SECOND-ORDER CONE REPRESENTATION
FOR CONVEX SUBSETS OF THE PLANE

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ABSTRACT. Semidefinite programming (SDP) is the task of optimizing a linear function over the common solution set of finitely many linear matrix inequalities (LMIs). For the running time of SDP solvers, the maximal matrix size of these LMIs is usually more critical than their number. The semidefinite extension degree \( sxdeg(K) \) of a convex set \( K \subseteq \mathbb{R}^n \) is the smallest number \( d \) such that \( K \) is a linear image of a finite intersection \( S_1 \cap \cdots \cap S_N \), where each \( S_i \) is a spectrahedron defined by a linear matrix inequality of size \( \leq d \). Thus \( sxdeg(K) \) can be seen as a measure for the complexity of performing semidefinite programs over the set \( K \). We give several equivalent characterizations of \( sxdeg(K) \), and use them to prove our main result: \( sxdeg(K) \leq 2 \) holds for any closed convex semialgebraic set \( K \subseteq \mathbb{R}^2 \). In other words, such \( K \) can be represented using the second-order cone.

INTRODUCTION

Semidefinite programming (SDP) is the task of optimizing a linear function over the solution set of a linear matrix inequality (LMI)

\[
A_0 + \sum_{i=1}^{n} x_i A_i \succeq 0
\]  

(1)

where \( A_0, \ldots, A_n \) are real symmetric matrices of some size, and \( A \succeq 0 \) means that \( A \) is positive semidefinite. Under mild conditions, semidefinite programs can be solved in polynomial time up to any prescribed accuracy. Thanks to the enormous expressive power of LMIs, semidefinite programming has numerous applications from a wide range of areas. See [1] for background on SDP.

Solution sets \( S \subseteq \mathbb{R}^n \) of LMIs \( \mathbf{1} \) are called spectrahedra. So the feasible sets of SDP are spectrahedra, and more generally linear images of spectrahedra (aka spectrahedral shadows). Generally, the performance of SDP solvers is strongly influenced by the matrix size of the LMI. It is therefore desirable to express a given feasible set by an LMI of smallest possible size. Both upper and lower bounds for the matrix size have been studied in a number of papers. Here we adopt a point of view that was introduced by Averkov [2]. It is motivated by the observation that it is often possible to represent a given convex set \( K \) by the combination of finitely many LMIs of small size \( d \). Practical experience shows that this size \( d \) is far more critical for the running time than the number \( N \) of the LMIs. Following Averkov, we define the semidefinite extension degree \( sxdeg(K) \) of a (convex) set \( K \subseteq \mathbb{R}^n \) as the smallest number \( d \) such that \( K \) is a linear image of a finite intersection \( S_1 \cap \cdots \cap S_N \) of spectrahedra that are all described by LMIs of size \( \leq d \). For example, \( sxdeg(K) \leq 1 \) if and only if \( K \) is a polyhedron, and \( sxdeg(K) \leq 2 \) if and only if \( K \) is second-order cone representable.
Fawzi [6] showed that the $3 \times 3$ psd matrix cone is not second-order cone representable, or in other words, that $\text{sxdeg}(S^3_3) = 3$. Soon after, Averkov found a general condition of combinatorial geometric nature that is an obstruction against $\text{sxdeg}(K) \leq d$, see [2] Main Thm 2.1 and Theorem 2.13 below. His proof generalizes Fawzi’s techniques and uses elaborate combinatorial techniques from Ramsey theory. As a consequence, he was able to prove for a variety of prominent cones (like sums of squares cones, psd matrix cones) that their semidefinite extension degrees are not smaller than indicated by their standard representations. Saunderson [18] generalized Averkov’s obstruction from $S^d d \times \cdots \times S^d d$-lifts of convex sets to $C \times \cdots \times C$-lifts, where $C$ can be an arbitrary cone without long chains of faces.

Our main result is:

**Theorem 0.1.** Any closed convex semialgebraic set $K \subseteq \mathbb{R}^2$ is second-order cone representable, i.e. we have $\text{sxdeg}(K) \leq 2$.

From [19] it is known that every convex semialgebraic subset of $\mathbb{R}^2$ is a spectrahedral shadow. So far, however, no general bounds for the size of representing LMIs are known. To prove the main theorem we first provide an alternative characterization of $\text{sxdeg}(K)$ that uses a different and more algebraic setup. Let $K \subseteq \mathbb{R}^n$ be a convex semialgebraic set, let $R$ be a real closed field that contains the real numbers $\mathbb{R}$, and let $K_R \subseteq R^n$ be the base field extension of $K$ (described by the same finite system of polynomial inequalities as $K$). Given a point $a \in K_R$ and a linear polynomial $f \in R[x_1, \ldots, x_n]$ with $f \geq 0$ on $K_R$, we define the tensor evaluation $f \otimes (a)$ as an element of the ring $R \otimes R \otimes_R R$. We show that $K$ is a spectrahedral shadow if and only if $f \otimes (a)$ is a sum of squares in $R \otimes R$ for any choice of $R, f$ and $a$ (Corollary 3.19). More precisely, $\text{sxdeg}(K) \leq d$ holds if and only if $f \otimes (a)$ can be written as a sum of squares of tensors of rank $\leq d$, for all $R, f$ and $a$ (Theorem 3.10). In this way, the task of proving Theorem 0.1 gets transformed into finding a suitable algebraic decomposition of the tangent to a plane algebraic curve at a general point (Theorem 4.5).

Although this approach appears to be highly abstract, we point out that it is essentially constructive. Given an explicit set $K \subseteq \mathbb{R}^2$ which is closed, convex and semialgebraic, one can in principle construct a second-order representation of $K$ in finitely many steps, see Section 6.

We expect that applications of this method are not confined to convex sets in the plane:

1. Let $K \subseteq \mathbb{R}^n$ be the closed convex hull of an algebraic curve, or more generally, of a one-dimensional semialgebraic set. From [19] it is known that $K$ is a spectrahedral shadow. We conjecture that always $\text{sxdeg}(K) \leq \left\lfloor \frac{n}{2} \right\rfloor + 1$ holds. The bound is reached (for even $n$) by the convex hull of the rational normal curve, see Averkov [2] Corollary 2.3. Note that Theorem 0.1 proves this conjecture for $n = 2$.

2. If $K \subseteq \mathbb{R}^n$ is a compact convex body whose boundary is nonsingular and has strictly positive curvature, then $K$ is known to be a spectrahedral shadow, by results of Helton and Nie [12]. Using the techniques developed in this paper, it can be shown that $\text{sxdeg}(K) = 2$ holds in this case.

**0.2. Notations and conventions.** By $S^d$ we denote the space of symmetric real $d \times d$ matrices, equipped with the standard inner product $(A, B) = \text{tr}(AB)$. We write $A \succeq B$ (resp. $A \succ 0$) to indicate that $A - B$ is positive semidefinite (resp. positive definite). The psd matrix cone is denoted by $S^d_+ = \{ A \in S^d : A \succeq 0 \}$. 
An \( \mathbb{R} \)-algebra is a ring \( A \) together with a specified ring homomorphism \( \mathbb{R} \to A \). If \( U \subseteq A \) is an \( \mathbb{R} \)-linear subspace then \( \Sigma U^2 \subseteq A \) denotes the set of all (finite) sums of squares of elements from \( U \). Moreover \( UU := \text{span}(\Sigma U^2) \) is the \( \mathbb{R} \)-linear subspace of \( A \) spanned by all products \( uu' \) (\( u, u' \in U \)).

Algebraic varieties need neither be irreducible nor reduced. Thus an affine \( \mathbb{R} \)-variety \( X \) is just given by a finitely generated \( \mathbb{R} \)-algebra \( A \). We write \( X = \text{Spec}(A) \) or \( A = \mathbb{R}[X] \), and call \( A = \mathbb{R}[X] \) the affine coordinate ring of \( X \), as usual. Any morphism \( \phi: X \to Y \) of affine \( \mathbb{R} \)-varieties determines the pull-back homomorphism \( \phi^*: \mathbb{R}[Y] \to \mathbb{R}[X] \) between their coordinate rings, and conversely is determined by \( \phi^* \). If \( \mathbb{R} \subseteq E \) is a field extension, the set of \( E \)-rational points of \( X = \text{Spec}(A) \) is \( X(E) = \text{Hom}_E(A, E) \) (set of homomorphisms \( A \to E \) of \( \mathbb{R} \)-algebras).

For a set \( K \) in \( \mathbb{R}^n \), the convex hull of \( K \) is \( \text{conv}(K) \), the conic hull of \( K \) is \( \text{cone}(K) = \{0\} \cup \{ \sum_{i=1}^r a_i x_i : r \geq 1, x_i \in K, a_i \geq 0 \} \). Throughout the paper, \( P_K := \{ f \in \mathbb{R}[x_1, \ldots, x_n] : \mathbb{E} f \geq 0, \mathbb{E} \deg(f) \leq 1 \} \) denotes the convex cone of all affine-linear functions that are non-negative on \( K \).

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1. Semidefinite extension degree: Basic properties

1.1. Let \( n \geq 1 \). For any semialgebraic set \( S \subseteq \mathbb{R}^n \) let \( \text{spdeg}(S) \) be the \textit{spectrahedral degree} of \( S \), defined as follows. If \( S \) is an affine subspace of \( \mathbb{R}^n \) put \( \text{spdeg}(S) = 0 \). Otherwise let \( \text{spdeg}(S) \) be the smallest \( d \geq 1 \) such that there are \( m \geq 1 \) and an affine-linear map \( \varphi: \mathbb{R}^n \to (S^d)^m = S^d \times \cdots \times S^d \) with \( S = \varphi^{-1}(S^d \times \cdots \times S^d) \). If no such \( d \) exists we put \( \text{spdeg}(S) = \infty \).

So \( \text{spdeg}(S) < \infty \) if and only if \( S \) is a spectrahedron, in which case \( \text{spdeg}(S) \) is the smallest \( d \) such that \( S \) is the common solution set of finitely many linear matrix inequalities of size \( d \times d \).

1.2. (See Averkov \cite{Averkov}) For a subset \( K \subseteq \mathbb{R}^n \) we define the \textit{semidefinite extension degree} \( \text{sxdeg}(K) \) as

\[
\text{sxdeg}(K) := \inf_{S, \pi} \text{spdeg}(S),
\]

with the infimum taken over all affine-linear maps \( \pi: \mathbb{R}^s \to \mathbb{R}^n \) (with \( s \geq 1 \)) and all spectrahedra \( S \subseteq \mathbb{R}^s \) with \( K = \pi(S) \).

Remarks 1.3.

1. Let \( K \subseteq \mathbb{R}^n \). By definition, \( \text{sxdeg}(K) \) is the smallest \( d \geq 0 \) for which there is a diagram \( \mathbb{R}^n \leftarrow \mathbb{R}^s \xrightarrow{\varphi} S^d \times \cdots \times S^d \) with affine-linear maps \( \varphi, \pi \), such that \( K = \pi(\varphi^{-1}(S^d \times \cdots \times S^d)) \). This almost agrees with Averkov’s definition \cite{Averkov, Definition 1.1}, except that \cite{Averkov} requires in addition that the map \( \varphi \) is injective. Both definitions agree whenever \( K \) does not contain an affine subspace of positive dimension.

2. If \( K \) is an affine space then \( \text{spdeg}(K) = \text{sxdeg}(K) = 0 \). If \( K \) is a polyhedron (and not an affine space) then \( \text{spdeg}(K) = \text{sxdeg}(K) = 1 \). In all other cases
spdeg(\(K\)) ≥ sxdeg(\(K\)) ≥ 2. By definition, \(K\) is a spectrahedral shadow if and only if sxdeg(\(K\)) < ∞.

3. Let \(K \subseteq \mathbb{R}^n\). By definition, sxdeg(\(K\)) ≤ \(d\) means that \(K\) is a linear image of a spectrahedron that can be described by finitely many LMI s of symmetric \(d \times d\)-matrices. So it means that \(K\) has a representation

\[
K = \left\{ x \in \mathbb{R}^n : \exists y \in \mathbb{R}^m \ \forall \nu = 1, \ldots, r \ A^{(\nu)} + \sum_i x_i B_i^{(\nu)} + \sum_j y_j C_j^{(\nu)} \geq 0 \right\}
\]

with all matrices real symmetric of size (at most) \(d \times d\).

We record some elementary properties of sxdeg(\(K\)).

**Lemma 1.4.** Let \(f : \mathbb{R}^n \to \mathbb{R}^m\) be an affine-linear map, let \(K \subseteq \mathbb{R}^n, K' \subseteq \mathbb{R}^m\) be subsets. Then

(a) sxdeg(\(f(K)\)) ≤ sxdeg(\(K\)),
(b) sxdeg(\(f^{-1}(K')\)) ≤ sxdeg(\(K'\)),
(c) sxdeg(\(K \times K'\)) ≤ \(\max\{sxdeg(\(K\)), sxdeg(\(K'\))\}\),
(d) (if \(m = n\)) sxdeg(\(K \cap K'\)), sxdeg(\(K + K'\)) ≤ \(\max\{sxdeg(\(K\)), sxdeg(\(K'\))\}\).

