Partial Regularity of Suitable Weak Solutions of the Model Arising in Amorphous Molecular Beam Epitaxy

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Abstract In this paper, we are concerned with the precise relationship between the Hausdorff dimension of possible singular point set $S$ of suitable weak solutions and the parameter $\alpha$ in the nonlinear term in the following parabolic equation

$$h_t + h_{xxxx} + \partial_x|h_x|^\alpha = f.$$  

It is shown that when $5/3 \leq \alpha < 7/3$, the $3\alpha - 5\alpha - 1$-dimensional parabolic Hausdorff measure of $S$ is zero, which generalizes the recent corresponding work of Ozaniski and Robinson in [SIAM J. Math. Anal., 51, 228–255 (2019)] for $\alpha = 2$ and $f = 0$. The same result is valid for a 3D modified Navier–Stokes system.

Keywords Surface growth model, modified Navier–Stokes equations, partial regularity, Hausdorff dimension

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1 Introduction

The fourth-order parabolic equation describing dynamic crystal growth in materials science is given by

$$h_t + h_{xxxx} + \partial_x|h_x|^\alpha = f, \quad \alpha > 1.$$  

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Here, $h$ represents the height of a crystalline layer. The surface growth model (1.1) plays an important role in molecular-beam-epitaxy (MBE) process and its physical background can be found in [2, 18, 23, 34, 36].

Considering the diffusion term $h_{xx}$ due to evaporation-condensation on the left hand side of (1.1), Stein and Winkler [34] constructed the global mild solution with $1 < \alpha \leq 5/3$ and established the global weak solutions with $5/3 < \alpha < 10/3$. As a special case of equation (1.1) with $\alpha = 2$ and $f = 0$,

$$h_t + h_{xxxx} + \partial_{xx}|h_x|^2 = 0,$$

this equation is known as the conserved Kardar–Parisi–Zhang (Kuramoto–Sivashinsky) equation. Recently, starting from the work of Blömker and Romito [5], the mathematical study of equation (1.2) attracts a lot of attention (see, e.g., [6, 9, 12, 30, 31] and references therein), since it shares similar features to the 3D Navier–Stokes equations. A celebrated result of the 3D Navier–Stokes system is that 1-dimensional Hausdorff measure of singular set of its suitable weak solutions is zero. This is so-called Caffarelli–Kohn–Nirenberg theorem [10]. Notice that most recent generalized Caffarelli–Kohn–Nirenberg theorems [11, 13, 32, 38] mainly interpret how the fractional dissipation $(-\Delta)^\alpha$ affects the regularity of suitable weak solutions in the 3D Navier–Stokes equations. It seems that there are few works involving how the nonlinear term affects the regularity of suitable weak solutions in the Navier–Stokes equations.

We switch our attention to the surface growth model. In [31], Ozański and Robinson first studied the partial regularity of suitable weak solution and successfully extended Caffarelli–Kohn–Nirenberg theorem to equation (1.2) via the following $\varepsilon$-regularity criterion involving dimensionless quantity

$$\limsup_{\varrho \to 0} \frac{1}{\varrho} \int_{t-\varrho^4}^{t+\varrho^4} \int_{x-\varrho}^{x+\varrho} |h_{yy}|^2 dyd\tau \leq \varepsilon,$$

which means that $h$ is Hölder continuous at point $(x, t)$. Based on this, it is shown that 1-dimensional Hausdorff measure of the set of potential singular points is zero. Here, a point is said to be a regular point to (1.2) if $h$ is Hölder continuous in some neighborhood of this point. The rest points will be called singular points and the responding set is denoted by $\mathcal{S}$. To this end, they applied the blow-up technology developed by Lin [28] and Ladyzenskaja and Seregin [27] to the suitable weak solutions of equation (1.2) to establish the $\varepsilon$-regularity criterion below

$$\frac{1}{\varrho^4} \int_{t-\varrho^4}^{t+\varrho^4} \int_{x-\varrho}^{x+\varrho} |h_y|^3 dyd\tau \leq \varepsilon.$$  

Subsequently, the higher regularity of suitable weak solutions under the $\varepsilon$-regularity criterion (1.4) was obtained by Burczak, Ozański and Seregin [9]. The generalization of $\varepsilon$-regularity criterion (1.4) was considered by Choi and Yang in [12].

Inspired by the works [11, 13, 31, 32, 34, 38], we consider the partial regularity in equation (1.1) to reveal how the nonlinear term affects the regularity of suitable weak solutions in equation (1.1). We formulate our theorems as follows. The notations appearing here can be found in Subsection 2.1.

**Theorem 1.1** Suppose that $h$ is a suitable weak solution to (1.1) with $1 < \alpha < 7/3$ and force $f$ belonging to Morrey spaces $\mathcal{M}^{m, \frac{\alpha+1}{\alpha}}(Q(1))$, where $m \geq \frac{\alpha+1}{\alpha}$. There exists an absolute constant
\( \varepsilon_{01} \) such that if \( h \) satisfy
\[
\iint_{Q(1)} |h_y|^{\alpha+1}dydt \leq \varepsilon_{01},
\]
then \( h \) is Hölder continuous in \([x - \frac{1}{2}, x + \frac{1}{2}] \times [t - \frac{1}{2r}, t + \frac{1}{2r}]\).

**Theorem 1.2** Assume that \( h \) is a suitable weak solution to (1.1) with \( 1 < \alpha < 7/3 \) and force \( f \) is in Morrey spaces \( \mathcal{M}^{\alpha, \frac{\alpha+1}{\alpha}}(Q(1)) \) with \( m \geq \frac{\alpha+1}{\alpha} \). There is a universal constant \( \varepsilon_{02} \) such that \( h \) is regular at point \((x, t)\) if
\[
\limsup_{r \to 0} \frac{1}{r^{\frac{\alpha+1}{\alpha}}} \iint_{Q(r)} |h_{yy}|^2dyd\tau \leq \varepsilon_{02}. \tag{1.5}
\]

**Remark 1.3** From (1.5), one immediately yields the regularity of suitable weak solutions of equation (1.1) with \( 1 < \alpha \leq 5/3 \), which is consistent with Stein and Winkler’s work [34].

The Vitali cover lemma allows us to estimate the Hausdorff dimensional of the singular points set of equation (1.1).

**Corollary 1.4** Let \( 5/3 \leq \alpha < 7/3 \) and \( f \) be in Morrey space \( \mathcal{M}^{\alpha, \frac{\alpha+1}{\alpha}} \) with \( m \geq \frac{\alpha+1}{\alpha} \). The \( \frac{3\alpha-5}{\alpha-1} \)-dimensional Hausdorff measure of the set of potential singular points of suitable weak solutions in (1.1) is zero.

**Remark 1.5** Further discussion involving partial regularity of suitable weak solution of equations (1.1) and (1.2) can be found in Section 5.

Compared with the Ozánski and Robinson’s work [31], the general nonlinear term and no-zero force are considered in above results. The precise relationship between the Hausdorff dimension of possible singular point set \( S \) and the parameter \( \alpha \) in the nonlinear term in the parabolic equation is presented in Corollary 1.4. It is worth remarking that the extension of Poincaré inequality of weak solutions to the parabolic equation is established. To the knowledge of authors, this type of Poincaré inequality of weak solutions of the parabolic equation is presented in Corollary 1.4. It is worth remarking that the extension of Poincaré inequality of weak solutions to the parabolic equation is established. To the knowledge of authors, this type of Poincaré inequality of weak solutions of the parabolic equation (system) appears frequently and plays an important role in partial regularity for parabolic systems by A-caloric approximation (see [7, 15, 16, 33, 35] and references). Three kinds of different approaches to prove this type inequality are provided. Let \( d \) denote the dimension of time direction and is determined by the parabolic equation (system). The first one is partially motivated by the proof of Poincaré inequality [24, Lemma 3.1, p. 458] by Krylov via applying the classical Poincaré inequality in time direction. Indeed, following the path of [24], one uses the Poincaré inequality in time direction to conclude that
\[
\int_{t-r^d}^{t+r^d} \left| \int_{B_k} u(t)dy \right| \left| \iint_{Q(r),\sigma} u(t)dyd\tau \right|^p ds \leq \int_{t-r^d}^{t+r^d} \left| \int_{B_k} u_{t}(t)dy \right|^p d\tau, \quad 1 < p < \infty,
\]
where
\[
\iint_{Q(r),\sigma} u(t)dyd\tau = \int_{t-r^d}^{t+r^d} \left( \int_{B_k} u(t)dyd\tau \right) / \int_{t-r^d}^{t+r^d} d\tau.
\]
Here, the new ingredient is an application of the following Poincaré–Wirtinger’s inequality (see [8, p. 233])
\[
\left| \int_{B_k} u(t)dy \right| - \iint_{Q(r),\sigma} u(t)dyd\tau \leq C \int_{t-r^d}^{t+r^d} \left| \int_{B_k} u_{t}(t)dy \right| d\tau,
\]
which allows us to slightly improve the corresponding Poincaré inequality in [24]. It is worth
pointing out that an alternative proof of the results in [1] can also be given, which is of inde-
pendent interest. The second one is modification of that in [31]. The last one originates from
aforementioned works involving $\mathcal{A}$-caloric approximation [7, 15, 16, 33, 35].

Inspired by the equation (1.1), we introduce the following modified Navier–Stokes equations
\begin{align}
\begin{cases}
  u_{it} - \Delta u_i + u \cdot \nabla u_i^{\alpha-1} + \partial_i \Pi = 0, \quad i = 1, 2, 3, \quad \alpha > 1, \\
  \text{div } u = 0, \\
  u|_{t=0} = u_0.
\end{cases}
\end{align}

Here, as the standard Navier–Stokes equations, the system (1.6) also shares the cancellation of
the nonlinear term in energy space $L^2$, that is,
\begin{align}
  \int_{\Omega} u_j \partial_j u_i^{\alpha-1} u_i dx = \frac{\alpha - 1}{\alpha} \int_{\Omega} u_j \partial_j u_i^{\alpha} dx = \frac{\alpha - 1}{\alpha} \int_{\Omega} \partial_j (u_j u_i^{\alpha}) dx = 0.
\end{align}
Hence, for the regular solutions of the modified Navier–Stokes equations (1.6), there holds the
energy equality
\begin{align}
  \|u(T)\|_{L^2(\Omega)}^2 + 2 \int_0^T \|\nabla u\|_{L^2(\Omega)}^2 ds = \|u_0\|_{L^2(\Omega)}^2.
\end{align}
The Leary–Hopf type weak solutions of equations (1.6) can be proved by Galerkin approxima-
tion. The next objective is to show how the nonlinear term affects the regularity of suitable
weak solutions in the modified Navier–Stokes equations (1.6). The notations appearing here
can be found in Subsection 2.2.

Theorem 1.6 Suppose that the pair $(u, \Pi)$ is a suitable weak solution to the modified Navier–
Stokes equations (1.6) with $1 < \alpha < 7/3$. Then $|u|$ can be bounded by $1$ on $[-\frac{1}{\alpha}, 0] \times B(\frac{1}{8})$
provided the following condition holds,
\begin{align}
  \left( \int_{Q(1)} |u|^{\alpha+1} dydt \right)^{\frac{1}{\alpha+1}} + \left( \int_{Q(1)} |\Pi|^{\alpha+1} dydt \right)^{\frac{1}{\alpha+1}} \leq \varepsilon_1,
\end{align}
for an absolute constant $\varepsilon_1 > 0$.

Theorem 1.7 Suppose that $u$ is a suitable weak solution to the modified Navier–Stokes equa-
tions (1.6) with $1 < \alpha < 7/3$ and for a universal constant $\varepsilon_2 > 0$,
\begin{align}
  \limsup_{r \to 0} \frac{1}{r^{\frac{n-5\alpha}{n-\alpha}}} \int_{Q(r)} |\nabla u|^2 dydt < \varepsilon_2.
\end{align}
Then $(0, 0)$ is a regular point for $u(x, t)$.

Corollary 1.8 $\frac{2\alpha - 5}{n-\alpha}$-dimensional Hausdorff measure of the set of potential singular points of
suitable weak solutions in the modified Navier–Stokes equations (1.6) with $5/3 \leq \alpha < 7/3$ is
zero.

It is coincident that partial regularity results of the surface growth model (1.1) and the
modified Navier–Stokes equations (1.6) are the same, however, the blow up argument in [6, 27,
28, 31] can not be directly applied to the modified Navier–Stokes equations due to the different
nonlinear term. Moreover, it seems that the inductive method developed in [10, 32, 38] works for
the modified Navier–Stokes equations (1.6) with only $\alpha \leq 2$ and De Giorgi technique introduced
in [43] breaks down for the general case $\alpha \neq 2$. We observe that the modified blow up procedure
together with the fractional integration theorem [25, 29] (Riesz potential estimate [19–21, 39]) involving parabolic Morrey spaces allows us to consider the partial regularity of the modified Navier–Stokes equations (1.6). The proof of Theorem 1.6 is close to the strategy owing to Wang [39] and his co-authors in [14, 19–21].

It is an interesting question to show the existence of suitable weak solution of the modified Navier–Stokes equations (1.6). The corresponding results at least can be seemed as the partial regularity of smooth solution at the first blow-up time. We state the well-posedness of the modified Navier–Stokes equations (1.6) with initial data in $H^1(\Omega)$.

