ALGEBRA OF OPERATORS AFFILIATED
WITH A FINITE TYPE I VON NEUMANN ALGEBRA

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Abstract. It is shown that the \( \ast \)-algebra of all (closed densely defined linear) operators affiliated with a finite type I von Neumann algebra admits a unique center-valued trace, which turns out to be, in a sense, normal. It is also demonstrated that for no other von Neumann algebras similar constructions can be performed.

1. Introduction

With every von Neumann algebra \( \mathfrak{A} \) one can associate the set \( \text{Aff}(\mathfrak{A}) \) of operators (unbounded, in general) which are affiliated with \( \mathfrak{A} \). In [10] Murray and von Neumann discovered that, surprisingly, \( \text{Aff}(\mathfrak{A}) \) turns out to be a unital \( \ast \)-algebra when \( \mathfrak{A} \) is finite. This was in fact the first example of a rich set of unbounded operators in which one can define algebraic binary operations in a natural manner. This concept was later adapted by Segal [15, 16] who distinguished a certain class of unbounded operators (namely, measurable with respect to a fixed normal faithful semi-finite trace) affiliated with an arbitrary semi-finite von Neumann algebra and equipped it with a structure of a \( \ast \)-algebra (for an alternative proof see e.g. [11] or §2 in Chapter IX of [19]). A more detailed investigations of algebras of the form \( \text{Aff}(\mathfrak{A}) \) were initiated by a work of Stone [17] who described their models for commutative \( \mathfrak{A} \) in terms of unbounded continuous functions densely defined on the Gelfand spectrum \( \mathfrak{X} \) of \( \mathfrak{A} \). Much later Kadison [5] studied this one-to-one correspondence between operators in \( \text{Aff}(\mathfrak{A}) \) and functions on \( \mathfrak{X} \). Recently Liu [8] established an interesting property of \( \text{Aff}(\mathfrak{A}) \) concerning Heisenberg uncertainty principle. Namely, she showed that the canonical commutation relation, which has the form \( AB - BA = I \), fails to hold for any \( A, B \in \text{Aff}(\mathfrak{A}) \) provided \( \mathfrak{A} \) is finite. We shall obtain her result for finite type I algebras as a simple corollary of our results. The main purpose of the paper is to show that whenever \( \mathfrak{A} \) is a finite type I von Neumann algebra, then \( \text{Aff}(\mathfrak{A}) \) has a uniquely determined center-valued trace, as shown by

1.1. Theorem. Let \( \mathfrak{A} \) be a finite type I von Neumann algebra and let \( \text{Aff}(\mathfrak{A}) \) be the \( \ast \)-algebra of all operators affiliated with \( \mathfrak{A} \). Then there is a unique linear map \( \text{tr}_{\text{Aff}}: \text{Aff}(\mathfrak{A}) \to \mathbb{C}(\text{Aff}(\mathfrak{A})) \) such that

\( \text{tr}_1 \) \( \text{tr}_{\text{Aff}}(A) \) is non-negative provided \( A \in \text{Aff}(\mathfrak{A}) \) is so;

\( \text{tr}_2 \) \( \text{tr}_{\text{Aff}}(X \cdot Y) = \text{tr}_{\text{Aff}}(Y \cdot X) \) for any \( X, Y \in \text{Aff}(\mathfrak{A}) \);

\( \text{tr}_3 \) \( \text{tr}_{\text{Aff}}(Z) = Z \) for each \( Z \in \mathbb{C}(\text{Aff}(\mathfrak{A})) \).

What is more,

\[ \mathbb{C}(\text{Aff}(\mathfrak{A})) = \text{Aff}(\mathbb{C}(\mathfrak{A})) \]

and

\[ \mathbb{C}(\text{Aff}(\mathfrak{A})) = \text{Aff}(\mathbb{C}(\mathfrak{A})) \]
(tr4) $\text{tr}_{\mathcal{A}}(A) \neq 0$ provided $A \in \mathcal{A}$ is non-zero and non-negative;

(tr5) $\text{tr}_{\mathcal{A}}(X \cdot Z) = \text{tr}_{\mathcal{A}}(X) \cdot Z$ for any $X \in \mathcal{A}$ and $Z \in \mathcal{Z}(\mathcal{A})$;

(tr6) every increasing net $(A_\sigma)_{\sigma \in \Sigma}$ of self-adjoint members of $\mathcal{A}$ which is majorized by a self-adjoint operator in $\mathcal{A}$ has its least upper bound in $\mathcal{A}$, and

\begin{equation}
\sup_{\sigma \in \Sigma} \text{tr}_{\mathcal{A}}(A_\sigma) = \text{tr}_{\mathcal{A}} \left( \sup_{\sigma \in \Sigma} A_\sigma \right).
\end{equation}

It is worth noting that (2) is a natural counterpart of normality (in the terminology of Takesaki—see Definition 2.1 in Chapter V of [18]) of center-valued traces in finite von Neumann algebras. It is natural to ask whether the above result may be generalised to a wider class of von Neumann algebras (e.g. for all finite). Our second goal is to show that the answer is negative, which is somewhat surprising. A precise formulation of the result is stated below. We recall that, in general, the set $\mathcal{A}$ admits no structure of a vector space, nevertheless, it is always homogeneous and for any $T \in \mathcal{A}$ and $S \in \mathcal{A}$ the operator $T + S$ is a (well defined) member of $\mathcal{A}$. Based on this observation, we may formulate our result as follows.

1.2. Proposition. Let $\mathfrak{A}$ be a von Neumann algebra and let $\mathcal{A} = (\mathfrak{A}, \mathcal{A})$ be the set of all operators affiliated with $\mathfrak{A}$. Assume there exists a function $\varphi : \mathcal{A} \to \mathfrak{A}$ with the following properties:

(a) if $A, B \in \mathfrak{A}$ are such that $\varphi(A) \in \mathfrak{A}$, then $\varphi(\alpha A + \beta B) = \alpha \varphi(A) + \beta \varphi(B)$ for any scalars $\alpha, \beta \in \mathbb{C}$;

(b) $\varphi(A)$ is non-negative provided $A \in \mathcal{A}$ is so;

(c) if $A \in \mathcal{A}$ and $B \in \mathfrak{A}$ are non-negative, and $\varphi(B) \in \mathfrak{A}$, then $\varphi(A + B) = \varphi(A) + \varphi(B)$;

(d) $\varphi(AB) = \varphi(BA)$ for all $A, B \in \mathfrak{A}$;

(e) $\varphi(Z) = Z$ for each $Z \in \mathcal{Z}(\mathfrak{A})$;

(f) if $A$ and $U$ are members of $\mathfrak{A}$ and $U$ is unitary, then $U^* \varphi(A) U = \varphi(A)$.

Then $\mathfrak{A}$ is finite and type $I$.

The paper is organized as follows. In the next section we establish an interesting property of finite type $I$ von Neumann algebras, which is crucial in this paper, since all other results, apart from Proposition 1.2, are its consequences. Its proof involves measure-theoretic techniques, which is in contrast to all other parts of the paper, where all arguments are, roughly speaking, intrinsic and algebraic. In Section 3 we establish most relevant properties of the set $\mathcal{A}$ (for a finite type $I$ algebra $\mathfrak{A}$), including a new proof of the fact that $\mathcal{A}$ admits a structure of a $\ast$-algebra. In the fourth part we introduce the center-valued trace on $\mathcal{A}$ and prove all items of Theorem 1.1 apart from (tr6), which is shown in Section 5, where we establish also other order properties of $\mathcal{A}$. The last, sixth, part contains a proof of Proposition 1.2.

