Spinon-Holon Attraction in the Supersymmetric $t-J$ Model with $1/r^2$-Interaction

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We derive the coordinate representation of the one-spinon one-holon wavefunction for the supersymmetric $t-J$ model with $1/r^2$-interaction. This result allows us to show that spinon and holon attract each other at short distance. The attraction gets stronger as the size of the system is increased and, in the thermodynamic limit, it is responsible for the square root singularity in the hole spectral function $\tilde{\mathcal{N}}$.

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The low-lying excitations of one-dimensional (1-d) strongly-correlated electron systems are not Landau’s quasiparticles or quasiholes carrying both charge and spin $\frac{1}{2}$, but rather collective modes carrying spin-1/2, but no charge (spinons), or charge 1, but no spin (holons) $\frac{1}{2}$. Spinons and holons are semions, i.e., particles with statistics half that of regular fermions $\frac{1}{2}$. In the thermodynamic limit, the physics of 1-d correlated electron systems is described by a gas of spinons and holons $\frac{1}{2}$. The corresponding energy is additive, being the sum of the energies of each isolated particle. Although this implies that the spinon-holon interaction energy is irrelevant in the thermodynamic limit, it gives no information about short-distance dynamics.

In this letter we carefully analyze the short-distance interaction between a spinon and a holon in an exact solution of the supersymmetric $t-J$ model with $1/r^2$ interaction (“Kuramoto-Yokohama” (KY)-model”) $\frac{1}{2}$. The KY-model is a system of electrons on a lattice with periodic boundary conditions. Double occupancy of a site is forbidden by strong Coulomb repulsion. Unoccupied sites are allowed and therefore holes can live on the lattice. Charge hopping, Coulomb, and spin-spin antiferromagnetic terms are inversely proportional to the square of the chord between the corresponding sites. The Hamiltonian is given by

$$H_{\text{KY}} = J \left( \frac{2\pi}{N} \right)^2 \sum_{\alpha<\beta} \frac{1}{|z_\alpha - z_\beta|^2} P \left\{ \vec{S}_\alpha \cdot \vec{S}_\beta - \frac{3}{4} \right\} P,$$

$$- \frac{1}{2} \sum_{\alpha} \left( c_{\alpha\uparrow}^\dagger c_{\alpha\downarrow} + \frac{1}{4} (n_\alpha + n_\beta) - \frac{1}{4} n_\alpha n_\beta - \frac{3}{4} \right) P,$$

where $z_\alpha = \exp(2\pi i \alpha/N)$ ($\alpha$ is a lattice site, $N$ is the number of sites), $\vec{S}_\alpha$ is the spin operator at site $\alpha$, $c_{\alpha\sigma}$ is the electron operator at site $\alpha$, $n_\alpha = c_{\alpha\uparrow}^\dagger c_{\alpha\downarrow}$, and $P$ is the Gutzwiller projector that annihilates configurations with doubly-occupied sites: $P = \prod_\alpha (1 - n_\alpha^\dagger n_\alpha)$.

At filling-1/2, the KY model coincides with the Haldane-Shastry (HS) model of 1-d antiferromagnet.

In recent papers $\frac{1}{2}$, we have worked out the real-space coordinate representation for two-spinon wavefunctions of the HS model. In this letter, we extend our technique to define and work out the coordinate representation for one-spinon one-holon eigenfunctions and the corresponding Schrödinger equation for $H_{\text{KY}}$ close to filling-1/2, where antiholon excitations, made out of a hole created in a holon sea, are ruled out $\frac{1}{2}$. Spinon-holon interaction and its nature follow from the behaviour of the exact solution of this equation. In Fig.1 we plot the result. The probability does not depend on the spinon-holon separation when they are far apart. This shows that the two particles are noninteracting at large distances. However, at short separations, the probability is largely enhanced, a clear evidence of a short-range attractive interaction between the two particles. As the size of the system is increased, the enhancement peaks up, although the interaction contribution to the total energy decreases and becomes vanishingly small in the

![Figure 1](https://example.com/figure1.png)

**FIG. 1.** Square of the spinon-holon wavefunction $|p_{mn}(z)|^2$ defined by Eq. (22) for $m = N/2 - 1$ and $N = 600$. The probability peaks up at short separation between spinon and holon, while it does not depend on the distance at large separations. The inset shows the function around the origin for $N = 200, 400, 600$. The value at the origin diverges as $N \to \infty$. 

