Decoupling of Translations from Homogeneous Transformations in Inhomogeneous Quantum Groups*

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Abstract

We briefly report on our result that, if there exists a realization of a Hopf algebra \( H \) in a \( H \)-module algebra \( A \), then their cross-product is equal to the product of \( A \) itself with a subalgebra isomorphic to \( H \) and \textit{commuting} with \( A \). We illustrate its application to the Euclidean quantum groups in \( N \geq 3 \) dimensions.

1 Introduction and main results

Let \( H \) be a Hopf algebra and \( A \) a unital \( H \)-module algebra. Here for the sake of brevity we stick to consider the case of a right \( H \)-module algebra. Then by definition there exists a bilinear map \( \triangleleft : (a, g) \in A \times H \rightarrow a \triangleleft g \in A \), called a right action, such that

\[
a \triangleleft (gg') = (a \triangleleft g) \triangleleft g'
\]

\[
(aa') \triangleleft g = (a \triangleleft g(1)) (a' \triangleleft g(2)).
\]

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We use a Sweedler-type notation with suppressed summation sign for the coproduct $\Delta(g)$ of $g$, namely the short-hand notation $\Delta(g) = g_{(1)} \otimes g_{(2)}$. The cross-product algebra $\mathcal{A} \bowtie \mathcal{H}$ is $\mathcal{H} \otimes \mathcal{A}$ as a vector space, whereas the product of two elements is given by

$$(g \otimes a)(g' \otimes a') = gg'_{(1)} \otimes (a \triangleleft g'_{(2)})a'$$

for any $a, a' \in \mathcal{A}$, $g, g' \in \mathcal{H}$. To simplify the notation, in the sequel we denote $g \otimes a$ by $ga$ and omit either unit $1_{\mathcal{A}}, 1_{\mathcal{H}}$ whenever multiplied by non-unit elements; consequently, for $g = 1_{\mathcal{H}}, a' = 1_{\mathcal{A}}$ (3) takes the form

$$ag' = g'_{(1)} (a \triangleleft g'_{(2)}).$$

$\mathcal{A} \bowtie \mathcal{H}$ itself is a $\mathcal{H}$-module algebra under the right action \triangleleft of $\mathcal{H}$ if we extend the latter on the elements of $\mathcal{H}$ as the adjoint action, $h \triangleleft g = Sg_{(1)}hg_{(2)}, g, h \in \mathcal{H}$. (5)

One very important situation in which one encounters cross products is in inhomogeneous (quantum) groups, that are Hopf algebras whose algebras are cross-products $\mathcal{A} \bowtie \mathcal{H}$, where $\mathcal{H}$ is the Hopf subalgebra of “homogeneous transformations” and $\mathcal{A}$ the (braided) Hopf subalgebra of “translations”.

In Ref. [3] we have found a sufficient condition for $\mathcal{A} \bowtie \mathcal{H}$ to be equal to a product $\mathcal{A} \mathcal{H}',$ where $\mathcal{H}' \subset \mathcal{H} \bowtie \mathcal{A}$ is a subalgebra isomorphic to $\mathcal{H}$ and commuting with $\mathcal{A}$: it is sufficient to assume that there exists an algebra map

$$\tilde{\varphi} : \mathcal{A} \bowtie \mathcal{H} \rightarrow \mathcal{A}$$

acting as the identity on the subalgebra $\mathcal{A}$, namely for any $a \in \mathcal{A}$

$$\tilde{\varphi}(a) = a.$$ (7)

Here we briefly recall this result, as well as a similar one requiring a weaker assumption, namely that $\mathcal{H}$ admits a Gauss decomposition into two Hopf subalgebras $\mathcal{H}^+, \mathcal{H}^-$ for each of which analogous maps $\tilde{\varphi}^+, \tilde{\varphi}^-$ exist. In the last section we illustrate the application of these general results to a class of inhomogeneous quantum groups, the Euclidean quantum groups in $N \geq 3$ dimensions, for which $\mathcal{A}$ is (“the algebra of functions on”) the quantum Euclidean $N$-dimensional space; the corresponding maps $\tilde{\varphi}, \tilde{\varphi}^+, \tilde{\varphi}^-$ have been determined in [1] and further studied in [4]. [Note that no such maps can exist for the (undeformed) Euclidean algebra, for which $\mathcal{A} \equiv$ the algebra of functions
on $\mathbb{R}^N$, which is abelian). Other applications include $U_qg$-covariant Heisenberg (or Clifford) algebras and the $q$-deformed 2-dimensional fuzzy sphere $S^2_{q,M}$.

