Estimation error for occupation time functionals of stationary Markov processes

Randolf Altmeyer, Jakub Chorowski

Institut für Mathematik, Humboldt-Universität zu Berlin, Unter den Linden 6, 10099 Berlin, Germany

Abstract

The approximation of integral functionals with respect to a stationary Markov process by a Riemann-sum estimator is studied. Stationarity and the functional calculus of the infinitesimal generator of the process are used to get a better understanding of the estimation error and to prove a general error bound. The presented approach admits general integrands and gives a unifying explanation for different rates obtained in the literature. Several examples demonstrate how the general bound can be related to well-known function spaces.

Keywords: Markov processes, integral functionals, occupation time, Sobolev spaces, infinitesimal generator, Ornstein-Uhlenbeck

2010 MSC: Primary 62M05; Secondary 60J55, 60J35

1. Introduction

Statistics for continuous-time Markov processes is usually based on the observation of a sample path. Typically, only discrete-time observations are available. An important task is the estimation of integral functionals such as

$$\Gamma_T(f) = \int_0^T f(X_r) \, dr, \quad T \geq 0.$$  

Here, $X = (X_r)_{r \geq 0}$ is an $S$-valued Markov process on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$ for a Polish space $S$ equipped with its Borel-$\sigma$-field and $f : S \to \mathbb{R}$ is a given function such that $\Gamma_T(f)$ is well-defined. Functional operators of this form appear in many problems. For instance, in mathematical finance they are used to model path dependent derivatives (see Hugonnier [19], Chesney et al. [6]). In evolutionary dynamics the value $f(x)$ is often associated with the cost of staying in state $x$ (Pollett [24]). The most important case for applications is $f = 1_A$ for a Borel set $A$. $\Gamma_T(f)$ is then known as the occupation time of $X$ in $A$ and measures the time $X$ spends in $A$ before $T$. For general $f$ and when $X$ is ergodic with invariant measure $\mu$, integral functionals are also important for studying the long term behavior of the process as $T \to \infty$, since $T^{-1} \Gamma_T(f) \to \int f \, d\mu$ by the ergodic theorem. Furthermore, the smoothness properties of $x \mapsto$
\[ \int_0^T f(x+X_r)dr \] play an important role for solving ordinary differential equations, for example in combination with the phenomenon of regularization by noise (Catellier and Gubinelli [4]).

Our goal is to approximate \( \Gamma_T(f) \) given the equidistant observations \( X_{k\Delta_n}, k = 0, \ldots, n-1 \), where \( \Delta_n = T/n \). A natural candidate for this is the Riemann-sum estimator

\[ \hat{\Gamma}_{T,n}(f) = \sum_{k=1}^n f(X(k-1)\Delta_n)\Delta_n. \]

Consistency of this estimator follows from Riemann-approximation already under weak assumptions on \( f \) and \( X \). The rate of convergence, however, depends on \( f \), \( X \), \( T \) and \( \Delta_n \). For deriving useful finite sample bounds it is crucial to make these dependencies explicit. The Riemann-sum estimator has appeared in many places in the literature, mostly for estimating the occupation time (Chorowski [8], Gobet and Matulewicz [16]), or as a proxy for approximating the local time of \( X \) in the diffusion case, such as in Hoffmann [18]. For general \( f \) see also Dion and Genon-Catalot [10]. Error bounds are usually derived ad-hoc, leading to suboptimal bounds or without explicit constants for the dependence on parameters.

It is clear that the approximation error depends on the smoothness of \( f \). Note, however, when measuring the approximation error in the \( L^2(\mathbb{P}) \) sense, it is not possible, in general, to have a faster rate of convergence than \( \sqrt{T\Delta_n} \), even with smooth \( f \). Indeed, for \( X \) being a Brownian motion and \( f \) the identity, we have \( \|\Gamma_T(f) - \hat{\Gamma}_{T,n}(f)\|_{L^2(\mathbb{P})} = \sqrt{T\Delta_n}/\sqrt{3} \). A systematic study of this approximation problem has started only recently. For one-dimensional diffusion processes with smooth coefficients and \( T = 1 \) Ngo and Ogawa [22] obtain the rate \( n^{-3/4} \) for the special case of indicator functions \( f = 1_{[K,\infty)} \), \( K \in \mathbb{R} \). Interestingly, they also provide a lower bound for the rate \( n^{-3/4} \) in the \( L^2(\mathbb{P}) \)-sense. The specific analysis for indicator functions, however, cannot explain which rates we can expect for more general \( f \). For Markov processes in \( \mathbb{R}^d \) with transition kernels satisfying certain heat kernel bounds Ganychenko et al. [14] prove for bounded \( f \) the \( L^p(\mathbb{P}) \)-bound \( \|f\|_{L^p(\mathbb{P})} \|\sqrt{T}\Delta_n \log n\|^{1/2} \leq 2 \). \( \alpha \)-Hölder functions for \( 0 < \alpha \leq 1 \) are studied by Kohatsu-Higa et al. [21], again for one-dimensional smooth diffusions, and by Ganychenko [13], for the same class of Markov processes as for Ganychenko et al. [14]. They essentially obtain the same upper bounds, \( n^{-(1+\alpha)/2} \) for Kohatsu-Higa et al. [21], and \( \|f\|_\alpha \sqrt{T}\Delta_n^{(1+\alpha)/2} \) for Ganychenko [13], both of them losing a log \( n \)-factor when \( \alpha = 1 \). Surprisingly, indicators obey the same bounds as \( 1/2 \)-Hölder functions.

The aim of this paper is twofold. First, we want to study the approximation of \( \Gamma_T(f) \) by \( \hat{\Gamma}_{T,n}(f) \) for more general functions \( f \) on arbitrary state spaces \( S \) and to find a unifying mathematical explanation for the different rates obtained in the literature. Second, we want to develop a better understanding of the key quantities driving the estimation error. For this, we focus on the special but important case of stationary Markov processes, because this allows us to calculate the error explicitly in terms of the associated semigroup. The main insight of our results is that the discretization error depends on the action of
fractional powers of the infinitesimal generator applied to $f$. We demonstrate in several examples how this can be related to more familiar $L^2$-Sobolev norms of $f$. These norms are the key idea to explain both the rates for Hölder and indicator functions by suitable interpolation. The dependence on $T$ in the error is explicit. This allows us, when $X$ is ergodic, to approximate integral functionals with respect to the invariant measure of $X$ under weaker conditions than the ones commonly used in the literature. Our approach is based on the functional calculus of the generator. We therefore consider only stationary Markov processes whose generators are normal operators.

The paper is organized as follows. In Section 2 we state a general upper bound for the $L^2(P)$-approximation error. We apply it to approximate integral functionals with respect to the invariant measure, when $X$ is ergodic. In Section 3 we study several concrete examples of processes and functions. We also discuss the important example of Brownian motion, which is not a stationary Markov process, but which can be approximated by reflected stationary diffusions. Proofs can be found in Section 4. The appendix contains a brief summary of the most important facts about semigroups and the functional calculus for normal operators.

2. A general upper bound

In the following, we assume that $X$ is a continuous-time Markov process on some probability space $(\Omega, \mathcal{F}, P)$ and with Polish state space $S$. For any measure $\mu$ on $S$ we denote by $L^2(\mu) := L^2(S, \mu)$ the space of square integrable functions $f : S \to \mathbb{R}$ with respect to $\mu$ and with norm $\|f\|_\mu = (\int f^2 d\mu)^{1/2}$. $\|\cdot\|_{\infty, \mu}$ denotes the sup-norm in $L^\infty(\mu)$ and $a \lesssim b$ for $a, b \in \mathbb{R}$ means $a \leq Cb$ for some constant $C > 0$. $Z_n = O_P(a_n)$ for a sequence of random variables $(Z_n)_{n \geq 1}$ and real numbers $(a_n)_{n \geq 1}$ means that $a_n^{-1} Z_n$ is tight. For basic concepts of semigroup theory and functional calculus refer to the appendix. Our main assumptions are the following:

**Assumption 2.1.** $X$ is a stationary Markov process with invariant measure $\mu$. The associated semigroup $(P_t)_{t \geq 0}$ is Feller and its infinitesimal generator $L$ is a normal operator on $L^2(\mu)$.

These assumptions are satisfied for many important processes. A leading example is the standard Ornstein-Uhlenbeck process. A process with normal but not necessarily self-adjoint generator will be discussed in Section 3.