**Proof.** (a) and (c) are obvious. For (b) let \(\pi : \mathbb{R}^s \to \mathbb{R}^m\) be affine-linear, and let \(S \subseteq \mathbb{R}^s\) a spectrahedron with \(\pi(S) = K\). Let

\[
W := \{ (u, w) \in \mathbb{R}^n \times \mathbb{R}^s : f(u) = \pi(w) \}
\]

(fibre sum, an affine-linear space), and let \(pr_1 : W \to \mathbb{R}^n, pr_2 : W \to \mathbb{R}^s\) be the canonical maps. Then \(S' := pr_2^{-1}(S)\) is a spectrahedron in \(W\) with spdeg(\(S'\)) ≤ spdeg(\(S\)), and \(pr_1(S') = f^{-1}(\pi(S)) = f^{-1}(K)\). (d) follows from (a)–(c). \(\square\)

The second part of (d) is also proved in [2] Lemma 5.5.

**Example 1.5.** (See [6], [2]) The Lorentz cone \(L_n = \{(x, t) \in \mathbb{R}^n \times \mathbb{R} : |x|_2 \leq t\}\) is a spectrahedral cone with spdeg(\(L_n\)) ≤ \(n + 1\). It is easy to see that \(L_n\) is a linear image of a linear section of \((L_2)^n-1 = L_2 \times \cdots \times L_2\) (see e.g. [3]), and therefore sxdeg(\(L_n\)) ≤ 2. A second-order cone program (SOCP) optimizes a linear function over a finite intersection of affine-linear preimages of Lorentz cones. By Lemma [1] any such intersection has sxdeg ≤ 2, and the same is true for linear images of such sets. So it follows that the feasible sets of SOCP are precisely the sets \(K\) with sxdeg(\(K\)) ≤ 2.

**Proposition 1.6.** Let \(K, L \subseteq \mathbb{R}^n\) be convex sets.

(a) sxdeg(\(cone(K)\)) ≤ \(\max\{2, sxdeg(\(K\))\}\).
(b) sxdeg(\(conv(\(K \cup L\))\)) ≤ \(\max\{2, sxdeg(\(K\)), sxdeg(\(L\))\}\).

When \(K\) is an unbounded polyhedron, the cone generated by \(K\) need not be closed. Therefore occurrence of the number 2 on the right hand sides of [1] cannot be avoided.

**Proof.** (a) Assume \(d = sxdeg(\(K\)) < \infty\). Then \(K\) can be written in the form

\[
K = \left\{ x \in \mathbb{R}^n : \exists y \in \mathbb{R}^m \ \forall \nu = 1, \ldots, r \ A^{(\nu)} + \sum_i x_i B_i^{(\nu)} + \sum_j y_j C_j^{(\nu)} \geq 0 \right\}
\]
with symmetric matrices $A^{(\nu)}, B^{(\nu)}_i, C^{(\nu)}_j$ of size $d \times d \ (1 \leq \nu \leq r)$. Then \( \text{cone}(K) \) is the set of \( x \in \mathbb{R}^n \) such that there exist \( (y, s, t) \in \mathbb{R}^m \times \mathbb{R} \times \mathbb{R} \) with

\[
sA^{(\nu)} + \sum_i x_i B^{(\nu)}_i + \sum_j y_j C^{(\nu)}_j \succeq 0
\]

\( (\nu = 1, \ldots, r) \) and

\[
\begin{bmatrix} s & x_i & t \end{bmatrix} \succeq 0, \quad i = 1, \ldots, n.
\]

(This elegant argument is due to Netzer and Sinn, see \cite{15} Proposition 2.1.)

(b) Let \( \bar{K} \) resp. \( \bar{L} \) be the convex hull of \( K \times \{1\} \) resp. \( L \times \{1\} \) in \( \mathbb{R}^n \times \mathbb{R} \). Then \( \text{conv}(K \cup L) = \{x \in \mathbb{R}^n: (x, 1) \in \bar{K} + \bar{L}\} \), so assertion (b) follows from (a) and \ref{14}b).

\[\square\]

**Proposition 1.7.** Let \( C \subseteq \mathbb{R}^n \) be a convex cone, and let \( C^* \) be its dual cone. Then \( \text{sxdeg}(C^*) \leq \text{sxdeg}(C) \).

(See Averkov \cite{2} p 135 for the case where \( C \) is closed and pointed.)

**Proof.** Let \( d = \text{sxdeg}(C) \). We first reduce to the case where the cone \( C \) is spectrahedral. There are a linear map \( f: \mathbb{R}^s \rightarrow \mathbb{R}^n \) and a spectrahedron \( S \subseteq \mathbb{R}^s \) such that \( f(S) = C \) and \( \text{spdeg}(S) = d \). Let \( S^h \subseteq \mathbb{R}^s \times \mathbb{R} \) be the homogenization of \( S \) (\cite{21} 1.13), i.e. \( S^h = \text{conv}(S \times 1) + \text{rc}(S) \times 0 \) where \( \text{rc}(S) \) is the recession cone of \( S \). Then \( S^h \) is a spectrahedral cone with \( \text{spdeg}(S^h) \leq d \), and \( C = g(S^h) \) for the linear map \( g: \mathbb{R}^s \times \mathbb{R} \rightarrow \mathbb{R}^n, g(x, t) = f(x) \). Therefore \( C^* \) is the preimage of the dual cone \( (S^h)^* \) under the dual linear map, and so \( \text{sxdeg}(C^*) \leq \text{sxdeg}((S^h)^*) \) by \ref{14}b).

If we have proved \( \text{sxdeg}((S^h)^*) \leq \text{sxdeg}(S^h) \), we are therefore done.

So let \( C \) be a spectrahedral cone with a representation \( C = \{x \in \mathbb{R}^n: A_j(x) \succeq 0, j = 1, \ldots, m\} \) where \( A_j(x) = \sum_{k=1}^n x_k A_{jk} \) are linear matrix pencils in \( S^d \). By a standard argument we can assume that the LMIs \( A_j(x) \succeq 0 \) are strictly feasible. Then if \( \phi: (S^d)^m \rightarrow \mathbb{R}^n \) is the linear map

\[
\phi(B_1, \ldots, B_m) = \left( \sum_{j=1}^m \langle B_j, A_{j1} \rangle, \ldots, \sum_{j=1}^m \langle B_j, A_{jn} \rangle \right),
\]

we have \( C^* = \phi(S^d_1 \times \cdots \times S^d_m) \). \( \square \)

**Corollary 1.8.** If \( C \subseteq \mathbb{R}^n \) is a closed convex cone then \( \text{sxdeg}(C^*) = \text{sxdeg}(C) \).

**Corollary 1.9.** Let \( K \subseteq \mathbb{R}^n \) be a convex set, let \( P_K \subseteq \mathbb{R}[x_1, \ldots, x_n] \) be the cone of all polynomials \( f \) with \( \deg(f) \leq 1 \) and \( f|_K \geq 0 \). Then

\[
\text{sxdeg}(K) \leq \text{sxdeg}(P_K) \leq \max\{1, \text{sxdeg}(K)\}.
\]

Similarly \( \text{sxdeg}(K^\circ) \leq \max\{1, \text{sxdeg}(K)\} \) where \( K^\circ \) is the polar of \( K \).

**Proof.** The assertion is true when \( K \) is a polyhedron, so we may assume \( \text{sxdeg}(K) \geq 2 \). Since \( P_K \) is identified with the dual of the cone \( \bar{K} = \text{cone}(K \times 1) = \{(tx: t \geq 0, x \in K) \} \) in \( \mathbb{R}^n \times \mathbb{R} \), the second inequality follows from \ref{16}(a). The first follows from \ref{16}(b) since \( K \) is an affine-linear section of the dual cone \( (P_K)^* \).

Similarly, \( K^\circ \) is an affine-linear section of the cone \( P_K \). \( \square \)
2. Equivalent characterizations of sxdeg

Let \( n \in \mathbb{N} \), write \( x = (x_1, \ldots, x_n) \) and \( L = \text{span}(1, x_1, \ldots, x_n) \subseteq \mathbb{R}[x] \) for the space of affine-linear polynomials.

2.1. Let \( K \subseteq \mathbb{R}^n \) be a convex set. By definition of sxdeg, \( K \) is a spectrahedral shadow if and only if \( \text{sosdeg}(K) < \infty \). In this section we relate the precise value of sxdeg(\( K \)) to the characterization of spectrahedral shadows that was given in [20]: if \( K \) is closed then (20 Theorem 3.4) \( K \) is a spectrahedral shadow if and only if there exists a morphism \( \phi: X \to \mathbb{A}^n \) of affine \( \mathbb{R} \)-varieties with \( \phi(X(\mathbb{R})) = K \) such that \( \phi^*(P_K) \subseteq U^2 \) holds for some finite-dimensional linear subspace \( U \subseteq \mathbb{R}[X] \).

2.2. Since it was somewhat hidden in [20], let us recall how such \( \phi \) and \( U \) can be found explicitly from a lifted LMI representation of \( K \). Let \( K \subseteq \mathbb{R}^n \) be a spectrahedral shadow, not necessarily closed. Replacing \( \mathbb{R}^n \) by the affine hull of \( K \) we assume that \( K \) has nonempty interior. Then \( K = \pi(\varphi^{-1}(S^d)) \), where \( \pi: \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n \), \( \pi(x, y) = x \) and \( \varphi: \mathbb{R}^n \times \mathbb{R}^m \to S^d \), \( \varphi(x, y) = M_0 + \sum_{i=1}^n x_i M_i + \sum_{j=1}^m y_j N_j \) for suitable matrices \( M_i, N_j \in S^d \). The LMI in this representation can be chosen to be strictly feasible, i.e. we can assume that \( \varphi(u, v) > 0 \) for some pair \((u, v) \in \mathbb{R}^n \times \mathbb{R}^m \).

Let \( X = \{(x, y, Z) \in \mathbb{A}^n \times \mathbb{A}^m \times \text{Sym}_d: Z^2 = \varphi(x, y)\} \), a closed subvariety of \( \mathbb{A}^n \times \mathbb{A}^m \times \text{Sym}_d \), and let \( \phi: X \to \mathbb{A}^n \) be defined by \( \phi(x, y, Z) = x \). Clearly \( \phi(X(\mathbb{R})) = K \). Given \( f \in P_K \), there are (by semidefinite duality [16]) a symmetric matrix \( B \geq 0 \) and a real number \( c \geq 0 \) with \( f(x) = c + \langle B, M_0 \rangle + \sum_i \langle B, M_i \rangle x_i \) and with \( \langle B, N_j \rangle = 0 \) for \( j = 1, \ldots, m \). Let \( V \) be a symmetric matrix with \( V^2 = B \), let \((x, y, Z) \in X(\mathbb{R}) \). Then

\[
f(x) = c + \langle B, \varphi(x, y) \rangle = c + \langle V^2, Z^2 \rangle = c + \langle ZV, ZV \rangle
\]

as elements of \( \mathbb{R}[X] \). (Here we write \( \langle M, M' \rangle = \text{tr}(M'M') \) for arbitrary \( d \times d \) matrices \( M, M' \).) Hence \( \phi^*(f) \) is a sum of squares of elements from the subspace \( U := \mathbb{R}1 + \text{span}(z_{ij}: 1 \leq i \leq j \leq d) \) of \( \mathbb{R}[X] \), where \( Z = (z_{ij}) \).

We are going to characterize \( \text{sosdeg}(K) \) in terms of the possible spaces \( U \) in 2.1. To this end we define:

**Definition 2.3.** For \( K \subseteq \mathbb{R}^n \) a convex semialgebraic set, let \( \text{sosdeg}(K) \) denote the smallest integer \( d \geq 0 \) such that there is a morphism \( \phi: X \to \mathbb{A}^n \) of affine \( \mathbb{R} \)-varieties, together with finitely many \( \mathbb{R} \)-linear subspaces \( U_1, \ldots, U_r \subseteq \mathbb{R}[X] \), satisfying:

1. \( K \) is contained in the convex hull of \( \phi(X(\mathbb{R})) \),
2. \( \dim(U_i) \leq d \) (\( i = 1, \ldots, r \)),
3. \( \phi^*(P_K) \subseteq \mathbb{R}1 + \langle \Sigma U_1^2 \rangle + \cdots + \langle \Sigma U_r^2 \rangle \) (in \( \mathbb{R}[X] \)).

If there is no such \( d \) we write \( \text{sosdeg}(K) = \infty \).

The goal of this section is to prove \( \text{sosdeg}(K) = \text{sosdeg}(K) \) whenever \( K \) is closed and convex (Theorem 2.4.10 below).

**Proposition 2.4.** Let \( K \subseteq \mathbb{R}^n \) be convex with \( \text{sosdeg}(K) = d < \infty \). Then there are \( \phi: X \to \mathbb{A}^n \) and subspaces \( U_1, \ldots, U_r \subseteq \mathbb{R}[X] \) as in 2.3 such that the stronger condition

1. \( K \subseteq \phi(X(\mathbb{R})) \)
Lemma 2.5, and $K$ convex cone $K$ contradicting.