Let $\tilde{C}$ be the commutant of $A$ within $A \rtimes H$, i.e. the subalgebra
\[ \tilde{C} := \{ c \in A \rtimes H \mid [c, a] = 0 \quad \forall a \in A \}. \tag{8} \]
Clearly $\tilde{C}$ contains the center $Z(A)$ of $A$. Let $\tilde{\zeta}: H \to A \rtimes H$ be the map defined by
\[ \tilde{\zeta}(g) := g(1) \tilde{\varphi}(Sg(2)). \tag{9} \]
Note that if we apply $\tilde{\varphi}$ to $\tilde{\zeta}$ and recall that $\tilde{\varphi}$ is both a homomorphism and idempotent we find $\tilde{\varphi} \circ \tilde{\zeta} = \varepsilon$. Now, $\varepsilon(g)$ is a complex number times $1_A$ and therefore trivially commutes with $A$. Since by definition the commutation relations between $A$ and either $H$ or $\tilde{\varphi}(H)$ are the same, we expect that also $\tilde{\zeta}(g)$ commutes with $A$. This is confirmed by

**Theorem 1** \[3\] Under the above assumptions the map $\tilde{\zeta}$ is an injective algebra homomorphism $\tilde{\zeta}: H \to \tilde{C}$; moreover $\tilde{C} = \tilde{\zeta}(H) Z(A)$ and $A \rtimes H = \tilde{\zeta}(H) A$. If, in particular $Z(A) = C$, then $\tilde{C} = \tilde{\zeta}(H)$ and $\tilde{\zeta}: H \leftrightarrow \tilde{C}$ is an algebra isomorphism. Moreover, the center of the cross-product $A \rtimes H$ is given by $Z(A \rtimes H) = Z(A) \tilde{\zeta}(Z(H))$. Finally, if $H_c, A_c$ are maximal abelian subalgebras of $H$ and $A$ respectively, then $A_c \tilde{\zeta}(H_c)$ is a maximal abelian subalgebra of $A \rtimes H$.

In other words, the subalgebra $H'$ looked for will be obtained by setting $H' := \tilde{\zeta}(H)$. For these reasons we shall call $\tilde{\zeta}$, as well as the other maps $\tilde{\zeta}^\pm$ which we shall introduce below, decoupling maps.

To introduce the latter we just need a weaker assumption, namely that, instead of a map $\tilde{\varphi}$, we just have at our disposal two homomorphisms $\tilde{\varphi}^+, \tilde{\varphi}^-$
\[ \tilde{\varphi}^\pm: H^\pm \triangleleft A \to A \tag{10} \]
fulfilling \[8\], where $H^+, H^-$ denote two Hopf subalgebras of $H$ such that Gauss decompositions $H = H^+ H^- = H^- H^+$ hold. (The typical case is when $H = U_qg$ and $H^+, H^-$ denote its positive and negative Borel subalgebras.) Then Theorem \[3\] will apply separately to $A \rtimes H^+$ and $A \rtimes H^-$, if we define corresponding maps $\tilde{\zeta}^\pm: H^\pm \to A$ by
\[ \tilde{\zeta}^\pm(g) := g(1) \tilde{\varphi}^\pm(Sg(2)), \tag{11} \]
where $g \in H^\pm$ respectively. What can we say about the whole $H \triangleleft A$?
Theorem 2. Under the above assumptions formulae (11) define injective algebra homomorphisms $\tilde{\zeta}^\pm : H^\pm \to \tilde{\mathcal{C}}$. Moreover,
\[ \tilde{\mathcal{C}} = \tilde{\zeta}^+(H^+) \tilde{\zeta}^-(H^-) \mathcal{Z}(\mathcal{A}) = \tilde{\zeta}^-(H^-) \tilde{\zeta}^+(H^+)(A) \tag{12} \]
and
\[ \mathcal{A} \rtimes H = \tilde{\zeta}^+(H^+) \tilde{\zeta}^-(H^-) = \tilde{\zeta}^-(H^-) \tilde{\zeta}^+(H^+)\mathcal{A}. \tag{13} \]
In particular, if $\mathcal{Z}(\mathcal{A}) = \mathbb{C}$, then the commutant is equal to $\tilde{\mathcal{C}} = \tilde{\zeta}^+(H^+) \tilde{\zeta}^-(H^-) = \tilde{\zeta}^-(H^-) \tilde{\zeta}^+(H^+)$. Furthermore, any element $c$ of the center $\mathcal{Z}(\mathcal{A} \rtimes H)$ of the cross-product $\mathcal{A} \rtimes H$ can be expressed in the form
\[ c = \tilde{\zeta}^+ \left( c^{(1)} \right) \tilde{\zeta}^- \left( c^{(2)} \right) c^{(3)}, \tag{14} \]
where $c^{(1)} \otimes c^{(2)} \otimes c^{(3)} \in H^+ \otimes H^- \otimes \mathcal{Z}(\mathcal{A})$ and $c^{(1)} c^{(2)} \otimes c^{(3)} \in \mathcal{Z}(H) \otimes \mathcal{Z}(\mathcal{A})$; vice versa any such object $c$ is an element of $\mathcal{Z}(\mathcal{A} \rtimes H)$. Finally, if $H_c \subset H^+ \cap H^-$ and $A_c$ are maximal abelian subalgebras of $H$ and $A$ respectively, then $A_c \tilde{\zeta}^+(H_c)$ (as well as $A_c \tilde{\zeta}^-(H_c)$) is a maximal abelian subalgebra of $\mathcal{A} \rtimes H$.

