Multivariate Alexander colorings

Lorenzo Traldi
Lafayette College
Easton Pennsylvania 18042, United States

Abstract

We extend the notion of link colorings with values in an Alexander quandle to link colorings with values in a module $M$ over the Laurent polynomial ring $\Lambda = \mathbb{Z}[t_1^{\pm 1}, \ldots, t_\mu^{\pm 1}]$. If $D$ is a diagram of a link $L$ then the colorings of $D$ with values in $M$ form a $\Lambda_\mu$-module $\text{Color}_A(D, M)$. Extending a result of Inoue [Kodai Math. J. 33 (2010), 116-122], we show that $\text{Color}_A(D, M)$ is isomorphic to the module of $\Lambda_\mu$-linear maps from the Alexander module of $L$ to $M$. In particular, suppose $M = F$ is a field, considered as a $\Lambda_\mu$-module via a homomorphism $\varphi : \Lambda_\mu \to F$ of rings with unity. Then $\text{Color}_A(D, M)$ is a vector space over $F$, and we show that its dimension is determined by the images under $\varphi$ of the elementary ideals of $L$. This result applies in the special case of Fox tricolorings, which correspond to $M = GF(3)$ and $\varphi(t_i) \equiv -1$. Examples show that even in this special case, the higher Alexander polynomials do not suffice to determine $|\text{Color}_A(D, M)|$; this observation corrects erroneous statements of Inoue [J. Knot Theory Ramifications 10 (2001), 813-821; op. cit.].

1 Introduction

This paper is concerned with link invariants defined from diagrams. We use standard notation and terminology: A (tame, classical) link $L = K_1 \cup \cdots \cup K_\mu$ has $\mu$ disjoint components, each of which is a knot, i.e., a piecewise smooth copy of $S^1$ in $S^3$. A diagram $D$ of $L$ in the plane is obtained from a projection with only finitely many singularities, all of which are double points called crossings. At each crossing, $D$ distinguishes the underpassing component by removing two short segments, one on each side of the crossing. Removing these segments splits $D$ into a finite number of arc components. The set of arc components is denoted $A(D)$, and the set of crossings of $D$ is denoted $C(D)$.

The idea of a quandle or distributive groupoid was introduced in the 1980s by Joyce [11] and Matveev [15]. In the intervening decades a sizable literature has developed, involving many different generalizations and special cases of the quandle idea. In this paper we generalize one of these special cases.

Definition 1 An Alexander quandle is a module $M$ over the ring $\Lambda = \mathbb{Z}[t^{\pm 1}]$ of Laurent polynomials in the variable $t$, with integer coefficients. The quandle
Figure 1: The arcs incident at a crossing.

Operation is given by

\[ a_2 \triangleright a_1 = (1 - t) \cdot a_1 + t \cdot a_2. \]

Notice that for an Alexander quandle, the quandle operation is determined by the module structure. As we do not refer to any non-Alexander quandles in this paper, we use notation and terminology for modules rather than quandles. For instance, the following definition is equivalent to the definition of Alexander quandle colorings in the literature, even though the definition does not include the word “quandle.”

**Definition 2** Let \( D \) be a link diagram, and \( M \) a \( \Lambda \)-module. An Alexander coloring of \( D \) with values in \( M \) is given by a function \( f : A(D) \to M \) such that at every crossing \( c \) as indicated in Figure 1, the following equation is satisfied:

\[ f(a_3) = (1 - t) \cdot f(a_1) + t \cdot f(a_2). \]

Here is a multivariate version of Definition 2.

**Definition 3** Let \( D \) be a diagram of a link \( L = K_1 \cup \cdots \cup K_\mu \), and let \( M \) be a module over the ring \( \Lambda_\mu = \mathbb{Z}[t_1^{\pm 1}, \ldots, t_\mu^{\pm 1}] \) of Laurent polynomials in the variables \( t_1, \ldots, t_\mu \), with integer coefficients. Let \( \kappa : A(D) \to \{1, \ldots, \mu\} \) be the map with \( \kappa(a) = i \) if and only if \( a \) is an arc of \( K_i \). Then a multivariate Alexander coloring of \( D \) with values in \( M \) is given by a function \( f : A(D) \to M \) such that at every crossing \( c \) as indicated in Figure 1, the following equation is satisfied:

\[ f(a_3) = (1 - t_{\kappa(a_2)}) \cdot f(a_1) + t_{\kappa(a_1)} \cdot f(a_2). \]

The set of all multivariate Alexander colorings of \( D \) with values in \( M \) is denoted \( \text{Color}_A(D, M) \).

Here are several remarks about these definitions.

