Open Problems in Quantum Information Theory
Institut für Mathematische Physik
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Problems 1–29

Some Open Problems in Quantum Information Theory

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For information about the QI open problems project at IMaPh refer to the web-pages
http://www.imaph.tu-bs.de/qi/problems/
Please support us by suggesting further interesting problems!

For questions, additional problems or other contributions please contact R.F. Werner at
http://www.imaph.tu-bs.de/qi
Abstract
This article is a snap-shot of a web site, which has been collecting open problems in quantum information for several years, and documenting the progress made on these problems. By posting it we make the complete collection available in one printout. We also hope to draw more attention to this project, inviting every researcher in the field to raise to the challenges, but also to suggest new problems.

All updates will appear on

http://www.imaph.tu-bs.de/qi/problems/
## Contents

| No | Title                                                | Contact           | Date        | Last progress | Solved by                  | Page |
|----|------------------------------------------------------|-------------------|-------------|---------------|---------------------------|------|
|    | Introduction                                         |                   |             |               |                           |      |
| 1  | All the Bell Inequalities                           | R. F. Werner      | 25 Oct 1999 | 22 Oct 2002   |                           | 9    |
| 2  | Undistillability implies ppt?                        | D. Bruß           | 02 Mar 2000 | 25 Oct 2002   |                           | 15   |
| 3  | Polynomial entanglement invariants                  | R. F. Werner      | 13 Oct 2000 | 18 Dec 2001   | A. Sudbery                | 18   |
| 4  | Catalytic majorization                              | M. B. Plenio      | 18 Dec 2000 |               |                           | 23   |
| 5  | Maximally entangled mixed states                    | K. Audenaert      | 08 Nov 2001 |               |                           | 25   |
| 6  | Nice error bases                                    | D. Schlingemann   | 08 Nov 2001 | 28 May 2003   | A. Klappenecker, M. Roetteler | 26   |
| 7  | Additivity of Entanglement of Formation             | K. G. H. Vollbrecht| 16 Nov 2001 | 11 Nov 2004   | (equivalent to problem 10)| 28   |
| 8  | Qubit formula for Relative Entropy of Entanglement  | J. Eisert         | 20 Jun 2003 |               |                           | 30   |
| 9  | Reduction criterion implies majorization?           | M. M. Wolf        | 12 Feb 2002 | 20 May 2003   | T. Hiroshima              | 32   |
| 10 | Additivity of classical capacity and related problems| A. S. Holevo      | 31 Jan 2003 | 11 Nov 2004   |                           | 34   |
| 11 | Continuity of the quantum channel capacity          | M. Keyl           | 20 Jun 2003 |               |                           | 41   |
| No | Title                                                                 | Contact          | Date       | Last progress | Solved by | Page |
|----|-----------------------------------------------------------------------|------------------|------------|---------------|-----------|------|
| 12 | Bell Inequalities for long range vacuum correlations                  | R. Verch         | 22 Jan 2002| –             | –         | 42   |
| 13 | Mutually unbiased bases                                               | B.-G. Englert    | 31 Jan 2003| 07 Jan 2004   | –         | 43   |
| 14 | Tough error models                                                    | E. Knill         | 31 Jan 2003| –             | –         | 43   |
| 15 | Separability from spectrum                                            | E. Knill         | 31 Jan 2003| 13 Aug 2003   | –         | 45   |
| 16 | Complexity of product preparations                                    | E. Knill         | 31 Jan 2003| –             | –         | 60   |
| 17 | Reversibility of entanglement assisted coding                         | P. Shor          | 31 Jan 2003| –             | –         | 61   |
| 18 | Qubit bi-negativity                                                   | K. G. H. Vollbrecht| 10 Feb 2003| –             | –         | 59   |
| 19 | Stronger Bell Inequalities for Werner states?                         | N. Gisin         | 20 Jun 2003| –             | –         | 59   |
| 20 | Reversible entanglement manipulation                                  | M. B. Plenio     | 08 Feb 2005| –             | –         | 65   |
| 21 | Bell violation by tensoring                                            | Y. C. Liang      | 08 Feb 2005| –             | –         | 58   |
| 22 | Asymptotic cloning is state estimation?                               | M. Keyl          | 10 Feb 2005| –             | –         | 65   |
| 23 | SIC POVMs and Zauner’s Conjecture                                      | D. Gross         | 17 Feb 2005| –             | –         | 60   |
| 24 | Secret key from all entangled states                                   | P. Horodecki     | 15 Mar 2005| –             | –         | 69   |
| 25 | Lockable entanglement measures                                        | P. Horodecki     | 15 Mar 2005| –             | –         | 65   |
| 26 | Bell inequalities holding for all quantum states                       | R. Gill          | 11 Apr 2005| –             | –         | 65   |
| No | Title                                           | Contact     | Date       | Last progress | Solved by | Page |
|----|-------------------------------------------------|-------------|------------|---------------|-----------|------|
| 27 | The power of CGLMP inequalities                 | R. Gill     | 15 Apr 2005 | –             | –         | 68   |
| 28 | Local equivalence of graph states               | D. Schlingemann | 21 Apr 2005 | –             | –         | 70   |
| 29 | Entanglement of formation for Gaussian states   | O. Krüger   | 21 Apr 2005 | –             | –         | 72   |
Introduction

Open problems are among the most important resources of a researcher. Often enough the key to a scientific discovery is to ask the right question. But, of course, most of the time things are not that easy: many problems do resist a serious effort, and time and again we all come to the point when a problem has essentially won the fight, and we would be just as happy if somebody else comes and finally settles it. This is precisely the sort of problems we want to collect on our problem pages. Of course, nobody would or should post a problem, for which he or she has a concrete, promising but untried approach in mind.

The difficulty of the problems in the collection range very widely, from problems that are, in fact, settled, and turned out to be easy, to major challenges, which the best people in the field have struggled with without complete success. A good example of the first kind is No. 3, which was solved by A. Sudbery, essentially by pointing out that there is a a Theorem in the literature which precisely does the job. I count this solution as a successful example for our page: a call for help that was answered by someone with a different kind of expertise from the proposer.

On the other hand, there are the big challenges, like the additivity problems 7 and 10, shown to be equivalent by Peter Shor. Probably every quantum information theorist worth his salt has had a go on that one. Such challenges are landmarks in any field. If you can make serious progress on one of them, you know you have really moved.

What makes a list of open problems so intriguing is that you never really know which class any one problem belongs to. Until it is too late.

Our problem page was created in 1999 and has been growing slowly over the years. However, it is still not very widely known in the community. We have therefore obtained the consent of the quant-ph moderators to post it in its current form. We might post an update after a couple of years. But you should always look to the site http://www.imaph.tu-bs.de/qi/problems for the up-to-date version.
Introduction

Procedures and Policies

- Anybody is invited to contribute problems. They should be stated concisely, and in a self-contained way, using only the current accepted terminology of the field.

- We make an effort to publish problems quickly, but reserve the right to reject problems we find less suitable.

- We occasionally also post problems that come up in the literature and satisfy our criteria.

- The best format for submissions is (simple) \( \text{LATEX} \) source code, with section names taken from a typical published problem (e.g. Problem, Background, Partial Results, Literature). The source for the whole collection is actually maintained in \( \text{LATEX} \).

- Every problem is assigned a contact person. This is not necessarily the proposer, or the person who formulated the problem. However, these colleagues have agreed to keep an eye on the problem, are requested to report major partial solutions, and will be asked to verify any proposed full solution.

- If you want to add a partial solution, or some other relevant remark, it is best to send an email both to the contact person and to us. If possible, please use \( \text{LATEX} \) for this purpose, too.

- Full and partial solutions are typically documented via citations. If there is no separate paper about the solution, we may also post it directly on these pages.

- No problem is ever deleted from the list. This is to ensure that the entries can be cited in a reliable way. It also helps to give due credit to the person who actually solved the problem.
Problem 1

All the Bell Inequalities

Remarks

The title was taken from a recent exposition by A. Peres [Pe].

Problem

Find all those linear inequalities characterizing the existence of joint probability distributions for all variables in a correlation experiment.

More specifically, suppose that measurements are made on systems, which are decomposed into $N$ subsystems. On each of these subsystems one out of $M$ observables is measured, producing $K$ outcomes each. Thus we consider $M^N$ different experimental setups, each of which may lead to $K^N$ different outcomes, so all in all $(MK)^N$ probabilities are measured. Classically (in a “realistic local theory”) these numbers would be generated by specifying probabilities for each “classical configuration”, i.e. every assignment of one of the $K$ values to each of the $NM$ observables. Thus the task is to characterize a convex polyhedron in $(MK)^N$ dimensions (minus a few for normalization constraints), which is generated by $K^{(NM)}$ explicitly known extreme points, in terms of linear inequalities.

For $(N,M,K) = (2,2,2)$ this is solved by the CHSH inequalities. A general solution for all $N,M,K$ is highly unlikely to exist. Therefore we pose the following more manageable tasks:

- Find complete solutions for other small values of $(N,M,K)$.
- Find efficient ways of generating new inequalities, i.e., inequalities which cannot be written as convex combinations of lower order ones.
- Find infinite families of new inequalities. These could be complete families of inequalities with certain additional symmetries.
- Restrict to “full correlation functions”, i.e., disregard constraints on marginal distributions.
Problem 1  

• Do the same for the special case of correlation inequalities. These belong to the case $K = 2$, and are unchanged, when, for an even number of subsystems, all measurement outputs are interchanged. Such inequalities are best written in terms of the expectations of $A_1 A_2 \ldots A_N$, where each $A_i$ takes values $+1, -1$, resp. $-1 \leq A_i \leq 1$.

• Decide by what margin these can be violated by quantum states, or by quantum states with special properties (e.g., fixed Hilbert space dimension, invariance under symmetry transformations or positive partial transposes).

Background

This is a special instance of a standard problem in convex geometry: compute the (maximal) faces of a polyhedron given in terms of its extreme points. That is: given $R$ vectors $e_k$ in a finite dimensional real vector space, find the extreme points of the convex set of vectors $f$ such that $f \cdot e_k \leq 1$ for all $k$. By the Bipolar Theorem [Sc] (or “Farkas’ Lemma”, a special case for polyhedral cones), $x$ then lies in the convex hull of the $e_k$ and the origin, if and only if $f \cdot x \leq 1$ for all extremal $f$. It is easy to decide when such a vector $f$ is extremal: in that case $f$ must be uniquely determined by the equations $f \cdot e_k = 1$ it satisfies.

To find some extreme point is not so difficult: there is a standard algorithm for maximizing an affine functional on a convex set given in this way known as the Simplex Algorithm, which runs into an extreme point. It is an entirely different matter, however, to ask for all extreme points. A straightforward method would be to list all subsets of $\{1, \ldots, R\}$ with ($\#$ elements) = ($\#$ dimensions), and to check for each whether the corresponding set of equations determines an inequality vector $f$. It is immediately clear that such a brute force approach to the above problem will end in an exponential-of-exponential explosion of computing time, and is bound to fail. There are more intelligent algorithms (e.g. the packages available on netlib C++, or in Mathematica), but they, too, all run into serious growth problems for very small $(N, M, K)$. In fact, there is a theorem by Pitovski to the effect that in a closely related problem finding the inequalities would also solve some known hard problems in computational complexity (e.g. to the notorious NP = P, resp. NP = coNP questions [Pi]).

So a solution of the problem as posed here necessarily makes use of the structure of these particular convex sets.

Partial Solutions

Constraints on the possible range of values of correlations in the form of inequalities have been investigated for many years (see the monograph by Frechet [Fred]), even before physicists developed an interest in that subject due to the work of Bell [Be].

The convex geometry aspect of the above problem was seen clearly by many authors
Problem 1  All the Bell Inequalities

in the last two decades (e.g. [Fr], [Ci], [GM], [Pi], [Pe]). Undoubtedly some of these have conducted numerical searches for new Bell inequalities. However, there is only little knowledge about inequalities beyond the case \((N,M,K) = (2,2,2)\). Posing this problem is intended as a focal point for putting together the compilations, and the existing general observations, so that the state of the art becomes accessible to a wider community.

- The first to consider all the possible correlation functions as a convex set surrounded by the faces of a polyhedron apparently was M. Froissart [Fr]. He identified these faces with extremal generalizations of Bell’s inequalities and gave some examples up to the case where \((N,M,K) = (2,3,2)\).

- The case \((2,2,2)\) was analyzed completely by Fine [Fi]. There are only two types of inequalities: one type just expresses positivity of measured probabilities, the second is the CHSH-inequality.

- Tsirelson took up Froissart’s idea and concentrated on the quantum analogue of Bell’s inequalities. He pointed out that quantum theory leads to a convex body which is in general not a polytope and thus cannot be described by a finite number of inequalities. His most complete results were on bipartite correlation inequalities \((N = K = 2)\), where the extremal quantum correlations are attained by states on Clifford algebras. The precise structure of the extremal quantum correlations remained unclear, though. For example, it is not known whether it admits a description by a finite number of analytic, or even polynomial, inequalities [Ci].

