Uniqueness in Weighted Lebesgue Spaces for an Elliptic Equation with Drift on Manifolds

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Abstract
We investigate the uniqueness, in suitable weighted Lebesgue spaces, of solutions to a class of elliptic equations with a drift posed on a complete, noncompact, Riemannian manifold $M$ of infinite volume and dimension $N \geq 2$. Furthermore, in the special case of a model manifold with polynomial volume growth, we show that the conditions on the drift term are sharp.

Keywords Uniqueness theorems · Weighted Lebesgue spaces · Riemannian manifolds · Elliptic equations with drift

Mathematics Subject Classification 35A02 · 35B53 · 35J10 · 58J05

1 Introduction

We are concerned with uniqueness of solutions, in suitable weighted Lebesgue spaces, to the following linear elliptic equation

$$\Delta u + g(b, \nabla u) - cu = 0 \quad \text{in} \ M,$$

where $M$ is an $N$-dimensional, $N \geq 2$, complete, noncompact, Riemannian manifold of infinite volume, endowed with a metric tensor $g$. Here, $\Delta$ and $\nabla$ denote, respectively, the Laplace-Beltrami operator and the gradient on $M$ with respect to $g$; $b : M \to M$ is a given vector field and $c$ is a nonnegative function defined in $M$. The huge interest around the Laplace-Beltrami operator with a drift comes from very different fields...
ranging from probability to analysis and geometry. A classical assumption that ensures the uniqueness property for harmonic functions

$$\Delta u = 0 \quad \text{in } M,$$

is to restrict $u$ to lie in some $L^p$-space. In this setting the uniqueness result for (1.2) is the following: if $u \in L^p(M)$, for some $1 \leq p \leq \infty$, then $u$ is identically constant. In two celebrated papers Yau proved the uniqueness result for $p = \infty$, under the assumption that the Ricci curvature of $M$ is non-negative, and for $p \in (1, \infty)$ without additional assumptions on $M$; we refer to [30] and to [29], respectively. These results have been extended to the case $p \in (0, 1)$ and to the case $p = 1$ under sharp lower bounds for the Ricci curvature of $M$, we refer to [11] and to [10], respectively. The simplest example of Laplace-Beltrami operator with a drift is the so-called weighted Laplacian

$$\Delta u - g(\nabla f, \nabla u) = e^f \text{div} \left( e^{-f} \nabla u \right),$$

i.e. when $b = -\nabla f$, for some smooth real-valued $f$ on $M$. In this setting of metric measure spaces $(M, g, e^{-f} d\mu)$, where $d\mu$ denotes the usual volume form induced by $g$, the aforementioned uniqueness results have been extended, e.g., in [14, 16, 17, 28] and we refer to these papers and to the references therein for further details. In this paper we consider the general equation (1.1) and we prove uniqueness results in weighted Lebesgue spaces, for the literature related to the weighted uniqueness results we refer to Subsection 1.2. Before presenting our main assumptions and results we need some notation; for all vector fields $X, Y$ belonging to the tangent space of $M$, we set

$$\langle X, Y \rangle := g(X, Y).$$

Moreover, for any $x_0 \in M$ and $R > 0$, let $B_R(x_0) := \{ x \in M : \text{dist}(x, x_0) < R \}$ denote the geodesic ball of radius $R$ and centred at $x_0$, where $\text{dist}(x, x_0)$ is the geodesic distance between $x$ and $x_0$. Furthermore, let $V(x_0, R)$ denote the Riemannian volume of $B_R(x_0)$. In what follows, we set

$$r(x) \equiv r := \text{dist}(x, x_0).$$

(1.3)

Let us define the following subsets of $M$

$$D_+ := \{ x \in M : \langle b(x), x \rangle > 0 \};$$
$$D_- := \{ x \in M : \langle b(x), x \rangle \leq 0 \}. $$

(1.4)
Then, concerning the vector field $b$, we will assume either that:

(i) $b : M \to M, \ b \in C^1(M)$;
(ii) there exist $K_0 > 0$ such that

$$|b(x)| \leq K_0 \text{ for all } x \in D_+;$$
(iii) $[\text{div } b(x)]_- \leq K_0 \text{ for all } x \in M; \tag{H_0}$

or that, for some $0 < \theta < 1$ that will be specified later,

(i) $b : M \to M, \ b \in C^1(M)$;
(ii) there exist $\sigma \leq 1 - \theta$ and $K_1 > 0$ such that

$$\langle b(x), \nabla r \rangle \leq K_1 (1 + r)^\sigma \text{ for all } x \in D_+;$$
(iii) $[\text{div } b(x)]_- \leq K_1 (1 + r)^{\sigma - 1} \text{ for all } x \in M; \tag{H_1}$

or that

(i) $b : M \to M, \ b \in C^1(M)$;
(ii) there exist $\sigma \leq 1$ and $K_2 > 0$ such that

$$\langle b(x), \nabla r \rangle \leq K_2 (1 + r)^\sigma \text{ for all } x \in D_+;$$
(iii) $[\text{div } b(x)]_- \leq K_2 (1 + r)^{\sigma - 1} \text{ for all } x \in M. \tag{H_2}$

Here, for a given function $v$, the negative part $[\cdot]_-$ is defined as

$$[v]_- := \max \{0; -v\}.$$ 

Moreover, the coefficient $c$ is such that

$$c \in C(M), \ c(x) \geq c_0 \text{ for all } x \in M, \tag{H_3}$$

for some $c_0 > 0$. We now summarize our main results and we refer to Section 2 for the precise statements.

### 1.1 Outline of Our Results

The main results of this paper will be given in detail in the forthcoming Theorems 2.2, 2.3 and 2.4. We give here a sketchy outline of these results, describing motivations and techniques of proofs.

For any $\phi \in C(M), \ \phi > 0, \ p > 1$, set

$$L^p_\phi(M) := \left\{ u : M \to \mathbb{R} \text{ measurable } \mid \int_M |u|^p \phi(x) dx < \infty \right\}. \tag{7}$$
We shall prove the following

- Let $M$ be a complete, noncompact, Riemannian manifold of infinite volume such that, for some $x_0 \in M$,

$$V(x_0, r) \leq e^{ar}, \quad \text{for some } \alpha > 0.$$  \hspace{1cm} (1.5)

If $b : M \rightarrow M$ satisfies assumption ($H_0$), then the solution to equation (1.1) is unique in the class $L^p_\psi(M)$ with $p > 1$ and

$$\psi(x) = \psi(r) := e^{-\beta r} \quad (r \in (0, +\infty)),$$  \hspace{1cm} (1.6)

for properly chosen $\beta > \alpha$ (see Theorem 2.2). Observe that, in general, our uniqueness class includes unbounded solutions. Thus, in particular, we get uniqueness of bounded solutions. More precisely, bounded solutions always belong to this class of uniqueness $L^p_\psi(M)$. Furthermore, also unbounded solutions can be considered if such solutions satisfy

$$u(x) \leq Ce^{\nu r} \quad \text{for } r > 1, \ \nu > 0,$$

provided that $\beta > \alpha + \gamma p$.

- Let $M$ be a complete, noncompact, Riemannian manifold of infinite volume such that, for some $x_0 \in M$,

$$V(x_0, r) \leq e^{ar^\theta}, \quad \text{for some } \alpha > 0, \quad \text{for some } 0 < \theta < 1.$$  \hspace{1cm} (1.7)

If $b : M \rightarrow M$ satisfies assumption ($H_1$) with $\theta$ as in (1.7), then the solution to equation (1.1) is unique in the class $L^p_\eta(M)$ with $p > 1$ and

$$\eta(r) := e^{-\beta r^\theta} \quad \text{for some } \beta > 0,$$  \hspace{1cm} (1.8)

for properly chosen $\beta > \alpha$ (see Theorem 2.3). Observe that, in particular, bounded solutions always belong to this class of uniqueness $L^p_\eta(M)$. Moreover, also unbounded solutions can be considered if such solutions satisfy

$$u(x) \leq Ce^{\nu r^\theta} \quad \text{for } r > 1, \ \nu > 0,$$

provided that $\beta > \alpha + \gamma p$.

- Let $M$ be a complete, noncompact, Riemannian manifold of infinite volume such that, for some $x_0 \in M$,

$$V(x_0, r) \leq r^\alpha, \quad \text{for some } \alpha > 0.$$  \hspace{1cm} (1.9)

If $b : M \rightarrow M$ satisfies assumption ($H_2$), then the solution to equation (1.1) is unique in the class $L^p_\xi(M)$ with $p > 1$ and

$$\xi(x) = \xi(r) := (1 + r)^{-\tau} \quad (r \in (0, +\infty)),$$  \hspace{1cm} (1.10)
for properly chosen $\tau > \alpha + N - 1$ (see Theorem 2.4). Similarly to the previous case, we observe that, our uniqueness class includes unbounded solutions. Thus, in particular, we get uniqueness of bounded solutions. More precisely, bounded solutions always belong to this class of uniqueness $L^p_{\xi}(M)$. Moreover, also unbounded solutions can be considered if such solutions satisfy

$$u(x) \leq Cr^\gamma \text{ for } r > 1, \gamma > 0,$$

provided that $\tau > \alpha + \gamma p + N - 1$.

