Context-Tree Weighting and Bayesian Context Trees: Asymptotic and Non-Asymptotic Justifications

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Abstract—The Bayesian Context Trees (BCT) framework is a recently introduced, general collection of statistical and algorithmic tools for modelling, analysis and inference with discrete-valued time series. The foundation of this development is built in part on some well-known information-theoretic ideas and techniques, including Rissanen’s tree sources and Willems et al.’s context-tree weighting algorithm. This paper presents a collection of theoretical results that provide mathematical justifications and further insight into the BCT modelling framework and the associated practical tools. It is shown that the BCT prior predictive likelihood (the probability of a time series of observations averaged over all models and parameters) is both pointwise and minimax optimal, in agreement with the same time as its importance for data compression was being recognised, it was also gradually becoming clear that the CTW algorithm can be best understood once it is properly rooted on a firm statistical foundation. Following a number of early relevant works [32], [49], [54], [59], [60], progress along this direction culminated in the recent development [24] of the Bayesian Context Trees (BCT) framework.

The BCT framework provides a general Bayesian foundation within which the CTW algorithm and its associated tools and techniques can be explored in a systematic and principled fashion. Indeed, in [24] it was shown that the CTW algorithm along with two generalisations of the context-tree maximising algorithm [54], [60] can be used very effectively for Bayesian inference with discrete time series, in particular for model selection and prediction. A Markov chain Monte Carlo sampler was also developed for the posterior distribution over both models and parameters, and a more efficient, simple Monte Carlo sampler was introduced in [34] and [36], where the application of the BCT framework to estimation problems was explored further. These ideas were extended to provide effective methods for segmentation and change-point detection of discrete time series in [26] and [27]. Finally, a perhaps somewhat surprising generalisation of the BCT framework for real-valued time series was introduced in [33] and [35]. Many of the algorithms described in these recent works are implemented in the publicly available R package ‘BCT’ [37]. Similar models have been considered in [17] and [18].
The main purpose of this paper is to derive a number of theoretical results that offer additional insight into the performance of statistical and methodological tools associated with the BCT framework, and to provide theoretical justification for their practical application. Some of these results are in the form of classical information-theoretic or statistical asymptotics, while others provide explicit finite-blocklength bounds.

In Section II we recall the BCT framework and collect the definitions and basic properties that will be used throughout the paper. Section III contains our main results, and Section IV contains their proofs.

II. PRELIMINARIES: BAYESIAN CONTEXT TREES

Here we collect the necessary definitions, assumptions, and basic results that will be used throughout the paper. All the results of this section (except for the straightforward observations in Proposition 2.2, proved in Section IV) can be found, along with more extensive discussion and details, in [24].

A. Variable-Memory Markov Chains

Let \( \{X_n\} \) be a discrete random source, understood as a random process taking values in a finite alphabet \( A \) of \( m := |A| \geq 2 \) symbols; without loss of generality, we assume throughout that \( A = \{0, 1, \ldots, m - 1\} \). The models we consider for \( \{X_n\} \) are variable-memory representations of Markov chains with memory length no greater than some fixed \( D \geq 0 \). These representations describe the conditional distribution of \( X_n \) given \( X_{n-D}^{n-1} = (X_{n-D}, X_{n-D+1}, \ldots, X_{n-1}) \) by specifying a model \( T \) and an associated parameter vector \( \theta \); throughout, we write \( X_i^j \) for a vector of random variables \( (X_i, X_{i+1}, \ldots, X_j) \) and similarly \( x_i^j \) for a string \( (x_i, x_{i+1}, \ldots, x_j) \in A^{j-i+1} \), for \( i \leq j \).

The model of \( \{X_n\} \) is represented by a tree \( T \) from the class \( T(D) \) of all proper \( m \)-ary trees of depth no greater than \( D \); \( T \) is proper if all of the \( m \) possible children of every internal node of \( T \) are also in \( T \). Viewing every node of \( T \) as a context, namely, a string of no more than \( D \) symbols from \( A \), and viewing \( T \) as the collection of its leaves, the parameter vector \( \theta \) associated with \( T \) is \( \theta = \{\theta_s; s \in T\} \), where each \( \theta_s \) is a discrete probability vector,

\[
\theta_s = (\theta_s(0), \theta_s(1), \ldots, \theta_s(m-1)),
\]

so that the \( \theta_s(j) \) are nonnegative and \( \sum_{j \in A} \theta_s(j) = 1 \), for each \( s \in T \).

A model \( T \in T(D) \) together with an associated parameter vector \( \theta = \{\theta_s; s \in T\} \) specify the conditional distribution of \( X_n \) given \( X_{n-D}^{n-1} \) as follows. Given \( X_{n-D}^{n-1} = x_{n-D}^{n-1} \), let \( s \) denote the unique leaf of \( T \) that is a suffix of \( x_{n-D}^{n-1} \). Then, for each \( a \in A \),

\[
P(X_n = a | X_{n-D}^{n-1} = x_{n-D}^{n-1}) = \theta_s(a).
\]

For example, consider the tree model \( T \) of a 5th order chain with alphabet \( A = \{0, 1, 2\} \), represented by the tree shown in Figure 1. Then, conditional on the past values \((X_{n-1}, X_{n-2}, X_{n-3}, X_{n-4}, X_{n-5}) = (0, 2, 2, 1, 2)\), the probability that \( X_n = 1 \) is,

\[
P(1|02212) := P(X_n = 1 | X_{n-5}^{n-1} = 02212) = \theta_{022}(1),
\]

where the relevant context now is \( s = 022 \). More generally, the probability of a block of observations \( x_n^m \) given an initial context \( x_0^{D+1} \) can be expressed, via the Markov property, as,

\[
P(x_1^n | x_0^{D+1}) := P(X_1^n = x_1^n | X_0^n = x_0^n) = \prod_{i=1}^n P(x_i | x_{i-D}^{i-1})
\]

\[
= \prod_{s \in T} \prod_{j \in A} \theta_s(j)^{a_s(j)},
\]

(1)

where the count vector \( a_s = (a_s(0), a_s(1), \ldots, a_s(m-1)) \) consists of the elements,

\[
a_s(j) := \# \text{ times symbol } j A \text{ follows context } s \text{ in } x_1^n.
\]

Given \( T \in T(D) \), the family of chains with that model can be parametrised by vectors \( \phi = \{\phi_s; s \in T\} \) with each \( \phi_s = (\phi_s(0), \ldots, \phi_s(m-2)) \), where the \( \phi \) belong to the set,

\[
\Omega(T, m) := \{\phi = \{\phi_s; s \in T\} \in [0, 1]^{T(m-1)} | \sum_{j=0}^{m-2} \phi_s(j) \leq 1, \text{ for each } s \in T\},
\]

(3)

which is a compact Euclidean subset of \( \mathbb{R}^{T(m-1)} \) with nonempty interior, and where \( |T| \) denotes the number of leaves of \( T \). To each \( \phi \in \Omega(T, m) \) we naturally associate a parameter vector \( \theta = \theta(\phi) \) by letting, for each \( s \in T \),

\[
\theta_s(j) = \phi_s(j), \text{ for } 0 \leq j \leq m-2,
\]

\[
\text{and } \theta_s(m-1) = 1 - \sum_{0 \leq j \leq m-2} \phi_s(j).
\]

(4)

B. Prior Structure

1) Model Prior: Given a fixed maximal depth \( D \geq 0 \) and an arbitrary \( \beta \in (0, 1) \), we define a prior distribution on the collection \( T(D) \) of models \( T \) of depth no more than \( D \), by,

\[
\pi(T) := \pi_D(T) := \pi_D(T; \beta) := \alpha^{|T|-1} \beta^{|T|-LD(T)},
\]

(5)
where $\alpha := (1 - \beta)^{1/(m - 1)}$, and $L_D(T)$ denotes the number of leaves $T$ has at depth $D$. Clearly $\pi(T)$ penalises larger models by an exponential amount, and the value of the hyperparameter $\beta$ controls the degree of this penalisation. The fact that (5) indeed defines a probability distribution and the specific way in which larger models are penalised are discussed in detail in [24].

2) Prior on Parameters: Given a model $T \in \mathcal{T}(D)$, we define the following prior distribution on the parameter vectors $\theta = \{\theta_s; s \in T\}$: We place an independent Dirichlet distribution with parameters $(1/2, \ldots, 1/2)$ (denoted Dir$(1/2, \ldots, 1/2)$) on each $\theta_s$ so that, $\pi(\theta|T) = \prod_{s \in T} \pi(\theta_s)$, where,

$$
\pi(\theta_s) = \pi(\theta_s(0), \theta_s(1), \ldots, \theta_s(m - 1)) = \frac{\Gamma(m/2)}{\pi^{m/2}} \prod_{j=0}^{m-1} \theta_s(j)^{-\frac{1}{2}} \propto \prod_{j=0}^{m-1} \theta_s(j)^{-\frac{1}{2}}.
$$

3) Likelihood: Given a model $T \in \mathcal{T}(D)$, the associated parameter vector $\theta = \{\theta_s; s \in T\}$, and observations $x^n_T$ with initial context $x_{-D+1}^0$, the likelihood is given as in (1),

$$
P(x^n_T|x_{-D+1}^0, \theta, T) = \prod_{s \in T} \prod_{j=0}^{m-1} \theta_s(j)^{a_s(j)}. \tag{6}
$$

In order to avoid cumbersome notation, in what follows we often write $x$ for the string $x^n_T$ and suppress the dependence on its initial context $x_{-D+1}^0$, so that, for example, we denote, $P(x|T) = P(x^n_T|x_{-D+1}^0, \theta, T)$.

C. Marginal Likelihood, Posterior, and Prior Predictive Likelihood

A useful property induced by the prior specification above is that the parameters $\theta$ can be integrated out, so that the marginal likelihood $P(x|T)$ can be expressed in closed form.

Lemma 2.1: The marginal likelihood $P(x|T)$ of the observations $x$ given a model $T$ is

$$
P(x|T) = \int P(x, \theta|T)d\theta = \int P(x, \theta|T)\pi(\theta|T)d\theta = \prod_{s \in T} P_c(a_s),
$$

where the count vectors $a_s$ are defined in (2) and the estimated probabilities $P_c(a_s)$ are,

$$
P_c(a_s) := \frac{\prod_{j=0}^{m-1} [(1/2)(3/2) \cdots (a_s(j) - 1/2)]}{(m/2)(m/2 + 1) \cdots (m/2 + M_s - 1)}.
$$

where $M_s := a_s(0) + \cdots + a_s(m - 1)$, with the convention that any empty product is equal to 1.

In terms of inference, the most important quantity is the posterior distribution,

$$
\pi(\theta, T|x) = \frac{P(x|\theta, T)\pi(\theta, T)}{P(x)},
$$

where the main obstacle in its computation is the prior predictive likelihood term,

$$
P(x) := P_D^\ast(x) := \sum_{T \in \mathcal{T}(D)} \pi_D(T)P(x|T) = \sum_{T \in \mathcal{T}(D)} \int \pi_D(T)P(x|\theta, T)\pi(\theta|T)\,d\theta. \tag{8}
$$

Nevertheless, the general version of the context-tree weighting (CTW) algorithm can be used to efficiently compute the exact value of $P_D^\ast(x)$.

