A NOTE ON THE STABILITY OF A SECOND ORDER FINITE DIFFERENCE SCHEME FOR SPACE FRACTIONAL DIFFUSION EQUATIONS

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Abstract. The unconditional stability of a second order finite difference scheme for space fractional diffusion equations is proved theoretically for a class of variable diffusion coefficients. In particular, the scheme is unconditionally stable for all one-sided problems and problems with Riesz fractional derivative. For problems with general smooth diffusion coefficients, numerical experiments show that the scheme is still stable if the space step is small enough.

1. Introduction. In this paper, we consider the following two-sided one dimensional fractional diffusion equation (FDE)

\[ \frac{\partial u(x,t)}{\partial t} = d^+(x) a D_{x}^\alpha u(x,t) + d^-(x) b D_{x}^{\alpha} u(x,t) + f(x,t), \]

\[ (x,t) \in (a, b) \times (0, T], \]

\[ u(a, t) = u(b, t) = 0, \quad t \in [0, T], \]

\[ u(x, 0) = u_0(x), \quad x \in [a, b], \]

where the diffusion coefficients \( d^\pm(x) \geq 0 \). Here \( a D_{x}^\alpha u(x,t) \) and \( b D_{x}^{\alpha} u(x,t) \) denote the \( \alpha \)-order left and right Riemann-Liouville (RL) fractional derivatives of \( u \), respectively, and they are defined as

\[ a D_{x}^\alpha u(x,t) = \frac{1}{\Gamma(2-\alpha)} \frac{\partial^2}{\partial x^2} \int_a^x \frac{u(\xi,t)}{(x-\xi)^{\alpha-1}} \, d\xi, \]

\[ b D_{x}^{\alpha} u(x,t) = \frac{1}{\Gamma(2-\alpha)} \frac{\partial^2}{\partial x^2} \int_x^b \frac{u(\xi,t)}{(\xi-x)^{\alpha-1}} \, d\xi, \]

where \( \Gamma(\cdot) \) is the gamma function, and \( \alpha \in (1, 2) \).

Fractional derivative can be naturally discretized by the standard Grünwald-Letnikov (GL) formula [11] with first order accuracy. However, the finite difference

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scheme based on the GL formula for time dependent FDE is unstable \[8\]. To overcome this problem, Meerschaert and Tadjeran propose to use the shifted GL approximation. For solving FDE (1), backward Euler scheme with shifted GL difference (BE-SGD scheme) is proposed in \[9\]. The BE-SGD scheme is proved to be unconditionally stable by using the Gershgorin theorem since the differentiation matrix for shifted GL discretization is diagonally dominant. Some efficient solvers for the BE-SGD scheme are provided in \[7, 17, 18, 19, 20, 10\]. However, the BE-SGD scheme is only first order accurate.

To achieve second order accuracy, Crank-Nicolson scheme with shifted GL discretization is proposed in \[14, 15\]. With Richardson’s extrapolation, a numerical solution with second order accuracy in both time and space can be obtained. However, the computational cost of this approach is high since an extra numerical solution on fine grid is required. Recently, second order discretizations for RL fractional derivatives are being studied. Sousa and Li \[12\], and Tian et al. \[16\] present two different second order discretizations which possess an advantage that their differentiation matrices preserve the Toeplitz structure as the one for the shifted GL discretization. Nevertheless, they are not diagonally dominant for all \(\alpha \in (1, 2)\). Due to this reason, the unconditional stability of those second order finite difference schemes proposed in \[12, 16, 5\] cannot be obtained analogously. In fact, those schemes are only proved to be unconditionally stable for the case with constant diffusion coefficients.

In this paper, stability of the Crank-Nicolson scheme with Sousa’s second order discretization is studied. The scheme is proved to be unconditionally stable if (i) the diffusion coefficients \(d^\pm(x) = k^\pm d(x)\) where \(k^\pm \geq 0\) and \(d(x) \geq 0\); or (ii) the order of the fractional derivative \(\alpha \in (\alpha_0, 2)\) where \(\alpha_0 \approx 1.5545\). In particular, the scheme is unconditionally stable for all one-sided problems, and problems with Riesz derivative \[1\] instead of the RL derivative. For FDE (1) with smooth diffusion coefficients, numerical examples reveal that the scheme is still stable when the number of mesh points in space is large enough even though no restriction is imposed on the time step.

