ON THE OSCILLATION BEHAVIOR OF SOLUTIONS TO THE
ONE-DIMENSIONAL HEAT EQUATION

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Abstract. We study the oscillation behavior of solutions to the one-dimensional heat equation and give some interesting examples. We also demonstrate a simple ODE method to find explicit solutions of the heat equation with certain particular initial conditions.

1. Introduction. Consider the one-dimensional heat equation with initial condition
\[
\begin{align*}
& \quad u_t(x,t) = u_{xx}(x,t), \quad x \in \mathbb{R}, \quad t > 0 \\
& u(x,0) = \varphi(x), \quad x \in \mathbb{R}.
\end{align*}
\]
(1)

It is known that if \( \varphi(x) \) is a bounded continuous function defined on \( \mathbb{R} \), then the function given by convolution integral
\[
u(x,t) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi t}} e^{-\frac{(x-y)^2}{4t}} \varphi(y) dy, \quad x \in \mathbb{R}, \quad t > 0
\]
(2)
is a smooth solution of the heat equation on \( \mathbb{R} \times (0, \infty) \) with \( \lim_{(x,t) \to (x_0,0)} u(x,t) = \varphi(x_0) \) for any \( x_0 \in \mathbb{R} \). We all know that the solution of the heat equation is not unique even if the initial data \( \varphi(x) \) is given (unless we impose the growth condition of \( u(x,t) \) as \( |x| \to \infty \)). From now on, when we say \( u(x,t) \) is “the solution” of the heat equation with \( u(x,0) = \varphi(x), \ x \in \mathbb{R} \), we always mean that it is given by the convolution integral (2).

For bounded continuous initial data \( \varphi(x) \), there is a nice property of determining, for fixed \( x_0 \), whether we have the convergence of \( u(x_0,t) \) to 0 as \( t \to \infty \). Without loss of generality, it suffices to look at the case \( x_0 = 0 \) and we have the well-known result:

Theorem 1. (See [2, 3, 6].) Assume \( \varphi \in C^0(\mathbb{R}) \cap L^\infty(\mathbb{R}) \) and \( u(x,t) \) is a solution of the heat equation (1) given by formula (2). Then
\[
\lim_{t \to \infty} u(0,t) = 0 \quad \text{if and only if} \quad \lim_{R \to \infty} \frac{1}{2R} \int_{-R}^{R} \varphi(\theta) d\theta = 0.
\]
(3)
Moreover,
\[ \lim_{t \to \infty} \left( \sup_{x \in \mathbb{R}} |u(x,t)| \right) = 0 \]  
(4)
if and only if
\[ \lim_{R \to \infty} \left( \sup_{x \in \mathbb{R}} \frac{1}{2R} \left| \int_{-R}^{x+R} \varphi(\theta) \, d\theta \right| \right) = 0. \]  
(5)

Remark 2. Note that, in order for \( \varphi(x) \) to satisfy the limit in (3), it does not have to decay to zero as \( |x| \to \infty \). Moreover, one can check that if \( \lim_{|x| \to \infty} \varphi(x) = 0 \), then it also satisfies (5) and so we have (4).

Remark 3. As the heat equation is linear, Theorem 1 also implies
\[ \lim_{t \to \infty} u(a,t) = b \text{ if and only if } \lim_{R \to \infty} \frac{1}{2R} \int_{a-R}^{a+R} \varphi(\theta) \, d\theta = b, \]  
(6)
where \( a, b \) are any two numbers.

Remark 4. As \( \varphi \) is bounded, if it satisfies the limit in (3), we also have
\[ \lim_{R \to \infty} \frac{1}{2R} \int_{a-R}^{a+R} \varphi(\theta) \, d\theta = 0 \]
for any number \( a \). By Remark 3, if we have \( \lim_{t \to \infty} u(0,t) = 0 \), we also have \( \lim_{t \to \infty} u(a,t) = 0 \) for any \( a \in \mathbb{R} \).

Remark 5. It is easy to find a bounded function \( \varphi(x) \) satisfying the limit in (3), but not (5). For example, take \( \varphi(x) = \tan^{-1} x, \ x \in \mathbb{R}. \)

Remark 6. There is a good motivation for the result (3) and is worth mentioning. If the initial data \( \varphi(x) \) is a 2\( \pi \)-periodic smooth function on \( \mathbb{R} \), then it satisfies the identity
\[ \lim_{R \to \infty} \frac{1}{2R} \int_{-R}^{R} \varphi(x) \, dx = \lim_{R \to \infty} \frac{1}{2R} \int_{0}^{2\pi} \varphi(\theta) \, d\theta = \frac{1}{2\pi} \int_{0}^{2\pi} \varphi(\theta) \, d\theta. \]  
(7)
On the other hand, the Fourier series expansion of the solution \( u(x,t) \) has the form
\[ u(x,t) = \frac{a_0(0)}{2} + \sum_{n=1}^{\infty} e^{-n^2t} \left( a_n(0) \cos nx + b_n(0) \sin nx \right), \quad (x,t) \in \mathbb{R} \times [0, \infty) \]  
(8)
where
\[
\left\{ \begin{array}{l}
a_n(0) = \frac{1}{\pi} \int_{0}^{2\pi} \varphi(\theta) \cos n\theta \, d\theta, \quad n = 0, 1, 2, 3, \ldots \\
b_n(0) = \frac{1}{\pi} \int_{0}^{2\pi} \varphi(\theta) \sin n\theta \, d\theta, \quad n = 1, 2, 3, \ldots .
\end{array} \right.
\]
From the expansion we see that \( u(x,t) \) converges to the constant \( \frac{1}{2\pi} \int_{0}^{2\pi} \varphi(\theta) \, d\theta \) for all \( x \in (-\infty, \infty) \).

As we shall demonstrate, the above theorem fails if the initial data \( \varphi(\theta), \ \theta \in \mathbb{R} \), is not bounded (see Remark 16 below). The purpose of this paper is to give some interesting examples (for both bounded and unbounded \( \varphi \)) of non-convergence of \( u(0,t) \) as \( t \to \infty \). When \( \varphi \) is unbounded but with bounded \( H \) (see (14)), the convergence of \( \lim_{R \to \infty} \int_{-R}^{R} \varphi(\theta) \, d\theta = 0 \) can imply \( \lim_{t \to \infty} u(0,t) = 0 \), but not the converse. See Lemma 13 below.
2. The case when \( \varphi \) is bounded. In this section, we assume \( \varphi \in C^0(\mathbb{R}) \cap L^\infty(\mathbb{R}) \). We first note the following relation between the value \( u(0, t) \) and the average integral \( \frac{1}{2\pi} \int_{-R}^{R} \varphi(\theta) \, d\theta \):

Lemma 7. Assume \( \varphi \in C^0(\mathbb{R}) \cap L^\infty(\mathbb{R}) \) and \( u(x, t) \) is given by (2). Then we have the identity

\[
 u(0, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-z^2} \varphi \left( \sqrt{4t}z \right) \, dz = \frac{4}{\sqrt{\pi}} \int_{0}^{\infty} z^2 e^{-z^2} \left( \frac{1}{2\sqrt{4tz}} \int_{-\sqrt{4tz}}^{\sqrt{4tz}} \varphi(\theta) \, d\theta \right) \, dz, \quad \forall \, t \in (0, \infty). \tag{9}
\]