Notation and terminology. In this paper $\mathfrak{A}$ is reserved to denote an arbitrary von Neumann algebra acting on a (complex) Hilbert space $\mathcal{H}$. All operators are linear, closed and densely defined in a Hilbert space, projections are orthogonal and non-negative operators are, by definition, self-adjoint. The algebra of all bounded operators on $\mathcal{H}$ is denoted by $\mathcal{B}(\mathcal{H})$. An operator $T$ in $\mathcal{H}$ is affiliated with $\mathfrak{A}$ if $UTU^{-1} = T$ for any unitary operator $U$ belonging to the commutant $\mathfrak{A}'$ of $\mathfrak{A}$. $\mathcal{A}$ stands for the set of all operators affiliated with $\mathfrak{A}$. If $\mathfrak{A}$ is finite, $\mathcal{A}$ may naturally be equipped with a structure of a $\ast$-algebra (see e.g. [10]). In that case we denote binary algebraic operations in $\mathcal{A}$ by ‘$+$’ (for addition), ‘$-$’ (for subtraction) and ‘$\cdot$’ (for multiplication). For any ring $\mathfrak{R}$, $\mathfrak{Z}(\mathfrak{R})$ stands for the center of $\mathfrak{R}$ (that is, $\mathfrak{Z}(\mathfrak{R})$ consists of all elements of $\mathfrak{R}$ which commute with any element of $\mathfrak{R}$).
of $\mathfrak{A}$. This mainly applies for $\mathfrak{A} = \mathfrak{A}$ and $\mathfrak{A} = \text{Aff}(\mathfrak{A})$ provided $\mathfrak{A}$ is finite. For any operator $S$, we use $D(S)$, $\mathcal{N}(S)$ and $\mathcal{R}(S)$ to denote, respectively, the domain, the kernel and the range of $S$. By $|S|$ we denote the operator $(S^*S)^{1/2}$. For any collection $\{T_s\}_{s \in S}$ of operators, $\bigoplus_{s \in S} T_s$ is understood as an operator with a maximal domain defined naturally; that is, $\bigoplus_{s \in S} T_s x_s$ belongs to the domain of $\bigoplus_{s \in S} T_s x_s$ for each $x_s \in D(T_s)$ for each $s \in S$, and $\sum_{s \in S} \|T_s x_s\|^2 < \infty$ (and, of course, $(\bigoplus_{s \in S} T_s) (\bigoplus_{s \in S} T_s) = \bigoplus_{s \in S} (T_s T_s) (T_s x_s)$). The center-valued trace on a finite von Neumann algebra $\mathfrak{M}$ is denoted by $\text{tr}_\mathfrak{M}$. The center-valued trace on the algebra of operators affiliated with a finite type I von Neumann algebra will be denoted by $\text{tr}_{\text{Aff}}$. All vector spaces are assumed to be over the field $\mathbb{C}$ of complex numbers. For two $C^*$-algebras $\mathfrak{C}_1$ and $\mathfrak{C}_2$, we write $\mathfrak{C}_1 \cong \mathfrak{C}_2$ when $\mathfrak{C}_1$ and $\mathfrak{C}_2$ are $*$-isomorphic. The direct product of a collection $\{\mathfrak{C}_s\}_{s \in S}$ of $C^*$-algebras is denoted by $\prod_{s \in S} \mathfrak{C}_s$, and it consists of all systems $(a_s)_{s \in S}$ with $a_s \in \mathfrak{C}_s$ and $\sup_{s \in S} \|a_s\| < \infty$ (cf. Definition II.8.1.2 in [1]). By $I$ we denote the identity operator on $\mathcal{H}$.

2. Key result

As we shall see in the sequel, all our main results depend on the following theorem, whose proof is the purpose of this section.

2.1. Theorem. Assume $\mathfrak{A}$ is finite and type I. Then for any $T \in \mathfrak{A}$ the following conditions are equivalent:

(a) $\|T\xi\| < \|\xi\|$ for each non-zero vector $\xi \in \mathcal{H}$;

(b) there is a sequence $Z_1, Z_2, \ldots \in \mathfrak{M}(\mathfrak{A})$ of mutually orthogonal projections such that $\sum_{n=1}^{\infty} Z_n = I$ and $\|TZ_n\| < 1$ for any $n \geq 1$.

We shall derive the above theorem as a combination of a classical result on classification of type I von Neumann algebras and a measure-theoretic result due to Maharam [9].

For any positive integer $n$, let $M_n$ be the $C^*$-algebra of all $n \times n$ complex matrices. Whenever $(X, \mathfrak{M}, \mu)$ is a finite measure space, we use $L^\infty(X, \mu, M_n)$ to denote the $C^*$-algebra of all $M_n$-valued essentially bounded measurable functions on $X$ (a function $f = [f_{jk}]: X \to M_n$ is measurable if each of the functions $f_{jk}: X \to \mathbb{C}$ is measurable; in other words, $L^\infty(X, \mu, M_n) \cong L^\infty(X, \mu) \otimes M_n$). The following result is well-known and may easily be derived from Theorems 1.22.13 and 2.3.3 in [14] (cf. also Theorem 6.6.5 in [8]).

2.2. Theorem. For every finite type I von Neumann algebra $\mathfrak{A}$ there are a collection $\{(X_j, \mathfrak{M}_{j}, \mu_j)\}_{j \in J}$ of probabilistic measure spaces and a corresponding collection $\{\nu_j\}_{j \in J}$ of positive integers such that

$$\mathfrak{A} \cong \prod_{j \in J} L^\infty(X_j, \mu_j, M_{\nu_j}).$$

We need a slight modification of (3) (see Theorem 2.4 below). To this end, let us introduce certain classical measure spaces, which we call canonical. Let $\alpha$ be an infinite cardinal and $S_\alpha$ be a fixed set of cardinality $\alpha$. We consider the set $D_\alpha = \{0, 1\}^{S_\alpha}$ (of all functions from $S_\alpha$ to $\{0, 1\}$) equipped with the product $\sigma$-algebra $\mathfrak{M}_\alpha$ and the product probabilistic measure $m_\alpha$; that is, $\mathfrak{M}_\alpha$ coincides with the $\sigma$-algebra on $D_\alpha$ generated by all sets of the form

$$\text{Cyl}(G) \overset{\text{def}}{=} \{u \in D_\alpha: |F| \cap u \in G\}$$

where $F$ is a finite subset of $S_\alpha$ and $G$ is any subset of $\{0, 1\}^F$, while $m_\alpha$ is a unique probabilistic measure on $\mathfrak{M}_\alpha$ such that $m_\alpha(\text{Cyl}(G)) = \text{card}(G)/2^{\text{card}(F)}$ for any such sets $G$ and $F$. It is worth noting that when $\alpha$ is uncountable and $D_\alpha$ is considered with the product topology, not every open set in $D_\alpha$ belongs
to \(\mathcal{M}_0\). Additionally, we denote by \((D_0, \mathcal{M}_0, m_0)\) a unique probabilistic measure space with \(D_0 = \{0\}\). For simplicity, let \(\text{Card}_\infty\) stand for the class of all infinite cardinal numbers. We call the measure spaces \((D_\alpha, \mathcal{M}_\alpha, m_\alpha)\) with \(\alpha \in \text{Card}_\infty \cup \{0\}\) canonical. In the sequel we shall apply the following consequence of a deep result due to Maharam [9]:

2.3. Theorem. For any probabilistic measure space \((X, \mathcal{M}, \mu)\) there is a sequence (finite or not) \(\alpha_1, \alpha_2, \ldots \in \text{Card}_\infty \cup \{0\}\) such that the C*-algebras \(L^\infty(X, \mu)\) and \(\prod_{\alpha \geq 1} L^\infty(D_\alpha, m_\alpha)\) are *-isomorphic.

The above result is not explicitly stated in [9], but may simply be deduced from Theorems 1 and 2 included there.

As a consequence of Theorems 2.2 and 2.3 (and the fact that \(L^\infty(X, \mu, M_\alpha) \cong L^\infty(X, \mu) \otimes M_\alpha\)), we obtain

2.4. Theorem. For every finite type I von Neumann algebra \(\mathfrak{A}\) there are collections \(\{\alpha_j\}_{j \in J} \subseteq \text{Card}_\infty \cup \{0\}\) and \(\{\nu_j\}_{j \in J} \subseteq \{1, 2, \ldots\}\) such that

\[
\mathfrak{A} \cong \prod_{j \in J} L^\infty(D_{\alpha_j}, m_{\alpha_j}, M_{\nu_j}).
\]

The following simple lemma will also prove useful for us.

2.5. Lemma. If \(T\) is an arbitrary member of \(\mathfrak{A}\), then \(\mathcal{N}(T) = \{0\}\) iff the mapping

\[
\mathfrak{A} \ni X \mapsto TX \in \mathfrak{A}
\]

is one-to-one.

Proof. If \(\mathcal{N}(T) = \{0\}\) and \(TX = 0\), then \(\mathcal{R}(X) \subseteq \mathcal{N}(T) = \{0\}\) and hence \(X = 0\). For the converse, let \(E : B(\mathfrak{B}(\mathfrak{H})) \rightarrow B(\mathfrak{H})\) be the spectral measure of \(|T|\), defined on the \(\sigma\)-algebra \(B(\mathfrak{B}(\mathfrak{H}))\) of all Borel subsets of \(\mathfrak{B}(\mathfrak{H})\). Then \(E(\sigma) \in \mathfrak{A}\) for any Borel set \(\sigma \subseteq \mathfrak{B}(\mathfrak{H})\). Since \(TE(\{0\}) = 0\), we conclude from the injectivity of \((5)\) that \(E(\{0\}) = 0\) and thus \(\mathcal{N}(T) = \mathcal{N}(|T|) = \{0\}\).

2.6. Proposition. Let \(\mathfrak{A}\) be a finite type I von Neumann algebra. Let \(\{\alpha_j\}_{j \in J}\) and \(\{\nu_j\}_{j \in J}\) be two collections as in the assertion of Theorem 2.4. Further, let \(\Phi : \mathfrak{A} \rightarrow \prod_{j \in J} L^\infty(D_{\alpha_j}, m_{\alpha_j}, M_{\nu_j})\) be any *-isomorphism. For an arbitrary operator \(T\) in \(\mathfrak{A}\) and \((f_j)_{j \in J} \equiv \Phi(T)\), the following conditions are equivalent:

(a) \(|T\|_\infty < \|\xi\|_\mathfrak{H}\) for each non-zero vector \(\xi \in \mathfrak{H}\);
(b) for each \(j \in J\), the set \(\{x \in D_{\alpha_j} : \|f_j(x)\| < 1\}\) is of full measure \(m_{\alpha_j}\).

