Signless Laplacian spectral radius and matchings in graphs*

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Abstract

The signless Laplacian matrix $Q(G)$ of a graph $G$ is defined as $Q(G) = D(G) + A(G)$, where $D(G)$ is the diagonal matrix of vertex degrees and $A(G)$ is the adjacency matrix of $G$. Let $q_1(G)$ denote the signless Laplacian spectral radius of $G$, i.e. the largest eigenvalue of the signless Laplacian matrix $Q(G)$. Let $r(n)$ be the largest root of the equation $x^3 - (3n - 7)x^2 + n(2n - 7)x - 2(n^2 - 7n + 12) = 0$. In this paper, we prove that for a positive even integer $n \geq 4$, if $G$ is an $n$-vertex connected graph with $q_1(G) > \begin{cases} r(n), & \text{for } n \geq 10 \text{ or } n = 4, \\ 4 + 2\sqrt{3}, & \text{for } n = 6, \\ 6 + 2\sqrt{6}, & \text{for } n = 8, \end{cases}$ then $G$ has a perfect matching. It is sharp in the sense that there exist graphs $H$ such that $H$ has no perfect matching and $q_1(H)$ equals the lower bound for $q_1(G)$ for every positive even integer $n \geq 4$.

Keywords: Signless Laplacian spectral radius; Perfect matching

1 Introduction

Let $G$ be a simple connected graph with vertex set $V(G)$ and edge set $E(G)$. The signless Laplacian matrix $Q(G)$ of $G$ is defined as $Q(G) = D(G) + A(G)$, where $D(G)$ is the diagonal matrix of vertex degrees and $A(G)$ is the adjacency matrix of $G$. The largest eigenvalue of $Q(G)$, written as $q_1(G)$, is called the signless Laplacian spectral radius of $G$. For a vertex subset $S \subset V(G)$, let $G[S]$ be the subgraph of $G$ induced by the vertex set $S$, and let $G - S$ be the graph obtained from $G$ by deleting the vertices in $S$ together with their incident edges. Let $G_1 \lor G_2$ denote the join of two graphs $G_1$ and $G_2$, which is the graph such that $V(G_1 \lor G_2) = V(G_1) \cup V(G_2)$ and $E(G_1 \lor G_2) = E(G_1) \cup E(G_2) \cup \{uv | u \in V(G_1), v \in V(G_2)\}$.

A matching in a graph is a set of disjoint edges, and a perfect matching in $G$ is a matching covering all vertices of $G$. The sufficient and necessary condition for the existence of perfect matchings in a graph is first given by the Tutte’s 1-factor theorem [22], which states $G$ has a perfect matching if and only if $o(G - S) - |S| \leq 0$ for any $S \subset V(G)$, where $o(G - S)$ is the

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number of odd components in $G - S$. In the past two decades, many researchers devoted to study the matchings in graphs, including maximum matching [7, 8, 14, 23], fractional matching [1, 12, 13, 16], rainbow matching [11], and matching energy [6, 15] and so on. Among these researches, relations between the eigenvalues and the matchings in graphs have been a hotpot. Chang [4] investigated the largest eigenvalue of trees with perfect matchings, while Chang and Tian [5] investigated the largest eigenvalue of unicyclic graphs with perfect matchings. In 2005, Brouwer and Haemers [2] found some sufficient conditions for the existence of perfect matchings in a graph in terms of the Laplacian matrix spectrum, and they also gave an improved result for an $r$-regular graph $G$ with order $n$ has a perfect matching when $n$ is even, and a matching of order $n - 1$ when $n$ is odd.

In 2016, O [18] investigated the existence of fractional perfect matchings in terms of the spectral radius of graphs. Later, Pan et al. [20] furtherly studied this topic from the signless Laplacian spectral radius of graphs. Very recently, O [19] obtained the relation between the spectral radius and perfect matchings in an $n$-vertex graph $G$ by firstly determining a lower bound on the number of edges of $G$ which guarantees the existence of a perfect matching in $G$.

