The Relativistic Avatars of Giant Magnons
and their S-Matrix

Timothy J. Hollowood

Department of Physics,
University of Wales Swansea,
Swansea, SA2 8PP, UK.
E-mail: t.hollowood@swansea.ac.uk

and J. Luis Miramontes

Departamento de Física de Partículas and IGFAE,
Universidad de Santiago de Compostela
15782 Santiago de Compostela, Spain
E-mail: jluis.miramontes@usc.es

ABSTRACT: The motion of strings on symmetric space target spaces underlies the
integrability of the AdS/CFT correspondence. Although these theories, whose ex-
citations are giant magnons, are non-relativistic they are classically equivalent, via
the Pollmeyer reduction, to a relativistic integrable field theory known as a sym-
metric space sine-Gordon theory. These theories can be formulated as integrable
deformations of gauged WZW models. In this work we consider the class of sym-
metric spaces $\mathbb{C}P^{n+1}$ and solve the corresponding generalized sine-Gordon theories
at the quantum level by finding the exact spectrum of topological solitons, or kinks,
and their S-matrix. The latter involves a trigonometric solution of the Yang-Baxter
equation which exhibits a quantum group symmetry with a tower of states that is
bounded, unlike for magnons, as a result of the quantum group deformation param-
eter $q$ being a root of unity. We test the S-matrix by taking the semi-classical limit
and comparing with the time delays for the scattering of classical solitons. We argue
that the internal $\mathbb{C}P^{n-1}$ moduli space of collective coordinates of the solitons in the
classical theory can be interpreted as a $q$-deformed fuzzy space in the quantum the-
ory. We analyse the $n = 1$ case separately and provide a further test of the S-matrix
conjecture in this case by calculating the central charge of the UV CFT using the
thermodynamic Bethe Ansatz.
1. Introduction

One of the many remarkable features of the AdS/CFT correspondence is the emergence of integrability. This is fortunate indeed, because it promises a quantitative investigation of the conjectured duality. On the CFT side, integrability is manifested by the appearance of integrable spin chains whose Hamiltonians provide the spectrum of exact scaling dimensions $\Delta$ [1–3]. In the particular limit where $\Delta$ and a conserved $R$-charge $J$ become infinite, with the difference $\Delta - J$ and the ’t Hooft coupling $\lambda$ held fixed, the string duals of the fundamental magnon excitations of those spin chains are lump-like solutions known as “giant magnons”, which propagate on an infinite long string [4]. Giant magnons describe the classical motion of (bosonic) strings on curved space-times of the form $\mathbb{R}_t \times \mathcal{M}$, with $\mathcal{M} = F/G$ a symmetric space. For example, the (original) giant magnon of AdS$_5$/CFT$_4$ and its dyonic generalization correspond to $S^n = SO(n+1)/SO(n)$ with $n = 2$ and $3$, respectively [4,5]. In a similar way, the basic giant magnons of AdS$_4$/CFT$_3$ are associated to $\mathbb{C}P^n = SU(n+1)/U(n)$ with $n = 1$ and $2$ [6–9].

The gauged-fixed worldsheet theory on $\mathbb{R}_t \times \mathcal{M}$ is a sigma model with target space $\mathcal{M}$ subject to additional constraints that preserve integrability, but break conformal and relativistic invariance on the worldsheet. The gauge fixing conditions are the Pohlmeyer constraints [10,11], and the giant magnons are the solitons of the resulting constrained theory. Their spectrum and S-matrix has been completely determined at the quantum level [12–15]. The S-matrix is complicated by the fact that, in comparison with the “usual” situation, the worldsheet theory is non-relativistic. It is remarkable, however, that there is a re-formulation of the sigma model with Pohlmeyer constraints as a conventional massive integrable field theory of a type that generalizes the sine-Gordon theory. These relativistic field theories are known as the symmetric space sine-Gordon (SSSG) theories [16–21]. They can formulated at the Lagrangian level as a gauged WZW model with an integrable deforming potential [22,23], which naturally leads to their description as perturbations of coset CFTs [24]. The equivalence between the gauged fixed worldsheet theory and the SSSG theory is a classical equivalence in which the non-relativistic magnons map to a relativistic soliton “avatar” in the SSSG theory. It does not seem possible that the equivalence can be maintained at the quantum level, since the two descriptions have a different Poisson structure [25]. However, in the context of AdS$_5$/CFT$_4$, it has been argued by Grigoriev and Tseytlin [26] and by Mikhailov and Schäfer-Nameki [27] that quantum equivalence may hold in the full theory with all the fermions included. Then, the Lagrangian formulation of the SSSG theory would be the starting point to find a novel, manifestly two-dimensional Lorentz invariant, formulation of the full AdS$_5 \times S^5$ superstring theory that would be an alternative to the usual formulation.
in the light-cone gauge. This conjecture has already passed a number of tests [28]. Nevertheless, the equivalence can only be properly judged once the SSSG theories have been solved at the quantum level. To date, the knowledge of the SSSG theories is extremely limited to the cases with no fermions and then only to the SSSG theories related to $S^2 = SO(3)/SO(2)$ and $S^3 = SO(4)/SO(3)$, since these are the well-known sine-Gordon [29] and complex sine-Gordon [30] theories, respectively. The aim of the present work is to begin to fill the gap in our knowledge by solving—in the sense of finding the spectrum and S-matrix—the theories corresponding to the symmetric spaces $\mathbb{C}P^{n+1}$. This is directly relevant to the AdS/CFT correspondence for AdS$_4 \times \mathbb{C}P^3$; namely, AdS$_4$/CFT$_3$ [35]. The extension to other symmetric spaces should now follow by similar methods.

The plan of the paper is as follows. In section 2 we review the Pohlmeyer reduction for the example $\mathbb{C}P^{n+1}$ focussing particularly on the algebraic approach that leads to the associated SSSG theory in a rather simple way. We explain how the SSSG theory can be formulated at the Lagrangian level as a gauged WZW model for $U(n+1)/U(n)$ with an integrable deforming potential. In section 3, we show that the non-relativistic magnons in the original sigma model can be constructed at the same time as the soliton avatar using the dressing method. We spend some time explaining how the magnons/solitons have a $\mathbb{C}P^{n-1}$ moduli space of internal collective coordinates. We also show how the soliton avatar is a kink carrying a topological charge. In section 4 we present our conjecture for the exact quantum S-matrix of the topological kinks of the deformed WZW model. Section 5 is devoted to a check of the S-matrix by taking the semi-classical limit. Section 6 focusses on the symmetric space $\mathbb{C}P^2$ which is somewhat different and simpler because the symmetry group is abelian. In this case we are also able to test our conjectured S-matrix by using the thermodynamic Bethe Ansatz. Finally in Section 7 we draw some conclusions.

2. The Symmetric Space Sine-Gordon Theories

The starting point is a sigma model with target space a symmetric space $F/G$. The group in the numerator $F$ admits an involution $\sigma$ whose stabilizer is the subgroup $G$. Acting on the Lie algebra of $F$, the involution gives rise to the canonical decom-

---

1Although they are not directly relevant in the context of the AdS/CFT correspondence, the homogeneous sine-Gordon theories provide another set of SSSG theories that have been solved at the quantum level [31–34]. They are Pohlmeyer reductions of the principal chiral model corresponding to a Lie group $G$, which can be formulated as the symmetric space $G \times G/G$ [23].
\[ f = g \oplus p \quad \text{with} \quad [g, g] \subset g, \quad [g, p] \subset p, \quad [p, p] \subset g, \quad (2.1) \]

where \( g \) and \( p \) are the +1 and −1 eigenspaces of \( \sigma \), respectively. This allows us to formulate the symmetric space in terms of a group element \( F \in F \) constrained via

\[ \sigma(F) = F^{-1}. \quad (2.2) \]

For the case of \( \mathbb{C}P^{n+1} = SU(n+2)/U(n+1) \) we can describe the target space in terms of \( n+2 \) complex homogeneous coordinates \( Z \) with the identification \( Z \sim \lambda Z \), \( \lambda \in \mathbb{C}^* \). The map from the space \( \mathbb{C}P^{n+1} \) to the group field realized in the \( n+2 \)-dimensional defining representation of \( SU(n+2) \) is given by

\[ F = \theta \left( I - 2 \frac{ZZ^\dagger}{|Z|^2} \right), \quad (2.3) \]

where \( \theta = \text{diag}(-1,1,\ldots,1) \) implements the involution

\[ \sigma(F) = \theta F \theta. \quad (2.4) \]

The subgroup \( G = U(n+1) \) then consists of elements \( F \) of the form

\[ F = \begin{pmatrix} e^{i\phi} & 0 \\ 0 & W \end{pmatrix} \in SU(n+2), \quad (2.5) \]

with \( W \in U(n+1) \) and \( e^{-i\phi} = \text{det}W \).

The Lagrangian of the sigma model is

\[ \mathcal{L} = -\text{Tr}(J_\mu J^\mu) \quad \text{with} \quad J_\mu = \partial_\mu FF^{-1}, \quad (2.6) \]

whose equations-of-motion are

\[ \partial_\mu J^\mu = 0. \quad (2.7) \]

The conserved current \( J_\mu \) corresponds to the global symmetry transformation\(^2\)

\[ F \rightarrow UF\sigma(U^{-1}), \quad U \in F, \quad (2.8) \]

with a conserved Noether charge

\[ Q = \int_{-\infty}^{+\infty} \partial_0 FF^{-1} \quad (2.9) \]

\(^2\)The Lagrangian density \( L \) is invariant under the global transformations \( F \rightarrow UFV \) for any \( U, V \in F \). However, this symmetry is reduced by the constraint \( (2.2) \) so that the symmetric space sigma model is invariant only under \( (2.8) \).
that takes values in the Lie algebra of $F$.

In the case of the $\mathbb{C}P^{n+1}$ sigma model, the Pohlmeyer reduction involves imposing the conditions $[23, 36]^3$

$$\partial_\pm FF^{-1} = \mu f_\pm \Lambda f_\pm^{-1}, \quad (2.10)$$

where $f_\pm \in F$ and

$$\Lambda = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \quad (2.11)$$

Here, $\mu$ is an arbitrary mass scale and in most of the following we shall set $\mu = 1$ with the understanding that it can be re-introduced in order to reconcile the dimensions. $\Lambda$ is the unique element, up to conjugation, of the the $-1$ eigenspace $p$ of the Lie algebra of $SU(n+2)$.

Notice that the Pohlmeyer constraints break the Lorentz and conformal invariance of the sigma model. The key observation is that the Lorentz invariance can be recovered by a re-formulation of the constrained system as a relativistically invariant, massive and integrable theory: this is the symmetric space sine-Gordon theory $[11, 22, 23]$. The degree-of-freedom of the SSSG theory is the $G$-valued field

$$\gamma = f_-^{-1} f_+ \quad (2.12)$$

which satisfies the SSSG equations

$$[\partial_+ + \gamma^{-1} \partial_+ \gamma + \gamma^{-1} A_+^{(L)} \gamma - \frac{1}{2} \Lambda, \partial_- + A_-^{(R)} - \frac{1}{2} \gamma^{-1} \Lambda \gamma] = 0. \quad (2.13)$$

Here, the quantities $A_+^{(L)}$ and $A_-^{(R)}$ can be interpreted as lightcone components of gauge fields associated to a $H_L \times H_R$ gauge symmetry under which

$$f_\pm \longrightarrow f_\pm h_\pm^{-1}, \quad (2.14)$$

where $h_\pm$ are local group elements in the subgroup $H \subset G$, the subgroup of $G \subset F$ that commutes with $\Lambda$. In the present case $H = U(n)$ and its elements are of the form

$$\begin{pmatrix} e^{i\phi} & 0 \\ 0 & M \end{pmatrix} \in SU(n+2) \quad (2.15)$$

---

$^3$In our notation, $x_\pm = t \pm x$ and $\partial_\pm = \frac{1}{2}(\partial_t \pm \partial_x)$. 

