Combinatorial structure of the parameter plane of the family $\lambda \tan z^2$

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Abstract: In this article we will discuss combinatorial structure of the parameter plane of the family $\mathcal{F} = \{\lambda \tan z^2 : \lambda \in \mathbb{C}^*, \ z \in \mathbb{C}\}$. The parameter space contains components where the dynamics are conjugate on their Julia sets. The complement of these components is the bifurcation locus. These are the hyperbolic components where the post-singular set is disjoint from the Julia set. We prove that all hyperbolic components are bounded except the four components of period one and they are all simply connected.

1 Introduction

Quasi-conformal mappings play an important role in studying parameter spaces of holomorphic dynamical systems because they are used to characterize when maps in the same family have similar dynamics. A very useful technique called quasi-conformal surgery was introduced by Douady\cite{Douady}. It is this flexibility that produces the basis for what is known as quasi-conformal surgery, in which we change mappings and sometimes also the spaces involved. When the construction is successful the final goal is to end with a holomorphic map with the desired properties, obtained via the Measurable Riemann Mapping Theorem.

Definition. Let $\mu$ be a Beltrami coefficient. The Beltrami equation associated to $\mu$ is the partial differential equation

$$f \overline{z} = \mu(z)f_z$$

Theorem 1.1. (Measurable Riemann Mapping Theorem on $\mathbb{C}$) Let $\mu$ be a Beltrami coefficient of $\mathbb{C}$. Then, there exists a unique quasi-conformal map $f : \mathbb{C} \to \mathbb{C}$ such that $f(0) = 0$, $f(1) = 1$
and $\mu_f = \mu$. Furthermore, if $\mu_t$ is a family of Beltrami coefficients such that $\mu_t(z)$ depends analytically on $t$ for any $z \in \mathbb{C}$, then $f_t$ depends analytically on $t$.

**Definition.** A meromorphic map $f$ is called *hyperbolic* if it is expanding on its Julia set; that is, there exist constants $c > 0$ and $K > 1$ such that for all $z$ in a neighborhood $V \supset J$, $|(f^n)'(z)| > cK^n$.

The key idea to investigate the combinatorial structure of the parameter plane is to understand the regions in which the dynamical systems are quasi-conformally conjugate on their Julia sets. For the maps in $\mathcal{F}$, the $\lambda$-plane is divided into center capture component and its complement. In the center capture component, the Julia set is a Cantor set. The complement is further divided into hyperbolic components in which $f_\lambda$ has connected Julia set and the Fatou set consists of simply connected components. Each of these components has a unique boundary point $\lambda^*$, called a *virtual center* such that for any sequence $\lambda_n$ inside the component converging to $\lambda^*$, the multiplier map $\rho(\lambda_n) \to 0$. These components are called hyperbolic *shell components* in the parameter plane of $\mathcal{F}$. We prove that the virtual center is the unique boundary point of a pair of hyperbolic shell components.

We denote the hyperbolic components in the parameter plane as follows:

- $C_0 = \{\lambda \in \mathbb{C}^* : \text{All singular values are in the immediate basin of 0}\}$
- $\mathcal{C} = \{\lambda \in \mathbb{C}^* : \text{All singular values eventually land in the immediate basin of 0 but do not belong to the immediate basin of 0}\}$
- $\mathcal{H} = \{\lambda \in \mathbb{C}^* : \text{Asymptotic values are attracted to an attracting (not super-attracting) periodic point}\}$

**Definition.** A hyperbolic component in $\mathcal{C} \cup \mathcal{C}_0$ is called a *capture component*. $\mathcal{C}_0$ is called the *central capture component* since it contains the parameter singularity, the origin. A hyperbolic component in $\mathcal{H}$ is called a *shell component*.

We use the following theorem in this context.
Theorem 1.2. If $W$ is a hyperbolic component of the interior of $M$, then the multiplier $\rho(c)$, $c \in W$ of the attracting cycle maps $W$ conformally onto the open unit disk $\mathbb{D}$. It extends continuously to $\partial W$ and maps $\overline{W}$ homeomorphically onto the closed disk $\overline{\mathbb{D}}$. The point in $W$ where $\rho(c) = 0$ is called the center of $W$.

The organization of this paper is as follows: In section 2, we discuss the topological structure of the hyperbolic capture components in the parameter space. We use the coding of the centers to give an indexing of the capture components. This coding describes the itinerary of the capture components in the parameter plane. Section 3 is devoted to giving combinatorial structure of the hyperbolic shell components in the parameter plane. The itinerary of a hyperbolic shell component is determined by its virtual center. We investigate how these components fit together at the virtual centers. We also prove that the shell components are bounded except the shell components of period one. We study the bifurcation and boundedness properties of the shell components in section 4. The bifurcation along the boundary of the shell components demonstrate the itinerary of the shell components. We also prove that all shell components except the period one components are bounded. We thank Linda Keen for the figure 1 and figure 2 of this article.

2 Capture components

The following two lemmas are preliminary results about the symmetry of the parameters. These are used later to prove main results of this article.

Lemma 2.1. For any $\lambda$ in a hyperbolic component the attracting (super-attracting) cycles of $f_\lambda$ and $f_{\bar{\lambda}}$ are complex conjugates; so are their multipliers (trivial for the super-attracting case).

Proof. For any $f_\lambda$ in $\mathcal{F}$ we have

$\overline{f_\lambda^k(\lambda i)} = f_\lambda^k(\overline{\lambda i})$

$= f_\lambda^k(-i\lambda)$

$= f_\lambda^k(i\bar{\lambda})$ for $k \in \mathbb{Z}^+$

Therefore the orbits of the asymptotic values of $f_\lambda$ and $f_{\bar{\lambda}}$ are conjugates.
Suppose \( \lambda \) is in a hyperbolic shell component and suppose \( z_p \) is a periodic point of \( f_\lambda \) with period \( p \). Then \( \bar{z}_p \) is a periodic point of \( f_\lambda \). Their multipliers are \( \rho(\lambda) = [f_\lambda^p(\bar{z}_p)]' \) and \( \rho(\lambda) = [f_\lambda^p(z_p)]' \). Thus it is clear that \( \rho(\bar{\lambda}) = \overline{\rho(\lambda)} \). \( \square \)

**Lemma 2.2.** Suppose \( \lambda \) is in a hyperbolic component and \( \{ z_p \} \), \( p \geq 1 \) is an attracting periodic cycle of \( f_\lambda \). Then \( f_{-\lambda}, f_{\pm \lambda} \) have attracting periodic cycles \( \{-z_p\}, \{ \mp z_p i \} \) respectively. If \( \rho(\lambda) \) is the multiplier of the attracting cycle of \( f_\lambda \) then \( (-1)^p \rho(\lambda), (\mp i)^p \rho(\pm \lambda i) \) are multipliers of the attracting cycles of \( f_{-\lambda}, f_{\pm \lambda} \).

**Proof.** Let \( z_0 \) be a periodic point of \( f_\lambda \) of period \( p \geq 1 \). Then \( f^p_{-\lambda}(-z_0) = f^p_\lambda(z_0) = -z_0 \). Suppose that there exists an integer \( q < p \) such that \( f^q_{\pm \lambda}(-z_0) = -z_0 \). Then \( f^q_{\lambda}(z_0) = -z_0 \) so that \( f^q_\lambda(z_0) = z_0 \) contradicting the hypothesis.

Now we prove that \( f_{\pm \lambda} \) has the attracting periodic cycle \( \{ \pm z_p i \} \). It is clear that \( f^p_\lambda(z_0) = -i f^p_\lambda(z_0) = -iz_0 \). Therefore \( f^p_{\pm \lambda}(-z_0 i) = -i f^p_{\lambda}(z_0) = 0 \), for \( p \geq 1 \). Suppose that there exists an integer \( q < p \) such that \( f^q_{\pm \lambda}(-z_0 i) = -z_0 i \). Thus \( f^q_{\pm \lambda}(-z_0 i) = -z_0 i \) implies that \( f^q_{\pm \lambda}(z_0) = z_0 \), contradicts the hypothesis. The proof for \( f_{-\lambda} \) follows similarly. For the multiplier map, the proof is the following.

\[
\rho(-\lambda) = \prod_{i=0}^{p-1} f'_{-\lambda}(z_i) = \prod_{i=0}^{p-1} f'_\lambda(z_i) = (-1)^p \rho(\lambda),
\]
\[
\rho(\pm \lambda i) = \prod_{i=0}^{p-1} f'_{\pm \lambda i}(z_i) = \prod_{i=0}^{p-1} (\mp i)^p \rho(\lambda).
\]

\( \square \)

**Proposition 2.3.** Let \( \mathcal{B}_n = \{ \lambda \in \mathbb{C}^* : f^n_\lambda(\lambda i) = 0 \} \) \( \mathcal{B} = \bigcup_{n=1}^\infty \mathcal{B}_n \). Then \( \mathcal{B} \) is the set of pre-zeros of the maps in \( \mathcal{F} \).

**Proof.** For \( n = 1 \), set \( \mathcal{B}_1 = \{ c_{1_k} = \sqrt{k\pi} : k = \pm 1, \pm 2, \ldots \} \) so that \( \pm \mathcal{B}_1 \) contains all pre-images of \( 0 \). For \( n = 2 \), set \( \mathcal{B}_2 = \{ \lambda : f^2_\lambda(\pm \lambda i) = 0 \} \). Therefore \( f_\lambda(\pm \lambda i) = \pm \sqrt{k\pi} \), for some \( k \in \mathbb{Z}^* \). Suppose that \( f_\lambda(\lambda i) = p_k = \sqrt{k\pi} \). Then \( \lambda \) can be determined by solving \( f_\lambda(\pm \lambda i) = \pm \sqrt{k\pi}, k \in \mathbb{Z}^* \). The numerical solution \( \lambda = (x, y) \) can be obtained by iterating \( \lambda_{m+1} = \phi(\lambda_m) = (\phi_1(\lambda_m), \phi_2(\lambda_m)) \)
where \( \phi \) is defined below. From \( \lambda_{m+1} = \pm \sqrt{\arctan \frac{p_k}{x_m}} \) we get the following formula.

Let \( X_{k,j} = \frac{1}{2} \arctan \frac{-2px_k}{x^2 + y^2 - p^2_y} + j\pi, \ j \in \mathbb{Z}, \ Y_k = \frac{1}{2} \ln \frac{\sqrt{4x^2y^2 + (p^2_y + x^2 - y^2)^2}}{x^2 + (p_y + y)^2} \).