**Theorem 1.1** (See Theorem 1.2 in [19]). Let $G$ be an $n$-vertex connected graph, where $n \geq 4$ is an even number. Then $G$ has a perfect matching if

$$|E(G)| > \begin{cases} \frac{1}{2}n^2 - \frac{5}{2}n + 5, & \text{for } n \geq 10 \text{ or } n = 4, \\ 9, & \text{for } n = 6, \\ 18, & \text{for } n = 8. \end{cases}$$

Moreover, O pointed out that this bound is sharp by calculating the number of edges of graphs $K_{n-3} \lor K_1 \lor \overline{K_2}$, $K_2 \lor \overline{K_4}$ and $K_3 \lor \overline{K_5}$, which has no perfect matchings.

Motivated by [18, 19, 20], it is natural to consider the relations between the signless Laplacian spectral radius and matchings in graphs. By following the proof of O [19], we obtain the main results as below.

**Theorem 1.2.** Let $G$ be an $n$-vertex connected graph, where $n \geq 4$ is an even number. The largest root of the equation $x^3 - (3n - 7)x^2 + n(2n - 7)x - 2(n^2 - 7n + 12) = 0$ is denoted by $r(n)$. Then $G$ has a perfect matching if

$$q_1(G) > \begin{cases} r(n), & \text{for } n \geq 10 \text{ or } n = 4, \\ 4 + 2\sqrt{3}, & \text{for } n = 6, \\ 6 + 2\sqrt{6}, & \text{for } n = 8. \end{cases}$$

Theorem 1.1 and 1.2 show that if $G$ has the maximum number of edges among the $n$-vertex graphs without a perfect matching, then $G$ has the maximum signless Laplacian spectral radius among them, except $n = 6$ or $n = 8$. The two special cases $n = 6$ and $n = 8$ says that even if graphs $G$ and $H$ share a certain graph property and $|E(G)| > |E(H)|$, we cannot guarantee that $q_1(G) > q_1(H)$.
2 Preliminaries

For a matrix $B$, let $\rho(B)$ be the largest eigenvalue of $B$.

**Lemma 2.1.** [10] Let $B$ and $B_1$ be real nonnegative matrices such that $B - B_1$ is nonnegative, then $\rho(B_1) \leq \rho(B)$.

We now explain the concepts of equitable matrices and equitable partitions.

**Definition 2.2.** [3] Suppose $B$ is a symmetric real matrix of order $n$ whose rows and columns are indexed by $P = \{1, 2, \ldots, n\}$. Let $\{P_1, P_2, \ldots, P_m\}$ be a partition of $P$. Denote $n_i = |P_i|$ and then $n = n_1 + n_2 + \cdots + n_m$. Let $B$ be partitioned according $\{P_1, P_2, \ldots, P_m\}$, that is

$$B = \begin{pmatrix} B_{1,1} & B_{1,2} & \cdots & B_{1,m} \\ B_{2,1} & B_{2,2} & \cdots & B_{2,m} \\ \vdots & \vdots & \ddots & \vdots \\ B_{m,1} & B_{m,2} & \cdots & B_{m,m} \end{pmatrix}_{n \times n},$$

where the blocks $B_{i,j}$ denotes the submatrix of $B$ formed by rows in $P_i$ and the $P_j$ columns. Let $c_{i,j}$ denote the average row sum of $B_{i,j}$. Then the matrix $C = (c_{i,j})$ is called the quotient matrix of $B$ w.r.t. the given partition. Particularly, if the row sum of each submatrix $B_{i,j}$ is constant then the partition is called equitable.

**Lemma 2.3.** [27] Let $C$ be an equitable quotient matrix of $B$ as defined in Definition 2.2. If $B$ is a nonnegative matrix, then $\rho(C) = \rho(B)$.

3 Proof of Theorem 1.2

Similar to the proof technique used by O [19], we now give the proof of Theorem 1.2.

