Synchronization of Random Linear Maps

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We study synchronization of random one-dimensional linear maps for which the Lyapunov exponent can be calculated exactly. Certain aspects of the dynamics of these maps are explained using their relation with a random walk. We confirm that the Lyapunov exponent changes sign at the complete synchronization transition. We also consider partial synchronization of nonidentical systems. It turns out that the way partial synchronization manifests depends on the type of differences (in Lyapunov exponent or in contraction points) between the systems. The crossover from partial synchronization to complete synchronization is also examined.

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I. INTRODUCTION

Synchronization of chaotic systems is a subject of current intensive study [1]. To a large extent this is due to its various applications, ranging from laser dynamics [2] to electronic circuits [3], chemical and biological systems [4], secure communications [5], etc. But there is also a pure theoretical interest in this phenomenon, that is related perhaps to its counterintuitive nature: how it is possible that chaotic (i.e., by definition unpredictable) systems can be synchronized and thus brought under some ‘control’. And, even more puzzling, noise can play the role of the synchronizing factor. Indeed, early reports [6] that sufficiently strong noise can completely synchronize two identical chaotic systems were initially met with scepticism and attributed to finite precision of computations [7] or to biased noise [8]. However, more recent examples show this effect even for unbiased noise [9].

Since real systems are typically nonidentical, complete synchronization is difficult to achieve. It is an interesting problem to examine whether noise can induce some sort of ‘weaker synchronization’ in nonidentical but relatively similar systems. Recent works do show the existence of such partial synchronization [10, 11].

One important problem of the theoretical and numerical studies of synchronization is how to detect it. This problem is essentially solved for the complete synchronization of two identical systems that are described by variables x and x’, respectively. In this case, for the synchronized state the difference |x – x’| equals zero, while it remains positive in the unsynchronized state. Moreover, the transition between these two states is accompanied by the change of the sign of the largest Lyapunov exponent, that becomes negative in the synchronized state.

However, for the partial synchronization this problem is much more subtle. In this case, the two systems are not identical and the difference |x – x’| always remains positive. It has already been noted for some models with continuous dynamics that partial synchronization manifests through changes in the probability distribution of the ‘phase difference’ [11]. In addition to that, in the partially synchronized phase the so-called zero Lyapunov exponent becomes negative [10, 11].

Studies of synchronization rely, to a large extent, on numerical calculations. Precise estimations of Lyapunov exponents or probability distributions (invariant measures) constitute very often demanding computational problems. To further test the already accumulated knowledge on synchronization, it would be desirable to find models where at least some of these properties could be computed analytically.

In the present paper we examine synchronization of random one-dimensional linear maps. For such maps one can easily find the exact Lyapunov exponent and locate the point where it changes sign. Numerical calculations for two identical systems confirm that this is also the point where a complete synchronization transition takes place. We briefly report on a correspondence between such maps and a random walk process, that allows for a simple interpretation of the initial stages of the evolution of the maps.

We also examine the partial synchronization of nonidentical maps. It is seen that the way partial synchronization manifests depends on the type of difference between the nonidentical systems. In a certain case, the difference in the location of contraction points of the maps is imprinted in the probability distribution at the partial synchronization transition. When the difference between two subsystems δ tends to vanish, partial synchronization approaches the complete synchronization. Due to the exact knowledge of the complete synchronization point, one can examine some details of this crossover. In particular, it is shown that for δ → 0, vanishing of the synchronization error is very slow ∼ (−1/log10 δ).

II. RANDOM 2-MAPS

First, let us consider the simplest example of a random linear map

\[ x_{n+1} = f_i(x_n), \text{ } i = 0, 1 \text{ and } n = 0, 1, \ldots \] (1)

where at each time step n one of the maps \( f_0 \) or \( f_1 \) is applied with a probability \( p \) and \( (1-p) \), respectively. The maps are defined as \( f_0(x) = ax \mod(1) \), and \( f_1(x) = bx \), where \( 0 < x < 1 \) and \( a > 1, 0 < b < 1 \). Related models have already been examined in the context of on-off intermittency [12, 13], advection of particles by chaotic flows [14], and others [15, 16]. Some aspects of synchronization were
also studied for piecewise linear random maps, but both the Lyapunov exponent and the location of the synchronization transition were determined only numerically [17].

