GROMOV–HAUSDORFF DISTANCE FOR
QUANTUM METRIC SPACES

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Abstract. By a quantum metric space we mean a $C^*$-algebra (or more generally an order-unit space) equipped with a generalization of the usual Lipschitz seminorm on functions which one associates to an ordinary metric. We develop for compact quantum metric spaces a version of Gromov–Hausdorff distance. We show that the basic theorems of the classical theory have natural quantum analogues. Our main example involves the quantum tori, $A_\theta$. We show, for consistently defined “metrics”, that if a sequence $\{\theta_n\}$ of parameters converges to a parameter $\theta$, then the sequence $\{A_{\theta_n}\}$ of quantum tori converges in quantum Gromov–Hausdorff distance to $A_\theta$.

1. Introduction

When one looks at the theoretical physics literature which deals with string theory and related topics, one finds various statements to the effect that some sequence of operator algebras converges to an other operator algebra. A mathematician specializing in operator algebras will immediately suspect that one is dealing here with continuous fields of operator algebras [19]. But closer inspection shows that in many situations the framework of the physicists involves various lengths (see references below), and that the physicists are quite careful about the bookkeeping for these lengths as they discuss the convergence of their sequences of algebras. (This is hardly surprising, since the physicists want action functionals, such as Yang–Mills functionals.) All of this suggests that there are metric considerations involved in their convergence of algebras, and that one is perhaps dealing with some kind of convergence for corresponding “quantum” metric spaces.

Within the mathematical literature, the only widely used notion of convergence of ordinary metric spaces of which I am aware is that

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given by the Gromov–Hausdorff distance between metric spaces [25], [26]. The aim of the present article is to introduce a corresponding “quantum Gromov–Hausdorff distance” for “quantum metric spaces”, and to develop its basic properties. As our main example we will consider the quantum tori [58] [60]. We will see that, for a consistent choice of “metrics”, if a sequence of parameters \( \{ \theta_n \} \) converges to a parameter \( \theta \), then the corresponding sequence of quantum tori, \( \{ A_{\theta_n} \} \), converges in quantum Gromov–Hausdorff distance to \( A_{\theta} \). We remark that recently quantum tori have found considerable employment in string theory ([15], [66], [42], [44], [67] and references therein).

In this article we will deal only with compact quantum metric spaces. (Most of the examples in the string-theory literature are compact.) I have already introduced the notion of a compact quantum metric space in [61], [62] (but without using that terminology), following up on glimpses of such a notion given by Connes [11], [12] in connection with his theory of quantum Riemannian geometry defined by Dirac operators. The basic definitions and facts will be reviewed in Section 2. But, very briefly, in the quantum case the role of the metric on an ordinary metric space is played by a generalization of the usual Lipschitz seminorm on functions which is defined by an ordinary metric. Our quantum spaces are unital \( C^* \)-algebras, or, more generally, order-unit spaces. It will be crucial for us that such a “Lipschitz seminorm” defines an ordinary metric on the state-space of our quantum space, in generalization of the Kantorovich metric [39] [40] on the probability measures on an ordinary metric space [61], [62].

I plan to discuss elsewhere further examples of quantum Gromov–Hausdorff convergence for situations pertinent to the quantum physics literature. One class of examples [63] involves sequences of matrix algebras of increasing dimension, equipped with consistent “metrics”, which converge to ordinary compact metric spaces. A number of examples in the quantum physics literature are of this type. The case in which the ordinary compact space is the 2-sphere appears in a number of places, and within this context the matrix algebras are often referred to as “fuzzy spheres”. See, for example [27], [5], sections 2.2 of [70], and references therein. Examples of bookkeeping with lengths is found, for example, in sections 7.2–3 of [50]. (A nice exposition of some of the relations between string theory and non-commutative geometry, with much bookkeeping of lengths, can be found in [44], but it contains little discussion of convergence of algebras.) Approximating the sphere by matrix algebras is popular because the symmetry group \( SU(2) \) acts on the matrix algebras as well as on the sphere, whereas traditional “lattice” approximations coming from choosing a
finite number of points on the sphere break that symmetry. In [63] we use the material of the present paper together with ideas from Berezin quantization to show how matrix algebras do indeed converge to the sphere for quantum Gromov-Hausdorff distance, with analogous results for any integral coadjoint orbit of any compact Lie group.

The quantum fuzzy sphere is discussed in [28]. The fuzzy 4-sphere also makes an occasional appearance — see [33] and references therein. The case in which the space is the 2-torus has also received considerable attention. See, for example, [20], [2], and references therein. My incomplete calculations make me optimistic that in a natural way one can show that matrix algebras converge to tori (of any dimension). What will distinguish these matrix algebras from those converging to the sphere is the “metric” structure which is placed on them. This is probably related to the ideas of “change of topology” which one finds in the string-theory literature [51] [3] [4] [35].

The case of higher-genus surfaces is mentioned fleetingly in the seminal paper [15], as well as in [8]. Within the mathematical literature, Berezin–Toeplitz quantization for compact Kähler manifolds has been extensively explored. (See [65] and references therein.) It leads to sequences of matrix algebras. It will be an interesting challenge to see how generally Berezin–Toeplitz quantization might mesh with our quantum metric-space theory.

Very recently, building on the present paper, Hanfeng Li has shown [46] for the Connes-Landi-DuboisViolette spheres \( \{ S_\theta \} \) (and related quantum manifolds) [17] [16] with their Dirac operators, that they form compact quantum metric spaces, and that if a sequence \( \theta_n \) of parameters converges to a parameter \( \theta \), then the sequence \( S_{\theta_n} \) converges to \( S_\theta \) for quantum Gromov-Hausdorff distance. Also, very recently David Kerr has developed [41] a matricial version of quantum Gromov-Hausdorff distance.

At an extremely speculative level, there is the popular BFSS conjecture [6] in string theory, which conjectures that the putative “M-theory” which is supposed to unify the various versions of string theory is a “suitable” limit of theories on matrix algebras. See [67] for relations with non-commutative geometry, including bookkeeping with lengths, especially in Section 7. One can wonder whether quantum Gromov–Hausdorff distance might have a bit to say in clarifying “suitable”.

When one goes back to the string-theory literature to see whether our notion of quantum Gromov-Hausdorff distance is of use, it is clear that, while it may clarify matters a bit, it is nevertheless quite inadequate. The reason is that the string-theorists need the whole apparatus
of bundles, connections, action functionals, etc. (including in the non-commutative setting). Thus what seems to be needed is a definition of when (quantum) spaces together with all of their apparatus are close together. No such definition seems to have been given so far, even for ordinary spaces, probably because it is not clear what apparatus to include once one goes beyond manifolds. (But see [31], [32], [29].) However, hints of what apparatus to include can be found in the literature on the collapsing of Riemannian manifolds. This involves bundles, and forms, over the (possibly singular) limit space. (See [23] [26] [64] [48] [49] [69] [43] and the references therein, especially to papers of Cheeger, Fukaya and Gromov.) However, no characterization has been given for the limit structures which occur in this Riemannian setting, so they do not have an autonomous existence yet. Thus much less is this so for quantum spaces. It is natural to speculate that Connes’ axioms for non-commutative Riemannian manifolds [14] [24] will play an important role in elucidating this matter.

The rest of the contents of this paper are as follows. In Section 3 we recall the definition of ordinary Gromov–Hausdorff distance, and, motivated by this definition, we develop the material concerning quotients (= “subsets”) which we will need in the quantum setting. Then in Section 4 we give our definition of quantum Gromov–Hausdorff distance, and prove that it satisfies the triangle inequality. We also discuss there the fact that when our definition is applied to ordinary metric spaces, it does not in general give the ordinary Gromov–Hausdorff distance. (But we show that the quantum distance is never greater than the ordinary distance.) A specific example of this failure, found by Hanfeng Li, is presented in Appendix 1.

In Section 5 we develop a useful technique for estimating quantum Gromov–Hausdorff distances, involving “bridges”, and we use it to present some simple examples. Bridges are used extensively in later sections.

In the situation of ordinary compact metric spaces, if the classical Gromov–Hausdorff distance between two metric spaces is 0, they need not be the same set-theoretically, but they will be isometric to each other. In Section 7 we show that the analogous fact holds for quantum Gromov–Hausdorff distance. In preparation for this we discuss isometries between compact quantum metric spaces in Section 6.

In [61] several ways of constructing “metrics” on $C^*$-algebras were described when one has an action of a compact group on a $C^*$-algebra together with a length function on the group. In Section 8, in preparation for our discussion of quantum tori, we give an important way of using this group-action construction to approximate the $C^*$-algebra
with metric, for Gromov–Hausdorff distance, by natural finite dimensional compact quantum metric spaces. Sections 9–11 then carry out the discussion of quantum tori. This involves, in particular, a discussion of continuous fields of quantum metric spaces.

A basic fact in the classical theory is that the metric space of isometry-classes of compact metric spaces, equipped with the Gromov–Hausdorff distance, is complete. In Section 12 we show that the same is true in the quantum case. In the classical case Gromov then gives an important necessary and sufficient criterion for when a subset of this complete space is compact. This criterion has had many applications in Riemannian geometry [26], [9], [64], [68]. Section 13 is devoted to formulating and demonstrating the corresponding quantum criterion.

Finally, in the brief second appendix we answer question 11.1 of [62] by observing that every “metric” for a compact quantum metric space can be obtained by the “Dirac operator” construction.

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2. COMPACT QUANTUM METRIC SPACES

As in [62] we will work with order-unit spaces. Typically they will arise as real linear subspace of the vector space of self-adjoint operators on a Hilbert space, which contain the identity operator (the order unit). In fact, any order-unit space can be realized in this way. We will be most interested in the order-unit spaces which arise as the space of all self-adjoint elements of a unital $C^*$-algebra (an algebra of operators on a Hilbert space which is closed under taking adjoints of operators, and closed for the operator norm). But we will see in Sections 8–11 that it is technically very useful to include order-unit spaces which do not arise in this way from $C^*$-algebras.

There is an attractive abstract characterization of order-unit spaces due to Kadison [37], [1]. An order-unit space is a real partially ordered vector space, $A$, with a distinguished element $e$ (the order unit) which satisfies:

1) (Order unit property) For each $a \in A$ there is an $r \in \mathbb{R}$ such that $a \leq re$.

2) (Archimedean property) If $a \in A$ and if $a \leq re$ for all $r \in \mathbb{R}$ with $r > 0$, then $a \leq 0$. 
The norm on an order-unit space is given by
\[ \|a\| = \inf \{ r \in \mathbb{R} : -re \leq a \leq re \}. \]

Thus \( A \) is a normed vector space, and we can consider its dual, \( A' \), consisting of the bounded linear functionals, equipped with the dual norm, \( \| \cdot \|' \).

By a state of an order-unit space \((A, e)\) we mean a \( \mu \in A' \) such that \( \mu(e) = 1 = \|\mu\|' \). States are automatically positive. We denote the collection of all the states of \( A \), i.e. the state-space of \( A \), by \( S(A) \). It is a bounded closed convex subset of \( A' \), and so is compact for the \( w^* \)-topology on \( A' \). Unless the contrary is specified, it will always be the \( w^* \)-topology which we use on \( S(A) \). Each \( a \in A \) defines a continuous affine function on \( S(A) \) by \( a(\mu) = \mu(a) \). It is a basic theorem of Kadison [37][1] that this representation of \( A \) as affine functions is isometric for the supremum norm on affine functions.

In [61], [62] we recalled that the metric on an ordinary compact metric space is determined by the Lipschitz seminorm it defines on functions. This suggested that for “non-commutative spaces”, that is \( C^* \)-algebras, the way to specify a “metric” is by means of a seminorm playing the role of a Lipschitz seminorm. For such a seminorm, \( L \), we defined an ordinary metric, \( \rho_L \), on \( S(A) \) by
\[ \rho_L(\mu, \nu) = \sup \{|\mu(a) - \nu(a)| : L(a) \leq 1\}. \]

(This may take value \(+\infty\) in the absence of further hypotheses.) This is a generalization of the Kantorovich metric on the probability measures on an ordinary compact metric space [39][40], and within the context of Dirac operators it was introduced into non-commutative geometry by Connes in [11], [12]. In [62] we extended these ideas to consider “Lipschitz seminorms” on order-unit spaces. Since \( C^* \)-algebras are over the complex numbers, while order-unit spaces are over the real numbers, we should point out that for \( C^* \)-algebras we require a Lipschitz seminorm, \( L \), to satisfy \( L(a^*) = L(a) \) for \( a \in A \). Under this condition it suffices to take the above supremum just over self-adjoint elements of \( A \) when defining \( \rho_L \). One sees this as follows. (See also lemma 1 of [36].) Let \( \mu, \nu \in S(A) \) and \( \delta > 0 \) be given. Then there is an \( a \in A \) such that, after multiplying it by a complex number of modulus 1, we have \( \mu(a) - \nu(a) \geq \rho_L(\mu, \nu) - \delta \) and \( L(a) \leq 1 \). Let \( b = (a + a^*)/2 \). Then \( L(b) \leq 1 \) since \( L(a^*) = L(a) \), while still \( \mu(b) - \nu(b) \geq \rho_L(\mu, \nu) - \delta \).

As in [62], we must require that \( L \) be such that \( \rho_L \) is nice. This is formulated by:
2.1 DEFINITION. Let \((A,e)\) be an order-unit space. By a **Lip-norm** on \(A\) we mean a seminorm, \(L\), on \(A\) with the following properties:

1) For \(a \in A\) we have \(L(a) = 0\) iff \(a \in \mathbb{R}e\).

2) The topology on \(S(A)\) from the metric \(\rho_L\) is the \(w^*\)-topology.

We remark that condition 2 implies that \(\rho_L\) takes only finite values on \(S(A)\). For if \(\rho(\mu_0, \nu_0) = +\infty\) for some \(\mu_0, \nu_0 \in S(A)\), then \(\{\mu : \rho(\mu, \nu_0) < +\infty\}\) is a proper subset of \(S(A)\) which is both open and closed, which is not possible since \(S(A)\), being convex, is always connected. In definition 5.1 of [62], where we first gave the definition of a Lip-norm, we also required that \(L\) be lower semi-continuous for the norm on \(A\). (Even for ordinary metric spaces \(L\) is usually not continuous. That is why we do not require that \(A\) be complete for its norm.) In theorem 4.2 of [62] we saw that if \(L\) is not lower semi-continuous, it can always be replaced by the largest lower semi-continuous seminorm smaller than \(L\), as this will give the same metric on \(S(A)\). We omit here the requirement of lower semi-continuity in Definition 2.1 only for the convenience that at many intermediate stages of argument we will then not need to verify, or adjust to obtain, this lower semi-continuity.

In definition 5.1 of [62], condition 2 of Definition 2.1 was stated in a form which appears quite different, but which is shown in theorem 1.9 of [61] to be equivalent, and which is more useful for verifying examples. We will recall this other form below in Theorem 4.5.

We are now prepared to make:

2.2 DEFINITION. By a **compact quantum metric space** we mean a pair \((A,L)\) consisting of an order-unit space \(A\) with a Lip-norm \(L\) defined on it.

We remark that the use of the word “quantum” here may seem a bit of a stretch. Given the \(C^*\)-algebraic origins, at first thought the terms “non-commutative” might seem more appropriate. But since we are using general order-unit spaces, which have no algebra structure in general, there is nothing present which can be “non-commutative”. On the other hand, the state space \(S(A)\) (with its metric \(\rho_L\)) will play the central role in our story, and states play a central role in quantum physics, so it seems to me that use of the term “quantum” is not unreasonable here.

Let \(A'\) denote again the Banach-space dual of \(A\). Let \(A^o\) denote the subspace of elements \(\lambda \in A'\) for which \(\lambda(e) = 0\). As seen in lemma 2.1 of [62], the ball of radius 2 about 0 in \(A^o\) coincides with \(\{\mu - \nu : \mu, \nu \in S(A)\}\). Define \(L'\) on \(A^o\) by the usual formula for a dual
norm, namely \( L'(\lambda) = \sup\{|\lambda(a)| : L(a) \leq 1\} \). A simple computation, given in lemma 4.3 of [62], shows that \( \rho_L(\mu, \nu) = L'(\mu - \nu) \). Since \( L \) is a Lip-norm so that \( \rho_L \) gives \( S(A) \) the \( w^* \)-topology, it follows that \( \rho_L \) is bounded on \( S(A) \), so that \( L' \) is bounded on the ball of radius 2. In other words, \( L' \) is bounded with respect to \( \| \cdot \|' \), and so there is a smallest constant, \( r_L \), such that \( L' \leq r_L \| \cdot \|' \). This constant is called the radius of \((A, L)\), because it is exactly half the diameter of the metric space \((S(A), \rho_L)\), as shown in proposition 2.2 of [62]. There we used the equivalent relation that \( \| \tilde{a} \| \sim \leq r_L \rho_L (\tilde{a}) \), where \( \tilde{a} \) denotes the image of \( a \) in \( \tilde{A} = A/\mathbb{R}e \), while \( \| \cdot \|' \) and \( L' \) denote the corresponding quotient norms. We often prefer to use the radius rather than the diameter so as to avoid factors of 2 in the above formulas. (The above observations work for any seminorm \( L \) for which \( L(e) = 0 \), though then we must allow \( r_A = +\infty \).) At times we will denote the diameter of \((A, L)\) by \( \text{diam}(A, L) \).

3. Quotients (= “subsets”)

We now recall the definition of classical Gromov–Hausdorff distance, since we will model our quantum version on it. Let \((Z, \rho)\) be an ordinary compact metric space, and let \( X \) be a closed subset of \( Z \). For any \( r \in \mathbb{R}, r > 0 \), we define the \( r \)-neighborhood, \( \mathcal{N}_r^\rho(X) \), of \( X \) for \( \rho \) by

\[
\mathcal{N}_r^\rho(X) = \{ z \in Z : \text{there is } x \in X \text{ with } \rho(z, x) < r \}.
\]

For closed subsets \( X \) and \( Y \) of \( Z \), the Hausdorff distance, \( \text{dist}_H^\rho(X, Y) \), between them for \( \rho \), is defined to be:

\[
\text{dist}_H^\rho(X, Y) = \inf\{ r : X \subseteq \mathcal{N}_r^\rho(Y) \text{ and } Y \subseteq \mathcal{N}_r^\rho(X) \}.
\]

Suppose now that \((X, \rho_X)\) and \((Y, \rho_Y)\) are independent compact metric spaces, not viewed as subsets of some larger metric space. There are several equivalent definitions of the Gromov–Hausdorff distance between them [25], [26], [9], [64]. The following is most convenient for our purposes. Let \( X \cup Y \) denote the disjoint union of \( X \) and \( Y \), with corresponding compact topology for which \( X \) and \( Y \) are closed subspaces. Let \( \mathcal{M}(\rho_X, \rho_Y) \) denote the set of all metrics on \( X \cup Y \) giving the topology of \( X \cup Y \) and whose restrictions to \( X \) and \( Y \) are \( \rho_X \) and \( \rho_Y \), respectively. Then the Gromov–Hausdorff distance between \((X, \rho_X)\) and \((Y, \rho_Y)\) is defined by

\[
\text{dist}_{GH}(X, Y) = \inf\{ \text{dist}_H^\rho(X, Y) : \rho \in \mathcal{M}(\rho_X, \rho_Y) \}.
\]

Note that for simplicity of notation we are not explicitly indicating the metrics \( \rho_X \) and \( \rho_Y \) on the left-hand side.
To formulate a version of the above definition for compact quantum metric spaces, we need first to see how to translate to the quantum situation the statement above that the restriction of $\rho$ to $X$ is $\rho_X$. Thus, let $(Z, \rho)$ be an arbitrary compact metric space, let $X$ be a closed subset of $Z$, and let $\rho_X$ be the restriction of $\rho$ to $X$. Let $C(Z)$ denote the algebra of real-valued continuous functions on $Z$, and similarly for $C(X)$. Let $L_X$ be the Lipschitz “seminorm” for $\rho_X$ on $C(X)$, defined for $g \in C(X)$ by

$$L_X(g) = \sup\{|g(x) - g(y)|/\rho_X(x, y) : x \neq y\}.$$ 

This supremum can have value $+\infty$. We can work with this, or with the dense subalgebra where $L_X$ is finite (the Lipschitz functions), as convenient. We will usually do the latter. Let $L$ be the corresponding Lipschitz seminorm for $\rho$ on $Z$. It is well-known [74] that for any $g \in C(X)$ there is an $f \in C(Z)$ such that $\pi(f) = g$ and $L(f) = L_X(g)$. (This latter fails for complex-valued functions, see example 1.5.7 of [74], thus providing one important reason for our emphasis on real vector spaces.) All of this says that $L_X$ is exactly the quotient seminorm from $L$ for $\pi$. For our purposes, the appropriate morphisms between order-unit spaces are linear positive maps which preserve the order-units, and consequently are of norm 1. (See proposition II.1.3 of [1], and surrounding text, for some other possibilities.)

Let $A$ and $B$ be order-unit spaces, and let $\varphi : A \to B$ be a morphism. We then have the dual mapping, $\varphi' : B' \to A'$, which is of norm 1 and carries $B'^\circ$ into $A'^\circ$. For any $\nu \in S(B)$ we have $\varphi'(\nu) \in S(A)$, and we obtain in this way a continuous (for the $w^*$-topologies) affine mapping, $S(\varphi)$, from $S(B)$ into $S(A)$. In particular, $S(\varphi)(S(B))$ will be a closed convex subset of $S(A)$.

For later purposes (e.g., Proposition 12.7), it is useful to treat seminorms $L$ on $A$ more general than Lip-norms; we will only require that $L(a) = 0$ exactly if $a \in \mathbb{R}e$. As in [62], we call these “Lipschitz seminorms”. They define a metric, $\rho_L$, on $S(A)$ by the same formula as before, except that $\rho_L$ can take the value $+\infty$. In particular, $\rho_L$ need not give the $w^*$-topology, though as seen in proposition 1.4 of [61] the $\rho_L$-topology is always finer than the $w^*$-topology.

3.1 Proposition. Let $A$ and $B$ be order-unit spaces, and let $\pi : A \to B$ be a morphism which is surjective, so that $S(\pi)$ is an injection.
of $S(B)$ into $S(A)$. Let $L$ be a Lipschitz seminorm on $A$, and let $L_B$ be the corresponding quotient seminorm on $B$, defined by

$$L_B(b) = \inf\{L(a) : \pi(a) = b\}.$$  

Then $\pi'$ is an isometry for the norms $L'_B$ and $L'$ on $B^\circ \cap A^\circ$, and $S(\pi)$ is an isometry for the corresponding metrics $\rho_{L_B}$ and $\rho_L$. If $L$ is actually a Lip-norm, then so is $L_B$.

It is important to note that in the setting of Proposition 3.1 the norm on $B$ (from the order-unit) will not, in general, be the quotient of that on $A$, though it is always less than or equal to the quotient norm. (See proposition II.1.6 of [1] and the example following it.) Put another way, in the classical setting described earlier, for any $g \in C(X)$ the extension theorem says that actually there is an $f \in C(Z)$ restricting to $g$ such that both $L(f) = L_X(g)$ and $\|f\|_\infty = \|g\|_\infty$. But we do not have this stronger statement in our present setting. However, this does not seem to cause us difficulties.

We also remark that in the setting of Proposition 3.1 it is not true in general that if $L$ is lower semi-continuous then $L_B$ is also, even if $L$ is a Lip-norm:

**3.2 EXAMPLE.** Let $L$ be the Lip-norm for the usual metric on the interval $Z = [0,3]$. Let $h$ be the function on $Z$ which interpolates linearly between the points $(0,0)$, $(1,0)$, $(2,1)$, $(3,-1)$. Notice that $L(h) = 2$. Let $A$ consist of the functions on $Z$ which are the sum of a polynomial and a scalar multiple of $h$. We view $L$ as defined just on $A$. It is easily seen that the restriction of any Lip-norm to a subspace containing the order-unit is again a Lip-norm (e.g., use Theorem 4.5), which is lower semi-continuous if the original Lip-norm is. Thus, $L$ on $A$ is a lower semi-continuous Lip-norm.

Let $B$ be the order-unit space of functions on $[0,2]$ obtained by restricting the functions in $A$ to $[0,2]$, and let $\pi$ be the corresponding restriction map. Let $L_B$ be the quotient of $L$ on $B$. Then $L_B$ is not lower semi-continuous. To see this, let $g = \pi(h)$. Note that $h$ is the only preimage of $g$ under $\pi$ in $A$. Thus, $L_B(g) = 2$. Somewhat as in example 3.5 of [62], for each $n \geq 1$ let $f_n$ be the continuous function on $Z$ with value 1 on $[1,3]$, value 0 on $[0,1-1/n]$, and linear in between. Let $q_n$ be a polynomial such that $\|f_n - q_n\|_\infty < 1/2n$ on $Z$, and let $p_n$ be the anti-derivative of $q_n$ for which $p_n(0) = 0$. Then

$$L(p_n) = \|p'_n\|_\infty \leq 1 + 1/2n.$$
Furthermore, for $t \in [0, 1]$ we have
\[ |p_n(t) - h(t)| = \left| \int_0^t q_n(s)ds \right| \leq 1/2n, \]
while for $t \in [1, 2]$ we have
\[ |p_n(t) - h(t)| \leq (1/2n) + \left| \int_1^t (q_n(s) - f_n(s))ds \right| \leq 1/n. \]
This says that $\{\pi(p_n)\}$ converges uniformly to $g$. But $L_B(\pi(p_n)) \leq L(p_n) \leq 1 + 1/2n$ while $L_B(g) = 2$. Thus, $L_B$ is not lower semi-continuous.

We can avoid the above pathology if we work with closed Lip-norms. (We recall from definition 4.5 of [62] that a Lip-norm $L$ on $A$ is closed if its “unit ball” $\{a : L(a) \leq 1\}$ is closed in the completion, $\bar{A}$, of $A$. Closed Lip-norms are automatically lower semi-continuous.) But before showing this, we give:

**Proof of Proposition 3.1.** The first part is basically just the familiar fact that when forming quotients of normed spaces, at the level of dual spaces one obtains isometries. Since here we work with seminorms, we recall the usual argument for this, using our present notation.

Let $\lambda \in B^\circ$. For any $a \in A$ we clearly have $L_B(\pi(a)) \leq L(a)$, and so if $L(a) \leq 1$ we have
\[ |\pi'(\lambda)(a)| = |\lambda(\pi(a))| \leq L_B'(\lambda). \]
Consequently, $L'(\pi'(\lambda)) \leq L_B'(\lambda)$. But let $\delta > 0$ be given, and let $b \in B$ with $L_B(b) \leq 1$. Then there is an $a \in A$ such that $\pi(a) = b$ and $L(a) \leq \alpha L(b)$, where $\alpha = 1 + \delta$. Thus, $L(a/\alpha) \leq L_B(b) \leq 1$. Consequently,
\[ L'(\pi'(\lambda)) \geq |\pi'(\lambda)(a/\alpha)| = |\lambda(b)|/\alpha. \]
Taking the supremum over $b \in B$ with $L_B(b) \leq 1$, we see that
\[ L'(\pi'(\lambda)) \geq L_B'(\lambda)/(1 + \delta). \]
Since $\delta$ is arbitrary, we see that $\pi'$ is indeed an isometry. But then $S(\pi)$ is also an isometry, since $\rho_L(\mu, \nu) = L'(\mu - \nu)$ by lemma 4.3 of [62], and similarly for $\rho_{L_B}$.

Suppose now that $L$ is a Lip-norm. Since $\pi'$ is $w^*$-continuous and injective, and $S(B)$ is compact, $\pi'$ is a homeomorphism of $S(B)$ onto $\pi'(S(B))$ in $S(A)$. Because $L$ is a Lip-norm, $\rho_L$ gives the $w^*$-topology on $S(A)$, and so the restriction of $\rho_L$ to $\pi'(S(B))$ gives the relative
topology of $\pi'(S(B))$. Since $S(\pi)$ is an isometry, it follows that $\rho_L$ gives the $w^*$-topology on $S(B)$. Thus $L_B$ is a Lip-norm.

For an example of the use of a quotient Lip-norm in a non-commutative context in the spirit of what we do here, see equation 3.19 of [75].

Because of Proposition 3.1 we will find it convenient in a number of places to identify $S(B)$ with its image in $S(A)$, and view the metric on $S(B)$ as just the restriction of $\rho_L$ to $S(B)$. Under the conditions of Proposition 3.1 we will say that $L$ induces $L_B$.

As promised earlier, we have:

3.3 PROPOSITION. Let $\pi : A \to B$ be a morphism of order-unit spaces which is surjective. Let $L$ be a Lip-norm on $A$ which is closed (and so lower semi-continuous). Then the quotient Lip-norm, $L_B$, is closed (so lower semi-continuous).

Proof. Let $(B^c, L_B^c)$ denote the closure of $(B, L_B)$, as in definition 4.5 of [62] (so that the “unit ball” for $L_B^c$ is the closure in $\overline{B}$ of that for $L_B$). Let $d \in B^c$ with $\|d\|_B \leq 1$. Then there is a sequence, $\{b_n\}$, in $B$ which converges in $\overline{B}$ to $d$ and for which $\{L_B(b_n)\}$ is bounded, say by $k$. We can assume that $\|b_n\| \leq \|b\| + k$ for each $n$. Pick some $\delta > 0$. Then we can find a sequence, $\{a_n\}$, in $A$ such that $\pi(a_n) = b_n$ and $L(a_n) \leq L_B(b_n) + \delta \leq k + \delta$ for each $n$. The following lemma will be useful later, as well as here.

