Semi-classical kinetic theory for massive spin-half fermions with leading-order spin effects

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(Dated: March 30, 2022)

We consider the quantum kinetic-theory description for interacting massive spin-half fermions using the Wigner function formalism. We derive general collision kernel involving local as well as non-local collisions between particles assuming that the spin effects can have both classical and quantum origins. To track the effect of such different contributions we use the semi-classical expansion method to obtain the generalized kinetic equations with spin. Employing the ansatz for the equilibrium distribution function we derive global equilibrium conditions, assuming the total angular momentum is conserved in the microscopic collisions.

I. INTRODUCTION

Relativistic fluid dynamics has been very successful in modeling the collective evolution of the strongly-interacting matter produced in relativistic heavy-ion collision experiments [1–7]. Considering such success, an attempt has been made to incorporate the spin degrees of freedom within the hydrodynamic framework to explain the recent measurements of spin polarization of particles emitted in these processes [8–15]. Such a formalism of relativistic fluid dynamics with spin was first proposed in Ref. [16] and further developed in Refs. [17–26]; for reviews see Refs. [27, 28]. Other similar approaches used concepts of thermodynamic equilibrium [29], effective action [30–32], entropy current [33–37], statistical operator [38], non-local collisions [39–47], chiral kinetic theory [39, 48], and holographic duality [49–51].

The understanding of global polarization (along the direction of the total angular momentum) and local polarization (along the beam direction) phenomena has become a subject of very intense investigations recently [52–54]. While the theoretical approaches assuming spin-vorticity coupling [55] can explain the global polarization measurements [55–62], they fail to describe differential observables [63–67]; though there are some recent advances in this respect [68–71].

On general thermodynamic grounds [72], the spin polarization effects are expected to be quantified by an antisymmetric tensor \( \omega^{\mu \nu} \) conjugated to the generators of the Lorentz transformations, which in local thermal equilibrium may be independent of the thermal vorticity [16–18]. The spin polarization tensor \( \omega^{\mu \nu} \) has been considered as a hydrodynamic variable that manifests the effects of spin at a macroscopic level. The hydrodynamic approach discussed in Refs. [16–18] implicitly assumes that the spin is a separately conserved quantity. However, in general, a microscopic collision process may also give rise to a transfer of angular momentum between the orbital and spin part, keeping their sum conserved. An example of such a process is a non-local collision [42, 43]. In order to account for such a possibility, the hydrodynamic formalism presented in Refs. [16–18] needs to be extended to incorporate the contributions of non-local collisions.

In the current work, following Refs. [42, 43, 73] we extend the kinetic theory framework presented in Ref. [27] to include the effects of local and non-local collisions. We use the Wigner function formalism to formulate a quantum kinetic theory for interacting spin-half Dirac particles [73, 74], and employ semi-classical approximation to derive transport equations of various components of the Wigner function [75–79]. Considering a systematic \( \hbar \) expansion of the Wigner function components, we assume that the spin polarization effects can be manifested at both leading and next-to-leading order, which goes beyond the situation discussed in Refs. [42, 43]. Using mapping of the Wigner function components to a classical distribution function with a phase-space extended to spin [80–82], we construct a classical counterpart of the resulting quantum kinetic equations. Considering the local and non-local collisions we also introduce the general collision kernel [42, 43]. Using an ansatz for the phase-space distribution function, we derive the conditions for global equilibrium, having the same form as found in Refs. [42, 43, 83].

The structure of the paper is as follows: We begin with the description of the Wigner function formalism in Sec. II. Using the semi-classical expansion approach, in Sec. III we derive the transport equations for the components of the Wigner function, while in Sec. IV we obtain mass-shell conditions at zeroth and first-order in \( \hbar \). In Sec. V, we derive kinetic equations for the scalar and axial-vector components, and in Sec. VI we formulate a general quantum kinetic equation and its classical counterpart. Details about the structure of the collision term are presented in Sec. VII. Using its explicit form and the

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generalized distribution function, in Sec. VIII we obtain the global equilibrium conditions. We summarize our findings in Sec. IX.

Notations and Conventions: In this work, we use the Cartesian coordinate system with \( x^\mu \equiv (t, \mathbf{x}) \) and the mostly-minus metric convention, \( g_{\mu\nu} = \text{diag}(+1, -1, -1, -1) \). The scalar product of two four-vectors \( a^\alpha \) and \( b^\beta \) reads \( a \cdot b = a^\alpha b^\beta - a^\beta b^\alpha \), where the three-vectors are denoted by bold font. We work with the convention \( e^{\mu\nu\alpha\beta} = +1 \) for the Levi-Civita symbol \( \epsilon^{\alpha\beta\gamma\delta} \). Also, for symmetrization and anti-symmetrization we use the notation \( A_{\mu\nu} = A_{\mu\nu} + A_{\nu\mu} \) and \( A_{[\mu\nu]} = A_{\mu\nu} - A_{\nu\mu} \), respectively. We assume \( c = \hbar = 1 \) throughout while keeping \( \hbar \) explicitly for our calculations. The dual form of the tensor \( A_{\mu\nu} \) is defined as \( \bar{A}^\mu_\alpha = \frac{1}{2} \epsilon^{\alpha\nu\beta\gamma} A_{\gamma\beta} \).

II. WIGNER FUNCTION AND ITS QUANTUM KINETIC EQUATION

In the case of spin-half massive particles and the absence of gauge fields, the Wigner function can be expressed as follows [73, 76]

\[
W_{\alpha\beta}(x, k) = \int \frac{d^4y}{(2\pi\hbar)^4} e^{-\frac{i}{\hbar} k \cdot y} \langle \tilde{\psi}_{\beta}(x_+) \psi_{\alpha}(x_-) \rangle, \tag{1}
\]

where the Dirac field operator \( \psi \) and its adjoint \( \tilde{\psi} \equiv \psi^\dagger \gamma^0 \) are defined at two different points in spacetime \( x_\pm \equiv x \pm y/2 \) with \( x \) and \( y \) being the center and relative position, respectively. Here, the angle brackets indicate the ensemble average and the colon denotes normal ordering.

In the presence of interactions, the Dirac equation is expressed as [73]

\[
(\not{D} - m) \psi(x) = \hbar \rho(x), \tag{2}
\]

where \( \rho(x) = -(1/\hbar) \partial \mathcal{L}_I / \partial \bar{\psi} \), with \( \mathcal{L}_I(x) \) denoting the interaction Lagrangian density, and \( \not{D} = i\hbar \gamma^\mu \partial_\mu \).

From the Lagrangian density \( \mathcal{L}(x) = \mathcal{L}_D(x) + \mathcal{L}_I(x) \) where

\[
\mathcal{L}_D(x) = \frac{1}{2} \bar{\psi}(x) \not{D} \psi(x) - m \bar{\psi}(x) \psi(x) \tag{3}
\]

is the Lagrangian density for the free Dirac field with mass \( m \) and \( \not{D} \equiv \gamma^\mu \partial_\mu \), we can derive the following transport equation for the Wigner function (1) [73],

\[
\left[ \gamma \cdot k + \frac{\not{D}}{2} - m \right] W(x, k) = \hbar \bar{C}[W(x, k)]. \tag{4}
\]

Here, the collision term \( \bar{C}[W(x, k)] \) is defined as [73]

\[
\bar{C}[W(x, k)] = \int \frac{d^4y}{(2\pi\hbar)^4} e^{-\frac{i}{\hbar} k \cdot y} \langle \rho(x_-) \tilde{\psi}(x_+) \rangle. \tag{5}
\]

We would like to point out here that, in global equilibrium, the collision term (5) must vanish regardless of its form [73, 74]. Here, we consider that the collision term describes the system away from equilibrium and gives rise to quantum corrections to the leading-order Wigner function, appearing at the 6th order or higher.

The Wigner function \( W(x, k) \) is a matrix in the Dirac space, therefore we can express it in terms of the generators of the Clifford algebra as

\[
W(x, k) = \frac{1}{4} \left[ \mathcal{F}(x, k) + i \gamma^5 \mathcal{P}(x, k) + \gamma^\mu \mathcal{V}_\mu(x, k) + \gamma^5 \gamma^\mu \mathcal{A}_\mu(x, k) + \Sigma_{\mu\nu} \mathcal{S}_{\mu\nu}(x, k) \right], \tag{6}
\]

with \( \Sigma_{\mu\nu} \equiv (1/2)\sigma_{\mu\nu} \equiv (i/4)[\gamma^\mu, \gamma^\nu] \) being the Dirac spin operator. Since the Wigner function is a complex matrix of order 4, it has 16 independent components: \( \mathcal{F}(x, k), \mathcal{P}(x, k), \mathcal{V}_\mu(x, k), \mathcal{A}_\mu(x, k), \) and \( \mathcal{S}_{\mu\nu}(x, k) \), which can be obtained by calculating the trace of \( W(x, k) \) after multiplying first by the matrices: \( \Gamma_X \in \{1, -i/\gamma^5, \gamma^\mu, \gamma^\mu \gamma^5, 2\Sigma_{\mu\nu}\} \), where \( X \in \{\mathcal{F}, \mathcal{P}, \mathcal{V}, \mathcal{A}, \mathcal{S}\} \), respectively.

Under Lorentz transformations the expansion coefficients of the Wigner function, \( \mathcal{F}, \mathcal{P}, \mathcal{V}_\mu, \mathcal{A}_\mu \) and \( \mathcal{S}_{\mu\nu} \) transform as a scalar, pseudo-scalar, vector, axial-vector, and tensor, respectively [76, 85]. The coefficients \( \mathcal{F} \) and \( \mathcal{P} \) have the interpretation of mass and pseudo-scalar condensate, respectively, whereas, \( \mathcal{V}_\mu \) and \( \mathcal{A}_\mu \) are known as the fermion number current density and the polarization density, respectively. Since \( \mathcal{S}_{\mu\nu} \) is antisymmetric, it has six independent components having the physical interpretation of electric and magnetic dipole moments.

