Evaluating structure learning algorithms with a balanced scoring function

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ABSTRACT: Several structure learning algorithms have been proposed towards discovering causal or Bayesian Network (BN) graphs, which is a particularly challenging problem in AI. The performance of these algorithms is evaluated based on the relationship the learned graph has with respect to the ground truth graph. However, there is no agreed scoring function to determine this relationship. Moreover, this paper shows that the commonly used metrics tend to be biased in favour of graphs that minimise the number of edges. The evaluation bias is inconsistent and may lead to evaluating graphs with no edges as superior to graphs with varying numbers of correct and incorrect edges; implying that graphs that minimise edges are often favoured over more complex graphs due to bias rather than overall accuracy. While graphs that are less complex are often desirable, the current metrics ‘encourage’ algorithms to optimise for simplicity, and to discover graphs with a limited number of edges that do not enable full propagation of evidence. This paper proposes a Balanced Scoring Function (BSF) that eliminates this bias by adjusting the reward function based on the difficulty of discovering an edge, or no edge, proportional to their occurrence rate in the ground truth graph. The BSF score can be used in conjunction with other traditional metrics to provide an alternative and unbiased assessment about the capability of structure learning algorithms in discovering causal or BN graphs.

Keywords: Bayesian Networks, causal discovery, probabilistic graphical models, scoring rules, structure learning.

1. Introduction

Causality is the process by which an event, the cause, connects to another event, the effect, where the former is understood to be at least partly responsible for the latter. A causal Bayesian Network (BN) is typically used to represent causal relationships, under the assumption that the arcs between nodes represent direct causation. The causal assumption is useful in cases where we seek to simulate the effect of interventions, or to perform counterfactual reasoning [1]. Otherwise, a non-causal BN is viewed as a model that captures the conditional dependencies between variables of interest.

Constructing a BN model involves determining a) its graphical structure which specifies the dependencies between variables, and b) its Conditional Probability Tables (CPTs) which specify the relationship between directly dependent variables. This paper focuses on the problem of evaluating the accuracy of machine learned graphs, independent of the type of parameter learning that could be used to parameterise the CPTs of the learned graph. Hence, the focus is fully oriented towards discovery rather than inference.
Determining the graphical structure of a BN is not exclusively a machine learning problem [2]. While automated structure learning is useful in disciplines such as bioinformatics, where automated discovery may reveal insights that would otherwise remain unknown [3] [4] [5] [6] knowledge-based graphs represent a popular alternative in areas with access to expertise or rule-based information not captured by data. Some examples include medical applications [7], law and forensics [8] [9], project management [10] [11], and property market [12]. Moreover, automated structure learning can be performed with knowledge-based constraints in terms of what can and cannot be discovered; effectively representing a combination of the two approaches.

Over the past three decades, several algorithms have been proposed that aim to discover the structure of a BN. These algorithms largely fall into two categories. First, the score-based algorithms which, as in traditional machine learning systems, search for different graphical structures and score them based on some metric. The graph that maximises the preferred score is the preferred graph. Popular algorithms that fall within this category include the Optimal Reinsertion [13], GES [14], Sparse Candidate [15], and K2 [16] algorithms. Second, the constraint-based algorithms aim to discover causal structures, under very specific assumptions, by performing a series of conditional independence tests. The PC [17] and IC [18] algorithms are the first to introduce this type of learning strategy, and the PC algorithm has become particularly popular in this area of research. Hybrid algorithms that combine the these two approaches have also been proposed, and include the L1-Regularisation paths [19] and the Max-Min Hill-Climbing (MMHC) [20] algorithms, with the latter illustrating a strong performance relative to many other algorithms.

Section 2 provides a review of the metrics and scoring functions that are used in the literature to evaluate learned graphs. The need for a balanced scoring function is justified in Section 3, which also describes the Balance Scoring Function (BSF). The concluding remarks are provided in Section 4.

