A generalization of the topological Brauer group

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Abstract

In the present paper we introduce some generalization of the topological Brauer group $H^3_{\text{tor}}(X; \mathbb{Z})$. We hope that this construction can be transferred to algebraic geometry and would provide a basis for introducing new birational invariants of varieties.

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Introduction

The aim of this paper is to introduce a homotopy functor $\text{GBr}$ which can be viewed as a generalization of the “topological” Brauer group $\text{Br}$.

Suppose $X$ is a finite $CW$-complex. The topological Brauer group $\text{Br}(X)$ can be defined as the group of equivalence classes of locally-trivial bundles $A_k$ over $X$ with fibers $M_k(\mathbb{C})$ (for arbitrary $k \in \mathbb{N}$) with respect to the following equivalence relation:

$$A_k \sim B_l \iff A_k \otimes \text{End}(\xi_m) \cong B_l \otimes \text{End}(\eta_n)$$

for some complex vector bundles $\xi_m, \eta_n$ of rank $m, n$ respectively (in particular, $km = ln$). Roughly speaking, we can say that $\text{Br}(X)$ is the group of obstructions to the lifting of locally trivial bundles $A_k$ over $X$ with fiber the matrix algebra $M_k(\mathbb{C})$ to bundles of the form $\text{End}(\xi_k)$ for some locally trivial $\mathbb{C}^k$-bundle $\xi_k$ over $X$.

In the case of our generalized Brauer group $\text{GBr}$, instead of bundles with fibers $M_k(\mathbb{C})$, $k \in \mathbb{N}$, we consider bundles $\mathfrak{A}_{k,l}$ whose fibers are “fans” $M_{k,l}$, $k, l \in \mathbb{N}$, $(k, l) = 1$. Similarly to the usual Brauer group, we define $\text{GBr}(X)$ as the group of obstructions for the lifting of bundles $\mathfrak{A}_{k,l}$ to bundles of the form $F_{k,l}(\text{End}(\xi_k))$, where $\xi_k$ is some locally trivial $\mathbb{C}^k$-bundle over $X$. Here $F_{k,l}$ is a functor from the category of $M_k(\mathbb{C})$-bundles to the category of $M_{k,l}$-bundles over $X$; this functor is induced by the map $M_k(\mathbb{C}) \hookrightarrow M_{k,l}$ (see Subsection 2.1). Compared to the classical case, a whole new step is added to the procedure of the lifting. At first, we have to lift locally trivial bundles $\mathfrak{A}_{k,l}$ with fibers $M_{k,l}$ to bundles of the form $F_{k,l}(A_k)$ for some $A_k$. The second step essentially coincides with the lifting of bundles $A_k$ to bundles of the form $\text{End}(\xi_k)$ for some locally trivial bundle $\xi_k$, i.e. with the classical case.
According to a classical theorem of J.-P. Serre, there is an isomorphism $\text{Br}(X) \cong H^3_{\text{tors}}(X; \mathbb{Z})$ [1]. In other words, all obstructions for the lifting of locally-trivial bundles $A_k$ over $X$ with fiber $M_k(\mathbb{C})$ to bundles of the form $\text{End}(\xi_k)$ are independent of the higher-dimensional integer cohomology of dimensions greater than 3. In contrast to the classical case, the generalized Brauer group $\text{GBr}(X)$ actually depends on the higher-dimensional cohomology of $X$.

On the other hand, just as in the classical group $\text{Br}(X)$, any element of $\text{GBr}(X)$ has finite order (moreover, the order of the equivalence class represented by a fan bundle $\mathcal{A}_{k,l}$ with fiber $M_{k,l}(\mathbb{C})$ is a power of $k$).

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1 Fans of subalgebras

1.1 Definition of $M_{k,l}$

Let us fix two coprime integers $k, l$. By $M_n(\mathbb{C})$ denote the algebra of all $n \times n$ matrices over $\mathbb{C}$. Let $\mathcal{A}_{k,l}'$ be the set of all central simple subalgebras in $M_{kl}(\mathbb{C})$ that are isomorphic to $M_k(\mathbb{C})$, and for any $\alpha \in \mathcal{A}_{k,l}'$ let $M_{k,\alpha} \subset M_{kl}(\mathbb{C})$ be the corresponding subalgebra.

Definition 1. The set of matrices

$$M_{k,l}' := \bigcup_{\alpha \in \mathcal{A}_{k,l}'} M_{k,\alpha} \subset M_{kl}(\mathbb{C})$$

is called a (general) fan of $k$-subalgebras in $M_{kl}(\mathbb{C})$.

Let us give another (equivalent) description of $M_{k,l}'$.

Noether-Scolem’s theorem claims that all subalgebras in $M_{kl}(\mathbb{C})$ isomorphic to $M_k(\mathbb{C})$ are conjugate to each other. In particular, any one of them is conjugate to the so-called “standard subalgebra” $M_k(\mathbb{C}) \otimes \mathbb{C} E_l \subset M_{kl}(\mathbb{C})$, where $E_l$ is the unit $l \times l$-matrix. Hence

$$M_{k,l}' = \bigcup_{g \in \text{GL}_{kl}(\mathbb{C})} g(M_k(\mathbb{C}) \otimes \mathbb{C} E_l)g^{-1} \subset M_{kl}(\mathbb{C}).$$
Consider the matrix Grassmannian \( \text{Gr}_{k,l} := \text{PGL}_{k,l}(\mathbb{C})/ \text{PGL}_k(\mathbb{C}) \otimes \text{PGL}_l(\mathbb{C}) \) (there is a canonical bijection \( \text{Gr}_{k,l} \cong \mathbb{A}_{k,l} \), see the Appendix). Let \( \mathcal{A}_{k,l} \rightarrow \text{Gr}_{k,l} \) be the tautological \( M \)-bundle over \( \text{Gr}_{k,l} \). There is an important relation between the fan \( M'_{k,l} \) and the bundle \( \mathcal{A}_{k,l} \). In order to describe it, note that \( \mathcal{A}_{k,l} \) is equipped with the canonical embedding \( \mu' \) into the trivial bundle \( \text{Gr}_{k,l} \times M_{kl}(\mathbb{C}) \):

\[
\begin{array}{ccc}
\mathcal{A}_{k,l} & \xrightarrow{\mu'} & \text{Gr}_{k,l} \times M_{kl}(\mathbb{C}) \\
\downarrow \pi' & & \downarrow \pi \\
M'_{k,l} & \to & M_{kl}(\mathbb{C})
\end{array}
\]

where for any \( x \in \text{Gr}_{k,l} \) the fiber \((\mathcal{A}_{k,l})_x\) is embedded by \( \mu' \) into the fiber \((\text{Gr}_{k,l} \times M_{kl}(\mathbb{C}))(x) = M_{kl}(\mathbb{C})\) as the central simple subalgebra corresponding to \( x \). Consider the composite map

\[
\mathcal{A}_{k,l} \xrightarrow{\mu'} \text{Gr}_{k,l} \times M_{kl}(\mathbb{C}) \xrightarrow{p_2} M_{kl}(\mathbb{C})
\]

(here \( p_2 \) denotes the projection onto the second factor). Clearly, \( M'_{k,l} = \text{im}(p_2 \circ \mu') \subset M_{kl}(\mathbb{C}) \). Thus, we have the canonical surjective map \( \pi' := p_2 \circ \mu' : \mathcal{A}_{k,l} \rightarrow M'_{k,l} \) such that the diagram

\[
\begin{array}{ccc}
\mathcal{A}_{k,l} & \xrightarrow{\mu'} & \text{Gr}_{k,l} \times M_{kl}(\mathbb{C}) \\
\downarrow \pi' & & \downarrow \pi \\
M'_{k,l} & \to & M_{kl}(\mathbb{C})
\end{array}
\]

commutes, where the horizontal arrows are the natural embeddings.

**Remark 2.** Notice the following property of fans. Consider the fans \( M'_{k,l} \) and \( M''_{l,k} \) as contained in the same algebra \( M_{kl}(\mathbb{C}) \). Then their intersection \( M'_{k,l} \cap M''_{l,k} \) coincides with the center \( CE_{kl} \) of \( M_{kl}(\mathbb{C}) \). Indeed, this result can be proved by using Jordan’s normal form of matrices and the assumption \((k,l) = 1\).

However, because \( \text{Gr}_{k,l} \) is a noncompact space, some problems arise when we deal with the general fan \( M'_{k,l} \). Therefore we also define a unitary version of fans.

Let \( \text{tr} : M_n(\mathbb{C}) \rightarrow \mathbb{C} \) be the trace. Consider the standard Hermitian metric \( (A, B) = \text{tr}(AB^*) \) on \( M_{kl}(\mathbb{C}) \). We say that a \( k \)-subalgebra \( M_{k,\alpha} \subset M_{kl}(\mathbb{C}) \), \( \alpha \in \mathbb{A}_{k,l} \) is *unitary* if there is a unitary transformation in \( \text{Aut}(M_{kl}(\mathbb{C})) \) which takes the standard subalgebra \( M_k(\mathbb{C}) \otimes CE_l \) onto the given one \( M_{k,\alpha} \), i.e. \( M_{k,\alpha} = g(M_k(\mathbb{C}) \otimes CE_l)g^{-1} \) for some \( g \in U(kl) \subset \text{GL}_{kl}(\mathbb{C}) \). Let \( \mathbb{A}_{k,l} \) be the set of unitary \( k \)-subalgebras in \( M_{kl}(\mathbb{C}) \).

**Definition 3.** The set of matrices in \( M_{kl}(\mathbb{C}) \)

\[
M_{k,l} := \bigcup_{\alpha \in \mathbb{A}_{k,l}} M_{k,\alpha} = \bigcup_{g \in U(kl)} g(M_k(\mathbb{C}) \otimes CE_l)g^{-1} \subset M'_{k,l}
\]

is called a *(unitary)* fan.
The defined above Hermitian metric on $M_{kl}(\mathbb{C})$ can be restricted to the unitary fan and determines there a concordant family of Hermitian structures on subalgebras $M_{k,\alpha} \subset \mathbb{M}_{k,l}$.

**Remark 4.** Note that any $g \in U(kl)$ determines a unitary isomorphism from $M_{k}(\mathbb{C}) \otimes \mathbb{C}E_l$ to the subalgebra $M_{k,\alpha} = g(M_{k}(\mathbb{C}) \otimes \mathbb{C}E_l)g^{-1}$, i.e. it preserves the Hermitian structure. This follows from the formula

$$\langle Ag^{-1}, gBg^{-1} \rangle = \text{tr}(gAg^{-1}gBg^{-1}) = \text{tr}(gABg^{-1}) = \langle A, B \rangle,$$

because for any $g \in U(kl)$ we have $g^{-1} = \overline{g}$. We see that it is sufficient to define the Hermitian metric only on $M_{k}(\mathbb{C}) \otimes \mathbb{C}E_l$, because it can be spreaded to the whole fan by $PU(kl)$-equivariance.

Let $Gr_{k,l} := PU(kl)/PU(k) \otimes PU(l)$ be the “unitary” matrix Grassmannian. Then we have the canonical bijection $Gr_{k,l} \cong A_{k,l}$.

Let $A_{k,l} \to Gr_{k,l}$ be the tautological $M_{k}(\mathbb{C})$-bundle over $Gr_{k,l}$. In the unitary version there is also a closed relation between the fan $M_{k,l}$ and the bundle $A_{k,l}$. In order to describe it notice that $A_{k,l}$ is defined together with the canonical embedding $\mu$ into the trivial bundle $Gr_{k,l} \times M_{kl}(\mathbb{C})$:

$$A_{k,l} \xrightarrow{\mu} Gr_{k,l} \times M_{kl}(\mathbb{C}) \xrightarrow{p_2} M_{kl}(\mathbb{C})$$

where for any $x \in Gr_{k,l}$ the corresponding fiber $(A_{k,l})_x$ is embedded by $\mu$ into the fiber $(Gr_{k,l} \times M_{kl}(\mathbb{C}))_x = M_{kl}(\mathbb{C})$ as a central simple subalgebra parametrized by $x$. Consider the composite map

$$A_{k,l} \xrightarrow{\mu} Gr_{k,l} \times M_{kl}(\mathbb{C}) \xrightarrow{p_2} M_{kl}(\mathbb{C})$$

(here $p_2$ is the projection onto the second factor). Obviously, $M_{k,l} = \text{im}(p_2 \circ \mu) \subset M_{kl}(\mathbb{C})$. Thus, we have the canonical surjective map $\pi := p_2 \circ \mu: A_{k,l} \to M_{k,l}$. Note that the bundle $A_{k,l}$ is equipped with the canonical Hermitian metric which is the restriction of the standard Hermitian metric from the trivial bundle $Gr_{k,l} \times M_{kl}(\mathbb{C})$ to the subbundle $A_{k,l}$, moreover the map $\pi: A_{k,l} \to M_{k,l}$ is an isometry in the natural sense.

**1.2 The topology and the algebraic structure on $\mathbb{M}_{k,l}$**

The fan $M_{k,l}'$ is a subset in the Euclidean space $M_{kl}(\mathbb{C})$. By $\mathcal{O}'$ denote the induced topology on $M_{k,l}'$. It follows immediately from the definition that $\mathcal{O}'$ is the weakest
topology on $M_{k,l}'$ such that the embedding $M_{k,l}' \hookrightarrow M_{k,l}(\mathbb{C})$ is continuous. Furthermore, for any $\alpha \in A_{k,l}'$ the embedding $M_{k,\alpha} \hookrightarrow M_{k,l}'$ is a continuous map.

The set $M_{k,l}'$ is not a $\mathbb{C}$-algebra, moreover, not even a vector space, in any natural sense. But for elements in the same $k$-subalgebra in $M_{k,l}'$ we have usual operations "·" and "+".

Since all $k$-subalgebras in $M_{k,l}'$ contain the identity element, their intersection includes the line $\mathbb{C}E_{kl}$. In particular, the vector space structures on different $k$-subalgebras are concordant to each other, because the multiplication by scalars is none other than the multiplication by the corresponding matrices from the center $\mathbb{C}E_{kl} \cong \mathbb{C}$.

**Proposition 5.** The intersection of all $k$-subalgebras in $M_{k,l}'$ is just the center $\mathbb{C}E_{kl} \subset M_{k,l}'$, where $E_{kl}$ is the unit $kl \times kl$-matrix. In the previous notation,

$$\bigcap_{\alpha \in A_{k,l}'} M_{k,\alpha} = \mathbb{C}E_{kl}.$$  

**Proof.** Clearly, $\cap_{\alpha} M_{k,\alpha}$ is a subalgebra in $M_{k,l}(\mathbb{C})$; moreover, it is invariant with respect to all automorphisms of the algebra $M_{k,l}(\mathbb{C})$. The required assertion follows from the well-known decomposition of the adjoint representation $Ad$ of $GL_n(\mathbb{C})$ into irreducible ones. □

**Remark 6.** Note that even $(M_k(\mathbb{C}) \otimes \mathbb{C}E_l) \cap (\mathbb{C}E_l \otimes M_k(\mathbb{C})) = \mathbb{C}E_{kl} \subset M_{k,l}(\mathbb{C})$. It can be proved by a direct computation with matrices.

We consider the unitary fan $M_{k,l}$ as a topological space with respect to the topology $\mathcal{O}$ induced by the embedding $M_{k,l} \subset M_{k,l}(\mathbb{C})$. All the mentioned results about general fan $M_{k,l}'$ can be transferred (after the appropriate modification) to the unitary fan $M_{k,l}$.

The proof of the following proposition is based on the specific property of the unitary case, precisely that the space $Gr_{k,l}$ is compact (note that $Gr_{k,l}'$ is a noncompact space).

**Proposition 7.** The fan $M_{k,l}$ is a closed subset in $M_{k,l}(\mathbb{C})$.

**Proof.** Suppose $X \in M_{k,l}(\mathbb{C})$, $X \notin M_{k,l}$. Define the function $d_X : Gr_{k,l} \to \mathbb{R}$ in the following way: for any $\alpha \in Gr_{k,l}$ $d_X(\alpha) = \text{“the distance between $M_{k,\alpha}$ and $X$”}$ (here we consider $M_{k,\alpha}$ as a vector subspace in the Euclidean space $M_{k,l}(\mathbb{C})$). Clearly, $d_X$ is continuous and everywhere positive. Since $Gr_{k,l}$ is a compact space, there is $\varepsilon > 0$ such that $d_X(\alpha) > \varepsilon \forall \alpha \in Gr_{k,l}$. □

Set $\hat{S} := S \cap M_{k,l}$, where $S$ is the unit sphere in $M_{k,l}(\mathbb{C})$. Let $S(A_{k,l}) \xrightarrow{S^{2n^2-1}} Gr_{k,l}$ be the spherical fibre bundle associated with $A_{k,l}$. Clearly, $\hat{S} = \pi(S(A_{k,l})) \subset M_{k,l}$.

**Corollary 8.** $\hat{S}$ is a compact subset in $M_{k,l}$.

**Proof.** $\hat{S}$ is simultaneously closed and bounded as a subset in the Euclidean space $M_{k,l}(\mathbb{C})$. □
1.3 Automorphisms of fans

**Definition 9.** An isomorphism of fans $\vartheta: M_{k,l}' \to M_{k,l}'$ is a homeomorphism of topological spaces $\{M_{k,l}', O'\}$ such that for any $k$-subalgebra $M_{k,\alpha} \subset M_{k,l}'$ its restriction $\vartheta |_{M_{k,\alpha}}$ is a $C$-algebras isomorphism with some $k$-subalgebra $M_{k,\beta} \subset M_{k,l}'$ ($\alpha, \beta \in A_{k,l}'$), i.e. for any $\alpha \in A_{k,l}'$ the corresponding diagram

$$
\begin{array}{ccc}
M_{k,l}' & \xrightarrow{\vartheta} & M_{k,l}' \\
\cup & \cong & \cup \\
M_{k,\alpha} & \xrightarrow{\cong} & M_{k,\beta}
\end{array}
$$

commutes (here $\beta = \bar{\vartheta}(\alpha)$, where $\bar{\vartheta}$ is the induced map on $A_{k,l}'$).

By $\mathcal{G}_{k,l}'$ denote the full group of automorphisms $\text{Aut}(M_{k,l}')$.

Acting on $M_{kl}(C)$, the group $\text{PGL}_{kl}(C)$ preserves the subset $M_{k,l}' \subset M_{kl}(C)$. Hence we have a group homomorphism

$$
\kappa_{k,l}': \text{PGL}_{kl}(C) \to \mathcal{G}_{k,l}'.
$$

**Proposition 10.** The homomorphism $\kappa_{k,l}'$ is injective.

**Proof.** Obviously, $Z_{\text{PGL}_{kl}(C)}(M_k(C) \otimes CE_l) = E_k \otimes \text{PGL}_l(C) \subset \text{PGL}_{kl}(C)$, where by $Z$ we denote a centralizer and by $E_k \otimes \text{PGL}_l(C)$ we denote the image of the embedding $\text{PGL}_l(C) \ni A \mapsto E_k \otimes A \in \text{PGL}_{kl}(C)$

induced by the Kronecker product of matrices. It is sufficient to prove that

$$
\bigcap_{g \in \text{GL}_{kl}(C)} g(E_k \otimes \text{PGL}_l(C))g^{-1} = \{e\}.
$$

It was proven in Proposition [5] that

$$
\bigcap_{g \in \text{GL}_{kl}(C)} g(C_E_k \otimes M_l(C))g^{-1} = C_{E_{kl}}.
$$

Hence

$$
\text{GL}_{kl}(C) \cap \bigcap_{g \in \text{GL}_{kl}(C)} g(C_E_k \otimes M_l(C))g^{-1} = \text{GL}_{kl}(C) \cap C_{E_{kl}} = C^*E_{kl}.
$$

Let us factorize the both sides by $C^*E_{kl}$. Then the right-hand side is clearly equal to

$$(\text{GL}_{kl}(C) \cap C_{E_{kl}})/C^*E_{kl} = C^*E_{kl}/C^*E_{kl} = \{e\}.$$

The input left-hand side is actually equal to

$$
\bigcap_{g \in \text{GL}_{kl}(C)} g(E_k \otimes \text{GL}_l(C))g^{-1},
$$

7
therefore when factorized by $C^*E_{kl}$, it gives us

$$
\{e\} = \left( \bigcap_{g \in \text{GL}_{kl}(\mathbb{C})} g(E_k \otimes \text{GL}_l(\mathbb{C}))g^{-1}/C^*E_{kl} \right) = \bigcap_{g \in \text{GL}_{kl}(\mathbb{C})} g(E_k \otimes \text{GL}_l(\mathbb{C}))g^{-1}/C^*E_{kl} = \bigcap_{g \in \text{GL}_{kl}(\mathbb{C})} g(E_k \otimes \text{PGL}_l(\mathbb{C}))g^{-1}.
$$

Thus, we get the desired equality. □

Consider the embedding $M_k(\mathbb{C}) \to M'_{k,l}$, which maps $M_k(\mathbb{C})$ isomorphically onto a $k$-subalgebra in $M'_{k,l}$.

**Proposition 11.** The embedding $M_k(\mathbb{C}) \to M'_{k,l}$ is functorial.

**Proof.** Any automorphism $\theta$ of a matrix $k$-subalgebra $A_k \subset M_{kl}(\mathbb{C})$ can be extended to the uniquely automorphism $\tilde{\theta}$ of the whole algebra $M_{kl}(\mathbb{C})$ such that $\tilde{\theta} |_{Z(A_k)} = \text{id}|_{Z(A_k)}$, where $Z(A_k) \cong M_I(\mathbb{C})$ is the centrilizer of the subalgebra $A_k \subset M_{kl}(\mathbb{C})$. The assignment $\theta \mapsto \tilde{\theta}$ corresponds to the group homomorphism $\text{PGL}_k(\mathbb{C}) \otimes E_l \to \text{PGL}_{kl}(\mathbb{C})$. Thus, we obtain the automorphism $\vartheta = \kappa'_{k,l}(\tilde{\theta})$ of the fan $M'_{k,l}$, where $\kappa'_{k,l} : \text{PGL}_{kl}(\mathbb{C}) \to \mathfrak{G}'_{k,l}$ is the group homomorphism as above. □

Now consider the unitary version.

**Definition 12.** An isomorphism of (unitary) fans $\vartheta : M_{k,l} \to M'_{k,l}$ is a homeomorphism of topological spaces $\{M_{k,l}, \mathcal{O}\}$ such that for any $k$-subalgebra $M_{k,\alpha} \subset M_{k,l}$ the restriction $\vartheta |_{M_{k,\alpha}}$ is a unitary (in the sense of the defined above Hermitian structure) $\mathbb{C}$-algebras isomorphism with some $k$-subalgebra $M_{k,\beta} \subset M_{k,l}$ ($\alpha, \beta \in A_{k,l}$), i.e. for any $\alpha \in A_{k,l}$ the corresponding diagram

$$
M_{k,l} \xrightarrow{\vartheta} M'_{k,l} \\
\cup \quad \cup \\
M_{k,\alpha} \cong \quad M_{k,\beta}
$$

commutes and the isomorphism $M_{k,\alpha} \cong M_{k,\beta}$ is unitary (here $\beta = \tilde{\vartheta}(\alpha)$, where $\tilde{\vartheta}$ is the induced map on $A_{k,l}$).

Note that since the restriction of the automorphism $\vartheta$ to a subalgebra $M_{k,\alpha} \subset M_{k,l}$ is a unitary map, $\vartheta$ preserves the subset $\hat{S} \subset M_{k,l}$.

By $\mathfrak{G}_{k,l}$ denote the full group of automorphisms $\text{Aut}(M_{k,l})$.

Acting on $M_{kl}(\mathbb{C})$, the group $\text{PU}(kl)$ preserves the subset $M_{k,l} \subset M_{kl}(\mathbb{C})$, moreover, it acts on it by unitary transformation. Hence we have the group homomorphism

$$
\kappa_{k,l} : \text{PU}(kl) \to \mathfrak{G}_{k,l}.
$$

The following proposition is an analog of Proposition 10.
Proposition 13. The homomorphism \( \kappa_{k,l} \) is injective.

Proof is similar to the one of Proposition 10. □

Now we want to define an appropriate topology on \( G_{k,l} \).

Recall that \( M_{kl}(\mathbb{C}) \) is an Euclidean space, by \( \rho \) denote the corresponding metric. Define the function \( d: G_{k,l} \times G_{k,l} \to \mathbb{R} \) in the following way. By definition, its value \( d(\vartheta_1, \vartheta_2) \) at a pair of automorphisms \( \vartheta_1, \vartheta_2 \in G_{k,l} \) is equal to \( \sup_{X \in \hat{S}} \rho(\vartheta_1(X), \vartheta_2(X)) \).