Proof. Let $\phi$ and the $U_i$ be as in 2.4. Then $S := \phi(X(\mathbb{R}))$ is a semialgebraic set with $K \subseteq \text{conv}(S)$. Construct a morphism $\psi: Y \to A^n$ as follows. Let $Z \subseteq A^{n+1}$ be the hypersurface $z_0^2 + \cdots + z_n^2 = 1$, so $Z(\mathbb{R})$ is the unit sphere in $\mathbb{R}^{n+1}$. Let $Y := X^{n+1} \times Z = X \times \cdots \times X \times Z$, and let $\psi: Y \to A^n$ be defined by

$$\psi(x_0, \ldots, x_n; z_0, \ldots, z_n) = \sum_{i=0}^n z_i^2 \phi(x_i).$$

Then $K \subseteq \text{conv}(S) = \psi(Y(\mathbb{R}))$ by Carathéodory’s theorem. The coordinate ring of $Y$ is $\mathbb{R}[Y] = \mathbb{R}[X] \otimes \cdots \otimes \mathbb{R}[X] \otimes \mathbb{R}[Z]$ ($n+1$ tensor factors $\mathbb{R}[X]$). For $0 \leq i \leq n$ and $1 \leq j \leq r$ define the subspace $V_{ij}$ of $\mathbb{R}[Y]$ by

$$V_{ij} := \mathbb{R}1 \otimes \cdots \otimes U_j \otimes \cdots \otimes \mathbb{R}1 \otimes \mathbb{R}z_i$$

with $U_j$ at position $i$. Then $\dim(V_{ij}) = \dim(U_j) \leq d$ for all $i, j$, and $\psi^*(P_K) \subseteq \mathbb{R}, 1 + \sum_{i=1}^r (\sum_{j=1}^m V_{ij}^2)$. Indeed, if $f \in P_K$ then for $j = 1, \ldots, r$ there are elements $g_{jk} \in U_j$ with $\phi^*(f) = c + \sum_{j=1}^r \sum_k g_{jk}^2$ for some $0 \leq c \in \mathbb{R}$, by (3). Therefore, if we evaluate the pullback $\psi^*(f) \in \mathbb{R}[Y]$ at a tuple $(\xi; \zeta) = (\xi_0, \ldots, \xi_n; \zeta_0, \ldots, \zeta_n) \in X^{n+1} \times Z$ (of geometric points), we get

$$\psi^*(f)(\xi; \zeta) = \sum_{i=0}^n f(\phi(\xi_i)) \cdot \zeta_i^2 = c + \sum_{i=0}^n \sum_{j=1}^m \sum_k g_{jk}(\xi_i)^2 \cdot \zeta_i^2$$

So, as an element of $\mathbb{R}[Y]$, we have

$$\psi^*(f) = c + \sum_{i=0}^n \sum_{j=1}^m \sum_k (1 \otimes \cdots \otimes g_{jk} \otimes \cdots \otimes 1 \otimes z_i)^2$$

and the tensor that gets squared in the $(i, j, k)$-summand lies in $V_{ij}$, for each triple $(i, j, k)$. Hence $\psi$ and the $V_{ij}$ satisfy Definition 2.3 with $(1')$ instead of (1). \qed

Lemma 2.5. Let $A$ be an $\mathbb{R}$-algebra, let $U_1, \ldots, U_r \subseteq A$ be linear subspaces with $\dim(U_i) \leq d$ ($i = 1, \ldots, r$). Then $C := \Sigma U_1^2 + \cdots + \Sigma U_r^2$ is a convex cone with $\text{sxsd}(C) \leq d$.

Proof. $C$ is a cone in the finite-dimensional subspace $\sum_{i=1}^r U_iU_i$ of $A$. By Lemma 1.4(d) it suffices to prove the claim for $r = 1$, i.e. for $C = \Sigma U_1^2$ where $\dim(U) \leq d$. If $u_1, \ldots, u_d$ is a system of linear generators of $U$ then the linear map

$$\pi: S^d \to UU, \quad (a_{ij}) \mapsto \sum_{i,j} a_{ij}u_iu_j$$

satisfies $\pi(S^d_+) = \Sigma U^2$. \qed

Lemma 2.6. Let $K \subseteq \mathbb{R}^n$ be convex and semialgebraic. Then $\text{sxsd}(K) \leq \text{sosdeg}(K)$.

Proof. Let $d = \text{sosdeg}(K) < \infty$. We can assume to have $\phi: X \to A^n$ and $U_i \subseteq \mathbb{R}[X]$ as in 2.4. If $d = 0$ then $\phi^*(P_K) \subseteq \mathbb{R}, 1$. This implies that $K$ is an affine subspace (and so $\text{sxsd}(K) = 0$). Indeed, otherwise there would exist $f \in P_K$ such that $f$ is not constant on $K$. But $\phi^*(f) = c$ is a constant, so $f \equiv c$ on the image of $\phi$, contradicting $K \subseteq \phi(X(\mathbb{R}))$.

Let now $d \geq 1$. By Proposition 1.9 it suffices to show $\text{sxsd}(P_K) \leq d$. The convex cone $C := \mathbb{R}_+ + \sum_{i=1}^r \Sigma U_i^2$ in $\sum_i U_iU_i \subseteq \mathbb{R}[X]$ satisfies $\text{sxsd}(C) \leq d$ by Lemma 2.5 and $\phi^*(P_K) \subseteq C$ holds by assumption. On the other hand, elements
of $C$ are nonnegative on $X(\mathbb{R})$. Therefore every linear $f \in \mathbb{R}[x]$ with $\phi^*(f) \in C$ is nonnegative on $K$. This shows $P_K = (\phi^*)^{-1}(C)$, so the proof is completed by Lemma 1.4(b). \hfill \Box

**Remark 2.7.** In Lemma 2.6 the inequality $sxdeg(K) \leq sosdeg(K)$ need not hold. For example $sosdeg(K) = 1$ but $sxdeg(K) \geq 2$ if $K$ is a dense but not closed convex subset of a polyhedron.

The next lemma is the analogue of Lemma 1.4 for the invariant sosdeg:

**Lemma 2.8.** Let $f : \mathbb{R}^m \to \mathbb{R}^n$ be an affine-linear map, let $K \subseteq \mathbb{R}^n$ and $L \subseteq \mathbb{R}^m$ be convex sets. Then

(a) $sosdeg f(L) \leq sosdeg(L)$,

(b) $sosdeg f^{-1}(K) \leq sosdeg(K)$,

(c) $sosdeg(K \times L) \leq \max\{sosdeg(K), sosdeg(L)\}$.

**Proof.** (a) and (c) are clear. For (b) let $\phi : X \to \mathbb{A}^n$ be a morphism of affine varieties with $K \subseteq \phi(X(\mathbb{R}))$ and $\phi^*(P_K) \subseteq \sum_{i=1}^m (\Sigma U_i^2)$ with subspaces $U_i \subseteq \mathbb{R}[X]$ of dimension $\leq d$ ($i = 1, \ldots, m$). In the cartesian square (fibre product)

$$
\begin{array}{ccc}
Y & \xrightarrow{g} & X \\
\downarrow \phi & & \downarrow \phi \\
\mathbb{A}^m & \xrightarrow{f} & \mathbb{A}^n \\
\end{array}
$$

we have $f^{-1}(K) \subseteq \psi(Y(\mathbb{R}))$. We can assume $f^{-1}(K) \neq \emptyset$. Then $P_{f^{-1}(K)} = f^*(P_K)$ holds. The subspaces $V_i := g^*(U_i)$ of $\mathbb{R}[Y]$ satisfy $\dim(V_i) \leq d$ ($i = 1, \ldots, m$), and

$$
\psi^*(P_{f^{-1}(K)}) = \psi^* f^*(P_K) = g^* \phi^*(P_K) \subseteq g^* \left( \sum_{i} \Sigma U_i^2 \right) \subseteq \sum_{i} \Sigma V_i^2,
$$

whence $sosdeg(f^{-1}(K)) \leq d$. \hfill \Box

**Lemma 2.9.** If $K \subseteq \mathbb{R}^n$ is convex then $sosdeg(K) \leq sxdeg(K)$.

**Proof.** Let $sxdeg(K) = d < \infty$, so there are affine-linear maps $\mathbb{R}^n \xleftarrow{\pi} \mathbb{R}^s \xrightarrow{\rho} (\mathbb{S}^d)^m$ such that $K = \pi(\phi^{-1}(C))$ for $C = (\mathbb{S}_+^d)^m$. By Lemma 2.8 it suffices to show $sosdeg(\mathbb{S}_+^d) \leq d$.

To this end consider the morphism $\phi : M_d \to \text{Sym}_d$ given by $\phi(A) = AA^t$. Let $x_{ij} = x_{ji}$ be the coordinates on $\text{Sym}_d$ and $y_{ij}$ those on $M_d$ ($1 \leq i, j \leq d$). The ring homomorphism $\phi^* : \mathbb{R}[\text{Sym}_d] \to \mathbb{R}[M_d]$ is given by $\phi^*(x_{ij}) = \sum_k y_{ik}y_{jk}$. For $k = 1, \ldots, d$ let

$$
U_k := \text{span}(y_{1k}, \ldots, y_{dk}) \subseteq \mathbb{R}[M_d].
$$

Since the cone $\mathbb{S}_+^d$ is self-dual, the linear forms on $\mathbb{S}_+^d$ that are nonnegative on $\mathbb{S}_+^d$ are precisely the linear forms $f_B = \sum_{i,j} b_{ij}x_{ij}$, where $B = (b_{ij}) \in \mathbb{S}_+^d$ is an arbitrary psd matrix. We claim that $\phi^*(f_B) \in (\Sigma U_1^2) + \cdots + (\Sigma U_d^2)$ for every $B \in \mathbb{S}_+^d$. To show this it suffices to consider $B \preceq 0$ with $\text{rk}(B) = 1$, so let $B = bb^t$ with $b \in \mathbb{R}^n$. Then

$$
\phi^*(f_B) = \sum_{i,j} b_i b_j \phi^*(x_{ij}) = \sum_{i,j,k} b_i b_j y_{ik}y_{jk} = \sum_k \left( \sum_i b_i y_{ik} \right)^2
$$

which shows the claim. \hfill \Box

Combining Lemmas 2.6 and 2.9 we have proved:
Remark 2.12. For every convex set $K \subseteq \mathbb{R}^n$ one has
\[ \text{sxdeg}(K) \leq \text{sosdeg}(K) \leq \text{sxdeg}(K). \]
In particular, $\text{sxdeg}(K) = \text{sosdeg}(K)$ if $K$ is closed. \hfill \Box

Remark 2.11. We used uniform sum of squares decompositions of elements $f \in P_K$ in algebraic varieties $X$ over $\mathbb{A}^n$, to characterize $\text{sxdeg}(K)$. Alternatively, one can phrase the above results, and in particular Theorem 2.10, in terms of uniform decompositions into sums of squares of semialgebraic (not necessarily continuous) functions, as was suggested by Fawzi [7]. Both setups are directly equivalent, since every surjective semialgebraic map between semialgebraic sets has a semialgebraic section.

Remark 2.12. Let $K \subseteq \mathbb{R}^n$ be a closed convex set with $\text{sxdeg}(K) \leq d < \infty$. By the preceding remark, there exist linear spaces $U_1, \ldots, U_m$ of semialgebraic functions on $\mathbb{R}^n$ with $\dim(U_i) = d$ for all $i$, such that every $f \in P_K$ lies in $(\Sigma U_1^2) + \cdots + (\Sigma U_m^2)$. Let $p_{i1}, \ldots, p_{id}$ be a basis of $U_i$, for $1 \leq i \leq m$. For $x \in K$ and $1 \leq i \leq m$ let $A_i(x) = (p_{ij}(x)p_{ik}(x))_{j,k}$, a psd symmetric matrix of rank $\leq 1$ and size $d \times d$. For $f \in P_K$, since $f \in \sum_{i=1}^m (\Sigma U_i^2)$, there are symmetric matrices $B_i(f), \ldots, B_m(f) \succeq 0$ of size $d \times d$ such that
\[ f = \sum_{i=1}^m \sum_{j,k=1}^d b_{ijk}p_{ij}p_{ik} \]
where $B_i(f) = (b_{ijk})_{j,k}$. These matrices constitute a $(\mathbb{S}^d)^m$-factorization of $K$ in the sense of Gouveia, Parrilo and Thomas [9]: One has
\[ f(x) = \sum_{i=1}^m \langle A_i(x), B_i(f) \rangle \]
for every $x \in K$ and every $f \in P_K$. Note that the existence of such a $(\mathbb{S}^d)^m$-factorization, for some $m$, is essentially equivalent to $\text{sxdeg}(K) \leq d$, by a particular case of the main result of [9].

We use our setup to re-prove Averkov’s main theorem (2 Theorem 2.1) in a somewhat more general setting. Given a set $S$ and an integer $k \geq 1$, let $\binom{S}{k}$ denote the set of all $k$-element subsets of $S$.

Theorem 2.13. (Averkov) Let $K \subseteq \mathbb{R}^n$ be a closed convex semialgebraic set, let $d \in \mathbb{N}$. Suppose that there exist subsets $S \subseteq K$ of arbitrarily large finite cardinality that have the following property:
\[ (*) \text{ For every } T \in \binom{S}{d} \text{ there exists } f \in P_K \text{ with } f = 0 \text{ on } T \text{ and } f > 0 \text{ on } S \setminus T. \]
Then $\text{sxdeg}(K) \geq d + 1$.

Proof. We copy Averkov’s elegant proof [2] and transfer it from the context of slack matrices to our setup. By way of contradiction, assume $\text{sxdeg}(K) \leq d$. By Theorem 2.10 (and Remark 2.11), there are linear spaces $U_1, \ldots, U_m$ of semialgebraic functions on $K$ with $\dim(U_i) = d$ $(i = 1, \ldots, m)$, such that every $f \in P_K$ can be written $f = \sum_{i=1}^m g_i$ with $g_i \in \Sigma U_i^2$ for $i = 1, \ldots, m$. For every $x \in K$ and $i = 1, \ldots, m$ let $\lambda_{x,i} \in U_i^\ast$ (dual space of $U_i$) be defined by $\lambda_{x,i}(g) := g(x)$ $(g \in U_i)$, and for every subset $T \subseteq K$ write $L_i(T) := \text{span}(\lambda_{x,i} : x \in T) \subseteq U_i^\ast$. From property $(*)$ we infer:
Lemma 3.4. The following properties of sosx are obvious:

Indeed, let $f \in P_K$ as in (*), and write $f = \sum_{i=1}^{m} g_i$ with $g_i \in \Sigma U_i$. Since $f(y) \neq 0$ there is $1 \leq i \leq m$ with $g_i(y) \neq 0$. On the other hand, $g_i(x) = 0$ for every $x \in T$, and so $\lambda_{y,i}$ is not a linear combination of the $\lambda_{x,i}$ ($x \in T$).

