Characterizing and Understanding the Generalization Error of Transfer Learning with Gibbs Algorithm

Yuheng Bu*, Gholamali Aminian*, Laura Toni, Miguel Rodrigues, Gregory Wornell,

Abstract

We provide an information-theoretic analysis of the generalization ability of Gibbs-based transfer learning algorithms by focusing on two popular transfer learning approaches, $\alpha$-weighted-ERM and two-stage-ERM. Our key result is an exact characterization of the generalization behaviour using the conditional symmetrized KL information between the output hypothesis and the target training samples given the source samples. Our results can also be applied to provide novel distribution-free generalization error upper bounds on these two aforementioned Gibbs algorithms. Our approach is versatile, as it also characterizes the generalization errors and excess risks of these two Gibbs algorithms in the asymptotic regime, where they converge to the $\alpha$-weighted-ERM and two-stage-ERM, respectively. Based on our theoretical results, we show that the benefits of transfer learning can be viewed as a bias-variance trade-off, with the bias induced by the source distribution and the variance induced by the lack of target samples. We believe this viewpoint can guide the choice of transfer learning algorithms in practice.

I. INTRODUCTION

A common assumption in supervised learning is that both the training and test data samples are generated from the same distribution. However, this assumption does not always hold in many applications, as we often have easy access to samples generated from a source distribution, and we want to use the hypothesis trained using source samples for a different target task, from which only limited data are available. Transfer learning and domain adaptation methods are developed to tackle this problem, and the state of the art transfer learning algorithms based on pre-trained models and fine tuning has led to significant improvements in various applications such as computer vision, natural language processing, etc [1, 2, 3, 4].

Many works try to explain the empirical success of transfer learning from different theoretical perspectives. The first theoretical analysis for domain adaptation is proposed by [5] for binary classification, where the authors provide a VC-dimension-based excess risk bound for the zero-one loss in terms of $d_A$-distance as a measure of discrepancy between source and target tasks. A new notion of discrepancy measure for transfer learning called transfer-exponent under covariate-shift assumption is proposed in [6]. A minimax lower bound of generalization error for transfer learning in neural networks is derived in [7]. Recently, an Empirical Risk Minimization (ERM) algorithm via representation learning is proposed in [8], and an upper bound on the excess risk of the new task is provided in terms of Gaussian complexity. [9] provides an upper bound on excess risk based on instance weighting. Using KL divergence as a measure of similarity between source and target data-generating distribution, an information-theoretic generalization error upper bound for transfer learning is proposed in [10].

However, these upper bounds on excess risk and generalization error may not entirely capture the generalization ability of a transfer learning algorithm. One apparent reason is the tightness issue, as the proposed bounds [9] can be loose or even vacuous when evaluated in practice. More importantly, the current definitions

* Equal Contribution.

Y. Bu and G. Wornell are with the Department of Electrical Engineering and Computer Science, Massachusetts Institute of Technology, Cambridge, MA 02139 (Email: buyuheng, gww@mit.edu). 
G. Aminian, L. Toni and M. Rodrigues are with the Electronic and Electrical Engineering Department at University College London, UK, (Email: g.aminian, l.toni, m.rodrigues@ucl.ac.uk).
of discrepancy metric do not fully characterize all the aspects that could influence the performance of a transfer learning problem, e.g., most discrepancy measures are either algorithm independent (KL divergence in [10]), or defined under specific assumption, e.g. transfer-exponent under covariate-shift assumption in [6], or only depend on the hypothesis class ($d_A$-distance in [5]), which cannot provide too much insight in selecting different transfer learning algorithms in practice.

To overcome these limitations, we study two Gibbs algorithms, i.e., $\alpha$-weighted Gibbs algorithm and two-stage Gibbs algorithm which can be viewed as randomized version of two ERM-based transfer learning algorithms, i.e., $\alpha$-weighted-ERM [11, 12] and two-stage-ERM [8, 13] using information-theoretic tools.

Our main contributions are as follows:

- We derive exact characterizations of the generalization errors for $\alpha$-weighted Gibbs algorithm and two-stage Gibbs algorithm using conditional symmetrized KL information. We also provide novel distribution-free upper bounds, which quantify how the number of samples from the source and target will influence the generalization error of these transfer learning algorithms.
- We further demonstrate how to use our method to characterize the asymptotic behavior of the generalization error for the Gibbs algorithms under large inverse temperature, where the Gibbs algorithms converge to the $\alpha$-weighted-ERM and Two-stage-ERM, respectively.
- By studying the excess risk of the $\alpha$-weighted-ERM and Two-stage-ERM algorithms in the asymptotic regime, we show that the benefits of transfer learning algorithms can be viewed as a bias-variance trade-off, which suggests that the choice of transfer learning algorithm should depend on both the bias induced by the source distribution and the number of target samples.

## II. Problem Formulation

Let $D_s = \{Z^s_i\}_{i=1}^n$ and $D_t = \{Z^t_j\}_{j=1}^m$ be the source and target training sets, respectively, where $Z^s_i$ and $Z^t_j$ are defined on the same alphabet $\mathcal{Z}$. Note that $D_s$ and $D_t$ are independent, but neither $D_s$ nor $D_t$ is required to be i.i.d generated from the data-generating distribution $P^*_Z$ or $P^t_Z$. We denote the joint distribution of all source training samples as $P_{D_s}$ and that of the target training samples as $P_{D_t}$. We denote the hypotheses by $w \in \mathcal{W}$, where $\mathcal{W}$ is a hypothesis class. The performance of any hypotheses is measured by a non-negative loss function $\ell : \mathcal{W} \times \mathcal{Z} \to \mathbb{R}_{+}^+$, and we can define the empirical risk and the population risk of a source task as

$$L_E(w, d_s) = \frac{1}{n} \sum_{i=1}^{n} \ell(w, z^s_i),$$

(1)

$$L_P(w, P_{D_s}) = \mathbb{E}_{P_{D_s}}[L_E(w, D_s)],$$

(2)

and the empirical risk and the population risk of target task

$$L_E(w, d_t) = \frac{1}{m} \sum_{j=1}^{m} \ell(w, z^t_j),$$

(3)

$$L_P(w, P_{D_t}) = \mathbb{E}_{P_{D_t}}[L_E(w, D_t)].$$

(4)

A transfer learning algorithm can be modeled as a randomized mapping from the source and target training sets $(D_s, D_t)$ onto a hypothesis $W \in \mathcal{W}$ according to the conditional distribution $P_{W|D_s,D_t}$. Thus, the expected transfer generalization error quantifying the degree of over-fitting on the target training data can be written as

$$\mathbb{E}_{\mathcal{W}}(P_{W|D_s,D_t}, P_{D_s}, P_{D_t}) = \mathbb{E}_{P_{W,D_s,D_t}}[L_P(W, P_{D_t}) - L_E(W, D_t)],$$

(5)

where the expectation is taken over the joint distribution $P_{W,D_s,D_t} = P_{W|D_s,D_t} \otimes P_{D_s} \otimes P_{D_t}$. 


A. Two Transfer learning Algorithms

We focus on the following two transfer learning approaches, including \( \alpha \)-weighted-ERM and Two-stage-ERM.

\textbf{\( \alpha \)-Weighted-ERM Transfer Learning:} We denote the hypotheses by \( w_\alpha \in \mathcal{W} \) as the output of \( \alpha \)-weighted-ERM learning algorithm. The hypothesis \( w_\alpha \) is trained by minimizing a convex combination of the source and target task empirical risks as in [11], i.e.,

\[
L_E(w_\alpha, d_s, d_t) = (1 - \alpha)L_E(w_\alpha, d_s) + \alpha L_E(w_\alpha, d_t),
\]

for \( 0 \leq \alpha \leq 1 \).

\textbf{Two-stage-ERM Transfer Learning:} Suppose that the hypothesis \( w \in \mathcal{W} \) can be written as \( w = (w_\phi, w_c) \), where \( w_\phi \in \mathcal{W}_\phi \) is the shared hypothesis (parameter) across both source and target tasks, and \( w_c \) denotes some task-specific hypothesis (parameter) for source and target tasks, i.e., \( w_c \in \mathcal{W}_c \) and \( w_\phi \in \mathcal{W}_\phi \). For example \( w_\phi \) collects parameters of first few layers of a neural network for both tasks and \( w_c^s \) and \( w_c^t \) collect the remaining parameters for source and target tasks respectively. The performance of the pair \((w_\phi, w_c)\) is measured by a non-negative loss function \( \ell : \mathcal{W}_c \times \mathcal{W}_\phi \times \mathcal{Z} \to \mathbb{R}_+^+ \). Now, we consider the following two-stage-ERM transfer learning algorithm inspired by [8].

\textbf{First Stage:} The algorithm first learns the shared hypothesis \( w_\phi \) and the source-specific hypothesis \( w_c^s \) by minimizing the following empirical risk function defined on the source data set at Stage 1:

\[
L_E^{S1}(w_\phi, w_c^s, d_s) = \frac{1}{n} \sum_{i=1}^{n} \ell(w_\phi, w_c^s, z_i^s).
\]

\textbf{Second Stage:} We fix the shared hypothesis \( w_\phi \) and learn the target-specific hypothesis \( w_c^t \) by minimizing the following empirical risk function defined on the target data set at Stage 2:

\[
L_E^{S2}(w_\phi, w_c^t, d_t) = \frac{1}{m} \sum_{j=1}^{m} \ell(w_\phi, w_c^t, z_j^t).
\]

B. Transfer Learning with Gibbs algorithms

We generalize the ERM-based transfer learning algorithms by considering their Gibbs versions. The \((\gamma, \pi(w), f(w, s))\)-Gibbs distribution, which was first proposed by [14] in statistical mechanics, is defined as:

\[
P_{\mathcal{W}|S}(w|s) \triangleq \frac{\pi(w)e^{-\gamma f(w, s)}}{V(s, \gamma)}, \quad \gamma \geq 0,
\]

where \( \gamma \) is the inverse temperature, \( \pi(w) \) is an arbitrary prior distribution on \( \mathcal{W} \), \( f(w, s) \) is energy function, and \( V(s, \gamma) \triangleq \int \pi(w)e^{-\gamma f(w, s)} dw \) is the partition function.

The \((\gamma, \pi(w), L_E(w, d_t))\)-Gibbs distribution can be viewed as a randomized version of an ERM algorithm using only target samples if we specify the energy function \( f(w, s) = L_E(w, d_t) \). As the inverse temperature \( \gamma \to \infty \), the prior distribution \( \pi(w) \) becomes negligible, and the Gibbs algorithm converges to the standard supervised-ERM algorithm.

Similarly, we define the following \( \alpha \)-weighted Gibbs algorithm and two-stage Gibbs algorithm, which can be viewed as randomized \( \alpha \)-weighted-ERM and randomized two-stage-ERM, respectively.
Fig. 2: Two-stage Gibbs Algorithm

\( D_s \)  
\( P_{W_s, W_s^0 | D_s} \)  
\( D_t \)  
\( P_{W_t^0 | D_t, W_s} \)  
\( W_φ \)  
\( W_t^0 \)  
\( W_φ \)

\[ D_s \] \[ P_{W_s, W_s^0 | D_s} \] \[ D_t \] \[ P_{W_t^0 | D_t, W_s} \] \[ W_φ \] \[ W_t^0 \] \[ W_φ \]

\( D_s \) \[ P_{W_s, W_s^0 | D_s} \] \[ D_t \] \[ P_{W_t^0 | D_t, W_s} \] \[ W_φ \] \[ W_t^0 \] \[ W_φ \]

\( \alpha \)-weighted Gibbs algorithm generalizes the \( \alpha \)-weighted-ERM by considering the \((\gamma, \pi(w_\alpha), L_E(w_\alpha, d_s, d_t))\)-Gibbs algorithm (see, Figure 1)

\[ P_{W_\alpha | D_s, D_t}(w_\alpha | d_s, d_t) = \frac{\pi(w_\alpha)e^{-\gamma L_E(w_\alpha, d_s, d_t)}}{V_\alpha(d_s, d_t, \gamma)}. \] (10)

The expected transfer generalization error of the \( \alpha \)-weighted Gibbs algorithm is denoted as

\[ \text{gen}_\alpha(D_s, D_t) \triangleq \mathbb{E}_{P_{W_\alpha | D_s, D_t}}[P_{W_\alpha | D_s, D_t}]. \] (11)

Two-stage Gibbs algorithm generalizes the two-stage-ERM by considering the \((\gamma, \pi(w_\phi^t), L_E^{S2}(w_\phi, w_c^t, d_t))\)-Gibbs algorithm

\[ P_{W_c^t | D_t, W_s}(w_c^t | d_t, w_\phi) = \frac{\pi(w_c^t)e^{-\gamma L_E^{S2}(w_\phi, w_c^t, d_t)}}{V_\beta(w_\phi, d_t, \gamma)}. \] (12)

in the second stage, where the learned shared hypothesis \( w_\phi \) is the output of the learning algorithm \( P_{W_s, W_c^0 | D_s} \) at the first stage. As shown in Figure 2, the two-stage Gibbs algorithm is constructed by concatenating two randomized mappings \( P_{W_c^t | D_s, W_s} \) and \( P_{W_c^t, W_c^0 | D_s} \).

The population risk for the target task is defined as:

\[ L_P(w_\phi, w_c^t, D_t) = \mathbb{E}_{P_{D_t}}[L_E^{S2}(w_\phi, w_c^t, D_t)], \] (13)

and the expected transfer generalization error under two-stage Gibbs algorithm can be denoted as

\[ \text{gen}_\beta(D_s, D_t) \triangleq \mathbb{E}_{P_{W_s, W_c^0, D_s, D_t}}[L_P(W_\phi, W_c^t, D_t) - L_E^{S2}(W_\phi, W_c^t, D_t)], \] (14)

where the expectation is taken over the joint distribution \( P_{W_s, W_c^0, D_s, D_t} = P_{W_c^0 | D_s} \otimes P_{D_t} \otimes P_{W_s} \otimes P_{D_s} \).

C. Information Measures

We will be characterizing the aforementioned generalization errors using various information measures. If \( P \) and \( Q \) are probability measures over space \( X \), and \( P \) is absolutely continuous with respect to \( Q \), the Kullback-Leibler (KL) divergence between \( P \) and \( Q \) is given by \( D(P || Q) \triangleq \int_X \log \left( \frac{dP}{dQ} \right) dP \). If \( Q \) is also absolutely continuous with respect to \( P \), the symmetrized KL divergence (a.k.a. Jeffrey’s divergence [15]) is

\[ D_{\text{SKL}}(P || Q) \triangleq D(P || Q) + D(Q || P). \] (15)

The mutual information between two random variables \( X \) and \( Y \) is the KL divergence between the joint distribution and product-of-marginal distribution \( I(X; Y) \triangleq D(P_{X,Y} || P_X \otimes P_Y) \), or equivalently, the conditional KL divergence between \( P_{Y|X} \) and \( P_Y \) averaged over \( P_X \), \( D(P_{Y|X} || P_Y | P_X) \triangleq \int_X D(P_{Y|X} \| P_Y ) dP_X (x) \). By swapping the role of \( P_{X,Y} \) and \( P_X \otimes P_Y \) in mutual information, we get the lautum information introduced by [16], \( L(X; Y) \triangleq D(P_X \otimes P_Y || P_{X,Y}) \). Finally, the symmetrized KL information between \( X \) and \( Y \) is given by [17]:

\[ I_{\text{SKL}}(X; Y) \triangleq D_{\text{SKL}}(P_{X,Y} || P_X \otimes P_Y ) = I(X; Y) + L(X; Y). \] (16)

Throughout the paper, upper-case letters denote random variables (e.g., \( Z \)), lower-case letters denote the realizations of random variables (e.g., \( z \)), and calligraphic letters denote sets (e.g., \( \mathcal{Z} \)). All the logarithms are the natural ones, and all information measure units are in nats. \( \mathcal{N}(\mu, \Sigma) \) denotes a Gaussian distribution with mean \( \mu \) and covariance matrix \( \Sigma \).
III. RELATED WORK

Other Interpretations for Gibbs Algorithm: Besides viewing the Gibbs algorithm as randomized ERM, there are additional interpretations for considering Gibbs algorithm in transfer learning.

SGLD: The Stochastic Gradient Langevin Dynamics (SGLD), which can be viewed as noisy version of Stochastic Gradient Descent (SGD), is defined as:

\[ W_{k+1} = W_k - \eta \nabla L_E(W_k, d_t) + \sqrt{\frac{2\beta}{\gamma}} \zeta_k, \quad k = 0, 1, \ldots, \]

where \( \zeta_k \) is a standard Gaussian random vector and \( \eta > 0 \) is the step size. In [18], it is proved that under some conditions on the loss function, the conditional distribution \( P_{W|D_t} \) induced by SGLD algorithm is close to \((\gamma, \pi(W)), L_E(w_k, d_t))-Gibbs distribution in 2-Wasserstein distance for sufficiently large \( k \).

Information Risk Minimization: The Gibbs algorithm also arises when conditional KL divergence is used as a regularizer to penalize over-fitting in the information risk minimization framework. It is shown in [19, 20, 21] that the solution to the regularized ERM problem

\[ P_{W|D_t} = \arg \inf_{P_{W|D_t}} \left( E_{P_{W|D_t}}[L_E(W, D_t)] + \frac{1}{\gamma} D(P_{W|D_t}||P(W)|P(D_t)) \right), \]  

(17)

corresponds to the \((\gamma, \pi(w)), L_E(w, d_t))-Gibbs distribution. The inverse temperature \( \gamma \) controls the regularization term and balances between over-fitting and generalization.