According to Corollary 8 \( \hat{S} \) is a compact space, therefore this function is well-defined. Notice the formula
\[
\sup_{X \in \hat{S}} \rho(\vartheta_1(X), \vartheta_2(X)) = \sup_{X \neq 0} \frac{\rho(\vartheta_1(X), \vartheta_2(X))}{\rho(X, X)}
\]
(in the right-hand side \( X \) runs over all nonzero elements of the fan \( M_{k,l} \)), which follows from the definition of a fan automorphism: by definition, for any \( \vartheta \in \text{Aut}(M_{k,l}) \) its restriction to arbitrary subalgebra \( M_{k,\alpha} \subset M_{k,l} \) is linear.

Proposition 14. The function \( d \) is a metric on \( G_{k,l} \).

Proof. If \( d(\vartheta_1, \vartheta_2) = 0 \), then \( \rho(\vartheta_1(X), \vartheta_2(X)) = 0 \) \( \forall X \in \hat{S} \), but since the restrictions of \( \vartheta_1, \vartheta_2 \) to any subalgebra \( M_{k,\alpha} \subset M_{k,l} \) are linear maps, the automorphisms \( \vartheta_1, \vartheta_2 \) can be recovered by their restrictions to the set \( \hat{S} \). The other properties of a metric can easily be checked. □

The group \( G_{k,l} \) we shall consider as a topological group with respect to the topology of the metric space \( (G_{k,l}, d) \).

Clearly, \( G_{k,l} \) is a Hausdorff topological space.

Proposition 15. The natural maps: multiplication \( G_{k,l} \times G_{k,l} \to G_{k,l}, (g_1, g_2) \mapsto g_1g_2 \), inversion \( G_{k,l} \to G_{k,l}, g \mapsto g^{-1} \) and action \( G_{k,l} \times M_{k,l} \to M_{k,l}, (g, x) \mapsto gx \) are continuous.

Proof is clear. □

Proposition 16. The defined above map \( \kappa_{k,l} \) embeds \( \text{PU}(kl) \) into \( G_{k,l} \) as a topological subgroup (we consider the standard topology on \( \text{PU}(kl) \)).

Proof. Clearly, the induced topology on \( \text{PU}(kl) \subset G_{k,l} \) coincides with the uniform topology, but it is well known that the uniform topology coincides with the standard one. □

We see that the group \( \text{PU}(kl) \) can be considered as a topological subgroup in \( G_{k,l} \) consisting of all those automorphisms of the fan \( M_{k,l} \) that are restrictions of unitary automorphisms of the matrix algebra \( M_{kl}(\mathbb{C}) \).
Remark 17. Note that the group embedding $\text{PU}(k) \hookrightarrow \mathfrak{G}_{k,l}$ corresponding to an unitary embedding $M_k(\mathbb{C}) \hookrightarrow M_{k,l}$ is continuous. Indeed, it is the composite homomorphism $\text{PU}(k) \overset{\kappa_{k,l}}{\hookrightarrow} \text{PU}(kl) \overset{\delta_{k,l}}{\hookrightarrow} \mathfrak{G}_{k,l}$ and the required assertion follows from the previous proposition.

Remark 18. It seems to be rather reliable that there is a group embedding $\mathfrak{G}_{k,l} \overset{\kappa_{k,l}}{\hookrightarrow} \mathfrak{G}_{k,l}'$. Indeed, this result we shall prove below by means of some kind of bundles.

1.4 The space of $k$-frames in $M_{kl}(\mathbb{C})$

Definition 19. A $k$-frame in $M_{kl}(\mathbb{C})$ is an ordered collection of $k^2$ linearly independent matrices $\{\alpha_{i,j} \mid 1 \leq i, j \leq k\}$ such that $\alpha_{i,j}\alpha_{m,n} = \delta_{jm}\alpha_{i,n}$ for all $1 \leq i, j, m, n \leq k$, where $\delta_{jm}$ is the Kronecker delta-symbol.

Clearly, every such $k$-frame is a basis in a certain (uniquely determined by the frame) $k$-subalgebra in $M_{kl}(\mathbb{C})$. We have the “standard” $k$-frame $\{e_{i,j} \mid 1 \leq i, j \leq k\}$, where $e_{i,j} := E_{ij} \otimes E_l$ is the Kronecker product of a “matrix unit” $E_{ij}$ of order $k$ with the unit $l \times l$ matrix $E_l$. It is a frame in the subalgebra $M_k(\mathbb{C}) \otimes \mathbb{C} E_l \subset M_{kl}(\mathbb{C})$.

Applying Noether-Scolem’s theorem, we see that all $k$-frames in $M_{kl}(\mathbb{C})$ are conjugate to each other. Thus, the set of $k$-frames in $M_{kl}(\mathbb{C})$ is the homogeneous space $\text{Fr}_{k,l}' := \text{PGL}_{kl}(\mathbb{C})/E_k \otimes \text{PGL}_l(\mathbb{C})$. It is the total space of the following principal bundle

$$\text{Fr}_{k,l}' \overset{\text{PGL}_k(\mathbb{C})}{\longrightarrow} \text{Gr}_{k,l}'$$

over the matrix Grassmannian $\text{Gr}_{k,l}'$. The fiber of the bundle over $x \in \text{Gr}_{k,l}'$ consists of all $k$-frames contained in the $k$-subalgebra $M_{k,x} \subset M_{kl}(\mathbb{C})$ parametrized by $x$. The tautological $M_k(\mathbb{C})$-bundle $\mathcal{A}_{k,l}$ over $\text{Gr}_{k,l}'$ is associated with this principal bundle.

There is also a unitary analog of this notion. Clearly, the set of all unitary $k$-frames in $M_{kl}(\mathbb{C})$ is the homogeneous space $\text{Fr}_{k,l} := \text{PU}(kl)/E_k \otimes \text{PU}(l)$. It is the total space of the following principal bundle

$$\text{Fr}_{k,l} \overset{\text{PU}(k)}{\longrightarrow} \text{Gr}_{k,l}$$

over the matrix Grassmannian $\text{Gr}_{k,l}$. The fiber of this bundle over $x \in \text{Gr}_{k,l}$ consists of all unitary $k$-frames contained in the $k$-subalgebra $M_{k,x} \subset M_{kl}(\mathbb{C})$ parametrized by $x$ (points $x \in \text{Gr}_{k,l}$ correspond to unitary $k$-subalgebras in $M_{kl}(\mathbb{C})$). The tautological $M_k(\mathbb{C})$-bundle $\mathcal{A}_{k,l}$ over $\text{Gr}_{k,l}$ is associated with this principal bundle.

Remark 20. Recall that the group $\text{U}(n)$ ($\text{PU}(n)$) is a strong deformation retract of the group $\text{GL}_n(\mathbb{C})$ ($\text{PGL}_n(\mathbb{C})$ respectively). This implies that there are homotopy equivalences $\text{Gr}_{k,l} \simeq \text{Gr}_{k,l}'$ and $\text{Fr}_{k,l} \simeq \text{Fr}_{k,l}'$. 

10
1.5 The action $\nu_{k,l}: \mathcal{G}_{k,l} \times \text{Fr}_{k,l} \rightarrow \text{Fr}_{k,l}$

The group $\mathcal{G}_{k,l}$ acts upon $\mathbb{M}_{k,l}$ and therefore upon the space of $k$-frames $\text{Fr}_{k,l}$. We shall consider $\text{Fr}_{k,l}$ as a topological space with respect to the standard topology of the homogeneous space $\text{PU}(kl)/E_k \otimes \text{PU}(l)$.

**Proposition 21.** The natural map $\nu_{k,l}: \mathcal{G}_{k,l} \times \text{Fr}_{k,l} \rightarrow \text{Fr}_{k,l}$ is continuous.

**Proof.** We have $k^2$ continuous maps $\tilde{\alpha}_{m,n}: \text{Fr}_{k,l} \rightarrow \mathbb{M}_{k,l}$, $1 \leq m, n \leq k$ which to any $k$-frame $\{\alpha_{i,j} \mid 1 \leq i, j \leq k\}$ assign its $m, n$-component $\alpha_{m,n} \in \mathbb{M}_{k,l}$. Clearly, the product map $\tilde{\alpha} := \tilde{\alpha}_{1,1} \times \ldots \tilde{\alpha}_{k,k}: \text{Fr}_{k,l} \rightarrow \mathbb{M}_{k,l}^{k^2}$ is injective. Obviously, the topology on $\text{Fr}_{k,l}$ coincides with the one induced by the inclusion $\text{Fr}_{k,l} \hookrightarrow \mathbb{M}_{k,l}^{k^2} \subset M_{kl}(\mathbb{C})^{k^2}$. Now the required assertion follows from the commutative diagram

$$
\begin{array}{ccc}
\mathcal{G}_{k,l} \times \text{Fr}_{k,l} & \xrightarrow{\nu_{k,l}} & \text{Fr}_{k,l} \\
\text{id}_{\mathcal{G}_{k,l}} \times \tilde{\alpha} & \downarrow & \tilde{\alpha} \\
\mathcal{G}_{k,l} \times \mathbb{M}_{k,l}^{k^2} & \longrightarrow & \mathbb{M}_{k,l}^{k^2},
\end{array}
$$

where $\mathcal{G}_{k,l} \times \mathbb{M}_{k,l}^{k^2} \rightarrow \mathbb{M}_{k,l}^{k^2}$ is the diagonal action. □

**Remark 22.** Note that the action $\nu_{k,l}: \mathcal{G}_{k,l} \times \text{Fr}_{k,l} \rightarrow \text{Fr}_{k,l}$ preserves the structure of the fiber bundle $\text{Fr}_{k,l} \xrightarrow{\text{PU}(k)} \text{Gr}_{k,l}$. Since the topology on $\text{Gr}_{k,l}$ coincides with the quotient topology, the corresponding action $\mathcal{G}_{k,l} \times \text{Gr}_{k,l} \rightarrow \text{Gr}_{k,l}$ is continuous. Furthermore, naturally acting on the direct product $\text{Gr}_{k,l} \times \mathbb{M}_{k,l}$, the group $\mathcal{G}_{k,l}$ preserves the tautological $M_{k}(\mathbb{C})$-bundle $\mathcal{A}_{k,l} \subset \text{Gr}_{k,l} \times \mathbb{M}_{k,l}$. Since the standard topology on $\mathcal{A}_{k,l}$ coincides with the topology induced by the natural embedding $\mathcal{A}_{k,l} \hookrightarrow \text{Gr}_{k,l} \times \mathbb{M}_{k,l}$, the induced action $\mathcal{G}_{k,l} \times \mathcal{A}_{k,l} \rightarrow \mathcal{A}_{k,l}$ is also continuous. Note that the diagram (in which the horizontal arrows are the considered actions)

$$
\begin{array}{ccc}
\mathcal{G}_{k,l} \times \mathcal{A}_{k,l} & \longrightarrow & \mathcal{A}_{k,l} \\
\text{id} \times \pi & \downarrow & \pi \\
\mathcal{G}_{k,l} \times \mathbb{M}_{k,l} & \longrightarrow & \mathbb{M}_{k,l}
\end{array}
$$

commutes by definition.

Suppose $\text{St}(x) \subset \mathcal{G}_{k,l}$ is the stabilizer of $x \in \text{Fr}_{k,l}$, then $\text{St}(y) = g \text{St}(x)g^{-1}$ for $y = gx$.

**Proposition 23.**

$$
\bigcap_{y \in \text{Fr}_{k,l}} \text{St}(y) = \bigcap_{g \in \mathcal{G}_{k,l}} g \text{St}(x)g^{-1} = \{e\} \subset \mathcal{G}_{k,l}.
$$

In other words, the action $\nu_{k,l}: \mathcal{G}_{k,l} \times \text{Fr}_{k,l} \rightarrow \text{Fr}_{k,l}$ is effective.

**Proof.** Suppose $h \in \bigcap_{g \in \mathcal{G}_{k,l}} g \text{St}(x)g^{-1}$, then $h$ keeps all points of $\text{Fr}_{k,l}$ fixed $\Rightarrow$ $h$ keeps all $k$-frames in $\mathbb{M}_{k,l}$ fixed $\Rightarrow$ for any $k$-subalgebra $A_k \subset \mathbb{M}_{k,l}$ the restriction $h \mid_{A_k}$ is the identity homomorphism $\text{id}_{A_k} \Rightarrow h$ is the identity map of the fan $\mathbb{M}_{k,l} \Rightarrow h = e$. □

Thus, the group $\mathcal{G}_{k,l}$ is a group of homeomorphisms of $\text{Fr}_{k,l}$. 

11
2 Fan bundles

In this section we study some properties of fan bundles of a special form and apply the obtained results to fans themselves. In particular, we construct a bifunctor generalizing the tensor product of matrix algebras.

Starting from this section we shall denote the general and the unitary fan by the same symbol $\mathbb{M}_{k,l}$. The same is valid for the groups $\mathfrak{G}'_{k,l}$, $\mathfrak{G}_{k,l}$, the matrix Grassmanians $\text{Gr}'_{k,l}$, $\text{Gr}_{k,l}$, and the tautological bundles $\mathcal{A}'_{k,l}$, $\mathcal{A}_{k,l}$ over them. As a rule, we deal with the general (nonunitary) case, but any obtained result has an obvious unitary analog. In order to obtain this analog one should equip any considered bundle with an Hermitian metric and consider as morphisms only bundle maps preserving such a metric, in particular, one has to substitute the group $\text{PU}(n)$ for $\text{PGL}_n(\mathbb{C})$. We shall not indicate a unitary analog if it is clear.

2.1 Fan bundles associated with algebra bundles

Let $X$ be a finite CW-complex. Let $A_k$, $C_l$ be bundles over $X$ with fibers $M_k(\mathbb{C})$ and $M_l(\mathbb{C})$ respectively, $B_{kl} := A_k \otimes C_l$ their tensor product which is an $M_{kl}(\mathbb{C})$-bundle over $X$. In this subsection we study fan bundles associated with algebra bundles of the form $A_k \otimes C_l$.

Consider the $\text{Gr}_{k,l}$-bundle $\text{Gr}_{k,l}(B_{kl})$ over $X$ associated with $B_{kl}$.

Proposition 24. $\text{Gr}_{k,l}(B_{kl})$ is isomorphic to the trivial bundle $X \times \text{Gr}_{k,l}$.

Proof. By $\text{Prin}(B_{kl})$, $\text{Prin}(A_k)$, $\text{Prin}(C_l)$ denote the principal bundles corresponding to the bundles $B_{kl}$, $A_k$, $C_l$, respectively. We have the embedding $\text{Prin}(A_k) \otimes \text{Prin}(C_l) \hookrightarrow \text{Prin}(B_{kl})$ corresponding to the morphism of bundles $A_k \times C_l \supset B_{kl}$. The groups $\text{PGL}_{kl}(\mathbb{C})$, $\text{PGL}_k(\mathbb{C})$, $\text{PGL}_l(\mathbb{C})$ act freely upon the principal bundles $\text{Prin}(B_{kl})$, $\text{Prin}(A_k)$, $\text{Prin}(C_l)$, respectively. Moreover, the following diagram

$$
\begin{array}{ccc}
\text{PGL}_{kl}(\mathbb{C}) \times \text{Prin}(B_{kl}) & \rightarrow & \text{Prin}(B_{kl}) \\
\uparrow & & \uparrow \\
(\text{PGL}_k(\mathbb{C}) \otimes \text{PGL}_l(\mathbb{C})) \times (\text{Prin}(A_k) \otimes \text{Prin}(C_l)) & \rightarrow & \text{Prin}(A_k) \otimes \text{Prin}(C_l)
\end{array}
$$

commutes, i.e. the action is compatible with the embedding of the subbundles. Let $\text{Prin}(B_{kl})/(\text{Prin}(A_k) \otimes \text{Prin}(C_l))$ be the homogeneous space of cosets $\{g \cdot (\text{Prin}(A_k) \otimes \text{Prin}(C_l)) \mid g \in \text{PGL}_{kl}(\mathbb{C})\}$. Consider $\text{PGL}_{kl}(\mathbb{C})$-equivariant map

$$
\text{Prin}(B_{kl})/(\text{Prin}(A_k) \otimes \text{Prin}(C_l)) \rightarrow \text{PGL}_{kl}(\mathbb{C})/(\text{PGL}_k(\mathbb{C}) \otimes \text{PGL}_l(\mathbb{C}))
$$

$$
g \cdot (\text{Prin}(A_k) \otimes \text{Prin}(C_l)) \mapsto g \cdot (\text{PGL}_k(\mathbb{C}) \otimes \text{PGL}_l(\mathbb{C})), \quad g \in \text{PGL}_{kl}(\mathbb{C}).
$$

Clearly, it gives us a trivialization of the bundle $\text{Gr}_{k,l}(B_{kl})$. □
Let $A_k \otimes C_l = B_{kl} \xrightarrow{\varphi} B'_{kl} = A'_k \otimes C'_l$ be an isomorphism (which is not necessarily identity on the base) which takes the subbundle $A_k \otimes 1_{C_l} \subset B_{kl}$ isomorphically to the subbundle $A'_k \otimes 1_{C'_l} \subset B'_{kl}$ (hence it maps $1_{A_k} \otimes C_l \subset B_{kl}$ to $1_{A'_k} \otimes C'_l \subset B'_{kl}$). The proof of the previous proposition implies the following corollary.

**Corollary 25.** There is a unique $\text{PGL}_{kl}(\mathbb{C})$-equivariant isomorphism $\text{Gr}_{k,l}(B_{kl}) \rightarrow \text{Gr}_{k,l}(B'_{kl})$ mapping the section of the bundle $\text{Gr}_{k,l}(B_{kl}) \rightarrow X$ corresponding to the subbundle $A_k \otimes 1_{C_l} \subset B_{kl}$ to the section of the bundle $\text{Gr}_{k,l}(B'_{kl}) \rightarrow X$ corresponding to the subbundle $A'_k \otimes 1_{C'_l} \subset B'_{kl}$. Moreover, this isomorphism preserves the trivializations $\text{Gr}_{k,l}(B_{kl}) \cong X \times \text{Gr}_{k,l}$ and $\text{Gr}_{k,l}(B'_{kl}) \cong X \times \text{Gr}_{k,l}$ defined in the previous proposition and therefore has the form $\tilde{\varphi} \times \tilde{\varphi}$, where $\varphi: X \rightarrow X$, $\tilde{\varphi}: \text{Gr}_{k,l} \rightarrow \text{Gr}_{k,l}$ are the induced maps.

**Proof** easily follows from the previous proposition. We just note that $\varphi$ induces a $\text{PGL}_{kl}(\mathbb{C})$-equivariant isomorphism $\tilde{\varphi}: \text{Prin}(B_{kl}) \rightarrow \text{Prin}(B'_{kl})$ of principal bundles. □

Consider the sequence

$$A_k \hookrightarrow A_k \otimes C_l \xrightarrow{R_{k,l}} \mathfrak{A}_{k,l},$$

where $R_{k,l}$ is the functor which to an $M_{kl}(\mathbb{C})$-bundle assigns the corresponding $M_{k,l}$-bundle.

In connection with the fan bundle $\mathfrak{A}_{k,l}$ Proposition 24 can be treated in the following way. Let $\sigma: X \rightarrow \text{Gr}_{k,l}(B_{kl})$ be the section corresponding to the subbundle $A_k \otimes 1_{C_l} \subset B_{kl}$. The group $\text{PGL}_{kl}(\mathbb{C})$ acts naturally on $\text{Gr}_{k,l}(B_{kl})$ and therefore on the set of sections $X \rightarrow \text{Gr}_{k,l}(B_{kl})$ conjugate with $\sigma$. Furthermore, there is an $M_k(\mathbb{C})$-subbundle in $\mathfrak{A}_{k,l}$ corresponding to any such a section. We see that any $k$-subalgebra in $\mathfrak{A}_{k,l}$ belongs to a unique such a subbundle, moreover, Proposition 24 establishes a one-to-one correspondence between the set of such subbundles and points of $\text{Gr}_{k,l}$. In particular, we see that the structure of the set of such subbundles does not depend on the choice of $C_l$. Now the following proposition does not seem suprisingly.

**Proposition 26.** The isomorphism class of the bundle $\mathfrak{A}_{k,l}$ does not depend on the choice of $C_l$.

**Proof.** Set $B_{kl} := A_k \otimes C_l$, $B'_{kl} := A'_k \otimes C'_l$, where $C'_l$ is some other $M_l(\mathbb{C})$-bundle over $X$. Consider the associated $\text{Gr}_{k,l}$-bundles $\text{Gr}_{k,l}(B_{kl}) \rightarrow X$, $\text{Gr}_{k,l}(B'_{kl}) \rightarrow X$. Let $\tilde{\mathfrak{A}}_{k,l} \rightarrow \text{Gr}_{k,l}(B_{kl})$ be the tautological $M_k(\mathbb{C})$-bundle. Note that it is a subbundle in

$$\text{Gr}_{k,l}(B_{kl}) \times B_{kl} = \text{Gr}_{k,l} \times B_{kl}$$

whose restriction to any fiber $\text{Gr}_{k,l}$ of the bundle $\text{Gr}_{k,l}(B_{kl}) \rightarrow X$ is the tautological bundle $A_k \rightarrow \text{Gr}_{k,l}$. Let $\tilde{C}'_{l,k} \rightarrow \text{Gr}_{k,l}(B'_{kl})$ be the $M_l(\mathbb{C})$-subbundle which is complementary to the tautological one. Note that it is a subbundle in

$$\text{Gr}_{k,l}(B'_{kl}) \times B'_{kl} = \text{Gr}_{k,l} \times B'_{kl}$$
such that $\widetilde{A}_{k,l} \otimes \widetilde{C}_{l,k} = Gr_{k,l} \times B_{kl} \}$ (where $\widetilde{A}_{k,l}$ is determined by $B_{kl}'$ in the same way as $\widetilde{A}_{k,l}$ by $B_{kl}$) and such that its restriction to any fiber $Gr_{k,l}$ of the bundle $Gr_{k,l}(B_{kl}') \to X$ is the subbundle $C_{l,k} \to Gr_{k,l}$ in $Gr_{k,l}(B_{kl}) \mid_{Gr_{k,l}} = Gr_{k,l} \times M_{kl}(C)$ complementary to the tautological one, i.e. such that $A_{k,l} \otimes C_{l,k} = Gr_{k,l} \times M_{kl}(C)$. According to Proposition 24 we have

$$Gr_{k,l}(B_{kl}) \cong X \times Gr_{k,l} \cong Gr_{k,l}(B_{kl}').$$

Identify $Gr_{k,l}(B_{kl})$ with $Gr_{k,l}(B_{kl}')$ by means of the $PGL_{kl}(C)$-equivariant isomorphism $Gr_{k,l}(B_{kl}) \cong Gr_{k,l}(B_{kl}')$ taking the section $\sigma$ of $Gr_{k,l}(B_{kl}) \to X$ corresponding to the subbundle $A_k \otimes 1_{C_l} \subset B_{kl}$ to the section $\sigma'$ of $Gr_{k,l}(B_{kl}') \to X$ corresponding to the subbundle $A_k \otimes 1_{C_l'} \subset B_{kl}'$. This enables us to consider $\widetilde{A}_{k,l}$ and $\widetilde{C}_{l,k}$ as bundles over the same base $Gr_{k,l}(B_{kl})$. Consider their tensor product $\widetilde{A}_{k,l} \otimes \widetilde{C}_{l,k}$. It has the following properties: first, for the section $\sigma$: $X \to Gr_{k,l}(B_{kl})$ as above we have

$$\sigma^*(\widetilde{A}_{k,l} \otimes \widetilde{C}_{l,k}) = A_k \otimes C_l' = B_{kl}'$$

(here we use the identification $A_k \otimes 1_{C_l} \subset B_{kl}$ with $A_k \otimes 1_{C_l'} \subset B_{kl}'$); second, its restriction to any fiber $Gr_{k,l}$ of $Gr_{k,l}(B_{kl}') \to X$ is the trivial bundle $A_{k,l} \otimes C_{l,k} \cong Gr_{k,l} \times M_{kl}(C)$. There is the canonical map

$$\varphi: A_{k,l} \times C_{l,k} \to M_{kl}(C)$$

given by the tensor product of matrix algebras, i.e. $\varphi_x: (A_{k,l})_x \times (C_{l,k})_x \to (A_{k,l})_x \otimes (C_{l,k})_x = M_{kl}(C)$ is the canonical bilinear map, for any $x \in Gr_{k,l}$. The map $\varphi$ determines the canonical trivialization of $A_{k,l} \otimes C_{l,k}$. Since the map $\varphi$ is canonical, it can be extended to bundles. For example, it defines the canonical maps

$$\widetilde{\varphi}: \widetilde{A}_{k,l} \times \widetilde{C}_{l,k} \to B_{kl}$$

and

$$\widetilde{\varphi}': \widetilde{A}_{k,l} \times \widetilde{C}_{l,k} \to B_{kl}'$$

($im(\widetilde{\varphi}') = B_{kl}'$ because of $\sigma^*(\widetilde{A}_{k,l} \otimes \widetilde{C}_{l,k}') = A_k \otimes C_l' = B_{kl}'$ as we have noticed above).