Observe that for a stationary Markov process and $f \in L^2(\mu)$ both $\Gamma_T(f)$ and $\Gamma_{T,n}(f)$ are $\mu$-a.s.-well-defined random variables in $L^2(P)$. We consider the spaces $\mathcal{D}^s(L) := \text{dom}(|L|^{s/2}) \subset L^2(\mu)$, $s \geq 0$, defined via the functional calculus of $L$, and with seminorm $\|f\|_{\mathcal{D}^s(L)} := \|\|L|^{s/2}f\|_\mu$. When $X$ is the Ornstein-Uhlenbeck process, then the related spaces $\text{dom}((I - L)^{s/2}) \subset \mathcal{D}^s(L)$ are known as Bessel potential spaces and play an important role in Malliavin calculus (Watanabe [27]). We are now ready to state the general upper bound.
**Theorem 2.2.** Let \( X \) be a Markov process satisfying Assumption \( \text{A.1} \). There exists a universal constant \( C \) such that for all \( 0 \leq s \leq 1 \) and \( f \in D^s(L) \)
\[
\left\| \Gamma_T(f) - \hat{\Gamma}_{T,n}(f) \right\|_{L^2(\mathbb{P})} \leq C \|f\|_{D^s(L)} \sqrt{T \Delta_n^{1+s}}.
\]

The proof of this theorem is remarkably short. For \( s = 0 \) we have \( D^0(L) = L^2(\mu) \) and the rate is \( \sqrt{T \Delta_n^{1/2}} \) which slightly improves the results of Ganychenko et al. \( \text{[14]} \), removing an additional \( \sqrt{\log n} \). Since \( D^s(L) \subset D^1(L) \) for \( s \geq 1 \), the rate is never better than \( \sqrt{T \Delta_n} \). For \( 0 < s < 1 \) the bound interpolates between the two extreme cases. A deeper understanding of the spaces \( D^s(L) \) requires more explicit knowledge about the generator. For example, if \( L \) is self-adjoint, then \( |L| = -L \) and thus \( \|f\|_{D^1(L)}^2 = \|(-L)^{1/2}f\|_{\mu}^2 = \langle -Lf, f \rangle_{\mu} \). This is the Dirichlet form associated with \( L \) and \( \mu \). It is typically easier to analyze than studying \( D^1(L) \) directly in terms of the functional calculus. An important example are diffusions on \( \mathbb{R}^d \) such that for sufficiently smooth functions \( f \) the Dirichlet form is bounded by \( \|\nabla f\|_{\mu}^2 \), the \( L^2(\mu) \)-norm of the gradient of \( f \). This immediately leads to upper bounds for H"older and indicator functions. Up to some additional conditions, we will show that \( \alpha \)-H"older functions lie in \( D^s(L) \) and indicator functions of certain cylinder sets of \( \mathbb{R}^d \) lie in \( D^{1/2}(L) \). This gives a unifying explanation for the different rates (see also Remark \( \text{3.8} \)). These and other examples will be discussed in Section \( \text{4} \). A simple corollary shows that the result of Theorem \( \text{2.2} \) remains valid if we relax the assumption of starting from the invariant distribution.

**Corollary 2.3.** Let \( X \) be a Markov process satisfying Assumption \( \text{A.1} \) except that \( X_0 \overset{\mathcal{d}}{\sim} \eta \) such that \( \eta \ll \mu \) with density \( d\eta/d\mu \). Then there exists a universal constant \( C \) such that for all \( 0 \leq s \leq 1 \) and \( f \in D^s(L) \)
\[
\left\| \Gamma_T(f) - \hat{\Gamma}_{T,n}(f) \right\|_{L^2(\mathbb{P})} \leq C \cdot \frac{\|d\eta\|_{\infty,\mu}^{1/2}}{\|d\mu\|_{\infty,\mu}} \cdot \|f\|_{D^s(L)} \sqrt{T \Delta_n^{1+s}}.
\]

Instead of \( \Gamma_T(f) \) a different target functional is often \( \int f d\mu \). It is well-known that \( T^{-1} \Gamma_T(f) \) is \( \sqrt{T} \)-consistent for \( \int f d\mu \), i.e. \( T^{-1} \Gamma_T(f) - \int f d\mu = O_{\mathbb{P}}(T^{-1/2}) \), when \( L \) is self-adjoint and \( f \in \text{dom}(L^{-1/2}) \) (see e.g. Kipnis and Varadhan \( \text{[20]} \)). By Theorem \( \text{2.2} \) we can now extend this to the estimator \( T^{-1} \hat{\Gamma}_{T,n}(f) \) and more general \( L^2(\mu) \)-functions.

**Theorem 2.4.** Let \( X \) be a Markov process satisfying Assumption \( \text{A.1} \). There exists a universal constant \( C \) such that for all \( f \in L^2(\mu) \) with \( f_0 \in \text{dom}(|L|^{-1/2}) \), \( f_0 = f - \int f d\mu \),
\[
\left\| T^{-1} \hat{\Gamma}_{T,n}(f) - \int_S f(x) d\mu(x) \right\|_{L^2(\mathbb{P})} \leq C \cdot \frac{\sqrt{T}}{\|L|_{\mu}^{1/2}} \left( \|f\|_{\mu|L|_{\mu}^{1/2}} + \|L|^{-1/2} f_0 \|_{\mu} \right).
\]

As an example for \( \text{dom}(|L|^{-1/2}) \) being non-trivial, assume that 0 is a simple eigenvalue of \( L \) and that \( L \) has a spectral gap, i.e. \( s_0 > 0 \), where \( s_0 = \sup \{ r > 0 \...
$B(0, r) \cap \sigma(L) = \{0\}$ and $B(0, r) = \{z \in \mathbb{C} : |z| \leq r\}$. In that case $X$ is ergodic and it can be shown that $f_0 \in \text{dom}(|L|^{-1/2})$ is satisfied whenever $f$ is non-constant (Bakry et al. [2]). Furthermore, the upper bound of the theorem simplifies, since

$$
\left\| |L|^{-1/2} f_0 \right\| \leq s_0^{-1/2} \| f_0 \|_\mu \leq s_0^{-1/2} \| f \|_\mu.
$$

A concrete example of a process with spectral gap is the Ornstein-Uhlenbeck process (Chapter 4 of Bakry et al. [2]). In general, Theorem 2.4 shows that in order to achieve the rate $\sqrt{T}$ as $n, T \to \infty$ there is essentially no gain in the high-frequency case, i.e. $\Delta_n \to 0$, compared to the low-frequency case with $\Delta_n$ fixed. The error bound improves on the commonly used condition in the literature that $T \Delta_n \ll 1$ to achieve $\sqrt{T}$-consistency (see e.g. Section 5 of Dion and Genon-Catalot [10]). The theorem also shows that the $T \to \infty$ case is controlled by negative powers of the generator, while the fixed $T$ case is controlled by positive powers of $|L|$ according to Theorem 2.2.

3. Examples

In this section, we apply the general bound from Theorem 2.2 to several important examples. Our goal is not to give an exhaustive list, but rather to demonstrate the theory in straightforward cases. We first study Markov jump processes, i.e. continuous time Markov processes with countable state spaces. We then consider a special class of diffusion processes for which the spaces $D^s(L)$ can be described easily via the Dirichlet form $\langle -Lf, f \rangle_\mu$. After that, we show for the one-dimensional Brownian motion how the assumption of stationarity can be removed. Finally, we discuss two infinite dimensional diffusions, where the second one is an example of a process whose infinitesimal generator is only a normal operator.

3.1. Markov-jump processes

Consider a continuous time Markov process $(X_t)_{t \geq 0}$ on a countable state space $S$. Such a process can always be realized as $X_t = Y_{N_t}$ for a Markov chain $(Y_s)_{s \in S}$ starting in some initial distribution $\mu$ with transition probabilities $(P_{xy})_{x,y \in S}$ and an independent Poisson process $(N_t)_{t \geq 0}$ with intensity $0 < \lambda < \infty$ (Chapter 4.2 of Ethier and Kurtz [12]). Observing a path of the process $X$ at discrete times $0, \Delta_n, 2\Delta_n, ..., (n-1)\Delta_n$, we can identify the jump times with $\Delta_n$ precision. Hence, if the function $f$ is bounded, then every jump contributes at most $2 \| f \|_\infty$ to the estimation error $\| \Gamma_T(f) - \hat{\Gamma}_{T,n}(f) \|$ which yields the bound

$$
\| \Gamma_T(f) - \hat{\Gamma}_{T,n}(f) \|_{L^2(\mathbb{P})} \leq 2 \| f \|_\infty \mathbb{E} \left[ N_T^2 \right]^{1/2} \Delta_n = 2 \| f \|_\infty \left( \lambda T + (\lambda T)^2 \right)^{1/2} \Delta_n.
$$

This already provides the optimal rate $\Delta_n$ but requires the function $f$ to be bounded. Moreover, the error grows linearly in $T$ as opposed to $\sqrt{T}$ as in Theorem 2.2. We can improve on this assuming in addition that $X$ is stationary with
invariant measure \( \mu \) and, for instance, reversible, i.e. \( P^T = P \). The infinitesimal generator is \( L = \lambda (P - I) \) (Ether and Kurtz [12]) which is a bounded, non-negative self-adjoint operator. Therefore \( \| f \|_{\mathcal{D}^s(L)} \leq \| (-L)^{1/2} \| \| f \|_{\mu} \) with operator norm \( \| (-L)^{1/2} \| \). It follows that \( \mathcal{D}^1(L) = L^2(\mu) \) and we obtain by Theorem 2.2

\[
\left\| \Gamma_T(f) - \hat{\Gamma}_{T,n}(f) \right\|_{L^2(\mathbb{P})} \leq C \sqrt{T} \| f \|_{\mu} \sqrt{T} \Delta_n.
\]

Note that the results of Ganychenko et al. [14], Ganychenko [13] do not apply here, because the state space is countable and therefore heat kernel bounds are not available.