Proof. The first identity is clear. For the second identity, let \( M > 0 \) be a constant with \( |\varphi(x)| \leq M \) for all \( x \in \mathbb{R} \). Define \( U(r) = \int_{0}^{r} \varphi(\theta) \, d\theta \), \( r \in \mathbb{R} \), which is a differentiable function with \( U(0) = 0 \) and \( |U'(r)| = |\varphi(r)| \leq M \) for all \( r \). We have

\[
 \left| \frac{1}{2R} \int_{-R}^{R} \varphi(\theta) \, d\theta \right| = \left| \frac{U(R) - U(- R)}{2R} \right| \leq M, \quad \forall \, R \in (0, \infty), \tag{10}
\]

and \( \lim_{R \to 0} \frac{U(R) - U(- R)}{2R} = \varphi(0) \). By integration by parts, we obtain for fixed \( t > 0 \) the following:

\[
 u(0, t) = \lim_{R \to \infty} \left( \frac{1}{4\pi t} \int_{-R}^{R} e^{-z^2/4t} \varphi(z) \, dz \right) = \lim_{R \to \infty} \left( \frac{1}{\sqrt{\pi t}} \int_{-R}^{R} e^{-z^2/\pi t} dU(\theta) \right)
\]

\[
 = \lim_{R \to \infty} \left[ \frac{1}{\sqrt{\pi t}} \int_{-R}^{R} e^{-z^2/\pi t} \left( \frac{U(R) - U(-R)}{2R} \right) + \frac{1}{\sqrt{\pi t}} \int_{-R}^{R} U(\theta) e^{-z^2/\pi t} \theta \, d\theta \right]
\]

\[
 = \lim_{R \to \infty} \left[ \frac{4}{\sqrt{\pi}} \int_{0}^{\infty} z^2 e^{-z^2} \left( \frac{1}{2\sqrt{4tz}} \int_{-\sqrt{4tz}}^{\sqrt{4tz}} \varphi(\theta) \, d\theta \right) \, dz \right], \quad \forall \, t \in (0, \infty). \tag{11}
\]

The proof is done. \( \square \)

By Lemma 7, we can prove the following:

Corollary 8. Assume \( \varphi \in C^0(\mathbb{R}) \cap L^\infty(\mathbb{R}) \) and the limit \( \lim_{R \to \infty} \frac{1}{2\pi} \int_{-R}^{R} \varphi(\theta) \, d\theta \) does not exist. Let \( u(x, t) \) be given by (2). Then we have

\[
 \lim_{R \to \infty} \frac{1}{2R} \int_{-R}^{R} \varphi(\theta) \, d\theta \leq \lim_{t \to \infty} \inf u(0, t) < \lim_{t \to \infty} \sup u(0, t) \leq \lim_{R \to \infty} \frac{1}{2R} \int_{-R}^{R} \varphi(\theta) \, d\theta. \tag{12}
\]

Remark 9. Since \( \varphi(x) \) is a bounded function and so is \( u(x, t) \), the above four limit values all exist and are all finite. From now on, we shall use \( p, \alpha, \beta, q \) to denote the four limit values in (12) respectively.
Proof. This is an easy consequence of representation formula (9). For each fixed \( z > 0 \), by (10) and Fatou’s lemma, we have
\[
\frac{4}{\sqrt{\pi}} \int_{0}^{\infty} z^2 e^{-z^2} \left( \liminf_{t \to \infty} \frac{1}{2 \sqrt{4t}} \int_{-\sqrt{4t}}^{\sqrt{4t}} \varphi (\theta) \, d\theta \right) \, dz \\
\leq \liminf_{t \to \infty} \frac{4}{\sqrt{\pi}} \int_{0}^{\infty} z^2 e^{-z^2} \left( \frac{1}{2 \sqrt{4t}} \int_{-\sqrt{4t}}^{\sqrt{4t}} \varphi (\theta) \, d\theta \right) \, dz = \liminf_{t \to \infty} u (0, t),
\]
where we note that
\[
\liminf_{t \to \infty} \frac{1}{2 \sqrt{4t}} \int_{-\sqrt{4t}}^{\sqrt{4t}} \varphi (\theta) \, d\theta = \liminf_{R \to \infty} \frac{1}{2R} \int_{-R}^{R} \varphi (\theta) \, d\theta, \quad \forall \ z > 0.
\]
Hence the first inequality in (12) follows if we note that \( \frac{4}{\sqrt{\pi}} \int_{0}^{\infty} z^2 e^{-z^2} \, dz = 1 \). The second inequality in (12) is due to Theorem 1. The proof of the third inequality in (12) is also due to Fatou’s lemma.

2.1. Some interesting examples for the bounded case. In the following we will give two bounded examples of non-convergence of \( u (0, t) \) as \( t \to \infty \). One has \( p = \alpha < \beta = q \) and the other has \( p < \alpha < \beta < q \). We first do some general discussion.

For given initial data \( \varphi \in \mathcal{C}^0 (\mathbb{R}) \cap L^\infty (\mathbb{R}), \ |\varphi| \leq M, \) let
\[
H (x) = \frac{1}{2x} \int_{-x}^{x} \varphi (\theta) \, d\theta, \quad x \in (0, \infty).
\]
We see that \( H (x) : (0, \infty) \to \mathbb{R} \) is a bounded differentiable function on \( (0, \infty) \), with \( \lim_{x \to 0} H (x) = \varphi (0), \ |H (x)| \leq M. \) \( H (x) \) is a continuous function on \( [0, \infty) \) if we define \( H (0) = \varphi (0). \) The right hand side of (14) can be defined for \( x \in (-\infty, 0) \).

Hence if we allow \( x \) to be negative in (14), the function \( H (x) \) becomes an even continuous function on \( \mathbb{R} \) with \( H (0) = \varphi (0). \)

By the identity \( \int_{-x}^{x} \varphi (\theta) \, d\theta = 2xH (x) \) and taking derivative with respect to \( x \), we find
\[
\varphi (x) + \varphi (-x) = 2H (x) + 2xH' (x), \quad \forall \ x \in (0, \infty).
\]
Hence the function \( H' (x) \) must satisfy \( H' (x) = O \left( \frac{1}{x^2} \right) \) as \( x \to \infty \), i.e.
\[
\lim_{x \to \infty} H' (x) = \lim_{x \to \infty} \frac{\varphi (x) + \varphi (-x) - 2H (x)}{2x} = 0.
\]
In general, \( H (x) \) cannot be differentiable at \( x = 0 \) unless \( \varphi (\theta) \) is differentiable at \( \theta = 0 \). For example, if we choose \( \varphi (\theta) = |\theta|, \ \theta \in \mathbb{R}, \) we get \( H (x) = |x|/2, \ x \in \mathbb{R}, \) which is not differentiable at \( x = 0 \). On the other hand, if \( \varphi (\theta) \) is differentiable at \( \theta = 0 \), then we can express \( \varphi (\theta) \) near \( \theta = 0 \) as \( \varphi (\theta) = \varphi (0) + \varphi' (0) \theta + h (\theta) \theta \), for some function \( h (\theta) \) with \( \lim_{\theta \to 0} h (\theta) = 0. \) Hence for small \( x > 0 \), by the mean value theorem for integrals, we have
\[
\frac{H (x) - H (0)}{x} = \frac{1}{2x^2} \int_{-x}^{x} [\varphi (\theta) - \varphi (0)] \, d\theta = \frac{1}{2x^2} \int_{-x}^{x} h (\theta) \, d\theta = h (\theta_*) \frac{\theta_*}{x},
\]
for some number \( \theta_* \in [-x, x] \). Hence we conclude
\[
\left| \frac{H (x) - H (0)}{x} \right| \leq |h (\theta_*)| \frac{\theta_*}{x} \leq |h (\theta_*)| \to 0 \quad \text{as} \quad x \to 0^+,
\]
and similar result holds for \( x \to 0^- \). Therefore \( H (x) \) is differentiable at \( x = 0 \) with \( H' (0) = 0. \) However, the converse is not true. One can take \( \varphi (\theta) \) for small
\(|\theta|\) to be the function \(\varphi (\theta) = \theta \sin \frac{1}{\theta}, \ \varphi (0) = 0\), then for small \(x > 0\) the following holds
\[
\frac{H (x) - H (0)}{x} = \frac{1}{2x^2} \int_{-x}^{x} \theta \sin \frac{1}{\theta} d\theta = \frac{1}{2x^2} \left( 2x^3 \cos \frac{1}{x} - \int_{-x}^{x} 3\theta^2 \cos \frac{1}{\theta} d\theta \right),
\]
which gives the same limit result as in (17) and similar result holds for \(x \to 0^+\).

Thus we conclude \(H' (0) = 0\), but the function \(\varphi (\theta)\) is not differentiable at \(\theta = 0\).