---
with \( M \in U(n) \) and \( e^{-2i\phi} = \det M \). Under this symmetry

\[
\gamma \rightarrow h_- \gamma h_+^{-1}
\]

and

\[
A_{-}^{(R)} \rightarrow h_+ (A_{-}^{(R)} + \partial_-) h_+^{-1}, \quad A_{+}^{(L)} \rightarrow h_- (A_{+}^{(L)} + \partial_+) h_-^{-1}.
\] (2.17)

For general non-abelian \( H \), a Lagrangian formalism can be found by identifying \( A_{-} = A_{-}^{(R)} \) and \( A_{+} = A_{+}^{(L)} \) as the two lightcone components of a gauge field, and by imposing the constraints \([22]\)

\[
\left[ \gamma^{-1} \partial_+ \gamma + \gamma^{-1} A_+ \gamma \right]_h = A_+ ,
\]

\[
\left[ - \partial_- \gamma^{-1} + \gamma A_- \gamma^{-1} \right]_h = A_- ,
\]

where the projection is onto the Lie algebra of \( H \). These conditions can be viewed as a set of partial gauge fixing conditions \([23, 26]\). They reduce the \( H_L \times H_R \) gauge symmetry \((2.16)\) to the \( H \) vector subgroup\(^4\)

\[
\gamma \rightarrow U \gamma U^{-1}, \quad U \in H ,
\] (2.19)

under which \( A_\mu \) transforms as a gauge connection:

\[
A_\mu \rightarrow U (A_\mu + \partial_\mu) U^{-1} .
\] (2.20)

The gauge-fixed equations-of-motion (with \( \mu \) re-introduced) are then

\[
\left[ \partial_+ + \gamma^{-1} \partial_+ \gamma + \gamma^{-1} A_+ \gamma, \partial_- + A_- \right] = \frac{\mu^2}{4} \left[ \Lambda, \gamma^{-1} \Lambda \gamma \right]
\] (2.21)

and these follow as the equations-of-motion of the Lagrangian density

\[
\mathcal{L} = \mathcal{L}_{WZW}(\gamma) + \frac{1}{2\pi} \text{Tr} \left( -A_+ \partial_- \gamma^{-1} + A_- \gamma^{-1} \partial_+ \gamma + \gamma^{-1} A_+ \gamma A_- - A_+ A_- - \frac{\mu^2}{4} \gamma^{-1} \Lambda \gamma \right) ,
\] (2.22)

where \( \mathcal{L}_{WZW}(\gamma) \) is the usual WZW Lagrangian density for \( \gamma \). In fact this theory is the gauged WZW model for \( G/H \) deformed by the last term which is a potential.

\(^4\)Note that it is also possible to gauge the axial vector subgroup of the overall \( U(1) \) subgroup of \( H \), whilst still gauging the vector subgroup of the non-abelian factor \( SU(n) \). This gives rise to a different Lagrangian formulation of the theory for which the solitons carry a \( U(1) \) Noether charge \([36]\). The two formulations are related by a kind of T-duality \([37]\).
Notice that the partial gauge-fixing constraints (2.18) now appear as the equations-of-motion of the gauge connection. The coupling of the theory is the level of the WZW part of the action which we denote by the integer \( k \).

At the classical level, the deformed WZW model has a degenerate vacuum which one can identify with constant elements \( \gamma_v \in H \) modulo gauge transformations: \( \gamma_v \sim U \gamma_v U^{-1}, U \in H \). In other words, there is a classical vacuum moduli space that is the Cartan torus of \( H = U(n) \). However, the putative massless fluctuations in field directions in the moduli space turn out to have singular kinetic terms.

As an example, we can consider the case of \( \mathbb{C}P^2 \) discussed in detail in [36]. In this case \( H = U(1) \) is abelian and we can use both vector or axial gauging to achieve a Lagrangian formulation. For present purposes we discuss the vector gauged model which generalizes to the non-abelian cases. We can gauge fix the vector symmetry by choosing a gauge slice of the form

\[
\gamma = \begin{pmatrix}
e^{i\psi/2} & 0 & 0 \\
0 & \cos \theta e^{i(\phi+\psi/2)} & \sin \theta e^{-i\psi/2} \\
0 & -\sin \theta e^{i\psi/2} & \cos \theta e^{-i(\phi+\psi)}
\end{pmatrix}.
\] (2.23)

We then solve the conditions (2.18) for \( A_\mu \) and then insert these into the Lagrangian to give an effective Lagrangian for the physical degrees-of-freedom

\[
\mathcal{L} = \partial_\mu \theta \partial^\mu \theta + \frac{1}{4} \partial_\mu \psi \partial^\mu \psi + \cot^2 \theta \partial_\mu (\psi + \varphi) \partial^\mu (\psi + \varphi) + 2 \cos \theta \cos \varphi.
\] (2.24)

Notice that the vacuum is degenerate with \( \theta = \varphi = 0 \) and \( 0 \leq \psi < 4\pi \), but note that the kinetic term for \( \psi \) is singular due to the \( \cot \theta \) pre-factor. At this stage we can only conclude that a conventional approach to quantization via perturbation theory is likely to be unconventional [26,38,39].

3. The Classical Magnons/Solitons

The non-relativistic system consisting of the original \( F/G \) sigma model subject to the Pohlmeyer constraints has lump-like solutions known as giant magnons. These solutions have a relativistic soliton “avatar” that satisfies the SSSG equations: these are solitons in the form of kinks carrying topological charges of the deformed WZW theory.

The map between the magnons and solitons is complicated. However, in [36] it was shown how the dressing method, applied to magnons in [40], can be used to
construct both the magnons and their soliton avatars simultaneously without the need to map one into the other. Here, we briefly review the construction for the $\mathbb{C}P^{n+1}$ in order to describe the solitons and, importantly, to reveal their internal structure in some detail.

The dressing transformation method makes use of the associated linear system
\[
\partial_\pm \Psi(x; \lambda) = \frac{\partial_\pm \mathcal{F} \mathcal{F}^{-1}}{1 \pm \lambda} \Psi(x; \lambda), \quad \Psi(x; \infty) = I, \quad \mathcal{F}(x) = \Psi(x; 0),
\]
whose integrability conditions are equivalent to the equations of motion of the sigma model. For $\mathbb{C}P^{n+1}$, the solutions $\Psi(x; \lambda)$ have to satisfy the two conditions
\[
\Psi^{-1}(x; \lambda) = \Psi^\dagger(x; \lambda^*) , \quad \Psi(x; 1/\lambda) = \mathcal{F} \theta \Psi(x; \lambda) \theta ,
\]
which ensure that $\mathcal{F}^{-1} = \mathcal{F}^\dagger$ and that the constraint (2.2) is satisfied. Then, the dressing transformation involves constructing a new solution $\Psi$ of the linear system of the form
\[
\Psi(x; \lambda) = \chi(x; \lambda) \Psi_0(x; \lambda)
\]
in terms of an old one $\Psi_0(x; \lambda)$, which can be chosen to be the “vacuum” solution
\[
\Psi_0(x; \lambda) = \exp \left[ \left( \frac{x_+}{1 + \lambda} + \frac{x_-}{1 - \lambda} \right) \Lambda \right].
\]
In terms of the homogeneous coordinates, it corresponds to $\mathbf{Z}_0 = (\cos t, -\sin t, \mathbf{0})$. This vacuum solution, on the sigma model side, represents the motion of a point-like string on the target space $\mathbb{C}P^{n+1}$ at the speed of light.

Following [41], the general form of the “dressing factor” is
\[
\chi(\lambda) = 1 + \sum_i \frac{Q_i}{\lambda - \lambda_i}, \quad \chi^{-1}(\lambda) = 1 + \sum_i \frac{R_i}{\lambda - \mu_i},
\]
where the residues are rank-1 matrices of the form
\[
Q_i = X_i F_i^\dagger, \quad R_i = H_i K_i^\dagger
\]
for vectors $X_i$, $F_i$, $H_i$, and $K_i$. For $\mathbb{C}P^{n+1}$, they are given by
\[
X_i \Gamma_{ij} = H_j , \quad K_i (\Gamma^\dagger)_{ij} = -F_j , \quad \Gamma_{ij} = \frac{F_i^\dagger H_j}{\lambda_i - \mu_j},
\]
\[
F_i = \Psi_0(\lambda_i^*) \varpi_i , \quad H_i = \Psi_0(\mu_i) \pi_i ,
\]
where $\varpi_i$ and $\pi_i$ are complex constant $n+2$ dimensional vectors. The allowed number of poles and their positions are constrained by the conditions (3.2). They imply that $\mu_i = \lambda_i^*$ and, moreover, that the poles $\{\lambda_i\}$ must come in pairs $(\lambda_i, \lambda_{i+1} = 1/\lambda_i)$. In addition, $\pi_i = \varpi_i$ and, for each pair,

$$\varpi_{i+1} = \theta \varpi_i .$$

One the main results of [36] is that the dressing transformation not only produces the giant magnons but also directly the soliton “avatars” of the related SSSG in the form

$$\gamma = F_0^{-1/2} \chi (+1)^{-1} \chi (-1) F_0^{1/2} ,$$

with $A_+^{(L)} = A_-^{(R)} = 0$. This expression automatically satisfies the constraints (2.18)

$$[\gamma^{-1} \partial_+ \gamma]_b = [\partial_- \gamma \gamma^{-1}]_b = 0 .$$

The basic soliton for the $\mathbb{C}P^{n+1}$ case is obtained by taking a solution with a single pair of poles $\{\xi, 1/\xi\}$ where we parametrize $\xi = re^{ip/2}$. The “dressing data” involves the complex $n+2$ vector $\varpi$, with $\varpi_1 = \varpi$ and $\varpi_2 = \theta \varpi$. The various choices that can be made are discussed at length in [8]. The basic magnon, or its soliton avatar, is obtained by taking

$$\varpi = (1, i, \Omega) ,$$

where $\Omega$ is a complex $n$-dimensional vector subject to $|\Omega| = 1$. The magnons, or solitons, only depend on $\Omega$ up to a phase and so the lump has an internal collective coordinate valued in $\mathbb{C}P^{n-1}$.

The data $\{\xi = re^{ip/2}, \Omega\}$ (where we will implicitly identify $\Omega \sim e^{i\alpha} \Omega$) determines the rapidity and the charges of the magnon and its soliton avatar. The rapidity of the magnon and of the soliton are of course equal—they are the same object viewed from two different perspectives—and is determined by

$$\tanh \vartheta = \frac{2r}{1 + r^2} \cos \frac{p}{2} .$$

The $SU(n+2)$ Noether charge of the magnon is

$$\Delta Q = J_\Lambda \Lambda + J_H h \Omega , \quad J_\Lambda = -\frac{1+r^2}{r} \left| \sin \frac{p}{2} \right| , \quad J_H = -\frac{1-r^2}{r} \left| \sin \frac{p}{2} \right| ,$$

The charge $\mathcal{Q}$ is defined relative to the vacuum $\Delta Q = Q - Q_0$. 

