CYLINDRICAL HOMOMORPHISMS AND LAWSON HOMOLOGY

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Abstract. We use the cylindrical homomorphism and a geometric construction introduced by J. Lewis to study the Lawson homology groups of certain hypersurfaces \( X \subset \mathbb{P}^{n+1} \) of degree \( d \leq n + 1 \). As an application, we compute the rational semi-topological K-theory of a generic cubic of dimension 5, 6 and 8 and, using the Bloch-Kato conjecture, we prove Suslin’s conjecture for these varieties. Using the generic cubic sevenfolds, we show that there are smooth projective varieties with the lowest non-trivial step in their s-filtration infinitely generated and undetected by the Abel-Jacobi map.

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Lawson homology of a projective variety \( X \) is defined as the homotopy groups of algebraic cycle spaces of a fixed dimension on \( X \). We write

\[ L_r H_n(X) := \pi_{n-2r}(Z_r(X)) \]

and, intuitively, think of an element in this group as a “family of \( r \)-cycles parametrized by a \((n-2r)\)-sphere” [15]. These groups, replete with information about the projective variety \( X \), are a combination of algebraic and topological information. For example, the algebraic equivalence class of an algebraic \( r \)-cycle can be expressed as a connected component of the topological space of algebraic \( r \)-cycles \( Z_r(X) \) [7]. In the same flavor, Dold-Thom theorem shows that “families of 0-cycles parametrized by a \( n \)-sphere” are the same as the topological classes in the singular homology of \( X^{an} \). The \( s \)-map is a map that “measures” how close Lawson homology groups are to algebraic geometry or topology. We have the following sequence of maps

\[ A_r(X) = L_r H_{2r}(X) \xrightarrow{s} L_{r-1} H_{2r}(X) \xrightarrow{s} \ldots \xrightarrow{s} L_1 H_{2r}(X) \xrightarrow{s} H_{2r}(X^{an}). \]

The composition of the above \( s \)-maps gives the usual cycle map between the Chow group of algebraic cycles modulo algebraic equivalence, denoted \( A_r(X) \), and the

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singular homology of the complex points of $X$, written $H_2^r(X^{an})$ \cite{ Lawson Suspension}. Encoded in the construction of the s-map is the celebrated suspension theorem for algebraic cycles proved by Lawson \cite{Lawson Suspension}, the starting point of Lawson homology.

Lawson homology with finite coefficients of a smooth projective variety $X$ is proved to be isomorphic, via a Poincare type duality \cite{Poincare Duality}, with the motivic cohomology with finite coefficients of $X$ \cite{Motivic Cohomology}. Through this isomorphism, we can study the torsion of the Lawson homology groups by motivic cohomology tools (see Proposition \ref{1.4}). In particular, the Beilinson-Lichtenbaum conjecture may be used to identify, for certain indices, Lawson homology groups with finite coefficients with singular homology groups with finite coefficients. As shown in \cite{Bloch-Kato}, the Beilinson-Lichtenbaum conjecture is equivalent to the Bloch-Kato conjecture, which has been proven by V. Voevodsky and M. Rost (see \cite{Voevodsky}, \cite{Rost}).

The torsion free subgroup of a Lawson homology group may be studied using methods from Hodge theory (see \cite{Hodge Theory}, \cite{Hodge Theory}). Some of these methods, mostly designed to study the niveau filtration of singular cohomology and the Generalized Hodge conjecture, may be adapted to work on Lawson homology groups. The generalized cycle maps between the Lawson homology groups of $X$ and the singular homology of $X^{an}$ factor through steps in the niveau filtration of the latter and it is conjectured that these steps give the entire image of these generalized cycle maps \cite{Generalized Hodge Conjecture}. With finite coefficients, this conjecture follows from the Bloch-Kato conjecture. With integer coefficients, this conjecture follows from the very far reaching Suslin’s conjecture (see Conjecture \ref{1.6}). Although Suslin’s conjecture and the Generalized Hodge conjecture are not known to be related in some direct way (i.e. neither is known to imply the other), all the varieties proved to satisfy Suslin’s conjecture (see \cite{Suslin}, \cite{Suslin}) were previously known to satisfy the Generalized Hodge conjecture. On the other hand, there are varieties that are known to fulfill the Generalized Hodge conjecture and for which Suslin’s conjecture is still not known. For example, abelian varieties as in \cite{Abelian Varieties} or generic cubics sevenfolds or elevenfolds as in Remark 4.13.

The main difficulty that appears in these particular examples lives in understanding the kernels of s-maps. The kernels of s-maps, in particular the kernels of the generalized cycle maps, are mostly mysterious and expected to be in general very huge (i.e. infinitely generated rational vector spaces). However, Suslin’s conjecture predicts that, in the range of indices as in the Beilinson-Lichtenbaum conjecture, these kernels are zero for any projective smooth variety.

In some cases (for example in those of certain projective varieties with “few” algebraic cycles like rationally connected varieties) these kernels can be understood, at least in some range of indices \cite{Rationally Connected Varieties}, \cite{Rationally Connected Varieties}. In this paper, we use a geometrical construction, introduced by J. Lewis \cite{Lewis Construction}, to study the Lawson homology groups of a generic hypersurface that has “enough” $k$-planes, $k \geq 1$. Except in the middle dimension, the Lawson homology of a hypersurface always surjects, at least rationally, onto its singular homology, as an application of the simple structure of its singular homology (see Proposition \ref{3.3}). The study of the kernels of s-maps of a smooth projective hypersurface is highly non-trivial, even for indices outside of the middle dimension (see Corollary 4.10). Applying the method of \cite{Suslin} (also \cite{Suslin}, \cite{Lewis Construction}) on Lawson homology groups allows us not only to obtain the results of \cite{Suslin}, but also...
computations of certain Lawson homology groups. As a consequence, we are also able to compute the semi-topological K-theory for certain generic cubic hypersurfaces. Semi-topological K-theory of a projective smooth variety $X$, denoted $K_{sst}(X)$, is a K-theory that lives between the algebraic K-theory of $X$ and the topological K-theory of $X^{an}$, i.e.

$$K_{alg}^n(X) \to K_{sst}^n(X) \to ku^{-n}(X^{an})$$

sharing important properties with these two K-theories [17]. The main tool to compute it is an “Atiyah-Hirzebruch” type of spectral sequence from the Lawson homology groups of the smooth projective variety $X$ [11].

The motivation of this paper was the study of Lawson homology of rationally connected varieties of dimension greater than four. In [31], we computed the Lawson homology of rationally connected varieties of dimension three and the rational Lawson homology of dimension four rationally connected varieties; in particular, we were able to check Suslin’s conjecture and Friedlander-Mazur’s conjecture in these cases. Some questions appeared: Could we check Suslin’s conjecture for rationally connected varieties of higher dimension? Are the s-maps always monomorphisms for a rationally connected variety of any dimension? The results of [31] anticipate that the Lawson homology of rationally connected hypersurfaces may provide answers to these questions, in particular the Lawson homology of a generic cubic sevenfold and eightfold. In [31], the cases of a cubic fivefold and sixfold were discussed by means of the decomposition of diagonal method and of a result of Esnault, Viehweg and Levine [6]. Using the natural and transparent geometric method of J. Lewis [23], we obtain more general results (the case of a cubic eightfold, see Corollary 4.7).

The paper is divided in four sections and an Appendix. In the first section, we introduce our notations and recall some of the results needed later in the paper.

In the second section, we recall the geometric construction from J. Lewis [23] and use it in the context of Lawson homology groups. We will also introduce here the cylindrical homomorphisms, which are the main objects of study in this paper. The results proved in this section give us the main tools that will be used in the fourth section. Part of the proofs needed in this section is given in the Appendix.

In the third section, we extend a “weak Lefschetz” type of theorem, proved for Chow groups in [23], to Lawson homology groups and discuss some applications of the weak Lefschetz theorem on homology to the s-maps on the Lawson homology of a hypersurface of any dimension and degree.

The fourth section is devoted to applications of the tools developed in the second and third sections. We compute, in a certain range of indices, the Lawson homology of some rationally connected generic hypersurfaces. As an application we compute the rational Lawson homology and semi-topological K-theory of a generic cubic eightfold (Corollary 4.7), obtaining, in particular, the validity of Suslin’s conjecture in this particular case. In the end of this section, based on the results obtained in the case of a generic cubic sevenfold, we remark that there are examples of varieties with the lowest nontrivial step in the s-filtration of a Griffiths group (see (2)) an infinitely generated $\mathbb{Q}$-vector space and undetected by the Abel-Jacobi map (see Corollary 4.14).
This paper uses in an essential way J. Lewis’s geometric construction from [22] and [23] in the context of Lawson homology. We thank J. Lewis for making this interesting construction available.

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1. Notations and Recollection

In this section, we will introduce the notations used in the paper and briefly state some of the results needed later on.

All algebraic varieties in this paper are smooth and irreducible over complex numbers. By \( L^p q(X) \), \( CH^r(X) \), \( A_r(X) \), \( K^{set}(X) \), \( H^BM_q \) and \( H_q(X^{an}) \) we define Lawson homology, Chow group of algebraic cycles modulo rational equivalence, Chow group of algebraic cycles modulo algebraic equivalence, semi-topological K-theory, Borel-Moore cohomology and singular homology with integer coefficients. By \( L^p q(X) \) we mean Lawson homology with rational coefficients (similar notation for rational semi-topological K-theory and rational singular homology). By an isomorphism written like \( \cong \) we mean an isomorphism of rational vector spaces. By a monomorphism written \( \hookrightarrow \) we mean a rational monomorphism. We will write \( \alpha . \beta \) for the intersection product of two algebraic cycles \( \alpha \), \( \beta \) in \( X \). We will write \( 1 \) for the identity map, when there is no confusion. We write \( [x] \) for the integer part of the real number \( x \).

We call a hypersurface of dimension \( n \) generic if it belongs to a point in a non-empty Zariski open subset of the variety of hypersurfaces of degree \( d \) in the projective space \( \mathbb{P}^{n+1} \).

For a smooth projective variety \( X \), we define \( Z_r(X) = (\mathcal{C}_r(X))^+ \), the naive group completions of \( \mathcal{C}_r(X) = \bigoplus d \mathcal{C}_{r,d}(X) \). Here \( \mathcal{C}_{r,d}(X) \) is the Chow variety of algebraic cycles of degree \( d \) and dimension \( r \). The topology on the group \( Z_r(X) \) is the quotient topology induced by the complex topology of the projective varieties \( \mathcal{C}_{r,d}(X) \). We call \( Z_r(X) \) the topological space of \( r \)-dimensional algebraic cycles. The empty cycle \( 0 \in \mathcal{C}_r(X) \) is the natural base point of \( Z_r(X) \). We define

\[
L^q H_n(X) = \pi_{n-2q}(Z_q(X))
\]

for any \( 0 \leq q \leq d \) and \( n \geq 2q \). For a quasi-projective variety \( U \), with a projective closure \( X \), we let (see [25])

\[
Z_r(U) := Z_r(X)/Z_r(X \setminus U).
\]

This is, up to isomorphism, a well defined object in the category of topological abelian groups that admit a structure of CW-complex with inverted homotopy equivalences ([25], [10], [27]). We call this category \( \mathcal{H}^{-1}AbTop \).