3.4 LEMMA. Let $(A, L)$ be a compact quantum metric space of radius $r$, and let $\pi : A \to B$ be a morphism of order-unit spaces. For any $a \in A$ we have

$$\|a\| \leq \|\pi(a)\| + 2rL(a).$$

Proof. Fix some $\nu \in S(B)$. Then for any $\mu \in S(A)$ we have

$$|\mu(a)| \leq |\mu(a) - \pi'(\nu)(a)| + |\pi'(\nu)(a)| \leq \rho_L(\mu, \pi'(\nu))L(a) + |\nu(\pi(a))| \leq 2rL(a) + \|\pi(a)\|.$$ 

We return to the proof of Proposition 3.3. It follows from Lemma 3.4 that $\{a_n\}$ is a bounded sequence. Because $L$ is a Lip-norm, any sequence in $A$ which is bounded for both $\|\cdot\|$ and $L$ is totally bounded. (See Theorem 4.5.) Thus, $\{a_n\}$ has a subsequence, which we still denote by $\{a_n\}$, which converges in $A$, say to $a_*$. Since $(A, L)$ is closed, $a_* \in A$. But $\pi$ is continuous, and so $\pi(a_*) = d$. Hence $d \in B$. Thus $B^c = B$, so that $(B, L_B)$ is closed.
But again, we do not insist that our Lip-norms be closed, since that is not how examples are usually presented to us.

We have seen, in essence, that a surjection \( \pi : A \to B \) of order-unit spaces gives a continuous affine injection, \( \pi' \), of \( S(B) \) into \( S(A) \), so that \( \pi(S(B)) \) is a closed convex subset of \( S(A) \). We now show the converse. (For the moment, Lip-norms are not involved.) Let \( A \) be an order-unit space, and let \( K \) be a closed convex subset of \( S(A) \). We denote the space of affine continuous functions on \( K \) by \( Af(K) \). View the elements of \( A \) as (affine) functions on \( S(A) \), and let \( B \) consist of their restrictions to \( K \), with \( \pi \) the restriction map of \( A \) into \( Af(K) \), and so onto \( B \). As a subspace of \( Af(K) \) containing the order-unit, \( B \) has a natural structure as an order-unit space.

**3.5 Proposition.** Let \( A \) be an order-unit space, and let \( K \) be a closed convex subset of \( S(A) \). Let \( \pi \) be the “restriction” map from \( A \) into \( Af(K) \), and let \( B = \pi(A) \), with its natural order-unit space structure. Then \( \pi'(S(B)) = K \).

**Proof.** Since \( \pi \) is surjective, \( \pi' \) is injective. By the definition of \( B \), every \( \mu \in K \) defines a state, \( \mu_B \), of \( B \). For \( a \in A \) we then have \( \pi'_{\mu_B}(a) = \mu_B(\pi(a)) = \mu(a) \). Thus, \( K \subseteq \pi'(S(B)) \).

Suppose now that \( \mu \in S(A) \) and \( \mu \notin K \). Because \( K \) is convex and closed, by the Hahn–Banach theorem there is an \( a \in A \) and a \( t \in \mathbb{R} \) such that \( \mu(a) < t \leq \mu_1(a) \) for all \( \mu_1 \in K \). Upon replacing \( a \) by \( a - te \), we obtain \( a \in A \) such that \( \mu(a) < 0 \leq \mu_1(a) \) for all \( \mu_1 \in K \). Thus, \( \pi(a) \geq 0 \) in \( B \). If we had \( \mu = \pi'(\nu) \) for some \( \nu \in S(B) \), we would then have \( \mu(a) = \nu(\pi(a)) \geq 0 \). Thus, \( \mu \notin \pi'(S(B)) \), and so \( \pi'(S(B)) = K \) as desired. \( \square \)

We remark that if \( K_1 \) and \( K_2 \) are two closed convex subsets of \( S(A) \) with \( K_1 \nsubseteq K_2 \) then the above Hahn–Banach argument shows that there will be an \( a \in A \) which is non-negative on \( K_1 \) but takes a strictly negative value on at least one point of \( K_2 \). In particular, suppose that \( K \) is a closed convex subset of \( S(A) \) which spans all of \( A' \). Then \( \pi : A \to B \) is bijective, but it will not be an isomorphism of order-unit spaces if \( K \neq S(A) \), because \( \pi^{-1} \) will not be positive.

Given an order-unit space \( A \), there is an evident notion of isomorphism between pairs \((\pi, B)\) such that \( B \) is an order-unit space and \( \pi \) is a surjective morphism from \( A \) onto \( B \). We will refer to such pairs as “quotients” of \( A \). The above comments can be combined to obtain:
3.6 PROPOSITION. Let $A$ be an order-unit space. There is a natural bijection between isomorphism classes of quotients of $A$ and closed convex subsets of $S(A)$.

All of this makes it natural to think of the closed convex subsets of $S(A)$ as corresponding to the “quantum closed subsets” of $A$. If $L$ is a Lip-norm on $A$, then $\operatorname{dist}_{\rho}^L$ gives a metric on these quantum subsets. It is easily seen (and well-known) that the Hausdorff limit of convex subsets is again convex. Thus, the set of quantum closed subsets of $A$ is closed in the space of all closed subsets of $S(A)$ for the Hausdorff metric. Since the latter space is compact (Proposition 6.1 of the appendices of [64]) we see that the space of quantum closed subsets of $A$ is a compact metric space for the Hausdorff metric.

At various points later we will have use for the following transitivity property, which is really just a (probably known) property of quotients of general seminorms.

3.7 PROPOSITION. Let $A$, $B$ and $C$ be order-unit spaces, let $\pi_1$ be a morphism of $A$ onto $B$, and let $\pi_2$ be a morphism of $B$ onto $C$. Set $\pi = \pi_2 \circ \pi_1$, so that $\pi$ is a morphism of $A$ onto $C$. Let $L$ be a Lipschitz seminorm on $A$, let $L_B$ be its quotient for $\pi_1$, and let $L_C$ be its quotient for $\pi$. Let $L_B^C$ be the quotient of $L_B$ for $\pi_2$. Then $L_B^C = L_C$.

Proof. Let $c \in C$. If $a \in A$ and $\pi(a) = c$, then $\pi_2(\pi_1(a)) = c$, so that $L_C^B(c) \leq L_B(\pi_1(a)) \leq L(a)$. It follows that $L_C^B \leq L_C$. But let $\varepsilon > 0$ be given. Then there exists $b \in B$ with $\pi_2(b) = c$ and $L_B(b) \leq L_B^C(c) + \varepsilon$. But then there exists $a \in A$ with $\pi_1(a) = b$ and $L(a) \leq L_B(b) + \varepsilon$. Then $\pi(a) = c$ and $L(a) \leq L_B^C(c) + 2\varepsilon$. Thus $L_C(c) \leq L_B^C(c) + 2\varepsilon$. Since $\varepsilon$ is arbitrary, we obtain the desired conclusion. □

We will need later a (partial) converse to Proposition 3.1. We will consider two compact quantum metric spaces, $(A, L_A)$ and $(B, L_B)$, and a surjection $\pi: A \to B$, and we will consider, in terms of the dual norms $L_A'$ and $L_B'$, how $L_B$ relates to the quotient of $L_A$. For this purpose we let $(L_A)_B$ denote the quotient of $L_A$. Where convenient we will identify $B^\circ$ with its image in $A^\circ$ via $\pi'$. We approach our goal in steps.

3.8 PROPOSITION. With notation as above, suppose that $L_A'(\lambda) \leq L_B'(\lambda)$ for all $\lambda \in B^\circ$. If $L_B$ is lower semi-continuous, then $(L_A)_B \geq L_B$. 

Proof. For any $a \in A$ and $\lambda \in B^o$ we have
\[ |\langle \pi(a), \lambda \rangle| = |\langle a, \pi'(\lambda) \rangle| \leq L_A(a)L'_A(\pi'(\lambda)) \leq L_A(a)L_B(\lambda). \]
Thus, if $L_A(a) \leq 1$, then $|\langle \pi(a), \lambda \rangle| \leq L_B(\lambda)$, so that $\pi(a) \in O(ball_{L_B})$, where $ball_{L_B} = \{ \lambda : L_B(\lambda) \leq 1 \}$, and where $O$ denotes “prepolar” [18]. By the bipolar theorem [18], $O(ball_{L_B}) = (ball_{L_B})^-$. Since $L_B$ is lower semi-continuous, $ball_{L_B}$ is closed in $B$, and so $L_B(\pi(a)) \leq 1$. Thus, $L_B(\pi(a)) \leq L_A(a)$. It follows that $L_B \leq (L_A)_B$. \hfill \Box

3.9 Proposition. With notation as above, suppose that $L'_A(\lambda) \geq L'_B(\lambda)$ for all $\lambda \in B^o$. If $L_A$ is closed, then $(L_A)_B \leq L_B$.

Proof. The hypothesis says that
\[ ball_{L'_A} \cap B^o \subseteq ball_{L'_B}. \]
Let $\lambda \in B^o$. Now $\lambda \in ball_{L'_A} \cap B^o$ iff for all $a \in A$ with $L_A(a) \leq 1$ we have
\[ |\langle \pi(a), \lambda \rangle| = |\langle a, \pi'(\lambda) \rangle| \leq 1, \]
iff $\lambda \in (\pi(ball_{L_A}))^O$, the polar [18]. That is,
\[ ball_{L'_A} \cap B^o = (\pi(ball_{L_A}))^O. \]
Consequently,
\[ O((\pi(ball_{L_A}))^O) \supseteq O(ball_{L'_A}), \]
and so by the bipolar theorem [18],
\[ (\pi(ball_{L_A}))^- \supseteq (ball_{L_B})^- \supseteq ball_{L_B}. \]
Let $b \in ball_{L_B}$. It follows that there is a sequence, $\{a_n\}$, in $ball_{L_A}$ such that $\|b - \pi(a_n)\| \leq 1/n$ for each $n$. Note that then $\|\pi(a_n)\| \leq \|b\| + 1$ for each $n$. By Lemma 3.4 the sequence $\{a_n\}$ is bounded. Because $L_A$ is a closed Lip-norm, there is a subsequence which converges to some $a \in A$ such that $L_A(a) \leq 1$. (See Theorem 4.5.) But $\pi$ is continuous, so $\pi(a) = b$. Thus, $(L_A)_B(b) \leq 1$. This shows that $(L_A)_B \leq L_B$. \hfill \Box

Upon combining these two propositions, we obtain:

3.10 Corollary. With notation as above, suppose that $L'_A(\lambda) = L'_B(\lambda)$ for all $\lambda \in B^o$. If $L_A$ is closed and $L_B$ is lower semi-continuous, then $L_B = (L_A)_B$.

We remark that it then follows from Proposition 3.3 that $L_B$ is closed.
4. QUANTUM GROMOV–HAUSDORFF DISTANCE

From our descriptions of the classical Gromov–Hausdorff distance given in the previous section, it is easy to guess how to proceed in the quantum case. Let $(A, L_A)$ and $(B, L_B)$ be compact quantum metric spaces. The natural generalization of forming the disjoint union $X \cup Y$ is to form the direct sum, $A \oplus B$, of vector spaces, with $(e_A, e_B)$ as order-unit, and with evident order structure, to obtain an order unit space. We have the evident projections from $A \oplus B$ onto $A$ and $B$, which are order-unit space morphisms. The natural generalization of metrics on $X \cup Y$ which restrict to $\rho_X$ and $\rho_Y$ consists of Lip-norms on $A \oplus B$ which induce $L_A$ and $L_B$.

4.1 NOTATION. We will denote by $\mathcal{M}(L_A, L_B)$ the set of Lip-norms on $A \oplus B$ which induce $L_A$ and $L_B$.

For any given $L \in \mathcal{M}(L_A, L_B)$ we have its metric, $\rho_L$, on $S(A \oplus B)$. As discussed somewhat before Proposition 3.1, we view $S(A)$ and $S(B)$ as (closed, convex) subsets of $S(A \oplus B)$. Thus we can consider the Hausdorff distance between them, that is, $\text{dist}^\rho_H(S(A), S(B))$. We are now ready for the main definition of this paper.

4.2 DEFINITION. Let $(A, L_A)$ and $(B, L_B)$ be compact quantum metric spaces. We define the quantum Gromov–Hausdorff distance between them, denoted $\text{dist}_q(A, B)$, by

$$\text{dist}_q(A, B) = \inf \{ \text{dist}^\rho_H(S(A), S(B)) : L \in \mathcal{M}(L_A, L_B) \}.$$  

Let us see in what way $\text{dist}_q$ has the properties of a distance. It is clearly symmetric in $A$ and $B$. We are about to show that it satisfies the triangle inequality. In Section 7 we will show that if $\text{dist}_q(A, B) = 0$, then $A$ and $B$ are isometrically isomorphic, just as in the classical case. Thus, $\text{dist}_q$ is really a metric on the isometric isomorphism classes. We will even show, in Section 12, that the set of isometric isomorphism classes is complete for this metric.

4.3 THEOREM (THE TRIANGLE INEQUALITY). Let $(A, L_A)$, $(B, L_B)$, and $(C, L_C)$ be compact quantum metric spaces. Then

$$\text{dist}_q(A, C) \leq \text{dist}_q(A, B) + \text{dist}_q(B, C).$$

Proof. Let $\varepsilon > 0$ be given. Then we can find an $L_{AB} \in \mathcal{M}(L_A, L_B)$ such that

$$\text{dist}^\rho_H(S(A), S(B)) \leq \text{dist}_q(A, B) + \varepsilon.$$
Similarly we can find a corresponding $L_{BC} \in \mathcal{M}(L_B, L_C)$. For the next lemma, and in many places later, the symbol $\vee$ will mean “maximum”.

**4.4 LEMMA.** Define $L$ on $A \oplus B \oplus C$ by

$$L(a, b, c) = L_{AB}(a, b) \vee L_{BC}(b, c).$$

Then $L$ is a Lip-norm, and it induces $L_{AB}$, $L_{BC}$, $L_A$, $L_B$, and $L_C$ for the evident quotient maps.

**Proof.** We first verify the inducing statements. Let $\delta > 0$ be given. For any given $(a, b) \in A \oplus B$ we can find $c \in C$ such that $L_{BC}(b, c) \leq L_B(b) + \delta$. But $L_B(b) \leq L_{AB}(a, b)$, and so

$$L(a, b, c) \leq L(a, b) + \delta.$$  

Thus $L$ induces $L_{AB}$. In the same way one sees that $L$ induces $L_{BC}$. The fact that $L$ then induces $L_A$, $L_B$, and $L_C$ follows from the transitivity established in Proposition 3.7.

We now show that $L$ is a Lip-norm. Condition 1 of Definition 2.1 is easily checked. We must verify Condition 2, which asserts that $\rho_L$ gives the $w^*$-topology on $S(A \oplus B \oplus C)$. For this purpose we use the criterion given in theorem 1.9 of [61]. Since we will also use this criterion later, we recall it here for the reader’s convenience. For this we use the definition of radius recalled at the end of Section 2. As in theorem 1.9 of [61], or section 5 of [62], let

$$\mathcal{B}_1 = \{a : L(a) \leq 1 \text{ and } \|a\| \leq 1\}.$$  

Then we have:

**4.5 THEOREM** (essentially theorem 1.9 of [61]). Let $L$ be a semi-norm on the order-unit space $A$ such that $L(a) = 0$ iff $a \in \mathbb{R}e$. Then $\rho_L$ gives $S(A)$ the $w^*$-topology exactly if

i) $(A, L)$ has finite radius, and
ii) $\mathcal{B}_1$ is totally bounded in $A$ for $\| \cdot \|_A$.

We now apply this criterion to complete the proof of Lemma 4.4. Let $d_{AB}$ and $d_{BC}$ denote the diameters of $S(A \oplus B)$ and $S(B \oplus C)$ for $L_{AB}$ and $L_{BC}$, respectively. Pick any $\nu \in S(B)$. Then for any $\mu \in S(A)$ and $\zeta \in S(C)$ we have

$$\rho_L(\mu, \zeta) \leq \rho_L(\mu, \nu) + \rho_L(\nu, \zeta) = \rho_{LAB}(\mu, \nu) + \rho_{LBC}(\nu, \zeta) \leq d_{AB} + d_{BC},$$

since $\rho_L$ restricts to $\rho_{LAB}$ and $\rho_{LBC}$ by Proposition 3.1. Thus, the $\rho_L$-diameter of $S(A) \cup S(B) \cup S(C)$ is no bigger than $d_{AB} + d_{BC}$. But $S(A \oplus B \oplus C)$ is the closed convex hull of $S(A) \cup S(B) \cup S(C)$, and $\rho_L$ is convex (definition 9.1 of [62]) since it just comes from the dual
norm $L'$ of $L$ by lemma 4.3 of [62]. It follows easily that the diameter of $S(A \oplus B \oplus C)$ for $\rho_L$ is no bigger than $d_{AB} + d_{BC}$. Thus $L$ has finite radius.

To apply Theorem 4.5 we must show next that

$$B_1 = \{(a, b, c) : L(a, b, c) \leq 1 \text{ and } \|(a, b, c)\| \leq 1\}$$

is totally bounded in $A \oplus B \oplus C$. But if $(a, b, c) \in B_1$ then $L_{AB}(a, b) \leq 1$ and $\|(a, b)\| \leq 1$, while $L_{C}(c) \leq L_{BC}(c) \leq 1$ and $\|c\| \leq 1$. That is,

$$B_1 \subseteq B_1^{AB} \times B_1^C$$

for the evident notation. But $B_1^{AB}$ and $B_1^C$ are totally bounded, from which it is easy to see that $B_1$ is also. \hfill \Box

From Propositions 3.1 and 3.7 we then immediately obtain:

**4.6 Lemma.** Let $L_{AC}$ be the quotient of the above $L$ for the evident quotient map from $A \oplus B \oplus C$ onto $A \oplus C$. Then $L_{AC}$ is a Lip-norm which induces $L_A$ and $L_C$, that is, $L_{AC} \in \mathcal{M}(L_A, L_C)$.

We complete the proof of Theorem 4.3 by showing that $L_{AC}$ gives the desired estimate for $\text{dist}_q(A, C)$. Now $\rho_L$ restricted to $S(A \oplus B)$ is $\rho_{L_{AB}}$, and so

$$\text{dist}^\rho_{L}(S(A), S(B)) \leq \text{dist}_q(A, B) + \varepsilon.$$ 

In the same way

$$\text{dist}^\rho_{L}(S(B), S(C)) \leq \text{dist}_q(B, C) + \varepsilon,$$

and so

$$\text{dist}^\rho_{L}(S(A), S(C)) \leq \text{dist}_q(A, B) + \text{dist}_q(B, C) + 2\varepsilon.$$ 

But by Proposition 3.1 the restriction of $\rho_L$ to $S(A \oplus C)$ is $\rho_{L_{AC}}$, and so

$$\text{dist}^\rho_{L_{AC}}(S(A), S(C)) \leq \text{dist}_q(A, B) + \text{dist}_q(B, C) + 2\varepsilon.$$ 

Since $L_{AC} \in \mathcal{M}(L_A, L_C)$, it follows that

$$\text{dist}_q(A, C) \leq \text{dist}_q(A, B) + \text{dist}_q(B, C) + 2\varepsilon.$$ 

Since $\varepsilon$ is arbitrary, we obtain the triangle inequality for $\text{dist}_q$. \hfill \Box

It is important to emphasize that when our definition of quantum Gromov-Hausdorff distance is applied to ordinary compact metric spaces it does not in general agree with ordinary Gromov-Hausdorff distance. By this we mean the following. Let $(X, \rho)$ be an ordinary compact metric space, let $A$ denote its subalgebra of real-valued Lipschitz functions, viewed as an order-unit space, and let $L$ denote the Lipschitz
norm on $A$. Thus $(A, L)$ can be considered to be a compact quantum metric space. Now suppose that we have two ordinary compact metric spaces, $(X_j, \rho_j)$ for $j = 1, 2$, with associated $(A_j, L_j)$’s. Let us compare $\text{dist}_{GH}(X_1, X_2)$ with $\text{dist}_q(A_1, A_2)$. Suppose that $\rho$ is a metric on $X_1 \cup X_2$ whose restrictions to $X_j$ for $j = 1, 2$ coincide with $\rho_j$. Let $L$ be the Lip-norm on $A_1 \oplus A_2$ corresponding to $\rho$, with its associated metric $\rho_L$ on $S(A_1 \oplus A_2)$. Now $X_j$ is naturally identified with the set of extreme points of $S(X_j)$. Let $d_\rho = \text{dist}_{GH}(X_1, X_2)$. Given $x \in X_1$ there is a $y \in X_2$ with $\rho_L(x, y) = \rho(x, y) \leq d_\rho$, and conversely. Because $\rho_L$ is a convex metric (definition 9.1 of [62]) it follows that $\text{dist}_{GH}^\rho(S(A_1), S(A_2)) \leq d_\rho$. Consequently, $\text{dist}_q(A_1, A_2) \leq d_\rho$. Since $\text{dist}_{GH}(X_1, X_2) = \inf_\rho \{ d_\rho \}$, we obtain:

4.7 PROPOSITION. Let $(X_j, \rho_j)$ be ordinary compact metric spaces for $j = 1, 2$, with associated quantum compact metric spaces $(A_j, L_j)$. Then

$$\text{dist}_q(A_1, A_2) \leq \text{dist}_{GH}(X_1, X_2).$$

But the above inequality can fail to be an equality. A specific example, found by Hanfeng Li, is presented in Appendix 1.

Let us explore a bit more what is happening here. View everything inside $S(A_1 \oplus A_2)$. Let $L$ be any Lip-norm on $A_1 \oplus A_2$ which induces $L_1$ and $L_2$ on $A_1$ and $A_2$, respectively; and let $\rho_L$ be the corresponding metric on $S(A_1 \oplus A_2)$. Then $\rho_L$ certainly restricts to a metric on $X_1 \cup X_2$ which agrees with $\rho_j$ on $X_j$. Thus, $\text{dist}_{GH}^\rho(X_1, X_2)$ makes sense, and the infimum of these Hausdorff distances over all $L$ will be $\text{dist}_{GH}(X_1, X_2)$. But here we are asking how close each extreme point of $S(A_1)$ is to an extreme point of $S(A_2)$, and conversely. However, as the example in Appendix 1 shows, it can happen that each extreme point of $X_1$ is close to some point of $S(A_2)$, but that some extreme points of $X_1$ are relatively far from all of the extreme points of $S(A_2)$. And similarly for $X_2$. The consequence will be that $\text{dist}_q(A_1, A_2) < \text{dist}_{GH}(X_1, X_2)$.

Let us make clear that this can only happen because we admit Lip-norms which need not come from metrics on $X_1 \cup X_2$. To be specific:

4.8 PROPOSITION. Let $(X_j, \rho_j)$ for $j = 1, 2$ be ordinary compact metric spaces, and let $(A_j, L_j)$ be their associated quantum compact metric spaces. Let $L$ be a Lip-norm on $A_1 \oplus A_2$ which comes from a metric, $\rho$, on $X_1 \cup X_2$ whose restriction to $X_j$ is $\rho_j$ for $j = 1, 2$. Then

$$\text{dist}_{GH}(X_1, X_2) \leq \text{dist}_{GH}^\rho(S(A_1), S(A_2)).$$
Proof. Let \( d = \text{dist}_{GH}(X_1, X_2) \). Then there must be at least one point, \( p \), in one of \( X_1 \) or \( X_2 \), say \( X_1 \), such that \( \rho(p, y) \geq d \) for all \( y \in X_2 \). Define \( h \) on \( X_1 \cup \cdot \cup X_2 \) by \( h(z) = \rho(p, z) \). Then \( L(h) = 1 \). (This is the step which fails if \( L \) does not come from a metric, as can be seen by examining example 7.1 of [62].) Then for any \( \mu \in \mathcal{S}(A_2) \) we have

\[
\rho_L(\delta_p, \mu) \geq |(\delta_p - \mu)(h)| = \mu(h) = \int \rho(p, y) d\mu(y) \geq d.
\]

Thus \( \text{dist}^p_{H}(S(A_1), S(A_2)) \geq d. \) \( \square \)

Let us put this another way. For any order-unit space \( A \), let \( S^e(A) \) denote the set of extreme points of \( S(A) \). This can be a quite strange set. In particular, often it is not closed. Nevertheless, given quantum compact metric spaces \( (A_j, L_j) \) for \( j = 1, 2 \), and a Lip-norm \( L \) on \( A_1 \oplus A_2 \) inducing \( L_1 \) and \( L_2 \), it makes sense to consider \( \text{dist}^p_{H}(S^e(A_1), S^e(A_2)) \).

We will get the same result if we use the closures, \((S^e(A_j))^-\). Because \( L_\rho \) is a convex metric, it follows just as in the proof of Proposition 4.7 that:

4.9 PROPOSITION. With notation as above,

\[
\text{dist}^p_{H}(S(A), S(B)) \leq \text{dist}^p_{H}(S^e(A), S^e(B)).
\]

Thus for arbitrary compact quantum metric spaces \( (A_j, L_j) \) we could define \( \text{dist}^e \) by

\[
\text{dist}^e_q(A_1, A_2) = \inf_{L} (\text{dist}^p_{H}(S^e(A_1), S^e(A_2))).
\]

We would then have a definition which clearly agrees with the classical definition for ordinary compact metric spaces. But for many \( C^* \)-algebras the set \( S^e(A) \) is quite elusive. This is certainly true for the quantum tori (which constitute our main example), when there is some irrationality in the structure constants. (Though Ed Effros has reminded me that a theorem of Glimm (see lemma 11.2.1 of [19]) says that for any unital simple infinite-dimensional \( C^* \)-algebra, such as the irrational rotation algebras, \( S^e(A) \) is dense in \( S(A) \).) Thus, it is not clear to me whether \( \text{dist}^e \) can be useful.

Only experience with more examples will reveal whether the fact that \( \text{dist}^e \) does not always agree with \( \text{dist}_{GH} \) for ordinary compact metric spaces is “a feature or a bug”.

5. Bridges

Before continuing with the general theory it seems appropriate to give a few simple examples. When dealing with specific examples, the challenge, of course, is to construct Lip-norms on \( A \oplus B \) which induce \( L_A \) and \( L_B \) and for which \( \text{dist}_{H}^{\rho}(S(A), S(B)) \) is appropriately small. In this section we will formulate a somewhat general approach to doing this. We then use it to discuss some simple examples. But we will find this approach very useful later in connection with our main examples.

If \( L \in \mathcal{M}(L_A, L_B) \), then for \( (a, b) \in A \oplus B \) we must have
\[
L(a, b) \geq L_A(a) \vee L_B(b).
\]
Thus, we can look for \( L \) of the form
\[
L(a, b) = L_A(a) \vee L_B(b) \vee N(a, b)
\]
for some seminorm \( N \) on \( A \oplus B \). Every \( L \) is trivially of this form by setting \( N = L \). But if we think of \( N \) as only seeing distances between \( A \) and \( B \), not within \( A \) or within \( B \), then \( N \) can have a technically attractive feature. Intuitively, because we are viewing \( A \) and \( B \) as "disjoint", for any given \( L \) there should be a gap between \( S(A) \) and \( S(B) \), that is, a strictly positive lower bound on distances between points of \( S(A) \) and points of \( S(B) \). This means that \( N \) should be bounded with respect to the norm on \( A \oplus B \), unlike \( L_A \) and \( L_B \). Note that we must have \( N(e_A, e_B) = 0 \) if \( L \) is to have the same (required) property. But note also that we cannot have \( N(e_A, 0) = 0 \), for otherwise we would have \( L(e_A, 0) = 0 \), which is not permitted as it gives infinite distances. (Equivalently, we can require \( N(0, e_B) \neq 0 \).) Of course, the most important requirement — the one which is most difficult to arrange for specific examples — is that \( N \) be such that \( L \) induces \( L_A \) and \( L_B \). These observations are summarized in:

5.1 Definition. Let \( (A, L_A) \) and \( (B, L_B) \) be compact quantum metric spaces. By a bridge between \( (A, L_A) \) and \( (B, L_B) \) we mean a seminorm, \( N \), on \( A \oplus B \) such that:

1) \( N \) is continuous for the norm on \( A \oplus B \).
2) \( N(e_A, e_B) = 0 \) but \( N(e_A, 0) \neq 0 \).
3) For any \( a \in A \) and \( \delta > 0 \) there is a \( b \in B \) such that
\[
L_B(b) \vee N(a, b) \leq L_A(a) + \delta,
\]
and similarly for \( A \) and \( B \) interchanged.

5.2 Theorem. Let \( N \) be a bridge between compact quantum metric spaces \( (A, L_A) \) and \( (B, L_B) \). Define \( L \) on \( A \oplus B \) by
\[
L(a, b) = L_A(a) \vee L_B(b) \vee N(a, b).
\]
Then $L$ is a Lip-norm which induces $L_A$ and $L_B$. That is, $L \in \mathcal{M}(L_A, L_B)$.

If $L_A$ and $L_B$ are lower semi-continuous, then so is $L$.

**Proof.** From Condition 2 of Definition 5.1 it is clear that $L(e_A, e_B) = 0$ and that if $L(a, b) = 0$ then $(a, b) \in \mathbb{R}(e_A, e_B)$. From Condition 1 of Definition 5.1 it is easily seen that $L$ is lower semi-continuous if $L_A$ and $L_B$ are. From Condition 3 of Definition 5.1 it follows immediately that $L$ induces $L_A$ and $L_B$. Thus, the one property which is not quickly evident is that $\rho_L$ gives the $w^*$-topology on $S(A \oplus B)$. We now verify this property. For this purpose we use the criterion given in Theorem 4.5. In contrast to what usually happens, it is the finite radius condition which is slightly tricky to verify here, while the total boundedness is easy. We deal first with the finite radius condition.

