1. Introduction

Of the geometric figures in a given family satisfying real conditions, some figures are real while the rest occur in complex conjugate pairs, and the distribution of the two types depends subtly upon the configuration of the conditions. Despite this difficulty, applications ([7],[28],[32]) may demand real solutions. Fulton [11] asked how many solutions of an enumerative problem can be real, and we consider a special case of his question: Given a problem of enumerative geometry, are there real conditions such that every figure satisfying them is real? Such an enumerative problem is fully real.

Bézout's Theorem, or rather the problem of intersecting hypersurfaces in $\mathbb{P}^n$, is fully real. This is readily seen for $\mathbb{P}^2$, and the argument generalizes to $\mathbb{P}^n$. Suppose $X_0$ consists of $d$ real lines, $Y_0$ of $e$ real lines, and $X_0$ meets $Y_0$ transversally in (necessarily) $d \cdot e$ real points. Let $X$ and $Y$ be defined by suitably small generic real deformations of the forms defining $X_0$ and $Y_0$. Then $X$ and $Y$ are smooth real plane curves of degrees $d$ and $e$ meeting transversally in $d \cdot e$ real points.

This argument used a degenerate case free of multiplicities; $X_0$ and $Y_0$ are reduced and meet transversally. While it is typical to introduce multiplicities (for example, in the proof of Bézout’s Theorem in [29]) to establish enumerative formulas, multiplicities may lead to complex conjugate pairs of solutions, complicating the search for real solutions.

All Schubert-type enumerative problems involving lines in $\mathbb{P}^n$ are fully real [37]. This follows from the existence of (multiplicity-free) deformations of generically transverse intersections of Schubert varieties into sums of Schubert varieties. Refining this method of multiplicity-free deformations [39] yields techniques for showing other enumerative problems are fully real. Fulton, and more recently, Ronga, Tognoli, and Vust [31], have shown the problem of 3264 conics tangent to five general plane conics is fully real. Their analysis utilizes degenerate conditions having multiplicities.

Enumerative problems that we know are not fully real share a common flaw: they do not involve intersecting general subvarieties. For example, Klein [20] showed...
that at most \( n(n - 2) \) of the \( 3n(n - 2) \) flexes on a real plane curve of degree \( n \) can be real. These flexes are the intersection of the curve with its Hessian determinant, not with a general curve of degree \( 3(n - 2) \). Also, Khovanskii [17] showed that if hypersurfaces in a complex torus are defined by polynomials with few monomials, then the real points of intersection are at most a fraction of all points of intersection. However, these are not generic hypersurfaces with given Newton polytope.

Little is known about fully real enumerative problems. For instance, we are unaware of a good theoretical framework for studying fully real enumerative problems. Also, it is not known how common it is for an enumerative problem to be fully real. Here are some examples of enumerative problems worth considering:

1. Is the Kouchnirenko-Bernstein Theorem (8, 24) for hypersurfaces in a torus fully real? That is, given lattice polytopes \( \Delta_1, \ldots, \Delta_n \) in \( \mathbb{Z}^n \) are there real polynomials \( f_1, \ldots, f_n \) where \( \Delta_i \) is the Newton polytope of \( f_i \) and all solutions to the system \( f_1 = \cdots = f_n = 0 \) in \( (\mathbb{C}^*)^n \) are real?

2. Generalize the results of [17] and [39]: Are other (all?) Schubert-type enumerative problems on flag varieties fully real?

3. All known examples involve spherical varieties (8, 21, 26). Which enumerative problems on other spherical varieties are fully real?

4. In [39] all problems of enumerating lines incident upon subvarieties of fixed dimension and degree in \( \mathbb{P}^n \) are shown to be fully real. What is the situation for rational curves of higher degree? (Degree 0 is Bézout’s Theorem.) For example, for which positive integers \( d \) do there exist \( 3d - 1 \) real points in \( \mathbb{P}^2 \) such that the Kontsevich number \( N_d \) of degree \( d \) rational curves passing through these points (23, 33) are all real? For an introduction to these questions of quantum cohomology, see the paper by Fulton and Pandharipande [12] in this volume.

This technique of multiplicity-free deformations may have applications beyond showing the existence of real solutions. When the deformations are explicitly described (which is the case in most known results), it may be possible to obtain explicit solutions to the enumerative problem using continuation methods of numerical analysis [1] to follow real points in the degenerate configuration backwards along the deformation. Algorithms to accomplish this have recently been developed in the case of intersecting hypersurfaces in a complex torus [9, 16].

This note is organized as follows: In §2 we discuss some examples of fully real enumerative problems for which multiplicity-free deformations play a central role. This technique is illustrated in §3, where we show that there are nine real Veronese surfaces in \( \mathbb{P}^5 \) such that the 11010048 planes meeting all nine are real. We conclude with a discussion of the work of Fulton and of Ronga, Tognoli, and Vust [31] on the problem of conics tangent to five conics and show that the multiplicities they introduce are unavoidable.