The CTW first builds an $m$-ary tree, $T_{\text{MAX}}$, whose leaves are all the contexts $x_{-D+1}^i$, $i = 1, 2, \ldots, n$, that appear in the observations string $x^n_D$, together with any additional leaves required so that $T_{\text{MAX}}$ is proper. Then the estimated probabilities $P_{c,s} := P_c(a_s)$ given by (7) are computed at each node $s$ of $T_{\text{MAX}}$, and finally the mixture or weighted probabilities are computed at each node $s$ of $T_{\text{MAX}}$,

$$
P_{w,s} := \begin{cases} P_{c,s}, & \text{if } s \text{ is a leaf}, \\ \beta P_{c,s} + (1 - \beta) \prod_{j=0}^{m-1} P_{w,sj}, & \text{otherwise}, \end{cases}
$$

where $sj$ denotes the concatenation of context $s$ and symbol $j$, corresponding to the $j$th child of node $s$. The mixture probability $P_{w,\lambda}$ at the root $\lambda$ is exactly $P_D^\ast(x)$.

Similarly, the BCT algorithm efficiently identifies the maximum a posteriori probability (MAP) context tree model $T_1$, that satisfies:

$$
\pi(T_1|x) = \max_{T \in \mathcal{T}(D)} \pi(T|x). \tag{10}
$$

Proceeding as in the CTW, after constructing the tree $T_{\text{MAX}}$ and computing the estimated probabilities, now the maximal probabilities are computed at each $s \in T_{\text{MAX}}$:

$$
P_{m,s} := \begin{cases} P_{c,s}, & \text{if } s \text{ is a leaf at depth } D, \\ \beta, & \text{if } s \text{ is a leaf at depth } < D, \\ \max \beta P_{c,s}, (1 - \beta) \prod_{j=0}^{m-1} P_{m,sj}, & \text{otherwise}. \end{cases}
$$

Then, proceeding recursively from the root to its descendants, for each node $s$: If the above maximum can be achieved by the first term, then prune all its descendants from $T_{\text{MAX}}$; otherwise, repeat the same process at each of the children of node $s$. The resulting tree, $T_1$, after all nodes have been exhausted satisfies (10).

Also we recall that the full conditional distribution of the parameters given the model and observations, $\pi(\theta|x, T)$, is given by the following product of Dirichlet distributions:

$$
\prod_{s \in T} \text{Dir}(a_s(0) + \frac{1}{2}, a_s(1) + \frac{1}{2}, \ldots, a_s(m - 1) + \frac{1}{2}), \tag{11}
$$

D. Maximum Likelihood and the Posterior Predictive Distribution

For a given data string $x$ and a fixed depth $D$, the first steps of the CTW algorithm can be used to compute the maximum likelihood estimates (MLEs) for the model and parameters, namely, the model $\hat{T}_{\text{MLE}}$ and the associated parameters $\hat{\theta}_{\text{MLE}} := \{\theta_s; s \in \hat{T}_{\text{MLE}}\}$ that achieve:

$$
\hat{P}_{\text{MLE}}^\ast(x) := \max_{T \in \mathcal{T}(D)} \sup_{\theta \in \{\theta_s; s \in \hat{T}_{\text{MLE}}\}} P(x|\theta, T). \tag{12}
$$
Proposition 2.2: Let $x = x_{n,D+1}^n$ be a given string of samples from $A$, let $D \geq 1$, and write $T_c(D)$ for the complete $m$-ary tree of depth $D$.

(i) The maximum likelihood over models $T \in T(D)$ in (12) is equivalent to the classical maximum likelihood among all Markov chains of memory depth $D$:

$$
\hat{P}_{\text{MLE}}(x) = \hat{P}_{\text{MLE}}(x|T_c(D)) := \sup_{\theta = \{\theta_s; s \in T_c(D)\}} P(x|\theta, T_c(D)).
$$

(ii) The maximum likelihood model $\hat{T}_{\text{MLE}}$ can always be taken to be $T_c(D)$. The corresponding maximum likelihood parameters $\theta_{\text{MLE}} = \{\theta_s; s \in T_{\text{MLE}}\}$ at each leaf $s$ are given by the empirical frequencies $a_s/M_s$, where $a_s$ is given in (2) and $M_s = \sum_j a_s(j)$, whenever $a_s$ is not the all-zero vector. If $a_s$ is zero, then $\theta_s$ can be taken arbitrary.

(iii) Equivalently, $\hat{T}_{\text{MLE}}$ can be taken to be the tree $T_{\text{MAX}}$ computed in the first step of CTW, with the parameter vector $\theta_{\text{MLE}} = \{\theta_s; s \in T_{\text{MAX}}\}$ defined as before.

(iv) The actual maximum likelihood can be expressed as:

$$
\hat{P}_{\text{MLE}}(x) = P(x|\theta_{\text{MLE}}^n, \hat{T}_{\text{MLE}}) = \prod_{s \in T_{\text{MAX}}} \prod_{j=0}^{m-1} \hat{\theta}_s(j)^{a_s(j)}. \quad (13)
$$

Finally we recall that, for the purposes of prediction, standard Bayesian methodology dictates that the canonical rule for predicting the next observation $x_{n+1}$ given the past $x_1^n$, is given by the posterior predictive distribution,

$$
P^*_D(x_{n+1}|x_1^n) = \sum_T \int_\theta P(x_{n+1}|x_1^n, \theta, T) \pi(\theta, T|x_1^n) d\theta. \quad (14)
$$

where again, for simplicity, we suppressed the dependence on the initial context $x_{-D+1}^0$. A key observation is that, using the CTW, $P^*_D(x_{n+1}|x_1^n)$ can be computed exactly and sequentially, as:

$$
P^*_D(x_{n+1}|x_1^n) = \frac{P^*_D(x_{n+1}|x_1^n)}{P^*_D(x_1^n)}. \quad (15)
$$

III. MAIN RESULTS: BOUNDS AND ASYMPOTICS

The statistical tools provided by the BCT framework have been found to provide efficient methods for very effective inference in a variety of applications [24, 27, 35, 36, 37]. In terms of the underlying theory, the Bayesian perspective adopted in [24] and this work is neither purely subjective, interpreting the prior and posterior as subjective descriptions of uncertainty pre- and post-data, respectively, nor purely objective, treating the resulting methods as simple black-box procedures [9]. For example, we think of the MAP model as the most accurate, data-driven representation of the regularities present in a given time series, but we inform our analysis of the resulting inferential procedures by simulation experiments on hypothetical models, and by examining their frequentist properties; see, e.g., [3, Chapter 6] or [14, Chapter 4] for broad discussions of the relationship between the Bayesian and classical outlook. This latter examination is the main purpose of this paper. Our main results, presented in this section, provide classical asymptotic results as well as nonasymptotic bounds, than can be viewed as partial justifications of the BCT framework.

A point of view which has had very significant influence in the development of the ideas presented in this work is Rissanen’s celebrated MDL principle. As should become apparent from the form of the results in this section, there is also a strong connection with Schwarz’s Bayesian Information Criterion (BIC) [43], and its familiar “(1/2) log $n$-per-degree-of-freedom” log-likelihood penalty; see [12, 22, 23, 29], and [50] for extensive discussions of the role of the BIC within Bayesian theory in general, and its use in conjunction with Markov chain models.

Our main results are Theorems 3.1–3.10 in Sections III-A–III-D. As some of them are simple consequences of known general results or generalisation of previously established special cases, a detailed bibliographical discussion is given in Section III-E. All proofs are deferred to Section IV.

A. The Prior Predictive Likelihood

The following three results show that the logarithm of the prior predictive likelihood (cf. (8)) of any data string of length $n$, is uniformly close to the log-likelihood of every variable-memory chain, up to the best possible penalty of order $\log n$. Specifically, for every $x_1^n$ of arbitrary length $n$, any initial context $x_{-D+1}^0$, and any model $T \in T(D)$ with parameters $\theta = \{\theta_s; s \in T\}$,

$$
\log P_D(x_1^n|x_{-D+1}^0) \approx \log P(x_1^n|x_{-D+1}^0, \theta, T) - \frac{|T|(m-1)}{2} \log n; \quad (16)
$$

recall that $m$ denotes the alphabet size and ‘log’ denotes the natural logarithm throughout this work. Moreover, this performance is in a strong sense best possible.

The first result states that the prior predictive likelihood indeed achieves the performance announced in (16), in a strong, nonasymptotic sense.

Theorem 3.1: For any variable-memory chain with model $T \in T(D)$ and associated parameters $\theta = \{\theta_s; s \in T\}$, for any sequence $x_1^n$ of arbitrary length $n$, and any initial context $x_{-D+1}^0$, the prior predictive likelihood for any $\beta$ satisfies,

$$
\log P_D^*(x_1^n|x_{-D+1}^0) 
\geq \log P(x_1^n|x_{-D+1}^0, \theta, T) - \frac{|T|(m-1)}{2} \log n + C,
$$

where the constant $C = C(T, m, \beta)$ is independent of $n$ and of $\theta$, and can be taken equal to

$$
\frac{|T|(m-1)}{2} \log |T| - |T| \log m + \log \pi_D(T; \beta),
$$

for $n \geq \varepsilon |T|$.

The next result shows that no other probability assignment can essentially outperform the prior predictive likelihood, even on the average, and even for a small fraction of processes, as defined by their relative volume in terms of the parametrisation in (3). Theorem 3.2 is a simple consequence of a fundamental result due to Rissanen [39, 41].
Theorem 3.2: Let \( \{X_n\} \) denote an arbitrary variable-memory chain with model \( T \in \mathcal{T}(D) \) and associated parameters \( \theta = \{ \theta_s; s \in T \} \), and suppose \( \{Q_n\} \) is any consistent sequence of probability distributions \( Q_n \) on \( A^n \), \( n \geq 1 \). Then, for every \( n \) and every \( \epsilon > 0 \),
\[
E_{\theta,T}[\log Q_n(X_1^n)] \\
\leq E_{\theta,T}[\log P(X_1^n|X_{D+1}^0, \theta, T)] \\
-(1-\epsilon)\frac{|T|(m-1)}{2}\log n,
\]
for all parameter vectors \( \theta \), except for those corresponding to a subset \( A_s(n) \) of \( \Omega(T,m) \), whose volume tends to zero as \( n \to \infty \), and where the expectation is taken with respect to the distribution of the chain \( \{X_n\} \).