- In the work of work of Garg and Mermin [GM] the case \(K > 2\) was considered, in order to study higher spin analogues of the standard spin-1/2 situation, and maybe find the signs of a classical limit. From the point of view of the problem stated here, the symmetry assumptions of Garg and Mermin are rather strong, so that the inequalities obtained describe only a low dimensional section of the convex body under investigation.

- Building on [GM], Peres recently claimed “a graphical method giving a large number of Bell inequalities of the Clauser-Horne type [Pe]”. Unfortunately, in that paper he merely applies it to show how to find inequalities for small \((N,M,K)\) again in larger systems, i.e., he does not give any new inequalities in the above technical sense. Peres agrees with Pitovsky that an algorithm for algebraic construction of these Farkas vectors runs into serious computational problems unless one does not use special symmetry properties of these particular convex sets in order to obtain a more efficient algorithm.

- Pitowsky and Svozil [PS] recently numerically derived a complete set of inequalities for \((N,M,K) = (3,2,2)\) and \((2,3,2)\) taking into account constraints on the marginal distributions. Their results (the coefficients of 53856 inequalities) can be found on their website \([3,2,2]\) and \([2,3,2]\).
• The complete set of correlation inequalities for all $N$ with $M = K = 2$ was recently computed by Werner and Wolf [WW]. This is somewhat surprising, since the worst growth of the problem is expected in the parameter $N$. There are $2^{(2^N)}$ inequalities on the $2^N$-dimensional set of correlations corresponding to the maximal faces of a hyper-octahedron, which can thus be characterized by a single albeit non-linear inequality. Any of these inequalities is maximally violated for the generalized GHZ state. Moreover, one can show that these inequalities are satisfied if all the partial transposes of the state are positive semi-definite operators. For the construction and algebraic manipulation of these inequalities a Mathematica 4.0 notebook is provided.

• For $N = 2, M = 4$, we get the following extremal correlation inequalities (E stands for expectation, $A$ for observables of the first and $B$ for observables of the second subsystem):

$$E(A_1(2B_1 + B_2 - B_3) + A_4(B_2 + B_3) + A_3(-B_1 + B_2 - B_3 + B_4)$$
$$+ A_2(B_1 - B_2 + B_3 + B_4)) \leq 6,$$

$$E(A_2(B_1 + 2B_2 + B_3 - 2B_4) + A_4(2B_1 - 2B_2 + B_3 - B_4)$$
$$+ A_3(2B_1 + B_2 - 2B_3 + B_4)$$
$$+ A_1(B_1 + B_2 + 2B_3 + 2B_4)) \leq 10.$$
Problem 1 All the Bell Inequalities

Literature

[Ac] A. Acin, Distillability, Bell inequalities and multiparticle bound entangle-
ment, Phys. Rev. Lett. 88, 027901 (2002) and quant-ph/0108029 (2001).

[ASWa] A. Acin, V. Scarani, M. M. Wolf, Violation of Bell’s inequalities implies

distillability for N qubits, quant-ph/0112102 (2001).

[ASWb] A. Acin, V. Scarani, M. M. Wolf, Bell inequalities and distillability in N-
quantum-bit systems, quant-ph/0206083 (2002).

[Be] J. S. Bell, On the Einstein Podolsky Rosen Paradox, Physics 1 (1964).

[BT] D. Bacon, B. F. Toner, Bell inequalities with communication,
quant-ph/0208057 (2002).

[CGLMP] D. Collins, N. Gisin, N. Linden, S. Massar, S. Popescu, Bell Inequalities for
Arbitrarily High-Dimensional Systems, Phys. Rev. Lett 88, 040404 (2002)
and quant-ph/0106024 (2001).

[Fr] M. Froissart, Constructive generalization of Bell’s inequalities, Nuovo Cimento B 64, 241 (1981).

[CB] M. Zukowski, C. Brukner, Bell’s theorem for general N-qubit states, Phys.
Rev. Lett. 88, 210401 (2002) and quant-ph/0102039 (2001).

[CI] B. S. Tsirelson, Quantum Analogues to the Bell Inequalities, J. Sov. Math. 36 (1987); B. S. Tsirelson, L. A. Khalfin, Quantum/Classical Correspondence in the Light of Bell’s Inequalities, Found. Phys. 22, 879 (1992).

[Du] W. Dür, Multiparticle bound entangled states that violate Bell’s inequality,
Phys. Rev. Lett. 87, 230402 (2001) and quant-ph/0107050 (2001).

[Fi] A. Fine, Hidden Variables, Joint Probability, and the Bell Inequalities, Phys.
Rev. Lett. 48, 291 (1982).

[GM] A. Garg, N. D. Mermin, Farkas’s lemma and the nature of reality: Statistical
implications of quantum correlations, Found. Phys. 14, 1 (1984).

[MPRG] S. Massar, S. Pironio, J. Roland, B. Gisin, A Zoology of Bell inequalities resistant to detector inefficiency, quant-ph/0205130 (2002).

[Pe] A. Peres, All the Bell Inequalities, Found. Phys. 29, 589 (1999) and
quant-ph/9807017 (1998).

[Pi] I. Pitovsky, Quantum Probability – Quantum Logic, Springer (Berlin) 1989.

[Fre] M. Fréchet, Les Probabilités Associées a un Système D’Événements Com-
patibles et Dépendants, Hermann (Paris) 1940.
Problem 1

All the Bell Inequalities

[Sc] H. H. Schaefer, *Topological Vector Spaces*, Springer (Berlin) 1980.

[PS] I. Pitowsky and K. Svozil, *New optimal tests of quantum nonlocality*, quant-ph/0011060 (2000).

[WW] R. F. Werner and M. M. Wolf, *All multipartite Bell correlation inequalities for two dichotomic observables per site*, quant-ph/0102024 (2001).

[WWa] R. F. Werner and M. M. Wolf, *Bell inequalities and Entanglement*, Quant. Inf. Comp. 1 (3), 1 (2002) and quant-ph/0107093 (2001).
Problem 2

Undistillability implies ppt?

Problem

A state on a bipartite quantum system is called distillable, if from sufficiently many pairs prepared in that state one can obtain a close approximation of a maximally entangled singlet state, using only local quantum operations and classical communication (LOCC). It is well-known that states with positive partial transpose (PPT) are not distillable. The problem is to decide the converse.

Background

This problem has been evident ever since it was shown in [HHH1] that entangled PPT states are undistillable. The two properties, PPT on the one hand and being undistillable on the other, are mathematically as different as they can be. Whereas the latter is a variational problem on an unbounded number of tensor products of density matrices, the first is a simple eigenvalue problem:

- A bipartite density operator $\rho$ is said to be PPT if its partial transpose $\rho^{TA}$, defined with respect to some product basis via $\langle ij | \rho^{TA} | kl \rangle = \langle kj | \rho | il \rangle$, is positive semi-definite, i.e., has only non-negative eigenvalues.

- A bipartite state characterized by a density matrix $\rho$ is distillable if there is a number $n$, such that $\rho^{\otimes n}$ can locally be projected onto an entangled two qubit state. That is, there are two dimensional projectors $Q$ and $P$ acting on the $n$-fold tensor product corresponding to Alice respectively Bob, such that

$$
\left( (P \otimes Q) \rho^{\otimes n} (P \otimes Q) \right)^{TA}
$$

has at least one negative eigenvalue. If $n$ copies of $\rho$ have such an entangled two qubit subspace, then the state is called $n$-distillable. There is yet no example of a state, which is distillable but not 1-distillable.
Using the above criterion of distillability, which was proven by the Horodeckis in [HH], the problem can be reformulated as [DSST]:

Given a completely positive map $S$ such that $TS$ is 2-positive (i.e. $id_2 \otimes TS$ is positive), where $T$ denotes the transpose map. Decide whether $TS \otimes TS$ is necessarily 2-positive.

### Partial Solutions

- For special cases like states on Hilbert spaces of dimension $2 \times m$ or Gaussian states it was proven in [DCLB], [HHH2] respectively [GDCZ], that every such state having a non-positive partial transpose (NPPT) is distillable.

- It was proven in [HH], that every NPPT state can be mapped onto an NPPT Werner state by means of LOCC operations. Hence, the matter can be decided considering the one-parameter family of Werner states only: if there exist any undistillable NPPT states, then there are undistillable entangled Werner states.

- In [DCLB], [DSST] numerical evidence has been presented, that there may be undistillable NPPT states. Moreover, it was proven analytically in [DSST], that for every fixed finite $n$ there is an interval of $n$-undistillable entangled Werner states. However, the parameter interval for which this statement has been proven, goes to zero for $n \to \infty$.

- It was proven in [EVWW] that if one enlarges the class of allowed operations from LOCC to PPT preserving maps, then every NPPT state becomes 1-distillable. For a proof using entanglement witnesses and the discussion of the tripartite case see [KLC]. Note that every PPT-preserving map can stochastically be implemented as LOCC operation with an additional PPT entangled state as a resource [CDKL].

- If an additional entangled PPT state $\sigma$ makes an NPPT state $\rho$, which is not 1-distillable itself, become 1-distillable, then we say that $\sigma$ activates the distillability of $\rho$. It has been proven in [VW] that there are PPT states $\sigma$, which are capable of activating every NPPT state. Moreover, the required amount of entanglement (measured in terms of any entanglement measure, which is continuous at the separable boundary) has been shown to be infinitesimally small [VW]. In [KLC] a formalism was introduced that connects entanglement witnesses and the activation properties of a state. Here it was shown that there exist tripartite NPPT states with the property that two copies can neither be distilled, nor activated.
Problem 2

Undistillability implies ppt?

Literature

[CDKL] J. I. Cirac, W. Dür, B. Kraus, and M. Lewenstein, *Entangling Operations and Their Implementation Using a Small Amount of Entanglement*, Phys. Rev. Lett. 86, 544 (2001) and quant-ph/0007057 (2000).

[DCLB] W. Dür, J. I. Cirac, M. Lewenstein, and D. Bruß, *Distillability and transposition in bipartite systems*, Phys. Rev. A 61, 062313 (2000) and quant-ph/9910022 (1999).

[DSST] D. P. DiVincenzo, P. W. Shor, J. A. Smolin, B. M. Terhal, and A. V. Thapliyal, *Evidence for bound entangled states with negative partial transpose*, Phys. Rev. A 61, 062312 (2000) and quant-ph/9910026 (1999).

[EVWW] T. Eggeling, K. G. H. Vollbrecht, R. F. Werner, and M. M. Wolf, *Distillability via Protocols Respecting the Positivity of Partial Transpose*, Phys. Rev. Lett. 87, 257902 (2001) and quant-ph/0104095 (2001).

[GDCZ] G. Giedke, L.-M. Duan, J. I. Cirac, and P. Zoller, *Distillability criterion for all bipartite Gaussian states*, Quant. Inf. Comp. 1(3), 79 (2001) and quant-ph/0104072 (2001).

[HH] M. Horodecki and P. Horodecki, *Reduction criterion of separability and limits for a class of protocols of entanglement distillation*, Phys. Rev. A 59, 4206-4216 (1999) and quant-ph/9708015 (1997).

[HHH1] M. Horodecki, P. Horodecki, and R. Horodecki, *Mixed-State Entanglement and Distillation: Is there a 'Bound' Entanglement in Nature?*, Phys. Rev. Lett. 80, 5239-5242 (1998) and quant-ph/9801069 (1998).

[HHH2] M. Horodecki, P. Horodecki, and R. Horodecki, *Inseparable Two Spin-1/2 Density Matrices Can Be Distilled to a Singlet Form*, Phys. Rev. Lett. 78, 574 (1997).

[KLC] B. Kraus, M. Lewenstein, and J. I. Cirac, *Characterization of distillable and activatable states using entanglement witnesses*, Phys. Rev. A 65, 042327 (2002) and quant-ph/0110174 (2001).

[VW] K. G. H. Vollbrecht and M. M. Wolf, *Activating Distillation with an Infinitesimal Amount of Bound Entanglement*, Phys. Rev. Lett. 88, 247901 (2002) and quant-ph/0201103 (2002).
Problem 3

Polynomial entanglement invariants

Problem

We say that two bipartite quantum states $\rho$ and $\sigma$ are “equally entangled” if they differ only by a choice of bases in Alice’s and Bob’s subspaces, i.e., if we can find unitaries $U_A, U_B$, such that

$$\rho = (U_A \otimes U_B)\sigma(U_A \otimes U_B)^*.$$ 

An entanglement invariant is by definition any real valued function on the space of bipartite density operators, which assigns the same value to equally entangled density operators. A polynomial invariant is an entanglement invariant, which can be computed as a polynomial in the matrix elements of $\rho$. Note that because we only consider hermitian operators, allowing polynomials in the matrix elements and their complex conjugates does not enlarge this class.

The basic problem is to decide the following question:

- Are the polynomial entanglement invariants complete?, i.e., if all polynomial invariants of $\rho$ and $\sigma$ agree, can we infer the existence of unitaries $U_A, U_B$ satisfying the above equation?