2. Effective Rational Equivalence

A common feature of many fully real enumerative problems is multiplicity-free deformations of intersection cycles. Effective rational equivalence is a precise formulation of this for Grassmannians and flag varieties.
2.1. Real effective rational equivalence. Varieties will be quasi-projective, reduced, complex and defined over the real numbers, \( \mathbb{R} \). Let \( X \) be a Grassmannian or flag variety, \( G \) a linear algebraic group which acts transitively on \( X \), and \( B \) a Borel subgroup of \( G \). The letters \( U \) and \( V \) denote smooth rational varieties. Let the real points \( Y(\mathbb{R}) \) of a variety \( Y \) be equipped with the classical topology.

A subvariety \( \Xi \subset U \times X \) (or \( \Xi \to U \)) with generically reduced equidimensional fibres over \( U \) is a family of (multiplicity-free) cycles on \( X \) over \( U \). We assume all families are \( G \)-stable; if \( Y \) is a fibre of \( \Xi \) over \( U \), then so are all translates of \( Y \).

Associating a point \( u \) of \( U \) to the fundamental cycle of the fibre \( \Xi_u \) determines a morphism \( \phi : U \to \text{Chow} X \). Here, \( \text{Chow} X \) is the Chow variety of \( X \) parameterizing cycles of the same dimension and degree as \( \Xi_u \) ([34], §I.9). A priori, \( \phi \) is only a function. However, if \( C \subset U \) is a smooth curve, then \( \Xi|_C \) is flat and the canonical map of the Hilbert scheme to the Chow variety ([30], §5.4) shows \( \phi|_C \) is a morphism. By Hartogs’ Theorem on separate analyticity, \( \phi \) is in fact a morphism. In fact, if \( U \) is normal, then \( \phi \) is a morphism ([22, §1] or [14, §3]). For a discussion of Chow varieties in the analytic category (which suffices for our purposes), see [3].

Any cycle \( Y \) on \( X \) is rationally equivalent to an integral linear combination of Schubert classes. As Hirschowitz [14] observed, this rational equivalence occurs within the closure of \( B \cdot Y \) in \( \text{Chow} X \) since \( B \)-stable cycles of \( X \) (\( B \)-fixed points in \( B \cdot Y \)) are integral linear combinations of Schubert varieties. If any coefficients in this linear combination exceed 1, this stable cycle has multiplicities.

A family \( \Xi \to U \) of multiplicity-free cycles on \( X \) has effective rational equivalence with witness \( Z \) if there is a cycle \( Z \in \overline{\phi(U)} \) which is a sum of distinct Schubert varieties, and hence multiplicity-free. An effective rational equivalence is real if \( Z \in \overline{\phi(U(\mathbb{R}))} \) and each component of \( Z \) is a Schubert variety defined by a real flag.

Suppose \( \Xi_1 \to U_1, \ldots, \Xi_b \to U_b \) are \( G \)-stable families of multiplicity-free cycles on \( X \). By Kleiman’s Transversality Theorem [38], there is a nonempty open set \( U \subset \prod_{i=1}^{b} U_i \) consisting of \( b \)-tuples \((u_1, \ldots, u_b)\) such that the fibres \((\Xi_i)_{u_1}, \ldots, (\Xi_b)_{u_b}\) meet generically transversally. Let \( \Xi \subset U \times X \) be the resulting family of intersection cycles and call \( \Xi \to U \) the intersection problem given by \( \Xi_1, \ldots, \Xi_b \).

Theorem 1. Any intersection problem given by families of Schubert varieties in the Grassmannian of lines in projective space has real effective rational equivalence.

We present a synopsis of the proof in [17]: Let \( X \) be the Grassmannian of lines in \( \mathbb{P}^n \) and suppose \( \Xi \to U \) is an intersection problem given by families of Schubert varieties. A sequence \( \Psi_0 \to V_0, \ldots, \Psi_c \to V_c \) of families of multiplicity-free cycles on \( X \) is constructed with each \( V_i \) rational, where \( \Psi_0 \to V_0 \) is the family \( \Xi \to U \), \( V_c \) is a point, and \( \Psi_c \) a union of distinct real Schubert varieties. For each \( i = 0, \ldots, c \), let \( \mathcal{G}_i \subset \text{Chow} X \) be \( \phi(V_i(\mathbb{R})) \), the set of fibres of the family \( \Psi_i \) over \( V_i(\mathbb{R}) \).