1. Definition 3 includes Definition 2. If \( M \) is a module over \( \Lambda = \mathbb{Z}[t^{\pm 1}] \) then \( M \) is also a \( \Lambda_\mu \)-module, with \( t_i \cdot m = t \cdot m \forall m \in M \forall i \in \{1, \ldots, \mu\} \). In particular, there is no difference between Definitions 2 and 3 when \( \mu = 1 \).

2. When we refer to Definition 2 we sometimes use the phrase “standard Alexander coloring” to emphasize that we are not discussing Definition 3.
3. Definition 3 does not seem to be associated with a notion of “multivariate Alexander quandles” analogous to the notion of standard Alexander quandles. There is no quandle structure on $M$ because $\kappa$ is defined on $A(D)$, not $M$.

4. Nosaka has pointed out that he mentioned the possibility of defining link colorings in $\Lambda_\mu$-modules in [16, Remark 2.6]. This idea was also mentioned by Manturov and Ilyutko [14, Theorem 3.14], in the more general context of virtual links. These authors did not develop the results we present below, though.

5. For each $m \in M$, the constant function $f(a) = m$ satisfies Definition 2 and the nonconstant function $f(a) = (1 - t_{\kappa(a)}) \cdot m$ satisfies Definition 3.

6. Color $\text{Color}_A(D, M)$ is itself a module over $\Lambda_\mu$, using pointwise addition and scalar multiplication. That is, if $f_1, f_2 \in \text{Color}_A(D, M)$ and $\lambda \in \Lambda_\mu$ then $(f_1 + f_2)(a) = f_1(a) + f_2(a)$ and $(\lambda \cdot f_1)(a) = \lambda \cdot f_1(a)$ $\forall a \in A(D)$.

In Section 2 we prove the following result, which we call the Fundamental Theorem of Alexander colorings.

**Theorem 4** Let $D$ be a diagram of $L = K_1 \cup \cdots \cup K_\mu$, and let $M$ be a module over $\Lambda_\mu$. If $M_A(L)$ is the Alexander module of $L$, then

$$\text{Color}_A(D, M) \cong \text{Hom}_{\Lambda_\mu}(M_A(L), M).$$

The fundamental theorem extends a theorem of Inoue [9] from standard Alexander colorings to multivariate Alexander colorings. Many other authors have discussed the fact that standard Alexander colorings are connected to the Alexander module, or to the Alexander polynomials [1, 4, 6, 11, 12, 13, 15, 16].

In our terminology, Inoue’s theorem is stated as follows.

**Corollary 5** ([9]) Let $D$ be a diagram of a link $L$, and let $M$ be a module over $\Lambda = \mathbb{Z}[t^{\pm 1}]$. If $M_A^{\text{red}}(L)$ is the reduced Alexander module of $L$, then the $\Lambda$-module of Alexander colorings of $D$ with values in $M$ is isomorphic to $\text{Hom}_\Lambda(M_A^{\text{red}}(L), M)$.

There is no easily computable set of complete invariants for modules over $\Lambda$ and $\Lambda_\mu$, so these modules are not particularly convenient structures to work with. Theorems 4 and Corollary 5 yield more concrete results when $M$ is both a module over $\Lambda_\mu$ and a vector space over a field, because the vector space is characterized up to isomorphism by its dimension. Before stating a result we establish some notation.

Suppose $F$ is a field and $\varphi : \Lambda_\mu \rightarrow F$ is a homomorphism of rings with unity. If $M$ is a vector space over $F$ then $M$ is also a module over $\Lambda_\mu$ via $\varphi$; the scalar product of $\lambda \in \Lambda_\mu$ and $m \in M$ is given by $\lambda \cdot m = \varphi(\lambda) \cdot m$. Also, the tensor product $F \otimes_{\Lambda_\mu} M_A(L)$ is a vector space over $F$. Moreover, we have an isomorphism

$$\text{Hom}_{\Lambda_\mu}(M_A(L), M) \cong \text{Hom}_F(F \otimes_{\Lambda_\mu} M_A(L), M).$$

In Section 3 we observe that the dimension of $F \otimes_{\Lambda_\mu} M_A(L)$ is determined by the images under $\varphi$ of certain classical invariants of $L$, the elementary ideals. To
be precise, if $j_0$ is the smallest index of an elementary ideal with $\varphi(E_j(L)) \neq 0$, then $j_0$ is equal to the dimension of $F \otimes_{\Lambda_\mu} M_A(L)$. We deduce the following special case of the fundamental theorem.

