Which 3-manifold groups are Kähler groups?

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Abstract. The question in the title, first raised by Goldman and Donaldson, was partially answered by Reznikov. We give a complete answer, as follows: if \( G \) can be realized as both the fundamental group of a closed 3-manifold and of a compact Kähler manifold, then \( G \) must be finite—and thus belongs to the well-known list of finite subgroups of \( O(4) \), acting freely on \( S^3 \).

Keywords. Kähler manifold, 3-manifold, fundamental group, cohomology ring, resonance variety, isotropic subspace

1. Introduction

1.1. As is well-known, every finitely presented group \( G \) occurs as the fundamental group of a smooth, compact, connected, orientable 4-dimensional manifold \( M \). As shown by Gompf \([14]\), the manifold \( M \) can be chosen to be symplectic. Requiring a complex structure on \( M \) is no more restrictive, as long as one is willing to go up to complex dimension 3 (see Taubes \([32]\)).

Suppose now \( G \) is the fundamental group of a compact Kähler manifold \( M \). Groups arising this way are called Kähler groups (or, projective groups, if \( M \) is actually a smooth projective variety). The Kähler condition puts strong restrictions on what \( G \) can be. For instance, the first Betti number, \( b_1(G) \), must be even, by classical Hodge theory. Moreover, \( G \) must be 1-formal, by work of Deligne, Griffiths, Morgan, and Sullivan \([9]\). Also, \( G \) cannot split non-trivially as a free product, by a result of Gromov \([17]\). On the other hand, every finite group is a projective group, by a classical result of Serre \([29]\). We refer to \([11]\) for a comprehensive survey of Kähler groups, and to the recent work of Delzant–Gromov \([10]\), Napier–Ramachandran \([25]\), and Delzant \([10]\) for further geometric restrictions imposed by the Kähler condition on a group \( G \).

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Requiring that $M$ be a 3-dimensional compact, connected manifold also puts severe restrictions on $G = \pi_1(M)$. For example, if $G$ is abelian, then $G$ is either $\mathbb{Z}/n\mathbb{Z}$, $\mathbb{Z}$, $\mathbb{Z} \oplus \mathbb{Z}_2$, or $\mathbb{Z}^3$ (see [20]).

1.2. A natural question—raised by Goldman and Donaldson in 1989, and independently by Reznikov in 1993—is then: what are the 3-manifold groups which are Kähler groups?

In [28], Reznikov proved the following result, which Simpson [31] calls “one of the deepest restrictions” on the homotopy types that may occur for Kähler manifolds: Let $M$ be an irreducible, atoroidal 3-manifold, and suppose there is a homomorphism $\rho: \pi_1(M) \to \text{SL}(2, \mathbb{C})$ with Zariski dense image. Then $G = \pi_1(M)$ is not a Kähler group. The same conclusion was reached by Hernández-Lamoneda in [19], under the assumption that $M$ is a geometrizable 3-manifold, with all pieces hyperbolic.

In this note, we answer the above question for all 3-manifold groups, as follows.

Theorem 1.1. Let $G$ be the fundamental group of a compact, connected 3-manifold. If $G$ is a Kähler group, then $G$ is finite.

By the 3-dimensional spherical space-form conjecture, now established by Perelman [26, 27], a closed 3-manifold $M$ has finite fundamental group if and only if it admits a metric of constant positive curvature (for a detailed proof, see Morgan and Tian [24, Corollary 0.2]). Thus, $M = S^3/G$, where $G$ is a finite subgroup of $\text{O}(4)$, acting freely on $S^3$. The list of such finite groups (essentially due to Hopf) is given by Milnor in [23].

