Equations that Describe Waves in Tubes with Elastic Walls and Analysis of Numerical Methods for Reversible and Weakly Dissipative Systems

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Abstract. Models of tube with elastic walls are presented: with controlled pressure, filled with incompressible fluid, filled with compressible gas. The non-linear theory of hyperelasticity is applied. The walls of a tube are described with complete membrane model. Equations are solved numerically. General approach for calculation of non-dissipative and weakly dissipative systems is given. Three-layer time and space centered reversible numerical scheme and similar two-layer space reversible numerical schemes with approximation of time derivatives by high-order Runge-Kutta methods are analysed. A method of correction of numerical schemes by inclusion of terms with high-order derivatives is developed. Additional terms with high-order derivatives may be included also in order to calculate non-dissipative and dissipative shocks at the same time.

1. Equations for waves in tubes
Models that describe waves in elastic tubes are of great interest for technical and biological applications. Here equations based on the hyperelastic model and the complete membrane model are analysed. Such equations were derived in [1] and analytically investigated in [2] and [3]. The purpose of this work is the application of the theory of reversible and weakly dissipative shocks [4, 5], the development of numerical methods.

The equations of wave motion of the walls of an elastic incompressible cylindrical tube with fixed internal and external pressure are given by [2]

\[
\left( R\sigma_1'/\lambda_1^2 \right)' - P^* r' = \rho R \ddot{z}, \quad \left( R\sigma_1 r'/\lambda_1^2 \right)' - \sigma_2/\lambda_2 + P^* r' = \rho R \ddot{r},
\]

\[
\lambda_1 = \sqrt{r^2 + z^2}, \quad \lambda_2 = r/R, \quad \lambda_3 = h/H, \quad \sigma_i = \lambda_i W \lambda_i - p, \quad P^* = P/H.
\]

Here the prime denotes the derivative with respect to the variable \( Z \), which stands for Lagrange initial longitudinal spatial coordinate, the upper dots denote derivative with respect to the time. Unknowns \( z \) and \( r \) determine the surface of the tube in the cylindrical system of coordinates, the \( z \) axis of this system coincides with the central line of the tube. The parameter \( P \) stands for the difference between internal and external pressures, the unknown \( p \) is the pressure in the material. The parameter \( \rho \) is the density of material per square unit. The thickness of the wall is denoted as \( h \). Here \( \lambda_i \) are the principal stretches and \( \sigma_i \) are the principal Cauchy stresses, \( W \)
is the stress function. It is assumed that in the absence of any external forces \( z = Z, r = R, h = H \). Due to the condition of incompressibility,

\[
\lambda_1 \lambda_2 \lambda_3 = 1, \quad \sigma_i = \lambda_i \tilde{W}_i, \quad \tilde{W}_i = \tilde{W}_{\lambda_i}, \quad \tilde{W} = W(\lambda_1, \lambda_2, 1/(\lambda_1 \lambda_2)).
\]

The stress function used in this paper for numerical calculations corresponds to the Gent material

\[
W = -\tilde{\mu} J_m \ln \left[ 1 - (\lambda_1^2 + \lambda_2^2 + \lambda_3^2 - 3)/J_m \right].
\]

It is considered that this stress function is applicable for rubber.

There are dispersion branches of longitudinal and transversal elastic waves in this model. Transversal branches do not intersect point of origin in \((\omega, \kappa)\) plane. Hence there are no shock structures of transversal waves.

For the case of a fluid-filled tube with arbitrary pressure we must include the equations [3]

\[
\dot{r} z' - r' \dot{z} + vr' + \frac{1}{2} rv' = 0, \\
\rho f (\dot{v} z' - v' \dot{z} + vv') + P' = 0.
\]