Let $\gamma = 1/N(e_A, 0)$. We call $\gamma$ the gap of $N$ because we have:

### 5.3 Proposition. For all $\mu \in S(A)$ and $\nu \in S(B)$ we have $\rho_L(\mu, \nu) \geq \gamma$.

**Proof.** Since $L((\gamma e_A, 0)) = 1$, we have

$$\rho_L(\mu, \nu) \geq |\mu(\gamma e_A, 0) - \nu(\gamma e_A, 0)| = \gamma \mu(e_A) = \gamma.$$

\[\square\]

Let $(a, b) \in A \oplus B$. As usual (see section 1 of [62]) we let $\max(a) = \inf \{r : a \leq re_A\}$, and similarly for $\min(a)$. Set $m_a = (\max(a) + \min(a))/2$, viewed as either a scalar or as $m_ae_A \in A$. Then

$$\|a\| = \|a - m_a\| = (\max(a) - \min(a))/2.$$  

Define $m_b$ similarly. For any $t \in \mathbb{R}$ we have

$$\|a - t(e_A, e_B)\| = \|(a - m_a, b - m_b) + (m_a - te_A, m_b - te_B)\|$$

$$\leq \|a - m_a\| \lor \|b - m_b\| + |m_a - t| \lor |m_b - t|.$$

We minimize the right-most summand by setting $t = (m_a + m_b)/2$. Then the above expression is

$$\leq \|a\| \lor \|b\| + |(m_a - m_b)/2|.$$

Consequently,

$$\|(a, b)\| \leq \|a\| \lor \|b\| + |m_a - m_b|/2.$$
But
\[ |m_a - m_b| = \gamma N((m_a - m_b)e_A, 0) = \gamma N(m_a, m_b) \leq \gamma(N(a, b) + N(a - m_a, b - m_b)) \leq \gamma(L(a, b) + \|N\|(a - m_a, b - m_b)) = \gamma(L(a, b) + \|N\|(\|a\| \sim \|b\|)).\]

Putting this together with the previous inequality, we obtain:
\[ \|a\| \sim \leq (1 + \gamma\|N\|/2)\|a\| \sim + \gamma/2)\|b\| \sim + \gamma/2)\].

This says exactly that
\[ \text{radius}(A \oplus B, L) \leq (1 + \gamma\|N\|/2)r_A \sim r_B + \gamma/2, \]
and so \((A \oplus B, L)\) has finite radius.

We now verify that Condition ii) of Theorem 4.5 holds in the present situation. We denote by \(B_A^1\) the \(B_1\) of Theorem 4.5 for our \(A\), and similarly for \(B_B^1\). Let \((a, b) \in B_1^1\) for \((A \oplus B, L)\). We know that \(L_A(a) \leq L(a, b) = 1\) and \(\|a\| \sim \leq \|(a, b)\| \leq 1\), so that \(a \in B_A^1\). In the same way \(b \in B_B^1\). That is, \(B_1 \subseteq B_A^1 \times B_B^1\). But \(B_A^1\) and \(B_B^1\) are totally bounded since \(L_A\) and \(L_B\) are Lip-norms. It follows easily that \(B_1\) is totally bounded. \[\] We now give four simple applications of bridges. The method for constructing bridges which we use in the first application will play an important role later. (See Proposition 11.1.) This first application shows that the distance between any two compact quantum metric spaces is always finite.

**5.4 PROPOSITION.** Let \((A, L_A)\) and \((B, L_B)\) be compact quantum metric spaces. Then
\[ \dist_q(A, B) \leq \text{diameter}(A) + \text{diameter}(B). \]

**Proof.** We can think of a proof of the corresponding fact for ordinary metric spaces as follows. The two spaces are islands, with distance being (dry) minimal walking distance. We can choose a point on each of these islands and build a bridge between these two points, of arbitrarily small strictly positive length (the “gap”). This will not change the minimal walking distance on each island individually.
Accordingly, given compact quantum metric spaces \((A, L_A)\) and \((B, L_B)\), we choose arbitrarily \(\mu_0 \in S(A)\), \(\nu_0 \in S(B)\), and \(\gamma > 0\). As bridge we set
\[
N(a, b) = \gamma^{-1}|\mu_0(a) - \nu_0(b)|.
\]
It is evident that \(N\) satisfies the first two conditions of Definition 5.1. To check the third condition, given \(a \in A\) it suffices to choose \(b\) to be \(\mu_0(a) e_B\); and similarly if we are given \(b \in B\).

We now find a bound for the corresponding distance between \(S(A)\) and \(S(B)\). Let \(\mu \in S(A)\) and \(\nu \in S(B)\). For any \((a, b) \in A \oplus B\) with \(L(a, b) \leq 1\) we have \(|\mu_0(a) - \nu_0(b)| \leq \gamma\), so that
\[
|\mu(a, b) - \nu(a, b)| = |\mu(a) - \nu(b)|
\leq |\mu(a) - \mu_0(a)| + |\mu_0(a) - \nu_0(b)| + |\nu_0(b) - \nu(b)|
\leq \rho_{LA}(\mu, \mu_0) + \gamma + \rho_{LB}(\nu_0, \nu).
\]
Since \(\gamma\) is arbitrarily small, it follows that
\[
\dist_q(A, B) \leq \text{diameter}(A) + \text{diameter}(B),
\]
where \(\text{diameter}(A)\) is just the usual diameter of the ordinary metric space \((S(A), \rho_{LA})\), or equivalently, twice the radius defined earlier. □

We consider next distances from the one-point space. That is, we take \(B = \mathbb{R}\) and \(L_B \equiv 0\). Just as in the classical case we have:

**5.5 Proposition.** Let \(B\) be the one-point order-unit space \(\mathbb{R}\). Then for any compact quantum metric space \((A, L_A)\) we have
\[
\dist_q(A, \mathbb{R}) = \text{radius}(A).
\]

**Proof.** Of course, \(S(B)\) consists of just one point. Let \(r = \text{radius}(A)\). Since we saw that \(r\) is the ordinary radius of \(S(A)\) for \(\rho_{LA}\), and since \(\dist_q\) is defined in terms of ordinary Hausdorff distance between \(S(A)\) and \(S(B)\), it is easily seen that \(\dist_q(A, \mathbb{R})\) cannot be strictly smaller than \(r\). But define a bridge by
\[
N(a, b) = r^{-1} \sup\{ |\mu(a) - b| : \mu \in S(A) \}.
\]
It is easily seen that \(N\) satisfies the first two conditions of Definition 5.1. We check the third. Given \(b \in B = \mathbb{R}\), it suffices to choose \(a = b e_A\). Suppose instead that we are given \(a \in A\). Define \(m_a\) as done shortly after the proof of Proposition 5.3, so that, as seen there, \(\|a\|_{\sim} = \|a - m_a\|\). Set \(b = m_a\). For any \(\mu \in S(A)\) we have
\[
|\mu(a) - b| = |\mu(a - m_a)| \leq \|a\|_{\sim}.
\]
Thus \(N(a, b) \leq r^{-1}\|a\| \leq L_A(a)\), as needed.
For the corresponding $L$ we estimate the Hausdorff distance. Let $\eta$ denote the unique element of $S(B)$. Then for $(a, b) \in A \oplus B$ with $L(a, b) \leq 1$ we have, for any $\mu \in S(A)$,

$$|\mu(a, b) - \eta(a, b)| = |\mu(a) - b| \leq r.$$  

\[ \square \]

5.6 **EXAMPLE.** Given a compact quantum metric space $(A, L)$, it is at times convenient to form a space consisting of two copies of it at Hausdorff distance some given $\varepsilon > 0$ from each other. (A typical application will be given in the next proposition.) To do this we construct a bridge, $N$, on $A \oplus A$ by

$$N(a, b) = \varepsilon^{-1}||a - b||.$$  

It is easily seen that this is indeed a bridge. To see that it gives the desired distance, let $\mu \in S(A)$ viewed as in the first copy, then choose $\nu = \mu$ viewed as in the second copy. If $L$ is the Lip-norm on $A \oplus A$ using $N$, then when $L(a, b) \leq 1$ we have $||a - b|| \leq \varepsilon$, and so

$$|\mu(a, b) - \nu(a, b)| = |\mu(a) - \nu(b)| = |\mu(a - b)| \leq ||a - b|| \leq \varepsilon.$$  

The roles of $\mu$ and $\nu$ can be reversed. Thus, indeed the two copies are at distance $\leq \varepsilon$ from each other. But the gap $\gamma$ is clearly $\varepsilon$, and so by Proposition 5.3 the two copies are at distance exactly $\varepsilon$.

We will later have use for the following fact about quotients, as defined in Section 3, especially as viewed in Proposition 3.5 and the discussion which followed.

5.7 **PROPOSITION.** Let $(A, L_A)$ be a compact quantum metric space, and let $K_1$ and $K_2$ be compact convex subsets of $S(A)$. Let $(B_j, L_j)$ for $j = 1, 2$ be the corresponding quotients. Then

$$\text{dist}_q(B_1, B_2) \leq \text{dist}^\rho_H(K_1, K_2).$$  

**Proof.** We remark first that the inequality can be strict, for example for one-point subsets. We also remark that $K_1$ and $K_2$ need not be disjoint here.

We need to construct suitable Lip-norms on $B_1 \oplus B_2$. Actually, we will construct a Lip-norm for each $\varepsilon > 0$, using the construction of Example 5.6. Let $\varepsilon > 0$ be given, and define $N$ on $A \oplus A$ as in that example. View $K_j$ as $K_j \times \{0\}$ in $S(A \oplus A)$, and set $K'_j = \{0\} \times K_j$ in $S(A \oplus A)$, for $j = 1, 2$. Let $K = \text{co}(K_1 \cup K'_2)$, and let $\pi$ be the restriction map from $A \oplus A$ into $Af(K)$ as discussed before Proposition 3.5. Let $C$ be the image of $A \oplus A$ under $\pi$, so that it is the corresponding quotient.
By Proposition 3.5 we can canonically identify \( S(C) \) with \( K \). Now \( K_1 \) and \( K'_2 \) are disjoint subsets of \( K \) (in fact, “split faces” as discussed in [1]), and it is evident that the kernel of \( \pi \) is just the direct sum of the kernels of the restriction maps of \( A \) onto \( K_1 \) and \( K'_2 \), respectively. In other words, there is a canonical identification \( C = B_1 \oplus B_2 \).

Let \( M \) denote the Lip-norm on \( A \oplus A \) for \( N \), and let \( L \) denote its quotient on \( C \). According to Proposition 3.1 the metric \( \rho_L \) on \( S(C) = K \) coincides with the restriction to \( K \) of \( \rho_M \).

Of course, \( S(B_1) \) and \( S(B_2) \) are naturally included in \( S(C) \), and coincide with the inclusions of \( K_1 \) and \( K'_2 \) into \( K \). Thus, \( \text{dist}_q(B_1, B_2) \leq \text{dist}^{\rho_L}_H(K_1, K'_2) \). Now from the justification given in Example 5.6 it is clear that \( \text{dist}^{\rho_M}_H(K_1, K'_2) \leq \varepsilon \). But \( \rho_M \) coincides on \( S(A) \) with \( \rho_{L_A} \), so \( \text{dist}^{\rho_M}_H(K_1, K_2) = \text{dist}^{\rho_{L_A}}_H(K_1, K_2) \). By the ordinary triangle inequality

\[
\text{dist}^{\rho_M}_H(K_1, K'_2) \leq \text{dist}^{\rho_{L_A}}_H(K_1, K_2) + \varepsilon.
\]

But \( \text{dist}^{\rho_M}_H(K_1, K'_2) = \text{dist}^{\rho_L}_H(K_1, K'_2) \). Thus,

\[
\text{dist}^{\rho_L}_H(S(B_1), S(B_2)) \leq \text{dist}^{\rho_{L_A}}_H(K_1, K_2) + \varepsilon.
\]

Since \( \varepsilon \) is arbitrary, we obtain the desired result. \( \square \)

6. Isometries

One of the basic facts about the classical Gromov–Hausdorff distance is that if the distance between two compact metric spaces is zero, then the spaces are isometric. In the next section we will prove the corresponding fact for compact quantum metric spaces. But in preparation for this we need to discuss what we mean by isometries in the quantum case.

Let \( (A, L_A) \) and \( (B, L_B) \) be compact quantum metric spaces. Let \( \varphi \) be an order-unit isomorphism from \( A \) onto \( B \) such that \( L_B(\varphi(a)) = L_A(a) \) for all \( a \in A \). Let \( \varphi' \) denote the corresponding \( w^* \)-continuous affine bijection of \( S(B) \) onto \( S(A) \). Then it is easily seen that \( \varphi' \) is an ordinary isometry from \( (S(B), \rho_{L_B}) \) onto \( (S(A), \rho_{L_A}) \). Thus, we certainly want to consider \( \varphi \) to be an isometry from \( (A, L_A) \) to \( (B, L_B) \).

But this is not quite sufficient for our purposes. For example, let \( T \) be the unit circle, and let \( A \) consist of the smooth functions on \( T \). View \( A \) as consisting of functions on \( \mathbb{R} \) periodic of period 1. Define \( L \) on \( A \) just as in example 3.5 of [62], that is, by

\[
L(f) = \| f' \|_{\infty} + |f'(0)|.
\]

It is not difficult to show that \( L \) is a Lip-norm for our present definition, and that \( \rho_L \) is the usual metric on \( T \) coming from that on \( \mathbb{R} \). Thus, we would expect that every rotation of \( T \) gives an isometry of \( (A, L) \).
But it is clear that the only rotation leaving $L$ invariant is the identity rotation. Now as seen in example 3.5 of [62], $L$ is not lower semi-continuous. It is not difficult to show that the largest lower semi-continuous seminorm, $L^s$, smaller than $L$, whose existence is assured by theorem 4.2 of [62], is given by

\[ L^s(f) = \|f'\|_{\infty}. \]

In accordance with theorem 4.2 of [62], $L^s$ gives the same metric as $L$, and we see that $L^s$ is invariant under all rotations of $T$. This suggests that we need to use lower semi-continuous Lip-norms.

But this is not enough. Choose some fixed $t_0 \in T$, and define $A$ to be, instead, the algebra of all smooth functions on $T$ which are constant on some neighborhood of $t_0$, where the neighborhood depends on the function. Define $L$ on this $A$ by *) above. It is easy to see that $L$ is still lower semi-continuous, and that $\rho_L$ still gives the usual metric on $T$. But it is clear that the only rotation of $T$ which carries $A$ into itself is the identity rotation.

It appears that the nicest solution to these difficulties is to require that $(A, L)$ be closed, as defined in definition 4.5 of [62], and recalled above in connection with Proposition 3.3. By proposition 4.4 of [62] every lower semi-continuous Lip-norm extends uniquely to a closed Lip-norm. We have the following attractive description of the closure, which was not given in [62]:

**6.1 Proposition.** Let $(A, L)$ be a compact quantum metric space. If $L$ is closed, then $A$ corresponds to the subspace of $Af(S(A))$ consisting of all affine functions on $S(A)$ which are ordinary Lipschitz functions for $\rho_L$. Conversely, if $L$ is lower semi-continuous and if every affine function on $S(A)$ which is Lipschitz for $\rho_L$ corresponds to an element of $A$, then $L$ is closed.

**Proof.** The proof of the first assertion is very similar to the proof of theorem 4.2 of [62]. We include the argument here for the reader’s convenience. Let $\mathcal{L}_1$ denote the “unit-ball” for $L$, that is, $\{a \in A : L(a) \leq 1\}$. We view its elements as functions contained in $Af(S(A))$, and examine its bipolar [18] there, where $Af(S(A))$ is equipped with its order-unit norm. As in [62], we denote the norm-dual of $Af(S(A))$, which coincides with that of $A$, by $A'$. Since $\mathbb{R}e \subseteq \mathcal{L}_1$, the polar, $(\mathcal{L}_1)^O$, of $\mathcal{L}_1$ will be contained in $A'^o$, the set of $\lambda \in A'$ for which $\lambda(e) = 0$. In fact, essentially by definition,

\[ (\mathcal{L}_1)^O = \{\lambda \in A'^o : |\lambda(a)| \leq 1 \text{ for all } a \text{ with } L(a) \leq 1\}. \]
But this just says that \((L_1)^O\) is the unit-ball for the “dual seminorm”, \(L'\), to \(L\) on \(A^\circ\). Thus the bipolar, \(O((L_1)^O)\), of \(L_1\) is given by

\[O((L_1)^O) = \{ b \in Af(S(A)) : |\lambda(b)| \leq 1 \text{ for all } \lambda \text{ with } L'(\lambda) \leq 1 \}.\]

Equivalently, \(b \in O((L_1)^O)\) exactly if \(|\lambda(b)| \leq L'(\lambda)\) for all \(\lambda \in A^\circ\). As in [62], let \(D_2 = \{\lambda \in A^\circ : \|\lambda\| \leq 2\}\). Then, clearly, \(b \in O((L_1)^O)\) exactly if \(|\lambda(b)| \leq L'(\lambda)\) for all \(\lambda \in D_2\). But by lemma 2.1 of [62] every \(\lambda \in D_2\) is (not uniquely) of the form \(\lambda = \mu - \nu\) for \(\mu, \nu \in S(A)\), and conversely. Furthermore, by lemma 4.3 of [62] we then have \(L'(\lambda) = \rho_L(\mu, \nu)\). Thus, we see that \(b \in O((L_1)^O)\) exactly if \(|\mu(b) - \nu(b)| \leq \rho_L(\mu, \nu)\) for all \(\mu, \nu \in S(A)\). But this says exactly that \(L_{\rho_L}(b) \leq 1\). Thus, the bipolar of \(L_1\) in \(Af(S(A))\) coincides with the elements of \(\rho_L\)-Lipschitz norm \(\leq 1\). Now, \(L_1\) is already convex and balanced, so the bipolar theorem [18] says that the bipolar of \(L_1\) is the norm-closure of \(L_1\). But we are assuming that \(L_1\) is norm-closed. Thus, \(L_1\) coincides with the set of elements of \(Af(S(A))\) of \(\rho_L\)-Lipschitz norm \(\leq 1\). Consequently, \(A\) coincides with the set of elements of \(Af(S(A))\) which are Lipschitz for \(\rho_L\), as desired.

Conversely, suppose \(A\) corresponds to the set of all affine functions on \(S(A)\) which are Lipschitz for \(\rho_L\) (and so necessarily continuous). It is easily seen that the set of \(a \in Af(S(A))\) with \(L_{\rho_L}(a) \leq 1\) is norm-closed. Since we assume now that \(L\) is lower semi-continuous, we have \(L = L_{\rho_L}\) by theorem 4.2 of [62]. Thus \(L_1\) for \(L\) is closed in \(\overline{A}\), and this means that \(L\) is closed.

\[\square\]

6.2 THEOREM. Let \((A, L_A)\) and \((B, L_B)\) be compact quantum metric spaces which are closed. For every affine map \(\alpha\) from \(S(B)\) onto \(S(A)\) which is isometric for \(\rho_B\) and \(\rho_A\) there is a (unique) order isomorphism, \(\varphi\), of \(A\) onto \(B\) such that \(\alpha = \varphi'\) (that is, \(\alpha(\nu)(a) = \nu(\varphi(a))\) for all \(\nu \in S(B)\) and \(a \in A\)) and such that \(L_A = L_B \circ \varphi\). The converse is also true.

\[\textbf{Proof}.\] We already saw at the beginning of this section that the converse holds. Now let \(\alpha\) be as above. Since \(\alpha\) is isometric, it is continuous. Thus, composing with \(\alpha\) gives an order isomorphism, \(\varphi\), of \(Af(S(A))\) onto \(Af(S(B))\), and so from \(A\) onto \(B\) since we are assuming that \(L_A\) and \(L_B\) are closed (and we use Proposition 6.1). Since \(\alpha\) is isometric, we will have \(L_{\rho_B}(\varphi(a)) = L_{\rho_A}(a)\) for all \(a \in A\). But because \(L_A\) and \(L_B\) are closed, they are lower semi-continuous, so that \(L_{\rho_A} = L_A\) on \(A\), and similarly for \(B\). Thus, \(L_A = L_B \circ \varphi\) as desired. \[\square\]
For any compact quantum metric space \((A, L_A)\) let \(L_A^s\) denote, as before, the greatest lower semi-continuous Lip-norm smaller than \(L_A\), and let \((A^e, L_A^e)\) denote the closure of \((A, L_A)\).

**6.3 Definition.** Let \((A, L_A)\) and \((B, L_B)\) be compact quantum metric spaces. By an *isometry* from \((A, L_A)\) to \((B, L_B)\) we mean an order isomorphism \(\varphi\) from \(A^e\) onto \(B^e\) such that \(L_A^e = L_B^e \circ \varphi\).

From Theorem 6.2 we immediately obtain:

**6.4 Corollary.** The isometries from \((A, L_A)\) to \((B, L_B)\) are in natural bijective correspondence with the affine isometries from \((S(B), \rho_L B)\) onto \((S(A), \rho_L A)\).

Isometries can be composed in the evident way. There is, of course, a good argument for saying that \(\varphi\) as in Definition 6.3 is “from \((B, L_B)\) onto \((A, L_A)\)” rather than the other way around.

By the comments at the beginning of this section, if we happen to be given an order isomorphism \(\varphi\) from \(A\) to \(B\) such that \(L_A = L_B \circ \varphi\), then it will lead to an isometry as defined in Definition 6.3. Of course, the reason we do not always work with Lip-norms which are closed is that the Lip-norms which arise in examples are often not closed, e.g., the case considered earlier in which \(A\) consists of the smooth functions on \(T\). In fact, we see that to a large extent our theory is about compact convex sets (e.g., \(S(A)\)) and metrics on them which are suitably convex in the sense discussed in section 9 of [62]. But again, this is not the form in which the examples arise.

We remark that our definition of isometries is more general than the definition given in [54], [55], since that definition assumes that \(L\) comes from a specific spectral triple, and that the isometry is unitarily implemented on the Hilbert space of that spectral triple. A definition of isometries in the context of spectral triples which is closer to our definition makes a brief appearance in the third paragraph after corollary III.1.5 of [47].

We now give a class of examples of isometries which is closely related to the considerations which we will see in Section 8.

**6.5 Example.** As in [61], let \(G\) be a compact group and let \(\ell\) be a continuous length-function on \(G\). Let \(\alpha\) be an ergodic action of \(G\) on a unital \(C^*\)-algebra \(\bar{A}\). For \(a \in \bar{A}\) let

\[
L(a) = \sup\{\|\alpha_x(a) - a\|/\ell(x) : x \neq e_G\},
\]
and let $A = \{ a \in \bar{A} : L(A) < \infty \}$ (or, more precisely, the real subspace of self-adjoint elements). Then $(A, L)$ is a compact quantum metric space for which $L$ is closed (Proposition 8.1). As remarked in section 2 of [61], it is not so clear whether $A$ is carried into itself by $\alpha$. However, if $\ell$ satisfies the additional hypothesis that $\ell(\alpha x \alpha^{-1}) = \ell(x)$ for all $x, z \in G$, then a little calculation given in section 2 of [61] shows that $\alpha$ does carry $A$ into itself; and furthermore, that $\alpha$ leaves $L$ invariant. In other words, $G$ acts as a group of isometries of $(A, L)$ when $\ell$ satisfies this extra condition.

Let $(A, L)$ be a compact quantum metric space, and let $G = \text{Iso}(A, L)$ be its group of isometries. It is natural to define a length function, $\ell$, on $G$ by

$$\ell(\varphi) = \sup\{ \| \varphi(a) - a \| : a \in A^c \text{ and } L^c(a) \leq 1 \}.$$ 

A quick calculation then shows that $\ell$ has the special property used in Example 6.5, namely, $\ell(\psi \varphi \psi^{-1}) = \ell(\varphi)$ for all $\varphi, \psi \in G$. Thus, this special condition is quite natural in our context. The following proposition extends a well-known fact about the isometry groups of compact manifolds, and the comments surrounding equation 1.30 of [13].

**6.6 Proposition.** Let $(A, L)$ be a compact quantum metric space for which $L$ is complete. Let $G = \text{Iso}(A, L)$ equipped with the length function $\ell$ defined above, and the corresponding left-invariant metric, $\rho$. Then $G$ is a compact group for the topology defined by $\rho$.

**Proof.** Let $r$ denote the radius of $(A, L)$ as defined at the end of Section 2, and let

$$B_r = \{ a \in A : L(a) \leq 1 \text{ and } \| a \| \leq r \}.$$ 

Because $\| a \| \leq r L(a)$ for all $a \in A$, and $\varphi(e) - \psi(e) = 0$ for all $\varphi, \psi \in G$, an equivalent definition of $\rho$ is

$$\rho(\varphi, \psi) = \sup\{ \| \varphi(a) - \psi(a) \| : a \in B_r \}.$$ 

Note that $B_r$ is carried into itself by any element of $G$. Because $L$ is closed, $B_r$ is closed in $\bar{A}$, and so is compact for the norm topology. But each element of $G$ is an isometry for the norm, and so $G$ is a bounded equicontinuous collection of maps from $B_r$ into itself. Thus $G$ is totally bounded by the Arzela-Ascoli theorem for the point-norm topology, which is exactly the topology determined by $\rho$. But it is easily verified that $G$ is closed in the set of all continuous maps from $B_r$ into itself. Thus $G$ is compact. Since $\rho$ comes from a length function on $G$, its
It would be interesting to develop and study the notion of “quantum isometry group” for quantum metric spaces as quantum subgroups of the quantum symmetry groups studied by Wang [73].

7. Distance zero

The aim of this section is to show that if $\text{dist}_q(A, B) = 0$ then there is an isometry from $A$ to $B$, in the sense defined in the previous section. We note, conversely, that if there is an isometry, $\psi$, from $A$ to $B$ (so defined on $A^c$), then indeed $\text{dist}_q(A, B) = 0$. To see this, for any $\gamma > 0$ define a bridge from $A^c$ to $B^c$ by

$$N(a, b) = \frac{\gamma}{1} \|\psi(a) - b\|.$$ 

It is easily seen that for the corresponding Lip-norm $L$ on $A^c \oplus B^c$ we have $\text{dist}_{GH}(S(A^c), S(B^c)) \leq \gamma$. We also need to clarify the situation with:

7.1 PROPOSITION. Let $(A, L)$ be a compact quantum metric space, let $L^s$ denote the largest lower semi-continuous seminorm smaller than $L$ and let $(A^c, L^c)$ denote the closure of $(A, L)$. Then

$$\text{dist}_q((A, L), (A, L^s)) = 0 = \text{dist}_q((A, L), (A^c, L^c)).$$

Proof. We consider first the case of $L^s$. Let $\gamma > 0$ be given, and define $N$ on $A \oplus A$ by

$$N(a, b) = \frac{\gamma}{1} \|a - b\|.$$ 

We must show that $N$ is a bridge for $L$ and $L^s$. The first two conditions of Definition 5.1 are clearly satisfied. We now check Condition 3. Let $A_1$ and $A_2$ denote $A$ as first or second copy in $A \oplus A$. For $a \in A_1$ set $b = a$ in $A_2$. This works since $L^s(a) \leq L(a)$. Suppose, however, that we are given $b \in A_2$. We recall from theorem 4.2 of [62] that the “unit $L^s$-ball” is the norm closure in $A$ (not $\bar{A}$) of the “unit $L$-ball”. Thus we can find a sequence $\{a_n\}$ of elements in $A$ such that $L(a_n) \leq L^s(b)$ and $\{a_n\}$ converges to $b$ in norm. Consequently, we can find an $a_n$, viewed as in $A_1$, such that $\frac{\gamma}{1} \|a_n - b\| \leq L^s(b)$. This shows that $N$ is a bridge. For the corresponding Lip-norm on $A \oplus A$ for $N$, say $M$ (briefly, $M = (L \lor L^s) \lor N$), it is now routine to verify that

$$\text{dist}_{GH}(S(A_1), S(A_2)) \leq \gamma.$$ 

Since $\gamma$ is arbitrary, we see that $\text{dist}_q(A_1, A_2) = 0$. 

As a consequence of the above, we have

$$\text{dist}_q((A, L), (A^c, L^c)) = 0.$$
The proof for the case of \((A^c, L^c)\) is very similar, but now we must recall from proposition 4.4 of [62] that the “unit \(L^c\)-ball” is the closure in \(A\) of the “unit \(L\)-ball”, and notice that the norm in the definition above of \(N\) must be that for \(A^c\).\(\square\)

In developing our proof that if \(\text{dist}_q(A, B) = 0\) then there is an isometry between \(A\) and \(B\), we find it useful to first give a proof of the corresponding theorem for ordinary compact metric spaces, but along lines which then can be used for the proof of the quantum case. A significant obstacle to the proof in either case is that there is, in general, no uniqueness to the isometry — it can be composed with isometries of either of the individual spaces to get other isometries. This means that some choice principle must be used, usually in the form of compactness. The compactness which we find convenient to use here is that involved in the Arzela–Ascoli theorem [18]. In my browsing through the extensive literature concerning Gromov–Hausdorff distance I have not noticed the use of the Arzela–Ascoli theorem for this specific purpose. But I would not be surprised if this use does appear somewhere, since I have seen the Arzela–Ascoli theorem used for closely related purposes in the literature.

Let \((X, \rho_X)\) and \((Y, \rho_Y)\) be ordinary compact metric spaces. We wish to work on one fixed space containing them, namely the disjoint union \(X \cup Y\). To compensate for this we must admit semi-metrics, that is, continuous functions which satisfy all the properties of a metric except that they may take value 0 on some pairs of distinct points. We denote by \(\Sigma(\rho_X, \rho_Y)\) the family of all semi-metrics on \(X \cup Y\) whose restrictions to \(X\) and \(Y\) are \(\rho_X\) and \(\rho_Y\), respectively.