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thermodynamic limit [3].

The spinon-holon interaction has important consequences on the functional form of the hole spectral density, \( A_h(\omega, q) \). \( A_h(\omega, q) \) does not exhibit Landau’s quasi-particle resonance [3]. It rather shows a sharp square-root threshold, followed by a broad branch-cut. Broad spectra have also been experimentally seen in ARPES experiments performed on quasi 1-d samples [1]. We prove that the divergent square-root threshold is the main effect of the attraction between a spinon and a holon. The attraction enhances the matrix element for decay of a hole into a spinon-holon pair, thus making the hole excitation fully unstable to decay into a spinon and a holon. In the thermodynamic limit, the enhancement turns into the sharp square-root threshold, followed by the branch cut.

In the thermodynamic limit, the enhancement turns into the sharp square-root threshold, followed by the branch cut. Broad particle resonance [2]. It rather shows a sharp square-root threshold, followed by a broad branch-cut. Broad particle resonance [2].

Let us start our analysis with some basic results from the KY-model. At filling 1/2, the ground state of the KY-model is the same as the ground state of the HS-model. For even \( N \), the corresponding wavefunction is given by

\[
\Psi_{GS}(z_1, \ldots, z_M) = \prod_{i<j} (z_i - z_j)^2 \prod_j z_j ,
\]

where \( M = N/2 \) and the \( \{j\} \)'s denote the locations of \( \uparrow \) spins, all the others being \( \downarrow \). The corresponding energy is given by \( E_{GS} = -J(\pi^2/24)(N+5/N) \) [4, 14]. A \( \downarrow \) spinon excitation localized at \( s \), at fixed filling, can be thought of as a singlet sea where the spin at \( s \) is constrained to be \( \downarrow \). It is described by the wavefunction

\[
\Psi_s(z_1, \ldots, z_M) = \prod_j (z_j - s) z_j \prod_{i<j} (z_i - z_j)^2 ,
\]

where now \( N \) is odd and \( M = (N-1)/2 \). One-spinon eigenstates of \( H_{KY} \) are given by spinon plane waves,

\[
\Psi^s_m(z_1, \ldots, z_M) = \frac{1}{N} \sum_s (s^s)^m \Psi_s(z_1, \ldots, z_M) ,
\]

where \( m = 0, \ldots, M \) and the total crystal momentum is \( q^s_m = (\pi/2)N - (2\pi/N)(m + 1/4) \) (mod 2\( \pi \)). The energy is given by \( E^s_m = -J(\pi^2/24)(N-1/N) + J/2(2\pi/N)^2m(M-m) \). In terms of \( q^s_m \), the energy with respect to the ground state is \( E(q^s_m) = (J/2)(\pi/2)^2 - (q^s_m)^2 \) (mod \( \pi \)).

One-holon states carry spin 0 and charge 1 with respect to the ground state. They can be constructed by removing an electron from the center of a spinon [3, 14, 13]. Unlike in the spinon case, the Brillouin zone for one-holon states is not halved [3]. The negative-energy part of the Brillouin zone is spanned by the propagating one-holon wavefunctions

\[
\Psi^h_n(z_1, \ldots, z_M|h) = h^n \prod_j (z_j - h) z_j \prod_{i<j} (z_i - z_j)^2 ,
\]

where \( h \) is the coordinate of the empty site and \( 0 \leq n \leq (N+1)/2 \). The total crystal momentum of the state in Eq. (3) is \( q^h_n = (\pi/2)N + (2\pi/N)(n-1/4) \) (mod 2\( \pi \)). Its total energy is given by \( E^h_n = -J(\pi^2/24)(N-1/N) + J/2(2\pi/N)^2n(-(N+1)/2+n) \). The energy with respect to the ground state is \( E(q^h_n) = -(J/2)((\pi/2)^2 - (q^h_n)^2) \) (mod \( \pi \)). Here we will not consider positive-energy one holon eigenstates [13, 14], since they are irrelevant to our analysis.

