Double Darboux method for the Taub continuum

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Summary. - The strictly isospectral double Darboux method is applied to the quantum Taub model in order to generate a one-parameter family of strictly isospectral potentials for this case. The family we build is based on a scattering Wheeler-DeWitt solution first discussed by Ryan and collaborators that we slightly modified according to a suggestion due to Dunster. The strictly isospectral Taub potentials possess different (attenuated) scattering states with respect to the original Taub potential.

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Quantum cosmology and its supersymmetric extension are an interesting “laboratory” for techniques of current use in nonrelativistic quantum mechanics. One such technique is the strictly isospectral double Darboux method (SIDD), which is a procedure within Witten’s supersymmetric quantum mechanics.

Previously, we have applied SIDD to closed, radiation-filled Friedmann-Robertson-Walker (FRW) quantum universes, obtaining a one-parameter family of strictly isospectral FRW quantum potentials and the corresponding wavefunctions. The Taub minisuperspace model is a separable quantum problem and therefore is well suited for SIDD. It is our purpose in this paper to develop the method for the continuous part of the spectrum of the Taub model. The motivation for doing this exercise was found in a paper by Sukhatme and collaborators.

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who obtained bound states in the continuum in quantum mechanics by SIDDM. However, for the cosmological Taub case the method does not produce such states as we shall show in the following.

SIDDM is a technique of deleting followed by reinstating an energy level of a one-dimensional (1D) potential $V(x)$ by which one can generate a one-parameter family of isospectral potentials $V_{iso}(x; \lambda)$, where $\lambda$ is a real, labeling parameter of each member potential in the set. As a matter of fact, Khare and Sukhatme [6] were also able to construct multiparameter families of isospectral potentials, but we shall be concerned only with the one-parameter case in the following. SIDDM is the physical formulation of a mathematical scheme based on the general Riccati solution, which has been introduced by Mielnik [7]. The strictly isospectral techniques are well established for nodeless bound states, resulting from any pair combination of the well-known Abraham-Moses procedure, the Pursey one, and the “supersymmetric” Darboux one [6]. Recently, Sukhatme and collaborators [8] extended the usage of the double Darboux construction to energy states in the continuum region of the spectrum and obtained several families of isospectral potentials with bound states in the continuum. In particular, they performed the double Darboux construction on the half line free particle wavefunction $u_0 = \sin kr$, obtaining bound states embedded in the continuum for this simple but relevant case.

Let us take a simple 1D Schrödinger equation for a wavefunction $u(x)$ at an arbitrary energy level $E$ in the standard form $-u''(x) + V_b(x)u(x) = Eu(x)$. Then, the strictly isospectral potentials with respect to $V_b$ obtained by means of SIDDM, by which one deletes and reinserts the level $E$ in the spectrum will be [9]

$$V_{iso}(x; \lambda) = V_b(x) - 2 \frac{d^2}{dx^2} \ln(\mathcal{I} + \lambda),$$

where $\mathcal{I}(x) = \int_{-\infty}^{x} u^2(y)dy$ and $\lambda$ is the so called isospectral parameter, which is a real quantity. The isospectral family of potentials should be understood in the sense that the whole family has the same supersymmetric partner “fermionic” potential given by $V_f(x) = V_b(x) - 2 \frac{d^2}{dx^2} \ln(u)$. When $u$ is a nodeless state there are no difficulties in applying the double Darboux technique. However, when one uses an eigenfunction with nodes, the corresponding “fermionic” potential will have singularities. Nevertheless, the resulting isospectral family of potentials is free of
singularities \[3\] and this makes the method viable. Another basic result of the scheme is that the \(u\) wavefunction is changed into \(u/\left[\mathcal{I}(x) + \lambda\right]\), i.e., a spatial damping is introduced.

