The Petrovskiĭ criterion and barriers for degenerate and singular $p$-parabolic equations

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Abstract In this paper we obtain sharp Petrovskiĭ criteria for the $p$-parabolic equation, both in the degenerate case $p > 2$ and the singular case $1 < p < 2$. We also give an example of an irregular boundary point at which there is a barrier, thus showing that regularity cannot be characterized by the existence of just one barrier.

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1 Introduction

In [16] Petrovskiĭ proved the following result.

Petrovskiĭ’s criterion. The origin $(0,0)$ is regular for the heat equation \( \partial_t u - \Delta u = 0 \) in \( \mathbb{R}^{n+1} \) with respect to the domain

\[
\{ (x, t) \in \mathbb{R}^{n+1} : |x| < K \sqrt{-t} \sqrt{\log |\log(-t)|} \text{ and } -1 < t < 0 \} \tag{1.1}
\]

if and only if \( K \leq 2 \).

In this paper we obtain similar results for the nonlinear $p$-parabolic equation

\[
\partial_t u - \Delta_p u := \frac{\partial u}{\partial t} - \text{div}(|\nabla u|^{p-2} \nabla u) = 0 \tag{1.2}
\]
both in the \textit{degenerate} case $p > 2$ and the \textit{singular} case $1 < p < 2$. For $p = 2$, (1.2) reduces to the usual heat equation. (The gradient $\nabla u$ and the $p$-Laplacian $\Delta_p$ are taken with respect to $x \in \mathbb{R}^n$.)

Boundary regularity for the $p$-parabolic equation has been studied by Lindqvist [15], Kilpeläinen–Lindqvist [9] and Björn–Björn–Gianazza–Parviainen [3]. Sufficient Petrovskii-type conditions were given in [15] and [3]. Boundary regularity has also been studied for the normalized $p$-parabolic equation $\partial_t u - |\nabla u|^2 - \frac{p}{\lambda^p} u = 0$ by Banerjee–Garofalo [1].

For the heat equation ($p = 2$), regular boundary points have been characterized by means of Wiener type criteria, see Evans–Gariepy [8], Landis [13], [14] and Fabes–Garofalo–Lanconelli [7]. For $p \neq 2$, no Wiener type criterion is known.

There are some significant differences in the theory of boundary regularity for $p \neq 2$ and for the heat equation ($p = 2$). The scaling argument in Sect. 4 shows that for $p \neq 2$ one cannot have a Petrovskii-type criterion where a parameter similar to $K$ in (1.1) dictates regularity. Instead we obtain the following result. See also Remark 4.3.

**Theorem 1.1** (Petrovskii-type criteria for $1 < p < \infty$) Let $K > 0$, $q > 0$ and

$$\Theta = \{(x, t) \in \mathbb{R}^{n+1} : |x| < K(-t)^q \text{ and } -1 < t < 0\}. \quad (1.3)$$

Then the following are true:

(a) If $p > 2$, then $(0, 0)$ is regular if and only if $q > 1/p$.

(b) If $p = 2$, then $(0, 0)$ is regular if and only if $q \geq 1/2$.

(c) If $1 < p < 2$, then $(0, 0)$ is regular if $q > 1/p$ and irregular if $q < 1/p$.

In all cases regularity is with respect to $\Theta$.

For $p = 2$ this follows quite directly from the Petrovskii criterion above. For $p < 2$ and $q = 1/p$ we do not know whether $(0, 0)$ is regular or not, but the case $p = 2$ shows that it is quite possible that $p = 2$ is a break point for this result and that $(0, 0)$ may be regular when $p < 2$ and $q = 1/p$.

The Petrovskii-type criterion in Lindqvist [15, Theorem, p. 571] and Björn–Björn–Gianazza–Parviainen [3, Theorem 6.1] gives regularity when

$$p > 2 \quad \text{and} \quad q \geq \frac{1}{p} + \frac{n(p - 2)^2}{\lambda p},$$

where from now on we use the shorthand $\lambda = n(p - 2) + p$. It was conjectured in [15, p. 572] that this would be sharp, which is now disproved by Theorem 1.1. For $p < 2$ and $q > 1/p$ regularity follows from Proposition 7.1 in [3] (and [3, Proposition 3.4] when $0 < K \leq 1$).

In Kilpeläinen–Lindqvist [9, pp. 676–677] it was shown that $(0, 0)$ is an irregular boundary point with respect to the so-called Barenblatt balls when $p > 2$, i.e. for $q = 1/\lambda < 1/p$, with $K$ dependent on $p$. Lindqvist [15, footnote p. 572] also states that “it is not too difficult to show” irregularity for $q = 1/p$ when $p > 2$. Theorem 1.1 extends these results and completes the picture. As a matter of fact, for $p > 2$ we provide more powerful criteria in Theorem 6.1 and Proposition 6.2. As we do not
know what happens when $p < 2$ and $q = 1/p$, we have refrained from giving such
criteria when $p < 2$.

We are also interested in barrier characterizations. Already Kilpeläinen–Lind-qvist [9] suggested that regularity can be characterized using one (traditional) barrier.
Such a criterion turned out to be problematic, and it has been an open problem since then whether a single (traditional) barrier guarantees regularity. A criterion using a family of barriers was obtained in [3, Theorem 3.3]. In this paper we prove the following result.

**Proposition 1.2** Let $1 < p < 2$, $K > 0$ and $0 < q < 1/p$. Then there is a traditional barrier at $(0, 0)$ for the domain

$$\Theta = \{(x, t) \in \mathbb{R}^{n+1} : |x| < K(-t)^q \text{ and } -1 < t < 0\}$$

despite the fact that $(0, 0)$ is irregular.

This shows that regularity cannot be characterized using only one barrier, at least not for $p < 2$. We conjecture that this is true also for $p > 2$, but we have not been able to find a counterexample in the degenerate range.

We end this introduction by mentioning that quite a lot of attention has been given to the study of nonlinear parabolic problems in the last 20–30 years, in particular for the $p$-parabolic equation as here. See, for example, Bögelein–Duzaar–Mingione [4], DiBenedetto [5], DiBenedetto–Gianazza–Vespri [6], Kuusi–Mingione [11] and Björn–Björn–Gianazza–Parviainen [3] for the recent history and many more references to the current literature.

## 2 Preliminaries

We will use the notation and several results from Björn–Björn–Gianazza–Parviainen [3]. Here we will be brief and only introduce and discuss what we really need, see [3] for a more extensive discussion.

From now on we will always assume that $\Theta \subset \mathbb{R}^{n+1}$ is a nonempty bounded open set and $1 < p < \infty$.