**Proof.** Assume to the contrary that $G$ has no perfect matching. By Tutte’s 1-factor theorem, there exists $S \subset V(G)$ such that $o(G - S) - |S| \geq 1$, and all components of $G - S$ are odd, otherwise, we can remove one vertex from each even component to the set $S$, in consequence, the number of odd component and the number of vertices in $S$ have the same increase, so that $o(G - S)$ is always larger than $|S|$ and all components of $G - S$ are odd. Since $n$ is an even positive integer, then we have $o(G - S)$ and $|S|$ have same parity. Let $k = o(G - S)$ and $s = |S|$, then $k \geq s + 2$.

Let $G_1, G_2, \ldots, G_k$ be the components of $G - S$ with $|V(G_1)| \geq |V(G_2)| \geq \cdots \geq |V(G_k)|$. Note that $n = s + n_1 + n_2 + \cdots + n_k$, where $n_i = |V(G_i)|$ ($i = 1, 2, \ldots, k$). To find the feasible maximum signless Laplacian spectral radius, we construct a new graph $G'$ by joining $S$ and $G - S$ and by adding edges in $S$ and in all components in $G - S$ so that $G[S]$ and all components in $G' - S$ are cliques. By Lemma 2.1 we get $q_1(G) \leq q_1(G')$. 

3
The quotient matrix of the signless Laplacian matrix $Q(G')$ of the graph $G'$ with the vertex partition $\{S, V(G_1), V(G_2), \cdots, V(G_k)\}$ can be expressed as

$$M_1 = \begin{pmatrix} n + s - 2 & n_1 & n_2 & \cdots & n_k \\ s & 2n_1 + s - 2 & 0 & \cdots & 0 \\ \vdots & 0 & 2n_2 + s - 2 & \cdots & 0 \\ s & \vdots & \vdots & \cdots & \vdots \\ s & 0 & 0 & \cdots & 2n_k + s - 2 \end{pmatrix}$$

Let $f(x)$ be the characteristic polynomial of the matrix $M_1$. By expanding the above matrix determinant on the first line, we have

$$f(x) = (x - n - s + 2)(x - 2n_1 - s + 2) \cdots (x - 2n_k - s + 2) - sn_1(x - 2n_2 - s + 2) \cdots (x - 2n_k - s + 2) + sn_2(x - 2n_1 - s + 2)(x - 2n_3 - s + 2) \cdots (x - 2n_k - s + 2) + \cdots + (\text{...}) + (-1)^{k}sn_k(x - 2n_1 - s + 2) \cdots (x - 2n_{k-1} - s + 2)(x - 2n_k - s + 2) + \cdots + (-1)^{k}sn_k(x - 2n_1 - s + 2) \cdots (x - 2n_{k-1} - s + 2).$$

Note that $M_1$ is an equitable quotient matrix of $Q(G')$. By Lemma 2.3, we obtain $q_1(G') = r_f$, where $r_f$ is the largest root of the equation $f(x) = 0$. Moreover, by Lemma 2.1, we have $r_f > n + s - 2$, and $r_f > 2n_1 + 2s - 2$.

If $n_k \geq 3$, similarly, we consider a new graph $G''$, which is obtained from $G'$ by deleting two vertices in $G_k$ and adding two vertices to $G_1$ by joining the two vertices to the vertices in $V(G_1)$ and $S$. For the partition $\{S, V(G_1'), V(G_2'), \cdots, V(G_{k-1}', V(G_k')\}$ of $G''$, the corresponding quotient matrix $M_2$ of $Q(G'')$ has the characteristic polynomial $\tilde{f}(x)$ obtained from $f(x)$ by replacing $n_1$ and $n_k$ by $n_1 + 2$ and $n_k - 2$, respectively. Thus,

$$\tilde{f}(x) = (x - n - s + 2)(x - 2n_1 - s - 2)(x - 2n_2 - s + 2) \cdots (x - 2n_k - s + 2) - s(n_1 + 2)(x - 2n_2 - s + 2) \cdots (x - 2n_k - s + 6) + sn_2(x - 2n_1 - s - 2)(x - 2n_3 - s + 2) \cdots (x - 2n_k - s + 6) + \cdots + (-1)^{k}s(n_k - 2)(x - 2n_1 - s - 2)(x - 2n_2 - s + 2) \cdots (x - 2n_{k-1} - s + 2)(x - 2n_k - s + 2) = f(x) + \text{...}$$