2. A second order finite difference scheme. Let \(N, M\) be positive integers, \(h = (b - a)/(N + 1)\) and \(\tau = T/M\) be the space step and time step, respectively. We define a spatial and temporal partition \(x_i = a + ih\) for \(i = 0, 1, \ldots, N + 1\), \(t_m = m\tau\) for \(m = 0, 1, \ldots, M\). In \[12, 13\], Sousa and Li proved that,

\[
\begin{align*}
_a D^\alpha_t u(x_i, t_m) &= \frac{c_\alpha}{h^\alpha} \sum_{k=0}^{i+1} q^{(\alpha)}_k u(x_{i-k+1}, t_m) + O(h^2), \\
_x D^\alpha_x u(x_i, t_m) &= \frac{c_\alpha}{h^\alpha} \sum_{k=0}^{N-i+2} q^{(\alpha)}_k u(x_{i+k-1}, t_m) + O(h^2),
\end{align*}
\]

(2)

where \(c_\alpha = \frac{1}{\Gamma(4-\alpha)}\) and

\[
q^{(\alpha)}_k = \begin{cases} 
1, & k = 0, \\
-4 + 2^3 - \alpha, & k = 1, \\
3^3 - \alpha - 4 \cdot 2^3 - \alpha + 6, & k = 2, \\
(k + 1)^3 - \alpha - 4k^3 - \alpha + 6(k - 1)^3 - \alpha - 4(k - 2)^3 - \alpha + (k - 3)^3 - \alpha, & k \geq 3.
\end{cases}
\]

(3)
Let \( u^m_i \approx u(x_i, t_m) \), \( d_x^i = d_x^i(x_i) \), \( f^m_{i+\frac{1}{2}} = f(x_i, t_{m+\frac{1}{2}}) \), and \( t_{m+\frac{1}{2}} = \frac{1}{2}(t_m + t_{m+1}) \) for \( i = 0, 1, \ldots, N + 1 \) and \( m = 0, 1, \ldots, M \). Using the Crank-Nicolson technique for the time discretization of (1) with the space discretization given in (2), we have the following second order finite difference scheme

\[
\frac{u^{m+1}_i - u^m_i}{\tau} = \frac{c_\alpha}{2} \left( \frac{d_x^+}{h^\alpha} \sum_{k=0}^{i+1} q_k^{(\alpha)} u^{m}_{i-k+1} + \frac{d_x^+}{h^\alpha} \sum_{k=0}^{i+1} q_k^{(\alpha)} u^{m+1}_{i-k+1} \right) \\
+ \frac{d_x^-}{h^\alpha} \sum_{k=0}^{N-i+2} q_k^{(\alpha)} u^{m}_{i+k-1} + \frac{d_x^-}{h^\alpha} \sum_{k=0}^{N-i+2} q_k^{(\alpha)} u^{m+1}_{i+k-1} \right) + f_{i+\frac{1}{2}}^{m+\frac{1}{2}} \tag{4}
\]

for \( i = 1, 2, \ldots, N \), and \( m = 0, 1, \ldots, M - 1 \).

Let \( \nu_{\tau,h,\alpha} = \frac{c_\alpha(1-\tau\alpha)}{2(1-\alpha)h^\alpha} \), \( u^m = [u^m_1, u^m_2, \ldots, u^m_N]^T \), \( f^m_{\frac{1}{2}} = [f^m_{1+\frac{1}{2}}, \ldots, f^m_{N+\frac{1}{2}}]^T \), and \( D^\pm = \text{diag}(d_x^1, d_x^2, \ldots, d_x^N) \) be a diagonal matrix. The scheme (4) is expressed in the matrix form

\[
(I - \nu_{\tau,h,\alpha} B)u^{m+1} = (I + \nu_{\tau,h,\alpha} B)u^m + \tau f^{m+\frac{1}{2}},
\tag{5}
\]

where \( B = D^+ Q + D^- Q^T \) and

\[
Q = \begin{bmatrix}
q_1^{(\alpha)} & q_0^{(\alpha)} & 0 & \cdots & 0 \\
q_2^{(\alpha)} & q_1^{(\alpha)} & q_0^{(\alpha)} & \cdots & 0 \\
\vdots & q_2^{(\alpha)} & q_1^{(\alpha)} & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & q_0^{(\alpha)} \\
q_N^{(\alpha)} & \cdots & \cdots & q_2^{(\alpha)} & q_1^{(\alpha)}
\end{bmatrix}.
\tag{6}
\]

Here \( I \) in (5) is the identity matrix, and the entries of \( Q \) are given in (3) with properties given in Lemma 3.1.