Let $U$ be an open set in $\mathbb{R}^n$. The parabolic boundary of the cylinder $U_{t_1, t_2} := U \times (t_1, t_2) \subset \mathbb{R}^{n+1}$ is

$$\partial_p U_{t_1, t_2} = (\overline{U} \times \{t_1\}) \cup (\partial U \times (t_1, t_2)).$$

By the parabolic Sobolev space $L^p(t_1, t_2; W^{1,p}(U))$, with $t_1 < t_2$, we mean the space of functions $u(x, t)$ such that the mapping $x \mapsto u(x, t)$ belongs to $W^{1,p}(U)$ for almost every $t_1 < t < t_2$ and the norm

$$\left( \int_{t_1}^{t_2} \int_U (|u(x, t)|^p + |\nabla u(x, t)|^p) \, dx \, dt \right)^{1/p}$$

is finite. The definition of the space $L^p(t_1, t_2; W^{1,p}_0(U))$ is similar. Analogously, by the space $C([t_1, t_2]; L^p(U))$, with $t_1 < t_2$, we mean the space of functions $u(x, t)$,
such that the mapping \( t \mapsto \int_U |u(x, t)|^p \, dx \) is continuous in the time interval \([t_1, t_2]\).
(The gradient \( \nabla \) and divergence \( \text{div} \) are always taken with respect to the \( x \)-variables in this paper.) We can now introduce the notion of weak solution.

**Definition 2.1** A function \( u : \Theta \to [-\infty, \infty] \) is a \textit{weak solution} to equation (1.2) if whenever \( U_{t_1, t_2} \subseteq \Theta \) is an open cylinder, we have \( u \in C([t_1, t_2]; L^p(U)) \cap L^p(t_1, t_2; W^{1,p}(U)) \), and \( u \) satisfies the integral equality

\[
\int_{t_1}^{t_2} \int_U |u|^{p-2} u \cdot \nabla \phi \, dx \, dt - \int_{t_1}^{t_2} \int_U u \frac{\partial \phi}{\partial t} \, dx \, dt = 0 \quad \text{for all } \phi \in C^\infty_0(U_{t_1, t_2}).
\]

A \( p \)-parabolic function is a continuous weak solution.

A function \( u \) is a \textit{weak supersolution} if whenever \( U_{t_1, t_2} \subseteq \Theta \) we have \( u \in L^p(t_1, t_2; W^{1,p}(U)) \) and the left-hand side above is nonnegative for all nonnegative \( \phi \in C^\infty_0(U_{t_1, t_2}) \). For simplicity, we will omit \textit{weak}, when talking of weak super solutions.

The most important \( p \)-parabolic function is the \textit{Barenblatt solution} \([2]\) \( B_p : \mathbb{R}^n \times (0, \infty) \to [0, \infty) \) defined by

\[
B_p(x, t) = t^{-n/\lambda} \left( C - \frac{p-2}{p} \frac{1}{\lambda} \left( \frac{|x|}{t^{1/\lambda}} \right)^{p/(p-1)} \right)^{(p-1)/(p-2)}, \quad \lambda = n(p-2) + p,
\]

where \( C > 0 \) is an arbitrary constant. Even though it was introduced in the context of degenerate equations for \( p > 2 \), it is well defined also for \( p < 2 \), provided that \( \lambda > 0 \), i.e. that \( 2n/(n+1) < p < 2 \). We will not directly use the Barenblatt solution in this paper, but some of our expressions are closely related to the Barenblatt solution.

**Definition 2.2** A function \( u : \Theta \to (-\infty, \infty] \) is \textit{p-superparabolic} if

\( i \) \( u \) is lower semicontinuous;

\( ii \) \( u \) is finite in a dense subset of \( \Theta \);

\( iii \) \( u \) satisfies the following comparison principle on each space-time box \( Q_{t_1, t_2} \subseteq \Theta \): If \( h \) is \( p \)-parabolic in \( Q_{t_1, t_2} \) and continuous on \( \overline{Q}_{t_1, t_2} \), and if \( h \leq u \) on \( \partial_p Q_{t_1, t_2} \), then \( h \leq u \) in the whole \( Q_{t_1, t_2} \).

A function \( v : \Theta \to [-\infty, \infty) \) is \textit{p-subparabolic} if \(-u\) is \( p \)-superparabolic.

Here \( Q_{t_1, t_2} \) is a \textit{space-time box} if it is of the form \( Q_{t_1, t_2} = Q \times (t_1, t_2) \), where \( Q = (a_1, b_1) \times \cdots \times (a_n, b_n) \).

The connection between \( p \)-superparabolic functions and supersolutions is a delicate issue. However, a continuous supersolution is \( p \)-superparabolic by the comparison principle of Korte–Kuusi–Parviainen \([10, \text{Lemma 3.5}]\).

We will need the following parabolic comparison principle.

**Theorem 2.3** (Parabolic comparison principle, \([3, \text{Theorem 2.4}]\)) Suppose that \( u \) is \( p \)-superparabolic and \( v \) is \( p \)-subparabolic in \( \Theta \). Let \( T \in \mathbb{R} \) and assume that

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\[
\limsup_{\Theta \ni (y,s) \to (x,t)} v(y,s) \leq \liminf_{\Theta \ni (y,s) \to (x,t)} u(y,s) \neq -\infty
\]
for all \((x,t) \in \{(x,t) \in \partial \Theta : t < T\} \). Then \(v \leq u\) in \(\{(x,t) \in \Theta : t < T\}\).

We now turn to the Perron method. For us it will be enough to consider Perron solutions for bounded functions, so for simplicity we restrict ourselves to this case.

**Definition 2.4** Given a bounded function \(f : \partial \Theta \to \mathbb{R}\), let the upper class \(U_f(\Theta)\) be the set of all \(p\)-superparabolic functions \(u\) on \(\Theta\) which are bounded below and such that

\[
\liminf_{\Theta \ni \eta \to \xi} u(\eta) \geq f(\xi) \quad \text{for all} \quad \xi \in \partial \Theta.
\]

Define the upper Perron solution of \(f\) by

\[
\overline{H}_f(\Theta) = \inf_{u \in U_f(\Theta)} u(\xi), \quad \xi \in \Theta.
\]

Similarly, let the lower class \(L_f(\Theta)\) be the set of all \(p\)-subparabolic functions \(u\) on \(\Theta\) which are bounded above and such that

\[
\limsup_{\Theta \ni \eta \to \xi} u(\eta) \leq f(\xi) \quad \text{for all} \quad \xi \in \partial \Theta,
\]

and define the lower Perron solution of \(f\) by

\[
\underline{H}_f(\Theta) = \sup_{u \in L_f(\Theta)} u(\xi), \quad \xi \in \Theta.
\]

If the domain under consideration is clear from the context, we will often drop \(\Theta\) from the notation above. It follows from the parabolic comparison principle (Theorem 2.3) that \(\overline{H}_f \leq \underline{H}_f\). Moreover \(\overline{H}_f = -\underline{H}(-f)\). Kipeläinen–Lindqvist [9, Theorem 5.1] showed that both \(\overline{H}_f\) and \(\underline{H}_f\) are \(p\)-parabolic.

The following simple lemma is easily proved by direct calculation.

