Non-Abelian Wilson Surfaces

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ABSTRACT: A definition of non-abelian genus zero open Wilson surfaces is proposed. The ambiguity in surface-ordering is compensated by the gauge transformations.
1. Introduction

A higher dimensional generalization of the non-abelian Wilson line is not known. Only recently the notion of a connection on a non-abelian 1-gerbe was introduced in the work of Breen and Messing [1].

A motivation for defining the Non-abelian Wilson Surfaces comes from the string theory. NWS are relevant to six dimensional theories on the world volumes of coincident five branes [2].

The main problem in defining NWS is the lack of a natural order on a 2-dimensional surface. A naive guess for the NWS is

\[ P \exp \left( \int_{\Sigma} B \right), \]  

(1.1)

where \( B \) is a non-abelian 2-form. The choice of a surface-ordering \( P \) involves a time-slicing of the 2-surface \( \Sigma \). A no-go theorem of Teitelboim [3] states that no such a choice is compatible with the reparametrization invariance.

Let us recall the notion of a connection on a non-abelian 1-gerbe [4]. A connection on a principal bundle (0-gerbe) can be thought of as follows. Let \( x_0 \) and \( x_1 \) be two infinitesimally close points. The fibers \( S_{x_0} \) and \( S_{x_1} \) over these points are sets and the connection is a function

\[ f_{01} : S_{x_1} \rightarrow S_{x_0}, \]  

(1.2)

The connection on a non-abelian 1-gerbe is defined by analogy with the 0-gerbe case [4]. The fibers are categories \( C_{x_0} \) and \( C_{x_1} \), and the connection is a functor

\[ \varepsilon_{01} : C_{x_1} \rightarrow C_{x_0}. \]  

(1.3)
Let $x_0$, $x_1$ and $x_2$ be three infinitesimally close points. A diagram of functors and natural transformations is shown in figure [1]. Let $\text{Aut}(G)$ be the group of automorphisms of a non-abelian group $G$. Let $\text{Lie}(G)$ be the Lie algebra of $G$. It is shown in [1] that 2-arrow $K$, 1-arrow $\kappa$ and 1-arrow $\varepsilon$ in the diagram correspond to a $\text{Lie}(G)$-valued 2-form $B$, a $\text{Lie}(\text{Aut}(G))$-valued 2-form $\nu$ and a $\text{Lie}(\text{Aut}(G))$-valued 1-form $\mu$ respectively.

The paper is organized as follows. In section 2 a definition of NWS is proposed. Section 3 is devoted to gauge transformations. Some comments are listed in section 4.

2. Definition

We interpret the infinitesimal 2-simplex in figure [1] as a transmuted form of an infinitesimal Wilson surface expressed in the language of category theory. The fibered category in the formulation of [1] can be thought of as an ‘internal symmetry space’ of a non-abelian string. Let $\Sigma$ be a 2-dimensional surface with the disk topology. Let $C$ be a clockwise oriented boundary of $\Sigma$ and $P$ a marked point on it (see figure 2). We associate group elements

\[ W[\Sigma, C, P] \in G \]

and

\[ V[\Sigma, C, P] \in \text{Aut}(G) \]

with the data $(\Sigma, C, P)$. We write $W[\Sigma]$ and $V[\Sigma]$ when the omitted arguments are obvious from the context. With an open curve $C$ we associate an element of $\text{Aut}(G)$:

\[ M[C] \in \text{Aut}(G). \] (2.1)

Let $C = C_2 \circ C_1$ be a composition of curves $C_2$ and $C_1$. We assume

\[ M[C] = M[C_2 \circ C_1] = M[C_2]M[C_1]. \] (2.2)

We now propose an equation relating $M[C]$, $W[\Sigma, C]$ and $V[\Sigma, C]$. For a group element $g \in G$ we denote by $i_g$ the inner automorphism

\[ i_g(h) = ghg^{-1}, \quad \forall h \in G. \] (2.3)
Figure 2: Composition of surfaces with the disk topology. (a) Surfaces $\Sigma_i$ with the marked points $P_i$ and the clockwise oriented boundaries $C_i$. (b) Surfaces are joined along the common boundary segment $C_5$. (c) The resulting surface $\Sigma_2 \circ \Sigma_1$ with the marked point $P_1$ and the clockwise oriented boundary $C \circ C_4 \circ C_3$.

