over classical sources of coherent states for the discrimination between general multiple composite thermal-loss channels.

The proof directly utilizes the C-D conversion module, and is achievable with the module plus optimal discrimination between multiple coherent states (see Appendix I).
This result immediately implies that the conversion module is also optimal in the discrete version of target ranging problem [14].

IV. COMPLETING THE RECEIVER DESIGNS

With the conversion module, the detection of cross-correlation in EA scenarios is reduced to the detection of semi-classical coherent states, where receiver designs have been extensively explored theoretically and experimentally [24, 25, 42–48]. Below, we present some examples based on only linear optics and photon detection to complete the optimal receiver design. We will also benchmark with practical schemes based on optical parametric amplifier receivers (OPAR) or phase conjugation receivers (PCR) [17, 30, 49] (see Appendix K).

For QI target detection, at the $N_S \ll 1$ limit the states output from the conversion module are coherent states with low noise, and therefore the Dolinar receiver [24] saturates the Helstrom limit and completes the optimum receiver design (see Appendix H). In Fig. 6(a), we evaluate the performance of the Dolinar receiver combined with a C-D conversion module (green), which indeed achieves the optimal error probability (red). Some discrepancy can be found when $M$ is too large, due to the small noise $E \simeq N_S$ being significant at low error probability. As expected, OPAR (yellow) and PCR (orange) give worse performance, although still better than the coherent-state homodyne scheme (black) [49].

For phase estimation, a simple homodyne detection on the C-D conversion output achieves the Fisher information in Eq. (5) (see Appendix F) and complete the receiver design. Although OPAR and PCR are also asymptotically optimal [30], however, we note that the C-D scheme has larger Fisher information in the non-asymptotic region, especially when $\kappa$ is close to unity and $N_B$ is small, as can be verified in Fig. 6(b).

For EA communication, after the conversion module, the rest of the receiver design problem reduces to achieving the Holevo information among an ensemble of noisy coherent state [25, 50]. To enable a near-term receiver, we consider binary PSK combined with the Hadamard code and Green machine [25, 51] (see Appendix K). The performance is shown in Fig. 6(c) (magenta), which achieves the optimal $\ln(1/N_S)$ scaling of $\chi_{C,D}$, while only relying on linear optics and photon counting.

V. CONCLUSION

We propose the correlation-to-displacement conversion module to resolve the optimal receiver design problem for a wide range of entanglement-assisted sensing and communication problems. The module provides a unique approach for solving quantum detection problems by reducing to semi-classical detection problems.

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QZ conceived the project. QZ and HS proposed the correlation-to-displacement conversion module, with calculations also verified by BZ. BZ analyzed quantum illumination with inputs from HS, and HS analyzed phase estimation and communication, both under the supervision of QZ. QZ proved Theorem 1, with inputs from BZ. QZ wrote the manuscript, with inputs from HS and BZ.

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Appendix A: Additional details

1. Ultimate bounds

Here we provide details on the benchmarks.

Nair and Gü derived a lower bound on the error probability of QI [31] target detection applicable to $M$ probes with mean photon number $N_S$ assisted by arbitrary form of entanglement,

$$P_e \geq P_{NG} \equiv \frac{1}{4} \exp[ - \beta M N_S ] ,$$  \hspace{1cm} (A1)

where $\beta \equiv - \log (1 - \kappa/(N_B + 1))$.

Ref. [36] derived an ultimate upper bound of Fisher information in noisy phase estimation applicable to any form of entangled input

$$\mathcal{F}_{UB} = \frac{4M\kappa N_S (\kappa N_S + (1 - \kappa) N_B^* + 1)}{(1 - \kappa) [\kappa N_S (2N_B^* + 1) - \kappa N_B^* (N_B^* + 1) + (N_B^* + 1)^2]^{1/2}} , \hspace{1cm} (A2)$$

where $N_B^* = N_B/(1 - \kappa)$ is the mean photon number of the thermal state at the environment mode in the Stinespring representation of the channel [36].

2. Comparing with ‘no-go’ results

Here, we discuss the relationship to some no-go type of results. In general, local operations and classical communication (LOCC) strategy on each copy is not optimal for the discrimination between a pair of identical copies of states [52, 53]. For finite $N_S$, our approach is not LOCC between copies because it requires joint detection of the $M$ idler modes, e.g. via a beamsplitter transform and then joint photodetection on output. At the $N_S \ll 1$ limit, the conversion module produces pure coherent states, and then LOCC in the form of Dolinar receiver can be adopted directly without the beamsplitter on all idlers to achieve the optimal. However, this does not contradict Refs. [52, 53] as they do not preclude special cases of optimal mixed-state discrimination to be achievable by LOCC between copies.

Appendix B: Gaussian states

We consider a system of $K$ modes, described by the annihilation operators $\{\hat{a}_\ell, 1 \leq \ell \leq K\}$ satisfying the canonical commutation relation $[\hat{a}_\ell, \hat{a}_\ell^\dagger] = \delta_{\ell k}$. We define the momentum and position quadratures [26] as

$$\hat{q}_\ell = \hat{a}_\ell + \hat{a}_\ell^\dagger , \hspace{0.5cm} \hat{p}_\ell = -i (\hat{a}_\ell - \hat{a}_\ell^\dagger) . \hspace{1cm} (B1)$$

Now we can introduce the vector operator

$$\hat{x} = (\hat{q}_1, \hat{p}_1, \cdots, \hat{q}_K, \hat{p}_K)^T , \hspace{1cm} (B2)$$

which satisfies the commutation relation

$$[\hat{x}_\ell, \hat{x}_k] = 2i\Omega_{\ell k} . \hspace{1cm} (B3)$$

Here $\Omega = \bigoplus_{k=1}^K iY$ with $Y$ being the Pauli-Y matrix. For a quantum state $\hat{\rho}$, one can define the mean and the covariance matrix as

$$\bar{x} \equiv \langle \hat{x} \rangle \hspace{1cm} (B4) \hspace{1cm} \text{V kl} = \frac{1}{2} \langle \{ \hat{x}_k - \langle \hat{x}_k \rangle, \hat{x}_l - \langle \hat{x}_l \rangle \} \rangle , \hspace{1cm} (B5)$$

where $\{\hat{a}, \hat{b}\} \equiv \hat{a}\hat{b} + \hat{b}\hat{a}$ is the anti-commutator and $\langle \hat{A} \rangle = \text{Tr} \left( \hat{A}\hat{\rho} \right)$ for any operator $\hat{A}$.

Gaussian states are entirely characterized by the mean and covariance matrix. In this paper, we consider entanglement from two-mode squeezed vacuum (TMSV), which is a zero-mean two-mode Gaussian state with the covariance matrix

$$\text{V}_{SI} = \left( \begin{array}{cc} (2N_S + 1)\mathbb{I} & 2\sqrt{N_S (N_S + 1)}Z \\ 2\sqrt{N_S (N_S + 1)}Z^\dagger & (2N_S + 1)\mathbb{I} \end{array} \right) , \hspace{1cm} (B6)$$

where $Z$ is the Pauli-Z matrix and $\mathbb{I}$ is $2 \times 2$ identity.

The TMSV state is still Gaussian after the signal mode is transmitted through a bosonic Gaussian channel. For the input-output relation in Eq. (1) of the main text, the covariance matrix of the return and the idler is

$$\text{V}_{RI} = \left( \begin{array}{cc} (2\kappa N_S + 2N_B + 1)\mathbb{I} & 2\sqrt{\kappa N_S (N_S + 1)}Z \mathbb{R} \\ 2\sqrt{\kappa N_S (N_S + 1)}Z^\dagger \mathbb{R}^T & (2N_S + 1)\mathbb{I} \end{array} \right) , \hspace{1cm} (B7)$$

where $\mathbb{R} = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix}$.

Appendix C: General-dyne measurement statistics

Consider a bipartite system, $A$ with $K_A$ modes and $B$ with $K_B$ modes, in a Gaussian state characterized by the mean and covariance matrix

$$\bar{x} \equiv (\bar{x}_A, \bar{x}_B)^T , \hspace{1cm} (C1) \hspace{1cm} \text{V} = \begin{pmatrix} \text{V}_A & \text{V}_{AB} \\ \text{V}_{AB}^T & \text{V}_B \end{pmatrix} . \hspace{1cm} (C2)$$


A general-dyne measurement on subsystem $B$ can be described by a projective measurement with a set of pure Gaussian states \( \{|\psi_{\Pi}\rangle\} \), each with mean \( \hat{\omega}_{\Pi} \) and covariance matrix \( V_{\Pi} \), where \( \Pi \) denotes the corresponding Gaussian measurement. Under a general-dyne measurement, the remaining subsystem results in a Gaussian state conditioned on the measurement outcome following a Gaussian distribution, as shown in Ref. [32]. Below, we provide a brief derivation of the measurement results and output Gaussian states.

Following the Fourier-Weyl relation, the Gaussian state \( \hat{\rho} \) can be written in the basis of displacement operator as

\[
\hat{\rho} = \frac{1}{(2\pi)^{K_A+K_B}} \int_{\mathbb{R}^{2(K_A+K_B)}} dz \chi(z) \hat{D}(-z) \tag{C3}
\]

where the displacement operator \( \hat{D}(\xi) \equiv \exp(i\hat{a}^T \hat{\Omega} \xi) \) and the Wigner characteristic function

\[
\chi(z) = \text{Tr}(\hat{\rho} \hat{D}(z)) = \exp \left[ -\frac{1}{2} z^T (\hat{\Omega} V \hat{\Omega}^T) z - i \langle \hat{\Omega} \hat{\Omega}^T z \rangle \right].
\]

From the state overlap between \( \hat{\rho} \) and \( |\psi_{\Pi}\rangle \), we have

\[
\langle \psi_{\Pi} | \hat{\rho} | \psi_{\Pi} \rangle = \frac{1}{\sqrt{\det(V_{\Pi})}} \int dz d\xi e^{-\frac{1}{2} z^T \hat{\Omega} V \hat{\Omega}^T z} \chi(z) \langle \psi_{\Pi} | \hat{D}(\xi) | \psi_{\Pi} \rangle
\]

where in the fourth line, we apply the formula of integral \( \int dz d\xi e^{z^T A z + \xi^T b} = \pi^{K_A} e^{b^T (A^{-1}) b/4} / \sqrt{\det(A)} \). From the above projection, the mean and covariance matrix of the unmeasured subsystem \( A \), and the measurement outcome distribution are directly found as

\[
V'_{\Pi} = V_{\Pi} - \frac{1}{V_{B} + V_{\Pi}} V_{AB}^T, \tag{C6a}
\]

\[
\hat{\omega}'_{\Pi} = \hat{\omega}_{\Pi} + \frac{1}{V_{B} + V_{\Pi}} (\hat{\omega}_{AB} - \hat{\omega}_{\Pi}), \tag{C6b}
\]

\[
p(\hat{\omega}_{\Pi}) = \frac{e^{-\frac{1}{4} \langle \hat{\omega}_{\Pi} - \hat{\omega}_{\Pi} \rangle^2}}{(2\pi)^{K_B} \sqrt{\det(V_{B} + V_{\Pi})}}. \tag{C6c}
\]

