Integrable generalized Heisenberg ferromagnet equations with self-consistent potentials and related Yajima-Oikawa type equations

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Abstract
We consider some nonlinear models describing interactions of long and short (LS) waves. Such LS models have been derived and proposed with various motivations, which mainly come from fluid and plasma physics. In this paper, we study some of integrable LS models, namely, the Yajima-Oikawa equation, the Newell equation, the Ma equation, the Geng-Li equation and etc. In particular, the gauge equivalent counterparts of these integrable LS models (equations) are found. In fact, these gauge equivalents of the LS equations are integrable generalized Heisenberg ferromagnet equations (HFE) with self-consistent potentials (HFESCP). The associated Lax representations of these HFESCP are given. We also presented several spin-phonon equations which describe nonlinear interactions of spin and lattice subsystems in ferromagnetic materials.

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1 Introduction

Nonlinear models of long wave-short wave resonant interactions, the so-called LS equations, play important role in modern physics as well as in modern mathematics [1]-[5]. Some of these models are integrable nonlinear partial differential equations in 1+1 and 2+1 dimensions (see, e.g. [6]-[16] and references therein). In 1+1 dimensions, such LS equations of long wave-short wave resonant interactions, formally in general, can be written as

\[ iq_t + q_{xx} + f_1(v, v_x, q, \bar{q}, ...) = 0, \]

\[ v_t + \delta |q|^2_x = 0, \]

where \( q(x,t) \) represents the envelope (complex-valued) of the short wave and \( v(x,t) \) represents the amplitude (real-valued) of the long wave (potential), \( \delta = \text{const} \). Here \( f_1 \) is some function of its arguments. The set of equations (1)-(2) can be considered as the nonlinear Schrödinger type equations with self-consistent potentials. It combines all well known integrable LS models, namely, the Yajima-Oikawa equation (YOE), the Newell equation (NE), the Geng-Li equation (GLE) etc. This outcome is similar to the one that proves that the Korteweg-de Vries and the modified-KdV equations are just two particular cases of the Gardner equation. There exists another interesting class of integrable systems, namely, integrable generalized spin systems or Heisenberg ferromagnet equations with self-consistent potentials (GHFESCP) (see, e.g. [17]-[21] and references therein). Such spin systems in general can be written as

\[ iS_t + f_2(S, S_x, S_{xx}, u, u_x, ...) = 0, \]

\[ u_t + \nu tr(S[S_x, S_t])_x = 0, \]

where the \( 2 \times 2 \) spin matrix \( S \) has the form

\[ S = \begin{pmatrix} S_3 & S^- \\ S^+ & -S_3 \end{pmatrix} \]

and satisfies the following condition

\[ S^2 = I. \]

Some time we use the following form of integrable generalized spin systems

\[ iR_t + f_3(R, R_x, R_{xx}, u, u_x, ...) = 0, \]

where in contrast to (5) and (6), \( R \) is the \( 3 \times 3 \) spin matrix satisfying the condition

\[ R^3 = \epsilon R, \quad (\epsilon = \pm 1). \]
In above equations, $f_2$ and $f_3$ are some matrix functions of their arguments, $u$ is a real function (potential) and $(\nu, \delta)$ are real constants.

The set of equations (1)-(2) is some kind generalizations of the nonlinear Schrödinger equation (NLSE):

$$i q_t + q_{xx} + 2\nu|q|^2 q = 0. \quad (9)$$

At the same time the set (3)-(4) and (7) are some extensions of the following Heisenberg ferromagnet equation (HFE)

$$i S_t + \frac{1}{2}[S, S_{xx}] = 0. \quad (10)$$

Both of the NLSE (9) and HFE (10) are integrable and admit some integrable extentions in 1+1 and 2+1 dimensions (see, e.g., [22]-[47] and references therein). It is well-known that between the NLSE (9) and the HFE (10) takes place the gauge and geometrical equivalence [42]-[43]. The aim of this paper is the finding such gauge equivalence between some particular reductions of the sets of equations (1)-(2), (3)-(4) and (7).

