FIELDS OF DEFINITION OF CYCLIC P-GONAL CURVES

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ABSTRACT. Let $S$ be cyclic $p$-gonal Riemann surface, that is, a closed Riemann surface of genus $g \geq 2$ admitting a conformal automorphism $\varphi$ of prime order $p$ so that $S/\langle \varphi \rangle$ has genus zero. In this article we provide a short argument of the fact that if $S$ is definable over a subfield $k$ of $\mathbb{C}$, then there is a subfield $k_0$ of $\mathbb{C}$, this being an extension of degree at most $2(p-1)/2$ of $k$, so that $S$ is definable by a curve $y^p = F(x)$, where $F(x)$ is a polynomial with coefficients in $k_0$. Moreover, if both $S$ and $\varphi$ are simultaneously definable over $k$, then $k_0$ can be chosen to be an extension of degree at most 2 of $k$.

In the particular case $p = 2$ the above means that every hyperelliptic Riemann surface, which is definable over $k$, is also hyperelliptically definable over an extension of degree at most 2 of it, extending results due to Mestre, Huggins and Lercier, Ritzenthaler and Sijslingit.

1. Introduction

A closed Riemann surface $S$, of genus $g \geq 2$, is called cyclic $p$-gonal, where $p$ is a prime integer, if it admits a conformal automorphism $\varphi$ of order $p$ so that the orbifold $S/\langle \varphi \rangle$ has genus zero. In this case, we say that $\varphi$ is a $p$-gonal automorphism and that the cyclic group $\langle \varphi \rangle$ is a $p$-gonal group of $S$. If $m$ denotes the number of fixed points of $\varphi$, then $g = (m-2)(p-1)/2$. Cyclic $2$-gonal Riemann surfaces are also known as hyperelliptic Riemann surfaces, in which case there is a unique 2-gonal automorphism; called the hyperelliptic involution.

In [1] R.D. Accola studied cyclic 3-gonal automorphisms and in [16] G. González-Diez proved that $p$-gonal groups are unique up to conjugation in the full group of conformal automorphisms. In general, the $p$-gonal group (for $p \geq 3$) is not unique [9, 10, 34]. In [34] A. Wootton studied the uniqueness and described those cases were the uniqueness fails. Real structures on $p$-gonal Riemann surfaces have been studied in [6, 7]. Other results concerning automorphisms of $p$-gonal Riemann surfaces can be found, for instance, in [4, 2, 3, 5, 8, 18, 19, 28, 29, 35].

As a consequence of Riemann-Roch’s theorem [13], every closed Riemann surface can be defined by irreducible complex algebraic curves. In the case of a cyclic $p$-gonal Riemann surface $S$, with $p$-gonal automorphism $\varphi$, such an algebraic curve can be constructed as follows. Consider a regular branched cover $\pi : S \to \mathbb{C}$, with $\langle \varphi \rangle$ as its deck group, and let $a_1, \ldots, a_m \in \mathbb{C}$ be its branch values. If none of the points $a_j$ is equal to $\infty$, then there exist integers $n_1, \ldots, n_m \in \{1, \ldots, p-1\}$, $n_1 + \cdots + n_m \equiv 0 \mod p$, so that $S$ is definable by the $p$-gonal curve

$$E = y^p = F(x) = \prod_{j=1}^{m} (x - a_j)^{n_j} \in \mathbb{C}[x],$$

and $\varphi$ and $\pi$ are defined, respectively, in the above model, by

$$\varphi(x, y) = (x, \omega_p y), \quad \text{where} \quad \omega_p = e^{2\pi i/p},$$

$$\pi(x, y) = x.$$
If one of the branch values is equal to $\infty$, say $a_m = \infty$, then in the above we need to delete the corresponding factor $(x - a_m)^{y_m}$ and to have $n_1 + \cdots + n_{m-1} \neq 0 \mod p$. In the case $p = 2$ (i.e., when $S$ is hyperelliptic) one has that $m \in \{2g + 1, 2g + 2\}$ and $n_j = 1$.

If $k_0$ is a subfield of $\mathbb{C}$ so that we may choose $F(x) \in k_0[x]$, then we say that $S$ is cyclically $p$-gonally defined over $k_0$. Note that the field $k_0$ is not uniquely determined by $S$ and it is not clear the existence of a minimal such field for which $S$ is cyclically $p$-gonally defined.

In this paper we provide a simple and short proof of the following fact.