3. Stability and convergence of the finite difference scheme. Suppose \( u^m \) and \( v^m \) are two solutions of the finite difference scheme (5), and denote \( \eta^m = u^m - v^m \). Then \( \eta^m \) satisfies \( \eta^{m+1} = H \eta^m \) where \( H = (I - \nu_{\tau,h,\alpha} B)^{-1}(I + \nu_{\tau,h,\alpha} B) \), and furthermore, we have \( \eta^m = H^m \eta^0 \). If the spectral radius of \( H \) is not larger than 1, then there exists a constant \( C' \) such that \( ||H^m|| \leq ||H||^m \leq C' < \infty \) for some norm, and hence we have

\[
||\eta^m|| \leq C' ||\eta^0||,
\]

which means that the scheme is stable in some sense.

Note that if \( \lambda \) is an eigenvalue of \( B \), \( (1 + \nu_{\tau,h,\alpha} \lambda)(1 - \nu_{\tau,h,\alpha} \lambda)^{-1} \) is an eigenvalue of \( H \), and \( \text{Re}(\lambda) \leq 0 \) implies \( |(1 + \nu_{\tau,h,\alpha} \lambda)(1 - \nu_{\tau,h,\alpha} \lambda)^{-1}| \leq 1 \). Therefore, the finite difference scheme is stable if each eigenvalue of \( B \) has nonpositive real part.

Recall that a matrix is said to be negative semi stable if each of its eigenvalue has nonpositive real part. Now, by showing that the matrix \( B \) is negative semi stable, two theorems are presented for the stability of the finite difference scheme (4). To prove the two theorems, we need the following lemmas.
Lemma 3.1. (See [12, 13]) For any $\alpha \in (1, 2)$, the sequence $\{q_k^{(\alpha)}\}$ defined in (3) satisfies the following properties:

$$
q_1^{(\alpha)} < 0, \quad q_0^{(\alpha)} \geq q_3^{(\alpha)} \geq q_4^{(\alpha)} \cdots \geq 0, \quad q_0^{(\alpha)} + q_2^{(\alpha)} \geq 0, \quad \sum_{k=0}^{\infty} q_k^{(\alpha)} = 0,
$$

and $q_2^{(\alpha)} \begin{cases} 
> 0, & \alpha \in (\alpha_0, 2) \\
\leq 0, & \alpha \in (1, \alpha_0) 
\end{cases}$

where $\alpha_0 \approx 1.5545$ is the root of the equation $3^{3-\alpha} - 4 \cdot 2^{3-\alpha} + 6 = 0$ on the interval $(1, 2)$.

Lemma 3.2. (See [6]) A square real matrix $G$ is negative semi stable if the matrix $G + G^T$ is negative semi definite.

Lemma 3.3. (See [5]) For any $\alpha \in (1, 2)$, the matrix $Q$ in (6) is negative semi stable, and the symmetric matrix $Q + Q^T$ is negative semi definite.

Theorem 3.4. For all $\alpha \in (1, 2)$, the finite difference scheme (4) for (1) is unconditionally stable if

$$
d^{\pm}(x) = k^{\pm}d(x),
$$

where $k^{\pm} \geq 0$ and $d(x) \geq 0$.

Proof. Since $d^{\pm}(x) = k^{\pm}d(x)$, we have $D^{\pm} = k^{\pm}D$ and then $B = D\Sigma$ where $\Sigma = k^{+}Q + k^{-}Q^T$. Let us firstly show that $\Sigma$ is negative semi stable.

If $k^+ = 0$ or $k^- = 0$, then $\Sigma$ is negative semi stable by Lemma 3.3. If $k^+ > 0$, then $\Sigma + \Sigma^T = (k^+ + k^-)(Q + Q^T)$. Since $k^+ + k^- > 0$ and $Q + Q^T$ is negative semi definite (see Lemma 3.3), we get that $\Sigma + \Sigma^T$ is negative semi definite and hence, by Lemma 3.2, the matrix $\Sigma$ is negative semi stable. Now, since $D$ is a diagonal matrix with nonnegative entries, by the Corollary 1.7.7 in [6], the matrix $B = D\Sigma$ is also negative semi stable.