**Theorem 6** Let $F$ be a field, and let $\varphi : \Lambda_\mu \rightarrow F$ be a homomorphism of rings with unity. Let $M$ be a vector space over $F$, considered as a module over $\Lambda_\mu$ via $\varphi$. Let $L$ be a link, and let $j_0$ be the smallest index with $\varphi(E_{j_0}(L)) \neq 0$. Then for any diagram $D$ of $L$,

$$\text{Color}_A(D, M) \cong \text{Hom}_F(F^{j_0}, M).$$

That is, $\text{Color}_A(D, M)$ is a vector space over $F$ of dimension $j_0 \cdot \text{dim}_F(M)$.

In case $M = F$, we have the following.

**Corollary 7** Let $F$ be a field, and let $\varphi : \Lambda_\mu \rightarrow F$ be a homomorphism of rings with unity. Consider $F$ as a module over $\Lambda_\mu$ via $\varphi$. Let $L$ be a link, and let $j_0$ be the smallest index with $\varphi(E_{j_0}(L)) \neq 0$. Then for any diagram $D$ of $L$,

$$\text{Color}_A(D, F) \cong \text{Hom}_F(F^{j_0}, F).$$

That is, $\text{Color}_A(D, F)$ is a vector space over $F$ of dimension $j_0$.

If $\varphi(t_i) = \varphi(t_j) \forall i, j$, then the colorings described in Corollary 7 are standard Alexander colorings. These colorings have been studied by Kauffman and Lopes [12], who refer to them as colorings by linear Alexander quandles. The most familiar instances are the Fox colorings, which correspond to homomorphisms with $\varphi(t_i) = -1 \forall i$.

As far as we know, the precise statement of Corollary 7 has not appeared before, although a version of the special case for Fox colorings was announced recently [18]. Inoue [8, 9] stated a similar result for standard Alexander colorings, with the elementary ideals replaced by the higher Alexander polynomials.

In Section 4 we show that Inoue’s version of Corollary 7 is incorrect even in the simplest case, i.e., Fox colorings of knots with $F = GF(3)$, the field of three elements.

After discussing examples in Sections 4 – 6, in Section 7 we outline the extension of Theorem 6 from fields to principal ideal domains.

## 2 The fundamental theorem

A proof of the fundamental theorem requires only the most basic information about Alexander modules. For more thorough discussions of these modules we refer to the literature [2, 3, 5, 7].

If $D$ is a diagram of an oriented link $L$ then there is an associated Alexander matrix $M(D)$. The columns of $M(D)$ are indexed by $A(D)$, and the rows of $M(D)$ are indexed by $C(D)$. Suppose $c$ is a crossing with the incident arcs indexed as in Figure 1 (N.b. The underpassing arcs are indexed using the
orientation of \( a_1 \): \( a_2 \) is on the right side of an observer facing forward on \( a_1 \), and \( a_3 \) is on the left side.) If \( a_2 \neq a_3 \), then the row of \( M(D) \) corresponding to \( c \) has these entries:

\[
M(D)_{ca} = \begin{cases} 
1 - t_{\kappa(a_2)}, & \text{if } a = a_1 \\
t_{\kappa(a_1)}, & \text{if } a = a_2 \\
-1, & \text{if } a = a_3 \\
0, & \text{if } a \notin \{a_1, a_2, a_3\}
\end{cases}
\]

If \( a_2 = a_3 \), then the row of \( M(D) \) corresponding to \( c \) has these entries:

\[
M(D)_{ca} = \begin{cases} 
1 - t_{\kappa(a_2)}, & \text{if } a = a_1 \\
t_{\kappa(a_1)} - 1, & \text{if } a = a_2 = a_3 \\
0, & \text{if } a \notin \{a_1, a_2\}
\end{cases}
\]

The reader familiar with the free differential calculus will recognize that the entries of the \( c \) row of \( M(D) \) are the images in \( \Lambda_\mu \) of the free derivatives of the relator \( a_1a_2a_1^{-1}a_3^{-1} \) corresponding to the crossing \( c \).

**Definition 8** The Alexander module \( M_A(L) \) is the \( \Lambda_\mu \)-module represented by any Alexander matrix \( A(D) \).

That is to say, if \( D \) is a diagram of \( L \) and \( \Lambda_\mu^{A(D)} \) is the free \( \Lambda_\mu \)-module on the set \( A(D) \), then \( M_A(L) \) is isomorphic to the quotient of \( \Lambda_\mu^{A(D)} \) by the submodule \( S \) generated by all elements of the form

\[
(1 - t_{\kappa(a_2)}) \cdot a_1 + t_{\kappa(a_1)} \cdot a_2 - a_3
\]

where the arcs \( a_1, a_2, a_3 \) appear at a crossing of \( D \) as in Figure 1.

