LAGRANGIAN DYNAMICS ON MATCHED PAIRS

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Abstract. Given a matched pair of Lie groups, we show that the tangent bundle of the matched pair group is isomorphic to the matched pair of the tangent groups. We thus obtain the Euler-Lagrange equations on the trivialized matched pair of tangent groups, as well as the Euler-Poincaré equations on the matched pair of Lie algebras. We show explicitly how these equations cover those of the semi-direct product theory. In particular, we study the trivialized, and the reduced Lagrangian dynamics on the group $SL(2,\mathbb{C})$.

1. Introduction

Lie groups are configuration spaces of many physical systems such as rigid body, fluid and plasma theories [34]. As a result, there are extensive studies investigating the geometry underlying both the Lagrangian and the Hamiltonian dynamics on Lie groups [3, 26, 37, 28].

The Lagrangian formulation of a system whose configuration space is a Lie group $G$ is available on the tangent bundle $TG$ whose (left) trivialization is a semi-direct product $G \ltimes \mathfrak{g}$ of the group $G$ and its Lie algebra $\mathfrak{g}$. On the trivialized tangent bundle, a real-valued Lagrangian function(al) $\mathcal{L} = \mathcal{L}(g, \xi)$ generates the trivialized Euler-Lagrange equations

$$
\frac{d}{dt} \frac{\delta \mathcal{L}}{\delta \xi} = T^*_e L_{\mathfrak{g}} \frac{\delta \mathcal{L}}{\delta g} - ad^*_\xi \frac{\delta \mathcal{L}}{\delta \xi}.
$$

Here, $ad^*_\xi : \mathfrak{g}^* \to \mathfrak{g}^*$ is the coadjoint action of $\xi \in \mathfrak{g}$ on the linear algebraic dual $\mathfrak{g}^*$ of $\mathfrak{g}$. For the trivialized Euler-Lagrange equations we refer the reader to, an incomplete list, [5, 11, 12, 14, 17, 18].

In the presence of a symmetry by the (left) action of $G$, we arrive at a reduced Lagrangian function(al) $\mathcal{L}$ which is free from the group variable. In this case, the first term in the right hand side of (1.1) drops, and the dynamics is governed by the Euler-Poincaré equations

$$
\frac{d}{dt} \frac{\delta \mathcal{L}}{\delta \xi} = -ad^*_\xi \frac{\delta \mathcal{L}}{\delta \xi},
$$

on the Lie algebra $\mathfrak{g}$. The Euler-Poincaré equation has been written for a wide spectrum of Lie groups; from matrix Lie groups to diffeomorphism groups, see for instance [26, 37, 15, 16, 23, 4, 24] and the references therein. It is also possible to find various different forms of the Euler-Poincaré equations in the literature. For instance, starting with a Lie groupoid instead of a Lie group, it has been achieved to recast a discrete version of the Euler-Poincaré equations [41, 33, 35]. Moreover, taking the Lie group to be a semi-direct product of two Lie groups, the application area of the theory is considerably enhanced including the dynamics of coupled systems, and the control theory, [6, 7, 13, 10, 8, 21, 22, 25, 36, 39].
The purpose of this work is to study the Lagrangian dynamics, both in trivialized and reduced forms, on a matched pair Lie group. A matched pair Lie group \( G \bowtie H \) is itself a Lie group that contains \( G \) and \( H \) as two non-intersecting Lie subgroups acting on each other along with certain compatibility conditions \([29, 31, 30, 32, 40]\). The Lie algebra \( g \bowtie h \) of the matched pair group \( G \bowtie H \) is the direct sum of the Lie algebras \( g \) and \( h \), as a vector space. The Lie algebra structure of \( g \bowtie h \) is determined by the mutual actions of the Lie algebras \( g \) and \( h \), which in turn satisfy the infinitesimal versions of the compatibility conditions that the group actions obey.

We shall denote the left action of \( H \) on \( G \) by \( \bowtie : H \times G \to G \), and similarly by \( \triangleright : H \times G \to H \) the right action of \( G \) on \( H \). In this setting, we show that the Euler-Poincaré equations (1.2) take the form

\[
\begin{align*}
\frac{d}{dt} \frac{\delta L}{\delta \xi} &= -\text{ad}^*_{\xi} \frac{\delta L}{\delta \xi} + \frac{\delta L}{\delta \xi} \triangleright \eta + \alpha^*_{\eta} \frac{\delta L}{\delta \eta}, \\
\frac{d}{dt} \frac{\delta L}{\delta \eta} &= -\text{ad}^*_{\eta} \frac{\delta L}{\delta \eta} - \xi \bowtie \frac{\delta L}{\delta \eta} - b^*_{\xi} \frac{\delta L}{\delta \xi},
\end{align*}
\]

(1.3)

generated by a reduced Lagrangian function(al) \( L = L(\xi, \eta) \) on the matched pair of Lie algebras \( g \bowtie h \). We call (1.3) the matched Euler-Poincaré equations. A closer look reveals that the first terms on the right hand sides of (1.3) are those in the individual Euler-Poincaré equations on the Lie algebras \( g \) and \( h \), respectively, whereas the second and the third terms are given by the dualizations of the infinitesimal actions of \( g \) and \( h \) on each other. Therefore, (1.3) may be considered as a non-trivial way of coupling two Euler-Poincaré equations in the form of (1.2).

Conversely, if a Lie group \( M \) has a matched pair decomposition, say \( M = G \bowtie H \), then its Lie algebra \( m \) is a matched pair Lie algebra \( m = g \bowtie h \). In this case, the Euler-Poincaré equations (1.2) can be written as the matched Euler-Poincaré equations in form of (1.3). In other words, the existence of a matched pair decomposition of the configuration space leads to a decoupling of the dynamics.

Moreover, the dynamics on the matched pairs can be realized as a generalization of the semidirect product theory. If, in particular, one of the group actions in \( G \bowtie H \) is trivial, then the matched Euler-Poincaré equations reduce to the semidirect product Euler-Poincaré equations. More explicitly, if the right action of \( H \) on \( G \) is trivial, then the equations (1.3) reduce to those considered in [6]. Furthermore, if the Lie group \( H \) is Abelian, then (1.2) recovers also the semidirect product theory in [7, 10, 8, 21, 22, 25, 36, 39]. We finally remark that, despite the notational resemblance, our formalism and that of the centered semidirect products studied recently in [13] are far from being equal. In other words, the Lagrangian dynamics on matched pair of Lie groups is a generalization of the semi-direct product theory in a different direction than the centered semi-direct products.

The organization of the paper is as follows. The second section below begins with the definitions of the matched pairs of Lie groups, and the matched pairs of Lie algebras. In this section we also compute the tangent lifts and the infinitesimal versions of the group actions. Using the dualizations between the tangent and the cotangent spaces, we list the cotangent lifts and the (linear algebraic) duals of the infinitesimal actions. We present the trivialization of the tangent bundle \( T(G \bowtie H) \) of the matched pair group \( G \bowtie H \) together with the actions of \( G \) and \( H \) on it. We also show in this section that the tangent bundle \( T(G \bowtie H) \) is isomorphic to the matched pair group \( TG \bowtie TH \). Finally we discuss several reductions of \( TG \bowtie TH \).
On the third section we first review the Lagrangian dynamics on Lie groups, and then we derive the matched Euler-Lagrange equations generated by a Lagrangian function(al) on the (left) trivialization of $T(G \rtimes H)$. We discuss in detail the reduction of the Lagrangian dynamics on $T(G \rtimes H)$ to a Lagrangian dynamics on the matched pair $H \rtimes (g \ltimes h)$ in the presence of a symmetry given by the (left) action of $G$. We showed that, under the symmetry of the left action of $H$, the Lagrangian dynamics on $H \rtimes (g \ltimes h)$ reduces further to the matched Euler-Poincaré equations (1.3) on $g \ltimes h$. In the literature, this is called the reduction by stages [10]. Finally, taking the action of $H$ on $G$ to be trivial, in which case the matched pair group $G \rtimes H$ reduces to the semi-direct product $G \ltimes H$, we show how the Lagrangian dynamics on matched pairs of Lie groups covers the Lagrangian dynamics on semi-direct products.

In the fourth section, we illustrate the theory on $SL(2, \mathbb{C})$. Using its matched pair decomposition $SU(2) \rtimes K$ into the special unitary group $SU(2)$ and its half-real form $K$ from [30], we exhibit the matched Euler-Lagrange equations on the (left) trivialized tangent bundle $T(SL(2, \mathbb{C}))$ and the matched Euler-Poincaré equations on the matched pair Lie algebra $sl(2, \mathbb{C})$.

Throughout the text $G$ and $H$ will denote two Lie groups, with Lie algebras $g$ and $h$, respectively. The generic elements will be denoted as follows:

\begin{equation}
(1.4) \quad g, g_1, g_2 \in G, \quad h, h_1, h_2 \in H, \quad \xi, \xi_1, \xi_2 \in g, \quad \eta, \eta_1, \eta_2 \in h, \quad U_g \in T_g G, \quad V_h \in T_h H, \quad \mu \in g^*, \quad \nu \in h^*, \quad \alpha_g \in T_g^* G, \quad \beta_h \in T_h^* H.
\end{equation}

Let $\phi : G \to H$ be a differentiable map. Then, its tangent lift is a map $T\phi : T G \to T H$. Using the dualization between tangent and cotangent spaces, the cotangent lift $T^*\phi : T^* H \to T^* G$ is given by

\begin{equation}
(1.5) \quad \langle T_g \phi(U_g), \beta_h \rangle = \langle U_g, T_g^* \phi(\beta_h) \rangle,
\end{equation}

where $\phi(g) = h$.

## 2. Matched Pairs

### 2.1. Matched pairs of Lie groups and matched pairs of Lie algebras

In this subsection we recall the basics on the matched pairs of Lie groups. Further details on the subject can be found in [29, 31, 30, 32, 40].

Let $H$ and $G$ be two Lie groups. Assume that the group $H$ acts on the group $G$ from the left, that is, there exists a differentiable mapping

\begin{equation}
(2.1) \quad \rho : H \times G \to G, \quad (h, g) \mapsto h \triangleright g
\end{equation}

satisfying

\begin{equation}
(2.2) \quad h_1 h_2 \triangleright g = h_1 \triangleright (h_2 \triangleright g) \quad \text{and} \quad e_H \triangleright g = g,
\end{equation}

where $h_1, h_2 \in H$, $g \in G$, and $e_H$ is the identity element in $H$. Assume also that $G$ acts on $H$ from the right, that is, there exists a differentiable mapping

\begin{equation}
(2.3) \quad \sigma : H \times G \to H, \quad (h, g) \mapsto h \triangleright g
\end{equation}

such that

\begin{equation}
(2.4) \quad h \triangleright g_1 g_2 = (h \triangleleft g_1) \triangleright g_2 \quad \text{and} \quad h \triangleright e_G = h,
\end{equation}

where $h \triangleright g$ denotes the right action of $H$ on $G$, $h \triangleright g$ denotes the left action of $H$ on $G$, and $e_G$ is the identity element in $G$. Then, the triple $(H, G, \triangleright)$ is called a matched pair of Lie groups.

