Oscillation for a Class of Fractional Differential Equations with Damping Term in the Sense of the Conformable Fractional Derivative

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Abstract—In this paper, we are concerned with oscillation for a class of fractional differential equations with damping term, where the fractional derivative is defined in the sense of the conformable fractional derivative. By certain inequality and integration average technique, some new oscillatory criteria for the equations are established. We also present one application for the results established.

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Index Terms—Oscillation; Fractional differential equations; Conformable fractional derivative; Damping term

1. Introduction

In the research of qualitative properties for fractional differential equations, research of existence, stability and oscillation has gained much attention by many authors in the last few decades [1-5]. Also some numerical methods have been presented so far [6-10]. In [11-24], oscillation of solutions of various differential equations and systems as well as dynamic equations on time scales were researched, and a lot of new oscillation criteria for these equations have been established therein. In these investigations, we notice that relatively less attention has been paid to the research of oscillation of fractional differential equations.

In [25], Chen researched oscillation of the following fractional differential equation:

\[ [r(t)D^\alpha y(t)]' - q(t)f\left(\int_0^\infty (v-t)^{-\alpha}g(v)dv\right) = 0, \quad t > 0, \]

where \( r, q \) are positive-valued functions, \( \eta \) is the quotient of two odd positive numbers, \( \alpha \in (0,1) \), \( D^\alpha y(t) \) denotes the Liouville right-sided fractional derivative of order \( \alpha \) of \( y \), and \( D^\alpha y(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_t^\infty (\xi-t)^{-\alpha}y(\xi)d\xi \). Then in [26], under similar conditions to [25], some new oscillatory criteria are established for the following fractional differential equation with damping term:

\[ D^{1+\alpha}y(t) - p(t)D^{\alpha}y(t) + q(t)f\left(\int_0^\infty (v-t)^{-\alpha}g(v)dv\right) = 0, \quad t > 0, \]

In [27], Han et al. investigated oscillation of a class of fractional differential equations as follows

\[ [r(t)g((D^\alpha y(t))(t))]' - p(t)f\left(\int_t^\infty (s-t)^{-\alpha}y(s)ds\right) = 0, \quad t > 0, \quad \alpha \in (0,1]. \]

For the research mentioned above, we note that the fractional differential equations concerned are all defined in Liouville right-sided fractional derivative.

In [28-30], the authors researched oscillation of several classes of fractional differential equations as follows

\[ D^\alpha_0 x + f_1(t,x) = v(t) + f_2(t,x), \]

\[ D^\alpha_0 x(t) + q(t)f(x(t)) = 0, \]

\[ (D^{1+\alpha}_0 y)(t) + p(t)(D^\alpha_0 y)(t) + q(t)f(y(t)) = 0, \]

where the fractional derivative is defined by the Riemann-Liouville derivative.

Recently, Khalil et al. proposed a new definition for fractional derivative named conformable fractional derivative [31]. The fractional derivative is defined as follows

\[ D^\alpha f(t) = \lim_{\varepsilon \to 0} \frac{f(t + \varepsilon t^{1-\alpha}) - f(t)}{\varepsilon}, \]

and satisfies the following properties:

(i). \( D^\alpha[a f(t) + b g(t)] = a D^\alpha f(t) + b D^\alpha g(t) \).

(ii). \( D^\alpha(t^\gamma) = \gamma t^{\gamma-\alpha} \).

(iii). \( D^\alpha[f(t)g(t)] = f(t)D^\alpha g(t) + g(t)D^\alpha f(t) \).

(iv). \( D^\alpha C = 0 \), where \( C \) is a constant.

(v). \( D^\alpha f[g(t)] = f'[g(t)]D^\alpha g(t) \).

(vi). \( D^\alpha \left( \frac{f}{g}(t) \right) = \frac{g(t)D^\alpha f(t) - f(t)D^\alpha g(t)}{g^2(t)} \).