The fundamental theorem follows immediately. If \( M \) is a \( \Lambda_\mu \)-module and \( f : A(D) \to M \) is an arbitrary function then \( f \) defines a \( \Lambda_\mu \)-linear map \( \hat{f} : \Lambda_\mu^{A(D)} \to M \). This map \( \hat{f} \) defines a \( \Lambda_\mu \)-linear map with domain \( M_A(L) \) if and only if \( S \subseteq \ker(\hat{f}) \).

**3 Proof of Theorem 6**

The elementary ideals of a link \( L \) are defined from an Alexander matrix \( M(D) \). If \( j \geq |A(D)| \), then \( E_j(L) = \Lambda_\mu \). If \( |A(D)| > j \geq \max\{0, |A(D)| - |C(D)|\} \), then \( E_j(L) \) is the ideal of \( R \) generated by the determinants of \(|A(D)| - j \times |A(D)| - j\) submatrices of \( M(D) \). If \( j < \max\{0, |A(D)| - |C(D)|\} \), then \( E_j(L) = (0) \).

Suppose \( F \) is a field and \( \varphi : \Lambda_\mu \to F \) is a homomorphism of rings with unity. If \( M \) is a vector space over \( F \) then as mentioned in the introduction \( M \) is a module over \( \Lambda_\mu \), \( F \otimes_{\Lambda_\mu} M_A(L) \) is a vector space over \( F \) and

\[
\text{Hom}_{\Lambda_\mu}(M_A(L), M) \cong \text{Hom}_F(F \otimes_{\Lambda_\mu} M_A(L), M).
\]
If $D$ is a diagram of a link $L$ then $D$ provides an Alexander matrix $M(D)$, which is a presentation matrix for $M_A(L)$ over $\Lambda_\mu$. It follows that $\varphi(M(D))$ is a presentation matrix for the $F$-vector space $F \otimes_{\Lambda_\mu} M_A(L)$. For a vector space, the only isomorphism-invariant information provided by a presentation matrix is the dimension: an $m \times n$ matrix of rank $r$ is a presentation matrix for a vector space of dimension $n - r$.

The rank $r$ of $\varphi(M(D))$ is the size of the largest square submatrix with nonzero determinant. Determinants are functorial, in the sense that every square $\Lambda_\mu$-matrix $X$ has $\varphi(\det X) = \det(\varphi(X))$. It follows that the rank $r$ of $\varphi(M(D))$ is the largest size of a square submatrix $X$ of $M(D)$ with $\varphi(\det X) \neq 0$.

If $j_0$ is the smallest index with $\varphi(E_{j_0}(L)) \neq 0$ then the largest size of a square submatrix $X$ of $M(D)$ with $\varphi(\det X) \neq 0$ is $|A(D)| - j_0$, so

$$\dim_F(F \otimes_{\Lambda_\mu} M_A(L)) = |A(D)| - r = |A(D)| - (|A(D)| - j_0) = j_0.$$ This is all we need to deduce Theorem 6 from Theorem 4.

4 Two knots

Inoue [8] asserted that “the number of all quandle homomorphisms of a knot quandle to an Alexander quandle is completely determined by Alexander polynomials of the knot.” Corollary 7 implies a similar assertion, with ‘Alexander polynomials’ replaced by ‘elementary ideals.’ In this section we observe that for Fox tricolorings of the knots pictured in Figure 2, Corollary 7 is correct and Inoue’s assertion is incorrect.

$$\varphi(2t^2 - 5t + 2) = \varphi(2 - t) = \varphi(1 - 2t) = 0.$$ We see that with respect to this homomorphism $\varphi$, $6_1$ has $j_0 = 2$ and $9_{46}$ has $j_0 = 3$. 

![Figure 2: The knots 6_1 and 9_{46}.]
A Fox tricoloring \cite{Exercise VI.6} of a link diagram $D$ is a function $f : A(D) \to GF(3)$. At each crossing as in Figure \ref{fig:toruslink} the sum $f(a_1) + f(a_2) + f(a_3)$ must be 0 in $GF(3)$. (This is simply the requirement that the coloring satisfies Definition \ref{def:foxtricoloring}, with $M = GF(3)$ considered as a $\Lambda$-module via the homomorphism $\varphi : \Lambda \to GF(3)$ with $\varphi(t) = -1$.) We leave it to the reader to verify the following descriptions of the spaces of Fox tricolorings of $6_{1}$ and $9_{46}$.

- Every Fox tricoloring of $6_{1}$ is given by arbitrary values of $f(u)$ and $f(v)$ in $GF(3)$, with $f(w) = f(z) = -f(u) - f(v)$, $f(x) = f(u)$, and $f(y) = f(v)$.
- Every Fox tricoloring of $9_{46}$ is given by arbitrary values of $f(a)$, $f(b)$ and $f(g)$ in $GF(3)$, with $f(c) = -f(a) - f(b)$, $f(d) = f(b)$, $f(e) = f(a)$, $f(h) = -f(b) - f(g)$, $f(i) = -f(a) - f(g)$ and $f(j) = f(g)$.