In the main text, we consider a pair of modes (denoted as ‘signal’ and ‘idler’ mode, \( K_A = K_B = 1 \)) in a TMSV state with mean photon number \( N_{\Pi} \), described by the covariance matrix Eq. (B6), resulting channel output with covariance matrix Eq. (B7). By performing the heterodyne measurement on the returned mode, it is mapped to a coherent state with mean \( \hat{\omega}_{\Pi} \equiv \langle \Omega_{\Pi}, p_{\Pi} \rangle \) and identity covariance matrix. From Eqs. (C6), the idler mode’s covariance matrix and mean, and the distribution of measurement outcome are

\[
V'_{\Pi} = \left( \frac{2(1 - \kappa + N_B)N_S}{\kappa N_S + N_B + 1} + 1 \right) \mathbb{I}, \tag{C7a}
\]

\[
\hat{\omega}'_{\Pi} = \left( \sqrt{\kappa N_{\Pi}(N_{\Pi} + 1)} \left( \frac{\cos \theta_{q_{\Pi}} - \sin \theta_{q_{\Pi}}}{} \right)^{-1} \right), \tag{C7b}
\]

\[
p(\hat{\omega}_{\Pi}) = \frac{e^{-\frac{1}{4} \langle \hat{\omega}_{\Pi} - \hat{\omega}_{\Pi} \rangle^2}}{4\pi^N (\kappa N_{\Pi} + N_{\Pi} + 1)}. \tag{C7c}
\]

One can directly realize that the idler mode is in a displaced thermal state with mean \( \hat{\omega}'_{\Pi} \) and thermal photon number \( E \equiv (1 - \kappa + N_B)N_S / (\kappa N_S + N_B + 1) \), as stated in the main text. Formally, a displaced thermal state with complex mean \( \alpha \) and mean thermal photon number \( E \) is defined as

\[
\hat{\rho}_{\alpha,E} \equiv \sum_{n=0} \hat{D}(\alpha) \frac{E^n}{(1 + E)^{n+1}} |n\rangle \langle n| \hat{D}^\dagger(\alpha), \tag{C8}
\]

where \( \hat{D}(\alpha) = \exp(\alpha \hat{a} - \alpha^* \hat{a}) \) is the complex displacement operator acting on a mode \( \hat{a} \) and \( |n\rangle \) is a number state. Note that the complex displacement \( \alpha = \langle \hat{a} \rangle = \langle \hat{q} + i\hat{p} \rangle / 2 \).

Using the distribution of quadratures Eq. (C7c), we can obtain the distribution of the complex heterodyne
readout on the $n$th returned mode $\mathcal{M}_m = (q_{R_m} + ip_{R_m})/2$ as

$$p(\mathcal{M}_m) = \frac{e^{-\frac{|\mathcal{M}_m|^2}{2}}}{\pi(N_S + N_B + 1)}.$$  

(C9)

At the same time, the complex displacement of idler conditioned on the measurement result is

$$d_m = \frac{\sqrt{\kappa N_S(N_S + 1)}}{\kappa N_S + N_B + 1} e^{i\theta_m} \mathcal{M}_m^*$$  

(C10)

where $\mathcal{M}_m^*$ denotes the complex conjugate.

Through the change of variables, one can write the total displacement amplitude square $|d_T|^2 = \sum_{m=1}^M |d_m|^2 = \xi \sum_{i=1}^{2M} z_i^2$, with $\xi = \kappa N_S(N_S + 1)/2(N_B + \kappa N_S + 1)$ and $z_i \sim \mathcal{N}(0,1)$ being a standard normal random variable, and thus we obtain the $\chi^2$ distribution specified in Eq. (2) of the main text.

At the end of the section, we discuss the Gaussian approximation to the distribution Eq. (2) in the main text to enable a more efficient numerical simulation when $M \geq 10^7$. Note that the following approximation is not utilized in any proof in the paper. First, we define $Z = \sum_{i=1}^{2M} z_i^2 \sim \chi^2(2M)$. By central limit theorem, at the limit of $M \gg 1$, $(Z - 2M)/\sqrt{4M} \sim \mathcal{N}(0,1)$ follows standard normal distribution. Therefore, when $M \gg 1$, we can approximate the distribution

$$|d_T|^2 \sim \mathcal{N}(2M\xi, 4M\xi^2)$$  

(C11)

as a Gaussian distribution with mean $2M\xi$ and variance $4M\xi^2$.

Appendix D: Helstrom limit and Quantum Chernoff bound

In general, given $m_s$ quantum states $\{p_j, \hat{\rho}_j\}_{j=1}^{m_s}$ with prior probability $p_j$ for each state, there exists a well-known lower bound of error probability in discrimination—the Helstrom limit [33–35],

$$P_H(\{\hat{\rho}_j\}_{j=1}^{m_s}) = 1 - \max_{\{\hat{\Pi}_j\}} \sum_{j=1}^{m_s} p_j \text{Tr}\left\{\hat{\Pi}_j \hat{\rho}_j\right\}$$  

(D1)

where $\{\hat{\Pi}_j\}$ is a set of POVM operators whose $j$th element corresponds to state $\hat{\rho}_j$ and $\sum_j \hat{\Pi}_j = \hat{1}$. In the case of $m_s = 2$ with equal prior probability $p_1 = p_2 = 1/2$, the Helstrom limit is

$$P_H(\hat{\rho}_1, \hat{\rho}_2) = \frac{1}{2} \left(1 - \frac{1}{2} \text{Tr}\{[\hat{\rho}_1 - \hat{\rho}_2]\}\right)$$  

(D2)

and for pure states $|\psi_1\rangle, |\psi_2\rangle$, it can be further simplified to $P_1 = \left(1 - \sqrt{1 - \frac{4}{\langle \psi_1 | \psi_2 \rangle^2}}\right)/2$.

In general, the Helstrom limit is hard to evaluate. A useful upper bound of the the Helstrom limit is the quantum Chernoff bound (QCB) [20]. For the binary state discrimination between $M$ identical copies of states $\hat{\rho}_0^{\otimes M}$ and $\hat{\rho}_1^{\otimes M}$, we have

$$P_H(\hat{\rho}_0^{\otimes M}, \hat{\rho}_1^{\otimes M}) \leq P_{\text{QCB}} = \frac{1}{2} \left(\inf_{x \in [0,1]} Q_s\right)^M,$$  

(D3)

with $Q_s = \text{Tr}\{\hat{\rho}_1^s \hat{\rho}_2^{1-s}\}$. QCB can be efficiently evaluated for Gaussian states [21]. For two $K$-mode Gaussian states $\{\hat{\rho}_h\}_{h=1}^2$ with mean quadrature $\mathcal{X}_h$ and covariance matrix $\mathcal{V}_h$, one can find the symplectic decomposition of the covariance matrix as $\mathcal{V}_h = S_h \mathcal{V}_h^B S_h^T$ where $\mathcal{V}_h^B = \oplus_{j=1}^K \nu_j^{(h)} S_j^T$ with the symplectic spectrum $\nu_j^{(h)}$. In this case, the QCB can be evaluated via

$$Q_s = \overline{Q}_s \exp\left\{-\frac{1}{2} \text{Tr}\left(\hat{V}_1(s) + \hat{V}_2(1-s)\right)^{-1} d\right\}$$  

(D4)

where $d = \overline{\mathcal{X}} - \overline{\mathcal{X}}$ and $\overline{Q}_s$ is defined as

$$\overline{Q}_s = \frac{2^K \prod_{j=1}^n G_s(\nu_j^{(1)}) \nu_j^{(2)}}{\sqrt{\det\left[\hat{V}_1(s) + \hat{V}_2(1-s)\right]}}$$  

(D5)

with $\hat{V}_h(s) = S_h \left[\oplus_{j=1}^K \Lambda_h(\nu_j^{(h)}) S_j^T \right]$. Here two known functions are introduced as

$$G_p(\nu) = \frac{\nu^p}{(\nu + 1)^{p+1} - (\nu - 1)^{p+1}},$$  

(D6a)

$$\Lambda_p(\nu) = \frac{\nu^p}{(\nu + 1)^{p} - (\nu - 1)^{p}}.$$  

(D6b)

Appendix E: Details on error probability analyses

Lemma 1 In general, we can obtain an upper bound

$$P_{\text{QCB}} \leq \frac{1}{2} \min_{s \in [0,1]} \left(1 + \frac{4\xi}{\Lambda_s(1 + 2N_S) + \Lambda_{1-s}(1 + 2E)}\right)^{-M},$$  

(E1)

where we have defined the function

$$\Lambda_p(\nu) = \frac{(\nu + 1)^{p} + (\nu - 1)^{p}}{(\nu + 1)^{p} - (\nu - 1)^{p}}.$$  

(E2)

By choosing $s = 1/2$, we can also obtain a slightly looser upper bound

$$P_{\text{QCB}} \leq \left(1 + \frac{4\xi}{h(N_S) + h(E)}\right)^{-M},$$  

(E3)

where $h(y) = \Lambda_{1/2}(1 + 2y) = (\sqrt{y + 1} + \sqrt{y})^2$. When $\xi \ll 1$, we can approximate the above upper bounds as exponential functions and obtain lower bounds
on the error exponent. So we obtain a lower bound of the conversion module

\[
 r_{\text{C-D}} \geq \frac{4\xi}{\max_{s \in [0,1]} \Lambda_s(1 + 2N_S) + \Lambda_{1-s}(1 + 2E)} \quad (E4)
\]

\[
 \geq \frac{2}{h(N_S) + h(E) N_B + \kappa N_S + 1} \quad (E5)
\]

\[
 \geq \left(\frac{\sqrt{N_S} + 1 - \sqrt{N_S}}{\sqrt{N_B} - \sqrt{N_B}}\right)^2 \frac{\kappa N_S (N_S + 1)}{N_B + \kappa N_S + 1} \quad (E6)
\]

where in the last step we used the fact that the noise \( E \leq N_S \). A comparison of the above three lower bounds, normalized by the coherent-state Chernoff exponent \( r_{\text{CS}} = \kappa N_S (\sqrt{N_B} + 1 - \sqrt{N_B})^2 \), is shown in Fig. 7. The lower bounds are shown to be always close to each other.

From the above lower bound, we have

\[
 \frac{r_{\text{C-D}}}{r_{\text{CS}}} \geq \frac{\left(\frac{\sqrt{N_S} + 1 - \sqrt{N_S}}{\sqrt{N_B} - \sqrt{N_B}}\right)^2 (N_S + 1)}{\left(\frac{\sqrt{N_B} + 1 - \sqrt{N_B}}{\sqrt{N_B} - \sqrt{N_B}}\right)^2 N_B + \kappa N_S + 1}. \quad (E7)
\]

It is easy to see that when \( \kappa \ll 1 \), the condition for advantage \( (r_{\text{C-D}}/r_{\text{CS}}) > 1 \) is true as long as \( N_S < N_B \).

Now we prove Lemma 1

**Proof.** To begin with, the Helstrom limit is upper bounded by the QCB for any number of copies of states [20, 21], therefore

\[
P_H(\hat{\rho}_{0,N_S}, \hat{\rho}_{\sqrt{\tau},E}) \leq \frac{1}{2} \inf_{s \in [0,1]} Q_s(\hat{\rho}_{0,N_S}, \hat{\rho}_{\sqrt{\tau},E}) \quad (E8)
\]

\[
\leq \frac{1}{2} \inf_{s \in [0,1]} \overline{Q}_s \exp\left\{-\frac{1}{2} d^T \left( \tilde{V}_1(s) + \tilde{V}_2(1-s) \right)^{-1} d \right\} \quad (E9)
\]

\[
= \frac{1}{2} \inf_{s \in [0,1]} \overline{Q}_s \exp\left\{-\frac{2x}{\Lambda_s(1 + 2N_S) + \Lambda_{1-s}(1 + 2E)} \right\} \quad (E10)
\]

where we utilize the definition of QCB in Appendix D, with \( d = (2\sqrt{\tau}, 0) \) and \( \tilde{V}_1(s) = \Lambda_s(1 + 2N_S) \), \( \tilde{V}_2(s) = \Lambda_{1-s}(1 + 2E) \).