2. The scoring functions used in the literature

Various metrics have been considered to evaluate the accuracy of the learned graph with respect to the ground truth, or some other, graph. The ground truth graph tends to be a Directed Acyclic Graph (DAG), or a Partially Directed Acyclic Graph (PDAG) [21]. A PDAG\(^1\) incorporates both directed and undirected edges and represents an equivalence class of DAGs. An undirected edge in a PDAG indicates that corresponding DAGs produce identical outputs irrespective of the directionality of an undirected edge in the PDAG. However, an assessment on PDAGs, rather than on DAGs, is not always desirable; i.e., the direction of influence is important in cases where we are interested in causal graphs or causal BNs for interventional analysis. Further, depending on the algorithm, a learned graph may include other type of edges with different endpoints to differentiate between different types of uncertainty for each edge [22]. For simplicity, in this paper we shall assume that an edge can be directed (i.e., an arc) or undirected (i.e., everything else that is not an arc).

Table 1 presents the different metrics used in 29 relevant papers from the literature. The metrics fall under four types of evaluation. These are:

a) metrics based on derivations from a confusion matrix, marked with ‘+’ in Table 1;

b) metrics based on the number of structural differences between graphs, marked with ‘Δ’;

\(^1\) PDAGs are also called patterns by Spirtes et al [17], essential graphs by Andersson et al [50], and maximally oriented graphs by Meek [51].
c) inference-based methods, serving as graph evaluators, marked with ‘×’;
d) theoretical proofs, used as an alternative to a metric, marked with ‘ο’.

Table 1 suggests that the category (a) metrics have become popular over the last decade, whereas the metrics that fall within categories (b) and (c) appear to have been popular over the whole period. Regarding (d), while theoretical proofs often accompany a proposed algorithm irrespective of the metric used to evaluate the performance of the algorithm, there were two papers in which theoretical proofs appear to have been used as an alternative to, and not in addition to, empirical evaluation with a metric. On this basis, we shall assume that at least two papers from those reviewed have based their evaluation criterion on theoretical proofs.

**Table 1.** The various metrics used in the literature to evaluate structure learning algorithms, where ‘+’ represent metrics that are based on derivations from a confusion matrix, ‘Δ’ represent metrics that focus on the number of structural differences between graphs, ‘×’ represent inference-based metrics, and ‘ο’ represent the cases where theoretical proof was used as an alternative to, and not in addition to, a metric.

| Year | Reference | Partial conf. | Precision | Recall/ Sensitivity | Stats on graph. diff. | SHD | DDM | BIC | BDeu | ROC/AUC | Other | Theoretical Proof |
|------|-----------|---------------|-----------|---------------------|------------------------|-----|-----|-----|------|---------|-------|------------------|
| 1990 | [18]      |               |           |                     |                        |     |     |     |      |         |       | ο                |
| 1991 | [23]      | Δ             |           |                     |                        |     |     |     |      |         |       |                  |
| 1992 | [16]      | +             |           |                     |                        |     |     |     |      |         |       |                  |
| 1995 | [24]      | Δ             |           |                     |                        |     |     |     |      |         |       |                  |
| 1999 | [15]      |               |           |                     |                        |     |     |     |      |         |       |                  |
| 2002 | [14]      | Δ             |           |                     |                        |     |     |     |      |         |       |                  |
| 2003 | [25]      |               |           |                     |                        |     |     |     |      |         |       |                  |
| 2003 | [26]      | +             | Δ         | ×                   |                        |     |     |     |      |         |       |                  |
| 2005 | [27]      |               |           |                     |                        |     |     |     |      |         |       |                  |
| 2006 | [28]      | Δ             |           |                     |                        |     |     |     |      |         |       |                  |
| 2006 | [20]      |               |           |                     |                        |     |     |     |      |         |       |                  |
| 2008 | [29]      |               |           |                     |                        |     |     |     |      |         |       | ο                |
| 2009 | [5]       | +             |           |                     |                        |     |     |     |      |         |       |                  |
| 2010 | [30]      | +             | Δ         | ×                   |                        |     |     |     |      |         |       |                  |
| 2011 | [31]      | +             | +         |                     |                        |     |     |     |      |         |       |                  |
| 2012 | [32]      | Δ             |           |                     |                        |     |     |     |      |         |       |                  |
| 2012 | [33]      | +             | +         | x                   |                        |     |     |     |      |         |       |                  |
| 2014 | [34]      | +             | Δ         | x                   |                        |     |     |     |      |         |       |                  |
| 2015 | [35]      | +             | +         |                     |                        |     |     |     |      |         |       |                  |
| 2016 | [36]      | +             | +         |                     |                        |     |     |     |      |         |       |                  |
| 2016 | [22]      | +             | +         |                     |                        |     |     |     |      |         |       |                  |
| 2016 | [37]      | +             | +         | Δ                   |                        |     |     |     |      |         |       |                  |
| 2017 | [38]      | +             | +         | ×                   |                        |     |     |     |      |         |       |                  |
| 2018 | [39]      | +             | +         |                     |                        |     |     |     |      |         |       |                  |
| 2018 | [40]      | +             | +         | Δ                   | Δ                     |     |     |     |      |         |       |                  |
| 2018 | [41]      |               |           |                     |                        |     |     |     |      |         |       |                  |
| 2018 | [42]      |               |           |                     |                        |     |     |     |      |         |       |                  |
| 2019 | [43]      | Δ             | Δ         | x                   | x                     |     |     |     |      |         |       |                  |
| 2019 | [44]      | Δ             |           |                     |                        |     |     |     |      |         |       |                  |