Then \( \mathcal{G}_i \subset \overline{\mathcal{G}_{i-1}} \): For any \( v \in V_i(\mathbb{R}) \) a family \( \Gamma \to C \) of cycles is constructed with \( C \) a smooth rational curve, the cycle \((\Psi_i)_v \) a fibre over \( C(\mathbb{R}) \), and all other fibres of \( \Gamma \) are fibres of \( \Psi_{i-1} \). This family induces a morphism \( \phi : C \to \text{Chow} X \), which shows \((\Psi_i)_v \in \overline{\mathcal{G}_{i-1}} \) since \( \phi(C(\mathbb{R})) - \{(\Psi_i)_v\} \subset \mathcal{G}_{i-1} \). It follows that \( \Psi_c \in \overline{\mathcal{G}_0 = \phi(U(\mathbb{R}))} \), showing \( \Xi \to U \) has real effective rational equivalence. \( \Box \)
It is typically difficult to describe an intersection of several Schubert varieties. While this task is easier when they are in special position, even this may be too hard. It is better yet to consider the limiting position of intersection cycles as the subvarieties being intersected degenerate to the point of attaining excess intersection. This is the aim of effective rational equivalence. For example, the ‘limit cycle’ $\Psi$ of the previous proof is generally not an intersection of Schubert varieties, however, it is a deformation of such cycles.

An enumerative problem of degree $d$ is an intersection problem $\Xi \to U$ with finite fibres of cardinality $d$. It is fully real if there is a fibre $\Xi_u$ with $u \in U(\mathbb{R})$ consisting entirely of real points. Here, $u = (u_1, \ldots, u_b)$ with $u_i \in U_i(\mathbb{R})$ and $\Xi_u$ is the transverse intersection of the cycles $(\Xi_{1u_1}, \ldots, \Xi_{bu_b})$.

The set $M \subset \text{Sym}^d X$ of degree $d$ zero cycles consisting of $d$ distinct real points of $X$ is an open subset of $(\text{Sym}^d X)(\mathbb{R})$. Thus $\Xi \to U$ is fully real if and only if it has real effective rational equivalence. Hence Theorem 1 has the following consequence:

**Corollary 2.** Any enumerative problem given by Schubert conditions on lines in projective space is fully real.

2.2. Products in $A^*X$. $X$ is the quotient $G/P$ of $G$ by a parabolic subgroup $P$. A Schubert subvariety $\Omega_{wF}$ of $X$ is given by a complete flag $F$ and a coset $w$ of the corresponding parabolic subgroup of the symmetric group (I, Ch. IV, §2.5). Call $w$ the type of $\Omega_{wF}$. A Schubert class $\sigma_w$ is the cycle class of $\Omega_{wF}$.

Let $\Xi_1 \to U_1, \ldots, \Xi_b \to U_b$ be families of cycles on $X$ giving an intersection problem $\Xi \to U$. Then fibres of $\Xi \to U$ have cycle class $\prod_i \beta_i$, where $\beta_i$ is the cycle class of fibres of $\Xi_i \to U_i$. Suppose $\Xi \to U$ has effective rational equivalence with witness $Z$. Let $c_w$ count the components of $Z$ of type $w$. Since $Z$ is rationally equivalent to fibres of $\Xi \to U$, we deduce the formula in $A^*X$.

$$\prod_{i=1}^b \beta_i = \sum_w c_w \cdot \sigma_w.$$

2.3. Pieri-type formulas. Given a such product formula with each $c_w \leq 1$, the action of a real Borel subgroup $B$ of $G$ shows that the family of intersection cycles $\Xi \to U$ has real effective rational equivalence: Let $Y$ be a fibre of $\Xi \to U$ over a real point of $U$. Then the closure of the orbit $B(\mathbb{R}) \cdot Y$ in Chow $X(\mathbb{R})$ contains a $B(\mathbb{R})$-fixed point $Z$, as Borel’s fixed point Theorem (I, III.10.4), holds for $B(\mathbb{R})$-stable real analytic sets. Moreover, $Z$ is multiplicity-free as $c_w \leq 1$.

In the Grassmannian of $k$-planes in $\mathbb{P}^n$, a special Schubert variety is the locus of $k$-planes having excess intersection with a fixed linear subspace. A special Schubert variety of a flag variety is the inverse image of a special Schubert variety in a Grassmannian projection. Pieri’s formula for Grassmannians (L3, L4) and the Pieri-type formulas for flag varieties (L25, L36) show that the coefficients $c_w$ in a product of a Schubert class with a special Schubert class are either 0 or 1. Thus any intersection problem given by a Schubert variety and a special Schubert variety has real effective rational equivalence. We use this to prove the following theorem.

**Theorem 3.** Any enumerative problem in any flag variety given by five Schubert varieties, three of which are special, is fully real.
Proof. First pair each non-special Schubert variety with a special Schubert variety. The associated families \( \Xi \to U \) and \( \Xi' \to U' \) of intersection cycles have real effective rational equivalence with witnesses \( Z \) and \( Z' \), respectively.