\footnote{The charge $\mathcal{Q}$ is defined relative to the vacuum $\Delta Q = Q - Q_0$.}
where

$$h_\Omega = i \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -2\Omega \Omega^\dagger \end{pmatrix}$$  \hspace{1cm} (3.14)$$

is one of the infinitesimal generators of $H = U(n)$, which is the subgroup of elements of $G = U(n + 1)$ that commute with $\Lambda$. These charges satisfy the relation

$$-J_\Lambda = \sqrt{J_H^2 + 4 \sin^2 \frac{p}{2}}.$$  \hspace{1cm} (3.15)$$

In the AdS/CFT context [4,5], $J_\Lambda$ and $J_H$ are identified, up to scaling, with $\Delta - \frac{1}{2} J$ and $Q$, respectively, where $\Delta$ is the scaling dimension of the associated operator in the CFT, and $J$ and $Q$ are two conserved $U(1)$ R-charges:

$$\Delta - \frac{1}{2} J = -\sqrt{\frac{\lambda}{2}} J_\Lambda, \quad \frac{1}{2} Q = \sqrt{\frac{\lambda}{2}} J_H,$$  \hspace{1cm} (3.16)$$

where $\lambda$ is the 't Hooft coupling. Then, (3.15) becomes the celebrated (non-relativistic) dispersion relation

$$\Delta - \frac{1}{2} J = \sqrt{\frac{1}{4} Q^2 + 2\lambda \sin^2 \frac{p}{2}}.$$  \hspace{1cm} (3.17)$$

In the SSSG theory, the soliton is a relativistic kink with a topological charge

$$\gamma(-\infty)^{-1} \gamma(\infty) = \exp(-2q h_\Omega),$$  \hspace{1cm} (3.18)$$

where $h_\Omega$ is the Lie algebra element (3.14) and

$$\tan q = \frac{2r}{1 - r^2 \sin \frac{p}{2}}.$$  \hspace{1cm} (3.19)$$

The mass of the soliton is then

$$m = \frac{2k\mu}{\pi} |\sin q|.$$  \hspace{1cm} (3.20)$$

The inequivalent solutions are obtained by restricting $0 \leq p \leq 2\pi$. The charge $q$ can then be chosen to lie $-\frac{\pi}{2} \leq q \leq \frac{\pi}{2}$ and solitons with $r < 1$ have charge $0 \leq q \leq \frac{\pi}{2}$ and those with $r > 1$ have charge $-\frac{\pi}{2} \leq q \leq 0$. There is a notion of charge conjugation that takes $r \to 1/r$, or $\xi \to 1/\xi^*$, and $q \to -q$. Notice that $q$ is only defined modulo

---

6To be specific, we use the same normalization as [9].

7If we define $X^\pm = r e^{\pm ip/2}$, which are convenient variables from the magnon side, and $Z^\pm = e^{\theta \pm i q}$, which are convenient variables from the soliton side, then $X^\pm = (Z^\pm - 1)/(Z^\pm + 1)$ and $Z^\pm = (1 + X^\pm)/(1 - X^\pm)$. 
Therefore, \( q = \frac{\pi}{2} \) and \( q = -\frac{\pi}{2} \) actually correspond to the same solutions, which are those obtained with \( r = 1 \).

The internal collective coordinate \( \Omega \) plays a different role for magnons and solitons. For the former, there is an \( H \subset F \) Noether symmetry under which \( \Omega \) transforms as a vector. One can think of \( \Omega \) as a kind of angular momentum and in the quantum theory one finds that states will come in representations of \( H \). For the solitons, \( \Omega \) encodes the topological kink charge rather than a Noether charge. Classically the perturbed gauged WZW model has a vacuum which we can identify with a constant element \( \gamma_v \in H \) modulo gauge transformations: \( \gamma_v \sim U \gamma_v U^{-1}, U \in H \); in other words, there is a vacuum moduli space that is the Cartan torus of \( H = U(n) \). Consequently, the topological charge should be thought of as taking values in a Cartan subalgebra of \( H \). In the quantum theory, we will find that the classical moduli space \( \mathbb{C}P^{n-1} \) becomes a “fuzzy” space with non-commuting coordinates and once again the quantum states form representations of \( H \), or more precisely its \( q \) deformation the quantum group \( U_q(H) \), where \( q \) is determined by \( k \).

For multi-soliton solutions it is not generally possible to use gauge transformations to take the topological charge of each soliton into the same Cartan subalgebra. From this perspective, the dependence on \( \Omega \) simply corresponds to the freedom to choose the Cartan subalgebra, and the scattering amplitudes will depend on \( \Omega \). The situation simplifies for the special choices \( \Omega^{(i)}_a = \delta_{ia}, \) for \( i = 1, \ldots, n \), or in vector language \( \Omega^{(i)} = e_i \), such that the corresponding solitons carry topological charges laying on the same Cartan subalgebra. These special solutions play an important role later in section 5.2 because it is particularly simple to relate their classical scattering to the semi-classical limit of the quantum S-matrix.

4. The Soliton S-matrix Conjecture

Finding the S-matrix of an integrable field theory is never a direct process: one has to use a variety of evidence in order to pin it down. In an ideal world one would like to quantize the perturbed gauge WZW model from first principles but this is not something that can be done with present understanding. Fortunately, there are plenty of clues and many other examples to guide us. Firstly, integrable deformations of WZW theories typically lead to S-matrices describing a system of kinks. For example, there are integrable deformations of WZW models associated to the symmetric spaces \( G/H \) [42] (unlike the present situation where \( G/H \) is not a symmetric space). The deformation is these cases is provided by the operator \( \sum_a J_a \bar{J}_a \), with a sum over the components of the currents in \( G \) but not in \( H \). Then, the spectrum
consists of a set of kinks which interpolate a finite set of vacua associated to the irreducible representations of the symmetry group $G$ of level $\leq k$, where $k$ is the level of the WZW model. In this case the kinks have topological charges which are weights of the fundamental (anti-symmetric) representations of $G$, and the S-matrix elements involve the trigonometric solution of the Yang-Baxter equation associated to the quantum group $U_q(G)$ with a deformation parameter $q = -\exp(i\pi/(k + h))$, where $h$ is the dual Coxeter number of $G$. The fact that the S-matrix involves kinks seems to be related at a fundamental level to a basis of quasi-particles known as spinons in the original coset CFT [43–45].

We will also find that the S-matrix of the symmetric space sine-Gordon theories described as deformations of the $G/H$ WZW model are kinks which interpolate a set of vacua which are associated to the representations of $H = U(n)$ of level $\leq k$. Notice that in this case the symmetry group is $H$ rather than $G$ since the latter symmetry is broken by the deforming potential. The S-matrix will also involve the trigonometric solution of the Yang-Baxter equation associated to $U_q(H)$. The main difference is that the spectrum will involve the symmetric representations rather than the anti-symmetric ones. It seems natural that these theories should have a quantum group symmetry much like the sine-Gordon theory itself whose kinks have an $U_q(SU(2))$ symmetry [46]. For generic $q$ the representations of $U_q(H)$ are simply deformations of those of $H$, however, in the present case $q$ is a root of unity and this means that the set of representations is restricted in a way that is crucial to the construction of the S-matrix. S-matrices associated to trigonometric solutions of the Yang-Baxter equation have been considered in the past [46–51] the main difference with the present case is that those S-matrices involved the anti-symmetric representations.

For $q$ a root of unity, it is most appropriate to use the restricted-solid-on-solid (RSOS) picture for which the states of the theory are kinks. The kinks interpolate between a discrete set of vacuum states which are identified with the irreducible representations of $SU(n)$ of level $\leq k$, which we denote $\Lambda^*(k)$. Concretely these are associated to Young Tableaux whose width is restricted to be $\leq k$, or the set of vectors $\sum_{i=1}^{n-1} a_i e_i$, with

$$k \geq a_1 \geq a_2 \geq \cdots \geq a_{n-1} \geq 0,$$

where the $e_i$'s provide the set of weights of the vector representation of $SU(n)$ (see [46]). A kink with rapidity $\vartheta$ is then denoted $K_{ab}(\vartheta)$ for $a, b \in \Lambda^*(k)$. The topological charges of a kink $a - b$ are weights associated to the Cartan elements of $U_q(SU(n))$. Note that these elements will commute with the S-matrix and the topological charge is conserved. Notice also that the set of vacua describe a kind of

\footnote{The overall $U(1)$ subgroup of $H$ is trivially represented on the kinks.}
discretization of the Cartan torus of $H$, the classical vacuum moduli space, that is recovered in the limit $k \to \infty$.

In an integrable field theory the complete S-matrix is then determined by the S-matrix for the $2 \to 2$ processes

$$K_{ac}(\vartheta_1) + K_{cd}(\vartheta_2) \to K_{ab}(\vartheta_2) + K_{bd}(\vartheta_1).$$

We will find that the topological charge $a - b$ of a kink $K_{ab}(\vartheta)$ have to be weights of one of the symmetric representations of $SU(n)$ with Young Tableau $[a]$, or their conjugates $[a, \ldots, a] = [a^{n-1}]$. With $q$ a root of unity, $q = -\exp(i\pi/(n+k))$, the quantum group restriction means that $a = 1, \ldots, k$ only. We denote the set of weights in the representation $[a]$ and $[a^{n-1}]$ as $\Sigma_{[a]}$ and $\Sigma_{[a^{n-1}]}$, respectively. We will identify the overall $U(1)$ kink charge (not to be confused with the quantum group deformation parameter) as equal to $q = \pm \pi a/N$, for kinks and anti-kinks, respectively, for an integer $N$ that will be identified as we proceed. The mass of a kink with a topological charge in $\Sigma_{[a]}$ or $\Sigma_{[a^{n-1}]}$ follows from the classical formula

$$m_a = M \sin \left( \frac{\pi a}{N} \right), \quad a = 1, 2, \ldots, k,$$

where $M$ is an overall renormalized mass scale.

The S-matrix elements are constructed from the trigonometric solutions of the Yang-Baxter equation associated to a certain deformation of the universal enveloping algebra of the Lie algebra known as a quantum group [52,53]: in the present context $U_q(SU(n))$. The solutions can be thought of as intertwiners between tensor products of representations of the algebra:

$$\hat{R}(\vartheta) : U(\vartheta_1) \otimes V(\vartheta_2) \to V(\vartheta_2) \otimes U(\vartheta_1),$$

where $\vartheta = \vartheta_1 - \vartheta_2$ is the (additive) spectral parameter which we will later identify with the rapidity. Such an $R$-matrix has a spectral decomposition [52]

$$\hat{R}(\vartheta) = \sum_{W \subset U \otimes V} \rho_{\lambda}(\vartheta) \mathbb{P}_W,$$

where $\mathbb{P}_W$ is a quantum group invariant homomorphism from $U \otimes V$ to $V \otimes U$ with the property that $\sigma \mathbb{P}_W$ is a projection onto $W \subset U \otimes V$, where $\sigma : v \otimes u \mapsto u \otimes v$, for $u \in U$ and $v \in V$, is the permutation. It is important that, in the context of the

---

9We use the label $[a_1, a_2, \ldots, a_{n-1}]$ for the representation of $SU(n)$ with highest weight $\sum_i a_i e_i$. In a Young Tableaux $a_i$ gives the number of boxes in $i^{th}$ row and $a_1 \geq a_2 \geq \cdots \geq a_{n-1} \geq 0$. The quantum group further restricts $k \geq a_1$. 

---
quantum group, the tensor product is a subset of the tensor product of the group. In the following we shall switch between the language of spectral decompositions and the RSOS picture where necessary.

The basic S-matrix elements

To start with we consider the solutions associated to the vector representation of the algebra $\tilde{R}_{11}(\vartheta)$. The set of weights of the vector representation are

$$
\Sigma_{[1]} = \{e_1, \ldots, e_n\},
$$

(4.6)

where the $e_i$'s are a set of vectors with $e_i \cdot e_j = \delta_{ij} - 1/n$ in an $n-1$-dimensional space, and $\sum_{i=1}^n e_i = 0$.