It follows that the space of Fox tricolorings of $6_{1}$ has dimension 2 over $GF(3)$, and the space of Fox tricolorings of $9_{46}$ has dimension 3 over $GF(3)$. We see that $6_{1}$ and $9_{46}$ have different numbers of Fox tricolorings, even though all of their Alexander polynomials are the same.

5 Two links

In this section we apply Corollary \ref{cor:foxtricoloring} to the torus link $T_{(2,8)}$ and Whitehead’s link $W$, pictured in Figure \ref{fig:toruslink}. With the indicated orientations, their elementary ideals are $E_j(T_{(2,8)}) = E_j(W) = \Lambda_2$ for $j > 1$, $E_j(T_{(2,8)}) = E_j(W) = (0)$ for $j < 1$, $E_1(W) = (\Delta_1(W)) \cdot (t_1 - 1, t_2 - 1) = (t_1 - 1)(t_2 - 1)(t_1 - 1, t_2 - 1)$, and $E_1(T_{(2,8)}) = (\Delta_1(T_{(2,8)})) \cdot (t_1 - 1, t_2 - 1) = (t_1^3 + t_1^2 t_2 + t_1 t_2^2 + t_2^3)(t_1 - 1, t_2 - 1)$.

The elementary ideals may be confirmed using the Alexander matrices obtained from Figure \ref{fig:toruslink} as described in Section 2. The Alexander polynomials $\Delta_1(T_{(2,8)})$ and $\Delta_1(W)$ may also be verified on the LinkInfo website (http://www.indiana.edu/~linkinfo/), where the two links are labeled L5a1{1} and L8a14{1}.

![Figure 3: T_{(2,8)} and Whitehead’s link.](image)

Both links have $E_0 = (0)$ and $E_2 = \Lambda_2$, so both links have $j_0 \in \{1, 2\}$ for every instance of Corollary \ref{cor:foxtricoloring}, a ring homomorphism $\varphi$ yields $j_0 = 2$ if and only if $\varphi(E_1) = (0)$. Table \ref{table:j0values} below gives the $j_0$ values for homomorphisms.

| Homomorphism | $j_0$ |
|--------------|-------|
| $\varphi_1$  | 1     |
| $\varphi_2$  | 2     |
| $\varphi_3$  | 1     |

Table 1: $j_0$ values for homomorphisms.
\( \varphi : \Lambda_2 \to GF(3) \). (All of the elementary ideals of both links are symmetric with respect to the transposition \( t_1 \leftrightarrow t_2 \), so we do not need to list the homomorphism with \( \varphi(t_1) = -1 \) and \( \varphi(t_2) = 1 \); it yields the same \( j_0 \) values as the homomorphism with \( \varphi(t_1) = 1 \) and \( \varphi(t_2) = -1 \).) We see that \( \text{Color}_A(T_{(2,8)}, GF(3)) \cong \text{Color}_A(W, GF(3)) \) for every homomorphism of rings with unity \( \varphi : \Lambda_2 \to GF(3) \).

\[
\begin{array}{cccc}
\varphi(t_1) & \varphi(t_2) & j_0(T_{(2,8)}) & j_0(W) \\
1 & 1 & 2 & 2 \\
1 & -1 & 2 & 2 \\
-1 & -1 & 1 & 1 \\
\end{array}
\]

Table 1: Values of \( j_0 \) for homomorphisms \( \varphi : \Lambda_2 \to GF(3) \).

The \( j_0 \) values for homomorphisms \( \varphi : \Lambda_2 \to GF(5) \) appear in Table 2. We see that \( \text{Color}_A(T_{(2,8)}, GF(5)) \cong \text{Color}_A(W, GF(5)) \) for every homomorphism of rings with unity \( \varphi : \Lambda_2 \to GF(5) \) that has \( \varphi(t_1) = \varphi(t_2) \), but there are homomorphisms with \( \text{Color}_A(T_{(2,8)}, GF(5)) \not\cong \text{Color}_A(W, GF(5)) \) and \( \varphi(t_1) \neq \varphi(t_2) \).

\[
\begin{array}{cccc}
\varphi(t_1) & \varphi(t_2) & j_0(T_{(2,8)}) & j_0(W) \\
1 & 1 & 2 & 2 \\
1 & 2 & 2 & 2 \\
1 & 3 & 2 & 2 \\
1 & 4 & 2 & 2 \\
2 & 2 & 1 & 1 \\
2 & 3 & 2 & 1 \\
2 & 4 & 2 & 1 \\
3 & 3 & 1 & 1 \\
3 & 4 & 2 & 1 \\
4 & 4 & 1 & 1 \\
\end{array}
\]

Table 2: Values of \( j_0 \) for homomorphisms \( \varphi : \Lambda_2 \to GF(5) \).