The following result is analogous to that of Theorem 3.2, except it states the prior predictive likelihood will outperform any other probability assignment \( \{Q_n\} \) not just on the average but on “most” sample strings \( x^n \): The bound (17) holds not only in expectation but in fact for most \( x^n \). In order to state it precisely, we need the following notation. Given a model \( T \in \mathcal{T}(D) \) and an initial context \( x^0_{D+1} \), we say that the strings \( x^n \) and \( y^n \) belong to the same \( T \)-type, if \( x^n_{D+1} \) and the concatenation of \( x^0_{D+1} \) and \( y^n \) induce the same count vectors \( a_s \) for all contexts \( s \in T \); cf. Lemma 2.1. Then it is easy to see, e.g., that \( A^n \) can be decomposed into polynomially many different \( T \)-types, each of which consists of exponentially many strings. Let \( M_n(T) \) denote the total number of \( T \)-types of strings of length \( n \).

Theorem 3.3 is a consequence of a general result due to Weinberger, Merhav and Feder [51].

Theorem 3.3: Let \( \{Q_n\} \) be any consistent sequence of probability distributions \( Q_n \) on \( A^n \), \( n \geq 1 \), and let \( \epsilon > 0 \). Then, for every model \( T \in \mathcal{T}(D) \), every parameter vector \( \theta = \{ \theta_s; s \in T \} \), every sample size \( n \), and every initial context \( x_0 \), we have,
\[
\log Q_n(x^n_1) \\
\leq \log P(x^n_1|x^0_{D+1}, \theta, T) - (1-\epsilon)\frac{|T|(m-1)}{2}\log n,
\]
for all strings \( x^n \in A^n \) except those in a set \( B_n(\epsilon) \subset A^n \) which is asymptotically small in the sense that the number \( N_n(\epsilon, T) \) of \( T \)-types \( \tau \) that contain a non-negligible part of \( B_n(\epsilon) \), i.e.,
\[
\frac{|B_n(\epsilon) \cap \tau|}{|\tau|} > n^{-\epsilon/3},
\]
is asymptotically negligible:
\[
N_n(\epsilon, T)/M_n(T) \to 0, \quad \text{as} \quad n \to \infty.
\]

Because the proofs of Theorems 3.2 and 3.3 both depend of general information-theoretic results that are not proved here, in Section III-D we give a different upper bound which can be proved directly, without relying on any external results.

B. The Posterior Predictive Distribution

Recall that a variable-memory chain \( \{X_n\} \) with model \( T \in \mathcal{T}(D) \) and associated parameters \( \theta = \{ \theta_s; s \in T \} \) is said to be ergodic, if the corresponding first order chain \( \{Z_n := X^n_{n-D+1}; n \geq 1\} \) is irreducible and aperiodic on a possibly strict subset of \( A^\infty \). For an ergodic chain \( \{X_n\} \) we write \( \pi \) for its unique stationary distribution, as long as this notation does not cause confusion with the similar notation used for the posterior distribution \( \pi(\theta, T|x) \). In order to avoid uninteresting technicalities, whenever we assume that \( \{X_n\} \) is ergodic, we implicitly also assume that its stationary distribution \( \pi \) gives strictly positive probability to all strings corresponding to contexts \( s \) in the model \( T \).

As discussed in Section II-D, an important aspect of the BCT framework is that the sequential nature of the CTW algorithm makes it possible to efficiently compute the posterior predictive distribution, cf. (14) and (15),
\[
P(X_{n+1} = j|x^n_{D+1}) = P_D^n(j|x^n_{D+1}) = \frac{P_D^n(j|x^n_{D+1}j)}{P_D^n(x^n_{D+1})},
\]
for all \( j \in A \). Our next result, Theorem 3.4, states that the posterior predictive distribution will in fact converge to the true underlying distribution, asymptotically with probability one.

Theorem 3.4: Suppose \( \{X_n\} \) is an ergodic variable-memory chain, with model \( T \in \mathcal{T}(D) \) and associated parameters \( \theta = \{ \theta_s; s \in T \} \). The posterior predictive distribution obtained from the prior predictive likelihood with an arbitrary \( \beta \) converges to the true conditional distribution of the process \( \{X_n\} \); For each \( j \in A \), almost surely (a.s.) as \( n \to \infty \):
\[
P_D^n(j|x^n_{D+1}) - P(j|x^n_{D+1}, \theta, T) \to 0.
\]

C. The Posterior: Asymptotic Consistency and Normality

Let \( \{X_n\} \) be a variable-memory chain with model \( T \in \mathcal{T}(D) \). As discussed in [24] the specific model \( T \) that describes the chain is typically not unique. Of course, the main goal in model selection is to identify the “minimal” model, that is, the smallest model that can fully describe the distribution of the chain.

We call a model \( T \in \mathcal{T}(D) \) minimal with respect to the parameter vector \( \theta = \{ \theta_s; s \in T \} \), if \( T \) is either equal to \( \Lambda := \{ \lambda \} \) or, if \( T \neq \Lambda \), then every \( m \)-tuple of leaves \( \{s_j; j = 0,1,\ldots,m-1\} \) in \( T \) contains at least two with non-identical parameters, i.e., there are \( j \neq j' \) such that \( \theta_{s_j} \neq \theta_{s_{j'}} \). It is easy to see that every \( D \)-th order Markov chain \( \{X_n\} \) has a unique minimal model \( T^* \in \mathcal{T}(D) \).

Throughout this section we will assume that \( \{X_n\} \) is an ergodic \( D \)-th order chain, with minimal model \( T^* \in \mathcal{T}(D) \), associated parameters \( \theta^* = \{ \theta^*_s; s \in T^* \} \), unique stationary distribution denoted by \( \pi^* \), and an arbitrary initial context \( x^0_{D+1} \).

Given a sample \( x = x^n_{D+1} \), a maximum a posteriori probability (MAP) model \( T^*(n) \) within \( T(D) \) is any \( T \in \mathcal{T}(D) \) which maximises the posterior probability \( \pi(T|x) \) over all \( T \in \mathcal{T}(D) \). Of course, the maximiser \( T^*(n) \) need not always be unique. Our next result says that, if the sample \( x = x^n_{D+1} \) is produced by an ergodic chain \( \{X_n\} \) with minimal model \( T^* \in \mathcal{T}(D) \), then the MAP model \( T^*(n) \) is eventually unique and \( T^*(n) = T^* \), with probability one.
Theorem 3.5: Let \( \{X_n\} \) be an ergodic variable-memory chain with minimal model \( T^* \in T(D) \). For any \( \beta \), the MAP model \( T^*(n) \) based on the random sample \( X_{n+1}^n \) is eventually a.s. unique and in fact:

\[
T^*(n) = T^*, \quad \text{eventually, a.s.}
\]

Using a different argument than that in the proof of Theorem 3.5, the following stronger consistency result can also be established.

Theorem 3.6: Under the assumptions of Theorem 3.5, the posterior distribution over models eventually concentrates on \( T^* \). For any \( \beta \),

\[
\pi(T^*|X_{n+1}^n) \to 1, \quad \text{as } n \to \infty.
\]

Next we show, that as long as the true model belongs to \( T(D) \), the posterior distribution on both the model and parameters eventually a.s. concentrates around the true underlying values \((\theta^*, T^*)\). The proof of Theorem 3.7 uses Theorem 3.6.

Theorem 3.7: Let \( \{X_n\} \) be an ergodic variable-memory chain with minimal model \( T^* \in T(D) \) and associated parameters \( \theta^* = \{\theta_s^*; s \in T^*\} \); let \( \beta \) be arbitrary. The posterior distribution \( \pi(\theta, T|X_{n+1}^n) \) asymptotically concentrates around the true model and parameters, i.e.,

\[
\pi(\cdot|X_{n+1}^n) \xrightarrow{D} \delta(\theta^*, T^*), \quad \text{a.s. as } n \to \infty,
\]

where \( \xrightarrow{D} \) denotes weak convergence of probability measures, and \( \delta(\theta^*, T^*) \) is the unit mass at the point \((\theta^*, T^*)\).

Our next asymptotic result states that the (appropriately centered and scaled) posterior distribution on the parameters is asymptotically normal. Recall that the density of the posterior can be decomposed as,

\[
\pi(\theta, T|X_{n+1}^n) = \pi(T|X_{n+1}^n) f_n(\theta|X_{n+1}^n, T),
\]

where \( f_n(\theta|X_{n+1}^n, T) \) is the full conditional density of the parameters given in (11). Since \( \pi(T|X_{n+1}^n) \) converges in distribution to \( \delta_T \), a.s., by Theorem 3.6, we concentrate on the asymptotic distribution \( \pi(\theta|X_{n+1}^n, T^*) \) of the parameters \( \theta \) on \( T^* \).

For the statement of Theorem 3.8 we will need the following notation. Given an ergodic chain with model \( T^* \), stationary distribution \( \pi \), and parameters \( \theta^* \), for each \( s \in T^* \) let \( J_s \) denote the \( m \times m \) matrix,

\[
J_s = \frac{1}{\pi(s)} \left[ (\Omega_s^*)^j (\theta_s^*)^j \right],
\]

where \( \Omega_s^* \) is the diagonal matrix with entries \( \theta_s^*(j), j \in A \), and \( \theta_s^* \) is viewed as a row vector in \( \mathbb{R}^m \). Then by \( J \) we denote the \( m|T^*| \times m|T^*| \) block-diagonal matrix consisting of all \( m \times m \) blocks \( J_s \), for \( s \in T^* \),

\[
J = \bigoplus_{s \in T^*} J_s.
\]

Theorem 3.8: Let \( \{X_n\} \) be an ergodic variable-memory Markov chain with stationary distribution \( \pi \), minimal model \( T^* \in T(D) \) and associated parameters \( \theta^* = \{\theta_s^*; s \in T^*\} \), with each \( \theta_s^*(j) > 0 \); let \( \beta \) be arbitrary. Suppose \( \theta(n) \) is distributed according to the posterior \( \pi(\cdot|X_{n+1}^n, T^*) \), and let \( \bar{\theta}(n) \) denote its mean. Then, as \( n \to \infty \),

\[
\sqrt{n} \left[ \theta(n) - \bar{\theta}(n) \right] \xrightarrow{D} Z \sim N(0, J), \quad \text{a.s.,}
\]

where \( N(0, J) \) is the multivariate normal on \( \mathbb{R}^{|T^*|} \) with zero mean and covariance matrix \( J \) given in (20). Moreover \( \bar{\theta}(n) \to \theta^* \) as \( n \to \infty \): With probability 1 as \( n \to \infty \),

\[
\bar{\theta}_s(n) = \mathbb{E} \left( \theta_s(n) | X_{n+1}^n, T^* \right) \to \theta_s^*(j),
\]

for all \( s \in T^* \), \( j \in A \).

D. An Explicit Minimax Bound

Finally we give a minimax version of Theorem 3.2. Theorem 3.10 gives a more precise bound which is near-optimal up to constant terms, not just the terms of order \( \log n \) as in Theorems 3.2 and 3.3. On the other hand, it is a weaker version of Theorem 3.3: It states that, for any probability assignment, there is at least one variable-memory chain and at least one string \( x^n \) on which it cannot outperform the prior predictive likelihood asymptotically.