Using the representation of the Wigner function in terms of the generators of the Clifford algebra (6) in the kinetic equation (4) gives rise to kinetic equations for different coefficients of the Wigner function [86], which, after separating the real and imaginary parts, yields two sets of equations for \( \mathcal{F}, \mathcal{P}, \mathcal{V}_\mu, \mathcal{A}_\mu \) and \( \mathcal{S}_{\mu\nu} \), where the real parts are

\[
k^\mu \mathcal{V}_\mu - m \mathcal{F} = \hbar \partial_\mathcal{F}, \tag{7}
\]

\[
\frac{\hbar}{2} k^\mu \mathcal{A}_\mu + m \mathcal{P} = -\hbar \partial_\mathcal{P}, \tag{8}
\]

\[
k_\mu \mathcal{F} - \frac{\hbar}{2} \partial_\mathcal{P} \mathcal{S}_{\mu\nu} - m \mathcal{V}_\mu = \hbar \partial_\mathcal{V}_\mu, \tag{9}
\]

\[-\frac{\hbar}{2} \partial_\mathcal{V}_\mu \mathcal{P} + k^\beta \partial_{\mathcal{P}}^\beta \mathcal{S}_{\mu\nu} + m \mathcal{A}_\mu = -\hbar \partial_\mathcal{A}_\mu, \tag{10}\]

\[
\frac{\hbar}{2} \partial_\mathcal{A}_\mu \mathcal{V}_\mu - \epsilon_{\mu\nu\alpha\beta} k^\alpha \mathcal{A}_\beta - m \mathcal{S}_{\mu\nu} = \hbar \partial_\mathcal{S}_{\mu\nu}, \tag{11}\]

1 Note that alternative definitions of the Wigner function without normal ordering have also been considered in the literature, e.g. in Ref. [84], where it has been argued that different definitions of the Wigner function may give rise to different results when the chiral anomaly is involved. However, we do not discuss such a situation here.
while the imaginary parts are expressed as
\[ \hbar \partial^\alpha \mathcal{V}_\mu = 2 \hbar C_x, \quad (12) \]
\[ k^\alpha A_\mu = \hbar C_p, \quad (13) \]
\[ \frac{\hbar}{2} \partial_\nu \mathcal{F} + k^\nu S_{\nu \mu} = \hbar C_{V, \mu}, \quad (14) \]
\[ k^\mu \mathcal{P} + \frac{\hbar}{2} \partial^\beta \mathcal{S}_{\beta \mu} = -\hbar C_{A, \mu}, \quad (15) \]
\[ k_{[\mu} \mathcal{V}_{\nu]} + \frac{\hbar}{2} \epsilon_{\mu \nu \alpha \beta} \partial^\alpha A^\beta = -\hbar C_{S, \mu \nu}. \quad (16) \]

In Eqs. (7)-(16), \( \mathcal{D}_X = \text{Re Tr}[\Gamma_x C[W(x, k)]] \) and \( C_X = \text{Im Tr}[\Gamma_x C[W(x, k)]] \).

Note that since Eq. (12) has \( \hbar \) on both sides, one can argue that this equation can be considered at the leading order (zeroth order in \( \hbar \)), however, in this work we consider Eq. (12) at the first order [76], keeping \( \hbar \) on both sides of the equation.

### III. SEMI-CLASSICAL EXPANSION

In general, quantum kinetic equations (7)–(16) are quite complicated because of the couplings between different components of the Wigner function. However, employing semi-classical expansion, we can decrease the complexity by breaking Eqs. (7)–(16) into a number of independent equations. The form of Eqs. (7)–(16) indicate that we can search for the solutions for various components of the Wigner function in the form of a series expansion in \( \hbar \), \( \mathcal{X} = \sum_n \hbar^n \mathcal{X}^{(n)} \). Similarly, for the collision terms \( C_X \) and \( \mathcal{D}_X \), we write \( C_X = \sum_n \hbar^n C_X^{(n)} \) and \( \mathcal{D}_X = \sum_n \hbar^n \mathcal{D}_X^{(n)} \). Below we analyse Eqs. (7)–(16) up to second-order in \( \hbar \); for the extension to the third-order, see Appendix A.

#### A. Zeroth order

In the leading order, i.e., the zeroth order in \( \hbar \), the real parts give [76]

\[ k^\mu \mathcal{V}_\mu^{(0)} - m \mathcal{F}^{(0)} = 0, \quad (17) \]
\[ m \mathcal{P}^{(0)} = 0, \quad (18) \]
\[ k^\nu S_{\nu \mu}^{(0)} - m \mathcal{V}_\mu^{(0)} = 0, \quad (19) \]
\[ k^\beta \mathcal{S}_{\mu \beta}^{(0)} + m A_\mu^{(0)} = 0, \quad (20) \]
\[ \epsilon_{\mu \nu \alpha \beta} k^\alpha A^\beta^{(0)} + m S_{\mu \nu}^{(0)} = 0, \quad (21) \]

while the imaginary parts yield [76]

\[ k^\mu A_\mu^{(0)} = 0, \quad (22) \]
\[ k^\nu S_{\nu \mu}^{(0)} = 0, \quad (23) \]
\[ k^\mu \mathcal{P}^{(0)} = 0, \quad (24) \]
\[ k_{[\mu} \mathcal{V}_{\nu]}^{(0)} = 0. \quad (25) \]

From Eqs. (17)–(25), we conclude that \( \mathcal{F}^{(0)} \) and \( A_\mu^{(0)} \) can be assumed as the basic independent coefficients in terms of which all other components of the Wigner function can be expressed, provided \( A_\mu^{(0)} \) satisfies orthogonality condition (22).

#### B. First order

In the first order in \( \hbar \), real parts give

\[ k^\mu \mathcal{V}_\mu^{(1)} - m \mathcal{F}^{(1)} = \mathcal{D}^{(0)}_\mathcal{F}, \quad (26) \]
\[ \frac{1}{2} \partial_\mu A_\mu^{(0)} + m \mathcal{P}^{(1)} = -\mathcal{D}^{(0)}_\mathcal{P}, \quad (27) \]
\[ k^\mu \mathcal{F}^{(1)} - \frac{1}{2} \partial_\nu S_{\nu \mu}^{(0)} - m \mathcal{V}_\mu^{(1)} = \mathcal{D}^{(0)}_{V, \mu}, \quad (28) \]
\[ -\frac{1}{2} \partial_\mu \mathcal{P}^{(0)} + k^\beta \mathcal{S}_{\mu \beta}^{(1)} + m A_\mu^{(1)} = -\mathcal{D}^{(0)}_{A, \mu}, \quad (29) \]
\[ \frac{1}{2} \partial_{[\mu} \mathcal{V}_{\nu]}^{(0)} - \epsilon_{\mu \nu \alpha \beta} k^\alpha A^\beta^{(1)} + m S_{\mu \nu}^{(1)} = \mathcal{D}^{(0)}_{S, \mu \nu}, \quad (30) \]

and the imaginary parts yield

\[ \partial^\mu \mathcal{V}_\mu^{(0)} = 2 \mathcal{C}^{(0)}_\mathcal{F}, \quad (31) \]
\[ k^\mu A_\mu^{(1)} = \mathcal{C}^{(0)}_\mathcal{P}, \quad (32) \]
\[ \frac{1}{2} \partial_\mu \mathcal{F}^{(0)} + k^\nu S_{\nu \mu}^{(1)} = \mathcal{C}^{(0)}_{V, \mu}, \quad (33) \]
\[ k^\mu \mathcal{P}^{(1)} + \frac{1}{2} \partial_\beta \mathcal{S}_{\mu \beta}^{(0)} = -\mathcal{C}^{(0)}_{A, \mu}, \quad (34) \]
\[ k_{[\mu} \mathcal{V}_{\nu]}^{(1)} + \frac{1}{2} \epsilon_{\mu \nu \alpha \beta} \partial^\alpha A^\beta^{(0)} + m S_{\mu \nu}^{(0)} = -\mathcal{C}^{(0)}_{S, \mu \nu}. \quad (35) \]

From Eq. (32) we can immediately see that due to the presence of the collisions, the first-order axial-vector coefficient \( A^{(1)} \) is not orthogonal to \( k \), cf. Eq. (22).

#### C. Second order

We need to have second-order equations of motion to derive the transport equations for the first-order coefficients of the Wigner function. Hence, in the second order, real parts give

\[ k^\mu \mathcal{V}_\mu^{(2)} - m \mathcal{F}^{(2)} = \mathcal{D}^{(1)}_\mathcal{F}, \quad (36) \]
\[ \frac{1}{2} \partial_\mu A_\mu^{(1)} + m \mathcal{P}^{(2)} = -\mathcal{D}^{(0)}_\mathcal{P}, \quad (37) \]
\[ k^\mu \mathcal{F}^{(2)} - \frac{1}{2} \partial_\nu S_{\nu \mu}^{(1)} - m \mathcal{V}_\mu^{(2)} = \mathcal{D}^{(1)}_{V, \mu}, \quad (38) \]
\[ -\frac{1}{2} \partial_\mu \mathcal{P}^{(1)} + k^\beta \mathcal{S}_{\mu \beta}^{(2)} + m A_\mu^{(2)} = -\mathcal{D}^{(1)}_{A, \mu}, \quad (39) \]
\[ \frac{1}{2} \partial_{[\mu} \mathcal{V}_{\nu]}^{(1)} - \epsilon_{\mu \nu \alpha \beta} k^\alpha A^\beta^{(2)} + m S_{\mu \nu}^{(2)} = \mathcal{D}^{(1)}_{S, \mu \nu}, \quad (40) \]
while the imaginary parts yield
\[
\partial^\mu \mathcal{V}_\mu^{(1)} = 2\mathcal{C}_F^{(1)},
\]
\[
k^\mu \mathcal{A}_\mu^{(2)} = \mathcal{C}_P^{(1)},
\]
\[
\frac{1}{2} \partial^\mu \mathcal{F}^{(1)} + k^\nu \mathcal{S}_{\mu\nu}^{(2)} = \mathcal{C}_{\mathcal{V}}^{(1)},
\]
\[
k^\mu \mathcal{P}^{(2)} + \frac{1}{2} \partial^\mu \mathcal{T}_{\mu\nu}^{(1)} = -\mathcal{C}_{\mathcal{A}},
\]
\[
k_{[\mu} \mathcal{V}_{\nu]}^{(2)} + \frac{1}{2} \mathcal{C}_{\mu\nu\rho} \partial^\rho \mathcal{A}^{(1)} = -\mathcal{C}_{\mathcal{S},\mu\nu}.
\]

IV. MASS-SHELL CONDITIONS

A. Zeroth order

From Eq. (18) we trivially obtain the leading-order pseudo-scalar component, \( \mathcal{P}^{(0)} = 0 \) \([18, 76, 87]\), while from Eq. (19) we find the expression for the leading-order vector coefficient in terms of the leading-order scalar coefficient as \([18, 76, 87]\)

\[
\mathcal{V}_\mu^{(0)} = \frac{k_\mu}{m} \mathcal{F}^{(0)}.
\]

Multiplying Eq. (25) with \( k^\mu \) and then using Eqs. (17) and (19) we obtain constraint equation for the leading-order vector coefficient

\[
(k^2 - m^2) \mathcal{V}_\mu^{(0)} = 0.
\]

Inserting Eq. (46) in Eq. (17) we get analogous constraint equation for the leading-order scalar coefficient \([87]\)

\[
(k^2 - m^2) \mathcal{F}^{(0)} = 0.
\]

From Eq. (21) one can find the definition of the leading-order tensor coefficient and its dual in terms of leading-order axial-vector coefficient \([18, 76]\), respectively, as

\[
\mathcal{S}_{\mu\nu}^{(0)} = \frac{1}{m} \mathcal{C}_{\mu\nu\beta} k^\beta \mathcal{A}_\mu^{(0)},
\]

\[
\tilde{\mathcal{S}}_{\mu\nu}^{(0)} = \frac{1}{m} k_{[\mu} \mathcal{V}_{\nu]}^{(0)}.
\]

Using Eq. (50) in Eq. (20) and employing Eq. (22) one can get the constraint equation for \( \mathcal{A}_\mu^{(0)} \) \([87]\)

\[
(k^2 - m^2) \mathcal{A}_\mu^{(0)} = 0.
\]

One should note here, that due to our assumption that polarization effects can come from both the leading and first order in \( \hbar \), \( \mathcal{A}_\mu^{(0)} \) does not vanish which implies that the zeroth-order tensor coefficient \( \mathcal{S}_{\mu\nu}^{(0)} \) in Eq. (49) does not vanish either. We would like to stress that in this respect our analysis goes beyond the approach of Refs. \([42, 43]\) and assumes that spin does not have to be only a dissipative effect, having both classical and quantum counterparts.