As a consequence of this theorem, for any $g^+ \in H^+$, $g^- \in H^-$ there exists a sum $c^{(1)} \otimes c^{(2)} \otimes c^{(3)} \in \mathcal{Z}(\mathcal{A}) \otimes H^- \otimes H^+$ (depending on $g^+, g^-$) such that
\[ \tilde{\zeta}^+(g^+) \tilde{\zeta}^-(g^-) = c^{(1)} \tilde{\zeta}^- (c^{(2)}) \tilde{\zeta}^+(c^{(3)}). \tag{15} \]
These will be the commutation relations between elements of $\tilde{\zeta}^+(H^+)$ and $\tilde{\zeta}^-(H^-)$. Their form will depend on the specific algebras considered.

Assume that $H$ is a Hopf $*$-algebra and $\mathcal{A}$ a $H$-module $*$-algebra. Then, as known, these two $*$-structures can be glued in a unique one to make $\mathcal{A} \rtimes H$ a $*$-algebra itself. What can we say about the behaviour of the decoupling maps under the latter $*$-structure?

Proposition 1. If $\tilde{\varphi} : \mathcal{A} \rtimes H \to \mathcal{A}$ is a $*$-homomorphism, then also the map $\tilde{\zeta} : H \to \tilde{\mathcal{C}}$ is. If $\varphi^{\pm}$ are $*$-homomorphism, then also the map $\tilde{\zeta}^\pm : H^\pm \to \tilde{\mathcal{C}}$ are. If $\tilde{\varphi}^\pm$ fulfill the condition $\tilde{\varphi}^\pm (\alpha^*) = [\tilde{\varphi}^\pm (\alpha)]^*$ for any $\alpha \in H^\mp \rtimes \mathcal{A}$, then $\tilde{\zeta}^\pm$ fulfill
\[ \tilde{\zeta}^\pm (g^*) = [\tilde{\zeta}^\pm (g)]^*, \quad g \in H^\mp. \tag{16} \]
2 Application to the Euclidean quantum group $R_q^N \rtimes U_q so(N)$

As an algebra $A$ we shall consider a slight extension of the quantum Euclidean space $R_q^N$ (the $U_q so(N)$-covariant quantum space), i.e. of the unital associative algebra generated by $p^i$ fulfilling the relations

$$P_{a_{hk}}^{jk} p^h p^k = 0,$$

(17)

where $P_a$ denotes the $q$-deformed antisymmetric projector [29], and the indices take the values $i = -n, \ldots, -1, 0, 1, \ldots n$ for $N$ odd, and $i = -n, \ldots, -1, 1, \ldots n$ for $N$ even; here $n := \left[ \frac{N}{2} \right]$ is the rank of $so(N)$. The multiplet $(p^i)$ carries the fundamental $N$-dim (or vector) representation $\rho$ of $U_q so(N)$: for any $g \in U_q so(N)$

$$p^i \triangleleft g = \rho^i_j (g) p^j.$$

(18)

