Solution generating theorems for the TOV equation

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The Tolman–Oppenheimer–Volkov [TOV] equation constrains the internal structure of general relativistic static perfect fluid spheres. We develop several “solution generating” theorems for the TOV equation, whereby any given solution can be “deformed” into a new solution. Because the theorems we develop work directly in terms of the physical observables — pressure profile and density profile — it is relatively easy to check the density and pressure profiles for physical reasonableness.

This work complements our previous article [Phys. Rev. D71 (2005) 124030; gr-qc/0503007] wherein a similar “algorithmic” analysis of the general relativistic static perfect fluid sphere was presented in terms of the spacetime geometry — in the present analysis the pressure and density are primary and the spacetime geometry is secondary. In particular, our “deformed” solutions to the TOV equation are conveniently parameterized in terms of $\delta p_c$ and $\delta p_\Sigma$, the finite shift in the central density and central pressure. We conclude by presenting a new physical and mathematical interpretation for the TOV equation — as an integrability condition on the density and pressure profiles.

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I. INTRODUCTION

The general relativistic static perfect fluid sphere has a long and venerable history that nevertheless continues to provide significant surprises [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12]. As emphasised in the review article by Delgaty and Lake [11], while it is often relatively easy to write down explicit spacetime metrics that solve the differential equation corresponding to a relativistic static perfect fluid sphere, it is often much more difficult to check whether the corresponding pressure profile and density profile is “physically reasonable” — indeed there is often some confusion as to what the phrase “physically reasonable” might entail.

This problem has if anything become even more acute with the recent introduction of “algorithmic” techniques that are capable of generating, and in principle classifying, all perfect fluid spheres [1, 3, 5, 6, 7, 8, 9, 10]. In view of this difficulty we have reformulated those “algorithmic” techniques in terms of several new “solution generating theorems” that apply directly to the TOV equation itself — and so directly give information about the pressure profile and density profile. Specifically, let us consider the well-known TOV system of equations, whose derivation is now a common textbook exercise [13, 14]:

\[
\frac{dp(r)}{dr} = - \left[ \frac{\rho(r) + p(r)}{r^2 [m(r) + 4\pi p(r)r^3]} \right], \\
\frac{dm(r)}{dr} = 4\pi \rho(r) r^2.
\]

We shall adopt units where $G = c = 1$, so that both $m/r$ and $p r^2$ are “dimensionless”.

Our basic strategy will be to assume that somehow we have obtained, or been given, some specific “physically reasonable” solution of the TOV equation in terms of the “initial” profiles $p_0(r)$ and $\rho_0(r)$, and then to ask how these initial profiles can be deformed while still continuing to satisfy the TOV equation. The output from the analysis will be several single-parameter, or sometimes multi-parameter, distorted profiles $p_2(r)$ and $\rho_2(r)$ that continue to solve the TOV equation for arbitrary values of the parameters $\Sigma$, along with some results regarding the “physical reasonableness” thereof.

We shall use notation such as $p_{0,c}$ and $p_c$, and $\rho_{0,c}$ and $\rho_c$, to denote the central pressure, and central density, before and after applying the distortion. Viewing the general relativistic static perfect fluid sphere as a first-approximation to a relativistic star, (though the discussion could just as easily be applied to a perfect fluid planet), we will concentrate on the region deep inside the stellar core, and specifically on the regularity conditions at the centre of the star, asking that the pressure and density remain positive and non-singular there. Observe that in seeking solutions of the TOV equation we are effectively only demanding pressure isotropy — we make no claims as to constraints on, or even the existence of, an equation of state.

We finally conclude by presenting a new physical and mathematical interpretation for the TOV equation — as
an integrability condition on the density and pressure profiles that allows us to explicitly write down the spacetime metric for a relativistic perfect fluid sphere in terms of an iterated integral involving the density and the pressure profiles.

II. LOCATING THE SURFACE

As usual, we shall take the location of the first pressure zero \( p(r_s) = 0 \) as defining the surface of the star, \( r_s \), though we should note a number of caveats:

1. The “star” might not have a well-defined surface if \( p(r) \neq 0 \) for all \( r \) — this corresponds to a “cosmological” configuration where a localized star-like object is surrounded by an infinite diffuse “atmosphere” which merges continuously into some asymptotic limit. While physically there is nothing necessarily wrong with such a configuration, they are certainly non-traditional [1].

2. If the “star” is embedded in an asymptotically de Sitter or anti-de Sitter spacetime then the surface of the star is located by asking that the material pressure be zero. The total pressure at the surface is then determined by the cosmological constant, so the total pressure is nonzero at the surface: \( p_{\text{total}}(r_s) = \rho \Lambda = -\rho \Lambda \neq 0 \). Though at first this may seem a little alarming, at worst this situation introduces some minor modifications of the standard analysis.

3. The situation we shall implicitly be most interested in is where the spacetime is asymptotically flat and the surface occurs at some finite radius \( r_s \). To avoid needing too many special case technical analyses we shall always take our initial density and pressure profiles \( \rho_0(r) \) and \( p_0(r) \) to mathematically extend to arbitrary \( r \), including beyond \( r_s \). When we specifically want the physical density and pressure profiles we shall explicitly truncate them at the surface — for instance (in the pure vacuum case) by setting

\[
m_{\text{truncated}}(r > r_s) = m(r_s) = M, \quad (3)
\]

and

\[
\rho_{\text{truncated}}(r > r_s) = 0; \quad (4)
\]

\[
p_{\text{truncated}}(r > r_s) = 0. \quad (5)
\]

These truncated profiles are certainly a solution to the TOV equation, and amount (in accordance with Birkhoff’s theorem) to continuously joining the surface of the “star” to a portion of the exterior Schwarzschild geometry.