Let us pass now to the quantum Taub model which has been studied in some detail by Ryan and collaborators \[4\] who found that it is separable. Indeed, the Taub Wheeler-DeWitt equation is

\[
\frac{\partial^2 \Psi}{\partial \alpha^2} - \frac{\partial^2 \Psi}{\partial \beta^2} + e^{4\alpha} V(\beta) \Psi = 0, \tag{2}
\]

where \(V(\beta) = \frac{1}{3} (e^{-8\beta} - 4e^{-2\beta})\). Eq. (2) can be separated in the variables \(x_1 = 4\alpha - 8\beta\) and \(x_2 = 4\alpha - 2\beta\). Thus, one gets two independent 1D problems for which the supersymmetric procedures are standard practice. The two equations are as follows

\[
-\frac{\partial^2 u_1}{\partial x_1^2} + \frac{1}{144} e^{x_1} u_1 = \frac{\omega^2}{4} u_1, \tag{3}
\]

and

\[
-\frac{\partial^2 u_2}{\partial x_2^2} + \frac{1}{9} e^{x_2} u_2 = \omega^2 u_2, \tag{4}
\]

where we have already multiplied both sides in Eqs. (3) and (4) by \(-1\) in order to get standard Schrödinger equations. The quantity \(\omega\) is mathematically the separation constant, which physically is related to the wavenumber of a positive energy level.

Martínez and Ryan have considered a wavepacket solution made of wavefunctions \(\Psi\) having the form of a product of modified Bessel functions of imaginary order. We shall slightly modify their \(\Psi\) as follows

\[
\Psi \equiv u_1 u_2 = K_{i\omega}(\frac{1}{6} e^{x_1/2})[L_{2i\omega}(\frac{2}{3} e^{x_2/2}) + K_{2i\omega}(\frac{2}{3} e^{x_2/2})] \tag{5}
\]

since, according to Dunster \[8\], the \(L\) function defined as

\[
L_{2i\omega} = \frac{\pi i}{2 \sinh(2\omega \pi)} (I_{2i\omega} + I_{-2i\omega}), \tag{6}
\]

contrary to the \(I_{2i\omega}\) function used by Martínez and Ryan, being real on the real axis is a better companion for the \(K\) function of imaginary order which is also real on the real axis. In order to introduce such a change, Dunster invoked the criteria on the choice of standard solutions for a homogeneous linear differential equation of the second order due to Miller \[9\]. In the following,
we shall make the double Darboux construction on the base of a $\Psi$ wavefunction of the type given in Eq. (5).

The isospectral potential for the $x_1$ variable will be

$$V_1(x_1; \lambda_1) = \frac{1}{144} e^{x_1} - 2 \frac{d^2}{dx_1^2} \ln[\lambda_1 + \mathcal{I}(x_1)]$$  \hspace{1cm} (7)$$

and for the $x_2$ one

$$V_2(x_2; \lambda_2) = \frac{1}{9} e^{x_2} - 2 \frac{d^2}{dx_2^2} \ln[\lambda_2 + \mathcal{I}(x_2)],$$  \hspace{1cm} (8)$$

where the integrals are

$$\mathcal{I}(x_1) = \int_{-\infty}^{x_1} K_i \omega \left(\frac{1}{6} e^{y/2}\right)dy$$  \hspace{1cm} (9)$$

and

$$\mathcal{I}(x_2) = \int_{-\infty}^{x_2} \left[ L_2 \omega \left(\frac{2}{3} e^{y/2}\right) + K_2 \omega \left(\frac{2}{3} e^{y/2}\right)\right]^2 dy.$$  \hspace{1cm} (10)$$

The total Taub isospectral wavefunction has the following form

$$\Psi^T_{iso}(x_1, x_2; \lambda_1, \lambda_2) \equiv u_{iso,1} u_{iso,2} = \frac{K_i \omega \left(\frac{1}{6} e^{x_1/2}\right) L_2 \omega \left(\frac{2}{3} e^{x_2/2}\right) + K_2 \omega \left(\frac{2}{3} e^{x_2/2}\right)}{\left[\mathcal{I}(x_1) + \lambda_1\right] \left[\mathcal{I}(x_2) + \lambda_2\right]}.$$  \hspace{1cm} (11)$$

We have plotted the Taub isospectral potentials and the corresponding isospectral wavefunction for several values of the parameters in Figs. 1, 2, 3, respectively. We confirm the previous findings on the strictly isospectral supersymmetric effects [5], except for the fact that the wavefunctions are still not normalizable (i.e., they are not bound states in the continuum).