The metrics that fall under categories (c) and (d) are out of the scope of this paper. Regarding (d), this is simply because theoretical proof is not metric. Regarding (c), the main purpose of inference-based methods is to serve as scoring functions during the structure learning process; i.e., to search and return the graph that maximises a particular inference-based score $X$. A structure learning algorithm that is designed to maximise $X$ is likely to outperform
an algorithm that is designed to maximise some other score \( Y \), and vice versa. Thus, inference-based scores are inherently biased in favour of the algorithms that maximise them and are not suitable to assess graphs generated by different algorithms. Regardless, the main reason type (c) methods are out of the score of this is because they are based on inference by design and hence, they consider information that goes beyond the graphical structure of the learned graph, such as the parameter learning process which is often independent of the structure learning algorithm under evaluation. On the other hand, the metrics and scoring functions that fall under categories (a) and (b) are fully oriented towards discovery rather than inference. The subsections 2.1 and 2.2 that follow provide a brief overview of the type (a) and (b) metrics respectively.

2.1. Metrics based on derivations from of a confusion matrix

For the purposes of structure learning, a confusion matrix is a simple two-dimensional table that captures the performance of the learned graph in relation to the ground truth graph, as illustrated in Table 2 based on a hypothetical example. Specifically, and assuming DAGs rather than PDAGs for simplicity,

i. **True Positives** (TP): the number of true arcs discovered in the learned graph.

ii. **False Positives** (FP): the number of false arcs or edges discovered in the learned graph.

iii. **True Negatives** (TN): the number of true direct independencies discovered in the learned graph.

iv. **False negatives** (FN): the number of false direct independencies discovered in the learned graph.

|                          | True arcs | True independ. |
|--------------------------|-----------|-----------------|
| Discovered arcs          | 7 (TP)    | 5 (FP)          |
| Discovered independ.     | 3 (FN)    | 30 (TN)         |

Table 2. An example of the simplest possible confusion matrix that could be considered by a structure learning algorithm. The hypothetical example assumes a ground truth graph with 10 arcs, and a learned graph with 12 arcs.

There are many different derivations that can be retrieved from confusion matrix, and they can be very confusing in their interpretation. The derivations discovered in the papers reviewed (refer to Table 1) are:

i. **Partial confusion matrix stats**: the cases where the authors reported some of the parameters of the confusion matrix; typically focusing on TP, FP, or FN, or any combination of the three.

ii. **Precision**: the cases where the authors reported the Precision rate, defined as \( \frac{TP}{TP + FP} \), and which represents the rate of accuracy in discovering direct dependencies. For example, the Precision rate in Table 1 is 58.3% because the hypothetical algorithm discovered 7 true arcs out of the 12 arcs discovered.

iii. **Recall (or Sensitivity)**: while both descriptions are well-known, Recall is the description more commonly used in the papers reviewed. It is defined as \( \frac{TP}{TP + FN} \), and
represents the rate of accuracy in discovering direct dependencies. For example, the Recall rate in Table 1 is 70% because the hypothetical algorithm discovered 7 out of the 10 true arcs.

