Gauge-invariant perturbation theory on the Schwarzschild background spacetime Part III: — Realization of exact solutions —

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This is the Part III paper of our series of papers on a gauge-invariant perturbation theory on the Schwarzschild background spacetime. After reviewing our general framework of the gauge-invariant perturbation theory and the proposal on the gauge-invariant treatments for \( l = 0 \), 1 mode perturbations on the Schwarzschild background spacetime in [K. Nakamura, arXiv:2110.13508 [gr-qc]], we examine the problem whether the \( l = 0 \), 1 even-mode solutions derived in the Part II paper [K. Nakamura, arXiv:2110.13512 [gr-qc]] are physically reasonable, or not. We consider the linearized versions of the Lemaître-Tolman-Bondi solution and the non-rotating C-metric. As the result, we show that our derived even-mode solutions to the linearized Einstein equations actually realize above two linearized solutions. This fact supports that our derived solutions are physically reasonable, which implies that our proposal on the gauge-invariant treatments for \( l = 0 \), 1 mode perturbations are also physically reasonable. We also briefly summarize our conclusions of our series of papers.

1. Introduction

Gravitational-wave observations are now on the stage where we can directly measure many events through the ground-based gravitational-wave detectors [1–4]. One of the future directions of gravitational-wave astronomy will be a precise science through the statistics of many events. Toward further development, the projects of future ground-based gravitational-wave detectors [3, 4] are also progressing to achieve more sensitive detectors and some projects of space gravitational-wave antenna are also progressing [2, 10]. Although there are many targets of these detectors, the Extreme-Mass-Ratio-Inspiral (EMRI), which is a source of gravitational waves from the motion of a stellar mass object around a supermassive black hole, is a promising target of the Laser Interferometer Space Antenna [3]. Since the mass ratio of this EMRI is very small, we can describe the gravitational waves from EMRIs through black hole perturbations [11]. Furthermore, the sophistication of higher-order black hole perturbation theories is required to support these gravitational-wave physics as a precise science. The motivation of our series of papers, Refs. [12–15] and this paper, is in this theoretical sophistication of black hole perturbation theories toward higher-order perturbations.

In black hole perturbation theories, further sophistications are possible even in perturbation theories on the Schwarzschild background spacetime. From the works by Regge and

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Wheeler [16] and Zerilli [17, 18], many studies on the perturbations in the Schwarzschild background spacetime are carried out [19–32]. In these works, perturbations are decomposed through the spherical harmonics $Y_{lm}$, because of the spherical symmetry of the background spacetime, and $l = 0, 1$ modes should be separately treated. Due to this separate treatments, “gauge-invariant” treatments for $l = 0$ and $l = 1$ modes were unclear.

Owing to this situation, in the previous papers [12, 14], we proposed the strategy of the gauge-invariant treatments of these $l = 0, 1$ mode perturbations, which is declared as Proposal 2.1 in this paper below. We have been developing the general formulation of a higher-order gauge-invariant perturbation theory on a generic background spacetime toward unambiguous sophisticated nonlinear general-relativistic perturbation theories [33–38]. Although we have been applied this general framework to cosmological perturbations [39–46], we applied it to black hole perturbations in the series of papers, i.e., Refs. [12–15] and this paper.

In the Part I paper [14], we also derived the linearized Einstein equations in a gauge-invariant manner following Proposal 2.1. Perturbations on the spherically symmetric background spacetime are classified into even- and odd-mode perturbations. In the same paper [14], we also gave the strategy to solve the odd-mode perturbations including $l = 0, 1$ modes. Furthermore, we also derived the explicit solutions for the $l = 0, 1$ odd-mode perturbations to the linearized Einstein equations following Proposal 2.1. In the Part II paper [15], we gave the strategy to solve the even-mode perturbations including $l = 0, 1$ modes and we also derive the explicit solutions for the $l = 0, 1$ even-mode perturbations following Proposal 2.1. This series of papers is the full paper version of our short paper [12].

In this paper, we check whether the solutions for even-mode perturbations derived in the Part II paper [15] are physically reasonable, or not. We consider the correspondence between our linearized solutions and two exact solutions. One of exact solutions discussed in this paper is the Lemaître-Tolman-Bondi (LTB) solution and the other is the non-rotating C-metric [47, 48].

The organization of this Part III paper is as follows. In Sec. 2, after briefly review the framework of the gauge-invariant perturbation theory, we summarize our proposal in Refs. [12, 14]. Then, we also summarize the linearized even-mode Einstein equations on the Schwarzschild background spacetime derived in Ref. [14]. These are derived based on Proposal 2.1. In Sec. 3, we discuss the realization of the LTB solution by our derived solutions for $l = 0, 1$-mode even-mode perturbations to the linearized Einstein equation following Proposal 2.1. In Sec. 4, we discuss the realization of the C-metric from our derived solutions in the Part II paper [15].

The final section (Sec. 5) devoted to our summary and discussion. This final section also includes a brief conclusion of our series of papers on a gauge-invariant perturbation theory on the Schwarzschild background spacetime.

We use the notation used in the previous papers [12–14] and the unit $G = c = 1$, where $G$ is Newton’s constant of gravitation and $c$ is the velocity of light.

2. Brief review of the general-relativistic gauge-invariant perturbation theory

In this section, we review the premises of our series of papers [12, 14, 15] which are necessary to understand the ingredients of this paper. In Sec. 2.1, we briefly review our framework of the gauge-invariant perturbation theory [33–34]. This is an important premise of the series of our papers [12, 14, 15] and this paper. In Sec. 2.2, we review the gauge-invariant perturbation
theory on spherically symmetric spacetimes which includes our proposal in Refs. \[12, 14\]. In Sec. 2.3, we summarize the even-mode linearized Einstein equations on the Schwarzschild background spacetime and their explicit solutions for \(l = 0, 1\) modes, which are necessary for the arguments in this paper.

2.1. General framework of gauge-invariant perturbation theory

In any perturbation theory, we always treat two spacetime manifolds. One is the physical spacetime \((\mathcal{M}_{\text{ph}}, g_{ab})\), which is identified with our nature itself, and we want to describe this spacetime \((\mathcal{M}_{\text{ph}}, \bar{g}_{ab})\) by perturbations. The other is the background spacetime \((\mathcal{M}, g_{ab})\), which is prepared as a reference by hand. Note that these two spacetimes are distinct. Furthermore, in any perturbation theory, we always write equations for the perturbation of the variable \(Q\) as follows:

\[
Q(“p”) = Q_0(p) + \delta Q(p).
\]

Equation (2.1) gives a relation between variables on different manifolds. Actually, \(Q(“p”)\) in Eq. (2.1) is a variable on \(\mathcal{M}_\epsilon = \mathcal{M}_{\text{ph}}\), whereas \(Q_0(p)\) and \(\delta Q(p)\) are variables on \(\mathcal{M}\). Because we regard Eq. (2.1) as a field equation, Eq. (2.1) includes an implicit assumption of the existence of a point identification map \(\mathcal{X}_\epsilon : \mathcal{M} \to \mathcal{M}_\epsilon : p \in \mathcal{M} \mapsto “p” \in \mathcal{M}_\epsilon\). This identification map is a gauge choice in general-relativistic perturbation theories. This is the notion of the second-kind gauge pointed out by Sachs \[49\]. Note that this second-kind gauge is a different notion from the degree of freedom of the coordinate transformation on a single manifold, which is called the first-kind gauge \[14, 44, 46\]. This distinction of the first- and the second-kind of gauges extensively explained in the Part I paper \[14\] and is also important to understand the results in Secs. 3 and 4 in this paper.

To compare the variable \(Q\) on \(\mathcal{M}_\epsilon\) with its background value \(Q_0\) on \(\mathcal{M}\), we use the pull-back \(\mathcal{X}_\epsilon^*\) of the identification map \(\mathcal{X}_\epsilon : \mathcal{M} \to \mathcal{M}_\epsilon\) and we evaluate the pulled-back variable \(\mathcal{X}_\epsilon^*Q\) on the background spacetime \(\mathcal{M}\). Furthermore, in perturbation theories, we expand the pull-back operation \(\mathcal{X}_\epsilon^*\) to the variable \(Q\) with respect to the infinitesimal parameter \(\epsilon\) for the perturbation as

\[
\mathcal{X}_\epsilon^*Q = Q_0 + \epsilon^{(1)} Q + O(\epsilon^2).
\]

Equation (2.2) are evaluated on the background spacetime \(\mathcal{M}\). When we have two different gauge choices \(\mathcal{X}_\epsilon\) and \(\mathcal{Y}_\epsilon\), we can consider the gauge-transformation, which is the change of the point-identification \(\mathcal{X}_\epsilon \to \mathcal{Y}_\epsilon\). This gauge-transformation is given by the diffeomorphism \(\Phi_\epsilon := (\mathcal{X}_\epsilon)^{-1} \circ \mathcal{Y}_\epsilon : \mathcal{M} \to \mathcal{M}\). Actually, the diffeomorphism \(\Phi_\epsilon\) induces a pull-back from the representation \(\mathcal{X}_\epsilon^*Q_\epsilon\) to the representation \(\mathcal{Y}_\epsilon^*Q_\epsilon\) as \(\mathcal{Y}_\epsilon^*Q_\epsilon = \Phi_\epsilon^*\mathcal{X}_\epsilon^*Q_\epsilon\). From general arguments of the Taylor expansion \[50\], the pull-back \(\Phi_\epsilon^*\) is expanded as

\[
\mathcal{Y}_\epsilon^*Q_\epsilon = \mathcal{X}_\epsilon^*Q_\epsilon + \epsilon \mathcal{L}_{\xi_\epsilon^{(1)}} \mathcal{X}_\epsilon^*Q_\epsilon + O(\epsilon^2),
\]

where \(\xi_\epsilon^{(1)}\) is the generator of \(\Phi_\epsilon\). From Eqs. (2.2) and (2.3), the gauge-transformation for the first-order perturbation \((1)\) is given by

\[
\frac{(1)}{\mathcal{Y}} Q - \frac{(1)}{\mathcal{X}} Q = \mathcal{L}_{\xi_\epsilon^{(1)}} Q_0.
\]

We also employ the order by order gauge invariance as a concept of gauge invariance \[42\]. We call the \(k\)th-order perturbation \((k)\) as gauge invariant if and only if

\[
\frac{(k)}{\mathcal{Y}} Q = \frac{(k)}{\mathcal{X}} Q
\]
for any gauge choice $X$ and $Y$.

Based on the above setup, we proposed a formulation to construct gauge-invariant variables of higher-order perturbations \([33, 34]\). First, we expand the metric on the physical spacetime $M$, which was pulled back to the background spacetime $M$ through a gauge choice $X$ as

$$X^* g_{ab} = g_{ab} + \epsilon X h_{ab} + O(\epsilon^2). \quad (2.6)$$

Although the expression \((2.6)\) depends entirely on the gauge choice $X$, henceforth, we do not explicitly express the index of the gauge choice $X$ in the expression if there is no possibility of confusion. The important premise of our formulation of higher-order gauge-invariant perturbation theory was the following conjecture \([33, 34]\) for the linear metric perturbation $h_{ab}$:

**Conjecture 2.1.** If the gauge-transformation rule for a perturbative pulled-back tensor field $h_{ab}$ to the background spacetime $M$ is given by $\nabla h_{ab} - \nabla h_{ab} = \mathcal{L}_\xi g_{ab}$ with the background metric $g_{ab}$, there then exist a tensor field $F_{ab}$ and a vector field $Y^a$ such that $h_{ab}$ is decomposed as $h_{ab} =: F_{ab} + \mathcal{L}_Y g_{ab}$, where $F_{ab}$ and $Y^a$ are transformed as $\nabla F_{ab} - \nabla F_{ab} = 0$ and $\nabla Y^a - \nabla Y^a = \xi^a$ under the gauge transformation, respectively.

We call $F_{ab}$ and $Y^a$ as the gauge-invariant and gauge-variant parts of $h_{ab}$, respectively. The proof of Conjecture \textit{2.1} is highly nontrivial \([35, 37]\), and it was found that the gauge-invariant variables are essentially non-local. Despite this non-triviality, once we accept Conjecture \textit{2.1}, we can decompose the linear perturbation of an arbitrary tensor field $(1)XQ$, whose gauge-transformation is given by Eq. \((2.4)\), through the gauge-variant part $Y_a$ of the metric perturbation in Conjecture \textit{2.1} as

$$^{(1)}XQ = ^{(1)}Q + \mathcal{L}_\xi Y^a Q^a; \quad (2.7)$$

where $^{(1)}Q$ is the gauge invariant part of the perturbation $(1)XQ$. As examples, the linearized Einstein tensor $^{(1)}\mathcal{G}_a^b$ and the linear perturbation of the energy-momentum tensor $^{(1)}\mathcal{T}_a^b$ are also decomposed as

$$^{(1)}\mathcal{G}_a^b = \mathcal{G}_a^b \left[ \mathcal{F} \right] + \mathcal{L}_\xi Y G_a^b, \quad ^{(1)}\mathcal{T}_a^b = ^{(1)}\mathcal{T}_a^b + \mathcal{L}_\xi Y T_a^b; \quad (2.8)$$

where $G_{ab}$ and $T_{ab}$ are the background values of the Einstein tensor and the energy-momentum tensor, respectively. The explicit form of the gauge-invariant part $^{(1)}\mathcal{G}_a^b$ of the linear-order perturbation of the Einstein tensor is not important within this paper. Using the background Einstein equation $G_a^b = 8\pi T_a^b$, the linearized Einstein equation $^{(1)}\mathcal{G}_a^b = 8\pi ^{(1)}\mathcal{T}_a^b$ is automatically given in the gauge-invariant form

$$^{(1)}\mathcal{G}_a^b \left[ \mathcal{F} \right] = 8\pi ^{(1)}\mathcal{T}_a^b \left[ \mathcal{F}, \phi \right] \quad (2.9)$$

even if the background Einstein equation is nontrivial. Here, “$\phi$” in Eq. \textit{2.9} symbolically represents the matter degree of freedom.

For the ingredients of this paper, it is important to note that the decomposition of the metric perturbation $h_{ab}$ into its gauge-invariant part $\mathcal{F}_{ab}$ and into its gauge-variant part $Y^a$ is not unique as noted in Refs. \([14, 42, 44]\). Actually, the decomposition of the metric
perturbation $h_{ab}$ is also given by

$$h_{ab} = \mathcal{F}_{ab} - \mathcal{L}_Z g_{ab} + \mathcal{L}_{Z^+} g_{ab} =: \mathcal{H}_{ab} + \mathcal{L}_X g_{ab},$$  \hspace{1cm} (2.10)$$

where $Z^a$ is gauge-invariant in the second kind, i.e., $\mathcal{F}Z^a - \mathcal{L}Z^a = 0$. The tensor field $\mathcal{H}_{ab} := \mathcal{F}_{ab} - \mathcal{L}_Z g_{ab}$ is also regarded as the gauge-invariant part (in the sense of the second-kind) of the perturbation $h_{ab}$ because $\mathcal{F}h_{ab} - \mathcal{L}h_{ab} = 0$. Similarly, the vector field $X^a := Z^a + Y^a$ is also regarded as the gauge-variant part of the perturbation $h_{ab}$ because $\mathcal{F}X^a - \mathcal{L}X^a = \xi^a_{(1)}$. The difference between the variables $\mathcal{H}_{ab}$ and $\mathcal{F}_{ab}$ is given by $\mathcal{L}_{-Z} g_{ab}$. Here, we note that if the gauge-invariant variable $\mathcal{F}_{ab}$ is a solution to the linearized Einstein equation, $\mathcal{H}_{ab} = \mathcal{F}_{ab} - \mathcal{L}_Z g_{ab}$ is also a solution to the linearized Einstein equation (2.9) due to a symmetry of the linearized Einstein equation (2.9) as explained in Part I paper [14]. This implies that the terms in the form $\mathcal{L}_Z g_{ab}$ may always appear in the solutions to the linearized Einstein equation due to the above symmetry of the linearized Einstein equation. Since our formulation already exclude the second-kind gauge completely, we should regard that the gauge-invariant term $\mathcal{L}_{-Z} g_{ab}$ as the first-kind gauge of the background spacetime induced by the infinitesimal coordinate transformations on the physical spacetime $\mathcal{M}_c$ as discussed in the Part I paper [14].