This paper is organized as follows. In Section 2, the integrable Tolkynay equation (TE) is presented. The Yajima-Oikawa-Mewell equation (YONE) is considered in Section 3 and its gauge equivalence with the TE was studied in Section 4. In the next two sections (Section 5 and Section 6), the gauge equivalent counterparts of the Yajima-Oikawa equation (YOE) and the Ma equation (ME) are presented. The relation between the TE and NE is established in Section 7. The same problem was studied in Section 8 for the Geng-Li equation (GLE). In Section 9, the M-XXXIV equation is investigated. In Section 10, the M-V equation and its relation with the LS equations were considered. The nonlinear magnon-phonon equations were presented in Section 11. The last section is devoted to some conclusions and discussions.

2 The Tolkynay equation

In this paper, in particular, we study the following Tolkynay equation (TE)

$$iR_t + [R^2, R_x]x = 0. \quad (11)$$

Here the spin matrix $R$ has the form

$$R = \begin{pmatrix} R_{11} & R_{12} & R_{13} \\ R_{21} & R_{22} & R_{23} \\ R_{31} & R_{32} & R_{33} \end{pmatrix}, \quad (12)$$

and satisfies the following conditions

$$R^3 = R, \quad tr(R) = 0, \quad det(R) = 0. \quad (13)$$

The TE is one of integrable generalized Heisenberg ferromagnet type equation. The Lax representation (LR) of the TE is given by

$$\Psi_x = U_1 \Psi, \quad (14)$$

$$\Psi_t = V_1 \Psi, \quad (15)$$

where

$$U_1 = i\lambda R, \quad (16)$$

$$V_1 = -i\lambda^2 \left( R^2 - \frac{2}{3} I \right) - \lambda[R^2, R_x]. \quad (17)$$
The YONE

One of most general integrable LS equations is the Yajima-Oikawa-Newell equation (YONE) \[10\]

\[
\begin{align*}
iq_t + q_{xx} + (i\alpha v + \alpha^2 v^2 - \beta v - 2\alpha|q|^2)q &= 0, \\
v_t - 2(|q|)_x &= 0,
\end{align*}
\]

where the parameters $\alpha$, $\beta$ are arbitrary real constants. These parameters may be considered as independent constants which are responsible for the long-short wave cross-interaction. This system reduces to the YOE for $\alpha = 0$, $\beta = 1$ and to the Newell equation as $\alpha = \sigma$, $\beta = 0$. The YONE (18)-(19) is integrable. Its LR has the form \[10\]

\[
\begin{align*}
\Phi_x &= U_2 \Phi, \\
\Phi_t &= V_2 \Phi,
\end{align*}
\]

where

\[
\begin{align*}
U_2 &= i\lambda \Sigma + Q, \\
V_2 &= -\lambda^2 B_2 + i\lambda B_1 + B_0.
\end{align*}
\]

Here

\[
\Sigma = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \\
Q = \begin{pmatrix} 0 & q & iv \\ \alpha \bar{q} & 0 & \bar{q} \\ i(\alpha^2 v - \beta) & \alpha q & 0 \end{pmatrix},
\]

\[
B_2 = \frac{i}{3} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \\
B_1 = \begin{pmatrix} 0 & iq & 0 \\ i\alpha \bar{q} & 0 & -i\bar{q} \\ 0 & -i\alpha q & 0 \end{pmatrix},
\]

\[
B_0 = \begin{pmatrix} -i\alpha|q|^2 & -avq + iq_x & i|q|^2 \\ -\alpha^2 v \bar{q} + \beta \bar{q} - i\alpha \bar{q} x & 2i\alpha|q|^2 & -avq - i\bar{q} \bar{q} \\ \alpha^2 |q|^2 & -\alpha^2 vq + \beta q + i\alpha q_x & -\alpha|q|^2 \end{pmatrix}.
\]