1.3. The paper is organized as follows. In §2 we discuss the characteristic and resonance varieties of a group $G$, and two notions of isotropy. In §3 we recall the Isotropic Subspace Theorem of Catanese, and a correspondence due to Beauville. In §4 we use these tools to prove a key result, tying the first resonance variety of a Kähler manifold to the rank of the cup-product map in low degrees. In §5 we investigate the first resonance variety of a closed, oriented 3-manifold; Poincaré duality and properties of Pfaffians yield a very different conclusion in this setting.

All this works quite well, provided the first Betti number of $G$ is positive. To deal with the remaining case, we need two theorems of Reznikov and Fujiwara, relating the Kähler, respectively the 3-manifold condition on a group to Kazhdan’s property $T$; we recall those in §6. Finally, we put everything together in §7 and give a proof of Theorem 1.1.

A natural question arises out of this work: Which 3-manifold groups are quasi-Kähler? (A group $G$ is quasi-Kähler if $G = \pi_1(M \setminus D)$, where $M$ is a compact Kähler manifold and $D$ is a divisor with normal crossings.) We have some partial results in this direction; those results will be presented elsewhere.

2. Cohomology jumping loci and isotropic subspaces

2.1. Let $X$ be a connected CW-complex with finitely many cells in each dimension. Let $G = \pi_1(X)$ be the fundamental group of $X$, and $T = \text{Hom}(G, \mathbb{C}^\ast)$ its character variety.
Every character \( \rho \in \mathbb{T} \) determines a rank 1 local system, \( \mathbb{C}_\rho \), on \( X \). The characteristic varieties of \( X \) are the jumping loci for cohomology with coefficients in such local systems:

\[
V^\rho_d(X) = \{ \rho \in \mathbb{T} \mid \dim H^i(X, \mathbb{C}_\rho) \geq d \}. \tag{1}
\]

The varieties \( V_d(X) = V^1_d(X) \) depend only on \( G = \pi_1(X) \), so we sometimes denote them as \( V_d(G) \).

2.2. Consider now the cohomology algebra \( A = H^*(X, \mathbb{C}) \). Left multiplication by an element \( x \in A^1 \) yields a cochain complex \((A, x): A^0 \xrightarrow{x} A^1 \xrightarrow{x} A^2 \xrightarrow{x} \cdots \) The resonance varieties of \( X \) are the jumping loci for the homology of this complex:

\[
R^i_d(X) = \{ x \in A^1 \mid \dim H^i(A, x) \geq d \}. \tag{2}
\]

The varieties \( R_d(X) = R^1_d(X) \) depend only on \( G = \pi_1(X) \), so we sometimes denote them by \( R_d(G) \). By definition, an element \( x \in A^1 \) belongs to \( R_d(X) \) if and only if there exists a subspace \( W \subset A^1 \) of dimension \( d + 1 \) such that \( x \cup y = 0 \) for all \( y \in W \).

Fix bases \( \{ e_1, \ldots, e_n \} \) for \( A^1 \) and \( \{ f_1, \ldots, f_m \} \) for \( A^2 \). Writing the cup-product as \( e_i \cup e_j = \sum_{k=1}^m \mu_{i,j,k} f_k \), we may define an \( m \times n \) matrix \( \Delta \) of linear forms in variables \( x_1, \ldots, x_n \), with entries

\[
\Delta_{k,j} = \sum_{i=1}^n \mu_{i,j,k} x_i. \tag{3}
\]

It is readily seen that \( R_d(X) = V(E_d(\Delta)) \), where \( E_d \) denotes the ideal of \((n-d) \times (n-d)\) minors. Note also that \( x \cup x = 0 \) for all \( x \in A^1 \) implies \( \Delta \cdot \bar{x} = 0 \), where \( \bar{x} \) is the column vector with entries \( x_1, \ldots, x_n \).

2.3. Foundational results on the structure of the cohomology support loci for local systems on compact Kähler manifolds were obtained by Beauville [2], Green–Lazarsfeld [15], Simpson [30], and Campana [5]: if \( G \) is the fundamental group of such a manifold, then \( V_d(G) \) is a union of (possibly translated) subtori of the algebraic group \( \mathbb{T} \).