Spinons and holons do not lose their identity in a many-spinon many-holon state. Hence, to construct one-holon one-spinon eigenstates of \( H_{KY} \) we can start with states in a mixed representation, where only the spinon is localized at \( s \). Let \( N \) be even and \( M = N/2 - 1 \). The state \( \Psi^s_n \) is defined as

\[
\Psi^s_n(z_1, \ldots, z_M|h) = h^n \prod_j (z_j - s)(z_j - h) z_j \prod_{i<j} (z_i - z_j)^2 ,
\]

where \( 1 \leq n \leq M + 2 \), the spin at \( s \) is constrained to be \( \downarrow \) and \( h \) is the coordinate of the empty site. States with a well-defined crystal momentum can be constructed by propagating the spinon and are given by

\[
\Psi_m(z_1, \ldots, z_M|h) = \sum_s (s^s)^m N \Psi^s_n(z_1, \ldots, z_M|h) ,
\]

The total crystal momentum is \( q = (\pi/2)(N-2) + q^s_m + q^h_n \) (mod 2\( \pi \)); \( q^s_m \) and \( q^h_n \) are the momenta of the single spinon and holon, respectively. One-spinon one-holon eigenstates of the KY-Hamiltonian are linear combinations of the states \( \Psi_m \) with fixed total momentum

\[
\Phi_m = \sum_{\ell=0}^m a_\ell \Psi_{m-\ell,-n-\ell} \quad \text{if } m-n+1 < 0 ,
\]

and

\[
\Phi_m = \sum_{\ell=0}^{M-m} a_\ell \Psi_{m+\ell,n+\ell} \quad \text{if } m-n+1 \geq 0 .
\]

The coefficients \( a_\ell \) are defined by recursion as

\[
a_\ell = -\frac{1}{2\ell} \sum_{k=0}^{\ell-1} a_k a_0 = 1 ,
\]

and the corresponding eigenvalue is

\[
E_{mn} = E_{GS} + E(q^s_m) + E(q^h_n) - \frac{\pi J}{N} \frac{|q^s_m - q^h_n|^2}{2} .
\]

The energy of the one-spinon one-holon state relative to the ground state is the sum of the energies of an isolated spinon and an isolated holon plus an interaction term that is vanishingly small in the thermodynamic limit.
In order to compute the norm of the states $\Phi_{mn}$, we employ the recursive technique introduced in [6]. $\Psi_s$ has the generic form $\Psi_s(z_1, \ldots, z_M|s) = \phi_s(z_1, \ldots, z_M|s)$, where $\phi_s$ is a symmetric polynomial in $z_1, \ldots, z_M$, and $\Psi_{GS}$ is the wavefunction introduced in Eq. (13). Define the operator $e_1(z_1, \ldots, z_M) = z_1 + \ldots + z_M$. It is straightforward to prove that

$$[H_{KY}, e_1] \Phi_{mn} = \Psi_{GS} J \frac{2\pi}{N} \left\{ \left( M + \frac{1}{2} \right) + M(s + h) - s^2 \frac{\partial}{\partial s^2} \right\} \phi_s + h \frac{\partial}{\partial s} \left[ \phi_s - \left( \frac{s}{h} \right)^n \phi_s^n \right].$$

(12)

Using Eq. (12), we find the following relations between matrix elements of $[H_{KY}, e_1]$ between energy eigenstates, calculated with respect to the inner product $\langle f | g \rangle = \sum_{s=1}^{z_{M+1}} f^s(z_1, \ldots, z_M|s)\bar{g}(z_1, \ldots, z_M|s)$ and valid in the case $m = n + 1 < 0$

$$\frac{\langle \Phi_{m-1,n+1} | e_1 | \Phi_{mn} \rangle}{\langle \Phi_{m-1,n} | \Phi_{m-1,n} \rangle} = - \frac{M - m + \frac{3}{2}}{2(M - m + 1)}.$$  

(13)

and

$$\frac{\langle \Phi_{m-1,n+1} | e_1 | \Phi_{mn} \rangle}{\langle \Phi_{m,n} | \Phi_{m,n} \rangle} = - \frac{m}{2(m - \frac{1}{2})}.$$  

(14)

From eqs. (13,14) one finds by recursion that

$$\frac{\langle \Phi_{mn} | \Phi_{mn} \rangle}{\langle \Psi_{GS} | \Psi_{GS} \rangle} = \frac{\Gamma[m + \frac{3}{2}]\Gamma[M - m + \frac{3}{2}]}{N \Gamma[m + 1] \Gamma[M - m + \frac{3}{2}]}.$$  

