Matrix method of polynomial solutions to constant coefficient PDE’s

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Abstract

In the paper, we introduce a matrix method to constructively determine spaces of polynomial solutions (in general, multiplied by exponentials) to a system of constant coefficient linear PDE’s with polynomial (multiplied by exponentials) right-hand sides. The matrix method reduces the funding of a polynomial subspace of null-space of a differential operator to the funding of the null-space of a block matrix. Using this matrix approach, we investigate some linear algebra properties of the spaces of polynomial solutions. In particular, for a solution space containing polynomials up to some arbitrarily large degree, we can determine dimension and basis of the space. Some examples of polynomial (multiplied by exponential, in general) solutions of the Laplace, Helmholtz, Poisson equations are considered.

Keywords: polynomial solution to linear constant coefficient PDE’s, exponential solution to nonhomogeneous PDE, null-space of matrix

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1 Introduction

The polynomial solutions to linear constant coefficient PDE’s is the well-known problem of algebra, see, for example, [1, 3, 7, 8, 9]. However, under the Fourier transform, a linear constant coefficient PDE is dual to an algebraic polynomial. Thus the problem to find a polynomial solution to linear constant coefficient PDE’s can be transformed to solve a linear algebraic system. So, the linear algebra approach allows to solve nonhomogeneous and induced by nonhomogeneous polynomials PDE’s. Hence the matrix method allows to find exponential solutions (in general, multiplied by algebraic polynomials) and to solve to PDE’s with polynomial (multiplied by exponentials) right-hand sides. Moreover, the matrix method enables to determine
some (linear algebra) characteristics such as dimension, basis, affinely-invariance, maximal total degrees of polynomials, etc., of a solution space. Note that the matrix method is valid for polynomials that induce PDE's with coefficients from any algebraically closed field. Note also that the matrix methods can be easily algorithmized.

In addition, the matrix approach can be generalized to find polynomial solutions to PDE's with polynomial coefficients. This will be discussed elsewhere.

Note that the discussed in this paper matrix approach to solve linear constant coefficient PDE's was stimulated by a generalization of the Strang-Fix conditions, see [2, 3]. In particular, the polynomial (multiplied by exponentials) solutions to the well-known differential equations (like Laplace’s equation), when we take a root of the operator symbol that the root is not the origin, were obtained, see [10].

The paper is organized as follows. Section 2 contains used in the paper notations and definitions. In particular, in Subsection 2.2, the lexicographically ordered sets of polynomials and derivatives are introduced. Section 3 is devoted to the matrix of the linear system; in Subsections 3.1, a method to construct the matrix is presented, and, in Subsection 3.2, some properties of the matrix are discussed. In Section 4, the matrix method to solve (in particular, nonhomogeneous and induced by nonhomogeneous polynomials) PDE's are discussed. Section 5, is devoted to examples to find polynomial solutions to some PDE’s.

2 Notations and definitions

2.1 Basic notations

Let \( \delta_{ij} := \begin{cases} 0 & \text{if } i \neq j, \\ 1 & \text{if } i = j \end{cases} \) be the Kronecker delta and \( \delta \) be the Dirac delta-distribution.

A multi-index \( \alpha \) is a \( d \)-tuple \((\alpha_1, \ldots, \alpha_d)\) with its components being nonnegative integers, i.e., \( \alpha \in \mathbb{Z}_{\geq 0}^d \). The length of a multi-index \( \alpha := (\alpha_1, \ldots, \alpha_d) \in \mathbb{Z}_{\geq 0}^d \) is defined as \( \alpha_1 + \cdots + \alpha_d \) and denoted by \( |\alpha| \). For \( \alpha = (\alpha_1, \ldots, \alpha_d) \), \( \beta = (\beta_1, \ldots, \beta_d) \), we write \( \beta \leq \alpha \) if \( \beta_j \leq \alpha_j \) for all \( j = 1, \ldots, d \). The factorial of \( \alpha \) is \( \alpha! := \alpha_1! \cdot \cdots \cdot \alpha_d! \). The binomial coefficient for multi-indices \( \alpha, \beta \) is

\[
\binom{\alpha}{\beta} := \binom{\alpha_1}{\beta_1} \cdot \cdots \cdot \binom{\alpha_d}{\beta_d} = \frac{\alpha!}{\beta! (\alpha - \beta)!}.
\]

By definition, put

\[
\binom{\alpha}{\beta} = 0 \quad \text{if } \beta \nleq \alpha. \tag{2.1}
\]

By \( x^\alpha \), where \( x = (x_1, \ldots, x_d) \), \( \alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{Z}_{\geq 0}^d \), denote a monomial \( x_1^{\alpha_1} \cdots x_d^{\alpha_d} \). Note that the total degree of \( x^\alpha \) is \( |\alpha| \). By \( \Pi_l, l \in \mathbb{Z}_{\geq 0} \), denote the space of (homogeneous) polynomials that the total degrees of the polynomials are equal to \( l \): \( \Pi_l := \text{span} \{ x^\alpha : \alpha \in \mathbb{Z}_{\geq 0}^d, |\alpha| = l \} \); and by \( \Pi_{\leq l} \) denote the space of polynomials that the total degrees of the polynomials are less than or equal to \( l \): \( \Pi_{\leq l} := \text{span} \{ x^\alpha : \alpha \in \mathbb{Z}_{\geq 0}^d, |\alpha| \leq l \} \) (here and in the sequel, ‘span’ means the linear span over \( \mathbb{C} \)).
Remark 2.1. Since the linear algebra definitions and assertions are valid for any field; we can consider polynomials with coefficients from an arbitrary field.

On the other hand, we would like to use some algebraically closed fields (a field $\mathbb{C}$, for example) or we must use algebraic extensions of fields.

The dot product of two vectors ($d$-tuples) $x = (x_1, \ldots, x_d)$, $y = (y_1, \ldots, y_d)$ is $x \cdot y := x_1y_1 + \cdots + x_dy_d$. If all the polynomials from the space $\Pi_l$ multiplied by an exponential $e^{i\bar{x}_0 \cdot x}$, where $x_0 \in \mathbb{C}^d$ is a given point; then we shall write $e^{i\bar{x}_0 \cdot x} \Pi_l$ (for $\Pi \leq l$, $e^{i\bar{x}_0 \cdot x} \Pi \leq l$).

Let $D^{\alpha}$ imply a differential operator $D^{\alpha_1}_1 \cdots D^{\alpha_d}_d$, where $D^{n}_n$, $n = 1, \ldots, d$, is the partial derivative with respect to the $n$th coordinate. Note that $D^{(0, \ldots, 0)}$ is the identity operator.

Abusing notations, for a function $f = f(x)$ and constant point $x_0$ we shall write everywhere $D^{\alpha} f(x_0)$, meaning, in fact, $D^{\alpha} f(x)|_{x=x_0}$.

The multi-dimensional version of the Leibniz rule is

$$(fg)^{(\alpha)} = \sum_{\beta \leq \alpha \in \mathbb{Z}^d \geq 0} \binom{\alpha}{\beta} f^{(\beta)} g^{(\alpha-\beta)}, \quad \alpha \in \mathbb{Z}^d \geq 0,$$  

(2.2)

where the functions $f(x)$, $g(x)$, $x = (x_1, \ldots, x_d)$, are sufficiently differentiable.

The Fourier transform of a function $f \in L^1(\mathbb{R}^d)$ is defined by

$$f(x) \mapsto \hat{f}(\xi) = (\hat{F}f)(\xi) := (2\pi)^{-d/2} \int_{\mathbb{R}^d} f(x) e^{-i \xi \cdot x} \, dx, \quad \xi \in \mathbb{R}^d.$$ 

Note that the Fourier transform can be extended to compactly supported functions (distributions) that the functions belong, for example, to the space of tempered distributions $S'(\mathbb{R}^d)$. Moreover, the domain of the Fourier transform can be extended (it is possible, in particular, for a compactly supported function) to the whole complex space $\mathbb{C}^d$. And the Schwartz space $S$ (of test functions), i.e., the space of functions that all the derivatives of the functions are rapidly decreasing, also can be extended to $\mathbb{C}^d$.