**Theorem 1.9** Suppose that $u_0 \in H^1(\Omega)$ and $\text{div} u_0 = 0$. There exist a constant $T > 0$ and a strong solution $u$ of system (1.6) with the initial datum $u_0$ such that

$$u \in L^\infty(0, T; H^1(\Omega)) \cap L^2(0, T; H^2(\Omega)).$$

The strong solution can be extended beyond $t = T$ if

$$u \in L^p(0, T; L^q(\Omega)) \quad \text{with} \quad \frac{2}{p} + \frac{3}{q} = \frac{1}{\alpha - 1}, \quad q > 3\alpha - 3.$$

**Corollary 1.10** For every initial datum $u_0 \in H^1(\Omega)$ with divergence-free condition, there exists a global regular solution $u$ to modified Navier–Stokes equations (1.6) with $1 < \alpha \leq 5/3$ in

$$L^\infty(0, T; H^1(\Omega)) \cap L^2(0, T; H^2(\Omega)).$$

Finally, we would like to state a result to improve $\varepsilon$-regularity criterion (1.3) in conserved Kardar–Parisi–Zhang equation (1.2).

**Theorem 1.11** There exists an absolute constant $\varepsilon_{03}$ with the following property. If $h$ is a suitable weak solution to (1.1) and

$$\limsup_{\varepsilon \to 0} \frac{1}{\varepsilon^4} \int_{t-\varepsilon^4}^{t+\varepsilon^4} \left( \int_{x-\varepsilon^4}^{x+\varepsilon^4} |\partial_y h|^2 dy \right)^{\frac{2}{3}} d\tau \leq \varepsilon_{03},$$

(1.8)

or

$$\limsup_{\varepsilon \to 0} \left( \sup_{-\varepsilon^4 \leq \tau - t \leq \varepsilon^4} \frac{1}{\varepsilon^4} \int_{x-\varepsilon^4}^{x+\varepsilon^4} |h(y)| dy \right) \leq \varepsilon_{03},$$

(1.9)

then $(x, t)$ is a regular point.

**Remark 1.12** The $\varepsilon$-regularity criterion (1.9) means that the sufficiently small $L^\infty_{t,x}$ norm of $h$ yields the regularity of suitable weak solutions of equation (1.2). This criterion is an improvement of corresponding results in [31].

The remaining paper is structured as follows. In Section 2, we present the dimensional analysis, some dimensionless quantities and auxiliary lemmas for the surface growth equation (1.1) and the modified Navier–Stokes equations, respectively. Then the generalized parabolic Poincaré inequality of solutions of equation (1.1) with three different proofs are given. Section 3 is devoted to the blow-up analysis applying to the equation (1.1) to show Theorem 1.1 and contains the proof of Theorem 1.2 and Theorem 1.11. In Section 4, we study the partial regularity of suitable weak solutions of the modified Navier–Stokes equations. In Section 5, we will mention some problems involving partial regularity of suitable weak solution of equations (1.1) and (1.2).
2 Notations and Some Auxiliary Lemmas

2.1 Auxiliary Results for Surface Growth Model

It is clear that if \( h(x,t) \) solves system (1.1) with \( f(x,t) \), then \( h_\lambda \) is also a solution of (1.1) with \( f_\lambda \) for any \( \lambda \in \mathbb{R}^+ \), where

\[
h_\lambda = \lambda^{\frac{2-\alpha}{1-\alpha}} h(\lambda x, \lambda^4 t), \quad f_\lambda = \lambda^{\frac{2-\alpha}{1-\alpha}} f(\lambda x, \lambda^4 t).
\]  

(2.1)

As [10], we can assign a “dimension” to each quantity as follows

| Quantity | Dimension |
|----------|-----------|
| \( x \)  | 1         |
| \( t \)  | 4         |
| \( h \)  | \( \frac{2-\alpha}{1-\alpha} \) |
| \( f \)  | \( \frac{2-\alpha}{1-\alpha} \) |
| \( \partial_x \) | -1 |
| \( \partial_t \) | -4 |

Hence, the quantities in (1.3)–(1.9) are all dimensionless. In addition, We will use the following quantities:

\[
E_s(r) = \text{ess sup}_{s \in (t-r^4,t+r^4)} \frac{1}{r^{\alpha-1}} \int_{B(r)} h(y,s)^2 \, dy,
\]

\[
\tilde{E}_s(r) = \text{ess sup}_{s \in (t-r^4,t+r^4)} \frac{1}{r^{\alpha-1}} \int_{B(r)} (h(s) - h_{Q(r)})^2 \, dy,
\]

\[
E(r) = \frac{1}{r^{\alpha-1}} \int_{Q(r)} |h_{yy}|^2 \, dy \, d\tau,
\]

\[
D_p(r) = \frac{1}{r^{\alpha(p+1)-2p-1}} \int_{Q(r)} |h|^p \, dy \, d\tau,
\]

\[
\tilde{D}_p(r) = \frac{1}{r^{\alpha(p+1)-2p-1}} \int_{Q(r)} |h - h_{Q(r)}|^p \, dy \, d\tau,
\]

\[
E_{\alpha+1}(r) = \frac{1}{r^{2(1-2\alpha)}} \int_{Q(r)} |h_y|^\alpha \, dy \, d\tau,
\]

\[
E_{12/7,2}(\varrho) = \frac{1}{\varrho^2} \int_{t-\varrho^4}^{t+\varrho^4} \left( \int_{x-\varrho}^{x+\varrho} |\partial_{yy} h|^2 \, dy \right) \frac{4}{\varrho^2} \, d\tau,
\]

\[
D_{\infty,1}(\varrho) = \sup_{-\varrho^4 \leq \tau \leq \varrho^4} \frac{1}{\varrho} \int_{x-\varrho}^{x+\varrho} |h(y)| \, dy,
\]

where we used the following notation

\[
B(x; \mu) = \{ y \in \mathbb{R} ||x-y| \leq \mu \}, \quad B(\mu) := B(x; \mu),
\]

\[
Q(x,t; \mu) = B(x, \mu) \times (t-\mu^4,t+\mu^4), \quad Q(\mu) := Q(x,t; \mu),
\]

\[
\tilde{B}(\mu) = B(x_0; \mu), \quad \tilde{Q}(\mu) := Q(x_0,t_0; \mu),
\]

\[
\hat{B}(\mu) = B(0; \mu), \quad \hat{Q}(\mu) := Q(0,0; \mu),
\]

and

\[
h_{B(r),\sigma}(t) = \int_{B(r),\sigma} h(y,t) \, dy = \frac{\int_{B(r)} h(y,t) \sigma \, dy}{\int_{B(r)} \sigma \, dy},
\]

\[
h_{Q(r),\sigma}(t) = \int_{Q(r),\sigma} h(y,t) \, dy \, dt = \frac{\int_{Q(r)} h(y,t) \sigma \, dy \, dt}{\int_{Q(r)} \sigma \, dy \, dt},
\]

\[
h_{B(r)}(t) = h_{B(r),1}(t),
\]

\[
h_{Q(r)} = h_{Q(r),1},
\]
Partial Regularity of the Surface Growth Model

here $\sigma(x)$ is a smooth cut off function such that $\sigma(x) = 1$ in $B(\vartheta_1 r)$ and $\sigma(x) = 0$ in $B^c(r)$, where $0 < \vartheta_1 < 1$.

For $p \in [1, \infty]$, the notation $L^p(0, T; X)$ stands for the set of measurable functions $f(x, t)$ on the interval $(0, T)$ with values in $X$ and $\|f(\cdot, t)\|_X$ belongs to $L^p(0, T)$. For $f \in L^1(T; L^p(0, T; X))$, we denote $\hat{f}(k) = \frac{1}{2\pi} \int_T e^{-ikx} f(x) dx$ for the $k$-th Fourier coefficient of $f$. The space $H^s$ is equipped with the norm $\|f\|_{H^s(T)} = (\sum_{k \in \mathbb{Z}} (1 + |k|^2 |\hat{f}(k)|^2)^{s/2})^{1/2}$. The homogeneous space $\dot{H}^s$ is given by

$$\dot{H}^s = \left\{ f \left| \|f\|_{\dot{H}^s} = \left( \sum_{k \in \mathbb{Z}} |k|^{2s} |\hat{f}(k)|^2 \right)^{1/2} \right. \right\}.$$ 

The Morrey space $\mathcal{M}^{p,l}(\Omega)$, with $1 \leq l < \infty$, $1 \leq p < \infty$ and a domain $\Omega \subset \mathbb{R}^d$, is defined as the space of all measurable functions $f$ on $\Omega$ for which the norm

$$\|f\|_{\mathcal{M}^{p,l}(\Omega)} = \sup_{R > 0} \sup_{x \in \Omega} R^d \left( \int_{B(x, R) \cap \Omega} |f(y)|^p dy \right)^{1/p} < \infty.$$ 

Similarly, one can define the parabolic Morrey space $\mathcal{M}^{p,l}(Q(r))$. We will use $C$ to denote an absolute constant which may be different from line to line unless otherwise stated. For simplicity, we write

$$\|f\|_{L^p L^q(Q(r))} := \|f\|_{L^p (t^{-r}x,t+r^{-1}x)} \text{ and } \|f\|_{L^p L^q(Q(r))} := \|f\|_{L^p L^q(Q(r))}.$$

**Definition 2.1** A function $h$ is called a suitable weak solution to the equation (1.1) provided the following conditions are satisfied,

1. $h \in L^\infty((0, T); L^2(T)) \cap L^2((0, T); \dot{H}^2(T))$;
2. $h$ solves (1.1) in the sense of distributions, for $\varphi(x, t) \in C_0^\infty(\mathbb{T} \times (0, T))$,

$$\int_0^T \int_\mathbb{T} (h(t) \varphi_t - h_{xx} \varphi_{xx} - |h_x|^\alpha \varphi_{xx}) dx dt = - \int_0^T \int_\mathbb{T} f \varphi(t) dx dt; \tag{2.2}$$

3. $h$ satisfies the following inequality, for a.e. $t \in [0, T]$,

$$\frac{1}{2} \int_\mathbb{T} |h(t)|^2 \varphi(t) dx + \int_0^t \int_\mathbb{T} |h_{xx}|^2 \varphi dx dt \leq \int_0^t \int_\mathbb{T} \left( \frac{1}{2} |\phi_t - \phi_{xxxx}||h|^2 + 2h^2 \phi_x \varphi - \frac{2\alpha + 1}{\alpha + 1} |h_x|\alpha \phi_x - |h_x|^\alpha \phi_{xx} + f \phi \right) dx dt, \tag{2.3}$$

for all nonnegative function $\phi \in C_0^\infty(\mathbb{T} \times (0, \infty))$.

**Remark 2.2** Just as $\alpha = 2$, equation (1.1) has the cancellation of the nonlinear term in energy space $L^2$, namely,

$$\int_\mathbb{T} \partial_{xx}(|h_x|^\alpha) dx = \int_\mathbb{T} |h_x|^\alpha h_{xx} dx = \frac{1}{\alpha + 1} \int_\mathbb{T} \partial_x(|h_x|^\alpha h_x) dx = 0.$$ 

Therefore, the existence of suitable weak solution of equation (1.1) with $\alpha < 7/3$ can be showed as the argument in [31]. It seems that the critical case $\alpha = 7/3$ corresponds to the existence of suitable weak solution of the 4D Navier–Stokes equations. Maybe a parabolic concentration-compactness method recently developed by Wu [42] for the 4D Navier–Stokes equations can deal with this case.

Next, we are concerned with the Poincaré inequality of weak solutions of the parabolic equation. To the knowledge of authors, Poincaré type inequality for solutions of parabolic
system plays an important role in partial regularity for parabolic systems, especially, by A-caloric approximation (see [7, 15, 16, 33, 35] and references).

**Lemma 2.3** Suppose that $h$ is a weak solution of equation (1.1) satisfying (2.2). Then, for $p \geq \alpha$, there holds

\[
\|h - h_{Q(\vartheta_1 r)}\|_{L^p(Q(\vartheta_1 r))}^p \leq C_1 \{ r^p \|h_y\|_{L^p(Q(r))}^p + r^{5+2p-5\alpha} \|h_y\|_{L^p(Q(r))}^{p_\alpha} + r^{5-p} \|f\|_{L^1(Q(r))}^p \}. \tag{2.4}
\]

**Remark 2.4** The key point for the parabolic Poincaré inequality (2.4) is the following estimate, for $r \in (t - r^4, t + r^4)$,

\[
|h_{B(r),\sigma}(\tau) - h_{Q(r),\sigma}| \leq Cr^{-4} \|h_y\|_{L^1(Q(r))} + Cr^{-3} \|h_y\|_{L^\alpha(Q(r))} + Cr^{-1} \|f\|_{L^1(Q(r))}. \tag{2.5}
\]

Here, we will provide three different methods to show it.

**Proof** Assume for a while we have proved (2.5). By the triangle inequality, the weighted Poincaré inequality and (2.5), we know that

\[
\int\int_{Q(r)} \left| h - \int\int_{Q(r),\sigma} h(t) \, dy \right|^p \sigma \, dy \, dt \\
\quad \leq \int\int_{Q(r)} \left| h - h_{\tilde{B}_k}(t) \right|^p \sigma \, dx \, dt + \int\int_{Q(r)} \left| h(t) \, dy - \int\int_{Q(r),\sigma} h(t) \, dy \right|^p \sigma \, dy \, d\tau \\
\quad \leq Cl^p \int\int_{Q(r)} \left| h_y \right| \, p \, dy \, dt + Cr^5 \left( |r^{-4} \|h_y\|_{L^1(Q(r))} + r^{-3} \|h_y\|_{L^\alpha(Q(r))} + r^{-1} \|f\|_{L^1(Q(r))} \right).^p.
\]

We further deduce from the Hölder inequality that

\[
\int\int_{Q(r)} \left| h - \int\int_{Q(r),\sigma} h(t) \, dx \right|^p \sigma \, dx \, dt \\
\quad \leq Cl^p \int\int_{Q(r)} \left| h_y \right| \, p \, dx \, dt + Cr^5 \left( \left( |r^{-4} \|h_y\|_{L^1(Q(r))} + r^{-3} \|h_y\|_{L^\alpha(Q(r))} + r^{-1} \|f\|_{L^1(Q(r))} \right)^p \right).
\]

Direct calculation means that

\[
\int\int_{Q(\vartheta_1 r)} \left| h - \int\int_{\vartheta_1 r} h(t) \, dy \right|^p \, dx \, dt \leq C \int\int_{Q(\vartheta_1 r)} \left| h - \int\int_{Q(r),\sigma} h(t) \, dy \right|^p \, \sigma \, dy \, dt,
\]

which yields that desired estimate.