3.2. Diffusions with generator in divergence form

Let \((X_t)_{t \geq 0}\) be a stationary diffusion with values in some closed subset \( U \subseteq \mathbb{S} := \mathbb{R}^d \) with invariant measure \( \mu \) that has support in \( U \). In case \( U \subseteq \mathbb{R}^d \) we think of \( \mu \) as a measure on \( \mathbb{R}^d \) and embed the domain of the infinitesimal generator \( \text{dom}(L) \subseteq L^2(U, \mu) \) canonically into \( L^2(\mathbb{R}^d, \mu) \) by letting \( Lf := L\hat{f} \) whenever \( f|_U = \hat{f} \) for \( f \in L^2(\mathbb{R}^d, \mu) \), \( \hat{f} \in L^2(U, \mu) \). We assume that \( L \) is an elliptic operator in divergence form (c.f. Bass [3, Chapter VII]) that satisfies

\[
\langle -Lf, g \rangle_{\mu} = \int_{\mathbb{R}^d} \langle A(x) \nabla f(x), \nabla g(x) \rangle_{\mathbb{R}^d} d\mu(x), \quad f, g \in \text{dom}(L) \cap C^2(\mathbb{R}^d), \tag{1}
\]

for a measurable function \( x \mapsto A(x) \in \mathbb{R}^{d \times d} \) such that \( A(x) \) is symmetric, positive definite for all \( x \in \mathbb{R}^d \) and such that \( \| A \|_{\infty, \mu} \) is finite, where \( \| \cdot \| \) is any matrix norm. Observe that the right hand side is also well-defined for \( L^2(\mathbb{R}^d, \mu) \)-integrable functions \( f, g \in C^1(\mathbb{R}^d) \). An operator \( L \) satisfying (1) is self-adjoint on \( \text{dom}(L) \cap C^2(\mathbb{R}^d) \) such that \( |L|^{1/2} = (-L)^{1/2} \). The starting point of our discussion is the observation that

\[
\| f \|_{\mathcal{D}^s(L)} = \| (-L)^{s/2}f \|_{\mu}^2 \leq \| (I - L)^{s/2}f \|_{\mu}^2 = \| (I - Lf, f) \|_{\mu} \leq \| f \|_{\mu}^2 + \| A \|_{\infty, \mu} \| \nabla f \|_{\mu}^2 \leq \max(1, \| A \|_{\infty, \mu}) \| f \|_{H^s(\mu)}^2, \tag{2}
\]

for \( f \in \text{dom}(L) \cap C^2(\mathbb{R}^d) \subseteq \text{dom}(L) \subset \text{dom}((-L)^{s/2}) = \mathcal{D}^s(L) \) and \( 0 \leq s \leq 1 \). The norm

\[
\| f \|_{H^s(\mu)} = \| f \|_{\mu} + \| \nabla f \|_{\mu}
\]

is the \( \mu \)-weighted Sobolev norm. Combining this with Theorem 2.2 yields

\[
\left\| \Gamma_T(f) - \hat{\Gamma}_{T,n}(f) \right\|_{L^2(\mathbb{P})} \leq \begin{cases} 
C \max \left( 1, \| A \|_{\infty, \mu}^{1/2} \right) \| f \|_{H^1(\mu)} \sqrt{T} \Delta_n, & \text{if } f \in \text{dom}(L) \cap C^2(\mathbb{R}^d), \\
C \| f \|_{\mu} \sqrt{T} \Delta_n^{1/2}, & \text{if } f \in L^2(\mu).
\end{cases} \tag{3}
\]
By interpolation between the two cases $f \in L^2(\mu)$ and $f \in \text{dom}(L) \cap C^2(\mathbb{R}^d)$ we will study Hölder and indicator functions, recovering previously obtained results. We will further give a unifying view on these results by using Sobolev spaces instead. Before doing this let us discuss some important examples where (1) holds.

**Example 3.1** (Ornstein-Uhlenbeck process). Assume that $(X_t)_{t \geq 0}$ satisfies the stochastic differential equation

$$dX_t = -X_t \, dt + \sqrt{2} \, dW_t$$

in $\mathbb{R}^d$ where $(W_t)_{t \geq 0}$ is a $d$-dimensional Brownian motion. If $X_0 \sim \mu$, where $\mu$ has Lebesgue density $d\mu(x)/dx = (2\pi)^{-d/2} \exp(-|x|^2/2)$, then $X$ is stationary with invariant measure $\mu$. The infinitesimal generator $L$ satisfies

$$Lf(x) = -\langle x, \nabla f(x) \rangle_{\mathbb{R}^d} + \Delta f(x), \quad x \in \mathbb{R}^d,$$

with $f \in \text{dom}(L) = H^2(\mu)$, the $\mu$-weighted Sobolev space of twice weak differentiable functions with all partial derivatives up to order two belonging to $L^2(\mu)$ (Chojnowska-Michalik and Goldys [7]). Using integration by parts it follows (c.f. Pavliotis [23], Section 4.4) that

$$\langle -Lf, g \rangle_{\mu} = \int_{\mathbb{R}^d} \langle \nabla f(x), \nabla g(x) \rangle_{\mathbb{R}^d} \, d\mu(x), \quad f, g \in C^2(\mathbb{R}^d),$$

Hence $L$ is a self-adjoint operator of the form (1) with $A_{jk} = 1$ (j = k) for all $1 \leq j, k \leq d$. This example can be generalized considerably (see Chojnowska-Michalik and Goldys [7] and Subsection 3.4 below).

**Example 3.2** (Scalar diffusion with possibly attracting boundaries). Fix boundaries $-\infty \leq \lambda < \rho \leq \infty$. Assume that $(X_t)_{t \geq 0}$ is a stationary diffusion process on $[\lambda, \rho]$ solving the one-dimensional stochastic differential equation

$$dX_t = b(X_t) \, dt + \sigma(X_t) \, dW_t,$$

for a continuous drift $b : [\lambda, \rho] \to \mathbb{R}$, strictly positive continuous volatility $\sigma : [\lambda, \rho] \to (0, \infty)$ and a one-dimensional Brownian motion $(W_t)_{t \geq 0}$. Sufficient conditions for the existence of such a process can be found in Hansen et al. [17]. In particular, stationarity is guaranteed if the speed density

$$m(x) = \frac{1}{\sigma^2(x)} \exp \left( \int_{x_0}^{x} \frac{2b(y)}{\sigma^2(y)} \, dy \right), \quad \lambda \leq x_0 \leq \rho, \lambda < x < \rho,$$

is integrable on $[\lambda, \rho]$. The stationary measure then has the density

$$\frac{d\mu(x)}{dx} = C_0 m(x) \mathbf{1}(\lambda < x < \rho),$$

where $C_0$ is a normalization constant.
where $C_0$ is a normalizing constant. The infinitesimal generator $L$ satisfies

$$Lf(x) = b(x)f'(x) + \frac{\sigma^2(x)}{2}f''(x)$$

$$= \frac{1}{2} \left( \frac{d\mu(x)}{dx} \right)^{-1} \left( f'(x)\sigma^2(x)\frac{d\mu(x)}{dx} \right)'$$

$\lambda < x < \rho$ with $f \in \text{dom}(L)$, where

$$\text{dom}(L) = \left\{ f \in L^2([\lambda, \rho], \mu) : f$ and $f'$ are absolutely continuous with $\lim_{x \searrow \lambda} f'(x)m(x)\sigma^2(x) = \lim_{x \nearrow \rho} f'(x)m(x)\sigma^2(x) = 0$ and $Lf \in L^2([\lambda, \rho], \mu) \right\}.$

For details see Section 3.3 of Hansen et al. [17]. Embedding the domain into $L^2(\mu)$ as mentioned before and integrating by parts it follows that

$$\left\langle -Lf, g \right\rangle_{\mu} = \int_{\mathbb{R}} f'(x)g'(x)\sigma^2(x)\frac{d\mu(x)}{dx}, f, g \in \text{dom}(L) \cap C^2(\mathbb{R}),$$

which is of the form (1) with $A = \sigma^2$. For $b(x) = -x$ and $\sigma(x) = \sqrt{2}$, $X$ is just the one-dimensional Ornstein-Uhlenbeck process.

**Example 3.3 (Reflected diffusion).** Fix boundaries $-\infty < \lambda < \rho < \infty$. Assume that $X$ is a one-dimensional reflected diffusion on $[\lambda, \rho]$. By this we mean that $X$ satisfies the Skorokhod type stochastic differential equation

$$dX_r = b(X_r)dr + \sigma(X_r)dW_r + dK_r,$$

for a bounded measurable drift $b : [\lambda, \rho] \to \mathbb{R}$, strictly positive continuous volatility $\sigma : [\lambda, \rho] \to (0, \infty)$, $(W_r)_{r \geq 0}$ is a Brownian motion and $(K_r)_{r \geq 0}$ is an adapted continuous process with finite variation starting from 0 and such that for every $r \geq 0$

$$\int_{0}^{r} 1_{(\lambda, \rho)}(X_s)dK_s = 0.$$

The invariant measure and the generator $L$ are as in the last example. Since $[\lambda, \rho]$ is compact, the domain simplifies to

$$\text{dom}(L) = \left\{ f \in L^2([\lambda, \rho], \mu) : f$ and $f'$ are absolutely continuous with $f'(\lambda) = f'(\rho) = 0$ and $Lf \in L^2([\lambda, \rho], \mu) \right\}.$$

Therefore (11) holds here, as well. For more details see Chorowski [8].
3.2.1. Hölder functions