This implies that we can identify $\widetilde{A}_{k,l} \otimes \widetilde{C}_{l,k}$ with $Gr_{k,l} \times B_{kl}'$.

Note that there is the commutative diagram

$$\begin{array}{ccc}
\widetilde{A}_{k,l} & \longrightarrow & Gr_{k,l} \times B_{kl} \\
\downarrow{\tilde{z}} & & \downarrow{p_2} \\
A_{k,l} & \longrightarrow & B_{kl}
\end{array}$$

corresponding to the diagram

$$\begin{array}{ccc}
A_{k,l} & \longrightarrow & Gr_{k,l} \times M_{kl}(C) \\
\downarrow{\pi} & & \downarrow{} \\
M_{k,l} & \longrightarrow & M_{kl}(C)
\end{array}$$
after the restriction to a fiber of \( \text{Gr}_{k,l}(B_{kl}) \rightarrow X \) (here \( \mathfrak{a}_{k,l} := R_{k,l}(B_{kl}) \)). From the other hand, it follows from the identification \( \tilde{\mathcal{A}}_{k,l} \otimes \tilde{C}_{l,k} = \text{Gr}_{k,l} \times B'_{kl} \) that there exists the commutative diagram

\[
\begin{array}{ccc}
\mathfrak{a}_{k,l} & \xrightarrow{\pi} & \text{Gr}_{k,l} \times B'_{kl} \\
\downarrow & & \downarrow \text{id} \\
\mathfrak{a}_{k,l} & \xrightarrow{p_2} & B'_{kl}.
\end{array}
\]

This implies that \( R_{k,l}(B_{kl}) = \mathfrak{a}_{k,l} = R_{k,l}(B'_{kl}) \).

Note that actually we have constructed a canonical isomorphism \( \tilde{\mathcal{A}}_{k,l} \cong \tilde{\mathcal{A}}'_{k,l} \). \( \square \)

**Remark 27.** Let us give some remarks concerning to the proof. 1) First, let us clarify the role of a fixed section \( \sigma \). Consider the matrix Grassmannian \( \text{Gr}_{k,l} \) with the bundles \( \mathcal{A}_{k,l} \) and \( \mathcal{C}_{l,k} \) over it. Suppose we have fixed a point \( x \in \text{Gr}_{k,l} \). Suppose \( M_{kl}(\mathbb{C}) = (\mathcal{A}_{k,l})_x \otimes (\mathcal{C}_{l,k})_x \) (in particular, \( (\mathcal{A}_{k,l})_x \) is identified with the subalgebra \( M_k(\mathbb{C}) \otimes \mathbb{C}E_l \subset M_{kl}(\mathbb{C}) \), and \( (\mathcal{C}_{l,k})_x \) with the subalgebra \( \mathbb{C}E_k \otimes M_l(\mathbb{C}) \subset M_{kl}(\mathbb{C}) \)). Then \( \mathcal{A}_{k,l} \otimes \mathcal{C}_{l,k} = \text{Gr}_{k,l} \times M_{kl}(\mathbb{C}) \) and the decomposition \( M_{kl}(\mathbb{C}) = (\mathcal{A}_{k,l})_x \otimes (\mathcal{C}_{l,k})_x \) is fixed.

2) Note that the obtained result is closely connected with the following fact: the fans \( \mathbb{M}_{k,l} \subset M_{kl}(\mathbb{C}) \supset M_{k,l} \) intersect only in the center \( \mathbb{C}E_{kl} \). This implies that the fan bundles \( R_{k,l}(B_{kl}) \) and \( R_{k,l}(B_{kl}) \) are actually independent of each other.

**Remark 28.** Note that actually we have proved that there is a canonical isomorphism \( R_{k,l}(B_{kl}) \cong R_{k,l}(B'_{kl}) \) taking the subbundle \( \mathcal{A}_k \otimes 1_{\mathcal{C}_l} \subset R_{k,l}(B_{kl}) \) to the subbundle \( \mathcal{A}_k \otimes \mathcal{C}'_{l,k} \subset R_{k,l}(B'_{kl}) \) (and covering the \( \text{PGL}_{kl}(\mathbb{C}) \)-equivariant isomorphism \( \text{Gr}_{k,l}(B_{kl}) \cong \text{Gr}_{k,l}(B'_{kl}) \) identifying the sections \( \sigma \) and \( \sigma' \)). Therefore we can choose the bundle \( R_{k,l}(\mathcal{A}_k \otimes \tilde{\mathcal{M}}_l) \), where \( \tilde{\mathcal{M}}_l \) is a trivial \( M_l(\mathbb{C}) \)-bundle over \( X \), as a canonical representative of the isomorphism class of fan bundles of the form \( R_{k,l}(\mathcal{A}_k \otimes \mathcal{C}_l) \) (where \( A_k \) is fixed, and \( C_l \) is a variable bundle). Note that this fan bundle is glued by the same \( \text{PGL}_k(\mathbb{C}) \)-cocycle as the algebra bundle \( \mathcal{A}_k \) (we use the natural representation \( \text{PGL}_k(\mathbb{C}) \rightarrow \text{PGL}_{kl}(\mathbb{C}) \subset \mathfrak{g}_{k,l} \)).

We have already defined the functor \( R_{k,l} \) which to any \( M_{kl}(\mathbb{C}) \)-bundle \( B_{kl} \) assigns the corresponding \( \mathbb{M}_{k,l} \)-bundle. This functor corresponds to the group embedding \( \text{PGL}_{kl}(\mathbb{C}) \leftarrow \mathfrak{g}_{k,l} \). Now define the functor \( \mathbf{F}_{k,l} \): \( A_k \mapsto A_k \otimes \tilde{\mathcal{M}}_l \mapsto R_{k,l}(A_k \otimes \tilde{\mathcal{M}}_l) \) which to any \( M_k(\mathbb{C}) \)-bundle \( A_k \) assigns the corresponding \( \mathbb{M}_{k,l} \)-bundle \( \mathbf{F}_{k,l}(A_k) \). Clearly, it corresponds to the composite map \( \text{PGL}_k(\mathbb{C}) \rightarrow \text{PGL}_{kl}(\mathbb{C}) \rightarrow \mathfrak{g}_{k,l} \). Moreover, it follows from the previous results that any fan bundle of the form \( R_{k,l}(A_k \otimes \mathcal{C}_l) \) is canonically isomorphic to \( \mathbf{F}_{k,l}(A_k) \).

**Corollary 29.** For any \( M_{kl}(\mathbb{C}) \)-bundle \( B_{kl} \) containing \( A_k \) as a matrix algebra subbundle (i.e. such that for any point \( x \in X \) the fiber \( A_k)_x \) is a central simple subalgebra in \( (B_{kl})_x \) we have \( R_{k,l}(B_{kl}) \cong \mathbf{F}_{k,l}(A_k) \).
Proof. Any $M_{kl}(\mathbb{C})$-bundle $B_{kl}$ containing $A_k$ as above can be represented in the form $A_k \otimes C_l$, where $C_l$ is the subbundle in $B_{kl}$ consisting of the centralizers for fibers of $A_k$. It follows from Proposition [26] that $R_{k,l}(B_{kl}) \cong F_{k,l}(A_k)$. □

Now let us introduce the following notation. Given an $M_k(\mathbb{C})$-bundle $A_k$ we denote by $\Gamma(X; A_k)$ the algebra of its global sections. By $GL_l(A_k)$ denote the group $GL_l(\Gamma(X; A_k))$ of all invertible $l \times l$-matrices with entries from the ring $\Gamma(X; A_k)$. In other words, $GL_l(A_k) = GL_1(M_l(\Gamma(X; A_k))) = GL_1(\Gamma(X; A_k \otimes \tilde{M}_l))$. We have the group monomorphism

$$GL_1(\Gamma(X; \mathbb{C})) \hookrightarrow GL_l(A_k), \quad \alpha \mapsto \alpha E_{kl},$$

identifying $GL_1(\Gamma(X; \mathbb{C}))$ with the center of $GL_l(A_k)$, where $GL_1(\Gamma(X; \mathbb{C})) = \Gamma(X; \mathbb{C}^\times)$ and $E_{kl}$ denotes the unit $kl \times kl$-matrix. By $PGL_l(A_k)$ denote the corresponding quotient group.

Clearly, there is a natural action of the group $PGL_l(A_k)$ on the fan bundle $F_{k,l}(A_k)$ such that its restriction to any fiber (isomorphic to $M_{kl}(\mathbb{C})$) coincides with the natural action $PGL_{kl}(\mathbb{C}) = Aut(M_{kl}(\mathbb{C}))$ upon the fan $M_{k,l} \subset M_{kl}(\mathbb{C})$. This shows that the fan bundle $F_{k,l}(A_k)$ can be written as

$$F_{k,l}(A_k) = \bigcup_{g \in GL_l(A_k)} g(A_k \otimes \mathbb{C}E_l)g^{-1}.$$ 

Note that on fibers this representation coincides with the representation

$$M_{k,l} = \bigcup_{g \in GL_{kl}(\mathbb{C})} g(M_k(\mathbb{C}) \otimes \mathbb{C}E_l)g^{-1}.$$ 

**Corollary 30.** Any automorphism $\lambda: A_k \to A_k$ of the bundle $A_k$ (not necessarily identity on the base) can be extended to a unique $PGL_l(A_k)$-equivariant automorphism $\tilde{\lambda}: F_{k,l}(A_k) \to F_{k,l}(A_k)$ of the corresponding fan bundle.

**Remark 31.** We mean that $\tilde{\lambda}$ is $PGL_l(A_k)$-equivariant with respect to the map $\lambda_*: \Gamma(X; A_k) \to \Gamma(X; A_k)$ induced by $\lambda$.

**Proof.** In the obvious notation $\tilde{\lambda} = R_{k,l}(\lambda \otimes \text{id}_{\tilde{M}_l})$. □

Let us also describe a unitary analog. Suppose an $M_k(\mathbb{C})$-bundle $A_k$ is equipped with an Hermitian metric. By $U_l(A_k)$ denote the group $U_l(\Gamma(X; A_k))$ of unitary $l \times l$-matrices with entries from the ring $\Gamma(X; A_k)$.

We have the group monomorphism

$$U_1(\Gamma(X; \mathbb{C})) \hookrightarrow U_l(A_k), \quad \alpha \mapsto \alpha E_{kl}$$

identifying $U_1(\Gamma(X; \mathbb{C}))$ with the center of $U_l(A_k)$. By $PU_l(A_k)$ denote the corresponding quotient group.
Here by $F_{k,l}$ denote the functor which to any $M_k(\mathbb{C})$-bundle equipped with an Hermitian metric assigns the corresponding bundle with fiber the unitary fan $\mathbb{M}_{k,l}$. Clearly, we have the natural action of the group $\text{PU}_l(A_k)$ on the bundle $F_{k,l}(A_k)$ such that it coincides with the natural action of $\text{PU}(kl)$ on the unitary fan $\mathbb{M}_{k,l} \subset M_{kl}(\mathbb{C})$ after the restriction to any fiber. This shows that the fan bundle $F_{k,l}(A_k)$ has the form

$$\bigcup_{g \in U(A_k)} g(A_k \otimes \mathbb{C}E_l) g^{-1},$$

which coincides with the representation

$$\mathbb{M}_{k,l} = \bigcup_{g \in U(kl)} g(M_k(\mathbb{C}) \otimes \mathbb{C}E_l) g^{-1}$$
on fibers.

There is also a unitary analog of Corollary 30.

**Corollary 32.** Any unitary automorphism $\lambda : A_k \to A_k$ of the bundle $A_k$ (not necessarily identity on the base) can be extended to a unique $\text{PU}_l(A_k)$-equivariant automorphism $\tilde{\lambda} : F_{k,l}(A_k) \to F_{k,l}(A_k)$ of the corresponding bundle of unitary fans.

**Proof** is rather clear; note only that Remark 4 is useful in it. □

**Remark 33.** One may ask a question: what happen if we replace the section (in the notation of Proposition 26) $\sigma : X \to \text{Gr}_{k,l}(B_{kl})$ by some other section $\varsigma$ of the bundle $\text{Gr}_{k,l}(B_{kl}) \to X$? Then, generally speaking, $\varsigma^*(\tilde{A}_{k,l}) = A'_k$ is not isomorphic to $A_k = \sigma^*(\tilde{A}_{k,l})$. From the other hand, it is clear that $\mathfrak{A}_{k,l} = F_{k,l}(A'_k) = F_{k,l}(A_k) = \tilde{\pi}(\tilde{A}_{k,l})$.

What is the relation between the bundles $A_k$ and $A'_k$? After study the stable theory we will see that to the stable equivalence class of the bundle $\mathfrak{A}_{k,l} = F_{k,l}(A_k)$ corresponds the class of bundles obtained from $A_k$ by taking the tensor product with the core of an arbitrary Fab over $X$. Consider the case of a trivial bundle $\mathfrak{A}_{k,l}$. Obviously, the bundle $\mathfrak{A}_{k,l}$ is trivial if and only if $\tilde{A}_{k,l} \cong X \times A_{k,l}$. In this case $A_k = \sigma^*(A_{k,l})$ is the core of some Fab.

In the next proposition we prove the mentioned fact.

**Proposition 34.** The bundle $F_{k,l}(A_k)$ is trivial if and only if $A_k$ is the core of some $\text{Fab}$ $(A_k, \mu, \tilde{M}_{kl})$.

**Remark 35.** Consider the assignment $(A_k, \mu, \tilde{M}_{kl}) \mapsto A_k$ to a Fab its core as a functor which we denote by $T_{k,l}$. We see that the functors $T_{k,l}$ and $F_{k,l}$ determine the exact sequence of pointed sets:

$$\{\text{isomorphism classes of FABs of the form } (A_k, \mu, \tilde{M}_{kl})\}$$

$$\mapsto \{\text{isomorphism classes of } M_k(\mathbb{C})\text{-bundles}\}$$

$$\mapsto \{\text{isomorphism classes of } \mathbb{M}_{k,l}\text{-bundles}\}$$

(all bundles are considered over the same base $X$).
Proof. Suppose $F_{k,l}(A_k) \cong X \times M_{k,l}$. By $\mu: A_k \hookrightarrow X \times M_{kl}(\mathbb{C})$ denote the composite map $A_k \hookrightarrow F_{k,l}(A_k) \cong X \times M_{k,l} \subset X \times M_{kl}(\mathbb{C})$. It is clear that the triple $(A_k, \mu, \widetilde{M}_{kl})$ is a FAB over $X$.

Conversely, suppose $A_k$ is the core of a FAB $(A_k, \mu, \widetilde{M}_{kl})$. Using Corollary 29 we see that $F_{k,l}(A_k) \cong R_{k,l}(\widetilde{M}_{kl})$ is a trivial bundle. □

2.2 The coincidence $\text{Aut}(A_{k,l}) = \text{Aut}(M_{k,l})$

Notice that the canonical map $\pi: A_{k,l} \to M_{k,l}$ is $\text{PGL}_{kl}(\mathbb{C})$-equivariant. In this subsection we show that the map $\pi: A_{k,l} \to M_{k,l}$ identifies $\text{Aut}(A_{k,l})$ with $G_{k,l} = \text{Aut}(M_{k,l})$.

The important consequence of this result is that the theory of locally trivial $M_{k,l}$-bundles coincides with the theory of locally-trivial $A_{k,l}$-bundles.

Proposition 36. Suppose $\vartheta: M_{k,l} \to M_{k,l}$ is a fan automorphism. Then there is a unique automorphism $\tilde{\vartheta}: A_{k,l} \to A_{k,l}$ of the core of the tautological FAB over $\text{Gr}_{k,l}$ such that $\tilde{\vartheta} |_{(A_{k,l})x} = \vartheta |_{M_{k,x}}$ for any $x \in \text{Gr}_{k,l}$, where $M_{k,x}$ is the $k$-subalgebra in $M_{k,l}$ parametrized by $x \in \text{Gr}_{k,l}$. Moreover, the assignment $\vartheta \mapsto \tilde{\vartheta}$ is a group homomorphism $G_{k,l} \to \text{Aut}(A_{k,l})$.

Proof. To a given $\vartheta: M_{k,l} \to M_{k,l}$ we assign the continuous map $\overline{\vartheta}: \text{Gr}_{k,l} \to \text{Gr}_{k,l}$ such that $\overline{\vartheta} |_{M_{k,x}}: M_{k,x} \cong (A_{k,l})_x \mapsto \vartheta(x)$ for any $x \in \text{Gr}_{k,l}$. Then $\tilde{\vartheta}: A_{k,l} \to A_{k,l}$ can be defined as a unique map such that the diagram

\[ \begin{array}{ccc}
\text{Gr}_{k,l} \times M_{k,l} & \xrightarrow{\overline{\vartheta} \times \vartheta} & \text{Gr}_{k,l} \times M_{k,l} \\
\downarrow & & \downarrow \\
A_{k,l} & \xrightarrow{\tilde{\vartheta}} & A_{k,l}
\end{array} \]

commutes (the vertical arrows are the canonical embeddings). Obviously, $\tilde{\vartheta}$ is a continuous map. □

Proposition 37. Conversely, suppose $\tau: A_{k,l} \to A_{k,l}$ is an automorphism of the tautological bundle over $\text{Gr}_{k,l}$. Then there is a unique fan automorphism $\hat{\tau}: M_{k,l} \to M_{k,l}$ such that $\hat{\tau} |_{M_{k,x}} = \tau |_{(A_{k,l})x}$ for any $x \in \text{Gr}_{k,l}$, if we identify $M_{k,x}$ with $(A_{k,l})_x$ by the canonical map $\pi: A_{k,l} \to M_{k,l}$. Moreover, the assignment $\tau \mapsto \hat{\tau}$ determines a group homomorphism $\text{Aut}(A_{k,l}) \to G_{k,l}$.

Proof. First, recall that there is the canonical surjective map

$\pi: A_{k,l} \to M_{k,l}$

induced by the map $\text{Gr}_{k,l} \to \text{pt}$ and identifying $M_{k,x}$ with $(A_{k,l})_x$. We will prove that for a given $\tau$ there is a map

$\hat{\tau}: M_{k,l} \to M_{k,l}$
such that the diagram
\[
\begin{array}{ccc}
A_{k,l} & \xrightarrow{\tau} & A_{k,l} \\
\downarrow{\pi} & & \downarrow{\pi} \\
M_{k,l} & \xrightarrow{\hat{\tau}} & M_{k,l}
\end{array}
\]  \hspace{1cm} (2)

commutes.

Note that the commutative diagram
\[
\begin{array}{ccc}
A_{k,l} & \xrightarrow{p \times \pi} & Gr_{k,l} \times M_{k,l} \\
\downarrow{\pi} & & \downarrow{p_2} \\
M_{k,l} & \xrightarrow{\mu} & M_{k,l}
\end{array}
\]  \hspace{1cm} (3)

where \(p_2\) is the projection onto the second factor, determines the canonical trivialization on the trivial \(M_{k,l}\)-bundle over \(Gr_{k,l}\). We equip the trivial bundle \(F_{k,l}(A_{k,l})\) with this trivialization: \(\mu: F_{k,l}(A_{k,l}) \cong Gr_{k,l} \times M_{k,l}\). Note that the diagram
\[
\begin{array}{ccc}
A_{k,l} & \xrightarrow{i} & F_{k,l}(A_{k,l}) \\
\downarrow{p \times \pi} & & \downarrow{\mu} \\
Gr_{k,l} \times M_{k,l}
\end{array}
\]

commutes, where \(i\) is the natural embedding.

Let \(\tilde{\tau}: F_{k,l}(A_{k,l}) \to F_{k,l}(A_{k,l})\) be a \(PGL_k(C)\)-equivariant isomorphism, as in Corollary 39. Note that the map \(\tilde{\tau}: M_{k,l} \to M_{k,l}\) making diagram (2) commutative is unique (if it exists). This implies that under our choice of trivialization \(\mu: F_{k,l}(A_{k,l}) \cong Gr_{k,l} \times M_{k,l}\) the restriction of \(\tilde{\tau}\) to any fiber of the trivial bundle \(Gr_{k,l} \times M_{k,l} \to Gr_{k,l}\) does not depend on \(x \in Gr_{k,l}\), i.e. \(\tilde{\tau}\) has the form \(\bar{\tau} \times \hat{\tau}\), where \(\bar{\tau}: M_{k,l} \to M_{k,l}\) is the required fan automorphism. \(\Box\)

**Remark 38.** Note that the map \(\pi: A_{k,l} \to M_{k,l}\) canonically identifies the principal \(PGL_k(C)\)-bundle associated with the tautological bundle \(A_{k,l} \to Gr_{k,l}\) with the space \(Fr_{k,l}\) of \(k\)-frames in \(M_{k,l}\). Furthermore, the map \(\tau: A_{k,l} \to A_{k,l}\) induces a \(PGL_k(C)\)-equivariant map of the principal bundles \(\bar{\tau}: Fr_{k,l} \to Fr_{k,l}\) (covering the map \(\bar{\tau}: Gr_{k,l} \to Gr_{k,l}\) induced by \(\tau\) on the base). Clearly, \(\bar{\tau}: Fr_{k,l} \to Fr_{k,l}\) is the map of frame spaces corresponding to \(\bar{\tau}: M_{k,l} \to M_{k,l}\). The map \(\bar{\tau}\) is continuous, because it corresponds to the continuous map \(\bar{\tau}\) of frame spaces and we consider the metric topologies on the spaces \(Fr_{k,l}\) and \(M_{k,l}\).

**Corollary 39.** The composition \(\vartheta \mapsto \bar{\tau}\), where \(\vartheta = \bar{\vartheta}\), is the identity map \(\text{id}_{\mathcal{G}_{k,l}}\). The composition \(\tau \mapsto \bar{\vartheta}\), where \(\bar{\tau} = \vartheta\), is the identity map \(\text{id}_{\text{Aut}(A_{k,l})}\). The canonical map \(\pi: A_{k,l} \to M_{k,l}\) is \(\text{Aut}(A_{k,l}) = \mathcal{G}_{k,l}\)-equivariant.

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Proof. Recall that the map \( \pi : A_{k,l} \to M_{k,l} \) is surjective. Hence it is sufficient to observe that the maps \( \vartheta \) and \( \hat{\tau} \) both make the diagrams

\[
\begin{array}{ccc}
A_{k,l} & \xrightarrow{\vartheta=\tau} & A_{k,l} \\
\downarrow \pi & & \downarrow \pi \\
M_{k,l} & \xrightarrow{\vartheta} & M_{k,l}
\end{array} \quad \begin{array}{ccc}
A_{k,l} & \xrightarrow{\vartheta=\tau} & A_{k,l} \\
\downarrow \pi & & \downarrow \pi \\
M_{k,l} & \xrightarrow{\tilde{\vartheta}=\tilde{\tau}} & M_{k,l}
\end{array}
\]

commutative. Furthermore, the maps \( \tau \) and \( \tilde{\vartheta} \) both make the diagrams

\[
\begin{array}{ccc}
M_{k,l} & \xrightarrow{\vartheta=\tau} & M_{k,l} \\
\downarrow \pi & & \downarrow \pi \\
A_{k,l} & \xrightarrow{\tau} & A_{k,l}
\end{array} \quad \begin{array}{ccc}
M_{k,l} & \xrightarrow{\vartheta=\tau} & M_{k,l} \\
\downarrow \pi & & \downarrow \pi \\
A_{k,l} & \xrightarrow{\tilde{\vartheta}} & A_{k,l}
\end{array}
\]

commutative. \( \square \)

Remark 40. Note that the proved result has some relation to the following argument. If we have an \( A_{k,l} \)-bundle (i.e. with fiber \( A_{k,l} \)) over some base \( X \), then the fiberwise application of the canonical map \( \pi : A_{k,l} \to M_{k,l} \) gives us the corresponding \( M_{k,l} \)-bundle \( A_{k,l} \) such that the total space of the tautological bundle (in the previous notation) \( \tilde{A}_{k,l} \subset \text{Gr}(A_{k,l}) \times X \) \( A_{k,l} \) over \( \text{Gr}(A_{k,l}) \) coincides with the input \( A_{k,l} \)-bundle. The same is true in the reverse order. Note also that since the map \( \pi : A_{k,l} \to M_{k,l} \) canonically identifies the principal \( \text{PGL}_k(C) \)-bundle over \( \text{Gr}_{k,l} \) associated with \( A_{k,l} \) with the space \( \text{Fr}_{k,l} \) of \( k \)-frames in \( M_{k,l} \), then the total space of the principal \( \text{PGL}_k(C) \)-bundle associated with \( \tilde{A}_{k,l} \to \text{Gr}(A_{k,l}) \) coincides with the frame space of the bundle \( A_{k,l} \).