Similarly to Eq. (49), we can arrive at the expression

\[
\mathcal{A}_\mu^{(0)} = -\frac{k_\lambda}{m} \mathcal{S}_{\lambda\mu}^{(0)} = -\frac{1}{2m} \epsilon_{\lambda\alpha\beta} k_\lambda \mathcal{S}_{\alpha\beta}^{(0)}.
\]

Putting Eq. (52) in Eq. (21), and then using Eq. (23) we get the constraint equation for \( \mathcal{S}_{\mu\nu}^{(0)} \)

\[
(k^2 - m^2) \mathcal{S}_{\mu\nu}^{(0)} = 0.
\]

Therefore, for a non-trivial solution to exist, all the leading-order coefficients have to satisfy the on-shell condition, \( i.e., k^2 = m^2 \), where \( k \) denotes the kinetic momentum. Hence, in the leading order, we obtain Eqs. (46), (49) and (50), in terms of independent quantities \( \mathcal{F}^{(0)} \) and \( \mathcal{A}_\mu^{(0)} \) \([18, 76]\) where on-shell conditions for \( \mathcal{F}^{(0)} \) and \( \mathcal{A}_\mu^{(0)} \) (Eq. (48) and Eq. (51), respectively), lead to \([87]\)

\[
\mathcal{F}^{(0)} = \delta(k^2 - m^2) \mathcal{F}^{(0)}, \quad \mathcal{A}_\mu^{(0)} = \delta(k^2 - m^2) \mathcal{A}_\mu^{(0)},
\]

with \( \mathcal{F}^{(0)} \) and \( \mathcal{A}_\mu^{(0)} \) being arbitrary scalar and axial-vector functions, respectively, which are non-singular at \( k^2 = m^2 \) and need to be determined by the kinetic equations. One can easily verify at this point that Eqs. (46), (49) and (50) satisfy Eqs. (17)–(25) if the axial-vector component of zeroth order fulfills the orthogonality condition (22).

B. First order

From Eqs. (27), (28) and (30) one obtains the first-order contributions to the pseudo-scalar, vector and tensor components of the Wigner function, respectively, as

\[
\mathcal{P}^{(1)} = -\frac{1}{2m} \left[ \partial^\mu \mathcal{A}_\mu^{(0)} + 2 \mathcal{D}^{(0)}_P \right],
\]

\[
\mathcal{V}_\mu^{(1)} = \frac{1}{m} \left[ k_\mu \mathcal{F}^{(1)} - \frac{1}{2} \partial^\nu \mathcal{S}_{\nu\mu}^{(0)} - \mathcal{D}^{(0)}_{\mathcal{V},\mu} \right],
\]

\[
\mathcal{S}_{\mu\nu}^{(1)} = \frac{1}{2m} \left[ \partial_{[\mu} \mathcal{V}_{\nu]}^{(0)} - 2 \mathcal{C}_{\mu\nu\beta} k^\beta \mathcal{A}_\mu^{(1)} - 2 \mathcal{D}^{(1)}_{\mathcal{S},\mu\nu} \right],
\]

where the dual form of \( \mathcal{S}_{\mu\nu}^{(1)} \) is obtained by contracting with Levi-Civita tensor,

\[
\tilde{\mathcal{S}}_{\mu\nu}^{(1)} = \frac{1}{m} \left[ \frac{1}{4} \epsilon_{\mu\beta\sigma\rho} \partial^\rho \mathcal{V}_{\nu]}^{(0)} + k_{[\mu} \mathcal{A}_{\nu]}^{(1)} - \frac{1}{2} \mathcal{C}_{\mu\nu\beta} \mathcal{D}^{\sigma\rho} \mathcal{S}_{\rho\sigma}^{(0)} \right].
\]

Using Eqs. (18) and (58) in Eq. (29) and subsequently using Eqs. (32) and (46) we get constraint condition for first-order axial-vector coefficient as

\[
(k^2 - m^2) \mathcal{A}_\mu^{(1)} = k_\mu \mathcal{C}_P^{(0)} - \frac{1}{2} \mathcal{C}_{\mu\beta\sigma\rho} \mathcal{D}^{\sigma\rho} \mathcal{S}_{\rho\sigma}^{(0)} + m \mathcal{D}^{(0)}_{\mathcal{A},\mu}.
\]

To obtain the constraint equation satisfied by the first-order scalar coefficient we contract Eq. (28) with \( k^\mu \) and use Eqs. (26) and (49), getting

\[
(k^2 - m^2) \mathcal{F}^{(1)} = k^\mu \mathcal{D}_{\mathcal{V},\mu}^{(0)} + m \mathcal{D}^{(0)}_F.
\]
Multiplying Eq. (35) with $\kappa_\mu$ and then using Eq. (26), as well as Eqs. (21) and (28) yields the constraint equation for first-order vector coefficient

$$(k^2 - m^2)\mathcal{V}^{(1)}_\lambda = m\mathcal{D}_\lambda^{(0)} + k^\rho\mathcal{D}_\lambda^{(1)} - k_\lambda\mathcal{C}^{\rho\lambda}_S^{(0)}.$$ \hfill (61)

On the other hand, multiplying Eq. (34) by $k_\rho$ we get

$$k^2\mathcal{P}^{(1)} + \frac{1}{2}k_\rho\partial_\lambda\tilde{S}^{\rho\lambda}_S^{(0)} = -k_\lambda\mathcal{C}^{\rho\lambda}_A^{(0)}.$$ \hfill (62)

After subtracting Eq. (27) multiplied by $m$ from Eq. (62), and using in the resulting formula Eq. (20), we get the constraint equation for first-order pseudo-scalar coefficient as

$$(k^2 - m^2)\mathcal{P}^{(1)} = -k_\lambda\mathcal{C}^{\rho\lambda}_A^{(0)} + m\mathcal{D}_\rho^{(0)}.$$ \hfill (63)

Combining Eqs. (18), (29) and (30), after some straightforward algebraic manipulations, gives us the following equation

$$m\partial^{(0)}_\lambda \mathcal{V}^{(1)}_\lambda - \epsilon^{\alpha\rho\lambda\sigma}\epsilon_{\gamma\beta\delta}\kappa_\sigma k^\beta S^{\gamma\delta}_S^{(1)} + 2\epsilon^{\rho\lambda\sigma}\kappa_\sigma\mathcal{D}^{(0)}_{\alpha\sigma} - 2m^2\mathcal{S}^{(1)}_\lambda = 2m\mathcal{D}^{\rho\lambda}_S.$$ \hfill (64)

Contracting Levi-Civita tensors and using Eq. (33) in Eq. (64), and subsequently using Eq. (19), one arrives at the constraint equation for first-order tensor coefficient $S^{(1)}_\lambda$.

$$(k^2 - m^2)S^{(1)}_\lambda = k^{[\rho}\mathcal{C}^{\lambda]}_S^{(0)} + m\mathcal{D}_S^{\rho\lambda} - \epsilon^{\rho\lambda\sigma}\kappa_\sigma\mathcal{D}^{(0)}_{A\alpha}.$$ \hfill (65)

Eqs. (59)–(61), (63) and (65) are the mass-shell conditions for all the first-order coefficients of the Wigner function. From these equations, one can observe that in the collisionless limit, all first-order coefficients remain on-shell. Furthermore, unlike in Refs. [42, 43], the above equations assume the most general structure of the interactions by considering all the collision terms to be non-vanishing, i.e. $\mathcal{D}_X \neq 0$ and $\mathcal{C}_X \neq 0$.

V. KINETIC EQUATIONS FOR SCALAR AND AXIAL-VECTOR COMPONENTS

Combining Eqs. (31) and (46) we find the kinetic equation to be satisfied by the leading-order scalar coefficient

$$k^\rho\partial_\rho\mathcal{F}^{(0)} = 2m\mathcal{C}_F^{(0)}.$$ \hfill (66)

while plugging Eq. (56) in Eq. (41) we obtain the kinetic equation for the first-order scalar coefficient

$$k^\rho\partial_\rho\mathcal{F}^{(1)} = 2m\mathcal{C}_F^{(1)} + \partial_\mu\mathcal{D}_\mu^{(0)}.$$ \hfill (67)

Using Eq. (27) together with Eq. (50) in Eq. (34), we get the kinetic equation to be satisfied by leading-order axial-vector coefficient

$$k^\beta\partial_\beta\mathcal{A}^{(0)}_\mu = 2m\mathcal{C}_{A\mu}^{(0)} - 2k_\mu\mathcal{D}_\rho^{(0)}.$$ \hfill (68)

Finally, using Eq. (58) and Eq. (A11) in Eq. (44) we get the kinetic equation to be satisfied by the first-order axial-vector coefficient

$$k^\beta\partial_\beta\mathcal{A}^{(1)}_\mu = 2m\mathcal{C}_{A\mu}^{(1)} - 2k_\mu\mathcal{D}_\rho^{(1)} - \frac{1}{2}\epsilon_{\mu\beta\gamma\delta}\partial_\beta\mathcal{D}^{\gamma\delta}_S.$$ \hfill (69)