Consider an $\alpha$-Hölder continuous function $f : \mathbb{R}^d \to \mathbb{R}$, i.e. $f$ has finite Hölder-norm

$$\|f\|_{\alpha} = \sup_{x \neq y \in \mathbb{R}^d} \frac{|f(x) - f(y)|}{|x - y|^\alpha},$$

$0 \leq \alpha \leq 1$, and such that $f \in L^2(\mu)$. We want to derive an upper bound for $\|\Gamma_T(f) - \hat{\Gamma}_{T,n}(f)\|_{L^2(\nu)}$ in terms of $\|f\|_{\alpha}$ using (3). Let $(\varphi_\varepsilon)_{\varepsilon \geq 0}$ be a non-negative smooth kernel, i.e. $\varphi_\varepsilon(x) = \varepsilon^{-1} \varphi(\varepsilon^{-1} x)$, $0 \leq \varphi \in C_c^\infty(\mathbb{R}^d)$, $\text{supp}(\varphi) \subset [-1,1]^d$, $\int_{\mathbb{R}^d} \varphi(x) \, dx = 1$. Then the convolution $f_\varepsilon = f * \varphi_\varepsilon$ lies in $C^\infty(\mathbb{R}^d)$ with bounded derivatives and therefore $f_\varepsilon \in L^2(\mu) \cap C^2(\mathbb{R}^d)$. In order to apply (3) to $f_\varepsilon$ we also need that $f_\varepsilon \in \text{dom}(L)$. From the examples above we see that the only issue might be possible boundary conditions for functions in $\text{dom}(L)$. In order to remove these conditions, we assume the following:

**Assumption 3.4.** $\text{dom}(L) \cap C^2(\mathbb{R}^d)$ is dense in $L^2(\mu) \cap C^1(\mathbb{R}^d)$ with respect to $\|\cdot\|_{H^1(\mu)}$.

This assumption is relatively weak. It is satisfied in all examples above. In particular, if functions $f \in \text{dom}(L)$ do not have to satisfy any boundary conditions, then $L^2(\mu) \cap C^2(\mathbb{R}^d) \subset \text{dom}(L) \cap C^2(\mathbb{R}^d)$, as is the case for the Ornstein-Uhlenbeck process. By approximation we can thus extend (3) to

$$\|\Gamma_T(f) - \hat{\Gamma}_{T,n}(f)\|_{L^2(\nu)} \leq \begin{cases} C \max \left(1, \|A\|_{L^\infty(\mu)}^{1/2} \right) \|f\|_{H^1(\mu)} \sqrt{T} \triangle_n, & f \in L^2(\mu) \cap C^1(\mathbb{R}^d), \\ C \|f\|_\mu \sqrt{T} \triangle_n^{1/2}, & f \in L^2(\mu). \end{cases}$$

(9)

We will extend this to Sobolev functions in Subsection 3.2.2. It follows that $f_\varepsilon \in L^2(\mu) \cap C^1(\mathbb{R}^d)$. Using $\int \varphi(x) \, dx = 1$ and $\int \nabla \varphi(x) \, dx = 0$, we have

$$\|f - f_\varepsilon\|_\mu^2 = \int \left| \int (f(x) - f(x + \varepsilon y)) \varphi(y) \, dy \right|^2 \, d\mu(x) \leq \|f\|_\alpha^2 \varepsilon^{2\alpha},$$

$$\|\nabla f_\varepsilon\|_\mu^2 = \int \left( \frac{f(x)}{\varepsilon} \int \nabla \varphi(y) \, dy - \nabla f_\varepsilon(x) \right)^2 \, d\mu(x)$$

$$= \frac{1}{\varepsilon^2} \int \left( \int (f(x) - f(x + \varepsilon y)) \nabla \varphi(y) \, dy \right)^2 \, d\mu(x)$$

$$\lesssim \|f\|_\alpha^2 \varepsilon^{2\alpha - 2}.$$ 

Hence $\|f_\varepsilon\|_{H^1(\mu)} \leq \|f\|_\mu + \|f\|_\alpha \varepsilon^{\alpha - 1}$. Together with $\|f_\varepsilon\|_\mu \leq \|f - f_\varepsilon\|_\mu + \|f\|_\mu$ and (9) this yields

$$\|\Gamma_T(f) - \hat{\Gamma}_{T,n}(f)\|_{L^2(\nu)} \leq \|\Gamma_T(f - f_\varepsilon) - \hat{\Gamma}_{T,n}(f - f_\varepsilon)\|_{L^2(\nu)} + \|\Gamma_T(f_\varepsilon) - \hat{\Gamma}_{T,n}(f_\varepsilon)\|_{L^2(\nu)}$$

$$\lesssim \|f\|_\alpha \sqrt{T} \triangle_n^{1/2} \varepsilon^\alpha + \|f\|_\alpha \sqrt{T} \triangle_n \varepsilon^{\alpha - 1} + \|f\|_\mu \sqrt{T} \triangle_n.$$
Choosing $\varepsilon = \Delta_n^{1/2}$ implies the bound $\|f\|_{\alpha} \sqrt{T \Delta_n^{\alpha}} + \|f\|_\mu \sqrt{T \Delta_n}$. Up to the second term, which is of smaller order as long as $\alpha < 1$, these are the rates obtained by Kohatsu-Higa et al. [21] and Ganychenko [13]. We sum up this discussion in the following theorem.

**Theorem 3.5.** Let $X$ be a stationary diffusion with values in $\mathbb{R}^d$, invariant measure $\mu$ and whose generator $L$ satisfies (1) and Assumption 3.4. Then we have for an $\alpha$-Hölder continuous function $f$, $0 \leq \alpha \leq 1$,

$$\left\| \Gamma_T(f) - \hat{\Gamma}_{T,n}(f) \right\|_{L^2(\mathbb{P})} \lesssim \|f\|_{\alpha} \sqrt{T \Delta_n^{\frac{1+\alpha}{2}}} + \|f\|_\mu \sqrt{T \Delta_n}.$$  

If $L$ satisfies a Poincaré type inequality as in (10) for some $c < \infty$, then the upper bound is just $\|f\|_{\alpha} \sqrt{T \Delta_n^{\frac{1+\alpha}{2}}}$. 

### 3.2.2. Indicator functions

We study now indicator functions $f = 1_{[K, \infty)}$, $K \in \mathbb{R}$, such that $f \in L^2(\mu)$. We will argue again with (9) by approximation. Let $(\varphi_\varepsilon)_{\varepsilon>0}$ be a non-negative smooth kernel as in the previous example, but this time in $\mathbb{R}$. Then $f_\varepsilon = f * \varphi_\varepsilon$ is bounded by 1 and lies in $L^2(\mu) \cap C^2(\mathbb{R})$. Moreover, $f - f_\varepsilon$ has support in $[K - \varepsilon, K + \varepsilon]$ such that

$$\|f - f_\varepsilon\|_\mu^2 \leq \int_{K-\varepsilon}^{K+\varepsilon} d\mu,$$

$$\|f'_\varepsilon\|_\mu^2 = \int_{-\varepsilon}^{\varepsilon} \int \varphi' (y) d\mu (x) = \int_{-\varepsilon}^{\varepsilon} \left( f (x) - f (x + \varepsilon y) \right) \varphi' (y) d\mu (x) \leq \frac{1}{\varepsilon^2} \int_{K-\varepsilon}^{K+\varepsilon} d\mu.$$
As before \( \| f_i \|_\mu \leq \| f - f_i \|_\mu + \| f \|_\mu \). If \( \mu \) is absolutely continuous with respect to the Lebesgue measure with bounded density, then \( \varepsilon^{-1} \int_{K - \varepsilon}^{K + \varepsilon} d\mu \) is bounded and in that case we have from (3) uniform in \( K \)

\[
\left\| \Gamma_T(f) - \hat{\Gamma}_{T,n}(f) \right\|_{L^2(\mathbb{P})} \lesssim \sqrt{T} (\Delta_n \varepsilon)^{1/2} + \sqrt{T} \Delta_n \varepsilon^{-1/2} + \sqrt{T} \Delta_n. \tag{12}
\]

The last term is of lower order compared to the first two. Hence choosing \( \varepsilon = \Delta_n^{1/2} \) yields the rate \( \sqrt{T} \Delta_n^{3/4} \) obtained by Ngo and Ogawa \( [22] \) (see also Kohatsu-Higa et al. \( [21] \)) for one-dimensional diffusions. However, now the rate is uniform in \( K \) with explicit dependence on \( T \). These arguments can easily be extended to general dimensions and we obtain the following theorem.

**Theorem 3.6.** Let \( X \) be a stationary diffusion with values in \( \mathbb{R}^d \), invariant measure \( \mu \) and whose generator \( L \) satisfies (1) and Assumption 3.4. We assume furthermore that \( \mu \) has bounded Lebesgue density. If \( f \) is an indicator function in \( \mathbb{R}^d \) of the form \( \{K_1, L_1 \} \times \cdots \times \{K_d, L_d \}, -\infty < K_j < L_j \leq \infty, 1 \leq j \leq d \), then

\[
\left\| \Gamma_T(f) - \hat{\Gamma}_{T,n}(f) \right\|_{L^2(\mathbb{P})} \lesssim \sqrt{T} \Delta_n^{3/4},
\]

uniformly in \( K_j, L_j \).

The same rate clearly holds up to constants for finite linear combinations of such indicators.