2.1.1. Example 1: \(p = \alpha < \beta = q\) for arbitrary values of \(\alpha, \beta\). There are two nice examples for the case \(p = \alpha = -1, \ \beta = q = +1\), one in [1] and the other in [4].

Our example is very different from that in [1] and in [4].

Recall the representation formula (9), which is
\[
\frac{d^2}{dt^2} u (0, t) = 4 \sqrt{\pi} \int_{0}^{\infty} z^2 e^{-z^2} H (\sqrt{4t} z) dz, \ \forall t \in (0, \infty).
\]

The key observation is the following. If the function \(H (\sqrt{4t} z)\) has no oscillation at all (i.e. \(H (x) \equiv C\) for some constant \(C\)), then \(u (0, t)\) is equal to the same constant \(C\) for all \(t \in (0, \infty)\). On the other hand, if \(H (\sqrt{4t} z)\) oscillates too fast as \(t \to \infty\), say it oscillates between \(-1\) and \(+1\) rapidly, then similar to the Reimann Lebesgue Lemma, we may have \(u (0, t) \to 0\) as \(t \to \infty\).

Since we want to have \(p = \alpha\) and \(\beta = q\), we must choose \(H (x)\) to be oscillatory (so that \(p < q\)), but as slow as possible (so that we can maintain \(p = \alpha\) and \(q = \beta\)), and it also has to satisfy the condition \(H' (x) = O \left( \frac{1}{x^2} \right)\) as \(x \to \infty\). Therefore, we choose the following function which has a very very slow oscillation as \(x \to \infty\) :
\[
H (x) = \frac{1}{2x} \int_{-x}^{x} \varphi (\theta) d\theta = \sin \left( \log \left( \log (x + 2) \right) \right), \quad x \in (0, \infty),
\]
where
\[
H' (x) = \frac{1}{(x + 2) \log (x + 2)} \cos \left( \log \left( \log (x + 2) \right) \right), \quad x \in (0, \infty),
\]
and by the identity (15), we can solve \(\varphi (\theta)\) to get
\[
\varphi (\theta) = \begin{cases} 
\sin \left( \log \left( \log |\theta| + 2 \right) \right) \\
+ \frac{|\theta|}{(\log |\theta| + 2)^2} \cos \left( \log \left( \log |\theta| + 2 \right) \right), \quad \theta \in (-\infty, \infty).
\end{cases}
\]

Note that the above function lies in the space \(\varphi \in C^0 (\mathbb{R}) \cap L^\infty (\mathbb{R})\). By (18), the solution \(u\) of the heat equation with initial data (20) satisfies
\[
u (0, t) = \frac{4}{\sqrt{\pi}} \int_{0}^{\infty} z^2 e^{-z^2} \sin \left[ \log \left( \log \left( \sqrt{4t} z + 2 \right) \right) \right] dz,
\]
\[
\quad = \frac{4}{\sqrt{\pi}} \int_{0}^{\infty} z^2 e^{-z^2} \sin \left[ \log \sqrt{4t} + \tau (t, z) \right] dz, \quad t \in (0, \infty),
\]
where
\[
\tau (t, z) = \log \left( 1 + \frac{1}{\log \sqrt{4t}} \log \left( z + \frac{2}{\sqrt{4t}} \right) \right), \quad t, \ z \in (0, \infty).
\]

It satisfies \(\lim_{t \to \infty} \tau (t, z) = 0\) for all \(z \in (0, \infty)\). Hence by Lebesgue Dominated Convergence Theorem, we have (note that \(\frac{4}{\sqrt{\pi}} \int_{0}^{\infty} z^2 e^{-z^2} dz = 1\))
\[
\lim_{t \to \infty} \left| u (0, t) - \sin \left( \log \left( \log \sqrt{4t} \right) \right) \right| = 0,
\]

(21)
where we note that the profile of \( u(0,t) \), as \( t \to \infty \), is exactly the same as the profile of \( H(x) = \sin[\log(\log(x+2))] \) as \( x \to \infty \).

For any two numbers \( \alpha < \beta \), if we pick

\[
\psi(\theta) = \frac{\beta - \alpha}{2} \varphi(\theta) + \frac{\beta + \alpha}{2}, \quad \theta \in (-\infty, \infty),
\]

where \( \varphi(\theta) \) is from (20), we will have

\[
H(x) = \frac{1}{2x} \int_{-x}^{x} \psi(\theta) d\theta = \frac{\beta - \alpha}{2} \sin[\log(\log(x+2))] + \frac{\beta + \alpha}{2}, \quad x \in (0, \infty)
\]

which gives

\[
\lim_{t \to \infty} \left| u(0,t) - \left[ \frac{\beta - \alpha}{2} \sin\left( \log\left( \log \sqrt{4t} \right) \right) + \frac{\beta + \alpha}{2} \right] \right| = 0.
\]

By (23) and (24), we clearly have \( p = \alpha, q = \beta \), where \( \alpha < \beta \) can be arbitrary values.

2.1.2. Example 2: \( p < \alpha < \beta < q \) for some particular values of \( p, \alpha, \beta, q \). The next example is to demonstrate that a strict inequality between \( p, \alpha \) can happen, and similarly for \( \beta, q \). Again, we look for \( H(x) \) first and then find the corresponding \( \varphi(\theta) \). We still need \( H(x) \) to satisfy the condition \( H'(x) = O\left(\frac{1}{x}\right) \) as \( x \to \infty \) and to oscillate (so that \( p < q \)) between two numbers, but not as slow as in the previous example (so that \( p \) increases to \( \alpha \) and \( q \) decreases to \( \beta \)). With this in mind, we now choose

\[
H(x) = \frac{1}{2x} \int_{-x}^{x} \varphi(\theta) d\theta = \sin(\log(x+1)), \quad x \in (0, \infty),
\]

where

\[
H'(x) = \frac{1}{x+1} \cos(\log(x+1)), \quad x \in (0, \infty),
\]

and by the identity (15), we can solve \( \varphi(\theta) \) to get

\[
\varphi(\theta) = \sin(\log(|\theta|+1)) + \frac{|\theta|}{|\theta|+1} \cos(\log(|\theta|+1)), \quad \forall \theta \in (-\infty, \infty),
\]

where \( \varphi \in C^0(\mathbb{R}) \cap L^\infty(\mathbb{R}) \). By (18), we have

\[
u(0,t) = \frac{4}{\sqrt{\pi}} \int_0^\infty z^2 e^{-z^2} \sin\left( \log\left( \sqrt{4t} z + 1 \right) \right) dz
\]

\[
= \frac{4}{\sqrt{\pi}} \int_0^\infty z^2 e^{-z^2} \left[ \log \sqrt{4t} + \log \left( z + \frac{1}{\sqrt{4t}} \right) \right] dz, \quad t \in (0, \infty),
\]

which implies

\[
u(0,t) = \left( \sin \left( \log \sqrt{4t} \right) \right) \cdot \left( \frac{4}{\sqrt{\pi}} \int_0^\infty z^2 e^{-z^2} \cos\left( \log \left( z + \frac{1}{\sqrt{4t}} \right) \right) dz \right), \quad \forall t > 0.
\]

Here the dot sign “·” in (27) means the inner product in \( \mathbb{R}^2 \). By Lebesgue Dominated Convergence Theorem, we have

\[
\lim_{t \to \infty} \frac{4}{\sqrt{\pi}} \int_0^\infty z^2 e^{-z^2} \cos\left( \log \left( z + \frac{1}{\sqrt{4t}} \right) \right) dz
\]

\[
= \frac{4}{\sqrt{\pi}} \int_0^\infty z^2 e^{-z^2} \cos(\log z) dz \approx 0.89225
\]

(28)
and
\[
\lim_{t \to \infty} \frac{4}{\sqrt{\pi}} \int_0^\infty z^2 e^{-z^2} \sin \left[ \log \left( z + \frac{1}{\sqrt{4t}} \right) \right] \, dz = \frac{4}{\sqrt{\pi}} \int_0^\infty z^2 e^{-z^2} \sin (z) \, dz \approx 0.03095.
\] (29)

Therefore, we conclude the limit
\[
\lim_{t \to \infty} |u(0, t) - \left[ 0.89225 \sin \left( \log \sqrt{4t} \right) + 0.03095 \cos \left( \log \sqrt{4t} \right) \right]| = 0
\] (30)
and get
\[
\alpha = \liminf_{t \to \infty} u(0, t) \approx -0.89279, \quad \beta = \limsup_{t \to \infty} u(0, t) \approx 0.89279.
\]

Combined with the fact that \( p = -1, \ q = 1 \), the example is complete.