The matched pair $(H, G, \triangleright)$ induces a matched pair $(h, g, \ltimes)$ of Lie algebras, where

\begin{equation}
(1.3) \quad \xi \ltimes \eta = \xi \triangleright \eta - \eta \triangleright \xi,
\end{equation}

and

\begin{equation}
(1.4) \quad \mu \ltimes \nu = \mu \triangleright \nu - \nu \triangleright \mu.
\end{equation}

The map $\rho : H \times G \to G$ is called the left action of $H$ on $G$, and the map $\sigma : H \times G \to H$ is called the right action of $H$ on $G$. The matched pair $(H, G, \triangleright)$ is said to be a matched pair for the action of $H$ on $G$ if $H$ acts on $G$ from the left and $G$ acts on $H$ from the right.

The matched pair decomposition of $SU(2) \rtimes K$ into the special unitary group $SU(2)$ and its half-real form $K$ from [30] provides a concrete example of a matched pair of Lie groups.

### 2.2. Matched pairs of Lie algebras

Let $g$ and $h$ be Lie algebras associated with Lie groups $G$ and $H$, respectively. A matched pair of Lie algebras $(g, h, \ltimes)$ is a pair of Lie algebras $g$ and $h$ and a bilinear product $\ltimes : g \times h \to g$ such that

\begin{equation}
(1.3) \quad \xi \ltimes \eta = \xi \triangleright \eta - \eta \triangleright \xi,
\end{equation}

and

\begin{equation}
(1.4) \quad \mu \ltimes \nu = \mu \triangleright \nu - \nu \triangleright \mu.
\end{equation}

The matched pair $(g, h, \ltimes)$ is said to be a matched pair for the action of $H$ on $G$ if $H$ acts on $G$ from the left and $G$ acts on $H$ from the right.

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where \( g_1, g_2 \in G \), \( h \in H \), and \( e_G \) is the identity element in \( G \). A matched pair of Lie groups \( G \bowtie H \) is the Cartesian product \( G \times H \) equipped with the actions (2.1) and (2.3) satisfying the compatibility conditions
\[
(2.5) \quad h \triangleright (g_1 g_2) = (h \triangleright g_1) ((h \triangleright g_1) \triangleright g_2), \\
(2.6) \quad (h_1 h_2) \lhd g = (h_1 \lhd (h_2 \triangleright g)) (h_2 \lhd g).
\]
The group structure on the matched pair \( G \bowtie H \) is given by
\[
\varpi_{G\bowtie H} ((g_1, h_1), (g_2, h_2)) = (g_1 (h_1 \triangleright g_2), (h_1 \lhd g_2) h_2) = (g_1 \rho (h_1, g_2), \sigma (h_1, g_2) h_2).
\]
The unit element is \((e_G, e_H)\), and the inversion is given by
\[
(2.7) \quad (g, h)^{-1} = (h^{-1} \triangleright g^{-1}, h^{-1} \lhd g^{-1}).
\]
We note that \( G \) and \( H \) are both subgroups of \( G \bowtie H \) by the inclusions
\[
G \hookrightarrow G \bowtie H : g \mapsto (g, e_H), \quad H \hookrightarrow G \bowtie H : h \mapsto (e_G, h).
\]
The converse is also true.

**Proposition 2.1.** [32, Prop. 6.2.15]. If a Lie group \( M \) is a Cartesian product of two subgroups \( G \leftarrow M \leftarrow H \), and if the multiplication on \( M \) is a bijection \( G \times H \cong M \), then \( M \) is a matched pair, that is, \( M \cong G \bowtie H \). In this case, the mutual actions are given by
\[
(2.8) \quad h \cdot g = (h \triangleright g) (h \lhd g),
\]
for any \( g \in G \), and any \( h \in H \).

In case the action (2.1), resp. (2.3), is trivial, the matched pair group \( G \bowtie H \) reduces to a semi-direct product \( G \rtimes H \), resp. \( G \ltimes H \).

The groups \( G \) and \( H \) act on \( G \bowtie H \) both from the left and the right as follows:
\[
G \times (G \bowtie H) \to (G \bowtie H), \quad (g_1, (g_2, h_2)) \mapsto (g_1 g_2, h_2), \\
H \times (G \bowtie H) \to (G \bowtie H), \quad (h_1, (g_2, h_2)) \mapsto (h_1 \triangleright g_2, (h_1 \lhd g_2) h_2), \\
(G \bowtie H) \times H \to (G \bowtie H), \quad ((g_1, h_1), h_2) \mapsto (g_1, h_1 h_2), \\
(G \bowtie H) \times (G \bowtie H) \to (G \bowtie H), \quad ((g_1, h_1), (g_2, h_2)) \mapsto (g_1 (h_1 \triangleright g_2), h_1 \lhd g_2).
\]
Finally, we record the (left) inner automorphism of \( G \bowtie H \) on itself for the later use,
\[
I_{(g_1, h_1)} (g_2, h_2) = (g_1, h_1) (g_2, h_2) (g_1, h_1)^{-1} = (g_1 (h_1 \triangleright g_2) ((h_1 \lhd g_2) h_2 h_1^{-1} \triangleright g_1^{-1}), (h_1 \lhd g_2) h_2 h_1^{-1} \lhd g_1^{-1}) .
\]
We proceed to the matched pairs of Lie algebras. Let \( \mathfrak{g} \) and \( \mathfrak{h} \) be two Lie algebras equipped with
\[
\triangleright : \mathfrak{h} \otimes \mathfrak{g} \to \mathfrak{g} \quad \text{and} \quad \lhd : \mathfrak{g} \otimes \mathfrak{h} \to \mathfrak{h}
\]
via which \( \mathfrak{g} \) is a left \( \mathfrak{h} \)-module,
\[
[\eta_1, \eta_2] \triangleright \xi = \eta_1 \triangleright (\eta_2 \triangleright \xi) - \eta_2 \triangleright (\eta_1 \triangleright \xi),
\]
and \( \mathfrak{h} \) is a right \( \mathfrak{g} \)-module
\[
\eta \lhd [\xi_1, \xi_2] = (\eta \lhd \xi_1) \lhd \xi_2 - (\eta \lhd \xi_2) \lhd \xi_1,
\]
in such a way that
\[
\eta \triangleright [\xi_1, \xi_2] = [\eta \triangleright \xi_1, \xi_2] + [\xi_1, \eta \triangleright \xi_2] + (\eta \lhd \xi_1) \triangleright \xi_2 - (\eta \lhd \xi_2) \triangleright \xi_1.
\]
and
\begin{equation}
[\eta_1, \eta_2] \triangleleft \xi = [\eta_1, \eta_2 \triangleleft \xi] + [\eta_1 \triangleleft \xi, \eta_2] + \eta_1 \triangleleft (\eta_2 \triangleright \xi) - \eta_2 \triangleleft (\eta_1 \triangleright \xi)
\end{equation}
are satisfied for any \(\eta, \eta_1, \eta_2 \in \mathfrak{h}\), and any \(\xi, \xi_1, \xi_2 \in \mathfrak{g}\). Then the pair \((\mathfrak{g}, \mathfrak{h})\) is called a matched pair of Lie algebras. In this case \(\mathfrak{g} \bowtie \mathfrak{h} := \mathfrak{g} \oplus \mathfrak{h}\) becomes a Lie algebra with the bracket
\begin{equation}
[(\xi_1, \eta_1), (\xi_2, \eta_2)] = ([\xi_1, \xi_2] + \eta_1 \triangleright \xi_2 - \eta_2 \triangleright \xi_1, [\eta_1, \eta_2] + \eta_1 \triangleleft \xi_2 - \eta_2 \triangleleft \xi_1).
\end{equation}

It is immediate that both \(\mathfrak{g}\) and \(\mathfrak{h}\) are Lie subalgebras of \(\mathfrak{g} \bowtie \mathfrak{h}\) via the obvious inclusions. Conversely, given a Lie algebra \(\mathfrak{m}\) with two subalgebras \(\mathfrak{g} \hookrightarrow \mathfrak{m} \hookleftarrow \mathfrak{h}\), if \(\mathfrak{m} \simeq \mathfrak{g} \oplus \mathfrak{h}\) via \((\xi, \eta) \mapsto \xi + \eta\), then \(\mathfrak{m} = \mathfrak{g} \bowtie \mathfrak{h}\) as Lie algebras. In this case, the mutual actions of \(\mathfrak{g}\) on \(\mathfrak{h}\) and \(\mathfrak{h}\) on \(\mathfrak{g}\) are uniquely determined by
\begin{equation}
[\eta, \xi] = (\eta \triangleright \xi, \eta \triangleleft \xi)
\end{equation}

We conclude this subsection with the infinitesimal adjoint action of the group \(G \bowtie H\) to its Lie algebra \(\mathfrak{g} \bowtie \mathfrak{h}\). Differentiating the inner automorphism (2.10), for any \((g, h) \in G \bowtie H\), and any \((\xi, \eta) \in \mathfrak{g} \bowtie \mathfrak{h}\), we obtain
\begin{equation}
Ad_{(g, h)^{-1}}(\xi, \eta) = (h^{-1} \triangleright \xi, T_h^{-1}R_h(h^{-1} \triangleleft \xi) + Ad_h^{-1}(\eta \triangleright g))
\end{equation}
where \(\zeta := Ad_h^{-1}\xi + T_gL_g^{-1}(\eta \triangleright g) \in \mathfrak{g}\).

