Functorial products for $GL_2 \times GL_3$ and the symmetric cube for $GL_2$

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Dedicated to Robert P. Langlands

Introduction

In this paper we prove two new cases of Langlands functoriality. The first is a functorial product for cusp forms on $GL_2 \times GL_3$ as automorphic forms on $GL_6$, from which we obtain our second case, the long awaited functorial symmetric cube map for cusp forms on $GL_2$. We prove these by applying a recent version of converse theorems of Cogdell and Piatetski-Shapiro to analytic properties of certain $L$-functions obtained from the method of Eisenstein series (Langlands-Shahidi method). As a consequence we prove the bound $5/34$ for Hecke eigenvalues of Maass forms over any number field and at every place, finite or infinite, breaking the crucial bound $1/6$ (see below and Sections 7 and 8) towards Ramanujan-Petersson and Selberg conjectures for $GL_2$. As noted below, many other applications follow.

To be precise, let $\pi_1$ and $\pi_2$ be two automorphic cuspidal representations of $GL_2(\mathbb{A}_F)$ and $GL_3(\mathbb{A}_F)$, respectively, where $\mathbb{A}_F$ is the ring of adeles of a number field $F$. Write $\pi_1 = \otimes_v \pi_{1v}$ and $\pi_2 = \otimes_v \pi_{2v}$. For each $v$, finite or otherwise, let $\pi_{1v} \boxtimes \pi_{2v}$ be the irreducible admissible representation of $GL_6(F_v)$, attached to $(\pi_{1v}, \pi_{2v})$ through the local Langlands correspondence by Harris-Taylor [HT], Henniart [He], and Langlands [La4]. We point out that, if $\varphi_{iv}$, $i = 1, 2$, are the two- and the three-dimensional representations of Deligne-Weil group, parametrizing $\pi_{iv}$, respectively, then $\pi_{1v} \boxtimes \pi_{2v}$ is attached to the six-dimensional representation $\varphi_{1v} \otimes \varphi_{2v}$. Let $\pi_1 \boxtimes \pi_2 = \otimes_v (\pi_{1v} \boxtimes \pi_{2v})$. Our first result (Theorem 5.1) is:

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Theorem A. The representation $\pi_1 \boxtimes \pi_2$ of $GL_6(\mathbb{A}_F)$ is automorphic, i.e., functorial products [La3] for $GL_2 \times GL_3$ exist. It is isobaric and more specifically, is irreducibly induced from unitary cuspidal representations, i.e., $\pi_1 \boxtimes \pi_2 = \text{Ind} \sigma_1 \otimes \cdots \otimes \sigma_k$, where $\sigma_i$'s are unitary cuspidal representations of $GL_{n_i}(\mathbb{A}_F)$, $n_i > 1$. Moreover, $k = 3$, $n_1 = n_2 = n_3 = 2$, or $k = 2$, $n_1 = 2, n_2 = 4$ occur if and only if $\pi_2$ is a twist of $\text{Ad}(\pi_1)$, the adjoint of $\pi_1$ (see below), by a gr"ossencharacter.

We remark that at the moment, we are unable to characterize the image of this functorial product or completely determine all the pairs for which the image is cuspidal.

Next, let $\pi$ be a cuspidal representation of $GL_2(\mathbb{A}_F)$. Write $\pi = \bigotimes_v \pi_v$. Let $\text{Ad}: GL_2(\mathbb{C}) \to GL_3(\mathbb{C})$ be the adjoint representation of $GL_2(\mathbb{C})$. Then $\text{Ad}(\text{diag}(a,b)) = \text{diag}(ab, a^{-1}, a^{-1}b)$. Let $\varphi_v$ be the two-dimensional representation of the Deligne-Weil group attached to $\pi_v$ ([Ku], [La4]). Then $\text{Ad}(\varphi_v) = \text{Ad} \cdot \varphi_v$ is a three-dimensional representation. Let $\text{Ad}(\pi_v)$ be the irreducible admissible representation of $GL_3(F_v)$ attached to $\text{Ad}(\varphi_v)$ ([He2], [La4]). Set $\text{Ad}(\pi) = \bigotimes_v \text{Ad}(\pi_v)$. Then in [GeJ], Gelbart and Jacquet proved that $\text{Ad}(\pi)$ is an automorphic representation of $GL_3(\mathbb{A}_F)$ which is cuspidal unless $\pi$ is monomial, i.e., $\pi \simeq \pi \otimes \eta$, where $\eta \neq 1$ is a gr"ossencharacter of $F$.

Observe that, if $\text{Sym}^2$ is the three-dimensional irreducible representation of $GL_2(\mathbb{C})$ on symmetric tensors of rank 2, this implies the same facts about $\text{Sym}^2(\pi) = \bigotimes_v \text{Sym}^2(\pi_v)$, where $\text{Sym}^2(\pi_v)$ is the irreducible admissible representation of $GL_3(F_v)$ attached to $\text{Sym}^2(\varphi_v) = \text{Sym}^2 \cdot \varphi_v$. Moreover $\text{Sym}^2(\pi) = \text{Ad}(\pi) \otimes \omega_\pi$, where $\omega_\pi$ is the central character of $\pi$.

We now proceed to the next case and, as before, let $\pi$ be a cuspidal representation of $GL_2(\mathbb{A}_F)$. Write $\pi = \bigotimes_v \pi_v$ and denote by $\text{Sym}^3: GL_2(\mathbb{C}) \to GL_4(\mathbb{C})$, the four-dimensional irreducible representation of $GL_2(\mathbb{C})$ on symmetric tensors of rank 3. Simply put, for each $g \in GL_2(\mathbb{C})$, $\text{Sym}^3(g) \in GL_4(\mathbb{C})$ can be taken to be the matrix that gives the change in coefficients of an arbitrary homogeneous cubic polynomial in two variables, under the change of variables by $g$. Again, as before, let $\varphi_v$ be the two-dimensional representation of the Deligne-Weil group attached to $\pi_v$. Then $\text{Sym}^3(\varphi_v) = \text{Sym}^3 \cdot \varphi_v$ is a four-dimensional representation. Let $\text{Sym}^3(\pi_v)$ be the irreducible admissible representation of $GL_4(F_v)$ attached to $\text{Sym}^3(\varphi_v)$ by the local Langlands correspondence (see references above). Set $\text{Sym}^3(\pi) = \bigotimes_v \text{Sym}^3(\pi_v)$ which we call the symmetric cube of $\pi$. Applying Theorem A to $\pi_1 = \pi$ and $\pi_2 = \text{Ad}(\pi)$, and using the classification theorem [JS1] for $GL_6(\mathbb{A}_F)$, we obtain (Theorem 6.1):

Theorem B. The representation $\text{Sym}^3(\pi)$ is an automorphic representation of $GL_4(\mathbb{A}_F)$. It is cuspidal, unless $\pi$ is either of dihedral or of tetrahedral type, i.e., those attached to dihedral and tetrahedral representations of
the Galois group of $\overline{F}/F$ (cf. [Ge], [La2]). In particular, if $F = \mathbb{Q}$ and $\pi$ is the automorphic representation attached to a nondihedral holomorphic form of weight $\geq 2$, then $\text{Sym}^3(\pi)$ is cuspidal.

Due to the large number of applications and resilience to previously available methods, the existence of the symmetric cube has fascinated a good number of experts in the field (cf. [KS1, 2, 3] and references therein, also see [BK]) ever since symmetric squares were established by Gelbart and Jacquet [GeJ]. Here the map is symplectic (cf. Section 9) and a much richer geometry is involved, which seems to be the pattern for odd symmetric powers, making them harder to get than even ones. Moreover, neither of the trace formulas, or for that matter, any other approach, can be used to prove either of these theorems.

Our present proof of Theorem B is quite surprising. In fact, we were originally planning to prove the existence of the symmetric cube for $\text{GL}_2$ by further twisting by forms on $\text{GL}_2(\mathbb{A}_F)$ which at the time was totally out of reach [KS1]. On the other hand, our present proof of the existence of the symmetric cube is an indirect consequence of the functorial product for $\text{GL}_2 \times \text{GL}_3$, from which the properties of $L$-functions attached to the above twisting immediately follow. We should point out that one does not need the full functorial product for $\text{GL}_2 \times \text{GL}_3$ to prove the existence of the functorial symmetric cube for cuspidal forms on $\text{GL}_2(\mathbb{A}_F)$ (Remark 6.7).

Theorem B is a very pleasant conclusion to a project the second author has pursued since 1978. In fact, following Langlands [La6] and his theory of Eisenstein series [La1], this has led him to develop a machinery [Sh1, 2, 4, 5, 7], whose full power and subtlety (for example cf. Sections 3-7 of [Sh1] and Sections 2–5 here) are necessary to prove Theorem A from which existence of the symmetric cube, i.e., Theorem B, follows. In fact, as explained in the next paragraph, this is not possible, unless we bring in two new and crucial results [Ki1], [GeS], but still using the same method.

Theorem A is proved by applying a recent and ingenious version [CP-S3] of converse theorems of Cogdell and Piatetski-Shapiro to analytic properties of $L$-functions obtained from our method (cf. Theorem 3.2 here). While functional equations and their subtle consequences were proved in full generality earlier in [Sh4], [Sh1], two new and important ingredients are needed. The first one is a crucial observation of Kim (Proposition 2.1 of [Ki1] or Lemma 7.5 of [La1]), allowing us to utilize holomorphy of highly ramified twists of certain $L$-functions obtained from our method. In fact, it was generally believed that it would not be possible to obtain the holomorphy of the $L$-functions using the Langlands-Shahidi method. But the fact that cuspidal representations which are not invariant under certain Weyl group elements do not contribute to the residual spectrum, combined with the Langlands-Shahidi method, gives the
holomorphy of corresponding $L$-functions. In view of recent powerful converse theorems [CP-S1], [CP-S3], this is sufficient. The second is a recent paper of Gelbart-Shahidi [GeS], where boundedness in finite vertical strips for every $L$-function obtained from our method, is proved. In the present situation, we need to apply the main theorem of [GeS] to four different cases in the lists in [La6] and [Sh2] (case (iii) of [La6] and the triple product $L$-function cases $D_5 - 2$, $E_6 - 1$, and $E_7 - 1$ of [Sh2]), all of which but one, have more than one $L$-function in their constant terms, requiring us to use the full power and subtlety of Theorem 4.1 of [GeS] (the number of $L$-functions $m = 1, 2, 3,$ and $4$, in these four cases, respectively).

As striking as this result is, it only allows us to prove the existence of a weak lift (Theorem 3.8). To prove the existence of a lift whose local components are everywhere those attached by Harris-Taylor [HT], Henniart [He], and Langlands [La4], through the local Langlands correspondence, a lot more work is needed (Theorem 5.1).

What we need to do is to prove equalities (3.2.1) and (3.2.2) of Section 3, i.e.,

\[ L(s, \pi_{1v} \times \pi_{2v} \times \sigma_v) = L(s, (\pi_{1v} \boxtimes \pi_{2v}) \times \sigma_v) \]

and

\[ \varepsilon(s, \pi_{1v} \times \pi_{2v} \times \sigma_v, \psi_v) = \varepsilon(s, (\pi_{1v} \boxtimes \pi_{2v}) \times \sigma_v, \psi_v), \]

for all irreducible admissible generic representations $\sigma_v$ of $GL_n(F_v)$, $1 \leq n \leq 4$. These equalities are not obvious at all since the two factors on the left and right are defined using completely different techniques. The factors on the left are those defined by the triple $L$-functions in our four cases [Sh1, 7], [La4], while the ones on the right are those of Rankin-Selberg for $GL_6(\mathbb{A}_F) \times GL_n(\mathbb{A}_F)$, $1 \leq n \leq 4$ [JP-SS], [Sh1]. By the local Langlands correspondence, they are Artin factors.

Using [Sh5] on the equality of $L$-functions defined from our method and those of Artin for $GL_k \times GL_4$, we can show the above equalities when $\pi_{1v}$, $\pi_{2v}$ are not both supercuspidal representations. If $v \nmid 2$, then any supercuspidal representation of $GL_2(F_v)$ is attached to an induced representation of the corresponding Weil group. Hence, using quadratic base change [AC], [La2], we can reduce to the case when $\pi_{1v}$ is not supercuspidal. We do the same when $v|2$ and $\pi_{1v}$ is not an extraordinary supercuspidal representation. Now, suppose $v|2$ and $\pi_{1v}$ is an extraordinary supercuspidal representation. Then $\pi_{2v}$ is attached to an induced representation from a cubic extension (normal or otherwise). If $n \leq 3$, then using either a normal cubic base change [AC], [La2], or a nonnormal one [JP-SS2, 3], we can reduce to the case when $\pi_{2v}$ is not supercuspidal. However, due to the fact that the theory of nonnormal cubic base change for $GL_4$ is not available at present, this does not work for $n = 4$. Namely, our argument cannot proceed when $v|2$, $\pi_{1v}$ is an extraor-
ordinary supercuspidal representation of $GL_2(F_v)$, and $\pi_{2v}$ is a supercuspidal representation of $GL_3(F_v)$, attached to an induced representation from a character of a nonnormal cubic extension. By using Proposition 5.1 of [Sh1], which allows us to produce cusp forms with prescribed supercuspidal components, we can obtain an irreducible admissible representation $\Pi_v$ of $GL_6(F_v)$, which satisfies the above equalities for $n \leq 3$, and which differs from $\pi_{1v} \boxtimes \pi_{2v}$ by at most a quadratic character. Then the self-contained appendix by Bushnell and Henniart [BH], which uses certain subtle results from the theory of types and conductor for pairs [BHK], [Sh5], proves that in fact $\Pi_v \simeq \pi_{1v} \boxtimes \pi_{2v}$. To complete the theorem, we need to apply the converse theorem twice again. (See the argument at the end of Section 5.)

On the other hand, we should point out that it is possible to show that $\text{Sym}^3(\pi)$ is functorial even at $v|2$ without resorting to the appendix [BH] (cf. Remark 6.7).

Section 4 is devoted to a verification of Assumption 1.1 and Conjecture 1.2 (cf. Section 1 here) in our four cases. They ought to be verified, if our machine is going to work, as they are needed both for our Proposition 2.1, as well as Theorem 4.1 of [GeS]. The proof relies heavily on repeated application of multiplicativity (Theorem 3.5.3 of [Sh1]) and certain results and ideas from [Ki1, 2, 5], [CS], [Z].

Our paper concludes with a large number of applications (Sections 7–10). Many more, some yet to be formulated, are expected (e.g., in dimension four generalization of Wiles’ program [W]).

Sections 7 and 8 are devoted to analytic number theory. The first result establishes a new bound towards Ramanujan-Petersson and Selberg conjectures for $GL_2$. More precisely, our Theorem 7.1, which is a consequence of applying Theorem B to main estimates in [LRS1], states:

**Theorem C.** Let $\pi$ be a cuspidal representation of $GL_2(\mathbb{A}_F)$. Let $\pi_v$ be a local (finite or infinite) spherical component of $\pi$, i.e., $\pi_v = \text{Ind}(\mathbb{A}^{s_j}_v, \mathbb{A}^{s_2}_v)$, $s_j \in \mathbb{C}$. Then $|\text{Re}(s_j)| \leq 5/34$, $j = 1, 2$.

When $F = \mathbb{Q}$ and $\lambda_1(\Gamma)$ is the smallest positive eigenvalue for Laplacian on $L^2(\Gamma \backslash \mathfrak{h})$, where $\mathfrak{h}$ denotes the upper half plane and $\Gamma$ is a congruence subgroup, then Theorem C implies: $\lambda_1(\Gamma) \geq 66/289 \cong 0.22837$. The earlier bound of 0.21 was due to Luo-Rudnick-Sarnak [LRS2]. Observe that $1/7 < 5/34 < 1/7 + 0.004$.

The fact that the estimate $5/34$ is sharper than $1/6$ is crucial and allows us to prove some fundamental results in analytic number theory. For example, our Propositions 8.1 and 8.2 on shifted sums [Go] and hyperbolic circle problem [I],

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yield as sharp a result as if one assumes the full Selberg Conjecture: $\lambda_1(\Gamma) \geq 0.25$. The next crucial bound is $1/12$. Section 8 was suggested to us by Peter Sarnak. Further applications to theory of automorphic forms (e.g., Jacquet’s conjecture), using the fact that $5/34 < 1/6$, are also expected.

In Section 9, following a suggestion of Joseph Shalika, we prove a conditional theorem (Theorem 9.2) on the existence of Siegel modular cusp forms of weight $3$ for $\text{GSp}_4(\mathbb{A}_Q)$. Here we need the validity of Arthur’s multiplicity formula for $\text{GSp}_4(\mathbb{A}_Q)$. As we remark there (Remark 9.3), this must follow from stabilization of the trace formula for $\text{GSp}_4(\mathbb{A}_Q)$ and the most general form of the twisted trace formula for $\text{GL}_4(\mathbb{A}_Q)$.

Section 10 is devoted to new examples of Artin’s conjecture and global Langlands correspondence. In view of the recent progress in Taylor’s program [Tay] on Artin’s conjecture for icosahedral representations of global Weil group $W_F$, our Theorem B immediately proves new cases of Artin’s conjecture for four-dimensional irreducible primitive representations of $W_F$. Together with certain other new cases, this is recorded as our Theorem 10.1. We refer to Theorem 10.2 for new cases of Artin’s conjecture coming from Theorem A, when the representations are of icosahedral type. The solvable cases of our theorem are already established by Ramakrishnan in [R3].

Finally, we refer to [Ki4] for other cases of functoriality obtained using this method, namely, the exterior square lift from $\text{GL}_4$ to $\text{GL}_6$ and the symmetric fourth powers of cusp forms on $\text{GL}_2$. In fact, when the exterior square lift from $\text{GL}_4$ to $\text{GL}_6$ is combined with our symmetric cube, it leads to the existence of symmetric fourth powers of cusp forms on $\text{GL}_2(\mathbb{A}_F)$. Further consequences of the existence of these two symmetric powers are collected in [KS3].

There are a number of mathematicians whose comments have influenced this paper. We first thank Hervé Jacquet whose comments as a Comptes Rendus editor on our announcement [KS2] led to the present form of Section 5. Next, we thank Peter Sarnak for his continued encouragement and support for this project. In particular, we thank him for suggesting the material in Section 8. The problem in Section 9 was suggested by Joseph Shalika for which we thank him. The authors thank him as well for a discussion which led to formulation of the existence of the symmetric cube in the form presented in Section 6. This paper owes much, as well, to Dinakar Ramakrishnan, particularly for many conversations the first author had with him during his stay at the Institute for Advanced Study during the 1999–2000 Special Year.

Thanks are also due to James Cogdell for insights provided by him on his converse theorems with Piatetski-Shapiro. Our paper concludes with an appendix by Colin Bushnell and Guy Henniart which allows us to remove the final hurdle in establishing our functorial product. We are grateful to them for providing it.
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The authors would like to dedicate this paper to Robert Langlands with admiration. Developing his ideas over the years made these surprising results possible.

1. Preliminaries

Let $F$ be a number field and denote by $\mathbb{A}_F$ its ring of adèles. For each place $v$ of $F$, let $F_v$ denote the corresponding completion of $F$. When $v < \infty$, we let $O_v$ denote the ring of integers of $F_v$. Let $\varpi_v$ be a uniformizing element for the maximal prime ideal $\mathcal{P}_v$ of $O_v$. Let $q_v$ be the cardinality of $O_v/\mathcal{P}_v$ and fix an absolute value $|.|_v$ in the class of $v$ such that $|\varpi_v|=q_v^{-1}$.

Let $G$ be a quasi-split connected reductive algebraic group over $F$. Fix a Borel subgroup $B$ of $G$ over $F$. Write $B=TU$ where $T$ is a maximal torus and $U$ is the unipotent radical of $B$. Let $P \supset B$ be a parabolic subgroup of $G$ over $F$. Write $P=MN$ for a Levi decomposition of $P$ with a Levi factor $M$ and unipotent radical $N$. Fix $M$ by assuming $M \supset T$. Observe that $N \subset U.$ We let $A_0$ be the maximal $F$-split subtorus of $T$. Let $W$ be the Weyl group of $A_0$ in $G$. The choice of $U$ determines a set of positive roots for $A_0$. Let $\Delta$ be the set of simple roots and denote by $\theta$ the subset of $\Delta$ generating $M$.

Throughout this paper, we shall assume $P$ is maximal. Then $\Delta \setminus \theta$ is a singleton. Let $\Delta \setminus \theta = \{\alpha\}$. The simple root $\alpha$ can be identified as the unique reduced root of $A$, the split component of the center of $M$, in $N$. If $\rho_P$ is half the sum of roots in $N$, we let $\tilde{\alpha} = \langle \rho_P, \alpha \rangle^{-1}\rho_P$ as in [Sh2].

There exists a unique element $\tilde{w}_0 \in W$ such that $\tilde{w}_0(\theta) \subset \Delta$ while $\tilde{w}_0(\alpha)$ is a negative root. We shall fix a representative $w_0$ for $\tilde{w}_0$ in $K \cap G(F)$ as in [Sh7], where $K$ is a fixed maximal compact subgroup of $G = G(\mathbb{A}_F)$. Similarly we use $B,T,U,P,M,N,A$, and $A_0$ to denote the corresponding adelic points. Finally, given $v$, and a group $H$ over $F_v$, we use $H_v$ to denote $H(F_v)$. We, therefore, have $G_v, B_v$, and so on.
Given a connected reductive algebraic group $H$ over $F$, let $LH$ be its $L$-group. We use $\hat{H}$ to denote the connected component $LH^0$ of $LH$. Considering $H$ as a group over each $F_v$, we then denote by $LH_v$ its $L$-group over $F_v$, and we have a natural homomorphism $LH_v \to LH$. Let $\eta_v: LH_v \to LM$ be this map for $M$.

Let $LN$ be the $L$-group of $N$ defined naturally in [Bor]. Let $L\mathfrak{n}$ be its (complex) Lie algebra, and let $r$ denote the adjoint action of $LM$ on $L\mathfrak{n}$. Decompose $r = \bigoplus_{i=1}^m r_i$ to its irreducible subrepresentations, indexed according to values $\langle \alpha, \beta \rangle = i$ as $\beta$ ranges among the positive roots of $T$. More precisely, $X_{\beta^i} \in L\mathfrak{n}$ lies in the space of $r_i$, if and only if $\langle \alpha, \beta \rangle = i$. Here $X_{\beta^i}$ is a root vector attached to the coroot $\beta^i$, considered as a root for the $L$-group. Moreover, $\langle \ , \ \rangle$ denotes the Killing form, i.e., for every pair of positive roots $\gamma$ and $\delta$ of $T$, $\langle \gamma, \delta \rangle = 2\langle \gamma, \delta \rangle / \langle \delta, \delta \rangle = \langle \gamma, \delta' \rangle$, where $\delta'$ is the coroot $2\delta / \langle \delta, \delta \rangle$ attached to $\delta$ (cf. [Sh2]).

Now, let $\pi = \otimes_v \pi_v$ be an irreducible unitary globally generic cuspidal representation of $M = M(\mathbb{A}_F)$. For each place $v$ and each $i, 1 \leq i \leq m$, let $L(s, \pi_v, r_i \cdot \eta_v)$ and $\varepsilon(s, \pi_v, r_i \cdot \eta_v, \psi_v)$ be the $L$-function and the root number defined in [Sh1], where $\psi = \otimes_v \psi_v$ is a nontrivial additive character of $F \setminus \mathbb{A}_F$. When $v = \infty$ or $\pi_v$ is unramified or has only an Iwahori-fixed vector, then they are exactly those defined by Langlands [La4, 5, 6], [Sh7]. Moreover, if

$$L(s, \pi, r_i) = \prod_v L(s, \pi_v, r_i \cdot \eta_v)$$

and

$$\varepsilon(s, \pi, r_i) = \prod_v \varepsilon(s, \pi_v, r_i \cdot \eta_v, \psi_v),$$

then

$$L(s, \pi, r_i) = \varepsilon(s, \pi, r_i) L(1 - s, \pi, r_i),$$

(1.1)

where $\bar{\pi}$ is the contragredient of $\pi$ (cf. [Sh1]). We set $r_{i,v} = r_i \cdot \eta_v$.