### 3.2.3. Sobolev functions

Our goal is to explain why the bound for indicator functions in Theorem 3.6 can be obtained by evaluating the bound for Hölder functions in Theorem 3.5 at \( \alpha = 1/2 \). The key idea is to study indicators as elements of certain Sobolev spaces and interpolate between the two extreme cases in (9). The closure of \( L^2(\mu) \cap C^1(\mathbb{R}^d) \) with respect to \( \| \cdot \|_{H^1(\mu)} \) yields the space \( H^1(\mu) \), a \( \mu \)-weighted Sobolev space. This is not a Banach space in general (V. Gol’dshtein [20]). In order to avoid this issue we assume that \( \mu \) has a bounded Lebesgue density \( d\mu/dx \). Then \( L^2(\mathbb{R}^d) := L^2(\mathbb{R}^d, \lambda) \subset L^2(\mathbb{R}^d, \mu) \), where \( \lambda \) is the Lebesgue measure, and we have

\[
\| f \|_{H^1(\mu)} \leq \| \frac{d\mu}{dx} \|_{\infty} \| f \|_{H^1}, \quad f \in L^2(\mathbb{R}^d) \cap C^1(\mathbb{R}^d).
\]

Here, \( \| \cdot \|_\infty := \| \cdot \|_{L^\infty} \) is the \( \lambda \)-sup-norm and \( \| f \|_{H^1} := \| f \|_{H^1(\lambda)} \) is the classical Sobolev norm with respect to \( \lambda \). Taking the closure of \( L^2(\mu) \cap C^2(\mathbb{R}^d) \) with respect to \( \| \cdot \|_{H^1} \) leads to the Sobolev space \( H^1(\mathbb{R}^d) \) of weakly differentiable functions. This yields then instead of (9)

\[
\left\| \Gamma_T(f) - \hat{\Gamma}_{T,n}(f) \right\|_{L^2(\mathbb{P})} \leq \begin{cases} C \max \left( 1, \| A \|_{\infty}^{1/2}, \frac{\| f \|_{\infty}}{\| f \|_{H^1}} \right) \left\| \frac{d\mu}{dx} \|_{\infty} \right\|_{\infty} \| f \|_{H^1} \sqrt{T} \Delta_n, \quad f \in H^1(\mathbb{R}^d), \\ C \frac{\| d\mu \|_{\infty}}{\| f \|_{H^1}} \| f \|_{\mu} \sqrt{T} \Delta_n^{1/2}, \quad f \in L^2(\mathbb{R}^d). \end{cases} \tag{13}
\]

11
Consider now the one-dimensional indicator norm \( f \). Then (12) remains true, but we also have
\[
\| y \| \leq C \max \left( 1, \| A \|^{1/2} \mu \right) \| f \|_{H^s(\mathbb{R}^d)} \| f \|_{H^s(\mathbb{R}^d)}^{1/2}. \]

We sum up this discussion in a theorem.

**Theorem 3.7.** Let \( X \) be a stationary diffusion with values in \( \mathbb{R}^d \), invariant measure \( \mu \) and whose generator \( L \) satisfies (1) and Assumption 3.4. We assume furthermore that \( \mu \) has bounded Lebesgue density. Then (11) holds for any \( f \in H^s(\mathbb{R}^d) \), \( 0 \leq s \leq 1 \).

Let us check what this bound implies for Hölder and indicator functions. If \( f \) is an \( \alpha \)-Hölder continuous function with \( 0 < \alpha \leq 1 \) and compact support in \( \mathbb{R}^d \), then it is well-known that \( f \in H^{\alpha-\delta}(\mathbb{R}^d) \) for any small \( \delta > 0 \). (14) thus implies the rate \( \sqrt{T} \triangle_n^{1/2-\delta} \). Using large deviation bounds for \( X \), we can extend this to general \( \alpha \)-Hölder continuous \( f \) losing a \( \log \triangle_n \)-factor in the rate. Consider now the one-dimensional indicator \( f = 1_{(K,L)} \), \( -\infty < K < L < \infty \). Then (12) remains true, but we also have \( f \in H^s(\mathbb{R}) \) for all \( s < 1/2 \) and (14) yields up to constants the bound \( \| f \|_{H^{1/2-s}_s(\mathbb{R})} \sqrt{T} \triangle_n^{3/4-\delta} \) for any small \( \delta > 0 \). This is not uniform in \( K,L \) anymore, but still describes well how the error depends on \( f \). Compared to Theorems 3.5 and 3.6 we see that explicit approximation via (9) yields in general sharper bounds than (14), but the latter one is easier to apply since we only need to bound the Fourier transform of \( f \).

**Remark 3.8.** (i) Assume that \( H^1(\mu) \) is a Banach space. This is true, for instance, in the Examples 3.1 and 3.2 when \( m(x) \) is uniformly bounded from above and below. In that case we can directly interpolate between \( H^1(\mu) \) and \( L^2(\mu) \) with a similar bound as in (14), but with \( \| \|_H^s \) replaced by an appropriate interpolation norm. The results in Theorems 3.5 and 3.6 are explicit cases of this. Up to boundary conditions this implies that \( \alpha \)-Hölder functions lie in \( D^\alpha(L) \), \( 0 \leq \alpha \leq 1 \), and indicator functions \( f = 1_{(K,L)} \) lie in \( D^{1/2-L} \), \( -\infty < K < L < \infty \).

(ii) Depending on the boundary conditions for functions \( f \in \text{dom}(L) \) and if \( \mu \) has bounded Lebesgue-density, in many examples it can be shown that \( H^1(\mathbb{R}^d) \) embeds continuously into \( \text{dom}((I-L)^{1/2}) \subset D^1(L) \). This holds, for instance, for the Ornstein-Uhlenbeck process and for the reflected diffusions in Example 6. Since \( L^2(\mathbb{R}^d) \subset D^0(L) = L^2(\mathbb{R}^d, \mu) \), we obtain by interpolation that \( H^s(\mathbb{R}^d) \) embeds continuously into \( \text{dom}((I-L)^{s/2}) \subset D^s(L) \). In particular, the indicator functions \( f = 1_{(K,L)} \) are elements of \( D^{1/2-\delta}(L) \) for any small \( \delta > 0 \).
Arguing like in the proof of Corollary 2.3 the strict stationarity assumption can be relaxed and we obtain the same results when \( X \) starts from some initial distribution \( \eta \) which is absolutely continuous with respect to \( \mu \) and has bounded density \( d\eta/d\mu \). This is also true for Theorems 3.5 and 3.6.

3.3. Brownian motion and related diffusions

We will demonstrate now how in some cases the procedure of the last subsection can be adapted if the generator satisfies (1), but the invariant measure is not a probability measure. The main example for this is the one-dimensional Brownian motion. Its generator is \( Lf = (1/2)\Delta f \), \( \text{dom}(L) = H^2(\mathbb{R}) \), and the invariant measure \( \mu \) is the Lebesgue measure on \( \mathbb{R} \). The key idea idea is to approximate the process by stationary diffusions with reflecting boundaries.

**Theorem 3.9.** Assume that \((X_r)_{r \geq 0}\) satisfies \( X_r = X_0 + B_r \), where \((B_r)_{r \geq 0}\) is a one-dimensional Brownian motion, \( X_0 \sim \eta \) is independent of \((B_r)_{r \geq 0}\) and \( \eta \) has Lebesgue density with compact support. Then Theorem 3.7 applies to \((X_r)_{r \geq 0}\), as well, with \( A = 1 \) and \( d\mu/dx = d\eta/dx \).

Following a similar program as in Ngo and Ogawa [22] or Kohatsu-Higa et al. [21] this result can be extended to more general one-dimensional diffusion processes, using the Lamperti and Girsanov transforms.

3.4. Infinite dimensional diffusions

Since the general state space \( \mathcal{S} \) of \( X \) is Polish, we can also study infinite dimensional diffusions. Note that the results of Ganychenko et al. [14], Ganychenko [13] do not apply here, because, in general, heat kernel bounds are not available in this setting. Example 3.1 can be generalized considerably. If \( X \) satisfies the stochastic differential equation

\[
dX_r = AX_r dr + Q^{1/2}dW_r,
\]

where \( A \) and \( Q \) are operators on a separable Hilbert space \( \mathcal{H} \), with \( Q \) being bounded self-adjoint, then \( X \) is a Gaussian Markov process and the generator \( L \) satisfies a similar formula as in (1) with \( \nabla \) and \( \Delta \) replaced by the corresponding Fréchet derivatives \( D \) and \( D^2 \). Under certain conditions on \( A \) and \( Q \) the generator is reversible and \( X \) has some invariant measure \( \mu \). The domain is again a \( \mu \)-weighted Sobolev space and the associated Dirichlet form is

\[
\langle -L f, g \rangle \mu = \frac{1}{2} \int_{\mathcal{H}} \left( Q^{1/2}Df(x), Q^{1/2}Dg(x) \right)_{\mathcal{H}} d\mu(x).
\]

The results of Section 3.2 therefore remain formally the same. For details see Chojnowska-Michalik and Goldys [7]. A different kind of example are infinite dimensional systems of the form

\[
dx_r^{(i)} = \left( pV' \left( X_r^{(i+1)} - X_r^{(i)} \right) - qV' \left( X_r^{(i)} - X_r^{(i-1)} \right) \right) dr + dW_r^{(i)},
\]
(r, i) ∈ [0, ∞) × ℤ, where \( \{W^{(i)}_r : i \in ℤ\} \) is an independent family of Brownian motions, \( p, q \geq 0 \) with \( p + q = 1 \) and \( V \) is some potential function (Diehl et al. [9]). The authors show stationarity of \( X = (X^{(i)}_r)_{r \geq 0, i \in ℤ} \), derive the infinitesimal generator and derive its Dirichlet form (for the symmetric part). A similar analysis can then be derived as in Section 3.2. Note that the generator in this example is not necessarily self-adjoint, but it is always normal (Lemma 2.1 of Diehl et al. [9]).

