On the geodesic form of non-relativistic dynamic equations

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Abstract. It is shown that any second order dynamic equation on a configuration bundle $Q \to \mathbb{R}$ of non-relativistic mechanics is equivalent to a geodesic equation with respect to a (non-linear) connection on the tangent bundle $TQ \to Q$. The case of quadratic dynamic equations is analyzed in detail. The equation for Jacobi vector fields is constructed and investigated by the geometric methods.

1 Introduction

We are concerned with non-relativistic mechanics on a configuration bundle $Q \to \mathbb{R}$, where $\mathbb{R}$ is the time axis. The corresponding velocity phase space is the first order jet manifold $J^1Q$ of sections of $Q \to \mathbb{R}$. A second order dynamic equation (called further simply a dynamic equation) on a fibre bundle $Q \to \mathbb{R}$ is defined as a first order dynamic equation on the jet bundle $J^1Q \to \mathbb{R}$, given by a holonomic connection $\xi$ on $J^1Q \to \mathbb{R}$ which takes its values in the second order jet manifold $J^2Q \subset J^1QJ^1Q$ (see, e.g., [11, 12, 13, 15]). This connection $\xi$ is also called a semispray vector field [12], a SODE field [2], a special vector field [6] because of the canonical imbedding $J^1J^1Q \to T^1J^1Q$.

The fact that $\xi$ is a flat connection places a limit on the geometric analysis of non-relativistic dynamic equations. Nevertheless, it was proved that every dynamic equation $\xi$ defines a connection $\gamma$ on the affine jet bundle $J^1Q \to Q$, and *vice versa* [3, 4, 12, 13]. For the sake of simplicity, we call $\gamma$ a dynamic connection, but this is not the terminology of [15], where this term stands for a linear connection on the tangent bundle $T^1J^1Q \to J^1Q$ associated with each dynamic equation $\xi$ too [3, 4, 13]. Here, we show that, due to the canonical imbedding $J^1Q \to TQ$, every dynamic connection yields a (non-linear) connection on the tangent bundle $TQ \to Q$, and *vice versa*. As a consequence, every dynamic equation on $Q$ gives rise to an equivalent geodesic equation on the tangent bundle $TQ \to Q$ in accordance with the following Proposition.

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Proposition 1. Given a configuration bundle $Q \to \mathbb{R}$ coordinated by $(q^0 = t, q^i)$ and its second order jet manifold $J^2Q$ coordinated by $(q^\lambda, q^i, q^i_\lambda)$, any dynamic equation

$$q^i_{tt} = \xi^i(t, q^j, q^j_t)$$

of non-relativistic mechanics on $Q \to \mathbb{R}$ is equivalent to the geodesic equation

$$\ddot{q}^0 = 0, \quad \ddot{q}^0 = 1,$$

$$\ddot{q}^i = \tilde{K}^i_0 + \tilde{K}^i_j \dot{q}^j$$

with respect to a connection $\tilde{K}$ on $TQ \to Q$ which fulfills the conditions

$$\tilde{K}^a_\lambda = 0, \quad \xi^i = \tilde{K}^i_0 + q^j \tilde{K}^i_j \big|_{q^a = 1, \dot{q}^i = q^i_t}.$$  \hspace{1cm} (3)

Remark 1. Recall that, in conservative mechanics, a second order dynamic equation on a configuration manifold $M$ is defined as a particular holonomic vector field $\Xi$ on the tangent bundle $TM$. This dynamic equation yields a connection on $TM \to M$, but fails to be a geodesic equation in general [16]. Nevertheless, every second order dynamic equation on $M$ gives rise to a dynamic equation on the fibre bundle $\mathbb{R} \times M \to \mathbb{R}$ (see Remark 2), and can be written as a geodesic equation in accordance with Proposition 1. Since a configuration bundle $Q \to \mathbb{R}$ is trivial, the existent formulations of non-relativistic mechanics often imply its preliminary splitting $Q = \mathbb{R} \times M$, and cannot be a repetition of mechanics on $\mathbb{R} \times M$, but implies additionally a connection on $Q \to \mathbb{R}$ which is a non-relativistic reference frame [13, 17] (see, e.g., Proposition 5). Proposition 1 shows that, considered independently on a trivialization of $Q \to \mathbb{R}$, non-relativistic dynamic equations make the geometric sense of geodesic equations. Treated in such a way, non-relativistic dynamic equations can be examined by means of the differential geometric methods. For instance, the curvature of the connection $\tilde{K}$ is called into play (see Propositions 11, 13, 14).