From above we have the following:

\[
\phi_{1,j}(\lambda_m) = \sqrt{\frac{X_{k,j} + \sqrt{X_{k,j}^2 + Y_k^2}}{2}}, \quad \phi_{2,j}(\lambda_m) = \frac{Y_k}{|Y_k|} \sqrt{\frac{X_{k,j} + \sqrt{X_{k,j}^2 + Y_k^2}}{2}}.
\]

We have introduced the index \( j \) to indicate the branch of the solution of the arctangent. We get \( B_2 = \{ \lambda_{k,j} \in \mathbb{C} \mid \lambda \text{ is a solution of } f_\lambda(\pm \lambda i) = \pm p_k, \ k \in \mathbb{Z}^* \} \). As \( k \to \infty, \ f_\lambda(\pm \lambda i) \to \infty \) implies that the solution of the above equation \( \lambda_{k,j} \to s_j = \sqrt{(2j+1)\pi}, \ j \in \mathbb{Z} \) as \( k \to \infty \).

Similarly \( B_3 \) consists of all solutions \( \lambda \) so that \( f_\lambda^2(\pm \lambda i) = \pm p_k, \ k \in \mathbb{Z}^* \). For a solution \( \lambda \) of the above equation, one more index has been introduced. In general any point in \( B_p \) can be coded as \( \lambda_{k,j_1,j_2,...,j_{p-1}} \) where the indices are determined by the branches of the solution in all intermediate steps. The point \( \lambda_{k,j_1,j_2,...,j_{p-1}} \) is in a neighborhood of \( s_{k,j_1,j_2,...,j_{p-1}} \) for large enough \( j_{p-1} \) where \( s_{k,j_1,j_2,...,j_{p-1}} \) is a pre-pole of order \( p \). Thus each point in \( B_p \) can be indexed in a suitable way to recognize the pre-zeros of this family of maps.

There is a unique point in each of the capture components such that the corresponding asymptotic values of \( f_\lambda \) are mapped to the origin by finite iterations of \( f_\lambda \). These points are called the centers of the components. The following proposition describes a characterization of these centers of the capture components. The result can be used to give an indexing of the capture components.

**Proposition 2.4.** For each \( n \) and \( c_{n_k} \in B_n, \ k \in \mathbb{Z} \), there are capture components \( C_{n_k} \) containing \( c_{n_k} \) so that \( f_\lambda^n(c_{n_k}i) = 0 \). The point \( c_{n_k} \) is called the center of the component \( C_{n_k} \).

**Proof.** At each point \( c_{1_k} \in B_1, \ f_\lambda = c_{1_k} \) \( (c_{1_k}i) = 0 \). Then \( f_\lambda \) has only one super-attracting periodic cycle and the asymptotic values \( \pm \lambda i = \pm (c_{1_k}i) \) are pre-periodic. 0 is always a super-attracting fixed point for \( \lambda \in \mathbb{C}^* \). For \( \lambda \) in a hyperbolic component in the parameter space, the forward orbit of the asymptotic values must be in the stable set (Fatou set). Using quasi-conformal conjugacy and the Böttcher map, there is an open set \( U \) such that for all \( \lambda' \in U, \lambda' \neq \lambda \), \( f_{\lambda'} \) is
quasi-conformally conjugate to \( f_\lambda \). Therefore the dynamical behavior of \( f_\lambda \) coincides with the dynamical behavior of \( f_\lambda \) on their Julia sets. Let \( C_{1k} \) be the largest neighborhood of \( c_{1k} \) where this quasi-conformal conjugacy can be extended. Then \( C_{1k} \) is a capture component with center at \( c_{1k} \). We see in Theorem \( 2.6 \) that \( c_{1k} \) is the unique point in \( C_{1k} \) such that \( c_{1k} i \) is pre-periodic. As \( n \to \infty \) the set of points in \( B_1 \) tends to \( \infty \) along the real and imaginary lines.

Using the indexing of the pre-zeros from proposition \( 2.3 \) we get that \( \lambda_{k,j_1,j_2,\ldots,j_p-1} \) is the solution of the equation \( f_\lambda^{p-1}(\lambda_i) = 0 \) or \( f_\lambda^{p-1}(-\lambda_i) = 0 \). Then \( c_{k,j_1,j_2,\ldots,j_p-1} = \lambda_{k,j_1,j_2,\ldots,j_p-1} \) is the center of the capture component \( C_{k,j_1,j_2,\ldots,j_p-1} \).

Using the indices that refer to the branches of the arctangent, one can actually give a coding to the capture components, based on how many iterations \( f_\lambda \) takes to map the asymptotic values to the immediate basin of zero. Using the proposition \( 2.4 \) we can give a more precise definition of a capture component as follows.

**Definition.** The capture components of depth \( i \geq 1 \) are the connected components \( C_{n_1,n_2,\ldots,n_i} \) of \( C \), where \( n_1,n_2,\ldots,n_i = \{ \lambda \in C : f_\lambda^i(\pm \lambda_i) \in A_\lambda(0) \text{ and } f_\lambda^{i-1}(\pm \lambda_i) \notin A_\lambda(0) \} \) and \( A_\lambda(0) \) is the immediate attracting basin of zero. The indices \( n_1,\ldots,n_i \) indicate the inverse branches of the arctangent that map 0 back to \( \pm \lambda_i \). \( C_0 \) is the only capture component of depth zero containing the origin.

**Lemma 2.5.** Let \( C_k \) be a capture component containing \( \lambda \in B_k, \ k \in \mathbb{N} \). Let \( A_\lambda(0) \) be the immediate attracting basin of zero corresponding to \( f_\lambda \). Then for all \( \lambda \in C_k \), \( f_\lambda(\lambda_i) \in A_\lambda(0) \).

**Proof.** Since \( \lambda_0 \) is the center of \( C_k \), we have \( f_\lambda(\lambda_0 i) = 0 \). Let us define \( g(\lambda) = f_\lambda(\lambda_i), \ \lambda \in C_k \). Then \( g(\lambda) \) is a well-defined holomorphic map from \( C_k \) to \( \bigcup_{\lambda \in C_k} A_\lambda(0) \). Thus \( g(C_k) \) is a connected component in \( \bigcup_{\lambda \in C_k} A_\lambda(0) \) containing 0. But the only connected component of \( \bigcup_{\lambda \in C_k} A_\lambda(0) \) containing 0 is the immediate attracting basin of 0 of \( f_{\lambda_0} \). Therefore \( f_\lambda(\lambda_i) \) is in the immediate basin of 0 for all \( \lambda \in C_k \).

The connectivity of capture components is proved in the following theorem. We use the technique of quasiconformal surgery introduced by Douady [1]. See Branner-Fagella [2] for a full
discussion about quasi-conformal surgery.

**Theorem 2.6.** Any capture component in $\mathcal{C}$ is simply connected.

**Proof.** Let $\lambda_0 \in \mathcal{C}$ so that $f_{\lambda_0}^k(\lambda_0i) = 0$. Let $\psi_{\lambda_0}$ be the Böttcher map $\psi_{\lambda_0} : \mathcal{A}_{\lambda_0}^0(0) \to \mathbb{D}$ which conjugates $f_{\lambda_0}$ on the immediate basin of 0 to the map $z \mapsto z^2$ in $\mathbb{D}$. Then $\lambda \mapsto \psi_{\lambda}(f_{\lambda}^k(\lambda_0i))$ is a holomorphic map from a neighborhood $U$ of $\lambda_0$ to a neighborhood of the origin for some $k$. Define the map $\Phi : U \to \mathbb{D}$ by $\Phi(\lambda) = \psi_{\lambda}(f_{\lambda}^k(\lambda_0i))$. Let $\lambda_0 \in U$ so that $0 = \Phi(\lambda_0)$. The idea of this surgery construction is the following: for any point $z$ near 0, we can build a map $f_{\lambda}(z)$ such that $\Phi(\lambda(z)) = z$, or in other words we can find a local inverse of $\Phi_U$. This proves that the component containing the center is open.

Let $A_{\lambda_0}(0)$ be the pre-image of $\mathcal{A}_{\lambda_0}^0(0)$ under $f_{\lambda_0}^{-k}$ and let $W_{\lambda_0}$ be the connected component of $A_{\lambda_0}(0)$ containing $f_{\lambda_0}^k(\lambda_0i)$. Let $V_{\lambda_0}$ be any small neighborhood of $f_{\lambda_0}^{k+1}(\lambda_0i)$ contained in $\mathcal{A}_{\lambda_0}^0(0)$. Consider $B_{\lambda_0} \subset W_{\lambda_0}$ to be the pre-image of $V_{\lambda_0}$ containing $f_{\lambda_0}^k(\lambda_0i)$. Note that by continuity it is the preimage given by the itinerary of the center $\lambda_0$. For any $0 < \epsilon < \min\{|z_0|, 1 - |z_0|\}$ we consider $D(z_0, \epsilon)$, the disk of radius $\epsilon$ centered at $z_0$. For any $z \in D(z_0, \epsilon)$, choose a diffeomorphism $\delta_z : B_{\lambda_0} \to V_{\lambda_0}$ with the following properties:

i) $\delta_{z_0} = f_{\lambda_0}$;

ii) $\delta_z$ coincides with $f_{\lambda_0}$ in a neighborhood of $\partial B_{\lambda_0}$ for any $z$;

iii) $\delta_z(f_{\lambda_0}^k(\lambda_0i)) = \psi_{\lambda_0}^{-1}(z)$;

We consider the following mapping for any $z \in D(z_0, \epsilon)$:

$$G_z : \mathbb{C} \to \mathbb{C} :$$

$$G_z(w) = \begin{cases} 
\delta_z(w) & \text{if } w \in B_{\lambda_0}, \\
 f_{\lambda_0}(w) & \text{if } w \notin B_{\lambda_0}.
\end{cases}$$

We construct an invariant almost complex structure $\sigma_z$ with bounded dilatation ratio. Let $\sigma_0$ be the standard complex structure of $\mathbb{C}$. We define a new almost complex structure $\sigma_z$ in $\mathbb{C}$ by
\[ \sigma_z(w) = \begin{cases} 
(\hat{\sigma}_z)^{-1} \sigma_0 & \text{on } B_{\lambda_0} \\
(f_{\lambda_0}^k)^{-1} \sigma_0 & \text{on } f_{\lambda_0}^{-k}(B_{\lambda_0}), \ \forall k \geq 1 \text{ (where defined)} \\
\sigma_0 & \text{on } \mathbb{C} \setminus \{ \cup_{n \geq 1} f_{\lambda_0}^{-k}(B_{\lambda_0}) \cup B_{\lambda_0} \} 
\end{cases} \]