It suffices to show (2.5). The argument can be made rigorous by standard approximation techniques or by the use of Steklov averages (see [15, 16, 33, 35] and references).

**Method (1)**: Note that

\[
h_{Q(r),\sigma} = \frac{\int_{t-r^4}^{t+r^4} h_{B(r),\sigma}(\tau) \, d\tau}{\int_{t-r^4}^{t+r^4} d\tau}.
\]

Hence, we conclude by the Poincaré–Wirtinger’s inequality that, for $\tau \in (t - r^4, t + r^4)$,

\[
|h_{B(r),\sigma} - h_{Q(r),\sigma}| \leq \int_{t-r^4}^{t+r^4} \left| h_{B(r),\sigma}'(\tau) \right| d\tau. \tag{2.6}
\]
By virtue of integration by parts, we arrive at
\[
    h'_{B(r),\sigma}(\tau) = \frac{\int_{B(r)} h_\tau(y, \tau) \sigma dy}{\int_{B(r)} \sigma dx} = \frac{\int_{B(r)} (-h_{yyyy} - \partial_{yy} |h_y|^\alpha + f) \sigma dy}{\int_{B(r)} \sigma dx} = \frac{\int_{B(r)} (h_x \sigma_{yyy} - |h_y|^\alpha \sigma_{yy} + f \sigma) dy}{\int_{B(r)} \sigma dx}.
\]

The above formal computations can be made rigorous by Steklov averages (see [7, pp. 216–218, Proof of Lemma 5.1]).

Plugging this into (2.6), we further get
\[
    |h_{B(r),\sigma} - h_{Q,r,\sigma}| \leq Cr^{-4} \|h_y\|_{L^1(Q(r))} + Cr^{-3} \|h_y\|_{L^\infty(Q(r))} + Cr^{-1} \|f\|_{L^1(Q(r))}.
\]

Method (2): Assume that \(\tau_1, \tau_2 \in (0, T)\). Without loss of generality, we suppose that \(\tau_1 < \tau_2\). Consider the Lipschitz continuous function which is defined by
\[
    \xi_\epsilon(t) = \begin{cases} 
    \frac{t - \tau_1}{\epsilon}, & \tau_1 < t \leq \tau_1 + \epsilon, \\
    1, & \tau_1 + \epsilon < t \leq \tau_2 - \epsilon, \\
    \frac{\tau_2 - t}{\epsilon}, & \tau_2 - \epsilon < t < \tau_2.
    \end{cases}
\]

Take \(\varphi(t) = \xi_\epsilon(t)(h_{B(r),\sigma} - h_{B(r),\sigma})(\tau_1)\) as the test function in (2.2).

According to Lebesgue’s differentiation theorem and the definition of \(h_{B(r),\sigma}\), we see that
\[
    \lim_{\epsilon \to 0} \int_0^T \int_T h(t) \varphi \, dy \, dt = \lim_{\epsilon \to 0} \left( \frac{1}{\epsilon} \int_{\tau_1}^{\tau_1 + \epsilon} \int_T h(t) \sigma \, dy \, dt - \frac{1}{\epsilon} \int_{\tau_2 - \epsilon}^{\tau_2} \int_T h(t) \sigma \, dy \, dt \right) (h_{B(r),\sigma}(\tau_1) - h_{B(r),\sigma}(\tau_2))
\]
\[
    = \left| \int_{\hat{B}_k} h(\tau_1) \, dy - \int_{\hat{B}_k} h(\tau_2) \, dy \right| \int_{B(r)} \sigma \, dy.
\]

The Hölder inequality guarantees that
\[
    \left| \lim_{\epsilon \to 0} \int_0^T \int_T h_{yy} \varphi_{yy} \, dy \, dt \right| = \left| (h_{B(r),\sigma}(\tau_1) - h_{B(r),\sigma}(\tau_2)) \int_{\tau_1}^{\tau_2} \int_{B(r)} h_y \sigma_{yyy} \, dy \, dt \right| \leq C r^{-1} \left| \int_{\hat{B}_k} h(\tau_1) \, dy - \int_{\hat{B}_k} h(\tau_2) \, dy \right| \|h_y\|_{L^1(Q(r))}.
\]

Likewise,
\[
    \left| \lim_{\epsilon \to 0} \int_0^T \int_T |h_y|^\alpha \varphi_{yy} \, dy \, dt \right| \leq C r^{-2} \left| \int_{\hat{B}_k} h(\tau_1) \, dy - \int_{\hat{B}_k} h(\tau_2) \, dy \right| \|h_y\|_{L^\infty(Q(r))},
\]
\[
    \left| \lim_{\epsilon \to 0} \int_0^T \int_T f \varphi \, dy \, dt \right| \leq C \left| \int_{\hat{B}_k} h(\tau_1) \, dy - \int_{\hat{B}_k} h(\tau_2) \, dy \right| \|f\|_{L^1(Q(r))}.
\]
Combining (2.7) and (2.8), we obtain the desired estimate
\[
\left| \int_{\tilde{B}_k} h(\tau_1)dy - \int_{\tilde{B}_k} h(\tau_2)dy \right| \leq C r^{-4}\|h_y\|_{L^1(Q(r))} + C r^{-3}\|h_y\|_{L^\alpha(Q(r))} + C r^{-1}\|f\|_{L^1(Q(r))},
\]
which implies that
\[
|h_{B(r),\sigma}(\tau) - h_{Q(r),\sigma}| = \frac{1}{2r^4} \int_{t-r^4}^{t+r^4} \left| \int_{\tilde{B}_k} h(\tau_1)dy - \int_{\tilde{B}_k} h(\tau_1)dy \right| d\tau_1
\leq C r^{-4}\|h_y\|_{L^1(Q(r))} + C r^{-3}\|h_y\|_{L^\alpha(Q(r))} + C r^{-1}\|f\|_{L^1(Q(r))},
\]
where the definition of $h_{Q(r),\sigma}$ was used.

Method (3): Formally, in the light of the equation (1.1) and integration by part, we know that
\[
\left| \int_{\tilde{B}_k} h(\tau_1)dy - \int_{\tilde{B}_k} h(\tau_2)dy \right| = \left| \int_{\tau_1}^{\tau_2} \int_{B(r)} h_y(y,\tau)\sigma d\tau dy \right|
\leq C r^{-4}\|h_y\|_{L^1(Q(r))} + C r^{-3}\|h_y\|_{L^\alpha(Q(r))} + C r^{-1}\|f\|_{L^1(Q(r))}.
\]
From (2.9), we also complete the proof of (2.5). 

Next, we establish some interpolation inequalities for proving Theorems 1.2 and 1.11.

**Lemma 2.5** Suppose that $h \in L^\infty L^1(Q(r)) \cap L^2 H^2(Q(r))$, then
\[
\|h_y\|_{L^3(Q(r))} \leq \|h\|_{L^\infty L^1(Q(r))}\|\partial_{yy} h\|_{L^2(Q(r))} + C r^{-\frac{4}{3}}\|h\|_{L^\infty L^1(Q(r))}, \tag{2.10}
\]
\[
\|h\|_{L^3(Q(r))} \leq C r^{-\frac{4}{3}}\|h\|_{L^\infty L^1(Q(r))}\|\partial_{yy} h\|_{L^2(Q(r))} + C r^{-\frac{4}{3}}\|h\|_{L^\infty L^1(Q(r))}. \tag{2.11}
\]
Assume that $h$ is spatial periodic function on $(x-r_0, x+r_0)$ and $h \in L^\infty L^2(Q(r_0)) \cap L^2 H^2(Q(r_0))$, then
\[
\|h_y\|_{L^{\alpha+1}(Q(r_0))} \leq C r_0^{-\frac{4}{3\alpha+1}}\|h\|_{L^\infty L^2(B(r_0))}\|\partial_{yy} h\|_{L^2(Q(r_0))} + C r_0^{-\frac{4}{3\alpha+1}}\|h\|_{L^\infty L^1(Q(r_0))}. \tag{2.12}
\]

**Proof** Thanks to the Gagliardo–Nirenberg inequality, we get
\[
\|h_y\|_{L^3} \leq C\|h\|_{L^1}^\frac{1}{3}\|\partial_{yy} h\|_{L^2}^\frac{2}{3} + C r^{-\frac{4}{3}}\|h\|_{L^1},
\]
\[
\|h\|_{L^3} \leq C\|h\|_{L^1}^\frac{1}{3}\|\partial_{yy} h\|_{L^2}^\frac{2}{3} + C r^{-\frac{4}{3}}\|h\|_{L^1}.
\]
Integrating over $\tau$ from $t-r^4$ to $t+r^4$, we arrive at
\[
\|h_y\|_{L^3(Q(r))} \leq C\|h\|_{L^\infty L^1(Q(r))}\|\partial_{yy} h\|_{L^2(Q(r))}^2 + C r^{-1}\|h\|_{L^3(Q(r))}^2,
\]
and
\[
\|h\|_{L^3(Q(r))} \leq C\|h\|_{L^\infty L^1(Q(r))}^\frac{1}{3}\int_{t-r^4}^{t+r^4}\|\partial_{yy} h\|_{L^2}^\frac{2}{3} d\tau + C r^{-2}\|h\|_{L^3(Q(r))}^2
\leq C\|h\|_{L^\infty L^1(Q(r))}^\frac{1}{3}\left(\int_{t-r^4}^{t+r^4}\|\partial_{yy} h\|_{L^2}^2 d\tau\right)^\frac{2}{3} + C r^{-2}\|h\|_{L^\infty L^1(Q(r))}^3,
\]
where the Hölder inequality was used. We get the desired estimates (2.10) and (2.11).
Since $\int_{x-r_0}^{x+r_0} h_y dy = 0$, we derive from the fractional Poincaré inequality that

$$\|h_y\|_{L^{\alpha+1}(B(r_0))} \leq C \|h\|_{H^{\frac{3\alpha+1}{2}}(B(r_0))}. \quad (2.13)$$

As [31], the latter inequality can be derived as follows

$$\|h_y\|_{L^{\alpha+1}(B(r_0))}^2 \leq C \|h_y\|_{H^{\frac{3\alpha+1}{2}}(B_1)} \leq C \sum_{k \in \mathbb{Z}} (1 + |k|^{\frac{\alpha+1}{\alpha+1}}) |\hat{h}_y(k)|^2$$

$$= C \sum_{k \neq 0} ((|k|^2 + |k|^{2+\frac{\alpha+1}{\alpha+1}})) |\hat{h}(k)|^2 \leq C \sum_{k \neq 0} |k|^{2+\frac{\alpha+1}{\alpha+1}} |\hat{h}(k)|^2$$

$$\leq C \|h\|_{H^{\frac{3\alpha+1}{2}}(B(r_0))},$$

where $\hat{f}(k)$ denotes the $k$-th Fourier mode in the Fourier expansion of $f$ on $(x - r_0, x + r_0)$.

By means of interpolation inequality, we obtain

$$\|h_y\|_{L^{\alpha+1}(B(r_0))} \leq C \|h\|_{H^{\frac{3\alpha+1}{2}}(B(r_0))} \leq C \|h\|_{L^2(B(r_0))} \|\partial_y h\|_{L^{\alpha+1}(B(r_0))}.$$ 

We further conclude by the Hölder inequality that

$$\|h_y\|_{L^{\alpha+1}(Q(r_0))} \leq C \|h\|_{L^\infty L^2(Q(r_0))} \int_{t-r_0^4}^{t+r_0^4} \|\partial_y h\|_{L^{\alpha+1}(B(r_0))} dr,$$

$$\leq C r_0^{\frac{2-\alpha}{\alpha}} \|h\|_{L^\infty L^2(Q(r_0))} \|\partial_y h\|_{L^{\alpha+1}(Q(r_0))}.$$ 

The proof of this lemma is complete. \hfill $\square$

Finally, we recall the regularity estimate of biharmonic heat equation and the general Morrey–Campanato integral characterization of Hölder spaces recently established in [31].

**Lemma 2.6** ([31]) Suppose that $0 < \vartheta_2 < 1$, $0 < \rho$, $H, H_x, H_{xx}, H_{xxx}$ are distributional functions in $Q(\vartheta_2 \rho)$ and that $H$ is a distributional solution to the biharmonic heat equation $H_t = -H_{xxxx}$ in $Q(\rho)$, that is

$$\iint_{Q(\rho)} H \phi_t dx dt = \iint_{Q(\rho)} H \phi_{xxxx} dx dt \quad (2.14)$$

for every $\phi \in C_0^\infty(Q(\vartheta_2 \rho))$. Then

$$\|H_x\|_{L^\infty(Q(\vartheta_2 \rho))} \leq C_2(\vartheta_2, \rho) \left(\|H\|_{L^2(Q(\rho))} + \|H_x\|_{L^2(Q(\rho))}\right)$$

for some $C_2(\vartheta_2, \rho) > 0$.