(vii). \( D^\alpha f(t) = t^{1-\alpha} f'(t) \).

Note that the properties above can be easily proved due to the definition of the conformable fractional derivative. Afterwards, many authors investigated various applications of the conformable fractional derivative [32-37].

Motivated by the analysis above, in this paper, we are concerned with oscillation of a class of fractional differential equations with damping term as follows:

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In Section 3, we present some applications for them. Some oscillatory. Eq. (1) is called oscillatory if all its solutions are derivative with respect to the variable where \( \lambda XY^{\lambda - 1} - X^\lambda \leq (\lambda - 1)Y^\lambda \) for all \( \lambda > 1 \).

**Theorem 2.** Let \( h_1, h_2, H \in C([\xi_0, \infty), R) \) satisfying \( H(\xi, \xi) = 0, \hat{H}(\xi, s) > 0, \xi > s \geq \xi_0, \) and \( H \) has continuous partial derivatives \( \hat{H}_t(\xi, s) \) and \( \hat{H}_s(\xi, s) \) on \( [\xi_0, \infty) \). Assume that

\[
\int_{\xi_0}^\infty \frac{1}{H(c, s)^{1/2}} ds = \infty,
\]

and for any sufficiently large \( T \geq \xi_0 \), there exist \( \phi \in C^1([t_0, \infty), R_+) \) and \( \varphi \in C^1([t_0, \infty), [0, \infty)) \), and \( a, b, c \) with \( T < a < c < b\) satisfying

\[
\frac{1}{H(c, a)} \int_{a}^{b} \hat{H}(s, a) [A(s)\hat{\phi}(s) + \hat{\phi}(s)] ds
\]

\[
\geq \frac{\theta^{1+\frac{1}{2}}(s)}{[\varphi(s)A(s)]^{\frac{1}{2}}} ds
\]

\[
\int_{\xi_0}^{\infty} \frac{1}{H(c, a)} \int_{a}^{b} \hat{H}(s, a) [A(s)\hat{\phi}(s) + \hat{\phi}(s)] ds
\]

\[
+ \frac{\theta^{1+\frac{1}{2}}(s)}{[\varphi(s)A(s)]^{\frac{1}{2}}} ds
\]

\[
> \frac{1}{H(c, a)} \int_{a}^{b} \hat{H}(s, a) \left\{ \frac{1}{(\gamma + 1)^{\gamma + 1}} \phi(s) + \phi(s) \hat{\varphi}(s) \right\} ds
\]

\[
+ \frac{\theta^{1+\frac{1}{2}}(s)}{[\varphi(s)A(s)]^{\frac{1}{2}}} ds
\]

where \( \hat{\phi}(\xi) = \phi(t), \hat{\varphi}(\xi) = \varphi(t) \). Then every solution of Eq. (1) is eventually positive solution of Eq. (6), and there exists \( \xi > \xi_0 \) such that \( \hat{\varphi}(\xi) > 0 \) on \( [\xi_1, \infty) \). Furthermore, we have

\[
[A(\xi)\hat{\varphi}(\xi)]^{\gamma} + A(\xi)\hat{\varphi}(\xi)\hat{\varphi}(\xi) \geq 0,
\]

and thus \( \hat{\varphi}(\xi) \) is strictly decreasing on \( \xi \in [\xi_1, \infty) \), and \( \hat{\varphi}(\xi) \) is eventually of one sign. We claim \( \hat{\varphi}(\xi) > 0 \) on \( [\xi_2, \infty) \), where \( \xi_2 > \xi_1 \) is sufficiently large. Otherwise, assume there exists a sufficiently large \( \xi_1 > \xi_2 \) such that \( \hat{\varphi}(\xi) \leq 0 \) on \( [\xi_3, \infty) \). Then for \( \xi \in [\xi_3, \infty) \) we have

\[
\hat{\varphi}(\xi) = \hat{\varphi}(\xi_3) = \int_{\xi_3}^{\xi} \hat{\varphi}(s) ds
\]

\[
= \int_{\xi_3}^{\infty} \frac{A(s)\hat{\varphi}(s)}{[A(s)\hat{\varphi}(s)]^{\frac{1}{2}}} ds
\]

\[
\leq [A(\xi_3)\hat{\varphi}(\xi_3)]^{\frac{1}{2}} \int_{\xi_3}^{\infty} \frac{1}{[A(s)\hat{\varphi}(s)]^{\frac{1}{2}}} ds.
\]