Up to the end of this section we return to the notation of Section 1.

Note that there is a unitary version of the proved results. For example, in this version instead of \( \text{PGL}_l(A_{k,l}) \)-equivariant maps we use \( \text{PU}_l(A_{k,l}) \)-equivariant ones.

The previous propositions enable us to prove the following result.

**Theorem 41.** Suppose \( \text{Aut}(M_{k,l})^0 \) is the connected component of \( e \in \text{Aut}(M_{k,l}) \). Then there is a group epimorphism \( \psi : \text{PU}_l(A_{k,l}) \to \text{Aut}(M_{k,l})^0 \) with kernel \( \text{PU}_l(\Gamma(\text{Gr}_{k,l}; \mathbb{C})) \subset \text{PU}_l(A_{k,l}) \) (we mean the identification \( \Gamma(\text{Gr}_{k,l}; \mathbb{C}) \) with the center of \( \Gamma(\text{Gr}_{k,l}; A_{k,l}) \)).

**Remark 42.** Below we prove the homotopy equivalence \( \text{Aut}(M_{k,l}) \simeq \text{Fr}_{k,l} \), this implies that \( \text{Aut}(M_{k,l}) = \text{Aut}(M_{k,l})^0 \).

**Proof.** First, let us give a construction of \( \psi \). Suppose \( g \in \text{PU}_l(A_{k,l}) \). Consider the composite map

\[
A_{k,l} \xrightarrow{\pi \times \pi} \text{Gr}_{k,l} \times M_{k,l} \xrightarrow{\mu^{-1}} \text{F}_{k,l}(A_{k,l}) \xrightarrow{g} \text{F}_{k,l}(A_{k,l}) \xrightarrow{\mu} \text{Gr}_{k,l} \times M_{k,l},
\]

where \( \mu : \text{F}_{k,l}(A_{k,l}) \xrightarrow{\simeq} \text{Gr}_{k,l} \times M_{k,l} \) is the trivialization as above. By \( \tilde{A}_{k,l} \subset \text{Gr}_{k,l} \times M_{k,l} \) denote the image of the previous composition. The composition takes
a fiber $(A_{k,l})_x, x \in \Gr_{k,l}$ to the fiber $(\hat{A}_{k,l})_x$. Using the canonical map $\pi: A_{k,l} \to \mathbb{M}_{k,l}$ we can identify $(A_{k,l})_x$ with the corresponding subalgebra $M_{k,x} \subset \mathbb{M}_{k,l}$; analogously $(\hat{A}_{k,l})_x$ is identified with $M_{k,\hat{g}(x)} \subset \mathbb{M}_{k,l}$ for some $\hat{g}(x) \in \Gr_{k,l}$. Thus, we have a map $\hat{g}: \Gr_{k,l} \to \Gr_{k,l}$ which is clearly a homeomorphism.

Consider the composite map

$$\Gr_{k,l} \times \mathbb{M}_{k,l} \overset{\mu^{-1}}{\longrightarrow} F_{k,l}(A_{k,l}) \gatemap{g} F_{k,l}(A_{k,l}) \overset{\mu}{\longrightarrow} \Gr_{k,l} \times \mathbb{M}_{k,l} \overset{\hat{g} \times \text{id}_{\mathbb{M}_{k,l}}}{\longrightarrow} \Gr_{k,l} \times \mathbb{M}_{k,l}.$$ 

It preserves the tautological subbundle $A_{k,l} \subset \Gr_{k,l} \times \mathbb{M}_{k,l}$, hence it induces a map $\tau = \tau(g): A_{k,l} \to A_{k,l}$. Therefore by Proposition 37 we obtain an isomorphism

$$\psi(g) := \hat{\tau}: \mathbb{M}_{k,l} \to \mathbb{M}_{k,l}.$$ 

Obviously, the assignment $g \mapsto \psi(g)$ determines a group homomorphism $\text{PU}_l(A_{k,l}) \to \text{Aut}(\mathbb{M}_{k,l})$. Its kernel coincides with $\text{PU}_l(\Gamma(\Gr_{k,l}; \mathbb{C})) \subset \text{PU}_l(A_{k,l})$ (moreover, $\text{PU}_l(\Gamma(\Gr_{k,l}; \mathbb{C}))$) corresponds to the subbundle $\mathcal{C}E_k \otimes \widetilde{M}_l \subset \mathbb{A}_{k,l} \otimes \mathbb{M}_l$ because the centralizer of $M_k(\mathbb{C}) \otimes \mathcal{C}E_l \subset M_{kl}(\mathbb{C})$ coincides with $\mathcal{C}E_k \otimes \mathcal{M}_l(\mathbb{C}) \subset M_{kl}(\mathbb{C})$.

Now let us prove that $\text{im}(\psi) = \text{Aut}(\mathbb{M}_{k,l})^0$. Suppose $\vartheta \in \text{Aut}(\mathbb{M}_{k,l})^0$. Consider the map

$$\Gr_{k,l} \times \mathbb{M}_{k,l} \overset{\text{id}_{\Gr_{k,l}} \times \vartheta}{\longrightarrow} \Gr_{k,l} \times \mathbb{M}_{k,l}.$$ 

Suppose $\hat{A}_{k,l}$ is the image of $A_{k,l} \subset \Gr_{k,l} \times \mathbb{M}_{k,l}$ under $\text{id}_{\Gr_{k,l}} \times \vartheta$, $\theta$ is the restriction of $\text{id}_{\Gr_{k,l}} \times \vartheta$ to $A_{k,l} \subset \Gr_{k,l} \times \mathbb{M}_{k,l}$. Then we have the isomorphism $\theta: A_{k,l} \cong \hat{A}_{k,l}$. Set $B_{kl} := A_{k,l} \otimes \widetilde{M}_l$. Identify $R_{k,l}(B_{kl}) = F_{k,l}(A_{k,l})$ with $\Gr_{k,l} \times \mathbb{M}_{k,l}$ by $\mu$. We can consider $A_{k,l}$ and $\hat{A}_{k,l}$ as subbundles in $B_{kl}$.

Consider $p^*(B_{kl}) = \Gr_{k,l}(B_{kl}) \times B_{kl}$, where $p: \Gr_{k,l}(B_{kl}) \to \Gr_{k,l}$ is the bundle projection. The bundle $\Gr_{k,l}(B_{kl}) \times B_{kl}$ contains the tautological subbundle $\hat{A}_{k,l} \subset \Gr_{k,l}(B_{kl}) \times B_{kl}$ and the complementary subbundle $\hat{C}_{l,k} \subset \Gr_{k,l}(B_{kl}) \times B_{kl}$ such that $\hat{A}_{k,l} \otimes \hat{C}_{l,k} = \Gr_{k,l}(B_{kl}) \times B_{kl}$. Let $\sigma, \hat{\sigma}: \Gr_{k,l} \to \Gr_{k,l}(B_{kl})$ be the sections corresponding to the subbundles $A_{k,l} \subset B_{kl}$ and $\hat{A}_{k,l} \subset B_{kl}$, respectively. Since $\vartheta \in \text{Aut}(\mathbb{M}_{k,l})^0$, we see that $\sigma \sim \hat{\sigma}$. We claim that the complementary subbundle $C_l$ for $\hat{A}_{k,l} \subset B_{kl}$ is trivial. Indeed, it is just the pull-back $\hat{\sigma}^*(\hat{C}_{l,k})$ and we have $\hat{\sigma}^*(\hat{C}_{l,k}) \cong \sigma^*(\hat{C}_{l,k}) = \widetilde{M}_l$ because of $\sigma \sim \hat{\sigma}$.

Choose a unitary isomorphism $\alpha: \widetilde{M}_l \to C_l$ and consider the automorphism

$$\theta \otimes \alpha: A_{k,l} \otimes \widetilde{M}_l = B_{kl} \to \hat{A}_{k,l} \otimes C_l = B_{kl}.$$ 

It follows from Noether-Scolem’s theorem that $\theta \otimes \alpha \in \text{PU}_l(A_{k,l})$. Moreover, the restriction of $\theta \otimes \alpha$ to $R_{k,l}(B_{kl}) \overset{\mu}{\cong} \Gr_{k,l} \times \mathbb{M}_{k,l}$ coincides with $\text{id}_{\Gr_{k,l}} \times \vartheta$ (because its restriction to $A_{k,l} \subset \Gr_{k,l} \times \mathbb{M}_{k,l}$ is $\theta$). 

□

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Remark 43. Let us remark that instead of the trivial bundle \( \tilde{M}_l \) we can take another \( M_l(\mathbb{C}) \)-bundle over \( \text{Gr}_{k,l} \), for example the complementary bundle \( C_{l,k} \), i.e. such that \( A_{k,l} \otimes C_{l,k} = \text{Gr}_{k,l} \times M_{kl}(\mathbb{C}) \). In this case we get the analogous result \( \text{Aut}(M_{k,l})^0 \cong \text{PU}_{kl}(\Gamma(\text{Gr}_{k,l}; \mathbb{C})) / \text{PU}_1(C_{l,k}) \).

Proposition 44. There is a natural embedding \( \mathfrak{G}_{k,l} \hookrightarrow \mathfrak{G}'_{k,l} \).

Proof. Suppose \( \vartheta \in \mathfrak{G}_{k,l} \) is an automorphism of the unitary fan \( M_{k,l} \). Let \( \tilde{\vartheta} : A_{k,l} \rightarrow A_{k,l} \) be the corresponding automorphism of the tautological bundle over the (compact) Grassmannian \( \text{Gr}_{k,l} \). Extend it to a \( \text{PGL}_l(A_{k,l}) \)-equivariant map \( \tilde{\tau}' : F'_{k,l}(A_{k,l}) \rightarrow F'_{k,l}(A_{k,l}) \) (here \( F'_{k,l} \) denotes the functor which to a \( M_{k,l}(\mathbb{C}) \)-bundle assigns the corresponding \( M'_{k,l}(\mathbb{C}) \)-bundle). The map \( \tilde{\tau}' \) is an extension of the \( \text{PU}_l(A_{k,l}) \)-equivariant map \( \tilde{\tau} : F_{k,l}(A_{k,l}) \rightarrow F_{k,l}(A_{k,l}) \). By analogy with the proof of Proposition 37 we see that \( \tilde{\tau}' \) preserves the canonical trivialization. Therefore we get the commutative diagram

\[
\begin{array}{ccc}
M'_{k,l} & \xrightarrow{\vartheta'} & M'_{k,l} \\
\downarrow & & \downarrow \\
M_{k,l} & \xrightarrow{\vartheta} & M_{k,l}
\end{array}
\]

where \( \vartheta = \tilde{\tau} \) in notation of Proposition 37. Clearly, the assignment \( \vartheta \mapsto \vartheta' \) defines the required embedding \( \mathfrak{G}_{k,l} \hookrightarrow \mathfrak{G}'_{k,l} \). □

2.3 The tensor product of fans

Suppose \( (km, ln) = 1 \). It is well known that for matrix algebras we have \( M_{kl}(\mathbb{C}) \otimes M_{mn}(\mathbb{C}) = M_{klmn}(\mathbb{C}) \). Set \( M_{k,l} \hat{\otimes} M_{m,n} := M_{km,ln} \).

We want to show that \( \hat{\otimes} \) is a bifunctor. In particular, we must produce a functorial construction which to any pair of fan automorphisms \( \vartheta_1 \in \mathfrak{G}_{k,l}, \vartheta_2 \in \mathfrak{G}_{m,n} \) assigns an automorphism \( \vartheta_1 \hat{\otimes} \vartheta_2 \in \mathfrak{G}_{km,ln} \) of their tensor product.

The construction of \( (\vartheta_1, \vartheta_2) \mapsto \vartheta_1 \hat{\otimes} \vartheta_2 \). The main ingredient of the construction is the following lemma. Note that the similar argument has already been used in the proof of Proposition 37.

Lemma 45. Suppose we have a trivial bundle \( B \times F \) with a fiber \( F \) and a base \( B \). Assume that there is a subbundle \( A \xrightarrow{i} B \times F \) such that the composition of the inclusion \( i \) with the projection onto the second factor \( p_2 : B \times F \rightarrow F \) is the surjection \( \pi : A \rightarrow F \). Assume also that a bundle automorphism \( \tau : A \rightarrow A \) can be extended to an automorphism \( \tilde{\tau} : B \times F \rightarrow B \times F \) such that the diagram

\[
\begin{array}{ccc}
B \times F & \xrightarrow{\tilde{\tau}} & B \times F \\
\downarrow & & \downarrow \\
A & \xrightarrow{\tau} & A
\end{array}
\]

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commutes. Then there is a unique automorphism \( \tilde{\tau}: F \to F \) such that the diagram

\[
\begin{array}{ccc}
A & \xrightarrow{\tau} & A \\
\downarrow{\pi} & & \downarrow{\pi} \\
F & \xrightarrow{\tilde{\tau}} & F
\end{array}
\]

commutes.

Proof. Equip the trivial bundle \( B \times F \) with the trivialization given by the projection \( p_2: B \times F \to F \) such that the diagram

\[
\begin{array}{ccc}
A & \xrightarrow{p \times \pi} & B \times F \\
\downarrow{\pi} & \downarrow{p_2} & \downarrow{\pi} \\
F & & F
\end{array}
\]

commutes (here \( p: A \to B \) is the bundle projection). It is easy to see that under our choice of trivialization the map \( \tilde{\tau}: B \times F \to B \times F \) has the form \( \bar{\tau} \times \hat{\tau} \), where \( \hat{\tau}: F \to F \) is the required fiber automorphism. \( \Box \)

Let us return to the construction. To a given \( \vartheta_1, \vartheta_2 \) as above let \( \tau_1: A_{k,l} \to A_{k,l} \) and \( \tau_2: A_{m,n} \to A_{m,n} \) be the corresponding automorphisms of the tautological bundles as in Proposition 36. Consider the exterior tensor product \( A_{k,l} \otimes A_{m,n} \). It is an \( M_{km}(\mathbb{C}) \)-bundle over \( \text{Gr}_{k,l} \times \text{Gr}_{m,n} \) and \( \tau_1 \otimes \tau_2: A_{k,l} \otimes A_{m,n} \to A_{k,l} \otimes A_{m,n} \) is its automorphism. Set \( \mathcal{A}_{km,ln} := F_{km,ln}(A_{k,l} \otimes A_{m,n}) \). It is a trivial \( M_{km,ln} \)-bundle over \( \text{Gr}_{k,l} \times \text{Gr}_{m,n} \). Let

\[
\tilde{\tau}_1 \otimes \tilde{\tau}_2: \mathcal{A}_{km,ln} \to \mathcal{A}_{km,ln}
\]

be the \( \text{PGL}_{ln}(A_{k,l} \otimes A_{m,n}) \)-equivariant automorphism which extends \( \tau_1 \otimes \tau_2 \). By \( \tilde{A}_{km,ln} \) denote the tautological subbundle

\[
\tilde{A}_{km,ln} \subset \text{Gr}(\mathcal{A}_{km,ln})_{Gr_{k,l} \times Gr_{m,n}} \times \mathcal{A}_{km,ln}
\]

(let us remark that

\[
\text{Gr}(\mathcal{A}_{km,ln}) \cong \text{Gr}_{k,l} \times \text{Gr}_{m,n} \times \text{Gr}_{km,ln}, \quad \text{Gr}(\mathcal{A}_{km,ln})_{Gr_{k,l} \times Gr_{m,n}} \times \mathcal{A}_{km,ln} \\
\cong \text{Gr}(\mathcal{A}_{km,ln}) \times M_{km,ln}, \quad \tilde{A}_{km,ln} \cong \text{Gr}_{k,l} \times \text{Gr}_{m,n} \times A_{km,ln}).
\]

Equip the bundle \( \mathcal{A}_{km,ln} \) with a trivialization in the following way. Note that it is sufficient to produce the corresponding projection \( \mathcal{A}_{km,ln} \to M_{km,ln} \) onto the second factor. Recall that the bundle \( \text{Gr}_{km,ln} \times M_{km,ln} \) has the canonical trivialization given by the commutative diagram

\[
\begin{array}{ccc}
A_{km,ln} & \xrightarrow{p \times \pi} & \text{Gr}_{km,ln} \times M_{km,ln} \\
\downarrow{\pi} & \downarrow{p_2} & \downarrow{\pi} \\
M_{km,ln} & & M_{km,ln}
\end{array}
\]
There is the map \( \lambda = \lambda_{k,l,m,n} : \text{Gr}_{k,l} \times \text{Gr}_{m,n} \to \text{Gr}_{km,ln} \) induced by the tensor product of matrix algebras \( M_{kl}(\mathbb{C}) \times M_{mn}(\mathbb{C}) \xrightarrow{\otimes} M_{km,ln}(\mathbb{C}) \). We can consider \( A_{km,ln} \) as the pull-back of \( \text{Gr}_{km,ln} \times \text{M}_{km,ln} \) under \( \lambda \), this provides \( A_{km,ln} \) with the canonical trivialization \( A_{km,ln} \cong \text{Gr}_{k,l} \times \text{Gr}_{m,n} \times A_{km,ln} \) such that the diagram

\[
\begin{array}{ccc}
\text{Gr}_{k,l} \times \text{Gr}_{m,n} \times \text{M}_{km,ln} & \xrightarrow{\lambda \times \text{id}_{\text{M}_{km,ln}}} & \text{Gr}_{km,ln} \times \text{M}_{km,ln} \\
& \downarrow{p_1} & \downarrow{p_2} \\
\text{M}_{km,ln} & & \\
\end{array}
\]

commutes, where \( p_i \) is the projection onto the \( i \)-th factor. Clearly, this also determines the trivializations \( \text{Gr}(A_{km,ln}) \times A_{km,ln} \cong \text{Gr}(A_{km,ln}) \times \text{M}_{km,ln} \) and \( \tilde{A}_{km,ln} \cong \text{Gr}_{k,l} \times \text{Gr}_{m,n} \times A_{km,ln} \) such that the diagram

\[
\begin{array}{ccc}
\tilde{A}_{km,ln} & \xrightarrow{\subset} & \text{Gr}(A_{km,ln}) \times \text{M}_{km,ln} \\
& \downarrow{\tilde{\pi}} & \downarrow{p_2} \\
\text{M}_{km,ln} & & \\
\end{array}
\]

commutes, where \( \tilde{\pi} \) is the composite map \( \tilde{A}_{km,ln} \cong \text{Gr}_{k,l} \times \text{Gr}_{m,n} \times A_{km,ln} \xrightarrow{p_3} A_{km,ln} \xrightarrow{\pi} \text{M}_{km,ln} \).

The automorphism \( \tau_1 \otimes \tau_2 \) induces the automorphism \( \tilde{\tau}_{1,2} \) of the trivial bundle \( \text{Gr}(A_{km,ln}) \times A_{km,ln} \). Equip \( \text{Gr}(A_{km,ln}) \times A_{km,ln} \) with the defined above trivialization \( \text{Gr}(A_{km,ln}) \times \text{M}_{km,ln} \). Clearly, \( \tilde{\tau}_{1,2} \) preserves the subbundle \( \tilde{A}_{km,ln} \subset \text{Gr}(A_{km,ln}) \times \text{M}_{km,ln} \).

If we put \( F = \text{M}_{km,ln} \), \( B = \text{Gr}(A_{km,ln}) \), \( A = \tilde{A}_{km,ln} \), \( \tau = \tilde{\tau}_{1,2} \), and \( \pi = \tilde{\pi} \), then all the conditions of the previous lemma are satisfied. Therefore there is the unique map \( \vartheta_1 \otimes \vartheta_2 : \text{M}_{km,ln} \to \text{M}_{km,ln} \) such that the diagram

\[
\begin{array}{ccc}
\tilde{A}_{km,ln} & \xrightarrow{\tilde{\tau}_{1,2} \mid A_{km,ln}} & \tilde{A}_{km,ln} \\
& \downarrow{\tilde{\pi}} & \downarrow{\tilde{\pi}} \\
\text{M}_{km,ln} \vartheta_1 \otimes \vartheta_2 & \xrightarrow{\vartheta_1 \otimes \vartheta_2 \mid \text{M}_{km,ln}} & \text{M}_{km,ln} \\
\end{array}
\]

commutes. This completes the construction. \( \square \)

Now let us study some properties of the tensor product \( \otimes \) of fans.

First, let us introduce one more notation. By \( \text{M}_{k,l} \otimes \text{M}_{m,n} \subset \text{M}_{km,ln} \) denote the proper subset obtaining by the tensor product of all possible \( k \) and \( m \)-subalgebras contained in \( \text{M}_{k,l} \) and \( \text{M}_{m,n} \), respectively.

There is the natural map \( \tilde{\lambda} = \tilde{\lambda}_{k,l,m,n} : A_{k,l} \otimes A_{m,n} \to A_{km,ln} \) covering the classifying map \( \lambda : \text{Gr}_{k,l} \times \text{Gr}_{m,n} \to \text{Gr}_{km,ln} \) induced by the tensor product of matrix algebras \( M_{kl}(\mathbb{C}) \) and \( M_{mn}(\mathbb{C}) \). Consider the composite map

\[
A_{k,l} \otimes A_{m,n} \xrightarrow{\tilde{\lambda}} A_{km,ln} \xrightarrow{\pi} \text{M}_{km,ln};
\]
where \( \pi: A_{km, ln} \to M_{km, ln} \) is the canonical map. Clearly, the image \( \text{im}(\pi \circ \tilde{\lambda}) \) coincides with the subset \( M_{k,l} \otimes M_{m,n} \subset M_{km, ln} \). Moreover, if \( p_2: A_{km, ln} \to M_{km, ln} \) is the projection corresponding to the chosen trivialization of \( A_{km, ln} \), then the diagram

\[
\begin{array}{ccc}
A_{k,l} \boxtimes A_{m,n} & \xrightarrow{i} & A_{km, ln} \\
\pi \circ \lambda & \downarrow & \vert \\
M_{km, ln} & \xrightarrow{p_2} & & \\
\end{array}
\]

commutes, where \( i \) is the natural inclusion.

Note that there are the embeddings

\[
M_{kl}(\mathbb{C}) \hookrightarrow M_{klmn}(\mathbb{C}), \quad X \mapsto X \otimes E_{mn}; \quad M_{mn}(\mathbb{C}) \hookrightarrow M_{klmn}(\mathbb{C}), \quad Y \mapsto E_{kl} \otimes Y.
\]

The same takes place for fans; by \( i_1, i_2 \) denote the corresponding embeddings

\[
M_{k,l} \hookrightarrow M_{km, ln}, \quad X \mapsto X \otimes E_{mn}; \quad M_{m,n} \hookrightarrow M_{km, ln}, \quad Y \mapsto E_{kl} \otimes Y.
\]

**Proposition 46.** \( \vartheta_1 \hat{\otimes} \vartheta_2 \mid_{\text{im} i_1} = \vartheta_1, \vartheta_1 \hat{\otimes} \vartheta_2 \mid_{\text{im} i_2} = \vartheta_2. \)

**Proof.** We shall prove even a little more general assertion. Recall that we have already defined the subset \( M_{k,l} \otimes M_{m,n} \subset M_{km, ln} \) which is the image of the composition \( \pi \circ \tilde{\lambda}: A_{k,l} \boxtimes A_{m,n} \to M_{km, ln} \). Note that any \( km \)-subalgebra \( M_{km, z} \subset M_{k,l} \otimes M_{m,n} \) is the tensor product of some \( k \)-subalgebra \( M_{k,x} \subset M_{k,l} \) by some \( m \)-subalgebra \( M_{m,y} \subset M_{m,n} \). We claim that \( \vartheta_1 \hat{\otimes} \vartheta_2 \mid_{M_{km, z}} = (\vartheta_1 \mid_{M_{k,x}}) \otimes (\vartheta_2 \mid_{M_{m,y}}) \). Indeed, let \( A_{k,l;m,n} \) be the image of the map \( A_{k,l} \boxtimes A_{m,n} \xrightarrow{p \times \lambda} \text{Gr}_{k,l} \times \text{Gr}_{m,n} \times A_{km, ln} = \tilde{A}_{km, ln} \), where \( p: A_{k,l} \boxtimes A_{m,n} \to \text{Gr}_{k,l} \times \text{Gr}_{m,n} \) is the bundle projection. It easily follows from our construction that the restriction of the map \( \tilde{\tau}_{1,2} \) to \( A_{k,l;m,n} \subset \tilde{A}_{km, ln} \) coincides with \( \tau_1 \boxtimes \tau_2 \) (under the natural identification \( A_{k,l;m,n} = A_{k,l} \boxtimes A_{m,n} \)). Now the required assertion follows from Corollary 39 \( \square \)

**Proposition 47.** The map \( \phi_{kl, mn}: \mathcal{G}_{k,l} \times \mathcal{G}_{m,n} \to \mathcal{G}_{km, ln} \) corresponding to the tensor product of fans is a group homomorphism.