2.2. Metrics or scoring functions based on structural differences

The metrics and scoring functions that fall within this category are easier to interpret since they are based on scores that directly correspond to some discrepancy between the learned graph and the ground truth graph. In Table 1, the papers under Stats on graphical dif would generally report some graphical discrepancies that are, in fact, often no different to the TP, FN, and FP terminologies discussed above.

Tsamardinos et al [20] appear to be the first to propose a scoring function, called the Structural Hamming Distance (SHD), with a particular focus on evaluating graphs generated by structure learning algorithms. The SHD is defined as nothing more than the minimum number of edge insertions, deletions, and arc reversals that are needed in order to transform the learned graph into the true graph (or the PDAG equivalent). Note that the number of insertions and deletions are equivalent to the FN and FP rates respectively, whereas for the number of arc reversals corresponds to the number of correct direct dependencies discovered but with incorrect direction; i.e., partially correct or incorrect arc (both views are present in the literature).

Similarly, in [43] the DAG Dissimilarity Metric (DDM) is used to determine the level of dissimilarity between two DAGs. The score ranges from $-\infty$ to 1, where a score of 1 indicates perfect agreement between the two graphs. The score moves to $-\infty$ the stronger the dissimilarity is between the two graphs. Specifically, if we compare how dissimilar the graph $A$ is with respect to graph $B$, then

$$\text{DDM} = \frac{\text{TP} + \frac{r}{t} - \text{FN} - \text{FP}}{t}$$

where $r$ is the number of arcs from $B$ reoriented in $A$, and $t$ is total number of arcs in $B$. Note that $r$ is equivalent to the number of arc reversals in SHD, but its penalty is halved in DDM to acknowledge the fact that an arc reversal is an event proceeded by a successfully discovery of a direct dependency.

It is also worth noting that Peters and Buhlmann proposed the Structural Intervention Distance (SID) metric [45], as an alternative to SHD. While SID is directly assessed with reference to the SHD [45], it is significantly more complex than SHD and aimed at causal BNs by focusing on evaluating causal effects (i.e., intervention distributions). This limitation may perhaps explain why SID has been absent by the relevant literature. Since we are interested in metrics and scoring functions that are suitable in assessing learned BNs that are not necessarily causal graphs, the SID is also out of the scope of this paper.

3. The need for a balanced scoring function

Section 2 described how each of the metrics relate to the parameters of a confusion matrix. The definition of a scoring function, however, also depends on how the different types of discrepancy between graphs are defined. For example, note how the standard definitions of TP, TN, FP, and FN in Section 2.1 do not differentiate between a) incorrect dependencies and b) correct dependencies with incorrect direction; i.e., both errors fall under FP. The first
illustration in Section 3.1 assumes no difference between (a) and (b), whereas Section 3.2 accounts for these differences.

3.1. A closer look at the current metrics

Table 3 presents the scores generated by the different metrics based on a series of hypothetical TP, FP, TN and FN parameters, and under the assumption that the true graph consists of 10 variables and 10 arcs. For simplicity, we shall assume that the learned graphs only incorporate arcs (e.g., no undirected or bi-directed dependencies).

The illustration is based on three different scenarios. In the first scenario, TP is the only parameter that varies between learned graphs. Similarly, in the second scenario FP is the only parameter that varies. On the other hand, the third scenario consists of a) a fully connected graph in which all correct dependencies have the correct directionality, b) a fully connected graph in which half of the correct dependencies have the correct directionality, representing the case of a randomly generated fully connected graph, c) a fully unconnected graph, d) the most inaccurate graph possible given the true graph, and e) the most accurate graph possible; i.e., the true graph.

Table 3. The evaluation scores generated by each metric when based on the hypothetical TP, FP, TN, and FN parameters, and assuming that the true DAG consists of 10 variables and 10 arcs.