In order to make the comparison between the upper and lower bounds on \( P_D^* \) more transparent, before stating Theorem 3.10 we give a simple corollary of Theorem 3.1. A close examination of its proof shows that the following more detailed bound is actually established there.

Corollary 3.9: For any variable-memory chain with model \( T \in T(D) \) and associated parameters \( \theta = \{\theta_s; s \in T\} \), for any sequence \( x^n \) of arbitrary length \( n \), and any initial context \( x_0^n \), let \( L(Q_n, x_0^n|\theta, T) \) denote the log-likelihood ratio between an arbitrary probability distribution \( Q_n \) on \( A^n \) and the true underlying distribution of the chain:

\[
L(Q_n, x^n|\theta, T) := \log Q_n(x^n) - \log P(x^n|x_0^n, \theta, T).
\]

Then, for any \( \beta \), the prior predictive likelihood \( P_D^* \) achieves,

\[
E(P_D^*, x^n|\theta, T) \geq - \left[ \sum_{s \in T: M_s \neq 0} \left( m - \frac{1}{2} \log M_s + \log m \right) - \log \pi_D(T; \beta) \right],
\]

where \( M_s \) are the sums of the count vectors \( a_s \) as in Lemma 2.1.

Our final result shows that the performance achieved by the prior predictive likelihood \( P_D^* \), as described in Corollary 3.9, cannot be improved upon by any sequence of probability distributions: The log-likelihood of any such choice will asymptotically be no better than that of \( P_D^* \) on at least one realisation produced by some variable-memory chain, up to a constant term that depends only on the alphabet size and the maximal memory length \( D \).

Theorem 3.10: Suppose \( \{Q_n\} \) is any (not necessarily consistent) sequence of probability distributions \( Q_n \) on \( A^n \), \( n \geq 1 \). Then, for any \( \beta \), any \( D \geq 1 \), and any initial context \( x_0^n \), the expression in equation (21), shown at the bottom of the next page, is always \( \leq 0 \), where \( \theta(\phi) \) for \( \phi \in \Omega(T, M) \) refers to the parametrisation of variable-memory chains given in (4), and \( S_n = S_n(\theta, T, x_0^n) \) denotes the collection of all
strings \( x^n \in A^n \) that have positive probability under \((\theta, T)\): 
\[
P(x^n_1 | x_{-D+1}^0, \theta, T) > 0.
\]

Observe that the difference between the log-likelihood achieved by \( P_D^0 \) in Corollary 3.9 and the minimax optimality bound in Theorem 3.10 is small, it depends only on the alphabet size \( m \), and it corresponds to a constant penalty \( \Delta_m \) per model leaf, where \( \Delta_m \) is simply,

\[
\Delta_m = \log m - \log \left( \frac{\sqrt{2\pi}}{2^{m/2} \Gamma(m/2)} \right).
\]

The bound in Theorem 3.10 can be viewed as a generalisation of Shtarkov’s minimax redundancy theorem in [44]. Sharp results in the same spirit, for both i.i.d. and more general Markov processes, are given in [20], [61], and [46]. And for any specific model \( T \in T(D) \), precise asymptotics for the ‘minimax regret,’

\[
\min_{Q_n} \sup_{\phi \in \Omega(T,m)} \max_{s \in S_n} \left[ -\mathcal{L}(Q_n, x^n_1 | \theta(\phi), T) \right],
\]

are developed in [45].

### E. History and Bibliographical Remarks

The lower bound on the prior predictive likelihood in Theorem 3.1, although essentially implicit in the existing literature, is new in the form presented here; it was established for the special case of binary data \( (m = 2) \) and \( \beta = 1/2 \) in [57] and [58], and a version for general \( m \) but only a specific value of \( \beta = \beta(m) \) was given in [47], [48], and [56]. The corresponding lower bound in expectation given in Theorem 3.2 is a straightforward corollary of Rissanen’s celebrated results in [39] and [41]. Similarly, the lower bound in Theorem 3.3 follows from the general results in [51]. A weaker, asymptotic version of the bounds in Theorems 3.1–3.3 is established, under stronger assumptions in [18]. The asymptotic consistency of the posterior predictive distribution stated in Theorem 3.4 was first given in the special case \( \beta = 1/2 \) in [21, Lemma 2]. Versions of the consistency result in Theorem 3.5 and the minimax lower bound in Theorem 3.10 for binary data and \( \beta = 1/2 \) are given in [55], and [57], and similar techniques were used to establish the eventual-a.s. consistency of the BIC estimator for context trees in [13]; the general results as stated and proved here are new. Theorems 3.6, 3.7 and 3.8 are new. An asymptotic relation similar to Theorem 3.5, and conditions for an implicit version of Theorem 3.8, under additional assumptions, are discussed in [18]. A stronger version of Theorem 3.6, including a rate of convergence, but under slightly stronger assumptions, was recently shown in [36]. Earlier versions of Theorems 3.6 and 3.8 are described, in terms of data compression, in [17].

### IV. Proofs

We note for later use the following simple property of the prior \( \pi_D(T) \).

#### Lemma 4.1

If \( T \in T(D) \), \( t \in T \) is at depth \( d < D \), \( S \in T(D - d) \) is nonempty, and \( T \cup S \) consists of the tree \( T \) with \( S \) added as a subtree rooted at \( t \), then,

\[
\pi_D(T \cup S) = \beta^{-1} \pi_D(T) \pi_D(D - d)(S).
\]

**Proof.** The result immediately follows from the simple observations that \( |T \cup S| = |T| + |S| - 1 \) and \( L_D(T \cup S) = L_D(T) + L_D(D - d)(S) \), together with the definition of the prior. \(\square\)

**Proof of Proposition 2.2.** For the equivalence in (i), observe that, in the present setting, a Markov chain with memory length \( D \) is equivalent to a variable-memory chain with tree model corresponding to the complete tree \( T_c(D) \). On the other hand, a variable-memory chain with model given by some tree \( T \) can be represented as a chain with memory length \( D \) by extending \( T \) to the complete tree and assigning to each new leaf the same parameter vector as its most recent ancestor in \( T \). The above argument also shows that we can always take \( T_{\text{MLE}} \) to be the complete tree \( T_c(D) \). Then, to maximise with respect to the parameters \( \theta \), recall from (6) that the log-likelihood is,

\[
\log P(x | \theta, T_c(D)) = \sum_{s,j=0}^{m-1} a_s(j) \log \theta_s(j)
\]

\[
= \sum_s M_s \sum_{j=0}^{m-1} \hat{a}_s(j) M_s \log \theta_s(j),
\]

where we sum over all leaves \( s \) of the complete tree for which \( a_s \) is not the all-zero count vector, and \( M_s = \sum_j a_s(j) \). Writing \( \hat{a}(j) = a_s(j) / M_s \), this becomes,

\[
\log P(x | \theta, T_c(D)) = \sum_s M_s \left\{ \sum_{j=0}^{m-1} \hat{a}_s(j) \log \left( \frac{\theta_s(j)}{\hat{a}_s(j)} \right) + \sum_{j=0}^{m-1} \hat{a}_s(j) \log \hat{a}_s(j) \right\}
\]

\[
= - \sum_s M_s D(\hat{a}_s || \theta_s) - \sum_s M_s H(\hat{a}_s),
\]

where \( H(p) \) and \( D(p||q) \) denote the entropy and relative entropy, respectively, in nats. Therefore, the likelihood is maximised by making each divergence above equal to zero, i.e., by taking each \( \theta_s = \hat{a}_s \) for leaves \( s \) with nonzero count vectors \( a_s \). This proves (ii).

The fact claimed in (iii), that we can take \( T_{\text{MLE}} = T_{\text{MAX}} \), is an immediate consequence of the above computation, combined with the observation that the only leaves \( s \) of \( T_{\text{MAX}} \) that are at depth strictly smaller than \( D \) will necessarily have all-zero count vectors \( a_s \). Finally, the result of part (iii) together with the simple expression for the likelihood in (6)

\[
\lim_{n \to \infty} \min_{T \in T(D)} \inf_{P \in \Omega(T,m)} \min_{s \in S_n} \left[ \mathcal{L}(Q_n, x^n_1 | \theta(\phi), T) + \sum_{s \in T : M_s \neq 0} \left( \frac{m-1}{2} \log M_s + \log \left( \frac{\sqrt{2\pi}}{2^{m/2} \Gamma(m/2)} \right) - \log \pi_D(T; \beta) \right) \right]
\]

(21)
show that the expression in (13) indeed computes the required maximised likelihood.

Proof of Theorem 3.1. For the sake of clarity, we adopt the simpler notation of Sections II-B and II-C. Our starting point is the following pair of bounds on the probabilities $P_e(a)$; they follow from rather involved but elementary computations and are stated here without proof. The results are implicit in [25], [47], and [48], and slightly different proofs are given in [8]; see also [61] for more detailed bounds.

Lemma 4.2: For any count vector $a$, the probabilities $P_e(a)$ defined in Lemma 2.1 satisfy,

$$\log P_e(a) \geq \sum_{j=0}^{m-1} a(j) \log \frac{a(j)}{M} - \frac{m-1}{2} \log M - \log m; \quad (22)$$

$$\log P_e(a) \leq \sum_{j=0}^{m-1} a(j) \log \frac{a(j)}{M} - \frac{m-1}{2} \log M - \frac{m/2}{\Gamma(m/2)}. \quad (23)$$

with $M = a(0) + \cdots + a(m-1)$.

We can now bound the marginal likelihoods $P(x|T)$ for any $T \in T(D)$ and any parameter vector $\theta = \{\theta_s; s \in T\}$ as,

$$\log P(x|T) \overset{(a)}{=} \sum_{e \in T} \log P_e(a_e) \overset{(b)}{\geq} \sum_{e \in T: M_e \neq 0} \left\{ \sum_{j=0}^{m-1} a_s(j) \log \frac{a_s(j)}{M_s} \right\}$$

$$- \frac{m-1}{2} \log M_s - \log m$$

$$= \sum_{e \in T: M_e \neq 0} \log \left( \prod_{j=0}^{m-1} \frac{a_s(j)}{M_s} \right) - \sum_{e \in T: M_e \neq 0} \left\{ \frac{m-1}{2} \log M_s + \log m \right\}$$

$$\overset{(c)}{\geq} \sum_{e \in T} \log \left( \prod_{j=0}^{m-1} \theta_s(j) a_s(j) \right) - \frac{m-1}{2} \sum_{e \in T: M_e \neq 0} \log M_s - |T_n| \log m,$$

where (a) follows by Lemma 2.1, (b) follows from (22) in Lemma 4.2, (c) follows form the fact that, as in the proof of Proposition 2.2, the empirical frequencies maximise the likelihood over all parameter choices, and $|T_n|$ denotes the number of $s \in T$ for which $M_s \neq 0$.