So, for $x_0 \in \mathbb{C}^d$, $\alpha \in \mathbb{Z}^d \geq 0$, we have the following formula

$$(\hat{F}e^{i\bar{x}_0 \cdot x^{\alpha}})(\xi) = i^{\vert \alpha \vert} D^{\alpha} \delta(\xi - x_0), \quad \xi \in \mathbb{C}^d.$$ 

Definition 2.1. Let $f, g \in L^2(\mathbb{C}^d)$ be complex functions. Then an inner product in the space $L^2(\mathbb{C}^d)$ is

$$\langle f, g \rangle := \int_{\mathbb{C}^d} f(x) \overline{g(x)} \, dx.$$  

(2.3)

Here and in the sequel, the overline $\overline{\cdot}$ denotes the complex conjugation.

In the following definition, we consider complex distributions and complex test functions, see for example [4, 6].
Definition 2.2. Let $\phi \in S(\mathbb{C}^d)$ be a complex test function. Let $f = f(x)$, $x \in \mathbb{C}^d$, be a locally integrable on $\mathbb{C}^d$ complex function. Then the function $f$ induces some distribution (continuous linear functional) on $S(\mathbb{C}^d)$ as follows
\[ T_f(\phi) := \int_{\mathbb{C}^d} f(x)\phi(x) \, dx = \langle f, \phi \rangle, \tag{2.4} \]
where $\langle \cdot, \cdot \rangle$, in the right-hand side of (2.4), is the inner product defined by (2.3).

Remark 2.2. Any functional defined by (2.4) is a linear functional; in particular, the functional is homogeneous:
\[ T_f(a\phi) = aT_f(\phi), \]
where $a$ is a complex valued function.

In the paper, we usually denote matrices by upper-case bold symbols or enclose the symbols of matrices in the square brackets. On the other hand, abusing notation slightly, we shall denote a vector of some linear space by plain lower-case symbol and interpret the vector of a linear space as a column vector.

Definition 2.3. Suppose $A := [a_{ij}]$ ($1 \leq i \leq n, 1 \leq j \leq m$, $a_{ij} \in \mathbb{C}$, is an $n \times m$ matrix. By definition, put
\[ \ker A := \{ v \in \mathbb{C}^m : Av = 0 \}. \]
We say that the linear space $\ker A$ is the (right) null-space of the matrix $A$.

Remark 2.3. However sometimes we shall treat a null-space $\ker A$ as $(\dim \ker A)$-column matrix of vectors that forms a basis for $\ker A$.

Now recall some block matrix notions. A block matrix is a matrix broken into sections called blocks or submatrices. A block diagonal matrix is a block matrix such that the main diagonal square submatrices can be non-zero and all the off-diagonal submatrices are zero matrices. The (block) diagonals can be specified by an index $k$ measured relative to the main diagonal, thus the main diagonal has $k = 0$ and the $k$-diagonal consists of the entries on the $k$th diagonal above the main diagonal. Note that all the $k$-diagonal submatrices, except submatrices on the main diagonal, can be non-square.

2.2 Ordered sets

By $<_\text{lex}$ we denote some lexicographical order and by $A_k$, $k \in \mathbb{Z}_{\geq 0}$, denote the lexicographically ordered set of all multi-indices of length $k$
\[ A_k := \{ q_\alpha \} = k, q = 1, \ldots , d(k), \]
where
\[ d(k) := \binom{d + k - 1}{k} = \frac{(d + k - 1)!}{k!(d - 1)!} \]
is the number of \( k \)-combinations with repetition from the \( d \) elements.

By \( \tilde{A}_k \) we denote a concatenated set of multi-indices
\[
\tilde{A}_k := (A_0, A_1, \ldots, A_k),
\]
where the comma must be considered as a concatenation operator to join 2 sets. Actually the order of \( \tilde{A}_k \) is the graded lexicographical order. By \( \tilde{d}(k) \) denote the length of a concatenated set like \( \tilde{A}_k \)
\[
\tilde{d}(k) := d(0) + d(1) + \cdots + d(k) = \frac{(d + k)!}{k!d!}.
\]

By \( P_k, k \in \mathbb{Z}_{\geq 0} \), denote the lexicographically ordered set of all monomials of total degree \( k \)
\[
P_k(x) := \left(x^{\alpha_1}, \ldots, x^{d(k)\alpha_d}\right), \quad x = (x_1, \ldots, x_d), \quad (\alpha_1, \ldots, d(k)\alpha_d) = A_k.
\]
For \( \beta \in \mathbb{Z}_{\geq 0}^d \), by \( P^\beta_k \) denote the following set of monomials
\[
P^\beta_k(x) := \left(\begin{array}{c}
\left(\begin{array}{c}
\beta_1 \\
\alpha_1
\end{array}\right)x^{\alpha_1-\beta_1}, \ldots, \left(\begin{array}{c}
\beta_d \\
\alpha_d
\end{array}\right)x^{d(k)\alpha_d-\beta_d}
\end{array}\right), \quad (\alpha_1, \ldots, d(k)\alpha_d) = A_k.
\] (2.5)

Similarly, define the ordered sets of differential operators as
\[
D_k := (-i)^k D^{\alpha_1}, \ldots, (-i)^k D^{d(k)\alpha_d},
\]
\[
D^\beta_k := (-i)^{k-|eta|} \left(\begin{array}{c}
\beta_1 \\
\alpha_1
\end{array}\right) D^{\alpha_1-\beta_1}, \ldots, (-i)^{k-|eta|} \left(\begin{array}{c}
\beta_d \\
\alpha_d
\end{array}\right) D^{d(k)\alpha_d-\beta_d},
\] (2.6)
where \( (\alpha_1, \ldots, d(k)\alpha_d) = A_k \). Note that if \( \beta \not\leq q\alpha \), then the \( q \)th entries of (2.5) and (2.6) are zeros. Moreover, if \( |eta| > k \); then sets (2.5), (2.6) are zero sets.

By \( \tilde{P}_k \) and \( \tilde{P}^\beta_k \) denote the following concatenated sets of monomials
\[
\tilde{P}_k := (P_0, P_1, \ldots, P_k),
\] (2.7)
\[
\tilde{P}^\beta_k := (P^\beta_0, P^\beta_1, \ldots, P^\beta_k).
\] (2.8)

The concatenated sets of derivatives are defined similarly to (2.7), (2.8)
\[
\tilde{D}_k := (D_0, D_1, \ldots, D_k),
\] (2.9)
\[
\tilde{D}^\beta_k := (D^\beta_0, D^\beta_1, \ldots, D^\beta_k).
\] (2.8)

3 The matrix of the linear system

As it has been said, we shall frequently interpret the ordered sets (for example, \( P_k, D_k \)) as row vectors and enclose their symbols in the square brackets \([P_k], [D_k]\)
3.1 Formation of the matrix

For some $k, l \in \mathbb{Z}_{\geq 0}$, $k \leq l$, define a $d(l) \times d(k)$ matrix $D_k$ as follows

$$D_k := \begin{bmatrix} D_k^0 \\ D_k^1 \\ \vdots \\ D_k^k \\ 0 \\ \vdots \\ 0 \end{bmatrix},$$

(3.1)

where $D_k^r$ are $d(r) \times d(k)$, $r = 0, 1, \ldots, k$, submatrices defined as

$$D_k^r := \begin{bmatrix} D_k^{ij} \\ D_k^{2j} \\ \vdots \\ D_k^{d(r)j} \end{bmatrix}, \quad (i \beta, \ldots, d(r)\beta) = A_r,$$

(3.2)

and the row vectors $[D_k^{qj}]$, $q = 1, \ldots, d(r)$, are given by (2.6). By definition, if $r > k$; then $D_k^r$ is a zero matrix.