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Moreover, for a closed embedding of projective varieties $Y \subset X$, the exact sequence of topological groups

$$0 \to Z_r(Y) \to Z_r(X) \to Z_r(X \setminus Y) \to 0$$

gives a long exact localization sequence of homotopy groups ([25], [10])

$$\ldots \to \pi_* Z_r(Y) \to \pi_* Z_r(X) \to \pi_* Z_r(X \setminus Y) \to \pi_{*-1} Z_r(Y) \to \ldots$$

$$\ldots \to \pi_0 Z_r(Y) \to \pi_0 Z_r(X) \to \pi_0 Z_r(X \setminus Y) \to 0.$$ 

For $q < 0$, we define the following topological cycle spaces (with the quotient topology):

$$Z_q(X) = Z_0(X \times \mathbb{A}^{-q}) := Z_0(X \times \mathbb{P}^{-q})/Z_0(X \times \mathbb{P}^{-q-1}).$$

The homotopy groups of these cycle spaces give the negative Lawson homology, i.e.

$$L_q H_n(X) := \pi_{n-2q}(Z_q(X))$$

for any $q < 0$. The following equalities ([13]) show that these groups are all isomorphic with the Borel-Moore homology of $X$. For any $q < 0$

$$(1) \quad L_q H_n(X) = \pi_{n-2q}(Z_0(X \times \mathbb{A}^{-q})) = H_n^{BM}(X \times \mathbb{A}^{-q}) \simeq H_n^{BM}(X^{an}) = L_0 H_n(X).$$

The space of t-cocycles for a smooth projective variety $X$ is defined to be the following naive completion

$$Z^t(X) = (\mathcal{M}or(X, C_0(\mathbb{A}^t))^{an}/\mathcal{M}or(X, C_0(\mathbb{A}^{t-1}))^{an})^+$$

where by $\mathcal{M}or(X, C_0(Y))^{an}$ we mean the abelian monoid of morphisms between $X$ and the Chow monoid $C_0(Y)$ provided with the compact-open topology ([12]).

The following “Poincare duality” type of theorem was proved in [13]:

**Theorem 1.1.** ([13]) There is a homotopy equivalence

$$D : Z^t(X) \to Z_d(X \times \mathbb{A}^t) \simeq Z_{d-t}(X)$$

for any smooth projective variety $X$ of dimension $d$ and for any $t \geq 0$.

Theorem 1.1 says that the cycle spaces $Z_q(X)$, with $q \leq 0$, are, up to homotopy, the t-cocycle spaces $Z^t(X)$, with $t \geq \text{dim}(X)$. In Proposition 1.3 we give a simple application of this remark.

For a proper map of quasi-projective varieties $f : X \to Y$ and any $r \in \mathbb{Z}$, we have the push-forward map $f_* : Z^r(X) \to Z^r(Y)$ ([10], [20]). For any locally complete intersection map $f : X \to Y$ of codimension $d$ ($d = \text{dim}(Y) - \text{dim}(X)$), we have a well-defined Gysin map (in $\mathcal{H}^{-1} \text{AbTop}$)

$$f^* : Z^r(Y) \to Z_{r-d}(X)$$

for any $r \in \mathbb{Z}$ ([10]). The Gysin map for a proper map of quasi-projective varieties $f : X \to Y$, with $Y$ a smooth quasi-projective variety, is defined to be the composition

$$Z_r(Y) \xrightarrow{pr^*_Y} Z_{s+\text{dim}(X)}(X \times Y) \xrightarrow{\delta^*} Z_{s+\text{dim}(X)-\text{dim}(Y)}(X)$$

where $pr^*_Y$ is the flat pull back of the projection map on $Y$ and $\delta^*$ is the Gysin map of the regular embedding of the graph of $f$. In the case of a regular embedding
for any $n$, This implies that the right vertical s-map is a homotopy equivalence. Then we have

$$Z_r(X) \xrightarrow{pr^*_s} Z_{r+1}(X \times \mathbb{A}^1) \xrightarrow{\delta} Z_r(N_Y X) \xrightarrow{\pi^*_s} Z_{r-d}(V).$$

The left map is a flat pull-back of the projection on $X$, the middle map is the “specialization map” given by the deformation of the normal cone $N_Y X$ and the last map is the inverse of the flat pull-back vector bundle isomorphism $\pi^*: Z_{r-d}(V) \rightarrow Z_r(N_Y X)$.

Using these Gysin maps one can construct a well defined intersection product on cycle spaces in $\mathcal{H}^{-1}AbTop$. For example, if $i_V: V \hookrightarrow X$ is a regular embedding of an $r$-dimensional subvariety $V$ in the smooth quasi-projective variety $X$ of dimension $d$, then the intersection with $V$ on $X$ is given by the composition

$$Z_s(X) \xrightarrow{i^*_V} Z_{r-s-d}(V) \xrightarrow{i^*_V} Z_{r-s-d}(X).$$

where $i^*_V: Z_s(X) \rightarrow Z_{s+r-d}(V)$ is the Gysin map associated to the regular embedding $V \hookrightarrow X$ and $r + s \geq d$. In general, we have the following theorem:

**Theorem 1.2.** ([10], Theorem 3.5) If $X$ is a smooth quasi-projective variety of dimension $d$ and if $r + s \geq d$ then there is an intersection pairing (in $\mathcal{H}^{-1}AbTop$)

$$Z_r(X) \otimes Z_s(X) \rightarrow Z_{r+s-d}(X)$$

which on $\pi_0$ gives the usual intersection pairing on Chow groups of algebraic cycles modulo algebraic equivalence.

According to ([13], [25]), there is an operation on Lawson groups, called s-map

$$s: L_r H_n(X) \rightarrow L_{r-1} H_n(X)$$

for any quasi-projective variety $X$. The inverse of the isomorphism ([11]) is given in the following proposition.

**Proposition 1.3.** The s-map on the negative Lawson homology of a smooth projective variety $X$ of dimension $d$ is an isomorphism, i.e.

$$L_0 H_n(X) \xrightarrow{s} L_{-1} H_n(X) \xrightarrow{s} L_{-2} H_n(X) \xrightarrow{s} ...$$

**Proof.** According to ([13], Proposition 2.6), for any smooth projective variety $X$ of dimension $d$ and for any $t \geq 0$ we have the following commutative diagram in $(\mathcal{H}^{-1}AbTop)$:

$$
\begin{array}{ccc}
Z^t(X) \wedge S^2 & \xrightarrow{D_{\mathbb{A}^1}} & Z_d(X \times \mathbb{A}^1) \wedge S^2 \\
\downarrow s & & \downarrow s \\
Z^{t+1}(X) & \xrightarrow{Z_{d-1}(X \times \mathbb{A}^t)} & Z_{d-1}(X \times \mathbb{A}^1).
\end{array}
$$

If $t \geq d$, then the left vertical arrow is a homotopy equivalence ([13], Theorem 5.8). This implies that the right vertical s-map is a homotopy equivalence. Then we have

$$\pi_n(Z_0(X)) \xrightarrow{s} \pi_{n+2}(Z_{-1}(X)) = L_{-1} H_n(X) \xrightarrow{s} \pi_{n+4}(Z_{-2}(X)) = L_{-2} H_n(X) \xrightarrow{s} ...$$

for any $n \geq 0$. \qed
The s-map is a natural map that commutes with push-forwards, flat pull-backs, Gysin maps, intersection with a cycle and with localization sequences ([8], Page 4 and Proposition 1.7; [13], Proposition 2.3).

We let $\text{cyc}_{p,q}$ denote the (generalized) cycle maps

$$\text{cyc}_{p,q} : L_p H_q(X) \to H^{BM}_q(X^{an})$$

for any quasi-projective variety $X$. We recall that if $X$ is a smooth projective variety then $H^{BM}_q(X^{an}) = H_q(X^{an})$. The generalized cycle maps are compositions of s-maps ([15], [8]), i.e.

$$\text{cyc}_{p,q} : L_p H_q(X) \to L_{p-1} H_q(X) \to \cdots \to L_1 H_q(X) \to L_0 H_q(X) = H^{BM}_n(X^{an})$$

for any $q \geq 2p \geq 0$. If $p < 0$, then the maps $\text{cyc}_{p,q}$ are isomorphisms (Proposition [13] and the isomorphism [11]).

The above decomposition gives a filtration on the kernel of $\text{cyc}_{p,q}$. The kernel of $\text{cyc}_{r,2r}$ is the Griffiths group of algebraic $r$-cycles [9]. Let

$$Z_r(X) \xrightarrow{i} L_r H_{2r}(X) = \pi_0(Z_r(X)) \xrightarrow{\delta} L_{r-1} H_{2r}(X) \to \cdots \to L_2 H_{2r}(X).$$

Define $S_r Z_r(X) = \text{Ker}(i \circ \pi)$. We know that

$$S_0 Z_r(X) = \{\text{algebraic cycles algebraically equivalent to zero}\}$$

and that

$$S_r Z_r(X) = \{\text{algebraic cycles homologically equivalent to zero}\}.$$ 

If we take the quotient of the above filtration by $\text{Ker}(\pi)$, we obtain a filtration on the $\text{Griff}_r(X)$ called the s-filtration of the Griffiths groups. This is

$$0 \subset S_1 Z_r(X)/S_0 Z_r(X) \subset \cdots \subset \text{Griff}_r(X) = S_r Z_r(X)/S_0 Z_r(X).$$

We let $L^\text{hom}_p H_q(X) = \text{Ker}(\text{cyc}_{p,q})$ and $C_{p,q}(X) = \text{Coker}(\text{cyc}_{p,q})$ to be the kernel and the cokernel of the maps $\text{cyc}_{p,q}$.

The following proposition was proved in [31]:

**Proposition 1.4.** ([31])

Let $X$ be a smooth projective variety of dimension $d$. Assume that the Bloch-Kato conjecture is valid for all the primes. Then:

a) Let $n \geq d + q - 1$. Then $L^\text{hom}_n H_n(X)$ is divisible and $C_{q,n}(X)$ is torsion free.

b) $L_q H_n(X)$ is uniquely divisible for $n > 2d$ and $L_q H_{2d}(X)$ is torsion free (for any $q \leq d$).

The following theorem is the projective bundle theorem in Lawson homology.

**Theorem 1.5.** ([10], [19]) Let $E$ be a rank $r+1$ vector bundle over a quasi-projective variety $Y$, let $p : P(E) \to Y$ be the canonical map and let $O_{P(E)}(1)$ be the canonical line bundle on $P(E)$. Let $h = c_1(O_{P(E)}(1))$. Then

$$\phi = \sum_{j=0}^{r} h^{r-j} \circ p^* : \bigoplus_{j=0}^{r} Z_{i-j}(Y) \to Z_i(P(E)).$$
is a homotopy equivalence for any $i \geq 0$. In particular
\[ \phi = \sum_{j=0}^{r} h^{r-j} \circ p^* : \bigoplus_{j=0}^{r} \pi_k Z_{i-j}(Y) \to \pi_k Z_i(P(E)) \]
is an isomorphism for any $k \geq 0, i \geq 0$

We will refer later in this paper to the following two conjectures.

