A NOTE ON FOCUS-FOCUS SINGULARITIES

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Abstract. We give a topological and geometrical description of focus-focus singularities of integrable Hamiltonian systems. In particular, we explain why the monodromy around these singularities is non-trivial, a result obtained before by J.J. Duistermaat and others for some concrete systems.

1. Introduction

Many integrable Hamiltonian systems in classical mechanics - from as simple as the spherical pendulum - contain focus-focus singularities (see Section 5). Thus the study of these singularities is important in order to understand the topology of integrable systems. We address this problem in the present note. It turns out that the topological structure of focus-focus singularities is quite simple, though very different from elliptic and hyperbolic cases. The affine structure of the orbit space near focus-focus singularities is also very simple (cf. Proposition 3). As a corollary, we obtain that the monodromy around these singularities is non-trivial. The notion of monodromy was first given by Duistermaat [6], and its non-triviality was observed by Duistermaat, Cushman, Knörrer, Bates, etc., for various systems, all of which turn out to be connected with focus-focus singularities (see Sections 3,5).

For simplicity of the exposition we will consider only systems with two degrees of freedom. The results remain unchanged for focus-focus codimension 2 singularities of integrable Hamiltonian systems with more degrees of freedom.

2. Local structure

Throughout this work, by an integrable system we will mean a Poisson \( \mathbb{R}^2 \) action on a real smooth symplectic 4-manifold \((M^4, \omega)\), given by a moment map \( F = (F_1, F_2) : M^4 \rightarrow \mathbb{R}^2 \). We will also assume that the level sets of \( F \) are compact, and hence they are disjoint unions of Liouville tori wherever non-singular.

Suppose that \( x_0 \in M^4 \) is a fixed point of the above Poisson action: \( dF_1(x_0) = dF_2(x_0) = 0 \). Let \( H_i \) be the quadratic part of \( F_i \) at \( x_0 \) (\( i = 1, 2 \)). Since \( F_1, F_2 \) are Poisson commuting, so are \( H_1 \) and \( H_2 \): \( \{H_1, H_2\} = 0 \).

We will assume that \( x_0 \) is a non-degenerate singular point, i.e. \( H_1 \) and \( H_2 \) form a Cartan subalgebra of the algebra of quadratic forms under the natural Poisson bracket. Then by a classical theorem of Williamson (see e.g. [13, 1, 9]), after a linear change of basis of the Poisson action (i.e. a linear change of the moment map \( F' = A \circ F \), \( A \) being a constant invertible matrix), one of the following four alternative cases happens:

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Proposition 2. focus-focus singularities.

To obtain some geometric invariants of torus singular Lagrangian foliations with

seems to omit the focus-focus case). In the analytic case, one can complexify the

Notice also the uniqueness of this Hamiltonian

The local analysis of focus-focus singular points was done by Lerma

Moreover, the local singular Lagrangian foliation given by the moment map

It turns out that near a focus-focus singular point near a

The following proposition is not needed for the rest of this note but will be used

Proposition 1. Let

Then there is a natural

The local analysis of focus-focus singular points was done by Lerman and Um-

The local singular Lagrangian foliation given by the moment map

In this paper we are interested in the fourth, focus-focus case. Notice that if

Here

Remark that this Hamiltonian

In fact, one can give a different proof of the above proposition, which relies only

Proposition 1. Let

Then

Remark that this Hamiltonian

Proposition 2. Let

and

Then we have

and the difference between

Proof. The proof follows directly from the results of Vey

In the analytic case, one can complexify the

Hence

In the smooth case, Taylor expansions will give

and

Here

They give rise to a periodic flow on

The local analysis of focus-focus singular points was done by Lerman and Uman-

In this paper we are interested in the fourth, focus-focus case. Notice that if

The following proposition is not needed for the rest of this note but will be used

