Input-to-state stability and Lyapunov functions with explicit domains for
SIR model of infectious diseases∗

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Abstract. This paper demonstrates input-to-state stability (ISS) of the SIR model of infectious diseases with respect to the disease-free equilibrium and the endemic equilibrium. Lyapunov functions are constructed to verify that both equilibria are individually robust with respect to perturbation of newborn/immigration rate which determines the eventual state of populations in epidemics. The construction and analysis are geometric and global in the space of the populations. In addition to the establishment of ISS, this paper shows how explicitly the constructed level sets reflect the flow of trajectories. Essential obstacles and keys for the construction of Lyapunov functions are elucidated. The proposed Lyapunov functions which have strictly negative derivative allow us to not only establish ISS, but also get rid of the use of LaSalle’s invariance principle and popular simplifying assumptions.

Key words. Epidemic models, input-to-state stability, Lyapunov functions, ordinary differential equations

AMS subject classifications. 93D30, 93C10, 93D09, 92D25, 34D23

1. Introduction. For infectious diseases, mathematical models play two major roles in helping epidemiologist and societies design schemes aiming to improve control or eradicate the infection from population [16]. One role is quantitative prediction in which its accuracy is the primary concern. The other is qualitative understanding of epidemiological processes. For the latter, analytical studies on simple models have been providing generic interpretations of behavior of diseases transmission and spread. This paper pursues this direction by focusing on the popular model called the SIR model [6, 17].

The SIR model has an endemic equilibrium and a disease-free equilibrium. If the newborn rate is large in the population, the endemic equilibrium emerges and the trajectory of populational behavior heads for the equilibrium. Here, the newborn rate is the external signal flowing into the SIR model, and it describes not only birth, but also the susceptible flux entering the area to which populations of interest belongs, i.e., immigration of susceptible individuals.

Stability is a fundamental concept that characterizes behavior of dynamics for each equilibrium. Roughly, asymptotic stability gives a guarantee that trajectories starting sufficiently near the target equilibrium converges to the equilibrium. Jacobian linearization, which is called Lyapunov’s first method, explains asymptotic stability of the two equilibria [16]. Drawing phase portraits has also visualized the behavior outside the sufficiently small neighborhood of each equilibrium [11]. For systematic analysis outside the small neighborhood, many studies constructed Lyapunov functions to invoke Lyapunov’s second method for the SIR model and its variants (see [21, 19, 9, 28, 8, 27, 29, 4] and references therein). However, it has not been successful satisfactorily. Unless reasonable sublevel sets of constructed Lyapunov functions are confirmed, computing negative derivative of the functions along the trajectories cannot go beyond the local analysis Jacobian linearization offers. Sublevel sets are the only means to estimate of the domain of attraction in Lyapunov’s second method

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Since achieving the negative derivative in reasonably large sublevel sets has been too hard for the SIR model, many preceding studies invoke LaSalle’s invariance principle to relax the negativity into non-positivity [18]. To use LaSalle’s invariance principle, the notable study [21] proposed to use a simplified model in which the newborn rate is endogenously determined to keep precise conservation of the total population. The key is that the simplification reduces the dimension of the system, and leads to an one-dimensional subspace for which the argument of LaSalle’s invariance principle is effective since oscillations are not possible. The approach has facilitated the use of Lyapunov functions in infectious diseases widely (see, e.g., [19, 20, 8] to name a few). However, it remains true that the simplifying assumption limits the use of models in prediction and understanding the disease transmission. In fact, the simplification ignores not only the actual newborn rate and its perturbation, but also individuals entering the area. Furthermore, LaSalle’s invariance principle is invalid in the presence of time-varying parameters. Indeed, the non-positivity of the derivative does not have margins to accommodate perturbations and external fluxes. Strict negativity of the derivative is useful, and such Lyapunov functions are called strict Lyapunov functions [23].

The first objective of this paper is to construct a strict Lyapunov function for the SIR model without the simplification and the invariance principle, and to investigate its sublevel sets for understanding the attractivity behavior of the two equilibria on the entire state space.

The second objective is to demonstrate robustness of the SIR model. Since the SIR model is nonlinear, asymptotic stability does not guarantee anything about behavior of trajectories in the presence of the variation of external parameters or signals [18].

This paper employs the notion of input-to-state stability (ISS) to evaluate robustness of the SIR model with respect to perturbation of the newborn/immigration rate [30]. To the best of the authors’ knowledge, this ISS property has not been investigated for models of infectious diseases. To this end, this paper constructs functions called ISS Lyapunov functions [33]. As a matter of fact, this construction leads to an answer to the first objective. When the newborn/immigration rate is constant, the ISS property reduces to the asymptotic stability. The constructed Lyapunov functions have negative derivative, and they address external variations by getting rid of LaSalle’s invariance principle. Recall that the SIR model has two equilibria, and a bifurcation occurs as the newborn/immigration rate changes. The paper demonstrates that the bifurcation takes place as a continuous change of the transient and the steady state with respect to the change of the newborn/immigration rate. The bifurcation is not a discontinuous phenomenon. This is true in both directions, from the disease-free equilibrium to the endemic equilibrium, and vice versa.

2. Preliminaries. This paper uses the symbols $\mathbb{R} := (-\infty, \infty)$, $\mathbb{R}_+ := [0, \infty)$ and $\mathbb{R}_+^n := [0, \infty)^n$. For $v \in \mathbb{R}^n$, the symbol $|v|$ denotes a norm which is selected consistently throughout the paper. It is the absolute value if $n = 1$. This paper writes $\Gamma \in \mathcal{P}$ if $\Gamma : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is continuous and satisfies $\Gamma(0) = 0$ and $\Gamma(s) > 0$ for all $s \in \mathbb{R}_+ \setminus \{0\}$. A function $\Gamma \in \mathcal{P}$ is said to be of class $\mathcal{K}$ and written as $\Gamma \in \mathcal{K}$ if it is strictly increasing. A class $\mathcal{K}$ function is said to be of class $\mathcal{K}_\infty$ if it is unbounded. A continuous function $\Phi : \mathbb{R}_+ \times \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is said to be of class $\mathcal{K}\mathcal{L}$ if, for each fixed $t \geq 0$, $\Phi(\cdot, t)$ is of class $\mathcal{K}$ and, for each fixed $s > 0$, $\Phi(s, \cdot)$ is decreasing and $\lim_{t \rightarrow \infty} \Phi(s, t) = 0$. The zero function of appropriate dimension is denoted by $0$.

Composition of the functions $\Gamma_1, \Gamma_2 : \mathbb{R} \rightarrow \mathbb{R}$ is expressed as $\Gamma_1 \circ \Gamma_2$.

For a continuous function $f : \mathbb{R}^n \times \mathbb{R}^p \rightarrow \mathbb{R}^n$ satisfying $f(0, 0) = 0$, a system of
the form
\begin{equation}
\dot{x}(t) = f(x(t), u(t))
\end{equation}
is said to be input-to-state stable (ISS) with respect to the input \( u \) if there exist \( \Phi \in KL \) and \( \Gamma \in K \cup \{0\} \) such that, for all measurable locally essentially bounded functions \( u : \mathbb{R}_+ \to \mathbb{R}^p \), all \( x(0) \in \mathbb{R}^n \) and all \( t \geq 0 \), its unique solution \( x(t) \) exists and satisfies
\begin{equation}
\forall t \in \mathbb{R}_+ \quad |x(t)| \leq \Phi(|x(0)|, t) + \Gamma(\text{ess sup}_{t \in \mathbb{R}_+} |u(t)|).
\end{equation}
The function \( \Gamma \) is called an ISS-gain function. ISS of \( f \) with the initial condition \( x = x_0 \) holds for all \( t \geq 0 \).

If \( x \) is said to be ISS on the set \( \Omega \) with respect to the input \( u \), then \( (2.1) \) implies globally asymptotic stability of the equilibrium \( x = 0 \) for \( u = 0 \). If a radially unbounded and continuously differentiable function \( V : \mathbb{R}^n \to \mathbb{R}_+ \) satisfies
\begin{equation}
V(x) \geq \chi(|u|) \Rightarrow \frac{\partial V}{\partial x}(x, u) f(x, u) \leq -\alpha(V(x))
\end{equation}
for some \( \chi \in \mathcal{K} \) and some \( \alpha \in \mathcal{P} \), the function \( V(x) \) is said to be an ISS Lyapunov function. The existence of an ISS Lyapunov function guarantees ISS of system \( (2.1) \) [33]. An ISS-gain function in \( (2.2) \) is obtained as \( \Gamma = \Phi^{-1} \circ \chi \), where \( \Phi \) is a class \( K_{\infty} \) function satisfying \( \Phi(|x|) \leq V(x) \) for all \( x \in \mathbb{R}^n \). ISS Lyapunov functions become conventional Lyapunov functions when \( u = 0 \). All the above are standard definitions given for sign-indefinite system \( (2.1) \). When the vector field \( f \) generates only non-negative \( x(t) \) in \( (2.1) \) defined with \( x(0) \in \mathbb{R}^n_+ \) and \( u(t) \in \mathbb{R}^p_+ \), all the above definitions and facts are valid by replacing \( \mathbb{R} \) with \( \mathbb{R}_+ \).

For scalar \( u \), one can define ISS with respect to the input \( u(t) \) restricted to a range \( (-\underline{u}, \overline{u}) \) for some constants \( \underline{u}, \overline{u} \in \mathbb{R}_+ \cup \{\infty\} \). To assess such ISS, one can just introduce a bijective function \( \zeta : \mathbb{R}_+ \to (-\underline{u}, \overline{u}) \) in \( f(x, u) \) as \( f(x, \zeta(r)) \), where \( \zeta(0) = 0 \). The standard restriction-free characterization \( (2.3) \) can be applied to \( f(x, \zeta(r)) \) with the auxiliary non-restricted input \( r \). In this paper, for a compact set \( \Omega \in \mathbb{R}^n \) satisfying \( 0 \in \Omega \), system \( (2.1) \) is said to be ISS on the set \( \Omega \) with respect to the input \( u \) satisfying \( u(t) \in (-\underline{u}, \overline{u}) \) if \( (2.2) \) holds for all \( x(0) \in \Omega \) and all \( u(t) \in (-\underline{u}, \overline{u}) \). To measure the magnitude of \( x \), the implication \( (2.3) \) employs \( V(x) \) instead of \( |x| \). Hence, ISS on \( \Omega \) is implied by \( (2.3) \) if \( x \in \mathbb{R}^n \) in \( (2.3) \) is replaced with a sublevel set
\begin{equation}
\overline{\Omega}(L) := \{ x \in \mathbb{R}^n : L \geq V(x) \}
\end{equation}
containing \( \Omega \) and satisfying \( L \geq \chi(|u|) \) for all \( u \).

If the function \( V \) is not continuously differentiable, but locally Lipschitz, \( \partial V/\partial x \cdot f \) in \( (2.3) \) is replaced by
\begin{equation}
D^+ V(x, u) := \liminf_{t \to u^+} \frac{(V(\psi(t, x, u)) - V(x))}{t},
\end{equation}
where \( \psi(t, x, u) \) is the solution of \( (2.1) \) with the initial condition \( x \) and the input function \( u \). Let \( \mathcal{N} \) denote the subset of \( \mathbb{R}^n \) where the gradient \( \partial V/\partial x \) does not exist. Rademacher’s theorem shows that the set \( \mathcal{N} \) has measure zero for a locally Lipschitz \( V \). Furthermore, the lower Dini derivative \( (2.5) \) for each fixed \( u \) agrees with \( \partial V/\partial x \cdot f \) except in \( \mathcal{N} \). The existence of an ISS Lyapunov function defined with \( (2.5) \) guarantees ISS of system \( (2.1) \) since \( f \) and \( \alpha \) continuous functions [1].