2.2. Linear perturbations on spherically symmetric background

Here, we consider the $2+2$ formulation of the perturbation of a spherically symmetric background spacetime, which originally proposed by Gerlach and Sengupta [24–27]. Spherically symmetric spacetimes are characterized by the direct product $\mathcal{M} = \mathcal{M}_1 \times S^2$ and their metric is

$$g_{ab} = y_{ab} + r^2 \gamma_{ab},$$  \hspace{1cm} (2.11)$$

where $x^A = (t, r), x^p = (\theta, \phi)$, and $\gamma_{pq}$ is the metric on the unit sphere. In the Schwarzschild spacetime, the metric (2.11) is given by

$$y_{ab} = -f(dt)a(dt)b + f^{-1}(dr)a(dr)b, \hspace{1cm} f = 1 - \frac{2M}{r},$$  \hspace{1cm} (2.12)$$

$$\gamma_{ab} = (d\theta)a(d\theta)b + \sin^2 \theta(d\phi)a(d\phi)b = \theta_a \theta_b + \phi_a \phi_b,$$  \hspace{1cm} (2.13)$$

$$\theta_a = (d\theta)a, \hspace{1cm} \phi_a = \sin \theta (d\phi)a.$$  \hspace{1cm} (2.14)$$

On this background spacetime ($\mathcal{M}, g_{ab}$), the components of the metric perturbation is given by

$$h_{ab} = h_{AB}(dx^A)a(dx^B)b + 2h_{Ap}(dx^A)a(dx^p)b + h_{pq}(dx^p)a(dx^q)b.$$  \hspace{1cm} (2.15)$$

Here, we note that the components $h_{AB}, h_{Ap}$, and $h_{pq}$ are regarded as components of scalar, vector, and tensor on $S^2$, respectively. In the Part I paper [14], we showed the linear-independence of the set of harmonic functions

$$\left\{ S_\delta, \hat{D}_p S_\delta, \epsilon_{pq} \hat{D}^q S_\delta, \frac{1}{2} \epsilon_{pq} \hat{D}_p \hat{D}_q S_\delta, \frac{1}{2} \gamma_{pq} S_\delta, 2\epsilon_{r[p} \hat{D}_q) \hat{D}^r S_\delta \right\},$$  \hspace{1cm} (2.16)$$

where $\hat{D}_p$ is the covariant derivative associated with the metric $\gamma_{pq}$ on $S^2$, $\hat{D}^p = \gamma^{pq} \hat{D}_q$, $\epsilon_{pq} = \epsilon_{[pq]} = 2\theta[p, \phi q]$ is the totally antisymmetric tensor on $S^2$. In the set of harmonic functions
\(2.17\), the scalar harmonic function \(S_\delta\) is given by

\[
S_\delta = \begin{cases} 
Y_{lm} & \text{for } l \geq 2; \\
k_{(\Delta+2)m} & \text{for } l = 1; \\
k_{(\tilde{\Delta})} & \text{for } l = 0.
\end{cases}
\]

(2.18)

Here, functions \(k_{(\Delta)}\) and \(k_{(\Delta+2)m}\) are the kernel modes of the derivative operator \(\hat{\Delta}\) and \([\Delta + 2]\), respectively, and we employ the explicit form of these functions as

\[
k_{(\Delta)} = 1 + \delta \ln \left( \frac{1 - \cos \theta}{1 + \cos \theta} \right)^{1/2}, \quad \delta \in \mathbb{R},
\]

(2.19)

\[
k_{(\Delta+2,m=0)} = \cos \theta + \delta \left( \frac{1}{2} \cos \theta \ln \frac{1 + \cos \theta}{1 - \cos \theta} - 1 \right), \quad \delta \in \mathbb{R},
\]

(2.20)

\[
k_{(\Delta+2,m=\pm 1)} = \left[ \sin \theta + \delta \left( \frac{1}{2} \sin \theta \ln \frac{1 + \cos \theta}{1 - \cos \theta} + \cot \theta \right) \right] e^{\pm i\phi}.
\]

(2.21)

Then, we consider the mode decomposition of the components \(\{h_{AB}, h_{Ap}, h_{pq}\}\) as follows:

\[
h_{AB} = \sum_{l,m} \tilde{h}_{AB} S_\delta,
\]

(2.22)

\[
h_{Ap} = r \sum_{l,m} \left[ \tilde{h}_{(e1)A} \hat{D}_p S_\delta + \tilde{h}_{(o1)A} \epsilon_{pq} \hat{D}^q S_\delta \right],
\]

(2.23)

\[
h_{pq} = r^2 \sum_{l,m} \left[ \frac{1}{2} \gamma_{pq} \tilde{h}_{(e2)} S_\delta + \tilde{h}_{(e2)} \left( \hat{D}_p \hat{D}_q - \frac{1}{2} \gamma_{pq} \hat{D}^r \hat{D}_r \right) S_\delta + 2 \tilde{h}_{(o2)} \epsilon_{r(p} \hat{D}_q) \hat{D}^r S_\delta \right].
\]

(2.24)

Since the linear-independence of each element of the set of harmonic function \(2.17\) is guaranteed, the one-to-one correspondence between the components \(\{h_{AB}, h_{Ap}, h_{pq}\}\) and the mode coefficients \(\{\tilde{h}_{AB}, \tilde{h}_{(e1)A}, \tilde{h}_{(o1)A}, \tilde{h}_{(e2)}, \tilde{h}_{(o2)}\}\) with the decomposition formulae \(2.22\), \(2.24\) is guaranteed including \(l = 0, 1\) mode if \(\delta \neq 0\). Then, the mode-by-mode analysis including \(l = 0, 1\) is possible when \(\delta \neq 0\). However, the mode functions \(2.19\), \(2.21\) are singular if \(\delta \neq 0\). When \(\delta = 0\), we have \(k_{(\Delta)} \propto Y_{00}\) and \(k_{(\Delta+2)m} \propto Y_{1m}\). Because of this situation, we proposed the following strategy:

**Proposal 2.1.** We decompose the metric perturbation \(h_{ab}\) on the background spacetime with the metric \(2.14\) through Eqs. \(2.22\) \(2.24\) with the harmonic function \(S_\delta\) given by Eq. \(2.18\). Then, Eqs. \(2.22\) \(2.24\) become invertible including \(l = 0, 1\) modes. After deriving the mode-by-mode field equations such as linearized Einstein equations by using the harmonic functions \(S_\delta\), we choose \(\delta = 0\) as regular boundary condition for solutions when we solve these field equations.

As shown in the Part I paper \([14]\), once we accept Proposal \(2.1\), the Conjecture \(2.1\) becomes the following statement:

**Theorem 2.1.** If the gauge-transformation rule for a perturbative pulled-back tensor field \(h_{ab}\) to the background spacetime \(\mathcal{M}\) is given by \(\xi h_{ab} - \xi h_{ab} = \mathcal{L}_\xi g_{ab}\) with the background metric \(g_{ab}\) with spherically symmetry, there then exist a tensor field \(\mathcal{F}_{ab}\) and a vector field \(Y^a\) such that \(h_{ab}\) is decomposed as \(h_{ab} = \mathcal{F}_{ab} + \mathcal{L}_Y g_{ab}\), where \(\mathcal{F}_{ab}\) and \(Y^a\) are transformed into \(\xi \mathcal{F}_{ab} - \xi \mathcal{F}_{ab} = 0\) and \(\xi Y^a - \xi Y^a = \xi_{(1)}^a\) under the gauge transformation, respectively.
Actually, the gauge-variant variable $Y_a$ is given by
\[
Y_a := \sum_{l,m} \tilde{Y}_A S_\delta (dx^A)_a + \sum_{l,m} \left( \tilde{Y}_{(e1)} \hat{D}_p S_\delta + \tilde{Y}_{(o1)} \epsilon_{pq} \hat{D}^q S_\delta \right) (dx^p)_a,
\] (2.25)
where
\[
\tilde{Y}_A := r \tilde{h}_{(e1)A} - r^2 \hat{D}_A \tilde{h}_{(e2)},
\] (2.26)
\[
\tilde{Y}_{(e1)} := \frac{r^2}{2} \tilde{h}_{(e2)},
\] (2.27)
\[
\tilde{Y}_{(o1)} := -r^2 \tilde{h}_{(o2)}.
\] (2.28)

Furthermore, including $l = 0, 1$ modes, the components of the gauge-invariant part $\mathcal{F}_{ab}$ of the metric perturbation $h_{ab}$ is given by
\[
\mathcal{F}_{AB} = \sum_{l,m} \hat{F}_{AB} S_\delta,
\] (2.29)
\[
\mathcal{F}_{Ap} = r \sum_{l,m} \hat{F}_{A \epsilon pq} \hat{D}^q S_\delta, \quad \hat{D}^p \mathcal{F}_{Ap} = 0,
\] (2.30)
\[
\mathcal{F}_{pq} = \frac{1}{2} \gamma_{pq} r^2 \sum_{l,m} \hat{F} S_\delta,
\] (2.31)
where $\hat{F}_{AB}$, $\hat{F}_A$, and $\hat{F}$ are given by
\[
\hat{F}_{AB} := \tilde{h}_{AB} - 2 \hat{D}_{(A} \tilde{Y}_{B)},
\] (2.32)
\[
\hat{F}_A := \tilde{h}_{(o1)A} + r \hat{D}_A \tilde{h}_{(o2)},
\] (2.33)
\[
\hat{F} := \tilde{h}_{(e0)} - \frac{4}{r} \tilde{Y}_A \hat{D}^A r + \tilde{h}_{(e2)} l (l + 1).
\] (2.34)

Thus, we have constructed gauge-invariant metric perturbations on the Schwarzschild background spacetime including $l = 0, 1$ modes.

To discuss the linearized Einstein equation (2.29) and the linear perturbation of the continuity equation
\[
\nabla_a \left( \mathcal{T}^a_b \right) = 0
\] (2.35)
of the gauge-invariant energy-momentum tensor $\left( \mathcal{T}^a_b \right) := g^{ac} (1) \mathcal{T}_{bc}$ on a vacuum background spacetime, we consider the mode-decomposition of the gauge-invariant part $\left( \mathcal{T}^a_b \right)$ of the linear perturbation of the energy-momentum tensor through the set (2.17) of the harmonics as follows:
\[
(1) \mathcal{T}_{ab} = \sum_{l,m} \tilde{T}_{AB} S_\delta (dx^A)_a (dx^B)_b + r \sum_{l,m} \left\{ \tilde{T}_{(e1)A} \hat{D}_p S_\delta + \tilde{T}_{(o1)A} \epsilon_{pq} \hat{D}^q S_\delta \right\} 2 (dx^A)_a (dx^p)_b
+ \sum_{l,m} \left\{ \tilde{T}_{(e0)2} \frac{1}{2} \gamma_{pq} S_\delta + \tilde{T}_{(e2)} \left( \hat{D}_p \hat{D}_q S_\delta - \frac{1}{2} \gamma_{pq} \hat{D}^r \hat{D}^r S_\delta \right) + \tilde{T}_{(o2)2} 2 \epsilon_{r(p} \hat{D}_q) \hat{D}^r S_\delta \right\} (dx^p)_a (dx^q)_b.
\] (2.36)
In terms of these mode coefficients, the components of the continuity equation (2.35) for the
gauge-invariant part of the linearized energy-momentum tensor are summarized as follows:

\[
\bar{D}^C \bar{T}_C^B + \frac{2}{r}(D^D r) \bar{T}_D^B - \frac{1}{r} l(l + 1) \bar{T}_{(e1)}^B - \frac{1}{r} (\bar{D}^B r) \bar{T}_{(e0)} = 0,
\]

(2.37)

\[
D^C \bar{T}_{(e1)C} + \frac{3}{r} (D^C r) \bar{T}_{(e1)C} + \frac{1}{2r} \bar{T}_{(e0)} - \frac{1}{2r} (l - 1)(l + 2) \bar{T}_{(e2)} = 0,
\]

(2.38)

\[
D^C \bar{T}_{(o1)C} + \frac{3}{r} (D^D r) \bar{T}_{(o1)D} + \frac{1}{r} (l - 1)(l + 2) \bar{T}_{(o2)} = 0.
\]

(2.39)

In the Part I paper [14], we derived the linearized Einstein equations, discussed the odd-

mode perturbation \( \bar{F}_{Ap} \) in Eq. (2.30), and derived the \( l = 1 \) odd-mode solutions to these
equations. The Einstein equation for even mode \( \bar{F}_{AB} \) and \( \bar{F} \) in Eqs. (2.29) and (2.31) also
derived in the Part I paper [14], and derived \( l = 0,1 \) even-mode solutions are derived in the
Part II paper [15]. These solutions include the Kerr parameter perturbation and the
Schwarzschild mass parameter perturbation of the linear order in the vacuum case and are
physically reasonable. Then, we conclude that our proposal is also physically reasonable.
The purpose of this paper is to check that our derived solutions include the linearized LTB
solution and non-rotating C-metric with the Schwarzschild background. For this purpose, the
even-mode solutions are necessary. Therefore, we review the strategy to derive the even-mode
solutions, below.

### 2.3. Even-mode linearized Einstein equations

The even-mode part of the linearized Einstein equation (2.9) is summarized as follows:

\[
\bar{F}_D^D = -16\pi r^2 \bar{T}_{(e2)},
\]

(2.40)

\[
\bar{D}^D \bar{F}_{AD} - \frac{1}{2} \bar{D}_A \bar{F} = 16\pi \left[ r \bar{T}_{(e1)A} - \frac{1}{2} r^2 \bar{D}_A \bar{T}_{(e2)} \right] = 16\pi S_{(ec)A},
\]

(2.41)

where the variable \( \bar{F}_{AB} \) is the traceless part of the variable \( \bar{F}_{AB} \) defined by

\[
\bar{F}_{AB} := \bar{F}_{AB} - \frac{1}{2} g_{AB} \bar{F}_C^C.
\]

(2.42)

We also have the evolution equations

\[
\left( \bar{D}_D \bar{D}^D + \frac{2}{r} (D^D r) \bar{D}_D - \frac{(l - 1)(l + 2)}{r^2} \right) \bar{F} - \frac{4}{r^2} (\bar{D}_D r) (D_r D) \bar{F}_{CD} = 16\pi S_{(F)},
\]

(2.43)

\[
S_{(F)} := \bar{T}_C^C + 4(\bar{D}_D r) \bar{T}_{(e1)}^D + 2r (\bar{D}_D r) \bar{D}^D \bar{T}_{(e2)} - (l(l + 1) + 2) \bar{T}_{(e2)}.
\]

(2.44)

\[
\left[ -\bar{D}_D \bar{D}^D - \frac{2}{r} (D^D r) \bar{D}^D + \frac{4}{r} (\bar{D}^D \bar{D}_D r) + \frac{l(l + 1)}{r^2} \right] \bar{F}_{AB} + \frac{4}{r} (D^D r) \bar{D}_A (\bar{F}_{B})_D - \frac{2}{r} (\bar{D}_A r) \bar{D}_B \bar{F} = 16\pi S_{(F)AB},
\]

(2.45)
of the even-mode part of the continuity equations (2.37) and (2.38) of the linearized energy-

where

Furthermore, we obtain the evolution equations for the variables \( \Phi \) and \( \tilde{S} \), where the source term

From Eqs. (2.41) and (2.45), we obtain the initial value constraints for the variable \( \tilde{S} \) and the Moncrief variable \( \Phi \) as follows:

where

From Eqs. (2.41) and (2.45), we obtain the initial value constraints for the variable \( \tilde{S} \) and \( Y \) as follows:

where

Furthermore, we obtain the evolution equations for the variables \( \Phi \) and \( \tilde{F} \) as follows:

where the potential function \( V_{\text{even}} \) in Eq. (2.54) is defined by

\[
V_{\text{even}} := \frac{1}{r^2} \left\{ \Lambda^3 - 2(2 - 3f)\Lambda^2 + 6(1 - 3f)(1 - f)\Lambda + 18f(1 - f)^2 \right\},
\]
and the source terms in Eq. (2.54) and (2.55) are given by

\[ S_{(\Phi_{(e)})} := \frac{1}{2} \left( \frac{\Lambda}{2f} - 1 \right) \partial_t \tilde{T}_t + \frac{1}{2} \left( (2f) - \frac{1}{2} \Lambda \right) f \dot{\tilde{T}}_t \partial_r - \frac{1}{2} f \partial_r \tilde{T}_t + \frac{1}{2} f^2 \partial_r \dot{\tilde{T}}_r, \]

\[ S_{(F)} := -\frac{1}{f} \partial_t \tilde{T}_t + f \partial_r \tilde{T}_r + 4f \partial_r \tilde{T}_{(e1)t} - 2r f \partial_r \tilde{T}_{(e2)} - (l(l+1)+2) \tilde{T}_{(e2)}, \]

respectively. The consistency of evolution equations (2.54) and (2.55) with the initial value constraint (2.60) leads the identity

\[ 0 = r^2 \partial_t^2 S_{(\Lambda \tilde{F})} + \left[ (5 - 3f) \Lambda + 3(1-f)(1+f) + 18 \frac{1}{\Lambda} f(1-f)^2 \right] f S_{(\Lambda \tilde{F})} + 2 \left[ (1 - 3f) + 2\Lambda \right] f^2 r \partial_r S_{(\Lambda \tilde{F})} - \Lambda r^2 f \partial_r \left[ f \partial_r S_{(\Lambda \tilde{F})} \right] + \frac{1}{4} \left[ (1 - 3f) - \Lambda \right] \Lambda^2 f S_{(F)} - 2r f^2 \Lambda \partial_r S_{(\Phi_{(e)})} - [\Lambda + (1 + 3f)] \Lambda f S_{(\Phi_{(e)})}. \]

Actually, we can confirm Eq. (2.55) from the definitions (2.52), (2.57), and (2.58), and the continuity equations (2.37) and (2.38), i.e.,

\[ -\partial_t \tilde{T}_t + f^2 \partial_r \tilde{T}_t - \frac{(1+f)f}{r} \tilde{T}_t = 0, \]

\[ -\partial_t \tilde{T}_r + \frac{1-f}{2rf} \tilde{T}_t + f^2 \partial_r \tilde{T}_r + \frac{(3+f)f}{2r} \tilde{T}_r = 0, \]

\[ -\partial_t \tilde{T}_{(e1)t} + f^2 \partial_r \tilde{T}_{(e1)t} + \frac{(1+f)f}{r} \tilde{T}_{(e1)t} = 0. \]

For the mode with \( l \neq 0 \), the master equation (2.54) is solved through appropriate boundary conditions for the Cauchy problem and obtain the Moncrieff variable \( \Phi_{(e)} \). Then, we obtain the variable \( \tilde{F} \) through Eq. (2.50). From the solution \( (\Phi_{(e)}, \tilde{F}) \), we obtain the component \( X_{(e)} \) through the definition (2.48) of the Moncrieff variable. Through the solution \( (\Phi_{(e)}, \tilde{F}, X_{(e)}) \), we obtain the component \( Y_{(e)} \) through Eq. (2.51). We can check the evolution equation (2.55) as a consistency check of solutions. Together with Eq. (2.40), we obtain the solution \( (\tilde{F}_{AB}, \tilde{F}) \) as a solution to the linearized Einstein equations when \( l \neq 0 \).

Actually, from the above strategy, for the \( l = 1 \)-mode perturbation, we can derive the solution to the linearized Einstein equation through the strategy for \( l \neq 0 \) mode perturbation described above. For \( m = 0 \) mode, in the Part II paper [15], we derived the following solution.
to the linearized Einstein equation

\begin{equation}
\mathcal{F}_{ab} = \mathcal{L} V_{ab} - \frac{16\pi r^2}{3(1-f)} \left[ f^2 \left\{ \frac{1}{2} T_{rr} + rf \partial_r T_{rr} - \bar{T}_{(e0)} - 4\bar{T}_{(e1)r} \right\} (dt)_a (dt)_b \\
+ 2r \left\{ \partial_t \bar{T}_{tt} - \frac{3f(1-f)}{2r} \bar{T}_{tr} \right\} (dt)_a (dr)_b \\
+ r \left\{ \partial_r \bar{T}_{tt} - \frac{3(3f-1)}{2rf} \bar{T}_{tt} \right\} (dr)_a (dr)_b \\
+ r^2 \bar{T}_{tt} \gamma_{ab} \right] \cos \theta,
\end{equation}

where the vector field \( V_a \) is given by

\begin{equation}
V_a := -r \partial_r \Phi_e \cos \theta (dt)_a + (\Phi_e - r \partial_r \Phi_e) \cos \theta (dr)_a - r \Phi_e \sin \theta (d\theta)_a.
\end{equation}

On the other hand, for the \( l = 0 \)-mode, we may choose \( \bar{T}_{(e1)A} = 0 \) and \( \bar{T}_{(e2)} = 0 \) and we may regard that the tensor \( \tilde{F}_{AB} \) is traceless. Furthermore, Eqs. \( 2.50 \) and \( 2.51 \) yield the \( r \)- and \( t \)-derivative of the Moncrief variable \( \Phi_e \), respectively. The integrability of these equations are guaranteed by the continuity equation \( 2.37 \). Then, we obtain the Moncrief variable \( \Phi_e \). In this case, the master equation \( 2.54 \) is trivial and the evolution equation \( 2.55 \) gives the variable \( \tilde{F} \). Then, we obtain the variable \( (\Phi_e, \tilde{F}) \). Through the definition \( 2.48 \) of the Moncrief variable \( \Phi_e \), we obtain the component \( X_e \). To obtain the component \( Y_e \), we regard the constraints \( 2.44 \) as the equation for the component \( Y_e \). Through this strategy, in the Part II paper \[15\], we derived the \( l = 0 \) mode solution

\begin{equation}
\mathcal{F}_{ab} = \frac{2}{r} \left( M_1 + 4\pi \int dr \frac{r^2}{f} T_{tt} \right) \left( (dt)_a (dt)_a + \frac{1}{f^2} (dr)_a (dr)_a \right) \\
+ 2 \left[ 4\pi r \int dt \left( \frac{1}{f} \tilde{T}_{tt} + f \bar{T}_{rr} \right) \right] (dt)_a (dr)_b + \mathcal{L} V_{ab},
\end{equation}

where

\begin{equation}
V_a = \left( \frac{f}{4} \tilde{Y} + \frac{rf}{4} \partial_r \bar{Y} - \frac{r\Xi(r)}{1-3f} + f \int dr \frac{2\Xi(r)}{f(1-3f)^2} \right) (dt)_a + \frac{1}{4f} r \partial_r \bar{Y} (dr)_a.
\end{equation}

Here, the variable \( \tilde{F} =: \partial_t \bar{Y} \) must satisfy Eq. \( 2.55 \) and \( \Xi(r) \) is an arbitrary function of \( r \).

