Mobile fluxon qubits in a long superconductor-ferromagnet-superconductor Josephson junction

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Abstract. We propose a new type of mobile qubit that utilizes a bound pair of half fluxons in a long superconductor-ferromagnet-superconductor (SFS) Josephson junction. The qubit states are composed of the lowest two levels of the quantized nonlinear internal oscillation of the bound pair. The energy levels are estimated by the numerical quantization based on a collective coordinate method. The qubit operation scheme is discussed, showing an estimate of the interaction strength between a bound pair and a microcircuit.

1. Introduction
A quantum computer is an innovative computer based on the principles of quantum mechanics. The fundamental unit of a quantum computer is a qubit, which is a quantum-mechanical superposition state of a two-state system. As regards solid-state qubits, the precise mechanism providing control of the interaction between an arbitrary pair of qubits remains unresolved. To overcome this problem, we proposed a mobile qubit [1, 2] using breather excitation in a conventional long Josephson junction. However, a breather is fragile with respect to background perturbations. In particular, a breather can disappear through energy loss with dissipation, because it is topologically equivalent to a vacuum. This feature makes it difficult to prepare the initial $|0\rangle$ state needed for quantum computing.

Here we propose a new type of mobile qubit that utilizes a bound pair of half fluxons in a long superconductor-ferromagnet-superconductor (SFS) Josephson junction. Transitions between usual (0) and $\pi$ junction states have been demonstrated in SFS junctions as a function of temperature [3] and barrier thickness [4]. The $\pi$ junction state is characterized by a ground state at a phase difference of $\pi$. In the vicinity of the crossover between the 0 and $\pi$ states, the second harmonic term in the current-phase relation is expected to dominate [5, 6, 7, 8]. This term would modify the electrodynamics of long SFS junctions, and the propagation of fluxons would be governed by the double sine-Gordon equation (DSGE) instead of the sine-Gordon equation that governs usual long junctions. Since a $2\pi$ kink in the double sine-Gordon system splits into two $\pi$ kinks (half fluxons) and forms a bound pair with nonlinear internal oscillation modes (called $\pi-\pi$ kink oscillation), the bound-half-fluxon pair in long SFS junctions can carry...
quantum information if the internal oscillations are quantized. This bound-half-fluxon qubit is robust against perturbations because of its topological stability compared with the breather qubit. Once the internal oscillation has attenuated in the presence of dissipation, zero-point oscillation, namely the initial $|0\rangle$ state, is realized because a kink pair with a topological twist cannot decay into a vacuum. Besides, it is easy to control the speed of a moving pair because the center of mass couples with the direct current bias as in the case of $2\pi$ fluxons in conventional long junction systems.

In the following sections we investigate the quantum energy levels of the $\pi-\pi$ kink oscillation by numerical quantization based on a collective coordinate method. We will also briefly discuss an experimental setting for observing and manipulating the quantum states of bound-fluxon qubits.

2. Double sine-Gordon equation for long SFS junction

We have recently shown that the phase difference of a long SFS junction obeys DSGE (unpublished paper). Consider two identical superconductors separated by a thin ferromagnetic insulator with thickness $a$ as shown in Fig.1. The width of the junction $w$ is so small that the junction is considered to be one-dimensional.

![Figure 1. Schematic diagram of a long Josephson junction with a ferromagnetic insulator.](image)

![Figure 2. Phase profile during the $\pi-\pi$ kink oscillation.](image)

In contrast to a usual long Josephson junction, the current density in a long Josephson junction with a ferromagnetic insulator is now given as $[5, 6, 7, 8]$

$$J = \frac{J_c}{1 + 4\eta} \left( \sin \phi + 2\eta \sin 2\phi \right),$$

where $J_c$ is the amplitude of the current density for the first harmonic when $\eta = 0$ and half of that for the second harmonic when $\eta \to \infty$. This is therefore a measure of the Josephson critical current. From Maxwell equations that take account of the spatiotemporal dependence of the phase difference $\phi$ across the junction caused by an electric field $E$ and magnetic field $B$ through the relations $E = (h/2ea)\partial_t \phi$ and $B = (h/2ed)\partial_x \phi$, with $\partial_t = \partial/\partial t$ and $\partial_x = \partial/\partial x$, we obtain

$$\partial_t^2 \phi - \partial_x^2 \phi + \frac{1}{1+4\eta} (\sin \phi + 2\eta \sin 2\phi) = 0.$$  