Observe that hypotheses $(H_1)$ and $(H_2)$ allow $b$ to be unbounded. In addition, such assumptions require a growth condition on $b$, when $\langle b(x), x \rangle > 0$. However, when $\langle b(x), x \rangle \leq 0$ no further conditions on $b$ are imposed.

- Furthermore, we show sharpness of the result of Theorem 2.4 in the special case of a model manifold $M$ such that (2.1) holds. More precisely, we show that, if the drift term $b$ violates, in an appropriate sense, assumption $(H_2)$ (that is, if it satisfies inequality (2.2) below), then infinitely many bounded solutions to problem (1.1) exist (see Proposition 2.5 and Corollary 2.6 below).

We shall now briefly recall some known results related to equation (1.1).

### 1.2 Summary of Known Results

Let us firstly consider the Euclidean setting. In [2] uniqueness of solutions to the Cauchy problem

$$\begin{cases} 
\partial_t u = Lu & \text{in } \mathbb{R}^N \times (0, T) \\
u = 0 & \text{in } \mathbb{R}^N \times \{0\},
\end{cases}$$

(1.11)

is investigated. Here $T > 0$ and

$$Lu := \sum_{i,j=1}^{N} \frac{\partial^2 [a_{ij}(x, t)u]}{\partial x_i \partial x_j} - \sum_{i=1}^{N} \frac{\partial [b_i(x, t)u]}{\partial x_i} + c(x, t)u,$$

the coefficients which appear in $L$, together with all their derivatives, are locally bounded functions and the matrix $A \equiv (a_{ij})$ is assumed to be positive semidefinite in $\mathbb{R}^N \times (0, T)$. Moreover the authors assume that

$$|a_{ij}(x, t)| \leq K_1(1 + |x|^2)^{\frac{\gamma+1}{2}},$$

$$|b_i(x, t)| \leq K_2(|x|^2 + 1)^{\frac{1}{2}},$$

$$|c(x, t)| \leq K_3(|x|^2 + 1)^{\frac{1}{2}},$$

for almost every $(x, t) \in \mathbb{R}^N \times (0, T)$, for some constants $\gamma \geq 0$ and $K_i > 0$, for $i = 1, 2, 3$. Given $\phi \in C(\mathbb{R}^N \times (0, T)), \phi > 0$, set
In [2, Theorem 1] it is shown that if \( u \) is a solution to problem (1.11) and \( u \in L^1_{\phi}(\mathbb{R}^N \times (0, T)) \), with

\[
\phi(x) = (|x|^2 + 1)^{-\alpha_0} \quad (x \in \mathbb{R}^N) \quad \text{if } \lambda = 0,
\]

or

\[
\phi(x) = e^{-\alpha_0(|x|^2+1)\frac{2-a}{2}} \quad (x \in \mathbb{R}^N) \quad \text{if } \lambda > 0,
\]

for some \( \alpha_0 > 0 \), then

\( u \equiv 0 \quad \text{in} \quad \mathbb{R}^N \times (0, T) \).

The result in [2] has attracted a lot of interests and analogue results has been proved in [3, 8, 9, 20] (we refer to these papers for further discussions and details) and generalized to the fractional heat equation in [12, 25, 26]. We also mention that similar results has been considered in the case of degenerate elliptic and parabolic problems in \( \mathbb{R}^N \) and also in bounded domains of \( \mathbb{R}^N \) (see e.g. [13, 19, 20, 23, 24] and the references therein for previous results).

Let us now consider the Riemannian setting. The study of uniqueness of solutions to the heat equation

\[
\begin{aligned}
\partial_t u &= \Delta u \quad \text{in} \quad M \times (0, T) \\
u &= u_0 \quad \text{in} \quad M \times \{0\},
\end{aligned}
\]

(1.14)

where \( M \) is a complete Riemannian manifolds, has been largely investigated in the literature. Indeed, it is well-known that uniqueness of solutions to problem (1.14) is equivalent to the stochastic completeness of the Riemannian manifold \( M \) (see e.g. [7, Theorem 6.2]). We refer the interested reader to [7] for a complete and exhaustive picture about these kind of results. More precisely, in [7, Theorem 9.2] it is shown that, if \( u \) is a classical solution to (1.14) with \( u_0 \equiv 0 \), then

\( u \equiv 0 \quad \text{in} \quad M \times (0, T) \),

provided that \( u \in L^2_{\phi}(M \times (0, T)) \), where

\[
L^2_{\phi}(M \times (0, T)) := \left\{ u : M \times (0, T) \to \mathbb{R} \text{ measurable} \mid \int_0^T \int_M u(x, t)^2 \phi(x) d\mu(x) dt < \infty \right\},
\]
with
\[ \phi(x) = e^{-f(r(x))} , \]
where \( r \) is given in (1.3) and \( f \) is a positive increasing continuous function defined in \((0, \infty)\) such that
\[ \int_{R_0}^{\infty} \frac{r}{f(r)} \, dr = \infty , \]
for some \( R_0 > 0 \). The previous result in [7] has been generalized in [22] in the following sense: if \( u \) is a classical solution to (1.14) with \( u_0 \equiv 0 \), then
\[ u \equiv 0 \text{ in } M \times (0, T) , \]
provided \( u \in L^p_\phi(M \times (0, T)) \), for \( 1 < p \leq 2 \), where
\[
L^p_\phi(M \times (0, T)) := \left\{ u : M \times (0, T) \to \mathbb{R} \text{ measurable} \mid \int_0^T \int_M |u(x, t)|^p \phi(x) \, d\mu(x) \, dt < \infty \right\} ,
\]
with \( \phi \) as before.

Concerning the elliptic case, in [6] (see also [7, Section 13.2]) the author proves that the only bounded solution to equation (1.1), with \( b \equiv 0 \) and \( c(x) \geq 0 \), is the trivial one provided that
\[ V(x_0, r) \leq kr^2 e^{kC^2(r/2)} , \text{ for all } r > 0 \text{ large enough} \]
where \( k > 0 \) and
\[ C(r) := \int_0^r \inf_{x \in \partial B_r(x_0)} c(x) \, ds . \]

Similar results have also been obtained for more general linear, elliptic, degenerate equations such as
\[ Lu - cu = f \text{ in } \Omega ; \quad (1.15) \]
where
\[
Lu := \sum_{i,j=1}^N a_{ij} \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^N b_i \frac{\partial u}{\partial x_i} ,
\]
with possibly unbounded coefficients \( a_{ij} \), \( b_i \), \( c \) and function \( f \). Well-posedness of problem (1.15) has been intensively investigated both in the case of \( \Omega = \mathbb{R}^N \) and
\(\Omega\) being a bounded domain where the problem is completed with suitable boundary conditions and the coefficients can be degenerate or singular at the boundary of the domain (see e.g. [21, 24] and the references therein). Similar uniqueness results have been studied also in the framework of nonlocal diffusion, we refer to [12, 25, 26].

In particular, in [21, Section 1.2] the author consider the problem (1.15) in a bounded and regular domain \(\Omega\) of \(\mathbb{R}^N\) where the coefficients \(a_{ij}\) and \(b_i\) satisfy suitable assumptions (see [21, \((H_1) - (H_3) - (H_4)\)]), \(c\) satisfies \((H_3)\) and \(f \in C(\Omega)\). Under these hypothesis in [21, Theorem 1.5] it is shown that there exist at most one solution \(u \in L^p_{\phi}(\Omega)\) of problem (1.15), for some suitable \(p > 1\). In this case,

\[
\phi(x) = [d(x)]^\alpha
\]

where \(\alpha > 0\) and \(d(x) = \text{dist}(x, \partial\Omega)\). Of course, if \(f \equiv 0\) then one obtains a uniqueness result in the weighted Lebesgue space in the same spirit of the results that we are going to present.

**Organisation of the paper** The paper is organized as follows. In Sect. 2 we state our main results and we recall some basic facts from Riemannian geometry that will be useful in the sequel. Sections 3, 4 and 5 are devoted to the proof of Theorem 2.2, Theorem 2.3 and Theorem 2.4, respectively. Finally, in Sect. 6, we show Proposition 2.5 and Corollary 2.6.

### 2 Mathematical Framework and Results

Throughout the paper we deal with classical solutions to equation (1.1), for this reason we recall their definition.

**Definition 2.1** We say that a function \(u\) is a classical solution to equation (1.1) if

(i) \(u \in C^2(M)\);

(ii) \(\Delta u + \langle b(x), \nabla u \rangle - c(x)u = 0\) for all \(x \in M\).

Furthermore, we say that \(u\) is a supersolution (subsolution) to equation (1.1), if in \((ii)\) instead of “=” we have “\(\geq\)” “\(<\)”.

We now state the main result where we consider those Riemannian manifolds \(M\) such that (1.5) is satisfied and we assume the vector field \(b\) to be bounded, see \((H_0)\).

