NON-PERTURBATIVE SCHOTTKY PROBLEM AND STABLE EQUATIONS
FOR THE HYPERELLIPTIC LOCUS

GIULIO CODOGNI

Abstract. Given an integer $g$ and an even unimodular positive definite lattice $\Lambda$, one can construct the classical Theta series $\Theta_{\Lambda,g}$, which is a degree $g$ modular form. If we fix the lattice and we package all these modular forms together, we may interpret this as a character $\Theta_{\Lambda}$ on the monoid $A_{\infty} = \bigcup A_0$. We then consider differences of Theta series. It is known that none of these vanish on $M_\infty = \bigcup M_g$, on the other hand, we are able to exhibit many non-trivial differences of Theta series vanishing on $Hyp_\infty = \bigcup Hyp_g$. Furthermore, we describe the behavior of differences of Theta series associated to rank 24 lattices. One of the main ingredients is a precise description of the tangent space to the boundary of the Satake compactification.

1. Introduction

To a smooth complex genus $g$ curve $C$ one can associate its Jacobian variety $Jac(C)$. This can be done in families, so we have the Torelli morphism

$$T: M_g \rightarrow A_g$$

$C \mapsto Jac(C)$

where $M_g$ denotes the moduli space of smooth genus $g$ curves and $A_g$ the moduli space of $g$ dimensional principally polarised abelian varieties. There is an important difference between the moduli stack and the associated coarse space. In this paper, we will mainly work on coarse spaces: this means that we will consider only general curves.

The Schottky problem is to understand when an abelian variety is the Jacobian of a curve. This problem has many aspects, we are interested in the modular forms viewpoint. A weight $k$ and degree $g$ modular form is a section of the line bundle $L_g^\otimes k$ on the stack $A_g$, where $L_g$ is the determinant of the Hodge bundle. We would like to find modular forms vanishing on $M_g$. In other words we are looking for the equations of $M_g$ in $A_g$. The same question makes sense for the locus of hyperelliptic curves $Hyp_g$.

There is a rather surprising way to construct modular forms: given an even unimodular positive definite lattice $\Lambda$, for every integer $g$, the associated Theta series $\Theta_{\Lambda,g}$ is a weight $\frac{1}{2}rk(\Lambda)$ modular form on $A_g$ (see section 7 for definitions and notations).

To give more structure to these Theta series, we first consider the Satake compactification $A_\infty$ of $A_g$, this is a normal variety constructed by using modular forms. The boundary of $A_\infty$ is isomorphic to $A_{g-1}$, thus we can construct the commutative ind-monoid

$$A_\infty := \bigcup_{g \geq 0} A_g,$$

where the operation is the product of abelian varieties. We fix a lattice $\Lambda$ and we consider all the Theta series at once: we obtain a stable modular form

$$\Theta_{\Lambda} := \bigcup_{g \geq 0} \Theta_{\Lambda,g}$$
As discussed in section 3, this stable modular form is a character for $\mathcal{A}_\infty$.

We then consider the Satake compactification $\mathcal{M}_g^S$ of $\mathcal{M}_g$ (respectively $Hyp^S_g$ of $Hyp_g$): this is the closure of $\mathcal{M}_g$ (resp. $Hyp_g$) inside $A^S_g$. The intersection of $\mathcal{M}_{g+m}^S$ with $\mathcal{A}_g$ is, as set, $\mathcal{M}_g^S$ and we can thus consider the ind-monoids

$$\mathcal{M}_\infty := \bigcup_{g \geq 0} \mathcal{M}_g^S \quad \text{and} \quad Hyp_\infty := \bigcup_{g \geq 0} Hyp_g^S.$$ 

These are sub-monoids of $\mathcal{A}_\infty$. Using lemma 2.3, we can show that the ideal of stable modular forms vanishing on them is generated as a vector space by differences of Theta series

$$\Theta_\Lambda - \Theta_\Gamma.$$ 

The natural question is to describe these ideals in greater detail. To tackle this problem we give a precise description of the local structure of $\mathcal{M}_g^S$ and $Hyp_g^S$. This is done in [CSB13] and in sections 4 and 5. We will show how the behaviours of $\mathcal{M}_\infty$ and $Hyp_\infty$ are completely different: compare theorem 1.1 with theorem 1.4, corollary 1.2 with theorem 1.7.

In section 3 we review the following:

**Theorem 1.1** (= [CSB13] Theorem 1.1). The intersection of $\mathcal{M}_{g+m}^S$ and $\mathcal{A}_g$ is not transverse, it contains the $m$-th infinitesimal neighbourhood of $\mathcal{M}_g$ in $\mathcal{A}_g$.

This result has the following corollary:

**Corollary 1.2.** The ideal of stable modular forms vanishing on $\mathcal{M}_\infty$ is trivial.

However, because of Lemma 3.4, it might be interesting to discuss further the behaviour of differences of Theta series on $\mathcal{M}_g$. The classical case is the Schottky form

$$\Theta_{E_8 \oplus E_8} - \Theta_{D_{16}}.$$ 

It is well known that this form vanishes on $\mathcal{M}_g$ for $g \leq 4$, more recently it has been proved that it cuts out a divisor of slope 8 on $\mathcal{M}_5$ ([GSM11]). We extend this result by considering rank 24 lattices.

**Theorem 1.3** (= Corollary 8.5). Let $\Lambda$ and $\Gamma$ be two even positive definite unimodular lattices of rank 24 with the same number of vectors of norm 2, then the stable modular form

$$F := \Theta_\Lambda - \Theta_\Gamma$$

vanishes on $\mathcal{M}_g$ for $g \leq 4$ and it cuts out a divisor of slope 12 on $\mathcal{M}_5$.

The key point of the proof is to show that, under certain assumptions, the form $F_{g+1}$ vanishes on $\mathcal{M}_{g+1}$ if and only if its restriction to $\mathcal{A}_g^S$ vanishes on $\mathcal{M}_g^S$ with multiplicity at least 2 (see theorem 8.4). Similar results are obtained in [GV09] and [GKV10], however the terminology and the techniques are rather different. We do not consider rank 32 lattices, but we expect much more different behaviours.

We now turn our attention to the hyperelliptic locus. It is proved in [Poo96] that the Schottky form defined in (1) is a stable equation for the hyperelliptic locus, i.e. it vanishes on $Hyp_g$ for every $g$, so corollary 1.2 does not hold for $Hyp_\infty$. In contraposition to Theorem 1.1, we can prove the following geometric result:

**Theorem 1.4** (= Theorem 4.1). The intersection of $\mathcal{A}_g^S$ and $Hyp_{g+1}^S$ is transverse.
This means that the existence of stable equations for the hyperelliptic locus is entirely possible. In fact, we are able to construct a number of explicit examples as follows. Given a lattice $\Lambda$, call $\mu_\Lambda$ the norm of the shortest non-trivial vectors.

**Theorem 1.5** (=$\text{Theorem 7.1}$). Let $\Lambda$ and $\Gamma$ be two even positive definite unimodular lattices of rank $N$ and $\mu_\Lambda = \mu_\Gamma =: \mu$, if

$$\frac{N}{\mu} \leq 8,$$

then

$$F := \Theta_\Lambda - \Theta_\Gamma$$

is a stable equation for the hyperelliptic locus.

The proof relies upon Theorem 1.4 and the well known criterion 6.1 based on projective invariants of hyperelliptic curves. The hypotheses on the lattices are quite restrictive, for example they imply that the rank is smaller than or equal to 48. However, this result holds for rank 32 lattices with $\mu = 4$ and in [Kin03] it is shown that there exist at least ten millions of such lattices.

Using the description of the tangent space given Theorem 5.4, we can prove that also the stable modular forms discussed in Theorem 1.3 are stable equations for the hyperelliptic locus, although they do not satisfy the hypotheses of Theorem 1.5.

**Theorem 1.6** (=$\text{Theorem 8.6}$). Let $\Lambda$ and $\Gamma$ be two even positive definite unimodular lattices of rank 24 with the same number of vectors of norm 2, then

$$F := \Theta_\Lambda - \Theta_\Gamma$$

is a stable equation for the hyperelliptic locus.

Summarising, if we combine Theorem 3.5 and Theorem 1.5 we can give the following picture:

**Theorem 1.7.** The ideal of stable equations for the hyperelliptic locus is generated, as vector space, by differences of Theta series. There are more than 10000000 linearly independent differences of Theta series vanishing on $Hyp_g$ for every $g$.

An open question is whether the ideal of stable equations for the hyperelliptic locus is big enough to define $Hyp_\infty$ in $A_\infty$.

The $n$-gonal locus, for $n > 2$, behaves similarly to $M_g$. In [SB13] there is a precise description of the tangent space to the boundary of the Satake compactification of the locus of $n$-gonal curves with total ramification. As a consequence, it is shown that the ideal of stable equations for the $n$-gonal locus is trivial.

The term “non-perturbative” in the title derives from string theory. Indeed, a non-perturbative bosonic string theory should take into account the spaces $M_g$ all at once (see e.g. [BNS96]), so it should be constructed on $M_\infty$. Moreover, stable modular forms could be non-perturbative partition functions. In this paper we are looking at how $M_\infty$ and $Hyp_\infty$ sit inside $A_\infty$.