Note that $f(r_f) = 0$, $r_f > n + s - 2$, $r_f > 2n_1 + 2s - 2$, and $n_1 \geq \cdots \geq n_k$. By plugging the value $r_f$ into $x$ of $\tilde{f}(x)$, we have

$$\tilde{f}(r_f) = 8(n_k - n_1 - 2)(r_f - n - s + 2)(r_f - 2n_2 - s + 2) \cdots (r_f - 2n_{k-1} - s + 2)$$
- 2(r_f + 2n_1 - 2n_k - s + 6)s(r_f - 2n_2 - s + 2) \cdots (r_f - 2n_{k-1} - s + 2)
+ 8(n_k - n_1 - 2)s_n_2(r_f - 2n_3 - s + 2) \cdots (r_f - 2n_{k-1} - s + 2)
- 8(n_k - n_1 - 2)s_n_3(r_f - 2n_2 - s + 2)(r_f - 2n_4 - s + 2) \cdots (r_f - 2n_{k-1} - s + 2) + \cdots
+ (-1)^q \cdot (-2)(r_f + 2n_1 - 2n_k - s - 2)s(r_f - 2n_2 - s + 2) \cdots (r_f - 2n_{k-1} - s + 2) < 0,

which implies that \( r_f < r_{f}' \), where \( r_{f}' \) is the largest root of the equation \( \tilde{f}(x) = 0 \). Since \( M_2 \) is an equitable quotient matrix of \( Q(G'') \), we have \( q_1(G') < q_1(G'') \). From the above discussion, we have \( G' \) has the maximum spectral radius if \( n_1 = n - s - k + 1 \) and \( n_i = 1 \) for all \( 2 \leq i \leq k \). Now, we suppose \( |E(G')| = \left( \frac{s + n_1}{2} \right) + s(k - 1) \), where \( n_1 = n - s - k + 1 \). There exists a partition \( \{V(G_1), S, V(G) - S - V(G_1)\} \) of \( V(G') \). The corresponding quotient matrix \( M_3 \) of \( Q(G') \) has the form

\[
M_3 = \begin{pmatrix}
2n_1 + s - 2 & s & 0 \\
n_1 & n + s - 2 & k - 1 \\
0 & s & s
\end{pmatrix}.
\]

The characteristic polynomial \( g(x) \) of the matrix \( M_3 \) can be calculated as below by expanding \( M_3 \)'s determinant on the first line:

\[
g(x) = (x - 2n_1 - s + 2)[(x - n - s + 2)(x - s) - s(k - 1)] - n_1s(x - s).
\]

Let \( r_g \) be the largest root of the equation \( g(x) = 0 \). By Lemma 2.8, we have \( r_g > 2n_1 + s - 2 \), \( r_g > n + s - 2 \) and \( r_g > s \). Combining \( g(r_g) = 0 \), yields \( (r_g - n - s + 2)(r_g - s) - s(k - 1) > 0 \).

Next, we construct a new graph \( G''' \) obtained from \( G' \) deleting the two components \( G_k \) and \( G_{k-1} \), which are single vertices, and adding two vertices to \( G_1 \) by joining the two vertices to the vertices in \( V(G_1) \) and \( S \). One can see that \( G''' - S \) only has \( k - 2 \) components and \( n'''_1 = |V(G'''_1)| = n_1 + 2 \). Consider the partition \( \{V(G'''), S, V(G) - S - V(G_1)\} \), the characteristic polynomial of the quotient matrix of signless Laplacian matrix \( Q(G''') \) of graph \( G''' \) equals

\[
g'(x) = (x - 2n_1 - s - 2)[(x - n - s + 2)(x - s) - s(k - 3)] - (n_1 + 2)s(x - s)
= g(x) - 4sn_1 - 4s - 4[(x - n - s + 2)(x - s) - s(k - 1)].
\]

