Noncoaxial multivortices in the complex sine-Gordon theory on the plane

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Abstract

We construct explicit multivortex solutions for the complex sine-Gordon equation (the Lund-Regge model) in two Euclidean dimensions. Unlike the previously found (coaxial) multivortices, the new solutions comprise $n$ single vortices placed at arbitrary positions (but confined within a finite part of the plane.) All multivortices, including the single vortex, have an infinite number of parameters. We also show that, in contrast to the coaxial complex sine-Gordon multivortices, the axially-symmetric solutions of the Ginzburg-Landau model (the stationary Gross-Pitaevskii equation) do not belong to a broader family of noncoaxial multivortex configurations.

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I. INTRODUCTION

Topological solutions of nonlinear partial differential equations on the plane have been a subject of intensive investigations in recent years, with applications ranging from nonlinear optics to cosmic strings. The simplest type of topological solitons arise in systems involving one complex field. In this case the solitons realise a $S^1 \rightarrow S^1$ map of the circle of a sufficiently large radius on the $(x,y)$-plane into a unit circle in the internal space $(\text{Re}\psi, \text{Im}\psi)$ and can be classified according to the Brouwer degree of the map, or the winding number,

$$Q = \lim_{R \to \infty} \frac{1}{2\pi} \oint_{C_R} \text{d} (\text{arg} \psi) = \frac{1}{2\pi} \int_{R^2} \epsilon_{nk} \partial_n \partial_k (\text{arg} \psi) \, d^2x. \quad (1)$$

One celebrated model of this kind is the Ginzburg-Landau model (also known as the stationary Gross-Pitaevskii equation) arising in the description of boson condensates (in particular, superconductors and superfluids) \[1,2\]:

$$\nabla^2 \psi + \psi(1 - |\psi|^2) = 0. \quad (2)$$

(Here $\nabla = i\partial_x + j\partial_y$.) Another system of a similar type which received considerable attention in literature, is the Heisenberg ferromagnet with easy-plane anisotropy \[3\]. In terms of the stereographically projected field this can be written as

$$\nabla^2 \psi - \frac{2 (\nabla \psi)^2 \overline{\psi}}{1 + |\psi|^2} + \psi \frac{1 - |\psi|^2}{1 + |\psi|^2} = 0. \quad (3)$$

In physics literature the winding number (1) is usually referred to as vorticity and the planar topological solitons as vortices. Both the Ginzburg-Landau and the easy-plane ferromagnet equation are well known to possess axially-symmetric solutions describing $n$ vortices sitting on top of each other, $\psi^{(n)} = e^{in\theta} \Phi_n(r)$, where the function $\Phi_n(r) \to 1$ as $r \to \infty$. (Here $r$ and $\theta$ are the polar coordinates on the $(x,y)$-plane.) Of fundamental importance, both physically and mathematically, is the question of whether noncoaxial multivortices exist. However, despite some encouraging recent insights \[4\], very little is known about this.
The present paper is devoted to another physically meaningful system which shares a lot of similarities with (2) and (3), the so-called complex sine-Gordon equation:

\[ \nabla^2 \psi + \frac{(\nabla \psi)^2 \overline{\psi}}{1 - |\psi|^2} + \psi(1 - |\psi|^2) = 0. \] (4)

The equation (4) is similar to (2) and (3) in that it is also an equation for one complex field on the plane, and that it also possesses coaxial multivortex solutions. An important difference, however, is that this model is integrable and hence the complex sine-Gordon multivortices are given by explicit analytic formulas whereas their Ginzburg-Landau and magnetic counterparts are available only numerically, even in the single-vortex case. The single-vortex solution of eq.(4) has the form

\[ \psi^{(1)}(r, \theta) = \frac{I_1(r)}{I_0(r)} e^{i\theta}, \] (5)

where \( I_0 \) and \( I_1 \) are the modified Bessel functions. Using a purely algebraic recursive procedure one can construct axially-symmetric solutions of arbitrarily high vorticity. For example, the \( Q = 2 \) and \( Q = 3 \) solutions are given by

\[ \psi^{(2)}(r, \theta) = \frac{I_1(r)^2 - I_0(r)I_2(r)}{I_0(r)^2 - I_1(r)^2} e^{2i\theta}, \] (6)

and

\[ \psi^{(3)}(r, \theta) = \frac{[I_2(r) - I_1(r)][I_0^2(r) - I_2^2(r)] + I_1(r)[I_0(r) - I_2(r)]^2}{[I_0(r) - I_2(r)][I_0(r)I_2(r) - 2I_1^2(r) + I_0^2(r)]} e^{3i\theta}, \] (7)

respectively.

The primary objective of this paper is to show that the complex sine-Gordon equation has an infinite-parameter family of exact, explicit, noncoaxial multivortices. This family includes the axially-symmetric \( n \)-vortex solution which, therefore, admits a continuous into \( n \) spatially separated single vortices. We calculate the energy (i.e. the Euclidean action)

\[ ^1 \text{Some references to the applications of the complex sine-Gordon theories can be found in the concluding section below.} \]
of the noncoaxial \( n \)-vortex configuration and demonstrate that it does not depend on the positions of the individual vortices. This implies that the individual vortices making up the \( n \)-vortex configuration are non-interacting.

Having established the existence of a general \( n \)-vortex solution for equation (4), a natural question to ask is: Do the Ginzburg-Landau model (2) and the easy-plane ferromagnet (3) have multivortex solutions other than the axially-symmetric ones? Confining ourselves to the case of the Ginzburg-Landau equation, we attempt to answer this question via the analysis of the spectra of linearised excitations of its symmetric solutions. If the axially-symmetric solution admitted a \( p \)-parameter nonsymmetric continuation, the corresponding linearised operator would have \( p \) zero eigenvalues. We study the linearised spectra of the Ginzburg-Landau axisymmetric solutions numerically; the upshot of this study is that they have only three zero modes related to obvious symmetries of the equation and therefore do not belong to a broader family of noncoaxial multivortices.

On the contrary, each of the coaxial multivortices of the complex sine-Gordon theory admits an infinite number of zero-frequency excitations. This is an expected fact, of course, given the existence of the infinite-parameter nonsymmetric configurations. However, as we show below, the presence of the infinite number of zero modes in this case can be established even without knowledge of the nonsymmetric generalisations. In other words, the possibility of the nonsymmetric continuation of the coaxial multivortex could have been predicted simply on the basis of the analysis of its linearisation.

The paper is organised as follows. In the next section we derive the noncoaxial \( n \)-vortex solution and in section 3 discuss some of its general properties. Section 4 deals with the simplest special case of the new solution when it depends only on one free parameter. We consider, in detail, the \( n = 1, n = 2 \) and \( n = 3 \)-solutions, and then extrapolate our conclusions to the situation of the general \( n \). The energies of the multivortices are evaluated in section 5. In section 6 we study, numerically, the linearised excitations of axially-symmetric multivortices of the Ginzburg-Landau equation and compare them to their complex sine-Gordon counterparts. Finally, some concluding remarks are made in section 7.
II. THE GENERAL MULTIVORTEX SOLUTION

As in [5], we start with rewriting the second-order equation (4) as a system of two first-order equations:

\[ \partial \psi^{(n-1)} + \psi^{(n)}(1 - |\psi^{(n-1)}|^2) = 0, \]
\[ \partial \psi^{(n)} - \psi^{(n-1)}(1 - |\psi^{(n)}|^2) = 0. \]

(8) \hspace{1cm} (9)

Here \( \partial = \partial/\partial z, \ \overline{\partial} = \partial/\partial \overline{z}, \) and \( z = (x + iy)/2, \ \overline{z} = (x - iy)/2. \) This first-order system has a field-theoretic interpretation of its own; it is nothing but the Euclidean version of the massive Thirring model [6,7]. Both \( \psi^{(n-1)} \) and \( \psi^{(n)} \) satisfy equation (4), hence eqs.(8)-(9) can be seen as the Bäcklund transformations relating two solutions of the complex sine-Gordon equation.

Let \( n = 1 \) in eqs.(8)-(9). For any \( \psi^{(1)} \) eq.(8) is solved by \( \psi^{(0)} = 1. \) Letting \( \psi^{(0)} = 1 \) in eq.(9), we get

\[ \partial \psi^{(1)} = 1 - |\psi^{(1)}|^2. \]

(10)

Decomposing \( \psi^{(1)} = f + ig, \) the imaginary part of (10) yields

\[ \partial_x g = \partial_y f, \]

whence we can define the potential \( \mathcal{F}(x, y) \) such that \( f = \partial_x \mathcal{F} \) and \( g = \partial_y \mathcal{F}, \) or, equivalently,

\[ \psi^{(1)} = \overline{\partial} \mathcal{F}. \]

(11)

The real part of (11) is then

\[ \nabla^2 \mathcal{F} + (\nabla \mathcal{F})^2 = 1. \]

(12)

Letting \( \mathcal{F} = \ln Z_0, \) this reduces to the (modified) Helmholtz equation

\[ \nabla^2 Z_0 - Z_0 = 0. \]

In polar coordinates, the general solution of (12), regular at the origin, is given by
where $I_m(r)$ is the modified Bessel’s function of order $m$ (of the first kind); $\sigma_m = \beta_m \cos(m\theta + \delta_m)$, and $\beta_m, \delta_m$ are arbitrary real constants. (In particular, all $\beta_m$ with $m$ greater than a certain $M$ can be set equal to zero in which case the series (13) becomes a finite sum.)