2.2. **Tangent lifts of the actions.** Freezing the arguments of the action (2.1), we arrive at two differential mappings
\[ \rho_g : H \to G \quad h \mapsto h \triangleright g, \quad \text{and} \quad \rho_h : G \to G \quad g \mapsto h \triangleright g, \]
derivatives of which are given by
\begin{equation}
T_h\rho_g : T_hH \to T_{h \triangleright g}G, \quad T_g\rho_h : T_gG \to T_{h \triangleright g}G,
\end{equation}
respectively. Over the identity, that is for the Lie algebras \(\mathfrak{h} = T_eH\) and \(\mathfrak{g} = T_{eG}G\), these mappings become
\[ T_e\rho_g : \mathfrak{h} \to T_gG, \quad \eta \mapsto \eta \triangleright g := T_e\rho_g(\eta), \]
\[ T_{eG}\rho_h : \mathfrak{g} \to \mathfrak{g}, \quad \xi \mapsto h \triangleright \xi := T_{eG}\rho_h(\xi). \]
Using the tangent lift \(T_h\rho_g : T_hH \to T_{h \triangleright g}G\), we define a mapping
\begin{equation}
\tilde{\rho} : TH \times G \to TG, \quad (V_h, g) \mapsto V_h \triangleright g := T_h\rho_g(V_h) \in T_{h \triangleright g}G,
\end{equation}
and similarly, using the infinitesimal action \(T_g\rho_h : T_gG \to T_{h \triangleright g}G\), we define the action of the group \(H\) on the tangent bundle \(TG\) as
\begin{equation}
\hat{\rho} : H \times TG \to TG, \quad (h, U_g) \mapsto h \triangleright U_g := T_g\rho_h(U_g) \in T_{h \triangleright g}G.
\end{equation}
In particular, we obtain the action of the group \(H\) on the Lie algebra \(\mathfrak{g}\),
\begin{equation}
\hat{\rho} : H \times \mathfrak{g} \to \mathfrak{g}, \quad (h, \xi) \mapsto h \triangleright \xi := T_e\rho_h(\xi).
\end{equation}
Differentiating (2.16) and (2.17) with respect to the group variables, we obtain the mappings

\[ T_g \hat{\rho}_V : T_g G \to T_{V^g} G, \quad v_g \mapsto T_g \hat{\rho}_V (U_g) =: V_h \triangleright U_g, \]

\[ T_h \hat{\rho}_V : T_h H \to T_{H^g} H, \quad V_h \mapsto T_h \hat{\rho}_V (V_h) =: V_h \triangleright U_g, \]

where (2.19) is the infinitesimal action of the group \( TH \) on \( TG \). We also note by

\[ T_g \hat{\rho}_V (V_g) = \left. \frac{d}{ds} \right|_{s=0} V_h \triangleright g^s = \left. \frac{d}{ds} \right|_{s=0} \left( \frac{d}{dt} \right)_{t=0} h^t \triangleright g^s, \]

\[ T_h \hat{\rho}_V (V_h) = \left. \frac{d}{dt} \right|_{t=0} h^t \triangleright V_g = \left. \frac{d}{ds} \right|_{s=0} \left( \frac{d}{dt} \right)_{t=0} h^t \triangleright g^s \]

that \( T_g \hat{\rho}_V (U_g) = T_h \hat{\rho}_V (V_h) \). As a result, we use the same notation \( V_h \triangleright U_g \) for both. Pictorially, we summarize these arguments by the following tangent rhombic [1].

\[ (V_h \triangleright U_g) \in TG \]

\[ (h \triangleright U_g) \in TG \]

\[ (h \triangleright g) \in G \]

\[ (V_h \triangleright g) \in TG \]

where the elements are shown in parenthesis. In case \( h = e_H \), we have

\[ T_g \hat{\rho}_\eta (U_g) = T_{e_H} \hat{\rho}_U (\eta) =: \eta \triangleright U_g, \]

whereas \( g = e_G \) yields

\[ T_{e_G} \hat{\rho}_V (\xi) = T_h \hat{\rho}_\xi (V_h) =: V_h \triangleright \xi. \]

Therefore, \( h = e_H \) and \( g = e_G \) together renders

\[ T_{e_G} \hat{\rho}_\eta : g \to g, \quad \xi \mapsto \eta \triangleright \xi; \quad b_\xi := T_{e_H} \hat{\rho}_\xi : h \to g, \quad \eta \mapsto \eta \triangleright \xi. \]

Similar arguments can be repeated for the action (2.3). To this end we introduce the mappings

\[ \sigma_h : G \to H, \quad g \mapsto h \triangleleft g, \]

\[ \sigma_g : H \to H, \quad h \mapsto h \triangleleft g, \]

and their derivatives

\[ T_g \sigma_h : T_g G \to T_{h \triangleleft g} H, \quad T_{e_G} \sigma_h : g \to T_h H, \quad \xi \mapsto h \triangleleft \xi. \]

Over the identities, these tangent mappings reduces to

\[ T_h \sigma_g : T_h H \to T_{h \triangleleft g} H, \quad T_{e_H} \sigma_g : h \to h, \quad \eta \mapsto \eta \triangleleft g. \]

We also define the mappings

\[ \tilde{\sigma} : H \times TG \to TH, \quad (h, U_g) \mapsto h \triangleleft U_g := T_g \sigma_h (U_g) \in T_{h \triangleleft g} H, \]

\[ \tilde{\sigma} : TH \times G \to TH, \quad (V_h, g) \mapsto V_h \triangleleft g := T_h \sigma_g (V_h) \in T_{h \triangleleft g} H, \]

\[ (h, U_g) \mapsto h \triangleleft U_g := T_g \sigma_h (U_g) \in T_{h \triangleleft g} H, \]

\[ (V_h, g) \mapsto V_h \triangleleft g := T_h \sigma_g (V_h) \in T_{h \triangleleft g} H, \]
from which, freezing the vector entries, we arrive at \( \tilde{\sigma}_{U_g} : H \to TH \) and similarly \( \tilde{\sigma}_{V_h} : G \to TH \) whose derivatives are

\[
(2.31) \quad T_h \tilde{\sigma}_{U_g} : T_hH \to T_{h \lt U_g}TH, \quad V_h \mapsto V_h \lt U_g, \\
(2.32) \quad T_g \tilde{\sigma}_{V_h} : T_gG \to T_{V_h \lt g}TH, \quad U_g \mapsto V_h \lt U_g.
\]

Then, similar to (2.21) we obtain \( T_h \tilde{\sigma}_{V_h} (V_h) = T_g \tilde{\sigma}_{V_h} (V_g) \), and employ the notation \( V_h \lt U_g \) for both. We summarize our discussion in the following tangent rhombic [1].

\[
(2.33) \quad \begin{array}{c}
(V_h \lt U_g) \in TTH \\
(h \lt U_g) \in TH \\
(h \lt g) \in TH \\
(h \lt g) \in H
\end{array}
\]

Furthermore, in case \( g = e_G \), the tangent mappings (2.31) and (2.32) produce

\[
T_h \tilde{\sigma}_{\xi} (V_h) = T_{e_G} \tilde{\sigma}_{V_h} (\xi) = V_h \mapsto V_h \lt \xi,
\]

and for \( h = e_H \)

\[
(2.34) \quad T_{e_H} \tilde{\sigma}_{U_g} (\eta) = T_g \tilde{\sigma}_{U_g} (\eta) = \eta \lt U_g.
\]

Therefore, setting \( g = e_G \) and \( h = e_H \), we arrive at

\[
(2.35) \quad T_{e_H} \tilde{\sigma}_{\xi} : h \mapsto h, \quad \eta \mapsto \eta \lt \xi; \quad a_\eta := T_{e_G} \tilde{\sigma}_{\eta} : g \mapsto h, \quad \xi \mapsto \eta \lt \xi.
\]

2.3. Cotangent lifts of the actions. In this subsection, we discuss briefly the cotangent lifts of the mappings considered in Subsection 2.1. In other words, in the present subsection we dualize the maps studied in Subsection 2.2.

Let us begin with the (right) action of \( H \) on the cotangent bundle \( T^*G \) as a cotangent lift of the mapping (2.17). We have,

\[
(2.36) \quad \hat{\rho}^* : T^*G \times H \to T^*G, \quad (\alpha_g, h) \mapsto \alpha_g \lt h := \hat{\rho}^*_h(\alpha_g) = T_{h \lt 1 > g} \rho_h (\alpha_g) \in T_{h \lt 1 > g}^* G,
\]

where

\[
(2.37) \quad \left< T_{h \lt 1 > g} \rho_h (\alpha_g), U_{h \lt 1 > g} \right> = \left< \alpha_g, T_{h \lt 1 > g} \rho_h (U_{h \lt 1 > g}) \right>.
\]

In particular, we have the action

\[
(2.38) \quad \hat{\rho}^* : g^* \times H \to g^*, \quad (\mu, h) \mapsto \mu \lt h := T_{e_H} \rho_h (\mu)
\]

of \( H \) on \( T_{e_G}^* G \cong g^* \). In short, (2.37) and (2.38) correspond to

\[
(2.39) \quad \left< \alpha_g \lt h, U_{h \lt 1 > g} \right> = \left< (\alpha_g), h \triangleright U_{h \lt 1 > g} \right>, \quad \left< \mu \lt h, \xi \right> = \left< \mu, h \triangleright \xi \right>,
\]

respectively. On the other hand, freezing the covector component of (2.36), and in particular of (2.38), we obtain

\[
(2.40) \quad T_h \hat{\rho}^*_{\alpha_g} : T_hH \to T_{\alpha_g \lt h}^* G, \quad V_h \mapsto \alpha_g \lt V_h,
\]

\[
(2.41) \quad T_h \hat{\rho}^*_{\mu} : T_hH \to g^*, \quad V_h \mapsto \mu \lt V_h.
\]
On the next step, taking \( h = e_H \) we obtain
\[
T_{e_H} \tilde{\rho}^*_\alpha_g : \mathfrak{h} \to T_{\alpha_g} T^* G, \quad \eta \to \alpha_g^* \ll \eta,
\]
(2.42)
\[
T_{e_H} \tilde{\rho}^*_\mu : \mathfrak{h} \to \mathfrak{g}^*, \quad \eta \to \mu^* \ll \eta.
\]
In terms of action duality, (2.41) and (2.42) correspond to
\[
\left\langle \mu^* \ll V_h, \xi \right\rangle = \left\langle \mu, V_h \rhd \xi \right\rangle \quad \left\langle \mu^* \ll \eta, \xi \right\rangle = \left\langle \mu, \eta \rhd \xi \right\rangle.
\]
(2.43)
Similarly, the dualization of (2.30) yields
\[
\hat{\sigma}^* : G \times T^* H \to T^* H, \quad (g, \alpha_h) \mapsto g \rhd \alpha_h := T^*_{h \rhd g^{-1}} \sigma_g (\alpha_h) \in T^*_{h \rhd g^{-1}} H,
\]
and in particular
\[
\hat{\sigma}^* : G \times \mathfrak{h}^* \to \mathfrak{h}^*, \quad (\nu, g) \mapsto g \rhd \nu := T^*_{e_H} \sigma_g (\nu).
\]
(2.45)
Thus, (2.44) and (2.45) dualizes (2.30) as
\[
\left\langle g \rhd \alpha_h, V_{h \rhd g^{-1}} \right\rangle = \left\langle \alpha_h, V_{h \rhd g^{-1}} \ll g \right\rangle, \quad \left\langle g \rhd \nu, \eta \right\rangle = \left\langle \nu, \eta \ll g \right\rangle.
\]
Fixing the covector component of the infinitesimal version of (2.44) we also have
\[
T_g \hat{\sigma}^*_{\beta_h} : T_g G \to T_{g \rhd \beta_h} T^* H, \quad U_g \mapsto U_g \rhd \beta_h,
\]
\[
T_{e_G} \hat{\sigma}^*_{\beta_h} : \mathfrak{g} \to T_{\beta_h} T^* H, \quad \xi \mapsto \xi \rhd \beta_h.
\]
In particular, for \( g = e_G \), we arrive at the infinitesimal actions
\[
T_{e_G} \hat{\sigma}^*_{\beta_h} : \mathfrak{g} \to T_{\beta_h} T^* H, \quad \xi \mapsto \xi \rhd \beta_h,
\]
(2.46)
As a result, we have the dual actions
\[
\left\langle \nu, \eta \ll \xi \right\rangle = \left\langle \xi \rhd \nu, \eta \right\rangle.
\]

2.4. Tangent bundle of a matched pair of Lie groups. In this subsection we discuss the tangent bundle \( T(G \bowtie H) \) of a matched pair group \( G \bowtie H \). We show that it is the matched pair group of \( (TG, TH) \). We also mention briefly the actions of the groups \( G \) and \( H \) on \( T(G \bowtie H) \), and the corresponding reductions.