4. If one wishes to embed a finite size star in de Sitter or anti-de Sitter spacetime then the relevant truncation conditions are

\[
m_{\text{truncated}}(r > r_s) = m(r_s) + \frac{4\pi}{3}\rho \Lambda (r^3 - r_s^3), \quad (6)
\]

and

\[
\rho_{\text{truncated}}(r > r_s) = \rho \Lambda; \quad (7)
\]

\[
p_{\text{truncated}}(r > r_s) = \rho \Lambda. \quad (8)
\]

These truncated profiles are again certainly a solution to the TOV equation, and amount (in accordance with a suitable generalization of Birkhoff’s theorem) to continuously joining the surface of the “star” to a portion of the Schwarzschild–de Sitter [Kottler] geometry.

We shall now use this general framework to develop several mathematical theorems regarding the TOV equation, and subsequently study their physical implications.

III. THE TOV EQUATION AS A RICCATI EQUATION

Viewed as a differential equation for \( p(r) \), with \( m(r) \) [and hence \( \rho(r) \)] specified, the TOV equation is a specific example of a Riccati equation, for which the number of useful solution techniques is rather limited. (See for instance, Bender & Orszag [15], or Polyanin and Zaitsev [16]. The most important key results are collected in the appendix to this article.) Using these standard references it is relatively easy to show the following.

**Theorem (P1).** Let \( p_0(r) \) and \( \rho_0(r) \) solve the TOV equation, and hold \( m_0(r) = 4\pi \int p_0(r) r^2 \, dr \) fixed. Define an auxiliary function \( g_0(r) \) by

\[
g_0(r) = \frac{m_0(r) + 4\pi p_0(r) r^3}{r^2[1 - 2m_0(r)/r]} \quad (9)
\]
Then the general solution to the TOV equation is \( p(r) = p_0(r) + \delta p(r) \) where

\[
\delta p(r) = \frac{\delta p_c \sqrt{1 - 2m_0/r} \exp \left\{ -2 \int_0^r g_0 \, dr \right\}}{1 + 4\pi \delta p_c \int_0^r \frac{1}{\sqrt{1 - 2m_0/r}} \exp \left\{ -2 \int_0^r g_0 \, dr \right\} \, r \, dr},
\]

and where \( \delta p_c \) is the shift in the central pressure.

**Proof.** By using equation (A2) from the appendix we know that for any arbitrary real parameter \( k \) the quantity

\[
\delta p(r) = \frac{k \exp \left[ -\int m_0 + 4\pi \rho_0 r^3 + 8\pi \rho_0 r^3}{r^2(1 - 2m_0/r)} \, dr \right]}{1 + 4\pi k \int_0^r \frac{1}{\sqrt{1 - 2m_0/r}} \exp \left[ -\int m_0 + 4\pi \rho_0 r^3 + 8\pi \rho_0 r^3}{r^2(1 - 2m_0/r)} \, dr \right]} \, dr
\]

is such that \( p(r) = p_0(r) + \delta p(r) \) also solves the TOV equation. But by using an integration by parts one can simplify this to establish

\[
\exp \left[ -\int m_0 + 4\pi \rho_0 r^3 + 8\pi \rho_0 r^3}{r^2(1 - 2m_0/r)} \, dr \right] = \sqrt{1 - 2m_0/r} \exp \left[ -2 \int m_0 + 4\pi \rho_0 r^3}{r^2(1 - 2m_0/r)} \, dr \right]
\]

\[
= \sqrt{1 - 2m_0/r} \exp \left[ -2 \int g_0 \, dr \right],
\]

Thus

\[
\delta p(r) = \frac{k \sqrt{1 - 2m_0/r} \exp \left[ -2 \int g_0 \, dr \right]}{1 + 4\pi k \int_0^r \frac{1}{\sqrt{1 - 2m_0/r}} \exp \left[ -2 \int g_0 \, dr \right] \, r \, dr}
\]

Finally, one fixes the constant \( k \) by first choosing the limits of integration to be 0 to \( r \), and then applying the boundary condition \( p(0) = p_c \) at \( r = 0 \) to deduce \( k = \delta p_c \). Thus

\[
\delta p(r) = \frac{\delta p_c \sqrt{1 - 2m_0/r} \exp \left[ -2 \int_0^r g_0 \, dr \right]}{1 + 4\pi \delta p_c \int_0^r \frac{1}{\sqrt{1 - 2m_0/r}} \exp \left[ -2 \int_0^r g_0 \, dr \right] \, r \, dr}
\]

as claimed.

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**A. Physically reasonable centre**

If the original \( p_0(r) \) and \( m_0(r) \) are “physically reasonable”, in particular if the central pressure \( p_{0,c} \) and density \( \rho_{0,c} \) are finite and positive, then \( m_0(r) = O(r^3) \), so \( m_0/r = O(r^2) \), and \( g_0(r) = O(r) \). Thus all the integrals used above are finite and well-behaved (convergent) at the centre of the star. Consequently, the new central pressure will likewise be finite and well behaved; and will even be positive so long as \( \delta p_c > -p_{0,c} \). So the central region of the new stellar configuration will automatically be “physically reasonable”. Since within the context of this theorem the density profile is unaffected, we also know that no horizon will be generated if we start from a horizon-free initial state. (Horizon formation is impossible in the current situation: If it were to happen, it would be signaled by the existence of an \( r_H \) such that \( 2m(r_H)/r_H = 1 \), but by hypothesis \( p(r) \), and hence \( m(r) \), are held fixed. So if there is no horizon in the original configuration, one cannot generate a horizon by applying the current theorem.)

**Comment:** With hindsight one can view this theorem P1 as a consequence of theorem T2 of reference [4]. Some tedious algebraic manipulations will actually show that they are identical, up to a specific regularity choice. (In the notation of [4], one must choose \( \sigma \neq 0 \).) This regularity condition is now forced upon us because we want the pressure and density at the center of the star to remain finite.