**Remark 10.** Since \(|\varphi|\) is bounded by a constant \( M \), by (2) it is easy to derive the gradient estimate
\[
|u_x(x, t)| \leq \frac{M}{\sqrt{\pi} t}, \quad \forall \ (x, t) \in \mathbb{R} \times (0, \infty),
\] (31)
which, for each fixed \( x \), implies
\[
\lim_{t \to \infty} |u(x, t) - U(t)| \leq \lim_{t \to \infty} (|u(x, t) - u(0, t)| + |u(0, t) - U(t)|) = 0,
\] (32)
where \( U(t) \) is the function in (30). Thus for each fixed \( x \), \( u(x, t) \) also approaches to the same function \( U(t) \) as \( t \to \infty \). Moreover, due to (31), the convergence is uniform in \( x \in K \) for any compact set \( K \subset \mathbb{R} \).

**Remark 11.** This is to give a comparison. The solution of the heat equation with the initial **very slow oscillation** profile \( H(x) \) in (25) converges to a slightly changed profile in (30). On the other hand, the solution with the initial **very very slow oscillation** profile \( H(x) \) in (19) converges to an unchanged profile in (21).

**Remark 12.** Is there an initial data \( \varphi \in C^0(\mathbb{R}) \cap L^\infty(\mathbb{R}) \) so that under the heat equation \( u_t = u_{xx} \) we have \( \lim_{t \to \infty} |u(0, t) - P(t)| = 0 \), where \( P(t) \) is a non-constant \( 2\pi \)-periodic function? The answer is **no**. Since, by (18), we have
\[
u_t(0, t) = \frac{4}{\sqrt{\pi}} \int_0^\infty z^2 e^{-z^2} H'(\sqrt{4t}z) \frac{z}{\sqrt{t}} \, dz, \quad \forall \ t \in (0, \infty),
\]
where \( H(x) \) satisfies
\[
H'(x) = \frac{\varphi(x) + \varphi(-x) - 2H(x)}{2x}, \quad x \in (0, \infty); \quad H'(x) = O \left( \frac{1}{x} \right) \quad \text{as} \quad x \to \infty,
\]
we must have \( u_t(0, t) = O \left( \frac{1}{\sqrt{t}} \right) \to 0 \) as \( t \to \infty \). Therefore, it is impossible for \( u(0, t) \) to converge to a non-constant \( 2\pi \)-periodic function as \( t \to \infty \). In view of this, it seems reasonable to see that, instead of converging to \( A \sin t + B \cos t \), we get convergence to \( A \sin (\log \sqrt{4t}) + B \cos (\log \sqrt{4t}) \) for some constants \( A, \ B \). On the other hand, if we allow the initial data \( \varphi \) to be **unbounded**, then the function
\[
u(x, t) = \lambda + Ae^{\frac{x}{\sqrt{2}}} \cos \left( t + \frac{x}{\sqrt{2}} \right) + Be^{\frac{x}{\sqrt{2}}} \sin \left( t + \frac{x}{\sqrt{2}} \right), \quad (x, t) \in \mathbb{R}^2,
\]
where $\lambda$, $A$, $B$ are arbitrary constants, is a time-periodic solution of the heat equation with

\[
\begin{align*}
    u(x, 0) &= \lambda + Ae^{\frac{x}{\sqrt{2}}} \cos \left( \frac{x}{\sqrt{2}} \right) + Be^{\frac{x}{\sqrt{2}}} \sin \left( \frac{x}{\sqrt{2}} \right), \quad x \in \mathbb{R} \\
    u(0, t) &= \lambda + A \cos t + B \sin t, \quad t \in \mathbb{R}.
\end{align*}
\]

The function $u(0, t)$ is a non-constant $2\pi$-periodic function and so is $u(x, t)$ for each fixed $x$.

3. The case when $\varphi$ is unbounded. When the initial data $\varphi(x)$ is unbounded (in this paper we always assume $\varphi(x)$ is, at least, a continuous function), Theorem 1 fails. In this section, we assume that the unbounded function $\varphi(x)$ satisfies the following growth condition

\[
|\varphi(x)| = o \left( e^{\lambda|x|^2} \right) \text{ as } |x| \to \infty, \quad \text{for any constant } \lambda > 0. \tag{33}
\]

With this, formula (2) still represents a smooth solution of the heat equation on $\mathbb{R} \times (0, \infty)$ with $\lim_{(x,t)\to(x_0,0)} u(x,t) = \varphi(x_0)$ for any $x_0 \in \mathbb{R}$.

Similar to Lemma 7, for the unbounded case, we have:

**Lemma 13.** Assume $\varphi \in C^0(\mathbb{R})$ is unbounded and satisfies (33) and $u(x,t)$ is given by (2). Then the representation formula (9) is still correct. Moreover, if the corresponding function $H(x)$ is bounded on $x \in (0, \infty)$, then we have

\[
\liminf_{R \to \infty} \frac{1}{2R} \int_{-R}^{R} \varphi(\theta) d\theta \leq \liminf_{t \to \infty} \sup_{t \to \infty} u(0,t) \leq \limsup_{t \to \infty} \sup_{t \to \infty} \frac{1}{2R} \int_{-R}^{R} \varphi(\theta) d\theta.
\]

In particular, if $\lim_{R \to \infty} \frac{1}{2R} \int_{-R}^{R} \varphi(\theta) d\theta = 0$, it implies $\lim_{t \to \infty} u(0,t) = 0$.

**Remark 14.** In the unbounded case, each term in (34) may be finite or infinite.

**Remark 15.** It is easy to find an unbounded function $\varphi$ with bounded $H$. See Example 1-4 in Section 3.1 below. For Example 5-7 in Section 3.1, the function $H(x)$ is unbounded with $\liminf_{x \to \infty} H(x) = -\infty$ and $\limsup_{x \to \infty} H(x) = \infty$; hence (34) is automatically satisfied.

**Proof.** If $\varphi$ satisfies (33), then by the mean value theorem for integrals, estimate (10) becomes

\[
\left| \frac{U(R) - U(-R)}{2R} \right| = \left| \frac{1}{2R} \int_{-R}^{R} \varphi(\theta) d\theta \right| \leq |\varphi(\theta(R))|,
\]

for some point $\theta(R) \in [-R, R]$. For any fixed $\lambda > 0$ and $\varepsilon > 0$, there exists number $N > 0$ such that if $|x| \geq N$, we have $|\varphi(x)| e^{-\lambda x^2} \leq \varepsilon$. As $R \to \infty$, we have either $|\theta(R)| < N$ or $|\theta(R)| \geq N$. In the first case, we have $|\varphi(\theta(R))| e^{-\lambda R^2} \leq (\max_{[-N,N]} |\varphi|) e^{-\lambda N^2} \leq \varepsilon$. In the second case, we have

\[
\frac{|\varphi(\theta(R))|}{e^{\lambda R^2}} = \frac{\varphi(\theta(R)) e^{\lambda \theta^2(R)}}{e^{\lambda R^2}} \leq \varepsilon.
\]

The above two estimates will imply

\[
\lim_{R \to \infty} \frac{1}{2R} e^{\frac{\theta^2}{2}} \int_{-R}^{R} \left( U(R) - U(-R) \right) = 0, \quad \forall \, t \in (0, \infty)
\]

and the identity (11) still holds and we still have the representation formula (9).
We choose \( \lim \) 

3.1. Some interesting examples for the unbounded case. Again, we construct some interesting examples. The examples given below all have the initial condition \( \varphi (x) \) satisfying the growth condition \((33)\) and we can evaluate the four limit values (call them \( p, \alpha, \beta, q \)) in \((34)\) directly. They all satisfy the inequality \((34)\).