The solution of the YBE is labelled by four weights of the algebra:

$$
\tilde{R}_{11}\left(\begin{array}{c} a \ b \\ c \ d \end{array}\vartheta\right), \quad a, b, c, d \in \Lambda^*(k),
$$

(4.7)

with the property that $\tilde{R}_{11}$ is only non-zero if $c-a, d-c, b-a$ and $d-b$ are in $\Sigma_{[1]}$. For completeness we now write down the explicit solutions following [54] (see also the review [55]). In the following $\omega$ is a constant which is related to the deformation parameter of the quantum group $q = -e^{i\omega}$ and so $\omega = \frac{\pi}{n+k}$. For convenience we introduce for $a \in \Lambda^*(k)$ and $\mu, \nu \in \Sigma_{[1]}$

$$
a_\mu = \omega (a + \rho) \cdot \mu, \quad a_{\mu \nu} = a_\mu - a_\nu,
$$

(4.8)

where $\rho$ is the sum of the fundamental weights of the algebra.$^{10}$

With a suitable choice of overall normalization, the solution is

$$
\tilde{R}_{11}\left(\begin{array}{c} a \\ a + e_i \end{array} \vartheta\right) = 1,
$$

$$
\tilde{R}_{11}\left(\begin{array}{c} a \\ a + e_i, a + 2e_i \end{array} \vartheta\right) = \frac{\sin(a_{e_i e_j} + i\lambda \vartheta) \sin \omega}{\sin(a_{e_i e_j}) \sin(\omega - i\lambda \vartheta)} ,
$$

$$
\tilde{R}_{11}\left(\begin{array}{c} a \\ a + e_i, a + e_i + e_j \end{array} \vartheta\right) = \frac{\sin(i\lambda \vartheta)}{\sin(\omega - i\lambda \vartheta)} \left(\frac{\sin(a_{e_i e_j} + \omega) \sin(a_{e_i e_j} - \omega)}{\sin^2(a_{e_i e_j})}\right)^{1/2}.
$$

(4.9)

The solution satisfies the unitarity condition

$$
\sum_e \tilde{R}_{11}\left(\begin{array}{c} a \ e \\ c \ d \end{array} \vartheta\right) \tilde{R}_{11}\left(\begin{array}{c} a \ b \\ e \ d \end{array} \vartheta\right) = \delta_{bc}.
$$

(4.10)

$^{10}$These are the vectors $\omega_i = e_1 + \cdots + e_i$, for $i = 1, \ldots, n-1$. 
The solution of the YBE equation written above naturally leads to an $S$-matrix for the two [1] kink process once multiplied by a suitable scalar factor,

$$S_{11} \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) = X_{11}(\vartheta)Y_{11}(\vartheta)\tilde{R}_{11} \left( \begin{array}{cc} a & b \\ c & d \end{array} \right).$$

(4.11)

The fact that we split the scalar factor into 2 pieces $X_{11}(\vartheta)$ and $Y_{11}(\vartheta)$ is for convenience: the first factor will contain all the bound-state poles on the physical strip while the second is needed to satisfy unitarity and crossing. Both factors must be chosen so that the $S$-matrix axioms are satisfied and the semi-classical limit of the $S$-matrix is consistent. For instance, assuming that $X_{ab}(\vartheta)$ satisfies unitarity and crossing separately, we see from (4.10) that unitarity requires

$$Y_{11}(\vartheta)Y_{11}(-\vartheta) = 1.$$  

(4.12)

**The bootstrap**

We now describe how to build up the full $S$-matrix from this basic one by applying the bootstrap. The idea is that simple poles on the physical sheet in rapidity space, $0 \leq \text{Im} \, \vartheta \leq \pi$, are interpreted as the exchange of a bound-state in either the direct or crossed channel. If we look at the spectral decomposition of the basic $R$-matrix then

$$\tilde{R}_{11}(\vartheta) = \mathbb{P}_{[2]} - \frac{\sinh(\lambda\vartheta - i\omega)}{\sinh(\lambda\vartheta + i\omega)}\mathbb{P}_{[1^2]},$$

(4.13)

where $\mathbb{P}_{[2]}$ and $\mathbb{P}_{[1^2]}$ are the quantum group invariant projectors which appear in the tensor product of two vector representations. The idea of the bootstrap is that kinks with topological charges in $\Sigma_{[2]}$ will appear as a bound-state provided $X_{11}(\vartheta)$ has a simple pole at a rapidity difference that is fixed by the formula

$$m_2^2 = m_1^2 + m_1^2 + 2m_1^2 \cos \vartheta,$$

(4.14)

giving $\vartheta = 2i\pi/N = i\omega/\lambda$. This fixes

$$\lambda = \frac{N}{2(k + n)},$$

(4.15)

so that the residue of the pole is proportional to $\mathbb{P}_{[2]}$. This condition on $X_{11}(\vartheta)$ is not enough to complete fix it. This kind of situation is common in an integrable field theory, $S$-matrix can often only be determined up to "CDD factors", that is functions which are analytic on the physical strip. In the present case, we will simply postulate...
a form for $X_{11}(\vartheta)$ which is consistent with the semi-classical limit that we discuss later: \(^{11}\)

$$X_{11}(\vartheta) = \frac{\sinh(\frac{\lambda \vartheta}{2} + i\omega) \cosh(\frac{\lambda \vartheta}{2} + i\omega)}{\sinh(\frac{\lambda \vartheta}{2} - i\omega) \cosh(\frac{\lambda \vartheta}{2} - i\omega)}. \quad (4.16)$$

The first quotient here is strictly-speaking the minimum that is necessary since it provides the simple pole. The second factor is a CDD factor that we will later find is necessary to produce the correct semi-classical limit. We cannot rule out further CDD factors that have a trivial semi-classical limit.

The S-matrix elements for the bound-state kinks with the fundamental kinks then follows from the bootstrap equations

$$S_{21}(\vartheta) = S_{11}(\vartheta + \frac{i\pi}{N})S_{11}(\vartheta - \frac{i\pi}{N}). \quad (4.17)$$

where the right-hand side is implicitly restricted to $[2] \times [1]$ in the tensor product $[1] \times [1] \times [1]$.

The bootstrap then proceeds in a similar fashion to generate all the particles with $a = 1, \ldots, k$ transforming in representations $[a]$ with S-matrix elements

$$S_{ab}(\vartheta) = X_{ab}(\vartheta)Y_{ab}(\vartheta)\tilde{R}_{ab}(\vartheta), \quad (4.18)$$

where $\tilde{R}_{ab}(\vartheta)$ is the RSOS solution of the Yang-Baxter equation for the product of representations $[a] \times [b]$. The tensor product $[a] \times [b]$ in the quantum group is only a subset of the tensor product in the group itself:

$$[a] \times [b] = \bigoplus_{j=\max(0,a+b-k)}^{\min(a,b)} [a+b-j,j]. \quad (4.19)$$

The lower limit in here involves the level $k$ and is a consequence of the quantum group structure at $q$ a root of unity. The masses of the kinks determine that $S_{ab}(\vartheta)$ should have a bound state pole at $\vartheta = i\pi(a+b)/N$ corresponding to kinks with topological charge in $[a+b]$. We must now verify that $S_{ab}(\vartheta)$ has this pole and also that the residue is proportional to $P_{[a+b]}$.

The bootstrap equations in general takes the form

$$S_{a+b,c}(\vartheta) = S_{ac}(\vartheta + \frac{i\pi}{N})S_{bc}(\vartheta - \frac{i\pi}{N}). \quad (4.20)$$

\(^{11}\)In principle, the semi-classical limit and bootstrap allows for a more general expression where the arguments of the hyperbolic functions are scaled by some function which $\to 1$ as $k \to \infty$. For example the scaling could be by $\lambda^{-1}$. However, such a choice would differ only by CDD factors from the one we chose. Such ambiguities would be pinned down by a TBA calculation of the central charge.
where the right-hand side is implicitly restricted to \([a + b] \times [c]\) in the tensor product \([a] \times [b] \times [c]\). Applying the bootstrap equation recursively to the scalar factor gives

\[
X_{ab}(\vartheta) = \prod_{j=0}^{a-1} \prod_{l=0}^{b-1} X_{11}(\vartheta + \frac{i\pi(a + b - 2j - 2l - 2)}{N})
\]

\[
= \prod_{j=1}^{\min(a,b)} \frac{(a + b - 2j + 2)_{\vartheta}(a + b - 2j)_{\vartheta}(a + b - 2j + 1)_{\vartheta}}{(2j - a - b - 2)_{\vartheta}(2j - a - b)_{\vartheta}(2j - a - 1)_{\vartheta}},
\]

where we have defined for later use

\[
(x)_{\vartheta} = \sinh\left(\frac{\lambda \vartheta}{2} + \frac{i\omega x}{4}\right), \quad [x]_{\vartheta} = \cosh\left(\frac{\lambda \vartheta}{2} + \frac{i\omega x}{4}\right).
\]

Notice that \(X_{ab}(\vartheta)\) does have a simple pole at \(\vartheta = i\pi(a + b)/N\) as required. It also has a simple pole at \(i\pi|a - b|/N\) whose significance will emerge, and also double poles at \(i\pi(a - b - 2j)/N, j = 1, \ldots, \min(a, b) - 1\).

Now we turn to the \(R\)-matrix for \([a] \times [b]\) which has a spectral decomposition to match (4.19) \(12\)

\[
\tilde{R}_{ab}(\vartheta) = \sum_{j=\max(0,a+b-k)}^{\min(a,b)} (-1)^j \rho^j_{ab}(\vartheta) \mathbb{P}_{[a+b-j,j]}
\]

with

\[
\rho^j_{ab}(\vartheta) = \prod_{l=0}^{j-1} \frac{\sinh(\lambda \vartheta - i\omega(a + b - 2l)/2)}{\sinh(\lambda \vartheta + i\omega(a + b - 2l)/2)}.
\]

Although the quantum group structure fixes the spectral decomposition of the \(R\)-matrix, it does not determine the overall normalization which we have in hindsight fixed by setting \(\rho^0_{ab}(\vartheta) = 1\). This must be fixed by solving the bootstrap equation. Fortunately in the present context, since we are dealing with symmetric representations, the normalization is easy to fix by the following simple argument. The two kinks with topological charge \(ae_i\) and \(be_j\) can only couple through the projector \(\mathbb{P}_{[a+b]}\) because \((a+b)e_i\) can only be in \(\Sigma_{[a+b]}\). Moreover, the basic \(\tilde{R}_{11}\)-matrix factor for the constituents \(e_i\) and \(e_j\) is from (4.13) unity and so applying the bootstrap equation to these special states only we see that the \(R\) matrix element for \(ae_i\) with \(be_j\) must also be unity and so, as we claimed, \(\rho^0_{ab}(\vartheta) = 1\).

Now that we have fixed the \(R\)-matrix, we can easily verify that at the simple pole \(\vartheta = i\pi(a + b)/N\) we have

\[
\tilde{R}_{ab}(\vartheta) = \begin{cases} \mathbb{P}_{[a+b]} & a + b \leq k \\ 0 & a + b > k \end{cases},
\]

\(12\)These decompositions follow from the general technology involving the tensor product graph described in [56]; see also Appendix A of [51].
due to the factor \( \sinh(\lambda \vartheta - i\omega(a + b)/2) \) in the numerator of (4.24).