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2. An introduction to ind-Monoids

We recall some definitions about ind-varieties and we specialize them for ind-monoid. A reference is [Kum02] chapter IV.

A (projective) ind-variety is a set $X$ with a filtration $X_n$ such that each $X_n$ is a (projective) finite dimensional algebraic variety and the inclusion of $X_n$ in $X_{n+1}$ is a closed embedding. A line bundle $L$ on $X$ is the data of a line bundle $L_n$ on each $X_n$ compatible with the restriction. A section $s$ is a collection of sections $s_n$ compatible with the restriction. The ring of sections of $L$ is thus defined as a projective limit in the category of graded rings

$$\mathcal{R}(X, L) := \lim_{\leftarrow} \mathcal{R}(X_n, L_n).$$

A (projective) ind-monoid is a (projective) ind-variety $M$ with an associative multiplication and an identity element $1_M$. A multiplication is a family of maps

$$m_{g,h} : M_g \times M_h \to M_{g+h}$$

compatible with the restrictions. $M$ is commutative if the multiplication is.

**Definition 2.1** (Split monoid). Let $M$ be a commutative ind-monoid and $L$ a line bundle on $M$, we say that $M$ is split with respect to $L$ if the following two conditions hold

1. For every $g$ and $h$
   $$m_{g,h}^* L_{g+h} \cong pr_1^* L_g \otimes pr_2^* L_h$$
   where $pr_i$ are the projections,

2. the ring of sections $\mathcal{R}(M, L)$ is generated as vector space by characters, where a section $\chi$ of $L$ is a character if
   $$m_{g,h}^* \chi_{g+h} = \chi_g \chi_h \quad \forall g, h$$

Remark that the definition of character makes sense only if condition (1) holds. We are going to write $\chi(\alpha \beta) = \chi(\alpha) \chi(\beta)$ instead than $\chi_{g+h}(\alpha \beta) = \chi_g(\alpha) \chi_h(\beta)$.

**Lemma 2.2.** Let $M$ be a commutative monoid, suppose it is split with respect to a line bundle $L$, then the characters are linearly independent.

*Proof.* This proof is standard. We argue by contradiction. Take $n$ minimal such that there exist $n$ distinct characters $\chi_1, \ldots, \chi_n$ linearly dependent, so we can write

$$\chi_n = \sum_{i=1}^{n-1} \lambda_i \chi_i \quad \lambda_i \in \mathbb{C}.$$ 

Take $\alpha \in M$ such that $\chi_1(\alpha) \neq \chi_n(\alpha)$, for any $\beta \in M$ we have

$$\sum_{i=1}^{n-1} \lambda_i \chi_i(\alpha) \chi_i(\beta) = \chi_n(\alpha) \chi_n(\beta) = \chi_n(\alpha) (\sum_{i=1}^{n-1} \lambda_i \chi_i(\beta))$$

Since $\beta$ is arbitrary we get

$$\sum_{i=1}^{n-1} \lambda_i (\chi_i(\alpha) - \chi_n(\alpha)) \chi_i = 0.$$

The coefficient $\chi_1(\alpha) - \chi_n(\alpha)$ is non-zero, so we have written a non-trivial linear relation among less than $n$ characters, this contradicts the minimality of $n$. $\square$

**Lemma 2.3.** Let $M$ be a commutative ind-monoid and $N$ a submonoid, suppose that $M$ is split with respect to a line bundle $L$, then the ideal $I_N$ in $\mathcal{R}(M, L)$ of sections vanishing on $N$ is generated as vector space by differences of characters

$$\chi_i - \chi_j$$
Proof. Take $s$ in $I_N$, since $R(M, L)$ is generated as a vector space by characters we can write

$$s = \lambda_1 \chi_1 + \cdots + \lambda_n \chi_n$$

where $\chi_i$ are characters and $\lambda_i$ are constants. Restricting $\chi_i$ to $N$ some of them might become equal. Up to relabel the $\chi_i$, we can fix integers $0 = m_0 < m_1 < \cdots < m_k = n$ and distinct characters $\theta_1, \ldots, \theta_k$ of $N$ such that

$$\chi_i |_{N} = \theta_j \iff m_{j-1} < i \leq m_j$$

For $j = 1, \ldots, k$, let us define

$$\mu_j := \sum_{i=m_{j-1}+1}^{m_j} \lambda_i .$$

By hypothesis we know that

$$0 = s |_{N} = \sum_{j=1}^{k} \mu_j \theta_j$$

because of lemma 2.2 we have $\mu_j = 0$ for every $j$, so

$$s = s - \sum_{j=1}^{k} \mu_j \chi_{m_j} = \sum_{j=1}^{k} \sum_{i=m_{j-1}+1}^{m_j} \lambda_i (\chi_i - \chi_{m_j})$$

and the differences $\chi_i - \chi_{m_j}$ vanish on $N$. □

3. Satake compactification and “non-perturbative” Schottky problem

We recall some facts about modular forms and the Satake compactification of $A_g$, a standard reference is [Fre83]. The line bundle $L_g$ of weight one modular form on the stack $A_g$ is the determinant of the Hodge bundle, it is ample. The Satake compactification $A^S_g$ is a normal projective variety defined as follows

$$A^S_g := \text{Proj}(\bigoplus_{n \geq 0} H^0(A_g, L^n_g))$$

The Siegel operator is a map of graded rings

$$(2) \quad \Phi : \bigoplus_{n \geq 0} H^0(A_g, L^n_g) \to \bigoplus_{n \geq 0} H^0(A_{g-1}, L^n_{g-1})$$

defined as

$$\Phi(F)(\tau) := \lim_{t \to +\infty} F(\tau \oplus it),$$

where we are using the fact that the universal cover of $A_g$ is the Siegel upper half space $\mathcal{H}_g$ and $\tau$ is an element of $\mathcal{H}_g$. The Siegel operator is surjective for $n$ even and larger than $2g$, so it defines a stratification

$$A^S_g = A_g \sqcup A^S_{g-1} = A_g \sqcup A_{g-1} \cdots \sqcup A_1 \sqcup A_0 .$$

In other words, the boundary $\partial A^S_g$ is isomorphic to $A^S_{g-1}$ and the Siegel operator is the restriction map from $A^S_g$ to $A^S_{g-1}$. Moreover, $L_g$ restricts to $L_{g-1}$.

We can now consider the commutative ind-monoid

$$A_\infty := \bigcup_{g \geq 0} A^S_g$$
the multiplication is the product of abelian varieties
\[ m_{g,h} : \mathcal{A}^S_g \times \mathcal{A}^S_h \to \mathcal{A}^S_{g+h} \]
the identity element is \( A_0 \). The line bundles \( L_g \) define a line bundle \( L \) on \( \mathcal{A}_\infty \) of weight one stable modular form. A \textbf{weight \( k \) stable modular form} is a section of \( L^k \). We have
\[ m^*_{g,h} L_{g+h} = pr_1^* L_g \otimes pr_2^* L_h. \]
We can associated to every even unimodular positive definite lattices \( \Lambda \) a Theta series \( \Theta_\Lambda \), this is a weight \( \frac{1}{2} rk(\Lambda) \) stable modular form. We will give the explicit definition in section 7, we now just want to recall some properties. Given \( X \in \mathcal{A}_g \) and \( Y \in \mathcal{A}_h \), we have the factorisation property
\[ \Theta_{\Lambda,g+h}([X \times Y]) = \Theta_{\Lambda,g}(X)\Theta_{\Lambda,h}(Y), \]
which means that the \textbf{Theta series are characters} for the monoid \( \mathcal{A}_\infty \). It is proved in [Fre77] theorem 2.2 that the ring of stable modular form is generated as a vector space by Theta series, so \( \mathcal{A}_\infty \) is split with respect to \( L \).

\textbf{Theorem 3.1} (= [Fre77] Theorem 2.5). \textit{The ring of stable modular forms} \( \mathcal{R}(\mathcal{A}_\infty,L) \) is a polynomial ring in the Theta series associated to irreducible lattices.

\textit{Proof.} Let \( P \) be the vector space with basis the Theta series associated to lattices. Given two lattice \( \Lambda \) and \( \Gamma \), one can check that
\[ \Theta_{\Gamma \oplus \Lambda} = \Theta_{\Gamma} \Theta_{\Lambda}, \]
so any Theta series is a monomial in Theta series associated to irreducible lattices and \( P \) is the polynomial ring in the Theta series associated to irreducible lattices.

We endow \( P \) with the co-multiplication \( \mu(\Theta_\Lambda) = \Theta_\Lambda \otimes \Theta_\Lambda \), so \( \text{Proj}(P) \) is a commutative monoid. Suppose that
\[ \mathcal{R}(\mathcal{A}_\infty,L) = P/I \]
where \( I \) is some ideal. This would mean that \( \mathcal{A}_\infty \) is a submonoid of \( \text{Proj}(P) \), so we can apply lemma 2.3 and show that \( I \) is generated by difference of Theta series. To show that the ideal \( I \) is trivial, it is enough to prove that, given any two lattices \( \Lambda \) and \( \Gamma \) of rank \( N \), their Theta series are different for \( g >> 0 \). To see this we can take \( g = N \) and look at the coefficients corresponding to \( \Gamma \) and \( \Lambda \) in the Fourier expansions. \( \square \)

The usual compactification of \( \mathcal{M}_g \) is the Deligne-Mumford compactification \( \overline{\mathcal{M}}_g \). The boundary is composed by divisors \( \delta_i \), for \( i = 0, \ldots, \lfloor \frac{2g}{3} \rfloor \). The general point of \( \delta_0 \) is a singular curve whose normalization is a genus \( g-1 \) curve, the general point of \( \delta_i \) is a curve whose normalization is the disjoint union of a genus \( i \) and a genus \( g-i \) curve.