Let us first recall from [2, 36] that the tangent bundle \( TG \) of a Lie group \( G \) is a Lie group with the multiplication
\[
\varpi_{TG} (U_{g_1}, U_{g_2}) = TL_{g_1} U_{g_2} + TR_{g_2} U_{g_1}.
\]
(2.47)
The group \( TG \) can be identified, as a Lie group, with the semi-direct product \( G \bowtie \mathfrak{g} \), (resp. \( \mathfrak{g} \bowtie G \)) via the left (resp. right) trivialization
\[
tr^L_{TG} : TG \to G \bowtie \mathfrak{g}, \quad U_g \mapsto (g, \xi = T_g L_{g^{-1}} U_g),
\]
(2.48)
\[
(\text{resp. } tr^R_{TG} : TG \to \mathfrak{g} \bowtie G, \quad U_g \mapsto (\xi = T_g R_{g^{-1}} U_g, g)).
\]
(2.49)
The trivialization maps, then, induces the Lie group structures given by
\[
\varpi_{G \bowtie \mathfrak{g}} ((g_1, \xi_1), (g_2, \xi_2)) = (g_1 g_2, \xi_2 + Ad_{g_2^{-1}} \xi_1),
\]
(2.50)
\[
\varpi_{\mathfrak{g} \bowtie G} ((\xi_1, g_1), (\xi_2, g_2)) = (Ad_{g_1} \xi_2 + \xi_1, g_1 g_2).
\]
Transferring the action of the group $G$ on its tangent bundle $T G$ via the trivialization maps (2.48) and (2.49), we obtain the actions

\begin{equation}
G \times (G \ltimes g) \rightarrow G \ltimes g, \quad (g_1, (g_2, \xi)) \mapsto (g_1 g_2, \xi),
\end{equation}

(2.51)

\begin{equation}
(g \times G) \times G \rightarrow (g \times G), \quad ((\xi, g_1), g_2) \mapsto (\xi, g_1 g_2),
\end{equation}

(2.52)

\begin{equation}
(G \ltimes g) \times G \rightarrow G \ltimes g, \quad ((g_1, \xi), g_2) = (g_1 g_2, \text{Ad}_{g_2}^{-1} \xi),
\end{equation}

(2.53)

\begin{equation}
G \times (g \times G) \rightarrow (g \times G), \quad (g_1, (\xi, g_2)) \mapsto (\text{Ad}_{g_1} \xi, g_1 g_2).
\end{equation}

We next discuss the matched pair decomposition of the tangent bundle

\[ T(G \bowtie H) = \{ (U_g, V_h) \mid V_g \in T_g G, \ V_h \in T_h H \} \]

of a matched pair group $G \bowtie H$. The group structure on the tangent bundle, is given by

\begin{equation}
\varpi_{T(G\bowtie H)} ((U_{g_1}, V_{h_1}), (U_{g_2}, V_{h_2})) = TL_{(g_1, h_1)} (U_{g_2}, V_{h_2}) + TR_{(g_2, h_2)} (U_{g_1}, V_{h_1}),
\end{equation}

(2.54)

where the tangent lifts of the left and the right regular actions of $G \bowtie H$ are given by

\[ T_{(g_2, h_2)} L_{(g_1, h_1)} (U_{g_2}, V_{h_2}) = (T_{h_1 \triangleright g_2} L_{g_1} (h_1 \triangleright U_{g_2}), T_{h_1 \triangleright g_2} R_{h_2} (h_1 \triangleright U_{g_2}) + T_{h_2} L_{(h_1 \triangleright g_2)} V_{h_2}), \]

(2.55)

and

\[ T_{(g_1, h_1)} R_{(g_2, h_2)} (U_{g_1}, V_{h_1}) = (T_{g_1} R_{(h_1 \triangleright g_2)} U_{g_1} + T_{h_1 \triangleright g_2} L_{g_1} (V_{h_1} \triangleright g_2), T_{h_1 \triangleright g_2} R_{h_2} (V_{h_1} \triangleright g_2)). \]

We next investigate the matched pair decomposition of the tangent bundle $T(G \bowtie H)$, given a matched pair $(G, H)$ of Lie groups. To this end let us first recall a key ingredient from [30].

**Theorem 2.2.** [30, Thm. 3.1]. Let $(\mathfrak{g}, \mathfrak{h})$ be a matched pair of Lie algebras, and let $G$ and $H$ be the simply-connected Lie groups of $\mathfrak{g}$ and $\mathfrak{h}$, respectively. Then, there is a unique smooth action of $\mathfrak{h}$ on $G$ such that

\begin{equation}
\eta \triangleright (gg') = T_g R_{g'} (\eta \triangleright g) + T_g L_g ((\eta \triangleright g) \triangleright g'), \quad \eta \triangleright e_G = 0.
\end{equation}

(2.56)

for any $\eta \in \mathfrak{h}$, and any $g, g' \in G$. Similarly, there is a unique smooth action of $\mathfrak{g}$ on $H$ such that

\begin{equation}
(h' \triangleright \xi) \triangleright h = T_h L_{h'} ((h' \triangleright \xi) \triangleright h) + T_{h'} R_{h'} (h' \triangleright (h \triangleright \xi)), \quad e_H \triangleright \xi = 0.
\end{equation}

(2.57)

We are now ready to prove the main result of the present subsection.

**Proposition 2.3.** Given a matched pair $(G, H)$ of Lie groups, $(TG, TH)$ is a matched pair of Lie groups, and $T(G \bowtie H) \simeq TG \bowtie TH$.

**Proof.** We shall use the proposition 2.1 to show that $T(G \bowtie H) \simeq TG \bowtie TH$. That $TG \hookrightarrow T(G \bowtie H) \hookrightarrow TH$, that is $TG$ and $TH$ are subgroups of $T(G \bowtie H)$ follows easily from (2.54) and (2.55) by

\[ U_g \hookrightarrow (U_g, 0), \quad (0, V_h) \hookrightarrow V_h. \]

On the next step, we note via (2.53) that the (group) multiplication

\[ (U_g, V_h) \mapsto U_g \cdot V_h = (U_g, 0) \cdot (0, V_h) \]

where

\[ (U_g, 0) \cdot (0, V_h) = T_{(e_G, h)} L_{(g, e_H)} (0, V_h) + T_{(g, e_H)} R_{(e_G, h)} (U_g, 0) = (U_g, V_h) \]
is indeed a bijection. □

In terms of trivialized tangent bundles, the explicit isomorphism is given by

\[(G \ltimes g) \cong (H \ltimes h) \to (G \ltimes H) \ltimes (g \ltimes h),\]

\[(g, \xi, (h, \eta)) \mapsto ((g, h), (h^{-1} \triangleright \xi, T_{h^{-1}} R_{h}(h^{-1} \triangleright \xi) + \eta)),\]  

(2.58)

together with the inverse

\[(G \ltimes H) \times (g \ltimes h) \to (G \ltimes g) \ltimes (H \ltimes h), \quad ((g, h), (\xi, \eta)) \mapsto ((g, h \triangleright \xi, (h, \eta - T_{h^{-1}} R_{h}(h^{-1} \triangleright (h \triangleright \xi)))) = ((g, h \triangleright \xi, (h, \eta + T_{h} L_{h^{-1}}(h \triangleright \xi))),\]

where the latter equality follows from the theorem (2.2), more precisely, taking \(h' = h^{-1}\) in (2.57). The inclusions \(G \ltimes g \hookrightarrow (G \ltimes H) \ltimes (g \ltimes h) \hookrightarrow H \ltimes h\) correspond to

\[(g, \xi) \mapsto ((g, e_{H}, (\xi, 0)), \quad ((e_{G}, h), (0, \eta)) \mapsto (h, \eta),\]

and thus it follows from

\[(h, \eta) \cdot (g, \xi) = ((e_{G}, h), (0, \eta)) \cdot ((e_{G}, h), (\xi, 0))\]

\[= ((h \triangleright g, h \triangleright g), (\xi, 0) + Ad_{(g,e_{H})^{-1}}(0, \eta))\]

\[= ((h \triangleright g, h \triangleright g), (\xi + T_{g} L_{g}^{-1}(\eta \triangleright g), \eta \triangleright g))\]

\[= ((h \triangleright g, e_{H}), ((h \triangleright g) \triangleright \xi + (h \triangleright g) \triangleright T_{g} L_{g}^{-1}(\eta \triangleright g), 0))\]

\[= ((e_{G}, h \triangleright g), (0, \eta \triangleright g - T_{(h \triangleright g)} L_{(h \triangleright g)^{-1}}(\eta \triangleright g) \triangleright ((h \triangleright g) \triangleright ((\xi + T_{g} L_{g}^{-1}(\eta \triangleright g))))\])\]

that the mutual actions of the trivialized tangent bundles are given by

\[(h, \eta) \triangleright (g, \xi) = (h \triangleright g, (h \triangleright g) \triangleright \xi + (h \triangleright g) \triangleright T_{g} L_{g}^{-1}(\eta \triangleright g)),\]

\[(h, \eta) \triangleleft (g, \xi) = (h \triangleleft g, h \triangleleft g \triangleleft \xi \triangleleft g - T_{(h \triangleleft g)}^{-1} R_{(h \triangleleft g)}((h \triangleleft g)^{-1} \triangleleft ((h \triangleleft g) \triangleleft (\xi + T_{g} L_{g}^{-1}(\eta \triangleright g)))),\]

where, once again, the latter equality is a result of (2.57).

