SIMULTANEOUS NON-VANISHING OF CENTRAL VALUES OF
\(GL(2) \times GL(3)\) AND \(GL(3)\) \(L\)-FUNCTIONS

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Abstract. We study simultaneous non-vanishing of 
\(L(\frac{1}{2}, f)\) and \(L(\frac{1}{2}, g \otimes f)\), when \(F\) runs over an orthogonal basis of the space of Hecke-Maass cusp forms for \(SL(3, \mathbb{Z})\) and \(g\) is a fixed \(SL(2, \mathbb{Z})\) Hecke cusp form of weight \(k \equiv 0 \pmod{4}\).

1. Introduction

Like the Birch and Swinnerton-Dyer conjecture which relates the order of vanishing of the Hasse-Weil \(L\)-function at the central point to the rank of an elliptic curve, the vanishing or non-vanishing of an automorphic \(L\)-function at the special points are related to several deep problems with great significance in number theory. Therefore, it is a profoundly interesting question to understand whether product of two or more \(L\)-functions are simultaneous non-vanishing at the central point. This type of question has been studied by many authors (see for example [8], [15], [10], [16], [13], [6], [14]). In 2014, Das & Khan [4] proved that \(GL(2) \times GL(1)\) and \(GL(1)\) \(L\)-functions are simultaneous non-vanishing. Ramakrishnan & Rogawski [18] in 2005, showed a simultaneous non-vanishing result for \(GL(2) \times GL(1)\) and \(GL(2)\) \(L\)-functions. Similar type of non-vanishing results for \(GL(2) \times GL(2)\) and \(G(2)\) \(L\)-functions were proved by Xu [19] and Liu [12] for Maass form and holomorphic Hecke cusp form with weight aspect respectively. Non-vanishing problem for \(GL(3) \times GL(2)\) and \(GL(2)\) \(L\)-functions was first studied by Li [11] in 2009. More precisely, let \(f\) be a fixed Hecke-Maass cusp form for \(SL(3)\). Li Proved that there are infinitely many \(SL(2)\) Hecke-maass cusp forms \(u_j\) such that \(L(\frac{1}{2}, f \times u_j) L(\frac{1}{2}, u_j) \neq 0\).

In this paper, we consider the first moment of the product of \(GL(2) \times GL(3)\) and \(GL(3)\) \(L\)-functions. More precisely, we fix an Hecke cusp form \(g\) for \(SL(2, \mathbb{Z})\). We study first moment of \(L(s, F \otimes g) L(s, F)\) at the central point as \(F\) runs over an orthogonal basis of the space of Hecke-Maass cusp forms for \(SL(3, \mathbb{Z})\).

Let \(\mu_f = (\mu_1, \mu_2, \mu_3)\) be the Langlands parameter and \(\nu_f = (\nu_1, \nu_3, \nu_3)\) be the spectral parameter of a Hecke-Maass cusp form \(f\) for \(SL(3, \mathbb{Z})\). As in Blomer-Buttcane [2], we consider the generic case in short interval. Let \(\mu_0 = (\mu_{0,1}, \mu_{0,2}, \mu_{0,3})\) be a fixed point in \(\Lambda_{1/2}\) (see (1)). So \(\nu_0 = (\nu_{0,1}, \nu_{0,2}, \nu_{0,3})\) satisfies the relations (2) and (3). We consider the case

\[|\mu_{0,j}| \asymp |\nu_{0,j}| \asymp ||\mu_0|| \asymp ||\nu_0|| := T, \quad 1 \leq j \leq 3.\]

Let us denote \(R = T^\theta\) for any fixed \(\theta\) in \((0, 1)\). We choose the test function \(h(\mu)\) so that it has the localizing effect at a ball of radius \(R\) about \(w(\mu_0)\), where \(w\) are elements in the

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Weyl group $\mathfrak{W}$ of $GL(3, \mathbb{R})$. It is defined by

$$h(\mu) := P(\mu)^2 \left( \sum_{w \in \mathfrak{W}} \psi \left( \frac{w(\mu) - \mu_0}{R} \right) \right)^2,$$

where $\psi(\mu) = \exp \left( -\left( \mu_1^2 + \mu_2^2 + \mu_3^2 \right) \right)$ and

$$P(\mu) = \prod_{1 \leq n \leq A_0} \prod_{j=1}^{3} \frac{\nu_j - \frac{1}{3}(1+2n)}{|\nu_{0,k}|^2}$$

for some fixed large $A_0 > 0$. Here

$$\mathfrak{W} := \left\{ I, w_2 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, w_3 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, w_4 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, w_5 = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, w_6 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \right\}$$

is the Weyl group for $SL(3, \mathbb{R})$. Let $d_{\text{spec}}(\mu) = \text{spec}(\mu)d\mu$ with

$$\text{spec}(\mu) = \prod_{j=1}^{3} \left( 3\nu_j \tan \left( \frac{3\pi}{2}\nu_j \right) \right) \quad \text{and} \quad d\mu = d\mu_1 d\mu_2 = d\mu_2 d\mu_3 = d\mu_3 d\mu_1.$$

Let us define

$$N_T = \|F\|^2 \prod_{j=1}^{3} \cos \left( \frac{3\pi}{2}\nu_j \right)$$

to be the normalizing factor. Now we state the main theorem of this article.

**Theorem 1.** Let $g$ be a Hecke cusp form for $SL(2, \mathbb{Z})$ of weight $k \equiv 0 \pmod{4}$. Let $\{F\}$ be a basis of the space of Hecke-Maass cusp forms for $SL(3, \mathbb{Z})$. Then we have

$$\sum_{f} \frac{h(\mu_f)}{N_f} L(\frac{1}{2}, g \otimes F)L(\frac{1}{2}, F) = \frac{1}{192\pi^5} \int_{\mathfrak{S}(\mu)=0} \frac{M(\mu, k)h(\mu)\text{spec}(\mu)d\mu}{\mathfrak{S}(\mu)} + O\left( T^{\frac{17}{6} + \varepsilon} R^2 \right),$$

where

$$M(\mu, k) = \zeta(\frac{3}{2}) + L(1, g) \prod_{j=1}^{3} \frac{\Gamma(\frac{1}{4} + \mu_j)}{\Gamma(\frac{1}{4} - \frac{\mu_j}{2})} + L(1, g) \prod_{j=1}^{3} \frac{\Gamma(\frac{1}{2} + \mu_j)}{\Gamma(\frac{1}{2} - \mu_j)}$$

$$+ \zeta(\frac{3}{2}) \prod_{j=1}^{3} \frac{\Gamma(\frac{k}{2} + \mu_j)\Gamma(\frac{1}{4} + \frac{\mu_j}{2})}{\Gamma(\frac{k}{2} - \mu_j)\Gamma(\frac{1}{4} - \frac{\mu_j}{2})}.$$ 

Note that, $\int_{\mathfrak{S}(\mu)=0} M(\mu, k)h(\mu)\text{spec}(\mu)d\mu \asymp T^3 R^2$.

**Remark 1.** We have used bounds $\zeta(1/2 + it) \ll t^{1/6}$ and $L(1/2 + it, g \otimes f) \ll t$ in the estimation of the error terms (see section 3.1.6). Note that by using best known bounds for $\zeta(1/2 + it)$ and $L(1/2 + it, g \otimes f)$ one can get slight improvement in the error term. Since we are only focusing on simultaneous non-vanishing results we have not incorporate such a small improvement in the error term.

As a corollary of Theorem 1, we have the following result.

**Corollary 1.** Let $g$ be a Hecke cusp form of weight $k \equiv 0 \pmod{4}$ for $SL(2, \mathbb{Z})$. Then there exist infinitely many Hecke-Maass cusp forms $F$ for $SL(3, \mathbb{Z})$ such that $L(\frac{1}{2}, F)L(\frac{1}{2}, g \otimes F) \neq 0$. 

Remark 2. If $g$ is a Hecke cusp form for $SL(2, \mathbb{Z})$ with $k \equiv 2 \mod 4$, then $L\left(\frac{1}{2}, g \otimes F\right) = 0$. In that case one can consider the following sum

$$
\sum \frac{h(\mu_2)}{N_f} L\left(\frac{1}{2}, g \otimes F\right)L\left(\frac{1}{2}, F\right)
$$

and get similar results.

2. Preliminaries

In this section we review some definitions, essential facts and tools that will be used in later development.