**Lemma 2.7** ([31]) Let $R \in (0,1)$, $f \in L^1(Q(R))$ and assume that there exist positive constants $\nu \in (0,1)$, $M > 0$, such that

$$\left(\iint_{Q(R)} |f(y) - f_{Q(r)}|^p dy ds\right)^{\frac{1}{p}} \leq M r^\nu$$

for any $z = (x,t) \in Q(R/4)$ and any $0 < r < R/4$. Then $f$ is Hölder continuous in $Q(R/4)$, namely, for any $z, w \in Q(R/4)$, $z = (x,t)$, $w = (y,s)$,

$$|f(x,t) - f(y,s)| \leq c M (|x - y| + |t - s|^{1/4})^\nu. \quad (2.15)$$
2.2 Preliminary Results for the Modified Navier–Stokes Equations

Notice that if the pair \((u(x,t), \Pi(x,t))\) solves the modified Navier–Stokes system (1.6), then the pair \((u_\lambda, \Pi_\lambda)\) is also a solution of (1.6) for any \(\lambda \in \mathbb{R}^+\), where

\[
  u_\lambda = \lambda^{\frac{1}{\alpha}} u(\lambda x, \lambda^2 t), \quad \Pi_\lambda = \lambda^{\frac{1}{\alpha - 1}} \Pi(\lambda x, \lambda^2 t). \tag{2.16}
\]

Based on this, we introduce some dimensionless quantities for the modified Navier–Stokes equations (1.6) as follows

\[
  E_p(u; r) = \frac{1}{r^{\frac{2\alpha - 1}{\alpha}} \int_{\Omega(r)} |u(y, t)|^2 dy dt, \quad P_{\alpha + 1}(r) = \frac{1}{r^{\frac{2\alpha - 1}{\alpha}} \int_{\Omega(r)} |\Pi(y, t)|^{\alpha + 1} dy dt, \tag{2.17}
\]

\[
  E(u; r) = \frac{1}{r^{\frac{2\alpha - 1}{\alpha}} \int_{\Omega(r)} |\nabla u(y, t)|^2 dy dt, \quad E_u(u; r) = \sup_{-r^2 \leq t < 0} \frac{1}{r^{\frac{2\alpha - 1}{\alpha}} \int_{B(r)} |u(y, t)|^2 dy. \tag{2.18}
\]

Here we used the following notation

\[
  B(z; r) = \{ y \in \mathbb{R}^3 ||x - y| \leq r \}, \quad B(r) = B(0; r), \quad \tilde{B}(r) = B(x_0; r), \quad Q(x, t; r) = B(x; r) \times (t - r^2, t), \quad Q(r) = Q(0, 0; r), \quad \tilde{Q}(r) = Q(x_0, t_0; r).
\]

Now we present the definition of the suitable weak solution to the modified Navier–Stokes equations (1.6).

**Definition 2.8** A pair \((u, \Pi)\) is called a suitable weak solution to the modified Navier–Stokes equations (1.6) provided the following conditions are satisfied,

1. \(u \in L^\infty(-T, 0; L^2(\Omega)) \cap L^2(-T, 0; \dot{H}^1(\Omega)), \Pi \in L^{\frac{\alpha + 1}{\alpha}}(-T, 0; L^\frac{\alpha + 1}{\alpha}(\Omega));\)

2. \((u, \Pi)\) solves the modified Navier–Stokes equations (1.6) in \(\Omega \times (-T, 0)\) in the sense of distributions;

3. \((u, \Pi)\) satisfies the following inequality for a.e. \(t \in [-T, 0]\)

\[
  \int_\Omega |u|^2 \varphi(x, t) + 2 \int_{-T}^t \int_\Omega \varphi(x, t) |\nabla u|^2 \leq \int_{-T}^t \int_\Omega |u|^2 (\partial_t \varphi + \Delta \varphi) + \frac{2}{\alpha} - \frac{1}{\alpha} \int_{-T}^t \int_\Omega u \cdot \nabla \varphi(u_1 + u_2 + u_3) + 2 \int_{-T}^t \int_\Omega u \cdot \nabla \Pi, \tag{2.19}
\]

where non-negative function \(\varphi(x, s) \in C_0^\infty(\Omega \times (-T, 0))\).

The pressure equation of the modified Navier–Stokes equations (1.6) reads

\[
  \Delta \Pi = -\phi \partial_i \partial_j (u_j u_i^{\alpha - 1}).
\]

The usual local technique of the pressure \(\Pi\) is to use the following equation

\[
  \partial_i \partial_j (\Pi \varphi) = -\phi \partial_i \partial_j (u_j u_i^{\alpha - 1}) + 2 \partial_i \phi \partial_j \Pi + \Pi \partial_i \partial_j \phi, \tag{2.20}
\]

where \(\phi\) is a standard smooth cut-off function. In the spirit of [10, Lemma 5.4, p. 802], [40, Lemma 2.4, p. 1236] and [32, Lemma 2.2, p. 11], we can establish the following decay estimates of dimensionless quantity involving pressure via local pressure equation (2.20) and the interior estimate of harmonic function. We omit the detail here. We leave this to the interested readers.

**Lemma 2.9** For \(0 < \mu \leq \frac{1}{4} \rho\), there exists an absolute constant \(C\) independent of \(\mu\) and \(\rho\) such that

\[
  P_{\alpha + 1}(\mu) \leq C \left( \frac{\mu}{\rho} \right)^{\frac{4\alpha - 4}{\alpha}} E_{\alpha + 1}(u; \rho) + C \left( \frac{\mu}{\rho} \right)^{\frac{4\alpha - 4}{\alpha}} P_{\alpha + 1}(\rho). \tag{2.21}
\]
Lemma 3.2 Then

\[
E_{\alpha+1}(u; \mu) \leq C \left( \frac{\mu}{\rho} \right)^{\frac{4\alpha-6}{3\alpha-2}} E_{\frac{5-\alpha}{\alpha}}^{\frac{5-\alpha}{\alpha}}(u; \rho) E_{\frac{3\alpha-3}{4}}^{\frac{3\alpha-3}{4}}(u; \rho) + C \left( \frac{\mu}{\rho} \right)^{\frac{3-\alpha}{\alpha}} E_{\alpha+1}(u; \rho). \tag{2.22}
\]

3 Partial Regularity of the Surface Growth Model

3.1 Regularity Criterion at One Scale

This section is devoted to the proof of Theorem 1.1 via blow up analysis developed in [27, 28, 31]. It suffices to prove the following proposition.

Proposition 3.1 Let \( h \) be a suitable weak solutions of equation (1.1) in \( Q(r) \) with \( f \in \mathcal{M}^{m, \frac{\alpha+1}{\alpha}}(Q(r)) \). There exist \( \varepsilon_{01} \) and \( R_1 > 0 \) such that if there holds \( r < R_1 \),

\[
\Phi(x, t; r) := \left( \frac{1}{r^{\frac{2(1-2\alpha)}{1-\alpha}}} \int_{Q(r)} |h_y|^{\alpha+1} dyd\tau \right)^{\frac{1}{\alpha+1}} < \varepsilon_{01},
\]

then \( h \) is Hölder continuous at point \((x, t)\).

To this end, we need the following decay lemma.

Lemma 3.2 Let \( h \) be a suitable weak solutions of (1.1) in \( Q(r) \) with \( f \in \mathcal{M}^{m, \frac{\alpha+1}{\alpha}}(Q(r)) \). For \( \theta \in (0, \frac{1}{2}) \), there exist \( \varepsilon_{02}, R_2 \) such that if \( r \leq R_2 \),

\[
\left( \frac{1}{r^{\frac{2(1-2\alpha)}{1-\alpha}}} \int_{Q(r)} |h_y|^{\alpha+1} dyd\tau \right)^{\frac{1}{\alpha+1}} + \|f\|_{\mathcal{M}^{m, \frac{\alpha+1}{\alpha}}(Q(r))} r^\beta < \varepsilon_{02},
\]

where \( m > \frac{5(\alpha-1)}{3\alpha-2} \) and \( m \geq \frac{(\alpha+1)}{\alpha} \), \( 0 < \beta < \frac{3\alpha-2}{\alpha-1} - \frac{5}{m} \), then there holds the decay estimate

\[
\Phi(x, t; \theta r) \leq C_3 \theta^\frac{1}{\alpha+1} (\Phi(x, t; r) + \|f\|_{\mathcal{M}^{m, \frac{\alpha+1}{\alpha}}(Q(r))} r^\beta). \tag{3.2}
\]

Proof We will prove this lemma by contradiction. Suppose there exist sequences \( r_k \to 0, \{\varepsilon_k\}, (x_k, t_k), f_k \in \mathcal{M}^{m, \frac{\alpha+1}{\alpha}}(Q(r_k)) \), and a sequence of suitable weak solution \( \{h_k\} \) such that

\[
\Phi(x_k, t_k; r_k) = \left( \frac{1}{r_k^{\frac{2(1-2\alpha)}{1-\alpha}}} \int_{Q(x_k, t_k; r_k)} |\partial_x h_k|^{\alpha+1} dxdt \right)^{\frac{1}{\alpha+1}} + \|f_k\|_{\mathcal{M}^{m, \frac{\alpha+1}{\alpha}}(Q(r_k))} r_k^\beta
\]

\[
= \varepsilon_k \to 0, \text{ as } k \to \infty,
\]

and

\[
\left( \frac{1}{(\theta r_k)^{\frac{2(1-2\alpha)}{1-\alpha}}} \int_{Q(x_k, t_k; \theta r_k)} |\partial_x h_k|^{\alpha+1} dxdt \right)^{\frac{1}{\alpha+1}} \geq C_3 \theta^\frac{1}{\alpha+1} \varepsilon_k.
\]

To proceed further, we set

\[
H_k(x, t) = \varepsilon_k^{-1} r_k^{\frac{\alpha-2}{\alpha-1}} \left[ h_k(x_k + x r_k, t_k + t r_k^4) - \int_{Q(x_k, t_k; \partial_3 r_k)} h_k dxdt \right],
\]

\[
g_k(x, t) = \varepsilon_k^{-1} r_k^{\frac{3-\alpha}{\alpha}} f(x_k + x r_k, t_k + t r_k^4).
\]
As a consequence, there holds
\begin{equation}
\left(\int_{Q(1)} |\partial_x H_k|^{\alpha+1} dx dt \right)^{\frac{1}{\alpha+1}} + \varepsilon_k^{-1} \|f_k\|_{\mathcal{M}^m, \frac{\alpha+1}{\alpha}(Q(r_k))} r_k^\beta = 1, \tag{3.3}
\end{equation}
\begin{equation}
\int_{Q(\vartheta_3)} H_k dx ds = 0, \tag{3.4}
\end{equation}
\begin{equation}
\left(\int_{Q(\vartheta)} |\partial_x H_k|^{\alpha+1} dx dt \right)^{\frac{1}{\alpha+1}} \geq C_3 \theta^{\frac{\alpha}{\alpha+1}}, \tag{3.5}
\end{equation}
\begin{equation}
\int_{Q(1)} (H_k \phi_r - \partial_{xx} H_k \phi_{xx} - \varepsilon_k^{\alpha-1} |\partial_x H_k|^\alpha \phi_{xx}) dx ds \tag{3.6}
\end{equation}
\begin{equation}
= - \int_{Q(1)} g_k \phi(s) dx ds, \quad \phi \in C_0^\infty (\hat{Q}(1)), \tag{3.7}
\end{equation}
and $h_k$ satisfies the local energy inequality
\begin{equation}
\frac{1}{2} \int_{B(1)} |H_k(t)|^2 \phi(t) dx + \int_{-1}^t \int_{B(1)} (\partial_{xx} H_k)^2 \phi \leq \int_{-1}^t \int_{B(1)} \left[ \frac{1}{2} (\phi_t - \phi_{xxxx}) (H_k)^2 + 2 (\partial_x H_k)^2 \phi_{xx}
\right. \left. - \frac{2 \alpha + 1}{\alpha + 1} \varepsilon_k^{\alpha-1} |\partial_x H_k|^\alpha H_k \phi_{xx} - \varepsilon_k^{\alpha-1} |\partial_x H_k|^\alpha H_k \phi_{xx} + g_k H_k \phi \right]. \tag{3.8}
\end{equation}

By virtue of (3.3), we infer that
\begin{equation}
\|g_k\|_{\mathcal{M}^m, \frac{\alpha+1}{\alpha}(Q(1))} \leq \varepsilon_k^{-1} \frac{4\alpha+5}{\alpha+1} r_k^{-\frac{5}{2}} \|f_k\|_{\mathcal{M}^m, \frac{\alpha+1}{\alpha}(Q(r_k))} \leq C \frac{4\alpha+5}{\alpha+1} r_k^{-\frac{5}{2}} \leq C. \tag{3.9}
\end{equation}
Thanks to (3.9) and (3.3), Poincaré inequality (2.4) with $\vartheta_1 = 7/8$ and (3.4) with $\vartheta_3 = 7/8$, we have
\begin{equation}
\left(\int_{Q(\hat{\vartheta})} |H_k|^{\alpha+1} \right) \leq C_1. \tag{3.10}
\end{equation}

Abusing notation slightly, we denote the subsequences of $\{H_k\}$ by $\{H_k\}$. Now, we choose a subsequences of $\{H_k\}$ such that
\begin{equation}
H_k \rightharpoonup H, \quad \partial_x H_k \rightharpoonup H_x \quad \text{in } L^{\alpha+1}(\hat{Q}(\hat{\vartheta})) \text{ as } k \to \infty. \tag{3.11}
\end{equation}
Let $k \to \infty$ in (3.7). Then
\begin{equation}
\int_{Q(1)} (H \phi_r - \partial_{xx} H \phi_{xx}) dx dt = 0, \quad \phi \in C_0^\infty \left( \hat{Q} \left( \frac{7}{8} \right) \right).
\end{equation}

With the help of Lemma 2.6 and (3.10), we get
\begin{equation}
\|H_x\|_{L^\infty(\hat{Q}(\hat{\vartheta}))} \leq C_2(\|H\|_{L^2(\hat{Q}(\hat{\vartheta}))} + \|H_x\|_{L^2(\hat{Q}(\hat{\vartheta}))}) \leq C(C_2, C_1).
\end{equation}
As a consequence,
\begin{equation}
\frac{1}{\theta^{\frac{\alpha}{\alpha+1}}} \int_{Q(\vartheta)} |\partial_x H|^\alpha dx \leq C(C_2, C_1). \tag{3.12}
\end{equation}
Assume for a while we have proved that
\begin{equation}
\partial_x H_k \rightarrow H_x \text{ in } L^{\alpha+1} \left( \hat{Q} \left( \frac{1}{2} \right) \right) \quad \text{with } \alpha < \frac{7}{3}. \tag{3.13}
\end{equation}
Taking the limit in (3.5), using (3.13) and (3.12), we deduce that
\[
C_3 \leq \left( \frac{1}{b^5} \int_{Q(\theta)} |\partial_x H_k|^{|\alpha+1|} \right) \leq C(C_2, C_1). \tag{3.14}
\]
We take \( C_3 = 2C(C_2, C_1) \) to get a contradiction.