So far, we have discussed mass-shell conditions and kinetic equations satisfied by different components of the Wigner function without imposing any special assumptions. At this point, it is useful to highlight the important differences between the present work and Refs. [42, 43]. The crucial assumption considered in Refs. [42, 43] is that the spin effects appear at first order in $\hbar$ in the semi-classical expansion of the Wigner function. Such an assumption is physically motivated when one considers relatively small spin polarization effects arising through scatterings in a vortical medium. Within the quantum kinetic theory approach, the polarization effects appear through the axial-vector component $\mathcal{A}_\mu$ [42, 43, 88]. The assumption that the spin polarization effect is at least of order $\hbar$ implies that $\mathcal{A}_\mu^{(0)}$ can be considered to be zero, and consequently, the tensor component of the Wigner function $\mathcal{S}_{\mu\nu}^{(0)} = 0$ (see Eq. (49)). Moreover, the leading-order pseudo-scalar component $\mathcal{P}^{(0)}$ is always vanishing (18). Furthermore, for the consistency of the framework, it has been argued that the leading-order collision terms involving the pseudo-scalar, axial-vector, and tensor components must vanish, i.e., $\mathcal{C}_\rho^{(0)} = 0$; $\mathcal{C}_A^{(0)} = 0$; $\mathcal{C}_S^{\mu\nu(0)} = 0$; $\mathcal{D}_\rho^{(0)} = 0$; $\mathcal{D}_A^{(0)} = 0$; $\mathcal{D}_S^{\mu\nu(0)} = 0$ [42, 43]. Such constraints on the collision terms also affect the on-shell conditions for various components of the Wigner function. It can be shown in this case that $\mathcal{A}_\mu^{(1)}$, not only is on-shell, see Eq. (59), but also remains orthogonal to momentum, $k^\mu\mathcal{A}_\mu = \mathcal{O}(\hbar^2)$ [42, 43]. Similarly, due to the assumptions that $\mathcal{A}_\mu^{(0)}$ and the collision term $\mathcal{D}_\rho^{(0)}$ are vanishing, the pseudo-scalar component $\mathcal{P}$ is at least of the order of $\hbar^2$, see Eq. (55).

The novelty of the current study in comparison to Refs. [42, 43], is that we have assumed that the polarization effects can also appear at the leading-order of the semi-classical approximation, i.e. $\mathcal{A}_\mu^{(0)} \neq 0$. The physical motivation behind our assumption is that the polarization effect can be manifested even at the classical level, and thus polarization can be generated in the presence as well as in the absence of specific collision processes in a vortical fluid. We also assume that all the collision terms are, in general, non-vanishing, hence, in contrast to Refs. [42, 43], we consider $\mathcal{P}^{(1)} \neq 0$, $(k^2 - m^2)\mathcal{A}_\mu^{(1)} \neq 0$, and $k^\mu\mathcal{A}_\mu = \mathcal{O}(\hbar)$. Such a different treatment of the axial-vector component of the Wigner function and various collision terms gives rise to significantly different on-shell conditions and kinetic equations for various components of the Wigner function.
VI. GENERAL KINETIC EQUATION AND ITS CLASSICAL COUNTERPART

In this section we combine the zeroth and first-order kinetic equations for $F$ and $A_{\mu}$ to derive the collision kernel in Sec. VII. This may be used to develop a hydrodynamic framework where spin effects arise at both $h^0$ and $h^1$ orders.

Combining Eqs. (66) and (67) we get the kinetic equation for the scalar coefficient as

$$ k^\mu \partial_\mu \tilde{F} = 2m \tilde{C}_F, $$

where

$$ \tilde{F} = F^{(0)} + h F^{(1)}, $$

$$ \tilde{C}_F = C_F^{(0)} + h \left( C_F^{(1)} + \frac{1}{2m} \partial^{\mu} D^{(0)}_{\nu,\mu} \right). $$

Similarly, from Eqs. (68) and (69), we obtain the following kinetic equation for axial-vector component

$$ k^0 \partial_0 \tilde{A}_\mu = 2m \tilde{C}_{A,\mu}, $$

where

$$ \tilde{A}_\mu = A^{(0)}_{\mu} + h A^{(1)}_{\mu}, $$

$$ \tilde{C}_{A,\mu} = C_{A,\mu}^{(0)} + h C_{A,\mu}^{(1)} - \frac{k_\mu}{m} \left( D^{(0)}_\mu + h D^{(1)}_\mu \right) - \frac{h}{4m} \epsilon_{\mu\beta\gamma} \partial^{\beta} D^{\delta}_{S^{(0)}}. $$

It is useful to introduce spin as an additional phase-space variable in the distribution function as $[^{27, 42, 80-82}]$

$$ f(x, k, s) = \frac{1}{2} \left( \tilde{F}(x, k) - s \cdot \tilde{A}(x, k) \right), $$

where $s^\alpha$ is the spin four-vector. Using Eq. (75) one can also obtain $\tilde{F}(x, k)$ and $\tilde{A}(x, k)$,

$$ \int dS(k) f(x, k, s) = \tilde{F}(x, k), $$

$$ \int dS(k) g^\mu f(x, k, s) = \tilde{A}^\mu(x, k), $$

where the spin measure $[^{42, 43}]$

$$ \int dS(k) = \frac{1}{\pi} \sqrt{\frac{k^2}{3}} \int d^4s \delta(s \cdot s + 3) \delta(k \cdot s), $$

satisfies the following identities $[^{42, 43}]$

$$ \int dS(k) = 2, $$

$$ \int dS(k) g^\mu = 0, $$

$$ \int dS(k) g^\mu g^\nu = -2 \left( g^{\mu\nu} - \frac{k^\mu k^\nu}{k^2} \right). $$

The relationship between $\tilde{F}(x, k)$ and $f(x, k, s)$ (76) can be easily established using the above identities. But it is rather non-trivial to express $\tilde{A}^\mu(x, k)$ in terms of $f(x, k, s)$. Using Eq. (75) on the left-hand side of Eq. (77) it can be shown that,

$$ \int dS(k) g^\mu f(x, k, s) = \tilde{A}^\mu(x, k) - \frac{k^\mu}{k^2} \left( k \cdot \tilde{A} \right). $$

Equation (77) is valid only when $k \cdot \tilde{A} = 0$. From Eqs. (22) and (32) one can observe that $k \cdot \tilde{A} = h C_P^{(1)}$. Thus the collision term $C_P^{(0)}$ must vanish to obtain Eq. (77). In Refs. $^{42, 43}$ it was considered that $C_P^{(0)} = 0$ since the spin polarization effects are assumed to be at least of order $h$. In the present investigation, although we do not use such an assumption, one can still show that $C_P^{(0)}$ vanishes. From Eqs. (10) and (13) one obtains

$$ k^\mu \partial_\mu \mathcal{P} = 2m C_P + 2k^\mu D_{A,\mu}. $$

Using the semi-classical expansion of $\mathcal{P}$, $C_P$ and $D_{A,\mu}$ with the condition $D^{(0)} = 0$ the above equation gives

$$ m C_P^{(0)} + k^\mu D_{A,\mu}^{(0)} = 0. $$

Note that the collision term $D_{A,\mu}^{(0)}$ transforms as an axial-vector under the Lorentz transformation. Therefore we can express $D_{A,\mu}^{(0)}$ in terms of the spin four-vector $s^\mu$, i.e., $D_{A,\mu}^{(0)} = s_\mu \delta A$, where $\delta A$ is a scalar function. Since $k \cdot s = 0$ is manifested by the Dirac delta function $\delta(k \cdot s)$ in the expression of generalized spin measure (78), Eq. (82) gives $C_P^{(0)} = 0$. Thus, Eqs. (76) and (77) provide a well defined prescription to obtain $\tilde{F}(x, k)$ and $\tilde{A}^\mu(x, k)$ in terms of the distribution function $f(x, k, s)$ which satisfies a kinetic equation. The inversion relations from $\tilde{F}(x, k)$ and $\tilde{A}^\mu(x, k)$ to $f(x, k, s)$ are crucial because in the quantum kinetic theory approach, energy-momentum and the spin tensors are expressed in terms of various independent components of the Wigner function. Therefore, with the help of inversion relations (76) and (77), those macroscopic currents can be written in terms of $f(x, k, s)$ $^{[16-18, 28]}$. From Eqs. (70), (72) and (75), we get the general Boltzmann equation to be solved $^{2}$

$$ k^\mu \partial_\mu f(x, k, s) = m \mathcal{C}(f), $$

where $\mathcal{C}(f) = \tilde{C}_F - s \cdot \tilde{C}_A$ is the collision term. Within the quasiparticle approximation, the generalized distribution function can be expressed as $f(x, k, s) = m \delta(k^2 - M^2) f(x, k, s)$ $^{[42, 43]}$. Here, the on-shell singularity for the quasiparticle has been expressed as $\delta(k^2 - M^2)$ and the function $f(x, k, s)$ is a function without singularity. Furthermore, $M$ denotes quasiparticle mass, which includes quantum corrections to the bare mass $m$. However, such quantum corrections to the particle mass do not contribute to the kinetic equation $^{[42, 43]}$.

$^{2}$ Note that Boltzmann equation (83) derived here, assumes that spin effects are at least of leading and first-order in $h$ which is not the case derived in Refs. $^{42, 43}$. 
VII. STRUCTURE OF COLLISION TERM

Collision term $\mathcal{C}[f]$ in Eq. (83) can have terms in general up to any order in $\hbar$ with contributions from both local and non-local collisions. In this paper, we consider the terms only up to the first order in $\hbar$ [28, 42, 43, 89]. We note here that the zeroth-order collision term has only the local contribution [89] while the first-order collision term has both the local and non-local contributions, where the latter part comes from the gradients. Hence, the structure of the collision term is as follows

$$\mathcal{C}[f] = \mathcal{C}_l[f] + \hbar \mathcal{C}_nl^{(1)}[f] = \mathcal{C}_l^{(0)}[f] + \hbar \mathcal{C}_l^{(1)}[f] + \hbar \mathcal{C}_nl^{(1)}[f] , \quad (84)$$

where, $l$ and $nl$ denote the local and non-local contributions to the collision term, respectively.