The conjectural equation reads

$$M[C] = i_{W[\Sigma]} V[\Sigma]. \quad (2.4)$$

An infinitesimal version of this equation was first derived in [1] from the requirement that $K$ in figure 1 is a natural transformation. We regard eq. (2.4) as a fundamental equation relating bulk and boundary of the non-abelian string world-sheet.

Eq. (2.4) can be used to find a composition rule for two NWS. Consider the 2-surfaces in figure 2. The identity

$$i_{W[\Sigma_2 \circ \Sigma_1, P_1]} V[\Sigma_2 \circ \Sigma_1] = M[C \circ C_4 \circ C_3] M[\Sigma] M[C \circ C_4 \circ C_5^{-1}] M[C^{-1}] M[C \circ C_5 \circ C_3] M[C]$$

suggestions the following composition rule for Wilson surfaces:

$$W[\Sigma_2 \circ \Sigma_1] = M[C] (W[\Sigma_2]) M[C] V[\Sigma_2] M[C^{-1}] (W[\Sigma_1]),$$

$$V[\Sigma_2 \circ \Sigma_1] = M[C] V[\Sigma_2] M[C^{-1}] V[\Sigma_1]. \quad (2.5)$$

An infinitesimal version of eq. (2.5) appeared implicitly in the category-theoretic definition of the curvature in [1].

Eq. (2.5) can be understood as follows. When the curve $C$ is absent, i.e. when the marked points of $\Sigma_1$ and $\Sigma_2$ coincide, eq. (2.6) simplifies to

$$W[\Sigma_2 \circ \Sigma_1] = W[\Sigma_2] V[\Sigma_2] (W[\Sigma_1]),$$

$$V[\Sigma_2 \circ \Sigma_1] = V[\Sigma_2] V[\Sigma_1]. \quad (2.6)$$

Thus when the marked points of the two surfaces coincide, the Wilson surfaces are composed as in eq. (2.7). If we think of $V[\Sigma, P]$ as an operator which acts on the
objects with the marked point $P$ and assume that only the objects with the same marked points can be multiplied, then the meaning of eq. $(2.6)$ becomes clear. The role of $M[C]$ in eq. $(2.6)$ is to transform the objects with the marked point $P_2$ to the objects with the marked point $P_1$.

Composition of three or more surfaces is in general ambiguous. Consider figure 3. Using the composition rule $(2.6)$ it can be shown that

$$W[\Sigma_3 \circ (\Sigma_2 \circ \Sigma_1)] \neq W[\Sigma_2 \circ (\Sigma_3 \circ \Sigma_1)],$$

$$V[\Sigma_3 \circ (\Sigma_2 \circ \Sigma_1)] \neq V[\Sigma_2 \circ (\Sigma_3 \circ \Sigma_1)].$$

(2.8)

Given

$$V[\delta \Sigma] \approx 1 + v[P] \equiv 1 + v_{\mu\nu}[P]\sigma^{\mu\nu}$$

(2.9)

for an infinitesimal surface $\delta \Sigma$ with the area element $\sigma^{\mu\nu}$, we want to find $V[\Sigma]$ for a finite-size surface $\Sigma$. This can be done using a trick similar to the one used in the context of the non-abelian Stokes formula [4]. Consider the contour $C'$ in figure 4.

From the relation

$$M[C'] = M[C_P^{-1}]M[\delta C]M[C_P]M[C]$$

(2.10)

and eq. (2.4) one finds

$$V[\Sigma'] = M[C_P^{-1}]V^{-1}[\delta \Sigma]M[C_P]V[\Sigma].$$

(2.11)

Thus we have

$$\delta V[\Sigma] = M[C_P^{-1}]v[P]M[C_P]V[\Sigma].$$

(2.12)

A solution of this equation involves a choice of ordering and it is given by

$$V[\Sigma] = \hat{P}_\tau \exp \left( \int_{\Sigma} M[C_P^{-1}]v[P]M[C_P] \right),$$

(2.13)

where $\hat{P}_\tau$ is the ordering in $\tau$ and the curve $C_P$ is defined in figure 4. Note that the expression eq. $(2.13)$ depends on the parametrization $x^\mu = x^\mu(\sigma, \tau)$ of the surface $\Sigma$. 

\[\text{Figure 3: } \Sigma_3 \circ (\Sigma_2 \circ \Sigma_1) \neq \Sigma_2 \circ (\Sigma_3 \circ \Sigma_1)\]
Figure 4: Contour \( C' = C P^{-1} \circ \delta C \circ C P \circ C \).