Since the coefficients \( c_w \) in the Pieri-type formulas are either 0 or 1, a zero-dimensional intersection of three real Schubert varieties in general position where one is special is a single real point. Considering components of \( Z \) and \( Z' \) separately, we see that if \( Z, Z' \), and the third special Schubert variety \( Y \) are in general position with \( Y \) real, then they intersect transversally with all points of intersection real. Suitably small deformations of \( Z \) and \( Z' \) into real fibres of \( \Xi \) and \( \Xi' \) preserve the number of real points of intersection, completing the proof. \( \Box \)

3. The Grassmannian of planes in \( \mathbb{P}^5 \)

The Grassmannian of planes in \( \mathbb{P}^5 \), \( G_{2,5} \), is a 9-dimensional variety. If \( K \) is a plane in \( \mathbb{P}^5 \), then the set \( \Omega(K) \) of planes which meet \( K \) is a hyperplane section of \( G_{2,5} \) in its Plücker embedding. Thus the number of planes which meet 9 general planes is the degree of \( G_{2,5} \), which is \( \frac{1209}{84} = 42 \) \([33]\). This variety is the smallest dimensional flag variety for which an analog of Corollary 2 is not known. We illustrate the methods of §2 to prove the following result:

**Theorem 4.** There are 9 real planes in \( \mathbb{P}^5 \) such that the 42 planes meeting all 9 are real.

The Veronese surface in \( \mathbb{P}^5 \) is the image of \( \mathbb{P}^2 \) under the embedding induced by the complete linear system \( |O(2)| \), and so it has degree 4.

**Corollary 5.** There are 9 real Veronese surfaces in \( \mathbb{P}^5 \) such that the \( 11010048 (=4^9 \cdot 42) \) planes meeting all 9 are real.

Proof. Let \( x_{ij}, 1 \leq i \leq j \leq 3 \), be real coordinates for \( \mathbb{P}^5 \). For \( t \neq 0 \)

\[
\langle x_{11}x_{33} - t^4 x_{13}^2, \ x_{11}x_{22} - t^2 x_{12}^2, \ x_{11}x_{23} - tx_{12}x_{13}, \ x_{12}x_{33} - tx_{13}x_{23}, \ x_{13}x_{22} - tx_{12}x_{23}, \ x_{22}x_{33} - t^2 x_{23}^2 \rangle
\]

(1)

generates the ideal of a Veronese surface, \( \mathcal{V}(t) \) (cf. \([11]\), p. 142), which is real for \( t \in \mathbb{R} \). This family of Veronese surfaces is induced by the (real) \( \mathbb{C}^\times \)-action on the space of linear forms on \( \mathbb{P}^5 \):

\[
x_{ij} \mapsto t^{i-j} x_{ij} \quad \text{for} \quad t \in \mathbb{C}^\times.
\]

The ideal of the special fibre \( \mathcal{V}(0) \) of this family is generated by the underlined terms, so \( \mathcal{V}(0) \) is the union of the four planes given by the ideals:

\[
\langle x_{11}, x_{22}, x_{33} \rangle, \quad \langle x_{ii}, x_{jj}, x_{ij} \rangle, \quad ij = 12, 13, 23.
\]

(2)

By Theorem \( \Box \), there exist 9 real planes \( K_1, \ldots, K_9 \) such that \( \bigcap_{i=1}^9 \Omega(K_i) \) is a transverse intersection consisting of 42 real planes. This property of \( K_1, \ldots, K_9 \) is preserved by small real deformations. So for each \( 1 \leq i \leq 9 \), there is a neighborhood \( W_i \) of \( K_i \) in \( G_{2,5}(\mathbb{R}) \) such that if \( K'_i \in V_i \) for \( 1 \leq i \leq 9 \), then \( \bigcap_{i=1}^9 \Omega(K'_i) \) is transverse and consists of 42 real planes.

For each \( 1 \leq i \leq 9 \), choose a set of real coordinates for \( \mathbb{P}^5 \) so that the four planes, \( K_{i,j} \), for \( j = 1, 2, 3, 4 \), defined by the ideals of (2) are in \( W_i \). In these
same coordinates, consider the family $V_i(t)$ of real Veronese surfaces given by the ideals (1) with special member $V_i(0) = K_{i,1} + K_{i,2} + K_{i,3} + K_{i,4}$. If the sets of coordinates are chosen sufficiently generally, there exists $\epsilon > 0$ such that whenever $t \in (0, \epsilon)$, there are exactly $4^9 \cdot 42$ real planes meeting each of $V_i(t), \ldots, V_9(t)$.