Next, with notation as in [Sh1, 2], we have the globally induced representation $I(s, \pi) = I(s, \pi_0)$ from $\pi \otimes \exp(s \alpha, H_F(\cdot))$ as well as the local ones $I(s, \pi_v)$, induced from $\pi_v \otimes \exp(s \alpha, H_{F_v}(\cdot))$, for each $v$. Let us point out that in our notation, $s = 0$ always corresponds to a unitarily induced representation. We finally recall the global intertwining operator $M(s, \pi)$ defined by

$$M(s, \pi)f(g) = \int_{N'} f(w^{-1}ng)dn \quad (g \in G),$$

(1.2)

where $f \in I(s, \pi)$ and $N'$ is the unipotent radical of the standard parabolic subgroup which has the Levi subgroup $M' \supset T$, generated by $\tilde{w}_0(\theta)$. Then
where the local operators are defined the same way. Finally at each \( v \), let \( N(s\tilde{\alpha}, \pi_v, w_0) \) be the normalized operator

\[
N(s\tilde{\alpha}, \pi_v, w_0) = \prod_{i=1}^{m} \varepsilon(is, \pi_v, \tilde{r}_{i,v}) L(is, \pi_v, \tilde{r}_{i,v})^{-1} M(s\tilde{\alpha}, \pi_v, w_0).
\]

To proceed, we need the following assumption, originally called Assumption A in [Ki1], and later Assumption 2.1 in [GS].

Given a place \( v \), the map \( f_0^v(g) \mapsto \exp(\langle s\tilde{\alpha}, H_{P_v}(g) \rangle) f_0^v(g), \ g \in G_v \), defines a bijection from the space of \( I(0, \pi_v) \) onto \( I(s, \pi_v) \). Set

\[
f_v(g) = \exp(\langle s\tilde{\alpha}, H_{P_v}(g) \rangle) f_0^v(g).
\]

**Assumption 1.1.** Fix a place \( v \) and assume \( \text{Re}(s) \geq 1/2 \). Then there exists a function \( f_0^v \in V(0, \pi_v) \) such that \( N(s\tilde{\alpha}, \pi_v, w_0)f_v(g) \) is holomorphic and nonzero at \( s \) for some \( g \in G_v \).

The usual duality arguments show that if Assumption 1.1 holds for \( \pi \), then it holds for \( \tilde{\pi} \).

**Remark.** The proof of the assumption (usually called Assumption A) in many cases is the subject matter of a work in preparation by Kim [Ki5]. As outlined there, besides the Standard Module Conjecture (cf. [CS], [V]), one needs the following conjecture (Conjecture 7.1 of [Sh1]).

**Conjecture 1.2.** Assume \( \pi_v \) is tempered, then each \( L(s, \pi_v, r_{i,v}) \) is holomorphic for \( \text{Re}(s) > 0 \).

We shall prove both the assumption and the conjecture in any of the cases which we need in this paper. We refer to [CS], [Ki5], and [As] for the progress made on Conjecture 1.2.

We conclude this section by recording the following well-known (cf. [Sh2], [La6]) equation for the sake of completeness.

\[
M(s, \pi) = \bigotimes_v M(s\tilde{\alpha}, \pi_v, w_0),
\]

(1.3) where the \( f = \bigotimes_v f_v \in I(s, \pi) \).
2. A general result

In this section we will generalize an idea from [Ki1, Prop. 2.1] and [La1, Lemma 7.5] to establish the holomorphy of each $L$-function $L(s, \pi, r_i)$ under highly ramified twists. This will be the global analogue of [Sh6]. We shall continue to make our Assumption 1.1 for $(G, M, \pi)$ as well as each $(G_i, M_i, \pi')$ and any other cusp form (such as $\pi_\chi$; see below) that appears (cf. [GS]).

**Proposition 2.1.** a) Given a pair $(G, M)$, where $G$ is a quasisplit connected reductive algebraic group over a number field $F$ and $M$ is an $F$-rational maximal Levi subgroup of $G$, there exists a rational character $\xi$ of $M$, i.e., $\xi \in X(M)_F$, with following property: Let $S$ be a nonempty finite set of finite places of $F$. For every globally generic cuspidal representation $\pi$ of $M = M(\mathbb{A}_F)$, there exist nonnegative integers $f_v$, $v \in S$, such that for every gr"ossencharacter $\chi = \otimes_v \chi_v$ of $F$ for which conductor of $\chi_v$, $v \in S$, is larger than or equal to $f_v$, every $L$-function $L(s, \pi_\chi, r_i)$, $i = 1, \ldots, m$, is entire, where $\pi_\chi = \pi \otimes (\chi \cdot \xi)$.

b) The integers $f_v$ depend only on the conductors of the local central characters of $\pi_v$ for all $v \in S$.

**Proof.** We first specify $\xi$. Let
\[
\xi(m) = \det(\text{Ad}(m)|n),
\]
m $\in M$, where $n$ is the Lie algebra of $N$, as in [Sh6].

Clearly $\xi \in X(M)_F$ and therefore $\xi \in X(M)_{F_0}$. Consequently, given a gr"ossencharacter $\chi = \otimes_v \chi_v$ of $F$, each $\chi_v \cdot \xi$ becomes a character of $M_v$. Later on in the proof we will replace $\xi$ with a power of $\xi$, if need be.

Next, let $\tilde{w}_0$ be the longest element in $W(A_0)$ modulo that of the Weyl group of $A_0$ in $M$. Then $\tilde{w}_0$ sends $\alpha$ to a negative root, while $\tilde{w}_0(\theta) \subset \Delta$.

We now apply Proposition 2.1 of [Ki1] (cf. Lemma 7.5 of [La1]) to equation (1.5) for $\pi_\chi$. It states that if $P$ is not self-conjugate, or if $P$ is self-conjugate but $w_0(\pi_\chi) \not\cong \pi_\chi$, then the constant term $M(s\tilde{\alpha}, \pi_\chi)$ of the corresponding Eisenstein series is holomorphic for $\text{Re}(s) \geq 0$. Then, together with Assumption 2.1 for $\pi_\chi$, this implies:

**Lemma 2.2.** Let $\chi$ be a gr"ossencharacter of $F$. Suppose $w_0(\pi_\chi) \not\cong \pi_\chi$ which is in particular valid if $P$ is not self-conjugate. Then
\[
\prod_{i=1}^m L(is, \pi_\chi, r_i)/L(1+is, \pi_\chi, r_i)
\]
is holomorphic for $\text{Re}(s) \geq 1/2$. Here $\pi_\chi = \pi \otimes (\chi \cdot \xi)$ and $w_0$ is a representative for $\tilde{w}_0$ in $G(F) \cap K$ as in Section 1.
It is clear that from now on we may assume $P$ is self-conjugate and therefore $w_0(A) = A$. We must first show that we can find $f_v, v \in S$, as demanded in the proposition. By Lemma 2.2, it would be enough to find a $v \in S$ and a positive integer $f_v$ for which $w_0(\pi_v \boxtimes (\chi_v \cdot \xi)) \neq \pi_v \boxtimes (\chi_v \cdot \xi)$, if $\text{Cond}(\chi_v) \geq f_v$. Let $\omega_v$ be the central character of $\pi_v$. Since the center of $M_v$ contains $A_v$, it is enough to show that for a highly ramified character $\chi_v, w_0(\omega_v(\chi_v \cdot \xi)) \neq \omega_v(\chi_v \cdot \xi)$, upon restriction to $A_v$.

Let $$A^1 = \{ a \in A | \tilde{w}_0(a) = a^{-1} \}.$$ It is a connected subgroup of $A$. The rational morphism $\alpha: A^1 \to G_m$ is surjective and consequently $\alpha(A^1_v)$ is open in $F_v^*$. Let $\omega_{v,1} = \omega_v|A^1_v$. By the conductor of $\omega_{v,1}$ we mean the smallest nonnegative integer $n_v$ such that $1 + \mathcal{P}_v^{n_v}$ is contained in $\alpha(\text{Ker}(\omega_{v,1}))$.

Next, observe that, by Lemma 2 of [Sh6], $\xi(A^1_v)$ is also open in $F_v^*$. Choose $\ell_v \in \mathbb{Z}^+$ such that $1 + \mathcal{P}_v^{\ell_v} \subset \xi(A^1_v)$. Now take $\chi_v$ so that the conductor of $\chi_v^2$ is larger than $\ell_v$ and that of $\omega_{v,1}^{-2}$. This guarantees that $\chi_v^2 \cdot \xi \neq \omega_{v,1}^{-2}$ upon restriction to $A^1_v$, implying $w_0((\chi_v \cdot \xi)\omega_v) \neq (\chi_v \cdot \xi)\omega_v$ on $A_v$ as desired. We summarize this as:

**Lemma 2.3.** Suppose $\chi$ is a gr"ossencharacter of $F$ which is appropriately highly ramified at one place $v \in S$. Then

$$\prod_{i=1}^m L(is, \pi_\chi, r_i)/L(1 + is, \pi_\chi, r_i)$$

is holomorphic for $\text{Re}(s) \geq 1/2$.

Next we need to show that under the assumption of Proposition 2.1, each $L(s, \pi_\chi, r_i)$ is holomorphic for $\text{Re}(s) \geq 1/2$. The statement of the proposition then immediately follows from functional equation (1.1). To do this we need to use the general induction from [Sh1, 2] (also see Proposition 3.1 of [GS]). Simply said: To use the fact that for each $i$, there exists a triple $(G_i, M_i, \pi')$ such that

$$L(s, \pi, r_i) = L(s, \pi', r'_i)$$

and $\varepsilon(s, \pi, r_i) = \varepsilon(s, \pi', r'_i)$,

where $r' = \bigoplus_{j=1}^{m'} r'_j$ is the corresponding decomposition for $(G_i, M_i)$, and $m' < m$.

This is necessary since we next need to show inductively that each $L(s, \pi_\chi, r_i)$, $i = 2, \ldots, m$, is nonzero for $\text{Re}(s) \geq 1$, if $\chi$ is highly ramified.

For our purposes it is more convenient to choose $G_i$ such that $M_i = M$ and $\pi' = \pi$. We will therefore use Arthur’s version of our induction which makes use of endoscopic groups ([A1, Prop. 5]).
In Arthur’s version, groups $G_i$ can be chosen to be among endoscopic groups of $G$ which share $M$, and where $\pi' = \pi$. Although we do not need this in our proof, it is worth observing that: Let $A_G$ and $A_{G_i}$ be connected components of centers of $G$ and $G_i$, respectively, and let $\overline{\chi}$ be a character of $M = M(\mathbb{A}_F)$. Then $\overline{\chi}|(A_G(\mathbb{A}_F)) = \overline{\chi}|A_{G_i}(\mathbb{A}_F)$ and therefore one restriction is trivial if and only if the other one is. In our setting $\chi \cdot \xi|A_{G_i}(\mathbb{A}_F)$ is trivial.

Next, let $\xi_i$ be the corresponding rational character defined for $(G_i, M)$; i.e.

$$\xi_i(m) = \det(\text{Ad}(m)|n_i).$$

Since $X(A^1) \mathbb{Q}$ is equal to the $\mathbb{Q}$-span of $\xi$, it is clear that there exist positive integers $n_i, i = 1, 2, \ldots, m$, such that

$$\xi^n_i = \xi^{n_1}.$$ 

Our earlier arguments are still valid if $\xi$ is changed to $\xi^{n_1}$. Then $\chi \cdot \xi^{n_1} = \chi \cdot \xi^n_i$ for each $i = 1, \ldots, m$, and if at a place $v \in S, \chi_v$ is ramified enough, then $w_{i,0}(\pi_{\chi}) \not\equiv \pi_{\chi}$, where $w_{i,0}$ is the corresponding longest element for each $i = 1, 2, \ldots, m$ and $w_{1,0} = w_0$.

We can now again appeal to Proposition 2.1 of [Ki1] (Lemma 7.5 of [La1]) to conclude that the corresponding Eisenstein series and consequently its Fourier coefficients are all holomorphic for $\text{Re}(s) \geq 0$, if $\chi_v$ is highly ramified at a place $v \in S$. It then immediately follows, say for example from Proposition 5.2 of [GS], that

$$\prod_{i=1}^{m} L(1 + is, \pi_{\chi}, r_i)$$

is nonzero for $\text{Re}(s) \geq 0$. (One needs to notice that local $L$-functions are never zero.)

Our induction hypothesis is now that: If $\chi$ is highly ramified at a place $v \in S$, then each $L(s, \pi_{\chi}, r_i), i = 2, \ldots, m$, is holomorphic for $\text{Re}(s) \geq 1/2$ and nonzero for $\text{Re}(s) \geq 1$.

We now assume $\chi$ is appropriately ramified such that $w_{i,0}(\pi_{\chi}) \not\equiv \pi_{\chi}$ for all $i = 1, 2, \ldots, m$. Applying our induction hypothesis for $L(s, \pi_{\chi}, r_i), i = 2, \ldots, m$, first to (2.1) to conclude that $L(s, \pi_{\chi}, r_1)$ is nonzero for $\text{Re}(s) \geq 1$, and next, applying this to Lemma 2.3, to get the holomorphy of $L(s, \pi_{\chi}, r_1)$ again for $\text{Re}(s) \geq 1/2$, complete the induction. Proposition 2.1 is now proved.

\textbf{Corollary 2.4 (of the proof).} Suppose $\chi$ is highly ramified for some $v \in S$. Then every $L$-function $L(s, \pi_{\chi}, r_i)$ is nonzero for $\text{Re}(s) \geq 1, 1 \leq i \leq m$. 

\begin{flushright} $\square$ \end{flushright}
Corollary 2.5. Suppose \( m = 1 \), but \( \pi \) is any cuspidal representation of \( M = \text{M}(\mathbb{A}_F) \), i.e., not necessarily globally generic. Then \( L(s, \pi, r_1) \) is analytic for \( \text{Re}(s) \geq 1/2 \), if \( \chi \) is unitary and highly ramified. Moreover, if it satisfies a standard functional equation, then it is entire. Similar statements are true if \( m = 2 \) and \( r_2 \) is one-dimensional.

3. Functorial products for \( \text{GL}_2 \times \text{GL}_3 \); The weak lift

In this section we shall start proving our main results. We will first establish Theorem 3.8 from which other results follow. To establish the lifts we shall use an appropriate version of converse theorems of Cogdell and Piatetski-Shapiro [CP-S1], and it is in fact quite surprising that analytic properties of appropriate \( L \)-functions that are needed are mainly established using our method [Sh1, 2, 4, 7], [Ki1], [GS] rather than that of Rankin-Selberg, where converse theorems have their roots.

We shall start by considering four special cases of our general machinery outlined in Section 1.

Let \( \pi_1 \) and \( \pi_2 \) be two cuspidal representations of \( \text{GL}_2(\mathbb{A}_F) \) and \( \text{GL}_3(\mathbb{A}_F) \), respectively. Here are the four cases. In each case we give a triple \( (G, M, \pi) \) as in Section 1 and in what follows \( \sigma \) always denotes a cuspidal representation of \( \text{GL}_n(\mathbb{A}_F) \), \( n = 1, 2, 3, \) and 4.

Case 1 (Case iii of [La6]). Here \( G = \text{GL}_5 \), \( M = \text{GL}_2 \times \text{GL}_3 \), \( \pi = (\pi_1 \otimes \sigma) \otimes \tilde{\pi}_2 \), \( n = 1 \) and therefore \( \sigma \) is a gr"ossencharacter of \( F \). The index \( m = 1 \). The theory of \( L \)-functions of Section 1 in this case is that of Rankin-Selberg \( L \)-functions \( L(s, (\pi_1 \otimes \sigma) \times \pi_2) \) which is very well developed. In particular the necessary analytic properties of these \( L \)-functions are all well-known (cf. [JS1], [JP-SS], [Sh4, 5], [MW2]).

Case 2 (\( D_5 - 2 \) of [Sh2]). In this case \( G \) is the simply connected type of \( D_5 \) and the pair \( (G, M) \) is as in Case \( D_5 - 2 \) of [Sh2], i.e., \( G = \text{Spin}_{10} \) and the derived group of \( M \) is \( \text{SL}_3 \times \text{SL}_2 \times \text{SL}_2 \). The index \( m = 2 \).

Case 3 (\( E_6 - 1 \) of [Sh2]). Here the group \( G \) is the simply connected \( E_6 \) and the derived group of \( M \) is \( \text{SL}_3 \times \text{SL}_2 \times \text{SL}_3 \). The index \( m = 3 \).

Case 4 (\( E_7 - 1 \) of [Sh2]). Finally we take \( G \) to be the simply connected \( E_7 \) and take the derived group of \( M \) equal to \( \text{SL}_3 \times \text{SL}_2 \times \text{SL}_4 \). Here \( m = 4 \).

We look at Case 4 in detail. The other cases are similar. We will use Bourbaki’s notation. Let \( \theta = \Delta - \{\alpha_4\} \). Let \( P = P_\theta = MN \) and let \( A \) be the connected component of the center of \( M \). Then

\[
A = \left( \bigcap_{\alpha \in \theta} \ker \alpha \right)^0
= \{ a(t) = H_{\alpha_1}(t^4)H_{\alpha_3}(t^6)H_{\alpha_4}(t^{12})H_{\alpha_5}(t^9)H_{\alpha_6}(t^6)H_{\alpha_7}(t^3) : t \in F^* \},
\]
Since $G$ is simply connected, the derived group $M_D$ of $M$ is simply connected, and consequently
\[ M_D = SL_2 \times SL_3 \times SL_4. \]

Moreover
\[ A \cap M_D = \{ H_{a_1}(t^4)H_{a_3}(t^8)H_{a_2}(t^6)H_{a_4}(t^{12})H_{a_5}(t^9)H_{a_6}(t^6)H_{a_7}(t^3) : t^{12} = 1 \}. \]

If we identify $A$ with $GL_1$, then
\[ M = (GL_1 \times SL_2 \times SL_3 \times SL_4)/(A \cap M_D). \]

We define a map $\tilde{f} : A \times M_D \to GL_1 \times GL_1 \times GL_1 \times SL_2 \times SL_3 \times SL_4$ by
\[ \tilde{f}(a(t), x, y, z) \mapsto (t^6, t^4, t^3, x, y, z). \]

It induces a map $f : M \to GL_2 \times GL_3 \times GL_4$ which is in fact an injection. We need to determine $f(H_{a_4}(t))$, $t \in F$. Write $H_{a_4}(t) = axyz$, $a \in A$, $x \in SL_2$, $y \in SL_3$, and $z \in SL_4$. Then, a glance at
\[ H_{a_4}(t^{12}) = a(t)H_{a_2}(t^{-6})H_{a_1}(t^{-4})H_{a_3}(t^{-8})H_{a_5}(t^{-9})H_{a_6}(t^{-6})H_{a_7}(t^{-3}) \]
shows that for a fixed 12th root $t^{1/12}$ of $t$, whose choice is irrelevant upon going to $f$, $x = H_{a_2}(t^{-1/2})$, $y = H_{a_1}(t^{-1/3})H_{a_3}(t^{-2/3})$, and
\[ z = H_{a_5}(t^{-3/4})H_{a_6}(t^{-1/2})H_{a_7}(t^{-1/4}), \]
where $t^{1/2} = (t^{1/12})^6$, $t^{1/3} = (t^{1/12})^4$, and $t^{1/4} = (t^{1/12})^3$. Moreover, $a = a(t^{1/12}).$

Using this, we see easily that
\[ f(H_{a_4}(t)) = (\text{diag}(1,t), \text{diag}(1,1,t), \text{diag}(1,1,1,t)). \]

Let $\pi_i, i = 1, 2$, be cuspidal representations of $GL_{1+i}(A_F)$ with central characters $\omega_{\pi_i}$, $i = 1, 2$, respectively. Let $\sigma$ be a cuspidal representation of $GL_4(A_F)$ whose central character is denoted by $\omega_\pi$.

Since $f$ is $F$-rational, it induces an injection
\[ f : M(A_F) \to GL_2(A_F) \times GL_3(A_F) \times GL_4(A_F), \]

Moreover $M(A_F)(A_F^*)^2$ is co-compact in $GL_2(A_F) \times GL_3(A_F) \times GL_4(A_F)$, where $(A_F^*)^2$ is embedded as a center of, say, the first two factors. Consequently $\pi_1 \otimes \pi_2 \otimes \sigma f(M)$, $M = M(A_F)$, decomposes to a direct sum of irreducible cuspidal representations of $M$. Let $\pi$ be any irreducible constituent of this direct sum. Then the central character of $\pi$ is $\omega_\pi = \omega_{\pi_1}^6\omega_{\pi_2}^4\omega_\pi^3$. As we shall see, choice of $\pi$ is irrelevant. Write $\pi = \otimes_v \pi_v$. (The fact that $f$ is an injection is not important. All that we need is that there is an $F$-rational map which is the identity on $M_D$.)
Now suppose each $\pi_{iv}$ is an unramified representation, given by

$$
\pi_{1v} = \pi(\eta_1, \eta_2), \quad \pi_{2v} = \pi(\nu_1, \nu_2, \nu_3), \quad \sigma_v = \pi(\mu_1, \mu_2, \mu_3, \mu_4).
$$

Then $\pi_v$ is an unramified representation of $M(F_v)$, induced from the character $\chi$ of the torus of $M(F_v)$ whose image under $f$ lies in the product of diagonal subgroups of corresponding $GL_i(F_v)$’s. Since $f$ is the identity on $M_D$,

$$
\chi \circ H_{\alpha_7}(t) = \mu_1 \mu_2^{-1}(t), \quad \chi \circ H_{\alpha_6}(t) = \mu_2 \mu_3^{-1}(t), \quad \chi \circ H_{\alpha_5}(t) = \mu_3 \mu_4^{-1}(t),
$$

Moreover, (3.1) implies

$$
\chi \circ H_{\alpha_4}(t) = \mu_4 \nu_3 \eta_2(t).
$$

We conclude that $m = 4$, and

$$
L(s, \pi_v, r_1) = L(s, \pi_{1v} \times \pi_{2v} \times \sigma_v),
$$

$$
L(s, \pi_v, r_2) = L(s, \pi_{2v} \otimes \sigma_v, (\rho_3 \otimes \omega_{\pi_{1v}}^{\omega_{\pi_{2v}}} \omega_{\pi_{2v}}^{\omega_{\pi_{2v}}} \omega_{\pi_{2v}}^{\omega_{\pi_{2v}}}) \otimes \rho_4),
$$

$$
L(s, \pi_v, r_3) = L(s, (\pi_{1v} \otimes \omega_{\pi_{1v}}, \omega_{\pi_{1v}}, \omega_{\pi_{1v}}) \times \sigma_v),
$$

$$
L(s, \pi_v, r_4) = L(s, \pi_{2v} \otimes \omega_{\pi_{1v}} \omega_{\pi_{2v}} \omega_{\pi_{2v}}).
$$

(By Proposition 2.1 we do not need to know the precise form of the last three $L$-functions.)

In Case 2, we do the same, namely, we construct a map $f : M \rightarrow GL_2 \times GL_3 \times GL_2$. Let $\pi_i, \ i = 1, 2$, be cuspidal representations of $GL_{1+i}(\mathbb{A}_F)$ with central characters $\omega_{\pi_i}, \ i = 1, 2$, resp. Let $\sigma$ be a cuspidal representation of $GL_2(\mathbb{A}_F)$ with central character $\omega_{\sigma}$. Finally, let $\pi$ be an irreducible cuspidal representation of $M(\mathbb{A}_F)$, induced by the map $f$ from $\pi_1, \pi_2, \sigma$ as before. Then the central character of $\pi$ is $\omega_\pi = \omega_{\pi_1}^{\omega_{\pi_2}^{\omega_{\pi_2}}}$ and for an unramified place $v$

$$
L(s, \pi_v, r_1) = L(s, \pi_{1v} \times \pi_{2v} \times \sigma_v),
$$

$$
L(s, \pi_v, r_2) = L(s, \pi_{2v} \otimes \omega_{\pi_{1v}}, \omega_{\pi_{2v}} \omega_{\pi_{2v}}).
$$

In Case 3, we again construct a map $f : M \rightarrow GL_2 \times GL_3 \times GL_3$ and proceed as before. Then the central character of $\pi$ is $\omega_\pi = \omega_{\pi_1}^{\omega_{\pi_2}^{\omega_{\pi_2}}} \omega_{\sigma}$ and for an unramified place $v$

$$
L(s, \pi_v, r_1) = L(s, \pi_{1v} \times \pi_{2v} \times \sigma_v),
$$

$$
L(s, \pi_v, r_2) = L(s, \pi_{2v} \otimes \omega_\pi) \times \sigma_v),
$$

$$
L(s, \pi_v, r_3) = L(s, \pi_{1v} \otimes \omega_v),
$$

where $\omega_v = \omega_{\pi_1}, \omega_{\pi_2}, \omega_{\pi_3}$.