The Satake compactification of \( \mathcal{M}_g \) (respectively of \( \text{Hyp}_g \)) is denoted by \( \mathcal{M}^S_g \) (\( \text{Hyp}^S_g \)): it is the closure of \( \mathcal{M}_g \) (\( \text{Hyp}_g \)) inside \( \mathcal{A}^S_g \). Set theoretically, \( \mathcal{M}^S_g \) is equal to the union of all products \( \mathcal{M}_{g_1} \times \cdots \times \mathcal{M}_{g_k} \) with \( \sum g_i \leq g \) (cf. [Hoy63]). Equivalently, as shown in [Nam73] theorem 6, we can extend the Torelli morphism
\[ T : \overline{\mathcal{M}}_g \to \mathcal{A}^S_g \]
\[ C \mapsto \text{Jac}(\hat{C}) \]
where \( \hat{C} \) is the normalization of \( C \). Since \( \overline{\mathcal{M}}_g \) is reduced, the image of \( T \) is equal, as a scheme, to \( \mathcal{M}^S_g \). The pull-back of \( L_g \) is usually denoted by \( \lambda \). This means that \( \lambda \) is semi-ample and the normalization of \( \mathcal{M}^S_g \) is given by the Itaka map of \( \lambda \). In [ACG11] it is shown that the normalization is bijective.
Theorem 3.2 (=[CSB13] Theorem 1.1) This is a consequence of a geometric feature of the Satake compactification: the slope of $M$. 

Corollary 3.3. Given a stable modular form concretely, the slope of $F$ it contains the particular, if we intersect $M_{k}\infty$ this could be a positive number or zero on $M_{S}$. Hyp where these are sub-monoid of $M$ components with $a, b$ is zero on $M_{g}$ even though the boundary of $M$ is not zero on $M_{g}\infty$. We can consider the commutative ind-monoid $M_{\infty} := \bigcup_{g \geq 0} M_{g}^{S}$ $Hyp_{g} := \bigcup_{g \geq 0} Hyp_{g}^{S}$ these are sub-monoid of $A_{\infty}$.

In [CSB13], it is shown that the ideal of stable modular forms vanishing on $M_{\infty}$ is trivial. This is a consequence of a geometric feature of the Satake compactification:

Theorem 3.2 (=[CSB13] Theorem 1.1). The intersection of $M_{g+m}^{S}$ and $A_{g}$ is not transverse, it contains the $m$-th infinitesimal neighbourhood of $M_{g}$ in $A_{g}$.

This result means that if a modular form $F_{g+m}$ on $A_{g+m}$ vanishes with order at least $k$ on $M_{g+m}$, then the restriction $F_{g}$ of $F_{g+m}$ to $A_{g}$ vanishes with order at least $k + m$ on $M_{g}$.

Corollary 3.3. Given a stable modular form $F$, if $F_{g}$ vanishes on $M_{g}$ with multiplicity exactly $k$, then $F_{g+k}$ does not vanish on $M_{g+k}$.

Still differences of Theta series might have an interesting behaviour. Let us recall the definition of slope given in [HM90]. The Picard group of $M_{g}$ is freely generated by $\lambda$ and the boundary divisors $\delta_{i}$. If an effective divisor $D$ can be written as $D = a\lambda - \sum b_{i}\delta_{i}$ with $a, b_{i} \geq 0$, its slope is defined as $s(D) := \max_{i} \frac{a}{b_{i}}$, this could be a positive number or $\infty$. If $D$ can not be written in this way then its slope is $\infty$. The slope of $M_{g}$ is the infimum of the slopes of effective divisors.

Let $F_{g}$ be a modular form, call $a$ the weight, $b_{0}$ its vanishing order along $A_{g-1} \cap M_{g}^{S}$ and $b_{i}$ its order along $A_{i} \times A_{g-i} \cap M_{g}^{S}$. We can write $\sum b_{i}\delta_{i} + E = \{F_{g} = 0\} = a\lambda$ where $E$ is the closure of an effective divisor on $M_{g}$. The slope of $F_{g}$ is the slope of $E$. More concretely, the slope of $F_{g}$ is the weight divided by the smallest of the $b_{i}$.

Lemma 3.4. Given any two lattices $\Lambda$ and $\Gamma$ of the same rank, there exists an integer $g > 0$ such that the modular form $F_{g} = \Theta_{\Lambda,g} - \Theta_{\Gamma,g}$ is zero on $M_{g}$ for $g < g$, it is not zero on $M_{g}$ and it cuts out a divisor of finite slope, it is not zero on $M_{g}$ for $g > g$ and cuts out a divisor of slope $\infty$.

Proof. Any difference of Theta series is zero on $A_{0}$, the existence of $g$ is guaranteed by corollary 3.3. We need to check the statement about the slope. For $g > g$, $F_{g}$, as a modular form on $M_{g}^{S}$, is not zero on $M_{g-1}$, so $b_{0} = 0$ and the slope is $\infty$. When $g = g$, we know that $F_{g}$ vanishes on $M_{g-1}$ so $b_{0} \neq 0$. We need to check that $b_{i} \neq 0$ for $i > 0$, in other words that $F_{g}$ vanishes on all components $M_{g-1} \times M_{i}$. Take a point $\begin{pmatrix} \tau_{i} & 0 \\ 0 & \tau_{g-i} \end{pmatrix} \in M_{i} \times M_{g-i}$
we argue as follows

\[ F_\bar{g} \left( \begin{pmatrix} \tau_i & 0 \\ 0 & \tau_{\bar{g} - i} \end{pmatrix} \right) = \Theta_{\Lambda,i}(\tau_i)\Theta_{\Lambda,\bar{g} - i}(\tau_{\bar{g} - i}) - \Theta_{\Gamma,i}(\tau_i)\Theta_{\Gamma,\bar{g} - i}(\tau_{\bar{g} - i}) = \Theta_{\Lambda,\bar{g} - i}(\tau_{\bar{g} - i})F_i(\tau_i) = 0 \]

where in the next to last equality we have used that \( F_{\bar{g} - i} \) is zero on \( \mathcal{M}_{\bar{g} - i} \) and in the last we have used that \( F_i \) is zero on \( \mathcal{M}_i \).

The previous argument also shows that \( F_2\bar{g} + 1 \) does not vanish on any boundary divisor of \( \overline{\mathcal{M}}_{2g + 1} \). The values of \( \bar{g} \) and of the slope are known for lattices of rank 16 (the Schottky form (1) discussed in the introduction) and 24 (Theorem 1.3).

The situation of the hyperelliptic locus is completely different. In [Poo96] it is shown that the Schottky form

\[ \Theta_{E_8 \oplus E_8} - \Theta_{D_{16}^+} \]

vanishes on \( Hyp_g \) for every \( g \), so the ideal of stable equations for the hyperelliptic locus is not trivial. We can apply Lemma 2.3 to prove the following result

**Theorem 3.5.** The ideal of stable modular forms vanishing on \( Hyp_{\infty} \) is generated as a vector space by differences of Theta series

\[ \Theta_{\Lambda} - \Theta_{\Gamma}. \]

Theorem 1.5 provides many new examples of differences of Theta series vanishing on \( Hyp_{\infty} \).

### 4. Transversality

This section is devoted to the proof of the following result:

**Theorem 4.1.** The intersection of \( \mathcal{A}_g^S \) and \( Hyp_{g + 1}^S \) is transverse. In other words, scheme theoretically, it is equal to \( Hyp_g^S \).

Let \( I_{Hyp_{g + 1}} \) be the ideal of modular forms on \( \mathcal{A}_g \) vanishing on \( Hyp_g \). The inclusion \( \mathcal{A}_g^S \hookrightarrow \mathcal{A}_{g + 1}^S \) is induced by the Siegel operator \( \Phi \). We have to prove that the map

\[ \Phi : I_{Hyp_{g + 1}} \rightarrow I_{Hyp_g} \]

is surjective. For technical reasons, we will first prove the theorem on the finite cover defined by the level structure \((4,8)\), the claim will follow because finite groups are linearly reductive in characteristic zero.

Let us recall a few facts about level structures, cf. e.g. [Fre83] II.6. The group \( \Gamma(4,8) \) is a normal co-finite subgroup of \( Sp(2g, \mathbb{Z}) \). Call \( G \) the finite quotient. The moduli space \( \mathcal{A}_g(4,8) \) is the quotient of the Siegel upper half space by \( \Gamma(4,8) \). A point of \( \mathcal{A}_g(4,8) \) represent a principally polarised abelian variety with extra structures. Among these extra data, we have an isomorphism \( \phi \) between the subgroup of two torsion elements and \((\mathbb{Z}/2\mathbb{Z})^{2g} \). On \( \mathcal{A}_g(4,8) \) there is the ample line bundle \( L \) of weight one modular forms, whose sections are holomorphic functions on \( H_g \) which transform appropriately under the action of \( \Gamma(4,8) \). Using this line bundle, we can construct the Satake compactification \( \mathcal{A}_g^S(4,8) \) of \( \mathcal{A}_g(4,8) \). The boundary is composed by many irreducible components \( X_i \), permuted by \( G \).