Here \( v \) is the fluid velocity, \( \rho f \) is the fluid density. For the calculations, the equations were transformed to the form with excluded pressure [8]

\[
\left[ \rho f \dot{z}' - \left( \frac{r' \rho R^2}{r z'^2 \xi} \right) \right] \dot{v}' - \left[ \frac{r' \rho R^2}{r z'^2 \xi} + \left( \frac{\rho R^2}{r z'^2} \right) \right] v' - \frac{\rho R^2}{2 z'^2} v'' = \rho f (v' q - v v') + \left[ -P_{b^*} + \frac{P}{\xi} \right]' - \{c_{4} v'''^m\},
\]

\[
\dot{r} z' - r' q + vr' + \frac{1}{2} rv' = 0 + \{c_{6} v'''''^m z'\}, \quad \dot{z} = q - \{c_{6} v'''^m\}, \quad \rho R q = \left( R \sigma_1 \frac{z'}{\lambda_2} \right)' - (P^* + P_b^*) rv',
\]

\[
P^* + P_b^* = \frac{P + P_b}{H} = -\frac{1}{\xi} \left[ \frac{r' \rho R}{r z'^2} \dot{v}' + \frac{\rho R}{2 z'^2} \ddot{v}' + \dddot{v}' \right], \quad \xi = 1 + \frac{r'^2}{z'^2},
\]

\[
P = \frac{1}{r z'} \left[ \left( R \sigma_1 \frac{z'}{\lambda_2} \right)' - \frac{r'}{z'} \left( R \sigma_1 \frac{z'}{\lambda_2} \right)' \frac{\sigma_2}{\lambda_2} - \frac{\rho R}{r z'^2} (q-v) \left( \frac{\dot{z}' q - \dot{v} r' - \frac{1}{2} r v'}{z'} \right) \right] \left( \frac{r' q - \dot{v} r' - \frac{1}{2} r v'}{2 z'} \right) \left( \frac{1}{2} \dot{v} - q' \right).
\]

Here terms in braces may be included for effectiveness of calculations (see below).

There are branches of longitudinal and transversal elastic waves in this model. All branches intersect point of origin.

The analysis shows that for the case of the gas-filled tube [8] the equation (2) must be replaced by

\[
(\rho f \dot{z}' - \rho f \dot{z}) r^2 + 2 \rho f r (\dot{r} z' - r' \dot{z}) + (\rho f v v') = 0.
\]

In the barotropic case only one additional equation of state is required. An additional equation for energy is required for the complete gas dynamics model.

There are branches of longitudinal and transversal elastic waves and gas dynamic waves in this model. Transversal branches do not intersect point of origin as in the case of controlled pressure.

The viscosity of the fluid or gas may be taken into account if the additional term \( c_{6} v'''^m \) is included in (3). The viscosity of the material of the walls is enforced within the Kelvin-Voight model [6]

\[
\sigma_{i} \rightarrow \sigma_{i} + \sigma_{vi}, \quad \sigma_{vi} = \nu_{vi} \dot{\lambda}_i, \quad i = 1, 2, 3.
\]
It can be shown that for some simplified version of the equations this model leads to the inclusion of the term $c_v \varepsilon''$ in the equation (3) also [8].

The compressibility of the material of the wall may be taken into account if a stress function (potential) without $p$ is used. A two dimensional potential (potential with hat) can be derived from the equation $\sigma_3 = p_e$, where $p_e$ is the external pressure.

In order to take into account the meridional bending resistance of the thin walls additional “bending resistance pressure” must be formally included in (1) but not in (3) [7, 8]

$$ P \rightarrow P + P_b, \quad P_b = -\cos \alpha \left[ \frac{\partial^2}{\partial x^2} \left( D \frac{\partial^2}{\partial x^2} r \right) \right] - \tan \alpha \left[ \frac{\partial^2}{\partial x^2} \left( D \frac{\partial^2}{\partial x^2} z \right) \right], \quad \frac{\partial}{\partial x} = \cos \alpha \frac{\partial}{\partial z}, \quad \tan \alpha = \frac{r'}{z'}.$$

Here $\alpha$ is the angle of inclination of the wall of the tube. This approach is based on the Germain-Lagrange formula. The bending resistance coefficient $D$ must be calculated for the anisotropic linearised state [8]

$$ D = \{ W_1 + \lambda_1(W_{11} - W_{13}) + [W_3 + \lambda_3(W_{33} - W_{31})]\lambda_3/\lambda_1 \} h^3/12.$$

If the geometric nonlinearity is not taken into account then $P_b = P_{bs} = -Dr''''$ and the bending pressure is included only in the equation for $r$. Only the term $-br''''$ is added to the left part in this case. A problem is not well-posed if $\sigma_1 < 0$ and initial equations are used. The inclusion of the bending resistance enforces well-posedness.