Using Proposition 1, we examine quadratic dynamic equations in details. In this case, the corresponding dynamic connection $\gamma$ on $J^1Q \to Q$ is affine, while the connection $\tilde{K}$ (3) on $TQ \to Q$ is linear. Then the equation for Jacobi vector fields along the geodesics of the connection $\tilde{K}$ can be considered. This equation coincides with the existent equation.
for Jacobi fields of a Lagrangian system \[3, 13\] in the case of non-degenerate quadratic Lagrangians, when they can be compared. We will consider more general case of quadratic Newtonian systems characterized by a pair \((\xi, m)\) of a quadratic dynamic equation \(\xi\) and a Riemannian mass tensor \(m\) which satisfy a certain compatibility condition. Given a reference frame, a Riemannian mass tensor \(m\) is extended to a Riemannian metric on the configuration space \(Q\). Then conjugate points of solutions of the dynamic equation \(\xi\) can be examined in accordance with the well-known geometric criteria.

2 Technical preliminaries

A configuration bundle \(Q \to \mathbb{R}\) of non-relativistic mechanics throughout is coordinated by \((t, q^i)\), where \(t\) is a Cartesian coordinate on the time axis \(\mathbb{R}\) with the transition functions \(t' = t + \text{const}\). We will use the compact notation \((q^{\lambda=0} = t, q^i)\), \(\partial_\lambda = \partial/\partial q^\lambda\), \(\hat{\partial}_\lambda = \partial/\partial \dot{q}^\lambda\). The velocity phase space \(J^1Q\) is provided with the adapted coordinates \((q^\lambda, q^i_t)\).

Recall that the first order jet manifold \(J^1Q\) comprises the equivalence classes \(j^1c\) of sections of \(Q \to \mathbb{R}\) which are identified by their values \(c^i(t)\) and the values of their partial derivatives \(\partial_t c^i(t)\) at points \(t \in \mathbb{R}\), i.e., \(q^i_t(j^1c) = \partial_t c^i(t)\) (see, e.g., [10, 7, 14, 18]). There is the canonical imbedding

\[
\lambda_1 : J^1Q \rightarrow TQ, \quad \lambda_1 = d_t = \partial_t + q^i_t \partial_i, \tag{4}
\]

where \(d_t\) denotes the total derivative. From now on, we will identify \(J^1Q\) with its image in \(TQ\). This is an affine bundle modelled over the vertical tangent bundle \(VQ\) of \(Q \to \mathbb{R}\).

As a consequence of (4), every connection

\[
\Gamma : Q \to J^1Q, \quad \Gamma = dt \otimes (\partial_t + \Gamma^i \partial_i), \tag{5}
\]

on a fibre bundle \(Q \to \mathbb{R}\) is identified with the nowhere vanishing vector field

\[
\Gamma : Q \to J^1Q \subset TQ, \quad \Gamma = \partial_t + \Gamma^i \partial_i, \tag{6}
\]

on \(Q\) \[13, 14\]. This is the horizontal lift of the standard vector field \(\partial_t\) on \(\mathbb{R}\) by means of the connection \(\Gamma\). Conversely, any vector field \(\Gamma\) on \(Q\) such that \(dt|\Gamma = 1\) defines a connection on \(Q \to \mathbb{R}\). Accordingly, the covariant differential associated with a connection \(\Gamma\) on \(Q \to \mathbb{R}\) reads

\[
D_\Gamma : J^1Q \rightarrow VQ, \quad \dot{q}^i \circ D_\Gamma = q^i_t - \Gamma^i. \tag{7}
\]
By $J^1 J^1 Q$ is meant the first order jet manifold of the jet bundle $J^1 Q \to \mathbb{R}$, coordinated by $(q^\lambda, q^i_t, q^i_t t)$. The second order jet manifold $J^2 Q$ of the fibre bundle $Q \to \mathbb{R}$ is the holonomic subbundle $q^i_t = q^i_{(t)}$ of $J^1 J^1 Q$, coordinated by $(q^\lambda, q^i_t, q^i_t t)$. There are the imbeddings

$$J^2 Q \xrightarrow{\lambda_2} T J^1 Q \xrightarrow{T \lambda_1} T^2 Q,$$

$$\lambda_2 : (q^\lambda, q^i_t, q^i_t t) \mapsto (q^\lambda, q^i_t, \dot{q}^0 = 1, \dot{q}^i = q^i_t, \ddot{q}^i = q^i_{tt}).$$

(8)

$$T \lambda_1 \circ \lambda_2 : (q^\lambda, q^i_t, q^i_t t) \mapsto (q^\lambda, \dot{q}^0 = 1, \dot{q}^i = q^i_t, \ddot{q}^0 = 0, \ddot{q}^i = q^i_{tt}).$$

(9)

where $(q^\lambda, \dot{q}^\lambda, \ddot{q}^\lambda)$ are holonomic coordinates on $T^2 Q$. By $J^1 J^1 Q$ is meant the first order jet manifold of the affine jet bundle $J^1 J^1 Q \to Q$. The adapted coordinates on $J^1 J^1 Q$ are $(q^\lambda, q^i_t, q^i_{tt})$.