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\(^3\) The original definition in [33] employs \( \alpha \in \mathcal{K} \). However, the function \( V \) can always be rescaled to modify \( \alpha \in \mathcal{P} \) into a class \( \mathcal{K} \) function.
Remark 2.1. This paper demonstrates ISS of an epidemic model. Here, it is worth recalling that for nonlinear systems, global asymptotic stability of an equilibrium cannot guarantee boundedness of the state with respect to input of bounded magnitude [18]. In fact, for example, the origin $I = 0$ of I-system in (3.1b) is globally asymptotic stable for the nil input $S = 0$, while the constant input $S > (\gamma + \mu)/\beta$ makes $I(t)$ unbounded. Therefore, I-system (3.1b) is not ISS.

3. SIR Model. Let $x(t) := [S(t), I(t), R(t)]^T \in \mathbb{R}_+^3$ and assume that it satisfies

\begin{align}
(3.1a) \quad & \dot{S} = B - \mu S - \beta IS \\
(3.1b) \quad & \dot{I} = \beta IS - \gamma I - \mu I \\
(3.1c) \quad & \dot{R} = \gamma I - \mu R
\end{align}

defined for any $x(0) := [S(0), I(0), R(0)]^T \in \mathbb{R}_+^3$ and any measurable and locally essentially bounded function $B : \mathbb{R}_+ \to \mathbb{R}_+$. In fact, for each $x(0)$ and $B$, the equation (3.1) admits a unique maximal solution $x(t)$ [18]. Equation (3.1), which is expressed compactly as

\begin{equation}
\dot{x}(t) = f(x(t), u(t))
\end{equation}

with the vector field $f = [f_1, f_2, f_3]^T$ and the input $u = B$, also guarantees $x_i(t) \geq 0$, $t \in \mathbb{R}_+$, for each $i = 1, 2, 3$ since $f_i(x, u) \geq 0$ holds at $x_i = 0$ for each $i = 1, 2, 3$. The variables $x_1(t)$, $x_2(t)$ and $x_3(t)$ are denoted by $S(t)$, $I(t)$ and $R(t)$, respectively, since (3.1) is the equation popular model called SIR model (with demography) for infectious diseases [6, 17, 16]. The variable $S(t)$ describes the (continuum) number of the susceptible population, $I(t)$ is that of the infected population, while $R(t)$ is of the population recovered with immunity. The variable $B(t)$ is the newborn/immigration rate. The positive numbers $\beta$, $\gamma$ and $\mu$ are the transmission rate, the recovery rate and the death rate, respectively. Define the total population $N(t) := S(t) + I(t) + R(t)$ as usual. Since

\begin{equation}
\dot{N}(t) = B - \mu N(t)
\end{equation}

follows from (3.1), $x(t)$ exists for all $t \in \mathbb{R}_+$, which is referred to the forward completeness of system (3.2). Property (3.3) also implies that system (3.2) is ISS with respect to the input $u$ [13]. Indeed, it is easy to see that

\begin{equation}
S(t) + I(t) + R(t) \leq e^{-t} \left( S(0) + I(0) + R(0) - \frac{B}{\mu} \right) + \frac{B}{\mu}
\end{equation}

\begin{equation}
\leq e^{-t}(S(0) + I(0) + R(0)) + \frac{B}{\mu}
\end{equation}

for all $t \in \mathbb{R}_+$ with respect to any $B(t) \in [0, B]$ (a.e.). As discussed in [13], I-system (3.1b) is not ISS with respect to its input $S$. The absence of ISS is characterized there as strong integral input-to-state stability on which this paper does not go into detail [32, 26, 3]. Interestingly, the absence of ISS of I-system provides a bifurcation selecting one of the two equilibria $x_e$ and $x_f$ depending on $R_0$ to be explained below. $S$-system (3.1a) compensates the weak stability of I-system so that the overall system (3.1) is ISS.
Clearly, if the newborn/immigration rate is constant, i.e., \( B(t) \equiv \hat{B} \geq 0 \), equation (3.1) has two equilibria
\[
\begin{align*}
    x_f &:= \begin{bmatrix} \frac{\hat{B}}{\mu} \\ 0 \\ 0 \end{bmatrix} ^T \\
    x_e &:= \begin{bmatrix} \gamma + \mu \\ \frac{\mu(\hat{R}_0 - 1)}{\beta} \\ \frac{\gamma(\hat{R}_0 - 1)}{\beta} \end{bmatrix} ^T,
\end{align*}
\]
where the non-negative number
\[
\hat{R}_0 := \frac{\beta \hat{B}}{\mu(\gamma + \mu)}
\]
is called the basic reproduction number [16]. The former state \( x_f \) is called the disease-free equilibrium, while the latter \( x_e \) is called the endemic equilibrium. When \( \hat{R}_0 < 1 \), the endemic equilibrium \( x_e \) disappears since \( x(t) \in \mathbb{R}^3_+ \). For \( \hat{R}_0 = 1 \), \( x_e \) coincides with \( x_f \). By local analysis based on Jacobian linearization\(^2\), the disease-free equilibrium \( x_f \) is asymptotically stable if \( \hat{R}_0 \leq 1 \) for the constant \( B(t) \equiv \hat{B} \geq 0 \) (see, e.g., [16]). The endemic equilibrium \( x_e \) is asymptotically stable if \( \hat{R}_0 > 1 \). Here, as in the fundamental of stability theory, the proved asymptotic stability is local in the sense that the estimated domain of attraction is a sufficiently small neighborhood of the equilibrium. Construction of a Lyapunov function has a potential to go beyond the local property [18]. If a Lyapunov function is found, an appropriate sublevel set of the function can be an estimate of the domain of attraction.

Once one of \( x_f \) and \( x_e \) is chosen as the target equilibrium, let \( \hat{x} \in \mathbb{R}_+^3 \) denote the chosen equilibrium and define
\[
\begin{align*}
    \hat{x}(t) &:= x(t) - \hat{x} \\
    \hat{u}(t) &:= B(t) - \hat{B}.
\end{align*}
\]
Then the SIR model (3.1) can be rewritten as
\[
\hat{x} = \hat{f}(\hat{x}, \hat{u}),
\]
where the function \( \hat{f} = [\hat{f}_1, \hat{f}_2, \hat{f}_3]^T \) satisfies \( \hat{f}(0,0) = 0 \). For brevity, let \( [-\hat{x}_1, \infty)^3 \) denote \( [-\hat{x}_1, \infty) \times [-\hat{x}_2, \infty) \times [-\hat{x}_3, \infty) \). System (3.10) is defined on \( [-\hat{x}_i, \infty)^3 \). The main objective of this paper is to prove that system (3.10) is ISS with respect to the newborn/immigration rate perturbation \( \hat{u} \) on the entire state space of \( \hat{x} \). This property is not obvious from (3.4) since ISS requires not only boundedness of the state \( \hat{x} \), but also a gain function that characterizes the boundedness as a continuous function \( \Gamma \) of the input \( \hat{u} \) so that asymptotic stability is included as a special case, i.e., (2.2). Importantly, another major objective is the construction of an ISS Lyapunov function which serves as an classical (but, strict) Lyapunov function when \( \hat{u} = 0 \).

**Remark 3.1.** This paper does not introduce assumptions on \( B \) to make the analysis simple. For example, if \( B = \mu(S+I+R) \) or an equivalent formulation is assumed, we have \( S(t) + I(t) + R(t) = N \) for all \( t \in \mathbb{R}_+ \) with a positive constant \( N \) [16]. This

\(^2\)In the field of nonlinear systems and control, the term “local” is used exclusively for the existence of a sufficiently small set in which a claimed property holds true. One cannot specify the set a priori.
dependence between variables allows one to remove one of the three variable from (3.1). Many analytical studies assume this simplification (e.g., [21, 20, 28]), and the equation is sometimes called the SIRS model. The same implication has also been employed for variants of the SIR model in some studies (e.g., [22, 19, 8, 34]). The simplification disallows one to consider perturbation and immigration, and $B$ becomes endogenous. The simplification prevents the robustness analysis.

4. Disease-Free Equilibrium. The first result in this paper is stated as the next theorem.

**Theorem 4.1.** Suppose that $\hat{B} > 0$ and

\begin{equation}
\hat{R}_0 < 1
\end{equation}

hold. Let $\hat{x} = x_f$. Then the disease-free equilibrium $\hat{x} = 0$ of the SIR model (3.1) is asymptotically stable, and the set $[-\hat{x}_1, \infty) \times \mathbb{R}^2_+$ is the domain of attraction. Moreover, the SIR model (3.1) is ISS on $[-\hat{x}_1, \infty) \times \mathbb{R}^2_+$ with respect to the newborn rate perturbation $\hat{u}$ satisfying

\begin{equation}
\forall t \in \mathbb{R}_+ \text{, } \hat{u}(t) \in [-\hat{B}, \infty).
\end{equation}

Furthermore, the function $\hat{V} : \mathbb{R} \times \mathbb{R}^2_+ \to \mathbb{R}_+$ defined by

\begin{equation}
\hat{V}(\hat{x}) = \begin{cases} 
-\frac{\mu_0 \hat{x}_1}{\beta \hat{x}_1} & \hat{x}_1 < -\frac{\beta \hat{x}_1}{\mu_0} (\hat{x}_2 + \lambda_3 \hat{x}_3) \\
\hat{x}_2 + \lambda_3 \hat{x}_3, & \hat{x}_1 + \hat{x}_2 + \lambda_3 \hat{x}_3, \quad 0 \leq \hat{x}_1 
\end{cases}
\end{equation}

is locally Lipschitz on $\mathbb{R} \times \mathbb{R}^2_+$, and an ISS Lyapunov function on $[-\hat{x}_1, \infty) \times \mathbb{R}^2_+$ with respect to (4.2).

**Proof.** First, recall that $\hat{B} > 0$ and (4.1) imply $\hat{x}_1 = \hat{B}/\mu > 0$, $\hat{x}_2 = \hat{x}_3 = 0$. Thus, $\hat{x}_2 = x_2 \geq 0$, $\hat{x}_3 = x_3 \geq 0$. Definition (4.6) and conditions (4.1) and (4.5) yield $\hat{R}_0 + \epsilon < 1$, and $0 < \gamma_0 < \gamma$. Hence, $\lambda_3 > 0$ in (4.7). Define

\begin{align}
(4.8a) \quad A &:= \{ \hat{x} \in \mathbb{R} \times \mathbb{R}^2_+ : 0 \leq \hat{x}_1 \} \\
(4.8b) \quad B &:= \{ \hat{x} \in \mathbb{R} \times \mathbb{R}^2_+ : -\frac{\beta \hat{x}_1}{\mu_0} (\hat{x}_2 + \lambda_3 \hat{x}_3) \leq \hat{x}_1 < 0 \} \\
(4.8c) \quad C &:= \{ \hat{x} \in \mathbb{R} \times \mathbb{R}^2_+ : \hat{x}_1 < -\frac{\beta \hat{x}_1}{\mu_0} (\hat{x}_2 + \lambda_3 \hat{x}_3) \}.
\end{align}

The partitioning of (4.8) clearly satisfies $A \cup B \cup C = \mathbb{R} \times \mathbb{R}^2_+$. By definition (4.3), $\hat{V}(\hat{x}) = 0$ holds if and only if $\hat{x} = 0$ in $\mathbb{R} \times \mathbb{R}^2_+$. We have $\hat{V}(\hat{x}) > 0$ for all $\mathbb{R} \times \mathbb{R}^2_+ \setminus \{0\}$. At $\hat{x}_1 = 0$, the function $\hat{V}$ is continuous and $\hat{V}(\hat{x}_1) = \hat{x}_2 + \lambda_3 \hat{x}_3$. It is also verified at
the point \( -\beta \tilde{x}_1(\tilde{x}_2 + \lambda_3 \tilde{x}_3) / \mu_0 \) that the function \( \tilde{V} \) is continuous and \( \tilde{V}(\tilde{x}_1) = \tilde{x}_2 + \lambda_3 \tilde{x}_3 \). Hence, the function \( \tilde{V} \) defined by (4.3) is locally Lipschitz on \( \mathbb{R} \times \mathbb{R}^2_+ \).