Therefore,

\[
P_{\text{C-D}} = \int dx P^{(M)}(x) P_H(\hat{\rho}_{0,N_S}, \hat{\rho}_{\sqrt{\tau},E}) \quad (E11)
\]

\[
\leq \int dx P^{(M)}(x) \frac{1}{2} \inf_{s \in [0,1]} \overline{Q}_s \exp\left\{-\frac{2x}{\Lambda_s(1 + 2N_S) + \Lambda_{1-s}(1 + 2E)} \right\} \quad (E12)
\]

\[
= \frac{1}{2} \inf_{s \in [0,1]} \overline{Q}_s \left(1 + \frac{4\xi}{\Lambda_s(1 + 2N_S) + \Lambda_{1-s}(1 + 2E)}\right)^{-M} \quad (E13)
\]

To complete the error probability analysis of our conversion module in the entire parameter region, in addition to the figures presented in the main text, we study the case where the parameters \( N_S, N_B \) are both large but \( N_S \leq N_B \). We see that quantum advantage still exists in Fig. 8(a)-(b). In the non-asymptotic region, we also see that quantum advantage can only be revealed by our module, not the known QCB. Note when the brightness is large, there is a relatively large gap to the Nair-Gu bound in Eq. (A1) of the main text, which is further confirmed in Fig. 8(c).

In the main text, we show the error probability ratio with fixed \( P_{H,CS} = 0.05 \); here, we extend to \( P_{H,CS} = 0.1, 0.01, 0.001 \) in Fig. 9(a)-(c) to cover both the non-asymptotic and asymptotic regions. With the conversion module, the error probability decreases with a smaller fixed \( P_{H,CS} \), indicating a larger quantum advantage. At the same time, the parameter region where quantum advantage can be predicted by QCB also increases. Note that there exist a region where \( P_{QCB} < P_{H,CS} < P_{\text{C-D}} \).
and quantum advantage can only be revealed by QCB, as shown in Fig. 9(c). The advantage can also be seen from Fig. 7(b),(c), when $N_B \ll N_S$, the error exponent lower bound of conversion module is smaller than the coherent state one, while the QCB error exponent is larger than it.

Appendix F: Quantum Fisher information

In this section, we will frequently utilize the formula of quantum Fisher information (QFI) of Gaussian states proposed in Ref. [54], summarized as the following. To be consistent with Ref. [54], we will adopt a different definition of mean and covariance matrix compared with Appendix B. For an arbitrary $n$-mode Gaussian state, define a vector of annihilation operators $\hat{a} = [\hat{a}_1, \hat{a}_1^\dagger, \ldots, \hat{a}_n, \hat{a}_n^\dagger]$. The state has mean $\vec{d} \equiv \langle \hat{a} \rangle$, and covariance matrix $\Sigma_{\mu\nu} \equiv \frac{1}{2} \langle (\hat{a}_\mu - d_\mu) (\hat{a}_\nu - d_\nu) + (\hat{a}_\nu - d_\nu) (\hat{a}_\mu - d_\mu) \rangle$. The commutation relation is $[\hat{a}_i, \hat{a}_j^\dagger] = \Omega_{ij}$, where $\Omega = \bigoplus_{k=1}^K iY$ with $Y$ being the Pauli-Y matrix. Given the mean $\vec{d}$ and covariance matrix $\Sigma$, Ref. [54] states the formula of the Gaussian-state QFI as

$$F = \frac{1}{2} \mathcal{R}^{-1}_{\alpha\beta\mu\nu} \partial_\theta \Sigma_{\alpha\beta} \partial_\theta \Sigma_{\mu\nu} + \Sigma^{-1}_{\mu\nu} \partial_\theta d_\mu \partial_\theta d_\nu,$$  

where $\Sigma_{\pm} \equiv \Sigma \pm \Omega/2$, $\mathcal{R} \equiv \Sigma \otimes \Sigma + \Omega \otimes \Omega/4$. Here $\theta$ can be an arbitrary parameter, while we focus on the estimation of the signal phase $\theta$ in this paper.

Let us evaluate Eq. (F1) for specific examples. A displaced thermal state $\rho_{\sqrt{\pi}e^{i\theta}x}$ defined in Eq. (C8) has the mean and covariance matrix

$$d = [\sqrt{xe^{i\theta}}, \sqrt{xe^{-i\theta}}]^T,$$

$$\Sigma = \begin{pmatrix} 0 & y + 1/2 \\ y + 1/2 & 0 \end{pmatrix}.$$  

Thus

$$F_{\text{DTS}} = \frac{4x}{1 + 2y}.$$  

Consider $M$ independent and identically distributed (i.i.d.) probes estimating the lossy channel $\Phi^{\kappa, \theta}$ (defined in Eq. (1) of the main text) with thermal noise $N_B$, each with mean photon number $N_S$. For a classical protocol using coherent-state probes $|\sqrt{N_S} \rangle^\otimes M$, observe that the channel output $[\Phi_{\kappa, \theta}(|\sqrt{N_S} \rangle |\sqrt{N_S} \rangle)]^\otimes M$ is a product of displaced thermal states. Then the $M$ channel outputs can be combined into a single mode in a displaced thermal state by a balanced $M$-port beamsplitter. This processing does not change the QFI, because the beamsplitter transform is a unitary and the output is again a product state, where the additional noise modes can be discarded. The output state has $x = M \kappa N_S$, $y = N_B$, thus

$$F_{\text{CS}} = \frac{4M \kappa N_S}{1 + 2N_B}.$$  

Similarly, for an entanglement-assisted protocol using correlation-to-displacement (C-D) module with TMSV probes, the outputs at the idler ports are combined into a displaced thermal state. The random readouts of heterodyne detection at the signal ports determines the squared mean $x$ of the displaced thermal state to be in the $\chi^2$ distribution $F_{\kappa}^{(M)}(x)$ defined as Eq. (2) in the main text. Thus

$$F_{\text{C-D}} \equiv \int dx F_{\kappa}^{(M)}(x) F_{\theta}(\rho_{\pi/2} \theta) = \frac{8M \xi}{1 + 2E}.$$  

where $F_{\theta}(\rho_{\pi/2} \theta)$ is the QFI of the displaced thermal state conditioned on a specific $x$. Plugging the definitions of $\xi$, $E$ in the main text, we obtain

$$F_{\text{C-D}} = \frac{4MN_kN_S(N_S + 1)}{1 + N_B + N_S(2N_B + 2 - \kappa)}.$$  

At the neighborhood of true value, the QFI of a displaced thermal state is achieved by homodyne measurement. This can be seen as follows. For $\hat{\rho}_{\sqrt{\pi}e^{i\theta}x}$, suppose one first apply a phase rotation of angle $\theta_c$, the state becomes $\hat{\rho}_{\sqrt{\pi}e^{i(\theta + \theta_c)}x}$. Then we apply homodyne detection, giving the random readout $Q$ subject to the distribution

$$p_Q(q) = \frac{1}{\sqrt{2\pi \sigma^2}} \exp \left\{ -\frac{(Q - \sqrt{2} \text{Re}(d))^2}{2\sigma^2} \right\}.$$  

where $\sigma^2 = 1/2 + y$, $d = \sqrt{xe^{i\theta}}$. Thus the Fisher information of homodyne measurement, depending on a phase compensation $\theta_c$, can be calculated from the distribution as

$$F_{\text{hom}}(x, E, \theta) = \frac{4x}{1 + 2y} \sin^2(\theta + \theta_c).$$  

Now we see that homodyne measurement achieves the QFI in Eq. (F3) locally, which is true only when the prior knowledge is sufficient such that $\theta + \theta_c$ is close to $\pi/2$, while its performance decays rapidly when $\theta_c$ deviates from the ideal compensation $\pi/2 - \theta$. When prior knowledge is insufficient, an adaptive policy can be designed to approach the ideal compensation, as the number of available probes is sufficiently large.

It is worthwhile to note that the C-D module is optimal for TMSV-based phase estimation: it achieves the QFI of the channel output of TMSV sources [30]

$$F_{\text{TMSV}} = \frac{4MN_kN_S(N_S + 1)}{1 + N_B(1 + 2N_S) + N_S(1 - \kappa)}$$  

at the limit of $N_B \gg 1$. It is verified in the numerical evaluation as shown in Fig. 10.

Appendix G: Classical capacity and Holevo information

First, we begin with a review of communication capacity.
Figure 8. Quantum illumination error probability versus number of identical copies $M$ with (a) $N_S = 1$, $N_B = 10$ and (b) $N_S = 0.3, N_B = 1$. Vertical dashed lines indicate corresponding error probability with $P_{NG} = 10^{-6}$ and $P_{NG} = 0.02$ separately. (c) Error probability ratio $P_{C-D}/P_{NG}$ versus $N_S$ for $P_{NG}$ and $N_B$ chosen from (a) and (b). In all cases $\kappa = 0.01$.

Figure 9. Quantum illumination error probability ratio $P_{C-D}/P_{H,CS}$ (plot in logarithmic scale) versus $N_S$, $N_B$ with $P_{H,CS} = 0.1, 0.01, 0.001$ (from left to right). Red dashed lines indicate the boundary of quantum advantage by the conversion module $N_S \leq N_B$ and orange dashed curves represent the quantum advantage boundary by QCB. The discontinuity of curve in the upper right of (c) is due to the integer requirement of optimal $M$ to hold $P_{H,CS} = 0.001$. In all cases $\kappa = 0.01$.

Figure 10. The ratio in quantum Fisher information of the correlation-to-displacement conversion module over the TMSV limit under various $N_S$, $N_B$. The red dashed diagonal line indicates the boundary of quantum-enhanced region $N_S = N_B/(1 - \kappa)$. $\kappa = 0.01$.

Here, $g(n) = (n + 1) \log_2(n + 1) - n \log_2 n$ is the entropy of a thermal state with mean photon number $n$. When $N_S \rightarrow 0$, we can expand $C$ to the leading order as

$$C = \kappa N_S \log_2(1 + \frac{1}{N_B}) + O(N_S^2). \quad (G2)$$

In the noisy scenario $N_B \gg 1$, the classical capacity $C \approx \kappa N_S/\ln(2) N_B$ is saturated by a heterodyne or a homodyne receiver [30].

Entanglement assistance boosts the communication capacity to [16]

$$C_E = g(N_S) + g(N'_S) - g(A_+) - g(A_-), \quad (G3)$$

where $A_\pm = (D - 1 \pm (N'_S - N_S))/(\sqrt{2})$, $N'_S = \kappa N_S + N_B$ and $D = (N_S + N'_S + 1)^2 - 4\kappa N_S(N_S + 1)$. When $N_S \rightarrow 0$, the leading order can be obtained as

$$C_E = \frac{\kappa N_S}{N_B + 1} \left[ \log_2 \left( \frac{1}{N_B N_S (N_B - \kappa + 1)} \right) + R \right] + O(N_S^2)$$

$$= \frac{\kappa N_S \log_2(1/N_S)}{N_B + 1} + O(N_S), \quad (G4)$$

For the channel $\Phi_{a,\theta}$ described by Eq. (1) of the main text, the classical capacity with energy constraint $\langle \hat{a}^\dagger \hat{a} \rangle \leq N_S$, without entanglement assistance, is known as [55]

$$C = g(\kappa N_S + N_B) - g(N_B). \quad (G1)$$
Figure 11. The ultimate EA capacity over lossy channel under various $N_B, N_S$, normalized by the unassisted capacity $C$. The red dashed diagonal line indicates $N_S = N_B$. Channel transmissivity $\kappa = 0.01$.

where $\mathcal{R} = \left( (N_B + 1) \log_2 (N_B - \kappa + 1) + \kappa + (-N_B + 2 \kappa - 1) \log_2 (N_B + 1) \right) / \kappa$ is independent of $N_S$. We see a diverging advantage $C_E/C \sim \ln \left( 1/N_S \right)$, in the $N_S \ll 1$ limit. Remarkably, such an advantage is not limited to the region $N_B \gg 1, N_S \ll 1$ considered in Ref. [16, 30], as shown in Fig. 11. In the main text such an extension of advantageous region is shown to hold for our conversion module as well. The EA capacity is known to be achieved by the Holevo information of phase encoding on TMSV at this limit [30].