Finally we shall discuss briefly the reductions of \(T(G \ltimes H)\) to \(g \ltimes h\). To this end, we next consider the left and right actions of \(G\) and \(H\) on the tangent bundle \(T(G \ltimes H)\), by lifting the actions (2.9) of \(G\) and \(H\) on \(G \ltimes H\). Accordingly,

\[G \times T(G \ltimes H) \to T(G \ltimes H), \quad (g_{1}, (U_{g_{2}}, V_{h_{2}})) \to (T_{g_{2}} L_{g_{2}}, U_{g_{2}}, V_{h_{2}}),\]

\[T(G \ltimes H) \times G \to T(G \ltimes H), \quad ((U_{g_{1}}, V_{h_{1}}), g_{2}) \to (T_{g_{1}} R_{(h_{1} \triangleright g_{2})} U_{g_{1}}, V_{h_{1}} \triangleleft g_{2}),\]

\[H \times T(G \ltimes H) \to T(G \ltimes H), \quad (h_{1}, (U_{g_{2}}, V_{h_{2}})) \to (h_{1} \triangleright U_{g_{2}}, T_{R_{h_{2}}} h_{1} \triangleleft U_{g_{2}}) h_{2} + T_{L_{h_{1} \triangleleft g_{2}}} V_{h_{2}}),\]

(2.60)

\[T(G \ltimes H) \times H \to T(G \ltimes H), \quad ((U_{g_{1}}, V_{h_{1}}), h_{2}) \to (U_{g_{1}}, T_{R_{h_{2}}} V_{h_{2}}).\]

On the level of the trivialized tangent bundles, the left action of \(G\) appears as

\[G \ltimes ((G \ltimes H) \ltimes (g \ltimes h)) \to (G \ltimes H) \ltimes (g \ltimes h),\]

\[(g_{1}, ((g_{2}, h_{2}), (\xi_{2}, \eta_{2}))) \mapsto ((g_{1} g_{2}, h_{2}), (\xi_{2}, \eta_{2})).\]

The space of orbits of this action is given by

\[G \backslash ((G \ltimes H) \ltimes (g \ltimes h)) \simeq H \ltimes (g \ltimes h).\]
The group $H$ being a subgroup of $H \ltimes (g \rtimes h)$, we have also the left action
\begin{equation}
H \times (H \ltimes (g \rtimes h)) \to H \ltimes (g \rtimes h), \quad (h_1, ((h_2, \xi_2), \eta_2)) \mapsto (h_1 h_2, (\xi_2, \eta_2)).
\end{equation}

Then the space of orbits of the action (2.62) is isomorphic to the matched pair Lie algebra $g \rtimes h$.

On the other hand, we could arrive at $g \rtimes h$ via the reduction of $T (G \rtimes H)$ by the right action of $H$ first, and then the right action of $G$. Furthermore, the left (resp. right) action of the matched pair group $G \rtimes H$ on the left (resp. right) trivialization of $T(G \rtimes H)$ would also yield the matched pair Lie algebra $g \rtimes h$. Finally, we summarize the reduction stages by the following diagram:
\begin{equation}
(G \rtimes H) \times (g \rtimes h) \xrightarrow{tr_{G}(G \rtimes H)} T(G \rtimes H) \xrightarrow{tr_{G}(G \rtimes H)} (g \rtimes h) \times (G \rtimes H)
\end{equation}

3. Lagrangian Dynamics

3.1. Lagrangian dynamics on Lie Groups. Let $G$ be a Lie group, and $\mathcal{L}$ a Lagrangian density on the tangent bundle $TG$. Let us then define on $G \ltimes g$,
\begin{equation}
\mathcal{L}(g, \xi) = \mathcal{L} \circ tr_{T G} (U_g) = \mathcal{L}(U_g),
\end{equation}

where $\xi = T_g L_{g^{-1}} U_g$. In order to determine the extremum of the action integral $\int \mathcal{L}(g, \xi) \, dt$ over the paths joining two fixed points $g(a)$ and $g(b)$, we compute the variation of the action integral
\begin{equation}
\delta \int_{b}^{a} \mathcal{L}(g, \xi) \, dt = \int_{b}^{a} \left( \left\langle \frac{\delta \mathcal{L}}{\delta g}, \delta g \right\rangle_g + \left\langle \frac{\delta \mathcal{L}}{\delta \xi}, \delta \xi \right\rangle_e \right) \, dt
\end{equation}

\begin{align*}
&= \int_{b}^{a} \left( \left\langle \frac{\delta \mathcal{L}}{\delta g}, \delta g \right\rangle_g + \left\langle \frac{\delta \mathcal{L}}{\delta \xi}, \dot{\xi} + [\xi, \eta] \right\rangle_e \right) \, dt \\
&= - \left\langle \frac{\delta \mathcal{L}}{\delta \xi}, \eta \right\rangle_e \bigg|_{b}^{a} + \int_{b}^{a} \left( \left\langle \frac{\delta \mathcal{L}}{\delta g}, \delta g \right\rangle_g + \left\langle - \frac{d}{dt} \frac{\delta \mathcal{L}}{dt} \frac{\delta \mathcal{L}}{\delta \xi} + ad_{\xi}^{*} \frac{\delta \mathcal{L}}{\delta \xi}, \eta \right\rangle_e \right) \, dt \\
&= - \left\langle \frac{\delta \mathcal{L}}{\delta \xi}, T_g L_{g^{-1}} \delta g \right\rangle_e \bigg|_{b}^{a} + \int_{b}^{a} \left( \left\langle \frac{\delta \mathcal{L}}{\delta g}, \delta g \right\rangle_g + \left\langle ad_{\xi}^{*} \frac{\delta \mathcal{L}}{\delta \xi} - \frac{d}{dt} \frac{\delta \mathcal{L}}{dt} \frac{\delta \mathcal{L}}{\delta \xi}, T_g L_{g^{-1}} \delta g \right\rangle_e \right) \, dt \\
&= - \left\langle T_g^{*} L_{g^{-1}} \frac{\delta \mathcal{L}}{\delta \xi}, \delta g \right\rangle_g \bigg|_{a}^{b} + \int_{b}^{a} \left\langle \frac{\delta \mathcal{L}}{\delta g} + T_g^{*} L_{g^{-1}} \left( ad_{\xi}^{*} \frac{\delta \mathcal{L}}{\delta \xi} - \frac{d}{dt} \frac{\delta \mathcal{L}}{dt} \frac{\delta \mathcal{L}}{\delta \xi} \right), \delta g \right\rangle_g \, dt.
\end{align*}

If $\delta g$ vanishes at boundaries, we arrive at the trivialized Euler-Lagrange dynamics
\begin{equation}
\frac{d}{dt} \frac{\delta \mathcal{L}}{\delta \xi} = T_g^{*} L_{g} \frac{\delta \mathcal{L}}{\delta g} - ad_{\xi}^{*} \frac{\delta \mathcal{L}}{\delta \xi}.
\end{equation}
see, for instance, [5, 11, 12, 14, 17, 18]. The variation with respect to the fiber (Lie algebra) variable \( \xi \) is achieved by the reduced variational principle \( \delta \xi = \dot{\eta} + [\xi, \eta] \), see, for example, [9, 18, 37, 26]. On the other hand, if the Lagrangian \( \mathcal{L} \) is independent of the group variable, that is \( \mathcal{L}(g, \xi) = L(\xi) \), then the term involving the derivatives with respect to the group variable drops, and (3.3) reduces to the Euler-Poincaré equations

\[
(3.4) \quad \frac{d}{dt} \frac{\delta l}{\delta \xi} = -ad^*_\xi \frac{\delta l}{\delta \xi}.
\]

We finally note that, here \( ad^* \) denotes the (left) infinitesimal coadjoint action, rather than the linear algebraic dual of infinitesimal adjoint action \( ad \).

### 3.2. Lagrangian dynamics on matched pairs

In this subsection we develop the explicit Euler-Lagrange and Euler-Poincaré equations on the tangent bundle \( T(G \bowtie H) \) of matched pair of groups \( G \bowtie H \).

Since the trivialized Euler-Lagrange equations (3.3) include the cotangent lift \( T^*L \) of the left translation, as well as the infinitesimal coadjoint action \( ad^* \), we start by deriving those from (2.54) and (2.13), respectively. The cotangent lift of the left trivialization then, dualizing (2.54), is given by

\[
T_{(g_2, h_2)}^* L_{(g_1, h_1)} (\alpha_{g_1(h_1 \bowtie g_2)}, \beta_{(h_1 \bowtie g_2)h_2}) =
\]

\[
(3.5) \quad \left( (\frac{T^*_{g_1 \bowtie g_2} L_{g_1} \alpha_{g_1(h_1 \bowtie g_2)})}{\delta h_1} + \frac{T^*_{g_2} (R_{h_2} \circ \sigma_{h_1}) \beta_{(h_1 \bowtie g_2)h_2}, T^*_{h_2} L_{h_1 < g_2} \beta_{(h_1 \bowtie g_2)h_2}) \right.
\]

for any \( g_1, g_2 \in G, h_1, h_2 \in H, \alpha_{g_1(h_1 \bowtie g_2)} \in T^*_{g_1(h_1 \bowtie g_2)} G \) and any \( \beta_{(h_1 \bowtie g_2)h_2} \in T^*_{(h_1 \bowtie g_2)h_2} H \). In particular, for \( (g_2, h_2) = (e_G, e_H) \), the mapping (3.5) reduces to

\[
(3.6) \quad T_{(e_G, e_H)}^* L_{(g, h)} (\alpha_g, \beta_h) = \left( (\frac{T^*_{e_G} L_{g_1} \alpha_g)}{\delta h} + \frac{T^*_{e_G} \sigma_h} \beta_h, T^*_{e_H} L_{h} \beta_h \right).
\]

On the other hand, it follows from

\[
\langle ad^*_\xi(\xi_1, \eta_1) (\mu, \nu), (\xi_2, \eta_2) \rangle = - \langle (\mu, \nu), [(\xi_1, \eta_1), (\xi_2, \eta_2)] \rangle
\]

and the formula (2.13) that

\[
(3.7) \quad ad^*_\xi(\mu, \nu) = \left( \frac{ad^*_\xi \mu - \mu \cdot \eta - a^*_\xi \nu, ad^*_\eta \nu + \xi \cdot \nu + b^*_\xi \mu} \right),
\]

for any \( \xi \in \mathfrak{g}, \eta \in \mathfrak{h}, \mu \in \mathfrak{g}^*, \) and any \( \nu \in \mathfrak{h}^* \). On the right hand side, the mapping \( a^*_\eta : \mathfrak{h}^* \to \mathfrak{g}^* \) is the transpose of (2.35), and \( b^*_\xi : \mathfrak{g}^* \to \mathfrak{h}^* \) is the transpose of (2.24).