For the map \( f \) it is elementary to calculate exactly its Lyapunov exponent \( \lambda \) defined as

\[
\lambda = \lim_{N \to \infty} \frac{1}{N} \sum_{n=0}^{N-1} \log_{10} \left| \frac{dx_{n+1}}{dx_n} \right| .
\]

Indeed, since both \( f_0 \) and \( f_1 \) have constant derivatives, one immediately obtains

\[
\lambda = p \log_{10} a + (1 - p) \log_{10} b .
\]

It follows that \( \lambda \) changes sign at

\[
p = p_c = \frac{-\log_{10} b}{\log_{10} (a/b)} .
\]

It is easy to understand this result. For \( p > p_c \), the expanding (chaotic) map \( f_0 \) prevails over the contracting map \( f_1 \) and the overall behaviour is chaotic with \( \lambda > 0 \). The opposite situation takes place for \( p < p_c \) and the map \( f_1 \) contracts to \( x = 0 \). At \( p = p_c \) and in its close vicinity the map \( f \) exhibits intermittent bursts of activity [12, 13], but we will not focus on such a behaviour in the present paper.

To study synchronization one can make two runs \( \{x_n\} \) and \( \{x'_n\} \) of iterations of map \( f \) starting each time from (slightly) different initial conditions \( x_0 \) and \( x'_0 \) but with the same realization of noise, i.e., with the same sequence of maps \( f_i \). Then one measures the synchronization error \( w_n \) defined as

\[
w_n = \langle |x_n - x'_n| \rangle ,
\]

where \( \langle ... \rangle \) represents the mean over the realizations of the noise. Moreover, we introduce the steady-state average \( w = \lim_{n \to \infty} w_n \). In the synchronized state \( w = 0 \) while it is positive in the unsynchronized state. Our numerical results for \( a = 1/b = 3/2 \) (not presented here) show that \( w \) vanishes at \( p = 1/2 \), i.e., the point where the Lyapunov exponent changes sign.

However, due to the fact that it contracts to \( x = 0 \) for \( p < p_c \), the map \( f \) is not quite suitable to study synchronization of chaotic systems. For such a purpose it would be desirable to have a map with a somehow more complex behaviour in the regime with a negative Lyapunov exponent.

Nevertheless, the simplicity of the map \( f \) allows us to get some additional insight into its dynamics. Let us fix the initial points of our maps as, e.g., \( x_0 < x'_0 < 1 \). As we shall see, such a choice results in a certain transient regime that can be understood using an analogy with a random walk process.

Numerical evaluation of \( w_n \) shows that it exhibits three types of behavior as a function of time \( n \) (Fig. 1).

(a) For \( p > p_c \) (positive Lyapunov exponent) after an initial exponential increase, \( w_n \) saturates and acquires a constant nonzero value.

(b) For \( p < p_c \), \( w_n \) decreases after an initial exponential increase. The asymptotic exponential decay is consistent with the (negative) Lyapunov exponent \( \lambda \).

(c) For even smaller values of \( p \) the synchronization error \( w_n \) decreases exponentially already from the beginning.

To understand the initial behaviour of \( w_n \), let us recall that the iteration of \( x_n \) and \( x'_n \) starts from very small values and for a certain number of iterations the \( \text{mod}(1) \) part of the map \( f \) does not play any role. Consequently, one has

\[
w_n = \langle \alpha \rangle^n w_0 ,
\]

where

\[
\langle \alpha \rangle = p a + (1 - p) b .
\]

Correspondingly, the initial decrease or increase of \( w_n \) depends whether \( p \) is smaller or greater than \( (1 - b)/(a - b) \).