Returning to the variable $\psi^{(1)}$, we obtain

$$\psi^{(1)} = e^{i\theta} \left\{ \frac{\sum I'_m(r) \sigma_m(\theta)}{\sum I_m(r) \sigma_m(\theta)} + \frac{i}{r} \sum I_m(r) \partial_\theta \sigma_m(\theta) \right\},$$

where $I'_m = dI_m/dr$ and all sums run over $m = 0, 1, \ldots \infty$. We will assume that $\beta_0 \neq 0$; otherwise the above solution is singular. Without loss of generality we can let $\sigma_0 = \beta_0 \cos \delta_0 = 1$ in (14) and this convention will be implied throughout this paper. At infinity, the solution (14) tends to $e^{i\theta}$; more precisely

$$\psi^{(1)}(r, \theta) = e^{i\theta} \left\{ 1 - \frac{1 - i\kappa}{2r} + \frac{\mu_1 + i\nu_1}{r^2} + O \left( \frac{1}{r^3} \right) \right\} \text{ as } r \to \infty,$$

where $\kappa, \mu_1$ and $\nu_1$ are functions of $\theta$:

$$\kappa = 2\partial_\theta \sum \sigma_m, \quad \mu_1 = \frac{\sum (4m^2 - 1) \sigma_m}{8 \sum \sigma_m}, \quad \nu_1 = -\partial_\theta \mu_1(\theta).$$

Therefore, equation (14) gives a solution with infinite number of parameters and vorticity $Q = 1$. For purposes of this paper we will be referring to (14) as the “general one-vortex solution”. Now the general solutions with $Q = 2$ and all higher vorticities can be obtained in a purely algorithmic way. We simply use eq.(8) to express $\psi^{(n)}$ via $\psi^{(n-1)}$:

$$\psi^{(n)} = -\frac{1}{1 - |\psi^{(n-1)}|^2} \bar{\partial} \psi^{(n-1)}, \quad n = 2, 3, \ldots$$

This recursive procedure can be made rather efficient by introducing auxiliary variables

$$\mathcal{Z}_k(r, \theta) = \sum_{m=0}^{\infty} (\gamma_m \xi_{k+m} + \bar{\tau}_m \xi_{k-m}), \quad k = 0, \pm 1, \pm 2, \ldots,$$

where

$$\xi_s(r, \theta) = I_s(r) e^{is\theta}, \quad \gamma_m = \frac{\beta_m}{2} e^{i\delta_m}.$$
Note that \( Z_{-k} = \overline{Z_k} \); also note that the function \( Z_0 \) has already been defined before (see eq. (13)). What makes the variables \( Z_k \) useful is that the differential operators \( \partial \) and \( \overline{\partial} \) act on them simply as index lowering and raising operators:

\[
\partial Z_k = Z_{k-1}, \quad \overline{\partial} Z_k = Z_{k+1}.
\]

Indeed, writing \( \partial \) as \( e^{-i\theta} (\partial_r - \frac{1}{r} \partial_\theta) \), \( \overline{\partial} \) as \( e^{i\theta} (\partial_r + \frac{1}{r} \partial_\theta) \) and using the identities \[8\]

\[
\frac{dI_m}{dr} = \frac{I_{m-1} + I_{m+1}}{2}, \quad \frac{mI_m}{r} = \frac{I_{m-1} - I_{m+1}}{2},
\]

one can readily verify that

\[
\partial \xi_m = e^{i(m-1)\theta} \left( \frac{dI_m}{dr} + \frac{m}{r} I_m \right) = I_{m-1} e^{i(m-1)\theta} = \xi_{m-1},
\]

\[
\overline{\partial} \xi_m = e^{i(m+1)\theta} \left( \frac{dI_m}{dr} - \frac{m}{r} I_m \right) = I_{m+1} e^{i(m+1)\theta} = \xi_{m+1}.
\]

From here the relations (20) are straightforward.

Recalling that \( F = \ln Z_0 \) and using equation (11), the general 1-vortex solution (14) can be written simply as

\[
\psi^{(1)}(r, \theta) = \frac{Z_1}{Z_0}.
\]

Now applying (17) and using (20) gives

\[
\psi^{(2)}(r, \theta) = \frac{Z_2 - Z_1 Z_0}{Z_0^2 - Z_{-1} Z_1},
\]

\[
\psi^{(3)}(r, \theta) = \frac{Z_3(Z_0^2 - Z_{-1} Z_1) + Z_{-1} Z_2^2 + Z_1^3 - 2 Z_0 Z_1 Z_2}{Z_0(Z_0^2 - 2 Z_{-1} Z_1 - Z_{-2} Z_2) + Z_{-1}^2 Z_2 + Z_1^2 Z_{-2}},
\]

and so on. Setting to zero all \( \beta_m \) with \( m \geq 1 \), eqs. (22), (23) and (24) reduce to the axially-symmetric vortex solutions (5), (6) and (7).

The explicit expressions of the multivortices with \( n \geq 4 \) become cumbersome and we restrict ourselves to producing only their asymptotic behaviours as \( r \to \infty \):
\[ \psi^{(n)}(r, \theta)e^{-in\theta} = 1 - \frac{n}{2r} + \frac{\mu_n(\theta)}{r^2} + i \left( \frac{n}{2r^2} \kappa(\theta) + \frac{\nu_n(\theta)}{r^2} \right) + O \left( \frac{1}{r^3} \right), \] (25)

where \( \mu_n \) and \( \nu_n \) are defined by recurrence relations

\[ \mu_n = \mu_{n-2} + \frac{4\mu_{n-1}}{n-1} + \frac{\partial_\theta \kappa}{2}, \]

\[ \nu_n = \frac{n+1}{n-1} \nu_{n-1} - \frac{n\kappa}{4} \partial_\theta \kappa; \quad n \geq 2, \] (26)

with \( \mu_0 = 0 \) and \( \mu_1, \nu_1 \) and \( \kappa \) as in (16). Equations (25)-(26) can be easily proved by induction with the help of the Bäcklund transformation (8)-(9). The recurrence relation for \( \mu_n \) can be easily resolved yielding an explicit expression

\[ \mu_n = n^2 \mu_1 + \frac{n(n-1)}{4} \partial_\theta \kappa, \quad n \geq 2. \] (27)

(Unfortunately, there are no similar closed formulas for \( \nu_n \).)

The relations (18) can be seen as expansions over the eigenfunctions of the angular momentum, with \( \beta_m \) being the coefficient of the eigenfunction associated with the orbital quantum number \( m \). Accordingly, solutions (22)-(25) can be interpreted as orbital deformations of the axially-symmetric multivortices. Below, in section IV, we will discuss several particular orbital deformations in more detail.

III. SOME GENERAL PROPERTIES

A. Regularity and convergence considerations

It is not difficult to realise that inequality

\[ \sum_{m=1}^{\infty} |\beta_m| \leq 1 \] (28)

is sufficient to ensure the regularity of the general 1-vortex solution (22). (It is not necessary though; see the next subsection.) Indeed, due to the positivity of \( I_m(r) \) for \( r > 0 \) and the second identity in (21), we have \( I_0(r) > I_{2m}(r) \) and \( I_1(r) > I_{2m+1}(r) \) for all \( m \geq 1 \). Combining these inequalities with \( I_0(r) > I_1(r) \) gives \( I_0(r) > I_m(r) \) for all \( m \geq 1 \) and
$r > 0$. This latter inequality, taken together with (28), guarantees that $Z_0 > 0$ and hence the solution (22) is bounded on the entire $(x,y)$-plane.

In case of infinitely many nonzero coefficients $\beta_m$ we need to make sure that the series in (28) converges. This can be accomplished by imposing, for example, that

$$|\beta_m| \leq q^m,$$

(29)

with some $0 < q < 1$. The inequalities (28) and (29) are also sufficient for the convergence of the series in the asymptotic formula (15).

Next, the 2-vortex solution resulting from the Bäcklund transformation (17) will only be regular if the seed 1-vortex solution is bounded by 1 in absolute value, that is, if $|Z_1| < Z_0$. That the latter inequality holds true can be easily verified using the representation

$$Z_k = \frac{1}{2\pi i} \oint_{|\ell|=1} \frac{d\ell}{\ell^{k+1}} G(\ell)e^{z\ell + \bar{z}/\ell}, \quad G(\ell) = \sum_{m=0}^\infty \left( \gamma_m \ell^{-m} + \bar{\tau}_m \ell^m \right)$$

(30)

which arises by replacing $\xi_m$ in eq.(18) by

$$\xi_m(r, \theta) = I_m(r) e^{im\theta} = \frac{1}{2\pi i} \oint_{|\ell|=1} \frac{d\ell}{\ell^{m+1}} \exp \left( z\ell + \bar{z}/\ell \right).$$

(31)

Eq.(31), in turn, follows from the integral formula for the modified Bessel function [8]:

$$I_m(r) = \frac{1}{2\pi i} \oint_{|\zeta|=1} \frac{d\zeta}{\zeta^{m+1}} \exp \left\{ \frac{r}{2} \left( \zeta + \frac{1}{\zeta} \right) \right\},$$

where we only need to set $\zeta = \ell e^{i\theta}$. (We remind the reader that $z = \frac{1}{2}r e^{i\theta}$.) To show that $|Z_1| < Z_0$, assume that the series $\sum |\beta_m|$ converges, with eq.(28) being in place. The function $G(e^{i\varphi}) = 1 + \sum_{m=1}^\infty \beta_m \cos(m\varphi + \delta_m)$ is obviously positive for all $\varphi$ and hence using equation (30) with $\ell = e^{i\varphi}$ we get, finally,

$$|Z_1| = \frac{1}{2\pi} \left| \int_0^{2\pi} d\varphi e^{-i\varphi} G(e^{i\varphi}) e^{r \cos(\varphi + \theta)} \right| < \frac{1}{2\pi} \int_0^{2\pi} d\varphi G(e^{i\varphi}) e^{r \cos(\varphi + \theta)} = Z_0.$$
It is interesting to note that for certain choices of parameters \( \beta_1, \beta_2, \ldots \), solutions \( \psi^{(1)}, \psi^{(2)}, \psi^{(3)} \) etc., describe pure translations of the corresponding coaxial multivortices. The translations are associated with infinite numbers of nonzero \( \beta \)'s. Consider, for example, the translation along the \( x \)-axis:

\[
\hat{x} = x - R, \quad \hat{y} = y.
\]

The transformation \((32)\) can be written as one formula,

\[
\hat{r} \cos(\hat{\theta} + \varphi) = r \cos(\theta + \varphi) - R \cos \varphi,
\]

which holds for an arbitrary fixed angle \( \varphi \). Hence, picking up \( G(e^{i\varphi}) = e^{-R \cos \varphi} \) in the representation \((30)\), we obtain

\[
Z_k = \frac{1}{2\pi} \int_0^{2\pi} d\varphi e^{-ik\varphi} e^{-R \cos \varphi} e^{r \cos(\theta + \varphi)} = \xi_k(\hat{r}, \hat{\theta}),
\]

where \( \xi_k \) were defined in eq.\((19)\): \( \xi_k(r, \theta) = I_k(r)e^{ik\theta} \). According to \((33)\), in the reference frame \((\hat{x}, \hat{y})\) our functions \( Z_k \) of which the solution \( \psi^{(n)} \) is to be built, have the form \((18)\) with all \( \beta_m = 0 \) except \( \beta_0 = 1 \). In other words, in the translated reference frame the solutions \( \psi^{(n)} \) constructed using \( G(e^{i\varphi}) = e^{-R \cos \varphi} \), have the form of coaxial \( n \)-vortices.