3 Geometry of non-relativistic mechanics

This Section is devoted to the proof of Proposition 1.

As was mentioned above, a dynamic equation on a configuration bundle $Q \to \mathbb{R}$ is defined as the geodesic equation $\text{Ker} D^\xi \subset J^2 Q$ for a holonomic connection $\xi$ on the jet bundle $J^1 Q \to \mathbb{R}$. It is given by the coordinate expression (1). Due to the morphism (8), a holonomic connection $\xi$ is represented by the horizontal vector field on $J^1 Q$

$$\xi = \partial_t + q^i_t \partial_i + \xi^i (q^\mu_t, q^i_t) \dot{q}^\mu_t.$$

(10)

Remark 2. A dynamic equation $\xi$ is said to be conservative if there exists a trivialization $Q \cong \mathbb{R} \times M$ such that the vector field $\xi$ (10) on $J^1 Q \cong \mathbb{R} \times TM$ is projectable onto $TM$. Then this projection

$$\Xi_\xi = \dot{q}^i \partial_i + \xi^i (q^\mu_t, \dot{q}^\mu_t) \dot{q}^\mu_t$$

(11)

is a second order dynamic equation

$$\ddot{q}^i = \Xi_\xi^i$$

(12)

on the typical fibre $M$ of $Q \to \mathbb{R}$. Conversely, every second order dynamic equation $\Xi$ (12) on a manifold $M$ can be seen as a conservative dynamic equation

$$\xi_\Xi = \partial_t + \dot{q}^i \partial_i + u^i \dot{q}^i$$

(13)

on the fibre bundle $\mathbb{R} \times M \to \mathbb{R}$. 

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Let us turn to the above mentioned relationship between the holonomic connections $\xi$ on $J^1Q \to R$ and the dynamic connections

$$\gamma = dq^\lambda \otimes (\partial_\lambda + \gamma^\lambda_i \partial^i_t)$$

(14)
on the affine jet bundle $J^1Q \to Q$ (see, e.g., [13]).

**Proposition 2.** Any dynamic connection $\gamma$ (14) defines the holonomic connection

$$\xi_\gamma = \partial_t + q^i_t \partial_i + (\gamma^i_0 + q^j_t \gamma^i_j) \partial^i_t$$

(15)
on $J^1Q \to R$. Conversely, any holonomic connection $\xi$ (11) on $J^1Q \to R$ defines the dynamic connection

$$\gamma_\xi = dt \otimes [\partial_t + (\xi^i - \frac{1}{2} q^i_t \partial^j_t \xi^j) \partial^i_t] + dq^i \otimes [\partial_j + \frac{1}{2} \partial^j_t \xi^i \partial^i_t].$$

(16)

It follows that every dynamic connection $\gamma$ (14) yields the non-relativistic dynamic equation

$$q^i_{tt} = \gamma^i_0 + q^j_t \gamma^i_j$$

(17)
on the configuration bundle $Q \to R$. Different dynamic connections may lead to the same dynamic equation (17). The dynamic connection $\gamma_\xi$ (16), associated with a dynamic equation, possesses the property

$$\gamma^k_i = \partial_i \gamma^k_0 + q^j_t \partial^j_t \gamma^k_j,$$

which implies the relation $\partial_j \gamma^k_i = \partial_i \gamma^k_j$. Such a dynamic connection is called symmetric. Let $\gamma$ be a dynamic connection (14) and $\xi_\gamma$ the corresponding dynamic equation (15). Then the connection (16), associated with $\xi_\gamma$, takes the form

$$\gamma^k_{\xi_\gamma} = \frac{1}{2} (\gamma^k_i + \partial_i \gamma^k_0 + q^j_t \partial^j_t \gamma^k_j), \quad \gamma^k_{\xi_\gamma} = \xi^k - q^i_t \gamma^k_i.$$

It is readily observed that $\gamma = \gamma_\xi$ if and only if $\gamma$ is symmetric.

Now let us turn to the proof of Proposition 4.