**Theorem 2.2** Let \(M\) be a complete, noncompact, Riemannian manifold such that (1.5) is satisfied. Let assumptions \((H_0)\) and \((H_3)\) be satisfied. Let \(u\) be a classical solution to equation (1.1). Let \(\psi\) be as in (1.6). Let \(p > 1\), with \(pc_0 > C = C(\beta, K_0)\) where \(C\) can be explicitly computed, see (3.24). If \(u \in L^p_{\psi}(M)\), then

\[u \equiv 0\text{ in }M.\]

The next two results show that, if one restricts the assumptions on the Riemannian manifold, thus (1.5) does not follows anymore, then \(b\) can also be unbounded in an appropriate sense that, somehow, is related to the volume growth condition made on \(M\).
Theorem 2.3 Let $M$ be a complete, noncompact, Riemannian manifold such that (1.7) is satisfied. Let assumptions $(H_1)$ and $(H_3)$ be satisfied. Let $u$ be a classical solution to equation (1.1). Let $\eta$ be as in (1.8). Let $p > 1$, with $p c_0 > C = C(\beta, K_1)$, where $C$ can be explicitly computed, see (4.6). If $u \in L^p_\eta(M)$, then

$$u \equiv 0 \quad \text{in} \quad M.$$ 

Theorem 2.4 Let $M$ be a complete, noncompact, Riemannian manifold such that (1.9) is satisfied. Let assumptions $(H_2)$ and $(H_3)$ be satisfied. Let $u$ be a classical solution to equation (1.1). Let $\xi$ be defined as in (1.10). Let $p > 1$, with $p c_0 > C = C(\beta, K_2)$ where $C$ can be explicitly computed, see (5.18). If $u \in L^p_\xi(M)$, then

$$u \equiv 0 \quad \text{in} \quad M.$$ 

In the special case of $b = 0$, our Theorems apply and agree with the already mentioned Yau’s results, see [29] and [30], concerning harmonic and positive subharmonic functions. Similarly, when $b$ is the gradient of some function similar results have been proved in the already mentioned [14, 16, 17, 28]. Our results cover also the case when $b$ is not a gradient, where the situation is different and more involved.

In the following we collect some observations regarding the hypothesis of the theorems:

(i) The hypothesis $p c_0$ large enough made in Theorem 2.2, Theorem 2.3 and Theorem 2.4 will be specified in their proofs.

(ii) Observe that assumption (1.7) in Theorem 2.3 is stronger than assumption (1.5) in Theorem 2.2. Although, this hypothesis enables us to relax the assumption on the vector field $b$, thus $b$ in Theorem 2.3 might be unbounded, see $(H_1)$.

(iii) We mention that thanks to the Laplacian comparison Theorems (see e.g. [5, Section 2] or [7, Section 15]) we have that assumptions (1.5), (1.7) and (1.9) are satisfied if the Ricci curvature of the Riemannian manifold $M$ is bounded from below. In particular, (1.5) and (1.7) are guaranteed if

$$\text{Ric} \geq -k \quad \text{for some} \quad k > 0.$$ 

Similarly, (1.9) is true if

$$\text{Ric} \geq 0.$$ 

Finally, in the special case of a model manifold $M^{N}_\varphi$ (see Definition 2.7) satisfying (1.9) with

$$\varphi(r) = r^\lambda, \quad \text{for any} \quad r > 1, \quad \text{and for some} \quad \lambda < \infty,$$ 

we show that the hypothesis $(H_2)$ on $b$, is sharp in the sense that, if it fails, then infinitely many bounded solutions to problem (1.1) exist; this is the content of the next proposition. In particular, we observe that due to (2.8), assumption (2.1) is compatible with (1.9), provided $\lambda < \frac{\beta - 1}{N - 1}$.
Proposition 2.5 Let $M^N_\phi$ be a model manifold such that (2.1) and (1.9) are satisfied. Suppose that assumption $(H_3)$ holds and let $b \in C^1(M^N_\phi)$ be such that

$$
\langle b(x), \nabla r \rangle \geq K r^\sigma \quad \text{for all } x \in M^N_\phi \setminus B_{R_0},
$$

for some

$$
R_0 > 1, \quad \sigma > 1, \quad \text{and } K > 1.
$$

Then equation (1.1) admits infinitely many bounded solutions in the sense of Definition 2.1.

Observe that, provided that $\lambda < \frac{\alpha-1}{N-1}$, then it is always true that $\lambda < \frac{\tau-1}{N-1}$, where $\tau$ is introduced in (1.10). Consequently any bounded solution to problem (1.1) given by Proposition 2.5 belongs also to the space $L^p_\xi(M^N_\phi)$ with $\xi$ as in (1.10).

In the following corollary we summarize the picture of uniqueness and non-uniqueness results in the special case of model manifolds.

Corollary 2.6 Let $M^N_\phi$ be a model manifold such that (2.1) and (1.9) are satisfied. Suppose that assumption $(H_3)$ holds and let $b \in C^1(M^N_\phi)$.

(i) Assume that $b$ satisfies assumption $(H_2)$. Let $u$ be a classical solution to equation (1.1). Let $p > 1$ and assume that $pc_0$ large enough. If $u \in L^p_\xi(M^N_\phi)$, then $u \equiv 0$ in $M^N_\phi$.

(ii) Assume that $b$ satisfies assumption (2.2). Then equation (1.1) admits infinitely many classical solutions.

2.1 Notation from Riemannian Geometry

In this subsection we recall and fix some basic notations and useful definitions from Riemannian geometry that we will use in the rest of the paper (we refer e.g. to [1, 4, 7]).

Throughout the paper, $M$ will denote an $N-$dimensional, complete, connected, noncompact, smooth Riemannian manifold without boundary and of infinite volume, endowed with a smooth Riemannian metric $g = \{g_{ij}\}$. As usual, we consider the volume form, denoted by $d\mu$, induced by $g$ and we denote by $\text{div}(X)$ the divergence of a smooth vector field $X$ on $M$, that is, in local coordinates

$$
\text{div}(X) = \frac{1}{\sqrt{|g|}} \sum_{i=1}^N \partial_i \left( \sqrt{|g|} X^i \right),
$$

where $|g| = \det(g_{ij})(\geq 0)$. 

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We also denote by $\nabla$ and $\Delta$ the Riemannian gradient and the Laplace-Beltrami operator on $M$ with respect to $g$, that is, in local coordinates,

$$(\nabla u)^i = \sum_{j=1}^{N} g^{ij} \partial_j u,$$

and

$$\Delta u = \text{div}(\nabla u) = \frac{1}{\sqrt{|g|}} \sum_{i,j=1}^{N} \partial_i \left( \sqrt{|g|} g^{ij} \partial_j u \right),$$

for any $C^2$-function $u : M \to \mathbb{R}$, where $\{g^{ij}\}$ denotes the inverse of the metric tensor $g$.

Due to the divergence structure of the Laplace-Beltrami operator we have the following integration by parts formula

$$\int_{M} v \Delta u \, d\mu = - \int_{M} \langle \nabla u, \nabla v \rangle \, d\mu, \quad (2.3)$$

for any $C^2$-functions $v, u : M \to \mathbb{R}$, with either $v$ or $u$ compactly supported and where $\langle \cdot, \cdot \rangle$ is the scalar product induced by $g$.

Moreover, given a vector field $X$, we denote

$$|X| = \sqrt{\langle X, X \rangle}.$$ 

Observe that, for any $C^2$-functions $u : M \to \mathbb{R}$ and $\psi : \mathbb{R} \to \mathbb{R}$ we have

$$\Delta[\psi(u)] = \psi'(u) \Delta u + \psi''(u) |\nabla u|^2. \quad (2.4)$$

Hence if $u$ is a classical solution of (1.1) then

$$\Delta[\psi(u)] + g(b, \nabla[\psi(u)]) - c\psi(u) = \psi''(u)|\nabla u|^2 - c[\psi(u) - \psi'(u)u]. \quad (2.5)$$

Lastly, we recall the definition of model manifold (see e.g. [5, 15, 18])

**Definition 2.7** An $N$-dimensional model manifold with warping function $\varphi$ is a Riemannian manifold $M_N^\varphi$ diffeomorphic to $\mathbb{R}^N$ and endowed with a rotationally symmetric Riemannian metric. The model $M_N^\varphi$ is realized as the quotient space $[0, +\infty) \times S^{N-1}/\sim$, where $\sim$ identifies $\{0\} \times S^{N-1}$ with the pole $o$ of the space, and the Riemannian metric has the expression

$$g = dr^2 + \varphi^2(r)d\theta^2,$$

where $r(x) = \text{dist}(x, o)$, $d\theta^2$ denotes the standard metric of $S^{N-1}$ and $\varphi : [0, +\infty) \to [0, +\infty)$ is a smooth function satisfying the following conditions:
• \( \varphi(r) > 0 \) for all \( r > 0 \),
• \( \varphi'(0) = 1 \),
• \( \varphi^{(2k)}(0) = 0 \) for every \( k \geq 0 \).

Observe that the Euclidean space \( \mathbb{R}^N \) and the (generalized) hyperbolic space \( \mathbb{H}^N \) of constant curvature \( -c < 0 \) can be seen as model manifolds (take \( \varphi(r) = r \) and \( \varphi(r) = \frac{\sinh(\sqrt{c}r)}{\sqrt{c}} \), respectively). Moreover, due to their structure the Laplace-Beltrami operator on \( \mathbb{M}_{\varphi}^N \) can be written in the following way

\[
\Delta = \frac{\partial^2}{\partial^2 r} + (N - 1) \frac{\varphi'}{\varphi} \frac{\partial}{\partial r} + \frac{1}{\varphi^2} \Delta_{\theta},
\]

(2.6)

where \( \Delta_{\theta} \) denotes the Laplace-Beltrami operator on \( S^{N-1} \). In particular,

\[
\Delta r = (N - 1) \frac{\varphi'}{\varphi}.
\]

(2.7)

Moreover, being the volume form on \( \mathbb{M}_{\varphi}^N \) given by

\[
d\mu = \varphi^{N-1} r \, dr \, d\theta,
\]

one has the following:

\[
V(o, r) = c_N \int_0^r \varphi^{N-1}(t) \, dt,
\]

(2.8)

where, \( c_N \) is the area of the \((N - 1)\)-dimensional unit sphere and \( o \) is the pole of \( \mathbb{M}_{\varphi}^N \).