A non-differential term related with the latitudinal (circular) bending resistance [9] may be included in pressure formula. But if the wall of the tube is weakly stretched (this is a typical case when inclusion of meridional bending resistance is required) or if $h/R$ is close to zero this term vanishes.

The term simplified equations is used in the theory of shocks to denote hyperbolic equations describing waves with slow variations. The simplified form of the controlled pressure equations is

$$ \dot{u} = q'; \quad \rho R \dot{q} = \left( R \sigma_1 u/\lambda_1^2 \right)' - P^* r', \quad \lambda_1 = u = z', \quad P^* = \sigma_2/(\lambda_2 ru).$$

No simplification is required for the equations (2), (3), (4). These equations describe non-wave zones for Riemann problem solutions.

Equations for waves in tubes in non-dissipative cases are reversible equations with dispersion caused by terms with derivatives of various order. Dispersion vanishes if wavenumber $\kappa \to 0$. Exception is the case when bending resistance is taken into account in simplified form. Hence we may assume that in some cases only generalized solutions with dissipative shock structures or other singularities may exist.

2. Numerical methods, general approach for non-dissipative and weakly-dissipative models

Standard non-dissipative three-layer time and space centered scheme and space centered two-layer schemes with approximation of time derivatives by Runge-Kutta method were used for solition of equatios of waves in tubes. These schemes are used for all models analysed in the theory of reversible and weakly dissipative shocks [4, 5]. The schemes are explicit usually. Similar implicit numerical schemes are used for the approximation of complex time-space derivatives. If three-layer scheme is used then dissipative terms are approximated on lower layer, by DuFort-Frankel type scheme or implicitly. According to the theory in non-dissipative case homogeneous states and waves zones are observed in solutions of Riemann problem. They are separated by reversible shock structures of solitary wave type and kink type. Length of wave zones is increased with time. In weakly dissipative case increase of wave zones is stopped after some
time and their length depends on values of dissipative coefficients. That is why it is essential to use non-dissipative or weakly dissipative numerical schemes.

For the case of controlled pressure the three-layer scheme was used [7]. But for cases of fluid-filled and gas-filled tubes boundary numerical instability is observed. For fluid-filled tubes this instability may be corrected by inclusion of dissipative zones near boundaries. An additional term with second-order derivative must be included in the equation for \( v \). If the space step is decreased then the coefficient before this term must be increased. Good results are achieved if this term is approximated by the DuFort-Frankel method. This is not convenient for calculations. That is why two-layer schemes were used for the cases of fluid-filled and gas-filled tube.

Results of analysis of methods based on Runge-Kutta approximation for time derivatives are given below in order to understand their properties. Generalised transport equation is considered

\[
\frac{\partial u}{\partial t} = Lu, \quad L = \sum_{j=1}^{M} a_j L_j, \quad L_j = \frac{\partial^j u}{\partial x^j}
\]

Terms with odd order derivatives are non-dissipative terms. Terms with even order derivatives are dissipative terms. Dissipation is correct if \( a_{4m-2} > 0 \) and \( a_{4m} < 0 \), \( m = 1, 2, \ldots \). This equation is similar to equation for \( v \) of the transformed system of equation for fluid-filled tubes.

Centered approximations for space derivatives are used

\[
u_x \rightarrow \frac{u_{k+1}^n - u_{k-1}^n}{2h}, \quad u_{xx} \rightarrow \frac{u_{k+1}^n + u_{k-1}^n - 2u_k^n}{h^2}, \quad u_{xxx} \rightarrow \frac{u_{k+2}^n - 3u_{k+1}^n + 3u_{k-1}^n - u_{k-2}^n}{2h^3}, \ldots
\]

There are many variants of Runge-Kutta method. It may be shown that in scalar case with fixed coefficients one step representation of this methods is

\[
u^{n+1} = \left( 1 + L + \frac{L^2}{2!} + \frac{L^3}{3!} + \frac{L^4}{4!} + \ldots \right) v^n
\]