Equivalently, if the original \( p_0(r) \) and \( m_0(r) \) are “phys-
ically reasonable”, then (provided the relevant integrals are chosen to run from 0 to r), the σ = 0 sub-case of theorem T2 of reference [4] yields a spacetime geometry for which gtt → 0 at the origin, which we deem to not be physically reasonable. The key difference in the physics of the current article is that we now have a physics reason for restricting the solutions generated by theorem T2 of reference [4], and now can thereby derive an explicit theorem directly in terms of the shift in the pressure profile.

B. Central redshift

If one writes the spacetime geometry in the form
\[ ds^2 = -\zeta_0(r)^2 dt^2 + \frac{dr^2}{1 - 2m_0(r)/r} + r^2 d\Omega, \]
then the Einstein equations imply (see, for instance, [4, 13, 14])
\[ \zeta_0(r) = \exp \left\{ \int_0^r g_0(r) dr \right\}. \]

(16)

It is important to realise that with current conventions \( g_0(r) \) is defined by equation (10), and that equation (16) is a result of the Einstein equations.

In particular, throughout the star
\[ \zeta_0(r) = \zeta_{0,c} \exp \left\{ \int_0^r g_0(r) dr \right\}, \quad (r \leq r_s), \]
and at the surface of the star
\[ \zeta_0(r_s) = \zeta_{0,c} \exp \left\{ \int_0^{r_s} g_0(r) dr \right\}, \]

(17)

where \( \zeta_{0,c} \) denotes the value of \( \zeta_0(r) \) at the centre of the fluid sphere — which is effectively the central redshift of the fluid sphere. In fact for an idealized freely propagating particle emitted from the centre
\[ \zeta_{0,c} = \frac{1}{1 + z_c}. \]

(19)

If one truncates the matter profile at \( r_s \), and normalizes using in the usual way using \( \zeta_{\text{truncated}}(\infty) = 1 \), then because of the Birkhoff theorem the geometry exterior to the star is Schwarzschild, and so one also has:
\[ \zeta_0(r_s) = \sqrt{1 - 2M/r_s}. \]

(20)

Consequently
\[ \zeta_{0,c} = \sqrt{1 - 2M/r_s} \exp \left\{ - \int_0^{r_s} g_0(r) dr \right\}. \]

(21)

Note that choosing the normalization \( \zeta_{\text{truncated}}(\infty) = 1 \) means that we are choosing the \( t \) coordinate to be proper time as measured at spatial infinity — which in general is well outside the surface \( r_s \) of the star. For \( r \leq r_s \) one could if one chooses rewrite equation (17) in the completely equivalent form
\[ \zeta_0(r) = \sqrt{1 - 2M/r_s} \exp \left\{ - \int_r^{r_s} g_0(r) dr \right\}, \quad (r \leq r_s). \]

(22)

These considerations now allow one to rewrite the result of theorem P1 as:
\[ \delta p(r) = \frac{\delta p_c \, \zeta_{0,c}^2 \sqrt{1 - 2m_0(r)/r}}{\zeta_0(r)^2 \left[ 1 + 4\pi \delta p_c \, \zeta_{0,c}^2 \int_0^r r \, dr \sqrt{1 - 2m_0/r} \, \zeta_0(r)^2 \right]} \]

(23)

This has allowed us to eliminate the inner integral (involving \( g_0 \)) in terms of the \( g_{tt} \) metric component.

C. Shift in the surface

From either equation (10) or (23), since all the other terms are positive, we see that the sign of \( \delta p(r) \) is the same as the sign of \( \delta p_c \), and so the same as the sign of the shift in the surface radius \( \delta r_s \). This is most obvious for \( \delta p_c > 0 \) where the positivity of \( \delta p(r) \) implies that the surface definitely moves outwards. For small \( \delta p_c < 0 \) the negativity of \( \delta p(r) \) implies that the surface moves inwards — the tricky point is that for negative \( \delta p_c \) one could in principle (mathematically) encounter a pole at some finite \( r_{\text{pole}} \), but this would then imply \( p(r_{\text{pole}}) = -\infty \), so the putative pole cannot occur inside the star — it is at worst a mathematical artifact located at \( r_{\text{pole}} > r_s \), outside the physical surface of the star. All in all,
\[ \text{sign}[\delta r_s] = \text{sign}[\delta p_c]. \]

(24)

We can say a little more if we restrict attention to infinitesimal shifts in the central pressure \( \delta p_c \) and take derivatives. The surface of the star is defined by
\[ p(p_c, r_s(p_c)) = 0, \]

(25)

where
\[ \delta p = p(p_c, r) - p_0(r) + \delta p(\delta p_c, r). \]

(26)

Thus
\[ \frac{\partial p}{\partial p_c} \bigg|_{r_s, p_0, c} + \frac{\partial p}{\partial r} \bigg|_{r_s, p_0, c} \frac{\partial r_s}{\partial p_c} \bigg|_{p_0, c} = 0. \]

(27)

That is
\[ \frac{\partial r_s}{\partial p_c} \bigg|_{p_0, c} = -\frac{\partial p/\partial p_c}{\partial p/\partial r} \bigg|_{r_s, p_0, c}. \]

(28)

But now
\[ \frac{\partial p}{\partial p_c} \bigg|_{r_s, p_0, c} = \frac{\partial (\delta p_c)}{\partial (\delta p_c)} \bigg|_{r_s, \delta p_c = 0}, \]

(29)

\[ = \sqrt{1 - 2M/r_c} \exp \left\{ -2 \int_0^{r_s} g_0 \, dr \right\}, \]

(30)

\[ = \zeta_{0,c}^2 \sqrt{1 - 2M/r_s}. \]

(31)
Furthermore, from the TOV equation
\[ \frac{\partial p}{\partial r}_{r_s, p_0, c} = -\frac{\rho_0(r_s) M}{r_s^2[1 - 2M/r_s]}, \]
(32)
Combining, we see
\[ \frac{\partial r_s}{\partial p_c} = \frac{\zeta_{0,c}^2 r_s^2 \sqrt{1 - 2M/r_s}}{\rho_0(r_s) M}. \]
(33)
All terms appearing above are manifestly positive.