From (4.21), we see that there is another simple pole at \( \vartheta = i\pi|a - b|/N \) which must be properly accounted for. This will be identified with a bound state in the crossed channel. A consistent \( S \)-matrix must satisfy crossing symmetry which requires that each kink \([a]\) has a charge conjugate anti-kink with minus the topological charge, of the same mass, and transforming in the conjugate representation with Young Tableau \([a^*-1]\). The crossing symmetry relation then gives the \( S \)-matrix elements for anti-kink/kink scattering as

\[
S_{ba} \left( \begin{array}{c|c|c}
  a & b \\
  c & d \\
\end{array} \right) \vartheta = S_{ab} \left( \begin{array}{c|c|c}
  c & a \\
  d & b \\
\end{array} \right) \left( i\pi - \vartheta \right),
\]

(4.26)

Notice here that on the right-hand side the topological charges \( b - d \) and \( a - c \in \Sigma(b) \) whereas on the left-hand side \( d - b \) and \( c - a \in \Sigma(b^{-1}) \). The cross-channel pole in \( S_{ab}(\vartheta) \) at \( \vartheta = i\pi|a - b|/N \) is then interpreted as a direct channel pole at \( \vartheta = i\pi - i\pi(a - b)/N \) for \([b^{-1}] \otimes [a]\) scattering. If \( a > b \) these must be kinks in representation \([a - b] \subset [b^{-1}] \times [a]\) which appears in the tensor product, while if \( a < b \) they are anti-kinks \([b - a^{-1}] \subset [b^{-1}] \times [a]\). For overall consistency, we require that the \( R \)-matrix for \([b^{-1}] \otimes [a]\), which we denote \( \hat{R}_{ba}(\vartheta) \), must be proportional to the projector \( P_{[a-b]} \), if \( a > b \), and \( P_{[(b-a)^{-1}]} \), if \( a < b \). The spectral decompositions are, firstly for \( a > b \),

\[
R_{ba}(\vartheta) = \Phi_{ba}(\vartheta) \sum_{j=0}^{b} (-1)^j \rho_{ba}^j (\vartheta) P_{[a-b+2j+n-2]},
\]

(4.27)

with

\[
\rho_{ba}^j (\vartheta) = \prod_{l=0}^{j-1} \frac{\sinh(\lambda \vartheta + i\omega(n + a - b + 2l)/2)}{\sinh(\lambda \vartheta - i\omega(n + a - b + 2l)/2)}.
\]

(4.28)

In the above \( \Phi_{ba}(\vartheta) \) is a scalar function which we will not need to specify for the following argument. Notice that, as required, \( \rho_{ba}^j (\vartheta) = 0 \) for \( j \neq 0 \) when \( \vartheta = i\pi - i\pi(a - b)/N \) due to the factor with \( l = 0 \) in the numerator of (4.28) as long we fix

\[
N = n + 2k.
\]

(4.29)

This is an interesting result because it is consistent with intuition from a completely different viewpoint. If we go back to the Lagrangian formulation of the SSSG equations it is possible to proceed in an alternative way by treating the abelian subgroup of \( H = U(n) \) differently form the non-abelian part. For the latter, we can only gauge the vector subgroup of \( H_L \times H_R \). However, for the \( U(1) \) part we can choose to gauge
the axial or the vector subgroup. This gives a different formulation of the SSSG theory which is related by T-duality to the “usual” formulation \[37\]. It is thought that T duality is an exact quantum equivalence between theories. What is interesting is that in this alternative theory there is a genuine $U(1)$ symmetry which is not broken by the vacuum corresponding to abelian vector transformations $\gamma \rightarrow U\gamma U^{-1}$, $U \in U(1)$. In this formulation the soliton charge $q$ is a genuine Noether charge and we may apply the Bohr-Sommerfeld quantization rule. This gives the condition that $q = \pi a/2k$, for $a \in \mathbb{Z}$. If T-duality is indeed an exact equivalence then this quantization of the charge $q$ is consistent with the semi-classical limit of $q = \pi a/N$ with $N = n + 2k$.

Returning the kink/anti-kink S-matrix, we can repeat the analysis with $a < b$, for which

$$R_{ba}(\vartheta) = \Phi_{ba}(\vartheta) \sum_{j=0}^{a} (-1)^j \rho_{ba}^j(\vartheta) \prod_{b-a+2j, (b-a+j) \neq 0} \sinh((n+2k)\pi t) \sinh((n+k)\pi t)$$ \[4.30\]

with

$$\rho_{ba}^j(\vartheta) = \prod_{l=0}^{j-1} \frac{\sinh(\lambda \vartheta + i\omega (n + b - a + 2l)/2)}{\sinh(\lambda \vartheta - i\omega (n + b - a + 2l)/2)}.$$ \[4.31\]

Once again, as required, $\rho_{ba}^j(\vartheta) = 0$ for $j \neq 0$ when $\vartheta = i\pi - i\pi (b - a)/N$ due to the factor with $l = 0$ in the numerator of \[4.31\].

Crossing leads to a non-trivial equation for the scalar factor $Y_{11}(\vartheta)$ which can be viewed as the unitarity constraint for $S_{ba}(\vartheta)$. It can be shown \[51\] that this leads to the requirement

$$Y_{11}(i\pi - \vartheta)Y_{11}(i\pi + \vartheta) = \frac{\sin(\omega + \pi \lambda - i\lambda \vartheta) \sin(\omega + \pi \lambda + i\lambda \vartheta)}{\sin(\pi \lambda - i\lambda \vartheta) \sin(\pi \lambda + i\lambda \vartheta)}.$$ \[4.32\]

The “minimal” solution—having no poles or zeros on the physical strip—can be written most succintly as a integral \[51\],

$$Y_{11}(\vartheta) = \exp \left[ 2i \int_0^\infty \frac{dt}{t} \sin((n + 2k)\vartheta t) \sinh((n + k)\pi t) \sinh((n + 2k)\pi t) \right].$$ \[4.33\]

The fusing rules

The fusing rules summarize the direct channel bound states. In the present
theory they are

\[ [a] \circ [b] = \begin{cases} [a + b] & a + b \leq k \\ 0 & a + b > k \end{cases} \]  

(4.34)

\[ [a] \circ [b^{n-1}] = \begin{cases} [a - b] & a > b \\ [(b - a)^{n-1}] & a < b \end{cases} \]

These are a subset of the fusing rules of the minimal \( A^{(1)}_{N-1} \) S-matrix. We have shown that the simple poles in the S-matrix can all be accounted for as bound-state poles in the either the direct or crossed channels. Notice that the solution of the bootstrap is much simpler than the one considered in [51] for which the kinks transformed in the anti-symmetric representations. The reason being that the bootstrap for the present case does not “bite its own tail” because crossing symmetry is much easier to implement arising from the fact that for the anti-symmetric representations the anti-kinks arise as bound states of the kinks, and hence non-trivial consistency conditions arise, whereas for the symmetric representations they do not.

The S-matrix elements \([118]\) also have a series of double poles. These will be interpreted exactly as for the \( A_{N-1} \) minimal S-matrix in terms of anomalous thresholds via the Coleman-Thun mechanism [57].

The quantum group symmetry

The S-matrix that we have constructed has an underlying quantum group structure. In fact, the appropriate algebraic context is the quantum loop group \( U_q(SU(n)^{(1)}) \) with \( e^{\lambda \vartheta} \) playing the role of the loop variable. Just like the ordinary group, the quantum group \( U_q(SU(n)) \) can be generated by the Chevalley generators \( \{ e_i, f_i, h_i \} \), \( i = 1, \ldots, n-1 \) associated to the simple roots. The affine quantum group involves adding in the generators for the highest root \( e_0 \) and \( f_0 \) with powers of the loop variable. The action of these generators on the basic representations \([1]\) and \([1^{n-1}]\) is identical to the ordinary group. What distinguishes a quantum group is how the generators act on a tensor product. This describes the quantum group as a Hopf algebra with a co-product. In contrast to the ordinary group, on a tensor product \( V \times U \) there is a non-trivial co-product

\[ \Delta(h_i) = h_i \otimes 1 + 1 \otimes h_i , \]

\[ \Delta(e_i) = e_i \otimes q^{-h_i} + q^{h_i} \otimes e_i , \]

\[ \Delta(f_i) = f_i \otimes q^{-h_i} + q^{h_i} \otimes f_i . \]

(4.35)

The normal action is recovered in the limit \( q \to 1 \). The S-matrix is invariant under this action

\[ \Delta(a) S(\vartheta) = S(\vartheta) \Delta(a) . \]

(4.36)
In addition, the $S$-matrix is invariant under a rapidity-dependent symmetry which manifests the fact that it is actually invariant under the affine symmetry $U_q(SU(n)^{(1)})$. Let $(e_0, f_0)$ be the raising and lowering operators associated to the highest root and $h_0 = - \sum_{i=1}^{n-1} h_i$. The $S$-matrix also commutes with the action
\[
\left(e^{\lambda \vartheta_2} e_0 \otimes q^{-h_0/2} + e^{\lambda \vartheta_1} q^{h_0/2} \otimes e_0\right) S(\vartheta) = S(\vartheta) \left(e^{\lambda \vartheta_1} e_0 \otimes q^{-h_0/2} + e^{\lambda \vartheta_2} q^{h_0/2} \otimes e_0\right),
\] (4.37)
with a similar relation for $f_0$ with $e^{\lambda \vartheta_1,2} \rightarrow e^{-\lambda \vartheta_1,2}$.

Notice that the action of the Cartan generators of the quantum group is identical to the ordinary group and so the $S$-matrix has a conventional $U(1)^n$ symmetry which is interpreted as a conserved vector-valued topological charge.

5. The Semi-Classical Limit

The scattering of solitons (or magnons) in an integrable field theory has a very characteristic feature, the individual momenta, or rapidities, of the solitons are conserved, however, a given soliton can experience a rapidity-dependent time delay. The semi-classical limit relates this time delay directly to the phase of the $S$-matrix and this provides a very stringent test of the $S$-matrix hypothesis. In the present case, the semi-classical limit involves the level $k \rightarrow \infty$ and, in this limit, the phase shift $\delta$, defined by $S = e^{2i\delta}$, is related to the classical time-delay $\Delta t(E)$ for two soliton scattering via the WKB formula derived by Jackiw and Woo [58]
\[
\delta = \frac{n_B \pi}{2} + \frac{1}{2} \int_{E_{Th}}^{E} dE' \Delta t(E') ,
\] (5.1)
where $E = m_1 \cosh \vartheta_1 + m_2 \cosh \vartheta_2$ is the energy in the COM frame and $E_{Th}$ is the threshold energy $E_{Th} = m_1 + m_2$. The integer $n_B$ is the number of bound states below threshold which will be 0 in the present context. In this section we will use this formula to test our $S$-matrix hypothesis. Note that the leading term of $\delta$ in the semi-classical limit scales like $k$ with corrections of order $k^{-j}$, $j = 0,1,\ldots$. In particular, the constant term in (5.1) only plays a role at leading order if the number of bound states scales like $k$ which does not happen for the $S$-matrix in question.

5.1 The classical time delay

In order to calculate the time delay experienced by a soliton as it scatters with another soliton, we need to specify the soliton’s space-time position in terms of the
dressing data. The key quantity is
\[
\beta = \mathbf{F}^\dagger \mathbf{F} = \varpi^\dagger \Psi_0(\xi)^{-1} \Psi_0(\xi^*) \varpi ,
\] (5.2)
which for a soliton in isolation is
\[
\beta = 2e^{4F(x,t)} + 1
\] (5.3)
where
\[
F(x,t) = \frac{\mu \sin q}{2} \left(x \cosh \vartheta - t \sinh \vartheta\right).
\] (5.4)
The spacetime position of the soliton can be identified with the place where \( F(x,t) = -\frac{1}{4} \log 2 \), i.e. \( x = t \tanh \vartheta + \text{const.} \).