3. Realization of LTB solution as a perturbation of Schwarzschild spacetime

3.1. Perturbative expression of the LTB solution on Schwarzschild background spacetime

Here, we consider the Lemaitre-Tolman-Bondi (LTB) solution \[51\] which is an exact solution to the Einstein equation with the matter field

\begin{equation}
T_{ab} = \rho u_a u_b, \quad u_a = -(d\tau)_a,
\end{equation}

and the metric

\begin{equation}
g_{ab} = -(d\tau)_a (d\tau)_b + \frac{(\partial_r R)^2}{1 + f(R)} (dR)_a (dR)_b + r^2 \gamma_{ab}, \quad r = r(\tau, R).
\end{equation}

This solution is a spherically symmetric solution to the Einstein equation. The function \( r = r(\tau, R) \) satisfies the differential equation

\begin{equation}
(\partial_r r)^2 = \frac{F(R)}{r} + f(R).
\end{equation}
Here, we note that $F(R)$ is an arbitrary function of $R$, which represents initial distribution of the dust matter. $f(R)$ is also an arbitrary function of $R$ which represents initial distribution of the energy of dust field in Newtonian sense. The solution to Eq. (3.3) is given in the three cases

(i) $f(R) > 0$ :

$$
\begin{align*}
    r &= \frac{F(R)}{2f(R)}(\cosh \eta - 1), \\
    \tau_0(R) - \tau &= \frac{F(R)}{2f(R)^{3/2}}(\sinh \eta - \eta),
\end{align*}
(3.4)
$$

(ii) $f(R) < 0$ :

$$
\begin{align*}
    r &= \frac{F(R)}{-2f(R)}(1 - \cos \eta), \\
    \tau_0(R) - \tau &= \frac{F(R)}{2(-f(R))^{3/2}}(\eta - \sin \eta),
\end{align*}
(3.5)
$$

(iii) $f(R) = 0$ :

$$
\begin{align*}
    r &= \left(\frac{9F(R)}{4}\right)^{1/3} \left[\tau_0(R) - \tau\right]^{2/3}.
\end{align*}
(3.6)
$$

The energy density $\rho$ is given by

$$
8\pi \rho = \frac{\partial_R F}{(\partial_R r)^2}.
(3.7)
$$

The LTB solution includes the three arbitrary functions $f(R)$, $F(R)$, and $\tau_0(R)$. Here, we consider the vacuum case $\rho = 0$. In this case, from Eq. (3.7), we have

$$
\partial_R F = 0.
(3.8)
$$

Furthermore, we consider the case $f = 0$. Here, we chose $\tau_0 = R$, i.e., $\partial_R \tau_0 = 1$. In this case, Eq. (3.0) yields

$$
\begin{align*}
    (dR)_a &= \left(\frac{9F}{4}\right)^{1/3} \frac{2}{3} [R - \tau]^{-1/3} [(dR)_a - (d\tau)_a] \\
    &= \left(\frac{F}{r}\right)^{1/2} [(dR)_a - (d\tau)_a],
\end{align*}
(3.9)
$$

$$
\begin{align*}
    (\partial_R r) &= \left(\frac{F}{r}\right)^{1/2},
\end{align*}
(3.10)
$$

and

$$
\begin{align*}
    (dR)_a &= (d\tau)_a + (dr)_{a}^{-1/2} (dr)_{a}.
\end{align*}
(3.11)
$$

Then, the metric (3.2) is given by

$$
\begin{align*}
    g_{ab} &= - (d\tau)_a (d\tau)_b + (\partial_R r)^2 (dR)_a (dR)_b + r^2 \gamma_{ab} \\
    &= - \left(1 - \frac{F}{r}\right) \left[(d\tau)_a - \left(1 - \frac{F}{r}\right)^{-1} \left(\frac{F}{r}\right)^{1/2} (dr)_a\right] \\
    &\quad \times \left[(d\tau)_b - \left(1 - \frac{F}{r}\right)^{-1} \left(\frac{F}{r}\right)^{1/2} (dr)_b\right] \\
    &\quad + \left(1 - \frac{F}{r}\right)^{-1} (dr)_a (dr)_b + r^2 \gamma_{ab}.
\end{align*}
(3.12)
$$
Here, we define the time function $t$ by

$$(dt)_a := (d\tau)_a - \left(1 - \frac{F}{r}\right)^{-1} \left(\frac{F}{r}\right)^{1/2} (dr)_a.$$  \hspace{1cm} (3.13)

Then, we obtain

$$g_{ab} = -f(dt)_a(dt)_b + f^{-1}(dr)_a(dr)_b + r^2 \gamma_{ab}, \quad f = 1 - \frac{2M}{r},$$  \hspace{1cm} (3.14)$$

with the identification $F = 2M$. This is the Schwarzschild metric with the mass parameter $M$.

Now, we consider the perturbation of the Schwarzschild spacetime which is derived by the exact LTB solution (3.2) so that

$$F(R) = 2\left[M + \epsilon m_1(R)\right] + O(\epsilon^2),$$  \hspace{1cm} (3.15)$$

$$f(R) = 0 + \epsilon f_1(R) + O(\epsilon^2),$$  \hspace{1cm} (3.16)$$

$$\tau_0(R) = R + \epsilon \tau_1(R) + O(\epsilon^2).$$  \hspace{1cm} (3.17)$$

Through these perturbations (3.15)–(3.17), we consider the perturbative expansion of the function $r$ which is determined by Eq. (3.3):

$$r(\tau, R) = r_s(\tau, R) + \epsilon r_1(\tau, R) + O(\epsilon^2).$$  \hspace{1cm} (3.18)$$

Here, the function $r_s(\tau, R)$ is given by Eq. (3.6), i.e.,

$$r_s(\tau, R) = r(\tau, R) = \left(\frac{9M}{2}\right)^{1/3} [R - \tau]^{2/3}.$$  \hspace{1cm} (3.19)$$

In Eqs. (3.17) and (3.19), we chose the background value of the function $\tau_0(R)$ to be $R$.

Through this perturbative expansion, we evaluate Eq. (3.3) and we obtain

$$O(\epsilon^0) : (\partial_\tau r_s)^2 - \frac{2M}{r_s} = 0$$  \hspace{1cm} (3.20)$$

$$O(\epsilon^1) : (\partial_\tau r_s)(\partial_\tau r_1) - \frac{m_1(R)}{r_s} + \frac{M}{r^2} r_1 - \frac{1}{2} f_1(R) = 0.$$  \hspace{1cm} (3.21)$$

Using Eq. (3.20), the linear perturbation (3.21) yields

$$(1 - f)^{1/2}(\partial_\tau r_1) + \frac{m_1(R)}{r} - \frac{M}{r^2} r_1 + \frac{1}{2} f_1(R) = 0,$$  \hspace{1cm} (3.22)$$

where we used

$$\partial_\tau r_s = -(1 - f)^{1/2}$$  \hspace{1cm} (3.23)$$

and the replacement $r_s \rightarrow r$. The solution to Eq. (3.22) is given by

$$r_1 = \left(\frac{M}{6}\right)^{1/3} \frac{m_1(R)}{M} [R - \tau]^{2/3} - \frac{3}{20} \left(\frac{6}{M}\right)^{1/3} f_1(R) [R - \tau]^{4/3}$$

$$+ B(R) [R - \tau]^{-1/3}.$$  \hspace{1cm} (3.24)$$

From the comparison with Eq. (3.19), $B(R)$ is the perturbation of the $\tau_1(R)$ as $\tau_0(R) = R + \tau_1(R)$ in the exact solution (3.4)–(3.6). Furthermore, the solution (3.24) can be also
derived from the exact solution (3.1)–(3.6). From Eq. (3.7), the perturbative dust energy density given by

$$8\pi\rho = \frac{2\partial_{RR}m_1(R)}{(\partial_{RR})^2 r^2}.$$  

Through the perturbative solution (3.24), the metric (3.2) is given by

$$g_{ab} = -(d\tau)_a (d\tau)_b + (\partial_{RR})^2 (dR)_a (dR)_b + r^2 \gamma_{ab}$$

$$+ \epsilon \left[ (2(\partial_{RR}r_1) - f_1(R)) (\partial_{RR}) (dR)_a (dR)_b + 2rr_1 \gamma_{ab} \right] + O(\epsilon^2)$$

$$=: g^{(0)}_{ab} + \epsilon \mathcal{H}_{ab} + O(\epsilon^2).$$  

(3.26)

As shown in Eq. (3.14), the background metric $g^{(0)}_{ab}$ is given by the Schwarzschild metric in the static chart. On the other hand, the linear order perturbation $\mathcal{H}_{ab}$ (in the gauge $\xi^a$) is given by

$$\mathcal{H}_{ab} = (2(\partial_{RR}r_1) - f_1(R)) (\partial_{RR}) (dR)_a (dR)_b + 2rr_1 \gamma_{ab}.$$  

(3.27)

Here, we fixed the second-kind gauge so that

$$\mathcal{H}_{(\tau,R,\theta,\phi)}: M_{ph} \mapsto (\tau,R,\theta,\phi) \in M.$$  

(3.28)

Of course, if we employ the different gauge choice $Y_\epsilon$ from the above gauge-choice $\mathcal{H}_\epsilon$, we obtain the different expression of the metric perturbation $Y_{ab} = \mathcal{H}_{ab} + \mathcal{L}_\epsilon g_{ab}$, where $\xi^a$ is the generator of second-rank gauge transformation $\mathcal{H}_{\epsilon} \rightarrow Y_{\epsilon}$.

### 3.2. Expression of the perturbative LTB solution in static chart

Here, we consider the expression of the linear perturbation $\mathcal{H}_{ab}$ given by Eq. (3.27). Here, the radial coordinate $r$ is related to the coordinates $\tau$ and $R$ through Eq. (3.19) as

$$R - \tau = \left(\frac{2}{9M}\right)^{1/2} r^{3/2}. $$

(3.29)

Then, we obtain

$$R = \tau + \frac{4M}{3} \left(\frac{r}{2M}\right)^{3/2}.$$  

(3.30)

Furthermore, the relation of the time function $t$ and coordinates $\tau$ and $r$ is given by Eq. (3.13) as

$$t = \tau + 4M \left[ \left(\frac{r}{2M}\right)^{1/2} + \ln \left\{ \left(\frac{r}{2M}\right)^{1/2} - 1 \right\} \right].$$

(3.31)

Then, we have obtained

$$\tau = t - 4M \left[ \left(\frac{r}{2M}\right)^{1/2} + \ln \left\{ \left(\frac{r}{2M}\right)^{1/2} - 1 \right\} \right],$$

(3.32)

$$R = t - 4M \left[ \frac{1}{3} \left(\frac{r}{2M}\right)^{3/2} + \left(\frac{r}{2M}\right)^{1/2} + \ln \left\{ \left(\frac{r}{2M}\right)^{1/2} - 1 \right\} \right].$$

(3.33)
Through the coordinates \((t, r)\), we express the linear perturbation \(\chi h_{ab}\) in Eq. (3.27). From Eqs. (3.11) and (3.13) with \(F = 2M\), we obtain

\[
\begin{align*}
(dR)_a &= (dt)_a + f^{-1}(1 - f)^{-1/2}(dr)_a, \quad f = 1 - \frac{2M}{r}, \quad (3.34) \\
(d\tau)_a &= (dt)_a + f^{-1}(1 - f)^{1/2}(dr)_a. \quad (3.35)
\end{align*}
\]

Before evaluating the metric perturbation \(\chi h_{ab}\), we consider the perturbation of the energy momentum tensor of the matter field. In the case of the LTB solution, the matter field is characterized by the dust field whose energy momentum tensor (3.1) is given by

\[
T_{ab} = \rho u_a u_b, \quad u_a = -d\tau, \quad u^a = (\partial_r)^a. \quad (3.36)
\]

In our case, the linearized Einstein equation gives Eq. (3.25), i.e.,

\[
8\pi\rho = \frac{2\partial R m_1(R)}{(\partial R r)r^2}. \quad (3.37)
\]

Since we have

\[
(\partial R r) = (1 - f)^{1/2} \quad (3.38)
\]

from Eq. (3.29), we obtain

\[
\rho = \frac{\partial R m_1(R)}{4\pi r^2} (1 - f)^{-1/2}. \quad (3.39)
\]

On the other hand, substituting Eq. (3.35) into Eq. (3.36), we obtain

\[
T_{ab} = \rho (dr)_a (d\tau)_b = \rho \left( (dt)_a + f^{-1}(1 - f)^{1/2}(dr)_a \right) \left( (dt)_b + f^{-1}(1 - f)^{1/2}(dr)_b \right) \]

\[
= \rho (dt)_a (dt)_b + \rho \frac{(1 - f)^{1/2}}{f} 2(dt)_a (dr)_b + \frac{1 - f}{f^2} (dr)_a (dr)_b. \quad (3.40)
\]

Then, we obtain the components of the energy-momentum tensor for the static coordinate \((t, r)\) as

\[
\begin{align*}
\tilde{T}_{tt} &= \rho, & \tilde{T}_{tr} &= \frac{(1 - f)^{1/2}}{f} \rho & \tilde{T}_{rr} &= \frac{1 - f}{f^2} \rho. \quad (3.41)
\end{align*}
\]

Here, we note that the function \(\rho\) is given by the Einstein equation (3.39) and \(R\) is given by Eq. (3.33).

Here, we check the components (2.60) and (2.61) with \(l = 0\) of the divergence of the energy-momentum tensor in the LTB case. Using Eqs. (3.34) and (3.41), we obtain

\[
\begin{align*}
\partial_t \tilde{T}_{tt} &= \partial_t \rho = \frac{\partial^2 m_1(R)}{4\pi r^2} (1 - f)^{-1/2} \left( \frac{\partial R}{\partial t} \right) = \frac{\partial^2 m_1(R)}{4\pi r^2} (1 - f)^{-1/2}, \\
-f^2 \partial_r \tilde{T}_{rt} &= -f^2 \partial_r \left( \frac{\partial R m_1(R)}{4\pi r^2} \right) = -\frac{\partial^2 m_1(R)}{4\pi r^2} (1 - f)^{-1/2} + \frac{\partial R m_1(R)}{4\pi r^3} (1 + f), \\
-\frac{(1 + f)}{r} \tilde{T}_{rt} &= -\frac{(1 + f)}{r} \frac{(1 - f)^{1/2}}{f} \rho = -\frac{\partial R m_1(R)}{4\pi r^3} (1 + f).
\end{align*}
\]
Then, we can confirm Eq. (2.60) with \( l = 0 \). Next, we check Eq. (2.61) with \( l = 0 \). Since we may choose \( \tilde{T}_{(e1)A} = 0 \) and \( \tilde{T}_{(e2)} = 0 \), Eq. (2.62) yields \( \tilde{T}_{(e0)} = 0 \). Furthermore, using

\[
\partial_t \tilde{T}_{tr} = \frac{(1 - f)^{1/2}}{f} \partial_t \rho - \frac{1}{2rf} \rho = \frac{(1 - f)^{1/2}}{8\pi r^3 f} \frac{\partial Rm_1(R)}{f},
\]

\[
- \frac{1 - f}{2rf} \tilde{T}_{tr} = \frac{1 - f}{2rf} \rho = - \frac{1}{4\pi r^2 f} \frac{\partial Rm_1(R)}{f},
\]

\[
- f^2 \partial_r \tilde{T}_{rr} = \frac{(2 - f)(1 - f)}{rf} \rho - (1 - f) \partial_r \rho = - \frac{\partial Rm_1(R)}{8\pi r^3 f} + (4 + f)(1 - f)^{1/2} \frac{\partial Rm_1(R)}{8\pi r^3 f},
\]

\[
- \frac{(3 + f)f}{2r} \tilde{T}_{rr} = \frac{(1 - f)(3 + f)}{2rf} \rho = -(3 + f)(1 - f)^{1/2} \frac{\partial Rm_1(R)}{8\pi r^3 f},
\]

we can confirm Eq. (2.61) with \( l = 0 \). Thus, the definitions (3.41) of the components \( \tilde{T}_{tt} \), \( \tilde{T}_{tr} \), \( \tilde{T}_{rr} \) and the result of the Einstein equation (3.39) are justified. We also note that the continuity equations (2.61) and (2.60) with \( l = 0 \) and \( \tilde{T}_{(e0)} = 0 \) are important premises of the solution (2.65) for \( l = 0 \) mode perturbations.