**Lemma 2.5** For any \(\alpha, C \in \mathbb{R}\) we have

\[
\Delta_p(C|x|^{\alpha}) = C|\alpha|C\alpha|p-2(n + (\alpha - 1)(p - 1) - 1)|x|^{(\alpha - 1)(p - 1) - 1}.
\]

In particular, if \(\alpha = p/(p - 2)\) and \(C\alpha > 0\) then

\[
\Delta_p(C|x|^{\alpha}) = (C\alpha)^{p-1}(n + \alpha)|x|^{\alpha} = \frac{(C\alpha)^{p-1}\lambda}{p - 2}|x|^{\alpha},
\]

and if \(\alpha = p/(p - 1)\) and \(C > 0\) then \(\Delta_p(C|x|^{\alpha}) = (C\alpha)^{p-1}n\).
3 Boundary regularity

**Definition 3.1** A boundary point \( \xi_0 \in \partial \Theta \) is regular with respect to \( \Theta \), if

\[
\lim_{\Theta \ni \xi \to \xi_0} H f(\xi) = f(\xi_0)
\]

whenever \( f : \partial \Theta \to \mathbb{R} \) is continuous.

Observe that since \( H f = -H(-f) \), regularity can equivalently be formulated using lower Perron solutions.

**Definition 3.2** Let \( \xi_0 \in \partial \Theta \). A family of functions \( w_j : \Theta \to (0, \infty] \), \( j = 1, 2, \ldots \), is a barrier family in \( \Theta \) at the point \( \xi_0 \) if for each \( j \),

(a) \( w_j \) is a positive \( p \)-superparabolic function in \( \Theta \);
(b) \( \lim_{\Theta \ni \zeta \to \xi_0} w_j(\zeta) = 0 \);
(c) for each \( k = 1, 2, \ldots \), there is a \( j \) such that

\[
\lim_{\Theta \ni \zeta \to \xi} \inf_{\Theta} w_j(\zeta) \geq k \quad \text{for all} \quad \xi \in \partial \Theta \quad \text{with} \quad |\xi - \xi_0| \geq 1/k.
\]

We also say that the family \( w_j \) is a strong barrier family in \( \Theta \) at the point \( \xi_0 \) if, in addition, the following conditions hold:

(d) \( w_j \) is continuous in \( \Theta \);
(e) there is a nonnegative function \( d \in C(\overline{\Theta}) \), with \( d(z) = 0 \) if and only if \( z = \xi_0 \), such that for each \( k = 1, 2, \ldots \), there is a \( j = j(k) \) such that \( w_j \geq kd \) in \( \Theta \).

**Theorem 3.3** ([3, Theorem 3.3]) Let \( \xi_0 \in \partial \Theta \). Then the following are equivalent:

1. \( \xi_0 \) is regular;
2. there is a barrier family at \( \xi_0 \);
3. there is a strong barrier family at \( \xi_0 \).

In classical potential theory, a barrier is a superharmonic (when dealing with the Laplace equation) or superparabolic (when dealing with the heat equation) function \( w \) such that

\[
\lim_{\zeta \to \xi_0} w(\zeta) = 0 \quad \text{and} \quad \liminf_{\zeta \to \xi} w(\zeta) > 0 \quad \text{for} \quad \xi \in \partial \Theta \setminus \{\xi_0\}.
\]

Existence of such a single barrier implies the regularity of a boundary point in these classical cases, since one can scale and lift the barrier (i.e. if \( u \) is a barrier, then also \( au + b \) is superharmonic/superparabolic, where \( a > 0 \) and \( b \in \mathbb{R} \)). A similar property holds also for the nonlinear \( p \)-Laplace equation \( \Delta_p u = 0 \). However, this is not the case for the \( p \)-parabolic equation, since it is not homogeneous: If \( u \) is a supersolution, then \( au \) (with \( a > 0 \)) is usually not a supersolution, even though, \( u + a \) is indeed still a supersolution.

We say that \( u \) is a traditional barrier at \( \xi_0 \in \partial \Theta \) if
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(a) $u$ is a positive $p$-superparabolic function in $\Theta$;
(b) $\lim_{\Theta \ni \xi \to \xi_0} u(\xi) = 0$;
(c) $\lim \inf_{\Theta \ni \xi \to \xi} u(\xi) > 0$ for all $\xi \in \partial \Theta \setminus \{\xi_0\}$.

It is clear that regularity implies the existence of a traditional barrier (this follows e.g. from 3.3 in Theorem 3.3 above). Conversely, as mentioned in the introduction, it has been an open problem whether the existence of a traditional barrier characterizes regularity, which we solve in the negative when $p < 2$.

The following results are important consequences of the barrier characterization in Theorem 3.3.

**Proposition 3.4** ([3, Proposition 3.4]) Let $\xi_0 \in \partial \Theta$ and let $G \subset \Theta$ be open and such that $\xi_0 \in \partial G$. If $\xi_0$ is regular with respect to $\Theta$, then $\xi_0$ is regular with respect to $G$.

**Proposition 3.5** ([3, Proposition 3.5]) Let $\xi_0 \in \partial \Theta$ and $B$ be a ball containing $\xi_0$. Then $\xi_0$ is regular with respect to $\Theta$ if and only if $\xi_0$ is regular with respect to $B \cap \Theta$.

It is easy to see that regularity is invariant under translations, and we therefore formulate most of our regularity results around the origin. See [3] for more on boundary regularity.

### 4 Scaling invariance

The main aim of this section is to prove the following result.

**Proposition 4.1** Let $p \neq 2$, $a > 0$ and $\Theta \subset \mathbb{R}^{n+1}$ be a domain with $(0, 0) \in \partial \Theta$. Set

$$
\tilde{\Theta} = \{(ax, t) \in \mathbb{R}^{n+1} : (x, t) \in \Theta\}.
$$

Then $(0, 0)$ is regular with respect to $\Theta$ if and only if it is regular with respect to $\tilde{\Theta}$.

A direct consequence, is that if $\theta : (-1, 0) \to (0, \infty)$ is a bounded continuous function, then $(0, 0)$ is regular for $\partial_t u - \Delta_p u = 0$, $p \neq 2$, with respect to

$$
\Theta_K = \{(x, t) \in \mathbb{R}^{n+1} : |x| < K\theta(t) \text{ and } -1 < t < 0\}
$$

if and only if it is regular with respect to $\Theta_1$, i.e. regularity is independent of $K > 0$. Thus, there is no Petrovskii-type criterion for $p \neq 2$ of the same type as for $p = 2$.

**Proof** Let $\tilde{u}$ be a function on $\tilde{\Theta}$ and set

$$
\tilde{u}(x, t) = K\tilde{u}(ax, t) \text{ for } (x, t) \in \Theta, \quad \text{where } K = a^{-p/(p-2)}.
$$

Then

$$
\partial_t u(x, t) = K\partial_t \tilde{u}(ax, t) \quad \text{and} \quad \Delta_p u(x, t) = K^{p-1} a^p \Delta_p \tilde{u}(ax, t) = K \Delta_p \tilde{u}(ax, t),
$$

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from which it follows that \( u \) is \( p \)-superparabolic in \( \Theta \) if and only if \( \tilde{u} \) is \( p \)-superparabolic in \( \tilde{\Theta} \). Next let \( \tilde{f} \in C(\partial \tilde{\Theta}) \) and set

\[
    f(x, t) = K \tilde{f}(ax, t) \quad \text{for} \quad (x, t) \in \partial \Theta.
\]

Then we see from the above that

\[
    \overline{H}_{\Theta} f(x, t) = \overline{H}_{\tilde{\Theta}}(K \tilde{f})(ax, t) \quad \text{for} \quad (x, t) \in \Theta.
\]

This shows that regularity of the origin with respect to \( \Theta \) implies regularity with respect to \( \tilde{\Theta} \). The converse implication follows by switching the roles of \( \Theta \) and \( \tilde{\Theta} \), and replacing \( a \) by \( 1/a \).