Therefore,

$$\log P(x|T) \overset{(d)}{\geq} \log P(x|\theta, T)$$

$$- |T_n|(m-1) \frac{1}{2} \sum_{e \in T: M_e \neq 0} \frac{1}{|T_n|} \log M_s - |T_n| \log m$$

$$\overset{(e)}{\geq} \log P(x|\theta, T) - |T_n|(m-1) \frac{1}{2} \log \left( \frac{n}{|T_n|} \right) - |T_n| \log m,$$

where (d) follows Jensen’s inequality and the concavity of the logarithm, and (e) follows from the observation that

$$\sum_{s \in T: M_s \neq 0} M_s = n.$$

Using the above inequality, the prior predictive likelihood $P(x) = \sum_T \pi_D(T; \beta) P(x|T)$ can now be trivially bounded as,

$$\log P(x) \overset{(d)}{\geq} \log (\pi_D(T; \beta) P(x|T))$$

$$\overset{(e)}{\geq} \log P(x|\theta, T) - |T|(m-1) \frac{1}{2} \log \left( \frac{n}{|T|} \right) - |T| \log m + \log \pi_D(T; \beta),$$

where the last inequality holds for $n \geq e|T|$, as required. □

Proof of Theorem 3.2: The result is a more or less immediate consequence of [41, Theorem 1], once we verify its assumptions. Recall the parametrisation of all chains with model $T$ given in (3). For any given chain with model $T$, its parameters $\phi = \{\phi_s; s \in T\}$ can be estimated from a sample $x_{n-D+1}^n$ by the maximum likelihood estimates, $\hat{\phi}_s(j) = a_s(j)/M_s$, for $s \in T, j = 0, 1, \ldots, m-2$. Then the collection of estimates $\hat{\phi}_s = \{\hat{\phi}_s; s \in T\}$ is asymptotically normal, as established, e.g., by Billingsley in [4] and [5]. The final condition requiring that, for the class of processes considered here,

$$\sum_{n \geq 1} P\{ \sqrt{n} \| \hat{\phi} - \phi \|_1 \geq \log n \} < \infty,$$

where $\| \cdot \|_1$ denotes the $L^1$ norm, is verified in [40]. □

Proof of Theorem 3.3: The result of the theorem is simply a special case of [51, Theorem 1]. The probability assignment $\{Q_n\}$ corresponds to scheme $M$, and the pair $(\theta, T)$ corresponds to a finite-state machine $F$ there. A simple computation like that performed in the proof of Proposition 2.2 shows that their conditional entropy $H(x_{n}^{|F|})$ is exactly the negative of the normalised maximum log-likelihood,

$$- \frac{1}{n} \log \hat{P}_{MLE}(x_{n}^{|F|} | x_{n-D+1}^0, T),$$

in the notation of Proposition 2.2. And noting that, by definition,

$$- \frac{1}{n} \log \hat{P}_{MLE}(x_{n}^{|F|} | x_{n-D+1}^0, \theta, T) \leq - \frac{1}{n} \log P(x_{n}^{|F|} | x_{n-D+1}^0, \theta, T),$$

The result of Theorem 3.3 is an immediate corollary of [51, Theorem 1]. □
Proof of Theorem 3.4: We will need the following asymptotic result on the ratios of the probabilities $P_e(a)$ to the mixture probabilities computed by CTW:

Lemma 4.3: Under the assumptions of Theorem 3.4, given a sample $x_{n,D+1}^n$ as in CTW, let $P_{e,s,n}$ and $P_{w,s,n}$ denote the probabilities at each node $s$ of $T$, as defined in (7) and (9), respectively. Then, for any internal node $s$ of $T$, almost surely (a.s.), as $n \to \infty$:

$$\zeta_{s,n} := \frac{P_{e,s,n}}{\prod_{j=0}^{m-1} P_{w,j,n}} \to 0.$$ 

The result of the lemma was first stated for the special case $\beta = 1/2$ in [7] and [21]; the proof of the general case follows along similar lines.

Proof outline. From the definitions of $\zeta_{s,n}$, $P_{e,s,n}$ and $P_{w,s,n}$, we have,

$$\frac{\zeta_{s,n}}{\beta \zeta_{s,n} + (1 - \beta)} = \frac{P_{e,s,n}}{P_{w,s,n}} + (1 - \beta) \prod_{j=0}^{m-1} P_{w,j,n} \leq (1 - \beta) \prod_{j=0}^{m-1} P_{w,j,n} \leq \frac{(1 - \beta)^{m+1} \prod_{j=0}^{m-1} P_{e,s,n}}{\beta \zeta_{s,n} + (1 - \beta)} \leq (1 - \beta)^{m-1} \times \exp \left\{ M_s \left[ \frac{1}{M_s} \log P_{e,s,n} - \frac{1}{M_s} \log \prod_{j=0}^{m-1} P_{w,j,n} \right] \right\}.$$ 

Therefore, to prove the claimed result it suffices to show that the exponential in the above right-hand-side converges to zero a.s., because that would imply that $\zeta_{s,n}/[\beta \zeta_{s,n} + (1 - \beta)] \to 0$ a.s., which would in turn prove the lemma. But that is exactly the content of the last part of the proof of [21, Lemma 12], where we observe that the stationarity assumption can be removed, in view of the classical ergodic theorem for Markov chains [10], [31].

In order to compute the ratio of the two prior predictive likelihoods that defines the posterior predictive distribution, we first recall the sequential updating procedure described in Section III-E of [24]: In the notation of Lemma 4.3, having computed the count vectors $a_{s,n}$, the probabilities $P_{e,s,n}$ and the mixture probabilities $P_{w,s,n}$ at each node of the tree $T_{\text{MAX}}$, based on $x_{n,D+1}^n$, and given an additional sample $x_{n+1} = j$, we observe that the contexts of length $D$, $D-1$, ..., 0, respectively, immediately preceding $x_{n+1} = j$. For the computation of $P_{w,\lambda,n+1} = P_D^e((x_1, \ldots, x_n, j)|x_D^{0,D+1})$:

- At each of the nodes $s_D, s_{D-1}, \ldots, s_0$, the count vectors are updated as $a_{s,n+1}(j) = a_{s,n}(j) + 1$ and $M_{s,n+1} = M_{s,n} + 1$; at all other nodes $s$, $a_{s,n+1}(j) = a_{s,n}(j)$ and $M_{s,n+1} = M_{s,n}$.
- At each of the nodes $s_D, s_{D-1}, \ldots, s_0$, the probabilities $P_e$ are updated by $P_{e,s,n+1} = P_{e,s,n} P_{e,s,n+1|n}$, where,

$$P_{e,s,n+1|n} := a_{s,n+1}(j) - 1/2 \quad m/2 + M_{s,n+1} - 1,$$

and we let $P_{e,s,n+1} = P_{e,s,n}$, at all other nodes $s$.

- Finally the mixture probabilities are updated: For $s = s_D$, which is necessarily a leaf, let $P_{w,s,n+1} = P_{e,s,n+1}$, so that,

$$P_{w,s,n+1|n} := \frac{P_{w,s,n+1}}{P_{w,s,n}} = \frac{P_{e,s,n+1}}{P_{e,s,n}} = P_{e,s,n+1|n}.$$ 

For the contexts $s = s_{D-1}, \ldots, s_0$ which correspond to internal nodes, let,

$$P_{w,s,n+1} = \beta P_{e,s,n+1} + (1 - \beta) \prod_{\ell=0}^{m-1} P_{w,s,\ell,n+1},$$

as in (9), so that,

$$\begin{align*}
P_{w,s,n+1|n} &:= \frac{P_{w,s,n+1}}{P_{w,s,n}} \\
&= \frac{\beta P_{e,s,n+1} + (1 - \beta) \prod_{\ell=0}^{m-1} P_{w,s,\ell,n+1}}{P_{w,s,n}}. \\
&= \left(\frac{\beta \zeta_{s,n}}{(1 - \beta + \beta \zeta_{s,n})} \right) P_{e,s,n+1|n} \\
&\quad + \left(1 - \beta \right) \left(1 - \beta + \beta \zeta_{s,n}\right) P_{w,s,a,n+1|n}, \quad \text{(24)}
\end{align*}$$

by the definition of $\zeta_{s,n}$ in Lemma 4.3.

Now, in order to estimate the posterior predictive probability,

$$P_D^e(j|x_{n,D+1}^n) = P_D^e((x_1, \ldots, x_n, j)|x_{n,D+1}^n) = \frac{P_D(x_n^D)}{P_D^e(x_{n,D+1}^n)} = \frac{P_{w,\lambda,n+1}}{P_{w,\lambda,n}} = P_{w,\lambda,n+1|n},$$

we observe that this can be done recursively, starting from the leaf $s_D$ where $P_{w,s_{D,n+1}|n} = P_{e,s_{D,n+1|n}}$, and then proceeding through $s_{D-1}, \ldots, s_1$ all the way to the root $\lambda = s_0$ via successive applications of (25), until $P_{w,\lambda,n+1|n}$ is expressed as a linear combination of the conditional probabilities, $P_{e,s,a,n+1|n}$, $t = D, D-1, \ldots, 0$. It is easy to see the coefficient of $P_{e,s,a,n+1|n}$ in this linear combination is,

$$\begin{align*}
0 \prod_{t=D-1}^{0} \frac{1 - \beta}{(1 - \beta + \beta \zeta_{s,n})}.
\end{align*} \quad \text{(26)}$$
while for all internal contexts \( s_T, T = D - 1, \ldots, 0 \), the coefficient of \( P_{e,sT,n+1}^{(n)} \) is,

\[
\frac{\beta_{s_T,t}}{(1 - \beta) + \beta_{s_T,n}^{(n)}} \prod_{t=0}^{T-1} \frac{1 - \beta}{1 - (1 - \beta) + \beta_{s_t,n}} = \frac{\beta_{s_T,t}}{(1 - \beta)} \prod_{t=T}^{0} \frac{1 - \beta}{1 - (1 - \beta) + \beta_{s_t,n}}. \tag{27}
\]

By Lemma 4.3, the coefficients of the form (26) tend to one while the coefficients of the form (27) tend to zero, therefore, a.s. as \( n \to \infty \),

\[
P_D^*(j|x_{-D+1}) - P_{e,sD,n+1}^{(n)}\to \frac{a_{sD,n+1}(j) - 1/2}{m/2 + M_{sD,n+1}} - 1 \to 0.
\]

Moreover, the ergodic theorem for Markov chains [10], [31] implies that for any context \( s \) of length \( D \), a.s., as \( n \to \infty \),

\[
a_{s,n}(j) = \frac{a_{s,n}(j)}{M_{s,n}} \to \frac{\pi(s)}{\pi(s)} = \theta_s(j), \tag{28}
\]

where with a slight abuse of notation we write \( \pi(s) \) for the probability assigned by the stationary distribution of the chain to the string corresponding to a context \( s \). The proof is completed upon noting that, since there are only finitely many contexts \( s \) of length no more than \( D \), the convergence in (28) occurs uniformly in \( s \).