The dressing transformation makes it simple to extract the classical time delay experienced by a magnon/soliton as it scatters with another magnon/soliton. The idea is to focus on the spacetime position of soliton 2 and think of it as dressed by soliton 1. As for the soliton in isolation, the position of soliton 2 is encoded in the quantity
\[
\beta^{(2)} = \varpi^{(2)\dagger} \Psi^{(1)}(\xi_2)^{-1} \Psi^{(1)}(\xi_2^*) \varpi^{(2)}
\] (5.5)
where now we have the dressed quantity
\[
\Psi^{(1)}(\lambda) = \chi^{(1)}(\lambda) \Psi_0(\lambda).
\] (5.6)

In order to calculate the time delay, or spacetime shift, we then need to take the limits of \( \chi^{(1)}(\lambda) \) as \( x \to \pm \infty \). This follows from
\[
\chi(\lambda) \xrightarrow{x \to \infty} 1 + \frac{\xi - \xi^*}{2 \lambda - \xi} \begin{pmatrix}
1 & -i & 0 \\
+i & 1 & 0 \\
0 & 0 & 0
\end{pmatrix} + \frac{\xi^{-1} - \xi^{*-1}}{2 \lambda - \xi^{-1}} \begin{pmatrix}
1 & +i & 0 \\
-i & 1 & 0 \\
0 & 0 & 0
\end{pmatrix}
\] (5.7)
and
\[
\chi(\lambda) \xrightarrow{x \to -\infty} 1 + A^{-1} \left[ \frac{1}{\lambda - \xi} \left( \frac{\xi}{|\xi|^2 - 1} - \frac{|\xi|^2}{\xi - \xi^*} \right) \\
+ \frac{1}{\lambda - \xi^{-1}} \left( \frac{1}{\xi - \xi^*} - \frac{\xi^*}{|\xi|^2 - 1} \right) \right] \begin{pmatrix}
0 & 0 \\
0 & 0 & \Omega \Omega^\dagger
\end{pmatrix}
\] (5.8)
with
\[
A = \frac{|\xi|^2}{(|\xi|^2 - 1)^2} - \frac{|\xi|^2}{(\xi - \xi^*)^2}.
\] (5.9)
From these we deduce that soliton 2 has $F_2(x,t)$ shifted by

$$\Delta F = \log \left[ \frac{1}{\xi_1 - \xi_2} \left| \frac{1 - \xi_1 \xi_2}{1 - \xi_1^* \xi_2^*} \cos^2 \Theta + \frac{\xi_1 - \xi_2}{\xi_1 - \xi_2^*} \sin^2 \Theta \right|^2 \right], \quad (5.10)$$

where we have taken $|\Omega^{(2)*} \cdot \Omega^{(1)}| = \cos \Theta$. The corresponding time delay is

$$\Delta t = \frac{\Delta F}{\mu |\sin q_2| \sinh \vartheta_2}. \quad (5.11)$$

The result matches that in [59] (see also [60]). In the COM frame $|\sin q_1| \sinh \vartheta_1 = -|\sin q_2| \sinh \vartheta_2$ and so we can write the formula for the phase shift in a manifestly relativistic way as

$$\delta = \frac{n_B \pi}{2} + \frac{k}{\pi} \int_0^\vartheta d\vartheta' \Delta F(\vartheta') , \quad (5.12)$$

where $\Delta F$ in (5.10) can be written in terms of the rapidity difference

$$\Delta F(\vartheta) = \log \left[ \frac{\sinh(\vartheta - i\frac{q_1 - q_2}{2})}{\sinh(\vartheta - i\frac{q_1 + q_2}{2})} \right]^4 \left| \frac{\cosh(\vartheta - i\frac{q_1 - q_2}{2})}{\cosh(\vartheta - i\frac{q_1 + q_2}{2})} \right|^2 \cos^2 \Theta$$

$$+ \left| \frac{\sinh(\vartheta - i\frac{q_1 - q_2}{2})}{\sinh(\vartheta - i\frac{q_1 + q_2}{2})} \right|^2 \sin^2 \Theta \right]. \quad (5.13)$$

5.2 Taking the semi-classical limit of the S-matrix

In order to take the semi-classical limit of our S-matrix we need to specify carefully which particular quantum states can be discussed. The states which have a good semi-classical limit are those with a fixed charge $q$ as $k \to \infty$. This means that for states in representation $[a]$, $a$ must also $\to \infty$ with $a/k$ fixed. In other words the good semi-classical states are in large symmetric representations. Actually this is very natural, the classical solitons have an internal collective coordinate $\Omega$ valued in $\mathbb{C}P^{n-1}$, since the phase of $\Omega$ is irrelevant. To semi-classically quantize this degree-of-freedom, one lets the collective coordinates become time-dependent and plugs this into the action to yield an effective quantum-mechanical action. This action turns out to be first order in the time derivatives and the resulting quantization is not conventional. Rather, as we shall argue, the classical moduli space itself should be thought of as a symplectic manifold in its own right and quantized accordingly.

---

The consistency of the classical time delays with the AdS$_4$/CFT$_3$ magnon S-matrix conjectured by Ahn and Nepomechie [15] has been checked in [59].
The classical moduli space then emerges from the semi-classical limit of a “fuzzy” geometry.\footnote{This will be explained in more detail in the companion paper [61].}

This classical moduli space has a description in terms of an adjoint orbit of $SU(n)$ since we can rotate $\Omega$ by means of a global $H$ transformation on the soliton solution: $\gamma \to U\gamma U^{-1}$ implies $\Omega \to U\Omega$. If we describe the classical moduli space in terms of $h_{\Omega}$, the infinitesimal generator of $H = U(n) \subset SU(n+2)$ defined in 3.14, the adjoint orbit is then of the form $h_{\Omega} = U\text{diag}(1,1,-2,0,\ldots,0)U^{-1}$. More intrinsically, if we project out the part of $h_{\Omega}$ lying inside the Lie algebra of $SU(n)$ (and re-scale appropriately) then the adjoint orbit is the one through the $SU(n)$ Lie algebra element $\text{diag}(-n+1,1,\ldots,1)$ which is another way to define the projective space $\mathbb{C}P^{n-1}$.

Quantization of the collective coordinate moduli space spanned by $\Omega$ involves pulling back the symplectic form of the WZW model to classical moduli space and leads to the quantization of the adjoint orbit (or co-adjoint orbit since these are the same for semi-simple Lie groups). In the case to hand, the quantization on the co-adjoint orbit is known as the fuzzy $\mathbb{C}P^{n-1}$ [62]. The coordinates on the space are to be thought of as quantum operators on a Hilbert space, and hence are non-commuting. The resulting quantum Hilbert space of states are the symmetric representations of $SU(n)$ and as the dimension of the representations becomes large the fuzzy $\mathbb{C}P^{n-1}$ becomes a closer approximation to the “ordinary” space. To see this, let $|e_i\rangle$, $i = 1,\ldots,n$, be a basis for the $n$-dimensional module of $SU(n)$. So $e_i$ is the soliton with topological charge $e_i$. Consider the special states $||\Omega, a\rangle\rangle = (\Omega_i|e_i\rangle)^\otimes a$. These states lie in the module $[a]$ and have an inner-product

$$
\langle\langle \Omega, a||\Omega', a\rangle\rangle = (\Omega^* \cdot \Omega')^a
$$

which goes to zero as $a \to \infty$ if $\Omega' \neq \Omega$. Consequently, as $a \to \infty$, the quantum state $||\Omega, a\rangle\rangle$ is a quasi-classical state (or coherent state) which approximates the classical configuration with collective coordinate $\Omega$. Actually, since the symmetry in the present context is a quantum group symmetry we expect that the quantization of the solitons involves a fuzzy $\mathbb{C}P^{n-1}$ with a $q$-deformation [63,64]. Notice that the $q$ deformation will only become apparent for states where $a$ is of order $k$.

In the following, we shall focus on the particular states $||e_i, a\rangle\rangle$ for simplicity. These states are associated to the special set of classical solitons with $\Omega = e_i$ described previously which have a topological charge that is aligned with the choice of gauge. The S-matrix elements for these states follow in a simple way by fusing the basic elements for the solitons with charges $e_i$. These elements are given by (4.11).
We define the scattering of $|e_i\rangle$ with $|e_i\rangle$, that is two kinks with topological charge $e_i$, as

$$S_1(\vartheta) = X_{11}(\vartheta)Y_{11}(\vartheta)\tilde{R}_{11}\left(\begin{array}{c} a \\ a + e_i \\ a + 2e_i \end{array} \right| \vartheta)$$

$$= Y_{11}(\vartheta)\sinh\left(\frac{\vartheta}{2} + i\omega\right)\cosh\left(\frac{\vartheta}{2} - i\omega\right) \sinh\left(\frac{\vartheta}{2} + i\omega\right) \cosh\left(\frac{\vartheta}{2} - i\omega\right).$$

(5.15)

The scattering of two kinks with topological charge $e_i$ and $e_j$, $i \neq j$ has both a transition amplitude and a reflection amplitude. In order to take the semi-classical limit we need only consider the transition amplitude which we define as

$$S_2(\vartheta) = X_{11}(\vartheta)Y_{11}(\vartheta)\tilde{R}_{11}\left(\begin{array}{c} a \\ a + e_i \\ a + e_i + e_j \end{array} \right| \vartheta)$$

$$= X_{11}(\vartheta)Y_{11}(\vartheta)\frac{\sinh(\lambda\vartheta)}{\sinh(\lambda\vartheta + i\omega)}Z$$

$$= Y_{11}(\vartheta)\sinh\left(\frac{\vartheta}{2} + i\omega\right)\cosh\left(\frac{\vartheta}{2} + i\omega\right) \sinh\left(\frac{\vartheta}{2} + i\omega\right) \cosh\left(\frac{\vartheta}{2} - i\omega\right) Z,$$

where $Z$ is the square root factor in [119] that depends on the vacuum state. Since this factor is a real number and we are only interested in the phase of the S-matrix, it will play no role in what follows. We have written the factor $Y_{11}(\vartheta)$ explicitly since as $k \to \infty$, $\log Y_{11}(\vartheta)$ is order $k^{-1}$ and therefore is subleading and can be ignored.

We can then calculate the scattering of the quasi-classical state $|\lbrace e_i, a \rbrace\rangle$ with $|\lbrace e_j, b \rbrace\rangle$ by applying the bootstrap equations. For $i = j$, and taking $a \geq b$

$$S_{ia,ib}(\vartheta) = \prod_{j=0}^{a-1} \prod_{l=0}^{b-1} S_1\left(\vartheta + \frac{i\pi(a + b - 2j - 2l - 2)}{N}\right)$$

$$= \prod_{j=1}^{b} \frac{(a + b - 2j + 2)\vartheta(a + b - 2j)\vartheta[a + b - 2j + 1]_{\vartheta}}{(2j - a - b - 2)\vartheta(2j - a - b)\vartheta[2j - a - b - 1]_{\vartheta}},$$

(5.17)

while for $i \neq j$

$$S_{ia,jb}(\vartheta) = \prod_{j=0}^{a-1} \prod_{l=0}^{b-1} S_2\left(\vartheta + \frac{i\pi(a + b - 2j - 2l - 2)}{N}\right)$$

$$= Z_{ia,jb}\prod_{j=1}^{b} \frac{(a + b - 2j)\vartheta[a + b - 2j + 1]_{\vartheta}[2j - a - b]_{\vartheta}}{(2j - a - b - 2)\vartheta[2j - a - b - 1]_{\vartheta}[a + b - 2j + 2]_{\vartheta}}.$$

(5.18)

where the functions $(x)_{\vartheta}$ and $[x]_{\vartheta}$ are defined in [112] and $Z_{ia,jb}$ is a real-valued vacuum-dependent factor.