We first note that if we add an unbounded odd function, say \( x \cos x \), to the initial data in the two examples in Section 2.1, it will not affect the values of \( \frac{1}{2\pi} \int_{-R}^{R} \varphi (\theta) d\theta \) and \( u (0, t) \). Hence the oscillation behavior of the two examples in Section 2.1 can also occur in the unbounded case. Thus, the following two examples can be attained:

**Example 1:** \( p = \alpha < \beta = q \) for arbitrary values of \( \alpha, \beta \).

**Example 2:** \( p < \alpha < \beta < q \) for some particular values of \( p, \alpha, \beta, q \).

3.1.1. Example 3: \( p = \alpha - \lambda < \alpha = \beta < \beta + \lambda = q \) for arbitrary values of \( \alpha, \lambda \), where \( \lambda > 0 \). For arbitrary numbers \( \alpha, \lambda > 0 \), we take \( \varphi (x) = \alpha + \lambda x \sin x, \ x \in \mathbb{R} \), which is an even unbounded function. We have

\[
\frac{1}{2\pi} \int_{-R}^{R} \varphi (\theta) d\theta = \alpha - \lambda \cos R + \frac{\lambda \sin R}{R}, \quad R > 0, \quad (35)
\]

which gives \( p = \alpha - \lambda, \ q = \alpha + \lambda \). The solution \( u (x, t) \) of the heat equation with initial data \( \varphi (x) = \alpha + \lambda x \sin x \) is given by

\[
u (x, t) = \alpha + \lambda \left[ e^{-t} x \sin x + 2te^{-t} \cos x \right], \quad x \in \mathbb{R}, \quad t > 0, \quad (36)
\]

which gives \( \lim_{t \to \infty} u (0, t) = \alpha \). To see the solution formula \((36)\), it suffices to explain how to derive the solution \( v (x, t) \) of the heat equation (given by \((2)\)) with initial data \( x \sin x \). Instead of using the integral formula \((2)\), we use a little trick here. Note that \((v + v_{xx}) (x, 0) = 2 \cos x \). As \( v (x, t) + v_{xx} (x, t) \) is also a solution of the heat equation, we have \( v + v_{xx} = w \), where \( w (x, t) \) is a solution of the heat equation with \( w (x, 0) = 2 \cos x \), i.e. \( w (x, t) = 2e^{-t} \cos x \). Hence we get

\[
e^{t} (v (x, t) + v_{xx} (x, t)) = e^{t} (v (x, t) + v_{t} (x, t)) = \frac{\partial}{\partial t} (e^{t} v (x, t)) = 2 \cos x,
\]

and derive \( v (x, t) = e^{-t} x \sin x + 2te^{-t} \cos x \). The solution \((36)\) is actually defined for all \((x, t) \in \mathbb{R}^{2} \) and it satisfies \( \lim_{t \to \infty} u (x, t) = \alpha \) for all \( x \in \mathbb{R} \). The convergence is uniform on any compact subset of \( x \). The example is finished.

**Remark 16.** In particular, if we choose \( \alpha = 0 \), then the limit \( \lim_{t \to \infty} u (0, t) = 0 \) cannot imply \( \lim_{R \to \infty} \frac{1}{2\pi} \int_{-R}^{R} \varphi (\theta) d\theta = 0 \). Thus Theorem 1 fails for the unbounded case.

**Remark 17.** In the above example, the limit \( \lim_{t \to \infty} u (0, t) = \alpha \) happens to be the average value of \( p = \alpha - \lambda \) and \( q = \alpha + \lambda \). Here we give a similar example with \( \lim_{t \to \infty} u (0, t) = \alpha \), but the average value of \( p \) and \( q \) can be as large as we want. We choose

\[
\varphi (x) = \alpha - 4\rho (x \sin x) - \rho (2x \sin 2x), \quad \rho > 0 \text{ is a constant,}
\]
which gives \( \lim_{t \to \infty} u(0, t) = \alpha \) and
\[
\frac{1}{2R} \int_{-R}^{R} \varphi(\theta) \, d\theta = \alpha + 4\rho \left( \cos R - \frac{\sin R}{R} \right) + \rho \left( \cos 2R - \frac{\sin 2R}{2R} \right), \quad R > 0. \tag{37}
\]
If we put \( x = \cos R \in [-1, 1] \), we get
\[
4\rho \cos R + \rho \cos 2R = 2\rho x^2 + 4\rho x - \rho, \quad x \in [-1, 1]
\]
and the maximum and minimum values of this polynomial on \([-1, 1]\) are \( 5\rho, -3\rho \) respectively. Thus we have \( p = \alpha - 3\rho, \ q = \alpha + 5\rho \) and \( (p + q)/2 = \alpha + \rho \), where \( \rho > 0 \) can be large.

3.1.2. Example 4: \( p = \alpha - \lambda < \alpha < \beta < \beta + \lambda = q \) for arbitrary values of \( \alpha, \beta, \lambda \), where \( \lambda > 0 \). For this example, we need the help of Example 1 in Section 2.1.1. Let \( \varphi(x) \) be the bounded continuous function in (20) and choose the initial data as
\[
\psi(x) = \frac{\beta - \alpha}{2} \varphi(x) + \frac{\beta + \alpha}{2} + \lambda x \sin x, \quad \lambda > 0, \quad x \in (-\infty, \infty). \tag{38}
\]
We have
\[
H(x) = \frac{1}{2x} \int_{-x}^{x} \psi(\theta) \, d\theta
\]
\[
= \frac{\beta - \alpha}{2} \sin [\log (\log (x + 2))] + \frac{\beta + \alpha}{2} - \lambda \cos x + \lambda \frac{x}{x}, \quad x \in (0, \infty)
\]
and it is not difficult to see that there exists a sequence \( x_k \to \infty \) so that
\[
\sin [\log (\log (x_k + 2))] \to 1 \quad \text{and} \quad \cos x_k \to -1
\]
as \( k \to \infty \). Thus we have \( q = \beta + \lambda \). Similarly, there exists a sequence \( x_j \to \infty \) so that
\[
\sin [\log (\log (x_j + 2))] \to -1 \quad \text{and} \quad \cos x_j \to 1
\]
as \( j \to \infty \). Thus we have \( p = \alpha - \lambda \). On the other hand, by (24) and (36), we also have
\[
\lim_{t \to \infty} \left| u(0, t) - \left[ \frac{\beta - \alpha}{2} \sin \left( \log (\sqrt{4t}) \right) + \frac{\beta + \alpha}{2} \right] \right| = 0,
\]
which gives
\[
\alpha = \lim \inf_{t \to \infty} u(0, t) < \lim \sup_{t \to \infty} u(0, t) = \beta.
\]
Hence we conclude that \( p = \alpha - \lambda < \alpha < \beta < \beta + \lambda = q \). The example is finished.

3.1.3. Example 5: \( -\infty = p < \alpha = \beta < q = \infty \) for arbitrary value of \( \alpha \). For arbitrary value of \( \alpha \), let \( \varphi(x) = \alpha + x^2 \cos x, \ x \in \mathbb{R} \). We have
\[
\frac{1}{2R} \int_{-R}^{R} \varphi(\theta) \, d\theta = \alpha + R \sin R + 2 \cos R - \frac{2 \sin R}{R}, \quad R > 0,
\]
which gives \( p = -\infty, \ q = \infty \). By Section 4 below, one can see that the solution \( u(x, t) \) of the heat equation with initial data \( \varphi(x) = \alpha + x^2 \cos x \) is given by
\[
u(x, t) = \alpha + \left[ e^{-t} x^2 \cos x + 2te^{-t} (\cos x - 2x \sin x) - 4t^2 e^{-t} \cos x \right], \tag{40}
\]
which satisfies \( \lim_{t \to \infty} u(0, t) = \alpha \). The example is finished.
3.1.4. Example 6: \(-\infty = p < \alpha < \beta < q = \infty\) for arbitrary values of \(\alpha, \beta\). In this example, we use the idea of **time-periodic solutions** of the heat equation to help us. Let \(v(x, t) = e^{x} \cos(2t + x)\) and \(w(x, t) = e^{x} \sin(2t + x)\). Both are time-periodic solutions of the heat equation \(u_t = u_{xx}\) with initial conditions \(e^{x} \cos x\) and \(e^{x} \sin x\). Hence the function

\[
 u(x, t) = \lambda + A e^{x} \cos(2t + x) + B e^{x} \sin(2t + x),
\]

where \(\lambda, A, B\) are constants to be chosen later on, is also a time-periodic solution of the heat equation with

\[
 u(0, t) = \lambda + A \cos 2t + B \sin 2t, \quad t \in \mathbb{R}.
\]