7.2 LEMMA. The family \(\Sigma(\rho_X, \rho_Y)\) of functions on \((X \cup Y)^2\) is equicontinuous.

Proof. Both \(X \times X\) and \(Y \times Y\) are open subsets of \((X \cup Y)^2\), and all the functions in \(\Sigma(\rho_X, \rho_Y)\) agree on these two subsets, so equicontinuity is obvious there. Suppose we are given \(x_0 \in X\) and \(y_0 \in Y\), as well as \(\varepsilon > 0\). Let

\[
N_{x_0} = \{x \in X : \rho_X(x, x_0) < \varepsilon/2\},
\]

and similarly for \(N_{y_0}\), so that \(N_{x_0} \times N_{y_0}\) is a neighborhood of \((x_0, y_0)\). If \((x, y) \in N_{x_0} \times N_{y_0}\), then for any \(\sigma \in \Sigma(\rho_X, \rho_Y)\) we have

\[
|\sigma(x, y) - \sigma(x_0, y_0)| \leq |\sigma(x, y) - \sigma(x, y_0)| + |\sigma(x, y_0) - \sigma(x_0, y_0)|
\leq \rho_Y(y, y_0) + \rho_X(x, x_0) < \varepsilon.
\] \(\square\)
7.3 Lemma. The family $\Sigma(\rho_X, \rho_Y)$ is uniformly closed in $C((X \cup Y)^2)$.

Proof. All of the conditions in the definition of a metric are closed conditions except the condition that the distance between distinct points must be non-zero. It is for this reason that we drop this condition and allow semi-metrics. \hfill \Box

7.4 Lemma. Let $\sigma \in \Sigma(\rho_X, \rho_Y)$. For each $x \in X$ there is at most one $y \in Y$ such that $\sigma(x, y) = 0$; and similarly for each $y \in Y$.

Proof. If $\sigma(x, y) = 0 = \sigma(x, y')$, then $\rho_Y(y, y') \leq \sigma(y, x) + \sigma(x, y') = 0$, so that $y' = y$. \hfill \Box

The condition that two points of $X \cup Y$ have $\sigma$-distance zero is an equivalence relation, $\sim$, on $X \cup Y$. By Lemma 7.4 each equivalence class contains either one point, or a pair $(x, y) \in X \times Y$. We set $X \cup_\sigma Y = (X \cup Y) / \sim$. Then $\sigma$ drops to a genuine metric, $\rho_\sigma$, on $X \cup_\sigma Y$. Furthermore, the map $i_X : X \to X \cup_\sigma Y$ defined by $i_X(x) = \hat{x}$ is clearly an isometry from $X$ into $X \cup_\sigma Y$, and similarly for $i_Y$. We clearly have $X \cup_\sigma Y = i_X(X) \cup i_Y(Y)$, not necessarily disjointly, and $X \cup_\sigma Y$ is clearly compact.

Suppose, conversely, that we have a compact metric space $(Z, \rho)$ together with isometries $j_X$ and $j_Y$ of $(X, \rho_X)$ and $(Y, \rho_Y)$ into $Z$ such that $Z = j_X(X) \cup j_Y(Y)$. It is evident how to define from this a semi-metric, $\sigma$, on $X \cup Y$ such that there is an isometry (unique) from $(X \cup_\sigma Y, \rho_\sigma)$ onto $(Z, \rho)$ which carries $(i_X, i_Y)$ to $(j_X, j_Y)$. From these considerations we obtain:

7.5 Proposition. Let $(X, \rho_X)$ and $(Y, \rho_Y)$ be compact metric spaces. There is a natural bijection between $\Sigma(\rho_X, \rho_Y)$ and the set of equivalence classes of objects $(Z, \rho, j_X, j_Y)$ where $(Z, \rho)$ is a compact metric space and $j_X$ and $j_Y$ are isometries of $X$ and $Y$ into $Z$ such that $Z = j_X(X) \cup j_Y(Y)$, under the equivalence relation of surjective isometries which preserve $(j_X, j_Y)$. Under this bijection $\sigma \in \Sigma(\rho_X, \rho_Y)$ is sent to the equivalence class of $(X \cup_\sigma Y, \rho_\sigma)$.

7.6 Theorem. Let $(X, \rho_X)$ and $(Y, \rho_Y)$ be compact metric spaces, and let $d = \text{dist}_{GH}(X, Y)$. Then there is a compact metric space $(Z, \rho)$, and isometric injections $i_X$ and $i_Y$ of $X$ and $Y$ into $Z$, such that

$$\text{dist}_{\rho}^H(i_X(X), i_Y(Y)) = d,$$

and $Z = i_X(X) \cup i_Y(Y)$. 
Proof. From the definition of \( d \), there is a sequence, \( \{ \rho_n \} \), of metrics in \( \Sigma(\rho_X, \rho_Y) \) such that within \( X \cup Y \)
\[
\text{dist}^{\rho_n}_H(X, Y) \leq d + 1/n.
\]
This sequence is easily seen to be uniformly bounded by
\[
(\text{diameter}(X) \lor \text{diameter}(Y)) + d + 1.
\]
Since the sequence is also equicontinuous by Lemma 7.2, we can apply the Arzela–Ascoli theorem to conclude that there will be a uniformly convergent subsequence. For simplicity we still denote this subsequence by \( \{ \rho_n \} \). Let \( \sigma \) be its limit. Since \( \Sigma(\rho_X, \rho_Y) \) is closed, \( \sigma \) must be a semi-metric. Set \( Z = X \cup \sigma Y \) and \( \rho = \rho_\sigma \) as constructed above, with \( i_X \) and \( i_Y \) the corresponding isometric inclusions.

Let us determine \( \text{dist}^{\rho}_H(i_X(X), i_Y(Y)) \). It cannot be smaller than \( d \) by the definition of \( d \). But let \( x \in X \). Then for each \( n \) there is a \( y_n \in Y \) such that \( \rho_n(x, y_n) \leq d + 1/n \). The sequence \( \{ y_n \} \) has a convergent subsequence, which for simplicity, we denote again by \( \{ y_n \} \). Relabel the \( \rho_n \)'s accordingly. Let \( y_0 \) be the limit of \( \{ y_n \} \). Given \( \varepsilon > 0 \), choose \( N \) such that if \( n > N \) then \( \| \rho_n - \sigma \|_\infty < \varepsilon/3 \), and \( 1/n < \varepsilon/3 \), and \( \rho_Y(y_n, y_0) < \varepsilon/3 \). Then for \( n > N \) we have
\[
\sigma(x, y_0) \leq \sigma(x, y_n) + \sigma(y_n, y_0) \leq \rho_n(x, y_n) + \varepsilon/3 + \rho_Y(y_n, y_0) \leq d + \varepsilon/3 + 2\varepsilon/3 = d + \varepsilon.
\]
Since \( \varepsilon \) is arbitrary, \( \sigma(x, y_0) \leq d \). In the same way, for each \( y \in Y \) there is an \( x_0 \in X \) such that \( \sigma(y, x_0) \leq d \). These inequalities pass to \( \rho \) on \( Z \), and show that \( \text{dist}^{\rho}_H(i_X(X), i_Y(Y)) = d \), as desired. \( \square \)

If \( d = 0 \) then \( \text{dist}^{\rho}_H(i_X(X), i_Y(Y)) = 0 \) under the circumstances of the above theorem, so that \( i_X(X) = i_Y(Y) \) since they are both closed. Thus, \( i_X \) and \( i_Y \) are surjective, and \( i_Y^{-1} \circ i_X \) is an isometry from \( X \) onto \( Y \). In this way we obtain:

7.7 THEOREM. Let \( (X, \rho_X) \) and \( (Y, \rho_Y) \) be compact metric spaces. If
\[
\text{dist}_{GH}(X, Y) = 0,
\]
then there is an isometry from \( X \) onto \( Y \).

We now turn to the quantum case. For the next theorem it is essential that by “isometry” we mean in the sense discussed in Section 6, involving closures.
7.8 THEOREM. Let \((A, L_A)\) and \((B, L_B)\) be compact quantum metric spaces. If
\[
dist_q(A, B) = 0,
\]
then there is an isometry between \((A, L_A)\) and \((B, L_B)\).

Proof. If \(\dist_q(A, B) = 0\), then there is a sequence, \(\{L_n\}\), of Lip-norms on \(A \oplus B\) inducing \(L_A\) and \(L_B\) such that
\[
dist_{H}^{\rho_{L_n}}(S(A), S(B)) < 1/n.
\]
For notational simplicity set \(\rho_n = \rho_{L_n}\), and \(\rho_A = \rho_{L_A}\), \(\rho_B = \rho_{L_B}\). As seen in Proposition 3.1, the restrictions of \(\rho_n\) to \(S(A)\) and \(S(B)\) are \(\rho_A\) and \(\rho_B\), respectively. Of course, \(S(A)\) and \(S(B)\) are disjoint subsets of \(S(A \oplus B)\). Thus, if we view \(\rho_n\) as restricted to their disjoint union, this says that \(\rho_n \in \Sigma(\rho_A, \rho_B)\) in the notation introduced early in this section. Exactly as in the proof of Theorem 7.6, there is a subsequence, which we still denote by \(\{\rho_n\}\), which converges uniformly on \(S(A) \cup S(B)\) to a semi-metric, \(\sigma\). Exactly as in the proof of Theorem 7.7 we see that \(\sigma\) determines an isometry, \(\alpha\), from \(S(A)\) onto \(S(B)\), by the condition that \(\sigma(\mu, \alpha(\mu)) = 0\). According to Theorem 6.2, in order to show that \(\alpha\) gives an isometry from \(B^c\) onto \(A^c\) it suffices to show that \(\alpha\) is affine. Thus, let \(\mu_1, \mu_2 \in S(A)\) and let \(t \in [0, 1]\). Let \(\varepsilon > 0\) be given, and find \(N\) such that if \(n \geq N\) then \(|\sigma - \rho_n|_{\infty} < \varepsilon/2\). Then for any \(n \geq N\) we have
\[
\sigma(t\mu_1 + (1 - t)\mu_2, t\alpha(\mu_1) + (1 - t)\alpha(\mu_2)) \\
\leq \rho_n(t\mu_1 + (1 - t)\mu_2, t\alpha(\mu_1) + (1 - t)\alpha(\mu_2)) + \varepsilon/2 \\
= L_n^t(t(\mu_1 - \alpha(\mu_1)) + (1 - t)(\mu_2 - \alpha(\mu_2))) + \varepsilon/2 \\
\leq tL_n^t(\mu_1 - \alpha(\mu_1)) + (1 - t)L_n^t(\mu_2 - \alpha(\mu_2)) + \varepsilon/2 \\
= t\rho_n(\mu_1, \alpha(\mu_1))(1 - t)\rho_n(\mu_2, \alpha(\mu_2)) + \varepsilon/2 \\
\leq t(\sigma(\mu_1, \alpha(\mu_1)) + \varepsilon/2) + (1 - t)(\sigma(\mu_2, \alpha(\mu_2)) + \varepsilon/2) + \varepsilon/2 \\
\leq \varepsilon.
\]

Since \(\varepsilon\) is arbitrary, we conclude that
\[
\alpha(t\mu_1 + (1 - t)\mu_2) = t\alpha(\mu_1) + (1 - t)\alpha(\mu_2),
\]
as needed.

We remark that a \(C^*\)-algebra and its opposite algebra need not be isomorphic; but their corresponding order-unit spaces of self-adjoint elements will, nevertheless, be isomorphic as order-unit spaces. Thus, in the above theorem, if \(A\) and \(B\) happen to be the order-unit spaces of self-adjoint elements of two \(C^*\)-algebras, it does not follow that those
\(C^*\)-algebras must be isomorphic. Interesting examples of \(C^*\)-algebras which are not isomorphic to their opposite algebra can be found in [56].

Well after this paper was submitted for publication, two approaches to dealing with this unsatisfactory feature have been given. David Kerr [41] has extended much of the present paper to operator systems with their matricial structure, where he can take advantage of the fact that if two \(C^*\)-algebras are completely order-isomorphic, in terms of their matricial norms, then they are isomorphic as \(C^*\)-algebras. (See corollary 5.2.3 of [21].) My former doctoral student Hanfeng Li has shown in his doctoral thesis [46] how to define a Gromov-Hausdorff distance between \(C^*\)-algebras with Lip-norms which explicitly uses the product in the \(C^*\)-algebras, and has the desired property.

For simplicity of notation in the next corollaries, we will not distinguish between \(C(X)\) and its dense subalgebras of Lipschitz functions, but all this should be interpreted as in the discussion following the proof of Lemma 4.6.

**7.9 Corollary.** Let \(X\) be a compact space, and let \(L\) be a Lip-norm on \(C(X)\), not necessarily coming from a metric on \(X\). Let \((B, L_B)\) be a compact quantum metric space. If
\[
\text{dist}_q((C(X), L), (B, L_B)) = 0,
\]
then the completion of \(B\) is order-isomorphic to \(C(X)\).

**Proof.** By Theorem 7.8 there will be an isometry between the two spaces. Now by Definition 6.3 an isometry is, in particular, an order-isomorphism between the closures. But any order isomorphism is norm-continuous, and so extends to an order-isomorphism of the norm completions. \(\square\)

**7.10 Corollary.** Let \((X, \rho_X)\) and \((Y, \rho_Y)\) be compact metric spaces. Let \(L_X\) and \(L_Y\) be the (closed) Lip-norms determined by \(\rho_X\) and \(\rho_Y\), so that \((C(X), L_X)\) and \((C(Y), L_Y)\) can be viewed as compact quantum metric spaces. If
\[
\text{dist}_q((C(X), L_X), (C(Y), L_Y)) = 0,
\]
then
\[
\text{dist}_{GH}((X, \rho_X), (Y, \rho_Y)) = 0.
\]

**Proof.** From Theorem 7.7 there is an isometry from \((C(X), L_X)\) to \((C(Y), L_Y)\). By the definition of an isometry, it must, in particular, extend as above to an order-isomorphism from \(C(X)\) to \(C(Y)\). But by
corollary 3.4.8 of [38], such an order-isomorphism is an algebra isomorphism, which then corresponds to a homeomorphism between $X$ and $Y$. It is easily seen that this homeomorphism is an isometry, so that $\text{dist}_{GH}(X, Y) = 0$ as desired. \hfill \Box

For further results along these lines see Theorem 13.16.

8. Actions of compact groups

Our aim now is to apply to quantum tori the theory which we have developed. But one of our main steps in doing this can be treated in a somewhat more general framework, useful in other situations. We will discuss this step in this section.

Much as in [61] and in Example 6.5, we consider a compact group $G$, equipped with a length function $\ell$ which satisfies the additional condition that $\ell(xyx^{-1}) = \ell(y)$ (which we saw in Example 6.5 insures that $G$ will act by isometries). We let $\alpha$ be an ergodic action of $G$ on a unital $C^*$-algebra $\hat{A}$ (where “ergodic” means simply that the fixed-point subalgebra is $C_1\hat{A}$). We define $L$ on $\hat{A}$ by

$$L(a) = \sup_G\{\|\alpha_x(a) - a\|/\ell(x) : x \neq e_G\},$$

and we set $A = \{a \in \hat{A} : L(a) < \infty\}$. As shown in [61], $A$ is a dense $*$-subalgebra of $\hat{A}$, and $L$ is a Lip-norm on $A$. To remain strictly within Definition 2.1 we should restrict further to the order-unit space of all self-adjoint elements of $A$. Since the above $L$ clearly satisfies $L(a^*) = L(a)$, this is equivalent, by the comments made just before Definition 2.1. Consequently, we will be a bit careless here about this distinction.

Notice further that the above $L$ is defined as a supremum of functions on $\hat{A}$ which are continuous, and so $L$ is lower semi-continuous, though it may take value $+\infty$ on $\hat{A}$. From this we easily see:

8.1 PROPOSITION. Let $A$ and $L$ be defined as above in terms of an ergodic action $\alpha$ of the compact group $G$ on the unital $C^*$-algebra $\hat{A}$ and a length function $\ell$ on $G$. Then $(A, L)$ is closed.

We will now use $\alpha$ to single out certain order-unit subspaces of $A$. For this we must assume that $G$ has a faithful finite-dimensional unitary representation, say $\pi_0$. Thus $G$ is a Lie group, possibly disconnected (or even a finite group). Adjust $\pi_0$ so that it contains the trivial representation, and set $\pi = \pi_0 \otimes \tilde{\pi}_0$, where $\tilde{\pi}_0$ denotes the contragradient representation. Thus $\pi$ is a faithful representation containing the trivial representation, and the character, $\chi$, of $\pi$ is a non-negative real-valued function on $G$. Let $\hat{G}$ denote the dual of $G$, that is, the set
of equivalence classes of irreducible unitary representations of $G$. Let $\hat{G}_n$ denote the finite subset of $\hat{G}$ consisting of those irreducible representations which occur in $\pi^\otimes n$. Notice that $\hat{G}_n$ is closed under taking contragradient representations, and contains the trivial representation.

For each $n$ let $B_n$ denote the direct sum of the (finite number of) $\alpha$-isotypic subspaces of $\hat{A}$ corresponding to all the elements of $\hat{G}_n$. Because $\hat{G}_n$ is closed under taking contragredients, $B_n$ is closed under taking adjoints. Because $\hat{G}_n$ contains the trivial representation, $B_n$ contains the identity element of $\hat{A}$. Thus, $B_n$ (or more precisely the real part of $B_n$) is an order-unit subspace of $\hat{A}$.

We now need the quite unobvious fact [34], [72] that because $\alpha$ is ergodic, each isotypic component of $\hat{A}$ is finite-dimensional. Thus each $B_n$ is finite-dimensional. Now $\alpha$ carries each $B_n$ into itself, and the Lipschitz elements for this action (for $\ell$) will form a dense subspace, and so be all of $B_n$. (The proof is much like that of proposition 2.2 of [61].) Thus, each $B_n$ is in $\hat{A}$, not just in $\hat{A}$. This says that our $L$ above is finite on $B_n$. Thus $(B_n, L)$ is a compact (in fact “finite”) quantum metric space. The main theorem of this section is:

8.2 THEOREM. Let notation be as above. Then there is a sequence, \( \{\delta_n\} \), of non-negative numbers, converging to 0, depending only on $\pi$ and $\ell$, but not on $A$ and $\alpha$, such that

$$\text{dist}_q(A, B_n) \leq \delta_n$$

for each $n$.

Proof. We begin by specifying the $\delta_n$’s. With $\chi$ the character of $\pi$ as before, $\chi^n$ (pointwise product) is the character of the inner tensor-power $\pi^\otimes n$. Fix a Haar measure on $G$, and set

$$\varphi_n = \chi^n/\|\chi^n\|_1,$$

where the norm on the right is the norm of $L^1(G)$. Set

$$\delta_n = \int_G \varphi_n(x)\ell(x)dx.$$

We show that this choice of $\delta_n$’s has the properties given in the statement of the theorem. The sequence $\{\delta_n\}$ clearly depends only on $\pi$ and $\ell$.

Since $\pi$ was chosen so that $\chi$ is non-negative, clearly $\varphi_n \geq 0$ and $\int_G \varphi_n(x)dx = 1$ for each $n$. In particular, $\delta_n \geq 0$. Because $\pi$ is a faithful representation of $G$, it assigns the identity operator only to $e_G$, the identity element of $G$. For any other $x \in G$ the unitary operator $\pi(x)$ will have some eigenvalues different from 1. It follows that $\chi(e_G) >$
\(\chi(x)\) for \(x \neq e_G\). Set \(d = \chi(e_G)\). Note that \(d \geq 2\) as long as \(G\) has more than one element. From this it follows that the “mass” of \(\varphi_n\) is increasingly concentrated near \(e_G\) as \(n\) increases. For completeness and for future use, we sketch the simple proof of this, following the ideas in the proof of lemma 4.1 of [30]. It does not use the fact that \(\chi\) is a character, but only the observations immediately above.

**Lemma:** Let \(\epsilon > 0\) be given, and let \(W_\epsilon = \{x : \chi(x) < (d-2\epsilon)\}\) and \(N_\epsilon = \{x : \chi(x) > (d-\epsilon)\}\). Let \(|W_\epsilon|\) and \(|N_\epsilon|\) denote the measures of \(W_\epsilon\) and \(N_\epsilon\). Then

\[
\|\chi^n\|_1 \geq \int_{N_\epsilon} \chi^n \geq |N_\epsilon|(d-\epsilon)^n,
\]

while

\[
\int_{W_\epsilon} \chi^n \leq |W_\epsilon|(d-2\epsilon)^n.
\]

Thus

\[
\int_{W_\epsilon} (\chi^n/\|\chi^n\|_1) \leq (|W_\epsilon|/|N_\epsilon|)((d-2\epsilon)/(d-\epsilon))^n,
\]

which goes to 0 as \(n\) grows. But the complements of the \(W_\epsilon\)'s form a neighborhood base for \(e_G\) as \(\epsilon\) goes to 0.

Thus \(\{\varphi_n\}\) is an approximate identity (of norm 1) for the convolution algebra \(L^1(G)\). It is an analogue of the classical Fejer kernel of harmonic analysis. Note that each \(\varphi_n\) is central in \(L^1(G)\) since characters are always central. (The construction of \(\{\varphi_n\}\) basically appears in [30], and probably in other places, but I have not seen it used in conjunction with Lipschitz seminorms as we do below.)

Because \(\ell(e_G) = 0\) and \(\ell\) is continuous, it follows from the above properties of \(\{\varphi_n\}\) that \(\{\delta_n\}\) converges to 0.

Fix \(n\). We define a bridge, \(N\), between \((A, L)\) and \((B_n, L)\) by

\[
N(a, b) = \delta_n^{-1} \|a - b\|,
\]

where the norm is that of \(A\). We check that \(N\) satisfies the conditions of Definition 5.1. It is clearly norm-continuous, and satisfies the required properties with respect to \(e_A = e_B\). Let \(b \in B_n\) be given. Then we can choose \(a = b\) to show that this part of Condition 3 is satisfied.

So the challenge is to see, given \(a \in A\), how to choose a corresponding \(b \in B\). For any \(f \in L^1(G)\) we define the operator \(\alpha_f\) on \(\hat{A}\) as usual by \(\alpha_f(a) = \int_G f(x)\alpha_x(a)dx\). Set \(P_n = \alpha_{\varphi_n}\). Now \(\varphi_n\) is a (finite) linear combination of the characters of the irreducible representations in \(\hat{G}_n\). Consequently, the range of \(P_n\) is contained in \(B_n\), by well-known properties [19], [22] of characters. (In fact, \(P_n\) is a unit-preserving completely positive map of \(\hat{A}\) onto \(B_n\).) We now view \(P_n\) as a map
from $A$ into $B_n$. We choose $b$ for $a$ by $b = P_n(a)$. We now start to show that this works.

**8.3 Lemma.** For any $a \in A$ we have $\|a - P_n(a)\| \leq \delta_n L(a)$.

**Proof.** For $a \in A$ we have

$$\|a - P_n(a)\| = \left\| a \int \varphi_n(x)dx - \int \varphi_n(x)\alpha_x(a)dx \right\|
\leq \int \varphi_n(x)\|a - \alpha_x(a)\|dx \leq \int \varphi_n(x)\ell(x)L(a)dx
= \delta_n L(a).$$

Consequently, for our choice of $b = P_n(a)$ we have $\delta_n^{-1}\|a - b\| \leq L(a)$ as needed.

**8.4 Lemma.** The map $P_n$ is $\alpha$-equivariant, and $\|P_n\| \leq 1$. Consequently, $L(P_n(a)) \leq L(a)$ for all $a \in A$.

**Proof.** Because $\|\varphi_n\|_1 \leq 1$ we have $\|P_n\| \leq 1$. A simple computation shows that because $\varphi_n$ is central $P_n$ is equivariant. Thus,

$$\|\alpha_x(P_n(a)) - P_n(a)\| = \|P_n(\alpha_x(a) - a)\| \leq \|\alpha_x(a) - a\|
$$

for any $x \in G$. From this we get $L(P_n(a)) \leq L(a)$. □

Consequently, for our choice of $b$ we have $L(b) \leq L(a)$, which is the other required conditions. Thus $N$ is a bridge.

We can carry out the rest of the proof of Theorem 8.2 within the more general framework of order-unit spaces as follows:

**8.5 Proposition.** Let $(A, L_A)$ be a compact quantum metric space, and let $B$ be a subspace of $A$ which contains $e_A$. Let $L_B$ denote the restriction of $L_A$ to $B$, so that $(B, L_B)$ is a compact quantum metric space. Let $P$ be a function (not necessarily even linear or continuous) from $A$ into $B$ for which there is a $\delta > 0$ such that:

1) $L_B(P(a)) \leq L_A(a)$ for all $a \in A$.

2) $\|a - P(a)\| \leq \delta L_A(a)$ for all $a \in A$.

Then $\text{dist}_q(A, B) \leq \delta$.

**Proof.** It is easily verified that $L_B$ is a Lip-norm on $B$. Exactly as above, we define a bridge, $N$, between $A$ and $B$ by

$$N(a, b) = \delta^{-1}\|a - b\|.$$
The conditions above insure that \( N \) is indeed a bridge, where we use \( P \) just as we used \( P_n \) above. We let \( L \) denote the corresponding Lip-norm on \( A \oplus B \).

Suppose we are given \( \mu \in S(A) \). Let \( \nu \in S(B) \) be the restriction of \( \mu \) to \( B \). Suppose that \((a, b) \in A \oplus B \) is such that \( L(a, b) \leq 1 \). Then \( \|a - b\| \leq \delta \), and so

\[
|\mu(a, b) - \nu(a, b)| = |\mu(a) - \nu(b)| = |\mu(a) - b| \leq \|a - b\| \leq \delta.
\]

Thus, we see that \( S(A) \) is in the \( \delta \)-neighborhood of \( S(B) \).

Suppose instead that we are given \( \nu \in S(B) \). Then we can extend it by the Hahn–Banach theorem to \( \mu \in S(A) \). Then the above argument works again. \( \square \)

It is shown in section 3 of [61] that if \( G \) happens to be a connected Lie group, then we can use norms on the Lie algebra, \( \mathfrak{g} \), of \( G \) to construct Lip-norms on \( A \). We now consider how this construction fits into the present context.

Let \( \| \cdot \|_{\mathfrak{g}} \) be some norm on \( \mathfrak{g} \). As in [61] we now take \( A \) to be the dense \( * \)-subalgebra of \( \dot{A} \) consisting of the elements which are once-differentiable for the action \( \alpha \). For \( X \in \mathfrak{g} \) we let \( \alpha_X \) denote the derivation of \( A \) corresponding to \( X \). For any \( a \in A \) we denote by \( da \) the linear map \( X \mapsto \alpha_X(a) \) from \( \mathfrak{g} \) to \( \dot{A} \). Since both \( \mathfrak{g} \) and \( \dot{A} \) have a norm, the operator norm \( \|da\| \) is defined. Then for \( a \in A \) we set \( L(a) = \|da\| \). It is shown in [61] that \( L \) is a Lip-norm. (But \( L \) is now not closed, though it is lower semi-continuous.)

By using \( \| \cdot \|_{\mathfrak{g}} \) in the usual way to define the lengths of smooth paths in \( G \), we obtain a continuous length function, \( \ell \), on \( G \), to which we can apply the discussion of the present section. But we need to require that \( \| \cdot \|_{\mathfrak{g}} \) is Ad-invariant in order that \( \ell \) satisfy the extra condition \( \ell(yxy^{-1}) = \ell(x) \) needed above. In [61] the argument is recalled for the fact that \( \|\alpha_x(a) - a\| \leq \|da\|\ell(x) \), so that \( L_0(a) \leq L(a) \), where \( L_0 \) is the Lip-norm from \( \ell \). But for any \( X \in \mathfrak{g} \) we have \( \ell(\exp(tX)) = |t|\|X\|_{\mathfrak{g}} \) for small \( t \), and so if \( \|X\|_{\mathfrak{g}} = 1 \) we have

\[
\frac{\|\alpha_{\exp(tX)}(a) - a\|}{t} = \frac{\|\alpha_{\exp(tX)}(a) - a\|}{\ell(\exp(tX))}.
\]

From this we see that \( \|d_Xa\| \leq L_0(a) \) for all \( X \) with \( \|X\|_{\mathfrak{g}} = 1 \). Thus, for later use, we can record:

**8.6 PROPOSITION.** For \( L \) defined as above by a norm on \( \mathfrak{g} \), and for \( L_0 \) defined in terms of the corresponding length function on \( G \), we have \( L_0 = L \).

Consequently Theorem 8.2 applies to \( L \) defined by a norm on \( \mathfrak{g} \).
In section 4 of [61] it is shown how to construct Lip-norms on \( A \) in terms of an inner product on the dual, \( g' \), of \( g \) and of a corresponding Dirac operator. Presumably the results of the present section can be applied to that situation too, as long as the inner product is invariant under the coadjoint representation, so that \( G \) again acts by isometries for the Lip-norm. But I have not verified this.

9. Quantum tori

We can now treat the main example of this paper. We first recall the basic facts about the quantum tori which we need, and establish our notation. The positive integer \( d \) will denote the dimension of the tori we work with. We let \( \Theta \) denote the vector space of all real skew-symmetric \( d \times d \) matrices. For any \( \theta \in \Theta \) we let \( \tilde{A}_\theta \) denote the corresponding quantum torus [58], [60]. It is defined as follows. Let \( \sigma_\theta \) denote the skew bicharacter on \( \mathbb{Z}^d \) defined by

\[
\sigma_\theta(p, q) = \exp(i\pi p \cdot \theta q).
\]

Equip \( C_c(\mathbb{Z}^d) \), the space of complex-valued functions on \( \mathbb{Z}^d \) of finite support, with the product consisting of convolution twisted by \( \sigma_\theta \). That is, for \( f, g \in C_c(\mathbb{Z}^d) \) we have

\[
(f \ast g)(p) = \sum f(q)g(p - q)\sigma_\theta(q, p).
\]

We also equip \( C_c(\mathbb{Z}^d) \) with the involution \( f^*(p) = \tilde{f}(-p) \), and the norm of \( \ell^1(\mathbb{Z}^d) \), so that \( C_c(\mathbb{Z}^d) \) is a \( \ast \)-normed algebra. It is important for us that the involution and norm do not depend on \( \theta \). We let \( \pi_\theta \) denote the \( \ast \)-representation of \( C_c(\mathbb{Z}^d) \) on the Hilbert space \( \ell^2(\mathbb{Z}^d) \) given by formula (9.1) except with \( g \) replaced by \( \xi \in \ell^2(\mathbb{Z}^d) \). We let \( \| \cdot \|_\theta \) be the \( C^* \)-norm on \( C_c(\mathbb{Z}^d) \) defined by \( \| f \|_\theta = \| \pi_\theta(f) \| \). Then \( \tilde{A}_\theta \) is defined to be the completion of \( C_c(\mathbb{Z}^d) \) for this norm.

