SEQUENCE SPACES OF FUZZY REAL NUMBERS
DEFINED BY ORLICZ FUNCTIONS

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ABSTRACT. In this article we study different properties of convergent, null and bounded sequence spaces of fuzzy real numbers defined by an Orlicz function, like completeness, solidness, symmetricity, convergence free etc. We prove some inclusion results, too.

1. Introduction

An Orlicz function $M$ is mapping $M: [0, \infty) \to [0, \infty)$ such that it is continuous, non-decreasing and convex with $M(0) = 0$, $M(x) > 0$ for $x > 0$ and $M(x) \to \infty$, as $x \to \infty$. If the convexity of Orlicz function $M$ is replaced by

$$M(x + y) \leq M(x) + M(y),$$

then this function is called the modulus function, introduced by Nakanouchi [6].

Remark 1. It is well known if $M$ is an Orlicz function, then $M(\lambda x) \leq \lambda M(x)$ for all $\lambda$ with $0 < \lambda < 1$.

Lindenstrauss and Tzafriri [7] used the idea of Orlicz function to construct the sequence space,

$$\ell_M = \left\{ (x_k) \in w : \sum_{k=1}^{\infty} M \left( \frac{|x_k|}{\rho} \right) < \infty, \text{ for some } \rho > 0 \right\}.$$
The space $\ell_M$ becomes a Banach space, with the norm

$$
\|x\| = \inf \left\{ \rho > 0 : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) \leq 1 \right\},
$$

which is called an Orlicz sequence space.

The space $\ell_M$ is closely related to the space $\ell^p$, which is an Orlicz sequence space with $M(x) = |x|^p$, for $1 \leq p < \infty$.

Later on Orlicz sequence spaces were investigated by Paras harass and Choudhary [8], Tripathy et al. [13], Savas and Rhoades [9] and many others.

Let $D$ denote the set of all closed and bounded intervals $X = [a_1, a_2]$ on $\mathbb{R}$, the real line. For $X, Y \in D$ we define

$$
\delta(X, Y) = \max(|a_1 - b_1|, |a_2 - b_2|),
$$

where $X = [a_1, a_2]$ and $Y = [b_1, b_2]$. It is known that $(D, \delta)$ is a complete metric space.

A fuzzy real number $X$ is a fuzzy set on $\mathbb{R}$, i.e. a mapping $X : \mathbb{R} \to [0,1]$ associating each real number $t$ with its grade of membership $X(t)$.

A fuzzy real number $X$ is called convex if

$$
X(t) \geq X(s) \wedge X(r) = \min(X(s), X(r)),
$$

where $s < t < r$.

If there exists $t_0 \in \mathbb{R}$ such that $X(t_0) = 1$, then the fuzzy real number $X$ is called normal.

A fuzzy real number $X$ is said to be upper-semi continuous if, for each $\varepsilon > 0$, $X^{-1}([0, a + \varepsilon))$, for all $a \in I$ is open in the usual topology of $\mathbb{R}$.

The set of all upper-semi continuous, normal, convex fuzzy real numbers is denoted by $R(I)$ and throughout the article, by a fuzzy real number we mean that the number belongs to $R(I)$.

The $\alpha$-level set $[X]^\alpha$ of the fuzzy real number $X$, for $0 < \alpha \leq 1$, is defined as $[X]^\alpha = \{ t \in \mathbb{R} : X(t) \geq \alpha \}$. If $\alpha = 0$, then it is the closure of the strong 0-cut.

The set $\mathbb{R}$ of all real numbers can be embedded in $R(I)$. For $r \in \mathbb{R}$, $r \in R(I)$ is defined by

$$
\tau(t) = \begin{cases} 
1, & \text{for } t = r, \\
0, & \text{for } t \neq r.
\end{cases}
$$

The arithmetic operations for $\alpha$-level sets are defined as follows:

Let $X, Y \in R(I)$ and the $\alpha$-level sets be $[X]^\alpha = [a_1^\alpha, b_1^\alpha]$, $[Y]^\alpha = [a_2^\alpha, b_2^\alpha]$, $\alpha \in [0, 1]$. Then

$$
[X \oplus Y]^\alpha = [a_1^\alpha + a_2^\alpha, b_1^\alpha + b_2^\alpha],
$$

$$
[X - Y]^\alpha = [a_1^\alpha - a_2^\alpha, b_1^\alpha - b_2^\alpha].
$$
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\[ [X \otimes Y]^{\alpha} = \left[ \min_{i,j \in \{1,2\}} a_i^\alpha b_j^\alpha, \max_{i,j \in \{1,2\}} a_i^\alpha b_j^\alpha \right] \]

and

\[ [Y^{-1}]^{\alpha} = \left[ \frac{1}{b_2}, \frac{1}{a_2} \right], \quad \text{if } 0 \notin Y. \]