Finally, for $l \in \mathbb{Z}_{\geq 0}$, define a $d(l) \times d(l)$ matrix $\tilde{D}_l$ as

$$\tilde{D}_l := [D_0 \ D_1 \ \ldots \ D_{l-1} \ D_l] = \begin{bmatrix} D_0^0 & D_0^1 & \ldots & D_{l-1}^0 & D_l^0 \\ 0 & D_1^0 & \ldots & D_{l-1}^1 & D_l^1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & D_{l-1}^{l-1} & D_l^{l-1} \\ 0 & 0 & \ldots & 0 & D_l^l \end{bmatrix}.$$

(3.3)

Note that, in formulas (3.1), (3.3), the symbol ‘0’ must be considered as a zero block of the corresponding size.

3.2 Some properties of the matrix

3.2.1 Single function

Definition 3.1. Let $l \in \mathbb{Z}_{\geq 0}$, a function $f : \mathbb{C}^d \to \mathbb{C}^d$ be sufficiently differentiable, and $x_0$ be a point of $\mathbb{C}^d$. By $V_l$ denote the null-space (kernel) of the matrix $\tilde{D}_l f(x_0)$:

$$V_l := \ker \tilde{D}_l f(x_0).$$

(3.4)
Theorem 3.1. Let \( l \in \mathbb{Z}_{\geq 0} \). Let the set \( \tilde{D}_l \) be given by (2.9), the matrix \( \tilde{D}_l \) be given by (3.3), and functions \( f, g \) be sufficiently differentiable. Then we have

\[
\begin{bmatrix} \tilde{D}_l(fg) \end{bmatrix} = \begin{bmatrix} \tilde{D}_lf \end{bmatrix} \begin{bmatrix} \tilde{D}_lg \end{bmatrix} = \begin{bmatrix} \tilde{D}_lg \end{bmatrix} \begin{bmatrix} \tilde{D}_lf \end{bmatrix},
\]

(3.7)

Here we omit the proof of formula (3.7) and note only that the formula is a direct consequence of formula (3.6) and the Leibniz rule, see (2.2).

Now we investigate ranks of submatrices in the upper right corner of the matrix \( \tilde{D}_lf(x_0) \) (the matrices \( D^m_m f(x_0), m' = 0, \ldots, m, m = 0, \ldots, l \), given by (3.2)); where the function \( f : \mathbb{C}^d \to \mathbb{C}^d \) is sufficiently differentiable and \( x_0 \in \mathbb{C}^d \) is a given point.

Proposition 3.2. Let \( l \in \mathbb{Z}_{\geq 0} \). The submatrix \( D^m_m f(x_0), m' = 0, \ldots, m, m = 0, \ldots, l \), contains only the derivatives of order \( m - m' \).

Corollary 3.3. All the submatrices on the \( m \)th block diagonal of the matrix \( \tilde{D}_l \), i.e., the submatrices \( D^m_m, D^m_{m+1}, \ldots, D^m_{l-m} \), contain the derivatives of order \( m \).

It easy to see that the \( q \)th, \( 1 \leq q \leq d(m) \), row of the matrix \( D^m_m f(x_0) \), \( m = 0, \ldots, l \), contains only one non-zero element \( f(x_0) \) (if \( f(x_0) \neq 0 \)), which is situated on the \( q \)th position. Thus \( D^m_m f(x_0) = \mathbf{1} f(x_0) \), where \( \mathbf{1} \) is the \( d(m) \times d(m) \) identity matrix. Since the matrix \( \tilde{D}_lf(x_0) \) is an upper triangular matrix, we can state a necessary and sufficient condition that the matrix \( \tilde{D}_lf(x_0) \) is singular.

Theorem 3.4. The matrix \( \tilde{D}_lf(x_0) \), \( l \in \mathbb{Z}_{\geq 0} \), is singular iff \( f(x_0) = 0 \).

Now state a theorem about ranks of all other blocks of the matrix \( \tilde{D}_lf(x_0) \).

Theorem 3.5. The \( d(m') \times d(m) \) submatrix \( D^m_{m'} f(x_0), m' = 0, \ldots, m - 1, m = 1, \ldots, l \), \( l \in \mathbb{Z}_{\geq 0} \), has full rank, i.e., the rank of \( D^m_{m'} f(x_0) \) is equal to \( d(m') \), if and only if there exists at least one non-zero derivative \( D^\gamma f(x_0), |\gamma| = m - m' \).
The proof of Theorem 3.5 is given in Appendix A.

Hence we see that each of the submatrices $D_{m'}f(x_0), m' = 0, \ldots, m, m = 0, \ldots, l$, is either a full rank matrix or zero matrix.

Finally the following theorem allows to determine the dimension of the null-space of $\tilde{D}_lf(x_0)$.

**Theorem 3.6.** Let $l \in \mathbb{Z}_{\geq 0}$. Suppose there exists a multi-index $\alpha \in \mathbb{Z}^d_{\geq 0}, |\alpha| > 0$, such that $D^{\alpha}f(x_0) \neq 0$ and, for any multi-index $\alpha' \in \mathbb{Z}^d_{\geq 0}$ such that $0 \leq |\alpha'| < |\alpha|$, the derivative $D^{\alpha'}f(x_0)$ vanishes. Then

$$\dim \ker \tilde{D}_lf(x_0) = \begin{cases} \bar{d}(l - |\alpha|) & \text{if } l \geq |\alpha| > 0; \\ \bar{d}(l) & \text{if } l < |\alpha|. \end{cases}$$

(3.8)

The theorem is a direct consequence of the following lemma.

**Lemma 3.7.** Under the conditions of Theorem 3.6, we have

$$\text{rank } \tilde{D}_lf(x_0) = \begin{cases} \bar{d}(l - |\alpha|) & \text{if } l \geq |\alpha| > 0; \\ 0 & \text{if } l < |\alpha|. \end{cases}$$

Sketch of the proof of Lemma 3.7. For the case $l \geq |\alpha| > 0$, by Theorem 3.5 and Corollary 3.3, each block on the $|\alpha|$th block diagonal of the matrix $\tilde{D}_lf(x_0)$ is a full rank matrix. However, since the blocks of $\tilde{D}_lf(x_0)$ are not, generally, square matrices; the problem to determine the rank of the matrix is not trivial.

Using an analog of the Gaussian elimination algorithm (applied to columns instead of rows) and moving (actually permutating) zero columns to the left, the matrix $\tilde{D}_lf(x_0)$ can always be transformed into a strictly upper triangular matrix, where the lowest non-zero diagonal goes to the lower right corner of the last submatrix $D_{l-|\alpha|}$. So rank $\tilde{D}_lf(x_0) = \bar{d}(l - |\alpha|)$.

Since, in the case $l < |\alpha|$, the matrix $\tilde{D}_lf(x_0)$ is a zero matrix; the case $l < |\alpha|$ is trivial. □

Introduce some notation. We always can consider $\mathbb{C}^{\bar{d}(l)}, l \in \mathbb{Z}_{\geq 0}$, as a space with a Cartesian coordinate system, where the dot product of the Cartesian coordinates is $x \cdot y := \sum_q x_q y_q$. Namely, we have

$$\mathbb{C}^{\bar{d}(l)} := \text{span} \left\{ e_q : 1 \leq q \leq \bar{d}(l) \right\},$$

where the span is over $\mathbb{C}$ and $e_q$ is the $q$th basis vector: $e_q := \left( \delta_{q1}, \ldots, \delta_{q,\bar{d}(l)} \right)$; and we can decompose $\mathbb{C}^{\bar{d}(l)}$ as follows

$$\mathbb{C}^{\bar{d}(l)} = \mathbb{C}^{\bar{d}(l)}_0 \oplus \mathbb{C}^{\bar{d}(l)}_1 \oplus \cdots \oplus \mathbb{C}^{\bar{d}(l)}_m,$$

(3.9)

where

$$\mathbb{C}^{\bar{d}(l)}_0 := \text{span} \left\{ e_1 \right\},$$

$$\mathbb{C}^{\bar{d}(l)}_m := \text{span} \left\{ e_q : \bar{d}(m - 1) + 1 \leq q \leq \bar{d}(m) \right\}, \quad m = 1, \ldots, l,$$

and the direct sums in (3.9) are orthogonal.
Remark 3.2. Decomposition (3.9) corresponds to the block structure of the matrix $\tilde{D}_l$, $l \in \mathbb{Z}_{\geq 0}$, (as well as structures of $\tilde{P}_l$ and $\tilde{D}_l$).