It remains to show (3.13) we have assumed. To this end, we require the uniform bound of the right hand side of (3.5) to apply Aubin–Lions lemma. This together with (3.3) and (3.5), (3.9) leads to
\[
\|H_k\|_{L^\infty(L^2(Q(3/4)))} + \|\partial_{xx} H_k\|_{L^2(Q(3/4))} \leq C, \tag{3.15}
\]
which turns out that
\[
\|\partial_x H_k\|_{L^{\frac{10}{3}}(Q(3/4))} \leq C, \tag{3.16}
\]
and
\[
\|H_k\|_{L^{\frac{2}{3}}(H^{\frac{4}{3}}(Q(3/4)))} \leq \|H_k\|_{L^\infty(L^2(Q(3/4)))} \|H_k\|_{L^2(H^2(Q(3/4)))} \leq C. \tag{3.17}
\]
It follows from (3.7) that
\[
\left| \int_{Q(3/4)} \partial_t H_k \phi dx dt \right| = \left| - \int_{Q(3/4)} \partial_{xx} H_k \phi_x dx dt - \varepsilon_k^{\alpha-1} \int_{Q(3/4)} |\partial_x h_k|^\alpha \phi_{xx} + g_k \phi dx dt \right|
\]
\[
\leq (\|\partial_x H_k\|_{L^{\frac{\alpha+1}{\alpha}}(Q(3/4))} + \|\partial_x H_k\|_{L^{\alpha+1}(Q(3/4))} + \|g_k\|_{M^{\frac{\alpha+1}{\alpha}}(Q(1))}) \times \|\phi\|_{L^{\alpha+1}(W^{2,\alpha+1}(\hat{Q}(3/4)))}
\]
\[
\leq C \|\phi\|_{L^{\alpha+1}(W^{2,\alpha+1}(\hat{Q}(3/4)))}, \tag{3.18}
\]
for all \( \phi \in C_0^\infty(\hat{Q}(3/4)) \).

Therefore, there holds \( \|\partial_t H_k\|_{L^{\frac{\alpha+1}{\alpha}}(t_{3/4}; W^{2,\alpha+1}(Q(3/4)))} \leq C \). Since
\[
H^{\frac{4}{3}} \subset H^{\frac{2}{3}} \subset (W^{2,\alpha+1})^*, \tag{3.19}
\]
Aubin–Lions lemma allows us to select a subsequence of \( \{H_k\} \) converging in \( L^3(H^{7/6}(Q(3/4))) \).

Sobolev embedding theorem helps us to get that \( \partial_x H_k \) converges in \( L^3(\hat{Q}(3/4)) \).

From (3.16), we infer that
\[
\partial_x H_{k_n} \rightarrow H_x \text{ in } L^{\alpha+1}(\hat{Q}(1/2)) \quad \text{with } \alpha < \frac{7}{3}. \]
Thus, we give the proof of assertion (3.13). The proof of this lemma is completed. \( \square \)

At this stage, iterating the above lemma and using the general parabolic Campanato Lemma 2.7 allow us to prove Proposition 3.1.

**Proof of Proposition 3.1** We set
\[
R_1 = \min \left\{ R_2, \left[ \frac{\varepsilon_{02}}{4\|f\|_{M^{\frac{\alpha+1}{\alpha}}(Q(r))}} \right]^\frac{1}{\alpha} \right\},
\]
therefore, \( \forall (x_0, t_0) \in Q(x, t; \rho) \) with \( \rho \leq \frac{1}{2} \), there holds
\[
\left( \frac{1}{\frac{1}{2}} \int_{Q(x_0, t_0; \frac{1}{2})} |h_x|^{\alpha+1} \right)^\frac{1}{\alpha+1} \leq \left( \frac{4^{\frac{\alpha+1}{\alpha}}}{\frac{1}{2}} \int_{Q(x_0, t_0; \frac{1}{2})} |h_x|^{\alpha+1} \right)^\frac{1}{\alpha+1} \leq 4^{\frac{\alpha+1}{\alpha}} \varepsilon_{01}. \]
We choose \( \varepsilon_{01} \) sufficiently small such that
\[
\left( \frac{1}{(L)^{2(3-x_0) / (1-x_1)}} \int_{Q(x_0,t_0; \frac{r}{2})} |h_x|^{\alpha+1} \right)^{\frac{1}{\alpha+1}} + \|f\|_{\mathcal{M}^m, \frac{a+1}{a} (Q(r/2))} \left( \frac{r}{2} \right) ^\beta < \frac{1}{2} \varepsilon_{02}. \tag{3.20}
\]
We claim that, for
\[
0 < \beta_1 < \frac{1}{\alpha - 1} \quad \text{and} \quad \beta_1 \leq \beta, \tag{3.21}
\]
there holds
\[
\begin{cases}
\Phi \left( x_0, t_0; \frac{r}{2} \theta^{k-1} \right) + \|f\|_{\mathcal{M}^m, \frac{a+1}{a} (Q(\theta^{k-1}) \left( \frac{r}{2} \theta^{k-1} \right) ^\beta \leq \varepsilon_{02}; \\
\Phi \left( x_0, t_0; \frac{r}{2} \theta^k \right) \leq \theta^{\beta_1} \left[ \Phi \left( x_0, t_0; \frac{r}{2} \right) + \|f\|_{\mathcal{M}^m, \frac{a+1}{a} (Q(r/2))} \left( \frac{r}{2} \right) ^\beta \right]. \tag{3.22}
\end{cases}
\]
We will prove this assertion by induction arguments. The case \( k = 1 \) is a direct application of Lemma 3.2. To illustrate the ideas in the proof of (3.22) for arbitrary \( k \), we also present the detail of (3.22) for \( k = 2 \). Then, we assume that there holds (3.22) with \( k - 1 \) and prove (3.22) for \( k \).

To proceed further, we take \( \theta \) sufficiently small such that
\[
2C_3 \theta^\frac{1}{2} (\frac{1}{\alpha - 1} - \beta_1) < 1. \tag{3.23}
\]
First, with (3.20) in hand, we can invoke Lemma 3.2 to obtain
\[
\Phi \left( x_0, t_0; \frac{r}{2} \theta^k \right) \leq C_3 \theta^\frac{1}{2} \left[ \Phi \left( x_0, t_0; \frac{r}{2} \right) + \|f\|_{\mathcal{M}^m, \frac{a+1}{a} (Q(r/2))} \left( \frac{r}{2} \right) ^\beta \right]
\]
\[
= C_3 \theta^\frac{1}{2} (\frac{1}{\alpha - 1} - \beta_1) \theta^\frac{1}{2} (\frac{1}{\alpha - 1} + \beta_1) \left[ \Phi \left( x_0, t_0; \frac{r}{2} \right) + \|f\|_{\mathcal{M}^m, \frac{a+1}{a} (Q(r/2))} \left( \frac{r}{2} \right) ^\beta \right]
\]
\[
\leq \theta^{\beta_1} \left[ \Phi \left( x_0, t_0; \frac{r}{2} \right) + \|f\|_{\mathcal{M}^m, \frac{a+1}{a} (Q(r/2))} \left( \frac{r}{2} \right) ^\beta \right],
\]
where we have used (3.23). This together with (3.20) implies that we have proved (3.22) with \( k = 1 \).

Second, we use (3.22) with \( k = 1 \) and (3.20) to get
\[
\Phi \left( x_0, t_0; \frac{r}{2} \theta^2 \right) + \|f\|_{\mathcal{M}^m, \frac{a+1}{a} (Q(\theta^2))} \left( \frac{r}{2} \theta^2 \right) ^\beta \leq \theta^{2\beta_1} \left[ \Phi \left( x_0, t_0; \frac{r}{2} \right) + \|f\|_{\mathcal{M}^m, \frac{a+1}{a} (Q(r/2))} \left( \frac{r}{2} \right) ^\beta \right]
\]
\[
+ \|f\|_{\mathcal{M}^m, \frac{a+1}{a} (Q(\theta^2))} \left( \frac{r}{2} \theta^2 \right) ^\beta \leq 2\theta^{\beta_1} \left[ \Phi \left( x_0, t_0; \frac{r}{2} \right) + \|f\|_{\mathcal{M}^m, \frac{a+1}{a} (Q(r/2))} \left( \frac{r}{2} \right) ^\beta \right]
\]
\[
\leq \theta^{3\beta_1} \varepsilon_{02}.
\]
By means of Lemma 3.2 again and (3.23), we conclude that
\[
\Phi \left( x_0, t_0; \frac{r}{2} \theta^2 \right) \leq C_3 \theta^\frac{1}{2} \left[ \Phi \left( x_0, t_0; \frac{r}{2} \theta^2 \right) + \|f\|_{\mathcal{M}^m, \frac{a+1}{a} (Q(r/2))} \left( \frac{r}{2} \theta^2 \right) ^\beta \right]
\]
\[
\leq C_3 \theta^\frac{1}{2} (\frac{1}{\alpha - 1} - \beta_1) \theta^\frac{1}{2} (\frac{1}{\alpha - 1} + \beta_1) \left[ \theta^{\beta_1} \left[ \Phi \left( x_0, t_0; \frac{r}{2} \right) + \|f\|_{\mathcal{M}^m, \frac{a+1}{a} (Q(r/2))} \left( \frac{r}{2} \right) ^\beta \right] + \|f\|_{\mathcal{M}^m, \frac{a+1}{a} (Q(r/2))} \left( \frac{r}{2} \right) ^\beta \right].
\]
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For any $k$ as a consequence, we show (3.22) with (3.20), we infer that
\[
\Phi(x_0, t_0; \frac{r}{2}) + \|f\|_{\mathcal{M}^{m, \frac{\alpha + 1}{\alpha}}(Q(r/2))} \left(\frac{r}{2}\right)^{\beta}
\]
\[
\leq C_3 \theta^{\frac{1}{2}} (\frac{1}{\alpha - 1} - \beta_1) \theta^{\frac{1}{2}} (\frac{1}{\alpha - 1} + \beta_1) \theta^{\beta_1} 2 \left[ \Phi(x_0, t_0; \frac{r}{2}) + \|f\|_{\mathcal{M}^{m, \frac{\alpha + 1}{\alpha}}(Q(r/2))} \left(\frac{r}{2}\right)^{\beta} \right]
\]
\[
\leq \theta^{2\beta_1} \left[ \Phi(x_0, t_0; \frac{r}{2}) + \|f\|_{\mathcal{M}^{m, \frac{\alpha + 1}{\alpha}}(Q(r/2))} \left(\frac{r}{2}\right)^{\beta} \right].
\]  
\tag{3.24}

As a consequence, we show (3.22) with $k = 2$.

Third, we assume that (3.22) is valid for $k = k - 1$. With the help of (3.22)$_2$ with $k = k - 1$ and (3.20), we infer that
\[
\Phi(x_0, t_0; \frac{r}{2}) + \|f\|_{\mathcal{M}^{m, \frac{\alpha + 1}{\alpha}}(Q(r/2))} \left(\frac{r}{2}\right)^{\beta}
\]
\[
\leq \theta^{\beta_1(k-1)} \left[ \Phi(x_0, t_0; \frac{r}{2}) + \|f\|_{\mathcal{M}^{m, \frac{\alpha + 1}{\alpha}}(Q(r/2))} \left(\frac{r}{2}\right)^{\beta} \right]
\]
\[
+ \|f\|_{\mathcal{M}^{m, \frac{\alpha + 1}{\alpha}}(Q(r/2))} \left(\frac{r}{2}\right)^{\beta}
\]
\[
\leq 2\theta^{\beta_1(k-1)} \left[ \Phi(x_0, t_0; \frac{r}{2}) + \|f\|_{\mathcal{M}^{m, \frac{\alpha + 1}{\alpha}}(Q(r/2))} \left(\frac{r}{2}\right)^{\beta} \right]
\]
\[
\leq \theta^{2\beta_1(k-1)} \varepsilon_{02}.
\]

Arguing the same manner as (3.24), we have
\[
\Phi(x_0, t_0; \frac{r}{2}) \leq C_3 \theta^{\frac{1}{2}} (\frac{1}{\alpha - 1} - \beta_1) \theta^{\frac{1}{2}} (\frac{1}{\alpha - 1} + \beta_1) \theta^{\beta_1} 2 \left[ \Phi(x_0, t_0; \frac{r}{2}) + \|f\|_{\mathcal{M}^{m, \frac{\alpha + 1}{\alpha}}(Q(r/2))} \left(\frac{r}{2}\right)^{\beta} \right]
\]
\[
\leq C_3 \theta^{\frac{1}{2}} (\frac{1}{\alpha - 1} - \beta_1) \theta^{\frac{1}{2}} (\frac{1}{\alpha - 1} + \beta_1) \theta^{\beta_1} 2 \left[ \Phi(x_0, t_0; \frac{r}{2}) + \|f\|_{\mathcal{M}^{m, \frac{\alpha + 1}{\alpha}}(Q(r/2))} \left(\frac{r}{2}\right)^{\beta} \right]
\]
\[
+ \|f\|_{\mathcal{M}^{m, \frac{\alpha + 1}{\alpha}}(Q(r/2))} \left(\frac{r}{2}\right)^{\beta}
\]
\[
\leq C_3 \theta^{\frac{1}{2}} (\frac{1}{\alpha - 1} - \beta_1) \theta^{\frac{1}{2}} (\frac{1}{\alpha - 1} + \beta_1) \theta^{\beta_1(k-1)} 2 \left[ \Phi(x_0, t_0; \frac{r}{2}) + \|f\|_{\mathcal{M}^{m, \frac{\alpha + 1}{\alpha}}(Q(r/2))} \left(\frac{r}{2}\right)^{\beta} \right]
\]
\[
\leq \theta^{2\beta_1} \left[ \Phi(x_0, t_0; \frac{r}{2}) + \|f\|_{\mathcal{M}^{m, \frac{\alpha + 1}{\alpha}}(Q(r/2))} \left(\frac{r}{2}\right)^{\beta} \right].
\]

Collecting the above estimates, we derive that (3.22) for $k = k$. Hence, it is shown that (3.22) is valid.