We start from the relation between the connections $\gamma$ (14) on the affine jet bundle $J^1Q \to Q$ and the connections

$$K = dq^\lambda \otimes (\partial_\lambda + \xi^\lambda_a \partial^a)$$

(18)
on the tangent bundle \( TQ \to Q \). Let us consider the diagram

\[
\begin{array}{ccc}
J^1Q & \overset{\lambda_1}{\longrightarrow} & J^1Q \\
\downarrow \gamma & & \downarrow \kappa \\
J^1Q & \overset{\lambda_1}{\longrightarrow} & TQ
\end{array}
\]  

(19)

where \( J^1Q \) is the first order jet manifold of the tangent bundle \( TQ \to Q \), coordinated by \((q^\lambda, \dot{q}^\lambda, \ddot{q}^\lambda)\). The jet prolongation over \( Q \) of the canonical imbedding \( \lambda_1 \) (4) reads

\[
J^1\lambda_1 : (q^\lambda, q^i_t, q^i_{\mu t}) \mapsto (q^\lambda, \dot{q}^\lambda = 1, \dot{q}^i = q^i_t, \dot{q}^0 = 0, \dot{q}^\mu = \gamma^\mu),
\]

We have

\[
J^1\lambda_1 \circ \gamma : (q^\lambda, q^i_t) \mapsto (q^\lambda, \dot{q}^\lambda = 1, \dot{q}^i = q^i_t, \dot{q}^0 = 0, \dot{q}^\mu = \gamma^\mu),
\]

\[
K \circ \lambda_1 : (q^\lambda, q^i_t) \mapsto (q^\lambda, \dot{q}^\lambda = 1, \dot{q}^i = q^i_0, \dot{q}^0 = K^0_\mu, \dot{q}^\mu = K^\mu). 
\]

It follows that the diagram (19) can be commutative only if the components \( K^0_\mu \) of the connection \( K \) on \( TQ \to Q \) vanish. Since the transition functions \( t \to t' \) are independent of \( q^i \), a connection \( \tilde{K} = dq^\lambda \otimes (\partial_\lambda + K^i_\lambda \partial_i) \) (20) with the components \( K^0_\mu = 0 \) can exist on the tangent bundle \( TQ \to Q \). It obeys the transformation law

\[
K'^i_\lambda = (\partial_j x'^\mu K^j_\mu + \partial_\mu x'^\lambda) \frac{\partial q^\mu}{\partial x'^\lambda}. 
\]

(21)

Now the diagram (19) becomes commutative if the connections \( \gamma \) and \( \tilde{K} \) fulfill the relation

\[
\gamma^i_\mu = K^i_\mu (q^\lambda, \dot{q}^\lambda = 1, \dot{q}^i = q^i_t),
\]

(22)

which holds globally since the substitution of \( \dot{q}^i = q^i_t \) into (21) restates the coordinate transformation law of \( \gamma \). In accordance with this relation, a desired connection \( \tilde{K} \) is an extension of the local section \( J^1\lambda_1 \circ \gamma \) of the affine bundle \( J^1Q \) over the closed submanifold \( J^1Q \subset TQ \) to a global section. Such an extension always exists, but is not unique. Thus, it is stated the following.

**Proposition 3.** Every non-relativistic dynamic equation (1) on the configuration bundle \( Q \to \mathbb{R} \) can be written in the form

\[
q^i_{tt} = K^i_0 \circ \lambda_1 + q^i_t K^j_\mu \circ \lambda_1, 
\]

(23)
where $\tilde{K}$ is a connection (20) on the tangent bundle $TQ \to Q$. Conversely, each connection $\tilde{K}$ (20) on $TQ \to Q$ defines the dynamic equation (23) on $Q \to R$.

Let us consider the geodesic equation (2) on $TQ$ with respect to the connection $\tilde{K}$. Its solution is a geodesic curve $c(t)$ which also satisfies the dynamic equation (1), and vice versa. It states Proposition 1.

4 Non-relativistic reference frames

Proposition 3 gives more than it is needed for Proposition 1, and we can prove a converse of Proposition 1.

Let us start from the notion of a reference frame in non-relativistic mechanics. From the physical viewpoint, a reference frame in non-relativistic mechanics on a configuration bundle $Q \to R$ sets a tangent vector at each point of $Q$ which characterizes the velocity of an "observer" at this point. Then any connection $\Gamma$ on $Q \to R$ is said to be such a reference frame [5, 13, 15, 17].

Lemma 4. [13, 14]. Each connection $\Gamma$ on a fibre bundle $Q \to R$ defines an atlas of local constant trivializations of $Q \to R$ whose transition functions are independent of $t$, and vice versa. One finds $\Gamma = \partial_t$ with respect to this atlas. In particular, there is one-to-one correspondence between the complete connections $\Gamma$ (6) on $Q \to R$ and the trivializations of this bundle.

By virtue of this Lemma, any coordinate atlas $(t, q^i)$ on $Q \to R$ whose transition functions are independent of time is also regarded as a reference frame. Using the notion of a reference frame, we can formulate a desired converse of Proposition 1.

Proposition 5. Given a reference frame $\Gamma$, any connection $K$ (18) on the tangent bundle $TQ \to Q$ defines the dynamic equation

$$\xi^i = (K^i_\lambda - \Gamma^i_\lambda K^0_\lambda)q^\lambda|_{q^\mu=1, \dot{q}^i=q^i_t}.$$  \hspace{1cm} (24)

The proof follows at once from Proposition 3 and the following assertion.