**Proof of Theorem 3.5:** Consider an alternative model \( T \in T(D) \), different from the true minimal model \( T^* \). We will show that the posterior probability of \( T \) will be strictly smaller than that of \( T^* \), eventually almost surely (a.s.), as the size \( n \) of the sample increases. Since there are only finitely many models in \( T(D) \), this suffices to prove the theorem.

We consider two cases, which are not necessarily mutually exclusive: Since \( T \neq T^* \), either there is a leaf \( t \in T \) such that the collection \( T^*(t) \) of contexts in \( T^* \) that are descendants of \( t \) in \( T \) is nonempty; or there is a \( t \in T^* \) such that the collection \( T(t) \) of contexts in \( T \) that are descendants of \( t \) in \( T^* \) is nonempty.

**Case 1.** Let \( t \in T \) be at level \( d < D \), and such that \( T^*(t) \neq \Lambda \). We will show that the posterior of the union \( T \cup T^*(t) \), which consists of \( T \) together with the subtree \( T^*(t) \) starting at \( t \), satisfies, \( \pi(T \cup T^*(t)|x) > \pi^*(T|x) \), that is, \( P(x|T \cup T^*(t)) \pi_D(T) > P(x|T) \pi_D(T^*) \), or equivalently, using Lemma 2.1,

\[
\sum_{s \in T \cup T^*(t)} \log P_e(a_s) + \log \pi_D(T \cup T^*(t)) > \sum_{s \in T} \log P_e(a_s) + \log \pi_D(T),
\]

eventually, a.s., which, using Lemma 4.1 is,

\[
\sum_{s \in T^*(t)} \log P_e(a_s) - \log P_e(a_t) + \pi_D - \log \beta > 0,
\]

eventually, a.s., or, equivalently,

\[
\sum_{s \in T^*(t)} \log \left( \frac{P_e(a_s)}{\prod_j \theta_s(j)^{a_s(j)}} \right) - \log \left( \frac{P_e(a_t)}{\prod_j \theta_t(j)^{a_t(j)}} \right) > \sum_{s \in T^*(t)} \pi(s) \log \theta_s(j).
\]

Using (22) of Lemma 4.2 as in the proof of Theorem 3.1, the first term above is bounded below by,

\[
-\sum_{s \in T^*(t)} \left( \frac{m-1}{2} \log M_s - \log m \right).
\]

Similarly, using (23) of Lemma 4.2, the second term above is bounded below by,

\[
\frac{m-1}{2} \log M_t + \log \frac{m/2}{\Gamma(m/2)}.
\]

And writing \( \hat{p}_t(j) = a_t(j)/M_t \) as in the proof of Proposition 2.2, the third term above equals,

\[
M_t \log \hat{p}_t + \sum_{s \in T^*(t)} \sum_j a_s(j) \log \theta_s(j).
\]

In fact, we will prove the stronger fact that, almost surely,

\[
\liminf_n \left\{ \frac{m-1}{2n} \log M_t - \frac{m-1}{2n} \sum_{s \in T^*(t)} \log M_s + \frac{M_t}{n} \log \hat{p}_t + \frac{1}{n} \sum_{s \in T^*(t)} \sum_j a_s(j) \log \theta_s(j) \right\} > 0.
\]

Since \( M_t \) is always no greater than \( n \), the first term goes to zero as \( n \to \infty \), and using Jensen’s inequality as in the proof of Theorem 3.1 we also have that the second term is bounded below by \( |T^*(t)|(m-1)/(2n) \log n/T^*(t)) = O((\log n)/n) \), which also goes to zero a.s. as \( n \to \infty \). Therefore, it suffices to show that, almost surely,

\[
\liminf_{n \to \infty} \left\{ \frac{M_t}{n} \log \hat{p}_t + \sum_{s \in T^*(t)} \frac{M_s}{n} \sum_j a_s(j) \log \theta_s(j) \right\} > 0.
\]

Using the same notation as in equation (28), in the proof of Theorem 3.4, the ergodic theorem [10], [31] implies that \( M_s/n \to \pi(s) \) and \( a_s(j)/M_s \to \theta_s(j) \), a.s., as \( n \to \infty \), so that the above \( \liminf \) is actually a limit, which equals,

\[
\pi(t) H(\theta_t) - \sum_{s \in T^*(t)} \pi(s) \theta_s = H(\theta_t). \tag{29}
\]

And the strict concavity of the entropy implies that,

\[
\sum_{s \in T^*(t)} \pi(s) H(\theta_s) \leq H \left( \sum_{s \in T^*(t)} \frac{\pi(s)}{\pi(t)} \theta_s \right) = H(\theta_t),
\]

with equality only if all \( \theta_s \) are equal, which is ruled out by the assumption that the model \( T^* \) is minimal. This implies that the difference in (29) is strictly positive and completes this case.
Case 2: Let \( t \in T^* \) be a leaf at level \( d \geq 1 \) such that the collection \( T(t) \) of contexts in \( T \) that are descendants of \( t \in T^* \) is nonempty, so that \( T \) can be expressed as the union \( T^p \cup T(t) \) of the tree \( T^p \) which is \( T \) pruned at \( t \) and \( T(t) \). Since \( \pi(t) > 0 \) by assumption, we must have that \( \pi(s) > 0 \) for at least one \( s \in T(t) \). If this holds for exactly only one \( s \in T(t) \), then a simple calculation shows that \( \pi(T|x) = \pi(T^p|x) \) and \( \pi(T|x) \) is \( T \) pruned at \( t \), and we will show that \( \pi(T|x) = \pi(T^p \cup T(t)|x) \), i.e., that \( P(x|T^p)\pi_D(T^p) > P(x|T^p \cup T(t))\pi_D(T(t)) \). As in Case 1, using Lemmas 2.1 and 4.1, this is easily seen to be the same as,

\[
\sum_{s \in T(t)} \log P_e(a_s) + \log P_e(a_t) - \log \pi_D - d(T(t)) + \log \beta > 0,
\]

eventually, a.s., or, equivalently,

\[
\sum_{s \in T(t)} \log \left( \frac{\prod_j (a_s(j)/M_s)^{a_s(j)}}{P_e(a_t)} \right) + \log \left( \frac{P_e(a_t)}{\prod_j \pi_D(j)^{a_s(j)}} \right)
\]

\[
- \log \left( \frac{\prod_j (a_s(j)/M_s)^{a_s(j)}}{\prod_j \pi_D(j)^{a_s(j)}} \right)
\]

\[
- \log \pi_D - d(T(t)) + \log \beta > 0,
\]

eventually, a.s.

Note that in the sums and products over \( s \in T(t) \), we can (and do) restrict attention to only those \( s \) with \( \pi(s) > 0 \). Again, using (22) of Lemma 4.2 like in the proof of Theorem 3.1, the second term above is bounded below by,

\[
- \frac{m - 1}{2} \log M_t - \log m,
\]

similarly, using (23) of Lemma 4.2, the first term is bounded below by,

\[
\sum_{s \in T(t)} \left( - \frac{m - 1}{2} \log \frac{M_s}{2\pi} + \log \frac{\pi^{m/2}}{\Gamma(m/2)} \right),
\]

and noting that \( a_s(j) = \sum_{s \in T(t)} a_s(j) \) for all \( j \), the third term is actually equal to,

\[
- \sum_{s \in T(t)} M_s \sum_j a_s(j) \log \frac{a_s(j)}{M_s} + \sum_j a_t(j) \log \theta_t(j)
\]

\[
= - \sum_{s \in T(t)} M_s D(\hat{p}_s||\theta_t),
\]

where, as before, \( \hat{p}_s(j) = a_s(j)/M_s \). Combining the above expressions, it suffices to show that, eventually a.s.,

\[
\sum_{s \in T(t)} M_s D(\hat{p}_s||\theta_t),
\]

is strictly bounded above by,

\[
- \frac{m - 1}{2} \left( \sum_{s \in T(t)} \log M_s - \log M_t \right)
\]

\[
+ |T(t)| \log \frac{\pi^{1/2}}{2^{(m-1)/2} \Gamma(m/2)}
\]

\[
- \log \pi_D - d(T(t)) + \log \beta,
\]

and since the last three terms above are uniformly bounded in \( n \), it suffices to show that,

\[
\limsup_{n \to \infty} \left\{ \frac{1}{\log n} \sum_{s \in T(t)} M_s D(\hat{p}_s||\theta_t)
\right\}
\]

\[
- \frac{m - 1}{2} \log n \left( \sum_{s \in T(t)} \log M_s - \log M_t \right) < 0, \text{ a.s. (30)}
\]

For the first term above we have,

\[
\sum_{s \in T(t)} M_s D(\hat{p}_s||\theta_t) \leq a \sum_{s \in T(t)} M_s \sum_j (\hat{p}_s(j) - \theta_t(j))^2
\]

\[
= \sum_{s \in T(t)} \frac{1}{M_s} \sum_j (a_s(j) - \theta_t(j)M_s)^2
\]

\[
= \sum_{s \in T(t)} \frac{1}{M_s} \sum_j \left( (a_s(j) - n\pi(s)) - \theta_t(j)(M_s - n\pi(s))^2 \right)
\]

\[
= (\log n) \sum_{s \in T(t)} \frac{1}{M_s} \sum_j \left[ \frac{a_s(j) - n\pi(s)}{\sqrt{n \log n}} \right]^2
\]

\[
- \theta_t(j)(M_s - n\pi(s))^2 \right],
\]

where \( a \) follows from the well-known bound for the relative entropy in terms of the \( \chi^2 \) distance [16], and \( b \) follows from the fact that, since \( T^* \) is minimal and \( s \) is a descendant of \( t \in T^* \), we have \( \pi(s)/\pi(t) = \theta_t(j) \). By the law of the iterated logarithm [10] applied to the chain \( \{ Z_n = X_n^n \}_{n \geq 1} \), each of the two fractions in the above square brackets is \( O(1) \) a.s., and by the ergodic theorem so is \( n/M_s \), so that the entire expression, as \( n \to \infty \),

\[
\sum_{s \in T(t)} M_s D(\hat{p}_s||\theta_t) = O(\log n), \text{ a.s. (31)}
\]

Moreover, the second term in (30) equals,

\[
- \frac{m - 1}{2} \frac{1}{\log n} \left( \sum_{s \in T(t)} \log \frac{M_s}{n} - \log \frac{M_t}{n} + |T(t)| - 1 \right) \log n \right),
\]

where \( |T(t)| \geq 2 \) denotes the number of \( s \in T(t) \) such that \( \pi(s) > 0 \). Since, by the ergodic theorem, \( M_s/n \to \pi(s) \) and \( M_t/n \to \pi(t) \), a.s., as \( n \to \infty \), the above expression converges a.s. to,

\[
\frac{(m - 1)(|T(t)| - 1)}{2} > 0.
\]

Combining this with (31) shows that (30) holds, completing the proof of this case and proving the theorem.