After a certain time, the \( \text{mod}(1) \) part of the map comes into play and the initial behaviour of \( w_n \) is replaced by a different one. Namely, for \( p > p_c \), \( w_n \) saturates at a nonzero value and it decays exponentially for \( p < p_c \). We shall return to this point at the end of this section.

To estimate the time scale \( \tau \) when the initial behaviour \( \langle \alpha \rangle \) changes, we relate our map to a random walk process. For simplicity, we consider the case \( b = 1/a \). In this case there is a one-to-one correspondence between the stochastic variable \( x_n \) and \( y_n = -\log_{10} (x_n) \). Multiplication of \( x_n \) by \( a \) or \( 1/a \) corresponds to the decrease or increase of \( y_n \) by unity. Consequently, \( y_n \) is nothing else but the position of a random walker on a lattice of unit spacing with transition probabilities \( p \) to the left and \((1 - p) \) to the right. The correspondence with the random walk holds as long as the \( \text{mod}(1) \) part of the map is not applied, i.e., the walker does not cross \( \log_{10}(1) = 0 \). The above random-walker problem has two characteristic time scales connected to the first-passage process [18] from its initial position to 0, and one can expect that one of them is related to \( \tau \) [which is, recall, the time scale on which the initial...
behavior of the map \( f \) is altered by the \( \mod(1) \) part of the map. (i) First, there is the mean first passage time \( \tau_M \). But for \( p \leq p_c \), \( \tau_M \) is infinite - contrary to \( \tau \), and therefore one cannot use \( \tau_M \) as a measure for \( \tau \). (ii) Secondly, there is the time moment \( \tau_P \) when the probability that the walker hits 0 for the first time is maximal. The probability distribution of the first hit is known to be [13]:

\[
P(n; p, y_0) = \frac{y_0}{n} \left( \frac{n - y_0}{2} \right) \binom{n}{y_0} p^{\frac{(n+y_0)}{2}} (1-p)^{\frac{(n-y_0)}{2}},
\]

(8)

where \( y_0 \) is the initial position of the walker and \( n \) is the number of steps; the binomial coefficient is to be interpreted as zero if \( (n - y_0)/2 \) is not an integer in the interval \([0, n]\).

Our numerical simulations show (see Fig. 1) that the value of \( n = \tau_P \) for which \( P(n; p, y_0) \) in Eq. (8) attains a maximum offers a reasonable estimation of \( \tau \). One cannot expect to get a more precise estimate of \( \tau \), since the change of the initial behavior \( f \) of the map towards the asymptotic one is a gradual process, that involves all the trajectories of the equivalent random walker that hit 0 (for the first time) before as well as after \( \tau_P \).

Let us notice that the bias of the random walk is related with the sign of the Lyapunov exponent and therefore with the asymptotic behaviour of our map. Indeed, for \( p < p_c = 1/2 \) the random walk is biased toward \(+\infty\), that translates into an exponential decay of \( w_n \). Our numerical simulations for the longer time regime suggest that this decay is governed by the Lyapunov exponent \( \lambda \). Recall that \( \lambda \) is known to govern the evolution of the typical value of \( |x_n - x'_n| \) (see, e.g., [14]). Thus in the long-time regime the mean \( w_n \) and the typical value of \( |x_n - x'_n| \) behave identically. On the other hand, they are clearly different in early-time regime. It means that in this regime \( w_n \) is strongly influenced by rare events, i.e., unlikely excursions of the random walker against the bias. For \( p > p_c = 1/2 \) the random walk is biased toward 0 and, since the map is bounded, \( w_n \) saturates at a positive value.