Remark that this Hamiltonian

In the analytic case, one can complexify the

Hence

In the smooth case, Taylor expansions will give

Remark that this Hamiltonian

In the analytic case, one can complexify the

Hence

In the smooth case, Taylor expansions will give

Remark that this Hamiltonian

In the analytic case, one can complexify the
the same result, up to flat functions. Note that even in the smooth case we have

\[ X_1 Y_2 - X_2 Y_1 = x_1 y_2 - x_2 y_1 \]

or

\[ X_1 Y_2 - X_2 Y_1 = -(x_1 y_2 - x_2 y_1) \]

since they are the Hamiltonian of a unique natural \( S^1 \) action discussed in the previous proposition. □

3. Stable case: topology and monodromy

Denote by \( N(x_0) \) the connected component of the preimage of the moment map \( \mathbf{F} \) which contains \( x_0 \). We will always assume that all singular points in \( N(x_0) \) of the Poisson action are non-degenerate. (See e.g. \( [5, 13] \) for the definition of nondegeneracy). Then \( N(x_0) \) is a non-degenerate singular leaf in the singular Lagrangian foliation by Liouville tori in a most natural sense (see \( [19] \) for more details). From the results of Lerman and Umanskii \( [9] \) it follows that singular points in \( N(x_0) \) either lie in one-dimensional closed singular hyperbolic orbits or are focus-focus fixed points.

By convention, we will say that a focus-focus singular leaf \( N(x_0) \) is \emph{topologically stable} if it does not contain singular hyperbolic orbits, i.e. if all singular points in it are focus-focus fixed points.

Suppose now that \( N(x_0) \) is topologically stable and contains exactly \( n \) focus-focus fixed points \( x_0, \ldots, x_{n-1} \). Then because of the Poisson \( \mathbb{R}^2 \) action, \( N(x_0) \setminus \{x_0, \ldots, x_{n-1}\} \) must be a non-empty disjoint union of annuli. It follows that \( N(x_0) \) consists of a chain of \( n \) Lagrangian spheres, each of which intersects transversally with two other. (This simple but important fact was observed by Bolsinov, and also by Lerman and Umanskii themselves). In particular, the fundamental group of a tubular neighborhood of it is \( \mathbb{Z} \). When \( n = 1 \), \( N(x_0) \) is just a sphere with one point of self-intersection. It is well-known that the orbit space of the singular Lagrangian foliation (by Liouville tori) has a unique natural integral affine structure outside the singularities (see e.g. \( [8] \)). We have the following:

**Proposition 3.** Let \( N(x_0) \) be a topologically stable non-degenerate focus-focus leaf with \( n \) fixed points as above. Then in a neighborhood of the image of this leaf in the orbit space of the singular Lagrangian foliation, the affine structure can be obtained from the standard flat structure in \( \mathbb{R}^2 = \{ (x, y) \} \) near the origin \( O \) by cutting out the angle \( \angle \{ (0,1), (-n,1) \} \) and gluing the edges of the rest together by the integral linear transformation \( (x, y) \mapsto (x + ny, y) \).

Before proving the above proposition let us now construct an algebraic model for this singularity.

Near the origin \( O \) in the local standard symplectic space \( \mathbb{R}^4, \omega = dp_1 \wedge dq_1 + dp_2 \wedge dq_2 \) we have two generating functions for a Poisson \( \mathbb{R}^2 \)-action with the singularity of the type focus-focus:

\[ f_1 = p_1 q_1 + p_2 q_2, f_2 = p_1 q_2 - p_2 q_1 \]

Set \( z_1 = p_1 - ip_2, z_2 = q_1 + iq_2 \). Here we define a complex structure:

\[ J : (p_1, p_2, q_1, q_2) \rightarrow (p_2, -p_1, -q_2, q_1) \]

Then \( f_1 \) and \( f_2 \) are the real and imaginary part of function \( z_1 z_2 \). In particular, the level sets of \( (f_1, f_2) \) are the level sets of \( z_1 z_2 \) in \( \mathbb{C}^2 \).