3 Proof of Theorem 2.2

Let \( M \) be a complete, noncompact, Riemannian manifold such that (1.5) is satisfied. Firstly, we state a general criterion for uniqueness of solutions to equation (1.1) in \( L^p(\varphi^s) \) for \( \varphi \) as in (1.6). We suppose that there exists a positive function \( \zeta \in C^1(M) \), which solves the following

\[
\delta |\nabla \zeta|^2 - \langle b, \nabla \zeta \rangle - \text{div}b - c \, p < 0 \text{ in } M;
\]

(3.1)

for some \( \delta > 0 \) sufficiently large and for some \( p > 1 \). We mention that, \( \delta = \delta(\varepsilon) \) where \( \varepsilon > 0 \) will be specified in the proof of Theorem 2.2. Such inequality is meant in the sense of Definition 2.1-(ii), where instead of “=“ we have “<“.

Proposition 3.1 Let \( M \) be a complete, noncompact, Riemannian manifold such that (1.5) is satisfied. Moreover, assume that \((H_0)\) and \((H_3)\) hold. Let \( u \) be a solution to
equation (1.1). Assume that there exists a positive function \( \zeta \in C^1(M) \) such that \( \phi = e^\zeta \) satisfies

\[
\phi(x) \leq C \psi(x) \quad \text{for all } x \in M;
\]

for some constant \( C > 0 \) and \( \psi \) as in (1.6). Moreover, assume that \( \zeta \) solves (3.1). If \( u \in L^p_\psi(M) \), then

\[
u \equiv 0 \quad \text{in } M.
\]

### 3.1 Proof of Proposition 3.1

In order to show Proposition 3.1 we need the following

**Lemma 3.2** Let \( M \) be a complete, noncompact, Riemannian manifold. Assume that \((H_0)\) and \((H_3)\) hold. Let \( \phi \in C^1(M) \), \( \phi > 0 \) be such that (3.2) holds. Let \( v \in L^1_\psi(M) \). Then

\[
\lim_{R \to \infty} \int_M |v(x)|\phi|\nabla \gamma_R(x)|^2 \, d\mu = 0,
\]

and

\[
\lim_{R \to \infty} \left[ \int_{D_+} |v(x)|\phi \gamma_R(x) \langle b(x), \nabla \gamma_R(x) \rangle \, d\mu \right] = 0;
\]

for a suitable family of cut-off functions \( \gamma_R \in C^\infty_c(B_R) \) and where \( D_+ \) defined as in (1.4).

**Remark 3.3** Observe that the quantity into the brackets of formula (3.4) is negative due to (1.4).

**Proof of Lemma 3.2** Let us consider a cut-off function \( \gamma \in C^\infty([0, +\infty)) \), \( 0 \leq \gamma \leq 1 \) such that

\[
\gamma(r) := \begin{cases} 
1 & 0 \leq r \leq \frac{1}{2} \\
0 & r \geq 1,
\end{cases}
\]

where \( r \) is the geodesic distance given by (1.3). Then, let us define

\[
\gamma_R(x) := \gamma \left( \frac{r}{R} \right) \quad \text{for all } x \in M.
\]

To show (3.3), let us observe that, for any \( x \in B_R \setminus B_{R/2} \), for some \( \tilde{C} > 0 

\[
|\nabla \gamma_R|^2 = \left[ \gamma'_R \left( \frac{r}{R} \right) \right]^2 \frac{1}{R^2} |\nabla r|^2 \leq \frac{\tilde{C}}{r^2} \leq \frac{\tilde{C}}{r^2}.
\]

\( \square \) Springer
Therefore, by combining (3.2) with (3.6), we obtain
\[
\int_M |v| \phi |\nabla \gamma R|^2 d\mu \leq \int_{B_R \setminus B_{R/2}} |v| \phi \frac{\tilde{C}}{r^2} d\mu \\
\leq \tilde{C} C \frac{4}{R^2} \int_{B_R \setminus B_{R/2}} |v| \psi d\mu.
\]

Then, since \(v \in L^1_{\psi}(M)\), from the latter we obtain (3.3) by letting \(R \to \infty\). On the other hand, to show (3.4), let us observe that, due to (H0), for any \(x \in D_+ \cap (B_R \setminus B_{R/2})\), for some \(\tilde{C} > 0\)
\[
- \langle b(x), \nabla \gamma R(x) \rangle = -\gamma'(\frac{r}{R}) \frac{1}{R} \langle b(x), \nabla r \rangle \leq \tilde{C} K_0,
\]
(3.7)
where, \(D_+\) has been defined in (1.4). Then, due to (3.2) together with (3.7), we get
\[
- \int_{D_+} |v(x)| \phi(x) \gamma_R \langle b(x), \nabla \gamma R \rangle d\mu \leq \int_{D_+ \cap (B_R \setminus B_{R/2})} |v(x)| \phi(x) \tilde{C} K_0 d\mu \\
\leq \tilde{C} C K_0 \int_{D_+ \cap (B_R \setminus B_{R/2})} |v(x)| \psi(x) d\mu.
\]
(3.8)
Then, since \(v \in L^1_{\psi}(M)\), we obtain from (3.8) that
\[
\lim_{R \to \infty} \left[ - \int_{D_+} |v(x)| \phi(x) \gamma_R \langle b(x), \nabla \gamma R(x) \rangle d\mu \right] = 0.
\]
This completes the proof of the Lemma.

**Proof of Proposition 3.1** For any \(\alpha > 0\) small enough, we define
\[
G_\alpha(u) := (u^2 + \alpha)^{\frac{p}{2}}.
\]
(3.9)
Then we take \(v \in C^1(M)\) such that
\[
v := G_\alpha(u)^2 e^\xi,
\]
where \(\gamma_R\) has been defined in (3.5). Now, we consider left-hand side of equation (1.1) with \(u = G_\alpha(u)\), i.e.
\[
\Delta [G_\alpha(u)] + \langle b, \nabla [G_\alpha(u)] \rangle - c G_\alpha(u).
\]
(3.10)
Then, we multiply it by \( v \), we integrate over \( M \) and we apply the integration by parts formula (2.3). We obtain

\[
\int_M \{ \Delta [G_\alpha(u)] + \langle b, \nabla [G_\alpha(u)] \rangle - c G_\alpha(u) \} \, v \, d\mu \\
= -\int_M |\nabla G_\alpha(u)|^2 \gamma_R^2 e^\xi \, d\mu - \int_M 2 \langle \nabla G_\alpha(u), \nabla \gamma_R \rangle G_\alpha(u) \gamma_R e^\xi \, d\mu \\
- \int_M \langle \nabla G_\alpha(u), \nabla \xi \rangle G_\alpha(u) \gamma_R^2 e^\xi \, d\mu \\
+ \frac{1}{2} \int_M \langle b, \nabla [G_\alpha(u)] \rangle G_\alpha(u) \gamma_R^2 e^\xi \, d\mu - \frac{1}{2} \int_M \langle b, \nabla [G_\alpha(u)] \rangle G_\alpha(u) \gamma_R e^\xi \, d\mu \tag{3.11}
\]

By using Young’s inequality, for every \( \varepsilon > 0 \), (3.11) reduces to

\[
\int_M \{ \Delta [G_\alpha(u)] + \langle b, \nabla [G_\alpha(u)] \rangle - c G_\alpha(u) \} \, v \, d\mu \\
\leq -\varepsilon \int_M |\nabla G_\alpha(u)|^2 \gamma_R^2 e^\xi \, d\mu - (1 - \varepsilon) \int_M |\nabla G_\alpha(u)|^2 \gamma_R^2 e^\xi \, d\mu \\
+ \frac{\varepsilon}{2} \int_M |\nabla G_\alpha(u)|^2 \gamma_R^2 e^\xi \, d\mu + \frac{2}{\varepsilon} \int_M |\nabla \gamma_R|^2 G_\alpha(u) e^\xi \, d\mu \\
+ \frac{\varepsilon}{2} \int_M |\nabla G_\alpha(u)|^2 \gamma_R^2 e^\xi \, d\mu + \frac{1}{2\varepsilon} \int_M |\nabla \xi|^2 G_\alpha(u)^2 \gamma_R^2 e^\xi \, d\mu \\
+ \frac{1}{2} \int_M \langle b, \nabla [G_\alpha(u)] \rangle G_\alpha(u) \gamma_R^2 e^\xi \, d\mu \\
- \frac{1}{2} \int_M \langle b, \nabla [G_\alpha(u)] \rangle G_\alpha(u) \gamma_R e^\xi \, d\mu \\
- \frac{1}{2} \int_M 2 \langle b, \nabla \gamma_R \rangle G_\alpha^2(u) \gamma_R e^\xi \, d\mu - \frac{1}{2} \int_M \langle b, \nabla \xi \rangle G_\alpha^2(u) \gamma_R^2 e^\xi \, d\mu \\
- \frac{1}{2} \int_M \text{div} \, b G_\alpha^2(u) \gamma_R^2 e^\xi \, d\mu - \int_M c G_\alpha^2(u) \gamma_R^2 e^\xi \, d\mu \\
\leq -(1 - \varepsilon) \int_M |G'_\alpha(u)|^2 |\nabla u|^2 \gamma_R^2 e^\xi \, d\mu + \frac{2}{\varepsilon} \int_M |\nabla \gamma_R|^2 G_\alpha^2(u) e^\xi \, d\mu \\
+ \frac{1}{2\varepsilon} \int_M |\nabla \xi|^2 G_\alpha^2(u) \gamma_R^2 e^\xi \, d\mu - \int_M \langle b, \nabla \gamma_R \rangle G_\alpha^2(u) \gamma_R e^\xi \, d\mu \\
- \frac{1}{2} \int_M \langle b, \nabla \xi \rangle G_\alpha^2(u) \gamma_R^2 e^\xi \, d\mu - \frac{1}{2} \int_M \text{div} \, b G_\alpha^2(u) \gamma_R^2 e^\xi \, d\mu \\
- \int_M c G_\alpha^2(u) \gamma_R^2 e^\xi \, d\mu \tag{3.12}
\]
On the other hand, formula (3.10) can be reduced as follows