As a set of generators of $H \equiv U_q so(N)$ it is convenient to introduce the FRT generators [3] $L^+_{ij}, L^-_{ij}$, together with the square roots of the diagonal elements $L^+_i, L^-_i$. The FRT generators and the braid matrix $\hat{R}$ are related to the so-called universal $R$-matrix $R \in H \otimes H$ by

$$L^+_i := \mathcal{R}^{(1)} \rho^0_i (\mathcal{R}^{(2)})$$

and

$$L^-_i := \rho^0_i (\mathcal{R}^{-1(1)}) \mathcal{R}^{-1(2)}$$

(19)

and $\hat{R}^i_{hk} = p^i_h (L^+_k) = (\rho^i_h \otimes \rho^i_k) (\mathcal{R})$. Since in our conventions $\mathcal{R} \in H^+ \otimes H^-$ ($H^+, H^-$ denote the positive, negative Borel subalgebras) we see that $L^+_i \in H^+$ and $L^-_i \in H^-.$

To define $\tilde{\varphi}$ or $\tilde{\varphi}^\pm$ one [1] slightly enlarges $R_q^N$ as follows. One introduces some new generators $\sqrt{P_a}$, with $1 \leq a \leq \frac{N}{2}$, together with their inverses $(\sqrt{P_a})^{-1}$, requiring that

$$P_a^2 = \sum_{h=-a}^a p^h p_h = \sum_{h,k=-a}^a g_{hk} p^h p^k.$$  

(20)

In the previous equation $g_{hk}$ denotes the ‘metric matrix’ of $SO_q(N)$:

$$g_{ij} = g_{ji} = q^{-\rho_i} \delta_{i,-j}.$$  

(21)

It is a $SO_q(N)$-isotropic tensor and is a deformation of the ordinary Euclidean metric. We have introduced the notation [3] $(\rho_i) = (n -
$\frac{1}{2}, \ldots, \frac{1}{2}, 0, -\frac{1}{2}, \ldots, -\frac{1}{2}, n)$ for $N$ odd, $(n-1, \ldots, 0, 0, \ldots, 1-n)$ for $N$ even. In the sequel we shall call $P^2_n$ also $P^2$. Moreover for odd $N$ we add also $\sqrt{p^0}$ and its inverse as new generators. The commutation relations involving these new generators can be fixed consistently, in particular one finds that

$$\sqrt{p^0} p^i = p^i \sqrt{p^0} \times \begin{cases} 1 & \text{if } |i| \leq a \\ \sqrt{q} & \text{if } i > a \\ 1/\sqrt{q} & \text{if } i < a \end{cases} \quad (22)$$

The center of $R_q^N$ is generated by $\sqrt{P}$ and, only in the case of even $N$, by $\sqrt{p^1/p^{-1}}$ and its inverse $\sqrt{p^{-1}/p^1}$. In the case of even $N$ one needs to include also the FRT generator $L^{-1}_1 = L^{+1}_{-1}$ and its inverse $L^{+1}_1 = L^{-1}_{-1}$ [which are generators of $U_q so(N)$ belonging to the natural Cartan subalgebra] among the generators of $A$. They satisfy nontrivial commutation relations with the generators of $A$, and the standard FRT relations with the rest of $U_q so(N)$. As a consequence, $\sqrt{p^{\pm 1}/p^{\mp 1}}$ are eliminated from the center of $A$ (in fact they do not $q$-commute with $L^{\pm 1}$).