Generalization Error of the Gibbs Algorithm: An exact characterization of the generalization error for Gibbs algorithm in terms of symmetrized KL information is provided by [Aminian* and Bu* et al., 2021]. The authors also provide a generalization error upper bound with the rate of \( O(\alpha/n) \) under the sub-Gaussian assumption. An information-theoretic upper bound with similar rate \( O(\alpha/n) \) is provided in [23] for the Gibbs algorithm with bounded loss function, and PAC-Bayesian bounds using a variational approximation of Gibbs posteriors are studied in [24, 25, 26] both focus on bounding the excess risk of the Gibbs algorithm.

Information-theoretic generalization error bounds for Supervised Learning: Recently, [19, 27] propose to use the mutual information between the input training set and the output hypothesis to upper bound the expected generalization error. Multiple approaches have been proposed to tighten these information-based bound. [28] provides tighter bounds by considering the individual sample mutual information, [25, 29] propose using chaining mutual information, and [30, 31, 32] advocate the conditioning and processing techniques. Information-theoretic generalization error bounds using other information quantities are also studied, such as, \( f \)-divergence [33], \( \alpha \)-Rényi divergence and maximal leakage [34, 35], and Jensen-Shannon divergence [36, 37]. Using rate-distortion theory, [38, 39, 40] provide information-theoretic generalization error upper bounds for model misspecification and model compression.

Other Analyses of Transfer Learning: In hypothesis transfer learning problem [41], where we only have access to the learned source hypotheses instead of the source training data, an upper bound on the leave-one-out error measured by square loss is provided. An extension of hypothesis transfer learning is studied in [42], where an algorithm combining the hypotheses from multiple sources based on regularized ERM principle is studied. There are also works focusing on the theoretical aspects of domain adaptation, see [5, 11, 43, 44, 45, 46], which are also related to our problem. Note that in domain adaptation, there is no labeled target data and only unlabeled target samples are available. Actually, having access to target labeled data would improve the performance of the learning algorithm for target task [9, 47].

Note that we provide an exact characterization of the generalization error for Gibbs algorithms in transfer learning scenarios, which differs from this body of research.

IV. GENERALIZATION ERROR OF TRANSFER LEARNING ALGORITHM

We now offer an exact characterizations of the expected transfer generalization errors in terms of symmetrized KL information for the \( \alpha \)-weighted and two-stage Gibbs algorithms, respectively. Then, combining the exact characterization of expected transfer generalization error for Gibbs algorithms with a conditional
mutual information-based generalization error upper bound, we derive novel distribution-free upper bounds for these two Gibbs algorithms. Finally, we provide another exact characterizations of the generalization errors in terms of symmetrized KL divergence, which is shown to be useful in the asymptotic analysis.

A. Exact Characterization of Generalization Error Using Symmetrized KL Information

One of our main results, which characterizes the exact expected transfer generalization error of the α-weighted Gibbs algorithm with prior distribution π(wα), is as follows:

**Theorem 1** (Proved in Appendix A). For the α-weighted Gibbs algorithm, 0 < α < 1 and γ > 0,

\[ P_{W_α|D_α,D_t}(w_α|d_s,d_t) = \frac{π(w_α)e^{-γL_E(w_α,d_s,d_t)}}{V_α(d_s,d_t,γ)}, \]  

(18)

the expected transfer generalization error is given by

\[ \mathbb{E}_{α}(P_{D_α,D_t}) = \frac{I_{\text{SKL}}(W_α;D_t|D_s)}{γα}. \]  

(19)

We also provide an exact characterization of the expected transfer generalization error for two-stage Gibbs algorithm using conditional symmetrized KL information.

**Theorem 2** (Proved in Appendix A). The expected transfer generalization error of the two-stage Gibbs algorithm in (12) is given by

\[ \mathbb{E}_β(P_{D_α,D_t}) = \frac{I_{\text{SKL}}(D_t;W_φ|W_α)}{γ}. \]  

(20)

To the best of our knowledge, these results are the first exact characterizations of the expected transfer generalization error for the α-weighted and two-stage Gibbs algorithm. Note that both Theorem 1 and Theorem 2 only assume that the loss function is non-negative and the training set of source and target are independent, and they hold even for non-i.i.d training samples in source and target training sets.

The expected transfer generalization errors are non-negative, i.e., \( \mathbb{E}_α(P_{D_α,D_t}) ≥ 0 \) and \( \mathbb{E}_β(P_{D_α,D_t}) ≥ 0 \), which follows by the non-negativity of the conditional symmetrized KL information.

B. Example: Mean Estimation

We now consider a simple mean estimation problem, where the symmetrized KL information can be computed exactly, to demonstrate the usefulness of our Theorems. All details are provided in Appendix B.

Consider the problem of learning the mean \( μ_t \in \mathbb{R}^d \) of the target task using \( n \) i.i.d. source samples \( D_s = \{Z_i^s\}_{i=1}^n \) and \( m \) i.i.d. target samples \( D_t = \{Z_j^t\}_{j=1}^m \). We assume that the samples from the source and target tasks satisfying \( \mathbb{E}[Z^s] = \mu_s \), \( \text{cov}[Z^s] = \sigma^2_s I_d \) and \( \mathbb{E}[Z^t] = \mu_t \), \( \text{cov}[Z^t] = \sigma^2_t I_d \), respectively. We adopt the mean-squared loss \( \ell(w,z) = \|z - w\|^2_2 \), and assume a Gaussian prior for the mean \( π(w) = \mathcal{N}(μ_0, σ_0^2 I_d) \).

For the α-weighted Gibbs algorithm, if we set inverse-temperature \( γ = \frac{m+n}{2σ^2} \) and \( α = \frac{m}{m+n} \), then the \( (\frac{m+n}{2σ^2}, \mathcal{N}(μ_0, σ_0^2 I_d), L_E(w_α,d_s,d_t))-\text{Gibbs algorithm} \) is given by the following posterior [48],

\[ P_{W_α|D_α,D_t}(w_α|D_s,D_t) \sim \mathcal{N}(m_α, σ^2_α I_d), \]  

(21)

with \( m_α = \frac{σ^2_s μ_0 + σ^2_t}{σ^2_s + σ^2_t} (\sum_{i=1}^n Z_i^s + \sum_{j=1}^m Z_j^t) \), and \( σ^2_α = \frac{σ^2_s σ^2_t}{(m+n)σ^2_s + σ^2_t} \). Since \( P_{W_α|D_α,D_t} \) is Gaussian, the conditional symmetrized KL information does not depend on the distribution \( P_{Z^t} \) when \( \text{cov}[Z^t] = σ^2_t I_d \), i.e.,

\[ I_{\text{SKL}}(W_α;D_t|D_s) = \frac{mdασ^2_0}{(m+n)(σ^2_0 + σ^2)} \]  

(22)

From Theorem 1, the expected transfer generalization error of this algorithm can be computed exactly as:

\[ \mathbb{E}_α(P_{D_α,D_t}) = \frac{I_{\text{SKL}}(W_α;D_t|D_s)}{γα} = \frac{2dασ^2_0}{(m+n)(σ^2_0 + σ^2)}. \]  

(23)
For the two-stage Gibbs algorithm, we learn the first $d_\phi$ components $\mu_\phi \in \mathbb{R}^{d_\phi}$ using source samples, and use the $(\frac{m}{2\pi}, \mathcal{N}(\mu_{0,c}, \sigma_0^2 I_d))$-Gibbs algorithm to learn the remain $d_c = d - d_\phi$ components. Following similar steps, by Theorem 2, we have
\[
\overline{\text{gen}}(P_{D_s}, P_{D_t}) = \frac{I_{\text{SKL}}(W_t' D_t | W_\phi)}{\gamma} = \frac{2d_c \sigma_0^2 \sigma^2_t}{m(\sigma^2_0 + \frac{1}{2\gamma})}.
\] 

**Remark 1** (Comparison with Supervised Learning). It is shown in [Aminian* and Bu* et al., 2021] that the generalization error of a supervised Gibbs algorithm is
\[
\overline{\text{gen}}(P_{W|D_s}, P_{D_t}) = \frac{2d \sigma_0^2 \sigma_0^2}{m(\sigma_0^2 + \frac{1}{2\gamma})},
\] 
where $P_{W|D_s}$ is $(\frac{m}{2\pi}, \mathcal{N}(\mu_0, \sigma_0^2 I_d), L_E(w, d_t))$-Gibbs algorithm. Comparing to the supervised learning algorithm, the $\alpha$-weighted Gibbs algorithm reduces the generalization error to $O(\frac{d}{m+n})$ by fitting $n$ source samples and $m$ target samples simultaneously, and the two-stage Gibbs algorithm achieves the rate of $O(\frac{d}{m})$ by only learning $w_\phi$ from $D_s$.

**Remark 2** (Effect of Source samples). As shown in (23) and (24), the transfer generalization errors of this mean estimation problem do not depend on the distribution of sources samples $D_\phi$. The reason is that the effect of sources samples is cancelled out in generalization error by subtracting the empirical risk from the population risk. Although different sources samples (distribution) do not change generalization error, they will influence the population risks and excess risks, and more detailed discussion is provided in Appendix B.

C. Distribution-free Upper Bounds

To understand the behaviour of expected transfer generalization error, we provide distribution-free upper bounds in this subsection. These bounds quantify how the generalization errors of $\alpha$-weighted and two-stage Gibbs algorithms depend on the number of target (source) samples $m$ ($n$), and can be applied when directly computing symmetrized KL information is hard.

We first provide a conditional mutual information based upper bound on the expected transfer generalization error for any general learning algorithm $P_{W|D_s, D_t}$ under i.i.d and $\sigma$-sub-Gaussian assumption.

**Theorem 3** (Proved in Appendix D). Suppose that the target training samples $D_t = \{Z_j\}_{j=1}^m$ are i.i.d generated from the distribution $P_t^d$, and the non-negative loss function $\ell(w, Z)$ is $\sigma$-sub-Gaussian\(^1\) under the distribution $P_t^d \otimes P_W$. Then the following upper bound holds
\[
|\overline{\text{gen}}(P_{W|D_s, D_t}, P_{D_s}, P_{D_t})| \leq \sqrt{\frac{2\sigma^2}{m} I(W; D_t | D_s)}.
\] 

The following distribution-free upper bound on the expected transfer generalization error for $\alpha$-weighted Gibbs algorithm can be obtained by combining the upper bound in Theorem 3 and the exact characterization in Theorem 1.

**Theorem 4** (Proved in Appendix G). Suppose that the target training samples $D_t = \{Z_j\}_{j=1}^m$ are i.i.d generated from the distribution $P_t^d$, and the non-negative loss function $\ell(w, z)$ is $\sigma_\alpha$-sub-Gaussian under the distribution $P_t^d \otimes P_W$. If we further assume $C_\alpha \leq \frac{I(W_t; D_t | D_s)}{I(W_t, D_t; D_s)}$ for some $C_\alpha \geq 0$, then for the $\alpha$-weighted Gibbs algorithm and $0 < \alpha < 1$,
\[
\overline{\text{gen}}_\alpha(P_{D_s}, P_{D_t}) \leq \frac{2\sigma_\alpha^2 \gamma_\alpha}{(1 + C_\alpha)m}.
\] 

\(^1\)A random variable $X$ is $\sigma$-sub-Gaussian if $\log \mathbb{E}[e^{\lambda(X - \mathbb{E}X)}] \leq \frac{\lambda^2}{2\sigma^2}$, $\forall \lambda \in \mathbb{R}$. 

Remark 3. Let $\alpha = \frac{m}{n+m}$, then we have
\[
\text{gen}_\alpha(P_{D_1}, P_{D_1}) \leq \frac{2\sigma^2 \gamma}{(1 + C_\alpha)(n + m)},
\]
which is lower than the distribution-free upper bound for $(\gamma, \pi(w), L_E(w, d_t))$-Gibbs algorithm $P_{W|D_t}^\gamma$ provided in [Aminian* and Bu* et al., 2021, Theorem 2], i.e., $\text{gen}_\alpha(P_{W|D_t}^\gamma, P_{D_t}) \leq \frac{2\sigma^2 \gamma}{(1 + C_E)m}$, if $C_E = C_\alpha$ and $\sigma^2 = \sigma_\alpha^2$.

Using similar approach, we can obtain a distribution-free upper bound on the expected transfer generalization error for the two-stage Gibbs algorithm.

Theorem 5 (Proved in Appendix G). Suppose that the target training samples $D_t = \{Z_j\}_{j=1}^m$ are i.i.d generated from the distribution $P_Z^t$, and the non-negative loss function $\ell(w, z)$ is $\gamma$-sub-Gaussian under distribution $P_{W|D_t}^t \otimes P_{W|D_t}^t$ for all $w, z \in W\phi$. If we further assume $C_\gamma \leq \frac{L(W; D_t|W_\phi)}{I(W; D_t|W_\phi)}$ for some $C_\gamma \geq 0$, then for the two-stage Gibbs algorithm in (12), we have
\[
\text{gen}_\beta(P_{D_1}, P_{D_1}) \leq \frac{2\sigma^2 \gamma}{(1 + C_\gamma)m}.
\]

Remark 4 (Choice of $C_\gamma$ and $C_\alpha$). Setting $C_\alpha = 0$ in Theorem 4 and $C_\gamma = 0$ in Theorem 5 is always valid since the lautum information is always positive whenever the mutual information is positive.

D. Exact Characterization of Generalization Error Using Symmetrized KL divergence

In this section, we provide exact characterizations of expected transfer generalization errors for $\alpha$-weighted and two-stage Gibbs algorithms using conditional symmetrized KL divergence by considering the Gibbs algorithm defined with the population risks. Such a result is very useful in the asymptotic analysis Section V-A.

Theorem 6 (Proved in Appendix G). The expected transfer generalization error of the $\alpha$-weighted Gibbs algorithm in (10) is given by:
\[
\text{gen}_{\alpha}(P_{D_1}, P_{D_1}) = D_{\text{SKL}}(P_{W, D_1}^{\gamma|D_1} \| P_{W, D_1}^{\gamma|D_1})_{\alpha},
\]
where $P_{W, D_1}^{\gamma|D_1}$ is $(\gamma, \pi(w), L_{\alpha}(w, d_s, P_{D_1}))$-Gibbs algorithm with $L_{\alpha}(w, d_s, P_{D_1}) \triangleq \alpha L_P(w, P_{D_1}) + (1 - \alpha) L_E(w, d_s)$.

Similar result can be obtained for the two-stage Gibbs algorithm.

Theorem 7 (Proved in Appendix G). The expected transfer generalization error of the two-stage Gibbs algorithm in (12) is given by:
\[
\text{gen}_{\gamma}(P_{D_1}, P_{D_1}) = D_{\text{SKL}}(P_{W, D_1}^{\gamma|D_1} \| P_{W, D_1}^{\gamma|D_1})_{\gamma},
\]
where $P_{W, D_1}^{\gamma|D_1}$ is the $(\gamma, \pi(w, d_t), L_P(w, d_t, P_{D_1}))$-Gibbs algorithm.

More discussions about the connection between the results obtained using symmetrized KL information and those of symmetrized KL divergence is provided in Appendix G.

V. ASYMPTOTIC BEHAVIOR OF GENERALIZATION ERROR AND EXCESS RISK

In this section, we first consider the asymptotic behavior of the generalization error for the two Gibbs algorithms as the inverse temperature $\gamma \to \infty$. Note that in this regime, both Gibbs algorithms converge to the corresponding ERM algorithms, and the distribution-free upper bounds obtained in the previous section would become vacuous. Then, we show that such results can be applied to characterize the excess risks of the two ERM algorithms as $m, n \to \infty$, which provides some intuitions for the selection of different transfer learning algorithms.
A. Generalization Error

α-weighted-ERM: We assume that there exists a unique $\hat{W}_\alpha(D_s, D_t)$ and a unique $\hat{W}_\alpha(D_s)$ that minimizes the risk $L_E(w, D_s, D_t)$ and $L_\alpha(w, D_s, P_{D_t})$, respectively, i.e.,

$$\hat{W}_\alpha(D_s, D_t) = \arg\min_{w \in W} L_E(w, D_s, D_t),$$

$$\hat{W}_\alpha(D_s) = \arg\min_{w \in W} L_\alpha(w, D_s, P_{D_t}).$$

(32)

(33)

It is shown in [49] that if the following Hessian matrices

$$H^*(D_s, D_t) \triangleq \nabla_w^2 L_E(w, D_s, D_t)_{|w = \hat{W}_\alpha(D_s, D_t)},$$

$$H^*(D_s) \triangleq \nabla_w^2 L_\alpha(w, D_s, P_{D_t})_{|w = \hat{W}_\alpha(D_s)}$$

(34)

(35)

are not singular, then, as $\gamma \to \infty$

$$P_{W_{\alpha}^\gamma|D_s, D_t} \to N(\hat{W}_\alpha(D_s, D_t), \frac{1}{\gamma} H^*(D_s, D_t)^{-1}),$$

and $P_{W_{\alpha}^\gamma|D_s} \to N(\hat{W}_\alpha(D_s), \frac{1}{\gamma} H^*(D_s)^{-1})$

(36)

in distribution. Thus, the conditional symmetrized KL divergence in Proposition 6 can be evaluated directly using Gaussian approximations.

**Proposition 1** (Proved in Appendix J). If the Hessian matrices $H^*(D_s, D_t) = H^*(D_s) = H^*$ are independent of $D_s$ and $D_t$, then the generalization error of the α-weighted-ERM algorithm is

$$\gen(\alpha)(P_{D_t}, P_{D_s}) = \frac{\mathbb{E}_{P_{D_t}, D_s} [||\hat{W}_\alpha(D_s, D_t) - \hat{W}_\alpha(D_s)||_H^2]}{\alpha},$$

where the notation $||W||_H^2 \triangleq W^\top HW$.