2.1. Automorphic forms for $SL(3, \mathbb{Z})$ and their $L$-functions. Let

$$
\mathbb{H}_3 = GL(3, \mathbb{R})/O(3, \mathbb{R})\mathbb{R}^+
$$

be the generalized upper half plane. For $0 \leq c \leq \infty$, let

$$
\Lambda'_c = \left\{ \mu = (\mu_1, \mu_2, \mu_3) \in \mathbb{C}^3, \quad |\Re(\mu_j)| \leq c, \quad \mu_1 + \mu_2 + \mu_3 = 0, \quad \{\mu_1, -\mu_2, -\mu_3\} = \{\mu_1, \mu_2, \mu_3\} \right\}. \quad (1)
$$

Consider $\mu$ to be the Langlands parameter of a Hecke-Maass form $f$ in $L^2(SL(3, \mathbb{Z}) \backslash \mathbb{H}_3)$. Let us define

$$
\nu_1 = \frac{1}{3}(\mu_1 - \mu_2), \quad \nu_2 = \frac{1}{3}(\mu_2 - \mu_3), \quad \nu_3 = -\nu_1 - \nu_2 = \frac{1}{3}(\mu_3 - \mu_1) \quad (2)
$$

where $\nu = (\nu_1, \nu_2, \nu_3)$ is known as the spectral parameter of $f$. So

$$
\mu_1 = 2\nu_1 + \nu_2, \quad \mu_2 = \nu_2 - \nu_1, \quad \mu_3 = -\nu_1 - 2\nu_2. \quad (3)
$$

Let $A_f(m_1, m_2)$ be the normalised Fourier coefficients of a $GL(3)$ Hecke Maass cusp form $F$ with Langlands parameters $\mu_f = (\mu_1, \mu_2, \mu_3)$. The stander $L$-function associated to $F$ is given by

$$
L(s, F) = \sum_{n \geq 1} \frac{A_f(1, n)}{n^s} \quad \text{for } \Re(s) > 1.
$$

The dual form of $F$ is denoted by $\bar{F}$ with the Langlands parameter $\mu_{\bar{F}} = (-\mu_1, -\mu_2, -\mu_3)$ and the coefficients $A_{\bar{F}}(n, 1) = A_f(1, n) = A_{\bar{f}}(1, n)$. Let us define

$$
\Lambda(s, F) := \gamma(s, F) L(s, F),
$$

where $\gamma(s, F) = \prod_{j=1}^{3} \Gamma_{\mathbb{R}}(s - \mu_j)$ and $\Gamma_{\mathbb{R}}(s) = \pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)$. $\Lambda(s, F)$ is called the completed $L$-function, which is an entire function and satisfies the functional equation

$$
\Lambda(s, F) = \Lambda(1 - s, \bar{F}).
$$

2.2. The maximal Eisenstein series. Let $u \in \mathbb{C}$ have sufficiently large real part. Let $f$ be a Hecke-Maass cusp form for $SL(2, \mathbb{Z})$ with the spectral parameter $it$, Hecke eigenvalues $\lambda_f(m)$ and $\|f\| = 1$. The maximal Eisenstein series and it’s Hecke eigenvalue at $(m, n)$ are denoted by $E_{u, f}^{\max}(z)$ and $B_{u, f}^{\max}(m, n)$, respectively. The Hecke eigenvalue $B_{u, f}^{\max}(1, m)$ is defined by (see Goldfeld [5])

$$
B_{u, f}^{\max}(1, m) = \sum_{d_1d_2=m} \lambda_f(d_1)d_1^{-u}d_2^{-2u}
$$
and satisfies the following Hecke relations
\[ B_{u,f}^{\text{max}}(m, 1) = B_{u,f}^{\text{max}}(1, m), \quad B_{u,f}^{\text{max}}(m, n) = \sum_{d \mid (m, n)} \mu(d) B_{u,f}^{\text{max}}(\frac{m}{d}, 1) B_{u,f}^{\text{max}}(1, \frac{n}{d}). \]

The \( L \)-function associated to \( E_{u,f}^{\text{max}}(z) \) is given by
\[ L(s, E_{u,f}^{\text{max}}) = \sum_{m \geq 1} \frac{B_{u,f}^{\text{max}}(1, m)}{m^s} = \zeta(s - 2u)L(s + u, f), \quad (4) \]
for sufficiently large \( \Re(s) \). It satisfies the functional equation
\[ \Lambda(s, E_{u,f}^{\text{max}}) = \prod_{j=1}^{3} \Gamma_{
abla}(s + \mu_j')L(s, E_{u,f}^{\text{max}}) = \Lambda(1 - s, E_{u,f}^{\text{max}}), \]
where \( \mu_1' = u + it, \mu_2' = u - it \) and \( \mu_3' = -2u \). The normalized factor for the maximal Eisenstein series is defined by
\[ N_{u,f}^{\text{max}} := 8L(1, \Ad^2 f)|L(1 + 3u, f)|^2. \]

2.3. The minimal Eisenstein series. Let \( \nu_1, \nu_2 \in \mathbb{C} \) and \( (\mu_1, \mu_2, \mu_3) \) be the Langlands parameter given by (\ref{LanglandsParameter}). We denote the minimal Eisenstein series by \( E_{\nu_1, \nu_2}^{\text{min}}(z) \). The Hecke eigenvalue \( B_{\nu_1, \nu_2}^{\text{min}}(m, n) \) of \( E_{\nu_1, \nu_2}^{\text{min}}(z) \) at \( (m, n) \) is defined by (see Goldfeld \cite{Goldfeld})
\[ B_{\mu}^{\text{min}}(1, m) = B_{\nu_1, \nu_2}^{\text{min}}(1, m) := \sum_{d_1d_2d_3=n} d_1^{-\mu_1}d_2^{-\mu_2}d_3^{-\mu_3} \]
and satisfies the following Hecke relations
\[ B_{\nu_1, \nu_2}^{\text{min}}(m, 1) = B_{\nu_1, \nu_2}^{\text{min}}(1, m), \quad B_{\nu_1, \nu_2}^{\text{min}}(m, n) = \sum_{d \mid (m, n)} \mu(d) B_{\nu_1, \nu_2}^{\text{min}}(\frac{m}{d}, 1) B_{\nu_1, \nu_2}^{\text{min}}(1, \frac{n}{d}). \]
The \( L \)-function associated to \( E_{\nu_1, \nu_2}^{\text{min}}(z) \) is given by
\[ L(s, E_{\nu_1, \nu_2}^{\text{min}}) = \sum_{m \geq 1} \frac{B_{\nu_1, \nu_2}^{\text{min}}(1, m)}{m^s} = \zeta(s + \mu_1)\zeta(s + \mu_2)\zeta(s + \mu_3), \quad (5) \]
for \( \Re(s) > 1 \). The normalized factor for the minimal Eisenstein series is defined by
\[ N_{\mu}^{\text{min}} = N_{\nu_1, \nu_2}^{\text{min}} := \frac{1}{16} \prod_{j=1}^{3} |\zeta(1 + 3\nu_j)|^2. \]

2.4. The Rankin-Selberg \( L \)-function on \( \text{GL}(2) \times \text{GL}(3) \). Let \( F \) be a \( \text{GL}(3) \) Hecke Maass cusp with Langlands parameters \( \mu = (\mu_1, \mu_2, \mu_3) \). Let \( g \) be a Hecke cusp form for \( \text{SL}(2, \mathbb{Z}) \) of weight \( k \) and \( \lambda_g(n) \) be the \( n \)-th Hecke eigenvalue. The Rankin-Selberg \( L \)-function of \( g \) and \( F \) is defined by
\[ L(s, g \otimes F) = \sum_{m,n \geq 1} \frac{\lambda_g(n) A_F(m,n)}{(nm^2)^s}, \quad \Re(s) > 1. \]
It is entire and satisfies the functional equation
\[ \Lambda(s, g \otimes F) = i^{3k} \Lambda(1 - s, g \otimes \tilde{F}), \]
where
\[ \Lambda(s, g \otimes F) = \gamma(s, g \otimes F) L(s, g \otimes F) \]
and
\[ \gamma(s, g \otimes F) = \prod_{j=1}^{3} \Gamma_{\mathbb{R}} \left( s + \frac{k-1}{2} - \mu_j \right) \Gamma_{\mathbb{R}} \left( s + \frac{k+1}{2} - \mu_j \right). \]

Let \( E_{u,f}^{\text{max}} \) be the maximal Eisenstein series as in \( \S 2.2 \) The Rankin-Selberg \( L \)-function \( L(s, g \otimes E_{u,f}^{\text{max}}) \) is defined by
\[ L(s, g \otimes E_{u,f}^{\text{max}}) = \sum_{m,n \geq 1} \frac{\lambda_g(n) E_{u,f}^{\text{max}}(m,n)}{(nm^2)^s}, \]

for sufficiently large \( \Re(s) \) and any \( A > 0 \).