**Theorem 1.1.** Let $S$ be a cyclic $p$-gonal Riemann surface of genus $g \geq 2$, definable over a subfield $k$ of $\mathbb{C}$, and let $\varphi$ be a $p$-gonal automorphism of $S$. Then

1. $S$ is cyclically $p$-gonally definable over an extension of degree at most $2(p - 1)$ of $k$.
2. If both $S$ and $\varphi$ are simultaneously defined over $k$, then $S$ is cyclically $p$-gonally definable over an extension of degree at most 2 of $k$.
3. If, in equation (1) we have that $n_1 = \cdots = n_m = n$, then $S$ is cyclically $p$-gonally definable over an extension of degree at most 2 of $k$.

Let us consider the case $p = 2$, that is, when $S$ is a hyperelliptic Riemann surface of genus $g$. In [27] J-F. Mestre proved that, if $g$ is even and $S$ is definable over $k$, then it is also hyperelliptically definable over $k$. If $g$ is odd, then this fact is in general false as can be seen from the examples in [25, 26]. B. Huggings [21] proved that if the reduced group of automorphisms of $S$ is neither trivial or a cyclic group, then $S$ is also hyperelliptically definable over $k$. In [26] R. Lercier, C. Ritzenthaler and J. Sijsling proved that if the reduced group is a non-trivial cyclic group, then $S$ is always hyperelliptically definable over an extension of degree at most 2 of $k$. Theorem 1.1 completes the above results to the case when the reduced group of automorphisms is trivial.

**Corollary 1.2.** Every hyperelliptic Riemann surface, which definable over a subfield $k$ of $\mathbb{C}$, is hyperelliptically definable over an extension of degree at most 2 of $k$.

## 2. A remark on fields of moduli

Let $S$ be a closed Riemann surface and let $C$ be a curve defining $S$. The field of moduli of $S$ is the fixed field of the group $\Gamma_C = \{\sigma \in \text{Aut}(C/\mathbb{Q}) : C^\sigma \cong C\}$; this field does not depend on the choice of $C$. In [22] S. Koizumi proved that the field of moduli coincides with the intersection of all fields of definitions of $S$ and in [33] J. Wolfart proved that if $S/\text{Aut}(S)$ is the Riemann sphere with exactly 3 cone points (i.e., $S$ is a quasiplatonic Riemann surface), then the field of moduli of $S$ is a field of definition of it.

Let us now assume that $S$ is a $p$-gonal Riemann surface of genus $g \geq 2$, with a $p$-gonal automorphism $\varphi$. In the case that the $p$-group $\langle \varphi \rangle$ is not normal (i.e., not unique), then $S$ can always be $p$-gonally defined over an extension of degree at most 2 of its field of moduli (see Section 3). Next, let us assume $\langle \varphi \rangle$ is a normal subgroup of $\text{Aut}(S)$. In this case, we may consider the quotient group $\text{Aut}(S)/\langle \varphi \rangle$; called the reduced group of $S$. A. Kontogeorgis [23] proved that if the reduced group is neither trivial or a cyclic group, then $S$ can always be defined over its field of moduli. So, a direct consequence of Theorem 1.1 is the following.
Corollary 2.1. If the reduced group of the cyclic \( p \)-gonal Riemann surface \( S \) is different from the trivial group or a cyclic group, then \( S \) is \( p \)-gonally definable over an extension of degree at most \( 2(p - 1) \) of its field of moduli.

Every hyperelliptic Riemann surface can be defined over an extension of degree at most 2 of its field of moduli. In the case that the reduced group is non-trivial, then the hyperelliptic Riemann surface can always be defined over an extension of degree at most 2 of its field of moduli \([21, 26]\). In this way, Theorem 1.1 asserts the following.

Corollary 2.2. Every hyperelliptic Riemann surface is hyperelliptically definable over an extension of degree at most 4 of its field of moduli. Moreover, if either (i) the genus is even or (ii) the genus is odd and the reduced group of automorphisms is neither trivial or a cyclic group, then the hyperelliptic Riemann surface is hyperelliptically defined over an extension of degree 2 of its field of moduli.

Examples of hyperelliptic Riemann surface with trivial reduced group which cannot be defined over their field of moduli were provided by Earle \([11, 12]\) and Shimura \([30]\). Same type of examples, but with non-trivial cyclic reduced group, were provided by Huggins \([21]\).