We can use Proposition 1 to obtain the generalization error of the maximum likelihood estimates (MLE) in the asymptotic regime $m, n \to \infty$. More specifically, suppose that we have $m$ and $n$ i.i.d. samples generated from the target distribution $P_{Z_t}^\gamma$ and source distribution $P_{Z_s}^\gamma$, respectively. We want to fit the training data with a parametric distribution family $\{f(z|w_\alpha)\}$ using the α-weighted-ERM algorithm, where $w_\alpha \in W \subset \mathbb{R}^d$ denotes the parameter. Here, the true data-generating distribution may not belong to the parametric family, i.e., $P_{Z_t}^\gamma, P_{Z_s}^\gamma \notin \{f(\cdot|w_\alpha)|w_\alpha \in W\}$.

If we use the log-loss $\ell(w_\alpha, z) = -\log f(z|w_\alpha)$ in the α-weighted Gibbs algorithm, and set $\alpha = \frac{m}{m+n}$, as $\gamma \to \infty$, it converges to the α-weighted-ERM algorithm, which is equivalent to the following MLE, i.e.,

$$\hat{W}_\alpha(D_s, D_t) = \arg\max_{w_\alpha \in W} \sum_{i=1}^n \log f(Z_t^s|w_\alpha) + \sum_{j=1}^m \log f(Z_t^t|w_\alpha).$$

(37)

If we further let $m, n \to \infty$, under regularization conditions for MLE (details in Appendix K) which guarantee that $\hat{W}_\alpha(D_s, D_t)$ and $\hat{W}_\alpha(D_s)$ are unique, we can show that

$$\hat{W}_\alpha(D_s, D_t) - \hat{W}_\alpha(D_s) \to N(0, \frac{m}{(m+n)^2} \bar{J}(w_\alpha^*)^{-1} \bar{I}_t(w_\alpha^*) \bar{J}(w_\alpha^*)^{-1}).$$

(38)

where

$$w_\alpha^* \triangleq \arg\min_{w_\alpha \in W} nD(P_{Z_t}^\gamma || f(\cdot|w)) + mD(P_{Z_s}^\gamma || f(\cdot|w)),$$

(39)

$$\bar{J}(w_\alpha^*)$$ is the weighted expectation of the Hessian matrix, and $\bar{I}(w_\alpha^*)$ is the weighted Fisher information matrix. Detailed definitions of $\bar{J}$ and $\bar{I}$ and proofs are provided in Appendix L.

In addition, the Hessian matrix $H^*(D_s, D_t) \to \bar{J}(w_\alpha^*)$ as $m, n \to \infty$, which is independent of the samples $D_s, D_t$. Thus, Proposition 1 gives

$$\gen(\alpha)(P_{D_t}, P_{D_s}) = \frac{\text{tr}(\bar{I}(w_\alpha^*) \bar{J}(w_\alpha^*)^{-1})}{n + m},$$

(40)
which scales as $O\left(\frac{d}{m+n}\right)$.

**Two-stage-ERM:** We assume that there exists one unique $\hat{W}_{c}^{t}(D_t, W_\phi)$ which minimize the empirical risk of stage 2,

$$\hat{W}_{c}^{t}(D_t, W_\phi) \triangleq \arg\min_{w_c \in W_c} L^{S2}_{E}(W_\phi, w_c, D_t),$$

and there is one unique $\hat{W}_{c}^{t}(W_\phi)$ which minimize the population risk by considering a fixed $W_\phi$.

$$\hat{W}_{c}^{t}(W_\phi) \triangleq \arg\min_{w_c \in W_c} L_{P}(W_\phi, w_c, P_{D_t}).$$

Similarly, if the following Hessian matrices

$$H^t_c(D_t, W_\phi) \triangleq \nabla^2_{w_c} L^{S2}_{E}(W_\phi, w_c, D_t)|_{w_c=\hat{W}_{c}^{t}(D_t, W_\phi)}$$

$$H^t_s(W_\phi) \triangleq \nabla^2_{w_c} L_{P}(W_\phi, w_c, P_{D_t})|_{w_c=\hat{W}_{c}^{t}(W_\phi)}$$

are not singular, we can obtain the following result by evaluating the conditional symmetrized KL divergence in Proposition 7 using similar Gaussian approximation as in (36).

**Proposition 2** (Proved in Appendix J). If Hessian matrices $H^t_c(D_t, W_\phi) = H^t_s(W_\phi) = H^t_c$ are independent of $D_s, D_t$, then the generalization error of the two-stage-ERM algorithm is

$$\mathcal{E}_{\text{gen}}(P_{D_t}, P_{D_s}) = \mathbb{E}_{D_s, D_t, W_\phi} [||\hat{W}_{c}^{t}(D_t, W_\phi) - \hat{W}_{c}^{t}(W_\phi)||^2_2].$$

Consider a similar MLE setting as we did for the $\alpha$-weighted-ERM algorithm, and now we want to fit data with a parametric distribution family $\{f(z_j|w_\phi, w_c^t)\}_{j=1}^n$ using the two-stage-ERM algorithm, where $w_\phi \in W_\phi \subset \mathbb{R}^{d_\phi}$, $w_c^t \in W_c \subset \mathbb{R}^{d_c}$ denote the shared and specific parameters, respectively.

If we use the log-loss $\ell(w_\phi, w_c^t, z) = -\log f(z|w_\phi, w^t_c)$ in the two-stage Gibbs algorithm, as $\gamma \to \infty$, it converges to the following two-stage MLE approach,

$$[\hat{W}_{\phi}(D_s), \hat{W}_{c}^{s}(D_s)] \triangleq \arg\max_{[w_\phi, w_c] \in W} \sum_{i=1}^{n} \log f(Z_i^t|w_\phi, w_c),$$

$$\hat{W}_{c}^{t}(D_t, \hat{W}_{\phi}) \triangleq \arg\max_{w_c \in W_c} \sum_{j=1}^{m} \log f(Z_j^t|\hat{W}_{\phi}, w_c).$$

As $m, n \to \infty$, under similar regularization conditions (details in Appendix K) which guarantee the uniqueness of these estimates, we can show that

$$\hat{W}_{c}^{t}(D_t, \hat{W}_{\phi}) - \hat{W}_{c}^{t}(\hat{W}_{\phi}) \to \mathcal{N}(0, \frac{J_{c}^{t}(w_{\phi}^{*}, w_{c}^{st})^{-1}J_{c}^{t}(w_{\phi}^{*}, w_{c}^{st})J_{c}^{t}(w_{\phi}^{st}, w_{c}^{st})^{-1}}{m}),$$

where

$$[w_{\phi}^{*}, w_{c}^{st}] \triangleq \arg\min_{[w_\phi, w_c] \in W} D(P_{Z^t}||f(\cdot|w_\phi, w_c)),$$

$$w_{c}^{st} \triangleq \arg\min_{w_c \in W_c} D(P_{Z^t}||f(\cdot|w_\phi^{*}, w_c)).$$
and $J_t^l(w^{st}_s, w^{st}_c), J_c^l(w^{st}_s, w^{st}_c)$ stands for the expected Hessian matrix and Fisher information matrix over $w_c$ under target distribution, respectively. Detailed proofs are provided in Appendix L. As the Hessian matrix $H^*_c(D_t, W_\phi) = H^*_c(W_\phi) \rightarrow J^*_c(w^{st}_s, w^{st}_c)$ as $m, n \rightarrow \infty$, by Proposition 2, we have
\[
\gen \beta(P_{D_t}, P_{D_c}) = \mathcal{O}(\frac{d}{m}).
\] (47)

B. Excess Risk in MLE setting

In this subsection, we further consider the excess risks of the $\alpha$-weighted-ERM algorithm and the two-stage-ERM algorithm in the aforementioned MLE setting when $m, n \rightarrow \infty$, and show that such analyses provide some intuitions in selecting different transfer learning algorithms. All the details are provided in Appendix M.

The excess risk [50] is defined as the difference between the population risk achieved by the learning algorithm and that achieved by the optimal hypothesis given the knowledge of the true target distribution $P_{Z_t}$, i.e.,
\[
\mathcal{E}_r(P_{W}) \triangleq \mathbb{E}_{P_{D_{s},D_{t}}}[L_P(W, P_{D_c})] - L_P(w^*_t, P_{D_c}),
\]
with $w^*_t \triangleq \arg\min_{w \in W} L_P(w, P_{D_c}),$ (48)

where $w^*_t = \arg\min_{w \in W} D(P^l_{\tilde{Z}}\| f(\cdot|w))$ also holds in the MLE setting considered here.

**$\alpha$-weighted-ERM:** In general, a proper transfer learning algorithm should have small excess risk $\mathcal{E}_r$, which justifies the following approximation of the excess risk
\[
\mathcal{E}_r(P_{W_{s}(D_{s},D_{t})}) \approx \frac{1}{2}\mathbb{E}_{P_{D_{s},D_{t}}}[\|\tilde{W}_\alpha(D_s, D_t) - w^*_t\|_{J_t(w^*_t)}^2] = \frac{1}{2}\|w^*_s - w^*_t\|_{J_t(w^*_t)}^2 + \frac{\text{tr}(J_t(w^*_t)\text{Cov}(\tilde{W}_\alpha(D_s, D_t)))}{2}.
\]
As we can see from the above expression, the excess risk can be decomposed into squared bias and variance terms. The bias is caused by learning from the mixture of the source and target distributions instead of just the target distribution $P^l_{\tilde{Z}}$. In addition, it can be shown that $\text{tr}(J_t(w^*_t)\text{Cov}(\tilde{W}_\alpha(D_s, D_t))) = \mathcal{O}(\frac{d}{m+n})$, which has the same order as the generalization error in (40).

**Two-stage-ERM:** In the two-stage algorithm, $w^{st}$ can be written as $w^{st} = [w^{st}_s, w^{st}_c]$, and using similar approximation, we have
\[
\mathcal{E}_r(P_{\tilde{W}_{s}(D_{s}),\tilde{W}_c(D_{t},\tilde{W}_{c})}) \approx \frac{1}{2}\|[w^{st}_s, w^{st}_c] - [w^{st}_s, w^{st}_c]\|_{J_t(w^*_s,w^*_c)}^2 + \frac{\text{tr}(J_t(w^{st}_s,w^{st}_c)\text{Cov}(\tilde{W}_\phi(D_s), \tilde{W}_c(D_t, \tilde{W}_\phi)))}{2}.
\] (49)

Here the bias is caused by sharing the parameter $w^{st}_\phi$ from the source distribution. If $w^{st}_s = w^{st}_\phi$, then $w^{st}_c = w^{st}_c$ and the bias is zero. It can be shown that the variance term scales as $O(\frac{d}{n} + \frac{d}{m})$. When $n \gg m$, it reduces to $O(\frac{d}{m})$, which is the same as the generalization error in (47).

In Table I, we summarize the excess risk, and generalization error results for the two transfer learning algorithms studied in the paper and those of the standard supervised learning under MLE setting [51] as $m, n \rightarrow \infty$. The improvement of the excess risk for transfer learning algorithms comes from trading the variance induced by the lack of target samples with the bias introduced by the source distribution, which suggests that the choice of learning algorithm should depend on both source distribution and the number of samples $m,n$.

The bias term in the excess risks can be interpreted as another notion of discrepancy measure, which is algorithm-dependent, as $w^{st}_s$ and $w^{st}_c, w^{st}_c$ are defined as the optimal parameters under different algorithms given the knowledge of both source and target distributions. Sometimes, these bias terms are more useful in choosing an algorithm than the discrepancy measure used in the literature. For example, consider the mean estimation example in Section IV-B, if we set $\mu_s = \mu_t, \sigma_s^2 \ll \sigma_t^2$, and let $m, n \rightarrow \infty$, then the bias term for both $\alpha$-weighted-ERM and two-stage-ERM should be zero, and transfer learning algorithms are preferred.
over the standard ERM. However, the KL divergence between the source and target distribution, which is proposed as a discrepancy measure in [10], would be large.

The generalization error can be interpreted as the variance of the excess risk when $n \gg m$, and the analysis provided in the paper could help us to find a good balance in the bias and variance trade-off.

VI. CONCLUSION

We provide an exact characterization of the generalization error for two Gibbs-based transfer learning algorithms, i.e., $\alpha$-weighted Gibbs algorithm and two-stage-ERM Gibbs algorithm, using conditional symmetrized KL information and divergence. Based on our results, we show that the benefits of transfer learning can be viewed as a bias-variance trade-off, and the bias term suggest a new notion of discrepancy measure, which requires further investigation.
REFERENCES

[1] W. Li, R. Zhao, and X. Wang, “Human reidentification with transferred metric learning,” in *Asian conference on computer vision*, pp. 31–44, Springer, 2012.

[2] M. Long, Y. Cao, J. Wang, and M. Jordan, “Learning transferable features with deep adaptation networks,” in *International conference on machine learning*, pp. 97–105, PMLR, 2015.

[3] J. Yosinski, J. Clune, Y. Bengio, and H. Lipson, “How transferable are features in deep neural networks?,” *arXiv preprint arXiv:1411.1792*, 2014.

[4] C. Raffel, N. Shazeer, A. Roberts, K. Lee, S. Narang, M. Matena, Y. Zhou, W. Li, and P. J. Liu, “Exploring the limits of transfer learning with a unified text-to-text transformer,” *arXiv preprint arXiv:1910.10683*, 2019.

[5] S. Ben-David, J. Blitzer, K. Crammer, F. Pereira, et al., “Analysis of representations for domain adaptation,” *Advances in neural information processing systems*, vol. 19, p. 137, 2007.

[6] S. Hanneke and S. Kpotufe, “On the value of target data in transfer learning,” *Advances in Neural Information Processing Systems*, vol. 32, pp. 9871–9881, 2019.

[7] M. Kalan and Z. Fabian, “Minimax lower bounds for transfer learning with linear and one-hidden layer neural networks,” *Neural Information Processing Systems (NeuRIPS 2020)*, 2020.

[8] N. Tripuraneni, M. Jordan, and C. Jin, “On the theory of transfer learning: The importance of task diversity,” *Advances in Neural Information Processing Systems*, vol. 33, 2020.

[9] B. Wang, J. Mendez, M. Cai, and E. Eaton, “Transfer learning via minimizing the performance gap between domains,” *Advances in Neural Information Processing Systems*, vol. 32, pp. 10645–10655, 2019.

[10] X. Wu, J. H. Manton, U. Aickelin, and J. Zhu, “Information-theoretic analysis for transfer learning,” in *2020 IEEE International Symposium on Information Theory (ISIT)*, pp. 2819–2824, IEEE, 2020.

[11] S. Ben-David, J. Blitzer, K. Crammer, A. Kulesza, F. Pereira, and J. W. Vaughan, “A theory of learning from different domains,” *Machine learning*, vol. 79, no. 1, pp. 151–175, 2010.

[12] C. Zhang, L. Zhang, and J. Ye, “Generalization bounds for domain adaptation,” *Advances in neural information processing systems*, vol. 4, p. 3320, 2012.

[13] J. Donahue, Y. Jia, O. Vinyals, J. Hoffman, N. Zhang, E. Tzeng, and T. Darrell, “Decaf: A deep convolutional activation feature for generic visual recognition,” in *International conference on machine learning*, pp. 647–655, PMLR, 2014.

[14] J. W. Gibbs, “Elementary principles of statistical mechanics,” *Compare*, vol. 289, p. 314, 1902.

[15] H. Jeffreys, “An invariant form for the prior probability in estimation problems,” *Proceedings of the Royal Society of London. Series A. Mathematical and Physical Sciences*, vol. 186, no. 1007, pp. 453–461, 1946.

[16] D. P. Palomar and S. Verdú, “Lautum information,” *IEEE transactions on information theory*, vol. 54, no. 3, pp. 964–975, 2008.

[17] G. Aminian, H. Arjmandi, A. Gohari, M. Nasiri-Kenari, and U. Mitra, “Capacity of diffusion-based molecular communication networks over lti-poisson channels,” *IEEE Transactions on Molecular, Biological and Multi-Scale Communications*, vol. 1, no. 2, pp. 188–201, 2015.

[18] M. Raginsky, A. Rakhlin, and M. Telgarsky, “Non-convex learning via stochastic gradient langevin dynamics: a nonasymptotic analysis,” in *Conference on Learning Theory*, pp. 1674–1703, PMLR, 2017.

[19] A. Xu and M. Raginsky, “Information-theoretic analysis of generalization capability of learning algorithms,” in *Advances in Neural Information Processing Systems*, pp. 2524–2533, 2017.

[20] T. Zhang, “Information-theoretic upper and lower bounds for statistical estimation,” *IEEE Transactions on Information Theory*, vol. 52, no. 4, pp. 1307–1321, 2006.

[21] T. Zhang et al., “From e-entropy to kl-entropy: Analysis of minimum information complexity density estimation,” *The Annals of Statistics*, vol. 34, no. 5, pp. 2180–2210, 2006.

[22] G. Aminian, Y. Bu, L. Toni, M. R. Rodrigues, and G. Wornell, “Characterizing the generalization error of gibbs algorithm with symmetrized kl information,” *ICML-21 Workshop on Information-Theoretic
Methods for Rigorous, Responsible, and Reliable Machine Learning, 2021.

[23] M. Raginsky, A. Rakhlin, M. Tsao, Y. Wu, and A. Xu, “Information-theoretic analysis of stability and bias of learning algorithms,” in *2016 IEEE Information Theory Workshop (ITW)*, pp. 26–30, IEEE, 2016.

[24] P. Alquier, J. Ridgway, and N. Chopin, “On the properties of variational approximations of gibbs posteriors,” *The Journal of Machine Learning Research*, vol. 17, no. 1, pp. 8374–8414, 2016.

[25] A. R. Asadi and E. Abbe, “Chaining meets chain rule: Multilevel entropic regularization and training of neural networks,” *Journal of Machine Learning Research*, vol. 21, no. 139, pp. 1–32, 2020.

[26] I. Kuzborskij, N. Cesa-Bianchi, and C. Szepesvári, “Distribution-dependent analysis of gibbs-erm principle,” in *Conference on Learning Theory*, pp. 2028–2054, PMLR, 2019.