Let \( E_{\nu_1,\nu_2}^{\text{min}} \) be the minimal Eisenstein series as in \( \S 2.3 \) The Rankin-Selberg \( L \)-function \( L(s, g \otimes E_{\nu_1,\nu_2}^{\text{min}}) \) is defined by
\[ L(s, g \otimes E_{\nu_1,\nu_2}^{\text{min}}) = \sum_{m,n \geq 1} \frac{\lambda_g(n) E_{\nu_1,\nu_2}^{\text{min}}(m,n)}{(nm^2)^s}, \]

for \( \Re(s) > 1 \).

From the above functional equation we deduce the approximate functional equation for \( L(s, F) \) and \( L(s, g \otimes F) \) at the central point \( s = \frac{1}{2} \) (See §5.2 of [7]). Let \( G(s) = e^{s^2} \). We define
\[ V_f(y) := \frac{1}{2\pi i} \int_{(3)} y^{-u} \frac{\gamma(\frac{1}{2} + u, F)}{\gamma(\frac{1}{2}, F)} G(u) \frac{du}{u}, \]
\[ \tilde{V}_f(y) := \frac{1}{2\pi i} \int_{(3)} y^{-u} \frac{\gamma(\frac{1}{2} + u, \tilde{F})}{\gamma(\frac{1}{2}, \tilde{F})} G(u) \frac{du}{u}, \]

and
\[ W_f(y) := \frac{1}{2\pi i} \int_{(3)} y^{-u} \frac{\gamma(\frac{1}{2} + u, g \otimes F)}{\gamma(\frac{1}{2}, g \otimes F)} G(u) \frac{du}{u}, \]
\[ \tilde{W}_f(y) := \frac{1}{2\pi i} \int_{(3)} y^{-u} \frac{\gamma(\frac{1}{2} + u, g \otimes \tilde{F})}{\gamma(\frac{1}{2}, g \otimes \tilde{F})} G(u) \frac{du}{u}. \]

Lemma 2.1. We have
\[ L(\frac{1}{2}, F) = \sum_{n \geq 1} \frac{A_F(1, l)}{l^{1/2}} V_f(l) + \sum_{l \geq 1} \frac{A_F(1, l)}{l^{1/2}} \tilde{V}_f(l). \]

Moreover, for \( \mu = (\mu_1, \mu_2, \mu_3) \) with \( \mu_j \gg T \), one has
\[ y^{j_1} \frac{\partial}{\partial y^{j_1}} V_f(y) \ll \left( \frac{y}{T^3} \right)^{-A}, \quad y^{j_1} \frac{\partial}{\partial y^{j_1}} \tilde{V}_f(y) \ll \left( \frac{y}{T^3} \right)^{-A} \]

for any \( A > 0 \) and any \( j_1 \in \mathbb{N} \cup \{0\} \). Also, for \( y \gg T^3 \)
\[ V_f(y) = 1 + O_A \left( \frac{T^3}{y} \right)^{-A}, \quad \tilde{V}_f(y) = \prod_{j=1}^{3} \frac{\Gamma \left( \frac{1}{4} + \frac{\mu_j}{2} \right)}{\Gamma \left( \frac{1}{4} - \frac{\mu_j}{2} \right)} + O_A \left( \frac{T^3}{y} \right)^{-A} \]

for any \( A > 0 \).
Lemma 2.2. We have

\[ L\left(\frac{1}{2}, g \otimes F\right) = \sum_{m,n \geq 1} \frac{\lambda_g(n)A_f(m,n)}{(nm^2)^{1/2}} W_f(nm^2) + i^{3k} \sum_{m,n \geq 1} \frac{\lambda_g(n)A_f(m,n)}{(nm^2)^{1/2}} \tilde{W}_f(nm^2), \]

Moreover, for \( \mu = (\mu_1, \mu_2, \mu_3) \) with \( \mu_j \gg T \), one has

\[ y^{j_1} \frac{\partial^{j_1}}{\partial y_1^{j_1}} W_f(y) \ll_k \left( \frac{y}{T^3} \right)^{-\lambda_1}, \quad y^{j_2} \frac{\partial^{j_2}}{\partial y_2^{j_2}} \tilde{W}_f(y) \ll_k \left( \frac{y}{T^3} \right)^{-\lambda_2} \]

for any \( A > 0 \) and any \( j_1 \in \mathbb{N} \cup \{0\} \). Also, for \( y \gg T^3 \)

\[ W_f(y) = 1 + O_{A,k} \left( \frac{T^3}{y} \right)^{-\lambda_1}, \quad \tilde{W}_f(y) = \prod_{j=1}^3 \Gamma \left( \frac{k}{2} - \mu_j \right) + O_{A,k} \left( \frac{T^3}{y} \right)^{-\lambda_2} \]

for any \( 0 < A < \frac{k-1}{2} \).

2.5. The Kloosterman sums. For \( n_1, n_2, m_1, m_2, D_1, D_2 \in \mathbb{N} \), we define the following Kloosterman sums.

\[ \tilde{S}(n_1, n_2, m_1; D_1, D_2) := \sum_{C_1(\text{mod } D_1), C_2(\text{mod } D_2)} \sum_{(C_1, D_1)=(C_2, D_2), D_1C_2+B_1C_1 \equiv 0(\text{mod } D_1, D_2)} e \left( \frac{n_2 C_1 C_2}{D_2} + m_1 \frac{C_2}{D_1} + n_1 \frac{C_1}{D_1} \right) \]

for \( D_1 | D_2 \), and

\[ S(n_1, m_2, m_1, n_2; D_1, D_2) := \sum_{B_1, C_1(\text{mod } D_1); B_2, C_2(\text{mod } D_2)} \sum_{D_1 C_2 + B_1 C_1 \equiv 0(\text{mod } D_1, D_2)} \sum_{(B_1, C_1, D_1) = 1} e \left( \frac{n_1 B_1 + m_1 (Y_1 D_2 - Z_1 B_2)}{D_1} + \frac{m_2 B_2 + n_2 (Y_2 D_1 - Z_2 B_1)}{D_2} \right), \]

where \( B_j Y_j + C_j Z_j \equiv 1(\text{mod } D_j) \) for \( j = 1, 2 \).

2.6. Integral Kernels. Following [3, Theorem 2 & 3], we define the integral kernel in terms of Mellin-Barnes representations. For \( s \in \mathbb{C} \), \( \mu = (\mu_1, \mu_2, \mu_3) \) define the meromorphic function

\[ \tilde{G}^{\pm}(s, \mu) := \frac{\pi^{-3s}}{12288\pi^{7/2}} \left( \prod_{j=1}^3 \Gamma \left( \frac{1}{2} (s - \mu_j) \right) \right) \left( \prod_{j=1}^3 \Gamma \left( \frac{1}{2} (1 - s + \mu_j) \right) \right) \pm i \left( \prod_{j=1}^3 \Gamma \left( \frac{1}{2} (1 + s - \mu_j) \right) \right) \left( \prod_{j=1}^3 \Gamma \left( \frac{1}{2} (2 - s + \mu_j) \right) \right), \]

and for \( s = (s_1, s_2) \in \mathbb{C}^2 \), \( \mu = (\mu_1, \mu_2, \mu_3) \) define the meromorphic function

\[ G(s, \mu) := \frac{1}{\Gamma(s_1 + s_2)} \prod_{j=1}^3 \Gamma(s_1 - \mu_j) \Gamma(s_2 + \mu_j). \]
We also define the following trigonometric functions

\[ S^{++}(s; \mu) := \frac{1}{24\pi^2} \prod_{j=1}^{3} \cos \left( \frac{3}{2} \pi \nu_j \right), \]

\[ S^{--}(s; \mu) := \frac{1}{32\pi^2} \frac{\cos \left( \frac{5}{2} \pi \nu_1 \right)}{\sin \left( \frac{5}{2} \pi \nu_1 \right)} \sin \left( \frac{3}{2} \pi \nu_2 \right) \sin \left( \frac{3}{2} \pi \nu_3 \right) \sin \left( \pi (s_1 + s_2) \right), \]

\[ S^{-+}(s; \mu) := \frac{1}{32\pi^2} \frac{\cos \left( \frac{5}{2} \pi \nu_1 \right)}{\sin \left( \frac{5}{2} \pi \nu_1 \right)} \sin \left( \frac{3}{2} \pi \nu_2 \right) \sin \left( \frac{3}{2} \pi \nu_3 \right) \sin \left( \pi (s_1 + s_2) \right), \]

\[ S^{+-}(s; \mu) := \frac{1}{32\pi^2} \frac{\cos \left( \frac{5}{2} \pi \nu_1 \right)}{\sin \left( \frac{5}{2} \pi \nu_1 \right)} \sin \left( \frac{3}{2} \pi \nu_2 \right) \sin \left( \frac{3}{2} \pi \nu_3 \right) \sin \left( \pi (s_1 + s_2) \right). \]