In ramified places, we take $L(s, \pi_v, r_i)$ to be the one defined in [Sh1] for each of these cases. Observe that in particular, if $v = \infty$, then $L(s, \pi_v, r_i)$ is the corresponding Artin $L$-function (cf. [La4, 5], [Sh7]) in each case.
Remark. One can also use similitude groups to get these $L$-functions since local coefficients depend only on derived groups, while the Levi subgroups are less complicated. For example, the corresponding Levi in $GE_7 = (GL_1 \times E_7^{sc})/Z(E_7^{sc})$ is isomorphic to $GL_3 \times M_0$, where $M_0$ is the standard Levi subgroup $SL_6 \cap (GL_2 \times GL_4)$ of $SL_6$. For $GE_6$, $M = GL_2 \times M_0$, where $M_0 = SL_6 \cap (GL_3 \times GL_3)$, while for $GSpin(10)$, $M = GL_3 \times GSpin(4)$ (cf. [As]). Note that in each case there is an $F$-rational injection $f$ from $M$ to $GL_2 \times GL_3 \times GL_k, k = 2, 3, 4$.

Next, fix a nontrivial additive character $\psi = \otimes_v \psi_v$ of $F \backslash \mathbb{A}_F$ and define local root numbers $\varepsilon(s, \pi_v, r_1, \psi_v)$ again as in [Sh1]. Similar comment applies when $v = \infty$.

To proceed, we must dispose of Assumption 1.1 in our four cases, as well as in the inductive cases $(G_i, M_i, \pi'_i)$ attached to them. We will prove it for components of arbitrary $\pi$ and all the corresponding $\pi'$. With notation as in Section 1, let $N(s\tilde{\alpha}, \pi_v, \omega_0)$ denote the normalized local intertwining operator (1.4) in each of our cases. In the next section we will prove:

**Proposition 3.1.** The normalized local intertwining operators

$$N(s\tilde{\alpha}, \pi_v, \omega_0)$$

are holomorphic and nonzero for $\text{Re}(s) \geq 1/2$ and for all $v$.

For each $v$, let $\pi_{1v} \boxtimes \pi_{2v}$ be the irreducible admissible representation of $GL_6(F_v)$ attached to $\pi_{1v} \otimes \pi_{2v}$ through the local Langlands correspondence. More precisely, if $\delta_{1v}$ and $\delta_{2v}$ are representations of the Deligne-Weil group, parametrizing $\pi_{1v}$ and $\pi_{2v}$ through the local Langlands correspondence for $GL_2(F_v)$ and $GL_3(F_v)$, respectively, then let $\pi_{1v} \boxtimes \pi_{2v}$ be the representation of $GL_6(F_v)$ attached to $\delta_{1v} \otimes \delta_{2v}$ ([HT], [He], [La4]).

Set

$$\pi_1 \boxtimes \pi_2 = \otimes_v (\pi_{1v} \boxtimes \pi_{2v}).$$

It is an irreducible admissible representation of $GL_6(\mathbb{A}_F)$. Langlands’ functoriality in our case is equivalent to the assertion that $\pi_1 \boxtimes \pi_2$ is an automorphic representation.

To proceed we need to state the converse theorem that applies to our situation.

**Theorem 3.2** (Theorem 2 of [CP-S1]). Suppose $\Pi = \otimes_v \Pi_v$ is an irreducible admissible representation of $GL_m(\mathbb{A}_F)$ whose central character is a grössencharacter. Let $S$ be a finite set of finite places of $F$ and let $T^S(n)$ be the set of cuspidal representations of $GL_m(\mathbb{A}_F)$ that are unramified at all places $v \in S$. Suppose for each $n \leq m - 2$ and every cuspidal representation
σ ∈ \mathcal{T}^S(n), L(s, Π × σ) is “nice” in the sense that it satisfies the following properties:

a) The L-function L(s, Π × σ) is entire,

b) it is bounded in every vertical strip of finite width, and

c) it satisfies a standard functional equation of type (1.1).

Then there exists an automorphic representation Π' of GL_m(\mathbb{A}_F) such that Π_v ≃ Π'_v for all v ∉ S and in particular for every archimedean place of F. We will then say that Π is quasi-automorphic with respect to S.

We apply the converse theorem to \pi_1 \boxtimes \pi_2 = \otimes_v (\pi_{1v} \boxtimes \pi_{2v}). For that, we need to consider Rankin triple product L-functions found in our cases 1–4. To show that they can be made “nice,” we will twist our representation \pi by an appropriately ramified gr\"ossencharacter of F to which we can apply Proposition 2.1 to verify condition a) of Theorem 3.2. Condition b) is proved in full generality in [GS]. Condition c) is delicate and requires multiplicativity of local factors (Theorem 3.5.3 of [Sh1]). We need to prove

\begin{align}
L(s, \pi_{1v} \times \pi_{2v} \times \sigma_v) &= L(s, (\pi_{1v} \boxtimes \pi_{2v}) \times \sigma_v) \\
\varepsilon(s, \pi_{1v} \times \pi_{2v} \times \sigma_v, \psi_v) &= \varepsilon(s, (\pi_{1v} \boxtimes \pi_{2v}) \times \sigma_v, \psi_v),
\end{align}

for all irreducible admissible generic representations \sigma_v of GL_n(F_v), 1 ≤ n ≤ 4.

These equalities are not obvious at all since the two factors on the left and right are defined using completely different techniques. The factors on the left are those defined by the triple L-functions of our four cases (to be recalled in Section 5), while the ones on the right are those of Rankin-Selberg for GL_6(\mathbb{A}_F) × GL_n(\mathbb{A}_F), 1 ≤ n ≤ 4 [JP-SS], [Sh5, 7]. By the local Langlands correspondence, they are Artin factors.

In this section we will apply the converse theorem to \pi_1 \boxtimes \pi_2 where S is a finite set of places of F which contain all the finite places v where at least one of \pi_{iv}, i = 1, 2, is ramified. This makes the application of the converse theorem simpler because if \sigma ∈ \mathcal{T}^S(n), one of \sigma_v, \pi_{1v}, \pi_{2v}, is in the principal series for \nu < \infty.

In Section 5, we shall prove that the equalities (3.2.1) and (3.2.2) hold for all v. Machinery of converse theorems then implies that \pi_1 \boxtimes \pi_2 is automorphic. (See the proof immediately after the proof of Proposition 5.8 as well as the paragraph before Proposition 5.4.) We start with the following definition:

**Definition 3.3.** Let \pi_1 (\pi_2, resp.) be irreducible cuspidal representations of GL_2(\mathbb{A}_F) (GL_3(\mathbb{A}_F), resp.). An automorphic representation \Pi = \otimes_v \Pi_v of GL_6(\mathbb{A}_F) is a **strong lift or transfer** of \pi_1 \otimes \pi_2, if for every v, \Pi_v is a local lift...
or transfer of \(\pi_1 \otimes \pi_2\), in the sense that

\[
L(s, \pi_1 \times \pi_2 \times \sigma_v) = L(s, \Pi_v \times \sigma_v)
\]

and

\[
\varepsilon(s, \pi_1 \times \pi_2 \times \sigma_v, \psi_v) = \varepsilon(s, \Pi_v \times \sigma_v, \psi_v),
\]

for all irreducible admissible generic representations \(\sigma_v\) of \(\text{GL}_n(F_v)\), \(1 \leq n \leq 4\). If these equalities hold for all \(v \notin S\), then \(\Pi\) is a weak lift of \(\pi_1 \otimes \pi_2\) with respect to \(S\).

Let \(S\) be a finite set of finite places of \(F\) which contain all the places \(v\) where at least one of \(\pi_{iv}, i = 1, 2\), is ramified. Fix \(v_0 \in S\). Take a gr"ossencharacter \(\chi = \otimes_v \chi_v\). We will assume \(\chi_{v_0}\) is appropriately highly ramified so that Proposition 2.1 can be applied to \(L\)-functions \(L(s, \pi_\chi, r_1)\) in each of our four cases.

Replacing \(\chi\) with an appropriate integral power of \(\chi\) if necessary, one can find gr"ossencharacters \(\chi_n, n = 1, 2, 3, 4\), such that

\[
L(s, (\pi_1 \otimes \chi) \times \pi_2 \times \sigma_v) = L(s, \pi_{\chi_n}, r_1)
\]

for \(n = 1, 2, 3, 4\), representing each of our four cases, or equivalently as \(\sigma \in \mathcal{T}^S(n)\), \(n = 1, 2, 3, 4\).

As \(\sigma\) ranges in \(\mathcal{T}^S(n)\), the conductor of the central character of \(\pi\) will not change and therefore in view of Proposition 3.1, Part b) of Proposition 2.1 applies, implying:

**Proposition 3.4.** There exists a positive integer \(f_0\) such that for every gr"ossencharacter \(\chi = \otimes_v \chi_v\) with \(\text{Cond}(\chi_{v_0}) \geq f_0\), \(L(s, (\pi_1 \otimes \chi) \times \pi_2 \times \sigma_v)\) is entire for every \(\sigma \in \mathcal{T}^S(n)\), \(n = 1, 2, 3, 4\).

Next, taking into account our Proposition 3.1 and applying Theorem 4.1 of [GS] to our four cases, we have:

**Proposition 3.5.** Fix \(f_0\) as in Proposition 3.4. Then each

\[
L(s, (\pi_1 \otimes \chi) \times \pi_2 \times \sigma_v)
\]

is bounded in every vertical strip of finite width.

Next we show:

**Proposition 3.6.** Notation being as in Proposition 3.5, let \(\sigma \in \mathcal{T}^S(n) \otimes \chi\). Then for each \(v\),

\[
\gamma(s, \pi_1 \times \pi_2 \times \sigma_v, \psi_v) = \gamma(s, (\pi_1 \otimes \pi_2) \times \sigma_v, \psi_v),
\]

\[
L(s, \pi_1 \times \pi_2 \times \sigma_v) = L(s, (\pi_1 \otimes \pi_2) \times \sigma_v).
\]

The equality of \(\varepsilon\)-factors follows from the above equalities.
Proof. As we noted before, if \( \sigma \in T^S(n) \otimes \chi \), then for each \( v < \infty \), one of \( \sigma_v, \pi_1 \otimes \pi_2 \), is in the principal series. When \( v = \infty \), as has been shown in [Sh7], the factors on the left are Artin factors, and hence we have the equalities. If \( v < \infty \), by multiplicativity of \( \gamma \)-factors and \( L \)-functions (Part 3 of Theorem 3.5 and Section 7 of [Sh1]; also see the discussions at the beginning of Section 5 here), the factors on the left-hand side are a product of those for \( GL_k \times GL_l \). Shahidi [Sh5] has shown that in the case of \( GL_k \times GL_l \), his factors are those of Artin. Since the same multiplicativity holds for the factors on the right-hand side, we have the equalities. \( \square \)

Remark. Multiplicativity of \( L \)-functions is particularly transparent in this case since the principal series representation among \( \pi_{iv} \) and \( \sigma_v \) at each place \( v \) is in fact its own standard module (cf. Section 7 of [Sh1] and Section 5 here).

Proposition 3.7. Fix \( S \) and \( \chi \) as in Proposition 3.4. Then each \( L \)-function \( L(s, ((\pi_1 \otimes \pi_2) \otimes \chi) \times \sigma) \) is “nice” as \( \sigma \) runs over the sets \( T^S(n) \), \( n = 1, 2, 3, 4 \).

Proof. This follows immediately from Propositions 3.4, 3.5, 3.6, and the functional equation satisfied by \( L(s, \pi_1 \times \pi_2 \times \sigma) \), proved in [Sh1]. \( \square \)

We now apply the Converse Theorem 3.2 to Proposition 3.7 to conclude:

Theorem 3.8. Let \( S \) be a finite set of finite places containing all the places \( v \) for which either \( \pi_1 \) or \( \pi_2 \) is ramified. Then there exists an automorphic representation \( \Pi = \otimes_v \Pi_v \) of \( GL_6(\mathbb{A}_F) \) such that \( \Pi_v \cong \pi_{1v} \otimes \pi_{2v} \) for \( v \not\in S \).

Proof. Using Proposition 3.7, we only need to apply Theorem 3.3 to \( (\pi_1 \otimes \pi_2) \otimes \chi \) for some \( \chi \) which is highly ramified on \( S \). Thus \( (\pi_1 \otimes \pi_2) \otimes \chi \) is quasi-automorphic with respect to \( S \) and therefore so is \( \pi_1 \otimes \pi_2 \). \( \square \)

Remark 3.9. Even if one does not have the local Langlands correspondence, one can still use the converse theorem. For each \( v \in S \), take \( \Pi_v \) to be arbitrary, up to its central character, which is predetermined by central characters of \( \pi_{iv} \), \( i = 1, 2 \). We would then need to use stability of Rankin-Selberg \( \gamma \)-functions under highly ramified twists (Proposition 4 of [JS2]) which is a deep result from the Rankin-Selberg method. We refer to [CP-S3] for more detail, as well as [CKPSS] for a very important application.

Now, let \( \Pi = \otimes_v \Pi_v \) denote a weak lift of \( \pi_1 \otimes \pi_2 \), with respect to \( S \), i.e. an automorphic representation for which \( \Pi_v \cong \pi_{1v} \otimes \pi_{2v} \), for all \( v \not\in S \). Choose real numbers \( r_i \) and (unitary) cuspidal representations \( \sigma_i \) of \( GL_{n_i}(\mathbb{A}_F) \),
i = 1, \ldots, k, such that \( \Pi \) is equivalent to a subquotient of
\[
I = \text{Ind} \, \sigma_1| \det( )|^{r_1} \otimes \cdots \otimes \sigma_k| \det( )|^{r_k}.
\]
Since the central character \( \omega_\Pi = \omega_1^3 \omega_2^2 \) is unitary,
\[
n_1 r_1 + \cdots + n_k r_k = 0.
\]
Observe that \( n_i > 1 \) since \( L_S(s, \Pi \otimes \mu) = L_S(s, (\pi_1 \otimes \pi_2) \times \pi_2) \) is entire for every gr\"ossencharacter \( \mu \). Here \( L_S \) is the partial \( L \)-function with respect to \( S \), i.e., the product of all the local \( L \)-functions for \( v \) outside of \( S \). We shall prove:

PROPOSITION 3.10. The exponents \( r_1 = \cdots = r_k = 0 \) and \( I = \text{Ind} \, \sigma_1 \otimes \cdots \otimes \sigma_k \) are irreducible. Therefore in the notation of \([La^3]\)
\[
(3.10.1) \quad \Pi = \sigma_1 \boxplus \cdots \boxplus \sigma_k.
\]
Moreover, \( \Pi \) is unique and each local component \( \Pi_v \) of \( \Pi \) is irreducible, unitary, and generic.

Proof. We need to use the weak Ramanujan property for a (unitary) cuspidal representation \( \pi = \otimes_v \pi_v \) of \( \text{GL}_n(\mathbb{A}_F) \), where \( n \) is a positive integer. Let \( \pi_v \) be an unramified component of \( \pi \). Write:
\[
(3.10.2) \quad \pi_v = \text{Ind} \mu_1| \det( )|^{s_1} \otimes \cdots \otimes \mu_\ell| \det( )|^{s_\ell} \otimes \cdots \otimes \mu_\nu| \det( )|^{-s_1} \otimes \cdots \otimes \mu_1| \det( )|^{-s_1},
\]
where \( \mu_i \) and \( \nu_j \) are unitary characters of \( \text{F}_v^* \) and \( 0 < s_\ell \leq \cdots \leq s_1 < 1/2 \), by classification of irreducible unitary generic representations of \( \text{GL}_n(\text{F}_v) \) (cf. \([Tad]\)). Here we have suppressed the dependence of all the factors on \( v \) for simplicity of notation. Then \( \pi \) is said to satisfy the \textit{weak Ramanujan property}, if given \( \varepsilon > 0 \), the set of places \( v \) for which \( s_1 \geq \varepsilon \) is of density zero (cf. \([CP^2]\) for the original idea; also see \([Ki^2]\)).

It then follows from Ramakrishnan (Lemma 3.1 of \([Ra^2]\)) that for \( n = 2 \) and 3 every irreducible cuspidal representation satisfies the weak Ramanujan property. Consequently, so does every weak lift \( \Pi \) of \( \pi_1 \otimes \pi_2 \) (with the same definition for an automorphic representation).

Suppose \( \tau_i = \sigma_i| \det( )|^{r_i}, 1 \leq i \leq k \). For an unramified pair \((\pi_{1v}, \pi_{2v})\), let \( t_v = \text{diag}(a_{1v}, \eta_v a_{1v}^{-1}, \ldots) \) denote the Hecke-Frobenius (Satake) parameter of \( \sigma_{1v}, |\eta_v| = 1 \), by the fact that \( \sigma_{1v} \) has the form \((3.10.2)\). We may assume
\[
|a_{1v}| \geq 1.
\]
By the equality \( n_1 r_1 + \cdots + n_k r_k = 0 \), if \( r_i \) are not all zero, then there is \( i \) such that \( r_i > 0 \). Using the fact that \( |\varpi_v|^{r_i} t_v \) defines the Hecke-Frobenius parameter of \( \tau_{1v} \), we have
\[
|\tilde{a}_{1v}| = |a_{1v}|^{-1} q_v^{-r_i} \leq q_v^{-r_i} < 1,
\]
where \( \tilde{a}_{1v} = \eta_v a_{1v}^{-1} |\varpi_v|^{r_i} \). Now take \( \varepsilon = r_i \) to contradict the weak Ramanujan property for \( \Pi \). Thus \( r_i = 0 \) for all \( i \).
Note that for each \( v \), \( \text{Ind} \sigma_{1v} \otimes \cdots \otimes \sigma_{kv} \) is irreducible, unitary, and generic [Tad]. Hence \( \Pi_v = \text{Ind} \sigma_{1v} \otimes \cdots \otimes \sigma_{kv} \) for all \( v \). Thus \( \Pi = \sigma_1 \boxplus \cdots \boxplus \sigma_k \).

From the classification theorem for \( GL(n) \) (cf. [JS1]) and (3.10.1) it is clear that the weak lift is unique. The proposition is now proved. \( \square \)

4. Proof of Proposition 3.1

In [Ki5], it was shown that in a fairly general setting, the normalized local intertwining operators \( N(s, \sigma_v, w_0) \) are holomorphic and nonzero for \( \text{Re} s \geq \frac{1}{2} \) for all \( v \). For the sake of completeness, we include a proof here in our cases. First we need:

**Lemma 4.1.** Conjecture 7.1 of [Sh1] is true in the following cases:

1. \( D_n - 2 \) \( (n \geq 4) \): i.e., the local \( L \)-function \( L(s, \pi_{1v} \times \pi_{2v} \times \sigma_v) \) is holomorphic for \( \text{Re} s > 0 \), where \( \pi_{1v}, \pi_{2v}, \) and \( \sigma_v \) are tempered representations of \( GL_2(F_v), GL_{n-2}(F_v), GL_2(F_v) \),

2. \( E_6 - 1 \) and \( E_7 - 1 \): i.e., the local \( L \)-functions \( L(s, \pi_{1v} \times \pi_{2v} \times \sigma_v) \) are holomorphic for \( \text{Re} s > 0 \), where \( \pi_{1v}, \pi_{2v}, \) and \( \sigma_v \) are tempered representations of \( GL_2(F_v), GL_3(F_v), \) and \( GL_n(F_v) \), \( n = 3, 4 \), resp.

3. \( D_6 - 3 \): i.e., the local \( L \)-function \( L(s, \pi_{2v} \otimes \sigma_v, \rho_3 \otimes \wedge^2 \rho_4) \) is holomorphic for \( \text{Re} s > 0 \), where \( \pi_{2v} \) and \( \sigma_v \) are tempered representations of \( GL_3(F_v) \) and \( GL_4(F_v) \), resp. Note that the case \( D_6 - 3 \) appears as the second \( L \)-function in the case \( E_7 - 1 \) (see Section 3).

**Proof.** For simplicity, we drop the subscript \( v \). For the case \( D_n - 2 \) \( (m = 2) \), we can use either Asgari’s thesis ([As]) in which many other results on Conjecture 7.1 of [Sh1] are proved, or use Ramakrishnan’s result [R1] on the local lift of \( GL_2(F) \times GL_2(F) \) to \( GL_4(F) \) to reduce it to a Rankin-Selberg \( L \)-function for \( GL_{n-2}(F) \times GL_4(F) \), namely,

\[
L(s, \pi_1 \times \pi_2 \times \sigma) = L(s, \pi_2 \times (\pi_1 \boxplus \sigma)).
\]

For the case \( D_6 - 3 \), we need to use [As]. In this case, the trouble is when \( \pi_{2v} \) is a Steinberg representation, for then factors cancel between numerator and denominator of \( \gamma \)-factors. If we use \( \text{Spin}_{12} \), the Levi subgroup is very complicated and it is difficult to use multiplicativity of \( \gamma \)-functions (part 3 of Theorem 3.5 of [Sh1]). Asgari’s idea is to use \( G\text{Spin}_{12} \), instead of \( \text{Spin}_{12} \). The Levi subgroup is \( GL_3 \times G\text{Spin}_6 \) and multiplicativity of \( \gamma \)-functions (part 3 of Theorem 3.5 of [Sh1]) becomes transparent.

The cases \( E_6 - 1 \) and \( E_7 - 1 \) are dealt with case by case analysis. We first consider the case \( E_6 - 1 \) \( (m = 3) \):
Case 1. If $\pi_1$ is not a discrete series, then by multiplicativity of $\gamma$-functions (part 3 of Theorem 3.5 of [Sh1]), $\gamma(s, \pi, r_1, \psi)$ is a product of $\gamma$-functions for rank-one situations for $\text{SL}_3 \times \text{SL}_3$. Hence $L(s, \pi, r_1)$ is a product of $L$-functions for $\text{SL}_3 \times \text{SL}_3$, and is therefore holomorphic for $\text{Re} \ s > 0$.

Case 2. If $\pi_1$ is a special representation, given as the subrepresentation of $\text{Ind} \, \mu | \left[ \frac{\tau}{2} \otimes \mu \right] | ^{-\frac{1}{2}}$, then by multiplicativity of $\gamma$-functions (part 3 of Theorem 3.5 of [Sh1]),

$$\gamma(s, \pi, r_1, \psi) = \gamma \left( s + \frac{1}{2}, \sigma_1, \psi \right) \gamma \left( s - \frac{1}{2}, \sigma_2, \psi \right),$$

where $\sigma_1, \sigma_2$ are tempered representations of $F$-points of $M'$ for which the derived group $M'_D$ is $\text{SL}_3 \times \text{SL}_3$, and the $\gamma(s, \sigma_i, \psi)$'s are the Rankin-Selberg $\gamma$-factors for $\text{GL}_3 \times \text{GL}_3$. Note that $L(s, \sigma_i)$ is holomorphic for $\text{Re} \ s > 0$ and hence the only possible poles of $L(s, \pi, r_1)$ are with $\text{Re} \ s = \frac{1}{2}$, which is excluded.

Case 3. Representation $\pi_1$ is supercuspidal.

Case 3.1. If $\pi_2$ is supercuspidal, then $L(s, \pi, r_2)$ is trivial unless $\sigma$ is also supercuspidal and we are reduced to a case of Proposition 7.2 of [Sh1].