The links \( T_{(2,8)} \) and \( W \) are of interest because of the fact that for every abelian group \( A \), they have isomorphic groups of Fox colorings in \( A \). This fact was verified using Goeritz matrices in [19, Section 6], but we can also deduce it from Corollary 10 below, because the homomorphism \( \varphi : \Lambda_2 \to \mathbb{Z} \) defined by \( \varphi(t_1) = \varphi(t_2) = -1 \) has \( \varphi(E_j(T_{(2,8)})) = \varphi(E_j(W)) \) \( \forall j \).

6 A non-invertible link

The Laurent polynomial ring \( \Lambda_\mu = \mathbb{Z}[t_1^{\pm 1}, \ldots, t_\mu^{\pm 1}] \) has an automorphism given by \( t_i \mapsto t_i^{-1} \) \( \forall i \). This automorphism is sometimes called \textit{conjugation}, and denoted by an overline. Here are two important properties of conjugation.
1. Let $L^{\text{inv}}$ be the inverse of an oriented link $L$, obtained by reversing the orientation of every component of $L$. Then $E_j(L) = E_j(L^{\text{inv}})$ $\forall j$.

2. If $K$ is a knot then $E_j(K) = E_j(K)$ $\forall j$.

To verify property 1, let $D$ be a diagram of $L$ and let $D^{\text{inv}}$ be the diagram of $L^{\text{inv}}$ obtained from $D$ by reversing the orientation of every component. The effect of the orientation reversals is to interchange the indices of the arcs $a_2$ and $a_3$ at every crossing as indicated in Figure 1. Observe that the effect of (a) interchanging $a_2$ and $a_3$ at every crossing and (b) replacing every $t_i$ with $t_i^{-1}$ in the resulting matrix is the same as the effect of (c) multiplying the $a$ column of $M(D)$ by $t_{\kappa(a)}$ for each $a \in A(D)$ and (d) multiplying the $c$ row of $M(D)$ by $-t_{\kappa(a_1)}^{-1} t_{\kappa(a_2)}^{-1}$ for each crossing as indicated in Figure 1. Property 1 follows because operations (c) and (d) involve multiplying rows and columns by units of $\Lambda_{\mu}$, and hence do not affect the elementary ideals.

Verifying property 2 is more difficult; see [3, Chapter IX].

Properties 1 and 2 indicate that the elementary ideals cannot detect non-invertibility of knots. However the elementary ideals can sometimes detect non-invertibility of links. An example is the two-component link $T$ pictured in Figure 4, which was discussed by Turaev [20]. With the indicated component indices and orientations, $T$ has the elementary ideals $E_3(T) = \Lambda_2$ and $E_2(T) = (t_1 - 3, t_2 - 1, 7)$. (We do not present detailed calculations.) Notice that if $\varphi : \Lambda_2 \to GF(7)$ is the ring homomorphism with $\varphi(t_1) = 3$ and $\varphi(t_2) = 1$ then
\( \varphi(E_2(T)) = 0 \) but \( \varphi(E_2(T)) \) includes the nonzero element \( \varphi(t_1^{-1} - 3) = 5 - 3 = 2 \). It follows that \( E_2(T) \neq E_2(T^{inv}) \), so \( T \) is not invertible.

Corollary 4 tells us that multivariate Alexander colorings detect the non-invertibility of \( T \): if \( GF(7) \) is considered as a \( \Lambda_2 \)-module via \( \varphi \) then the dimension of \( \text{Color}_A(D, GF(7)) \) over \( GF(7) \) is 3, but the dimension of \( \text{Color}_A(D^{inv}, GF(7)) \) is no more than 2. We leave it to the reader to verify the following explicit descriptions of these spaces.

- Every \( f \in \text{Color}_A(D, GF(7)) \) is given by arbitrary values of \( f(a), f(b) \) and \( f(i) \) in \( GF(7) \), with \( f(c) = f(b) - 2f(a), f(d) = f(j) = f(k) = -2f(a), f(a) = f(e) = f(h) = f(u) = f(o) = f(r) = f(a), f(g) = 2f(a) + f(b), f(l) = 4f(a) - 2f(b) + 3f(i), f(m) = f(i), f(p) = 2f(a) + f(i), f(q) = 4f(a) + f(i) \) and \( f(s) = f(a) - 2f(b) + 3f(i) \).

