$K\bar{K}$ scattering amplitude to one loop in chiral perturbation theory, its unitarization and pion form factors

Francisco Guerrero and José Antonio Oller
Departament de Física Teòrica, Universitat de València and IFIC, CSIC - Universitat de València, C/ del Dr. Moliner 50, E-46100 Burjassot (València), Spain

Abstract
We have calculated the $K\bar{K} \rightarrow K\bar{K}$ scattering amplitude to next to leading order in Chiral Perturbation Theory. Then, making use of a unitarization procedure with one or several coupled channels ($\pi\pi$ in our case) we have calculated the $\pi\pi \rightarrow \pi\pi$, $\pi\pi \rightarrow K\bar{K}$ and $K\bar{K} \rightarrow K\bar{K}$ $S$ and $P$ waves in good agreement with the experiment up to $\sqrt{s} \simeq 1.2$ GeV.

The $\pi\pi$ scattering lengths with isospin and spin $(I,J)$ equal to $(0,0)$, $(1,1)$ and $(2,0)$ are also calculated in agreement with experiment and former Chiral Perturbation Theory calculations.

Finally we have employed these amplitudes, making use of an Omnès representation, to calculate the scalar and the vector pion form factors, obtaining a good agreement with the available experimental data.

PACS numbers: 12.39.Fe, 13.75.Lb, 13.40.Gp, 11.80.Gw
Keywords: Chiral Symmetry, Unitarization, Coupled channels, Form Factors

May 1998
1. Introduction

The low energy effective theory of the strong interactions (QCD) is Chiral Perturbation Theory (ChPT). The chiral symmetry constraints [1] are a powerful tool, enough to determine the low energy matrix elements in a systematic way [2, 3, 4, 5, 6]. In this way one can evaluate the scattering amplitudes corresponding to the lightest octet of pseudoscalar mesons \((\pi, K, \eta)\) in a loop expansion. This has already been done for the \(\pi\pi \rightarrow \pi\pi\) scattering amplitude for the \(SU(2)\) case in [4] and extended to \(SU(3)\) in [6]. The \(K\pi \rightarrow K\pi\) scattering amplitude is also evaluated in [6].

This loop expansion can also be seen as an expansion in powers of the masses and external tetramomenta of the pseudoscalars over some typical scale around 1 GeV, so that the series is, in principle, only useful for low energy. Furthermore, in the meson-meson scattering, it is common the appearance of resonances with a typical mass around 1 GeV, which obviously can not be reproduced in a power expansion because they correspond to poles in the T matrix. The region of validity of the series depends on the channel \((I,J)\) considered, where \(I\) accounts for the isospin and \(J\) for the angular momentum. A way to get a hint about this convergence problem is to calculate the next to leading contribution \((O(p^4))\) and compare it with the lowest order contribution \((O(p^2))\). This seems logical and is the reason advocated in [7] to justify their \(K\pi O(p^4)\) calculation even though the threshold for this reaction is 0.64 GeV and typically the chiral expansions break around \(\sqrt{s} \sim 0.5 - 0.6\) GeV.

In this paper we first calculate the \(K\bar{K} \rightarrow K\bar{K}\) scattering amplitude at \(O(p^4)\) in ChPT. This implies to evaluate two amplitudes due to the fact that both \(I=0,1\) are possible. In this way we calculate the \(K^+K^- \rightarrow K^+K^-\) and \(K^+K^- \rightarrow K^0\bar{K}^0\) scattering amplitudes. It is not necessary to calculate the \(K^0\bar{K}^0 \rightarrow K^0\bar{K}^0\) amplitude because in the isospin limit \(T(K^+K^- \rightarrow K^+K^-) = T(K^0\bar{K}^0 \rightarrow K^0\bar{K}^0)\).

This calculation seems in principle to be unreasonable since the \(K\bar{K}\) threshold is almost 1 GeV, and furthermore, for \(I=0,1\) at this energy the \(f_0(980), a_0(980)\) resonances appear which couple strongly to the \(K\bar{K}\) channel. In fact, after making this \(O(p^4)\) calculation, we will see that, at this energy, the \(O(p^4)\) result is larger than the \(O(p^2)\) lowest order calculation. Obviously, what all this is telling us is that one cannot rely in the perturbative chiral expansion at these energies. This implies that one cannot compare the predictions directly with the experiment, using the \(O(p^2)\) plus \(O(p^4)\) ChPT amplitudes.
In the last years, there have been several attempts to extend ChPT to higher energies. One of them uses an effective field theory with explicit resonance fields as degrees of freedom \[8, 9\]. This resonance chiral effective theory has one drawback: applying only the constraints coming from the symmetry the number of free parameters grows so fast with higher orders that in practice predictions become impossible. However, using this lagrangian at \(O(p^4)\) and imposing some short distance QCD constraints \[9\] one can obtain interesting and good results. For example, the study of the vector pion form factor done in \[10\].