For \( y \in \mathbb{R}^* \) with \( \text{sgn}(y) = \epsilon \), let

\[ K_{w_1}(y; \mu) := \int_{-i\infty}^{i\infty} |y|^{-s} \mathcal{G}(s, \mu) \frac{ds}{2\pi i}. \]

For \( y = (y_1, y_2) \in (\mathbb{R}^*)^2 \) with \( \text{sgn}(y_1) = \epsilon_1, \text{sgn}(y_2) = \epsilon_2 \), let

\[ K_{w_6}^{\epsilon_1, \epsilon_2}(y; \mu) := \int_{-i\infty}^{i\infty} \int_{-i\infty}^{i\infty} \left| 4\pi^2 y_1 \right|^{-s_1} \left| 4\pi^2 y_2 \right|^{-s_2} \mathcal{G}(s, \mu) S^{\epsilon_1, \epsilon_2}(s; \mu) \frac{ds_1 ds_2}{(2\pi i)^2}. \]

2.7. The Kuznetsov formula. Define the spectral measure on the hyperplane \( \mu_1 + \mu_2 + \mu_3 = 0 \) by \( d_{\text{spec}}\mu = \text{spec}(\mu) d\mu \), where

\[ \text{spec}(\mu) = \prod_{j=1}^{3} \left( 3\nu_j \tan \left( \frac{3\pi}{2} \nu_j \right) \right) \quad \text{and} \quad d\mu = d\mu_1 \ d\mu_2 = d\mu_2 \ d\mu_3 = d\mu_3 \ d\mu_1. \]

Now we state the Kuznetsov trace formula in the version of Buttcane [3, Theorem 2, 3, 4].

Lemma 2.3. Let \( n_1, n_2, m_1, m_2 \in \mathbb{N} \) and \( h \) be a holomorphic function on

\[ \Lambda_{\frac{1}{2} + \delta} = \left\{ \mu = (\mu_1, \mu_2, \mu_3) \in \mathbb{C}^3, \mu_1 + \mu_2 + \mu_3 = 0, \Re(\mu_j) \leq \frac{1}{2} + \delta \right\} \]

for some \( \delta > 0 \), symmetric under the Weyl group \( W \), of rapid decay when \( |\Im(\mu_j)| \to \infty \) and satisfies

\[ h(3\nu_1 \pm 1, 3\nu_2 \pm 1, 3\nu_3 \pm 1) = 0. \]

Then we have

\[ \mathcal{C} + \mathcal{E}_{\text{min}} + \mathcal{E}_{\text{max}} = \Delta + \Sigma_4 + \Sigma_5 + \Sigma_6, \]

where

\[ \mathcal{C} := \sum_f \frac{h(\mu_f)}{N_f} A_f(m_1, m_2) A_f(n_1, n_2), \]

\[ \mathcal{E}_{\text{min}} := \frac{1}{24(2\pi i)^2} \int_{\Re(\mu) = 0} \frac{h(\mu)}{N^\text{min}_\mu B^\text{min}_\mu(m_1, m_2) B^\text{min}_\mu(n_1, n_2)} \ d\mu_1 d\mu_2, \]

\[ \mathcal{E}_{\text{max}} := \sum_f \frac{1}{2\pi i} \int_{\Re(u) = 0} \frac{h(u + it_f, u - it_f, -2u)}{N^\text{max}_{u,f} B^\text{max}_{u,f}(m_1, m_2) B^\text{max}_{u,f}(n_1, n_2) du,} \]
and
\[
\Delta := \delta_{m_1,n_1} \delta_{m_2,n_2} \frac{1}{192\pi^5} \int_{\mathbb{R}(\mu) = 0} h(\mu) \, d_{\text{spec}} \mu,
\]
\[
\Sigma_4 := \sum_{\epsilon = \pm 1} \sum_{D_2|D_1} \sum_{m_2D_1 = n_1D_2^2} \tilde{S}(-\epsilon n_2,m_2,m_1;D_2,D_1) \frac{\Phi_{w_4}(\frac{\epsilon n_1m_2n_2}{D_1D_2})}{D_1D_2},
\]
\[
\Sigma_5 := \sum_{\epsilon = \pm 1} \sum_{m_1|D_2} \sum_{m_1D_2 = n_1D_1^2} \tilde{S}(\epsilon n_1,m_1,m_2;D_1,D_2) \frac{\Phi_{w_5}(\frac{\epsilon n_1m_2n_1}{D_1D_2})}{D_1D_2},
\]
\[
\Sigma_6 := \sum_{\epsilon_1,\epsilon_2 = \pm 1} \sum_{D_1,D_2} \tilde{S}(\epsilon_2n_2,\epsilon_1n_1,m_1,m_2;D_1,D_2) \frac{\Phi_{w_6}(\frac{-\epsilon_2n_1m_2D_2}{D_1^2}, \frac{-\epsilon_1n_2m_1D_1}{D_2^2})}{D_1D_2}
\]
with
\[
\Phi_{w_4}(y) := \int_{\mathbb{R}(\mu) = 0} h(\mu) K_{w_4}(y;\mu) \, d_{\text{spec}} \mu,
\]
\[
\Phi_{w_5}(y) := \int_{\mathbb{R}(\mu) = 0} h(\mu) K_{w_4}(-y,-\mu) \, d_{\text{spec}} \mu,
\]
\[
\Phi_{w_6}(y) := \int_{\mathbb{R}(\mu) = 0} h(\mu) K_{w_6}^{\text{sgn}(y_1),\text{sgn}(y_2)}(y;\mu) \, d_{\text{spec}} \mu.
\]

Here we quote Lemma 8 and Lemma 9 of [2] which are used in truncating summation in geometric terms after the application of the Kuznetsov formula.

**Lemma 2.4.** Let \(0 < |y| \leq T^{3-\epsilon}\). Then for any constant \(B > 0\), we have
\[
\Phi_{w_4}(y) \ll_{\epsilon,B} T^{-B}.
\]

If \(|y| > T^{3-\epsilon}\), then
\[
|y| \frac{d}{dy} \Phi_{w_4}(y) \ll_{\epsilon,B} T^{3+\epsilon} R^2 (T + |y|^{1/3})^j
\]
for any \(j \in \mathbb{N} \cup \{0\}\).

**Lemma 2.5.** Let \(\Upsilon := \min \{|y_1|^{1/3}|y_2|^{1/6}, |y_2|^{1/3}|y_1|^{1/6}\}\). If \(\Upsilon \ll T^{1-\epsilon}\), then for any constant \(B > 0\)
\[
\Phi_{w_6}(y_1,y_2) \ll_{\epsilon,B} T^{-B}.
\]

If \(\Upsilon \gg T^{1-\epsilon}\), then we have
\[
|y_1|^{j_1} |y_2|^{j_2} \frac{\partial^{j_1}}{\partial y_1^{j_1}} \frac{\partial^{j_2}}{\partial y_2^{j_2}} \Phi_{w_6}(y_1,y_2)
\ll_{\epsilon,j_1,j_2} T^3 R^2 (T + |y_1|^{1/2} + |y_1|^{1/3}|y_2|^{1/6})^{j_1} (T + |y_2|^{1/2} + |y_2|^{1/3}|y_1|^{1/6})^{j_2}
\]
for any \(j_1, j_2 \in \mathbb{N} \cup \{0\}\).