The absolute value, \(|X|\) of \(X \in R(I)\) is defined by (see for instance Kaleva and Seikkala [2])

\[ |X|(t) = \begin{cases} \max\{X(t), X(-t)\}, & \text{if } t \geq 0, \\ 0, & \text{if } t < 0. \end{cases} \]

A fuzzy real number \(X\) is called non-negative if \(X(t) = 0\), for all \(t < 0\). The set of all non-negative fuzzy real numbers is denoted by \(R^*(I)\).

Let \(d: R(I) \times R(I) \to \mathbb{R}\) be defined by

\[ d(X,Y) = \sup_{0 \leq \alpha \leq 1} d([X]^\alpha, [Y]^\alpha). \]

Then \(d\) defines a metric on \(R(I)\).

The additive identity and multiplicative identity in \(R(I)\) are denoted by 0 and 1 respectively.

2. Definitions and preliminaries

After the introduction of \(R(I)\), different classes of sequences were introduced and studied by Tripathy and Nanda [12], Savas [9], Das and Choudhury [1], Subrahmaniam [11] and many others.

A sequence \((X_k)\) of fuzzy real numbers is said to be convergent to the fuzzy real number \(X_0\) if, for every \(\varepsilon > 0\), there exists \(n_0 \in \mathbb{N}\) such that \(d(X_k, X_0) < \varepsilon\), for all \(k \geq n_0\).

A fuzzy real number sequence \((X_k)\) is said to be bounded if \(\sup_k |X_k| \leq \mu\), for some \(\mu \in R^*(I)\).

For \(r \in \mathbb{R}\) and \(X \in R(I)\) we define

\[ rX(t) = \begin{cases} X(r^{-1}t), & \text{if } r \neq 0, \\ 0, & \text{if } r = 0. \end{cases} \]

Throughout the article \(w^F, \ell^F_\infty, c^F\) and \(c^F_0\) denote the classes of all, bounded, convergent and null fuzzy real number sequences respectively.

A sequence space \(E^F\) is said to be normal (or solid) if \((Y_k) \in E^F\), whenever \(|Y_k| \leq |X_k|\), for all \(k \in \mathbb{N}\), for some \((X_k) \in E^F\).
Let $K = \{k_1 < k_2 < k_3 < \ldots \} \subseteq \mathbb{N}$ and $E^F$ be a sequence space. A \textit{K-step space} of $E^F$ is a sequence space $\lambda^E_{K} = \{(X_{k_n}) \in w^F : (X_n) \in E^F\}$.

A \textit{canonical pre-image} of a sequence $(X_{k_n}) \in \lambda^E_{K}$ is a sequence $(Y_n) \in w^F$ defined as follows:

$$Y_n = \begin{cases} X_n, & \text{if } n \in K, \\ \bar{0}, & \text{otherwise}. \end{cases}$$

A \textit{canonical pre-image} of a step space $\lambda^E_{K}$ is a set of canonical pre-images of all elements in $\lambda^E_{K}$, i.e., $Y$ is in canonical pre-image $\lambda^E_{K}$ if and only if $Y$ is canonical pre-image of some $X \in \lambda^E_{K}$.

A sequence space $E^F$ is said to be \textit{monotone} if $E^F$ contains the canonical pre-images of all its step spaces.

From the above definitions we have the following remark.

\textbf{Remark 2.} A sequence space $E^F$ is solid $\implies E^F$ is monotone.

A sequence space $E^F$ is said to be \textit{symmetric} if $(X_{\pi(n)}) \in E^F$, whenever $(X_k) \in E^F$, where $\pi$ is a permutation of $\mathbb{N}$.

A sequence space $E^F$ is said to be \textit{convergence free} if $(Y_k) \in E^F$, whenever $(X_k) \in E^F$ and $X_k = \bar{0}$ implies $Y_k = \bar{0}$.