Now, it follows from (3.22)$_2$ that, $\forall \rho \in (0, \frac{r}{2})$,
\[
\Phi(x_0, t_0; \rho) \leq C \theta^{\beta_1} (\frac{1}{2})^\beta_1 \left[ \Phi(x_0, t_0; \frac{r}{2}) + \|f\|_{\mathcal{M}^{m, \frac{\alpha + 1}{\alpha}}(Q(r/2))} \left(\frac{r}{2}\right)^{\beta} \right].
\]  
\tag{3.25}

For any $\varphi \in (0, \frac{r}{2})$, it is obvious that $\varphi < 2\varphi < \frac{r}{2}$. Therefore, employing Lemma 2.3 with $\vartheta_1 = \frac{1}{2}$,
(3.25), and $\alpha > 1$, we observe that
\[
\left( \iint_{Q(\varepsilon)} |h(t) - h_{\varepsilon}|^{\alpha+1} \, dx \, dt \right)^{\frac{1}{\alpha+1}} \leq C(2\beta)^{\frac{(\alpha+1)}{2}} \left\{ \frac{1}{(2\beta)^{\left(\frac{1}{2\beta}\right)}} \left( \iint_{Q(\varepsilon)} |h_x|^{\alpha+1} \, dx \, dt \right)^{\frac{1}{2}} \right\}^{\frac{1}{\alpha+1}}
\]
\[
+ C(2\beta)^{4-\frac{\beta}{\alpha}} \|f\|_{\mathcal{M}_{m+ \frac{\alpha+1}{2}}(Q(\varepsilon))}
\]
\[
\leq C(2\beta)^{\frac{(\alpha-2)}{\alpha-1}} \left[ \phi \left( \frac{1}{\alpha-1} \right) + \|f\|_{\mathcal{M}_{m+ \frac{\alpha+1}{2}}(Q(\varepsilon))} \right] + C(2\beta)^{4-\frac{\beta}{\alpha}} \|f\|_{\mathcal{M}_{m+ \frac{\alpha+1}{2}}(Q(\varepsilon))}
\]
\[
\leq C \beta_2,
\]
where $\beta_2 = \min\left( \frac{(\alpha-2)}{\alpha-1} + \frac{\alpha}{4} - \frac{\alpha}{5} \right)$. If $\alpha \geq 2$, it is clear that $\beta_2 > 0$. If $1 < \alpha < 2$, the choice of $\beta_1$ and $\beta_2$ (that is, $\frac{\alpha}{\alpha-1} + \frac{(\alpha-2)}{\alpha-1} > 0$ and $\frac{3\alpha-2}{\alpha-1} - \frac{\alpha}{5} + \frac{(\alpha-2)}{\alpha-1} > 0$) also ensures that $\beta_2 > 0$. From (3.26) and Lemma 2.7, we complete the proof of Proposition 3.1.

3.2 Regularity Criterion at Infinite Scales

**Proof of Theorem 1.2** The condition (1.5) yields that there exists a positive constant $r_0$ such that
\[
E(r) \leq \varepsilon_0, \quad \text{for any } r \leq r_0.
\]
(3.27)

It follows from the local energy inequality (2.3) and the Hölder inequality that
\[
E_s(r) + E(r)
\]
\[
\leq C \left[ D_{a+1}(2r) + E_{a+1}(2r) + E_{a+1}(2r) D_{a+1}(2r) \right.
\]
\[
+ r^{\frac{3\alpha-2}{\alpha-1}} \|f\|_{\mathcal{M}_{m+ \frac{\alpha+1}{2}}(Q(2r))} \left. D_{a+1}^{\frac{1}{\alpha+1}}(2r) \right] + r^{\frac{3\alpha-2}{\alpha-1}} \|f\|_{\mathcal{M}_{m+ \frac{\alpha+1}{2}}(Q(2r))} \left. D_{a+1}^{\frac{1}{\alpha+1}}(2r) \right].
\]
(3.28)

Since $h - C$ is also the solution of (1.1), we can replace $h$ by $h_{Q(r)}$ or $h_{Q(r), \sigma}$ in (3.28). We will take $h - h_{Q(r)}$ in (3.28). It is worth remarking that the following proof works for both $h - h_{Q(r)}$ and $h - h_{Q(r), \sigma}$. Hence, we reformulate inequality (3.28) as
\[
\tilde{E}_s(r) + E(r)
\]
\[
\leq C \left[ \tilde{D}_{a+1}^{\frac{3}{\alpha+1}}(r) + E_{a+1}^{\frac{3}{\alpha+1}}(r) + E_{a+1}(2r) + E_{a+1}(2r) \tilde{D}_{a+1}^{\frac{1}{\alpha+1}}(2r) \right.
\]
\[
+ r^{\frac{3\alpha-2}{\alpha-1}} \|f\|_{\mathcal{M}_{m+ \frac{\alpha+1}{2}}(Q(2r))} \tilde{D}_{a+1}^{\frac{1}{\alpha+1}}(2r) \left. \right].
\]
(3.29)

Taking advantage of Lemma 2.3, we have
\[
\tilde{D}_{a+1}(r) \leq E_{a+1}(4r) + E_{a+1}(4r) + r^{(\alpha+1)}(\frac{2-3\alpha}{\alpha-1}) \|f\|_{\mathcal{M}_{m+ \frac{\alpha+1}{2}}(Q(2r))}.
\]
(3.30)
From the interpolation inequality (2.12), we see that
\[ E_{\alpha+1}(r) \leq \tilde{E}_s^{\alpha/3} (r) E_{\frac{2\alpha}{3}+1}^{\frac{2}{3}} (r), \quad E_{\alpha+1}^{\alpha} (r) \leq \tilde{E}_s^{\frac{(\alpha+3)}{(3a+1)}} (r) E_{\frac{2\alpha}{3}+1}^{\frac{1}{3}} (r). \] (3.31)

Combining (3.30) and (3.31), we know that
\[
\begin{aligned}
\tilde{D}_{\alpha+1}^{3/2}(r) &\leq CE_{\alpha+1}^{\alpha} (4r) + CE_{\alpha+1}^{\alpha} (2r) + Cr^{\frac{2-3\alpha}{1-\alpha}} \|f\|_{M^{\alpha,1}(Q(2r))} \\
&\leq CE_{\alpha+1}^{\alpha} (2r) E_{\frac{2\alpha}{3}+1}^{\frac{2}{3}} (2r) + Cr^{\frac{2-3\alpha}{1-\alpha}} \|f\|_{M^{\alpha,1}(Q(2r))} \\
&\leq CE_{\alpha+1}^{\alpha} (2r) E_{\frac{2\alpha}{3}+1}^{\frac{2}{3}} (2r) + Cr^{\frac{2-3\alpha}{1-\alpha}} \|f\|_{M^{\alpha,1}(Q(2r))},
\end{aligned}
\] (3.32)

and
\[
\begin{aligned}
\tilde{D}_{\alpha+1}^{3/2}(r) &\leq CE_{\alpha+1}^{\alpha} (2r) E_{\frac{2\alpha}{3}+1}^{\frac{2}{3}} (2r) + Cr^{\frac{2-3\alpha}{1-\alpha}} \|f\|_{M^{\alpha,1}(Q(2r))} \\
&\leq CE_{\alpha+1}^{\alpha} (2r) E_{\frac{2\alpha}{3}+1}^{\frac{2}{3}} (2r) + Cr^{\frac{2-3\alpha}{1-\alpha}} \|f\|_{M^{\alpha,1}(Q(2r))}.
\end{aligned}
\] (3.33)

As a consequence, we arrive at
\[
\begin{aligned}
\tilde{E}_s^{\alpha} (2r) \tilde{D}_{\alpha+1}^{3/2}(r) &\leq CE_{\alpha+1}^{\alpha} (2r) E_{\frac{2\alpha}{3}+1}^{\frac{2}{3}} (2r) + CE_{\alpha+1}^{\alpha} (2r) E_{\frac{2\alpha}{3}+1}^{\alpha} (2r) \\
&+ Cr^{\frac{2-3\alpha}{1-\alpha}} \|f\|_{M^{\alpha,1}(Q(2r))} \tilde{E}_s^{\frac{(\alpha+3)}{(3a+1)}} (r) E_{\frac{2\alpha}{3}+1}^{\frac{1}{3}} (r).
\end{aligned}
\] (3.34)

Plugging (3.31), (3.32), (3.33) and (3.34) into (3.29), we end up with
\[
\begin{aligned}
\tilde{E}_s^{\alpha} (r) + E(r) &\leq CE_{\alpha+1}^{\alpha} (r) + CE_{\alpha+1}^{\alpha} (2r) + E_{\alpha+1}^{\alpha} (2r) \\
&+ Cr^{\frac{2-3\alpha}{1-\alpha}} \|f\|_{M^{\alpha,1}(Q(2r))} \tilde{E}_s^{\frac{(\alpha+3)}{(3a+1)}} (r) E_{\frac{2\alpha}{3}+1}^{\frac{1}{3}} (r).
\end{aligned}
\] (3.35)

Since \( \frac{\alpha+1}{\alpha} > \frac{5(\alpha-1)}{3a-2} \) with \( 1 < \alpha < 7/3 \), there holds \( r^{\frac{2-3\alpha}{1-\alpha}} \|f\|_{M^{\alpha,1}(Q(2r))} \to 0 \) as \( r \to 0 \).

Notice that \( 1 < \alpha < 7/3 \) guarantees \( \max\{\frac{\alpha+3}{3a+1}, \frac{\alpha(\alpha+3)}{3(3a+1)}, \frac{\alpha+3}{3a+1}, \frac{\alpha(\alpha+3)}{3(3a+1)}\} < 1 \). In addition, in view of (3.35), the iteration method as [10] together with (3.27) helps us to get the smallness of \( \tilde{E}_s(r) + E(r) \) for \( 0 < r < r_2 < r_1 \). The interpolation inequality implies the smallness of \( E_{\alpha+1}(r) \). We conclude the proof by Theorem 1.1. \( \square \)
3.3 Improvement of Regularity Criterion

*Proof of Theorem 1.11*  First, we focus on the proof of (1.8). We rewrite (2.12) with $\alpha = 2$ and $f = 0$ in Lemma 2.2 as

$$E_3(4r) \leq CE_3^2(4r)E_3^{\frac{5}{2}}(4r)E_{4,2}^{\frac{2}{3}}(4r).$$  (3.36)

With (3.36) in hand, arguing in the same manner as (3.32), we observe that

$$\tilde{D}_3^\frac{1}{2}(2r) \leq E_{3}^{\frac{1}{2}}(4r) + E_3(4r)
\leq (E_3^{\frac{1}{2}}(4r)E_3^{\frac{5}{2}}(4r)E_{4,2}^{\frac{2}{3}}(4r))^{\frac{1}{2}} + (E_3^{\frac{1}{2}}(4r)E_3^{\frac{5}{2}}(4r)E_{4,2}^{\frac{2}{3}}(4r))^{\frac{1}{2}}.$$  (3.37)

From (3.29) with $\alpha = 2$ and $f = 0$, we conclude that

$$\tilde{E}_s(r) + E(r) \leq C[D_3^\frac{1}{2}(2r) + E_3^2(2r) + E_3^2(2r)\tilde{D}_3^\frac{1}{2}(2r)].$$  (3.38)

Plugging (3.36) and (3.37) into (3.38), we infer that

$$\tilde{E}_s(r) + E(r) \leq C[E_3^{\frac{1}{2}}(4r)E_3^{\frac{5}{2}}(4r)E_{4,2}^{\frac{2}{3}}(4r)]^{\frac{1}{2}} + C[E_3^{\frac{1}{2}}(4r)E_3^{\frac{5}{2}}(4r)E_{4,2}^{\frac{2}{3}}(4r)]^{\frac{1}{2}}
+ [E_3^{\frac{1}{2}}(4r)E_3^{\frac{5}{2}}(4r)E_{4,2}^{\frac{2}{3}}(4r)]^{\frac{1}{2}} + C[E_3^{\frac{1}{2}}(4r)E_3^{\frac{5}{2}}(4r)E_{4,2}^{\frac{2}{3}}(4r)]^{\frac{1}{2}}
+ (E_3^{\frac{1}{2}}(4r)E_3^{\frac{5}{2}}(4r)E_{4,2}^{\frac{2}{3}}(4r))^{\frac{1}{2}}.$$  (3.39)

Set $F(r) = \tilde{E}_s(r) + E(r)$, hence,

$$F(r) \leq F_3^\frac{1}{2}(4r)E_{4,2}^{\frac{2}{3}}(4r) + F(4r)E_{4,2}^{\frac{2}{3}}(4r) + F(4r)E_{4,2}^{\frac{2}{3}}(4r)
+ F_3^\frac{1}{2}(4r)E_{4,2}^{\frac{2}{3}}(4r)[F_3^\frac{1}{2}(4r)E_{4,2}^{\frac{2}{3}}(4r) + F_3^\frac{1}{2}(4r)E_{4,2}^{\frac{2}{3}}(4r)]
\leq CF_3^\frac{1}{2}(4r)E_{4,2}^{\frac{2}{3}}(4r) + CF(4r)E_{4,2}^{\frac{2}{3}}(4r) + CF(4r)E_{4,2}^{\frac{2}{3}}(4r).$$

This together with iteration method mentioned above implies the smallness of $F(r)$ under the smallness of $E_{4,2}^{\frac{2}{3}}(4r)$. Combining this with Theorem 1.1, we complete the proof of this part.