\[
\Delta [G_\alpha(u)] + \langle b, \nabla [G_\alpha(u)] \rangle - c \ G_\alpha(u) \\
= G'_\alpha(u) \Delta u + G''_\alpha(u) |\nabla u|^2 + G'_\alpha(u) \langle b, \nabla u \rangle - c \ G_\alpha(u) + c \ G'_\alpha(u) u - c \ G''_\alpha(u) u \\
= G'_\alpha(u) [\Delta u + \langle b, \nabla u \rangle - c u] + G''_\alpha(u) |\nabla u|^2 - c \ G_\alpha(u) + c \ G'_\alpha(u) u \\
= G''_\alpha(u) |\nabla u|^2 + G'_\alpha(u) c u - c \ G_\alpha(u),
\]

(3.13)

where we used (2.4), (2.5) and (1.1). From (3.12) and (3.13),

\[
\int_M \left[ G''_\alpha(u) |\nabla u|^2 + G'_\alpha(u) c u - c \ G_\alpha(u) \right] G_\alpha(u) \gamma^2_R e^\xi \, d\mu \\
\leq -(1 - \varepsilon) \int_M |G'_\alpha(u)|^2 |\nabla u|^2 \gamma^2_R e^\xi \, d\mu + \frac{2}{\varepsilon} \int_M |\nabla \gamma_R|^2 G''_\alpha(u) e^\xi \, d\mu \\
+ \frac{1}{2\varepsilon} \int_M |\nabla \xi|^2 G'_\alpha(u) \gamma^2_R e^\xi \, d\mu - \int_M \langle b, \nabla \gamma_R \rangle G'_\alpha(u) \gamma_R e^\xi \, d\mu \\
- \frac{1}{2} \int_M \langle b, \nabla \xi \rangle G'_\alpha(u) \gamma^2_R e^\xi \, d\mu - \frac{1}{2} \int_M \text{div} \ b G''_\alpha(u) \gamma^2_R e^\xi \, d\mu \\
- \int_M c G'_\alpha(u) \gamma^2_R e^\xi \, d\mu.
\]

(3.14)

By rearranging the terms in (3.14), we get

\[
\int_M \left[ G'_\alpha(u) c u - c \ G_\alpha(u) \right] G_\alpha(u) \gamma^2_R e^\xi \, d\mu \\
\leq - \int_M \left\{ (1 - \varepsilon)|G'_\alpha(u)|^2 + G''_\alpha(u) G_\alpha(u) \right\} |\nabla u|^2 \gamma^2_R e^\xi \, d\mu \\
+ \frac{2}{\varepsilon} \int_M |\nabla \gamma_R|^2 G''_\alpha(u) e^\xi \, d\mu + \frac{1}{2\varepsilon} \int_M |\nabla \xi|^2 G'_\alpha(u) \gamma^2_R e^\xi \, d\mu \\
- \int_M \langle b, \nabla \gamma_R \rangle G'_\alpha(u) \gamma_R e^\xi \, d\mu - \frac{1}{2} \int_M \langle b, \nabla \xi \rangle G'_\alpha(u) \gamma^2_R e^\xi \, d\mu \\
- \frac{1}{2} \int_M \text{div} \ b G''_\alpha(u) \gamma^2_R e^\xi \, d\mu - \int_M c G'_\alpha(u) \gamma^2_R e^\xi \, d\mu.
\]

(3.15)

On the left-hand side of (3.15), by substituting the definition of \( G_\alpha \) given in (3.9), we have

\[
\int_M \left[ G'_\alpha(u) c u - c \ G_\alpha(u) \right] G_\alpha(u) \gamma^2_R e^\xi \, d\mu \\
= \int_M (u^2 + \alpha)^{\frac{p}{2} - 1} c \left[ \left( \frac{p}{2} - 1 \right) u^2 - \alpha \right] (u^2 + \alpha)^{\frac{p}{2}} \gamma^2_R e^\xi \, d\mu.
\]

(3.16)
Similarly, on the right-hand side of (3.16), by exploiting the definition of $G_\alpha$ given in (3.9) and by choosing $\varepsilon > 0$ sufficiently small, for any $p > 1$, we have that

$$ -\int_M \left\{ (1 - \varepsilon)|G'_\alpha(u)|^2 + G''_\alpha(u)G_\alpha(u) \right\} |\nabla u|^2 \gamma^2 e^\varepsilon \gamma R e \mu \leq 0, \quad (3.17) $$

since, by choosing $0 < \varepsilon \leq 2 - \frac{2}{p}$,

$$(1 - \varepsilon)|G'_\alpha(u)|^2 + G''_\alpha(u)G_\alpha(u)$$

$$= (1 - \varepsilon) \frac{p^2}{4}(u^2 + \alpha)^{\frac{p}{2}-2}u^2 + \frac{p}{2}(u^2 + \alpha)^{\frac{p}{2}-1} + p \left( \frac{p}{4} - 1 \right)(u^2 + \alpha)^{\frac{p}{2}-2}u^2$$

$$= (2 - \varepsilon) \frac{p^2}{4}(u^2 + \alpha)^{\frac{p}{2}-2}u^2 + \frac{p}{2}(u^2 + \alpha)^{\frac{p}{2}-1} - p(u^2 + \alpha)^{\frac{p}{2}-2}u^2$$

$$= (u^2 + \alpha)^{\frac{p}{2}-2} \left[ u^2 \left( (2 - \varepsilon) \frac{p}{4} - \frac{p}{2} \right) + \frac{p}{2} \alpha \right] \geq 0. $$

Hence, (3.15), combined with (3.16) and (3.17), reduces to

$$ \int_M (u^2 + \alpha)^{\frac{p}{2}-1} c \left[ \left( \frac{p}{2} - 1 \right) u^2 - \alpha \right] (u^2 + \alpha)^{\frac{p}{2}} \gamma^2 e^\varepsilon \gamma R e \mu \leq + \frac{2}{\varepsilon} \int_M |\nabla \gamma R|^2 G_\alpha^2(u) e^\varepsilon \gamma R e \mu + \frac{1}{2\varepsilon} \int_M |\nabla \xi|^2 G_\alpha^2(u) \gamma^2 R e^\varepsilon \gamma R e \mu$$

$$- \int_M \langle b, \nabla \gamma R \rangle G_\alpha^2(u) \gamma R e^\varepsilon \gamma R e \mu - \frac{1}{2} \int_M \langle b, \nabla \xi \rangle G_\alpha^2(u) \gamma R e^\varepsilon \gamma R e \mu - \frac{1}{2} \int_M \text{div} b G_\alpha^2(u) \gamma R e^\varepsilon \gamma R e \mu - \int_M c G_\alpha^2(u) \gamma R e^\varepsilon \gamma R e \mu \quad (3.18) $$

Letting $\alpha \to 0^+$

$$(u^2 + \alpha)^{\frac{p}{2}-1} c \left[ \left( \frac{p}{2} - 1 \right) u^2 - \alpha \right] (u^2 + \alpha)^{\frac{p}{2}} \to \left( \frac{p}{2} - 1 \right) |u|^p,$$

and

$$ G_\alpha(u) = (u^2 + \alpha)^{\frac{p}{2}} \to |u|^p. $$
Hence, by the dominated convergence theorem, (3.18) becomes

\[
\left( \frac{p}{2} - 1 \right) \int_M c |u|^p \gamma_R^2 e^\xi \, d\mu \\
\leq + \frac{2}{\varepsilon} \int_M |\nabla \gamma_R|^2 |u|^p e^\xi \, d\mu + \frac{1}{2\varepsilon} \int_M |\nabla \xi|^2 |u|^p \gamma_R^2 e^\xi \, d\mu \\
- \int_M \langle b, \nabla \gamma_R \rangle |u|^p \gamma_R e^\xi \, d\mu - \frac{1}{2} \int_M \langle b, \nabla \xi \rangle |u|^p \gamma_R^2 e^\xi \, d\mu \\
- \frac{1}{2} \int_M \text{div} b |u|^p \gamma_R^2 e^\xi \, d\mu - \int_M c |u|^p \gamma_R^2 e^\xi \, d\mu,
\]

which, defining \( \delta := \frac{1}{\varepsilon} \), is equivalent to

\[
\frac{1}{2} \int_M |u|^p \gamma_R^2 e^\xi \left[ -\delta |\nabla \xi|^2 + \langle b, \nabla \xi \rangle + \text{div} b + c \, p \right] \, d\mu \\
\leq \int_M |u|^p e^\xi \left[ 2\delta |\nabla \gamma_R|^2 - \langle b, \nabla \gamma_R \rangle \gamma_R \right] \, d\mu. \tag{3.19}
\]