4 Gauge equivalence between the TE and the YONE

In this section, we establish the gauge equivalence between the TE (11) and the YONE (18)-(19). To this purpose, let us consider the gauge transformation

\[
\Psi = g^{-1} \Phi, \quad g = \Phi_{\lambda=0}.
\]

We obtain

\[
\begin{align*}
\Psi_x &= U_1 \Psi, \\
\Psi_t &= V_1 \Psi,
\end{align*}
\]

where

\[
U_1 = g^{-1}(U_2 - g_x g^{-1})g, \quad V_1 = g^{-1}(V_2 - g_t g^{-1})g.
\]

As results, we have

\[
U_1 = i\lambda R, \quad V_1 = -\lambda^2 g^{-1} B_{2g} + i\lambda g^{-1} B_1 g,
\]
where
\[ R = g^{-1} \Sigma g. \] (32)

After some calculations, we obtain
\[ Q \Sigma + \Sigma Q = \begin{pmatrix} 0 & q & 0 \\ \frac{\alpha \bar{q}}{q} & 0 & -\bar{q} \\ 0 & -\alpha q & 0 \end{pmatrix} = -iB_1, \] (33)

or
\[ g^{-1}Qgg^{-1}\Sigma g + g^{-1}\Sigma gg^{-1}Qg = g^{-1} \begin{pmatrix} 0 & q & 0 \\ \frac{\alpha \bar{q}}{q} & 0 & -\bar{q} \\ 0 & -\alpha q & 0 \end{pmatrix} = -ig^{-1}B_1g = g^{-1}QgR + Rg^{-1}Qg. \] (34)

On the other hand, we obtain
\[ R_x = g^{-1}[\Sigma, Q]g = g^{-1} \begin{pmatrix} 0 & q & 2i\nu \\ -\frac{\alpha \bar{q}}{q} & 0 & \bar{q} \\ -2i(\alpha^2 v - \beta) & -\alpha q & 0 \end{pmatrix} g \] (35)

Hence we get
\[ [R, R_x] = g^{-1} \begin{pmatrix} 0 & q & 4i\nu \\ \frac{\alpha \bar{q}}{q} & 0 & \bar{q} \\ 4i(\alpha^2 v - \beta) & \alpha q & 0 \end{pmatrix} g = g^{-1}Qg + 3ig^{-1} \begin{pmatrix} 0 & 0 & v \\ 0 & 0 & 0 \\ (\alpha^2 v - \beta) & 0 & 0 \end{pmatrix} g \] (36)

so that we have
\[ g^{-1}Qg = [R, R_x] - 3ig^{-1} \begin{pmatrix} 0 & 0 & v \\ 0 & 0 & 0 \\ (\alpha^2 v - \beta) & 0 & 0 \end{pmatrix} g \] (37)

Taking into account the formula
\[ \Sigma \begin{pmatrix} 0 & 0 & v \\ 0 & 0 & 0 \\ (\alpha^2 v - \beta) & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & v \\ 0 & 0 & 0 \\ (\alpha^2 v - \beta) & 0 & 0 \end{pmatrix} \Sigma = 0 \] (38)

we obtain
\[ g^{-1}B_1g = i(g^{-1}Qgg^{-1}\Sigma g + g^{-1}\Sigma gg^{-1}Qg) = i([R, R_x]R + R[R, R_x]) = i[R^2, R_x] \] (39)

Thus, we have shown that
\[ g^{-1}B_2g = -\frac{2i}{3}I + iR^2, \] (40)
\[ g^{-1}B_1g = i[R^2, R_x]. \] (41)

Finally, we have the following Lax pair for the TE (11)
\[ U_1 = i\lambda R, \] (42)
\[ V_1 = -i\lambda^2(R^2 - \frac{2}{3}I) - \lambda[R^2, R_x]. \] (43)
Let us present some useful formulas

\[ tr(R_x^2) = -4\alpha|q|^2 + 8\nu(\alpha^2v - \beta), \quad det(R_x) = 2i\beta|q|^2. \] (44)