Case 3.2. Suppose $\pi_2$ is not a discrete series. Then by multiplicativity of $\gamma$-functions (part 3 of Theorem 3.5 of [Sh1]), $\gamma(s, \pi, r_1, \psi)$ is a product of $\gamma$-functions for rank-one situations for either $D_5 - 2$ or $\text{SL}_2 \times \text{SL}_3$. Hence $L(s, \pi, r_1)$ is a product of $L$-functions for either $D_5 - 2$ or $\text{SL}_2 \times \text{SL}_3$, and it is therefore holomorphic for $\text{Re} \ s > 0$.

Case 3.3. Suppose $\pi_2$ is a special representation, given as the subrepresentation of $\text{Ind} \, \mu | \left[ \frac{\tau}{2} \otimes \mu \otimes \mu \right] | ^{-1}$. Then again by multiplicativity of $\gamma$-functions (part 3 of Theorem 3.5 of [Sh1]),

$$\gamma(s, \pi, r_1, \psi) = \gamma(s + 1, \sigma_1, \psi) \gamma(s, \sigma_2, \psi) \gamma(s - 1, \sigma_3, \psi),$$

where the $\sigma_i$'s are tempered representations of $F$-points of $M'$ whose derived group is $\text{SL}_2 \times \text{SL}_3$, and the $\gamma(s, \sigma_i, \psi)$'s are the Rankin-Selberg $\gamma$-factors for $\text{GL}_2 \times \text{GL}_3$. Note that $L(s, \sigma_i) = 1$ unless $\sigma$ is of the form $\sigma = \text{Ind} \, \tau \otimes \eta$, where $\tau$ is a supercuspidal representation of $\text{GL}_2(F)$. In this case, we use the argument in case 3.2 when $\pi_2$ is not in the discrete series.

Next we look at the case $E_7 - 1$ ($m = 4$):

Case 1. If $\pi_1$ is not a discrete series, then by multiplicativity of $\gamma$-functions (part 3 of Theorem 3.5 of [Sh1]), $\gamma(s, \pi, r_1, \psi)$ is a product of $\gamma$-functions for rank-one situations for $\text{SL}_3 \times \text{SL}_4$. Hence $L(s, \pi, r_1)$ is a product of $L$-functions for $\text{SL}_3 \times \text{SL}_4$, and is consequently holomorphic for $\text{Re} \ s > 0$.

Case 2. Suppose $\pi_1$ is a special representation, given as the subrepresentation of $\text{Ind} \, \mu | \left[ \frac{\tau}{2} \otimes \mu \right] | ^{-\frac{1}{2}}$; then

$$\gamma(s, \pi, r_1, \psi) = \gamma \left( s + \frac{1}{2}, \sigma_1, \psi \right) \gamma \left( s - \frac{1}{2}, \sigma_2, \psi \right),$$
where $\sigma_i$'s are tempered representations of $F$-points of $M'$ for which the derived group $M'_D$ is $SL_3 \times SL_4$, and the $\gamma(s, \sigma_i, \psi)$'s are the Rankin-Selberg $\gamma$-factors for $GL_3 \times GL_4$. Since $L(s, \sigma_i)$ is holomorphic for $Re\, s > 0$, the only poles of $L(s, \pi, r_1)$ are at $Re\, s = \frac{1}{2}$, which are excluded.

Case 3. Representation $\pi_1$ is supercuspidal.

Case 3.1. Suppose $\pi_2$ is not a discrete series. Then by multiplicativity of $\gamma$-functions (part 3 of Theorem 3.5 of [Sh1]), $\gamma(s, \pi, r_1, \psi)$ is a product of $\gamma$-functions for rank-one situations for $D_6 - 2$, or $SL_2 \times SL_4$. Hence $L(s, \pi, r_1)$ is a product of $L$-functions for either $D_6 - 2$ or $SL_2 \times SL_4$, and is therefore holomorphic for $Re\, s > 0$.

Case 3.2. Suppose $\pi_2$ is supercuspidal. If $\sigma$ is not a discrete series, then by multiplicativity of $\gamma$-functions (part 3 of Theorem 3.5 of [Sh1]), $\gamma(s, \pi, r_1, \psi)$ is a product of $\gamma$-functions for rank-one situations for either $E_6 - 1$, $D_5 - 2$, or $SL_2 \times SL_3$. Hence $L(s, \pi, r_1)$ is a product of $L$-functions for either cases $E_6 - 1$, $D_5 - 2$, or $SL_2 \times SL_3$, and is consequently holomorphic for $Re\, s > 0$ by the above result for the case $E_6 - 1$.

If $\sigma$ is supercuspidal, we are reduced to a case of Proposition 7.2 of [Sh1]. Suppose $\sigma$ is given as the subrepresentation of $\text{Ind} \rho | \text{det}^{-\frac{1}{2}} \otimes \rho | \text{det}^{-\frac{1}{2}}$, where $\rho$ is a supercuspidal representation of $GL_2(F)$. Then

$$\gamma(s, \pi, r_1, \psi) = \gamma(s + \frac{1}{2}, \sigma_1, \psi) \gamma(s - \frac{1}{2}, \sigma_2, \psi),$$

where $\sigma_i$'s are tempered representations of $F$-points of $M'$ for which the derived group $M'_D$ is $SL_2 \times SL_2 \times SL_2$. This is case $D_4 - 2$, and by the above result, $L(s, \sigma_i)$ is holomorphic for $Re\, s > 0$. Hence $L(s, \pi, r_1)$ can have a pole at most with $Re\, s = \frac{1}{2}$, which is excluded.

If $\sigma$ is given as the subrepresentation of $\text{Ind} \mu | \frac{1}{2} \otimes \mu | \frac{1}{2} \otimes \mu | -\frac{1}{2} \otimes \mu | -\frac{1}{2}$, then all rank-one situations are that of $SL_3 \times SL_2$, which is non-self-conjugate. Hence $L(s, \pi, r_1) = 1$.

Case 3.3. Suppose $\pi_2$ is a special representation, given as the subrepresentation of $\text{Ind} \mu | \otimes \mu \otimes \mu | -1$. Then

$$\gamma(s, \pi, r_1, \psi) = \gamma(s + 1, \sigma_1, \psi) \gamma(s, \sigma_2, \psi) \gamma(s - 1, \sigma_3, \psi),$$

where $\sigma_i$'s are tempered representations of $F$-points of $M'$ for which the derived group $M'_D$ is $SL_2 \times SL_4$, and $\gamma(s, \sigma_i, \psi)$'s are the Rankin-Selberg $\gamma$-factors for $GL_2 \times GL_4$. Note that $L(s, \sigma_i)$ is not trivial only when $\sigma$ is of the form $\text{Ind} \tau \otimes \tau'$, where $\tau, \tau'$ are either supercuspidal representations of $GL_2(F)$, or are in the discrete series, given each as the irreducible subrepresentation of $\text{Ind} \rho | \text{det}^{-\frac{1}{2}} \otimes \rho | \text{det}^{-\frac{1}{2}}$, where $\rho$ is a supercuspidal representation of $GL_2(F)$. If $\sigma$ is of the first form, then we are in the first part of Case 3.2. If $\sigma$ is in the
discrete series, then the \( L \)-function is given by \((1 - uq^{-\frac{1}{2} - s})^{-1}\), where \( u \) is a complex number with absolute value 1. Hence \( L(s, \sigma, r_1) \) can have a pole only at \( \text{Re} \, s = \frac{1}{2} \), which is excluded. \( \square \)

**Proposition 4.2.** The normalized local intertwining operators

\[ N(s\tilde{\alpha}, \pi_v, w_0) \]

are holomorphic and nonzero for \( \text{Re} \, s \geq \frac{1}{2} \) and for all \( v \).

**Proof.** We proceed as in [Ki2, Prop. 3.4]. If \( \pi_v \) is tempered, then the unnormalized operator is holomorphic and nonzero for \( \text{Re} \, s > 0 \). By Lemma 4.1, \( L(s, \pi_v, r_i) \) is holomorphic for \( \text{Re} \, s > 0 \). Hence the normalized operator \( N(s\tilde{\alpha}, \pi_v, w_0) \) is holomorphic and nonzero for \( \text{Re} \, s > 0 \).

If \( \pi_v \) is nontempered, we write \( I(s, \pi_v) \) as in [Ki1, p. 841],

\[ I(s, \pi_v) = I(s\tilde{\alpha} + \Lambda_0, \pi_0) = \text{Ind}^{G(F)}_{M_0(F_0)N_0(F_0)} \pi_0 \otimes q^{(s\tilde{\alpha} + \Lambda_0, H_{F_0}(\cdot))}, \]

where \( \pi_0 \) is a tempered representation of \( M_0(F_0) \) and \( P_0 = M_0N_0 \) is another parabolic subgroup of \( G \). Moreover, \( (\Lambda_0, \pi_0) \) is the Langlands parameter of \( \pi_v \). We can identify the normalized operator \( N(s\tilde{\alpha}, \pi_v, w_0) \) with the normalized operator \( N(s\tilde{\alpha} + \Lambda_0, \pi_0, w_0) \), which is a product of rank-one operators attached to tempered representations (cf. [Z, Prop. 1]).

By direct observation, we see that all the rank-one operators are operators attached to tempered representations of a parabolic subgroup whose Levi subgroup has a derived group isomorphic to \( SL_k \times SL_l \) inside a group whose derived group is \( SL_{k+l} \), except one case. It is in the case \( E_7 - 1 \), when \( \pi_1, \pi_2 \) are tempered, and \( \sigma \) is the nontempered representation, given by the quotient of \( \text{Ind} | \det |^r \rho \otimes | \det |^{-r} \rho \), where \( \rho \) is a tempered representation of \( GL_2(F) \) and \( 0 < r < \frac{1}{2} \). The rank-one operator is that of the case \( D_5 - 2 \), attached to \( \pi_1, \pi_2 \) and \( \rho \). However in this case, \( s\tilde{\alpha} + \Lambda_0 \) is in the corresponding positive Weyl chamber for \( \text{Re} \, s \geq \frac{1}{2} \), and hence \( N(s\tilde{\alpha} + \Lambda_0, \pi_0, w_0) \) is holomorphic for \( \text{Re} \, s \geq \frac{1}{2} \) [Ki1, Lemma 2.4].

The rank-one operators attached to tempered representations of \( SL_k \times SL_l \) are then restrictions to \( SL_{k+l} \) of corresponding standard operators for \( GL_{k+l} \). By [MW2, Prop. I.10] one knows that these rank-one operators are holomorphic for \( \text{Re} \, s > -1 \). Hence by identifying roots of \( G \) with respect to a parabolic subgroup, with those of \( G \) with respect to the maximal torus, it is enough to check \( (s\tilde{\alpha} + \Lambda_0, \beta^\vee) > -1 \) for all positive roots \( \beta \) if \( \text{Re} \, s \geq \frac{1}{2} \). We do this case by case as follows:

**Case \( D_5 - 2 \).** In the notation of Bourbaki [Bou], \( \tilde{\alpha} = e_1 + e_2 + e_3; \Lambda_0 = r_1e_1 - r_1e_3 + r_2(e_4 - e_5) + r_3(e_4 + e_5) \), where \( \frac{1}{2} > r_1, r_2, r_3 \geq 0 \). Here \( \pi_{1v} \) is tempered if \( r_1 = 0 \). Hence

\[ s\tilde{\alpha} + \Lambda_0 = (s + r_1)e_1 + se_2 + (s - r_1)e_3 + (r_2 + r_3)e_4 + (-r_2 + r_3)e_5. \]
Ad (It is irreducibly induced from cuspidal representations, \( v \pi \) and \( \pi \) correspond. Let \( \pi \) be a presentation of \( GL_6 \).) Let \( \beta = e_3 - e_4 \). It is larger than \( -1 \) if \( Re s \geq \frac{1}{2} \). Consequently, \( N(s\alpha + \Lambda_0, \pi_0, w_0) \) is holomorphic for \( Re s \geq \frac{1}{2} \). By Zhang’s lemma (cf. [Ki2, Lemma 1.7] and [Z]), it is nonzero as well.

**Case** \( E_6 - 1 \). In the notation of Bourbaki [Bou], \( \alpha = w_4 = 2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 6\alpha_4 + 4\alpha_5 + 2\alpha_6 \); \( \Lambda_0 = r_1\alpha_1 + r_1\alpha_3 + r_2\alpha_5 + r_2\alpha_6 + r_3\alpha_2 \), where \( \frac{1}{2} > r_1, r_2, r_3 \geq 0 \). Hence

\[
\begin{align*}
\tilde{s\alpha} + \Lambda_0 &= (2s + r_1)\alpha_1 + (3s + r_3)\alpha_2 + (4s + r_1)\alpha_3 \\
&+ 6\alpha_4 + (4s + r_2)\alpha_5 + (2s + r_2)\alpha_6.
\end{align*}
\]

We observe that the least value of \( \text{Re}(s\alpha + \Lambda_0, \beta^\vee) \) is \( Re s - (r_1 + r_2 + r_3) \) when \( \beta = \alpha_4 \). It is larger than \( -1 \) if \( Re s \geq \frac{1}{2} \). Consequently, \( N(s\alpha + \Lambda_0, \pi_0, w_0) \) is holomorphic for \( Re s \geq \frac{1}{2} \). By Zhang’s lemma (cf. [Ki2, Lemma 1.7] and [Z]), it is nonzero as well.

**Case** \( E_7 - 1 \). In the notation of Bourbaki [Bou], \( \alpha = w_4 = 4\alpha_1 + 6\alpha_2 + 8\alpha_3 + 12\alpha_4 + 9\alpha_5 + 6\alpha_6 + 3\alpha_7 \); \( \Lambda_0 = r_1\alpha_1 + r_1\alpha_3 + r_2\alpha_5 + (r_2 + r_3)\alpha_6 + r_2\alpha_7 + r_4\alpha_2 \), where \( \frac{1}{2} > r_1, r_4 \geq 0 \) and \( \frac{1}{2} > r_2, r_3 \geq 0 \). Hence

\[
\begin{align*}
\tilde{s\alpha} + \Lambda_0 &= (4s + r_1)\alpha_1 + (6s + r_4)\alpha_2 + (8s + r_1)\alpha_3 + 12\alpha_4 \\
&+ (9s + r_2)\alpha_5 + (6s + r_2 + r_3)\alpha_6 + (3s + r_2)\alpha_7.
\end{align*}
\]

We observe that the least value of \( \text{Re}(s\alpha + \Lambda_0, \beta^\vee) \) is \( Re s - (r_1 + r_2 + r_4) \) when \( \beta = \alpha_4 \). It is larger than \( -1 \) if \( Re s \geq \frac{1}{2} \). Consequently, \( N(s\alpha + \Lambda_0, \pi_0, w_0) \) is holomorphic for \( Re s \geq \frac{1}{2} \). By Zhang’s lemma (cf. [Ki2, Lemma 1.7]), it is nonzero as well.

\[\Box\]

**5. Functorial products for \( GL_2 \times GL_3 \)**

In this section we will prove our main theorem. Recall that \( \pi_1 = \bigotimes_v \pi_{1v} \) and \( \pi_2 = \bigotimes_v \pi_{2v} \) are cuspidal representations of \( GL_2(A_F) \) and \( GL_3(A_F) \), respectively. Moreover, for each \( v \), \( \pi_{1v} \otimes \pi_{2v} \) is the irreducible admissible representation of \( GL_6(F_v) \) attached to \( \pi_{1v} \otimes \pi_{2v} \) through the local Langlands correspondence. Let \( \pi_1 \boxtimes \pi_2 = \bigotimes_v (\pi_{1v} \boxtimes \pi_{2v}) \). It is an irreducible admissible representation of \( GL_6(A_F) \).

**Theorem 5.1.** The representation \( \pi_1 \boxtimes \pi_2 \) of \( GL_6(A_F) \) is automorphic. It is irreducibly induced from cuspidal representations, i.e., \( \Pi = \text{Ind}_{\sigma_1 \otimes \cdots \otimes \sigma_k} \), where \( \sigma_i \)’s are cuspidal representations of \( GL_{n_i}(A_F), n_i > 1 \). The cases \( k = 3, n_1 = n_2 = n_3 = 2 \), or \( k = 2, n_1 = 2, n_2 = 4 \), occur if and only if \( \pi_2 \) is a twist of \( Ad(\pi_1) \) by a grüssencharacter.
We will first prove the last statement, assuming the earlier ones. Suppose
\( n_i = 2 \) for some \( i \) and consider the partial \( L \)-function
\[
L_S(s, \Pi \times \tilde{\sigma}_i)
\]
which has a pole at \( s = 1 \) (cf. [JS1] and [Sh4]), where \( S \) is a finite set of places
outside of which everything is unramified. It equals
\[
L_S(s, (\pi_1 \boxtimes \pi_2) \times \tilde{\sigma}_i) = L_S(s, \pi_2 \times (\pi_1 \boxtimes \tilde{\sigma}_i)).
\]
Since any quadratic base change of \( \pi_2 \) is cuspidal (Theorem 4.2 of [AC]),
we may assume that \( \pi_1 \boxtimes \tilde{\sigma}_i \) is not an automorphic induction from a quadratic
extension of \( F \). For then \( L_S(s, \pi_2 \times (\pi_1 \boxtimes \tilde{\sigma}_i)) \) would never have a pole. Consequently,
by the discussion in the proof of Part II of Lemma 3.1.1 of [R1], we may assume neither
\( \pi \) nor \( \tilde{\sigma} \), is monomial. By [R1, Th. M, pg. 54], \( \pi \boxtimes \tilde{\sigma} \) is cuspidal unless \( \sigma \cong \pi_1 \otimes \eta \) for some gr"oschencharacter \( \eta \) and only then. In this
case, \( \pi \boxtimes \tilde{\sigma} = \text{Ad}(\pi_1) \otimes \eta^{-1} \boxplus \eta^{-1} \). Since \( L_S(s, \pi_2 \times (\text{Ad}(\pi_1) \otimes \eta^{-1})) \) must now
have a pole at \( s = 1 \), we have \( \pi_2 \cong \text{Ad}(\pi_1) \otimes \eta \). Conversely, if \( \pi_2 \cong \text{Ad}(\pi_1) \otimes \eta \),
then
\[
L_S(s, \Pi \times (\pi_1 \otimes \eta^{-1})) = L_S(s, \pi_2 \times (\text{Ad}(\pi_1) \otimes \eta^{-1})) L_S(s, \pi_2 \otimes \eta^{-1})
\]
will have a pole at \( s = 1 \). Thus \( \sigma = \pi_1 \otimes \eta \) appears in the inducing data for
\( \Pi = \pi_1 \boxtimes \pi_2 \).

We now proceed to prove the main part of Theorem 5.1 and it is appropriate to remind the reader of the definitions of \( L \)-functions and root numbers using our method ([Sh1]). We shall freely use definitions and results from
[Sh1, 4, 7]. We start by applying Theorem 3.5 of [Sh1] to each of our four cases
explained in Section 3. This allows us to define a \( \gamma \)-factor \( \gamma(s, \pi_{1v} \times \pi_{2v} \times \sigma_v, \psi_v) \)
at each place \( v \). When \( v = \infty \), we will define \( L(s, \pi_{1v} \times \pi_{2v} \times \sigma_v) \) and
\( \varepsilon(s, \pi_{1v} \times \pi_{2v} \times \sigma_v, \psi_v) \) using parametrization ([La4] and [Sh7]). They satisfy
\[
\gamma(s, \pi_{1v} \times \pi_{2v} \times \sigma_v, \psi_v)
= \varepsilon(s, \pi_{1v} \times \pi_{2v} \times \sigma_v, \psi_v) L(1 - s, \pi_{1v} \times \pi_{2v} \times \sigma_v)/L(s, \pi_{1v} \times \pi_{2v} \times \sigma_v).
\]
Now, suppose \( v < \infty \) and \( \pi_{iv}, i = 1, 2 \), and \( \sigma_v \) are all tempered. As explained in
[Sh1, §7], we then define \( L(s, \pi_{1v} \times \pi_{2v} \times \sigma_v) \) as the inverse of a polynomial in \( q_v^{-s} \)
whose constant term is 1 and which has the same zeros as \( \gamma(s, \pi_{1v} \times \pi_{2v} \times \sigma_v, \psi_v) \). The root number \( \varepsilon(s, \pi_{1v} \times \pi_{2v} \times \sigma_v, \psi_v) \) is now defined using (5.1). It is
now clear that if \( \pi_{iv}, i = 1, 2 \), and \( \sigma_v \) are all tempered, then the \( \gamma \)-function
determines the root number and the \( L \)-function uniquely.

Defining the \( L \)-function for nontempered ones is more delicate. We need to use Langlands classification and multiplicativity (Part 3 of Theorem 3.5 in
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Then $\gamma(s, \pi_{1v} \times \pi_{2v} \times \sigma_v, \psi_v)$ can be written as a product of $\gamma$-functions defined by the quasi-tempered data which gives the Langlands parameter for $\pi_{1v} \otimes \pi_{2v} \otimes \sigma_v$ as its unique Langlands subrepresentation. (To apply multiplicativity, one must consider the representation $\pi_{1v} \otimes \pi_{2v} \otimes \sigma_v$ as a subrepresentation, in fact the unique one, of the induced representation with a parameter in the negative Weyl chamber. But even if one uses the standard module, i.e., the one with a parameter in the positive Weyl chamber, then the two defining local coefficients differ only by a local coefficient which is defined by a standard intertwining operator for the group $M$, rather than one for $G$. Consequently, the corresponding local coefficient is independent of $s$ and can be used to normalize the defining operator, which can have a pole or a zero, leading to a local coefficient, now defined by this normalized operator, equal to 1.) More precisely, each of these $\gamma$-functions is defined by means of a maximal Levi in a connected reductive group via Theorem 3.5 of [Sh1]. For each of them an $L$-function is defined by means of analytic continuation of the $L$-function from the tempered data, as explained earlier, to the quasi-tempered ones. In our case, they are all triple product $L$-functions coming from groups of lower rank. The $L$-function $L(s, \pi_{1v} \times \pi_{2v} \times \sigma_v)$ is then the product of these $L$-functions. This is what we call multiplicativity for $L$-functions. This definition agrees completely with parametrization and in particular when $v = \infty$ (cf. [La4]). The root numbers are now defined by means of (5.1). We observe that, as discussed above, both the $L$-function and the root number are independent of whether the representation is considered as a Langlands subrepresentation or a Langlands quotient. More generally, local coefficients are defined to depend only on the equivalence classes of inducing data, by setting any local coefficient which can be defined by means of an intertwining operator between different realizations of inducing data, which in fact does not depend on the complex parameter, equal to 1. Moreover, as discussed in Section 9 of [Sh1], the same definition can be used to define the factors even for Langlands quotients which are not generic. We refer to Sections 7 and 9 of [Sh1] for more detail. The reader can consult our proof of Lemma 4.1 in Section 4 to see how multiplicativity is applied and what the lower rank factors are.

**Proposition 5.2.** Suppose either $\pi_{1v}$, or $\pi_{2v}$, is not supercuspidal. Then the equalities (3.2.1) and (3.2.2) hold. More precisely,

\begin{align}
L(s, \pi_{1v} \times \pi_{2v} \times \sigma_v) &= L(s, (\pi_{1v} \boxtimes \pi_{2v}) \times \sigma_v) \\
\varepsilon(s, \pi_{1v} \times \pi_{2v} \times \sigma_v, \psi_v) &= \varepsilon(s, (\pi_{1v} \boxtimes \pi_{2v}) \times \sigma_v, \psi_v)
\end{align}

for every irreducible admissible generic representation $\sigma_v$ of $GL_n(F_v)$, $n = 1, 2, 3, 4$.  

functional equations for $L_w < \text{Ramakrishnan} [R1]$. Consider the case $D_σ \text{L}_3 \text{percuspidal representation of GL}_π\text{tations}\text{Langlands correspondence. By [Sh1, Prop. 5.1], we can find cu spidual representations}$

$\pi_1 = \text{an extraordinary supercuspidal representation of GL}_π\text{ of Artin [Sh7].}$

Since $π_1 \otimes \psi_1$, or $π_π\text{γ of corresponding }\gamma\text{-factors,}$

$\gamma(s, π_1 \times π_2 \times σ_π, \psi_π) = \gamma(s, π_1 \times ρ \times σ_π, \psi_π)\gamma(s, (π_1 \otimes ρ) \times σ_π, \psi_π)$.