**Conjecture 1.6. (Suslin’s conjecture)** The generalized cycle map
\[ \text{cyc}_{q,n} : L_q H_n(X) \to H_n(X^{an}) \]
is an isomorphism for $n \geq d + q$ and a monomorphism for $n \geq d + q - 1$ for any smooth projective variety $X$ of dimension $d$.

We notice that this conjecture contains a conjecture due to E. Friedlander and B. Mazur [15].

**Conjecture 1.7. (Friedlander-Mazur conjecture)** For any complex smooth projective variety $X$ of dimension $d$
\[ L_q H_n(X) = 0 \]
for any $n > 2d$.

These highly non-trivial conjectures have been checked on certain varieties, like curves, surfaces, rationally connected threefolds and fourfolds, smooth projective toric varieties ([11], [31]).

The (singular) semi-topological K-theory of a complex projective variety $X$ was introduced in [16]. This is defined by
\[ K^\text{sst}_*(X) = \pi_*(\text{Mor}(X, \text{Grass})^+) \]
where $\text{Grass} = \Pi_{n,N} \text{Grass}_n(\mathbb{P}^N)$. By $\text{Mor}(X, \text{Grass})^+$, we define the topological group given by the homotopy completion of the space of algebraic maps between $X$ and $\text{Grass}$. The main tool for computing $K^\text{sst}_*(X)$ is a spectral sequence with $E^2$-term given by the Lawson homology of $X$, when $X$ is a smooth variety.

**Theorem 1.8. ([11])** For any smooth, projective complex variety $X$ and any abelian group $A$, there is a natural map of strongly convergent spectral sequences
\[ E_2^{p,q}(\text{sst}) = L^{-q} H^{p-q}(X, A) \quad \Rightarrow \quad K^\text{sst}_{-p+q}(X, A) \]
\[ E_2^{p,q}(\text{top}) = H^{p-q}(X^{an}, A) \quad \Rightarrow \quad ku^{p+q}(X^{an}, A). \]
inducing the usual maps on both $E_2$-terms and abutments.
2. About the Fano variety of k-planes

In this section, we describe a geometrical construction, introduced in [23], for generic hypersurfaces \( X \) with “enough” \( k \)–planes and prove Theorem 2.5 which will be the main tool in the fourth section.

We start this section with the definition of the Fano variety of \( k \)-planes on a projective variety.

**Definition 2.1.** Let \( Z \subset \mathbb{P}^N \) be a variety. We let \( \Omega_Z(k) = \{ \mathbb{P}^{k'} \subset Z \} \subset \text{Grass}_k \mathbb{P}^N \) be the set of all \( k \)-planes included in \( Z \) and call it the Fano variety of \( k \)-planes of \( Z \).

The following theorem describes \( \Omega_X(k) \) for a generic hypersurface \( X \subset \mathbb{P}^{n+1} \) of degree \( d \leq n+1 \).

**Theorem 2.2.** (Borcea [5], Corollary 2.2)

Let \( X \subset \mathbb{P}^{n+1} \) be a generic hypersurface of degree \( d \leq n+1 \) and let \( k = \left\lfloor \frac{n+1}{d} \right\rfloor \). Then \( \Omega_X(k) \) is non-empty and smooth of pure dimension

\[
\gamma = (k+1)(n+1-k) - \binom{d+k}{k}
\]

provided that \( \gamma \geq 0 \) and \( X \) is not a quadric. In the case \( X \) is a quadric we require \( n \geq 2k \). Furthermore, if \( \gamma > 0 \) or if in the case \( X \) a quadric with \( n > 2k \), then \( \Omega_X(k) \) is irreducible.

We will study below only generic hypersurfaces \( X \subset \mathbb{P}^{n+1} \) of degree \( 3 \leq d \leq n+1 \) with the property that

\[
\dim(\Omega_X(k)) = (k+1)(n+1-k) - \binom{d+k}{k} \geq n-2k
\]

where \( k = \left\lfloor \frac{n+1}{d} \right\rfloor \). Theorem 2.2 says that such varieties have \( \Omega_X(k) \) non-empty, smooth and irreducible. Consider the homogeneous polynomial \( F(X_0, X_1, \ldots, X_{n+1}) \) of degree \( d \) that gives our \( X \). J. Lewis ([23], [22]) introduced the following construction: let \( G(X_0, X_1, \ldots, X_{n+2}) = X_{n+2}^d + F(X_0, X_1, \ldots, X_{n+1}) \) be a homogeneous polynomial of degree \( d \) in \( n+3 \) variables that defines a smooth hypersurface \( Z \subset \mathbb{P}^{n+2} \). We notice that \( X \) is a hyperplane section of \( Z \) if we consider the embedding

\[
\mathbb{P}^{n+1} = V(X_{n+2}) \subset \mathbb{P}^{n+2}
\]

where by \( V(H) \) we understand the set of zeros of the homogenous polynomial \( H \). Moreover \( \Omega_Z(k) \) is smooth, irreducible and of dimension \( (n-k+1) + l \) ([23]), where \( l = \gamma - (n-2k) \geq 0 \).

Let \( \Omega_Z \) be the subvariety cut out by \( l \) general hyperplane sections of the projective variety \( \Omega_Z(k) \) in a projective embedding and let \( \Omega_X = \Omega_X(k) \cap \Omega_Z \). Using Bertini’s theorem we conclude that \( \Omega_Z \) and \( \Omega_X \) are smooth varieties of pure dimension \( (\Omega_Z \) is an irreducible variety) with \( \dim(\Omega_Z) = n-k+1 \) and \( \dim(\Omega_X) = n-2k \).
Below we will recall the main known properties of these varieties and fix some notations that we use later in the text (we use, for consistency, the same notations as in \[23\], \[22\], \[3\]).

Let \( \pi_X : P(X) \to X \) and \( \pi_Z : P(Z) \to Z \) the projections from the incidence varieties \( P(X) = \{(c, x) \in \Omega_X \times X \text{ s.t. } x \in \mathbb{P}^k_c\} \), \( P(Z) = \{(c, x) \in \Omega_Z \times Z \text{ s.t. } x \in \mathbb{P}^k_c\} \) to the variety \( X \), respectively \( Z \).

We denote by \( \rho_Z : P(Z) \to \Omega_Z \) and \( \rho_X : P(X) \to \Omega_X \) the natural projections. We have a cartesian diagram

\[
\begin{array}{ccc}
X' = X \times Z P(Z) & \xrightarrow{j_2} & P(Z) \\
\downarrow \pi & & \downarrow \pi_Z \\
X & \xrightarrow{j} & Z \\
\end{array}
\]

If \( k = 1 \), then \( X' = Bl_{\Omega_X}(\Omega_Z) \), the blow-up of \( \Omega_X \subset \Omega_Z \) (see \[22\]). Denote \( \rho = \rho_Z \circ j_2 : X' \to \Omega_Z \). We also have a commutative diagram

\[
\begin{array}{ccc}
P(X) & \xrightarrow{i} & X' \\
\downarrow \rho_X & & \downarrow \rho \\
\Omega_X & \xrightarrow{j_0} & \Omega_Z
\end{array}
\]

with the embedding maps \( j_0, i \). We have the following properties (\[22\], \[23\]):

1) \( \pi \) and \( \pi_Z \) are generically finite.
2) \( \rho_X : P(X) \to \Omega_X \) and \( \rho_Z : P(Z) \to \Omega_Z \) are \( \mathbb{P}^k \)-bundles.
3) \( X' \) is smooth by Bertini’s theorem.
4) \( \rho_{Z/X} := \rho|_{X' \setminus P(X)} : X' \setminus P(X) \to \Omega_Z \setminus \Omega_X \) is a \( \mathbb{P}^{k-1} \)-bundle.
5) \( \dim(X) = \dim(X') = n \), \( \dim(Z) = \dim(P(Z)) = n + 1 \), \( \dim(P(X)) = n - k \) and all varieties are smooth.

Let \( H_Z = \mathbb{P}^{n+1} \cap Z \) and \( H_X = H_Z \cap X \) be generic hyperplane sections of \( Z \), respectively \( X \). Then we denote \( \mu = \pi^{-1}(H_X) \) and \( \mu_Z = \pi_Z^{-1}(H_Z) \) and we use the same notations for their algebraic cycle classes.

The following definition introduces one of the objects of study in this paper. In the above notation, we have

**Definition 2.3.** We call the map

\[
\pi_* \circ i_* \circ \rho_X^* : L_{r-k}H_{s+2(r-k)}(\Omega_X) \to L_rH_{s+2r}(X)
\]

the cylindrical homomorphism on the Lawson homology groups of \( X \).

For a motivation behind the Definition 2.3 see \[24\]. For a related discussion, see the remarks after the proof of Theorem 4.3.

**Remark 2.4.** (\[24\])

Cylindrical homomorphism (Definition 2.3) coincides with the action of the incidence correspondence in \( \Omega_X \times X \) on the Lawson homology.
Theorem 2.5. With notations as above, there is a homotopy equivalence
\[ Z_r(X') \simeq \bigoplus_{j=0}^{k-1} Z_{r-j}(\Omega Z) \oplus Z_{r-k}(\Omega X) \]
for any \( r \geq 0 \).

**Proof.** We prove this theorem following the lines of the proof of ([23], Proposition 3.2). We will only focus on the parts of the proof that require an adaptation to our topological groups.

In Appendix, Proposition 5.1 we prove that
\[ (\rho \times 1)_* \circ pr_1^* \mu^{k-1-l} \circ (\rho \times 1)^*: Z_r(\Omega Z \times \mathbb{P}^m) \to Z_{r+l}(\Omega Z \times \mathbb{P}^m) \]
is the zero map for any \( 0 < l \leq k - 1 \) and the identity map for \( l = 0 \). Here \( pr_1: X' \times \mathbb{P}^m \to X' \) is the canonical projection map on \( X' \) and \( m \geq 0 \).

For any \( r, m \geq 0 \), we have the following commutative diagram of fibration sequences:

\[
\begin{array}{ccc}
Z_r(\Omega Z \times \mathbb{P}^{m-1}) & \longrightarrow & Z_r(\Omega Z \times \mathbb{P}^m) \\
g \downarrow & & \downarrow g \\
Z_{r+l}(\Omega Z \times \mathbb{P}^{m-1}) & \longrightarrow & Z_{r+l}(\Omega Z \times \mathbb{P}^m)
\end{array}
\]

where \( g = (\rho \times 1)_* \circ pr_1^* \mu^{k-1-l} \circ (\rho \times 1)^* \). This implies that the induced map
\[ g = (\rho \times 1)_* \circ pr_1^* \mu^{k-1-l} \circ (\rho \times 1)^*: Z_r(\Omega Z \times \mathbb{P}^m) \to Z_{r+l}(\Omega Z \times \mathbb{A}^m) \]
is zero for any \( r, m \geq 0 \) and \( 0 < l \leq k - 1 \) and it is identity for any \( r, m \geq 0 \) and \( l = 0 \). We conclude that for any \( r \in \mathbb{Z} \), the map
\[ \rho_* \circ \mu^{k-1-l} \circ \rho^*: Z_r(\Omega Z) \to Z_{r+l}(\Omega Z) \]
is zero for \( l > 0 \) and is identity for \( l = 0 \).