### III. RANDOM 3-MAPS

As already mentioned, the map (6) has a trivial behaviour for \( p < p_c \) and is not suitable to study synchronization of chaotic systems. In this context the following 3-map version is more interesting:

\[
x_{n+1} = f_i(x_n), \quad i = 0, 1, 2,
\]

(9)

where \( f_0(x) = ax \mod(1) \), \( f_1(x) = bx \mod(1) \), and \( f_2(x) = bx + (1-b) \). The maps \( f_0 \), \( f_1 \), and \( f_2 \) are applied at random with probabilities \( p \), \( (1-p)/2 \), and \( (1-p)/2 \), respectively. It is easy to show that for such a random map Eqs. (3)-4 still hold.

Numerical evaluation of the synchronization error \( w = \lim_{n \to \infty} w_n \) for the map (9) with \( a = 1/b = 3/2 \), based on \( N = 10^5 \) iterations, is shown in Fig. 2. For \( p > p_c = 1/2 \) the Lyapunov exponent \( \lambda \) is positive and the system is not synchronized \( (w > 0) \). For \( p < 1/2 \) we have \( \lambda < 0 \) and the system synchronizes \( (w = 0) \). But this time the behaviour for \( p < 1/2 \) is much more complex. The two maps \( f_1 \) and \( f_2 \) are still contracting ones, but to two different points \( (0 \) and \( 1) \). Since they are applied randomly, the system, while remaining synchronized, irregularly wanders throughout the whole interval \((0, 1)\). The probability density \( P(x) \) of visiting a given point \( x \), that is shown in the inset of Fig. 2 confirms such a behaviour.

Random linear maps can be also used to study partial synchronization that might occur for two dynamical systems that are not identical, although relatively similar. Generally, one considers the following pair of maps

\[
x_{n+1} = f_i(x_n), \quad x'_{n+1} = f'_i(x'_n), \quad i = 0, 1, 2,
\]

(10)

for which \{ \( f_0 \), \( f'_0 \) \}, \{ \( f_1 \), \( f'_1 \) \}, and \{ \( f_2 \), \( f'_2 \) \} are applied at random with probabilities \( p \), \( (1-p)/2 \), and \( (1-p)/2 \), respectively. To complete the definition we have to specify the functions \( f_i \) and \( f'_i \), \( i = 0, 1, 2 \). We present below the results for two particular choices. These ones correspond, respectively, to a perturbation of the Lyapunov spectrum - case (A) - and of the attractor - case (B).

Case (A): We choose

\[
f'_0(x) = a' x \mod(1), \quad f'_1(x) = b' x, \quad f'_2(x) = b' x + (1-b')
\]

(11)

where \( a' = 3/2 + \delta, b' = 1/a' \) and \( \delta = 10^{-3} \). Functions \( f_0 \), \( f_1 \), and \( f_2 \) are defined as for the map (9) with \( a = 1/b = 3/2 \). For such a choice, the Lyapunov exponent of the map in Eq. (11) is a linear function of \( p \) that also changes sign at \( p = p_c = 1/2 \) albeit with a different slope. Assuming that a partial synchronization transition is also governed by the Lyapunov exponent, we might expect that such a transition, if it exists, takes place at \( p = 1/2 \). For the pair of maps (10),

FIG. 2: The synchronization error \( w \) as a function of \( p \) calculated for the map (9) with \( a = 1/b = 3/2 \). The unnormalized probability distribution \( P(x) \) of visiting a point \( x \) for \( p = 0.4 \) (synchronized phase) is shown in the inset. The data were collected in \( 10^5 \) bins. Complex probability distributions are known to appear also for some other random linear maps [15].
probability distribution of slightly smaller value $P_i$; i.e., values that are relatively far from each other. For smaller differences between the values of $x$, there is a qualitative change in the probability distribution $P(x-x')$ that could locate precisely the partial synchronization transition. If not, it could mean that in this case either there is not a well-defined partial synchronization transition, or we are not looking at the right quantity to detect it.

For further comparison, we also calculated the probability $s$ that the difference $|x-x'|$ remains smaller (if initially so) than a given value $\varepsilon = 10^{-4}$ during the iterations of the system (12). One finds that $s$ is a monotonic function of $p$ (Fig. 3).