As integrable equation, the YONE admits the infinity number of integrals of motion. For example, here we present the following integral of motion for the YONE (18)-(19):

\[ P = \int Jdx, \] (45)

where \(a, b\) are some constants and

\[ J = 4a[\alpha(5\beta - 1)]|q|^2 + 2\nu(\alpha^2v - \beta)] = atr(S_x^2) + bdet(S_x), \quad b = \frac{10\alpha a}{i\beta}, \quad a = const, \quad b = const. \] (46)

In fact, the quantity \(J\) satisfies the following conservation equation

\[ J_t = 16a[i\alpha(q_xq - \bar{q}q_x) - \beta|q|^2 + 2\alpha^2v|q|^2] \] (47)

so that

\[ P_t = \left( \int Jdx \right)_t = 0 \] (48)

for the boundary conditions

\[ \lim q(x, t) \to 0, \quad \lim v(x, t) \to 0 \quad as \quad x \to \pm \infty. \] (49)

5 Gauge equivalence between the YOE and the MM-IIE

The first example of integrable long-short waves interactions models is the following Yajima-Oikawa equation (YOE) \[1\]

\[ iq_t + \frac{1}{2}q_{xx} - uq = 0, \] (50)

\[ u_t + u_x + |q|^2 = 0. \] (51)

Its Lax representation reads as

\[ U_3 = A_0 + 2i\lambda \Sigma + (2\lambda)^{-1}A_{-1}, \] (52)

\[ V_3 = -U + 2i\lambda^2 A_1^2 + \lambda B_1 + B_0 + i(4\lambda)^{-1} \begin{pmatrix} |\Phi|^2 & 0 & |\Phi|^2 \\ \Phi_x & 0 & \Phi_x \\ -|\Phi|^2 & 0 & -|\Phi|^2 \end{pmatrix}, \] (53)

where

\[ \Sigma = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad A_0 = \begin{pmatrix} 0 & -\Phi^* & 0 \\ 0 & 0 & 0 \\ 0 & \Phi^* & 0 \end{pmatrix}, \quad A_{-1} = \begin{pmatrix} -ni & 0 & -ni \\ \Phi & 0 & \Phi \\ ni & 0 & ni \end{pmatrix}, \] (54)

\[ B_1 = \begin{pmatrix} 0 & -\Phi^* & 0 \\ 0 & 0 & 0 \\ \Phi^* & 0 & 0 \end{pmatrix}, \quad B_0 = \begin{pmatrix} 0 & \frac{i}{2}\Phi_x + \Phi^* & 0 \\ \frac{i}{2}\Phi_x & 0 & -\frac{i}{2}\Phi \\ 0 & -\frac{i}{2}\Phi_x - \Phi^* & 0 \end{pmatrix}, \] (55)

\[ B_{-1} = \begin{pmatrix} |\Phi|^2 & 0 & |\Phi|^2 \\ \Phi_x & 0 & \Phi_x \\ -|\Phi|^2 & 0 & -|\Phi|^2 \end{pmatrix}, \quad \Phi = qe^{i(\frac{1}{2}t - x)}. \] (56)
Consider a gauge transformation $\phi = g \bar{\phi}$, where $g(x, t, \lambda_0) = \phi(x, t, \lambda)|_{\lambda=\lambda_0} \subset GL(\xi, C)$ as $\lambda = \lambda_0$.