By (2) we deduce that \( \lim_{t \to \infty} \hat{\varphi}(\xi) = -\infty \), which contradicts to the fact that \( \hat{\varphi}(\xi) \) is a eventually positive solution of Eq. (6). So it holds that \( \hat{\varphi}(\xi) > 0 \) on \( [\xi_2, \infty) \).

Define the generalized Riccati transformation function:

\[
\omega(t) = \hat{\phi}(t) \frac{A(t)\hat{\varphi}(t)(\hat{\varphi}(t))^{\gamma}}{\hat{\varphi}(t)}.
\]

Let \( \omega(t) = \hat{\varphi}(t) \). Then \( D^\alpha_t \omega(t) = \hat{\varphi}(t) \), and \( D^\alpha_t \hat{\varphi}(t) = \hat{\varphi}(t) \). So for \( \xi \in [\xi_2, \infty) \), we have

\[
\hat{\varphi}(t) = \hat{\varphi}(t) \frac{A(t)\hat{\varphi}(t)(\hat{\varphi}(t))^{\gamma}}{\hat{\varphi}(t)}
\]
by a combination of Lemma 1 and (11) we get that
\[ \int_c^\xi \tilde{H}(\xi, s)\{A(s)\tilde{\phi}(s)\tilde{q}(s) - \tilde{\phi}(s)s\tilde{e}^2(s) + \tilde{\phi}(s)\tilde{q}(s)\tilde{p}(s)\tilde{r}(s)A(s)\} \frac{\gamma}{[\tilde{\phi}(s)\tilde{r}(s)A(s)]^{1\over 2}} ds \]
\[ + \tilde{\phi}(s)s\tilde{e}^2(s) + \tilde{\phi}(s)[\tilde{r}(s)A(s)]^{1\over 2}\tilde{H}_s(\xi, s)\tilde{w}(s) ds. \]
\[ \text{(11)} \]

Setting
\[ \lambda = 1 + \frac{1}{\gamma}, \quad X^\lambda = \frac{\gamma}{[\tilde{\phi}(s)\tilde{r}(s)A(s)]^{1\over 2}} \tilde{w}(s), \]
\[ Y^{\lambda - 1} = \frac{\gamma^{1\over 2}}{[1 + 1\gamma \tilde{H}_s(\xi, s)]^{1\over 2}} \tilde{H}_s(\xi, s), \]

by a combination of Lemma 1 and (11) we get that
\[ \int_c^\xi \tilde{H}(\xi, s)\{A(s)\tilde{\phi}(s)\tilde{q}(s) - \tilde{\phi}(s)s\tilde{e}^2(s) + \tilde{\phi}(s)\tilde{q}(s)\tilde{p}(s)\tilde{r}(s)A(s)\} \frac{\gamma}{[\tilde{\phi}(s)\tilde{r}(s)A(s)]^{1\over 2}} ds \]
\[ + \tilde{\phi}(s)s\tilde{e}^2(s) + \tilde{\phi}(s)[\tilde{r}(s)A(s)]^{1\over 2}\tilde{H}_s(\xi, s)\tilde{w}(s) ds. \]
\[ \text{(12)} \]
Dividing both sides of the inequality (12) by $\tilde{H}(\xi, c)$ and let $\xi \to b^-$, we obtain
\[
\frac{1}{\tilde{H}(b,c)} \int_c^b \tilde{H}(b,s) \{ A(s) \tilde{\phi}(s) \tilde{q}(s) - \tilde{\phi}(s) \tilde{q}'(s) + \frac{\tilde{\gamma}^{+} (s) \tilde{\phi}(s)}{[\tilde{r}(s) A(s)]^{\frac{1}{2}}} \} ds
\leq \tilde{w}(c) + \frac{1}{\tilde{H}(b,c)} \int_c^b \tilde{H}(b,s) \frac{1}{(\gamma + 1)^{\gamma + 1} \tilde{\phi}'(s) \tilde{r}(s) A(s)} \left\{ \left\{ \frac{1}{(\gamma + 1)^{\gamma + 1} \tilde{\phi}'(s) \tilde{r}(s) A(s)} \right\} \tilde{H}_e'(b,s) \right\} ds
\]
which contradicts to (3). So the proof is complete.