Another attempt (the one we are going to use here) is to unitarize the ChPT amplitudes. In \[11, 12, 13\] the Inverse Amplitude Method (IAM) was developed and used to study the \(\pi\pi\) and \(K\pi\) scattering. This method has the problem that only elastic unitarization is employed, so that, for those cases where inelastic thresholds are important, as the \(KK\) one in \(I = J = 0\), the improvement over ChPT is poor. But for those other channels predominantly elastic, as the \(I = J = 1\), the improvement is remarkable and, in fact, in this channel the \(\rho\) resonance is obtained. In order to use the IAM one needs the \(O(p^4)\) ChPT amplitude. More recently, in \[14\], a resummation of the chiral series was done inspired in the Bethe-Salpeter (BS) equations for the \(J = 0\) sector making use only of the \(O(p^2)\) ChPT amplitudes and imposing unitarity with coupled channels, \(\pi\pi\) and \(KK\). The agreement with the \(I = J = 0\) and \(I = 1, J = 0\) experimental data was very good, reproducing the presence of the \(f_0(980)\) and \(a_0(980)\) resonances respectively. However, with this method the higher orders are obtained through unitarity loops in the \(s\) channel which are subleading in the large \(N_c\) limit and then fails in the vector sector where the large \(N_c\) limit works very well. The leading part in this latter limit are tree level contributions appearing through the chiral lagrangians. In fact in \[7\] it is seen that in the \(J = 1\) \(K\pi\) scattering the \(O(p^4)\) corrections are clearly dominated by the polynomial counterterms contribution coming from the \(O(p^4)\) chiral lagrangian. This same phenomenon is also seen in the P-wave \(\pi\pi\) scattering and in the vector form factor \[2, 10, 15, 16, 17\]. It is clear then that a unitarization procedure which could include the success of the IAM and BS approaches, or equivalently, a method that could handle unitarization in coupled channels incorporating both the \(O(p^2)\) and \(O(p^4)\) chiral amplitudes would be very welcome. This in fact has been recently done in \[18\] and the resonances appearing in both \(J = 1\) (\(\rho(770)\) and \(K^*(890)\)) and \(J = 0\) (\(a_0(980), f_0(980)\)) channels were reproduced. This is in fact the method we are going to use here. The novelties are that while in \[18\] the \(O(p^4)\) chiral amplitudes were approximated in a way inspired in \[14\], here we are going to use the exact \(\pi\pi\)
and $K\pi$ scattering amplitudes $[3,7]$ and the $K\bar{K}$ ones, which we calculate here to $O(p^4)$. With these ingredients we obtain the scattering amplitudes for the channels $(I,J)=(0,0)$, $(1,1)$ and $(2,0)$, generating dynamically the $f_0$ and $\rho$ resonances. The agreement with experiment is good up to around $\sqrt{s} \simeq 1.2$ GeV as it will show.

Finally, making use of the calculated $\pi\pi \rightarrow \pi\pi$ phase shifts we derive an Omnès representation $[13,20]$ for the scalar and vector pion form factors based on the Watson final state theorem $[21]$. For the case of the vector form factor experimental data are available and, with our calculation, the agreement is rather good. In the case of the scalar form factor we compare the calculation making the unitarization with and without including the $K\bar{K}$ channel and the differences are very significative, even at energies around $|\sqrt{s}| = 500$ MeV. The $f_0(980)$ resonance appears clearly in the scalar form factor of the pion.

2. The $K\bar{K} \rightarrow K\bar{K}$ scattering amplitude to next to leading order in ChPT

As it was stated before, in order to calculate the $K\bar{K}$ amplitude, we need to evaluate two amplitudes. These two independent isospin amplitudes cannot be connected by crossing symmetry because they have different absolute values for the strangeness. This is contrary to what happens in $K\pi$ scattering with $I = 3/2$, $I = 1/2$.

We calculate the amplitudes $K^+K^- \rightarrow K^+K^-$ and $K^+K^- \rightarrow K^0\bar{K}^0$ which we denote by $T_{cc}$ and $T_{cn}$ respectively.

The scattering amplitudes with definite isospin $T^{(I)}$ can be written in terms of $T_{cc}$ and $T_{cn}$ in the following way:

$$T^{(0)}(s,t,u) = T_{cc}(s,t,u) + T_{cn}(s,t,u)$$

$$T^{(1)}(s,t,u) = T_{cc}(s,t,u) - T_{cn}(s,t,u) \quad (1)$$

We now proceed to describe the calculation scheme for these amplitudes up to $O(p^4)$.

At lowest order one has the ChPT lagrangian at $O(p^2)$

$$\mathcal{L}_2 = \frac{f_0^2}{4} \left( \partial_\mu U^\dagger \partial^\mu U + \mathcal{M} \left( U + U^\dagger \right) \right)$$

where $\langle \rangle$ stands for the trace of the $3 \times 3$ matrices built from $U(\Phi)$ and $\mathcal{M}$,
\[ U(\Phi) = \exp \left( \frac{i\sqrt{2}}{f_0} \Phi \right) \]  

(3)

where \( \Phi \) is expressed in terms of the Goldstone boson fields as

\[ \Phi(x) = \begin{bmatrix} \frac{1}{\sqrt{2}} \pi^0 + \frac{1}{\sqrt{6}} \eta \\ \pi^- \\ K^- \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} \eta + \frac{1}{\sqrt{6}} \eta \\ \pi^0 \\ K^0 \end{bmatrix} \]

(4)

The mass matrix \( \mathcal{M} \) is given by

\[ \mathcal{M} = \begin{bmatrix} \hat{m}_\pi^2 & 0 & 0 \\ 0 & \hat{m}_\eta^2 & 0 \\ 0 & 0 & 2\hat{m}_K^2 - \hat{m}_\pi^2 \end{bmatrix} \]

(5)

in the isospin limit. Where \( \hat{m} \) means bare masses.

From this lagrangian we can evaluate the lowest order contribution to the \( K\bar{K} \) scattering amplitudes

\[ T_{cc,2}(s, t, u) = \frac{1}{f_0^2} \left[ \frac{2}{3}\hat{m}_K^2 + \frac{4}{3}\hat{m}_K^2 - u \right] \]

\[ T_{cn,2}(s, t, u) = \frac{1}{2f_0^2} \left[ \frac{2}{3}\hat{m}_K^2 + \frac{4}{3}\hat{m}_K^2 - u \right] \]

(6)

where the subindex 2 means \( \mathcal{O}(p^2) \).

At \( \mathcal{O}(p^2) \) \( f_0 = f_\pi = 93.3 \) MeV and \( \hat{m}_K = m_K = 495.7 \) MeV. But when we go to next order these equalities do not hold. This is the reason why we keep the distinction between bare and physical masses.