Now, we consider the problem whether the form the perturbation \( \mathcal{G}_{ab} \) given by Eq. (3.27) is described by the solution (2.65), or not. Substituting Eq. (3.33) into Eq. (3.27), we obtain

\[
\mathcal{G}_{ab} = \left( 2(\partial_Rr_1)(1 - f)^{1/2} - f_1(R)(1 - f) \right) (dt)_a (dt)_b
\]

\[
+ \frac{1}{f} \left( 2(\partial_Rr_1) - f_1(R)(1 - f)^{1/2} \right) 2(dt)_a (dr)_b
\]

\[
+ \left( 2(\partial_Rr_1)(1 - f)^{1/2} - f_1(R) \right) f^{-2} (dr)_a (dr)_b
\]

\[
+ 2rr_1\gamma_{ab}.
\]

(3.42)

Here, we used Eq. (3.38). Comparing Eq. (3.42) with Eq. (2.65), we easily see that the last term \( 2rr_1\gamma_{ab} \) should be included in the term of the Lie derivative of the background metric \( g_{ab} \). Then, we consider the components of \( \mathcal{L}_{V(1)}g_{ab} \) with the generator \( V(1)_a = V(1)_r (dr)_a \). The components of \( \mathcal{L}_{V(1)}g_{ab} \) are summarized as

\[
\mathcal{L}_{V(1)}g_{tt} = -ff'V(1)_r, \quad \mathcal{L}_{V(1)}g_{tr} = \partial_t V(1)_r, \quad \mathcal{L}_{V(1)}g_{rr} = 2\partial_r V(1)_r + \frac{f'}{f} V(1)_r, \quad \mathcal{L}_{V(1)}g_{\theta\theta} = 2rfV(1)_r, \quad \mathcal{L}_{V(1)}g_{\phi\phi} = 2rf \sin^2 \theta V(1)_r.
\]

(3.43)

(3.44)

If the last term \( 2rr_1\gamma_{ab} \) in Eq. (3.42) is included in the term \( \mathcal{L}_{V(1)}g_{ab} \), we should choose the component \( V(1)_r \) as

\[
V(1)_r = \frac{r_1}{f}.
\]

(3.45)

Substituting Eq. (3.45) into Eq. (3.33), we obtain

\[
\mathcal{L}_{V(1)}g_{tt} = -\frac{1 - f}{r} r_1, \quad \mathcal{L}_{V(1)}g_{tr} = \frac{1}{f} \partial_r r_1, \quad \mathcal{L}_{V(1)}g_{rr} = -\frac{1 - f}{rf^2} r_1 + \frac{2}{f} (\partial_r r_1).
\]

(3.46)
Then, we have

\[ \mathcal{L} h_{ab} = \left( 2(\partial_R r_1)(1 - f)^{1/2} - f_1(R)(1 - f) + \frac{1 - f}{r} r_1 \right) (dt)_a(dt)_b \]

\[ + \frac{1}{f} \left( 2(\partial_R r_1) - f_1(R)(1 - f)^{1/2} - \partial_t r_1 \right) 2(dt)_a(dr)_b \]

\[ + \left( 2(\partial_R r_1)(1 - f)^{-1/2} - f_1(R) + \frac{1 - f}{r} r_1 - 2f(\partial_t r_1) \right) f^{-2}(dr)_a(dr)_b \]

\[ + \mathcal{L}_{V_1} g_{ab}. \] (3.47)

Here, we note the inverse relation of Eqs. (3.34) and (3.35) as follows:

\[ (dt)_a = \frac{1}{f} (d\tau)_a - \frac{1 - f}{f} (dR)_a, \] (3.48)

\[ (dr)_a = -(1 - f)^{1/2} (d\tau)_a + (1 - f)^{1/2} (dR)_a. \] (3.49)

From Eqs. (3.48) and (3.49), we obtain

\[ (\partial_R r_1) = \frac{\partial t}{R} (\partial_t r_1) + \frac{\partial r}{R} (\partial_r r_1) \]

\[ = \frac{1 - f}{f} (\partial_t r_1) + (1 - f)^{1/2} (\partial_r r_1). \] (3.50)

Then, we obtain

\[ (\partial_t r_1) = - \frac{f}{1 - f} (\partial_R r_1) + \frac{f}{1 - f} (1 - f)^{1/2} (\partial_r r_1). \] (3.51)

Substituting Eq. (3.51) into Eq. (3.47), we obtain

\[ \mathcal{L} h_{ab} = \left( 2(\partial_R r_1)(1 - f)^{1/2} - f_1(R)(1 - f) + \frac{1 - f}{r} r_1 \right) (dt)_a(dt)_b \]

\[ + \frac{1}{f} (1 - f)^{-1/2} \left( +(2 - f)(1 - f)^{-1/2} (\partial_R r_1) - f_1(R)(1 - f) - f(\partial_t r_1) \right) 2(dt)_a(dr)_b \]

\[ + \left( 2(\partial_R r_1)(1 - f)^{-1/2} - f_1(R) + \frac{1 - f}{r} r_1 - 2f(\partial_t r_1) \right) f^{-2}(dr)_a(dr)_b \]

\[ + \mathcal{L}_{V_1} g_{ab}. \] (3.52)

Here, we also note that, apart from the term \( \mathcal{L}_{V_1} g_{ab} \), the solution (2.65) is traceless. Therefore, the trace part of \( (t, r) \) components in Eq. (3.52) should be included in the term of the Lie derivative of the background metric \( g_{ab} \). To see this, we consider the components \( \mathcal{L}_{V_2} g_{ab} \) with the generator \( V_2(t) = V_2(t)(dt)_a \) as follows:

\[ \mathcal{L}_{V_2} g_{tt} = 2\partial_t V_2(t), \quad \mathcal{L}_{V_2} g_{tr} = \partial_r V_2(t) - \frac{f'}{f} V_2(t). \] (3.53)
Substituting Eq. (3.53) into Eq. (3.52), we obtain

\[ \mathcal{L}_{ab} = \left( 2(\partial_R r_1)(1 - f)^{1/2} - f_1(R)(1 - f) + \frac{1 - f}{r} r_1 - 2\partial_t V_{(2)t} \right) (dt)_a (dt)_b + \frac{(1 - f)^{-1/2}}{f} \left( (2 - f)(1 - f)^{-1/2}(\partial_R r_1) - f_1(R)(1 - f) - f(\partial_r r_1) 
\right.

\[ - f(1 - f)^{1/2} \partial_r V_{(2)t} + \frac{1 - f}{r}(1 - f)^{1/2} V_{(2)t} \right) 2(dt)_a (dr)_b \\
+ \left( 2(\partial_R r_1)(1 - f)^{-1/2} - f_1(R) + \frac{1 - f}{r} r_1 - 2f(\partial_r r_1) \right) f^{-2}(dr)_a (dr)_b \\
+ \mathcal{L}_{V_{(1)} + V_{(2)}, g_{ab}}. \quad (3.54) \]

Apart from the term \( \mathcal{L}_{V_{(1)} + V_{(2)}, g_{ab}} \), the remaining term in \( \mathcal{L}_{ab} \) should be traceless. Then, we obtain

\[ 0 = g^{ab} \left[ \mathcal{L}_{ab} - \mathcal{L}_{V_{(1)} + V_{(2)}, g_{ab}} \right] \\
= \frac{1}{f} \left( +2f(\partial_R r_1)(1 - f)^{-1/2} - 2f(\partial_r r_1) - ff_1(R) + 2\partial_t V_{(2)t} \right). \quad (3.55) \]

Here, we choose \( V_{(2)t} \) so that

\[ \partial_t V_{(2)t} = -f(\partial_R r_1)(1 - f)^{-1/2} + f(\partial_r r_1) + \frac{1}{2} ff_1(R). \quad (3.56) \]

Through this expression of \( (\partial_R r_1) \) given by Eq. (3.50), Eq. (3.56) is given by

\[ \partial_t V_{(2)t} = \partial_t ((1 - f)^{1/2} r_1) + \frac{1}{2} ff_1(R) \quad (3.57) \]

and

\[ V_{(2)t} = (1 - f)^{1/2} r_1 + \frac{1}{2} f \int dt f_1(R), \quad (3.58) \]

where we choose the arbitrary function \( r \) to be zero. From Eq. (3.58), we obtain

\[ -f(1 - f)^{1/2} \partial_r V_{(2)t} + \frac{1 - f}{r}(1 - f)^{1/2} V_{(2)t} \\
= \frac{1}{2r}(1 - f)(2 - f)r_1 - f(1 - f)\partial_r r_1 - \frac{1}{2} f^2(1 - f)^{1/2} \int dt \partial_r f_1(R). \quad (3.59) \]

Here, we note that \( f_1 = f_1(R) \) and its derivative with respect to \( r \) is given by

\[ \partial_r f_1(R) = \frac{\partial R}{\partial r} \frac{d}{dR} f_1(R) = \frac{1}{f(1 - f)^{1/2}} \frac{d}{dR} f_1(R). \quad (3.60) \]

On the other hand, the derivative of \( f_1(R) \) with respect to \( t \) is given by

\[ \partial_t f_1(R) = \frac{\partial R}{\partial t} \frac{d}{dR} f_1(R) = \frac{d}{dR} f_1(R). \quad (3.61) \]

Then, we obtain

\[ \partial_r f_1(R) = \frac{1}{f(1 - f)^{1/2}} \partial_t f_1(R). \quad (3.62) \]
Substituting Eq. (3.62) into Eq. (3.59), we obtain

\[-f(1-f)^{1/2} \partial_t V_{(2)t} + \frac{1-f}{r}(1-f)^{1/2}V_{(2)t} \]

\[= \frac{1}{2r}(1-f)(2-f)r_1 - f(1-f)\partial_r r_1 - \frac{1}{2}ff_1. \tag{3.63} \]

Furthermore, the substitution of Eqs. (3.56) and (3.63) into Eq. (3.54), we obtain

\[\mathcal{X}_{ab} = 2\left((\partial_R r_1)(1-f)^{-\frac{1}{2}} + \frac{1-f}{2r}r_1 - f(\partial_r r_1) - \frac{1}{2}f_1(R)\right)\]

\[\times \left((dt)_a(dt)_b + \frac{1}{f^2}(dr)_a(dr)_b\right)\]

\[+ \frac{(2-f)(1-f)^{-\frac{1}{2}}}{f}\left((\partial_R r_1)(1-f)^{-1/2} + \frac{1-f}{2r}r_1 - f(\partial_r r_1) - \frac{1}{2}f_1(R)\right)\]

\[\times 2(dt)_a(dr)_b\]

\[+ \mathcal{L}_{V_{(t)+V_{(2)g}}}. \tag{3.64} \]

Here, we consider the information from the solution (3.24) of the linearized LTB solution. To consider the necessary information from the solution (3.24), we consider the derivative \(\partial_r r_1\) in Eq. (3.64). Using Eqs. (3.34) and (3.35), we obtain

\[\partial_r r_1 = \frac{\partial R}{\partial r}(\partial_R r_1) + \frac{\partial \tau}{\partial r}(\partial_r r_1)\]

\[= \frac{1}{f}(1-f)^{-1/2}(\partial_R r_1) + \frac{1}{f}(1-f)^{1/2}(\partial_r r_1). \tag{3.65} \]

From Eqs. (3.65) and (3.22), we obtain

\[2(\partial_R r_1)(1-f)^{-1/2} - 2f\partial_r r_1 + \frac{1-f}{r}r_1 - f_1(R) = \frac{2m_1(R)}{r}. \tag{3.66} \]

Through Eq. (3.66), we obtain

\[\mathcal{X}_{ab} = \frac{2m_1(R)}{r}\left[(dt)_a(dt)_b + \frac{1}{f^2}(dr)_a(dr)_b\right] + \frac{2-f}{f(1-f)^{1/2}}\frac{m_1(R)}{r}2(dt)_a(dr)_b\]

\[+ \mathcal{L}_{V_{(LTB)}}g_{ab}. \tag{3.67} \]

where \(V_{(LTB)}\) are given by

\[V_{(LTB)} := V_{(1)a} + V_{(2)a} = \left[(1-f)^{1/2}r_1 + \frac{1}{2}f \int dt f_1(R)\right](dt)_a + \frac{r_1}{f}(dr)_a. \tag{3.68} \]

Now, we check whether the linear-order perturbative solution (3.67) have the form of the general solution (2.65) for the \(l = 0\) mode perturbations, or not. Here, we only consider the case \(M_1 = 0\) in Eq. (2.65), because we can add the term \(M_1\) with an appropriate term of the Lie derivative of the background metric. First, we consider the first term in Eq. (3.67).
From Eq. (2.65), the expression
\[ m_1(t, r) = 4\pi \int dr \left[ \frac{r^2}{f} \tilde{T}_{tt} \right] \] (3.69)
should appear in Eq. (3.67). From Eqs. (3.41) and (3.39), we obtain
\[ 4\pi \int dr \left[ \frac{r^2}{f} \tilde{T}_{tt} \right] = 4\pi \int dr \left[ \frac{r^2}{f} f \right] = \int dr \left[ f^{-1}(1 - f)^{-1/2}\partial_R m_1(R) \right]. \] (3.70)
Here, we note that
\[ \partial_r m_1(R) = \frac{\partial R}{\partial r} \partial_R m_1(R) + \frac{\partial \tau}{\partial r} \partial_\tau m_1(R) = f^{-1}(1 - f)^{-1/2}\partial_R m_1(R). \] (3.71)
Substituting Eq. (3.71) into Eq. (3.70), we obtain
\[ 4\pi \int dr \left[ \frac{r^2}{f} \tilde{T}_{tt} \right] = \int dr \partial_r m_1(R) = m_1(R). \] (3.72)
Thus, we may regard that the first term in Eq. (2.65) realizes the first term in Eq. (3.67) of the linearized LTB solution.

Next, we consider the second term in Eq. (2.65). In this case, we evaluate the integration
\[ 4\pi r \int dt \left( \frac{1}{f} \tilde{T}_{tt} + f \tilde{T}_{rr} \right). \] (3.73)
From Eq. (3.41), we obtain
\[ \frac{1}{f} \tilde{T}_{tt} + f \tilde{T}_{rr} = \frac{1}{f} f + f \frac{1 - f}{f^2} f = \frac{2 - f}{f} = \frac{1}{4\pi r^2} f (2 - f)(1 - f)^{-1/2}\partial_R m_1(R). \] (3.74)
Here, we consider \( \partial_t m_1(R) \) as
\[ \partial_t m_1(R) = \frac{\partial R}{\partial t} \partial_R m_1(R) + \frac{\partial \tau}{\partial t} \partial_\tau m_1(R) = \partial_R m_1(R). \] (3.75)
Then, we obtain
\[ 4\pi r \int dt \left( \frac{1}{f} \tilde{T}_{tt} + f \tilde{T}_{rr} \right) = 4\pi r \int dt \left( \frac{1}{4\pi r^2} f (2 - f)(1 - f)^{-1/2}\partial_R m_1(R) \right) \]
\[ = \frac{2 - f}{r f(1 - f)^{1/2}} \int dt \partial_t m_1(R) = \frac{2 - f}{r f(1 - f)^{1/2}} \frac{m_1(R)}{r}. \] (3.76)
Thus, we confirmed that the second term in Eq. (2.65) realizes the second term in Eq. (3.67) in the linearized LTB solution.

The remaining term in Eq. (3.67) is the Lie derivative of the background spacetime. Here, we note that there is always ambiguity of the gauge-choice \( \mathcal{L}_V g_{ab} \) with an arbitrary vector field \( V^a \) in the linear perturbation \( \partial h_{ab} \). Through this degree of freedom, we can always adjust the solution \( \partial h_{ab} \) so that the last term in Eq. (3.67) is identical with the last term in Eq. (2.65).

Thus, the linear perturbation version (3.67) of the LTB exact solution with Schwarzschild background spacetime is realized from the solution (2.65) of the \( l = 0 \) mode perturbations. In this sense, the solutions (2.65) of the \( l = 0 \) mode perturbations are justified by the LTB solutions. It is important to note that the arbitrary functions \( f_1(R) \) and \( \tau_1(R) \) of the perturbative LTB solution is included only in the vector field \( V^{(LTB)}_{(L_{TB})} \) in Eq. (3.67). Therefore, we may say that the term \( \mathcal{L}_V g_{ab} \) includes physical information of the LTB solution, i.e., this term has its physical meaning.
4. Realization of the linearized non-rotating C-metric

4.1. The linearized non-rotating C-metric

Here, we consider the non-rotating vacuum C-metric \([48]\), in which conical singularities may occur both in the axis \(\theta = 0\) and \(\theta = \pi\). The C-metric is well-known as the solution describing uniformly accelerating black holes which are pulled or pushed by the straight string at \(\theta = 0\) or \(\theta = \pi\). The C-metric is described by the metric

\[
g_{ab} = \frac{1}{(1 + \alpha r \cos \theta)^2} \left( -Q(dt)_a(dt)_b + \frac{1}{Q}(dr)_a(dr)_b 
+ \frac{r^2}{P}(d\theta)_a(d\theta)_b + Pr^2 \sin^2 \theta(d\varphi)_a(d\varphi)_b \right), \tag{4.1}
\]

where

\[
P = 1 + 2\alpha m \cos \theta, \quad Q = (1 - \alpha^2 r^2) \left( 1 - \frac{2m}{r} \right), \quad \varphi \in (-C\pi, +C\pi) \tag{4.2}
\]

includes the singularities both in the axis \(\theta = 0\) and \(\theta = \pi\). To see this, we note that the metric given by Eqs. (4.1) and (4.2) includes three positive real parameters \(m, \alpha\) (satisfying \(2\alpha m < 1\)), and \(C\) (which is hidden in the range of the rotational coordinate \(\varphi \in (-C\pi, +C\pi)\)).

Here, we consider the two-dimensional section of the spacetime with the metric (4.1) as

\[
(1 + \alpha r \cos \theta)^2 g_{ab}|_{r=\text{const.},t=\text{const.}} =: \frac{r^2}{P} \gamma_{ab} = \frac{r^2}{P} ((d\theta)_a(d\theta)_b + P r^2 \sin^2 \theta(d\varphi)_a(d\varphi)_b). \tag{4.3}
\]

Besides the conformal factor \(r^2/P\), the “radius” which is the proper distance along the \((\partial/\partial \theta)^a\) is given by

\[
\int_0^\theta \sqrt{\gamma_{ab}(\partial/\partial \theta)^a(\partial/\partial \theta)^a} d\theta = \theta, \tag{4.4}
\]

\[
\int_\theta^\pi \sqrt{\gamma_{ab}(\partial/\partial \theta)^a(\partial/\partial \theta)^a} d\theta = \pi - \theta. \tag{4.4}
\]

On the other hand, the “circumference”, which is the proper distance along \((\partial/\partial \varphi)^a\) from \(\varphi = -C\pi\) to \(\varphi = C\pi\), for any \(\theta\) is given by

\[
\int_{-C\pi}^{+C\pi} \sqrt{\gamma_{ab}(\partial/\partial \varphi)^a(\partial/\partial \varphi)^a} d\varphi = \int_{-C\pi}^{+C\pi} (1 + 2\alpha m \cos \theta) \sin \theta d\varphi = 2\pi C (1 + 2\alpha m \cos \theta) \sin \theta. \tag{4.5}
\]

From Eqs. (4.4) and (4.5), we obtain the following results: in the neighborhood of \(\theta = 0\),

\[
\text{circumference at } \theta = 0 \quad \text{radius from } \theta = 0 \quad = \frac{2\pi C (1 + 2\alpha m \cos \theta) \sin \theta}{\theta}; \tag{4.6}
\]

and in the neighborhood of \(\theta = \pi\),

\[
\text{circumference at } \theta = \pi \quad \text{radius from } \theta = \pi \quad = \frac{2\pi C (1 + 2\alpha m \cos \theta) \sin \theta}{\pi - \theta}
\]

\[
= \frac{2\pi C (1 - 2\alpha m \cos (\pi - \theta)) \sin (\pi - \theta)}{\pi - \theta}. \tag{4.7}
\]
Then, we obtain
\[
\lim_{\theta \to 0} \frac{\text{circumference at } \theta}{\text{radius from } \theta = 0} = 2\pi C (1 + 2\alpha m),
\]
\[
\lim_{\theta \to \pi - 0} \frac{\text{circumference at } \theta}{\text{radius from } \theta = \pi} = 2\pi C (1 - 2\alpha m).
\] (4.8)

These imply the existence of a conical singularity with a different conicity (unless \(\alpha m = 0\)). The deficit or excess angle of either of these two conical singularity can be removed in an appropriate choice of the constant \(C\), but not both simultaneously. In general, the constant \(C\) can thus be seen to determine the balance between the deficit/excess angles on the two halves of the axis. In particular, one natural choice is to remove the conical singularity at \(\theta = 0\) by setting \(C = (1 + 2\alpha m)^{-1}\). In this choice, the deficit angle at the poles \(\theta = 0, \pi\) are respectively
\[
\delta_0 = 0, \quad \delta_\pi = 2\pi - \frac{2\pi (1 - 2\alpha m)}{1 + 2\alpha m} = 2\pi \frac{4\alpha m}{1 + 2\alpha m}.
\] (4.9)

To compare the Schwarzschild spacetime, it is convenient to rescale the range of the rotational coordinate is \(2\pi\). This can be achieved by the simple rescaling
\[
\varphi = C \phi,
\] (4.10)
where \(\phi \in (-\pi, \pi)\). For this choice, the metric (4.1) is given by
\[
g_{ab} = \frac{1}{(1 + \alpha r \cos \theta)^2} \left( -Q(dt)_a (dt)_b + \frac{1}{Q} (dr)_a (dr)_b \\
+ r^2 (d\theta)_a (d\theta)_b + PC^2 r^2 \sin^2 \theta (d\phi)_a (d\phi)_b \right),
\] (4.11)
where \(P\) and \(Q\) are still given by Eq. (4.2).