Remark 1. We want to emphasize that our proof here works for the case that $D$ has zero entries on its main diagonal. If all diagonal entries of $D$ are strictly positive, $D^{-\frac{1}{2}}$ exists. Then we can prove that $B = D\Sigma$ is negative semi stable by the following way. Note that $B$ is similar to

$$
C = D^{-\frac{1}{2}}BD^{\frac{1}{2}} = D^{\frac{1}{2}}\Sigma D^{\frac{1}{2}},
$$

and the matrix $C$ is negative semi stable since

$$
C + C^T = D^{\frac{1}{2}}(\Sigma + \Sigma^T)D^{\frac{1}{2}}
$$

is negative semi definite. Therefore $B$ is negative semi stable. A similar proof is also given in the Theorem 4.1 in [2]. However, if $D$ has zero entries on its main diagonal, the above proof cannot work since $D^{-\frac{1}{2}}$ does not exists.

Remark 2. The result in Theorem 3.4 reveals that the finite difference scheme when applied to one-sided FDE problems ($k^+ = 0$ or $k^- = 0$), and FDE problems with Riesz fractional derivative ($k^+ = k^-$) is unconditionally stable for any diffusion coefficients.

By noting that $Q$ is diagonally dominant if $\alpha \in (\alpha_0, 2)$, we have another theorem for the unconditional stability of the finite difference scheme.

Theorem 3.5. The finite difference scheme (4) for FDE (1) with any diffusion coefficients is unconditionally stable if $\alpha \in (\alpha_0, 2)$, where $\alpha_0 \approx 1.5545$. 
Remark 3. For the case $\alpha \in (\alpha_0, 2)$, from Lemma 3.1, we have $q_0^{(\alpha)}, q_2^{(\alpha)}, q_3^{(\alpha)}, \ldots \geq 0$, and according to the Gerschgorin theorem, all eigenvalues of $B$ are in the disks centered at $c_i = (d^+_i + d^-_i)q_1^{(\alpha)} < 0$, $i = 1, \ldots, N$, with radius

$$r_1 = d^+_1 |q_0^{(\alpha)}| + d^-_1 \sum_{k=2}^N |q_k^{(\alpha)}| < -c_1,$$

$$r_N = d^+_N \sum_{k=2}^N |q_k^{(\alpha)}| + d^-_N |q_0^{(\alpha)}| < -c_N,$$

and

$$r_i = d^+_i \sum_{k=0,k\neq i}^i |q_k^{(\alpha)}| + d^-_i \sum_{k=i-1}^{N-i+1} |q_k^{(\alpha)}| < (d^+_i + d^-_i) \left( \sum_{k=0}^{\infty} |q_k^{(\alpha)}| - q_1^{(\alpha)} \right) = -c_i$$

for $i = 2, \ldots, N - 1$. Therefore, every disk is contained in the left-half complex plane, and hence $B$ is negative semi stable.

**Proof.** For $\alpha \in (\alpha_0, 2)$, from Lemma 3.1, we have $q_0^{(\alpha)}, q_2^{(\alpha)}, q_3^{(\alpha)}, \ldots \geq 0$, and according to the Gerschgorin theorem, all eigenvalues of $B$ are in the disks centered at $c_i = (d^+_i + d^-_i)q_1^{(\alpha)} < 0$, $i = 1, \ldots, N$, with radius

$$r_1 = d^+_1 |q_0^{(\alpha)}| + d^-_1 \sum_{k=2}^N |q_k^{(\alpha)}| < -c_1,$$

$$r_N = d^+_N \sum_{k=2}^N |q_k^{(\alpha)}| + d^-_N |q_0^{(\alpha)}| < -c_N,$$

and

$$r_i = d^+_i \sum_{k=0,k\neq i}^i |q_k^{(\alpha)}| + d^-_i \sum_{k=i-1}^{N-i+1} |q_k^{(\alpha)}| < (d^+_i + d^-_i) \left( \sum_{k=0}^{\infty} |q_k^{(\alpha)}| - q_1^{(\alpha)} \right) = -c_i$$

for $i = 2, \ldots, N - 1$. Therefore, every disk is contained in the left-half complex plane, and hence $B$ is negative semi stable.