We conclude this section by briefly comparing the linear and nonlinear cases regarding multiplied equations.

**Definition 4.2** Let \( 1 < p < \infty \). A boundary point \( \xi_0 \in \partial \Theta \) is completely regular with respect to \( \Theta \), if whenever \( f : \partial \Theta \to \mathbb{R} \) is continuous and \( a > 0 \),

\[
    \lim_{\Theta \ni \xi \to \xi_0} \overline{H}^a f(\xi) = f(\xi_0)
\]

(where \( \overline{H}^a \) denotes the upper Perron solution with respect to the equation \( a \partial_t u = \Delta_p u \)), i.e. whenever \( \xi_0 \) is simultaneously regular for all the multiplied equations.

**Remark 4.3** By Theorem 3.6 in Björn–Björn–Gianazza–Parviainen [3] regularity and complete regularity are the same when \( p \neq 2 \). On the contrary, it follows from the classical Petrovskii criterion that complete regularity is a strictly stronger condition when \( p = 2 \). The Petrovskii criterion also shows that one may replace “regular” by “completely regular” in Theorem 1.1 for \( p = 2 \) as well, thus providing examples of completely regular boundary points for \( p = 2 \).

More generally, consider

\[
    \Theta = \{(x, t) \in \mathbb{R}^{n+1} : |x| < \sqrt{-t} \sqrt{\log |\log(-t)|} h(t) \text{ and } -1 < t < 0\},
\]

where \( h \) is a positive continuous function. A scaling argument, similar to the proof of Proposition 4.1 (together with Petrovskii’s criterion), shows that when \( h(t) := K > 0 \) is constant, then \( (0, 0) \) is regular for \( a \partial_t u = \Delta u \) if and only if \( K \leq 2/\sqrt{a} \). Thus, for nonconstant \( h \), \( (0, 0) \) is completely regular for \( p = 2 \) if \( \lim_{t \to 0^-} h(t) = 0 \), while it is not completely regular if \( \lim \inf_{t \to 0^-} h(t) > 0 \). Moreover, if \( \lim_{t \to 0^-} h(t) = \infty \) then \( (0, 0) \) is not regular for any \( a \partial_t u = \Delta u \).

The Petrovskii criterion and the classical barrier characterization for the heat equation show that the existence of a (traditional) barrier for the heat equation does not imply complete regularity. Lanconelli [12, Theorem 1.1] showed that if a point is regular for \( a_1 \partial_t u = \Delta u \) and \( 0 < a_2 < a_1 \), then it is also regular for \( a_2 \partial_t u = \Delta u \). Thus the existence of a countable barrier family with one barrier for each \( a = 1, 2, \ldots \), is equivalent to the complete regularity when \( p = 2 \).
All of this suggests that regularity for $p \neq 2$ rather corresponds to complete regularity for $p = 2$ than to regularity for $p = 2$. Also Proposition 4.1 holds for $p = 2$ and complete regularity.

By Fabes–Garofalo–Lanconelli [7, Corollary 1.4], complete regularity for $p = 2$ is equivalent to simultaneous regularity for all linear parabolic equations of the form $\partial_t u = \text{div}(A(x, t) \nabla u)$, where $A(x, t)$ is a symmetric uniformly elliptic matrix with $C^1$-Dini continuous coefficients.

5 The singular case $1 < p < 2$

We start this section by proving Theorem 1.1 in the singular range.

Proof of Theorem 1.1 for $1 < p < 2$. When $q > 1/p$ and $K > 1$, regularity was obtained in Björn–Björn–Gianazza–Parviainen [3, Proposition 7.1]. It follows from Proposition 3.4, that any $K > 0$ will do; this follows also from Proposition 4.1.

Now assume that $0 < q < 1/p$. By Proposition 4.1 we can assume that $K = 1$. We shall construct an irregularity barrier (in the terminology of [9] and Petrovskii [16, p. 389]). Let

$$u(x, t) = \begin{cases} |x|^{p/(p-1)} \left( \frac{p}{1-pq} \right)^{p-1} \left( -t \right)^{1-pq}, & \text{if } (x, t) \in \bar{\Theta} \setminus \{(0, 0)\}, \\ 1, & \text{if } (x, t) = (0, 0). \end{cases}$$

Using Lemma 2.5 we see that in $\Theta$,

$$\partial_t u = \frac{pq}{p-1} \frac{|x|^{p/(p-1)}}{(-t)^{1+pq/(p-1)}} + \frac{n}{(-t)^{pq}} \left( \frac{p}{p-1} \right)^{p-1} \geq \frac{n}{(-t)^{pq}} \left( \frac{p}{p-1} \right)^{p-1} = \Delta_p u.$$

Hence, $\partial_t u - \Delta_p u \geq 0$ in $\Theta$, which shows that $u$ is $p$-superparabolic in $\Theta$.

Let $f = u|_{\partial \Theta} \in C(\partial \Theta)$ and let $v \in L_f(\Theta)$. By the parabolic comparison principle (Theorem 2.3), with $T = 0$, we see that $v \leq u$ in $\Theta$, and thus we also have $Hf \leq u$. But then

$$\liminf_{\Theta \ni (x, t) \to (0, 0)} Hf(x, t) \leq \liminf_{\Theta \ni (x, t) \to (0, 0)} u(x, t) \leq \liminf_{t \to 0-} u(0, t) = 0 < 1 = f(0, 0).$$

Hence $(0, 0)$ is irregular for $\Theta$. $\square$

Next, we turn to Proposition 1.2. First, we formulate it in a different form which also gives regularity for small boundary data.

Proposition 5.1 Let $1 < p < 2$ and $0 < q \leq 1/p$. Then there is a traditional barrier $u$ at $(0, 0)$ for the domain

$$\Theta = \{(x, t) \in \mathbb{R}^{n+1} : |x| < (-t)^q \text{ and } -1 < t < 0\}.$$
In particular if \( f \in C(\partial \Theta) \) satisfies \(|f - f(0, 0)| \leq g \) on \( \partial \Theta \), where
\[
g(x, t) = \frac{B}{2} \min \left\{ -t, \left( \frac{B}{2} \right)^{(p-1)/pq} \right\}^{1/(2-p)} \quad \text{and } B = \min \left\{ n(2 - p) \left( \frac{p}{p - 1} \right)^{p-1}, 1 \right\},
\]
then
\[
\lim_{\Theta \ni (x, t) \to (0, 0)} H f(x, t) = \lim_{\Theta \ni (x, t) \to (0, 0)} H f(x, t) = f(0, 0).
\] (5.1)

Of course, we have a traditional barrier also when \( q > 1/p \). The point here is that we obtain a traditional barrier even at an irregular boundary point. Note that for \( q = 1/p \) we find one traditional barrier, but we do not know whether the origin is regular or not.