Let $F: \binom{S}{d} \rightarrow \{0, \ldots, d\}^m$ be the map defined by

$$F(T) := (\dim L_1(T), \ldots, \dim L_m(T)).$$

If $|S|$ is sufficiently large then, by Ramsey’s theorem for hypergraphs, there is a set $W \in \binom{S}{d+1}$ such that $F$ is constant on $\binom{W}{d}$, see [2] Theorem 3.4 and [11]. As in [2] (claim on p 142), one shows for any $T, T' \in \binom{W}{d}$ and $1 \leq i \leq m$, that the subspaces $L_i(T)$ and $L_i(T')$ of $U_i$ have not only the same dimension, but that they do in fact coincide. This implies $L_i(T) = L_i(W)$ for every $T \in \binom{W}{d}$. But this contradicts (**), as we see by taking $T \in \binom{W}{d}$ and $y \in W \setminus T$.

3. Local characterization of sxdeg

In this section we use Theorem 2.10 to prove another characterization of sxdeg($K$) which is of local nature (Theorem 3.10). Even though it appears to be very “abstract”, it will be essential for the proof of our main result, see Sections 4 and 5.

3.1. Let $R$ be a real closed field that contains the field $\mathbb{R}$ of real numbers. If $\phi: X \rightarrow Y$ is a morphism of affine $\mathbb{R}$-varieties then $\phi_R: X_R \rightarrow Y_R$ denotes the base extension of $\phi$ by $R \rightarrow R$. Given a semialgebraic set $M \subseteq \mathbb{R}^n$, let $M_R$ denote the base field extension of $M$ to $R$ (see [1] Sect. 5.1). This is the subset of $R^n$ that is defined by the same finite boolean combination of polynomial inequalities as $M$.

3.2. By $B \subseteq R$ we denote the canonical valuation ring of $R$, which is the convex hull of $\mathbb{R}$ in $R$, i.e. $B = \{b \in R : \exists a \in \mathbb{R} - a < b < a\}$. The maximal ideal of $B$ is $\mathfrak{m}_B = \{b \in R : -\frac{1}{n} < b < \frac{1}{n} \text{ for every } n \in \mathbb{N}\}$. The residue field of $B$ is $\mathbb{R}$, and the residue map $B \rightarrow \mathbb{R}$ will be written $b \mapsto \overline{b}$.

We work in the $\mathbb{R}$-algebra $R \otimes R := R \otimes_{\mathbb{R}} R$ and its subring $B \otimes B = B \otimes_{\mathbb{R}} B$. The composite ring homomorphism $B \otimes B \rightarrow B \rightarrow \mathbb{R}$, $b_1 \otimes b_2 \mapsto \overline{b_1 b_2}$ will be denoted by $\theta \mapsto \overline{\theta}$.

Given $\theta \in R \otimes R$, let $\text{rk}(\theta)$ denote the tensor rank of $\theta$, i.e. the minimal number $r \geq 0$ such that $\theta$ can be written as a sum of $r$ elementary tensors $a_i \otimes b_i$ (with $a_i, b_i \in R$). Clearly we have $\text{rk}(\theta_1 + \theta_2) \leq \text{rk}(\theta_1) + \text{rk}(\theta_2)$ and $\text{rk}(\theta_1 \theta_2) \leq \text{rk}(\theta_1) \cdot \text{rk}(\theta_2)$. We sometimes refer to tensors of rank 1, 2, . . . as monomial, binomial etc. tensors.

Definition 3.3. Given a tensor $\theta \in R \otimes R$ which is a sum of squares in $R \otimes R$, we define sosx($\theta$) to be the smallest $d \geq 0$ such that $\theta$ has a representation $\theta = 1 \otimes c + \sum_{i=1}^{N} \theta_i^2$ with $0 \leq c \in R$ and $\theta_i \in R \otimes R$ such that $\text{rk}(\theta_i) \leq d$ for $i = 1, \ldots, N$. If $\theta$ is not a sum of squares in $R \otimes R$ we put sosx($\theta$) = $\infty$.

In particular, sosx($\theta$) = 0 if and only if $\theta = 1 \otimes c$ with $0 \leq c \in R$. We introduced this extra case only to make Theorem 3.10 below work in the $d = 0$ case as well.

The following properties of sosx are obvious:

Lemma 3.4. Let $\theta, \theta_1, \theta_2 \in R \otimes R$.

(a) sosx($\theta$) $\leq 1$ iff there are $a_i, b_i \geq 0$ in $R$ with $\theta = \sum_i a_i \otimes b_i$. 
(b) \( \text{sos}(\theta_1 + \theta_2) \leq \max\{\text{sos}(\theta_1), \text{sos}(\theta_2)\} \).

(c) If \( \text{sos}(\theta_1), \text{sos}(\theta_2) \geq 1 \) then \( \text{sos}(\theta_1 \theta_2) \leq \text{sos}(\theta_1) \cdot \text{sos}(\theta_2) \).

The following simple observation is important:

**Proposition 3.5.** Let \( \theta \in B \otimes B \). If \( \theta \in \mathbb{R} \) is strictly positive then \( \theta \) can be written in the form

\[
\theta = \sum_{i=1}^{m} u_i \otimes v_i
\]

with \( u_i, v_i \in B \) and \( \pi_i, \nu_i > 0 \) for every \( i \). In particular, \( \text{sos}(\theta) \leq 1 \).

**Proof.** Let \( \theta = \sum_{i=1}^{n} a_i \otimes b_i \) with \( a_i, b_i \in B \). Write (uniquely) \( a_i = c_i + \alpha_i, \) \( b_i = d_i + \beta_i \) with \( c_i, d_i \in \mathbb{R} \) and \( \alpha_i, \beta_i \in \mathbb{m}_B \) (\( i = 1, \ldots, n \)). Choose strictly positive real numbers \( r, s \) and \( r_1, s_i (i = 1, \ldots, n) \) with \( r + s + \sum_{i=1}^{n} r_i s_i = \theta = \sum_{i=1}^{n} c_i d_i \), which is possible since \( \theta = \sum_{i=1}^{n} a_i b_i > 0 \). Then \( \theta \) is equal to

\[
\sum_{i=1}^{n} (r_i + \alpha_i) \otimes (s_i + \beta_i) + \left( r + \sum_{i=1}^{n} (d_i - s_i) \alpha_i \right) \otimes 1 + 1 \otimes \left( s + \sum_{i=1}^{n} (c_i - r_i) \beta_i \right)
\]

and this decomposition has the desired form. \( \square \)

**Remark 3.6.** The subset \( T := \{ \sum_{i=1}^{r} a_i \otimes b_i : r \geq 0, a_i, b_i \in B, \pi_i, \vartheta_i > 0 \} \) of \( B \otimes B \) is a subsemiring of \( B \otimes B \). It is easy to see that \( T \) is archimedean, i.e. \( \mathbb{Z} \cap T = B \otimes B \). Indeed, if \( a, b \in B \), choose \( m, n \in \mathbb{N} \) with \( m \pi < m \) and \( m \vartheta < n \), then

\[
3mn + a \otimes b = (m - a) \otimes (n - b) + (m + a) \otimes n + m \otimes (n + b),
\]

and the right hand side lies in \( T \). This gives an alternative (but less explicit) proof of Proposition 3.5. Every \( \theta \in B \otimes B \) with \( \theta > 0 \) is strictly positive on the entire real spectrum of \( B \otimes B \), therefore \( \theta \in T \) by the archimedean Positivstellensatz (e.g. [[34] Theorem 5.4.4]).

3.7. Let \( V \) be an affine \( \mathbb{R} \)-variety, and let \( R \supseteq \mathbb{R} \) be a real closed field. We write \( R[V] := \mathbb{R}[V] \otimes R = \mathbb{R}[V] \otimes \mathbb{R} \) for the extension of the coordinate ring of \( V \) from \( R \) to \( R \). Recall that \( V(\mathbb{R}) \), the set of \( R \)-points of \( V \), is identified with the set of \( \mathbb{R} \)-homomorphisms \( \mathbb{R}[V] \to R \), by associating with an \( R \)-point the evaluation homomorphism at this point. Given \( f \in R[V] \) and \( a \in V(\mathbb{R}) \) we define \( f^\otimes(a) \), the “outer” or “tensor evaluation” of \( f \) at \( a \), to be the image of \( f \) under the ring homomorphism

\[
R[V] = \mathbb{R}[V] \otimes R \xrightarrow{\alpha \otimes 1} R \otimes R.
\]

For example, for affine \( n \)-space \( V = \mathbb{A}^n \), for \( a \in \mathbb{R}^n \) and any \( R \)-polynomial \( f = \sum_{\alpha} c_\alpha x^\alpha \in R[x] \) (with \( x = (x_1, \ldots, x_n) \) and \( c_\alpha \in R \)) we get

\[
f^\otimes(a) = \sum_{\alpha} a^\alpha \otimes c_\alpha \in R \otimes R.
\]

From the definition it is clear that \( (f + g)^\otimes(a) = f^\otimes(a) + g^\otimes(a) \) and \( (fg)^\otimes(a) = f^\otimes(a) \cdot g^\otimes(a) \) hold. If \( V = \mathbb{A}^n \) and \( f = c_0 + \sum_i c_i x_i \) is a linear polynomial (with \( c_i \in \mathbb{R} \)) then

\[
f^\otimes(sa + tb) = (s \otimes 1) \cdot f^\otimes(a) + (t \otimes 1) \cdot f^\otimes(b)
\]
Proof. Let \( R \supseteq \mathbb{R} \) be real closed, let \( S \subseteq V(\mathbb{R}) \) be a semialgebraic set, let \( f \in R[\mathbb{R}] \) with \( f \geq 0 \) on \( S_R \), and let \( a \in S_R \). Then \( f^\circ(a) \) is a positive semidefinite (psd) element in \( R \otimes R \), i.e. for any two homomorphisms \( \varphi_1, \varphi_2 : R \to E \) into a real closed field \( E \), the image of \( f^\circ(a) \) under \( \phi : R \otimes R \to E, a_1 \otimes a_2 \mapsto \varphi_1(a_1)\varphi_2(a_2) \) is nonnegative.

**Proof.** \( \phi(f^\circ(a)) = g^\circ(b) \), where \( g = \varphi_2(f) \in E[V] \) satisfies \( g \geq 0 \) on \( S_E \), and \( b = \varphi_1(a) \in S_E \). So \( \phi(f^\circ(a)) \geq 0 \). \( \square \)

**Lemma 3.8.** Let \( R \supseteq \mathbb{R} \) be real closed, let \( S \subseteq V(\mathbb{R}) \) be a semialgebraic set, let \( f \in R[\mathbb{R}] \) with \( f \geq 0 \) on \( S_R \), and let \( a \in S_R \). Then \( f^\circ(a) \) is a psd element in \( R \otimes R \), i.e. for any two homomorphisms \( \varphi_1, \varphi_2 : R \to E \) into a real closed field \( E \), the image of \( f^\circ(a) \) under \( \phi : R \otimes R \to E, a_1 \otimes a_2 \mapsto \varphi_1(a_1)\varphi_2(a_2) \) is nonnegative.

**Remark 3.9.** The ring \( R \otimes R \) is an integral domain (by [3] V.17.2, Corollaire), and it is an easy exercise to show that its quotient field is real, i.e. has a unique ordering. Therefore, in the situation of Lemma 3.8 the element \( -f^\circ(a) \) is not psd in \( R \otimes R \), and in particular is not a sum of squares, unless it is zero. This argument will be used in the proof of the main theorem in 5.5.

Recall the notation \( P_K = \{ f \in \mathbb{R}[x_1, \ldots, x_n] : \deg(f) \leq 1, f|_K \geq 0 \} \) for \( K \subseteq \mathbb{R}^n \).

The main result of this section is

**Theorem 3.10.** Let \( K \subseteq \mathbb{R}^n \) be a closed and convex semialgebraic set, let \( P = P_K \), and let \( d \geq 0 \) be an integer. Moreover let \( S \subseteq K \) and \( E \subseteq P \) be semialgebraic subsets with \( K = \text{conv}(S) \) and \( P = \text{cone}(E) \). Then the following are equivalent:

(i) \( \text{sxd} \deg(K) \leq d \);
(ii) \( \text{sosx} f^\circ(a) \leq d \) holds for every real closed field \( R \supseteq \mathbb{R} \), every \( f \in P_R \) and every \( a \in K_R \);
(iii) \( \text{sosx} f^\circ(a) \leq d \) holds for every real closed field \( R \supseteq \mathbb{R} \), every \( f \in E_R \) and every \( a \in S_R \).

Obviously, condition (iii) is a weakening of (ii). It proves useful if we want to get a bound on \( \text{sxd} \deg(K) \) through an analysis of the tensors \( f^\circ(a) \). Typically, \( E \) may be the union of all extreme rays of \( P \) (assuming that \( K \) has non-empty interior in \( \mathbb{R}^n \)), and \( S \) may be the set of extreme points of \( K \) (in the case when \( K \) is compact).