Here, the coordinate $x$ is normalized by the Josephson penetration depth, i.e., $\lambda_J = \sqrt{h/2e\mu d}J_c$ and the time $t$ is normalized by $\omega_0^{-1} = \lambda_J/c$ with $c = \sqrt{a/\epsilon \mu}$, where $\epsilon$ is the dielectric constant of the ferromagnetic insulator, $\mu$ is the magnetic permeability and $d = a + 2\lambda_L$ with $\lambda_L$ being the London penetration depth. The phase difference in an SFS hybrid junction thus obeys the DSGE.

The DSGE is not integrable and exact soliton solutions do not exist. However, it is possible to find a topologically stable $2\pi$-kink solution for a range of parameter $\eta$, i.e., $\eta > -1/4$. For
\( \eta \geq 0 \), the \( 2\pi \)-kink solution can be constructed by a superposition of \( \pi \)-kink solutions when \( \eta \to \infty \), i.e., \( \phi_\pi(x) = 2 \tan^{-1} \exp(x) \) \cite{9, 10, 11, 12}, and expressed as

\[
\phi^K_\pi(x, R_0) = \phi_\pi(x + R_0) + \phi_\pi(x - R_0),
\]

where \( 2R_0 \) is the equilibrium distance between the \( \pi \)-kinks and is given by \( \eta = \frac{1}{4} \sinh^2 R_0 \).

The potential energy density is expressed as

\[
u_1(\phi) = \frac{1}{\cosh^2 R_0} \left( 1 - \cos \phi \right) + \frac{\sinh^2 R_0}{4} \left( 1 - \cos 2\phi \right),
\]

Here, the energy is normalized by \( E_0 = \epsilon_J \lambda J_0 \) with \( \epsilon_J = hJ_c/2e \) being the Josephson coupling energy per unit area. When \( \eta = 0 \) \((R_0 = 0)\), Eq. (2) is simply the sine-Gordon equation, and Eq. (3) reduces to the exact \( 2\pi \)-soliton solution. The shape of the \( 2\pi \) soliton is determined by the competition between the potential energy density \( u_1(\phi) \) and the elastic energy density, \( u_E(\phi) = 1/2(\partial \phi / \partial x)^2 \), which measures the rigidity of the Josephson phase difference. Also when \( \eta > 0 \) \((R_0 > 0)\), there is a \( 2\pi \)-kink solution where the phase changes from 0 to \( \pm 2\pi \) as \( x \) passes from \(-\infty \) to \( \infty \), since this potential has minima at \( \phi = 0 \) \( (\text{mod} \ 2\pi) \). However, the balance between \( u_J \) and \( u_E \) is changed around \( \phi = \pi \) by the effect of the second harmonic, the \( 2\pi \) kink splits into two separate \( \pi \) kinks owing to the gain of the Josephson energy \( u_J \) against the elastic energy \( u_E \). With a dynamical \( \pi \)-\( \pi \) kink pair, these two energies compete in time, resulting in a new type of dynamics. The \( \pi \)-\( \pi \) kink pair exhibits an internal oscillation related to the relative oscillations of two \( \pi \) kinks around the equilibrium separation \cite{9, 10, 11, 12}. It has been shown that the frequency of the \( \pi \)-\( \pi \) kink oscillation is below the lower edge of the continuum phonon band \cite{10, 13}. Figure 2 shows a typical profile of phase difference \( \phi \) during the \( \pi \)-\( \pi \) kink oscillation.

3. Collective coordinate for \( \pi \)-\( \pi \) kink oscillation

It is known that the dynamics of the \( \pi \)-\( \pi \) kink oscillation is well described by a single degree of freedom, namely, the separation between the \( \pi \) kinks, \( 2R \), when the equilibrium separation, \( R_0 \), is sufficiently large \cite{10, 11, 12}. We assume the solution to DSGE in the form

\[
\phi_K(x, R) = \phi_\pi(x + R) + \phi_\pi(x - R),
\]

and substitute it into the Lagrangian of the DSGE. We then obtain the Lagrangian for \( R \) as \cite{11}

\[
L(R, \dot{R}) = \frac{M(R)}{2} \dot{R}^2 - U(R),
\]

\[
M(R) = 4\{1 - f(R)\},
\]

\[
U(R) = 4 \left\{ 1 + \frac{\cosh^2 R}{\cosh^2 R_0} \right\} - 2\{1 - f(R)\} \left[ 1 + 2 \frac{\cosh^2 R}{\cosh^2 R_0} - \frac{\tanh^2 R_0}{\tanh^2 R} \right],
\]

where \( f(R) = 2R/\sinh(2R) \) and the dot denotes the differentiation with respect to \( t \).