[27] D. Russo and J. Zou, “How much does your data exploration overfit? controlling bias via information usage,” *IEEE Transactions on Information Theory*, vol. 66, no. 1, pp. 302–323, 2019.

[28] Y. Bu, S. Zou, and V. V. Veeravalli, “Tightening mutual information-based bounds on generalization error,” *IEEE Journal on Selected Areas in Information Theory*, vol. 1, no. 1, pp. 121–130, 2020.

[29] A. Asadi, E. Abbe, and S. Verdú, “Chaining mutual information and tightening generalization bounds,” in *Advances in Neural Information Processing Systems*, pp. 7234–7243, 2018.

[30] T. Steinke and L. Zakyntinou, “Reasoning about generalization via conditional mutual information,” *arXiv preprint arXiv:2001.09122*, 2020.

[31] H. Hafez-Kolahi, Z. Golgooni, S. Kasaei, and M. Soleymani, “Conditioning and processing: Techniques to improve information-theoretic generalization bounds,” *Advances in Neural Information Processing Systems*, vol. 33, 2020.

[32] M. Haghifam, J. Negrea, A. Khisti, D. M. Roy, and G. K. Dziugaite, “Sharpened generalization bounds based on conditional mutual information and an application to noisy, iterative algorithms,” *Advances in Neural Information Processing Systems*, 2020.

[33] J. Jiao, Y. Han, and T. Weissman, “Dependence measures bounding the exploration bias for general measurements,” in *2017 IEEE International Symposium on Information Theory (ISIT)*, pp. 1475–1479, IEEE, 2017.

[34] I. Issa, A. R. Esposito, and M. Gastpar, “Strengthened information-theoretic bounds on the generalization error,” in *2019 IEEE International Symposium on Information Theory (ISIT)*, pp. 582–586, IEEE, 2019.

[35] A. R. Esposito, M. Gastpar, and I. Issa, “Generalization error bounds via rényi-, f-divergences and maximal leakage,” *IEEE Transactions on Information Theory*, 2021.

[36] G. Aminian, L. Toni, and M. R. Rodrigues, “Jensen-shannon information based characterization of the generalization error of learning algorithms,” *2020 IEEE Information Theory Workshop (ITW)*, 2020.

[37] G. Aminian, L. Toni, and M. R. Rodrigues, “Information-theoretic bounds on the moments of the generalization error of learning algorithms,” *arXiv preprint arXiv:2102.02016*, 2021.

[38] M. S. Masiha, A. Gohari, M. H. Yassaee, and M. R. Aref, “Learning under distribution mismatch and model misspecification,” in *IEEE International Symposium on Information Theory (ISIT)*, 2021.

[39] Y. Bu, W. Gao, S. Zou, and V. Veeravalli, “Information-theoretic understanding of population risk improvement with model compression,” in *Proceedings of the AAAI Conference on Artificial Intelligence*, vol. 34, pp. 3300–3307, 2020.

[40] Y. Bu, W. Gao, S. Zou, and V. V. Veeravalli, “Population risk improvement with model compression: An information-theoretic approach,” *Entropy*, vol. 23, no. 10, p. 1255, 2021.

[41] I. Kuzborskij and F. Orabona, “Stability and hypothesis transfer learning,” in *International Conference on Machine Learning*, pp. 942–950, PMLR, 2013.

[42] I. Kuzborskij and F. Orabona, “Fast rates by transferring from auxiliary hypotheses,” *Machine Learning*, vol. 106, no. 2, pp. 171–195, 2017.

[43] Y. Mansour, M. Mohri, and A. Rostamizadeh, “Domain adaptation: Learning bounds and algorithms,” *Conference on Learning Theory, (COLT)*, 2009.

[44] Y. Mansour, M. Mohri, and A. Rostamizadeh, “Multiple source adaptation and the rényi divergence,” in *Proceedings of the Twenty-Fifth Conference on Uncertainty in Artificial Intelligence*, pp. 367–374, 2009.
[45] P. Germain, A. Habrard, F. Laviolette, and E. Morvant, “A new pac-bayesian perspective on domain adaptation,” in *International conference on machine learning*, pp. 859–868, PMLR, 2016.

[46] S. B. David, T. Lu, T. Luu, and D. Pál, “Impossibility theorems for domain adaptation,” in *Proceedings of the Thirteenth International Conference on Artificial Intelligence and Statistics*, pp. 129–136, JMLR Workshop and Conference Proceedings, 2010.

[47] Y. Mansour, M. Mohri, J. Ro, A. T. Suresh, and K. Wu, “A theory of multiple-source adaptation with limited target labeled data,” in *International Conference on Artificial Intelligence and Statistics*, pp. 2332–2340, PMLR, 2021.

[48] K. P. Murphy, “Conjugate bayesian analysis of the gaussian distribution,” *def*, vol. 1, no. 2σ2, p. 16, 2007.

[49] C.-R. Hwang, “Laplace’s method revisited: weak convergence of probability measures,” *The Annals of Probability*, pp. 1177–1182, 1980.

[50] M. Mohri, A. Rostamizadeh, and A. Talwalkar, *Foundations of machine learning*. MIT press, 2018.

[51] A. W. Van der Vaart, *Asymptotic statistics*, vol. 3. Cambridge university press, 2000.

[52] S. Boucheron, G. Lugosi, and P. Massart, *Concentration inequalities: A nonasymptotic theory of independence*. Oxford university press, 2013.

[53] H. Zhang and S. X. Chen, “Concentration inequalities for statistical inference,” *arXiv preprint arXiv:2011.02258*, 2020.
A. \(\alpha\)-weighted Gibbs Algorithm

**Theorem 1. (re-stated)** For the \(\alpha\)-weighted Gibbs algorithm, \(0 < \alpha < 1\) and \(\gamma > 0\),

\[
P_{W_\alpha|D_s, D_t}^\gamma(w_\alpha|d_s, d_t) = \frac{\pi(w_\alpha)e^{-\gamma L(w_\alpha, d_s, d_t)}}{V_\alpha(d_s, d_t, \gamma)},
\]
the expected transfer generalization error is given by

\[
\overline{\text{gen}}_\alpha(P_{D_s}, P_{D_t}) = \frac{I_{\text{SKL}}(W_\alpha; D_t|D_s)}{\gamma \alpha}.
\]

**Proof.** By the definition of conditional symmetrized KL information, we have

\[
I_{\text{SKL}}(W_\alpha; D_t|D_s) = \mathbb{E}_{P_{D_t}}[\mathbb{E}_{P_{W_\alpha, D_t|D_s}}[\log(\frac{P_{W_\alpha|D_s}^\gamma}{P_{W_\alpha|D_s}^\gamma})]] + \mathbb{E}_{P_{W_\alpha, D_t|D_s}}[\log(\frac{P_{W_\alpha|D_s}^\gamma}{P_{W_\alpha|D_s}^\gamma})]\]
\]

Combining with fact that \(D_s\) and \(D_t\) are independent, and plug in the posterior of \(\alpha\)-weighted Gibbs algorithm, we have

\[
I_{\text{SKL}}(W_\alpha; D_t|D_s) = \mathbb{E}_{P_{D_t}}[\mathbb{E}_{P_{W_\alpha, D_t|D_s}}[L_E(W_\alpha, D_s, D_t)] - \mathbb{E}_{P_{W_\alpha, D_t, P_{D_t}}}[L_E(W_\alpha, D_s, D_t)]]
\]

\[
= \mathbb{E}_{P_{D_t}}[\mathbb{E}_{P_{W_\alpha, D_t|D_s}}[(1-\alpha)L_E(w_\alpha, d_s) + \alpha L_E(w_\alpha, d_t)] - \mathbb{E}_{P_{W_\alpha, D_t, P_{D_t}}}[L_E(w_\alpha, d_t)]
\]

\[
= \mathbb{E}_{P_{D_t}}[\mathbb{E}_{P_{W_\alpha, D_t|D_s}}[(1-\alpha)L_E(w_\alpha, d_s) + \alpha L_E(w_\alpha, d_t)] - \mathbb{E}_{P_{W_\alpha, D_t, P_{D_t}}}[L_E(w_\alpha, d_t)]
\]

Due to the symmetry of the \(\alpha\)-weighted Gibbs algorithm, if we use \(\overline{\text{gen}}_\alpha(P_{D_s}, P_{D_t})\) to denote the generalization error of treating \(P_{D_t}\) as source task and \(D_s\) as the target, we can obtain that \(\overline{\text{gen}}_\alpha(P_{D_s}, P_{D_t}) = \frac{I_{\text{SKL}}(W_\alpha; D_t|D_s)}{\gamma \alpha}\).

It is also worthwhile to mention that the \(\alpha\)-weighted expected generalization error of both source and target tasks can be characterized in terms of symmetrized KL information as shown in the following Proposition.

**Proposition 3.** For \((\gamma, \pi(w_\alpha), L_E(w_\alpha, d_s, d_t))\)-Gibbs algorithm and \(0 < \alpha < 1\), we have

\[
\overline{\text{gen}}_\alpha(P_{D_s}, P_{D_t}) + (1-\alpha)\overline{\text{gen}}_\alpha(P_{D_t}, P_{D_s}) = \frac{I_{\text{SKL}}(W_\alpha; D_t|D_s)}{\gamma}.
\]

**Proof.** The symmetrized KL information can be written as

\[
I_{\text{SKL}}(W_\alpha; D_t, D_s) = \mathbb{E}_{P_{W_\alpha, D_t, D_s}}[\log(\frac{P_{W_\alpha|D_s}^\gamma}{P_{W_\alpha|D_s}^\gamma})] - \mathbb{E}_{P_{W_\alpha, D_t, D_s}}[\log(\frac{P_{W_\alpha|D_s}^\gamma}{P_{W_\alpha|D_s}^\gamma})].
\]

Plug in the posterior of \(\alpha\)-weighted Gibbs algorithm,

\[
I_{\text{SKL}}(W_\alpha; D_t, D_s) = \mathbb{E}_{P_{W_\alpha, D_t, D_s}}[-\gamma L_E(w_\alpha, d_s, d_t)] + \mathbb{E}_{P_{W_\alpha, P_{D_t}, D_s}}[\gamma L_E(w_\alpha, d_s, d_t)]
\]

\[
= -\gamma \mathbb{E}_{P_{W_\alpha, D_t, D_s}}[\alpha L_E(w_\alpha, d_t) + (1-\alpha)L_E(w_\alpha, d_s)] + \gamma \mathbb{E}_{P_{W_\alpha, P_{D_t}, D_s}}[\alpha L_E(w_\alpha, d_t) + (1-\alpha)L_E(w_\alpha, d_s)]
\]

\[
= \alpha \overline{\text{gen}}_\alpha(P_{D_t}, P_{D_s}) + (1-\alpha)\overline{\text{gen}}_\alpha(P_{D_s}, P_{D_t}).
\]

Note that the Proposition 3 holds even for dependent source \(D_s\) and target \(D_t\) samples.
B. Two-stage Gibbs Algorithm

**Theorem 2. (restated)** The expected transfer generalization error of the two-stage Gibbs algorithm,

\[
P_{c|D,phi}^\gamma(w_c^t|d_t, w_\phi) = \frac{\pi(w_c^t) e^{-\gamma L_E^{g}(w_\phi, w_c^t, d_t)}}{V_\beta(w_\phi, d_t, \gamma)},
\]

is given by

\[
\text{gen}_\beta(P_{D,1}, P_{D,1}) = \frac{I_{\text{SKL}}(D_t; W_c^t|W_\phi)}{\gamma}.
\]

**Proof.** In the second stage we freeze the share parameters \(W_\phi\), and we will update the specific target task parameter. Thus,

\[
I_{\text{SKL}}(W_c^t; D_t|W_\phi) = \mathbb{E}_{P_{W_\phi}} [\mathbb{E}_{P_{W_c^t}, D_t|w_\phi} [\log(P_{W_c^t|D_t, W_\phi})] - \mathbb{E}_{P_{W_c^t}, D_t|w_\phi} [\log(P_{W_c^t|D_t, W_\phi})]] = \gamma \left( \mathbb{E}_{P_{W_\phi}} [\mathbb{E}_{P_{W_c^t}, D_t|w_\phi} [L_E^{g}(W_\phi, W_c^t, D_t)] - \mathbb{E}_{P_{W_c^t}, D_t} [L_E^{g}(W_\phi, W_c^t, D_t)]] \right)
\]

(55)

\[
= \gamma \text{gen}_\beta(P_{D,1}, P_{D,1}).
\]

\[\square\]

C. Symmetrized KL Divergence

The following lemma from [16] characterizes the mutual and lautum information for the Gaussian channel.

**Lemma 1. [16, Theorem 14]** Consider the following model

\[
Y = AX + N_G,
\]

(56)

where \(X \in \mathbb{R}^{d_x}\) denotes the input random vector with zero mean (not necessarily Gaussian), \(A \in \mathbb{R}^{d_y \times d_x}\) denotes the linear transformation undergone by the input, \(Y \in \mathbb{R}^{d_y}\) is the output vector, and \(N_G \in \mathbb{R}^{d_y}\) is a Gaussian noise vector independent of \(X\). The input and the noise covariance matrices are given by \(\Sigma\) and \(\Sigma_{N_G}\). Then, we have

\[
I(X; Y) = \frac{1}{2} \text{tr}(\Sigma_{N_G}^{-1} A \Sigma A^\top) - D(P_Y || P_{N_G}),
\]

(57)

\[
L(X; Y) = \frac{1}{2} \text{tr}(\Sigma_{N_G}^{-1} A \Sigma A^\top) + D(P_Y || P_{N_G}).
\]

(58)

In the \(\alpha\)-weighted Gibbs algorithm, the output \(W_\alpha\) can be written as

\[
W_\alpha = \frac{\sigma_0^2}{\sigma_0^2} \mu_0 + \frac{\sigma_1^2}{\sigma_0^2} \left( \sum_{i=1}^{n} Z_i^s + \sum_{j=1}^{m} Z_j^c \right) + N = \frac{\sigma_0^2}{\sigma_0^2} \sum_{j=1}^{m} (Z_j^c - \mu_c) + \frac{\sigma_1^2}{\sigma_0^2} \mu_0 + \frac{\sigma_1^2}{\sigma_0^2} \mu_c + \frac{\sigma_1^2}{\sigma_0^2} \sum_{i=1}^{n} Z_i^s + N,
\]

(59)

where \(N \sim \mathcal{N}(0, \sigma_1^2 I_d)\), and \(\sigma_1^2 = \frac{\sigma_0^2 \sigma_2}{(m+n)\sigma_0 + \sigma_2}\). For fixed sources training sample \(d_s\), we can set \(P_{N_G} \sim \mathcal{N}(\sigma_1^2 \mu_0, \frac{m \sigma_1^4}{\sigma_0^2} \mu_c + \frac{n \sigma_1^2}{\sigma_0^2} \sum_{i=1}^{n} z_i^s, \sigma_1^2 I_d)\) and \(\Sigma = \sigma_1^2 I_{nd}\) in Lemma 1 gives

\[
I_{\text{SKL}}(W_\alpha; D_t|D_s = d_s) = \text{tr}(\Sigma_{N_G}^{-1} A \Sigma A^\top) = \text{tr} \left( \frac{\sigma_1^2}{\sigma_1^2} \frac{A A^\top} {A A^\top} \right).
\]

(60)

Noticing that \(A A^\top = \frac{m \sigma_1^4}{\sigma_0^2} I_d\) and taking expectation over \(P_S\), we have

\[
I_{\text{SKL}}(W_\alpha; D_t|D_s) = \frac{m \sigma_1^2 \sigma_0^2}{(m+n) \sigma_0^2 + \sigma_2^2 \sigma_0^2}.
\]

(61)

For the two-stage Gibbs algorithm, the output \(W_c^t\) can be written as

\[
W_c^t = \frac{\sigma_0^2}{\sigma_0^2} \mu_0 + \frac{\sigma_c^2}{\sigma_0^2} \sum_{j=1}^{m} Z_j^c + N_c = \frac{\sigma_0^2}{\sigma_0^2} \sum_{j=1}^{m} (Z_j^c - \mu_c) + \frac{\sigma_c^2}{\sigma_0^2} \mu_0 + \frac{\sigma_c^2}{\sigma_0^2} \mu_c + N_c,
\]

(62)
where \( N_c \sim \mathcal{N}(0, \sigma_c^2 I_{d_c}) \), \( \sigma_c^2 = \frac{\sigma_\phi^2 \sigma_\theta^2}{m \sigma_\phi^2 + \sigma^2} \), and subscript \( c \) stands for the task-specific component of the parameters. Since \( W'_c \) is independent of the source samples, setting \( P_{N_G} \sim \mathcal{N}(\frac{\sigma_c^2}{\sigma_\theta^2} \mu_{0,c} + \frac{n \sigma_c^2}{\sigma_\theta^2} \mu_{t,c}, \sigma_c^2 I_{d_c}) \) and \( \Sigma = \sigma_t^2 I_{nd_c} \) in Lemma 1 gives

\[
I_{\text{SKL}}(W'_c; D_t|W_\phi) = \text{tr}(\Sigma_{N_c}^{-1} A \Sigma A^T) = \text{tr}(\frac{\sigma_t^2}{\sigma_c^2} A A^T) = \frac{md_c \sigma_0^2 \sigma_\phi^2}{(m \sigma_\phi^2 + \sigma^2) \sigma_c^2},
\]

where the last step follows due to the fact \( A A^T = \frac{m \sigma^2}{\sigma_c^2} I_{d_c} \) in this case.

\[ \text{D. Effect of Source samples} \]

As shown in (23) and (24), the transfer generalization errors of this mean estimation problem only depend on the number of samples of \( D_s \), and do not depend on the distribution \( P_{D_s} \). In this subsection, we will show that, though different sources samples (distribution) do not change generalization error, they will influence the population risks and excess risks.