**3.11.** Let us first dispose of the case \( d = 0 \). If \( K \) is an affine space, i.e. \( \text{sxd} \deg(K) = 0 \), then every \( f \in P \) is a nonnegative constant on \( K \), and so \( f^\circ(a) = 1 \otimes c \) with \( c \geq 0 \) for every \( f, a \) as in (ii). If \( K \) is not an affine space, there is \( f \in E \) which is not constant on \( K \), and so for \( R \supseteq \mathbb{R} \) there is \( a \in S_R \) with \( f(a) \notin \mathbb{R} \). Hence \( f^\circ(a) = f(a) \otimes 1 \) is not of the form \( 1 \otimes c \), so (iii) doesn’t hold with \( d = 0 \).

**3.12.** In the rest of the proof we assume \( d \geq 1 \). To show (i) \( \Rightarrow \) (ii), let \( K \subseteq \mathbb{R}^n \) be a convex semialgebraic set with \( \text{sxd} \deg(K) = d \). By Theorem 2.13 (and Proposition 2.4) there is a morphism \( \phi : X \to \mathbb{A}^n \) of affine \( \mathbb{R} \)-varieties with \( K \subseteq \phi(X(\mathbb{R})) \), together with linear subspaces \( U_1, \ldots, U_m \) of \( \mathbb{R}[X] \) with \( \text{dim}(U_i) \leq d \), such that \( \phi^\star(P) \subseteq \mathbb{R}_+ + \sum_{i=1}^m (\Sigma U_i) \) holds. By Tarski’s transfer principle, the analogue of this inclusion holds over \( R \) as well. So there exist elements \( u_{ij} \in U_i \otimes R \) (for \( i = 1, \ldots, m \) and \( j = 1, \ldots, d \)) such that

\[
\phi^\star(f) = c + \sum_{i=1}^m \sum_{j=1}^d u_{ij}^2.
\]
Let $K$ steps. For Lemmas 3.15 to 3.18 below let

\begin{align*}
\text{Lemma 3.15.} \quad & \text{(Assumptions as in 3.14) Given } R = R[X] \text{ with } \phi(b) = a, \text{ and we conclude} \\
& f^\otimes(a) = (\phi^* f)^\otimes(b) = 1 \otimes c + \sum_{i=1}^m \sum_{j=1}^d u_{ij}^\otimes(b)^2. \\
\end{align*}

Since $\dim(U_i) \leq d$ we have $\text{rk}(u_{ij}^\otimes(b)) \leq d$ for all $i, j$, which proves the lemma.

3.13. The implication (ii) $\Rightarrow$ (iii) in 3.10 is trivial. To prove the converse, assume that (iii) holds. Let $R \supseteq \mathbb{R}$ be real closed, let $f \in P_R$ and $a \in K_R$. There are $f_1, \ldots, f_r \in E_R$ (with $r = n + 1$, if we want) and $0 \leq t_1, \ldots, t_r \in R$ with

\[ f = \sum_{j=1}^r t_j f_j. \]

So

\[ f^\otimes(b) = \sum_{j=1}^s (1 \otimes t_j) \cdot f_j^\otimes(b) \]

for every $b \in R^n$. On the other hand, there are $a_1, \ldots, a_m \in S_R$ (again with $m = n + 1)$ and $0 \leq s_1, \ldots, s_m \in R$ with $\sum_{i=1}^m s_i = 1$ and $a = \sum_{i=1}^m s_i a_i$. Therefore

\[ g^\otimes(a) = \sum_{i=1}^m (s_i \otimes 1) \cdot g^\otimes(a_i) \]

for every linear polynomial $g \in R[x]$ (see 3.7). Altogether

\[ f^\otimes(a) = \sum_{i,j} (s_i \otimes t_j) \cdot f_j^\otimes(a_i) \]

which shows $\text{sos} \ f^\otimes(a) \leq d$ by assumption (iii).

3.14. The proof of the remaining implication (ii) $\Rightarrow$ (i) in 3.10 requires several steps. For Lemmas 3.15 to 3.18 below let $K \subseteq \mathbb{R}^n$ be a convex semialgebraic set, write $P = P_K$, and assume that $\text{sos} \ f^\otimes(a) \leq d$ holds for every real closed field $R \supseteq \mathbb{R}$, every $a \in K_R$ and every $f \in P_R$ (with $d \geq 1$).

Lemma 3.15. (Assumptions as in 3.13) Given $R \supseteq \mathbb{R}$, a point $a \in K_R$ and a linear polynomial $f \in P_R$, there exists a morphism $\phi: X \rightarrow \mathbb{A}^n$ of affine $\mathbb{R}$-varieties, together with linear subspaces $U_1, \ldots, U_m \subseteq \mathbb{R}[X]$ of dimension $\leq d$, such that $a \in \phi(X(R))$ and

\[ \phi^\ast_R(f) \in \Sigma(U_1 \otimes R)^2 + \cdots + \Sigma(U_m \otimes R)^2. \]

Proof. Let $a = (a_1, \ldots, a_n)$. By definition of $\text{sos} \ f^\otimes(a)$, there exist finitely many linear $\mathbb{R}$-subspaces $U_i \subseteq R$ with $\dim(U_i) \leq d$ ($i = 1, \ldots, m$) such that $f^\otimes(a) \in \Sigma(U_1 \otimes R)^2 + \cdots + \Sigma(U_m \otimes R)^2$ in $R \otimes R$. Let $A$ be the $\mathbb{R}$-subalgebra of $R$ that is (finitely) generated by $a_1, \ldots, a_n \in R$ and by $U_1 + \cdots + U_m \subseteq R$, and let $\varphi: \mathbb{R}[x_1, \ldots, x_n] \rightarrow A$ be the homomorphism of $\mathbb{R}$-algebras defined by $x_i \mapsto a_i$ ($i = 1, \ldots, n$). Let $X = \text{Spec}(A)$, let $\phi = \varphi^\ast: X \rightarrow \mathbb{A}^n$ be the morphism of $\mathbb{R}$-varieties defined by $\varphi$. The $U_i$ are $\mathbb{R}$-linear subspaces of $\mathbb{R}[X] = A$ with $\dim(U_i) \leq d$. Moreover $a$ lies in $\phi(X(R))$, corresponding to the inclusion homomorphism $i: A \subseteq R$. Under the inclusion $i \otimes 1: A \otimes R \subseteq R \otimes R$, the element $\phi^\ast_R(f) \in A \otimes R$ is mapped to $f^\otimes(a)$. Therefore $\phi^\ast_R(f)$ has a representation of the desired form. \[\square\]}
Lemma 3.16. (Assumptions as in [3,14]) Given \( R \supseteq \mathbb{R} \) and \( f \in P_R \), there is a morphism \( \phi: X \to \mathbb{A}^n \) of affine \( \mathbb{R} \)-varieties with \( K \subseteq \phi(X(\mathbb{R})) \), and there are \( \mathbb{R} \)-linear subspaces \( U_i \subseteq \mathbb{R}[X] \) with \( \dim(U_i) \leq d \) (\( i = 1, \ldots, m \)), such that
\[
\phi_R(f) \in \Sigma(U_1 \otimes \mathbb{R})^2 + \cdots + \Sigma(U_m \otimes \mathbb{R})^2.
\]

Proof. For every real closed field \( R' \supseteq \mathbb{R} \) and \( a \in K_{R'} \), \([3,16]\) has shown that there exists an \( \mathbb{R} \)-morphism \( \phi: X \to \mathbb{A}^n \) with \( a \in \phi(X(R')) \), together with \( \mathbb{R} \)-subspaces \( U_j \subseteq \mathbb{R}[X] \) satisfying \( \dim(U_j) \leq d \) and \( \phi'_R(f) \in \sum_j \Sigma(U_j R)^2 \). For each such \( \phi \), the image set \( \phi(X(\mathbb{R})) \) is a semialgebraic subset of \( \mathbb{R}^n \). By well-known compactness properties of the real spectrum this implies that there exist finitely many \( \mathbb{R} \)-morphisms \( \phi_i: X_i \to \mathbb{A}^n \) (\( i = 1, \ldots, N \)) such that \( K \subseteq \bigcup_{i=1}^N \phi_i(X_i(\mathbb{R})) \), and for every \( i = 1, \ldots, N \) finitely many \( \mathbb{R} \)-subspaces \( U_{ij} \subseteq \mathbb{R}[X_i] \) (\( j = 1, \ldots, m_i \)) with \( \dim(U_{ij}) \leq d \), such that for each \( i = 1, \ldots, N \) we have
\[
\phi_i^R(f) \in \Sigma(U_{i1} \otimes \mathbb{R})^2 + \cdots + \Sigma(U_{im_i} \otimes \mathbb{R})^2.
\]

From \( \phi_1, \ldots, \phi_N \) we can fabricate a single \( \phi \), as follows. Let \( X := \bigsqcup_{i=1}^N X_i \) (disjoint sum), and let \( V_1, \ldots, V_N \subseteq \mathbb{R}[X] \) be the \( \mathbb{R} \)-subspaces
\[
\{0\} \times \cdots \times U_{ij} \times \cdots \times \{0\} \subseteq \mathbb{R}[X] = \mathbb{R}[X_1] \times \cdots \times \mathbb{R}[X_N]
\]
for \( 1 \leq i \leq N \) and \( 1 \leq j \leq m_i \), where \( U_{ij} \) stands at position \( i \) in the direct product. Then \( \dim(V_0) \leq d \) for all \( \nu \). If \( \phi: X \to \mathbb{A}^n \) denotes the morphism which restricts to \( \phi_i \) on \( X_i \), we clearly have \( K \subseteq \phi(X(\mathbb{R})) \). Moreover the element \( \phi^*_R(f) = (\phi_1^*(f), \ldots, \phi_N^*(f)) \in R[X] \) lies in
\[
\Sigma(V_1 \otimes \mathbb{R})^2 + \cdots + \Sigma(V_N \otimes \mathbb{R})^2.
\]
Indeed, this is clear by writing \( \phi^*_R(f) \) as
\[
(\phi_1^*(f), 0, \ldots, 0) + (0, \phi_2^*(f), 0, \ldots, 0) + \cdots + (0, \ldots, 0, \phi_N^*(f))
\]
and using (2) for \( i = 1, \ldots, N \). \( \Box \)

Lemma 3.17. (Assumptions as in [3,14]) There is a morphism \( \phi: X \to \mathbb{A}^n \) of affine \( \mathbb{R} \)-varieties, together with \( \mathbb{R} \)-linear subspaces \( U_1, \ldots, U_m \subseteq \mathbb{R}[X] \) with \( \dim(U_i) \leq d \), such that \( K \subseteq \phi(X(\mathbb{R})) \) and \( \phi^*(P) \subseteq (\Sigma U_1^2) + \cdots + (\Sigma U_m^2) \).

Proof. By Lemma 3.16 there exists, for every \( R \supseteq \mathbb{R} \) and every \( f \in P_R \), a morphism \( \phi: X \to \mathbb{A}^n \) of affine \( \mathbb{R} \)-varieties with \( K \subseteq \phi(X(\mathbb{R})) \), together with \( \mathbb{R} \)-subspaces \( U_j \subseteq \mathbb{R}[X] \) with \( \dim(U_j) \leq d \) (\( j = 1, \ldots, m \)), such that \( (\phi_j^*(f)) \in \sum_j \Sigma(U_j R)^2 \). For each such \( \phi \), the subset
\[
\{g \in P: \phi^*(g) \in \Sigma U_1^2 + \cdots + \Sigma U_m^2\}
\]
of \( P \) is semialgebraic. Again, we conclude that there exist finitely many \( \phi_i: X_i \to \mathbb{A}^n \) (\( i = 1, \ldots, N \)), each satisfying \( K \subseteq \phi_i(X_i(\mathbb{R})) \), and for each index \( i \) there exist finitely many \( \mathbb{R} \)-subspaces \( U_{ij} \subseteq \mathbb{R}[X_i] \) (\( j = 1, \ldots, m_i \)) of dimension \( \dim(U_{ij}) \leq d \), such that the following is true: For every \( f \in P \) there exists an index \( i \in \{1, \ldots, N\} \) with
\[
\phi_{ij}^*(f) = \Sigma U_{1i}^2 + \cdots + \Sigma U_{mi}^2.
\]
Again we construct a single \( \phi \) from \( \phi_1, \ldots, \phi_N \). Let \( X := X_1 \times \mathbb{A}^n \times \cdots \times \mathbb{A}^n \times X_N \) (fibre product over \( \mathbb{A}^n \) via the morphisms \( \phi_i: X_i \to \mathbb{A}^n \)), so \( \mathbb{R}[X] \) is the tensor product
\[ \mathbb{R}[X_1] \otimes \mathbb{R}[X_2] \cdots \otimes \mathbb{R}[X_n] = \mathbb{R}[X_N] \] via the homomorphisms \( \phi^+_i : \mathbb{R}[x] \to \mathbb{R}[X_i] \). The natural morphism \( \phi : X \to \mathbb{A}^n \) satisfies \( K \subseteq \phi(X(\mathbb{R})) \), and for \( f \in \mathbb{R}[x] \) we have
\[
\phi(f) = \phi_1^+(f) \otimes 1 \otimes \cdots \otimes 1 = \cdots = 1 \otimes \cdots \otimes 1 \otimes \phi_N^+(f)
\]
in \( \mathbb{R}[X] \). Let \( V_1, \ldots, V_t \subseteq \mathbb{R}[X] \) be the subspaces
\[
\mathbb{R}1 \otimes \cdots \otimes U_{ij} \otimes \cdots \otimes \mathbb{R}1
\]
for \( 1 \leq i \leq N \) and \( 1 \leq j \leq m_i \), where \( U_{ij} \) stands at position \( i \) in the tensor product. Then \( \dim(V_\nu) \leq d \) for each \( \nu \). Given \( f \in P \), let \( 1 \leq i \leq N \) be an index with \( \phi^+_i \).