Lemma 6. Given a connection $\Gamma$ on the fibre bundle $Q \to R$ and a connection $K$ on the tangent bundle $TQ \to Q$, there is the connection $\tilde{K}$ on $TQ \to Q$ with the components

$$\tilde{K}^0_\lambda = 0, \quad \tilde{K}^i_\lambda = K^i_\lambda - \Gamma^i_\lambda K^0_\lambda.$$  

It is proved by the inspection of transition functions.
5 Quadratic dynamic equations

From the physical viewpoint, the most interesting dynamic equations are the quadratic ones, i.e.,

$$\xi^i = a^i_{jk}(q^\mu)q_j^t q_k^t + b^i_j(q^\mu)q_j^t + f^i(q^\mu). \tag{25}$$

This property is coordinate-independent due to the affine transformation law of coordinates $q^t_i$. Then, it is readily observed that the corresponding dynamic connection $\gamma_\xi$ is affine:

$$\gamma = dq^\lambda \otimes [\partial_\lambda + (\gamma^i_{\lambda 0}(q^\mu) + \gamma^i_{\lambda j}(q^\mu)q_j^t)\partial^t_i],$$

and vice versa. This connection is symmetric if and only if $\gamma^i_{\lambda \mu} = \gamma^i_{\mu \lambda}$.

**Lemma 7.** There is one-to-one correspondence between the affine connections $\gamma$ on the affine jet bundle $J^1Q \to Q$ and the linear connections $\tilde{K}$ on the tangent bundle $TQ \to Q$.

This correspondence is given by the relation (22) which takes the form

$$\gamma^i_\mu = \gamma^i_{\mu 0} + \gamma^i_{\mu j}q_j^t, \gamma^i_{\mu \lambda} = K^i_{\mu \lambda}.$$ 

In particular, if an affine dynamic connection $\gamma$ is symmetric, so is the corresponding linear connection $\tilde{K}$.

Then we come to the following corollaries of Propositions 1, 5.

**Corollary 8.** Any quadratic dynamic equation

$$q^i_{tt} = a^i_{jk}(q^\mu)q_j^t q_k^t + b^i_j(q^\mu)q_j^t + f^i(q^\mu) \tag{26}$$

is equivalent to the geodesic equation

$$\ddot{q}^0 = 0, \quad \dot{q}^0 = 1, \quad \ddot{q}^i = a^i_{jk}(q^\mu)\dot{q}^j\dot{q}^k + b^i_j(q^\mu)\dot{q}^j\dot{q}^0 + f^i(q^\mu)\dot{q}^0 \dot{q}^0. \tag{27}$$

for the symmetric linear connection $\tilde{K} = dq^\lambda \otimes (\partial_\lambda + K^\mu_{\lambda \nu}(q^\alpha)\dot{q}^\alpha \dot{\mu})$ on $TQ \to Q$, given by the components

$$K^0_{\lambda \nu} = 0, \quad K_0^i j = K^j_0 i = 1/2 b^j_i, \quad K^i j k = a^i_{jk}. \tag{28}$$
Corollary 9. Conversely, any linear connection \( K \) on the tangent bundle \( TQ \to Q \) defines the quadratic dynamic equation

\[
q_{it} = K_{j\, k} q_{it}^j q_{it}^k + (K_{0\, j} + K_{j\, 0}) q_{it}^j + K_{0\, 0},
\]

written with respect to a given reference frame \((t, q^i)\).

The geodesic equation (27) however is not unique for the dynamic equation (26).

Proposition 10. Any quadratic dynamic equation (25), being equivalent to the geodesic equation with respect to the linear connection \( \tilde{K} \) (28), is also equivalent to the geodesic equation with respect to an affine connection \( K' \) on \( TQ \to Q \) which differs from \( \tilde{K} \) (28) in a soldering form \( \sigma \) on \( TQ \to Q \) with the components

\[
\sigma_\lambda^0 = 0, \quad \sigma_k^i = h_k^i + (s - 1) h_k^0 q^0, \quad \sigma_0^i = -s h_k^i q^k - h_0^i q^0 + h_0^0,
\]

where \( s \) and \( h_\lambda^\lambda \) are local functions on \( Q \).

6 Free motion equation

Let us point out the following interesting class of dynamic equations which we agree to call the free motion equations.