In this article we introduce the following sequence spaces of fuzzy real numbers defined by Orlicz function:

$$(\ell_\infty)_F(M) = \left\{ (X_k) : \sup_k M \left( \frac{d(X_k, \bar{0})}{\rho} \right) < \infty, \text{ for some } \rho > 0 \right\}.$$

$$c_F(M) = \left\{ (X_k) : \lim_{k \to \infty} M \left( \frac{d(X_k, L)}{\rho} \right) = 0, \text{ for some } \rho > 0 \text{ and } L \in R(I) \right\}.$$

$$(c_0)_F(M) = \left\{ (X_k) : \lim_{k \to \infty} M \left( \frac{d(X_k, \bar{0})}{\rho} \right) = 0, \text{ for some } \rho > 0 \right\}.$$

\section{Main results}

\textbf{Theorem 3.1.} The spaces $(\ell_\infty)_F(M), c_F(M)$ and $(c_0)_F(M)$ are complete metric spaces under the metric

$$f(X, Y) = \inf \left\{ \rho > 0 : \sup_k M \left( \frac{d(X_k, Y_k)}{\rho} \right) \leq 1 \right\}.$$
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Proof. Consider the sequence space \((\ell_\infty)_F(M)\). Let \((X^i)\) be a Cauchy sequence in \((\ell_\infty)_F(M)\). Let \(\varepsilon > 0\) be given. For a fixed \(x_0 > 0\), choose \(r > 0\) such that \(M\left(\frac{rx_0}{2}\right) \geq 1\). Now there exists \(m_0 \in \mathbb{N}\) such that

\[
f(X^i, X^j) < \frac{\varepsilon}{rx_0}, \quad \text{for all } i, j \geq m_0.
\]

By the definition of \(f\) we have,

\[
\sup_k M\left(\frac{d(X^i_k, X^j_k)}{\rho}\right) \leq 1 \quad \text{for all } i, j \geq m_0
\]

(1)

\[\Rightarrow \quad M\left(\frac{d(X^i_k, X^j_k)}{f(X^i_k, X^j_k)}\right) \leq 1 \leq M\left(\frac{rx_0}{2}\right) \quad \text{for all } i, j \geq m_0 \text{ and for each } k \in \mathbb{N}.
\]

\[\Rightarrow \quad \frac{d(X^i_k, X^j_k)}{f(X^i_k, X^j_k)} < \frac{\varepsilon}{2} \quad \text{for all } i, j \geq m_0 \text{ and for each } k \in \mathbb{N}.
\]

Thus \((X^i_k)\) is a Cauchy sequence of fuzzy real numbers. So there exists a fuzzy real number \(X_k\) such that \(\lim_{j \to \infty} X^j_k = X_k\) for each \(k \in \mathbb{N}\).

Since \(M\) is continuous, so taking \(j \to \infty\) in (1) we get

\[
\sup_k M\left(\frac{d(X^i_k, X_k)}{\rho}\right) \leq 1.
\]

Taking infimum of such \(\rho\)'s we get

\[
f(X^i, X) < \varepsilon, \quad \text{for all } i \geq m_0.
\]

Let \(i \geq m_0\), then using the triangle inequality, \(f(X, \overline{0}) \leq f(X, X^i) + f(X^i, \overline{0})\), we have \(X \in (\ell_\infty)_F(M)\).

Hence \((\ell_\infty)_F(M)\) is complete. Similarly it can be shown that the other spaces are also complete. \(\square\)

**Theorem 3.2.** The sequence spaces \((\ell_\infty)_F(M)\) and \((c_0)_F(M)\) are solid and hence monotone.

Proof. Consider the sequence space \((\ell_\infty)_F(M)\). Let \((X_k) \in (\ell_\infty)_F(M)\) and \((Y_k)\) be such that \(\overline{d}(Y_k, \overline{0}) \leq \overline{d}(X_k, \overline{0})\) for each \(k \in \mathbb{N}\). Since \(M\) is increasing, we have

\[
\sup_k M\left(\frac{\overline{d}(Y_k, \overline{0})}{\rho}\right) \leq \sup_k M\left(\frac{\overline{d}(X_k, \overline{0})}{\rho}\right).
\]

Hence the space \((\ell_\infty)_F(M)\) is solid. Similarly \((c_0)_F(M)\) is also solid. \(\square\)

**Property 3.3.** The sequence space \(c_F(M)\) is not monotone and hence not solid.

The result follows from the following example:
Example 3.1. Let $J = \{k : k \text{ is even}\} \subseteq \mathbb{N}$. Let the sequence $(X_k)$ be defined as: For all $k \in \mathbb{N}$,

$$X_k(t) = \begin{cases} 
t + 1, & \text{if } -1 \leq t \leq 0, \\
-t + 1, & \text{if } 0 \leq t \leq 1, \\
0, & \text{otherwise.}
\end{cases}$$

Then $(X_k) \in c_F(M)$. Let $(Y_k)$ be the canonical pre-image of $(X_k)_J$ for the subsequence $J$ of $\mathbb{N}$. Then

$$Y_k = \begin{cases} 
X_k, & \text{for } k \in J, \\
0, & \text{otherwise.}
\end{cases}$$

Then $(Y_k) \notin c_F(M)$. Hence $c_F(M)$ is not monotone.