But we may add some further, closely related problems:

- Given the dimensions of Alices’s and Bob’s Hilbert spaces, name a finite set of invariants which is already complete.

- Do all this for multi-partite states. In this case even the case of pure states is not obvious.

- Decide whether the set of separable states can be described in terms of a polynomial invariant $f$, such that $f(\rho) \geq 0$ is equivalent to separability. There are many weaker versions of this statement, which may be of interest. For example, we might merely ask for a sufficient or a necessary separability criterion, and we might allow $f$ to depend on the dimensions.
Background

1. When \(d_1\) and \(d_2\) are the dimensions of Alice’s and Bob’s Hilbert space, respectively, the state space is \((d_1 d_2)^2 - 1\) dimensional. Since phases for \(U_A, U_B\) drop out of the transformation equation, we may fix their determinant to be 1, and hence get \((d_1^2 - 1) + (d_2^2 - 1)\) for the dimension of the symmetry group. Subtracting we get an expected manifold dimension of \((d_1^2 - 1)(d_2^2 - 1)\) for the quotient manifold, i.e., the manifold of all invariants. Of course, it may happen that no set of this many differentiable invariants is sufficient to pin down each equivalence class uniquely, and more invariants are needed to rule out some discrete choices.

2. It is perhaps useful to recall the “unipartite” version of this problem, i.e., the characterization of density operators up to unitary equivalence. The well-known complete set of invariants in that case is the spectrum of the density operator (including multiplicities). The eigenvalues are not polynomial, but the coefficients of the characteristic polynomial (i.e., the elementary symmetric functions of the eigenvalues) or, equivalently the numbers \(a_n = tr(\rho^n)\) are polynomial, and from these the eigenvalues can be determined. Hence a complete set of invariants are the \(a_n\) for \(n = 1, \ldots, \text{dimension}\).

3. A basis for the ring of invariant polynomials (even in the multi-partite case, and for arbitrary Hilbert space dimensions) was given in [GRB] and [R]. Note that any homogeneous polynomial of degree \(k\) in \(\rho\) can be written as an expectation value of the \(k\text{th}\) tensor power of \(\rho\), i.e., as \(tr(\rho^k X)\), with a uniquely determined \(X\). For an \(n\)-partite system this is an operator on a tensor product of \(nk\) Hilbert spaces. Invariance requires that it commutes with all unitaries of the form \(U_1^\otimes k \otimes \cdots \otimes U_n^\otimes k\), where \(U_m\) is an operator on the Hilbert space of the \(m\text{th}\) type of systems (\(m = 1, \ldots, n\)). Then the commutation theorem of von Neumann algebras, and the corresponding result for \(n = 1\), imply that \(X\) must be a tensor product of \(n\) permutation operators, each one permuting the \(k\) tensor factors belong to one of the \(n\) system types.

Partial Solutions

- Y. Makhlin [M] has shown completeness in the bipartite qubit case. Moreover, he has identified a set of 18 invariants, which is sufficient in that case, and has shown that none of these may be omitted without destroying completeness.

- A. Sudbery [S] has solved the case of pure three qubit states, finding 8 polynomial invariants (6 being the dimension of the manifold of all invariants).
Problem 3  
Polynomial entanglement invariants

Solution

The basic question of principle (are the polynomial entanglement invariants complete?) is answered in Onishchik and Vinberg’s book *Lie Groups and Algebraic Groups*, which contains the theorem [OV].

The orbits of a compact linear group acting in a real vector space are separated by the polynomial invariants.

In other words (those of quantum information theory), if two states of a multipartite system are not related by local unitary transformations, then they have different values for some polynomial entanglement invariant.

It follows that the space of entanglement types of states, i.e. the space of orbits factored by normalisation, can be identified with the space of polynomial invariants (more precisely, the ring of polynomial functions on this space is isomorphic to the ring of polynomial invariants). The dimension of this space is known in full generality for pure states [CHS]. For two parties it is one less than the dimension of the smaller state space (a complete set of invariants is the set of Schmidt coefficients, which sum to 1 by normalisation). For $n > 2$, if the parties have state spaces with dimensions $d_1, \ldots, d_n$ in increasing order, then the space of orbits of normalised states has dimension

$$D_{\text{pure}} = 2 \prod_{r=1}^{n} d_r - \sum_{r=1}^{n} d_r^2 + n - 2 + \Delta^2$$

where $\Delta = d_n - d_1 \ldots d_{n-1}$ if this is positive, otherwise $\Delta = 0$. If all the parties are qudits ($d_1 = \cdots = d_n = d$) this becomes

$$D_{\text{pure}} = 2d^n - nd^2 + n - 2.$$ 

The corresponding dimension for mixed states is

$$D_{\text{mixed}} = d^{2n} - nd^2 + n - 1$$

which is probably correct, though a careful treatment has never appeared in the literature. The general case for mixed states has not been discussed.

The number of invariants needed to uniquely specify a state up to local unitary transformations is not the same as the dimension $D$ of the space of entanglement types; this is in general a curved space, with complicated geometry. Makhlin’s work [M] shows that the space of entanglement types of mixed states of two qubits is a nine-dimensional manifold in $\mathbb{R}^{18}$ (the ring of polynomial invariants has 18 generators subject to 9 relations). For pure states of three qubits, which have $D = 6$ (including the norm), a complete set of invariants \[AAJT\] consists of the six independent invariants given in \[S\] together with one more found by Grassl. Thus the space of orbits of non-normalised state vectors is a hypersurface in $\mathbb{R}^7$; normalising, the space of entanglement types of pure states of three qubits is a hypersurface in real projective 6-space.
The above theorem was used by Hilary Carteret and myself in our proof \cite{CS} that on an orbit whose dimension is exceptionally low, some entanglement invariant has an extreme value. We classified these exceptional orbits for pure states of three qubits.

The condition that the group should be compact is essential, as is shown by the example of the general linear group $GL(n, \mathbb{C})$ acting on $n \times n$ complex matrices by the similarity transformation $X \mapsto GXG^{-1}$ where $G \in GL(n, \mathbb{C})$. The polynomial invariants here are the coefficients in the characteristic equation of $X$, so two matrices have the same values of the invariants if and only if they have the same eigenvalues. But having the same eigenvalues is not sufficient for two matrices to be similar; if some of the eigenvalues are repeated, there are different possible Jordan normal forms which are not related by similarity.

An even simpler example, and one which is relevant to quantum information theory, is the action of $GL(m, \mathbb{C}) \times GL(n, \mathbb{C})$ on $m \times n$ matrices by $X \mapsto PXQ^T$ where $P \in GL(m, \mathbb{C})$ and $Q \in GL(n, \mathbb{C})$. In this case there are no polynomial invariants, but matrices can only be transformed into each other by such a transformation if they have the same rank. (The rank is a non-polynomial invariant.) If we take $X$ to be an element of $\mathbb{C}^m \otimes \mathbb{C}^n$ representing a pure state of a bipartite system, two states are related by this action if there are local operations which will convert them into each other with non-zero probability. This generalises the deterministic (unitary) local operations which define equally entangled states in the statement of the problem. The corresponding orbits for three qubits have been determined by Dür, Vidal and Cirac \cite{DVC}, and for four qubits by Verstraete, Dehaene, De Moor and Verschelde \cite{MVDV}.

**Literature**

\cite{AAJT} A. Acin, A. Andrianov, E. Jane, and R. Tarrach, *Three-qubit pure-state canonical forms*, J. Phys. A \textbf{34}, 6725 (2001) and quant-ph/0009107 (2000).

\cite{CHS} H. A. Carteret, A. Higuchi, and A. Sudbery, *Multipartite generalisation of the Schmidt decomposition*, J. Math. Phys. \textbf{41} (2000) and quant-ph/0006125 (2000).

\cite{CS} H. A. Carteret and A. Sudbery, *Local symmetry properties of pure states of three qubits*, J. Phys. A \textbf{33}, 4981 (2000) and quant-ph/0001091 (2000).

\cite{DVC} W. Dür, G. Vidal, and J. I. Cirac, *Three qubits can be entangled in two inequivalent ways*, Phys. Rev. A \textbf{62}, 062314 (2000) and quant-ph/0005115 (2000).

\cite{GRB} M. Grassl, M. Rötteler, and T. Beth, *Computing local invariants of qubit systems*, Phys. Rev. A \textbf{58}, 1833 (1998) and quant-ph/9712040 (1997).

\cite{M} Y. Makhlin, *Nonlocal properties of two-qubit gates and mixed states and optimization of quantum computations*, quant-ph/0002045 (2000).
Problem 3  Polynomial entanglement invariants

[MVDV] B. De Moor, F. Verstraete, J. Dehaene, and H. Verschelde, *Four qubits can be entangled in nine inequivalent ways*, quant-ph/0109033 (2001).

[OV] A. L. Onishchik and E. B. Vinberg, *Seminar on Lie groups and algebraic groups*, Springer (Berlin) 1990, p.144 (in Russian); English translation *Lie groups and algebraic groups*, Springer (Berlin) 1990, Chap. 3, Paragraph 4, Theorem 3.

[R] E. M. Rains, *Polynomial invariants of quantum codes*, quant-ph/9704042 (1997).

[S] A. Sudbery, *On local invariants of pure three-qubit states*, J. Phys. A 34, 643 (2001) and quant-ph/0001116 (2000).
Problem

Catalytic majorization

Problem

With a Theorem by Nielsen [N], we have a completely explicit criterion to decide, when one pure bipartite state can be converted to another such state, using only local quantum operations and classical communication. Using Nielsen’s criterion one can show [JP1] that the following strange situation can happen: state A cannot be converted to state B, but $A \otimes C$ can be converted to $B \otimes C$, where C is a suitably chosen entangled state, the ”catalyst”.

The problem is to give a similarly efficient criterion to decide which pure bipartite states can be converted into each other using a catalyst.

Background

Here is Nielsen’s criterion, which is a surprisingly direct rendering of the intuition that a “more entangled” pure state has a “more mixed” restriction. Thus A can be converted to B if and only if the eigenvalue sequence of the restriction of A is more mixed than that of B in the sense of majorization of probability vectors [Ma]. We say that one probability vector $p = (p_1, \ldots, p_n)$ is more mixed than another, $q = (q_1, \ldots, q_n)$ in the sense of majorization, if one and hence all of the following equivalent statements hold:

- For all $k$ : $\sum_{i=k} p_i \leq \sum_{i=k} q_i$, provided both $p$ and $q$ are first brought into decreasing order.
- There is a doubly stochastic matrix $D$ (positive entries, sum of all rows and all columns = 1) such that $p = Dq$.
- For every convex function $f : \mathbb{R} \to \mathbb{R}$ : $\sum_i f(p_i) \leq \sum_i f(q_i)$

The above problem can be rephrased completely in this context of majorization of classical probability vectors, since tensoring pure bipartite states means again tensoring of probability vectors for the eigenvalues of the reduced density operators. Thus we would like to characterize the order relation “catalytic majorization”:
For some $r$, $(p \otimes r)$ is more mixed than $(q \otimes r)$ in the sense of majorization.

The above list of equivalent characterizations of majorization points to a way a characterization might look like: we might look for convex functions $f$, such that $p \mapsto \sum_i f(p_i)$ is monotone with respect to catalytic majorization, and hope to characterize the relation by such a set. One class of functions $f$ with this monotonicity property is $f(t) = t^x$, for $x > 1$, because the corresponding functionals on probability vectors are multiplicative with respect to tensor products.

There is some further literature on the use of majorization for the characterization of pure state entanglement [V1], [JP1], [VJN1], [N2] and on catalysis [EW1] that may be useful.

**Literature**

[N] M. A. Nielsen, Phys. Rev. Lett. 83, 436-439 (1999) and quant-ph/9811053 (1998). The original proof in this paper can be simplified considerably.

[V1] G. Vidal, Phys. Rev. Lett. 83, 1046-1049 (1999) and quant-ph/9902033 (1999).

[JP1] D. Jonathan and M. B. Plenio, Phys. Rev. Lett. 83, 3566-3569 (1999) and quant-ph/9905071 (2000).

[JP2] D. Jonathan and M. B. Plenio, Phys. Rev. Lett. 83, 1455-1458 (1999) and quant-ph/9903054 (1999).

[VJN1] G. Vidal, D. Jonathan, and M. A. Nielsen, Phys. Rev. A 62, 012304 (2000).

[N2] M. Nielsen, quant-ph/0008073 (2000).

[EW1] J. Eisert and M. Wilkens, Phys. Rev. Lett. 85, 437-440 (2000).

[Maj] We have avoided the use of a comparison symbol, or the terminology “$p$ is majorized by $q$”, because there are different conventions in the literature. There is a rich literature on the subject, starting with the still to be recommended classic

G. H. Hardy, J. E. Littlewood, and G. Polya, *Inequalities*, Cambridge UP (1934).

Further standard references are

A. W. Marshal and I. Olkin, *Inequalities: Theory of Majorization and Its Applications*, Academic Press (1979)

and, in the quantum context,

R. Bhatia, *Matrix Analysis*, Springer (1996).
Problem 5

Maximally entangled mixed states

Problem

Among all density operators of two qubits with the same spectrum one may look for those maximizing some measure of entanglement. It turns out that for ‘entanglement of formation’, ‘relative entropy of entanglement’ and ‘negativity’ one gets the same “maximally entangled states”.