In this mean estimation example, the population risk of any \( W \) can be decomposed into

\[
L_P(W, P_{D_s}) = \mathbb{E}_Z(||W - Z||^2) = \mathbb{E}_Z(||W - \mathbb{E}[W] + \mathbb{E}[W] - \mu_t + \mu_t - Z||^2)
\]

\[
= ||\mathbb{E}[W] - \mu_t||^2 + \text{tr}(\text{Cov}[W]) + d \sigma_t^2,
\]

where the first term, \( ||\mathbb{E}[W] - \mu_t||^2 \), is the squared bias, and the second term, \( \text{tr}(\text{Cov}[W]) \), is the variance. It is easy to verify that the optimal \( w^* = \arg \min L_P(W, P_{D_s}) \) is just the target mean \( \mu_t \), and \( L_P(w^*, P_{D_s}) = d \sigma_t^2 \), then the excess risk defined in (48) can be written as,

\[
\mathcal{E}_t(P_W) = ||\mathbb{E}[W] - \mu_t||^2 + \text{tr}(\text{Cov}[W]).
\]

For the \( \alpha \)-weighted Gibbs algorithm in (62), it can be shown that

\[
\text{Bias} = \mathbb{E}[W_\alpha] - \mu_t = \frac{\sigma^2(\mu_0 - \mu_t) + n \sigma_0^2(\mu_s - \mu_t)}{(m + n) \sigma_0^2 + \sigma^2},
\]

\[
\text{tr}(\text{Cov}[W_\alpha]) = \frac{d \sigma_t^4}{\sigma_1^4} (n \sigma_s^2 + m \sigma_\phi^2) + d \sigma_1^2.
\]

The Bias term will be zero if \( \mu_0 = \mu_s = \mu_t \). Thus, the excess risk of \( \alpha \)-weighted Gibbs algorithm will be minimized when \( \mu_s = \mu_t \) and \( \sigma_s^2 = 0 \), which is equivalent to the case that the target mean \( \mu_t \) is known.

For the two-stage Gibbs algorithm, if we learn the first \( d_\phi \) components \( \mu_\phi \in \mathbb{R}^{d_\phi} \) using the Gibbs algorithm with \( (\frac{n}{2} m, \mathcal{N}(\mu_{1,\phi}, \sigma_\phi^2 I_{d_\phi}), L_{E}^{(1)}(\omega_\phi, w_\phi, d_s)) \), and use the \( (\frac{m}{2} m, \mathcal{N}(\mu_{2,\phi}, \sigma_\phi^2 I_{d_\phi}), L_{E}^{(2)}(\mu_\phi, w_\phi, d_s)) \)-Gibbs algorithm to learn the remain \( d_c \) components in the second stage, it can be shown that

\[
\text{Bias}_\phi = \mathbb{E}[W_\phi] - \mu_{t,\phi} = \frac{\sigma^2(\mu_{1,\phi} - \mu_{t,\phi}) + n \sigma_0^2(\mu_{s,\phi} - \mu_{t,\phi})}{n \sigma_0^2 + \sigma^2},
\]

\[
\text{Bias}_c = \mathbb{E}[W'_c] - \mu_{t,c} = \frac{\sigma^2(\mu_{2,c} - \mu_{t,c})}{m \sigma_0^2 + \sigma^2},
\]

\[
\text{tr}(\text{Cov}[W_\phi]) = \frac{md_\phi \sigma_\phi^4 \sigma_\phi^2}{\sigma_1^4} + d_\phi \sigma_\phi^2,
\]

\[
\text{tr}(\text{Cov}[W'_c]) = \frac{md_c \sigma_t^4 \sigma_c^2}{\sigma_1^4} + d_c \sigma_c^2,
\]

with \( \sigma_\phi^2 = \frac{\sigma_\theta^2 \sigma_\phi^2}{m \sigma_\phi^2 + \sigma^2} \) and \( \sigma_c^2 = \frac{\sigma_\theta^2 \sigma_\phi^2}{m \sigma_\phi^2 + \sigma^2} \). The excess risk of the two-stage Gibbs algorithm will be minimized when \( \mu_{s,\phi} = \mu_{t,\phi} \) and \( \sigma_s^2 = 0 \), i.e., the optimal shared parameter \( \mu_{t,\phi} \) is known.
E. Preliminaries

We first provide some preliminaries for our proofs in this section by introducing the notion of cumulant generating function, which characterizes different tail behaviors of random variables.

**Definition 1.** The cumulant generating function (CGF) of a random variable $X$ is defined as

$$\Lambda_X(\lambda) \triangleq \log \mathbb{E}[e^{\lambda X - \mathbb{E}[X]}].$$

(72)

Assuming $\Lambda_X(\lambda)$ exists, it can be verified that $\Lambda_X(0) = \Lambda'_X(0) = 0$, and that it is convex.

**Definition 2.** For a convex function $\psi$ defined on the interval $[0, b)$, where $0 < b \leq \infty$, its Legendre dual $\psi^*$ is defined as

$$\psi^*(x) \triangleq \sup_{\lambda \in [0, b)} (\lambda x - \psi(\lambda)).$$

(73)

The following lemma characterizes a useful property of the Legendre dual and its inverse function.

**Lemma 2.** [52, Lemma 2.4] Assume that $\psi(0) = \psi'(0) = 0$. Then $\psi^*(x)$ defined above is a non-negative convex and non-decreasing function on $[0, \infty)$ with $\psi^*(0) = 0$. Moreover, its inverse function $\psi^{-1}(y) = \inf\{x \geq 0 : \psi^*(x) \geq y\}$ is concave, and can be written as

$$\psi^{-1}(y) = \inf_{\lambda \in [0, b)} \left(\frac{y + \psi(\lambda)}{\lambda}\right), \quad b > 0.$$  

(74)

We consider the distributions with the following tail behaviors in the appendices:

- **Sub-Gaussian:** A random variable $X$ is $\sigma$-sub-Gaussian, if $\psi(\lambda) = \frac{\sigma^2 \lambda^2}{2}$ is an upper bound on $\Lambda_X(\lambda)$, for $\lambda \in \mathbb{R}$. Then by Lemma 2,

  $$\psi^{-1}(y) = \sqrt{2\sigma^2 y}.$$  

- **Sub-Exponential:** A random variable $X$ is $(\sigma^2, b)$-sub-Exponential, if $\psi(\lambda) = \frac{\sigma^2 \lambda^2}{2}$ is an upper bound on $\Lambda_X(\lambda)$, for $0 \leq |\lambda| \leq \frac{1}{b}$ and $b > 0$. Using Lemma 2, we have

  $$\psi^{-1}(y) = \begin{cases} \sqrt{2\sigma^2 y}, & \text{if } y \leq \frac{\sigma^2}{2b}; \\ by + \frac{\sigma^2}{2b}, & \text{otherwise}. \end{cases}$$

- **Sub-Gamma:** A random variable $X$ is $\Gamma(\sigma^2, c_s)$-sub-Gamma [53], if $\psi(\lambda) = \frac{\lambda^2 \sigma^2}{2(1-c_0)}$ is an upper bound on $\Lambda_X(\lambda)$, for $0 < |\lambda| < \frac{1}{c_s}$ and $c_s > 0$. Using Lemma 2, we have

  $$\psi^{-1}(y) = \sqrt{2\sigma^2 y} + c_s y.$$  

F. Proof of Theorem 3

We prove a more general form of Theorem 3 as follows:

**Theorem 8.** Suppose that the target training samples $D_t = \{Z_j^t\}_{j=1}^m$ are i.i.d generated from the distribution $P_Z^t$ and the loss function $\ell(w, Z)$ satisfies $\Lambda_{\ell(w, Z)}(\lambda) \leq \psi(-\lambda)$, for $\lambda \in (-b, 0)$ and $\Lambda_{\ell(w, Z)}(\lambda) \leq \psi(\lambda)$, for $\lambda \in (0, b)$ and $b > 0$ under the distribution $P_Z^t \otimes P_W$. The following upper bound holds:

$$|\mathbb{E}_{P_W[D_t, D_t, P_{D_t}, P_{D_t}]}[I(W; D_t)|D_S]| \leq \psi^{-1}\left(\frac{I(W; D_t)|D_S)}{m}\right).$$

(75)

**Proof.** The generalization error can be written as

$$|\mathbb{E}_{P_W[D_t, D_t, P_{D_t}, P_{D_t}]|D_S}| \leq \frac{1}{m} \sum_{i=1}^{m} \mathbb{E}_{P_{W[Z_i^t]}}[\ell(W, Z_i^t)] - \mathbb{E}_{P_W \otimes P_Z^t}[\ell(W, Z_i^t)]].$$

(76)

Using the Donsker–Varadhan variational representation [52], for all $\lambda \in (-b, +b)$,

$$D(P_{W[Z_i^t]|D_S} P_W | D_t \otimes P_Z^t) \geq \mathbb{E}_{P_{W[Z_i^t]|D_S}}[\lambda \ell(W, Z_i^t)] - \log \left(\mathbb{E}_{P_{W[Z_i^t]|D_S}}[e^{\lambda \ell(W, Z_i^t)}]\right).$$

(77)
Then, the following upper bound holds
\[ \mathbb{E}_{P_{w,z_i}}[\lambda(W, Z_i)] - \mathbb{E}_{P_{D_s}}[\log(\mathbb{E}_{P_{W|D_s} \otimes P_Z}[e^{\lambda(W, Z_i)}])] \]
\[ \geq \mathbb{E}_{P_{w,z_i}}[\lambda(W, Z_i)] - \log(\mathbb{E}_{P_{W} \otimes P_Z}[e^{\lambda(W, Z_i)}])] \]
\[ \geq \lambda(\mathbb{E}_{P_{w,z_i}}[\ell(W, Z_i)] - \mathbb{E}_{P_{W} \otimes P_Z}[\ell(W, Z_i)]) - \psi(\lambda). \] (78)

Using similar approach as in [28, Theorem 1],
\[ |\mathbb{E}_{P_{w,z_i}}[\ell(W, Z_i)] - \mathbb{E}_{P_{W} \otimes P_Z}[\ell(W, Z_i)]| \leq \psi^{-1}(I(W; Z_i | D_s)). \] (79)

Now by combining (76) and (79), we have:
\[ |\mathbb{E}_{P_{w,z_i}}[\ell(W, Z_i)] - \mathbb{E}_{P_{W} \otimes P_Z}[\ell(W, Z_i)]| \leq m \sum_{i=1}^{m} \psi^{-1}(I(W; Z_i | D_s)) \]
\[ \leq \psi^{-1}\left(\frac{1}{m} \sum_{i=1}^{m} I(W; Z_i | D_s)\right) \]
\[ \leq \psi^{-1}\left(\frac{I(W; D_t | D_s)}{m}\right), \] (80)

where the inequality follows due to the concavity of \( \psi^{-1} \) function and the independence between \( Z_i \).

**Theorem 3. (restated)** Suppose that the target training samples \( D_t = \{Z_j^t\}_{j=1}^{m} \) are i.i.d generated from the distribution \( P_Z^t \), and the non-negative loss function \( \ell(w, Z) \) is \( \sigma \)-sub-Gaussian under the distribution \( P_Z^t \otimes P_W \). Then, the following upper bound holds
\[ |\mathbb{E}_{P_{w,z_i}}[\ell(W, Z_i)] - \mathbb{E}_{P_{W} \otimes P_Z}[\ell(W, Z_i)]| \leq \sqrt{\frac{2\sigma^2}{m} I(W; D_t | D_s)}. \]

**Proof.** For \( \sigma \)-subgaussian assumption, we have \( \psi^{-1}(y) = \sqrt{2\sigma^2 y} \) in Theorem 8 and this completes the proof. \( \square \)

**Remark 5.** Similar upper bound on the expected transfer generalization error in Theorem 3 holds by considering a different assumption that the loss function \( \ell(w, Z) \) is \( \sigma \)-sub-Gaussian under the distribution \( P_Z^t \) for all \( w \in W \).

**G. Other Tail Distributions**

Using Theorem 8, we can also provide upper bounds on the expected transfer generalization error for any general learning algorithms under sub-Exponential and sub-Gamma assumptions.

**Corollary 1** (Sub-Exponential). Suppose that the target training samples \( D_t = \{Z_j^t\}_{j=1}^{m} \) are i.i.d generated from the distribution \( P_Z^t \), and the non-negative loss function \( \ell(w, Z) \) \( (\sigma^2, b) \)-sub-Exponential under distribution \( P_Z^t \otimes P_W \). Then the following upper bound holds
\[ |\mathbb{E}_{P_{w,z_i}}[\ell(W, Z_i)] - \mathbb{E}_{P_{W} \otimes P_Z}[\ell(W, Z_i)]| \leq \left\{ \begin{array}{ll} \sqrt{\frac{2\sigma^2}{b} I(W; D_t | D_s)}, & \text{if } \frac{I(W; D_t | D_s)}{m} \leq \frac{\sigma^2}{2b}; \\
\frac{\sigma^2}{b} + \frac{a_s}{b}, & \text{otherwise}. \end{array} \right. \] (81)

**Corollary 2** (Sub-Gamma). Suppose that the target training samples \( D_t = \{Z_j^t\}_{j=1}^{m} \) are i.i.d generated from the distribution \( P_Z^t \), and the non-negative loss function \( \ell(w, Z) \) is \( \Gamma(\sigma^2, c_s) \)-sub-Gamma under distribution \( P_Z^t \otimes P_W \). Then the following upper bound holds
\[ |\mathbb{E}_{P_{w,z_i}}[\ell(W, Z_i)] - \mathbb{E}_{P_{W} \otimes P_Z}[\ell(W, Z_i)]| \leq \sqrt{\frac{2\sigma^2}{c_s} I(W; D_t | D_s)} + c_s \frac{I(W; D_t | D_s)}{m}. \] (82)
Theorem 4. (restated) Suppose that the target training samples \( D_t = \{Z^t_j\}_{j=1}^m \) are i.i.d generated from the distribution \( P^t_2 \), and the non-negative loss function \( \ell(w, z) = \sigma_\alpha \)-sub-Gaussian under the distribution \( P^t_2 \otimes P_{W_t} \). If we further assume \( C_\alpha < \frac{L(W_t; D_t|D_s)}{I(W_t; D_t|D_s)} \) for some \( C_\alpha > 0 \), then for the \( \alpha \)-weighted Gibbs algorithm and \( 0 < \alpha < 1 \),

\[
\mathrm{gen}_\alpha(P_{D_t}, P_{D_s}) \leq \frac{2\sigma_\alpha^2\gamma\alpha}{(1 + C_\alpha)m}.
\]

Proof. By equation (26) in Theorem 3, we have

\[
\mathrm{gen}_\alpha(P_{D_t}, P_{D_s}) = \frac{I_{SKL}(W_t; D_t|D_s)}{\gamma\alpha} \leq \sqrt{\frac{2\sigma^2 I(W_t; D_t|D_s)}{m}}.
\]

As we have \( I(W_t; D_t|D_s)(1 + C_\alpha) \leq I_{SKL}(W_t; D_t|D_s) \) in the assumption, the following upper bound holds:

\[
I(W_t; D_t|D_s)(1 + C_\alpha) \leq \sqrt{\frac{2\sigma^2 I(W_t; D_t|D_s)}{m}},
\]

which implies that

\[
I(W_t; D_t|D_s) \leq \frac{2\sigma^2\gamma^2\alpha^2}{(1 + C_\alpha)^2m}.
\]

Combining (85) with (83) completes the proof. \( \square \)

Theorem 5. (restated) Suppose that the target training samples \( D_t = \{Z^t_j\}_{j=1}^m \) are i.i.d generated from the distribution \( P^t_2 \), and the non-negative loss function \( \ell(w, z) = \sigma_\beta \)-sub-Gaussian under distribution \( P^t_2 \otimes P_{W_t} \otimes W_t \) for all \( w_t \in W_t \). If we further assume \( C_\beta \leq \frac{L(W_t; D_t|w_t)}{I(W_t; D_t|D_s)} \) for some \( C_\beta \geq 0 \), then for the two-stage Gibbs algorithm,

\[
P_{W_t|D_t, w_t}^{\gamma}(w_t|D_t, w_t) = \frac{\pi(w_t^t) e^{-\gamma L^t_{2\gamma}(w_t^t,w_t^s)}}{V_{\beta}(w_t^s, d_t, \gamma)},
\]

we have

\[
\mathrm{gen}_\beta(P_{D_t}, P_{D_s}) \leq \frac{2\sigma^2\gamma}{(1 + C_\beta)m}.
\]

Proof. Using Theorem 3 by considering \( W = (W_t^t, W_t) \),

\[
\|[\mathrm{gen}_\beta(P_{D_t}, P_{D_s})]\| \leq \sqrt{\frac{2\sigma^2}{m} I(W_t^t; W_t^s; D_t|D_s)}.
\]

Now, based on chain rule for mutual information we have

\[
I(W_t^t; W_t^s; D_t|D_s) = I(W_t^s; D_t|D_s) + I(W_t^t; D_t|D_s, W_t^s)
\]

\[
= I(W_t^t; D_t|W_t^s),
\]

where \( I(W_t^s; D_t|D_s) = 0 \) due to the fact that \( W_t^s \) is independent from \( D_t \) given \( D_s \), and \( I(W_t^t; D_t|W_t^s, D_s) = I(W_t^t; D_t|W_t^s) \) since \( D_s \perp (W_t^t, D_t)|W_t^s \).