Now, we perform a coordinate transformation, which eliminates the \( R \) dependence of the mass. We can see from Fig. 3 that \( M(R) \) can be well approximated by the form

\[
M_{ap}(R) = 4 \tanh^2 (\Gamma R),
\]

where \( \Gamma = \sqrt{2/3} \), which is chosen so that the derivatives of \( M_{ap}(R) \) and \( M(R) \) are coincident at \( R = 0 \). Using this approximation and adopting a new coordinate \( X \), which is given by

\[
X(R) = \frac{1}{\Gamma} \log(\cosh \Gamma R),
\]
we obtain the Lagrangian for $X$ as

$$L_{ap}(X) = 2\dot{X}^2 - V(X),$$

$$V(X) = U\left(\frac{1}{R} \cosh^{-1}(e^{\Gamma X})\right).$$

Figure 4 shows the potential $V(X)$ for the new collective coordinate $X$ when $R_0 = 2$. We can see that the potential has characteristics of Toda potential [14] with an exponential repulsive barrier and a long-range linear attraction. Thus, the $\pi$-$\pi$ kink oscillation can be regarded as the oscillation of a particle with a mass 4 around the equilibrium position $X_0 = \log(\cosh(\Gamma R_0))/\Gamma$ on a Toda-like potential.

Figure 5 shows the frequency of a small amplitude $\pi$-$\pi$ kink oscillation as a function of $R_0$. The open circles denotes the results of numerical simulation for DSGE, while the solid line is obtained from the analytical expression of the frequency for a small oscillation of $X$ around $X_0$, namely, $\omega(R_0) = \sqrt{\frac{d^2 V(X_0)/dX_0^2}{4}}$. We can see that the results of collective coordinate approach are in good agreement with the results of a numerical simulation for DSGE in the region of $R_0 > 1$ where the separation between the $\pi$ kinks is larger than the size of each kink.

4. Quantization of $\pi$-$\pi$ kink oscillation

We are now ready to quantize the $\pi$-$\pi$ kink oscillation. The Schrödinger equation to be solved is

$$\left[-\frac{\hbar^2}{8} \frac{\partial^2}{\partial X^2} + V(X)\right] \Psi_n(X) = E_n \Psi_n(X),$$

where $\hbar = \hbar\omega_1/E_0$ is the normalized Planck constant. The eigenenergy $E_n$ and eigenfunction $\Psi_n$ are numerically calculated using the Numerov method. We adopt the boundary condition, $\Psi_n(0) = 0$, which implies that the eigenfunctions for the relative coordinate between the $\pi$ kinks have odd parity, in other words, the $\pi$ kinks obey Fermi-Dirac statistics. However, there was no noticeable difference between the eigenenergies of odd and even parity states in the parameter region of $R_0 > 1.5$, where our collective coordinate treatment is adequate.

In Fig. 6, we show the eigenfunctions for $R_0$, $\Phi_n(R) = \Psi_n(X(R))\sqrt{\tanh(\Gamma R)}$ for $R_0 = 2$ and $\hbar = 0.3$. We see that most of each wave function is in the region of $R > 1$. This shows the validity of our collective coordinate approach for this parameter range.

In Fig. 7 we show the energy level spacing $\Delta E_{nm} = E_n - E_m$ as a function of $\hbar$ when $R_0 = 2$. The transition frequency between the two lowest energy levels is $\omega_{10} = (\Delta E_{10}/\hbar)\omega_1 \simeq 0.35\omega_1$ for $R_0 = 2$ and $\hbar = 0.3$, and the separation of the two lowest transition frequencies is sufficient for...
to distinguish them since $\omega_{10} - \omega_{21} \simeq 0.051\omega_0$, which is the same as the value of a phase qubit. Thus, the quantum $\pi$-$\pi$ kink oscillation functions as a qubit.