For example a boundary-preserving reparametrization will change \( C P \) to a \( C P' \) (see figure 3). Thus \( V[\Sigma] \) and \( W[\Sigma] \) depend on the parametrization of \( \Sigma \):

\[
V = V[\Sigma, x^\mu(\sigma, \tau)], \quad W = W[\Sigma, x^\mu(\sigma, \tau)].
\]

(2.14)

In section 3 we will see that if \((\sigma, \tau)\) and \((\tilde{\sigma}, \tilde{\tau})\) are two different parametrizations of a surface \( \Sigma \), then

\[
(V[\Sigma, x^\mu(\sigma, \tau)], W[\Sigma, x^\mu(\sigma, \tau)])
\]

and

\[
(V[\Sigma, x^\mu(\tilde{\sigma}, \tilde{\tau})], W[\Sigma, x^\mu(\tilde{\sigma}, \tilde{\tau})])
\]

are related by the gauge transformation. In other words, the non-abelian internal symmetry and the reparametrization symmetry mix.

3. Gauge transformations

In this section we introduce the gauge transformations which compensate the ambiguity in the composition of NWS. Suppose that a surface \( \Sigma \) is composed out of three or more smaller surfaces. Let \((W[\Sigma], V[\Sigma])\) and \((\tilde{W}[\Sigma], \tilde{V}[\Sigma])\) correspond to two different compositions resulting in the surface \( \Sigma \). We have

\[
M[C] = i_{W[\Sigma]} V[\Sigma] = i_{\tilde{W}[\Sigma]} \tilde{V}[\Sigma].
\]

(3.1)

Since \( W \) and \( \tilde{W} \) are elements of a group \( G \), there is a group element \( R[\Sigma] \in G \) such that

\[
\tilde{W}[\Sigma] = W[\Sigma](R[\Sigma])^{-1}.
\]

(3.2)

Let us decompose \( W \) and \( \tilde{W} \) into the abelian and non-abelian factors:

\[
W = W_{ab} \cdot W_{nonab}, \quad \tilde{W} = \tilde{W}_{ab} \cdot \tilde{W}_{nonab}.
\]

(3.3)

It is clear that the ambiguity in the composition does not affect the abelian part. Thus we have

\[
\tilde{W}_{ab}[\Sigma] = W_{ab}[\Sigma].
\]

(3.4)
Combining this equation with eq. (3.2) we find
\[ \tilde{W}_{\text{nonab}}[\Sigma] = W_{\text{nonab}}[\Sigma](R[\Sigma])^{-1}. \] (3.5)

We propose that eq. (3.4) and eq. (3.5) define the gauge transformation of \( W \). In order for this gauge transformation of \( W \) to be compatible with eq. (3.1), \( V \) should transform as
\[ \tilde{V}[\Sigma] = i_{R[\Sigma]} V[\Sigma]. \] (3.6)

It can be checked that the gauge transformations (3.4–3.6) are compatible with the composition rule (2.4) provided that the composition rule for \( R \) is the same as that of \( W \), namely
\[ R[\Sigma_2 \circ \Sigma_1] = M[C](R[\Sigma_2])M[C]V[\Sigma_2]M[C^{-1}](R[\Sigma_1]). \] (3.7)

More generally, consider a surface \( \Sigma \) divided into \( n \) smaller surfaces \( \Sigma_1, \ldots, \Sigma_n \). Let \( C \) be the boundary of \( \Sigma \). Repeating the reasoning leading to eq. (2.6) we have
\[ M[C] = M[C_1]i_{W[\Sigma_1]} V[\Sigma_1] M[C_1] M[C_2] i_{W[\Sigma_2]} V[\Sigma_2] M[C_2] \cdots \] (3.8)
for some curves \( C_1, C_2, \ldots \). From this equation we find
\[ W[\Sigma] = M[C_1](W[\Sigma_1])M[C_1] V[\Sigma_1] M[C_2](W[\Sigma_2]) \cdots, \]
\[ V[\Sigma] = M[C_1] V[\Sigma_1] M[C_2] V[\Sigma_2] M[C_3] \cdots. \] (3.9)