Now, we evaluate the derivative of \( \tilde{V}(\tilde{x}(t)) \) along the solution \( \tilde{x}(t) \) of (3.1) region by region. In region A, by virtue of \( B - \mu \tilde{x}_1 - \beta \tilde{x}_2 \tilde{x}_1 = B - \mu \tilde{x}_1 = 0 \), from (4.7) we obtain

\[
\frac{\partial \tilde{V}}{\partial \tilde{x}} \tilde{\dot{x}} = B - \mu S - \beta IS + \beta IS - \gamma I - \mu I + \lambda_3(\gamma I - \mu R)
= \tilde{u} - \mu \tilde{x}_1 - (\gamma_0 + \mu) \tilde{x}_2 - \lambda_3 \mu \tilde{x}_3
\leq -\mu \tilde{V}(\tilde{x}) + \tilde{u}.
\] (4.9)

In region B we have

\[
\beta \tilde{x}_1 < \beta \tilde{x}_1 = \frac{\beta B}{\mu} = (\gamma + \mu) \tilde{R}_0.
\]

Due to (4.6), in region B,

\[
\frac{\partial \tilde{V}}{\partial \tilde{x}} \tilde{\dot{x}} = \beta IS - \gamma I - \mu I + \lambda_3(\gamma I - \mu R)
= - (\gamma_0 + \mu - \beta \tilde{x}_1) \tilde{x}_2 - \lambda_3 \mu \tilde{x}_3
\leq -\varepsilon (\gamma_0 + \mu) \tilde{x}_2 - \lambda_3 \mu \tilde{x}_3
\leq -\xi \tilde{V}(\tilde{x})
\] (4.10)

is obtained, where \( \xi := \min \{ \varepsilon (\gamma_0 + \mu), \mu \} \). In region C, since the definition of C yields

\[
\tilde{x}_1 < -\frac{\beta \tilde{x}_1}{\mu_0} (x_2 + \lambda_3 x_3) \leq -\frac{\beta \tilde{x}_1}{\mu_0} x_2 < -\frac{\beta}{\mu_0} x_1 x_2
\]

it is verified that

\[
\frac{\partial \tilde{V}}{\partial \tilde{x}} \tilde{\dot{x}} = -\frac{\mu_0}{\beta \tilde{x}_1} (B - \mu S - \beta SI)
= -\frac{\mu_0}{\beta \tilde{x}_1} (\tilde{u} - \mu - \mu_0) \tilde{x}_1 - \mu_0 \tilde{x}_1 - \beta \tilde{x}_1 \tilde{x}_2
\leq \frac{\mu_0}{\beta \tilde{x}_1} ((\mu - \mu_0) \tilde{x}_1 - \tilde{u}).
\] (4.11)

Note that \( \tilde{x}_1 < 0 \) in C.

Due to (4.4), combining (4.9), (4.10) and (4.11), for an arbitrarily given \( \delta \in (0, 1) \), we obtain

\[
\tilde{V}(\tilde{x}) \geq \frac{1}{\delta (\mu - \mu_0)} |\tilde{u}| \Rightarrow \frac{\partial \tilde{V}}{\partial \tilde{x}} \tilde{\dot{x}} \leq -(1 - \delta) (\mu - \mu_0) \tilde{V}(\tilde{x})
\] (4.12)

for all \( \tilde{x} \in \mathbb{R} \times \mathbb{R}^2_+ \) and \( \tilde{u} \in [-\tilde{B}, \infty) \). Equation (3.1) by itself guarantees \( \tilde{x}(t) \in [-\tilde{x}_1, \infty) \times \mathbb{R}^2_+ \) for all \( t \in \mathbb{R}_+ \) with respect to all \( \tilde{x}(0) \in [-\tilde{x}_1, \infty) \times \mathbb{R}^2_+ \) and \( \tilde{u} \in [-\tilde{B}, \infty) \). Therefore, all the claims are proved.

Let the perturbed basic reproduction number \( R_0(t) \) be defined with \( B = \tilde{u}(t) + \tilde{B} \), while the (nominal) basic reproduction number \( \tilde{R}_0 \) has been defined with the nominal rate \( \tilde{B} \). If \( \lim_{t \to \infty} R_0(t) > 1 \), the state \( x(t) \) does not converge to \( x_f \) even for (4.1).
In the same way, if \( \lim_{t \to \infty} R_0(t) < 1 \) holds, the state \( x \) does not converge to \( x_e \) even for (5.4) to be presented in the next section. The established ISS property does not override the mechanism of the basic reproduction number. Note that \( \hat{R}_0 = 1 \) holds if and only if \( x_f = x_e \). The ISS property obtained in Theorem 4.1 not only guarantees the boundedness of \( \hat{x}(t) \) with respect to bounded \( \hat{u}(t) \), but also continuous variation of the bound with respect to the maximum magnitude of \( \hat{u}(t) \). Interestingly, the continuous transition holds true although the change of \( \hat{u} \) causes a bifurcation. The obtained property (4.12) together with definition (4.3) establishes that the bound of the state variable \( \hat{x} \) is a linear function of the magnitude of the variation \( \hat{u} \). Figure 1 illustrates level sets of the ISS Lyapunov function (4.3) for \( V = 10, 30, 60, 100, 180, 260, 340, ..., 500 \). The parameters are \( \beta = 0.0002, \mu = 0.015, \gamma = 0.032, \hat{B} = 3 \) and \( \mu_0 = 0.0149 \), and they satisfy \( \hat{R}_0 = 0.851 < 1 \), (4.4) and (4.5) with \( \epsilon = 0.0745 \) and \( \mu_0 = 0.0148 \).

**Remark 4.2.** The preceding study [13] demonstrated ISS of the SIR model (3.1) irrespective of the value \( R_0 \) by treating the entire amount \( B \) as the input of the ISS property. It means that in [13], the whole \( R_0 \) is a disturbance, and its nominal value is \( R_0 = 0 \). Hence, the focused equilibrium was \( \hat{x} = [0, 0, 0]^T \) in [13], instead of \( x_f = [\hat{B}_0/\mu, 0, 0]^T \). The ISS of (3.1) for \( \hat{x} = 0 \) does not conclude that the state \( x \) converges to the point \( x_f = [\hat{B}_0/\mu, 0, 0]^T \) when \( B(t) = \hat{B} \) and \( \hat{R}_0 < 1 \). The ISS property of (3.1) with respect to the non-zero equilibrium is not obvious from the ISS property of (3.1) with the zero equilibrium \( x = \hat{x} = 0 \) either.

### 5. Endemic Equilibrium.

This section constructs a Lyapunov function dealing with the endemic equilibrium \( x_e \). For this purpose, we set \( \hat{x} = x_e \). When the disease-free equilibrium \( x_f \) was of interest, the component \( x_2 \) of the trajectories \( x(t) \) of the SIR model (3.1) could not go below \( \hat{x}_2 \). Thus, the level contours of the Lyapunov function (4.3) were sheared off at the plane of \( x_2 = \hat{x}_2 \) in the three-dimensional space of \( x \). Since \( x_2 \) can go below \( \hat{x}_2 \) for the endemic equilibrium \( x_e \), an end of each level contour of the Lyapunov function (4.3) needs to be placed more carefully at the plane of \( x_2 = 0 \) in order be able to connect the other end to form a loop. In addition to closing the contours, the influence of equation (3.1c) is not as simple as that in the case of the disease-free equilibrium. In fact, the endemic equilibrium also allows \( x_3 \) to...
be go below \( \hat{x}_3 \). The term of \( x_2 \) in (3.1c) needs to be taken care of depending on the sign of \( \hat{x}_2 \) and \( \hat{x}_3 \) to make the Lyapunov function decrease along the trajectory \( x(t) \).

Let \( x_{1,2}(t) = [x_1(t), x_2(t)]^T \), and define

\[
\begin{align*}
\Omega & := \{ \hat{x} \in [-\hat{x}_1, \infty)^3 : \hat{x}_2 \neq -\hat{x}_2 \} \\
G_{1,2} & := \{ \hat{x}_{1,2} \in [-\hat{x}_1, \infty)^2 : \hat{x}_1 + \hat{x}_2 > -\hat{x}_2, \ \hat{x}_2 \neq -\hat{x}_2 \} \\
G & := G_{1,2} \times [-\hat{x}_3, \infty).
\end{align*}
\]

The set \( \Omega \) is the domain on which we want to establish stability properties. The situation \( \hat{x}_2 = -\hat{x}_2 \) is and must be removed from \( \Omega \) since \( x_f \) remains an equilibrium of the SIR model (3.1) independently of \( B \), i.e., \( \hat{R}_0 \). Indeed, the point \( I = 0 \), i.e., \( \hat{x}_2 = -\hat{x}_2 \), remains an equilibrium of (3.1b) irrespective of \( x_1 \) and \( x_3 \). The set \( G \) is the domain on which a Lyapunov function is to be constructed. The following summarizes stability properties established in this section.

**Theorem 5.1. Assume that \( \hat{B} > 0 \) and**

\[
\hat{R}_0 > \frac{\gamma}{\mu} + 2
\]

**hold. Let \( \hat{x} = x_e \). Then the endemic equilibrium \( \hat{x} = 0 \) of the SIR model (3.1) is asymptotically stable, and any compact subset in \( \Omega \) belongs to the domain of attraction. Furthermore, for an arbitrarily given compact set \( \overline{G} \) contained in the interior of \( G \), there exists a compact set \( \overline{G} \supset \overline{G} \) such that the SIR model (3.1) is ISS on \( \overline{G} \) with respect to the newborn/immigration rate perturbation \( \hat{u} \) satisfying (a.e.)**

\[
\forall t \in \mathbb{R}_+ \quad \hat{u}(t) \in [-\hat{B}, \infty)
\]

The above theorem is established by the construction of the following ISS Lyapunov function.

**Theorem 5.2. Assume that \( \hat{B} > 0 \) and (5.4) are satisfied. Define the function**

\[
\hat{V}(\hat{x}) = \hat{V}_{1,2}(\hat{x}_{1,2}) + \hat{V}_3(\hat{x}_3)
\]