3. **Proof of Theorem 1**

Let us define
\[
S(T) := \sum_{\mathcal{F}} \frac{h(\mu_r)}{N_r} L(1/2, g \otimes \mathcal{F}) L(1/2, \mathcal{F}).
\]
By using the approximate functional equations of $L(1/2, g \otimes F)$ and $L(1/2, F)$ (Lemma 2.2 and Lemma 2.1), we get

$$S(T) = \sum_{m,n \geq 1} \sum_{l \geq 1} \frac{\lambda_g(n)}{m(nl)^{1/2}} \sum_{r} h_1(\mu_r) \frac{A_{\mathcal{F}}(m,n)}{N_{\mathcal{F}}} A_r(1,l)$$

$$+ \sum_{m,n \geq 1} \sum_{l \geq 1} \frac{\lambda_g(n)}{m(nl)^{1/2}} \sum_{r} h_2(\mu_r) \frac{A_{\mathcal{F}}(m,n)}{N_{\mathcal{F}}} A_r(1,l)$$

$$+ \sum_{m,n \geq 1} \sum_{l \geq 1} \frac{\lambda_g(n)}{m(nl)^{1/2}} \sum_{r} h_3(\mu_r) \frac{A_{\mathcal{F}}(m,n)}{N_{\mathcal{F}}} A_r(1,l)$$

$$+ \sum_{m,n \geq 1} \sum_{l \geq 1} \frac{\lambda_g(n)}{m(nl)^{1/2}} \sum_{r} h_4(\mu_r) \frac{A_{\mathcal{F}}(m,n)}{N_{\mathcal{F}}} A_r(1,l)$$

$$= S_1(T) + S_2(T) + S_3(T) + S_4(T)$$

Here

$$h_1(\mu_r) = h(\mu_r) W_r(nm^2) \tilde{V}_r(l), \quad h_2(\mu_r) = h(\mu_r) W_r(nm^2) \tilde{V}_r(l),$$

$$h_3(\mu_r) = h(\mu_r) \tilde{W}_r(nm^2) \tilde{V}_r(l) \quad \text{and} \quad h_4(\mu_r) = h(\mu_r) \tilde{W}_r(nm^2) \tilde{V}_r(l).$$

Now, Theorem 1 follows immediately by the proposition given below.

**Proposition 1.** We have

$$S_1(T) = L(1,g) \frac{1}{192 \pi^5} \int_{\mathbb{R}(\mu)=0} h(\mu) \prod_{j=1}^{3} \frac{\Gamma\left(\frac{1}{2} + \frac{\mu_j}{2}\right)}{\Gamma\left(\frac{1}{2} - \frac{\mu_j}{2}\right)} \operatorname{spec}(\mu) d\mu + O(T^{\frac{17}{6} + \varepsilon} R^2),$$

$$S_2(T) = \zeta\left(\frac{3}{2}\right) \frac{1}{192 \pi^5} \int_{\mathbb{R}(\mu)=0} h(\mu) \operatorname{spec}(\mu) d\mu + O(T^{\frac{17}{6} + \varepsilon} R^2),$$

$$S_3(T) = L(1,g) \frac{1}{192 \pi^5} \int_{\mathbb{R}(\mu)=0} h(\mu) \prod_{j=1}^{3} \frac{\Gamma\left(\frac{1}{2} + \frac{\mu_j}{2}\right)}{\Gamma\left(\frac{1}{2} - \frac{\mu_j}{2}\right)} \operatorname{spec}(\mu) d\mu + O(T^{\frac{17}{6} + \varepsilon} R^2),$$

$$S_4(T) = \zeta\left(\frac{3}{2}\right) \frac{1}{192 \pi^5} \int_{\mathbb{R}(\mu)=0} h(\mu) \prod_{j=1}^{3} \frac{\Gamma\left(\frac{1}{2} + \frac{\mu_j}{2}\right)}{\Gamma\left(\frac{1}{2} - \frac{\mu_j}{2}\right)} \operatorname{spec}(\mu) d\mu + O(T^{\frac{17}{6} + \varepsilon} R^2).$$

3.1. **Proof of the Proposition** We only prove the first identity ($S_1(T)$) of the Proposition as the proof of other identities is the same as the first identity.

Applying the Kuznetsov’s trace formula (Lemma 2.3), one has

$$S_1(T) = \sum_{m,n \geq 1} \sum_{l \geq 1} \frac{\lambda_g(n)}{m(nl)^{1/2}} \left(\Delta^{(1)} + \Sigma_{4}^{(1)} + \Sigma_{5}^{(1)} + \Sigma_{6}^{(1)} - \mathcal{E}_{\min}^{(1)} - \mathcal{E}_{\max}^{(1)}\right),$$

(9)
where
\[
\Delta^{(1)} := \delta_{m,1} \delta_{n,t} \frac{1}{192\pi^5} \oint_{\mathcal{R}(\mu)=0} h_1(\mu) d_{\text{spec}} \mu,
\]
\[
\Sigma_4^{(1)} := \sum_{\epsilon=\pm 1} \sum_{D_2|D_1, lD_1=mD_2^2} \frac{\tilde{S}(\epsilon n, l; D_1, D_2)}{D_1 D_2} \Phi_{w_4}^{(1)} \left( \frac{\epsilon l n}{D_1 D_2} \right),
\]
\[
\Sigma_5^{(1)} := \sum_{\epsilon=\pm 1} \sum_{D_2|D_1, lD_2=mD_2^2} \frac{\tilde{S}(\epsilon m, 1; D_1, D_2)}{D_1 D_2} \Phi_{w_5}^{(1)} \left( \frac{\epsilon l m}{D_1 D_2} \right),
\]
\[
\Sigma_6^{(1)} := \sum_{\epsilon_1,\epsilon_2=\pm 1} \sum_{D_1, D_2} S(\epsilon_2 n, \epsilon_1 m, 1; D_1, D_2) \Phi_{w_6}^{(1)} \left( \frac{-\epsilon_2 n D_2}{D_1^2}, \frac{-\epsilon_1 l m D_1}{D_2} \right),
\]
with \( \Phi_{w_4}^{(1)}(y) \), \( \Phi_{w_5}^{(1)}(y) \) and \( \Phi_{w_6}^{(1)}(y) \) defined as in (8) by using the new test function \( h_1(\mu_r) = h(\mu_r) W_f (nm^2) \gamma (l) \) respectively; and
\[
\epsilon_{\text{min}} := \frac{1}{24(2\pi)^2} \oint_{\mathcal{R}(\mu)=0} \frac{h_1(\mu) B_{\text{min}}^{(1, l)} B_{\text{min}}^{(m, n)}}{N_{\mu}^{\text{min}}} d\mu_1 d\mu_2,
\]
\[
\epsilon_{\text{max}} := \sum_{f} \frac{1}{2\pi i} \oint_{\mathcal{R}(u)=0} \frac{h_1(u + it_1, u - it_1, -2a)}{N_{u,f}^{\text{max}}} B_{u,f}^{(1, l)} B_{u,f}^{(m, n)} du,
\]

3.1.1. The diagonal term. Let us denote \( D^{(1)} \) to be contribution of \( \Delta^{(1)} \) to the sum in the equation (8). Thus we have
\[
D^{(1)} = \frac{1}{192\pi^5} \oint_{\mathcal{R}(\mu)=0} h_1(\mu) D(\mu) d_{\text{spec}} \mu,
\]
where
\[
D(\mu) := \frac{1}{(2\pi i)^2} \int_{(3)} \int_{(3)} L(1 + u_1 + u_2, g) \frac{\gamma(\frac{1}{2} + u_1, \tilde{F})}{\gamma(\frac{1}{2}, \tilde{F})} \frac{\gamma(\frac{1}{2} + u_2, g \otimes F)}{\gamma(\frac{1}{2}, g \otimes F)} \times G(u_1) G(u_2) \frac{du_1}{u_1} \frac{du_2}{u_2}
\]
with \( G(u) = e^{u^2} \), \( \gamma(u, \tilde{F}) = \prod_{j=1}^{3} \Gamma_{\mathcal{R}}(u + \mu_j) \), \( \gamma(u, F) = \prod_{j=1}^{3} \Gamma_{\mathcal{R}}(u - \mu_j) \) and
\[
\gamma(u, g \otimes F) = \prod_{j=1}^{3} \Gamma_{\mathcal{R}}(u + \frac{k-1}{2} - \mu_j) \Gamma_{\mathcal{R}}(u + \frac{k+1}{2} - \mu_j).
\]