Then
\begin{align*}
g_x &= U_3(x, t, \lambda_0)g, \\
g_t &= V_3(x, t, \lambda_0)g.
\end{align*}

We obtain
\begin{align*}
U_4 &= 2i(\lambda - \lambda_0)g^{-1}A_1g - \frac{\lambda - \lambda_0}{4\lambda_0 \lambda} g^{-1}A_{-1}g, \\
V_4 &= 2i(\lambda - \lambda_0)g^{-1}A_1g - U_4 - \frac{i(\lambda - \lambda_0)}{4\lambda_0 \lambda} g^{-1}B_{-1}g + (\lambda - \lambda_0)B_1.
\end{align*}

Now let us introduce two new matrices $R$ and $\sigma$ as
\begin{align*}
R &= g^{-1}\Sigma g, \\
\sigma &= g^{-1}A^{-1}g, \quad R^3 = R.
\end{align*}

This Lax pair gives the following Makhankov-Myrzakulov-II equation (MM-IIIE) [27]
\begin{align*}
R_t + R_x + (\frac{i}{2}RR^2_x - 2\lambda_0 R^2_x)_x + (4i\lambda_0)^{-1}h &= 0, \\
\sigma_t + \sigma_x + (\frac{1}{2}[\sigma, R^2]_x - i\lambda_0[\sigma, R^2])_x + i\lambda_0 h &= 0,
\end{align*}
where
\begin{align*}
h &= [\sigma(R_x - 2i\lambda_0 I), R^2] + ([R^2, \sigma]R)_x, \\
tr(U_4) &= tr(V_4) = tr(U_3) = tr(V_3) = tr(R) = tr(\sigma) = 0.
\end{align*}

6 The MM-IE and its relation with the Ma equation

Let us consider the following Makhankov-Myrzakulov-I equation (MM-IE) [27]
\begin{align*}
iR_t + 2[R, R_{xx}] - 4(R_x R)_x = 0,
\end{align*}
where the spin matrix $R$ satisfies the following condition
\begin{align*}
R^3 = R.
\end{align*}

The MM-IE is integrable. The corresponding LR has the form
\begin{align*}
\Psi_x &= U_5 \Psi, \\
\Psi_t &= V_5 \Psi,
\end{align*}
where $U_5$ and $V_5$ are constants.
where
\[ U_5 = i\lambda R, \quad V_5 = 2i\lambda^2 R^2 + 2\lambda(R^2)_x. \] (72)
\[ V_5 = 2i\lambda_2 R^2 + 2\lambda(R^2)_x. \] (73)

Note that the MM-IE is gauge equivalent to the following Ma equation (ME) [27]
\[ iq_t - 2q_{xx} + 2uq = 0, \quad u_t + |q|^2 = 0, \] (74)
(75)
In fact, after some simple scale transformations, the ME takes the form of the YOE (50)-(51). In contrast to the YOE, for the ME the LR takes the more simple form [15]
\[ \Phi_x = U_6\Phi, \quad \Phi_t = V_6\Phi, \] (76)
(77)
where
\[ U_6 = C_0 + i\lambda\Sigma, \quad V_6 = D_0 + \lambda D_1 + 2i\lambda^2 D_2. \] (78)

Here
\[ C_0 = \begin{pmatrix} 0 & \frac{E}{E^*} & \imath n \\ 0 & 0 & \frac{E}{E^*} \\ -\imath & 0 & 0 \end{pmatrix}, \quad \Sigma = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \] (79)
\[ D_0 = \begin{pmatrix} 0 & -\imath E_x & -\frac{\imath |E|^2}{2} \\ -E^* & 0 & \imath E_x^* \\ 0 & E & 0 \end{pmatrix}, \quad D_1 = \begin{pmatrix} 0 & E & 0 \\ 0 & 0 & -E^* \\ 0 & 0 & 0 \end{pmatrix}, \quad D_2 = \Sigma^2. \] (80)

As we mentioned, the MM-IE and the ME is gauge equivalent to each other. To prove this, let us consider the gauge transformation \( \Psi = \omega^{-1}\Phi \), where \( \omega = \Phi(x, t, \lambda) \big|_{\lambda=0} \). Then we have
\[ \omega_x = U_{60}\omega, \quad \omega_t = V_{60}\omega, \] (81)
(82)
where
\[ U_{60} = U_1 \big|_{\lambda=0}, \quad V_{60} = V_1 \big|_{\lambda=0}. \]