We say that the dynamic equation (1) is a free motion equation if there exists a reference frame \((t, \bar{q}^i)\) on the configuration bundle \( Q \to \mathbb{R} \) such that this equation reads

\[
\bar{q}_{it}^i = 0.
\]

With respect to arbitrary bundle coordinates \((t, q^i)\), a free motion equation takes the form

\[
q_{it}^i = d_t \Gamma^i + \partial_j \Gamma^i (q_{it}^j - \Gamma^j) - \frac{\partial q^i}{\partial \bar{q}^m} \frac{\partial \bar{q}^m}{\partial q^j} (q_{it}^j - \Gamma^j) (q_{it}^k - \Gamma^k),
\]

where \( \Gamma^i = \partial_t q^i(t, \bar{q}^j) \) is the connection associated with the initial frame \((t, \bar{q}^i)\). One can think of the right hand side of the equation (30) as being the general coordinate expression of an inertial force in non-relativistic mechanics. The corresponding dynamic connection \( \gamma \) on the affine jet bundle \( J^1 Q \to Q \) reads

\[
\gamma_k^i = \partial_k \Gamma^i - \frac{\partial q^i}{\partial \bar{q}^m} \frac{\partial \bar{q}^m}{\partial q^j} \partial_k q^j (q_{it}^j - \Gamma^j), \quad \gamma_0^i = \partial_t \Gamma^i + \partial_j \Gamma^i q_{it}^j - \gamma_k^i \Gamma^k.
\]
It is affine. By virtue of Lemma 7, this dynamic connection defines a linear connection \( K \) on the tangent bundle \( TQ \to Q \) whose curvature is necessarily equal to 0. Thus, we come to the following criterion of a dynamic equation to be a free motion equation.

**Proposition 11.** If \( \xi \) is a free motion equation, it is quadratic and the corresponding linear symmetric connection (28) on the tangent bundle \( TQ \to Q \) is flat.

This criterion fails to be a sufficient condition since it may happen that the components of a curvature-free linear symmetric connection on \( TQ \to Q \) vanish with respect to the coordinates on \( Q \) which are not compatible with the fibration \( Q \to \mathbb{R} \). Nevertheless, one can formulate the necessary and sufficient condition of the existence of a free motion equation on a configuration space \( Q \).

**Proposition 12.** [2, 13]. A free motion equation on an a configuration bundle \( Q \to \mathbb{R} \) exists if and only if the typical fibre \( M \) of \( Q \) admits a curvature-free linear symmetric connection.

### 7 Quadratic Lagrangian and Newtonian systems

A Lagrangian of a mechanical system on \( Q \to \mathbb{R} \) is defined as a function on the velocity phase space \( J^1Q \). Let us consider a non-degenerate quadratic Lagrangian

\[ L = \frac{1}{2}m_{ij}(q^\mu)q^i_t q^j_t + k_i(q^\mu)q^i_t + f(q^\mu), \]  

(32)

where \( m_{ij} \) is a Riemannian fibre metric in the vertical tangent bundle \( VQ \), called a mass metric. As for quadratic dynamic equations, this property is coordinate-independent. Similarly to Lemma 7, one can show that any quadratic polynomial on \( J^1Q \subset TQ \) is extended to a bilinear form on \( TQ \). Then the Lagrangian \( L \) (32) can be written as

\[ L = \frac{1}{2}g_{\alpha\nu}q^\alpha_t q^\nu_t, \quad q^0_t = 1, \]

where \( g \) is the (degenerate) fibre metric

\[ g_{00} = 2f, \quad g_{0i} = k_i, \quad g_{ij} = m_{ij} \]

(33)

in the tangent bundle \( TQ \). The associated Lagrange equation takes the form

\[ q^{i\mu} = (m^{-1})^{ik}\{\lambda \kappa \nu \} q^\lambda_t q^\nu_t, \quad q^0_t = 1, \]

(34)
where
\[ \{\lambda_{\mu\nu}\} = -\frac{1}{2}(\partial_\lambda g_{\mu\nu} + \partial_\nu g_{\mu\lambda} - \partial_\mu g_{\lambda\nu}) \]
are the Christoffel symbols of the metric (33). The corresponding geodesic equation (27) on \( TQ \) reads
\[ \ddot{q}^0 = 0, \quad \dot{q}^0 = 1, \]
\[ \ddot{q}^i = (m^{-1})^{ik}\{\lambda_{k\nu}\}\dot{q}^\lambda\dot{q}^\nu, \quad (35) \]
where \( \tilde{K} (3) \) is a linear connection with the components
\[ \tilde{K}_{\lambda}^0_{\nu} = 0, \quad \tilde{K}_{\lambda}^i_{\nu} = (m^{-1})^{ik}\{\lambda_{k\nu}\}. \quad (36) \]
We have the relation
\[ \dot{q}^\lambda(\partial_\lambda m_{ij} + K^i_{\lambda\nu}\dot{q}^\nu) = 0. \quad (37) \]

