Segmentation of Time Series: Parameter Dependence of Blake-Zisserman and Mumford-Shah Functionals and the Transition from Discrete to Continuous

A. Kempe\(^1\)* V. Liebscher\(^1\) G. Winkler\(^1\)
O. Wittich\(^2\)

\(^1\)Institute of Biomathematics and Biometry
GSF-National Research Center for Environment and Health
Ingolstädter Landstr.1, D - 85764 Neuherberg
kempe,liebscher,winkler@gsf.de

\(^2\)M 12, Centre for Mathematical Science, TU München
Boltzmannstr. 3, D - 85747 Garching bei München
wittich@ma.tum.de

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Abstract

The paper deals with variational approaches to the segmentation of time series into smooth pieces, but allowing for sharp breaks. In discrete time, the corresponding functionals are of Blake-Zisserman type. Their natural counterpart in continuous time are the Mumford-Shah functionals. Time series which minimise these functionals are proper estimates or representations of the signals behind recorded data. We focus on consistent behaviour of the functionals and the estimates, as parameters vary or as the sampling rate increases.

For each time continuous time series \( f \in L^2([0,1]) \) we take conditional expectations w.r.t. \( \sigma \)-algebras generated by finer and finer partitions of the time domain into intervals, and thereby construct a sequence \( (f_n)_{n \in \mathbb{N}} \) of discrete time series. As \( n \) increases this amounts to sampling the continuous time series with more and more accuracy.

Our main result is consistent behaviour of segmentations w.r.t. to variation of parameters and increasing sampling rate.

Keywords: Segmentation, Blake - Zisserman and Mumford - Shah functional, \( \Gamma \)–Convergence, Hausdorff - metric

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1 Introduction

We will first introduce and discuss the adopted concept of segmentation, and define the functionals we are dealing with. Then the main result will be stated, and finally, we sketch the plan of the paper.

1.1 Segmentations

A fundamental variational ansatz for the segmentation of time series is the minimisation of functionals which penalise undesired properties of the estimate against fidelity to the data. The latter will be measured by the $L^2$-distance of the estimate and data. The penalty should depend on

(i) the measure of the set of ‘jumps’ or ‘breaks’ between regions of ‘smoothness’,

(ii) a notion of smoothness which restricts the behaviour of the signal between two subsequent jumps.

The notion of ‘jump’ and ‘smoothness’ will be made precise shortly. Let us illustrate the concepts by way of two examples. Suppose that the segmentation is a square integrable function $g$ on the closed unit interval $U = [0, 1]$. As segmentations, we allow functions $f = t + F$ with a right continuous step function $t$ and a Sobolev differentiable function $F$. These functions are of special bounded variation, see Section 2.4. The Mumford-Shah functionals in time dimension one are defined as

$$\text{MS}_{\gamma, \mu, g}(f) := \gamma j(t) + \frac{1}{\mu^2} \int_U |f'|^2 ds + \|g - f\|^2$$

where $j(t)$ is the number of discontinuities of $t$ in $(0, 1)$, $f'$ is the Sobolev derivative of $F$, and the parameters $\gamma$ and $\mu$ control the number of breaks and the degree of smoothness. This is the one-dimensional version of the functionals introduced in [12] and [13] for space dimension two.

In this model, jumps are the discontinuities of $t$ which by Sobolev’s embedding theorem can be identified with the discontinuities of $f$. ‘Smoothness’ is measured by the $L^2$-norm of the derivative of $F$ within the intervals between subsequent jumps; it coincides there with the derivative of $f$. Fidelity to the data, finally, is measured by the $L^2$-distance of the segmentation to data.
The second example – which is in fact a special case of the Mumford - Shah functional if $\mu = 0$ – is called Potts - functional, inspired by the functional introduced in [14] as a generalization of the Ising model ([7]) in statistical mechanics. It is defined for step functions $t$ and given by

$$P_{\gamma,g}(t) := \gamma j(t) + \|t - g\|^2.$$ 

The notions of jump and of fidelity to the data is the same as for the Mumford - Shah functional, whereas the notion of smoothness is considerably stronger, only functions which are even constant on the intervals between consecutive jumps are possible outcomes of the segmentation procedure. The Mumford - Shah functional (and hence the Potts-functional as well) can be extended to $L^2(U)$ in the sense that it is given by the expression above, if the equivalence class of $f \in L^2(U)$ contains a function which can be written as $f = t + F$, and that it is equal to $\infty$, if not (cf. Definition 10).

In contrast to the continuous setting, there are no ‘obvious’ notions of smoothness or jumps for discrete time - series $g = (g^0, ..., g^{n-1}) \in \mathbb{R}^n$. One possibility is to consider those points $\kappa \in \{0, ..., n-2\}$ as jumps, where the difference $|f^{\kappa+1} - f^\kappa|$ exceeds a given threshold (cf. Definition 6). One particular functional, where this notion is immanent is the Blake - Zisserman functional (see [3], [4]). For $\gamma, \mu \geq 0$ it is given by

$$BZ_{\gamma,\mu,g}(f) := \sum_{\kappa=0}^{n-2} \min\{|f^{\kappa+1} - f^\kappa|^2/\mu^2, \gamma\} + \sum_{\kappa=0}^{n-1} (f^\kappa - g^\kappa)^2.$$

An equivalent definition (cf. Lemma 3) focussing more closely on the nature of jumps (edges’) in this functional by showing that the min’ in the penalizing term arises from a minimization with respect to a larger space of covariables including explicit information about jumps’, was given in [6]. As for the Mumford - Shah functional, the special case $\mu = 0$, i.e.

$$P_{\gamma,g}(f) := \gamma |\{\kappa = 0, ..., n-1 : f^{\kappa+1} \neq f^{\kappa}\}| + \sum_{\kappa=0}^{n-1} (f^\kappa - g^\kappa)^2$$

is called (discrete) Potts model. It also favors segmentations $f$ which are constant between consecutive jump points.

In the variational ansatz, we consider as segmentations those functions or vectors, respectively, which minimize the given functionals. That immediately leads to the following questions:
(i) Is there always a minimizer of the given functional?

(ii) How does it depend on the (free) parameters $\gamma$ and $\mu$?

In view of the discussion about jumps and smoothness above, we may also ask a question about the relation between the discrete and the continuous setup:

(iii) Is there an embedding of the discrete situation into the continuous one, i.e. a way of discrete sampling from a continuous ‘truth’, such that the discrete segmentations converge to the segmentations of the continuous signal?

One exact formulation of question (iii) and an affirmative answer about the relation of Mumford-Shah and Blake-Zisserman functional is the main subject of this paper and is provided by Theorem 1 in the next subsection. As a byproduct of our analysis, we obtain in that case answers to questions (i) and (ii) as well.

1.2 The Main Result

For an exact formulation of the third question, we have to specify the embedding of the discrete into the continuous situation. Since we are considering signals $g \in L^2(U)$, it is not suitable to consider vectors arising from the evaluation of a signal $g$ at several distinct time points. Instead, we consider its conditional expectation with respect to the $\sigma$-algebra generated by a fixed partition of $U = [0, 1]$. The intuition behind this procedure is, that the output of our measuring device is truly an average of the signal over a short period of time. Sampling with more and more accuracy thus means to decrease the length of these periods. We adopt the following conventions:

Definition 1 Let $n \in \mathbb{N}$ and the equidistant setup be given by

$$\mathcal{S}(n) := \{ \frac{\kappa}{n} : \kappa = 0, ..., n \} \subset U$$

and $\sigma_n$ denote the $\sigma$-algebra

$$\sigma_n := \sigma \left( [0, \frac{1}{n}), [\frac{1}{n}, \frac{2}{n}), ..., [\frac{n-2}{n}, \frac{n-1}{n}), [\frac{n-1}{n}, 1] \right).$$

The conditional expectation of a signal $g \in L^2(U)$ with respect to $\sigma_n$ will be shortly denoted by $g_n := E(g | \sigma_n)$. 
Remark. The fact that the rightmost interval is closed has no further significance. We found that it is most convenient to deal with the right boundary point in the way that we consider only step functions which are continuous at $s = 1$.

As an appropriate discrete input for the Blake-Zisserman functional we consider now the signal $g_n = \pi_n g := (g_0, \ldots, g_{n-1}) \in \mathbb{R}^n$ given by

$$g_n^\kappa := E(g \mid \sigma_n)(\frac{\kappa}{n}) = n \int_{\frac{\kappa}{n}}^{\frac{\kappa+1}{n}} g(s)ds.$$ 

A one-sided inverse to this discretization map is provided for $f \in \mathbb{R}^n$ by the step function

$$\tau_n f(s) := f^{n-1} \chi_{[n \frac{\kappa-1}{n}, 1]}(s) + \sum_{\kappa=0}^{n-2} f^\kappa \chi_{[\frac{\kappa}{n}, \frac{\kappa+1}{n})}(s).$$

Remark. Note that for the conditional expectation of a signal $g \in L^2(U)$, we have $g_n = E(g \mid \sigma_n) = \tau_n \circ \pi_n g$. This fact will be used frequently in the sequel.

As a consequence of our treatment of the boundary point $s = 1$ – which implies that all $\sigma_n$-measurable functions are right-continuous step functions with left limits which are additionally continuous at $s = 1$ – we will assume this property as well for all step functions considered in this paper. Hence we arrive at the following definition.

**Definition 2** (1) We denote by $\mathcal{T}(U)$ the set of all right-continuous step functions on $U$ with left limits which are additionally continuous at $1 \in U$. (2) We denote by $\mathcal{T}_n(U) \subset \mathcal{T}(U)$ the set of $\sigma_n$-measurable functions on $U$.

With these conventions in mind, we can now define an embedded version of the Blake-Zisserman functional and of the discrete Potts-Functional on $L^2(U)$. Let $f_n := \tau_n f = (f^0, \ldots, f^{n-1}) \in \mathbb{R}^n$ given by $f^\kappa := f(\frac{\kappa}{n})$ if $f \in \mathcal{T}_n(U)$ and $g_n$ given by the conditional expectation above. Then

$$\text{BZ}^n_{\gamma, \mu, g}(f) := \begin{cases} \text{BZ}_{\gamma, \mu/n, g_n}/\sqrt{n}(\pi_n f/\sqrt{n}) & \text{if } f \in \mathcal{T}_n(U) \\ \infty & \text{else} \end{cases}$$
and
\[
P_{\gamma,g}^n(f) := \begin{cases} \frac{P_{\gamma,g}}{\sqrt{n}}(\pi_n f / \sqrt{n}) & \text{if } f \in \mathcal{T}_n(U) \\ \infty & \text{else} \end{cases},
\]
respectively (cf. Definition 11).

To be more precise, we embed the functionals defined above into a family of functionals depending on two real and one additional rational parameter which reflects the transition from discrete to continuous. Let
\[
T := \{1/n : n \in \mathbb{N}\} \cup \{0\} \subset \mathbb{R}
\]
be equipped with its relative topology as a subset of \(\mathbb{R}\), i.e. 0 is the only accumulation point. Consider the (pseudo-) cube \(Q = \mathbb{R}_0^+ \times \mathbb{R}_0^+ \times T\). To each point of the cube corresponds one functional as follows:

Let \((\gamma, \mu, t) \in Q\) and \(g \in L^2(U)\). We adopt the convention that \(t = 1/n\), \(0 = 1/\infty\), respectively. Then, the functions \(F(\gamma, \mu, t) : L^2(U) \to \mathbb{R}\) are given by

\[
F(\gamma, \mu, t) := \begin{cases} 
BZ_{\gamma,\mu,g}^n & \gamma > 0, \mu > 0, t > 0 \\
MS_{\gamma,\mu,g} & \gamma > 0, \mu > 0, t = 0 \\
P_{\gamma,g}^n & \gamma > 0, \mu = 0, t > 0 \\
P_{\gamma,g} & \gamma > 0, \mu = 0, t = 0 \\
d_{\gamma,g}^n & \gamma = 0, \mu > 0, t > 0 \\
d_{\gamma,g} & \gamma = 0, \mu > 0, t = 0 
\end{cases}.
\]

Here \(BZ_{\gamma,\mu,g}^n\), \(MS_{\gamma,\mu,g}\) are the Mumford-Shah- and Blake-Zisserman functionals which were shortly discussed above and which precise form extended to functionals on \(L^2(U)\) is given in the Definitions 10, 11 below. The Potts-functionals \(P_{\gamma,g}^n\) and \(P_{\gamma,g}\) are as above, the discrete distance functional \(d_{\gamma,g}^n : L^2(U) \to \mathbb{R}\) is given by

\[
d_{\gamma,g}^n(f) := \begin{cases} \frac{1}{n} \sum_{k=0}^{n-1} (f - g_n)^2(\ell_n) & f \in \mathcal{T}_n(U) \\ \infty & \text{else} \end{cases},
\]
and finally the continuous distance functional \(d_{\gamma,g} : L^2(U) \to \mathbb{R}\) given by

\[
d_{\gamma,g}(f) := \int_U (f - g)^2 ds.
\]

With these definitions, the main result of this paper reads as follows:

**Theorem 1** For the family \(F(q) : L^2(U) \to \mathbb{R}, q \in Q\), the following statements hold:
(i) For all $q \in Q$ there is a minimizer of $F(q)$.

(ii) Let $q_s = (\gamma_s, \mu_s, t_s), \ s \in \mathbb{N}$ converge to $q = (\gamma, \mu, t)$ in $Q$ as $s$ tends to infinity. Denote by $f_{s,s}^{*} \in \mathbb{N}$ a sequence of minimizers of $F(q_s)$. Then:

(a) Every convergent subsequence of $f_{s,s}^{*} \in \mathbb{N}$ converges to a minimizer of $F(q)$.

(b) $f_{s,s}^{*} \in \mathbb{N}$ always contains a convergent subsequence.

That means, segmentation of a time series using the variational ansatz with these functionals behaves consistently under variation of the parameters and under sampling the true signal with more and more accuracy.

\subsection*{1.3 Plan of the Paper}

The proof of our main result is mainly based on the fact, that the minimisation of the functionals under consideration can be splitted up: Once a set of jumps is fixed, there is a minimizer of the functional among all admissible functions whose jump set is contained in the given one. This minimizer can be computed rather explicitly. The minimization of the functional over all possible segmentations reduces thus to the minimization with respect to all jump sets, which will be identified with \textit{partitions} of $U$ or, respectively, its associated minimizers. In the sequel, this common feature of the functionals will be called the \textit{reduction principle}. The minimizers associated to a fixed jump set will be called \textit{partition solvers}.