Consider first the simplest case, when \( n = 1 \). We can construct the model as follows: Take the conformal map \( \Phi : (z_1, z_2) \mapsto (z_2^{-1}, z_1 z_2^2) \). Note that \( \omega = \text{Red} z_1 \wedge d z_2, \) and \( \Phi \) is a complexification of a real area-preserving map. Hence \( \Phi \), where it is well-defined, is a symplectic mapping. Consider a small neighborhood \( D \times \mathbb{C}P^1 \) of the sphere \( 0 \times \mathbb{C}P^1 \) (\( z_2 \) lies in \( \mathbb{C}P^1 \)). Gluing the points near \( (0,0) \) to the
Denote these generators by $\gamma, \delta$ homotopically, and in the end come back to some new cycles moves along the circle in the positive direction (anti-clockwise), $\gamma$ tion on $T$ in our model. On the submanifold $\{|z| = \epsilon \}$ to see it for any $n$. Since the assertion is topological, then it is enough to prove it 

$1)$ The curves $\{z_1z_2 = \epsilon \}$ are straight lines in the affine structured orbit space.

2) $\begin{pmatrix} \gamma_{\text{new}} \\ \delta_{\text{new}} \end{pmatrix} = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \gamma \\ \delta \end{pmatrix}$

Proof. 1) follows from the fact that the flow of $X_g$ is periodic with constant period. We prove 2) for $n = 1$. After that one can use the $n$-covering argument to see it for any $n$. Since the assertion is topological, then it is enough to prove it in our model. On the submanifold $\{|z_1z_2| = \epsilon \}$ in $U$ denote by $\theta$ the cycle where

1erratum: homeomorphic, not necessarily diffeomorphic
Then $\delta$ singular orbits. As before, 2-dimensional orbits in $z$ and $\arg z$ by the rule: $(z_1^{\text{new}}, z_2^{\text{new}}) = (z_2^{-1}, z_1 z_2^e)$. Fix a point $A$ in one representative of $\gamma$. Moving $A$ along $\lambda$ by some angle $e$ means increasing arg $z_1$ by this angle. Making things go homotopically around $\gamma$, what we get is that arg $z_1^{\text{new}}$ increases by $e$, $z_2^{\text{new}}$ remains constant. By the above rule, in the old coordinates arg $z_1$ increases by $2e$, and arg $z_2$ decreases by $e$. That yields that after going around $\gamma$, $A$ becomes to move on $\lambda + \delta$ with the same angle. It follows that $\gamma_{\text{new}} = \gamma + \delta$. □

Proof of Proposition 3. It follows directly from Lemma 4. □

Let $V^m$ be an affine structured manifold. Then in the tangent bundle of $V$ there is a unique natural flat connection, and fixing a point $x \in V$ there is a monodromy linear representation of $\pi_1(V)$ in $T_x V$, defined as usual (cf. [6]). From Proposition 3 we immediately get:

**Corollary 1.** The local monodromy near every topologically stable focus-focus point in the orbit space is nontrivial (and is generated by $\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$).

### 4. $S^1$ action and reduction

Consider now the non-stable case as well, i.e. allow $N(x_0)$ to contain hyperbolic singular orbits. As before, 2-dimensional orbits in $N(x_0)$ are annuli. It follows that $N(x_0)$ is a union of immersed closed Lagrangian surfaces which intersect transversally at hyperbolic orbits and focus-focus points. Again, it can be easily seen that the local $S^1$ action discussed before can be extended naturally to a Hamiltonian $S^1$ action in a saturated neighborhood $U(N(x_0))$ of $N(x_0)$, which preserves the moment map. Near (possible) hyperbolic orbits in $N(x_0)$, this $S^1$ action coincides with another Hamiltonian $S^1$ action, which is defined in a natural way in a tubular neighborhood of each closed hyperbolic orbit of the Poisson action. Notice that the natural $S^1$ action defined near a hyperbolic closed orbit has isotropy group at most $Z_2$ (the cyclic group of two elements) at this and some nearby hyperbolic orbits and is free outside them (cf. [14]). In other words, we have:

**Proposition 4.** i) In a saturated neighborhood $U(N(x_0))$ there is a unique Hamiltonian $S^1$ action, generated by a function $g$, $g(x_0) = 0$, which preserves the moment map. In particular, it leaves $N(x_0)$ and hyperbolic singular orbits invariant.

ii) This action is trivial at focus-focus points, may have isotropy group $Z_2$ at hyperbolic orbits, and is free elsewhere. □

Consider the moment map $g : U(N) \to \mathbb{R}$ of the above $S^1$ action. At each small value $s$ denote by $P_s$ the symplectic 2-dimensional space obtained by the Marsden-Weinstein reduction at $g = s$.