As result, we get
\[ U_5 = \omega^{-1}U_{60}\omega - \omega^{-1}\omega_x, \quad V_5 = \omega^{-1}V_{60}\omega - \omega^{-1}\omega_t, \] (83)
or
\[ U_6 = i\lambda\omega^{-1}\Sigma\omega, \quad V_6 = \lambda\omega^{-1}D_1\omega + 2i\lambda^2\omega^{-1}D_2\omega. \]

After some calculation we obtain
\[ \omega^{-1}D_1\omega = 2(R^2)_x, \quad \omega^{-1}D_2\omega = R^2, \]
where
\[ R = \omega^{-1}\Sigma\omega. \] (84)

The matrix function \( R \) satisfies the following conditions
\[ \text{tr}(R) = 0, \quad \det(R) = \frac{i}{2}|E|^2, \quad \det(R_x) = \det(\omega^{-1}[C_1, C_0]\omega) = \frac{i}{2}|E|^2. \] (85)

It is not difficult to verify that the matrix function \( R \) satisfies the MM-IE (68). This is proves that the MM-IE (68) and the ME (74)-(75) is gauge equivalent to each other.
7 The TE and its relation with the Newell equation

The TE has the form
\[ iR_t + [R^2, R_x]_x = 0, \]  
(86)

where the spin matrix \( R \) has the form
\[
R = \begin{pmatrix}
R_{11} & R_{12} & R_{13} \\
R_{21} & R_{22} & R_{23} \\
R_{31} & R_{32} & R_{33}
\end{pmatrix} = \begin{pmatrix}
R_{11} & R_{12} & iv \\
-\sigma R_{12} & 0 & \sigma R_{12} \\
-iv & -R_{12} & -R_{11}
\end{pmatrix} = \begin{pmatrix}
R_3 & R^- & iv \\
-\sigma R^+ & 0 & \sigma R^+ \\
-iv & -R^- & -R_3
\end{pmatrix},
\]  
(87)

and satisfies the following conditions
\[ R^3 = R, \quad tr(R) = 0, \quad det(R) = 0. \]  
(88)

Here \( \sigma = \pm 1 \), \( R_{12} = R^- = R_1 - iR_2 \) is a complex function and \( R_{11} = R_3, R_1, R_2, v \) are real functions. Let us calculate some useful expressions
\[
R^2 = \begin{pmatrix}
R_{11}^2 - \sigma|R_{12}|^2 + v^2 & (R_{11} - iv)R_{12} & \sigma|R_{12}|^2 \\
-(R_{11} + iv)\sigma R_{12} & -2\sigma|R_{12}|^2 & -(R_{11} + iv)\sigma R_{12} \\
\sigma|R_{12}|^2 & (R_{11} - iv)R_{12} & R_{12}^2 - \sigma|R_{12}|^2 + v^2
\end{pmatrix},
\]  
(89)

\[
R^3 = R^2R = R = (R_{11}^2 - 2\sigma|R_{12}|^2 + v^2) \begin{pmatrix}
R_{11} & R_{12} & iv \\
-\sigma R_{12} & 0 & \sigma R_{12} \\
-iv & -R_{12} & -R_{11}
\end{pmatrix} = (R_{11}^2 - 2\sigma|R_{12}|^2 + v^2)R. \]  
(90)

Hence we obtain the following condition
\[
R_{11}^2 - 2\sigma|R_{12}|^2 + v^2 = R_3^2 - 2\sigma|R^+|^2 + v^2 = R_3^2 - 2\sigma(R_1^2 + R_2^2) + v^2 = 1. \]  
(91)