Proof. Assume first that (b) holds. Observe that then \(|T| = \|\Phi(T)\| \leq 1\) and \(\Phi((I - T^*TS)) = (1 - \Phi(T)^* \Phi(T)) \Phi(S) \neq 0\) for any non-zero operator \(S \in \mathfrak{A}\). So, Lemma 2.3 ensures us that \(I - T^*T\) is one-to-one. Consequently, (a) is fulfilled.

Now assume that (a) is satisfied, or, equivalently, that \(I - T^*T \geq 0\) and the mapping \(\mathfrak{A} \ni X \mapsto (I - T^*T)X \in \mathfrak{A}\) is one-to-one. This means that

\[
1 - \Phi(T)^* \Phi(T) \geq 0
\]

and \((1 - \Phi(T)^* \Phi(T))g \neq 0\) for each non-zero \(g \in \mathfrak{L} \defeq \prod_{j \in J} L^\infty(D_{\alpha_j}, m_{\alpha_j}, M_{\nu_j})\). Suppose, on the contrary, that

\[
m_{\alpha_k}(\{x \in D_{\alpha_k} : \|f_k(x)\| < 1\}) < 1
\]

for some \(k \in J\). To get a contradiction, it is enough to find a bounded measurable function \(u : D_{\alpha_k} \rightarrow M_{\nu_k}\) such that \(u\) is a non-zero vector in \(L^\infty(D_{\alpha_k}, m_{\alpha_k}, M_{\nu_k})\) and \((1 - f_k^* f_k)u = 0\) (because then it suffices to put \(y_k = u\) and \(g_j = 0\) for \(j \neq k\) in order to obtain a non-zero vector \(g \equiv (g_j)_{j \in J} \in \mathfrak{L}\) for which \((1 - \Phi(T)^* \Phi(T))g = 0\)). It is now a moment when we shall make use of the form of the measurable space
(\(D_{\alpha_k}, \mathfrak{M}_{\alpha_k}\)). If \(\alpha_k = 0\), the existence of \(u\) is trivial. We therefore assume that \(\alpha_k\) is infinite. Since \(f_k: \{0,1\}^{S_{\alpha_k}} \rightarrow M_{\alpha_k}\) is measurable and \(\mathfrak{M}_{\alpha_k}\) is the product \(\sigma\)-algebra, we conclude that there exist a countable infinite set \(F \subset S_{\alpha_k}\) and a measurable function \(f: \{0,1\}^F \rightarrow M_{\alpha_k}\) such that
\[
 f_k(\eta) = f(\eta|_F)
\]
for any \(\eta \in \{0,1\}^{S_{\alpha_k}}\). For simplicity, we put \(\Omega = \{0,1\}^F\), \(\mathfrak{M} = \{G \subset \Omega: \text{Cyl}(G) \in \mathfrak{M}_{\alpha_k}\}\) (cf. \(\text{(4)}\)) and a measure \(\lambda: \mathfrak{M} \rightarrow [0,1]\) by \(\lambda(G) = m_{\alpha_k}(\text{Cyl}(G))\) for any \(G \in \mathfrak{M}\). Note that the probabilistic measure space \((\Omega, \mathfrak{M}, \lambda)\) is naturally isomorphic to \((\mathcal{D}_{\alpha_k}, \mathfrak{M}_{\alpha_k}, m_{\alpha_k})\) and hence it is a standard measure space (which is relevant for us). It suffices to find a measurable bounded function \(v: \Omega \rightarrow M_n\) such that \(v\) is a non-zero vector in \(L^\infty(\Omega, \lambda, M_n)\) and
\[
 (1 - f^* f)v \equiv 0
\]
(because then \(u\) may be defined by \(u(\eta) \overset{\text{def}}{=} v(\eta|_F)\)). Let \(G \overset{\text{def}}{=} \{\omega \in \Omega: \|f(\omega)\| = 1\}(\in \mathfrak{M})\). It follows from \(\text{(9)}, \text{(4)}\) and \(\text{(3)}\) that
\[
 \lambda(G) > 0.
\]
Since we deal with (finite-dimensional) matrices, we see that
\[
 \forall \omega \in G: N(I - f(\omega)^* f(\omega)) \neq \{0\}.
\]
Now we consider a multifunction \(\Psi\) on \(\Omega\) which assigns to each \(\omega \in \Omega\) the kernel of \(I - f(\omega)^* f(\omega)\). Equipping the set of all linear subspaces of \(\mathbb{C}^n\) with the Effros-Borel structure (see \(\text{[2]}\) or §6 in Chapter V in \(\text{[13]}\) and Appendix there), we conclude that \(\Psi\) is measurable (this is a kind of folklore; it may also be simply deduced e.g. from a combination of Proposition 2.4 in \(\text{[4]}\) and Corollary A.18 in \(\text{[18]}\)). So, it follows from Effros’ theory that there exist measurable functions \(h_1, h_2, \ldots: \Omega \rightarrow \mathbb{C}^n\) such that the set \(\{h_k(\omega): k \geq 1\}\) is a dense subset of \(\Psi(\omega)\) for each \(\omega \in \Omega\) (to convince of that, consult e.g. subsection A.16 of Appendix in \(\text{[19]}\)). We infer from \(\text{(10)}\) that there is \(k \geq 1\) such that the set \(D \overset{\text{def}}{=} \{\omega \in G: h_k(\omega) \neq 0\}\) has positive measure \(\lambda\). Finally, we define \(v: \Omega \rightarrow M_n\) as follows: for \(\omega \in D, v(\omega)\) is the matrix which corresponds (in the canonical basis of \(\mathbb{C}^n\)) to a linear operator
\[
 C^n \ni \xi \mapsto \frac{\langle \xi, h_k(\omega) \rangle}{\|h_k(\omega)\|^2} h_k(\omega) \in \mathbb{C}^n
\]
(where \(\langle \cdot, \cdot \rangle\) denotes the standard inner product in \(\mathbb{C}^n\), and \(v(\omega) = 0\) otherwise. It is readily seen that \(v\) is measurable and bounded. What is more, since \(\lambda(D) > 0\), we see that \(v\) is a non-zero element of \(L^\infty(\Omega, \lambda, M_n)\). Finally, \(\text{(9)}\) holds, because \(h_k(\omega) \in \Psi(\omega) = N(I - f^* f(\omega))\) for each \(\omega\). This completes the proof. \(\square\)

Now we are ready to give

**Proof of Theorem 2.3** It is clear that (b) is followed by (a). Now assume (a) holds. Let collections \(\{\alpha_j\}_{j \in J}\) and \(\{\nu_j\}_{j \in J}\) and a \(\ast\)-isomorphism
\[
 \Phi: \mathfrak{A} \rightarrow \prod_{j \in J} L^\infty(D_{\alpha_j}, m_{\alpha_j}, M_{\nu_j})
\]
be as in Proposition 2.6. Define \((f_j)_{j \in J} \in \mathfrak{L} \overset{\text{def}}{=} \prod_{j \in J} L^\infty(D_{\alpha_j}, m_{\alpha_j}, M_{\nu_j})\) as \(\Phi(T)\). We infer from Proposition 2.6 that for any \(j \in J,\)
\[
 m_{\alpha_j}(\{x \in D_{\alpha_j}: \|f_j(x)\| < 1\}) = 1.
\]
We put \(W_{j,n} = \{x \in D_{\alpha_j}: 1 - 2^{1-n} \leq \|f_j(x)\| < 1 - 2^{-n}\}\) and let \(w_{j,n} \in L^\infty(D_{\alpha_j}, m_{\alpha_j}, M_{\nu_j})\) be (constantly) equal to the unit \(\nu_j\times\nu_j\) matrix on \(W_{j,n}\) and 0 off \(W_{j,n}\). Observe that \(w_n \overset{\text{def}}{=} (w_{j,n})_{j \in J}\) is a central projection in \(\mathfrak{L}\) and \(\sum_{n=1}^\infty w_n = 1\)
(thanks to (12)). What is more, it follows from the definition of the sets $W_{j, n}$’s that $\|\Phi(T)w_n\| \leq 1 - 2^{-n}$ for any $n \geq 1$. Thus, it remains to define $Z_n$ as $\Phi^{-1}(w_n)$ to finish the proof. □

For simplicity, let us introduce

2.7. Definition. A partition (in $\mathfrak{A}$) is an arbitrary collection $\{Z_s\}_{s \in S}$ of mutually orthogonal projections such that $\sum_{s \in S} Z_s = I$ and $Z_s \in \mathfrak{J}(\mathfrak{A})$ for any $s \in S$.