In addition, Theorem A from [12] establishes a strong relationship between the characteristic and resonance varieties of a Kähler group \( G \): the tangent cone to \( V_d(G) \) at the identity of \( \mathbb{T} \) equals \( R_d(G) \) for all \( d \geq 1 \).

2.4. A non-zero subspace \( E \subset H^1(X, \mathbb{C}) \) is (totally) isotropic if the restriction of the cup-product map \( \cup_X : H^1(X, \mathbb{C}) \otimes H^1(X, \mathbb{C}) \to H^2(X, \mathbb{C}) \) to \( E \otimes E \) is identically zero. By analogy, we say \( E \) is 1-isotropic if the restriction of \( \cup_X \) to \( E \otimes E \) has 1-dimensional image.

Note that these properties of \( E \) depend only on \( G = \pi_1(X) \). Indeed, let \( h : X \to K(G, 1) \) be a classifying map. Then \( h_* : H_1(X, \mathbb{Z}) \to H_1(G, \mathbb{Z}) \) is an isomorphism, and \( h_* : H_2(X, \mathbb{Z}) \to H_2(G, \mathbb{Z}) \) is an epimorphism. Using Kronecker duality and the functoriality of the cup-product, it is readily seen that \( E \) is a (1-) isotropic subspace of \( H^1(G, \mathbb{C}) \) for \( \cup_G \) if and only if \( h^*(E) \) is a (1-) isotropic subspace of \( H^1(X, \mathbb{C}) \) for \( \cup_X \).
3. The Isotropic Subspace Theorem

By a fibration we mean a surjective morphism \( f : M \to N \) with connected fibers between two compact complex manifolds \( M \) and \( N \). Two fibrations \( f : M \to C \) and \( f' : M \to C' \) over projective curves \( C \) and \( C' \) are said to be equivalent if there is an isomorphism \( \phi : C \to C' \) such that \( f' = \phi \circ f \). We denote by \( \mathcal{E}(M) \) the set of equivalence classes of fibrations \( f : M \to C \), with \( C \) a projective curve of genus \( g \geq 2 \).

Let \( M \) be a compact Kähler manifold. Beauville’s work \([2] \) establishes a bijection between the set \( \mathcal{E}(M) \) and the set of irreducible components of the first characteristic variety \( V_1(M) \) passing through the identity of the algebraic group \( \mathbb{T} = \text{Hom}(\pi_1(M), \mathbb{C}^*) \).

In particular, the set \( \mathcal{E}(M) \) must be finite.

The Isotropic Subspace Theorem, due to Catanese \([6, \text{Theorem 1.10}] \), establishes a relation between the set of equivalence classes of fibrations of a Kähler manifold \( M \) over curves of genus \( g \geq 2 \), and the maximal isotropic subspaces in \( H^1(M, \mathbb{C}) \).

**Theorem 3.1 (Catanese \([6]\)).** Let \( M \) be a compact Kähler manifold. Then, for any maximal isotropic subspace \( E \subset H^1(M, \mathbb{C}) \) of dimension \( g \geq 2 \), there is a fibration \( f : M \to C \) onto a smooth curve of genus \( g \) and a maximal isotropic subspace \( E' \subset H^1(C, \mathbb{C}) \) such that \( E = f^*E' \).

For more information on this correspondence, see \([7]\).

4. The first resonance variety of a Kähler manifold

**Theorem 4.1.** Let \( M \) be a compact Kähler manifold with \( b_1(M) \neq 0 \). If \( R_1(M) = H^1(M, \mathbb{C}) \), then \( H^1(M, \mathbb{C}) \) is 1-isotropic.