Now we evaluate the double integral in the first term on the right side of the equation (10). We first shift the contour to \( \mathcal{R}(u_1) = \epsilon, \mathcal{R}(u_2) = \epsilon \) without encountering a pole, for any \( \epsilon > 0 \). Then we move the contour \( \mathcal{R}(u_2) = \epsilon \) to \( \mathcal{R}(u_2) = -\frac{1}{2} \), in doing so we encounter a simple pole at \( u_2 = 0 \). Since, \( G(u) \ll e^{-u^2} \) and \( L(1/2 + it, g) \ll t^{1/3} \), the integral on \( \mathcal{R}(u_1) = \epsilon, \mathcal{R}(u_2) = -\frac{1}{2} \) is bounded by \( O \left( \prod_{j=1}^{3} |\mu_j|^{-1/2} \right) \). The contribution from the residue at \( u_2 = 0 \) is
\[
\frac{1}{2\pi i} \int_{(\epsilon)} L(1 + u_1, g) \frac{\gamma(\frac{1}{2} + u_1, \tilde{F})}{\gamma(\frac{1}{2}, \tilde{F})} G(u_1) \frac{du_1}{u_1} \quad (11)
\]
Now shift the contour in (11) to \( \Re(u_1) = -\frac{1}{2} \) encountering a simple pole at \( u_1 = 0 \). The integral on the line \( \Re(u_1) = -\frac{1}{2} \) is bounded by \( O \left( \prod_{j=1}^{3} |\mu_j|^{-1/4} \right) \). The contribution from the residue at \( u_1 = 0 \) is \( L(1,g) \gamma \left( \frac{1}{2}, \hat{F} \right) \). Note that, \( \int_{\Re(\mu)=0} h(\mu) \text{spec}(\mu) d\mu \asymp T^3R^2 \). Therefore,

\[
D^{(1)} = L(1,g) \frac{1}{192\pi^5} \int_{\Re(\mu)=0} h(\mu) \prod_{j=1}^{3} \frac{\Gamma \left( \frac{1}{4} + \frac{\mu_j}{2} \right) \text{spec}(\mu) d\mu}{\Gamma \left( \frac{1}{4} - \frac{\mu_j}{2} \right)} + O(T^{2+\varepsilon}R^2).
\]

3.1.2. Contribution of \( \Sigma_4^{(1)} \). Let \( E_4^{(1)} \) be the contribution of \( \Sigma_4 \) to the sum in the equation (11). So

\[
E_4^{(1)} = \sum_{m,n \geq 1} \sum_{l \geq 1} \frac{\lambda_g(n)}{m(nl)^{1/2}} \sum_{\varepsilon \pm 1} \sum_{D_1|D_2, D_1 = mD_2} \frac{\tilde{S}(\mp ln, l, 1; D_1, D_2) \Phi_{\omega_4}^{(1)} \left( \mp l\ln D_1 D_2 \right)}{D_1 D_2},
\]

Let \( U_i(x) \) (\( i = 1, 2 \)) be smooth functions which are compactly supported in \([1, 2]\), satisfy \( x^j U_i^{(j)}(x) \ll 1 \). By partition of unity, to get a bound for \( E_4^{(1)} \) it is enough to estimate the following sum

\[
\sum_{m,n \geq 1} \sum_{l \geq 1} \frac{\lambda_g(n)U_1 \left( \frac{nm^2}{N} \right) U_2 \left( \frac{l}{L} \right)}{(nm^2)^{1/2}(l)^{1/2}} \sum_{\delta D = mD} \frac{\tilde{S}(\mp ln, l; D, \delta D) \Phi_{w_4}^{(1)} \left( \mp l\ln \delta D^2 \right)}{\delta D^2}, \tag{12}
\]

where \( 1 \leq N \leq T^{3+\varepsilon} \) and \( 1 \leq L \leq T^{3/2+\varepsilon} \). By Lemma 2.3, \( \Phi_{w_4}^{(1)} \left( \mp l\ln \delta D^2 \right) \) is negligibly small unless

\[
\frac{\ln \delta D^2}{\delta D^2} \ll T^{3-\varepsilon}. \tag{13}
\]

Note that \( \delta l = mD \). By (13), we deduce that

\[
1 \leq l\delta^3 \leq \frac{nm^2}{T^{3-\varepsilon}} \leq T^\varepsilon,
\]

which implies, \( \delta, L, mD \leq T^\varepsilon \) and \( T^{3-\varepsilon} \leq n \leq N \leq T^{3+\varepsilon} \).

We recall the Kloosterman sum

\[
\tilde{S}(\mp n, l; D, \delta D) = \sum_{C_1(\text{mod } D), C_2(\text{mod } \delta D)} e \left( \frac{lC_1 C_2}{D} \mp \frac{nC_1}{D} \right).
\]

In (12), the only non-trivial sum is the sum over \( n \), which is given by

\[
\sum_{n \geq 1} \lambda_g(n) e \left( \mp \frac{nC_1}{D} \right) \theta(n),
\]

where

\[
\theta(y) = \frac{1}{\sqrt{y}} U_1 \left( \frac{\ln y}{N} \right) \Phi_{\omega_4}^{(1)} \left( \mp \frac{ly}{\delta D^2} \right).
\]

Now the \( GL(2) \)-Voronoi summation formula transforms the above sum into

\[
\frac{1}{D} \sum_{n \geq 1} \lambda_g(n) e \left( \pm \frac{nC_1}{D} \right) \Theta(n),
\]
where
\[ \Theta(y) = 2\pi i^k \int_0^\infty \theta(x) J_{k-1} \left( \frac{4\pi \sqrt{xy}}{D} \right) \, dx. \]

We now analyse the integral transform \( \Theta(n) \). Using the properties of Bessel function we arrive at the following expression
\[ \frac{N^{1/4} D^{1/2}}{m^{1/2} n^{1/4}} \int_0^\infty \frac{1}{x^{3/4}} U_1(x) \Phi_{w_4}^{(1)} \left( \pm \frac{l N x}{\delta (mD)^2} \right) e \left( \pm \frac{2 \sqrt{nN x}}{md} \right) \, dx. \] (14)

By Lemma 2.4, we have
\[ |x|^j \frac{d^j}{dx^j} \Phi_{w_4}^{(1)} (x) \ll_{\epsilon,j} T^{3+\epsilon} R^2 (T + |x|^{1/3})^j. \]

First, we change the variable \( x \to x^2 \) in the integral of equation (14). Then by repeated integration by parts we see that
\[ \Theta(n) \ll_{\epsilon,j} \frac{N^{1/4} D^{1/2} T^{3+\epsilon} R^2}{m^{1/2} n^{1/4}} \left( T + \left( \frac{l N}{\delta (mD)^2} \right)^{1/3} \right)^j \left( \frac{mD}{\sqrt{nN}} \right)^j. \]

Since \( 1 \leq \delta, L, m, D \leq T^\epsilon \) and \( T^{3-\epsilon} \leq N \leq T^{3+\epsilon} \). Therefore,
\[ E_4^{(1)} \ll_A T^{-A}, \]
for any \( A > 0 \).

3.1.3. Contribution of \( \Sigma_5^{(1)} \). Let \( E_5^{(1)} \) be the contribution of \( \Sigma_5 \) to the sum in the equation (11). As in the previous case it is enough to estimate the following sum
\[ \sum_{m,n \geq 1} \sum_{l \geq 1} \sum_{\delta \geq 1} \lambda_g(n) U_1 \left( \frac{nm^2}{l^{1/2}} \right) U_2 \left( \frac{l}{D} \right) \frac{\tilde{S}(\pm m, 1, l; D, \delta D)}{\delta D^2} \Phi_{w_5}^{(1)} \left( \pm \frac{lm}{\delta D^2} \right) \] (15)

to get a bound for \( E_5^{(1)} \). Here \( 1 \leq nm^2 \leq N \leq T^{3+\epsilon} \) and \( 1 \leq l \leq L \leq T^{3/2+\epsilon} \). Since \( \Phi_{w_5}^{(1)} \left( \pm \frac{lm}{\delta D^2} \right) \) is negligibly small unless
\[ T^{3-\epsilon} \ll \frac{lm}{\delta D^2}, \]

Note that \( \delta = nD \), together with above inequality we get
\[ nD^3 \leq \frac{lm}{T^{3-\epsilon}} \leq T^\epsilon. \]

Therefore \( n, \delta, D \leq T^\epsilon \) and \( T^{3/2-\epsilon} \leq m, l \leq T^{3/2+\epsilon} \). Recall the Kloosterman sum
\[ \tilde{S}(\pm m, 1, l; D, \delta D) = \sum_{(C_1 \mod D), (C_2 \mod \delta D)} \sum_{(C_1, D) = (C_2, \delta) = 1} e \left( \frac{\bar{C}_1 C_2}{D} + \frac{l C_2}{\delta} \right). \]

Now, the \( l \)-sum in the equation (15) is given by
\[ \sum_{l \geq 1} e \left( \frac{\bar{C}_2}{\delta} \right) \frac{1}{\sqrt{l}} U_2 \left( \frac{l}{L} \right) \Phi_{w_5}^{(1)} \left( \pm \frac{lm}{\delta D^2} \right). \]