---

15In the following we do not indicate the $Y_{ab}(\vartheta)$ factors because they are subleading.
We now have the S-matrix elements in a form that is suitable for taking the semi-classical limit. As $k \to \infty$, keeping the charges $q_1 = a\omega/2\lambda$ and $q_2 = b\omega/2\lambda$ fixed, the products over $j$ in (5.17) and (5.18) can be expressed as an integral over a continuous variable:

$$
\prod_{j=1}^{b} f\left(\frac{\lambda \vartheta}{2} \pm \frac{i\omega (a + b - 2j + l)}{4}\right) \rightarrow \exp \left[ \frac{4k}{\pi} \int_{q_1 - q_2}^{q_1 + q_2} d\eta \log f\left(\frac{\vartheta}{2} \pm \frac{i\eta}{2}\right) \right]. \quad (5.19)
$$

In the above $l$ is arbitrary as long as it is fixed as $k \to \infty$. We now write the integral over $\eta$ as an integral over $\vartheta$ using the identity

$$
i \int_{q_1 - q_2}^{q_1 + q_2} d\eta \log \left[ \frac{f\left(\frac{\vartheta}{2} - \frac{i\eta}{2}\right)f\left(\frac{i\eta}{2}\right)}{f\left(\frac{\vartheta}{2} + \frac{i\eta}{2}\right)f\left(-\frac{i\eta}{2}\right)} \right] = \int_{0}^{\vartheta} d\vartheta' \log \left| \frac{f\left(\frac{\vartheta'}{2} - \frac{i(q_1 - q_2)}{2}\right)}{f\left(\frac{\vartheta'}{2} - \frac{i(q_1 + q_2)}{2}\right)} \right| \quad (5.20)
$$

One can then check that in the semi-classical limit

$$\text{Im} \log S_{ia,ib}(\vartheta) = \frac{2k}{\pi} \int_{0}^{\vartheta} d\vartheta' \Delta F(\vartheta') \bigg|_{\Theta = 0} \quad (5.21)$$

and

$$\text{Im} \log S_{ia,jb}(\vartheta) = \frac{2k}{\pi} \int_{0}^{\vartheta} d\vartheta' \Delta F(\vartheta') \bigg|_{\Theta = \frac{\pi}{2}}. \quad (5.22)$$

This completes our check of the S-matrix via the semi-classical limit.

6. The Case $\mathbb{C}P^2$

Strictly speaking our analysis only applies to the case where the group $H$ is non-abelian and so this excludes the case $\mathbb{C}P^2$ for which $G/H = U(2)/U(1)$. In this section we consider this case and find some similarities but also some differences. Importantly, in this case we are able to test the S-matrix against both the semi-classical limit but also against the Thermodynamic Bethe Ansatz (TBA) via which one can calculate the central charge of the UV CFT: the $U(2)_{k}/U(1)$ gauged WZW model.

Our conjectured S-matrix for $\mathbb{C}P^2$ is based on a spectrum of states which matches (4.3), so that $N = 2k + 1$. The S-matrix elements are then conjectured to be

$$S_{ab}(\vartheta) = \tilde{\eta}(a, b) X_{ab}(\vartheta) \quad (6.1)$$

$\tilde{\eta}(a, b)$ is a constant phase that is needed to satisfy crossing and bootstrap (we will fix it below). In the above, $X_{ab}(\vartheta)$ is defined as in (4.21) but with $\lambda = 1$ and $\omega = 2\pi/N$.
where we interpret the labels $a, b$ as defined modulo $N = 2k + 1$. The S-matrix has a pole structure on the physical strip that matches the minimal S-matrix associated to $A_{2k}^{(1)}$. This means that the fusing rules allow more bound states than (4.34), with $\overline{a} = a + b \mod N$. Moreover, the particle labelled by $\overline{a} = N - a$ is identified with the anti-particle of the particle labelled by $a$. In fact if we write the (diagonal) S-matrix as

$$S_{ab}(\vartheta) = \tilde{\eta}(a, b) S_{ab}^{\text{min}}(\vartheta) S_{ab}^{\text{CDD}}(\vartheta),$$

then $S_{ab}^{\text{min}}$ is the minimal S-matrix associated to $A_{2k}^{(1)}$ and the CDD part is related to the CCD part of the S-matrix of the homogeneous sine-Gordon theory models at level $N$ (see [34], section 4):

$$S_{ab}^{\text{CDD}}(\vartheta) = \left[ S_{ab}^{\text{F}}(\vartheta) \right]^{-1},$$

where

$$S_{ab}^{\text{F}}(\vartheta) = \prod_{j=1}^{\min(a,b)} \frac{(a + b - 2j + 1)\vartheta}{(2j - a - b - 1)\vartheta}. \quad (6.4)$$

Then, the resulting set of TBA equations is

$$\varepsilon_a(\vartheta) = \nu_a(\vartheta) - \sum_{b=1}^{N-1} \left( \phi_{ab} + \psi_{ab} \right) * L_b(\vartheta),$$

with

$$\nu_a = m_a r \cosh \vartheta, \quad L_a = \log \left( 1 + e^{-\varepsilon_a} \right),$$

$$\phi_{ab} = -i \frac{d}{d \vartheta} S_{ab}^{\text{min}}(\vartheta), \quad \psi_{ab} = +i \frac{d}{d \vartheta} S_{ab}^{\text{F}}(\vartheta), \quad (6.6)$$

whose scaling function is

$$c(r) = \frac{3}{\pi^2} \sum_{a=1}^{N-1} \int_{-\infty}^{+\infty} d\vartheta \nu_a(\vartheta) L_a(\vartheta). \quad (6.7)$$

Taking into account that $\nu_a = \nu_{\overline{a}}$, $\phi_{ab} = \phi_{\overline{a}b}$ and $\psi_{ab} = \psi_{\overline{a}b}$, it follows that $\varepsilon_a = \varepsilon_{\overline{a}}$, and the system of equations (6.3) can be written in the equivalent way

$$\varepsilon_a(\vartheta) = \nu_a(\vartheta) - \sum_{b=1}^{N-1} \left( \phi_{ab} + \psi_{ab} \right) * L_b(\vartheta). \quad (6.8)$$

Taking advantage of the fact that $\phi_{ab}$ and $\psi_{ab}$ are the kernels that enter the TBA equations of the HSG models, we can relate our set of TBA equations to those of
the $SU(3)_N$ HSG model:

$$
\varepsilon^1_a(\vartheta) = \nu^1_a(\vartheta) - \sum_{b=1}^{N-1} (\phi_{ab} * L^1_b(\vartheta) + \psi_{ab} * L^2_b(\vartheta - \sigma_{21})) \\
\varepsilon^2_a(\vartheta) = \nu^2_a(\vartheta) - \sum_{b=1}^{N-1} (\phi_{ab} * L^2_b(\vartheta) + \psi_{ab} * L^1_b(\vartheta - \sigma_{12})),
$$

(6.9)

where $\nu^i_a = M_i m_a r \cosh \vartheta$. For any non-vanishing value of the mass scales $M_1$ and $M_2$, and any value of the resonance parameters $\sigma_{12} = -\sigma_{21}$, it was shown in [33] that

$$
c(r) = \frac{3}{2} \frac{2}{\pi^2} \sum_{i=1}^{N} \int_{-\infty}^{+\infty} d\vartheta \nu^i_a(\vartheta) L^i_a(\vartheta) \rightarrow \frac{6(N-1)}{N+3},
$$

(6.10)

when $r \rightarrow 0$, which is the central charge of the $SU(3)_N/U(1)^2$ coset CFT.

Let us consider the particular choice of parameters $M_1 = M_2 = 1$, and $\sigma_{12} = 0$. Then the $SU(3)_N$ TBA equations simplify to

$$
\varepsilon^1_a(\vartheta) = \nu^1_a(\vartheta) - \sum_{b=1}^{N-1} (\phi_{ab} * L^1_b(\vartheta) + \psi_{ab} * L^2_b(\vartheta)) \\
\varepsilon^2_a(\vartheta) = \nu^2_a(\vartheta) - \sum_{b=1}^{N-1} (\phi_{ab} * L^2_b(\vartheta) + \psi_{ab} * L^1_b(\vartheta)).
$$

(6.11)

Obviously, $\varepsilon^1_a(\vartheta) = \varepsilon^2_a(\vartheta)$, and we obtain two identical copies of the system [33]

$$
\varepsilon_a(\vartheta) = \nu_a(\vartheta) - \sum_{b=1}^{N-1} (\phi_{ab} * L^1_b(\vartheta) + \psi_{ab} * L^2_b(\vartheta)),
$$

(6.12)

which is just (6.8). Therefore,

$$
c(r) = \frac{3}{\pi^2} \sum_{a=1}^{N} \int_{-\infty}^{+\infty} d\vartheta \nu_a(\vartheta) L_a(\vartheta) \rightarrow \frac{3(N-1)}{N+3};
$$

(6.13)

namely, one half the UV central charge of the $SU(3)_N/U(1)^2$ coset CFT.

Our hypothesis is that $N = 2k+1$ and so

$$
c_{CFT} = \frac{3k}{k+2},
$$

(6.14)

which is precisely the central charge of the $U(2)_k/U(1)$ coset CFT. Finally, let us fix the overall phases in $S_{ab}$. As pointed out in [32],

$$
S^F_{ab}(i\pi - \vartheta) = (-1)^a S^F_{ba}(\vartheta).
$$

(6.15)
Therefore, since $N = 2k + 1$ is odd, it is not difficult to check that

$$\tilde{\eta}(a, b) = (-1)^{ab}$$

(6.16)

ensures that the S-matrix satisfies the usual crossing and bootstrap relations. Obviously, the overall constant phase plays no rôle in the TBA equations.

7. Discussion

The purpose of this paper has been to begin the programme of solving the SSSG theories at the quantum level with the goal of seeing to what extent the spectrum and S-matrix of these relativistic theories is related to their non-relativistic Pohlmeyer cousins that describe giant magnons in string theory. It is not expected that there will be an exact equivalence of any kind unless the full problem for the supergroup symmetric space models is considered. However, we have seen that the soliton theory does have certain things common with the magnon theory in that states transform in symmetric representations of the symmetry. However, even at the classical level, there is a non-trivial rapidity dependent mapping between the charges of the magnons and solitons. In addition, in the soliton case, the symmetry is an affine quantum group symmetry, whereas in the magnon case it is a “normal” symmetry. Both solitons and magnons come in a tower of states; however, for the solitons the tower is truncated by the quantum group structure. It is clear that if there is some kind of equivalence for the cases involved in the AdS/CFT correspondence then we can expect some surprises for the supergroup extensions of the SSSG theories.