In the sequel we shall need a strengthening of Theorem 2.1 stated below. Since its proof is a slight modification of the argument used in the proof of Theorem 2.1 we skip it and leave it to the reader.

2.8. Corollary. Let $\Lambda$ be a countable infinite set of indices and $\{a_\lambda : \lambda \in \Lambda\}$ be a set of positive real numbers such that

$$\sup_{\lambda \in \Lambda} a_\lambda = 1. \tag{13}$$

For $T \in \mathfrak{A}$, the following conditions are equivalent:

(a) $\|T\xi\| < \|\xi\|$ for every non-zero vector $\xi \in \mathcal{H}$;
(b) there exists a partition $\{Z_\lambda\}_{\lambda \in \Lambda}$ such that $\|TZ_\lambda\| \leq a_\lambda$ for every $\lambda \in \Lambda$.

3. Algebra of affiliated operators

The aim of this part is to show the following result.

3.1. Theorem. Let $\mathfrak{A}$ be finite and type $I$, and $\{c_\lambda\}_{\lambda \in \Lambda}$ be a countable and unbounded set of positive real numbers. For any operator $T$ in $\mathcal{H}$ the following conditions are equivalent:

(a) $T \in \text{Aff}(\mathfrak{A})$;
(b) there is $S \in \mathfrak{A}$ and a partition $\{Z_\lambda\}_{\lambda \in \Lambda}$ for which $T = \sum_{\lambda \in \Lambda} c_\lambda S Z_\lambda$.

To make the above result more precise (and understood), let us introduce the following

3.2. Definition. Let $\{Z_\lambda\}_{\lambda \in \Lambda}$ be a partition and $\{S_\lambda\}_{\lambda \in \Lambda}$ be any collection of operators in $\mathfrak{A}$. An operator $\sum_{\lambda \in \Lambda} S_\lambda Z_\lambda$ is defined as follows:

$$D\left(\sum_{\lambda \in \Lambda} S_\lambda Z_\lambda\right) = \left\{\xi \in \mathcal{H} : \sum_{\lambda \in \Lambda} \|S_\lambda Z_\lambda \xi\|^2 < \infty\right\}$$

and $(\sum_{\lambda \in \Lambda} S_\lambda Z_\lambda)\xi = \sum_{\lambda \in \Lambda} (S_\lambda Z_\lambda)\xi$ (notice that $S_\lambda Z_\lambda \xi = Z_\lambda S_\lambda \xi$ and thus the vectors $S_\lambda Z_\lambda \xi, \lambda \in \Lambda, \xi \in \mathcal{H}$, are mutually orthogonal).

The following simple result will find many applications in the sequel.

3.3. Lemma. Let $(Z_\lambda)_{\lambda \in \Lambda}$ be a partition. Denote by $\mathcal{H}_\lambda$ the range of $Z_\lambda$. Then there exists a unitary operator $U : \mathcal{H} \to \bigoplus_{\lambda \in \Lambda} \mathcal{H}_\lambda$ such that for any collection $\{S_\lambda\}_{\lambda \in \Lambda} \subset \mathfrak{A}$,

$$U \left(\sum_{\lambda \in \Lambda} S_\lambda Z_\lambda\right)U^{-1} = \bigoplus_{\lambda \in \Lambda} S_\lambda |_{\mathcal{H}_\lambda} : \mathcal{H}_\lambda \to \mathcal{H}_\lambda.$$  

Proof. For each $\xi \in \mathcal{H}$, it is enough to define $U\xi$ as $\oplus_{\lambda \in \Lambda} Z_\lambda \xi$. □

Now we list only the most basic consequences of Lemma 3.3. Below $\{Z_\lambda\}_{\lambda \in \Lambda}$ is a partition in $\mathfrak{A}$, $\{S_\lambda\}_{\lambda \in \Lambda}$ is an arbitrary collection of operators in $\mathfrak{A}$ and $\mathcal{E}$ stands for the linear span of $\bigcup_{\lambda \in \Lambda} \mathcal{R}(Z_\lambda)$. Notice that $\mathcal{E}$ is dense in $\mathcal{H}$.

(\Sigma 1) $\sum_{\lambda \in \Lambda} S_\lambda Z_\lambda$ is closed;
(\Sigma 2) $\mathcal{E} \subset D(\sum_{\lambda \in \Lambda} S_\lambda Z_\lambda)$ and $\mathcal{E}$ is a core of $\sum_{\lambda \in \Lambda} S_\lambda Z_\lambda$;
(\Sigma 3) $(\sum_{\lambda \in \Lambda} S_\lambda Z_\lambda)^* = \sum_{\lambda \in \Lambda} S_\lambda^* Z_\lambda$;
(\Sigma 4) $\sum_{\lambda \in \Lambda} S_\lambda Z_\lambda \in \text{Aff}(\mathfrak{A})$;
(Σ5) if \( S_\lambda = c_\lambda S \) with \( c_\lambda > 0 \) for each \( \lambda \in \Lambda \), then \( \sum_{\lambda \in \Lambda} S_\lambda Z_\lambda \) is self-adjoint (resp. non-negative; normal) iff \( S \) is so.

In the proof of Theorem 3.1 we shall make use of a certain transformation which assigns to any closed densely defined operator a contraction. In the existing literature there are at least two such transformations. The first was studied e.g. by Kaufman [7] and it associates with every closed densely defined operator \( T \) the operator \( T(I + T^* T)^{-\frac{1}{2}} \). The second, quite similar to the first, is the so-called \( b \)-transform introduced in [12] and given by \( b(T) = T(I + |T|)^{-1} \). We will use the following properties of the latter transform.

3.4. Lemma. Let \( T \) and \( T_s \), \( s \in S \), be closed densely defined operators in \( \mathcal{H} \) and \( \mathcal{H}_s \), respectively. Then:

(b1) the \( b \)-transform establishes a one-to-one correspondence between the set of all closed densely defined operators in \( \mathcal{H} \) and the set of all bounded operators \( S \) on \( \mathcal{H} \) such that \( \|S\| < \|\xi\| \) for each non-zero vector \( \xi \in \mathcal{H} \); the inverse transform is given by \( S \mapsto ub(S) \overset{\text{def}}{=} S(I - |S|)^{-1} \).

(b2) \( T \) is bounded iff \( \|b(T)\| < 1 \); conversely, if \( S \in \mathcal{B}(\mathcal{H}) \) and \( \|S\| < 1 \), then \( \text{ub}(S) \in \mathcal{B}(\mathcal{H}) \);

(b3) \( T \in \text{Aff}(\mathfrak{A}) \iff b(T) \in \mathfrak{A} \);

(b4) \( b(UTU^{-1}) = UB(T)U^{-1} \) for any unitary operator \( U : \mathcal{H} \to \mathcal{K} \);

(b5) \( \text{b}(\bigoplus_{s \in S} T_s) = \bigoplus_{s \in S} b(T_s) \).

Below we use the \( b \) - and \( \text{ub} \)-transforms also as complex-valued functions defined on \( \mathbb{C} \), given by appropriate analogous formulas.

Proof of Theorem 3.1. Property (Σ4) shows that (a) is implied by (b). Now assume that \( T \in \text{Aff}(\mathfrak{A}) \). Then \( b(T) \in \mathfrak{A} \) and \( \|b(T)\| < \|\xi\| \) for each \( \xi \neq 0 \) (see (b1) and (b3)). Using Corollary 2.3 with \( a_\lambda \overset{\text{def}}{=} b(c_\lambda) = \frac{c_\lambda}{1 + c_\lambda} \), we obtain a partition \( \{Z_\lambda\}_{\lambda \in \Lambda} \) such that \( \|b(T)Z_\lambda\| \leq a_\lambda < 1 \). We now infer from (b2) that there exist operators \( S_\lambda \in \mathcal{B}(\mathcal{H}), \lambda \in \Lambda \), such that

\[
(14) \quad b(S_\lambda) = b(T)Z_\lambda.
\]

We can express \( S_\lambda \) directly as \( S_\lambda = (b(T)Z_\lambda(I - |b(T)Z_\lambda|))^{-1} \) and this formula clearly implies that \( S_\lambda \in \mathfrak{A} \). It is a well-known property of the functional calculus for self-adjoint (bounded) operators that \( \|ub(A)\| = ub(\|A\|) = \frac{A}{1 + A} \) for any non-negative operator \( A \) of norm less than 1. We shall apply this property for \( A = |b(T)S_\lambda| \)
We have:

\[
(15) \quad S_\lambda Z_\lambda = b(T)Z_\lambda^2(I - |b(T)Z_\lambda|)^{-1} = b(T)Z_\lambda(I - |b(T)Z_\lambda|)^{-1} = S_\lambda
\]

and

\[
(16) \quad \|S_\lambda\| = \|b(T)Z_\lambda(I - |b(T)Z_\lambda|)^{-1}\| = \|b(T)Z_\lambda(I - |b(T)Z_\lambda|)^{-1}\| = \|ub(|b(T)Z_\lambda|)\| = ub(\|b(T)Z_\lambda\|) \leq ub(a_\lambda) = c_\lambda.
\]