Then from (4) we see that
\[
\phi^+(f) \in \Sigma V_1^2 + \cdots + \Sigma V_t^2.
\]
Altogether this shows that \( \phi^+(P) \) is contained in the right hand cone, which proves the lemma. \( \square \)

3.18. Proof of (ii) \( \Rightarrow \) (i) in Theorem 3.10. Let \( K \subseteq \mathbb{R}^n \) be closed convex and semialgebraic, and assume (ii) (see 3.14). Then Lemma 3.17 says that sosdeg(K) \( \leq d \). Combining this with Theorem 2.10 we conclude that sxdeg(K) \( \leq d \) since \( K \) is closed. This completes the proof of Theorem 3.10. \( \square \)

We record an obvious relaxation of Theorem 3.10.

Corollary 3.19. Let \( K \subseteq \mathbb{R}^n \) be a convex semialgebraic set. Then \( K \) is a spectrahedral shadow if and only if \( f^+(a) \) is a sum of squares in \( R \otimes R \), for every real closed field \( R \supseteq \mathbb{R} \), every \( f \in (P_K)_R \) and every \( a \in K_R \).

Proof. For the “if” direction, assume that \( f^+(a) \) is a sum of squares for all choices of \( R \), \( f \) and \( a \). Following the proof of Theorem 3.10 (ii) \( \Rightarrow \) (i) (see 3.14), one sees that there exists a morphism \( \phi : X \to \mathbb{A}^n \) together with a linear subspace \( U \subseteq \mathbb{R}[X] \) of finite dimension such that \( K \subseteq \phi(X(\mathbb{R})) \) and \( \phi^+(P) \subseteq \Sigma U^2 \). By Theorem 2.10 this implies sxdeg(K) \( \leq \dim(U) < \infty \). The “only if” direction is obvious from Theorem 3.10. \( \square \)

Our proof of Theorem 0.1 depends on Theorem 3.10 in an essential way. The next section will provide the necessary algebraic background.

4. Tensor decomposition

4.1. The setup in this section is somewhat technical. Before we go into the details, we give an informal outline.

Let \( K \subseteq \mathbb{R}^2 \) be a closed convex semialgebraic set, let \( P = P_K \), the cone of linear functions nonnegative on \( K \). To prove sxdeg(K) \( \leq 2 \), we have to show (by Theorem 3.10) that sosx \( f^+(a) \) \( \leq 2 \) for every \( a \in K_R \) and \( f \in P_R \), where \( R \supseteq \mathbb{R} \) is a real closed field. To describe the essential case, fix an irreducible plane algebraic curve \( C \subseteq \mathbb{A}^2 \) over \( \mathbb{R} \). Take two arbitrary \( R \)-rational points \( a \neq b \) on \( C \), and let \( f = \tau_b \) be the equation of the tangent to \( C \) at \( b \) (we assume that \( b \) is a nonsingular \( R \)-point). When \( \tau_b \) \( \neq 0 \), we need to show sosx \( \tau_b^+(a) \) \( \leq 2 \). This in turn will follow from Theorem 1.5 which is the main result of this section. See Section 5 for a rigorous proof of the main result 0.1 from this theorem.

From a (reduced) equation \( F(x, y) = 0 \) for \( C \) we get a uniform choice for an equation \( \tau_v \) of the tangent at nonsingular points \( v \) of \( C \). This gives a regular function \( T : (u, v) \to \tau_v(u) \) of \( C \times C \), i.e. an element \( T \in \mathbb{R}[C] \otimes \mathbb{R}[C] \). If \( a, b \in C(R) \) are nonsingular \( R \)-points, the tensor evaluation \( \tau_b^+(a) \in R \otimes R \) is the image of \( T \).
under the map $\mathbb{R}[C] \otimes \mathbb{R}[C] \to R \otimes R$, $p \otimes q \mapsto p(a) \otimes q(b)$. Roughly, Theorem 4.4 establishes a decomposition of $T$ in (a localization of) $\mathbb{R}[X] \otimes \mathbb{R}[X]$ where $X \to C$ is the normalization of $C$. When read in $R \otimes R$, this decomposition yields the desired conclusion $\tau \lesssim (a) \leq 2$.

In this section we work with a plane curve $C$ over $\mathbb{R}$ and with its normalization. Throughout we could work over an arbitrary base field $k$ of characteristic zero, except that this would require a slightly different formulation of Theorem 4.4. Since we have no need for this greater generality, we stick to $k = \mathbb{R}$.

4.2. We now present the details. Let $C \subseteq \mathbb{A}^2$ be an irreducible (reduced) curve over $\mathbb{R}$, and let $\pi: X \to C$ be its normalization. Let $P \in X(\mathbb{R})$ be a point, fixed for the entire discussion, and let $Q = \pi(P) \in C(\mathbb{R})$. Let $X_0 \subseteq X$ be an (affine) open neighborhood of $P$ that we will shrink further according to our needs, and write $A = \mathbb{R}[X_0]$. Always consider $A \otimes A = A \otimes \mathbb{R}$ as an $A$-algebra via the second embedding $i_2: A 	o A \otimes A$, $a \mapsto 1 \otimes a$. So for $f \in A$ and $\theta \in A \otimes A$, the notation $f \theta$ means $(1 \otimes f) \cdot \theta$. Let $\text{mult}: A \otimes A \to A$ be the product map, let $I$ be its kernel. For $f \in A$ the element $\delta(f) := f \otimes 1 - 1 \otimes f$ lies in $I$.

We choose $X_0$ so small that the $A$-module $\Omega = \Omega_{A/\mathbb{R}}$ of Kähler differentials is freely generated by $ds$, for some $s \in A$. For $f \in A$ define $\frac{df}{ds} \in A$ by $df = \frac{df}{ds} ds$, as usual, and let inductively $\frac{d^i df}{ds^i} = \frac{d}{ds}\left(\frac{d^{i-1} df}{ds^{i-1}}\right)$ for $i \geq 1$. The isomorphism $\Omega \cong I/I^2$, $df \mapsto \delta(f) + I^2$ of $A$-modules induces $A$-linear isomorphisms $\text{Sym}_A^d(\Omega) \to I^d/I^{d+1}$ for all $d \geq 0$ (17.12.4, 16.9.4). Hence, for any $f \in A$, there are unique elements $p_i \in A$ ($i \geq 0$) such that for every $d \geq 0$ the congruence

$$f \otimes 1 = \sum_{i=0}^d \frac{p_i}{i!} \delta(s)^i \pmod{I^{d+1}}$$

holds in $A \otimes A$, and we have $p_i = \frac{d^i df}{ds^i}$ for all $i \geq 0$. Hence the congruence

$$\delta(f) = \sum_{i=1}^d \frac{1}{i!} \frac{d^i f}{ds^i} \delta(s)^i \pmod{I^{d+1}} \quad (5)$$

holds in $A \otimes A$ for every $f \in A$ and every $d \geq 1$.

4.3. Via $\pi$ we consider the affine coordinates $x, y$ of $\mathbb{A}^2$ as elements of $A$. Assume that $C$ is not a line, i.e. that 1, $x, y$ are $\mathbb{R}$-linearly independent in $A$. Let $\text{val}_P: \mathbb{R}(X)^* \to \mathbb{Z}$ be the discrete (Krull) valuation of the function field $\mathbb{R}(X)$ that is centered at $P$. Since $\text{val}_P$ has residue field $\mathbb{R}$, there are (unique) integers $1 \leq m_P < n_P$ such that $\{0, m_P, n_P\} = \{\text{val}_P(f): 0 \neq f \in \mathbb{R} + \mathbb{R}x + \mathbb{R}y\}$. Note that $Q$ is a nonsingular point of $C$ if and only if $m_P = 1$, and that $n_P = 2$ holds if and only if $Q$ is nonsingular and the tangent at $Q$ is simple.

4.4. Recall that $s \in A$ is such that $ds$ is a free generator of $\Omega_{A/\mathbb{R}}$. We call

$$T_s := T_s(x, y) = \frac{dx}{ds} \cdot \delta(y) - \frac{dy}{ds} \cdot \delta(x) \in \text{span}_\mathbb{R}(1, x, y) \otimes A \subseteq A \otimes A$$

the tangent tensor of $C$ (relative to $s$). Changing $s$ results in multiplying $T_s(x, y)$ with a unit of $A$. Note that $T_s(x, y)$ is $\mathbb{R}$-bilinear in $x$ and $y$ and satisfies $T_s(y, x) = -T_s(x, y)$ and $T_s(x, 1) = 0$. Moreover $T_s(x, y) \in I^2$ since $\frac{d^2 f}{ds^2} dy = \frac{d^2 f}{ds^2} dx$ in $\Omega$. 

To explain the terminology, note that if $R \to E$ is a field extension and $b \in X(E)$ is such that $\pi(b)$ is a nonsingular $E$-point of $C$, then the image of $T_s(x, y)$ under
$$1 \otimes b : \text{span}_R(1, x, y) \otimes A \to \text{span}_R(1, x, y) \otimes E$$
is an equation for the tangent to the curve $C$ at the $E$-point $\pi(b)$ of $C$. The main result of this section is:

**Theorem 4.5.** Let $A_P = \mathcal{O}_{X, P}$, and consider the tangent tensor $T = T_s(x, y)$ as an element of $A_P \otimes A_P$. Let $(m, n) = (m_P, n_P)$ as in 4.3 Then, for any local uniformizer $t \in A_P$, there is a choice of sign $\pm$ such that
$$\pm T = (1 \otimes t^{n-1}) \cdot \sum_{i=0}^{n-2} (t^i \otimes t^{n-2-i}) \cdot (\alpha_i \delta(u_1)^2 + \beta_i \delta(u_2)^2) \quad (6)$$
in $A_P \otimes A_P$, with elements $u_1, u_2 \in A_P$ and $\alpha_i, \beta_i \in A_P \otimes A_P$, such that $\overline{\alpha}_i, \overline{\beta}_i > 0$ in $R$ for all $i$.

Here, if $\alpha \in A_P \otimes A_P$, we denote by $\overline{\alpha} \in R$ the evaluation of $\alpha$ at $(P, P) \in (X \times X)(R)$. So $\overline{\alpha}$ is the image of $\alpha$ under $A \otimes A \mult \rightarrow \overline{A} \times \overline{A}$. The essential point in 4.5 is that an identity (6) can be chosen such that the residues $\overline{\alpha}_i, \overline{\beta}_i$ are all strictly positive.

It is worthwhile to isolate the generic situation $(m, n) = (1, 2)$:

**Corollary 4.6.** In Theorem 4.5 assume that $Q = \pi(P)$ is a nonsingular point of $C$ with simple tangent. Then there is an identity
$$\pm T = \alpha \cdot \delta(u_1)^2 + \beta \cdot \delta(u_2)^2$$
in $A_P \otimes A_P$ with $\overline{\alpha}, \overline{\beta} > 0$ in $R$.

\[\square\]

4.7. If Theorem 4.5 has been proved for one choice of uniformizers $s, t$ at $P$, then it holds for any choice. We’ll prove the identity for $s = t$ with $t$ chosen according to the next lemma. This lemma allows us to assume that $A$ is generated by two elements as an $R$-algebra.

**Lemma 4.8.** Let $X$ be a nonsingular affine curve over $R$. Given any point $P \in X(R)$, there is an open affine neighborhood $U$ of $P$ on $X$ such that there are $t, u, s \in \mathbb{R}[X]$ with $\text{val}(t) = 1$ and $\mathbb{R}[U] = \mathbb{R}[t, u]_s$.

**Proof.** Choose an open neighborhood $V$ of $P$ on $X$ and a morphism $\pi : V \to \mathbb{A}^2$ which is birational onto $Y : = \pi(V)$ such that $\pi(P) = O$ is a nonsingular point of $Y$ (see e.g. 8 Problem 7.21). Then $\mathbb{R}[Y]$ is generated over $\mathbb{R}$ by two elements $t, u$, and we can assume that $t$ is a local parameter of $Y$ at $O$. Since suitable neighborhoods of $P$ (on $X$) and $O$ (on $Y$) are isomorphic under $\pi$, we are done.

\[\square\]

4.9. Assume from now on that $A = \mathbb{R}[X] = \mathbb{R}[t, u]_s$ with $s(P) \neq 0$ and $\text{val}(t) = 1$ (we may do so by Lemma 1.8). Clearly, we can also assume $\text{val}(u) \geq 2$. By changing $s$ we can assume in addition that the $A$-module $\Omega = \Omega_A/\mathbb{R}$ is freely generated by $dt$, and that $t$ generates the maximal ideal $m_P$ of $A$. Writing $(m, n) := (m_P, n_P)$ (so $1 \leq m < n$), we may assume $\text{val}(x) = m$ and $\text{val}(y) = n$. Having arranged matters in this way, we’ll establish a decomposition (6) for the tangent tensor
$$T := T_s(x, y) = \frac{dx}{dt} \delta(y) - \frac{dy}{dt} \delta(x)$$
in $A \otimes A$, with $u_1 = t$ and $u_2 = u$. 
Lemma 4.10. The ideal \( I = \ker(A \otimes A \xrightarrow{\text{mult}} A) \) of \( A \otimes A \) is generated by \( \delta(t) \) and \( \delta(u) \).