It is enough to prove that

$\gamma(s, π_1 \times ρ \times σ_π, \psi_π) = \gamma(s, (π_1 \boxtimes ρ) \times σ_π, \psi_π)$,

where $π_1 \boxtimes ρ$ is an irreducible representation of $GL_4(F_v)$, given by the local Langlands correspondence. By [Sh1, Prop. 5.1], we can find cuspidal representations $π_1 = \otimes_w π_1w$, $π′_1 = \otimes_w π′_1w$ of $GL_2(A_F)$ and a cuspidal representation $σ = \otimes_w σ_w$ of $GL_n(A_F)$ such that $π_1w$, $π′_1w$, and $σ_w$ are all unramified for $w \neq v$, $w < \infty$, and $π′_1w = ρ$. Let $π_1 \boxtimes π′$ be the functorial product, constructed by Ramakrishnan [R1]. Consider the case $D_{π+2} = 2$ in [Sh2], and compare the functional equations for $L(s, π_1 \times π′ \times σ)$ and $L(s, (π_1 \boxtimes π′) \times σ)$:

$L(s, π_1 \times π′ \times σ) = ε(s, π_1 \times π′ \times σ)L(1 - s, π_1 \times π′ \times σ)$,

$L(s, (π_1 \boxtimes π′) \times σ) = ε(s, (π_1 \boxtimes π′) \times σ)L(1 - s, (π_1 \boxtimes π′) \times σ)$.

Since $π_1w$, $π′_1w$, and $σ_w$ are all unramified for $w \neq v, w < \infty$,

$\gamma(s, π_1w \times π′_1w \times σ_w, \psi_π) = \gamma(s, (π_1w \boxtimes π′_1w) \times σ_w, \psi_π)$,

for all $w \neq v$. Hence we have

$\gamma(s, π_1v \times π′_1v \times σ_π, \psi_π) = \gamma(s, (π_1v \boxtimes π′_1v) \times σ_π, \psi_π)$,

since similar inequalities hold at archimedean places, the factors being those of Artin [Sh7].

It is more difficult to prove the equalities (3.2.1) and (3.2.2) if both $π_1v$ and $π_2v$ are supercuspidal. Let $T$ be the finite set of places $v \mid 2$ such that $π_1v$ is an extraordinary supercuspidal representation of $GL_2(F_v)$, while $π_2v$ is a supercuspidal representation of $GL_3(F_v)$ attached to a character of a nonnormal cubic extension $L$ of $F_v$.

**Proposition 5.3.** Suppose $π_1v$ and $π_2v$ are both supercuspidal. Then:

a) Equalities (5.2.1) and (5.2.2) hold for every irreducible admissible generic representation $σ_π$ of $GL_n(F_v)$, $n = 1, 2, 3$.

b) Suppose $v \notin T$. Then (5.2.1) and (5.2.2) hold for every irreducible admissible generic representation $σ_π$ of $GL_n(F_v)$, $n = 1, 2, 3, 4$. 
Proof. By multiplicativity of $\gamma$-factors, it is enough to prove the equality of corresponding $\gamma$-functions for supercuspidal representations $\sigma_v$.

Let $\pi_{1v}, \pi_{2v}, \sigma_v$ be supercuspidal representations of $GL_2(F_v)$, $GL_3(F_v)$, $GL_n(F_v)$, resp., where $n = 2, 3, 4$. Let $\rho_{1v}, \rho_{2v}, \rho_{3v}$ be the corresponding Weil group representations. We shall prove the equivalent equality:

$$
(5.3.1) \quad \gamma(s, \pi_{1v} \times \pi_{2v} \times \sigma_v, \psi_v) = \gamma(s, \rho_{1v} \otimes \rho_{2v} \otimes \rho_{3v}, \psi_v).
$$

Here $\gamma(s, \pi_{1v} \times \pi_{2v} \times \sigma_v, \psi_v)$ is the $\gamma$-factor defined in [Sh1], while $\gamma(s, \rho_{1v} \otimes \rho_{2v} \otimes \rho_{3v}, \psi_v)$ is the corresponding Artin $\gamma$-factor. Since

$$
\gamma(s, (\pi_{1v} \boxtimes \pi_{2v}) \times \sigma_v, \psi_v) = \gamma(s, (\rho_{1v} \otimes \rho_{2v}) \times \rho_{3v}, \psi_v),
$$

and $\pi_{1v} \boxtimes \pi_{2v}$ corresponds to $\rho_{1v} \otimes \rho_{2v}$ by the local Langlands correspondence, Proposition 5.3 then follows, namely,

$$
\gamma(s, \pi_{1v} \times \pi_{2v} \times \sigma_v, \psi_v) = \gamma(s, (\pi_{1v} \boxtimes \pi_{2v}) \times \sigma_v, \psi_v).
$$

Proof of (5.3.1). We first consider the case where $v \mid 2$. Then $v \mid 3$ and therefore $\pi_{2v}$ is tame and by classification of tame supercuspidal representations there exists a cubic extension $L$ of $F_v$ such that $\pi_{2v}$ corresponds to $\text{Ind}(W_{F_v}, W_L, \eta)$, where $\eta$ is a character of $L^\times$. To proceed, we embed $\pi_{1v}, \sigma_v$ as local components of cuspidal representations $\pi_1, \sigma$ with unramified finite components everywhere else.

Next, we choose a cubic extension $\mathbb{L}/F$ such that $\mathbb{L}_u = L$, $u \mid v$; we also choose a gr"ossencharacter $\chi$ of $\mathbb{L}$ satisfying $\chi_w = \eta$. We assume further that $\chi_u$ is unramified for every $u < \infty$, $u \neq w$. Let $\pi_2$ be the cuspidal representation of $GL_3(\mathbb{A}_F)$ corresponding to $\rho_2 = \text{Ind}(W_F, W_L, \chi)$ [AC], [JP-SS2]. Observe that $\mathbb{L}/F$ need not be normal.

Let $\pi_2^\mathbb{L}$ and $\sigma_2^\mathbb{L}$ be the base changes of $\pi_1$ and $\sigma$ to $\mathbb{L}$. Observe that when $v \in T$, we are assuming $n \leq 3$ so that the nonnormal base change of $\sigma$ to $\mathbb{L}$ exists [JP-SS2]. We compare the functional equations for $L(s, \pi_1 \times \pi_2 \times \sigma)$ and $L(s, (\pi_1 \boxtimes \chi) \times \sigma_2^\mathbb{L})$. More precisely,

$$
L(s, \pi_1 \times \pi_2 \times \sigma) = \varepsilon(s, \pi_1 \times \pi_2 \times \sigma)L(1 - s, \pi_1 \times \pi_2 \times \sigma)
$$

and

$$
L(s, (\pi_1 \boxtimes \chi) \times \sigma_2^\mathbb{L}) = \varepsilon(s, (\pi_1 \boxtimes \chi) \times \sigma_2^\mathbb{L})L(1 - s, (\pi_1 \boxtimes \chi^{-1}) \times \sigma_2^\mathbb{L}).
$$

Let $L_u = \mathbb{L} \otimes_F F_u$. Then either $L_u/F_u$ is a cubic field extension, $L_u \simeq F_u \oplus L_w$ with $L_w/F_u$ a quadratic extension, or $L_u \simeq F_u \oplus F_u \oplus F_u$. Thus if $w$ is a place of $\mathbb{L}$ over $u$, then the local base changes to $L_w$ are either cubic, quadratic, or trivial.

To continue we need a general discussion as follows.
Let $K/F$ be a finite separable extension of local fields. Fix a nontrivial additive character $\psi_F$ of $F$ and let $\psi_{K/F} = \psi_F \cdot Tr_{K/F}$. Let $\lambda(K/F, \psi_F)$ be the corresponding Langlands $\lambda$-function (cf. [La5], [AC]). Let $\pi_1$ and $\pi_2$ be two irreducible admissible representations of $GL_m(F)$ and $GL_n(F)$, respectively. Fix continuous representations $\rho_1$ and $\rho_2$ of the Deligne-Weil group $W_F$ attached to $\pi_1$ and $\pi_2$ by the local Langlands correspondence [HT], [He], respectively. Assume $\rho_2 = \text{Ind}(W_F, W_K, \tau_2)$ with $\dim \tau_2 = p$. Let $\pi^K$ be the base change of $\pi_1$, i.e. the representation attached to $\rho_1|W_K$. Then

$$\varepsilon(s, \rho_1 \otimes \rho_2, \psi_F) = \lambda(K/F, \psi_F)^{mp} \varepsilon(s, \rho_1|W_K \otimes \tau_2, \psi_{K/F}).$$

Consequently

$$\varepsilon(s, \pi_1 \times \pi_2, \psi_F) = \lambda(K/F, \psi_F)^{mp} \varepsilon(s, \pi^K_1 \times \beta_2, \psi_{K/F})$$

where $\beta_2$ is the representation of $GL_n(K)$, $p = \dim \tau_2$, attached to $\tau_2$. The identities for $L$-functions hold without any $\lambda$-functions.

In our setting $p = 1$, $\tau_2 = \eta$, and $\rho_{2v} = \text{Ind}(W_{F_v}, W_{L_v}, \eta)$. Thus

$$\varepsilon(s, (\rho_{1v} \otimes \rho_{3v}) \otimes \rho_{2v}, \psi_v) = \lambda(L/F_v, \psi_v)^{2n} \varepsilon(s, \text{Res}_{W_v} (\rho_{1v} \otimes \rho_{3v}) \otimes \eta, \psi_{L/F_v})$$

since $\dim(\rho_{1v} \otimes \rho_{3v}) = 2n$. By the local Langlands correspondence,

$$\varepsilon(s, \rho_{1v} \otimes \rho_{2v} \otimes \rho_{3v}, \psi_v) = \lambda(L/F_v, \psi_v)^{2n} \varepsilon(s, (\pi_{1v} \boxtimes \sigma_v)^L \otimes \eta, \psi_{L/F_v})$$

$$= \lambda(L/F_v, \psi_v)^{2n} \varepsilon(s, (\pi_{1v}^L \otimes \eta) \times \sigma_v^L, \psi_{L/F_v}).$$

On the other hand for $v_1$, a place of $F$ different from $v$, we note that $\rho_2 = \text{Ind}(W_F, W_{L_v}, \chi_v)$, as a representation of $W_{v_1} = W_{F_{v_1}}$, is equal to

$$\bigoplus_{w_1|v_1} I(\chi_{w_1}),$$

where

$$I(\chi_{w_1}) = \text{Ind}(W_{w_1}, W_{w_1}, \chi_{w_1}),$$

with $W_{w_1} = W_{L_{w_1}}$. Thus

$$\varepsilon(s, (\rho_{1v_1} \otimes \rho_{3v_1}) \otimes \rho_{2v_1}, \psi_{v_1})$$

$$= \prod_{w_1|v_1} \varepsilon(s, (\rho_{1v_1} \otimes \rho_{3v_1}) \otimes I(\chi_{w_1}), \psi_{v_1})$$

$$= \prod_{w_1|v_1} \lambda(L_{w_1}/F_{v_1}, \psi_{v_1})^{2n} \varepsilon(s, (\pi_{1v_1} \boxtimes \sigma_{v_1})^{L_{w_1}} \otimes \chi_{w_1}, \psi_{L_{w_1}/F_{v_1}})$$

$$= \prod_{w_1|v_1} \lambda(L_{w_1}/F_{v_1}, \psi_{v_1})^{2n} \varepsilon(s, (\pi_{1v_1}^{L_{w_1}} \otimes \chi_{w_1}) \times \sigma_{v_1}^{L_{w_1}}, \psi_{L_{w_1}/F_{v_1}})$$

by the local Langlands correspondence. Similarly

$$L(s, \rho_{1v_1} \otimes \rho_{2v_1} \otimes \rho_{3v_1}) = \prod_{w_1|v_1} L(s, (\pi_{1v_1}^{L_{w_1}} \otimes \chi_{w_1}) \times \sigma_{v_1}^{L_{w_1}}).$$
To prove (5.3.1), it would then be enough to show that \( \varepsilon(s, \pi_{1v_1} \times \pi_{2v_1} \times \sigma_{v_1}, \psi_{v_1}) \) and \( L(s, \pi_{1v_1} \times \pi_{2v_1} \times \sigma_{v_1}) \) are equal to the right-hand sides of (5.3.2) and (5.3.3), for every \( v_1 \neq v \), for then (5.3.1) would follow immediately by comparison of functional equations for \( L(s, \pi_1 \times \pi_2 \times \sigma) \) and \( L(s, (\pi_1^\varepsilon \otimes \chi) \times \sigma^L) \). But these last equalities are obvious since \( \pi_{iv}, i = 1, 2, \) and \( \sigma_{v_i} \) are either unramified or archimedean.

Since when \( v \notin T \), we can use cubic cyclic base change, which is available for all \( n \), part b) follows as well.

If \( v \nmid 2 \), then \( \pi_{1v} \) is dihedral, and the argument goes as before except that this time we use a quadratic base change. \( \square \)

It remains to prove (5.3.1) when \( \sigma_v \) is a supercuspidal representation of \( \text{GL}_4(F_v) \) for \( v \in T \). Namely, \( v \mid 2, \pi_{1v} \) is an extraordinary supercuspidal representation of \( \text{GL}_2(F_v) \), and \( \pi_{2v} \) is a supercuspidal representation of \( \text{GL}_3(F_v) \) attached to a character of a nonnormal cubic extension \( L \) of \( F_v \). The difficulty is that the theory of nonnormal cubic base change for \( \text{GL}_4 \) is not available at present. We need to proceed as follows. We first construct a local lift \( \Pi_v \) in the sense of Definition 3.3, and show that it differs from \( \pi_{1v} \boxtimes \pi_{2v} \) by at most a quadratic character. Then the appendix by Bushnell and Henniart [BH] proves that, in fact, \( \Pi_v \simeq \pi_{1v} \boxtimes \pi_{2v} \). We start with:

**Proposition 5.4.** For each \( v \in T \), there exists a local lift \( \Pi_v \) of \( \pi_{1v} \otimes \pi_{2v} \) in the sense of Definition 3.3.

**Proof.** We start by letting \( \pi_1 \otimes \pi_2 = \bigotimes_w (\pi_{1w} \otimes \pi_{2w}) \) be a cuspidal representation of \( \text{GL}_2(\mathbb{A}_F) \times \text{GL}_3(\mathbb{A}_F) \) such that \( \pi_{1w} \otimes \pi_{2w} \) is unramified for all \( w < \infty, w \neq v \) (cf. [Sh1]). Let \( \Pi = \bigotimes_w \Pi_w \) be the weak lift of \( \pi_1 \otimes \pi_2 \) with respect to the set \( S = \{v\} \) obtained from Proposition 3.10. We will show that \( \Pi_v \) is a local lift of \( \pi_{1v} \otimes \pi_{2v} \), i.e., that it satisfies (3.3.1) and (3.3.2). To proceed, we must be extra careful. In fact, we no longer know, *a priori*, that \( \Pi_v \) is tempered, and therefore the equality of \( \gamma \)-functions does not directly imply that of \( L \) and \( \varepsilon \) factors which we used occasionally earlier, as it will require the knowledge of its Langlands parameter. All that we know is that \( \Pi_v \) is irreducible, unitary, and generic. But fortunately this is enough. In fact, \( \Pi_v \) is of the form

\[
\text{Ind}_{\tau_1} \bigotimes_{\ell} \bigotimes \tau_\ell \bigotimes \tau_{\ell+1} \bigotimes \cdots \bigotimes \tau_{\ell+u} \bigotimes \tau_{\ell+u+1} \bigotimes \cdots \bigotimes \tau_{\ell+u+j} \bigotimes \cdots \bigotimes \tau_{\ell+u+j+k} \bigotimes \cdots \bigotimes \tau_{\ell+u+j+k+1} \bigotimes \cdots \bigotimes \tau_{\ell+u+j+k+1+m} \bigotimes \cdots \bigotimes \tau_{\ell+u+j+k+1+m+n}
\]

where the \( \tau_\ell \)'s are discrete series representations of smaller \( \text{GL}_\ell \)'s, and \( 0 < \ell \leq s_1 \leq \cdots \leq s_k < \frac{1}{2} \) (cf. [Tad]).

For \( \sigma_v \) in the discrete series of \( \text{GL}_n(F_v) \), \( n = 1, 2, 3, 4 \), the \( L \)-function \( L(s, \Pi_v \times \sigma_v) \) is equal to

\[
\prod_{k=1}^\ell L(s - s_k, \tau_k \times \sigma_v) L(s + s_k, \tau_k \times \sigma_v) \prod_{j=1}^u L(s, \tau_{\ell+j} \times \sigma_v).
\]
By strict inequalities $0 < s_k < 1/2$ and the holomorphy of each $L(s, \tau_k \times \sigma_v)$ for $\text{Re}(s) > 0$, it is easy to see that as a function of $q_{v}^{-s}$, $L(s, \Pi_v \times \sigma_v)^{-1}$ has the same zeros as $\gamma(s, \Pi_v \times \sigma_v, \psi_v)$ and therefore the equalities

$$L(s, \Pi_v \times \sigma_v) = L(s, \pi_{1v} \times \pi_{2v} \times \sigma_v)$$

and

$$\varepsilon(s, \Pi_v \times \sigma_v, \psi_v) = \varepsilon(s, \pi_{1v} \times \pi_{2v} \times \sigma_v, \psi_v),$$

follow from (5.4.1)

$$\gamma(s, \Pi_v \times \sigma_v, \psi_v) = \gamma(s, \pi_{1v} \times \pi_{2v} \times \sigma_v, \psi_v),$$

since $\pi_{iv}$ and $\sigma_v$ are tempered (cf. Section 7 of [Sh1] as well as the beginning of Section 5 here).

It is therefore enough to prove (5.4.1) for $\sigma_v$ in the discrete series. The case of an irreducible admissible generic $\sigma_v$ follows from this by multiplicativity.

To prove (5.4.1), we take a cuspidal representation $\sigma$ of $GL_n(k_F)$, $n = 1, 2, 3, 4$, which has $\sigma_v$ as its $v$th component [AC]. Although we know little about other components of $\sigma$, we still have

$$\gamma(s, \Pi_w \times \sigma_w, \psi_w) = \gamma(s, \pi_{1w} \times \pi_{2w} \times \sigma_w, \psi_w)$$

for every $w \neq v$; for $\pi_{iw}$ and $\Pi_w$ are now unramified, if $w \neq v$, $w < \infty$, and therefore $\gamma$-functions on both sides are the same products as those of Godement-Jacquet by multiplicativity (cf. [Sh1]). We recall that for $w = \infty$, the factors on both sides are those of Artin [Sh7], and are therefore automatically equal. The equality (5.4.1) is now an immediate consequence of a comparison of the functional equations for $L(s, \Pi \times \sigma)$ and $L(s, \pi_1 \times \pi_2 \times \sigma)$. We should point out that it would not be much harder to conclude the equality of root numbers and $L$-functions, if one establishes (5.4.1) only for supercuspidal $\sigma_v$.

Proposition 5.5. Let $K/k$ be a quadratic extension of local fields. Let $\rho_1$ and $\rho_2$ be supercuspidal representations of $GL_2(k)$ and $GL_3(k)$, resp., and let $\Omega$ be the local lift of $\rho_1 \otimes \rho_2$, constructed in Proposition 5.4. Let $\rho^K_1$ and $\rho^K_2$ be the base changes of $\rho_1$ and $\rho_2$, respectively, to $K$. We shall assume that $\rho^K_1$ and $\rho^K_2$ are both supercuspidal, and let $\Omega_K$ be the local lift of $\rho^K_1 \otimes \rho^K_2$.

Proposition 5.5. Let $\Omega^K$ be the base change of $\Omega$ to $K$. Then

$$\Omega_K = \Omega^K,$$

i.e., our local lift commutes with the base change.
Proof. By [Sh1, Prop. 5.1], we can take a number field $F$ with $F_v = k$, and
$$\pi_1 = \otimes_u \pi_{1u}, \pi_2 = \otimes_u \pi_{2u},$$
cuspidal representations of $GL_2(\mathbb{A}_F), GL_3(\mathbb{A}_F)$, resp. such that $\pi_{1v} = \rho_1, \pi_{2v} = \rho_2$ and $\pi_{1u}, \pi_{2u}$ are unramified for $w \neq v, w < \infty$. Let $E/F$ be a quadratic extension such that $E_v = K$ ($v$ is inert). Let $\Sigma_v$ be a supercuspidal representation of $GL_n(K), n = 1, 2, 3, 4$, and $\Sigma = \otimes_u \Sigma_u$, a cuspidal representation of $GL_n(\mathbb{A}_E)$ such that $\Sigma_u$ is unramified for $u \neq v, u < \infty$. By the definition of the local lift, we have
$$\gamma(s, \Sigma_v \times \Omega_K, \psi_v) = \gamma(s, \Sigma_v \times \rho_1^K \times \rho_2^K, \psi_v).$$
We only need to prove that
$$\gamma(s, \Sigma_v \times \Omega^K, \psi_v) = \gamma(s, \Sigma_v \times \rho_1^K \times \rho_2^K, \psi_v).$$

Since $\rho_1^K$ and $\rho_2^K$ are supercuspidal, $\pi_1^E$ and $\pi_2^E$ are cuspidal representations. Let $\Pi$ be the weak lift constructed in the proof of Proposition 5.4 such that $\Pi_v = \Omega$. Compare functional equations for $L(s, \Sigma \times \Pi^E)$ and $L(s, \Sigma \times \pi_1^E \times \pi_2^E)$, and note that for $u \neq v$,
$$L(s, \Sigma_u \times (\Pi^E)_u) = L(s, \Sigma_u \times (\pi_1^E)_u \times (\pi_2^E)_u).$$
Our equality now follows. \hfill \square

We continue with the same notation as before Proposition 5.4.

PROPOSITION 5.6. Suppose $v \in T$. Fix a normal closure $K$ of $L$. Let $E/F_v$ be the unique quadratic extension of $F_v$ inside $K$. Then
$$L(s, \pi_{1v}^E \times \pi_{2v}^E \times \sigma) = L(s, (\pi_{1v} \boxtimes \pi_{2v})^E \times \sigma)$$
and
$$\varepsilon(s, \pi_{1v}^E \times \pi_{2v}^E \times \sigma, \psi_{E/F_v}) = \varepsilon(s, (\pi_{1v} \boxtimes \pi_{2v})^E \times \sigma, \psi_{E/F_v}),$$
for every irreducible admissible generic representation $\sigma$ of $GL_n(E), n = 1, 2, 3, 4$. Here for each representation $\tau$ of $GL_m(F_v)$, $\tau^E$ denotes its base change to $GL_m(E)$, $m = 2, 3$, and $\psi_{E/F_v} = \psi_v \cdot Tr_{E/F_v}$.

Proof. Observe that $\pi_{1v}^E, i = 1, 2,$ is still supercuspidal since $v \in T$ (and moreover $(\pi_{1v} \boxtimes \pi_{2v})^E = \pi_{1v}^E \boxtimes \pi_{2v}^E$). Consequently, $\pi_{1v}^E$ is attached to Ind$(W_E, W_K, \eta)$, where $\eta$ is a character of $W_K$ and $K/E$ is a cubic cyclic extension. We can now apply Proposition 5.3.b) to the pair $(\pi_{1v}^E, \pi_{2v}^E)$ to conclude the proof of the proposition. \hfill \square

COROLLARY 5.7. Suppose $v \in T$. Then there exists a local lift $\Pi_v$ of $\pi_{1v} \otimes \pi_{2v},$ such that $\Pi_v \simeq (\pi_{1v} \boxtimes \pi_{2v}) \otimes \eta,$ where $\eta^2 = 1.$
Proof. By Propositions 5.4, 5.5, and 5.6, applied to \( \pi_1^E \) and \( \pi_2^E \), and using the equality \((\pi_1 \boxtimes \pi_2)^E = \pi_1^E \boxtimes \pi_2^E\), one concludes that
\[(\Pi_v)^E = (\pi_1 \boxtimes \pi_2)^E,\]
by appealing to the local converse theorem proved in [CP-S1, §7], and [Ch]. Our result follows with \( \eta \) whose kernel contains \( N_{E/F_v}(E^*) \).