By construction \( \sigma \) is \( G_z \)-invariant, i.e. \((G_z)^* \sigma = \sigma \) and \( \sigma \) has bounded distortion since \( \hat{\sigma}_z \) is a diffeomorphism and \( f_{\lambda_0} \) is holomorphic in the attracting basin. Applying the Measurable Riemann Mapping Theorem we obtain a quasi-conformal map \( \phi_z : \mathbb{C} \rightarrow \mathbb{C} \) such that \( \phi_z \) preserves the complex structure \( \sigma_z \), i.e. \((\phi_z)^* \sigma = \sigma \). The map \( \phi_z \) is uniquely determined up to an affine transformation; therefore it can be determined by what it does to two points. We assume \( \phi \) fixes the origin and maps the two asymptotic values to a pair of points symmetric with respect to the origin. Then the map \( F_z = \phi_z \circ G_z \circ \phi_z^{-1} \) is meromorphic with \( 0 \) as a fixed critical point of multiplicity two. \( F_z \) respects the dynamics: it has a super-attracting periodic cycle. Moreover \( F_z \) is quasi-conformally conjugate to \( G_z \) in the respective basins of attraction and is conformally conjugate to \( G_z \) everywhere else. Then \( F_z \) is a meromorphic map of the form \( \nu \tan(a_2 z^2 + a_0) \) for some \( a_2, a_0 \in \mathbb{C}, \ a_2 \neq 0 \) [3]. Doing a suitable change of variable and composing with an affine transformation, if necessary, we can get a \( \lambda \in \mathbb{C}^* \) such that \( f_{\lambda(z)} = (k \phi_z) \circ G_z \circ (k \phi_z)^{-1} \).

By construction \( \phi_{z_0} \) is the identity map for \( z = z_0 \). Therefore there exists a continuous function \( z : D(z_0, \epsilon) \mapsto \lambda(z) \in U \) such that \( \lambda(z_0) = z_0 \) and \( F_{\lambda(z)} = \phi_z \circ G_{\lambda(z_0)} \circ \phi_z^{-1} \). Moreover \( \phi_z \) is holomorphic on \( A_{\lambda_0}^0(0) \) conjugating \( F_{\lambda_0} \) to \( F_{\lambda(z)} \). Hence from the following commutative diagram:

\[
\begin{array}{ccc}
A_{\lambda_0}^0(0) & \xleftarrow{\phi_z} & A_{\lambda_0}^0(0) \\
\downarrow{f_{\lambda(z)}} & & \downarrow{f_{\lambda_0}} \\
A_{\lambda(z)}^0(0) & \xleftarrow{\phi_z} & A_{\lambda_0}^0(0) \\
\end{array}
\]

we have that \( \psi_{\lambda(z)} = \psi_{\lambda_0} \circ \phi_z^{-1} \) is the Böttcher Coordinate of \( A_{\lambda(z)}^0(0) \).

Finally we conclude that \( \Phi(\lambda(z)) = \psi_{\lambda(z)}(f_{\lambda(z)}^{\phi^{-1}_{\lambda}(\lambda z)}) = z \), since \( f_{\lambda(z)}^{\phi^{-1}_{\lambda}(\lambda z)} = \phi_z \circ G_z^{2 \phi^{-1}_{\lambda}(\lambda z)} \circ \phi_z^{-1} = \phi_z \circ G_z(f_{\lambda_0}^{\phi^{-1}_{\lambda}(\lambda z)}) = \phi_z \circ \psi_{\lambda_0}^{-1}(z) = \phi_z \circ \phi_z^{-1} \circ \psi_{\lambda(z)}^{-1}(z) = \psi_{\lambda(z)}^{-1}(z) \).

By the Riemann-Hurwitz formula, \( \Phi \) is a degree one covering map. Therefore \( \Phi^{-1}(z) \) is a compact
set and \( \Phi : C_k \to \mathbb{D} \) is a proper map. That completes the proof.

\[ \square \]

**Lemma 2.7.** Let us assume \( \lambda \in \mathbb{R} \) or \( \lambda \in \mathbb{I} \) and \( |\lambda| < \sqrt{\frac{4}{\pi}} \). Then \( \lambda \in C_0 \).

**Proof.** We will prove the lemma when \( \lambda \) is in the positive real axis. For the imaginary axis the proof follows similarly. We see that \( \lambda = \sqrt{\frac{4}{\pi}} \) is a repelling fixed point of \( f_\lambda \). Suppose \( 0 < \lambda < \sqrt{\frac{4}{\pi}} \).

It follows that \( |f_\lambda(\lambda i)| = |\lambda \tan \frac{x}{2}| < |\lambda| \) and

\[
|f_\lambda^2(\lambda i)| < |\lambda \tan \lambda^2| < |\lambda| \\
\vdots \\
|f_\lambda^n(\lambda i)| < |\lambda|.
\]

Therefore the post singular orbit of \( f_\lambda \) is bounded in the dynamic plane for \( \lambda \) in the interval \((-\pi/4, \pi/4)\). For any small neighborhood \( I_\lambda \subset \mathbb{R} \) in the parameter space with \( |\lambda| < \sqrt{\frac{4}{\pi}} \), \( f_\lambda^k(\lambda i) \) form a normal family in \( I_\lambda \). Therefore \( I_\lambda \) must be in a hyperbolic component in the parameter space. In proposition 3.1 of the next section we prove that there is no shell component intersecting \( \mathbb{R} \) and \( \mathbb{I} \). Thus \( I_\lambda \) must be in one of the capture components. As \( \lambda \) can be chosen arbitrarily close to the origin, \( I_\lambda \subset C_0 \). Hence \( \lambda \in C_0 \).

\[ \square \]

3 **Arrangement of the hyperbolic shell components at a virtual center**

We investigate the combinatorial structure of the hyperbolic components that are not capture components in the parameter space. We denote shell components as \( \Omega_p \), where the period of the attracting cycle is \( p \).

**Proposition 3.1.** Suppose \( \lambda \in \mathbb{R} \) or \( \lambda \in \mathbb{I} \). Then \( \lambda \) is not in any hyperbolic shell component in the parameter space.

**Proof.** We prove it by contradiction. Suppose \( \lambda \in \mathbb{R} \), and \( \lambda \) is in a shell component. Then \( f_\lambda \) has an attracting periodic cycle of period \( p \geq 1 \) and one of the asymptotic values is attracted to
the attracting periodic fixed point (not the super-attracting fixed point). We label the attracting periodic components \( \{ U_i \}_{i=0}^{p-1} \) so that the asymptotic value \( \lambda i \in U_1 \) and denote the periodic fixed point in \( U_i \) by \( z_i \). By our assumption \( \lambda i \in \Im \) and \( f_{\lambda}^{n}(\lambda i) = \lambda \tan(\lambda i)^2 = -\lambda \tan \lambda^2 \in \mathbb{R} \).

This implies that \( f_{\lambda}^{n}(\lambda i) \in \mathbb{R}, \forall n \in \mathbb{N} \). As \( f_{\lambda}^{2n}(\lambda i) \to z_1 \) as \( n \to \infty \) all periodic points in this cycle are on \( \mathbb{R} \). As \( f_{\lambda}(\lambda i) \in U_2 \cap \mathbb{R}, f_{\lambda}^{2n}(f_{\lambda}(\lambda i)) \in U_2 \cap \mathbb{R} \) for all \( n \) and \( f_{\lambda}^{2n}(f_{\lambda}(\lambda i)) \to z_2 \) as \( n \to \infty \). Since \( U_2 \) is simply connected and symmetric about the real line, there is an interval \( I \subset U_2 \) containing both \( f_{\lambda}(\lambda i) \) and \( z_2 \). We take a branch \( g \) of \( f_{\lambda}^{-1} \) so that \( g(I) \) is an interval that contains \( \lambda i \) in \( U_1 \). Thus \( U_1 = U_1 \). Let \( \gamma \) be a path in \( U_1 \) joining \( z_1 \) and \( \lambda i \). Using the symmetry of \( f_{\lambda} \) with respect to the real and imaginary axes we have \(-U_1, -U_1, -U_1 \) are also in the stable domain. Therefore they have a non-empty intersection with \( U_1 \). Therefore \( U_1 = -U_1 = -U_1 = U_1 \). Thus the degree of \( f_{\lambda} : U_1 \to U_2 \) is at least two. Since \( U_1 \) is a bounded periodic component, \( U_1 \) must contain the critical point, the origin. Contradiction! We can use similar argument to get a contradiction when \( \lambda \in \Im \).\( \square \)

**Definition.** Let \( \rho_{\lambda} \) denote the multiplier of an attracting or neutral periodic cycle of \( f_{\lambda} \). If \( \Omega_p \) is an arbitrary shell component and \( \Delta^* \) is the unit disk punctured at the origin, the multiplier map \( \rho : \Omega_p \to \Delta^* \) is defined by \( \lambda \mapsto \rho_{\lambda} \). For each \( \alpha \in \mathbb{R} \) the internal ray \( R(\alpha) \) is defined by \( R(\alpha) = \rho_{\lambda}^{-1}(re^{2\pi i \alpha}), 0 < r < 1 \).

The following two theorems describes very important topological properties of the hyperbolic shell components. The theorems are proved in Fagella-Keen [3].

**Theorem 3.2.** For each shell component \( \Omega_p \) of \( H \), the multiplier map \( \rho_{\lambda} : \Omega_p \to \Delta^* \) is a covering map.

**Theorem 3.3.** For any \( \lambda^* \) and \( \lambda \) in the component \( \Omega_p \), there exists a unique quasi-conformal map \( g \) such that \( f_{\lambda^*} \circ g = g \circ f_{\lambda} \).

Let \( H_l \) denotes the right half plane. From Theorem 3.2 we have that the multiplier map \( \rho_{\lambda} : \Omega \to \mathbb{D}^* \) is a universal covering. Hence there is a conformal homeomorphism \( \phi : H_l \to \Omega \), unique up to precomposition by a Mobious transformation such that \((\rho_{\lambda} \circ \phi)(w) = e^w \). Under
the map \( \phi : H_l \rightarrow \Omega \), the boundary of \( \Omega \) corresponds to the imaginary axis.