Using Theorem 2, it can be shown that

\[
\mathrm{gen}_\beta(P_{D_t}, P_{D_s}) = \frac{I_{SKL}(D_t; W_t^t|W_t^s)}{\gamma} \leq \sqrt{\frac{2\sigma^2}{m} I(W_t^t; D_t|W_t^s)}.
\]

As we have \( I(W_t^t; D_t|W_t^s)(1 + C_\beta) \leq I_{SKL}(W_t^t; D_t|W_t^s) \), the following bound holds:

\[
\frac{I(W_t^t; D_t|W_t^s)(1 + C_\beta)}{\gamma} \leq \frac{2\sigma_\beta I(W_t^t; D_t|W_t^s)}{m},
\]
which implies that
\[ I(W_c^t; D_t|W_\phi) \leq \frac{2\sigma^2 \gamma^2}{(1 + C_\beta)^2 m} \] (88)
Combining (88) with (86) completes the proof. \(\square\)

We could provide distribution-free upper bounds under sub-Exponential and sub-Gamma assumption using similar approach as in Theorem 4 and Theorem 5 for \(\alpha\)-weighted Gibbs algorithm and two-stage Gibbs algorithm, respectively.

**sub-Exponential:** For \(\alpha\)-weighted Gibbs algorithm, we assume that the loss function is \((\sigma^2_{\alpha,e}, b_\alpha)\)-sub-Exponential under distribution \(P^*_Z \otimes P_{W_\phi}\). And for two-stage Gibbs algorithm, we assume that the loss function is \((\sigma^2_{\beta,e}, b_\beta)\)-sub-Exponential under distribution \(P^a_{Z} \otimes P_{W|w=\phi} W_\phi = w_\phi\) for all \(w_\phi \in W_\phi\). We provide the results in Table II. Denote \(B_\alpha \triangleq \left[ \frac{\gamma \alpha}{1 + C_\alpha} \right]\), \(B_\beta \triangleq \left[ \frac{\gamma \beta}{1 + C_\beta} \right]\), \(I_\alpha \triangleq \frac{2b_\alpha I(W_\phi; D_t|D_\alpha)}{\sigma_{\alpha}^2}\) and \(I_\beta \triangleq \frac{2b_\beta I(W_\phi; D_t|W_\phi)}{\sigma_{\beta}^2}\) in Table II.

**sub-Gamma:** For \(\alpha\)-weighted Gibbs algorithm, we assume that the loss function is \(\Gamma(\sigma^2_{\alpha,s}, c_{\alpha,s})\)-sub-Gamma under distribution \(P^*_Z \otimes P_{W_\phi}\) and \(m > \frac{\gamma \alpha c_{\alpha,s}}{(1 + C_\alpha)}\). For two-stage Gibbs algorithm, we assume that the loss function is \(\Gamma(\sigma^2_{\beta,s}, c_{\beta,s})\)-sub-Gamma under distribution \(P^a_{Z} \otimes P_{W|w=\phi} W_\phi = w_\phi\) for all \(w_\phi \in W_\phi\) and \(m > \frac{\gamma c_{\beta,s}}{(1 + C_\beta)}\). We provide the results in Table II.

**TABLE II: Distribution-free Upper Bounds under different Tail Distributions.**

| sub-Exponential | sub-Gamma |
|-----------------|-----------|
| \(\alpha\)-weighted Gibbs Algorithm | \[\begin{align*}
2\sigma^2_{\alpha} & \hfill \frac{\gamma \alpha}{1 + C_\alpha} + 1 \hfill \\
\sigma^2_{\alpha} & \hfill \frac{\gamma \alpha}{m(1 + C_\alpha) - \gamma \alpha} + 1
\end{align*}\] if \(m \geq I_\alpha\); \(\frac{2\sigma^2 \gamma}{(1 + C_\alpha) m - \gamma \alpha c_{\alpha,s}} \hfill \frac{2\sigma^2 \gamma}{(1 + C_\alpha) m - \gamma \alpha c_{\alpha,s}}\) if \(B_\alpha < m < I_\alpha\) |
| Two-stage Gibbs Algorithm | \[\begin{align*}
2\sigma^2_{\beta} & \hfill \frac{\gamma \beta}{1 + C_\beta} + 1 \hfill \\
\sigma^2_{\beta} & \hfill \frac{\gamma \beta}{m(1 + C_\beta) - \gamma \beta} + 1
\end{align*}\] if \(m \geq I_\beta\); \(\frac{2\sigma^2 \gamma}{(1 + C_\beta) m - \gamma \beta c_{\beta,s}} \frac{2\sigma^2 \gamma}{(1 + C_\beta) m - \gamma \beta c_{\beta,s}}\) if \(B_\beta < m < I_\beta\) |

We first present the following Lemma to prove the results related to symmetrized KL divergence.

**Lemma 3.** Denote the \((\gamma, \pi(w), L_E(w, d_t))\)-Gibbs algorithm as \(P^\gamma_{W|D_t}\) and the \((\gamma, \pi(w), L_P(w, P_{D_t}))\)-Gibbs algorithm as \(P^\gamma \triangleq L_{P_{D_t}}\). Then, the following equality holds for these two Gibbs distributions with the same inverse temperature and prior distribution
\[
\mathbb{E}_{\Delta(P^\gamma_{W|D_t=d_t}, P^\gamma_{W|L_{P_{D_t}}})} [L_P(W, P_{D_t}) - L_E(W, d_t)] = \frac{D_{\text{SKL}}(P^\gamma_{W|D_t=d_t}, P^\gamma_{W|L_{P_{D_t}}})}{\gamma},
\] (89)
where \(\mathbb{E}_{\Delta(P^\gamma_{W|D_t=d_t}, P^\gamma_{W|L_{P_{D_t}}})} [f(W)] = \mathbb{E}_{P^\gamma_{W|D_t=d_t}} [f(W)] - \mathbb{E}_{P^\gamma_{W|L_{P_{D_t}}}} [f(W)]\).

**Proof.**
\[
D_{\text{SKL}}(P^\gamma_{W|D_t=d_t}, P^\gamma_{W|L_{P_{D_t}}}) = \int_W \left( P^\gamma_{W|D_t=d_t} - P^\gamma_{W|L_{P_{D_t}}} \right) \log \left( \frac{P^\gamma_{W|D_t=d_t}}{P^\gamma_{W|L_{P_{D_t}}}} \right) dW
\]
\[
= \int_W \left( P^\gamma_{W|D_t=d_t} - P^\gamma_{W|L_{P_{D_t}}} \right) \log (e^{-\gamma(L_E(w, d_t) - L_P(w, P_{D_t}))}) dW
\]
\[
= \gamma \mathbb{E}_{\Delta(P^\gamma_{W|D_t=d_t}, P^\gamma_{W|L_{P_{D_t}}})} [L_P(W, P_{D_t}) - L_E(W, d_t)].
\] \(\square\)

Using Lemma 3, we provide different characterizations of \(\alpha\)-weighted Gibbs algorithm and two-stage Gibbs algorithm using symmetrized KL divergence.
H. α-weighted Gibbs Algorithm

**Theorem 6. (restated)** The expected transfer generalization error of the α-weighted Gibbs algorithm in (10) is given by:

\[
\Delta_{\alpha}(P_{D_s}, P_{D_t}) = D_{\text{SKL}}(P_{W_a[D_s,D_t]}^\gamma || P_{W_a[D_s]}^\gamma L_a(w_a, d_s, P_{D_t})) / \gamma \alpha,
\]

where \( P_{W_a[D_s]}^\gamma \) is the \((\gamma, \pi(w_a), L_a(w_a, d_s, P_{D_t}))\)-Gibbs algorithm with \( L_a(w, d_s, P_{D_t}) = \alpha L_P(w_a, P_{D_t}) + (1 - \alpha) L_E(w_a, d_s) \).

**Proof.** Applying Lemma 3 to the α-weighted Gibbs algorithm and \((\gamma, \pi(w_a), L_a(w, d_s, P_{D_t}))\)-Gibbs algorithm gives

\[
D_{\text{SKL}}(P_{W_a[D_s,D_t]}^\gamma || P_{W_a[D_s]}^\gamma L_a(w_a, d_s, P_{D_t})) = E \Delta(P_{W_a[D_s,D_t]}^\gamma || P_{W_a[D_s]}^\gamma L_a(w_a, d_s, P_{D_t})) = \alpha E \Delta(P_{W_a[D_s,D_t]}^\gamma || P_{W_a[D_s]}^\gamma L_a(w_a, d_s, P_{D_t}))
\]

Notice the fact that

\[
E_{P_{W_a[D_s = d_s]}}[L_P(W_a, P_{D_t})] = E_{P_{W_a[D_s = d_s]}}[E_{P_{D_t}}[L_P(W_a, P_{D_t})]]
\]

and taking expectation over \( D_s \) and \( D_t \), we have

\[
D_{\text{SKL}}(P_{W_a[D_s,D_t]}^\gamma || P_{W_a[D_s]}^\gamma L_a(w_a, d_s, P_{D_t})) = E_{P_{D_s}, P_{D_t}}[D_{\text{SKL}}(P_{W_a[D_s,D_t]}^\gamma || P_{W_a[D_s]}^\gamma L_a(w_a, d_s, P_{D_t}))]
\]

\[= \gamma \alpha \Delta_{\alpha}(P_{D_s}, P_{D_t}). \]

In the following, we provide an explanation for the existence of two different characterizations of the expected transfer generalization error, i.e., Theorem 6 and Theorem 1.

For an arbitrary conditional distribution on hypothesis space \( Q_{W_a[D_s]} \), we can write

\[
I(W_a; D_t | D_s) = D(P_{W_a[D_s]} || Q_{W_a[D_s] \otimes P_{D_t}}) - D(P_{W_a[D_s]} || Q_{W_a[D_s]}) + D(P_{W_a[D_s]} || Q_{W_a[D_s]}) - D(P_{W_a[D_s]} || Q_{W_a[D_s]} \otimes P_{D_t}),
\]

\[
L(W_a; D_t | D_s) = E_{P_{D_s}}[E_{P_{D_t} \otimes P_{W_a[D_s]}}[\log(Q_{W_a[D_s]/P_{W_a[D_s,D_t]})]]] + D(P_{W_a[D_s]} || Q_{W_a[D_s]} \otimes P_{D_t}).
\]

Thus, the symmetrized KL information can be written as

\[
I_{\text{SKL}}(W_a; D_t | D_s) = I(W_a; D_t | D_s) + L(W_a; D_t | D_s)
\]

\[
= D(P_{W_a[D_s,D_t]} || Q_{W_a[D_s] \otimes P_{D_t}}) + E_{P_{D_s}}[E_{P_{D_t} \otimes P_{W_a[D_s]}}[\log(Q_{W_a[D_s]/P_{W_a[D_s,D_t]})]]],
\]

which holds for all \( Q_{W_a[D_s]} \). We compare this expression with the following representation:

\[
D(P_{W_a[D_s,D_t]} || Q_{W_a[D_s] \otimes P_{D_t}}) + D(Q_{W_a[D_s]} || Q_{W_a[D_s,D_t]} \otimes P_{D_t}).
\]

The difference between these two expressions is as follows:

\[
I_{\text{SKL}}(W_a; D_t | D_s) - (D(P_{W_a[D_s,D_t]} || Q_{W_a[D_s] \otimes P_{D_t}}) + D(Q_{W_a[D_s]} || Q_{W_a[D_s,D_t]} \otimes P_{D_t}))
\]

\[
= E_{P_{D_s}}[E_{P_{D_t} \otimes P_{W_a[D_s]}}[\log(Q_{W_a[D_s]/P_{W_a[D_s,D_t]})]]] - D(Q_{W_a[D_s]} || Q_{W_a[D_s,D_t]} \otimes P_{D_t})
\]

\[
= E_{P_{D_s}}[E_{P_{D_t} \otimes P_{W_a[D_s]}}[\log(Q_{W_a[D_s]/P_{W_a[D_s,D_t]})]]] - E_{P_{D_s} \otimes Q_{W_a[D_s]}}[\log(Q_{W_a[D_s]/P_{W_a[D_s,D_t]})]]
\]

\[
= E_{P_{D_s}}[E[D_{\text{SKL}}(P_{W_a[D_s,D_t]} || Q_{W_a[D_s]}), E_{P_{D_t}}[\log(Q_{W_a[D_s]/P_{W_a[D_s,D_t]})]]].
\]

Thus, if \( Q_{W_a[D_s]} \) satisfies the following condition

\[
E_D(P_{W_a[D_s,D_t]} || Q_{W_a[D_s]}), E_{P_{D_t}}[\log(Q_{W_a[D_s]/P_{W_a[D_s,D_t]})]] = 0,
\]

we can write
then we have
\[ I_{\text{SKL}}(W_\alpha; D_t) = D(P_{W_\alpha,D_t}|D_s) \| Q_{W_\alpha}|D_s \| P_{D_t} \| P_{D_s}) + D(Q_{W_\alpha}|D_s \| P_{D_t} \| P_{W_\alpha,D_t}|D_s \| P_{D_s}). \] (99)

Now, if we set \((\gamma, \pi(w), L_E(w, d_s, d_t))\)-Gibbs algorithm as \(P_{W_\alpha,D_t}|D_s\), then it can be verified that using \((\gamma, \pi(w), L_\alpha(w_\alpha, d_s, P_{D_t}))\)-Gibbs algorithm as \(Q_{W_\alpha}|D_s\) would satisfy the condition in (98). Thus, we can represent the expected transfer generalization error using both symmetrized KL information and divergence.

I. Two-stage Gibbs Algorithm

**Theorem 7. (restated)** The expected transfer generalization error of the two-stage Gibbs algorithm in (12) is given by:

\[ \overline{\text{gen}}_{\beta}(P_{D_s}, P_{D_t}) = \frac{D_{\text{SKL}}(P_{W_\gamma|D_s,D_t}|D_s \| W \| P_{W_\gamma|D_s,D_t}|D_s \| P_{D_s})}{\gamma}, \]

where \(P_{W_\gamma|D_s,D_t}|D_s \| W \| P_{W_\gamma|D_s,D_t}|D_s \| P_{D_s})\) is the \((\gamma, \pi(w_c^t), L_P(w_\phi, w_c^t, P_{D_t}))\)-Gibbs algorithm.

**Proof.** Applying Lemma 3 to the two-stage Gibbs algorithm and \((\gamma, \pi(w_c^t), L_P(w_\phi, w_c^t, P_{D_t}))\)-Gibbs algorithm, we have

\[ D_{\text{SKL}}(P_{W_\gamma|D_s,D_t}|D_s \| W \| P_{W_\gamma|D_s,D_t}|D_s \| P_{D_s}) \]

\[ = \mathbb{E}_{D_t} \left( P_{W_\gamma|D_s,D_t}|D_s \| W \| P_{W_\gamma|D_s,D_t}|D_s \| P_{D_s} \right) \left[ L_P(W_c^t, w_\phi, P_{D_t}) - L_E(W_c^t, w_\alpha, d_t) \right]. \] (100)

Notice the fact that

\[ \mathbb{E}_{P_{W_\gamma|D_s,D_t}|D_s \| W \| P_{W_\gamma|D_s,D_t}|D_s \| P_{D_s}} \left[ L_P(W_c^t, w_\phi, P_{D_t}) \right] = \mathbb{E}_{P_{D_t}} \left[ \mathbb{E}_{P_{W_\gamma|D_s,D_t}|D_s \| W \| P_{W_\gamma|D_s,D_t}|D_s \| P_{D_s}} \left[ L_E(W_c^t, w_\phi, d_t) \right] \right], \]

and taking expectation over \(W_\phi\) and \(D_t\) completes the proof. \(\square\)

J. Generalization Error

**Proposition 1. (restated)** If the Hessian matrices \(H^*(D_s, D_t) = H^*(D_s) = H^*\) are independent of \(D_s\) and \(D_t\), then the generalization error of the \(\alpha\)-weighted-ERM algorithm is

\[ \overline{\text{gen}}_{\alpha}(P_{D_s}, P_{D_t}) = \mathbb{E}_{P_{D_s,D_t}} \left[ \| \hat{W}_\alpha(D_s, D_t) - \hat{W}_\alpha(D_s) \|^2_{H^*} \right], \]

where the notation \(\| W \|^2_H \triangleq W^\top H W\).