(15)

where $\langle \Psi_{GS} | \Psi_{GS} \rangle = N^{M+1}(2M+2)!/2^{M+1}$. As $m - n + 1 \geq 0$, by following the same steps we find

$$\frac{\langle \Phi_{mn} | \Phi_{mn} \rangle}{\langle \Psi_{GS} | \Psi_{GS} \rangle} = \frac{\Gamma[m + 1] \Gamma[M - m + \frac{3}{2}]}{N \Gamma[m + 2] \Gamma[M - m + \frac{3}{2}]}.$$  

(16)

The state for a localized spinon at $s$ and a localized holon at $h_0$, $\Psi_{sh}$, is defined as the Fourier-transform of $\Psi_s$ back to coordinate space

$$\Psi_{sh} = \sum_{n=1}^{M+2} h_0^{-n} \phi_s^n.$$  

(17)

As in [9,3], we define the real-space coordinate representation for a spinon-holon pair, $s^m h_0^{-n} \Psi_{mn}(s/h_0)$, as

$$\Psi_{sh} = \sum_{n=1}^{M+2} \sum_{m=0}^{M} s^m h_0^{-n} \phi_s^n \phi_m.$$  

(18)

Notice that, from Eqs. (17,18), we find that the relative wavefunction, $\Psi_{sh}$, is a polynomial in the variable $s/h_0$ if $m - n + 1 < 0$, and a polynomial in the variable $h_0/s$ if $m - n + 1 \geq 0$. Since $\Phi_{mn}$ is an eigenstate of the KY-Hamiltonian, we obtain

$$(E_{mn} - E_{GS}) \langle \Psi_{mn} | \Psi_{sh} \rangle = \langle \Phi_{mn} | (H_{KY} - E_{GS}) | \Psi_{sh} \rangle =$$

$$J \left( \frac{2\pi}{N} \right)^2 \left[ \left( M - s \frac{\partial}{\partial s} \right) s \frac{\partial}{\partial s} + h_0 \frac{\partial}{\partial h_0} \left( 1 + \frac{N}{2} + h_0 \frac{\partial}{\partial h_0} \right) + \frac{1}{2} \left( h_0 + s \right) s \frac{\partial}{\partial s} + h_0 \frac{\partial}{\partial h_0} \right] \langle \Phi_{mn} | \Psi_{sh} \rangle + \frac{h_0}{s - h_0} \left( s \frac{\partial}{\partial s} \right)^{\nu} \langle \Phi_{mn} | \Psi_{sh} \rangle,$$

(19)

where $\nu = M$ if $m - n + 1 < 0$, $\nu = 0$ otherwise.

For $m - n + 1 < 0$, Eq. (19) turns into the following equation of motion for $p_{mn}$

$$\frac{2d}{dz} - \frac{1}{(1 - z)^2} \left( p_{mn}(z) + \frac{z^{M-m-1}}{(1 - z)} p_{mn}(1) = 0 \right)$$

(20)

while, if $m - n + 1 \geq 0$, it becomes

$$\frac{2d}{dz} - \frac{1}{(1 - z)^2} \left( p_{mn}(z) + \frac{(z^{M-m})}{(1 - z)} p_{mn}(1) = 0. \right.$$  

(21)

Eqs. (20,21) are Dirac-like first order differential equations. They contain a short-range interaction potential that diverges at short distances as the first power of the separation between spinon and holon. Its effects can be analyzed by studying the corresponding exact solutions. The solution to Eq. (20) is a polynomial in $z$

$$p_{mn}(z) = \sum_{k=0}^{M-m-1} \frac{\Gamma[k + \frac{3}{2}]}{\Gamma[k + 1]} z^k.$$  

(22)

while the solution to Eq. (21) is a polynomial in $1/z$

$$p_{mn}'(z) = \sum_{k=0}^{m} \frac{\Gamma[k + \frac{3}{2}]}{\Gamma[k + 1]} \frac{1}{z^k}.$$  

(23)

In Fig. 1 we plot $|p_{mn}(e^{i\theta})|^2$ vs. $\theta$. The sharp maximum at small spinon-holon separation is the main effect of the strong attractive potential in Eqs. (20,21). It is worth stressing that the interaction between a spinon and a holon has, for large $N$, exactly the same shape as the interaction between two spinons [9,3], although the differential equation for the two-spinon wavefunction is second order, while Eqs. (20,21) are first order.