Now we are in a position to characterize the virtual center in the boundary of a shell component. Here we give a formal definition of the virtual center of a shell component. We use the following definition from \([3]\).

**Definition.** Let \( \Omega \) be a shell component and \( \rho : \Omega \rightarrow \mathbb{D} \) be the multiplier map. A point \( \lambda \in \partial \Omega \) is called a **virtual center** if for any sequence \( \lambda_n \in \Omega \) with \( \lambda_n \rightarrow \lambda \), the multiplier map \( \rho(\lambda_n) \rightarrow 0 \).

Let \( T_k = \{ w \in H_l | 2k\pi < \Im w < 2(k+1)\pi \} \) where \( k = 0, \pm 1, \pm 2, \ldots \). Every open set \( T_k \) is a horizontal strip of \( H_l \). Let \( V_k = \phi(T_k) \). Then \( V_k \) is an open subset of \( \Omega \) obtained by cutting \( \Omega \) along \( \mathcal{R}(k) \) for all integers \( k \) where \( \mathcal{R}(k) \) is the image of the boundary of the horizontal strip \( T_k \) under \( \phi \). The boundary of \( V_k \), \( \partial V_k \) consists of three curves \( \mathcal{R}(k), \mathcal{R}(k+1) \), and \( \{ \phi(2\pi\alpha) : k < \alpha < k+1 \} \) together with their endpoints. These curves are all regular simple arcs; hence \( \partial V_k \) is a Jordan curve. By the Uniformization Theorem and the Carathéodory theorem the conformal isomorphism \( \phi|_{V_k} \) extends to a homeomorphism of \( \overline{V_k} \) onto \( \overline{T_k} \).

The boundary piece of \( \Omega \), \( \{ \phi(2\pi\alpha) : k < \alpha < k+1 \} \) is a regular arc. It may not be regular at the endpoints \( \phi(2k\pi) \) and \( \phi(2(k+1)\pi) \). The points where the boundary of \( \Omega \) fails to be smooth, are called the cusps of \( \Omega \). Each cusp is mapped under \( \phi \) to a point \( 2k\pi i \) for some integer \( k \). Computer pictures show that the cusps of the unbounded shell components (see Proposition 4.5) lie in each quadrant which contains the component. The pictures show that there are saddle node bifurcation points along the boundary of any component \( \Omega \in \mathcal{H} \) and there are components attached to \( \Omega \) at these points.

We show that if \( p > 1 \), the asymptotic values of the functions corresponding to the virtual centers of \( \Omega_p \) are pre-poles of order \( p - 1 \). For \( p = 1 \) the virtual center of \( \Omega_1 \) is infinity. First we prove the lemma under the assumption that the components are bounded.

**Lemma 3.4.** For any bounded hyperbolic shell component \( \Omega_p \) with \( p > 1 \), the virtual center \( \lambda^* \) is finite and \( f_\lambda^{(p-1)}(\lambda^*i) = \infty \); that is, \( \lambda^*i \) is a pre-pole of order \( p - 1 \).
Proof. Let \( \lambda \in \Omega_p \) and \( f_{\lambda} \) has an attracting periodic cycle of period \( p \). Let \( U_0 \) be the unbounded component containing asymptotic tract and \( z_0 \in U_0 \) be the periodic fixed point. Suppose \( z_i = f_{\lambda}(z_{i-1}) \) be the attracting periodic cycle. This implies that \( U_1 \) contains \( \lambda i \) and \( z_1 \). Denote the pre-image of \( z_0 \) in the periodic cycle by \( z_{p-1} \). There exists \( n \in \mathbb{Z} \) such that \( f_{\lambda,n}^{-1}(z_0) = z_{p-1} \) and \( f_{\lambda,n}^{-1}(U_0) = U_{p-1} \) where \( n \) is the index to denote the inverse branch of \( \arctan \). Since \( U_0 \) contains an asymptotic tract, \( \partial U_{p-1} \) contains a pole \( s_n \). We note that there is a pre-asymptotic tract at \( s_n \) in \( U_{p-1} \) containing a pre-image of either \( z = \pm i \sqrt{t} \) or \( \pm \sqrt{it} \) for large \( t > 0 \). If \( \partial U_{p-1} \) would contain any other pole there would be another pre-asymptotic tract in \( U_{p-1} \) at this pole containing the pre-image of the same segment and \( f_{\lambda}|_{U_{p-1}} \) would not be injective.

Since the maps \( f_{\lambda} \) for \( \lambda \in \Omega_p \) are quasi-conformally conjugate on their Julia sets, the pre-pole varies continuously with \( \lambda \) in \( \Omega_p \). Suppose \( \lambda \) moves along an internal ray \( R(\alpha) \) to the virtual center \( \lambda^* \) as \( r \to 0 \), so that

\[
\lim_{\lambda \to \lambda^*} \rho_{\lambda} = 0.
\]

Since \( (\tan z)^{2} = 2 \sec z \) and \( \lambda = \frac{z_i}{\tan z_{p-1}-1} \), it follows that

\[
\rho_{\lambda} = [f_{\lambda}^{p}(z_0(\lambda))]^{2} = \prod_{i=1}^{p} f_{\lambda}^{i}[f_{\lambda}^{i-1}(z_0(\lambda))] = 2^{p} \prod_{i=1}^{p} \frac{2z_i z_{i-1}}{\sin 2z_{i-1}^{2}}.
\]

The only way some factor may tend to 0 as \( \lambda \to \lambda^* \) is for \( \sin 2z_{i-1}^{2} \to \infty \) for some \( i \), or equivalently \( \Im z_{i-1}^{2} \to \infty \). Since \( z_0 \) is in the asymptotic tract, we conclude that \( \Im z_{0}^{2} \to \infty \). By hypothesis \( p > 1 \) and \( \lambda^* \neq \infty \) so that \( z_{p-i} \neq z_{p-1} \) for \( i \neq 1 \). Therefore

\[
\lim_{\lambda \to \lambda^*} \frac{\lambda \tan z_{p-1}(\lambda)}{\lambda \tan z_{0}(\lambda)} = \lim_{\lambda \to \lambda^*} \frac{z_{0}(\lambda)}{\lambda} = \infty.
\]
We can say further that
\[
\lim_{\lambda \to \lambda^*} z_{p-1}(\lambda) = \lim_{\lambda \to \lambda^*} f_{\lambda,n}(z_0(\lambda)) = s_n.
\]
and
\[
\lim_{\lambda \to \lambda^*} z_1(\lambda) = \lambda^* i
\]
so that \(\lambda^* i\) is a prepole of order \(p-1\).

**Proposition 3.5.** Let \(\Omega_p\) be a hyperbolic shell component such that for \(\lambda \in \Omega_p\), \(f_\lambda\) has an attracting \(p\)-periodic cycle \(\{z_0, z_1, z_2, \ldots, z_{p-1}\}\). If the virtual center \(\lambda^* = \infty\), then for \(j = 0, 1, \ldots, p-1\) as \(\lambda\) varies along some internal ray \(R(\alpha)\) in \(\Omega_p\),
\[
z_j^* = \lim_{\lambda \to \lambda^*} z_j(\lambda) = \infty.
\]

**Proof.** For \(\lambda \in \Omega_p\), let \(z_0 = z_0(\lambda)\) belongs to the component \(U_0\) that contains an asymptotic tract of \(\lambda i\). Then \(\lambda i\) and \(z_1(\lambda i)\) both belong to the component \(U_1\). Let us assume that \(\lambda\) moves along the internal ray \(R(\alpha)\) in \(\Omega_p\) to the limit point \(\lambda^* = \infty\). Because we assumed \(\lambda^*\) is a virtual center, \(\lim_{\lambda \to \lambda^*} \rho(\lambda) = 0\). It follows that \(\Im z_0^2(\lambda) \to +\infty\). We need to show that \(z_j^* = \lim_{\lambda \to \lambda^*} z_j(\lambda) = \infty\). If not, \(\exists\) a sequence \(\lambda_n \to \lambda^*\) in \(\Omega_p\) such that \(\lim_{\lambda \to \lambda^*} z_1(\lambda) = c \neq \infty\). Then \(z_1(\lambda_n) = \lambda_n \tan(z_0(\lambda_n))^2\) implies
\[
\lim_{\lambda \to \lambda^*} \tan(z_0(\lambda_n))^2 = \lim_{\lambda \to \lambda^*} \frac{z_1(\lambda_n)}{\lambda_n} = 0.
\]
Therefore the curve \(z_0(\lambda), \lambda \in R(\alpha)\) is bounded and \(\lim_{\lambda \to \lambda^*} z_0(\lambda) = m\pi\) for some integer \(m\) or \(z_0(\lambda)\) is unbounded but comes arbitrary close to infinitely many integral multiples of \(\pi\). Either possibility contradicts \(\Im z_0^2(\lambda) \to +\infty\). Thus for \(z_j, j = 2, 3, \ldots, p-1\) we can argue as follows:

If \(z_j \neq \infty\) as \(\lambda \to \lambda^*\), \(j = 2, 3, \ldots, p-1\), then \(\exists\) a sequence \(\lambda_n \to \lambda^*\) in \(\Omega_p\) such that \(\lim_{\lambda \to \lambda^*} z_j(\lambda) = c \neq \infty\), \(j = 2, \ldots, p-1\). Arguing as above we get either \(z_{j-1}(\lambda)\) is bounded and \(\lim_{\lambda \to \lambda^*} z_{j-1}(\lambda) = m\pi\) for some integer \(m\) or \(z_{j-1}(\lambda)\) is unbounded but comes arbitrary close to \(\lambda \to \lambda^*\).
infinitely many integral multiples of \( \pi \). In either case an arbitrary small neighborhood of \( m\pi \) contains an attracting fixed point of order \( p > 1 \) for infinitely many \( \lambda_n \). That is an arbitrary small neighborhood of zero contains \( z_j(\lambda_n) \) for infinitely many \( \lambda_n \). But zero is a critical point and a super-attracting fixed point. The hypothesis \( \lambda \in \Omega_p \) and the fact that the immediate basin of zero contains an attracting fixed point of order \( p > 1 \) give a contradiction. \( \square \)

The following lemma is proved for more general families in [3]. We state them for our families.