Let’s consider the following homomorphisms:
\[ \lambda_1 = \bigoplus_{l=0}^{k-1} \rho_* \circ \mu^{k-1-l}: \pi_* Z_r(X') \to \bigoplus_{l=0}^{k-1} \pi_* Z_{r-k+1+l}(\Omega Z) \]
and
\[ \lambda_2 = \sum_{l=0}^{k-1} \mu^l \circ \rho^*: \bigoplus_{l=0}^{k-1} \pi_* Z_{r-k+1+l}(\Omega Z) \to \pi_* Z_r(X') \]
and
\[ T = \lambda_1 \circ \lambda_2 : \bigoplus_{l=0}^{k-1} \pi_* Z_{r-k+1+l}(\Omega Z) \to \bigoplus_{l=0}^{k-1} \pi_* Z_{r-k+1+l}(\Omega Z). \]

We can prove that \( \lambda_1 \) is surjective and \( \lambda_2 \) is injective. To see this consider \( \xi = (*, *, \ldots, *) \in \bigoplus_{l=0}^{k-1} \pi_* Z_{r-k+1+l}(\Omega Z) \). Then \( T(\xi) - \xi = (*, *, \ldots, *, 0) \). This is because the last component of \( T \) is
\[ \rho_* \circ \left( \sum_{l=0}^{k-1} \mu^l \circ \rho^* \right) = \sum_{l=0}^{k-1} \rho_* \circ \mu^l \circ \rho^* = \text{identity} \]
according to the above results. It implies that
\[ (T - I)^k = T^k + a_{k-1} T^{k-1} + \ldots + a_1 T \pm I = 0 \]
on \( \bigoplus_{l=0}^{k-1} \pi_* Z_{r-k+1+l}(\Omega Z) \).
This shows that there is a polynomial \( f \in \mathbb{Z}[X] \) such that \( f(T)T = Tf(T) = \pm I \) on \( \oplus_{l=0}^{k-1} \pi_\ast Z_{r-k+1+l}((\Omega_Z)\). This implies that \( \lambda_1 \) is surjective and \( \lambda_2 \) is injective.

We have the following commutative diagram with exact rows:

\[
\begin{array}{cccccc}
\pi_{+1}(Z_r(P(X))) & \longrightarrow & \pi_{+1}(Z_r(X')) & \longrightarrow & \pi_{+1}(Z_r(X'\setminus P(X))) & \\
\downarrow f(T) \circ \lambda_1 & & \downarrow f(T) \circ \lambda_1 & & \downarrow f(T) \circ \lambda_1 & \\
\oplus_{l=0}^{k-1} \pi_\ast Z_{r-k+1+l}((\Omega_X) & \longrightarrow & \oplus_{l=0}^{k-1} \pi_\ast Z_{r-k+1+l}((\Omega_Z) & \longrightarrow & \oplus_{l=0}^{k-1} \pi_\ast Z_{r-k+1+l}((\Omega_Z\setminus \Omega_X) & \\
(4) & & (2) & & & \\
\end{array}
\]

The exact rows are given by the long exact localization sequences applied to the closed embeddings \( i : P(X) \hookrightarrow X' \) and \( j_0 : \Omega_X \hookrightarrow \Omega_Z \).

The vertical rows are constructed from the maps given by the projective bundle theorem (Theorem \ref{thm:projective_bundle}) applied to the \( \mathbb{P}^k \)-bundle \( \rho_X : P(X) \to \Omega_X \) and to the \( \mathbb{P}^k \)-bundle \( \rho_Z/X : X' \setminus P(X) \to \Omega_Z \setminus \Omega_X \).

In Appendix, Proposition \ref{prop:diagram_chase} a diagram chase in Diagram (4) gives us the following isomorphism

\[
\lambda_2 + i_\ast \circ \rho_X^\ast : \oplus_{l=0}^{k-1} \pi_\ast Z_{r-k+1+l}((\Omega_Z) \oplus \pi_\ast Z_{r-k}((\Omega_X) \simeq \pi_\ast(Z_r(X'))
\]

for any \( r \geq 0 \). We notice that the map \( \lambda_2 + i_\ast \circ \rho_X^\ast \) could be defined on the cycle spaces. We may rewrite the indices \( (j = k-l) \) such that \( \oplus_{l=0}^{k-j} \pi_\ast Z_{r-k+1+l}((\Omega_Z) = \oplus_{j=0}^{k-j} \pi_\ast Z_{r-j}((\Omega_Z)\).

Because the topological groups \( Z_p(X) \) are C.W. complexes for any projective variety \( X \), the isomorphism (5) gives us a homotopy equivalence

\[
Z_r(X') \simeq \oplus_{j=0}^{k-j} Z_{r-j}((\Omega_Z) \oplus Z_{r-k}((\Omega_X)
\]

for any \( r \geq 0 \).

In particular, for \( r = 0 \), the isomorphism (5) gives the decomposition of the singular homology of \( X' \) proven by J. Lewis in (23, Proposition 3.2).

**Corollary 2.6.** (23) There is an isomorphism

\[
\{\oplus_{l=0}^{k-1} H^{n-2l}((\Omega_Z, \mathbb{Z})) \oplus H^{n-2k}((\Omega_X, \mathbb{Z})) \simeq H^n(X', \mathbb{Z})\).
\]

Moreover the s-map respects the decomposition (5) because it commutes with the maps involved in the above decomposition (see Proposition \ref{prop:diagram_chase} and the discussion after Proposition \ref{prop:diagram_chase}). This means that the cycle maps

\[
s : L_r H_{s+r}((X') \to H_{s+2r}((X')
\]

respect the decomposition (5). This gives the next corollary.

**Corollary 2.7.** For any \( r \geq 0 \), the isomorphism

\[
\lambda_2 + i_\ast \circ \rho_X^\ast : \oplus_{l=0}^{k-1} \pi_\ast Z_{r-k+1+l}((\Omega_Z) \oplus \pi_\ast Z_{r-k}((\Omega_X) \simeq \pi_\ast(Z_r(X'))
\]

for any \( r \geq 0 \).
commutes with the s-map operation. In particular, the kernel and the cokernel of the cycle maps from Lawson homology of $X'$ to the singular homology of $X'$ have the decompositions

$$L_r^\text{hom} H_{*+2r}(X') \simeq \oplus_{l=0}^{k-1} L_{r-k+l+1}^\text{hom} H_{*+2r-2k+2l+2}(\Omega Z) \oplus L_{r-k}^\text{hom} H_{*+2r-2k}(\Omega X).$$

and

$$C_{r,*+2r}(X') \simeq \oplus_{l=0}^{k-1} C_{r-k+l+1,*+2r-2k+2l+2}(\Omega Z) \oplus C_{r-k,*+2r-2k}(\Omega X).$$

Corollary 2.7 will be essential in proving certain properties of the cylindrical homomorphism on Lawson homology (see Theorem 4.1 and Theorem 4.3).

### 3. Some remarks on the Lawson homology groups of a generic hypersurface

In this section we extend to Lawson homology groups a weak Lefschetz result ([22], Proposition 2.1) proved on Chow groups and discuss some applications of the classical weak Lefschetz theorem on Lawson homology (Proposition 4.3). The results in this section will be used as technical tools in the next section.

Let $X \subset \mathbb{P}^{n+1}$ be a nonsingular hypersurface given by a homogeneous polynomial $F(X_0, X_1, \ldots, X_{n+1})$ of degree $d$. Consider

$$G(X_0, X_1, \ldots, X_n, X_{n+1}, X_{n+2}) = X_{n+2}^d + F(X_0, X_1, \ldots, X_{n+1})$$

a homogeneous polynomial of degree $d$. Then $G$ defines a smooth hypersurface $Z \subset \mathbb{P}^{n+2}$ with the property that $Z \cap \mathbb{P}^{n+1} = X$, where we considered the embedding $\mathbb{P}^{n+1} = V(X_{n+2}) \subset \mathbb{P}^{n+2}$. Consider the map $v_p : \mathbb{P}^{n+2} \to \mathbb{P}^{n+1}$, the projection from the point $p = [0, 0, \ldots, 0, 1]$ and the inclusions $k : X \hookrightarrow \mathbb{P}^{n+1}$ and $j : X \hookrightarrow Z$. It is obvious that $v_p \circ j = k$, i.e.

$$k : X \overset{j}{\hookrightarrow} Z \overset{v_p}{\hookrightarrow} \mathbb{P}^{n+1}.$$

Consider the graphs of these maps $W_j, W_k, W_v$ as correspondences in their corresponding Chow groups. The graphs have the property (see [23]) that

$$d^t W_j = i^* W_k \circ W_v \in CH_n(Z \times X)$$

where on the right side of the equality we have a composition of correspondences. We may view this equality in $A_n(Z \times X)$, the Chow group of algebraic cycles modulo algebraic equivalence.

The action of correspondences and its properties [as in ([18], Chapter 16)] can be word for word extended on Lawson homology because of the good properties this theory enjoys (see [20]).

**Theorem 3.1.** Let $X \subset \mathbb{P}^{n+1}$ be a nonsingular hypersurface of degree $d$ and $Z \subset \mathbb{P}^{n+2}$ be the associated smooth hypersurface defined above. Let $m \geq 2p \geq 0$. Consider the Gysin map

$$j^* : L_p H_m(Z) \to L_{p-1} H_{m-2}(X)$$

defined by the regular embedding $j : X \hookrightarrow Z$. Then $dj^*$ is zero on the kernel and on the cokernel of the cycle map $L_p H_m(Z) \to H_m(Z^{an})$. 

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\textit{Proof}. Consider the maps \( j, k, v_p \) defined as above and the relation between the correspondences given by the graphs of these maps: \( d^iW_j = ^tW_k \circ W_{v_p} \). The action of these correspondences on the Lawson homology groups gives the equality

\[ dj^* = k^*v_p^* \]

The fact that the action of the correspondence \(^tW_j\) has the same effect as the Gysin map \( j^* \) is a direct consequence of the projection formula and base change formula for Gysin maps proved in [27] (for the formalism see [18] Proposition 16.1.2 and Proposition 16.1.1 b),c)).

Applying the s-map \( p \) times gives the following commutative diagram:

\[
\begin{array}{ccc}
L_pH_m(Z) & \xrightarrow{v_p^*} & L_pH_m(\mathbb{P}^{n+1}) \\
\downarrow \text{cyc}_{p,m} & & \downarrow \text{cyc}_{p,m} \quad \downarrow \text{cyc}_{p-1,m-2} \\
H_m(Z^{an}) & \xrightarrow{v_p^*} & H_m(\mathbb{P}^{n+1}) & \xrightarrow{k^*} & H_m-2(X^{an}) \\
\end{array}
\]

for any \( p, m \geq 0, m \geq 2p \). The middle cycle map of the above diagram is an isomorphism for any \( p, m \geq 0 \) [21]. The composition of the horizontal maps is given by the Gysin map \( d_j^* \). It implies that \( d_j^*(L_p^{hom}H_m(Z)) = 0 = d_j^*(C_{p,m}(Z)) \). \( \square \)

\textbf{Corollary 3.2.} Consider the Gysin map \( j^* : H_m(Z^{an}) \to H_{m-2}(X^{an}) \) and fix \( p > 0 \). Then for any \( x \notin \text{Im}(\text{cyc}_{p,m}) \) we have \( d_j^*(x) \in \text{Im}(\text{cyc}_{p-1,m-2}) \).