D. Shift in the compactness

If we now consider the “compactness”, defined as
\[ \chi = \frac{2M}{r_s} = \frac{2m(r_s)}{r_s}, \]
(34)
then, since \( m(r) = m_0(r) \) is invariant under the current hypotheses,
\[ \delta \chi = \frac{8\pi}{3} \{3\rho_0(r_s) - \bar{\rho}_0(r_s)\} r_s \delta r_s + \mathcal{O}([\delta r_s]^2), \]
(35)
where \( \bar{\rho}_0(r_s) \) is the “average density” defined by \( m_0(r_s)/(4\pi r_s^3) \). Unfortunately, while (as we have seen above) the sign of \( \delta r_s \) is controlled, there is in general no a priori constraint on the sign of \( 3\rho_0(r_s) - \bar{\rho}_0(r_s) \), so we cannot (without further assumptions) constrain the sign of the shift in compactness. This means that theorem P1 by itself will not tell you if you are approaching or receding from the Buchdahl–Bondi bound \( \chi_{\text{maximum}} = 8/9 \).

The best we can say is
\[ \text{sign}\{\delta \chi\} = \text{sign}\{3\rho_0(r_s) - \bar{\rho}_0(r_s)\} \times \text{sign}\{\delta r_s\}. \]
(36)
Note that the function \( \{3\rho_0(r) - \bar{\rho}_0(r)\} = \{3\rho(r) - \bar{\rho}(r)\} \) remains invariant under the conditions of theorem P1. Whether or not this quantity is positive or negative depends on one’s a priori arbitrary choice of the initial solution profile \( \{\rho_0(r), p_0(r)\} \) used as input into the theorem. If we now consider infinitesimal shifts in the central pressure we see
\[ \frac{\partial \chi}{\partial r_s} = \frac{8\pi}{3} \{3\rho_0(r_s) - \bar{\rho}_0(r_s)\} r_s, \]
(37)
and via the chain rule
\[ \frac{\partial \chi}{\partial p_c} = \frac{8\pi}{3} \{3\rho_0(r_s) - \bar{\rho}_0(r_s)\} r_s^2 \zeta_{0,c}^2 \sqrt{1 - 2M/r_s}. \]
(38)

E. Shift in the spacetime geometry

One might also reasonably ask what happens to the spacetime metric itself? As regards the radial \( g_{rr} \) part of the spacetime metric, \( m_0(r) \) is by construction fixed, and so \( g_{rr} \) is unaffected. To calculate the “deformed” value of the the \( g_{tt} = -\zeta(r)^2 \) component we note that from equation [9] we have
\[ \delta g(r) = \frac{4\pi \delta p(r) r}{1 - 2m_0(r)/r}, \]
(39)
whence
\[ \delta g(r) = \frac{4\pi \delta p_c r \exp\left\{ -2 \int_0^r g_0 \, dr \right\}}{\sqrt{1 - 2m_0/r} \exp\left\{ -2 \int_0^r g_0 \, dr \right\}}. \]
(40)
That is
\[ \delta g(r) = \frac{d}{dr} \ln \left\{ 1 + 4\pi \delta p_c \int_0^r \frac{r \, dr}{\sqrt{1 - 2m_0/r} \exp\left\{ -2 \int_0^r g_0 \, dr \right\}} \right\}. \]
(41)
Note that the Einstein equations imply, vide [10]–[17],
\[ \ln(\zeta/\zeta_0) = \ln(\zeta_c/\zeta_{0,c}) + \int_0^r \delta g \, dr. \]
(42)
Integrating,
\[ \ln(\zeta/\zeta_0) = \ln(\zeta_c/\zeta_{0,c}) + \ln \left\{ 1 + 4\pi \delta p_c \int_0^r \frac{r \, dr}{\sqrt{1 - 2m_0/r} \exp\left\{ -2 \int_0^r g_0 \, dr \right\}} \right\}. \]
(43)
The only tricky part of the computation lies in getting the overall normalization correct. Since \( g_0(r) \) is defined by (9) directly in terms of mass and pressure profiles, the above formula provides a completely explicit formula for the (multiplicative) shift in \( \zeta \), and so for the (multiplicative) shift in \( g_{tt} \). Explicitly,

\[
\frac{\zeta(r)}{\zeta_c} = \frac{\zeta_0(r)}{\zeta_{0,c}} \times \left\{ 1 + 4\pi \delta p_c \frac{\zeta_0^2}{\zeta_0} \int_0^r \frac{r \, dr}{\sqrt{1 - 2m_0/r} \, \zeta_0(r)^2} \right\}.
\]

(44)

In view of these formulae for \( \zeta(r) \) we can rewrite the shift in pressure in the more compact manner

\[
\delta p(r) = \frac{\delta p_c}{\sqrt{1 - 2m_0/r} \, \zeta_0(r) \, \zeta_c(r)}.
\]

(45)

Though significantly more compact, the trade-off in this formula is that \( \zeta(r) \) and \( \zeta_0(r) \) are themselves quite complicated functions of the mass and pressure profiles.

**Normalization of theorem P1 versus theorem T2:** With the above normalization results now in hand, we see that theorem P1 corresponds to theorem T2 of [4] with the integrals running from 0 to \( r \), with \( \sigma \to \zeta_c/\zeta_{0,c} \), while the other parameter \( \epsilon \) occurring in T2 can be physically identified by taking \( \epsilon/\sigma \to 4\pi \delta p_c \zeta_c^2 \).