- Every \( f \in \text{Color}_A(D^{inv}, GF(7)) \) is given by arbitrary values of \( f(b) \) and \( f(i) \) in \( GF(7) \), with \( f(a) = f(d) = f(e) = f(h) = f(j) = f(k) = f(n) = f(o) = f(r) = f(u) = 0, f(c) = f(g) = f(b), f(l) = f(s) = 3f(b) + 5f(i), \) and \( f(m) = f(p) = f(q) = f(i) \).

## 7 Principal ideal domains

The theory of the Smith normal form is explained in many algebra books, like [10][17]. We summarize the ideas briefly.

Suppose \( R \) is a principal ideal domain and \( X \) is an \( m \times n \) matrix with entries from \( R \). Define the elementary ideals \( E_j(X) \) as in Section 3. That is: if \( j \geq n \), then \( E_j(X) = R \); if \( n > j \geq \max\{0, n - m\} \), then \( E_j(X) \) is the ideal of \( R \) generated by the determinants of \( (n - j) \times (n - j) \) submatrices of \( X \); and if \( j < \max\{0, n - m\} \), then \( E_j(X) = \{0\} \). As \( R \) is a principal ideal domain, for each integer \( j \) there is an \( e_j(X) \in R \) such that \( E_j(X) \) is the principal ideal generated by \( e_j(X) \). Determinants satisfy the Laplace expansion property, so these elements \( e_j(X) \) form a sequence of divisors: \( e_{j+1}(X) \mid e_j(X) \forall j \). The quotients \( d_j(X) = e_j(X)/e_{j+1}(X) \) are the \textit{invariant factors} of \( X \). Like the \( e_j \), the \( d_j \) are well-defined only up to associates, i.e. the principal ideals \( (d_j(X)) \) are invariants of \( X \), but the particular elements \( d_j(X) \) are not. The invariant factors also form a sequence of divisors: \( d_{j+1}(X) \mid d_j(X) \forall j \). The \textit{Smith normal form} of \( X \) is the \( m \times n \) matrix obtained from the diagonal matrix

\[
\begin{pmatrix}
d_0(X) & 0 & 0 & 0 \\
0 & d_1(X) & 0 & 0 \\
0 & 0 & \ddots & 0 \\
0 & 0 & 0 & d_{n-1}(X)
\end{pmatrix}
\]

by adjoining \( m - n \) rows of zeroes if \( n < m \), and removing \( n - m \) rows of zeroes if \( n > m \).

The Smith normal form of \( X \) is equivalent to \( X \), i.e., there are invertible matrices \( P, Q \) such that \( PXQ \) is equal to the Smith normal form of \( X \). It follows
that if $X$ is a presentation matrix for the $R$-module $M$, then the Smith normal form of $X$ is also a presentation matrix for $M$. That is, if $X$ is a presentation matrix for $M$ then
\[ M \cong \bigoplus_{j=0}^{n-1} R/(d_j(X)). \] (1)

The fact that the $d_j(X)$ form a sequence of divisors implies that $(d_j(X)) \subseteq (d_{j+1}(X)) \forall j$. In particular, if $(d_i(X)) = R$ then $(d_j(X)) = R \forall j \geq i$. Notice that values of $j$ with $(d_j(X)) = R$ contribute nothing of significance to the direct sum of (1).

Now, let $\varphi : \Lambda_\mu \to R$ be a homomorphism of rings with unity. The first two paragraphs of Section 3 still apply, with $F$ and vector spaces over $F$ replaced with $R$ and modules over $R$. In the third paragraph we must replace the sentence “For a vector space, the only isomorphism-invariant information provided by a presentation matrix is the dimension” with “For a module over a principal ideal domain, the only isomorphism-invariant information provided by a presentation matrix $X$ is the Smith normal form of $X$, ignoring diagonal entries with $(d_j(X)) = R$.” Considering the isomorphism (1), we deduce the following.

**Theorem 9** Let $R$ be a principal ideal domain, and let $\varphi : \Lambda_\mu \to R$ be a homomorphism of rings with unity. Let $M$ be an $R$-module, considered as a module over $\Lambda_\mu$ via $\varphi$. Let $D$ be a diagram of a link $L$, and let $d_0, d_1, \ldots, d_{|A(D)|−1}$ be the invariant factors of $\varphi(M(D))$. Then
\[ \text{Color}_A(D, M) \cong \bigoplus_{j=0}^{|A(D)|−1} \text{Hom}_R(R/(d_j), M). \]

The direct sum of Theorem 9 seems to vary from one diagram to another, but the invariance of the Alexander module guarantees that if $D$ and $D'$ are diagrams of the same link and $|A(D)| < |A(D')|$ then the invariant factors $d'_j$ of $\varphi(M(D'))$ with $j \geq |A(D)|−1$ all generate the same principal ideal, $(d'_j) = R$. It follows that these invariant factors contribute nothing of significance to the direct sum of Theorem 9.