Here \( L/\tau \) is operator of the finite-difference approximation of \( L \), \( \tau \) is time step. Order of expansion of \( \exp L \) corresponds to order of method. Let function \( F(A) \) is a result of substitution \( u_k^n = q^n \exp ik\phi \) to some operator \( A \), \( F(AB) = F(A)F(B) \), \( F(A + B) = F(A) + F(B) \). For non-dissipative case we derive

\[
|q|^2 = F\left( 1 + \frac{L^2}{2!} + \frac{L^4}{4!} + \ldots \right)^2 - F\left( L + \frac{L^3}{3!} + \frac{L^5}{5!} + \ldots \right) = 1 - 2G(L^2) + \frac{2G(L)^2}{2!} + \frac{2G(L^2)^2}{4!} - \frac{G(L^3)^2}{4!} + \frac{G(L^4)^4}{4!} + \ldots + G(L^2) = -F(L^2) > 0
\]

Results of calculation for 1-5 order methods are given below

\[
|q|^2 = 1 + G(L^2); \quad \tau^2 c_{2m-1}^{-1} 2^{(2m-2)-1} (-1)^m \frac{\partial^{2m} u}{\partial x^{2m}}
\]

\[
|q|^2 = 1 + \frac{G(L^2)^2}{4}; \quad \tau^4 c_{2m-1}^{-1} 2^{(2m-2)-1} (-1)^m \frac{\partial^{2m} u}{\partial x^{2m}}
\]

\[
|q|^2 = 1 - G(L^2)^2 \left( \frac{1}{12} - \frac{G(L)^2}{3!} \right); \quad \tau^6 c_{2m-1}^{-1} 2^{(2m-2)-1} (-1)^m \frac{\partial^{2m} u}{\partial x^{2m}}
\]

\[
|q|^2 = 1 - G(L^2)^3 \left( \frac{1}{72} - \frac{G(L)^2}{4!} \right); \quad \tau^6 c_{2m-1}^{-1} 2^{(2m-2)-1} (-1)^m \frac{\partial^{2m} u}{\partial x^{2m}}
\]
\[ |q|^2 = 1 + G(L^2)^3 \left[ \frac{1}{360} - G(L^2) \left( \frac{1}{960} - \frac{G(L^2)}{512} \right) \right] \cdot \frac{\tau^5}{C_{2m-1}} \left( \frac{1}{2} \right)^{2(2m-2)-1} (-1)^m \frac{\partial^{2m} u}{\partial^{2m} x^2} \]

In addition to \(|q|^2\) here dissipative term with the lowest order derivative in the differential approximation of numerical scheme for \(L = L_{2m-1}\) is given. For 1, 2, 5 order it is non-correct (for 6, 7 order it is non-correct also). For 3 and 4 order methods conditions of non-growing of disturbances are given also. Order of growing or decrease of disturbances is \(1 + O(\tau^{2s}/h^{2s}(2m-1))\). Here \(s = 1\) for first-order method, \(s = 2\) for 2 and 3 order methods, \(s = 3\) for 4 and 5 order methods and so on. According to spectral method scheme is stable if \(|q| < 1 + ct\), \(c > 0\). In the case non-correct dissipation stability is achieved with \(\tau \sim h^{2s(2m-1)/(2s-1)}\), but for fixed values of \(\tau\) long-time calculations are stopped due to growth of disturbances. Calculations must carried out for the interval \(0 < t < T\), \(T \sim \tau^{-1}\).

Growing may be eliminated by adding of dissipative term \(-(-1)^m a_{2j} \partial^{2j} u/\partial x^{2j}\), \(a_{2j} > 0\), \(a_{2j} \sim \tau^{2s-1}/h^{2s(2m-1)-2}\), \(\tau < Ch^{2m-1}\). Here \(C\) is some value.