Now, we consider the situation where the black hole mass \(m\) is finite, and the acceleration \(\alpha\) is infinitesimally small. In this case, the Rindler horizon \(r = 1/\alpha\) is larger than the black hole horizon \(r = 2m\). Therefore, this situation is naturally given by the inequality
\[
1/\alpha > 2m, \quad \text{i.e.,} \quad 2m\alpha < 1.
\] (4.12)
Furthermore, we consider the situation
\[
2m\alpha \ll 1.
\] (4.13)
This situation is appropriate for the consideration of the linearized C-metric spacetime around the Schwarzschild spacetime. Moreover, we consider the situation where the constant \(C\) is finite. We regard that the metric on the physical spacetime is given by
\[
\bar{g}_{ab}(\bar{M}, \bar{\alpha}, \bar{C}; \bar{x}) = \bar{g}_{\mu\nu}(\bar{M}, \bar{\alpha}, \bar{C}; \bar{x})(d\bar{x}^\mu)_a (d\bar{x}^\nu)_b.
\] (4.14)

This metric is given by the replacements \(m \to \bar{M}, \alpha \to \bar{\alpha}, C \to \bar{C}\), and \(\{t, r, \theta, \phi\} \to \{\bar{t}, \bar{r}, \bar{\theta}, \bar{\phi}\}\) in Eqs. (4.1) and (4.2).

As a gauge choice of the second kind, we consider the point-identification between the background spacetime with the metric (2.11)–(2.14) in terms of the coordinates \(\{x^\mu\} = \{t, r, \theta, \phi\}\).
{t, r, θ, φ} and the physical spacetime with the metric (4.14) by

\[ \bar{x}^\mu \equiv x^\mu. \]  

(4.15)

We call this gauge choice \( \mathcal{X}_\epsilon \). We also consider the situation of perturbation

\[ \bar{M} := M + \epsilon M_1, \]  

(4.16)

\[ \bar{\alpha} := \alpha + \epsilon \alpha_1, \]  

(4.17)

\[ \bar{C} := C + \epsilon C_1. \]  

(4.18)

Then, the pull-back \( \mathcal{X}_\epsilon^* \) of the metric \( \bar{g}_{ab} \) on the physical spacetime with the metric (4.14) to the background spacetime with the metric (2.11)–(2.14) is given by

\[ \mathcal{X}_\epsilon^\ast \bar{g}_{ab} = g_{ab}(M, \alpha, C : x) + \epsilon M_1 \partial_M \bar{x} \bar{g}_{ab}(M, \alpha, C; x) \]

\[ + \epsilon \alpha_1 \partial_\alpha \bar{x} \bar{g}_{ab}(M, \alpha, C; x) + \epsilon C_1 \partial_C \bar{x} \bar{g}_{ab}(M, \alpha, C; x) \]

\[ + O(\epsilon^2) \]  

(4.19)

in the second-kind gauge choice \( \mathcal{X}_\epsilon \). Since the linear-order perturbations \( \mathcal{X}^\ast h_{ab} \) under the gauge-choice \( \mathcal{X}_\epsilon \) is defined by Eq. (2.6), we obtain the representation of the linear perturbation \( \mathcal{X}^\ast h_{ab} \) under the gauge choice \( \mathcal{X}_\epsilon \) as

\[ \mathcal{X}^\ast h_{ab} = M_1 \partial_M \bar{x} \bar{g}_{ab}(M, \alpha, C; x) + \alpha_1 \partial_\alpha \bar{x} \bar{g}_{ab}(M, \alpha, C; x) \]

\[ + C_1 \partial_C \bar{x} \bar{g}_{ab}(M, \alpha, C; x). \]  

(4.20)

On the other hand, if we apply the other gauge choice \( \mathcal{Y}_\epsilon \), we have other representation of the linear-order perturbation

\[ \mathcal{Y}^\ast g_{ab} = g_{ab} + \epsilon \mathcal{Y}^\ast h_{ab} + O(\epsilon^2). \]  

(4.21)

As the gauge choice \( \mathcal{Y}_\epsilon \), we consider the point-identification

\[ \bar{x}^\mu \equiv x^\mu. \]  

(4.22)

from the background spacetime with the metric (4.11) to the physical spacetime with the metric (4.14). We assume that the coordinates \( \{x^\mu\} \) in the gauge choice \( \mathcal{Y}_\epsilon \) is related to the coordinate \( \{\bar{x}^\mu\} \) as

\[ x^\mu = x^\mu + \epsilon \xi^\mu + O(\epsilon^2). \]  

(4.23)

This is the coordinate transformation induced by the second-kind gauge-transformation \( \Phi_\epsilon = \mathcal{X}_\epsilon \circ \mathcal{Y}^{-1} \). The metric on the physical spacetime pulled-back by the second-kind
Comparing Eqs. (2.11)–(2.14) and (4.24), the perturbation \( \mathcal{Y} \) of the functions \( P \) given by Schwarzschild spacetime, we obtain

\[
\mathcal{Y}^\epsilon \tilde{g}_{ab}(\tilde{M}, \bar{\alpha}, \bar{C}; x'^\mu) = g_{ab} + \epsilon (M_1 \partial_M \bar{g}_{\mu
u}(M, \alpha, C; x) + \alpha_1 \partial_\alpha \bar{g}_{\mu
u}(M, \alpha, C; x) + C_1 \partial_C \bar{g}_{\mu
u}(M, \alpha, C; x)) (dx^\mu)_a(dx^\nu)_b + \epsilon \mathcal{L} \xi g_{ab} + O(\epsilon^2). \tag{4.24}
\]

Comparing Eqs. (2.11)–(2.14) and (4.24), the perturbation \( \mathcal{Y} h_{ab} \) in the gauge choice \( \mathcal{Y} \) is given by

\[
\mathcal{Y} h_{ab} = (M_1 \partial_M \tilde{g}_{\mu\nu}(M, \alpha, C; x) + \alpha_1 \partial_\alpha \tilde{g}_{\mu\nu}(M, \alpha, C; x) + C_1 \partial_C \tilde{g}_{\mu\nu}(M, \alpha, C; x)) (dx^\mu)_a(dx^\nu)_b + \mathcal{L} \xi g_{ab}. \tag{4.25}
\]

Together with Eq. (4.20), we obtain the gauge transformation

\[
\mathcal{Y} h_{ab} - \mathcal{Y} h_{ab} = \mathcal{L} \xi g_{ab}. \tag{4.26}
\]

Now, we consider the explicit expression of the perturbation \( h_{ab} \). From the definitions (4.2) of the functions \( P \) and \( Q \), we obtain

\[
\partial_M P = 2\alpha \cos \theta, \quad \partial_\alpha P = 2M \cos \theta, \tag{4.27}
\]

\[
\partial_M Q = (1 - \alpha^2 r^2) \left( -\frac{2}{r} \right), \quad \partial_\alpha P = -2\alpha r^2 \left( 1 - \frac{2m}{r} \right). \tag{4.28}
\]

Through these formulae and Eq. (4.11), we obtain

\[
\partial_M \bar{g}_{ab} = \frac{1}{(1 + \alpha r \cos \theta)^2} \left[ (1 - \alpha^2 r^2) \left( \frac{2}{r} \right) ((dt)_a(dt)_b + \frac{1}{Q^2}(dr)_a(dr)_b) + 2\alpha \cos \theta \frac{r^2}{P^2} (-((d\theta)_a(d\theta)_b + C^2 P^2 \sin^2 \theta (d\phi)_a(d\phi)_b)) \right], \tag{4.29}
\]

\[
\partial_\alpha \bar{g}_{ab} = -2 \frac{r \cos \theta}{(1 + \alpha r \cos \theta)} g_{ab} + \frac{2r^2}{(1 + \alpha r \cos \theta)^2} \left[ +\alpha \left( 1 - \frac{2M}{r} \right) ((dt)_a(dt)_b + \frac{1}{Q^2}(dr)_a(dr)_b) + M \cos \theta \left( -\frac{1}{P^2} (d\theta)_a(d\theta)_b + C^2 \sin^2 \theta (d\phi)_a(d\phi)_b) \right) \right], \tag{4.30}
\]

\[
\partial_C \bar{g}_{ab} = \frac{1}{(1 + \alpha r \cos \theta)^2} 2PCr^2 \sin^2 \theta (d\phi)_a(d\phi)_b. \tag{4.31}
\]

In the case of \( \alpha = 0 \) and \( C = 1 \), that are background value of these parameter for the Schwarzschild spacetime, we obtain

\[
\partial_M \bar{g}_{ab}|_{\alpha=0,C=1} = \left( \frac{2}{r} \right) ((dt)_a(dt)_b + f^{-2}(dr)_a(dr)_b), \quad f = 1 - \frac{2M}{r}, \tag{4.32}
\]

\[
\partial_\alpha \bar{g}_{ab}|_{\alpha=0,C=1} = -2r \cos \theta g_{ab} + 2Mr^2 \cos \theta (-(d\theta)_a(d\theta)_b + \sin^2 \theta (d\phi)_a(d\phi)_b). \tag{4.33}
\]
and
\[ \partial C \tilde{g}_{ab} \bigl|_{a=0,c=1} = 2r^2 \sin^2 \theta (d\phi)_a (d\phi)_b. \] (4.34)

Thus, the linear-order perturbation \( \mathcal{X}_h \) defined by Eq. (4.32) in the second-kind gauge choice \( \mathcal{X}_r \) is given by
\[ \mathcal{X}_h_{ab} = \frac{2M_1}{r} \left( (dt)_a (dt)_b + f^{-2} (dr)_a (dr)_b \right) \]
\[ + 2\alpha_1 \left[ -r \cos \theta g_{ab} + M r^2 \cos \theta \left( -(d\theta)_a (d\theta)_b + \sin^2 \theta (d\phi)_a (d\phi)_b \right) \right] \]
\[ + 2C_1 r^2 \sin^2 \theta (d\phi)_a (d\phi)_b \] (4.35)

and that in the gauge \( \mathcal{Y} \) is given by
\[ \mathcal{Y}_h_{ab} = \frac{2M_1}{r} \left( (dt)_a (dt)_b + f^{-2} (dr)_a (dr)_b \right) \]
\[ + 2\alpha_1 \left[ -r \cos \theta g_{ab} + M r^2 \cos \theta \left( -(d\theta)_a (d\theta)_b + \sin^2 \theta (d\phi)_a (d\phi)_b \right) \right] \]
\[ + 2C_1 r^2 \sin^2 \theta (d\phi)_a (d\phi)_b \]
\[ + \mathcal{L}_\xi g_{ab}. \] (4.36)

Here, we note that the first lines in Eqs. (4.35) and (4.36), which corresponds to the mass perturbations, are included in the perturbation \( \mathcal{F}_{AB} \) in Eqs. (2.29). In the previous papers [12, 15], we already saw this term in the analyses of the \( l = 0 \) mode vacuum perturbations. On the other hand, the second and third lines in Eqs. (4.35) and (4.36), which corresponds to the acceleration perturbations and perturbations of the deficit/excess angle are not so simple. We also note that the deficit/excess angle perturbation of the third line in Eqs. (4.35) and (4.36) may depends on the mass perturbation and acceleration perturbations.

4.2. Components of metric perturbation of the linearized non-rotating C-metric

Here, we consider the components \( \mathcal{X}_h_{ab} \) which is given by Eq. (4.35). We omit the gauge-index \( \mathcal{X}_r \) in the notation of \( \mathcal{X}_h_{ab} \). The first term is the additional mass parameter perturbation of the Schwarzschild spacetime shown in the papers [12, 15], which is also described in Eq. (2.65) for \( l = 0 \) mode perturbation. The \( (t-r) \)-part \( h_{ab} \) is given by
\[ h_{AB} = \frac{2M_1}{r} \left( (dt)_A (dt)_B + f^{-2} (dr)_A (dr)_B \right) - 2\alpha_1 r \cos \theta y_{AB}. \] (4.37)

In the expression of \( h_{ab} \) in Eq. (4.35), there is no component of \( h_{Ap} \). Furthermore, the angular part of \( h_{ab} \) is given by
\[ h_{pq} = 2\alpha_1 \left[ -r \cos \theta r^2 \gamma_{pq} + M r^2 \cos \theta \left( -(d\theta)_p (d\theta)_q + \sin^2 \theta (d\phi)_p (d\phi)_q \right) \right] \]
\[ + 2C_1 r^2 \sin^2 \theta (d\phi)_a (d\phi)_b. \] (4.38)

This component \( h_{pq} \) is decomposed as Eq. (2.24). The trace of \( h_{pq} \) is given by
\[ \sum_{l,m} \tilde{h}_{(e0)} S_l = \frac{1}{r^2} \gamma_{pq} h_{pq} = -4\alpha_1 r \cos \theta + 2C_1. \] (4.39)

On the other hand, the traceless part of \( h_{pq} \) is given by
\[ \frac{1}{r^2} h_{pq} = -\frac{1}{2} \gamma_{pq} \frac{1}{r^2} \gamma^{rs} h_{rs} = (2\alpha_1 M \cos \theta + C_1) \left( -(d\theta)_p (d\theta)_q + \sin^2 \theta (d\phi)_p (d\phi)_q \right). \] (4.40)
4.3. Harmonic decomposition of the perturbative non-rotating C-metric

Here, we choose the mode function \( S_l \) as the Legendre function \( P_l(\cos \theta) \) as

\[
S_l = P_l(\cos \theta). \tag{4.41}
\]

This choice corresponds to the fact that we concentrate only on the \( m = 0 \) modes for arbitrary \( l \). Then, the vector harmonics is defined by

\[
\hat{D}_p S_l = -\sqrt{1-z^2} \frac{d}{dz} P_l(z)(d\theta)_p, \quad z := \cos \theta, \tag{4.42}
\]

\[
\epsilon_{pq} \hat{D}^q S_l = (1-z^2) \frac{d}{dz} P_l(z)(d\phi)_p. \tag{4.43}
\]

Furthermore, the tensor harmonics consists of the trace part

\[
\frac{1}{2} \gamma_{pq} S_l = \frac{1}{2} \gamma_{pq} P_l(z), \tag{4.44}
\]

the traceless even part

\[
\left( \hat{D}_p \hat{D}_q - \frac{1}{2} \gamma_{pq} \hat{D}^r \hat{D}_r \right) S_l = P_l^2(z) \frac{1}{2} (\theta_p \theta_q - \phi_p \phi_q), \quad (l \geq 2), \tag{4.45}
\]

and the traceless odd part

\[
2\epsilon_{rpq} \hat{D}_q \hat{D}_r S_l = P_l^2(z) 2\theta_{(p} \phi_{q)}, \quad (l \geq 2). \tag{4.46}
\]

These are derived from the formula

\[
\hat{D}_q \hat{D}_r S_l = \left[ (1-z^2) \frac{d^2}{dz^2} P_l(z) - z \frac{d}{dz} P_l(z) \right] \theta_q \theta_r + \left[ -z \frac{d}{dz} P_l(z) \right] \phi_q \phi_r. \tag{4.47}
\]

For \( l = 0, 1 \) modes, tensor harmonics which correspond to Eqs. (4.45) and (4.46) vanish. These correspond to the fact that we have already impose the regularity \( \delta = 0 \) in Proposal 2.1.