Is this true for arbitrary entanglement monotones?

Obvious variants of this problem are for higher dimensional systems and weaker constraints on the spectrum, e.g., largest eigenvalue or entropy.

Background

(Refer to definitions of the measures of entanglement and ‘entanglement monotone’.)

Literature

[VAM] F. Verstraete, K. Audenaert, and B. De Moor, Maximally entangled mixed states of two qubits, quant-ph/0011110 (2000).
Problem 6

Nice error bases

Problem

There are two special constructions to obtain orthogonal bases of unitaries, i.e., collections of unitary operators $U_i$, $i = 1, \ldots, d^2$, on a $d$-dimensional Hilbert space, such that $\text{tr}(U_i^*U_j) = d\delta_{ij}$.

On the one hand one can require in addition that the product of any two unitaries in the basis gives another one up to a phase, i.e., $U_iU_j = \text{phase} \cdot U_k$. The composition of labels $(i,i) \mapsto k$ then defines a group, the “index group” of the basis. Bases of this kind have been called nice error bases.

On the other hand, one may require that, in a suitable basis of the Hilbert space, the unitaries are obtained as the products of a collection of $d$ permutation operators and $d$ multiplication operators. Bases constructed in this way are called shift and multiply type.

The question that arises here is to decide whether every nice error basis is of shift and multiply type.

Background

Orthogonal bases are precisely what is needed to construct schemes for entanglement assisted teleportation or dense coding. For qubits ($d = 2$) there is only one such basis up to left and right multiplication by fixed unitaries, namely the Pauli matrices together with the identity.

The shift and multiply constructions can be classified further: for the “shift” part one precisely needs a Latin square, whereas the multiplication part requires the construction of $d$ complex Hadamard matrices.

A finite group $H$ is called of central type if it possesses an irreducible representation in $d = \sqrt{|H|/|Z(H)|}$ dimensions, where $Z(H)$ is the center of $H$. 
Solution

An answer to the question, given above, has recently been found by Andreas Klappenecker and Martin Roetteler. They show in their article “On the monomiality of nice error basis” [4] that there is in fact a nice error basis which is not of shift and multiplier type.

Roughly their argumentation is based on the following: First one observes that every nice error basis which is of shift and multiplier type is monomial, i.e. each of its unitary matrices has in every row and column precisely one non-vanishing entry. An abstract error group is one which is generated by nice error bases (central extension of the index group). Such a group is of central type with cyclic center. Employing the theory of characters for these groups, which has been studied by P. Fergusson, I.M. Isaacs (see references given in [4]), an abstract error group can be constructed which has a non-monomial irreducible representation.

Literature

[1] A. Klappenecker and M. Roetteler, Beyond Stabilizer Codes I: Nice Error Bases, quant-ph/0010082 (2000).
[2] E. Knill, Group Representations, Error Bases and Quantum Codes, quant-ph/9608049 (1996).
[3] R. F. Werner, All teleportation and dense coding schemes, quant-ph/0003070 (2000).
[4] A. Klappenecker and M. Roetteler, On the Monomiality of Nice Error Bases, quant-ph/0301078 (2003).
Problem 7
Additivity of Entanglement of Formation

Problem

The entanglement of formation [BD96] is one of the standard measures of entanglement. It is defined, for any density operator \( \rho \) on a bipartite system, as

\[
E_F(\rho) = \inf \left\{ \sum_i r_i S(\rho_i|A) \left| \sum_i r_i \rho_i = \rho \right. \right\},
\]

where \( S(\cdot) \) denotes the von Neumann entropy and \( \rho|A \) denotes the restriction of a density operator \( \rho \) to the “Alice” subsystem (partial trace over the other subsystem), the \( \rho_i \) are density operators and the \( r_i \) are positive, adding up to one. Since \( S(\cdot) \) is concave, the infimum is attained at a convex decomposition of \( \rho \) into pure states, and the definition is often given as this restricted infimum.

Consider now a pair \( \rho^{(i)}, i = 1, 2 \) of bipartite density operators, and their tensor product \( \rho = \rho^{(1)} \otimes \rho^{(2)} \), which lives on a tensor product of four Hilbert spaces, but can be considered as a bipartite state when the two Alice subspaces and the two Bob subspaces are grouped together. Then it is easy to show (by plugging the tensor product of the optimal decompositions of the factors into the variational expression and using the additivity of the entropy) that \( E_F(\rho) \leq E_F(\rho^{(1)}) + E_F(\rho^{(2)}) \).

The problem is to show that equality always holds here.

Background

This inequality is crucial to settle the interpretation of \( E_F \) as a resource quantity. The typical kind of tensor products appearing in the theory are pairs created by (maybe different) sources of entangled states, and kept for later use.
Partial Solutions

The additivity of entanglement of formation could be proven for several examples of states by Vidal et al. [VDC02].

This problem has been shown to be equivalent to problem 10.

Literature

[BD96] C. H. Bennett, D. P. DiVincenzo, J. A. Smolin, and W. K. Wootters, *Mixed-state entanglement and quantum error correction*, Phys. Rev. A 54, 3824 (1996) and quant-ph/9604024 (1996).

[VDC02] G. Vidal, W. Dür, and J. I. Cirac, *Entanglement cost of mixed states*, Phys. Rev. Lett. 89, 027901 (2002) and quant-ph/0112131 (2001).
Problem 8
Qubit formula for Relative Entropy of Entanglement

Problem

The relative entropy of entanglement is an entanglement monotone that quantifies to what extent a given state can be operationally distinguished (in the sense of Stein’s Lemma) from the closest state which is either separable or has a positive partial transpose (PPT). For a state $\rho$ it is defined as

$$E_R(\rho) = \inf_{\sigma \in D} S(\rho||\sigma),$$

where $D$ stands for the convex sets of separable or PPT states, and $S(\cdot||\cdot)$ is the quantum relative entropy. The problem is to find a closed formula for this quantity for systems consisting of two qubits.

Background

The interpretation of the relative entropy of entanglement is a geometrical one: it is related to the error probability with which a state is mistakenly assumed to be merely classically correlated or PPT in quantum hypothesis testing. This entanglement monotone is an upper bound to the distillable entanglement, and in its asymptotic version conjectured to be identical to the Rains’ bound for distillable entanglement. As most other monotones of entanglement, and all other known monotones that are provably asymptotically continuous, the actual evaluation of this quantity amounts to solving an optimization problem. In the case at hand, it is a convex optimization problem.

The entanglement of formation is a monotone which is also defined as an optimization problem. If it turned out that the entanglement of formation was in fact additive (see problem 7), then this quantity could be interpreted as the entanglement cost, which fleshes out the resource character of entanglement. Historically, it was very important that for systems consisting of two qubits, the entanglement of formation
can (quite astonishingly) be evaluated: the Wootters formula \[2\] is a closed formula for the entanglement of formation for two-qubit systems. The proof exploits a number of the particular properties that are available for two-qubit systems \[3\] — and only for them. The task is to explicitly solve the convex optimization problem posed by the relative entropy of entanglement.

**Partial Solution**

So far, there is no published solution to the problem. Ref. \[4\] presents the solution to a related problem: for a two-qubit system, given a state on the boundary of separable states \(\sigma\), it characterizes the states \(\rho\) for which \(E_R(\rho) = S(\rho || \sigma)\).

**Literature**

[1] V. Vedral and M.B. Plenio, Phys. Rev. A **57**, 1619 (1998).

[2] W. Wootters, Phys. Rev. Lett. **78**, 5022 (1997).

[3] K.G.H. Vollbrecht, and R.F. Werner, J. Math. Phys. **41**, 6772 (2000).

[4] S. Ishizaka, Phys. Rev. A **67**, 060301(R) (2003).
Problem 9

Reduction criterion implies majorization?

Problem

The density matrix of any separable state is majorized by its reductions (the density matrix reduced to one subsystem, e.g. $\rho_A = \text{Tr}_B \rho_{AB}$). This is in fact the strongest separability criterion based on the spectra of a state and one of its reductions. However, it is not known how it is related to other separability criteria like PPT, undistillability or the reduction criterion. The problem is to find out how majorization enters into the known implication chain of separability criteria.

Background

One of the remarkable properties of entangled states is that they can exhibit locally more disorder than globally. The simplest example is the maximally entangled state, which is pure as a whole but it has maximally chaotic reductions. A powerful tool comparing the order/disorder of two systems is majorization and in fact it is a more stringent notion of order/disorder than entropy.

It was proven in [1] that the density matrix of a separable state is majorized by both of its reductions. Hence, majorization yields a separability criterion, which is merely based on the spectra of a state and its reductions.

There are many important separability/entanglement criteria or properties and in most cases the relations between them are well known: Separability $\Rightarrow$ positivity of the partial transpose [2] $\Rightarrow$ undistillability [3] $\Rightarrow$ reduction criterion [4].

The intuition may be, that all these criteria a strictly stronger than majorization, however the matter is not decided yet.
Partial Solutions

Apart from inconclusive numerical search for counterexamples for the implication: reduction criterion $\Rightarrow$ majorization, the only partial result is derived in [5], where it was shown, that the reduction criterion implies positivity for conditional Renyi entropies for every value of the entropic parameter. Although conditional entropies also measure the proportion between global and local disorder, this result cannot be extended directly to majorization.

Solution

The answer to the question is contained in [6], stating that the reduction criterion does imply majorization.

The key idea of the proof is that $\rho_A \otimes I_B \geq \rho_{AB}$ implies $\rho_{AB}^{1/2} = (\rho_A^{1/2} \otimes I_B) R$ with $||R|| \leq 1$, where $\rho_{AB}$ is a bipartite density matrix, $\rho_A = T_B \rho_{AB}$, and $||\cdot||$ is the operator norm. By virtue of this, we can derive the existence of the substochastic matrix $S$ such that $\lambda(\rho_{AB}) = S \lambda(\rho_A)$, where $\lambda(\rho_{AB}) [\lambda(\rho_A)]$ is the eigenvalue (column) vector of $\rho_{AB} [\rho_A]$. This last equation is equivalent to the weak submajorization relation $\lambda(\rho_{AB}) \preceq_w \lambda(\rho_A)$ which is none other than $\lambda(\rho_{AB}) < \lambda(\rho_A)$ in this problem.

Literature

[1] M. A. Nielsen and J. Kempe, Separable States Are More Disordered Globally than Locally, Phys. Rev. Lett. 86, 5184 (2001) and quant-ph/0011117 (2000).

[2] A. Peres, Separability Criterion for Density Matrices, Phys. Rev. Lett. 77, 1413 (1996) and quant-ph/9604005 (1996).

[3] M. Horodecki, P. Horodecki, and R. Horodecki, Mixed-State Entanglement and Distillation: Is there a “Bound” Entanglement in Nature?, Phys. Rev. Lett. 80, 5239 (1998) and quant-ph/9801063 (1998).

[4] M. Horodecki and P. Horodecki, Reduction criterion of separability and limits for a class of distillation protocols, Phys. Rev. A 59, 4206 (1999) and quant-ph/9708015 (1997);

N. J. Cerf, C. Adami, and R. M. Gingrich, Reduction criterion for separability, Phys. Rev. A 60, 898 (1999) and quant-ph/9710001 (1997).

[5] K. G. H. Vollbrecht and M. M. Wolf, Conditional entropies and their relation to entanglement criteria, quant-ph/0202058 (2002).

[6] T. Hiroshima, Majorization criterion for distillability of a bipartite quantum state, quant-ph/0303057 (2003).
Problem 10

Additivity of classical capacity and related problems

Problem

For each quantum channel $T$ (in the Schrödinger picture), define

$$
\chi(T) = \sup_{p,\rho} \left( H \left( \sum_i p_i T(\rho_i) \right) - \sum_i p_i H (T(\rho_i)) \right),
$$

where the supremum is over all probability vectors $p = (p_1, \ldots, p_n)$, and all collections of input states $\{\rho_1, \ldots, \rho_n\}$, and $H$ denotes the von Neumann entropy.

Show that $\chi(T_1 \otimes T_2) = \chi(T_1) + \chi(T_2)$, or else give a counterexample. The problem can be traced back to [BFS], see also [Ho].