**with**

\[
\begin{align*}
\hat{V}_{1,2}(\hat{x}_{1,2}) &= \begin{cases} \\
P^{-1} \left( -\lambda_1 \hat{x}_1 + \lambda_2 \hat{x}_2 \right), & 0 \leq \hat{x}_2, \ \hat{x}_1 < \nu(\hat{x}_2) \\
(\lambda_2 - k\lambda_1)\hat{x}_2, & 0 \leq \hat{x}_2, \ \nu(\hat{x}_2) \leq \hat{x}_1 < -k\hat{x}_2 \\
\lambda_1 \hat{x}_1 + \lambda_2 \hat{x}_2, & 0 \leq \hat{x}_1, -k\hat{x}_2 \leq \hat{x}_1 \\
P^{-1}(-\lambda_1 \hat{x}_1 - \lambda_2 \hat{x}_2), & \hat{x}_2 < 0, \ \hat{x}_1 \leq -k\hat{x}_2 \\
P^{-1}((k\lambda_1 - \lambda_2)\hat{x}_2), & \hat{x}_2 < 0, \ -k\hat{x}_2 < \hat{x}_1 \leq \theta^{-1}(-\hat{x}_2) \\
\lambda_1 \hat{x}_1 - \lambda_2 \hat{x}_2, & \hat{x}_2 < 0, \ \theta^{-1}(-\hat{x}_2) < \hat{x}_1
\end{cases}
\]

\[
\hat{V}_3(\hat{x}_3) = \lambda_3 |\hat{x}_3|
\]
and
\[
\theta(s) = \hat{x}_2 - \frac{\hat{x}_1 \hat{x}_2}{\hat{x}_1 + s}, \quad s \in (-\hat{x}_1, \infty)
\]
(5.9)

\[
\theta^{-1}(s) = \frac{\hat{x}_1 \hat{x}_2}{\hat{x}_2 - s} - \hat{x}_1, \quad s \in (-\infty, \hat{x}_2)
\]
(5.10)

\[
0 < \lambda_1 = \lambda_2
\]
(5.11)

\[
0 < \kappa < \min \left\{ 1 - \frac{\gamma + \mu}{\mu(R_0 - 1)}, \frac{\lambda_2 \theta^{-1}(\gamma \omega^{-1})}{\lambda_1 \theta^{-1}(\gamma \omega^{-1})} \right\}
\]
(5.12)

\[
0 < \lambda_3 < \min \left\{ \kappa \lambda_1 (R_0 - 1)(1 - k), \frac{\hat{\lambda}_2^2}{(1 - k) \lambda_1}, \frac{\beta \lambda_2 (\hat{\theta}_2 - \theta \circ \omega^{-1})}{\lambda_1}, \hat{\lambda}_2 \right\}
\]
(5.13)

\[
\omega(s) = \lambda_1 s + \hat{\lambda}_2 \theta(s), \quad s \in (-\hat{x}_1, \infty)
\]
(5.14)

\[
P(s) = (\lambda_2 - k \lambda_1) \theta \circ \omega^{-1}(s), \quad s \in \mathbb{R}
\]
(5.15)

\[
P^{-1}(s) = \lambda_1 \theta^{-1} \left( \frac{s}{\lambda_2 - k \lambda_1} \right) + \frac{\hat{\lambda}_2 s}{\lambda_2 - k \lambda_1}, \quad s \in (-\infty, (\lambda_2 - k \lambda_1) \hat{x}_2)
\]
(5.16)

\[
\nu(s) = \frac{1}{\lambda_1} \left( \lambda_2 s - P((\lambda_2 - k \lambda_1)s) \right), \quad s \in \mathbb{R}
\]
(5.17)

for \( \overline{L} > 0 \) and \( \hat{\lambda}_2 > 0 \). If \( \overline{L} > 0 \) and \( \hat{\lambda}_2 > 0 \) satisfy
\[
\forall L \in [0, \overline{L}] \; \nu \left( \frac{L}{\lambda_2 - k \lambda_1} \right) \leq \theta^{-1} \left( \frac{-L}{\lambda_2 - k \lambda_1} \right),
\]
(5.18)

the function \( \hat{V} \) is locally Lipschitz on the set
\[
H(\hat{\lambda}_2, k, \overline{L}) := \left\{ \hat{x} \in \mathbb{R}^3 : \begin{array}{l}
-\hat{x}_2 < \hat{x}_2 \leq \frac{\overline{L}}{\lambda_2(1 - k)}, \\
-\hat{x}_1 - \hat{x}_2 < (1 - k) \hat{x}_2, \\
-\lambda_1 \hat{x}_1 + \hat{\lambda}_2 \hat{x}_2 < \lambda_2(1 - k) \hat{x}_2
\end{array} \right\},
\]
(5.19)

and the function \( \hat{V} \) is an ISS Lyapunov function on
\[
G(\hat{\lambda}_2, k, \overline{L}) := \left\{ \hat{x} \in [-\hat{x}_1, \infty)^3 : \hat{V}(\hat{x}) \leq \overline{L} \right\}
\]
(5.20)

with respect to the input \( \hat{u} \) satisfying
\[
\forall t \in \mathbb{R}^+ \quad \hat{u}(t) \in \left( -\delta \frac{P(\overline{L})}{\lambda_1}, \frac{\delta \mu \overline{L}}{\lambda_1} \right)
\]
(5.21)

for an arbitrarily given \( \delta \in (0, 1) \).

The parameter \( \hat{\lambda}_2 > 0 \) introduced in (5.7) copes with the both-sided variables \( \hat{x}_2 \) and \( \hat{x}_3 \). The following lemma shows that the sublevel sets of the ISS Lyapunov function \( \hat{V} \) can always cover\(^3\) the set \( G \) entirely as \( \overline{L} \to \infty \) and \( \hat{\lambda}_2 \to 0 \), which is the key to the establishment of Theorem 5.1 from Theorem 5.2. In fact, it forms a central and unique idea of this paper.

\(^3\)cover any bounded sets in
Lemma 5.3. Assume that (5.4) is satisfied. Suppose that (5.6)-(5.17) and (5.19)-(5.20) are defined and given. Then the following hold true:

(i) For any compact set \( G \) contained in the interior of \( G \), there exist \( \lambda_2 > 0 \), \( \mathcal{T} \geq 0 \) and \( k \in (0, k_0) \) such that (5.18) and

\[
G \subset \mathcal{G}(\lambda_2, k, \mathcal{T}) \subset G
\]

are satisfied for all \( \lambda_2 \in (0, \lambda_2] \) and all \( k \in (0, \mathcal{K}] \).

(ii) For each \( k \) satisfying (5.12),

\[
0 \leq a \leq b \Rightarrow \mathcal{G}(b, k, \mathcal{T}) \subset \mathcal{G}(a, k, \mathcal{T})
\]

holds for all \( L \in [0, \mathcal{T}] \) if (5.18) holds.

(iii) For each \( \lambda_2 \geq 0 \),

\[
0 \leq a \leq b \Rightarrow \mathcal{G}(\lambda_2, b, \mathcal{T}) \cap \left\{ \bar{x}_2 \leq \frac{L}{\lambda_2} \right\} \subset \mathcal{G}(\lambda_2, a, \mathcal{T}) \cap \left\{ \bar{x}_2 \leq \frac{L}{\lambda_2} \right\}
\]

holds for all \( L \in [0, \mathcal{T}] \) if (5.18) holds.

(iv) Property (5.18) holds for any all \( \mathcal{T} \in \mathbb{R}_+ \) if \( \lambda_2 = 0 \).

As demonstrated in Remark 6.2 in section 6, the ISS-gain function from \( \hat{u} \) to \( \hat{x} \) is bounded from above by a linear function. Recall that if a negative value \( \hat{u} \) goes below the threshold determined by the basic reproduction number, a bifurcation occurs. The ISS property established by Theorem 5.1 establishes a linear transition globally in spite of the bifurcation.

Level sets of the ISS Lyapunov function (5.6) are shown in Figure 2 for \( \hat{V} = 20, 100, 180, 260, \ldots, 340 \). The parameters are \( \beta = 0.0002 \), \( \mu = 0.015 \), \( \gamma = 0.032 \) and \( \hat{B} = 17 \), and they satisfy \( \hat{R}_0 = 4.82271 > 4.1333 = \gamma/\mu + 2 \). It can be verified that \( \hat{\lambda}_2 = 0.01 \), \( k = 0.0902 \) and \( \mathcal{T} = 340 \) fulfill (5.12) and (5.18). The level sets can be expanded further by using smaller \( \hat{\lambda}_2 \), \( k \) and \( 1/\mathcal{T} \).

Remark 5.4. In contrast to \( \bar{x}_2 = -\hat{x}_2 \) which is an equilibrium of (3.1b) irrespective of \( x_1 \) and \( x_3 \), the equilibrium of \( x_1 \)-equation (3.1a) depends on its input \( x_2 \), and the equilibrium of \( x_3 \)-equation (3.1c) is influenced by its input \( x_2 \). Therefore, excluding the two-dimensional spaces \( \hat{x}_1 = -\hat{x}_1 \) and \( \hat{x}_3 = -\hat{x}_3 \) from the domain of \( \hat{V} \) is not necessary. In fact, the function chosen in (5.6) is not forced to be unbounded at \( \hat{x}_1 = -\hat{x}_1 \) and \( \hat{x}_3 = -\hat{x}_3 \). The popular logarithmic function [21] excludes \( \hat{x}_1 = -\hat{x}_1 \) and \( \hat{x}_3 = -\hat{x}_3 \), and becomes unbounded there.

6. Proofs for the Endemic Equilibrium.

6.1. Proof of Lemma 5.3. Since (5.4) implies \( (\gamma + \mu)/\mu < \hat{R}_0 - 1 \), we have

\[
0 < \frac{\gamma + \mu}{\mu(\hat{R}_0 - 1)} < 1
\]

and \( \hat{x}_1 < \hat{x}_2 \). Property (5.12) yields \( k \in (0, 1) \), and property (5.11) guarantees \( \lambda_2 - k\lambda_1 > 0 \). The choice (5.12) yields

\[
\theta^{-1}\left( -\frac{\mathcal{T}}{\lambda_2} \right) < \frac{\mathcal{T}}{\lambda_2/k - \lambda_1}.
\]
Since the function \( \theta^{-1} \) satisfies \( \theta^{-1}(0) = 0 \) and \( (\theta^{-1})'(s) > 0 \) for all \( s \in (-\infty, \hat{x}_2) \), we have
\[
\theta^{-1}\left(-\frac{L}{\lambda_2 - k\lambda_1}\right) < \theta^{-1}\left(-\frac{L}{\lambda_2}\right).
\]
From (5.10) it is verified that \( (\theta^{-1})''(s) > 0 \) holds for all \( s \in (-\infty, \hat{x}_2) \). Thus,
\[
\forall L \in [0, \overline{\lambda}] \quad \theta^{-1}\left(-\frac{L}{\lambda_2 - k\lambda_1}\right) \leq \frac{-kL}{\lambda_2 - k\lambda_1}
\]
is achieved. Combining this with (5.18) yields
\[
\forall L \in [0, \overline{\lambda}] \quad \nu\left(\frac{L}{\lambda_2 - k\lambda_1}\right) \leq \frac{-kL}{\lambda_2 - k\lambda_1}.
\]
Therefore, the partitioning in (5.7) is well-defined as long as \( \hat{x} \in H(\hat{\lambda}_2, k, \overline{\lambda}) \). Define

\begin{align*}
\text{(6.2a)} & \quad A := \{ \hat{x} \in H(\hat{\lambda}_2, k, \overline{\lambda}) : 0 \leq \hat{x}_2, \ -k\hat{x}_2 \leq \hat{x}_1 \} \\
\text{(6.2b)} & \quad B := \{ \hat{x} \in H(\hat{\lambda}_2, k, \overline{\lambda}) : 0 \leq \hat{x}_2, \ \nu(\hat{x}_2) \leq \hat{x}_1 < -k\hat{x}_2 \} \\
\text{(6.2c)} & \quad C := \{ \hat{x} \in H(\hat{\lambda}_2, k, \overline{\lambda}) : 0 \leq \hat{x}_2, \ \hat{x}_1 < \nu(\hat{x}_2) \} \\
\text{(6.2d)} & \quad D := \{ \hat{x} \in H(\hat{\lambda}_2, k, \overline{\lambda}) : \hat{x}_2 < 0, \ \hat{x}_1 \leq -k\hat{x}_2 \} \\
\text{(6.2e)} & \quad E := \{ \hat{x} \in H(\hat{\lambda}_2, k, \overline{\lambda}) : \hat{x}_2 < 0, \ -k\hat{x}_2 < \hat{x}_1 \leq \theta^{-1}(-\hat{x}_2) \} \\
\text{(6.2f)} & \quad F := \{ \hat{x} \in H(\hat{\lambda}_2, k, \overline{\lambda}) : \hat{x}_2 < 0, \ \theta^{-1}(-\hat{x}_2) < \hat{x}_1 \}.
\end{align*}

Clearly, we have
\[
A \cup B \cup C \cup D \cup E \cup F = H(\hat{\lambda}_2, k, \overline{\lambda}).
\]
By the definition of (5.6), (5.7), and (5.8), we have \( \bar{V}(\hat{x}) < \infty \) for all \( \hat{x} \in H(\hat{\lambda}_2, k, \overline{L}) \). On the set of \( \hat{x}_{1,2} \) belonging to \( H(\hat{\lambda}_2, k, \overline{L}) \), \( \bar{V}_{1,2}(\hat{x}_{1,2}) = 0 \) implies \( \hat{x}_{1,2} = 0 \). Due to (5.8) and (5.6),