**Corollary 10** Let $R$ be a principal ideal domain, and let $\varphi : \Lambda_\mu \to R$ be a homomorphism of rings with unity. Let $M$ be an $R$-module, considered as a module over $\Lambda_\mu$ via $\varphi$. Then for any diagram $D$ of a link $L$, the $R$-module $\text{Color}_A(D, M)$ is determined up to isomorphism by $M$ and the images under $\varphi$ of the elementary ideals of $L$.

The examples of Section 4 show that if we replace “elementary ideals” by “Alexander polynomials” in Corollary 10 then the resulting statement is false, in general.

We close with the observation that Theorem 9 implies Theorem 6 in two different ways. (i) Suppose $F$ is the field of quotients of a principal ideal domain
If $M$ is a vector space over $F$, then $\text{Hom}_R(R/(d_j),M)$ is isomorphic to either $(0)$ (if $d_j \neq 0$) or $M$ (if $d_j = 0$). (ii) Suppose $I$ is a maximal ideal of a principal ideal domain $R$, $F = R/I$ and $M$ is a vector space over $F$. Then $\text{Hom}_R(R/(d_j),M)$ is isomorphic to either $(0)$ (if $d_j \notin I$) or $M$ (if $d_j \in I$).

References

[1] Y. Bae, Coloring link diagrams by Alexander quandles, J. Knot Theory Ramifications 21 (2012), 1250094.

[2] G. Burde and H. Zieschang, Knots, 2nd edn. deGruyter Studies in Mathematics Vol. 5 (Walter de Gruyter, Berlin and New York, 2003).

[3] R. H. Crowell and R. H. Fox, Introduction to Knot Theory, Graduate Texts in Mathematics, Vol. 57 (Springer, New York, 1977).

[4] M. Elhamdadi and S. Nelson, Quandles, Student Mathematical Library Vol. 74 (Amer. Math. Soc., Providence, R.I., 2015).

[5] R. H. Fox, A quick trip through knot theory, in Topology of 3-Manifolds and Related Topics (Proc. The Univ. of Georgia Institute, 1961) (Prentice-Hall, Englewood Cliffs, N.J., 1962), pp. 120-167.

[6] C. Hayashi, M. Hayashi and K. Oshiro, On linear $n$-colorings for knots, J. Knot Theory Ramifications 21 (2012), 1250123.

[7] J. A. Hillman, Algebraic Invariants of Links, 2nd edn. Series on Knots and Everything, Vol. 52 (World Scientific, Singapore, 2012).

[8] A. Inoue, Quandle homomorphisms of knot quandles to Alexander quandles, J. Knot Theory Ramifications 10 (2001), 813-821.

[9] A. Inoue, Knot quandles and infinite cyclic covering spaces, Kodai Math. J. 33 (2010), 116-122.

[10] N. Jacobson, Basic Algebra I, 2nd edn. (W. H. Freeman and Company, New York, 1985).

[11] D. Joyce, A classifying invariant of knots, the knot quandle, J. Pure Appl. Algebra 23 (1982), 37-65.

[12] L. H. Kauffman and P. Lopes, Colorings beyond Fox: the other linear Alexander quandles, Linear Alg. Appl. 548 (2018), 221-258.

[13] R. A. Litherland, Quadratic quandles and their link invariants, arXiv: math/0207099.

[14] V. O. Manturov and D. P. Ilyutko, Virtual Knots: The State of the Art, Series on Knots and Everything, Vol. 51 (World Scientific, Singapore, 2013).
[15] S. V. Matveev, Distributive groupoids in knot theory, Mat. Sb (N.S.) 119 (1982), 78-88.

[16] T. Nosaka, Twisted cohomology pairings of knots I; diagrammatic computation, Geom. Dedicata 189 (2017), 139-160.

[17] D. Serre, Matrices, 2nd edn. Graduate Texts in Mathematics, Vol. 21 (Springer, New York, 2010).

[18] D. A. Smith, L. Traldi and W. Watkins, A note on Dehn colorings and invariant factors, [arXiv:1804.02700](https://arxiv.org/abs/1804.02700)

[19] L. Traldi, Link colorings and the Goeritz matrix, J. Knot Theory Ramifications 26 (2017), 1750045.

[20] V. G. Turaev, Elementary ideals of links and manifolds: symmetry and asymmetry, Algebra i Analiz 1 (1989), 223-232; translation in Leningrad Math. J. 1 (1990), 1279-1287.