We let \( G = \mathbb{T}^d \), where \( \mathbb{T} \) is the circle group. Thus \( G \) is the dual group of \( \mathbb{Z}^d \). We denote the duality by \( \langle p, x \rangle \). For us, a crucial fact is that \( G \) has a natural action, \( \alpha \), on \( \tilde{A}_\theta \), defined on \( C_c(\mathbb{Z}^d) \) by

\[
(\alpha_x(f))(p) = \langle p, x \rangle f(p).
\]

Furthermore, this action is ergodic, so that we are in the setting of the previous section. It is important for us that the definition of the action \( \alpha \) on functions is independent of \( \theta \).

In [61] we gave three different methods for defining Lip-norms on quantum tori, namely, by length functions on the dual group, by norms on the Lie algebra of the dual group, and by a Dirac operator construction. The simplest technically is in terms of length functions. We discuss this case first.
We fix a continuous length function, $\ell$, on $G$. (Because $G$ is Abelian, the condition $\ell(xyx^{-1}) = \ell(y)$ is automatic.) For each $\theta \in \Theta$ we define the Lip-norm $L_\theta$ on $A_\theta$ in terms of $\alpha$ and $\ell$ exactly as we did early in Section 8. We let $A_\theta$ denote the dense $*$-subalgebra where $L_\theta$ is finite. Thus each $(A_\theta, L_\theta)$ is a compact quantum metric space (at least if we take its self-adjoint part), and the $L_\theta$’s are defined consistently in terms of one fixed length function $\ell$ as $\theta$ varies.

We will actually obtain uniform estimates of quantum Gromov–Hausdorff distance between the $A_\theta$’s in terms of the parameter $\theta$. For this purpose we fix an arbitrary norm on the vector space $\Theta$. We can now state our main theorem.

9.2 THEOREM. Let notation be as above. For any $\varepsilon > 0$ there is a $\delta > 0$ such that if $\|\theta - \psi\| < \delta$ for $\theta, \psi \in \Theta$, then

$$\text{dist}_q(A_\theta, A_\psi) < \varepsilon.$$ 

Proof. Choose a finite-dimensional faithful unitary representation, $\pi$, of $G$ as done in the previous section, so that, in particular, its character, $\chi$, is non-negative. Notice that the dual group, $\hat{G}$, of $G$ is just $\mathbb{Z}^d$. Thus, each $\hat{G}_n$, as defined in the previous section, is just a finite subset of $\mathbb{Z}^d$, containing 0, closed under $p \to -p$, and generating $\mathbb{Z}^d$. For each $p \in \hat{G}$ the $\alpha$-isotypic component of $A_\theta$ for $p$ clearly consists of the 1-dimensional subspace of functions on $\mathbb{Z}^d$ supported at $p$. Thus the spaces $B_n$ of the previous section all consist exactly of the functions supported on $\hat{G}_n \subset \mathbb{Z}^d$, independently of $\theta$. What depends on $\theta$ is how these functions act as operators on $\ell^2(\mathbb{Z}^d)$, and so their operator norms and whether they are positive (but not whether they are self-adjoint). We denote these spaces with that structure by $B^n_\theta$.

For each $n$ define $\varphi_n \in L^1(G)$ in terms of $\pi$ as done in the previous section, and then define $\delta_n$ in terms of $\varphi_n$ and $\ell$ as done there. Choose $n$ such that $\delta_n < \varepsilon/3$. We hold $n$ fixed for the rest of the proof. Then for simplicity of notation we write $B$ for $B_n = L^1(\hat{G}_n)$, and we write $B_\theta$ instead of $B^n_\theta$. Each $B_\theta$ is equipped with the restriction to it of $\| \|_\theta$ and $L_\theta$. We know from Theorem 8.2 that for each $\theta$

$$\text{dist}_q(A_\theta, B_\theta) < \varepsilon/3.$$ 

From the triangle inequality of Theorem 4.3 it follows that, in order to complete the proof, it suffices to find a $\delta > 0$ such that if $\|\theta - \psi\| < \delta$ then

$$\text{dist}_q(B_\theta, B_\psi) \leq \varepsilon/3.$$
We are in a favorable situation for doing this because each $B_\theta$ is the same finite-dimensional space $B$ of functions, but with different operator norms and Lip-norms (but same self-adjoint part and order-unit).

We need to construct a bridge, $N$, between $B_\theta$ and $B_\psi$. For this purpose we let $\pi_\theta$ denote the representation of $A_\theta$ on $\ell^2(\mathbb{Z}^d)$ defined earlier, restricted to $B_\theta$; and similarly for $\pi_\psi$. We now collect the facts which we need. What we will be dealing with is the subject of continuous fields of compact quantum metric spaces. In the interest of brevity we will not develop its general theory here. In particular, we will not strive for minimal hypotheses. But we remark that much of the theory of continuous fields of $C^*$-algebras, as presented, for example, in [10], [19], extends to a theory of continuous fields of order-unit spaces. One must then add to that the topic of continuous fields of Lip-norms. Many of our maneuvers below can be placed in this general framework.

The first fact which we need is that the $C^*$-algebras $\hat{A}_\theta$ form a continuous field of $C^*$-algebras over $\Theta$. Specifically, we need the fact that for any $f \in \ell^1(\mathbb{Z}^d)$ the function $\theta \mapsto \|f\|_\alpha$ is continuous. See corollary 2.8 of [59]; but note from the proof of theorem 2.5 of [59] that while $\theta \mapsto \pi_\theta(f)$ is strongly continuous, it cannot be expected to be norm continuous. (Also, the hypotheses of theorem 2.5 should have included the requirement that $\Omega$ be first countable.) We denote by $\| \cdot \|_\star$ (rather than $\| \cdot \|_1$) the usual norm on $\ell^1(\mathbb{Z}^d)$. It is a standard fact [22] that $\|f\|_\theta \leq \|f\|_\star$ for $f \in \ell^1(\mathbb{Z}^d)$.

Let $S$ denote some finite subset of $\mathbb{Z}^d$, such as our earlier $G_n$’s. Note that $\alpha$ carries the finite-dimensional vector space $\ell^1(S)$ into itself. By finite-dimensionality we have $L_\theta(f) < \infty$ for all $f \in \ell^1(S)$. (See the proof of proposition 2.2 of [61].) Of course, each $f \in \ell^1(S)$ is “Lipschitz” for $\| \cdot \|_\star$ also. We denote the corresponding Lip-norm by $L_\star$.

9.3 LEMMA. Let $f \in \ell^1(\mathbb{Z}^d)$. If $f$ has finite support, then the function $\theta \mapsto L_\theta(f)$ is continuous. Furthermore, $L_\theta(f) \leq L_\star(f)$ for each $\theta \in \Theta$.

Proof. Let $f$ be supported on the finite set $S$. Let

$$D_f = \{(\alpha x(f) - f)/\ell(x) : x \neq e_G\}.$$ 

Note that $D_f \subseteq \ell^1(S)$, and that $D_f$ is bounded for $\| \|_\star$ since $L_\star(f) < \infty$. Now $L_\theta(f)$ is the supremum of $\|g\|_\theta$ for $g \in D_f$, and similarly for $L_\star(f)$. Thus $L_\theta(f) \leq L_\star(f)$, since $\| \cdot \|_\theta \leq \| \cdot \|_\star$.

For any $g \in \ell^1(S)$ let $F_g$ be the continuous function on $\Theta$ defined by $F_g(\theta) = \|g\|_\theta$. Since $L_\theta(f) = \sup\{\|g\|_\theta : g \in D_f\}$, it follows that $\theta \mapsto L_\theta(f)$ is the supremum of the functions $F_g$ for $g \in D_f$. As a
supremum of continuous functions, \( \theta \mapsto L_\theta(f) \) must be lower semi-continuous.

We now show that \( \theta \mapsto L_\theta(f) \) is actually continuous. For any \( h, k \in \ell^1(S) \) we have
\[
|F_h(\theta) - F_k(\theta)| = ||h||_\theta - ||k||_\theta \leq ||h - k||_\theta \leq ||h - k||_*. 
\]
Let \( U \) be any compact subset of \( \Theta \), and view \( F_h \) and \( F_k \) as restricted to \( U \), and so as elements of \( C(U) \) with the usual norm \( || \cdot ||_\infty \). The above inequality says that \( ||F_h - F_k||_\infty \leq ||h - k||_* \). Thus \( F \), as a mapping from \( \ell^1(S) \) into \( C(U) \), is Lipschitz. Now \( D_f \), as a bounded subset of a finite-dimensional normed vector space, is totally bounded. Consequently, \( F(D_f) \) is totally bounded in \( C(U) \). But by part of the Arzela–Ascoli theorem [18] it follows that \( F(D_f) \) is equicontinuous. In other words, \( \{F_g : g \in D_f\} \) is an equicontinuous family of functions on \( U \). But the supremum of an equicontinuous family is continuous. Thus \( \theta \mapsto L_\theta(f) \) is continuous on \( U \).

\[\Box\]

For each \( \theta \) we let \( r_\theta \) denote the radius of \( (A_\theta, L_\theta) \). As an immediate consequence of lemma 2.4 of [61] we have:

**9.4 Lemma.** There is a constant, \( R \), which depends only on the length function \( \ell \) and the choice of Haar measure on \( G \), such that \( r_\theta \leq R \) for all \( \theta \in \Theta \).

We now need an ample supply of continuous fields of states. We let \( T_1^+ \) denote the space of positive trace-class operators of trace 1 on \( \ell^2(\mathbb{Z}^d) \), that is, the “density matrices”. We denote the trace by \( \text{tr} \). For each \( T \in T_1^+ \) and each \( \theta \in \Theta \) we let \( \omega_\theta^T \) denote the state on \( A_\theta \) defined by
\[
\omega_\theta^T(a) = \text{tr}(\pi_\theta(a)T).
\]
Because \( \pi_\theta \) is a faithful representation of \( A_\theta \), it follows from proposition VII.5.4 of [22] that this set of states is \( w^* \)-dense in \( S(A_\theta) \).

**9.5 Lemma.** For each \( f \in \ell^1(\mathbb{Z}^d) \) and \( T \in T_1^+ \) the function \( \theta \mapsto \omega_\theta^T(f) \) is continuous.

**Proof.** As mentioned above (with caveat), the function \( \theta \mapsto \pi_\theta(f) \) is strong-operator continuous by the proof of theorem 2.5 of [59]. It follows easily that \( \theta \mapsto \omega_\theta^T(f) \) is continuous when \( T \) has rank 1; and hence when \( T \) is of finite rank. But the finite rank operators in \( T_1^+ \) are dense for the trace-norm. A uniform convergence argument completes the proof. \[\Box\]
We have now collected the facts which we need. Before continuing with the proof, we turn to considering briefly the method discussed at the end of Section 8 for defining Lip-norms in terms of norms on the Lie algebra. Accordingly, let $\mathfrak{g}$ denote the Lie algebra of $\mathbb{T}^d$, so that $\mathfrak{g}$ can be identified with $\mathbb{R}^d$. Let $\| \cdot \|_\mathfrak{g}$ denote some norm on $\mathfrak{g}$. As done near the end of Section 8, we now take $A_\theta$ to be the dense $\ast$-subalgebra of $\bar{A}_\theta$ consisting of the elements which are once-differentiable for the action $\alpha$. For $X \in \mathfrak{g}$ we let $\alpha_X$ denote the derivation of $\bar{A}_\theta$ corresponding to $X$. Thus for $f \in C_c(\mathbb{Z}^d)$ and $p \in \mathbb{Z}^d$ we have

\[(\alpha_X(f))(p) = 2\pi i (p \cdot X)f(p),\]

where $p \cdot X$ denotes the standard inner product on $\mathbb{R}^d$ and we use $\mathbb{Z}^d \subseteq \mathbb{R}^d$. We define $da$ as near the end of Section 8, and set $L_\theta(a) = \|da\|$. In the same way we define $L_*$. The proof of Lemma 9.3 for this setting is actually easier than the proof given above:

**9.6 Lemma.** Let $A_\theta$ and $L_\theta$ be defined as just above in terms of a norm on $\mathfrak{g}$. For each $f \in C_c(\mathbb{Z}^d)$ and $p \in \mathbb{Z}^d$ we have

\[(\alpha_X(f))(p) = 2\pi i (p \cdot X)f(p),\]

where $p \cdot X$ denotes the standard inner product on $\mathbb{R}^d$ and we use $\mathbb{Z}^d \subseteq \mathbb{R}^d$. We define $da$ as near the end of Section 8, and set $L_\theta(a) = \|da\|$. In the same way we define $L_*$. The proof of Lemma 9.3 for this setting is actually easier than the proof given above:

**Proof.** Changing the notation given in the proof of Lemma 9.3, we now set

\[D_f = \{\alpha_X(f) : \|X\|_\mathfrak{g} \leq 1\},\]

so that $L_\theta(f) = \sup\{\|g\|_\mathfrak{g} : g \in D_a\}$. From this we immediately have $L_\theta(f) \leq L_*(f)$. Define $F_\theta$ exactly as in the proof of Lemma 9.3, so that again $\theta \mapsto L_\theta(f)$ is the supremum of the functions $F_\theta$ for $g \in D_f$. Since $\{X : \|X\|_\mathfrak{g} \leq 1\}$ is a compact subset of a finite-dimensional vector space, and $X \mapsto \alpha_X(f)$ is linear, it is clear that $D_f$ is a compact subset of $\ell^1(\mathbb{Z}^d)$. Then the argument in the proof of Lemma 9.3 shows that $F(D_f)$ is an equicontinuous family of functions on compact subsets of $\Theta$, so that $\theta \mapsto L_\theta(f)$ is continuous.

From the discussion at the end of Section 8 it is clear that the analogue of Lemma 9.4 holds.

In order to try to make the rest of the proof of Theorem 9.2 as clear as possible, we find it useful to treat the situation developed in this section in a somewhat axiomatic framework. We carry this out in the next two sections. But we remark first that because the $(A_\theta, L_\theta)$’s are unchanged when integers are added to the entries of $\theta$, it is sufficient to prove that our main theorem holds for any compact subset, $\Theta_0$, of...
Θ. For simplicity of notation, in the next sections we will let Θ denote a compact metric space.

10. Continuous fields of order-unit spaces

As said earlier, we will not strive for a full theory of continuous fields here. Rather we will take a fairly direct route to what we need. Throughout this section $V$ will be a finite-dimensional real vector space equipped with a distinguished element $e$ (for example, the self-adjoint part of $\ell^1(\hat{G}_n)$ with $e = 1 = \delta_0$). We let $\Theta$ be a compact set with metric $d$. We assume that we are given, for each $\theta \in \Theta$, a norm, $\| \cdot \|_\theta$, on $V$ such that for each $v \in V$ the function $\theta \mapsto \|v\|_\theta$ is continuous. We will call such a family of norms a continuous field of norms. We will shortly make further hypotheses on the $\| \cdot \|_\theta$’s.

If we pick a basis $\{v_j\}$ for $V$ and use compactness and continuity to find a constant $K$ such that $\|v_j\|_\theta \leq K$ for all $\theta \in \Theta$ and all $j$, we see quickly that there is a norm, $\| \cdot \|_*$, such that $\| \cdot \|_\theta \leq \| \cdot \|_*$ for all $\theta$. It will be convenient for us to fix such a norm.

We let $V'$ denote the vector-space dual of $V$, and we let $\| \cdot \|'_\theta$ and $\| \cdot \|'_*$ denote the corresponding dual norms. We let $\Sigma_\theta$ denote the unit $\| \cdot \|_\theta$-sphere (not ball), and similarly for $\Sigma_*$, etc.

10.1 Lemma. The function $(\theta, v) \mapsto \|v\|_\theta$ is jointly continuous on $\Theta \times V$. The family $\{\| \cdot \|'_\theta\}$ is a continuous field of norms on $V'$. There is a strictly positive constant, $k$, such that for all $\theta \in \Theta$

$$k\| \cdot \|_* \leq \| \cdot \|_\theta \leq \| \cdot \|_*$$

and

$$k^{-1}\| \cdot \|'_* \geq \| \cdot \|'_\theta \geq \| \cdot \|'_*.$$ 

Proof. The joint continuity at $(w, \psi)$ follows from the inequalities

$$\|\|v\|_\theta - \|w\|_\psi\| \leq \|\|v\|_\theta - \|w\|_\theta\| + \|\|w\|_\theta - \|w\|_\psi\| \leq \|v - w\|_\theta + \|\|w\|_\theta - \|w\|_\psi\| \leq \|v - w\|_* + \|\|w\|_\theta - \|w\|_\psi\|.$$ 

Let $\lambda \in V'$, and for each $v \in \Sigma_*$ define a function, $F_v$, on $\Theta$ by

$$F_v(\theta) = |\langle v, \lambda \rangle|/\|v\|_\theta.$$ 

From the above joint continuity we see that the function $(\theta, v) \mapsto F_v(\theta)$ is jointly continuous on $\Theta \times \Sigma_*$. It follows from the compactness of $\Theta \times \Sigma_*$ that this function is uniformly continuous. But this implies that the family $\{F_v\}$ is equicontinuous. Thus, $\theta \mapsto \|\lambda\|'_\theta$, being the supremum of this equicontinuous family, is continuous.
The function \((\theta, v) \mapsto \|v\|_\theta\) is continuous on \(\Theta \times \Sigma_\ast\) and never takes value 0 there. Thus, by compactness there is a strictly positive constant, \(k\), such that \(k \leq \|v\|_\theta\) for all \(\theta\) and all \(v\) with \(\|v\|_\ast = 1\), as needed. The inequalities for the dual norms follow.

The metric on \(V\) from the norm \(\|\cdot\|_\ast\) give a corresponding Hausdorff metric on the compact subsets of \(V\), which we denote by \(\text{dist}_H^\ast\). We will not explicitly need the following lemma and its corollary later. But we will need the calculation which constitutes most of the proof of the lemma; and the corollary is a good preview of what comes later.

10.2 Lemma. Let \(\|\cdot\|_1\) and \(\|\cdot\|_2\) be norms on \(V\), and let \(k\) be a constant such that \(k\|v\|_\ast \leq \|v\|_j\) for \(j = 1, 2\) and all \(v \in V\). Let \(\varepsilon > 0\) be given, and suppose that \(\|\|v\|_1 - \|v\|_2\| < \varepsilon k^2\|v\|_\ast\) for all \(v\). Then \(\text{dist}_H^\ast(\Sigma_1, \Sigma_2) < \varepsilon\).

\textbf{Proof.} Let \(u \in \Sigma_1\) be given. Set \(v = u/\|u\|_2\), so that \(v \in \Sigma_2\). Then

\[
\|u - v\|_\ast = \|(1 - \|u\|_2^{-1})u\|_\ast \\
\quad = \||u\|_2^{-1} - \|u\|_2^{-1}\|u\|_\ast \leq \||u\|_2 - \|u\|_1|k^{-1} \\
\quad \leq \varepsilon k\|u\|_1 \leq \varepsilon\|u\|_1 = \varepsilon.
\]

Thus, \(\Sigma_1\) is in the \(\varepsilon\)-neighborhood of \(\Sigma_2\) for \(\|\cdot\|_\ast\). But we can reverse the roles of \(\Sigma_1\) and \(\Sigma_2\). \(\square\)

10.3 Corollary. For a continuous field \(\{\|\cdot\|_\theta\}\) of norm as above, the function \(\theta \mapsto \Sigma_\theta\) is (uniformly) continuous for \(\text{dist}_H^\ast\).

\textbf{Proof.} Let \(\varepsilon > 0\) be given. Let \(k\) be as in Lemma 10.1, and let \(E_\ast\) denote the unit \(\|\cdot\|_\ast\)-ball of \(V\). According to Lemma 10.1 the function \((v, \theta) \mapsto \|v\|_\theta\) restricted to the compact set \(\Theta \times E_\ast\) is jointly continuous, and so uniformly continuous. Consequently, the family of functions \(\theta \mapsto \|v\|_\theta\) for \(v \in E_\ast\) is uniformly equicontinuous, that is, we can find \(\delta > 0\) such that if \(d(\theta, \psi) < \delta\), then for each \(v \in E_\ast\) we have \(\|\|v\|_\theta - \|v\|_\psi\| \leq \varepsilon k^2\). We are then exactly in position to use Lemma 10.2 to conclude that \(\text{dist}_H^\ast(\Sigma_\theta, \Sigma_\psi) < \varepsilon\). \(\square\)

In accordance with the situation for \(B\) and the norms \(\|\cdot\|_\theta\) of the previous section, we will now assume further that each \(\|\cdot\|_\theta\) is an order-unit norm on \((V, e)\), where:

10.4 Definition. Let \(V\) be a vector space and let \(e\) be a distinguished element of \(V\). Let \(\|\cdot\|\) be a norm on \(V\), and set

\[
V^+ = \{v \in V : \|(\|v\|e - v)\| \leq \|v\|\}.
\]
We say that $\| \cdot \|$ is an order-unit norm on $(V,e)$ if $\|e\| = 1$ and $V^+$ is the positive cone for an ordering on $V$ for which $(V,e)$ is an order-unit space with order-unit norm equal to $\| \cdot \|$. When $\| \cdot \|$ is an order-unit norm, we will feel free to refer to $(V,e,\| \cdot \|)$ as an order-unit space, with the ordering from $V^+$ understood.

We remark that the above definition of $V^+$ is motivated, for example, by corollary VI.7.7 of [22], or by part 2 of the proof of proposition II.1.3 of [1].

Since the norms $\| \cdot \|_\theta$ are now order-unit norms, it is reasonable to consider the corresponding state spaces. They will all consist of linear functionals on our fixed vector space $V$. We denote the state space for $\| \cdot \|_\theta$ by $S_\theta$. Our eventual aim is, roughly speaking, to show that the $\varepsilon$-density of a family of continuous fields of states propagates. This will permit us to define suitable bridges $N$. For this purpose we need to examine the continuous-field structure of the duals of the order-unit spaces $(V,e,\| \cdot \|_\theta)$. By a theorem of Ellis (theorem II.1.15 of [1]) the dual of an order-unit space is a base-norm space. This means the following. Let $(V,e,\| \cdot \|)$ be an order-unit space. Let $V'$ be its Banach-space dual. Let $\eta$ denote the linear functional which $e$ defines on $V'$, and let $\| \cdot \|'$ denote the dual norm. The base, $S$, is defined by

$$S = \{ \lambda \in V' : \eta(\lambda) = 1 = \|\lambda\|' \}.$$  

We recognize this as just the state space. Then the unit $\| \cdot \|'$-ball of $V'$ is equal to $\text{co}(S \cup -S)$. (See proposition II.1.7 of [1].) We emphasize that because $S$ is $w^*$-compact, “co” here means “convex hull”, not “closed convex hull”. It is easy to see then that if $\lambda \in V'$ with $\|\lambda\|' \leq 1$, then there are $\mu, \nu \in S$ (not unique) and $t \in [0,1]$ such that $\lambda = t\mu - (1-t)\nu$. This is closely related to lemma 2.1 of [62] mentioned in the last paragraph of Section 1. Let $C$ be the cone in $V'$ generated by $S$ (so that $S$ is a “base” for $C$ because $\eta(S) = 1$). Then $C$ is the positive cone for an order on $V'$ which satisfies the requirements for $(V',C,S)$ to be a base-norm space, as defined just after proposition II.1.12 of [1].

All this suggests the following:

**10.5 Definition.** Let $X$ be a vector space, and let $\eta$ be a distinguished linear functional on $X$. By a base-norm on $(X,\eta)$ we mean a norm, $\| \cdot \|$, on $X$ such that when we set

$$S = \{ \lambda \in X : \eta(\lambda) = 1 = \|\lambda\| \},$$

the unit $\| \cdot \|$-ball coincides with $\text{co}(S \cup -S)$. 


Much as above, it is easily seen that if \( \| \cdot \| \) is a base-norm on \((X, \eta)\), then \((X, S)\) is a base-norm space [1] when \(X\) is ordered by the cone generated by \(S\).

Returning to the situation where we have a continuous field of order-unit norms on a finite-dimensional vector space \(V\), we see that \(\{\| \cdot \|_\theta\}\) is a continuous field of base-norm on \((V', \eta)\). We can now momentarily forget where \(V'\) and the \(\| \cdot \|_\theta\) come from, and just consider a finite-dimensional vector space, \(X\), with linear functional \(\eta\), and a continuous family \(\{\| \cdot \|_\theta\}\) of base-norms. In view of Lemma 10.1, we now assume that we have a norm \(\| \cdot \|_*\) and a constant \(k\) such that
\[
k^{-1}\|x\|_* \geq \|x\|_\theta \geq \|x\|_*
\]
for \(x \in X\). The metric from \(\| \cdot \|_*\) defines a Hausdorff metric on the compact subsets of \(X\), which we again denote by \(\text{dist}^*_H\).

**10.6 THEOREM.** Let \(X\) be a finite-dimensional vector space with distinguished linear functional \(\eta\). Let \(\{\| \cdot \|_\theta\}\) be a continuous field of base-norms for \((X, \eta)\), and let \(S_\theta\) denote the base for \((V, \eta, \| \cdot \|_\theta)\). Let \(\| \cdot \|_*\) be a norm on \(X\) such that \(\| \cdot \|_\theta \geq \| \cdot \|_*\) for all \(\theta\). Then the function \(\theta \mapsto S_\theta\) on \(\Theta\) is (uniformly) continuous for \(\text{dist}^*_H\).

**Proof.** It suffices to show that for every \(\varepsilon > 0\) such that \(\varepsilon < 2\) there is a \(\delta > 0\) such that if \(d(\theta, \psi) < \delta\) then for every \(\mu \in S_\psi\) there is a \(\nu \in S_\theta\) with \(\|\mu - \nu\|_* < \varepsilon\), since then we can reverse the roles of \(\theta\) and \(\psi\). For each \(\theta\) let \(\Sigma_\theta\) denote the unit \(\| \cdot \|_\theta\)-sphere of \(X\), much as we did earlier in the dual situation. Much as in the proof of Corollary 10.3, we let \(E_*\) denote the unit \(\| \cdot \|_*\)-ball in \(X\), so that \(\Sigma_\theta \subset E_*\) for all \(\theta\). Note that \(S_\theta \subset \Sigma_\theta\).

Given \(\varepsilon > 0\), we choose \(\delta\) as follows. The function \((\theta, x) \mapsto \|x\|_\theta\) restricted to the compact set \(\Theta \times E_*\) is jointly continuous by Lemma 10.1, and so is uniformly continuous. Thus, the family of functions \(\theta \mapsto \|x\|_\theta\) for \(x \in E_*\) is uniformly equicontinuous, and so we can find \(\delta > 0\) such that if \(d(\theta, \psi) < \delta\) then for each \(x \in E_*\) we have
\[
\|\|x\|_\theta - \|x\|_\psi\| < \varepsilon/4.
\]
This choice of \(\delta\) works, as we see from the following lemma, which we will also need later.

**10.7 KEY LEMMA.** Let \(\| \cdot \|_1\) and \(\| \cdot \|_2\) be base-norms on \((X, \eta)\), and let \(\| \cdot \|_*\) be a norm on \(X\) such that \(\|x\|_* \leq \|x\|_j\) for \(j = 1, 2\) and all \(x \in X\). Let \(\varepsilon > 0\) be given such that \(\varepsilon < 2\), and suppose that
\[
\|\|x\|_1 - \|x\|_2\| < (\varepsilon/4)\|x\|_*
\]
for all $x \in X$. Then $\dist^*_H(S_1, S_2) < \varepsilon$.

**Proof.** Let $\mu \in S_1$. Let $x = \mu/\|\mu\|_2$, so that $x \in \Sigma_2$. By the calculation in the proof of Lemma 10.2 we see that $\|\mu - x\|_* < \varepsilon/4$. Since $\| \cdot \|_2$ is a base-norm and $\|x\|_2 = 1$, we can find $\nu, \nu_0 \in S_2$ such that $x = t\nu - (1 - t)\nu_0$ for some $t \in [0, 1]$. When we apply $\eta$ to this equation we obtain $\|\mu\|_2^{-1} = 2t - 1$.

Thus, $\|\mu\|_2^{-1} - 1 = 2(t - 1).

But $\varepsilon < 2$ and $\|\mu\|_* \leq \|\mu\|_1 = 1$, and so

$$|1 - \|\mu\|_2| = \|\mu\|_1 - \|\mu\|_2 < (\varepsilon/4)\|\mu\|_* \leq 1/2.$$ 

Thus $\|\mu\|_2^{-1} < 2$, so that, because $1 = \|\mu\|_1$,

$$|\|\mu\|_2^{-1} - 1| = |\|\mu\|_1 - \|\mu\|_2|\|\mu\|_2^{-1} < (\varepsilon/2)\|\mu\|_* \leq \varepsilon/2.$$ 

Consequently, $2|t - 1| < \varepsilon/2$, and so

$$\|x - \nu\|_* = \|(t - 1)(\nu + \nu_0)\|_* \leq |t - 1|\|\nu + \nu_0\|_2 \leq 2|t - 1| < \varepsilon/2.$$ 

It follows that $\|\mu - \nu\|_* < \varepsilon$. Thus $S_1$ is in the $\varepsilon$-neighborhood of $S_2$ for $\dist^*_H$. But we can reverse the roles of $S_1$ and $S_2$. \qed

If the requirement that the $\| \cdot \|_j$’s be base-norms is dropped, the conclusions of the above lemma and theorem can easily fail. An example could involve strictly convex norms converging to a base-norm.