**Lemma 3.6.** Let \( \Omega_p, p \geq 1 \) be a hyperbolic shell component and \( \lambda_n \in \Omega_p \) be such that \( \lambda_n \to \lambda^* \) where \( \lambda^* \in \partial \Omega_p \). Let \( \{a_n^0, a_n^1, \ldots, a_n^{p-1}\} \) be the attracting periodic cycle of \( f_{\lambda_n} \) such that \( a_n^1 \) is in the component of the immediate attracting basin that contains the asymptotic value \( \lambda_n i \). Suppose \( |a_n^j| \to \infty \) as \( n \to \infty \). Then \( j = 0 \) and

a) \( a_n^0 \to \lambda_n i \) as \( n \to \infty \).

b) \( a_n^{p-1} \) tends to a pole of \( f_{\lambda}^p \).

c) \( a_n^{k-i} \) tends to a pre-pole of \( f_{\lambda}^k \).

d) The multiplier map \( \rho_n \to 0 \) as \( n \to \infty \).

**Proposition 3.7.** If \( \lambda^* i \) is a pre-pole of \( f_{\lambda^*} \) of order \( p - 1, p > 1 \), then \( \exists \lambda \) near to \( \lambda^* \) such that \( f_{\lambda}^{(p-1)}(\lambda i) \in \mathcal{A} \), for a given asymptotic tract \( \mathcal{A} \).

**Proof.** Let \( \mathcal{A} \) be an asymptotic tract such that \( \mathcal{A} = \{z : \Im z^2 > r, \Re z, \Im z > 0\} \) for large enough \( r, r > 0 \). Let \( U \) be a small neighborhood around \( \lambda^* i \) and \( V \) be the corresponding neighborhood around \( \lambda^* \) such that \( \lambda \in V \) iff \( \lambda i \in U \). Suppose the assumption in the proposition is not true. Then \( \forall \lambda \in V, f_{\lambda}^{(p-1)}(\lambda i) \notin \mathcal{A} \). Now we can define a map \( g : V \to \mathbb{C} \) by \( g(\lambda) = f_{\lambda}^{(p-2)}(\lambda i) \) so that \( g \) is bijective and \( g(V) \) is an open set around \( s_n = f_{\lambda}^{(p-2)}(\lambda i) \). Choose the branch of \( f_{\lambda}^{-1} \) such that \( f_{\lambda}^{-1}(\mathcal{A}) \) is an open set attached at \( s_n \). We get \( U_\lambda = ig^{-1}(g(V) \cap f_{\lambda}^{-1}(\mathcal{A})) \) is an open set attached at \( \lambda^* i \) but \( \lambda i \) is not in \( U_\lambda \) and this is true for all \( \lambda \in V \). So \( \forall \lambda \in V, f_{\lambda}^{-1}(U_\lambda) \) is bounded. But because \( \lambda^* i \) is in a virtual cycle of \( f_{\lambda^*} \), it follows that \( f_{\lambda^*}^{-1}(U_{\lambda^*}) \) is unbounded. Contradiction! \( \square \)

The proof of the following Lemma is modeled on the proof for the tangent family in Keen-Kotus.
Lemma 3.8. Let $\lambda^*i$ be a pre-pole of $f_\lambda$, of order $p - 1$ for $\lambda^*$ in the parameter space of $\mathcal{F} = \{\lambda \in \mathbb{C}^* : f_\lambda(z) = \lambda \tan(z)^2\}$. Then there exists $\lambda$ near $\lambda^*$ such that $f_\lambda$ has an attracting periodic cycle of period $p > 1$.

Proof. Let us choose an arbitrary small $\epsilon > 0$ and a set $U = B(\lambda^*i, \epsilon)$, a disk with center $\lambda^*i$ and radius $\epsilon$. Let $V$ be the corresponding neighborhood of $\lambda^*$ in parameter space such that $\lambda \in V$ iff $\lambda i \in U$. Let us choose an asymptotic tract $A$ for $f_\lambda$ large enough $r > 0$, such that $A = \{z : \Re z > r, \Re, \Im z > 0\}$. For $\lambda \in V$ consider the common pre-asymptotic tracts,

$$\mathcal{I}_n = \cap_{\lambda \in V} f_{n,\lambda}^{-1}(A)$$

attached to the pole $s_n$. We can find $0 < \eta = \eta(r)$ such that $|\arg \lambda - \arg \lambda^*| < \eta$ for $\lambda \in V$. Hence the angle between $f_{n,\lambda}^{-1}(\mathbb{R})$ and $\mathbb{R}$ or $\Im$ is bounded and $\mathcal{I}_n$ contains some triangular domain with one vertex at $s_n$. Let $g : V \to \mathbb{C}$ be a map defined by $g(\lambda) = f^{-1}(\lambda i)$. Then $g(V)$ is an open set containing $s_n$ and there exists open set $V^+ \subset V$ such that $V^+ = g^{-1}(\mathcal{I}_n)$. For any $\lambda \in V^+$, $f^{-1}(\lambda i)$ belongs to an asymptotic tract $A = \{z : \Im z > r^*, \Re z, \Im z > 0\}$ where possibly $r^* < r$. Moreover for the inverse branch such that

$$f_{n,\lambda}^{-1}(s_n) = \lambda^*i$$

we have the property $v_\lambda = f_{n,\lambda}^{-1}(s_n) \neq \lambda i$ by Hurwitz’s Theorem. Then $v_\lambda$ is defined by choosing the branch by analytic continuation and the imaginary part of the square of the pre-image $w_{\lambda,k} = f_{\lambda}^{-1}(v_\lambda)$ lies in the upper half plane and continuously depends on $\lambda$ and $\Re w_{\lambda,k}^2$ goes to $\infty$ as $\lambda \to \lambda^*$.

Now consider $\zeta_\lambda = |v_\lambda - \lambda i|$ and $B_\lambda = B(v_\lambda, \zeta_\lambda)$. Then $f^{-1}(B_\lambda)$ is a neighborhood around $s_n$ and taking the principal part we get $f_{\lambda}^{-1}(B_\lambda)$ is a subset of $\mathbb{C} \setminus D_{R_\lambda}$ where $D_{R_\lambda}$ is a disk around the origin of radius $|f^{-1}(\lambda i)| \approx R_\lambda$ with $R_\lambda \to \infty$ as $\lambda \to \lambda^*$. We need to prove that $\Im (f^{-1}(\lambda i))^2(\lambda i) > \Re w_{\lambda,k}^2$ for $k \leq k_0$ for some $k_0 \in \mathbb{Z}$. Let

$$M = \max_{z \in \mathcal{U}, \lambda \in V} |(f^{-1}(\lambda i))(z)|.$$
As \(|\lambda| \gg |\zeta|\), we have \(w_{\lambda,k} = f_{\lambda}^{-1}(v_\lambda) = f_{\lambda}^{-1}(\lambda i + \zeta_\lambda) = \sqrt{\frac{1}{2\pi} \log \frac{\lambda e^{i\zeta_\lambda}}{2\lambda - i\lambda \zeta_\lambda}}\).

We do the following calculation to estimate \(\Im w_{\lambda,k}\):

\[
\log \frac{\lambda e^{i\zeta_\lambda}}{2\lambda - i\lambda \zeta_\lambda} = \log |\frac{\lambda e^{i\zeta_\lambda}}{2\lambda - i\lambda \zeta_\lambda}| + i(\theta_\lambda + \pi k), \ k \in \mathbb{Z}
\]

\[
\approx \log |\zeta_\lambda| + i(\theta_\lambda + \pi k), \ k \in \mathbb{Z}.
\]

We choose a branch of the square root function so that \(\Re w_{\lambda,k}, \ \Im w_{\lambda,k} > 0\).

So \(w_{\lambda,k} \approx \frac{1}{\sqrt{2}} \sqrt{(\theta_\lambda + \pi k) - i \log |\zeta_\lambda|}, \ k \in \mathbb{Z}\)

\[
\approx R_{k_\lambda} e^{i \arcsin R_{k_\lambda}} \text{ where } R_{k_\lambda} = \frac{1}{\sqrt{2}} \sqrt{(\theta_\lambda + \pi k)^2 + (\log |\zeta_\lambda|)^2} \text{ and } P_{k_\lambda} = (\theta_\lambda + \pi k) - i \log |\zeta_\lambda|.
\]

Therefore \(|f_{n_\lambda,1}^{-1}(w_{\lambda,k}) - s_n| \approx \sqrt{\frac{1}{R_{k_\lambda}}}, \ k \in \mathbb{Z}\)

\[
|f_{n_{p-1},\lambda}^{-1}(w_{\lambda,k}) - \lambda i| \geq |f_{n_\lambda,1}^{-1}(w_{\lambda,k}) - s_n| \cdot \min_{z \in U, \lambda \in V} |(f_{\lambda}^{p-2})'(z)|
\]

\[
|f_{n_{p-1},\lambda}^{-1}(w_{\lambda,k}) - \lambda i| \geq \frac{1}{\pi} \sqrt{\frac{1}{R_{k_\lambda}}},
\]

Since \(\zeta_\lambda\) is assumed small we can get \(k_0 \in \mathbb{Z}\) such that \(\frac{1}{\pi} \sqrt{\frac{1}{R_{k_\lambda}}} \geq \zeta_\lambda, \ \forall k \leq k_0\).

Therefore \(|f_{\lambda}^{(p-2)}(\lambda i) - s_n| \leq |f_{n_\lambda,1}(w_{\lambda,k}) - s_n|, \ \forall k \leq k_0\)

so that \(|f_{\lambda}^{p-1}(\lambda i)| > |w_{\lambda,k}|, \ \forall k \leq k_0\). It follows that

\[
\Im (f_{\lambda}^{p-1})^2(\lambda i) > \Im w_{\lambda,k}^2.
\]

Now we will construct a domain \(\mathcal{T}\) for some fixed \(\lambda \in V^+\) inside the asymptotic tract \(\mathcal{A}\) such that \(f_{\lambda}^{p}(\mathcal{T}) \subset \mathcal{T}\). Let \(\tilde{R}_\lambda = \frac{1}{2}(\log |\zeta_\lambda|) - \epsilon\). Take \(\mathcal{A} = \{z : \Im z^2 > \tilde{R}_\lambda\}\) so that \(v_\lambda \in f_{\lambda}(\mathcal{A})\). Let \(I^\pm\) be the two rays meeting at \(s_n\) such that the triangular domain \(T\) between them is contained in \(f_{\lambda,1}^{-1}(\mathcal{A})\) and such that \(f_{\lambda}^{p-2}(\lambda i) \in T\). Let \(\mathcal{S}\) be the triangular region with vertex at \(v_\lambda\) bounded by \(\mathcal{J}^\pm = f_{n_{p-2},\lambda}(I^\pm)\) and an arc of the boundary of \(f_{\lambda}(\mathcal{A})\) so that \(\lambda i \in \mathcal{S}\).
Finally let $S = \cup_{k \in \mathbb{Z}} f_{\lambda, k}^{-1}(S)$. Then $S$ is an asymptotic tract whose boundary is formed by pre-images of $\partial S$, i.e it is made up of arcs $f_{\lambda, k}^{-1}(\mathcal{J}^\pm)$ that meet at $w_{\lambda, k}$. Now consider $\tilde{S} = f_{\lambda}^{(p-1)}(S)$. This is a triangle with a vertex at infinity, the sides meeting there are rays and the third side is an arc of a circle centered at the origin and the radius is slightly smaller than $|f_{\lambda}^{(p-1)}(\lambda i)|$. To prove $f_{\lambda}^p(S) \subset S$, we need to check:

1. $\Im(f_{\lambda}^{(p-1)}(\lambda i))^2 > \tilde{R}_\lambda$ and
2. $f_{\lambda}(I^\pm) \subset A$.