(6.3) \[ \bar{V}(\hat{x}) = 0 \iff \hat{x} = 0. \]

By virtue of \( \lambda_2 - k\lambda_1 > 0 \), the implications

\[-k\hat{x}_2 \leq \hat{x}_1 \Rightarrow \lambda_1\hat{x}_1 + \lambda_2\hat{x}_2 \geq (\lambda_2 - k\lambda_1)\hat{x}_2 \]
\[-k\hat{x}_2 \geq \hat{x}_1 \Rightarrow -\lambda_1\hat{x}_1 - \lambda_2\hat{x}_2 \geq -(\lambda_2 - k\lambda_1)\hat{x}_2 \]

yield

(6.4) \[ \hat{x} \neq 0 \Rightarrow \bar{V}(\hat{x}) > 0. \]

For \( \hat{x}_2 \geq 0 \), the function \( \bar{V}_{1,2} \) is continuous at \( \hat{x}_1 = -k\hat{x}_2 \), and \( \bar{V}_{1,2}(\hat{x}_{1,2}) = -k\lambda_1\hat{x}_2 + \lambda_2\hat{x}_2 \). At \( \hat{x}_1 = \nu(\hat{x}_2) \) for \( \hat{x}_2 \geq 0 \), the function \( \bar{V}_{1,2} \) is continuous since

\[
\bar{V}_{1,2}(\hat{x}_{1,2}) = P^{-1}(\lambda_1\hat{x}_1 + \lambda_2\hat{x}_2)
\]
\[
= P^{-1}(\lambda_1\nu(\hat{x}_2) + \lambda_2\hat{x}_2)
\]
\[
= P^{-1}(\lambda_2\hat{x}_2 + P((\lambda_2 - k\lambda_1)\hat{x}_2) + \lambda_2\hat{x}_2)
\]
\[
= (\lambda_2 - k\lambda_1)\hat{x}_2.
\]

For \( \hat{x}_2 < 0 \), the function \( \bar{V}_{1,2} \) is continuous at \( \hat{x}_1 = -k\hat{x}_2 \), and \( \bar{V}_{1,2}(\hat{x}_{1,2}) = P^{-1}((k\lambda_1 - \lambda_2)\hat{x}_2) \). At \( \hat{x}_1 = \theta^{-1}(-\hat{x}_2) \) for \( \hat{x}_2 < 0 \), the function \( \bar{V}_{1,2} \) is continuous since

\[
P^{-1}((k\lambda_1 - \lambda_2)\hat{x}_2) = P^{-1}(\lambda_2 - k\lambda_1)\theta(\hat{x}_1)
\]
\[
= \lambda_1\hat{x}_1 + \lambda_2\theta(\hat{x}_1)
\]
\[
= \lambda_1\hat{x}_1 - \lambda_2\hat{x}_2.
\]

For \( \hat{x}_1 < 0 \), the function \( \bar{V}_{1,2} \) is continuous at \( \hat{x}_2 = 0 \), and \( \bar{V}_{1,2}(\hat{x}_{1,2}) = P^{-1}(-\lambda_1\hat{x}_1) \).

At \( \hat{x}_2 = 0 \) for \( \hat{x}_1 > 0 \), the function \( \bar{V}_{1,2} \) is continuous and \( \bar{V}_{1,2}(\hat{x}_{1,2}) = \lambda_1\hat{x}_1 \). These arguments verify

(6.5) \[ \overline{G}(\lambda_2, k, \overline{L}) \subset H(\hat{\lambda}_2, k, \overline{L}). \]

Define

\[ \overline{G}_{1,2}(\lambda_2, k, L) := \{ \hat{x} \in [-\hat{x}_1, \infty)^2 : \bar{V}_{1,2}(\hat{x}_{1,2}) \leq L \} \]

for \( L \in \mathbb{R}_+ \). Since for each \( s > 0 \), \( P(s) \) defined with \( \hat{\lambda}_2 = a \) is larger than \( P(s) \) defined with \( \hat{\lambda}_2 = b \) for \( 0 \leq a \leq b \), the definition (5.7) yields

(6.6) \[ 0 \leq a \leq b \Rightarrow \overline{G}_{1,2}(b, k, L) \subset \overline{G}_{1,2}(a, k, L) \]

for all \( L \in \mathbb{R}_+ \). The definitions (5.6) and (5.8) proves (5.23) in (ii). Property (5.24) in (iii) is also verified from (5.7).

Since \( \omega(s) \) is increasing in \( \hat{\lambda}_2 \geq 0 \) for each \( s > 0 \) by definition, \( P(s) \) is decreasing in \( \hat{\lambda}_2 \). Thus the function \( \nu(L) \) is increasing in \( \hat{\lambda}_2 \geq 0 \) for each \( L > 0 \). Hence, for each
\( T \geq 0 \), there always exists \( \lambda_2 > 0 \) such that (5.18) holds. In fact, for \( \lambda_2 = 0 \) and 
\( \lambda_0 := \lambda_2 - k \lambda_1 > 0 \) we have
\[
\lambda_1 \vartheta^{-1}(-s) - \lambda_1 \nu(s) = \frac{\lambda_1 \dot{x}_1 \lambda_0 \dot{x}_2 - \lambda_1 \dot{x}_1 + \lambda_0 \dot{x}_2 - \frac{\lambda_1 \dot{x}_1 \lambda_0 \dot{x}_2}{\lambda_1 \dot{x}_1 + \lambda_0 s}}{\lambda_0 s + \lambda_1 \dot{x}_1 + \lambda_0 \dot{x}_2} = \frac{\lambda_0 s(\lambda_0 s + \lambda_1 \dot{x}_1 + \lambda_0 \dot{x}_2)(\lambda_0 \dot{x}_2 - \lambda_1 \dot{x}_1)}{(\lambda_1 \dot{x}_1 + \lambda_0 s)(\lambda_0 \dot{x}_2 + \lambda_0 s)} \geq 0
\]
for all \( s \in \mathbb{R}_+ \). The last inequality follows from (5.11) and
\[
1 - k = \frac{\lambda_0}{\lambda_1} \geq \frac{\dot{x}_1}{\ddot{x}_2} = \frac{\gamma + \mu}{\mu(R_0 - 1)}
\]
guaranteed by (5.12). Thus, property (5.18) holds for all \( T \in \mathbb{R}_+ \) if \( \overline{T} = 0 \). Item (iv) is proved.

Continuity of the functions guarantees the existence of \( \lambda_2 > 0 \) and \( T \geq 0 \) satisfying (5.18) for all \( \lambda_2 \in (0, \overline{\lambda}_2) \). By virtue of (5.7) and (6.6), for any \( \hat{x}_{1,2} \in G_{1,2} \), there exist \( \lambda_2 > 0 \), \( T \geq 0 \) and \( k \in (0, \lambda_0) \) such that \( \hat{x}_{1,2} \in \overline{G}_{1,2}(\lambda_2, k, T) \), (6.1) and (5.18) are satisfied for all \( \hat{\lambda}_2 \in (0, \lambda_2) \) and all \( k \in (0, \overline{k}) \). Therefore, (5.6) and (5.8) prove the claim (i).

### 6.2. Proof of Theorem 5.2.
First, recall that equation (3.1) is forward complete, and by itself guarantees the forward invariance of the set \( [-\hat{x}_1, \infty)^3 \). i.e., \( \hat{x}(t) \in [-\hat{x}_1, \infty)^3 \) for all \( t \in \mathbb{R}_+ \) with respect to all \( \hat{x}(0) \in [-\hat{x}_1, \infty)^3 \) and \( \hat{u}(t) \) satisfying \( \hat{u}(t) \in [-\hat{B}, \infty) \). As demonstrated in the proof of Lemma 5.3, for given \( \hat{\lambda}_2 \), \( \overline{T} \), \( k \) under the stated assumptions, the function \( \hat{V} \) is defined and continuous on \( H(\lambda_2, k, T) \), and satisfies (6.3) and (6.4). We also have \( \lambda_2 - k \lambda_1 > 0 \). Since \( \vartheta^{-1} \) and \( P^{-1} \) defined in (5.10) and (5.16) are locally Lipschitz, the function \( \hat{V}_{1,2} \) defined by (5.7) is locally Lipschitz, since \( \hat{V}_3 \) defined by (5.8) is locally Lipschitz, so is \( \hat{V} \). Since \( \lambda_2 \) and \( \hat{\lambda}_2 \) are positive, the definitions (5.10) and (6.9) imply \( (P^{-1})'(s) > 0 \) for all \( s \in (-\infty, (\lambda_2 - k \lambda_1) \hat{x}_2) \). In fact,
\[
(P^{-1})'(s) = \frac{d}{ds} \left( \lambda_1(\lambda_2 - k \lambda_1)^2 \dot{x}_1 \dot{x}_2 + \hat{\lambda}_2 \right) = \frac{\lambda_2(\lambda_2 - k \lambda_1)}{\lambda_2 - k \lambda_1}.
\]
From (5.16) and the above,
\[
\lim_{s \to (\lambda_2 - k \lambda_1) \hat{x}_2^-} P^{-1}(s) = \infty
\]
\[
\lim_{s \to (\lambda_2 - k \lambda_1) \hat{x}_2^-} (P^{-1})'(s) = \infty.
\]
Now, we evaluate the derivative of \( V(x(t)) \) along the solution \( x(t) \) of (3.1) region by region in accordance with (6.2). In the region \( A \cap \{ \hat{x}_3 \in [0, \infty) \} \), by virtue of (5.11) and \( f(\hat{x}, \bar{B}) = 0 \), we have
\[
\frac{\partial \hat{V}}{\partial \hat{x}} \hat{f} = \lambda_1(B - \mu S - \beta IS) + \lambda_2(\beta IS - \gamma I - \mu I) + \lambda_3(\gamma I - \mu R)
\]
\[
= \lambda_1(\hat{u} - \mu \hat{x}_1 - (\gamma A + \mu) \hat{x}_2) - \lambda_3 \mu \hat{x}_3
\]
\[
\leq -\mu \hat{V}(\hat{x}) + \lambda_1 \hat{u},
\]
\[(6.10)\]
where $\gamma_\lambda = \gamma(1 - \lambda_3/\lambda_2)$. In the region $A \cap \{\hat{x}_3 \in [-\hat{x}_3, 0]\}$,
\[
\frac{\partial \hat{V}}{\partial \hat{x}} \hat{f} = \lambda_1(B - \mu S - \beta IS) + \lambda_2(\beta IS - \gamma I - \mu I) + \lambda_3(\mu R - \gamma I)
\leq \lambda_1(\hat{u} - \mu \hat{x}_1 - (\gamma + \mu)\hat{x}_2) + \lambda_3 \mu \hat{x}_3
\leq -\mu \hat{V}(\hat{x}) + \lambda_1 \hat{u}.
\]
In the set $B \cap \{\hat{x}_3 \in [0, \infty]\}$, due to
\[-k\hat{x}_2 > x_1 - \hat{x}_1 = x_1 - \frac{\gamma + \mu}{\beta},
\]
we have
\[
\frac{\partial \hat{V}}{\partial \hat{x}} \hat{f} = (\lambda_2 - k_1)(\beta IS - \gamma I - \mu I) + \lambda_3(\gamma I - \mu R)
\leq (\lambda_2 - k_1)(\gamma + \mu - k_\beta \hat{x}_2 - \gamma - \mu) x_2 + \lambda_3(\gamma \hat{x}_2 - \mu \hat{x}_3)
\leq -k(\lambda_2 - k_1)\beta \hat{x}_2 \hat{x}_2 + \lambda_3(\gamma \hat{x}_2 - \mu \hat{x}_3)
= -\left(k(\lambda_2 - k_1)\mu(\hat{R}_0 - 1) - \lambda_3 \gamma\right) \hat{x}_2 - \lambda_3 \mu \hat{x}_3
= -\left(k\mu(\hat{R}_0 - 1) - \frac{\lambda_3 \gamma}{\lambda_2 - k_1}\right) \hat{V}_{1,2}(\hat{x}_{1,2}) - \lambda_3 \mu \hat{x}_3
\leq -a_B \hat{V}(\hat{x}).
\]
for some $a_B > 0$ since $k\mu(\hat{R}_0 - 1) - \lambda_3 \gamma/\lambda_2 - k\lambda_1$ is guaranteed by (5.11) and (5.13).
In the set $\mathbf{B} \cap \{\hat{x}_3 \in [-\hat{x}_3, 0]\}$, we obtain
\[
\frac{\partial \hat{V}}{\partial \hat{x}} \hat{f} = (\lambda_2 - k_1)(\beta IS - \gamma I - \mu I) + \lambda_3(\mu R - \gamma I)
\leq -k(\lambda_2 - k_1)\beta \hat{x}_2 \hat{x}_2 - \lambda_3 \gamma \hat{x}_2 + \lambda_3 \mu \hat{x}_3
= -\left(k\lambda_2 \mu(\hat{R}_0 - 1) + \lambda_3 \gamma\right) \hat{x}_2 + \lambda_3 \mu \hat{x}_3
= -\left(k\mu(\hat{R}_0 - 1) - \frac{\lambda_3 \gamma}{\lambda_2 - k_1}\right) \hat{V}_{1,2}(\hat{x}_{1,2}) + \lambda_3 \mu \hat{x}_3
\leq -a_B \hat{V}(\hat{x}).
\]
Since $\hat{x}_2 \leq -\theta(\hat{x}_1)$ is equivalent to $x_1 x_2 \leq \hat{x}_1 \hat{x}_2$, property (5.18) guarantees $x_1 x_2 \leq \hat{x}_1 \hat{x}_2$ for all $\hat{x}_{1,2}$ in $\mathbf{C}$. Hence, in the region $\mathbf{C} \cap \{\hat{x}_3 \in [0, \infty]\}$, we have
\[
\frac{\partial \hat{V}}{\partial \hat{x}} \hat{f} = (P^{-1})'(v) \left(\lambda_1(\mu \hat{x}_1 - \hat{u}) - \hat{\lambda}_2(\gamma_C + \mu) \hat{x}_2 - \lambda_3 \mu \hat{x}_3
\right.
\leq (P^{-1})'(v) \left(\lambda_1(\mu \hat{x}_1 - \hat{u}) - \hat{\lambda}_2(\gamma_C + \mu) \hat{x}_2 - \lambda_3 \mu \hat{x}_3
\right) - \lambda_3 \mu \hat{x}_3,
\]
by virtue of (6.7), where $v = P(\hat{V}_{1,2}(\hat{x}_{1,2})) = -\lambda_1 \hat{x}_1 + \hat{\lambda}_2 \hat{x}_2$. Note that
\[
\gamma_C := \gamma \left(1 - \frac{\lambda_3(\lambda_2 - k_1)}{\lambda_2^2}\right) \geq 0,
\]
due to (5.11) and (5.13). Property (6.9) implies the existence of \( \alpha_{c0} \in \mathcal{K}_\infty \) such that