In conclusion, SIDDM allows the introduction of a one-parameter family of isospectral quantum Taub potentials having more attenuated states in the continuum region of the spectrum in comparison to the original potential.

The double Darboux method appears to be a quite general and useful method to generate new sets of quantum cosmological solutions. This is so because any potential in the Schrödinger equation has a classical continuum of positive energy non-normalizable solutions. Ryan and collaborators [4] were among the first to pay attention to the continuum part of the Wheeler-DeWitt spectrum. Selecting by means of some preliminary physical arguments that are of quantum scattering type one of the continuum solutions [4], one can perform the double Darboux construction on that state and generate by this means strictly isospectral families of cosmological
potentials as well as isospectral cosmological wavefunctions. The quantum Taub model just illustrates this nice feature of the double Darboux method. If one pushes further the picture, one might say that the incipient universes were nothing else but sets of strictly isospectral states of a quantum continuum. The parameter $\lambda$ looks like a decoherence parameter embodying a sort of “quantum” cosmological dissipation (or damping) distance. Finally, one can easily apply the other procedures of deleting and reinserting energy levels, i.e., combinations of any pairs of Abraham-Moses procedure, Pursey’s one, and the Darboux one. However, only the double Darboux method used here leads to reflection and transmission amplitudes identical to those of the original potential.

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Appendix

We quote here some important properties of the modified Bessel functions of imaginary order.

Their behavior at $x \to 0^+$ is as follows

$$K_{i\omega}(x) = -\left[\frac{\pi}{\omega \sinh(\omega \pi)}\right]^{1/2} \left[\sin(\omega \ln(x/2) - \phi) + O(x^2)\right]$$

(A1)

and

$$L_{i\omega}(x) = -\left[\frac{\pi}{\omega \sinh(\omega \pi)}\right]^{1/2} \left[\cos(\omega \ln(x/2) - \phi) + O(x^2)\right]$$

(A2)

where the phase $\phi = \arg[\Gamma(1 + i\omega)]$. The amplitudes of oscillation of the two functions become unbounded as $\omega \to 0$ in the neighborhood of the origin. In the paper we worked at fixed, small positive $\omega$ for computational reasons.

In the complex plane, as $z \to \infty$, the asymptotics are the following

$$K_{i\omega}(z) = \left[\frac{\pi}{2z}\right]^{1/2} e^{-z} [1 + O(1/z)], \quad |\arg z| \leq 3\pi/2 - \delta$$

(A3)
and

\[ L_{i\omega}(z) = \frac{1}{\sinh(\omega\pi)} \left[ \frac{\pi}{2z} \right]^{1/2} e^{z[1 + 0(1/z)]}, \quad |\arg z| \leq \pi/2 - \delta \]  

(A4)

where \( \delta \) is an arbitrary small positive constant.

As for the zeros of these functions, it is known that \( K_{i\omega}(x) \) has an infinite number of simple positive zeros in \( 0 < x < \omega \) and no zeros in \( \omega \leq x < \infty \), whereas \( L_{i\omega}(x) \) has an infinite number of simple positive zeros. Up to \( x = \omega \) the two sets of zeros are interlaced.

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The original potentials $V_1(x_1)$ and $V_2(x_2)$ (left side) and the corresponding isospectral members for

$\lambda_1 = \lambda_2 = 1$ (right side) and $\omega = 1/4$. 

Fig. 1
The wave functions $u_1$ (the smaller amplitude one) and $u_2$ (the larger amplitude one) of the original Taub universe for $\omega = 1/4$. 
Isospectral wavefunctions \( u_{iso,1} \) for \( \lambda_1 = 1 \) (larger amplitude) and \( \lambda_1 = 10 \) (smaller amplitude); \( u_{iso,2} \) for \( \lambda_2 = 1 \), respectively, and \( \omega = 1/4 \).
Fig. 4

The original Taub potential in the $x_1$, $x_2$ variables.