**Proposition 5.8.** For \( v \in T \), \( \Pi_v \simeq \pi_1 \boxtimes \pi_2 \). In particular, equalities (5.2.1) and (5.2.2) hold for \( n = 1, 2, 3, 4 \) at every place \( v \) of \( F \).

**Proof.** We only need to apply Corollary 5.7 and Proposition 5.3.a) to the main theorem of the appendix [BH]. The proposition follows.

**Proof of Theorem 5.1.** Let \( \Pi = \bigotimes_v \Pi_v \), where \( \Pi_v = \pi_1 \boxtimes \pi_2 \). It is an irreducible admissible representation of \( \text{GL}_6(\mathbb{A}_F) \). Pick two finite places \( v_1, v_2 \), where \( \pi_j v_1, \pi_j v_2 \) are unramified for \( j = 1, 2 \). Let \( S_i = \{ v_i \} \), \( i = 1, 2 \). We apply the converse theorem twice to \( \Pi = \bigotimes_v \Pi_v \) with \( S_1 \) and \( S_2 \). We find two automorphic representations \( \Pi_1, \Pi_2 \) of \( \text{GL}_6(\mathbb{A}_F) \) such that \( \Pi_{1v} \simeq \Pi_{v} \) for \( v \neq v_1 \), and \( \Pi_{2v} \simeq \Pi_{v} \) for \( v \neq v_2 \). Hence \( \Pi_{1v} \simeq \Pi_{2v} \) for all \( v \neq v_1, v_2 \). By Proposition 3.10, \( \Pi_1, \Pi_2 \) are of the form \( \sigma_1 \boxtimes \cdots \boxtimes \sigma_k \), where \( \sigma_i \)'s are (unitary) cuspidal representations of \( \text{GL} \). By the classification theorem [JS1], \( \Pi_1 \simeq \Pi_2 \), in particular, \( \Pi_{1v_1} \simeq \Pi_{2v_1} \simeq \Pi_{v_1} \) for \( i = 1, 2 \). Thus \( \Pi \) is automorphic.

6. **Functorial symmetric cubes for \( \text{GL}_2 \)**

Let \( \text{Sym}^m : \text{GL}_2(\mathbb{C}) \rightarrow \text{GL}_{m+1}(\mathbb{C}) \) be the map given by the \( m \)th symmetric power representation of \( \text{GL}_2(\mathbb{C}) \) on the space of symmetric tensors of rank \( m \). Let \( \pi = \bigotimes_v \pi_v \) be a cuspidal representation of \( \text{GL}_2(\mathbb{A}_F) \) with central character \( \omega_\pi \). By the local Langlands correspondence, \( \text{Sym}^m(\pi_v) \) is well-defined for all \( v \). More precisely, it is the representation of \( \text{GL}_{m+1}(\mathbb{F}_v) \) attached to \( \text{Sym}^m(\rho_v) \), where \( \rho_v \) is the two-dimensional representation of the Deligne-Weil group attached to \( \pi_v \). Hence Langlands' functoriality is equivalent to the assertion that \( \text{Sym}^m(\pi) = \bigotimes_v \text{Sym}^m(\pi_v) \) is an automorphic representation of \( \text{GL}_{m+1}(\mathbb{A}_F) \). It is convenient to introduce \( A^m(\pi) = \text{Sym}^m(\pi) \otimes \omega_\pi^{-1} \) (called \( \text{Ad}^m(\pi) \) in [Sh3]). If \( m = 2 \), \( A^2(\pi) = \text{Ad}(\pi) \). Gelbart and Jacquet [GeJ] showed that \( \text{Ad}(\pi) \) is an automorphic representation of \( \text{GL}_3(\mathbb{A}_F) \), which is cuspidal unless \( \pi \) is monomial; i.e., \( \pi \simeq \pi \otimes \eta \), where \( \eta \neq 1 \) is a grössencharacter of \( F \). We prove:

**Theorem 6.1.** The representation \( \text{Sym}^3(\pi) \) is an automorphic representation of \( \text{GL}_4(\mathbb{A}_F) \). It is cuspidal, unless either \( \pi \) is a monomial representation, or \( \text{Ad}(\pi) \simeq \text{Ad}(\pi) \otimes \eta \), for a nontrivial grössencharacter \( \eta \). Equivalently, \( \text{Sym}^3(\pi) \) is cuspidal, unless \( \pi \) is either dihedral or it has a cubic cyclic base.
change which is dihedral, i.e., \( \pi \) is of tetrahedral type. In particular, if \( F = \mathbb{Q} \) and \( \pi \) is the automorphic cuspidal representation attached to a nondihedral holomorphic form of weight \( \geq 2 \), then \( \text{Sym}^3(\pi) \) is cuspidal.

We first consider:

6.1. \( \pi \) is a monomial cuspidal representation. That is, \( \pi \otimes \eta \simeq \pi \) for a nontrivial grössencharacter \( \eta \). Then \( \eta^2 = 1 \) and \( \eta \) determines a quadratic extension \( E/F \). According to [LL], there is a grössencharacter \( \chi \) of \( E \) such that \( \pi = \pi(\chi) \), where \( \pi(\chi) \) is the automorphic representation whose local factor at \( v \) is the one attached to the representation of the local Weil group induced from \( \chi_v \). Let \( \chi' \) be the conjugate of \( \chi \) by the action of the nontrivial element of the Galois group. Then \( \text{Ad}(\pi) \) is given by

\[
\text{Ad}(\pi) = \pi(\chi\chi'^{-1}) \boxtimes \eta.
\]

There are two cases:

Case 1. \( \chi\chi'^{-1} \) factors through the norm; i.e., \( \chi\chi'^{-1} = \mu \circ N_{E/F} \) for a grössencharacter \( \mu \) of \( F \). Then \( \pi(\chi\chi'^{-1}) \) is not cuspidal. In fact, \( \pi(\chi\chi'^{-1}) = \mu \boxplus \mu \eta \). In this case,

\[
A^3(\pi) = \pi(\chi\chi'^{-1}) \boxtimes \pi = (\mu \otimes \pi) \boxplus (\mu \eta \otimes \pi).
\]

Case 2. \( \chi\chi'^{-1} \) does not factor through the norm. In this case, \( \pi(\chi\chi'^{-1}) \) is a cuspidal representation. Then

\[
A^3(\pi) = \pi(\chi\chi'^{-1}) \boxtimes \pi = \pi(\chi^2 \chi'^{-1}) \boxtimes \pi.
\]

Here we used the fact that \( \pi(\chi)_E = \chi \boxplus \chi' \) ([R1, Prop. 2.3.1]) and furthermore that \( \pi' \boxtimes \pi = I^E_\pi (\pi'_E \otimes \chi) \), if \( \pi = \pi(\chi) \) ([R1, §3.1]). The index \( E \) signifies the base change to \( E \). Observe that we are now using subscripts to denote the base change rather than superscripts used in Section 5. A superscript seemed to be a more appropriate notation for that section.

6.2. \( \pi \) is not monomial. Then \( \text{Ad}(\pi) \) is a cuspidal representation of \( \text{GL}_3(A_F) \). We first prove:

**Lemma 6.2.** Let \( \sigma \) be a cuspidal representation of \( \text{GL}_2(A_F) \). Then the triple \( L \)-function \( L_S(s, \text{Ad}(\pi) \times \pi \times \sigma) \) has a pole at \( s = 1 \) if and only if \( \sigma \simeq \pi \otimes \chi \) and \( \text{Ad}(\pi) \simeq \text{Ad}(\pi) \otimes (\omega \pi \chi) \) for some grössencharacter \( \chi \). Here \( S \) is a finite set of places for which \( v \notin S \) implies that both \( \pi_v \) and \( \sigma_v \) are unramified.
Proof. Suppose \( L_S(s, \text{Ad}(\pi) \times \pi \times \sigma) \) has a pole at \( s = 1 \). Consider \( \pi \boxtimes \sigma \). It is an automorphic representation of \( \text{GL}_4(\mathbb{A}_F) \). As argued at the beginning of the proof of Theorem 5.1, we may assume that \( \sigma \) is not monomial. By [R1, Th. M, p. 54] \( \pi \boxtimes \sigma \) is then cuspidal, unless \( \sigma \simeq \pi \otimes \chi \) for some grössencharacter \( \chi \). If \( \pi \boxtimes \sigma \) is cuspidal, then \( L_S(s, \text{Ad}(\pi) \times \pi \times \sigma) \) is entire. Hence \( \sigma \simeq \pi \otimes \chi \) for some grössencharacter \( \chi \). Consider the following \( L \)-function identity:

\[
L_S(s, \text{Ad}(\pi) \times \pi \times (\pi \otimes \chi)) = L_S(s, \text{Ad}(\pi) \times (\text{Ad}(\pi) \otimes (\omega \pi \chi))) L_S(s, \text{Ad}(\pi) \otimes (\omega \pi \chi)).
\]

Since \( L_S(s, \text{Ad}(\pi) \otimes (\omega \pi \chi)) \) has no zero at \( s = 1 \), \( L_S(s, \text{Ad}(\pi) \times (\text{Ad}(\pi) \otimes (\omega \pi \chi))) \) has a pole at \( s = 1 \). Hence \( \text{Ad}(\pi) \simeq \text{Ad}(\pi) \otimes (\omega \pi \chi) \) since \( \text{Ad}(\pi) \) is self-contragredient. The converse is clear from the above identity.

We consider the functorial product \( \pi \boxtimes \text{Ad}(\pi) \) as in Theorem 5.1. By Lemma 6.2 and the classification theorem [JS1],

\[
\pi \boxtimes \text{Ad}(\pi) = \tau \boxplus \pi,
\]

where \( \tau \) is an automorphic representation of \( \text{GL}_4(\mathbb{A}_F) \). Since \( \pi_v \boxtimes \text{Ad}(\pi_v) = A^3(\pi_v) \boxplus \pi_v \), we conclude \( \tau_v \simeq A^3(\pi_v) \) for all \( v \). Hence we have:

**Proposition 6.3.** The representation \( A^3(\pi) \) is an automorphic representation of \( \text{GL}_4(\mathbb{A}_F) \). It is not cuspidal if and only if there exists a nontrivial grössencharacter \( \eta \) such that \( \text{Ad}(\pi) \simeq \text{Ad}(\pi) \otimes \eta \). In this case

\[
A^3(\pi) = (\pi \otimes \eta) \boxplus (\pi \otimes \eta^2).
\]

**Proof.** We only need to prove the last assertion. Clearly

\[
L_S(s, (\pi \boxtimes \text{Ad}(\pi)) \times \overline{\tau})
\]

has a pole at \( s = 1 \). Thus \( \pi \boxtimes \text{Ad}(\pi) = \pi \boxplus \tau \), where \( \tau \) is an automorphic representation of \( \text{GL}_4(\mathbb{A}_F) \). If \( \tau \) is not cuspidal, then \( \tau = \sigma_1 \boxplus \sigma_2 \), where \( \sigma_i \), \( i = 1, 2 \), are cuspidal representations of \( \text{GL}_2(\mathbb{A}_F) \). Then \( L_S(s, (\pi \boxtimes \text{Ad}(\pi)) \times \overline{\sigma_i}) \) must have a pole at \( s = 1 \). We can now proceed as in either Lemma 6.2 or as in the proof of the last statement of Theorem 5.1, to conclude that \( \sigma_i = \pi \otimes \eta^i \), where \( \text{Ad}(\pi) \cong \text{Ad}(\pi) \otimes \eta, \eta \neq 1 \).

**Corollary 6.4.** The representation \( \text{Sym}^3(\pi) \) is an automorphic representation of \( \text{GL}_4(\mathbb{A}_F) \). It is not cuspidal if and only if there exists a nontrivial grössencharacter \( \eta \) such that \( \text{Sym}^2(\pi) \cong \text{Sym}^2(\pi) \otimes \eta \). In this case

\[
\text{Sym}^3(\pi) = (\pi \otimes \eta \omega \eta^i) \boxplus (\pi \otimes \eta^2 \omega \eta).
\]

To complete the proof of Theorem 6.1, we need
**Lemma 6.5.** Suppose $\pi$ is not monomial, but $\text{Ad}(\pi)$ is, i.e., there exists a nontrivial grössencharacter $\eta$, necessarily cubic, such that $\text{Ad}(\pi) \otimes \eta \cong \text{Ad}(\pi)$. Then $\pi$ is of tetrahedral type, i.e., there exists a Galois representation $\sigma$ of tetrahedral type such that $\pi = \pi(\sigma)$.

**Proof.** Let $E/F$ be the cubic cyclic extension defined by $\eta$. Then as observed in [Sh8], the $E/F$-base change $\pi_E$ of $\pi$ is monomial. Let $\sigma_E$ be the two-dimensional dihedral representation of $W_E$ attached to $\pi_E$. Since $\sigma_E$ is invariant under the action of $\text{Gal}(E/F)$, it extends to a two-dimensional representation $\sigma$ of $W_F$ which is now of tetrahedral type (cf. [La2], [Ge]). Let $\pi'$ be the cuspidal representation of $\text{GL}_2(A_F)$ attached to $\sigma$, i.e. $\pi' = \pi(\sigma)$. Observe that $\pi'_E \cong \pi_E$ as they both correspond to $\sigma_E$ and therefore $\pi' \cong \pi \otimes \eta^a$ for some $a = 0, 1, 2$. But the lift $\sigma$ is unique only up to twisting by a power of $\eta$, and therefore changing the choice of $\sigma$ if necessary, we have $\pi \cong \pi' = \pi(\sigma)$.

The last assertion follows from the fact that holomorphic forms of weight $\geq 2$ can never be of tetrahedral type. \qed

**Remark 6.6.** For the proof of the functoriality of $\pi \boxtimes \text{Ad}(\pi)$, and hence $\text{Sym}^3(\pi)$, we do not need the appendix [BH]. The appendix is needed only for the general case of functoriality for $\text{GL}_2 \times \text{GL}_3$. The reason is the following. By Proposition 5.4, we can construct a strong lift $\Pi = \bigotimes_v \Pi_v$ of $\pi \otimes \text{Ad}(\pi)$. Then we still have $\Pi = \tau \boxtimes \pi$. We only need to prove that $\tau_v \simeq A^3(\pi_v)$ for all $v$. For that, it is enough to prove $\gamma(s, \sigma_v \times \tau_v, \psi_v) = \gamma(s, \sigma_v \times A^3(\pi_v), \psi_v)$ for every irreducible generic representation $\sigma_v$ of $\text{GL}_n(F_v)$, $n = 1, 2, 3$. Let $\varphi_v$ be the corresponding Weil group representation attached to $\pi_v$. Then $\varphi_v \otimes \text{Ad}(\varphi_v) \simeq A^3(\varphi_v) \oplus \varphi_v$. Now the equality of $\gamma$-factors follows immediately from (5.3.1) and (5.1.4). We should remark that since we are able to twist up to $n = 3$, the local converse theorem proved in [Ch] and [CP-S1] is not necessary in this case. We should point out that in proving Theorem 5.1, we could have confined ourselves to more conventional converse theorems, both local and global, if we were also to use case $E_8 - 1$ of [Sh2], which would allow us to twist by $\text{GL}_5$ as well. In view of [CP-S1], we did not pursue this. We should remind the reader that in our present approach, we have used the local converse theorem of [Ch] and [CP-S1] only in the proof of Corollary 5.7.

### 7. New estimates towards the Ramanujan and Selberg conjectures

An immediate consequence of the existence of $\text{Sym}^3(\pi)$ is a new estimate on Hecke eigenvalues of a Maass form. More precisely, let $\pi$ be a cuspidal representation of $\text{GL}_2(A_F)$. Write $\pi = \bigotimes_v \pi_v$. Assume $\pi_v$ is spherical, i.e.,

$$\pi_v = \text{Ind}(|i^{s_1v}, |i^{s_2v}),$$
When \( v < \infty \), we set \( \alpha_{1v} = |\varpi_v|^{s_{1v}} \) and \( \alpha_{2v} = |\varpi_v|^{s_{2v}} \), so that \( \text{diag}(\alpha_{1v}, \alpha_{2v}) \) represents the semisimple conjugacy class in \( \text{GL}_2(\mathbb{C}) \) attached to \( \pi_v \). Next, suppose \( F = \mathbb{Q} \). Let \( \pi \) be attached to a Maass form \( f \) with respect to a congruence subgroup \( \Gamma \). Denote by \( \lambda_1(\Gamma) = \lambda_1(\Gamma) \) the smallest positive eigenvalue of Laplace operator \( \Delta = -y^2 \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) \) on \( L^2(\Gamma \setminus \mathfrak{h}) \), where \( \mathfrak{h} \) denotes the upper half plane. It depends on \( f \). More precisely,

\[
\Delta f = \frac{1}{4}(1 - s^2)f,
\]

with \( \lambda_1(\Gamma) = \frac{1}{4}(1 - s^2) \), where \( s = 2\text{Re}(s_{1\infty}) = -2\text{Re}(s_{2\infty}) \). Then, one expects

**Selberg’s Conjecture**. \( \lambda_1(\Gamma) \geq \frac{1}{4} \).

**Ramanujan-Petersson’s Conjecture**. \( |\alpha_{1v}| = |\alpha_{2v}| = 1 \).

We now prove

**Theorem 7.1**. Let \( \pi \) be a cuspidal representation of \( \text{GL}_2(\mathcal{A}_F) \). Let \( \pi_v \) be a local (finite or infinite) spherical component of \( \pi \), i.e. \( \pi_v = \text{Ind}(| \cdot |_{\mathfrak{m}_v}^{s_{1v}}, | \cdot |_{\mathfrak{m}_v}^{s_{2v}}) \). Then \( \text{Re}(s_{1jv}) \leq \frac{5}{34} \), \( j = 1, 2 \).

**Proof.** Consider \( \text{Sym}^3(\pi) \). If it is not cuspidal, then \( \pi \), being of dihedral or tetrahedral type, satisfies Ramanujan’s conjecture. Hence we may assume that it is cuspidal. We apply [LRS1] to \( \text{Sym}^3(\pi) \). It states that if \( \Pi = \bigotimes_v \Pi_v \) is a cuspidal representation of \( \text{GL}_n(\mathcal{A}_F) \) and if \( \Pi_v \) is the spherical constituent of \( \text{Ind}_{B_n(\mathcal{O}_v)}^{\text{GL}_n(\mathcal{O}_v)} \bigotimes_{j=1}^n | t_{jv} |^{s_{1jv}}, t_{jv} \in \mathbb{C}, j = 1, \ldots, n \), then \( \text{Re}(t_{jv}) \leq \frac{1}{2} - \frac{1}{n^2 + 1} \).

In our case \( n = 4 \) and

\[
3\text{Re}(s_{iv}) \leq \frac{1}{2} - \frac{1}{4^2 + 1}.
\]

Our result follows. \( \square \)

**Remark 7.2.** Theorem 7.1 establishes a new estimate towards Selberg’s conjecture. It is \( \lambda_1 = \frac{1}{4}(1 - s^2) \geq 66/289 \approx 0.22837 \). We refer to Luo-Rudnick-Sarnak [LRS2] for the estimate \( \lambda_1 \geq 0.21 \) obtained earlier. When \( v < \infty \), Theorem 7.1 can be written as

\[
|\alpha_{jv}| \leq q_v^{5/34}.
\]

The earlier exponents of 1/5 and 5/28 for arbitrary \( F \) and \( F = \mathbb{Q} \) are due to [Sh2] and [LRS1], and [BDHI], respectively.

### 8. Applications to analytic number theory

This section has been suggested to us by Peter Sarnak. We would like to thank him for his suggestion and helpful advice.
In this section let $F = \mathbb{Q}$. The bounds towards the Ramanujan conjectures

\[(8.1) \quad |\alpha_{jp}| \leq p^{\frac{1}{4}}\]

and especially their archimedean counterparts

\[(8.2) \quad |\text{Re}(s_{j\infty})| \leq \frac{5}{34}\]

have numerous applications, see for example [I] and [IS]. They lead to improvements of exponents in a number of places, namely those where the question of small eigenvalues of the Laplacian on $\Gamma_0(N)\backslash \mathfrak{h}$ enters (see [Se]). One such example is that of cancellation in Kloosterman sums (see [Sa]).

There are also applications where the fact that (8.1) and (8.2) are sharper than $\frac{1}{6}$ has more fundamental consequences. Let $\Gamma$ be a congruence subgroup of $\text{SL}_2(\mathbb{Z})$. Let $f$ be either a holomorphic cusp form of weight $k$, or a Maass cusp form for $\Gamma \backslash \mathfrak{h}$. In the latter case, $f$ is an eigenfunction of the Laplacian;

\[(8.3) \quad y^2 \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) f(z) + \left( \frac{1}{4} - t^2 \right) f(z) = 0,\]

with $t$ imaginary or $0 < t < \frac{1}{2}$ and $z = x + iy$, $y > 0$. More precisely, we take $t = s_{1\infty}$. In both cases, $f$ has a Fourier expansion

\[(8.4) \quad f(z) = \sum_{n \neq 0} a_f(n)|n|^\frac{k-1}{2}W(nz),\]

where $a(1) = 1$. Here if $f$ is a holomorphic cusp form, $W(z) = e^{2\pi iz}, n > 0$ and $k$ is the weight. If $f$ is a Maass form, $k = 0$ and

\[W(z) = \sqrt{y}K_i(2\pi y)e^{2\pi ix},\]

where $K_i$ is a Bessel function and $W(z) = W(\overline{z})$, if $z$ is in the lower half plane. With this normalization, the Ramanujan conjectures for $f$ (at the finite places) are equivalent to

\[(8.5) \quad a_f(n) = O_\varepsilon(|n|^{\varepsilon}), \quad \varepsilon > 0.\]

The problem of cancellations in sums of shifted coefficients is as follows. Fix $h \in \mathbb{Z}$, $h \neq 0$, and for $X$ large, set

\[(8.6) \quad D_{f,h}(X) := \sum_{|n| \leq X} a_f(n)a_f(n + h).\]

This sum has been studied extensively ([Go], [M], to name a few). (We may include here the case of shifted divisor function sums.) It is known [Go] that if the exceptional eigenvalues of the Laplacian on $\Gamma \backslash \mathfrak{h}$ are denoted by $t_j$, then

\[\sum_{n \leq X} a_f(n)a_f(n + h) = O_\varepsilon(|n|^{\varepsilon}), \quad \varepsilon > 0.\]
0 < t_j < \frac{1}{2}, then as X \to \infty,
\begin{align}
D_{f,h}(X) &= \sum_{0 \leq t_j < \frac{1}{2}} b_{f,j} X^{\frac{1}{2}+t_j} + O(X^{\frac{2}{3}}) \\
&= \sum_{\frac{1}{6} < t_j < \frac{1}{2}} b_{f,j} X^{\frac{1}{2}+t_j} + O(X^{\frac{2}{3}})
\end{align}

for suitable constants b_{f,j} depending on f and t_j.

When f is holomorphic, equation (8.7) is proved in [Go]. (See Part i of Theorem 1 of [Go]; with our normalization of Fourier coefficients in (8.4), \(O(X^{k-1/3})\) in [Go] becomes \(O(X^{2/3})\).) One expects similar arguments to apply, and (8.7) must be valid, even if f is a Maass form. But, so far as we know, no reference to that is available.