Definition 3.3. By $\mathcal{P}_m$, denote the orthogonal projection on the subspace $m\mathbb{C}^{\tilde{d}(l)}$, $m = 0, \ldots, l$; and define subspaces of the null-space $V_l$, see Definition 3.1, as

$$mV_l := \mathcal{P}_m V_l, \quad m = 0, 1, \ldots, l.$$  \hfill (3.10)

Remark 3.3. Note that generally $V_l$ is not a sum, like (3.9), of $mV_l$.

Now we can formulate the following theorem.

Theorem 3.8. Let $l \in \mathbb{N}$. Let the matrix $\tilde{D}_lf(x_0)$ be singular and $V_l := \ker \tilde{D}_lf(x_0)$. Then the subspace $V_l := \mathcal{P}_l V_l$ is non-zero.

Proof. If the matrix $\tilde{D}_lf(x_0)$ is a zero matrix, there is nothing to prove.

Assume the converse, i.e., suppose $V_l$ is a zero space; then

$$\dim \ker \tilde{D}_lf(x_0) = \dim \ker \tilde{D}_{l-1}f(x_0).$$

Since the matrix $\tilde{D}_lf(x_0)$ is singular and non-zero; there exists a number $r \in \mathbb{N}$, $1 \leq r \leq l$, that the block $r$-diagonal is the lowest non-zero block diagonal. By Theorem 3.6, we have

$$\dim \ker \tilde{D}_{l-1}f(x_0) = \tilde{d}(l-1) - \tilde{d}(l-1-r) = \tilde{d}(l) - \tilde{d}(l-r) = \dim \ker \tilde{D}_lf(x_0).$$

This contradiction proves the theorem. \hfill $\square$

3.2.2 Several functions

For several functions $f_1, f_2, \ldots, f_n$, we must consider the following concatenated matrix

$$\begin{bmatrix} \tilde{D}_lf_1(x_0) \\ \vdots \\ \tilde{D}_lf_n(x_0) \end{bmatrix}.$$  \hfill (3.11)

For concatenated matrix (3.11), an analog of Theorem 3.8 is not valid. We can state only the following proposition.

First we must restructure matrix (3.11) and introduce a notation.

Let $k, l \in \mathbb{Z}_{\geq 0}$, $k \leq l$. Define the $nd(l) \times d(k)$ matrix $\tilde{D}_k$, where $n$ is the number of matrices (functions), as follows

$$\tilde{D}_k := \begin{bmatrix} \tilde{D}_k^0 \\ \tilde{D}_k^1 \\ \vdots \\ \tilde{D}_k^k \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad \text{where} \quad \tilde{D}_k^r := \begin{bmatrix} D_k^r f_1(x_0) \\ D_k^r f_2(x_0) \\ \vdots \\ D_k^r f_n(x_0) \end{bmatrix}, \quad r = 0, 1, \ldots k.$$  \hfill (3.12)
Secondly formulate an analog of matrix (3.3)

\[
\DM_l := \begin{bmatrix}
\tilde{D}_0 & \tilde{D}_1 & \cdots & \tilde{D}_l \\
0 & \tilde{D}_0 & \cdots & \tilde{D}_{l-1} \\
& \vdots & \ddots & \vdots \\
0 & 0 & \cdots & \tilde{D}_l^{-1} & \tilde{D}_l^{-1} \\
0 & 0 & \cdots & 0 & \tilde{D}_l
\end{bmatrix}
\]  \hspace{1cm} (3.13)

Finally we present a proposition.

**Proposition 3.9.** Let \( l \in \mathbb{N} \). Let the matrices \( \DM_l \), \( \DM_{l-1} \) be defined by (3.13). Let \( V_l := \ker \DM_l \), \( V_{l-1} := \ker \DM_{l-1} \) and subspaces \( V_l \), \( V_{l-1} \) be given by (3.10). Suppose the subspace \( V_l \) is non-zero, then the subspace \( V_{l-1} \) is also non-zero.

The proof is given in Appendix B.

**Remark 3.4.** Note that Proposition 3.9 is also valid for single matrix (3.3).

## 4 Solution methods

### 4.1 Homogeneous PDE’s

In this subsection, we consider homogeneous PDE’s with constant coefficients from the field \( \mathbb{C} \). Note the polynomials that induce the differential equations are not necessary homogeneous; i.e., the polynomials are, in general, sums of terms of different degree.

#### 4.1.1 One equation

**Theorem 4.1.** Let \( P \) be a polynomial with constant coefficients from the field \( \mathbb{C} \). Let \( x_0 \) be a point of \( \mathbb{C}^d \). Let the matrix \( \DM_l \), \( l \in \mathbb{Z}_{\geq 0} \), be given by (3.3). Then the following non-zero function \( e^{ix_0 \cdot x} \left[ \tilde{P}_l(x) \right] v, v \in \mathbb{C}^{d(l)} \), belongs to \( \ker P(-iD) \) if and only if \( P(x_0) = 0 \) and \( v \in \ker \DM_l P(x_0) \).

**Proof.** As it has been said, see Theorem 3.4 a necessary and sufficient condition for the matrix \( \DM_l P(x_0) \) to be singular is that the point \( x_0 \in \mathbb{C}^d \) be a root of the polynomial \( P \).

Consider the function

\[
f(x) := P(-iD) \left( e^{ix_0 \cdot x} \left[ \tilde{P}_l(x) \right] v \right).
\]

Taking the Fourier transform of the previous function, we obtain

\[
\hat{f}(\xi) := P(\xi) \left[ \tilde{D}_l \delta(\xi - x_0) \right] v, \quad \xi \in \mathbb{C}^d.
\]  \hspace{1cm} (4.1)

The adjoint operator (set of operators) \( \DM_l^* \) satisfies a property

\[
\DM_l^* = \overline{\DM_l}
\]  \hspace{1cm} (4.2)
(the adjunction, like the complex conjugation, is distributive over the comma). Using Definition \[2.2\] Theorem \[3.1\] and property \[4.2\]; for any test function \(\phi \in S(\mathbb{C}^d)\), the functional \(\hat{T}_f(\phi)\) (where \(\hat{f}\) is distribution \[4.1\]) is of the form

\[
T_{\hat{f}}(\phi) = \left\langle P(\cdot) \left[ \overline{\mathcal{D}_l \delta(\cdot - x_0)} \right] v, \phi \right\rangle = \left\langle \delta(\cdot - x_0), \left[ \overline{\mathcal{D}_l (P\phi)} \right] \right\rangle = \left[ \mathcal{D}_l \overline{P(x_0)\phi(x_0)} \right] v, \phi \rangle = \left[ \overline{\mathcal{D}_l P(x_0)} \right] v. \tag{4.3}
\]

Using the expression in the right-hand side of \[4.3\], the proof is trivial. \(\square\)

**Remark 4.1.** Theorem 4.1 can be proved for other (linear and linear-conjugate, see for example \[6\]) functionals like \[2.4\].