At \( \mathcal{O}(p^4) \) one has to calculate the diagrams schematically shown in Fig. 1. \( T_4^T \) represents contributions coming from the \( \mathcal{L}_2 \) ChPT lagrangian with six fields and a tadpole loop. The \( T_4^U \) represents the loops constructed from
the $L_2$ amplitudes with four fields appearing in the vertices of the loop. We will call this contribution unitarity loops because it makes the amplitude unitary at $O(p^4)$. These loops include contributions from loops in the $s,t$ and $u$ channels, as shown in Fig. 2.

\begin{align*}
L_4 = & L_1 \langle \partial_\mu U^\dagger \partial^\mu U \rangle^2 + L_2 \langle \partial_\mu U^\dagger \partial_\nu U \rangle \langle \partial^\mu U^\dagger \partial^\nu U \rangle \\
& + L_3 \langle \partial_\mu U^\dagger \partial^\mu U \partial_\nu U^\dagger \partial^\nu U \rangle + L_4 \langle \partial_\mu U^\dagger \partial^\mu U \rangle \langle U^\dagger M + M^\dagger U \rangle \\
& + L_5 \langle \partial_\mu U^\dagger \partial^\mu U \left( U^\dagger M + M^\dagger U \right) \rangle + L_6 \langle U^\dagger M + M^\dagger U \rangle^2 \\
& + L_7 \langle U^\dagger M - M^\dagger U \rangle^2 + L_8 \langle M^\dagger U M^\dagger U + U^\dagger M U + M \rangle
\end{align*}

When taking into account the wave function renormalization, Fig. 3, and the relation between bare and physical masses and decay constants, other $O(p^4)$ contributions come from the lowest order amplitudes.

\begin{align*}
\text{Figure 3: Wave function renormalization}
\end{align*}

The relation between $m_K$ and $\hat{m}_K$ and the one between $f_0$ and $f_\pi$ at $O(p^4)$ can be obtained from [3].
The final amplitude up to next to leading order is then obtained summing all the former contributions. We express it divided in four parts: $T_2$, $T_4^T$, $T_4^U$ and $T_4^P$ where the subindices indicate the order in powers of momentum.

In $T_4^T$ and $T_4^P$ we have also included, in addition to the one coming from Fig. 1, the $O(p^4)$ contributions from the renormalization of wave function, masses and decay constants. The ones with $L_i$ parameters are included in $T_4^P$ and the rest, of tadpole type, in $T_4^T$.

The amplitude for $K^+K^- \rightarrow K^+K^-$ is:

$$T_{cc, 2}(s, t, u) = \frac{2m_K^2 - u}{f_\pi^2}$$ \hspace{1cm} (8)

$$T_{cc, 4}^{T} = \frac{A_r}{288f_\pi^4 \pi^2} (8m_K^2 + m_\pi^2 - 3u) + \frac{A_r}{288f_\pi^4 \pi^2} (8m_K^2 + 3u) \hspace{1cm} (9)$$

$$T_{cc, 4}^{U} = -\frac{20m_K^4 - 2s^2 - 2su + u(3m_\pi^2 + u) + m_K^2 (8s + 11u - 4m_\pi^2)}{192f_\pi^4 \pi^2} \hspace{1cm} (10)$$

$$T_{cc, 4}^{P} = L_1 \frac{8}{f_\pi^4} (-8m_K^4 + s^2 + t^2 + 4m_K^2 u) + L_2 \frac{4}{f_\pi^4} (s^2 + t^2 + 2u^2 - 4m_K^2 u)$$
\[ +L_3 \frac{4}{f_\pi^4} (-8m_K^4 + s^2 + t^2 + 4m_K^2 u) \]
\[ -L_4 \frac{16m_K^2 u}{f_\pi^4} - L_5 \frac{8m_K^2 u}{f_\pi^4} + (2L_6 + L_8) \frac{32m_K^4}{f_\pi^4} \quad (11) \]

The amplitude for \( K^+ K^- \rightarrow K^0 \bar{K}^0 \) is
\[ T_{cn,2} = \frac{2m_K^2 - u}{2f_\pi^2} \quad (12) \]

\[ T_{cn,4}^T = \frac{A_r^r}{576f_\pi^4 \pi^2} (-4m_K^2 + m_\pi^2 + 6t) + \frac{A_r^r}{288f_\pi^4 \pi^2} (10m_K^2 - 3t) \]
\[ + \frac{A_r^\eta}{576f_\pi^4 \pi^2} (20m_K^2 - m_\pi^2 - 6u) \quad (13) \]

\[ T_{cn,4}^U = \frac{24m_K^4 - 6m_\pi^2 s - 2s^2 - 2s u + u^2 + 2m_K^2 (4m_\pi^2 - 7t)}{384f_\pi^4 \pi^2} \]
\[ + \frac{A_r^r}{1152f_\pi^4 \pi^2} (38m_K^2 - 2m_\pi^2 - 6s - 15u) \]
\[ + \frac{A_r^r}{576f_\pi^4 \pi^2} (10m_K^2 + 3s - 6u) + \frac{A_r^\eta}{1152f_\pi^4 \pi^2} (22m_K^2 + 2m_\pi^2 - 15u) \]
\[ + \frac{B_r^s(s)}{1536f_\pi^4 \pi^2} (8m_K^2 (s - 4m_\pi^2) + s(7s - 4t) + 8m_\pi^2 (s + 2t)) \]
\[ + \frac{B_r^s(t)}{384f_\pi^4 \pi^2} (s - u)(t - 4m_\pi^2) + \frac{B_r^K(s)}{96f_\pi^4 \pi^2} (-8m_K^2 + s(s - u) + 4m_K^2 (s + u)) \]
\[ + \frac{B_r^K(t)}{384f_\pi^4 \pi^2} (-8m_K^2 + t(t - u) + 4m_K^2 (t + u)) + \frac{B_r^K(u)}{64f_\pi^4 \pi^2} (u - 2m_K^2)^2 \]
\[ + \frac{B_r^\pi(s)}{4608f_\pi^4 \pi^2} (8m_K^2 - 9s)^2 - \frac{B_r^\pi(s)}{768f_\pi^4 \pi^2} (3s - 4m_K^2)^2 \]
\[ + \frac{B_r^\pi(t)}{384f_\pi^4 \pi^2} (3t - 4m_K^2)^2 \quad (14) \]