Remark 2.3. In \([14, 15]\), for the case \( k = \mathbb{Q} \), the authors have provided the following hyperelliptic curves (of genus \( g = 4\gamma - 3 \))

\[ C : y^2 = \prod_{d=4}^{2\gamma+2} \left( x^4 - 2 \left( 1 - \frac{r_3 - r_1 q_d - r_2}{r_3 - r_2 q_d - r_1} \right) x^2 + 1 \right), \]

where \( r_1, r_2, r_3 \) are the zeroes of \( x^3 - 3x + 1 \) and \( q_4, ..., q_{2\gamma+2} \in \mathbb{Q} \) are chosen so that its reduced group is \( H = \langle A(x) = -x, B(x) = 1/x \rangle \cong \mathbb{Z}_2^2 \). None of the branched values of the two-fold branched cover \( \pi : C \to \hat{C}, \pi(x, y) = x \), is a fixed points of an involution of \( H \). In \([15]\) it is proved that the field of moduli of \( C \) is \( \mathbb{Q} \) and claimed that \( C \) cannot be hyperelliptic defined over \( \mathbb{Q}(\sqrt{d}) \), where \( d > 0 \) is an integer. This is not in a contradiction to Corollary 1.2 as the degree 2 extension provided by it may be non-real, i.e., of the form \( \mathbb{Q}(\sqrt{d}) \), with \( d < 0 \). The results in \([14, 15]\) seem to be in contradiction to the results of Huggings \([21]\) as mentioned to the author by C. Ritzenthaler.

Consider the hyperelliptic curve, of genus \( 4\gamma \),

\[ D : y^2 = Q(x) = x(x^4 - 1) \prod_{d=4}^{2\gamma+2} \left( x^4 - 2 \left( 1 - \frac{r_3 - r_1 q_d - r_2}{r_3 - r_2 q_d - r_1} \right) x^2 + 1 \right), \]

which has \( H \) as subgroup of its reduced group of automorphisms. The 6 added points are all the fixed points of all the involutions in \( H \). It can be checked that the field of moduli of \( D \) is contained in \( k \). By results due to Huggins \([21]\), \( D \) can be defined (maybe not in an hyperelliptic form) over its field of moduli, so over \( k \). As the genus of \( D \) is even, by Mestre’s result in \([27]\), \( D \) is hyperelliptically definable over \( k \), that is, there is a Möbius transformation \( T : \hat{C} \to \hat{C} \) so that the image under \( T \) of the zeroes of \( Q \) is invariant under \( \text{Gal}(\overline{k}/k) \). The image of the 6 extra points are
sent to the 6 fixed points of $THT^{-1}$. If $H$ is the reduced group of $D$, then these 6 points are also invariant under $\text{Gal}(\overline{k}/k)$. In that case, it follows that the set of the other points is also invariant under $\text{Gal}(\overline{k}/k)$; but these points provide a hyperelliptic model of $C$ as desired. The problem with the above argument is that the reduced group of $D$ may be larger than $H$.

3. Automorphisms of $p$-gonal Riemann surfaces

Let us consider a cyclic $p$-gonal Riemann surface $S$, of genus $g \geq 2$, together a $p$-gonal automorphism $\varphi$. Let us denote by $m$ the number of fixed points of $\varphi$.

In [16] G. González-Diez proved that the $p$-gonal group $\langle \varphi \rangle$ is unique up to conjugation, that is, if $\vartheta$ is another $p$-gonal automorphism of $S$, then there exists a conformal automorphism $\tau$ of $S$ so that $\tau(\varphi)\tau^{-1} = \langle \vartheta \rangle$.

We say that the $p$-gonal group $\langle \varphi \rangle$ is unique if for any other $p$-gonal automorphism $\vartheta$ of $S$ one has that $\langle \varphi \rangle = \langle \vartheta \rangle$. The uniqueness property has been used by many authors to describe the full group of conformal automorphisms of cyclic $p$-gonal Riemann surfaces [5, 3, 34] and the types of symmetries they may have [6].

Castelnuovo-Severi’s inequality [1, 31] ensures that if $g > (p-1)^2$, equivalently $p < m/2$ (so $m \geq 5$), then $\langle \varphi \rangle$ is unique. Let us note that for each fixed $m$ there are only a finite number of possible primes $p$ satisfying such an inequality (for instance, (i) if $m = 3, 4$, then there are no primes $p$ solving that inequality; (ii) for $m = 5$ the inequality obligates to have $p = 2$, but this case is impossible as involutions have an even number of fixed points and (iii) for $m = 6$ the only possibility is $p = 2$, given by the hyperelliptic involution).

Examples, for $p \geq m/2$, in which the uniqueness of $\langle \varphi \rangle$ is not longer true are known [9, 10, 17] (we will be back to these examples below). In [20] we obtained that, for each $m \geq 3$, there is an integer $q(m)$ so that if $p \geq q(m)$, then the $p$-gonal group is unique. By results due to Lefschetz [24] it can be seen that $q(3) = 11$. In [20] it was noticed that $q(4) = 7$.