Here we check the formulae for the orthogonality of the harmonics which are necessary later. First, we point out that the orthogonality of the scalar harmonics (4.41) for \( l, l' \geq 0 \) modes:

\[
\int d^2 \Omega \sin \theta d\theta P_l(\cos \theta) P_{l'}(\cos \theta) = 2\pi \int_{-1}^{1} dz P_l(z) P_{l'}(z) = \frac{4\pi}{2l+1} \delta_{ll'}. \tag{4.48}
\]

Next, we consider the orthogonality of the even tensor harmonics (4.45) for \( l, l' \geq 2 \) modes:

\[
\int d^2 \Omega \left( \hat{D}_p \hat{D}_q - \frac{1}{2} \gamma_{pq} \hat{D}^r \hat{D}_r \right) S_l \left( \hat{D}^p \hat{D}^q - \frac{1}{2} \gamma_{pq} \hat{D}^r \hat{D}_r \right) S_{l'} \quad = \quad \frac{2\pi (l-1)(l+1)(l+2)}{(2l+1)} \delta_{ll'}. \tag{4.49}
\]

Now, we consider the perturbative C-metric given by Eqs. (4.37), (4.38), and \( h_{Ap} = 0 \). The angular part \( h_{pq} \) given by Eq. (4.38) is also decomposed into the trace and the traceless part as Eqs. (4.39) and (4.40). Since there is no \( h_{Ap} \) component, the perturbative C-metric does not have any vector part. Furthermore, the traceless part of the angular components (4.40) does not have \( (\theta - \phi) \) component. Therefore, the perturbative C-metric does not have any tensor odd mode. Furthermore, we do not have any vector- and tensor-modes in \( l = 0, 1 \) modes.
For our convention, we introduce the notation $|K_{pq}^l\rangle$ to consider the orthogonality Eq. (4.49) of the even-mode tensor harmonics by

$$|K_{pq}^l\rangle := \left[\hat{D}_p\hat{D}_q - \frac{1}{2}\gamma_{pq}\hat{D}_r\hat{D}^r\right]S_l.$$  \hspace{1cm} (4.50)

The orthogonality condition (4.49) is denoted as

$$\langle K_{pq}^l|K_{pq}^{l'}\rangle := \int d^2\Omega \left[\hat{D}_p\hat{D}_q - \frac{1}{2}\gamma_{pq}\hat{D}_r\hat{D}^r\right]P^l_l\left[\hat{D}_p\hat{D}_q - \frac{1}{2}\gamma_{pq}\hat{D}_s\hat{D}^s\right]P^{l'}_l = \frac{2\pi(l-1)(l+1)(l+2)}{2l+1}\delta_{ll'}.$$ \hspace{1cm} (4.51)

When an arbitrary traceless tensor $f_{pq}$, which represent the vector $|f_{pq}\rangle$ of the function space, is given by

$$|f_{pq}\rangle = \sum_l g_l |K_{pq}^l\rangle,$$ \hspace{1cm} (4.52)

applying $\langle K_{pq}^l|f_{pq}\rangle$ from the left, we obtain

$$\langle K_{pq}^{l'}|f_{pq}\rangle = \sum_{l\geq2} g_l \langle K_{pq}^{l'}|K_{pq}^l\rangle = \sum_{l\geq2} g_l \frac{2\pi(l-1)(l+1)(l+2)}{2l+1}\delta_{ll'}.$$ \hspace{1cm} (4.53)

Then, we have

$$g_l = \frac{2l+1}{2\pi(l-1)(l+1)(l+2)} \langle K_{pq}^l|f_{pq}\rangle \quad (l \geq 2).$$ \hspace{1cm} (4.54)

Since we note that the traceless part (4.40) in $h_{pq}$ does not have the odd-mode part, we obtain

$$\frac{1}{r^2} x h_{pq} - \frac{1}{2} \gamma_{pq} \frac{1}{r^2} \gamma^{rs} x h_{rs} = \sum_{l\geq2} \tilde{h}_{(e2)}^l |K_{pq}^l\rangle = (2\alpha_1 M \cos \theta + C_1) (-\theta_p\theta_q + \phi_p\phi_q).$$ \hspace{1cm} (4.55)

Then, we obtain

$$\tilde{h}_{(e2)}^l = -\frac{2l+1}{2\pi(l-1)(l+1)(l+2)} 2\pi \int_{-1}^1 dx \left(2\alpha_1 M x + C_1\right) P^2_l(x)$$

$$= -\frac{2l+1}{2\pi(l-1)(l+1)(l+2)} 2\pi \int_{-1}^1 dx \left(2\alpha_1 M x + C_1\right) (1 - x^2) \frac{d^2}{dx^2} P_l(x)$$

$$= -\frac{2l+1}{(l-1)(l+1)(l+2)} \left[2\alpha_1 M + C_1 + (-2\alpha_1 M + C_1)(-1)^l\right].$$ \hspace{1cm} (4.56)

In summary, the perturbative C-metric $h_{ab}$ given by Eqs. (4.37)–(4.38) is decomposed based on the scalar harmonic function $S_l = P^l_l(\cos \theta)$. Together with corresponding gauge-invariant and gauge-variant variables, these are summarized as follows: For $l = 0$ mode, the
Components of gauge-variant part $\tilde{Y}_A$, $\tilde{Y}_r$, and $\tilde{Y}_o$ of the metric perturbation for $l = 0$ mode are given by Eqs. (2.26), (2.28), and (2.27). Substituting Eqs. (4.64)–(4.66) into these equations, we obtain

$$\tilde{Y}_A := r\tilde{h}(e)_1A - \frac{r^2}{2}D_A\tilde{h}(e)_2 = 0, \quad \tilde{Y}_r := -r^2\tilde{h}(o)_2 = 0, \quad \tilde{Y}_o := \frac{r^2}{2}\tilde{h}(e)_2 = 0. \quad (4.60)$$

Components of the gauge-invariant variables $\tilde{F}_A$, $\tilde{F}$, and $\tilde{F}_{AB}$ defined by Eqs. (2.33)–(2.32) are given by

$$\tilde{F}_A := \tilde{h}(k\Delta,o)_A + rD_A\tilde{h}(k\Delta,o)_1 = 0, \quad (4.61)$$

$$\tilde{F} := \tilde{h}(k\Delta,e)_0 - \frac{4}{r}\tilde{h}_A\tilde{D}_A r = 2C_1, \quad (4.62)$$

$$\tilde{F}_{AB} := \tilde{h}(k\Delta)_A - 2\tilde{D}_A\tilde{Y}(k\Delta)_B = \frac{2M_1}{r}((dt)_A(dt)_B + f^{-2}(dr)_A(dr)_B). \quad (4.63)$$

For $l = 1$ mode, the mode coefficients of harmonic decomposition are given by

$$h_{AB} = -2\alpha_1 r(-f(dt)_A(dt)_B + f^{-1}(dr)_A(dr)_B), \quad (4.64)$$

$$\tilde{h}(e)_1A = 0, \quad \tilde{h}(o)_1A = 0, \quad (4.65)$$

$$\tilde{h}(e)_0 = -4\alpha_1 r, \quad \tilde{h}(e)_2 = 0, \quad \tilde{h}(o)_2 = 0. \quad (4.66)$$

Components of gauge-variant part $\tilde{Y}_A$, $\tilde{Y}_r$, and $\tilde{Y}_o$ of the metric perturbation for $l = 1$ modes are given by Eqs. (2.26), (2.28), and (2.27). Substituting Eqs. (4.64)–(4.66) into these equations, we obtain

$$\tilde{Y}_A := r\tilde{h}(e)_1A - \frac{r^2}{2}D_A\tilde{h}(e)_2 = 0, \quad \tilde{Y}_r := -r^2\tilde{h}(o)_2 = 0, \quad \tilde{Y}_o := \frac{r^2}{2}\tilde{h}(e)_2 = 0. \quad (4.67)$$

Components of the gauge-invariant variables $\tilde{F}_A$, $\tilde{F}$, and $\tilde{F}_{AB}$ defined by Eqs. (2.33)–(2.32) are given by

$$\tilde{F}_A := \tilde{h}(o)_1A + rD_A\tilde{h}(o)_2 = 0, \quad (4.68)$$

$$\tilde{F} := -4\alpha_1 r, \quad (4.69)$$

$$\tilde{F}_{AB} := -2\alpha_1 r(-f(dt)_A(dt)_B + f^{-1}(dr)_A(dr)_B). \quad (4.70)$$

For $l \geq 2$ modes, the mode coefficients of harmonic decomposition are given by

$$h_{AB} = 0, \quad \tilde{h}(e)_1A = 0, \quad \tilde{h}(o)_1A = 0, \quad (4.71)$$

$$\tilde{h}(e)_0 = 0, \quad \tilde{h}(o)_2 = 0, \quad (4.72)$$

$$\tilde{h}(e)_2 = -\frac{2(2l + 1)}{(l - 1)(l + 1)(l + 2)} \left[(+2\alpha_1 M + C_1) + (-2\alpha_1 M + C_1)(-1)^l\right]. \quad (4.73)$$

Components of gauge-variant part $\tilde{Y}_A$, $\tilde{Y}_r$, and $\tilde{Y}_o$ of the metric perturbation for $l \geq 2$ mode given by Eqs. (2.26), (2.28), and (2.27). Substituting Eqs. (4.71)–(4.73) into these
equations, we obtain
\[
\tilde{Y}_A := r \tilde{h}_{(e1)A} - \frac{r^2}{2} \tilde{D}_A \tilde{h}_{(e2)} = 0, \quad \tilde{Y}_{(e)} := -r^2 \tilde{h}_{(e2)} = 0, \tag{4.74}
\]
\[
\tilde{Y}_{(o)} := -r^2 \tilde{h}_{(e2)} = 0, \quad (4.75)
\]

Components of the gauge-invariant variables \(\tilde{F}_A, \tilde{F}, \) and \(\tilde{F}_{AB}\) defined by Eqs. (2.33)–(2.32) are given by
\[
\tilde{F}_A := \tilde{h}_{(e0)A} + r \tilde{D}_A \tilde{h}_{(e2)} = 0, \quad \tilde{F}_{AB} := \tilde{h}_{AB} - 2 \tilde{D}_A \tilde{Y}_B = 0, \tag{4.76}
\]
\[
\tilde{F} := \tilde{h}_{(e0)} - \frac{4}{r} \tilde{Y}_A \tilde{D}^A r + \tilde{h}_{(e2)} l(l + 1) = -\frac{2(2l + 1)}{(l - 1)(l + 2)} \left[ (+2\alpha_1 M + C_1) + (-2\alpha_1 M + C_1)(-1)^l \right]. \tag{4.77}
\]

### 4.4. Realization of \(l \geq 2\) mode perturbations
As shown in above, the perturbative expression of the C-metric does not include odd-mode perturbation as in the first equation in Eq. (4.76). Furthermore, we also obtain \(\tilde{F}_{AB} = 0\) from the second equation in Eq. (4.76) and the gauge-invariant variable \(\tilde{F}\) is constant given by Eq. (4.77). Therefore, we obtain
\[
\tilde{F}_{AB} = 0, \quad \partial_r \tilde{F} = \partial_t \tilde{F} = 0, \tag{4.78}
\]
for \(l \geq 2\) mode perturbations.

For \(l \geq 2\) mode, the linearized Einstein equations for even-mode perturbations are given in Sec. 2.3. From Eq. (2.40) and the first condition (4.78), we obtain
\[
\tilde{T}_{(e2)} = 0. \tag{4.79}
\]
Since Eqs. (4.78) implies \(X_{(e)} = Y_{(e)} = 0\) and \(\tilde{F}\) is constant, we obtain
\[
\tilde{T}_{(e1)A} = 0 \tag{4.80}
\]
from Eqs. (2.41) and (4.79). Furthermore, from Eq. (4.78) and Eq. (2.45), we obtain
\[
S_{(F)AB} = 0. \tag{4.81}
\]
Together with Eqs. (4.79) and (4.80), Eq. (4.81) yields
\[
\tilde{T}_{tr} = 0, \tag{4.82}
\]
\[
\tilde{T}_{tt} + f^2 \tilde{T}_{rr} = 0. \tag{4.83}
\]
Moreover, from Eqs. (2.48) and (4.78), Eq. (2.55) with the source term (2.58) is given by
\[
\tilde{F} = 16\pi \frac{r^2}{(l - 1)(l + 2)} \left( \frac{1}{f} \tilde{T}_{tt} - f \tilde{T}_{rr} \right) = \frac{32\pi r^2}{(l - 1)(l + 2)f} \tilde{T}_{tt} = \text{constant}. \tag{4.84}
\]
Finally, from Eqs. (4.79) and (4.80) the component (2.62) yields
\[
\tilde{T}_{(e0)} = 0 \tag{4.85}
\]
for \( l \geq 2 \) modes. Thus, from the definition of these components (2.36), for \( l \geq 2 \) mode, we obtain

\[
(1) \mathcal{F}_{ac} = \sum_{l=2}^{\infty} P_l(\cos \theta) \left( \tilde{T}_{tt}(dt)_a(dr)_c + \tilde{T}_{rr}(dr)_a(dr)_c \right)
\]

\[
= -\frac{1}{r^2} y_{ab} \sum_{l=2}^{\infty} \lambda_l P_l(\cos \theta),
\]

where we defined

\[
16\pi \lambda_l := 16\pi r^2 \tilde{T}_{tt} = -(2l + 1) \left[ (+2\alpha_1 M + C_1) + (-2\alpha_1 M + C_1)(-1)^l \right]
\]

from Eqs. (4.84) and (4.77).

We check the other components of the linearized Einstein equation. To carry out this, we see that the Moncrief variable \( \Phi_{(e)} \) in the case of Eq. (4.78) is given by

\[
\Phi_{(e)} = -\frac{r}{4} \tilde{F},
\]

through the definition (2.48). Through Eqs. (4.79) and (4.82), and from Eqs. (2.52) and (2.53), the source terms \( S_{(\Lambda \tilde{F})} \) and \( S_{(Y)} \) are given by

\[
S_{(\Lambda \tilde{F})} = \tilde{T}_{tt}, \quad S_{(Y)} = 0.
\]

From \( S_{(Y)} = 0 \) and the vanishing components \( (X_{(e)} Y_{(e)}) \) of \( \tilde{F}_{AB} \) and the derivative of \( \tilde{F} \) yields that Eq. (2.51) is trivial. On the other hand, substituting Eqs. (2.48) and \( S_{(\Lambda \tilde{F})} = \tilde{T}_{tt} \) into Eq. (2.50), we obtain the same result as Eq. (4.84).

Finally, we check the master equation (2.54) with the source term \( S_{(\Phi_{(e)})} \) given by Eq. (2.57). Substituting Eqs. (4.79), (4.80), (4.83), (4.84), and (4.85),

\[
S_{(\Phi_{(e)})} = \frac{1}{2f} \Lambda \tilde{T}_{tt} + \frac{2f - 1}{f} \tilde{T}_{tt} - \frac{3(1 - f)}{\Lambda} \tilde{T}_{tt}.
\]

Substituting Eqs. (4.88), (4.90), (2.56), and (4.84) into Eq. (2.54), and using Eq. (2.49), we can confirm that Eq. (2.54) is trivial.

### 4.5. Realization of \( l = 0 \) mode perturbations

The solutions to the Einstein equation for \( l = 0 \) mode with a generic matter field are extensively discussed in the Part II paper [13]. Following Proposal 2.3, we impose the regularity to the harmonic \( S_3 \) and the components \( \tilde{T}_{(e1)A} \) and \( \tilde{T}_{(e2)} \) do not appear in the expression of the linear perturbations of the energy-momentum tensor. Therefore, we may safely choose,

\[
\tilde{T}_{(e1)A} = 0, \quad \tilde{T}_{(e2)} = 0.
\]

From Eqs. (4.91) and (2.62), we have

\[
\tilde{T}_{(e0)} = 0,
\]

and Eqs. (2.60) and (2.61) are given by

\[
-\partial_t \tilde{T}_{tt} + f^2 \partial_r \tilde{T}_{rr} + \left( 1 + \frac{f}{r} \right) \tilde{T}_{rr} = 0,
\]

\[
-\partial_t \tilde{T}_{tr} + \frac{1 - f}{2rf} \tilde{T}_{tt} + f^2 \partial_r \tilde{T}_{rr} + \frac{3 + f}{2r} \tilde{T}_{rr} = 0.
\]
On the other hand, the metric perturbation of \( l = 0 \) mode summarized in Eq. (4.57)–(4.59) is given by

\[
h_{ab}(l = 0) = \frac{2M_1}{r} ((dt)_a(dt)_b + f^{-2}(dr)_a(dr)_b) + r^2C_1\gamma_{ab}.
\] (4.95)

The mass perturbation \( M_1 \) is discussed in the Part II paper [15]. To include this mass perturbation parameter \( M_1 \) as the solution to the Einstein equation in our formulation, we have to introduce the term

\[
\mathcal{L}h_{ab}(l = 0) = \mathcal{L}h_{ab}(l = 0) + \mathcal{L}_Vg_{ab},
\] (4.96)

where the generator \( V_a \) is given

\[
V_a = \left( \frac{f}{4} \Upsilon + \frac{rf}{4} \partial_r \Upsilon - \frac{r}{1 - 3f} \Xi(r) + f \int dr \frac{2}{f(1 - 3f)^2} \Xi(r) \right) (dt)_a + \frac{r}{4f} \partial_r (dr)_a.
\] (4.97)

Here, the function \( \Upsilon(t, r) \) is the solution to the equation

\[
- \frac{1}{f} \partial^2_t \Upsilon + \partial_r (f \partial_r \Upsilon) + \frac{1}{r^2} 3(1 - f) \Upsilon - \frac{4}{r^3} 2M_1 t = 0,
\] (4.98)

and \( \Xi(r) \) in Eq. (4.97) is an arbitrary function of \( r \).

Since we always introduce the mass parameter perturbation \( M_1 \) if we introduce the last term in Eq. (4.96), we ignore this mass perturbation at this moment. Then, we may concentrate on the perturbation \( C_1 \) in the metric perturbation

\[
h_{ab}(l = 0) = \frac{r^2}{2f} C_1\gamma_{ab}.
\] (4.99)

Since the \( \theta\theta \)- and \( \phi\phi \)-components in the solution (2.65) is described by the term \( \mathcal{L}_Vg_{ab} \), we consider the components \( \mathcal{L}_{V(1)}g_{ab} \) with the generator \( V_{(1)}a = V_{(1)}r(dr)_a \). In this case, the non-vanishing components of \( \mathcal{L}_{V(1)}g_{ab} \) are summarized as

\[
\mathcal{L}_{V(1)}g_{tt} = - f f'V_{(1)r}, \quad \mathcal{L}_{V(1)}g_{rr} = 2 \partial_r V_{(1)r} + \frac{f'}{f} V_{(1)r},
\]

\[
\mathcal{L}_{V(1)}g_{\theta\theta} = 2rfV_{(1)r}, \quad \mathcal{L}_{V(1)}g_{\phi\phi} = 2rf \sin^2 \theta V_{(1)r}.
\] (4.100)

Here, we choose

\[
V_{(1)r} = \frac{r^2}{2rf} C_1
\] (4.101)

so that \( \theta\theta \)- and \( \phi\phi \)-components are described by the term \( \mathcal{L}_{V(1)}g_{ab} \). Then, we have

\[
h_{ab}(l = 0) = \frac{1 - f}{2} C_1(dt)_a(dt)_b + \frac{1 - 3f}{2f^2} C_1(dr)_a(dr)_b + \mathcal{L}_{V(1)}g_{ab}.
\] (4.102)

As in the case of LTB solution, we make \((t, r)\) part in Eq. (4.102) to be traceless through the introduction of the term \( \mathcal{L}_{V(2)}g_{ab} \) with \( V_{(2)}a = V_{(2)t}(dt)_a \) with the condition \( \partial_t V_{(2)t} = \).
∂ₙV(2)ₜ = 0. In this choice of V(2)ₜ, the nonvanishing components of \( \mathcal{L}_{V(2)} g_{ab} \) are given by

\[
\mathcal{L}_{V(2)} g_{tt} = 2 \partial_t V(2)_t, \quad \mathcal{L}_{V(2)} g_{tr} = \partial_r V(2)_t - \frac{f'}{f} V(2)_t.
\]

Through this expression (4.103), \( h_{ab}(l = 0) \) is given by

\[
h_{ab}(l = 0) = \left( \frac{1 - f}{2} C_1 - 2 \partial_t V(2)_t \right) (dt)_a (dt)_b + \frac{1 - 3f}{2} C_1 f^{-2} (dr)_a (dr)_b
\]

and

\[
\partial_r V(2)_t - \frac{f'}{f} V(2)_t = 0,
\]

and

\[
\mathcal{L}_{V(2)} g_{ab} = \frac{1 - 3f}{2} C_1 ((dt)_a (dt)_b + f^{-2} (dr)_a (dr)_b) + \mathcal{L}_{V(2)} g_{ab}.
\]

Here, we choose \( V(2)_t \) so that

\[
V(2)_t = \frac{1}{2} ft C_1.
\]

This choice (4.105) yields

\[
\partial_r V(2)_t - \frac{f'}{f} V(2)_t = 0,
\]

and

\[
\mathcal{L}_{V(2)} g_{ab} = \frac{1 - 3f}{2} C_1 ((dt)_a (dt)_b + f^{-2} (dr)_a (dr)_b) + \mathcal{L}_{V(2)} g_{ab}.
\]

Now, we compare the metric perturbation (4.107) with the derived \( l = 0 \) solution (2.65) with the generator (2.66). Since we ignore the mass parameter perturbation \( M_1 \), we obtain the relations

\[
4\pi \int dr \left[ \frac{r^2}{f} \tilde{T}_{tt} \right] = \frac{1 - 3f}{4} r C_1,
\]

\[
4\pi r \int dt \left( \frac{1}{f} \tilde{T}_{tt} + f \tilde{T}_{rr} \right) = 0.
\]

If the condition (4.110) is satisfied for an arbitrary time \( t \), we obtain

\[
\tilde{T}_{tt} + f^2 \tilde{T}_{rr} = 0.
\]

Since we ignore the integration constant \( M_1 \), the condition (4.109) gives

\[
\frac{r^2}{f} \tilde{T}_{tt} = \partial_r \left( \frac{1 - 3f}{16\pi} r \right) C_1 = -\frac{1}{8\pi} C_1.
\]

Furthermore, we may choose \( \tilde{T}_{tr} = 0 \) without contradiction to the linear perturbation of the continuity equations (2.60) and (2.61) with \( l = 0 \). From the definition (2.36) of the

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components $\tilde{T}_{tt}$ and $\tilde{T}_{rr}$, for $l = 0$ mode, we obtain
\[(1)\mathcal{J}_{ac} = -\frac{1}{r^2} y_{ab} \lambda_{l=0} \quad (4.113)\]
from Eq. (4.111) and (4.112). Here, we defined
\[\lambda_{l=0} := \frac{r^2}{f} \tilde{T}_{tt} = -\frac{C_1}{8\pi}. \quad (4.114)\]

Next, we compare the generator $V_{(C_1)a}$ defined by Eq. (4.108) and the generator $V_a$ given by Eq. (2.66) in the $l = 0$ mode solution (2.65). Comparing the $r$-component of Eq. (4.108) with Eq. (2.66), we choose
\[\frac{1}{4} f \partial_t \Upsilon = \frac{r}{2f} C_1, \quad (4.115)\]
and obtain
\[\Upsilon = 2C_1 t. \quad (4.116)\]
Here, we ignore the integration constant in the integration of Eq. (4.115). Substituting this result (4.116) into Eq. (2.66)
\[V_a = \left( \frac{f}{4} 2C_1 t - \frac{r\Xi(r)}{(1 - 3f)} + f \int dr \frac{2\Xi(r)}{f(1 - 3f)^2} \right) (dt)_a + \frac{r}{2f} C_1 (dr)_a. \quad (4.117)\]
Choosing $\Xi(r) = 0$, the generator (4.117) coincides with the generator $V_{(C_1)a}$ given by Eq. (4.108). Thus, the $l = 0$ mode solution (2.65) realizes the $l = 0$ mode part (4.107) of the C-metric.