**Proof** Let
\[
v(x, t) = (-t)^{1/(2-p)}(B - |x|^{p/(p-1)}), \quad (x, t) \in \Theta.
\]

By Lemma 2.5, we have in \( \Theta \),
\[
\Delta_p v = -n \left( \frac{p}{p - 1} \right)^{p-1} (-t)^{(p-1)/(2-p)}
\]
and
\[
\partial_t v = -\frac{1}{2 - p} (-t)^{1/(2-p)-1} (B - |x|^{p/(p-1)}) \geq -\frac{B}{2 - p} (-t)^{(p-1)/(2-p)}.
\]

Hence
\[
\partial_t v - \Delta_p v \geq \left( n \left( \frac{p}{p - 1} \right)^{p-1} - \frac{B}{2 - p} \right) (-t)^{(p-1)/(2-p)} \geq 0,
\]
and thus \( v \) is \( p \)-superparabolic in \( \Theta \). Next, let
\[
\Theta' = \{ (x, t) \in \Theta : |x|^{p/(p-1)} < \frac{1}{2} B \},
\]
and
\[
M = \inf_{(x, t) \in \Theta \cap \Theta'} v(x, t) = \left( \frac{B}{2} \right)^{1+(p-1)/pq(2-p)}
\]

Then
\[
u(x, t) = \begin{cases} 
\min\{v(x, t), M\}, & \text{if } (x, t) \in \Theta', \\
M, & \text{if } (x, t) \in \Theta \setminus \Theta',
\end{cases}
\]
is \( p \)-superparabolic in \( \Theta \), by the pasting lemma in Björn–Björn–Gianazza–Parviainen [3, Lemma 2.9]. It is also easily seen that \( u \) satisfies the remaining properties required of a traditional barrier.

Finally, if \( |f - f(0,0)| \leq g \) on \( \partial \Theta \), then for \( (x_0, t_0) \in \partial \Theta \),

\[
\limsup_{\Theta \ni (x,t) \to (x_0,t_0)} (f(0,0) - u(x,t)) \leq f(x_0,t_0) \leq \liminf_{\Theta \ni (x,t) \to (x_0,t_0)} (f(0,0) + u(x,t))
\]

and hence

\[
f(0,0) - u \leq Hf \leq \overline{H}f \leq f(0,0) + u \quad \text{in} \quad \Theta,
\]

from which (5.1) follows directly, as \( \lim_{(x,t) \to (0,0)} u(x,t) = 0. \) \( \square \)

**Proof of Proposition 1.2** The existence of a traditional barrier follows from Proposition 5.1 if \( K = 1 \). For general \( K > 0 \) the existence follows from the scaling argument in the proof of Proposition 4.1. The irregularity is a direct consequence of Theorem 1.1. \( \square \)

### 6 The degenerate case \( p > 2 \)

The following theorem and its proof refine the results in Lindqvist [15, Theorem, p. 571] and Björn–Björn–Gianazza–Parviainen [3, Theorem 6.1].

**Theorem 6.1** Let \( p > 2 \), \( t_0 < 0 \) and

\[
\Theta = \{(x, t) \in \mathbb{R}^{n+1} : |x| < \zeta(t) \text{ and } t_0 < t < 0\},
\]

where \( \zeta \) is a positive continuous function on \((t_0, 0)\) such that

\[
\lim_{t \to 0^-} (-t)^{-1/p} \zeta(t) = 0. \quad (6.1)
\]

Then the origin \((0, 0)\) is regular with respect to \( \Theta \).

In the converse direction we have the following result, which shows that Theorem 6.1 is essentially sharp. That \( \zeta = (-t)^{1/p} \) with \( p > 2 \) implies irregularity was mentioned as a footnote already in [15, p. 572], with no further details. Here we strengthen the statement and provide a full proof of the result.

**Proposition 6.2** Let \( p > 2 \), \( t_0 < 0 \) and

\[
\Theta = \{(x, t) \in \mathbb{R}^{n+1} : |x| < \zeta(t) \text{ and } t_0 < t < 0\},
\]

where \( \zeta \) is a positive continuous function on \((t_0, 0)\) such that

\[
\liminf_{t \to 0^-} (-t)^{-1/p} \zeta(t) > 0.
\]
Then the origin \((0, 0)\) is irregular with respect to \(\Theta\). Moreover, there is no traditional barrier at \((0, 0)\).

As an irregular borderline case one might at first think that this could provide a counterexample showing that our conjecture after Proposition 1.2 is true. However, the last part of Proposition 6.2 shows that this is not possible in this case.

**Proof of Theorem 1.1** for \(p > 2\). This follows directly from Theorem 6.1 and Proposition 6.2. \(\square\)

**Proof of Proposition 6.2** By assumption there is \(m > 0\) and \(t_1\) such that \(t_0 \leq t_1 < 0\) and

\((-t)^{-1/p} \xi(t) > m\) for \(t_1 < t < 0\).

Let

\[\Theta' = \{(x, t) \in \mathbb{R}^{n+1} : |x| < m(-t)^{1/p} \text{ and } t_1 < t < 0\} \subset \Theta.\]

If we show that \((0, 0)\) is irregular with respect to \(\Theta'\), then by Proposition 3.4, \((0, 0)\) is irregular with respect to \(\Theta\) as well. By Proposition 4.1 we may assume that \(m = 1\), and by Proposition 3.5 we may assume that \(t_1 = -1\).

As in Sect. 5, we construct an irregularity barrier. Let

\[u(x, t) = C \left( \frac{|x|^p}{(-t)^{1/(p-2)}} \right)^{1/(p-2)} (n + \alpha)|x|^\alpha, \quad (x, t) \in \overline{\Theta'} \setminus \{(0, 0)\},\]

where \(C\) is a positive constant that will be determined later. Lemma 2.5 with \(\alpha = p/(p-2)\) shows that

\[\Delta_p u = \left( \frac{C\alpha}{(-t)^{1/(p-2)}} \right)^{p-1} (n + \alpha)|x|^\alpha \quad \text{and} \quad \partial_t u = \frac{C}{p-2} \frac{|x|^\alpha}{(-t)^{(p-1)/(p-2)}}.\]

Thus, it follows that in \(\Theta'\) we have

\[\partial_t u - \Delta_p u = \frac{C|x|^\alpha}{(-t)^{(p-1)/(p-2)}} \left( \frac{1}{p-2} - C^{p-2}(n + \alpha)\alpha p^{-1} \right) \geq 0,\]

provided that

\[0 < C^{p-2} \leq \frac{1}{(p-2)(n + \alpha)\alpha p^{-1}} = \frac{(p-2)^{p-1}}{\lambda p^{p-1}},\]

where \(\lambda = n(p-2) + p = (p-2)(n + \alpha)\). This makes \(u\) into a positive \(p\)-superparabolic function in \(\Theta'\). Next, it is easy to see that \(f : \partial \Theta' \to \mathbb{R}\) given by

\[f(x, t) = \begin{cases} u(x, t), & \text{if } (x, t) \in \partial \Theta' \setminus \{(0, 0)\}, \\ C, & \text{if } (x, t) = (0, 0) \end{cases}\]
is continuous.