Next, we evaluate the Holevo information [11, 37] of the output ensemble of our C-D module, using phase-encoded TMSV source. The Holevo information is a tight upper bound on the information rate of a channel given a specific encoding ensemble $\{\rho_\theta, \hat{\rho}_\theta\}$, which is achieved by the optimum receiver. The ultimate capacity can be obtained by optimizing the Holevo information over $\{\rho_\theta, \hat{\rho}_\theta\}$. In general $\theta$ can be an arbitrary parameter, while we specify it to be the phase shift in this paper. We consider repetition coding that yields $M$ i.i.d. copies of the output ensemble. Given $\theta$, the output state is $\hat{\rho}_{\sqrt{x}e^{i\theta}, E}$ defined by Eq. (C8), where $X$ is a random readout under $\chi^2$ distribution $P_{\chi^2}(x)$ defined in Eq. (2) in the main text. Let the encoding phase be a random variable $\Theta$ subject to probability distribution $P_\Theta$. Denote the output quantum system as $O$. In the communication protocol, the readouts $X$, the output quantum system $O$ along with the input symbol $\Theta$ is in a classical-quantum state

$$\sigma_{XO\Theta} = \int d\theta P_\Theta(\theta) \int dx P_{\chi^2}(x) |x\rangle \! \langle x|_X \otimes \hat{\rho}_{\sqrt{x}e^{i\theta}, E}_O \otimes |\theta\rangle \! \langle \theta|_\Theta.$$  \hspace{1cm} (G5)

The overall Holevo information about the input symbol is

$$\chi_{C-D} \equiv \frac{1}{M} \left[ S(\chi_{XO}) - S(\chi_{XO|\Theta}) \right]$$

$$= \frac{1}{M} \int d\theta P_\Theta(\theta) \int dx P_{\chi^2}(x) \left[ S(\chi_{XO}) - S(\chi_{XO|\Theta}) \right]$$

$$= \frac{1}{M} \int dx \int d\theta P_{\chi^2}(x) \chi \left( \{ \rho_\theta, \hat{\rho}_{\sqrt{x}e^{i\theta}, E} \} \right).$$  \hspace{1cm} (G6)

The second equality is due to the joint entropy theorem [56] given the orthogonality of $\{ |x \rangle \}$. Here $\chi \left( \{ \rho_\theta, \hat{\rho}_{\sqrt{x}e^{i\theta}, E} \} \right)$ can be efficiently evaluated in the following example.

Consider continuous PSK (CPSK) modulation on TMSV sources with $P_\Theta(\theta) = 1/2\pi, \theta \in [0, 2\pi)$. The output ensemble of the C-D module yields the Holevo information

$$\chi \left( \{ \rho_\theta, \hat{\rho}_{\sqrt{x}e^{i\theta}, E} \} \right)$$

$$= S \left( \int d\theta P_\Theta(\theta) \rho_{\sqrt{x}e^{i\theta}, E} \right) - \int d\theta P_\Theta(\theta) S(\rho_{\sqrt{x}e^{i\theta}, E})$$

$$= H \left( \{ P(n|X = x) \} \right) - g(E).$$  \hspace{1cm} (G7)

The third line follows from the following. The conditional states $\{ \rho_{\sqrt{x}e^{i\theta}, E} \}$ are Gaussian states with identical entropy, $S(\rho_{\sqrt{x}e^{i\theta}, E}) = g(E)$, where $g(n) = (n + 1) \log_2 (n + 1) - n \log_2 n$ is the entropy of a thermal state with mean photon number $n$. Thus $\int d\theta P_\Theta(\theta) S(\rho_{\sqrt{x}e^{i\theta}, E}) = g(E)$. Meanwhile, the unconditional state $\int d\theta P_\Theta(\theta) \rho_{\sqrt{x}e^{i\theta}, E}$ is completely dephased due to $[0, 2\pi)$ uniform phase encoding thus its eigenbasis is the photon number Fock basis. Its distribution on the Fock basis is $P(n|X = x) = E^n(E + 1)^{-n-1} e^{-\frac{x}{E(E + 1)}} L_n \left( \frac{x}{E(E + 1)} \right)$, where $L_n(x)$ is the $n$th Laguerre polynomial. Thus the unconditional entropy reduces to the Shannon entropy of the photon number distribution

$$H \left( \{ P(n|X = x) \} \right) = - \sum_{n=0}^{\infty} P(n|X = x) \log (P(n|X = x)).$$  \hspace{1cm} (G8)

Combining Eqs. (G6) and (G7), we have the Holevo information for CPSK

$$\chi_{CPSK} = \frac{1}{M} \left[ \int dx \int d\theta P_{\chi^2}(x) \chi \left( \{ P(n|X = x) \} \right) - g(E) \right].$$  \hspace{1cm} (G10)

We can adopt Eq. (G10) for efficient numerical evaluation. Below, we further obtain some asymptotic results.

At the limit $M \rightarrow \infty$, $x$ converges to $2M\xi$ with probability by law of large numbers. Then we have a closed-
form formula

$$
\chi_{\text{CP}} = \frac{1}{M} [H[(P(n|X = 2M\xi)] - g(E)]
$$

$$
= \kappa N_S \left[ \ln \left( \frac{1}{n_S} \right) + \mathcal{R}_{\text{C-D}} \right] + O(N_S^2)
$$

$$
= \kappa N_S \ln(1/N_S) + O(N_S),
$$

where \( \mathcal{R}_{\text{C-D}} = \frac{2(\kappa - \kappa M)}{\kappa M} \right) \tan^{-1} \left( \frac{\kappa M (2N - \kappa M + 2)}{2N - \kappa M} \right) + N_B + 1 \right) \ln 2 - \ln (N_B + \kappa (M - 1) + 1) \right) \right) + O(N_S^2)
$$

Note that the above scaling at \( N_S \to 0 \) saturates the EA classical capacity in Eq. (G4), therefore is asymptotically optimal. At the same time, the information per symbol is strictly higher than the case of \( M \gg 1 \), therefore the optimal scaling applies to any finite \( M \).

One may follow a similar route to solve the binary PSK (BPSK) case, where \( P_{\theta}(0) = P_{\theta}(\pi) = 1/2 \). The conditional entropy is the same as that in the CPSK, \( S(\rho_{\sqrt{\tau}e^{i\theta}}) = g(E) \). The evaluation of the unconditional entropy is more challenging; it is now a Von Neumann entropy where eigenvalues of the density operator \( \tilde{\rho} = \int d\theta P_{\theta}(\theta) \rho_{\sqrt{\tau}e^{i\theta}} \) are to be solved. Nevertheless, we find that a closed-form formula is still available at the limit of \( M \gg 1 \) (such that \( x \to 2M\xi \)) and \( N_S \to 0 \). Indeed, the performance of BPSK is almost identical to the CPSK case in the parameter region of Fig. 5 in the main text. Below, we approximate the eigenvalues via matrix perturbation theory. We consider the representation in Fock basis

$$
\rho_{mn} = \langle m| \tilde{\rho} |n \rangle = \sqrt{m!} e^{m/m^2/2} \frac{1}{(E + 1)^{m + 1} \sqrt{n!}} F_1 \left( m + 1; m - n + 1; \frac{x}{E + E} \right) e^{-x + 10(m-n)}
$$

where \( \tilde{F}_1(a; b; z) \) is the regularized confluent hypergeometric function. In the numerical evaluation, we truncate \( \rho \) in finite dimension \( d \times d \).

In the final approximation of eigenvalues, we expect to keep the infinitesimal terms up to \( O(N_S) \). We find that \( d = 3 \) is sufficient, the error analysis is deferred to the end of this section. Now, we apply Taylor expansion to each matrix entry \( \rho_{mn} \) as

$$
\rho_{mn} = \tilde{\rho}_{mn} + \delta \rho_{mn},
$$

where the approximation \( \tilde{\rho}_{mn} \sim O(N_S^2) \) omits higher order term \( \delta \rho_{mn} \sim O(N_S^{d+1}) \). With \( d = 3 \), the eigenvalues of \( \tilde{\rho} \) can be solved analytically, and thus we obtain the Holevo information

$$
\chi_{\text{BPSK}} = \frac{1}{M} [S(\rho) - g(E)] + O(\delta_d)
$$

$$
= \kappa N_S \left[ \ln \left( \frac{1}{n_S} \right) + \mathcal{R}_{\text{C-D}} \right] + O(\delta_{NS}) + O(\delta_d)
$$

$$
= \kappa N_S \ln(1/N_S) + O(N_S) + O(\delta_{NS}) + O(\delta_d),
$$

where \( \delta_d \) is the maximal error in eigenvalues from the matrix truncation and \( \delta_{NS} \) is from the matrix Taylor expansion. The residue \( \mathcal{R}_{\text{C-D}} \) is the same as that defined below the CPSK case Eq. (G11). The leading term of the matrix Taylor expansion at the second equality coincides with Eq. (G11).

At last, we analyze the errors \( \delta_d, \delta_{NS} \) in Eq. (G14).

First, let us consider \( \delta_d \). Define the true eigenvalues of the operator \( \tilde{\rho} \) as \( \mu_1 \geq \mu_2 \geq \ldots \) and the eigenvalues of the truncated representation \( \rho \) as \( \lambda_1 \geq \lambda_2 \geq \ldots \). Then \( \delta_d \equiv \max_{i=1}^{d} |\mu_i - \lambda_i| \). Using Theorem 4.14 in Ref. [57], we have

$$
\|\delta_d\| = \|\text{diag}(\mu_i - \lambda_i)\|_2 \leq \|X\|_2,
$$

where \( X = \rho^{(\infty)}I_d - I_d\rho \). \( I_d \) is a \( d \times d \) matrix representation of projector that implements the cutoff, \( \rho^{(\infty)} \) is the exact infinite-dimensional matrix representation of the operator \( \tilde{\rho} \). Here the matrix 2-norm is defined using vector 2-norm: for \( d \times d \) matrix \( A \in \mathbb{R}^{d \times d} \), \( \|A\|_2 \equiv \sup_{x \neq 0} ||Ax||_2/||x||_2, \forall x \in \mathbb{R}^d \). Observe that the 2-norm \( \|X\|_2 = O(N_S^{d/2}) \). Thus \( d = 3 \) is sufficient to suppress the error to \( O(N_S^{d/2}) \). Next, we consider \( \delta_{NS} \). Define the eigenvalues of \( \rho \) as \( \{\lambda_i\}_{i=1}^{d} \), and the eigenvalues of \( \tilde{\rho} \) as \( \{\tilde{\lambda}_i\}_{i=1}^{d} \). The error in eigenvalues is equal to the Hausdorff distance \( \|\delta_{NS}\|_2 \equiv \max_i |\lambda_i - \tilde{\lambda}_i| = \|h(\rho, \tilde{\rho})\|_2 \). When the perturbation is small such that the eigenvalues are still pairwise matched: \( j = \arg\min_j |\lambda_j - \tilde{\lambda}_j| \). Note that \( \|h(\rho, \tilde{\rho})\|_2 = O(N_S^{d+1/d}) \). According to Elsner’s theorem[57], the error is upper bounded by

$$
|\delta_{NS}| \leq (\|\rho\|_2 + ||\tilde{\rho}||_2)^{1-1/d} ||\delta\rho||_2^{1/d} = O(N_S^{1+1/d}),
$$

which is much smaller than \( O(N_S) \). Finally, we see that the overall error \( |\delta_d + \delta_{NS}| \ll O(N_S) \) when \( d = 3, N_S \to 0 \).
Appendix II: Summary of coherent state discrimination

In this section, we summarize coherent-state discrimination, including the Helstrom limit [33–35], homodyne detection, heterodyne detection, Kennedy receiver [58] and Dolinar receiver.