Consider $P_0$. It contains the image of $x_i$, denoted by $p_i$ ($i = 0, \ldots, n-1$). Let $(x_1, y_1, x_2, y_2)$ be a canonical system of coordinates at $x_0$. Then each orbit of the $S^1$ action, which lies in $\{g = 0\}$, intersects the symplectic plane $x_1 = y_1 = 0$, and the intersection is a pair of points of the type $\{(x_2, y_2), (-x_2, -y_2)\}$. It follows that $P_0$ is an orbifold of order 2 at $p_i$. (Homeomorphically we can ‘desingularize’ the points $p_i$, but not symplectically). Let $q_1, \ldots, q_k, k \geq 0$ denote the image of normally-nonorientable hyperbolic orbits (i.e. orbits on which the $S^1$ action is not free) of $N(x_0)$ in $P_0$. Then $P_0$ is also an orbifold of order 2 at these points. Thus,
$P_0$ is a topological surface, but symplectically it is a quotient of a symplectic surface by a $\mathbb{Z}_2$ action.

Notice that, since $g$ can be viewed as a function on the orbit space of the original $\mathbb{R}^2$ action, the restriction of the moment map on $\{g = s\}$ will give rise to a circle foliation on $P_s$. On $P_0$ this foliation is singular, with the singular leaf being the image of $N(x_0)$. In the topologically stable case this singular leaf is just a circle which contains all of the points $p_i$. In the non-stable case, $P_0$ with the singular foliation looks like a hyperbolic codimension 1 singularity (cf. [19]), only it contains some special (focus-focus) points in the singular leaf. In the topologically stable case, $P_s$ ($s \neq 0$) with the circle foliation on it is regular. In the non-stable case it can be obtained from $P_0$ by smoothening the points $p_i$ and perturbing the foliation a little bit. In particular, each $P_s$, $s \neq 0$ now represents a codimension 1 singularity.

The above description of the Marsden-Weinstein reduction for the distinguished $S^1$ action gives a better understanding of the topology of topologically stable and non-stable focus-focus points. On the other hand, by a small perturbation of the Poisson action we can always split out focus-focus singularities from hyperbolic codimension 1 singularities. In other words, we have:

**Proposition 5.** If $N(x_0)$ contains hyperbolic orbits, then there is an arbitrarily $C^\infty$ small perturbation $F'$ of $F$, such that $F'$ is again the moment map of some Poisson $\mathbb{R}^2$ action, which has $x_0$ as a focus-focus point, and the singular leaf $N'(x_0)$ with respect to $F'$ contains the same number of focus-focus points as $N(x_0)$ but does not contain any hyperbolic orbit.

**Proof.** By a local diffeomorphism of $\mathbb{R}^2$, we can assume that $F(x_0) = 0$ and $F_2 = g$, i.e. it generates the distinguished Hamiltonian $S^1$ action.

If $\gamma$ is a closed 1-dimensional hyperbolic orbit in $N(x_0)$, and the $S^1$ action on $\gamma$ is free, then it is easy to construct a system of symplectic coordinates $(x_1, y_1, x_2, y_2)$, $y_2 \mod 1$, near $\gamma$, such that $\gamma = \{x_1 = x_2 = y_2 = 0\}$, and $x_2 = g$. In this canonical system of coordinates, $F_1$ is a function depending only on 3 variables $x_1, x_2, y_2$. Moreover, we can make so that in a small tubular neighborhood of $\gamma$, $N(x_0)$ is given by $N(x_0) \cap \mathcal{U}(\gamma) = \{F_1 = F_2 = 0\} \cap \mathcal{U}(\gamma) = \{x_1 = 0, x_2y_2 = 0\}$. Then we can slightly perturb $F_1$, as a function of three variables $x_1, x_2, y_2$, so that it remains unchanged outside $\mathcal{U}(\gamma)$, and $\{F_1 = F_2 = 0\} \cap \mathcal{U}(\gamma)$ becomes smooth.