\[ T_{cn,4}^P = L_1 \frac{8}{f_\pi^4} (s - 2m_K^2)^2 \]
\[ + L_2 \frac{4}{f_\pi^4} (-8m_K^4 + t^2 + u^2 + 4m_K^2 s) \]
\[ +L_3 \frac{2}{f_\pi^2} (-8m_K^4 + s^2 + t^2 + 4m_K^2u) \\
+L_4 \frac{4}{3f_\pi^4} (-24m_K^4 + 12m_K^2s) - L_5 \frac{4m_K^2u}{f_\pi^4} \\
+(2L_6 + L_8) \frac{16m_K^4}{f_\pi^4} \]  

(15)

In the above formulas we have used the quantities

\[ A_P^r = -m_P^2 \left[ -1 + \ln \left( \frac{m_P^2}{\mu^2} \right) \right] \]  

(16)

\[ B_P^{rPQ}(s) = \frac{\lambda^{1/2}(s, m_P^2, m_Q^2)}{2s} \ln \left( \frac{m_P^2 + m_Q^2 - s + \lambda^{1/2}(s, m_P^2, m_Q^2)}{m_P^2 + m_Q^2 - s - \lambda^{1/2}(s, m_P^2, m_Q^2)} \right) \\
+2 - \ln \left( \frac{m_P^2}{\mu^2} \right) + \frac{m_P^2 - m_Q^2 + s}{2s} \ln \left( \frac{m_Q^2}{m_P^2} \right) \]  

(17)

In the equal mass limit (17) reduces to

\[ B_P(s) = 2 - \ln \left( \frac{m_P^2}{\mu^2} \right) - \sigma(m_P^2, s) \ln \left( \frac{\sigma(m_P^2, s) + 1}{\sigma(m_P^2, s) - 1} \right) \]  

(18)

where

\[ \lambda(s, m_P^2, m_Q^2) = [s - (m_P + m_Q)^2][s - (m_P - m_Q)^2] \]

\[ \sigma(m_P^2, s) = \sqrt{1 - \frac{4m_P^2}{s}} \]  

(19)

The functions (16) and (17) come from the Passarino-Veltman integrals with one and two propagators [22].

It is interesting to note that \( L_7 \) does not appear in \( T_{4P}^P \). This also happens in \( \pi\pi \to \pi\pi \) and \( K\pi \to K\pi \mathcal{O}(p^4) \) scattering amplitudes. \( L_6 \) and \( L_8 \) appear in the combination \( 2L_6 + L_8 \) as a consequence of the Kaplan and Manohar symmetry [23].

As it was stated before, the corrections coming from the \( \mathcal{O}(p^4) \) calculation are, at least, as large as the lowest order contribution itself. For example in the \( KK \) threshold:

\[ T_{cc,2} = 56.5 \]
\[ T_{cc,A} = 73.6 + i \, 36.74 \]

Unambiguously this means that a perturbative calculation is useless in this region and that some non perturbative scheme should be used in order to compare with the experimental phenomenology.

3. Unitarization of the $\pi\pi$ and $K\bar{K}$ amplitudes

We already stated in the introduction that we are going to use a unitarization procedure recently developed in [18] and thoroughly used and applied to phenomenology in [24], where the S and P waves meson-meson amplitudes were reproduced successfully. However, in both works, the $\mathcal{O}(p^4)$ amplitudes were approximated. Here we are going to use our calculated $\mathcal{O}(p^4)$ $K\bar{K}$ amplitude together with the $\pi\pi$ and $K\pi$ $\mathcal{O}(p^3)$ scattering amplitudes given in [7].

In [18] a rather general scheme was derived to obtain final unitarized amplitudes from the $\mathcal{O}(p^2)$ and $\mathcal{O}(p^4)$ ChPT scattering amplitudes. For our case with two channels, labelled as 1 for $\pi\pi$ and 2 for $K\bar{K}$, the amplitude with definite $I,J$ is given by

\[ T^{(I,J)} = T_2^{(I,J)} \cdot \left[ T_2^{(I,J)} - T_4^{(I,J)} \right]^{-1} \cdot T_2^{(I,J)} \]  \hspace{1cm} (20)

where the different amplitudes are actually $2 \times 2$ matrices for $I=0, 1$, according to the above labelling of the states, and just numbers for $I=2$.

Note that (20) was obtained from an expansion of the $1/T$ matrix amplitude, which has a zero when $T$ has a pole. In this way one expects that the $1/T$ expansion in powers of masses and momenta is meaningful even above the resonance region, where the low energy expansion of $T$ has no sense.

The projection in definite angular momentum is given by

\[ T^{(I,J)} = \frac{1}{32N\pi} \int_{-1}^{1} d(cos \theta) T^{(I)}(s,t) P_J(cos \theta) \]  \hspace{1cm} (21)

where $N = 1$ for $K\bar{K} \rightarrow K\bar{K}$, $N = \sqrt{2}$ for $\pi\pi \rightarrow K\bar{K}$ and $N = 2$ for $\pi\pi \rightarrow \pi\pi$, since in the isospin formalism the pions are identical particles.