Now $\lambda$ was chosen so that $f_{\lambda}^{(p-1)}(\lambda i)$ is in the asymptotic tract of the asymptotic value $\lambda i$, hence changing the argument we can insure 1 holds. 2 can be insured by decreasing the angle between $I^\pm$ if necessary.

Lemma 3.8 shows that there is a hyperbolic component attached to the point $\lambda^*$ in the parameter plane. The proof shows that it is a shell component and therefore every virtual cycle parameter is a virtual center.

Remark 3.1. For a shell component, although both asymptotic values are attracted to the periodic cycle, one is preferred in the sense that one is contained in the periodic component of Fatou set while the other is contained in a pre-periodic component. The above construction finds the preferred asymptotic value.

We have just shown that if $\lambda^*$ is a virtual center then it is virtual cycle parameter. Thus the set

$$\{\infty, \pm \lambda^* i, f_{\lambda^*}(\pm \lambda^* i), f_{\lambda^*}^2(\pm \lambda^* i), \ldots, f_{\lambda^*}^p(\pm \lambda^* i)\}$$

is a cycle, considered with appropriate limits. This cycle behaves like a super-attractive cycle with a singular value, namely the asymptotic value.

Set

$$\mathcal{D}_p = \{\lambda^* : f_{\lambda^*}^p(\lambda^* i) = \infty\}, \quad \mathcal{D} = \cup_p \mathcal{D}_p.$$
Now we can prove that for $\lambda^* \in D_{p-1}$ we can find a quadruplet $\{\Omega_p^i\}_{i=1}^4$ such that $\lambda^*$ is a virtual center for the quadruplet. We will first show that each virtual cycle parameter corresponds to a virtual center of a shell component.

**Theorem 3.9.** Let $\Omega$ be a shell component of period $p \geq 2$ and $\lambda^* \in \partial \Omega$. Then $\lambda^*$ is a virtual center if and only if $\lambda^*$ is a virtual cycle parameter.

**Proof.** Let $\lambda^*$ be the virtual center. Let $\lambda_n$ be a sequence of parameters in $\Omega$ such that $\lambda_n \to \lambda^*$ as $n \to \infty$ and let $\{a_0^n, \ldots, a_{p-1}^n\}$ be the corresponding attracting cycle of $f_{\lambda_n}$. If one of the $a_j^n$ tends to infinity as $n \to \infty$, then by Lemma 3.8 we are done. Now suppose that all points of the periodic cycle converge to finite points and the multiplier tends to 0. It follows that the periodic cycle is super-attracting and the cycle contains a critical point. That contradicts the assumption that $\lambda^*$ is in $\partial \Omega$. By the definition of a virtual cycle parameter at least one of the points of the cycle is the point at infinity. The multipliers of the cycles of $\lambda_n$ tend to the multiplier of the virtual cycle of $\lambda^*$ and $\lambda^*$ is a virtual center. \qed

**Proposition 3.10.** If $\lambda^* i$ is a prepole of $f_{\lambda^*}$ of order $p-1$ then there are four hyperbolic components $\{\Omega_p^i\}_{i=1}^4$ attached at $\lambda^*$ such that $f_{\lambda^*} \in \Omega_p^i$, has an attracting cycle of period $p$.

**Proof.** We saw in Proposition 3.7 and in Lemma 3.8 that given a virtual center at $\lambda^*$ of order $p-1$ and an asymptotic tract, we can have an unbounded periodic component containing the asymptotic tract. Given an asymptotic tract $\mathcal{A}$, we can choose $\Omega_p$ at $\lambda^*$ uniquely so that $f_{\lambda^*}^{p-1}(\lambda^* i) \in \mathcal{A}$. As there are four asymptotic tracts, there are four hyperbolic components of order $p$ attached to a virtual center of order $p-1$. \qed

**Proposition 3.11.** Suppose $\lambda^* \in D_{p-1}$ so that for $\lambda \in \Omega_p^i$, $f_{\lambda^*}^{(p-2)}(\lambda^* i) = s_n$. That is $\lambda^*$ is the virtual center of a quadruplet $\{\Omega_p^i\}_{i=1}^4$ so that $f_{\lambda}$ has attracting $p$-periodic cycle for a $\lambda \in \Omega_p^i$. Then there exists a sequence of component quadruplets $\{\Omega_p^i, k\}_{i=1}^4$, $k \in \mathbb{Z}$ with virtual centers $\lambda_k^* \in D_p$ where $f_{\lambda_k^*}^{(p-1)}(\lambda_k^* i) = s_k$ and $\lambda_k^* \to \lambda^*$ as $|s_k| \to \infty$.

**Proof.** Choose an arbitrary small $\epsilon > 0$ and let $U = B(\lambda^* i, \epsilon)$ be a small neighborhood around $\lambda^* i$
in the dynamic plane and let $V = D(\lambda^*, \epsilon)$ be the corresponding open set in the parameter space. Consider $g(\lambda) = f^{(p-2)}(\lambda i)$, $\lambda \in V$. Then $g(U)$ is an open set containing $s_n$. Taking the principal part of $f_{\lambda^n}$, we get $f^{(p-1)}_{\lambda^n}(U)$ is an unbounded set and $\exists k_0 \in \mathbb{Z}$ such that $\pm s_k \in f^{(p-1)}_{\lambda^n}(U)$, $\forall k \geq k_0$. The $s_k$ are pre-poles of higher order converging to $s_n$ so they are in $f^{(p-2)}_{\lambda^n}(U)$. Thus there are $\lambda_k \in D(\lambda^*, \epsilon)$ such that $f^{(p-1)}_{\lambda_k}(\lambda_k i) = \pm s_k$. So $\lambda_k \in D_p$. Using Lemma 3.8 Proposition 3.9 we get $\lambda_k \to \lambda^*$ as $|s_k| \to \infty$. □

**Proposition 3.12.** Let $\lambda_n \in D_p$. Then $\lambda_n$ is a virtual center for a sequence of components $\{\Omega^p_i\}_{i=1}^{N}$ with itineraries $n_p = (n_1, n_2, \ldots, n_p)$.

(a) If $(n_1, n_2, \ldots, n_p-1)$ are the same for all $\lambda_n$ and $n_p = n$ then the sequence $\lambda_n$ has accumulation point in $D_0 = \{\infty\}$.

(b) If $(n_2, n_3, \ldots, n_p)$ are the same for all $\lambda_n$ and $n_1 = n$ then the accumulation point of $\lambda_n$ is $\lambda \in D_{p-1}$ where $\lambda$ is a virtual center with itinerary $n_{p-1} = (n_2, \ldots, n_p)$ with $f^{(p-2)}_{\lambda_n}(\lambda)$ is well-defined and holomorphic in a neighborhood of $\lambda_n$.

**Proof.** Consider the set $S = \mathbb{C} \setminus \cup_{k=1}^{p-1} D_{p-1}$. Define a map $g : S \to \hat{\mathbb{C}}$ by $g(\lambda) = f^1_{\lambda}(\lambda)$. We have removed the set $\cup_{k=1}^{p-1} D_{p-1}$ because $g$ would have essential singularities at those points and $g$ is well-defined in $S$. From the construction of the set $S$ we see that $g$ has poles for $\lambda \in D_p$ and $g$ is holomorphic elsewhere. Suppose $\lambda'$ is an accumulation point of $\lambda_n \in S$. If $\lambda' \not\in \cup_{k=1}^{p-1} D_k \cup \{\infty\}$ then $g(\lambda)$ is well-defined and holomorphic in a neighborhood of $\lambda'$. On the other hand $\lambda'$ is an accumulation point of poles of $g$ and then $g$ has a non-removable singularity at $\lambda'$. Thus we arrive at a contradiction. We claim that $\lambda' \in D_0 = \{\infty\}$. Thus $\lambda_n = f^{-1}_{\lambda_n} \circ (f^{-1}_{n_{p-1}} \circ (\ldots f^{-1}_{n_1}(\infty)))$ implies that $\lambda_n^2$ is in $L_n$ where $L_n$ is the half open vertical strip between $l_{n-1} = (n-1/2)\pi/2 + it$ and $l_n = (n + 1/2)\pi/2 + it$, $t \in \mathbb{R}$, $n \in \mathbb{Z}$ containing the line $l_{n-1}$. So the only accumulation point $\lambda_n$ can have is at $\infty$.

As $n_1$ varies, we can write $\lambda_n = (f^{-1}_{n_p} \circ f^{-1}_{n_{p-1}} \circ \ldots \circ f^{-1}_{n_1}) \circ f^{-1}(\infty)$. Therefore $\lambda_n \to \lambda^*$ implies $\lambda^* = (f^{-1}_{n_p} \circ f^{-1}_{n_{p-1}} \circ \ldots \circ (\lim_{|n| \to \infty} f^{-1}(\infty)) = (f^{-1}_{n_p} \circ f^{-1}_{n_{p-1}} \circ \ldots)(\lim_{|n| \to \infty} s_n) = (f^{-1}_{n_p} \circ f^{-1}_{n_{p-1}} \circ \ldots f^{-1}_{n_1}(\infty) \in D_{p-1}$.