In order to prove Theorem 3.6 we need the simple asymptotic result of Proposition 4.4, which can be seen as a version of the Shannon-McMillan-Breiman theorem in this setting; see [1] and [11]. For an ergodic chain \( \{ X_n \} \) with minimal model \( T^* \), parameters \( \theta^* \) and stationary distribution \( \pi \) we will use the following notation. As before, for any context
of Theorem 3.4 above, that and they are where the (possibly random) constants

\[ T \]

Finally, suppose that

\[ \theta(s) \]

are leaves of

\[ T \]

are proper subtrees of

\[ T \]

This will be seen to be a simple consequence of the

\[ \bar{H}(X|T^*) \]

and using the log-sum inequality [11],

\[ \bar{H}(X|T^*) \]

Now, from the definitions we obviously have

\[ \bar{H}(X|T^*) \]

Finally, we would have equality in (a) only if for all

\[ \theta(s) \]

independent of

\[ s \]

So, the inequality is necessarily strict.

Proof of Theorem 3.6. We will show that

\[ \pi(T|X_n^0) \rightarrow 0 \quad a.s., \forall T \in T(D), T \neq T^* \]

Case 1. First, suppose that there is an internal node

\[ t \]

which is a leaf of

\[ T \]

By successively operations of adding and/or removing subtrees from

\[ T \]

as in the two cases considered in the proof of Theorem 3.5, we see that the tree

\[ T' \]

is exactly the same as

\[ T^* \]

but pruned at

\[ t \]

has posterior probability greater than

\[ T \]

eventually a.s. So we can assume, without loss of generality, that

\[ T \]

is of that form.

Then, the posterior probability of

\[ T \]

can easily be bounded above,

\[ \pi(T|X_n^0) \]

so that, using Proposition 4.4, we have a.s. as

\[ n \rightarrow \infty, \]

and recalling, as noted in equation (28) in the proof of

Theorem 3.4 above, that

\[ M_s/n \rightarrow \pi(s) \]

and

\[ a_s(j)/M_s \rightarrow \theta_s(j) \]

as in

\[ s \]

and

\[ j \]

the asymptotic result follows.

The nonnegativity of

\[ \bar{H}(X|T) \]

is obvious from its definition. Finally, suppose that

\[ T \]

is a proper subtree of

\[ T^* \]

for each

\[ t \]

let

\[ T^*(t) \]

denote the collection of descendants of

\[ t \]

that are leaves of

\[ T^* \]

or

\[ T^*(t) = \{ t \} \]

if

\[ t \]

is a leaf of both

\[ T \]

and

\[ T^* \]

We can write,

\[ \bar{H}(X|T^*) \]

\[ \bar{H}(X|T^*) \]

which is strictly negative because of our assumption that

\[ T \]

is a proper subtree of

\[ T^* \]

Therefore,

\[ \pi(T|X_n^0) \rightarrow 0 \quad a.s., \]

exponentially fast as

\[ n \rightarrow \infty. \]

Case 2: Alternatively, if no internal node

\[ t \]

is a leaf of

\[ T \]

then

\[ T^* \]

is a proper subtree of

\[ T \]

Again by successively repeating the pruning operation as in Case 2 in the proof of

Theorem 3.5, which increases the posterior probability of

\[ T \]

eventually a.s., we can assume without loss of generality that

\[ T \]

consists of exactly

\[ T^* \]

together with

\[ m \]

additional leaves

\[ \{ tk : k \in A \} \]

stemming from a specific
t

\[ t \]

in

\[ T^* \]

Then proceeding as in Case 1 above we have,

\[ \log \pi(T(X_n^0) \]

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by the ergodic theorem we know that
and by exactly the same argument as the one that led to (31)
and using Lemma 2.1 and the bounds from Lemma 4.2,
\[
\log \pi(T|X^n_{-D+1}) \\
\leq \sum_{s \in T: M_s \neq 0} \left[ \sum_{j=0}^{m-1} a_s(j) \log \frac{a_s(j)}{M_s} - \frac{m-1}{2} \log M_s - C_1 \right] \\
+ \sum_{s \in T^*: M_s \neq 0} \left[ \sum_{j=0}^{m-1} a_s(j) \log \frac{a_s(j)}{M_s} - \frac{m-1}{2} \log M_s - C_2 \right] + C_3,
\]
where \( C_1 = \log \left( \frac{\sqrt{2\pi}}{2^m m!} \right) \), \( C_2 = \log m \) and \( C_3 = \log \left( \frac{\pi(T)}{\pi(T^*)} \right) \). Since \( t \in T^* \) and we assume that \( \pi(t) > 0 \), by the ergodic theorem we know that \( M_t \neq 0 \) eventually a.s., therefore, we have,
\[
\log \pi(T|X^n_{-D+1}) \\
\leq \sum_{k \in A: M_{tk} \neq 0} \left[ \sum_{j=0}^{m-1} a_{tk}(j) \log \frac{a_{tk}(j)}{M_{tk}} - \frac{m-1}{2} \log M_{tk} \right] \\
- \sum_{j=0}^{m-1} a_{t}(j) \log \frac{a_{t}(j)}{M_{t}} - \frac{m-1}{2} \log M_{t} + C_4 \\
\leq \sum_{k \in A: M_{tk} \neq 0} \left[ \sum_{j=0}^{m-1} a_{tk}(j) \log \frac{a_{tk}(j)}{M_{tk}} \right] \\
- \sum_{j=0}^{m-1} a_{t}(j) \log \theta^*_t(j) \\
- \frac{m-1}{2} \left( \sum_{k \in A: M_{tk} \neq 0} \log M_{tk} - \log M_{t} \right) + C_4,
\]
eventually a.s., for a finite constant \( C_4 \), and where we used, as in the proofs of Theorems 3.1 and 3.5 the fact that the empirical frequencies \( \hat{p}_{tk}(j) = a_{tk}(j)/M_{tk} \) maximise the likelihood. Now recalling that \( \sum_k a_{tk}(j) = a_t(j) \), we have, eventually a.s.,
\[
\log \pi(T|X^n_{-D+1}) \\
\leq \sum_{k \in A: M_{tk} \neq 0} M_{tk} D(\hat{p}_{tk} || \theta^*_t) \\
- \frac{m-1}{2} \left( \sum_{k \in A: M_{tk} \neq 0} \log M_{tk} - \log M_{t} \right) + C_4,
\]
and by exactly the same argument as the one that led to (31) in the proof of Theorem 3.5,
\[
\log \pi(T|X^n_{-D+1}) \\
\leq -\frac{m-1}{2} \left( \sum_{k \in A: M_{tk} \neq 0} \log M_{tk} - \log M_{t} \right) + O(\log \log n),
\]
almost surely. Finally, again by the same argument that led to (32) and (33) in the proof of Theorem 3.5, we have, that
\[
\log \pi(T|X^n_{-D+1}) \leq -C_5 \log n + o(\log n), \text{ a.s.,}
\]
which implies that \( \log \pi(T|X^n_{-D+1}) \to -\infty \) and hence \( \pi(T|X^n_{-D+1}) \to 0 \) a.s., as \( n \to \infty \), completing the proof. 

Although superficially somewhat technical, Theorems 3.7 and 3.8 proved next are simple consequences of the exact form (11) of the full conditional density of \( \theta \) given \( x^n_{-D+1} \) and \( T \), combined with Theorem 3.6 and with some simple convergence properties of the Dirichlet distribution [15].

**Proof of Theorem 3.7:** Let \( x = x_{x_{-D+1}} \) be a semi-infinite sample realisation. For each \( n \), the posterior distribution \( \pi(\theta, T|x^n_{-D+1}) \) can formally be described as probability measure \( \mu \) on the space \( S \) consisting of elements \( (\theta, T) \), where \( T \in T(D) \) and \( \theta = \{\theta_s; s \in T\} \) with each \( \theta_s \in [0,1]^m \). We endow \( S \) with the \( \sigma \)-algebra \( \mathcal{F} \) consisting of all sets \( S \) of the form,
\[
S = \bigcup_{T \in T(D)} (B_T \times \{T\}), \quad B_T \in \mathcal{B}^{[m]}[T],
\]
where each \( \mathcal{B}^{[m]}[T] \) denotes the Borel \( \sigma \)-algebra of \([0,1]^{m}\). Then the probability of any such \( S \) can be decomposed as,
\[
\mu(S) = \sum_{T \in T(D)} \pi(T|x^n_{-D+1}) \pi(B_T|T, x^n_{-D+1}),
\]
and from Theorem 3.6 we know that, for almost all realisations \( x \), \( \pi(T|x^n_{-D+1}) \) asymptotically concentrates on \( T^* \), so that,
\[
\lim \frac{1}{n} \mu(S) = \lim \frac{1}{n} \mu(B_T^*|T^*, x^n_{-D+1}).
\]
Therefore, writing \( \theta^{(n)} \) for a random vector with distribution \( \pi(\cdot|x^n_{-D+1}, T^*) \), in order to establish the required result it suffices to show that, for any \( B \in \mathcal{B}^{[m]}[T] \) and almost all \( x \),
\[
\lim \frac{1}{n} \mathbb{P}\left( \theta^{(n)} \in B \big| x^n_{-D+1}, T^* \right) = \mathbb{I}\{\theta^* \in B\},
\]
or, equivalently, that \( \theta^{(n)} \) converges in probability to \( \theta^* \), for almost all \( x \), where \( \mathbb{I}\{\cdot\} \) denotes the indicator function of the event \( \{\cdot\} \).

Given the sample string \( x^n_{-D+1} \) up to time \( n \), as in the proof of Theorem 3.4 we write \( a_{s,n} \) and \( M_{s,n} \) for the induced count vectors and \( \hat{p}_{s,n}(j) = a_{s,n}(j)/M_{s,n}, j \in A \) for the corresponding empirical frequencies corresponding to each context \( s \in T^* \). By the same reasoning as in equation (28) earlier, we have that \( \hat{p}_{s,n} \to \theta^*_s(j) \) and \( M_{s,n}/n \to \pi(s) \) a.s., as \( n \to \infty \), for each \( s \in T^* \). Let \( A \) denote the set of all realisations \( x \) such that the result of Theorem 3.6 as well as all the above asymptotics hold, so that \( A \) has probability 1.