4. Proofs

4.1. Proof of Theorem 2.2

Proof. We first assume that \( f \in L^2(μ) \). Expanding the squared error yields

\[
\|Γ_+(f) - 1 \|_{L^2(μ)}^2 = \mathbb{E}\left[ \sum_{k=1}^n \int_{(k-1)Δ_n}^{kΔ_n} (f(X_r) - f(X_{(k-1)Δ_n})) \, dr \right]^2
\]

We bound the diagonal \( (l = k) \) and off-diagonal \( (l \neq k) \) terms separately. Consider first the diagonal case and \( (k-1)Δ_n \leq r \leq h \leq kΔ_n \). By the Markov property and stationarity of \( X \) we can calculate the expectation above explicitly. Indeed, we see that

\[
\mathbb{E}\left[ (f(X_r) - f(X_{(k-1)Δ_n})) (f(X_h) - f(X_{(k-1)Δ_n})) \right] = (P_h - f, f)_μ - (P_{(k-1)Δ_n} - f, f)_μ + (f, f)_μ
\]

Consequently, by symmetry in \( r, h \)

\[
\sum_{k=1}^n \int_{(k-1)Δ_n}^{kΔ_n} \int_{(k-1)Δ_n}^{kΔ_n} \mathbb{E}\left[ (f(X_r) - f(X_{(k-1)Δ_n})) (f(X_h) - f(X_{(k-1)Δ_n})) \right] \, dr \, dh
\]

We bound the diagonal \( (l = k) \) and off-diagonal \( (l \neq k) \) terms separately.
Since the generator $L$ is normal, by the functional calculus (see 4.4) we can write this as

$$\langle \Psi(L)f, f \rangle_\mu = \int_{\sigma(L)} \Psi(\lambda) d \langle E_\lambda f, f \rangle_\mu$$

for the measurable function

$$\Psi(\lambda) = 2n \left( \int_0^{\Delta_n} \int_0^h (e^{\lambda(h-r)} - 1) drdh + \Delta_n \int_0^{\Delta_n} (1 - e^{\lambda h}) dh \right), \quad \lambda \in \mathbb{C}.$$  

Fix now $0 \leq s \leq 1$ such that for $0 \leq |\lambda| \leq 2|z|^s$. Since $L$ is the generator of a Feller semigroup, we know that $\sigma(L) \subseteq \{ \lambda \in \mathbb{C} : Re(\lambda) \leq 0 \}$. We therefore conclude that $|\Psi(\lambda)| \leq 4n\Delta_n^{2+|\lambda|^s} |\lambda|^s, \; \lambda \in \sigma(L)$. Hence the diagonal terms are bounded by

$$\int_{\sigma(L)} |\Psi(\lambda)| d\langle E_\lambda f, f \rangle_\mu \leq 4T\Delta_n^{1+s}\int_{\sigma(L)} |\lambda|^s d\langle E_\lambda f, f \rangle_\mu = 4T\Delta_n^{1+s} ||L||^{s/2}f||^2_\mu,$$

which is true as long as $f \in \text{dom}(||L||^{s/2})$. For the off-diagonal terms consider $(l-1)\Delta_n \leq r \leq (k-1)\Delta_n \leq h$. Then, similar as before

$$\mathbb{E} \left[ (f(X_h) - f(X_{(k-1)\Delta_n})) (f(X_r) - f(X_{(l-1)\Delta_n})) \right] = \langle P_{h-r}f, f \rangle_\mu - \langle P_{h-(l-1)\Delta_n}f, f \rangle_\mu - \langle P_{(k-1)\Delta_n-r}f, f \rangle_\mu + \langle P_{(k-1)\Delta_n}f, f \rangle_\mu - \langle P_{(k-1)\Delta_n-r}(P_{h-(k-1)\Delta_n} - I)(I - P_{r-(l-1)\Delta_n})f, f \rangle_\mu.$$

If the generator is self-adjoint, then the $P_u, u \geq 0$, are as well. Then $P_u$ is positive, $P_u - I$ negative and $I - P_u$ again positive semidefinite. We can thus conclude that $P_{((k-1)\Delta_n-r)}(P_{h-(k-1)\Delta_n} - I)(I - P_{r-(l-1)\Delta_n})$ is negative semidefinite and that the off-diagonal terms do not contribute to the estimation error. In the more general case of a normal operator $L$, we have to explicitly bound the off-diagonal terms. First, the off-diagonal terms are equal to

$$2 \sum_{k,l=1}^n \int_{(k-1)\Delta_n}^{k\Delta_n} \int_{(l-1)\Delta_n}^{l\Delta_n} \mathbb{E} \left[ (f(X_r) - f(X_{(k-1)\Delta_n})) \cdot (f(X_h) - f(X_{(l-1)\Delta_n})) \right] drdh$$

$$= 2 \sum_{k,l=1}^n \left< \left( \int_{(k-1)\Delta_n}^{k\Delta_n} \int_{(l-1)\Delta_n}^{l\Delta_n} P_{(k-l)\Delta_n-(l-(l-1)\Delta_n)} \right) \left( P_{h-(k-1)\Delta_n} - I \right) (I - P_{r-(l-1)\Delta_n}) drdh \right> f, f_\mu$$

$$= \left< \left( \int_0^{\Delta_n} \int_0^{\Delta_n} \sum_{k,l=1}^n P_{(k-l)\Delta_n-r} \left( P_h - I \right) (I - P_r) drdh \right) f, f_\mu \right>$$

$$= \int_{\sigma(L)} \tilde{\Psi}(\lambda) d \langle E_\lambda f, f \rangle_\mu,$$
where we argue as before, but this time the function $\tilde{\Psi}$ is given by

$$
\tilde{\Psi}(\lambda) = 2 \int_0^{\Delta_n} \int_0^{\Delta_n} \left( \sum_{k,l=1}^{n} e^{\lambda(k-l)\Delta_n - r} \right) \left( e^{\lambda h} - 1 \right) \left( 1 - e^{\lambda r} \right) dr dh, \quad \lambda \in \mathbb{C}.
$$

We will show that there exists a universal constant $\tilde{C} < \infty$ such that

$$
\left| \tilde{\Psi}(\lambda) \right| \leq \tilde{C} T |\lambda|^s \Delta_n^{1+s}, \quad \lambda \in \sigma(L).
$$

As in (15) we then can bound the off-diagonal terms by

$$
\int_{\sigma(L)} |\Psi(\lambda)| d\langle E\lambda f, f \rangle_{\mu} \leq \tilde{C} T \Delta_n^{1+s} \int_{\sigma(L)} |\lambda|^s d\langle E\lambda f, f \rangle_{\mu} = \tilde{C} T \| |L|^{s/2} f \|_2^2 \Delta_n^{1+s}
$$

for $f \in \text{dom}(|L|^{s/2})$. Combining this with (14) yields the claim. For $\lambda = 0$ we also have $\Psi(\lambda) = 0$. It is therefore sufficient to consider $\lambda \neq 0$. In order to bound $\tilde{\Psi}$ in that case we calculate

$$
\sum_{k>l=1}^{n} e^{\lambda(k-l-1)\Delta_n} = \sum_{l=1}^{n} \frac{1 - e^{\lambda(n-l)\Delta_n}}{1 - e^{\lambda \Delta_n}} = \frac{n}{1 - e^{\lambda \Delta_n}} - \frac{1 - e^{\lambda n \Delta_n}}{(1 - e^{\lambda \Delta_n})^2}.
$$

Hence

$$
\left| \Delta_n^2 (1 - e^{\lambda \Delta_n})^2 \sum_{k>l=1}^{n} e^{\lambda(k-l-1)\Delta_n} \right| \leq 2n \Delta_n^2 |\lambda\Delta_n|^s + 2\Delta_n^2 |\lambda n \Delta_n|^s
$$

$$
\leq 4T \Delta_n^{1+s} |\lambda|^s.
$$

Therefore (16) follows if we can show that

$$
\left| \Delta_n^{-2} (1 - e^{\lambda \Delta_n})^{-2} \int_0^{\Delta_n} \int_0^{\Delta_n} e^{\lambda(\Delta_n - r)} \left( e^{\lambda h} - 1 \right) \left( 1 - e^{\lambda r} \right) dr dh \right|
$$

is bounded by a universal constant. To show this, let $z = \lambda \Delta_n$ and note that

$$
\Delta_n^{-1} \left| \int_0^{\Delta_n} \frac{(1 - e^{\lambda h})}{1 - e^{\lambda \Delta_n}} dh \right| = \Delta_n^{-1} \left| \frac{\Delta_n}{1 - e^{\lambda \Delta_n}} - \lambda^{-1} \right| = \frac{1}{z} \left| \frac{1}{e^z - 1} \right|,
$$

$$
\Delta_n^{-1} \left| \int_0^{\Delta_n} \frac{e^{\lambda(\Delta_n - r)} - e^{\lambda \Delta_n}}{1 - e^{\lambda \Delta_n}} dr \right| = \left| \frac{1}{z} - \frac{e^z}{1 - e^z} \right| = \frac{1}{z} \left| \frac{1}{e^z - 1} + 1 \right|.
$$