Correction is made only for shot waves if \(2j > 2s(2m-1)\). Optimal value for \(j\) is \(j = s(2m-1)\). It was shown for 1, 2 order methods that the given estimates are applicable also for generalized wave equation

\[ \frac{\partial^2 u}{\partial t^2} = c_{2m} \frac{\partial^2 u}{\partial x^{2m}}, \quad \omega^2 = (-1)^m c_{2m} \kappa^{2m} \]

Dispersion relation is given here also. Equations for walls of tube are similar to this equation. Non-scalar form of equation is used for calculations

\[ \frac{\partial u}{\partial t} = b_1 \frac{\partial v}{\partial x} - \left( (-1)^m a_{2j} \frac{\partial^{2j} u}{\partial x^{2j}} \right), \quad \frac{\partial v}{\partial t} = b_{2m-l} \frac{\partial^{2m-l} u}{\partial x^{2m-l}} - \left( (-1)^m a_{2j} \frac{\partial^{2j} v}{\partial x^{2j}} \right), \quad c_{2m} = b_l b_{2m-l} l = 0, 1, ... 2m \]

For \(m = 1, l = 1\) these equations are similar to gas dynamics equations. Estimates are applicable for well-posed cases \((\omega \in R)\) if \(2m-1\) is replaced by \(m\).

Note that the second-order Runge-Kutta method may be treated as method of predictor-corrector type. Half of time step predictor is used. Growth of disturbances may be excluded if the predictor is applied for the entire time step. But such numerical scheme possesses first-order \(\tau\) numerical dissipation and its application for non-dissipative and weakly dissipative problems is not convenient.

For calculation of generalized solution additional dissipative or non dissipative terms may be added. For term with derivative of order \(m\) the order of coefficient must be \(h^{m-1}\). In this case number of cells for shock structure will be not changed when \(h \rightarrow 0\). Hence there is essential difference between order of these terms and order of terms for correction of numerical schemes that are proportional to some order of \(\tau\). Inclusion of high-order derivatives affects mainly on long waves but do not affects on long waves. That is why it is appropriate to use dissipative terms with high-order derivatives if non-dissipative and dissipative shock structures are calculated at the same time.

3. Riemann problem

Numerical solutions of the Riemann problem are found. Results of calculations correspond to the theory of reversible and weakly dissipative shocks. Typical types of shock structures [5] are found for all cases. Examples of graphs for the case of controlled pressure are given in [7] and for fluid-filled tube they are given in [8]. One more example for fluid-filled tube is presented in fig. 1a. Second-order Runge-Kutta method with correction is used. Graphs for great density, \(\rho_f = 10\), curve 1, and low density, \(\rho_f = 0.25\), curve 2, are given, \(t = 1200\), \(v_{\|0} = 0.4\), \(c_v = c_v^4 = 10^{-5}\), \(\mu = 2\), \(J_m = 30\), \(R = 1\), \(\rho = 1\), \(\rho^* = 1\). Left longitudinal elastic waves left transversal elastic waves, kink structure, right transversal elastic and right longitudinal elastic
waves are observed. In the case of low density no homogeneous regions between shock structures are observed. The simplified equations badly describe such solution. These equations may be derived from the transformed equations of fluid-filled tube and it is obvious that in equation for \(v\) we cannot withdraw dispersive terms if \(\rho_f \to 0\). Model of fluid-filled tube do not transforms to model with controlled pressure. Note also that for the case of great density right shock structure of longitudinal elastic waves is calculated as dissipative shock due to correction of scheme.

Results obtained by fourth-order Runge-Kutta method for gas-filled tube are demonstrated in fig. 1b, \(t = 5500\), isothermal case, \(p = cp_f, c/H = 200, p_e = 0, v|_{t=0} = -0.25\). Left longitudinal elastic waves left gas dynamic waves, kink structure, right gas dynamic waves and right longitudinal elastic waves are observed. Growth of disturbances is not found.

![Figure 1. Solutions of Riemann problem](image)

4. Conclusions and discussion

Results of calculations show that the theory of reversible and weakly dissipative shock structures may be applied to all considered models for waves in tubes. Three-layer centered scheme is effective for calculation of such equations with complex dispersion. But number of branches of numerical dispersion relation for this scheme is two times greater than number of branches of calculated equations. It may lead to numerical instability. Two-layer schemes with Runge-Kutta approximation of time derivatives are effective in these cases. These schemes are stable but dissipative properties of these schemes may lead to some dissipative changes in solutions or to growth of disturbances. This effects are readily decreased if order of method is increased.

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