**Proposition 3.13.** Let $\lambda_n \in D_p \cap \mathbb{R}$ (or $D_p \cap \mathbb{R}$). Then $\lambda_n$ is a virtual center for a sequence of components $\{\Omega^p_i\}_{i=1}^{N}$ with itineraries $n_p = (n_1, n_2, \ldots, n_p)$.
(a) If \((n_1, n_2, \ldots, n_{p-1})\) are the same for all \(\lambda_n\) and \(n_p = n\) then the sequence \(\lambda_n\) has accumulation point in \(D_0 = \{\infty\}\).

(b) If \((n_1, \ldots, n_{j-1}, n_j+1, \ldots, n_p)\) are the same for all \(\lambda_n\) and \(n_j = n\) for \(1 \leq j \leq n_{p-1}\), then the accumulation point of \(\lambda_n\) is \(\lambda \in D_{j-1}\) where \(\lambda\) is a virtual center with itinerary \(n_{p-j} = (n_{j+1}, \ldots, n_p)\) where \(f^{(p-j-1)}(\lambda_i) = s_{n_{j+1}}\).

**Proof.** The proof of (a) is similar to the proof of Proposition 3.12 (a). To prove (b) we see that, if \(n_j = n\), we can write

\[
\lambda_n = f_{n_p}^{-1} \circ f_{n_{p-1}}^{-1} \circ \ldots \circ f_{n_j}^{-1} \circ f_{n_{j+1}}^{-1} \circ \ldots \circ f_{n_1}^{-1}(\infty). \]

Therefore \(\lambda_n \to \lambda^*\) implies

\[
\lambda^* = (f_{n_p}^{-1} \circ f_{n_{p-1}}^{-1} \circ \ldots \lim_{n \to \infty} f_{n_j}^{-1} \circ f_{n_{j+1}}^{-1} \circ \ldots \circ f_{n_1}^{-1}(\infty)) = (f_{n_p}^{-1} \circ f_{n_{p-1}}^{-1} \circ \ldots \circ f_{n_{j+1}}^{-1})(\infty) \quad \text{(By part (a))}. \]

In other words, \(f_{\lambda^*}^{(p-j-1)}(\lambda^* i) = s_{n_{j+1}}\). \(\square\)

## 4 Bifurcation at the boundaries and the boundedness of shell components

We have proved in Theorem 3.2 that there is a universal covering map, namely the multiplier map, \(\rho_{\lambda} : \Omega_p \to \mathbb{D}^*\) which can be lifted to a conformal isomorphism \(\phi : \mathbb{H}_l \to \Omega_p\), where \(\mathbb{H}_l\) denotes the right half plane so that \((\rho_{\lambda} \circ \phi)(c) = \exp^{2\pi i c} : \mathbb{H}_l \to \mathbb{D}^*\) and the map \(\phi\) extends continuously to the boundary of \(\mathbb{H}_l\).

**Definition.** We define a boundary point \(\lambda \in \partial \Omega_p\) to be a point of internal angle \(\alpha\) if \(\lambda = \rho_{\lambda}^{-1}(e^{2\pi i \alpha})\).

Suppose \(\lambda_0\) is a boundary point of \(\Omega_p\) where \(f_{\lambda_0}\) has a parabolic periodic cycle. If there is another component \(\Omega_q\), with boundary point \(\lambda_0\), and if \(p|q\), then \(\Omega_q\) is called a *bud* of \(\Omega_p\) and if \(q|p\) then \(\Omega_q\) is called a *root* of \(\Omega_p\). In a standard period doubling bifurcation each attractive cycle of period \(p\) bifurcates to an attractive periodic cycle of period \(q = 2p\). For maps in \(\lambda \tan z\) family, (in [5]) it is shown that a non-standard period doubling bifurcation occurs where a single attractive cycle bifurcates to two distinct attractive cycles of the same period. This kind of bifurcation is called cycle doubling bifurcation.
Keeping this in mind, we see that each of the lines \( P(\alpha) = \{ \phi(t + 2\pi i \alpha) | t \in (-\infty, 0), \alpha \in (0, 1) \} \) corresponds to an internal ray with the multiplier having a real value and one end of the ray corresponds to a virtual center while the other corresponds to the multiplier taking the value 1 or -1.

Furthermore we will see that all period doubling bifurcations occur along internal rays with \( \alpha = \frac{q}{p} \) with \( \gcd(q, p) = 1 \), \( p \neq 0 \) and any period \( p \) cycle bifurcates into a period of \( qp \) cycle. Since both asymptotic values have the same forward orbit, there is only one periodic attractive cycle for \( \lambda \) in a shell component. Therefore there can be no occurrence of cycle doubling bifurcations in this family.

The proof of the following results follows the text in [5]. We summarize the results here. We will see that the bud components are again shell components.

**Theorem 4.1.** Let \( \Omega \) be a shell component of \( \mathcal{F} \). Let \( \lambda \in \partial \Omega \), \( \rho_\lambda = e^{2\pi i \frac{q}{p}} \) with \( \gcd(q, p) =
1, \( p \neq 0, 1 \) and \( f_p^1(z_0) = z_0 \). Then there is a map \( \tilde{f}_\lambda \) such that \( \tilde{f}_\lambda \) has one repelling cycle at \( z_0 \) and one attracting cycle of period \( p \).

**Proposition 4.2.** Let \( \Omega_n \) be an arbitrary shell component of period \( n \). Suppose \( \lambda_0 \in \partial \Omega_n \) such that \( m_{\lambda_0} \) is a \( p \)-th root of unity and let \( f(z) \) be analytic in a neighborhood of the periodic point. Then there is a map \( \tilde{f}_\lambda \) such that \( \tilde{f}_\lambda \) has one repelling cycle at \( z_0 \) and one attracting cycle of period \( p \).

**Theorem 4.3.** For a given shell component \( \Omega_n \) of period \( n \), there are components \( \Omega_{np} \) called bud components attached to \( \Omega_n \) at the point of internal argument \( q/p, p \neq 0, 1 \) and \( \gcd(p, q) = 1 \).

The period of \( \Omega_{np} \) is \( np \).

Let \( \Omega_{np} \) be a bud component attached to \( \Omega_n \) at the boundary point \( \lambda^* \) of \( \Omega_n \) of internal argument \( \frac{q}{p} \). The point \( \lambda^* \) is the root of \( \Omega_{np} \). Let \( m_\lambda : \Omega_{np} \to \mathbb{D}^* \) be the conformal covering map induced by the multiplier. There are \( n \) periodic points \( z_i, i = 1, 2, \ldots, n \) of \( f_{\lambda^*} \) of period \( n \) with \( \prod_{i=1}^{n} f_{\lambda^*}^i(z_i) = e^{2\pi q/p} \). For \( \lambda \in \Omega_{np} \), for each \( i = 1, 2, \ldots, p \) in the \( n \) disjoint neighborhoods \( N_i \) of \( z_i \), there are \( p \) periodic points \( \xi_{ij} \) of \( f_\lambda \) of period \( np \) and \( \xi_{ij} \to z_i \) as \( \lambda \to \lambda^* \).

Therefore in the bud component \( \Omega_{np} \), the multiplier of the attracting cycle of period \( np \) satisfies

\[
\prod_{i=1}^{n} \prod_{j=1}^{p} f_{\lambda^*}^i(\xi_{ij}) \to \prod_{i=1}^{n} \prod_{j=1}^{p} f_{\lambda^*}^i(z_i) = \prod_{j=1}^{p} e^{2\pi q/p} = e^{2\pi q} \text{ as } \lambda \to \lambda^*.
\]

Therefore like the cusps, the root \( \lambda^* \) of a component \( \Omega \) is mapped under \( \phi \) (as defined earlier \( \phi : \mathcal{H}_1 \to \Omega_p \)) to a point \( 2k\pi \) for some integer \( k \). The computer picture shows that the boundary of \( \Omega \) is smooth at its root \( \lambda^* \).

Using Proposition 4.2, we see that the buds in turn have buds. For any component \( \Omega_p \) we can locate the bud components attached to \( \Omega_p \) by following the internal rays of rational arguments.

Let \( \Omega_p \) be an unbounded shell component. We can denote the bud components of \( \Omega_p \) attached to it at its boundary points of internal argument \( \frac{q_1}{p_1} \). The component has period \( p_1 \). We can then locate the bud components of \( \Omega_{\frac{q_1}{p_1}} \). The one attached to it at its boundary points with internal argument \( \frac{q_2}{p_2} \) is denoted by \( \Omega_{\frac{q_1q_2}{p_1p_2}} \). This bud component has period \( p_1p_2 \). Suppose we are at a component \( \Omega_{\frac{q_1q_2}{p_1p_2} \ldots p_k} \) of period \( p_1p_2 \ldots p_k \) where \( \frac{q_i}{p_1p_2 \ldots p_k} \) are all in \( Q/\mathbb{Z} \). Following the internal
Figure 2: Arrangement of the capture and shell components along the imaginary axis, CC = Capture Component, Number = Period of the component.

ray of \( \Omega q_1q_2\ldots q_k \) of argument \( \frac{q_k+1}{p_k+1} \in Q/Z \), we can locate a bud component attached to it at the point of internal argument \( \frac{q_k+1}{p_k+1} \). The period of this bud is \( p_1p_2\ldots p_kp_{k+1} \). This gives us a way to code the components.

However we may need to locate a component attached to the current component at the virtual center. In this case, we see in Proposition \ref{prop:virtual_center} that the period of that component is same as the period of the current component. We proved that all shell components appear in quadruplet and each quadruplet has a unique virtual center. Since the four components are attached at their shared virtual center, coding the virtual centers give a coding of the component quadruplets. By Proposition \ref{prop:virtual_center}, given an asymptotic tract, we can choose a shell component from a quadruplet at a given virtual center. Therefore the coding can be done by adding another subscript \( i = 1, 2, 3, 4 \) and by choosing the corresponding asymptotic tract that are in the periodic domain.
### 4.1 Unbounded components

**Proposition 4.4.** For $\lambda$ of the form, $\lambda = \pm i\sqrt{t}$ or $\lambda = \pm \sqrt{it}$ there exists some $s > 0$ such that for all $t > s > 0$, $f_\lambda$ has only one attracting fixed point (the origin is always a super-attracting fixed point). These $\lambda$'s belong to unbounded shell components.

**Proof.** First we will show that there is a periodic cycle for such $t$ and hence the multiplier map $\rho(\lambda) < 1$.