(18) converges to 1 and (19) converges to 0 as $|z| \to \infty$. We further see that (18) converges to $1/2$ and (19) converges to $3/2$ when $|z| \to 0$ and when considering only $z \in \{ \lambda \in \mathbb{C} : \text{Re}(\lambda) \leq 0 \}$. This implies a universal constant bounding (17). This proves (16) and thus the claim. \qed
4.2. Proof of Corollary 2.3

Proof. Recall that \(X_0 \overset{d}{\sim} \eta\) and \(\eta\) is absolutely continuous with respect to \(\mu\) with bounded density \(d\eta/d\mu\). We indicate by \(\nu\) for a measure \(\nu\) the initial distribution of \(X\). By conditioning on \(X_0\) we see that

\[
\left\| \tilde{\Gamma}_T(f) - \hat{\Gamma}_{T,n}(f) \right\|_{L^2(\mu)}^2 = E_\eta \left[ \left\| \Gamma_T(f) - \hat{\Gamma}_{T,n}(f) \right\|_{X_0=x}^2 \right]
\]

\[
= \int_S E \left[ \left\| \Gamma_T(f) - \hat{\Gamma}_{T,n}(f) \right\|_{X_0=x}^2 \right] d\eta(x)
\]

\[
\leq \left\| \frac{d\eta}{d\mu} \right\|_{\infty,\mu} \int_S E \left[ \left\| \Gamma_T(f) - \hat{\Gamma}_{T,n}(f) \right\|_{X_0=x}^2 \right] d\mu(x)
\]

\[
= \left\| \frac{d\eta}{d\mu} \right\|_{\infty,\mu} E_\nu \left[ \left\| \Gamma_T(f) - \hat{\Gamma}_{T,n}(f) \right\|_{X_0=x}^2 \right].
\]

The conclusion follows immediately from Theorem 2.2. \(\Box\)

4.3. Proof of Theorem 2.4

Proof. By the triangle inequality we have for \(f \in L^2(\mu)\) and Theorem 2.2 that

\[
\left\| T^{-1}\tilde{\Gamma}_{T,n}(f) - \int_S f(x) d\mu(x) \right\|_{L^2(\mu)} \leq T^{-1} \left\| \tilde{\Gamma}_{T,n}(f) - \Gamma_T(f) \right\|_{L^2(\mu)} + \left\| T^{-1}\Gamma_T(f) - \int_S f(x) d\mu(x) \right\|_{L^2(\mu)}
\]

\[
\leq C \sqrt{T} \left\| f \right\|_{L^2(\mu)} \Delta_n^{1/2} + \left\| T^{-1}\Gamma_T(f) - \int_S f(x) d\mu(x) \right\|_{L^2(\mu)}
\]

for a universal constant \(C\). The claimed bound for the second term is well-known, but we give the proof here to complement the proof of Theorem 2.2. Consider \(f\) such that \(f_0 = f - \int f d\mu \in \text{dom}(\|L\|^{-1/2})\). By linearity of the occupation time functional it holds

\[
T^{-1}\Gamma_T(f) - \int_S f d\mu = T^{-1}\Gamma_T(f_0).
\]

Fubini’s theorem yields

\[
E \left[ \left\| T^{-1}\Gamma_T(f_0) \right\| \right]^2 = T^{-2} \int_0^T \int_0^T E \left[ f_0(X_r) f_0(X_h) \right] drdh
\]

\[
= 2T^{-2} \int_0^T \int_0^h \langle P_{h-r} f_0, f_0 \rangle_{\mu} drdh
\]

\[
= \int_{\sigma(L)} \Psi(\lambda) d\langle E_\lambda f_0, f_0 \rangle_{\mu},
\]

where the measurable function \(\Psi\) is defined by

\[
\Psi(\lambda) = 2T^{-2} \int_0^T \int_0^h e^{\lambda(h-r)} drdh = 2 \frac{e^{\lambda T} - 1 - \lambda T}{\lambda^2 T^2} = 2 \frac{(\lambda T)^{-1}(e^{\lambda T} - 1) - 1}{\lambda T},
\]

17
and where \( \Psi(0) = 1 \) by continuous extension. Since \( z \to z^{-1}(e^z - 1) - 1 \) is bounded on the left half-plane \( \{ z \in \mathbb{C} : \Re(z) \leq 0 \} \), there exists a constant \( C < \infty \) such that
\[
|\Psi(\lambda)| \leq \frac{\tilde{C}}{|\lambda|^T}, \quad \lambda \in \sigma(L).
\]
We conclude that
\[
\mathbb{E}\left[ T^{-1} \Gamma_T(f_0) \right]^2 \leq \frac{C}{T} \int_{\sigma(L)} |\lambda|^{-1} d\langle E_{\lambda}f_0, f_0 \rangle_\mu = \frac{C}{T} |L|^{-1/2} f_0||^2_\mu. \quad \square
\]

4.4. Proof of Theorem 3.9

Proof. We consider only the case that \( f \in H^s(\mathbb{R}^d) \). The case of an indicator function is analogous. The main idea of the proof is to approximate the process \( X \) by reflected processes for which Theorem 3.7 can be applied. Choose \( M \) big enough such that \( \text{supp}(\eta) \subset [-M,M] \) and let \( \tau_M = \inf\{ r > 0 : |X_r| > M \} \) the first time \( X \) exits from \( [-M,M] \). By dominated convergence, for any fixed \( T > 0 \), it holds that
\[
\left\| \Gamma_T(f) - \hat{\Gamma}_{T,n}(f) \right\|_{L^2(P)} = \lim_{M \to \infty} \left\| \left( \Gamma_T(f) - \hat{\Gamma}_{T,n}(f) \right) \mathbf{1}(T < \tau_M) \right\|_{L^2(P)}. \tag{20}
\]
We will show that there exists a universal constant \( C \), independent of \( M \), such that
\[
\left\| \left( \Gamma_T(f) - \hat{\Gamma}_{T,n}(f) \right) \mathbf{1}(t < \tau_M) \right\|_{L^2(P)} \leq C \left\| \frac{d}{d\tau} \left\|_{H^s \triangle_n^{1/2}} \right\|^2_{H^s} \right\|_{L^2(P)}. \tag{21}
\]
For this define
\[
f_M(x) = \begin{cases} x - 4kM & : (4k - 1)M \leq x < (4k + 1)M \\ (4k + 2)M - x & : (4k + 1)M \leq x < (4k + 3)M \end{cases}.
\]
Applying the Itô-Tanaka formula one can check that the process \( X^{(M)}_r := f_M(X_r) \) is a reflected Brownian motion with barriers at \(-M, M\), i.e. \( X^{(M)}_r \) satisfies \( [8] \) with \( L = -M, R = M, \sigma = 0 \) and some process \( \hat{K}_r \geq 0 \). See Gihman and Skorohod [12, Chapter I.23] for a similar construction of reflected diffusion processes. Furthermore, for all \( 0 \leq r \leq T \leq \tau_M \) we have
\[
B_r = X^{(M)}_r. \tag{22}
\]
In the following denote by \( \Gamma^{(M)}_T, \hat{\Gamma}^{(M)}_{T,n} \) the integral functional and the Riemann estimator with respect to the process \( X^{(M)}_r \). From (22) we obtain that
\[
\left\| \left( \Gamma_T(f) - \hat{\Gamma}_{T,n}(f) \right) \mathbf{1}(T < \tau_M) \right\|_{L^2(P)} = \left\| \left( \Gamma^{(M)}_T - \hat{\Gamma}^{(M)}_{T,n}(f) \right) \mathbf{1}(T < \tau_M) \right\|_{L^2(P)} \leq \left\| \Gamma^{(M)}_T - \hat{\Gamma}^{(M)}_{T,n}(f) \right\|_{L^2(P)}.
\]

18
Since supp(η) ⊂ [−M, M], in particular \( X_0 = X_0^{(M)} \), i.e. \( X_0^{(M)} \) has distribution \( η \). Note that \( η \) is in general not the stationary distribution of \( (X_t^{(M)})_{t \geq 0} \). From Example 3.3 we actually know that the stationary distribution \( µ_M \) of the reflected Brownian motion on \([−M, M]\) has Lebesgue density \( dµ_M/dx = (2M)^{-1}1_{[−M,M]} \). Theorem 3.7 with \( A = 1 \), therefore implies together with the argument from the proof of Corollary 2.3 that

\[
\left\| \Gamma_T^{(M)} - \hat{\Gamma}_{T,n}^{(M)}(f) \right\|_{L^2(\mathbb{P})} \leq C \left\| \frac{dµ_M}{dx} \right\|_{∞,µ}^{1/2} \left\| \frac{dη}{dµ_M} \right\|_{∞,η}^{1/2} \left\| f \right\|_{H^r} \cdot \sqrt{T} \cdot \Delta_n^{\frac{r+1}{2r}}.
\]