In summary, we have obtained the $l = 0$ metric perturbation
\[\tilde{x}h_{ab}(l = 0) = \mathcal{L}_{\tilde{x}h_{ab}} + \mathcal{L}_{(M_1)g_{ab}} + \mathcal{L}_{(C_1)g_{ab}} + \mathcal{L}_{V_{(M_1)}g_{ab}} + \mathcal{L}_{V_{(C_1)}g_{ab}}. \quad (4.118)\]
where
\[V_{(M_1)a} = \frac{1}{4} f (\Upsilon_{M_1} + r \partial_r \Upsilon_{M_1}) (dt)_a + \frac{1}{4f} \partial_r \Upsilon (dr)_a, \quad (4.119)\]
\[V_{(C_1)a} = \frac{1}{2} ft C_1 (dt)_a + \frac{r}{2f} C_1 (dr)_a. \quad (4.120)\]
Here, the function $\Upsilon_{M_1}(t, r)$ is the solution to the equation (2.55) with $\tilde{F} =: \partial_t \Upsilon$, i.e.,
\[-\frac{1}{f} \partial_t^2 \Upsilon_{M_1} + \partial_r (f \partial_r \Upsilon_{M_1}) + \frac{1}{r^2} 3(1 - f) \Upsilon_{M_1} - \frac{4}{r^3} 2M_1 t = 0. \quad (4.121)\]
The $l = 0$ mode energy-momentum tensor for C-metric is given by
\[(1)\mathcal{J}_{ac} = -\frac{1}{r^2} y_{ab} \lambda_{l=0} \quad (4.122)\]
with
\[\lambda_{l=0} = -\frac{C_1}{8\pi}. \quad (4.123)\]
Here, we note that the result (4.123) is also realized by the substitution $l = 0$ into Eq. (4.87), although Eq. (4.87) is derived only in the case of $l \geq 2$ modes. This indicates that the formula (4.87) is also valid even for $l = 0$ mode perturbations.
4.6. Realization of $l = 1$ mode perturbations

The $l = 1$ mode of the C-metric is summarized as Eqs. (4.64)–(4.70). We consider the continuity equation of the energy-momentum tensor (2.60)–(2.62) with $l = 1$. As in the $l \geq 2$ and $l = 0$ cases, we choose $\tilde{T}_{(e1)A} = 0 = \tilde{T}_{(e2)}$. These and Eqs. (2.62) with $l = 1$ yield

$$\tilde{T}_{(e0)} = 0.$$  \hspace{1cm} (4.124)

Furthermore, we also assume that

$$\tilde{T}_{rt} = 0,$$  \hspace{1cm} (4.125)

inspecting the $l \geq 2$ and $l = 0$ cases. Then, Eqs. (2.60) and (2.61) with $l = 1$ are given by

$$\partial_t \tilde{T}_{tt} = 0,$$  \hspace{1cm} (4.126)

$$\partial_r \left( \tilde{T}_{tt} + f^2 \tilde{T}_{rr} \right) - \frac{f}{r^2} \partial_r \left( \frac{r^2}{f} \tilde{T}_{tt} \right) + \frac{5f - 1}{2rf} \left( \tilde{T}_{tt} + f^2 \tilde{T}_{rr} \right) = 0.$$  \hspace{1cm} (4.127)

Equation (4.126) indicates that we have the static energy density $\tilde{T}_{tt}$. As in the case of $l \geq 2$ and $l = 0$ modes, we define $\lambda_{l=1}$ by

$$\lambda_{l=1} := \frac{r^2}{f} \tilde{T}_{tt}$$  \hspace{1cm} (4.128)

and we assume that $\lambda_{l=1}$ is constant and

$$\tilde{T}_{tt} + f^2 \tilde{T}_{rr} = 0.$$  \hspace{1cm} (4.129)

Due to these assumptions, Eq. (4.127) is trivial. Through the above components of the energy-momentum tensor, Eq. (2.63) is given by

$$F_{ab} = \mathcal{L}_{V_{(vac)}} g_{ab}$$

$$- \frac{16\pi \lambda_{l=1}}{3(1 - f)} \left[ \frac{1 + f}{2} (dt)_a (dt)_b - \frac{1 - 3f}{2f^2} (dr)_a (dr)_b + r^2 \gamma_{ab} \right] \cos \theta.$$  \hspace{1cm} (4.130)

Here, we note that the $l = 1$ mode solution (4.68)–(4.70) is summarized as

$$\mathcal{F}_{ab} = \tilde{F}_{AB} \cos \theta (dx^A)_a (dx^B)_b + \frac{1}{2} \gamma_{pq} \tilde{F} \cos \theta (dx^p)_a (dx^q)_b$$

$$= -2\alpha_1 r \cos \theta g_{ab}.$$  \hspace{1cm} (4.131)

Comparing (4.130) and (4.131), we rewrite (4.130) as

$$\mathcal{F}_{ab} = \mathcal{L}_{V_{(vac)}} g_{ab}$$

$$- \frac{16\pi \lambda_{l=1}}{3(1 - f)} \left[ \frac{1 - f}{2} (dt)_a (dt)_b + \frac{1 + 3f}{2f^2} (dr)_a (dr)_b + \frac{1 + f}{f} r^2 \gamma_{ab} \right] \cos \theta$$

$$+ \frac{16\pi \lambda_{l=1}}{3(1 - f)} \cos \theta g_{ab}.$$  \hspace{1cm} (4.132)

Here, we explain the choice of the coefficients of the last term in Eq. (4.132). If the expression (4.87) of the $\lambda_l$ for $l \geq 2$ is also valid even for $l = 1$, Eq. (4.87) is given by

$$16\pi \lambda_{l=1} = -12\alpha_1 M$$  \hspace{1cm} (4.133)

and the last term in Eq. (4.132) is given by

$$\frac{16\pi \lambda_{l=1}}{3(1 - f)} \cos \theta g_{ab} = -2\alpha_1 r \cos \theta g_{ab}.$$  \hspace{1cm} (4.134)

This is the $l = 1$ mode solution described by Eqs. (4.68)–(4.70). As the remaining problem, we have to consider the problem whether the middle term in Eq. (4.132) has the form $\mathcal{L}_W g_{ab}$.
or not. If the middle term in Eq. (4.132) does have the form $\mathcal{L} g_{ab}$, we may say that our $l = 1$ mode solution (2.63) does describe the linearized C-metric apart from the term of the Lie derivative of the background metric $g_{ab}$.

Now, we concentrate on the problem whether the middle term in Eq. (4.132) has the form $\mathcal{L} W g_{ab}$, or not. To show this, we consider the components of $\mathcal{L} W g_{ab}$ for an appropriate vector field $W_a$. We consider the generator $W_a$ which satisfies $W_\phi = 0$, $\partial_\phi W_\theta = \partial_\phi W_r = 0$. Furthermore, we assume that $W_t =: w_t \cos \theta$, $W_r =: w_r \cos \theta$, and $W_\theta =: w_\theta \sin \theta$ using $\mathcal{L} g_{t\theta} = 0$. Then, the non-trivial components of $\mathcal{L} W g_{ab}$ are summarized as follows:

\begin{align*}
\mathcal{L} W g_{tt} &= \left( 2\partial_t w_t - f' w_r \right) \cos \theta, \quad (4.135) \\
\mathcal{L} W g_{tr} &= \left( \partial_t w_r + \partial_r w_t - \frac{f'}{f} w_t \right) \cos \theta, \quad (4.136) \\
\mathcal{L} W g_{t\theta} &= \left( \partial_t w_\theta - w_t \right) \sin \theta, \quad (4.137) \\
\mathcal{L} W g_{rr} &= \left( 2\partial_r w_r + \frac{f'}{f} w_r \right) \cos \theta, \quad (4.138) \\
\mathcal{L} W g_{r\theta} &= \left( \partial_r w_\theta - w_r - \frac{2}{r} w_\theta \right) \sin \theta, \quad (4.139) \\
\mathcal{L} W g_{\phi\phi} &= 2 \left( w_\theta + r f w_r \right) \cos \theta, \quad (4.140) \\
\mathcal{L} W g_{\phi\theta} &= 2 \left( w_\theta + r f w_r \right) \sin^2 \theta \cos \theta. \quad (4.141)
\end{align*}

The middle term in Eq. (4.132) has only its diagonal components, we may choose $\mathcal{L} W g_{t\theta} = \mathcal{L} W g_{r\theta} = 0$. From these equations, Eqs. (4.137) and (4.139) yield

\begin{align*}
w_t &= \partial_t w_\theta, \quad w_r = \partial_r w_\theta - \frac{2}{r} w_\theta. \quad (4.142)
\end{align*}

Furthermore, from Eqs. (4.140), (4.141), the second equation in Eq. (4.142), and the term proportional to $\gamma_{ab}$ in the second line of Eq. (4.132), we have

\begin{align*}
rf \partial_r w_\theta + (1 - 2f) w_\theta &= -\frac{r^2(1 + f)}{6(1 - f)} 16\pi \lambda_{l=1}. \quad (4.143)
\end{align*}

A solution to Eq. (4.143) is given by

\begin{align*}
w_\theta &= -\frac{r^2}{6(1 - f)} 16\pi \lambda_{l=1}. \quad (4.144)
\end{align*}

Then, from Eqs. (4.142), we obtain

\begin{align*}
w_t &= 0, \quad w_r = -\frac{r}{6(1 - f)} 16\pi \lambda_{l=1}. \quad (4.145)
\end{align*}

Substituting Eqs. (4.144) and (4.145) into Eqs. (4.135)–(4.141), the non-vanishing components of $\mathcal{L} W g_{ab}$ is given by

\begin{align*}
\mathcal{L} W g_{ab} &= \frac{16\pi \lambda_{l=1}}{3(1 - f)} \left[ \frac{1 - f}{2} (dt)_a (dt)_b - \frac{1 + 3f}{2f^2} (dr)_a (dr)_b - \frac{1 + f}{f} r^2 \gamma_{ab} \right] \cos \theta. \quad (4.146)
\end{align*}

Through Eq. (4.146), Eq. (4.132) is given by

\begin{align*}
\mathcal{F}_{ab} &= \mathcal{L} V_{(l=1)} g_{ab} + \frac{16\pi \lambda_{l=1}}{3(1 - f)} \cos \theta g_{ab}, \quad (4.147)
\end{align*}

where $V_{(l=1)} := V_{(vac)} + W_a$. Comparing with Eqs. (4.69) and (4.70), we obtain Eqs. (4.133) and (4.134) as expected. This also indicates that the coefficient (4.87) for $l \geq 2$ is also valid.
not only for \( l = 0 \) mode but also \( l = 1 \) mode, i.e., the coefficient (4.87) is valid for any \( l \geq 0 \). Furthermore, we have seen above the equation of state (4.129) with (4.128) yields

\[
(1) T_{ac}(l = 1) = -\frac{1}{r^2} y_{ab} \lambda_{l=1} P_1(\cos \theta),
\]

where \( \lambda_{l=1} \) is constant which is given by Eq. (4.133). At the same time, we may say that our \( l = 1 \) mode solution (2.63) does describe the linearized C-metric apart from the term of the Lie derivative of the background metric \( g_{ab} \).

4.7 Source term of the linearized C-metric

Here, we summarize the energy-momentum tensor \( T_{ab} \) for the linearized C-metric as follows

\[
(1) T_{ac} = -\frac{1}{r^2} y_{ab} \sum_{l=0}^{\infty} \lambda_l P_l(\cos \theta),
\]

where the constant \( \lambda_l \) is given by

\[
16\pi \lambda_l = -(2l + 1) \left[ (+2\alpha_1 M + C_1) + (-2\alpha_1 M + C_1)(-1)^l \right]
\]

(4.150)

for \( l \geq 0 \). Substituting Eq. (4.150) into Eq. (4.149), we obtain

\[
(1) T_{ac} = \frac{1}{4r^2} y_{ab} \left[ (+2\alpha_1 M + C_1) \left( \frac{1}{4\pi} \sum_{l=0}^{\infty} (2l + 1)P_l(\cos \theta) \right) \right.
\]

\[
+ (-2\alpha_1 M + C_1) \left( \frac{1}{4\pi} \sum_{l=0}^{\infty} (2l + 1)(-1)^l P_l(\cos \theta) \right) \right].
\]

(4.151)

Inspecting the work by Kodama [54], we consider the mode decomposition of the \( \delta \)-function on \( S^2 \) [52, 53]

\[
\delta^{(2)}(\mathbf{n} - \mathbf{n}') = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} Y_{lm}(\mathbf{n})Y_{lm}^*(\mathbf{n}'),
\]

(4.152)

where \( \mathbf{n} \) and \( \mathbf{n}' \) are the position vectors which point to the points on \( S^2 \) in embedded in \( \mathbb{R}^3 \), respectively. The summation over \( m \) is given by [52, 53]

\[
\sum_{m=-l}^{l} Y_{lm}(\mathbf{n})Y_{lm}^*(\mathbf{n}') = \frac{2l + 1}{4\pi} C_l^{1/2}(\mathbf{n} \cdot \mathbf{n}'),
\]

(4.153)

where \( C_l^{1/2}(x) \) is the Gegenbauer polynomial. In \( \mathbb{R}^3 \), any point of the unit sphere \( S^2 \) is specified by the orthogonal coordinates \((x, y, z)\) with the center of \( S^2 \) in \( \mathbb{R}^3 \)

\[
x = \sin \theta \cos \phi, \quad y = \sin \theta \sin \phi, \quad z = \cos \theta.
\]

(4.154)

The north pole is specified as \((x, y, z) = (0, 0, 1)\) and the south pole is specified as \((x, y, z) = (0, 0, -1)\). The inner product \( \mathbf{n} \cdot \mathbf{n}' \) in the case where \( \mathbf{n}' \) is the north pole or the south pole
is given by
\[ \mathbf{n} \cdot \mathbf{n}_{\text{north}} = \cos \theta, \quad \mathbf{n} \cdot \mathbf{n}_{\text{south}} = - \cos \theta, \quad (4.155) \]
respectively, and we have
\[ \sum_{m=-l}^{l} Y_{lm}(\mathbf{n}) Y_{lm}^*(\mathbf{n}_{\text{north}}) = \frac{2l+1}{4\pi} C_l^{1/2}(\cos \theta) = \frac{2l+1}{4\pi} P_l(\cos \theta), \quad (4.156) \]
\[ \sum_{m=-l}^{l} Y_{lm}(\mathbf{n}) Y_{lm}^*(\mathbf{n}_{\text{south}}) = \frac{2l+1}{4\pi} C_l^{1/2}(-\cos \theta) = \frac{2l+1}{4\pi} P_l(-\cos \theta) = \frac{2l+1}{4\pi} (-1)^l P_l(\cos \theta). \quad (4.157) \]

Then, from Eq. (4.152), we obtain
\[ \delta^{(2)}(\mathbf{n} - \mathbf{n}_{\text{north}}) = \sum_{l=0}^{\infty} \sum_{m=-l}^{m} Y_{lm}(\mathbf{n}) Y_{lm}^*(\mathbf{n}_{\text{north}}) = \frac{1}{4\pi} \sum_{l=0}^{\infty} (2l+1) P_l(\cos \theta), \quad (4.158) \]
\[ \delta^{(2)}(\mathbf{n} - \mathbf{n}_{\text{south}}) = \sum_{l=0}^{\infty} \sum_{m=-l}^{m} Y_{lm}(\mathbf{n}) Y_{lm}^*(\mathbf{n}_{\text{south}}) = \frac{1}{4\pi} \sum_{l=0}^{\infty} (2l+1)(-1)^l P_l(\cos \theta). \quad (4.159) \]

Through these expressions of the \( \delta \)-functions, the first-order perturbation of the energy-momentum tensor (4.151) yields
\[ (1) \mathcal{T}_{ac} = -\frac{1}{4\pi^2} y_{ab} \left[ (2\alpha_1 M + C_1) \delta^{(2)}(\mathbf{n} - \mathbf{n}_{\text{north}}) + (2\alpha_1 M - C_1) \delta^{(2)}(\mathbf{n} - \mathbf{n}_{\text{south}}) \right] \]
\[ = -\frac{1}{r^2} y_{ab} \left[ \mu_n \delta^{(2)}(\mathbf{n} - \mathbf{n}_{\text{north}}) + \mu_s \delta^{(2)}(\mathbf{n} - \mathbf{n}_{\text{south}}) \right], \quad (4.160) \]
where
\[ \mu_n = -\frac{1}{4} (2\alpha_1 M + C_1), \quad \mu_s = \frac{1}{4} (2\alpha_1 M - C_1). \quad (4.161) \]
The first-order perturbation of the energy-momentum tensor (4.160) coincides with the energy-momentum tensor for a half-infinite string with constant line densities \( \mu_n \) on the north half of the symmetry axis and \( \mu_s \) on the south half of the symmetry axis. If we impose the regularity at the north pole, i.e., \( \mu_n = 0 \), we have \( \mu_s = \alpha_1 M > 0 \) which corresponds to the positive energy density of string. On the other hand, if we impose the regularity at the south pole, i.e., \( \mu_s = 0 \), we have \( \mu_n = -\alpha_1 M < 0 \) which corresponds to the negative energy density of string. These results are consistent with the stringy interpretation of the singularity of the C-metric [54]. Finally, we have to emphasize that the treatment of \( l = 1 \) mode perturbations in our derivation of the energy density of the C-metric is essentially different from those in Ref. [54].