The error term of \(X^{\frac{2}{3}}\) above is present because of real analysis issues involved with the form of the sharp cutoff in the sum defining \(D_{f,h}(X)\) (i.e., the sum gives weight 1 for \(n \leq X\) and weight 0 for \(n > X\)). Insomuch as the main results, (8.1) and (8.2) above, yield \(|t_j| \leq \frac{1}{6}\), we have:

**Proposition 8.1.** Let \(\Gamma\) and f be as above. Suppose f is a holomorphic cusp form. Fix \(h \neq 0\). Then as \(X \to \infty\),
\[D_{f,h}(X) = O(X^{\frac{2}{3}}).\]

Another application is to that of the hyperbolic circle problem [I, p. 190]. Let \(z, w \in \mathfrak{h}\). The hyperbolic distance is
\[\rho(z, w) = \log \frac{|z - \overline{w}| + |z - w|}{|z - \overline{w}| - |z - w|}.
\]

Let
\[2 \cosh \rho(z, w) = e^{\rho(z, w)} + e^{-\rho(z, w)} = 2 + 4u(z, w), \quad u(z, w) = \frac{|z - w|^2}{4\text{Im}z\text{Im}w}.
\]

For \(X \geq 2\), set
\[P(X) = \#\{\gamma \in \Gamma \mid 4u(\gamma z, w) + 2 \leq X\}.
\]

**Proposition 8.2.** Let \(|F|\) be the volume of the fundamental domain of \(\Gamma \backslash \mathfrak{h}\). Then
\[P(X) = \pi|F|^{-1}X + O(X^{\frac{2}{3}}).
\]

This follows from [I, Th. 12.1], and again from the fact that there are no eigenvalues between \(\frac{1}{6}\) and \(\frac{1}{2}\). To be precise, note that in the notation of [I], the eigenvalue \(\lambda_j = s_j(1 - s_j) = \frac{1}{4} - t_j^2 = 2/9\) corresponds to \(t_j = 1/6\) and \(s_j = 2/3\), and therefore in the equation (12.6) of [I] we may disregard the interval \(2/3 \leq s_j < 1\).
Thus, the point is that with our present understanding of the (sharp cut-off) sums (8.6), such results yield as sharp a result as one would get if one assumes the full Ramanujan conjectures ($\text{Re}(t_j) = 0$).

9. Siegel cusp forms of weight three

In this section, using the existence of symmetric cubes which we proved in Section 6 and a conjecture of Arthur [A2], we prove the existence of illusive Siegel cusp forms of weight 3. We thank Joseph Shalika for suggesting the problem and start with the following unpublished result:

**Theorem 9.1** (Jacquet, Piatetski-Shapiro, and Shalika). Let $\sigma = \otimes_v \sigma_v$ be a cuspidal automorphic representation of $GL_4(\mathbb{A}_F)$ for which there exists a grössencharacter $\chi$ and a finite set of places $S$ for which every $\sigma_v$ with $v \notin S$ is unramified, such that $L_S(s, \sigma, \Lambda^2 \otimes \chi^{-1})$ has a pole at $s = 1$. Then there exists a globally generic cuspidal automorphic representations $\tau$ of $GSp_4(\mathbb{A}_F)$ with central character $\chi$ such that $\sigma$ is the functorial lift of $\tau$ under the embedding $GSp_4(\mathbb{C}) \hookrightarrow GL_4(\mathbb{C})$.

Let $\pi = \otimes_v \pi_v$ be a cuspidal representation of $GL_2(\mathbb{A}_F)$ whose central character is $\omega_\pi$. Assume $\pi$ is neither of dihedral, nor of tetrahedral type. Then $\text{Sym}^3(\pi)$ is a cuspidal representation of $GL_4(\mathbb{A}_F)$. It is easy to see that if $S$ is a finite set of places outside of which every $\pi_v$ is unramified, $L_S(s, \text{Sym}^3(\pi), \Lambda^2 \otimes \omega_\pi^{-3})$ has a pole at $s = 1$. Consider the identity $L_S(s, \text{Ad}(\pi)) = L_S(s, A^3(\pi), \Lambda^2 \otimes \omega_\pi^{-1})$. Since $\text{Ad}(\pi)$ is self-contragredient, the left-hand side has a pole at $s = 1$. But $L_S(s, \text{Ad}(\pi))$ has no zeros at $s = 1$ and consequently $L_S(s, A^3(\pi), \Lambda^2 \otimes \omega_\pi^{-1})$ has a pole at $s = 1$. Now observe that $L_S(s, A^3(\pi), \Lambda^2 \otimes \omega_\pi^{-1}) = L_S(s, \text{Sym}^3(\pi), \Lambda^2 \otimes \omega_\pi^{-3})$. Applying Theorem 9.1, we may consider $\text{Sym}^3(\pi)$ as a cuspidal representation of $GSp_4(\mathbb{A}_F)$.

Now, assume $F = \mathbb{Q}$, and let $\pi$ correspond to a non-CM type holomorphic cuspidal form of weight 2. Then $\pi_\infty$ is a (holomorphic) discrete series parametrized by a two-dimensional representation $\varphi_\infty$ of the Weil group $W_{\mathbb{R}}$. We assume $\omega_\pi$ is trivial. Then $\varphi_\infty|\mathbb{C}^*$ is given by

$$z \mapsto \left( \begin{array}{cc} z/|z| & 0 \\ 0 & \overline{z}/|z| \end{array} \right).$$

It is now clear that the $L$-packet of $\text{Sym}^3(\pi_\infty)$ is parametrized by the homomorphism $\psi_\infty: W_{\mathbb{R}} \to GSp_4(\mathbb{C})$ for which $\psi_\infty|\mathbb{C}^*$ is given by

$$z \mapsto \text{diag}(z^2/|z|^3, z/|z|, \overline{z}/|z|, \overline{z}^3/|z|^3).$$

Now, let $L_Q$ be the conjectural Langlands group whose two-dimensional representations parametrize automorphic forms on $GL_2(\mathbb{A}_Q)$. Let $\varphi: L_Q \to$
GL_2(\mathbb{C}) be the two-dimensional representation of L_Q which parametrizes \pi. Then \psi = \text{Sym}^3(\varphi) = \text{Sym}^3 \cdot \varphi factors through GSp_4(\mathbb{C}) and \psi|_{W_R} = \psi_\infty.

Since \pi is nondihedral, \varphi is surjective, and therefore the image of \text{Sym}^3(\varphi) in GSp_4(\mathbb{C}) is the same as that of Sym^3. Consequently, by Schur’s lemma, the centralizer of \text{Im}(\text{Sym}^3(\varphi)) in GSp_4(\mathbb{C}) consists of only scalars \mathbb{C}^*.

Thus, since the centralizer is connected, the component group S_\varphi of Arthur [A2] is trivial. Now, his multiplicity formula in [A2], implies that every member of the L-packet of Sym^3(\pi), i.e., the packet parametrized by \psi, is automorphic. In particular, we can change the component \pi_\infty in its L-packet from the generic to the holomorphic one [V], so as to pick up a weight 3 Siegel modular cusp form. Thus we have proved:

**Theorem 9.2.** Let \pi be a cuspidal representation of GL_2(\mathbb{A}_\mathbb{Q}) attached to a non-CM holomorphic form of weight 2. Assume the validity of Arthur’s multiplicity formula for GSp_4(\mathbb{A}_\mathbb{Q}). Then every member of the L-packet of Sym^3(\pi) is automorphic. In particular, Siegel modular cusp forms of weight 3 exist.

**Remark 9.3.** While the group L_Q is out of reach at present, one expects to prove Arthur’s multiplicity formulas using his trace formula. One can then also expect to define the group S_\varphi, using the trace formula and necessary local results (Langlands, Shelstad), without any knowledge of L_Q and \varphi. For GSp_4, the stable trace formula is in very good shape, since all the fundamental lemmas are already established. On the other hand, to be able to use Arthur’s trace formula so as to prove Theorem 9.2, one actually needs to show that the transfer of Sym^3(\pi) to GSp_4(\mathbb{A}_F) belongs to a stable L-packet. In general, this will require a comparison of the regular trace formula for GSp_4(\mathbb{A}_F) with the most general type of twisted trace formula for GL_4(\mathbb{A}_F), i.e., the one which picks up \Pi satisfying \Pi(t^g^{-1}) \cong \Pi(g)\omega(\det g) for a grössencharacter \omega. (As was pointed out to us by Ramakrishnan, this may be accomplished by considering GL_4 \times GL_1.) This does not seem to be as in good shape as the stable trace formula for GSp_4(\mathbb{A}_F) alluded to before.

### 10. Applications to global Langlands correspondence and Artin’s conjecture

In view of recent consequences of Taylor’s program [Tay] on Artin’s conjecture (Buzzard, Dickinson, Shepherd-Barron, Taylor), and the work of Langlands [La2] and Tunnell [Tu], with few exceptions, to every two-dimensional continuous representation \sigma of W_F, the Weil group of \overline{F}/F, one can attach an automorphic representation of GL_2(\mathbb{A}_F), preserving root numbers and L-functions. We remark that, if \sigma is of icosahedral type, we need to assume F = \mathbb{Q} and \sigma is odd [Tay].
Given a two-dimensional irreducible continuous representation $\sigma$ of $W_F$, let $\pi(\sigma)$ be the corresponding cuspidal representation of $GL_2(\mathbb{A}_F)$. On the other hand, $\text{Sym}^3(\sigma)$ defines a continuous four-dimensional representation of $W_F$. By Theorem 6.1, $\text{Sym}^3(\pi(\sigma))$ is an automorphic representation of $GL_4(\mathbb{A}_F)$, and the map $\text{Sym}^3(\sigma) \mapsto \text{Sym}^3(\pi(\sigma))$ preserves root numbers and $L$-functions. In fact, if $\tau$ is another one- or two-dimensional irreducible continuous representation of $W_F$ for which $\pi(\tau)$ exists, then this correspondence preserves the factors for pairs $\text{Sym}^3(\sigma) \otimes \tau$ and $\text{Sym}^3(\pi(\sigma)) \times \pi(\tau)$. We record this as:

**Theorem 10.1.** Let $\sigma$ be a continuous irreducible two-dimensional representation of $W_F$ for which $\pi(\sigma)$ exists (see discussion above). Then, to the continuous four-dimensional representation $\text{Sym}^3(\sigma)$ of $W_F$, there is attached an automorphic representation of $GL_4(\mathbb{A}_F)$, the symmetric cube $\text{Sym}^3(\pi(\sigma))$ of $\pi(\sigma)$, which preserves root numbers and $L$-functions. In particular, if $\sigma$ is not of dihedral type, then the Artin $L$-function $L(s, \text{Sym}^3(\sigma) \otimes \tau)$ is entire for every one-dimensional representation $\tau$ of $W_F$. Similarly, if $\sigma$ is of tetrahedral type, then the same statement is also true for any two-dimensional irreducible continuous representation $\tau$ of $W_F$, provided that $\pi(\sigma)$ exists. Moreover, if $\sigma$ is a representation of icosahedral type for which $\pi(\sigma)$ exists, then $L(s, \text{Sym}^3(\sigma))$ is a degree-four irreducible primitive Artin $L$-function which is entire.

It is easy to see, by the material in Section 6, that even when $\sigma$ is of octahedral type, $\text{Sym}^3(\sigma)$, although irreducible, is not primitive. In fact, there exists a quadratic extension $E/F$ such that $\sigma_E = \sigma|W_E$ is of tetrahedral type, to which there is attached a cubic character $\eta$. Then by Corollary 6.4, $\text{Sym}^3(\sigma)|W_E = \text{Sym}^3(\sigma_E) = (\sigma_E \otimes \eta \det(\sigma_E)) \oplus (\sigma_E \otimes \eta^2 \det(\sigma_E))$.

This immediately implies that $\text{Sym}^3(\sigma)$ is induced. But those attached to icosahedral type representations will always be primitive since $A_5$ has no proper subgroup of index less than 5.

Next, let $\sigma$ and $\tau$ be two continuous two-dimensional irreducible representations of $W_F$ such that $\pi(\sigma)$ and $\pi(\tau)$ exist. Then by Theorem 5.1, the representation $\sigma \otimes \text{Sym}^2(\tau)$ corresponds to $\pi(\sigma) \boxtimes \text{Sym}^2(\pi(\tau))$, i.e., to the six-dimensional representation $\sigma \otimes \text{Sym}^2(\tau)$ of $W_F$, there is attached an automorphic representation of $GL_6(\mathbb{A}_F)$, preserving root numbers and $L$-functions.

Suppose $\eta$ is another two-dimensional continuous representation of $W_F$ which is not of dihedral type. We will assume that $\sigma$ and $\eta$, have nonconjugate images in $PGL_2(\mathbb{C})$. Then by [R1], $\pi(\sigma) \boxtimes \pi(\eta)$, is a cuspidal representation of $GL_4(\mathbb{A}_F)$. Thus

$$(10.1) \quad L(s, \pi(\sigma) \times \text{Sym}^2(\pi(\tau)) \times \pi(\eta))$$
is entire. Next assume $\sigma$ and $\eta$ have conjugate images in $\text{PGL}_2(\mathbb{C})$. One can check quickly that by [JS1], (10.1) has a pole if $\text{Sym}^2(\pi(\tilde{\tau}))$ is a twist of $\text{Sym}^2(\pi(\sigma))$, or equivalently $\text{Ad}(\pi(\tau))$ is a twist of $\text{Ad}(\pi(\sigma))$. We therefore see that (10.1) is entire if $\text{Ad}(\tau)$ and $\text{Ad}(\sigma)$ have nonconjugate images in $\text{GL}_3(\mathbb{C})$.

We state this as:

**Theorem 10.2.** a) Let $\sigma$ and $\tau$ be two continuous two-dimensional representations of $W_F$ for which $\pi(\sigma)$ and $\pi(\tau)$ exist. Then to the six-dimensional representation $\sigma \otimes \text{Sym}^2(\tau)$, there is attached an automorphic representation of $\text{GL}_6(\mathbb{A}_F)$, which preserves root number and $L$-functions for pairs.

b) Let $\eta$ be another such representation and assume $\sigma$ and $\eta$ have nonconjugate images in $\text{PGL}_2(\mathbb{C})$. Then $L(s, \sigma \otimes \text{Sym}^2(\tau) \otimes \eta)$ is entire.

c) Assume $\sigma$ and $\eta$ have conjugate images in $\text{PGL}_2(\mathbb{C})$ and neither $\sigma$, nor $\tau$, nor $\eta$ is dihedral. Moreover assume $\text{Ad}(\sigma)$ and $\text{Ad}(\tau)$ have nonconjugate images in $\text{PGL}_3(\mathbb{C})$. Then again $L(s, \sigma \otimes \text{Sym}^2(\tau) \otimes \eta)$ is entire.

**Remark.** We note that Theorem 10.2 is only new if either $\sigma$ or $\tau$ is of icosahedral type. In fact, Theorem 10.2 does not give anything new beyond [R3] in the solvable cases.

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Appendix:

On certain dyadic representations

By Colin J. Bushnell and Guy Henniart

In this appendix, $F$ denotes a dyadic local field, that is, $F$ is a finite extension field of the 2-adic rational field $\mathbb{Q}_2$. We fix an algebraic closure $\overline{F}$ of $F$, and write $W_F$ for the Weil group of $\overline{F}/F$. If $E/F$ is a finite extension (always assumed to be contained in $\overline{F}$) we denote by $W_E$ the Weil group of $\overline{F}/E$. Also, if $\tau$ is a finite-dimensional continuous semisimple representation of $W_E$, we write $\text{Ind}_{E/F}(\tau)$ for the representation of $W_F$ induced by $\tau$.

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Let \( \psi \) be a nontrivial additive character of \( F \). Let \( n_1, n_2 \) be positive integers and let \( \pi_1, \pi_2 \) be irreducible smooth representations of \( \text{GL}_{n_1}(F) \), \( \text{GL}_{n_2}(F) \) respectively. We form

\[
\gamma(\pi_1 \times \pi_2, s, \psi) = \varepsilon(\pi_1 \times \pi_2, s, \psi) \frac{L(\pi_1 \times \pi_2, 1-s)}{L(\pi_1 \times \pi_2, s)},
\]

as in [6], [8], where \( s \) is a complex variable.

We now state our main result. This makes no pretence at generality: it is designed to serve only the paper [7] to which this is an appendix. We need the following data:

1. An irreducible continuous representation \( \rho_1 \) of \( \mathcal{W}_F \), which is primitive and of dimension 2,
2. an irreducible continuous representation \( \rho_2 \) of \( \mathcal{W}_F \), of dimension 3 and satisfying:
   a. There are a noncyclic cubic extension \( L/F \) and a quasicharacter \( \theta \) of \( \mathcal{W}_L \) such that \( \rho_2 \cong \text{Ind}_{L/F}(\theta) \), and
   b. \( \rho_2 \not\cong \chi \otimes \rho_2 \) for any unramified quasicharacter \( \chi \neq 1 \) of \( \mathcal{W}_F \).

We put \( \rho = \rho_1 \otimes \rho_2 \), and we let \( \pi \) be the irreducible smooth representation of \( \text{GL}_6(F) \) corresponding to \( \rho \) via the Langlands correspondence of [4], [5].

**Main Theorem.** Let \( \pi' \) be an irreducible smooth representation of \( \text{GL}_6(F) \) satisfying the following conditions:

1. Let \( E/F \) be an unramified quadratic extension, and let \( \pi_E, \pi'_E \) be the representations of \( \text{GL}_6(E) \) obtained by base change from \( \pi, \pi' \) respectively. Then
   \( \pi_E \cong \pi'_E. \)
2. \( \gamma(\pi \times \sigma, s, \psi) = \gamma(\pi' \times \sigma, s, \psi) \) for every irreducible supercuspidal representation \( \sigma \) of \( \text{GL}_m(F) \), \( m = 1, 2, 3. \)

Now

\( \pi \cong \pi'. \)

We start the paper with a discussion of the structure of \( \rho \) in Section 1. Then, in Section 2, we give a consequence of the conductor formula of [2] applicable to the present situation. We can then prove the main theorem in Section 3.

1. **Galois representations**

In this section, we are only concerned with the representations \( \rho_1, \rho_2 \) introduced above. By definition, the representation \( \rho_1 \) is not of the form \( \text{Ind}_{F'/F}(\chi) \),
for any quadratic extension $F'/F$ and any quasicharacter $\chi$ of $W_{F'}$. Also, $\rho_1$ remains irreducible on restriction to $W_K$, for any finite, tamely ramified extension $K/F$.

We need some notation attached to the representation $\rho_2 = \text{Ind}_{L/F}(\theta)$. Let $K/F$ be the normal closure of $L/F$, and let $E/F$ be the maximal unramified sub-extension of $K/F$. Then $[E : F] = 2$, $K = LE$, and $\text{Gal}(K/F)$ is isomorphic to the symmetric group $S_3$. We let $g \in \text{Gal}(K/F)$ have order 3, and we let $h$ be the nontrivial involution which fixes $L$.

We write $\text{Ad}^0$ for the adjoint action of $\text{GL}_2(\mathbb{C})$ on the Lie algebra of $\text{SL}_2(\mathbb{C})$. Thus if $\tau$ is a two-dimensional representation of, for example, a finite group $G$, we have

$$\tau \otimes \check{\tau} = 1_G \oplus \text{Ad}^0(\tau),$$

where $1_G$ denotes the trivial representation of $G$. We note that the representation $\text{Ad}^0(\tau)$ is reducible if and only if there is a nontrivial one-dimensional representation $\alpha$ of $G$ such that $\alpha \otimes \tau \cong \tau$. In that case, $\alpha$ is a component of $\text{Ad}^0(\tau)$.

We can now state and prove the main result of this section.

**Theorem 1.** (1) The representation $\rho = \rho_1 \otimes \rho_2$ is reducible if and only if there is a quasicharacter $\chi$ of $W_F$ such that

$$\rho_2 \cong \chi \otimes \text{Ad}^0(\rho_1).$$

(2) When condition (1.1) is satisfied,

$$\rho \cong (\chi \otimes \rho_1) \oplus (\chi \otimes \rho_1 \otimes \lambda),$$

where $\lambda$ denotes the unique irreducible two-dimensional representation of $\text{Gal}(K/F)$.

**Proof.** If $\tau$ is a representation of $W_F$ and $F'/F$ is a finite extension, we write $\tau_{F'}$ for the restriction of $\tau$ to $W_{F'}$. With this notation,

$$\rho = \rho_1 \otimes \rho_2 = \text{Ind}_{L/F}(\theta \otimes \rho_{1L}).$$

Let $\langle , \rangle$ denote the standard inner product of semisimple representations. The representation $\rho$ is thus reducible if and only if

$$\langle \rho, \rho \rangle = \langle \rho \otimes \check{\rho}, 1 \rangle \geq 2;$$

that is, $\rho$ is reducible if and only if $\rho \otimes \check{\rho}$ contains the trivial representation with multiplicity at least two.

With this in mind, the Mackey restriction-induction formula gives

$$\rho \otimes \check{\rho} = \text{Ind}_{L/F} \left( \theta \otimes \rho_{1L} \otimes \theta^{-1} \otimes \check{\rho}_{1L} \right) \oplus \text{Ind}_{K/F} \left( \theta_K \otimes \rho_{1K} \otimes \theta_K^{-1} \otimes \check{\rho}_{1K} \right).$$
The representation \( \rho_{1L} \) is irreducible. It follows that the first term in this expression contains the trivial character with multiplicity one. Thus \( \rho \otimes \hat{\rho} \) is reducible if and only if the second term contains the trivial representation. But, since \( \rho_{2E} = \text{Ind}_{K/E}(\theta_K) \) is irreducible, \( \theta_K^g \neq \theta_K \), and so it follows that:

**Lemma 1.** The representation \( \rho \) is reducible if and only if \( \theta_K^{1-g} \otimes \text{Ad}^0(\rho_{1K}) \) contains the trivial representation of \( W_K \).

The representation \( \rho_{1K} \) is irreducible, since \( \rho_1 \) is primitive and \( K/F \) is tame. By [9] therefore, there are just two, mutually exclusive possibilities: either \( \rho_{1K} \) is primitive, or else it is *triply imprimitive*. This means there is a Galois extension \( K'/K \), with \( \text{Gal}(K'/K) \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \), such that \( \rho_{1K} \) is induced from each of the three quadratic sub-extensions of \( K'/K \).

**Lemma 2.** (1) If \( \rho_{1K} \) is primitive, then \( \text{Ad}^0(\rho_{1K}) \) is irreducible.
(2) Suppose that \( \rho_{1K} \) is triply imprimitive, attached to a quartic extension \( K'/K \) as above. Let \( \eta \) be a nontrivial character of \( \text{Gal}(K'/K) \). Then

\[
\text{Ad}^0(\rho_{1K}) = \eta \oplus \eta^g \oplus \eta^{g^2}.
\]

The characters \( \eta, \eta^g \) and \( \eta^{g^2} \) are distinct.

**Proof.** The first assertion, and the fact that \( \eta \) is a component of \( \text{Ad}^0(\rho_{1K}) \) in part (2), both follow from earlier remarks. Indeed, these remarks show that, in part (2), \( \text{Ad}^0(\rho_{1K}) \) is the direct sum of the nontrivial characters of \( \text{Gal}(K'/K) \). The element \( g \) certainly acts on this set of characters: we have to show that this action is nontrivial. If the action were trivial then, since \( K/E \) is tame, each of these characters would be the restriction of a character of \( W_E \). It would follow that \( \text{Ad}^0(\rho_{1E}) \) is a direct sum of abelian characters, as well as being the restriction to \( W_E \) of the irreducible 3-dimensional representation \( \text{Ad}^0(\rho_1) \). Since \( W_E \) is a normal subgroup of \( W_F \) of index 2, this is impossible.

Let us deal with the theorem in the case where \( \rho_{1K} \) is primitive. By Lemmas 1 and 2, the representation \( \rho \) is irreducible. On the other hand, \( \rho_{2K} \) is certainly reducible, so (1.1) can never hold. Part (2) is vacuous in this case, so this proves the theorem for \( \rho_{1K} \) primitive.

From now on, therefore, we assume that \( \rho_{1K} \) is *triply imprimitive*.