**Proof.** By Hodge theory, we must have \( b_1(M) \geq 2 \). The equality \( R_1(M) = H^1(M, \mathbb{C}) \) says that, for any non-zero cohomology class \( x \in H^1(M, \mathbb{C}) \), there is a class \( y \in H^1(M, \mathbb{C}) \setminus \mathbb{C} : x \) such that \( x \cap y = 0 \). Consequently, the vector space spanned by \( x \) and \( y \) is a (2-dimensional) isotropic subspace containing \( x \).

Let \( U_x \) be a maximal isotropic subspace of \( H^1(M, \mathbb{C}) \) containing \( x \); we must then have \( \dim U_x \geq 2 \). Thus, by Theorem 3.1 there is a fibration \( f_x : M \to C_x \) onto a smooth projective curve \( C_x \) of genus \( g_x = \dim U_x \), with \( x \in f_x^*(H^1(C_x, \mathbb{C})) \).

Recall now that the set \( \mathcal{E}(M) \) of equivalence classes of fibrations of \( M \) over curves of genus at least 2 is finite. Thus, we may write the first cohomology group of \( M \) as a finite union of linear subspaces,

\[
H^1(M, \mathbb{C}) = \bigcup_{[f] \in \mathcal{E}(M)} f^*(H^1(C_f, \mathbb{C})),
\]

where \( f = f_x \) for some \( x \in H^1(M, \mathbb{C}) \), and \( C_f := C_x \). This is possible only if there is a fibration \( f_1 : M \to C_1 \) such that \( H^1(M, \mathbb{C}) = f_1^*(H^1(C_1, \mathbb{C})) \).

Since \( f_1 \) is a fibration, the induced morphism \( f_1^* : H^1(C_1, \mathbb{C}) \to H^1(M, \mathbb{C}) \) is injective. The defining property of \( f_1 \) implies that \( f_1^* : H^1(C_1, \mathbb{C}) \to H^1(M, \mathbb{C}) \) is an isomorphism.
On the other hand, the induced morphism $f_1^* : H^2(C_1, \mathbb{C}) \to H^2(M, \mathbb{C})$ is also injective. To prove this claim, first note that any cohomology class in $H^1(M, \mathbb{C})$ is primitive. Using the Hodge–Riemann bilinear relations (see e.g. [11], p. 123), it follows that, for any non-zero $(1, 0)$-class $\alpha \in H^1(M, \mathbb{C})$, the product $\beta = \sqrt{-1} \alpha \cup \overline{\alpha}$ is a non-zero, real, $(1, 1)$-class in $H^2(M, \mathbb{C})$. Since $f_1^* : H^1(C_1, \mathbb{C}) \to H^1(M, \mathbb{C})$ is an isomorphism, there is an element $a \in H^1(C_1, \mathbb{C})$ such that $f_1^*(a) = \alpha$. Hence, $f_1^*(\sqrt{-1} a \wedge \overline{a}) = \beta$, and the claim is proved.

Consider now the commuting diagram

$$
\begin{array}{c}
H^1(M, \mathbb{C}) \wedge H^1(M, \mathbb{C}) \\
\downarrow f_1^* \wedge f_1^* \\
H^1(C_1, \mathbb{C}) \wedge H^1(C_1, \mathbb{C})
\end{array}
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Proof. To prove (1), suppose dim im (\( \cup M \)) = 1. This means there is a hyperplane \( E \subset H := H^1(M, \mathbb{C}) \) such that \( x \cup y \cup z = 0 \) for all \( x, y \in H \) and \( z \in E \). Hence, the skew 3-form \( \mu : \bigwedge^3 H \to \mathbb{C} \) factors through a skew 3-form \( \bar{\mu} : \bigwedge^3 (H/E) \to \mathbb{C} \). But \( \dim H/E = 1 \) forces \( \bar{\mu} = 0 \), and so \( \mu = 0 \), a contradiction.