Background

This problem can also be paraphrased as “Can entanglement between signal states help to send classical information on quantum channels?”. Recall that the capacities of a memoryless channel are defined as the maximal transmission rate per use of the channel, with coding and decoding chosen for increasing number $n$ of parallel and independent uses of the channel

$$T^{\otimes n} = T \otimes \cdots \otimes T_n$$

such that the error probability goes to zero as $n \to \infty$. There are many different capacities, because one may consider sending different kinds (classical or quantum) information, restrict the admissible coding and decoding operations, and/or allow the use of additional resources. Here we only look at the transmission of classical information with no additional resources. Then one can distinguish four capacities [BS], according to whether for each block length $n$ we are allowed to use arbitrary entangled
Problem 10  Additivity of classical capacity and related problems

quantum operations on the full block of input (resp. output) systems, or if for each of the parallel channels we have to use a separate quantum coding (resp. decoding), and combine these only by classical pre (resp. post)-processing:

\[ C_{\infty}\infty : \text{full capacity, arbitrary (de)coding} \]

\[ C_{1\infty} = \chi : \text{unentangled coding, quantum block decoding} \]

\[ \geq \]

\[ C_{\infty}1 : \text{quantum block coding, separate decoding} \]

\[ \geq \]

\[ C_{11} : \text{one-shot capacity or accessible information, separate quantum (de)coding, block (de)coding only classical} \]

The equality in the lower right was established independently by several authors, see e. g. [KRb]. That \( C_{1\infty} \) on the left coincides with the quantity \( \chi \) given in the statement of the problem was shown in [HSW]. The inequality in the lower left is known to be strict sometimes [Ho], which means that entangling decodings indeed can increase the classical capacity. See [SKIH] for investigation of the corresponding information gain.

The full capacity and \( \chi \) are connected by the limit formula

\[ C_{\infty}\infty(T) = \lim_{n}(1/n)\chi(T^\otimes n) \]

Since \( \chi \) is easily seen to be superadditive (i. e., \( \chi(T_1 \otimes T_2) \geq \chi(T_1) + \chi(T_2) \)), we immediately get \( C_{\infty}\infty \geq \chi \). If additivity holds, then we will even have equality, i. e., “???” in the table can be replaced by “=” . While such a result would be very much welcome from a mathematical (and practical) point of view, giving a “single-letter” expression for the classical capacity, it would call for a physical explanation of strange asymmetry between the roles of entanglement in encoding and decoding procedures.

**Partial results**

Validity of the additivity conjecture was established if one of the channels is

- the identity channel [AHW, SWa];
- a unital qubit channel [Kib];
- the depolarizing channel [Kic];
- an entaglement breaking channel [Ho, Kia] (both for “c-q/q-c” channels), [Sha] (general entaglement breaking channel).
Some further more recent partial results will be mentioned below. Whether the additivity holds “globally”, i.e. for all quantum channels, is still an open problem. No counterexample was found despite extensive numerical search by groups in IBM, IMaPh, see also [ON]. If the conjecture is valid, then the additivity of $\chi$ tentatively relies upon yet another hypothetical property of multiplicativity of norms of the completely positive mappings

$$T : \ell_1(\mathcal{H}) \to \ell_p(\mathcal{H}); \quad p \geq 1,$$

where

$$\ell_p(\mathcal{H}) = \{X : X = X^*, \quad \|X\|_p \equiv (\text{Tr} |X|^p)^{1/p}\}$$

is a noncommutative analog of the space $\ell_p$ – the so called Schatten class. Namely, the conjecture [AHW] is that for $p$, sufficiently close to 1

$$\|T_1 \otimes T_2\|_p \approx \|T_1\|_p \|T_2\|_p,$$

(10.1)

where $\|T\|_p = \max_\rho \|T(\rho)\|_p$. By letting $p \downarrow 1$ this implies additivity of the minimal output entropy

$$H_{\text{min}}(T) = \min_\rho H(T(\rho)),$$

one of a whole number of properties equivalent, as it was shown in [Shb], to the additivity of $\chi$. The relation [GGLMSY] can be re-expressed as the additivity of the minimal output Renyi entropy of order $p$.

In all cases listed above where the additivity conjecture is proved, the multiplicativity of $p$–norms (for all $p \geq 1$) also holds, moreover, it underlies the proof of additivity in [Kib], [Kic]. The multiplicativity of $p$–norms holds for arbitrary bounded maps of the classical spaces $\ell_p$, where its proof can be based on a Minkowsky inequality. Therefore quite intriguing is counterexample of the channel

$$T(\rho) = \frac{1}{d-1} [I - \rho^T],$$

for which [MB] with $T_1 = T_2 = T$ fails to hold for sufficiently large $p$ ($p \geq 4,7823$ if $d = \dim \mathcal{H} = 3$ [WH]). Nevertheless, the additivity of $H_{\text{min}}$ and of $\chi$ holds for such channels, as shown in [MY], [DHS], [AF]. The standing conjecture is that multiplicativity holds globally at least for $1 \leq p \leq 2$, but even the case $p = 2$ is difficult, see [KNR], [KRc]. For some results concerning integer $p$ see [GLR].

In [AB] it was shown that proving the multiplicativity would solve another important open problem – superadditivity of the entanglement of formation (EoF). Earlier [MSW] brought attention to a simple correspondence between $\chi$ and EoF, and obtained several concrete results on additivity of EoF by using this correspondence. It was also remarked that superadditivity of EoF would imply additivity of $\chi$ for channels with linear additive input constraints. By combining the MSW correspondence and the convex duality technique of [AB] with an original and powerful channel extension technique, which allows to use effectively arbifariness of channels in question, [Shh] had shown equivalence of the global properties of additivity of the minimal output entropy,
Problem 10  Additivity of classical capacity and related problems

Figure 10.1: Equivalence of additivity properties. Bold (thin) arrows indicate nontrivial (obvious) implications for individual (ind.) or global (glob.) relations.

$\chi$, EoF and of superadditivity of EoF. The last equivalence for two fixed channels was also established in [Po].

In [HSA] several equivalent formulations of the additivity conjecture for channels with arbitrarily constrained inputs, which formally is substantially stronger than additivity of the unconstrained $\chi$, were given. It was shown that the additivity conjecture for channels with constrained inputs holds true for certain nontrivial classes of channels, e.g. a direct sum mixture of the identity channel and an entanglement breaking channel (such as erasure channel). The channel extension technique was used to show that additivity for two fixed constrained channels can be reduced to the same problem for unconstrained channels, and hence, the global additivity for channels with arbitrary input constraints is equivalent to the global additivity without constraints.

The additivity problem is still open for the minimal dimension 2: it is not known if the additivity holds for all nonunital qubit channels, although a strong numerical evidence in favour of this was given in [HIMRS]. Nevertheless there are several reasons to consider the problem in infinite dimensions. There is a good chance that both the additivity and the multiplicativity for all $p \geq 1$ hold for important and interesting class of Gaussian channels that act in infinite dimensional Hilbert space. However the only
instance where the additivity of $\chi$ and the multiplicativity for integer $p$ was proved is the pure loss channel, having the very special property $H_{\text{min}}(T) = 0$ [GGLMSY].

It was observed recently that Shor’s proof of equivalence of different forms of the global additivity conjecture for finite dimensional channels is related to weird discontinuity of the $\chi$--capacity as a function of channel in infinite dimensions. This also calls for a mathematically rigorous treatment of the entropic quantities related to the classical capacity of infinite dimensional channels [HSb]. In particular it is possible to show that additivity for all finite dimensional channels implies additivity of the constrained $\chi$--capacity with constraints fulfilling finiteness of the output entropy [Shi].

**Literature**

[AB] K. M. R. Audenaert, S. L. Braunstein, *On strong superadditivity of the entanglement of formation*, quant-ph/0303043 (2003).

[AF] R. Alicki, M. Fannes, *Note on multiple additivity of Renyi entropy output for Werner-Holevo channel*, quant-ph/0407033 (2004).

[AHW] G. G. Amosov, A. S. Holevo, and R. F. Werner, *On some additivity problems in quantum information theory*, Probl. Inform. Transm. 36 (4), 25 (2000) and math-ph/0003002 (2000); G. G. Amosov and A. S. Holevo, *On the multiplicativity conjecture for quantum channels*, math-ph/0103015 (2001).

[BFS] C. H. Bennett, C. A. Fuchs, J. A. Smolin, *Entanglement-enhanced classical communication on a noisy quantum channel*, in: Quantum Communication, Computing and Measurement, Proc. QCM96, ed. by O. Hirota, A. S. Holevo and C. M. Caves, New York: Plenum 1997, pp. 79-88 and quant-ph/9611006 (1996).

[BS] C. H. Bennett, P. W. Shor, *Quantum information theory*, IEEE Trans. Inform. Theory 44, 2724 (1998).

[DHS] N. Datta, A. S. Holevo, Y. M. Suhov. *A quantum channel with additive minimum output entropy*, quant-ph/0408177 (2004).

[GGLMSY] V. Giovannetti, S. Lloyd, L. Maccone, J. H. Shapiro, B. J. Yen, *Minimum Renyi and Wehrl entropies at the output of bosonic channels*, quant-ph/0404037 (2003).

[GL] V. Giovannetti, S. Lloyd, *Additivity properties of a Gaussian channel*, quant-ph/0403075 (2004).

[GLR] V. Giovannetti, S. Lloyd, M. B. Ruskai, *Conditions for the multiplicativity of maximal $l_p$-norms of channels for fixed integer $p$*, quant-ph/0408103 (2004).
Problem 10

Additivity of classical capacity and related problems

[HIMRS] M. Hayashi, H. Imai, K. Matsumoto, M. B. Ruskai, T. Shimono, Qubit channels which require four inputs to achieve capacity: implications for additivity conjectures, quant-ph/0403176 (2004).

[Ho] A. S. Holevo, Quantum coding theorems, Russ. Math. Surv. 53, 1295 (1998) and quant-ph/9809023 (1998).

[HSW] A. S. Holevo, The Capacity of the Quantum Channel with General Signal States, IEEE Trans. Inform. Theory 44, 269 (1998) and quant-ph/9611023 (1996); B. Schumacher and M. D. Westmoreland, Sending classical information via noisy quantum channels, Phys. Rev. A 56, 131 (1997).

[HSa] A. S. Holevo, M. E. Shirokov, On Shor’s channel extension and constrained channels, Commun. Math. Phys. 249, 417 (2004) and quant-ph/0306190 (2003); M. E. Shirokov, On the additivity conjecture for channels with arbitrary constrains, quant-ph/0308168 (2003).

[HSb] A. S. Holevo, M. E. Shirokov, Continuous ensembles and the $\chi$-capacity of infinite-dimensional channels, quant-ph/0403072 (2004).

[Kia] C. King, Maximization of capacity and $l_p$ norms for some product channels, J. Math. Phys. 43, 1247 (2002) and quant-ph/0103086 (2001).

[Kib] C. King, Additivity for a class of unital qubit channels, quant-ph/0103156 (2001).

[Kic] C. King, The capacity of the quantum depolarizing channel, quant-ph/0204172 (2002).

[KNR] C. King, M. Nathanson, M. B. Ruskai, Multiplicativity properties of entrywise positive maps on matrix algebras, quant-ph/0409181 (2004).

[KRa] C. King and M. B. Ruskai, Minimal Entropy of States Emerging from Noisy Quantum Channels, IEEE Trans. Info. Theory 47, 192 (2001) and quant-ph/9911073 (1999).

[KRb] C. King and M. B. Ruskai, Capacity of Quantum Channels Using Product Measurements, J. Math. Phys. 42, 87 (2001) and quant-ph/0004062 (2000).

[KRe] C. King, M. B. Ruskai, Comments on multiplicativity of maximal p-norms when p = 2, quant-ph/0401026 (2004).

[MSW] K. Matsumoto, T. Shimono, A. Winter, Remarks on additivity of the Holevo channel capacity and of the entanglement of formation, quant-ph/0206148 (2002).

[MY] K. Matsumoto, F. Yura, Entanglement cost of antisymmetric states and additivity of capacity of some quantum channel, quant-ph/0306009 (2003).
Problem 10  Additivity of classical capacity and related problems

[ON]  S. Osawa, H. Nagaoka, Numerical experiments on the capacity of quantum channel with entangled input states, quant-ph/0007115 (2000).

[Po]  A. A. Pomeransky, Strong superadditivity of the entanglement of formation follows from its additivity, quant-ph/0305056 (2003).

[SKIH]  M. Sasaki, K. Kato, M. Izutsu, O. Hirota, Quantum channels showing superadditivity in capacity, quant-ph/9801012 (1998).

[Shi]  M. E. Shirokov, The Holevo capacity of infinite dimensional channels, quant-ph/0408009 (2004).

[Sha]  P. W. Shor, Additivity of the classical capacity of entanglement-breaking quantum channels, Commun. Math. Phys. 246, 453 (2004) and quant-ph/0201149 (2002).

[Shb]  P. W. Shor, Equivalence of additivity questions in quantum information theory, quant-ph/0305035 (2003).

[SWa]  B. Schumacher and M. D. Westmoreland, Relative entropy in quantum information theory, quant-ph/0004045 (2000).

[SWb]  B. Schumacher and M. D. Westmoreland, Optimal signal ensembles, Phys. Rev. A 63, 022308 (2001) and quant-ph/9912122 (1999).

[WH]  R. F. Werner and A. S. Holevo, Counterexample to an additivity conjecture for output purity of quantum channels, J. Math. Phys., 43, 4353 (2002).
Problem 11

Continuity of the quantum channel capacity

Problem

The quantum capacity of a noisy quantum channel can be regarded as a function on the space of all channels. Is this function continuous? In other words: If the distance (e.g. with respect to the cb-norm) between two channels is small, is the distance between the corresponding capacities small as well?