With this normalization, in our case unitarity reads, for $(I,J) = (0,0)$

\[ 4m_{\pi}^2 < s < 4m_K^2 : \hspace{1cm} \text{Im} T_{11} = \sigma(m_{\pi}^2, s) |T_{11}|^2 \]  \hspace{1cm} (22)
\[ 4m_K^2 < s < 4m_{\eta}^2 : \hspace{1cm} \text{Im} T_{12} = \sigma(m_{\pi}^2, s) T_{11} T_{12}^* + \sigma(m_K^2, s) T_{12} T_{22}^* \]
\[ \text{Im} T_{22} = \sigma(m_{\pi}^2, s) |T_{12}|^2 + \sigma(m_K^2, s) |T_{22}|^2 \]
\[ \text{Im} T_{11} = \sigma(m_{\pi}^2, s)|T_{11}|^2 + \sigma(m_{K}^2, s)|T_{12}|^2 \quad (23) \]

For \((I,J)=(1,1)\)

\[ 4m_{\pi}^2 < s < 4m_{K}^2 : \quad \text{Im} T_{11} = \sigma(m_{\pi}^2, s)|T_{11}|^2 \quad (24) \]

\[ 4m_{K}^2 < s : \quad \text{Im} T_{12} = \sigma(m_{\pi}^2, s)T_{11}T_{12}^* + \sigma(m_{K}^2, s)T_{12}T_{22}^* \]

\[ \text{Im} T_{22} = \sigma(m_{\pi}^2, s)|T_{12}|^2 + \sigma(m_{K}^2, s)|T_{22}|^2 \]

\[ \text{Im} T_{11} = \sigma(m_{\pi}^2, s)|T_{11}|^2 + \sigma(m_{K}^2, s)|T_{12}|^2 \quad (25) \]

And for \((I,J)=(2,0)\)

\[ 4m_{\pi}^2 < s : \quad \text{Im} T_{11} = \sigma(m_{\pi}^2, s)|T_{11}|^2 \quad (26) \]

with \(\sigma\) given by (19).

In matrix notation (23) and (25) become

\[ \text{Im} T = T \sigma T^* \quad (27) \]

with \(\sigma = (\sigma(m_{\pi}^2, s), \sigma(m_{K}^2, s))\) a diagonal matrix.

In deriving (21) in [18] one was concerned essentially with the right hand cut, responsible for unitarity in the corresponding channel. As a result, the imaginary part of the amplitudes to be used in (20), above the lightest threshold (in our case the \(\pi\pi\) one, \(s = 4m_{\pi}^2\)) and below the highest one (\(s = 4m_{K}^2\)), was restricted to come only from unitarity, (23), neglecting the left hand cut contribution to the imaginary part that appears in \(T_{cc,4}\) and \(T_{cn,4}\) for \(s < 4m_{K}^2 - 4m_{\pi}^2\).

However, we have maintained the left hand cut contribution to the imaginary part of \(T_{cc,4}\) and \(T_{cn,4}\) below the \(KK\) threshold. One way to see how large is the resulting deviation from unitarity is to check the value of the inelasticity in the energy region \(4m_{\pi}^2 < s < 4m_{K}^2\) for \((I,J)=(0,0)\) and \((1,1)\) where two channels appear. In both cases the deviation from 1 is smaller than 1%.

The \(\eta\eta\) intermediate state has been included in the \(O(p^4)\) ChPT amplitudes. However, for the \((0,0)\) channel for \(\sqrt{s} > 2m_{\eta}\) this state gives further contribution to the imaginary part of our amplitudes in addition to the one expressed in (23). This means that (21) with only the \(\pi\pi\) and \(KK\) states does not fulfill unitarity strictly for \(\sqrt{s} > 2m_{\eta} \simeq 1.1\) GeV. The influence of the \(\eta\eta\) state is particularly significative in the \(KK \rightarrow \pi\pi\) S-wave phase shifts, Fig. 6. We will come back to this point later.
Omitting the $I, J$ labels, the relation between the $T$ and the $S$ matrix elements for a two channel process, in our case $\pi\pi$ and $K\bar{K}$ for $(I,J) = (0,0)$ and $(1,1)$, is given by

\begin{align}
S_{11} &= 1 + 2i \sigma(m_{\pi}^2, s)T_{11} \\
S_{22} &= 1 + 2i \sigma(m_{K}^2, s)T_{22} \\
S_{12} &= 2i \sqrt{\sigma(m_{\pi}^2, s)\sigma(m_{K}^2, s)}T_{12}
\end{align}

(28)

To accomplish unitarity the $2 \times 2$ $S$-matrix can be written [25] as

\begin{align}
S &= \begin{bmatrix}
\eta e^{2i\delta_1} & i(1 - \eta^2)^{1/2}e^{i(\delta_1 + \delta_2)} \\
i(1 - \eta^2)^{1/2}e^{i(\delta_1 + \delta_2)} & \eta e^{2i\delta_2}
\end{bmatrix}
\end{align}

(29)

From (28) and (29) we obtain the phase shifts for $\pi\pi \rightarrow \pi\pi$ ($\delta_1$) and $\pi\pi \rightarrow K\bar{K}$ ($\delta_1 + \delta_2$) for $(I,J) = (0,0)$ and $(1,1)$. For $(I,J) = (2,0)$ only the $\pi\pi$ channel is necessary, when omiting multipion states. In this case, it is enough to consider the first equality of (28) to obtain the phase shifts with $S_{11} = e^{i2\delta}$.

4. Fit, phase shifts and inelasticity

In ChPT the experimental values for the $L_i$ coefficients come from $O(p^4)$ fits to low energy experimental data. Here we fit the $L_i$ constants to experiment in a much broader energy interval and with an expression valid to all orders. Hence, differences are expected between our fitted $L_i$ parameters and the values quoted from ChPT.

Furthermore, our approach is not cross symmetric. This implies that contributions from the left hand cut of order higher than $O(p^4)$ are effectively reabsorbed in the values of our $L_i$ coefficients. This point has been studied in [28] with the conclusion that the value of the $L_i$ obtained from a non cross symmetric method are influenced by this reabsorption procedure of the left hand cut. In this way, the value we quote for our $L_i$ constants has to be taken with care when comparing with the values of the $L_i$ from ChPT.