Similarly, plugging the value of \( r_g \) into \( g'(x) \) yields

\[
\tilde{g}(r_g) = -4sn_1 - 4s - 4[(r_g - n - s + 2)(r_g - s) - s(k - 1)] < 0,
\]

which implies that \( r_g < \tilde{r}_g \), where \( \tilde{r}_g \) is the largest root of the equation \( \tilde{g}(x) = 0 \). Since these partitions are equitable, we have \( q_1(G') = r_g \) and \( q_1(G''') = \tilde{r}_g \). Hence, \( q_1(G') < q_1(G''') \) and \( G' \) has the maximum spectral radius if \( k = s + 2 \), i.e., \( n_1 = n - 2s - 1 \). Similarly, we suppose that \( |E(G')| = \left( \frac{s + n_1}{2} \right) + s(s + 1) \), where \( n_1 = n - 2s - 1 \). The quotient matrix of \( Q(G') \) according
to the partition \( \{ V(G_1), S, V(G) - S - V(G_1) \} \) is

\[
M_4 = \begin{pmatrix}
2n_1 + s - 2 & s & 0 \\
n_1 & n + s - 2 & s + 1 \\
0 & s & s
\end{pmatrix}.
\]

The characteristic polynomial of \( M_4 \) equals

\[
h(x) = (x - 2n_1 - s + 2)((x - n - s + 2)(x - s) - s(s + 1)) - n_1s(x - s) \\
= (x - 2n + 3s + 4)((x - n - s + 2)(x - s) - s(s + 1)) - (n - 2s - 1)s(x - s) \\
= x^3 + (s - 3n + 6)x^2 + (2n^2 + ns - 8n - 4s^2 - 4s + 8)x - 2s(n^2 - 2ns - 5n + s^2 + 5s + 6).
\]

Next, we discuss the difference between \( q_1(G') \) and \( q_1(K_{n-1} \cup K_1 \cup K_2) \). For \( n \geq 4 \), one can easily check that \( q_1(K_{n-1} \cup K_1 \cup K_2) = r(n) \). Furthermore, we have

\[
r(n) = n + \frac{2^{\frac{5}{2}} \cdot \left(63n + 3\sqrt{3} \cdot \sqrt{-4n^6 + 84n^5 - 781n^4 + 4074n^3 - 12633n^2 + 2232n - 17376 - 9n^2 - 38} \right)^{\frac{1}{3}}}{6} \\
+ \frac{2^{\frac{1}{2}} \cdot (3n^2 - 21n + 49)}{3 \left(63n + 3\sqrt{3} \cdot \sqrt{-4n^6 + 84n^5 - 781n^4 + 4074n^3 - 12633n^2 + 2232n - 17376 - 9n^2 - 38} \right)^{\frac{1}{3}}} - \frac{7}{3}.
\]

(3.3)

Plugging the value \( r(n) \) into \( h(x) \) yields

\[
h(r(n)) = (s - 1)r^2(n) + (ns - n - 4s^2 - 4s + 8)r(n) - 2(sn^2 - 2ns^2 - 5ns + s^3 + 5s^2 + 6s - n^2 + 7n - 12) \\
= (s - 1) \left[ r^2(n) + (ns - 4s - 8)r(n) - 2(n^2 - 2ns - 7n + s^2 + 6s + 12) \right].
\]

Note that \( n \geq 2s + 4 \) (or \( n_1 \geq 3 \)), then \( h(r(n)) \geq (s - 1) \left[ r^2(n) - (2s + 4)r(n) - 2s^2 \right] \). Together with (3.3), we obtain that \( r^2(n) - (2s + 4)r(n) - 2s^2 \geq 4.2843 \) with equality holding if and only if \( s = 1 \) and \( n_1 = 3 \), which implies that \( h(r(n)) \geq 0 \).