For each components \( X_i \), we have a Siegel operator \( \Phi_i \).

\[ \Phi_i : H^0(\mathcal{A}_g(4,8), L^k) \rightarrow H^0(\mathcal{A}_{g - 1}(4,8), L^k) \]
which realise an isomorphism between \( X_i \) and \( \mathcal{A}_{g-1}^S (4,8) \). There is a component, say \( X_0 \), called the “standard component”, where the Siegel operator is given by the usual formula

\[
\Phi_0 (F) (\tau) := \lim_{t \to \infty} F(\tau \otimes it)
\]

The others Siegel operators are obtained by letting \( G \) act.

References about the hyperelliptic locus with level structure \((4,8)\) are \[Igu67b\], \[Tsu91\] and \[SM03\]. We recall a few facts. The space \( \text{Hyp}_{p_g} (4,8) \) inside \( \mathcal{A}_g (4,8) \) is the preimage of \( \text{Hyp}_p \) under the quotient map. This space splits in many irreducible components \( Y_j \) permuted by \( G \). Call \( \text{Hyp}_{p_g} (4,8)^S \) the closure of \( \text{Hyp}_{p_g} (4,8) \) in \( \mathcal{A}_g^S (4,8) \). The intersection of \( \text{Hyp}_{p_g} (4,8)^S \) with any of the \( X_i \) is, set-theoretically, equal to \( \text{Hyp}_{p_g-1} (4,8)^S \). We shall show that the equality is true as schemes.

A way to specify an irreducible component \( Y_j \) is to fix a special fundamental system of Theta characteristics \( \mathfrak{m} = \{ m_0, \ldots, m_{2g+1} \} \). This is a subset of \((\mathbb{Z}/2\mathbb{Z})^{2g}\) with some additional properties, see \[SM03\] for the definition. The relation between special fundamental system and irreducible components is the following. Call \( W \) the set of Weierstrass points of a hyperelliptic curve \( C \). For any \( w \) in \( W \), call \( AJ_w \) the Abel-Jacobi map with base point \( w \). The set \( AJ_w(W) \) is a subset of the 2-torsion subgroup of \( \text{Jac}(C) \). The choice of a special fundamental system of Theta characteristic \( \mathfrak{m} \), determines the component \( Y_i = Y_m \) of abelian varieties \( (\text{Jac}(C), \Theta, \phi) \) such that there exists a \( w \) in \( W \) for which \( \phi(AJ_w(W)) = \mathfrak{m} \). Call \( Y_m^S \) its closure in \( \mathcal{A}_g^S (4,8) \). Different choice of \( \mathfrak{m} \) may determine the same component \( Y_i \), this because of the freedom in the choice of the base point of the Abel-Jacobi map.

Fix a system of Theta characteristic \( \mathfrak{m} \), so we have an irreducible component \( Y_m \) of \( \text{Hyp}_{p_g} (4,8) \). Let \( b \) be the sum of odd \( m_i \) in \( \mathfrak{m} \). For every Theta characteristic \( m \), the classical Theta-nullerwerte \( \theta_m \) is a well defined modular form on \( \mathcal{A}_g (4,8) \) (but not on \( \mathcal{A}_g \), this is the reason why we are using the level structure), see e.g. \[Igu67b\] or \[SM03\]. Our proof relies upon the following result.

**Theorem 4.2** \([SM03]\) Theorem 1. The scheme \( Y_m^S \) is ideal theoretically defined by the vanishing of Theta-nullerwerte \( \theta_{m+b} \) with \( m = m_{i_1} + \cdots + m_{i_k} \), where \( k \leq g \).

We still need some more notations. As usual, we write a Theta characteristic as two vectors of size \( g \). Call \( \varnothing \) the \( g \) dimensional zero vector. Define two \( g + 1 \) dimensional Theta characteristics

\[
p := \begin{bmatrix} q & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}, \quad q := \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}
\]

For every \( g \) dimensional Theta characteristic \( m = [\epsilon, \epsilon'] \), let us define the \( g + 1 \) dimensional Theta characteristic

\[
\overline{m} := \begin{bmatrix} \epsilon & 0 \\ \epsilon' & 0 \end{bmatrix},
\]

Moreover, for every special fundamental system of \( g \) dimensional Theta characteristic \( \mathfrak{m} \), we pose

\[
\overline{\mathfrak{m}} := p \cup q \cup \bigcup_{m \in \mathfrak{m}} \overline{m}.
\]

This is a special fundamental system of \( g + 1 \) dimensional Theta characteristics.

Let \( I_{\text{Hyp}_{p+1} (4,8)} \) be the ideal of \( \text{Hyp}_{p+1} (4,8) \) in \( \mathcal{A}_{g+1} (4,8) \), by \( I_{(\text{Hyp}_{p+1} (4,8), X_i)} \) we denote the ideal of \( \text{Hyp}_{p+1} (4,8) \) in the boundary component \( X_i \) of \( \mathcal{A}_{g+1} (4,8) \).

**Lemma 4.3.** Scheme theoretically, the intersection of \( Y_m^S \) and \( X_0 \) is isomorphic to \( Y_m \).

**Proof.** By direct computation one sees that

\[
\Phi_0(\theta \begin{bmatrix} \epsilon & 0 \\ \epsilon' & 0 \end{bmatrix}) = \theta \begin{bmatrix} \epsilon \\ \epsilon' \end{bmatrix}
\]
for $\delta$ equal either to 0 or 1. Suppose that a $g$ dimensional Theta characteristic $m$ is of the form prescribed by theorem 4.2 for the special fundamental system $Y$ of $Gyp_g(4,8)$, and the modular form $\theta_{m+b}$ vanishes on the irreducible component $Y_m$ of $Gyp_g(4,8)$. We have
$$\Phi_0(\theta_{m+b}) = \theta_{m+b},$$
so, because of theorem 4.2, the map
$$\Phi_0 : I_{Gyp_g+1}(4,8) \to I(Hyp_g(4,8), X_0)$$
is surjective.

**Proposition 4.4.** The intersection of $Hyp_g+1(4,8)^S$ and $X_i$ is transverse for every $i$.

**Proof.** For $i = 0$, the proposition is the previous lemma. For a general $i$, it is enough to notice that $G$ acts transitively on the boundary components and preserves the hyperelliptic locus. □

We have a $G$ equivariant map
$$\bigoplus_i \Phi_i : I_{Gyp_g+1}(4,8) \to \bigoplus_i I(Hyp_g(4,8), X_i)$$
This map is surjective because of the previous proposition. If we take $G$ invariants, we get a map
$$\bigoplus_i \Phi_i : I_{Gyp_g+1} \to \bigoplus_i I_{Gyp_g}$$
which is still surjective because $G$ is finite and the base field has characteristic zero. We obtain the theorem projecting onto one of the factor.

## 5. Description of the Tangent Spaces

Let us start recalling some facts about the structure of $A_{g+1}^S$ at a general point $X$ of $A_g$, see [Igu67a] for more details. The local ring of $(A_{g+1}^S, X)$ is normal and it is isomorphic to the ring of convergent power series
$$\sum_{n=0}^{\infty} f_n(\tau, z) q^n,$$
where now $X$ is identified with $\mathbb{C}^g/\mathbb{Z}^g \oplus \tau \mathbb{Z}^g$, $\tau$ is in the Siegel upper half space $\mathcal{H}_g$, $z$ belongs to $\mathbb{C}^g$, $f_n$ is a section of $H^0(X, 2n\Theta)$ and $q$ belongs to a small disc around zero in the complex plane. The image of a modular form $F_{g+1}$ of degree $g+1$ in this local ring is the Fourier-Jacobi expansion of $F_{g+1}$ at $X$. Let us be more explicit. For any element $T \in \mathcal{H}_{g+1}$ write
$$T = \begin{pmatrix} \tau & z \\ t & \bar{z} \end{pmatrix},$$
with $t$ in $\mathcal{H}_1$ and $\tau$ in $\mathcal{H}_g$. Let $q := \exp(2\pi it)$, the Fourier-Jacobi expansion of $F_{g+1}$ is
$$F_{g+1}(T) = f_0(\tau) + \sum_{n \geq 1} f_n(\tau, z) q^n,$$
where $f_0 = \Phi(F_{g+1}) = F_g$ is the restriction of $F_{g+1}$ to $A_g$. The function $f_n$ is the $n$-th Fourier-Jacobi coefficient of $F_{g+1}$, it belongs to $H^0(X, 2n\Theta)$. The normal bundle exact sequence of the inclusion of $A_g$ in $A_{g+1}^S$ is
$$0 \to T_X A_g \to T_X A_{g+1}^S \to H^0(X, 2\Theta)^C \to 0$$
Concretely, given a modular form $F_{g+1}$ on $A_{g+1}$, the tangent vectors in $T_X A_g$ act on the restriction $F_g$ of $F_{g+1}$ to $A_g$, the tangent vectors in $H^0(X, 2\Theta)^C$ act on the first Fourier-Jacobi
coefficient of $F_{g+1}$. Moreover, if we slice $A_g$ at $X$, the isolated singularity we get is the affine cone over the Kummer variety of $X$, i.e. the cone over the image of the morphism

$$|2\Theta|: X \to \mathbb{P}H^0(X, 2\Theta)^\vee$$

Let now $X$ be the Jacobian of a generic genus $g$ curve $C$, call $S^2C$ the second symmetric product of $C$. We consider the subtraction map

$$\delta: S^2C \to X \quad \delta(a, b) \mapsto AJ(a-b)$$

where $AJ$ is the Abel-Jacobi map. The image of this morphism is usually denoted in the literature by $C - C$ and it is customary to define the linear system

$$\Gamma_{00} := \{s \in H^0(X, 2\Theta) \text{ s.t. } \text{mult}_0X(s) \geq 4\}.$$ 

Its base locus has been studied in [vGvdG86] and [Wel86].