**Theorem 3.** If (2) holds, and for any sufficiently large $l \geq \xi_0$,
\[
\lim_{\xi \to \infty} \sup \int_l^c \tilde{H}(\xi,s) \{ A(s) \tilde{\phi}(s) \tilde{q}(s) - \tilde{\phi}(s) \tilde{q}'(s) + \frac{\tilde{\gamma}^{+} (s) \tilde{\phi}(s)}{[\tilde{r}(s) A(s)]^{\frac{1}{2}}} \} ds
\leq \frac{1}{\tilde{H}(c,a)} \int_a^c \tilde{H}(c,s) \frac{1}{(\gamma + 1)^{\gamma + 1} \tilde{\phi}'(s) \tilde{r}(s) A(s)} \left\{ \left\{ \frac{1}{(\gamma + 1)^{\gamma + 1} \tilde{\phi}'(s) \tilde{r}(s) A(s)} \right\} \tilde{H}_e'(c,s) \right\} ds
\]
then Eq. (1) is oscillatory.

**Proof:** For any sufficiently large $T \geq \xi_0$, let $a = T$.
In (17) we choose $l = a$. Then there exists $c > a$ such that
\[
\int_a^c \tilde{H}(c,s) \frac{1}{(\gamma + 1)^{\gamma + 1} \tilde{\phi}'(s) \tilde{r}(s) A(s)} \left\{ \left\{ \frac{1}{(\gamma + 1)^{\gamma + 1} \tilde{\phi}'(s) \tilde{r}(s) A(s)} \right\} \tilde{H}_e'(c,s) \right\} ds > 0.
\]
In (18) we choose $l = c > a$. Then there exists $b > c$ such that
\[
\int_c^b \tilde{H}(b,s) \frac{1}{(\gamma + 1)^{\gamma + 1} \tilde{\phi}'(s) \tilde{r}(s) A(s)} \left\{ \left\{ \frac{1}{(\gamma + 1)^{\gamma + 1} \tilde{\phi}'(s) \tilde{r}(s) A(s)} \right\} \tilde{H}_e'(b,s) \right\} ds > 0.
\]
Combining (19) and (20) we obtain (3). So according to Theorem 2, Eq. (1) is oscillatory.

In Theorems 2-3, if we choose \( \tilde{H}(\xi, s) = (\xi - s)^\lambda \), \( \xi \geq s \geq \xi_0 \), where \( \lambda > 1 \) is a constant, then we obtain the following two corollaries.

**Corollary 4.** Under the conditions of Theorem 2, if for any sufficiently large \( T \geq \xi_0 \), there exist \( a, b, c \) with \( T \leq a < c < b \) satisfying