### F. Two additional theorems

In the specific case considered in this section, where \( m(r) \) is held fixed, one can obtain two additional related theorems by using equations (A3) and (A4) of the appendix. Specifically:

**Theorem (P1b).** For a fixed specification of \( \rho_0(r) \), let \( p_1(r) \) and \( p_2(r) \) be two distinct solutions of the TOV equation. Then the general solution (for \( \lambda \) an arbitrary real parameter) is

\[
p(r) = \lambda \exp \left\{ -\int_0^r \frac{4\pi p_1}{1 - 2m_0/r} \, dr \right\} p_1(r) + (1 - \lambda) \exp \left\{ -\int_0^r \frac{4\pi p_2}{1 - 2m_0/r} \, dr \right\} p_2(r)
\]

\[
= \lambda \exp \left\{ -\int_0^r \frac{4\pi p_1}{1 - 2m_0/r} \, dr \right\} p_1(r) + (1 - \lambda) \exp \left\{ -\int_0^r \frac{4\pi p_2}{1 - 2m_0/r} \, dr \right\} p_2(r).
\]

(46)

The central pressure is

\[
p_c = \lambda p_{c,1} + (1 - \lambda) p_{c,2}.
\]

(47)

Note that this theorem no longer requires any nested integrations, at the cost of needing two specific solutions as input. A minor technical issue is that \( p_1(r) \) and \( p_2(r) \) have to solve exactly the same Riccati equation over the entire domain of interest — this leads to technical problems if we truncate the density and pressure profiles at the surface of the star. We will either be forced to work on the intersection of the truncation regions, that is \( r \in [0, \min\{r_{s,1}, r_{s,2}\}] \), or work with un-truncated pressure profiles \( p_1(r) \) and \( p_2(r) \) that have been extended into the region where pressure is permitted to become negative.

**Theorem (P1c).** For a fixed specification of \( \rho_0(r) \), let \( p_1(r) \), \( p_2(r) \), and \( p_3(r) \) be three distinct solutions of the TOV equation. Then the general solution (for \( \lambda \) an arbitrary real parameter) is

\[
p(r) = \lambda \frac{p_1(r) [p_3(r) - p_2(r)] + (1 - \lambda) p_2(r) [p_3(r) - p_1(r)]}{\lambda [p_3(r) - p_2(r)] + (1 - \lambda) [p_3(r) - p_1(r)]}.
\]

(48)

The central pressure is then

\[
p_c = \frac{\lambda p_{c,1} [p_{c,3} - p_{c,2}] + (1 - \lambda) p_{c,2} [p_{c,3} - p_{c,1}]}{\lambda [p_{c,3} - p_{c,2}] + (1 - \lambda) [p_{c,3} - p_{c,1}]}. \tag{49}
\]

Note that this last theorem no longer requires any integrations whatsoever, though now at the cost of needing three specific solutions as input. As in the preceding theorem, \( p_1(r) \), \( p_2(r) \), and \( p_3(r) \) have to solve exactly the same Riccati equation over the entire domain of interest — for stars with a well defined surface we will either be forced to work on the intersection of the truncation regions, that is \( r \in [0, \min\{r_{s,1}, r_{s,2}, r_{s,3}\}] \), or work with un-truncated pressure profiles \( p_1(r) \), \( p_2(r) \), and \( p_3(r) \) that have been extended into the region where pressure is permitted to become negative.

### IV. CORRELATED CHANGES IN DENSITY AND PRESSURE

The second main theorem we develop can be obtained by looking for correlated changes in the mass and pres-
Theorem (P2). Let \( p_0(r) \) and \( \rho_0(r) \) solve the TOV equation, and hold \( g_0 \) fixed, in the sense that
\[
g_0(r) = \frac{m_0(r) + 4\pi p_0(r) r^3}{r^2[1 - 2m_0(r)/r]} = \frac{m(r) + 4\pi p(r) r^3}{r^2[1 - 2m(r)/r]}.
\]
Then the general solution to the TOV equation is given by \( p(r) = p_0(r) + \delta p(r) \) and \( m(r) = m_0(r) + \delta m(r) \) where
\[
\delta m(r) = \frac{4\pi r^3 \delta p_c}{3[1 + \rho_0]} \exp \left\{ 2 \int_0^r \frac{1 - r \rho_0}{1 + r \rho_0} \, dr \right\},
\]
and
\[
\delta p(r) = \frac{\delta p_c}{1 + 2\rho_0} \frac{1 + 8\pi \rho_0 r^2}{1 + 2m_0/r}.
\]
Here \( \delta p_c \) is the shift in the central density.

Explicitly combining these formulae we have
\[
\delta p(r) = \frac{\delta p_c}{[1 + \rho_0]} \frac{1 + 8\pi \rho_0 r^2}{1 - 2m_0/r} \exp \left\{ 2 \int_0^r \frac{1 - r \rho_0}{1 + r \rho_0} \, dr \right\}.
\]

Proof. Since
\[
m(r) = \frac{g_0(r) r^2 - 4\pi p(r) r^3}{1 + 2\rho_0(r)},
\]
we have in particular
\[
\delta m(r) = -\frac{4\pi \delta p(r) r^3}{1 + 2\rho_0(r)} = -4\pi \delta p(r) r^3 \left\{ \frac{1 - 2m_0/r}{1 + 8\pi \rho_0 r^2} \right\},
\]
so that
\[
\delta p(r) = -\frac{1}{r^2} \frac{d}{dr} \left( \frac{\delta p(r) r^3}{1 + 2\rho_0(r)} \right).
\]

Now consider the TOV equation, or more precisely, the change in the TOV equation:
\[
\frac{d\delta p(r)}{dr} = -[\delta p(r) + \delta p(r)] g_0(r).
\]

Combining these last two differential equations yields a linear homogeneous differential equation for \( \delta p(r) \), which is easily integrated. The quoted form of the theorem results after an integration by parts.

\[\square\]

A. Physically reasonable centre

If the original density and pressure profiles are “physically reasonable”, then likewise the new density and pressure profiles will be well behaved — at least for a finite region including the origin. In particular, since regularity of the original solution implies \( g_0(r) = O(r) \) at the origin, the only integral we have to do for this theorem is guaranteed to converge at the lower limit \( r = 0 \). So the central region of the new stellar configuration will automatically be “physically reasonable”. Note that for theorem P2
\[
\delta p_c = -\frac{\delta p_c}{3}.
\]

If the central pressure increases \( \delta p_c > 0 \), then for the situation envisaged in this theorem the mass profile always decreases, implying that an event horizon never forms.