This is because there are $4^9 \cdot 42$ real planes meeting each of $V_i(0), \ldots, V_9(0)$, as

$$
\bigcap_{i=1}^{9} (\Omega(K_{i,1}) + \Omega(K_{i,2}) + \Omega(K_{i,3}) + \Omega(K_{i,4}))
$$

is a transverse intersection consisting of $4^9 \cdot 42$ real planes: Since $K_{i,j} \in W_i$ for $1 \leq i \leq 9$ and $1 \leq j \leq 4$, this follows if the $4^9$ sets of 42 planes $\bigcap_{i=1}^{9} \Omega(K_{i,l_i})$ given by all sequences $l_i$, where $1 \leq l_i \leq 4$ for $1 \leq i \leq 9$, are pairwise disjoint. But this may be arranged when choosing the sets of coordinates. \qed

**Lemma 6.** The intersection problem of planes meeting 4 given planes in $\mathbb{P}^5$ has real effective rational equivalence.

**Proof of Theorem 4 using Lemma 6**. Partition the 9 planes into two sets of 4, and a singleton. Apply Lemma 6 to the intersection problems $\Xi \to U$, $\Xi' \to U'$ given by each set of 4, obtaining witnesses $Z$ and $Z'$. Arguing as for Theorem 3 completes the proof. \qed

**Proof of Lemma 6**. We use an economical notation for Schubert varieties. A partial flag $A_0 \subsetneq A_1 \subsetneq A_2 \subset \mathbb{P}^5$ determines a Schubert subvariety of $\mathbb{G}_{2,5}$:

$$
\Omega(A_0, A_1, A_2) := \{ H \in \mathbb{G}_{2,5} \mid \dim H \bigcap A_i \geq i, \text{ for } i = 0, 1, 2 \}.
$$

If $A_j$ is a hyperplane in $A_{j+1}$ or if $A_j = \mathbb{P}^5$, then it is no additional restriction for $\dim H \bigcap A_j \geq j$. We omit such inessential conditions. Thus, if $\mu \subsetneq M \subsetneq \Lambda$ is a partial flag, then $\Omega(\mu, \Omega(\cdot, M))$ and $\Omega(\mu, \cdot, \Lambda)$ are, respectively, those planes $H$ which meet $\mu$, those $H$ with $\dim H \cap M \geq 1$, and those $H \subset \Lambda$ which also meet $\mu$.

Let $\Xi \subset U \times \mathbb{G}_{2,5}$ be the intersection problem of planes meeting four given planes. Then $U \subset (\mathbb{G}_{2,5})^4$ is the set of 4-tuples of planes $(K_1, K_2, K_3, K_4)$ such that $\bigcap_{i=1}^{4} \Omega(K_i)$ is a generically transverse intersection and the fibre of $\Xi$ over $(K_1, K_2, K_3, K_4)$ is $\bigcap_{i=1}^{4} \Omega(K_i)$. We show $\Xi \to U$ has real effective rational equivalence by exhibiting a family $\Psi \subset V \times \mathbb{G}_{2,5}$, satisfying the four conditions:

(a) $V$ is rational. In fact $V$ is a dense subset of Magyar’s configuration variety $\mathcal{F}_D$ [27], where $D$ is the diagram

(b) $V$ has a dense open subset $V^\circ$ such that the fibres of $\Psi|_{V^\circ}$ are also fibres in the family $\Xi \to U$.

(c) $V$ has a rational subset $V'$ such that the fibres of $\Psi|_{V'}$ are unions of distinct Schubert varieties, real for real points of $V'$.

(d) $V'(\mathbb{R}) \subset V^\circ(\mathbb{R})$. Hence $\phi(V'(\mathbb{R})) \subset \phi(U(\mathbb{R}))$. Together with (c), this shows $\Xi \to U$ has effective rational equivalence.
Let $V \subset (\mathbb{G}_{1,5})^3 \times (\mathbb{G}_{3,5})^3$ be the locus of sextuples $(\mu_1, \mu_2, \lambda; M_1, M_2, L)$ such that $\mu_i \subset M_i$, $i = 1, 2$, $\mu_1, \mu_2 \subset L$, $\lambda \subset M_1 \cap M_2$, and $\mu_i \not\subset M_j, i \neq j$. We illustrate the inclusions:

Let $V^o \subset V$ be the dense locus where $\langle \mu_i, M_j \rangle = \mathbb{P}^5$ for $i \neq j$. Then $\lambda = M_1 \cap M_2$ and $L = \langle \mu_1, \mu_2 \rangle$. Let $V' \subset V$ be the locus where $\mu_1 \cap \mu_2$ is a point, so that $\langle M_1, M_2 \rangle$ is a hyperplane. Then $V'$ is rational and $V'(\mathbb{R}) \subset V^o(\mathbb{R})$, proving (d).