We can write the full collision term, following the method used in Refs. [42, 43, 73, 74], including the non-local effects (coming from the scattering of particles at different spacetime points) as

$$\bar{\mathcal{C}}[f] = \int \! d\Gamma_1 d\Gamma_2 d\Gamma' \tilde{W}_{ks} \times$$

$$\left[ f (x + \Delta_1, k_1, s_1) \ f (x + \Delta_2, k_2, s_2) 
\ - f (x + \Delta, k, s) \ f (x + \Delta', k', s') \right]$$

$$+ \int \! d\Gamma_2 dS_1(k) dS'(k_2) W_s \times$$

$$\left[ f (x + \Delta_1, k_1, s_1) \ f (x + \Delta_2, k_2, s_2) 
\ - f (x + \Delta, k, s) \ f (x + \Delta', k_2, s') \right] , \quad (85)$$

where the definitions of $\tilde{W}_{ks}$ and $W_s$ are given in Appendix B and the phase space measure is [42, 43]

$$\int \! d\Gamma = \int \! d^3k \delta \left( k^2 - m^2 \right) \int \! dS(k) . \quad (86)$$

Eq. (85), non-local collision part comes from the gradients of the distribution function $f$ which is evaluated at different spacetime points. These spacetime shifts ($\Delta^\alpha$) are of order $\hbar$ and are functions of momentum and spin [42, 43, 73], implying that non-local collision contribution is of quantum origin coming from order $\hbar$. Note that the explicit form of the spacetime shifts can play an important role in the derivation of the hydrodynamic equations based on the kinetic equation (83) and collision term (85). However, to obtain the equilibrium conditions from the collision term, we do not require an explicit form of $\Delta$.

VIII. COLLISION TERM WITH EQUILIBRIUM DISTRIBUTION FUNCTION

In this section we will derive the collision term using the local equilibrium distribution function mentioned in Ref. [27] for the case of small polarization

$$f_{eq}(x, k, s) = \exp(-\beta (x) \cdot k) \left[ 1 + \frac{1}{2} \omega_{\mu\nu}(x) s^{\mu\nu} \right], \quad (89)$$

where $\beta^\mu$ is the ratio of fluid flow four-vector $U^\mu$ over temperature $T$, $\omega_{\alpha\beta}$ is the spin polarization tensor and $s^{\mu\nu} = (1/m) \epsilon^{\mu\nu\alpha\beta} k_\alpha s_\beta$ is the internal angular momentum [90] for spin-half massive particles in terms of spin four-vector $s^\alpha$ and on-shell particle four-momentum $k^\alpha$ [86]. The equilibrium distribution function (87) that includes spin as an additional phase space variable is only valid for small values of $\omega_{\alpha\beta}$, therefore it can be considered as a small $\omega_{\alpha\beta}$ expansion of the equilibrium distribution function.

To include the effect of non-local collisions of the particles scattering with a non-zero impact parameter, we need to evaluate the distribution function (89) at different points of spacetime. This can be done by including the position shift ($\Delta$) as

$$f_{eq}(x + \Delta, k, s) = \exp \left[ -\beta (x + \Delta) \cdot k \right] \times \left[ 1 + \frac{1}{2} \omega_{\mu\nu}(x + \Delta) s^{\mu\nu} \right]. \quad (88)$$

After Taylor expanding we have,

$$f_{eq}(x + \Delta, k, s) = \exp \left[ -\beta_\mu (x) k^\mu \right] \exp \left[ -k^\mu \Delta^\nu \partial_\nu \beta_\mu \right]$$

$$\left[ 1 + \frac{1}{2} (\omega_{\mu\nu}(x) + \Delta^\alpha \partial_\alpha \omega_{\mu\nu}) s^{\mu\nu} \right]. \quad (89)$$

One may notice that the equilibrium distribution function (87) used here is different from the one mentioned in Refs. [42, 43] where authors restore $\hbar$ explicitly in their definition. The apparent differences between these two forms of distribution functions are also manifested in the definition of the total angular momentum of individual participants in the microscopic collision processes. For the rest of our calculation we identify the total angular momentum as $J^{\mu\nu} = L^{\mu\nu} + g^{\mu\nu} = \Delta^{\alpha} k^\alpha + g^{\mu\nu}$ which is different from the one used in Refs. [42, 43] by a factor of $\hbar/2$ associated with the spin part of the total angular momentum. However, this difference does not affect the global equilibrium conditions derived below. Moreover, the $\hbar/2$ factor can be absorbed within the definition of the spin polarization tensor and $g^{\mu\nu}$. We emphasize that the position shift $\Delta^\mu$ which encodes the non-locality of the collision is implicitly of order $\hbar$ [42, 43].

We can write the second exponential in (89) as $\exp \left[ -\Delta^\nu k^\mu \partial_\nu \beta_\mu \right] \approx (1 - \Delta^\nu k^\mu \partial_\nu \beta_\mu)$ to keep terms linear in $\Delta$. This implies,

$$f_{eq}(x + \Delta, k, s) = \exp \left[ -\beta_\mu (x) k^\mu \right] \left[ 1 - k^\nu \Delta^\mu \partial_\mu \beta_\nu \right]$$

$$- \frac{1}{2} \delta^{\alpha\gamma} \omega_{\alpha\gamma} k^\nu \Delta^\mu \partial_\mu \beta_\nu + \frac{1}{2} \omega_{\mu\nu} g^{\mu\nu} + \frac{1}{2} g^{\mu\nu} \Delta^\alpha \partial_\alpha \omega_{\mu\nu} . \quad (90)$$

Using the local equilibrium distribution function including non-local effects (90) in Eq. (85), we obtain

$$\bar{\mathcal{C}}[f_{eq}] = \mathcal{J}_1 + \mathcal{J}_2 , \quad (91)$$
where
\[ \mathcal{J}_1 = -\int d\Gamma_1 d\Gamma_2 d\Gamma W_{k_2} \exp \left( -\beta \cdot (k_1 + k_2) \right) \times \]
\[ \frac{1}{2} \left[ 2\partial_{\mu} \beta_\mu \left( k_1^\mu k_2^\nu - k_2^\mu k_1^\nu \right) \right. \]
\[ -\omega_{\mu\nu} \left( s_1^{\mu\nu} + s_2^{\mu\nu} - g^{\mu\nu} - g^{\mu\nu} \right) \]
\[ -\partial_\alpha \omega_{\mu\nu} \left( s_1^{\mu\nu} \Delta^\alpha \right) \]
\[ + \omega_{\alpha\beta} \partial_\nu \beta_\mu \left( s_1^{\mu\nu} k_2^\alpha + s_2^{\mu\nu} k_1^\alpha - g^{\mu\nu} \Delta^\alpha \right) \]
\[ \left. - s^{\alpha\beta} k^\mu \Delta^\nu + s_1^{\alpha\beta} k_1^\nu + s_2^{\alpha\beta} k_2^\nu \right] \right] \tag{92} \]
and
\[ \mathcal{J}_2 = -\int d\Gamma_2 dS_1(k) dS(k) W_k \exp \left( -\beta \cdot (k + k_2) \right) \times \]
\[ \frac{1}{2} \left[ 2\partial_{\mu} \beta_\mu \left( k_2^\mu k_1^\nu - k_1^\mu k_2^\nu \right) \right. \]
\[ -\omega_{\mu\nu} \left( s_1^{\mu\nu} + s_2^{\mu\nu} - g^{\mu\nu} - g^{\mu\nu} \right) \]
\[ -\partial_\alpha \omega_{\mu\nu} \left( s_1^{\mu\nu} \Delta^\alpha \right) \]
\[ + \omega_{\alpha\beta} \partial_\nu \beta_\mu \left( s_1^{\mu\nu} k_2^\alpha + s_2^{\mu\nu} k_1^\alpha - g^{\mu\nu} \Delta^\alpha \right) \]
\[ \left. - s^{\alpha\beta} k^\mu \Delta^\nu + s_1^{\alpha\beta} k_1^\nu + s_2^{\alpha\beta} k_2^\nu \right] \right] \tag{93} \]

Physically the collision term \( \mathcal{J}_1 \) (92) represents momentum exchange as well as spin exchange in a microscopic collision process. On the other hand, the collision term \( \mathcal{J}_2 \) (93) represents only spin exchange [42, 73]. Equation (91) along with Eqs. (92) and (93) are our main results which include both local and non-local collisions. From Eq. (91) we obtain the global equilibrium conditions (see Appendix C for details), as
\[ \partial_\mu \beta_\mu + \partial_\nu \beta_\mu = 0, \tag{94} \]
\[ \omega_{\mu\nu} = -\frac{1}{2} \left( \partial_\mu \beta_\nu - \partial_\nu \beta_\mu \right) = \omega_{\mu\nu}, \tag{95} \]
having the same form as in Refs. [42, 83]. Equation (94) is the Killing condition whereas Eq. (95) implies that in global equilibrium spin polarization tensor equals thermal vorticity \( \omega_{\mu\nu} \). In global equilibrium the Killing condition (94) also implies that \( \omega_{\mu\nu} \) (95) is constant.

IX. SUMMARY AND CONCLUSIONS

In this work we have extended the formalism presented in Ref. [27] to include the effect of collisions (both local and non-local) following the methodology of Refs. [42, 43, 73]. We have used the Wigner function formalism and employed semi-classical expansion to derive generalized quantum kinetic equations for the components of the Wigner function of massive spin-half Dirac particles. In our calculations, we have considered that, unlike in Refs. [42, 43], the spin polarization effects may arise from both the zeroth and first-order contributions in the \( \hbar \) expansion. We have derived a general quantum kinetic equation for the independent Wigner function components and used the ansatz for the generalized phase-space distribution function, including the spin degrees of freedom, to obtain its classical counterpart. We have shown that the latter has the same Boltzmann-like form like the one found in Refs. [42, 43], and the classical distribution function can be mapped back to the components of the Wigner function. The kinetic theory formulation derived here can be used to develop a general spin-hydrodynamic formalism in a way presented in Ref. [45].

Finally, we have considered the resulting classical kinetic theory description for the generalized phase-space distribution function, assuming the presence of local as well as non-local collisions. As, in this case, spin cannot be considered as a separately conserved quantity, we have assumed the conservation of total angular momentum in microscopic interactions. Using the ansatz for the local equilibrium distribution function studied in Ref. [18], we have found that resulting global equilibrium conditions are of the same form as in Refs. [42, 83]. Hence, we have confirmed that their form remains unaffected by the chosen counting scheme of the spin-polarization effects.

ACKNOWLEDGMENTS

We thank S. Bhadury, A. Jaiswal, D. Rischke, E. Speranza and N. Weickgenannt for insightful discussions. RS, RR, WF, and AD were supported in part by the Polish National Science Centre Grants No. 2016/23/B/ST2/00717 and No. 2018/30/E/ST2/00432. R.S. also acknowledges the support of Polish NAWA PROM program and the hospitality of the Institute for Theoretical Physics, Goethe University Germany where part of this work was completed.

Appendix A: Third-order kinetic equations for the Wigner function coefficients

In this appendix, we derive the third-order kinetic equations for the coefficients of the Wigner function. Mass-shell conditions and transport equations for \( \mathcal{F}^{(2)} \) and \( \mathcal{A}^{(2)} \) are also shown.