Due to (3.19), and the definitions of \( D_+ \) and \( D_- \) in (1.4), we obtain

\[
\frac{1}{2} \int_M |u|^p \gamma_R^2 e^\xi \left[ -\delta |\nabla \xi|^2 + \langle b, \nabla \xi \rangle + \text{div} b + c \, p \right] \, d\mu \\
\leq \int_M 2\delta |u|^p e^\xi |\nabla \gamma_R|^2 \, d\mu - \int_{D_+} |u|^p e^\xi \langle b, \nabla \gamma_R \rangle \gamma_R \, d\mu \tag{3.20} \\
- \int_{D_-} |u|^p e^\xi \langle b, \nabla \gamma_R \rangle \gamma_R \, d\mu.
\]

Observing that, for all \( x \in D_- \)

\[-\langle b(x), \nabla \gamma_R(x) \rangle \leq 0,\]

we can rewrite (3.20) as follows

\[
\frac{1}{2} \int_M |u|^p \gamma_R^2 e^\xi \left[ -\delta |\nabla \xi|^2 + \langle b, \nabla \xi \rangle + \text{div} b + c \, p \right] \, d\mu \\
\leq \int_M 2\delta |u|^p e^\xi |\nabla \gamma_R|^2 \, d\mu - \int_{D_+} |u|^p e^\xi \langle b, \nabla \gamma_R \rangle \gamma_R \, d\mu. \tag{3.21}
\]

We now set \( \phi(x) := e^{\xi(x)} \) for any \( x \in M \). Observe that, \( \phi \) satisfies assumption (3.2). Hence, we can apply Lemma 3.2 with this choice of \( \phi \). From Lemma 3.2 and the monotone convergence theorem, sending \( R \to \infty \) in (3.21) we get

\[
\frac{1}{2} \int_M |u|^p e^\xi \left[ -\delta |\nabla \xi|^2 + \langle b, \nabla \xi \rangle + \text{div} b + c \, p \right] \, d\mu \leq 0. \tag{3.22}
\]
From (3.22) and (3.1), since $|u|^p \geq 0$, we can infer that $u \equiv 0$ in $M$. This completes the proof. □

We are now ready to prove Theorem 2.2, due to Proposition 3.1 it suffices to construct a subsolution to (3.1).

**Proof of Theorem 2.2** Let $\phi = \phi(r) := \psi(r)$ for any $r > 0$, where $\psi$ has been defined in (1.6), with $\beta > \alpha$. Moreover, we set $\zeta(r) := -\beta r$. Hence, $\phi(r) \equiv e^{\zeta(r)}$. At first observe that $\phi$ satisfies (3.2). Moreover, $\zeta$ solves (3.1), for properly chosen $\beta > \alpha$. Indeed, due to $(H_0)$ and $(H_3)$, we have

$$
\delta |\nabla \zeta|^2 - \langle b, \nabla \zeta \rangle - \text{div} b - c p
= \delta \zeta'(r)^2 |\nabla r|^2 - \zeta'(r) \langle b, \nabla r \rangle - \text{div} b - c p
\leq - \left( -\beta^2 \delta - \beta K_0 - K_0 + p c_0 \right). 
$$

(3.23)

Then, by (3.23), one has

$$
\delta |\nabla \zeta|^2 - \langle b, \nabla \zeta \rangle - \text{div} b - c p < 0,
$$

provided that

$$
p c_0 > \beta^2 \delta + \beta K_0 + K_0.
$$

(3.24)

Thus, by Proposition 3.1 the conclusion follows. □

**4 Proof of Theorem 2.3**

Similarly to Theorem 2.2, we want to state a general criterion for uniqueness by means of assumption $(H_1)$. Therefore, we state the next

**Proposition 4.1** Let $M$ be a complete, noncompact, Riemannian manifold such that (1.7) is satisfied. Moreover, assume that $(H_1)$ and $(H_3)$ hold. Let $u$ be a solution to equation (1.1) and $\eta$ be as in (1.8). Assume that there exists a positive function $\zeta \in C^1(M)$ such that $\phi = e^{\zeta}$ satisfies

$$
\frac{\phi^2(x)}{1 + r^2} \leq C \eta(x) \quad \text{for all} \quad x \in M;
$$

(4.1)

and

$$
\frac{\langle b(x), \nabla r \rangle}{(1 + r)^\beta} \phi^2(x) \leq C \eta(x) \quad \text{for all} \quad x \in D_+.
$$

(4.2)

Moreover, assume that $\zeta$ solves (3.1). If $u \in L^p_\eta(M)$, then

$$
u \equiv 0 \quad \text{in} \quad M.
$$
Analogously to Lemma 3.2 the next lemma can be shown. The proof is similar and for this reason we only sketch it.

**Lemma 4.2** Let $M$ be a complete, noncompact, Riemannian manifold such that (1.7) holds. Assume $(H_1)$ and $(H_3)$. Let $\phi \in C^1(M)$, $\phi > 0$ be such that (4.1) and (4.2) hold. Let $v \in L^1_{\eta}(M)$.

Then

$$
\lim_{R \to \infty} \int_M |v(x)|\phi^2 |\nabla \gamma_R(x)|^2 d\mu = 0,
$$

and

$$
\lim_{R \to \infty} \left[ \int_{D_+} |v(x)|\phi^2 \gamma_R(x) \langle b(x) , \nabla \gamma_R(x) \rangle d\mu \right] = 0;
$$

for a suitable family of cut-off functions $\gamma_R \in C^\infty_c(\mathbb{B}_R)$ and where $D_+$ defined as in (1.4).

**Proof** Equality (4.3) can be obtained as (3.3) by using assumption (4.1). To show (4.4) it is sufficient to argue as in the proof of Lemma 3.2 by replacing assumptions $(H_0)$ and (3.2) with $(H_1)$ and (4.2) respectively. \(\square\)

**Proof of Proposition 4.1** To prove Proposition 4.1 it is sufficient to argue as in the proof of Proposition 3.1 by using Lemma 4.2 instead of Lemma 3.2.

**Proof of Theorem 2.3** Let $\phi = \phi(r) := \eta(r)$ for any $r > 0$, where $\eta$ has been defined in (1.8). Moreover, we set $\xi(r) := -\beta r^\theta$, $0 < \theta < 1$ as in $(H_1)$. Hence, $\phi(r) \equiv e^{\xi(r)}$. At first observe that $\phi$ satisfies (4.1) and (4.2). Moreover, $\xi$ solves (3.1), for properly chosen $\beta > \alpha$. Indeed, due to $(H_1)$ and $(H_3)$, we have

$$
\delta |\nabla \xi|^2 - \langle b , \nabla \xi \rangle - \text{div} b - c p

\leq \beta^2 \theta^2 \delta r^{2\theta - 2} + \beta \theta K_1 r^{\theta - 1 - \sigma} + K_1 r^{\sigma - 1} - c_0 p

\leq - \left( -\beta^2 \theta^2 \delta - \beta K_1 \theta - K_1 + p c_0 \right).
$$

Then, by (4.5), one has

$$
\delta |\nabla \xi|^2 - \langle b , \nabla \xi \rangle - \text{div} b - c p < 0,
$$

provided that

$$
p c_0 > \beta^2 \theta^2 \delta + \beta K_1 \theta + K_1.
$$

Thus, by Proposition 4.1 the conclusion follows.
5 Proof of Theorem 2.4

Let $M$ be a complete, noncompact, Riemannian manifold such that (1.9) is satisfied. Firstly, we state a general criterion for uniqueness of solutions to equation (1.1) in $L^p_\xi(M)$ for $\xi$ as in (1.10). We suppose that there exists a positive function $\phi \in C^1(M)$, which solves, the following

$$\hat{\delta} |\nabla \phi|^2 - \langle b, \nabla \phi \rangle \phi - \frac{1}{2}(\text{div} b + p\phi^2) < 0 \text{ in } M; \quad (5.1)$$

for some $\hat{\delta} > 0$ sufficiently large and for some $p > 1$. We mention that, $\hat{\delta} = \hat{\delta}(\varepsilon)$ where $\varepsilon > 0$ will be specified in the proof of Theorem 2.4. Such inequality is meant in the sense of Definition 2.1-(ii), where instead of $"="$ we have $"<"$.

Proposition 5.1 Let $M$ be a complete, noncompact, Riemannian manifold such that (1.9) is satisfied. Moreover, assume that $(H_2)$ and $(H_3)$ hold. Let $u$ be a solution to equation (1.1) and $\xi$ be as in (1.10). Assume that there exists a positive function $\phi \in C^1(M)$, which solves (5.1) and such that

$$\frac{\phi^2(x)}{1+r^2} \leq C\xi(x) \text{ for all } x \in M; \quad (5.2)$$

and

$$\frac{\langle b(x), \nabla r \rangle}{1+r} \phi^2(x) \leq C\xi(x) \text{ for all } x \in D_+ \quad (5.3)$$

If $u \in L^p_\xi(M)$, then

$$u \equiv 0 \text{ in } M.$$

5.1 Proof of Proposition 5.1

Analogously to Lemma 3.2 and Lemma 4.2 the next lemma can be shown.