In [34] A. Wootton proved that the $p$-gonal group is a normal subgroup (so unique by Gabino’s result in [16]) in most of the cases.

**Theorem 3.1** (Wotton [34]). Let $S$ be a cyclic $p$-gonal Riemann surface of genus $g \geq 2$, let $\vartheta$ be a $p$-gonal automorphism of $S$ and let $m$ be the number of fixed points of $\vartheta$. If either $(m, p)$ is different from (i) $(3, 7)$, (ii) $(4, 3)$, (iii) $(4, 5)$, (iv) $(5, 3)$, (v) $(p, p)$, $p \geq 5$ and (vi) $(2p, p)$ with $p \geq 3$, then $\langle \vartheta \rangle$ is unique.

The cyclic $p$-gonal Riemann surfaces where the above non-normality occur (so, non-uniqueness) are described in the same paper [34]; below we describe them.

1. Case $(m, p) = (3, 7)$ corresponds to quartic Klein’s surface, $x^3y + y^3z + z^3x = 0$, whose group of automorphisms is $\text{PGL}_2(7)$ (of order 168). This surface can be defined by the 7-gonal curve $y^7 = x^2(z - 1)$.

2. Case $(m, p) = (4, 3)$ corresponds to the genus two surface $y^2 = x(x^4 - 1)$, whose group of automorphisms is $\text{GL}_2(3)$ (of order 48). This surface can be defined by the 3-gonal curve $y^3 = (x^2 - 1)(x^2 - 15\sqrt{3} + 26)^2$.

3. Case $(m, p) = (4, 5)$ corresponds to the genus four non-hyperelliptic Riemann surface, called the Bring curve, which is the complete intersection of the quadric $x_1x_4 + x_2x_3 = 0$ and the cubic $x_1^3x_3 + x_2^3x_1 + x_3^3x_4 + x_4^3x_2 = 0$ in the 3-dimensional complex projective space.
Its group of automorphisms is $\mathfrak{S}_5$, the symmetric group in five letters $\mathfrak{S}_5$. This surface can be represented by the 5-gonal curve $y^5 = (x^2 - 1)(x^2 + 1)^4$.

(4) Case $(m, p) = (5, 3)$ corresponds to the genus three non-hyperelliptic closed Riemann surface $x^4 + y^4 + z^4 + 2i\sqrt{3}z^2y^2 = 0$, whose group of automorphisms has order 48. The quotient of that surface by its group of automorphisms has signature $(0; 2, 3, 12)$. This surface can be represented 3-gonally as $y^3 = x^3(x^4 - 1)$.

(5) Case $(m, p) = (p, p)$, where $p \geq 5$, corresponds to the Fermat curve $x^p + y^p + z^p = 0$, whose group of automorphisms is $\mathbb{Z}^2_p \rtimes \mathfrak{S}_3$. This is already in a $p$-gonal form (take $z = 1$) as $y^p = -1 - x^p$.

(6) Case $(m, p) = (2p, p)$, where $p \geq 3$. There is a 1-dimensional family with group of automorphisms $\mathbb{Z}^2_p \rtimes \mathbb{Z}^2_2$ (the quotient by that group has signature $(0; 2, 2, 2, p)$). Also, there is a surface with group of automorphisms $\mathbb{Z}^2_p \rtimes D_3$ (the quotient by that group has signature $(0; 2, 4, 2p)$). These surfaces are $p$-gonally given as $y^p = (x^p - a^p)(x^p - 1/a^p)$.

We should note that in all the above exceptional cases the surface $S$ is $p$-gonally defined over an extension of degree at most 2 over the field of moduli.

4. Proof of Theorem 1.1

As we are assuming that $S$ is definable over the subfield $k$ of $\mathbb{C}$, we may assume that it is represented by a curve $C$ defined over $k$ and that $\varphi$ be a $p$-gonal automorphism of $C$ (defined over the algebraic closure $\overline{k}$ of $k$ inside $\mathbb{C}$). We only need to assume that $\langle \varphi \rangle$ is unique (by the observations in the previous section).

Let us take a regular branched covering, say $\pi : C \to \mathbb{P}^1_\overline{k}$, with $\langle \varphi \rangle$ as its deck group and whose branch values are $a_1, \ldots, a_m$, and let the integers $n_1, \ldots, n_m \in \{1, \ldots, p-1\}$, $n_1 + \cdots + n_m \equiv 0 \mod p$, so that $C$ is isomorphic to the $p$-gonal curve $E$ as in (1).