\[
\frac{1}{(c-a)^\lambda} \int_a^c (s-a)^{-\lambda} \{ A(s)\tilde{\varphi}(s)\tilde{q}(s) - \tilde{\varphi}(s)\tilde{\varphi}'(s)
+ \tilde{\varphi}^{1+\frac{1}{\lambda}}(s)\tilde{q}(s) \} ds
+ \frac{1}{(b-c)^\lambda} \int_c^b (b-s)^{-\lambda} \{ A(s)\tilde{\varphi}(s)\tilde{q}(s) - \tilde{\varphi}(s)\tilde{\varphi}'(s)
+ \tilde{\varphi}^{1+\frac{1}{\lambda}}(s)\tilde{q}(s) \} ds
> \frac{1}{(c-a)^\lambda} \int_a^c (s-a)^{-\lambda} \{ (\gamma + 1)\tilde{\varphi}(s)\tilde{\varphi}'(s) + \tilde{\varphi}'(s)\tilde{\varphi}'(s) \} ds
+ \frac{1}{(b-c)^\lambda} \int_c^b (b-s)^{-\lambda} \{ (\gamma + 1)\tilde{\varphi}(s)\tilde{\varphi}'(s) + \tilde{\varphi}'(s)\tilde{\varphi}'(s) \} ds
\]

then Eq. (1) is oscillatory.

**Corollary 5.** Under the conditions of Theorem 3, if for any sufficiently large \( l \geq \xi_0 \),

\[
l \limsup_{\xi \to \infty} \int_l^\xi \frac{(s-l)^{-\lambda} \{ A(s)\tilde{\varphi}(s)\tilde{q}(s) - \tilde{\varphi}(s)\tilde{\varphi}'(s)
+ \tilde{\varphi}^{1+\frac{1}{\lambda}}(s)\tilde{q}(s) \} ds
+ \frac{1}{(c-a)^\lambda} \int_a^c (s-a)^{-\lambda} \{ (\gamma + 1)\tilde{\varphi}(s)\tilde{\varphi}'(s) + \tilde{\varphi}'(s)\tilde{\varphi}'(s) \} ds
\]

then Eq. (1) is oscillatory.

**Theorem 6.** Under the conditions of Theorem 2, furthermore, suppose (2) does not hold. If for any \( T \geq \xi_0 \), there exist \( a, b \) with \( b > a \geq T \) such that for any \( u \in C[a, b], u'(t) \in L^2[a, b], u(a) = u(b) = 0 \), the following inequality holds:

\[
\int_a^b u^2(s) \{ A(s)\tilde{\varphi}(s)\tilde{q}(s) - \tilde{\varphi}(s)\tilde{\varphi}'(s)
+ \tilde{\varphi}^{1+\frac{1}{\lambda}}(s)\tilde{q}(s) \} ds
- \frac{1}{(b-a)^\lambda} \int_a^b \{ (\gamma + 1)\tilde{\varphi}(s)\tilde{\varphi}'(s) + \tilde{\varphi}'(s)\tilde{\varphi}'(s) \} ds
\]

where \( \tilde{\varphi}, \tilde{\varphi}' \) are defined as in Theorem 2, then Eq. (1) is oscillatory.

**Proof:** Assume (2) has a non-oscillatory solution \( x \) on \([a, b]\), where \( \xi_1 \) is sufficiently large. Let \( x(t) = x(\xi), \xi \) and \( \tilde{\varphi} \tilde{\varphi}'(\xi) \) be defined as in Theorem 2. Then similar to the proof of Theorem 2, it holds that \( \tilde{\varphi}'(\xi) \geq 0 \), where \( \xi_2 \) is sufficiently large, and we can obtain (10). Select \( a, b \) arbitrarily in \([\xi, \xi] \), with \( b > a \) such that \( u(a) = u(b) = 0 \). Substituting \( x \) with \( x \), multiplying both sides of (10) by \( u^2(s) \), integrating it with respect to \( s \) from \( a \) to \( b \), we get that

\[
\int_a^b u^2(s) \{ A(s)\tilde{\varphi}(s)\tilde{q}(s) - \tilde{\varphi}(s)\tilde{\varphi}'(s)
+ \tilde{\varphi}^{1+\frac{1}{\lambda}}(s)\tilde{q}(s) \} ds
\]

and

\[
+ \frac{1}{(c-a)^\lambda} \int_a^c (s-a)^{-\lambda} \{ (\gamma + 1)\tilde{\varphi}(s)\tilde{\varphi}'(s) + \tilde{\varphi}'(s)\tilde{\varphi}'(s) \} ds
\]

then Eq. (1) is oscillatory.