\[
\dot{u} = 0 \Rightarrow \frac{\partial \bar{V}}{\partial \bar{x}} \bar{f} \leq -\alpha_{c0}(\bar{V}).
\]

In the region \( \mathbf{C} \cap \{ \bar{x}_3 \in [-\bar{x}_3, 0] \} \),

\[
\frac{\partial \bar{V}}{\partial \bar{x}} \bar{f} = (P^{-1})'(v) \left( \lambda_1(\mu \bar{x}_1 - \bar{u}) - \dot{\lambda}_2(\gamma + \mu)\bar{x}_2 \right) + \lambda_3 \mu \bar{x}_3 \\
+ (P^{-1})'(v)(\lambda_1 + \dot{\lambda}_2)\beta(x_1 x_2 - \dot{x}_1 \dot{x}_2) \\
\leq (P^{-1})'(v) \left( \lambda_1(\mu \bar{x}_1 - \bar{u}) - \dot{\lambda}_2(\gamma + \mu)\bar{x}_2 \right) + \lambda_3 \mu \bar{x}_3
\]

(6.16)

\[
\leq (P^{-1})'(v)(-\mu v - \lambda_1 \bar{u}) + \lambda_3 \mu \bar{x}_3,
\]

and (6.15). In the region \( \mathbf{D} \cap \{ \bar{x}_3 \in [-\bar{x}_3, 0] \} \), we obtain

\[
\frac{\partial \bar{V}}{\partial \bar{x}} \bar{f} = (P^{-1})'(w) \left[ \lambda_1(\mu S + \beta IS - B) \\
+ \lambda_2(\gamma I + \mu I - \beta IS) \right] + \lambda_3(\mu R - \gamma I) \\
\leq (P^{-1})'(w)\lambda_1(\mu \bar{x}_1 - \bar{u} + (\gamma D + \mu)\bar{x}_2) + \lambda_3 \mu \bar{x}_3 \\
\leq (P^{-1})'(w)(-\mu w - \lambda_1 \bar{u}) + \lambda_3 \mu \bar{x}_3,
\]

(6.17)

by virtue of (6.7), where \( w = P(\hat{V}_{1,2}(\bar{x}_{1,2})) = -\lambda_1 \bar{x}_1 - \lambda_2 \bar{x}_2 \) and

\[
\gamma_D := \gamma \left( 1 - \frac{\lambda_3(\lambda_2 - k\lambda_1)}{\lambda_2 \lambda_3} \right) \geq 0,
\]

Here, \( \gamma_D \geq 0 \) follows from \( k \in (0, 1) \), (5.11) and (5.13). The existence of \( \alpha_{D0} \in \mathcal{K}_\infty \) such that

\[
\dot{u} = 0 \Rightarrow \frac{\partial \bar{V}}{\partial \bar{x}} \bar{f} \leq -\alpha_{D0}(\bar{V})
\]

also follows from (6.9). In the case of \( \bar{x} \in \mathbf{D} \cap \{ \bar{x}_3 \in [0, \infty) \} \), we have

\[
\frac{\partial \bar{V}}{\partial \bar{x}} \bar{f} = (P^{-1})'(w) \left[ \lambda_1(\mu S + \beta IS - B) \\
+ \lambda_2(\gamma I + \mu I - \beta IS) \right] + \lambda_3(\mu R - \gamma I) \\
\leq (P^{-1})'(w)\lambda_1(\mu \bar{x}_1 - \bar{u} + (\gamma + \mu)\bar{x}_2) - \lambda_3 \mu \bar{x}_3 \\
\leq (P^{-1})'(w)(-\mu w - \lambda_1 \bar{u}) - \lambda_3 \mu \bar{x}_3,
\]

(6.18)

and (6.18). In the case of \( \bar{x} \in \mathbf{E} \cap \{ \bar{x}_3 \in [-\bar{x}_3, 0] \} \), from

\[
-k \bar{x}_2 < x_1 - \hat{x}_1 = x_1 - \frac{\gamma + \mu}{\beta}
\]

and (6.9) we obtain

\[
\frac{\partial \bar{V}}{\partial \bar{x}} \bar{f} = (P^{-1})'(z)(\lambda_2 - k\lambda_1)(\gamma I + \mu I - \beta IS) + \lambda_3(\mu R - \gamma I) \\
\leq (P^{-1})'(z)(\lambda_2 - k\lambda_1)(\gamma + \mu + k\beta \bar{x}_2 - \gamma - \mu) x_2 + \lambda_3(\mu \bar{x}_3 - \gamma \bar{x}_2) \\
\leq (P^{-1})'(z)(\lambda_2 - k\lambda_1)(\bar{x}_2 - \theta \circ \omega^{-1}(\mathcal{L}))\beta \bar{x}_2 - \lambda_3 \gamma \bar{x}_2 + \lambda_3 \mu \bar{x}_3 \\
\leq -(P^{-1})'(z)(\bar{x}_2 - \theta \circ \omega^{-1}(\mathcal{L})\beta \gamma E z + \lambda_3 \mu \bar{x}_3 \\
\leq -\alpha_E(\bar{V}(\bar{x}))
\]

(6.20)
for some $\alpha \in \mathcal{K}_\infty$, where $z = P(\hat{V}_{1,2}(\hat{x}_{1,2})) = (k\lambda_1 - \lambda_2)(\hat{x}_2)$ and

$$\gamma := 1 - \frac{\lambda_3 \gamma}{\lambda_2(\hat{x}_2 - \theta \circ \omega^{-1}(\hat{T}))} > 0.$$ 

Here, (5.11) and (5.13) imply the above inequality. In the case of $\hat{x} \in E \cap \{\hat{x}_3 \in [0, \infty)\}$, we have

$$\frac{\partial \hat{V}}{\partial \hat{x}} f = (P^{-1})'(z)(\lambda_2 - k\lambda_1)(\gamma I + \mu I - \beta IS) + \lambda_3(\gamma I - \mu R)$$

$$\leq (P^{-1})'(z)(\lambda_2 - k\lambda_1)(\hat{x}_2 - \theta \circ \omega^{-1}(\hat{T}))\beta \hat{x}_2 + \lambda_3 \gamma \hat{x}_2 - \lambda_3 \mu \hat{x}_3$$

$$\leq - (P^{-1})'(z)(\hat{x}_2 - \theta \circ \omega^{-1}(\hat{T}))\beta \hat{x}_2 - \lambda_3 \mu \hat{x}_3$$

(6.21)

$$\leq - \alpha_E(\hat{V}(\hat{x})).$$

In the region $F \cap \{\hat{x}_3 \in [-\hat{x}_3, 0]\}$, since $\theta^{-1}(-\hat{x}_2) < \hat{x}_1$ is equivalent to $x_1x_2 > \hat{x}_1\hat{x}_2$, we obtain

$$\frac{\partial \hat{V}}{\partial \hat{x}} f = \lambda_1(\hat{u} - \mu \hat{x}_1) + \lambda_2(\gamma F + \mu)\hat{x}_2 + \lambda_3 \mu \hat{x}_3 - (\lambda_1 + \lambda_2)\beta (x_1x_2 - \hat{x}_1\hat{x}_2)$$

$$\leq \lambda_1(\hat{u} - \mu \hat{x}_1) + \lambda_2(\gamma F + \mu)\hat{x}_2 + \lambda_3 \mu \hat{x}_3$$

(6.22)

$$\leq - \mu \hat{V} + \lambda_1 \hat{u}.$$

where $\gamma F := \gamma(1 - \lambda_3/\lambda_2) \geq 0$ is implied by (5.11) and (5.13). In the case of $F \cap \{\hat{x}_3 \in [0, \infty)\}$, we have

$$\frac{\partial \hat{V}}{\partial \hat{x}} f \leq \lambda_1(\hat{u} - \mu \hat{x}_1) + \lambda_2(\gamma + \mu)\hat{x}_2 - \lambda_3 \mu \hat{x}_3 - (\lambda_1 + \lambda_2)\beta (x_1x_2 - \hat{x}_1\hat{x}_2)$$

(6.23)

$$\leq - \mu \hat{V} + \lambda_1 \hat{u}$$

Therefore, since (6.10), (6.11), (6.12), (6.13), (6.15), (6.18), (6.20), (6.21), (6.22) and (6.23) cover $\partial \hat{V}/\partial \hat{x} \hat{f}$ on the entire $H(\lambda_2, k, \hat{T})$, the equilibrium $\hat{x} = 0$ is asymptotically stable for $\hat{u} = 0$. The inclusion (6.5) and the forward invariance of $[-\hat{x}_1, \infty)^3$ imply that the set $G(\lambda_2, k, \hat{T})$ is forward invariant and belongs to the domain of attraction for $\hat{u} = 0$.