The TE is one of integrable generalized Heisenberg ferromagnet type equation. The LR of the TE is given by
\[
\Psi_x = U_\tau \Psi, \quad \Psi_t = V_\tau \Psi,
\]  
(92, 93)

where
\[
U_\tau = i\lambda R, \quad V_\tau = -i\lambda^2 \left( R^2 - \frac{2}{3}I \right) - \lambda[R^2, R_x].
\]  
(94, 95)

It is not difficult to verify that the gauge equivalent of TE (86) is the Newell equation (NE) which reads as \[2\]
\[
iq_t + q_{xx} + (iu_x + u^2 - 2\sigma|q|^2)q = 0 \quad (96)
iu_t - 2\sigma(|q|^2)_x = 0, \]  
(97)

where \( \sigma = \pm 1 \). Note that in addition to a long wave-short wave coupling, the short wave has the same self-interaction as the NLS equation (9). The NE is integrable. Its Lax representation looks like \[2, 13\]
\[
\Phi_x = U_8 \Phi, \quad \Phi_t = V_8 \Phi,
\]  
(98, 99)
where
\[
U_8 = \begin{pmatrix} i\lambda & q & iv \\ \sigma q & 0 & \sigma q \\ iv & q & -i\lambda \end{pmatrix},
\]
\[
V_8 = \begin{pmatrix} -\frac{i}{3}i\lambda^2 - i\sigma|q|^2 & -\lambda q + iq_x - vq & i\sigma|q|^2 \\ -\sigma(\lambda q + \bar{q}_x + vq) & \frac{2}{3}i\lambda^2 + 2i\sigma|q|^2 & \sigma(\lambda q - iq_x - vq) \\ i\sigma|q|^2 & \lambda q + iq_x - vq & -\frac{1}{3}i\lambda^2 - i\sigma|q|^2 \end{pmatrix}. \tag{100}
\]

It is not difficult to verify that these matrices satisfy the following conditions
\[
U^+(\lambda) = -AU(\bar{\lambda})A, \quad V^+(\lambda) = -AV(\bar{\lambda})A, \quad U^+(\lambda) = -BV(-\lambda)B, \quad U^+(\lambda) = -BV(-\lambda)B, \tag{101}
\]
where
\[
A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -\sigma & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}. \tag{102}
\]

8 Geng-Li equation

Again we consider TE
\[
iR_t + [R^2,R_x]_x = 0. \tag{103}
\]

In contrast to the previous cases, now we assume that the spin matrix \( R \) satisfies the conditions
\[
R^3 = -R, \quad tr(R) = 0, \quad det(R) = 0. \tag{104}
\]

Then the LR for the TE (103) takes the form
\[
\Psi_x = U_9\Psi, \quad \tag{105}
\]
\[
\Psi_t = V_9\Psi, \quad \tag{106}
\]
where
\[
U_9 = \lambda R, \quad \tag{107}
\]
\[
V_9 = -i\lambda^2 R^2 + i\lambda[R^2,R_x]. \tag{108}
\]

Now we assume that the matrix \( R \) can be written as
\[
R = g^{-1}Jg, \tag{109}
\]
where
\[
J = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{110}
\]

Consider the gauge transformation
\[
\Phi = g\Psi. \tag{111}
\]

Hence and from the equations (105)-(106) follows that the new matrix function \( \Phi \) satisfies the equations
\[
\Phi_x = U_8\Phi, \quad \tag{112}
\]
\[
\Phi_t = V_8\Phi, \quad \tag{113}
\]

\[10\]
where
\[
U_8 = \begin{pmatrix} iv & -\lambda & 0 \\ \lambda & 0 & -\bar{q} \\ 0 & q & 0 \end{pmatrix}, \quad V_8 = \begin{pmatrix} i\lambda^2 - 2i|q|^2 & 0 & -i\lambda\bar{q} \\ 0 & i\lambda^2 - i|q|^2 & iq_x + v\bar{q} \\ -i\lambda q & iq_x - vq & i|q|^2 \end{pmatrix}.
\] (114)

The compatibility condition
\[
U_{8t} - V_{8x} + [U_8, V_8] = 0
\] (115)
gives the Geng-Li equation (GLE) \[6\]
\[
iq_t + q_{xx} + 2|q|^2q + i(vq)_x = 0,
\] (116)
\[
v_t + 2(|q|^2)_x = 0.
\] (117)

This proves that the TE (103) with the conditions (104) and LR (105)-(106) is gauge equivalent to the GLE (116)-(117).