In the following section, we begin by fixing the notion of \textit{partition} which is basically the exact manifestation of jump set', together with some of its properties as finite sets. Furthermore, we make precise our point of view by defining what we understand as a \textit{segmentation}. As its most important manifestation in this paper, we consider \textit{functions of special bounded variation}. In Section \ref{sec3}, we give an exact statement of the \textit{reduction principle} and compute the \textit{partition solvers}. Continuity properties of the latter ones are proved in the subsequent section. The first important implication of these considerations is relative compactness of the set of minimizers derived in Section \ref{sec5}. As explained at the beginning of Section \ref{sec6}, $\Gamma$-convergence together with this compactness result implies Theorem \ref{thm1}. To establish the $\Gamma$-convergence result Theorem \ref{thm6} in the final section, the basic ingredient is an associated result of independent interest, given by the Lemmas \ref{lem13} and \ref{lem14} on the interchange-ability of Sobolev differentiation and approximation by step functions. This
is proved in the remainder of Section 6. We end up with a (then) short proof of our main result Theorem 1.

2 Segmentations and Partitions

In this section, we introduce what we mean by the segmentation of a signal which depends on one parameter, for instance a discrete or continuous time-series.

2.1 The Continuous Case

In that case, the signal depends on a continuous set of parameters. We restrict ourselves to square integrable signals.

**Definition 3** Let $U := [0,1]$ and $L^2(U)$ denote the Hilbert space of (equivalence classes) of square integrable functions on $U$ with respect to Lebesgue measure. A signal is a function $g \in L^2(U)$. A segmentation class on $U$ is a class $S(U)$ of (equivalence classes) of right-continuous functions with left limits, i.e. there is an injective map $S(U) \subset D(U)$ where $D(U)$ denotes the Skorohod space (see e.g. [2]).

We have the following examples for segmentation classes:

**Example.** (1) $S(U) = D(U)$. (2) $S(U) = T(U)$, the space of all right-continuous step functions with left limits which are even continuous at $1 \in U$. (3) $S(U) = SBV_2(U)$, the functions of 2-bounded special variation discussed below.

Intuitively, a segmentation provides a decomposition of a signal into homogeneous parts which are separated by abrupt changes (jumps). The decomposition idea is represented by the concept of partitions. For reasons of technical convenience, we decided that all partitions contain the boundary points of $I$.

**Definition 4** Let $|M|$ denote the cardinality of a subset $M \subset U$. (1) The partitions of the interval $U$ are given by the set

$$\mathcal{P}(U) := \{ p \subset [0,1] : |p| < \infty ; 0,1 \in p \}.$$
(2) The special closed subsets of the interval $I$ are given by the set
\[ C(U) := \{ c \subset [0,1] : c \text{ closed}; 0, 1 \in c \}. \]

(3) The partition $p(f)$ associated to a given $f \in D(U)$ with finite set of discontinuities $\text{disc}(f)$ is $p(f) := \text{disc}(f) \cup \{0,1\}$.

In the sequel, the points in $\text{disc}(f)$ are frequently called jumps.

2.2 The Discrete Case

In the discrete case, we still think of a continuous signal in the sense defined above. In contrast to the continuous case, we can only sample its values at finitely many (equidistant) time points. Hence, partitions into continuous parts separated by discontinuities make no longer sense and we have to substitute it by something else. Recall the definitions 1 and 2 of the equidistant setup and the step functions associated to it.

**Definition 5** By

\[ \mathcal{P}_n(U) := \{ p \in \mathcal{P}(U) : p \subset \mathcal{S}(n) \} \]

we denote the partitions compatible with the equidistant setup $\mathcal{S}(n)$.

So far, we did not say anything about a decomposition of the signal into more and less homogeneous parts. In the continuous case, this was achieved with the help of a partition associated to the segmentation function. We will do the same now by introducing a suitable threshold.

**Definition 6** A threshold is a function $T : \mathcal{S}(n) \to \mathbb{R}_0^+$. The $T$-partition of a function $f \in \mathcal{T}_n(U)$ is given by

\[ p_T(f) := \{ \frac{k}{n} : |f(\frac{k+1}{n}) - f(\frac{k}{n})| > T(k); k = 1, ..., n - 1 \} \cup \{0,1\} \in \mathcal{P}_n(U). \]

Points in $p_T(f)$ are called discrete discontinuities of $f$ (with respect to $T$).

In that sense, we consider those points as points of discontinuity where the difference of values at adjacent sampling points exceeds a given threshold. Note that the threshold may as well be adapted, i.e. depend on the function under consideration.
Example. (1) For \( T = 0 \), all points where \( f(\frac{\kappa + 1}{n}) \neq f(\frac{\kappa}{n}) \) are discrete discontinuities. This choice corresponds to the consideration of the minimizers of the Potts functional, which are constant off the jump set. (2) For the Blake-Zisserman functional to parameters \( \gamma, \mu, n \) (see below), the appropriate choice is \( T(\kappa) = \mu \sqrt{\gamma/n} \).

2.3 Partitions and Hausdorff Metric

A topology on the set of closed subsets of \( U \) is provided by the Hausdorff metric.

Definition 7 Let \( c, c' \) be closed non-empty subsets of \( U \). Then the distance

\[
d_H(c, c') := \max \left\{ \max_{x \in c} \min_{y \in c'} |x - y|, \max_{x \in c'} \min_{y \in c} |x - y| \right\}
\]

is called Hausdorff distance.

The properties of the Hausdorff distance are summarized by the following proposition, the proof follows from standard facts, available for instance in [1] or [11].

Proposition 1 (1) The closed subsets of \( U \), provided with Hausdorff distance, form a compact (hence complete) metric space. (2) The set \( \mathcal{C}(U) \) is a closed subspace, hence as well compact. (3) The subset \( \mathcal{P}(U) \subset \mathcal{C}(U) \) is dense, i.e. \( \overline{\mathcal{P}(U)} = \mathcal{C}(U) \). (4) The subset \( \mathcal{P}_n(U) \subset \mathcal{C}(U) \) is finite, hence as well closed and compact.

In the sequel, we will need another characterization of Hausdorff convergence of partitions focussing on the decomposition of \( U \) into intervals.

Definition 8 Let \( p \in \mathcal{P}(U) \). By \( p^c := U - p \) we denote the complement of \( p \) in \( U \). The complement of \( p = \{ 0 = x_0 < x_1 < ... < x_m = 1 \} \) is a disjoint union of open subintervals \( \Theta \subset U \). The collection of these subintervals will be denoted by

\[
\iota(p) := \{ \Theta \subset p^c : \Theta = (x_k, x_{k+1}) : k = 0, ..., m - 1 \}.
\]
Now the characterization of Hausdorff convergence in terms of the intervals reads as follows.

**Lemma 1** Let $p_n \in \mathcal{P}(U)$ converge to $p = \{0 = x_0 < x_1 < ... < x_m = 1\} \in \mathcal{P}(U)$ in Hausdorff metric. Then we have

(i) For all $\Theta \in \iota(p)$ there is a sequence $\Theta_n \in \iota(p_n)$ such that $\overline{\Theta_n}$ converges to $\overline{\Theta}$ in Hausdorff metric.

(ii) $\overline{\Theta_n} = [a_n, b_n]$ converges to $\overline{\Theta} = [a, b]$ if and only if $a_n \to a$ and $b_n \to b$.

(iii) Let $\Theta, \Theta' \in \iota(p)$ two adjacent intervals, i.e $\Theta = (x, y)$, $\Theta' = (y, z)$. Let $\Theta_n = (x_n, y_n), \Theta'_n = (y'_n, z'_n)$ the sequences of intervals from (i). Then $[y_n, y'_n] \to \{y\}$ in Hausdorff metric and $y_n, y'_n \to y$ uniformly for all adjacent $\Theta, \Theta' \in \iota(p)$.

**Proof:** Let $\delta_0 := 1/3 \min_{x \neq y \in \rho} |x - y|$ and $\Theta = (a, b) \in \iota(p)$ be fixed. Without loss of generality, assume that $n$ is so large that $d_H(p_n, p) < \delta_0$. (ii) By $a < b$, $a_n < b_n$ and by the assumption, we have necessarily $|a - a_n|, |b - b_n| < \delta_0$ and thus by definition of the Hausdorff metric

$$d_H([a, b], [a_n, b_n]) = \max \left\{ \max_{x \in \{a, b\}} \min_{y \in \{a_n, b_n\}} |x - y|, \max_{y \in \{a_n, b_n\}} \min_{x \in \{a, b\}} |x - y| \right\} = \max \{|a_n - a|, |b_n - b|\}.$$

(i) We now construct the sequence $\Theta_n$ by letting $\Theta_n = (a_n, b_n)$ where

$$a_n := \max \{x \in p_n : x \leq a + \delta_0\} \quad b_n := \min \{x \in p_n : x \geq b - \delta_0\}.$$

By the assumption made above, both sets are non-empty and by Hausdorff convergence we have $a_n \to a$, $b_n \to b$. Thus, by (ii), $\overline{\Theta_n}$ converges to $\overline{\Theta}$.

(iii) By construction of the sequence in (i), always $y_n \leq y'_n$. But by (ii), $y_n, y'_n \to y$ and

$$|y_n - y'_n| \leq |y_n - y| + |y - y'_n| \leq 2d_H(p_n, p).$$

As a first application, we derive that the counting function is lower semi-continuous.
Corollary 1 The function $| - | : \mathcal{P}(U) \to \mathbb{N}$ is lower semi-continuous if $\mathcal{P}(U)$ is equipped with the Hausdorff topology.

Proof: The intervals $\Theta_n \in \iota(p_n)$ approximating a given interval $\Theta \in \iota(p)$ constructed in Lemma 1 can be chosen disjoint for different $\Theta$. The left boundary points of the approximating intervals are elements of $p_n$ and all different. Thus $|p_n| \geq |p|$.

2.4 Functions of Special Bounded Variation

Now we consider a special segmentation class, namely functions which are of special bounded variation. They are defined as follows: Recall that a right-continuous function of bounded variation $f \in BV(U)$ with left limits defines a signed measure $\nu \in \mathcal{M}(U)$ which is uniquely determined by its values $\nu([a,b)) := f(b) - f(a)$ on half-open intervals. Recall further, that by Lebesgue decomposition (see for instance [17], Theorem I.13, I.14, p. 22) there are three uniquely determined measures $\nu_S, \nu_R, \nu_\perp$ where $\nu = \nu_S + \nu_R + \nu_\perp$ and

(i) $\nu_S = \sum c_x \delta_x$ is a sum of point measures,

(ii) $\nu_R << \lambda$ is absolutely continuous with respect to Lebesgue measure with Radon-Nikodym derivative $f' = d\nu_R/d\lambda$,

(iii) $\nu_\perp \perp \lambda$ is singular to Lebesgue measure without point measures, i.e. $\nu_\perp(\{x\}) = 0$ for all $x \in U$.

Definition 9 (1) A function $f \in BV(U)$ is called to be of special bounded variation, i.e. $f \in SBV(U)$, if in the decomposition above we have $\nu_\perp = 0$. It is called to be of p-special bounded variation, i.e. $f \in SBV_p(U)$, if in addition, $f' \in L^p(U)$ and – in contrast to the usual convention – if the support $\text{supp} (\nu_S)$ is a finite set. (2) The partition associated to a function $f \in SBV_2(U)$ is given by $p(f) := \text{supp} (\nu_S) \cup \{0,1\}$.

It is not yet obvious that these functions form indeed a segmentation class in the sense that they are equivalent to piecewise continuous functions as assumed in Definition 8. We will show this for $p = 2$ and will characterize them as piecewise Sobolev-functions.
Lemma 2 (1) Let \( f \in SBV_2(U) \). Then there are (up to an additive constant) uniquely determined functions \( t \in T(U) \), the space of step-functions introduced above, and \( F \in H^1(\Omega) \), the Hilbert-Sobolev space of one time generalized differentiable functions with square integrable derivative on the open interval \( \Omega := (0,1) \), such that \( f = t + F \). (2) In the equivalence class of \( F \in H^1(\Omega) \), there is a continuous representative as well denoted by \( F \in C(U) \). (3) The partition \( p(f) \) associated to \( f \in SBV_2(U) \) coincides with \( \text{disc}(t) \cup \{0,1\} \), where \( \text{disc}(t) \) denotes the points of discontinuity of the corresponding step function.

Proof: (1) Let \( Df \) denote the distributional derivative (measure) of \( f \). By assumption \( Df = f' \cdot \lambda + \nu_S \). Then the distribution function of the singular part \( \nu_S \) is a right continuous step function with left limits \( t \in T(U) \). Hence \( D(f - t) = f' \cdot \lambda \) with \( \int_U |f'|^2 ds < \infty \). Hence \( F := f - t \in H^1(\Omega) \). (2) The second statement follows from Sobolev’s embedding lemma (see [15], Thm. IX.24, p. 52). (3) This follows from the fact that the step function is the distribution function of the point measure.

Thus, functions of special bounded variation are indeed right-continuous with left limits and hence form a segmentation class.

3 The Reduction Principle

The reduction principle consists of the fact that both, the Mumford - Shah and the Blake - Zisserman functional − except for the degenerate case \( \gamma = 0 \) where this property is still true in a restricted sense − have the following property:

Among all segmentations associated to a fixed partition of the interval, the functional assumes a unique minimum. The minimizer for a fixed partition can be computed by separately minimizing independent functionals associated to the intervals of the given partition.

Hence − by the first property − the problem of minimizing the whole functional can be reduced to the problem of minimizing a reduced functional which is a function of partitions rather than segmentations. Then, simple a priori bounds on the number of jumps can be used to show that this partition
space is essentially compact. That, in particular, provides the existence of global minimizers. Considerations like that are the subject of this paper.

The second property is more important from the algorithmic point of view. For discrete time-series and the Potts model this property was used to establish an efficient algorithm to compute the minimizers, see e.g. [23] or the PhD-Thesis [9]. In the very recent PhD-Thesis [5], this property is used to construct efficient algorithms in 2D when the admissible partitions are restricted to certain subclasses (cf. the formulation of the reduction principle in [5], Definition 1.2.1).

3.1 The Reduction Principle for Mumford - Shah

We start with the statement of the reduction principle for Mumford - Shah. First of all, we give the exact definition of the functional already discussed above extending it to a functional on $L^2(U)$. Recall that functions of special bounded variation provide a segmentation class as explained in Section 2.4.