**Proof.** In view of Corollary 30 the proof easily follows from the fact that the composition of \( \text{PGL}_{ln}(A_{k,l} \boxtimes A_{m,n}) \)-equivariant maps \( \tau_1 \boxtimes \tau_2, \tau_1' \boxtimes \tau_2' \) corresponding to \( (\vartheta_1, \vartheta_2) \) and \( (\vartheta_1', \vartheta_2') \) is the \( \text{PGL}_{ln}(A_{k,l} \boxtimes A_{m,n}) \)-equivariant map corresponding to the composition \( (\vartheta_1' \circ \vartheta_1, \vartheta_2' \circ \vartheta_2) \). \( \square \)

Note that the tensor product of fans has the standard properties of the tensor product of algebras such as the associativity, in some sense commutativity, etc.

Now we want to prove that in a certain sense the tensor product of fans is concordant with the tensor product of algebras.
Recall that we defined two functors $F_{k,l}$ and $R_{k,l}$ which correspond to the assignments $M_k(C) \mapsto M_{k,l}$ and $M_{kl}(C) \mapsto M_{k,l}$, respectively; moreover, $F_{k,l}$ is the composition of the functor $M_k(C) \mapsto M_k(C) \otimes M_l(C) = M_{kl}(C)$ with $R_{k,l}$. Clearly, the functor $M_k(C) \mapsto M_k(C) \otimes M_l(C) = M_{kl}(C)$ preserves the tensor product. Therefore it is sufficient to prove that the functor $R_{k,l}$ is concordant with the tensor product, this implies the analogous assertion for $F_{k,l}$.

The tensor product of matrix algebras induces the group homomorphism $\rho_{k,l} : \text{PGL}_k(C) \times \text{PGL}_l(C) \to \text{PGL}_{kl}(C)$ given by the Kronecker product of matrices. Moreover, there is the group homomorphism $\kappa_{k,l} : \text{PGL}_{kl}(C) \to \mathcal{G}_{k,l}$ related to the embedding $M_{k,l} \subset M_{kl}(C)$.

**Proposition 48.** The diagram

$$
\begin{array}{ccc}
\mathcal{G}_{k,l} \times \mathcal{G}_{m,n} & \xrightarrow{\phi_{kl, mn}} & \mathcal{G}_{km, ln} \\
\kappa_{k,l} \times \kappa_{m,n} \downarrow & & \kappa_{km, ln} \downarrow \\
\text{PGL}_{kl}(C) \times \text{PGL}_{mn}(C) & \xrightarrow{\rho_{kl, mn}} & \text{PGL}_{klmn}(C),
\end{array}
$$

consisting of the defined above homomorphisms commutes. In other words, the functor $R_{k,l}$ which to any $M_{kl}(C)$-bundle assigns the corresponding $M_{k,l}$-bundle preserves the tensor product: $R_{km, ln}(B_{kl} \otimes B_{mn}') = R_{k,l}(B_{kl}) \otimes R_{m,n}(B_{mn}')$ (therefore for the functor $F_{k,l}$ assigning to any $M_k(C)$-bundle the $M_{k,l}$-bundle we have $F_{km, ln}(A_k \otimes A_m') = F_{k,l}(A_k) \otimes F_{m,n}(A_m')$).

The proof of the proposition is based on the following lemma.

**Lemma 49.** Suppose $\theta : M_{kl}(C) \to M_{kl}(C)$ is an isomorphism, $\tau : A_{k,l} \to A_{k,l}$ is the induced by $\theta$ isomorphism of the tautological bundle. Then the isomorphism $\hat{\tau} : \mathcal{M}_{k,l} \to \mathcal{M}_{k,l}$ constructed in Proposition 37 coincides with the restriction of $\theta$ to the fan $\mathcal{M}_{k,l} \subset M_{kl}(C)$.

**Proof of the lemma.** Recall that one way to define the fan $\mathcal{M}_{k,l}$ is to consider the image $\text{im}(p_2 \circ \mu) :$

$$
\begin{array}{ccc}
A_{k,l} & \xrightarrow{\mu} & \text{Gr}_{k,l} \times M_{kl}(C) \\
\downarrow & & \downarrow p_2 \\
\mathcal{M}_{k,l} & \subset & M_{kl}(C).
\end{array}
$$

Now lemma’s assertion follows from the commutative diagrams

$$
\begin{array}{ccc}
A_{k,l} & \xrightarrow{\tau} & A_{k,l} \\
\downarrow p_2 \circ \mu & & \downarrow p_2 \circ \mu \\
M_{kl}(C) & \xrightarrow{\theta} & M_{kl}(C)
\end{array}
\quad
\begin{array}{ccc}
A_{k,l} & \xrightarrow{\tau} & A_{k,l} \\
\downarrow & & \downarrow \\
\mathcal{M}_{k,l} & \xrightarrow{\hat{\tau}} & \mathcal{M}_{k,l}.
\end{array}
$$

\[\Box\]
Proof of the proposition. Let \( \theta_1: M_{kl}(\mathbb{C}) \to M_{kl}(\mathbb{C}) \), \( \theta_2: M_{mn}(\mathbb{C}) \to M_{mn}(\mathbb{C}) \) be automorphisms, \( \theta_1 \otimes \theta_2: M_{klmn}(\mathbb{C}) \to M_{klmn}(\mathbb{C}) \) their tensor product. By \( \vartheta_1, \vartheta_2, \vartheta \) denote the automorphisms \( \bar{\mathbb{M}}_{k,t} \to M_{k,t}, M_{m,n} \to \bar{\mathbb{M}}_{m,n} \) and \( \bar{\mathbb{M}}_{km,ln} \to \bar{\mathbb{M}}_{km,ln} \) induced by \( \theta_1, \theta_2 \) and \( \theta_1 \otimes \theta_2 \), respectively. We must show that \( \vartheta_1 \circ \vartheta_2 = \vartheta: \bar{\mathbb{M}}_{km,ln} \to \bar{\mathbb{M}}_{km,ln} \).

Let \( \tau_1: \mathcal{A}_{k,t} \to \mathcal{A}_{k,t}, \tau_2: \mathcal{A}_{m,n} \to \mathcal{A}_{m,n} \) be the automorphisms of the tautological bundles induced by \( \theta_1, \theta_2 \) respectively; by \( \tau_1 \otimes \tau_2: \mathcal{A}_{k,t} \boxtimes \mathcal{A}_{m,n} \to \mathcal{A}_{k,t} \boxtimes \mathcal{A}_{m,n} \) denote their tensor product (clearly, if we take \( \vartheta_1, \vartheta_2 \) instead of \( \theta_1, \theta_2 \) we obtain the same maps \( \tau_1, \tau_2 \) and therefore \( \tau_1 \otimes \tau_2 \)).

By \( t \) denote the automorphism \( \mathcal{A}_{km,ln} \to \mathcal{A}_{km,ln} \) induced by \( \theta_1 \otimes \theta_2 \). Let \( \tilde{t}: F_{km,ln}(\mathcal{A}_{km,ln}) \to F_{km,ln}(\mathcal{A}_{km,ln}) \) be the \( \text{PGL}_{ln}(\mathcal{A}_{km,ln}) \)-equivariant map which extends \( t \). It follows from the previous lemma that under our choice of the trivialization \( F_{km,ln}(\mathcal{A}_{km,ln}) \cong \mathcal{M}_{km,ln} \) the map \( \tilde{t}: F_{km,ln}(\mathcal{A}_{km,ln}) \to F_{km,ln}(\mathcal{A}_{km,ln}) \) has the form \( \tilde{\vartheta} \times \tilde{\vartheta} \), where \( \tilde{\vartheta}: \mathcal{M}_{km,ln} \to \mathcal{M}_{km,ln} \) is the map induced by \( \theta_1 \otimes \theta_2 \).

We have the map \( F_{km,ln}(\mathcal{A}_{k,t} \boxtimes \mathcal{A}_{m,n}) \to F_{km,ln}(\mathcal{A}_{km,ln}) \) which extends \( \tilde{\lambda}: \mathcal{A}_{k,t} \boxtimes \mathcal{A}_{m,n} \to \mathcal{A}_{km,ln} \). Moreover, the diagrams

\[
\begin{array}{ccc}
\mathcal{A}_{k,t} \boxtimes \mathcal{A}_{m,n} & \xrightarrow{\tau_1 \otimes \tau_2} & \mathcal{A}_{k,t} \boxtimes \mathcal{A}_{m,n} \\
\lambda \downarrow & & \downarrow \lambda \\
\mathcal{A}_{km,ln} & \xrightarrow{t} & \mathcal{A}_{km,ln}
\end{array}
\quad
\begin{array}{ccc}
F_{km,ln}(\mathcal{A}_{k,t} \boxtimes \mathcal{A}_{m,n}) & \xrightarrow{\tau_1 \otimes \tau_2} & F_{km,ln}(\mathcal{A}_{k,t} \boxtimes \mathcal{A}_{m,n}) \\
\downarrow & & \downarrow \\
F_{km,ln}(\mathcal{A}_{km,ln}) & \xrightarrow{\tilde{t}} & F_{km,ln}(\mathcal{A}_{km,ln})
\end{array}
\]

commute. The conclusion of the proof follows from the definition of the trivialization \( F_{km,ln}(\mathcal{A}_{k,t} \boxtimes \mathcal{A}_{m,n}) \cong \mathcal{M}_{km,ln} \) given by diagram \( 4 \). Indeed, by repitition of the construction of \( \vartheta_1 \circ \vartheta_2 \) we see that \( \vartheta_1 \circ \vartheta_2 \) coincides with the restriction of \( \tilde{t} \) to the second factor \( \mathcal{M}_{km,ln} \) (under the canonical choice of the trivialization \( F_{km,ln}(\mathcal{A}_{km,ln}) \cong \mathcal{M}_{km,ln} \)), i.e. with \( \vartheta \). \( \square \)

Remark 50. By definition, put \( \mathcal{A}_{km,ln} := \mathcal{A}_{km,ln} \). It follows from Corollary \( 39 \) that we also have the bifunctor \( (\mathcal{A}_{k,t}, \mathcal{A}_{m,n}) \mapsto \mathcal{A}_{k,t} \boxtimes \mathcal{A}_{m,n} \). Moreover, the functors \( M_k(\mathbb{C}) \mapsto \mathcal{A}_{k,t} \) and \( M_{kl}(\mathbb{C}) \mapsto \mathcal{A}_{k,t} \) (which are analogs of \( F_{k,t} \) and \( R_{k,t} \) respectively) are tensor, i.e. they preserve the tensor product.

Note that there are unitary analogs of all proved results. In the unitary version the following result holds (we return to the notation of Section 1).

**Proposition 51.** The group homomorphism \( \phi_{kl,mn}: \mathcal{G}_{k,t} \times \mathcal{G}_{m,n} \to \mathcal{G}_{km,ln} \) from (the unitary analog of) Proposition \( 47 \) induced by the tensor product \( \mathbb{M}_{k,t} \times \mathbb{M}_{m,n} \to \mathbb{M}_{km,ln} \) of unitary fans is continuous.

**Proof** is clear. \( \square \)
3 The classifying space of $\mathbb{M}_{k,l}$-bundles

In this section we consider only the unitary version. We assume that any bundle is equipped with an Hermitian metric which is preserved by morphisms. The notation coincides with the notation of Section 1.

3.1 Fibre bundles associated with fan bundles

To an arbitrary $\mathbb{M}_{k,l}$-bundle $\mathfrak{A}_{k,l}$ over $X$ (not necessarily of the form $F_{k,l}(A_k)$ or even of the form $R_{k,l}(B_{kl})$) we can associate the fiber bundle $\text{Gr}(\mathfrak{A}_{k,l}) \xrightarrow{p} X$ over $X$ with fiber $\text{Gr}_{k,l}$. Note that in the general case $\text{Gr}(\mathfrak{A}_{k,l})$ is not isomorphic to the product bundle $X \times \text{Gr}_{k,l}$ as it was in the case of bundles of the form $F_{k,l}(A_k)$.

Let $\mathfrak{A}_{k,l}$ be the core of the tautological FAB over $\text{Gr}_{k,l}$. We have the canonical $M_{k}(\mathbb{C})$-bundle $\tilde{\mathfrak{A}}_{k,l}$ over $\text{Gr}(\mathfrak{A}_{k,l})$ which is the tautological subbundle $\tilde{\mathfrak{A}}_{k,l} \subset \text{Gr}(\mathfrak{A}_{k,l}) \times \mathfrak{A}_{k,l}$ (in particular, for any fiber $\text{Gr}_{k,l}$ of the bundle $\text{Gr}(\mathfrak{A}_{k,l}) \xrightarrow{p} X$ the restriction $\tilde{\mathfrak{A}}_{k,l} \mid_{\text{Gr}_{k,l}}$ coincides with $\mathfrak{A}_{k,l} \subset \text{Gr}_{k,l} \times M_{k,l}$).

**Proposition 52.** The pullback $p^* (\mathfrak{A}_{k,l})$ has the form $F_{k,l}(\tilde{\mathfrak{A}}_{k,l})$.

**Proof.** By definition, the pullback $p^* (\mathfrak{A}_{k,l})$ is the fibered product $\text{Gr}(\mathfrak{A}_{k,l}) \times \mathfrak{A}_{k,l}$; in particular, $p^* (\mathfrak{A}_{k,l})$ contains $\tilde{\mathfrak{A}}_{k,l}$ as an $M_{k}(\mathbb{C})$-subbundle. Note that the bundle $\mathfrak{A}_{k,l}$ can be obtained from $\tilde{\mathfrak{A}}_{k,l}$ by applying the canonical surjective map $\tilde{\pi}: \tilde{\mathfrak{A}}_{k,l} \rightarrow \mathfrak{A}_{k,l}$ whose restriction to a fiber coincides with the canonical map $\pi: \mathfrak{A}_{k,l} \rightarrow M_{k,l}$. This is clear from the diagram

\[ \begin{array}{ccc}
\tilde{\mathfrak{A}}_{k,l} & \subset & \text{Gr}(\mathfrak{A}_{k,l}) \times \mathfrak{A}_{k,l} \\
\downarrow \quad \tilde{i} & & \downarrow p_2 \\
\mathfrak{A}_{k,l} & & \\
\end{array} \]  

(5)

Let $\tilde{M}_l$ be a trivial $M_l(\mathbb{C})$-bundle over $\text{Gr}(\mathfrak{A}_{k,l})$; consider the tensor product $\tilde{B}_{kl} := \tilde{\mathfrak{A}}_{k,l} \otimes \tilde{M}_l$ and the corresponding fan bundle $F_{k,l}(\tilde{\mathfrak{A}}_{k,l}) = R_{k,l}(\tilde{B}_{kl})$. For this bundle there also exists the canonical embedding

\[ \tilde{\mathfrak{A}}_{k,l} \hookrightarrow R_{k,l}(\tilde{B}_{kl}) \]

whose restriction to any fiber of the fiber bundle $\text{Gr}(\mathfrak{A}_{k,l}) \rightarrow X$ coincides with the embedding $\mathfrak{A}_{k,l} \hookrightarrow R_{k,l}(B_{kl})$, where $B_{kl} = \mathfrak{A}_{k,l} \otimes \tilde{M}_l$ is the restriction of $\tilde{B}_{kl}$ to the fiber (in the last formula $\tilde{M}_l$ denotes the trivial bundle over $\text{Gr}_{k,l}$). According to Proposition 34 the fan bundle $R_{k,l}(B_{kl})$ is trivial. Moreover, the bundle $R_{k,l}(B_{kl})$ has the canonical trivialization $\lambda: R_{k,l}(B_{kl}) \cong \text{Gr}_{k,l} \times M_{k,l}$ determined by the requirement
that the diagram

\[
\begin{array}{ccc}
A_{k,l}^{i=p \times \pi} & \rightarrow & Gr_{k,l} \times M_{k,l} \\
\pi & \downarrow & p_2 \\
M_{k,l} & & \\
\end{array}
\]

commutes. Now applying Corollary 39 we see that the trivialization \( \lambda \) can be extended to the isomorphism of fan bundles \( \tilde{\lambda}: R_{k,l}(B_{kl}) \cong Gr(A_{k,l}) \times A_{k,l} \) such that the diagram

\[
\begin{array}{ccc}
\tilde{A}_{k,l}^{i=\tilde{p} \times \tilde{\pi}} & \rightarrow & X(Gr(A_{k,l}) \times A_{k,l}) \\
\tilde{\pi} & \downarrow & p_2 \\
A_{k,l} & & \\
\end{array}
\]

commutes (here \( \tilde{p}: \tilde{A}_{k,l} \rightarrow Gr(A_{k,l}) \) is the bundle projection).

If the previous argument does not look reliable we give another reasoning.

Note that the assignment \( A_{k,l} \mapsto R_{k,l}(B_{kl}) \) is functorial (as a composition of functors \( A_{k,l} \mapsto A_{k,l} \otimes \tilde{M}_{l} \mapsto R_{k,l}(A_{k,l} \otimes \tilde{M}_{l}) \)), and the same is true for the map \( A_{k,l}^{p \times \pi} \mapsto Gr_{k,l} \times M_{k,l} \) (this follows from Corollary 39). Therefore the trivialization \( \lambda: R_{k,l}(B_{kl}) \cong Gr_{k,l} \times M_{k,l} \) is preserved by the structure group \( Aut(A_{k,l}) \) of the bundle \( \tilde{A}_{k,l} \rightarrow X \) with fiber \( A_{k,l} \) and therefore \( \lambda \) can be lifted to the global isomorphism

\[
\begin{array}{ccc}
\tilde{A}_{k,l}^{i=\tilde{p} \times \tilde{\pi}} & \rightarrow & X(Gr(A_{k,l}) \times A_{k,l}) \\
\tilde{\pi} & \downarrow & p_2 \\
A_{k,l} & & \\
\end{array}
\]

Conversely, if \( A_{k,l} = F_{k,l}(A_k) \), then \( A_k \) is a \( M_k(\mathbb{C}) \)-subbundle in \( A_{k,l} \). Clearly, any such a subbundle determines a section \( \sigma \).

Proposition 53. An \( M_{k,l}(\mathbb{C}) \)-bundle \( A_{k,l} \) over \( X \) has the form \( A_{k,l} = F_{k,l}(A_k) \) for some \( M_k(\mathbb{C}) \)-bundle \( A_k \) if and only if the fiber bundle \( Gr(A_{k,l}) \xrightarrow{p} X \) has a section \( \sigma \).

Proof. It follows from the previous proposition that \( p^*(A_{k,l}) = F_{k,l}(A_k) \). Consequently, to a given section \( \sigma \) we can define \( A_k = \sigma^*(A_{k,l}) \). Indeed, \( p \circ \sigma = id_X \), hence \( A_{k,l} = (\sigma^* \circ p^*)(A_{k,l}) = \sigma^*(F_{k,l}(A_{k,l})) = F_{k,l}(\sigma^*(A_{k,l})) = F_{k,l}(A_k) \) (here we have used the obvious fact that the functor \( F_{k,l} \) commutes with the pullback).

By comparison of the previous proposition with Proposition 26 we get the following result.

Corollary 54. The following conditions on \( M_{k,l}(\mathbb{C}) \)-bundle \( A_{k,l} \) are equivalent:

(i) \( A_{k,l} \) has the form \( F_{k,l}(A_k) \) for some \( M_k(\mathbb{C}) \)-bundle \( A_k \);
Remark 55. In the case $\mathcal{A}_{k,l} = R_{k,l}(B_{kl})$ the unexpected implication (ii) $\Rightarrow$ (iii) of the previous corollary is clear from the following arguments. Suppose $\mathcal{A}_{k,l} = R_{k,l}(B_{kl})$. Then it follows from the existence of a section of the bundle $Gr(\mathcal{A}_{k,l}) \xrightarrow{p} X$ that $B_{kl} = A_k \otimes C_l$. Therefore the bundle $Gr(\mathcal{A}_{k,l}) \xrightarrow{p} X$ is trivial by Proposition 24.

To a given $M_k,l$-bundle $\mathcal{A}_{k,l}$ over $X$ let $Fr(\mathcal{A}_{k,l}) \xrightarrow{\tilde{p}} X$ be the corresponding fiber bundle of $k$-frames over $X$.

**Proposition 56.** An $M_{k,l}$-bundle $\mathcal{A}_{k,l}$ is trivial if and only if the fiber bundle $Fr(\mathcal{A}_{k,l}) \xrightarrow{\tilde{p}} X$ has a section $\tilde{\sigma}$.

**Proof.** Consider the following three fibrations: $Gr(\mathcal{A}_{k,l}) \xrightarrow{p} X$ with fiber $Gr_{k,l}$, $Fr(\mathcal{A}_{k,l}) \xrightarrow{\tilde{p}} X$ with fiber $Fr_{k,l}$, and $Fr(\mathcal{A}_{k,l}) \xrightarrow{\tilde{p}} Gr(\mathcal{A}_{k,l})$ with fiber $PU(k)$. Clearly, the lifting $\tilde{p}^*(\tilde{A}_{k,l})$ is a trivial $M_k(\mathbb{C})$-bundle $\tilde{M}_k$ over $Fr(\mathcal{A}_{k,l})$. Furthermore, note that the bundle $\tilde{p}^*(\mathcal{A}_{k,l})$ has the form $F_{k,l}(\tilde{p}^*(\tilde{A}_{k,l}))$.

Suppose $\tilde{p}$ has a section $\tilde{\sigma}$; then $\mathcal{A}_{k,l} = F_{k,l}(\tilde{\sigma}^*(\tilde{M}_k))$ is a trivial bundle. The converse implication is clear. $\square$

### 3.2 An exact sequence of functors

Let $B \mathfrak{G}_{k,l}$ be the classifying space of the topological group $\mathfrak{G}_{k,l}$ (recall that we consider the unitary version). Consider the universal $M_{k,l}$-bundle $\mathcal{A}_{k,l}^{univ}$ over $B \mathfrak{G}_{k,l}$ which is glued by the universal $\mathfrak{G}_{k,l}$-cocycle. Let us associate the $Gr_{k,l}$-bundle $Gr(\mathcal{A}_{k,l}^{univ})$ with $\mathcal{A}_{k,l}^{univ}$ ($Gr(\mathcal{A}_{k,l}^{univ})$ is glued by the same cocycle by using the natural action $\mathfrak{G}_{k,l} \times Gr_{k,l} \rightarrow Gr_{k,l}$); by $p_{k,l} : Gr(\mathcal{A}_{k,l}^{univ}) \rightarrow B \mathfrak{G}_{k,l}$ denote its projection. Consider the tautological $M_k(\mathbb{C})$-bundle

$$\tilde{\mathcal{A}}_{k,l}^{univ} \subset Gr(\mathcal{A}_{k,l}^{univ}) \times \mathcal{A}_{k,l}^{univ} \quad \text{(6)}$$

over $Gr(\mathcal{A}_{k,l}^{univ})$ whose restriction to any fiber $Gr_{k,l}$ of the bundle $Gr(\mathcal{A}_{k,l}^{univ}) \xrightarrow{p_{k,l}} B \mathfrak{G}_{k,l}$ is the tautological bundle $\mathcal{A}_{k,l} \subset Gr_{k,l} \times M_{k,l}$ over the matrix Grassmannian. Let $\tilde{\pi}_{k,l} : \tilde{\mathcal{A}}_{k,l}^{univ} \rightarrow \mathcal{A}_{k,l}^{univ}$ be the composition of the embedding (6) with the projection onto the second factor

$$Gr(\mathcal{A}_{k,l}^{univ}) \times \mathcal{A}_{k,l}^{univ} \xrightarrow{p_{k,l}} B \mathfrak{G}_{k,l} \rightarrow \mathcal{A}_{k,l}^{univ}. $$

Note that the restriction of $\tilde{\pi}_{k,l}$ to a fiber $Gr_{k,l}$ coincides with the canonical map $\pi_{k,l} : \mathcal{A}_{k,l} \rightarrow M_{k,l}$, in particular it is surjective.

By definition of the pullback we have $p^*_{k,l}(\mathcal{A}_{k,l}^{univ}) = Gr(\mathcal{A}_{k,l}^{univ}) \times \mathcal{A}_{k,l}^{univ}$; from the other hand, it follows from Proposition 24 that $Gr(\mathcal{A}_{k,l}^{univ}) \times \mathcal{A}_{k,l}^{univ} = F_{k,l}(\tilde{\mathcal{A}}_{k,l}^{univ})$, 

30
therefore the map $p_{k,l}: \text{Gr}(A_{k,l}^{\text{univ}}) \to B \mathfrak{G}_{k,l}$ can be considered as a classifying map for the bundle $F_{k,l}(A_{k,l}^{\text{univ}})$ as an $M_{k,l}$-bundle.