\[
\limsup_{\xi \to \infty} \int_l^\xi \frac{(s-l)^{-\lambda} \{ A(s)\tilde{\varphi}(s)\tilde{q}(s) - \tilde{\varphi}(s)\tilde{\varphi}'(s)
+ \tilde{\varphi}^{1+\frac{1}{\lambda}}(s)\tilde{q}(s) \} ds
+ \frac{1}{(c-a)^\lambda} \int_a^c (s-a)^{-\lambda} \{ (\gamma + 1)\tilde{\varphi}(s)\tilde{\varphi}'(s) + \tilde{\varphi}'(s)\tilde{\varphi}'(s) \} ds
\]

Setting

\[
\lambda = 1 + \frac{1}{\gamma}, \quad \lambda \geq 1, \quad X^\lambda = \frac{\gamma}{(\phi(s)\tilde{\varphi}(s)\tilde{A}(s))^{1+\frac{1}{\lambda}}(s)}\tilde{\varphi}'(s), \quad Y^{\lambda-1} = \frac{\gamma^{1+\frac{1}{\lambda}}}{(\phi(s)\tilde{\varphi}(s)\tilde{A}(s))^{1+\frac{1}{\lambda}}}
\]

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by a combination of Lemma 1 and (25) we get that

\[ \int_1^t u^2(s)\left[ (\gamma + 1)^{\frac{1}{\gamma + 1}}(s) \frac{\tilde{\phi}(s)}{|\tilde{r}(s)|A(s)} \right] \frac{ds}{\alpha} \leq \int_c^t \frac{1}{(\gamma + 1)^{\frac{1}{\gamma + 1}}(s) |\tilde{r}(s)|A(s)} \frac{ds}{\alpha} \]

\[ + \left( \int_c^t \frac{1}{(\gamma + 1)^{\frac{1}{\gamma + 1}}(s) \tilde{r}(s) A(s)} \frac{ds}{\alpha} \right)^{\frac{1}{\gamma + 1}}. \]

We have investigated oscillation for a class of fractional differential equations with damping term, where the fractional derivative is defined in the sense of the conformable fractional derivative. Some new oscillatory criteria were presented. The validation of the main results have been verified by one application.

**IV. CONCLUSIONS**

We have investigated oscillation for a class of fractional differential equations with damping term, where the fractional derivative is defined in the sense of the conformable fractional derivative. Some new oscillatory criteria were presented. The validation of the main results have been verified by one application.

**REFERENCES**

[1] G. Feng and Y. Yang, “Existence of Solutions for the Four-point Fractional Boundary Value Problems Involving the p-Laplacian Operator,” *IAENG International Journal of Applied Mathematics*, vol. 49, no. 2, pp.234-238, 2019.

[2] L. Xu and T. Zhang, “Multiple Almost Periodic Solutions and Local Asymptotical Stability in a Harvesting System of Facultative Mutualism with Time Delays,” *IAENG International Journal of Applied Mathematics*, vol. 49, no. 4, pp.434-440, 2019.

[3] A. M. A. Abou-El-Ela, A. I. Sadek and A. M. Mahmoud, “Existence and Uniqueness of a Periodic Solution for Third-order Delay Differential Equation with Two Deviating Arguments,” *IAENG International Journal of Applied Mathematics*, vol. 42, no. 1, pp.7-12, 2012.

[4] B. Zheng and Q. Feng, “New Oscillatory Criteria for a Class of Fractional Differential Equations,” *Engineering Letters*, vol. 28, no. 3, pp.970-977, 2020.