5. Qubit operation

Let us consider the qubit operations required for a quantum computer. We can construct a one-qubit unitary gate with a classical alternating-current (AC) circuit and a two-qubit controlled-NOT gate with a quantum LC (QLC) circuit [15] as shown in Fig. 8. The operation scheme is the same as that for a breather qubit [1, 2] or for two-level atoms with microcavities [16] except for the qubit-circuit (atom-cavity) interaction. The qubit state is also read out by projecting it to the QLC state.

We now estimate the interaction between a bound half-fluxon ($\pi$-$\pi$ kink) pair and an AC (or a QLC) circuit. Consider that the magnetic field $B(x)$ is created at a point $x$ on the junction by the current in the circuit. If we assume that each fluxon is so small that any change in $B(x)$ within the fluxons can be disregarded, the interaction potential for the fluxon pair can be written as [17]

$$H_{\text{int}}(R) = -\int_{-\infty}^{\infty} 2 \frac{B(x) \partial \phi_K}{B_{c1}} \, dx \simeq -4\pi \frac{B(R)}{B_{c1}} \simeq -4\pi \frac{4\pi}{B_{c1}} \left\{ B(R_0) + B'(R_0)(R - R_0) \right\} ,$$

where $B_{c1}$ is the first critical field of the SFS Josephson junction and the prime denotes the differentiation with respect to $x$. Thus, the matrix element of $r = R - R_0$ for the eigenfunctions $\Phi_0$ and $\Phi_1$, i.e., $r_{01} = \int \Phi_0^*(R) r \Phi_1(R) dR$, determines the effect of the circuit on the qubit state in the rotating wave approximation.

Regarding the circuit as a magnetic dipole, we assume $B(x) = -\mu S I (t^2 + x^2)^{-3/2} / 4\pi$, where $S$ and $I$ are the area and current of the circuit, respectively, and $l$ is the normalized
distance between the circuit and the junction. The strength of the interaction between the qubit and the circuit can be estimated from \( \alpha I \) with
\[
\alpha = \frac{3|\langle 0 |S| 0 \rangle| \mu SR_0 E_0}{2B_{c1} \lambda J(2 + R_0^2)^{5/2}}. \tag{15}
\]
For an AC circuit with a current \( I = I_{AC} \cos \omega t \), Rabi oscillation will occur between the ground state and the first excited state with a frequency \( \Omega = \alpha I_{AC}/2 \sim \). In the vicinity of the 0-\( \pi \) crossover, the Josephson critical current would be very small, say two orders of magnitude smaller than that of a usual junction [6, 8], and thus \( \lambda J \sim 50 \mu m, \omega J \sim 100 \text{ GHz} \) and \( \tilde{h} \sim 0.3 \).

When \( I_{AC} = 10^{-4} \text{ A}, S = 10^{-10} \text{ m}^2, l = 0.5, R_0 = 2, \) and \( B_{c1} = \tilde{h}/ed\lambda J \) with \( d = 1 \mu m \), the matrix element \( r_{01} \) is calculated as \(-0.35 \), and thus the Rabi frequency is estimated as \( \Omega \sim 40 \text{ MHz} \).

6. Summary
We have studied the quantum energy levels of the nonlinear internal oscillation of a bound pair of half fluxons in a long superconductor-ferromagnet-superconductor (SFS) Josephson junction, and have proposed a new type of mobile qubit. Our proposed mobile qubit is superior to the previously proposed breather qubit [1, 2] as regards preparing the initial \( |0 \rangle \) state and controlling the moving speed, and it also can solve the unavoidable problem related to the coherent manipulation of the interaction between an arbitrary pair of stationary qubits built into solid-state quantum circuits.

We must estimate the decoherence time if we are to confirm that our proposed qubit can be realized. We think that the decoherence time may be comparable to that of other Josephson qubits, since it would be of the same order as the decoherence time of the vortex qubits estimated in [18]. A detailed estimate will be obtained in the near future.

Acknowledgments
This work was supported in part by a Grants-in-Aid for Scientific Research (17740267 and 18540352) from the Ministry of Education Culture, Sports, Science and Technology of Japan, and a Grant-in-Aid for JSPS Fellows (195836).

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