The orbital coefficients \( \beta_m \) associated with the translation can be found by expanding the function \( G = e^{-R \cos \varphi} \) in the Fourier series:

\[
e^{-R \cos \varphi} = \sum_{m=0}^{\infty} (\gamma_m e^{-im\varphi} + \gamma_m^* e^{im\varphi}),
\]

where

\[
\gamma_0 = \frac{1}{2} I_0(R); \quad \gamma_m = (-1)^m I_m(R), \quad m \geq 1.
\]

It is not difficult to check that the series \((34)\) converges. Indeed, the large-\( m \) asymptotic behaviour of the modified Bessel’s function is given by Horn’s formula \([8]\):
\[ I_m(R) = \frac{1}{\sqrt{2\pi}} \exp \left\{ m \left( 1 + \ln \frac{R}{2} \right) - \left( m + \frac{1}{2} \right) \ln m \right\} \left( 1 + O \left( \frac{1}{m} \right) \right), \quad m \to \infty. \quad (36) \]

Using (36), one can easily verify that the coefficients (35) pass the ratio test:

\[ \lim_{m \to \infty} \frac{\gamma_{m+1}(R)}{\gamma_m(R)} = 0. \quad (37) \]

We can normalise the coefficients according to our convention that \( \beta_0 \) be equal to 1. This is done simply by replacing \( \gamma_m \) in eq.(35) with

\[ \gamma_0 = \frac{1}{2}; \quad \gamma_m = (-1)^m I_m(R)/I_0(R), \quad m \geq 1. \quad (38) \]

(Note that the corresponding \( \beta_m, \beta_m = 2(-1)^m I_m(R)/I_0(R), \) do not satisfy the sufficient condition (28). Despite that, the translated vortex is perfectly regular.)

Thus we conclude that the recursion procedure (17)-(18) with an infinite sequence of nonzero orbital coefficients \( \gamma_m \) defined by eq.(38), gives rise to the coaxial multivortex centred at the point \( x = R, \ y = 0 \). In the next section we will show that choosing a finite number of nonzero \( \gamma \)'s may also result in a shift of the vortex; however that shift will always be accompanied by a deformation. On the contrary, the infinite sequence (38) produces a pure translation.

Our final remark in this section is on the convergence of yet another series:

\[ \sum_{m=0}^{\infty} m^k |\gamma_m(R)| < \infty. \]

(This is a useful by-product of eq.(37).) The fact that the above series converges for all \( k \) allows one to use the asymptotic formula (15) in the case of the translated 1-vortex. Restricting ourselves to terms of order \( r^{-1} \), we find that the axially-symmetric (undeformed) vortex (5) moved to the point \( (x = R, y = 0) \), has the asymptotic behaviour

\[ \psi^{(1)}(r, \theta) = e^{i\theta} \left\{ 1 - \frac{1}{2r} + \frac{iR \sin \theta}{r} + O \left( \frac{1}{r^2} \right) \right\} \quad \text{as} \quad r \to \infty. \quad (39) \]
IV. ONE-PARAMETER DEFORMATIONS OF THE AXIALLY-SYMMETRIC MULTIVORTICES

A. The $n = 1$ vortex

In this subsection we analyse in detail the simplest situation of a single nonzero orbital perturbation: $\beta_k \equiv \beta \neq 0$ for $k$ equal some fixed $m$, and $\beta_k = 0$ for all other $k$. Without loss of generality we can set $\delta_m = 0$ and consider $\beta$ to be non-negative: $0 \leq \beta \leq 1$. Eq. (22) gives

$$\psi^{(1)}(r) = \frac{I_1(r)e^{i\theta} + (\beta/2)[I_{m+1}(r)e^{i(m+1)\theta} + I_{m-1}(r)e^{-i(m-1)\theta}]}{I_0(r) + \beta I_m(r) \cos(m\theta)}.$$  \hspace{1cm} (40)

The first three orbital perturbations ($m = 1, 2, 3$) of the one-vortex solution are shown in figure 1. The unperturbed vortex (i.e. eq. (40) with $\beta = 0$) is also reproduced for comparison.

The one-parameter solution (40) is symmetric with respect to the rotation $\theta \rightarrow \theta + 2\pi/m$ in the $(x, y)$-plane. This accounts for the number of symmetric folds in the modulus of $\psi^{(1)}$ seen in figures 1(b-d). The solution (40) with $m > 3$ is different from figures 1(b-d) only in that it will have $m$ symmetric folds.

It also follows from the cyclic symmetry that out of all one-parameter perturbations, only the $m = 1$ perturbations give rise to the shift of the vortex from the origin. This shift breaks the rotational $\mathbb{Z}_m$-symmetry completely and therefore, is compatible only with $m = 1$. This can be easily verified by the Taylor’s expansion at the origin,

$$\psi^{(1)}(r, \theta) = \frac{\beta_1}{2} e^{-i\delta_1} + O(r).$$  \hspace{1cm} (41)

According to (41), the value of $\psi^{(1)}|_{r=0}$ is not equal to zero — unless $\beta_1 = 0$. 

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FIG. 1. (a): the modulus squared $|\psi^{(1)}|^2$ of the unperturbed $n = 1$ vortex. (b-d): its $m = 1$, $m = 2$ and $m = 3$ orbital deformations. Note that in panel (b) we used eq. (40) with $\delta_1 = \pi/2$ for the better visualisation. This gives rise to the $y$-shift of the vortex from the origin (and not the $x$-shift as in the case of $\delta_1 = 0$.) Similarly, in panel (d) we set $\delta_3 = \pi$.

B. The $n = 2$ multivortex

The one-parameter deformations of the coaxial two-vortex with $m = 1, 2, 3$ are shown in figure 2. Each of these three perturbations splits the repeated zero of $\psi^{(2)}(x, y)$ into several single zeros, or, equivalently, splits the coaxial two-vortex into several monovortices. The perturbations with $m = 1$ and $m = 2$ give rise to two single vortices. For $m = 2$, the indices of the newly born zeros are equal due to the discrete rotation symmetry $\theta \rightarrow \theta + 2\pi/m = \theta + \pi$; hence in this case the perturbed solution consists of two individual
vortices of vorticity $Q_1 = Q_2 = +1$. Here the vorticity of the $k$-th vortex is defined as the index of the corresponding zero of the field:

$$Q_k = \frac{1}{2\pi} \oint_{C_k} \! \! d(\arg \psi),$$

where $C_k$ is a closed contour enclosing the $k$-th zero of the solution but no other zeros.

FIG. 2. The modulus squared of the $n = 2$-solutions. (a) the coaxial 2-vortex ($\beta = 0$); (b) the $m = 1$ perturbation (here $\delta_1 = \pi/2$); (c) $m = 2$; (d) $m = 3$. In (b,c,d) we set $\beta = 1$ to ensure the maximum possible separation of individual vortices.
FIG. 3. The level curves of $|\psi^{(2)}|^2$ for the coaxial two-vortex (a) and its first three orbital perturbations: $m = 1$ (here $\delta_1 = \pi/2$) (b); $m = 2$ (c); $m = 3$ (d). In each of these plots $\beta = 1$. Only level curves with sufficiently low values of $|\psi|^2$ are shown for visual clarity.

For $m = 3$ there are four zeros. They are not clearly visible in figure 2(d) but become evident if one plots the level curves of $|\psi^{(2)}|^2$, figure 3. First of all, we have an antivortex with $Q_0 = -1$ sitting at the origin. In addition, there are three vortices with vorticities $Q_{1,2,3} = +1$ placed symmetrically around it. The fact that the origin is a zero with index $-1$, can be readily concluded from the Taylor expansion for small $r$:

$$\psi^{(2)}(r, \theta) \bigg|_{m=3} = -\frac{\beta r}{4} e^{-i\theta} + \mathcal{O}(r^2).$$  

(43)
Perturbations with higher orbital numbers do not produce the splitting of the 2-vortex. For all \( m \geq 4 \) the solution has a zero only at the origin. For example, for \( m = 4 \) and 5 the Taylor expansions about \( r = 0 \) are:

\[
\psi^{(2)}(r, \theta) \bigg|_{m=4} = \frac{r^2}{8} \left( 1 - \frac{\beta}{2} e^{-4i\theta} \right) e^{2i\theta} + \mathcal{O}(r^4), \tag{44}
\]

\[
\psi^{(2)}(r, \theta) \bigg|_{m=5} = \frac{r^2}{8} e^{2i\theta} + \mathcal{O}(r^3). \tag{45}
\]

Since \( 0 \leq \beta \leq 1 \), the argument of the term in brackets in (44) is a periodic function of \( \theta \) and hence the index at the origin equals +2. The sole effect of perturbations with higher orbital numbers on the 2-vortex solution amounts to the symmetric deformations, similarly to the effect of \( m \geq 2 \)-perturbations on the 1-vortex solution.