**Lemma 5.1.** The line bundle $\delta^*2\Theta$ is isomorphic to the canonical bundle $K$ of $S^2C$ and the map

$$\delta^*: H^0(X, 2\Theta) \to H^0(S^2C, K)$$

is surjective.

**Proof.** The first assertion is classical, it is equivalent to say that on $C \times C$ we have

$$\delta^*2\Theta = 2(K_1 + K_2 + \Delta)$$

where $\Delta$ is the diagonal and $K_i$ is the pull-back of the canonical bundle via the $i$-th projection. This last assertion follows from Riemann’s theorem.

The second fact is because of dimensional reasons. The dimension of $H^0(X, 2\Theta) = 2^g$. The kernel of $\delta^*$ is $\Gamma_{00}$, see lemma 8.1, so it is defined by the $\frac{1}{2}g(g+1)+1$ conditions

$$s(0) = 0, \quad \frac{\partial^2 s}{\partial z_i \partial z_j}(\tau, 0) = 0, \quad \forall i, j.$$

These conditions are linearly independent for non-decomposable abelian variety (see the proof of lemma 11 page 188 of [Igu72]), so the dimension of $\Gamma_{00}$ is $2^g - \frac{1}{2}g(g+1) - 1$. See also [vGvdG86] proposition 1.1. □

We are now ready to write out the normal bundle exact sequence of the inclusion of $A_g \cap M_{g+1}^S$ in $A_{g+1}^S$.

**Theorem 5.2.** Let $C$ be a generic genus $g$ curve, we have the following exact sequence

$$0 \to T_{C, A_g} \to T_{C, M_{g+1}^S} \to H^0(S^2C, K)^\vee \to 0$$

**Proof.** There are two things we have to prove, first

$$T_{C}(M_{g+1}^S \cap A_g) = T_{C, A_g}$$

this is theorem 1.1 when $m = 1$, see also the remark on page 13 of [CSB13].

To describe the co-kernel we have to show that, after blowing up $A_{g+1}^S$ at $A_g$, the proper transform of $M_{g+1}^S$ meets the Kummer variety of $X$ in $\delta(S^2C)$. This is proved in [Nam73] theorem 6. □

Let us point out that tangent cone of $M_{g+1}^S$ at $M_g$ is the cone over $\delta(S^2C)$, but neither the singularity nor its normalization are cone.

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We turn our attention to the hyperelliptic locus. Let \( X \) be the Jacobian of a generic genus \( g \) hyperelliptic curve \( C \), consider the morphism.

\[
(6)\quad \Psi : C \overset{\delta}{\rightarrow} C \times C \overset{\iota}{\rightarrow} X \overset{[2g]}{\rightarrow} \mathbb{P}H^0(X, 2\Theta)
\]

\[
(p, \iota(p)) \mapsto (a, b) \mapsto AJ(a - b)
\]

Call \( K \) the canonical bundle and \( W \) the divisor of Weierstrass points on \( C \).

**Lemma 5.3.** The line bundle \( \Psi^*2\Theta \) is isomorphic to \( 2(K + W) \) and the map

\[
\Psi^*: H^0(X, 2\Theta) \rightarrow H^0(C, 2(K + W))
\]

is not surjective, it has rank \( 2g \).

**Proof.** The line bundle \( \delta^*2\Theta \) is isomorphic to \( 2(K_1 + K_2 + \Delta) \). The pull back via \( f \) of \( \Delta \) is the locus defined by \( p = \iota(p) \), so it is \( W \). The pull back of \( K_1 + K_2 \) is \( K \).

Let us now prove the second assertion. We know that \( \delta^* \) is surjective. The pull back of \( H^0(S^2C, K) \) to \( C \times C \) is generated by \( S^2H^0(C, K_C) \) and the Szegö kernel \( \omega(a, b) \) (a symmetric differential with a pole of order two and no residue along the diagonal). In particular, \( S^2H^0(C, K) \) is the pull back of the sections of \( 2\Theta \) vanishing on the origin \( 0x \). Since \( 0x \) is contained in \( \Psi(C) \), the kernel of \( f^* \) is contained in \( S^2H^0(C, K_C) \). We have now to compute the rank of

\[
f^*: S^2H^0(C, K) \rightarrow H^0(C, 2K)
\]

\[
\omega_1 \otimes \omega_2 \mapsto \omega_1 \iota^* \omega_2
\]

Remark that \( \omega_1\iota^*\omega_2 \) is equal to \( -\omega_1\omega_2 \), so the rank of \( f^* \) is equal to the rank of the usual multiplication map from \( S^2H^0(C, K) \) to \( H^0(C, 2K) \). This rank, being \( C \) hyperelliptic, is \( 2g - 1 \) (see [ACG11] page 223).

The normal bundle exact sequence of \( A_g \cap Hyp^g_{g+1} \) in \( Hyp^g_{g+1} \) is the following:

**Theorem 5.4.** Let \( C \) be a generic hyperelliptic curve of genus \( g \) and \( \Psi \) the map defined in (6), the following exact sequence holds

\[
0 \rightarrow TC Hyp_g \rightarrow TC Hyp^g_{g+1} \rightarrow PC \rightarrow 0
\]

where \( PC \) is the image of \( \Psi^* \), in other words it is the span of the affine cone over \( \Psi(C) \).

**Proof.** Again, there are two things we have to prove, first

\[
TC Hyp^g_{g+1} \cap TC A_g = TC Hyp_g
\]

this is equivalent to theorem 4.1.

To describe the co-kernel we have to show that, after blowing up \( A_g^S \) in \( A^S_{g+1} \), the proper transform of \( Hyp^g_{g+1} \) meets the Kummer variety of \( X \) in \( \Psi(C) \). This is proved in [Nam73] theorem 6, just remark that to obtain a generic irreducible nodal hyperelliptic curve we need to glue two points conjugated under the hyperelliptic involution.

In this case, if we slice the \( Hyp_g \) in \( Hyp^g_{g+1} \) at \( X \) and normalise, we really get a cone. To prove this we look at the resolution of the singularity provided by the Torelli map (3). The fibre over \( X \) is \( C \) mod the hyperelliptic involution, so \( \mathbb{P}^1 \); the co-normal bundle has degree \( 4g \) and we can thus apply a classical result by Grauert ([Gra62] corollary page 363, see also [CM03]) which guarantees that the normalization of the singularity is a cone over the \( 4g \)-Veronese embedding of \( \mathbb{P}^1 \).
6. Projective invariants of hyperelliptic curves

In this section we introduce the projective invariants of a hyperelliptic curve and we prove the following well-known criterion.

**Criterion 6.1.** Let $F_g$ be a weight $n$ and degree $g$ modular form. Suppose that it vanishes on $A_{g-1} \cap Hyp_g^S$ with multiplicity at least $k$. If

$$\frac{n}{k} < 8 + \frac{4}{g}$$

then $F_g$ vanishes on $Hyp_g$.

This criterion could be proved using the slope of the hyperelliptic locus, see [CH88] theorem 4.12. We will rather use projective invariants, references are [Igu67b], [Poo96], [AL02] and [Pas11] Chapter 2.

Let $C$ be a smooth genus $g$ hyperelliptic curve. Fix a two to one map $\pi$ from $C$ to $\mathbb{P}^1$, this morphism is unique up to projective transformations of $\mathbb{P}^1$, it ramifies at $2g + 2$ points. A point $p$ is called a Weierstrass point if it is a ramification point for $\pi$.

**Definition 6.2.** (Projective invariants) The projective invariants of $C$ are the image of the Weierstrass points under $\pi$, considered up to permutations and projective automorphisms of $\mathbb{P}^1$.

Starting from $2g + 2$ distinct points on $\mathbb{P}^1$, one can construct a smooth genus $g$ hyperelliptic curve with the prescribed projective invariants. As an aside, let us recall that Thomae’s formula permits to write the cross-ratios of the projective invariants in term of second order Theta functions evaluated at the period matrix.

Call $B_g$ the moduli space of $2g + 2$ points on $\mathbb{P}^1$, up to permutation and projectivity. This space is a GIT quotient, the semi-stable locus (i.e. the $2g + 2$-tuples that $B_g$ parametrises) consist of all the $2g + 2$-tuples such that no more than $g + 1$ points coincide. $B_g$ can be defined as the $Proj$ of the ring $S(2, 2g + 2)$. This is the ring of symmetric functions in $2g + 2$ variables, which are invariant under the natural action of $SL(2, \mathbb{C})$. See the references for more details. The discriminant $\Delta$ is an element of $S(2, 2g + 2)$ of degree $4g + 2$, it cuts the divisor $D$ parametrising the $2g + 2$-tuples of points where at least two entries coincide.