**Theorem 4.2.** Let \(l \in \mathbb{Z}_{\geq 0}\). Let \(P\) be a polynomial. Let \(x_0 \in \mathbb{C}^d\) be a root of the polynomial \(P\). Let the matrices \(\mathcal{D}_l, \mathcal{D}_{l+1}\) be given by \[3.3\]. Let the polynomial spaces be of the form

\[
V_l := \left\{ \left[ \mathcal{P}_l \right] v : v \in \ker \mathcal{D}_l P(x_0) \right\},
\]

\[
V_{l+1} := \left\{ \left[ \mathcal{P}_{l+1} \right] v : v \in \ker \mathcal{D}_{l+1} P(x_0) \right\}.
\]

Then

\[
V_l = \Pi_{\leq l} \cap V_{l+1}. \tag{4.4}
\]

The proof of Theorem 4.2 is given in Appendix C.

**Remark 4.2.** The previous theorem reflects a property of differentiation to commutate with translation.

Below we state a corollary of Theorem 3.8 and Theorem 4.2.

**Corollary 4.3.** Under the conditions of Theorem 4.2 we see that the null-space of the operator \(P(-iD)\) contains polynomials (multiplied by the exponential \(e^{ix_0 \cdot x}\)) up to an arbitrary large total degree.

**Remark 4.3.** Corollary 4.3 expresses a fundamental property, see \[1\], of polynomial solutions to a single PDE with constant coefficients.

**Theorem 4.4.** Let \(l \in \mathbb{Z}_{\geq 0}\). Let \(P\) be an algebraic polynomial, let \(x_0\) be a root of \(P\). Let \(D^\alpha P(x_0), \alpha \in \mathbb{Z}^d_{\geq 0}\), be a non-zero derivative of the least order. Then we have

\[
\dim \left( e^{ix_0 \cdot x} \Pi_{\leq l} \cap \ker P(-iD) \right) = \begin{cases} \tilde{d}(l) - \tilde{d}(l - |\alpha|) & \text{if } l \geq |\alpha| > 0; \\ \tilde{d}(l) & \text{if } l < |\alpha|. \end{cases}
\]

Moreover, if \(l < |\alpha|\); then

\[
e^{ix_0 \cdot x} \Pi_{\leq l} \cap \ker P(-iD) = e^{ix_0 \cdot x} \Pi_{\leq l}.
\]

The previous theorem is a direct consequence of Theorem 3.6.
4.1.2 System of equations

For the case of a system of PDE’s, i.e., if we have several algebraic polynomials $P_n'$, $n' = 1, 2, \ldots, n$, the polynomial solution of the corresponding system

$$
\begin{cases}
P_1(-iD)\cdot = 0, \\
\vdots \\
P_n(-iD)\cdot = 0 
\end{cases}
$$

is the intersection of the null-spaces $\ker P_n'(-iD)$, $n' = 1, \ldots, n$.

Namely we have the following proposition.

**Proposition 4.5.** Let $P_n'$, $n' = 1, 2, \ldots, n$, be algebraic polynomials. The non-zero expression $e^{ix_0} \cdot [\tilde{P}_l] v, l \in \mathbb{Z}_{\geq 0},$ where $v \in \mathbb{C}^{\tilde{d}(0)}$, belongs to the null-space of system (4.5) iff the vector $v$ belongs to the null-space of the block matrix

$$
[\tilde{D}_lP_1(x_0)] \\
\vdots \\
[\tilde{D}_lP_n(x_0)]
$$

and $P_1(x_0) = \cdots = P_n(x_0) = 0$.

4.2 PDE with polynomial right-hand side

In the previous subsection, we have considered polynomial solutions of homogeneous constant coefficient PDE’s; i.e., actually, PDE’s have zero right-hand sides. Nevertheless the matrix approach allows generalizing PDE’s to polynomial right-hand sides (multiplied by an exponential).

**Theorem 4.6.** Let $l \in \mathbb{Z}_{\geq 0}$. Let $P, F$ be algebraic polynomials with constant coefficients from $\mathbb{C}$, $x_0 \in \mathbb{C}^d$ be a root of $P$, $\alpha \in \mathbb{Z}_{\geq 0}^d$ be a multi-index that defines the least order derivative such that $D^\alpha P(x_0) \neq 0$. Let the polynomial $F$ be defined as follows: $F(x) := \left[\tilde{P}_{\deg F}(x)\right] w$, $w \in \mathbb{C}^{\tilde{d}(\deg F)}$. Let $l \geq \deg F + |\alpha|$, the matrix $\tilde{D}_l$ be given by (3.3). Let $v \in \mathbb{C}^{\tilde{d}(l)}$ be a column vector and $p := [\tilde{P}_l] v$ be the corresponding polynomial. Then the algebraic polynomial $p$ is a solution to PDE

$$
P(-iD) \left(e^{ix_0 \cdot x} p(x)\right) = e^{ix_0 \cdot x} F(x)
$$

iff the vector $v$ is a solution to linear algebraic equation

$$
[\tilde{D}_lP(x_0)] v = \left[w^T \underbrace{0 \cdots 0}_{\tilde{d}(l)-\tilde{d}(\deg F)}\right]^T
$$
Remark 4.4. Under the conditions of the previous theorem, introduce notation. Denote by $V_l$ a linear space such that any vector $v \in V_l$ is a solution to linear algebraic equation (4.7) and by $\mathcal{V}_l := \left\{ \tilde{P}_l \right\}$ denote the corresponding space of polynomials that the polynomials are solutions to PDE (4.6). So we can state 3 remarks:

1. $\dim V_l = \tilde{d}(l) - \tilde{d}(l - |\alpha|)$;

2. for any polynomial $F$ such that $\deg F \leq l - |\alpha|$, linear system (4.7) is consistent;

3. there exists a polynomial of an arbitrary large degree to satisfy PDE (4.6).

However, in the previous theorem, the point $x_0 \in \mathbb{C}^d$ can be no root of $P$. Below we state the corresponding remark.

Remark 4.5. Under the conditions of the previous theorem and suppose $P(x_0) \neq 0$; we see that the polynomial $p$ is defined uniquely and does not depend on the choice of $l$.

The proof of the previous theorem and remarks is left to the reader. Note only that the theorem and remarks are based on Theorem 3.6, Theorem 4.1, and the classical Rouché-Capelli theorem.

5 Examples

Below, for a function of 2 variables $f = f(x, y), x, y \in \mathbb{C}$, we present a component-wise form of the matrix $\tilde{D}_3f(x_0, y_0), (x_0, y_0) \in \mathbb{C}^2$:

$$\tilde{D}_3f(x_0, y_0) = \begin{bmatrix}
\begin{array}{cccccccc}
0 & f & 0 & -2if_x & -2if_y & 0 & -3if_{xx} & -3if_{xy} & -3if_{yy} \\
0 & 0 & f & 0 & -if_x -2if_y & 0 & -f_{xx} -2f_{xy} -3f_{yy} \\
0 & 0 & 0 & f & 0 & 0 & -3if_x -if_y & 0 & 0 \\
0 & 0 & 0 & 0 & f & 0 & 0 & -2if_x -2if_y & 0 \\
0 & 0 & 0 & 0 & 0 & f & 0 & 0 & -if_x -3if_y \\
0 & 0 & 0 & 0 & 0 & 0 & f & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & f & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & f
\end{array}
\end{bmatrix}(x, y) = (x_0, y_0)$$

5.1 Homogeneous equations

First we consider 3 examples of polynomial solution to PDE’s, presented in the paper [7].
Example 1  The first example is 2D Laplace operator

\[ L_1 := \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}, \quad x, y \in \mathbb{R}. \quad (5.1) \]

So the polynomial that induces the operator is \( P_1(x, y) := -x^2 - y^2, x, y \in \mathbb{C} \); Thus the 10 \( \times \) 10 matrix \( \tilde{D}_3 P_1(0, 0) \) is of the form

\[
\begin{bmatrix}
0 & 0 & 0 & 2 & 0 & 2 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 6 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix} = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -3 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 3 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0
\end{bmatrix}.
\quad (5.2)