We define the family $\Psi$. For $v = (\mu_1, \mu_2, \lambda; M_1, M_2, L) \in V$, let $\Psi_v$ be the cycle

$$
\Omega(\mu_1) \cap \Omega(\mu_2) + \Omega(\mu_1) \cap \Omega(\mu_2) + \{H \in \Omega(\mu_1, L) \mid H \cap \mu_2 \neq \emptyset\} + \{H \in \Omega(\lambda, M_1) \mid \dim H \cap \mu_2 \geq 1\}.
$$

Let $\Psi \subset \mathbb{G}_{2,5} \times V$ be the subvariety with fibre $\Psi_v$ over points $v \in V$.

Suppose $v \in V^o$. Since $L = \langle \mu_1, \mu_2 \rangle$ and $\mu_1 \cap \mu_2 = \emptyset$,

$$
\Omega(\mu_1) \cap \Omega(\mu_2) = \{H \in \Omega(\mu_1, L) \mid H \cap \mu_2 \neq \emptyset\},
$$

as any plane meeting both $\mu_1$ and $\mu_2$ must intersect their span $L$ in at least a line. Similarly, $\Omega(\mu_1, M_1) \cap \Omega(\mu_2, M_2)$ is the fourth term of the cycle (3): If $l_i$ is a line in $H \cap M_i$ for $i = 1, 2$, then $l_1 \cap l_2 \subset \lambda = M_1 \cap M_2$. Thus $\Psi_v = \bigcap_{i=1}^2 (\Omega(\mu_i) + \Omega(\mu_i, M_i))$. Since the pairs of subspaces $(\mu_1, \mu_2)$, $(M_1, M_2)$, and $(\mu_i, M_i)$ for $i \neq j$ are in general position, this intersection is generically transverse.

We claim $\Psi_v$ is a fibre of $\Xi \to U$: Let $K_i, K'_i$ for $i = 1, 2$ be planes such that $\mu_i = K_i \cap K'_i$ and $M_i = \langle K_i, K'_i \rangle$. Then $\Omega(K_i) \cap \Omega(K'_i) = \Omega(\mu_i) + \Omega(\mu_i, M_i)$: If a plane $H$ meets both $K_i$ and $K'_i$, either it meets their intersection $\mu_i$, or else it intersects their span $M_i$ in at least a line. Moreover, while $K_i, K'_i$ are not in general position, this intersection is generically transverse as a proper intersection of a Schubert variety with a special Schubert variety is necessarily generically transverse (Š, §2.7). Thus $\Psi_v = \Xi_{(K_1, K'_1, K_2, K'_2)}$, proving (b).

To show (c), let $v = (\mu_1, \mu_2, \lambda; M_1, M_2, L) \in V'$. Set $p = \mu_1 \cap \mu_2$, a point and $\Lambda = \langle M_1, M_2 \rangle$, a hyperplane. Then $\langle \mu_1, \mu_2 \rangle$ is a plane $\nu$ contained in $L$ and $M_1 \cap M_2$ is a plane $N$ containing $\lambda$. We illustrate these inclusions:
To complete the proof, we show $\Psi_v$ is the sum of Schubert varieties

$\Omega(\mu_1, \cdot, \Lambda) + \Omega(p, M_2) + \Omega(\mu_2, \cdot, \Lambda) + \Omega(p, M_1) + \Omega(\cdot, \nu) + \Omega(p, L) + \Omega(\lambda, \cdot, \Lambda) + \Omega(\cdot, N)$.

First note that

$$\Omega(\mu_1) \cap \Omega(\cdot, M_2) = \Omega(\mu_1, \cdot, \Lambda) + \Omega(p, M_2).$$

If $H \in \Omega(\mu_1) \cap \Omega(\cdot, M_2)$, then either $H \cap \mu_1 \not\subset M_2$, so that $H \subset \langle \mu_1, M_2 \rangle = \Lambda$, or else $p \in H$ so that $H \in \Omega(p, M_2)$. Similarly, we have $\Omega(\mu_2) \cap \Omega(\cdot, M_1) = \Omega(\mu_2, \cdot, \Lambda) + \Omega(p, M_1)$. These intersections are generically transverse, as they are proper.

Furthermore,

$$\{ H \in \Omega(\mu_1, L) \mid H \cap \mu_2 \neq \emptyset \} = \Omega(\cdot, \nu) + \Omega(p, L).$$

Either $H \cap \mu_1 \cap \mu_2 = \emptyset$, thus $\dim H \cap \langle \mu_1, \mu_2 \rangle \geq 1$, and so $H \in \Omega(\cdot, \nu)$, or else $p \in H$, so that $H \in \Omega(p, L)$. Finally,

$$\{ H \in \Omega(\lambda, M_1) \mid \dim H \cap M_2 \geq 1 \} = \Omega(\lambda, \cdot, \Lambda) + \Omega(\cdot, N).$$

Either $H \cap M_1 \not\subset M_2$ thus $H \subset \langle M_1, M_2 \rangle = \Lambda$ and so $H \in \Omega(\lambda, \cdot, \Lambda)$, or else $\dim H \cap M_1 \cap M_2 \geq 1$, so that $H \in \Omega(\cdot, N)$.