\textbf{Proof.} This is just a reformulation of the statement \( d_j^*(C_{p,m}(Z)) = 0 \) proved above. \( \square \)

The classical weak Lefschetz theorem, together with the Poincare duality theorem [32], says that a smooth hypersurface \( X \subset \mathbb{P}^{n+1} \) has the following cohomology:

- \( H^k(X, \mathbb{Z}) = 0 \), for \( k \) odd, \( k \neq \text{dim}(X) \);
- \( H^{2k}(X, \mathbb{Z}) \simeq \mathbb{Z} \alpha \) for \( 2k > \text{dim}(X) \). Here \( \alpha \) is a class of a topological cycle with the intersection \( < \alpha, h^{n-1-k} > = 1 \). By \( h \) we denote the algebraic class of a hyperplane section of \( X \). Moreover, there is a positive integer \( a \in \mathbb{N} \) such that \( a\alpha \) is an algebraic cycle.
- \( H^{2k}(X, \mathbb{Z}) \simeq \mathbb{Z} h^k \) for \( 2k < \text{dim}(X) \), with \( h \) being the algebraic class of a hyperplane section of \( X \).

If \( X \subset \mathbb{P}^{n+1} \) is a smooth generic hypersurface of small degree \( 3 \leq d \leq n+1 \) that fulfills the condition \( \text{dim}(\Omega_X(k)) \geq 0 \) or equivalently \( (k+1)(n+1-k) - \binom{d+k}{k} \geq n-2k \), with \( k = \frac{n+1}{d} \), then we have at least one \( k \)-plane included in \( X \) [5]. This implies that \( H^{2n-2i}(X, \mathbb{Z}) = \mathbb{Z} \alpha = \mathbb{Z} \) for any \( 0 \leq i \leq k \), because we may take the generator \( \alpha \) to be an \( i \)-plane included in \( X \). We remark that a generic rationally connected hypersurface has always a line included in it.

We can conclude now the following proposition:

\textbf{Proposition 3.3.} For any smooth generic hypersurface \( X \subset \mathbb{P}^{n+1} \) of degree \( 3 \leq d \leq n+1 \), the cycle maps

\[ \text{cyc}_{p,m} : L_pH_m(X) \to H_m(X^{an}) \]
are

1) rationally surjective for any \( m \neq \dim(X) \).
2) surjective with integer coefficients for any \( m \neq \dim(X) \), \( m \) odd.
3) surjective with integer coefficients for any \( m = 2r > \dim(X) \).
4) surjective with integer coefficients for any \( 0 < m \leq 2k \) if \( \dim(\Omega_X(k)) \geq 0 \).

We notice that the first three points of the Proposition 3.3 apply to any smooth projective hypersurface. We will need later the following corollary:

**Corollary 3.4.** Let \( X \) be a smooth generic projective cubic hypersurface of dimension \( n = 5, 6 \) or \( 8 \). Then the cycle maps

\[
cyc_{p,m} : L_pH_m(X) \to H_m(X^{an})
\]

are surjective for any \( m \neq \dim(X) \).

**Proof.** If \( n = 5 \) or \( 6 \) then \( k = \left\lceil \frac{n+1}{2} \right\rceil = 2 \). Proposition 3.3 and Theorem 2.2 imply that all the cycle maps \( cyc_{p,m} \), with \( m \neq \dim(X) \), are surjective. For a generic cubic eightfold, we have \( k = 3 \) and again the conclusion follows from Propositions 3.3 and Theorem 2.2. \( \square \)

### 4. Applications

In this section we will show some applications of the results proved in the previous sections. Our main theorems are Theorem 4.1 and Theorem 4.3.

We proved in Theorem 2.5 that we have the following isomorphism:

\[
\lambda_2 + i_* \circ \rho^*_X : \oplus_{l=0}^{k-1} \pi_*Z_{r-k+1+l}(\Omega_Z) \oplus \pi_*Z_{r-k}(\Omega_X) \cong \pi_*(Z_r(X')).
\]

Because \( \pi : X' \to X \) is a generically finite map we have

\[
(6) \quad \pi_* \circ \pi^* = (\deg\pi)\text{id} : L_*H_*(X) \to L_*H_*(X)
\]

as a result of the projection formula for Gysin maps proved by C. Peters in [27] (see also [12]). This implies that \( \pi_* \otimes \mathbb{Q} : L_*H_*(X')\mathbb{Q} \to L_*H_*(X)\mathbb{Q} \) is a surjective map. From this and the corresponding in singular homology of the Equality (6), we can conclude that the restriction

\[
\pi_* \otimes \mathbb{Q} : L_*^\text{hom}H_*(X')\mathbb{Q} \to L_*^\text{hom}H_*(X)\mathbb{Q}
\]

is a surjective map.

Then the composition

\[
\pi_* \circ (\lambda_2 + i_* \circ \rho^*_X) = \pi_* \circ \lambda_2 + \pi_* \circ i_* \circ \rho^*_X : \oplus_{l=0}^{k-1} \pi_*Z_{r-k+1+l}(\Omega_Z) \oplus \pi_*Z_{r-k}(\Omega_X) \to \pi_*Z_r(X)
\]

is a rationally surjective map. Using Corollary 2.7 we conclude that

\[
(7) \quad \pi_* \circ (\lambda_2 + i_* \circ \rho^*_X) : \oplus_{l=0}^{k-1} L_r^{\text{hom}}H_{r-k+l+1}H_{s+2r-2k+2l} + \Omega_Z\mathbb{Q} \oplus L_r^{\text{hom}}H_{s+2r-2k}(\Omega_X)\mathbb{Q} \to L_r^{\text{hom}}H_{s+2r}(X)\mathbb{Q}
\]

is a surjective map.
The image of the map \( \pi_* \circ \lambda_2 : \bigoplus_{l=0}^{k-1} \pi_* Z_{r-k+l+1} (\Omega Z) \oplus \pi_* Z_{r-k} (\Omega X) \to Z_r (X) \) is included in the image of \( j^* : Z_{r+1} (Z) \to Z_r (X) \) because

\[
(8) \quad \pi_* \circ \lambda_2 = \pi_* \circ j_2^* (\sum_{l=0}^{k-1} \mu_l^Z \circ \rho_{Z}^*) = j^* \circ \pi_{Z*} \circ (\sum_{l=0}^{k-1} \mu_l^Z \circ \rho_{Z}^*)
\]

using the base change in the cartesian diagram

\[
\begin{array}{ccc}
X' & \xrightarrow{j_2} & P(Z) \\
\downarrow & & \downarrow \pi_Z \\
X & \xrightarrow{j} & Z.
\end{array}
\]

The following theorem is one of the main theorems of this section:

**Theorem 4.1.** Let \( X \subset \mathbb{P}^{n+1} \) be a generic hypersurface of degree \( 3 \leq d \leq n + 1 \) with \( \text{dim}(\Omega_X (k)) \geq n - 2k \). Then, the restriction of the cylindrical homomorphism

\[
\pi_* \circ i_* \circ \rho_X^* : L_{r-k}^{\text{hom}} H_{s+2(r-k)} (\Omega_X) \to L_{r}^{\text{hom}} H_{s+2r} (X) \mathbb{Q}
\]

is surjective for any \( r \geq 0 \). In particular

\[
L_{r}^{\text{hom}} H_{s+2r} (X) \mathbb{Q} = 0
\]

for any \( r \leq k \) and any \( r \geq n - k - 1 \). Moreover, we may fix a constant nonzero \( N \) such that \( NL_{r}^{\text{hom}} H_{s+2r} (X) = 0 \) for such \( r \).

**Proof.** According to (7)

\[
\pi_* (\lambda_2 + i_* \circ \rho_X^* ) : \bigoplus_{l=0}^{k-1} L_{r-k+l+1}^{\text{hom}} H_{s+2r-2k+2l+2} (\Omega Z) \oplus L_{r-k}^{\text{hom}} H_{s+2r-2k} (\Omega X) \to L_{r}^{\text{hom}} H_{s+2r} (X) \mathbb{Q}
\]

is a surjective map. From the Equality (8) we conclude that the composition

\[
L_{r-k}^{\text{hom}} H_{s+2r-2k} (\Omega X) \mathbb{Q} \xrightarrow{\pi_* \circ i_* \circ \rho_X^*} L_{r}^{\text{hom}} H_{s+2r} (X) \mathbb{Q} \to L_{r}^{\text{hom}} H_{s+2r} (X) \mathbb{Q} / \text{Im}(j^*) \mathbb{Q}
\]

is a surjective map. But Theorem 5.1 shows that \( dj^* \) is the zero map on the restriction to the kernel of \( \text{cyc}_{q,n} \) for any \( q, n \geq 0 \). It implies that

\[
\pi_* \circ i_* \circ \rho_X^* : L_{r-k}^{\text{hom}} H_{s+2(r-k)} (\Omega X) \mathbb{Q} \to L_{r}^{\text{hom}} H_{s+2r} (X) \mathbb{Q}
\]

is a surjective map.

Because \( \Omega_X \) is a smooth variety of pure dimension \( n - 2k \) we know that

\[
L_{n-2k} H_{2(n-2k)} (\Omega_X) \simeq H_{2(n-2k)} (\Omega_X) \simeq \mathbb{Z}^c
\]

where \( c \) is the number of connected components of \( \Omega_X \). We also know that the cycle maps

\[
\text{cyc}_{n-2k-1,q} : L_{n-2k-1} H_q (\Omega_X) \to H_q (\Omega_X)
\]

are isomorphisms for \( q \geq 2(n-2k) - 1 \) and monomorphisms for \( q = 2(n-2k) - 1 \) (4). This implies that \( L_{r-k}^{\text{hom}} H_{s+2(r-k)} (\Omega X) \mathbb{Q} = 0 \) for \( r \leq k \) and \( r \geq n - k - 1 \). Because \( \pi_* \circ i_* \circ \rho_X^* \) is a surjective map we conclude that \( L_{r}^{\text{hom}} H_{s+2r} (X) \mathbb{Q} = 0 \) for any \( r \leq k \) or \( r \geq n - k - 1 \).
We proved in Theorem 3.1 that $d_j^* = 0$ on $L_{h^0}^\text{hom}H_n(Z)$. We can see that the constant $N = d(\deg \pi)$ has the property that $NL_{r}^\text{hom}H_{s+2r}(X) = 0$ for any $r \leq k$ or any $r \geq n - k - 1$.

Theorem 4.1 together with Proposition 3.3 gives the following corollary:

**Corollary 4.2.** For any smooth generic hypersurface $X \subset \mathbb{P}^{n+1}$ of degree $3 \leq d \leq n + 1$ with $\dim(\Omega_X(k)) \geq n - 2k$ we have

$$L_rH_s(X)_\mathbb{Q} \cong H_s(X^{an})_\mathbb{Q}$$

for any $r \leq k$ and $r \geq n - k - 1$ and $\ast \neq \dim(X) = n$

The following theorem studies the surjectivity of the rational generalized cycle maps of $X$ into the middle dimension homology.