Define \( S \overset{\text{def}}{=} \sum_{\lambda \in \Lambda} \frac{1}{c_\lambda} S_\lambda \). From (15) and (16) we infer that the series converges in the strong operator topology. Consequently, \( S \in \mathfrak{A} \). Moreover, \( c_\lambda S Z_\lambda = S_\lambda Z_\lambda \). In order to prove that \( T = \sum_{\lambda \in \Lambda} c_\lambda S Z_\lambda \), it is enough to show that \( b(T) = b(\sum_{\lambda \in \Lambda} c_\lambda S Z_\lambda) \). Using Lemma 3.3, a unitary operator \( U \) and subspaces \( \mathcal{H}_\lambda \) appearing there, properties (b4) and (b5) formulated in Lemma 3.4, and (14), we
get
\[
\begin{align*}
    b\left( \sum_{\lambda \in \Lambda} c_{\lambda} S \lambda Z \right) &= U^{-1} \left( \bigoplus_{\lambda \in \Lambda} b(c_{\lambda} S|_{\mathcal{H}_{\lambda}}) \right) U = U^{-1} \left( \bigoplus_{\lambda \in \Lambda} b(S_{\lambda}|_{\mathcal{H}_{\lambda}}) \right) U \\
    &= U^{-1} \left( \bigoplus_{\lambda \in \Lambda} b(T_{\lambda}|_{\mathcal{H}_{\lambda}}) \right) U = \bigoplus_{\lambda \in \Lambda} b(T)|_{\mathcal{H}_{\lambda}} = b(T)
\end{align*}
\]
and we are done.

As a first application of Theorem 3.1 we obtain

3.5. Corollary. Let \( \mathfrak{A} \) be finite and type I, and \( \Lambda \) defined as \( \{ \nu = (\nu_1, \ldots, \nu_k) : \nu_1, \ldots, \nu_k \geq 1 \} \). For any collection \( T_1, \ldots, T_k \in \text{Aff}(\mathfrak{A}) \) there exist a partition \( \{ Z_{\nu} \}_{\nu \in \Lambda} \) in \( \mathfrak{A} \) and operators \( S_1, \ldots, S_k \in \mathfrak{A} \) such that for each \( j \in \{1, \ldots, k\} \),

\[
T_j = \sum_{\nu \in \Lambda} \nu_j S_j Z_{\nu}.
\]

Proof. Using Theorem 3.1 write each \( T_j \) as \( \sum_{n=1}^{\infty} n S_j Z_n^{(j)} \) and put \( Z_{\nu} = Z_n^{(1)} \cdot \ldots \cdot Z_n^{(k)} \).

3.6. Remark. Corollary 3.5 gives an alternative proof of the Murray-von Neumann algebra theorem 10 that \( \text{Aff}(\mathfrak{A}) \) can naturally be equipped with a structure of a *-algebra provided \( \mathfrak{A} \) is finite and type I (the assumption that \( \mathfrak{A} \) is type I is superfluous; however, our proof works only in that case). Indeed, if \( T_1, \ldots, T_k \) are arbitrary members of \( \text{Aff}(\mathfrak{A}) \), and \( \{ Z_{\nu} \}_{\nu \in \Lambda} \) and \( S_1, \ldots, S_k \in \mathfrak{A} \) are as in (17), then for any polynomial \( p(x_1, \ldots, x_n) \) in \( n \) non-commuting variables we may define \( p(T_1, \ldots, T_k) \) as \( \sum_{\nu \in \Lambda} p(\nu_1 S_1, \ldots, \nu_k S_k) Z_{\nu} \). With such a definition, the linear span of \( \bigcup_{\nu \in \Lambda} \mathcal{R}(Z_{\nu}) \) is a core for each operator of the form \( p(T_1, \ldots, T_k) \). Furthermore, the representation (17) enables us to prove briefly that \( T_1 = T_2 \) provided \( T_1 \subset T_2 \). It is now easy to conclude from all these observations that \( \text{Aff}(\mathfrak{A}) \) admits a structure of a *-algebra (in particular, all algebraic laws for an algebra, such as associativity, are satisfied). We leave the details to interested readers.

4. Trace

We now turn to the concept of a center-valued trace on \( \text{Aff}(\mathfrak{A}) \). In this section \( \mathfrak{A} \) is assumed to be finite and type I. We recall that ‘+’, ‘−’ and ‘∗’ denote the binary operations in \( \text{Aff}(\mathfrak{A}) \). Our main goal is to prove all items of Theorem 1.1 apart from (tr6), which will be shown in the next section.

We begin with

4.1. Proposition. \( \text{Aff}(3(\mathfrak{A})) = 3(\text{Aff}(\mathfrak{A})) \).

Proof. Take \( T \in \text{Aff}(3(\mathfrak{A})) \subset \text{Aff}(\mathfrak{A}) \). Since \( 3(\mathfrak{A}) \) is also finite and type I, it follows from Theorem 3.1 that \( T \) has the form \( T = \sum_{n=1}^{\infty} n S_{\lambda} Z_n \) with \( S_{\lambda}, Z_n \in 3(\mathfrak{A}) \). Similarly, any \( X \in \text{Aff}(\mathfrak{A}) \) has the form \( X = \sum_{n=1}^{\infty} n Y W_n \) with \( Y \in \mathfrak{A} \) and \( W_n \in 3(\mathfrak{A}) \). Then \( SY = YS \) and it follows from Remark 3.1 that for \( \Lambda = \{ \nu = (\nu_1, \nu_2) : \nu_1, \nu_2 \geq 1 \} \) and \( Z_{\nu} = Z_\nu W_{\nu_2}, T \cdot X = \sum_{\nu \in \Lambda} \nu_1 \nu_2 S Y Z_{\nu} = \sum_{\nu \in \Lambda} \nu_2 \nu_1 Y S Z_{\nu} = X \cdot T \), which shows that \( T \in 3(\text{Aff}(\mathfrak{A})) \). In particular, \( 3(\mathfrak{A}) \subset 3(\text{Aff}(\mathfrak{A})) \).

Conversely, take \( T \in 3(\text{Aff}(\mathfrak{A})) \) of the form \( T = \sum_{n=1}^{\infty} n S_{\lambda} Z_n \) (with \( S \in \mathfrak{A} \) and \( Z_n \in 3(\mathfrak{A}) \)). Then \( S Z_n = \frac{1}{n} T \cdot Z_n \) belongs to \( 3(\text{Aff}(\mathfrak{A})) \cap \mathfrak{A} \subset 3(\mathfrak{A}) \) and thus \( S = \sum_{n=1}^{\infty} S Z_n \in 3(\mathfrak{A}) \). Another application of Theorem 3.1 (for the von Neumann algebra \( 3(\mathfrak{A}) \)) yields \( T \in \text{Aff}(3(\mathfrak{A})) \).
4.2. **Lemma.** Let \( \{Z_\lambda\}_{\lambda \in \Lambda} \) and \( \{W_\gamma\}_{\gamma \in \Gamma} \) be two partitions in \( \mathfrak{A} \) and let \( \{T_\lambda\}_{\lambda \in \Lambda} \) and \( \{S_\gamma\}_{\gamma \in \Gamma} \) be two collections of operators in \( \mathfrak{A} \) such that

\[
\sum_{\lambda \in \Lambda} T_\lambda Z_\lambda = \sum_{\gamma \in \Gamma} S_\gamma W_\gamma.
\]

Then

\[
\sum_{\lambda \in \Lambda} \text{tr}_\Lambda(T_\lambda) Z_\lambda = \sum_{\gamma \in \Gamma} \text{tr}_\Lambda(S_\gamma) W_\gamma.
\]

**Proof.** For \( P_{\lambda,\gamma} \overset{\text{def}}{=} Z_\lambda W_\gamma \), we have, by (18), \( T_\lambda P_{\lambda,\gamma} = S_\gamma P_{\lambda,\gamma} \) for any \( \lambda \in \Lambda \) and \( \gamma \in \Gamma \). Consequently, \( \text{tr}_\Lambda(T_\lambda P_{\lambda,\gamma}) = \text{tr}_\Lambda(S_\gamma P_{\lambda,\gamma}) = \text{tr}_\Lambda(P_{\lambda,\gamma}) \). So,

\[
\sum_{\lambda \in \Lambda} \sum_{\gamma \in \Gamma} \text{tr}_\Lambda(T_\lambda P_{\lambda,\gamma}) = \sum_{\lambda \in \Lambda} \sum_{\gamma \in \Gamma} \text{tr}_\Lambda(S_\gamma P_{\lambda,\gamma}) = \sum_{\gamma \in \Gamma} \sum_{\lambda \in \Lambda} \text{tr}_\Lambda(S_\gamma P_{\lambda,\gamma})
\]

and we are done. \( \square \)

Now we are ready to introduce

4.3. **Definition.** The **center-valued trace** in \( \text{Aff}(\mathfrak{A}) \) is a mapping

\[
\text{tr}_{\text{Aff}} : \text{Aff}(\mathfrak{A}) \rightarrow \mathfrak{J}(\text{Aff}(\mathfrak{A}))
\]

defined as follows. For any partition \( \{Z_\lambda\}_{\lambda \in \Lambda} \) in \( \mathfrak{A} \) and a collection \( \{S_\lambda\}_{\lambda \in \Lambda} \subset \mathfrak{A} \),

\[
\text{tr}_{\text{Aff}} \left( \sum_{\lambda \in \Lambda} S_\lambda Z_\lambda \right) = \sum_{\lambda \in \Lambda} \text{tr}_\Lambda(S_\lambda) Z_\lambda.
\]

Theorem 4.1 and Lemma 4.2 ensure us that the definition is full and correct, while Proposition 4.1 (and its proof) shows that indeed \( \text{tr}_{\text{Aff}}(T) \) belongs to \( \mathfrak{J}(\text{Aff}(\mathfrak{A})) \) for any \( T \in \text{Aff}(\mathfrak{A}) \).