In the following discussion, we consider the case of discrimination between a vacuum state $|0\rangle$ and a coherent state $|\alpha\rangle$, where in general $\alpha = \alpha_R + i\alpha_I$. The noisy version of this discrimination problem is exactly the sub-task necessary to complete the receiver design for the entanglement-assisted applications after the conversion module.

We begin with the noiseless version, in which case the Helstrom limit of error probability has a closed-form solution

$$P_H(|0\rangle, |\alpha\rangle) = \frac{1}{2} \left( 1 - \sqrt{1 - e^{-|\alpha|^2}} \right). \quad \text{(H1)}$$

Now we discuss the performance of homodyne detection. Homodyne detection consists of measuring a single quadrature of the mode, for example the position quadrature $q$. For binary discrimination, we can write out the POVM element as

$$\hat{\Pi}_{0,\text{homo}} = \int_{-\infty}^{B} dq \langle q | \langle q |$$

$$\hat{\Pi}_{1,\text{homo}} = \hat{1} - \hat{\Pi}_{0,\text{homo}}, \quad \text{(H2)}$$

where $B$ determines the decision threshold. With the POVM elements, the error probability using homodyne detection is

$$P_{E,\text{homo}} = \min \frac{1}{2} \left( \langle \alpha | \Pi_{0,\text{homo}} | \alpha \rangle + \langle 0 | \Pi_{1,\text{homo}} | 0 \rangle \right)$$

$$= \min \frac{1}{B} \left( \int_{-\infty}^{B} dq \langle q | \langle q | + 1 - \int_{-\infty}^{B} dq \langle 0 | \langle 0 | \right)$$

$$= \frac{1}{2} \left( \int_{-\infty}^{B} dq e^{-\alpha_R q^2} + \int_{-\infty}^{B} dq e^{-q^2} \right)$$

$$= \frac{1}{4} \left( \text{Erfc} \left( \sqrt{2} \alpha_R \right) + \text{Erfc} (B) \right)$$

$$= \frac{1}{2} \text{Erfc} \left( \alpha_R \right), \quad \text{(H3)}$$

where in the last line the error probability is minimized at the threshold $B = \alpha_R/\sqrt{2}$. When the amplitude of coherent state $\alpha_R \gg 1$, as $\text{Erfc}(x) \sim e^{-x^2/\sqrt{\pi}x}$, we have $P_{E,\text{homo}} \sim e^{-|\alpha_R|^2/2}$.

Similarly, heterodyne detection projects the modes to coherent states with the POVM for a binary discrimination task

$$\hat{\Pi}_{0,\text{het}} = \frac{1}{\pi} \int_{A} d\beta \langle \beta |$$

$$\hat{\Pi}_{1,\text{het}} = \hat{1} - \hat{\Pi}_{0,\text{het}}, \quad \text{(H4)}$$

where $A$ denotes a decision region in the complex plane (denoted as $x$ and $y$ axes in the following discussion). The error probability applying heterodyne detection is thus

$$P_{E,\text{het}} = \frac{1}{2} \left( \langle \alpha | \Pi_{0,\text{het}} | \alpha \rangle + \langle 0 | \Pi_{1,\text{het}} | 0 \rangle \right)$$

$$= \frac{1}{2} \left( \int_{-\infty}^{\infty} d\beta \langle \beta | \langle \beta | + 1 - \int_{-\infty}^{\infty} d\beta \langle \beta | \langle \beta | \right)$$

$$= \frac{1}{2} + \frac{1}{2\pi} \left( \int_{-\infty}^{\infty} d\beta e^{-\alpha_R \beta^2} - \int_{-\infty}^{\infty} d\beta e^{-|\beta|^2} \right). \quad \text{(H5)}$$

Through simple geometry analysis, we can find the region $A$ to achieve the optimal error probability as $A \equiv \{(x, y)|y \leq -2\alpha_R x + |\alpha|^2\}$, with boundary denoted as $l_A$ for simplification. Therefore, we have the optimal error probability with heterodyne as

$$P_{E,\text{het}} = \frac{1}{2} + \frac{1}{2\pi} \int_{-\infty}^{\infty} dx \int_{-\infty}^{l_A} dy e^{-\alpha_R x^2 - y^2}$$

$$= \frac{1}{2} + \frac{1}{4} \left( 1 - \text{Erf} \left( \frac{|\alpha|}{2} \right) \right) - \frac{1}{4} \left( 1 + \text{Erf} \left( \frac{|\alpha|}{2} \right) \right)$$

$$= \text{Erfc} \left( \frac{|\alpha|}{2} \right). \quad \text{(H6)}$$

When $|\alpha| \gg 1$, we have $P_{E,\text{het}} \sim e^{-|\alpha|^2/4}$.

For coherent state discrimination, there exists a well-known nulling receiver called Kennedy receiver [58]. For two arbitrary coherent states $|\alpha_1\rangle$, $|\alpha_2\rangle$, the Kennedy receiver performs a displacement $-\alpha_1$, such that one of the state results in vacuum state $|\alpha_1\rangle \rightarrow |0\rangle$ while the other is $|\alpha_2\rangle \rightarrow |\alpha_2 - \alpha_1\rangle$. The unknown state is considered to be $|\alpha_1\rangle$ when there is no photon detected and $|\alpha_2\rangle$ otherwise. Therefore the error probability with Kennedy receiver is

$$P_{E,\text{Kennedy}} = \frac{1}{2} \left( |\langle 0 |\alpha_2 - \alpha_1 \rangle|^2 \right) = \frac{1}{2} e^{-|\alpha_2 - \alpha_1|^2}, \quad \text{(H7)}$$

since there is no error in predicting the state $|\alpha_1\rangle$. For the case under consideration, $|0\rangle$ versus $|\alpha\rangle$, as one of the state is always in vacuum state, the Kennedy receiver is equivalent to the a direct photon counting with error probability

$$P_{E,\text{Kennedy}} = \frac{1}{2} e^{-|\alpha|^2} \quad \text{(H8)}$$

which is approximately $P_{E,\text{Kennedy}} \approx 2P_H$ when the mean photon number $|\alpha|^2 \gg 1$.

Note that Kennedy receiver is only sub-optimal in the coherent states discrimination, an adaptive receiver, Dolinar receiver [24], has been proposed to approach the Helstrom limit in the noiseless case. The Dolinar receiver splits the input coherent state into $S$ slices and
makes a decision in terms of the prior probability of the original candidate states becoming displaced thermal state, and then updating the decision. We introduce $h$ to denote the true state, $g$ as the current decision $(h, g \in \{0, 1\})$ and $p_h^{(k)}$ as the prior probability for $k$th slice to be state $\hat{\rho}_h$. The number of photons measured from $k$th slice is denoted as $N(k)$ following a distribution $p_h^{(k)}(N(k), g|h)$. We use $p(N(k), g|h)$ to represent the Bayesian conditional probability for obtaining $N(k)$ photons when the $k$th slice is determined to be $\hat{\rho}_g$ while it is actually $\hat{\rho}_h$. For $0$ and $|\alpha\rangle$ with equal prior probability $p_0^{(0)} = p_1^{(1)}$, the Dolinar receiver works as the following.

**Algorithm 1** Dolinar receiver

$$S, h, \gamma = \sqrt{|\alpha|^2/2S}$$

$$k \leftarrow 1, g \leftarrow None$$

while $k \leq S$ do

$$u(k) = \gamma/\sqrt{1-\exp(-|\alpha|^2(k-1)/S)}$$

if $p_h^{(k)} > p_1^{(k)}$ then

$$g \leftarrow 0$$

else if $p_h^{(k)} < p_1^{(k)}$ then

$$g \leftarrow 1$$

else

$g \leftarrow \{0, 1\}$ with equal probability

end if

if $g = 0$ then

Perform displacement $-\gamma + u(k)$

else

Perform displacement $-\gamma - u(k)$

end if

Measure the photon number $N(k)$ with probability $p_N(N(k), g|h)$

Update prior probability $p_h^{(k+1)} \leftarrow p_h^{(k)} p(N(k), g|0)/\sum_{h'=0}^{1} p_h^{(k)} p(N(k), g|h')$

$p_1^{(k+1)} \leftarrow p_1^{(k)} p(N(k), g|1)/\sum_{h'=0}^{1} p_h^{(k)} p(N(k), g|h')$

end while

if $p_0^{S+1} > p_1^{S+1}$ then

$g \leftarrow 0$

else if $p_0^{S+1} < p_1^{S+1}$ then

$g \leftarrow 1$

else

$g \leftarrow \{0, 1\}$ with equal probability

end if

return $g$

The Dolinar receiver gives a prediction on the unknown state and we perform Monte-Carlo simulation to evaluate the error probability.

For noiseless case $|0\rangle$ versus $|\alpha\rangle$, the measured photon distribution on $k$th slice follows Poisson distribution $N(k) \sim p_N(n, g|h) = \begin{cases} \text{Pois}(n; (\gamma - u(k))^2), & \text{if } g = 0 \\ \text{Pois}(n; (\gamma + u(k))^2), & \text{otherwise} \end{cases}$ (H9)

where Pois$(\mu; \lambda)$ is the Poisson probability mass function.

The conditional Bayesian probability of getting $N(k)$ photons is $p(N(k), g|h) = p_N(N(k), g|h)$.