Now, let \( v \in \mathcal{L}_f(\Theta') \). By the parabolic comparison principle (Theorem 2.3), with \( T = 0 \), we see that \( v \leq u \) in \( \Theta' \), and thus we also have \( H_{\Theta'} f \leq u \). But then

\[
\liminf_{\Theta' \ni (x, t) \to (0, 0)} H_{\Theta'} f(x, t) \leq \liminf_{\Theta' \ni (x, t) \to (0, 0)} u(x, t) = 0 < C = f(0, 0),
\]
as \( u(0, t) = 0 \) for \( t_1 < t < 0 \). Hence, \((0, 0)\) is irregular for \( \Theta' \) and thus for \( \Theta \).

Next we turn to the existence of a traditional barrier. As in the beginning of the proof, we can reduce to \( \Theta' \) with \( m = 1 \) here as well; if \( \Theta \) had a traditional barrier at the origin, then its restriction to \( \Theta' \) would be a traditional barrier, and after scaling we would have a traditional barrier with \( m = 1 \).

Assume that \( w \) is a traditional barrier at \( (0, 0) \) for \( \Theta' \) with \( m = 1 \). Extending \( w \) to \( \partial \Theta' \) by letting

\[
w(\xi_0) = \liminf_{\Theta' \ni \xi \to \xi_0} w(\xi) \quad \text{for all} \quad \xi_0 \in \partial \Theta',
\]
makes \( w \) into a lower semicontinuous function on \( \Theta' \). Moreover, \( w \) is positive in \( \Theta' \setminus \{(0, 0)\} \) and \( w(0, 0) = 0 \). Let

\[
C = \min \left\{ \min_{(x, t_1) \in \partial \Theta'} w(x, t_1), \left( \frac{(p - 2)^{p-1}}{\lambda p^{p-1}} \right)^{1/(p-2)} \right\},
\]
and let \( u > 0 \) be the \( p \)-superparabolic irregularity barrier constructed above with this (admissible) \( C \). Then \( C - u \) is a \( p \)-subparabolic function in \( \Theta' \) such that

\[
\limsup_{\Theta' \ni (x, t) \to (x_0, t_0)} (C - u(x, t)) \leq \liminf_{\Theta' \ni (x, t) \to (x_0, t_0)} w(x, t)
\]
for all \( (x_0, t_0) \in \partial \Theta' \setminus \{(0, 0)\} \). Hence, by the parabolic comparison principle (Theorem 2.3) again, \( C - u \leq w \) in \( \Theta' \), and thus

\[
\limsup_{\Theta' \ni (x, t) \to (0, 0)} w(x, t) \geq C - \liminf_{\Theta' \ni (x, t) \to (0, 0)} u(x, t) = C,
\]
which contradicts the fact that \( w \) is a traditional barrier. Hence, there is no traditional barrier at \( (0, 0) \). \( \square \)

**Proof of Theorem 6.1** It will be convenient to rewrite \( \Theta \) as

\[
\Theta = \left\{ (x, t) \in \mathbb{R}^{n+1} : \left( \frac{|x|}{(-t)^{1/k}} \right)^{p/(p-1)} < \delta(t) \text{ and } t_0 < t < 0 \right\},
\]
where

\[
\delta(t) = \left( \frac{\zeta(t)}{(-t)^{1/k}} \right)^{p/(p-1)}.
\]
Let
\[ \beta = \frac{n(p-2)}{\lambda} \quad \text{and} \quad \gamma = \frac{n(p-2)}{\lambda(p-1)} = \frac{\beta}{p-1} < \beta. \]

Then it follows directly that (6.1) is equivalent to
\[ \lim_{t \to 0^-} (-t)^{-\gamma} \delta(t) = 0. \] (6.2)

In the proof we will use two additional properties of the function \( \delta \), namely that
\( \delta \) is smooth and \( t \mapsto (-t)^{-\beta} \delta(t) \) is nondecreasing. (6.3)

First, let us show how we can assume this without loss of generality. Let
\[ h(t) = (-t)^{-\beta} \delta(t), \quad \tilde{h}(t) = \sup_{t_0 < s \leq t} h(s) \quad \text{and} \quad \tilde{\delta}(t) = (-t)^{\beta} \tilde{h}(t) \quad \text{for} \quad t_0 < t < 0. \]

Then \( \tilde{\delta} \geq \delta \) and \( (-t)^{-\beta} \tilde{\delta}(t) = \tilde{h}(t) \) is nondecreasing. We also need that
\[ 0 = \lim_{t \to 0^-} (-t)^{-\gamma} \tilde{\delta}(t) = \lim_{t \to 0^-} (-t)^{\beta-\gamma} \tilde{h}(t). \] (6.4)

Assume that this is false. Then there is \( \varepsilon > 0 \) and \( t_j \not\to 0 \) so that \( (-t_j)^{\beta-\gamma} \tilde{h}(t_j) > \varepsilon \) for \( j = 1, 2, \ldots \). As \( \beta - \gamma > 0 \), we have \( \limsup_{j \to \infty} \tilde{h}(t_j) = \infty \), and so we can for each \( j \) find a \( k_j > j \) such that \( \tilde{h}(t_{k_j}) > \tilde{h}(t_j) \). By the definition of \( \tilde{h} \), and the continuity of \( h \), there is some \( s_j \) such that
\[ t_j < s_j \leq t_{k_j} \quad \text{and} \quad \tilde{h}(t_{k_j}) = h(s_j). \]

Thus
\[ (-s_j)^{-\gamma} \delta(s_j) = (-s_j)^{\beta-\gamma} h(s_j) \geq (-t_{k_j})^{\beta-\gamma} \tilde{h}(t_{k_j}) > \varepsilon. \]

But this contradicts (6.2), and hence (6.4) is true. Finally we can find a smooth \( \hat{\delta} \) such that \( \hat{\delta} < \tilde{\delta} < 2\delta \) and \( (-t)^{-\beta} \hat{\delta}(t) \) is nondecreasing. Note that
\[ \lim_{t \to 0^-} (-t)^{-\gamma} \hat{\delta}(t) = 0. \]

If we define \( \hat{\Theta} \supset \Theta \) in the same way as \( \Theta \), but using \( \hat{\delta} \) instead of \( \delta \), and \( \hat{\Theta} \) is regular, then also \( \Theta \) is regular, by Proposition 3.4. We have thus shown that we may assume (6.3) without loss of generality.

By Theorem 3.3, it is enough to show that there exists a barrier family \( \{w_C\} \supset C_0 \) in \( \Theta \) at the origin \( \xi_0 = (0, 0) \). The family \( \{w_C\} \supset C_0 \) we construct will be smooth in
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In the calculations below, we will for simplicity drop the subscript $C$.