[5] Q. Feng, “Oscillation for a class of conformable fractional dynamic equations on time scales,” *Engineering Letters*, vol. 28, no. 2, pp.363-366, 2020.

[6] L. Wang, H. Li, and Y. Meng, “Numerical Solution of Coupled Burgers’ Equation Using Finite Difference and Sin Collocation Method,” *Engineering Letters*, vol. 29, no. 2, pp.668-674, 2021.

[7] Y. Yang, “Explicit Error Estimate for the Nonconforming Crouzeix-Raviart Finite Element,” *IAENG International Journal of Applied Mathematics*, vol. 50, no. 1, pp.107-113, 2020.

[8] F. Salah and M. H. Elhafian, “Numerical Solution for Heat Transfer of Non-Newtonian Second-Grade Fluid Flow over Stretching Sheet via Successive Linearization Method,” *IAENG International Journal of Applied Mathematics*, vol. 49, no. 4, pp.505-512, 2019.

[9] X. Zhang, P. Zhang and Y. Ding, “A Reduced High-order Compact Finite Difference Scheme Based on Proper Orthogonal Decomposition for the Generalized Kuramoto-Sivashinsky Equation,” *IAENG International Journal of Applied Mathematics*, vol. 49, no. 2, pp.165-174, 2019.

[10] G. Guo and Y. Zhai, “Unconditionally Stable High Accuracy Alternating Difference Parallel Method for the Fourth-order Heat Equation,” *Engineering Letters*, vol. 28, no. 1, pp.56-62, 2020.

[11] L. Li, F. Meng and Z. Zheng, “Some new oscillation results for linear Hamiltonian systems,” *Appl. Math. Comput.*, vol. 208, pp.219-224, 2009.

[12] Q. Feng and F. Meng, “Oscillation of solutions to nonlinear forced fractional differential equations,” *Electron. J. Differ. Equ.*, vol. 2013, no. 169, pp. 1-10, 2013.

[13] Y. Huang and F. Meng, “Oscillation criteria for forced second-order nonlinear differential equations with damping,” *J. Comput. Appl. Math.*, vol. 224, pp.339-345, 2009.

[14] F. Meng and Y. Huang, “Interval oscillation criteria for a forced second-order nonlinear differential equations with damping,” *Appl. Math. Comput.*, vol. 218, pp.1857-1861, 2011.

[15] H. Liu, F. Meng and P. Liu, “Oscillation and asymptotic analysis on a new generalized Emden-Fowler equation,” *Appl. Math. Comput.*, vol. 219, no. 5, pp. 2739-2748, 2012.

[16] Z. Zheng and F. Meng, “On Oscillation Properties for Linear Hamiltonian Systems,” *Rocky Mount. J. Math.*, vol. 39, no. 1, pp. 343-358, 2009.

[17] Z. Zheng, X. Wang and H. Han, “Oscillation Criteria for Forced Second Order Differential Equations with Mixed Nonlinearities,” *Appl. Math. Lett.*, vol. 22, pp. 1096-1101, 2009.

[18] Z. Zheng, “Oscillation Criteria for Matrix Hamiltonian Systems via Summability Method,” *Rocky Mount. J. Math.*, vol. 39, no. 5, pp. 1751-1766, 2009.

[19] J. Shao, Z. Zheng and F. Meng, “Oscillation criteria for fractional differential equations with mixed nonlinearities,” *Adv. Differ. Equ.*, vol. 2013:323, pp. 1-9, 2013.

[20] H. Liu and F. Meng, “Interval oscillation criteria for second-order nonlinear forced differential equations involving variable exponent,” *Adv. Differ. Equ.*, vol. 2016:291, pp. 1-14, 2016.

[21] L. Liu and Y. Bai, “New oscillation criteria for second-order nonlinear neutral delay differential equations,” *J. Comput. Appl. Math.*, vol. 231, pp. 657-663, 2009.
[22] H. Liu and F. Meng, “Oscillation criteria for second order linear matrix differential systems with damping,” *J. Comput. Appl. Math.*, vol. 229, no. 1, pp. 222-229, 2009.