Choose and fix any one of the (almost all) realisations \( x = x^n_{-D+1} \in A \) for the remainder of the proof. As noted in equation (11), the distribution \( \pi(\cdot|x^n_{-D+1}, T^*) \) of \( \theta^{(n)} \) has a density \( f_n(\theta) \) with respect to Lebesgue measure, given by the product,
\[
f_n(\theta) = \prod_{s \in T^*} f_{n,s}(\theta_s), \quad (34)
\]
where each \( f_{n,s} \) denotes the Dir(\( a_s(0) + 1/2, \ldots, a_s(m-1) + 1/2 \)) density, so that, in particular, it is easy to compute the
corresponding means,
\begin{equation}
\tilde{\theta}_n^{(s)}(j) := \mathbb{E} \left( \theta_n^{(s)}(j) | x_{D + 1}^n, T^s \right) = \alpha_{s,n}(j) + 1/2 \frac{1}{M_s + m/2} \rightarrow \theta^*_s(j), \quad \text{as } n \rightarrow \infty,
\end{equation}
and variances,
\begin{equation}
\text{Var} \left( \theta_n^{(s)}(j) \right) | x_{D + 1}^n, T^s = \frac{(\alpha_{s,n}(j) + 1/2)(M_s - \alpha_{s,n}(j) + (m - 1)/2)}{(M_s + m/2)^2(M_s + m/2 + 1)} \rightarrow 0,
\end{equation}
as \( n \rightarrow \infty \). Then a simple application of Chebyshev’s inequality implies that \( \theta^{(n)} \) converges in probability to \( \theta^* \), completing the proof. \( \Box \)

**Proof of Theorem 3.8.** We follow the same reasoning and adopt the same notation as in the first part of the proof of Theorem 3.7, and again we choose and fix an arbitrary \( x = x_{D + 1}^\infty \in \mathcal{A} \). Then, for each \( n \), the density \( f_n(\theta) \) of \( \pi(\theta|x_{D + 1}^n, T^s) \) is given by the product of \( \theta^{(n)} \), and the claim \( (18) \) has already been established in \( (35) \).

In order to establish the asymptotic normality of \( \theta^{(n)} \), for each \( s \in T^* \), let \( \phi_j(.) \) denote the \( N(0, J_s) \) density on \( \mathbb{R}^m \), with \( J_s \) defined in \( (19) \). Since the collection of all sets of the form,
\begin{equation}
\prod_{s \in T^*} \left\{ [0, x_s(0)] \times [0, x_s(1)] \times \cdots \times [0, x_s(m - 1)] \right\},
\end{equation}
for \( x_s(j) \in [0, 1] \), \( s \in T^* \), \( j \in A \), form a \( \pi \)-system for the Borel \( \sigma \)-algebra of \( [0, 1]^{|T^*|} \), and also since for each \( n \) the components \( \theta^{(n)}_s \) of \( \theta^{(n)} \) for different \( s \in T^* \) are independent, in order to prove the theorem it suffices to show \( [6] \) that for each \( s \in T^* \),

\begin{equation}
\frac{1}{\sqrt{n}} \tilde{f}_{s,n} \left( \frac{z}{\sqrt{n}} + \tilde{\theta}^{(n)}_s \right) \rightarrow \phi_{J_s}(z), \quad \text{as } n \rightarrow \infty,
\end{equation}
where the convergence is uniform on compact subsets of \( \mathbb{R}^m \).

From Theorems 4.2 and 4.3 of \( (15) \) we have that, uniformly on compact sets,

\begin{equation}
\frac{1}{\sqrt{\nu_{s,n}}} \tilde{f}_{s,n} \left( \frac{z}{\sqrt{\nu_{s,n}}} + \tilde{\theta}^{(n)}_s \right) \rightarrow \phi_{I_s}(z), \quad \text{as } n \rightarrow \infty,
\end{equation}
where \( \nu_{s,n} = M_s + m/2 \),
\begin{equation}
\tilde{\theta}^{(n)}_s(j) = \frac{\alpha_{s,n}(j) - 1/2}{M_s - m/2}, \quad j \in A,
\end{equation}
and \( I_s = \Theta^* - (\tilde{\theta}^*_s)^{(n)} \). But from our assumptions we have that \( I_s = \Theta^* - (\tilde{\theta}^*_s)^{(n)} \). Then, as \( n \rightarrow \infty \), \( \nu_{s,n}/n \rightarrow \pi(s) \) and \( \tilde{\theta}^{(n)}_s \rightarrow 0 \). These together with \( (37) \) and the continuous mapping theorem \( [6] \) imply \( (36) \) as required. \( \Box \)

**Proof of Theorem 3.10.** The proof follows roughly along the same lines as the one for the special case of binary data and \( \beta = 1/2 \) given in \( [55] \) and \( [57] \), which in turn is a generalisation of Shtrikov’s original argument in \( [44] \).

For each \( n \), each string \( x_n^T \), each initial context \( x_{D + 1}^n \), and any \( T \in T(D) \), we denote by \( \Sigma(x_{D + 1}^n, x_n^T) \) the expression,
\begin{equation}
\Sigma(x_{D + 1}^n, x_1^n, T) = \sum_{s \in T: M_s \neq 0} \left( \frac{m - 1}{2} \log \frac{M_s}{2\pi} + \log \left( \frac{\pi_m/2}{\Gamma(m/2)} \right) \right) 
\end{equation}
\begin{equation}
\sum_{s \in T: M_s \neq 0} \left( \frac{m - 1}{2} \log M_s + \log \left( \frac{\pi_m/2}{\Gamma(m/2)} \right) \right),
\end{equation}
where \( M_s \) are the sums of the count vectors \( a_s \) corresponding to \( x_{D + 1}^n \), and we define a (conditional) probability measure \( \mu \) on \( \mathcal{A}^n \) as,
\begin{align*}
\mu(x_1^n | x_{D + 1}^0) &= \frac{1}{Z(x_{D + 1}^0)} \\
&\times \max_{T \in T(D)} \sup_{\phi \in \Theta(T,m)} \left[ \frac{P(y_1^n | x_{D + 1}^0, \theta(\phi), T)}{\exp \{ \Sigma(T) - \log \pi_D(T; \beta) \}} \right].
\end{align*}
where \( \Sigma(T) = \Sigma(x_{D + 1}^0, x_1^n, T) \) and \( Z(x_{D + 1}^0) \) is simply the normalising constant.

\begin{equation}
\Sigma(x_{D + 1}^0) = \sum_{y_1^n \in \mathcal{A}^n} \sup_{T \in T(D)} \left[ \frac{\max_{\phi \in \Theta(T,m)} \left\{ \sum_{x_{D + 1}^0, y_1^n} \pi(\phi, T) \right\}}{\exp \{ \Sigma(x_{D + 1}^0, y_1^n, T) - \log \pi_D(T; \beta) \}} \right].
\end{equation}

As we saw in Proposition 2.2, the supremum in the numerator above is achieved by the choice of parameters \( \tilde{\theta}_s = a_s/M_s \), for all \( s \in T \), so that,

\begin{align*}
\sup_{\phi \in \Theta(T,m)} &\left\{ \sum_{s \in T: M_s \neq 0} \left( \sum_{j \in A} \left( \frac{a_s(j)}{M_s} \right) a_s(j) \right) \right\} \\
&\sum_{y_1^n \in \mathcal{A}^n} \max_{T \in T(D)} \left( \sum_{x_{D + 1}^0, y_1^n} \frac{\max_{\phi \in \Theta(T,m)} \left\{ \sum_{x_{D + 1}^0, y_1^n} \pi(\phi, T) \right\}}{\exp \{ \Sigma(x_{D + 1}^0, y_1^n, T) - \log \pi_D(T; \beta) \}} \right).
\end{align*}

Further, using the bound \( (23) \) in Lemma 4.2, and the expression for the marginal likelihood in terms of the count vectors \( a_s \) in Lemma 2.1, we have,

\begin{equation}
Z(x_{D + 1}^0) \geq \sum_{y_1^n \in \mathcal{A}^n} \max_{T \in T(D)} \left( \sum_{s \in T: M_s \neq 0} \log P_e(a_s) + \log \pi_D(T; \beta) \right)
\end{equation}
\begin{equation}
\sum_{y_1^n \in \mathcal{A}^n} \max_{T \in T(D)} \left[ P(y_1^n | x_{D + 1}^0, T) \pi_D(T; \beta) \right].
\end{equation}

Now, by the definition of \( \mu \), and noting that a likelihood ratio cannot be uniformly smaller than 1, after some simple algebra we have the expression in \( (39) \), shown at the top of the next page. Therefore, in view of \( (38) \), in order to prove the theorem it suffices to show that the limit \( \lim_{n \rightarrow \infty} \) of
\begin{equation}
\log \left( \sum_{y_1^n \in \mathcal{A}^n} \max_{T \in T(D)} \left[ P(y_1^n | x_{D + 1}^0, T) \pi_D(T; \beta) \right] \right).
\end{equation}
is nonnegative. To that end we observe that, replacing the
maximum over $T$ by the expectation with respect to the
posterior of $T$, the above logarithm is,

$$
\log \left( \sum_{y_1^n \in A^n} \max_{T \in \mathcal{D}} P(y_1^n, T|x_{D+1}^0) \right) \\
\geq \log \left( \sum_{y_1^n \in A^n} \sum_{T \in \mathcal{D}} \pi(T|y_1^n, x_{D+1}^0) P(y_1^n, T|x_{D+1}^0) \right) \\
\geq \sum_{y_1^n \in A^n} \sum_{T \in \mathcal{D}} P(y_1^n, T|x_{D+1}^0) \log \pi(T|y_1^n, x_{D+1}^0),
$$

where the second inequality follows from Jensen’s inequality.

But the last term above can be seen to equal the negative of the
conditional entropy $-H(T|Y_1^n, x_{D+1}^0)$, where we recall that, for three discrete random variables $X, Y$ and $Z$, the conditional entropy of $X$ given $Y$ and $Z = z$ is defined, in the obvious notation, as,

$$H(X|Y, Z = z) = \sum_{x, y, z} P_{X,Y,Z}(x, y, z) \log P_{X|Y,Z}(x|y, z).$$

Now, since the MAP model $T^*(n)$ is a function of $Y_1^n, x_{D+1}^0$, by the data processing property of conditional entropy [11],

$$H(T|Y_1^n, x_{D+1}^0) = H(T|T^*(n), Y_1^n, x_{D+1}^0) \leq H(T|T^*(n), x_{D+1}^0),$$

where the inequality follows from the fact that conditioning reduces the entropy [11].

Now let $P_{e,n}$ denote the probability $P(T^*(n) \neq T|x_{D+1}^0)$, and note that it tends to zero by Theorem 3.5 and dominated convergence. Then, by Fano’s inequality [11],

$$H(T|Y_1^n, x_{D+1}^0) \leq H(T|T^*(n), x_{D+1}^0) \leq h(p) + P_{e,n} \log |\mathcal{D}|,$$

where $h(p) := -p \log p - (1-p) \log (1-p), p \in (0,1)$, denotes the binary entropy function. And letting $n \to \infty$ we have that,

$$\lim_{n \to \infty} \inf \log \left( \sum_{y_1^n \in A^n} \max_{T \in \mathcal{D}} P(y_1^n, T|x_{D+1}^0) \right) \\
\geq \lim_{n \to \infty} \inf \left[ -H(T|Y_1^n, x_{D+1}^0) \right] \\
\geq \lim_{n \to \infty} \left[ -h(P_{e,n}) - P_{e,n} \log |\mathcal{D}| \right] = 0,$$

establishing the required asymptotic bound on (40) and completing the proof. □