Note that for $v \in V'$, the fibre $\Psi_v$ is not an intersection of four Schubert varieties of type $\Omega(K)$, for $K$ a plane: The Schubert subvariety $\Omega(\cdot, N)$ is the locus of planes which contain a line $l \subset N$ and hence it consists of all planes of the form $\langle q, l \rangle$, where $l \subset N$ is a line and $q \in \mathbb{P}^5 \setminus l$ is a point. Suppose $\Psi_v \subset \Omega(K)$ so that $\Omega(\cdot, N) \subset \Omega(K)$. Then for every line $l \subset N$ and point $q \in \mathbb{P}^5 \setminus l$, we have $K \cap \langle q, l \rangle \neq \emptyset$. This implies that $K \cap l \neq \emptyset$ for every line $l \subset N$, and hence that $\dim K \cap N \geq 1$. Similarly, $\dim K \cap \nu \geq 1$, and so $K \cap N \cap \nu \neq \emptyset$, thus $p \in K$. This shows $\Omega(p) \subset \Omega(K)$ and so if $\Psi_v \subset \Omega(K_1) \cap \Omega(K_2) \cap \Omega(K_3) \cap \Omega(K_4)$, then this intersection must contain $\Omega(p)$. Hence $\Psi_v$ is a proper subset of the intersection.

If $a_i = \dim A_i$, then $\sigma_{a_1a_2a_3}$ is the rational equivalence class of $\Omega(A_0, A_1, A_2)$. By the observation of §2.2, Lemma 3 implies the formula in $A^*G_{2,5}$:

$$(\sigma_{245})^4 = 3 \cdot \sigma_{035} + 2 \cdot \sigma_{125} + 3 \cdot \sigma_{134},$$

which may be determined by other means from the classical Schubert calculus.

4. Real Plane Conics

In 1864 Chasles [8] showed there are 3264 plane conics tangent to five general conics. Fulton [11] asked how many of the 3264 conics tangent to five general (real) conics can be real. He later determined that all can be real, but did not publish that result. More recently, Ronga, Tognoli, and Vust [31] rediscovered this result. We conclude this note with an outline of their work. The author is grateful to Bill Fulton and Felice Ronga for explaining these ideas.

Let $X$ be the variety of complete plane conics, a smooth variety of dimension 5. Let the hypersurfaces $H_p$, $H_l$, and $H_C$, be, respectively those conics containing a
point \( p \), those tangent to a line \( l \), and those tangent to a conic, \( C \). If \( \hat{p}, \hat{l}, \) and \( \hat{C} \) are, respectively, their cycle classes in \( A^1X \), then

\[
\hat{C} = 2\hat{p} + 2\hat{l},
\]

which may be seen by degenerating a conic into two lines. Then the number of conics tangent to five general conics is the degree of

\[
\hat{C}^5 = 32(\hat{p}^5 + 5\hat{p}^4 \cdot \hat{l} + 10\hat{p}^3 \cdot \hat{l}^2 + 10\hat{p}^2 \cdot \hat{l}^3 + 5\hat{p} \cdot \hat{l}^4 + \hat{l}^5).
\]

The monomials \( \hat{p}^j \cdot \hat{l}^{5-j} \) for \( j = 0, \ldots, 5 \), have degrees \( 1, 2, 4, 4, 2, 1 \), giving Chasles’ number of \( 32(1 + 10 + 40 + 40 + 10 + 1) = 3264 \) [19, §9].

**Theorem 7** (Fulton, Ronga-Tognoli-Vust). There are five real conics in general position such that all of the 3264 conics tangent to the five conics are real.

**Proof.** The strategy is to realize the five conics as a deformation of five degenerate conics giving a maximal number of real conics. The first step is to show that for each \( j \), there are \( j \) lines and \( 5 - j \) points such that the \( 2^{\min\{j, 5-j\}} \) conics tangent to the lines and containing the points are real. In [31], this step is done explicitly with a precise determination of which configurations of points and lines are ‘maximal’; that is, have all solutions real. Remarkably, there are five lines \( l_1, \ldots, l_5 \) and five real points \( p_1, \ldots, p_5 \) with \( p_i \in l_i \) such that each of the 32 terms in

\[
\bigcap_{i=1}^{5} (H_{p_i} + H_{l_i})
\]

is a transverse intersection with all points of intersection real. Such a configuration is illustrated in Figure 1.