It is easy to see that the gauge transformations (3.4–3.6) are compatible with eq. (3.4) provided that \( R[\Sigma] \) is composed out of \( R[\Sigma_i] \) as follows:
\[ R[\Sigma] = M[C_1](R[\Sigma_1])M[C_1] V[\Sigma_1] M[C_2](R[\Sigma_2]) \cdots \] (3.10)

Thus \( R \) should be composed by the rule of composition of \( W \).

We now introduce new gauge transformations. These are the transformations of \( M, V \) and \( W \) compatible with eq. (2.4).
Let $Λ[P]$ be an $\text{Aut}(G)$-valued function of point $P$. Let $C$ be a directed path from $P_1$ to $P_2$. The gauge transformation of $M[C]$ reads

$$\tilde{M}[C] = Λ[P_2]M[C]Λ[P_1]^{-1}. \quad (3.11)$$

When $P_1 = P_2 = P$ this equation becomes

$$\tilde{M}[C] = Λ[P]M[C]Λ[P]^{-1}. \quad (3.12)$$

From this equation and

$$\tilde{M}[C] = i_\tilde{W} \tilde{V} \quad (3.13)$$

one finds

$$i_\tilde{W} V = Λ^{-1}i_\tilde{W} \tilde{V} Λ = i_{Λ^{-1}(i\tilde{W})Λ^{-1}} \tilde{V} Λ. \quad (3.14)$$

Thus we propose the gauge transformations:

$$\tilde{V} [Σ, P] = Λ[P]V[Σ, P]Λ[P]^{-1}, \quad (3.15)$$

$$\tilde{W} [Σ, P] = Λ[P](W[Σ, P]). \quad (3.15)$$

We now consider a new gauge transformation which is a finite generalization of the infinitesimal transformation considered in [1]. The transformation reads

$$\tilde{M}[C] = i_\tilde{Z}[C] M[C], \quad (3.16)$$

where $Z[C]$ is a $G$-valued functional of $C$. The composition rule for $Z$ can be inferred from the following chain of equations:

$$i_{Z[C_2 ∘ C_1]} M[C_2 ∘ C_1] = \tilde{M}[C_2 ∘ C_1]$$

$$= \tilde{M} C_2 \tilde{M} C_1$$

$$= i_{Z[C_2]} M[C_2] i_{Z[C_1]} M[C_1]$$

$$= i_{Z[C_2]} i_{M[C_2]} (Z[C_1]) M[C_2] ∘ C_1. \quad (3.17)$$

This equation suggests the following composition rule for $Z$:

$$Z[C_2 ∘ C_1] = Z[C_2] M[C_2] (Z[C_1]). \quad (3.18)$$

If a Lie($G$)-valued 1-form $ζ$ is given, $Z[C]$ for an open path $C$ can be constructed as follows. Let us divide $C$ into $n$ small subpaths as in figure 6(a). Applying eq. (3.18) we find

$$Z[C] = Z[C_n] · M[C_n](Z[C_{n-1}]) · M[C_n ∘ C_{n-1}](Z[C_{n-2}]) · · ·$$

$$M[C_n ∘ C_{n-1} · · · C_2](Z[C_1])$$

$$≈ (1 + ζ[ΔP]dx^μ)(1 + M[C_n](ζ[ΔP_{n-1}])dx^μ) · · ·$$

$$· · · (1 + M[C_n ∘ C_{n-1} · · · C_2](ζ[ΔP])dx^μ). \quad (3.19)$$
Figure 6: (a) The path $C$ is divided into $n$ small subpaths: $C = C_n \circ C_{n-1} \cdots \circ C_1$. (b) The point $P$ divides $C = C'' \circ C'$.

In the large $n$ limit we thus find

$$Z[C] = \hat{P} \exp \left( \int_C M[C''](\zeta_{\mu}[P])dx^\mu \right),$$

(3.20)

where $C''$ and $P$ are as in figure 6(b), and $\hat{P}$ is the path ordering operator.