Lemma 5.2 Let $M$ be a complete, noncompact, Riemannian manifold such that (1.9) holds. Assume $(H_2)$ and $(H_3)$. Let $\phi \in C^1(M)$, $\phi > 0$ be such that (5.2) and (5.3) hold. Let $v \in L^1_\xi(M)$. Then

$$\lim_{R \to \infty} \int_M |v(x)|\phi^2 |\nabla \gamma_R(x)|^2 \, d\mu = 0, \quad (5.4)$$

and

$$\lim_{R \to \infty} \left[\int_{D_+} |v(x)|\phi^2 \gamma_R(x) \langle b(x), \nabla \gamma_R(x) \rangle \, d\mu \right] = 0; \quad (5.5)$$
for a suitable family of cut-off functions $\gamma_R \in C_c^\infty(B_R)$ and where $D_+$ defined as in (1.4).

**Proof** Equality (5.4) can be obtained as (3.3) by using assumption (5.4). To show (5.5) it is sufficient to argue as in the proof of Lemma 3.2 by replacing assumptions $(H_0)$ and (3.2) with $(H_2)$ and (5.3) respectively. □

**Proof of Proposition 5.1** For any $\alpha > 0$ small enough, let $G_\alpha$ be defined as in (3.9). Then we take $v \in C^1(M)$ such that

$$v := G_\alpha(u)\gamma_R^2 \phi^2, \quad \text{for all } x \in M$$

where $\gamma_R$ has been defined in (3.5). Now, we consider the left-hand side of equation (1.1) with $u = G_\alpha(u)$, i.e.

$$\Delta[G_\alpha(u)] + \langle b, \nabla G_\alpha(u) \rangle - c G_\alpha(u). \quad (5.6)$$

Then, we multiply it by $v$, we integrate over $M$ and we use the integration by parts formula (2.3)

$$\int_M \{\Delta[G_\alpha(u)] + \langle b, \nabla G_\alpha(u) \rangle - c G_\alpha(u)\} \, v \, d\mu$$

$$\begin{align*}
&= -\int_M \langle \nabla G_\alpha, \nabla v \rangle \, d\mu + \frac{1}{2} \int_M \langle b, \nabla G_\alpha(u) \rangle \, v \, d\mu - \frac{1}{2} \int_M \langle b, \nabla v \rangle \, G_\alpha(u) \\
&\quad - \int_M \left( \frac{1}{2} \text{div} \, b + c \right) \, v \, d\mu \\
&= -\varepsilon \int_M |\nabla G_\alpha(u)|^2 \gamma_R^2 \phi^2 \, d\mu - (1 - \varepsilon) \int_M |\nabla G_\alpha(u)|^2 \gamma_R^2 \phi^2 \, d\mu \\
&\quad - 2 \int_M \langle \nabla G_\alpha(u), \nabla \gamma_R \rangle G_\alpha(u) \gamma_R \phi^2 \, d\mu - 2 \int_M \langle \nabla G_\alpha(u), \nabla \phi \rangle G_\alpha(u) \gamma_R^2 \phi \, d\mu \\
&\quad + \frac{1}{2} \int_M \langle b, \nabla[G_\alpha(u)] \rangle G_\alpha(u) \gamma_R^2 \phi^2 \, d\mu - \frac{1}{2} \int_M \langle b, \nabla \phi \rangle G_\alpha(u) \gamma_R^2 \phi \, d\mu \\
&\quad - \frac{1}{2} \int_M 2 \langle b, \nabla \gamma_R \rangle G_\alpha^2(u) \gamma_R \phi^2 \, d\mu - \frac{1}{2} \int_M 2 \langle b, \nabla \phi \rangle G_\alpha^2(u) \gamma_R^2 \phi \, d\mu \\
&\quad - \frac{1}{2} \int_M \text{div} \, b G_\alpha^2(u) \gamma_R^2 \phi^2 \, d\mu - \int_M c G_\alpha^2(u) \gamma_R^2 \phi^2 \, d\mu.
\end{align*}$$

\[\square\]
By Young’s inequality we obtain

\[
\int_M \{ \Delta [G_\alpha(u)] + \langle b, \nabla G_\alpha(u) \rangle - c G_\alpha(u) \} \, v \, d\mu
\leq \left( -\varepsilon + \frac{\varepsilon}{2} \right) \int_M |\nabla G_\alpha(u)|^2 \gamma_R^2 \phi^2 \, d\mu - (1 - \varepsilon) \int_M |\nabla G_\alpha(u)|^2 \gamma_R^2 \phi^2 \, d\mu
+ \frac{2}{\varepsilon} \int_M |\nabla \gamma_R|^2 G_\alpha^2(u) \phi^2 \, d\mu + \frac{2}{\varepsilon} \int_M |\nabla \phi|^2 G_\alpha^2(u) \gamma_R^2 \phi \, d\mu
\]

(5.7)

On the other hand, arguing as in (3.13), (5.6) can be reduced as follows

\[
\Delta [G_\alpha(u)] + \langle b, \nabla [G_\alpha(u)] \rangle - c G_\alpha(u)
= G_\alpha'(u) \Delta u + G_\alpha''(u) |\nabla u|^2 + G_\alpha'(u) \langle b, \nabla u \rangle - c G_\alpha(u)
\]

(5.8)

\[
= G_\alpha''(u) |\nabla u|^2 + G_\alpha'(u) c u - c G_\alpha(u),
\]

where we have used (2.4), (2.5) and (1.1). Similarly to the proof of Proposition 3.1, by combining together (5.7) and (5.8), due to (3.9), we get

\[
\int_M \left[ G_\alpha'(u) c u - c G_\alpha(u) \right] G_\alpha(u) \gamma_R^2 \phi^2 \, d\mu
\leq - (1 - \varepsilon) \int_M |G_\alpha'(u)|^2 |\nabla u|^2 \gamma_R^2 \phi^2 \, d\mu - \int_M G_\alpha''(u) |\nabla u|^2 G_\alpha(u) \gamma_R^2 \phi^2 \, d\mu
+ \frac{2}{\varepsilon} \int_M |\nabla \gamma_R|^2 G_\alpha^2(u) \phi^2 \, d\mu + \frac{2}{\varepsilon} \int_M |\nabla \phi|^2 G_\alpha^2(u) \gamma_R^2 \phi \, d\mu
\]

(5.9)

On the left-hand side of (5.9), by substituting the definition of \( G_\alpha \) given in (3.9), we have

\[
\int_M \left[ G_\alpha'(u) c u - c G_\alpha(u) \right] G_\alpha(u) \gamma_R^2 \phi^2 \, d\mu
= \int_M \left( u^2 + \alpha \right)^{\frac{p}{2} - 1} c \left( \frac{p}{2} - 1 \right) u^2 - \alpha \left( u^2 + \alpha \right)^{\frac{p}{2}} \gamma_R^2 \phi^2 \, d\mu.
\]

(5.10)

Similarly, on the right-hand side of (5.9), by exploiting the definition of \( G_\alpha \) given in (3.9), arguing as in (3.17), we can choose \( \varepsilon > 0 \) sufficiently small such that, for any
\[ p > 1, \]
\[
- \int_M \left\{ (1 - \varepsilon)|G^\varepsilon(u)|^2 + G^\varepsilon(u)G(u) \right\} |\nabla u|^2 \gamma^2 \phi^2 \, d\mu \leq 0. \tag{5.11}
\]

Hence, (5.9), combined with (5.10) and (5.11), reduces to
\[
\int_M (u^2 + \alpha)^{p-1} \left[ \left( \frac{p}{2} - 1 \right) u^2 - \alpha \right] (u^2 + \alpha)^{p} \gamma^2 \phi^2 \, d\mu 
\]
\[
\leq + \frac{2}{\varepsilon} \int_M |\nabla \gamma R|^2 G^2(u) \phi^2 \, d\mu + \frac{2}{\varepsilon} \int_M |\nabla \phi|^2 G^2(u) \gamma^2 \phi \, d\mu 
\]
\[
- \int_M \langle b \cdot \nabla \gamma R \rangle G^2(u) \gamma^2 \phi \, d\mu - \int_M \langle b \cdot \nabla \phi \rangle G^2(u) \gamma^2 \phi \, d\mu 
\]
\[
- \frac{1}{2} \int_M \text{div} b G^2(u) \gamma^2 \phi \, d\mu - \int_M c G^2(u) \gamma^2 \phi \, d\mu. \tag{5.12}
\]

Letting \( \alpha \to 0^+ \) in (5.12), by the dominated convergence and by choosing \( \hat{\delta} = \frac{2}{\varepsilon} \), we obtain the analogue of (3.19), i.e.
\[
\int_M |u|^p \gamma^2 \left[ -\hat{\delta} |\nabla \phi|^2 + \langle b \cdot \nabla \phi \rangle \phi + \frac{1}{2}(\text{div} b + p c) \phi^2 \right] \, d\mu 
\]
\[
\leq \int_M |u|^p \left[ \hat{\delta} |\nabla \gamma R|^2 \phi^2 - \langle b \cdot \nabla \gamma R \rangle \gamma R \phi^2 \right] \, d\mu. \tag{5.13}
\]