**Definition 10** Let $g \in L^2(U)$, $\gamma, \mu \geq 0$. The Mumford - Shah functional $MS_{\gamma, \mu, g} : L^2(U) \to \mathbb{R}$ to signal $g$ and parameters $\gamma, \mu$ is given by

$$MS_{\gamma, \mu, g}(f) := \begin{cases} \gamma j(f) + \frac{1}{\mu^2} \int_U |f'|^2 ds + \|f - g\|^2 & f \in SBV_2(U) \\ \infty & \text{else} \end{cases}$$

(1)

where $j(f) := |p(f)| - 2$ is the number of jumps.

As proper segmentations of the signal $g$, we consider minimizers of the functional, i.e.

$$f^*_{\gamma, \mu, \infty, g} := \arg\min_{f \in L^2(U)} MS_{\gamma, \mu, g}(f).$$

The minimizer is not necessarily unique. The starting point for the reduction principle is the fact that we may split up the minimization procedure by the following observation: Let

$$f^*_{\mu, \infty, g}(p) = \arg\min_{\{f : p(f) \subseteq p\}} \left[ \frac{1}{\mu^2} \int_U |f'|^2 ds + \|f - g\|^2 \right].$$

(2)

The key point, however, is that this minimizer for a fixed partition exists, is unique and can be computed explicitly due to a decoupling of the minimization procedure for the different intervals in the complement of the partition. Then, the global minimizer is given by

$$f^*_{\gamma, \mu, \infty, g} = \arg\min_{f \in \{f^*_{\mu, \infty, g}(p) : p \in \mathcal{P}(U)\}} MS_{\gamma, \mu, g}(f).$$

(3)
That means, the minimization of the functional can be reduced to the minimization on a much smaller subspace of the space of all segmentations. This subspace is an image of the space of partitions under the (in general not injective) map \( p \mapsto f_{\mu,\infty,g}(p) \).

First of all, we compute the unique minimizer in equation (2).

**Proposition 2** Let \( \gamma, \mu \geq 0, p \in \mathcal{P}(U) \) be fixed. Then the unique minimizer

\[
f_{\mu,\infty,g}^*(p) := \arg\min_{f : p(f) \subset p} \left[ \frac{1}{\mu^2} \int_U |f'|^2 ds + \|f - g\|^2 \right]
\]

can be constructed as follows: Let \( \Theta \in \iota(p) \). Denote by \( \Delta_{\Theta} \) the Laplacian on \( \Theta \) with Neumann boundary conditions. Then the function

\[
f_{\Theta} := -\mu^2 R(\Delta_{\Theta}, \mu^2)g
\]

where \( R(\Delta_{\Theta}, \mu^2) \) denotes the resolvent of the Laplacian, is continuous on \( \Theta \). Denote by \( \Theta^+ \), the closed interval \( \Theta \) with right boundary point removed. Then

\[
f_{\mu,\infty,g}^*(p) = \sum_{\Theta \in \iota(p)} f_{\Theta} \chi_{\Theta^+}.
\]

**Remark.** (i) Note that \( p(f_{\mu,\infty,g}^*(p')) \subset p' \) but that \( p(f_{\mu,\infty,g}^*(p')) \) can be strictly smaller. That explains why the map \( p \mapsto f_{\mu,\infty,g}^*(p) \) is not injective and was the reason to put \( \{f : p(f) \subset p\} \) in equation (2). (ii) Having in mind the decomposition \( L^2(U) = \bigoplus_{\Theta \in \iota(p)} L^2(\Theta^+) \) we could write shortly

\[
f_{\mu,\infty,g}^*(p) = -\mu^2 \bigoplus_{\Theta \in \iota(p)} R(\Delta_{\Theta}, \mu^2)g
\]

where we identify \( R(\Delta_{\Theta}, \mu^2)g \in H^1(\Theta) \) with its image in \( C(\Theta^+) \) according to the Sobolev embedding theorem. The second property of the reduction principle is reflected by the fact that the operator assigning to the signal \( g \) the minimizer \( f_{\mu,\infty,g}^*(p) \) is decomposable (see for instance [16], p. 284 ff.) with respect to the orthogonal sum decomposition \( L^2(U) = \bigoplus_{\Theta \in \iota(p)} L^2(\Theta^+) \).

(3) Note that the case \( \mu = 0 \) is included in the preceding considerations in the following sense: If \( \mu = 0 \), the penalization for SBV-functions with non-vanishing derivative tends to infinity, hence constant functions \( f_{\Theta} \) are favored in this case. According to this, the operator \( -\mu^2 R(\Delta_{\Theta}, \mu^2) \) tends to
Proof: Minimizing the Mumford - Shah functional for a fixed partition means minimization of the expression

$$\sum_{\Theta \in \iota(p)} \int_\Theta dx \left( \mu^{-2} |f'_\Theta|^2 + |f_\Theta - g|^2 \right)$$

where we first consider $f_{\Theta, \Theta \in \iota(p)}$ to be a tuple of functions $f_{\Theta} \in C^\infty(\Theta)$. Let $h_{\Theta, \Theta \in \iota(p)}$ be another such tuple and $\tau_{\Theta, \Theta \in \iota(p)} \in \mathbb{R}^{\iota(p)}$. The minimization condition reads

$$0 = \frac{\partial}{\partial \tau_\Theta} \int_\Theta dx \left( \mu^{-2} |f'_\Theta + \tau_\Theta h'_\Theta|^2 + |f_\Theta + \tau_\Theta h_\Theta - g|^2 \right) \bigg|_{\tau_\Theta = 0}$$

$$= 2 \int_\Theta dx \left( \mu^{-2} f'_\Theta h'_\Theta + f_\Theta - g \right)$$

$$= 2 \left( -\mu^{-2} \langle f'_\Theta, h'_\Theta \rangle + \langle f_\Theta, h_\Theta \rangle - \langle g, h_\Theta \rangle \right).$$

Hence, adopting the notation from [19], Section 5.7, p. 345 ff, we may consider the extended map $-\mu^2 L_N + 1$ from $H^1(\Theta)$ to the dual space $H^1(\Theta)^*$ defined by the relation

$$\langle (-\mu^2 L_N + 1)f_{\Theta}, h_{\Theta} \rangle := \mu^{-2} \langle f'_\Theta, h'_\Theta \rangle + \langle f_\Theta, h_\Theta \rangle.$$ 

By [19], Proposition 7.2, p. 346, for $\mu > 0$, $g \in L^2(U)$, the equation is solved by a unique $f_{\Theta} \in H^2(\Theta)$ satisfying

$$-\mu^2 f''_{\Theta} + f_{\Theta} = g \quad \text{on} \; \Theta$$
$$f'_\Theta |_{\partial \Theta} = 0$$

where the second equation is understood in terms of the trace map (see [19], Proposition 1.6, p. 273. Thus, the Euler equation can be equivalently described by $(\Delta_{\Theta} - \mu^2)f_{\Theta} = -\mu^2 g$ where $\Delta_{\Theta}$ is the Neumann laplacian on $\Theta$. Hence

$$f_{\Theta} = -\mu^2 (\Delta_{\Theta} - \mu^2)^{-1} g = -\mu^2 R(\Delta_{\Theta}, \mu^2) g$$

where $R$ denotes the resolvent. By Sobolev’s embedding theorem,

$$f_{\Theta} \in \text{dom}(\Delta_{\Theta}) \subseteq H^1(\Theta) \subset C(\overline{\Theta}),$$
and $f^*$ provides indeed a right-continuous solution. In the case $\mu = 0$, the minimum can only be assumed by locally constant functions, i.e. $f'_\Theta = 0$. Hence, in that case the only non-trivial variations of $f_\Theta$ are given by constant functions $h_\Theta$ as well. Thus, the solution to the variational problem is given by the constant function assuming the mean value of $g$ on $\Theta$, i.e.

$$\langle f_\Theta, h_\Theta \rangle = f_\Theta h_\Theta(1,1) = \langle g, h_\Theta \rangle = h_\Theta \langle g, 1 \rangle$$

and therefore $f_\Theta = \langle g, 1 \rangle / \langle 1, 1 \rangle$. The minimizer depends thus even continuously from $\mu$, since the resolvent function considered above tends to the projection onto the kernel of the Dirichlet Laplacian as $\mu$ tends to zero. Since the kernel consists of constant functions, this coincides with the mean value.

Hence, the precise formulation of the reduction principle in the case of the Mumford - Shah functional is given by

**Corollary 2 (Reduction for Mumford-Shah)** Let $\gamma, \mu \geq 0$. The minimization of the Mumford - Shah functional is equivalent to the minimization of the reduced Mumford - Shah functional $\text{ms}_{\gamma, \mu, g} : \mathcal{P}(U) \to \mathbb{R}$ given by

$$\text{ms}_{\gamma, \mu, g}(p) := \text{MS}_{\gamma, \mu, g}(f^*_{\mu, \infty, g}(p)) = \gamma j(p) - \langle u, f^*_{\mu, \infty, g}(p) \rangle + \|g\|^2,$$

more precisely, $p$ is a minimizer of $\text{ms}_{\gamma, \mu, g}$ if and only if $f^*_{\mu, \infty, g}(p)$ is a minimizer of $\text{MS}_{\gamma, \mu, g}$.

### 3.2 The Reduction Principle for Blake - Zisserman

The reduction principle for the Blake - Zisserman functional is similar to the one for Mumford - Shah. We just have to identify the respective quantities in the discrete setting. Again, we start by giving our definition of the functional. Recall the notions of *equidistant setup* and *conditional expectation* from Definition 1 and the classes of considered step-functions from Definition 2.

**Definition 11** Let $n \in \mathbb{N}$, $g \in L^2(U)$ and $\gamma, \mu \geq 0$ be fixed. The Blake-Zisserman functional $\text{BZ}^n_{\gamma, \mu, g} : L^2(U) \to \mathbb{R}$ is given by

$$\text{BZ}^n_{\gamma, \mu, g}(f) := \begin{cases} \Phi^n_{\gamma, \mu}(f) + \frac{1}{n} \sum_{\kappa=0}^{n-1} (f - g_n)^2(\kappa) & f \in \mathcal{T}_n(U) \\ \infty & \text{else} \end{cases} (5)$$
where

\[ \Phi_n^{\gamma, \mu}(f) := \sum_{\kappa=0}^{n-2} \min\{ \frac{n}{\mu} (f^{\kappa+1} - f^{\kappa})^2, \gamma \} \]

and \( g_n = E(g \mid \sigma_n) \) denotes conditional expectation.

Again, we seek for minimizers. The starting point for the reduction principle in this case will as well consist of an observation concerning the minimization of the functional. Let

\[ f^{\ast*}_{\gamma, \mu, n, g} := \text{argmin}_{B\mathcal{Z}} B\mathcal{Z}_{\gamma, \mu, g}(f) \]

be a (not necessarily unique) minimizer. Then we have

**Lemma 3** The minimization of \( B\mathcal{Z}_{\gamma, \mu, g}^n \) is equivalent to the minimization of the functional \( H_{\gamma, \mu, g}^n : Z_2^{n-1} \times \mathcal{T}_n(U) \to \mathbb{R} \) given by

\[ H_{\gamma, \mu, g}^n(e, f) := \Psi_{\gamma, \mu}^n(e, f) + \frac{1}{n} \sum_{\kappa=0}^{n-1} \frac{n}{\mu^2} (f - g_n)^2(\zeta) \]

where \( Z_2 := \{0, 1\} \), \( e = (e_1, \ldots, e_{n-1}) \) and

\[ \Psi_{\gamma, \mu}^n(e, f) := \sum_{\kappa=0}^{n-2} \frac{n}{\mu^2} (f^{\kappa+1} - f^{\kappa})(1 - e_{\kappa+1}) + \gamma e_{\kappa+1}. \]

**Proof:** See [22], p. 36 f. \[\square\]

The points \( \frac{\kappa}{n} \) such that \( e_\kappa = 1 \) correspond to edges in the segmentation of \( u_n \) (see again the presentation in [22], p. 36 f). They correspond exactly to those points \( \frac{\kappa}{n} \), where the minimizer fulfills \( |f_{\gamma, \mu, n, g}^{\ast*}(\frac{\kappa+1}{n}) - f_{\gamma, \mu, n, g}^{\ast*}(\frac{\kappa}{n})| > \mu \sqrt{\gamma/n} \), i.e. they represent the discrete discontinuities of \( f_{\gamma, \mu, n, g}^{\ast*} \) with respect to the threshold \( T \equiv \mu \sqrt{\gamma/n} \) as defined in Definition 6. Furthermore, there is a bijection onto associated partitions

\[ p(e) := \{ \frac{\kappa}{n} : e_\kappa = 1 \} \cup \{0, 1\} \in \mathcal{P}_n(U) \]

and these form the discrete analogue of the partitions in the continuous case. Using the lemma above, the starting point for the reduction principle is provided as in the continuous case by the following observation: Let

\[ Q_n^\mu(e, f) := \sum_{\kappa=0}^{n-2} \frac{n}{\mu^2} (f^{\kappa+1} - f^\kappa)^2(1 - e_\kappa) + \frac{1}{n} \sum_{\kappa=0}^{n-1} (f^\kappa - g_n(\frac{\kappa}{n}))^2 \]
and
\[ f_{\mu,n,g}^* (p) = \arg\min_{f : p_{\sqrt{n}/n} (f) \subset p} Q_{\mu}^n (e, f). \] (6)
Again, the minimizer for a fixed partition exists, is unique and can be computed explicitly due to a decoupling of the minimization procedure for the different intervals in the complement of the partition. Then, the global minimizer is given by
\[ f_{\gamma,\mu,n,g}^* = \arg\min_{f \in \{ f_{\mu,n,g}^* (p) : p \in \mathcal{P}_n (U) \}} BZ_{\gamma,\mu,g}^n (f). \] (7)

First of all, we recall the decomposition of the conditional expectation map that was already introduced in Section 1.2.

**Definition 12** The map \( \pi_n : L^2 (U) \to \mathbb{R}^n \) is given by the row
\[ \pi_n g := (g_n (0), g_n (1/n), \ldots, g_n (n - 1/n)) \]
where \( g_n := E (g \mid \sigma_n) \) denotes conditional expectation. The map \( \tau_n : \mathbb{R}^n \to L^2 (U) \) is given by
\[ \tau_n f (s) := \sum_{\kappa = 0}^{n-2} f_\kappa \chi_{[\kappa \cdot n, \kappa + 1 \cdot n]} (s). \]

**Remark.** Clearly, \( g_n = \tau_n \circ \pi_n g \).

As in the continuous case, we start by constructing the unique minimizer associated to a fixed partition.