We have used simultaneously the phase shifts of the $\pi\pi \rightarrow \pi\pi$ with $I=0$ and 1, Figs. 4 and 5, to fit the value of our free parameters: $L_1, L_2, L_3, L_4, L_5$ and $2L_6 + L_8$. The fit has been done using MINUIT. In the energy region $\sqrt{s} = 500$-950 MeV the data from different experiments for $S$-wave $\pi\pi$ phase shifts are incompatible. Given that situation, we have taken as central value for each energy the mean value between the different experimental results [29-34]. For $\sqrt{s} = 0.95$-1 GeV, the mean value comes from [31, 33]. In both
cases the error is the maximum between the experimental errors and the
largest distance between the experimental points and the average value.

The quoted errors in the value we have obtained for the $L_i$ coefficients
is just the statistical one.

The fit is pretty good, as can be seen in Figs. 4 and 5, with $\chi^2 = 1.3$
per degree of freedom. The values we obtain at the $M_\rho$ scale and in units of
$10^{-3}$ are

\[
\begin{align*}
L_1 &= 0.72^{+0.03}_{-0.02} \\
L_2 &= 1.36^{+0.02}_{-0.05} \\
L_3 &= -3.24 \pm 0.04 \\
L_4 &= 0.20 \pm 0.10 \\
L_5 &= 0.0^{+0.8}_{-0.4} \\
2L_6 + L_8 &= 0.00^{+0.26}_{-0.20}
\end{align*}
\]

The small errors for $L_1$, $L_2$ and $L_3$ are due to the strong constraints imposed
by the small errors in the experimental data of the $\delta_{11,\pi\pi}$ phase shift, Fig. 4.

The values of ChPT are

\[
\begin{align*}
L_1 &= 0.4 \pm 0.3 \\
L_2 &= 1.4 \pm 0.3 \\
L_3 &= -3.5 \pm 1.1 \\
L_4 &= -0.3 \pm 0.5 \\
L_5 &= 1.4 \pm 0.5 \\
2L_6 + L_8 &= 0.5 \pm 0.7
\end{align*}
\]

We can see that our values, taking errors into account, are compatible
with those from ChPT, and then we can guarantee a good behaviour at low
energies for our predictions.

Using these values for $L_i$ we also describe correctly phase shifts, scattering
lengths and form factors for the $I = J = 0$ and $I = J = 1$ channels, as
we will see later.

In Fig. 4 we show our fitted $\delta_1$ for $I = J = 1$, from the two pion
threshold up to 1.2 GeV and we see a good agreement with the experimental
data which are dominated by the presence of the $\rho(770)$ that we reproduce
nicely. In this channel we obtain that the influence of the $K\bar{K}$ channel coupled to $\pi\pi$ is negligible.

Figure 4: Phase shift for $\pi\pi \rightarrow \pi\pi$ in $I = J = 1$. Data: [34].

Figure 5: Phase shift for $\pi\pi \rightarrow \pi\pi$ in $I = J = 0$. Data: empty pentagon [29], empty circle [30], full square [33], full triangle [31], full circle represents the average explained above.
In Fig. 5 we show $\delta_1$ for $I = J = 0$, also from two pion threshold to 1.2 GeV. The agreement with experiment is quite satisfactory showing clearly the presence of the $f_0(980)$ resonance as a strong jump in the phase shift around 1 GeV. To get this resonance it is essential to include the kaons, and then to unitarize with coupled channel as we do. In this figure we also have plotted data from [29], but since no error is quoted we have not included this data in the fit.

Now, once we have fixed the $L_i$ from the fit we predict other magnitudes.

![Figure 6: Phase shift for $\pi\pi \rightarrow K\bar{K}$ in $I = J = 0$. Data: full square [35], full triangle [36].](image)

In Fig. 6 the phase shift for the $K\bar{K} \rightarrow \pi\pi$ scattering, $\delta_1 + \delta_2$, is shown for $I = J = 0$. In this figure one sees clearly the $\eta\eta$ threshold. This process is the most sensible to the $\eta\eta$ intermediate state, contrary to what happens with the $\pi\pi$ phase shifts, Fig. 5, where its inclusion is almost negligible. One way to realize the influence of this channel in the former phase shifts is to cancel the imaginary part to $T_4$ coming from the intermediate $\eta\eta$ s-channel loop for $\sqrt{s} > 2m_\eta \simeq 1.1$ GeV. In this way the dashed line curve in Fig. 6 is obtained, which agrees very well with data. This is telling us that the inclusion of the $\eta\eta$ channel in the unitarization procedure for the $(0,0)$ channel is important for studying the $K\bar{K} \rightarrow \pi\pi$ scattering. In any case, as explained above, the threshold is very close to 1.2 GeV where other intermediate states, as four pions, are also important and should be included as well. Hence, we think that the inclusion of the $\eta\eta$ threshold in eq. (20)
should be done when going to higher energy, that is, when extending the model for energies higher than 1.2 GeV.

In Fig. 7, \( (1 - (\eta_{00})^2)/4 \) is shown. Our results display the same tendency as the experimental data, particularly when taking into account the large experimental errors.

![Figure 7: \( (1 - (\eta_{00})^2)/4 \), where \( \eta_{00} \) is the inelasticity in \( I = J = 0 \). Data: starred square [29], full square [35], full triangle [36], full circle [37].](image)

In Fig. 8 we show the \( \pi \pi \) phase shift with \( I=2, J=0 \). The agreement with experimental data is fair. Contrary to the other channels we have shown, in this case no resonances appear and there is only the \( \pi \pi \) channel. So, in this case our result (apart of differences on the \( L_i \) values) is the same than in the IAM [13].