**Theorem 4.3.** Let $X \subset \mathbb{P}^{n+1}$ be a smooth hypersurface of degree $3 \leq d \leq n + 1$ with $\dim(\Omega_X(k)) \geq n - 2k$. Then

$$\text{cyc}_{r,n} \otimes \mathbb{Q} : L_rH_n(X)_\mathbb{Q} \cong H_n(X^{an})_\mathbb{Q}$$

for any $r \leq k$ or $r \geq n - k$. Moreover, we can fix a nonzero constant $N$ such that $NC_{r,n}(X) = 0$ for such $r$.

*Proof.* From Corollary 3.7 we have that

$$C_{r,s+2r}(X') \cong \bigoplus_{l=0}^{k-1}C_{r-k+l+1,s+2r-2k+2l+2}(\Omega_k) \oplus C_{r-k,s+2r-2k}(\Omega_X)$$

As above, the Equality (8) gives that the induced map

$$C_{r-k,s+2r-2k}(\Omega_X)_\mathbb{Q} \xrightarrow{\pi_{s,\ast}^\text{oi},\rho^X_{\ast}} C_{r,s+2r}(X)_\mathbb{Q} \rightarrow C_{r,s+2r}(X)_\mathbb{Q}/\text{Im}(j^*) \otimes \mathbb{Q}$$

is a surjective map. But Corollary 3.2 shows that $d_j^* = 0$ is the zero map on the cokernel of the cycle map $\text{cyc}_{q,n}$. This implies that the map

$$C_{r-k,s+2r-2k}(\Omega_X)_\mathbb{Q} \xrightarrow{\pi_{s,\ast}^\text{oi},\rho^X_{\ast}} C_{r,s+2r}(X)_\mathbb{Q}$$

is a surjective map. Let $\ast = n - 2r \geq 0$. Then we get the surjection $C_{r-k,n-2k}(\Omega_X)_\mathbb{Q} \rightarrow C_{r,n}(X)_\mathbb{Q}$. Because for $r \leq k$ or $r \geq n - k$ we have $C_{r-k,n-2k}(\Omega_X)_\mathbb{Q} = 0$, we also have $C_{r,n}(X)_\mathbb{Q} = 0$. As in Theorem 4.1, $N = d(\deg \pi)$ has the property that $NC_{r,n}(X) = 0$ for any $r \leq k$ or $r \geq n - k$.

We sketch now another way to prove that the cycle maps $\text{cyc}_{k,n} : L_kH_n(X)_\mathbb{Q} \rightarrow H_n(X^{an})_\mathbb{Q}$ are surjective maps for any $X \subset \mathbb{P}^{n+1}$ generic hypersurface of degree $3 \leq d \leq n + 1$ and with the property that $\dim(\Omega_X(k)) \geq n - 2k$. We can notice below that the cylindrical homomorphism on homology appears in a very natural way in the context of Lawson homology. We will not insist on all the details of this remark as this alternative proof is not important for the results of this paper.

We may identify $\Omega_X(k)$ with $C_{k,1}(X)$, the Chow variety of $k-$ dimensional subvarieties of degree $1$. We can see $P(X) \subset C_{k,1}(X) \times X$ as an algebraic cycle of dimension $n - k$. Its action gives a map

$$\phi^\ast : L_{r-k}H_{s+2(r-k)}(\Omega_X(k)) \rightarrow L_rH_{s+2r}(X).$$
Remark 4.4. The above composition of maps (9) points toward a possible strategy to prove the Generalized Hodge conjecture for any generic hypersurface $X$. The map $r$ is a section of the Hurewitz map of the topological group $Z_k(X)$. Moreover

$$H_{n-2k}(\Omega_X)_Q \xrightarrow{i_*} H_{n-2k}(\Omega_X(k))_Q$$

$$\pi_{X_s} \circ \phi_X \downarrow \quad \quad \downarrow \phi_s$$

$$H_n(X^{an})_Q \rightarrow H_n(X^{an})_Q$$

is a commutative diagram with $i_*$ surjective. We see that if $\pi_{X_s} \circ \phi_X$ is surjective then $\phi_s$ is surjective. From the composition (9) we see that if $\phi_s$ is surjective then the cycle map

$$\text{cyc}_{k,n} : L_k H_n(X)_Q = \pi_{n-2k}(Z_k(X))_Q \rightarrow H_n(X)_Q$$

is a surjective map. But $\pi_{X_s} \circ \phi_X$ is surjective for any generic smooth hypersurface $X$ with the property that $\dim(\Omega_X(k)) \geq n - 2k$ (23).

**Remark 4.4.** The above composition of maps (9) points toward a possible strategy to prove the Generalized Hodge conjecture for any generic hypersurface $X \subset \mathbb{P}^{n+1}$ of degree $d \leq n + 1$, not only for those fulfilling the condition $\dim(\Omega_X(k)) \geq n - 2k$. We see that it would be enough, for any generic hypersurface $X \subset \mathbb{P}^{n+1}$ of degree $d \leq n + 1$, to prove the much weaker condition (weaker than the condition $\phi_s$ surjective): the map

$$\oplus_d H_{n-2k}(C_{k,d}(X)) \rightarrow H_n(X^{an})$$

is a surjective map.

We may apply the above results to study semi-topological invariants of generic cubic hypersurfaces.

**Corollary 4.5.** Let $X$ be a smooth generic cubic fivefold. Then:

1) $L_1 H_2(X) \cong H_2(X^{an}) \cong \mathbb{Z}$.
2) $L_1 H_3(X) \cong H_3(X^{an}) = 0$.
3) $L_i H_4(X) \cong H_4(X^{an}) \cong \mathbb{Z}$ for any $0 \leq i \leq 2$.
4) $L_2 H_5(X) \cong L_1 H_5(X) \cong H_5(X^{an})$.
5) $L_i H_6(X) \cong H_6(X^{an}) \cong \mathbb{Z}$ for any $0 \leq i \leq 3$.
6) $L_i H_7(X) \cong H_7(X^{an}) = 0$ for any $0 \leq i \leq 3$.
7) $L_i H_8(X) \cong H_8(X^{an}) \cong \mathbb{Z}$ for any $0 \leq i \leq 4$.
8) $L_i H_9(X) \cong H_9(X^{an}) = 0$ for any $0 \leq i \leq 4$.
9) $L_i H_{10}(X) \cong H_{10}(X^{an}) \cong \mathbb{Z}$ for any $0 \leq i \leq 5$.
10) $L_n H_n(X) = 0$ for any $n > 2 \dim(X) = 10$.

In particular, any generic smooth cubic fivefold fulfills Suslin’s conjecture.
Proof. For a generic smooth cubic fivefold we have \( k = \left\lfloor \frac{n+1}{2} \right\rfloor = 2 \). For this \( k \), we may apply the same arguments as in the proof of the case of a generic cubic eightfold given below.

**Corollary 4.6.** Let \( X \) be a smooth generic cubic fivefold. Then
\[
K^{\text{sat}}_*(X) \cong ku^{-}_Q(X^{\text{an}})
\]
for any \( \ast \geq 0 \).

**Proof.** The corollary follows directly from Corollary 4.5 and Theorem 1.8.

**Corollary 4.7.** Let \( X \) be a generic smooth cubic hypersurface of dimension 8. Then:

1) \( L_1 H_2(X) \cong H_2(X^{\text{an}}) \cong \mathbb{Z} \).
2) \( L_1 H_3(X) \cong H_3(X^{\text{an}}) = 0 \).
3) \( L_i H_4(X) \cong H_4(X^{\text{an}}) \cong \mathbb{Z} \) for any \( 0 \leq i \leq 2 \).
4) \( L_i H_5(X) \cong H_5(X^{\text{an}}) = 0 \) for any \( 0 \leq i \leq 2 \).
5) \( L_i H_6(X) \cong H_6(X^{\text{an}}) \cong \mathbb{Z} \) for any \( 0 \leq i \leq 3 \).
6) \( L_3 H_7(X) \cong H_7(X^{\text{an}}) = 0 \) for any \( 0 \leq i \leq 3 \).
7) \( L_4 H_8(X) \cong L_7 H_8(X) \cong L_8 H_8(X) \cong H_8(X^{\text{an}}) \) for any \( 0 \leq i \leq 3 \).
8) \( L_4 H_9(X) \cong L_9 H_9(X) \cong L_3 H_9(X) \cong L_1 H_9(X) \cong H_9(X^{\text{an}}) = 0 \).
9) \( L_5 H_{10}(X) \cong L_4 H_{10}(X) \cong L_7 H_{10}(X) \cong H_{10}(X^{\text{an}}) \cong \mathbb{Z} \) for any \( 0 \leq i \leq 3 \).
10) \( L_5 H_{11}(X) \cong L_i H_{11}(X) \cong H_{11}(X^{\text{an}}) \) for any \( 0 \leq i \leq 4 \).
11) \( L_i H_{12}(X) \cong H_{12}(X^{\text{an}}) \cong \mathbb{Z} \) for any \( 0 \leq i \leq 6 \).
12) \( L_i H_{13}(X) \cong H_{13}(X^{\text{an}}) = 0 \) for any \( 0 \leq i \leq 6 \).
13) \( L_i H_{14}(X) \cong H_{14}(X^{\text{an}}) \cong \mathbb{Z} \) for any \( 0 \leq i \leq 7 \).
14) \( L_i H_{15}(X) \cong H_{15}(X^{\text{an}}) = 0 \) for any \( 0 \leq i \leq 7 \).
15) \( L_i H_{16}(X) \cong H_{16}(X^{\text{an}}) \cong \mathbb{Z} \) for any \( 0 \leq i \leq 8 \).
16) \( L_i H_n(X) = 0 \) for any \( n > 2\text{dim}(X) = 16 \).

In particular, any smooth generic cubic eightfold fulfills Suslin’s conjecture.

**Proof.** For a generic cubic eightfold we have \( k = \left\lfloor \frac{n+1}{2} \right\rfloor = 3 \). Using Corollary 4.2 we can conclude that the cycle maps
\[
cyc_{q,n} \otimes \mathbb{Q} : L_q H_n(X) \otimes \mathbb{Q} \to H_n(X^{\text{an}}) \otimes \mathbb{Q}
\]
with \( n \neq 8 \), are isomorphisms for any \( 0 \leq q \leq 8 \), \( n \geq 2q \). Because \( n \neq 8 \) (=middle dimension), \( cyc_{q,n} \) are surjective maps with integer coefficients (Corollary 3.4). Applying Theorem 1.1 to \( k = 3 \), we conclude that the kernels of
\[
cyc_{q,n} : L_q H_n(X) \to H_n(X^{\text{an}})
\]
have finite torsion for any \( n \geq 2q \).

