An absolute characterisation of locally determined $\omega$-colimits

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Abstract—Characterising colimiting $\omega$-cocones of projection pairs in terms of least upper bounds of their embeddings and projections is important to the solution of recursive domain equations. We present a universal characterisation of this local property as $\omega$-cocontinuity of locally continuous functors. We present a straightforward proof using the enriched Yoneda embedding. The proof can be generalised to Cattani and Fiore’s notion of locality for adjoint pairs.

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In the category theoretic solution of recursive domain equations [SP82], several technical results hinge upon the fact that the universality of $\omega$-cocones of projection pairs can be characterised locally in terms of least upper bounds (lubs) of their embeddings and projections. To fix terminology and notation, consider an $O$-category $K$. Let $K_{PR}$ be the $O$-category consisting of projection pairs $f : A \rightarrow B$ given by $f = \langle f^L : A \rightarrow B, f^R : B \rightarrow A \rangle$ where $f^R \circ f^L = \text{id}_A$ and $f^L \circ f^R \leq \text{id}_B$.

Definition ([SP82, Definition 8]). We say that a cocone $\langle C, c \rangle$ for an $\omega$-chain of projection pairs is locally determined when $\bigvee_{n \in \mathbb{N}} c^L_n \circ c^R_n = \text{id}_C$.

When all colimiting $\omega$-cocones of projection pairs are locally determined, we say that the $O$-category has locally determined $\omega$-cocones of projection pairs.

For example, the category $\omega$-CPO of (not necessarily pointed) $\omega$-cpos and continuous functions has locally determined $\omega$-cocones.

The importance of these cocones lies in the fact that every locally determined cocone is colimiting. As any locally continuous functor $F : K \rightarrow L$ gives a continuous functor $F_{PR} : K_{PR} \rightarrow L_{PR}$, given by $F_{PR}f := \langle Ff^L, Ff^R \rangle$, and locally determined $\omega$-cocones are preserved by these functors. Our contribution is to show the converse:

Theorem. An $\omega$-colimiting cocone of projection pairs is locally determined if and only if it is preserved by every locally continuous functor.

Let $\hat{K}$ be the $O$-category of $O$-presheaves, namely locally continuous functors and natural transformations from $K^{op}$ to $\omega$-CPO. Let $y : K \rightarrow \hat{K}$ be the enriched Yoneda embedding $yx := \omega$-CPO($\cdot, x$). Then, following from general principles [Kel82 Section 2.4], $y$ is locally continuous and fully faithful.

As is well-known, lubs and colimits in $O$-functor categories are given pointwise. The same argument shows that $\omega$-colimits of projection pairs are also given componentwise in $O$-functor categories. Therefore:

Proposition. If $K$, $L$ are $O$-categories and $L$ has locally determined $\omega$-colimits of projection pairs, then so does the $O$-functor category $L^{\omega}$. In particular, every $O$-presheaf category $\hat{K}$ has locally determined $\omega$-cocones.

We complete the proof of our theorem. Let $\langle C, c \rangle$ be any colimiting cocone that is preserved (in particular) by the locally continuous Yoneda embedding. As $\hat{K}$ has locally determined $\omega$-cocones:

$$y \left( \bigvee_n c^L_n \circ c^R_n \right) = \bigvee_n y(c^L_n) \circ y(c^R_n) = y(\text{id})$$

By the faithfulness of the Yoneda embedding we deduce that $\langle C, c \rangle$ is locally determined.

Corollary. An $O$-category has locally determined $\omega$-cocones of projection pairs if and only if every locally continuous functor from it yields an $\omega$-cocontinuous functor on projection pairs.

Much of the theory of recursive domain equations generalises to adjoint pairs $\langle f^L, f^R \rangle$ where $f^L \circ f^R \leq \text{id}$ and $\text{id} \leq f^R \circ f^L$. Cattani et al. [CFW98], [CF07] generalised locally determined cocones as follows:

Definition (cf. [CF07, Theorem 1.5]). We say that a cocone $\langle C, c \rangle$ for an $\omega$-chain $\Delta$ of adjoint pairs is locally determined when $\bigvee_{n \in \mathbb{N}} c^L_n \circ c^R_n = \text{id}_C$ and, for all $n \in \mathbb{N}$:

$$\bigvee_{m \geq n} \Delta_{R}^{m \leq n} \circ \Delta_{L}^{n \geq m} = c^R_n \circ c^L_n$$

When all colimiting $\omega$-cocones of adjoint pairs are locally determined, we say that the $O$-category has locally determined $\omega$-cocones of adjoint pairs.

As $\omega$-CPO has locally determined $\omega$-cocones of adjoint pairs, almost identical proofs show the following:

Theorem. An $\omega$-colimiting cocone of adjoint pairs is locally determined if and only if it is preserved by every locally continuous functor.

Corollary. An $O$-category has locally determined $\omega$-cocones of adjoint pairs if and only if every locally continuous functor from it yields an $\omega$-cocontinuous functor on adjoint pairs.
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