Comment: With hindsight one can also view this theorem P2 as a consequence of theorem T1 of reference [3]. The key difference now is that we have an explicit statement directly in terms of the shift in the pressure profile. To see this, write \( g_0 = \zeta_0^2/\zeta_0 \), substitute into T1 and apply boundary conditions to deduce \( \lambda \to 8\pi \delta p_c/3 \).

B. Shift in the surface

Note that under the conditions of theorem P2 we have
\[
\text{sign}[\delta p(r)] = -\text{sign}[\delta m(r)] = -\text{sign}[\delta \rho_c].
\]

So if the central density goes up, the pressure everywhere goes down, and the radius of the surface of the star (being defined by the first pressure zero) must decrease. That is
\[
\text{sign}[\delta r_s] = -\text{sign}[\delta \rho_c] = +\text{sign}[\delta p_c].
\]

If we now consider infinitesimal changes in central pressure we see that equation (28) is unaltered. Furthermore, under the hypotheses of theorem P2 we have
\[
\frac{\partial p}{\partial p_c}\bigg|_{r_s, \rho_0, \rho_c} = \frac{\partial (\delta p)}{\partial (\delta p_c)}\bigg|_{r_s, \delta \rho_c = 0} = \frac{1}{[1 + r_s \rho_0(r_s)]^2 [1 - 2M/r_s]} \times \exp \left\{ 2 \int_0^{r_s} \frac{1 - r \rho_0}{1 + r \rho_0} \, dr \right\}.
\]

which is manifestly positive. Thanks to the TOV equation we again have equation (52), so combining the above leads to
\[
\frac{\partial r_s}{\partial \rho_c}\bigg|_{\rho_0, \rho_c} = \frac{r_s^2}{[1 + r_s \rho_0(r_s)]^2 \rho_0(r_s) M} \times \exp \left\{ 2 \int_0^{r_s} \frac{1 - r \rho_0}{1 + r \rho_0} \, dr \right\}.
\]

All terms appearing above are manifestly positive, as was the case for theorem P1.

C. Shift in the compactness

The compactness \( \chi = 2m(r_s)/r_s \) now changes both due to changes in \( m(r) \) and due to changes in \( r_s \). For small
but finite changes we have
\[ \delta \chi = \frac{2 \delta m(r_s)}{r_s} + \frac{8 \pi \rho_0(r_s) - \bar{\rho}_0(r_s)}{3} \frac{r_s}{r_s} \delta r_s + \ldots \]  \tag{64} \]

It is more efficient to re-write the compactness as
\[ \chi(p_c) = \frac{2m(p_c, r_s(p_c))}{r_s(p_c)} \]  \tag{65}

and then differentiate. We see
\[ \frac{\partial \chi}{\partial p_c} \bigg|_{p_0,c} = \frac{2 \delta m(p_c, r)}{r_s} \frac{\partial m(p_c, r)}{\partial p_c} \bigg|_{r_s(p_0,c), p_0,c} \]
\[ + \frac{8 \pi}{3} \left\{ 3\rho_0(r_s) - \bar{\rho}_0(r_s) \right\} r_s \frac{\partial r_s(p_c)}{\partial p_c} \bigg|_{p_0,c}. \]

But under the hypotheses of theorem P2 we have
\[ \frac{\partial m(p_c, r_s)}{\partial p_c} \bigg|_{p_0,c} = -\frac{4 \pi r^3}{M[1 + r_s g_0]^2} \]
\[ \times \exp \left\{ 2 \int_0^{r_s} g_0 \frac{1 - r g_0}{1 + r g_0} \, dr \right\}. \tag{67} \]

Now, combining this with the results of the previous subsection
\[ \frac{\partial \chi}{\partial p_c} \bigg|_{p_0,c} = \frac{8 \pi r^3}{M[1 + r_s g_0]^2} \exp \left\{ 2 \int_0^{r_s} g_0 \frac{1 - r g_0}{1 + r g_0} \, dr \right\} \]
\[ \times \left\{ 1 - \frac{M}{r_s} - \frac{\bar{\rho}_0}{3 \rho_0(r_s)} \right\}. \tag{68} \]

While the first few terms are guaranteed positive, there is no such guarantee as to the sign of the last term. Thus without further hypotheses we again cannot constrain the sign of the shift in the compactness \( \chi = 2m(r_s)/r_s \).

D. Shift in the spacetime geometry

What happens to the spacetime geometry under the application of theorem P2? By construction the quantity \( g(r) = g_0(r) \) is unaffected, and therefore
\[ \frac{\zeta(r)}{\zeta_c} = \frac{\zeta_0(r)}{\zeta_{0,c}}. \tag{69} \]

It is only the \( g_{rr} \) component of the spacetime metric that changes significantly:
\[ [g_0]_{rr} \rightarrow g_{rr} = \frac{1}{[g_0]_{rr}^{-1} - \delta[2m(r)/r]}, \tag{70} \]

where we already know
\[ \delta \left[ \frac{2m(r)}{r} \right] = \frac{8 \pi r^2 \delta \rho_c}{3 \left[ 1 + r g_0 \right]^2} \exp \left\{ 2 \int_0^r g_0 \frac{1 - r g_0}{1 + r g_0} \, dr \right\}. \tag{71} \]

V. COMBINING THE PREVIOUS THEOREMS

One can obtain additional and more complicated theorems by iteratively applying theorems P1 and P2, in a manner similar to the discussion of reference [4]. This iteration will yield 2-parameter generalizations of theorems P1 and P2.