Let $\tilde{t}_{k,l}: \text{Gr}(A_{k,l}^{\text{univ}}) \to \text{BPU}(k)$ be a classifying map for $\tilde{A}_{k,l}^{\text{univ}}$ as an $M_{k}(\mathbb{C})$-bundle (note that the restriction of the map $\tilde{t}_{k,l}$ to any fiber coincides with $t_{k,l}: \text{Gr}_{k,l} \to \text{BPU}(k)$ which is a classifyfing map for $A_{k,l}$). Let $f_{k,l}: \text{BPU}(k) \to B \mathfrak{G}_{k,l}$ be a classifying map for the $M_{k,l}$-bundle $F_{k,l}(A_{k,l}^{\text{univ}})$, where $A_{k,l}^{\text{univ}}$ is the universal $M_{k}(\mathbb{C})$-bundle over $\text{BPU}(k)$.

**Proposition 57.** The diagram

$$\begin{array}{ccc}
\text{Gr}(A_{k,l}^{\text{univ}}) & \xrightarrow{\tilde{t}_{k,l}} & \text{BPU}(k) \\
p_{k,l} & & f_{k,l} \\
B \mathfrak{G}_{k,l} & \xleftarrow{t_{k,l}} & 
\end{array}$$

consisting of the classifying maps is commutative (up to homotopy).

**Proof.** It follows from $\tilde{t}_{k,l}^*: A_{k}^{\text{univ}} = \tilde{A}_{k,l}^{\text{univ}}$ that $\tilde{t}_{k,l}^*(F_{k,l}(A_{k}^{\text{univ}})) = F_{k,l}(\tilde{A}_{k,l}^{\text{univ}})$. Hence $\tilde{t}_{k,l}^*(f_{k,l}^*(A_{k,l}^{\text{univ}})) = F_{k,l}(\tilde{A}_{k,l}^{\text{univ}}) = p_{k,l}^*(A_{k,l}^{\text{univ}})$. □

Now let us prove the following main theorem.

**Theorem 58.** The map $\tilde{t}_{k,l}: \text{Gr}(A_{k,l}^{\text{univ}}) \to \text{BPU}(k)$ is a homotopy equivalence.

**Proof.** We shall prove the following assertion which is equivalent to theorem’s statement. Suppose $X$ is a finite CW-complex, then the map $\tilde{t}_{k,l}^*: [X; \text{Gr}(A_{k,l}^{\text{univ}})] \to [X; \text{BPU}(k)]$ is a bijection.

Let $A_{k}$ be an $M_{k}(\mathbb{C})$-bundle over $X$. Consider the $\text{Gr}_{k,l}$-bundle $\text{Gr}(F_{k,l}(A_{k}))$ which is the base of the tautological bundle $\tilde{A}_{k,l} \subset \text{Gr}(F_{k,l}(A_{k})) \times_{X} F_{k,l}(A_{k})$. Note that the subbundle $A_{k} \subset F_{k,l}(A_{k})$ determines a section $\sigma: X \to \text{Gr}(F_{k,l}(A_{k}))$ such that $\sigma^*(\tilde{A}_{k,l}) = A_{k}$.

Now suppose $\varphi: X \to B \mathfrak{G}_{k,l}$ is a classifying map for $F_{k,l}(A_{k})$, then since $\text{Gr}$ is a functor, we also have the map $\text{Gr}(\varphi): \text{Gr}(F_{k,l}(A_{k})) \to \text{Gr}(A_{k,l}^{\text{univ}})$. Then $(\text{Gr}(\varphi) \circ \sigma)^*(\tilde{A}_{k,l}^{\text{univ}}) = A_{k}$, i.e. $(\text{Gr}(\varphi) \circ \sigma)$ is a classifying map for $A_{k}$, and therefore $\tilde{t}_{k,l}^*: [X; \text{Gr}(A_{k,l}^{\text{univ}})] \to [X; \text{BPU}(k)]$ is surjective.

Now let us prove that the map $\tilde{t}_{k,l}^*: [X; \text{Gr}(A_{k,l}^{\text{univ}})] \to [X; \text{BPU}(k)]$ is injective. It requires some preparation.

Let $\varphi: X \to \text{Gr}(A_{k,l}^{\text{univ}})$ be a classifying map for a bundle $A_{k}$ over $X$. Then we have the pullback diagram

$$\begin{array}{ccc}
A_{k} & \xrightarrow{\tilde{\varphi}} & \tilde{A}_{k,l}^{\text{univ}} \\
\downarrow & & \downarrow \\
X & \xrightarrow{\varphi} & \text{Gr}(A_{k,l}^{\text{univ}})
\end{array}$$

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(the vertical arrows are the projections of bundles). The map $\tilde{\varphi}$ can be extended to a map of fan bundles $F_{k,l}(\tilde{\varphi}) : F_{k,l}(A_k) \to F_{k,l}(\tilde{A}_{k,l}^{univ})$. According to Proposition 52 we have $F_{k,l}(\tilde{A}_{k,l}^{univ}) = p_{k,l}^*(\mathfrak{A}_{k,l}^{univ})$, i.e. we have the pullback diagram

$$
\begin{array}{c}
F_{k,l}(\tilde{A}_{k,l}^{univ}) \\
\downarrow \\
\text{Gr}(\mathfrak{A}_{k,l}^{univ}) \\
\downarrow \\
B \mathfrak{G}_{k,l}.
\end{array}
$$

Consider the compositions $\psi := \tilde{\mu}_{k,l} \circ F_{k,l}(\tilde{\varphi}) : F_{k,l}(A_k) \to \mathfrak{A}_{k,l}^{univ}$ and $\varphi := p_{k,l} \circ \varphi : X \to B \mathfrak{G}_{k,l}$. Clearly, $\tilde{\varphi}$ is a classifying map for $F_{k,l}(A_k)$ as an $\mathbb{M}_{k,l}$-bundle and the diagram

$$
\begin{array}{c}
F_{k,l}(A_k) \\
\downarrow \psi \\
\mathfrak{A}_{k,l}^{univ} \\
\downarrow \\
X \\
\varphi \\
\downarrow \\
B \mathfrak{G}_{k,l}
\end{array}
$$

is a pullback diagram.

We stress that by definition the map $\psi$ can be lifted to the map $F_{k,l}(\varphi) : F_{k,l}(A_k) \to F_{k,l}(\tilde{A}_{k,l}^{univ})$, whose restriction to the subbundle $A_k \subset F_{k,l}(A_k)$ (clearly, we can consider $A_k$ as a subbundle in $F_{k,l}(A_k)$), coincides with the map $\tilde{\varphi} : A_k \to \tilde{A}_{k,l}^{univ}$ covering the classifying map $\varphi : X \to \text{Gr}(\mathfrak{A}_{k,l}^{univ})$.

Now we can conclude theorem’s prove. Suppose there are two classifying maps $\varphi_i : X \to \text{Gr}(\mathfrak{A}_{k,l}^{univ})$, $i = 1, 2$ for the bundle $A_k$ over $X$; we have to prove that they are homotopic to each other. Since the corresponding maps $\tilde{\varphi}_i : X \to B \mathfrak{G}_{k,l}$, $i = 1, 2$ are classifying maps for $F_{k,l}(A_k)$, and $B \mathfrak{G}_{k,l}$ is a classifying space for locally trivial bundles with fiber $\mathbb{M}_{k,l}$, then the maps $\tilde{\varphi}_1$ and $\tilde{\varphi}_2$ are homotopic. Let $\Phi : X \times I \to B \mathfrak{G}_{k,l}$ be a homotopy between them. Let $p_1 : X \times I \to X$ be the projection onto the second factor. By $\tilde{A}_k$ denote the bundle $p_1^*(A_k)$ over $X \times I$. Consider the corresponding pullback diagram

$$
\begin{array}{c}
F_{k,l}(\tilde{A}_k) \\
\downarrow \psi \\
\mathfrak{A}_{k,l}^{univ} \\
\downarrow \\
X \times I \\
\Phi \\
\downarrow \\
B \mathfrak{G}_{k,l}.
\end{array}
$$

Note that its restrictions to $X \times \{0\}$ and $X \times \{1\} \subset X \times I$ coincide with the corresponding diagrams (7) for $\varphi_i$, $i = 1, 2$. Now in order to obtain a homotopy between $\varphi_1$ and $\varphi_2$ it is sufficient to apply the functor $\text{Gr}$ to the map $\Psi$. Indeed, suppose $\tilde{\sigma} : X \times I \to \text{Gr}(F_{k,l}(\tilde{A}_k))$ is the section corresponding to the subbundle $\tilde{A}_k \subset F_{k,l}(\tilde{A}_k)$, then $\text{Gr}(\Psi) \circ \tilde{\sigma} : X \to \text{Gr}(\mathfrak{A}_{k,l}^{univ})$ determines the required homotopy. □

Remark 59. Note that the homotopy equivalence $\tilde{\iota}_{k,l}$ identifies the universal $M_k(\mathbb{C})$-bundle $A_k^{univ}$ over $\text{BPU}(k)$ with the tautological bundle $\tilde{A}_{k,l}^{univ}$ over $\text{Gr}(\mathfrak{A}_{k,l}^{univ})$. In particular, the bundle $A_k^{univ}$ is glued by means of the universal $\mathfrak{G}_{k,l}$-cocycle over $B \mathfrak{G}_{k,l}$.
by using the natural action $G_{k,l} \times A_{k,l} \to A_{k,l}$. It is interesting to compare this result with Corollary 39.

Consider the sequence

$$
\begin{array}{c}
\text{Gr}_{k,l} \xrightarrow{t_{k,l}} \text{BPU}(k) \xrightarrow{f_{k,l}} \text{B}G_{k,l},
\end{array}
$$

(8)

where $t_{k,l}$ denotes a classifying map for the core $A_{k,l}$ of the canonical FAB over $\text{Gr}_{k,l}$ as a PU($k$)-bundle.

**Corollary 60.** Sequence (8) is a fibration.

*Proof* follows from the commutative diagram

![Diagram](image)

where $i_{k,l}: \text{Gr}_{k,l} \to \text{Gr}(A_{\text{univ}}_{k,l})$ is the embedding of a fiber of the bundle $\tilde{p}_{k,l}: \text{Gr}(A_{\text{univ}}_{k,l}) \to \text{B}G_{k,l}$. □

**Remark 61.** In Remark 35 we have already noticed that there is the exact sequence of pointed sets:

$$
\{\text{isomorphism classes of FABs of the form } (A_k, \mu, \tilde{M}_{kl})\} \xrightarrow{} \{\text{isomorphism classes of } \text{M}_k(\mathbb{C})-\text{bundles}\} \xrightarrow{} \{\text{isomorphism classes of } \text{M}_{k,l}-\text{bundles}\},
$$

which is induced by the functors $T_{k,l}$ and $F_{k,l}$ (where the functor $T_{k,l}$ to any FAB $(A_k, \mu, \tilde{M}_{kl})$ assigns the $\text{M}_k(\mathbb{C})$-bundle $A_k$). Since the matrix Grassmannian $\text{Gr}_{k,l}$ is a classifying space for FABs of the form $(A_k, \mu, \tilde{M}_{kl})$ (at least over finite CW-complexes of dimension $\leq 2 \min\{k, l\}$, see the Appendix), and analogously, the spaces $\text{BPU}(k)$, $\text{B}G_{k,l}$ are classifying spaces for locally trivial bundles with fiber $\text{M}_k(\mathbb{C})$ and $\text{M}_{k,l}$ respectively, then fibration (8) corresponds to this exact sequence.

**Corollary 62.** There is a weak homotopy equivalence between the topological group $G_{k,l}$ and the frame space $\text{Fr}_{k,l}$.

*Proof* follows from the fact that $\text{Fr}_{k,l}$ is a fiber of the fibration $t_{k,l}: \text{Gr}_{k,l} \to \text{BPU}(k)$. □
4 The stable theory of $\mathbb{M}_{k,l}$-bundles

4.1 A stable equivalence relation

Let $X$ be a finite CW-complex. Since the tensor product of fans $\mathbb{M}_{k,l}$ and $\mathbb{M}_{m,n}$, $(km,ln) = 1$ is a continuous functor (Proposition 51), we see that it can be transferred to fan bundles. Thus, the tensor product of fan bundles with fibers $\mathbb{M}_{k,l}$, $\mathbb{M}_{m,n}$, $(km,ln) = 1$ is a well-defined fan bundle with fiber $\mathbb{M}_{km,ln}$.

By $\mathbb{M}_{k,l}$ denote a trivial $\mathbb{M}_{k,l}$-bundle over $X$.

**Definition 63.** Two fan bundles $\mathcal{A}_{km,l^n}$ and $\mathcal{B}_{k^n,l^t}$ over $X$ are said to be stable equivalent if there are positive integers $t, u, v, w$ such that

$$\mathcal{A}_{km,l^n} \otimes \mathbb{M}_{k,l} \cong \mathcal{B}_{k^n,l^t} \otimes \mathbb{M}_{k,l}$$

(of course, $m + t = r + v$, $n + u = s + w$).

By $[\mathcal{A}_{km,l^n}]$ denote the stable equivalence class of a bundle $\mathcal{A}_{km,l^n}$.

By $\text{Fan}_{k,l}^{t,n}(X)$ denote the set of stable equivalence classes of fan bundles over $X$ with fibers of the form $\mathbb{M}_{km,l^n}$ (for arbitrary $m, n$).

Using the group homomorphism $\phi_{kl, mn} : \mathcal{G}_{k,l} \times \mathcal{G}_{m,n} \to \mathcal{G}_{km,ln}$ from Proposition 47 we define the homomorphism $\tau_{kl, mn} : \mathcal{G}_{k,l} \to \mathcal{G}_{km,ln}$ as $g \mapsto \phi_{kl, mn}(g, e) \forall g \in \mathcal{G}_{k,l}$, where by $e$ we denote the unit of the group $\mathcal{G}_{k,l}$. (It follows from Proposition 48 that in fact it is a group embedding). It follows from Proposition 50 that the homomorphism $\tau_{kl, mn}$ is continuous. Therefore $\tau_{kl, mn} : \mathcal{G}_{k,l} \to \mathcal{G}_{km,ln}$ induces a continuous map of the classifying spaces $B(\tau_{kl, mn}) : B\mathcal{G}_{k,l} \to B\mathcal{G}_{km,ln}$. By $\mathcal{G}_{k^\infty,l^\infty}$, $B\mathcal{G}_{k^\infty,l^\infty}$ denote the direct limits $\lim_{\rightarrow n} \mathcal{G}_{k^n,l^n}$, $\lim_{\rightarrow n} B\mathcal{G}_{k^n,l^n}$, respectively.

By $\text{Fr}_{k^\infty,l^\infty}$ denote the direct limit $\lim_{\rightarrow n} \text{Fr}_{k^n,l^n}$ with respect to the following maps $\alpha_n$:

$$
\begin{array}{ccc}
PU(l^n) & & \text{PU}((kl)^n) \\
\downarrow_{\otimes E_l} & & \downarrow_{\otimes E_l} \\
\text{Fr}_{k^n,l^n} & & \text{Fr}_{k^\infty,l^\infty}
\end{array}
$$

(the rows are fibrations).

**Proposition 64.** The homotopy groups of the space $B\mathcal{G}_{k^\infty,l^\infty}$ are following:

$$\pi_r(B\mathcal{G}_{k^\infty,l^\infty}) = \lim_{\rightarrow n} \mathbb{Z}/k^n\mathbb{Z} \quad \text{for even} \ r \quad \text{and} \ 0 \quad \text{otherwise}.$$  

**Proof.** One can easily verify that in the stable dimensions $r \pi_r(\text{Fr}_{k^n,l^n}) = \mathbb{Z}/k^n\mathbb{Z}$ for odd $r \geq 1$ and $0$ otherwise. Furthermore, it follows from the above diagram that $\alpha_{n+1}$ induces the embedding

$$\pi_{2s-1}(\text{Fr}_{k^n,l^n}) \hookrightarrow \pi_{2s-1}(\text{Fr}_{k^{n+1},l^{n+1}}), \quad \mathbb{Z}/k^n\mathbb{Z} \hookrightarrow \mathbb{Z}/k^{n+1}\mathbb{Z}, \quad s \geq 1.$$
Hence $\pi_1(F_{k,\infty}^n, t^\infty) = \lim_{n \to \infty} \mathbb{Z}/k^n\mathbb{Z}$ for odd $r$ and 0 otherwise.

It follows from Corollary 62 that the spaces $\mathcal{G}_{k, l}$ and $F_{k, l}$ are weakly homotopy equivalent. Moreover, it is clear that the diagram

$$
\begin{array}{ccc}
F_{k,n+1, l,n+1} & \xrightarrow{\sim} & \mathcal{G}_{k,n+1, l,n+1} \\
\alpha_{n+1} & | & \downarrow \iota_{n+1} \\
F_{k,n, l,n} & \xrightarrow{\sim} & \mathcal{G}_{k,n, l,n}
\end{array}
$$

commutes, where we set $\iota_{n+1} := \iota_{(kl)^n,(kl)^{n+1}} : \mathcal{G}_{k,n, l,n} \to \mathcal{G}_{k,n+1, l,n+1}$. Now the equivalence $\Omega(B\mathcal{G}_{k, l}) = \mathcal{G}_{k, l}$ completes the proof. □

We can simplify our notation with the help of the following lemma.

**Lemma 65.** The homotopy type of the space $F_{k,\infty}^\infty$ does not depend on the choice of $l$, $(k, l) = 1$.

**Proof.** Clearly, $F_{k,\infty}^\infty$ is a fiber of the map

$$
\text{Gr}_{k,\infty}^\infty \xrightarrow{t_{k,\infty}^\infty} \text{BPU}(k^\infty),
$$

where $t_{k,\infty}^\infty$ is a classifying map for the core $A_{k,\infty}^\infty$ of the universal FAB over $\text{Gr}_{k,\infty}^\infty$. But the homotopy type of $\text{Gr}_{k,\infty}^\infty$ does not depend on the choice of $k, l$, $(k, l) = 1$, moreover, $\text{Gr}_{k,\infty}^\infty \simeq \text{BSU}$. The map $t_{k,\infty}^\infty$ actually does not depend on $l$, because for $k, l, m$, $(k, lm) = 1$ the diagram

$$
\begin{array}{ccc}
\text{Gr}_{k,\infty}^\infty & \xrightarrow{\sim} & \text{Gr}_{k,\infty}^\infty \\
\downarrow t_{k,\infty}^\infty & & \downarrow t_{k,\infty}^\infty \\
\text{Gr}_{k,\infty}^\infty & \xrightarrow{\sim} & \text{Gr}_{k,\infty}^\infty
\end{array}
$$

commutes. Thus, we see that $\text{Gr}_{k,\infty}^\infty$ and $t_{k,\infty}^\infty$ in (9) do not depend on $l$ under the condition $(l, k) = 1$, and therefore this is also true for the fiber $F_{k,\infty}^\infty$.

□

**Remark 66.** Another proof of the previous lemma can be obtained in the following way. Suppose $(k, lm) = 1$, then the maps in the diagram

$$
\begin{array}{ccc}
F_{k, lm} & \xrightarrow{\sim} & F_{k, lm} \\
\downarrow & & \downarrow \\
F_{k, l} & \xrightarrow{\sim} & F_{k, l} \\
\downarrow & & \downarrow \\
F_{k, m} & \xrightarrow{\sim} & F_{k, m}
\end{array}
$$

are homotopy equivalences in small dimensions. Thus, $F_{k,\infty}^\infty \simeq F_{k,\infty}^\infty (lm) \simeq F_{k,\infty}^\infty (m)$.  

Corollary 67. The homotopy types of the spaces $\mathcal{G}_{k,\infty}, B\mathcal{G}_{k,\infty}$ do not depend on the choice of $l$, $(k, l) = 1$.

Proof follows from the previous lemma and Corollary 62. □

Therefore we can omit $l$ in notation $Fr_{k,\infty}, \mathcal{G}_{k,\infty},$ and etc.

The proof of the following theorem is standard.

Theorem 68. The functor $X \mapsto \text{Fan}^{k,l}(X)$ is represented by the space $B\mathcal{G}_{k,\infty}$.

Corollary 69. The functors $X \mapsto \text{Fan}^{k,l}(X)$ corresponding to different $l$, $(k, l) = 1$ are naturally equivalent to each other.

Thus, we can write just $\text{Fan}^{k}(X)$ instead of $\text{Fan}^{k,l}(X)$.

Define the product of two stable equivalence classes $[\mathcal{A}_{km,tn}], [\mathcal{B}_{kn,ts}]$ over $X$ as

$$[\mathcal{A}_{km,tn}] \cdot [\mathcal{B}_{kn,ts}] := [\mathcal{A}_{km,tn} \otimes \mathcal{B}_{kn,ts}].$$

Clearly, the product is well defined.

Lemma 70. The order of any stable equivalence class of fan bundles over a finite CW-complex $X$ with respect to the above-defined operation is finite, i.e. there is $w \in \mathbb{N}$ such that $[\mathcal{A}_{km,tn}]^{k^{w}}$ is the stable equivalence class of a trivial bundle.

Proof follows from Theorem 68 and the fact that $\pi_r(B\mathcal{G}_{k,\infty}) = \varprojlim_{n} \mathbb{Z}/k^n\mathbb{Z}$ for even $r$ and 0 otherwise. □

It follows from the previous lemma that any stable equivalence class of fan bundles is invertible.

Corollary 71. The product of stable equivalence classes induces a structure of Abelian group on the set $\text{Fan}^{k}(X)$. In other words, $B\mathcal{G}_{k,\infty}$ is an $H$-space.

Remark 72. Note that the product on the $H$-space $B\mathcal{G}_{k,\infty}$ is defined by the maps

$$B(\hat{\phi}_{km,tn,kn},kn,tn) : B\mathcal{G}_{km,tn} \times B\mathcal{G}_{kn,ts} \rightarrow B\mathcal{G}_{km+tn,kn+ts}$$

which classify the tensor product of universal fan bundles over $B\mathcal{G}_{km,tn}$ and $B\mathcal{G}_{kn,ts}$ as an $M_{km,tn} \otimes M_{kn,ts} = M_{km+tn,kn+ts}$-bundle.

Recall that in Corollary 60 we have considered the fibration

$$Gr_{kn,tn} \rightarrow BPU(k^n) \rightarrow B\mathcal{G}_{kn,tn}.$$

Taking the limit as $n \rightarrow \infty$, we get the following fibration

$$Gr_{k\infty} \rightarrow BPU(k) \rightarrow B\mathcal{G}_{k\infty}, \quad (11)$$

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where $\text{Gr} := \text{Gr}_{k, \infty}$, $\text{BPU}(k^\infty) := \lim_{\rightarrow n}\text{BPU}(k^n)$ with respect to the maps induced by the group homomorphisms $\text{PU}(k^n) \to \text{PU}(k^{n+1})$, $A \mapsto A \otimes E_k$, and $t_{k^\infty} := \lim_{\rightarrow n}t_{k^n, l^n}$, $f_{k^\infty} := \lim_{\rightarrow n}f_{k^n, l^n}$.

Recall that $\text{Gr}$ and $\text{BPU}(k^\infty)$ are $H$-spaces. More precisely, $\text{Gr}$ is a classifying space for stable equivalence classes of FABs with the operation induced by the tensor product of FABs (see the Appendix). The $H$-space $\text{BPU}(k^\infty)$ is a classifying space for classes of locally trivial bundles with fibers $M_{k^m}(\mathbb{C})$, $m \in \mathbb{N}$ with respect to the following equivalence relation:

$$A_{k^m} \sim B_{k^n} \Leftrightarrow \exists r, s \text{ such that } A_{k^m} \otimes \tilde{M}_{k^r} \sim B_{k^n} \otimes \tilde{M}_{k^s},$$

where $\tilde{M}_{k^m}$ is a trivial bundle with fiber $M_{k^m}(\mathbb{C})$. The operation on this equivalence classes is induced by the tensor product of bundles. Thus, we have obtained the interpretation of the groups $\tilde{\text{AB}}^1(X) := [X; \text{Gr}]$ and $\tilde{\text{AB}}^k(X) := [X; \text{BPU}(k^\infty)]$.

**Theorem 73.** For any finite CW-complex $X$ the sequence of Abelian group

$$\tilde{\text{AB}}^1(X) \xrightarrow{t_{k^\infty}} \tilde{\text{AB}}^k(X) \xrightarrow{f_{k^\infty}} \text{Fan}^k(X)$$

is exact.