Now multiplying row vector \( \begin{bmatrix} \tilde{P}_3 \end{bmatrix} \), see (2.7), on the right by a 7-column matrix (null-space) \( \begin{bmatrix} \text{ker} \tilde{D}_3 P_1(0, 0) \end{bmatrix} \); we obtain the well-known result

\[
\begin{bmatrix}
1 & x & y & x^2 & xy & y^2 & x^3 & x^2y & xy^2 & y^3 \\
1 & x & y & x^2 & xy & y^2 - x^2 & 3xy^2 - x^3 & y^3 - 3x^2y
\end{bmatrix}.
\quad (5.3)
\]
Example 2  The polynomial $P_2(x, y) := -x^2 - iy$ that induces the operator of this example
$L_2 := \frac{\partial^2}{\partial x^2} - \frac{\partial}{\partial y}$ is not homogeneous. The matrix $\tilde{D}_3 P_2(0, 0)$ is of the form

$$
\begin{bmatrix}
0 & 0 & -1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & 6 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -2 & 0 & 2 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -3 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
$$

and a basis of the corresponding polynomial space $P_2(-iD) \cap \Pi_{\leq 3}$ is

$$
\{1, x, x^2 + 2y, x^3 + 6xy\}.
$$

Example 3  The third example taken from the paper [7] is interesting from two points of view. Namely we have a system of two operators: an elliptic (the Laplace operator) $L_3 := \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$ and hyperbolic $L_4 := \frac{\partial}{\partial x} \frac{\partial}{\partial y} + \frac{\partial}{\partial x} \frac{\partial}{\partial z} + \frac{\partial}{\partial y} \frac{\partial}{\partial z}$. And it is a 3D example.

Since, in [7], the 3rd degree polynomials are considered only; therefore, we use the last (block) columns $D_3 P_i(0, 0) := 
\begin{bmatrix}
D^0_3 P_i(0, 0) \\
\vdots \\
D^3_3 P_i(0, 0)
\end{bmatrix}, i = 3, 4, \text{ (see (3.1), (3.2))}$ of the matrices

$\tilde{D}_3 P_3(0, 0), \tilde{D}_3 P_4(0, 0)$, where $P_3, P_4$ are the algebraic polynomials that corresponds to the operators $L_3, L_4$, respectively. (It is possible because of affinely invariance of the polynomial solution.) Below we present the following $6 \times 10$ matrix (because other blocks are zero matrices):

$$
\begin{bmatrix}
D^1_3 P_3(0, 0) \\
D^1_3 P_4(0, 0)
\end{bmatrix} := 
\begin{bmatrix}
6 & 0 & 2 & 0 & 0 & 0 & 0 & 2 & 0 & 0 \\
0 & 2 & 0 & 6 & 0 & 0 & 0 & 0 & 2 & 0 \\
0 & 0 & 0 & 0 & 2 & 0 & 2 & 0 & 0 & 6 \\
0 & 2 & 0 & 0 & 2 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 2 & 0 & 0 & 1 & 2 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 2 & 0 & 0
\end{bmatrix} \quad \text{(5.4)}
$$

Using the null-space of matrix (5.4), we get a basis of the space $P_3(-iD) \cap P_4(-iD) \cap \Pi_3$:

$$
\{3x^2 y - 3x^2 z - y^3 + z^3, -x^3 + 3x^2 y + 3xy^2 - 6xyz - 2y^3 + 3y^2z, \\
-2x^3 + 3x^2 y + 3xy^2 - 6xyz + 3xz^2 - y^3, \\
x^3 + 3x^2 y - 3x^2 z - 3xy^2 - y^3 + 3y^2z\}.
$$

Note that, in the paper [7], another basis is presented. But it is not hard to see that our own and Pedersen’s, see [7], basis are bases of the same space.

Secondly we present two examples, where other (not the origin) roots of polynomials are used.
**Example 4** This example is taken from the paper [10]. Since the null-space of the symbol of 2D Laplace operator \((5.1)\) is a 2D, if we shall consider the null-space in \(\mathbb{C}^2\), manifold; we can take another root of \(P_1\). Here we take, as an example, a root \((1, i)\); and we obtain the following matrix

\[
\tilde{D}_3 P_1(1, i) :=
\begin{bmatrix}
0 & -2i & 2 & -2 & 0 & -2 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -4i & 2 & 0 & -6 & 0 & -2 & 0 & -6 \\
0 & 0 & 0 & 0 & 0 & -2i & 4 & 0 & -2 & 0 & -6 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -6i & 2 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -4i & 4 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2i & 6 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

and subspace of Laplace’s operator null-space:

\[
e^{ix - iy} \Pi_{\leq 3} \cap \ker L_1 = e^{ix - iy} \text{span} \{1, x + iy, x^2 + 2ixy - y^2, x^3 + 3ix^2y - 3xy^2 - iy^3\}.
\] (5.5)

Note the real or imaginary parts of the polynomials in \((5.5)\) (cf. \((5.3)\) and \((5.5)\)) multiplied by an exponential, do not become solutions of the Laplace operator.

**Example 5** This example is also taken from the paper [10]. Namely we shall consider Laplace operator \((5.1)\) shifted by a vector \((1, i)\):

\[
L_5 := \left( \frac{\partial}{\partial x} - i \right)^2 + \left( \frac{\partial}{\partial y} - i \right)^2,
\]

where \(I\) is the identity operator.

However the corresponding polynomial \(P_5\) will be of the form

\[
P_5(x, y) := (ix - 1)^2 + (iy - i)^2
\]

and the matrix \(\widetilde{D}_3 P_5(-1, 1)\) coincides with matrix \((5.2)\). Thus we have

\[
e^{x + iy} \Pi_{\leq 3} \cap L_5 = e^{x + iy} (\Pi_{\leq 3} \cap L_1),
\]

where \(L_1\) is Laplace operator \((5.1)\).

### 5.2 PDE with polynomial right-hand side

Finally we discuss PDE’s with polynomial (multiplied, in general, by an exponential) right-hand sides.
Example 6  Consider the Helmholtz operator

\[ L_5 := \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} - I, \]  

(5.6)

where I is the identity operator; also consider PDE with a polynomial right-hand side

\[ F(x, y) := 2 + 3x - 2xy + y^2 = \left[ \tilde{P}_2(x, y) \right] \begin{bmatrix} 2 & 3 & 0 & 0 & -2 & 1 \end{bmatrix}^T. \]  

(5.7)

Let \((x_0, y_0) := (0, 0)\). However the origin is not a root of the corresponding polynomial

\[ P_5(x, y) := -x^2 - y^2 - 1; \]  

(5.8)

and, according to Theorem 4.6, we can take \(l = 2\). So the linear algebraic equation is of the form

\[
\begin{bmatrix}
-1 & 0 & 0 & 2 & 0 & 2 \\
0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 \\
\end{bmatrix}
\begin{bmatrix}
v_1 \\
v_2 \\
v_3 \\
v_4 \\
v_5 \\
v_6 \\
\end{bmatrix}
= \begin{bmatrix} 2 \\ 3 \\ 0 \\ 0 \\ -2 \\ 1 \end{bmatrix}, \quad v \in \mathbb{C}^{\tilde{d}(2)}.  
\]  

(5.9)

Since the matrix \(\tilde{D}_2 P_5(0, 0)\) in the left-hand side of (5.9) is not singular, it follows that PDE \(L_5 \cdot F = \) has only the unique polynomial solution

\[ 2xy - 3x - y^2 - 4. \]

Example 7  Here we consider Helmholtz operator (5.6) and polynomial (5.7) again, but take another point \((x_0, y_0) = (i, 0)\) that is a root of polynomial (5.8). Since \(\text{deg } F = 2\) and \(|\alpha| = 1\), we shall use \(l = 3\); and the dimension of polynomial space to solve PDE

\[ L_5 (e^{-ix \cdot \cdot \cdot}) = e^{-ix} \left( -2xy + 3x + y^2 + 2 \right) \]  

(5.10)

is \(\tilde{d}(3) - \tilde{d}(2) = d(3) = 4\).