We turn our attention to the proof of (1.9). From Lemma 2.5, we see that

$$D_3(r) \leq CD_{\infty,1}^3(r)E_3^2(r) + CD_{\infty,1}^3(r),$$  (3.40)

$$E_3(r) \leq CD_{\infty,1}^3(r)E(r) + CD_{\infty,1}^3(r).$$

Substituting (3.28) with $\alpha = 2$ and $f = 0$ into (3.40), we infer that

$$D_3(r) \leq CD_{\infty,1}^3(r)[D_3^\frac{1}{2}(2r) + E_3^2(2r) + E_3^2(2r)D_3^\frac{1}{2}(2r)]^{\frac{1}{2}} + CD_{\infty,1}^3(r),$$

$$E_3(r) \leq CD_{\infty,1}^3(r)[D_3^\frac{1}{2}(2r) + E_3^2(2r) + E_3^2(2r)D_3^\frac{1}{2}(2r)] + CD_{\infty,1}^3(r).$$

Hence,

$$D_3(r) + E_3(r)
\leq CD_{\infty,1}^3(r)[D_3^\frac{1}{2}(2r) + E_3^2(2r) + E_3^2(2r)D_3^\frac{1}{2}(2r)]^{\frac{1}{2}}
+ CD_{\infty,1}^3(r)[D_3^\frac{1}{2}(2r) + E_3^2(2r) + E_3^2(2r)D_3^\frac{1}{2}(2r)] + CD_{\infty,1}^3(r).$$  (3.41)
Before going further, we write
\[ G(\mu) = D_3(\mu) + E_3(\mu). \]

As a consequence, we get
\[ G(r) \leq CD_{\infty,1}(r)[G^{\frac{4}{3}}(2r) + G_3(2r)]^{\frac{4}{3}} + CD_{\infty,1}(r)[G^{\frac{4}{3}}(2r) + G_3(2r)] + CD_{\infty,1}^3(r). \]

An iteration argument leads to the smallness of \( G(r) \) under the smallness of \( D_{\infty,1}^3(r) \). With this in hand, Theorem 1.1 entails us to achieve the proof of this case. \( \square \)

4 Partial Regularity of the Modified Navier–Stokes Equations

In this section, we study the partial regularity of suitable weak solutions of the modified Navier–Stokes equations (1.6). In Step 1, in the spirit of blow up technique in [19–21, 39], we prove Theorem 1.6 by the fractional integration theorem [25, 29] involving parabolic Morrey spaces. Step 2 is devoted to optimal Hausdorff dimension of possible singular point set \( S \) in the equations (1.6).

4.1 Proof of Theorem 1.6

**Lemma 4.1** There exist \( \varepsilon_1 > 0 \) and \( \theta \in (0, \frac{1}{2}) \) such that if \( (u, \Pi) \) is a suitable weak solutions of the modified Navier–Stokes equations in \( Q(1) \) satisfying
\[
\left( \int_{Q(1)} |u|^{\alpha+1} dx \right)^{\frac{1}{\alpha+1}} + \left( \int_{Q(1)} |\Pi|^{\frac{\alpha+1}{\alpha}} dx \right)^{\frac{\alpha}{\alpha+1}} \leq \varepsilon_1,
\]
then
\[
\left( \theta \frac{\alpha-4n}{\alpha} \int_{Q(1)} |u|^{\alpha+1} dx \right)^{\frac{1}{\alpha+1}} + \left( \theta \frac{\alpha-4n}{\alpha} \int_{Q(1)} |\Pi|^{\frac{\alpha+1}{\alpha}} dx \right)^{\frac{\alpha}{\alpha+1}} \leq \frac{1}{2} \left( \int_{Q(1)} |u|^{\alpha+1} dx \right)^{\frac{1}{\alpha+1}} + \left( \int_{Q(1)} |\Pi|^{\frac{\alpha+1}{\alpha}} dx \right)^{\frac{\alpha}{\alpha+1}}.
\]

**Proof** We suppose that the statement is invalid. Then, for any \( \theta \in (0, \frac{1}{2}) \), there exists a sequence of suitable weak solutions of the modified Navier–Stokes equations and a sequence \( \varepsilon_{1k} \) such that
\[
\left( \int_{Q(1)} |u_k|^{\alpha+1} dx \right)^{\frac{1}{\alpha+1}} + \left( \int_{Q(1)} |\Pi_k|^{\frac{\alpha+1}{\alpha}} dx \right)^{\frac{\alpha}{\alpha+1}} = \varepsilon_{1k} \to 0 \quad \text{as} \quad k \to \infty,
\]
\[
\left( \theta \frac{\alpha-4n}{\alpha} \int_{Q(1)} |u_k|^{\alpha+1} dx \right)^{\frac{1}{\alpha+1}} + \left( \theta \frac{\alpha-4n}{\alpha} \int_{Q(1)} |\Pi_k|^{\frac{\alpha+1}{\alpha}} dx \right)^{\frac{\alpha}{\alpha+1}} > \frac{1}{2} \varepsilon_{1k}.
\]

Before going further, we write
\[
(v_i)_k = \varepsilon_{1k}^{-1} (u_i)_k, \quad \pi_k = \varepsilon_{1k}^{-1} \Pi_k, \quad i = 1, 2, 3.
\]

Hence, we get
\[
\left( \int_{Q(1)} |v_k|^{\alpha+1} dx \right)^{\frac{1}{\alpha+1}} + \left( \int_{Q(1)} |\pi_k|^{\frac{\alpha+1}{\alpha}} dx \right)^{\frac{\alpha}{\alpha+1}} = 1,
\]
\[
\left( \theta \frac{\alpha-4n}{\alpha-1} \int_{Q(1)} |v_k|^{\alpha+1} dx \right)^{\frac{1}{\alpha+1}} + \left( \theta \frac{\alpha-4n}{\alpha-1} \int_{Q(1)} |\pi_k|^{\frac{\alpha+1}{\alpha}} dx \right)^{\frac{\alpha}{\alpha+1}} > \frac{1}{2}. \quad (4.1)
\]
In addition, there holds

\[
\int_{Q(1)} \left[ -v_k \partial_t \phi - v_k \Delta \phi - \varepsilon_1^{\nu - 1} \sum_{i=1}^{3} (v_i)_k^{\nu - 1} \mathbf{v} \cdot \nabla \phi_i - \pi_k \text{div} \phi \right] dx \tau = 0, \tag{4.2}
\]

and the local energy inequality below

\[
\int |v_k|^2 \varphi dx + 2 \int_{-T}^t \int_{B(1)} |\nabla v_k|^2 \varphi \leq \int_{-T}^t \int_{B(1)} |v_k|^2 (\varphi_t + \Delta \varphi) + 2\varepsilon_1^{\nu - 1} \frac{\alpha - 1}{\alpha} \int_{-T}^t \int_{B(1)} v_k \cdot \nabla \varphi \left[ (v_1)_k^\alpha \right. + (v_2)_k^\alpha + (v_3)_k^\alpha \right]

+ 2 \int_{-T}^t \int_{B(1)} v_k \cdot \nabla \varphi \pi_k dx dt. \tag{4.3}
\]

We conclude by (4.1) that there exist the subsequences of \( v_k \) and \( \pi_k \) satisfying

\[
v_k \rightharpoonup v \ \text{in} \ L^{\alpha + 1}(Q(1)), \quad \pi_k \rightharpoonup \pi \ \text{in} \ L^{\frac{\alpha + 1}{\alpha}}(Q(1)).
\]

In addition, the lower semicontinuity ensures that

\[
\|v\|_{L^{\alpha + 1}(Q(1))} + \|\pi\|_{L^{\frac{\alpha + 1}{\alpha}}(Q(1))} \leq C. \tag{4.4}
\]

Combining the Holder inequality and (4.3), we arrive at

\[
\|v_k\|_{L^\infty(L^2)(Q(\frac{T}{2}))} + \|\nabla v_k\|_{L^2(L^2)(Q(\frac{T}{2}))} \leq C. \tag{4.5}
\]

As a consequence, taking \( k \to \infty \) in (4.2), we infer that

\[
\int \int_{Q(1)} [-v \partial_t \phi - v \Delta \phi - \pi \text{div} \phi] dx \tau = 0.
\]

Thanks to the regularity of the Stokes equations and (4.4), we know that

\[
\theta^{\frac{6 - 4\alpha}{\alpha - 1}} \int_{Q(\theta)} |v|^\alpha dx dt \leq \theta^{\frac{6 - 4\alpha}{\alpha - 1}} \int_{Q(1)} \int_{Q(1)} |v|^\alpha dx dt \leq C \theta^{\frac{1 + \alpha}{\alpha - 1}}.
\]

Since the pressure equations of \( \pi_k \) is given by

\[
\Delta \pi_k = -\varepsilon_1^{\nu - 1} \partial_i \partial_j [(v_j)_k (v_i)_k^{\nu - 1}],
\]

by a slight modification the derivation of Lemma 2.9 and (4.1), we further get

\[
\theta^{\frac{6 - 4\alpha}{\alpha - 1}} \int_{Q(\theta)} \|\pi_k\|^{\alpha + 1} dx dt \leq C \varepsilon_1^{2 \nu - 1} \theta^{\frac{6 - 4\alpha}{\alpha - 1}} \int_{Q(1)} |v_k|^\alpha dx dt + C \theta^{\frac{6 - 4\alpha}{\alpha - 1}} \int_{Q(1)} \|\pi_k\|^{\alpha + 1} dx dt

\leq C \varepsilon_1^{2 \nu - 1} \theta^{\frac{6 - 4\alpha}{\alpha - 1}} + C \theta^{\frac{6 - 4\alpha}{\alpha - 1}},
\]

that is

\[
\left( \theta^{\frac{6 - 4\alpha}{\alpha - 1}} \int_{Q(\theta)} \|\pi_k\|^{\alpha + 1} dx dt \right)^{\frac{\alpha + 1}{\alpha - 1}} \leq C \varepsilon_1^{\nu - 1} \theta^{\frac{6 - 4\alpha}{\alpha - 1}} + C \theta^{\frac{6 - 4\alpha}{\alpha - 1}}.
\]

Substituting this into (4.1), we observe that

\[
\frac{1}{2} < \left( \theta^{\frac{6 - 4\alpha}{\alpha - 1}} \int_{Q(\theta)} |v_k|^\alpha dx dt \right)^{\frac{\alpha + 1}{\alpha - 1}} + C \varepsilon_1^{\nu - 1} \theta^{\frac{6 - 4\alpha}{\alpha - 1}} + C \theta^{\frac{6 - 4\alpha}{\alpha - 1}}. \tag{4.6}
\]
It follows from the Hölder inequality and (4.5) that
\[
\left| \int \int_{Q(\frac{\rho}{s})} v_{kt} \phi dx dt \right|
= \left| - \int \int_{Q(1)} \nabla v_k \nabla \phi + \varepsilon_k^{\alpha - 1} \sum_{i=1}^{3} (v_i)_k^{\alpha - 1} v_k \cdot \nabla \phi_i + \pi_k \text{div} \phi \right|
\leq (\|\nabla v_k\|_{L^2(Q(\frac{\rho}{s}))} + \|v_k\|_{\frac{\alpha + 1}{\alpha}}(Q(\frac{\rho}{s})) + \|\pi_k\|_{\frac{\alpha + 1}{\alpha}}(Q(\frac{\rho}{s}))) \|\nabla \phi\|_{L^{\alpha + 1}(Q(\frac{\rho}{s}))},
\]
Hence, we find that
\[
\|v_{kt}\|_{L^{\frac{\alpha + 1}{\alpha}}(W^{1,\alpha + 1}(B(\frac{\rho}{s})))} \leq C.
\]
This together with \(\|v\|_{L^2(I^1)} \leq C\) and the classical Aubin–Lions lemma implies that
\[
v_k \to v \quad \text{in} \quad L^2\left(Q\left(\frac{7}{8}\right)\right).
\]
Notice that (4.5) implies that \(\|v_k\|_{L^{\frac{7}{8}}(Q(\frac{\rho}{s}))} \leq C\), we obtain
\[
v_k \to v \quad \text{in} \quad L^{\alpha + 1}\left(Q\left(\frac{7}{8}\right)\right), \quad \alpha < \frac{7}{3}.
\]
Passing the limit in (4.6), we know that
\[
\frac{1}{2} < C \theta^{\frac{\alpha - 1}{\alpha - 1}} + C \varepsilon_k^{\alpha - 1} \theta^{\frac{\alpha(3 - 4\alpha)}{\alpha - 1}} + C \theta^{\frac{\alpha(3 - 4\alpha)}{\alpha - 1}}.
\]
First choosing \(\theta\) sufficiently small and then taking \(k\) sufficiently large, we get a contradiction. \(\square\)