Proof. \( I \) is generated by all elements \( \delta(f), f \in A \), and \( A \) is a localization of \( \mathbb{R}[[t,u]] \).

For \( a, b \in A \) one has \( \delta(ab) = a\delta(b) + b\delta(a) + \delta(a)\delta(b) \). If \( s \in A \) is a unit of \( A \) then \( \delta(\frac{1}{s}) = -(\frac{1}{s} \otimes \frac{1}{s})\delta(s) \). From these remarks the lemma follows.

Let \( J \) denote the kernel of the ring homomorphism \( A \otimes A \xrightarrow{\text{mult}} A \xrightarrow{P} A/\mathfrak{m}_P = \mathbb{R} \), \( \alpha \mapsto \pi \), and note that \( I \subseteq J \). Recall \( m = \text{val}_P(x) \) and \( n = \text{val}_P(y) \). For notational convenience we abbreviate \( t_1 := t \otimes 1 \) and \( t_2 := 1 \otimes t \in A \otimes A \), so \( \delta(t) = t_1 - t_2 \).

Since \( n > m \) (8) and (9), we get (recall \( t \) the ideal \( J \) is generated by \( t_1 \) and \( t_2 \).

Lemma 4.11. Let \( a = \alpha t^{-m} \) and \( b = \beta t^{-n} \). Then \( 0 \neq a, b \in \mathbb{R} \) and

\[
T = abt_2^{m-1} \cdot (\delta(t)^2 (S + w) + \delta(u)^2 w')
\]

with

\[
S := \frac{m\delta(t^n) - nt_2^{n-m}\delta(t^m)}{\delta(t)^2} = \sum_{j=2}^{\infty} \left( m\binom{n}{j} - n\binom{m}{j} \right) \cdot t_2^{n-j}\delta(t)^{j-2}
\]

and suitable \( w, w' \in J^{n-1} \).

Proof. Let \( \mathfrak{m} = \mathfrak{m}_P \subseteq A \), the maximal ideal corresponding to \( P \). The local expansions of \( x, y \in A \) with respect to the local parameter \( t \) are \( x = at^m + \cdots, y = bt^n + \cdots \).

So

\[
d^ix \over dt^i = a\binom{n}{i}!t^{m-i} \mod \mathfrak{m}^{m-i+1} \quad (1 \leq i \leq m),
\]

\[
d^iy \over dt^j = b\binom{n}{j}!t^{n-j} \mod \mathfrak{m}^{n-j+1} \quad (1 \leq j \leq n).
\]

By (5) we have

\[
\delta(x) = \sum_{i=1}^{m} \frac{1}{i!} d^ix \over dt^i \delta(t)^{i} \mod I^{m+1}
\]

and

\[
\delta(y) = \sum_{j=1}^{n} \frac{1}{j!} d^iy \over dt^j \delta(t)^{j} \mod I^{n+1}
\]

in \( A \otimes A \). Substituting these into \( T \) and observing that the terms linear in \( \delta(t) \) cancel, this gives

\[
T \equiv \sum_{i=2}^{m} \frac{1}{i!} d^ix \over dt^i \delta(t)^{i} - \sum_{i=2}^{n} \frac{1}{i!} d^iy \over dt^i \delta(t)^{i}
\]

modulo \( \frac{dt}{t}I^{m+1} + \frac{dt}{t}I^{m+1} \subseteq t_2^{m-1}I^{n+1} + t_2^{n-1}I^{m+1} \). Further, using approximations (8) and (9), we get (recall \( n > m \))

\[
T \equiv ab \left( mt_2^{m-1} \sum_{j=2}^{n} \binom{n}{j} \cdot t_2^{n-j}\delta(t)^{j} - nt_2^{n-1} \sum_{i=2}^{m} \binom{m}{i} \cdot t_2^{m-i}\delta(t)^{i} \right)
\]

\[
= abt_2^{m-1} \delta(t)^2 \left( m\sum_{j=2}^{n} \binom{n}{j} \cdot t_2^{n-j}\delta(t)^{j-2} - n\sum_{i=2}^{m} \binom{m}{i} \cdot t_2^{n-i}\delta(t)^{i-2} \right)
\]

\[
= abt_2^{m-1} \delta(t)^2 \cdot S
\]
modulo $t_2^{m-1}M'$ where

$$M' := I_2^{n+1} + \ell_2^{n-m}I_2^{m+1} + \langle t_2^{n+1-j}\delta(t)^j, j = 2, \ldots, n \rangle.$$

Recall $I = \langle \delta(t), \delta(u) \rangle \cup \{ \langle t_1^2, t_2^j \rangle, j = 2, \ldots, n \}$. This implies $I^3 = \langle \delta(t)^2, \delta(u)^2 \rangle \cdot I \cup \langle \delta(t)^2, \delta(u)^2 \rangle J_n^2$, and therefore

$$I' \subseteq \langle \delta(t)^2, \delta(u)^2 \rangle \cdot J_n^{r-2}, \quad r \geq 3.$$  \hfill (12)

Let $M := \langle \delta(t)^2, \delta(u)^2 \rangle J_n^{n-1}$. By the previous remark, all summands of $M'$ are contained in $M$ except $\ell_2^{n-m}I_2^{m+1}$ in case $m = 1$. So the lemma is already proved if $m > 1$. To deal with the case $m = 1$, replace (10) by the finer approximation

$$\delta(t) \equiv \frac{dx}{dt} \delta(t) + \frac{1}{2} \frac{d^2x}{dt^2} \delta(t)^2 \pmod{I^3}$$

which again holds by (5). Proceeding otherwise as before, we get $T = abt_2^{m-1}\delta(t)^2 \cdot S$ modulo $M = t_2^{m-1}M$, since the additional term $\frac{dx}{dt} \frac{d^2x}{dt^2} \delta(t)^2$ lies in $\delta(t)^2 J_n^{n-1} \subseteq M$(note $n - 1 = m + n - 2$). This proves the lemma in all cases.

Recall $\delta(t) = t_1 - t_2$.

**Lemma 4.12.** Let $1 \leq m < n$, let $S$ be defined as in (7). Then $S$ is equal to

$$(n - m) \sum_{i=0}^{m-2} (i + 1) t_1 i t_2^{n-i} + m(n - m) t_1 t_2^{m-1} + m \sum_{j=0}^{n-m-1} (j + 1) t_1 i t_2^{n-i-1} t_2^j.$$  \hfill (13)

**Proof.** Let $S_1$ denote the expression (13). It suffices to prove $\delta(t)^2 S_1 = m\delta(t^n) - nt_2^{m-}\delta(t^m)$ (see (4)), which is the identity

$$(t_1 - t_2)^2 S_1 = m t_1^n - nt_1^{n-m} t_2^m + (n - m) t_2^n$$

of binary forms. This can be checked coefficient-wise. \hfill \Box

**4.13.** We now complete the proof of Theorem 4.15. Note that (13) is a linear combination of all the products $t_1^i t_2^j$ (where $i, j \geq 0$ and $i + j = n - 2$) with strictly positive (integer) coefficients. So, by Lemmas 4.11 and 4.12 we can write $T = abt_2^{m-1}T'$ with

$$T' = \delta(t)^2 \sum_{i=0}^{n-2} c_i t_1^i t_2^{n-2-i} + \delta(t)^2 w + \delta(u)^2 w'$$

where $0 < c_i \in \mathbb{R}$ and $w, w' \in J_n^{n-1}$. Further, since $val_P(u) \geq 2$ (see 4.10), we have $\frac{dx}{dt} \in m$, so $\delta(u) \in t_2 \delta(t) + I^2$ by (5). This gives $\delta(u)^2 \in t_2^2 \delta(t)^2 + t_2 I^2 + I^3 \subseteq \delta(t)^2 J_n^2 + I^3 J_n$, and hence

$$\delta(u)^2 t_1^i t_2^{n-2-i} \in J_n^2 \cdot (\delta(t)^2 J_n^2 + I^3 J_n) = \delta(t)^2 J_n + I^3 J_n^2 \subseteq \langle \delta(t)^2, \delta(u)^2 \rangle J_n^2$$

for every $0 \leq i \leq n - 2$ (use (12) again). So we can as well write

$$T' = \delta(t)^2 \cdot \left( w + \sum_{i=0}^{n-2} c_i t_1^i t_2^{n-2-i} \right) + \delta(u)^2 \cdot \left( w' + \sum_{i=0}^{n-2} t_1^i t_2^{n-2-i} \right)$$

with new elements $w, w' \in J_n^{n-1}$.

Now the essential point is, the products $t_1^i t_2^{n-2-i}$ $(0 \leq i \leq n - 2)$ generate the ideal $J_n^{n-2}$ of $A \otimes A$. So we can express $w$ resp. $w'$ as

$$w = \sum_{i=0}^{n-2} w_i t_1^i t_2^{n-2-i}, \quad w' = \sum_{i=0}^{n-2} w'_i t_1^i t_2^{n-2-i}.$$
with suitable elements \( w_i, w'_i \in J \) \((0 \leq i \leq n - 2)\). Combining these with \( T = \delta(t)^2 \sum_{i=0}^{n-2} (c_i + w_i)t_1^it_2^{n-i-2} + \delta(w)^2 \sum_{i=0}^{n-2} (1 + w'_i)t_1^it_2^{n-i-2} \) (15)

which shows that \( T \) has the form asserted in \( \text{5.3} \) \( \square \)

5. Proof of the main theorem

5.1. Let \( K \subseteq \mathbb{R}^2 \) be a closed convex semialgebraic set. Ultimately we want to prove \( \text{sxdeg}(K) \leq 2 \) by applying Theorems 3.10 and 4.5. To do this we start by making a series of reductions. We can assume that \( K \) is not contained in a line and does not contain a half-plane. Then \( K \) is the convex hull of its boundary \( \partial K \) (Theorem 17, Theorem 18.4), and \( \partial K \) is a semialgebraic set of dimension one. So it suffices to prove \( \text{sxdeg}(\text{conv}(S)) \leq 2 \) for every closed semialgebraic set \( S \subseteq \mathbb{R}^2 \) with \( \dim(S) = 1 \). If \( S \) is decomposed as a finite union \( S = S_1 \cup \cdots \cup S_r \) of semialgebraic sets \( S_i \) then, by Theorems 1.6 and 1.9, it is enough to show \( \text{sxdeg}(\text{conv}(S_i)) \leq 2 \) for \( i = 1, \ldots, r \). In this way we can reduce to the case where \( C \subseteq \mathbb{R}^2 \) is an irreducible curve of degree \( 1 \), and \( S \subseteq C(\mathbb{R}) \) is a closed subset homeomorphic either to a circle or to a closed interval in the line. Since the curve \( C \) has only finitely many singular points or points with a higher order tangent, we can in addition assume that \( S \) contains no such point except possibly as a boundary point of \( S \). We can also assume that any \( f \in P_S \) vanishes in at most one point of \( S \).

5.2. For \( S \) as in 5.1 let \( \overline{\text{conv}(S)} \), and let \( P = P_K = \{ f \in \mathbb{R}[x,y] \colon \text{deg}(f) \leq 1 \} \). Let \( E \) be the union of the extreme rays of the convex cone \( P \), so \( E \) consists of all \( f \in P \) for which \( f = f_1 + f_2 \) and \( f_1, f_2 \in P \) implies \( f_1, f_2 \in \mathbb{R}, f \). Then \( E \) is a semialgebraic subset of \( P \) and \( P = \text{cone}(E) \), the conic hull of \( E \), by the Krein-Milman theorem.

Let \( f \in E \), and assume that \( f \) is not constant. Then \( \inf f(S) = 0 \). In addition, if there is \( b \in S \) with \( f(b) = 0 \), then \( f \) is tangent to the curve \( C \) at \( b \), or else \( b \) is a boundary point of \( S \). If \( f > 0 \) on \( S \) then the line \( f = 0 \) is an asymptote of \( C \) at infinity. Note that \( C \) has only finitely many such asymptotes.

5.3. For proving \( \text{sxdeg}(K) \leq 2 \) it is enough to show \( \text{sosx} f^\otimes(a) \leq 2 \) for every real closed field \( R \supseteq \mathbb{R} \), every \( f \in E_R \subseteq R[x,y] \) and every \( a \in S_R \subseteq C(R) \) (Theorem 3.10). When \( a \) or \( f \) has coordinates in \( R \) this holds trivially, since then the tensor \( f^\otimes(a) \) lies in \( R \otimes 1 \) resp. in \( 1 \otimes R \). Therefore we only need to consider the case where \( f = \tau_b \) is an equation of the tangent to \( C \) at a point \( b \in S_R \) which is not \( \mathbb{R} \)-rational. In particular, \( b \) is a nonsingular \( R \)-point of \( C \).