According to Proposition 1.4, \( \text{Ker}(cyc_{q,n}) \) is divisible for \( n \geq 7 + q \), so we can conclude that the cycle maps with integer coefficients
\[
cyc_{q,n} : L_q H_n(X) \to H_n(X^{\text{an}})
\]
are isomorphisms for any \( n \geq 7 + q, n \neq 8 \) and
\[
cyc_{1,8} : L_1H_8(X) \rightarrow H_8(X^{an})
\]
is an injective map with integer coefficients. Looking to the rational cycle maps
\[
cyc_{r,8} : L_rH_8(X)_Q \rightarrow H_8(X^{an})_Q
\]
we notice that they are isomorphisms for any \( r \leq 3 \) (Theorem 4.3) and monomorphism for \( r = 4 \) (Theorem 4.1). Moreover, according to Proposition 4.4 and Theorem 4.3 the cycle map
\[
cyc_{1,8} : L_1H_8(X) \rightarrow H_8(X^{an})
\]
is surjective. Thus \( L_1H_8(X) \simeq H_8(X^{an}) \). To conclude the theorem, we only need to prove that
\[
L_6H_{12}(X) \simeq L_5H_{12}(X).
\]
This map is injective because on codimension 2 cycles the algebraic equivalence coincides with the homological equivalence for smooth projective rationally connected varieties ([4], Theorem 1). It is also surjective, according to Corollary 3.4 because
\[
cyc_{6,12} : L_6H_{12}(X) \rightarrow L_5H_{12}(X) \simeq H_{12}(X^{an})
\]
is a surjective map. \(\Box\)

The case of a generic smooth cubic sixfold can be treated similarly.

**Corollary 4.8.** Let \( X \) be a generic cubic sixfold or eightfold. Then
\[
K^*_s(X)_Q \simeq ku_Q^{-s}(X^{an})
\]
for any \( s \geq 1 \) and
\[
K^*_0(X)_Q \rightarrow ku_Q^0(X^{an}).
\]

**Proof.** Let \( X \) be a generic cubic hypersurface. Then the corollary follows from Corollary 4.7 and the following theorem:

**Theorem 4.9.** ([31], [11])

Let \( X \) be a smooth quasi-projective complex variety of dimension \( d \). Let \( A \) be an abelian group and \( r \leq 0 \). Then if
\[
cyc_{A,n} : L^qH^n(X, A) \rightarrow H^n(X^{an}, A)
\]
is an isomorphism for \( n - 2q \leq r - 1 \) and a monomorphism for \( n - 2q \leq r \), then the map
\[
K^*_i(X, A) \rightarrow ku^{-i}(X^{an}, A)
\]
is an isomorphism for \( i \geq -r + 1 \) and a monomorphism for \( i = -r \).

The case of a cubic sixfold can be treated similarly. \(\Box\)
Recall that for a smooth projective variety $X$ we have the following maps, starting from the cycle spaces and ending in the singular homology groups of $X$:

$$Z_r(X) \xrightarrow{s} L_rH_{2r}(X) = \pi_0(Z_r(X)) \xrightarrow{s} L_{r-1}H_{2r}(X) \xrightarrow{s} \cdots \xrightarrow{s} H_{2r}(X^{an}).$$

If $S_1Z_r(X) = \text{Ker}(s^i \circ \pi)$ then

$$0 \subset S_1Z_r(X)/S_0Z_r(X) \subset \cdots \subset \text{Griff}_r(X) = S_rZ_r(X)/S_0Z_r(X)$$

gives the $s$-filtration \cite{2} recalled in the first section of the paper.

We use the above results to show examples of varieties for which the lowest step in the $s$-filtration of $\text{Griff}_r(X) \otimes \mathbb{Q}$ is an infinitely generated rational vector space.

**Corollary 4.10.** Let $X$ be a generic cubic sevenfold. Then we have

$$L_3H_6(X)_\mathbb{Q} \xrightarrow{s} L_2H_6(X)_\mathbb{Q} \cong L_1H_6(X)_\mathbb{Q} \cong H_6(X^{an})_\mathbb{Q} \cong \mathbb{Q}$$

where the only non-isomorphism $s$-map from the sequence has infinitely generated kernel. Consider $Y$ to be a sixfold of large degree obtained from intersecting $X$ with a generic hypersurface of large degree. Then

$$\text{Ker}(L_2H_4(Y)_\mathbb{Q} \xrightarrow{s} L_1H_4(Y)_\mathbb{Q})$$

is infinitely generated. Moreover, the Abel-Jacobi map restricted to this kernel is zero.

Notice that this kernel is the lowest step in the $s$-filtration of $\text{Griff}_2(Y) \otimes \mathbb{Q}$.

**Proof.** For a generic cubic sevenfold we have $k = \lceil \frac{7+1}{3} \rceil = 2$. Using Corollary 4.2 we obtain that

$$L_2H_6(X)_\mathbb{Q} \cong L_1H_6(X)_\mathbb{Q} \cong H_6(X^{an})_\mathbb{Q}.$$ 

This shows that

$$\text{Griff}_3(X)_\mathbb{Q} = \text{Ker}(L_3H_6(X)_\mathbb{Q} \xrightarrow{\text{cyc.}} H_6(X^{an})_\mathbb{Q})$$

$$= \text{Ker}(L_3H_6(X)_\mathbb{Q} \xrightarrow{s} L_2H_6(X)_\mathbb{Q}).$$

Let

$$\Phi_X^4 : \text{Griff}_3(X) \to J^4(X)$$

be the Abel-Jacobi map on dimension three algebraic cycles of $X$ with values in the intermediate Jacobian $J^4(X)$ \cite{32}. Albano and Collino \cite{2}, Theorem 1) proved that $\text{Griff}_3(X)_\mathbb{Q}$ is infinitely generated in the case of a cubic sevenfold showing that the image of this Abel-Jacobi map $\Phi_X^4(\text{Griff}_3(X)_\mathbb{Q})$ is infinitely generated. This result implies the first part of our corollary.

Let $Y$ be the intersection of $X$ with a hypersurface of high degree in $\mathbb{P}^8$. In \cite{2}, Theorem 2, it is proved that $\text{Griff}_2(Y)_\mathbb{Q}$ is infinitely generated, but the Abel-Jacobi map on this Griffiths group is zero. We show that

$$\text{Ker}(L_2H_4(Y)_\mathbb{Q} \xrightarrow{s} L_1H_4(Y)_\mathbb{Q}) \subset \text{Griff}_2(Y)_\mathbb{Q}$$

is infinitely generated. Let

$$\xi \in \text{Ker}_X(s \circ \pi : CH_3(X)_\mathbb{Q} \xrightarrow{\pi} L_3H_6(X)_\mathbb{Q} \xrightarrow{s} L_2H_6(X)_\mathbb{Q}) = CH_3(X)_{\text{hom}} \otimes \mathbb{Q}$$
and let $R$ denote the map given by the following composition:

$$R : \text{Ker}_X(s \circ \pi) \rightarrow \text{Ker}_X(s \circ \pi) \rightarrow \text{Ker}_Y(s \circ \pi)/\text{Ker}_Y \pi = \text{Ker}(L_2H_4(Y)_Q \xrightarrow{\delta} L_1H_4(Y)_Q).$$

Nori [26] defined the following groups for a smooth projective variety $X$:

$$A_rCH_r(X) = \text{subgroup in } Z_r(X) \text{ generated by those algebraic cycles that are rationally equivalent to cycles of the form } pr_X^*(pr_Y^*(W)Z),$$

where $Y$ is a smooth projective variety, $W \in S_jZ_j(Y)$ and $Z \in Z_{r+\dim(Y)-j}(X \times Y)$.

We have the following theorem of M.Nori [26].

**Theorem 4.11.** (Nori [26]) Let $X \subset \mathbb{P}^n$ be a smooth, projective variety and let $i : Y \hookrightarrow X$ be the intersection of $X$ with $h$ general hypersurfaces of sufficiently large degrees. Let $\xi \in CH^d(X)$ and let $\mu = i^*(\xi)$. Assume that $r + d < \dim(Y)$. If $\mu \in A_rCH_{\dim(Y)-d}(Y)_Q$, then

1. The cohomology class of $\xi$ vanishes in $H^{2d}(X, \mathbb{Q})$, and
2. the Abel-Jacobi image of a non-zero multiple of $\xi$ belongs to $J^d_r(X)$

We use this theorem in the case $h = 1, d = 4$ and $r = 0$. In this case $A_0CH_2(Y) = CH^4_{\text{alg}}(Y) = \{\text{algebraic cycles on } Y \text{ of dimension } 2 \text{ algebraically equivalent to zero}\}$.

If $R(\xi) = \mu = 0$ then $\xi \in J^4_0(X)_Q$ by the Theorem 4.11 $J^4_0(X)$ is by definition the “largest” abelian subvariety of the intermediate Jacobian $J^4(X)$. For a generic cubic sevenfold $X$, we have $J^4_0(X) = 0$ [32]. This shows that

$$\text{Ker}(R) \subset \text{Ker}(\Phi_X^4 : CH^4_{\text{hom}}(X)_Q \rightarrow J^4(X)_Q).$$

But

$$\text{Im}(\Phi_X^4) \simeq CH^4_{\text{hom}}(X)/\text{Ker}(\Phi_X^4) \subset CH^4_{\text{hom}}(X)/\text{Ker}(R) \simeq \text{Im}(R) \subset \text{Ker}(L_2H_4(Y)_Q \xrightarrow{\delta} L_1H_4(Y)_Q).$$

The image $\Phi_X^4(\text{Ker}_X(s \circ \pi)_Q)$ is infinitely generated because $\Phi_X^4(\text{Griff}_3(X)_Q)$ is infinitely generated [2]. We obtain that

$$\text{Ker}(L_2H_4(Y) \otimes \mathbb{Q} \xrightarrow{\delta} L_1H_4(Y) \otimes \mathbb{Q})$$

is infinitely generated. □

**Remark 4.12.** The above corollary supports the conjecture that for high degree complete intersection varieties all $s$-maps, except those in Suslin’s conjecture range of indices, have infinitely generated kernel.

**Remark 4.13.** Homotopy groups of topological spaces of cycles of dimension 3 on cubic sevenfolds are interesting particular cases in which Suslin’s conjecture and the Friedland-Mazur conjecture are not yet checked. For a generic cubic sevenfold $X$, Suslin’s conjecture predicts that $L_3H_r(X) = \pi_{r-6}Z_3(X)$ is a finitely generated group for any $r \geq 9$.

Our methods check the Friedland-Mazur conjecture for any space of algebraic cycles of a generic cubic sevenfold $X$, except for the space of algebraic cycles on $X$.
of dimension 3. That is, for a generic cubic sevenfold \( X \), \( L_r H_n(X) = 0 \) for any \( r \neq 3 \) and \( n > 2\dim(X) = 14 \).

Similar arguments show that homotopy groups of topological spaces of cycles of dimension 5 on generic cubic elevenfolds are interesting particular cases in which these two conjectures are not yet checked. For example, for a generic cubic elevenfold, our methods show that \( L_r H_n(X) = 0 \) for any \( r \neq 5 \) and \( n > 2\dim(X) = 22 \).

In the end we make the following remark.

**Corollary 4.14.** For any \( r \geq 2 \), there is a smooth projective variety \( X \) such that \( S_1 Z_2(X) \otimes \mathbb{Q}/S_0 Z_2(X) \otimes \mathbb{Q} \) is an infinitely generated \( \mathbb{Q} \)-vector space and the restriction of the Abel-Jacobi map on this step of the \( s \)-filtration of \( \text{Griff}_r(X) \otimes \mathbb{Q} \) is zero.