As a result, given a Lagrangian functional \( \mathcal{L} = \mathcal{L}(g, h, \xi, \eta) \) on the trivialized tangent bundle \( (G \bowtie H) \times (\mathfrak{g} \bowtie \mathfrak{h}) \), the matched Euler-Lagrange equations

\[
(3.8) \quad \frac{d}{dt} \left( \frac{\delta \mathcal{L}}{\delta \dot{\xi}} \right) = T_{(e_G, e_H)}^* L_{(g, h)} \left( \frac{\delta \mathcal{L}}{\delta g}, \frac{\delta \mathcal{L}}{\delta h} \right) - ad^*_\xi \left( \frac{\delta \mathcal{L}}{\delta \xi}, \frac{\delta \mathcal{L}}{\delta \eta} \right)
\]

are given, in view of (3.6) and (3.7), by

\[
(3.9) \quad \frac{d}{dt} \frac{\delta \mathcal{L}}{\delta \xi} = T_{(e_G, e_H)}^* L_{g_1} \left( \frac{\delta \mathcal{L}}{\delta g}, \eta \right) + T_{(e_G, e_H)}^* \sigma_{h_1} \left( \frac{\delta \mathcal{L}}{\delta h}, \beta_{(h_1 \bowtie g_2)h_2} \right) - ad^*_\xi \left( \frac{\delta \mathcal{L}}{\delta \xi}, \eta \right) + a^*_\eta \left( \frac{\delta \mathcal{L}}{\delta \eta} \right),
\]

\[
(3.10) \quad \frac{d}{dt} \frac{\delta \mathcal{L}}{\delta \eta} = T_{(e_G, e_H)}^* L_{h_2} \left( \frac{\delta \mathcal{L}}{\delta h}, \xi \right) - ad^*_\eta \left( \frac{\delta \mathcal{L}}{\delta \eta} \right) - \xi \cdot \left( \frac{\delta \mathcal{L}}{\delta \eta} - b^*_\xi \left( \frac{\delta \mathcal{L}}{\delta \xi} \right) \right).
\]
Alternatively, for the action integral $\int_a^b \mathcal{L}(g, h, \xi, \eta) \, dt$, let $\delta (g, h, \xi, \eta) = (\delta g, \delta h, \delta \xi, \delta \eta)$, where for any $(\xi_2, \eta_2) \in g \triangleright h$,

$$
\delta (g, h) = TL_{(g,h)} (\xi_2, \eta_2) \quad \text{and} \quad \delta (\xi, \delta \eta) = \frac{d}{dt} (\xi_2, \eta_2) + [(\xi, \eta), (\xi_2, \eta_2)],
$$
or more explicitly,

\begin{equation}
\delta g = T_{eG} (L_g \circ \rho_h) (\xi_2), \quad \delta h = T_{eG} \sigma_h (\xi_2) + T_{eH} L_h (\eta_2),
\end{equation}

$$
\delta \xi = \dot{\xi}_2 + [\xi, \xi_2] + \eta \triangleright \xi_2 - \eta_2 \triangleright \xi, \quad \delta \eta = \dot{\eta}_2 + [\eta, \eta_2] + \eta \triangleright \xi_2 - \eta_2 \triangleright \xi.
$$

Substituting the variations in (3.11) into the variation of the action integral we obtain

$$
\delta \int_a^b \mathcal{L}(g, h, \xi, \eta) \, dt
= \int_a^b \left( \left\langle \frac{\delta \mathcal{L}}{\delta g}, \delta g \right\rangle_g + \left\langle \frac{\delta \mathcal{L}}{\delta h}, \delta h \right\rangle_h + \left\langle \frac{\delta \mathcal{L}}{\delta \xi}, \delta \xi \right\rangle_e + \left\langle \frac{\delta \mathcal{L}}{\delta \eta}, \delta \eta \right\rangle_e \right) \, dt.
$$

Then, a straightforward calculation similar to (3.2) renders the matched Euler-Lagrange equations (3.9) and (3.10).

**Remark 3.1.** We note that, the matched Euler-Lagrange equations (3.9) and (3.10) are not preserved when the group variables of the Lagrangian are interchanged. The first reason of that is the trivialization we choose, see for instance (3.11). The second reason is the direction of the actions, that $H$ acts on $G$ from the left, and $G$ acts on $H$ from the right. In other words, if the actions are changed then so does the structure.

As summarized in the diagram (2.63), there are several possible reductions of the matched Euler-Lagrange equations (3.9) and (3.10). In case the Lagrangian $\mathcal{L} = \mathcal{L}(g, h, \xi, \eta)$ is invariant under the left action of $G$ on $(G \ltimes H) \ltimes (g \ltimes h)$, then the equations (3.9) and (3.10) reduce to

\begin{align}
\frac{d}{dt} \frac{\delta \mathcal{L}}{\delta \xi} &= T_{eG}^{*} \sigma_h \left( \frac{\delta \mathcal{L}}{\delta h} \right) - ad_{h}^{*} \frac{\delta \mathcal{L}}{\delta \xi} + \frac{\delta \mathcal{L}}{\delta \xi} \triangleright \eta + a_{h}^{*} \frac{\delta \mathcal{L}}{\delta \eta}, \\
\frac{d}{dt} \frac{\delta \mathcal{L}}{\delta \eta} &= T_{eH}^{*} L_h \left( \frac{\delta \mathcal{L}}{\delta h} \right) - ad_{h}^{*} \frac{\delta \mathcal{L}}{\delta \eta} - \xi \triangleright \frac{\delta \mathcal{L}}{\delta \eta} - b_{h}^{*} \frac{\delta \mathcal{L}}{\delta \xi},
\end{align}

encoding the Lagrangian dynamics on $H \ltimes (g \ltimes h)$ for the reduced Lagrangian $\mathcal{L} = \mathcal{L}(h, \xi, \eta)$.

If furthermore the Lagrangian $\mathcal{L} = \mathcal{L}(h, \xi, \eta)$ on $H \ltimes (g \ltimes h)$ is independent of the group variable $h \in H$, the equations (3.12 - 3.13) reduce to the matched Euler-Poincaré equations

\begin{align}
\frac{d}{dt} \frac{\delta \mathcal{L}}{\delta \xi} &= -ad_{h}^{*} \frac{\delta \mathcal{L}}{\delta \xi} + \frac{\delta \mathcal{L}}{\delta \xi} \triangleright \eta + a_{h}^{*} \frac{\delta \mathcal{L}}{\delta \eta}, \\
\frac{d}{dt} \frac{\delta \mathcal{L}}{\delta \eta} &= -ad_{h}^{*} \frac{\delta \mathcal{L}}{\delta \eta} - \xi \triangleright \frac{\delta \mathcal{L}}{\delta \eta} - b_{h}^{*} \frac{\delta \mathcal{L}}{\delta \xi}.
\end{align}

We note that as oppose to the matched Euler-Lagrange equations (3.9) and (3.10), the matched Euler-Poincaré equations (3.14) and (3.15) are symmetric with respect to the Lie algebra variables.

Let us finally note that the Euler-Poincaré equations follow directly from the the Euler-Lagrange equations if the Lagrangian $\mathcal{L} = \mathcal{L}(g, h, \xi, \eta)$ is independent of both of the group variables $(g, h) \in$...
$G \rtimes H$. In this case, (3.8) reduces to
\[
\frac{d}{dt} \left( \frac{\delta \mathcal{L}}{\delta \dot{\xi}}, \frac{\delta \mathcal{L}}{\delta \dot{\eta}} \right) = -ad^*_\xi (\mathcal{L}_{\xi}) - ad^*_\eta (\mathcal{L}_{\eta}).
\]

3.3. Lagrangian dynamics on semi-direct products. In this subsection, we show how matched Euler-Lagrange equations (3.9) and (3.10) generalize the semi-direct product theory, [7, 9, 10, 25, 36].

To this end we write the actions, variations, Euler-Lagrange and Euler-Poincaré equations on the tangent bundles of the semi-direct product groups $G \ltimes H$ and $G \rtimes H$ [6]. In particular, the case of one of the groups being a vector space is studied extensively in the literature [6, 7, 13, 10, 8, 21, 22, 25, 36, 39].

For the semi-direct product $G \ltimes H$, the left action of $H$ on $G$ is trivial, therefore, the coadjoint action (3.7) reduces to
\[
ad^*_\xi (\mu, \nu) = \left( ad^*_\xi \mu - \mu \triangleright \eta - a^*_\eta \nu, ad^*_\eta \nu \right),
\]
whereas the variations (3.11) becomes
\[
\delta g = T_{g \circ \rho_h}^* L_g (\xi), \quad \delta h = T_{\rho_h}^* L_h \eta_2,
\]
\[
\delta \xi = \xi_2 + [\xi, \xi_2], \quad \delta \eta = \eta_2 + [\eta, \eta_2] + \eta \triangleright \xi - \xi_2 - \eta_2 \triangleright \xi.
\]

In this case, we obtain from the matched Euler-Lagrange equations (3.9) and (3.10) the semi-direct product Euler-Lagrange equations
\[
\frac{d}{dt} \frac{\delta \mathcal{L}}{\delta \dot{\xi}} = T_{\rho_h}^* L_h \left( \frac{\delta \mathcal{L}}{\delta \dot{\eta}} \right) - T_{\rho_h}^* \left( \frac{\delta \mathcal{L}}{\delta h} \right) - ad^*_\xi \frac{\delta \mathcal{L}}{\delta \xi} - ad^*_\eta \frac{\delta \mathcal{L}}{\delta \eta},
\]
\[
\frac{d}{dt} \frac{\delta \mathcal{L}}{\delta \dot{\eta}} = ad^*_\eta \frac{\delta \mathcal{L}}{\delta \eta} - \xi \triangleright \frac{\delta \mathcal{L}}{\delta \eta},
\]
on the space $(G \ltimes H) \rtimes (g \ltimes h) \simeq (G \ltimes g) \rtimes (H \ltimes h)$.

If the Lagrangian $\mathcal{L} = \mathcal{L} (g, h, \xi, \eta)$ on $(G \ltimes H) \rtimes (g \ltimes h)$ is independent of the group variable $g \in G$, then we obtain the dynamical equations on $H \ltimes (g \ltimes h)$. If, in addition, $\mathcal{L} = \mathcal{L} (h, \xi, \eta)$ is independent of $h \in H$, then we obtain the semi-direct product Euler-Poincaré equations
\[
\frac{d}{dt} \frac{\delta L}{\delta \dot{\xi}} = ad^*_\xi \frac{\delta L}{\delta \xi} + a^*_\eta \frac{\delta L}{\delta \eta}, \quad \frac{d}{dt} \frac{\delta L}{\delta \dot{\eta}} = ad^*_\eta \frac{\delta L}{\delta \eta} - \frac{\delta L}{\delta \eta} \triangleright \xi
\]
on $g \ltimes h$.

The dynamics on $G \ltimes H$ is similar. In this case, the right action of $G$ on $H$ is assumed to be trivial, therefore, the coadjoint action reduces to
\[
ad^*_\xi (\mu, \nu) = \left( ad^*_\xi \mu, ad^*_\nu + \xi \triangleright \nu + \nu \triangleright \mu \right),
\]
whereas the variations become
\[
\delta g = T_{e_G} (L_g \circ \rho_h) (\xi), \quad \delta h = (\xi_2) + T_{e_H} L_h (\eta_2),
\]
\[
\delta \xi = \xi_2 + [\xi, \xi_2] + \eta \triangleright \xi_2 - \eta_2 \triangleright \xi, \quad \delta \eta = \eta_2 + [\eta, \eta_2].
\]
As a result, the semi-direct product Euler-Lagrange equations take the form

\[ \frac{d}{dt} \delta L \bigg|_{\delta \xi} = T^*_{\xi} L_g \left( \frac{\delta L}{\delta g} \bigg|_{\delta \xi} \right) + h \circ \xi \circ \eta, \]

\[ \frac{d}{dt} \delta L \bigg|_{\delta \eta} = T^*_{\eta} L_h \left( \frac{\delta L}{\delta h} \bigg|_{\delta \eta} \right) - \xi \circ \eta \circ \xi, \]

on the space \((G \rtimes H) \ltimes (g \ltimes h) \simeq (G \ltimes g) \rtimes (H \ltimes h)\). Similarly, the matched Euler-Poincaré equations (3.14) and (3.15) reduce to the semi-direct product Euler-Poincaré equations

\[ \frac{d}{dt} \delta L \bigg|_{\delta \xi} = ad^*_{\xi} \delta L \bigg|_{\delta \xi} + \eta \circ \xi, \quad \frac{d}{dt} \delta L \bigg|_{\delta \eta} = ad^*_{\eta} \delta L \bigg|_{\delta \eta} - b^* \xi \delta L, \]

on \(g \ltimes h\).