**Remark 3.** For the case $\alpha \in (1, \alpha_0]$ with general diffusion coefficients $d^\pm(x)$, the matrix $B$ may not be negative semi stable. For example, consider $N = 3$ and $\alpha = 1.01$, with

$$Q = \begin{bmatrix} q_1^{(\alpha)} & q_0^{(\alpha)} & 0 \\ q_2^{(\alpha)} & q_1^{(\alpha)} & q_0^{(\alpha)} \\ q_3^{(\alpha)} & q_2^{(\alpha)} & q_1^{(\alpha)} \end{bmatrix}, \quad D^+ = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}, \quad \text{and} \quad D^- = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

where $q_0^{(\alpha)} = 1$, $q_1^{(\alpha)} \approx -0.02763$, $q_2^{(\alpha)} \approx -0.9878$, and $q_3^{(\alpha)} \approx 0.007279$, the eigenvalues of the matrix $B = D^+ Q + D^- Q^T$ are $\lambda_1 = -1.0684 < 0$, $\lambda_2 = -0.0480 < 0$, and $\lambda_3 = 0.8953 > 0$. However, numerical examples in next section show that, if $d^\pm(x)$ are smooth and the matrix size of $B$ is large enough, $B$ will be negative semi stable.

Now, one can prove the second order convergence of the finite difference scheme (4) by using the similar technique given in [2, 3].

**Theorem 3.6.** Let $U^m_i$ be the exact solution of problem (1) and $u^m_i$ be the numerical solution of the finite difference scheme (4) at $(x_i, t_m)$. If (i) $d^\pm(x) = k^\pm d(x)$, where $k^\pm$ and $d(x)$ are positive; or (ii) $\alpha \in (\alpha_0, 2)$, $\alpha_0 \approx 1.5545$, there exists a positive constant $\tilde{C}$ and a norm $||\cdot||$ such that $||u^m - U^m|| \leq \tilde{C}(r^2 + h^2)$, for all $m = 1, \ldots, M$.

4. **Numerical examples.** In this section, numerical examples are given to support the results in Theorems 3.4, 3.5, and 3.6. By using Matlab, all eigenvalues $\lambda_i$, $i = 1, \ldots, N$, of the $N \times N$ matrix $B = D^+ Q + D^- Q^T$ are computed and the maximum of their real parts

$$\vartheta = \max_{i=1,\ldots,N} \text{Re}(\lambda_i)$$

are shown for different $N$. Note that if $\vartheta \leq 0$, the matrix $B$ is negative semi stable. Maximum errors between numerical solutions and the exact solutions are also given to show the second order convergence of the finite difference scheme (4). Now, let us present an example in which the diffusion coefficients are in the form of (7).
Example 1. Consider FDE (1) on space interval \([a, b] = [0, 1]\) with diffusion coefficients \(d^+(x) = x^{\alpha}\), \(d^-(x) = 0.5d^+(x)\), initial condition \(u_0(x) = x^2(1 - x)^2\), and source term
\[
f(x, t) = -e^{-t} \left\{ x^2(1 - x)^2 + \frac{\Gamma(3)}{\Gamma(3 - \alpha)}[x^2 + 0.5x^\alpha(1 - x)^2 - \alpha] \\
- 2\frac{\Gamma(4)}{\Gamma(4 - \alpha)}[x^3 + 0.5x^\alpha(1 - x)^3 - \alpha] + \frac{\Gamma(5)}{\Gamma(5 - \alpha)}[x^4 + 0.5x^\alpha(1 - x)^4 - \alpha] \right\}.
\]
The exact solution of this example is \(u(x) = e^{-t}x^2(1 - x)^2\).

From Figure 1, it is clear that \(\vartheta \leq 0\) for all \(N\), and hence the matrix \(B\) is negative semi stable. This supports the theoretical analysis in Theorem 3.4. Also, Table 1 shows the second order convergence of the finite difference scheme for \(\alpha = 1.01, 1.1, 1.5,\) and 1.9. Now, we present other two examples in which the diffusion coefficients are not in the form of (7).