Due to (5.13), and the definitions of \( D_+ \) and \( D_- \) in (1.4), we obtain
\[
\int_M |u|^p \gamma^2 \left[ -\hat{\delta} |\nabla \phi|^2 + \langle b \cdot \nabla \phi \rangle \phi + \frac{1}{2}(\text{div} b + p c) \phi^2 \right] \, d\mu 
\]
\[
\leq \hat{\delta} \int_M |u|^p \phi^2 |\nabla \gamma R|^2 \, d\mu - \int_{D_+} |u|^p \phi^2 \langle b \cdot \nabla \gamma R \rangle \gamma R \, d\mu 
\]
\[
- \int_{D_-} |u|^p \phi^2 \langle b \cdot \nabla \gamma R \rangle \gamma R \, d\mu. \tag{5.14}
\]

Observing that, for all \( x \in D_- \)
\[
- \langle b(x) \cdot \nabla \gamma R(x) \rangle \leq 0,
\]
we rewrite (5.14) as follows
\[
\int_M |u|^p \gamma^2 \left[ -\hat{\delta} |\nabla \phi|^2 + \langle b \cdot \nabla \phi \rangle \phi + \frac{1}{2}(\text{div} b + p c) \phi^2 \right] \, d\mu 
\]
\[
\leq \hat{\delta} \int_M |u|^p \phi^2 |\nabla \gamma R|^2 \, d\mu - \int_{D_+} |u|^p \phi^2 \langle b \cdot \nabla \gamma R \rangle \gamma R \, d\mu. \tag{5.15}
\]
We now set $\phi(x) := e^{\zeta(x)}$ for any $x \in M$. Observe that, $\phi$ satisfies assumptions (5.2) and (5.3). Hence, we can apply Lemma 5.2 with this choice of $\phi$. From Lemma 5.2 and the monotone convergence theorem, sending $R \to \infty$ in (5.15) we get

$$\int_M |u|^p \left[ -\delta |\nabla \phi|^2 + \langle b, \nabla \phi \rangle \phi + \frac{1}{2} (\text{div} b + c \phi)^2 \right] d\mu \leq 0. \quad (5.16)$$

From (5.16) and (5.1), since $|u|^p \geq 0$, we can infer that $u \equiv 0$ in $M$. This completes the proof. \hfill \Box

Proof of Theorem 2.4  Let $\phi = \phi(r) := \xi(r)^{1/2}$ for any $r > 0$, where $\xi$ has been defined in (1.10), with $\tau > 0$. At first observe that $\phi^2$ satisfies (5.2) and (5.3). Moreover, $\phi$ solves (5.1). Indeed, due to (H2) and (H3), we have

$$\delta |\nabla \phi|^2 - \langle b, \nabla \phi \rangle \phi - \frac{1}{2} (\text{div} b + c \phi)^2 \leq \tau \delta^2 \left( \frac{\tau}{2} + 1 \right)(1 + r)^{-\tau} |\nabla r|^2$$

$$+ \frac{\tau}{2} (1 + r)^{-\tau - 1} \langle b, \nabla r \rangle - \frac{1}{2} (\text{div} b + c \phi) (1 + r)^{-\tau}$$

$$\leq \frac{\tau^2}{4} (\tau + 2)(1 + r)^{-\tau - 4} + \frac{\tau}{2} K_2(1 + r)^{-\tau - 1 + \sigma}$$

$$+ \frac{K_2}{2} (1 + r)^{-\tau + \sigma - 1} - \frac{c_0 p}{2} (1 + r)^{-\tau}$$

$$\leq -\frac{1}{2} (1 + r)^{-\tau} \left( c_0 p - \frac{\tau^2}{2} (\tau + 2)(1 + r)^{-4} - K_2 (\tau + 1)(1 + r)^{\sigma - 1} \right). \quad (5.17)$$

Then, by (5.17), one has

$$\delta |\nabla \phi|^2 - \langle b, \nabla \phi \rangle \phi - \frac{1}{2} (\text{div} b + c \phi)^2 \leq 0,$$

provided that

$$pc_0 > \frac{\tau \delta}{2} (\tau + 2) + K_2(\tau + 1). \quad (5.18)$$

Thus, by Proposition 5.1 the conclusion follows. \hfill \Box

6 Proof of Proposition 2.5 and Corollary 2.6

We now show nonuniqueness for problem (1.1) with $b$ as in (2.2) in the special case of $M$ being a model manifold satisfying (2.1). From standard results, see e.g. [24, Theorem 2.5 and Proposition 2.7], the following Proposition can be proved.
Proposition 6.1 Let $\mathbb{M}_\varphi^N$ be a model manifold such that (2.1) and (1.9) are satisfied. Let $b : \mathbb{M}_\varphi^N \to \mathbb{R}^N$, $b \in C^1(\mathbb{M}_\varphi^N)$ be such that assumption (2.2) holds. Moreover assume $(H_3)$. Suppose that there exist

(i) a bounded supersolution $h$ of equation

$$
\Delta h + \langle b, \nabla h \rangle - ch = -1 \quad \text{in} \quad \mathbb{M}_\varphi^N \setminus B_{R_0},
$$

(6.1)

for some $R_0 > 0$, such that

$$
h > 0 \quad \text{in} \quad M, \quad \lim_{r \to +\infty} h(x) = 0.
$$

(ii) a positive bounded supersolution $W$ of the equation

$$
\Delta W + \langle b, \nabla W \rangle - cW = -1 \quad \text{in} \quad \mathbb{M}_\varphi^N.
$$

(6.2)

Then there exist infinitely many bounded solutions $u$ of problem (1.1). In particular, for any $\gamma \in \mathbb{R}$, there exists a solution $u$ to equation (1.1) such that

$$
\lim_{r \to \infty} u(x) = \gamma.
$$

Proposition 6.1 shows that infinitely many solutions to equation (1.1) exist, if a supersolution $h$ to (6.1) exists and if a superolution $W$ to (6.2) exists. Therefore the following existence results, combined with the above proposition, imply nonuniqueness.

Lemma 6.2 Let assumptions (2.1), (1.9), $(H_3)$ and (2.2) hold. Then there exists a supersolution $h > 0$ of equation (6.1).

Proof We consider the function

$$
h(x) := Cr^{-\beta} \quad \text{for any} \quad x \in \mathbb{M}_\varphi^N \setminus B_{R_0},
$$

where $C > 0$ and $\beta > 0$ have to be chosen. Note that, due to (2.6) and (2.1)

$$
\Delta h(x) = \beta(\beta + 1)C r^{-\beta - 2} - \frac{\varphi'}{\varphi}(N - 1) \beta C r^{-\beta - 1}
$$

$$
= \beta C (\beta + 1 - (N - 1) \lambda) r^{-\beta - 2} \quad \text{for all} \quad x \in \mathbb{M}_\varphi^N \setminus B_{R_0}.
$$

Thus, in view of (2.2), we have, for some $C_\beta > 0$.

\begin{align}
\Delta h(x) + \langle b(x), \nabla h(x) \rangle - c h(x) & \leq \beta C (\beta + 1 - (N - 1) \lambda) r^{-\beta - 2} - \beta C r^{-\beta - 1} \langle b(x), \nabla r \rangle - cC r^{-\beta} \\
& \leq \beta C (\beta + 1 - (N - 1) \lambda) r^{-\beta - 2} - \beta C K r^{\sigma - \beta - 1} - cC r^{-\beta} \\
& \leq -C_\beta r^{\sigma - 1 - \beta}.
\end{align}

(6.3)
for all $x \in \mathbb{M}_\varphi^N \setminus B_{R_0}$, with $R_0$ as in (2.2). Now, from (6.3) it follows that

$$\Delta h(x) + \langle b(x), \nabla h(x) \rangle - c h(x) \leq -1$$

for all $x \in \mathbb{M}_\varphi^N \setminus \bar{B}_{R_0}$;

provided that

$$\sigma - 1 - \beta > 0.$$ 

Hence, by the assumption $\sigma > 1$, for $\beta$ small enough, we get the thesis. \hfill \Box

**Lemma 6.3** Let assumption $(H_3)$ holds. Then there exists a bounded supersolution $W > 0$ of equation (6.2).

**Proof** We define the function

$$W(x) := \frac{1}{c_0},$$

for $c_0$ as in $(H_3)$. Then, due to $(H_3)$

$$\Delta W(x) + \langle b(x), \nabla W(x) \rangle - c W(x) \leq - \frac{c}{c_0} \leq -1,$$

for all $x \in \mathbb{M}_\varphi^N$. The Lemma is proved. \hfill \Box

**Proof of Proposition 2.5** By simply combining Proposition 6.1, Lemmas 6.2 and 6.3, the result follows. \hfill \Box

**Proof of Corollary 2.6** (i) Due to (2.1), it follows that, for any $B_R \subset M$,

$$\text{Vol}(x_0, r) \leq C r^{(N-1)\lambda + 1}.$$ 

Consequently, $M$ satisfies assumption (1.9) for $\alpha = (N - 1)\lambda + 1$. Therefore, since $b$ satisfies $(H_2)$, the thesis follows by Theorem 2.4.

(ii) The thesis directly follows by applying Proposition 2.5. \hfill \Box

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