Neither of the points \( a, b \in S_R \) needs to have bounded coordinates in general. But this can be rectified by making a suitable projective coordinate change over \( \mathbb{R} \) (we consider \( \mathbb{A}^2 \subseteq \mathbb{P}^2 \) in the standard way). So we can assume that \( a, b \) have coordinates in \( B \), the canonical valuation ring of \( R \) (see 4.2). Let \( \overline{a}, \overline{b} \in S \subseteq C(\mathbb{R}) \) be their specializations. By scaling we can also assume that the coefficients of \( f = \tau_b \) lie in \( B \), and not all lie in \( \mathfrak{m}_B \). Then \( \tau_b(a) \in B \) and \( \tau_b(a) \geq 0 \). If \( \tau_b(a) > 0 \) then \( \text{sosx} \tau_b^\otimes(a) = 1 \) by Proposition 3.5. So we can assume \( \tau_b(a) = 0 \). In this case, the reduced linear polynomial \( \overline{a} \in \mathbb{R}[x,y] \) is nonnegative on \( S \) and vanishes in both \( \overline{a}, \overline{b} \in S \). By our assumptions (see 5.1) we therefore have \( \overline{a} = \overline{b} \).
5.4. In summary we can assume that a, b ∈ C(B) are not \( \mathbb{R} \)-rational but have the same specialization \( \overline{\alpha} = \overline{\beta} := Q ∈ C(\mathbb{R}) \), and that \( f ∈ B[x, y] \) is the tangent to \( C \) at the point \( b \). Let \( π : X → C \) be the normalization of \( C \), write \( A = \mathbb{R}[X] \), and let \( a', b' ∈ X(B) \) be the preimages of \( a, b \) under \( π \). We have \( \overline{a'} = \overline{b'} \) in \( X(\mathbb{R}) \). Indeed, this can only fail if \( Q \) is a singular point of \( C \). But if \( Q \) is singular, then \( S \) contains only one half-branched center at \( Q \), by the initial assumptions 5.1, and so we still have \( \overline{a'} = \overline{b'} \). Denote this point by \( P \), and write \( A_P := \mathcal{O}_{X, P} \) for the local ring of \( X \) at \( P \), as in 4.3. The evaluation homomorphism \( A → B, p → p(a') \) at \( a' \) extends to a ring homomorphism \( A_P → B \), since \( p(a') = p(\overline{a}) = p(P) \) for every \( p ∈ A \). Similarly we have an evaluation homomorphism \( A_P → B, q → q(b') \) at \( b' \). So there is a well-defined ring homomorphism
\[
φ : A_P ⊗ A_P → B ⊗ B, \quad p ⊗ q → p(a') ⊗ q(b').
\]
Let \( T = T_s(x, y) ∈ A_P ⊗ A_P \) be the tangent tensor (for some local parameter \( s \) at \( P \)), as in 4.4. According to 4.4 \( f ⊗ (a) \) is the image of \( T \) under the homomorphism (16), up to a scaling factor of the form \( c ⊗ 1 \). Hence the decomposition of \( ±T \) established in Theorem 4.5 induces a corresponding decomposition of the tensor \( f ⊗ (a) \) in \( B ⊗ B \), via the homomorphism (16).

5.5. Let \( (m, n) = (m_P, n_P) \) as in 4.3, and assume first that \( (m, n) = (1, 2) \). By Corollary 4.6 combined with Proposition 3.5 we have \( sxdeg(±φ(T)) ≤ 2 \) for one choice of the sign \( ± \). Therefore \( sxdeg(f ⊗ (a)) ≤ 2 \), see Remark 3.9.

Now assume \( (m, n) ≠ (1, 2) \). Then, by the assumptions in 5.1 \( Q \) is an endpoint of \( S \). So both \( a, b \) lie on the same local real halfbranch of \( C \) centered at \( Q \). Therefore we can assume in Theorem 4.5 that the local uniformizer \( t \) is positive in \( a \) and \( b \) (otherwise replace \( t \) by \( -t \)). Reading the right hand side of (3) in \( B ⊗ B \) via the homomorphism (16), we see again that this element is a sum of squares of binomial tensors. This completes the proof of Theorem 0.1.

Remark 5.6. One may wonder whether Theorem 0.1 extends to convex semialgebraic sets \( K ⊆ \mathbb{R}^2 \) that are not closed. It is known that any such \( K \) is a spectrahedral shadow [19]. However, we were not able to decide whether always \( sxdeg(K) ≤ 2 \) holds. Given closed convex subsets \( T ⊆ S \) of \( \mathbb{R}^2 \), the question is whether the convex set \( (T → S) \) (see [14] Theorem 3.8 and [19], proof of Theorem 6.8) has \( sxdeg ≤ 2 \). From Netzer’s argument in [14] (proof of Theorem 3.8), we only seem to get the bound \( sxdeg(T → S) ≤ 4 \). So \( sxdeg(K) ≤ 4 \) holds for every convex semialgebraic set \( K ⊆ \mathbb{R}^2 \), but it is not clear whether this bound is sharp.

6. Constructive aspects

The proof of Theorem 0.1 in Sections 4 and 5 is essentially constructive. That is, given a closed convex semialgebraic set \( K ⊆ \mathbb{R}^2 \), one can (in principle) find an explicit second-order cone representation of \( K \). We first illustrate this in the case of a particular example. After this we’ll sketch the general procedure.

6.1. Consider the polynomial function \( f(t) = t^2 − t^6 \) on \( \mathbb{R} \), which is convex on \(-a ≤ t ≤ a \) for small \( a > 0 \) (more precisely, for \( a ≤ 1/\sqrt[4]{15} ≈ 0.5081 \) ). Fix such \( a \), let \( K = K_a ⊆ \mathbb{R}^2 \) be the convex hull of \( S := \text{graph}(f|_{[-a, a]} = \{(t, t^2 − t^6) : |t| ≤ a \}. \) We show how to find an explicit second-order cone representation of \( K \). The question of determining \( sxdeg(K_a) \), for small \( a > 0 \), was raised by Gennadiy Averkov (Oberwolfach, June 2019).
The cone $P_K \subseteq \mathbb{R}+\mathbb{R}x+\mathbb{R}y$ of linear polynomials nonnegative on $K$ is generated by the tangent
\[ \tau_v = y - (2v - 6v^5)x + (v^2 - 5v^6) \quad (-a \leq v \leq a) \]
at $(v, f(v))$ for $|v| \leq a$, together with $\tau := f(a) - y$ (see the discussion in 4.2). Let us make the procedure of Theorem 4.6 explicit for this example, in a neighborhood of the origin. The curve $X$ figuring in 4.5 is the affine line, so $A = \mathbb{R}[t]$. For $x = t$ and $y = f(t) = t^2 - t^6$, the tangent tensor in $A \otimes A$ is
\[ T = T(x, y) = \frac{dx}{dt} \delta(y) - \frac{dy}{dt} \delta(x) = \delta(t^2 - t^6) - (2t - 6t^5) \delta(t). \]
To simplify notation, write $A \otimes A = \mathbb{R}[u, v]$ where $u = t \otimes 1$, $v = 1 \otimes t$. Then $\delta(t) = u - v$, and expanding the above expression we get
\[ T = \delta(t)^2 \cdot \left(1 - u^4 - 2u^3v - 3u^2v^2 - 4uv^3 - 5v^4\right). \tag{17} \]
(which is an explicit version of Corollary 4.6 in this case). If we read $u, v$ as elements of $R$, then 17 is the tensor evaluation $\tau^0_0(a) \in R \otimes R$ where $a = (u, f(u))$. To arrive at an explicit representation of this element as a sum of squares of binomial tensors, we need to decompose the second factor in (17) as
\[ \sum_{i} p_i(u)q_i(v) \text{ in such a way that } p_i(0), q_j(0) > 0 \text{ (compare Proposition 3.5). There are many ways to do this. For example, we can write} \]
\[ 1 - u^4 - 2u^3v - 3u^2v^2 - 4uv^3 - 5v^4 = 2p_3q_1 + 3p_2q_2 + 4p_1q_3 + p_4 + q_4 \tag{18} \]
with $p_i(u) = a^i + u^i$, $q_j(v) = a^j - v^j$ ($i, j = 1, 2, 3$), $p_4(u) = \frac{1}{2} - 4a^4 - 4a^3u - 3a^2u^2 - 2au^3 - u^4$ and $q_4(v) = \frac{1}{2} - 5a^4 + 4a^3v + 3a^2v^2 + 4av^3 - 5v^4$. In this specific decomposition we have $p_i(u) \geq 0, q_j(v) \geq 0$ for $|u|, |v| \leq a$ and $i, j = 1, \ldots, 4$, as long as $a \leq 1/\sqrt{28} \approx 0.4347$.

6.2. We illustrate how a semidefinite representation of $K = K_a$ can be obtained from the preceding discussion. Starting with (18), let $a = 1/\sqrt{28}$ (or any smaller positive real number), and construct $\phi : V \to \mathbb{A}^2$ as follows. Let $V$ be the affine curve with $\mathbb{R}[V] = \mathbb{R}[t, z_0, \ldots, z_4]/a$, where the ideal $a$ is generated by $z_0^2 + f(t) - f(a)$ and $z_i^2 - p_i(t) (i = 1, \ldots, 4)$. In other words, $\mathbb{R}[V]$ is obtained by adjoining square roots of $f(a) - f(t), p_1(t), \ldots, p_4(t)$ to $\mathbb{R}[t]$. Let $\phi$ be defined by the ring homomorphism $\phi^* : \mathbb{R}[x, y] \to A'$ with $\phi^*(x) = t$ and $\phi^*(y) = f(t) = t^2 - t^6$. Then $\phi(V(\mathbb{R})) = S$. We have $\phi^*(\tau_v) = (t - v)^2 \cdot \left(2q_1(v)z_3^2 + 3q_2(v)z_2^2 + 4q_3(v)z_1^2 + z_4^2 + q_4(v)\right)$.

\[ \phi^*(\tau_v) = (t - v)^2 \cdot \left(2q_1(v)z_3^2 + 3q_2(v)z_2^2 + 4q_3(v)z_1^2 + z_4^2 + q_4(v)\right) \tag{19} \]
in $\mathbb{R}[V]$ by (17), (18). If $-a \leq v \leq a$ then $q_i(v) \geq 0$ for $i = 1, \ldots, 4$. So $\phi^*(P_K)$ consists of sums of squares in $\mathbb{R}[V]$. More precisely, let $U_i = \text{span}(z_i, t z_i)$ ($i = 1, \ldots, 4$) and $U_5 = \text{span}(1, t)$, $U_0 = \text{span}(z_0)$. Then $U_0, \ldots, U_5$ are linear subspaces of $\mathbb{R}[V]$ of dimension $\leq 2$, and
\[ \phi^*(P_K) \subseteq \sum U_0^2 + \sum U_1^2 + \cdots + \sum U_5^2. \]
Therefore, if $A, B, C$ are real numbers, then $Ax + By + C \in P_K$ if and only if there is an identity
\[ At + Bf(t) + C = a_0 \cdot (f(a) - f(t)) + \sum_{i=1}^5 g_i(t) \cdot p_i(t) \]
in $\mathbb{R}[t]$ with $p_5(t) = 1$, $0 \leq a_0 \in \mathbb{R}$ and nonnegative quadratic polynomials $g_i(t) = a_i t^2 + 2b_i t + c_i$ (i.e. with $\left(\frac{b_i}{c_i}\right) \geq 0$, $i = 1, \ldots, 5$. This is a semidefinite representation for the cone $P_K$ that shows $\text{sxdeg}(P_K) = 2$. Dualizing this representation (c.f. Proposition 1.7, Corollary 1.9) we obtain a second-order cone representation for $K$.

6.3. From the above decomposition one can read off an $(S_2^2)^m$-factorization of $K$, see 2.12. In particular, for $u, v \in K$ and $\tau_v \in P_K$ as above, the matrices

$$A_i(u) := p_i(u) \begin{pmatrix} 1 & u \\ u & u^2 \end{pmatrix}$$

$(1 \leq i \leq 5$, with $p_5(u) = 1)$ and

$$B_i(v) := h_i(v) \begin{pmatrix} v^2 & -v \\ -v & 1 \end{pmatrix}$$

$(1 \leq i \leq 5$, with $h_5(v) = 4q_3(v), h_5(v) = 3q_2(v), h_3(v) = 2g_1(v), h_4(v) = 1$ and $h_5(v) = q_4(v))$ are psd of rank $\leq 1$ and satisfy

$$\tau_v(u) = \sum_{i=1}^{5} \langle A_i(u), B_i(v) \rangle$$

by (19).

6.4. Suppose we want to find an explicit second-order cone representation for an arbitrary given closed convex semialgebraic set $K \subseteq \mathbb{R}^2$. We can assume that $K$ is the closed convex hull of a semialgebraic set $S \subseteq C(\mathbb{R})$ as in 5.1 where $C \subseteq \mathbb{A}^2$ is an irreducible curve. Let $\pi: X \rightarrow C$ be the normalization, and let $P \in X(\mathbb{R})$ with $Q = \pi(P) \in S$. Write $A = \mathbb{R}[X]$. Since the proof of Theorem 1.5 was constructive, we can find a decomposition (6) of the tensor $T(x,y) \in A_P \otimes A_P$ as in Theorem 1.5 with explicit elements $\alpha_i, \beta_i \in A_P \otimes A_P$ and $u_1, u_2 \in A_P$. Each of the $\alpha_i, \beta_i$ can be written (explicitly) as a sum of tensors $a_v \otimes b_v$ with $a_v, b_v \in A_P$ and $a_v(P), b_v(P) > 0$, see Proposition 5.5 and its proof. Let $S' \subseteq S$ be a closed neighborhood of $Q$ inside $S$ on which all the $a_v$ and the $b_v$ are strictly positive. Extend the ring $A$ by adjoining square roots of all the (finitely many) elements $a_v$, let $\psi: V \rightarrow X$ be the morphism so defined, and let $\phi = \pi \circ \psi: V \rightarrow C$. Similar to the arguments in 6.2 we see that we obtain an explicit second-order cone representation for the closed convex hull of $S'$.

Working locally around every point $Q$ of $S$ in this way, the set $S$ is covered by finitely many local patches. Patching together these local representations à la Proposition 1.6 one can then arrive at a global representation for $K$.

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