4.1. Proof of Part (1). It follows, from the uniqueness of $\langle \varphi \rangle$, that for each $\sigma \in \Gamma = \text{Aut}(\overline{k}/k)$ it holds that $\varphi^\sigma$ is a power of $\varphi$; in particular, that $\varphi$ is defined over an extension $k_1$ of $k$ of degree at most $p - 1$. Set $\Gamma_1 = \text{Gal}(\overline{k}/k_1)$.

If $\sigma \in \Gamma_1$, then the identity $I : C \to C = C^\sigma$ defines a M"{o}bius transformation $g_\sigma \in \text{PGL}_2(\overline{k})$ such that $\pi^\sigma = g_\sigma \circ \pi$. The transformation $g_\sigma$ is uniquely determined by $\sigma$; so the collection $\{g_\sigma\}_{\sigma \in \Gamma_1}$ satisfies the co-cycle relations

$$g_{\sigma \tau} = g_\tau \circ g_\sigma, \quad \sigma, \tau \in \Gamma_1.$$  

Weil’s descent theorem [32] ensures the existence of a genus zero irreducible and non-singular algebraic curve $B$, defined over $k_1$, and an isomorphism $R : \mathbb{P}^1_{\overline{k}} \to B$, defined over $\overline{k}$, so that

$$g_\sigma \circ R^\sigma = R, \quad \sigma \in \Gamma_1.$$  

Clearly, for $\sigma \in \Gamma_1$, we have the equality $\{\sigma(a_1), \ldots, \sigma(a_m)\} = \{g_\sigma(a_1), \ldots, g_\sigma(a_m)\}$, so it follows that $\{R(a_1), \ldots, R(a_m)\}$ is $\Gamma_1$-invariant.

Let us denote by $A(n_j)$ the set of those $a_l$’s for which $n_k = n_j$.

If, for $\sigma \in \Gamma_1$, we consider the cover $g_\sigma \circ \pi : C \to \overline{C}$, then we may see that the set $g_\sigma(A(n_j))$ corresponds to the set of those $\sigma(a_k)$ having the same $n_l$ (for some $l$); that is, $g_\sigma(A(n_j)) = \sigma(A(n_j))$.

As $\varphi^\sigma = \varphi$, we must have $n_1 = n_j$; that is, $g_\sigma$ and $\sigma$ both sends $A(n_j)$ to the same set. In particular, the set $R(A(n_j))$ is also $\Gamma_1$-invariant.
Let $\omega$ be a $k_1$-rational meromorphic 1-form in $B$. Since the canonical divisor $K = (\omega)$ is $k_1$-rational of degree $-2$, then for a suitable positive integer $d$, the divisor $D = R(a_1) + \cdots + R(a_m) + dK$ is $k_1$-rational of degree 1 or 2. Riemann-Roch’s theorem ensures the existence of an isomorphism $f : B \to \mathbb{P}^1_k$, defined over $k_1$, so that $U = (f) + D \geq 0$. As the divisor $U$ is $k_1$-rational of degree 1 or 2 and $U > 0$, the divisor $U$ has three possibilities:

1. $U = s$, where $s \in B$ is $k_1$-rational; or
2. $U = 2p$, where $p \in B$ is $k_1$-rational; or
3. $U = r + q$, where $r, q \in B$, $r \neq q$, and $(r, q)$ is $\Gamma_1$-invariant.

In cases (1) and (2) we have the existence of a $k_1$-rational point in $B$. In this case, we set $k_2 = k_1$.

In case (3) we have a point (say $r$) in $B$ which is rational over an extension $k_2$ of degree 2 of $k_1$. The existence of a $k_2$-rational point ensures the existence (again by Riemann-Roch’s theorem) of an isomorphism $L : B \to \overline{\mathbb{C}}$, defined over $k_2$. The sets $\{L(R(a_1)), \ldots, L(R(a_m))\}$ and $L(R(A(n_j)))$ are $\Gamma_2$-invariant, where $\Gamma_2 = \text{Gal}(\overline{k}/k_2)$. It follows that $q(x) = \prod_{j=1}^{m} \left(x - L(R(a_j)) \right) \in k_2[x]$ and that $C$ is isomorphic to $y^0 = q(x)$.

4.2. Proof of Parts (2) and (3). If in Equation (1) we have that $n_1 = \cdots = n_m = n$, then we work as above but with $\Gamma$ instead of $\Gamma_1$. As in this case, there will be only one set $A(n)$, we may proceed similarly as above to obtain the desired result. Similarly if we assume $\varphi$ is already defined over $k$.

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