Partial Solutions

In [1] it was shown that the quantum capacity as a function of the channel is lower semi-continuous.

Literature

[1] M. Keyl, R.F. Werner, How to correct small quantum errors, in: A. Buchleitner, K. Hornberger (eds.), Coherent Evolution in Noisy Environment, Springer, Lecture Notes in Physics 611, 263 (2002) and quant-ph/0206086 (2002).
Problem 12

Bell Inequalities for long range vacuum correlations

Problem

It is well known [SW] that vacuum fluctuations maximally violate the CHSH-Bell inequalities for suitable spacelike separated observables, and that this violation goes to zero as the two localization regions are moved apart. Decide whether some (necessarily small) violation of the inequalities is possible for regions arbitrarily far apart. For definiteness, consider a massive scalar free relativistic Bose field.

Background

It is known [HC] that the vacuum is not separable at any distance. More recently [VW], it has been shown that an analogue of the “positive partial transpose” condition fails for arbitrary regions at any distance. But the problem as stated above remains open.

Literature

[SW] S. J. Summers and R. F. Werner, The vacuum violates Bell’s inequalities, Phys. Lett. A 110, 257-259 (1985).

[HC] H. Halvorson and R. Clifton, Generic Bell correlation between arbitrary local algebras in quantum field theory, J. Math. Phys. 41, 1711-1717 (2000) and math-ph/9909013 (1999).

[VW] R. Verch and R. F. Werner, Distillability and positivity of partial transposes in general quantum field systems, quant-ph/0403089 (2004).
Problem 13

Mutually unbiased bases

Problem

Determine the maximal number $K$ of orthonormal bases in a $D$-dimensional Hilbert space, which are mutually unbiased in the following sense: If $e^k_i$ denotes the $i$th vector of the $k$th basis, all scalar products $\langle e^k_i, e^n_j \rangle$ with $k \neq n$ have the same absolute value (namely $D^{-1/2}$).

It is known that if $D$ is the power of a prime, $K = D+1$ can be reached, but this is not known for any other composite number. So the problem is already to decide whether there exist $K = 7$ mutually unbiased bases in $D = 6$ dimensions.

Background

The problem comes up in at least three (related) contexts:

(1) **State determination** [IV], [WF]

Suppose we want to determine the density operator of a source by measuring $K$ observables (with $D$ one-dimensional projections each). Each such measurement allows us to determine $D-1$ independent parameters, so we can determine $K(D-1)$ out of $D^2 - 1$ parameters in the density operator. Hence $K = D + 1$ should suffice. In order to achieve best estimation results, the measurements should duplicate no information already contained in other measurements, i.e., the observables should be pairwise complementary, or the bases mutually unbiased in the above sense.

(2) **Cryptography**

Suppose Alice sends $D$-level systems prepared in one of the $D$ pure states $e^k_i$ belonging to a set of $K$ orthonormal bases agreed between Alice and Bob. If Bob measures in the same basis, he can decode the value $i$ perfectly. In cryptography one also wants that if an eavesdropper measures the system in any one of the other bases, she can extract no information whatsoever about $i$. This requires the bases to be mutually unbiased.
Problem 13 Mutually unbiased bases

It is known that a higher error level can be tolerated in the channel for protocols using maximal families of mutually unbiased bases (e.g., the “six state protocol”, $D = 2$, $K = 3$) rather than non-maximal ones (e.g., BB84, using $D = 2$, $K = 2$).

(3) The Mean King [AE]

A ship-wrecked physicist gets stranded on a far-away island that is ruled by a mean king who loves cats and hates physicists since the day when he first heard what happened to Schrödinger’s cat. A similar fate is awaiting the stranded physicist. Yet, mean as he is, the king enjoys defeating physicists on their own turf, and therefore he maliciously offers an apparently virtual chance of rescue.

He takes the physicist to the royal laboratory, a splendid place where experiments of any kind can be performed perfectly. There the king invites the physicist to prepare a certain silver atom in any state she likes. The king’s men will then measure one of the three cartesian spin components of this atom – they’ll either measure $\sigma_x$, $\sigma_y$, or $\sigma_z$ without, however, telling the physicist which one of the measurements is actually done. Then it is again the physicist’s turn, and she can perform any experiment of her choosing. Only after she’s finished with it, the king will tell her which spin component had been measured by his men. To save her neck, the physicist must then state correctly the measurement result that the king’s men had obtained.

Much to the king’s frustration, the physicist rises to the challenge – and not just by sheer luck: She gets the right answer any time the whole procedure is repeated. How does she do it?

More generally, the king’s men might be allowed to perform one out of $K$ complete von Neumann measurements on a $D$-dimensional system. The problem first came up in [VA+], together with a solution for $D = 2$. Solutions involving mutually unbiased bases are presented in [AE], [Ara], [Arb], [EA]. Confer also the experimental realization in [SS+].

Partial Results

H. Barnum [Ba] points out a close connection of this problem with “spherical 2-designs”, which are collections of pure states such that the average of a polynomial of degree 2 on these states equals the integral of the polynomial over all pure states.

For the case $D = 6$ there are a number of different but equivalent formulations of this problem.

A. Pittenger and M. Rubin [PR] give a constructive proof for the case of prime power dimension [WF]. They also address the question of separability and provide an appendix on the necessary parts of algebraic field extensions. Another proof can be found in [KR].
C. Archer [Arc] shows that even generalizations of these constructions do not extend the results beyond prime power dimension.

**Literature**

[AE] Y. Aharonov, B.-G. Englert, *The mean king’s problem: Spin 1*, Z. Naturforsch. 56a, 16 (2001) and quant-ph/0101065 (2001).

[Ara] P.K. Aravind, *Solution to the King’s Problem in prime power dimensions*, Z. Naturforsch. 58a, 2212 (2003) and quant-ph/0210007 (2002).

[Arb] P.K. Aravind, *Best conventional solutions to the King’s Problem*, quant-ph/0306119 (2003).

[Arc] C. Archer, *There is no generalization of known formulas for mutually unbiased bases*, quant-ph/0312204 (2003).

[Ba] H. Barnum, *Information-disturbance tradeoff in quantum measurement on the uniform ensemble and on the mutually unbiased bases*, quant-ph/0205155 (2002).

[EA] B.-G. Englert, Y. Aharonov, *The mean king’s problem: Prime degrees of freedom*, Phys. Lett. A 284, 1 (2001) and quant-ph/0101134 (2001).

[Iv] I. D. Ivanovic, *Geometrical description of quantal state determination*, J. Phys. A 14, 3241 (1981).

[KR] A. Klappenecker, M. Roetteler, *Constructions of Mutually Unbiased Bases*, quant-ph/0309120 (2003).

[PR] A. O. Pittenger and M. H. Rubin, *Mutually Unbiased Bases, Generalized Spin Matrices and Separability*, quant-ph/0308142 (2003).

[SS+] O. Schulz, R. Steinh"ubl, M. Weber, B.-G. Englert, C. Kurtsiefer, H. Weinfurter, *Ascertaining the Values of $\sigma_x$, $\sigma_y$, and $\sigma_z$ of a Polarization Qubit*, quant-ph/0209127 (2002).

[VA+] L. Vaidman, Y. Aharonov, and D. Z. Albert, *How to ascertain the values of $\sigma_x$, $\sigma_y$, and $\sigma_z$ of a spin-1/2 particle*, Phys. Rev. Lett. 58, 1385 (1987).

[WF] W.K. Wootters, B.D. Fields, *Optimal state-determination by mutually unbiased measurements*, Ann. Phys. 191, 363 (1989).
Problem 14

Tough error models

Problem

An error model $E$ is an $e$-dimensional vector space of operators acting on an $n$-dimensional Hilbert space $H$. A quantum code is a subspace $C \subset H$, and is said to correct $E$, if the projector $P_C$ onto $C$ satisfies $P_C A^* B P_C = \lambda(A,B) P_C$ for all $A,B \in E$, and suitable scalars $\lambda(A,B)$.

- Given $e$ and $n$, find the largest $c = c(e,n)$ such that we can assert the existence of a code $C$ of dimension $c$ without further information about $E$.
- Find “tough error models” for which this bound is (nearly) tight.

Background

For an introduction to quantum error-correction see, for example, [KL02].

Partial results

See [KL00], where a lower bound of $c(e,n) > n/(e^2(e^2 + 1))$ is given.

A trivial upper bound on $c(e,n)$ comes from taking orthogonal projections of roughly equal dimension $n/e$ as the error model. Since the channel with these Kraus operators (a Lüders-von Neumann projective measurement) has capacity at most $n/e$, it is impossible to find larger code spaces. Hence $c(e,n) \leq \lceil n/e \rceil$.

Literature

[KL00] E. Knill, R. Laflamme, and L. Viola, Theory of Quantum Error Correction for General Noise, Phys. Rev. Lett. 84, 2525 (2000) and quant-ph/9908066 (1999).
[KL02] E. Knill, R. Laflamme, A. Ashikhmin, H. Barnum, L. Viola, and W. H. Zurek, *Introduction to Quantum Error Correction*, quant-ph/0207170 (2002) and [http://www.c3.lanl.gov/~knill/qip/ecprhtml](http://www.c3.lanl.gov/~knill/qip/ecprhtml).
Problem 15

Separability from spectrum

Problem

For a mixed state $\rho$ on an $NM$-dimensional Hilbert space: Are there any factorizations into an $N$ tensor an $M$ dimensional space with respect to which the state is not separable? This depends only on the spectrum of $\rho$ and the problem is to characterize the spectra for which the answer is "no".

Background

The question arises in the context where we are given a highly mixed state on two quantum systems and the ability to apply any unitary operator. Can an inseperable state be obtained? For sufficiently mixed states, this is not possible.

This problem is different from No. 9, because only the spectrum of $\rho$ and not the spectra of the reductions are to be part of the criterion.

Partial results

See the generic bounds on how close a state has to be to the completely mixed state to be guaranteed not to have entanglement. The paper of Leonid Gurvits and Howard Barnum [GB02] has further relevant results.

For the case of two qubits, the question is solved in [VA01]: Exactly the states with eigenvalues $x_1, x_2, x_3, x_4$ (arranged in decreasing order) obeying $x_1 - x_3 - 2\sqrt{x_2x_4} \leq 0$ cannot be transformed into a state with non-zero entanglement of formation by applying any unitary operator (Theorem 1).

Source

Howard Barnum, Leonid Gurvits, E. K.
Literature

[BC99] S. L. Braunstein, C. M. Caves, R. Jozsa, N. Linden, S. Popescu, and R. Schack, *Separability of very noisy mixed states and implications for NMR quantum computing*, Phys. Rev. Lett. **83**, 1054 (1999) and quant-ph/9908012 (1999).

[GB02] L. Gurvits and H. Barnum, *Size of the Separable Neighborhood of the Maximi-

ally Mixed Bipartite Quantum State*, quant-ph/0204159 (2002).

[VA01] F. Verstraete, K. Audenaert, and B. De Moor, *Maximally entangled mixed

states of two qubits*, Phys. Rev. A **64**, 012316 (2001) and (together with T. De

Bie) quant-ph/0011110 (2000).

[ZH98] K. Zyczkowski, P. Horodecki, A. Sanpera, and M. Lewenstein, *Volume

of the Set of Mixed Entangled States*, Phys. Rev. A **58**, 883 (1998) and

quant-ph/9804024 (1998).
Problem 16

Complexity of product preparations

Problem

What can be said about the algorithmic complexity of preparing $|\psi\rangle|\psi\rangle \ldots$ ($n$ times), asymptotically, as a function of $n$ and the algorithmic complexity of preparing $|\psi\rangle$? Take $|\psi\rangle$ to be a state of $m$ qubits. By algorithmic complexity I mean the number of gates required to prepare the state from $|0\rangle$. This depends on the gate set used so the question concerns asymptotics. For the present purposes, one can take as a gate set all rotations $e^{i\phi\sigma_u}$ where $\sigma_u$ is a product of Pauli matrices. The complexity of this gate is $|\phi|$. It might be useful to consider a version of this question involving an approximation parameter also.

Remark

It may be possible to clone $|\psi\rangle$ more efficiently than to prepare it, given that one knows $|\psi\rangle$.

Source

E. K., Gerardo Ortiz, Rolando Somma.

Literature

The literature on optimal cloning is relevant.
Problem 17

Reversibility of entanglement assisted coding

Problem

For any two quantum channels $S$ and $T$, define the entanglement assisted capacity $C_E(T,S)$ of $T$ for $S$-messages as the supremum of all rates $r$ such that, for large $n$, $rn$ parallel copies of $T$ may be simulated by $n$ copies of $S$, where the simulation involves arbitrary coding and decoding operations using (if necessary) arbitrarily many entangled pairs between sender and receiver, and where the errors go to zero as $n \to \infty$.

Show that $C_E(T,S) = C_E(S,T)^{-1}$.