### C. The \( n = 3 \) multivortex

We conclude our discussions of one-parameter deformations with the case of the \( (n = 3) \)-solution. The absolute values of \( \psi^{(3)} \) are shown in figure 4 for \( m = 1, 2, 3 \) and in figure 6 for \( m = 4 \) and \( m = 5 \). (As we explain below, all higher orbital numbers cannot produce the splitting of the 3-vortex and are less interesting, therefore.) The location of the positions of individual vortices is facilitated by plotting the level curves of \( |\psi^{(3)}|^2 \), figures 5 and 7. A poor visibility of zeros in figure 6(b) is due to a significant disproportion in the widths of the central and surrounding vortices. In figure 7(b) we have attempted to improve the visualisation by contouring the central vortex at lower levels of \( |\psi|^2 \) than the surrounding ones.
FIG. 4. The modulus squared of the \((n = 3)\)-solutions. (a) the coaxial 3-vortex \((\beta = 0)\); (b) its \(m = 1\) deformation (here \(\delta_1 = -\pi/2\)); (c) \(m = 2\); (d) \(m = 3\). In (b-d), \(\beta = 1\).
FIG. 5. Level curves of $|\psi|^2$ for the coaxial three-vortex (a) and its lowest orbital deformations. (b) $m = 1$ (here $\delta_1 = -\pi/2$); (c) $m = 2$; (d) $m = 3$. In (b-d) $\beta = 1$.

For $m \geq 2$, the indices of the surrounding zeros can be found from the symmetry considerations provided the index of the zero at the origin is known. The Taylor expansion about $r = 0$ gives:

$$\psi^{(3)}(r, \theta)|_{m=2} = -\frac{\beta}{4} \left( e^{i\theta} + \frac{\beta/2}{4-\beta^2} e^{-i\theta} \right) r + O(r^3), \quad (46)$$

$$\psi^{(3)}(r, \theta)|_{m=4} = \frac{\beta r}{4} e^{-i\theta} + O(r^3), \quad (47)$$

$$\psi^{(3)}(r, \theta)|_{m=5} = \frac{\beta r^2}{16} e^{-2i\theta} + O(r^3). \quad (48)$$
For higher orbital quantum numbers, \( m \geq 6 \), the \((n = 3)\)-solution consists of just one vortex (of vorticity +3) sitting at the origin.

FIG. 6. The modulus squared of the \( m = 4 \) (a) and \( m = 5 \) (b) orbital deformations of the coaxial three-vortex. Here \( \beta = 1 \).

FIG. 7. The level curves of \(|\psi|^2\) for the two solutions shown in figure 6. (a) \( m = 4 \); (b) \( m = 5 \).
D. The one-parameter deformations of the coaxial multivortices with general $n$

Having analysed a number of particular combinations of $n$ and $m$, we can now formulate general conjectures on the indices of one-parameter multivortex solutions. We have identified three distinct cases depending on the relation between $n$ and $m$.

(i.) The simplest situation occurs if $m \geq 2n$. In this case, it is sufficient to determine the index of the vortex at the origin. We conjecture that the Taylor’s expansion, as $r \to 0$, is

$$
\psi^{(n)} = \frac{r^n}{2^n n!} \left[ 1 - (-1)^n \delta_{m,2n} \frac{\beta}{2} e^{-2in\theta} \right] e^{in\theta} + O(r^{n+1}), \quad m \geq 2n.
$$

(49)

(Here $\delta_{m,2n}$ is Kronecker’s delta.) According to (49), in this case we have a single multivortex sitting at the origin, with vorticity $Q_0 = +n$.

(ii.) Next, let $n < m < 2n$. In this case we suggest the following expansion, as $r \to 0$:

$$
\psi^{(n)} = \frac{\beta (-1)^{n+1} r^{m-n}}{2^{m-n+1} (m-n)!} e^{-i(m-n)\theta} + O(r^{n+1}), \quad n < m < 2n.
$$

(50)

Eq.(50) shows that there is a vortex with vorticity $Q_0 = -(m-n)$ at the origin and $m$ vortices with indices $Q_{1,2,...m} = +1$ placed symmetrically about the origin in accordance with the rotation symmetry $\theta \to \theta + 2\pi/m$.

(iii.) Finally, it remains to consider the case $m \leq n$. This is the most nontrivial situation as in this case there may be no vortices at the origin at all. [See e.g. figures 3(b),(c) and 4(b),(d)]. The analysis of several combinations of $n$ and $m$ suggests that the index $Q_0$ at the origin is the smallest-modulus remainder (positive or negative) from the division of $n$ by $m$. That is, $Q_0$ is equal to the smallest of all positive and negative integers $q$ satisfying

$$
n = km + q, \quad |q| < m,
$$

(51)

where $k$ is a positive integer. In case there is a positive and negative remainder of equal modulus (i.e. if $m = 2l$ and $n = km \pm l$ with $l > 0$), the index equals the positive remainder: $Q_0 = l$. Each of the $km$ surrounding vortices has index $+1$. These are placed symmetrically
about the origin (except for the case of $m = 1$, of course) in accordance with the rotation symmetry $\theta \rightarrow \theta + 2\pi/m$. For instance, the $n = 4$, $m = 2$ solution consists of four vortices centred on the $x$-axis and grouped in two pairs symmetrically with respect to the origin.

All particular solutions considered so far verify equations (49)-(51). However we do not yet have proofs of these formulae in the case of general $n$ and $m$.

V. THE ENERGY OF THE MULTIVORTICES

In this section we calculate the action integral for the vortices. In the phenomenological theory of phase transitions this integral characterises the free energy of the system. (Apparently for this reason this quantity is also referred to as energy in mathematics literature [10–12] — although it gives the stationary Hamiltonian only for particular (2+1)-dimensional extensions of our (2+0)-dimensional model (3).) Besides its fundamental role in physical applications, the action can be used as a powerful variational tool in a purely mathematical analysis of stationary and moving topological solitons (see e.g. [10–13]).

The action integral for equation (4) has the form

$$E[\psi] = \int_{D_R} \left( \frac{|\nabla \psi|^2}{1 - |\psi|^2} + 1 - |\psi|^2 \right) d^2 x = \int_{D_R} \left( \frac{\overline{\partial \psi} \partial \overline{\psi} + \overline{\partial \psi} \partial \psi}{2(1 - |\psi|^2)} + 1 - |\psi|^2 \right) d^2 x, \quad (52)$$

where the integration is over a disc of a large radius $R$ centered at the origin. The massive Thirring model (8)-(9), in its turn, is derivable from the action

$$S[\psi^{(n-1)}, \psi^{(n)}] = \int_{D_R} \left\{ \overline{\psi^{(n-1)}} \partial^{(n)} \psi^{(n-1)} - \overline{\psi^{(n-1)}} \overline{\partial^{(n-1)} \psi^{(n-1)}} + (|\psi^{(n-1)}|^2 - 1)(|\psi^{(n)}|^2 - 1) + c.c. \right\} d^2 x. \quad (53)$$

The two systems are equivalent; expressing $\psi^{(n-1)}$ through $\psi^{(n)}$ from eq.(9) and substituting into (8) produces eq.(4) for $\psi^{(n)}$. Accordingly, the integrands in (52) and (53) differ only by a divergence:

$$S[\psi^{(n-1)}, \psi^{(n)}] = 2E[\psi^{(n)}] - \int_{D_R} \nabla \cdot \mathbf{A}^{(n)} d^2 x; \quad n \geq 1, \quad (54)$$

where
\[ A^{(n)} = -\nabla \ln(1 - |\psi^{(n)}|^2) + 2W(|\psi^{(n)}|^2) \nabla \times \arg \psi^{(n)}, \quad (55) \]

and

\[ W(\rho) = \frac{\rho}{1-\rho} + \ln(1-\rho). \]

Alternatively, we can express \( \psi^{(n)} \) through \( \psi^{(n-1)} \); substituting this into (45) produces eq.(46) for \( \psi^{(n-1)} \). This equivalence is, again, reflected by the corresponding actions. We have

\[ S[\psi^{(n-1)}, \psi^{(n)}] = 2E[\psi^{(n-1)}] - \int_{D_R} \nabla \cdot B^{(n-1)} d^2x; \quad n \geq 2, \quad (56) \]

where

\[ B^{(n-1)} = -\nabla \ln(1 - |\psi^{(n-1)}|^2) - 2W(|\psi^{(n-1)}|^2) \nabla \times \arg \psi^{(n-1)}. \quad (57) \]

Both \( A^{(n)} \) and \( B^{(n-1)} \) are regular in the finite part of the \((x,y)\)-plane and therefore the double integrals in (54) and (56) can be transformed to contour integrals over the boundary of the disc \( D_R \):

\[ S[\psi^{(n-1)}, \psi^{(n)}] = 2E[\psi^{(n)}] - \oint_{\partial D_R} A^{(n)} \cdot \hat{r} \, dl, \quad n \geq 1, \quad (58) \]

\[ S[\psi^{(n-1)}, \psi^{(n)}] = 2E[\psi^{(n-1)}] - \oint_{\partial D_R} B^{(n-1)} \cdot \hat{r} \, dl, \quad n \geq 2. \quad (59) \]

Here \( \hat{r} = r/r \). For large \( r \) eq.(23) gives

\[ |\psi^{(n)}|^2 = 1 - \frac{n}{r} + \frac{1}{r^2} \left[ n^2 \left( 2\mu_1 + \frac{1 + \kappa^2}{4} \right) + \frac{n(n-1)}{2} \kappa \theta \right] + O \left( \frac{1}{r^3} \right). \quad (60) \]