Observe that

\[
\int g dη = \int_{−M}^{M} g dµ_M/dx = 2M \int g \frac{dη}{dx} dµ_M
\]

for any bounded continuous function \( g \), i.e. \( dη/dµ_M = 2M \ dη/dx \). We conclude that

\[
\left\| \Gamma_T^{(M)} - \hat{\Gamma}_{T,n}^{(M)}(f) \right\|_{L^2(\mathbb{P})} \leq C \left\| \frac{dη}{dx} \right\|_{∞,η}^{1/2} \left\| f \right\|_{H^r} \cdot \sqrt{T} \cdot \Delta_n^{\frac{r+1}{2r}},
\]

which is (21). \( \square \)

**Appendix. A short review of semigroup theory and functional calculus**

We will briefly recall the basic objects needed in the theory of semigroups and the functional calculus for normal operators. For more details see Engel and Nagel [11] and Rudin [25]. Let \( µ \) be any probability measure on \((S, B(S))\). On the induced Hilbert space \( L^2(µ) \), denote by \( (P_t)_{t \geq 0} \) the Markov semigroup associated with \( X \) which satisfies \( P_t f(x) = E[f(X_t)|X_0 = x] \) for \( f \in L^2(µ) \), \( x \in S \) and \( P_{r+s} = P_r P_s \), \( r, s \geq 0 \). The infinitesimal generator of the semigroup is defined as

\[
Lf = \lim \limits_{r \to 0} \frac{P_r f - f}{r}, \quad f \in \text{dom}(L),
\]

where \( \text{dom}(L) \subset L^2(µ) \) is the set of all functions \( f \) for which this limit exists. If \( (P_t)_{t \geq 0} \) is strongly continuous, i.e. \( P_t f \to f \) in \( L^2(µ) \) as \( r \to 0 \) for all \( f \in L^2(µ) \), then the semigroup is Feller. This is true for most Markov processes in practice, including Lévy processes and many diffusions. In the Feller case, \( L \) is a densely defined closed linear and usually unbounded operator on its domain with spectrum \( σ(L) \subset \{ \lambda \in \mathbb{C} : \text{Re}(λ) \leq 0 \} \). In order to define fractional powers of the generator we further need the operator \( L \) (and thus the operators \( P_r \)) to be normal, i.e. \( LL^* = L^*L \) where \( L^* \) is the Hilbert space adjoint of \( L \). In that case the spectral theorem (Theorem 13.33 of Rudin [25]) guarantees the existence of a resolution of the identity or spectral measure \( (E_A)_{A \in B(\mathbb{C})} \) on \( L^2(µ) \). This means that \( (E_A)_{A \in B(\mathbb{C})} \) is a family of orthogonal projections \( E_A : L^2(µ) \to L^2(µ) \) for Borel sets \( A \subset \mathbb{C} \) such that for every \( f, g \in L^2(µ) \)
the map \( A \mapsto \langle E_A f, g \rangle_\mu \) is a complex measure, supported on \( \sigma(L) \). Moreover, \( A \mapsto \langle E_A f, f \rangle_\mu \) is a positive measure with total variation \( \langle C f, f \rangle_\mu = \|f\|_\mu \). By the spectral theorem we can associate to any measurable function \( \Psi : \mathbb{C} \mapsto \mathbb{C} \) a densely defined closed operator \( \Psi(L) \) with domain \( \text{dom}(\Psi(L)) := \{ f \in L^2(\mu) : \int_{\sigma(L)} |\Psi(\lambda)|^2 d\langle E_\lambda f, f \rangle_\mu < \infty \} \) by the relation

\[
\langle \Psi (L) f, g \rangle_\mu = \int_{\sigma(L)} \Psi (\lambda) d\langle E_\lambda f, g \rangle_\mu, \quad f \in \text{dom}(\Psi(L)), g \in L^2(\mu).
\]

It satisfies \( \|\Psi(L)f\|_\mu = \int_{\sigma(L)} |\Psi(\lambda)|^2 d\langle E_\lambda f, f \rangle_\mu \). In particular, we can define the fractional operators \( |L|^{s/2} \) on \( \mathcal{D}^s(L) := \text{dom}(|L|^{s/2}) \), for \( 0 \leq s \leq 1 \). By the spectral theorem for normal semigroups (Theorem 13.38 of Rudin [25]) we realize the semigroup in its usual exponential form, i.e. \( P_r = \Psi(L) \) with \( \Psi(x) = e^{rx} \), \( r \geq 0 \).

**Acknowledgements**

Both authors gratefully acknowledge the financial support of the DFG Research Training Group 1845 "Stochastic Analysis with Applications in Biology, Finance and Physics".

**References**

[1] Adams, R. A. and Fournier, J. J. F. (2003). *Sobolev Spaces*. Pure and Applied Mathematics. Elsevier Science.

[2] Bakry, D., Gentil, I., and Ledoux, M. (2013). *Analysis and Geometry of Markov Diffusion Operators*. Grundlehren der mathematischen Wissenschaften. Springer International Publishing.

[3] Bass, R. F. (2006). *Diffusions and Elliptic Operators*. Probability and Its Applications. Springer New York.

[4] Catellier, R. and Gubinelli, M. (2016). Averaging along irregular curves and regularisation of ODEs. *Stochastic Processes and their Applications*, 126(8):2323–2366.

[5] Chen, M. F. (2006). *Eigenvalues, Inequalities, and Ergodic Theory*. Probability and Its Applications. Springer London.

[6] Chesney, M., Jeanblanc-Picqué, M., and Yor, M. (1997). Brownian Excursions and Parisian Barrier Options. *Source: Advances in Applied Probability Adv. Appl. Prob*, 29(29).

[7] Chojnowska-Michalik, A. and Goldys, B. (2002). Symmetric Ornstein-Uhlenbeck semigroups and their generators. *Probability Theory and Related Fields*, 124(4):459–486.
[8] Chorowski, J. (2015). Nonparametric volatility estimation in scalar diffusions: Optimality across observation frequencies. *arXiv:1507.07139*.

[9] Diehl, J., Gubinelli, M., and Perkowski, N. (2016). The Kardar-Parisi-Zhang equation as scaling limit of weakly asymmetric interacting Brownian motions. *arXiv:1606.02331*.

[10] Dion, C. and Genon-Catalot, V. (2016). Bidimensional random effect estimation in mixed stochastic differential model. *Statistical Inference for Stochastic Processes*, 19(2):131–158.

[11] Engel, K.-J. and Nagel, R. (1999). *One-Parameter Semigroups for Linear Evolution Equations*. Graduate Texts in Mathematics. Springer New York.

[12] Ethier, S. N. and Kurtz, T. G. (1986). *Markov processes : characterization and convergence*. Wiley.

[13] Ganychenko, I. (2015). Fast L 2-approximation of integral-type functionals of Markov processes. *Modern Stochastics: Theory and Applications*, 2:165–171.

[14] Ganychenko, I. V., Knopova, V. P., and Kulik, A. M. (2015). Accuracy of discrete approximation for integral functionals of Markov processes. *Modern Stochastics: Theory and Applications*, 2(4):401–420.

[15] Gihman, I. and Skorohod, A. (2015). *Stochastic Differential Equations*. Springer.

[16] Gobet, E. and Matulewicz, G. (2016). Parameter estimation of Ornstein–Uhlenbeck process generating a stochastic graph. *Statistical Inference for Stochastic Processes*, pages 1–25.

[17] Hansen, L. P., Alexandre Scheinkman, J., and Touzi, N. (1998). Spectral methods for identifying scalar diffusions. *Journal of Econometrics*, 86(1):1–32.

[18] Hoffmann, M. (1999). L p Estimation of the Diffusion Coefficient. *Bernoulli*, 5(3):447.

[19] Hugonnier, J.-N. (1999). The Feynman–Kac formula and pricing occupation time derivatives. *International Journal of Theoretical and Applied Finance*, 2(02):153–178.

[20] Kipnis, C. and Varadhan, S. R. S. (1986). Central limit theorem for additive functionals of reversible Markov processes and applications to simple exclusions. *Communications in Mathematical Physics*, 104(1):1–19.

[21] Kohatsu-Higa, A., Makhlouf, A., and Ngo, H. (2014). Approximations of non-smooth integral type functionals of one dimensional diffusion processes. *Stochastic Processes and their Applications*, 124(5):1881–1909.
[22] Ngo, H.-L. and Ogawa, S. (2011). On the discrete approximation of occupation time of diffusion processes. *Electronic Journal of Statistics*, 5:1374–1393.

[23] Pavliotis, G. A. (2014). *Stochastic Processes and Applications: Diffusion Processes, the Fokker-Planck and Langevin Equations*. Texts in Applied Mathematics. Springer New York.

[24] Pollett, P. (2003). Integrals for continuous-time Markov chains. *Mathematical Biosciences*, 182(2):213–225.

[25] Rudin, W. (2006). *Functional Analysis*. International series in pure and applied mathematics. McGraw-Hill.

[26] V. Gol’dshtein, A. U. (2009). Weighted Sobolev Spaces and Embedding Theorems. *Transactions of the American Mathematical Society*, 361(7):3829–3850.

[27] Watanabe, S. (1984). *Lectures on Stochastic Differential Equations and Malliavin Calculus*. Lectures on mathematics and physics / Tata institute of fundamental research: Mathematics. Springer.