Example 2. Consider FDE (1) on space interval \([a, b] = [0, 1]\) with diffusion coefficients \(d^+(x) = x^\alpha\), \(d^-(x) = (1 - x)^\alpha\), initial condition \(u_0(x) = x^3(1 - x)^3\), and source term
\[
f(x, t) = -e^{-t} \left\{ x^3(1 - x)^3 + \frac{\Gamma(4)}{\Gamma(4 - \alpha)}[x^3 + (1 - x)^3] \\
- 3\frac{\Gamma(5)}{\Gamma(5 - \alpha)}[x^4 + (1 - x)^4] + 3\frac{\Gamma(6)}{\Gamma(6 - \alpha)}[x^5 + (1 - x)^5] \\
- \frac{\Gamma(7)}{\Gamma(7 - \alpha)}[x^6 + (1 - x)^6] \right\}.
\]
The exact solution of Example 2 is \(u(x, t) = e^{-t}x^3(1 - x)^3\).

Example 3. Consider FDE (1) on space interval \([a, b] = [0, 1]\) with diffusion coefficients \(d^+(x) = \frac{1}{x}\), \(d^-(x) = \frac{1}{x - 1}\), initial condition \(u_0(x) = x^3(1 - x)^3\), and

| \(\alpha\) | \(\|u\|_\infty\) | Rate | \(\|u\|_\infty\) | Rate | \(\|u\|_\infty\) | Rate | \(\|u\|_\infty\) | Rate |
|---|---|---|---|---|---|---|---|---|
| \(2^6\) | 6.8392E-05 | - | 3.7056E-05 | - | 2.8702E-05 | - | 2.8365E-05 | - |
| \(2^7\) | 1.7361E-05 | 1.98 | 9.1155E-06 | 2.00 | 7.1565E-06 | 2.00 | 7.3417E-06 | 1.95 |
| \(2^8\) | 4.3563E-06 | 1.99 | 2.3040E-06 | 2.03 | 1.7703E-06 | 2.02 | 1.8850E-06 | 1.96 |
| \(2^9\) | 1.0873E-06 | 2.00 | 5.4472E-07 | 2.03 | 4.3699E-07 | 2.02 | 4.7206E-07 | 1.97 |
| \(2^{10}\) | 2.7113E-07 | 2.00 | 1.3301E-07 | 2.03 | 1.0723E-07 | 2.02 | 1.2259E-07 | 1.98 |

Table 1. Maximum errors and observed orders for Example 1 with \(\tau = h = 1/(N + 1)\), \(t = 1\), and different \(\alpha\).
The exact solution of this example is $u(x) = e^{-t}x^3(1-x)^3$.

In Figures 2 and 3, we plot the curves of $\vartheta$ defined in (8) versus $N$ for $\alpha = 1.01$, 1.1, 1.5 and 1.9. One can see that $\vartheta \leq 0$ for all $N$ when $\alpha = 1.9 \in (\alpha_0, 2)$. This supports the theoretical result of Theorem 3.5. For other values of $\alpha$ being tested, we cannot guarantee that the matrix $B$ is negative semi stable theoretically. Numerically, as shown in the left of Figures 2 and 3, when $\alpha = 1.01$ and 1.1, it really occurs that $\vartheta > 0$ for some $N$. However, in the right of Figures 2 and 3, it is clear that when $N$ is large enough ($N \geq 100$), we have $\vartheta \leq 0$, i.e., $B$ is negative semi stable. Therefore, we guess that, for FDEs with smooth diffusion coefficients, the finite difference scheme (4) is stable provided that the number of node points in space is large enough even though no restriction is imposed on the time step $\tau$. Tables 2 and 3 also show the second order convergence of the scheme.
5. Conclusion. A second order finite difference scheme for one-dimensional FDE (1) is studied in this paper. The scheme is proved to be unconditionally stable and convergent if (i) $d^k(x) = k^2d(x)$ where $k \geq 0$ and $d(x) \geq 0$; or (ii) $\alpha \in (\alpha_0, 2)$ where $\alpha_0 \approx 1.5545$. Numerical examples are shown to support the theoretical results. Moreover, for FDEs with general smooth diffusion coefficients, numerical experiments reveal that the scheme is still stable if the number of node points in space is large enough ($\geq 100$ in our numerical tests), even though no restriction is imposed on the time step.

Besides, we mention that our stability analysis for scheme (4) also works if the second order discretization for fractional derivatives are replaced by the one proposed in [16]. Similar theoretical and numerical results on the stability and convergence can be obtained analogously. Furthermore, we remark that the stability of finite difference method for a fractional-order differential linear complementarity problem arising in pricing option was studied in [4] recently. However, the approach in our paper is different from that used in [4].

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