As one can easily check using (16), the \( O(r^{-2}) \) term in (60) is a total derivative:

\[ 2\mu_1 + \frac{1 + \kappa^2}{4} = -\partial_\theta \ln \sum |\sigma_m|, \]

and so (60) can be written as

\[ |\psi^{(n)}|^2 = 1 - \frac{n}{r} + \frac{1}{r^2} \partial_\theta \lambda^{(n)}. \quad (61) \]

Therefore
\[ W(|\psi^{(n)}|^2) = \frac{r}{n} - \ln \frac{r}{n} + \frac{\partial \lambda^{(n)}}{n} - 1 + O \left( \frac{1}{r} \right) \quad \text{as } r \to \infty, \]  
\[ \text{(62)} \]

and using this in (58)-(59) we obtain, finally,

\[ S[\psi^{(n-1)}, \psi^{(n)}] = 2E[\psi^{(n)}] - 4\pi \left( R - n \ln \frac{R}{n} - n + \frac{1}{2} \right), \quad n \geq 1, \]  
\[ \text{(63)} \]

\[ S[\psi^{(n-1)}, \psi^{(n)}] = 2E[\psi^{(n-1)}] + 4\pi \left( R - (n-1) \ln \frac{R}{n-1} - n + \frac{1}{2} \right), \quad n \geq 2. \]  
\[ \text{(64)} \]

Subtracting (64) from (63) we arrive at the formula relating the sine-Gordon actions of the solutions with vorticities \( n \) and \( (n-1) \):

\[ E^{(n)} - E^{(n-1)} = 2\pi \left( 2R - n \ln \frac{R}{n} - (n-1) \ln \frac{R}{n-1} - 2n + 1 \right); \quad n \geq 2. \]  
\[ \text{(65)} \]

This equation can be used to calculate, recursively, the energies of all vortices; the only outstanding ingredient is the action of the vortex with \( n = 1 \). The \( E^{(1)} \) can be found from (63), provided we know the Thirring action \( S(\psi^{(0)}, \psi^{(1)}) \). This action is obtained directly from eq.(53) where we only need to set \( \psi^{(0)} = 1 \):

\[ S(\psi^{(0)}, \psi^{(1)}) = \int_{D_R} \left( \overrightarrow{\partial} \psi^{(1)} + \partial \psi^{(1)} \right) d^2x = \oint_{\partial D_R} \left( \psi^{(1)} e^{i\theta} + \psi^{(1)} e^{-i\theta} \right) dl. \]  
\[ \text{(66)} \]

Making use of the asymptotic expansion (25), eq.(66) yields

\[ S(\psi^{(0)}, \psi^{(1)}) = 2\pi (2R - 1) + O(R^{-1}). \]  
\[ \text{(67)} \]

Finally, the sine-Gordon action of the single-vortex solution is

\[ E^{(1)} = 2\pi (2R - \ln R - 1) + O(R^{-1}). \]  
\[ \text{(68)} \]

The formulas (63)+(68) provide the actions for vortices with any \( n \). As one could have expected, the actions do not depend on any of the parameters \( \beta_m, \delta_m \) of the solution (14).

Note also that we can use equations (63) and (68) to define the energy of the vorticity-free state \( \psi^{(0)} \equiv 1 \). Consistently with one’s physical intuition, eq.(65) with \( n \to 1 \) yields \( E^{(0)} = 0. \)
Solving the recursion relation (65) with initial condition (68) we can obtain the action \( E(n) \) in closed form:

\[
E(n) = 2\pi \left[ 2nR - n^2 \ln R + n(\ln n - 1) + 2 \sum_{k=1}^{n-1} k(\ln k - 1) \right], \quad n \geq 2. \tag{69}
\]

Note that eq. (69) remains valid for \( n = 1 \) (in which case one should simply disregard the sum \( \sum_{k=1}^{n-1} \)) and for \( n = 0 \) (in which case one should also set \( n \ln n = 0 \)).

For completeness, we also evaluate the Thirring action. Using (69) in (63), we have

\[
S[\psi^{(n-1)}, \psi^{(n)}] = 2\pi \left[ 2(2n-1)R - 2n(n-1) \ln R + 4 \sum_{k=1}^{n-1} k(\ln k - 1) - 1 \right]; \quad n \geq 2. \tag{70}
\]

Like the previous formula, this expression remains valid for \( n = 1 \).

The main conclusion of this section is that the action (or “energy”) of an \( n \)-vortex solution does not depend on parameters \( \beta_m, \delta_m \) and therefore, on the relative positions of the individual vortices. This implies that in any \((2 + 1)\)-dimensional (i.e. time-dependent) extension of the planar complex sine-Gordon theory, vortices may form non-interacting configurations.

VI. COMPLEX SINE-GORDON VS GINZBURG-LANDAU: ANALYSIS OF ZERO MODES

The aim of this section is to find out whether the Ginzburg-Landau axially-symmetric vortices admit non-symmetric deformations similar to those arising in the complex sine-Gordon equation. Let \( \psi^{(n)} \) be an axially-symmetric solution of equation (2): \( \psi^{(n)}(r, \theta) = \Phi_n(r)e^{in\theta} \). If this \( \psi^{(n)} \) is a member of a broader family of solutions parameterised by \( p \) continuous parameters \( \alpha_1, \alpha_2, \ldots, \alpha_p \), i.e.

\[
\Phi_n(r)e^{in\theta} = \psi(r, \theta; \alpha_1, \ldots, \alpha_p) \bigg|_{\alpha_1=\alpha_2=\ldots=\alpha_p=0}, \tag{71}
\]

the equation (2) linearised about the symmetric multivortex will have \( p \) solutions of the form

\[
\delta \psi_j(r, \theta) = \frac{\partial \psi(r, \theta; \alpha_1, \ldots, \alpha_p)}{\partial \alpha_j} \bigg|_{\alpha_1=\alpha_2=\ldots=\alpha_p=0}; \quad j = 1, \ldots, p. \tag{72}
\]
Therefore our strategy will be to examine the spectrum of linearised perturbations of the Ginzburg-Landau vortices. We will also be considering the linearisation of the complex sine-Gordon equation (4); the comparison of the corresponding sets of zero modes for the two systems will give rise to some interesting observations.

A. Linearised eigenvalue problem

Technically, it is convenient to treat the linearised boundary-value problem as an eigenvalue problem. With this purpose in mind, we consider the (2 + 0)-dimensional Ginzburg-Landau equation (2) as a time-independent reduction of a (2 + 1)-dimensional Higgs-field equation

\[ \psi_{tt} - \nabla^2 \psi - (1 - |\psi|^2)\psi = 0. \] (73)

(Alternatively, we could have considered it as a reduction of the Gross-Pitaevskii equation \[ i\psi_t + \nabla^2 \psi + (1 - |\psi|^2)\psi = 0, \] but the relativistic generalisation (73) is computationally advantageous as it gives rise to a symmetric eigenvalue problem.) In a similar way we can define a (2 + 1)-dimensional generalisation of the planar complex sine-Gordon (4):

\[ \psi_{tt} - \nabla^2 \psi - \frac{(\nabla \psi)^2}{1 - |\psi|^2} - \psi(1 - |\psi|^2) = 0. \] (74)

(Note that (74) is not relativistically invariant and can hardly claim any physical relevance; we are introducing this equation just for auxiliary purposes here.) Assuming a solution of the form

\[ \psi(r, \theta, t) = \Phi_n(r)e^{i\theta} + \delta\psi(r, \theta, t) \equiv [\Phi_n(r) + \epsilon\phi(r, \theta)\cos\omega t]e^{i\theta} \] (75)

and linearising (73) in small \( \epsilon \), we obtain

\[ - \nabla^2 \phi - \frac{1}{r^2}(\partial_\theta + in)^2 \phi - \phi + 2\Phi_n^2\phi + \Phi_n^2\phi = \omega^2 \phi, \] (76)

where \( \nabla^2 \phi = \phi_{rr} + r^{-1}\phi_r \). In a similar way, the linearisation of equation (74) gives
\[-\nabla_r^2 \phi - \frac{1}{r^2} \partial_\theta^2 \phi - \frac{2\Phi_n'}{1 - \Phi_n^2} \partial_r \phi - \frac{2in}{r^2} \frac{1}{1 - \Phi_n^2} \partial_\theta \phi \]

\[+ \left[ \frac{(n^2/r^2) - (\Phi_n')^2}{(1 - \Phi_n^2)^2} + 2\Phi_n^2 \right] \phi + \left[ \Phi_n^2 + \frac{(n^2/r^2)\Phi_n^2 - (\Phi_n')^2}{(1 - \Phi_n^2)^2} \right] \phi = \omega^2 \phi, \quad (77)\]

where the prime over \(\Phi_n\) denotes the derivative with respect to \(r\). Equations (76) and (77) can be regarded as eigenvalue problems, with \(\omega^2\) being an eigenvalue and \((\phi, \overline{\phi})\) the associated eigenvector. Expanding \(\phi\) in the Fourier series in \(\theta\):

\[\phi(r, \theta) = \sum_{m=-\infty}^{\infty} \phi_m(r) e^{im\theta} = \sum_{m=-\infty}^{\infty} \{a_m(r) + ib_m(r)\} e^{im\theta}, \quad (78)\]

and transforming to

\[u_m(r) = a_m + a_{-m}, \quad v_m(r) = a_m - a_{-m}, \quad (79)\]

we obtain a sequence of one-dimensional eigenvalue problems, one for each value of the azimuthal number \(m\):

\[\mathcal{L}_{n,m} \begin{pmatrix} u_m \\ v_m \end{pmatrix} = \omega^2 \begin{pmatrix} u_m \\ v_m \end{pmatrix}. \quad (80)\]