If there is noise $N_{B,h}$ for both states, original candidate states become displaced thermal state, $\hat{\rho}_0 = \hat{\rho}_{0,N_{B,0}}$ and $\hat{\rho}_1 = \hat{\rho}_{\alpha,N_{B,1}}$. For simplicity, we only consider the case with equal thermal noise $N_{B,0} \simeq N_{B,1} = N_B$. Note the thermal noise part is irrelevant to each other in the slicing process. Alternatively, we make use of the fact that $\hat{\rho}_h$ has positive P-function, and therefore can be realized by generating random coherent state $|\alpha_h\rangle$ where $\alpha_0 = r_0$ and $\alpha_1 = \alpha + r_1$ with $[r_h]_{h=0}^1$ are real random number. The module $|r_h|$ follows exponential distribution $|r_h|^2 \sim \text{Exp}(1/N_B)$ and the argument is uniformly random as $\text{arg}(r_h) \sim U(0,2\pi)$ [59]. As the measured states are still coherent states, the probability distribution of photons also follows Poisson distribution $N(k) \sim p_N(n, g|h)$,

$$p_N(n, g|h) = \begin{cases} \text{Pois}(n; \frac{\alpha_0^2}{\sqrt{S}} - \gamma + u(k)^2), & \text{if } g = 0 \\ \text{Pois}(n; \frac{\alpha_1^2}{\sqrt{S}} - \gamma - u(k)^2), & \text{otherwise} \end{cases}$$

(H10)

Recall that photon number probability distribution for a displaced thermal state $\hat{\rho}_{\alpha,N_B}$ is [59]

$$P_{\alpha,N_B}(n) = e^{-\frac{\alpha^2}{N_B}} \frac{N_B^n}{(1+N_B)^{n+1}} \times 1_{F1}(n, 1, -\frac{|\alpha|^2}{N_B(N_B+1)}),$$

(H11)

where $1_{F1}(a, b; z)$ is the confluent hypergeometric function of the first kind. The conditional Bayesian probability of getting $N(k)$ photon is thus

$$p(N(k), g|h) = \begin{cases} P_{\gamma-u(k),N_B}/S(N(k)), & \text{if } g = h \\ P_{\gamma+u(k),N_B}/S(N(k)), & \text{otherwise} \end{cases}$$

(H12)

**Appendix I: Proof of error exponent advantage in general composite channel discrimination**

As the error exponent of multiple-hypothesis testing is given by the worst case of binary hypothesis testing between any of the two hypotheses involved [20, 22, 23], we can focus on the quantum channel discrimination (QCD) between just two channels $\Phi_{m,h}(g,h)$, with $h = 1, 2$. Due to the convexity of the Helstrom limit and the quantum Chernoff bound (see ref [14] supplemental materials), the optimal classical strategy is to utilize a product of coherent state $\hat{\rho}_h = \sum_{m=1}^{M} |\alpha_m\rangle \langle \alpha_m|$, as the probe input, which leads to the output of a product of displaced thermal states $\sum_{m=1}^{M} \hat{\rho} \exp(i\delta_m^{(h)}) \sqrt{\kappa_m^{(h)}} |\alpha_m\rangle_{N_B}$ for the two channels $h = 1, 2$. For the two displaced thermal state, we have the mean

$$\bar{X}_h = \begin{bmatrix} q_1^{(h)} & p_1^{(h)} & \cdots & p_M^{(h)} \end{bmatrix}^T$$

(ID)

where $q_m^{(h)} = 2\text{Re}\left\{e^{i\delta_m^{(h)}} \sqrt{\kappa_m^{(h)}} |\alpha_m\rangle \right\}$ and $p_m^{(h)} = 2\text{Im}\left\{e^{i\delta_m^{(h)}} \sqrt{\kappa_m^{(h)}} |\alpha_m\rangle \right\}$.

The covariance matrix is diag-
onal $B\ln B$ where $B = 2N_B + 1$. Now we evaluate the quantum Chernoff bound according to Appendix D. First, the quantity

$$Q_s = Q_s e^{-\frac{1}{2} \int d^2 \left[ \tilde{V}_1(s) + \tilde{V}_2(1-s) \right]^{-1} d}$$

where we define $\delta_m = \left| e^{i\theta_m^{(1)}} \sqrt{\kappa_m^{(1)}} - e^{i\theta_m^{(2)}} \sqrt{\kappa_m^{(2)}} \right|^2$ and

$$Q_s = \left( \frac{G_s(B)G_{1-s}(B)}{\Lambda_s(B) + \Lambda_{1-s}(B)} \right)^M = \frac{1}{2^M}.$$  

Therefore, the quantum Chernoff bound is

$$P^{QC}_{QCB} = \frac{1}{2} \inf_{s \in [0,1]} Q_s$$

$$\approx \frac{1}{2} Q_{1/2}$$

$$\approx \frac{1}{2} \exp \left[ -\frac{1}{\Lambda_{1/2}(B)} \sum_{m=1}^M \delta_m |\alpha_m|^2 \right]$$

$$= \frac{G_{1/2}(B)2^{2M}}{2\Lambda_{1/2}(B)^{2M}} \exp \left[ -\frac{1}{\Lambda_{1/2}(B)} \sum_{m=1}^M \delta_m |\alpha_m|^2 \right]$$

$$= \frac{1}{2} \exp \left[ -\sum_{m=1}^M \delta_m |\alpha_m|^2 (\sqrt{N_B} + 1) - \sqrt{N_B} \right],$$

where in the second line we utilize the fact that the minimum of $Q_s$ takes place at $s = 1/2$. This is because $Q_s$ in Eq. (13) is independent on $s$ and the exponent $1/(\Lambda_s(B) + \Lambda_{1-s}(B))$ in Eq. (12) is symmetric on $s$ and strictly concave as its second order derivative is negative

$$\delta^2_s \left( \frac{1}{\Lambda_s(B) + \Lambda_{1-s}(B)} \right)$$

$$= -\frac{1}{4} \frac{(B + 1)^{2s-1} + (B - 1)^{2s-1}}{(B^2 - 1)^{s-1}} \log^2 \left( B + 1 \right) < 0$$

due to $B > 1$. Therefore max$_{s \in [0,1]} \left\{ 1/(\Lambda_s(B) + \Lambda_{1-s}(B)) \right\} = 1/2\Lambda_{1/2}(B)$ where inf$_{s \in [0,1]} Q_s = Q_{1/2}$.

To conclude, we solve the error exponent between the discrimination of any two channels via the coherent state input as

$$P^{QC}_{CS} \sim \exp \left[ -\sum_{m} \delta_m |\alpha_m|^2 \left( \sqrt{N_B} + 1 - \sqrt{N_B} \right) \right]$$

$$\approx \exp \left[ -\sum_{m} \delta_m |\alpha_m|^2 /4N_B \right]$$

where we approximate $\left( \sqrt{N_B + 1} - \sqrt{N_B} \right)^2 = N_B (\sqrt{1 + N_B} - 1) = N_B (1/2N_B + O(N_B^3/2)) \sim 1/4N_B$ at the $N_B \gg 1$ limit.

For the entangled strategy, one inputs a product of TMSV, each mode pair with mean photon number $N_{S,m} = |\alpha_m|^2$ matching that of the classical input, via the conversion module, when the measurement result is $M_m$ we also arrive at a product of displaced thermal state displaced thermal state with mean

$$d_m = \zeta_m e^{i\vartheta_m^{(h)} M_m},$$

and mean thermal photon number

$$E_m = N_{S,m} \left( N_B - \kappa_m^{(h)} + 1 \right).$$

The measurement result $M_m = q M + i p M_m$, with each quadrature output obeying a zero-mean Gaussian distribution with variance $(N_B + \kappa_m^{(h)} N_S + 1)/2$. We consider the $N_B \gg 1$ limit, then $E_m \sim N_{S,m} \approx N_S$ is a constant noise background, therefore conditioned on the measurement result, the error probability follows Eq. (14)

$$P^{QC}_{CD} (\{M_m\})$$

$$\approx \frac{1}{2} \exp \left[ -\sum_{m} \left| d_m^{(1)} - d_m^{(2)} \right|^2 \left( \sqrt{N_S} + 1 - \sqrt{N_S} \right)^2 \right]$$

$$\approx \frac{1}{2} e^{-\sum_{m} \delta_m N_S (N_S + 1) M_m^*} \left( 2 \sqrt{\kappa_m^{(h)} N_S + 1} - \sqrt{\kappa_m^{(h)} N_S + 1} \right)^2 |\alpha_m|^2$$

We can solve the average error probability from Eq. (3) in the main text as

$$P^{QC}_{CD} = \frac{1}{2} \prod_{m} d M_m p(M_m) P_{CD} (\{M_m\})$$

$$= \frac{1}{2} \prod_{m} d M_m \frac{e^{-\frac{1}{2} \frac{M_m^2}{N_S + N_B + 1}}}{\pi (\kappa_m^{(h)} N_S + N_B + 1)}$$

$$\times e^{-\frac{1}{2} \delta_m N_S (N_S + 1) \left( \sqrt{\kappa_m^{(h)} N_S + 1} - \sqrt{\kappa_m^{(h)} N_S + 1} \right)^2 |\alpha_m|^2}$$

$$\approx \frac{1}{2} \prod_{m} \left( 1 + \kappa_m^{(h)} N_S + N_B + 1 \right) \frac{1}{\left( N_B + 1 \right)^2} \frac{N_S (N_S + 1) \left( \sqrt{N_S + 1} - \sqrt{N_S} \right)^2 \delta_m}{\left( N_B + 1 \right)^2 \delta_m}$$

where in the last line we approximate by $(\kappa_m^{(h)} N_S + N_B + 1) N_S (N_S + 1) \left( \sqrt{N_S + 1} - \sqrt{N_S} \right)^2 \approx N_S / N_B + O(N_S^3/2)$. The er-
ror exponent is thus

$$P_{\text{QCD}} = \exp \left[ -\log 2 + \log \left( \prod_{m} \frac{1}{1 + \frac{N_{E,m}}{N_{B}} \delta_{m}} \right) \right]$$

$$= \exp \left[ -\log 2 - \sum_{m} \log \left( 1 + \frac{N_{E,m} \delta_{m}}{N_{B}} \right) \right]$$

$$\simeq \exp \left[ -\log 2 - \sum_{m} N_{E,m} \delta_{m} \right] \quad (I11)$$

$$= \frac{1}{2} \exp \left[ -\sum_{m} \delta_{m} |\alpha_{m}|^2 / N_{B} \right].$$

Therefore, in terms of general pattern classification, entanglement combined with our conversion module enables a 6dB advantage in the error exponent.

**Appendix J: Achieving the optimum scaling of communication capacity**

The ‘Green machine’ is a receiver that attains a rate very close to the classical communication capacity with phase modulation of coherent states [25]. Here we show that a design concatenating it with our C-D module (see Fig. 13 for a schematic plot) achieves the same scaling of the ultimate EA capacity. The sender jointly encodes a block of $n$ signal modes by BPSK modulation according to an $n$-codeword Hadamard code, where $n$ is a power of 2 and each codeword contains $n$ symbols $\{\theta_{k}\}_{k=1}^{n}$. Fig. 12 shows an example with $n = 8$. The information rate can be further improved by repetitive encoding over $M$ i.i.d. copies of signal modes with identical symbols, where $M$ is to be optimized. After an $n$-block of signals goes through the channel $\Phi_{\kappa,0}$, the receiver obtains an $n$-block of returned signals. Then the receiver applies the C-D conversion module to the $n$-block of return-idler pairs, which yields $n$ displaced thermal states as defined in Eq. (C8), with quadrature phase subject to the $n$-codeword Hadamard code. We note that, the C-D conversion module combines the $M$ copies together, thereby the brightness of the $n$ displaced thermal states is increased such that the thermal background $E$ is negligible and the states resemble coherent states. The $n$ quasi-coherent-state outputs are input into the Green machine. The Green machine consists of a beamsplitter array. Denote the input modes as a vector $\hat{\alpha} = [\hat{a}_{1}, \ldots, \hat{a}_{n}]^{T}$, with mean $\langle \hat{\alpha} \rangle \propto [e^{i\theta_{1}}, \ldots, e^{i\theta_{n}}]^{T}$, where we have left out the amplitude to focus on the phase. The beamsplitter array fulfills a Bogoliubov transform $\hat{\alpha} \rightarrow S \hat{\alpha}$ with the unitary matrix

$$S = \frac{1}{\sqrt{n}} \left( \begin{array}{cc} 1 & 1 \\ -1 & 1 \end{array} \right)^{\otimes \log_{2} n}. \quad (J1)$$

Finally the receiver makes zero-or-not photon counting on the $n$ output modes of the Green machine individually. At the limit of weak thermal background $E \rightarrow 0$, each output mode yields non-zero photon count iff the input modes are constructively interfered by the beamsplitter array. Concretely, the Green machine converts one of the $n$ codewords of BPSK Hadamard code (in the quadrature phase) into one of the $n$ codewords of coherent-state pulse position modulation (PPM) (in the photon count), i.e. the photons of $n$ input modes are merged into one output mode.