We have also calculated the scattering lengths for the three channels unitarized in this work, \( (I,J) = (0,0), (1,1) \) and \( (2,0) \). We denote them by \( a_J^I \). In Table I we show the value we obtain for \( a_J^I \) together with the experimental and the ChPT values to \( \mathcal{O}(p^4) \). We see in this table that a good agreement with experiment is accomplished. Our values are also close to the ones from ChPT as one should expect because for low energies we recover the chiral expansion.
Figure 8: Phase shift for $\pi \pi \rightarrow \pi \pi$ in $I = 2, J = 0$. Data: cross [38], empty square [39].

Table I: Comparison of scattering lengths in different channels

| $a'_J$ | ChPT   | Our results | Experiment |
|-------|--------|-------------|------------|
| $a_0^1$ | 0.20 ± 0.01 | 0.210 ± 0.002 | 0.26 ± 0.05 |
| $a_1^1$ | 0.037 ± 0.002 | 0.0356 ± 0.0008 | 0.038 ± 0.002 |
| $a_0^2$ | −0.041 ± 0.004 | −0.040 ± 0.001 | −0.028 ± 0.012 |

5. Calculation of the scalar and vector pion form factors

The scalar and vector form factors of the pion are defined respectively as

$$\langle \pi^a(p')\pi^b(p) \text{ out} | \bar{m} (\bar{u}u + \bar{d}d) | 0 \rangle = \delta^{ab} \Gamma(s)$$

(32)

and

$$\langle \pi^i(p')\pi^l(p) \text{ out} | \bar{q} \gamma_\mu \left( \frac{\tau^k}{2} \right) q | 0 \rangle = i \epsilon^{ikl}(p' - p)_\mu F_V(s)$$

(33)

with $\bar{m} = (m_u + m_d)/2$ and $\epsilon^{ijk}$ the total antisymmetric tensor with three indices.

Assuming elastic unitarity (valid up to the $K\bar{K}$ threshold and neglecting multipion states) and making use of the Watson final state theorem [21] the
phase of $\Gamma(s)$ and $F_V(s)$ is fixed to be the one of the corresponding partial wave strong amplitude:

\[
\text{Im } \Gamma(s + i\epsilon) = \tan \delta_0^0 \text{Re } \Gamma(s)
\]
\[
\text{Im } F_V(s + i\epsilon) = \tan \delta_1^1 \text{Re } F_V(s)
\]

The solution of (34) is well known and corresponds to the Omnès type [19, 20]:

\[
\Gamma(s) = P_0(s) \Omega_0(s) \\
F_V(s) = P_1(s) \Omega_1(s)
\]  

(35)

With

\[
\Omega_i(s) = \exp \left\{ \frac{\pi}{m^2} \int_{4m^2_s}^{\infty} ds' \frac{\delta_i^i(s')}{s'^n - s - i\epsilon} \right\}
\]  

(36)

In (35) $P_0(s)$ and $P_1(s)$ are polynomials of degree fixed by the number of subtractions done in $\ln\{\Omega_0(s)\}$ and $\ln\{\Omega_1(s)\}$ minus one, and the zeros of $F_V$ and $\Gamma$. For $n = 1$, $P_i(s) = 1$. This follows from the normalization requirement that $\Gamma(0) = F_V(0) = 1$ and the absence of zeros for those quantities.

Since we have a prediction for the phase shifts we can calculate the dispersion integral (36) and obtain the pion form factors for both the scalar and vector cases. The results are shown in Figs. 9 and 10. The Omnès solution assumes the phase of the form factor to be that of the scattering amplitude, and that is true exactly only until the first inelastic threshold. The first inelastic threshold is the $4\pi$ one. However, as it was already said, its influence, in a first approach, is negligible. The first important inelastic threshold is the $K\bar{K}$ one around 1 GeV. This is essential in $I = J = 0$ but negligible in $I = J = 1$. This inelastic threshold, as discussed above, has been included in our approach and it is responsible for the appearance of the $f_0(980)$ resonance, as it is clearly seen in Figs. 5 and 10.

Due to the presence of this important $K\bar{K}$ threshold, overall for the $I = J = 0$ sector, eqs. (35) are strictly correct up to $\sqrt{s} = 2m_K$, which is indicated as a dashed-dotted vertical line in Figs. 9 and 10. Up to this energy, we see the clear appearance in Figs. 9 and 10 of the $\rho(770)$ and $f_0(980)$ respectively. In the case of the $F_V(s)$ the agreement with existing data is quite satisfactory. Above the $K\bar{K}$ threshold one expects deviations from (35) due to the opening of this inelastic channel. However, for the
vector form factor we still see a rather good agreement with data and the
development should be ascribed to the presence of the $\rho'$ resonance above
1.2 GeV. On the other hand, the result obtained for the vector form factor
is similar to the one recently obtained in [17] using another phase shift
expression, taking into account possible uncertainties coming from orders
higher than $p^4$ in ChPT. For the $I = J = 0$ channel the most dramatic
influence of the opening of the $K\bar{K}$ threshold is the appearance of the $f_0(980)$
resonance, what happens a little below $K\bar{K}$ threshold. Hence its appearance
in Fig. 10 is well accommodated in our assumption of elastic unitarity for
the Watson theorem, which we have used to evaluate the form factors. So
that we do not expect large deviations from our results even above the $K\bar{K}$
threshold up to the appearance of new $I = J = 0, f_0$, resonances higher in
energy, typically around $\sqrt{s} \sim 1.3$ GeV. In Fig. 10 the dashed line represents
the scalar form factor unitarizing only with pions to obtain the $\delta_{00,\pi\pi}$ phase
shift, in the line of the works [11, 26, 27] and we see a very large influence of
the $K\bar{K}$ channel through the $f_0$ resonance which is even substantial around
$|\sqrt{s}| = 500$ MeV.