The above gives

\[
 \liminf_{t \to \infty} u(0, t) = \lambda - \sqrt{A^2 + B^2} = \alpha, \quad \limsup_{t \to \infty} u(0, t) = \lambda + \sqrt{A^2 + B^2} = \beta
\]

if we choose \(\lambda, A, B\) satisfying

\[
 \lambda = \frac{\beta + \alpha}{2}, \quad \sqrt{A^2 + B^2} = \frac{\beta - \alpha}{2} > 0.
\]

Next, we evaluate

\[
 \limsup_{R \to \infty} \frac{1}{2R} \int_{-R}^{R} u(y, 0) \, dy
\]

\[
 = \limsup_{R \to \infty} \left\{ \lambda + \frac{A}{\pi} \left[ e^R \cos R - e^{-R} \cos R \right] + \frac{B}{\pi} \left[ e^R \sin R - e^{-R} \sin R \right] \right\}
\]

\[
 = \limsup_{R \to \infty} \left\{ \lambda + \frac{e^R}{4R} \left( \sqrt{2(A^2 + B^2)} \right) \right\}
\]

\[
 = \lim_{R \to \infty} \left( \lambda + \frac{e^R}{4R} \sqrt{2(A^2 + B^2)} \right) = \lim_{R \to \infty} \left( \lambda + \frac{\beta - \alpha}{2} \right) = +\infty
\]

and similarly

\[
 \liminf_{R \to \infty} \frac{1}{2R} \int_{-R}^{R} u(y, 0) \, dy = -\infty.
\]

The example is completed.

**Remark 18.** Using Fourier series expansion, it can be shown that, up to parabolic scaling, any \(2\pi\) **time-periodic** solution of the heat equation is an infinite superposition of the six solutions:

\[
 1, \quad x, \quad e^{\pm \frac{x}{\sqrt{2}}} \cos \left( t \pm \frac{x}{\sqrt{2}} \right), \quad e^{\pm \frac{x}{\sqrt{2}}} \sin \left( t \pm \frac{x}{\sqrt{2}} \right)
\]

We call them the **generating functions** for the \(2\pi\) time-periodic solutions of the heat equation.

3.1.5. Example 7: \(-\infty = p = \alpha < \beta = q = \infty\). Similar to the previous example, now we use the idea of **space-time periodic solutions** of the heat equation to help us. Consider the function

\[
 u(x, t) = e^{-2 + \sqrt{2\sqrt{65} - 1}(x-t)} \sin \left( \frac{\sqrt{2}}{4} \sqrt{65 - 1} (x-t) - 2t \right).
\]
One can check that it is a solution of the heat equation satisfying the space-time periodic condition $u(x + \pi, t + \pi) = u(x, t)$ for all $(x, t) \in \mathbb{R}^2$. We have

$$u(x, 0) = e^{-\frac{2 + \sqrt{65} - 1}{4} x} \sin \left( \frac{\sqrt{2}}{4} \sqrt{65} - 1 \right) x$$

and

$$u(0, t) = e^{\frac{2 + \sqrt{65} - 1}{4} t} \sin \left( - \frac{\sqrt{2}}{4} \sqrt{65} - 1 + 2 \right) t$$

and we can easily see that $p = \alpha = -\infty$ and $q = \beta = \infty$. The example is finished.

### 3.1.6. Some remarks about the unbounded case.

We have given several interesting examples for values of $p$, $\alpha$, $\beta$, $q$. However, there are some situations which we think they probably cannot be attained. The first situation is that for arbitrary three finite numbers $p < \alpha = \beta < q$, we probably cannot find an unbounded initial data $\varphi(x)$ so that these three values are attained. We feel that there should be some balance between $p$ and $q$ in order to achieve the limit value $\alpha$ (see Remark 17 for an example of balance). For example, if $p = -\varepsilon$ ($\varepsilon > 0$ is small), $\alpha = \beta = 0$ and $q \gg 0$ is sufficiently large, we think it cannot be attained.

The second situation is that for arbitrary four finite numbers $p < \alpha < \beta < q$, we think it is impossible to find an initial condition $\varphi(x)$ (bounded or unbounded) so that these four values are attained (however, by the Example 4, we can prescribe either $p$, $\alpha$, $\beta$ or $\alpha$, $\beta$, $q$).

### 4. Using ODE method to solve the heat equation.

In the example in Section 3.1.1, we use an ODE method to find explicit solution of the heat equation with initial data $x \sin x$. Here we give a more general discussion on it and look at some interesting examples.

Let $\varphi(x) \in C^2(\mathbb{R})$ be a function such that $|\varphi(x)|$, $|\varphi'(x)|$ and $|\varphi''(x)|$ all satisfy the growth condition (33). Let $u(x, t)$ be the solution of the heat equation with initial data $\varphi(x)$, $x \in (-\infty, \infty)$. Assume $\varphi(x)$ satisfies an ODE of the form

$$au''(x) + b\varphi'(x) + c\varphi(x) = h(x), \quad \forall x \in (-\infty, \infty)$$  \hspace{1cm} (43)

for some constants $a \neq 0$, $b$, $c$ and some function $h(x) \in C^0(\mathbb{R})$. Let $v(x, t)$ be the solution of the heat equation with initial data $v(x, 0) = h(x)$, $x \in (-\infty, \infty)$. By the growth condition of $|\varphi(x)|$, $|\varphi'(x)|$ and $|\varphi''(x)|$, we see that $au_{xx}(x, t) + bu_x(x, t) + cu(x, t)$ is also a solution of the heat equation with initial condition given by

$$au_{xx}(x, 0) + bu_x(x, 0) + cu(x, 0) = h(x), \quad \forall x \in (-\infty, \infty).$$

Hence, by uniqueness, we have

$$au_{xx}(x, t) + bu_x(x, t) + cu(x, t) = v(x, t), \quad (x, t) \in (-\infty, \infty) \times (0, \infty).$$  \hspace{1cm} (44)

As a consequence, we see that $u(x, t)$ must satisfy the first order PDE (with constant coefficients)

$$\begin{cases}
    au(x, t) + bu_x(x, t) + cu(x, t) = v(x, t), & (x, t) \in (-\infty, \infty) \times (0, \infty) \\
    u(x, 0) = \varphi(x), & x \in (-\infty, \infty).
\end{cases}$$  \hspace{1cm} (45)

(45) can be solved explicitly if we know the function $v(x, t)$ explicitly and know how to integrate the function given below in (46).
More precisely, one can convert (45) into an ODE by change of variables. If \( b \neq 0 \), we let \( z = x \), \( w = bt - ax \), which is a change of variables. Denote the function corresponding to \( u(x,t) \) and \( v(x,t) \) by \( \tilde{u}(z,w) \) and \( \tilde{v}(z,w) \). The equation for \( \tilde{u}(z,w) \) is \( \tilde{b}\tilde{u}_z(z,w) + \tilde{c}\tilde{u}(z,w) = \tilde{v}(z,w) \) and its general solution is
\[
\tilde{u}(z,w) = e^{-\frac{w}{b}} \left( \frac{1}{b} \int e^{\frac{w}{b}} \left( z, \frac{w + az}{b} \right) dz + C(w) \right), \quad w = bt - ax, \quad z = x, (46)
\]
where \( C(w) \) is an arbitrary function of \( w \). By (46), one can find the general solution \( u(x,t) \) of the equation in (45). Finally, we choose suitable \( C(w) \) so that the initial condition \( u(x,0) = \varphi(x) \) is satisfied. This solution \( u(x,t) \) will be the explicit solution of the heat equation with initial data \( \varphi(x) \). If \( b = 0 \), then (45) is already an ODE and we know how to solve it.