A choice of transformation of $V$ and $W$ compatible with eq. (2.4) and eq. (3.16) is

$$\tilde{V}[\Sigma, C] = V[\Sigma, C],$$

$$\tilde{W}[\Sigma, C] = Z[C]W[\Sigma, C].$$

(3.21)

Infinitesimal versions of these transformations agree with the transformations that can be derived from [1]. Let us consider an infinitesimal surface $\delta \Sigma$ with the area element $\sigma_{\mu\nu}$. Assume that $M[C] \in \text{Aut}(G)$ is an inner automorphism given by

$$M[C](g) = \hat{P}\exp \left( \int_C \mu \right) g \hat{P}\exp \left( -\int_C \mu \right) = \hat{P}\exp \left( \int_C \mu_{\text{adjoint}} \right) (g), \quad \forall g \in G,$$

(3.22)

where $\mu$ is a Lie($G$)-valued 1-form. From eq. (3.21) and

$$W[\delta \Sigma] \approx 1 + B_{\mu\nu}\sigma^{\mu\nu},$$

(3.23)

one can find the transformation of the 2-form $B$:

$$\tilde{B} = B + d\zeta - \frac{1}{2}[\zeta, \zeta] - [\mu, \zeta].$$

(3.24)

The transformation of $B$ corresponding to eqs. (3.4, 3.5) reads

$$\tilde{B}_{ab} = B_{ab}, \quad \tilde{B}_{\text{nonab}} = B_{\text{nonab}} - \rho,$$

(3.25)

where $\rho$ is a Lie($G$)-valued 2-form defined in

$$R[\delta \Sigma] \approx 1 + \rho_{\mu\nu}\sigma^{\mu\nu}.$$

(3.26)

Eq. (3.25) agrees with the transformations that can be derived from [1].
Unlike the gauge transformations (3.4–3.6, 3.15), the transformation (3.21) is not compatible with the composition rule (2.6). To find the correct transformation, $Z[C]$ in eq. (3.21) should be ‘smeared’ over the surface $\Sigma$. We give an explicit formula for the gauge transformation of $V[\Sigma]$. It reads

$$\tilde{V}[\Sigma] = \hat{P}_\tau \exp \left( \int_{\Sigma} i_{Z[C_P]} M[C_P] v[P] M[C_P^{-1}] i_{Z[C_P]}^{-1} \right). \quad (3.27)$$

4. Comments

- We found three kinds of gauge transformations of $M$, $V$ and $W$. These are $\Lambda[P]$-transformations (3.11,3.15), $R[\Sigma]$-transformations (3.4–3.6) and $Z[C]$-transformations (3.16,3.21). Eq. (3.21) is valid only for infinitesimal surfaces and should be replaced by a ‘smeared’ version eq. (3.27).
- The ambiguity in surface-ordering necessitates the introduction of gauge transformations which compensate the ambiguity. Locally this amounts to the transformation eq. (3.25). The number of gauge degrees of freedom present in a NWS is enormous. Thus NWS may be relevant to a topological string theory describing topological sectors of the non-abelian string of [2].
- Infinitesimal version of eq. (2.6) can be derived from the composition rule for the natural transformation $K$ in figure 1.
- We defined NWS on a local trivial patch. To define NWS globally one should cover the manifold with an atlas $\{U_\alpha\}$ and introduce $W_\alpha, V_\alpha, M_\alpha$ for each patch $U_\alpha$. As usual the quantities on the overlaps $U_{\alpha\beta} = U_\alpha \cap U_\beta$ are related by the gauge transformations. An analysis of global issues will be carried out elsewhere.
- We defined NWS with the disk topology. A generalization to higher-genus surfaces will be discussed elsewhere.

Note added

After submitting the original version of this paper to hep-th, the work [5] was brought to our attention. In [5] an equation similar to eq. (2.13) was taken as a definition of Wilson surface. The case considered in [5] corresponds, in our notation, to the $C$-independent $M[C]$. The surface-ordering ambiguities are absent in this case. For a list of miscellaneous work on non-abelian 2-form theories, see [6].

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