Now we evaluate the performance of the above protocol. The magnitudes of the means at the output modes of the C-D module depend on the squared mean $x$ of the random heterodyne readout, which is subject to the $\chi^2$ distribution $P_{x}(x)$ as defined in Eq. (2) in the main text. For an $M$-copy, $n$-codeword Green machine, the per-symbol rate given squared mean $x$ is [51]

$$R_{\text{GM}}(x) = \frac{1}{Mn \ln 2} \left[ (n-1) P_{d}(x) \ln \left( \frac{n P_{d}(x)}{P_{E}(x)} \right) - (n-1) P_{d}(x) + P_{E}(x) \right] \ln \left( \frac{(n-1) P_{d}(x)}{P_{E}(x)} + 1 \right) + P_{E}(x) \ln (n) \quad (J2)$$

where $P_{d}(x) = [1 - P_{c}(x)] P_{b}(1 - P_{b})^{n-2}$, $P_{c}(x) = P_{c}(1 - P_{b})^{n-1}$ and

$$P_{c}(x) = 1 - \frac{e^{-x}}{1 + E} \quad , \quad P_{b} = 1 - \frac{1}{1 + E}. \quad (J3)$$

Here the thermal background $E$ is defined above Eq. (2) in the main text. Thus, the overall rate is

$$R_{\text{GM}} = \int_{0}^{\infty} dx P_{x}(x) R_{\text{GM}}(x). \quad (J4)$$

Observe that $R_{\text{GM}}$ depends on $M, n$. We numerically optimize $R_{\text{GM}}$ over integer $M$ for each value of $N_{S}$ in $n$-block of returned signals. Then the receiver applies the C-D conversion module to the $n$-block of return-idler pairs, which yields $n$ displaced thermal states as defined in Eq. (C8), with quadrature phase subject to the $n$-codeword Hadamard code. We note that, the C-D conversion module combines the $M$ copies together, thereby the brightness of the $n$ displaced thermal states is increased such that the thermal background $E$ is negligible and the states resemble coherent states. The $n$ quasi-coherent-state outputs are input into the Green machine. The Green machine consists of a beamsplitter array. Denote the input modes as a vector $\hat{\alpha} = [\hat{a}_{1}, \ldots, \hat{a}_{n}]^{T}$, with mean $\langle \hat{\alpha} \rangle \propto [e^{i\theta_{1}}, \ldots, e^{i\theta_{n}}]^{T}$, where we have left out the amplitude to focus on the phase. The beamsplitter array fulfills a Bogoliubov transform $\hat{\alpha} \rightarrow S \hat{\alpha}$ with the unitary matrix

$$S = \frac{1}{\sqrt{n}} \left( \begin{array}{cc} 1 & 1 \\ -1 & 1 \end{array} \right)^{\otimes \log_{2} n}. \quad (J1)$$

Finally the receiver makes zero-or-not photon counting on the $n$ output modes of the Green machine individually. At the limit of weak thermal background $E \rightarrow 0$, each output mode yields non-zero photon count iff the input modes are constructively interfered by the beamsplitter array. Concretely, the Green machine converts one of the $n$ codewords of BPSK Hadamard code (in the quadrature phase) into one of the $n$ codewords of coherent-state pulse position modulation (PPM) (in the photon count), i.e. the photons of $n$ input modes are merged into one output mode.

Now we evaluate the performance of the above protocol. The magnitudes of the means at the output modes of the C-D module depend on the squared mean $x$ of the random heterodyne readout, which is subject to the $\chi^2$ distribution $P_{x}(x)$ as defined in Eq. (2) in the main text. For an $M$-copy, $n$-codeword Green machine, the per-symbol rate given squared mean $x$ is [51]

$$R_{\text{GM}}(x) = \frac{1}{Mn \ln 2} \left[ (n-1) P_{d}(x) \ln \left( \frac{n P_{d}(x)}{P_{E}(x)} \right) - (n-1) P_{d}(x) + P_{E}(x) \right] \ln \left( \frac{(n-1) P_{d}(x)}{P_{E}(x)} + 1 \right) + P_{E}(x) \ln (n) \quad (J2)$$

Fig. 13, while choosing the block size $n^*$ to be the asymptotic optimum Eq. (J9). The results in the main text is obtained similarly.

Below, we provide asymptotic analyses to obtain more insights. Note that our numerical results above are evaluated via the exact formula Eq. (J4) without asymptotic approximations, using only the value of $n^*$ derived below, and therefore represents an exact achievable rate. The optimized $M^* \equiv \arg \max_{M} R_{\text{GM}}$ is numerically found $\sim 10^4$, sufficiently large for all parameters being considered. Thus it is fair to invoke the law of large numbers, that the squared mean $x$ converges in probability to the
Figure 12. Entanglement-assisted communication protocol using BPSK Hadamard code and Green machine. The Green machine, a beamsplitter array defined in Eq. (J1), converts the BPSK Hadamard code in phase to pulse position modulation in photon count.

Figure 13. The information rate per symbol of the Green machine using BPSK-encoded TMSV state under various $N_B, N_S$, normalized by the unassisted capacity $C$. Plotted with numerical optimized repetition $M$ and asymptotically optimal codeword length $n$ Eq. (J9). The black diagonal line indicates $N_S = N_B$. $\kappa = 0.01$.

Below we optimize $R_{GM}$ with respect to $n$, in the asymptotic regime $N_S \to 0$. Let the optimal $n$ be

$$n^* \equiv \arg\max_{n \in \mathbb{N}} R_{GM},$$

where $\mathbb{N} = \{2^k | k \in \{1, 2, \ldots\}\}$ is the set of positive powers of 2. In this case $P_E \gg P_d$, thus the rate in Eq. (J2) is dominated by $P_E \ln(n)/Mn$. Then one can obtain the second-order expansion of the rate

$$MR_{GM} \simeq \frac{P_E \ln(n)}{n \ln 2} = (u + vn) \log_2(n) + O(N_S^3) \quad (J7)$$

where

$$u = -N_S \left[ -\kappa (N_B+1) (M+2N_S) + (N_B+1)^2 N_S + \kappa^2 N_S^2 \right] / (N_B+1)^2,$$

$$v = -\kappa MN_S^2 \left[ 2(N_B+1) + \kappa (M-2) \right] / [2(N_B+1)^2].$$

By solving $dR_{GM}/dn = 0$, we obtain

$$n^* \simeq -\frac{u}{v W(-ue/v)}, \quad (J9)$$

where $W$ is the principal branch of the Lambert $W$ function which satisfies $W(xe^x) = x$ for $x \geq -1$. Using the relation $\ln W(x) = \ln(x) - W(x)$ and the asymptotic expansion $W(x) = \ln(x) - \ln \ln(x) + O(1)$ as $\ln(x) \to \infty$, the rate $R_G$ converges to $C$.
\[
\ln n^* \simeq -1 + W\left(\frac{-ue}{v}\right)
\]
\[
= - \ln \left[ \ln \left( \frac{-2e (-\kappa (N_B + 1) (M + 2N_S) + (N_B + 1)^2 N_S + \kappa^2 N_S)}{\kappa M N_S (2N_B - 2\kappa + \kappa M + 2)} \right) \right]
+ \ln \left( \frac{-2 (-\kappa (N_B + 1) (M + 2N_S) + (N_B + 1)^2 N_S + \kappa^2 N_S)}{\kappa M N_S (2N_B + 1 + \kappa (M - 2))} \right). \tag{J10}
\]

Plugging Eqs. (J9) and (J10) in Eq. (J7), we have the asymptotic rate
\[
R_{GM} = \frac{\kappa N_S}{(N_B + 1) \ln 2} \ln \left( \frac{2(N_B + 1)}{N_B(2(N_B + 1) + \kappa (M - 2))} \right) - 1 + O\left(\frac{N_S^2}{(N_B + 1) \ln 2}\right). \tag{J11}
\]
At the limit of \(N_S \to 0\), it achieves the optimal scaling of the ultimate EA capacity \(C_E\) [16]
\[
R_{GM} = \frac{\kappa N_S}{(N_B + 1) \ln 2} (\ln(1/N_S) - O(1)) \propto N_S \ln(1/N_S). \tag{J12}
\]
Remarkably,
\[
\frac{R_{GM}}{C_E} = 1 - O\left(\frac{1}{\ln(1/N_S)}\right) \tag{J13}
\]
which goes to 1 as \(N_S \to 0\).

**Appendix K: Known sub-optimal receivers for target detection, phase sensing, and communication**

1. **Optical parametric amplifier receiver (OPAR)**

Fig. 14 shows the protocol of the OPAR. The OPAR applies parametric amplification across all the \(M\) return-idler mode pairs \(\{a_R^{(m)}, a_I^{(m)}\}\) to recast the cross-correlations between the input modes into photon-number differences. The amplification produces \(M\) output modes \(\hat{c}^{(m)} = \sqrt{G} a_R^{(m)} + \sqrt{N_S + 1} a_I^{(m)},\) \(1 \leq m \leq M\). For two-mode Gaussian states with zero mean and covariance matrix specified by Eq. (B7), each output mode is in a thermal state with mean photon number
\[
\bar{N}(\theta, \kappa) \equiv \langle \hat{c}^{(m)} \hat{c}^\dagger^{(m)} \rangle
= GN_S + (G - 1)(\kappa N_S + N_B + 1)
+ 2\sqrt{G(G - 1)} \kappa N_S (1 + N_S) \cos \theta. \tag{K1}
\]
We collect the total photon number \(\hat{N} = \sum_{m=1}^{M} \langle \hat{c}^{(m)} \hat{c}^\dagger^{(m)} \rangle\) across the \(M\) modes. The probability mass function of the random-variable readout \(N\) is [30]
\[
P_{N|\theta, \kappa}(n|\theta, \kappa) = \binom{n+M-1}{n} \left( \frac{\bar{N}(\theta, \kappa)}{1 + \bar{N}(\theta, \kappa)} \right)^n \left( \frac{1}{1 + \bar{N}(\theta, \kappa)} \right)^{M-n}, \tag{K2}
\]
where \(\binom{n}{b}\) is the binomial coefficient \(a\) choose \(b\). Below we utilize the above measurement statistics to evaluate the performance of quantum illumination, phase sensing, and communication.

In quantum illumination scenario, the task is to discriminate between two channel hypotheses, \(H_0 : \Phi_{0,0}\) and \(H_1 : \Phi_{0,1}\). When \(M \gg 1\), due to the central limit theorem, Eq. (K2) approximates to a Gaussian distribution, with mean and variance \(\mu_0 = GN_S + (G - 1)(1 + N_B),\) \(\sigma_0^2 = \mu_0 \sigma_0 + 1)\) for \(H_0\), and \(\mu_1 = GN_S + (G - 1)(1 + N_B + \kappa N_S) + 2\sqrt{G(G - 1)} \kappa N_S (1 + N_S),\) \(\sigma_1^2 = \mu_1 \sigma_1 + 1)\) for \(H_1\). One can make a near-optimum decision using a threshold detector that decides in favor of hypothesis \(H_0\) if \(N < N_{th}\), and \(H_1\) otherwise, with \(N_{th} \equiv [M(\sigma_1 \mu_1 + \sigma_0 \mu_0)/(\sigma_0 + \sigma_1)]\) [49]. The error probability for target detection is
\[
P_{E,\text{OPAR}} = \frac{1}{2} \text{Erfc} \left( \sqrt{\frac{R_{QI,\text{OPAR}} M}{2}} \right) \tag{K3}
\]
where \(R_{QI,\text{OPAR}} \approx \mu_1 - \mu_0)^2/2(\sigma_0 + \sigma_1)^2\). At the limit of \(N_S \ll 1, \kappa \ll 1, N_B \gg 1, R_{QI,\text{OPAR}} \approx \kappa N_S/(2N_B)\). Note that the exact optimal decision threshold is lengthy and only change the results slightly.