**Proposition 3** Let \( \gamma, \mu \geq 0 \) and \( p \in \mathcal{P}_n (U) \) be fixed. Then the minimizer
\[ f_{\gamma,\mu,n,g}^* (p) = \arg\min_{f : p_{\sqrt{n}/n} (f) \subset p} Q_{\mu}^n (e, f) \]
is unique and can be constructed as follows: Let \( p = \{ 0 = : \kappa_0 / n < \kappa_1 / n < \kappa_2 / n < \ldots < \kappa_l / n < 1 = : \kappa_{l+1} / n \} \in \mathcal{P}_n (U) \). Consider the block matrix
\[ A(p) := \begin{pmatrix} B(1) & 0 & \cdots & 0 \\ 0 & B(l + 1) \end{pmatrix} \]
where the blocks $B(r)$ are given by the $\kappa_r - \kappa_{r-1} \times \kappa_r - \kappa_{r-1}$-band matrices

$$B(r) := \begin{pmatrix}
-1 & 1 & 0 \\
1 & -2 & 1 \\
\vdots & \ddots & \ddots \\
1 & -2 & 1 \\
0 & 1 & -1
\end{pmatrix}.$$ 

Then

$$f_{\mu,n,g}(p) := -\mu^2 \tau_n \circ R(n^2 A(p), \mu^2) \circ \pi_n g^t$$

where $R(n^2 A(p), \mu^2)$ denotes the resolvent and $\pi_n g^t$ the transpose vector.

**Remark.** As in the continuous case, the block structure of $A(p)$ corresponds to a direct sum decomposition of underlying space and operator.

**Proof:** Analogous to the proof of Proposition 2, minimization of BZ for fixed partition $e$, writing shortly $f^\kappa = f(\frac{\kappa}{n})$, equivalent to

$$0 = \frac{\partial}{\partial f^\kappa} \sum_{\kappa=0}^{n-2} \mu^2 (f^{\kappa+1} - f^\kappa)^2 (1 - e_\kappa) + \gamma e_\kappa + \frac{1}{n} \sum_{\kappa=0}^{n-1} (f^\kappa - g_n(\frac{\kappa}{n}))^2$$

$$= \frac{\mu^2}{n} 2((f^{\kappa+1} - f^\kappa)(1 - e_\kappa) + (f^\kappa - f^{\kappa-1})(1 - e_{\kappa-1})) + \frac{2}{n} (f^\kappa - g_n(\frac{\kappa}{n}))$$

for all $\kappa = 0, ..., n-1$. This system of linear equations can be written as

$$(n^2 A(p) - \mu^2) f = -\mu^2 \pi_n g^t,$$

hence

$$f_{\mu,n,g} = \tau_n(f) = -\mu^2 \tau_n (n^2 A(p) - \mu^2)^{-1} \pi_n g^t = -\mu^2 \tau_n R(n^2 A(p), \mu^2) \pi_n g^t.$$ 

Hence, we obtain in analogy to Corollary 2.
Corollary 3 (Reduction for Blake-Zisserman) Let $\gamma, \mu \geq 0$. The minimization of the Blake - Zisserman functional is equivalent to the minimization of the reduced Blake - Zisserman functional $bZ^n_{\gamma,\mu,g} : \mathcal{P}(U) \to \mathbb{R}$ given by

$$bZ^n_{\gamma,\mu,g}(p) := \begin{cases} BZ^n_{\gamma,\mu,u}(f^*_{\mu,n,g}(p)) & p \in \mathcal{P}_n(U) \\ \infty & \text{else} \end{cases}.$$  

For $p \in \mathcal{P}_n(U)$, the reduced functional is given by

$$bZ^n_{\gamma,\mu,g}(p) = \gamma j(p) - \langle g_n, f^*_{\mu,n,g}(p) \rangle + \| g_n \|^2.$$

4 Continuity of the Reduced Functionals

So far, we have seen that the minimization on the function space can be reduced to a minimization on partition space. However, it is not yet clear whether the reduced functionals $mZ$ and $bZ$ depend continuously on the parameters and/or the partitions. In order to prove this, we first have to investigate the continuity properties of the minimizers for a fixed partition – from now on shortly denoted partition solvers – introduced in the preceding section.

4.1 The Partition Solvers

We consider continuity properties of the partition solvers. It turns out that they depend continuously on the parameters. Since the partition solvers for the Blake - Zisserman functional can not be applied to all partitions, we first have to define a proper domain.

Definition 13 Recall the definition of

$$T := \{1/n : n \in \mathbb{N} \} \cup \{0\} \subset \mathbb{R}$$  

from Theorem 1. Adopting the convention that $\mathcal{P}_{1/0}(U) = \mathcal{P}(U)$ let

$$E(T) := \{(t,p) \in T \times \mathcal{P}(U) : p \in \mathcal{P}_{1/t}(U)\} \subset \mathbb{R} \times \mathcal{P}(U)$$  

equipped with its relative topology, i.e. all accumulation points are of the form $(0,p), p \in \mathcal{P}(U)$.
With these conventions, the result about the parameter dependence of the partition solvers reads as follows.

**Theorem 2** For fixed \( g \in L^2(U) \), the partition solver map \( f^* : \mathbb{R}_0^+ \times E(T) \to L^2(U) \), given by
\[
f^*(\mu, t, p) := f_{\mu, 1/t, g}(p)
\]
is continuous.

**Proof:** The statement follows from the series of Lemmas proved below. By Lemma 5, \( f^* \) is continuous in \( E(T) \) for fixed \( \mu \). By Lemma 4, \( f^* \) is equicontinuous in \( \mu \) for all converging sequences of partitions. That implies joint continuity.

Before we come to the lemmas implying Theorem 2, we apply the theorem to prove lower semi-continuity of the reduced Mumford-Shah functional.

**Corollary 4** The reduced functional \( m_{s, \gamma, \mu, g} \) is lower semi-continuous.

**Proof:** By Corollary 1, the function \( j : \mathcal{P}(U) \to \mathbb{N} \) is lower semi-continuous. By Theorem 2, \( f^* \) depends for fixed \( \gamma, \mu, t = 0 \) continuously on \( p \in \mathcal{P}(U) \). By the explicit form of \( m_{s, \gamma, \mu, g} \) given in Corollary 2, that implies the statement.

The first lemma states that a family of partition solvers associated to a convergent sequence of partitions depends equicontinuously on the parameter \( \mu \geq 0 \).

**Lemma 4** (i) Let \( p \in \mathcal{P}(U) \) be fixed. Then the map \( G_\infty : \mathbb{R}_0^+ \to L^2(U) \) given by \( G_\infty(\mu) := f_{\mu, \infty, g}(p) \) is Lipschitz continuous. (ii) Let \( p \in \mathcal{P}_n(U) \) be fixed. Then the map \( G_n(\mu) := f_{\mu, \infty, g}(p) : \mathbb{R}_0^+ \to L^2(U) \) is Lipschitz continuous. (iii) Let \( p_{n, n} \in \mathcal{P}_n \) be a family of partitions such that \( p_n \in \mathcal{P}_n(U) \) converges to \( p \in \mathcal{P}(U) \) with respect to Hausdorff metric. Then the associated family \( G_{n, n=1,2,...,\infty} \) is uniformly Lipschitz.
Proof: (i) Recall the notations from Proposition 2. By definition, all partitions are finite. Thus it is enough to show Lipschitz continuity for one single interval, i.e. for a subinterval $\Theta := (x, y) \in \iota(p)$, the map

$$f_\Theta(\mu) := -\mu^2 R(\Delta_\Theta, \mu^2) g$$

is Lipschitz continuous. Since the Neumann laplacian is self-adjoint and has a discrete spectrum with semisimple non-positive eigenvalues and its kernel consists exactly of constant functions, we have the spectral decomposition

$$-\mu^2 R(\Delta_\Theta, \mu^2) g = E_0 g + \sum_{s \geq 1} \frac{\mu^2}{\mu^2 - \lambda_s(\Theta)} E_s g,$$

where $E_0 g$ denotes the mean value of $g$ in $\Omega$, i.e. the orthogonal projection of $g$ onto constant functions. Hence, by $0 > \lambda_1 > \lambda_2 ...$, we obtain

$$\|f_\Theta(\mu) - f_\Theta(\mu')\|^2 = \left\| \sum_{s \geq 1} \left[ \frac{\mu^2}{\mu^2 - \lambda_s(\Theta)} - \frac{\mu'^2}{\mu'^2 - \lambda_s(\Theta)} \right] E_s g \right\|^2 = \sum_{s \geq 1} \frac{\lambda_s^2(\Theta)(\mu'^2 - \mu^2)^2}{(\mu^2 - \mu'^2) + \lambda_s^2(\Theta))^2} \|E_s g\|^2 \leq \sum_{s \geq 1} \left( \frac{\mu'^2 - \mu^2}{\lambda_s^2(\Theta)} \right)^2 \|E_s g\|^2 \leq \frac{(\mu'^2 - \mu^2)^2}{\lambda_1^2(\Theta)} \|g\|^2.$$

Hence $f_\Theta$ is Lipschitz continuous with Lipschitz constant $C_\Theta := \|g\|/|\lambda_1(\Theta)|$ depending on the norm of $g$ and on the spectral gap $|\lambda_1(\Theta)|$ of the Neumann Laplacian on $\Theta$. That implies

$$\|G_\infty(\mu) - G_\infty(\mu')\| \leq C|\mu^2 - \mu'^2|$$

where $C := \max_{\Theta \in \iota(p)} C_\Theta < \infty$. (ii) Recall the notation from Proposition 3. By definition, all partitions are finite. Thus it is enough to show Lipschitz continuity for one single interval, i.e. for a subinterval $\Theta_r := [\kappa_{r-1}/n, \kappa_r/n) \subset U$ and the corresponding map

$$f_{\Theta_r}(\mu) := -\mu^2 \tau_n R(n^2 B(r), \mu^2) \pi_n g^t.$$
\( B(r) \) is a symmetric matrix and its kernel consists exactly of constant vectors. Thus, we have the spectral decomposition

\[
-\mu^2 R(n^2 B(r), \mu^2) \pi_n g^t = P_0 \pi_n g^t + \sum_{s=1}^{\kappa_{r+1}-\kappa_r-1} \frac{\mu^2}{\mu^2 - n^2 \lambda_s(B(r))} P_s \pi_n g^t
\]

where \( P_0 u \) denotes the orthogonal projection of \( u \) onto constant vectors. Hence we obtain using the fact that conditional expectation is a projection (which implies \( \|g_n\| \leq \|g\| \))

\[
\|f_{\Theta_n}(\mu) - f_{\Theta_n}(\mu')\|^2 = \sum_{s=1}^{\kappa_{r+1}-\kappa_r-1} \left[ \frac{\mu^2}{\mu^2 - n^2 \lambda_s(B(r))} - \frac{\mu^2}{\mu^2 - n^2 \lambda_s(B(r))} \right]^2 \|\pi_n P_s \pi_n g^t\|^2 \leq \frac{(\mu^2 - \mu'^2)^2}{n^4 \lambda_1^2(B(r))} \|g\|^2
\]

where \( |\lambda_1(B(r))| \) is the spectral gap of \( B(r) \). By the same argument as above, \( G_n \) is Lipschitz continuous. (iii) By the preceding parts of the proof and by Lemma 1, it is enough to show the following:

Let \( \Theta_n := [\kappa^{(n)}_r/n, \kappa^{(n)}_{r+1}/n] \) be a sequence of intervals tending to \( \Theta := [\kappa^-, \kappa^+] \) in Hausdorff metric. Then the spectral gap \( n^2 |\lambda_1(B^{(n)}(r))| \) of the associated block matrices is uniformly bounded below.

Note that the case \( \kappa^+ = \kappa^- \), i.e. \( \Theta = \{ \kappa^+ \} \), is included in the following considerations. First of all, by Lemma 1 Hausdorff convergence \( \Theta_n \to \Theta \) is equivalent to \( \lim \kappa^{(n)}_r/n = \kappa^- \), \( \lim \kappa^{(n)}_{r+1}/n = \kappa^+ \). That implies

\[
\frac{\kappa^{(n)}_{r+1} - \kappa^{(n)}_r}{n} = \kappa^+ - \kappa^- + o(n^{-1}).
\]

On the other hand, the eigenvalues of \( B^{(n)}(r) \) are well known to be

\[
\lambda_s := 2 \left( \cos \left( \frac{\pi(s-1)}{\kappa_r-\kappa_{r-1}} \right) - 1 \right)
\]

where \( s = 1, ..., \kappa_r - \kappa_{r-1} \) (see e.g. [10], Theorem 1.3). Thus, the largest non-zero eigenvalue is given by

\[
\lambda_1(B^{(n)}(r)) = 2 \left( \cos \left( \frac{\pi}{\kappa_r-\kappa_{r-1}} \right) - 1 \right)
\]

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and hence
\[
2n^2 \lambda_1(B(n)(r)) = 2n^2 \left( \cos \left( \frac{\pi}{n(n+\kappa-r+1)} \right) - 1 \right) = -\frac{\pi^2}{(\kappa^+ - \kappa^-)^2} + o(n^{-1}).
\]
That implies finally that for all \( \epsilon > 0 \) there is some \( n_0 \) with
\[
C_n \leq \frac{\|g\| L^2}{\pi^2} + \epsilon
\]
for all \( n > n_0 \) where \( L \) is the maximal length of an interval in \( \iota(p) \). Hence, the family \( G_n \) is uniformly Lipschitz.

The next result shows that the partition solver associated to the Mumford-Shah functional depends continuously on the given partition.

**Lemma 5** The function \( f^*_{\mu,\infty,g} : \mathcal{P}(U) \to \text{SBV}_2(U) \subset L^2(U) \) is continuous if \( \mathcal{P}(U) \) is equipped with the Hausdorff topology.