**Proof.** It is shown in [49] that if the following Hessian matrices

\[ H^*(D_s, D_t) \triangleq \nabla_w^2 L_E(w, D_s, D_t)|_{w=\hat{W}_\alpha(D_s, D_t)} \] (101)

\[ H^*(D_s) \triangleq \nabla_w^2 L_\alpha(w, D_s, P_{D_t})|_{w=\hat{W}_\alpha(D_s)} \] (102)

are not singular, then, as \(\gamma \to \infty\)

\[ P_{W_\alpha|D_s,D_t} \to \mathcal{N}(\hat{W}_\alpha(D_s, D_t), \frac{1}{\gamma} H^*(D_s, D_t)^{-1}) \]

and

\[ P_{W_\alpha|D_s} \to \mathcal{N}(\hat{W}_\alpha(D_s), \frac{1}{\gamma} H^*(D_s)^{-1}) \] (103)

in distribution, and we use \(P_{W_\alpha|D_s} \| D_s \), to denote \(P_{W_\alpha|D_s} \| D_s \).
Thus, the conditional symmetrized KL divergence in Theorem 6 can be evaluated directly using Gaussian approximations under the assumption that $H^*(D_s, D_t) = H^*(D_s) = H^*$,

$$D_{\text{SKL}}(P_{W_{\alpha}^{\gamma}}^{\gamma}_{D_s,D_t} \mid P_{W_{\alpha}^{\gamma,L_P}}^{\gamma,L_P}_{D_s,D_t} | P_{D_s}, P_{D_t})$$

$$= \mathbb{E}_{P_{D_s}, P_{D_t}} \left[ \mathbb{E}_{P_{W_{\alpha}^{\gamma,L_P}}^{\gamma,L_P_{W_{\alpha}^{\gamma,L_P}}}} \left[ \log \frac{P_{W_{\alpha}^{\gamma,L_P}}^{\gamma,L_P}_{W_{\alpha}^{\gamma,L_P}}}{P_{W_{\alpha}^{\gamma,L_P}}^{\gamma,L_P}_{W_{\alpha}^{\gamma,L_P}}(D_s, D_t)} \right] \right]$$

$$= \gamma \mathbb{E}_{P_{D_s}, P_{D_t}} \left[ \mathbb{E}_{\Delta P_{W_{\alpha}^{\gamma,L_P}}^{\gamma,L_P}_{W_{\alpha}^{\gamma,L_P}}(D_s, D_t)} \left[ W_{\alpha}^{T}H^*_{\alpha}(D_s, D_t) - W_{\alpha}^{T}H^*_{\alpha}(D_s) \right] \right]$$

Thus,

$$\mathbb{g}_{\alpha}(P_{D_s}, P_{D_t}) = \frac{D_{\text{SKL}}(P_{W_{\alpha}^{\gamma,L_P}}^{\gamma,L_P}_{D_s,D_t} \mid P_{W_{\alpha}^{\gamma,L_P}}^{\gamma,L_P}_{D_s,D_t})}{\gamma \alpha} = \mathbb{E}_{P_{D_s}, P_{D_t}} \left[ \mathbb{E}_{\Delta P_{W_{\alpha}^{\gamma,L_P}}^{\gamma,L_P}_{W_{\alpha}^{\gamma,L_P}}(D_s, D_t)} \left[ W_{\alpha}^{T}H^*_{\alpha}(D_s, D_t) - W_{\alpha}^{T}H^*_{\alpha}(D_s) \right] \right].$$

Proposition 2. (restated) If Hessian matrices $H^*_c(D_t, W_\phi) = H^*_c(W_\phi) = H^*_c$ are independent of $D_s, D_t$, then the generalization error of the two-stage-ERM algorithm is

$$\mathbb{g}_{\alpha}(P_{D_s}, P_{D_t}) = \mathbb{E}_{D_s, D_t, W_\phi} \left[ \mathbb{E}_{\Delta P_{W_{\alpha}^{\gamma,L_P}}^{\gamma,L_P}_{W_{\alpha}^{\gamma,L_P}}(D_s, D_t)} \left[ W_{\alpha}^{T}H^*_{\alpha}(D_s, D_t) - W_{\alpha}^{T}H^*_{\alpha}(D_s) \right] \right].$$

Proof. It is shown in [49] that if the following Hessian matrices

$$H^*_c(D_t, W_\phi) \triangleq \nabla^2_{w_c}L^2_{W_{\alpha}^{\gamma,L_P}}(W_\phi, w_c, D_t) \bigg|_{w_c = W^*_c(D_t, W_\phi)}$$

$$H^*_c(W_\phi) \triangleq \nabla^2_{w_c}L_P(W_\phi, w_c, P_{D_t}) \bigg|_{w_c = W^*_c(W_\phi)}$$

are not singular, then, as $\gamma \to \infty$

$$P_{W_{\alpha}^{\gamma,L_P}}^{\gamma,L_P}_{W_{\alpha}^{\gamma,L_P}} \to N(\hat{W}_{\alpha}^{T}(D_t, W_\phi), \frac{1}{\gamma}H^*_c(D_t, W_\phi)^{-1}),$$

$$P_{W_{\alpha}^{\gamma,L_P}}^{\gamma,L_P}_{W_{\alpha}^{\gamma,L_P}} \to N(\hat{W}_{\alpha}^{T}(W_\phi), \frac{1}{\gamma}H^*_c(W_\phi)^{-1}),$$

(107) where we use $P_{W_{\alpha}^{\gamma,L_P}}^{\gamma,L_P}_{W_{\alpha}^{\gamma,L_P}}$ to denote $P_{W_{\alpha}^{\gamma,L_P}}^{\gamma,L_P}_{W_{\alpha}^{\gamma,L_P}}(D_t, W_\phi)$. Thus, the conditional symmetrized KL divergence in Theorem 7 can be evaluated directly using Gaussian approximations under the assumption that $H^*_c(D_t, W_\phi) = H^*_c(W_\phi) = H^*_c$. 

$$D_{\text{SKL}}(P_{W_{\alpha}^{\gamma,L_P}}^{\gamma,L_P}_{D_s,D_t} \mid P_{W_{\alpha}^{\gamma,L_P}}^{\gamma,L_P}_{D_s,D_t} | P_{D_s}, P_{W_\phi})$$

$$= \mathbb{E}_{P_{D_s}, W_\phi} \left[ \mathbb{E}_{P_{W_{\alpha}^{\gamma,L_P}}^{\gamma,L_P}_{D_s,D_t}} \left[ \log \frac{P_{W_{\alpha}^{\gamma,L_P}}^{\gamma,L_P}_{D_s,D_t}}{P_{W_{\alpha}^{\gamma,L_P}}^{\gamma,L_P}_{D_s,D_t}(D_s, D_t)} \right] \right]$$

$$= \gamma \mathbb{E}_{P_{D_s}, W_\phi} \left[ \mathbb{E}_{\Delta P_{W_{\alpha}^{\gamma,L_P}}^{\gamma,L_P}_{W_{\alpha}^{\gamma,L_P}}(D_s, D_t)} \left[ W_{\alpha}^{T}H^*_c(D_t, W_\phi) - W_{\alpha}^{T}H^*_c(W_\phi) \right] \right]$$

Thus,

$$\mathbb{g}_{\alpha}(P_{D_s}, P_{D_t}) = \frac{D_{\text{SKL}}(P_{W_{\alpha}^{\gamma,L_P}}^{\gamma,L_P}_{D_s,D_t} \mid P_{W_{\alpha}^{\gamma,L_P}}^{\gamma,L_P}_{D_s,D_t})}{\gamma \alpha} = \mathbb{E}_{P_{D_s}, W_\phi} \left[ \mathbb{E}_{\Delta P_{W_{\alpha}^{\gamma,L_P}}^{\gamma,L_P}_{W_{\alpha}^{\gamma,L_P}}(D_s, D_t)} \left[ W_{\alpha}^{T}H^*_c(D_t, W_\phi) - W_{\alpha}^{T}H^*_c(W_\phi) \right] \right].$$
ERM satisfies

\[
-W_c'(W_\phi)'^TH_cW_c'(D_t, W_\phi) - W_c'(W_\phi)'^TH_cW_c'(W_\phi)
\]

Thus,

\[
\gamma E_{P_{D_t}, W_\phi} [(W_c'(D_t, W_\phi) - W_c'(W_\phi))'^TH_c(W_c'(D_t, W_\phi) - W_c'(W_\phi))].
\]  

(108)

K. Regularity Conditions for MLE

In this section, we present the regularity conditions required by the asymptotic normality [51] of maximum likelihood estimates.

**Assumption 1. Regularity Conditions for MLE:**

1) \( f(z|w) \neq f(z|w') \) for \( w \neq w' \).

2) \( \mathcal{W} \) is an open subset of \( \mathbb{R}^d \).

3) The function \( \log f(z|w) \) is three times continuously differentiable with respect to \( w \).

4) There exist functions \( F_1(z) : \mathcal{Z} \to \mathbb{R}, F_2(z) : \mathcal{Z} \to \mathbb{R} \) and \( M(z) : \mathcal{Z} \to \mathbb{R} \), such that

\[
E_{Z \sim f(z|w)}[M(Z)] < \infty,
\]

and the following inequalities hold for any \( w \in \mathcal{W} \),

\[
\left| \frac{\partial \log f(z|w)}{\partial w_i} \right| < F_1(z), \quad \left| \frac{\partial^2 \log f(z|w)}{\partial w_i \partial w_j} \right| < F_1(z),
\]

\[
\left| \frac{\partial^3 \log f(z|w)}{\partial w_i \partial w_j \partial w_k} \right| < M(z), \quad i, j, k = 1, 2, \cdots, d.
\]

5) The following inequality holds for an arbitrary \( w \in \mathcal{W} \),

\[
0 < E_{Z \sim f(z|w)} \left[ \frac{\partial \log f(z|w)}{\partial w_i} \frac{\partial \log f(z|w)}{\partial w_j} \right] < \infty, \quad i, j = 1, 2, \cdots, d.
\]

L. Generalization error in MLE

**\( \alpha \)-weighted ERM:** We use the following notations to denote the expectation of the Hessian matrices and the Fisher information matrices,

\[
J_\alpha(w_\alpha) \triangleq E_{P_2}[ - \nabla^2_{w_{\alpha}} \log f(Z|w_\alpha) ], \quad J_t(w_\alpha) \triangleq E_{P_2}[ - \nabla^2_{w_{\alpha}} \log f(Z|w_\alpha) ],
\]

\[
I_\alpha(w_\alpha) \triangleq E_{P_2}[ \nabla_{w_{\alpha}} \log f(Z|w_\alpha) \nabla_{w_{\alpha}} \log f(Z|w_\alpha)' ], \quad I_t(w_\alpha) \triangleq E_{P_2}[ \nabla_{w_{\alpha}} \log f(Z|w_\alpha) \nabla_{w_{\alpha}} \log f(Z|w_\alpha)' ];
\]

\[
\bar{J}(w_\alpha) = \frac{n}{m+n} J_\alpha(w_\alpha) + \frac{m}{m+n} J_t(w_\alpha), \quad \bar{I}(w_\alpha) = \frac{n}{m+n} I_\alpha(w_\alpha) + \frac{m}{m+n} I_t(w_\alpha).
\]

**Lemma 4.** Under Assumption 1, for any fixed source samples \( d_s \), if we let \( m \to \infty \), then the \( \alpha \)-weighted ERM satisfies

\[
\sqrt{m} (\hat{W}_\alpha(d_s, D_t) - \bar{W}_\alpha(d_s)) \to N(0, \alpha^2 \bar{J}(\hat{W}_\alpha(d_s))^{-1} I_t(\hat{W}_\alpha(d_s)) \bar{I}(\hat{W}_\alpha(d_s))^{-1}),
\]

(109)

where \( \bar{J}(\hat{W}_\alpha(d_s)) \triangleq \alpha J_t(\hat{W}_\alpha(d_s)) + (1-\alpha) \nabla_w L_E(w, d_s)\big|_{w=\hat{W}_\alpha(d_s)} \) and \( I_t(\hat{W}_\alpha(d_s)) \) is the covariance matrix of \( \nabla_w \log f(Z|\hat{W}_\alpha(d_s)) \).

**Proof.** By using a Taylor expansion of the first derivative of the weighted log-likelihood \( L_E(\hat{W}_\alpha(d_s, D_t), d_s, D_t) \) around \( \hat{W}_\alpha(d_s) \), we obtain

\[
0 = \nabla_w L_E(w, d_s, D_t)\big|_{w=\hat{W}_\alpha(d_s, D_t)}
\]

\[
\approx \nabla_w L_E(w, d_s, D_t)\big|_{w=\hat{W}_\alpha(d_s)} + \nabla^2_w L_E(w, d_s, D_t)\big|_{w=\hat{W}_\alpha(d_s)}(\hat{W}_\alpha(d_s, D_t) - \hat{W}_\alpha(d_s)).
\]

(110)
From the Taylor series expansion formula, the following approximation can be obtained

\[- \nabla_w^2 L_E(w, d_s, D_t)\big|_{w = \hat{W}_\alpha(d_s)} (\hat{W}_\alpha(d_s, D_t) - \hat{W}_\alpha(d_s)) \approx \nabla_w L_E(w, d_s, D_t)\big|_{w = \hat{W}_\alpha(d_s)}. \tag{111}\]

By the law of large numbers, when \(m \to \infty\), it can be shown that

\[- \nabla_w^2 L_E(\hat{W}_\alpha(d_s), D_t) = \frac{1}{m} \sum_{i=1}^{m} \nabla_w^2 \log f(Z_i^t | \hat{W}_\alpha(d_s)) \to -J_t(\hat{W}_\alpha(d_s)). \tag{112}\]

Thus, the LHS of (111) can be written as

\[- \nabla_w^2 L_E(w, d_s, D_t)\big|_{w = \hat{W}_\alpha(d_s)} = \nabla_w [\alpha L_E(w, D_t) + (1 - \alpha) L_E(w, d_s)]\big|_{w = \hat{W}_\alpha(d_s)} \to \bar{J}(\hat{W}_\alpha(d_s)), \tag{113}\]

where \(\bar{J}(\hat{W}_\alpha(d_s)) = \alpha J_t(\hat{W}_\alpha(d_s)) + (1 - \alpha) \nabla_w^2 L_E(w, d_s)\big|_{w = \hat{W}_\alpha(d_s)}\).

As for the RHS of (111), note that

\[\sqrt{m} \nabla_w L_E(w, D_t)\big|_{w = \hat{W}_\alpha(d_s)} = - \frac{1}{\sqrt{m}} \sum_{i=1}^{m} \nabla_w \log f(Z_i^t | \hat{W}_\alpha(d_s)), \tag{114}\]

by multivariate central limit theorem

\[\frac{1}{\sqrt{m}} \sum_{i=1}^{n} \left(- \nabla_w \log f(Z_i^t | \hat{W}_\alpha(d_s)) + \mathbb{E}_{Z_i} [\nabla_w \log f(Z_i^t | \hat{W}_\alpha(d_s))]\right) \to \mathcal{N}(0, \mathcal{I}_t(\hat{W}_\alpha(d_s))), \tag{115}\]

where \(\mathcal{I}_t(\hat{W}_\alpha(d_s))\) is the covariance matrix of \(\nabla_w \log f(Z_i^t | \hat{W}_\alpha(d_s))\).

Due to the definition of \(\hat{W}_\alpha(d_s)\), we have \(\nabla_w L_E(w, d_s, P_{D_t})\big|_{w = \hat{W}_\alpha(d_s)} = 0\), i.e.,

\[(1 - \alpha) \nabla_w L_E(\hat{W}_\alpha(d_s), d_s) = \alpha \mathbb{E}_{Z_i} [\nabla_w \log f(Z_i^t | \hat{W}_\alpha(d_s))]. \tag{116}\]

Thus, the RHS of (111) will converge to

\[\sqrt{m} \nabla_w L_E(w, D_s, D_t)\big|_{w = \hat{W}_\alpha(d_s)} \to \mathcal{N}(0, \alpha^2 \mathcal{I}_t(\hat{W}_\alpha(d_s))). \tag{117}\]

Combining with (112), when \(m \to \infty\), we obtain

\[\sqrt{m}(\hat{W}_\alpha(d_s, D_t) - \hat{W}_\alpha(d_s)) \to \mathcal{N}(0, \alpha^2 \bar{J}(\hat{W}_\alpha(d_s))^{-1} \mathcal{I}_t(\hat{W}_\alpha(d_s)) \bar{J}(\hat{W}_\alpha(d_s))^{-1}). \tag{118}\]

In the main body of the paper, we further let \(n \to \infty\), then \(\hat{W}_\alpha(d_s) \to \hat{w}_\alpha^*\), and \(\bar{J}(\hat{W}_\alpha(d_s)) \to \bar{J}(\hat{w}_\alpha^*)\), \(\mathcal{I}_t(\hat{W}_\alpha(d_s)) \to \mathcal{I}_t(\hat{w}_\alpha^*)\). For \(\alpha = \frac{m}{m+n}\), using Lemma 4, we can show that

\[\hat{W}_\alpha(D_s, D_t) - \hat{w}_\alpha^* \to \mathcal{N}(0, \frac{m}{(m+n)^2} \bar{J}(\hat{w}_\alpha^*)^{-1} \mathcal{I}_t(\hat{w}_\alpha^*) \bar{J}(\hat{w}_\alpha^*)^{-1}). \tag{119}\]

In addition, the Hessian matrix \(H^*(D_s, D_t) \to \bar{J}(\hat{w}_\alpha^*)\) as \(m, n \to \infty\), which is independent of the samples \(D_s, D_t\). Proposition 1 gives

\[\text{gen}_\alpha(P_{D_s}, P_{D_t}) = \frac{\text{tr}(\mathcal{I}_t(\hat{w}_\alpha^*), \bar{J}(\hat{w}_\alpha^*)^{-1})}{n+m} = O\left(\frac{d}{m+n}\right). \tag{119}\]

**Two-stage ERM:**

We use the following notations to denote the expectation of the Hessian matrix and the Fisher information matrix with respect to \(w_c\),

\[J_c^t(w_\phi, w_c) \triangleq \mathbb{E}_{P_Z^t}\left[ - \nabla_w^2 \log f(Z | [w_\phi, w_c]) \right], \]

\[I_c^t(w_\phi, w_c) \triangleq \mathbb{E}_{P_Z^t}(\nabla_w \log f(Z | [w_\phi, w_c])) \nabla_w^\top \log f(Z | [w_\phi, w_c]). \]
Lemma 5. Under Assumption 1, for any fixed $\hat{w}_\phi$, if we let $m \to \infty$, then the two-stage ERM satisfies
\[
\sqrt{m}((\hat{W}_c^t(D_t, \hat{w}_\phi) - \hat{w}_c^t(\hat{w}_\phi))) \to \mathcal{N}(0, J_c^t(\hat{w}_\phi, \hat{w}_c^t(\hat{w}_\phi))^{-1}I_c^t(\hat{w}_\phi, \hat{w}_c^t(\hat{w}_\phi))J_c^t(\hat{w}_\phi, \hat{w}_c^t(\hat{w}_\phi))^{-1}).
\] (120)

Proof. For any fixed $\hat{w}_\phi$, using a Taylor expansion of the gradient with respect to $w_c$ of the log-likelihood $L^S_{E}(\hat{w}_\phi, W_c^t(D_t, \hat{w}_\phi), D_t)$ around $\hat{w}_c^t(\hat{w}_\phi)$, we obtain
\[
0 = \nabla_w L^S_{E}(\hat{w}_\phi, W_c^t(D_t, \hat{w}_\phi), D_t)
\approx \nabla_w L^S_{E}(\hat{w}_\phi, \hat{w}_c^t(\hat{w}_\phi), D_t) + \nabla^2_w L^S_{E}(\hat{w}_\phi, \hat{w}_c^t(\hat{w}_\phi), D_t)(\hat{W}_c^t(D_t, \hat{w}_\phi) - \hat{w}_c^t(\hat{w}_\phi)).
\]

From the Taylor series expansion formula, the following approximation can be obtained
\[
-\nabla^2_w L^S_{E}(\hat{w}_\phi, \hat{w}_c^t(\hat{w}_\phi), D_t)(\hat{W}_c^t(D_t, \hat{w}_\phi) - \hat{w}_c^t(\hat{w}_\phi)) \approx \nabla_w L^S_{E}(\hat{w}_\phi, \hat{w}_c^t(\hat{w}_\phi), D_t). 
\] (121)

By the law of large numbers, when $m \to \infty$, it can be shown that
\[
-\nabla^2_w L^S_{E}(\hat{w}_\phi, \hat{w}_c^t(\hat{w}_\phi), D_t) = \frac{1}{m} \sum_{i=1}^{m} \nabla^2_w \log f(Z_i^t|\hat{w}_\phi, \hat{w}_c^t(\hat{w}_\phi))) \to -J_c^t(\hat{w}_\phi, \hat{w}_c^t(\hat{w}_\phi)).
\] (122)

As for the RHS of (121), note that $\mathbb{E}_{P_2}[\nabla_w \log f(Z|[\hat{w}_\phi, \hat{w}_c^t(\hat{w}_\phi)])] = 0$ due to the definition of $\hat{w}_c^t(\hat{w}_\phi)$, by multivariate central limit theorem, we have
\[
\frac{1}{\sqrt{m}} \sum_{i=1}^{m} (-\nabla_w \log f(Z_i^t|[\hat{w}_\phi, \hat{w}_c^t(\hat{w}_\phi)])) \to \mathcal{N}(0, I_c^t(\hat{w}_\phi, \hat{w}_c^t(\hat{w}_\phi))),
\] (123)

where $I_c^t(\hat{w}_\phi, \hat{w}_c^t(\hat{w}_\phi)) = \mathbb{E}_{P_2}[\nabla_w \log f(Z|[\hat{w}_\phi, \hat{w}_c^t(\hat{w}_\phi)])\nabla_w \log f(Z|[\hat{w}_\phi, \hat{w}_c^t(\hat{w}_\phi)]])$.