Figure 9: Vector pion form factor. The vertical line shows the opening of
the $K\bar{K}$ threshold. Data: [40].
Figure 10: Scalar form factor. The dashed curve is the result unitarizing only with pions. The solid line is the full result with both pions and kaons in the intermediate state. The vertical line shows the opening of the $K\bar{K}$ threshold.

6. Conclusions

In this work we have calculated the $K\bar{K}$ scattering amplitude to next to leading order in ChPT. We have seen that from the $K\bar{K}$ threshold, due to the large kaon mass, very large corrections appear. This in principle should imply that this $O(p^4)$ calculation is unlikely to be useful. However, we see that this is not true and that one can obtain accurate results when the ChPT calculations are supplied with some suitable non perturbative unitarization scheme as the one used here. In this way, we have successfully described the $\pi\pi \to \pi\pi$ phase shifts for $(I,J)=(0,0)$, $(1,1)$ and also the phase shift for $\pi\pi \to K\bar{K}$ in $(I,J)=(0,0)$ and the inelasticity in good agreement with experiment up to $\sqrt{s} = 1.2$ GeV. The scattering lengths for $\pi\pi$ with $I=0,1$ and 2 are also calculated in agreement with experiment and former ChPT calculations. We have finally computed the scalar and vector form factor of the pion making use of the above calculated $\pi\pi$ phase shifts in an Omnès representation and the results have also been satisfactory. The scalar form factor strongly shows that the $K\bar{K}$ channel is essential in order to reproduce the $f_0$ resonance.
Acknowledgements

We would like to acknowledge a critical reading and fruitful discussions with E. Oset and A. Pich. We also thank M.C. González García for her kind introduction to PAW and J. Fuster for his appreciated advices about the statistical behaviour of a fit. The work of F. Guerrero and J. A. Oller has been supported by an FPI scholarship of the Spanish Ministerio de Educación y Cultura and of the Generalitat Valenciana respectively.

References

[1] S. Weinberg, Physica A 96 (1979) 327.
[2] J. Gasser and H. Leutwyler, Ann. Phys. (NY) 158 (1984) 142.
[3] J. Gasser and H. Leutwyler, Nucl. Phys. B 250 (1985) 465, 517, 539.
[4] A. Pich, Rep. Prog. Phys. 58 (1995) 563.
[5] G. Ecker, Prog. Part. Nucl. Phys. 35 (1995) 1.
[6] U. G. Meissner, Rep. Prog. Phys. 56 (1993) 903.
[7] V. Bernard, N. Kaiser and U. G. Meissner, Nucl. Phys. B (1991) 129.
[8] G. Ecker, J. Gasser, A. Pich, E. de Rafael, Nucl. Phys. B 321 (1989) 311.
[9] G. Ecker, J. Gasser, H. Leutwyler, A. Pich, E. de Rafael, Phys. Lett. B 233 (1989) 425.
[10] F. Guerrero and A. Pich, Phys. Lett. B 412 (1997) 382.
[11] T. N. Truong, Phys. Rev. Lett. 61 (1988) 2526, ibid D67 (1991) 2260.
[12] A. Dobado, M. J. Herrero and T. N. Truong, Phys. Lett. B235 (1990) 134.
[13] A. Dobado, J. R. Peláez, Phys. Rev. D56 (1997) 3057.
[14] J. A. Oller and E. Oset, Nucl. Phys. A620 (1997) 438.
[15] J. Gasser and U. G. Meissner, Nucl. Phys. B357 (1991) 90.
[16] J. F. Donoghue, C. Ramírez and G. Valencia, Phys. Rev. D38 (1988) 2195.
[17] F. Guerrero, Phys. Rev. D57 (1998) 4136.
[18] J. A. Oller, E. Oset and J. R. Peláez, Phys. Rev. Lett. 80 (1998) 3452.
[19] N. I. Muskhelishvili, Singular Integral Equations (Noordhoof, Groningen, 1953)
[20] R. Omnès, Nuovo Cimento 8 (1958) 316.
[21] K. M. Watson, Phys. Rev. 95 (1955) 228.
[22] G. Passarino and M. Veltman, Nucl. Phys. B160 (1979) 151.
[23] D.B. Kaplan and A.V. Manohar, Phys. Rev. Lett. 56 (1986) 2004.
[24] J. A. Oller, E. Oset and J. R. Peláez, submitted to Phys. Rev. D.
[25] J. Weinstein and N. Isgur, Phys. rev. Lett. 48 (1982) 659.
[26] L. Beldjoudi and T.N. Truong, hep-ph/9403348.
[27] T. Hannah, Phys. Rev. D55 (1997) 5613.
[28] M. Boglione and M.R. Pennington, Z. Phys. C75 (1997) 113.
[29] C.D. Frogatt and J.L. Petersen, Nucl. Phys. B129 (1977) 89.
[30] R. Kaminski, L. Lesniak and K. Rybicki, Z. Phys. C74 (1997) 79.
[31] B. Hyams et al., Nucl. Phys. B64 (1973) 134.
[32] P. Estabrooks et al., AIP Conf. Proc. 13 (1973) 37.
[33] G. Grayer et al., Proc. 3rd Philadelphia Conf. on Experimental Meson Spectroscopy, Philadelphia, 1972 (American Institute of Physics, New York, 1972) 5.
[34] Protopopescu and M. Alson-Granjost, Phys. Rev. D7 (1973) 1279.
[35] D. Cohen, Phys. Rev. D22 (1980) 2595.
[36] A.D. Martin and E.N. Ozmuth, Nucl. Phys. B158 (1979) 520.
[37] W. Ochs, University of Munich thesis, 1974.

21
[38] L. Rosselet et al., Phys. Rev. D15 (1977) 574.

[39] A. Schenk, Nucl. Phys. B363 (1991) 97.

[40] L.M. Barkov, Nucl. Phys. B256 (1985) 365.