The above ODE method can be applied to the following initial conditions:

1. \( \varphi(x) = x^n, \ n \in \mathbb{N} \cup \{0\}, \ x \in (-\infty, \infty). \)
2. \( \varphi(x) = x^n e^{\alpha x}, \ n \in \mathbb{N} \cup \{0\}, \ \alpha \neq 0 \in \mathbb{R}, \ x \in (-\infty, \infty). \)
3. \( \varphi(x) = x^n \sin \beta x \) (or \( x^n \cos \beta x \)), \( n \in \mathbb{N} \cup \{0\}, \ \beta \neq 0 \in \mathbb{R}, \ x \in (-\infty, \infty). \)
4. \( \varphi(x) = x^n e^{\alpha x} \sin \beta x \) (or \( x^n e^{\alpha x} \cos \beta x \)), \( n \in \mathbb{N} \cup \{0\}, \ \alpha \neq 0, \ \beta \neq 0 \in \mathbb{R}, \ x \in (-\infty, \infty). \)

**(Case 1.)** \( \varphi(x) = x^n, \ n \in \mathbb{N} \cup \{0\}. \)

We know the solutions with initial condition \( x \) and \( x^2 \) are given respectively by \( u(x,t) = x \) and \( u(x,t) = x^2 + 2t \). For \( u(x,0) = x^3 \), we see that the function \( \frac{1}{6}u_{xx}(x,t) \) is a solution with initial condition \( \frac{1}{6}u_{xx}(x,0) = x \). Hence we have \( u_t(x,t) = 6x \) for all \( (x,t) \) and get \( u(x,t) = 6xt + C(x) \) for some integration function \( C(x) \). By \( u(x,0) = x^3 \), we get \( u(x,t) = 6xt + x^3 \).

Keep going and one can obtain the solutions:
\[
\begin{align*}
&\begin{cases}
u (x,t) = x^4 + 12x^2t + 12t^2, & u(x,0) = x^4 \\
u (x,t) = x^5 + 20x^3t + 60xt^2, & u(x,0) = x^5
\end{cases} \\
&\ldots
\end{align*}
\]
Each \( u(x,t) \) is a polynomial solution with homogeneous terms (view \( t \) as \( x^2 \)).

**(Case 2.)** \( \varphi(x) = x^n e^{\alpha x}, \ n \in \mathbb{N} \cup \{0\}, \ \alpha \neq 0 \in \mathbb{R}. \)

For \( n = 0 \), the solution with initial condition \( e^{\alpha x} \) is \( e^{\alpha^2 t + \alpha x} \). For \( u(x,0) = xe^{\alpha x} \), we have
\[
u_{xx}(x,0) - \alpha^2 u(x,0) = 2\alpha e^{\alpha x}.
\]
Hence we get the equation
\[
u_t(x,t) - \alpha^2 u(x,t) = \nu_{xx}(x,t) - \alpha^2 u(x,t) = 2\alpha e^{\alpha^2 t + \alpha x},
\]
which gives
\[
u(x,t) = (2\alpha t + x) e^{\alpha^2 t + \alpha x}.
\]
Repeating the same process, one can obtain the following solutions
\[
\begin{align*}
&\begin{cases}
u (x,t) = (2t + 4\alpha^2 t^2 + 4ax + x^3) e^{\alpha^2 t + \alpha x}, & u(x,0) = x^2 e^{\alpha x} \\
u (x,t) = (12\alpha t^2 + 8x^3 t^2 + 6tx + 12\alpha^2 t^2 x + 6ax^2 + x^3) e^{\alpha^2 t + \alpha x}, & u(x,0) = x^3 e^{\alpha x} \\
\ldots
\end{cases}
\end{align*}
\]
Case 3. $\varphi(x) = x^n \sin \beta x$, $n \in \mathbb{N} \cup \{0\}, \beta \neq 0 \in \mathbb{R}$.

For $n = 0$, the solution with $u(x, 0) = \sin \beta x$ is given by $u(x, t) = e^{-\beta^2 t} \sin \beta x$. For $u(x, 0) = x \sin \beta x$, by (36) we have

$$u(x, t) = e^{-\beta^2 t} x \sin \beta x + 2\beta t e^{-\beta^2 t} \cos \beta x$$  \hspace{1cm} (49)

and similarly for $u(x, 0) = x \cos \beta x$ we have

$$u(x, t) = e^{-\beta^2 t} x \cos \beta x - 2\beta t e^{-\beta^2 t} \sin \beta x.$$ \hspace{1cm} (50)

For $u(x, 0) = x^2 \sin \beta x$, we note that $u_{xx}(x, 0) + \beta^2 u(x, 0) = 2 \sin \beta x + 4\beta x \cos \beta x$ and so we get the equation

$$u_t(x, t) + \beta^2 u(x, t) = u_{xx}(x, t) + \beta^2 u(x, t)$$

$$= 2 e^{-\beta^2 t} \sin \beta x + 4\beta \left[ e^{-\beta^2 t} x \cos \beta x - 2\beta t e^{-\beta^2 t} \sin \beta x \right].$$

The above implies

$$u(x, t) = e^{-\beta^2 t} x^2 \sin \beta x + 2t e^{-\beta^2 t} (\sin \beta x + 2\beta x \cos \beta x) - 4\beta^2 t^2 e^{-\beta^2 t} \sin \beta x,$$ \hspace{1cm} (51)

which is the solution with $u(x, 0) = x^2 \sin \beta x$. Similarly, the solution with $u(x, 0) = x^2 \cos \beta x$ is given by

$$u(x, t) = e^{-\beta^2 t} x^2 \cos \beta x + 2t e^{-\beta^2 t} (\cos \beta x - 2\beta x \sin \beta x) - 4\beta^2 t^2 e^{-\beta^2 t} \cos \beta x.$$ \hspace{1cm} (52)

For $u(x, 0) = x^3 \sin \beta x$, we have $u_{xx}(x, 0) + \beta^2 u(x, 0) = 6 \sin \beta x + 6\beta x^2 \cos \beta x$ and by (49) and (52), we conclude

$$u_t(x, t) + \beta^2 u(x, t)$$

$$= \begin{cases} 
6 \left( e^{-\beta^2 t} x \sin \beta x + 2\beta t e^{-\beta^2 t} \cos \beta x \right) \\
+6\beta \left[ e^{-\beta^2 t} x^2 \cos \beta x + 2t e^{-\beta^2 t} (\cos \beta x - 2\beta x \sin \beta x) - 4\beta^2 t^2 e^{-\beta^2 t} \cos \beta x \right].
\end{cases}$$

Hence we get

$$u(x, t) = \begin{cases} 
e^{-\beta^2 t} x^3 \sin \beta x + 6t e^{-\beta^2 t} \left( x \sin \beta x + \beta x^2 \cos \beta x \right) \\
+6t^2 e^{-\beta^2 t} \left( 2\beta \cos \beta x - 2\beta^2 x \sin \beta x \right) - 8t^3 e^{-\beta^2 t} \beta^3 \cos \beta x.
\end{cases}$$ \hspace{1cm} (53)

Similarly, for $u(x, 0) = x^3 \cos \beta x$, we have $u_{xx}(x, 0) + \beta^2 u(x, 0) = 6 \cos \beta x - 6\beta x^2 \sin \beta x$ and get

$$u(x, t) = \begin{cases} 
e^{-\beta^2 t} x^3 \cos \beta x + 6t e^{-\beta^2 t} \left( x \cos \beta x - \beta x^2 \sin \beta x \right) \\
+6t^2 e^{-\beta^2 t} \left( -2\beta \sin \beta x - 2\beta^2 x \cos \beta x \right) + 8t^3 e^{-\beta^2 t} \beta^3 \sin \beta x.
\end{cases}$$ \hspace{1cm} (54)

One can repeat the same process to the solutions with initial data $x^4 \sin \beta x$, $x^4 \cos \beta x$, ... Case 3 is finished.