An application of the Poisson summation formula to the above sum yields
\[ \sqrt{L} \sum_{l \in \mathbb{Z}} \sum_{l \equiv -\bar{C}_2 (\mod \delta)} \int_\mathbb{R} \frac{1}{\sqrt{y}} U_2(y) \Phi_{w_5}^{(1)} \left( \pm \frac{Lmy}{\delta D^2} \right) e \left( \frac{-llN}{\delta} \right) \, dy. \] (16)
Note that, for \( l \neq 0 \) (non-zero frequency), by repeated integration by parts we have the inner integral in (16) is
\[
\ll_{\epsilon,j} T^{3+\epsilon} R^2 \left( T + \left( \frac{mL}{\delta D^2} \right)^{1/3} \right)^j \left( \frac{\delta}{|l|L} \right)^j.
\]
Thus we need to have \( \delta = 1 \), otherwise there will be no zero frequency. Therefore, we have \( D = 1 \) and \( n = 1 \). In this case the contribution of zero frequency is given by
\[
\sqrt{L} \sum_{m \geq 1} \frac{1}{m} U_1 \left( \frac{m^2}{N} \right) \int \frac{1}{\sqrt{y}} U_2(y) \Phi^{(1)}_{w_5} (\pm Lmy) \, dy.
\]
By the definition of \( \Phi^{(1)}_{w_5} (y) \) and \( K_{w_4}(y; \mu) \), the above \( y \)-integral becomes
\[
\int_{\mathbb{R}} \frac{1}{\sqrt{y}} U_2(y) \int_{\mathbb{R}(\mu) = 0} h_1(\mu) K_{w_4} (\mp Lmy; -\mu) \, d_{\text{spec}} \mu \, dy
\]
\[
= \int_{\mathbb{R}(\mu) = 0} h_1(\mu) \int_{-\infty}^{\infty} \left( \int_{\mathbb{R}} y^{-s} \frac{1}{2} U_2(y) \, dy \right) |mL|^{-s} \hat{G}^{\pm}(s, -\mu) \frac{ds}{2\pi i} \, d_{\text{spec}} \mu
\]
\[
= \int_{\mathbb{R}(\mu) = 0} h_1(\mu) \int_{-\infty}^{\infty} |mL|^{-s} \hat{U}_1 \left( \frac{1}{2} - s \right) \hat{G}^{\pm}(s, -\mu) \frac{ds}{2\pi i} \, d_{\text{spec}} \mu,
\]
(17)
where \( \hat{U}_1(s) \) is the Mellin transform of \( U_1(y) \), which is entire and rapidly decaying. Then we can restrict the \( s \)-integral to \( |\text{Im}(s)| \leq T^\epsilon \). Recall the definition of \( \hat{G}^{\pm}(s, \mu) \)
\[
\hat{G}^{\pm}(s, \mu) := \frac{\pi^{-3s}}{12288\pi^{7/2}} \left( \prod_{j=1}^{3} \frac{\Gamma \left( \frac{1}{2} (s - \mu_j) \right)}{\Gamma \left( \frac{1}{2} (s + \mu_j) \right)} \right) \pm i \prod_{j=1}^{3} \frac{\Gamma \left( \frac{1}{2} (1 + s - \mu_j) \right)}{\Gamma \left( \frac{1}{2} (2 - s + \mu_j) \right)}.
\]
Since \( \mu_0 \sim \mu_{0,j} \sim T \), then the \( \mu \)-integral in (17) is bounded by \( T^{3/2+\epsilon} R^2 \).
Thus,
\[
E_5^{(1)} \ll T^{4+\epsilon} R^2
\]
\[3.1.4. \textbf{Contribution of } \Sigma_6^{(1)} \text{.} \]
In this case we have to bound the following sum
\[
\sum_{m,n \geq 1} \sum_{l \geq 1} \sum_{D_1, D_2} S(\pm n, \pm m, 1, l; D_1, D_2) \frac{\Phi^{(1)}_{w_6} (\mp nD_2 \frac{D_1}{D_2}, \mp lmD_1 \frac{D_2}{D_1})}{D_1 D_2}.
\]
Using the property \( \Phi_{w_6} \) (Lemma 2.5), we have
\[
T^{1-\epsilon} \leq \left( \frac{lmn^2}{D_1} \right)^{1/6} \text{ and } T^{1-\epsilon} \leq \left( \frac{n^2m^2}{D_2} \right)^{1/6}.
\]
From these conditions we infer that \( 1 \leq D_2 \leq T^\epsilon, 1 \leq D_1 \leq T^{1/2+\epsilon}/m \) and \( 1 \leq m \leq T^{1/2+\epsilon} /D_2 \), also \( T^{3-\epsilon} \leq N \leq T^{3+\epsilon} \) and \( T^{3/2-\epsilon} \leq L \leq T^{3/2+\epsilon} \). The Kloosterman sum \( S(\pm n, \pm m, 1, l; D_1, D_2) \) is given by
\[
\sum_{B_i, C_i (\text{mod } D_i); \sum B_i = 0 (\text{mod } D_1 D_2)} \sum_{D_1C_i = 0 (\text{mod } D_1 D_2)} e \left( \pm nb_1 + \left( Y_1 D_2 - Z_1 B_2 \right) \frac{D_1}{D_2} + \pm mb_2 + l(Y_2 D_1 - Z_2 B_1) \right)
\]
where \( B_i Y_j + C_j Z_j \equiv 1 \) (mod \( D_j \)) for \( j = 1, 2 \). Note that \( (B_1, D_1) \mid D_2 \), so \( (B_1, D_1) \leq T^\epsilon \) as \( D_2 \leq T^\epsilon \). Let \( B_1 = B'_1(B_1, D_1) \) and \( D_1 = D'_1(B_1, D_1) \) with \( (B'_1, D'_1) = 1 \).
Now we apply $GL(2)$-Voronoi summation and Poisson summation formulae on $n$ and $l$-sums in the equation (18) respectively. Then the $n$ and $l$-sums in (18) transform into

$$2 \pi i \frac{m \sqrt{L}}{\sqrt{N D_1}} \sum_{l \geq \pm (Y_2 D_1 - Z_2 B_1) \text{mod } D_2} \sum_{n \geq 1} \lambda_g(n) e \left( \frac{\mp B_1 n}{D_1} \right) \int_{\mathbb{R}} \int_{\mathbb{R}} \Phi_{i u_0}^{(1)} \left( \mp \frac{N D_2 x}{D_1^2 m^2}, \mp m L D_1 y \right) J_{k-1} \left( \frac{4 \pi \sqrt{N n x}}{m D_1^2} \right) e \left( \frac{-i l y}{D_2} \right) U_1(x) U_2(y) dx dy.$$ 

(19)

By repeated integration by parts, the $y$-integral in (19) is

$$\ll \varepsilon \cdot T^{3+\varepsilon} R^2 \left( T + \left( \frac{m D_1 L}{D_2^2} \right)^{1/2} + \left( \frac{m D_1 L}{D_2^2} \right)^{1/3} \left( \frac{N D_2}{D_2^2 m^2} \right)^{1/6} \right)^j \left( \frac{D_2}{|l| L} \right)^j,$$

for $l \neq 0$. Therefore the non-zero frequency $(l \neq 0)$ in (19) contributes $O(T^{-A})$. In the zero frequency $(l = 0)$ case, we must have $D_2 \mid (Y_2 D_1 - Z_2 B_1)$. In this case equation (19) becomes

$$2 \pi i \frac{m \sqrt{L}}{\sqrt{N D_1}} \sum_{n \geq 1} \lambda_g(n) e \left( \frac{\mp B_1 n}{D_1} \right) \int_{\mathbb{R}} \int_{\mathbb{R}} \Phi_{i u_0}^{(1)} \left( \mp \frac{N D_2 x}{D_1^2 m^2}, \mp m L D_1 y \right) \times J_{k-1} \left( \frac{4 \pi \sqrt{N n x}}{m D_1^2} \right) U_1(x) U_2(y) dx dy.$$ 

Inserting the properties of Bessel function in the above equation, we have

$$\frac{s}{N^{3/4} D_1^{1/2}} \sum_{n \geq 1} \lambda_g(n) \frac{e}{n^{1/4}} \left( \frac{\mp B_1 n}{D_1} \right) \int_{\mathbb{R}} \int_{\mathbb{R}} \Phi_{i u_0}^{(1)} \left( \mp \frac{N D_2 x}{D_1^2 m^2}, \mp m L D_1 y \right) \times e \left( \frac{2 \pi \sqrt{N n x}}{m D_1^2} \right) U_1(x) U_2(y) dx dy.$$ 