**Proof** follows from the fact that the composition (11) is a fibration and $F_{km, ln}(A_k \otimes A'_m) = F_{k, l}(A_k) \otimes F_{m, n}(A'_m)$ for all $M_k(\mathbb{C})$-bundles $A_k$ and $M_m(\mathbb{C})$-bundles $A'_m$. □

### 4.2 A unitary version

Consider the spaces $\hat{\text{Gr}}_{k, l}$ and $\hat{\text{Gr}}$ defined in the Appendix. Let $\hat{t}_{k, l}: \hat{\text{Gr}}_{k, l} \to \text{BU}(k)$ be a classifying map for the tautological $U(k)$-bundle over $\hat{\text{Gr}}_{k, l}$. Let $\hat{f}_{k, l}: \text{BU}(k) \to \text{BG}_k$ be a classifying map for the fan bundle $F_{k, l}(\text{End}(\xi_{k}^{\text{univ}}))$, where $\xi_{k}^{\text{univ}}$ is the universal $\mathbb{C}^k$-bundle over $\text{BU}(k)$. In particular, the diagram

$$\begin{array}{ccc}
\text{BU}(k^\infty) & \xrightarrow{\hat{f}_{k^\infty}} & \text{BG}_k \\
\downarrow \quad \quad \quad & & \quad \quad \quad \downarrow \\
\text{BPU}(k^\infty) & \xrightarrow{\hat{f}_{k^\infty}} & \text{BG}_k
\end{array}$$

commutes.

It is easy to prove that the sequence

$$\hat{\text{Gr}}_{k, l} \xrightarrow{\hat{t}_{k, l}} \text{BU}(k) \xrightarrow{\hat{f}_{k, l}} \text{BG}_k$$

is a fibration. Taking the limit, we get the fibration

$$\hat{\text{Gr}} \xrightarrow{\hat{t}_{k^\infty}} \text{BU}(k^\infty) \xrightarrow{\hat{f}_{k^\infty}} \text{BG}_k.$$  \hspace{1cm} (12)
Proposition 74. The sequence (12) is a fibration.

Proof is trivial. □

It has already been mentioned in the Appendix that the map \( \hat{t}_k : \hat{\text{Gr}} \to \text{BU}(k^\infty) \) is a localization at \( k \) (in particular, \( \text{BU}(k^\infty) \) is a \( Z[\frac{1}{k}] \)-local space). The existence of fibration (12) agrees with the fact that \( \text{Fr}_k^\infty \simeq \Omega(\text{B} \mathcal{G}_k^\infty) \) is a fiber of the localization map \( \hat{t}_k : \hat{\text{Gr}} \to \text{BU}(k^\infty) \).

Finally note that fibration (12) is a \( U \)-analog of fibration (11):

\[
\text{Gr} \xrightarrow{t_k^\infty} \text{BPU}(k^\infty) \xrightarrow{f_k^\infty} \text{B} \mathcal{G}_k^\infty.
\]

4.3 \( 0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0 \)

Let us introduce the following notation for direct limits

\[
\text{B} \mathcal{U} := \lim_k \text{BU}(k), \quad \text{B} \mathcal{P} \mathcal{U} := \lim_k \text{BPU}(k), \quad \text{B} \mathcal{G} := \lim_k \text{B} \mathcal{G}_k,
\]

where \( k \) runs over all positive integers and the limits are taken with respect to the maps corresponding to the tensor product of bundles (in particular, \( \text{B} \mathcal{U} \) is a \( \mathbb{Q} \)-space).

Remark 75. In this remark we give a more detailed definition of the direct limit \( \text{B} \mathcal{G} = \lim_{(k,l)\to(1)} \text{B} \mathcal{G}_{k,l} \).

Let \( p_1, p_2, p_3, \ldots \) be the sequence of primes \( p_1 = 2, p_2 = 3, p_3 = 5, \ldots \). For any \( r \in \mathbb{N} \) we can choose \( q_r \) such that \( (p_1 \ldots p_r, q_r) = 1 \). Suppose \( (p_1 \ldots p_r p_{r+1}, q_r) > 1 \). It follows from Proposition 62 and (10) that there are homotopy equivalences

\[
\text{B} \mathcal{G}_{(p_1 \ldots p_r)^\infty, q_r^\infty} \xrightarrow{\simeq} \text{B} \mathcal{G}_{(p_1 \ldots p_r)^\infty, (q_r q_{r+1})^\infty} \xrightarrow{\simeq} \text{B} \mathcal{G}_{(p_1 \ldots p_r)^\infty, q_{r+1}^\infty}.
\]

Therefore we have the composite map

\[
\text{B} \mathcal{G}_{(p_1 \ldots p_r)^\infty, q_r^\infty} \xrightarrow{\simeq} \text{B} \mathcal{G}_{(p_1 \ldots p_r)^\infty, q_{r+1}^\infty} \to \text{B} \mathcal{G}_{(p_1 \ldots p_r p_{r+1})^\infty, q_{r+1}^\infty},
\]

and we can take the direct limit with respect to such maps.

The sequences (11) and (12) give the following fibrations:

\[
\text{Gr} \xrightarrow{i} \text{B} \mathcal{P} \mathcal{U} \xrightarrow{f} \text{B} \mathcal{G} \quad (13)
\]

and

\[
\text{Gr} \xrightarrow{i} \text{B} \mathcal{U} \xrightarrow{f} \text{B} \mathcal{G}, \quad (14)
\]

respectively.

The relation between (13) and (14) is very clear. More precisely, recall that

\[
\text{Gr} = \text{Gr} \times \mathbb{C}P^\infty, \quad \text{B} \mathcal{U} = \prod_{q \geq 1} K(2q, \mathbb{Q}), \quad \text{B} \mathcal{P} \mathcal{U} = K(2, \mathbb{Q}/\mathbb{Z}) \times \prod_{q \geq 2} K(2q, \mathbb{Q}), \quad (15)
\]
\[ CP^\infty = K(2, \mathbb{Z}), \text{ and we have the fibration} \]
\[ CP^\infty \hookrightarrow K(2, \mathbb{Q}) \to K(2, \mathbb{Q}/\mathbb{Z}) \]

Corresponding to the exact sequence of groups in the title of this subsection. Therefore (13) can be obtained from (14) with the help of the “factorization” by \( CP^\infty \).

Note that sequence (14) relates something “integer” (\( \hat{\text{Gr}} \)), something “rational” (\( B\mathcal{U} \)) and something “finite” (\( B\mathfrak{S} \)). Indeed, the sequence of homotopy groups
\[ \pi_r(\hat{\text{Gr}}) \overset{\text{ch}}{\to} \pi_r(B\mathcal{U}) \overset{\text{ch}}{\to} \pi_r(B\mathfrak{S}) \]

coincides with
\[ \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \]

for even \( r \) and \( 0 \to 0 \to 0 \) for odd \( r \).

5 The generalization of the topological Brauer group

5.1 More about \( H \)-spaces \( B\mathcal{U} \) and \( B\mathcal{P}\mathcal{U} \)

Recall that we consider the spaces \( B\mathcal{U} \) and \( B\mathcal{P}\mathcal{U} \) as \( H \)-spaces with respect to the multiplication induced by the tensor product of vector \( \mathcal{U} \)-bundles and \( \mathcal{P}\mathcal{U} \)-bundles with fiber a matrix algebra, respectively.

Consider \( \prod_{q \geq 1} K(2q, \mathbb{Q}) \) as an \( H \)-space with respect to the multiplication induced by the product of formal series of the form \( 1 + x_1 + x_2 + \ldots, \ x_i \in H^{2i}(X; \mathbb{Q}) \). We claim that \( H \)-spaces \( B\mathcal{U} \) and \( \prod_{q \geq 1} K(2q, \mathbb{Q}) \) are isomorphic to each other. Indeed, it follows from the identity \( \text{ch}(\xi \otimes \eta) = \text{ch}(\xi)\text{ch}(\eta) \) for any vector bundles \( \xi, \eta \), where \( \text{ch} \) is the Chern character.

Note that for the special unitary group \( SU \) we have the similar isomorphism
\[ B\mathcal{SU} \overset{\text{ch}}{\to} \prod_{q \geq 2} K(2q, \mathbb{Q}), \]

where \( \prod_{q \geq 2} K(2q, \mathbb{Q}) \) is considered as an \( H \)-space with respect to the multiplication induced by the product of formal series of the form \( 1 + x_2 + x_3 + \ldots, \ x_i \in H^{2i}(X; \mathbb{Q}) \).

Lemma 76. The \( H \)-space \( B\mathcal{P}\mathcal{U} \) is isomorphic to the product of \( H \)-spaces \( K(2, \mathbb{Q}/\mathbb{Z}) \times \prod_{q \geq 2} K(2q, \mathbb{Q}) \) (we consider \( K(2, \mathbb{Q}/\mathbb{Z}) \) as an Eilenberg-MacLane space, i.e. with respect to the \( H \)-space structure related to the addition of cohomology classes).

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Proof. Consider the commutative diagram of $H$-spaces:

$$
\begin{array}{ccc}
BPU & \xrightarrow{ch} & \prod_{q \geq 2} K(2q, \mathbb{Q}) \\
\uparrow & \cong & \uparrow \\
B SU. & \end{array}
$$

It follows from the last isomorphism in (15) that $\ker(ch)$ coincides with the $H$-space $K(2, \mathbb{Q}/\mathbb{Z})$. $\square$

5.2 Reminder: the classical Brauer group

Let $X$ be a finite $CW$-complex. Consider the set of isomorphism classes of locally trivial bundles over $X$ with fiber $M_k(\mathbb{C})$ with an arbitrary integer $k > 1$. On the set of such bundles consider the following equivalence relation:

$$A_k \sim B_l \iff A_k \otimes \widetilde{M}_m \cong B_l \otimes \widetilde{M}_n \quad \text{for some } m, n > 1, \quad (16)$$

where by $\widetilde{M}_r$ we denote the trivial bundle over $X$ with fiber $M_r(\mathbb{C})$. By $\{C_m\}$ denote the stable equivalence class of $C_m$. It can easily be checked that the product $\{A_k\} \circ \{B_l\} := \{A_k \otimes B_l\}$ is well defined and equips the set of stable equivalence classes with the structure of Abelian group. We denote this group by $\widetilde{\text{AB}}(X)$.

The $H$-space $BPU$ represents the homotopy functor $X \mapsto \widetilde{\text{AB}}(X)$ on the category of finite $CW$-complexes. Since we take the direct limit $BPU = \lim_{\longrightarrow} BPU(k)$ by the maps of classifying spaces $BPU(k) \xrightarrow{i^k_m} BPU(km)$ such that $i^*_{k,m}(A^\text{univ}) = A^\text{univ} \otimes \widetilde{M}_m$, where by $A^\text{univ}_r$ we denote the universal $M_r(\mathbb{C})$-bundle over $BPU(r)$, the stable equivalence relation (16) appears.

Now consider the following more coarse stable equivalence relation:

$$A_k \sim B_l \iff \exists \xi_m, \eta_n \text{ such that } A_k \otimes \text{End}(\xi_m) \cong B_l \otimes \text{End}(\eta_n), \quad (17)$$

where by $\xi_m, \eta_n$ we denote vector bundles over $X$ of rank $m, n$, respectively. By $[C_m]$ denote the stable equivalence class of $M_m(\mathbb{C})$-bundle $C_m$. The tensor product of bundles induces a group structure on the set of such stable equivalence classes. It is just the classical topological Brauer group $\text{Br}(X)$.

Let us give a homotopic description of $\text{Br}(X)$. Consider the fibration

$$
\begin{array}{ccc}
\mathbb{C}P & \hookrightarrow & BU(k) \\
\downarrow & & \downarrow \\
BPU(k) & \end{array}
$$

corresponding to the exact sequence of groups

$$1 \to U(1) \to U(k) \to PU(k) \to 1.$$
After taking the limit with \( k \to \infty \), the above fibration gives us the fibration

\[
\begin{align*}
\mathbb{C}P^\infty & \overset{i}{\hookrightarrow} BU \\
\downarrow p & \\
BPU & \text{ i.e. } K(\mathbb{Q}/\mathbb{Z}, 2) \times \prod_{q \geq 2} K(\mathbb{Q}, 2q) \to \end{align*}
\]

\[ \text{(19)} \]

It is easy to see that \( Br(X) = \text{coker}\{p_* : [X; BU] \to [X; BPU]\} \), i.e. \( Br(X) = \text{coker}\{[X; K(\mathbb{Q}, 2)] \to [X; K(\mathbb{Q}/\mathbb{Z}, 2)]\} = \text{coker}\{H^2(X; \mathbb{Q}) \to H^2(X; \mathbb{Q}/\mathbb{Z})\} \). Using the exact sequence of cohomology groups induced by the sequence of coefficients

\[
0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0,
\]

we see that \( \text{coker}\{H^2(X; \mathbb{Q}) \to H^2(X; \mathbb{Q}/\mathbb{Z})\} = \text{im}\{H^2(X; \mathbb{Q}/\mathbb{Z}) \xrightarrow{\delta} H^3(X; \mathbb{Z})\} \) (here \( \delta \) is the coboundary homomorphism), i.e. \( Br(X) = H^3_{\text{tors}}(X; \mathbb{Z}) \).

The explicit form of the isomorphism \( Br(X) \cong H^3_{\text{tors}}(X; \mathbb{Z}) \) can be described as follows. Recall that the structure group of a bundle \( A_k \) with fiber \( M_k(\mathbb{C}) \) is \( \text{Aut}(M_k(\mathbb{C})) = \text{PGL}_k(\mathbb{C}) \) and \( \text{PGL}_k(\mathbb{C}) \) contains \( \text{PU}(k) \) as a strong deformation retract. The obstruction theory asserts that the first (and unique!) obstruction for the lifting of \( \text{PU}(k) \)-bundle to a \( \text{U}(k) \)-bundle belongs to the group \( H^3(X; \pi_2(\mathbb{C}P^\infty)) \) (see fibration (18)). Therefore the assignment

\[
A_k \mapsto \{\text{the first obstruction for the lifting}\}
\]

gives us the required description of the isomorphism \( Br(X) \cong H^3_{\text{tors}}(X; \mathbb{Z}) \).

Thus, any element of the group \( H^3_{\text{tors}}(X; \mathbb{Z}) \) can be realized as an obstruction for the lifting of some \( \text{PU} \)-bundle to a \( \text{U} \)-bundle (or equivalent a \( \text{PGL} \)-bundle to a \( \text{GL} \)-bundle).

**Remark 77.** Now suppose \( A_k \) has a lifting \( \xi_k \) (i.e. \( \text{End}(\xi_k) = A_k \)). Then the obstruction theory says that \( \xi_k \) is determined by \( A_k \) up to taking the tensor product with a line bundle \( \zeta \) over \( X \). Indeed, \( H^2(X; \pi_2(\mathbb{C}P^\infty)) = H^2(X; \mathbb{Z}) \) is isomorphic to the group of line bundles with respect to the tensor product; from the other hand, for any line bundle \( \zeta \) we have \( \text{End}(\xi_k \otimes \zeta) = \text{End}(\xi_k) \).

**Remark 78.** Note that for any fixed integer \( k > 1 \) one can develop the corresponding theory of bundles with fibers of the form \( M_{kn}(\mathbb{C}) \) for arbitrary \( n \in \mathbb{N} \). In this way one can define the \( k \)-component of the Brauer group \( Br_k(X) = \text{coker}\{p_{k*} : [X; \text{BU}(k^\infty)] \to [X; \text{BPU}(k^\infty)]\} \) (which is the \( k \)-primary subgroup in the “whole” group \( Br(X) \)).

### 5.3 The generalized Brauer group

Let us return to fibration (14):

\[
\widehat{\text{Gr}} \overset{i}{\to} BU \overset{j}{\to} B\mathfrak{S}.
\]
By Fan we denote the homotopy functor to the category of Abelian groups represented by the $H$-space $B\mathfrak{G}$, i.e. $\text{Fan}(X) = [X; B\mathfrak{G}]$.

Thus, we have the exact sequence of groups

$$[X; \hat{\text{Gr}}] \xrightarrow{\hat{i}} [X; B\mathcal{U}] \xrightarrow{\hat{f}_*} \text{Fan}(X).$$

**Definition 79.** The generalized topological Brauer group $\text{GBr}(X)$ of $X$ is $\text{coker} \hat{f}_* = \text{Fan}(X)/\text{im}(\hat{f}_*)$.

**Remark 80.** Since for a finite CW-complex $X$ $\text{Fan}(X)$ is a finite (Abelian) group, so is $\text{GBr}(X)$. Moreover, its $k$-primary component $\text{GBr}_k(X)$ is just $\text{coker} \hat{f}_{k*} = \text{Fan}^k(X)/\text{im}(\hat{f}_{k*})$.

Let us also remark that fibration (14) can be considered as an analog of fibration (19):

$$\mathbb{C}P^\infty \xrightarrow{i} B\mathcal{U} \xrightarrow{p} B\mathcal{P}U.$$  

In particular, the space $\hat{\text{Gr}} \cong B\mathcal{U}_\otimes$ represents the homotopy functor

$$X \mapsto \{ \text{the group of virtual U-bundles of virtual dimension 1 over } X \},$$

where the group operation is the tensor product of virtual bundles of virtual dimension 1, therefore it is an analog of the $H$-space $\mathbb{C}P^\infty$, which represents the group (of isomorphism classes) of geometric line bundles with respect to the operation induced by the tensor product.

Now consider fibration (13):

$$\text{Gr} \xrightarrow{l} B\mathcal{P}U \xrightarrow{f} B\mathfrak{G}.$$  

By definition, put $\overline{\text{GBr}}(X) := \text{coker}\{ f_* : [X; B\mathcal{P}U] \to [X; B\mathfrak{G}] \}$. Using the fibrations (14) and (13), one can easily prove the following proposition.

**Proposition 81.** There is the exact sequence of functors

$$0 \to \text{Br} \to \text{GBr} \to \overline{\text{GBr}} \to 0.$$  

In other words, $\text{GBr}$ is an extension of the classical Brauer group $\text{Br}$ by $\overline{\text{GBr}}$.

**Proof** follows from the commutative diagram

$$\begin{array}{ccc}
\mathbb{C}P^\infty & \xrightarrow{\hat{f}} & B\mathcal{P}U \\
\downarrow & & \downarrow \\
B\mathcal{P}U & \xrightarrow{f} & B\mathfrak{G}.
\end{array}$$

Now we want to obtain a description of the generalized Brauer group in terms of equivalence classes of fan bundles with respect to some equivalence relation which is an analog of relation (17).
Recall that the $H$-space $B \mathcal{G}_k \infty$ is a classifying space for equivalence classes of fan bundles with fibers of the form $\mathcal{M}_{k^m, l^n}$ (for some $l$ coprime with $k$) with respect to the following equivalence relation:

$$\mathcal{A}_{k^m, l^n} \sim \mathcal{B}_{k^r, l^s} \iff \mathcal{A}_{k^m, l^n} \hat{\otimes} \mathcal{M}_{k^t, l^u} \cong \mathcal{B}_{k^r, l^s} \hat{\otimes} \mathcal{M}_{k^v, l^w},$$

(20)

where by $\mathcal{M}_{k^m, l^n}$ we denote the trivial $\mathcal{M}_{k^m, l^n}$-bundle over $X$. This equivalence relation is an analog of relation (16).

The analog of relation (17) is the following one:

$$\mathcal{A}_{k^m, l^n} \sim \mathcal{B}_{k^r, l^s} \iff \text{there are vector bundles } \xi_{k^t}, \eta_{k^v}$$

with fibers $\mathbb{C}^{k^t}, \mathbb{C}^{k^v}$, respectively, and integers $u, w \geq 1$ such that

$$\mathcal{A}_{k^m, l^n} \hat{\otimes} F_{k^t, l^u}(\text{End}(\xi_{k^t})) \cong \mathcal{B}_{k^r, l^s} \hat{\otimes} F_{k^v, l^w}(\text{End}(\eta_{k^v})).$$

(21)

(recall that $F_{k, l}$ is the functor to a $M_k(\mathbb{C})$-bundle assigning the corresponding $\mathcal{M}_{k, l}$-bundle, in particular, $F_{k, l}(\text{End}(\xi_k))$ is a fan bundle with fiber $\mathcal{M}_{k, l}$). It is natural to denote the composite functor $\xi_k \mapsto \text{End}(\xi_k) \mapsto F_{k, l}(\text{End}(\xi_k))$ by $\hat{F}_{k, l}$.

Note that the tensor product of fan bundles induces a well-defined operation on equivalence classes of equivalence relation (21). Note also that the equivalence classes of bundles of the form $F_{k^t, l^u}(\text{End}(\xi_{k^t}))$ form the image of the homomorphism $\hat{f}_{k^{\infty}} : [X; BU(k^{\infty})] \rightarrow [X; B \mathcal{G}_k \infty]$. Now the proof of the following proposition is clear.

**Proposition 82.** The $k$-primary component $\text{GBr}_k(X)$ of the generalized Brauer group $\text{GBr}(X)$ coincides with the group of equivalence classes of equivalence relation (21).

The similar description can be given for the group $\text{GBr}(X)$. More precisely, consider the following equivalence relation:

$$\mathcal{A}_{k^m, l^n} \sim \mathcal{B}_{k^r, l^s} \iff \text{there are bundles } A_{k^t}, B_{k^v}$$

with fibers $M_k(\mathbb{C}), M_{k^v}(\mathbb{C})$, respectively, and integers $u, w \geq 1$ such that

$$\mathcal{A}_{k^m, l^n} \hat{\otimes} F_{k^t, l^u}(A_{k^t}) \cong \mathcal{B}_{k^r, l^s} \hat{\otimes} F_{k^v, l^w}(B_{k^v}).$$

(22)

Note that the equivalence classes of bundles of the form $F_{k^t, l^u}(A_{k^t})$ form the image of the homomorphism $f_{k^{\infty}} : [X; BU(k^{\infty})] \rightarrow [X; B \mathcal{G}_k \infty]$.

**Proposition 83.** The $k$-primary component $\text{GBr}_k(X)$ of the group $\text{GBr}(X)$ coincides with the group of equivalence classes of relation (22).
Remark 84. (compare with Remark 77). Suppose a fan bundle $\mathfrak{A}_{k,l}$ over $X$ has a lifting $\mathfrak{A}_k$ (i.e. $\mathfrak{A}_{k,l} = F_{k,l}(A_k)$ for some $M_k(\mathbb{C})$-bundle $A_k$). Then the bundle $A_k$ is determined by the equivalence class (20) of $\mathfrak{A}_{k,l}$ up to taking the tensor product with the core of some FAB. In particular, liftings of the stable equivalence class (20) of a trivial fan bundle are precisely cores of FABs over $X$; thus, we see that FABs play the same role as (geometric) line bundles. This result follows from fibration (13). The similar result is true for fibration (14).

Remark 85. Recall that we defined the functor $\mathbf{R}_{k,l}$ which to any $M_{kl}(\mathbb{C})$-bundle $B_{kl}$ assigns the corresponding $M_{k,l}$-bundle. It determines the map $r_{kl}: BPU(kl) \to B\mathfrak{S}_{k,l}$ which is a fibration with fiber $BPU(l)$. Therefore one can study the obstructions for the lifting of an $M_{k,l}$-bundle to a $M_{kl}(\mathbb{C})$-bundle. We want to show that is the stable case this problem can be reduced to the one studied above.

It is sufficient to prove that in the case of a finite CW-complex $X$ for sufficiently large $n$ every $M_{(kl)^n}(\mathbb{C})$-bundle $B_{(kl)^n}$ has the form $A_{k^n} \otimes C_{l^n}$ for some $M_{k^n}(\mathbb{C})$-bundle $A_{k^n}$ and $M_{l^n}(\mathbb{C})$-bundle $C_{l^n}$.

First, applying the obstruction theory to the fibration

$$
\xymatrix{ \text{Gr}_{k^n,l^n} \ar[r] & \text{BSU}(k^n) \times \text{BSU}(l^n) \\
 & \text{BSU}((kl)^n), }
$$

we obtain that the map $[X, \text{BSU}(k^n) \times \text{BSU}(l^n)] \to [X, \text{BSU}((kl)^n)]$ is surjective (for $n$ such that $\dim(X) \leq 2 \min\{k^n, l^n\}$), i.e. the required result holds for bundles whose structure group reduces to the special linear group.