Because of the previous discussion, we have an isomorphism

$$f_g : Hyp_g \rightarrow B_g \setminus D$$

mapping a curve to its projective invariants. Following [AL02], this isomorphism extend to a map

$$f_g : \overline{Hyp}_g \rightarrow B_g$$

where $\overline{Hyp}_g$ is the Deligne-Mumford compactification of $Hyp_g$. This map is a birational isomorphism between the boundary divisors $\mathcal{X}_0$ and $D$, it contracts all the other boundary divisors of $\overline{Hyp}_g$ to subvariety of co-dimension greater than 1. The divisor $\mathcal{X}_0$ parametrises curves of compact type, i.e. curves obtained starting with a genus $g - 1$ hyperelliptic curve $C'$ and gluing two points conjugated under the hyperelliptic involution. Its image is the set of $2g + 2$ points of the form $\{p_1, \ldots, p_{2g}, p, p\}$, the projective invariants of $C'$ are $\{p_1, \ldots, p_{2g}\}$, the glued points are the preimages of $p$ under $\pi$.

We can consider the rational inverse of $f_g$, call $\bar{\rho}$ the composition

$$\bar{\rho} : B_g \xrightarrow{f_g^{-1}} \overline{Hyp}_g \xrightarrow{\pi} Hyp_g^S \hookrightarrow A_g^S$$
this map is the geometric version of the Igusa morphism of projective invariants $\rho$ defined in [Igu67b], which is a map of graded rings

$$\rho : \bigoplus_{n=0}^{\infty} H^0(A_g, L_g^n) \to S(2, 2g + 2)$$

whose kernel is exactly the ideal of modular forms vanishing on the hyperelliptic locus. The degree of $\rho$ is $\frac{4}{2}g$.

The closure of the image of the divisor $\Xi_0$ in $Hyp^{S}_{g}$ is $Hyp^{S}_{g} \cap A^{S}_{g-1}$, so the image of $D$ under $\bar{\rho}$ is $Hyp^{S}_{g} \cap A^{S}_{g-1}$. We can now prove the criterion.

**Proof.** (of criterion 6.1) Suppose $F_g$ vanishes with multiplicity at least $k$ on $Hyp^{S}_{g} \cap A^{S}_{g-1}$. This means that $\bar{\rho}^* F_g$ vanishes with multiplicity at least $k$ on $D$. In other words, $\Delta^k$ divides $\rho(F_g)$. The degree of the discriminant in $S(2g, 2g + 2)$ is $4g + 2$, the degree of $\rho(F_g)$ is $1g + 2$. Since, by hypothesis,

$$k(4g + 2) \geq \frac{1}{2} gn$$

we obtain that $\rho(F_g)$ is equal to zero, so the claim. □

7. **Stable equations for the hyperelliptic locus**

Let us start recalling the definitions of lattices and Theta series. A lattice is a couple $(\Lambda, Q)$ where $\Lambda$ is a free group and $Q$ is a $\mathbb{Z}$-valued quadratic form. The rank of the lattice is the rank of $\Lambda$, elements of $\Lambda$ are called vectors, the norm of a vector $v$ is $Q(v, v)$. We always assume $Q$ to be even (i.e. $Q(v, v)$ is even for every $v$), unimodular and positive definite. Often, we will denote a lattice just by $\Lambda$, forgetting $Q$.

Given a lattice $(\Lambda, Q)$ and an integer $g$, we can define the associated Theta series $\Theta_{\Lambda, g}$ as follows

$$\Theta_{\Lambda, g}(\tau) := \sum_{x_1, \ldots, x_g \in \Lambda} \exp(\pi i \sum_{i,j} Q(x_i, x_j) \tau_{ij}),$$

where $\tau$ belongs to the Siegel upper half space. Studying how this function transform under the action of $Sp(2g, \mathbb{Z})$, one shows that it is a weight $\frac{1}{2}rk(\Lambda)$ modular form.

By applying the Siegel operator to $\Theta_{\Lambda, g+1}$, one sees that the restriction of $\Theta_{\Lambda, g+1}$ to $A^S_g$ is $\Theta_{\Lambda, g}$, so we can define the stable modular form

$$\Theta_{\Lambda} := \bigcup_{g \geq 0} \Theta_{\Lambda, g}.$$

Let us write out the $n$-th Fourier-Jacobi coefficient of a Theta series $\Theta_{\Lambda, g+1}$ (we keep the notations of equations (4) and (5)), we have

$$f_n(\tau, z) = \sum_{x_1, \ldots, x_g \in \Lambda} \sum_{y \in R_{2n}(\Lambda)} \exp(\pi i \sum_{i,j} Q(x_i, x_j) \tau_{ij} + 2\pi i \sum_i Q(y_i, x_i) z_i)$$

where $R_{2n}(\Lambda)$ is the set of vectors of $\Lambda$ of norm $2n$. In particular, if $R_{2n}(\Lambda)$ is empty then $f_n$ is trivial. Let us define the following basic invariant

$$\mu_\Lambda := \min\{Q(v, v) \mid v \in \Lambda; v \neq 0\} = \min\{2n \mid R_{2n}(\Lambda) \neq \emptyset\}.$$

We write stable equations for the hyperelliptic locus as differences of Theta series, we first look for a necessary condition. Suppose that the stable modular form

$$\Theta_{\Lambda, g} - \Theta_{\Gamma, g}$$
vanishes on $Hyp_g$ for every $g$. This, in particular, means that it vanishes on $Hyp_1 = A_1$, so
\[ \Theta_{\Lambda,1} = \Theta_{\Gamma,1}. \]
Looking at the Fourier-Jacobi expansion for $g = 1$, the previous equality means that the two lattices have the same number of vectors of any given norm. In particular, we have
\[ \mu_{\Lambda} = \mu_{\Gamma}. \]

Combining theorem 4.1 and criterion 6.1 we can prove the following:

**Theorem 7.1.** Let $\Lambda$ and $\Gamma$ be two even positive definite unimodular lattices of rank $N$ and $\mu_{\Lambda} = \mu_{\Gamma} =: \mu$, if
\[ \frac{N}{\mu} \leq 8, \]
then
\[ F := \Theta_{\Lambda} - \Theta_{\Gamma} \]
is a stable equation for the hyperelliptic locus. In other words, $F_g$ vanishes on $Hyp_g$ for every $g$.

**Proof.** The proof is by induction on $g$. The difference of two Theta series vanishes on $A_0$. Suppose the statement true for $g$, we want to apply criterion 6.1 to $F_{g+1}$. Call $k := \frac{\mu}{2}$, we need to prove that $F_{g+1}$ vanishes at the boundary component $A_g \cap Hyp_g^{S+1}$ with multiplicity at least $k$.

To stress why theorem 4.1 is important, let us first give the proof when $k = 2$. The argument is local, take a generic point $\tau$ of $A_g \cap Hyp_g^{S+1}$. By induction we know that $F_{g+1}(\tau) = 0$, we want to prove that for every derivative $D$ in $T_\tau Hyp_g^{S+1}$ we have $DF_{g+1}(\tau) = 0$. Since $\mu = 4$, the first Fourier-Jacobi coefficient is trivial and the Fourier-Jacobi expansion of $F_{g+1}$ looks like
\[ F_{g+1} = F_g(\tau) + o(q^2) \]
so $DF_{g+1}(\tau) = DF_g(\tau)$. We can thus assume that $D$ is tangent to $A_g$. Now, we need to use theorem 4.1 to assume that $D$ is tangent to $Hyp_g$. By inductive hypothesis $F_g$ is zero on $Hyp_g$, so $DF_g(\tau) = 0$. (Remark that we can not run the same argument for $A_g$.)

For a general $k$ the argument is pretty much the same, we just need to enhance the notations. Suppose $F_{g+1}$ vanishes on $Hyp_g^{S+1} \cap A_g$ with order at least $s$ smaller than $k$, we want to prove it vanishes with order at least $s + 1$. In the local ring $(Hyp_g^{S+1}, \tau)$, consider the ideal $I$ of elements vanishing on $Hyp_g^{S+1} \cap A_g$. We know $F_{g+1}$ belongs to $I^s$, we want to show that its class in $I^s/I^{s+1}$ is trivial. The elements of $I^s/I^{s+1}$ are symmetric $s$-linear forms on $T_\tau Hyp_g$, restricting them to $T_{\tau}(A_g \cap Hyp_g^{S+1}) = T_\tau Hyp_g$ we get an exact sequence
\[ H^0(\tau, 2s\Theta) \to I^s/I^{s+1} \to \text{Sym}^s(T_\tau Hyp_g). \]
Where $\Phi$ is the restriction from $A_g^{S+1}$ to $A_g$. The class of $F_{g+1}$ is in the kernel of $\Phi$, because by inductive hypothesis $F_g$ vanishes identically on $Hyp_g$. Moreover, it is zero in $H^0(\tau, 2s\Theta)$, because $s < k$, so the conclusion. (We have used theorem 4.1 to replace $T_\tau(A_g \cap Hyp_g^{S+1})$ with $T_\tau Hyp_g$.)