(20)

Integrating by parts we see that the $x$-integral in (20) is negligibly small unless $n \leq T^\varepsilon$ (Dual length). Therefore the sum in (18) is bounded above by

$$\frac{T^\varepsilon \sqrt{L}}{N^{3/4}} \sum_{n \leq T^\varepsilon} \lambda_g(n) \frac{1}{n^{1/4}} \sum_{D_1 \leq T^{1/2+\varepsilon} D_2 \leq T^\varepsilon} \frac{1}{D_1^{3/2} D_2} \sum_{m \leq \frac{T^\varepsilon}{m_{1/2+\varepsilon}}} m^{1/2} C(D_1, D_2, n, m) I(D_1, D_2, n, m),$$

where

$$C(D_1, D_2, n, m) = \sum_{B_1, C_1 \text{(mod } D_1)} \sum_{B_2, C_2 \text{(mod } D_2)} e \left( \frac{\mp B_1 n}{D_1} + \frac{Y_1 D_2 - Z_1 B_2}{D_1} \pm \frac{m B_2}{D_2} \right),$$

and

$$I(D_1, D_2, n, m) = \int_{\mathbb{R}} \int_{\mathbb{R}} \Phi_{i u_0}^{(1)} \left( \mp \frac{N D_2 x}{D_1^2 m^2}, \mp m L D_1 y \right) \times e \left( \frac{2 \pi \sqrt{N n x}}{m D_1^2} \right) U_1(x) U_2(y) dx dy.$$ 

Moreover, trivially $C(D_1, D_2, n, m)$ is bounded by $O(T^{3+\varepsilon} D_1)$ and the integral

$$I(D_1, D_2, n, m) \ll T^{3+\varepsilon} R^2.$$

Hence, the contribution of $\Sigma_6^{(1)}$ is $O(T^{4+\varepsilon} R^2)$. 

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3.1.5. **Contribution of \( E_{\min}^{(1)} \)**. Let us denote
\[
E_{\min}^{(1)} := \frac{1}{24(2\pi i)^2} \int_{\Re(\mu)=0} \frac{h(\mu)}{\mathcal{N}_\mu^{\text{min}}} \sum_{m,n \geq 1} \frac{\lambda_\mu(n)}{mn^{1/2}} B_{\mu}^{\text{min}}(m,n) W_f(nm^2) \\
\times \sum_{l \geq 1} \frac{1}{l^{1/2}} B_{\mu}^{\text{min}}(1,l) \tilde{V}_l(l) \, d\mu_1 d\mu_2.
\]
Inserting the definition of \( V_f(y), \tilde{W}(y) \) and using \((7), (5)\) we have
\[
E_{\min}^{(1)} = \frac{1}{24(2\pi i)^2} \int_{\Re(\mu)=0} \frac{h(\mu)}{\mathcal{N}_\mu^{\text{min}}} \mathcal{T}_{\min}^{(1)}(\mu) d\mu_1 d\mu_2,
\]
where
\[
\mathcal{T}_{\min}^{(1)}(\mu) = \frac{1}{2(2\pi i)^2} \int_{(3)} \int_{(3)} G(s_2) \prod_{j=1}^3 \frac{\Gamma_R \left( \frac{1}{2} + \frac{j}{2} + \mu_j \right)}{\Gamma_R \left( \frac{1}{2} - \mu_j \right)} \zeta(s_2 + \frac{1}{2} - \mu_j) \\
\times G(s_1) \prod_{j=1}^3 \frac{\Gamma_R \left( \frac{1}{2} + \frac{j}{2} + \mu_j \right)}{\Gamma_R \left( \frac{1}{2} - \mu_j \right)} \Gamma_R \left( \frac{1}{2} + \frac{k+1}{2} - \mu_j \right) L(s_1 + \frac{1}{2} + \mu_j, g) ds_1 ds_2.
\]
Now we move the line of integration from \( \Re(s_1) = \Re(s_2) = 3 \) to \( \Re(s_1) = \Re(s_2) = \varepsilon \). Since \( L(1/2 + it, g) \ll t^{1/3}, \zeta(1/2 + it) \ll t^{1/6} \) and \( G(s) \ll e^{-\varepsilon^2} \), therefore
\[
\mathcal{T}_{\min}^{(1)}(\mu) \ll \prod_{j=1}^3 (1 + |\mu_j|)^{1/2}.
\]
Finally, using the bound
\[
\mathcal{N}_\mu^{\text{min}} = \mathcal{N}_{\nu_1, \nu_2}^{\text{min}} := \frac{1}{16} \prod_{j=1}^3 |\zeta(1 + 3\nu_j)|^2 \gg \prod_{j=1}^3 \left( \frac{1}{\log(1 + 3\Re(\nu_j))} \right)^2,
\]
we obtain
\[
E_{\min}^{(1)} \ll T^{\varepsilon + R^2}.
\]

3.1.6. **Contribution of \( E_{\max}^{(1)} \)**. Let us define
\[
E_{\max}^{(1)} := \sum_f \frac{1}{2\pi i} \int_{\Re(u)=0} \frac{h(u + it_f, u - it_f, -2u)}{\mathcal{N}_{u,f}^{\max}} \\
\times \sum_{m,n \geq 1} \frac{\lambda_\mu(n)}{mn^{1/2}} B_{u,f}^{\text{max}}(m,n) W_f(nm^2) \sum_{l \geq 1} \frac{1}{l^{1/2}} B_{u,f}^{\text{max}}(1,l) \tilde{V}_l(l) \, du.
\]
By similar argument as above one has
\[
E_{\max}^{(1)} = \sum_f \frac{1}{2\pi i} \int_{\Re(u)=0} \frac{h(u + it_f, u - it_f, -2u)}{\mathcal{N}_{u,f}^{\max}} \mathcal{I}(u + it_f, u - it_f, -2u) \, du,
\]
where \( \mathcal{I}_{\max}^{(1)}(u + it_f, u - it_f, -2u) \) is given by
\[
\frac{1}{(2\pi i)^2} \int_{(3)} \int_{(3)} G(s_1) G(s_2) L(s_1 + \frac{1}{2} - 2u, g) L(s_1 + \frac{1}{2} + u, g \otimes f) L(s_2 + \frac{1}{2} - u, g) \times \\
\zeta(s_2 + \frac{1}{2} + 2u) \prod_{j=1}^3 \frac{\Gamma_R \left( s_1 + \frac{1}{2} + \frac{k-1}{2} - \alpha_j \right) \Gamma_R \left( s_1 + \frac{1}{2} + \frac{k+1}{2} - \alpha_j \right) \Gamma_R \left( s_2 + \frac{1}{2} + \alpha_j \right)}{\Gamma_R \left( \frac{1}{2} + \frac{k-1}{2} - \alpha_j \right) \Gamma_R \left( \frac{1}{2} + \frac{k+1}{2} - \alpha_j \right) \Gamma_R \left( \frac{1}{2} - \alpha_j \right)} ds_1 ds_2.
\]
Here $\alpha_1 = u + it_f$, $\alpha_2 = u - it_f$ and $\alpha_3 = -2u$. By the definition of $h$ we have $h(u + it_f, u - it_f, -2u)$ is negligibly small unless

$$\left| u + it_f - \mu_{0,1} \right| \leq R, \quad \left| u - it_f - \mu_{0,2} \right| \leq R, \quad \left| -2u - \mu_{0,3} \right| \leq R.$$

Note that $\mu_{0,j} \asymp T$ and $N_{u,f} \max := 8L(1, \Ad^2 f)[L(1 + 3u, f)]^2 \gg (1 + \log |u|)^{-1}$.

We first shift the line of integration to $\Re(s_1) = \Re(s_2) = \varepsilon$ and use the bounds

$$\zeta(1/2 + it) \ll |t|^{1/6}, \quad L(1/2 + it, f) \ll |t|^{1/3}, \quad L(1/2 + it, g) \ll |t|^{1/3}$$

and $L(1/2 + it, g \otimes f) \ll |t|$ to get

$$E_{\max}^{(1)} \ll T^{11/6 + \varepsilon} R \sum_{T - R \leq t_f \leq T + R} 1.$$

Using the Weyl law for the number of eigenvalues associated to $GL(2)$ cusp forms we have

$$E_{\max}^{(1)} \ll T^{17/6 + \varepsilon} R^2.$$

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