Background

As for other capacities, the two-step coding inequality $C_E(T,S) \leq C_E(S,R)$ is easy to show. Hence $C_E(T,S) C_E(S,T) \leq 1$. Equality means here, that the two channels are essentially equivalent as a resource for simulating other channels $R$ (apart from a constant factor): $C_E(R,S) = \text{const} C_E(R,T)$ (with const $= C_E(T,S)$). In this case we call $S$ and $T$ reversible for entanglement assisted coding.

For ordinary capacity $C(T,S)$ (without entanglement assistance) reversibility fails in general: When $S$ is an ideal classical 1 bit channel, and $T$ is an ideal 1 qubit quantum channel, we have $C(S,T) = 1$, but $C(T,S) = 0$, because quantum information cannot be sent on classical channels. On the other hand, with entanglement assistance we have $C(S,T) = 2$ by superdense coding and $C(T,S) = 1/2$ by teleportation.

Because all ideal channels $S$ are equivalent as reference channels, we can define $C_E(T) = C_E(T,S_1)$, with $S_1$ the ideal classical 1 bit channel as the entanglement assisted capacity of $T$. For this quantity there is an explicit formula (coding theorem) by [BSST1].

The problem stated above appears in [BSST2] as the “Reverse Shannon Theorem”.

51
Partial Solutions

The problem is solved for the special case of a known “tensor power source”, i.e. a source emitting the same, known, density matrix at each time step. Recent efforts by P. Shor focus on the unknown tensor power source and the known “tensor product source” where the density matrix of the source is a tensor product \[ \text{SH}. \]

Literature

[BSST1] C. H. Bennett, P. W. Shor, J. A. Smolin, and A. V. Thapliyal, *Entanglement-assisted classical capacity of noisy quantum channels*, Phys. Rev. Lett. **83**, 3081 (1999) and quant-ph/9904023 (1999).

[BSST2] C. H. Bennett, P. W. Shor, J. A. Smolin, and A. V. Thapliyal, *Entanglement-assisted capacity of a quantum channel and the reverse Shannon theorem*, quant-ph/0106052 (2001).

[SH] P. W. Shor, private communication (2003).
Problem 18

Qubit bi-negativity

Problem

A little problem introduced in [AMVW02] is the bi-negativity on two qubits: Prove that

$$|\sigma^{T_2}|^{T_2} \geq 0$$

holds for every two-qubit state $\sigma$. Here, $T_2$ denotes the partial transpose with respect to the second system (see also problem 2) and $|.|$ is the operator absolute value, $|x| = \sqrt{x^* x}$.

Literature

[AMVW02] K. Audenaert, B. De Moor, K. G. H. Vollbrecht, and R. F. Werner, Asymptotic Relative Entropy of Entanglement for Orthogonally Invariant States, Phys. Rev. A 66, 032310 (2002) and quant-ph/0204143 (2002).
Problem 19

Stronger Bell Inequalities for Werner states?

| contact: N. Gisin | solved by: – |
| date: 20 Jun 2003 | last progress: – |

Problem

Find Bell Inequalities which are stronger than the CHSH inequalities in the sense that they are violated by a wider range of Werner states.

Background

Recently, Daniel Collins and Nicolas Gisin [CG] found a Bell Inequality and states which violate the new but not the CHSH inequalities. Alas, the range of Werner states violating the new inequality is smaller than that for the CHSH setting.

Literature

[CG] D. Collins, N. Gisin, A Relevant Two Qubit Bell Inequality Inequivalent to the CHSH Inequality, quant-ph/0306129 (2003).
Problem 20

Reversible entanglement manipulation

Problem

The concept of entanglement as a resource motivates the study of its transformation properties under certain classes of operations such as local operations and classical communication (LOCC).

For a finite number of identically prepared quantum systems the manipulation of entanglement under LOCC is generally irreversible, both for pure and mixed states. In the asymptotic limit of infinitely many identical copies of a pure state, in contrast, pure bi-partite entanglement can be interconverted reversibly \cite{BBPS96}. For mixed states, however, this asymptotic reversibility under LOCC operations is lost \cite{VC02, HSS02}.

However, there are more general sets of operations for which entanglement manipulation might become reversible again. One such example is the set of positive partial transpose preserving operations (ppt-operations) \cite{Ra00} which are all those completely positive maps that map the set of ppt-states into itself. It has been shown that under ppt-operations there are some mixed states that can be reversibly converted into pure singlet states in the asymptotic limit \cite{APE03}. This has been proven for the totally anti-symmetric Werner state and weak numerical evidence suggests that this is true for all Werner states \cite{Pl}. On the other hand in \cite{HOH02} it was shown that under certain conditions and for a set of operations (denoted Hyper-set in \cite{HOH02}) that is smaller than ppt-operations and strictly larger than LOCC asymptotic irreversibility persists.

Asymptotic reversibility under a class of operations would lead to a unique entanglement measure and impose a unique ordering on entangled states thereby playing a role similar to entropy in thermodynamics.

The following are open questions:

- Are ppt-operations sufficient to ensure asymptotically reversibly interconversion of all, i.e. pure and mixed, bi-partite entangled states \cite{BFC}?

- What is the smallest non-trivial class of operations that permits asymptotically reversibly interconversion of all, i.e. pure and mixed, bi-partite entangled states \cite{Bett}?
Literature

[BBPS96] C.H. Bennett, H.J. Bernstein, S. Popescu and B. Schumacher, *Concentrating partial entanglement by local operations*, Phys. Rev. A 53, 2046 (1996) and quant-ph/9511030 (1995).

[VC02] G. Vidal and J.I. Cirac, *Irreversibility in asymptotic manipulations of entanglement*, Phys. Rev. Lett. 86, 5803 (2002) and quant-ph/0102036 (2001).

[HSS02] M. Horodecki, A. Sen, and U. Sen, *Rates of asymptotic entanglement transformations for bipartite mixed states: maximally entangled states are not special*, Phys. Rev. A 67, 062314 (2003) and quant-ph/0207031 (2002).

[Ra00] E.M. Rains, *A semidefinite program for distillable entanglement*, IEEE T. Inform. Theory 47, 2921 (2001) and quant-ph/0008047 (2000).

[APE03] K. Audenaert, M.B. Plenio and J. Eisert, *Entanglement cost under positive-partial-transpose-preserving operations*, Phys. Rev. Lett. 90, 027901 (2003) and quant-ph/0207146 (2002).

[HOH02] M. Horodecki, J.Oppenheim and R. Horodecki, *Are the laws of entanglement theory thermodynamical?*, Phys. Rev. Lett. 89, 240403 (2002) and quant-ph/0207177 (2002).

[Pl] M.B. Plenio, unpublished

[BFC] This was boldly conjectured by the author and is in certain circles known as the Big-Fat-Conjecture.

[Bet] The existence of such a class is the subject of a bet between Michal Horodecki and Reinhard Werner.
Problem 21

Bell violation by tensoring

| contact: Y. C. Liang | solved by: – |
|----------------------|--------------|
| date: 08 Feb 2005    | last progress: – |

Problem

Can one find bipartite density operators $\rho_{1,2}$, neither of which violates any CHSH Bell inequality, with the property that $\rho_1 \otimes \rho_2$ does?
Problem 22

Asymptotic cloning is state estimation?

Problem

Fix an arbitrary probability measure on the pure states of a $d$-dimensional quantum system. Let $F(N,M)$ be the optimal single copy fidelity for $N$-to-$M$ cloning transformations, averaged with respect to the given probability measure and over all $M$ clones.

On the other hand, let $F(N,\infty)$ be the best mean fidelity achievable by measuring on $N$ input copies of the state, and repreparing a state according to the measured data. The problem is to decide whether one always gets

$$\lim_{M \to \infty} F(N,M) = F(N,\infty).$$

It is clear that the limit exists, because $F(N,M)$ is non-increasing in $M$. Moreover, the limit will be larger or equal than the right hand side, because estimation with repreparation is a particular cloning method. A weaker, but still interesting version of the problem is whether the above equation becomes true in the limit $N \to \infty$.

Background

In the examples [KW99, RCDM00], where optimal cloner and estimator have been computed, the formula is true. The limit formula is a piece of folklore, partly based on the idea that if one has many clones, one could make a statistical measurement on them and thereby obtain a good estimation. This reasoning is faulty, however, because it neglects the correlations, and possibly the entanglement between the clones.

Literature

[KW99] M. Keyl and R.F. Werner, *Optimal Cloning of Pure States, Judging Single Clones*, J. Math. Phys. 40, 3283 (1999) and quant-ph/9807010 (1998).
Problem 22  Asymptotic cloning is state estimation?

[BCDM00] D. Bruss, M. Cinchetti, G. M. D’Ariano, and C. Macchiavello, *Phase covariant quantum cloning*, Phys. Rev. A 62, 12302 (2000) and quant-ph/9909046 (1999).
Problem 23
SIC POVMs and Zauner’s Conjecture

Problem

We will give three variants of the problem, each being stronger than its predecessor. The terminology of problems 1 and 2 is taken mainly from [1]. For problem 3 see [2] and [3].

Problem 1: SIC-POVMs

A set of \(d^2\) normed vectors \(\{\phi_i\}\), in a Hilbert space of dimension \(d\) constitutes a set of equiangular lines if their mutual inner products

\[|\langle \phi_i | \phi_j \rangle|^2\]

are independent of the choice of \(i \neq j\). It can be shown [1] that

- the associated projection operators sum to a multiple of unity and thus induce a POVM (up to normalization) and that
- these operators are linearly independent and hence any quantum state can be reconstructed from the measurement statistics \(p_i := \text{tr} (|\phi_i\rangle\langle \phi_i| \rho)\) of the POVM.

A POVM that arises in this way is called symmetric informationally complete, or a SIC-POVM for short.

The most general form of the problem is: decide if SIC-POVMs exist in any dimension \(d\).

Problem 2: Covariant SIC-POVMs

For a given basis \(\{|q\rangle\}_{q=0,\ldots,d-1}\) of the Hilbert space, define the shift operator \(X\) and clock operator \(Z\) respectively by the relations

\[
X|q\rangle := |q + 1\rangle \\
Z|q\rangle := e^{i2\pi q/d}|q\rangle,
\]
where arithmetic is modulo $d$. Further, define the Weyl operators

$$w(p, q) = Z(p)X(q)$$

(23.1)

for all $p, q \in \mathbb{Z}_d$. We will refer to the group generated by (23.1) as the Heisenberg group. It is also known as the Weyl-Heisenberg group or Generalized Pauli group.

A vector $|\phi\rangle$ is called a fiducial vector with respect to the Heisenberg group if the set

$$\{w(p, q)|\phi\rangle\langle\phi|w(p, q)^*\}_{p,q=0...d−1}$$

(23.2)

induces a SIC-POVM. Such a SIC-POVM is said to be group covariant. The definition makes sense for any group of order at least $d^2$. However, we will focus on the Heisenberg group in what follows.

The problem: decide if group covariant SIC-POVMs exist in any dimension $d$.

**Problem 3: Zauner’s Conjecture**

The normalizer of the Heisenberg group within the unitaries $U(d)$ is called the Clifford group. There exists an element $z$ of the Clifford group which is defined via its action on the Weyl operators as

$$zw(p, q)z^* = w(q - p, -p).$$

Zauner’s conjecture, as formulated in [3], runs: in any dimension $d$, a fiducial vector can be found among the eigenvectors of $Z$.

**Background**

Besides their mathematical appeal, SIC-POVMs have obvious applications to quantum state tomography. The symmetry condition assures that the possible measurement outcomes are in some sense maximally complementary.

**Partial Results and History**

- In the context of quantum information, the problem seems to have been tackled first by Gerhard Zauner in his doctoral thesis [2] in 1999. To our knowledge, the results were neither published nor translated into English, which caused some confusion in the English literature, as to what Zauner had actually conjectured\(^1\). Zauner analyzed the spectrum of $z$. He listed analytical expressions for fiducial vectors in dimension 2, 3, 4, 5 and numerical expressions for $d = 6, 7$. He noted that for dimension 8 an analytic SIC-POVM is known, which is covariant under the action of the threefold tensor product of the two dimensional Heisenberg group.

\(^1\)Refer e.g. to the first vs. the second version of [2] on the arXiv server.
• Wide interest in the problem arose with the 2003 paper by Renes et. al. [1]. Building on concepts from frame theory, the authors reduced the task of numerically finding fiducial vectors to a non-convex global optimization problem. Using this method, they presented numerical fiducial vectors for all dimensions up to 45 and counted the number of distinct covariant SIC-POVMs up to dimension 7. The question of whether those vectors were eigenstates of a Clifford operation was left open (but see below). Further, four groups other than the Heisenberg group were numerically found to induce SIC-POVMs in the sense of Zauner.

The authors showed that a SIC-POVM corresponds to a spherical 2-design
$$^2$$.

The same assertion was proven by Klappenecker and Rötteler in [4] and was apparently known to Zauner (see Remark 3 in [4]).