In our earlier concrete situation of $\ell^1(\mathbb{Z}^d)$ we saw that we had a large supply of continuous fields of states, namely the $\omega_T^\theta$’s. For each $\theta$ they were $w^*$-dense in $S(A_\theta)$. But when restricted to $B = \ell^1(\hat{G}_n)$ they will then fill out all of $S(B_\theta)$. Thus, in our present abstract setting of a finite-dimensional vector space $V$ with distinguished element $e$ and order-unit norms $\| \cdot \|_\theta$ we can assume that we have a large family, $\mathcal{S}$, of continuous fields of states, where:

**10.8 Definition.** With notation as above, by a **continuous field of states** we will mean a function, $\Omega$, from $\Theta$ to $V'$, such that $\Omega_\theta \in S_\theta$ for each $\theta \in \Theta$, and such that $\theta \mapsto \langle v, \Omega_\theta \rangle$ is continuous for each $v \in V$. When $V$ is finite-dimensional, the latter condition is equivalent to $\Omega$ being continuous for $\| \cdot \|_\theta$ on $V'$.

We remark that related definitions of continuous fields of states for continuous fields of $C^*$-algebras are given in [10], [45], [52], [53]. If one does not already know that one has enough continuous fields of states, one can, in the infinite-dimensional case, try to prove their existence by means of selection theorems, as done for theorem 3.3 of [10]. Although we do not need it for our application to quantum tori, it
seems appropriate to point out that in our finite-dimensional situation there are always plenty of continuous fields of states. We can again work directly with fields of base-norms. In that setting, by a “continuous field of states” we will mean simply a continuous function \( \Omega \) from \( \Theta \) to \( X \) such that \( \Omega_\theta \in S_\theta \) for each \( \theta \in \Theta \).

10.9 PROPOSITION. Let \( X \) be finite-dimensional, and let \( \{ \| \cdot \|_\theta \} \) be a continuous field of base-norms on \((X, \eta)\). Then for each \( \psi \in \Theta \) and each \( \mu \in S_\psi \) there is a continuous field of states, \( \Omega \), such that \( \Omega_\psi = \mu \).

Proof. Because \( X \) is finite-dimensional, we can choose a norm, \( \| \cdot \| \), on \( X \) which comes from an inner-product, and so satisfies the parallelogram law. We denote the Hausdorff metric which it defines on compact subsets of \( X \) by \( \text{dist}^P_H \). Again, by finite-dimensionality, the norm \( \| \cdot \| \) will be equivalent to \( \| \cdot \|_\star \), and from this it is easily seen that the metric \( \text{dist}^P_H \) is equivalent to \( \text{dist}^\star_H \). From Theorem 10.6 we conclude that the function \( \theta \mapsto S_\theta \) is continuous on \( \Theta \) for \( \text{dist}^P_H \).

Let \( \psi \in \Theta \) and \( \mu \in S_\psi \) be given. Each \( S_\theta \) is a compact subset of \( X \), and so we can find a point of \( S_\theta \) which is closest to \( \mu \) for \( \| \cdot \|_\star \). We denote this point by \( \Omega_\theta \). Clearly \( \Omega_\psi = \mu \). We wish to show that the function \( \Omega \) is continuous. But this is an immediate consequence of the following proposition, which may well appear in the literature somewhere.

10.10 PROPOSITION. Let \( X \) be a Hilbert space with norm \( \| \cdot \| \). Let \( \mathcal{C} \) denote the collection of compact convex subsets of \( X \), with corresponding Hausdorff metric \( \text{dist}^P_H \). Fix \( z \in X \), and for each \( C \in \mathcal{C} \) let \( \Omega_C \) denote an element of \( C \) closest to \( z \). (This point is unique, but we don’t explicitly need that fact.) Then the function \( \Omega \) is continuous from \( \mathcal{C} \) to \( X \) for \( \text{dist}^P_H \) and \( \| \cdot \| \).

Proof. By translation we can, and do, assume that \( z = 0 \). Let \( D \in \mathcal{C} \). We show continuity of \( \Omega \) at \( D \). Let \( m \) be a constant such that \( \| x \| \leq m \) for all \( x \in D \). Let \( \delta > 0 \) be given, and suppose that \( C \in \mathcal{C} \) with \( \text{dist}^P_H(C, D) < \delta \). Then there is a \( c \in C \) with \( \| c - \Omega_D \| < \delta \), and a \( d \in D \) with \( \| d - \Omega_C \| < \delta \). By the parallelogram law

\[
2\| c \|^2 + 2\| \Omega_C \|^2 = \| c + \Omega_C \|^2 + \| c - \Omega_C \|^2
= 4\| (c + \Omega_C)/2 \|^2 + \| c - \Omega_C \|^2.
\]

Now \( (c + \Omega_C)/2 \in C \), and so \( \| (c + \Omega_C)/2 \| \geq \| \Omega_C \| \), while

\[
\| c \| \leq \| \Omega_D \| + \delta \leq \| d \| + \delta \leq \| \Omega_C \| + 2\delta.
\]
When we combine these two inequalities with the above application of the parallelogram law, we obtain
\[2(\|\Omega C\| + 2\delta)^2 + 2\|\Omega C\|^2 \geq 4\|\Omega C\|^2 + \|c - \Omega C\|^2.\]
Simplifying, we obtain
\[8\delta(\|\Omega C\| + \delta) \geq \|c - \Omega C\|^2.\]
But \(\|\Omega C\| \leq m + \delta\), and so
\[\|c - \Omega C\|^2 \leq 8\delta(m + 2\delta).\]
Since
\[\|\Omega C - \Omega D\| \leq \|\Omega C - c\| + \|c - \Omega D\| \leq \|c - \Omega C\| + \delta,\]
it is now clear that by making \(\delta\) sufficiently small we can arrange that \(\|\Omega C - \Omega D\|\) be as small as desired. \(\square\)

10.11 LEMMA. With notation as above, let \(\psi \in \Theta\) and \(\varepsilon > 0\) be given. Suppose that \(\mathcal{S}\) is a family of continuous fields of states such that \(\{\Omega_\psi : \Omega \in \mathcal{S}\}\) is \(\varepsilon\)-dense in \(S_\psi\) for \(\| \|_\ast\). Then there is a \(\delta > 0\) such that if \(d(\theta, \psi) < \delta\) then \(\{\Omega_\theta : \Omega \in \mathcal{S}\}\) is \(3\varepsilon\)-dense in \(S_\theta\) for \(\| \cdot \|_\ast\).

**Proof.** If \(\mathcal{S}\) is not already finite, we replace it by a finite subset, since \(S_\psi\) is compact. Each \(\Omega \in \mathcal{S}\) is continuous for \(\| \cdot \|_\ast\), and since there are only a finite number of them, we can find \(\delta_1\) such that if \(d(\theta, \psi) < \delta_1\) then for each \(\Omega \in \mathcal{S}\) we have \(\|\Omega_\theta - \Omega_\psi\|_\ast < \varepsilon\). By Theorem 10.6 we can find \(\delta \leq \delta_1\) such that for \(d(\theta, \psi) < \delta\) we have \(\text{dist}_H^*(S_\theta, S_\psi) < \varepsilon\). For such a \(\theta\) let \(\mu \in S_\theta\) be given. Then, by this Hausdorff metric closeness, there is a \(\nu \in S_\psi\) such that \(\|\mu - \nu\|_\ast < \varepsilon\). By the condition on \(\mathcal{S}\) there is an \(\Omega \in \mathcal{S}\) such that \(\|\nu - \Omega_\psi\|_\ast < \varepsilon\). In view of the choice of \(\delta_1\) we have \(\|\Omega_\theta - \Omega_\psi\|_\ast < \varepsilon\). Thus, \(\|\Omega_\theta - \mu\|_\ast < 3\varepsilon\). \(\square\)

10.12 PROPOSITION. Let \(\{\| \cdot \|_\theta\}\) be a continuous field of base-norms on \(X\). Let \(\mathcal{S}\) be a family of continuous fields of states such that \(\{\Omega_\theta : \Omega \in \mathcal{S}\} = S_\theta\) for each \(\theta \in \Theta\). Given any \(\varepsilon > 0\) there is a finite subset, \(\mathcal{S}_\varepsilon\), of \(\mathcal{S}\) such that \(\{\Omega_\theta : \Omega \in \mathcal{S}_\varepsilon\}\) is \(\varepsilon\)-dense for \(\| \cdot \|_\ast\) in \(S_\theta\) for every \(\theta \in \Theta\).

**Proof.** Let \(\varepsilon > 0\) be given. For each \(\psi \in \Theta\) we can choose a finite subset, \(S_\psi\), of \(\mathcal{S}\) such that \(\{\Omega_\psi : \Omega \in S_\psi\}\) is \(\varepsilon/3\)-dense in \(S_\psi\) for \(\| \cdot \|_\ast\). By Lemma 10.11 there is a neighborhood, \(U_\psi\), of \(\psi\) such that for \(\theta \in U_\psi\) we have \(\{\Omega_\theta : \Omega \in S_\theta\}\) is \(\varepsilon\)-dense in \(S_\theta\). By the compactness of \(\Theta\) a finite number of the \(U_\psi\)'s cover \(\Theta\). The union of the corresponding \(S_\psi\)'s provides the desired \(\mathcal{S}_\varepsilon\). \(\square\)

In view of Proposition 10.7 and the equivalence of norms on finite-dimensional vector spaces, we then obtain:
10.13 THEOREM. Let $X$ be finite-dimensional, and let $\{\| \cdot \|_\theta\}$ be a continuous field of base-norms on $(X, \eta)$. Let $\| \cdot \|_*$ be a norm on $X$. For each $\varepsilon > 0$ there is a finite set, $\mathcal{S}$, of continuous fields of states such that $\{\Omega_\theta : \Omega \in \mathcal{S}\}$ is $\varepsilon$-dense in $S_\theta$ for $\| \cdot \|_*$ for each $\theta \in \Theta$.

11. Continuous fields of Lip-norms

We must now bring Lip-norms into the picture. Throughout this section we again assume that $V$ is finite-dimensional with norm $\| \cdot \|_*$, and that $\theta \mapsto \| \cdot \|_\theta$ is a continuous field of order-unit norms for $(V, e)$ such that $\| \cdot \|_\theta \leq \| \cdot \|_*$ for all $\theta \in \Theta$. We will denote the order-unit space $(V, e, \| \cdot \|_\theta)$ by $V_\theta$. Motivated by the considerations of Section 9, especially Lemma 9.3, we now assume, in addition, that for each $\theta \in \Theta$ we are given a Lip-norm $L_\theta$ on $(V, e)$ such that $\theta \mapsto L_\theta(v)$ is continuous for each $v \in V$. We will refer to such a family $\{L_\theta\}$ as a continuous field of Lip-norms. Since $V$ is finite-dimensional, “Lip-norm” here means just that its null-space is spanned by $e$.

11.1 PROPOSITION. Let $\mathcal{S}$ be a finite, non-empty, set of continuous fields of states for the $V_\theta$'s. Let $\varepsilon > 0$ be given, and for any $\theta, \psi \in \Theta$ define $N_{\theta\psi}$ on $V \oplus V$ by

$$N_{\theta\psi}(u, v) = \varepsilon^{-1} \max\{|\Omega_\theta(u) - \Omega_\psi(v)| : \Omega \in \mathcal{S}\}.$$ 

Then there is a $\delta > 0$ such that $N_{\theta\psi}$ is a bridge between $(V_\theta, L_\theta)$ and $(V_\psi, L_\psi)$ whenever $d(\theta, \psi) < \delta$.

Proof. We remark that $N_{\theta\psi}$ is constructed from a finite number of spans each of which is a version of the bridge built in Proposition 5.4. It is clear that $N$ is continuous for the norms. Because $\Omega$ is not empty, Condition 2 in Definition 5.1 for a bridge is clearly satisfied. We must show that we can choose $\delta > 0$ such that Condition 3 is satisfied.

Since the situation is symmetric in $\theta$ and $\psi$, it suffices to show that for any $u \in V_\theta$ there is a $v \in V_\psi$ satisfying Condition 3. If $u \in \mathbb{R}e$ we can always take $v = u$. So we now show how to choose a $\delta > 0$ which works for all $u$'s not in $\mathbb{R}e$. Let $W$ be a subspace of $V$ which is complimentary to $\mathbb{R}e$, and let $\Sigma_W$ denote the unit $\| \cdot \|_*$-sphere of $W$. Each $L_\theta$ restricts to a norm on $W$ satisfying the conditions of Lemma 10.1. Thus, $(\theta, w) \mapsto L_\theta(w)$ is jointly continuous, and there is a constant, $c$, such that $c \leq L_\theta(w)$ for all $\theta \in \Theta$ and $w \in \Sigma_W$. By joint continuity and compactness, we can choose $\delta_1$ such that if $d(\theta, \psi) < \delta_1$ then $|L_\theta(w) - L_\psi(w)| < \varepsilon c^2/2$ for all $w \in \Sigma_W$. By uniform continuity and the finiteness of $\mathcal{S}$ we can find $\delta \leq \delta_1$ such that if $d(\theta, \psi) < \delta$ then
$|\Omega_\theta(w) - \Omega_\psi(w)| < \varepsilon c/2$ for all $\Omega \in \mathcal{S}$ and all $w \in \Sigma_W$. We show that this choice of $\delta$ works.

To each $u \in V$, viewed as an element of $V_\theta$, we must find a $v \in V$, viewed as in $V_\psi$, satisfying the requirement of Condition 3 of Definition 5.1. We assume that $u \notin \mathbb{R}e$. We argue much as in the proof of Lemma 10.2. We consider first the special case in which $u = w \in \Sigma_W$. Then clearly $L_\psi(w) \neq 0$, so that we can set $z = (L_\theta(w)/L_\psi(w))w$. Then $L_\psi(z) = L_\theta(w)$, so that we only need to check that $N_{\theta\psi}(w, z) \leq L_\theta(w)$. We assume that $d(\theta, \psi) < \delta$. Then for each $\Omega \in \mathcal{S}$ we have

$$|\Omega_\theta(w) - \Omega_\psi(z)| \leq |\Omega_\theta(w) - \Omega_\psi(w)| + |\Omega_\psi(w) - \Omega_\psi(z)|$$
$$\leq \varepsilon c/2 + |1 - (L_\theta(w)/L_\psi(w))||\Omega_\psi(w)|$$
$$= \varepsilon c/2 + |L_\psi(w) - L_\theta(w)|L_\psi(w)^{-1}|\Omega_\psi(w)|$$
$$\leq \varepsilon c/2 + (\varepsilon^2/2)c^{-1}\|w\|_* = \varepsilon c.$$ 

If we now take $u = te + w$ for some $t \in \mathbb{R}$, and set $v = te + z$, we again get $L_\psi(v) = L_\theta(u)$ and $|\Omega_\theta(u) - \Omega_\psi(v)| \leq \varepsilon c$. Thus

$$N(u, v) \leq c \leq L_\theta(u).$$

Now the two ends of this inequality are homogeneous in $(u, v)$ for positive scalars, as is the equality $L_\psi(v) = L_\theta(u)$. But every element of $V$ not in $\mathbb{R}e$ is a positive scalar multiple of an element of our special form $te + w$. Thus, for every $u \in V_\theta$ we can find $v \in V_\psi$ satisfying these two relations. Consequently, $N_{\theta\psi}$ is a bridge. \hfill \Box

Our main theorem here, which applies immediately to the $B_\theta$’s of Section 9, is:

**11.2 THEOREM.** Let $V$ be a finite-dimensional vector space with distinguished element $e$, and let $\Theta$ be a compact space with metric $d$. Let $\{\| \cdot \|_\theta\}$ be a continuous field of order-unit norms on $(V, e)$, and let $V_\theta$ denote $V$ equipped with $\| \cdot \|_\theta$. Let $\{L_\theta\}$ be a continuous field of Lip-norms on $(V, e)$. Then for every $\varepsilon > 0$ there is a $\delta > 0$ such that if $d(\theta, \psi) < \delta$ then

$$\text{dist}_q((V_\theta, L_\theta), (V_\psi, L_\psi)) < \varepsilon.$$ 

**Proof.** Let $\tilde{V} = (V/\mathbb{R}e)$. Each $L_\theta$ drops to a norm $\tilde{L}_\theta$ on $\tilde{V}$. It is simple to check that $\tilde{L}_\theta$ is a continuous field of norms. According to Lemma 10.1 the field $\{\tilde{L}_\theta\}$ of dual norms is continuous. But the dual of $\tilde{V}$ is canonically identified with the subspace $V^{*o}$ of $V^*$ consisting of functionals annihilating $\mathbb{R}e$. Thus, each $\tilde{L}_\theta$ gives a norm $\tilde{L}'_\theta$ on $V^{*o}$, and the field $\{\tilde{L}'_\theta\}$ is continuous. Again, according to Lemma 10.1, there
is a norm, \( \| \cdot \|_*' \), on \( V^\theta' \) such that \( L'_\theta \leq \| \cdot \|_*' \) for all \( \theta \in \Theta \). Thus, \( \rho_\theta(\mu, \nu) \leq \| \mu - \nu \|_*' \) for all \( \theta \in \Theta \) and \( \mu, \nu \in S_\theta \).

Let \( \varepsilon > 0 \) be given. According to Theorem 10.11 we can find a finite family, \( S \), of continuous fields of states such that \( \{ \Omega_\theta : \Omega \in S \} \) is \((\varepsilon/2)\)-dense in \( S_\theta \) for \( \| \|_*' \) for every \( \theta \in \Theta \). Define \( N_{\theta \psi} \) as in the statement of Proposition 11. Let \( \Omega_\theta \in S \theta \) and \( \mu, \nu \in S_\theta \). Define \( \rho_{L_{\theta \psi}}(\mu, \nu) \leq \| \mu - \nu \|_*' \) whenever \( d(\theta, \psi) < \delta \).

Let \( \theta, \psi \in \Theta \) with \( d(\theta, \psi) < \delta \). Let \( L_{\theta \psi} \) be the Lip-norm on \( V_\theta \oplus V_\psi \) defined as at the beginning of Section 5 using \( N_{\theta \psi} \). For simplicity write \( L \) for \( L_{\theta \psi} \). We show that \( \rho_{\| \|_*'}(S_\theta, S_\psi) < \varepsilon \).

Let \( \mu \in S_\theta \). By the choice of \( S \) there is an \( \Omega_\mu \in S \) such that

\[
\rho_{L_\theta}(\mu, \Omega_\mu) = \rho_{L_\theta}(\mu, \Omega_\mu) \leq \| \mu - \Omega_\mu \|_*' < \varepsilon/2.
\]

We now show that \( \rho_L(\mu, \Omega_\mu) < \varepsilon \), thus showing that \( S_\theta \) is in the \( \varepsilon \)-neighborhood of \( S_\psi \) for \( \rho_{L_{\theta \psi}} \).

Let \( u, v \in V \), with \( (u, v) \) viewed as in \( V_\theta \oplus V_\psi \), and suppose that \( L(u, v) \leq 1 \). Then \( N_{\theta \psi}(u, v) \leq (\varepsilon/2) \), so that

\[
|\Omega_\theta^\mu(u) - \Omega_\psi^\mu(v)| < \varepsilon/2.
\]

Since this is true for all \( (u, v) \) with \( L(u, v) \leq 1 \), this means that

\[
\rho_L(\Omega_\theta^\mu, \Omega_\psi^\mu) \leq \varepsilon/2.
\]

Thus

\[
\rho_L(\mu, \Omega_\mu) \leq \rho_L(\mu, \Omega_\mu) + \rho_L(\Omega_\mu, \Omega_\mu) < \varepsilon,
\]

as desired.

By reversing the roles of \( \theta \) and \( \psi \) we see that \( S_\psi \) is in the \( \varepsilon \)-neighborhood of \( S_\theta \).

When the above theorem is combined with the considerations of Section 9, we see that we have completed the proof of Theorem 9.2, our main example.

## 12. Completeness

One of the fundamental facts about the metric space of isometry classes of ordinary compact metric spaces with the Gromov–Hausdorff metric is that it is complete. We now show that the quantum analogue of this fact is true. (We will show in Theorem 13.15 that for the quantum case this metric space is also separable.) We parallel a common classical approach to this matter by showing that, under suitable conditions, everything can be assembled inside one compact quantum metric...
space where we can then use the fact that the set of closed subsets of a compact metric space is complete for the Hausdorff metric.

We must consider Cauchy sequences of compact quantum metric spaces. But in preparation for this we first consider sequences of order-unit spaces. (Much of what we do works for more general index sets than $\mathbb{N}$, but we do not need this greater generality here.) If $\{A_j\}$ is such a sequence, we denote by $\Pi^bA_j$ the subspace of the full product which consists of sequences $\{a_j\}$ for which the sequence $\{\|a_j\|\}$ is bounded. This certainly includes $e = \{e_j\}$. We say that a sequence is non-negative if each of its terms is non-negative. Then $\Pi^bA_j$ is clearly an order-unit space. We let $\oplus A_j$ denote the subspace of sequences which are 0 except for a finite number of terms. It is not an order-unit space, and $e$ is not in it.

We consider “compactifications” of $\oplus A_j$, namely subspaces $C$ of $\Pi^bA_j$ which contain both $\oplus A_j$ and $e$, and so are order-unit spaces. There is an evident projection of $\Pi A_j$, and so of $C$, onto each $A_n$. These projections give evident injections of each $S(A_j)$ into $S(C)$. Without further comment we will view each $S(A_j)$ as a subset of $S(C)$ when convenient.

Let $Z_0 = \bigcup S(A_j)$ inside $S(C)$. Notice that an element $c$ of $C$ is non-negative exactly if $\mu(c) \geq 0$ for all $\mu \in Z_0$. Let $Z = \text{co}(Z_0)$ be the convex hull of $Z_0$ in $S(C)$. Then we can apply the bipolar theorem almost exactly as done in the proof of proposition VII.5.4 of [22] to obtain:

12.1 Lemma. With notation as above, $Z$ is $w^*$-dense in $S(C)$.

For any positive $n$ let $B_n = \oplus_{j=1}^n A_j$. In the evident way each $B_n$ is an order-unit space. There are evident projections of $C$ onto each $B_n$, and of $B_n$ onto each $A_j$ and $B_j$ for $j \leq n$. Thus, for the evident identifications, we have

$$S(A_j) \subseteq S(B_j) \subseteq S(B_n) \subseteq S(C)$$

for $j \leq n$. Furthermore, $S(B_n) = \text{co}(\bigcup^n S(A_j))$, and so $Z = \bigcup^\infty S(B_n)$, an increasing union.

Suppose now that we have a sequence $\{(A_j, L_j)\}$ of compact quantum metric spaces. Suppose further that we have a family $M = \{M_j\}$ where each $M_j$ is a Lip-norm on $A_j \oplus A_{j+1}$ which induces $L_j$ and $L_{j+1}$. Define $L \equiv L_M$ on $\Pi A_j$, the full product, by

$$L(\{a_j\}) = \sup_j \{M_j(a_j, a_{j+1})\}.$$
Of course $L(e) = 0$. We set

$$C_M = \{ \{a_j\} \in \prod^b A_j : L(\{a_j\}) < \infty \}.$$ 

It is easy to check that $C_M$ is an order-unit subspace of $\prod^b A_j$ containing $\oplus^\infty A_j$, and that $L$ is a seminorm on $C_M$ which takes value 0 only on $Re$. But without further hypotheses, $L$ need not induce the $L_j$’s, nor be a Lip-norm.

We remark that the above discussion, and much of what follows, generalizes to the situation where one has a graph, with a compact quantum metric space associated to each vertex, and a Lip-norm generalizing the above $M_j$’s associated to each edge. But we will not discuss this possibility further here.

Much as for $L$, define a seminorm, $J_n$ on $B_n$ by

$$J_n(\{a_j\}) = \sup \{ M_j(a_j, a_{j+1}) : 1 \leq j \leq n - 1 \}.$$ 

By a simple adaptation of the proof of Lemma 4.4 we obtain:

12.2 LEMMA. With notation as above, $J_n$ is a Lip-norm on $B_n$. For the evident projections of $B_n$ onto $A_m$ for $m \leq n$, and onto $A_m \oplus A_{m+1}$ and $B_m$ for $m \leq n - 1$, the Lip-norm $J_n$ induces $L_m$, $M_m$, and $J_m$.

Let us denote the evident projection of $\prod A_j$ onto $B_n$ by $\pi_n$.

12.3 LEMMA. Let $b \in B_n$, and let $\varepsilon > 0$ be given. Then there is a $c \in \prod A_j$, the full product, such that $\pi_n(c) = b$ and $L(c) \leq J_n(b) + \varepsilon$.

Proof. We choose $b_k \in B_k$ for $k \geq n$ by induction. Set $b_n = b$. Suppose that $b_k$ has been chosen. Since $J_{k+1}$ induces $J_k$ by Lemma 12.1, we can find $b_{k+1} \in B_{k+1}$ such that $\pi_k(b_{k+1}) = b_k$ and $J_{k+1}(b_{k+1}) < J_k(b_k) + \varepsilon/2^k$. Consequently, for each $k > n$ we have $J_k(b_k) < J_n(b) + \varepsilon$. We set $c$ to be the unique element of $\prod A_j$ such that $\pi_k(c) = b_k$ for each $k > n$. Then $L(c) \leq J_n(b) + \varepsilon$ as desired. 

The above lemma goes in the direction of saying that $L$ induces $J_n$.

The difficulty is that in general the above $c$ will not be a bounded sequence. To ensure that it is bounded we need further hypotheses.

Because of the profusion of indices to follow, we will permit ourselves to write $\rho(S, T)$ instead of $\text{dist}_\rho^b(S, T)$ when $S$ and $T$ are closed subsets of some metric space with metric $\rho$.

12.4 LEMMA. For each $m$ and $n$ with $m < n$ we have

$$\rho_{J_n}(S(B_m), S(B_n)) \leq \sum_{j=m}^{n-1} \rho_{M_j}(S(A_j), S(A_{j+1})).$$
Proof. This is an adaptation of a calculation done in the proof of Lemma 4.4, so we will be a bit succinct here. Let \( \mu_i \in S(A_i) \) for some \( i \) with \( m < i \leq n \). Then we can inductively find \( \mu_{i-1}, \ldots, \mu_m \) with \( \mu_j \in S(A_j) \) and
\[
\rho_{J_n}(\mu_j, \mu_{j+1}) \leq \rho_{M_j}(S(A_j), S(A_{j+1}))
\]
for \( m \leq j \leq i-1 \). Consequently,
\[
\rho_{J_n}(\mu_i, \mu_m) \leq \sum_{j=m}^{i-1} \rho_{M_j}(S(A_j), S(A_{j+1})).
\]
Thus, \( \rho_{J_n}(S(B_n), S(A_i)) \) satisfies the same bound. But \( S(B_n) = \text{co}(\cup_{j=1}^{n-1} S(A_j)) \), and \( \rho_{J_n} \) is convex (definition 9.1 of [62]), and so \( \rho_{J_n}(S(B_n), S(B_n)) \) satisfies this bound too for \( i = n \). \( \square \)

When we apply the above lemma twice with \( m = 1 \) so that \( B_m = A_1 \), we obtain:

12.5 COROLLARY. For any \( n \) we have
\[
\text{diam}(B_n, J_n) \leq \text{diam}(A_1, L_1) + 2 \sum_{j=1}^{n-1} \rho_{M_j}(S(A_j), S(A_{j+1})).
\]

This lemma and corollary suggest correctly that the key condition for us (as in the classical case) will be that \( \sum_{j=1}^{\infty} \rho_{M_j}(S(A_j), S(A_{j+1})) < \infty \). To begin with, the corollary tells us that in this case there is a number, say \( \Delta \), such that \( \text{diam}(B_n, J_n) \leq \Delta \) for all \( n \). (There is no way that \( (C_M, L) \) can have finite diameter without such a bound.) We now use this bound to control infinite sequences.

12.6 LEMMA. Let \( \{a_j\} \in \prod_{j=1}^{\infty} A_j, \) the full product, and suppose that \( L(\{a_j\}) < \infty \). If there is a constant, \( \Delta \), such that \( \text{diam}(B_n, J_n) \leq \Delta \) for all \( n \), then \( \{a_j\} \in C_M, \) that is, \( \sup_j \{\|a_j\|\} < \infty \). Furthermore, \( L \) on \( C_M \) has finite diameter, no greater than \( \Delta \). That is, for any \( c \in C_M \) we have
\[
\|c\| \leq (\Delta/2)L(c).
\]

Proof. Set \( h = L(\{a_j\}) \), and let \( b_n = (a_1, \ldots, a_n) \) for each \( n \). Clearly, \( J_n(b_n) \leq h \). Then by proposition 2.2 of [62], recalled near the end of Section 2, \( \|b_n\| \leq h \Delta/2 \). Set \( e_n = e_{B_n} \). Then this means that there is a \( t \in \mathbb{R} \) such that for each \( n \)
\[
\|b_n - te_n\| \leq h \Delta/2.
\]
For each \( n \) set \( G_n = \{ t : \|b_n - te_n\| \leq h\Delta/2 \} \). Then \( G_n \) is a non-empty closed bounded subset of \( \mathbb{R} \). Furthermore, it is easily seen that \( G_n \supseteq G_{n+1} \). By compactness there is a \( t \in \cap_{\infty} G_n \). For this \( t \) we have \( \|a_j - te_j\| \leq h\Delta/2 \) for all \( j \), so that \( \|a_j\| \leq t + h\Delta/2 \) for all \( j \). Thus, \( \{a_j\} \in C_M \), and \( \|c\| \leq (\Delta/2)L(c) \) where \( c = \{a_j\} \).