Finally, let \( n_1 = 1 \), i.e., \( n = 2s + 2 \). The quotient matrix of \( Q(G') \) according to the partition \( \{ S, V(G') - S \} \) is

\[
M_5 = \begin{pmatrix}
n + s - 2 & s + 2 \\
\quad & \quad \\
n & s
\end{pmatrix}.
\]

The characteristic polynomial of \( M_5 \) equals

\[
l(x) = x^2 + (2 - 2s - n)x + (sn - 4s).
\]

Let \( r_1 \) be the largest root of the equation \( l(x) = 0 \). By calculation, we have

\[
r_1 = \frac{n + 2s - 2 + \sqrt{n^2 - 4n + 4s^2 + 8s + 4}}{2} = \frac{2n - 4 + \sqrt{2n(n - 2)}}{2}.
\]

From the above consideration, it’s not difficult to find the relation between \( r(n) \) and \( r_1 \): (i) For \( n \geq 10 \) (or \( s \geq 3 \)), then \( r(n) > r_1 \); (ii) For \( n = 6 \) and \( n = 8 \), (or \( s = 2 \) and \( s = 3 \)), then
For $n = 4$ (or $s = 1$), then $r(n) = r_l$. 

4 Extremal graphs

Finally, we determine the signless Laplacian spectral radius of $K_{n-3} \vee K_1 \vee \overline{K_2}$, $K_2 \vee \overline{K_4}$ and $K_3 \vee \overline{K_5}$ to show that the bound on the signless Laplacian spectral radius shown in Theorem 1.2 is sharp. In addition, the graph $K_{n-3} \vee K_1 \vee \overline{K_2}$, $K_2 \vee \overline{K_4}$ also can be used to prove that the bound given by Zhao et al. [25] is sharp.

**Theorem 4.1.** Let $r(n)$ be the largest root of the equation $x^n - (3n-7)x^2 + n(2n-7)x - 2(n^2 - 7n + 12) = 0$. Then $q_1(K_{n-3} \vee K_1 \vee \overline{K_2}) = r(n)$ holds for any even positive integer $n \geq 4$. In addition, we have $q_1(K_2 \vee \overline{K_4}) = 4 + 2\sqrt{3}$ and $q_1(K_3 \vee \overline{K_5}) = 6 + 2\sqrt{6}$.

**Proof.** Consider the vertex partition $\{V(K_{n-3}), V(K_1), V(\overline{K_2})\}$ of the graph $K_{n-3} \vee K_1 \vee \overline{K_2}$. The corresponding quotient matrix of $Q(K_{n-3} \vee K_1 \vee \overline{K_2})$ equals

$$
\begin{pmatrix}
2n-7 & 1 & 0 \\
n-3 & n-1 & 2 \\
0 & 1 & 1
\end{pmatrix}.
$$

It’s easy to calculate that the characteristic polynomial of the above matrix is $x^3 - (3n - 7)x^2 + n(2n - 7)x - 2(n^2 - 7n + 12) = 0$. Since the above quotient matrix is equitable, then we have $q_1(K_{n-3} \vee K_1 \vee \overline{K_2}) = r(n)$.

Analogously, for the vertex partition $\{V(K_2), V(\overline{K_4})\}$ of the graph $K_2 \vee \overline{K_4}$ and $\{V(K_3), V(\overline{K_5})\}$ of the graph $K_3 \vee \overline{K_5}$, the corresponding quotient matrix of $Q(K_2 \vee \overline{K_4})$ and $Q(K_3 \vee \overline{K_5})$ equal

$$
\begin{pmatrix}
6 & 4 \\
2 & 2
\end{pmatrix},
$$

and

$$
\begin{pmatrix}
9 & 5 \\
3 & 3
\end{pmatrix}.
$$

Since these partitions are equitable, calculating their spectral radius yields that $q_1(K_2 \vee \overline{K_4}) = 4 + 2\sqrt{3}$ and $q_1(K_3 \vee \overline{K_5}) = 6 + 2\sqrt{6}$. 

Note that $q_1(K_2 \vee \overline{K_4}) = 4 + 2\sqrt{3} \approx 7.4641 > 6.9095 = r(6)$ and $q_1(K_3 \vee \overline{K_5}) = 6 + 2\sqrt{6} \approx 10.8990 > 10.5136 = r(8)$. This verifies the conclusions in Theorem 1.2.

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