Moreover, since $Q \geq C$ and $f < 0$, we have by assumption (6.3) that

$$w(x, t) \leq \rho(t) = -C^{1/(p-2)} \delta(t) f(t) = C^{1/(p-2)} \delta(t)^{(p-1)/(p-2)} (-t)^{-n/\lambda} \to 0,$$

as $t \to 0^-$. Thus, (b) in Definition 3.2 holds.

In order to prove (a) in Definition 3.2, we need to show that the domain defined by $Q(x, t)^{(p-1)/(p-2)} - C^{(p-1)/(p-2)} < -\frac{\rho(t)}{f(t)} = C^{1/(p-2)} \delta(t).$ (6.5)

Moreover, since $Q \geq C$ and $f < 0$, we have by assumption (6.2) that

$$w(x, t) \leq \rho(t) = -C^{1/(p-2)} \delta(t) f(t) = C^{1/(p-2)} \delta(t)^{(p-1)/(p-2)} (-t)^{-n/\lambda} \to 0,$$

as $t \to 0^-$. Thus, (b) in Definition 3.2 holds.

In order to prove (a) in Definition 3.2, we need to show that the domain defined by (6.5) contains $\Theta$. Indeed, in $\Theta$ we have

$$\left( \frac{|x|}{(-t)^{1/\lambda}} \right)^{p/(p-1)} < \delta(t).$$

The elementary inequality $(1+s)^\alpha < 1 + \alpha s (1+s)^{\alpha-1}$ with $\alpha = (p-1)/(p-2) > 1$ then yields that for sufficiently large $C$ we have in $\Theta$,

$$Q^{(p-1)/(p-2)} - C^{(p-1)/(p-2)} \leq C^{(p-1)/(p-2)} \left[ \left( 1 + \frac{(p-2)\delta(t)}{pC^{1/(p-1)}} \right)^{(p-1)/(p-2)} - 1 \right]$$

$$\leq C^{(p-1)/(p-2)} \frac{(p-1)\delta(t)}{pC^{1/(p-1)}} \left( 1 + \frac{(p-2)\delta(t)}{pC^{1/(p-1)}} \right)^{1/(p-2)}$$

$$\leq \frac{p-1}{p} C^{1/(p-2)} \delta(t).$$ (6.6)

since $\lambda > 1$. Thus $\Theta$ is contained in the $1$-domain defined by (6.5) if $C$ is large enough.

Next we show that $w$ is $p$-superparabolic in $\Theta$. We have

$$\nabla Q = \frac{p-2}{(p-1)\lambda^{1/(p-1)}} \frac{|x|^{(2-p)/(p-1)} x}{(-t)^{p/\lambda(p-1)}}.$$
\[ \partial_t Q = \frac{p - 2}{(p - 1)\lambda^p/(p-1)} \left( \frac{|x|}{(-t)^{1/\lambda}} \right)^{p/(p-1)} \frac{1}{-t} = \frac{p(Q - C)}{\lambda(p - 1)(-t)}. \]

Since \( w(x, t) = (Q^{(p-1)/(p-2)} - C^{(p-1)/(p-2)}) f(t) + \rho(t) \), we have

\[ \nabla w = \frac{p - 1}{p - 2} f Q^{1/(p-2)} \nabla Q = \frac{Q^{1/(p-2)} f}{\lambda^{1/(p-1)}} \left( \frac{|x|}{(-t)^{1/\lambda}} \right)^{p/(p-1)} x. \]

\[ |\nabla w|^{p-2} \nabla w = \frac{Q^{(p-1)/(p-2)} |f|^{p-2} f}{\lambda} \frac{x}{(-t)^{p/\lambda}}. \]

Therefore,

\[ \Delta_p w = \frac{Q^{(p-1)/(p-2)} |f|^{p-2} f}{\lambda} \frac{n}{(-t)^{p/\lambda}} \]

\[ + \frac{Q^{1/(p-2)} |f|^{p-2} f}{\lambda^{p/(p-1)}} \left( \frac{|x|}{(-t)^{1/\lambda}} \right)^{p/(p-1)} 1 \]

\[ \leq \frac{Q^{(p-1)/(p-2)} |f|^{p-2} f}{\lambda} \frac{n}{(-t)^{p/\lambda}}. \]

since \( f < 0 \). Moreover,

\[ \partial_t w = \left( Q^{(p-1)/(p-2)} - C^{(p-1)/(p-2)} \right) f' + \rho' + \frac{p - 1}{p - 2} f Q^{1/(p-2)} \partial_t Q \]

\[ = \left( Q^{(p-1)/(p-2)} - C^{(p-1)/(p-2)} \right) f' + \rho' + \frac{p f Q^{1/(p-2)} (Q - C)}{\lambda(p - 2)(-t)}. \]

Combining the previous expressions yields

\[ \partial_t w - \Delta_p w \geq \rho' + \left( Q^{(p-1)/(p-2)} - C^{(p-1)/(p-2)} \right) f' \]

\[ + \frac{p f Q^{1/(p-2)} (Q - C)}{\lambda(p - 2)(-t)} - \frac{Q^{(p-1)/(p-2)}}{\lambda} \frac{n}{(-t)^{p/\lambda}} |f|^{p-2} f. \]

In the domain given by (6.5) (and thus in \( \Theta \)) we have

\[ 0 \leq Q^{(p-1)/(p-2)} - C^{(p-1)/(p-2)} < -\frac{\rho(t)}{f(t)}, \]

and as \( f' \leq 0 \), this yields

\[ \partial_t w - \Delta_p w \geq \rho' - \frac{\rho f'}{f} + \frac{p f Q^{1/(p-2)} (Q - C)}{\lambda(p - 2)(-t)} - \frac{Q^{(p-1)/(p-2)}}{\lambda} \frac{n}{(-t)^{p/\lambda}} |f|^{p-2} f \]

\[ = \left( \left( \frac{\rho}{f} \right) + \frac{p f Q^{1/(p-2)} (Q - C)}{\lambda(p - 2)(-t)} - \frac{n Q^{(p-1)/(p-2)}}{\lambda} \frac{|f|^{p-2}}{(-t)^{p/\lambda}} \right) f(t) \]

\[ =: H(x, t) f(t). \]
Since \( f < 0 \), the last expression will be nonnegative if \( H(x, t) \leq 0 \). We have \( \rho(t)/f(t) = -C^{1/(p-2)}\delta(t) \), and in \( \Theta \) we have for sufficiently large \( C \),

\[
C \leq Q \leq C + \frac{(p-2)\delta(t)}{p\lambda^{1/(p-1)}} \leq 2C.
\]

This yields

\[
H(x, t) \leq -C^{1/(p-2)}\delta'(t) + \frac{Q^{1/(p-2)}\delta(t)}{\lambda^{p/(p-1)}} - \frac{nQ^{(p-1)/(p-2)}}{\lambda} \frac{|f|^{p-2}}{(-t)^p/\lambda} \\
\leq -C^{1/(p-2)}\delta'(t) + \frac{(2C)^{1/(p-2)}\delta(t)}{\lambda^{p/(p-1)}} - \frac{nC^{(p-1)/(p-2)}}{\lambda} \frac{|f|^{p-2}}{(-t)^p/\lambda}.
\]

Now,

\[
\frac{|f|^{p-2}}{(-t)^{p/\lambda}} = \frac{\delta(t)(-t)^{-n(p-2)/\lambda}}{(-t)^{p/\lambda}} = \frac{\delta(t)}{-t},
\]

from which it follows that for sufficiently large \( C \),

\[
H(x, t) \leq C^{1/(p-2)} \left[ -\delta'(t) + \left( \frac{2^{1/(p-2)}}{\lambda^{p/(p-1)}} - \frac{nC}{\lambda} \right) \frac{\delta(t)}{-t} \right] \\
\leq C^{1/(p-2)} \left[ -\delta'(t) - \beta \frac{\delta(t)}{-t} \right] \leq 0,
\]

since \((-t)^{-\beta}\delta(t)\) is nondecreasing by (6.3). Thus we have shown that \( w_C \) is a supersolution in \( \Theta \), if \( C \) is large enough.