The hypothesis
\[ \frac{rk(\Lambda)}{\mu_{\Lambda}} \leq 8 \]
is quite restrictive. Indeed, given any even unimodular lattice $\Lambda$, there is an upper bound
\[ \mu_{\Lambda} \leq 2 \left[ \frac{rk(\Lambda)}{24} \right] + 2 \]
where “⌊ ⌋” is the round down (see [CS99] section 7.7 corollary 21); moreover μ is even and the rank, since Λ is positive definite, is divisible by 8. We conclude that if an even unimodular positive definite lattice Λ satisfies hypothesis (8), then the only possibilities for the couple $(rk(Λ), μ_Λ)$ are (8,2),(16,2),(24,4),(32,4) and (48,6). All these lattices are extremal, which means

$$\mu_Λ = 2\lfloor \frac{rk(Λ)}{24} \rfloor + 2.$$  

On the other hand, given two extremal lattices Λ and Γ, it is not true that $\Theta_Λ,g - \Theta_Γ,g$ vanishes on the hyperelliptic locus for every g. See [Oze88] for an example of 3 extremal lattices of rank 40 whose Theta series are different for $g = 2$. In the proof of theorem 7.1 we have not used the hypothesis $\mu_Λ = \mu_Γ$, however, as we have seen, hypothesis (8) and $rk(Λ) = rk(Γ)$ imply this fact.

There exist only two lattices of rank 16 and $μ = 2$, their difference gives the Schottky form discussed in the introduction. There exists exactly one lattice of rank 8, $E_8$, and one lattice of rank 24 and $μ = 4$, the Leech lattice, so in these cases we do not get any stable equation for the hyperelliptic locus

In [Kin03] corollary 5, using a generalization of the mass formula, it is shown that there exist at least ten millions of lattices of rank 32 and $μ = 4$ (in King’s paper every lattice is tacitly assumed to be positive definite), however just 15 of them are known explicitly.

The situation for lattices of type (48,6) is not clear, believably there exist many of them, see [Kin03] page 15, but there is not any lower bound and just 3 of them are known explicitly.

8. More stable equations, lattices of rank 24

In this section we deal with the weight 12 stable modular form

$$F := \Theta_Λ - \Theta_Γ$$

where Λ and Γ are rank 24 lattices with the same number of roots. By root we mean a vector of norm 2. Equivalently, the two lattices have the same Coxeter number. More specifically, we are dealing with the following 5 pairs of lattices

| Lattices | $A_4^2D_4$, $D_5^2$ | $A_5^2D_6$, $D_7^2$ | $E_8^2$, $A_{11}D_7E_6$ | $A_{11}E_7$, $D_{10}E_8^2$ | $E_8D_{16}$, $E_3^2$ |
|----------|---------------------|---------------------|------------------------|------------------------|------------------------|
| Number of roots | 144 | 240 | 248 | 432 | 720 |

where, as usual, a lattice of rank 24 is labelled by its root system. See e.g. [Ebe13] section 3 for more details. Let us point out that the couple $E_8^2$, $D_{16}E_8$ corresponds to the modular form $\Theta_{E_8}(\Theta_{E_8} \oplus E_8 - \Theta_{D_{16}})$, so its behaviour must be the same of he Schottky form defined in (1). The others cases are not covered by previous results. On the other hand, if the number of roots of Λ is different from the number of roots of Γ, the modular form F is already non-zero on $M_1 = A_1$, so these five are really the only interesting pairs of rank 24 lattices. Before proving our results we need to recall two classical facts.

A formula for sections of $2\Theta$ Let $s$ be a section of $2\Theta$ on the Jacobian of a curve $C$ with period matrix $τ$, then, for every period of points $a$ and $b$ of $C$ the following classical formula holds:

$$(9) \quad s(τ, a - b) = E(a, b)^2[s(τ, 0)\omega(a, b) + \sum_{i,j} \frac{\partial^2 s}{\partial z_i \partial z_j}(τ, 0)\omega_i(a)\omega_j(b)]$$

where $E$ is the Prime form, $\{\omega_i\}$ is the basis of the holomorphic differentials on $C$ corresponding to the basis $\{\frac{1}{2\pi i} \}$ of the tangent space at the origin of the Jacobian, $\omega(a, b)$ is the Szegö Kernel,
and everything is trivialised with respect to a choice of local co-ordinates $z_a$ and $z_b$. To prove this formula one remarks that the ratio
\[
\frac{s(\tau, a - b)}{E(a, b)^2}
\]
is a section of the canonical bundle of $S^2C$. The space $H^0(S^2C, K)$ is generated by $S^2H^0(C, K)$ and the Szegö kernel, so
\[
\frac{s(\tau, a - b)}{E(a, b)^2} = c_0\omega(a, b) + \sum_{i,j} c_{ij}\omega_i(a)\omega_j(b)
\]
for some constant coefficients $c_0$ and $c_{ij}$. To compute the coefficients one fixes $a$ and write out the Taylor expansion of both sides of the equation at $b = a$. The expansions of $\omega(a, b)$ and $E(a, b)$ are given in [Fay73] page 16-18.

Let us recall the subtraction map defined in (5)
\[
\delta : S^2C \rightarrow Jac(C) \quad (a, b) \mapsto AJ(a - b)
\]
and the linear system
\[
\Gamma_00 := \{ s \in H^0(X, 2\Theta) \text{ s.t. } \text{mult}_x(s) \geq 4 \},
\]
we have the following classical lemma

**Lemma 8.1.** The kernel of the map
\[
\delta^* : H^0(X, 2\Theta) \rightarrow H^0(S^2C, K)
\]
is $\Gamma_{00}$

**Proof.** The inclusion $\Gamma_{00} \subseteq \text{Ker}(\delta^*)$ follows directly from formula (9). Let us prove the reverse inclusion. Take $s$ in the kernel of $\delta^*$ and consider the quadric
\[
Q := \sum_{i,j} \frac{\partial^2 s}{\partial z_i \partial z_j}(\tau, 0)\omega_i\omega_j
\]
in $\mathbb{P}H^0(C, K)$. We know that $s(\tau, 0) = 0$, applying formula (9) we get
\[
Q(a, b) = 0 \quad \forall a, b \in C.
\]
In particular, $Q$ contains the canonical model of $C$. Polarizing the bilinear form and varying the local co-ordinates we show that $Q$ contains the secant variety of the canonical model of $C$: this means that the quadric is trivial (see [CSB13] page 12) and so
\[
\frac{\partial^2 s}{\partial z_i \partial z_j}(\tau, 0) = 0 \quad \forall i, j.
\]
\[\square\]
See also [vGvdG86] proposition 2.1, [Wel86] proposition 4.8 and [MV10] appendix A.

**The “heat equation” for rank 24 lattices** Let $(\Lambda, Q)$ be a lattice of rank 24. The classification of these lattices is due to Niemeier, but it has been simplified by Venkov proving and using the following identity
\[
r_2(\Lambda)Q(v, v) = 48 \sum_{y \in R_2(\Lambda)} Q(y, v)^2 \quad \forall v \in \Lambda
\]
where \( r_2(\Lambda) \) is the number of roots, \( \mathcal{R}_2(\Lambda) \) is the set of roots and 48 is the rank of the lattice times 2. The proof relies upon the theory of degree 1 modular forms with harmonic coefficients, it is explained in [Ebe13] section 3. Polarizing this identity one gets

\[
(11) \quad r_2(\Lambda) Q(v, w) = 48 \sum_{y \in \mathcal{R}_2(\Lambda)} Q(y, w) Q(y, v) \quad \forall v, w \in \Lambda.
\]

For any element \( T \) in the Siegel upper half space \( \mathcal{H}_{g+1} \) we write

\[
T = \begin{pmatrix} \tau & z \\ t^* & \tau \end{pmatrix},
\]

with \( t \) in \( \mathcal{H}_1 \) and \( \tau \) in \( \mathcal{H}_g \). Let \( q := \exp(2\pi it) \), the Fourier-Jacobi expansion of \( \Theta_{\Lambda,g+1} \) is

\[
\Theta_{\Lambda,g+1}(T) = \Theta_{\Lambda,g}(\tau) + \sum_{n \geq 1} f_n(\tau, z) q^n,
\]

where \( f_n \) is the \( n \)-th Fourier-Jacobi coefficient of \( \Theta_{\Lambda,g+1} \).

**Proposition 8.2** (Heat equation). The first Fourier-Jacobi coefficient \( f_1 \) of a Theta series associated to a rank 24 lattice \( \Lambda \) satisfies the following \textit{heat equation}:

\[
r_2(\Lambda) 2\pi i (1 + \delta_{ij}) \frac{\partial f_1}{\partial \tau_{ij}}(\tau, 0) = 48 \frac{\partial^2 f_1}{\partial \tau \partial z_j}(\tau, 0),
\]

where \( r_2(\Lambda) \) is the number of roots of \( \Lambda \).