Theorem (P3). Let \( p_0(r) \) and \( \rho_0(r) \) solve the TOV equation. Apply theorem P1 followed by theorem P2. Let us define intermediate quantities
\[ \delta p_1(r) = \frac{\Delta p}{\sqrt{1 - 2m_0/r}} \exp \left\{ -2 \int_0^r g_0 \, dr \right\}, \]
\[ 1 + 4 \pi \Delta p \int_0^r \sqrt{1 - 2m_0/r} \, dr \exp \left\{ -2 \int_0^r g_0 \, dr \right\} \tag{72} \]

and
\[ g_1(r) = g_0(r) + \frac{4 \pi \delta p_1(r)}{1 - 2m_0/r}. \tag{73} \]

Then \( p(r) = p_0(r) + \delta p(r) \) and \( m(r) = m_0(r) + \delta m(r) \) are also solutions of the TOV equation, where
\[ \delta m(r) = \frac{4 \pi r^3 \delta \rho_c}{3 \left[ 1 + r g_1 \right]^2} \exp \left\{ 2 \int_0^r g_1 \frac{1 - r g_1}{1 + r g_1} \, dr \right\}, \tag{74} \]
and
\[ \delta p(r) = \delta p_1(r) - \frac{\delta m}{4 \pi r^3} \left[ 1 + 8 \pi [p_0 + \delta p_1]\right]. \tag{75} \]

Here \( \delta \rho_c \) is the shift in the central density, and the total shift in the central pressure is given by
\[ \delta p_c = \Delta p - \frac{\delta \rho_c}{3}. \tag{76} \]

The two parameters \( \delta \rho_c \) and \( \Delta p \) can be specified independently.

Theorem (P4). Let \( p_0(r) \) and \( \rho_0(r) \) solve the TOV equation. Apply theorem P2 followed by theorem P1. Then \( p(r) = p_0(r) + \delta p(r) \) and \( m(r) = m_0(r) + \delta m(r) \) are also solutions of the TOV equation, where
\[ \delta m(r) = \frac{4 \pi r^3 \delta \rho_c}{3 \left[ 1 + r g_0 \right]^2} \exp \left\{ 2 \int_0^r g_0 \frac{1 - r g_0}{1 + r g_0} \, dr \right\}, \tag{77} \]
and
\[ \delta p(r) = \frac{\delta m}{4\pi r^3} \left( \frac{1 + 8\pi p_0 r^2}{1 - 2m_0/r} \right) \frac{\Delta p \zeta_0^2}{\zeta_0(r)^2} \left[ 1 - 2|m_0 + \delta m|/r \right]. \]  

(78)

Here \( \delta \rho_c \) is the shift in the central density, and the total shift in the central pressure is given by

\[ \delta p_c = \Delta p - \frac{\delta \rho_c}{3}. \]  

(79)

The two parameters \( \delta \rho_c \) and \( \Delta p \) can be specified independently.

Comment: Note that these two theorems provide two-parameter generalizations of the TOV solution \((\rho_0, p_0)\) that one starts from. These theorems are closely related to theorems \(T_4\) and \(T_3\) of reference [3]. Technically they are equivalent to the \( \sigma \neq 0 \) case of theorems \(T_4\) and \(T_3\) of reference [4], where \( \sigma \neq 0 \) is now a physical restriction we place on the solution by demanding that the density and pressure be well behaved at the centre of the fluid sphere.

VI. THE TOV EQUATION AS AN ABEL EQUATION

If, on the other hand, we think of the pressure profile \( p(r) \) as fixed, then we can rearrange the TOV system of equations as a differential equation for \( m(r) \), specifically:

\[ \frac{dm(r)}{dr} = -4\pi p(r)r^2 - \frac{4\pi r^2[1 - 2m(r)/r]}{m(r) + 4\pi p(r)r^3} \frac{dp}{dr}. \]  

(80)

This is now an Abel equation (2nd type, class A).

If we were able to develop a solution generating theorem based on this equation it would in many ways be the most natural companion to theorem \(P_1\). Unfortunately, despite numerous attempts at developing solution generating theorems based on this observation, we have no progress to report. This is ultimately due to the fact that mathematically Abel equations are considerably more difficult to deal with than Riccati equations.

VII. THE TOV EQUATION AS AN INTEGRABILITY CONDITION

We now develop an alternative point of view regarding the physical and mathematical significance of the TOV equation: In any spherically symmetric spacetime there are formally three algebraically independent components of the Einstein equations. Let us now formally solve two of the Einstein equations by building them into the chosen form of the line element, and use the remaining Einstein equation to derive (in the case of perfect fluids) an integrability condition — this provides us with a new and different way of interpreting the TOV equation.

Specifically, let us now consider two arbitrarily specified functions \( \rho(r) \) and \( p_r(r) \), and define

\[ m(r) = \int_0^r 4\pi \rho(r)r^2 dr. \]  

(81)

Now consider the line element

\[ ds^2 = -\exp \left\{ -2 \int_r^\infty \frac{4\pi[p(r) + p_r(r)]r}{1 - 2m(r)/r} dr \right\} \times \left[ 1 - \frac{2m(r)}{r} \right] \frac{d\theta^2}{\sin^2 \theta} + \frac{dr^2}{1 - 2m(r)/r} + r^2(d\theta^2 + \sin^2 \theta d\phi^2). \]  

(82)

That is, the line element is specified completely by a double iterated integral involving the arbitrary functions \( \{\rho(r), p_r(r)\} \). Then two of the Einstein equations are trivial, they are “built in” to the chosen line element:

\[ G_{ii} \equiv 8\pi G_N \rho; \]  

(83)

\[ G_{rr} \equiv 8\pi G_N p_r. \]  

(84)

These two equations permit us to physically interpret \( \rho(r) \) as the density and \( p_r(r) \) as the radial pressure.