In the \( \hbar^3 \) order, comparing the real and imaginary parts of the coefficients of the Wigner function in the Clifford-algebra basis we arrive at two sets of equations,
where the real part gives,

\[ k^\mu V^{(3)}_\mu - m F^{(3)} = D^{(2)}_F, \]  

(A1)

\[ \frac{1}{2} \partial^\mu A^{(2)}_\mu + m D^{(3)}_F = -D^{(2)}_P, \]  

(A2)

\[ k_\mu F^{(3)} = \frac{1}{2} \partial^\nu S^{(2)}_{\nu \mu} - k^\nu S^{(3)}_{\nu \mu} = D^{(2)}_V, \]  

(A3)

\[ -\frac{1}{2} \partial_\mu D^{(2)} = + k^\beta \bar{S}^{(3)} + m A^{(3)} = -D^{(2)}_A, \]  

(A4)

\[ \frac{1}{2} \partial_\mu V^{(2)}_\mu - \epsilon_{\mu \nu \alpha \beta} k^\alpha A^{(3)} \]  

and from the imaginary parts we obtain

\[ \partial^\mu V^{(2)}_\mu = 2 C^{(2)}_F, \]  

(A6)

\[ k^\mu A^{(3)}_\mu = C^{(2)}_A, \]  

(A7)

\[ \frac{1}{2} \partial_\mu F^{(2)} + k^\nu S^{(3)} = C^{(2)}_V, \]  

(A8)

\[ k_\mu F^{(3)} + \frac{1}{2} \partial^\nu S^{(2)}_{\nu \mu} - k^\nu S^{(3)}_\mu = -C^{(2)}_A, \]  

(A9)

\[ k_\mu V^{(3)}_\mu + \frac{1}{2} \epsilon_{\mu \nu \alpha \beta} k^\alpha A^{(2)} - 2 D^{(1)}_{S, \mu \nu} = -C^{(2)}_{S, \mu \nu}. \]  

(A10)

From Eqs. (37), (38) and (40) we obtain the second-order contributions to the pseudo-scalar, vector and tensor components of the Wigner function, respectively,

\[ P^{(2)} = -\frac{1}{2m} \left[ \partial^\mu A^{(1)}_\mu + 2 D^{(1)}_P \right], \]  

(A11)

\[ V^{(2)}_\mu = \frac{1}{m} \left[ k_\mu F^{(3)} - \frac{1}{2} \partial^\nu S^{(1)}_{\nu \mu} - D^{(1)}_V \right], \]  

(A12)

\[ S^{(2)}_\mu = C^{(2)}_V \]  

\[ \frac{1}{2m} \left[ \partial_\mu V^{(1)}_\mu - 2 \epsilon_{\mu \nu \alpha \beta} k^\alpha A^{(2)} - 2 D^{(1)}_{S, \mu \nu} \right]. \]  

(A13)

Again, the dual form of \( S^{(2)}_{\mu \beta} \) can be expressed as,

\[ \bar{S}^{(2)}_{\mu \beta} = \frac{1}{m} \left[ \frac{1}{4} \epsilon_{\mu \nu \alpha \beta} \delta^{(3)}(\gamma^0)^{(1)} + k_{\mu \nu \alpha \beta} A^{(2)}_\beta - \frac{1}{2} \epsilon_{\mu \nu \alpha \beta} D^{(3)}_S \right]. \]  

(A14)

Contracting Eq. (38) with \( k^\mu \) and using Eqs. (33) and (36) we get constraint equation for the second-order scalar coefficient,

\[ (k^2 - m^2) F^{(2)} = \frac{1}{4} \partial^\mu \partial_\mu F^{(0)} + k^\mu D^{(1)}_{V, \mu} + m D^{(1)}_F \]  

\[ -\frac{1}{2} \partial^\nu C^{(0)}_{V, \nu \nu}. \]  

(A15)

Using Eqs. (55) and (A14) in Eq. (39), we obtain the constraint equation for second-order axial-vector coefficient,

\[ (k^2 - m^2) A^{(2)}_\mu = \frac{1}{4} \partial_\mu \partial_\mu A^{(0)}_\mu + \frac{1}{2} \partial_\mu D^{(0)}_F + k_\mu C^{(1)}_F \]  

\[ + \frac{1}{4} \epsilon_{\mu \nu \alpha \beta} k^\beta \left( \partial^\nu \gamma^{(3)}_{\nu \mu} - D^{(3)}_S \right) \]  

\[ + m D^{(1)}_{A, \mu}. \]  

(A16)

Combining Eqs. (A6) and (A12) we get the kinetic equation for \( F^{(2)} \) as

\[ k^\mu \partial_\mu F^{(2)} = 2 m C^{(2)}_F + \partial^\mu D^{(1)}_{V, \mu}. \]  

(A17)

Finally, to arrive at the kinetic equation for \( A^{(2)}_\mu \), we put Eqs. (A2) and (A14) in Eq. (A9), getting

\[ k^\beta \partial_\beta A^{(2)}_\mu = 2 m C^{(2)}_A + 2 k_\mu D^{(2)}_P - \frac{1}{2} \epsilon_{\mu \nu \alpha \beta} \partial^\beta D^{(3)}_S \]  

(A18)

Appendix B: Definitions of \( \bar{W}_{ks} \) and \( W_s \)

In the derivation of the collision term (85), following the method used in Refs. [42, 43, 73, 74], the definitions of \( \bar{W}_{ks} \) and \( W_s \) are

\[ \bar{W}_{ks} = \frac{1}{8} s^4 (k + k' - k_1 - k_2) \sum_{s, r} [h_{sr}(k, s)] \times \]  

\[ \sum_{s', r', s_1, s_2, r_1, r_2} \]  

\[ \langle k; k', r', r'; | k_1, k_2; s_1, s_2 \rangle \times \langle k_1, k_2; r_1, r_2 | t^\dagger | k, k', s, s' \rangle, \]  

\[ W_s = \frac{\pi \hbar}{4 m} \sum_{s_1, s_2, r_1, r_2} \epsilon_{\mu \nu \alpha \beta} s^\alpha s^\nu r_1^\beta \]  

\[ \langle k, k_2; r_2 | t^\dagger + t | k, k_1, s_1, s_2 \rangle. \]  

(B2)

where

\[ h_{sr}(k, s) = \delta_{sr} + s \cdot n_{sr}(k), \]  

\[ n^\mu_s(k) = \frac{1}{2m} u_s(k) \gamma^\mu u_s(k), \]  

(B3)

with \( u_s(k) \), \( u_r(k) \) being the Dirac bispinors. Here, the scattering amplitude of two particles is denoted by \( \langle k, k'; r, r'; | k_1, k_2; s_1, s_2 \rangle \), where particles with momenta \( k_1, k_2 \) and with spin \( s_1, s_2 \) go into two particles with momenta and spin, \( k, k' \) and \( r, r' \), respectively.

Appendix C: Conditions for global equilibrium

In this section we give details on the derivation of the global equilibrium conditions (94)–(95) from Eq. (91). We start with the total angular momentum \( j^{\mu \nu} \), consisting of orbital angular momentum \( j^{\mu \nu} \) and spin angular momentum \( g^{\mu \nu} \)

\[ j^{\mu \nu} = L^{\mu \nu} + g^{\mu \nu} = \Delta^{(\mu \nu)} + g^{\mu \nu}, \]  

(C1)

and we assume that the total angular momentum is conserved in collisions of two particles, therefore, we can write

\[ j_1^{\mu \nu} + j_2^{\mu \nu} = j^{\mu \nu} + j_1^{\mu \nu}. \]  

(C2)

Then by using the identity \( k^{\mu} \Delta^{\nu} \partial_\mu \beta_\nu = (k^{[\mu} \Delta^{\nu]} \partial_\mu \beta_\nu] + k^{[\mu} \Delta^{\nu]} \partial_\mu \beta_\nu] )/4 \), along with Eqs. (C1)–(C2), we obtain from Eqs. (91)–(93)

\[ \tilde{c}[f_{eq}] = J_1 + J_2, \]  

(C3)
\[ \mathcal{J}_1 = - \int d\Gamma_1 d\Gamma_2 d\Gamma_3 W_{ks} \exp \left( -\beta \cdot (k_1 + k_2) \right) \times \left[ \frac{1}{4} \partial_\nu \beta_\mu \left( k^{(\mu}_1 \Delta_2^{\nu)} + k^{(\mu}_2 \Delta_1^{\nu)} - k^{(\mu}_1 \Delta_1^{\nu)} - k^{(\mu}_2 \Delta_2^{\nu)} \right) 
right. 
\left. - \frac{1}{2} \omega_{\mu\nu} + \frac{1}{4} \partial_{\nu \beta_\mu} \right) \left( s^\mu_1 + s^\mu_2 - s^{\nu\mu} - s^{\nu\mu} \right) - \frac{1}{2} \partial_\alpha \omega_{\mu\nu} \left( s^\nu_1 + s^\nu_2 - s^{\alpha\nu} + s^{\alpha\nu} \right) 
\left. + \frac{1}{16} \partial_{\nu \beta_\mu} \left( s^{\mu\alpha}_1 k^{(\beta}_2 \nu^{\alpha)} - s^{\mu\alpha}_2 k^{(\beta}_1 \nu^{\alpha)} - s^{\mu\alpha}_1 k^{(\beta}_2 \nu^{\alpha)} + s^{\mu\alpha}_2 k^{(\beta}_1 \nu^{\alpha)} \right) + \frac{1}{8} \partial_{\nu \beta_\mu} \left( s^{\mu\alpha}_2 k^{(\beta}_2 \nu^{\alpha)} + s^{\mu\alpha}_2 k^{(\beta}_1 \nu^{\alpha)} \right) 
\left. + \frac{1}{8} \partial_{\nu \beta_\mu} \left( s^{\mu\alpha}_2 k^{(\beta}_2 \nu^{\alpha)} + s^{\mu\alpha}_2 k^{(\beta}_1 \nu^{\alpha)} \right) \right], \quad (C4) 
\mathcal{J}_2 = - \int d\Gamma_2 dS_1 (k) dS_2 (k_2) W_s \exp \left( -\beta \cdot (k + k_2) \right) \times \left[ \frac{1}{4} \partial_\nu \beta_\mu \left( k^{(\mu}_1 \Delta_2^{\nu)} + k^{(\mu}_2 \Delta_1^{\nu)} - k^{(\mu}_1 \Delta_1^{\nu)} - k^{(\mu}_2 \Delta_2^{\nu)} \right) 
\right. 
\left. - \frac{1}{2} \omega_{\mu\nu} + \frac{1}{4} \partial_{\nu \beta_\mu} \right) \left( s^\mu_1 + s^\mu_2 - s^{\nu\mu} - s^{\nu\mu} \right) - \frac{1}{2} \partial_\alpha \omega_{\mu\nu} \left( s^\nu_1 + s^\nu_2 - s^{\alpha\nu} + s^{\alpha\nu} \right) 
\left. + \frac{1}{8} \partial_{\nu \beta_\mu} \left( s^{\mu\alpha}_1 k^{(\beta}_2 \nu^{\alpha)} - s^{\mu\alpha}_2 k^{(\beta}_1 \nu^{\alpha)} - s^{\mu\alpha}_1 k^{(\beta}_2 \nu^{\alpha)} + s^{\mu\alpha}_2 k^{(\beta}_1 \nu^{\alpha)} \right) + \frac{1}{8} \partial_{\nu \beta_\mu} \left( s^{\mu\alpha}_2 k^{(\beta}_2 \nu^{\alpha)} + s^{\mu\alpha}_2 k^{(\beta}_1 \nu^{\alpha)} \right) 
\left. + \frac{1}{8} \partial_{\nu \beta_\mu} \left( s^{\mu\alpha}_2 k^{(\beta}_2 \nu^{\alpha)} + s^{\mu\alpha}_2 k^{(\beta}_1 \nu^{\alpha)} \right) \right]. \quad (C5) 