Remark 19. By the above, we see that the solution of the heat equation with oscillating initial data $u(x, 0) = x^n \sin \beta x$ (or $x^n \cos \beta x$), $n \in \mathbb{N} \cup \{0\}$, $\beta \neq 0$, satisfies $\lim_{t \to \infty} u(x, t) = 0$ for all $x \in (-\infty, \infty)$. Moreover, the convergence is uniform on any compact subset of $x$. 
Case 4. \( \varphi (x) = x^n e^{\alpha x} \sin \beta x, \) \( n \in \mathbb{N} \cup \{0\}, \) \( \alpha \neq 0, \beta \neq 0 \in \mathbb{R}. \)

For \( n = 0, \) the solution with initial condition \( u(x, 0) = e^{\alpha x} \sin \beta x \) satisfies
\[ u_{xx} (x, 0) - 2 \alpha u_x (x, 0) + (\alpha^2 + \beta^2) u(x, 0) = 0, \]
which gives
\[ u_t (x, t) - 2 \alpha u_x (x, t) + (\alpha^2 + \beta^2) u(x, t) = u_{xx} (x, t) - 2 \alpha u_x (x, t) + (\alpha^2 + \beta^2) u(x, t) = 0, \quad u(x, 0) = e^{\alpha x} \sin \beta x. \]

Thus, if we do the change of variables \( z = x, \) \( w = 2 \alpha t + x, \) the equation for \( \tilde{u} (z, w) \) is
\[ 2 \alpha \tilde{u}_z (z, w) = (\alpha^2 + \beta^2) \tilde{u} (z, w), \]
which gives \( \tilde{u} (z, w) = C(w) e^{\frac{\alpha^2 + \beta^2}{\alpha^2} z} \) for some integration function \( C(w). \)

Taking the initial condition into consideration, we conclude
\[ u(x, t) = e^{\alpha x + \beta t} \sin (\beta x + qt), \quad \text{where} \quad p = \alpha^2 - \beta^2, \quad q = 2 \alpha \beta, \quad (55) \]
which is the solution of the heat equation with \( u(x, 0) = e^{\alpha x} \sin \beta x. \) Similarly, the solution of the heat equation with \( u(x, 0) = e^{\alpha x} \cos \beta x \) is given by
\[ u(x, t) = e^{\alpha x + \beta t} \cos (\beta x + qt), \quad \text{where} \quad p = \alpha^2 - \beta^2, \quad q = 2 \alpha \beta. \quad (56) \]

For \( n = 1, \) the solution with initial condition \( u(x, 0) = xe^{\alpha x} \sin \beta x \) satisfies
\[ u_{xx} (x, 0) - 2 \alpha u_x (x, 0) + (\alpha^2 + \beta^2) u(x, 0) = 2 \beta e^{\alpha x} \cos \beta x. \]

Hence by (56) we have
\[ u_t (x, t) - 2 \alpha u_x (x, t) + (\alpha^2 + \beta^2) u(x, t) = 2 \beta e^{\alpha x + \beta t} \cos (\beta x + qt). \quad (57) \]

By the change of variables \( z = x, \) \( w = 2 \alpha t + x, \) the above becomes
\[ \tilde{u}_z (z, w) - \frac{\alpha^2 + \beta^2}{2 \alpha} \tilde{u} (z, w) = -\frac{\beta}{\alpha} e^{\frac{\alpha^2 + \beta^2}{\alpha^2} z} \sin \beta w, \]
which gives
\[ e^{-\frac{\beta}{\alpha} (\alpha^2 + \beta^2) z} \tilde{u} (z, w) = \left( -\frac{\beta}{\alpha} e^{\frac{\alpha^2 + \beta^2}{\alpha^2} w} \cos \beta w \right) z + C(w) \]
and so
\[ \tilde{u} (z, w) = -\frac{\beta}{\alpha} e^{\frac{\alpha^2 + \beta^2}{\alpha^2} z} \sin \beta w + C(w) e^{\frac{\alpha^2 + \beta^2}{\alpha^2} z}. \]

Now we conclude
\[ u(x, t) = -\frac{\beta}{\alpha} e^{\frac{\alpha^2 + \beta^2}{\alpha^2} z} \sin \beta x + C(2 \alpha t + x) e^{\frac{\alpha^2 + \beta^2}{\alpha^2} z} \]
and the initial condition \( u(x, 0) = xe^{\alpha x} \sin \beta x \) gives
\[ u(x, 0) = -\frac{\beta}{\alpha} xe^{\alpha x} \cos \beta x + C(x) e^{\frac{\alpha^2 + \beta^2}{\alpha} z} = xe^{\alpha x} \sin \beta x. \]

Hence, the function \( C(x) \) is given by
\[ C(x) = \frac{\alpha \sin \beta x + \beta \cos \beta x}{\alpha} xe^{\alpha x} e^{-\frac{\beta}{\alpha} (\alpha^2 + \beta^2) x}. \]
and then
\[ C (2\alpha t + x) e^{\frac{1}{2}(\alpha^2 + \beta^2)x} = \frac{\alpha \sin (\beta x + qt) + \beta \cos (\beta x + qt)}{\alpha \beta} (\beta x + qt) e^{(\alpha x + pt)}. \]

Now we can obtain the final explicit form
\[
u (x,t) = -\frac{\beta}{\alpha} x e^{(\alpha x + pt)} \cos (\beta x + qt) + \frac{\alpha \sin (\beta x + qt) + \beta \cos (\beta x + qt)}{\alpha \beta} (\beta x + qt) e^{(\alpha x + pt)},
\]
which is the solution of the heat equation with \( \nu (x,0) = xe^{\alpha x} \sin \beta x \). Similarly, the explicit solution of the heat equation with \( \nu (x,0) = xe^{\alpha x} \cos \beta x \) is given by
\[
u (x,t) = \frac{\beta}{\alpha} x e^{(\alpha x + pt)} \sin (\beta x + qt) + \frac{\alpha \cos (\beta x + qt) - \beta \sin (\beta x + qt)}{\alpha \beta} (\beta x + qt) e^{(\alpha x + pt)}.
\]

Repeating the same process, one can obtain the solutions of the heat equation with \( \nu (x,0) = x^n e^{\alpha x} \sin \beta x \) and \( \nu (x,0) = x^n e^{\alpha x} \cos \beta x \) for each \( n = 2, 3, ..., \). Case 4 is finished.

**Remark 20.** One can generalize the above method to linear equation with constant coefficients, given by
\[ u_t = u_{xx} + Au_x + Bu, \quad u = u (x,t), \quad u (x,0) = \varphi (x),
\]
where \( A, B \) are two real constants. We see that if we let
\[ r (x,t) = e^{\frac{1}{2}x - (B - \frac{A^2}{2})t} u (x,t),
\]
then \( u \) satisfies (61) if and only if \( r \) satisfies the heat equation \( r_t = r_{xx} \) with initial condition \( r (x,0) = e^{Ax/2} \varphi (x) \). Therefore, if \( e^{Ax/2} \varphi (x) \) has one of the four forms in the above, we can still obtain the explicit solution \( u (x,t) \) satisfying (61). In particular, we see that if \( \varphi (x) \) has one of the four forms in the above, the method is still applicable.

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**REFERENCES**

[1] P. Collet and J.-P. Eckmann, Space-time behavior in problems of hydrodynamic type: A case study, *Nonlinearity*, 5 (1992), 1265–1302.
[2] S. D. Eidel’man, *Parabolic System*, North-Holland, Amsterdam, 1969.
[3] S. Kamin, On stabilization of solutions of the Cauchy problem for parabolic equations, *Proc. Roy. Soc. Edinburgh Sect. A*, 76/77 (1976), 43–53.
[4] M. Nara and M. Taniguchi, The condition on the stability of stationary lines in a curvature flow in the whole plane, *J. Diff. Eq.*, 237 (2007), 61–76.
[5] W.-M. Ni, *The Mathematics of Diffusion*, CBMS-NSF Regional Conference Series in Applied Mathematics, v. 82, SIAM, 2011.
[6] V. D. Repnikov and S. D. Eidel’man, *A new proof of the theorem on the stabilization of the solution of the Cauchy problem for the heat equation*, *Math. USSR Sb.*, 2 (1967), 135–139.

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