![Figure 1. A Maximal Configuration](image)

The maximality of such a configuration is stable under small real deformations of its points and lines. Thus we may choose real lines \( l_1', \ldots, l_5' \) where

1. \( p_i \in l_i' \) and \( l_i' \) is distinct from \( l_i \), for \( i = 1, \ldots, 5 \),
2. Any configuration obtained from a maximal configuration by substituting some primed lines for the corresponding unprimed lines is maximal.
3. The lines \( l_i \) and \( l_i' \) partition the real tangent directions at \( p_i \) into two intervals.

The configurations described in condition (2) give finitely many real conics
passing through $p_i$. We require that all tangent directions to these conics at $p_i$ lie within the interior of \textit{one} of these two intervals.

The relation $\tilde{C} = 2\tilde{p} + 2\tilde{l}$ may be obtained by considering a conic $C$ near a degenerate conic consisting of two lines $l, l'$ meeting at a point $p$, and a pencil of conics.

For any conic $q$ in that pencil tangent to one of the lines, there is a nearby conic $q'$ in that pencil tangent to $C$. However, for every conic $Q$ in the pencil containing $p$, there are \textit{two} nearby conics $Q', Q''$ in that pencil tangent to $C$. Moreover, if $Q$ is real, then $Q'$ and $Q''$ are real if and only if the real tangent line to $Q$ at $p$ does not intersect $C$.

By condition (3), we may choose real conics $C_1, \ldots, C_5$ with $C_i$ near the degenerate conic $l_i + l'_i$ and, if $Q$ is a conic in

$$\bigcap_{i=1}^{5} (H_{p_i} + H_{l_i} + H_{l'_i})$$

containing $p_i$, the the real tangent line to $Q$ at $p_i$ does not intersect $C_i$. If, in addition, the conics $C_i$ are sufficiently close to each degenerate conic, then there will be 3264 real conics tangent to each of $C_1, \ldots, C_5$.

Indeed, suppose $H_{C_1}$ replaces $H_{p_1} + H_{l_1} + H_{l'_1}$ in the intersection \textup{(4)}. Then for any conic $q$ in \textup{(4)} that is tangent to either $l_1$ or $l'_1$, there is a nearby real conic $q'$ tangent to $C$ which satisfies the other conditions on $q$ (since these other conditions determine a pencil of conics). If $Q$ is a conic in \textup{(4)} containing $p_1$, then there are two nearby real conics $Q'$ and $Q''$ tangent to $C$ which satisfy the other conditions on $Q$. Similarly, if $H_{C_2}$ now replaces $H_{p_2} + H_{l_2} + H_{l'_2}$ in the new intersection $H_{C_1} \cap \bigcap_{i=2}^{5} (H_{p_i} + H_{l_i} + H_{l'_i})$, then each conic tangent to $l_2$ and $l'_2$ gives a conic tangent to $C_2$, but each conic through $p_2$ gives two conics tangent to $C_2$. Replacing $H_{C_3}, H_{C_4}$, and $H_{C_5}$ in turn completes the argument.

This proof used the effective rational equivalence:

$$H_C \sim 2H_p + H_l + H_{l'},$$

where $l, l'$ form a degenerate conic with $p = l \cap l'$. This deformation to a cycle having multiplicities (the coefficient 2 of $H_p$) is unavoidable: The variety $X$ and thus \textit{Chow}$X$ has an action of $G = PGL(3, \mathbb{C})$. The locus of hypersurfaces $H_C$ on \textit{Chow}$X$ is a single 5-dimensional $G$-orbit. This family cannot have effective rational equivalence. If $Z$ is a cycle in the closure of this locus, then $Z$ is in a $G$-orbit of dimension at most 4. Thus if $Z = H_p + H_{p'} + H_l + H_{l'}$, then the dimension of the $G$-orbit of $(p, p', l, l')$ in the product of $\mathbb{P}^2$’s and their duals is at most 4. But this is impossible unless either $p = p'$ or $l = l'$.

\footnote{This version of this manuscript does not contain all of the postscript files of the original, in particular, it does not have an illustration of this degenerate conics $C$ and the nearby conics. To visualize this, think of a real hyperbola in $\mathbb{R}^2$ defined by $x^2 - y^2 = t$, where $t$ is a small positive real number. Then the two lines are $x = \pm y$ and the point $p$ is the origin. The condition that the real tangent line to $Q$ at $p$ does not intersect $C$ means the absolute value of its slope exceeds 1.}
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Mathematical Sciences Research Institute, 1000 Centennial Drive, Berkeley, CA 94720, USA

Department of Mathematics, University of Toronto, 100 St. George St., Toronto, Ontario M5S 3G3, Canada (on leave)

\textit{E-mail address: sottile@math.toronto.edu}