Case (B): We choose

$$f_0'(x) = a' x \mod(1), \quad f_1'(x) = b' x, \quad f_2'(x) = b' x + (1 - b') - b'',$$

where $a' = 3/2$, $b' = 2/3$ and $b'' = 10^{-3}$. Functions $f_0$, $f_1$, and $f_2$ are defined as in case (A). For such a choice the Lyapunov exponents of both systems are the same and they change sign at $p = p_c = 1/2$. The only difference is that the function $f_2'(x)$ has a contracting point at 1, while $f_2'(x')$ at a slightly smaller value $x' = 0.997$.

In this case the synchronization error $w$ behaves similarly to case (A) (Fig. 4). However, the probability distribution of $P(|x-x'|)$ has a little bit different shape. In particular, even at $p = 1/2$ it has a certain peak which increases with decreasing $p$. The maximum of the peak at $p = 1/2$ is most likely located at $|x-x'| = 0.003$ (Fig. 5) and this is related with the difference of the contracting points of the functions $f_2$ ($x = 1$) and $f_2'$ ($x' = 0.997$). We checked that off the $p = 1/2$ point the maximum shifts away from $|x-x'| = 0.003$. Moreover, the probability $s$ that the difference $|x-x'|$ is smaller than $\varepsilon = 10^{-4}$ shows a maximum at $p = 1/2$ (Fig. 4).

Comparison of partial synchronization in cases (A) and (B) revealed important differences between them. In case (A), where we perturbed the Lyapunov exponent, the probability distribution $P(|x-x'|)$ develops a peak around $|x-x'| = 0$ in the partially synchronized state that presumably exists for $p < 1/2$. Such a feature is similar to those already reported in...
the literature [11]. However, it is not clear to us whether in this case such a change can be characterized more quantitatively so that a well-defined transition point exists. A different behaviour takes place in the case (B) with perturbed contracting points. In this case the peak of the probability distribution is shifted by a value that at the partial synchronization transition \( p = 1/2 \) (as deduced from the vanishing of the Lyapunov exponent) curiously matches the shift of the contracting points. In addition to that, the probability of being in a state with a small difference \( |x - x'| \) has a maximum at \( p = 1/2 \) – in drastic contrast with case (A).

For simple maps as those examined in the present paper one has a complete knowledge of the Lyapunov exponents and fixed points. This is usually not the case for more complicated dynamical systems like Lorentz or R"ossler equations. Perturbing some parameters of these equations one usually modifies both the Lyapunov spectrum and the attractors. It is henceforth possible that partial synchronization in such systems combines some features of both cases (A) and (B) we discussed.

When the difference between nonidentical systems vanishes, partial synchronization is replaced by complete synchronization. We shall now briefly examine such a situation. In particular we consider two nonidentical systems as those described in the case (A) above, with varying difference \( \delta \). We calculated the synchronization error \( w \) as a function of \( \delta \) for \( p = 0.4 \) and \( p = 1/2 \). In both cases we expect that \( w \) vanishes for \( \delta \to 0 \). Figure 6 confirms such an expectation, but it also reveals that for \( p = 1/2 \) the convergence to zero is much slower than for \( p = 0.4 \). While for \( p = 0.4 \) the synchronization error seems to vanish as \( w \sim \delta \), the inset in Fig. 6 suggests that at the synchronization transition the vanishing is most likely logarithmic \( w \sim (-1/\log_{10} \delta) \).

IV. CONCLUSIONS

In the present paper we examined synchronization of one-dimensional random maps. Our results confirm that a complete synchronization transition coincides with the change of sign of the Lyapunov exponent. We also showed that the way the partial synchronization manifests depends on the type of difference between the two nonidentical systems. It would be desirable to explain the origin of the logarithmically slow crossover of the partial synchronization to the complete synchronization. It would be also interesting to explain why the location of the maximum of the probability distribution at the partial synchronization transition discussed in case (B) above matches exactly the shift of the contracting points.

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