Going over to the general case, note that it follows from the described theory of the classical Brauer group that the map

$$[X, BPU(k^n)] \to [X, K(\mu_{k^n}, 2)]$$

inducing by the projection of the fibration

$$\text{BSU}(k^n) \hookrightarrow \text{BPU}(k^n) \to K(\mu_{k^n}, 2)$$

is surjective for sufficiently large $n$ (here $\mu_{k^n} \cong \mathbb{Z}/k^n\mathbb{Z}$ is the group of degree $k^n$ roots of unity). Now the required assertion can be proved by the diagram search using the
exactness of rows and columns in the following diagram:

\[
\begin{array}{cccccccc}
[X, \text{Gr}_{kn}, l_n] & \longrightarrow & [X, \text{BSU}(k^n) \times \text{BSU}(l^n)] & \longrightarrow & [X, \text{BSU}((kl)^n)] & \longrightarrow & 0 \\
\cong & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
[X, \text{BSU}(k^n) \times \text{BSU}(l^n)] & \longrightarrow & [X, \text{BPU}(k^n) \times \text{BPU}(l^n)] & \longrightarrow & [X, \text{BPU}((kl)^n)] & \longrightarrow & 0 \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
[X, \text{K}(\mu_k^n, 2) \times \text{K}(\mu_l^n, 2)] & \cong & [X, \text{K}(\mu_{kl}^n, 2)] & \longrightarrow & 0 \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
[X, \text{K}(\mu_{kl}^n, 2)] & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 \\
\end{array}
\]

Suppose a finite CW-complex $X$ is obtained by applying the forgetful functor to some algebraic variety $\tilde{X}$ over $\mathbb{C}$. Then the group $\text{Br}(X)$ is not only a homotopy invariant of $X$ but also a birational invariant of $\tilde{X}$.

**Conjecture 86.** The group $\text{GBr}(X)$ is a birational invariant of $\tilde{X}$.

### 6 Appendix: The functor represented by the $H$-space $\text{Gr}$

In this appendix we give a brief survey of the stable theory of floating algebra bundles, for more details see [3].

#### 6.1 Floating algebra bundles

Let $X$ be a finite CW-complex. By $\tilde{M}_n$ denote a trivial bundle (over $X$) with fiber $M_n(\mathbb{C})$, where $M_n(\mathbb{C})$ is the algebra of all $n \times n$ matrices over $\mathbb{C}$.

**Definition 87.** Let $A_k$ ($k > 1$) be a locally trivial bundle over $X$ with fiber $M_k(\mathbb{C})$. Assume that there is a bundle map $\mu$

\[
\begin{array}{ccc}
A_k & \longrightarrow & \tilde{M}_{kl} \\
\downarrow & \downarrow & \downarrow \\
X & \longrightarrow & X
\end{array}
\]

such that for any point $x \in X$ it embeds the fiber $(A_k)_x \cong M_k(\mathbb{C})$ into the fiber $(\tilde{M}_{kl})_x \cong M_{kl}(\mathbb{C})$ as a central simple subalgebra, and the positive integers $k, l$ are relatively prime (i.e. their greatest common divisor $(k, l) = 1$). Then the triple $(A_k, \mu, \tilde{M}_{kl})$ is called a floating algebra bundle (abbrev. FAB) over $X$.

**Remark 88.** Let $A$ be a central simple algebra over a field $\mathbb{K}$, $B \subset A$ a central simple subalgebra in $A$. It is well known that the centralizer $Z_A(B)$ of $B$ in $A$ is a central

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simple subalgebra in $A$ again, moreover, the equality $A = B \otimes Z_A(B)$ holds. Taking centralizers for all fibers of the subbundle $A_k \subset \widetilde{M}_{kl}$ in the corresponding fibers of the trivial bundle $\widetilde{M}_{kl}$, we get the complementary subbundle $B_l$ with fiber $M_l(\mathbb{C})$ together with its embedding $\nu: B_l \hookrightarrow \widetilde{M}_{kl}$. Thus, we have the complementary subbundle $\nu(B_l) \subset \widetilde{M}_{kl}$, where $B_l$ is a locally trivial bundle with fiber $M_l(\mathbb{C})$, and $\nu$ is its evident embedding into $\widetilde{M}_{kl}$. Moreover, $A_k \otimes B_l = \widetilde{M}_{kl}$.

Conversely, to a given pair $(A_k, B_l)$ consisting of $M_k(\mathbb{C})$-bundle $A_k$ and $M_l(\mathbb{C})$-bundle $B_l$ over $X$ such that $A_k \otimes B_l = \widetilde{M}_{kl}$, we can construct a unique triple $(A_k, \mu, \widetilde{M}_{kl})$, where $\mu$ is the embedding $A_k \hookrightarrow A_k \otimes B_l$, $a \mapsto a \otimes 1_{B_l}$.

**Definition 89.** A morphism from a $\text{FAB}(A_k, \mu, \widetilde{M}_{kl})$ to a $\text{FAB}(C_m, \nu, \widetilde{M}_{mn})$ over $X$ is a pair $(f, g)$ of bundle maps $f: A_k \hookrightarrow C_m$, $g: \widetilde{M}_{kl} \hookrightarrow \widetilde{M}_{mn}$ such that

- $f, g$ are fiberwise homomorphisms of algebras (i.e. they are actually embeddings);
- the square diagram

$$
\begin{array}{ccc}
\widetilde{M}_{kl} & \xrightarrow{g} & \widetilde{M}_{mn} \\
\mu \downarrow & & \downarrow \nu \\
A_k & \xrightarrow{f} & C_m
\end{array}
$$

commutes;
- let $B_l \subset \widetilde{M}_{kl}$, $D_n \subset \widetilde{M}_{mn}$ be the complementary subbundles for $A_k$, $C_m$, respectively (see the remark above), then $g$ maps $B_l$ into $D_n$.

Note that a morphism $(f, g): (A_k, \mu, \widetilde{M}_{kl}) \to (C_m, \nu, \widetilde{M}_{mn})$ exists only if $k|m$, $l|n$.

In particular, an isomorphism between $\text{FABs}(A_k, \mu, \widetilde{M}_{kl})$ and $(C_k, \nu, \widetilde{M}_{kl})$ is a pair of bundle maps $f: A_k \to C_k$, $g: \widetilde{M}_{kl} \to \widetilde{M}_{kl}$ which are fiberwise isomorphisms of algebras such that the diagram

$$
\begin{array}{ccc}
\widetilde{M}_{kl} & \xrightarrow{g} & \widetilde{M}_{kl} \\
\mu \downarrow & & \downarrow \nu \\
A_k & \xrightarrow{f} & C_k
\end{array}
$$

commutes.

Clearly, $\text{FABs}$ over $X$ with morphisms just defined form a category $\mathfrak{FAB}(X)$.

For a continuous map $\varphi: X \to Y$ we have the natural transformation $\varphi^*: \mathfrak{FAB}(Y) \to \mathfrak{FAB}(X)$. 

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6.2 Classifying spaces

For any pair $k, l > 1$ there is the “matrix Grassmannian” $\text{Gr}_{k,l}'$ parametrizing $k$-subalgebras (i.e. those isomorphic to $M_k(\mathbb{C})$) in the fixed matrix algebra $M_{kl}(\mathbb{C})$. As a homogeneous space it can be represented as follows:

$$\text{Gr}_{k,l}' = \text{PGL}_{kl}(\mathbb{C}) / \text{PGL}_k(\mathbb{C}) \otimes \text{PGL}_l(\mathbb{C}),$$

where by $\text{PGL}_k(\mathbb{C}) \otimes \text{PGL}_l(\mathbb{C})$ we denote the image of the embedding $\text{PGL}_k(\mathbb{C}) \times \text{PGL}_l(\mathbb{C}) \hookrightarrow \text{PGL}_{kl}(\mathbb{C})$ induced by the Kronecker product of matrices.

By $\text{M}_{kl}'$ denote the trivial bundle $\text{Gr}_{k,l}' \times M_{kl}(\mathbb{C})$. There is a tautological FAB $(A_{k,l}', \mu', \text{M}_{kl}')$ over $\text{Gr}_{k,l}'$ which can be defined as follows: the fiber $(A_{k,l}')_x$ over $x \in \text{Gr}_{k,l}'$ is the $k$-subalgebra in $M_{kl}(\mathbb{C}) = (\text{M}_{kl}')_x$ corresponding to this point.

Let $\Psi_{k,l}(X)$ be the set of isomorphism classes of FABs of the form $(A_k, \mu, \text{M}_{kl})$ over $X$.

**Proposition 90.** If $\dim X \leq 2 \min\{k, l\}$, then the assignment

$$[X, \text{Gr}_{k,l}'] \to \Psi_{k,l}(X), \quad \varphi \mapsto \varphi^*(A_{k,l}', \mu', \text{M}_{kl}')$$

is a bijection.

The noncompact space $\text{Gr}_{k,l}'$ can be replaced by a homotopy equivalent compact one $\text{Gr}_{k,l}$. More precisely, let $\text{PU}(n)$ be the projective unitary group, i.e. the quotient group $\text{U}(n)/\{\exp(i\varphi)E_n \mid \varphi \in \mathbb{R}\}$. Then

$$\text{Gr}_{k,l} = \text{PU}(kl)/\text{PU}(k) \otimes \text{PU}(l)$$

(which is a subspace in $\text{Gr}_{k,l}'$ corresponding to the “unitary” subalgebras) with a tautological FAB $(A_{k,l}, \mu, \text{M}_{kl})$ over it.

Because of the homotopy equivalence $\text{Gr}_{k,l} \simeq \text{Gr}_{k,l}'$ it makes no difference between $\text{Gr}_{k,l}$ and $\text{Gr}_{k,l}'$ in this Appendix.

6.3 Stabilization

Define the product $\circ$ of two FABs over $X$ $(A_k, \mu, \text{M}_{kl})$, $(B_m, \nu, \text{M}_{mn})$ such that $(km, ln) = 1$ as

$$(A_k, \mu, \text{M}_{kl}) \circ (B_m, \nu, \text{M}_{mn}) = (A_k \otimes B_m, \mu \otimes \nu, \text{M}_{kl} \otimes \text{M}_{mn})$$

(notice that $\text{M}_{kl} \otimes \text{M}_{mn} = \text{M}_{klmn}$).

**Definition 91.** A FAB of the form $(\text{M}_k, \tau, \text{M}_{kl})$ is called trivial if the map $\tau: \text{M}_k \to \text{M}_{kl}$ acts as follows:

$$X \times M_k(\mathbb{C}) \to X \times M_{kl}(\mathbb{C}), \quad (x, T) \mapsto (x, T \otimes E_l)$$
(under some choice of trivializations on $\widetilde{M}_k$ and $\widetilde{M}_{kl}$) for any point $x \in X$, where $E_l$ is the unit $l \times l$-matrix and $T \otimes E_l$ denotes the Kronecker product of matrices. In other words, the bundle $\widetilde{M}_k$ is embedded into $\widetilde{M}_{kl}$ as a fixed subalgebra.

**Definition 92.** Two FABs $(A_k, \mu, \widetilde{M}_{kl})$ and $(B_m, \nu, \widetilde{M}_{mn})$ over $X$ are said to be **stable equivalent** if there is a sequence of pairs $\{t_i, u_i\} \in \mathbb{N}^2$, $1 \leq i \leq s$ such that

- $\{t_1, u_1\} = \{k, l\}$, $\{t_s, u_s\} = \{m, n\}$;
- $(t_i t_{i+1}, u_i u_{i+1}) = 1$ if $s > 1$, $1 \leq i \leq s - 1$,

and a corresponding sequence of FABs $(A_{t_i}, \mu_i, \widetilde{M}_{t_i u_i})$ over $X$ such that

- $(A_{t_1}, \mu_1, \widetilde{M}_{t_1 u_1}) = (A_k, \mu, \widetilde{M}_{kl})$, $(A_{t_s}, \mu_s, \widetilde{M}_{t_s u_s}) = (B_m, \nu, \widetilde{M}_{mn})$;
- $(A_{t_i}, \mu_i, \widetilde{M}_{t_i u_i}) \circ (\widetilde{M}_{t_{i+1}}, \tau, \widetilde{M}_{t_{i+1} u_{i+1}}) \cong (A_{t_{i+1}}, \mu_{i+1}, \widetilde{M}_{t_{i+1} u_{i+1}}) \circ (\widetilde{M}_{t_i}, \tau, \widetilde{M}_{t_i u_i})$,

where $1 \leq i \leq s - 1$ and $(\widetilde{M}_{t_i}, \tau, \widetilde{M}_{t_i u_i})$ are trivial FABs.

By $\widetilde{\text{AB}}^1(X)$ denote the set of stable equivalence classes of FABs over $X$.

The following theorem justifies the previous definition.

**Theorem 93.** 1) For all sequences of pairs of positive integers $\{k_j, l_j\}_{j \in \mathbb{N}}$ such that

- (i) $k_j, l_j \to \infty$; (ii) $k_j | k_{j+1}$, $l_j | l_{j+1}$; (iii) $(k_j, l_j) = 1$ for every $j$,

the corresponding direct limits $\lim_{j} \text{Gr}_{k_j, l_j}$ are homotopy-equivalent. This unique homotopy type we denote by $\text{Gr}$.

2) $\text{Gr}$ is a classifying space for stable equivalence classes of FABs over a finite CW-complex $X$. In other words, the functor $X \mapsto \widetilde{\text{AB}}^1(X)$ from the homotopy category of finite CW-complexes to the category $\mathbf{Sets}$ is represented by the space $\text{Gr}$.

The proof is based on Proposition 90 and on the following lemma (recall that we consider embeddings $\text{Gr}_{k_1, l_1} \to \text{Gr}_{m, n}$ induced by homomorphisms of algebras $M_{kl}(\mathbb{C}) \hookrightarrow M_{mn}(\mathbb{C})$; in particular, $k | m, l | n$).

**Lemma 94.** If $(m, n) = 1$ ($\Rightarrow (k, l) = 1$), then the embedding

$$\text{Gr}_{k, l} \to \text{Gr}_{m, n}$$

is a homotopy equivalence in dimensions $\leq 2 \min\{k, l\}$. 

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6.4 The group structure

Let \((A_k, \mu, \widetilde{M}_{kl})\) be a FAB over \(X\). By \([\(A_k, \mu, \widetilde{M}_{kl}\)]\) we denote its stable equivalence class. Define the product \(\odot\) of two classes \([\(A_k, \mu, \widetilde{M}_{kl}\)], [\(B_m, \nu, \widetilde{M}_{mn}\)]\) as

\[
[\(A_k, \mu, \widetilde{M}_{kl}\)] \odot [\(B_m, \nu, \widetilde{M}_{mn}\)] = [\((A_k, \mu, \widetilde{M}_{kl}) \odot (B_m, \nu, \widetilde{M}_{mn})\)]
\]

Clearly, this product is well defined if \((km, ln) = 1\). The following lemma allows us to reject this restriction.

**Lemma 95.** For any pair \(\{k, l\}\) such that (i) \((k, l) = 1\), (ii) \(2 \min\{k, l\} \geq \dim X\), any stable equivalence class of FABs over \(X\) has a representative of the form \([\(A_k, \mu, \widetilde{M}_{kl}\)]\).

Clearly, the product \(\odot\) is associative, commutative, and has identity element \([\(\widetilde{M}_k, \tau, \widetilde{M}_{kl}\)]\), where \([\(\widetilde{M}_k, \tau, \widetilde{M}_{kl}\)]\) is a trivial FAB. Moreover, for any class \([\(A_k, \mu, \widetilde{M}_{kl}\)]\) there exists the inverse element. In order to find it, let us recall the following fact. The centralizer \(Z_P(Q)\) of a central simple subalgebra \(Q\) in a central simple algebra \(P\) (over some field \(K\)) is a central simple subalgebra again, moreover, the equality \(P = Q \otimes_k Z_P(Q)\) holds. Therefore by taking centralizers for every fiber of the subbundle \(A_k\) in \(\widetilde{M}_{kl}\), we obtain the complementary subbundle \(B_l\) with fiber \(M_l(\mathbb{C})\) together with its embedding \(\nu: B_l \hookrightarrow \widetilde{M}_{kl}\) into the trivial bundle. Moreover, \(A_k \otimes B_l = \widetilde{M}_{kl}\). It is not hard to prove that \([\(B_l, \nu, \widetilde{M}_{kl}\)]\) is the inverse element for \([\(A_k, \mu, \widetilde{M}_{kl}\)]\). Thus, the functor \(X \mapsto \widetilde{\text{AB}}^1(X)\) takes values in the category of Abelian groups \(\text{Ab}\).

**Proposition 96.** The space \(\text{Gr}\) can be equipped with an \(H\)-space structure such that there is a natural equivalence of functors \(X \mapsto [X, \text{Gr}]\) and \(X \mapsto \widetilde{\text{AB}}^1(X)\) taking values in the category \(\text{Ab}\).

6.5 One interesting property of the core of a FAB

**Definition 97.** For a FAB \((A_k, \mu, \widetilde{M}_{kl})\), the locally trivial \(\text{Aut}(M_k(\mathbb{C})) \cong \text{PGL}_k(\mathbb{C})\) (or \(\text{PU}(k)\))-bundle \(A_k\) is said to be a core of \((A_k, \mu, \widetilde{M}_{kl})\).

**Lemma 98.** If \(A_k\) is the core of some FAB \((A_k, \mu, \widetilde{M}_{kl})\), then its structure group can be reduced from \(\text{Aut} M_k(\mathbb{C}) \cong \text{PGL}_k(\mathbb{C})\) to \(\text{SL}_k(\mathbb{C})\) (or equivalent from \(\text{PU}(k)\) to \(\text{SU}(k)\)).

**Lemma 99.** Let \(X\) be a finite \(CW\)-complex. Suppose \(\dim X \leq 2 \min\{k, m\}\); then the following conditions are equivalent:

- \(A_k\) is the core of some FAB over \(X\);
• for arbitrary $m$ such that $2m \geq \dim X$ there is a bundle $B_m$ with fiber $M_m(\mathbb{C})$ such that $A_k \otimes \tilde{M}_m \cong B_m \otimes \tilde{M}_k$;

• $A_k \otimes \tilde{M}_m \cong B_m \otimes \tilde{M}_k$ for some locally trivial bundle $B_m$ with fiber $M_m(\mathbb{C})$, where $(k,m) = 1$.

Moreover, for any pair of bundles $A_k, B_m$ such that $(k,m) = 1$ and $A_k \otimes \tilde{M}_m \cong B_m \otimes \tilde{M}_k$, there exists a unique stable equivalence class of FABs over $X$ which has (for sufficiently large $n$, $(km,n) = 1$) FABs of the forms $(A_k, \mu, \tilde{M}_{kn})$, $(B_m, \nu, \tilde{M}_{mn})$ as representatives (for some embeddings $\mu, \nu$).

### 6.6 Localization

Let $X$ be a finite CW-complex, let $k \geq 2$ be a fixed integer. The set of isomorphism classes of bundles of the form $A_{km}$ (for arbitrary $m \in \mathbb{N}$) over $X$ with fiber $M_{km}(\mathbb{C})$ is a monoid with respect to the operation $\otimes$ (with the identity element $M_{k0}(\mathbb{C}) \cong \mathbb{C}$).

Let us consider the following equivalence relation

$$A_{km} \sim B_{kn} \iff \exists r, s \in \mathbb{N} \text{ such that } A_{km} \otimes \tilde{M}_{k^r} \cong B_{kn} \otimes \tilde{M}_{k^s} \quad (24)$$

($\Rightarrow m + r = n + s$). The set of equivalence classes $[A_{km}]$ of such bundles is a group with respect to the operation induced by $\otimes$. This group we denote by $\widetilde{AB}^k(X)$.

Let us consider the direct limit $\lim_{\longrightarrow} \text{BPU}(k^n)$ with respect to the maps induced by $\text{PU}(k^n) \twoheadrightarrow \text{PU}(k^{n+1})$, $A \mapsto A \otimes E_k$.

Clearly, the functor $X \mapsto \widetilde{AB}^k(X)$ is represented by $\lim_{\longrightarrow} \text{BPU}(k^n)$.

According to Lemma 95, for any stable equivalence class of FABs over $X$ there is a representative of the form $(A_{km}, \mu, \tilde{M}_{(kl)m})$. Therefore for any $k$ we have the forgetful functor

$$T_{k^\infty, l^\infty} : \widetilde{AB}^1(X) \to \widetilde{AB}^k(X), \quad [(A_{km}, \mu, \tilde{M}_{(kl)m})] \mapsto [A_{km}]$$

This functor corresponds to the following diagram

$$\begin{array}{ccc}
\text{Gr}_{k^2, l^2} & \longrightarrow & \text{BPU}(k^2) \\
\uparrow & & \uparrow \\
\text{Gr}_{k, l} & \longrightarrow & \text{BPU}(k),
\end{array}$$

where the horizontal arrows are classifying maps for the cores $A_{k, l}$ and $A_{k^2, l^2}$ as $\text{PU}(k)$ and $\text{PU}(k^2)$-bundles, respectively.

The kernel of the homomorphism $\widetilde{AB}^1(X) \to \widetilde{AB}^k(X)$ is just the $k$-torsion subgroup in $\widetilde{AB}^1(X)$. 
6.7 Relation between $\tilde{AB}^1$ and $\tilde{KSU}$-theory

Recall that $BSU_\otimes$ is the space $BSU$ with the structure of $H$-space related to the tensor product of virtual SU-bundles of virtual dimension 1.

**Theorem 100.** There is an $H$-space isomorphism $Gr \cong BSU_\otimes$.

By $\tilde{KSU}(X)$ denote the reduced $K$-functor constructed by means of SU-bundles over $X$. Recall that $\tilde{KSU}(X)$ is a ring with the multiplication induced by the tensor product of bundles.

The previous theorem shows that the group $\tilde{AB}^1(X)$ is isomorphic to the multiplicative group of the ring $\tilde{KSU}(X)$, i.e. to the group ($X$ is a finite $CW$-complex) of elements of $\tilde{KSU}(X)$ with respect to the operation $\xi \star \eta = \xi + \eta + \xi \eta$ ($\xi, \eta \in \tilde{KSU}(X)$).

This gives us a geometric description of the $H$-structure on $BSU_\otimes$. For example, the construction of the inverse stable equivalence class $[(B_m, \nu, \tilde{M}_{mn})]$ for a given one $[(A_k, \mu, \tilde{M}_{kl})]$ is closely connected with taking centralizer for a subalgebra in a fixed matrix algebra.

6.8 A $U$-version

Consider the canonical map $BU(k) \to BPU(k)$ induced by the group homomorphism $U(k) \to PU(k)$. By $\tilde{Gr}_{k,l}$ denote the total space of the $Fr_{k,l}$-fibration (recall that $Fr_{k,l}$ denotes the space of (unitary) $k$-frames in $M_{kl}(\mathbb{C})$, see Section 1) induced by the fibration $Gr_{k,l} \to BPU(k)$ and the map $BU(k) \to BPU(k)$ (as ever, the integers $k, l$ are assumed to be coprime).

It follows easily that there is a $\mathbb{C}P^\infty$-fibration $\tilde{Gr}_{k,l} \to Gr_{k,l}$.

Consider the following morphism of $U(k)$-fibrations:

$$
\begin{array}{ccc}
U(k) & \to & \ast \\
\downarrow & & \downarrow \\
Fr_{k,l} & \to & BU(k) \\
\downarrow & & \downarrow \\
\tilde{Gr}_{k,l} & & \\
\end{array}
$$

where $\tilde{t}_{k,l}$ is a classifying map for the canonical $U(k)$-bundle over $\tilde{Gr}_{k,l}$ and by $\ast$ we denote a contractible space. A simple computation with homotopy sequences of the fibrations shows that $\tilde{t}_{k,l}^* : \pi_{2r}(\tilde{Gr}_{k,l}) \to \pi_{2r}(BU(k))$, $r \leq \min\{k, l\}$ is just the monomorphism $\mathbb{Z} \to \mathbb{Z}$, $1 \mapsto k \cdot 1$ (note that the odd-dimensional stable homotopy groups of both spaces are equal to 0). This implies that the limit map $\tilde{t}_{k,\infty} : \tilde{Gr} \to BU(k^\infty)$ is just the localization at $k$ (in the sense that $k$ is invertible; in particular, $BU(k^\infty)$ is a $\mathbb{Z}[\frac{1}{k}]$-local space), where $\tilde{Gr} := \lim_{(k,l)=1} \tilde{Gr}_{k,l}$.
The space $\widehat{\text{Gr}}$ is an $H$-space with respect to the multiplication induced by the tensor product of bundles. It can be proved that $\widehat{\text{Gr}} \cong \text{BU}_\otimes$ as $H$-spaces. Let us also recall that $\text{BU}_\otimes \cong \text{BSU}_\otimes \times \mathbb{C}P^\infty$ and $\text{Gr} \cong \text{BSU}_\otimes$ as $H$-spaces, hence $\widehat{\text{Gr}} \cong \text{Gr} \times \mathbb{C}P^\infty$. In particular, the $H$-space $\widehat{\text{Gr}}$ represents the functor of “multiplicative group” of the ring $\widetilde{K}_C$, i.e. the functor $X \mapsto \widetilde{K}_C(X)$, where $\widetilde{K}_C(X)$ is considered as a group with respect to the operation $\xi \ast \eta = \xi + \eta + \xi\eta$, $\xi, \eta \in \widetilde{K}_C(X)$ (here $\widetilde{K}_C$ is the reduced complex $K$-functor).

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