[23] S. S. Negi, S. Abbas, M. Malik and Y. Xia, “New oscillation criteria of special type second-order non-linear dynamic equations on time scales,” *Adv. Diff. Equ.*, vol. 2018:241, pp. 1-15, 2018.

[24] M. Zhang, W. Chen, M. El-Sheikh, R. Sallam, A. Hassan and T. Li, “Oscillation criteria for second-order nonlinear delay dynamic equations of neutral type,” *Adv. Diff. Equ.*, vol. 2018:26, pp. 1-9, 2018.

[25] D. X. Chen, “Oscillation criteria of fractional differential equations,” *Adv. Differ. Equ.*, vol. 2012:33, pp. 1-18, 2012.

[26] D. X. Chen, “Oscillatory behavior of a class of fractional differential equations with damping,” *U.P.B. Sci. Bull. Series A*, vol. 75, no. 1, pp. 107-118, 2013.

[27] Z. Han, Y. Zhao, Y. Sun, and C. Zhang, “Oscillation for a class of fractional differential equation,” *Discrete Dyn. Nat. Soc.*, vol. 2013:390282, pp. 1-6, 2013.

[28] S. R. Grace, R. P. Agarwal, J. Y. Wong, and A. Zafer, “On the oscillation of fractional differential equations,” *Frac. Calc. Appl. Anal.*, vol. 15, pp. 222-231, 2012.

[29] Y. Wang, Z. Han, P. Zhao and S. Sun, “On the oscillation and asymptotic behavior for a kind of fractional differential equations,” *Adv. Differ. Equ.*, vol. 2014:50, pp. 1-11, 2014.

[30] J. Yang, A. Liu and T. Liu, “Forced oscillation of nonlinear fractional differential equations with damping term,” *Adv. Differ. Equ.*, vol. 2015:1, pp. 1-7, 2015.

[31] R. Khalil, M. Al-Horani, A. Yousef and M. Sababheh, “A new definition of fractional derivative,” *J. Comput. Appl. Math.*, vol. 264, pp. 65-70, 2014.

[32] W. S. Chung, “Fractional Newton mechanics with conformable fractional derivative,” *J. Comput. Appl. Math.*, vol. 290, pp. 150-158, 2015.

[33] K. Hosseini, A. Bekir and R. Ansari, “New exact solutions of the conformable time-fractional Cahn-Allen and Cahn-Hilliard equations using the modified Kudryashov method,” *Optik*, vol. 132, pp. 203-209, 2017.

[34] Y. Çenesiz and A. Kurt, “The solutions of time and space conformable fractional heat equations with conformable Fourier transform,” *Acta Univ. Sapientiae, Mathematica*, vol. 7, no. 2, pp. 130-140, 2015.

[35] O. Tasbozan, Y. Çenesiz, A. Kurt and D. Baleanu, “New analytical solutions for conformable fractional PDEs arising in mathematical physics by exp-function method,” *Open Phys.*, vol. 15, pp. 647-651, 2017.

[36] H.W. Zhou, S. Yang and S.Q. Zhang, “Conformable derivative approach to anomalous diffusion,” *Physica A*, vol. 491, pp. 1001-1013, 2018.

[37] S. Yang, L. Wang and S. Zhang, “Conformable derivative: Application to non-Darcian flow in low-permeability porous media,” *Appl. Math. Lett.*, vol. 79, pp. 105-110, 2018.

[38] G. H. Hardy, J. E. Littlewood and G. Pólya, *Inequalities*, “Second edition,” *Cambridge Univ. Press*, Cambridge, UK, 1988.

[39] T. S. Hassan, “Oscillation of third order nonlinear delay dynamic equations on time scales,” *Math. Comput. Modelling*, vol. 49, pp. 1573-1586, 2009.