In the phase estimation scenario, the task is to estimate the parameter \(\theta\) of quantum channel \(\Phi_{\kappa, \theta}\). The Fisher

![Figure 14. Schematic of the OPAR. M i.i.d. return-idler pairs \(\{a_R^{(m)}, a_I^{(m)}\}_{m=1}^{M}\) are input to the receiver. An OPA component combines the idler and the conjugate of the signal, while only one of the output port is detected with photodetection. The total photon number over the \(M\) pairs is collected.](image-url)
information of OPAR is
\[
\mathcal{F}_{\text{OPAR}} = \sum_{n=0}^{\infty} \left( \partial_{\theta} \log P_{N[\theta,\kappa]}^{(M)}(n|\theta,\kappa) \right)^2 P_{N[\theta,\kappa]}^{(M)}(n|\theta,\kappa) .
\]
Plugging in Eq. (K2), we find that the Fisher information depends on amplification gain \( G \) as
\[
\mathcal{F}_{\text{OPAR}}(G) = \frac{4M(G - 1)G\kappa N_S(1 + N_S)\sin^2 \theta}{N(1 + \kappa N_S)} . \tag{K5}
\]
We derive the optimal gain as
\[
G_{\text{opt}} = \arg\max G \mathcal{F}_{\text{OPAR}}(G) = \max\{G^* , 1\} \tag{K6}
\]
where
\[
G^* = 1 + \sqrt{N_S(N_S + 1)(N_B' - 1)N_B^2 + N_S(N_S + 1)} \left( \frac{N_B - N_B(1 - \kappa)}{(N_B' + N_B)} \right),
\]
and \( N_B' \equiv N_B + \kappa N_S + 1 \). Here it is necessary to take the maximum between \( G^* \) and 1, because when \( N_S > N_B/(1 - \kappa) \), the optimum \( G^* \) falls below 1, which is not physical. As a result, at the limit \( N_S \ll 1 \) we have \( G^* = 1 + \sqrt{N_S} / \sqrt{N_B(1 + N_B)} + O(N_S) \). In this regime, the optimum Fisher information is \( \mathcal{F}_{\text{OPAR}} = 4\kappa N_S \sin^2 \theta / (1 + N_B) + O(\kappa^{3/2}) \).

In the communication scenario, let us consider the BFSK modulation where \( \theta \in \{0, \pi\} \) with equal probability 1/2. Then the conditional statistics of Eq. (K2) leads to the unconditional statistics \( P_{N}[\theta,\kappa]^\ast(n) = \sum_{\theta \in \{0, \pi\}} P_{N[\theta,\kappa]}^{(M)}(n|\theta,\kappa) / 2 \). Using these two distributions, we obtain the Shannon information
\[
I(N; \theta) = H(N) - H(N|\theta) , \tag{K8}
\]
where
\[
H(N|\theta) = - \sum_{\theta \in \{0, \pi\}} \frac{1}{2} \sum_{n=0}^{\infty} P_{N\theta,\kappa}^{(M)}(n|\theta,\kappa) \log_2 P_{N\theta,\kappa}^{(M)}(n|\theta,\kappa) ,
\]
\[
H(N) = - \sum_{n=0}^{\infty} P_{N}^{(M)}(n) \log_2 P_{N}^{(M)}(n) . \tag{K9}
\]
For simplicity of the description, in our simulation, we choose \( M = 1000 \) to match the choice of PCR in quantum illumination where Gaussian approximation requires large \( M \). Indeed, we find that it achieves a performance almost identical to the optimum choice of \( M = 1 \). The optimality is due to the fact that data processing, e.g. summing over \( M \) photon counts here, never increases Shannon information.

2. Phase conjugate receiver (PCR)

Fig. 15 shows the protocol of PCR. The inputs \( M \) i.i.d. return-idler pairs \( \{\hat{a}_R^{(m)} , \hat{a}_I^{(m)}\} \) are input to the receiver. A phase conjugator recasts \( \{\hat{a}_R^{(m)}\} \) to their phase conjugates \( \{\hat{a}_C^{(m)}\} \). Then the receiver recombines each phase conjugate \( \hat{a}_C^{(m)} \) with the paired idler \( \hat{a}_I^{(m)} \) on a 50 : 50 beamsplitter. Finally, the total photon number difference \( \hat{N} = \hat{N}_X - \hat{N}_Y \) is detected between the two arms \( X,Y \) over the \( M \) modes, where \( \hat{N}_X = \sum_{m=1}^{M} \hat{a}_X^{(m)*} \hat{a}_X^{(m)} , \hat{N}_Y = \sum_{m=1}^{M} \hat{a}_Y^{(m)*} \hat{a}_Y^{(m)} \). The random readout \( \hat{N} \) is approximately a Gaussian random variable at the limit of \( M \gg 1 \) due to the central limit theorem, subject to the Gaussian probability density function
\[
P_{\text{G}}^{(M)}(n|\theta,\kappa) = \frac{1}{2\sqrt{\pi\sigma^2(\theta,\kappa)}} \exp \left( \frac{- (n - \mu(\theta,\kappa))^2}{2\sigma^2(\theta,\kappa)} \right) , \tag{K10}
\]
with mean and variance
\[
\mu(\theta,\kappa) = M \cdot 2C_{CI}(\cos(\theta)) , 
\sigma^2(\theta,\kappa) = M \cdot (N_1 + 2N_C N_1 + N_C + 2C_{CI}^2 \cos(2\theta)) \tag{K11}
\]
where \( N_C = (G - 1)(\kappa N_S + N_B + 1) , N_I = N_S \) and \( C_{CI} = \sqrt{(G - 1)\kappa N_S(1 + N_S)} \).

In the quantum illumination scenario, as a reminder, the task is to discriminate between two channel hypotheses, \( H_0 : F_{\kappa,0} \) and \( H_1 : F_{\kappa,\theta} \). The mean and variance of the Gaussian statistics are different in the two hypotheses: \( \mu_0 = \mu(0,0) = \sigma_0^2 = \sigma_0^2(0,0) \) for \( H_0 \), and \( \mu_1 = \mu(0,\kappa) , \sigma_1^2 = \sigma_1^2(0,\kappa) \) for \( H_1 \). Using the near-optimum threshold detector with threshold \( N_{th} = \left[ M(\sigma_1 \mu_0 + \sigma_0 \mu_1) / (\sigma_0 + \sigma_1) \right] \) [49], the error probability for target detection is
\[
P_E^{\text{PCR}} = \frac{1}{2} \text{Erfc} \left( \sqrt{R_{\text{CR}}^M} \right) , \tag{K12}
\]

Figure 15. The schematic of PCR. \( M \) i.i.d. correlated return-idler pairs \( \{\hat{a}_R^{(m)} , \hat{a}_I^{(m)}\} \) are input to the receiver. The receiver applies a joint measurement on the returned signals \( \{\hat{a}_R^{(m)}\} \) and the idlers \( \{\hat{a}_I^{(m)}\} \): first \( \hat{a}_C^{(m)} \) are phase conjugated to produce \( \hat{a}_C^{(m)} \), and then an interferometry is applied to \( \hat{a}_C^{(m)} \) and \( \hat{a}_I^{(m)} \) by a balanced beamsplitter, finally the total photon number difference over the \( M \) pairs is collected.
where \( R_{\text{PCR}}^Q = \kappa N_S (N_S + 1) / (2N_B + 4N_S N_B + 6N_S + 4\kappa N_S^2 + 3\kappa N_S + 2) \). At the \( N_S \ll 1, \kappa \ll 1, N_B \gg 1 \) limit, its performance \( R_{\text{PCR}}^Q \approx \kappa N_S / (2N_B) \) becomes identical to the OPAR. Away from the asymptotic parameter region, PCR typically has a slightly better performance than OPAR.

In the phase estimation scenario, one estimates the parameter \( \theta \) of the channel \( \Phi_{\theta, \kappa} \), the PCR yields Fisher information

\[
\mathcal{F}_{\text{PCR}}(G) = \int_{-\infty}^{\infty} \left[ \partial_\theta \ln \left( p_{\text{G}}(n|\theta, \kappa) \right) \right]^2 p_{\text{G}}(n|\theta, \kappa) = \left[ \partial_\theta \mu(\theta, \kappa) \right]^2 / [\sigma^2(\theta, \kappa)/M].
\]

Substituting with Eq. (K11), the Fisher information is dependent on the conjugator gain \( G \) as

\[
\mathcal{F}_{\text{PCR}}(G) = M \cdot \frac{4(G - 1)\kappa N_S(N_S + 1) \sin^2 \theta}{(N_I + N_C) + (2N_C N_I + 2C_I^2 \cos(2\theta))}.
\]

It is easy to check that \( \mathcal{F}_{\text{PCR}}(G) \) monotonically increases with \( G \), while the gradient decays rapidly. As a result, one may regard the case of \( G = 2 \) as almost saturating the large gain limit, and obtain a performance sufficiently close to the optimum

\[
\mathcal{F}_{\text{PCR}}(G = 2) = \frac{4M\kappa N_S (N_S + 1) \sin^2 \theta}{N_B (1 + 2N_S) + N_S (2\kappa N_S + \kappa + 3) + 2\kappa \cos(2\theta) N_S (N_S + 1) + 1}.
\]

In practice, the gain can be limited because the photon-photon interaction is intrinsically weak. When \( N_S \) is sufficiently small, we can obtain a less stringent condition for \( G \) to saturate the quantum advantage. Consider the weak gain limit \( G - 1 \ll 1 \), we have

\[
\mathcal{F}_{\text{PCR}}(G) = \frac{4M\kappa N_S (N_S + 1) \sin^2 \theta}{1 + N_B + N_S/(G - 1) + O(N_S)}.
\]

The term \( N_S/(G - 1) \) in the denominator will be negligible when

\[
(G - 1)(1 + N_B) \gg N_S.
\]

Indeed, as long as this condition holds, the Fisher information of PCR reduces to the zero-order asymptotic formula \( \mathcal{F}_{\text{PCR}} \approx 4M\kappa N_S \sin^2 \theta / (1 + N_B) \), which saturates the optimum 3dB entanglement-assisted advantage over the classical coherent-state approach locally at \( \theta = \pi/2 \).

In the communication scenario, we consider the BPSK modulation such that \( \theta \in \{0, \pi\} \) with equal probability \( 1/2 \). Then the conditional statistics of Eq. (K10) leads to the unconditional statistics \( p_{\text{G}}^{(M)}(n) = \sum_{\theta \in \{0, \pi\}} p_{\text{G}}^{(M)}(n|\theta, \kappa) / 2 \), and thereby the Shannon information is also given by Eq. (K8), substituting \( p_{\text{G}}^{(M)}(n|\theta, \kappa) \) with Eq. (K10). In the simulation, we choose \( M = 1000 \) to validate the Gaussian approximation of Eq. (K10). Similar to the OPAR, we numerically find that the information rate does not decay significantly as \( M \) increases up to 1000 in the parameter region of interest.