One can show that an arbitrary Lagrangian system on a configuration bundle \( Q \to \mathbb{R} \) is a particular Newtonian system on \( Q \to \mathbb{R} \). The latter is defined as a pair \((\xi, m)\) of a dynamic equation \( \xi \) and a (degenerate) fibre metric \( m \) in the fibre bundle \( V_QJ^1Q \to J^1Q \) which satisfy the symmetry condition \( \partial_\xi k m_{ij} = \partial_j m_{ik} \) and the compatibility condition
\[ \xi]\left[d m_{ij} + m_{ik}\gamma_j^k + m_{jk}\gamma_i^k \right] = 0, \quad (38) \]
where \( \gamma_\xi \) is the dynamic connection (16) [13, 14]. Note that the compatibility condition (38) can be written in an intrinsic way as \( \nabla_\xi m = 0 \), where \( \nabla \) is the covariant derivative with respect to the canonical prolongation of the connection \( \gamma_\xi \) onto the vertical cotangent bundle \( J^*_QJ^1Q \to J^1Q \).

We will restrict our consideration to non-degenerate quadratic Newtonian systems when \( \xi \) is a quadratic dynamic equation (25) and \( m \) is a Riemannian fibre metric in \( V_Q \), i.e., \( m \) is independent of \( q_i \) and the symmetry condition becomes trivial. In this case, the dynamic equation (26) is equivalent to the geodesic equation (27) with respect a symmetric linear connection \( \tilde{K} (28) \), while the compatibility condition (38) takes the form (37).

Given a symmetric linear connection \( \tilde{K} (28) \) on the tangent bundle \( TQ \to Q \), one can consider the equation for Jacobi vector fields along geodesics of this connection, i.e., along solutions of the non-relativistic dynamic equation (26). If \( Q \) is provided with a Riemannian metric, the conjugate points of these geodesic can be investigated.
8 Non-relativistic Jacobi fields

Let us consider the quadratic dynamic equation (26) and the equivalent geodesic equation (27) with respect to the symmetric linear connection $\tilde{K}$ (28). Its curvature

$$R_{\lambda\mu}^{\alpha\beta} = \partial_\lambda K_{\mu}^{\alpha}_{\beta} - \partial_\mu K_{\lambda}^{\alpha}_{\beta} + K_{\lambda}^{\gamma\beta} K_{\mu}^{\alpha}\gamma - K_{\mu}^{\gamma\beta} K_{\lambda}^{\alpha}\gamma$$

has the temporal component

$$R_{\lambda\mu}^{0\beta} = 0. \quad (39)$$

Then the equation for a Jacobi vector field $u$ along a geodesic $c$ reads

$$\dot{q}^\beta \dot{q}^\mu (\nabla_\beta (\nabla_\mu u^\alpha) - R_{\lambda\mu}^{\alpha\beta} u^\lambda) = 0, \quad \nabla_\beta \dot{q}^\alpha = 0, \quad (40)$$

where $\nabla_\mu$ denote the covariant derivatives relative to the connection $\tilde{K}$. Due to the relation (39), the equation (40) for the temporal component $u^0$ of a Jacobi field takes the form

$$\dot{q}^\beta \dot{q}^\mu (\partial_\mu \partial_\beta u^0 + K_{\mu}^{\gamma\beta} \partial_\gamma u^0) = 0.$$

We chose its solution

$$u^0 = 0 \quad (41)$$

because all non-relativistic geodesics obey the constraint $\dot{q}^0 = 0$.

Note that, in the case of a quadratic Lagrangian $L$, the equation (40) coincides with the Jacobi equation

$$w^j d_0 (\partial_j \dot{q}_i) L + d_0 (\dot{u}^j \partial_j L) - w^j \partial_i \partial_j L = 0$$

for a Jacobi field on solutions of the Lagrange equations for $L$. This equation is the Lagrange equation for the vertical extension $L_V$ of the Lagrangian $L$ [8, 13, 14] (see also [3]).

Let us consider a quadratic Newtonian system with a Riemannian mass tensor $m_{ij}$. Given a reference frame $(t, q^i)$, this mass tensor is extended to the Riemannian metric

$$\bar{g}_{00} = 1, \quad \bar{g}_{0i} = 0, \quad \bar{g}_{ij} = m_{ij} \quad (42)$$
on $Q$. However, its covariant derivative with respect to the connection $\widetilde{K}$ (28) does not vanish in general. Nevertheless, due to the relations (37) and (41), the well-known formula

$$\int_{a}^{b} \left( \mathcal{G}_{\lambda\mu}(\dot{q}^\alpha \nabla_{\alpha} u^\lambda)(\dot{q}^\beta \nabla_{\beta} u^\mu) + R_{\lambda\mu\alpha\nu} u^\lambda u^\beta \dot{q}^\mu \dot{q}^\nu \right) dt +$$

(43)

$$\mathcal{G}_{\lambda\mu}\dot{q}^\alpha \nabla_{\alpha} u^\lambda \big|_{t=a} - \mathcal{G}_{\lambda\mu}\dot{q}^\alpha \nabla_{\alpha} u^\lambda \big|_{t=b} = 0$$

for a Jacobi vector field $u$ along a geodesic $c$ takes place. Accordingly, the following assertions also remain true [9].