For transparency, let us isolate the uniqueness part of Theorem 4.1 in the following

4.4. **Lemma.** If \( \text{tr}' : \text{Aff}(\mathfrak{A}) \rightarrow \mathfrak{J}(\text{Aff}(\mathfrak{A})) \) is a linear mapping which satisfies axioms (tr1)–(tr3) (with \( \text{tr}_{\text{Aff}} \) replaced by \( \text{tr}' \)), then \( \text{tr}' = \text{tr}_{\text{Aff}} \).

**Proof.** Fix a partition \( \{Z_n\}_{n=1}^\infty \) in \( \mathfrak{A} \) and consider the map

\[
f : \mathfrak{A} \ni S \mapsto \text{tr}' \left( \sum_{n=1}^\infty nSZ_n \right) \cdot \left( \sum_{n=1}^\infty \frac{1}{n} Z_n \right) \in \mathfrak{J}(\text{Aff}(\mathfrak{A})).
\]

It is clear that \( f \) is linear. Moreover, for any \( S_1, S_2 \in \mathfrak{A} \), using (tr2), we get

\[
f(S_1 S_2) = \text{tr}' \left( \sum_{n=1}^\infty nS_1 S_2 Z_n \right) \cdot \left( \sum_{n=1}^\infty \frac{1}{n} Z_n \right)
\]

\[
= \text{tr}' \left( \sum_{n=1}^\infty \sqrt{n} S_1 Z_n \cdot \sum_{n=1}^\infty \sqrt{n} S_2 Z_n \right) \cdot \left( \sum_{n=1}^\infty \frac{1}{n} Z_n \right)
\]

\[
= \text{tr}' \left( \sum_{n=1}^\infty \sqrt{n} S_2 Z_n \cdot \sum_{n=1}^\infty \sqrt{n} S_1 Z_n \right) \cdot \left( \sum_{n=1}^\infty \frac{1}{n} Z_n \right)
\]

\[
= \text{tr}' \left( \sum_{n=1}^\infty nS_2 S_1 Z_n \right) \cdot \left( \sum_{n=1}^\infty \frac{1}{n} Z_n \right) = f(S_2 S_1).
\]

Further, if \( S \in \mathfrak{A} \) is non-negative, then \( T \overset{\text{def}}{=} \sum_{n=1}^\infty nSZ_n \) is non-negative as well (by (Σ5)). Consequently, \( \text{tr}'(T) \) is non-negative and therefore so is \( f(S) \). Also
for $C \in \mathcal{F}(\mathfrak{A})$ we have $\sum_{n=1}^{\infty} nCZ_n \in \text{Aff}(\mathcal{F}(\mathfrak{A})) = \mathcal{F}(\text{Aff}(\mathfrak{A}))$ (cf. the proof of Proposition 4.1), thus, thanks to (tr3),

$$f(C) = \sum_{n=1}^{\infty} nCZ_n \cdot \sum_{n=1}^{\infty} \frac{1}{n}Z_n = \sum_{n=1}^{\infty} CZ_n = C.$$ 

Finally, we claim that $f(S)$ is bounded for any $S \in \mathfrak{A}$. (This will imply that $f(\mathfrak{A}) \subseteq \mathcal{F}(\text{Aff}(\mathfrak{A})) \cap \mathfrak{A} = \mathcal{F}(\mathfrak{A})$.) To see this, it is enough to assume that $S \in \mathfrak{A}$ is non-negative. Then the operator $\|S\|I - S$ is non-negative as well and hence both $f(S)$ and $f(\|S\|I - S)$ are non-negative. But $f(S) + f(\|S\|I - S) = f(\|S\|I) = \|S\|I$. Consequently, $f(S)$ is bounded, as we claimed.

As $f: \mathfrak{A} \to \mathcal{F}(\mathfrak{A})$ satisfies all axioms of the center-valued trace in $\mathfrak{A}$ (cf. e.g. Theorem 8.2.8 in [6]), we have $f = \text{tr}_\mathfrak{A}$ and consequently for each $S \in \mathfrak{A}$:

$$\text{tr}\left(\sum_{n=1}^{\infty} nSZ_n\right) = f(S) \cdot \sum_{n=1}^{\infty} nZ_n = \text{tr}_\mathfrak{A}(S) \cdot \sum_{n=1}^{\infty} nZ_n = \sum_{n=1}^{\infty} n \text{tr}_\mathfrak{A}(S)Z_n = \text{tr}_\mathfrak{A}\left(\sum_{n=1}^{\infty} nSZ_n\right).$$

Since the partition was arbitrary, an application of Theorem 5.1 completes the proof. \hfill \Box

Proof of Theorem 1.1. As we announced, property (tr6) shall be established in the next section. The linearity of $\text{tr}_{\text{Aff}}$ follows from Corollary 5.5 and the very definition of $\text{tr}_{\text{Aff}}$ (see also Remark 5.6 and Lemma 1.2). Conditions (tr1) and (tr4) are immediate consequences of (55). Property (tr3) follows from the fact that each $C \in \mathcal{F}(\text{Aff}(\mathfrak{A}))$ may be written in the form $C = \sum_{n=1}^{\infty} nWZ_n$ where $W \in \mathcal{F}(\mathfrak{A})$ and $\{Z_n\}_{n=1}^{\infty}$ is a partition (see the proof of Proposition 1.1). Further, (tr2) and (tr5) are implied by suitable properties of $\text{tr}_\mathfrak{A}$ and the way the multiplication in $\text{Aff}(\mathfrak{A})$ is defined (below we use Corollary 5.4 with $\Lambda = \{\nu = (\nu_1, \nu_2): \nu_1, \nu_2 \geq 1\}$): if $T = \sum_{\nu \in \Lambda} \nu_1SZ_\nu$ and $X = \sum_{\nu \in \Lambda} \nu_2YZ_\nu$ (with $Y \in \mathcal{F}(\mathfrak{A})$ provided $X \in \mathcal{F}(\text{Aff}(\mathfrak{A}))$), then $T \cdot X = \sum_{\nu \in \Lambda} \nu_1\nu_2SYZ_\nu$ and hence

$$\text{tr}_{\text{Aff}}(T \cdot X) = \sum_{\nu \in \Lambda} \nu_1\nu_2 \text{tr}_\mathfrak{A}(SY)Z_\nu = \sum_{\nu \in \Lambda} \nu_1\nu_2 \text{tr}_\mathfrak{A}(YS)Z_\nu = \text{tr}_{\text{Aff}}(X \cdot T);$$

and if $X \in \mathcal{F}(\text{Aff}(\mathfrak{A}))$, we get

$$\text{tr}_{\text{Aff}}(T \cdot X) = \sum_{\nu \in \Lambda} \nu_1\nu_2 \text{tr}_\mathfrak{A}(S)YZ_\nu = \sum_{\nu \in \Lambda} \nu_1 \text{tr}_\mathfrak{A}(S)Z_\nu \cdot \sum_{\nu \in \Lambda} \nu_2 YZ_\nu = \text{tr}_{\text{Aff}}(T) \cdot X.$$ 

Finally, uniqueness of $\text{tr}_{\text{Aff}}$ was already established in Lemma 5.1 and (II) is just the assertion of Proposition 4.1. \hfill \Box

It is worth noting that $\text{tr}_{\text{Aff}}(S) = \text{tr}_\mathfrak{A}(S)$ for each $S \in \mathfrak{A}$, the proof of which is left as a simple exercise.

As an immediate consequence of Theorem 1.1 we get the following

4.5. Corollary. Suppose that $\mathfrak{A}$ is finite and type I. There are no $X, Y \in \text{Aff}(\mathfrak{A})$ such that $X \cdot Y - Y \cdot X = I$. \hfill \Box

Proof. Apply the trace for both sides.