If $\lambda = \sqrt{it}, \ t > 0$, then

\[
f_\lambda(\lambda i) = \sqrt{it} \tan(it) = -i\sqrt{it} \tanh t.
\]

\[
f_\lambda^2(\lambda i) = \sqrt{it} \tan(it \tanh^2 t) = -i\sqrt{it} \tanh(t \tanh^2 t)
\]

\[
\vdots
\]

Therefore

\[
|f_\lambda^n(\lambda i)| = |-i\sqrt{it} \tanh(t \tanh^2 \ldots (t \tanh^2 t))\ldots)|
\]

\[
\leq |\sqrt{it}|
\]

Thus $f_\lambda^n(\lambda i)$ is on the line $l = -i\sqrt{iy}, \ y > 0$, for all $n$ and the line $l = -i\sqrt{iy}, \ y > 0$, is forward invariant under $f_\lambda$ for $\lambda = \sqrt{it}$. Moreover $|f_\lambda^n(i\sqrt{it})| < |\sqrt{it}|$ implies that the sequence $f_\lambda^n$ forms a normal family and is bounded by $|\lambda|$ and therefore $\lambda = \sqrt{it}$ is in a shell component for some $t > 0$. Also the orbit of the asymptotic values is bounded by $|\lambda| = \sqrt{t}$. Therefore the periodic point $z_i$ of the limit function satisfies $|z_i| < |\lambda|$ and $z_j = -i\sqrt{ix_j}$, for some $x_j > 0$. 

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Let \( \{ z_j = -i \sqrt{t} x_j \}_{j=0}^{p-1}, p > 1 \) be the set of periodic points and \( U_j \) be the corresponding periodic components labeled such that \( U_0 \) contains an asymptotic tract. The component \( U_1 \) containing the asymptotic value \( \lambda i \) contains the periodic point \( z_1 \) such that \( |z_1| > |z_j| \) for \( j \neq 1 \). This implies the asymptotic value is in the component containing the asymptotic tract and therefore the periodic component is invariant.

Since the central capture component is a simply connected component containing the origin, the component meets the line \( l \), and there is some \( s > 0 \) such that the above holds for all \( t > s > 0 \).

If \( \lambda = -\sqrt{it}, \ t > 0 \), imitating the above calculation we get the fixed point \( i \sqrt{it}, \ x > 0 \). For \( \lambda = i \sqrt{it} \) or \( \lambda = -i \sqrt{it} \) the proof follows by similar argument.

In the rest of this section our main goal is to prove that the shell components of period greater than one are bounded. To prove this we need to discuss the boundaries of the unbounded hyperbolic components. The next result describes the asymptotic behavior of the boundaries of the unbounded hyperbolic components. In Theorem 4.7, we will conclude the section with the final result.

**Proposition 4.5.** Let \( \Omega_{1}^j, j = 1, 2, 3, 4 \) be the unbounded shell components containing \( \lambda = \pm \sqrt{it}, \ t > s > 0 \) for some \( s \) (see 3.5 for the coding of shell components). The index \( j \) denotes the asymptotic tract, contained in the periodic domain of \( f_{\lambda} \). Then the boundary of \( \Omega_{1}^j \) is asymptotic to the curve \( \pm \sqrt{|t|} \pm ie^2\sqrt{|t|} \) as \( \Re \lambda = |t| \to \infty \).

**Proof.** We will prove this only for \( \Omega_{1}^1 \), the unbounded shell component in the first quadrant. The proof for other components follows by the symmetry. Let \( z = z(\lambda) \) be an attracting fixed point of \( f_{\lambda} \) for \( \lambda \in \Omega_{1}^1 \). Then \( f_{\lambda}(z) = z \) implies that \( \lambda \tan z^2 = z \). The multiplier map \( \rho_{\lambda} \) is given by
\[ \rho_\lambda = 2\lambda \sec^2 z^2 \text{ with } |\rho_\lambda| < 1 \text{ or equivalently } |2\lambda \sec^2 z^2| < 1. \]

So we have,

\[
2\lambda \sec^2 z^2 = \frac{2\lambda \sin z^2}{\sin z^2 \cos z^2 \cdot \cos z^2} = \frac{4\lambda \tan z^2}{\sin 2z^2} = \frac{2.2z^2}{\sin 2z^2} = \frac{2u}{\sin u}, \quad u = 2z^2.
\]

So the above condition can be written as \( \left| \frac{2u}{\sin u} \right| < 1. \)

Let \( H(u) = \frac{2u}{\sin u}. \)

In \( u = x + iy \) plane the curve \( |H(u)| = 1 \) has two branches symmetric about the \( x \)-axis and contained in the upper half and lower half regions. The boundary of \( |H(u)| = 1 \) is asymptotic...
to the curve $|x| \pm ie^{2|x|}$ as $|x| \to \infty$ and $|H(u)| < 1$ in the upper and lower half regions. Looking at these curves in the $z$-plane, we have $\pm \sqrt{|t|} \pm ie^{2\sqrt{|t|}}$ as $|t| \to \infty$.

Let $S(u) = \frac{\sqrt{u}}{(\sqrt{2} \tan \frac{u}{2})}$. The set of $u$ that satisfies $|S(u)| \geq 1$ is unbounded and contains region in the upper and lower half planes. The regions meet the upper and lower half planes in two unbounded, simply connected domains. If we set $\lambda = S(u)$, then the function maps each of these unbounded regions to some domain $\Omega$ in the $\lambda$-plane so that $f_\lambda$ has an attracting fixed point. That implies these unbounded regions are mapped to a hyperbolic shell component $\Omega$ of period one. Since $S$ maps the lines $x = 0$, $y \neq 0$ to $\lambda = \sqrt{t}$ for $t > s > 0$ for some $s$, then $\Omega = \Omega_1$.

The asymptotic behaviour of $\Omega_1$ directly follows from the asymptotic behavior of the curves $|H(u)| = 1$. By the symmetry in $F$, the boundary of the other unbounded shell components behave in the same way.

**Proposition 4.6.** Let $\Omega_2$ be the bud component tangent to $\Omega_1$ at $\lambda_k$ as above. The virtual center $\lambda^*$ is equal to $s_k i$ where $s_k$ denotes the pole of $f_\lambda(z)$.
Proof. We claim that $\lambda^*$ is finite. If not, there will be a sequence of $\lambda_j$ in some internal ray in $\Omega_2$ with an end point at $\lambda_k$ and the other end point tending to $\lambda^*$. For simplicity, we omit the subscript $j$ for both sequences in parameter space and use it for the corresponding sequence for periodic points. Consider $\lambda = \lambda_1 + i\lambda_2$, $z_j = x_j + iy_j$, $j = 0, 1$ where $z_j$ are the corresponding periodic points of period 2. We denote $X_j = \Re z_{2j}$ and $Y_j = \Im z_{2j}$. From the equation $z_1 = \lambda \tan z_0^2$ we get

$$X_1 = \frac{\lambda_1 \sin(2X_0) - \lambda_2 \sinh(2Y_0)}{\cos(2X_0) + \cosh(2Y_0)} \quad (A)$$

$$Y_1 = \frac{\lambda_1 \sinh(2Y_0) + \lambda_2 \sin(2X_0)}{\cos(2X_0) + \cosh(2Y_0)} \quad (B)$$

As $\lambda \to \infty$, $Y_0 \to \infty$. We have that $\lambda_2 \geq e^{2\sqrt{7}}$. Then $|X_1| \geq |\lambda_1 + \lambda_2 \sinh(2Y_0)|$ implies that $|X_1| \approx \lambda_2 \to \infty$. Using periodicity we can interchange $X_1$, $Y_1$ by $X_0$, $Y_0$ in the equations (A) and (B). As $|X_1| \to \infty$, the term $\lambda_2 \sin(2X_1)$ in $Y_0$ oscillates. Since $Y_0 \to \infty$ the term $|\lambda_1 \sinh(2Y_0)|$ must grow faster than $|\lambda_2 \sin(2X_0)|$which implies that $2Y_0 \approx \pm \lambda_1 \to \infty$. Using periodicity, we get $|X_0| \approx \pm \lambda_2 \to \infty$ and $2Y_1 \approx \pm \lambda_2 \to \infty$ similarly. Therefore we can estimate the multiplier map as
so that $|\rho| \approx 4\lambda_2^2 e^{4\lambda_{1}}$ or $|\rho| \approx 4\lambda_1^2$. So the multiplier map grows with $4\lambda_2^2$ along the internal ray as $\lambda$ tends to $\lambda^*$. Therefore the multiplier cannot tend to zero.

Thus $\lambda^*$ is finite and it is a pre-pole of $f_{\lambda^*}$ of order one. For each bifurcation parameter $\lambda_k \in \partial\Omega_{i} \cap \partial\Omega_{j}$, there is some internal curve $\gamma_k$ in $\Omega_{j}$ with one end at $\lambda_k$ and the other end at $s_n$ for some $n$. From the discussion of the section 4, we have that the curves $\gamma_k$ are all disjoint and lie in order. From the Lemma 3.4 we get that each of these $\gamma_k$ has one-to-one correspondence with $s_n \in \partial\Omega_2$. Thus by renaming the virtual center, if needed, we get the conclusion.

**Theorem 4.7.** The hyperbolic shell components $\Omega_{p}$ are bounded component for $p > 1$.

**Proof.** For each integer $n$, we can choose parameters $\pm s_n, \pm s_n i$; Choose the period two shell components $\pm \Omega_{2,n}, \pm \Omega_{2,n} i, \pm \Omega_{2,n} i$ budding off the shell component of period one and are attached to the respective virtual centers. Choose curves $\pm \gamma_n, \pm \gamma_n i, \pm \gamma_n i \pm \gamma_n i$ in $\pm \Omega_{2,n}, \pm \Omega_{2,n} i, \pm \Omega_{2,n} i$ respectively such that the curves $\pm \gamma_n, \pm \gamma_n i, \pm \gamma_n i \pm \gamma_n i$ with the boundary arcs of $\Omega_{j}, j = 1, 2, 3, 4$ enclose a region. Any shell component $\Omega_p, p > 1$ except $\pm \Omega_{2,n}, \pm \Omega_{2,n} i, \pm \Omega_{2,n} i$ lies in one of these bounded region as defined above. That proves $\Omega_p$ is bounded.

**Corollary 4.8.** All capture components are bounded.

**Proof.** Given a capture component, $C_{n_k}$ locate the center $c_{n_k}$. Now we can choose $\pm s_{n_{k+1}}, \pm s_{n_{k+1}}$, and follow the technique used in the proof of 4.7 to find a region encloses $C_{n_k}$. That proves the result.
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