The sole remaining Einstein equation depends on the \( \delta \theta = G_{\phi\phi} \) component of the Einstein tensor. Now, it is a purely geometrical statement that

\[ G_{\theta\theta} = G_{\phi\phi}; \]  

(85)

\[ = G_{\phi\phi} + 4\pi \left\{ \frac{dp_r}{dr} + \frac{(\rho + p_r)(m + 4\pi p_r r^3)}{r^2(1 - 2m/r)} \right\}. \]

If we enforce isotropy, \( G_{\theta\theta} = G_{\phi\phi} = G_{\phi\phi} \), then without further ado we deduce the TOV equation

\[ \frac{dp_r}{dr} = \frac{(\rho + p_r)(m + 4\pi p_r r^3)}{r^2(1 - 2m/r)}, \]  

(86)

which we now see can be interpreted as the integrability condition allowing us to solve the two equations [38]–[43] for a perfect fluid source. Of course once we have the TOV equation, however derived, we can apply the Riccati equation analysis developed earlier in this article.

A bonus of setting things up this way is that it is immediately clear what happens for anisotropic fluids. The third Einstein equation now yields

\[ p_t = p_r + \frac{r}{2} \left\{ \frac{dp_r}{dr} + \frac{(\rho + p_r)(m + 4\pi p_r r^3)}{r^2(1 - 2m/r)} \right\}. \]  

(87)
This can either be viewed as a way of calculating the transverse pressure \( p_t \) in an anisotropic fluid, or can be re-arranged to yield the well-known anisotropic TOV equation

\[
\frac{dp}{dr} = \frac{(\rho + p)(m + 4\pi p^3)}{r^2(1 - 2m/r)} + \frac{2(p_t - p_r)}{r}. \tag{88}
\]

VIII. DISCUSSION

The purpose of this article has been to develop several “physically clean” solution generating theorems for the TOV equation — where by “physically clean” we mean that it is relatively easy to understand what happens to the pressure and density profiles, especially in the vicinity of the stellar core. In many ways this article serves as a companion paper to reference [4], where related results were derived in terms of the spacetime geometry. These new results refine our previous results, in the sense that it is now clearer when a mathematical solution of the isotropy equations are “physically reasonable”, at least in the region of the stellar core.

Indeed one important message is that in theorems T2, T3, and T4 of reference [4] one should take \( \sigma \neq 0 \) if one wishes the pressure and density to remain well behaved at the center the of the fluid sphere.

Reference [4] also contains a number of other interesting results regarding the use of “solution generating” theorems to classify perfect fluid spheres — sometimes new solutions are obtained, sometimes old solutions are recovered in a new context, and theorems T1 - T4 can be used to generate a web of interconnections between known and new perfect fluid spheres. These comments also apply mutatis mutandis to the present theorems, and we direct interested readers to reference [4] for further details.

In closing, we reiterate that the general relativistic static perfect fluid sphere, despite its venerable history, continues to provide interesting surprises.

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APPENDIX A: KEY RESULTS ON RICCATI EQUATIONS

The general Riccati equation is

\[
\frac{dp(r)}{dr} = \alpha(r) + \beta(r) p(r) + \gamma(r) p(r)^2, \tag{A1}
\]

for specified functions \( \alpha(r) \), \( \beta(r) \), \( \gamma(r) \). While no completely general solution from first principles exists, if we are given one specific solution \( p_0(r) \) then the one-parameter general solution (parameterized by an arbitrary real number \( k \)) may be written [15, 16]:

\[
p(r) = p_0(r) + \frac{k \exp \left\{ \int [2\gamma(r) p_0(r) + \beta(r)] dr \right\}}{1 - k \int \gamma(r) \exp \left\{ \int [2\gamma(r) p_0(r) + \beta(r)] dr \right\} dr}. \tag{A2}
\]

This can easily be derived, for instance, from the argument sketched in the standard reference Bender & Orzag [15], alternatively an equivalent explicit statement be found in the reference handbook by Polyanin and Zaitsev [16]. A second useful result is that if we know two specific solutions \( p_1(r) \) and \( p_2(r) \) then the one-parameter general solution (now parameterized by an arbitrary real number \( \lambda \)) may be written [16]:

\[
p(r) = \frac{\lambda \exp \left\{ -\int \gamma(r) p_1(r) dr \right\} p_1(r) + (1 - \lambda) \exp \left\{ -\int \gamma(r) p_2(r) dr \right\} p_2(r)}{\lambda \exp \left\{ -\int \gamma(r) p_1(r) dr \right\} + (1 - \lambda) \exp \left\{ -\int \gamma(r) p_2(r) dr \right\}}. \tag{A3}
\]
Finally, if we know three specific solutions $p_1(r)$, $p_2(r)$, and $p_3(r)$ then the one-parameter general solution (again parameterized by an arbitrary real number $\lambda$) may be written \[16\]:

$$p(r) = \frac{\lambda p_1(r) \left[ p_3(r) - p_2(r) \right] + (1 - \lambda) p_2(r) \left[ p_3(r) - p_1(r) \right]}{\lambda \left[ p_3(r) - p_2(r) \right] + (1 - \lambda) \left[ p_3(r) - p_1(r) \right]}.$$  \hfill (A4)

Note that these three forms of the general solution require two, one, or zero integrations respectively, at the cost of needing one, two, or three pre-specified input solutions.

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