The above collision term is expanded up to the order \( \Delta \) and vanishes if the conditions (94)–(95) are satisfied. The term corresponding to \( 1/8 \omega_{\alpha\beta} \partial_{(\nu \beta_\mu)} \) is of the order \( O(\Delta^2) \) and therefore can be neglected due to small polarization limit considered herein. The term corresponding to \( 1/8 \omega_{\alpha\beta} \partial_{(\nu \beta_\mu)} \) automatically vanishes because of the Killing equation (94). The conditions (94)–(95), obtained by using the equilibrium distribution function (90) in Eq. (91) prove that, even if we keep the spin polarization tensor only up to leading order, we will obtain the global equilibrium.

[1] W. Florkowski, *Phenomenology of ultra-relativistic heavy-ion collisions*, World Scientific, Singapore, 2010. [https://cds.cern.ch/record/1321594](https://cds.cern.ch/record/1321594).

[2] C. Gale, S. Jeon, and B. Schenke, “Hydrodynamic Modeling of Heavy-Ion Collisions,” *Int. J. Mod. Phys. A* 28 (2013) 1340011, arXiv:1301.5893 [nucl-th].

[3] A. Jaiswal and V. Roy, “Relativistic hydrodynamics in heavy-ion collisions: general aspects and recent developments,” *Adv. High Energy Phys.* 2016 (2016) 9623034, arXiv:1605.08694 [nucl-th].

[4] P. Romatschke and U. Romatschke, *Relativistic Fluid Dynamics In and Out of Equilibrium*, Cambridge Monographs on Mathematical Physics. Cambridge University Press, 5, 2019. arXiv:1712.05815 [nucl-th].

[5] W. Florkowski, M. P. Heller, and M. Spalinski, “New theories of relativistic hydrodynamics in the LHC era,” *Rept. Prog. Phys.* 81 no. 4, (2018) 046001, arXiv:1707.02282 [hep-ph].

[6] S. Schlichting and D. Teaney, “The First fm/c of Heavy-Ion Collisions,” *Ann. Rev. Nucl. Part. Sci.* 69 (2019) 447–476, arXiv:1908.02113 [nucl-th].

[7] Y. Hidaka, S. Pu, Q. Wang, and D.-L. Yang, “Foundations and Applications of Quantum Kinetic Theory,” arXiv:2201.07644 [hep-ph].

[8] STAR Collaboration, L. Adamczyk et al., “Global \( \Lambda \) hyperon polarization in nuclear collisions: evidence for the most vortical fluid,” *Nature* 548 (2017) 62–65, arXiv:1701.06657 [nucl-ex].

[9] STAR Collaboration, J. Adam et al., “Global polarization of \( \Lambda \) hyperons in Au+Au collisions at \( \sqrt{s_{NN}} = 200 \) GeV,” *Phys. Rev. C* 98 (2018) 014910, arXiv:1805.04400 [nucl-ex].

[10] STAR Collaboration, J. Adam et al., “Polarization of \( \Lambda \) (\( \bar{\Lambda} \)) hyperons along the beam direction in Au+Au collisions at \( \sqrt{s_{NN}} = 200 \) GeV,” *Phys. Rev. Lett.* 123 no. 13, (2019) 132301, arXiv:1905.11917 [nucl-ex].

[11] ALICE Collaboration, S. Acharya et al., “Evidence of
Spin-Orbital Angular Momentum Interactions in Relativistic Heavy-Ion Collisions.

Phys. Rev. Lett. 125 no. 1, (2020) 012301, arXiv:1910.14408 [nucl-ex].

[12] ALICE Collaboration, S. Acharya et al., “Global polarization of Λ hyperons in Pb-Pb collisions at √sNN = 2.76 and 5.02 TeV,” Phys. Rev. C 101 no. 4, (2020) 044611, arXiv:1909.01281 [nucl-ex].

[13] F. J. Kornas, “Λ polarization in au + au collisions at √sNN = 2.4 gev measured with hades,” in The XVIII International Conference on Strangeness in Quark Matter (SQM 2019), D. Elia, G. E. Bruno, P. Colangelo, and L. Cosmai, eds., pp. 435–439. Springer International Publishing, Cham, 2020.

[14] STAR Collaboration, M. S. Abdallah et al., “Global Λ-hyperon polarization in Au+Au collisions at √sNN=3 GeV,” Phys. Rev. C 104 no. 6, (2021) L061901, arXiv:2108.00044 [nucl-ex].

[15] ALICE Collaboration, S. Acharya et al., “Polarization of Λ and ¯Λ hyperons along the beam direction in Pb-Pb collisions at √sNN = 5.02 TeV,” arXiv:2107.11183 [nucl-ex].

[16] W. Florkowski, B. Friman, A. Jaiswal, and E. Speranza, “Relativistic fluid dynamics with spin,” Phys. Rev. C97 no. 4, (2018) 041901, arXiv:1705.00587 [nucl-th].

[17] W. Florkowski, B. Friman, A. Jaiswal, R. Ryblewski, and E. Speranza, “Spin-dependent distribution functions for relativistic hydrodynamics of spin-1/2 particles,” Phys. Rev. D97 no. 11, (2018) 116017, arXiv:1712.07676 [nucl-th].

[18] W. Florkowski, A. Kumar, and R. Ryblewski, “Thermodynamic versus kinetic approach to polarization-vorticity coupling,” Phys. Rev. C98 (2018) 044906, arXiv:1806.02616 [hep-ph].

[19] W. Florkowski, A. Kumar, R. Ryblewski, and R. Singh, “Spin polarization evolution in a boost invariant hydrodynamical background,” Phys. Rev. C99 no. 4, (2019) 044910, arXiv:1901.09655 [hep-ph].

[20] R. Singh, G. Sophys, and R. Ryblewski, “Spin polarization dynamics in the Gubser-expanding background,” Phys. Rev. D 103 no. 7, (2021) 074024, arXiv:2011.14907 [hep-ph].

[21] S. Bhadury, W. Florkowski, A. Jaiswal, A. Kumar, and R. Ryblewski, “Relativistic dissipative spin dynamics in the relaxation time approximation,” Phys. Lett. B814 (2021) 136096, arXiv:2002.03937 [hep-ph].

[22] S. Bhadury, W. Florkowski, A. Jaiswal, A. Kumar, and R. Ryblewski, “Dissipative Spin Dynamics in Relativistic Matter,” Phys. Rev. D103 no. 1, (2021) 014030, arXiv:2008.10976 [nucl-th].

[23] R. Singh, M. Shokri, and R. Ryblewski, “Spin polarization dynamics in the Bjorken-expanding resistive MHD background,” Phys. Rev. D 103 no. 9, (2021) 094034, arXiv:2103.02592 [hep-ph].

[24] W. Florkowski, R. Ryblewski, R. Singh, and G. Sophys, “Spin polarization dynamics in the non-boost-invariant background,” Phys. Rev. D 105 no. 5, (2022) 054007, arXiv:2112.01856 [hep-ph].

[25] V. E. Ambrus, R. Ryblewski, and R. Singh, “Spin waves in spin hydrodynamics,” arXiv:2202.03952 [hep-ph].

[26] R. Singh, M. Shokri, and S. M. A. T. Mehr, “Relativistic magnetohydrodynamics with spin,” arXiv:2202.11504 [hep-ph].

[27] W. Florkowski, R. Ryblewski, and A. Kumar, “Relativistic hydrodynamics for spin-polarized fluids,” Prog. Part. Nucl. Phys. 108 (2019) 103709, arXiv:1811.04409 [nucl-th].

[28] E. Speranza and N. Weickgenannt, “Spin tensor and pseudo-gauges: from nuclear collisions to gravitational physics,” Eur. Phys. J. A 57 no. 5, (2021) 155, arXiv:2007.00138 [nucl-th].

[29] F. Becattini and L. Tinti, “The Ideal relativistic rotating gas as a perfect fluid with spin,” Annals Phys. 325 (2010) 1566–1594, arXiv:0911.0864 [gr-qc].

[30] D. Montenegro and G. Torrieri, “Causality and dissipation in relativistic polarizable fluids,” Phys. Rev. D 100 no. 5, (2019) 056011, arXiv:1907.02796 [hep-th].

[31] D. Montenegro and G. Torrieri, “Linear response theory and effective action of relativistic hydrodynamics with spin,” Phys. Rev. D 102 no. 3, (2020) 036007, arXiv:2004.10185 [hep-th].

[32] A. D. Gallegos, U. Gürsoy, and A. Yarom, “Hydrodynamics of spin currents,” SciPost Phys. 11 (2021) 041, arXiv:2101.04759 [hep-th].

[33] K. Hattori, M. Hongo, X.-G. Huang, M. Matsuo, and H. Taya, “Fate of spin polarization in a relativistic fluid: An entropy-current analysis,” Phys. Lett. B795 (2019) 100–106, arXiv:1901.06615 [hep-th].

[34] K. Fukushima and S. Pu, “Spin hydrodynamics and symmetric energy-momentum tensors – A current induced by the spin vorticity –,” Phys. Lett. B 817 (2021) 136346, arXiv:2010.01608 [hep-th].

[35] S. Li, M. A. Stephanov, and H.-U. Yee, “Nondissipative Second-Order Transport, Spin, and Pseudogauge Transformations in Hydrodynamics,” Phys. Rev. Lett. 127 no. 8, (2021) 082302, arXiv:2011.12318 [hep-th].