The above result for arbitrary finite von Neumann algebras was proved in [8].
### 5. Ordering

Throughout this section, $\mathfrak{A}$ continues to be finite and type I; and $\langle \cdot, \cdot \rangle$ stands for the inner product of $\mathcal{H}$. Similarly as in $C^*$-algebras, we may distinguish real part of $\text{Aff}(\mathfrak{A})$ and introduce a natural ordering in it. To this end, we introduce

#### 5.1. Definition

The real part $\text{Aff}_s(\mathfrak{A})$ of $\text{Aff}(\mathfrak{A})$ is the set of all self-adjoint operators in $\text{Aff}(\mathfrak{A})$. Additionally, we put $\mathfrak{A}_s = \mathfrak{A} \cap \text{Aff}_s(\mathfrak{A})$. For $A \in \text{Aff}_s(\mathfrak{A})$ we write $A \geq 0$ if $A$ is non-negative (that is, if $\langle A\xi, \xi \rangle \geq 0$ for each $\xi \in \mathcal{D}(A)$; or, equivalently, if the spectrum of $A$ is contained in $[0, \infty)$). For two operators $A_1, A_2 \in \text{Aff}_s(\mathfrak{A})$ we write $A_1 \leq A_2$ or $A_2 \geq A_1$ if $A_1 - A_2 \geq 0$.

The least upper bound (in $\text{Aff}_s(\mathfrak{A})$) of a collection $\{B_s\}_{s \in S} \subset \text{Aff}_s(\mathfrak{A})$ is denoted by $\sup_{s \in S} B_s$ provided it exists.

The following simple result gives another description of the ordering defined above.

#### 5.2. Lemma

Let $A$ and $B$ be arbitrary members of $\text{Aff}_s(\mathfrak{A})$.

(a) If both $A$ and $B$ are non-negative, so is $A + B$.

(b) The following conditions are equivalent:

(i) $A \leq B$;

(ii) $\langle A\xi, \xi \rangle \leq \langle B\xi, \xi \rangle$ for any $\xi \in \mathcal{D}(A) \cap \mathcal{D}(B)$.

**Proof.** All properties follow from the fact that $\mathcal{D}(A) \cap \mathcal{D}(B)$ is a core of self-adjoint operators $B - A$ and $A + B$. □

It is now readily seen that the ordering `$\leq$' in $\text{Aff}_s(\mathfrak{A})$ is reflexive, transitive and antisymmetric (which means that $A = B$ provided $A \leq B$ and $B \leq A$), and that it is compatible with the linear structure of $\text{Aff}_s(\mathfrak{A})$. Another property is established below.

#### 5.3. Proposition

If $A, B \in \text{Aff}_s(\mathfrak{A})$ are non-negative and $A \cdot B = B \cdot A$, then $A \cdot B$ is non-negative as well.

**Proof.** By Corollary 5.5 and (Σ5), we may express $A$ and $B$ in the forms $A = \sum_{\nu \in \Lambda} \nu_1 S\nu$ and $B = \sum_{\nu \in \Lambda} \nu_2 T\nu$, where $S, T \in \mathfrak{A}$ are non-negative. Moreover, we know that then

\[(19) \quad A \cdot B = \sum_{\nu \in \Lambda} \nu_1 \nu_2 ST\nu,\]

and $B \cdot A = \sum_{\nu \in \Lambda} \nu_2 \nu_1 TS\nu$. We now deduce from these connections and the assumption that $ST = TS$ and, consequently, $ST \geq 0$. Now the assertion follows from (19) and (Σ5). □

For transparency, we isolate a part of (tr6) (in Theorem 1.1) below.

#### 5.4. Lemma

Let $A = \{A_{\sigma}\}_{\sigma \in \Sigma} \in \text{Aff}_s(\mathfrak{A})$ be an increasing net (indexed by a directed set $\Sigma$), bounded above by $A \in \text{Aff}_s(\mathfrak{A})$. Then $A$ has a least upper bound in $\text{Aff}_s(\mathfrak{A})$.

**Proof.** First of all, we may and do assume that $A_{\sigma} \geq 0$ for any $\sigma \in \Sigma$. (Indeed, fixing $\sigma_0 \in \Sigma$ and putting $\Sigma' \overset{\text{def}}{=} \{\sigma \in \Sigma : \sigma \geq \sigma_0\}$, $A' \overset{\text{def}}{=} \{A_{\sigma'} : \sigma' \in \Sigma'$ with $A_{\sigma'} = A_{\sigma} - A_{\sigma_0}$ and $A' \overset{\text{def}}{=} A - A_{\sigma_0}$, it is easy to verify that $A'$ is an increasing net of non-negative operators upper bounded by $A'$, and $\sup_{\sigma \in \Sigma} A_{\sigma} = A_{\sigma_0} + \sup_{\sigma' \in \Sigma'} A_{\sigma'}$.)

Using Theorem 5.1 and (Σ5), we may express $A$ in the form $A = \sum_{n=1}^{\infty} nBZ_n$ where $B \in \mathfrak{A}$ is non-negative and $\{Z_n\}_{n=1}^{\infty}$ is a partition in $\mathfrak{A}$. Fix $k \geq 1$. It follows from Proposition 5.3 that the operators $(A - A_{\sigma}) \cdot Z_k$ and $A_{\sigma} \cdot Z_k$ are non-negative for any $\sigma \in \Sigma$. So, $0 \leq A_{\sigma} \cdot Z_k \leq A \cdot Z_k$. Since $A \cdot Z_k = kBZ_k$ is
bounded, we now conclude (e.g. from Lemma 5.2) that $A_\sigma \cdot Z_k$ is bounded as well. Moreover, the same argument shows that the net $\{A_\sigma \cdot Z_k\}_{\sigma \in \Sigma} \subset \mathcal{A}_s$ is increasing and upper bounded by $A \cdot Z_k \in \mathcal{A}_s$. From a classical property of von Neumann algebras we infer that this last net has a least upper bound in $\mathcal{A}_s$, say $G_k$. We now put $G = \sum_{k=1}^{\infty} G_k Z_k$. Note that $G \in \text{Aff}_s(\mathcal{A})$ (see (34) and (35)). Since $0 \leq A_\sigma \cdot Z_k \leq A \cdot Z_k = (A \cdot Z_k) Z_k$, we have $G_k = G_k Z_k (= G \cdot Z_k)$ and $A_\sigma \cdot Z_k = (A_\sigma \cdot Z_k) Z_k$, and thus $A_\sigma \cdot Z_k \leq G \cdot Z_k \leq A \cdot Z_k$ for any $\sigma \in \Sigma$ and $k \geq 1$. These inequalities imply that

$$A_\sigma \leq G \leq A \quad (\sigma \in \Sigma)$$

(because for $X \in \{A_\sigma, G, A\}$, $X = \sum_{k=1}^{\infty} (X \cdot Z_k) Z_k$ in the sense of Definition 3.2, then apply Lemma 5.2). We shall check that $G = \sup_{\sigma \in \Sigma} A_\sigma$. To this end, take an arbitrary upper bound $A' = \sum_{n=1}^{\infty} n B' Z_n'$ (where $B' \in \mathcal{A}$ is self-adjoint) of $A$. It remains to check that $G \leq A'$. In what follows, to avoid misunderstandings, `$\sup_{\mathcal{A}}$' will stand for the least upper bound in $\mathcal{A}_s$ of suitable families of bounded operators.

For arbitrary positive $n$ and $m$ we have $0 \leq A_\sigma \cdot (Z_n Z_m') \leq A' \cdot (Z_n Z_m') = m B' Z_n Z_m' \in \mathcal{A}_s$, which yields

$$G \cdot (Z_n Z_m') = G_n Z_m' = \sup_{\sigma \in \Sigma} [A_\sigma \cdot Z_n] Z_m' = \sup_{\sigma \in \Sigma} [(A_\sigma \cdot Z_n) Z_m']$$

$$= \sup_{\sigma \in \Sigma} [A_\sigma \cdot (Z_n Z_m')] \leq A' \cdot (Z_n Z_m').$$

Now, as before, it suffices to note that $X = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} (X \cdot (Z_n Z_m')) Z_n Z_m'$ for $X \in \{G, A'\}$ and then apply Lemma 5.2.

The argument presented above contains a proof of the following convenient property.

5.5. **Corollary.** If $T$ is an increasing net in $\mathcal{A}_s$ which is upper bounded in $\mathcal{A}_s$, then its least upper bounds in $\mathcal{A}_s$, and $\text{Aff}_s(\mathcal{A})$ coincide.

We need one more simple lemma.

5.6. **Lemma.** Let $T$ be any member of $\text{Aff}(\mathcal{A})$ and $\{Z_\lambda\}_{\lambda \in \Lambda}$ be a partition in $\mathcal{A}$. If $T_\lambda \overset{\text{def}}{=} T \cdot Z_\lambda$ is a bounded operator for any $\lambda \in \Lambda$, then $T_\lambda Z_\lambda = T_\lambda$ for all $\lambda \in \Lambda$ and $T = \sum_{\lambda \in \Lambda} T_\lambda Z_\lambda$.