**Proof:** First of all, recall that there is a strongly continuous representation of the affine group \( \text{Aff}(\mathbb{R}) := \{ L_{a,b}(x) := ax + b : a > 0, b \in \mathbb{R} \} \) on \( L^2(\mathbb{R}) \) given by \( \rho(L_{a,b})f := f(ax+b) \). In particular, given a sequence \((a_s, b_s)\) tending to \((1,0)\) as \( s \) tends to infinity, we obtain
\[
\lim_{s \to \infty} \|\rho(L_{a_s,b_s})f - f\| = 0
\]
for all \( f \in L^2(\mathbb{R}) \). By Lemma 11, it is again sufficient to prove that for a sequence \( \Theta := (\kappa_-^{(n)}, \kappa_+^{(n)}) \) of intervals such that \( \Theta_n \to \Theta := [\kappa_-, \kappa_+] \) in Hausdorff distance, we have
\[
\lim_{n \to \infty} \|\mu^2 R(\Delta_\Theta, \mu^2)g - \mu^2 R(\Delta_\Theta, \mu^2)g\| = 0.
\]
(i) Let \( \kappa_+ \neq \kappa_- \) and therefore \( L_n(x) := w_n x + v_n : \mathbb{R} \to \mathbb{R} \) the unique linear map such that \( L_n(\kappa_-^{(n)}) = \kappa_- \) and \( L_n(\kappa_+^{(n)}) = \kappa_+ \). Now \( L_n \) maps the domains of the corresponding Neumann laplacians, i.e. we have \( g \circ L_n \in \mathcal{D}(\Delta_\Theta) \iff g \in \mathcal{D}(\Delta_\Theta) \), and furthermore
\[
\Delta_\Theta(g \circ L_n) = w_n^2 (\Delta_\Theta g) \circ L_n.
\]
In the spirit of the remark made above about the representation of the affine group, we write \( \rho_n(f) := f \circ L_n \). In order to avoid difficulties with the domains, we will write \( g_\Theta \) for the restriction \( g \chi_\Theta \) of a function \( g \) to the interval \( \Theta \) having in mind that due to the support of the resolvent kernel, we always have that

\[
R(\Delta_\Theta, \mu^2)g = R(\Delta_\Theta, \mu^2)g_\Theta
\]

is supported on \( \Theta \). Now

\[
R(\Delta_\Theta_n, \mu^2)(g_\Theta \circ L_n) = w_n^{-2} (R(\Delta_\Theta, w_n^{-2}, \mu^2)g_\Theta) \circ L_n
\]

and by \( \|g_\Theta \circ L_n\| = w_n^{-1/2}\|g_\Theta\| \) and \( \|\mu^2 R(\Delta_\Theta, \mu^2)\| = 1 \) for all \( \mu \geq 0 \), \( \Theta \in \iota(p) \) we therefore obtain letting \( g(n)_\Theta := g_\Theta \circ L_n^{-1} = \rho_n^{-1}(g_\Theta) \)

\[
\|\mu^2 R(\Delta_\Theta_n, \mu^2)g_\Theta - \mu^2 R(\Delta_\Theta_n, \mu^2)g_\Theta_n\| \\
= \|\mu^2 R(\Delta_\Theta, \mu^2)g_\Theta - (\mu^2 w_n^{-2} R(\Delta_\Theta, \mu^2 w_n^{-2}) g(n)_\Theta) \circ L_n\| \\
\leq \|\mu^2 R(\Delta_\Theta, \mu^2)g_\Theta - (\mu^2 R(\Delta_\Theta, \mu^2)g_\Theta \circ L_n)\| \\
+ \|\mu^2 R(\Delta_\Theta, \mu^2)(g_\Theta - g(n)_\Theta)) \circ L_n\| \\
+ \|((\mu^2 R(\Delta_\Theta, \mu^2) - \mu^2 w_n^{-2} R(\Delta_\ Theta, \mu^2 w_n^{-2})) g(n)_\Theta) \circ L_n\| \\
\leq \|\mu^2 R(\Delta_\Theta_n, \mu^2)g_\Theta - \rho_n(\mu^2 R(\Delta_\Theta, \mu^2)g_\Theta)\| \\
+ \|\rho_n (\mu^2 R(\Delta_\Theta, \mu^2)(g_\Theta - g(n)_\Theta))\| \\
+ \|\rho_n ((\mu^2 R(\Delta_\Theta, \mu^2) - \mu^2 w_n^{-2} R(\Delta_\Theta, \mu^2 w_n^{-2})) g(n)_\Theta)\| \\
\leq \|\mu^2 R(\Delta_\Theta_n, \mu^2)g_\Theta - \rho_n(\mu^2 R(\Delta_\Theta, \mu^2)g_\Theta)\| \\
+ \|\rho_n\| \|g_\Theta - g(n)_\Theta\| + 2\mu^2 \|\rho_n\| \|g(n)_\Theta\||1 - w_n^{-2}|
\]

using the resolvent identity for the final step. Note that

\[
\|g_\Theta - g(n)_\Theta\| = \|g_\Theta - \rho_n^{-1}(g_\Theta) + \rho_n^{-1}(g_\Theta) - \rho_n^{-1}(g_\Theta_n)\| \\
\leq \|g_\Theta - \rho_n^{-1}(g_\Theta)\| + \|\rho_n^{-1}(g_\Theta - g_\Theta_n)\| \\
\leq \|g_\Theta - \rho_n^{-1}(g_\Theta)\| + \|\rho_n^{-1}\| \|\chi_\Theta - \chi_\Theta_n\| g\|
\]

and that implies the statement by \( (w_n, v_n) \to (1, 0) \) and strong continuity of the representation. (ii) If, on the other hand, \( \kappa_- = \kappa_+ \), i.e. the sequence \( \Theta_n \) tends to a single point, we have, again by contractivity of \( \mu^2 R(\Delta_\Theta_n, \mu^2) \),

\[
\|\mu^2 R(\Delta_\Theta_n, \mu^2)g_\Theta_n\| = \|\mu^2 R(\Delta_\Theta_n, \mu^2)g_\Theta_n\| \leq \|g_\Theta_n\|
\]

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which tends to zero as \( n \) tends to infinity.

Since the sets \( P_n(U) \) are discrete, we do not have to prove a corresponding result for fixed \( n \neq \infty \). It remains to show convergence for partition solvers to partitions \( p_n \in P_n(U) \) that converge to a partition \( p \in P(U) \) which is not contained in \( P_n(U) \) for any \( n \).

**Lemma 6** Let \( p_n \in P_n(U) \) be a sequence of partitions converging to \( p \in P(U) \) with respect to Hausdorff metric. Then

\[
\lim_{n \to \infty} f^*_{\mu, n, g}(p_n) = f^*_{\mu, \infty, g}(p).
\]

**Proof:** Recall the notations from Corollary 2.4 respectively. By Lemma 1, the proof consists of considering the following two cases: (i) Let \( \Theta = (\kappa_-, \kappa_+ \in \iota(p) \) such that \( \Theta_n \to \Theta \). Then \( \Theta_n = (\kappa_{n+1}^{n+1}, \kappa_{n+1}^{n+1}) \) and we have to show that for all \( f \in D(\Delta_n) \), we have

\[
-\mu^2 \tau_n R(n^2 B(r_n), \mu^2) \pi_n g \to -\mu^2 R(\Delta, \mu^2)g
\]

as \( n \) tends to infinity. We observe that, by cancelling \(-\mu^2\), this can be done by proving strong resolvent convergence of the operator

\[
\Delta_n := \tau_n n^2 B(r_n) \circ \pi_n
\]

to the Neumann laplacian \( \Delta_\Theta \). That follows from

\[
\mu^{-2}(\Delta_n - \mu^2) = \mu^{-2}(\tau_n n^2 B(r_n) \pi_n - \mu^2 \tau_n \pi_n + \mu^2 (\tau_n \pi_n - 1))
\]

\[
= \tau_n \mu^{-2}(n^2 B(r_n) - \mu^2) \pi_n - (1 - \tau_n \pi_n)
\]

where \( \tau_n \pi_n \) is nothing else but conditional expectation with respect to \( \sigma_n \) on the subinterval \( [\frac{\kappa_n}{n}, \frac{\kappa_{n+1}}{n}] \). Hence for all \( f \in D(\Delta_n) \), we have by \( \pi_n(1 - \tau_n \pi_n) = 0 \)

\[
\tau_n R(n^2 B(r_n), \mu^2) \pi_n (\Delta_n - \mu^2) f
\]

\[
= \tau_n \pi_n f - \mu^2 \tau_n R(n^2 B(r_n), \mu^2) \pi_n (1 - \tau_n \pi_n) f
\]

\[
= \tau_n \pi_n f.
\]

Together with \( \tau_n \pi_n f \to f_\Theta \) this implies that the resolvents \( R(\Delta_n, \mu^2) \) tend to the same limit as the operators \( \tau_n R(n^2 B(r_n), \mu^2) \pi_n \) strongly as \( n \) tends to infinity.
It remains to prove strong resolvent convergence of $\Delta_n$ to $\Delta_\Theta$. By Corollary 1.6, p. 429, and since $\{f \in C^\infty(\overline{\Theta}) : f'|_{\partial \Theta} = 0\}$ forms a core for the Neumann Laplacian, we just have to prove that

$$\lim_{n \to \infty} \|\Delta_n f - \Delta_\Theta f\| = 0$$

for all $f$ in the core. Let thus

$$f(x) = f(x_0) + (x - x_0)f'(x_0) + \frac{1}{2} f''(x_0)(x - x_0)^2 + R(x)$$

be a Taylor expansion of $f$ with remainder $|R(x)| = O(|x|^3)$ uniformly for $x$ in a compact interval containing $\overline{\Theta}$. Then, controlling the error by Taylor expansion yields

$$\Delta_n f = \sum_{k=\kappa_n+1}^{\kappa_n+1-2} (f''(\frac{x}{n}) + O(1/n)) \chi_{[\frac{k}{n}, \frac{k+1}{n})} - (nf'(\kappa_-) + O(1)) \chi_{[\frac{\kappa_n}{n}, \frac{\kappa_n+1}{n})} + (nf'(\kappa_+) + O(1)) \chi_{[\frac{\kappa_n+1-1}{n}, \frac{\kappa_n+1}{n})}.$$ 

Hence, the boundary conditions $f'(\kappa_-) = f'(\kappa_+) = 0$ imply strong convergence of $R(\Delta_n, \mu^2)$ to $R(\Delta_\Theta, \mu^2)$ uniformly on compact sets $\mu \in [a, b] \subset \mathbb{R}^+$, $b > a > 0$, contained in the resolvent set. By Lipschitz continuity established in Lemma 4 this extends to uniform convergence of $-\mu^2 R(\Delta_n, \mu^2) g$ to $-\mu^2 R(\Delta_\Theta, \mu^2) g$ for $\mu \in [0, b]$. (2) For a sequence of intervals collapsing to a point, uniform contractivity $\|\mu^2 R(\Delta_n, \mu^2)\| \leq 1$ implies convergence of the corresponding partition solvers to $0 \in L^2(U)$ as in the proof of the preceding lemma.

5 Compactness of the Set of Minimizers

By the reduction principle, the set of partition solvers to a fixed function $g \in L^2(U)$ contains the set of minimizers of $F$. Now we will prove, that the set of partition solvers is compact, which implies the same for the set of minimizers. Let $g \in L^2(U)$, $b > 0$ and

$$\mathcal{M}_b(g) := \{f_{\mu,n,g}^* : p \in \mathcal{P}_n(U), \mu \in [0, b], n \in \mathbb{N} \cup \{\infty\}\}.$$ 

In this section, will prove
Theorem 3 For all $b > 0$ and $g \in L^2(U)$, the set $\mathcal{M}_b(g) \subset L^2(U)$ is compact.

This result implies the existence of global minimizers.

Theorem 4 (Existence of Minimizers) Let $\gamma, \mu \geq 0$ and $u \in L^2(U)$. Then:

(i) The set of global minimizers $f^*$ of $\text{MS}_{\gamma, \mu, g}$ is non-empty.

(ii) The set of global minimizers $f_n^*$ of $\text{BZ}_{\gamma, \mu, g}^n$ is non-empty.

Proof: (i) Recall that the reduced Mumford - Shah functional is given by

$$\text{ms}_{\gamma, \mu, g}(p) = \gamma j(p) + \|g\|^2 - \langle g, f^*_{\mu, \infty}, g(p) \rangle.$$ 

By Theorem 3 minimizing $\text{MS}_{\gamma, \mu, g}$ on $L^2(U)$ is hence equivalent to the minimization of

$$\gamma j(p(f)) + \|g\|^2 - \langle f, g \rangle$$

for functions $f$ in the compact set $\mathcal{M}_\mu(g)$. For $\gamma = 0$, this function is continuous and hence assumes its minimum on the compact set. For $\gamma > 0$, the function is lower semi - continuous by Corollary 1 and assumes hence as well its minimum on a compact set. (ii) The set $\mathcal{P}_n(U)$ is finite, hence the minimization can be reduced to the minimization with respect to a finite number of partition solvers which implies that the functional assumes its minimum.

Now we start with the proof of Theorem 3.

5.1 A Preliminary Lemma

First of all, we will prove a lemma that will simplify the discussion considerably. It states that a proof of compactness for all $g \in L^2(U)$ can be reduced to a proof for a total subset of signals in $L^2(U)$. Denote therefore by

$$\mathcal{M}_b := \{g \in L^2(U) : \mathcal{M}_b(g) \subset L^2(U) \text{ compact}\}$$

the set of all signals, such that $\mathcal{M}_b(g)$ is compact in $L^2(U)$. Then we have the following statement:
Lemma 7 For all $b > 0$, the set $\mathcal{M}_b \subset L^2(U)$ is a closed linear subspace.

Proof: (1) Let $g_1, g_2 \in \mathcal{M}_b$, $\lambda_1, \lambda_2 \in \mathbb{R}$. Then, by linearity of the partition solver

$$f_{\mu, \lambda_1 g_1 + \lambda_2 g_2}(p) = \lambda_1 f_{\mu, g_1}(p) + \lambda_2 f_{\mu, g_2}(p),$$

and hence $\mathcal{M}_b(\lambda_1 g_1 + \lambda_2 g_2) \subset \lambda_1 \mathcal{M}_b(g_1) + \lambda_2 \mathcal{M}_b(g_2)$. The right hand side is compact by continuity of the linear operations on $L^2(U)$. (2) Let $g_k \in \mathcal{M}_b$ be a sequence of signals converging to $g \in L^2(U)$. The mappings $g \mapsto f_{\mu, n, g}(p)$ are contractions. Hence

$$\|f_{\mu, n, g_k}(p) - f_{\mu, n, g}(p)\| \leq \|g_k - g\|,$$

i.e. the sets $\mathcal{M}_b(g_k)$ converge to $\mathcal{M}_b(g)$ as closed subsets of $L^2(U)$ with respect to Hausdorff metric. That implies that $\mathcal{M}_b(g)$ is compact, and therefore $\mathcal{M}_b$ is closed.

The preceding lemma implies immediately, that if $\mathcal{M}_b$ contains a total set, i.e. a set of functions such that its closed linear hull equals $L^2(U)$, it already equals the whole space. Such a set is provided by the Heaviside functions on $U$.