| Scenario               | Arcs discovered (TP) | False dep. discovered (FP) | Independent discovered (TN) | Arcs not discovered (FN) | Precision | Recall | F1 | SHD | DDM |
|------------------------|----------------------|-----------------------------|----------------------------|--------------------------|-----------|-------|----|-----|-----|
| Discrepancies in TP (and naturally FN) | 10       | 20              | 15                         | 0                        | 0.33      | 1     | 0.5 | 20  | -1  |
|                        | 5        | 20              | 15                         | 5                        | 0.2       | 0.5   | 0.29 | 25  | -2  |
|                        | 0        | 20              | 15                         | 10                       | 0         | 0     | n/a | 30  | -3  |
| Discrepancies in FP (and naturally TN) | 5        | 15              | 20                         | 5                        | 0.25      | 0.5   | 0.33 | 20  | -1.5|
|                        | 5        | 10              | 25                         | 5                        | 0.33      | 0.5   | 0.4  | 15  | -1  |
|                        | 5        | 5               | 30                         | 5                        | 0.5       | 0.5   | 0.5  | 10  | -0.5|
| Fully connected graph A| 10       | 35              | 0                          | 0                        | 0.22      | 1     | 0.36 | 35  | -2.5|
| Fully connected graph B| 5        | 35              | 0                          | 5                        | 0.125     | 0.5   | 0.2  | 40  | -3.5|
| Fully unconnected graph| 0        | 35              | 10                         | n/a                      | 0         | n/a   | 0    | 45  | -4.5|
| Most inaccurate graph  | 0        | 35              | 10                         | 0                        | 0         | n/a   | 45   | -4.5|
| Most accurate graph    | 10       | 0               | 35                         | 0                        | 1         | 1     | 1    | 0   | 1   |

To begin with, it is important to highlight that the TP, FP, TN, and FN parameters should not be used independently to judge an algorithm. As shown in Table 3, this is because TP remains invariant when FP varies, and FP and TN remain invariant when TP varies. As obvious as this may seem, the TP, FP and TN stats are often used in the literature to judge a learned graph. While these stats are often used in addition to some other metric, the level of emphasis put on some of these individual indicators varies across papers and the risk that the reader will receive the wrong message is evident. In brief, while it is reasonable to refer to these statistical indicators in order to explain how or why the graphs differ or not, these outputs should not form part of the assessment process independently.

Precision and Recall are popular metrics in evaluating learned graphs. Yet, independently, these metrics are also inadequate. This is because each of these metrics will only consider two out of the four confusion matrix parameters. Because of this reason, the F1 score is generally used in other areas as a ‘fairer’ metric that provides the harmonic mean of...
Precision and Recall. Interestingly, none of the 29 papers reviewed have considered the F1 score. Regardless, the F1 score has its own limitations, which are already well documented in other disciplines [46]. For the purposes of this paper, the limitations of the F1 score include a) it does not take into account the TN parameter, b) it assumes that Precision and Recall are equally important, and c) the F1 score cannot be computed when the learned graph fails to capture one of the true arcs (refer to Table 3) and this is a problem when the number of variables is low.

Finally, the SHD and DDM scoring functions address many of the limitations discussed above and are able to correctly order each of the learned graphs within each of the three scenarios in Table 3. However, note how both scoring functions strongly favour the fully unconnected graph over the fully connected graphs in the third scenario. At first, this outcome may seem reasonable. After all, a fully unconnected graph is ‘closer’ to the true graph than a fully connected graph is. However, an unconnected graph is useless since it does not enable any propagation of evidence. Furthermore, the score assigned to the unconnected graph not only is significantly superior to those assigned to the fully connected graphs, but it is also the shared best score over all graphs and over all scenarios (excluding the ‘Most accurate graph’).

### 3.2. The Balanced Scoring Function (BSF)

The metrics discussed above generally suffer from two issues. The first issue has to do with the way discrepancies are defined, which often remain unweighted. The second, and primary, issue has to do with the bias in the reward/penalty function that generally leads to a strong scoring preference towards simplified graphs. This happens because the true graph will generally incorporate a higher number of independencies that arcs. The bias is proportional to the ratio of independencies over arcs in the true graph.