**Proof.** Since $T_\lambda$ is bounded, we get $T_\lambda Z_\lambda = T_\lambda \cdot Z_\lambda = T \cdot Z_\lambda = T_\lambda$. Express $T$ in the form $T = \sum_{n=1}^{\infty} n S W_n$ with $S \in \mathcal{A}$ and $W_n \in \mathcal{A}(\mathcal{A})$. Then $T_\lambda = \sum_{n=1}^{\infty} T_\lambda W_n = \sum_{n=1}^{\infty} (T \cdot Z_\lambda) \cdot W_n = \sum_{n=1}^{\infty} n B(W_n Z_\lambda)$ and hence

$$\sum_{\lambda \in \Lambda} T_\lambda Z_\lambda = \sum_{\lambda \in \Lambda} \sum_{n=1}^{\infty} n B(W_n Z_\lambda) = \sum_{n=1}^{\infty} n B(W_n Z_\lambda) = \sum_{n=1}^{\infty} n B W_n = T$$

and we are done. \hfill \square

Now we are ready to give

**Proof of item (tr6) in Theorem 1.1** We already know from Lemma 5.4 and (tr1) that both $A \overset{\text{def}}{=} \sup_{\sigma \in \Sigma} A_\sigma$ and $A' \overset{\text{def}}{=} \sup_{\sigma \in \Sigma} \text{trAff}(A_\sigma)$ are well defined. As in the proof of Lemma 5.4, we may and do assume that each operator $A_\sigma$ is non-negative. As usual, we express $A$ in the form $A = \sum_{n=1}^{\infty} n B Z_n$ where $B \in \mathcal{A}$. Then, from the very definition of $\text{trAff}$ we deduce that $\text{trAff}(A) = \sum_{n=1}^{\infty} n \text{tr}(B) Z_n$. Further, the proof of Lemma 5.4 combined with Corollary 5.5 yields that

$$A = \sum_{n=1}^{\infty} [\sup_{\sigma \in \Sigma} (A_\sigma \cdot Z_n)] Z_n.$$
Consequently, \( n B Z_n = A \cdot Z_n = \sup_{\sigma \in \Sigma} (A_\sigma \cdot Z_n) \). Now the normality of \( \text{tr}_A \) implies that \( n \text{tr}_A(B)Z_n = \sup_{\sigma \in \Sigma} \text{tr}_A(A_\sigma \cdot Z_n) \). But \( \text{tr}_A(A_\sigma \cdot Z_n) = \text{tr}_A(A_\sigma) \cdot Z_n \) (see (tr5)). We claim that \( \sup_{\sigma \in \Sigma} \text{tr}_A(A_\sigma) \cdot Z_n = A' \cdot Z_n \). (To convince of that, first note that inequality ‘\( \leq \)’ is immediate. To see the reverse inequality, denote by \( A'_1 \) and \( A'_2 \), respectively, \( \sup_{\sigma \in \Sigma} (A_\sigma \cdot Z_n) \) and \( \sup_{\sigma \in \Sigma} (A_\sigma \cdot (I - Z_n)) \), and observe that \( A_n \leq A'_1 + A'_2 \) and consequently \( A' \leq A'_1 + A'_2 \), from which one infers that \( A' \cdot Z_n \leq A'_1 \cdot Z_n + A'_2 \cdot Z_n \), but \( A'_2 \leq A' \cdot (I - Z_n) \) and thus \( A'_2 \cdot Z_n = 0 \). These observation lead us to \( A' = \sum_{n=1}^\infty n \text{tr}_A(B)Z_n = \text{tr}_A(A) \).

As we noted in the introductory part, condition (tr6) is a counterpart of normality (in the terminology of Takesaki; see Definition 2.1 in Chapter V of [18]) of the center-valued trace. Thus, the question of whether it is possible to equip \( \text{Aff}(\mathfrak{A}) \) with a topology (defined naturally) with respect to which the center-valued trace \( \text{tr}_A \) is continuous naturally arises. This will be a subject of further investigations.

6. Trace-like mappings in \( \text{Aff}(\mathfrak{A}) \) and the type of \( \mathfrak{A} \)

As Proposition 1.2 shows, finite type I von Neumann algebras may be characterised (among all von Neumann algebras) as those whose (full) sets of affiliated operators admit mappings which resemble center-valued traces. The aim of the section is to prove Proposition 1.2, which we now turn to.

Proof of Proposition 1.2. First observe that if \( A \in \mathfrak{A} \), then \( \varphi(A) \) is bounded and consequently \( \varphi(A) \in \mathfrak{A} \). Indeed, it is enough to show this for non-negative \( A \in \mathfrak{A} \). Such \( A \) satisfies \( \|A\| I - A \geq 0 \), therefore \( \varphi(\|A\| I - A) = \varphi(A) \) is non-negative (by (b)). But it follows from (e) and (a) that

\[
\varphi(\|A\| I - A) = \varphi(\|A\| I) - \varphi(A) = \|A\| I - \varphi(A),
\]

which means that \( 0 \leq \varphi(A) \leq \|A\| I \) and hence \( \varphi(A) \) is bounded. As \( \varphi(A) \) commutes with each unitary operator in \( \mathfrak{A} \) (by (f)), we conclude that \( \varphi(A) \in \mathfrak{A}(\mathfrak{A}) \) for each \( A \in \mathfrak{A} \). So, \( \psi = \varphi|_{\mathfrak{A}} : \mathfrak{A} \to \mathfrak{A}(\mathfrak{A}) \) is linear (thanks to (a)) and satisfies all axioms of a center-valued trace (see (b), (d) and (e)), hence \( \mathfrak{A} \) is finite. Note also that \( \varphi(X) = \text{tr}_A(X) \) for each \( X \in \mathfrak{A} \).

Suppose that \( \mathfrak{A} \) is not type I. Then one can find a non-zero projection \( Z \in \mathfrak{A}(\mathfrak{A}) \) such that \( \mathfrak{A}_0 \overset{\text{def}}{=} \mathfrak{A} Z \) is type \( \Pi_1 \). Recall that \( \text{tr}_A|_{\mathfrak{A}_0} \) coincides with the center-valued trace \( \text{tr}_{\mathfrak{A}_0} \)

Every type \( \Pi_1 \) von Neumann algebra \( \mathfrak{M} \) has the following property: for each projection \( P \in \mathfrak{M} \) and an operator \( C \in \mathfrak{A}(\mathfrak{M}) \) such that \( 0 \leq C \leq \text{tr}_{\mathfrak{M}}(P) \) there exists a projection \( Q \in \mathfrak{M} \) for which \( Q \leq P \) and \( \text{tr}_{\mathfrak{M}}(Q) = C \) (to convince of that, see Theorem 8.4.4 and item (vii) of Theorem 8.4.3, both in [16]). Involving this property, by induction we define a sequence \( (P_n)_{n=1}^\infty \) of projections in \( \mathfrak{A}_0 \) as follows: \( P_1 \in \mathfrak{A}_0 \) is arbitrary such that \( \text{tr}_{\mathfrak{A}_0}(P_1) = \frac{1}{2} Z \); and for \( n > 1 \), \( P_n \in \mathfrak{A}_0 \) is such that \( P_n \leq Z - \sum_{k=1}^{n-1} P_k \) and \( \text{tr}_{\mathfrak{A}_0}(P_n) = \frac{1}{2^n} Z \). Observe that the projections \( P_n, n \geq 1 \), are mutually orthogonal and for any \( N \geq 1 \),

\[
\text{tr}_A \left( \sum_{k=1}^{N} 2^k P_k \right) = \sum_{k=1}^{N} 2^k \text{tr}_{\mathfrak{A}_0}(P_k) = NZ.
\]

Now for \( N \geq 0 \), put \( T_N \overset{\text{def}}{=} \sum_{k=1}^{\infty} 2^k P_k \) (the series understood pointwise, similarly as in Definition 1.2). As each \( P_k \) belongs to \( \mathfrak{A} \), we see that \( T_N \in \text{Aff}(\mathfrak{A}_0) \). What is
more, $T_N$ is non-negative and we infer from axiom (c) that

$$\varphi(T_0) = \varphi\left(\sum_{k=1}^{N} 2^k P_k\right) + \varphi(T_N) = \text{tr}\left(\sum_{k=1}^{N} 2^k P_k\right) + \varphi(T_N) = NZ + \varphi(T_N).$$

Therefore, for $\xi \in D(\varphi(T_0)) = D(\varphi(T_N))$, we get:

$$\langle \varphi(T_0)\xi, \xi \rangle = N\|Z\xi\|^2 + \langle \varphi(T_N)\xi, \xi \rangle \geq N\|Z\xi\|^2$$

(here $\langle \cdot, \cdot \rangle$ denotes the inner product in $H$). Since $N$ can be arbitrarily large, the above implies that $Z\xi = 0$ for every $\xi \in D(\varphi(T_0))$. But this is impossible, because $Z \neq 0$ and $D(\varphi(T_0))$ is dense in $H$. The proof is complete. □

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