5.2 Compactness

Denote by

$$\mathcal{H}(U) := \{\chi_a := \chi_{[a, 1]} : 0 \leq a < 1\} \subset L^2(U)$$

the set of Heaviside functions on the interval $U$. The linear hull of $\mathcal{H}(U)$ is provided by the step functions which are dense in $L^2(U)$, hence $\mathcal{H}(U)$ forms a total set. The key result for compactness reads

Proposition 4 For all $\chi_a \in \mathcal{H}(U)$ we have

$$\mathcal{M}(\chi_a) \subset \{f \in \mathcal{M}_b(\chi_a) : |p(f)| \leq 6\},$$

and $\mathcal{M}_b(\chi_a)$ is compact.

Proof: Let $p = \{0 < x_1 < ... < x_r < 1\}$ be an arbitrary partition such that $a \in [x_k, x_{k+1})$. Then, $\chi_a$ and $\chi_{a,n} := E(\chi_a|\sigma_n)$ are constant on $[0, x_k)$ (equal
to zero) and $[x_{k+1}, 1]$ (equal to one). Since the kernels of $\Delta_\Theta, \Delta_n$ are provided by constant functions, vectors, respectively, the respective resolvents map the conditional expectation $\chi_{a,n}$ onto functions which are constant (and equal to zero) on $[0, x_{n-1})$ and constant (equal to one) on $[x_{k+2}, 1]$. That implies, they have at most six jumps points, contained in the set $\{x_{k-1}, x_k, x_{k+1}, x_{k+2}\} \cup \{0, 1\}$. By Lemma 1 (2) and Corollary 1, the set $\{p \in P(U) : |p| \leq 6\}$ is compact as a closed subset of a compact set and hence the closed subset

$$C := E(T) \cap T \times \{p \in P(U) : |p| \leq 6\}$$

is as well compact. That implies finally by Theorem 2 compactness of

$$\mathcal{M}_b(\chi_a) \subset f^*([0, b] \times C).$$

\[\blacksquare\]

**Remark.** Recall that by Definition 6 the notion of jump differs from the notion of discontinuity in the discrete setting.

By Lemma 7 that completes the proof of Theorem 3.

### 6 Γ-Convergence and Weak Derivatives

The notion of Γ-convergence is important for the investigation of minimizers. It provides some rather general sufficient condition for the possibility to approximate minimizers of a given functional by minimizers of a sequence of approximating functionals.

**Definition 14** Let $X$ be a metric space and $F_j : X \to \mathbb{R}$ be a sequence of functions. Then, $F_j$ converges to $F$ in the sense of Γ-convergence – in the sequel we will write shortly $F_j \rightharpoonup F_\infty$ – if

(i) For all $x \in X$ and all sequences $x_j \overset{X}{\to} x$ we have

$$F_\infty(x) \leq \liminf_{j \to \infty} F_j(x_j). \quad (8)$$
(ii) For all \(x \in X\) there is a sequence \(\hat{x}_j \xrightarrow{X} x\) such that

\[
F_\infty(x) \geq \limsup_{j \to \infty} F_j(\hat{x}_j).
\]  

(9)

Essentially, \(\Gamma\)-convergence is important due to the following facts (cf. Theorem 5.3.6 of [1])

**Theorem 5** Suppose \(F_j \xrightarrow{\Gamma} F_\infty\) and denote by \(\text{argmin} F\) the set of minimizers of \(F\). Then

(i) For any converging sequence \(x_j, j \in \mathbb{N}\), \(x_j \in \text{argmin} F_j\), we have necessarily

\[
\lim_{j \to \infty} x_j \in \text{argmin} F_\infty.
\]

(ii) If there is a compact subset \(K \subset X\) such that \(\emptyset \neq \text{argmin} F_j \subset K\) for large enough \(j\), then \(\text{argmin} F_\infty \neq \emptyset\) and

\[
\lim_{j \to \infty} d(x_j, \text{argmin} F_\infty) = 0
\]

for any sequence \(x_j, j \in \mathbb{N}\), \(x_j \in \text{argmin} F_j\).

(iii) If, additionally, \(\text{argmin} F_\infty\) is a singleton \(\{x_\infty\}\) then

\[
\lim_{j \to \infty} x_j = x_\infty
\]

for any sequence \(x_j, j \in \mathbb{N}\) with \(x_j \in \text{argmin} F_j\).

In the sequel, we will prove that the Blake-Zisserman functionals \(BZ^n\) converge to the associated Mumford-Shah functionals \(MS\) in \(\Gamma\)-sense as \(n\) tends to infinity. By the compactness results established in the preceding section, this implies convergence of the associated minimizers.

The crucial step will be the understanding of the behavior of the Blake-Zisserman penalty \(\Phi^n_{\gamma, \mu}\) (see Definition [1]) as \(n\) tends to infinity. For reasons that will become clear in the sequel, we prove a parameter dependent result.

**Proposition 5** Let \(\gamma_n, \mu_n\) sequences of non-negative numbers converging to \(\gamma, \mu \geq 0\). Then the following two statements are valid which imply \(\Gamma\)-convergence of \(\Phi^n_{\gamma_n, \mu_n, g}\) to the Mumford-Shah penalty to parameters \(\gamma, \mu\):
(i) Let \( f = t + F \in \text{SBV}_2(U) \). Then there is a sequence \( \tilde{f}_{n,n} \in \mathbb{N} \) converging to \( f \) in \( L^2(U) \) such that
\[
\limsup_{n \to \infty} \sum_{\kappa=0}^{n-2} \min \left\{ \frac{\mu}{\kappa} (\tilde{f}_{n}(\frac{\kappa+1}{n}) - \tilde{f}_{n}(\frac{\kappa}{n}))^2, \gamma_n \right\} \leq \gamma_j(f) + \mu^{-2} \int_0^1 |f'(x)|^2 dx.
\]

(ii) Let \( f \in L^2(U) \). For all sequences \( f_{n,n} \in \mathbb{N} \) converging to \( f \) in \( L^2(U) \) we have
\[
\liminf_{n \to \infty} \sum_{\kappa=0}^{n-2} \min \left\{ \frac{\mu}{\kappa} (f_{n}(\frac{\kappa+1}{n}) - f_{n}(\frac{\kappa}{n}))^2, \gamma_n \right\} \geq \begin{cases} 
\gamma_j(t) + \mu^{-2} \int_0^1 |f'(x)|^2 dx & f \in \text{SBV}_2(U) \\
\infty & \text{else}
\end{cases}
\]
Note that in the case \( \mu = 0 \), the right hand side of both inequalities is only finite for \( f \in \mathcal{T}(U) \subset \text{SBV}_2(U) \), since \( f' \equiv 0 \) exactly for those functions.

To prove Proposition 5 we have to collect several facts about the relation of weak differentiation and approximation by step functions. This will be done in the following two subsection. The final proof of Proposition 5 is given in 6.3.

As an introductory step, we discuss weak differentiability in \( L^2(U) \).

### 6.1 A Characterization of Weak Differentiability

A function \( f \) is Sobolev - differentiable on the one-sphere, i.e. \( f \in H^1(S^1) \), if and only if \( f \in L^2(S^1) \) and its Fourier coefficients \( \hat{f}(k) \), \( k \in \mathbb{Z} \) with respect to the orthonormal base \( \phi_k(t) := e^{2\pi i kt} \) satisfy
\[
\sum_{k \in \mathbb{Z}} k^2 |\hat{f}(k)|^2 < \infty
\]
(see for instance [18], Section 3). Although \( L^2(U) \) and \( L^2(S^1) \) can be identified by considering \( S^1 \) as the identification \( U/\{0,1\} \), the situation is different for the Sobolev spaces \( H^1(\Omega) \) and \( H^1(S^1) \) (\( \Omega = (0,1) \)), cf. Definition 2.4. By Sobolev’s embedding theorem, \( H^1(\Omega) \subset C(U) \), and \( f \in H^1(\Omega) \) can be identified with some \( \tilde{f} \in H^1(S^1) \) if and only if \( f(0) = f(1) \) for the continuous
representative of \( f \). Hence, \( H^1(S^1) \subset H^1(\Omega) \) is a linear subspace of codimension one. In order to prove \( \Gamma \)-convergence for the Blake-Zisserman functional, the first step will be to find a suitable analogue of the characterization \([10]\) for \( H^1(\Omega) \).

**Lemma 8** Equivalent are

(i) \( f \in H^1(\Omega) \),

(ii) there is some \( \alpha \in \mathbb{R} \) such that \( f - g_\alpha \in H^1(S^1) \) where \( g_\alpha(t) = \alpha t \),

(iii) there is some \( \alpha \in \mathbb{R} \) such that

\[
\sum_{k \in \mathbb{Z}} \left| 2\pi ik \hat{f}(k) - \alpha \right|^2 < \infty.
\]

where \( \hat{f}(k) \) denote the Fourier coefficients of \( f \).

In case \( f \in H^1(\Omega) \) we have for \( \alpha \in \mathbb{R} \) as above

\[
\sum_{k \in \mathbb{Z}} \left| 2\pi ik \hat{f}(k) - \alpha \right|^2 = \int_0^1 \|(f - g_\alpha)'\|^2 dt = \int_0^1 |f'|^2 dt - (f(1) - f(0))^2,
\]

hence \( \alpha = f(1) - f(0) \).

**Proof:** (1) First, we show that (i) implies (ii). Let \( f \in H^1(\Omega) \). By Sobolev’s embedding theorem, there is a representative \( f_0 \) of \( f \) such that \( f_0 \in C(U) \). Let \( \phi \in C^\infty(U) \) with \( d^k\phi/dt^k(0) = d^k\phi/dt^k(1) \) for all \( k = 0, 1, ... \) and \( \alpha := f(1) - f(0) \). Then

\[
\int_0^1 (f' - \alpha) \phi dt = \phi(0)(f - g_\alpha)|_0^1 - \int_0^1 (f - g_\alpha) \phi' dt
\]

and hence \( f' - \alpha \) is the weak derivative of \( f - g_\alpha \) in \( H^1(S^1) \). Since \( U \) is compact, square-integrability of \( f' \) implies square-integrability of \( f' - \alpha \). (2) To see that (ii) implies (i), let \( f - g_\alpha \in H^1(S^1) \) and \( \phi \in C^\infty_0(\Omega) \subset C^\infty(S^1) \). Then, since \( g_\alpha \in H^1(\Omega) \),

\[
\int_0^1 f \phi' dt = \int_0^1 (f - g_\alpha + g_\alpha) \phi' dt = -\int_0^1 ((f - g_\alpha)' + \alpha) \phi dt.
\]
Hence, $f$ is weakly differentiable on $\Omega$ with square integrable weak derivative $f' = (f - g_\alpha)' + \alpha$. (3) The equivalence of (ii) and (iii) follows from

$$\hat{f}'(k) = \int_0^1 dt e^{-2\pi i k t} f'(t) = f(1) - f(0) + 2\pi i k \hat{f}(k)$$

and the equivalence of (i) and (ii).

**Lemma 9** Let a family of mollifiers be given by

$$h_a(x) := \begin{cases} 
\frac{1}{a} x & 0 < x \leq a \\
1 & a < x \leq 1 - a \\
\frac{1}{a} (1 - x) & 1 - a \leq x < 1
\end{cases}$$

where $0 < a < 1/2$. Then, if $f \in L^2(U)$, we have $fh_a \in L^2(U)$ for all $a > 0$ with

$$\|fh_a\| \leq \|f\|$$

and $fh_a$ converges to $f$ in $L^2(U)$ as $a$ tends to zero.

**Proof:** Clearly $0 \leq h_a \leq 1$ and $h_a \to 1$ in $L^2(U)$ as $a$ tends to zero. Hence

$$\|fh_a\|^2 = \int_0^1 |f|^2 h_a^2 dt \leq \int_0^1 |f|^2 dt = \|f\|^2$$

and

$$\|f(h_a - 1)\|^2 \leq \int_0^a |f|^2 dt + \int_{1-a}^1 |f|^2 dt$$

which tends to zero as $a$ tends to zero. 

These simple observations are the starting point for the constructions in the next sections.

**6.2 Approximation and Weak Differentiability**

Given $f^{(n)} \in T_n(U)$, we consider

$$F^{(n)}(x) := n(f^{(n)}(x + 1/n) - f^{(n)}(x))$$
where we understand \( f^{(n)} \) to be continued periodically to the real axis. Then
\[
\int_0^{1 - \frac{1}{n}} F^{(n)}(s)^2 ds = \sum_{\kappa=0}^{n-2} n(f^{(n)}(\frac{\kappa+1}{n}) - f^{(n)}(\frac{\kappa}{n}))^2.
\]

Now we rewrite the integral in terms of the Fourier coefficients of \( F^{(n)} \) with respect to the orthonormal base
\[
\phi_k^{(n)}(t) := \left\{ \begin{array}{ll}
\sqrt{\frac{n}{n-1}} e^{\frac{2\pi i k t}{n-1}}, & 0 < t \leq 1 - \frac{1}{n} \\
0, & 1 - \frac{1}{n} < t < 1
\end{array} \right.
\]
on \( L^2([0,1 - \frac{1}{n}]) \). As \( n \) tends to infinity, the individual base vectors tend to \( \phi_k(t) := e^{2\pi i k t} \) and even the Fourier coefficients converge as we will show in the next lemma.