**Proof.** If \( r = 2 \), we can take \( X \) to be the above constructed generic complete intersection of dimension 6. Let \( r > 2 \) and let \( X \) be the generic complete intersection of dimension 6 defined above and \( P \) a smooth projective variety of dimension \( m \geq 1 \). Let \( \{ \gamma_i \in S_1 Z_2(X) \otimes \mathbb{Q}/S_0 Z_2(X) \otimes \mathbb{Q} \} \) be a set of linearly independent algebraic cycles. Suppose

\[
\{ \gamma_i \times P \in S_1 Z_{2+m}(X \times P) \otimes \mathbb{Q}/S_0 Z_{2+m}(X \times P) \otimes \mathbb{Q} \}
\]

is a set of linearly dependent algebraic cycles. This implies that there are some \( a_i \in \mathbb{Q} \) such that

\[
\sum_i a_i (\gamma_i \times P) = 0 \in S_1 Z_{2+m}(X \times P) \otimes \mathbb{Q}/S_0 Z_{2+m}(X \times P) \otimes \mathbb{Q}.
\]

According to ([9], Proposition 3.5) we can conclude that

\[
\sum_i a_i \gamma_i = 0 \in S_1 Z_{2+m}(X) \otimes \mathbb{Q}/S_0 Z_{2+m}(X) \otimes \mathbb{Q}
\]

which is a contradiction with our choice of \( X \) and \( \gamma_i \). It implies that \( S_1 Z_{2+m}(X \times P) \otimes \mathbb{Q}/S_0 Z_{2+m}(X \times P) \otimes \mathbb{Q} \) is an infinitely generated \( \mathbb{Q} \)-vector space. It is obvious that the Abel-Jacobi map applied to the cycles \( \gamma_i \times P \) is zero, because it is zero on the cycles \( \gamma_i \). This implies the corollary for \( r > 2 \).

\[\square\]

5. **Appendix**

In this appendix we will prove two technical propositions needed in the paper. We will use the same notations introduced in the beginning of the second section.

We prove the following proposition (in \( \mathcal{H}^{-1}\text{AbTop} \)):

**Proposition 5.1.** The map

\[
(\rho \times 1)_* \circ pr_1^* \mu^{k-1-l} \circ (\rho \times 1)^* : Z_r(\Omega_Z \times \mathbb{P}^m) \to Z_r(\Omega_Z \times \mathbb{P}^m)
\]

is the identity map if \( l = 0 \) and the zero map if \( l > 0 \).

Here \( pr_1 : X' \times \mathbb{P}^m \to X' \) is the canonical projection on \( X' \) and \( m \geq -1 \).
Proof. According to the projection formula for Gysin maps \[27\] we have
\[(\rho \times 1)_* \circ pr^*_1 \mu^{k-1-l} \circ (\rho \times 1)^*(\alpha) = [(\rho \times 1)_* pr^*_1 \mu^{k-1-l}] \cdot \alpha\]
for any \(\alpha \in Z_r(\Omega_Z \times \mathbb{P}^m)\) and \(r \geq 0\).

This means that the map \[(10)\] is the action of the cycle
\[\beta := (\rho \times 1)_* pr^*_1 \mu^{k-1-l}\]
on \(Z_r(\Omega_Z \times \mathbb{P}^m)\). Recall that \(dim(X') = n\) and \(dim(\Omega_Z) = n - k + 1\). We have
\[pr^*_1 \mu^{k-1-l} \in Z_{n-k+1+l+m}(X' \times \mathbb{P}^m)\]
so \(\beta \in Z_{n-k+1+l+m}(\Omega_Z \times \mathbb{P}^m)\). This implies that \(\beta = 0\) if \(l > 0\) and \(\beta = a[\Omega_Z \times \mathbb{P}^m]\), for some integer \(a\), in case \(l = 0\). Thus, we conclude that the map \[(10)\] is zero if \(l > 0\) and \(a(id)\) if \(l = 0\). In particular, the map \[(10)\] in case \(l = 0\) induces the multiplication by \(a\) on all homotopy groups of \(Z_r(\Omega_Z \times \mathbb{P}^m)\), for any choice of \(r \geq 0\).

In particular, for \(r = m + n - k + 1\), we have that
\[(\rho \times 1)_* pr^*_1 \mu^{k-1-l}(\rho \times 1)^* = a(id) : A_r(\Omega_Z \times \mathbb{P}^m) \to A_r(\Omega_Z \times \mathbb{P}^m)\]
where \(A_r(\Omega_Z \times \mathbb{P}^m) = \pi_0(Z_r(\Omega_Z \times \mathbb{P}^m))\) is the Chow group of algebraic cycles modulo algebraic equivalence. To avoid any possible confusions, we will write below \(pr_{X'}\) for the above map \(pr_1\). Looking to the following cartesian diagram
\[
\begin{array}{ccc}
X' \times \mathbb{P}^m & \xrightarrow{pr_{X'}} & X' \\
\rho \times 1 \downarrow & & \downarrow \rho \\
\Omega_Z \times \mathbb{P}^m & \xrightarrow{pr_{\Omega_Z}} & \Omega_Z
\end{array}
\]
we can conclude the following equalities on Chow groups:
\[(\rho \times 1)_* \circ pr^*_{X'} \mu^{k-1} \circ (\rho \times 1)^* \circ pr^*_{\Omega_Z}() = (\rho \times 1)_* [pr^*_{X'} \mu^{k-1},(pr_{\Omega_Z} \circ (\rho \times 1))^*()]
= (\rho \times 1)_* \circ pr^*_{X'} [\mu^{k-1},\rho^*()]
\equiv (3) \circ pr^*_{\Omega_Z} \circ \rho_*[\mu^{k-1},\rho^*()] = pr^*_{\Omega_Z} \circ \rho_* \circ \mu^{k-1} \circ \rho^*().
\]
The equality (3) comes from the flat base-change formula \([18]\). Thus, we have the following commutative diagram
\[
\begin{array}{ccc}
CH_{n-k+1}(\Omega_Z) & \xrightarrow{pr^*_\Omega Z} & CH_{m+n-k+1}(\Omega_Z \times \mathbb{P}^m) \\
\rho_* \circ \mu^{k-1} \circ \rho^* \downarrow & & \downarrow (\rho \times 1)_* \circ pr^*_{X'} \circ \mu^{k-1} \circ (\rho \times 1)^* \\
CH_{n-k+1}(\Omega_Z) & \xrightarrow{pr^*_\Omega Z} & CH_{m+n-k+1}(\Omega_Z \times \mathbb{P}^m)
\end{array}
\]
According to \([23\text{, Page } 216]\) the left vertical arrow is the identity map. Because the right vertical arrow is \(a(id)\) we have the following equality
\[pr^*_{\Omega Z}([\Omega_Z]) = apr^*_{\Omega Z}([\Omega_Z]).\]
From the injectivity of \(pr^*_{\Omega Z}\) \([18\text{, Corollary } 3.1]\), we conclude that \(a = 1\). \(\square\)
We have the following diagram of abelian groups with exact rows and surjective vertical maps (see diagram (25)):

\[
\begin{array}{cccccc}
\longrightarrow & \pi_{*+1}(Z_r(P(X))) & \xrightarrow{i_*} & \pi_{*+1}(Z_r(X')) & \longrightarrow & \pi_{*+1}(Z_r(X' \setminus P(X))) \\
\downarrow & \alpha & & \downarrow & \beta \cong & \downarrow \\
\oplus_{l=0}^{k-1} \pi_{*+1}Z_{r-k+1+l}(\Omega_X) & \xrightarrow{\oplus_{l=0}^{k-1} \pi_{*+1}Z_{r-k+1+l}(\Omega_Z)} & \oplus_{l=0}^{k-1} \pi_{*+1}Z_{r-k+1+l}(\Omega_X) \\
\end{array}
\]

Notice that, according to Theorem [1.5] for the \(P^k\)-bundle \(\rho_X : P(X) \to \Omega_X\) we have

\[
\pi_*Z_r(P(X)) = \sum_{j=0}^{k} \lambda_{k-j}(\rho_X)^*(\pi_*Z_r(\Omega_X)) \cong \oplus_{j=0}^{k-1} \pi_*Z_{r-j}(\Omega_X) \oplus \pi_*Z_{r-k}(\Omega_X)
\]

The Equality (1) is from rewriting the indices (making \(l = k - 1 - j\)). Thus, for \(\alpha : \pi_*Z_r(P(X)) \to \oplus_{l=0}^{k-1} \pi_*Z_{r-k+1+l}(\Omega_X)\) we have

\[
(11) \quad Ker(\alpha) = \rho_X^*(\pi_*Z_{r-k}(\Omega_X)).
\]

The following proposition is part of the proof of Theorem [2.5].

**Proposition 5.2.** The map

\[
\lambda_2 + i_* \circ \rho_X^* : \oplus_{l=0}^{k-1} \pi_*Z_{r-k+1+l}(\Omega_Z) \oplus \pi_*Z_{r-k}(\Omega_X) \cong \pi_*Z_r(X')
\]

is an isomorphism.

**Proof.** The proof is identical with the proof of Proposition 3.2 in [23], with the only change that instead of singular cohomology, we use homotopy of cycle spaces. For the sake of completeness, we will sketch below the details. We will prove the following four statements:

1) \(\pi_*Z_r(X') = \lambda_2(\oplus_{l=0}^{k-1} \pi_*Z_{r-k+1+l}(\Omega_Z)) + i_*\pi_*Z_r(P(X))).
2) \(i_* : \rho_X^*(\pi_*Z_{r-k}(\Omega_X)) \to \pi_*Z_r(X')) is injective.
3) \(\lambda_2(\oplus_{l=0}^{k-1} \pi_*Z_{r-k+1+l}(\Omega_Z)) \cap i_*\pi_*Z_r(X') = 0.
4) \(\pi_*Z_r(X') \cong \lambda_2(\oplus_{l=0}^{k-1} \pi_*Z_{r-k+1+l}(\Omega_Z)) \oplus i_*\rho_X^*(\pi_*Z_{r-k}(\Omega_X))

The conclusion of our proposition will follow directly from 4) and 2).

**Proof of 1)**

1) follows from the split short exact sequence (with \(\lambda_2\) section map):

\[
0 \to Ker(\mp f(T)\lambda_1) \to \pi_*Z_r(X') \to \oplus_{l=0}^{k-1} \pi_*Z_{r-k+1+l}(\Omega_Z) \to 0
\]

and from the fact that \(Ker(\mp f(T)\lambda_1) \subset Im(i_*)

**Proof of 2)**
The statement follows from a simple diagram chase in diagram (1) and (2), using the Equality (11) and that \( f(T) \lambda_1 : \pi_{r+1}(Z_r(X')) \rightarrow \bigoplus_{l=0}^{k-1} \pi_{r+1}(Z_{r-k+1+l}(\Omega_Z)) \) is a surjective map.

**Proof of 3)**

Let \( y \in \pi_{r-k}(\Omega_Z) \) and suppose there is \( z \in \bigoplus_{l=0}^{k-1} \pi_{r-k+1+l}(\Omega_Z) \) such that \( \lambda_2(z) = i_\ast \rho_X(y) \). Applying \( f(T) \lambda_1 \) on this equality gives \( z = 0 \).

**Proof of 4)**

The proof of 4) is word for word the proof of Lemma 3.6 from [23] so we skip this diagram chase. \( \square \)

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