5. Summary and Discussions

In summary, after reviewing our general framework of the gauge-invariant perturbation theory and its application to the perturbation theory on the Schwarzschild background spacetime developed in Refs. [12, 14, 15], we checked the fact that our linearized solutions
derived in Refs. [12, 14, 15] realizes the linearized LTB solution and the linearized C-metric around the Schwarzschild background spacetime. These facts yield that our derived \( l = 0, 1 \) solutions to the linearized Einstein equation following Proposal 2.1 are physically reasonable. Then, we may say that Proposal 2.1 itself is also physically reasonable. Our general framework of the gauge-invariant perturbation theory developed in Refs. [33–38] was applied to the cosmological perturbation theory in Refs. [39–43, 45, 46]. On the other hand, in this series of our papers [12–15], we apply our general framework of the gauge-invariant perturbation theory to the perturbations on the Schwarzschild background spacetime. Thus, we may say that the applicability of our general framework of the gauge-invariant perturbation theory developed in Refs. [33–38] is very wide.

Our general-framework is based on the single non-trivial Conjecture 2.1. This conjecture is almost proved in Ref. [37] except for the “zero-mode problem.” In the proof in Ref. [37], we introduced the Green functions for some elliptic derivative operators. This means that the kernel modes of these elliptic derivative operators were out of our considerations. We call these kernel modes as “zero modes” and the problem to find gauge-invariant treatments for these kernel modes as “zero-mode problem.” To carry out the application to the perturbations on the Schwarzschild background spacetime, we have to propose a gauge-invariant treatment of \( l = 0, 1 \) mode perturbation on the Schwarzschild background spacetime, because these modes correspond to the above “zero modes” and the gauge-invariant treatments was unclear until our proposal in Refs. [12, 14, 15]. We should also note that such “zero-mode problem” exists even in the cosmological perturbation theory which are developed in Refs. [39–43, 45, 46].

In conventional perturbation theory on spherically symmetric background spacetimes, we use the spherical harmonics \( S = Y_{lm} \) as the scalar-harmonics and construct vector and tensor harmonics from the derivative of this scalar harmonics. However, in this construction of tensor harmonics, the set (2.17) of the scalar-, vector- and tensor-harmonics loses its linear-independence as the basis of the tangent space on \( S^2 \) in \( l = 0, 1 \) mode. To recover this linear-independence of the set (2.17), we introduced the singular harmonics for \( l = 0, 1 \) modes at once and proposed the strategy to construct gauge-invariant variables and derive the Einstein equation as Proposal 2.1. The conventional expansion using the spherical harmonic functions \( Y_{lm} \) is the restriction of the function space to the \( L^2 \) space on \( S^2 \). This restriction corresponds to the imposition of the regular boundary condition for the functions on \( S^2 \) at the starting point. On the other hand, our introduction of the singular harmonic functions at once and Proposal 2.1 state that the boundary condition on \( S^2 \) should be imposed when we solve the linearized Einstein equations. Owing to Proposal 2.1, we could prove Conjecture 2.1 for perturbations on the spherically symmetric background spacetime. Then, we reached to the statement Theorem 2.1.

Actually, following Proposal 2.1 we could construct gauge-invariant variables not only for \( l \geq 2 \) modes but also for \( l = 0, 1 \) modes. Furthermore, in Ref. [14], we derive the solution of \( l = 1 \) odd-mode perturbations and, in Ref. [15], we derive the solutions of \( l = 0, 1 \) even-mode perturbations.

In this paper, we also reviewed the strategy to solve even-mode perturbation on the Schwarzschild background spacetime including \( l = 0, 1 \) modes which was discussed in the Part II paper [15], and then, we showed that it is possible to confirm the realizations of the
LTB solutions and non-rotating C-metric through the even-mode solutions derived in the Part II paper \[15\]. Because the LTB solution is a spherically symmetric solution, its linearized version should be realized \(l = 0\) even-mode perturbations. On the other hand, non-rotating C-metric includes all \(l \geq 0\) even-mode perturbations. This implies that the realization of the C-metric perturbation supports the fact that our derived solutions in the Part II paper \[15\] are reasonable, and then, we may say that Proposal 2.1 is also physically reasonable.

The LTB solutions is a spherically symmetric exact solution which describes the expanding universe with the dust matter or the dust matter collapse to a black hole. It is well-known that the LTB solutions include the Schwarzschild spacetime as a special case. For this reason, we can regard this LTB solution as a black hole solution with the perturbative collapsing dust matter. After reviewing the LTB exact solution, we considered the vacuum black hole solution, i.e., the Schwarzschild spacetime with the perturbative dust matter and examine our \(l = 0\) even-mode solution derived in the Part II paper \[15\] describes this perturbative solution at linear level. From the perturbative treatment, we confirmed the linearized continuity equations of the dust matter in terms of the static Schwarzschild coordinate. We also considered the linear metric perturbation on the Schwarzschild background spacetime of the LTB exact solution. In this realization, the perturbative arbitrary functions \(f(R)\) and \(\tau_0(R)\) in the LTB solutions, which have their physical meanings, included in the term of the Lie derivative of the background metric \(g_{\alpha\beta}\). Therefore, we should regard that such terms of the Lie derivative of the background metric \(g_{\alpha\beta}\) are physical. Thus, we confirmed that the \(l = 0\) even-mode solution derived in the Part II paper \[15\] does describe this perturbative LTB solution.

Next, we considered the linearized C-metric with the Schwarzschild background spacetime, which may have \(l = 1\) even-mode perturbations. Since this linearized C-metric actually includes \(l = 1\) even-mode perturbations, this solution is appropriate for check whether our derived \(l = 1\) even-mode perturbation physically reasonable, or not. After reviewing the non-rotating vacuum C-metric \[48\], in which conical singularities may occur both in the axis \(\theta = 0\) and \(\theta = \pi\), we considered the perturbative form of this solution on the Schwarzschild background spacetime. To consider the perturbative expression of the C-metric on the Schwarzschild background spacetime, we considered the situation where the acceleration parameter \(\alpha_1\) is sufficiently small. Furthermore, we have to consider the deficit/excess angle perturbation \(C_1\). We have to keep in our mind that the fact that we may always change the point-identification between the physical C-metric spacetime and the background Schwarzschild spacetime, i.e., we may change the second-kind gauge at any time as in the LTB case.

Although we follow Proposal 2.1, we compare the result after imposing the regularity \(\delta = 0\). We only consider the even \(m = 0\)-mode perturbations, because the non-rotating C-metric does not have the \(\phi\)-dependence nor the odd-mode components of perturbations. We consider the mode-decomposition of the linearized C-metric with the Schwarzschild background spacetime through the use of Legendre function \(P_1(\cos \theta)\) as the scalar harmonics. Then, we could identify the linearized C-metric with the \(l = 0,1\)- and \(l \geq 2\)-mode perturbations on the Schwarzschild background spacetime.

From the \(l \geq 2\) metric perturbation of the linearized C-metric, we identify the components of the linear perturbations of the energy-momentum tensor for \(l \geq 2\) modes. Then, we have obtained the linear perturbation of the energy-momentum tensor \[1,8,6\] with the constant
\( \lambda \) which given by (1.87). Furthermore, we also checked the consistency of all components of the Einstein equation for \( l \geq 2 \) modes.

For \( l = 0 \) modes of the linearized C-metric, we have the additional mass parameter perturbation of the Schwarzschild spacetime and the deficit/excess angle perturbation. The additional mass parameter perturbation can be always added as the vacuum solution with the term of the Lie derivative of the background metric \( g_{ab} \). In this case, the term of the Lie derivative of the background metric \( g_{ab} \) play important roles. On the other hand, the deficit/excess perturbation is proportional to the metric on \( S^2 \). This perturbation is also expressed as the traceless part of \((t, r)\)-component and the term of the Lie derivative of the background metric \( g_{ab} \). From this expression, we obtain the equation of state of the linear perturbations of the energy momentum tensor and the connection of the deficit/excess angle perturbation and the energy density. Then, we showed the formula of the deficit/excess angle perturbation and the energy density for \( l \geq 2 \) mode perturbations is also valid for \( l = 0 \) mode perturbations.

For \( l = 1 \) modes of the linearized C-metric, the equation of state for the linear perturbations of the energy-momentum tensor and the formula which gives the relation between the energy density and the deficit/excess perturbation in the cases \( l \geq 2 \) modes and \( l = 0 \) modes is also consistent even in the \( l = 1 \)-mode case. Furthermore, we showed that the linearized C-metric perturbation is given from the \( l = 1 \)-mode solution obtained in the Part II paper [15] apart from the Lie derivative of the background metric \( g_{ab} \). Note that the term of the Lie derivative of the background metric \( g_{ab} \) play quite important roles even in \( l = 1 \) mode solution.

As the results of the above \( l \geq 2 \)-mode, \( l = 0 \)-mode, and \( l = 1 \)-mode analyses, we have obtained the expression (4.149) of the linear-perturbation of the energy-momentum tensor with the relation (4.150) between the energy density and the acceleration parameter perturbation and the deficit/excess angle perturbation. Substituting Eq. (4.150) into Eq. (4.149), we obtain the \( \delta \)-function expression at the north and south pole of \( S^2 \) which is given by Eq. (4.160) with the string tension formulae (4.161). Thus, we have confirmed that the linearized C-metric is realized by the linear perturbative solutions obtained in our Part II paper [15].

As the summary of the three papers, Refs. [14, 15] and this paper, we have formulated a gauge-invariant formulation of the linear perturbations on the Schwarzschild background spacetime. Our formulation includes gauge-invariant treatments not only for \( l \geq 2 \) modes but also for \( l = 0, 1 \) mode perturbations. To construct gauge-invariant formulation for \( l = 0, 1 \)-mode perturbations, we introduce the singular harmonic function at once and propose Proposal 2.1 as the strategy to solve the linearized Einstein equations on the Schwarzschild background spacetime, which state that we eliminate the singular behavior of introduced singular harmonics when we solve the linearized Einstein equations. Following Proposal 2.1, we showed the strategy to solve the odd-mode perturbation in the Part I paper [14], that of the even-mode perturbation in the Part II paper [15].

We also derived the \( l = 0, 1 \) mode solution of the linearized Einstein equations for the odd and even-mode perturbations in the Part I [14] and in the Part II [15] papers, respectively. Furthermore, in this paper, we showed that the solutions for \( l = 0, 1 \)-mode perturbations derived in the Part II paper [15] realize two exact solutions. One is the LTB solutions and the other is the non-rotating C-metric. Thus, the results in this paper support our solutions
derived in the Part II paper [15] and our proposal in the Part I paper [14]. In this sense, we may say that our strategy to solve the linearized Einstein equations on the Schwarzschild background spacetime proposed as Proposal 2.1 is physically reasonable. We also note that the gauge-invariant solutions derived in the Part I [14] and the Part II [15] papers includes the terms of the Lie derivative of the background metric $g_{ab}$. In many literature, it is well-known that we have "residual gauge-degree of freedom" if we employ the Regge-Wheeler gauge. The terms of the Lie derivative of the background metric $g_{ab}$ seems to correspond to this "residual gauge." On the other hand, in this series of paper, we distinct the notion of the gauge of the first kind and the notion of the gauge of the second kind. Furthermore, we declare that the purpose of our gauge-invariant perturbation theory is to exclude not the gauge of the first kind but the gauge of the second kind. Moreover, our formulation completely excludes the gauge of the second kind. Therefore, the term of the Lie derivative of the background metric $g_{ab}$ in our derived solution should be regarded as the gauge degree of freedom of the first kind. Even in this Part III paper, these term of Lie derivative played crucial role. Therefore, we may say that to take into account of these term of Lie derivative is also important when we compare the two metric perturbations.

Owing to Proposal 2.1, we could treat $l = 0, 1$-mode perturbations on the Schwarzschild background spacetime in a gauge-invariant manner. This implies that the "zero mode problem" on our general framework of the gauge-invariant perturbation theory was resolved at least in the perturbations on the spherically symmetric background spacetime. This also implies that we can apply our general framework of higher-order gauge-invariant perturbation theory to any-order perturbations on the spherically symmetric background spacetime. This extension to any-order perturbations was briefly discussed in our companion paper [13]. We leave further detailed discussions as future works.

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References
[1] LIGO Scientific Collaboration 2021 home page: https://ligo.org
[2] Virgo 2021 home page: https://virgo.gw.eu
[3] KAGRA 2021 home page: https://gwcenter.icrr.u-tokyo.ac.jp
[4] LIGO INDIA 2021 home page: https://ligo-india.in
[5] Einstein Telescope 2021 home page : https://www.et-gw.eu
[6] Cosmic Explorer 2021 home page : https://cosmicexplorer.org
[7] LISA home page : https://lisa.nasa.gov
[8] S. Kawamura, et al., Prog. Theor. Exp. Phys. 2021 (2021), 05A105.
[9] J. Mei, et al., Prog. Theor. Exp. Phys. 2020 (2020), 05A107.
[10] Z. Luo, et al., Prog. Theor. Exp. Phys. 2020 (2020), 05A108.
[11] L. Barack and A. Pound, Rep. Prog. Phys. 82 (2019) 016904.
[12] K. Nakamura, Class. Quantum Grav. 38, (2021), 145010.
[13] K. Nakamura, Letters in High Energy Physics 2021 (2021), 215.
[14] K. Nakamura, “A gauge-invariant perturbation theory on the Schwarzschild background spacetime Part I : — Formulation and odd-mode perturbations —,” preprint arXiv:2110.13508 [gr-qc].
[15] K. Nakamura, “A gauge-invariant perturbation theory on the Schwarzschild background spacetime Part II : — Even-mode perturbations —,” preprint arXiv:2120.13512 [gr-qc].
[16] T. Regge and J. A. Wheeler, Phys. Rev. 108 (1957), 1063.
[17] F. Zerilli, Phys. Rev. Lett. 24 (1970), 737.
[18] F. Zerilli, Phys. Rev. D 2 (1970), 2141.
[19] H. Nakano, Private note on “Regge-Wheeler-Zerilli formalism” (2019).
[20] V. Moncrief, Ann. Phys. (N.Y.) 88 (1974), 323.
[21] V. Moncrief, Ann. Phys. (N.Y.) 88 (1974), 343.
[22] C. T. Cunningham, R. H. Price, and V. Moncrief, Astrophys. J. 224 (1978), 643.
[23] S. Chandrasekhar, *The mathematical theory of black holes* (Oxford: Clarendon Press, 1983).
[24] U.H. Gerlach and U.K. Sengupta, Phys. Rev. D 19 (1979), 2268.
[25] U.H. Gerlach and U.K. Sengupta, Phys. Rev. D 20 (1979), 3009.
[26] U.H. Gerlach and U.K. Sengupta, J. Math. Phys. 20 (1979), 2540.
[27] U.H. Gerlach and U.K. Sengupta, Phys. Rev. D 22 (1980), 1300.
[28] T. Nakamura, K. Oohara, Y. Kojima, Prog. Theor. Phys. Suppl. No. 90 (1987), 1.
[29] C. Gundlach and J.M. Martín-García, Phys. Rev. D61 (2000), 084024.
[30] J.M. Martín-García and C. Gundlach, Phys. Rev. D64 (2001), 024012.
[31] A. Nagar and L. Rezzolla, Class. Quantum Grav. 22 (2005), R167; Erratum *ibid.* 23 (2006), 4297.
[32] K. Martel and E. Poisson, Phys. Rev. D 71 (2005), 104003.
[33] K. Nakamura, Prog. Theor. Phys. 110, (2003), 723.
[34] K. Nakamura, Prog. Theor. Phys. 113 (2005), 481.
[35] K. Nakamura, Class. Quantum Grav. 28 (2011), 122001.
[36] K. Nakamura, Int. J. Mod. Phys. D 21 (2012), 124004.
[37] K. Nakamura, Prog. Theor. Exp. Phys. 2013 (2013), 043E02.
[38] K. Nakamura, Class. Quantum Grav. 31, (2014), 335013.
[39] K. Nakamura, Phys. Rev. D 74 (2006), 101301(R).
[40] K. Nakamura, Prog. Theor. Phys. 117 (2007), 17.
[41] K. Nakamura, ““Gauge” in General Relativity: – Second-order general relativistic gauge-invariant perturbation theory –”, in *Lie Theory and its Applications in Physics VII* ed. V. K. Dobrev et al, (Heron Press, Sofia, 2008)
[42] K. Nakamura, Phys. Rev. D 80 (2009), 124021.
[43] K. Nakamura, Prog. Theor. Phys. 121 (2009), 1321.
[44] K. Nakamura, Advances in Astronomy, 2010 (2010), 576273.
[45] A. J. Christopherson, K. A. Malik, D. R. -Matravers, K. Nakamura, Class. Quantum Grav. 28 (2011), 225024.
[46] K. Nakamura, “Second-order Gauge-invariant Cosmological Perturbation Theory: Current Status updated in 2019”, Chapter 1 in *“Theory and Applications of Physical Science vol.3,”* (Book Publisher International, 2020). DOI:10.9734/bpi/taps/v3. (Preprint [arXiv:1912.12805].
[47] W. Kinnersley and M. Walker, Phys. Rev. D 2 (1970), 1359.
[48] J. B. Griffiths, P. Krtous, and J. Podolsky, Class. and Quantum Grav. 23 (2006), 6745.
[49] R. K. Sachs, “Gravitational Radiation”, in *Relativity, Groups and Topology* ed. C. DeWitt and B. DeWitt, (New York: Gordon and Breach, 1964).
[50] M. Bruni, S. Matarrese, S. Mollerach and S. Sonego, Class. Quantum Grav. 14 (1997), 2585.
[51] L. Landau and E. Lifshitz, *The Classical Theory of Fields* (Addison-Wesley, Reading, Mass., 1962).
[52] R. Szmytkowski, J. Math. Phys. 47 (2006), 063506.
[53] R. Szmytkowski, J. Phys. A: Math. Theor. 40 (2007), 995-1009; and references therein.
[54] H. Kodama, Prog. Theor. Phys. 120 (2008), 371.