The operator \(\mathcal{L}_{n,m}\) is defined by

\[\mathcal{L}_{n,m} \equiv \begin{pmatrix} -\nabla_r^2 + \frac{n^2 + m^2}{r^2} + 3\Phi_n^2(r) - 1 \quad \frac{2mn}{r^2} \\ -\nabla_r^2 + \frac{2mn}{r^2} + \Phi_n^2(r) - 1 \end{pmatrix} \quad (81)\]

in the Ginzburg-Landau case, and by

\[\mathcal{L}_{n,m} \equiv \begin{pmatrix} -\nabla_r^2 + \mathcal{B}_n(r) \frac{d}{dr} + \frac{m^2}{r^2} + \mathcal{C}_n(r) + \mathcal{D}_n(r) \\ m\mathcal{A}_n(r) \end{pmatrix} \quad (82)\]

in the complex sine-Gordon case. In (82) we have introduced the notations

\[\mathcal{A}_n(r) = \frac{2n}{r^2} \frac{1}{1 - \Phi_n^2}, \quad \mathcal{B}_n(r) = \frac{2n}{r} \frac{\Phi_n^2}{1 - \Phi_n^2} - 2\Phi_n \Phi_{n-1}, \quad \mathcal{C}_n(r) = 2\Phi_n^2 - \Phi_n^2 \Phi_{n-1}^2 - 1 + \frac{n^2 1 + \Phi_n^2}{r^2 1 - \Phi_n^2} + \frac{2n \Phi_{n-1} \Phi_n^3}{r 1 - \Phi_n^2}, \quad \mathcal{D}_n(r) = \Phi_n^2 - \Phi_{n-1}^2 + \frac{2n \Phi_n \Phi_{n-1}}{r 1 - \Phi_n^2}. \quad (83)\]

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The imaginary parts of the Fourier coefficients $\phi_m$ satisfy the same eigenvalue problem (80) with $L_{n,m}$ as in (81) or (82), where the eigenfunctions $(u_m, v_m)$ should only be defined by

$$u_m(r) = b_m - b_{-m}, \quad v_m(r) = b_m + b_{-m}. \quad (84)$$

(Note that the eigenfunctions $(u_m, v_m)$ do depend on the vorticity $n$ but we are omitting the corresponding subscript to keep the notation simpler.)

Thus the spectrum of linearised excitations of the symmetric multivortex $\psi^{(n)}$ is given by eigenvalues of the operator $L_{n,m}$ with $m$ varying from $-\infty$ to $\infty$. In fact in view of equations (79) and (84) we can restrict ourselves to nonnegative $m$ only. If $(u_m, v_m)$ is an eigenvector of the operator $L_{n,m}$ associated with an eigenvalue $\omega^2$, then $(u_m, -v_m)$ is an eigenvector of the operator $L_{n,-m}$ associated with the same eigenvalue $\omega^2$.

We solved the eigenvalue problem (80)-(81) numerically. Before discussing the results of the numerical analysis, it is instructive to compare the spectrum structure of the Ginzburg-Landau operator (81) with that of its complex sine-Gordon counterpart.

**B. The spectrum structure**

For any positive $n$ and $m \geq 0$ we introduce two bases of solutions of the linear system (80)-(81). One basis can be defined by the asymptotic behaviours at the origin. To this end, we rewrite the system in terms of $\tilde{u} = (u_m + v_m)/2$ and $\tilde{v} = (u_m - v_m)/2$ (where the subscript $m$ is omitted for simplicity of notation):

$$
\begin{pmatrix}
L_{m+n} + 2\Phi_n^2 & \Phi_n^2 \\
\Phi_n^2 & L_{m-n} + 2\Phi_n^2
\end{pmatrix}
\begin{pmatrix}
\tilde{u} \\
\tilde{v}
\end{pmatrix}
= \omega^2
\begin{pmatrix}
\tilde{u} \\
\tilde{v}
\end{pmatrix}.
\quad (85)
$$

Here $L_s = -\nabla^2 + (s^2/r^2) - 1$. The advantage of the formulation (85) is in that for $r \to 0$, the cross-coupling potentials $\Phi_n^2 \sim r^{2n}$ are small and the equation for $\tilde{u}(r)$ decouples from the equation for $\tilde{v}(r)$. Expanding each of the $\tilde{u}(r)$ and $\tilde{v}(r)$ in power series in $r$ (supplemented by logarithmic terms where necessary), e.g.

$$
\tilde{u}(r) = (\tilde{u}_0 + \tilde{u}_0 \ln r)r^p + (\tilde{u}_1 + \tilde{u}_1 \ln r)r^{p+2} + ...
$$

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and substituting into (85), one can easily verify that for each pair \((n, m)\) there are four solutions with the following asymptotic behaviour:

\[
\begin{align*}
\tilde{Z}_1 &= \left( r^{m+n}[1 + o(r)] \atop O(r^{m+n+2}) \right); \\
\tilde{Z}_2 &= \left( O(r^{m-n}+2n+2) \atop r^{m-n}[1 + o(r)] \right); \\
\tilde{Z}_3 &= \left( r^{-(m+n)}[1 + o(r)] \atop o(r^{-(m+n)+2n+1}) \right); \\
\tilde{Z}_4 &= \left( o(r^{-m-n}+2n+1) \atop r^{-m-n}[1 + o(r)] \right) \text{ (for } m \neq n); \\
\tilde{Z}_4 &= \left( O(r^{2n+2} \ln r) \atop \ln r \cdot [1 + o(r)] \right) \text{ (for } m = n).
\end{align*}
\]

(Here \(\hat{Z}\) denotes the column \((\hat{u}, \hat{v})^T\), of course.) Transforming back to \(u_m\) and \(v_m\) and introducing the notation \(Z = (u_m, v_m)^T\), we have four linearly independent solutions of the system (80)-(81) (as \(r \to 0\)):

\[
\begin{align*}
Z_1 &= r^{m+n}[1 + o(r)] \left( \begin{array}{c} 1 \\ 1 \end{array} \right); \\
Z_2 &= r^{m-n}[1 + o(r)] \left( \begin{array}{c} 1 \\ -1 \end{array} \right); \\
Z_3 &= r^{-(m+n)}[1 + o(r)] \left( \begin{array}{c} 1 \\ 1 \end{array} \right); \\
Z_4 &= r^{-|m-n|}[1 + o(r)] \left( \begin{array}{c} 1 \\ -1 \end{array} \right) \text{ (for } m \neq n), \\
Z_4 &= \ln r \cdot [1 + o(r)] \left( \begin{array}{c} 1 \\ -1 \end{array} \right) \text{ (for } m = n). \\
\end{align*}
\]  

(86)

The solutions \(Z_1, Z_2\) are bounded and \(Z_3, Z_4\) unbounded for all \(m\) and \(n\).

In a similar way one can show that the linearised complex sine-Gordon (80)-(82) also has two bounded and two unbounded solutions near the origin. In this case one should only take into account that as \(r \to 0\),

\[
\begin{align*}
\mathfrak{A} &= \frac{2n}{r^2} + \frac{2n}{(2^n n!)^2} r^{2n-2} + O(r^{2n}), \\
\mathfrak{B} &= -\frac{1}{r} - \frac{2n}{(2^n n!)^2} r^{2n-1} + O(r^{2n+1}), \\
\mathfrak{C} &= \frac{n^2}{r^2} - 1 + \frac{2n^2}{(2^n n!)^2} r^{2n-2} + O(r^{2n}), \\
\mathfrak{D} &= O(r^{2n}).
\end{align*}
\]

The four basis solutions are given by the same equations (80) as in the case of the Higgs field. We should emphasise here that eqs. (86) are valid for all \(\omega\) (including \(\omega = 0\)), both in the Higgs and sine-Gordon cases.
The second basis is defined by the asymptotic behaviours as $r \to \infty$. Consider, first, the Higgs system (80)-(81) and let $0 < \omega < \sqrt{2}$. The four solutions are given by

$$Y_{1,2} = \frac{e^{\pm i\omega r}}{\sqrt{r}} \left( 1 \pm \frac{\frac{mn}{r^2}}{2} + \mathcal{O}\left(\frac{1}{r^3}\right) \right); \quad \text{(87)}$$

$$Y_{3,4} = \frac{e^{\pm \sqrt{2-\omega^2}r}}{\sqrt{r}} \left( 1 \pm \frac{\sqrt{2-\omega^2}r - \frac{2m^2 - 1 + \frac{np + 1}{4\omega^2}}{2(2-\omega^2)r} + \mathcal{O}\left(\frac{1}{r^3}\right)}{\frac{mn}{r^2} + \mathcal{O}\left(\frac{1}{r^3}\right)} \right). \quad \text{(88)}$$

The solutions $Y_1$, $Y_2$, and $Y_4$ are bounded and $Y_3$ unbounded as $r \to \infty$. Similarly, for all $0 < \omega < 2$ the linearised complex sine-Gordon (80)-(82) has three bounded and one unbounded solution as $r \to \infty$:

$$Y_{1,2} = \frac{e^{\pm i\omega r}}{r} \left( 1 \pm \frac{\frac{m^2}{16} \pm \frac{m^2}{16} - \frac{1}{4\omega^2} + \mathcal{O}\left(\frac{1}{r^3}\right)}{\frac{mn}{r^2} + \mathcal{O}\left(\frac{1}{r^3}\right)} \right); \quad \text{(89)}$$

$$Y_{3,4} = e^{\pm \sqrt{2-\omega^2}r^p} \left( 1 \pm \frac{\sqrt{2-\omega^2}r - \frac{2m^2 - 1 + \frac{np + 1}{4\omega^2}}{2(2-\omega^2)r} + \mathcal{O}\left(\frac{1}{r^3}\right)}{\frac{mn}{r^2} + \mathcal{O}\left(\frac{1}{r^3}\right)} \right), \quad \text{(90)}$$

where $p = \mp 2n(4-\omega^2)^{-1/2} - 1$. In the derivation of (89)-(90) we made use of the asymptotic expansions, as $r \to \infty$, of the coefficient functions in (83):

$$\mathfrak{A}_n(r) = \frac{2}{r} - \frac{1}{4r^3} - \frac{n}{2r^4} + \mathcal{O}\left(\frac{1}{r^5}\right),$$

$$\mathfrak{B}_n(r) = -\frac{2}{r} - \frac{1}{4r^3} + \mathcal{O}\left(\frac{1}{r^4}\right),$$

$$\mathfrak{C}_n(r) = 2 - \frac{2n}{r} - \frac{1}{2r^2} - \frac{3n}{4r^3} + \mathcal{O}\left(\frac{1}{r^4}\right),$$

$$\mathfrak{D}_n(r) = 2 - \frac{2n}{r} - \frac{1}{2r^2} - \frac{3n}{4r^3} + \mathcal{O}\left(\frac{1}{r^4}\right). \quad \text{(91)}$$