**Lemma 3.** Suppose that \( \rho_{1K} \) is triply imprimitive and that \( \rho \) is reducible. There exists a nontrivial character \( \eta \) of \( \text{Gal}(K'/K) \) such that

\[
\eta \otimes \rho_{1K} \cong \rho_{1K},
\]

\[
\eta^g = \eta,
\]

\[
(\theta_K/\eta)^g = \theta_K/\eta.
\]
Proof. We have $\rho_{1K} \cong \chi \otimes \rho_{1K}$ for every character $\chi$ of the noncyclic 4-group $\text{Gal}(K'/K)$, so the first property holds for any choice of $\eta$. Since $\text{Ad}^0(\rho_{1K}) = \eta \oplus \eta^g \oplus \eta^{g^2}$, Lemma 1 implies that $\theta_{K}^{g^{-1}}$ is one of $\eta, \eta^g, \eta^{g^2}$. In particular, $\theta_{K}^{g^{-1}}$ has order 2. By definition, $\theta_K$ is invariant under $h$, so that

$$(\theta_{K}^{(g^{-1})g})^h = \theta_{K}^{g^{-2}} = (\theta_{K}^{g^{-2}})^{-1} = \theta_{K}^{(g^{-1})g}.$$ 

Thus, $\theta_{K}^{(g^{-1})g}$ is also invariant under $h$. Changing notation if necessary, we can assume $\theta_{K}^{(g^{-1})g} = \eta$, which is therefore fixed by $h$. Since $\eta^g \eta^{g^2} = 1$, we also have $\eta = \eta^{(g-1)g}$. It follows that $\theta_K/\eta$ is fixed by $g$ as required.

We assume that $\rho_{1K}$ is reducible. Since $\theta_K/\eta$ (as in Lemma 3) is fixed by $g$, there exists a quasicharacter $\chi$ of $\mathcal{W}_E$ such that $\theta_K = \eta \cdot \chi_K$. Since $\theta_K$ and $\eta$ are fixed by $h$, so is $\chi_K$. Thus $\chi^{h^{-1}}$ is a character of $\text{Gal}(K/E)$. Moreover, $\chi$ is only determined modulo the character group of $\text{Gal}(K/E)$, on which the element $h$ acts nontrivially. We can therefore choose $\chi$ to satisfy $\chi^h = \chi$. Thus $\chi = \chi_E'$, for some character $\chi'$ of $\mathcal{W}_F$. Likewise, $\eta = \eta'_{K'}$ for some character $\eta'$ of $\mathcal{W}_L$. We have $\theta_K = \eta'_{K'} \chi_K$, so

$$\theta = \eta'_{K'} \chi_L' \quad \text{or} \quad \theta = \omega_{K'/L} \eta'_{K'} \chi_L',$$

where $\omega_{K'/L}$ is the nontrivial character of $\text{Gal}(K/L)$. This, however, is the restriction to $\mathcal{W}_L$ of the nontrivial character $\omega_{E/F}$ of $\text{Gal}(E/F)$, so we can absorb this into $\chi'$ and assume $\theta = \eta'_{K'} \chi_L'$.

Thus, when $\rho$ is reducible,

$$\rho_2 = \chi' \otimes \text{Ind}_{L/F}(\eta'),$$

for some quasicharacter $\chi'$ of $\mathcal{W}_F$. On the other hand, $\text{Ad}^0(\rho_{1L})$ contains either $\eta'$ or $\omega_{K'/L} \eta'$. It follows that either $\text{Ad}^0(\rho_1)$ or $\omega_{E/F} \otimes \text{Ad}^0(\rho_1)$ is equivalent to $\text{Ind}_{L/F}(\eta')$, so we conclude that, when $\rho$ is reducible, $\rho_2$ is a twist of $\text{Ad}^0(\rho_1)$.

We now treat the converse statement in part (1) of the theorem. We assume that $\rho_2 = \chi \otimes \text{Ad}^0(\rho_1)$ (and that $\rho_{1K}$ is triply imprimitive). Since, by definition, $\rho_1 \otimes \check{\rho}_1 = 1 \oplus \text{Ad}^0(\rho_1)$,

$$\langle \rho_1, \rho_1 \otimes \text{Ad}^0(\rho_1) \rangle = \langle \rho_1 \otimes \check{\rho}_1, \text{Ad}^0(\rho_1) \rangle \geq 1,$$

and so $\rho_1 \otimes \text{Ad}^0(\rho_1)$ contains $\rho_1$. Write $\text{Ad}^0(\rho_1) = \text{Ind}_{L/F}(\eta')$, for some quasicharacter $\eta'$ of $\mathcal{W}_L$ such that $\eta'_{K'} = \eta$. Then

$$\rho_1 \otimes \text{Ad}^0(\rho_1) = \text{Ind}_{L/F}(\eta' \otimes \rho_{1L}).$$

We have $(\eta' \otimes \rho_{1L})_K = \eta \otimes \rho_{1K} = \rho_{1K}$, so that $\eta' \otimes \rho_{1L}$ is either $\rho_{1L}$ or $\omega_{K'/L} \rho_{1L}$. In the first case, we get $\rho_1 \otimes \text{Ad}^0(\rho_1) = \rho_1 \oplus (\rho_1 \otimes \lambda)$, and in the second, $\omega_{E/F} \rho_1 \oplus (\rho_1 \otimes \lambda)$. (In particular, $\rho$ is reducible, which completes the proof of (1).) Since $\rho_1$ is a component of $\rho_1 \otimes \text{Ad}^0(\rho_1)$, it is the first case which
must hold. This gives the formula required for part (2), and we have completed the proof of Theorem 1.

**Variant.** Assume the hypotheses and notation of Theorem 1, except that $L/F$ is now cyclic and totally ramified. Then:

1. The representation $\rho$ is reducible if and only if there is a quasicharacter $\chi$ of $W_F$ such that
   
   $$\rho_2 \cong \chi \otimes \text{Ad}^0(\rho_1).$$

2. When (1.2) is satisfied, we have
   
   $$\rho \cong \chi \otimes \rho_1 \otimes \text{Ind}_{L/F} 1_L,$$

   where $1_L$ denotes the trivial character of $W_L$.

**Proof.** This is parallel to that of the theorem, setting $E = F$, $K = L$, and $h = 1$. □

We revert to the hypotheses of Theorem 1. It will be useful to have some further control of the representation $\rho = \rho_1 \otimes \rho_2$. Recall that an irreducible representation $\sigma$ of $W_F$ is totally ramified if $\sigma \otimes \chi \not\cong \sigma$ for any unramified quasicharacter $\chi \neq 1$ of $W_F$. Equivalently, $\sigma_{F'}$ is irreducible (and totally ramified) for any finite unramified extension $F'/F$. For example, the representations $\rho_1$, $\rho_2$ are totally ramified.

**Proposition 1.** In the situation of Theorem 1, suppose that $\rho$ is irreducible. Then $\rho$ is totally ramified.

**Proof.** Suppose otherwise; then there is an unramified quasicharacter $\alpha \neq 1$ of $W_F$ such that $\alpha \otimes \rho \cong \rho$. The quasicharacter $\alpha$ has finite order dividing 6, so we assume it has order $d = 2$ or 3. We may view $\alpha$ as a character of $\text{Gal}(M/F)$, where $M/F$ is unramified of degree $d$.

**Lemma 4.** The representation $\rho_{1M}$ is primitive.

**Proof.** Suppose that $\rho_{1M}$ is triply imprimitive, and take first the case $d = 3$. Thus there is a quadratic extension $M'/M$ and a quasicharacter $\xi$ of $W_{M'}$ such that $\rho_{1M} = \text{Ind}_{M'/M}(\xi)$. Next, $\rho_{2M'}$ is irreducible, so $\rho_{M'}$ is the direct sum of two irreducible 3-dimensional representations. On the other hand, the relation $\chi \otimes \rho \cong \rho$ implies that $\rho$ is induced from $W_M$, and that $\rho_M$ is the direct sum of three irreducible 2-dimensional representations. This case can therefore not arise.

Take next the case $d = 2$. The representation $\text{Ad}^0(\rho_1)$ is irreducible of dimension 3, so that $\text{Ad}^0(\rho_1)_M = \text{Ad}^0(\rho_{1M})$ is irreducible. On the other hand, if $\rho_{1M}$ is triply imprimitive, the representation $\text{Ad}^0(\rho_{1M})$ is the direct sum of three quasicharacters. Again, this case cannot arise. □
Now we prove the proposition. Suppose first that $M/F$ is of degree 3. The hypotheses of the theorem apply to the representations $\rho_{1M}$ and so we get a quasicharacter $\chi$ of $W_M$ such that $\rho_{2M} \cong \chi \otimes \text{Ad}^0(\rho_{1M})$. If $y$ generates $\text{Gal}(M/F)$, we have $\chi^y - 1 \otimes \rho_{2M} \cong \rho_{2M}$. The representation $\rho_{2M}$ is totally ramified and induced from the noncyclic cubic extension $LM/M$, so that $\chi^y = \chi$. Thus there is a quasicharacter $\chi'$ of $W_F$ with $\chi'_M = \chi$ and $\rho_2 \cong \chi' \otimes \text{Ad}^0(\rho_1)$. This is impossible, by Theorem 1.

This reduces to the case where $M/F$ is quadratic. We can apply the variant of the theorem to $\rho_{1M}$ and $\rho_{2M}$ to get a character $\chi$ of $W_M$ such that $\rho_{2M} \cong \chi \otimes \text{Ad}^0(\rho_{1M})$. If $h$ is a generator of $\text{Gal}(M/F)$, the character $\chi^h - 1$ fixes $\rho_{2M}$, and so is a character of $\text{Gal}(LM/M)$. Since $\chi$ is only determined modulo the character group of $\text{Gal}(LM/M)$, we can in fact assume that $\chi$ is fixed by $h$ and finish the proof as in the last case.

\section{Conductors of pairs}

In this section only, $F$ denotes an arbitrary non-Archimedean local field, with finite residue field of $q$ elements. Let $N$ denote a positive integer; we write $G_N = GL_N(F)$ and also $A_N = M_N(F)$, the algebra of $N \times N$-matrices over $F$.

We shall use the classification theory of $[3]$. In particular, we need the notion of a simple stratum in $A_N$, as in [3, 1.5]. Attached to the simple stratum $[A, n, 0, \beta]$, we have the compact open subgroups $H^r(\beta, A)$, $r \geq 1$, of $G_N$. Next, we fix a continuous character $\psi_0$ of $F$ nontrivial on the discrete valuation ring $\mathfrak{o}$ in $F$, but trivial on the maximal ideal $p$ of $\mathfrak{o}$. Such a choice of $\psi_0$ gives rise to a set $\mathcal{C}(A, 0, \beta) = \mathcal{C}(A, \beta)$ of characters of $H^1(\beta, A)$, called simple characters. (The terminology and notation are as in [3, Ch. 3].)

Let $\pi$ be an irreducible supercuspidal representation of $G_N$. If $\chi$ is a quasicharacter of $F^\times$, we write $\chi \cdot \pi$ for the representation $g \mapsto \chi(\det g)\pi(g)$ of $G_N$. We say that $\pi$ is totally ramified if $\chi \cdot \pi \not\cong \pi$ for any unramified quasicharacter $\chi \neq 1$ of $F^\times$.

Proposition 2. Let $\pi$ be an irreducible supercuspidal representation of $G_N$, $N \geq 2$. The following are equivalent:

1. $\pi$ is totally ramified;
2. There is a simple stratum $[A, n, 0, \beta]$ in $A_N$ and a simple character $\theta \in \mathcal{C}(A, \beta)$ such that:
   (a) $\theta$ occurs in $\pi$,
   (b) the field extension $F[\beta]/F$ is totally ramified of degree $N$.

Proof. This follows immediately from [3, (6.2.5), (8.4.1)].
The integer \( n = n(\pi) \) is an invariant of the totally ramified representation \( \pi \).

**Lemma 5.** Let \( \pi \) be a totally ramified, irreducible, supercuspidal representation of \( G_N, N \geq 2 \). The following are equivalent:

1. There is a quasicharacter \( \chi \) of \( F^\times \) such that \( n(\chi \cdot \pi) < n(\pi) \);
2. \( n(\pi) \equiv 0 \pmod{N} \).

**Proof.** Take a simple stratum \([\mathfrak{A}, n, 0, \beta]\) and a simple character \( \theta \) occurring in \( \pi \), as in Proposition 2. The integer \( n = n(\pi) \) is then \( -\nu_{F[\beta]}(\beta) \), where \( \nu \) denotes the normalized additive valuation.

We choose a simple stratum \([\mathfrak{A}, n, n-1, \alpha]\) in \( A_N \) which is equivalent to \([\mathfrak{A}, n, n-1, \beta]\). The algebra \( F[\alpha] \) is a field. The ramification index \( e(F[\alpha] | F) \) and residue class degree \( f(F[\alpha] | F) \) divide the corresponding invariants of \( F[\beta]/F \). In particular, \( F[\alpha]/F \) is totally ramified.

Since \([\mathfrak{A}, n, n-1, \alpha]\) is simple, the integer
\[
\nu_{F[\alpha]}(\alpha) = -e(F[\beta] | F)^{-1}e(F[\alpha] | F) n = -N^{-1}e(F[\alpha] | F) n
\]
is relatively prime to \( e(F[\alpha] | F) = [F[\alpha] : F] \). If \( N \) divides \( n \) therefore, \( e(F[\alpha] | F) \) must divide \( \nu_{F[\alpha]}(\alpha) \), and this forces \( F[\alpha] = F \).

By definition \([3, 3.2]\), the restriction of the character \( \theta \) to the congruence unit group \( U^n(\mathfrak{A}) \) of the order \( \mathfrak{A} \) takes the form
\[
x \mapsto \psi_0(\text{tr}(\alpha(x-1))), \quad x \in U^n(\mathfrak{A}),
\]
where \( \text{tr} \) denotes the matrix trace \( A_N \to F \). Since \( \alpha \in F \), there is a quasicharacter \( \chi \) of \( F^\times \) such that \( \chi(x) = \psi_0(\alpha(x-1)) \), for \( x \in U^n_F/N \). The representation \( \pi_1 = \chi^{-1} \cdot \pi \) then satisfies \( n(\pi_1) < n \).

Thus (2) \( \Rightarrow \) (1); the converse follows similarly. \( \square \)

Let \( N, N' \) be positive integers, and let \( \pi, \pi' \) be irreducible smooth representations of \( G_N, G_{N'} \) respectively. We form the local constant \( \varepsilon(\pi \times \pi', s, \psi) \) of the pair \((\pi, \pi')\), as in \([6]\) or \([8]\). Here, \( s \) is a complex variable and \( \psi \) is a nontrivial character of the additive group of \( F \). The local constant takes the form
\[
\varepsilon(\pi \times \pi', s, \psi) = q^{(\frac{1}{2}-s)f(\pi \times \pi', \psi)} \varepsilon(\pi \times \pi', \frac{1}{2}, \psi),
\]
for a certain integer \( f(\pi \times \pi', \psi) \). Let \( c(\psi) \) denote the conductor of \( \psi \); thus \( c(\psi) \) is the least integer \( k \) such that \( \psi \) is trivial on \( p^k \). There is then an integer \( a(\pi \times \pi') \), independent of \( \psi \), such that
\[
f(\pi \times \pi', \psi) = a(\pi \times \pi') - NN'c(\psi).
\]
We can now give the main result of the section.
Theorem 2. Let \( \pi \) be an irreducible, totally ramified supercuspidal representation of \( G_0 \). There are a positive integer \( m \) dividing 3, and an irreducible, totally ramified supercuspidal representation \( \pi' \) of \( G_m \) such that
\[
a(\pi \times \pi') \not\equiv 0 \pmod{2}.
\]

Proof. We choose a simple stratum \([A, n, 0, \beta]\) and a simple character \( \theta \in \mathcal{C}(A, \beta) \) which occurs in \( \pi \). If \( \chi \) is a quasicharacter of \( F^\times \),
\[
a(\chi : \pi \times \pi') = a(\pi \times \chi \cdot \pi').
\]
Therefore, by Lemma 5, we can replace \( \pi \) by \( \chi \cdot \pi \), for a suitable \( \chi \), to ensure that \( n \not\equiv 0 \pmod{6} \). That done, if we choose a simple stratum \([A, n, n-1, \gamma]\) equivalent to \([A, n, n-1, \beta]\), we have \([F[\gamma] : F] > 1\). The quantity
\[
-\nu_{F[\gamma]}(\gamma) = e(F[\gamma] \mid F) n/6
\]
is relatively prime to \( e(F[\gamma] \mid F) = [F[\gamma] : F] \) so, if \([F[\gamma] : F] \) is even, the integer \( n \) is odd and there is nothing to prove. We therefore assume \([F[\gamma] : F] = 3\).

Let \( r \) denote the integer \(-k_{0}(\beta, A)\) (notation of [3, 1.4.5]). By the assumption on \( F[\gamma] \), we have \( n > r \geq 1 \). We choose a simple stratum \([A, n, r, \gamma']\) equivalent to \([A, n, r, \beta]\). The field extension \( F[\gamma']/F \) is again of degree 3 (since it divides 6 properly, and is divisible by \([F[\gamma] : F] = 3\)). We could have taken \( \gamma = \gamma' \) above, and so we now simplify notation by doing so.

Lemma 6. The integer \( r \) is odd.

Proof. Let \( B \) denote the centralizer of \( \gamma \) in \( A = A_6 \), and put \( \mathfrak{B} = A \cap B \).
Let \( s_\gamma : A \to B \) be a tame corestriction ([3, 1.3]) relative to \( F[\gamma]/F \), and write \( \beta = \gamma + c \). Then \([\mathfrak{B}, r, r-1, s_\gamma(c)]\) is equivalent (in \( B \)) to a simple stratum \([\mathfrak{B}, r, r-1, \alpha]\) (ibid. (2.4.1)). Now,
\[
e(F[\gamma, \alpha] \mid F) = e(F[\beta] \mid F) = 6,
\]
\[
f(F[\gamma, \alpha] \mid F) = f(F[\beta] \mid F) = 1,
\]
and so the field extension \( F[\alpha, \gamma]/F[\gamma] \) is totally ramified of degree 2. However, by the definition of simple stratum, the integer \( r = -\nu_{F[\alpha, \gamma]}(\alpha) \) is relatively prime to \( e(F[\alpha, \gamma] \mid F[\gamma]) \), and so \( r \) is odd, as required. \( \square \)

We now start the construction of the desired representation \( \pi' \). First, \( \theta \in \mathcal{C}(A, \beta) \) is a character of the group \( H^1(\beta, A) \). We can also form \( H^1(\gamma, A) \), and we have \( H^{r+1}(\gamma, A) = H^{r+1}(\beta, A) \) [3, (3.1.9)]. There is a simple character \( \theta_0 \in \mathcal{C}(A, \gamma) \) such that
\[
\theta_0 \mid H^{r+1}(\beta, A) = \theta \mid H^{r+1}(\beta, A)
\]
(ibid. (3.3.21)). We now embed the field \( F[\gamma] \) in \( A_3 = M_3(F) \); we then get a simple stratum \([A', n', 0, \gamma]\) in \( A_3 \), for a uniquely determined principal order.
$A'$, with $n' = n/2$. We have a canonical bijection between $\mathcal{C}(A, \gamma)$ and $\mathcal{C}(A', \gamma)$ [3, (3.6.14)]: let $\theta_1 \in \mathcal{C}(A', \gamma)$ correspond to $\theta_0$.

Let $\pi_1$ be an irreducible representation of $G_3$ which contains $\theta_1$. Then $\pi_1$ is supercuspidal and totally ramified. In the language of [2, §6], the pair $(\pi, \pi_1)$ has a best common approximation of the form $([A, n, 0, \gamma], r, \vartheta)$, with $r$ and $n$ as above. We apply the formula of [2, 6.5(iii)] to get

$$a(\pi \times \pi_1) = 18 \left(1 + \frac{c(\gamma)}{9} + \frac{r}{18}\right),$$

for a certain integer $c(\gamma)$, and this yields

$$a(\pi \times \pi_1) \equiv r \equiv 1 \pmod{2}.$$

The representation $\pi' = \pi_1$ thus has the required property. \qed

3. Proof of the Main Theorem

We now prove the Main Theorem stated in the introduction. We therefore revert to the notation there and in §1. We treat first the case where $\rho$ is irreducible. Thus $\pi$ is supercuspidal. The representation $\rho_E$ is irreducible by Proposition 1, and $\pi_E$ corresponds to $\rho_E$ under the Langlands correspondence. The relation $\pi_E \cong \pi'_E$ implies that $\pi'_E$ is supercuspidal, and hence that $\pi'$ is supercuspidal.

Since $\pi$ is supercuspidal, we have $\gamma(\pi \times \sigma, s, \psi) = \varepsilon(\pi \times \sigma, s, \psi)$, for any irreducible smooth representation $\sigma$ of $GL_m(F), m \leq 5$. That is similar for $\pi'$, and the relation $\varepsilon(\pi \times \sigma, s, \psi) = \varepsilon(\pi' \times \sigma, s, \psi)$ implies $a(\pi \times \sigma) = a(\pi' \times \sigma)$. The relation $\pi_E \cong \pi'_E$ further implies that either $\pi \cong \pi'$ or $\pi \cong \omega_{E/F} \pi'$, where $\omega_{E/F}$ is the character of $F^\times$ corresponding to the nontrivial character of $Gal(E/F)$ [1, I Prop. 6.7]. By Theorem 2, we can find an irreducible supercuspidal representation $\sigma$ of $GL_m(F), m = 1$ or 3, such that $a(\pi \times \sigma) = a(\pi' \times \sigma)$ is odd. Then

$$\varepsilon(\omega_{E/F} \pi' \times \sigma, s, \psi) = (-1)^{a(\pi' \times \sigma)} \varepsilon(\pi' \times \sigma, s, \psi) = -\varepsilon(\pi \times \sigma, s, \psi).$$

It is therefore only the case $\pi \cong \pi'$ which arises, as desired.

We now assume that $\rho$ is reducible. Thus $\rho_2 = \chi \otimes \text{Ad}^0(\rho_1)$, for some quasicharacter $\chi$ of $W_F$ (Theorem 1). Clearly, it is enough to treat the case $\chi = 1$, so that $\rho = \rho_1 \oplus (\rho_1 \otimes \lambda)$, as in Theorem 1(2). The representation $\lambda$ is of the form $\text{Ind}_{E/F}(\phi)$, where $\phi$ is a nontrivial character of $Gal(K/E)$. It follows easily that $\rho_1 \otimes \lambda$ is irreducible, and satisfies

$$\rho_1 \otimes \lambda \otimes \omega_{E/F} \cong \rho_1 \otimes \lambda.$$

On the other hand, $\rho_1$ is irreducible and totally ramified.
Let \( \pi_1 \) be the irreducible supercuspidal representation of \( \text{GL}_2(F) \) corresponding to \( \rho_1 \), and \( \Pi \) that of \( \text{GL}_4(F) \) corresponding to \( \rho_1 \otimes \lambda \). Thus \( \pi_1 \) is totally ramified, while \( \omega_{E/F} \cdot \Pi \cong \Pi \) (where \( \omega_{E/F} \) now denotes the character of \( F^{\times} \) corresponding to the nontrivial character of \( \text{Gal}(E/F) \)). We have

\[
(3.1) \quad \gamma((\pi \times \pi_1, s, \psi)) = \frac{1 - q^{-s}}{1 - q^{-s-1}} \varepsilon(\pi_1 \times \pi_1, s, \psi) \varepsilon(\Pi \times \pi_1, s, \psi).
\]

The representation \( \Pi_E \) corresponds, via the Langlands correspondence, to \( (\rho_1 \otimes \lambda)_{E} = (\phi \otimes \rho_{1E}) \oplus (\phi^2 \otimes \rho_{1E}) \). Thus

\[
\pi E \cong \pi'_E \cong \pi_1 \oplus \phi \cdot \pi_{1E} \oplus \phi^2 \cdot \pi_{1E},
\]

for some integer \( a \). We now have

\[
\gamma((\pi' \times \pi_1, s, \psi)) = \frac{1 - (-1)^a q^{-s}}{1 - (-1)^a q^{-s-1}} \varepsilon(\omega_{E/F}^a \pi_1 \times \pi_1, s, \psi) \varepsilon(\Pi \times \pi_1, s, \psi).
\]

Comparing the location of the poles of this expression with those of (3.1), we get \( (-1)^a = 1 \) and \( \omega_{E/F}^a = 1 \), as desired.

\[\square\]

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