Since the linear algebraic system \(\tilde{D}_3 P_5(i, 0)v = \begin{bmatrix} 2 & 3 & 0 & 0 & -2 & 1 \end{bmatrix}^T, v \in \mathbb{C}^{\tilde{d}(2)},\) is undetermined and consistent; using the standard technics, we obtain the solution of PDE (5.10)

\[ v_1 (3xy + y^3) + v_2 (x + y^2) + v_3 y + v_4 \left( -\frac{x^2 y}{2} + x^2 + \frac{xy^2}{2} - \frac{xy}{2} - 2y^2 \right), \]

\[ v_1, v_2, v_3, v_4 \in \mathbb{C}. \]

Example 8  In this example, we consider Poisson’s equation. However we use a point \((1, 1)\) that is not root of \(P_1\). We solve the following PDE:

\[ L_1 (e^{ix+iy \cdot \cdot \cdot}) = e^{ix+iy}(3 + x - 2y), \]  

(5.11)
where $L_1$ is Laplace operator (5.1). The value $l = 1$ will suffice and the linear system is very simple
\[
\begin{bmatrix}
-2 & 2i & 2i \\
0 & -2 & 0 \\
0 & 0 & -2
\end{bmatrix} \begin{bmatrix} v 
\end{bmatrix} = \begin{bmatrix}
3 \\
1 \\
-2
\end{bmatrix}
\]
and the unique polynomial solution to PDE (5.11) is of the form
\[-\frac{x}{2} + y - \left(\frac{3}{2} - \frac{i}{2}\right).\]

A Proof of Theorem 3.5

First consider the matrix $D_m^0 f(x_0)$, $m = 1, \ldots, l$. We see that $D_m^0 f(x_0)$ is a row vector $[(-i)^m D^\alpha f(x_0)]$ ($1 \leq q \leq d(m)$). Consequently the matrix $D_m^0 f(x_0)$ has non-zero rank iff there exists at least one multi-index $\alpha \in A_m$ such that $D^\alpha f(x_0) \neq 0$.

Secondly consider the matrices $D_m^m f(x_0)$, $m' = 1, \ldots, m - 1$, $m = 1, \ldots, l$. Note that $D_m^m f(x_0)$ is a $d(m') \times d(m)$ matrix. By $\varrho_\alpha \varrho_\beta$, $1 \leq q \leq d(m')$, $1 \leq r \leq d(m)$, denote the difference $\varrho_\alpha - \varrho_\beta$, where $\{\varrho_\alpha, \ldots, d(m')\} = A_m$ and $\{\varrho_\beta, \ldots, d(m')\} = A_m'$. Define auxiliary $d(m') \times d(m)$ matrices $G_m^m$, $m' = 1, \ldots, m - 1$, $m = 1, \ldots, l$, as follows
\[
[G_m^m]_{qr}, 1 \leq q \leq d(m'), 1 \leq r \leq d(m) = \varrho_\gamma.
\]
We suppose that some entries of the matrix $G_m^m$ do not belong to $\mathbb{Z}_{\geq 0}^d$, i.e., a tuple $\varrho_\gamma$ can contain negative components.

Lemma A.1. Any multi-index $\gamma \in A_{m-m'}$ appears once in each row and no more than once in each column of the matrix $G_m^m$. However not every column of $G_m^m$ contains the multi-index $\gamma$.

Proof of Lemma A.1. For any multi-index $\gamma \in A_{m-m'}$ and any row $q$, $q = 1, \ldots, d(m')$, of the matrix $G_m^m$, we can always define a $d$-tuple $\alpha \in \mathbb{Z}_{\geq 0}^d$ as $\alpha := \gamma + \varrho_\beta$, where $\varrho_\beta \in A_m'$. Since $\varrho_\beta, \gamma \in \mathbb{Z}_{\geq 0}^d$ and $|\varrho_\beta| = m'$, $|\gamma| = m - m'$; therefore, $\alpha \in \mathbb{Z}_{\geq 0}^d$, $|\alpha| = m$. Consequently there exists a unique (column) number $r \in \{1, \ldots, d(m')\}$ such that $\alpha = \varrho_\gamma \in A_m$.

Fix a column number $r \in \{1, \ldots, d(m')\}$. Consider some $\gamma \in A_{m-m'}$ and define a $d$-tuple $\beta$ as $\beta := \varrho_\gamma - \varrho_\gamma$, where $\varrho_\gamma \in A_m$. If $\gamma \leq \varrho_\gamma$, then $\beta = \varrho_\beta \in A_m$ for a unique (row) number $q \in \{1, \ldots, d(m')\}$, else the column $r$ does not contain the multi-index $\gamma$. \hfill \Box

Lemma A.2. For $m' < m$, let $\{\varrho_\beta, \ldots, d(m')\} = A_{m'}$, $\{\varrho_\gamma, \ldots, d(m-m')\} = A_{m-m'}$. Consider an entry $\varrho_\gamma$ of the matrix $G_m^m$ and suppose that $\varrho_\gamma$ is equal to a multi-index $\varrho_\beta \in A_{m-m'}$, $j \in \{1, \ldots, d(m-m')\}$; then any entry of the $r$th column of $G_m^m$ below than $\varrho_\gamma$, i.e., the entry $\varrho_\gamma$, $q' \in \{q+1, \ldots, d(m')\}$, belongs either to the set of the multi-indices $\{\varrho_\gamma, \ldots, \varrho_{m-m'}\}$ or does not belong to $\mathbb{Z}_{\geq 0}^d$.

Proof of Lemma A.2. Since the multi-indices $\varrho_\beta, q = 1, \ldots, d(m')$, are lexicographically ordered, i.e., $1\beta <_{\text{lex}} 2\beta <_{\text{lex}} \cdots <_{\text{lex}} d(m')\beta$; using (A.1), we obtain the order
\[
\varrho_1 \gamma >_{\text{lex}} \varrho_2 \gamma >_{\text{lex}} \cdots >_{\text{lex}} \varrho_{d(m')} \gamma.
\]

(A.2)
Consider any \( q' \gamma, q' \in \{ q + 1, \ldots, d(m') \} \), assume the converse: \( q' \gamma \notin \{ \gamma, \ldots, q^{-1} \gamma \} \) and \( q' \gamma \in \mathbb{Z}^d_{\geq 0} \). Then there exists a unique number \( j' \in \{ j, \ldots, d(m - m) \} \) such that \( j' \gamma = q' \gamma \). Since \( j' \geq j \), we have \( j' \gamma = q' \gamma \geq \text{lex} \gamma = q \gamma \). By (A.2), we get \( q' \leq q \). This contradiction concludes the proof. \( \square \)

Finally let us prove the theorem.

Proof of Theorem 3.5. Obviously that at least one non-zero derivative \( \partial^\gamma f(x_0), |\gamma| = m - m' \), is necessary for full rank of the matrix \( D^m f(x_0) \).

Sufficiency. For some \( \gamma \in \mathcal{A}_{m-m'}, j \in \{ 1, \ldots, d(m - m') \} \), suppose that the derivative \( \partial^\gamma f(x_0) \neq 0 \). By Lemma A.1 from the matrix \( D^m f(x_0) \), we can take a \( d(m') \times d(m') \) submatrix with the non-zero main diagonal \( (D^\gamma f(x_0), \ldots, D^\gamma f(x_0)) \) and we denote this submatrix by \( S_j \).

Suppose, for \( \gamma \in \mathcal{A}_{m-m'}, D^\gamma f(x_0) \neq 0 \); then, by Lemma A.2 and property (2.1), all the entries below the main diagonal of the submatrix \( S_1 \) vanish. So the matrix \( S_1 \) is not singular, consequently \( \operatorname{rank} D^m f(x_0) = d(m') \).