**Proposition 13.** If the sectional curvature $R_{\lambda\mu\alpha\nu} u^\lambda u^\beta \dot{q}^\mu \dot{q}^\nu$ is positive on a geodesic $c$, this geodesic has no conjugate points.

**Proposition 14.** If the sectional curvature $R_{\lambda\mu\alpha\nu} u^\lambda u^\beta v^\mu v^\nu$, where $u, v$ are arbitrary unit vectors on a Riemannian manifold $Q$ less than $k < 0$, then, for every geodesic, the distance between two consecutive conjugate points is at most $\pi/\sqrt{k}$.

For instance, let us consider a one-dimensional motion described by the Lagrangian

$$L = \frac{1}{2}(\dot{q}^1)^2 - \phi(q^1),$$

where $\phi$ is a potential. The corresponding Lagrange equation is equivalent to the geodesic one on the 2-dimensional space $\mathbb{R}^2$ with respect to the connection $\widetilde{K}$ whose non-zero component is $\widetilde{K}_{0}^{0} = -\partial_{1} \phi$. The curvature of $\widetilde{K}$ has the non-zero component

$$R_{0101} = \partial_{1} \widetilde{K}_{0}^{1} = -\partial_{1}^{2} \phi.$$  

Choosing the Riemannian metric (42) as

$$g_{11} = 1, \quad g_{01} = 0, \quad g_{00} = 1,$$

we come to the formula (13)

$$\int_{a}^{b} \left[ (\dot{q}^\mu \partial_{\mu} u^1)^2 - \partial_{1}^{2} \phi(u^1)^2 \right] dt = 0.$$  

for a Jacobi vector field $u$ which vanishes at points $a$ and $b$. Then we obtain from Proposition 13 that, if $\partial_{1}^{2} \phi < 0$ at points of $c$, this motion has no conjugate points. In particular, let us consider the oscillator $\phi = k(q^1)^2/2$. In this case, the sectional curvature is $R_{0101} = -k$, while the half-period of this oscillator is exactly $\pi/\sqrt{k}$ in accordance with Proposition 14.
References

[1] Cariñena J and Fernández-Núñez J 1993 Fortschr. Phys. 41 517

[2] Crampin M, Martínez E and Sarlet W 1996 Ann. Inst. H. Poincaré 65A 223.

[3] Dittrich W and Reuter M 1992 Classical and Quantum Dynamics (Springer: Berlin)

[4] Echeverría Enríquez A, Muñoz Lecanda and Román Roy N 1991 Rev. Math. Phys. 3 301

[5] Echeverría-Enríquez A, Muñoz-Lecanda M and Román-Roy N 1995 J. Phys. A. 28 5553

[6] Giachetta G 1992 J. Math. Phys. 33 1652

[7] Giachetta G, Mangiarotti L and Sardanashvily G 1997 New Lagrangian and Hamiltonian Methods in Field Theory (Singapore: World Scientific)

[8] Giachetta G, Mangiarotti L and Sardanashvily G 1999 J. Math. Phys. 40 1376.

[9] Kobayashi S and Nomizu K 1969 Foundations of Differential Geometry, V.II (N.Y.: Interscience Publishers)

[10] Kolář I, Michor P and Slovák J 1993 Natural Operations in Differential Geometry (Berlin: Springer-Verlag)

[11] Krupkova O 1997 The Geometry of Ordinary Variational Equations (Berlin: Springer-Verlag)

[12] De León M and Rodrigues P 1989 Methods of Differential Geometry in Analytical Mechanics (Amsterdam: North-Holland)

[13] Mangiarotti L and Sardanashvily G 1998 Gauge Mechanics (Singapore: World Scientific)

[14] Mangiarotti L, Obukhov Yu and Sardanashvily G 1999 Connections in Classical and Quantum Field Theory (Singapore: World Scientific)

[15] Massa E and Pagani E 1994 Ann. Inst. Henri Poincaré 61 17
[16] Morandi G, Ferrario C, Lo Vecchio G, Marmo G and Rubano C 1990 Phys. Rep. 188 147

[17] Sardanashvily G 1998 J. Math. Phys. 39 2714

[18] Saunders D 1989 The Geometry of Jet Bundles (Cambridge: Cambr. Univ. Press)