**Proof of Theorem 1.6** With the aid of Lemma 4.1, arguing as the same manner as in (3.25), we have, for any \((x, t) \in Q(x, t; \rho)\) with \(\rho \leq \frac{3}{\lambda}\), there exists a constant \(0 < \beta_3 < 1\) such that
\[
\left(\rho^{\frac{1}{\alpha - 1}} \int_{Q(\rho)} |u|^\alpha dx dt \right)^{\frac{1}{\alpha}} + \left(\rho^{\frac{1}{\alpha - 1}} \int_{Q(\rho)} \|\Pi\|_{\frac{\alpha + 1}{\alpha}} dx dt \right)^{\frac{1}{\alpha}} \leq C \rho^{\frac{\alpha}{\alpha + 1}} \varepsilon_1.
\]
This means that \(\Pi, |u|^\alpha \in \mathcal{M}^{\frac{5(\alpha + 1)}{4\alpha - 1 - \beta_3}}(Q(\frac{1}{4}))\) and breaks the scaling of the equations. Then one can apply the fractional integration theorem [25, 29] (Riesz potential estimate [19–21, 39]) involving parabolic Morrey spaces to get that, for any \(q < \infty, u \in L^q(Q(\frac{1}{4}))\). The desired boundness of \(|v|\) can be improved by bootstrapping arguments (see [14, 20]). The rigorous proof can be found in these works. Here we just outline the proof of \(u \in L^q(Q(\frac{1}{4}))\). The fractional integration theorem due to [25, 29] reads
\[
\left\| \int \int_{\mathbb{R}^3} \left( \frac{f}{|x - \xi| + \sqrt{|t - \tau|}} \right)^4 d\xi d\tau \right\|_{L^p(I \times I)} \leq \|f\|_{L^m(I \times I)} \|f\|_{M^{\frac{m}{2} - 1}}^{\frac{m}{2} - 1} \mathcal{M}^{\frac{m}{2} - 1},
\]
with
\[
\frac{1}{p} = \frac{q}{m} \left( \frac{1}{q} - \frac{1}{5} - \lambda \right).
\]
Roughly speaking, notice that
\[
|u| \approx \int \int \left( \frac{|u|^\alpha + |\Pi|}{(|x - \xi| + \sqrt{|t - \tau|})^4} d\xi d\tau, \right.
\]
then, applying the fractional integration theorem mentioned above, we see that $u \in L^p$ with

$$p < \frac{1}{m} \left( 1 - \frac{q}{\alpha(5 - \lambda)} \right) = \frac{m}{\alpha - \frac{q}{5 - \lambda}}.$$  

We start with $q \equiv \alpha + 1$, $m_0 = \alpha + 1$, to obtain

$$p_1 < \frac{\alpha + 1}{1 + \alpha - \beta_3(\alpha - 1)\alpha}.$$  

Then, we set $m_1 = p_1$ to derive that

$$p_2 < \frac{m_1}{1 + \alpha - \beta_3(\alpha - 1)\alpha} = \frac{p_{k-1}}{1 + \alpha - \beta_3(\alpha - 1)\alpha}.$$  

Likewise,

$$p_k < \frac{m_k}{1 + \alpha - \beta_3(\alpha - 1)\alpha} = \frac{p_{k-1}}{1 + \alpha - \beta_3(\alpha - 1)\alpha}.$$  

We see that

$$\alpha + 1 < p_1 < p_2 < \cdots < p_k < \cdots < \infty.$$  

Therefore, for any $\alpha + 1 \le q < \infty$, we get $u \in L^p$. The proof of this theorem is completed. \qed

4.2 Proof of Theorem 1.7

Next, we turn our attention to the proof of Theorem 1.7. From the hypothesis of this theorem, we know that there exists a constant $r_0$ such that $E_1(u; r) \le \varepsilon_2$ for $r \le r_0$. It seems that, unlike the Navier–Stokes equations, the pressure equations of the modified Navier–Stokes equations

$$\Delta \Pi = -\phi \partial_i \partial_j (u_j - C)(u_i^{\alpha - 1} - C),$$

is invalid. To this end, by means of the local energy inequality and the hypothesis, we first show the smallness of $E_{\alpha + 1}(u; r)$ for $0 < r \le r_1 < r_0$. Then, using the local energy inequality once again, we can complete the proof by Theorem 1.6.

Proof of Theorem 1.7 By means of the usual test function and Hölder’s inequality, we derive from the local energy inequality (2.19) that

$$E(u; r) + E_1(u; r) \le C[E_2(u; 2r) + E_{\alpha + 1}(u; 2r) + P_{\frac{\alpha + 1}{\alpha}}(2r)E_{\alpha + 1}(u; 2r)].$$  

(4.7)

Multiplying (4.7) by $\varepsilon^{\frac{3}{16}}$ and using the Hölder and Young inequalities, we see that

$$\varepsilon^{\frac{3}{16}}E(u; r) + \varepsilon^{\frac{3}{16}}E_1(u; r) \le C\varepsilon^{\frac{3}{16}}[E_2(u; 2r) + E_{\alpha + 1}(u; 2r) + P_{\frac{\alpha + 1}{\alpha}}(2r)E_{\alpha + 1}(u; 2r)]$$

$$\le C(E_{\alpha + 1}(u; 2r))^{\frac{2}{\alpha + 1}} \varepsilon^{\frac{3}{16}} + C\varepsilon^{\frac{3}{16}}[E_{\alpha + 1}(u; 2r) + P_{\frac{\alpha + 1}{\alpha}}(2r)E_{\alpha + 1}(u; 2r)]$$

$$\le C\varepsilon^{\frac{3}{16} - \frac{\alpha}{2\alpha + 1}} + CE_{\alpha + 1}(u; 2r) + \varepsilon^{\frac{3}{16} - \frac{\alpha}{2\alpha + 1}}P_{\frac{\alpha + 1}{\alpha}}(2r).$$  

(4.8)

From Lemma 2.10 and Lemma 2.9 with $\mu = 2r$, we infer that

$$E_{\alpha + 1}(u; 2r) \le C \left( \frac{\rho}{r} \right)^{\frac{4\alpha - 6}{\alpha}} E_{\alpha + 1}(u; \rho) \varepsilon^{\frac{3\alpha - 3}{\alpha}}(u; \rho) + C \left( \frac{r}{\rho} \right)^{\frac{3 - \alpha}{\alpha - \frac{3}{2}}} E_{\alpha + 1}(u; \rho),$$

$$\varepsilon^{\frac{3}{16}}P_{\frac{\alpha + 1}{\alpha}}(2r) \le C\varepsilon^{\frac{1}{4}} \left( \frac{\rho}{r} \right)^{\frac{4\alpha - 6}{\alpha}} E_{\alpha + 1}(u; \rho) + C \left( \frac{r}{\rho} \right)^{\frac{3 - \alpha}{\alpha - \frac{3}{2}}} \varepsilon^{\frac{3}{16}}P_{\frac{\alpha + 1}{\alpha}}(\rho).$$  

(4.9)
Before going further, we write

\[ G(r) := E_{\alpha+1}(u; r) + \varepsilon^{3/16} E(u; r) + \varepsilon^{3/16} E_\ast(u; r) + \varepsilon^{1/4} P_{\alpha+1}(r). \]  

(4.10)

Plugging (4.8) and (4.9) into (4.10), we arrive at

\[ G(r) \leq C \varepsilon^{3(\alpha+1)/(n-1)} + C E_{\alpha+1}(u; 2r) + C \varepsilon^{3(\alpha+1)/(n-1)} P_{\alpha+1}(2r) + C \varepsilon^{3/4} P_{\alpha+1}(r) \]

\[ \leq C \varepsilon^{3(\alpha+1)/(n-1)} + C \left( \frac{\rho}{r} \right) \varepsilon^{1/2} E_{\alpha+1}(u; \rho) E_{\ast}(u; \rho) + C \left( \frac{\rho}{r} \right) \varepsilon^{1/4} P_{\alpha+1}(\rho). \]

(4.11)

Since \( \frac{5-n}{4} < 1 \), one can apply the iteration method as [10] to show that there exists a positive constant \( r_1 \), such that \( E_{\alpha+1}(u; r) \leq \varepsilon \) for any \( r \leq r_1 \).

Using the local energy inequality (4.7) and the decay estimates (4.9), we get

\[ E(u; r) + E_\ast(u; r) + P_{\alpha+1}(r) \]

\[ \leq C(E_{\alpha+1}(u; 2r))^{\frac{2}{\alpha+1}} + C E_{\alpha+1}(u; 2r) + C P_{\alpha+1}(2r) \]

\[ \leq C \left[ \left( \frac{\rho}{r} \right)^{\frac{4n-6}{\alpha+1}} E_{\alpha+1}(u; \rho) \right]^{\frac{2}{\alpha+1}} + C \left( \frac{\rho}{r} \right)^{\frac{4n-6}{\alpha+1}} E_{\alpha+1}(u; \rho) \]

\[ + C \left( \frac{\rho}{r} \right)^{\frac{4n-6}{\alpha+1}} E_{\alpha+1}(u; \rho) + C \left( \frac{\rho}{r} \right)^{\frac{3n-3}{\alpha-1}} P_{\alpha+1}(r). \]  

(4.12)

Now, we can apply the iteration argument as above once again to show that, there exists a positive constant \( r_2 \leq r_1 \) such that \( E(u; r) + E_\ast(u; r) + P_{\alpha+1}(r) \leq \varepsilon_0 \) for any \( r \leq r_2 \). Finally, we can invoke Theorem 1.6 to finish the proof of Theorem 1.7. □

4.3 Proof of Theorem 1.9

The key estimate for the proof of Theorem 1.9 is to show

\[ \frac{d}{dt} \int_\Omega |\nabla u|^2 \, dx \leq C \|u\|_{H^q(\Omega)}^{2+2q} \|\nabla u\|_{L^2(\Omega)}^2, \quad q > 3\alpha - 3. \]  

(4.13)

Taking \( q = 6 \) in the latter inequality and using the Sobolev inequality, we get

\[ \frac{d}{dt} \int_\Omega |\nabla u|^2 \, dx \leq C \|\nabla u\|_{L^2(\Omega)}^{2+2\alpha}, \]

which leads to the local well-posedness for initial data in \( H^1(\Omega) \). In addition, the Gronwall inequality and estimate (4.13) mean the rest result of this theorem.

Proof of Estimate (4.13) Multiplying the modified Navier–Stokes system (1.6) by \( \partial_k \partial_k u \), integrating over \( \Omega \), using \( \text{div} \, u = 0 \), and integrating by parts, we have

\[ \frac{1}{2} \frac{d}{dt} \int_\Omega |\nabla u|^2 \, dx + \int_\Omega |\nabla^2 u| \, dx = \sum_{i=1}^3 \int_\Omega u \cdot \nabla u_i^{\alpha-1} \partial_k \partial_k u_i \, dx. \]  

(4.14)

From Hölder’s inequality, we have

\[ I = \sum_{i=1}^3 \int_\Omega u \cdot \nabla u_i^{\alpha-1} \partial_k \partial_k u_i \, dx \leq C \|u\|_{L^\alpha(\Omega)}^{\alpha-1} \|\nabla u\|_{L^{\frac{n}{\alpha-1}}(\Omega)} \|\nabla^2 u\|_{L^2(\Omega)} \|\nabla^2 u\|_{L^2(\Omega)}. \]
It follows from the interpolation inequality and Sobolev’s embedding that
\[
\|\nabla u\|_{L^{q/2-\alpha+2}(\Omega)} \leq \|\nabla u\|_{L^0(\Omega)}^{\frac{3\alpha-3}{q}} \|\nabla u\|_{L^2(\Omega)}^{\frac{\alpha-3}{q}} \leq C \|\nabla^2 u\|_{L^2(\Omega)}^{\frac{3\alpha-3}{q}} \|\nabla u\|_{L^2(\Omega)}^{\frac{\alpha-3}{q}}. \tag{4.15}
\]
We derive from Young’s inequality and the latter inequality that
\[
I \leq C \|u\|_{L^0(\Omega)}^{\alpha-1} \|\nabla^2 u\|_{L^2(\Omega)}^{1+\frac{3\alpha-3}{q}} \|\nabla u\|_{L^2(\Omega)}^{\frac{\alpha+3-3\alpha}{q}} \leq C \|u\|_{L^0(\Omega)}^{\frac{2\alpha(\alpha-1)}{2q-3\alpha+3}} \|\nabla u\|_{L^2(\Omega)}^2 + \frac{1}{32} \|\nabla^2 u\|_{L^2(\Omega)}^2, \tag{4.16}
\]
where the fact that \(q > 3\alpha - 3\) were used.

Collecting above estimates and absorbing the terms containing \(\|\nabla^2 u\|_{L^2(\Omega)}^2\) by the left-hand side in (4.14), we deduce that
\[
\frac{d}{dt} \int_{\Omega} |\nabla u|^2 dx \leq C \|u\|_{L^0(\Omega)}^{\frac{2\alpha(\alpha-1)}{2q-3\alpha+3}} \|\nabla u\|_{L^2(\Omega)}^2. \tag{4.17}
\]
The desired estimate is derived.

5 Conclusion

Inspired by the work of Ozánski–Robinson [31] and Stein–Winkler [34], we follow the path of [31] to study the partial regularity of suitable weak solution of a surface growth equation (1.1) with the general nonlinear term and no-zero force. The precise relationship between the Hausdorff dimension of potential singular point set \(S\) and the parameter \(\alpha\) in this equation is presented in Corollary 1.4. The index range of Corollary 1.4 is restricted to the torus due to the interpolation inequality (2.12).

Now we give some remarks:

1. Partial regularity of the 3D stochastic Navier–Stokes equations was obtained by Flandoli–Romito in [17]. Can one obtain the partial regularity of the original equation (1.2) or (1.1) with the noise term?

2. It is an interesting question to prove the existence of suitable weak solution and related partial regularity theory for the critical case \(\alpha = 7/3\) in (1.1) as well as the 3D modified Navier–Stokes system (1.6). A probable approach involving existence is to use the strategy introduced by Wu in [42].

3. Is there any other approach to study the partial regularity of equation (1.1) besides blow up analysis?

4. Whether the same conclusion of Corollary 1.4 is valid on bounded domains and the whole space? The corollary heavily relies on the interpolation inequality on periodic boundary conditions.

5. What is optimal hypothesis of force to obtain the partial regularity of equation in (1.2) or (1.1)? Similar research for the 3D Navier–Stokes can be found in [26].

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