Next, define

$$Q(\hat{T}) = \{[\hat{V}_{1,2}, \hat{V}_3]^T \in \mathbb{R}^2_+ : \exists L \leq [\hat{T}, \infty) \hat{V}_{1,2} + \hat{V}_3 = L\}$$

$$\eta(\hat{T}) = \min_{[\hat{V}_{1,2}, \hat{V}_3]^T \in Q(\hat{T})} P(\hat{V}_{1,2}) + \frac{\lambda_3 \hat{V}_3}{(P^{-1})'(P(\hat{V}_{1,2}))}$$

for $\hat{T} \in \mathbb{R}_+$. By definition, $\eta$ is of class $\mathcal{P}$ and non-decreasing. Furthermore, the definition (5.15) gives

(6.24)

$$\lim_{s \to \infty} \eta(s) = \lim_{s \to \infty} P(s) = (\lambda_2 - k\lambda_1)\hat{x}_2$$

since $\hat{V}_{1,2} < \hat{V}_{1,2} + \hat{V}_3 = \infty$ implies $\hat{V}_3 = \infty$ and $\lambda_3 \hat{V}_3/(P^{-1})'(P(\hat{V}_{1,2})) = \infty$. From (6.14) and (6.16), in region $C$,

$$\hat{u} \in [-\delta \mu \hat{V}/\lambda_1, \infty) \Rightarrow$$

(6.25)

$$\frac{\partial \hat{V}}{\partial \hat{x}} \hat{f} \leq - (1 - \delta)(\mu \hat{V}_3 + \lambda_3 \mu |\hat{x}_3|) \leq - \alpha_C(\hat{V})$$
Lemma 5.3. Let $\hat{\mu} \in [-\delta \mu \eta(\hat{V})/\lambda_1, \infty)$ imply the existence of $\hat{\theta}_1$ and $\hat{\theta}_2$ with $G$ and the forward invariance of the set $\Omega$ implies that $\hat{\theta}_1 \leq -\delta \mu \eta(\hat{V})$ with $\hat{\alpha}_D = \alpha_C$ for region $D$. On the other hand, due to (6.10), (6.11), (6.22) and (6.23), in $A$ and $F$ we have

$$\hat{\theta}_1 \in (-\delta \mu \eta(\hat{V})/\lambda_1, 0) \Rightarrow \frac{\partial \hat{V}}{\partial \hat{\theta}_1} \leq -(1 - \delta)\mu \hat{V}.$$  

(6.27)

Let $\zeta : \mathbb{R} \to (-\delta \mu P(\hat{\theta}_2)/\lambda_1, \delta \mu \hat{\theta}_2/\lambda_1)$ be a bijective continuous function satisfying $\zeta(0) = 0$. Define $r = \zeta^{-1}(\hat{\theta}_1)$. Properties (6.27), (6.12), (6.13), (6.25), (6.26), (6.20) and (6.21) imply the existence of $\chi \in K$ and $a \in K_\infty$ such that

$$\hat{V}(\hat{\theta}_2) \geq \chi(|r|) \Rightarrow \frac{\partial \hat{V}}{\partial \hat{\theta}_1} \leq -\alpha(\hat{V}(\hat{\theta}_2))$$  

(6.28a)

$$\mathcal{T} \geq \chi(|r|)$$  

(6.28b)

are satisfied for all $\hat{\theta}_2 \in H(\hat{\lambda}_2, k, \hat{\theta}_3)$ and all $r(t) \in \mathbb{R}$ with any given $\delta \in (0, 1)$. Here, $\hat{u} \in (-\delta \mu P(\hat{\theta}_2)/\lambda_1, \delta \mu \hat{\theta}_2/\lambda_1)$ guarantees the achievement of (6.28b). With the help of the forward invariance of $[-\hat{x}_1, \infty)^3$, property (6.28) implies that $\hat{x}(0) \in \mathcal{G}(\hat{\lambda}_2, k, \hat{\theta}_3)$ yields $\hat{x}(t) \in \mathcal{G}(\hat{\lambda}_2, k, \hat{\theta}_3)$ for all $t \in \mathbb{R}_+$ as long as $\hat{u}$ satisfies (5.21). Property (6.28) also imply ISS of the SIR model (3.1) with respect to the input $\hat{u}$ satisfying (5.21) [33]. In fact, the function $\hat{V}(\hat{x})$ defined in (5.6) is an ISS Lyapunov function on the compact set $\mathcal{G}(\hat{\lambda}_2, k, \hat{\theta}_3)$ for the given $\hat{\lambda}_2, \hat{\theta}_3, k > 0$.

6.3. Proof of Theorem 5.1. Define

$$T := \{\hat{x} \in \Omega : \hat{x}_1 \leq -k \hat{x}_2, \hat{x}_2 \leq 0\}$$

$$W(\hat{x}) := -\hat{x}_1 - \hat{x}_2 + |\hat{x}_3|.$$  

When $\hat{x} \in T$, $\hat{x}_3 < 0$ and $\hat{u} = 0$ hold, the function $W(\hat{x})$ satisfies

$$\frac{\partial W}{\partial \hat{x}} \hat{f} = \mu S + \beta IS - B + \gamma I + \mu I - \beta IS + \mu R - \gamma I$$

$$= \mu \hat{x}_1 + (\gamma - \gamma + \mu) \hat{x}_2 + \mu \hat{x}_3$$

$$= -\mu W(\hat{x})$$

When $\hat{x} \in T$, $\hat{x}_3 \geq 0$ and $\hat{u} = 0$ hold, we have

$$\frac{\partial W}{\partial \hat{x}} \hat{f} = \mu S + \beta IS - B + \gamma I + \mu I - \beta IS + \gamma I - \mu R$$

$$= \mu \hat{x}_1 + (2\gamma + \mu) \hat{x}_2 - \mu \hat{x}_3$$

$$\leq -\mu W(\hat{x})$$

By virtue of (3.4) with $\mathcal{G} = \hat{\theta}$, Lemma 5.3 and the forward invariance of the set $[-\hat{x}_1, \infty)^3$, for each $x(0) \in T$, there exists $t_T \in [0, \infty)$, $\hat{\lambda}_2$, $\hat{\theta}_3$, $k > 0$ such that $x(t_T) \in \mathcal{G}(\hat{\lambda}_2, k, \hat{\theta}_3)$. Therefore, Theorem 5.2 with $\hat{u} = 0$ shows that any compact set in $\Omega$ is contained in the domain of attraction.
Next, writing $\overline{G}(\lambda_2, k, \underline{L})$ as $\overline{G}$, Lemma 5.3 guarantees that for any given compact set $\overline{G}$ contained in the interior of $G$, there exist sufficiently small $\lambda_2$, $1/\underline{L}$, $k > 0$ such that $\overline{G} \supset G$ is satisfied. As proved in Theorem 5.2, there exist $\Psi_G \in KL$ and $\Gamma_G \in K$ such that

\begin{align}
(6.29) & \quad \forall t \in \mathbb{R}_+ \quad |\hat{x}(t)| \leq \Phi_G(|\hat{x}(0)|, t) + \Gamma_G(\text{ess sup}_{t \in [0, t_\tau]} |\hat{u}(t)|) \\
(6.30) & \quad \hat{x}(t) \in \overline{G}(\lambda_2, k, \underline{L})
\end{align}

are satisfied for all $\hat{x}(0) \in \overline{G}(\lambda_2, k, \underline{L})$ and (5.21). Choose $|\cdot|$ as 1-norm for consistency. Recall that (3.4) holds for all $x(0) \in \mathbb{R}_+$ and all $B(t) \in [0, \overline{B}]$ (a.e.) with respect to an arbitrarily given constant $\overline{B} \geq 0$. Pick any $\Phi \in KL$ and $\Gamma \in K$ satisfying

\begin{align}
(6.31) & \quad \forall t \in \mathbb{R}_+ \quad \forall s \in \mathbb{R}_+ \quad \Phi(s, t) \geq \max \{\Phi_G(s, t), (s + |\hat{x}|)e^{-t}\} \\
(6.32) & \quad \forall t \in \mathbb{R}_+ \quad \forall s \in [0, \overline{\pi}] \quad \Gamma(s) \geq \min \{\Gamma_G(s), s + \hat{u} + |\hat{x}|\} \\
(6.33) & \quad \forall t \in \mathbb{R}_+ \quad \forall s \in [\overline{\pi}, \infty) \quad \Gamma(s) \geq s + \hat{u} + |\hat{x}|,
\end{align}

where $\overline{\pi} := \min\{\delta\mu P(\underline{L})/\lambda_1, \delta\mu \underline{L}/\lambda_1\}$. Using $|x| \leq |\hat{x}| + |\hat{x}|$ and $|\hat{x}| \leq |x| + |\hat{x}|$ one arrives at

\begin{align}
\forall t \in \mathbb{R}_+ \quad |\hat{x}(t)| \leq \Phi(|\hat{x}(0)|, t) + \Gamma(\text{ess sup}_{t \in \mathbb{R}_+} |\hat{u}(t)|)
\end{align}

for all $\hat{x}(0) \in \overline{G}(\lambda_2, k, \underline{L})$ and all $\hat{u}(t) \in [-\hat{B}, \infty)$.

Remark 6.1. As seen in (5.12) and (5.13), the parameters $k$ and $\lambda_3$ approach zero as $\underline{L}$ tends to $\infty$. Hence, the sublevel sets are expanded significantly in the $x_3$-direction. It allows the recovered population to increase, which is not bad in the control of infectious diseases. However, it is only an upper bound, and the recovered population does not necessarily swell that much. Indeed, we have the estimate (3.4).

Remark 6.2. For large magnitude of the input $\hat{u}$, an ISS-gain function obtained in the proof of Theorem 5.1 is bounded from above by a linear function as in (6.33). A linear bound of the ISS-gain function $\Gamma$ can also be verified for small magnitude of $\hat{u}$ in (6.32). In fact, the property $0 < (P^{-1})'(0) < \infty$ obtained from (6.7) implies that $\eta^{-1}$ can be bounded from above by a linear function in a neighborhood of the origin. Combining (6.27), (6.12), (6.13), (6.25), (6.26), (6.20) and (6.21) leads to (6.29) with a function $\Gamma_G$ which is bounded from above by a linear function in a neighborhood of the origin. Thus, a linear bound of $\Gamma$ in a neighborhood of the origin follows from (6.32). Therefore, for all magnitude of the input $\hat{u}$, the ISS-gain function of the SIR model (3.1) is bounded from above by a linear function.

7. Difficulties and Keys for Lyapunov Construction. The Lyapunov functions (4.3) and (5.6) proposed in this paper depict geometric structure with slopes and regions which the SIR model (3.1) requires. Note that the switching with sharp edges causing non-differentiability is not essential, but for simply highlighting the geometrical structure of sublevel sets. In fact, if one admits complexity sacrificing explicit analytical expression, numerical computation can help smooth out the edges to obtain differentiable Lyapunov functions. This section explains some of major components of the geometric structure, and elucidates points having hampered previous studies, and how this paper addresses those points to estimate reasonable domains of attraction without resorting to LaSalle’s invariance principle. In the previous sections, all the derivatives of the constructed Lyapunov functions along trajectories $\partial V / \partial \hat{x} \cdot \hat{f}$
are negative except at the target equilibrium in the absence of perturbation \( \tilde{u} \). Such functions are referred to strict Lyapunov functions in the field of control [23]. The strict negativity has allowed us to prove ISS of the SIR model in the presence of the perturbation.