Finally, we show that (c) in Definition 3.2 is satisfied. For \((x, t) \in \overline{\Theta} \setminus \{(0, 0)\}\) we have

\[
\left( \frac{|x|}{(-t)^{1/\lambda}} \right)^{p/(p-1)} \leq \delta(t),
\]

and hence by (6.6),

\[
Q(x, t)^{(p-1)/(p-2)} - C^{(p-1)/(p-2)} \leq \frac{p-1}{p} C^{1/(p-2)}\delta(t).
\]

Since \( f < 0 \), this implies that

\[
w_C(x, t) \geq \frac{p-1}{p} C^{1/(p-2)}\delta(t)f(t) - C^{1/(p-2)}\delta(t)f(t) \\
= -\frac{1}{p} C^{1/(p-2)}\delta(t)f(t) = \frac{1}{p} C^{1/(p-2)}\delta(t)^{(p-1)/(p-2)}(-t)^{-n/\lambda}.
\]
From (6.3) we conclude that \((-t)^{-\beta} \delta(t) \geq \theta\) for \(t_0/2 < t < 0\) and some \(\theta > 0\). Hence,

\[
w_C(x, t) \geq \frac{1}{p} C^{1/(p-2)} \theta^{(p-1)/(p-2)} (-t)^{(p-2)n/p} > 0
\]

for those \(t\).

As \((x, t) \in \partial \Theta\) with \(|(x, t)| \geq 1/k\) implies that \(-t \geq \varepsilon_k\) for some \(\varepsilon_k > 0\), this shows that \(\{w_C\}^\infty_{C=C_0}\) is a barrier family for the domain \(\Theta^* = \{(x, t) \in \Theta : t > t_0/2\}\), provided that \(C_0\) is large enough. It thus follows from Theorem 3.3 that \((0, 0)\) is regular with respect to \(\Theta^*\) and thus with respect to \(\Theta\), by Proposition 3.5.

Even though the domain \(\Theta\) in (6.7) below is irregular (by Theorem 1.1) and does not have a traditional barrier at the origin, we can still obtain regularity for some small functions vanishing at \((0, 0)\) as well as at \((0, -1)\).

**Proposition 6.3** Let \(0 < q \leq 1/p\),

\[
\Theta = \left\{(x, t) \in \mathbb{R}^{n+1} : |x| < (-t)^q \text{ and } -1 < t < 0\right\}. \quad (6.7)
\]

and

\[
u(x, t) = \begin{cases} 
A \left(\frac{|x|^p}{(-t)^p}\right)^{1/(p-2)}, & \text{if } (x, t) \in \overline{\Theta} \setminus \{(0, 0)\}, \\
0, & \text{if } (x, t) = (0, 0),
\end{cases}
\]

where

\[0 < \beta < pq \leq 1 \text{ and } A = \left(\frac{\beta}{\lambda} \left(1 - \frac{2}{p}\right)^{p-1}\right)^{1/(p-2)}\]

If \(f \in C(\partial \Theta)\) satisfies

\[|f(x, t) - f(0, 0)| \leq u(x, t) \text{ for } (x, t) \in \partial \Theta,\]

then

\[
\lim_{\Theta \ni (x, t) \to (0, 0)} H f(x, t) = \lim_{\Theta \ni (x, t) \to (0, 0)} \overline{H} f(x, t) = f(0, 0). \quad (6.8)
\]

It is easy to see that it is preferable to choose \(\beta\) as large as possible. However, we cannot choose \(\beta = pq\) as then \(u\) would not be continuous at the origin.

The function \(u\) constructed above fails to be a traditional barrier only in one respect, namely \(\lim_{\Theta \ni (x, t) \to (0, -1)} u(x, t) = u(0, -1) = 0\). Thus, one requirement on \(f\) is that \(f(0, -1) = f(0, 0)\). Moreover, it also follows from the proof below that

\[
\lim_{\Theta \ni (x, t) \to (0, -1)} H f(x, t) = \lim_{\Theta \ni (x, t) \to (0, -1)} \overline{H} f(x, t) = f(0, -1) = f(0, 0).
\]
Obviously one can obtain similar results for
\[ \Theta = \{ (x, t) \in \mathbb{R}^{n+1} : |x| < K(-t)^q \text{ and } t_0 < t < 0 \}. \]
when \( K > 0 \) and \( t_0 < 0 \).

**Proof** By Lemma 2.5 with \( \alpha = p/(p-2) \) we have
\[
\Delta_p u = \left( \frac{A \alpha}{(-t)^{\beta/(p-2)}} \right)^{p-1} \frac{\lambda}{p-2} |x|^{\alpha}
\]
and
\[
\partial_t u = \frac{A \beta}{(p-2)(-t)^{\beta/(p-2)+1}} |x|^{\alpha}.
\]

Thus, in \( \Theta \),
\[
\partial_t u - \Delta_p u = \frac{A |x|^\alpha}{(p-2)(-t)^{\beta/(p-2)+1}} \left( \beta - A^p \alpha p^{-2} \lambda (-t)^{1-\beta} \right)
\geq \frac{A |x|^\alpha}{(p-2)(-t)^{\beta/(p-2)+1}} \left( \beta - A^p \alpha p^{-2} \lambda \right) = 0,
\]
where we have used that \( \beta < 1 \). Hence, \( u \) is \( p \)-superparabolic in \( \Theta \). Moreover, as \( \beta < pq \), we see that \( u \in C(\overline{\Theta}) \). So \( u + f(0, 0) \in \mathcal{U}_f \) and \( -u + f(0, 0) \in \mathcal{L}_f \). Hence
\[
f(0, 0) = \lim_{\Theta \ni (x, t) \to (0, 0)} (f(0, 0) - u(x, t)) \leq \liminf_{\Theta \ni (x, t) \to (0, 0)} \mathcal{H} f(x, t)
\]
and
\[
\limsup_{\Theta \ni (x, t) \to (0, 0)} \mathcal{H} f(x, t) \leq \lim_{\Theta \ni (x, t) \to (0, 0)} (f(0, 0) + u(x, t)) = f(0, 0),
\]
which together with the inequality \( \mathcal{H} f \leq \mathcal{H} f \) yield (6.8).

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