To prove (2), recall \( R_1(M) = V(E_1(\Delta)) \). Since \( \Delta \) is a skew-symmetric matrix of even size, it follows from Buchsbaum–Eisenbud [4, Corollary 2.6] that \( V(E_1(\Delta)) = V(E_0(\Delta)) \) (see [8, eq. (6.9)]). But \( \Delta \cdot \vec{x} = 0 \) implies \( \det \Delta = 0 \), and so \( V(E_0(\Delta)) = H \).

\( \square \)

Remark 5.2. As noted by S. Papadima, the following holds. Suppose \( M \) is a closed, orientable 3-manifold, with \( b_1(M) \) odd. Then \( R_1(M) \neq H^1(M, \mathbb{C}) \) if and only if \( \mu_M \) is generic, in the sense of [3].

6. Kazhdan’s property \( T \)

The following question is due to J. Carlson and D. Toledo (see J. Kollár [22]): For a Kähler group \( G \), is \( b_2(G) \neq 0 \)? This question was answered in the affirmative by A. Reznikov in [28], under an additional assumption, as follows.

Theorem 6.1 (Reznikov [28]). Let \( G \) be a Kähler group. If \( G \) does not satisfy Kazhdan’s property \( T \), then \( b_2(G) \neq 0 \).

Recall that a discrete group \( G \) satisfies Kazhdan’s property \( T \) (for short, \( G \) is a Kazhdan group) if and only if \( H^1(G, \mathcal{H}) = 0 \) for all orthogonal or unitary representations of \( G \) on a Hilbert space \( \mathcal{H} \) (see de la Harpe and Valette [18, p. 47]). In particular, if \( b_1(G) \neq 0 \), then \( G \) is not Kazhdan. (For a simple proof of Theorem 6.1 in this case, see [21].)

We will also need the following relationship between 3-manifold groups and Kazhdan’s property \( T \), established by K. Fujiwara in [13].

\[ \text{Theorem 6.2 (Fujiwara [13]). Let } G \text{ be the fundamental group of a closed, orientable 3-manifold. If } G \text{ satisfies Kazhdan’s property } T, \text{ then } G \text{ is finite.} \]

In fact, the theorem is valid for any subgroup \( G < \pi_1(M) \), where \( M \) is a compact (not necessarily boundaryless), connected, orientable 3-manifold. Fujiwara further assumes that each piece of the canonical decomposition of \( M \) along embedded spheres, disks and tori admits one of the eight geometric structures in the sense of Thurston, but this is now guaranteed by the work of Perelman [26, 27].

7. Kähler 3-manifold groups

We are now in a position to prove Theorem 1.1 from the introduction.

Let \( G \) be the fundamental group of a compact, connected 3-manifold \( M \). Suppose \( G \) is a Kähler group, and \( G \) is not finite.
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Step 1. A finite-index subgroup of a Kähler group is again a Kähler group (see [11 Example 1.10]). Passing to the orientation double cover of $M$ if necessary, we may as well assume $M$ is orientable.

Step 2. Since $G$ is an infinite, orientable 3-manifold group, $G$ is not Kazhdan, by Fujisawa’s Theorem [6.2]. Since $G$ is Kähler and not Kazhdan, $b_2(G) \neq 0$, by Reznikov’s Theorem [6.1].

Step 3. Since $b_2(M) \geq b_2(G)$, we must also have $b_2(M) \neq 0$. By Poincaré duality, $b_1(M) = b_2(M)$. Hence, $b_1(G) = b_1(M)$ is not zero.

Step 4. Since $G$ is Kähler, $b_1(G)$ must be even. Since $M$ is a closed, orientable 3-manifold with $G = \pi_1(M)$, Proposition [5.1] tells us that $R^1(G) = H^1(G, \mathbb{C})$ and $H^1(G, \mathbb{C})$ is not 1-isotropic. Since, on the other hand, $G$ is Kähler, Theorem [4.1] tells us that $b_1(G) = 0$.

Our assumptions have led us to a contradiction. Thus, the theorem is proved.

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