Everyone notices the conservation of populations taking place in between (3.1a) and (3.1b) through \( \beta IS \). In the two regions

\[
\begin{align*}
\hat{B} &:= \{ x \in \mathbb{R}^3_+ : x_1 < \hat{x}_1, \hat{x}_1 \hat{x}_2 < x_1 x_2 \} \\
\hat{E} &:= \{ x \in \mathbb{R}^3_+ : x_1 > \hat{x}_1, \hat{x}_1 \hat{x}_2 > x_1 x_2 \},
\end{align*}
\]

the bilinear term \( \beta IS \) in (3.1a) generates force to let \( x_1 \) stay away from the equilibrium \( \hat{x}_1 \) of interest. Hence, in \( \hat{B} \) and \( \hat{E} \), \( x_2 \) and \( x_3 \) should dominate the Lyapunov function in making its derivative negative. In the case \( \hat{R}_0 < 1 \) of the disease-free equilibrium, region \( \hat{E} \) disappears since \( \hat{x}_2 = 0 \). This structure of \( \hat{B} \) and \( \hat{E} \) is incorporated in the definition of \( \tilde{\nu} \) and the partitioning functions in (4.3) and (5.7). To define a set taking care of \( \hat{B} \), the disease-free case can use a linear function in (4.3) since \( \tilde{x}_2 \) is non-negative as discussed at the beginning of section 5.

Functions in the form of

\[
\tilde{V}(\tilde{x}) = \tilde{V}_1(\tilde{x}_1) + \tilde{V}_2(\tilde{x}_2) + \tilde{V}_3(\tilde{x}_3)
\]

have been widely used as Lyapunov functions in stability analysis and design of dynamical systems. They are often referred to as sum-separable (Lyapunov) functions or scalar (Lyapunov) functions [7, 24]. In this paper, let a function \( \tilde{V}(\tilde{x}) : \mathbb{R}^3_+ \to \mathbb{R}_+ \) be said to be separable if

\[
j \neq i \Rightarrow \forall \tilde{x} \frac{\partial^2 \tilde{V}}{\partial \tilde{x}_j \partial \tilde{x}_i} = 0.
\]

Clearly, continuously differentiable functions in the form of (7.3) are separable\(^4\). The structure (7.4) is very popular and useful for constructing a Lyapunov function since the negativity of its derivative can be assessed by looking at components separately as

\[
\frac{\partial \tilde{V}}{\partial \tilde{x}}(\tilde{x}, \tilde{u}) \tilde{f}(\tilde{x}, \tilde{u}) = \sum_{i=1}^3 \frac{\partial \tilde{V}}{\partial \tilde{x}_i}(\tilde{x}_i) \tilde{f}_i(\tilde{x}, \tilde{u}).
\]

and focusing on the interaction between subsystems \( \tilde{x}_i = \tilde{f}_i(\tilde{x}, \tilde{u}), i = 1, 2, 3 \) (see [12, 14, 5, 25] and references therein). In fact, for popular models of infectious diseases, many preceding studies use the sum-separable form (7.3) (e.g., [21, 19, 20, 28, 8, 29, 28, 2, 9]).

There is a major difference between the endemic equilibrium and the disease-free equilibrium in constructing a Lyapunov function. The endemic case exhibits spiral trajectories around the equilibrium on the \( S-I \) plane, i.e., the origin \( \hat{x}_{1,2} = 0 \) of the \( (\hat{x}_1, \hat{x}_2) \)-plane. If

\[
x_1 = \hat{x}_1 = \frac{\gamma + \mu}{\beta},
\]

\( \text{The max-separable functions which are also popular in the literature [15, 24, 5, 7] are not separable in the sense of (7.4) since the switching depends on the whole } \hat{x} \text{ instead of the individual } \hat{x}_i. \)
then the SIR model (3.1) gives \( \dot{x}_2 = 0 \) and

(7.7a) \[ \dot{x}_2 < x_2 \Rightarrow \dot{x}_1 < 0 \]

(7.7b) \[ \dot{x}_2 > x_2 \Rightarrow \dot{x}_1 > 0. \]

No matter how far and close \( x_2 \) is to \( \dot{x}_2 \), this anti-parallel structure (7.7) of flows takes place. It disappears only at the equilibrium \( x_2 = \dot{x}_2 \). Since \( \dot{x}_2 = \dot{x}_2 = 0 \) hold for (7.6), a function \( \hat{V}(\dot{x}) \) of the form (7.4) exhibits the decrease \( \partial \hat{V} / \partial \dot{x} \cdot \dot{f} < 0 \) for \( \dot{x}_2 \neq 0 \) (i.e., \( x_2 \neq \dot{x}_2 \)) only if

(7.8a) \[ \dot{x}_3 < x_3 \leq \frac{\gamma}{\mu} x_2, \ \dot{x}_2 < x_2 \Rightarrow \left. \frac{\partial \hat{V}}{\partial \dot{x}_1}(\dot{x}_1) \right|_{\dot{x}_1=0} > 0 \]

(7.8b) \[ \dot{x}_3 > x_3 \geq \frac{\gamma}{\mu} x_2, \ \dot{x}_2 > x_2 \Rightarrow \left. \frac{\partial \hat{V}}{\partial \dot{x}_1}(\dot{x}_1) \right|_{\dot{x}_1=0} < 0, \]

provided that

(7.9a) \[ \dot{x}_3 < x_3 \Rightarrow \frac{\partial \hat{V}}{\partial \dot{x}_3}(\dot{x}_3) \geq 0 \]

(7.9b) \[ \dot{x}_3 > x_3 \Rightarrow \frac{\partial \hat{V}}{\partial \dot{x}_3}(\dot{x}_3) \leq 0. \]

The two conclusions in (7.8) contradict each other. This situation is illustrated by Figure 3 (a) on \((x_1, x_2)\)-plane. The positive definiteness of \( \hat{V} \) requires (7.9) at least locally at \( \dot{x}_3 = 0 \), i.e., in a neighborhood of \( \dot{x}_3 = 0 \). Thus, any (piecewise) continuously differentiable function \( \hat{V}(\dot{x}) \) which is separable (7.4) cannot be a Lyapunov function in the sense of \( \partial \hat{V} / \partial \dot{x} \cdot \dot{f} < 0 \). It is worth mentioning that property (7.9) is usually employed in the region of interest, instead of the existence of a small neighborhood of \( \dot{x}_3 = 0 \). In obtaining reasonable level sets to secure an estimate of domain of attraction, violating (7.9) is usually too hard. The Lyapunov function \( \hat{V}(\dot{x}) \) constructed in (5.6) is not separable. In fact, the conditions of the partitioning in (5.7) require both \( x_1 \) and \( x_2 \). Importantly, the second case (7.8b) disappears from (7.8) in the disease-free case since \( \dot{x}_2 = \dot{x}_3 = 0 \). Thus, the contradiction does not rise in the disease-free case. This is why (5.7) employed the slope \( k > 0 \), while (4.3) does not.

As seen in the definition (5.3) of \( G \), Theorem 5.1 dealing with the endemic equilibrium \( x_f \) does not cover a triangle region at the corner of \( x_1 \)-axis and \( x_2 \)-axis. No matter how much one modifies Lyapunov functions, there remains an uncovered region of non-zero volume at that corner along the \( x_1 \)-axis. To see this, notice that (3.1a) and (3.1b) satisfy the implication

(7.10) \[ x_1 < \dot{x}_1, \ x_2 > 0 \Rightarrow \dot{x}_2 < 0 \]

(7.11) \[ x_2 = 0, \ x_1 < x_{f,1} \Rightarrow \dot{x}_2 = 0, \ \dot{x}_1 > 0. \]

Here, \( \dot{x}_1 = (\gamma + \mu)/\beta \) and \( x_{f,1} = B/\mu \). The relationship \( \dot{x}_1 < x_{f,1} \) follows from \( R_0 > 1 \). Define

(7.12) \[ \tilde{D} := \{ \tilde{x} \in \mathbb{R}_+^3 : \dot{x}_1 < 0, \ -\dot{x}_2 < \dot{x}_2 < 0 \} \].

Consider an initial state \( x(0) \in \tilde{D} \) which is arbitrarily close to a point \([x_1(0), 0, 0]^T\) for some \( x_1(0) \in (0, \dot{x}_1) \). According to (7.10) and (7.11), the trajectory \( x(t) \) flows along
the plane of $x_2 = 0$ ($x_1$-axis on $(x_1, x_2)$-plane) by decreasing its distance to the plane ($x_1$-axis) further. The level set of a Lyapunov function passing through the point $x = x(0)$ must be intersected transversally by the trajectory $x(t)$ inward. Hence, the level set must intersect the plane (the $x_1$-axis). Due to (7.11), that level set crossing over $x_1$-axis on $(x_1, x_2)$-plane can never cross $x_1$-axis again as long as $x_1 < x_{f,1}$. This implies the existence of a sublevel set to which the equilibrium $x_f$ belongs. At the non-target equilibrium $x_f$, the derivative of any Lyapunov function candidate $\tilde{V}$ along the trajectory is zero. Hence, the function $\tilde{V}$ is not a strict Lyapunov function for the target equilibrium $x_e$. This mechanism is illustrated in Figure 3 (b). In this way, independently of methods of constructing a Lyapunov function, there is an area remaining uncovered by any sublevel sets along $x_1$-axis in region $\hat{D}$. Theorem 5.1 achieves the construction of a Lyapunov function by avoiding that prohibited region intentionally.

8. Concluding Remarks. This paper has proved ISS of the SIR model with respect to perturbation of the newborn/immigration rate in both the endemic and the disease-free scenarios. The establishment is based on the construction of ISS Lyapunov functions. The functions play the role of traditional Lyapunov functions when the newborn/immigration rate is constant. It has been discussed that the proposed Lyapunov functions give the largest possible estimate of the domain of attraction and the ultimate boundedness in a qualitative sense. The developments do not rely on the simplifying assumptions which are often employed in the literature. The derivative of the proposed Lyapunov functions is strictly negative everywhere in sublevel sets of the Lyapunov functions except at the target equilibrium. This has allowed us to bypasses LaSalle’s invariance principle, and to establish ISS addressing the perturbation. This paper has elaborated the construction of Lyapunov functions by distilling essential difficulties posed by the SIR model.

It seems that no attention had been paid to ISS of the SIR model with respect to perturbation of the newborn/immigration rate, i.e., robustness of the endemic equilibrium and the disease-free equilibrium. Proving the ISS property had not been possible either since Lyapunov functions were not strict, due to the reason clarified in section 7. The robustness of the endemic equilibrium may sound undesirable in
view of preventing disease spread. Nevertheless, controlling the peak and lowering the steady-state level of the infected population are beneficial to societies. The derivative of the ISS Lyapunov functions developed in this paper confirms that the increase of the death rate $\mu$ is the only almighty parameter that can not only reduce the peak and result in faster convergence, but also reduce the fluctuation of the state with respect to the perturbation of the newborn/immigration rate. It is also estimated that although the reduction of the transmission rate $\beta$ does not have such mighty effect, it can simply avoid the endemic equilibrium or lower the steady-state level of the infected population. These are already known by using traditional local analysis and phase portraits. Nevertheless, the geometric structure revealed by the region partitioning and slopes of the proposed Lyapunov functions gives an insight into the flow of the populations in the SIR model globally in the state space. Importantly and interestingly, the ISS property proved in this paper has confirmed a linear transition of the magnitude of the state variables with respect to the perturbation magnitude of the newborn/immigration rate globally in spite of the bifurcation from the disease-free equilibrium to the endemic equilibrium and vice versa.

Needless to say, Lyapunov functions are known to be useful for designing controllers, and investigating control design for the SIR model is the most important direction of the future research. To this end, the proposed Lyapunov functions aiming at geometric understanding the SIR model can be modified into functions which ease the construction of controllers by smoothing out the edges of switching [15]. In fact, the gradient-type design [31, 10] based on a non-smooth Lyapunov function results in a discontinuous controller, and the notion of the system solution and the derivative need to be adjusted mathematically [1]. Bypassing such technicalities would be practically advantageous.

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