It is interesting to consider how the quantum solution of the deformed WZW model relates to the field theory in the classical limit. Classically, the theory has a degenerate vacuum that can be identified with the Cartan torus of $H$. In the quantum theory, the set of vacua is the discrete set $\Lambda^*(k)$. However, as $k \rightarrow \infty$ there is an obvious sense in which this discrete set can be described by a continuum taking values in the Cartan torus. The solitons in the quantum theory are kinks whose topological charge takes values in the set of weights of the symmetric representations, which again as $k \rightarrow \infty$ become arbitrary vectors in the Cartan space. In fact, we have already mentioned that the internal $\mathbb{C}P^{n-1}$ moduli space of the classical soliton can be viewed as becoming a $q$-deformed fuzzy $\mathbb{C}P^{n-1}$ in the semi-classical approximation. It is important to emphasize that the S-matrix we have written down is subject to the CDD ambiguities and the semi-classical limit only partially constrains these. In order to pin them down definitively, one should perform a TBA analysis for all the $\mathbb{C}P^{n+1}$ cases; a task that will be pursued elsewhere.
It would be interesting to compare our S-matrix with the approach to quantizing the deformed WZW model adopted in [39]. In that reference the approach taken is essentially perturbative, in that fields are expanded around the vacuum in a particular gauge which involves setting $A_+ = 0$ and then integrating out $A_-$ to give a non-local form of the action. This non-local action then has an equivalent local form and the tree-level $S$-matrix can be computed. In our approach, we expect that the perturbative excitations of the theory correspond to states with lowest $U(1)$ charge, $q = \pm \pi/(2k + n)$. Indeed, in the semi-classical limit, these states have a perturbative mass $M = \mu$. In our approach these states are kinks but with vanishing small topological charge in the semi-classical limit.

This paper only presents the first step in understanding the SSSG theories at the quantum level. Generalizations to other symmetric spaces are currently under way. A particularly important class of examples are the symmetric spaces $F = SO(n+2)/SO(n+1) \simeq S^{n+1}$, for which the associated SSSG equations involve the WZW theory for coset $G/H = SO(n+1)/SO(n) \simeq S^n$. The solitons in this case, have a classical moduli space which has an adjoint orbit of $SO(n)$ identified with the real oriented Grassmannian $SO(n)/SO(2) \times SO(n-2)$. The quantum states in this case correspond to symmetric representations of $SO(n)$. The S-matrices for these theories will be described in a companion paper [61].

Acknowledgments

JLM acknowledges the support of MICINN (Spain) and FEDER (FPA2008-01838 and FPA2008-01177), Xunta de Galicia (INCITE09.296.035PR), and the Spanish Consolider-Ingenio 2010 Programme CPAN (CSD2007-00042).

TJH would like to thank Nick Dorey for useful conversations and the organizers of the conference “16 Supersymmetries” at City University London in May at which these results were first presented. TJH would also like to acknowledge the support of STFC grant ST/G000506/1.

We would both like to thank Arkady Tseytlin for discussions and comments on a draft of this paper.
References

[1] K. Zarembo, Comptes Rendus Physique 5 (2004) 1081 [Fortsch. Phys. 53 (2005) 647] [arXiv:hep-th/0411191].

[2] J. A. Minahan, J. Phys. A 39 (2006) 12657.

[3] J. A. Minahan and K. Zarembo, JHEP 0809, 040 (2008) [arXiv:0806.3951 [hep-th]].

[4] D. M. Hofman and J. M. Maldacena, J. Phys. A 39 (2006) 13095 [arXiv:hep-th/0604135].

[5] N. Dorey, J. Phys. A 39, 13119 (2006) [arXiv:hep-th/0604175]; H. Y. Chen, N. Dorey and K. Okamura, JHEP 0609, 024 (2006) [arXiv:hep-th/0605155].

[6] D. Gaiotto, S. Giombi and X. Yin, JHEP 0904 (2009) 066 [arXiv:0806.4589 [hep-th]].

[7] G. Grignani, T. Harmark and M. Orselli, Nucl. Phys. B 810 (2009) 115 [arXiv:0806.4959 [hep-th]].

[8] T. J. Hollowood and J. L. Miramontes, JHEP 0908 (2009) 109 [arXiv:0905.2534 [hep-th]].

[9] M. C. Abbott, I. Aniceto and O. O. Sax, arXiv:0903.3365 [hep-th].

[10] A. A. Tseytlin, arXiv:hep-th/0311139.

[11] K. Pohlmeyer, Commun. Math. Phys. 46 (1976) 207.

[12] M. Staudacher, JHEP 0505 (2005) 054 [arXiv:hep-th/0412188].

[13] N. Beisert, Adv. Theor. Math. Phys. 12 (2008) 945 [arXiv:hep-th/0511082]; N. Beisert, J. Stat. Mech. 0701 (2007) P017 [arXiv:nlin/0610017].

[14] G. Arutyunov, S. Frolov and M. Zamaklar, JHEP 0704 (2007) 002 [arXiv:hep-th/0612229].

[15] C. Ahn and R. I. Nepomechie, JHEP 0809 (2008) 010 [arXiv:0807.1924 [hep-th]].

[16] H. Eichenherr and K. Pohlmeyer, Phys. Lett. B 89 (1979) 76.

[17] H. Eichenherr and J. Honerkamp, J. Math. Phys. 22 (1981) 374.

[18] R. D’Auria, T. Regge and S. Sciuto, Phys. Lett. B 89, 363 (1980).

[19] R. D’Auria, T. Regge and S. Sciuto, Nucl. Phys. B 171, 167 (1980).

[20] R. D’Auria and S. Sciuto, Nucl. Phys. B 171, 189 (1980).
[21] V. E. Zakharov and A. V. Mikhailov, “Relativistically Invariant Two-Dimensional Models In Field Theory” Sov. Phys. JETP 47, 1017 (1978) [Zh. Eksp. Teor. Fiz. 74, 1953 (1978)].

[22] I. Bakas, Q. H. Park and H. J. Shin, Phys. Lett. B 372 (1996) 45 [arXiv:hep-th/9512030].

[23] J. L. Miramontes, JHEP 0810 (2008) 087 [arXiv:0808.3365 [hep-th]].

[24] O. A. Castro Alvaredo and J. L. Miramontes, Nucl. Phys. B 581, 643 (2000) [arXiv:hep-th/0002219].

[25] A. Mikhailov, arXiv:hep-th/0511069;
A. Mikhailov, arXiv:hep-th/0609108;
A. Mikhailov, J. Geom. Phys. 56 (2006) 2429 [arXiv:hep-th/0504035];

[26] M. Grigoriev and A. A. Tseytlin, Nucl. Phys. B 800 (2008) 450 [arXiv:0711.0155 [hep-th]].

[27] A. Mikhailov and S. Schafer-Nameki, JHEP 0805 (2008) 075 [arXiv:0711.0195 [hep-th]].

[28] R. Roiban and A. A. Tseytlin, JHEP 0904 (2009) 078 [arXiv:0902.2489 [hep-th]].

[29] A. B. Zamolodchikov and A. B. Zamolodchikov, Annals Phys. 120, 253 (1979).

[30] N. Dorey and T. J. Hollowood, Nucl. Phys. B 440 (1995) 215 [arXiv:hep-th/9410140].

[31] C. R. Fernandez-Pousa, M. V. Gallas, T. J. Hollowood and J. L. Miramontes, Nucl. Phys. B 484, 600 (1997) [arXiv:hep-th/9606032];
C. R. Fernandez-Pousa, M. V. Gallas, T. J. Hollowood and J. L. Miramontes, Nucl. Phys. B 499 (1997) 673 [arXiv:hep-th/9701109].

[32] J. L. Miramontes and C. R. Fernandez-Pousa, Phys. Lett. B 472, 392 (2000) [arXiv:hep-th/9910218].

[33] O. A. Castro-Alvaredo, A. Fring, C. Korff and J. L. Miramontes, Nucl. Phys. B 575 (2000) 535 [arXiv:hep-th/9912196].

[34] P. Dorey and J. L. Miramontes, Nucl. Phys. B 697 (2004) 405 [arXiv:hep-th/0405275].

[35] O. Aharony, O. Bergman, D. L. Jafferis and J. Maldacena, JHEP 0810 (2008) 091 [arXiv:0806.1218 [hep-th]]; M. Benna, I. Klebanov, T. Klose and M. Smedback, JHEP 0809 (2008) 072 [arXiv:0806.1519 [hep-th]].

[36] T. J. Hollowood and J. L. Miramontes, JHEP 0904 (2009) 060 [arXiv:0902.2405 [hep-th]].
[37] J. L. Miramontes, Nucl. Phys. B 702 (2004) 419 [arXiv:hep-th/0408119].

[38] B. Hoare, Y. Iwashita and A. A. Tseytlin, J. Phys. A 42 (2009) 375204
   [arXiv:0906.3800 [hep-th]].

[39] B. Hoare and A. A. Tseytlin, JHEP 1002 (2010) 094 [arXiv:0912.2958 [hep-th]].

[40] M. Spradlin and A. Volovich, JHEP 0610 (2006) 012 [arXiv:hep-th/0607009];
    A. Jevicki, C. Kalousios, M. Spradlin and A. Volovich, JHEP 0712 (2007) 047
    [arXiv:0708.0818 [hep-th]];
    C. Kalousios, M. Spradlin and A. Volovich, JHEP 0703 (2007) 020
    [arXiv:hep-th/0611033].

[41] J. P. Harnad, Y. Saint Aubin and S. Shnider, Commun. Math. Phys. 92 (1984) 329.

[42] P. Fendley, Phys. Rev. Lett. 83 (1999) 4468 [arXiv:hep-th/9906036].

[43] P. Bouwknegt, L. Chim and D. Ridout, Nucl. Phys. B 572 (2000) 574
    [arXiv:hep-th/9903176].

[44] S. Guruswamy and K. Schoutens, Nucl. Phys. B 556 (1999) 530
    [arXiv:cond-mat/9903045].

[45] P. Bouwknegt, A. W. W. Ludwig and K. Schoutens, Phys. Lett. B 359 (1995) 304
    [arXiv:hep-th/9412108].

[46] C. Ahn, D. Bernard and A. LeClair, Nucl. Phys. B 346, 409 (1990).

[47] H. J. de Vega and V. A. Fateev, Int. J. Mod. Phys. A 6, 3221 (1991).

[48] D. Gepner, Phys. Lett. B 313, 45 (1993) [arXiv:hep-th/9302115].

[49] T. J. Hollowood, “A Quantum group approach to constructing factorizable S
    matrices,” unpublished (1990).

[50] T. J. Hollowood, Int. J. Mod. Phys. A 8 (1993) 947 [arXiv:hep-th/9203076].

[51] T. J. Hollowood, Nucl. Phys. B 414 (1994) 379 [arXiv:hep-th/9305042].

[52] M. Jimbo, Lett. Math. Phys. 10 (1985) 63; Lett. Math. Phys. 11 (1986) 247; Int. J.
    Mod. Phys. A4 (1989) 3759.

[53] V.G. Drinfel’d, Sov. Math. Dokl. 32 (1985) 254.

[54] M. Jimbo, T. Miwa and M. Okado, Comm. Math. Phys. 116 (1988) 507.

[55] M. Wadati, T. Deguchi and Y. Akutsu, Phys. Rep. 180 (1989) 247.

[56] R. b. Zhang, M. D. Gould and A. J. Bracken, Nucl. Phys. B 354 (1991) 625.

[57] S. R. Coleman and H. J. Thun, Commun. Math. Phys. 61 (1978) 31.
[58] R. Jackiw and G. Woo, Phys. Rev. D 12 (1975) 1643.

[59] Y. Hatsuda and H. Tanaka, JHEP 1002 (2010) 085 [arXiv:0910.5315 [hep-th]].

[60] C. Kalousios and G. Papathanasiou, arXiv:1005.1066 [hep-th].

[61] T. J. Hollowood and J. L. Miramontes, “The Relativistic Avatars of Giant Magnons and their S-Matrix II”, to appear.

[62] A. P. Balachandran, S. Kurkuoglu and S. Vaidya, arXiv:hep-th/0511114.

[63] H. Grosse, J. Madore and H. Steinacker, J. Geom. Phys. 38 (2001) 308 [arXiv:hep-th/0005273].

[64] J. Pawelczyk and H. Steinacker, Nucl. Phys. B 638 (2002) 433 [arXiv:hep-th/0203110].