Otherwise, \( D^\gamma f(x_0) = 0 \). Suppose there exists a multi-index \( \gamma \in \mathcal{A}_{m-m'}, j \in \{ 2, \ldots, d(m - m') \} \), such that \( D^\gamma f(x_0) \neq 0 \) and all the derivatives \( \partial^{j' \gamma} f(x_0), j' = 1, \ldots, j - 1 \), vanish. Since all the entries below the main diagonal of the matrix \( S_j \) vanish, it follows that \( \det S_j \neq 0 \). This concludes the proof of the sufficiency. \( \square \)

**B Proof of Proposition 3.9**

Proof. Suppose a vector \( v \in \mathbb{C}^{\tilde{d}(l)} \) belongs to \( \ker \tilde{D}_l \), then we have the following system of equalities
\[
\sum_{j=1}^{\tilde{d}(l)} \binom{i_\alpha}{q_\beta} D^{\alpha - q_\beta} f_r(x_0) v_j = 0, \quad r = 1, \ldots, n, \quad q = 1, \ldots, d(l), \quad i_\alpha, q_\beta \in \tilde{A}_l. \tag{B.1}
\]
Consider some \( \tau \in \mathbb{Z}^d_{\geq 0}, |\tau| = 1 \), with a non-zero component \( \tau_s = 1, s \in \{ 1, \ldots, d \} \); then, for multi-indices \( q_\beta, i_\alpha \) such that \( \tau \leq q_\beta, \tau \leq i_\alpha \), we have
\[
\binom{i_\alpha}{q_\beta} = \binom{i_\alpha}{q_\beta} \binom{i_\alpha - \tau}{q_\beta - \tau},
\]
where \( i_\alpha_s, q_\beta_s \) are the \( s \)th components of \( i_\alpha, q_\beta \), respectively. For some \( j' \alpha, q' \beta \) such that \( \tau \leq j' \alpha \) and \( \tau \leq q' \beta \), the binomial coefficients \( \binom{j' \alpha - \tau}{q' \beta - \tau} \) vanish; thus, by (B.1), for all \( q_\beta \in \mathcal{A}_m \) such that \( \tau \leq q_\beta \), we have
\[
\sum_{j \in \{ 1, \ldots, d(l) \}, \tau \leq i_\alpha} \binom{i_\alpha}{q_\beta} \binom{i_\alpha - \tau}{q_\beta - \tau} D^{i_\alpha - q_\beta} f_r(x_0) v_j = 0. \tag{B.2}
\]
In the previous formula, renumbering the indices \( j \) and \( q \) one after another, since a lexicographical order respects subtraction; we can write

\[
\sum_{j=1}^{\tilde{d}(l-1)} \left( \begin{array}{c}
\sigma(j) \\
\sigma(q)
\end{array} \right) D^{\sigma(j)\alpha - \sigma(q)\beta} f_r(x_0) \tilde{v}_j = 0, \\
\begin{cases}
1\alpha, \ldots, \tilde{d}(l-1)\alpha = \tilde{A}_{l-1}, \\
1\beta, \ldots, d(m-1)\beta = A_{m-1}.
\end{cases}
\]

(B.3)

where

\[
\tilde{v}_j := \sigma(j)\alpha_{s} v_{\sigma(j)}
\]

(B.4)

and \( \sigma \)s stand for the permutations of multi-indices. (Note that \( \sigma(j) \) and \( \sigma(q) \) in formula (B.3) are different permutations.)

Suppose that the vector \( v \) contains at least one non-zero component \( v_t \neq 0, \) \( t \in \{1, \ldots, d(l)\} \).

In formula (B.2), taking some \( \tau \in \mathbb{Z}^{d}_{\geq 0}, |\tau| = 1, \) such that \( \tau \leq t \alpha \); then sums in (B.2) and (B.3) contain the non-zero component \( v_t \).

Now it is not hard to see that the vector \( \tilde{v} \in \mathbb{C}^{d(l-1)} \) is a non-zero vector from the null-space of \( \tilde{D}_{l-1} f(x_0) \).

\[\square\]

C  Proof of Theorem 4.2

Proof. Suppose a polynomial \( p \in \mathcal{V}_l \). Then there exists a vector \( v \in \ker \tilde{D}_l P(x_0) \) such that \( p = \left[ \tilde{P}_l \right] v \). Obviously,

\[
p \in \Pi_{\leq l}.
\]

(C.1)

The matrix \( \tilde{D}_{l+1} P(x_0) \) can be presented as a block matrix

\[
\tilde{D}_{l+1} P(x_0) := \begin{bmatrix}
\tilde{D}_l P(x_0) & D^0_{l+1} P(x_0) \\
0 & \vdots & \vdots & \vdots \\
0 & D^0_{l+1} P(x_0) & 0 & D^{l+1}_{l+1} P(x_0)
\end{bmatrix},
\]

(C.2)

where the submatrices \( D^0_{l+1}, \ldots, D^{l+1}_{l+1} \) are given by (3.2). Introduce an auxiliary column vector as follows \( v^\sharp := \begin{bmatrix} v^T \\ 0 \\ \vdots \\ 0 \\ \underbrace{0 \ldots 0}_{d(l-1)} \end{bmatrix}^T \).

Then, using block form (C.2) of the matrix \( \tilde{D}_{l+1} P(x_0) \), we get \( v^\sharp \in \ker \tilde{D}_{l+1} P(x_0) \). Since \( p = \left[ \tilde{P}_{l+1} \right] v^\sharp \), it follows that \( p \in \mathcal{V}_{l+1} \); and, by (C.1), we obtain \( p \in \Pi_{\leq l} \cap \mathcal{V}_{l+1} \). Thus we have \( \mathcal{V}_l \subseteq \Pi_{\leq l} \cap \mathcal{V}_{l+1} \).

Contrary. Suppose a polynomial \( p \in \Pi_{\leq l} \cap \mathcal{V}_{l+1} \). Since \( p \in \Pi_{\leq l} \), it follows that the polynomial can be presented as follows \( p = \left[ \tilde{P}_{l+1} \right] v^\sharp \), \( v^\sharp := \begin{bmatrix} v^T \\ 0 \\ \vdots \\ \underbrace{0 \ldots 0}_{d(l-1)} \end{bmatrix}^T \), where \( v \in \mathbb{C}^{d(l)} \). Since \( v^\sharp \in \ker \tilde{D}_{l+1} P(x_0) \) and, arguing as above; \( v \in \ker \tilde{D}_l P(x_0) \). Thus \( p \in \mathcal{V}_l \); i.e., we have \( \mathcal{V}_l \supseteq \Pi_{\leq l} \cap \mathcal{V}_{l+1} \).

This concludes the proof. \[\square\]
References

[1] Abramov S., Petkovšek M. On polynomial solutions of linear partial differential and $(q)$-difference equations. In: Computer Algebra in Scientific Computing. 14th International workshop LNCS 7442, 1–11 (2012)

[2] de Boor C. The polynomials in the linear span of integer translates of a compactly supported function. Constr. Approx. 3, 199–208 (1987)

[3] Dahmen W., Micchelli Ch. Translates of multivariate splines. Linear Algebra Appl. 52, 217–234 (1983)

[4] Gel’fand I.M., Shilov G.E. Generalized Functions, Vol. 1: Properties and Operators. AMS Chelsea Publishing (2016)

[5] Horvás J. Basic sets of polynomial solutions for partial differential equations. Proc. Amer. Math. Soc. 9, 569–575 (1958)

[6] Kolmogorov A.N., Fomin S.V. Elements of the Theory of Functions and Function Analysis. Dover Books, New York (1957)

[7] Pedersen P. A basis for polynomial solutions to systems of linear constant coefficient PDE’s. Adv. Math. 117, 157–163 (1996)

[8] Reznick B. Homogeneous polynomial solutions to constant coefficient PDE’s. Adv. Math. 117, 179–192 (1996)

[9] Smith S.P. Polynomial solutions to constant coefficient differential equations. Trans. Amer. Math. Soc. 329, 551–569 (1992)

[10] Zakharov V.G. Reproducing solutions to PDEs by scaling functions. Int. J. Wavelets Multiresolut. Inf. Process. 3, 2050017 (2020)