#### 3.2.1. Penalty function

The simplicity of SHD assumes that the penalty of discovering a false dependency is equivalent to discovering the true dependency but with no or incorrect direction (in both cases the discrepancy is 1). Tsamardinos et al [20] also acknowledged that the SHD is biased towards the sensitivity of identifying edges versus specificity. Colombo and Maathuis [34] tried to improve SHD by introducing the variant \(SHD\) edge marks that explicitly counts the number of errors in edge marks; i.e., penalising the change from \(A \leftrightarrow B\) to \(A \rightarrow B\) by 1, and the change from \(A \rightarrow B\) to \(A \perp B\) by 2. However, the double-counting of edge marks does not resolve the issue with the penalty function. On the other hand, the DDM function addresses this issue with the \(r\) parameter (refer to Section 2.2) by assuming that the reward of discovering a true dependency without the correct orientation is half relative to the reward of discovering a dependency with the correct orientation [43]. A comparison of the different properties of structural distance metrics, which can be seen as SHD variants (e.g., counting only missing arcs), but with a particular focus on causal BNs, is provided by de Jongh and Druzdzel [47].

Table 4 presents the proposed penalty weights for any scoring function, where an undirected edge, a bi-directed edge, or a reversed arc are assumed to fall under the same category of a ‘Partial match’. A possible argument here is that the penalty for reversing an arc should be higher than the penalty of directing an undirected or a bi-directed edge, similar to what Colombo and Maathuis [34] proposed. However, a counterargument is that we should not reward algorithms that make no attempt to determine the orientation of a dependency higher than we reward algorithms that go ahead and orient them, since what we expect from these algorithms is to output a DAG or a PDAG that can be converted into functioning BN model, rather than an incomplete graph.
Table 4. Proposed penalty weights for the any scoring function. Rule 6 is optional for cases where PDAGs are preferred over DAGs.

| Rule | True graph | Learned graph | Penalty | Justification |
|------|------------|---------------|---------|---------------|
| 1    | A → B     | A → B         | 0       | Complete match|
| 2    | A → B     | A ↔ B, A − B or A ← B | 0.5     | Partial match |
| 3    | A → B     | A ⊥ B         | 1       | No match      |
| 4    | A ⊥ B     | A ⊥ B         | 0       | Complete match|
| 5    | A ⊥ B     | Any edge/arc  | 1       | Incorrect dependency discovered |
| 6    | A → B     | A ↔ B, A − B or A ← B | 0       | If PDAG equivalence rules are met |

3.2.2. Eliminating the bias between arcs and independencies

Under the assumption that the difficulty of discovering an arc is directly dependent on the number of arcs that exist in the true graph, relative to the number of independencies, a balanced scoring function would need to weight the difficulty of discovering arcs $w_a$ and independencies $w_i$ as follows:

$$w_a = \frac{1}{a}, \quad w_i = \frac{1}{i}$$

where $a$ and $i$ are simply the number of arcs and independencies in the true graph respectively. If the number of arcs in the true graph are $a$, then

$$i = \frac{n \times (n - 1)}{2} - a$$

where $n$ is the number of nodes in the true graph. For example, in Table 3 $a = 10$ and $i = 45 - 10 = 35$. For this example, discovering $a$ is 3.5 times more difficult to discover than is to discover $i$ (i.e., $w_a = \frac{1}{10}$ versus $w_i = \frac{1}{35}$). To obtain a Balanced Scoring Function (BSF), we need to adjust each parameter $k$ in the confusion matrix to its associated difficulty of discovery as follows:

$$BSF = \frac{2}{k} \sum_k \left\{ \begin{array}{ll} k \times w_a, & k = TP \\ k \times w_i, & k = TN \\ -k \times w_a, & k = FP \\ -k \times w_i, & k = FN \end{array} \right.$$  

This ensures that BSF will return a score between -1 and 1, with a score of -1 corresponding to the worse possible learned graph and a score of 1 corresponding to the best possible learned graph that matches the true DAG or PDAG. Table 5 is a revised version of Table 3 that accounts for the partial arcs discovered (refer to Section 3.2.1), and also includes the BSF scores.
Table 5. Scores revised from Table 3 to account for “Partial arcs discovered”, and the BSF scores. The confusion matrix parameters assume a DAG that consists of 10 variables and 10 arcs, as in Table 3.