Each of the solutions $Z_i$ ($i = 1, \ldots, 4$) can be expanded over the basis $Y_j$:

$$Z_i(r) = \sum_{j=1}^{4} T_{ij}^{(n,m)}(\omega) Y_j(r).$$

For all $0 < \omega < \sqrt{2}$ in the Higgs case and all $0 < \omega < 2$ in the complex sine-Gordon case, the linear combination $T_{23}^{(n,m)} Z_1(r) - T_{13}^{(n,m)} Z_2(r)$ represents a solution which is bounded both
as $r \to 0$ and $r \to \infty$. Therefore, in both cases small nonzero $\omega$ belong to the continuous spectrum. The question that is of concern to us, of course, is whether $\omega = 0$ belongs to the continuum; in other words, is there a bounded solution for $\omega = 0$? Surprisingly, the answers for the Higgs (alias Ginzburg-Landau) and complex sine-Gordon linearisation, are different.

Let us start with the Higgs case and let $r \to \infty$. Two asymptotic solutions, $Y_{3,4}$, are given by eq.(88) where we only need to set $\omega = 0$. The other two asymptotic formulas, eq.(87), cannot be used for $\omega = 0$. Instead, we have two solutions with the asymptotics

$$Y_{1,2} = r^{\pm m} \left( \begin{array}{c} -\frac{m}{r^2} + O\left(\frac{1}{r}\right) \\ 1 + O\left(\frac{1}{r}\right) \end{array} \right) \quad (\text{for } m \neq 0);$$

$$Y_1 = \ln r \cdot \left[ 1 + O\left(\frac{1}{r}\right) \right] \left( \begin{array}{c} 0 \\ 1 \end{array} \right); \quad Y_2 = \left[ 1 + O\left(\frac{1}{r}\right) \right] \left( \begin{array}{c} 0 \\ 1 \end{array} \right) \quad (\text{for } m = 0). \quad (92)$$

Therefore, for $\omega = 0$ we only have two, not three, solutions bounded as $r \to \infty$: $Y_2$ and $Y_4$. The solutions $Z_1$ and $Z_2$, bounded at the origin, can still be expanded over the basis $Y_j$ ($j = 1, \ldots, 4$); however, this time in order for the linear combination $c_1 Z_1 + c_2 Z_2$ to remain bounded as $r \to \infty$, the constants $c_1$ and $c_2$ have to satisfy two conditions: $c_1 T_{11}^{(n,m)}(0) + c_2 T_{21}^{(n,m)}(0) = 0$ and $c_1 T_{13}^{(n,m)}(0) + c_2 T_{23}^{(n,m)}(0) = 0$. This imposes a requirement on the matrix $T^{(n,m)}(\omega)$ at the point $\omega = 0$:

$$\Delta^{(n,m)} \equiv \begin{vmatrix} T_{11}^{(n,m)}(0) & T_{21}^{(n,m)}(0) \\ T_{13}^{(n,m)}(0) & T_{23}^{(n,m)}(0) \end{vmatrix} = 0, \quad (93)$$

which is not a priori satisfied for all values of $n$ and $m$. Consequently, the zero mode (i.e. a solution $u_m(r), v_m(r)$ pertaining to $\omega = 0$ and bounded for all $0 \leq r < \infty$) can only arise for some pairs $(n, m)$.

Consider now the linearised complex sine-Gordon (80)+(82). As in the Higgs case, the asymptotic solutions $Y_{3,4}(r)$ (eqs.(81)) are valid for $\omega = 0$ whereas the formulas (88) are not. Instead, for $\omega = 0$ the two solutions $Y_{1,2}(r)$ have the asymptotics

$$Y_1 = \frac{1}{r} \left( \begin{array}{c} -\frac{m}{2r} + O\left(\frac{1}{r}\right) \\ 1 + O\left(\frac{1}{r}\right) \end{array} \right); \quad Y_2 = \left( \begin{array}{c} -\frac{m}{2r} + O\left(\frac{\ln r}{r}\right) \\ 1 + m^2 n \frac{\ln r}{r} + O\left(\frac{\ln r}{r}\right) \end{array} \right), \quad r \to \infty. \quad (94)$$
Thus in the complex sine-Gordon case we have three bounded solutions as $r \to \infty$: $Y_1$, $Y_2$, and $Y_4$ (and not two as in the Higgs case.) Accordingly, there always is a solution bounded for all $r \in [0, \infty)$: $(u_m, v_m) = T^{(n,m)}_{23} Z_1(r) - T^{(n,m)}_{13} Z_2(r)$, and so we may conclude that there is a zero mode for any pair $(n,m)$ — without having to find this solution analytically or numerically. In this sense, in the complex sine-Gordon case the value $\omega = 0$ belongs to the continuous spectrum whereas in the Ginzburg-Landau case zero modes can only appear as discrete eigenvalues. In the latter case we had to resort to the help of computer.

C. Ginzburg-Landau zero modes: a numerical search

To find eigenvalues and eigenfunctions numerically, we replaced derivatives in eq.(81) (and eqs.(95)-(96) below) with the second order-accurate finite differences. Typically we took a grid with 1000 points on an interval $0 < r \leq 80$ (i.e. with the step size $\Delta r = 0.08$). The coaxial multivortices $\Phi_n(r)$ of which the potentials in (81) are formed, were pre-computed using Newton’s method.

The appropriate boundary conditions at the origin follow from the asymptotic behaviours (86):

\[
\begin{align*}
    u_r(0) &= v_r(0) = 0, \quad \text{for } m \neq n \pm 1; \\
    u(0) &= v(0) = 0, \quad \text{for } m = n \pm 1.
\end{align*}
\]

At infinity, the boundary conditions are chosen to accommodate the bounded solutions in eqs.(88)+(92):

\[
\begin{align*}
    u_r \to 0, \quad v_r \to 0 \quad \text{as } r \to \infty.
\end{align*}
\]

Numerically, zero modes appear as small nonzero eigenvalues and one still has to distinguish them from small eigenvalues arising from the continuous spectrum when the infinite line is approximated by a finite interval $[0, 80]$. The genuine zero modes can be discerned by considering the large-$r$ behaviours of the associated eigenvectors. The “true” zero modes
are allowed to decay exponentially (as in (88)) or as \( r^{-m} \) (as in (92)) whereas the continuous spectrum solutions will generically decay as \( 1/\sqrt{r} \) (see eq.(87)).

Before calculating the eigenvalues of the Higgs vortices, we tested our numerical scheme on the complex sine-Gordon operator (82). As expected, we obtained one zero mode for each \( n \) and \( m \). (We tested \( m = 0, 1, ..., 5 \) for each of \( n = 1, 2, 3 \).) The associated eigenfunctions were found to coincide with the derivatives of the general \( n \)-vortex solution \( \psi^{(n)} \) w.r.t. the azimuthal parameters \( \beta_m \) (see the next subsection.)

Proceeding to the Higgs system, we examined the one-, two and three-vortex solutions, i.e., \( n = 1, 2 \) and 3. The azimuthal quantum number of the analysed perturbation ranged from \( m = 0 \) to 5 in each case. Zero eigenvalues were only found for \( m = 0 \) and 1; both result from obvious symmetry properties of the Ginzburg-Landau/Higgs equation.

For \( m = 0 \), the complex eigenfunction \( \phi(r) \) of the operator (76) associated with the numerically-found zero eigenvalue, was found to coincide with the function \( i\Phi_n(r) \) (i.e., \( u_0 = 0, \, v_0 = 2\Phi_n \)). This zero mode is related to the \( U(1) \) invariance of the Ginzburg-Landau equation (2):

\[
\phi(r)e^{in\theta} = \frac{\partial}{\partial \alpha}\Phi_n(r)e^{in\theta + i\alpha}\bigg|_{\alpha=0}.
\]

For \( m = 1 \), the numerical eigenvector \((u_1, v_1)\) associated with the zero eigenvalue was found to be equal to \((\Phi'_n, -\frac{2}{r}\Phi_n)\). (There is, of course, a zero mode for \( m = -1 \) as well: \((u_{-1}, v_{-1}) = (\Phi'_n, \frac{2}{r}\Phi_n)\).) The corresponding complex perturbations (75) are the translation modes, one \( \phi e^{in\theta} \) given by \( \partial_x [\Phi_n(r)e^{in\theta}] \) and the other one by \( \partial_y [\Phi_n(r)e^{in\theta}] \).

Both the translational and the \( U(1) \)-zero modes are well known to workers in this field. (See e.g. [11]). There is a simple analytical argument showing that zero modes cannot arise for \( m \geq 2n \); this fact is also known to specialists. However, the nonexistence of zero modes for \( 1 < m < 2n \) does not seem to have appeared in literature before.

Since the only zero-frequency excitations of the axially-symmetric vortices are those associated with the phase shifts and translations, we conclude that the coaxial multivortices of the stationary Ginzburg-Landau model do not admit continuous nonsymmetric defor-
mations. In particular, two vortices sitting, symmetrically, on top of each other cannot be continuously separated. This does not mean, of course, that the Ginzburg-Landau model does not admit noncoaxial multivortex solutions at all. Multivortex configurations with finite separations between individual vortices may exist (and in fact there are indications that they do exist [4]). However, the intervortex separations will only admit discrete sets of values, or will be allowed to vary continuously but be bounded from below by certain finite distances.