The homomorphisms $\tilde{\varphi}^\pm : A \cong U_q^\pm so(N) \to A$ take the simplest and most compact expression on the FRT generators of $U_q^\pm so(N)$. Let us introduce the short-hand notation $[A, B]_x = AB - xBA$. The images of $\tilde{\varphi}^-$ on the FRT generators read

$$\tilde{\varphi}^-(L_{-1}^i) = g_i^h [\bar{\mu}_h, p^k]_q g_{kj}, \quad (23)$$

$$\tilde{\varphi}^+(L_1^i) = g_i^h [\bar{\mu}_h, p^k]_{q^{-1}} g_{kj}, \quad (24)$$

where

$$\mu_0 = \gamma_0(p^0)^{-1}, \quad \bar{\mu}_0 = \bar{\gamma}_0(p^0)^{-1} \quad \text{for } N \text{ odd,}$$
$$\mu_{\pm 1} = \gamma_{\pm 1}(p^{\pm 1})^{-1} L_{\pm 1}^1, \quad \bar{\mu}_{\pm 1} = \bar{\gamma}_{\pm 1}(p^{\pm 1})^{-1} L_{\mp 1}^1 \quad \text{for } N \text{ even,}$$
$$\mu_a = \gamma_a P_{[a]}^{-1} P_{[a]-1} p^{-a}, \quad \bar{\mu}_a = \bar{\gamma}_a P_{[a]}^{-1} P_{[a]-1} p^{-a} \quad \text{otherwise,} \quad (25)$$

and $\gamma_a, \bar{\gamma}_a \in \mathbb{C}$ are normalization constants fulfilling the conditions

$$\gamma_0 = -q^{\frac{1}{2}} h^{-1}, \quad \bar{\gamma}_0 = q^{\frac{1}{2}} h^{-1} \quad \text{for } N \text{ odd,}$$
$$\gamma_{\pm 1} = -k^{-1}, \quad \bar{\gamma}_{\pm 1} = k^{-1} \quad \text{for } N \text{ even,}$$
$$\gamma_1 \gamma_{-1} = -q^{-1} h^{-2}, \quad \bar{\gamma}_1 \bar{\gamma}_{-1} = -q h^{-2} \quad \text{for } N \text{ odd,}$$
$$\gamma_a \gamma_{-a} = -q^{-1} k^{-2} \omega_a \omega_{-a}, \quad \bar{\gamma}_a \bar{\gamma}_{-a} = -q k^{-2} \omega_a \omega_{-a} \quad \text{for } a > 1. \quad (26)$$
Here \( h := q^{1/2} - q^{-1/2} \), \( k := q - q^{-1} \), \( \omega_a := (q^{\rho_a} + q^{-\rho_a}) \). One can choose the \( \gamma_a \)'s, \( \bar{\gamma}_a \)'s: 1) only for odd \( N \), in such a way that \( \tilde{\varphi}^-, \tilde{\varphi}^+ \) can be “glued” into a unique homomorphism \( \varphi \) of the type (1), (2); 2) for \( |q| = 1 \) or \( q \in \mathbb{R}^+ \) respectively, in such a way that either condition necessary for the application of Proposition 1 is fulfilled. From definition (9), using (30), (23), (24), we find

\[
\tilde{\zeta}^- (L^{-i,j}) = L^-_{h} \tilde{\varphi}^- (SL^{-h}_{j}) = L^-_{h} [\mu_h, p^j]_q,
\]

\[
\tilde{\zeta}^+ (L^{+i,j}) = L^+_{h} \tilde{\varphi}^+ (SL^+_{j}) = L^+_{h} [\bar{\mu}_h, p^j]_{q^{-1}}.
\]  

In Ref. [3] we have determined the commutation relations (17) between them.

### Appendix

The braid matrix of \( U_q so(N) \) is given by

\[
\hat{R} = q \sum_{i \neq 0} e^i_i \otimes e^i_i + \sum_{i \neq j, -j \text{ or } i = j = 0} e^i_j \otimes e^j_j + q^{-1} \sum_{i \neq 0} e^{-i}_i \otimes e^{-i}_i \quad (28)
\]

\[+ k \left( \sum_{i<j} e^i_i \otimes e^j_j - \sum_{i<j} q^{-\rho_i + \rho_j} e^{-j}_i \otimes e^{-i}_j \right). \]

It admits the orthogonal projector decomposition

\[
\hat{R} = q P_s - q^{-1} P_a + q^{1-N} P_t; \quad (29)
\]

\( P_a, P_t, P_s \) are \( q \)-deformed antisymmetric, trace, trace-free symmetric projectors. The coproduct and antipode of the FRT generators are given by

\[
\Delta(L_{\pm i,j}^\pm) = L_{\pm h}^\pm \otimes L_{\pm h}^\pm, \quad S L_{\pm i}^\pm = g_{ih} L_{\mp k}^\pm k^j. \quad (30)
\]

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