Thus, the RHS of (121) will converge to
\[
\sqrt{m} \nabla_w L^S_{E}(\hat{w}_\phi, w_c, D_t)|_{w_c = \hat{w}_c^t(\hat{w}_\phi)} \to \mathcal{N}(0, I_c^t(\hat{w}_\phi, \hat{w}_c^t(\hat{w}_\phi)))
\] (124)

When $m \to \infty$, we obtain
\[
\sqrt{m}((\hat{W}_c^t(D_t, \hat{w}_\phi) - \hat{w}_c^t(\hat{w}_\phi))) \to \mathcal{N}(0, J_c^t(\hat{w}_\phi, \hat{w}_c^t(\hat{w}_\phi))^{-1}I_c^t(\hat{w}_\phi, \hat{w}_c^t(\hat{w}_\phi))J_c^t(\hat{w}_\phi, \hat{w}_c^t(\hat{w}_\phi))^{-1}).
\] (125)

In the main body of the paper, we further let $n \to \infty$, then $\hat{w}_\phi \to w^{*\Phi}$, and $\hat{w}_c^t(\hat{w}_\phi) \to w^{*c\phi}$. Using Lemma 5, we can show that
\[
\hat{W}_c^t(D_t, \hat{W}_\phi) \to \hat{w}_c^t(\hat{W}_\phi) \to \mathcal{N}(0, J_c^t(\hat{w}_\phi, w^{*c\phi})^{-1}I_c^t(\hat{w}_\phi, w^{*c\phi})J_c^t(\hat{w}_\phi, w^{*c\phi})^{-1}m).
\]

As the Hessian matrix $H_c^*(D_t, \hat{W}_\phi) = H_c^*(\hat{W}_\phi) \to J_c^t(\hat{w}_\phi, w^{*c\phi})$ as $m, n \to \infty$. By Proposition 2, we have
\[
\text{gen}_{\beta}(P_{D_t}, P_{D_c}) = \frac{\text{tr}(J_c^t(\hat{w}_\phi, w^{*c\phi})J_c^t(\hat{w}_\phi, w^{*c\phi})^{-1}m)}{m} = \mathcal{O}(\frac{d_c}{m}).
\] (126)

M. Excess risk

$\alpha$-weighted ERM: In the following lemma, we characterize the variance of the $\alpha$-weighted ERM algorithm.

Lemma 6. Under Assumption 1, if we let $m, n \to \infty$, then the $\alpha$-weighted ERM satisfies
\[
\sqrt{m + n}(\hat{W}_\alpha(D_s, D_t) - w^{*\alpha}) \to \mathcal{N}(0, J_\alpha(\hat{w}_\alpha, w^{*\alpha})^{-1}I_\alpha(\hat{w}_\alpha, w^{*\alpha})J_\alpha(\hat{w}_\alpha, w^{*\alpha})^{-1}).
\] (127)

Proof. By using a Taylor expansion of the first derivative of the weighted log-likelihood $L_E(\hat{W}_\alpha(D_s, D_t), D_s, D_t)$ around $w^{*\alpha}$, we obtain
\[
0 = \nabla_w L_E(w, D_s, D_t)|_{w = \hat{w}_\alpha(D_s, D_t)} \approx \nabla_w L_E(w, D_s, D_t)|_{w = w^{*\alpha}} + \nabla^2_w L_E(w, D_s, D_t)|_{w = w^{*\alpha}}(\hat{W}_\alpha(D_s, D_t) - w^{*\alpha}).
\]
From the Taylor series expansion formula, the following approximation can be obtained
\[
- \nabla_w^2 L_E(w, D_s, D_t) \big|_{w=w_\alpha^*} (\hat{W}_\alpha(D_s, D_t) - w_\alpha^*) \approx \nabla_w L_E(w, D_s, D_t) \big|_{w=w_\alpha^*},
\]
(128)
By the law of large numbers, when \( m, n \to \infty \), it can be shown that
\[
- \nabla_w^2 L_E(w_\alpha^*, D_t) = \frac{1}{m} \sum_{i=1}^{m} \nabla_w^2 \log f(Z_i^t | w_\alpha^*) \to -J_t(w_\alpha^*),
\]
(129)
\[
- \nabla_w^2 L_E(w_\alpha^*, D_s) = \frac{1}{n} \sum_{i=1}^{n} \nabla_w^2 \log f(Z_i^s | w_\alpha^*) \to -J_s(w_\alpha^*).
\]
(130)
Thus, the LHS of (128) converges to
\[
\nabla_w^2 L_E(w, D_s, D_t) \big|_{w=w_\alpha^*} \to J_\alpha(w_\alpha^*),
\]
(131)
where \(J_\alpha(w_\alpha^*) \triangleq \alpha J_t(w_\alpha^*) + (1-\alpha) J_s(w_\alpha^*) \).

As for the RHS of (128), by multivariate central limit theorem
\[
\frac{1}{\sqrt{m}} \sum_{i=1}^{m} \left( - \nabla_w \log f(Z_i^t | w_\alpha^*) + E_Z [\nabla_w \log f(Z_i^t | w_\alpha^*)] \right) \to N(0, \mathcal{I}_t(w_\alpha^*)),
\]
(132)
\[
\frac{1}{\sqrt{n}} \sum_{i=1}^{n} \left( - \nabla_w \log f(Z_i^s | w_\alpha^*) + E_Z [\nabla_w \log f(Z_i^s | w_\alpha^*)] \right) \to N(0, \mathcal{I}_s(w_\alpha^*)),
\]
(133)
where \( \mathcal{I}_t(w_\alpha^*) \) and \( \mathcal{I}_s(w_\alpha^*) \) are the covariance matrix of \( \nabla_w \log f(Z^t | w_\alpha^*) \) and \( \nabla_w \log f(Z^s | w_\alpha^*) \), respectively.

Due to the definition of \(w_\alpha^* \), we have
\[
(1-\alpha) E_Z [\nabla_w \log f(Z^t | w_\alpha^*)] + \alpha E_Z [\nabla_w \log f(Z^s | w_\alpha^*)] = 0.
\]
(134)
Thus, the RHS of (128) will converge to
\[
\nabla_w L_E(w, D_s, D_t) \big|_{w=w_\alpha^*} \to N(0, \frac{\alpha^2}{m} \mathcal{I}_t(w_\alpha^*) + \frac{(1-\alpha)^2}{n} \mathcal{I}_s(w_\alpha^*)).
\]
(135)
When \( m, n \to \infty \), we obtain
\[
(\hat{W}_\alpha(D_s, D_t) - w_\alpha^*) \to N(0, \frac{\alpha^2}{m} \mathcal{I}_t(w_\alpha^*) + \frac{(1-\alpha)^2}{n} \mathcal{I}_s(w_\alpha^*)).J(w_\alpha^*)^{-1}.
\]
(136)

For \( \alpha = \frac{m}{m+n} \), if we denote \( \mathcal{I}(w_\alpha) = \frac{n}{m+n} \mathcal{I}_s(w_\alpha) + \frac{m}{m+n} \mathcal{I}_t(w_\alpha) \), we have
\[
(\hat{W}_\alpha(D_s, D_t) - w_\alpha^*) \to N(0, \frac{1}{m+n} J(w_\alpha^*)^{-1} \mathcal{I}(w_\alpha^*) J(w_\alpha^*)^{-1}).
\]
(137)

Thus, the variance term in the excess risk can be computed as:
\[
\text{tr}(J_t(w_\alpha^*) \text{Cov}(\hat{W}_A(D_s, D_t))) = \frac{\text{tr}(J_t(w_\alpha^*) J(w_\alpha^*)^{-1} \mathcal{I}(w_\alpha^*) J(w_\alpha^*)^{-1})}{m+n} = O\left(\frac{d}{m+n}\right).
\]
(138)

**Two-stage ERM**: We use the following notations to denote the expectation of the Hessian matrix and the Fisher information matrix with respect to \(w_\phi\),
\[
J^t_{c, \phi}(w_\phi, w_c) \triangleq E_{P_\phi^c} \left[ - \nabla_{w_c, w_\phi}^2 \log f(Z|[w_\phi, w_c]) \right],
J^s_{\phi}(w_\phi) \triangleq E_{P_\phi^c} \left[ - \nabla_{w_\phi} \log f(Z|[w_\phi, w_c]) \right],
\mathcal{I}^s_{\phi}(w_\phi, w_c) \triangleq E_{P_\phi^c} [\nabla_{w_c} \log f(Z|[w_\phi, w_c])]\nabla_{w_c}^\top \log f(Z|[w_\phi, w_c])].
\]
In the following lemma, we characterize the variance of the two-stage ERM algorithm.
Lemma 7. Under Assumption 1, if we let \( m, n \to \infty \), then the two-stage ERM satisfies

\[
\begin{align*}
(W_c^t(\hat{W}_\phi, D_t) - w_c^{sts}) & \to N\left(0, J_{c,\phi}^t(w_{\phi}^{sts}, w_c^{sts})^{-1}\right) \\
\left(\frac{1}{m} \mathcal{L}_c(w_{\phi}^{sts}, w_c^{sts}) + \frac{1}{n} J_{c,\phi}^t(w_{\phi}^{sts}, w_c^{sts}) J_{\phi}(w_{\phi}^{sts})^{-1} \mathcal{I}_\phi(w_{\phi}^{sts}), J_{\phi}(w_{\phi}^{sts})^{-1} J_{c,\phi}^t(w_{\phi}^{sts}, w_c^{sts}) J_{c,\phi}^t(w_{\phi}^{sts}, w_c^{sts})^{-1}\right).
\end{align*}
\]  

(139)

Proof. By using a Taylor expansion of the gradient with respect to \( w_c \) of \( L_{E}^2(\hat{W}_\phi(D_s), \hat{W}_c^t(\hat{W}_\phi, D_t), D_t) \) around \([w_{\phi}^{sts}, w_c^{sts}]\), we obtain

\[
0 = \nabla_{w_c} L_{E}^2(\hat{W}_\phi(D_s), \hat{W}_c^t(\hat{W}_\phi, D_t), D_t)
\approx \nabla_{w_c} L_{E}^2(\hat{W}_\phi(D_s), \hat{W}_c^t(\hat{W}_\phi, D_t), D_t) + \nabla_{w_c}^2 L_{E}^2(\hat{W}_\phi(D_s), \hat{W}_c^t(\hat{W}_\phi, D_t), D_t)(\hat{W}_c(D_s) - w_c^{sts})
+ \nabla_{w_c}^2 L_{E}^2(\hat{W}_\phi(D_s), \hat{W}_c^t(\hat{W}_\phi, D_t), D_t)(\hat{W}_c^t(\hat{W}_\phi, D_t) - w_c^{sts}).
\]

From the Taylor series expansion formula, the following approximation can be obtained

\[
- \nabla_{w_c}^2 L_{E}^2(\hat{W}_\phi(D_s), \hat{W}_c^t(\hat{W}_\phi, D_t), D_t)(\hat{W}_c(D_s) - w_c^{sts})
\approx \nabla_{w_c} L_{E}^2(\hat{W}_\phi(D_s), \hat{W}_c^t(\hat{W}_\phi, D_t), D_t) + \nabla_{w_c}^2 L_{E}^2(\hat{W}_\phi(D_s), \hat{W}_c^t(\hat{W}_\phi, D_t), D_t)(\hat{W}_c(D_s) - w_c^{sts}).
\]

(140)

By the law of large numbers, when \( m \to \infty \), it can be shown that

\[
- \nabla_{w_c} L_{E}^2(\hat{W}_\phi(D_s), \hat{W}_c^t(\hat{W}_\phi, D_t), D_t) = \frac{1}{m} \sum_{i=1}^{m} \nabla_{w_c} \log f(Z_i^t[\hat{w}_{\phi}^{sts}], \hat{w}_c^{sts})\to -J_{c,\phi}^t(w_{\phi}^{sts}, w_c^{sts}),
\]

(141)

\[
- \nabla_{w_c}^2 L_{E}^2(\hat{W}_\phi(D_s), \hat{W}_c^t(\hat{W}_\phi, D_t), D_t) = \frac{1}{m} \sum_{i=1}^{m} \nabla_{w_c} \log f(Z_i^t[\hat{w}_{\phi}^{sts}], \hat{w}_c^{sts})\to -J_{c,\phi}^t(w_{\phi}^{sts}, w_c^{sts}).
\]

(142)

As for the first term in the RHS of (140), note that \( \mathbb{E}_{P_{\hat{Z}}}[\nabla_{w_c} \log f(Z[\hat{w}_{\phi}^{sts}, \hat{w}_c^{sts}])] = 0 \), by multivariate central limit theorem, we have

\[
\frac{1}{\sqrt{m}} \sum_{i=1}^{m} \left( - \nabla_{w_c} \log f(Z_i^t[\hat{w}_{\phi}^{sts}], \hat{w}_c^{sts}) \right) \to N(0, I_{c,\phi}^t(w_{\phi}^{sts}, w_c^{sts})).
\]

(143)

When \( n \to \infty \), due to the asymptotic normality of maximum likelihood estimate, we have

\[
\sqrt{n}(\hat{W}_\phi(D_s) - w_{\phi}^{*}) \to N(0, J_{\phi}^t(w_{\phi}^{*})^{-1} \mathcal{I}_{\phi}(w_{\phi}^{*}) J_{\phi}^t(w_{\phi}^{*})^{-1}),
\]

(144)

where \( \mathcal{I}_{\phi}(w_{\phi}^{*}) = \mathbb{E}_{P_{\hat{Z}}}[\nabla_{w_c} \log f(Z[\hat{w}_{\phi}^{*, w_c^{*}}])] \nabla_{w_c} \log f(Z[\hat{w}_{\phi}^{*, w_c^{*}}])^T \).

Thus, the RHS of (140) converges to

\[
N\left(0, \frac{1}{m} I_{c,\phi}^t(w_{\phi}^{*, w_c^{*}}) + \frac{1}{n} J_{c,\phi}^t(w_{\phi}^{*, w_c^{*}}) J_{\phi}^t(w_{\phi}^{*})^{-1} \mathcal{I}_{\phi}(w_{\phi}^{*}) J_{\phi}^t(w_{\phi}^{*})^{-1} J_{c,\phi}^t(w_{\phi}^{*, w_c^{*}}) \right)
\]

(145)

when \( m, n \to \infty \).

Thus, we obtain

\[
(W_c^t(\hat{W}_\phi, D_t) - w_c^{sts}) \to N\left(0, J_{c,\phi}^t(w_{\phi}^{sts}, w_c^{sts})^{-1}\right)
\]

(146)

\[
\left(\frac{1}{m} \mathcal{L}_c(w_{\phi}^{sts}, w_c^{sts}) + \frac{1}{n} J_{c,\phi}^t(w_{\phi}^{sts}, w_c^{sts}) J_{\phi}(w_{\phi}^{sts})^{-1} \mathcal{I}_{\phi}(w_{\phi}^{sts}), J_{\phi}(w_{\phi}^{sts})^{-1} J_{c,\phi}^t(w_{\phi}^{sts}, w_c^{sts}) J_{c,\phi}^t(w_{\phi}^{sts}, w_c^{sts})^{-1}\right).
\]

Note that \( \text{Cov}(\hat{W}_\phi(D_s)) \) can be characterized by the asymptotic normality of maximum likelihood estimate. Thus, the variance term in the excess risk can be computed as:

\[
\text{tr}(J_{c}(w_{\phi}^{*, w_c^{*}}) \text{Cov}(\hat{W}_\phi(D_s), \hat{W}_c^t(D_t, \hat{W}_\phi))) = O\left(\frac{d_c}{m} + \frac{d}{n}\right).
\]

(147)