**Lemma 10** Let \( f_{n}^{(n)} \in \mathbb{N} \) be a sequence of step functions with \( f^{(n)} \in T_n(U) \) converging to \( f \) in \( L^2(U) \), and \( F^{(n)} \) as above. For fixed \( k \in \mathbb{Z}, a \in (0,1/2) \) we have
\[
\lim_{n \to \infty} \langle F^{(n)} h_a, \phi_k^{(n)} - \phi_k \rangle = 0.
\]

**Proof:** (1) First of all, we show that
\[
\lim_{n \to \infty} \langle F^{(n)} h_a, \phi_k^{(n)} - \tilde{\phi}_k^{(n)} \rangle = 0
\]
where
\[
\tilde{\phi}_k^{(n)}(t) := \sqrt{\frac{n}{n-1}} e^{\frac{2\pi i k t}{n-1}}.
\]
Assume without loss of generality that \( n \) is so large that \( 1/n < a \). Then by Cauchy - Schwarz inequality
\[
|\langle F^{(n)} h_a, \phi_k^{(n)} - \tilde{\phi}_k^{(n)} \rangle| = |\langle F^{(n)} h_a \chi_{[1-\frac{1}{n},1)} \tilde{\phi}_k^{(n)} \rangle| \leq \|F^{(n)}\| \|h_a \chi_{[1-\frac{1}{n},1)} \tilde{\phi}_k^{(n)}\| \leq 2n \|f^{(n)}\| \sqrt{\frac{n \int_{1-\frac{1}{n}}^1 dt (1-t)^2}{a^2(n-1)}} \leq \frac{2 \|f^{(n)}\|}{\sqrt{3(n-1)}}.
\]
This expression tends to zero since, by assumption, \( \|f^{(n)}\| \to \|f\| < \infty \). (2)

Now we have by substitution of \( u := t + 1/n \), using periodicity and \(|h_a| \leq 1\)

\[
|\langle F^{(n)}h_a, \tilde{\phi}_k^{(n)} - \phi_k \rangle |
\leq n \int_0^1 dt |(f^{(n)}(t + 1/n) - f^{(n)}(t))h_a(\tilde{\phi}_k^{(n)} - \phi_k)(t)|
\]

\[
\leq n \int_0^1 du |f^{(n)}(u)(\tilde{\phi}_k^{(n)}(u - 1/n) - \tilde{\phi}_k^{(n)}(u) + \phi_k(u) - \phi_k(u - 1/n))|.
\]

But now

\[
|\tilde{\phi}_k^{(n)}(u - 1/n) - \tilde{\phi}_k^{(n)}(u) + \phi_k(u) - \phi_k(u - 1/n)|^2
= \left| \sqrt{\frac{n}{n-1}}(1 - e^{\frac{2\pi itk}{n}})(1 - e^{-\frac{2\pi itk}{n}})e^{2\pi itk} \right|^2
= \frac{4n}{n-1} \left( 1 - \cos \frac{2\pi kt}{n} \right) \left( 1 - \cos \frac{2\pi ik}{n} \right) = O\left( \frac{1}{n^4} \right).
\]

Hence

\[
|\langle F^{(n)}h_a, \tilde{\phi}_k^{(n)} - \phi_k \rangle | = O\left( \frac{1}{n} \right)
\]
tends to zero as \( n \) tends to infinity. \( \blacksquare \)

**Lemma 11** Let \( f^{(n)} \in \mathbb{N} \) be a sequence of step functions with \( f^{(n)} \in \mathcal{T}_n(U) \) converging to \( f \) in \( L^2(U) \), and \( F^{(n)} \) as above. Let \( \tilde{f}(k) \) be the Fourier expansion of \( f \) with respect to the orthonormal base \( \phi_k(t) := e^{2\pi i kt} \) and \( h_a \) as above. Let

\[
\alpha_k(a, n) := \frac{1}{a} \left( \int_1^a f^{(n)}(t)\phi_k(t)dt - \int_0^a f^{(n)}(t)\phi_k(t)dt \right)
\]

and \( \alpha_k(a) := \lim_{n \to \infty} \alpha_k(a, n) \). Then

\[
\lim_{n \to \infty} \widehat{F^{(n)}h_a}(k) = 2\pi ik\hat{f}h_a(k) + \alpha_k(a).
\]

**Proof:** The function \( h_a\phi_k \) is everywhere left-differentiable with left differential

\[
(h_a\phi_k)'_-(t) := \lim_{s \to 0} \frac{h_a\phi_k(t) - h_a\phi_k(t - s)}{s}.
\]

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Hence, by substituting $t + 1/n = u$, we have

$$\hat{F}(n)h_a(k) = n\int_0^1 (f^{(n)}(t + 1/n) - f^{(n)}(t))h_a\phi_k(t)dt$$

$$= -n\int_0^1 f^{(n)}(u)(h_a\phi_k(u) - h_a\phi_k(u - 1/n))du.$$  

The function $h_a\phi_k$ is uniformly Lipschitz with Lipschitz constant $C_{a,k}$. Hence $|n(h_a\phi_k(u) - h_a\phi_k(u - 1/n))| \leq C_{a,k}$ and thus by dominated convergence

$$\lim_{n \to \infty} \hat{F}(n)h_a(k)$$

$$= -\lim_{n \to \infty} \int_0^1 f^{(n)}(h_a\phi_k)' du$$

$$= -\lim_{n \to \infty} \int_0^1 f^{(n)}(2\pi i h_a - h_a')\phi_k du$$

$$= -2\pi i \lim_{n \to \infty} \int_0^1 f^{(n)}h_a\phi_k du + \lim_{n \to \infty} \int_0^1 f^{(n)}\frac{1}{a}(\chi_{[0,a]} - \chi_{[1-a,1]})\phi_k du.$$  

Convergence $f^{(n)} \to f$ finally implies the statement.

\[\blacksquare\]

**Lemma 12** Let $\alpha_k(a)$ be as above. Let $a_m \to 0$ be a sequence of positive numbers such that $\lim_{m \to \infty} \alpha_k(a_m) = \alpha_k \neq \pm \infty$ exists for some $k \in \mathbb{Z}$. Then this limit exists for all $k \in \mathbb{Z}$ and equals $\alpha_0$.

**Proof:** $f \in L^2(U)$ implies $f \in L^1(U)$. On the other hand, there is a constant $C_k > 0$ such that $|1 - \phi_k(t)| \leq C_k \min \{|t|, |1 - t|\}$ for all $t \in [0, a] \cup [1 - a, 1]$. Using (13), the statement follows now from

$$|\alpha_0(a) - \alpha_k(a)| \leq \frac{1}{a} \left[ \int_0^a |f(1 - \phi_k)|dt + \int_{1-a}^1 |f(1 - \phi_k)|dt \right]$$

$$\leq C_k \left[ \int_0^a |f|dt + \int_{1-a}^1 |f|dt \right]$$

which tends to zero as $a \to 0$.  

\[\blacksquare\]

From this considerations, we obtain, having in mind inequality (8):
Lemma 13 Let $f^{(n)} \in T_n(U)$ a sequence of step functions converging to $f$ in $L^2(U)$, $F^{(n)}$ as above. Then we have

$$\liminf_{n \to \infty} \sum_{k=0}^{n-2} n(f^{(n)}(\frac{n+k}{n}) - f^{(n)}(\frac{k}{n}))^2 \geq \left\{ \begin{array}{ll} \int_0^1 |f'(x)|^2 \, dx & \text{if } f \in H^1(\Omega) \\ \infty & \text{if } f \notin H^1(\Omega) \end{array} \right.$$ 

Proof: We have by Lemma 9 for all $a > 0$

$$\int_0^{1-\frac{1}{n}} (F^{(n)}(x))^2 \, dx \geq \int_0^{1-\frac{1}{n}} (F^{(n)}(a)(x))^2 \, dx = \sum_{k \in \mathbb{Z}} \left| \langle F^{(n)} \alpha, \phi_k^{(n)} \rangle \right|^2$$

and all summands are non-negative. Hence by Fatou’s Lemma and Lemma 10 and 11 we have for all $a > 0$:

$$\liminf_{n \to \infty} \sum_{k \in \mathbb{Z}} \left| \langle F^{(n)} \alpha, \phi_k^{(n)} \rangle \right|^2 \geq \sum_{k \in \mathbb{Z}} \liminf_{n \to \infty} \left| \langle F^{(n)} \alpha, \phi_k^{(n)} \rangle \right|^2 = \sum_{k \in \mathbb{Z}} \left| \widehat{F^{(n)}} \alpha(k) \right|^2 = |\alpha_0(a)|^2 + \sum_{k \neq 0} \left| 2\pi ik \widehat{f} \alpha(k) - \alpha_k(a) \right|^2.$$

By Lemma 12 either $\liminf_{n \to 0} |\alpha_0(a)| = \infty$, or for all sequences $a_m \to 0$ for which the limit $\lim_{m \to \infty} |\alpha_0(a_m)|^2 = |a_0|^2$ exists, we have as well $\lim_{m \to \infty} |\alpha_k(a_m)|^2 = |a_k|^2$ for all $k \in \mathbb{Z}$. Taking such a subsequence, we obtain by Fatou’s Lemma and Lemma 9

$$\liminf_{n \to \infty} \int_0^{1-\frac{1}{n}} (F^{(n)}(x))^2 \, dx \geq \liminf_{m \to \infty} \left( |\alpha_0(a_m)|^2 + \sum_{k \neq 0} \left| 2\pi ik \widehat{f} \alpha_m(k) - \alpha_k(a_m) \right|^2 \right)$$

$$\geq \lim_{m \to \infty} |\alpha_0(a_m)|^2 + \sum_{k \neq 0} \lim_{m \to \infty} \left| 2\pi ik \widehat{f} \alpha_m(k) - \alpha_k(a_m) \right|^2$$

$$\geq \lim_{m \to \infty} |\alpha_0(a_m)|^2 + \sum_{k \neq 0} \lim_{m \to \infty} \left| 2\pi ik \widehat{f} (k) - \alpha_k(a_m) \right|^2 = |\alpha_0|^2 + \sum_{k \neq 0} \left| 2\pi ik \widehat{f} (k) - \alpha_0 \right|^2.$$
By Lemma 8, \( f \in H^1(\Omega) \) if and only if there is some \( \alpha \in \mathbb{R} \) such that the sum on the right hand side is finite. Hence, if \( f \notin H^1(\Omega) \), the right hand side is always infinite. If \( f \in H^1(\Omega) \), \( f \) has a continuous version and thus \( \lim_{a \to 0} \alpha_0(a) = f(0) - f(1) \) which implies by Lemma 8, that the limes inferior equals \( \int |f'|^2 dt \).

The result corresponding to inequality (9) reads as follows:

**Lemma 14** Let \( f \in H^1(\Omega) \), \( \widetilde{f}^{(n)} := E(f|\sigma_n) \) the conditional expectation with respect to the sigma-algebra \( \sigma_n \) and \( \tilde{F}^{(n)} = n(f^{(n)}(x+1/n) - f^{(n)}(x)) \) as above. Then

\[
\limsup_{n \to \infty} \int_{0}^{1-\frac{1}{n}} |\tilde{F}^{(n)}(x)|^2 dx \leq \int_{0}^{1} |f'(x)|^2 dx.
\]

**Proof:** By the definition of conditional expectation and Jensen’s inequality (applied to the probability measure \( n \, dx \) on \( [\frac{x}{n}, \frac{x+1}{n}] \)) we obtain

\[
\sum_{\kappa=0}^{n-2} n(\widetilde{f}^{(n)}(\frac{x}{n}) - \widetilde{f}^{(n)}(\frac{x+1}{n}))^2 = \sum_{\kappa=0}^{n-2} n \left[ \int_{\frac{x}{n}}^{\frac{x+1}{n}} (f(x+1/n) - f(x)) dx \right]^2 \leq \sum_{\kappa=0}^{n-2} \int_{\frac{x}{n}}^{\frac{x+1}{n}} (n(f(x+1/n) - f(x))^2 dx = \int_{0}^{1-\frac{1}{n}} (n(f(x+1/n) - f(x))^2 dx.
\]

By Lebesgue’s differentiation theorem ([20], (7.2) Theorem, p. 100) we have convergence

\[
n(f(x+1/n) - f(x)) = \frac{1}{\lambda([x, x+1/n])} \int_{x}^{x+1/n} f'(u) du
\]

to \( f'(x) \) for Lebesgue-almost all \( x \in I \). On the other hand

\[
|n(f(x+1/n) - f(x))| \leq 2S^*(x) := 2 \sup_V \frac{1}{|V|} \int_{V} |f'(u)| du,
\]

where \( V \) is any open sub-interval \( V \subset I \) such that \( x \in V \). The function \( S^* \) is not integrable, except for \( f = 0 \) almost surely (see [20], p. 105). But by the Lemma of Hardy-Littlewood ([20], (7.9) Theorem, p. 105) there is a constant \( c > 0 \), such that

\[
\lambda(\{x \in U : S^*(x) > \alpha\}) \leq \frac{c}{\alpha} \int_{U} |f'(u)| du < \infty,
\]

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since $L^2(U) \subset L^1(U)$. That implies
\[
\lambda(\{x \in U : |n(f(x+1/n) - f(x))| > \alpha\}) \leq \lambda(\{x \in U : 2S^*(x) > \alpha\}) \leq \frac{2c}{\alpha} \int_U |f'(u)|\,du
\]
and the sequence of difference functions $n(f(x+1/n) - f(x))$ is therefore \textit{uniformly integrable}. Thus, we may interchange limit and integration which implies the statement. 

\[\blacksquare\]

### 6.3 Convergence of the Smoothness Penalty

As a consequence of the considerations in the preceding subsection, we prove now Proposition 5, the corresponding $\Gamma$-convergence result for the Blake-Zisserman penalty $\Phi_{\gamma,\mu,u}^n$. Note that the case $\mu = 0$ requires some care.