In case the $S^1$ action is free on all hyperbolic orbits of $N(x_0)$, we can apply the above procedure to all these hyperbolic orbits to obtain the required result. In case there are some orbits with isotropy group $\mathbb{Z}_2$, we can use a double covering and make everything $\mathbb{Z}_2$ invariant to obtain the same result. □

The above proposition gives a justification for the word stable. Unlike the case of complicated hyperbolic (codimension 1) singularities, we don’t fear that when we perturb the things using the above proposition some finite symmetry breaks up, since focus-focus points and hyperbolic orbits are clearly of different natures and have different codimensions. One can also ‘split’ focus-focus points, i.e. make them lie on different levels of the moment map, by a similar $S^1$-invariant perturbation. But then some good finite symmetry may break up.

5. Examples and remarks

We have given the topological classification, and the affine structure of the orbit space, for focus-focus singularities. The geometrical classification (i.e. up to foliation-preserving symplectomorphisms) is discussed in work [3], where it is shown that there arises some formal Taylor series in the set of invariants, like in [3].
In [19] we have shown that (topological) 2-domains of orbit spaces of integrable Hamiltonian systems with two degrees of freedom can have fundamental group at most \( \mathbb{Z}_2 \), and 2-domains with non-trivial fundamental group can appear only in very special systems. Thus in general, at least for systems with two degrees of freedom, non-trivial monodromy is most probably connected with focus-focus singularities. (Recall that topological 2-domains of orbit spaces can contain focus-focus points).

It seems that the study of action-angle variables plays an important role in classical mechanics (see e.g. a survey by Marle [11]). A particular attention is given to the ‘phenomenon’ of non-triviality of the monodromy. As we speculated above, this phenomenon is almost for sure connected to the existence of topologically stable focus-focus singularities (but see [19]). We list here some known examples:

1. **Spherical pendulum** (cf. [6]). The spherical pendulum has a \( S^1 \) group of symmetries (rotations), hence it is an integrable system with two degrees of freedom. It has 2 stationary points: the lowest and highest positions with zero velocity. The lowest position is stable, and indeed it is an elliptic singular point. The highest position is unstable dynamically, and one can see that it is a focus-focus singular point, by just looking at the trajectories having this position as the limit. It follows that we have a stable (in our topological sense) focus-focus singularity with one fixed point.

2. **Lagrange top**. A detailed analysis of this classical spinning top is given in [4], together with the nontrivial monodromy. The existence of a focus-focus singularity was also observed by many people (see e.g. [6, 13]).

3. **Champagne bottle** (cf. [2]). The Hamiltonian is \( \frac{1}{2}(p_x^2 + p_y^2) - (x^2 + y^2) + (x^2 + y^2)^2 \), i.e. a special case of Garnier systems. Bates showed that there is a focus-focus singularity, and the monodromy is generated by the same matrix \( \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} \) as in the previous examples (or \( \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \) if one permutes the basis).

4. **Clebsch’s equation** (motion of a rigid body in a fluid). The bifurcation diagram of this system was constructed by Pogosyan [14], from where the existence of a focus-focus singularity is clear.

5. **Euler’s equation on \( so(4) \)**. The bifurcation diagram of some integrable Euler’s equations in \( so(4) \) was constructed by Oshemkov (see, e.g., [3]). These bifurcation diagrams also contain some isolated singular points, i.e. focus-focus points! One can suspect that Euler’s equations in many other Lie (co)algebras will also possess focus-focus singularities.

After this note was written, I found two relevant papers [10] and [18]. Lerman and Umanskiii [10] also studied the topology of extended neighborhoods of focus-focus singularities, but their description is rather complicated. Zou [18] already proved Corollary 1, but only for the case \( n = 1 \). His proof is based on an interesting observation that in case \( n = 1 \), the situation resembles the simplest case of Picard-Lefschetz theory. (In fact, our model in Section 3 is holomorphic so one can apply Picard-Lefschetz theory). In [18] Zou also mentioned a focus-focus singularity with \( n = 2 \) (in a system studied by him and Larry Bates), where his theorem does not apply.

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*erratum: the system must not contain some types of degenerate singularities*
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