We can now apply this lemma to the context of Lemma 12.3. It assures us that if \( \text{diam}(B_n, J_n) \leq \Delta \) for all \( n \), then the \( c \) of Lemma 12.3 is bounded. Of course, we always have \( J_n(\pi_n(c)) \leq L(c) \). When all of this is combined with Proposition 3.1, we obtain:

**12.7 PROPOSITION.** Suppose that there is a constant, \( \Delta \), such that \( \text{diam}(B_j, J_j) \leq \Delta \) for all \( j \). Then for any \( b_n \in B_n \) and \( \varepsilon > 0 \) there is a \( c \in C_M \) such that \( \pi_n(c) = b_n \) and \( L(c) \leq J_n(b_n) + \varepsilon \). In other words, \( L \) on \( C_M \) induces \( J_n \) via \( \pi_n \). Thus, the natural inclusion of \( S(B_n) \) into \( S(C_M) \) is isometric for \( \rho_{J_n} \) and \( \rho_L \).

Note that \( L \) still need not be a Lip-norm.

**12.8 LEMMA.** Suppose now that \( \sum_{j=1}^{\infty} \rho_{M_j}(S(A_j), S(A_{j+1})) < \infty \). Then the metric space \((Z, \rho_L)\) is totally bounded.

**Proof.** Let \( \varepsilon > 0 \) be given. Then there is an \( m \) such that

\[
\sum_{j=m}^{\infty} \rho_{M_j}(S(A_j), S(A_{j+1})) < \varepsilon.
\]

It follows from Lemma 12.4 that for each \( n \geq m \) we have

\[
\rho_L(S(B_m), S(B_n)) \leq \varepsilon.
\]

This says that \( S(B_m) \) is \( \varepsilon \)-dense for \( \rho_L \) in \( Z \). But \( S(B_m) \) is compact for the topology from \( \rho_L \), since \( \rho_{J_m} \) is a Lip-norm and \( \rho_L = \rho_{J_m} \) on \( S(B_m) \). Thus, \( S(B_m) \) is totally bounded for \( \rho_L \), and a finite subset \( \varepsilon \)-dense in \( S(B_m) \) will be \( 2\varepsilon \)-dense in \( Z \).

Let \( \hat{Z} \) denote the abstract completion of \( Z \) for \( \rho_L \). We let \( \rho_L \) denote also the extension of \( \rho_L \) to \( \hat{Z} \). Under the hypothesis of the above lemma, \( \hat{Z} \) is compact. Each \( c \in C_M \) is Lipschitz when viewed as a function on \( Z \), and so extends to give a Lipschitz function, \( \hat{c} \), on \( \hat{Z} \). It is clear that if \( c \geq 0 \) in \( C_M \) then \( \hat{c} \geq 0 \) as function, and that \( \hat{c} \equiv 1 \). Thus, \( C_M \) is realized isometrically isomorphically as an order-unit space of continuous functions on \( \hat{Z} \). Each \( x \in \hat{Z} \) determines an element \( \mu_x \) of \( S(C_M) \) by evaluation. By the definition of the \( \mbox{w}^* \)-topology the map \( x \mapsto \mu_x \) is continuous from the metric topology on \( \hat{Z} \). But each element
of $Z \subseteq \hat{Z}$ is carried to itself in $S(C_M)$, and $Z$ is dense in $S(C_M)$. Since $\hat{Z}$ is compact, its image is closed, and so is all of $S(C_M)$.

Let us see now that the map $x \mapsto \mu_x$ is injective. Let $x, y \in \hat{Z}$ with $x \neq y$, and set $\gamma = \rho_L(x, y)$. We can find $\mu, \nu \in Z$ such that $\rho_L(x, \mu) < \gamma/4$ and $\rho_L(y, \nu) < \gamma/4$. Thus $\rho_L(\mu, \nu) > \gamma/2$, so that we can find $c \in C_M$ with $L(c) \leq 1$ and $|\mu(c) - \nu(c)| > \gamma/2$. But $L_{\rho_L}(\hat{c}) \leq 1$ so that $|\hat{c}(x) - \hat{c}(\mu)| < \gamma/4$ and $|\hat{c}(y) - \hat{c}(\nu)| < \gamma/4$. Thus $|\hat{c}(x) - \hat{c}(y)| > 0$.

It follows that $x \mapsto \mu_x$ is a homeomorphism of $\hat{Z}$ onto $S(C_M)$. From this it is clear that the metric topology on $S(C_M)$ from $\rho_L$ agrees with the $w^*$-topology. We have thus obtained:

12.9 THEOREM. Let $\{(A_j, L_j)\}$ be a sequence of compact quantum metric spaces, and for each $j$ let $M_j$ be a Lip-norm on $A_j \oplus A_{j+1}$ which induces $L_j$ and $L_{j+1}$. Let $L$ and $C_M$ be defined as above. If

$$\sum_{j}^{\infty} \rho_{M_j}(S(A_j), S(A_{j+1})) < \infty,$$

then $L$ is a Lip-norm on $C_M$.

When we view the above situation within $S(C_M)$, we have

$$\sum_{j}^{\infty} \rho_L(S(A_j), S(A_{j+1})) < \infty,$$

which clearly implies that $\{S(A_j)\}$ is a Cauchy sequence for $\rho_L$ ($= \text{dist}_{H}^{\rho_L}$). Since the space of compact convex subsets is complete for Hausdorff distance, this sequence has a limit, say $K$, in $S(C_M)$. We let $\pi_\infty$ denote the process of restricting elements of $C_M$ to $K$, and we let $D = \pi_\infty(C_M)$. We let $L_D$ be the quotient of $L$ for $\pi_\infty$. Thus, $L_D$ is a Lip-norm by Proposition 3.1. Now, according to Proposition 5.7, we have

$$\text{dist}_q(A_j, D) \leq \text{dist}_{H}^{\rho_L}(S(A_j), K).$$

From this we obtain:

12.10 COROLLARY. Under the hypotheses of Theorem 12.9 there is a compact quantum metric space $(D, L_D)$ to which the sequence $\{(A_j, L_j)\}$ converges for $\text{dist}_q$.

12.11 THEOREM. The metric space of isometry classes of compact quantum metric spaces, with the metric $\text{dist}_q$, is complete.

Proof. Let $\{(A_j, L_j)\}$ be a Cauchy sequence. It suffices to show that a subsequence converges. But we can choose a subsequence, still denoted...
by \{ (A_j, L_j) \}, such that \( \sum_{j=1}^{\infty} \text{dist}_q(A_j, A_{j+1}) < \infty \). Then for each \( j \) we can find a Lip-norm \( M_j \) on \( A_j \oplus A_{j+1} \) which induces \( L_j \) and \( L_{j+1} \), and is such that

\[
\rho_{M_j}(S(A_j), S(A_{j+1})) \leq \text{dist}_q(A_j, A_{j+1}) + 1/2^j.
\]

It follows that \( \sum \rho_{M_j}(S(A_j), S(A_{j+1})) < \infty \). Then according to Corollary 12.10 there is a compact quantum metric space to which \{ (A_j, L_j) \} converges for \( \text{dist}_q \). □

13. Finite approximation and compactness

An elementary property of ordinary compact metric spaces is that for every \( \varepsilon > 0 \) there is a finite set which is \( \varepsilon \)-dense, and thus approximates within \( \varepsilon \) for Gromov–Hausdorff distance. We begin this section by showing that the quantum analogue of this is true. (Of course, in Sections 8 to 11 we were already using a special instance of this in which the “quantum finite sets” were chosen in a particularly useful way.) Afterward we use this finite approximation and some of our earlier results to prove a quantum analogue of a fundamental compactness criterion of Gromov for families of ordinary compact metric spaces for Gromov–Hausdorff distance. In addition to its spectacular application in [25], Gromov’s compactness theorem has been widely used in Riemannian geometry. See, for example, the exposition and references in [26], [23], [64], [68]. It remains to be seen what applications will be found for our quantum version.

Our finite approximation theorem is:

13.1 THEOREM. Let \((A, L)\) be a compact quantum metric space. For every \( \varepsilon > 0 \) there is a compact quantum metric space, \((B, L_B)\), such that \( B \) has finite dimension and \( \text{dist}_q(A, B) < \varepsilon \). In fact, we can take \((B, L_B)\) to be a quotient of \((A, L)\). Alternatively, or if \( A \) is infinite-dimensional, we can take \( B \) to be order-isomorphic to \( C(X) \) for some finite set \( X \).

Proof. Since \( S(A) \) with \( \rho_L \) is a compact metric space, we can find a finite subset, \( F \), of \( S(A) \) which is \( \varepsilon \)-dense in \( S(A) \) for \( \rho_L \). Let \( K = \text{co}(F) \), which is closed. Let \( B \) consist of the restrictions to \( K \) of elements of \( A \), and let \( L_B \) be the quotient Lip-norm, as discussed in Section 3. Note that \( B \) has finite dimension since its elements are determined by their values on \( F \). Also, \( K \) is naturally identified with \( S(B) \) by Proposition 3.5. Then

\[
\text{dist}_q(A, B) \leq \text{dist}^{\rho_B}_{H}(S(A), K) < \varepsilon
\]
by Proposition 5.7. If \( A \) is infinite-dimensional, then we can perturb the elements of \( F \) very slightly so that they are linearly independent. Then \( K \) will be a simplex, and \( B \) will be order-isomorphic to \( C(F) \). If \( A \) is finite-dimensional, then we must use a simple generalization of the doubling construction of Example 5.6 to make sufficiently many very close copies of \( A \) so that the elements of \( F \) can be chosen to be linearly independent. \( \square \)

We emphasize that the Lip-norms on \( C(X) \) will often not come from metrics on \( X \). In contrast to this, see Theorem 13.16, where in effect the \( L_B \)'s are required to come from ordinary metrics on the \( X \)'s.

A simple observation which seems important is that the topology of a compact topological space can not be recovered from the data consisting of an increasing dense union (with inclusion maps) of finite subsets with the relative topology (discrete), whereas the metric on a compact metric space (and so the space itself by completion) can be recovered from an increasing dense union of finite subsets with the relative metrics. It is easy to see that this latter is still true for compact quantum metric spaces if one uses finite-dimensional quotients.

We now recall the statement of the classical Gromov compactness theorem, along the lines given in his original formulation of the theorem [25], or as in theorem 6.3 of the appendix of [64]. Let \( \mathcal{M} \) denote the set of isometry equivalence classes of compact metric spaces, equipped with the metric \( \text{dist}_{GH} \). Our notation below will not distinguish between equivalence classes and their representatives.

13.2 Definition. Let \((X, \rho)\) be a compact metric space. We define a function, \( \text{Cov}_\rho \), from the strictly positive real numbers, \( \mathbb{R}^+ \), to the positive integers, \( \mathbb{N}^+ \), by setting \( \text{Cov}_\rho(\varepsilon) \) to be the smallest number of \( \varepsilon \)-balls for \( \rho \) which are needed to cover \( X \). We call the function \( \text{Cov}_\rho \) the covering growth of \( \rho \).

Of course, the most interesting aspect of \( \text{Cov}_\rho \) is how it grows as \( \varepsilon \to 0 \).

13.3 Gromov’s Compactness Theorem. Let \( \mathcal{S} \) be a subset of \( \mathcal{M} \). Then \( \mathcal{S} \) is totally bounded for \( \text{dist}_{GH} \) (hence has compact closure) if and only if

1) there is a constant \( D \) such that \( \text{diam}(X, \rho) \leq D \) for all \((X, \rho) \in \mathcal{S} \); and

2) there is a function, \( G \), from \( \mathbb{R}^+ \) to \( \mathbb{N}^+ \) such that \( \text{Cov}_\rho(\varepsilon) \leq G(\varepsilon) \) for every \((X, \rho) \in \mathcal{S} \) and every \( \varepsilon > 0 \).
We now formulate an analogue of this theorem for compact quantum metric spaces. For this purpose we need to mention that already in the literature on ordinary metric spaces there are several alternative measures of growth besides $\text{Cov}_\rho$, which are nevertheless equivalent for the purposes of the compactness theorem. So it is to be expected that this happens also in the quantum setting. In fact, the proof of Theorem 13.1 already suggests three such alternatives, which we now formalize.

13.4 DEFINITION. Let $(A, L)$ be a compact quantum metric space. For each $\varepsilon > 0$ we set:

1) $\text{Fin}_L(\varepsilon)$ is the smallest integer $n$ such that there is a compact quantum metric space $(B, L_B)$ such that $\text{dist}_q(A, B) < \varepsilon$ and $\text{dim}(B) \leq n$ (where $\text{dim}(B)$ is the vector-space dimension of $B$).

2) $\text{Cov}_L(\varepsilon)$ is the smallest integer $n$ such that there is a compact quantum metric space $(B, L_B)$ and a surjection $\pi$ of $A$ onto $B$ such that $L$ induces $L_B$, while $\text{dist}_H^{\rho_L}(S(B), S(A)) < \varepsilon$ and $\text{dim}(B) \leq n$.

3) $\text{Scv}_L(\varepsilon)$ is the smallest integer $n$ such that there is a subset of $S(A)$ which is $\varepsilon$-dense for $\rho_L$ and has only $n$ elements.

We will sometimes write $\text{Scv}_A$ instead of $\text{Scv}_L$, etc., when this seems helpful and $L$ is understood.

Of course, $\text{Scv}_L(\varepsilon)$ is always finite, and it is easily seen from the proof of Theorem 13.1 that

$$\text{Scv}_L(\varepsilon) \geq \text{Cov}_L(\varepsilon) \geq \text{Fin}_L(\varepsilon).$$

All this suggests, of course, that it would be interesting to study the Kolmogorov $\varepsilon$-entropy of $(S(A), \rho_L)$ for specific examples, or “quantum” versions of it, say $\ln_2(\text{Fin}_L(\varepsilon))$. See the discussion of fractal and Hausdorff-Besicovich dimension in [7].

Let now $Q$ denote the set of isometry equivalence classes of compact quantum metric spaces, equipped with the metric $\text{dist}_q$.

13.5 THE QUANTUM GROMOV COMPACTNESS THEOREM. Let $S$ be a subset of $Q$. If $S$ is totally bounded for $\text{dist}_q$ then

1) there is a constant, $D$, such that $\text{diam}(A, L) \leq D$ for all $(A, L) \in S$; and

2) there is a function, $G$, from $\mathbb{R}^+ \to \mathbb{N}^+$ such that $G(\varepsilon) \geq \text{Scv}_L(\varepsilon) \geq \text{Fin}_L(\varepsilon)$ for every $(A, L) \in S$ and every $\varepsilon > 0$.

Conversely, if
1) there is a constant, $D$, such that $\text{diam}(A, L) \leq D$ for all $(A, L) \in S$; and

2) there is a function, $G$, from $\mathbb{R}^+$ to $\mathbb{N}^+$ such that $G(\varepsilon) \geq \text{Fin}_L(\varepsilon)$ for every $(A, L) \in S$ and every $\varepsilon > 0$,

then $S$ is totally bounded in $\mathcal{Q}$.

Before launching into the proof of this theorem, we point out that it is relevant to the situation studied in Sections 8–11. Fix a compact Lie group $G$ (possibly not connected), and fix a length function, $\ell$, on $G$. Let $\mathcal{E}(G, \ell)$ denote the subset of $\mathcal{Q}$ consisting of the equivalence classes of pairs $(A, L_A)$ where $A$ is a unital $C^*$-algebra and $L_A$ is a Lip-norm defined, using $\ell$, by an ergodic action of $G$ on $A$, using the formula at the beginning of Section 8. Then Theorem 13.5 will tell us that $\mathcal{E}(G, \ell)$ is a totally bounded subset of $\mathcal{Q}$. To see this, note that by lemma 2.4 of [61] every element of $\mathcal{E}(G, \ell)$ has radius no larger than $\int_G \ell(x) dx$, where we use the Haar measure which gives $G$ mass 1. Thus condition 1 of Theorem 13.5 is satisfied. We now show how to define the function $G$ (not to be confused with the group $G$) for condition 2. Let $\varepsilon > 0$ be given. Choose a representation, $\pi$, of $G$ in the way discussed after Proposition 8.1, and let $\{\delta_n\}$ be the sequence obtained in Theorem 8.2. Choose $n$ such that $\delta_n < \varepsilon$. Let $\hat{G}_n$ be as defined after Proposition 8.1. We set

$$G(\varepsilon) = \sum \{(\dim(\rho))^2 : \rho \in \hat{G}_n\}.$$ 

This works for the following reasons. Let $\alpha$ be an ergodic action of $G$ on a unital $C^*$-algebra $A$, and let $B_n$ be defined as above. Now a key assertion of the main theorem of [34] (see also [72]) states that any irreducible representation of $G$ occurs in $A$ with multiplicity no greater than its dimension. Thus $\dim(B_n) \leq G(\varepsilon)$, while $\text{dist}_q(A, B_n) \leq \varepsilon$ according to Theorem 8.2. Consequently condition 2 of Theorem 13.5 is satisfied. Theorem 9.2 is consistent with this observation, and the results we will give in [63] will be also.

**Proof of Theorem 13.5.** Suppose first that $S$ is totally bounded. Although Condition 1 can be dealt with more directly, the following approach seems interesting.

13.6 Lemma. The function $(A, L) \mapsto \text{diam}(A, L)$ from $\mathcal{Q}$ to $\mathbb{R}$ is Lipschitz. In fact,

$$|\text{diam}(A, L_A) - \text{diam}(B, L_B)| \leq 2 \text{dist}_q(A, B)$$

for all $(A, L_A)$ and $(B, L_B)$ in $\mathcal{Q}$.
Proof. Let \((A, L_A)\) and \((B, L_B)\) ∈ \(Q\), and let \(d = \text{dist}_q(A, B)\). Given \(\varepsilon > 0\) there is a Lip-norm \(L\) on \(A \oplus B\) inducing \(L_A\) and \(L_B\) and such that \(\text{dist}^{\rho_L}(S(A), S(B)) < d + \varepsilon\). If \(\mu_1, \mu_2 \in S(A)\), then there are \(\nu_1, \nu_2 \in S(B)\) with \(\rho_L(\mu_j, \nu_j) < d + \varepsilon\). Thus
\[
\rho_{L_A}(\mu_1, \mu_2) \leq \text{diam}(B, L_B) + 2(d + \varepsilon).
\]
Because \(\varepsilon\) is arbitrary, it follows that
\[
\text{diam}(A, L_A) \leq \text{diam}(B, L_B) + 2d.
\]
But we can reverse the roles of \(A\) and \(B\) to obtain the desired inequality.

It follows that since \(\mathcal{S}\) is a totally bounded subset of \(Q\), the set of diameters of elements of \(\mathcal{S}\) will be a totally bounded subset of \(\mathbb{R}\), and so Condition 1 is satisfied.

We show now how to obtain a function \(G\) for Condition 2 of the first part. Let \(\varepsilon > 0\) be given. Since \(\mathcal{S}\) is totally bounded, there is a finite subset, \(F\), of \(\mathcal{S}\) which is \((\varepsilon/3)\)-dense in \(\mathcal{S}\) for \(\text{dist}_q\). We set
\[
G(\varepsilon) = \max\{\text{Scv}_L(\varepsilon/3) : (A, L) \in F\}.
\]
From the triangle inequality it follows that this choice of \(G(\varepsilon)\) works.

We now show that the conditions of the second part of Theorem 13.5 are sufficient. Thus we assume that \(D\) and the function \(G\) are given. Let \(\mathcal{S}_G^D\) consist of all (equivalence classes of) compact quantum metric spaces \((A, L)\) of diameter \(\leq D\) for which \(\text{Fin}_L \leq G\). It suffices to show that \(\mathcal{S}_G^D\) is totally bounded. Let \(\varepsilon > 0\) be given. Let \((A, L) \in \mathcal{S}_G^D\). Since \(\text{Fin}_L(\varepsilon) \leq G(\varepsilon)\), there is a compact quantum metric space \((B, L_B)\) with \(\text{dim}(B) \leq G(\varepsilon)\) such that \(\text{dist}_q(A, B) < \varepsilon\). It follows from Lemma 13.6 that
\[
\text{diam}(B, L_B) \leq D + 2\varepsilon.
\]
Let \(\mathcal{S}_n(d)\) denote the subset of \(Q\) consisting of elements of dimension \(\leq n\) and diameter \(\leq d\). Thus we have just seen that \(\mathcal{S}_G^D\) is contained in the \(\varepsilon\)-neighborhood of \(\mathcal{S}_G(\varepsilon)(D + 2\varepsilon)\). It thus suffices to show that each \(\mathcal{S}_n(d)\) is totally bounded. By a simple scaling argument we can reduce to the case of \(d = 2\), so radius \(\leq 1\). We set \(\mathcal{S}_n = \mathcal{S}_n(2)\).

Let \(\mathcal{S}^j\) denote the subset of \(Q\) consisting of elements of dimension exactly \(j\) and radius \(\leq 1\). Then \(\mathcal{S}_n\) is the disjoint union of the \(\mathcal{S}^j\) for \(j \leq n\). Thus we see that the crux of the matter is to show:

13.7 Proposition. For every integer \(n\) the set \(\mathcal{S}^n\) is totally bounded for \(\text{dist}_q\).
Proof. We fix \( n \) for the rest of the discussion. We proceed by first finding a kind of “standard position” for order-unit spaces of dimension \( n \). Fix a real vector space \( V \) of dimension \( n \), an inner product on it, and its corresponding Euclidean norm, \( \| \cdot \|_E \). According to a theorem of F. John (proposition 9.12 of [71]) for any other normed vector space, \((U, \| \cdot \|_U)\), of dimension \( n \) there is a linear operator, \( T \), from \( U \) onto \( V \) such that \( \| T \| \| T^{-1} \| \leq \sqrt{n} \). (I am indebted to Ed Effros for suggesting to me that this theorem of F. John might be useful for the present considerations.) Fix further a vector \( e \in V \) such that \( \| e \|_E = 1 \). Suppose now that \((U, e_U, \| \cdot \|_U)\) is an order-unit space of dimension \( n \). Let \( T : U \to V \) be as above. By multiplying \( T \) by a constant we can arrange that \( \| Te_U \| = 1 \). By then composing \( T \) with an orthogonal transformation we can arrange that \( Te_U = e \). Since it follows that \( T^{-1} e = e_U \), we see that \( \| T \| \geq 1 \) and \( \| T^{-1} \| \geq 1 \), so that \( \| T \| \leq \sqrt{n} \) and \( \| T^{-1} \| \leq \sqrt{n} \). We can now use \( T \) to transfer to \( V \) the norm of \( U \). Thus we see that every order-unit space of dimension \( n \) is (isometrically order) isomorphic to one coming from an order-unit norm, \( \| \cdot \| \), on \((V, e)\) such that

\[
\frac{1}{\sqrt{n}} \| v \|_E \leq \| v \| \leq \sqrt{n} \| v \|_E.
\]

It is convenient to relate this to our earlier notation by defining a new Euclidean norm \( \| \cdot \|_* \) by \( \| \cdot \|_* = \sqrt{n} \| \cdot \|_E \). We then summarize the above by:

**13.8 Lemma.** Let \( V \) be a vector space of dimension \( n \), equipped with a Euclidean norm \( \| \cdot \|_* \) and a distinguished vector \( e \) such that \( \| e \|_* = \sqrt{n} \). Then every order-unit space of dimension \( n \) is isomorphic to one coming from an order-unit norm, \( \| \cdot \| \), on \((V, e)\) such that for all \( v \in V \)

\[
n^{-1} \| v \|_* \leq \| v \| \leq \| v \|_*.
\]

Let \( O_n \) denote the set of order-unit norms on \((V, e)\) satisfying the above inequalities with respect to \( \| \cdot \|_* \). Let \( V' \) be the dual vector space to \( V \) with dual norm \( \| \cdot \|'_* \) and distinguished functional \( \eta \) from \( e \). Let \( O'_n \) denote the set of base-norms on \((V', \eta)\) which are the duals of the order-unit norms in \( O_n \). For \( \| \cdot \|' \in O'_n \) we will have

\[
n \| x \|'_* \geq \| x \|' \geq \| x \|'_*.
\]

for \( x \in V' \). Let \( B'_* \) denote the unit ball in \( V' \) for \( \| \cdot \|'_* \). View the elements of \( O'_n \) as functions on \( B'_* \). Because these functions are dominated by \( n \| \cdot \|_*' \), it is easily seen that \( O'_n \) is a bounded equicontinuous family of functions on \( B'_* \). Thus, by the Arzela–Ascoli theorem, \( O'_n \) is totally bounded for the supremum norm. From this we obtain:
13.9 LEMMA. For any $\delta > 0$ we can find a finite subset, $F_\delta$, of $\mathcal{O}_n'$ such that if $\| \cdot \|_1 \in \mathcal{O}_n'$ then there is a $\| \cdot \|_2$ in $F_\delta$ such that
\[ \| x \|_1' - \| x \|_2' \leq \delta \| x \|_2' \]
for all $x \in V'$.

We are then in position to apply Lemma 10.7, with $\delta = \varepsilon/4$. We obtain:

13.10 LEMMA. Let $\varepsilon > 0$ be given, with $\varepsilon < 2$. Then for any $\| \cdot \|_1 \in \mathcal{O}_n'$ there is a $\| \cdot \|_2 \in F_{\varepsilon/4}$ (chosen as in Lemma 13.9) such that
\[ \text{dist}^*_H(S_1, S_2) < \varepsilon. \]

We must now bring Lip-norms into the picture. Let $L_0$ be a Lip-norm on $(V,e)$. Again, “Lip-norm” here simply means that the null-space of $L$ is spanned by $e$. Let us consider two order-unit norms, $\| \cdot \|_1$ and $\| \cdot \|_2$ in $\mathcal{O}_n$. We will write $(V_1, L_1)$ when we think of $V$ as equipped with $\| \cdot \|_1$ and with $L_0$ as Lip-norm, and similarly for $(V_2, L_2)$. The corresponding state spaces will be denoted by $S_1$ and $S_2$, etc.

13.11 LEMMA. Let $\| \cdot \|_1$ and $\| \cdot \|_2 \in \mathcal{O}_n$, and let $L_0$ be a Lip-norm on $(V,e)$ giving $V_1$ and $V_2$ radius $\leq 1$. Suppose that
\[ \text{dist}^*_H(S_1, S_2) < \delta. \]

Then $\text{dist}_q(V_1, V_2) \leq n\delta$.

**Proof.** Let $\gamma > 0$ be given. By hypothesis we can find a finite subset, $F$, of $S_1 \times S_2$ such that if $(\mu, \nu) \in F$ then $\| \mu - \nu \|'_* < \delta$, the first coordinates of elements of $F$ are $\gamma$-dense in $S_1$ for $\rho_{L_1}$, and the second coordinates of elements of $F$ are $\gamma$-dense in $S_2$ for $\rho_{L_2}$. Define $N$ on $V \oplus V$ by
\[ N(u, v) = (n\delta)^{-1} \max \{ |\mu(u) - \nu(v)| : (\mu, \nu) \in F \}. \]

We show that $N$ is a bridge between $(V_1, L_1)$ and $(V_2, L_2)$. The first two conditions of Definition 5.1 clearly hold. For the third condition, let $u \in V_1$ be given. Set $v = u$ viewed as element of $V_2$. Clearly $L_2(v) = L_1(u)$. For $(\mu, \nu) \in F$ we have
\[ |\mu(u) - \nu(v)| = |(\mu - \nu)(u)| \leq \| \mu - \nu \|'_* \| u \|_1^* \leq \delta (n \| u \|_1^*) \leq \delta n L_1(u). \]
Thus $N(u, v) \leq L_1(u)$. We can do the same calculation with the roles of $V_1$ and $V_2$ reversed. Thus $N$ is a bridge.

Define $L$ on $V_1 \oplus V_2$ as before in terms of $N$ (briefly, by $L = (L_1 \vee L_2) \vee N$). Let $(\mu, \nu) \in F$. For every $(u, v) \in V_1 \oplus V_2$ such that $L(u, v) \leq 1$ we have $|\mu(u) - \nu(v)| \leq n\delta$, so that $\rho_L(\mu, \nu) \leq n\delta$. For any $\mu_0 \in S_1$
we can find \((\mu, \nu) \in F\) such that \(\rho_L(\mu_0, \mu) = \rho_L(\mu_0, \mu) < \gamma\). Thus \(\rho_L(\mu_0, \nu) < n\delta + \gamma\). In the same way we see that for every \(\nu_0 \in S_2\) there is a \(\mu \in S_1\) such that \(\rho_L(\mu, \nu_0) < n\delta + \gamma\). Thus \(\text{dist}^*_H(S_1, S_2) < n\delta + \gamma\). Since \(\gamma\) is arbitrary, it follows that \(\text{dist}_q(V_1, V_2) \leq n\delta\). \(\square\)

Let us now combine the above lemma with Lemma 13.10. Let \(\varepsilon > 0\) be given, with \(\varepsilon < 2\), and find \(F_{\varepsilon/4} \subset \mathcal{O}_n\) as in Lemma 13.9. Let \(\| \cdot \|_1 \in \mathcal{O}_n\), and let \(L_1\) be a Lip-norm on \(V_1\) of radius \(\leq 1\). By the definition of \(F_{\varepsilon/4}\) we can find \(\| \cdot \|_2 \in F_{\varepsilon/4}\) such that both

\[
\|x\|_1' - \|x\|_2' \leq (\varepsilon/4)\|x\|_2'
\]

for all \(x \in V'\), and \(\text{dist}^*_H(S_1, S_2) < \varepsilon\). Now let \(L_2\) be \(L_1\), but viewed as a Lip-norm on \(V_2\). The small difficulty with applying our previous lemma is that \(L_2\) may not have radius \(\leq 1\). However, for any \(x \in V'\) we do have

\[
L_2(x) = L_1(x) \leq \|x\|_1' \leq \|x\|_2' + (\varepsilon/4)\|x\|_2' \leq (1 + \varepsilon/4)\|x\|_2'.
\]

In other words, \(L_2\) has radius \(\leq 1 + (\varepsilon/4)\). Let \(r = 1 + (\varepsilon/4)\), and set \(M = rL_1\). Then \(M\) is a Lip-norm, and when viewed on either \(V_1\) or \(V_2\) it has radius \(\leq 1\). Thus we can apply the previous lemma to conclude that

\[
\text{dist}_q((V_1, M_1), (V_2, M_2)) \leq n\varepsilon.
\]

But changing from \(M\) to \(L\) clearly just multiplies distances by \(r\). We thus obtain

\[
\text{dist}_q((V_1, L_1), (V_2, L_2)) \leq n\varepsilon(1 + \varepsilon/4).
\]

By changing the meaning of \(\varepsilon\) and \(F_{\varepsilon}\) accordingly, we see that we obtain:

\[\textbf{13.12 LEMMA.}\] For any \(\varepsilon > 0\) there is a finite subset, \(F_{\varepsilon}\), of \(\mathcal{O}_n\) such that if \(\| \cdot \|_1 \in \mathcal{O}_n\) and if \(L_1\) is a Lip-norm on \(V_1\) of radius \(\leq 1\), then there is a \(\| \cdot \|_2 \in F_{\varepsilon}\) and a Lip-norm \(L_2\) on \(V_2\) such that

\[
\text{dist}_q(V_1, V_2) < \varepsilon
\]

(and radius\((L_2) \leq 1 + \varepsilon\)).