**Proof.** Let us write out the first Fourier-Jacobi coefficient

\[
f_1(\tau, z) = \sum_{x_1, \ldots, x_g \in \Lambda} \sum_{y \in \mathcal{R}_2(\Lambda)} \exp(\pi i \sum_{i,j} Q(x_i, x_j) \tau_{ij}) + 2\pi i \sum_{i} Q(y, x_i) z_i.
\]

Fix two indexes \( i \) and \( j \), by explicit computation we have

\[
\frac{\partial^2 f_1}{\partial z_i \partial z_j}(\tau, 0) = (2\pi i)^2 \sum_{x_1, \ldots, x_g \in \Lambda} \sum_{y \in \mathcal{R}_2(\Lambda)} Q(y, x_i) Q(y, x_j) \exp(\pi i \sum_{i,j} Q(x_i, x_j) \tau_{ij}),
\]

On the other hand

\[
(1 + \delta_{ij}) \frac{\partial f_1}{\partial \tau_{ij}}(\tau, 0) = 2\pi i \sum_{x_1, \ldots, x_g \in \Lambda} Q(x_i, x_j) \exp(\pi i \sum_{i,j} Q(x_i, x_j) \tau_{ij}),
\]

the coefficient \((1 + \delta_{ij})\) is because the variables on \( \mathcal{H}_g \) are \( \tau_{ij} \) with \( i \leq j \), so when we compute the derivative with respect to \( \tau_{ij} \) we need to derive both \( \tau_{ij} \) and \( \tau_{ji} \). Applying the identity (11) we obtain the proposition. \( \square \)

See also [MV10] page 16. This result could be generalized to higher order Fourier-Jacobi coefficients and higher rank lattices, but we do not need it.

Let us now look at the behaviour of \( F_g \) on \( \mathcal{M}_g \). We are going to use the notion of slope defined at the end of section 3 and we will make the following assumption:

**Assumption 8.3.** Let \( F_{g+1} \) be a modular form on \( \mathcal{A}_g \), if its weight divided by its vanishing order on \( \mathcal{A}_g \cap \mathcal{M}_{g+1} \) is smaller than or equal to 6, then \( F_{g+1} \) vanishes on \( \mathcal{M}_{g+1} \).

This assumption is true for low values of \( g \). If \( g \leq 23 \) we can apply corollary 1.2 of [FP05], which guarantees that the slope of \( F_{g+1} \) is the weight divided by the vanishing order on \( \mathcal{A}_g \cap \mathcal{M}_{g+1} \). If \( g \) is small (e.g. \( g \leq 4 \), see [HM90] theorem 0.3) the slope of \( \mathcal{M}_{g+1} \) is bigger than 6, so we obtain that \( F_{g+1} \) vanishes on \( \mathcal{M}_{g+1} \). The previous argument could possibly hold for any \( g \), but this is still an open problem.
Theorem 8.4. Let $\Lambda$ and $\Gamma$ be two even positive definite unimodular lattices of rank $24$ with the same number of roots, suppose that assumption 8.3 holds, then the stable modular form

$$F := \Theta_\Lambda - \Theta_\Gamma$$

is zero on $M_{g+1}$ if and only if its restriction $F_g$ to $A_g$ is zero with order at least two on $M_g$.

Proof. One direction is a general fact, it has nothing to do with lattices: Theorem 1.1 guarantees that if a modular form is zero with order at least $k$ on $M_{g+1}$, then its restriction to $A_g$ is zero on $M_g$ with order at least $k + 1$.

For the other direction, we need to use all the hypotheses. Because of assumption 8.3, it is enough to show that $F_{g+1}$ vanishes with order at least 2 along $A_g \cap M^S_{g+1}$. Let $C$ be a general element of $M_g$, call $\tau$ its period matrix. We use the description of the tangent space $T_C M^S_{g+1}$ given in Theorem 5.2. Since $F_g$ vanishes with order 2 on $M_g$, we get that $F_{g+1}$ vanishes along $T_C A_g$. We want to show that $F_{g+1}$ vanishes along $H^0(S^2 C, K)$, in other words we need to show that the first Fourier-Jacobi coefficient $f_1$ of $F_{g+1}$ is zero on $C - C$, or more formally $f_1$ is in the Kernel of $\delta^*$, where the subtraction map $\delta$ is defined in (10). Thanks to lemma 8.1, we have to show that

$$f_1(\tau, 0) = 0, \quad \frac{\partial^2 f_1}{\partial \tau_i \partial \tau_j}(\tau, 0) = 0 \quad \forall i, j.$$

We have $f_1(\tau, 0) = F_g(\tau) = 0$. Now we use the heat equation 8.2: since $r_2(\Lambda) = r_2(\Gamma) = r$ we have

$$48 \frac{\partial^2 f_1}{\partial \tau_i \partial \tau_j}(\tau, 0) = r 2 \pi i (1 + \delta_{ij}) \frac{\partial f_1}{\partial \tau_j}(\tau, 0),$$

moreover

$$\frac{\partial f_1}{\partial \tau_j}(\tau, 0) = \frac{\partial F_g}{\partial \tau_j}(\tau),$$

the right hand side is zero because $F_g$ vanishes with order at least two on $M_g$. \qed

Corollary 8.5. Let $\Lambda$ and $\Gamma$ be two even positive definite unimodular lattices of rank $24$ with the same number of roots, then the stable modular form

$$F := \Theta_\Lambda - \Theta_\Gamma$$

is zero on $M_g$ for $g \leq 4$, and it cuts a divisor of slope 12 on $M_5$.

Proof. Any difference of Theta series is zero on $A_0$. For $g \leq 3$, $M_3^S$ is equal to $A_3^g$ and hypothesis 8.3 holds, so applying theorem 8.4 we prove that $F_g$ is zero on $M_g$ for $g \leq 4$.

To show that $F_5$ does not vanish on $M_5$ we show that $F_4$ does not vanish with order two on $M_4$. The weight of $F_4$ is 12, $M_4$ is defined by a form of weight 8, so $F_4$ vanishes with order 2 on $M_4$ if and only if it is trivial on $A_4$. This is not possible because $\Theta_{\Lambda, 4}$ and $\Theta_{\Gamma, 4}$ have different Fourier coefficients with respect to the quadratic form $A_4$, as shown in [BFW98] table page 146.

To prove the statement about the slope, using the notations and the result of lemma 3.4, it is enough to know that $F_4$ vanishes with order exactly 1 on $A_4$, i.e. $b_0 = 1$. \qed

As a by-product, for any couple of lattices $\Lambda$ and $\Gamma$ satisfying the hypotheses of corollary 8.5, we get the following identity on $A_4$

$$\Theta_{\Lambda, 4} - \Theta_{\Gamma, 4} = k \Theta_{E_8, 4}(\Theta_{E_8 \oplus E_8, 4} - \Theta_{D_{16}, 4})$$

where $k$ is a non-zero constant which can be determined by looking at one Fourier coefficient. This formula is also obtained in [GV09] section 4.2.2, working on explicit generators of rings of modular forms.
Theorem 8.6. Let $\Lambda$ and $\Gamma$ be two even positive definite unimodular lattices of rank $24$ with the same number of roots, then

$$F := \Theta_\Lambda - \Theta_\Gamma$$

is a stable equation for the hyperelliptic locus.

Proof. The proof is pretty much the same of the proof of theorem 7.1 when $k = 2$. The only difference is that now the first Fourier-Jacobi coefficient $f_1$ is not trivial: we need to show that it vanishes when restricted to $P_C$, i.e. when it is restricted to points of the form $(\tau, p - \iota(p))$, where $\tau$ is the period matrix of a generic hyperelliptic curve $C$, and $p$ is a point of $C$ (see the description of tangent space given in 5.4). To do this we argue as follows. First remark that

$$f_1(\tau, 0) = F_\mathfrak{g}(\tau) = 0$$

Then we apply the formula (9), trivializing everything with respect to co-ordinates $z_p$ and $\iota^*z_p$ and recalling that

$$\frac{\omega}{dz_p}(p) = \frac{\omega}{\iota^*dz_p}(\iota(p))$$

we get

$$f_1(\tau, p - \iota(p)) = E(p, \iota(p))^2 \sum_{i,j} \frac{\partial^2 f_1}{\partial z_i \partial z_j}(\tau, 0) \omega_i(p) \omega_j(p)$$

Now the heat equation 8.2 comes into the game: since $r_2(\Lambda) = r_2(\Gamma) =: r$ we have

$$48 \sum_{i,j} \frac{\partial^2 f_1}{\partial z_i \partial z_j}(\tau, 0) \omega_i(p) \omega_j(p) = 2\pi r i \sum_{i,j} (1 + \delta_{ij}) \frac{\partial^2 F_\mathfrak{g}}{\partial \tau_i \partial \tau_j}(\tau) \omega_i(p) \omega_j(p) = (4r \pi i) dF_\mathfrak{g}(\tau)(p)$$

Let us recall that the fibre of the cotangent bundle of $A_p$ at $Jac(C)$ is isomorphic to $Sym^2 H^0(C, K_C)$, so $dF_\mathfrak{g}(\tau)$ is a quadric in $PH^0(C, K_C)^\vee$ and we can evaluate it on the image of $p$ under the canonical map. The co-normal bundle of $Hyp_p$ in $A_p$ is given by the quadric vanishing on the image of $C$ under the canonical map, so, since $F_\mathfrak{g}$ vanishes on $Hyp_p$, we conclude that $dF_\mathfrak{g}(\tau)(p)$ is zero for every $p$ in $C$. \hfill $\square$

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DPMMS, UNIVERSITY OF CAMBRIDGE

*E-mail address: g.codogni@dpmms.cam.ac.uk*