To rigorously prove that the spinon-holon attraction generates the square root singularity followed by a branch cut in the hole spectral function, $A_h(q, \omega)$ [1], we now calculate the contribution to $A_h(q, \omega)$ from one-spinon one-holon states. Since $H_{KY}$ acts on Gutzwiller-projected states, matrix elements of $H_{KY}$ between states with at least a doubly-occupied site are zero. Accordingly, at half filling $A_h(\omega, q)$ takes contributions only from hole states propagating forward. It is given by

3
where $|X\rangle$ is an exact eigenstate of $H_{KY}$, $E_X$ is its energy and $q = 2\pi k/N$. In order to calculate the contribution to $A_h(q, \omega)$ from one-spinon one-holon states, we sum over $|X\rangle = |\Phi_{mn}\rangle$. From Eq. (26) we obtain

$$A_{h}^{sp\ ho}(q, \omega) = 3\pi m \left\{ \sum_{l=2}^{M+2} \sum_{m=0}^{2} \frac{\delta_{k-m-l}(p_{ml})^2(1)}{\omega + i\eta - (E_{mn} - E_{GS})} \right\} \left\langle \Phi_{ml} | \Phi_{ml} \right\rangle$$

This proves that the contribution to the hole spectral density from one-spinon one-holon states is fully determined by the probability enhancement due to the short-range spinon-holon attraction.

The thermodynamic limit of Eq. (23) is defined as $M \to \infty$, with $m/M$ and $n/M$ constant. By using the identity

$$\sum_{k=0}^{L} \frac{\Gamma[k + \frac{1}{2}]}{\Gamma[k + 1]} = 2\pi \frac{\Gamma[L + \frac{3}{2}]}{\Gamma[L + 1]} ,$$

and by approximating the gamma functions with Stirling’s formula, the thermodynamic limit of Eq. (23) can be written as an integral over the one-spinon one-holon Brillouin zone

$$A_{h}^{sp\ ho}(q, \omega) = 2\pi m \int_{0}^{\pi} dq_{ho} \int_{0}^{q_{ho}} dq_{sp} \left\{ \frac{q_{sp}}{\sqrt{q_{sp}}} \right\} \left( \frac{\pi - q_{sp}}{q_{sp}} \right) \left( \frac{\pi - q_{sp} + q_{ho}}{q_{sp} - q_{ho}} \right) \left( \frac{\omega - \mu + i\eta - E(q_{sp}, q_{ho})}{\omega - \mu + i\eta - E(q_{sp}, q_{ho})} \right).$$

In Eq. (27) we have added the chemical potential $\mu = J\pi^2/4$, in order to fix the filling at $(N - 1)/2N$. $q_{ho}$ and $q_{sp}$ are the spinon and holon momenta.) Eq. (27) has been first worked out in [1]. $A_{h}^{sp\ ho}(q, \omega)$ is $\neq 0$ for $0 \leq \omega \leq \pi$. In this region of values of $q$, integration of Eq. (27) is straightforward. It provides:

$$A_{h}^{sp\ ho}(q, \omega) = \frac{1}{\pi^2 J^2} \left\{ J[q + \frac{\pi^2}{4}] - \omega \right\} \left\{ J[\frac{\pi^2}{4} + q(\pi - q)] - \omega \right\} \left\{ J[\frac{\pi^2}{4} + q(\pi - q)] - \omega \right\} \left\{ J[\frac{\pi^2}{4} + q(\pi - q)] - \omega \right\} .$$

From Eq. (28) we see that, in the thermodynamic limit, the short-distance probability enhancement in the spinon-holon wavefunction becomes the square-root singularity in $A_{h}^{sp\ ho}(q, \omega)$ at the threshold energy for creation of a spinon-holon pair, followed by the broad branch cut. Spinon-holon attraction enhances the matrix element for decay of a hole into a spinon-holon pair. The physical consequence is the instability of the hole, which is no longer a legitimate excitation of the system.

In conclusion, we have studied the spinon-holon interaction in an exact solution of the $t - J$ model with $1/r^2$ interaction. A spinon and a holon interact by means of a short-range attraction. Although unable to bind the two particles, it generates a probability enhancement as they are close to each other. It corresponds to an enhancement in the matrix element for decay of a hole into a spinon-holon pair, which makes the hole excitation unstable. Hence, our result shows that spinon-holon attraction is what makes Landau’s Fermi liquid theory break down in 1-d strongly correlated electron systems and, consequently, makes the quasiparticle resonance disappear.

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