If we combine this with the earlier discussion, we have arrived at the point where we see that, given \(G\) and \(\varepsilon > 0\), there is a finite set \(F\) of order-unit spaces of dimension \(\leq G(\varepsilon)\) such that if \((A, L) \in S^1_1\) then there is a \(B \in F\) and a Lip-norm \(L_2\) on \(B\) such that \(\text{dist}_q(A, B) < \varepsilon\) and \(\text{diameter}(B, L) \leq 1 + \varepsilon\). To conclude the proof of Theorem 14.5 it thus suffices to prove:

\[\textbf{13.13 PROPOSITION.}\] Let \((A, e, \| \cdot \|)\) be a finite-dimensional order-unit space, and let \(r \in \mathbb{R}^+\). Let \(\mathcal{Q}(A, r)\) denote the set of all elements
of \( Q \) represented by \( A \) equipped with a Lip-norm of radius \( \leq r \). Then \( Q(A, r) \) is totally bounded for \( \text{dist}_q \).

**Proof.** By a simple scaling argument we can assume that \( r = 1 \). The Lip-norms on \( A \) of radius \( \leq 1 \) correspond by duality to the norms \( L' \) on \( A^\circ \) such that \( L' \leq \| \cdot \|' \). We will denote this collection of norms by \( \mathcal{N}_1(A) \). For \( L' \in \mathcal{N}_1(A) \) and \( \lambda_1, \lambda_2 \in A^\circ \) we have

\[
|L'(\lambda_1) - L'(\lambda_2)| \leq L'(|\lambda_1 - \lambda_2|) \leq \|\lambda_1 - \lambda_2\|'.
\]

Thus, when the elements of \( \mathcal{N}_1(A) \) are viewed as functions on the unit \( \|\cdot\|' \)-ball, \( \mathcal{N}_1(A) \) is a bounded equicontinuous family of functions. Because the unit ball is compact since \( A \) is of finite dimension, we can apply the Arzela–Ascoli theorem to conclude that \( \mathcal{N}_1(A) \) is totally bounded. (We remark that usually \( \mathcal{N}_1(A) \) is not closed, reflecting the fact that elements of \( Q(A, 1) \) can converge to quantum compact metric spaces of strictly lower dimension.) Thus, given \( \varepsilon > 0 \), we can find a finite subset, \( F_\varepsilon \), of \( \mathcal{N}_1(A) \) such that if \( L' \in \mathcal{N}_1(A) \) then there is an \( L'_1 \in F_\varepsilon \) such that

\[
|L'(\lambda) - L'_1(\lambda)| \leq \varepsilon \|\lambda\|'
\]

for all \( \lambda \in A^\circ \). We now see that we need the following quantum analogue of corollary 6.24 of [9]. For this analogue we do not need our spaces to be finite-dimensional.

**13.14 Proposition.** Let \( (A, e, \| \cdot \|) \) be an order-unit space, and let \( L_1 \) and \( L_2 \) be two Lip-norms on \( A \). If there is a \( \delta > 0 \) such that

\[
|L'_1(\lambda) - L'_2(\lambda)| \leq \delta \|\lambda\|'
\]

for all \( \lambda \in A^\circ \), then

\[
\text{dist}_q((A, L_1), (A, L_2)) \leq \delta.
\]

**Proof.** We reduce first to the finite-dimensional situation. Let \( \varepsilon \geq 0 \) be given. We argue along the lines of the proof of Theorem 13.1. We can find a finite subset, \( F \), of \( S(A) \) which is \( \varepsilon \)-dense in \( S(A) \) for both \( \rho_{L_1} \) and \( \rho_{L_2} \). We let \( K = \text{co}(F) \), and we let \( B \) consist of the restrictions to \( K \) of the elements of \( A \). Thus \( B \) is finite-dimensional. We let \( L^1_B \) and \( L^2_B \) be the quotient Lip-norms on \( B \) from \( L_1 \) and \( L_2 \). Then

\[
\text{dist}_q((A, L_j), (B, L^j_B)) \leq \varepsilon
\]

for \( j = 1, 2 \). Since \( \varepsilon \) is arbitrary, it follows from the triangle inequality that it suffices to show that \( \text{dist}_q((B, L^1_B), (B, L^2_B)) \leq \delta \). Let \( \pi \) be the projection of \( A \) onto \( B \). Then from Proposition 3.1 we know that \( \pi' \) is
an isometry for both \((L^1_B)', L^1_1\) and \((L^2_B)', L^2_2\). Thus for any \(\lambda \in B^o\) we have
\[
|((L^1_B)'(\lambda) - (L^2_B)'(\lambda))| = |L^1_1(\pi'(\lambda)) - L^2_2(\pi'(\lambda))| \leq \delta_{\|\pi'(\lambda)\|'} \leq \delta_{\|\lambda\|'}.
\]
Thus the hypothesis of our proposition is satisfied, and we see that we have reduced matters to proving the proposition for \(A\) finite-dimensional.

We now assume that \(A\) is finite-dimensional, and we attempt to define a bridge, \(N\), on \(A \oplus A\) by
\[
N(a, b) = \delta^{-1}_{\|a - b\|}.
\]
Accordingly, let \(L\) be defined as earlier by
\[
L(a, b) = L_1(a) \lor L_2(b) \lor N(a, b).
\]
Because our hypotheses are in terms of the dual seminorms \(L'_j\), we must examine \(L'\). Now \(L'\) should be defined on \((A \oplus A)^o\). Notice that \(A^o \oplus A^o\) is of codimension 1 in \((A \oplus A)^o\), and that for any \(\xi \in A'\) we have \((\xi, -\xi) \in (A \oplus A)^o\). Thus any element, \((\xi_1, \xi_2)\), of \((A \oplus A)^o\) can be expressed (in many ways) as \((\lambda_1, \lambda_2) + (\xi, -\xi)\) for \(\lambda_1, \lambda_2 \in A^o\) and \(\xi \in A'\). We claim that
\[
L'(\xi_1, \xi_2) = \inf \{L'_1(\lambda_1) + L'_2(\lambda_2) + \delta_{\|\xi\|'}\},
\]
where the inf is taken over all such expressions of \((\xi_1, \xi_2)\). To see this, consider first the seminorm defined on \((A \oplus A) \oplus (A \oplus A)\) by
\[
((L_1 \lor L_2) \lor N)(a, b, c, d) = L_1(a) \lor L_2(b) \lor N(c, d).
\]
The “dual seminorm” to \(L_1 \lor L_2\) on \(A \oplus A\) has value \(+\infty\) off of the annihilator of the null-space of \(L_1 \lor L_2\), and that null-space is spanned by \((e, 0)\) and \((0, e)\). Thus \((L_1 \lor L_2)'\) is finite exactly on \(A^o \oplus A^o\), and there it is \(L'_1 + L'_2\). The “dual seminorm” of \(N\) has value \(+\infty\) off of the annihilator of the null-space of \(N\), and that null-space is \(\{(a, a) : a \in A\}\). The annihilator is \(\{\xi, -\xi : \xi \in A'\}\), and on this annihilator we have \(N'(\xi, -\xi) = \delta_{\|\xi\|'}\). Thus the dual of \((L_1 \lor L_2) \lor N\) is defined on \(\{(\lambda_1, \lambda_2, \xi, -\xi)\}\) and
\[
((L_1 \lor L_1) \lor N)'(\lambda_1, \lambda_2, \xi, -\xi) = L'_1(\lambda_1) + L'_2(\lambda_2) + \delta_{\|\xi\|'}.
\]
But \(A \oplus A\) can be viewed as the subspace of \((a, b, a, b)\)'s in \(A^4\), and our \(L\) defined above is just the restriction of \((L_1 \lor L_1) \lor N\) to this subspace. Because we are in the finite-dimensional situation so that \(L\) is continuous, it follows that \(L'\) is the quotient of the above seminorm \(((L_1 \lor L_2) \lor N)'\). (I thank Hanfeng Li for pointing out to me that this is difficult to justify in the infinite-dimensional case.) But this gives exactly the formula for \(L'\) given above.
We must check that $L$ induces $L_1$ and $L_2$, and we wish to do this by applying Corollary 3.10. Because $A$ is finite-dimensional, $L$, $L_1$ and $L_2$ are all closed. Then according to Corollary 3.10 it suffices to show that the restriction of $L'$ to $A_1^\circ \subset (A_1 \oplus A_2)^\circ$ coincides with $L_1'$, and similarly for $L_2'$. Let $\lambda \in A^\circ$, and suppose that we have an expression for $(\lambda, 0)$ as

$$(\lambda, 0) = (\lambda_1, \lambda_2) + (\xi, -\xi)$$

as above. Then $\xi = \lambda_2$ and $\lambda = \lambda_1 + \lambda_2$, and so $\|\xi\|'$ becomes $\|\lambda_2\|'$. But by hypothesis

$|L_1'(\lambda_2) - L_2'(\lambda_2)| \leq \delta \|\lambda_2\|'$. Thus

$L_1'(\lambda) = L_1'(\lambda_1 + \lambda_2) \leq L_1'(\lambda_1) + L_1'(\lambda_2) \leq L_1'(\lambda_1) + L_2'(\lambda_2) + \delta\|\xi\|'$. and so $L_1'(\lambda) \leq L'(\lambda, 0)$. But we can always take the decomposition with $\lambda_1 = \lambda$ and $\lambda_2 = 0 = \xi$. It follows that $L_1'(\lambda) = L'(\lambda, 0)$. In the same way we see that $L_2'(\lambda) = L'(0, \lambda)$. It now follows that $N$ is a bridge and that $L$ is a Lip-norm.

Suppose now that $\mu \in S(A)$. Then

$$\rho_L((\mu, 0) - (0, \mu)) = L'(\mu, -\mu).$$

In the formula above for $L'$ we can take $\lambda_1 = 0 = \lambda_2$ and $\xi = \mu$. We thus find that

$L'(\mu, -\mu) \leq \delta \|\mu\|' = \delta$.

From this it is clear that

$$\text{dist}_{H'}^\rho(S_1, S_2) \leq \delta,$$

so that $\text{dist}_q((A, L_1), (A, L_2)) \leq \delta$. □

When we combine this with the earlier considerations, we see that we have completed the proof of Theorem 13.5.

We can use some of the facts accumulated above to prove:

13.15 THEOREM. The space $Q$ of isometry equivalence classes of compact quantum metric spaces, with the metric $\text{dist}_q$, is separable.

Proof. From Theorem 13.1 we see that the subset of $Q$ consisting of finite dimensional spaces is dense. It thus suffices to show that for each integer $n$ the set of spaces of dimension $n$ is separable. But then it suffices to show that for each integer $D$ the set of spaces of dimension $n$ and diameter $\leq D$ is separable. But this follows from Proposition 13.7 by a scaling argument. □
As with Corollaries 7.9 and 7.10, our notation for the next corollary will not distinguish between $C(X)$ and the domain of a Lip-norm on it.

13.16 THEOREM. The set of (equivalence classes of) ordinary compact metric spaces is a closed subset of $Q$. That is, if $\{(X_n, \rho_n)\}$ is a sequence of ordinary compact metric spaces, with corresponding Lip-norms $L_{\rho_n}$, and if $(A, L)$ is a compact quantum metric space to which the sequence $(C(X_n), L_{\rho_n})$ converges for $\text{dist}_q$, then there is an ordinary compact space $Y$ such that the completion of $A$ is order-isomorphic to $C(Y)$

**Proof.** From Lemma 13.6 we see that there is a constant $D$ such that $\text{diam}(C(X_n), L_{\rho_n}) \leq D$ for all $n$. Since $X_n$ is identified with the extreme points of $S(C(X_n))$, it follows that $\text{diam}(X_n, \rho_n) \leq D$ for all $n$. From Theorem 13.5 there is a function $G$ such that $\text{Scv}_{C(X_n)}(\varepsilon) \leq G(\varepsilon)$ for each $n$. A simple argument shows that when we view $X_n$ as the subset of extreme points of $S(C(X_n))$, we have $\text{Cov}_{\rho_n}(\varepsilon) \leq \text{Scv}_{C(X_n)}(\varepsilon/2)$. From Gromov’s compactness and completeness theorems it follows that a subsequence of the $C(X_n)$’s converges for $\text{dist}_{GH}$ to some ordinary compact metric space, say $(Y, \rho)$. From Proposition 4.7 it follows that as quantum metric spaces this subsequence converges to $(Y, \rho)$ also for $\text{dist}_q$. Thus

$$\text{dist}_q((A, L_A), ((C(Y), L_\rho)) = 0.$$  

Then the completion of $A$ is isomorphic to $C(Y)$ as order-unit spaces, and under this isomorphism we have $L_A = L_\rho$ by Theorem 7.7. □

Finally, we give more quantitative relations between the three measures of growth defined in Definition 13.4. We begin with:

13.17 PROPOSITION. Let $(A, L)$ be a compact quantum metric space of dimension $n$, and let $D = \text{diam}(A, L)$. Then for any $\varepsilon > 0$, there is a subset of $S(A)$ which is $\varepsilon$-dense in $S(A)$ for $\rho_L$ and contains no more than $((D/\varepsilon) + 1)^{n-1}$ points.

**Proof.** Assume first that $D = 1$, so that $L' \leq (1/2)\|\cdot\|'$. Fix $\mu_0 \in S(A)$ and set $K = S(A) - \mu_0$, a subset of $A^\circ$. Let $B$ denote the open unit $L'$-ball about $0$ in $A^\circ$, and $\bar{B}$ its closure. Thus $K \subseteq \bar{B}$ because $D = 1$. We now use a standard argument. (See the proof of lemma 4.10 of [57]. I am indebted to Bernd Sturmfels for steering me toward this argument.) Let $\{\lambda_1, \ldots, \lambda_N\}$ be a subset of $K$ such that $\|\lambda_j - \lambda_k\|' \geq 2\varepsilon$ for all $j \neq k$. Then the open balls $\lambda_j + \varepsilon B$ are disjoint from each other, and are all contained in $(1 + \varepsilon)B$. Since $A^\circ$ is finite-dimensional, there is
a translation-invariant volume on $A^\circ$. When we apply it to the above situation, we see that
\[ N \operatorname{vol}(\varepsilon B) \leq \operatorname{vol}((1 + \varepsilon)B). \]
Since $A^\circ$ has dimension $n - 1$, it follows that
\[ N \leq ((1/\varepsilon) + 1)^{n-1}. \]
Now let $F$ be a maximal subset $K$ with respect to the above property that $\|\lambda_j - \lambda_k\|' \leq 2\varepsilon$. Then $F$ is $2\varepsilon$-dense in $K$ for $\| \cdot \|'$, and so $\varepsilon$-dense in $K$ for $L'$. It follows that $F + \mu_0$ is $\varepsilon$-dense in $S(A)$ for $\rho_L$. Thus we obtain the desired conclusion when $D = 1$.

If $D \neq 1$, then $D^{-1}L'$ has diameter 1, and so there is a subset $F$ of $S(A)$ which is $(\varepsilon/D)$-dense for $D^{-1}L'$, and has no more than $((D/\varepsilon) - 1)^{n-1}$ elements. Then $DF$ is $\varepsilon$-dense for $L'$.

The important aspect for us of the bound given in Proposition 13.16 is that it depends only on $n$ and $D$, and not on other features of $A$ or $L$.

Suppose now that $(A, L_A)$ is a general compact quantum metric space, and let $\varepsilon > 0$ be given. Let $D = \operatorname{diam}(A, L_A)$, and let $n = \operatorname{Fin}_L(\varepsilon)$. Thus we can find a compact quantum metric space $(B, L_B)$ such that $\operatorname{dist}(A, B) < \varepsilon$ and $\operatorname{dim}(B) = n$. Thus there is a Lip-norm $L$ on $A \oplus B$ inducing $L_A$ and $L_B$ such that $\operatorname{dist}(S(A), S(B)) < \varepsilon$. It is easily seen that $\operatorname{diam}(B, L_B) \leq D + 2\varepsilon$. Thus by Proposition 13.16 we can find a finite subset, $F$, of $S(B)$ which is $\varepsilon$-dense in $S(B)$, and for which
\[ |F| \leq (((D + 2\varepsilon)/\varepsilon) + 1)^{n-1}, \]
where $| \cdot |$ denotes “number of elements in”. For each point in $F$ choose a point in $S(A)$ within distance $\varepsilon$ of it, and let $F_A$ denote the set of these points. It is easily seen that $F_A$ is $3\varepsilon$-dense in $S(A)$. Thus $\operatorname{Scv}_{L_A}(3\varepsilon) \leq |F_A| \leq |F|$. Upon simplifying the earlier bound for $|F|$, we obtain:

13.18 PROPOSITION. Let $(A, L)$ be a compact quantum metric space of diameter $D$. Then for every $\varepsilon > 0$ we have
\[ \operatorname{Scv}_L(3\varepsilon) \leq ((D/\varepsilon) + 5)^{\operatorname{Fin}_L(\varepsilon)}-1. \]

From this proposition we see that if $S$ is a subset of $Q$ for which there is a constant $D$ and function $G$ such that every element of $S$ is of diameter $\leq D$, and that $\operatorname{Fin}_L(\varepsilon) \leq G(\varepsilon)$ for all $(A, L) \in S$, and $\varepsilon > 0$, then there is a function $H$ from $\mathbb{R}^+$ to $\mathbb{N}^+$ such that $\operatorname{Scv}_L(\varepsilon) \leq H(\varepsilon)$ for
all \((A, L) \in \mathcal{S}\) and all \(\varepsilon > 0\). This is the equivalence for the purposes of the compactness theorem which we had in mind.

**Appendix 1. An example where \(\text{dist}_{GH} > \text{dist}_q\)**

by Hanfeng Li

Let \(Y = \{y_1, y_2, y_3\}\), with metric \(\rho_Y(y_1, y_2) = 1 = \rho_Y(y_2, y_3)\) and \(\rho_Y(y_1, y_3) = 2\). Let \(Z = \{z_1, z_2\}\) with metric \(\rho_Z(z_1, z_2) = 3\). Let \(C(Y)\) and \(C(Z)\) be the algebras of real-valued functions on \(Y\) and \(Z\), with the Lip-norms \(L_{\rho_Y}\) and \(L_{\rho_Z}\). We will show that \(\text{dist}_{GH}(Y, Z) = 1\) but that \(\text{dist}_q((C(Y), L_{\rho_Y}), (C(Z), L_{\rho_Z})) = 1/2\). Hence this is an example for which \(\text{dist}_{GH} > \text{dist}_q\).

We first show that \(\text{dist}_{GH}(Y, Z) = 1\). Let \(\rho\) be a metric on the union of \(Y\) and \(Z\) which restricts to the given metrics. Consider the distance from \(y_2\) to \(Z\). Suppose that \(\rho(y_2, z_1) \leq 1\). Since for all \(j\) we have

\[3 = \rho(z_1, z_2) \leq \rho(z_1, y_2) + \rho(y_2, y_j) + \rho(y_j, z_2),\]

we must then have \(\rho(y_j, z_2) \geq 1\), that is, \(\rho(z_2, Y) \geq 1\). In the same way, if \(\rho(y_2, z_2) \leq 1\), then \(\rho(z_1, Y) \geq 1\). Thus \(\text{dist}_{GH}(Z, Y) \geq 1\). But we can isometrically embed \(Y\) and \(Z\) in \(\mathbb{R}\) as \(\{0, 1, 2\}\) and \(\{0, 3\}\), from which we see that \(\text{dist}_{GH}(Y, Z) = 1\).

To show that \(\text{dist}_q((C(Y), L_{\rho_Y}), (C(Z), L_{\rho_Z})) = 1/2\), we need some preparation. Notice first that Proposition 5.7 says in effect that for compact quantum metric spaces \((B_i, L_i)\), if their state spaces are affinely isometrically embedded into the state-space \(S(A)\) of some other compact quantum metric space \((A, L)\), then

\[\text{dist}_q(B_1, B_2) \leq \text{dist}_{GH}^0(S(B_1), S(B_2)).\]

This provides a powerful way of getting upper bounds for quantum Gromov-Hausdorff distance.

Here we will use the special case in which we are given \((A, L)\) and we just enlarge the state space within \(A'\). Let \(H(A)\) be the hyperplane \(\{\eta \in A' : \eta(\varepsilon_A) = 1\}\). Then \(H(A)\) is a convex set containing \(S(A)\). It is clear that \(H(A) - H(A) \subseteq A'^0\). Thus the usual formula \(\rho_L(\mu, \nu) = L'(\mu - \nu)\) actually defines \(\rho_L\) as a metric on all of \(H(A)\). It is easy to see that the proof of theorem 1.8 of [61] actually shows that the \(\rho_L\)-topology coincides with the \(w^*\)-topology on any \(\| \cdot \|'\)-bounded subset of \(H(A)\).

Let \(K\) be a closed convex \(\| \cdot \|'\)-bounded (so \(w^*\)-compact) subset of \(H(A)\) which contains \(S(A)\), and let \(\pi\) denote the evident restriction map from \(A\) into \(Af(K)\), much as in Proposition 3.5. Let \(B = \pi(A)\). Evidently \(B\) is an order-unit space. From corollary I.1.5 of [1] it is
easily seen that $B$ is dense in $Af(K)$ for the supremum norm. Note that $\pi$ is usually not a morphism, but $\pi$ is a bijection from $A$ onto $B$ because $K \supseteq S(A)$; and it is $\pi^{-1}$ which is a morphism. In particular, $B$ is just the vector space $A$ with the same distinguished element, but different order structure and norm. However, we will use the same $L$ on both $A$ and $B$.

Let $\| \cdot \|_B$ denote the supremum norm on $Af(K)$, and so on $B$. Then $\| \cdot \| \leq \| \cdot \|_B$ because $S(A) \subseteq K$. Since $K$ is $\| \cdot \|^2$-bounded, there is an $s \in \mathbb{R}$ such that $\| \cdot \| \leq s \| \cdot \|$. In particular, every state on $B$ will be continuous for $\| \cdot \|$, and so will be in $A'$. A simple argument quite similar to that in the proof of Proposition 3.5 now shows that $S(B)$ is naturally identified with $K$. Since $K \supseteq S(A)$, we are then in the situation of Proposition 3.5 but with the roles of $A$ and $B$ reversed.

For any $\mu, \nu \in K$ and any $a \in A$ we have

$$|\mu(\pi(a)) - \nu(\pi(a))| = |(\mu - \nu)(a)| \leq L'(\mu - \nu)L(a) = \rho_L(\mu, \nu)L(a).$$

Thus each element of $B$ is Lipschitz for $\rho_L$, and $L_{\rho_L}(a) = L(a)$ as long as $L$ is lower semi-continuous.

Suppose now that $K_1$ is another compact convex subset of $K$, and let $(C, L_C)$ be the corresponding quotient of $(B, L)$. Then from Proposition 5.7 we know that

$$\text{dist}_q(A, C) \leq \text{dist}_{H'}^\rho(S(A), K_1).$$

We apply this fact to the example introduced at the beginning of this appendix. Let $A = C(Y)$. Let $w_1$ and $w_2$ be the points of $H(A)$ defined by

$$w_1 = y_1 + (y_2 - y_3)/2, \quad w_2 = y_3 + (y_2 - y_1)/2,$$

for the evident meaning of the notation. Let $K \subseteq A^\circ$ be defined by

$$K = \text{co}\{y_1, y_2, y_3, w_1, w_2\}.$$ 

Thus $K \supseteq S(A)$, and the above discussion applies. We let $B$ denote $A$ but with the order and norm coming from viewing its elements as affine functions on $K$. Thus for $f \in B = C(Y)$ we have $f \geq 0$ exactly if $f \geq 0$ on $Y$ and $f(w_j) \geq 0$ for $j = 1, 2$. We equip $B$ with the Lip-norm $L$. Thus $\rho_L$ is defined on $K$ by $L'$, and its restriction to $S(A)$ is the original metric coming from that on $Y$.

Now it is easy to see that for $f = \{f_1, f_2, f_3\} \in C(Y)$ we have

$$L(f_1, f_2, f_3) = |f_1 - f_2| \vee |f_2 - f_3|.$$ 

It follows that for $\lambda = \{\lambda_1, \lambda_2, \lambda_3\} \in A^\circ$ we have

$$L'(\lambda_1, \lambda_2, \lambda_3) = |\lambda_1| + |\lambda_3|.$$
Then we calculate that
\[ \rho_L(w_1, w_2) = L'(3/2)(y_1 - y_3) = 3. \]
Consequently we can identify \( Z \) isometrically with a subset of \( K \) by sending \( z_j \) to \( w_j \) for \( j = 1, 2 \). We let \( K_1 \) be the closed line-segment in \( K \) joining \( w_1 \) and \( w_2 \). Then \( C(Z) \) becomes identified with the space \( C \) of affine functions on \( K_1 \), and \( S(C) = K_1 \). Because \( C \) is 2-dimensional, there is only one possible Lip-norm on \( C \) giving distance 3 between \( w_1 \) and \( w_2 \), and so it must be the quotient of \( L \) on \( C \). We are now in position to apply Proposition 5.7. From it we conclude that
\[ \text{dist}_{q}(A, C) \leq \dist_{\rho_L}^{\rho}(S(A), K_1). \]
Now
\[ \rho_L(y_1, w_1) = L'((y_3 - y_2)/2) = 1/2, \]
and similarly \( \rho_L(y_3, w_2) = 1/2 \). Furthermore, \( (w_1 + w_2)/2 \in K_1 \), and
\[ \rho_L(y_2, (w_1 + w_2)/2) = L'((-y_1 + 2y_2 - y_3)/4) = 1/2. \]
Because \( \rho_L \) is convex, it follows that \( \dist_{\rho_L}^{\rho}(S(A), K_1) \leq 1/2 \). Consequently \( \dist_q(C(Y), C(Z)) \geq 1/2 \), so that \( \dist_q(C(Y), C(Z)) = 1/2 \) as desired.

APPENDIX 2. DIRAC OPERATORS ARE UNIVERSAL

In this brief section we answer question 11.1 of [62]. Namely, we show that, in a suitable sense, every lower semi-continuous Lip-norm on an order-unit space can be obtained from a “Dirac” operator. This is seen as follows.

Let \( A \) be an order-unit space, and let \( L \) be a lower semi-continuous Lip-norm on \( A \). According to theorem 4.2 of [62], because \( L \) is lower semi-continuous, we can recover \( L \) from \( \rho_A \) by the formula
\[ L(a) = \sup\{\|\mu(a) - \nu(a)\|/\rho_A(\mu, \nu) : \mu \neq \nu\}. \]
(Notice that the right-hand side is always lower semi-continuous.)

Let \( C(S(A)) \) denote the algebra of complex-valued continuous functions on the compact space \( S(A) \), and let \( \mathcal{L} \) denote the dense \(*\)-subalgebra of \( C(S(A)) \) consisting of the Lipschitz functions for \( \rho_L \). Let \( L_0 \) denote the ordinary Lipschitz seminorm on \( \mathcal{L} \) for \( \rho_L \). We view \( A \) as a real subspace of \( C(S(A)) \) in the usual way. Then the formula above for \( L \) shows that \( L \) is the restriction of \( L_0 \) to \( A \).

Now \( S(A) \), as a compact metric space, is separable, and so we can find (in many ways) a positive finite Radon measure, \( m \), on \( S(A) \) whose
support is all of $S(A)$. We now use the construction described late in section 11 of [62], or on page 274 of [74]. We let $\Delta$ denote the diagonal of $S(A) \times S(A)$ and set $Y = (S(A) \times S(A)) \setminus \Delta$. We restrict $m \times m$ to $Y$, and let $\mathcal{H} = L^2(Y, m \times m)$. We represent $C(S(A))$ on $\mathcal{H}$ by

$$(f\xi)(\mu, \nu) = f(\mu)\xi(\mu, \nu)$$

for $f \in C(S(A))$ and $\xi \in \mathcal{H}$. We let $D$ denote the (usually unbounded) operator on $\mathcal{H}$ defined by

$$(D\xi)(\mu, \nu) = \xi(\nu, \mu)/\rho L(\mu, \nu),$$

with domain those $\xi$’s for which $D\xi \in \mathcal{H}$. It is easily seen that $D$ is self-adjoint. Furthermore, as seen by simple calculations (given in the references above), for any $f \in \mathcal{L}$, viewed as an operator on $\mathcal{H}$, we find that $[D, f]$ is a bounded operator, and that $L_0(f) = \|[D, f]\|$.

In particular, we see that $A$ is represented isomorphically as an order-unit space of self-adjoint operators on $\mathcal{H}$. Since $A \subseteq \mathcal{L}$, each $[D, a]$ is a bounded operator on $\mathcal{H}$, and $L(a) = L_0(a) = \|[D, a]\|$. In this way $L$ is obtained from the “Dirac” operator $D$.

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