[36] D. She, A. Huang, D. Hou, and J. Liao, “Relativistic Viscous Hydrodynamics with Angular Momentum,” arXiv:2105.04060 [nucl-th].

[37] A. Daher, A. Das, W. Florkowski, and R. Ryblewski, “Equivalence of canonical and phenomenological formulations of spin hydrodynamics,” arXiv:2202.12609 [nucl-th].

[38] J. Hu, “Kubo formulae for first-order spin hydrodynamics,” Phys. Rev. D 103 no. 11, (2021) 116015, arXiv:2101.08440 [hep-ph].

[39] Y. Hidaka and D.-L. Yang, “Nonequilibrium chiral magnetic/vortical effects in viscous fluids,” Phys. Rev. D 98 no. 1, (2018) 016012, arXiv:1801.08253 [hep-th].

[40] D.-L. Yang, K. Hattori, and Y. Hidaka, “Effective quantum kinetic theory for spin transport of fermions with collisional effects,” JHEP 07 (2020) 070.
F. Becattini, I. Karpenko, M. Lisa, I. Upsal, and S. Voloshin, “Global hyperon polarization at local thermodynamic equilibrium with vorticity, magnetic field and feed-down,” *Phys. Rev. C95* no. 5, (2017) 054902, arXiv:1610.02506 [nucl-th].

L.-G. Pang, H. Petersen, Q. Wang, and X.-N. Wang, “Vortical Fluid and A Spin Correlations in High-Energy Heavy-Ion Collisions,” *Phys. Rev. Lett. 117* no. 19, (2016) 192301, arXiv:1605.04024 [hep-ph].

Y. Xie, D. Wang, and L. P. Csernai, “Global Lambda polarization in high energy collisions,” *Phys. Rev. C95* no. 3, (2017) 031901, arXiv:1703.03770 [nucl-th].

Y. Sun and C. M. Ko, “A hyperpolarization in relativistic heavy ion collisions from a chiral kinetic approach,” *Phys. Rev. C96* no. 2, (2017) 024906, arXiv:1706.09467 [nucl-th].

H. Li, L.-G. Pang, and X.-L. Wang, Qun wand Xia, “Global Λ polarization in heavy-ion collisions from a transport model,” *Phys. Rev. C96* no. 5, (2017) 054908, arXiv:1704.01507 [nucl-th].

D.-X. Wei, W.-T. Deng, and X.-G. Huang, “Thermal vorticity and spin polarization in heavy-ion collisions,” *Phys. Rev. C99* no. 1, (2019) 014905, arXiv:1810.00151 [nucl-th].

I. Karpenko and F. Becattini, “Study of Λ polarization in relativistic nuclear collisions at √sNN = 7.7–200 GeV,” *Eur. Phys. J. C77* no. 4, (2017) 213, arXiv:1610.04717 [nucl-th].

F. Becattini and I. Karpenko, “Collective Longitudinal Polarization in Relativistic Heavy-Ion Collisions at Very High Energy,” *Phys. Rev. Lett. 120* no. 1, (2018) 012302, arXiv:1707.07984 [nucl-th].

X.-L. Xia, H. Li, Z.-B. Tang, and Q. Wang, “Probing vorticity structure in heavy-ion collisions by local Λ polarization,” *Phys. Rev. C98* (2018) 024905, arXiv:1803.00867 [nucl-th].

Y. Sun and C. M. Ko, “Azimuthal angle dependence of the longitudinal spin polarization in relativistic heavy ion collisions,” *Phys. Rev. C99* no. 1, (2019) 011903, arXiv:1810.10359 [nucl-th].

W. Florkowski, A. Kumar, R. Ryblewski, and A. Mazeliauskas, “Longitudinal spin polarization in a thermal model,” *Phys. Rev. C100* no. 5, (2019) 054907, arXiv:1904.00002 [nucl-th].

F. Becattini, M. Buzzegoli, and A. Palermo, “Spin-thermal shear coupling in a relativistic fluid,” *Phys. Lett. B820* (2021) 136519, arXiv:2103.10917 [nucl-th].

F. Becattini, M. Buzzegoli, G. Inghirami, I. Karpenko, and A. Palermo, “Local Polarization and Isothermal Local Equilibrium in Relativistic Heavy Ion Collisions,” *Phys. Rev. Lett. 127* no. 27, (2021) 272302, arXiv:2103.14621 [nucl-th].

B. Fu, S. Y. F. Liu, L. Pang, H. Song, and Y. Yin, “Shear-Induced Spin Polarization in Heavy-Ion Collisions,” *Phys. Rev. Lett. 127* no. 14, (2021) 142301, arXiv:2103.10403 [hep-ph].

W. Florkowski, A. Kumar, A. Mazeliauskas, and Y. Xie, D. Wang, and L. P. Csernai, “Global Lambda polarization in high energy collisions,” *Phys. Rev. C95* no. 3, (2017) 031901, arXiv:1703.03770 [nucl-th].

J.-H. Gao, Z.-T. Liang, Q. Wang, and X.-N. Wang, “From chiral kinetic theory to relativistic viscous spin hydrodynamics,” *Phys. Rev. C103* no. 4, (2021) 044906, arXiv:2008.09618 [nucl-th].

A. D. Gallegos and U. Gursoy, “Holographic spin fluids and Lovelock Chern-Simons gravity,” *JHEP 11* (2020) 151, arXiv:2004.05148 [hep-th].

M. Garbiso and M. Kaminski, “Hydrodynamics of simply spinning black holes & hydrodynamics for spinning quantum fluids,” *JHEP 12* (2020) 112, arXiv:2007.04345 [hep-th].

A. D. Gallegos, U. Gursoy, and A. Yarom, “Hydrodynamics, spin currents and torsion,” arXiv:2203.05044 [hep-th].

J.-H. Gao, Z.-T. Liang, Q. Wang, and X.-N. Wang, “Global polarization effect and spin-orbit coupling in strong interaction,” *Lect. Notes Phys. 987* (2021) 195–246, arXiv:2009.04803 [nucl-th].

F. Becattini and M. A. Lisa, “Polarization and Vorticity in the Quark–Gluon Plasma,” *Ann. Rev. Nucl. Part. Sci. 70* (2020) 395–423, arXiv:2003.03640 [nucl-ex].

J. L. Francesco Becattini and M. Lisa, eds., *Strongly Interacting Matter under Rotation*. Lecture Notes in Physics. Springer, 2021.

F. Becattini, V. Chandra, L. Del Zanna, and E. Grossi, “Relativistic distribution function for particles with spin at local thermodynamical equilibrium,” *Annals Phys. 338* (2013) 32–49, arXiv:1303.3431 [nucl-th].

F. Becattini, L. Csernai, and D. J. Wang, “A polarization in peripheral heavy ion collisions,” *Phys. Rev. C88* no. 3, (2013) 034905, arXiv:1304.4427 [nucl-th]. [Erratum: Phys. Rev.C93,no.6,069901(2016)].
13

R. Ryblewski, “Effect of thermal shear on longitudinal spin polarization in a thermal model,”
arXiv:2112.02799 [hep-ph].

[72] F. Becattini, W. Florkowski, and E. Speranza, “Spin tensor and its role in non-equilibrium thermodynamics,” Phys. Lett. B789 (2019) 419–425, arXiv:1807.10994 [hep-th].

[73] S. De Groot, W. Van Leeuwen, and C. Van Woert, Relativistic Kinetic Theory. Principles and Applications. North Holland, 1, 1980.

[74] R. Hakim, Introduction to relativistic statistical mechanics: classical and quantum. World scientific, Singapore, 2011. https://cds.cern.ch/record/1379544.

[75] H. T. Elze, M. Gyulassy, and D. Vasak, “Transport Equations for the \{QCD\} Quark Wigner Operator,” Nucl. Phys. B 276 (1986) 706–728.

[76] D. Vasak, M. Gyulassy, and H. T. Elze, “Quantum Transport Theory for Abelian Plasmas,” Annals Phys. 173 (1987) 462–492.

[77] H.-T. Elze and U. W. Heinz, “Quark - Gluon Transport Theory,” Phys. Rept. 183 (1989) 81–135. [117(1989)].

[78] F. Becattini, “Covariant statistical mechanics and the stress-energy tensor,” Phys. Rev. Lett. 108 (2012) 244502, arXiv:1201.5278 [gr-qc].

[79] J.-H. Gao, Z.-T. Liang, and Q. Wang, “Dirac sea and chiral anomaly in the quantum kinetic theory,” Phys. Rev. D 101 no. 9, (2020) 096015, arXiv:1910.11060 [hep-ph].

[80] X.-L. Sheng, Wigner Function for Spin-1/2 Fermions in Electromagnetic Fields. PhD thesis, Frankfurt U., 2019.

[81] R. Ekman, F. A. Asenjo, and J. Zamanian, “Fully relativistic kinetic equation for spin-1/2 particles in the long scale-length approximation,” Phys. Rev. E 96 no. 2, (2017) 023207, arXiv:1702.00722 [physics.plasm-ph].

[82] R. Ekman, H. Al-Naseri, J. Zamanian, and G. Brodin, “Relativistic kinetic theory for spin-1/2 particles: Conservation laws, thermodynamics, and linear waves,” Phys. Rev. E 100 no. 2, (2019) 023201, arXiv:1904.08727 [physics.plasm-ph].

[83] F. Becattini, “Covariant statistical mechanics and the stress-energy tensor,” Phys. Rev. Lett. 108 (2012) 244502, arXiv:1201.5278 [gr-qc].

[84] J.-H. Gao, Z.-T. Liang, and Q. Wang, “Dirac sea and chiral anomaly in the quantum kinetic theory,” Phys. Rev. D 101 no. 9, (2020) 096015, arXiv:1910.11060 [hep-ph].

[85] C. Itzykson and J. B. Zuber, Quantum Field Theory. International Series In Pure and Applied Physics. McGraw-Hill, New York, 1980. http://dx.doi.org/10.1063/1.2916419.

[86] J.-H. Gao, Z.-T. Liang, and Q. Wang, “Quantum kinetic theory for spin-1/2 fermions in Wigner function formalism,” Int. J. Mod. Phys. A 36 no. 01, (2021) 2130001, arXiv:2101.02629 [hep-ph].

[87] S. Li and H.-U. Yee, “Quantum Kinetic Theory of Spin Polarization of Massive Quarks in Perturbative QCD: Leading Log,” Phys. Rev. D 100 no. 5, (2019) 056018, arXiv:1902.06513 [hep-ph].

[88] M. Mathisson, “Neue mechanik materieller systemes,” Acta Phys. Polon. 6 (1937) 163–2900.