This is in sharp contrast with the coaxial multivortices of the complex sine-Gordon theory which can be continuously split and moved apart. As we have shown in previous sections, each axially-symmetric solution of this model is a member of an infinite-parameter family of solutions corresponding to a specific choice of azimuthal deformation parameters: \( \beta_1 = \beta_2 = \ldots = 0 \). We conclude this section by demonstrating that each of these continuous parameters gives rise to a zero mode in the spectrum of the corresponding linearised operator (82).

D. Zero modes of the complex sine-Gordon multivortices

The derivative of the general \( n \)-vortex solution w.r.t. the azimuthal parameter \( \beta_m \) \((m \geq 1)\), is given by

\[
\frac{\partial}{\partial \beta_m} \psi^{(n)} \bigg|_{\beta_1=\beta_2=\ldots=0} = \frac{1}{2} \sum_{k=1-n}^{n} \left( e^{i\delta_m \xi_{k+m}} + e^{-i\delta_m \xi_{k-m}} \right) \frac{\partial \psi^{(n)}}{\partial Z_k} \bigg|_{\beta_1=\beta_2=\ldots=0}. \tag{98}
\]

To calculate the derivative \( \partial \psi^{(n)}/\partial Z_k \), we notice that each of the multivortices can be written as a rational function,

\[
\psi^{(n)} = \frac{N^{(n)}}{M^{(n)}}, \tag{99a}
\]

where the numerator and denominator are homogeneous polynomials in \( Z_{1-n}, Z_{2-n}, \ldots, Z_n \), of degree \( P \) \((P \leq 2^{n-1})\). These polynomials have the form
\[
N^{(n)} = \sum_{1 - n \leq i_1, i_2, \ldots, i_P \leq n} C_{i_1i_2\ldots i_P} \delta_{i_1 + i_2 + \ldots + i_P, n} Z_{i_1}Z_{i_2}\ldots Z_{i_P},
\]
(99b)

\[
M^{(n)} = \sum_{1 - n \leq i_1, i_2, \ldots, i_P \leq n - 1} D_{i_1i_2\ldots i_P} \delta_{i_1 + i_2 + \ldots + i_P, 0} Z_{i_1}Z_{i_2}\ldots Z_{i_P},
\]
(99c)

where \( C_{i_1i_2\ldots i_P} \) and \( D_{i_1i_2\ldots i_P} \) are real coefficients and \( \delta_{s,l} \) is Kronecker’s delta. Equations (99) are straightforward from the recurrence relation (17). Note that in eq. (99b), products \( Z_{i_1}Z_{i_2}\ldots Z_{i_P} \) have their indices summing up to \( n \); we will be referring to this property by saying that the polynomial \( N^{(n)} \) has level \( n \). In this sense, the polynomial \( M^{(n)} \) (eq. (99c)) has level 0. When a polynomial of level \( l \) is differentiated w.r.t. \( Z_k \), its level is lowered down to \( l - k \). Therefore, the derivative \( \frac{\partial \psi^{(n)}}{\partial Z_k} \) is a rational function whose numerator is a polynomial of level \( n - k \) and denominator is a polynomial of level 0. Since \( Z_s|_{\beta_1=\beta_2=\ldots=0} = \xi_s = I_{s}(r) e^{i\alpha \theta} \), one can easily check that

\[
\left. \frac{\partial \psi^{(n)}}{\partial Z_k} \right|_{\beta_1=\beta_2=\ldots=0} = g_k^{(n)}(r)e^{i(n-k)\theta}
\]

with some real function \( g_k^{(n)}(r) \). Substituting into (98) we obtain the zero-frequency eigenfunctions of the operator (77) pertaining to the orbital quantum numbers \( m \) and \(-m\), respectively:

\[
\phi_m(r) = \sum_{k=1-n}^{n} g_k^{(n)}(r) I_{k+m}(r); \quad \phi_{-m}(r) = \sum_{k=1-n}^{n} g_k^{(n)}(r) I_{k-m}(r)
\]
(100)

\( m \geq 1 \). The zero modes (100) translate into zero-frequency eigenvectors of the operator (72):

\[
\begin{pmatrix}
    u_m \\
    v_m
\end{pmatrix} = \sum_{k=1-n}^{n} g_k^{(n)}(r) \begin{pmatrix}
    I_{k+m}(r) + I_{k-m}(r) \\
    I_{k+m}(r) - I_{k-m}(r)
\end{pmatrix},
\]
(101)

VII. CONCLUDING REMARKS

In this paper we have constructed families of explicit solutions of the complex sine-Gordon equation. In literature, the complex sine-Gordon model is usually considered in the (1+1)-dimensional Minkowski space where it has the form
\[
\psi_{xx} - \psi_{tt} + \frac{\left(\psi_x^2 - \psi_t^2\right)}{1 - |\psi|^2} \overline{\psi} + \psi(1 - |\psi|^2) = 0.
\]

(102)

This equation was introduced in the late 1970s in three independent field-theoretic contexts: (i) as a reduction of the \(O(4)\) nonlinear \(\sigma\)-model \[14\]; (ii) in the description of relativistic strings in a uniform antisymmetric tensor field \[13\], and (iii) in the theory of massless fermions with a scalar contact interaction \[16\]. Later on, eq.(102) reappeared in an entirely unrelated physical context — it turned out to be equivalent to the Maxwell-Bloch and self-induced transparency equations as well as the system governing stimulated Raman scattering in nonlinear optics \[17\]. In mathematics literature it is common to call eq.(102) the Lund-Regge model and write it as

\[
\alpha_{xx} - \alpha_{tt} - \frac{\sin \alpha}{\cos^3 \alpha} (\chi_x^2 - \chi_t^2) + \sin \alpha \cos \alpha = 0,
\]

\[
(\chi_x \tan^2 \alpha)_x = (\chi_t \tan^2 \alpha)_t,
\]

(103)

where \(\sin \alpha\) and \(\chi\) are the modulus and argument of the complex function \(\psi(x, t)\) in (102): \(\psi = \sin \alpha e^{i\chi}\). Geometrically, the Lund-Regge model gives the Gauss-Codazzi equations for the embedding of pseudospherical surfaces into a flat three-dimensional Euclidean space \[18\].

More recent studies of the Lorentzian complex sine-Gordon theory \(102\) and its Euclidean counterpart given by eq.(1), were motivated by the fact that eqs.(102) and (1) define integrable deformations of 2D conformal field theories, more specifically of the \(SU(2)/U(1)\) coset model and \(Z_n\) parafermions \[19\]. Other (not unrelated) sources of interest have been the search for exactly solvable conformal theories with black-hole background metrics \[20\] and exact factorisable \(S\)-matrices on the quantum level \[21\]. In the current mathematics literature, the hyperbolic Lund-Regge equation (103) is being discussed in connection with the localised induction hierarchy describing the motion of vortex filaments in an inviscid incompressible fluid \[22\]. Its applications to pseudospherical surfaces continue to attract attention (see e.g. \[23\]) while the elliptic equation (1) has been derived in the description of the “middle surfaces” of generalised Weingarten surfaces \[24\].

The (1+1)-dimensional system (102) was shown to be integrable via the inverse scattering
transform \( \mathbb{14,25,26} \) and, by virtue of this approach, broad classes of its exact solutions have become available \( \mathbb{27,28,6,7} \). The inverse scattering framework for the (2+0)-dimensional solitons was formulated in \( \mathbb{6} \). Solutions obtained in this way describe nonlinear superpositions of one-dimensional fronts intersecting at arbitrary angles on the \((x,y)\)-plane \( \mathbb{6,7} \).

In Ref. \( \mathbb{5} \) axially symmetric solutions were constructed describing \( n \) vortices \((n \geq 1)\) sitting on top of each other. In the present paper we have shown that each of these coaxial multivortices belongs to an infinite-parameter family of nonsymmetric solutions, which includes, in particular, configurations of \( n \) single vortices located at separate points of the plane. (The constellation of \( n \) separated monovortices is not the only splitting possibility for the coaxial \( n \)-vortex though; there can also be various combinations of vortices and antivortices with indices summing up to \( n \), the total topological charge characterising that particular family.) Our current construction, as well as the one of Ref. \( \mathbb{5} \), is based on the Bäcklund transformation of the complex sine-Gordon equation. (In the axially-symmetric case there is also an alternative technique employing the Painlevé reduction \( \mathbb{5} \).) For the understanding of the structure of the phase space, however, it would be instructive to embed the vortex solutions in the inverse scattering formalism; we are planning to return to this issue in future publications. The fact that even a one-vortex solution is characterised by an infinite number of parameters seems to indicate that vortices should be associated with the continuous spectrum rather than discrete eigenvalues (which account for front-like solitons.)

We have computed the energy (more precisely, the Euclidean action) for each family of multivortices. The action is found to be entirely determined by the total topological charge and independent of parameters specifying a particular solution within each topological class. This is not an unexpected result; the nontrivial dependence of the action on some of the continuous parameters entering the solution would contradict the stationarity of this configuration.

That axially-symmetric multivortices of the complex sine-Gordon theory admit noncoaxial continuations, follows already from their spectra of linearised perturbations. As we have shown in this work, the existence of a zero mode with any azimuthal number \( m = 0, 1, 2, 3... \)
is ensured by the availability of the necessary number of bounded asymptotic solutions of the linearised equations as \( r \to 0 \) and \( r \to \infty \). On the contrary, the axially symmetric solutions of the Ginzburg-Landau model do not have enough bounded linearised perturbations as \( r \to \infty \); consequently, the existence of a zero mode for a particular choice of \( n \) and \( m \) cannot be guaranteed \( a \ priori \). Our numerical analysis has demonstrated that the Ginzburg-Landau coaxial multivortices have only translational and rotational zero modes and therefore do not admit continuous deformations and/or splitting into separate vortices.

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