| Scenario                  | Arcs discovered | Partial arcs discovered (TP0.5) | False dep. discovered (FP) | Indep. discovered (TN) | Arcs not discovered (FN) | Precision | Recall | SHD | DDM | BSF |
|---------------------------|-----------------|---------------------------------|----------------------------|------------------------|--------------------------|-----------|--------|-----|-----|-----|
| Discrepancies in TP (and FN) | 8               | 2                               | 20                         | 15                     | 1                        | 0.31      | 0.9    | 0.46| 0.26| 21  |
| Discrepancies in FP (and TN) | 4               | 1                               | 15                         | 15                     | 5.5                      | 0.18      | 0.45   | 0.26| 0.31| 25.5|
| Fully connected A         | 10              | 0                               | 35                         | 0                      | 0                        | 0.22      | 1      | 0.36| 0.3 | 35  |
| Fully connected B         | 5               | 5                               | 35                         | 0                      | 2.5                      | 0.18      | 0.75   | 0.29| 0.37| 37.5|
| Fully unconnected graph   | 0               | 0                               | 35                         | 10                     | n/a                      | n/a       | n/a    | 10  | -1  | 0   |
| Most inaccurate graph     | 0               | 0                               | 35                         | 0                      | 10                       | 0         | 0      | 45  | -4.5| -1  |
| Most accurate graph       | 10              | 0                               | 35                         | 0                      | 0                        | 1         | 1      | 0   | 1   | 1   |

The addition of ‘partial arcs discovered’ has not led to any significant changes from the results in Table 3; i.e., the issues discussed in Section 3 are still present. The proposed BSF function is in agreement with the SHD and DDM functions in terms of the ranking of the the learned graphs within each of the scenarios. However, there are clear disagreements when comparing the ranking of the learned graphs across different scenarios.

3.3. Limitations and benefits of the BSF

A possible limitation of the BSF score is that, in its mission to eliminate bias, it will generally eliminate bias in favour of simplified graphs. This is because true graphs generally incorporate a lower number of direct dependencies than they incorporate independencies. However, this is precisely the purpose of BSF; to eliminate this bias. The BSF score can be used in conjunction with other metrics to offer an alternative and a balanced evaluation that encompasses the following advantages:

i. It considers fully connected (with correct true arcs) and fully unconnected graphs to be equivalent in terms of accuracy in their attempt to discover a graph; both cases will generate a score of 0. The alternative metrics are generally expected to generate scores that are heavily biased in favour of simple graphs, and are at high risk of determining graphs that fail to provide full propagation of evidence as superior to graphs that enable improved or full propagation of evidence.

ii. As a balanced scoring function, BSF produces a meaningful score that ranges from -1 to 1, where -1, 0, and 1 correspond to the most inaccurate, a semi-ignorant\(^2\), and most accurate graphs respectively (refer to Table 5, third scenario). A BSF score instantly reveals whether the learned graph is superior to an ignorant graph, as well as to how far the learned graph is from the most inaccurate and most accurate graphs in relation to an ignorant graph. This is an important advantage that improves the interpretation of a structure learning algorithm’s scoring performance, and allows for a more accurate

\(^2\) The case when the learned graph discovers either a) all of the true arcs but none of the true independencies, or b) all of the true independencies but none of the true arcs.
cross-study comparisons of scores derived from different ground truth and learned graphs.

iii. Assuming that a balanced score is more desirable then, importantly, the BSF score offers a more accurate ranking in evaluating different structure learning algorithms.

4. Concluding remarks

In the absence of an agreed or adequate evaluation method it is difficult to reach a consensus about which of the competing structure learning algorithms is ‘best’, but also to determine whether a particular algorithm is satisfactory. The different evaluation methods increase the risk of inconsistency whereby one evaluator determines algorithm $A$ to be superior to algorithm $B$, and another evaluator concludes the opposite. In such situations, the selection of an evaluation method is as important as the algorithm itself, since the evaluation method practically judges the performance of that algorithm.

Korb and Nicholson [48] acknowledged that there is no agreed evaluation process due to inadequate experimental survey of the main competing algorithms of the field, and stated that "every publication in the field attempts to make some kind of empirical case for the particular algorithm being described in that publication". Motivated by this claim, this paper provided a brief review of the metrics used in the relevant literature. The review revealed issues in relation to consistency, accuracy, and bias in evaluating the accuracy of learned graphs. On this basis, the BSF score is proposed that balances the reward function proportional to the rate of arcs over independencies in the true graph. The BSF score is consistent and unbiased, and offers a novel and an alternative way of evaluating the performance of structure learning algorithms.

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