\textbf{Proof:} (i) Let $f = F + t \in \text{SBV}_2(U)$. From the sequence $f_{n,n} \in \mathbb{N}$, $f_n \in \mathcal{T}_n(U)$ we construct the decomposition $f_n^F := f_n - f_n^t$ where
\[
f_n^t := \sum_{\kappa=0}^{n-1} t(\frac{\kappa}{n})\chi_{[\frac{\kappa}{n}, \frac{\kappa+1}{n})} \in \mathcal{T}_n(U).
\]
We thus have $f_n = f_n^F + f_n^t$ and $f_n^t \to t$ by boundedness of $t$, hence as well $f_n^F \to F$. Furthermore $f_n^t(\frac{\kappa}{n}) - f_n^t(\frac{\kappa+1}{n}) = 0$ if the interval $(\frac{\kappa}{n}, \frac{\kappa+1}{n})$ contains no jump of the step function $t$ and there are only finitely many intervals that
contain a jump, namely at most \( j(t) \). Hence

\[
\sum_{\kappa=0}^{n-2} \min\left\{ \frac{\kappa}{\mu_n} (f_n(\frac{\kappa+1}{n}) - f_n(\frac{\kappa}{n}))^2, \gamma_n \right\}
= \sum_{\{\kappa: (\frac{\kappa}{n}, \frac{\kappa+1}{n}] \cap \mathcal{P}(t) = \emptyset\}} \min\left\{ \frac{\kappa}{\mu_n} (f_n(\frac{\kappa+1}{n}) - f_n(\frac{\kappa}{n}))^2, \gamma_n \right\}
+ \sum_{\{\kappa: (\frac{\kappa}{n}, \frac{\kappa+1}{n}] \cap \mathcal{P}(t) \neq \emptyset\}} \min\left\{ \frac{\kappa}{\mu_n} (f_n(\frac{\kappa+1}{n}) - f_n(\frac{\kappa}{n}))^2, \gamma_n \right\}
= \sum_{\{\kappa: (\frac{\kappa}{n}, \frac{\kappa+1}{n}] \cap \mathcal{P}(t) = \emptyset\}} \min\left\{ \frac{\kappa}{\mu_n} (f_n(\frac{\kappa+1}{n}) - f_n^F(\frac{\kappa}{n}))^2, \gamma_n \right\}
+ \sum_{\{\kappa: (\frac{\kappa}{n}, \frac{\kappa+1}{n}] \cap \mathcal{P}(t) \neq \emptyset\}} \min\left\{ \frac{\kappa}{\mu_n} (f_n^F(\frac{\kappa+1}{n}) - f_n^F(\frac{\kappa}{n}))^2, \gamma_n \right\}
\]

Now we consider the sequence of conditional expectations \( \tilde{f}_n^F := E(f - t \mid \sigma_n) \). Then \( \tilde{f}_n^F := E(F \mid \sigma_n) \) and we have by Lemma 14

\[
\limsup_{n \to \infty} \sum_{\kappa=0}^{n-2} \min\left\{ \frac{\kappa}{\mu_n} (\tilde{f}_n^F(\frac{\kappa+1}{n}) - \tilde{f}_n^F(\frac{\kappa}{n}))^2, \gamma_n \right\}
\leq \lim_{n \to \infty} \sum_{\kappa=0}^{n-2} \frac{\mu_n}{\mu_n} (\tilde{f}_n^F(\frac{\kappa+1}{n}) - \tilde{f}_n^F(\frac{\kappa}{n}))^2 = \mu_n^{-2} \int_0^1 |f'(x)|^2 dx.
\]

For the analysis of the exceptional intervals, we use the fact that \( F \) is absolutely continuous (see [20], p. 115). Therefore, for all \( \epsilon > 0 \) there is an \( n_0 \), such that for all \( n \geq n_0 \) and \( \kappa = 0, ..., n - 1 \) we have

\[
\sup_{x, x' \in [\frac{\kappa}{n}, \frac{\kappa+1}{n}]} |F(x) - F(x')| \leq \epsilon.
\]

By the contraction property of conditional expectation that implies for all \( n \geq n_0 \), \( \kappa = 0, ..., n - 1 \) that \( |\tilde{f}_n^F(\frac{\kappa+1}{n}) - \tilde{f}_n^F(\frac{\kappa}{n})| \leq 2\epsilon \). Choose now \( n_0 \) so large that \( \epsilon < \delta/4 \) where \( \delta := \min_{x \in J(t)} |t(x) - t(x^-)| \) is the height of the smallest jump of the step function and additionally, such that all exceptional intervals contain exactly one discontinuity of \( t \). That implies for the exceptional intervals

\[
|\tilde{f}_n^F(\frac{\kappa+1}{n}) - \tilde{f}_n^F(\frac{\kappa}{n}) + f_n^F(\frac{\kappa+1}{n}) - f_n^F(\frac{\kappa}{n})| \geq \delta/4.
\]

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Thus for all $n$ with $n \geq n_0$ and $n\delta^2/16\mu_n^2 > \gamma_n$ we have
\[
\sum_{\{\kappa : (\frac{\kappa}{n}, \kappa + 1) \cap p(t) \neq \emptyset\}} \min\{\frac{n}{\mu_n} (f_n^F(\frac{\kappa+1}{n}) - f_n^F(\frac{\kappa}{n}))^2, \gamma_n\}
= \sum_{\{\kappa : (\frac{\kappa}{n}, \kappa + 1) \cap p(t) \neq \emptyset\}} \min\{n\delta^2/16\mu_n^2, \gamma_n\} = \gamma_n j(t).
\]

(ii) Let $f \in L^2(U)$, $f_n \in T_n(U)$ with $f_n \to f$. Then
\[
\liminf_{n \to \infty} \sum_{\kappa=0}^{n-2} \min\{\frac{n}{\mu_n} (f_n(\frac{\kappa+1}{n}) - f_n(\frac{\kappa}{n}))^2, \gamma_n\}
= \sum_{\kappa \in a_n} \frac{n}{\mu_n} (f_n(\frac{\kappa+1}{n}) - f_n(\frac{\kappa}{n}))^2 + \sum_{\kappa \in a_n^c} \gamma_n
= \gamma_n |a_n| + \sum_{\kappa \in a_n^c} \frac{n}{\mu_n} (f_n(\frac{\kappa+1}{n}) - f_n(\frac{\kappa}{n}))^2,
\]
where $a_n := \{\kappa \leq n-2 : \min\{\frac{n}{\mu_n} (g_n(\frac{\kappa+1}{n}) - g_n(\frac{\kappa}{n}))^2, \gamma\} = \gamma\}$ and $a_n^c := \{0, \ldots, n-2\} - a_n$. Without loss of generality, we consider the subsequence $f_n^{\inf}$ of $f_n$ with
\[
\liminf_{n \to \infty} \sum_{\kappa=0}^{n-2} \min\{\frac{n}{\mu_n} (f_n^{\inf}(\frac{\kappa+1}{n}) - f_n^{\inf}(\frac{\kappa}{n}))^2, \gamma_n\}
= \lim_{n \to \infty} \sum_{\kappa=0}^{n-2} \min\{\frac{n}{\mu_n} (f_n^{\inf}(\frac{\kappa+1}{n}) - f_n^{\inf}(\frac{\kappa}{n}))^2, \gamma_n\}
\]
with corresponding exceptional sets $a_n^{\inf}$. The sets $p_n^{\inf} := \{\kappa/n : \kappa \in a_n^{\inf}\} \cup \{0, 1\}$ are finite and hence closed in $[0, 1]$. Passing to another subsequence of $f_n^{\inf}$ if necessary, the compactness of the set of closed subsets of $[0, 1]$ with respect to Hausdorff distance implies, that the sequence of sets $p_n^{\inf}$ converges to a closed subset $c \subset [0, 1]$. By (i), convergence to a function $f \in SBV_2(U)$ implies by the absolute continuity of $f$ off the jump points that the exceptional set contains only finitely many points. Thus $|c| = \infty$ implies $f \notin SBV_2(U)$.

Therefore assume $|c| = K < \infty$. Hence $c \in \mathcal{P}(U)$. In that case, Lemma 13 yields the following alternative: Either $f \in SBV_2(U)$, then $f \in H^1(\Theta)$ for
all $\Theta \in \iota(c)$ and the limit of the subsequence is greater or equal to

$$\gamma K + \mu^{-2} \sum_{\Theta \in \iota(c)} |f'_\Theta|^2 dx$$

where $f'_\Theta = f'|_{\Theta}$, or $f \notin SBV_2(U)$ which implies that the limit is $\infty$. That implies the statement, in particular for $\mu = 0$.

7 Dependence on the Parameters

In the final section, we will prove the Theorem 1. According to Theorem 5, the proof follows from $\Gamma$-convergence together with the fact – already established in Section 5 – that the set of minimizers is compact. Thus, we start by showing that the functionals in question depend continuously on the respective parameters in an appropriate sense.

7.1 $\Gamma$-Continuity of the Segmentation Family

We consider the three-dimensional (pseudo-) cube $Q$ given by

$$Q := \mathbb{R}^+_0 \times \mathbb{R}^+_0 \times T$$

and the corresponding family of functionals $F(q) : L^2(U) \rightarrow \mathbb{R}, q \in Q$ defined in Theorem 1. The statement about $\Gamma$-continuity on the cube now reads as follows:

**Theorem 6** Let $q_{s,s} \in \mathbb{N}$ with $q_s := (\gamma_s, \mu_s, t_s) \in Q$ a sequence of parameters converging to $q := (\gamma, \mu, t) \in Q$. Then

$$F(q_s) \rightharpoonup F(q)$$

as $s$ tends to infinity.

To prove this, the crucial point is the statement about $\Gamma$-convergence of the penalizers established in Proposition 5. However, we will need two additional lemmas, the first of which states that the discrete $L^2$-distance used in the Blake - Zisserman functional converges to the continuous $L^2$-distance.
**Lemma 15** Let \( f, g \in L^2(U) \), \( f_n, n \in \mathbb{N} \) with \( f_n \in T_n(U) \) be a sequence of step functions converging to \( f \) in \( L^2(U) \) and \( g_n, n \in \mathbb{N} \) with \( g_n := E(g | \sigma_n) \) the sequence of conditional expectations. Then
\[
\lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} (f_n - g_n)^2(\frac{k}{n}) = \| f - g \|^2
\]

**Proof:** First of all, \( E(f_n | \sigma_n) = f_n \) by a standard property of conditional expectation. Furthermore, martingale convergence (see e.g. [21], Ch. 12) implies
\[
\lim_{n \to \infty} E(f_n - f | \sigma_n) = 0.
\]
Hence
\[
\lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} (f_n - g_n)^2(\frac{k}{n}) = \lim_{n \to \infty} \int_0^1 E(f_n - g | \sigma_n)^2 dx
\]
\[
= \lim_{n \to \infty} \int_0^1 (E(f_n - f | \sigma_n) + E(f - g | \sigma_n))^2 dx = \lim_{n \to \infty} \int_0^1 E(f - g | \sigma_n)^2 dx
\]
Again by martingale convergence, \( E(f - g | \sigma_n) \) tends to \( f - g \). \( \blacksquare \)

By the second lemma, we prove a result about approximation of \( L^2 \)-functions under additional constraints about the location of the jumps of the approximating step functions.

**Lemma 16** Let \( f \in L^2(U) \) and \( k_m \) a sequence of non negative integers such that \( k_m \leq m \) and \( k_m \to \infty \). Then there is a sequence of functions
\[
f_m \in \{ f \in T_m(U) : j(f_m) \leq k_m \}
\]
such that \( f_m \to f \) in \( L^2(U) \).

**Proof:** We consider three families associated to \( f \):

(i) \( F_k := \arg \min_{t \in \mathcal{T}(U), j(t) \leq k} \| t - f \|, \quad k = 1, 2, \ldots \) – the minimum is not necessarily unique.
(ii) \( F_{k,m} := E(F_k | \sigma_m) \) – we have \( j(F_{k,m}) \leq 2k \), \( k, m = 1, 2, \ldots \).

(iii) \( f_{k,m} := \arg\min_{t \in T_m(U)} \|t - f\|, k, m = 1, 2, \ldots \) – again, the minimum is not necessarily unique.

By \( k_m \to \infty \), we have for all \( K > 0 \) some \( M_0 > 0 \) such that \( k_m > 2K \) for all \( m \geq M_0 \). Then

\[
\|f_{k_m,m} - f\| \leq \|F_{K,m} - f\| \leq \|F_{K,m} - F_K\| + \|F_K - f\|.
\]

By construction

\[
\lim_{m \to \infty} \|F_{K,m} - F_K\| = 0, \quad \lim_{K \to \infty} \|F_K - f\| = 0
\]

and thus

\[
\limsup_{m \to \infty} \|f_{k_m,m} - f\| \leq \|F_K - f\|
\]

arbitrarily small and non negative. That implies the statement.

Now we can prove the theorem stated above.

**Proof:** For the proof, we have to consider three different cases depending on the location of \( q \).

1st case. \( \gamma, \mu \geq 0, t > 0 \): In that case \( q_s \to q \) if and only if \( \gamma_s \to \gamma, \mu_s \to \mu \) and \( t_s = t := 1/n \) for \( s \geq s_0 \). That implies for \( s \geq s_0 \)

\[
F(\gamma_s, \mu_s, t_s)(f) - F(\gamma, \mu, t)(f) = BZ_{\gamma_s,\mu_s,n}(f) - BZ_{\gamma,\mu,n}(f) = \left\{
\begin{array}{ll}
\sum_{n=0}^{s-1} \min\left\{ \frac{n}{t_s} (f(\frac{\gamma s}{n}) - f(\frac{s}{n}))^2, \gamma_s \right\} & f \in T_n(U) \\
- \min\left\{ \frac{n}{t s^2} (f(\frac{\gamma s}{n}) - f(\frac{s}{n}))^2, \gamma \right\} & \text{else}
\end{array}
\right.
\]

For step functions \( f \in T_n(U) \), \( \|f\|_{L^2(U)} \leq C \) implies \( \sup |f| \leq C \sqrt{n} \). Thus \( F(\gamma_s, \mu_s, t_s) \) converges to \( F(\gamma, \mu, t) \) uniformly on balls \( \{\|f\|_{L^2(U)} \leq C\} \). That implies \( \Gamma \)-convergence of the functionals.

2nd case. \( \gamma > 0, \mu \geq 0, t = 0 \): In that case, \( \Gamma \)-convergence follows from Proposition 5 for the penalizer and Lemma 15 for the distance term.

3rd case. \( \gamma = 0, \mu \geq 0, t = 0 \): In that case we have \( n_s := 1/t_s \to \infty \) and for \( f_s \in T_{n_s}(U) \)

\[
\tilde{d}_{g}^{n_s}(f_s) \leq BZ_{\gamma_s,\mu_s,g}^{n_s}(f_s) \leq n_s \gamma_s + \tilde{d}_{g}^{n_s}(f_s).
\]
Let now \( f \in L^2(U) \) and \( f_s \in T_{n_s}(U) \) a sequence of functions with \( f_s \to f \) in \( L^2(U) \). Then

\[
\operatorname{lim inf}_{s \to \infty} \mathbb{B}Z_{\gamma_s, \mu_s, \nu_s}(f_s) \geq \operatorname{lim inf}_{s \to \infty} \hat{d}^n_{\gamma_s}(f_s) = \lim_{s \to \infty} \hat{d}^n_{\gamma_s}(f_s) = d_g(f).
\]

That is condition (8) for \( \Gamma \)-convergence. Let now \( \hat{f}_s \in T_{n_s}(U) \) a sequence of step functions such that each \( \hat{f}_s \) is some best approximation of \( f \) in the set \( M_s := \{ t \in T_{n_s}(U) : j(t) \leq 1/\gamma_s \} \). Then \( \gamma_s j(\hat{f}_s) \to 0, \hat{f}_s \to f \) and by Lemma 16 this sequence fulfills condition (9).

### 7.2 Proof of the Main Theorem

The proof of Theorem 11 is now finally a consequence of Theorem 5 together with the compactness of the set of minimizers: Existence (i) of the minimizers is provided by Theorem 4. Convergence (iiia) of the minimizers follows from Theorem 5 (i) together with Theorem 6 the result on \( \Gamma \)-convergence established above. For all sequences \( q_s = (\gamma_s, \mu_s, t_s) \) converging to \( q = (\gamma, \mu, t) \in Q \), there is some \( b > 0 \) with \( \mu_s \leq b \) for all \( s \). Since, by the reduction principle, all possible minimizers are thus contained in the set \( M_b(g) \) (see Section 5), the existence of a convergent subsequence (iib) follows from the compactness result Theorem 8.

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