4. An example: \(SL(2, \mathbb{C})\)

In this section we study the Lagrangian dynamics on the tangent bundle of \(SL(2, \mathbb{C})\). It is discussed in [30] in detail that \(SL(2, \mathbb{C})\) is a matched pair

\[ SU(2) \bowtie K, \]

of \(SU(2)\), and its half-real form \(K\). Lie algebra \(\mathfrak{su}(2, \mathbb{C})\) is then a matched pair of Lie algebras

\[ \mathfrak{su}(2) \bowtie \mathfrak{r}, \]

corresponding to the Iwasawa decomposition as a real Lie algebra, where \(\mathfrak{su}(2)\) is the Lie algebra of \(SU(2)\) and \(\mathfrak{r}\) is the Lie algebra of the group \(K\).

We shall make use of the notation

\[ A \in SU(2), \quad B \in K, \quad X, X_1, X_2 \in \mathfrak{su}(2) \simeq \mathbb{R}^3, \quad Y, Y_1, Y_2 \in \mathfrak{r} \simeq \mathbb{R}^3, \]

\[ \Phi \in \mathfrak{su}(2)^* \simeq \mathbb{R}^3, \quad \Psi \in \mathfrak{r}^* \simeq \mathbb{R}^3, \quad \kappa = (0, 0, 1) \in \mathbb{R}^3, \]

for the generic elements of the groups \(SU(2)\) and \(K\), and their Lie algebras \(\mathfrak{su}(2)\) and \(\mathfrak{r}\). Here \(\mathfrak{su}(2)^*\) and \(\mathfrak{r}^*\) are the linear duals of \(\mathfrak{su}(2)\) and \(\mathfrak{r}\), respectively.

4.1. Representations. In this subsection we recall, from [30], the various presentations of the matched pair groups \(SU(2)\) and \(K\), as well as their Lie algebras \(\mathfrak{su}(2)\) and \(\mathfrak{r}\).

The group \(SU(2)\) is the matrix Lie group of the special unitary matrices

\[ SU(2) = \left\{ \begin{pmatrix} \omega & \theta \\ -\bar{\theta} & \bar{\omega} \end{pmatrix} \in SL(2, \mathbb{C}) : |\omega|^2 + |\theta|^2 = 1 \right\}. \]

It is the universal double cover of the group \(SO(3)\) of isometries of the Euclidean space \(\mathbb{R}^3\). The Lie algebra \(\mathfrak{su}(2)\) of the group \(SU(2)\) is then the matrix Lie algebra

\[ \mathfrak{su}(2) = \left\{ \begin{pmatrix} -t & t-sr \\ r+ts & -t \end{pmatrix} : r, s, t \in \mathbb{R} \right\} \]

of traceless skew-hermitian matrices. Following [30] we fix three matrices

\[ e_1 = \begin{pmatrix} 0 & -t/2 \\ -t/2 & 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 & -1/2 \\ 1/2 & 0 \end{pmatrix}, \quad e_3 = \begin{pmatrix} -t/2 & 0 \\ 0 & t/2 \end{pmatrix} \]
as a basis of the Lie algebra \( \mathfrak{su}(2) \). We further make use of this basis to identify the matrix Lie algebra \( \mathfrak{su}(2) \) with the Lie algebra \( \mathbb{R}^3 \) by cross product,

\[
re_1 + se_2 + te_3 \in \mathfrak{su}(2) \leftrightarrow X = (r, s, t) \in \mathbb{R}^3.
\]

(4.6)

We also identify the dual space \( \mathfrak{su}(2)^* \) of \( \mathfrak{su}(2) \cong \mathbb{R}^3 \) with \( \mathbb{R}^3 \) using the Euclidean dot product. Using this dualization, we express the coadjoint action of the Lie algebra \( \mathfrak{su}(2) \cong \mathbb{R}^3 \) on \( \mathfrak{su}^*(2) \cong \mathbb{R}^3 \) as

\[
ad^* : \mathfrak{su}(2) \times \mathfrak{su}^*(2) \to \mathfrak{su}^*(2), \quad (X, \Phi) \mapsto ad^*_X \Phi = X \times \Phi,
\]

for \( X \in \mathfrak{su}(2) \cong \mathbb{R}^3 \) and \( \Phi \in \mathfrak{su}^*(2) \cong \mathbb{R}^3 \).

The simply-connected group \( K \) is a subgroup of \( GL(3, \mathbb{R}) \) of the form

\[
K = \left\{ \begin{pmatrix} 1 + c & 0 & 0 \\ 0 & 1 + c & 0 \\ -a & -b & 1 \end{pmatrix} \in GL(3, \mathbb{R}) \mid a, b, c \in \mathbb{R} \text{ and } c > -1 \right\},
\]

(4.8)

where the group operation is the matrix multiplication. The Lie algebra \( \mathfrak{k} \) of \( K \) is given by

\[
\mathfrak{k} = \left\{ \begin{pmatrix} c & 0 & 0 \\ 0 & c & 0 \\ -a & -b & 0 \end{pmatrix} \in \mathfrak{gl}(3, \mathbb{R}) \mid a, b, c \in \mathbb{R} \right\},
\]

(4.9)

where the Lie bracket is the matrix commutator. A faithful representation of the group \( K \) as the subgroup of \( SL(2, \mathbb{C}) \) is given by [30, Lemma 2.3]. Accordingly, we may also consider

\[
K = \left\{ \frac{1}{\sqrt{1 + c}} \begin{pmatrix} 1 + c & 0 \\ a + ib & 1 \end{pmatrix} \in SL(2, \mathbb{C}) \mid a, b, c \in \mathbb{R} \text{ and } c > -1 \right\},
\]

(4.10)

the group operation being the matrix multiplication. In this case the Lie algebra \( \mathfrak{k} \) is given by

\[
\mathfrak{k} = \left\{ \begin{pmatrix} \frac{1}{2}c & 0 \\ a + ib & \frac{1}{2}c \end{pmatrix} \in \mathfrak{sl}(2, \mathbb{C}) \mid a, b, c \in \mathbb{R} \right\},
\]

(4.11)

with matrix commutator as the Lie algebra bracket. The group \( K \) can, alternatively, be identified with the subspace

\[
K = \left\{ (a, b, c) \in \mathbb{R}^3 \mid a, b \in \mathbb{R} \text{ and } c > -1 \right\}
\]

(4.12)

of \( \mathbb{R}^3 \) with the non-standard multiplication

\[
(a_1, b_1, c_1) \ast (a_2, b_2, c_2) = (a_1, b_1, c_1)(1 + c_2) + (a_2, b_2, c_2).
\]

In this case the Lie algebra \( \mathfrak{k} \) is \( \mathbb{R}^3 \), and the Lie bracket is

\[
[Y_1, Y_2] = k \times (Y_1 \times Y_2),
\]

(4.13)

where \( k \) is the unit vector \((0, 0, 1)\). In this case, using the dot product, we identify the dual space \( \mathfrak{k}^* \) with \( \mathbb{R}^3 \) as well. Then, the coadjoint action of the Lie algebra \( \mathfrak{k} \cong \mathbb{R}^3 \) on its dual space \( \mathfrak{k}^* \cong \mathbb{R}^3 \) can be computed as

\[
ad^* : \mathfrak{k} \times \mathfrak{k}^* \to \mathfrak{k}^*, \quad (Y, \Psi) \mapsto ad^*_Y \Psi = (k \cdot Y) \cdot \Psi - (\Psi \cdot Y) \cdot k,
\]

(4.14)

for any \( Y \in \mathfrak{k} \cong \mathbb{R}^3 \), and any \( \Psi \in \mathfrak{k}^* \cong \mathbb{R}^3 \).

The group isomorphisms relating (4.10), (4.8) and (4.12) are given by

\[
\frac{1}{\sqrt{1 + c}} \begin{pmatrix} 1 + c & 0 \\ a + ib & 1 \end{pmatrix} \leftrightarrow \begin{pmatrix} 1 + c & 0 & 0 \\ 0 & 1 + c & 0 \\ -a & -b & 1 \end{pmatrix} \leftrightarrow (a, b, c).
\]

(4.15)
The Lie algebra isomorphisms

\[
\begin{pmatrix}
\frac{1}{2}c \\
0 \\
-\frac{1}{2}c
\end{pmatrix} \leftrightarrow \begin{pmatrix}
c & 0 & 0 \\
0 & c & 0 \\
-a & -b & 0
\end{pmatrix} \leftrightarrow (a, b, c)
\]

between (4.11), (4.9) and (4.13) are then obtained by differentiating (4.15).

4.2. Actions. In this subsection we recall from [30] the mutual actions of the matched pair of Lie groups \((SU(2), K)\), as well as the matched pair of Lie algebras \((\mathfrak{su}(2), \mathfrak{R})\).

Given any \(A \in SU(2)\), and any \(B \in K \subset SL(2, \mathbb{C})\), the left action of \(K\) on \(SU(2)\) is given by

\[
B \triangleright A = \left\| BA \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\|^{-1} M \left( BA \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} + B^{-\dagger} A \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \right),
\]

where \(B^{-\dagger}\) denotes the inverse of the conjugate transpose of \(B \in K\), and \(\|B\|_{M}^{2} = \text{tr}(B^\dagger B)\) refers to the matrix norm on \(SL(2, \mathbb{C})\).

Differentiating (4.17) with respect to the \(A \in SU(2)\), and regarding \(B \in K \subset GL(3, \mathbb{R})\) via (4.15), we obtain

\[
\triangleright : K \times \mathfrak{su}(2) \rightarrow \mathfrak{su}(2), \quad \begin{pmatrix} B, X \end{pmatrix} \mapsto \triangleright B \triangleright X = BX,
\]

for any \(X \in \mathfrak{su}(2) \simeq \mathbb{R}^{3}\). On the next step, the derivative of (4.18) with respect to \(B \in K\) yields

\[
\triangleright : \mathfrak{R} \times \mathfrak{su}(2) \rightarrow \mathfrak{su}(2), \quad \begin{pmatrix} Y, X \end{pmatrix} \mapsto Y \triangleright X = Y \times (X \times k)
\]

for any \(Y \in \mathfrak{R} \simeq \mathbb{R}^{3}\) and any \(X \in \mathfrak{su}(2) \simeq \mathbb{R}^{3}\). Freezing \(X\) in (4.19), we define the mapping

\[
\mathfrak{b}_{X} : \mathfrak{R} \rightarrow \mathfrak{su}(2), \quad Y \mapsto Y \times (X \times k).
\]

The right action of \(SU(2)\) on \(K \subset \mathbb{R}^{3}\), on the other hand, is

\[
\triangleleft : K \times \mathfrak{su}(2) \rightarrow TK, \quad \begin{pmatrix} B, X \end{pmatrix} \mapsto \triangleleft B \triangleleft X = T_{e_{K}} R_{B} \left( X \times \tilde{B} \right),
\]

where \(X \in \mathfrak{su}(2) \simeq \mathbb{R}^{3}\), and

\[
\tilde{B} := \frac{1}{c+1} \begin{pmatrix} B \\ -\|B\|_{E}^{2} k \end{pmatrix},
\]

identifying once again \(B \in K \subset SL(2, \mathbb{C})\) with \(B \in K \subset \mathbb{R}^{3}\) via (4.15). Here, \(T_{e_{K}} R_{B}\) is the tangent lift of the right translation \(R_{B} : K \rightarrow K\) by \(B \in K\), and it acts simply by the matrix multiplication regarding \(X \times \tilde{B} \in \mathfrak{R} \simeq \mathbb{R}^{3} \cong \mathfrak{gl}(3, \mathbb{R})\) via (4.16).