• In [5] Grassl used a computer algebra system capable of symbolic calculations to prove Zauner’s conjecture for $$d = 6$$. He remarked that elements of the Clifford group map fiducial vectors onto fiducial vectors. Building on that observation, he could account for all 96 covariant SIC-POVMs that were reported to exist for $$d = 6$$ in [1].

• Appleby in [3] gave a detailed description of the Clifford group and extended it by allowing for anti-unitary operators. He verified that the numeric solutions of [1] were compatible with Zauner’s conjecture and analyzed their stability groups inside the Clifford group
$$^3$$. Appleby goes on to present analytical expressions for fiducial vectors in dimension 7 and 19 and specifies an infinite sequence of dimensions for which he conjectures that solutions can be found more easily.

• Inspired by a construction that links finite geometries to MUBs, there have been some speculations by Wootters about whether SIC-POVMs can be linked to finite affine planes [7]. The same line of thought was pursued by Bengtsson and Ericsson in [8]. However, the existence of such a construction remains an open problem. The results by Grassl are of some relevance here, as it is known that affine planes of order 6 do not exist.

Literature

[1] J. M. Renes, R. Blume-Kohout, A. J. Scott, and C. M. Caves, Symmetric Informationally Complete Quantum Measurements, J. Math. Phys. 45, 2171 (2004) and quant-ph/0310075 (2003).

[2] G. Zauner, Quantendesigns – Grundzüge einer nichtkommutativen Design-theorie, Doctorial thesis, University of Vienna, 1999 (available online at http://www.mat.univie.ac.at/~neum/papers/physpapers.html).

$$^2$$A finite set $$X$$ of unit vectors is a $$t$$-design if the average of any $$t$$-th order polynomial over $$X$$ is the same as the average of that polynomial over the entire unit sphere.

$$^3$$A similar analysis can be performed using discrete Wigner functions, as will be reported in [9].
[3] D. M. Appleby, *SIC-POVMs and the Extended Clifford Group*, quant-ph/0412001 (2004).

[4] A. Klappenecker, and M. Rötteler, *Mutually Unbiased Bases are Complex Projective 2-Designs*, quant-ph/0502031 (2005).

[5] M. Grassl, *On SIC-POVMs and MUBs in dimension 6*, quant-ph/0406175 (2004).

[6] D. Gross, Diploma thesis, University of Potsdam, 2005.

[7] W. K. Wootters, *Quantum measurements and finite geometry*, quant-ph/0406032 (2004).

[8] I. Bengtsson and Å. Ericsson, *Mutually Unbiased Bases and The Complementarity Polytope*, quant-ph/0410120 (2004).
Problem 24

Secret key from all entangled states

| contact: | P. Horodecki |
| solved by: | — |
| date: | 15 Mar 2005 |
| last progress: | — |

Problem

Can all bipartite entangled states be used to generate secret keys?

Background

In [HHHO02], it is shown that some bound entangled states do allow the extraction of a secret key. This is an extreme counterexample to the idea that secret key is best generated from an entangled state by first distilling pure singlets, and using these to get the key. In principle, this provides a new distinction among bipartite states in those which allow key generation and those which do not. The problem asks whether this is really a new distinction.

Literature

[HHHO02] K. Horodecki, M. Horodecki, P. Horodecki, J. Oppenheim, Secure key from bound entanglement, quant-ph/0309110 (2003).
Problem 25

Lockable entanglement measures

Problem

Are two-way distillable entanglement and secret key rate lockable?

Background

An entanglement measure is lockable, if it is extremely sensitive to the loss of a single qubit by one of the partners [HHHO04]. In this paper it is shown that entanglement of formation, entanglement cost, logarithmic negativity, and all convex, asymptotically discontinuous entanglement measures are lockable. More recently [CW05], also squashed entanglement has been shown to be lockable.

Literature

[HHHO04] K. Horodecki, M. Horodecki, P. Horodecki, J. Oppenheim, Locking entanglement measures with a single qubit, quant-ph/0404096 (2004).

[CW05] M. Christandl, A. Winter, Uncertainty, Monogamy, and Locking of Quantum Correlations, quant-ph/0501099 (2005).
Problem 26

Bell inequalities holding for all quantum states

Problem

The setting for this problem is the same as for Problem 1. We consider correlations between \( N \) parties, each of which can perform \( M \) different measurements yielding one of \( K \) possible outcomes each. We can reduce the number of dimensions by considering only those correlation data satisfying the no-signalling constraint, i.e., the choice of a measuring device by one party \( A \) never changes the (joint) probabilities seen by all the other parties, unless results are selected with respect to the outcomes of \( A \). Only obeying no-signalling and positivity constraints, we get the no-signalling polytope \( P \). Contained in it is the convex body \( Q \) of correlations obtainable from a multipartite quantum state with quantum mechanical POVM measurements, and inside \( Q \) the polytope \( C \) of correlations realizable by a classical realistic theory (see Figure).
Problem 26. A:

Consider the part of the boundary of $Q$, which is not already contained in the boundary of $P$. Can one reach all these points by choosing each one of the local Hilbert spaces to be $K$-dimensional, and each measurement as a complete von Neumann measurement (with $K$ orthogonal projectors) on pure states with minimal dimension?

Problem 26. B:

Consider a maximal face of the polytope $C$, which is not also a face of $P$ (a blue line in the above figure). In other words, consider a ”proper Bell inequality”, i.e., a tight linear inequality for local classical correlations, which does not follow from positivity and no-signalling. Then can we find points of $Q$ outside the face? Or, phrased in terms of Bell inequalities, can every proper Bell inequality be violated by quantum correlation data? In the above figure, this asks whether or not a face like the dashed red/blue line can occur.
Problem 27

The power of CGLMP inequalities

Problem

In the setting of problem 26, consider especially the case \((N, M, K) = (2, 2, d)\).

Problem 27.A:

Show that every face of the local polytope \(C\), which is not already contained in a face of the no-signalling polytope \(P\) is of CGLMP type, i.e., an inequality of the form first written out in \[\text{CGLMP}\], but possibly lifted from lower dimensions by fusing together some outcomes.

Problem 27.B:

Numerically, the observables maximally violating the CGLMP inequality on a maximally entangled state are of a very specific form \[\text{DKZ}\], involving measurements in computational basis, transformed by only discrete Fourier transformation and diagonal unitaries \[\text{CGLMP}\]. Show that this is necessarily the case. Show also that these measurements realize the highest resistance of violation to noise, and the best discrimination against classical realism in the sense of Kullback-Leibler divergence \[\text{Gill1}\].

Background

According to the setting \((N, M, K) = (2, 2, d)\), the CGLMP inequality features two parties, \(X\) and \(Y\), with two observables each: \(X_1, X_2\) and \(Y_1, Y_2\), respectively. Each observable has \(d\) possible outcomes. In order to simplify notation, we use the function \(m(x) = x \mod d\) where \(m(x) \in \{0, 1, \ldots, d-1\}\) for integer \(x\) and we denote expectation values by \(E\). The inequality can then be written \[\text{Gill2}\]:

\[
E(m(X_1 - Y_1)) + E(m(Y_1 - X_2)) + E(m(X_2 - Y_2)) + E(m(Y_2 - X_1 - 1)) \geq d - 1.
\]

This statement also suggests a very elegant proof of the inequality \[\text{Gill2}\]: Note that \((X_1 - Y_1) + (Y_1 - X_2) + (X_2 - Y_2) + (Y_2 - X_1 - 1) = -1\). Apply the function \(m\) to both sides, and use \(m(a) + m(b) + m(c) + m(d) \geq m(a + b + c + d)\).
Literature

[CGLMP] D. Collins, N. Gisin, N. Linden, S. Massar, and S. Popescu, *Bell inequalities for arbitrarily high dimensional systems*, Phys. Rev. Lett. **88**, 040404 (2002) and quant-ph/0106024 (2001).

[DKZ] T. Durt, D. Kaszlikowski, and M. Zukowski, *Violations of local realism with quantum systems described by N-dimensional Hilbert spaces up to N = 16*, Phys. Rev. A **64**, 024101 (2001) and quant-ph/011084 (2001).

[Gill1] W. van Dam, P. Grunwald, and R. Gill, *The statistical strength of nonlocality proofs*, quant-ph/0307125 (2003).

[Gill2] R. Gill, private communication.
Problem 28

Local equivalence of graph states

Problem

Decide whether two graph states, which can be mapped into each other by a local unitary, can also be mapped into each other by a local unitary from the Clifford group.

Background

Graph states [Schl] are multiparticle states which are associated with graphs. Each vertex of the graph corresponds to a qubit. The links describe contributions to the phase of the vector components in computational basis:

$$\langle q_1, q_2, \ldots, q_n | \psi \rangle = 2^{-n/2} \prod_{\text{edges } i, j} (-1)^{q_i q_j},$$

where each $q_i = 0, 1$. They can also be characterized [HEB04] by eigenvalue equations of stabilizer form

$$X^{(i)} \prod_{j: \text{edge } i, j} Z^{(j)} \psi = \psi,$$

where $X^{(i)}$ and $Z^{(i)}$ stand for the x- and z-Pauli matrices at vertex $i$. The Pauli operators which leave the vector $\psi$ invariant generate the stabilizer group of the graph state.

The Clifford group consists of those unitary operators $U$, such that $UPU^*$ is a multiple of a Pauli matrix, whenever $P$ is a Pauli matrix. This group is generated by the Hadamard matrix

$$1/\sqrt{2} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix},$$

and the Pauli matrices themselves.

The notion of graph states and the Clifford group can be generalized in a natural way to non-binary systems [Schl].
Partial results

A complete set of invariants for locally unitary equivalent qubit graph states is given by Van den Nest et al. [NDM04b]. A complete set of invariants of the local Clifford group is also known [NDM04d, NDM04e]. However, the relation between these two invariants is still unclear. At least for the qubit case it is conjectured that a complete family of local Clifford invariants exists which is contained in the class of local unitary invariants.

It has been shown by Van den Nest et al. [NDM04c] that for a particular class of qubit graph states local unitary equivalence implies local Clifford equivalence. This class consists of graph states for which stabilizer group $S$ has a particular structure. For instance, all stabilizer states that can be derived from a $GL(4)$-linear code belong to this class.

Examples which do not belong to this class are generalized GHZ states that correspond to star shaped graphs, i.e. one of the qubits is connected with each of the remaining qubits. Nevertheless it has been shown in [NDM04c] that local unitary equivalence implies local Clifford equivalence also in this case.

As outlined by Hein et al. [HEB04], numerical results show that local Clifford equivalence coincides with local unitary equivalence for qubit graph states associated with connected graphs up to 7 vertices.

Literature

[HEB04] M. Hein, J. Eisert, W. Dür and H. J. Briegel, *Multi-party entanglement in graph states*, Phys. Rev. A 69, 062311 (2004) and quant-ph/0307130 (2003).

[NDM04b] M. Van den Nest, J. Dehaene and B. De Moor, *Local invariants of stabilizer codes*, Phys. Rev. A 70, 032323 (2004) and quant-ph/0404106 (2004).

[NDM04c] M. Van den Nest, J. Dehaene and B. De Moor, *On local unitary versus local Clifford equivalence of stabilizer states*, quant-ph/0411115 (2004).

[NDM04d] M. Van den Nest, J. Dehaene and B. De Moor, *Finite set of invariants to characterize local Clifford equivalence of stabilizer states*, quant-ph/0410165 (2004).

[NDM04e] M. Van den Nest, J. Dehaene and B. De Moor, *An efficient algorithm to recognize local Clifford equivalence of graph states*, Phys. Rev. A 70, 034302 (2004) and quant-ph/0405023 (2004).

[Schl] D.-M. Schlingemann, *Cluster states, graphs and algorithms*, Quant. Inf. Comp. 4, 287 (2004) and quant-ph/0305177 (2003).
Problem 29

Entanglement of formation for Gaussian states

Problem

Entanglement of formation is defined as a minimum over all convex decompositions of a bipartite state into pure states (see problem 7). It has been shown for certain two-mode Gaussian states this minimum can be taken over decompositions of the given state into pure states, all of which are translates of the same squeezed Gaussian state, with Gaussian weights.

Show (or disprove) that this is true for all Gaussian states.

Background

If the optimization over convex decompositions of a bipartite state is restricted to decompositions into Gaussian states, entanglement of formation becomes a new entanglement measure, the Gaussian entanglement of formation introduced in [1]. With this, the above question reads: Does Gaussian entanglement of formation equal entanglement of formation for all Gaussian states?

Partial Results

It has been shown in [2] that Gaussian entanglement of formation equals entanglement of formation for two-mode Gaussian states which are symmetric with respect to interchange of the modes.

Literature

[1] M. M. Wolf, G. Giedke, O. Krüger, R. F. Werner, and J. I. Cirac, Gaussian Entanglement of Formation, Phys. Rev. A 69, 052320 (2004) and quant-ph/0306177 (2003).
[2] G. Giedke, M. M. Wolf, O. Krüger, R. F. Werner, and J. I. Cirac, *Entanglement of formation for symmetric Gaussian states*, Phys. Rev. Lett. 91, 107901 (2003) and quant-ph/0304042 (2003).