Theorem 3.4. The sequence spaces $(\ell_\infty)_F(M)$, $c_F(M)$ and $(c_0)_F(M)$ are symmetric.

Proof. It can be proved following the technique used for establishing the crisp set cases. \hfill \Box

The proof of the following two results are obvious.

Property 3.5. $Z(M_1) \cap Z(M_2) \subseteq Z(M_1 + M_2)$, for $Z = \ell^F_\infty(M)$, $c_F(M)$, $(c_0)_F(M)$.

Property 3.6. If $M_1(x) \leq M_2(x)$ for all $x \in [0, \infty)$, then $Z(M_2) \subseteq Z(M_1)$ for $Z = (\ell_\infty)_F$, $c_F$ and $(c_0)_F$.

Theorem 3.7. Let $M$ and $M_1$ be two Orlicz functions, then $Z(M_1) \subseteq Z(M \circ M_1)$, for $Z = (\ell_\infty)_F$, $c_F$ and $(c_0)_F$.

Proof. We prove it for the case $Z = (c_0)_F$, the other cases can be proved following similar technique. Let $\varepsilon > 0$ be given. Since $M$ is continuous, so there exists $\eta > 0$ such that $M(\eta) = \varepsilon$. Let $(X_k) \in (c_0)_F(M)$. Then there exists $k_0 \in \mathbb{N}$ such that

$$M_1\left(\frac{d(X_k, 0)}{\rho}\right) < \eta \quad \text{for all} \quad k \geq k_0.$$

$$\Rightarrow M \circ M_1\left(\frac{d(X_k, 0)}{\rho}\right) < \varepsilon \quad \text{for all} \quad k \geq k_0.$$

$$\Rightarrow (X_k) \in (c_0)_F(M \circ M_1).$$

Hence $Z(M_1) \subseteq Z(M \circ M_1)$, for $Z = (\ell_\infty)_F$, $c_F$ and $(c_0)_F$. \hfill \Box

Property 3.8. $Z(M) \subseteq (\ell_\infty)_F(M)$, for $Z = c_F$ and $(c_0)_F$. The inclusions are strict.
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Proof. The result \( Z(M) \subseteq (\ell_\infty)_F(M) \), for \( Z = c_F \) and \( (c_0)_F \) is obvious. To show that the inclusions are proper, consider the following example. □

Example 3.2. Let the sequence \((X_k)\) be defined by:
For \( k \) even,
\[
X_k(t) = \begin{cases} 
  t, & \text{if } 0 \leq t \leq 1, \\
  -t + 2, & \text{if } 1 \leq t \leq 2, \\
  0, & \text{otherwise}
\end{cases}
\]
For \( k \) odd,
\[
X_k(t) = \begin{cases} 
  1, & \text{if } 0 \leq t \leq 1, \\
  -t + 2, & \text{if } 1 \leq t \leq 2, \\
  0, & \text{otherwise}
\end{cases}
\]
Then \((X_k) \in (\ell_\infty)_F(M)\), but \((X_k)\) neither belongs to \( c_F(M) \) nor to \((c_0)_F(M)\).

Theorem 3.9. The sequence spaces \((\ell_\infty)_F(M), c_F(M)\) and \((c_0)_F(M)\) are not convergence free.

The result follows from the following example.

Example 3.3. Consider the sequence space \( c_F(M) \). Let the sequence \((X_k)\) be defined by:
For \( k = 1 \), \( X_k = \overline{0} \).
For \( k \neq 1 \),
\[
X_k(t) = \begin{cases} 
  t + 1, & \text{if } 0 \leq t \leq 1, \\
  -k(k + 1)^{-1}t + 1, & \text{if } 1 \leq t \leq 1 + k^{-1}, \\
  0, & \text{otherwise}
\end{cases}
\]
Let the sequence \((Y_k)\) be defined by:
For \( k = 1 \), \( Y_k = \overline{0} \).
For \( k \neq 1 \),
\[
Y_k(t) = \begin{cases} 
  1, & \text{if } 0 \leq t \leq 1, \\
  -(2k - 1)^{-1}t + 2k(2k - 1)^{-1}, & \text{if } 1 \leq t \leq 2k, \\
  0, & \text{otherwise}
\end{cases}
\]
Here \((X_k) \in c_F(M)\) but \((Y_k) \notin c_F(M)\). Hence \( c_F(M) \) is not convergent free.

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