Finally, differentiating (4.22) with respect to \(B \in K\), we arrive at the right action

\[
\triangleleft : \mathfrak{R} \times \mathfrak{su}(2) \rightarrow \mathfrak{R}, \quad \begin{pmatrix} Y, X \end{pmatrix} \mapsto \triangleleft Y \triangleleft X = X \times Y,
\]
for any $X \in \mathfrak{su}(2) \simeq \mathbb{R}^3$ and any $Y \in \mathfrak{r} \simeq \mathbb{R}^3$. Freezing $Y$ in (4.2), we define the mapping

$$a_Y : \mathfrak{su}(2) \rightarrow \mathfrak{r}, \quad X \mapsto X \times Y. \quad (4.24)$$

We conclude this subsection with a brief discussion on the dual actions. Let us begin with dualizing (4.18) to obtain a right action of $K$ on the dual space $\mathfrak{su}(2)^*$ of $\mathfrak{su}(2) \simeq \mathbb{R}^3$. Identifying the dual space $\mathfrak{su}(2)^*$ with $\mathbb{R}^3$ by the dot product, we obtain

$$\rhd : \mathfrak{su}(2)^* \times K \rightarrow \mathfrak{su}(2)^*, \quad (\Phi, Y) \mapsto \Phi \rhd Y = (Y \cdot k) \Phi - (\Phi \cdot k) Y,$$

where $k = (0, 0, 1)$. On the other hand, the right action (4.2) dualizes to the left action of $\mathfrak{su}(2)$ on $\mathfrak{r}^*$ as

$$\rhd : \mathfrak{su}(2) \times \mathfrak{r}^* \rightarrow \mathfrak{r}^*, \quad (X, \Psi) \mapsto X \rhd \Psi = \Psi \times X,$$

for any $\Psi \in \mathfrak{r}^* \simeq \mathbb{R}^3$.

Freezing the $\mathfrak{su}(2)$-component of (4.19) we dualize (4.20) to

$$b_X^* : \mathfrak{su}(2)^* \rightarrow \mathfrak{r}^*, \quad \Phi \mapsto b_X^* \Phi = (\Phi \cdot k) X - (\Phi \cdot X) k$$

for any $\Phi \in \mathfrak{su}(2)^* \simeq \mathbb{R}^3$, and any $X \in \mathfrak{su}(2) \simeq \mathbb{R}^3$. Similarly, we dualize (4.24) to

$$a_Y^* : \mathfrak{r}^* \rightarrow \mathfrak{su}(2)^*, \quad \Psi \mapsto a_Y^*(\Psi) = Y \times \Psi.$$

We note finally the cotangent lift $T_{eSU(2)}^* \sigma_B : T_B^* K \rightarrow \mathfrak{su}(2)^*$ of the mapping (2.25). Explicitly,

$$T_{eSU(2)}^* \sigma_B (\Psi_B) = \tilde{B} \times (\Psi_B B^T), \quad (4.25)$$

where $\tilde{B} \in \mathbb{R}^3$ was defined in (4.23), and $(\Psi_B B^T)$ is the vector representation of $(\Psi_B B^T)$ which is an element of $T_{eK}^* K$.

### 4.3. Lagrangian Dynamics.

In this subsection we specialize the content of Subsection 3.2 to the group $SL(2, \mathbb{C})$.

Let us recall first that both of the Lie algebras $\mathfrak{su}(2)$ and $\mathfrak{r}$ are isomorphic to $\mathbb{R}^3$ as introduced in (4.6) and (4.16), respectively. As a result, we have the matched pair decomposition $\mathfrak{sl}(2, \mathbb{C}) = \mathbb{R}^3 \bowtie \mathbb{R}^3$.

Following the previous notation, let $\mathcal{L} = \mathcal{L}(A, B, X, Y)$ be a Lagrangian density on the trivialized tangent bundle $(SU(2) \bowtie K) \times (\mathbb{R}^3 \bowtie \mathbb{R}^3)$. Substituting the dual actions of the previous
subsection, we obtain the matched Euler-Lagrange equations (3.9) and (3.10) as
\[
\frac{d}{dt} \frac{\delta \mathcal{L}}{\delta \dot{X}} = \left[ B^T \left( A_1 \frac{\delta \mathcal{L}}{\delta A} \right) + \tilde{B} \times (\Psi_B B^T) \right] - \dot{X} \times \frac{\delta \mathcal{L}}{\delta X} + (Y \cdot k) \frac{\delta \mathcal{L}}{\delta Y} - \left( \frac{\delta \mathcal{L}}{\delta X} \cdot k \right) Y + Y \times \frac{\delta \mathcal{L}}{\delta Y},
\]

(4.26)
\[
\frac{d}{dt} \frac{\delta \mathcal{L}}{\delta \dot{Y}} = \left[ B^T \frac{\delta \mathcal{L}}{\delta B} - (k \cdot Y) \frac{\delta \mathcal{L}}{\delta Y} - \left( \frac{\delta \mathcal{L}}{\delta Y} \cdot Y \right) k - \frac{\delta \mathcal{L}}{\delta Y} \times X - \left( \frac{\delta \mathcal{L}}{\delta X} \cdot k \right) X + \left( \frac{\delta \mathcal{L}}{\delta X} \cdot X \right) k,
\]

(4.27)
where the hat denotes the isomorphic image in \( \mathbb{R}^3 \).

In case the Lagrangian \( \mathcal{L} = \mathcal{L}(A, B, X, Y) \) is free of the group variable \( A \in SU(2, \mathbb{C}) \), that is, if the system is left invariant under the left action of \( SU(2, \mathbb{C}) \), the term involving \( \delta \mathcal{L}/\delta A \) in (4.27) drops. Then, the reduced Lagrangian \( \mathcal{L} = \mathcal{L}(B, X, Y) \) can be defined on the quotient space \( K \ltimes (\mathbb{R}^3 \rtimes \mathbb{R}^3) \) which is isomorphic to \( \mathbb{R}^3 \ltimes (K \ltimes \mathbb{R}^3) \). This corresponds to the reduction presented on the left hand side of (2.63). As indicated in the diagram, a second step of reduction can be achieved by reducing \( K \ltimes (\mathbb{R}^3 \rtimes \mathbb{R}^3) \) to \( \mathbb{R}^3 \rtimes \mathbb{R}^3 \), which eventually results with a Lagrangian \( \mathcal{L} = \mathcal{L}(X, Y) \). As a result, we obtain the Euler-Poincaré equations on \( \mathbb{R}^3 \rtimes \mathbb{R}^3 \) as

\[
\frac{d}{dt} \frac{\delta \mathcal{L}}{\delta \dot{X}} = -\dot{X} \times \frac{\delta \mathcal{L}}{\delta X} + (Y \cdot k) \frac{\delta \mathcal{L}}{\delta X} - \left( \frac{\delta \mathcal{L}}{\delta X} \cdot k \right) Y + Y \times \frac{\delta \mathcal{L}}{\delta Y},
\]

(4.28)
\[
\frac{d}{dt} \frac{\delta \mathcal{L}}{\delta \dot{Y}} = - (k \cdot Y) \frac{\delta \mathcal{L}}{\delta Y} - \left( \frac{\delta \mathcal{L}}{\delta Y} \cdot Y \right) k - \frac{\delta \mathcal{L}}{\delta Y} \times X - \left( \frac{\delta \mathcal{L}}{\delta X} \cdot k \right) X + \left( \frac{\delta \mathcal{L}}{\delta X} \cdot X \right) k.
\]

(4.29)

In particular, for the Lagrangian \( \mathcal{L} \) as the total kinetic energy
\[
\mathcal{L}(X, Y) = I_1 X \cdot X + I_2 Y \cdot Y,
\]
where \( I_1 \) and \( I_2 \) are three by three matrices corresponding to momentum inertia tensors, the matched Euler-Poincaré equations (4.28) and (4.29) become
\[
I_1 \dot{X} = -\dot{X} \times I_1 X + (Y \cdot k) I_1 X - (I_1 X \cdot k) Y + Y \times I_2 Y,
\]
\[
I_2 \dot{Y} = -(k \cdot Y) I_2 Y - (I_2 Y \cdot Y) k - I_2 Y \times X - (I_1 X \cdot k) X + (I_1 X \cdot X) k.
\]

5. Conclusion and Discussions

Given a matched pair of Lie groups \( G \bowtie H \), we obtained an isomorphism between the tangent bundle \( T(G \bowtie H) \) and the matched pair of tangent groups \( TG \bowtie TH \). Accordingly, we derived the matched Euler-Lagrange equations (3.9) on the (left) trivialization of \( TG \bowtie TH \). Using proper reductions, we further obtained the matched Euler-Poincaré equations (3.15) on \( g \bowtie \mathfrak{h} \). As an example, we considered the Iwasawa decomposition \( SU(2) \bowtie K \) of \( SL(2, \mathbb{C}) \). We then observed that, on the left trivialization of \( T(SU(2) \bowtie K) \), the matched Euler-Lagrange equations take the form of (4.27), whereas the matched Euler-Lagrange equations on the Lie algebra \( \mathfrak{sl}(2, \mathbb{C}) \) become (4.29).

As a natural complement of the Lagrangian dynamics on a matched pair of Lie groups, on the next step we plan to study the Hamiltonian dynamics on a matched pair of Lie groups, [19]. Being a cotangent bundle, \( T^*(G \bowtie H) \) carries a symplectic two-form which enables one to write
the canonical Hamilton’s equations. Applying the symplectic and the Poisson reductions to the symplectic framework on $T^*(G \bowtie H)$ we will be able to obtain several reduced Hamiltonian formulations.

On the other hand, since the linear algebraic dual $(g \bowtie h)^*$ of the Lie algebra $g \bowtie h$ is the matched pair of Lie coalgebras, it will be interesting to discuss the dual pair setting of [20] from the point of view of the Lie-Poisson formulation on the matched pairs of Lie coalgebras.

From the decomposition point of view, another natural follow up to our study in the present paper is to consider the Kac decomposition [27, 38] on diffeomorphism groups, to investigate the plasma dynamics.

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