9 The M-XXXIV equation

One of integrable HFE with self-consistent potentials (HFESCP) is the following Myrzakulov-XXXIV (M-XXXIV) equation
\[
S_t + S \wedge S_{xx} - uS = 0,
\] (118)
\[
u_t + \frac{1}{2}(S^2)_x = 0,
\] (119)

where \(S = (S_1, S_2, S_3)\) is the unit spin vector that is \(S^2 = S_1^2 + S_2^2 + S_3^2 = 1\) and \(u\) is a real function (potential). The M-XXXIV equation is integrable. The corresponding LR has the form
\[
\alpha \Psi_y = \frac{1}{2} [S + I]\Psi_x,
\] (120)
\[
\Psi_t = \frac{i}{2} [S + (2b + 1)I]\Psi_{xx} + \frac{i}{2} W\Phi_x,
\] (121)

where \(W = W_1 + W_2\) and
\[
W_1 = (2b + 1)E + (2b + \frac{1}{2})SS_x + (2b + 1)FS,
\] (122)
\[
W_2 = FI + \frac{1}{2}S_x + ES + \alpha SS_y, \quad S^\pm = S_1 \pm iS_2,
\] (123)
\[
E = -\frac{i}{2\alpha}u_x, \quad F = \frac{i}{2} \left( \frac{u_x}{\alpha} - 2u_y \right), \quad S = \begin{pmatrix} S_3 \\ S^+ \\ -S_3 \end{pmatrix}.
\] (124)

In fact, the compatibility condition \(\Psi_{yt} = \Psi_{ty}\) gives the following set of equations
\[
is_t + \frac{1}{2}[S, S_{\xi}] - iwS_\xi = 0,
\] (125)
\[
w_\eta - \frac{1}{4i} tr(S[S_\xi, S_\eta]) = 0,
\] (126)
\[ \xi = x + \frac{1}{\alpha} y, \quad \eta = -x, \quad w = u_\xi. \]  
\[ (127) \]

Hence after the simple transformation \( \eta = t, w \rightarrow u, \xi \rightarrow x \), we obtain
\[ iS_t + \frac{1}{2}[S, S_{xx}] - iuS_x = 0, \]
\[ u_t - \frac{1}{4i} tr(S[S_x, S_t]) = 0, \]
\[ (128) \]
\[ (129) \]

The following equations are correct:
\[ [S_x, S_t] = -i(S_x^2)xS, \quad S[S_x, S_t] = -i(S_x^2)xI, \quad S_x^2 = S_x^2I, \quad S[S_x, S_t] = -i(S_x^2)xI, \]
\[ (130) \]
and
\[ tr(S[S_x, S_t]) = -2i(S_x^2)x. \]
\[ (131) \]

Hence the M-XXXIV equation (128)-(129) can be written as
\[ iS_t + \frac{1}{2}[S, S_{xx}] - iuS_x = 0, \]
\[ u_t + \frac{1}{2}(S_x^2)x = 0. \]
\[ (132) \]
\[ (133) \]

Let us now find the equation which is gauge equivalent to the M-XXXIV equation (132)-(133). To this end, we consider the following transformation
\[ \Phi = g\Psi, \]
\[ (134) \]
where \( \Psi \) is the matrix solution of linear problem (120)-(121), \( \Phi \) and \( g \) are a temporally unknown matrix functions. Substituting (134) into (120)-(121) after some calculations we get
\[ \alpha \Phi_y = B_1 \Phi_x + B_0 \Phi, \]
\[ \Phi_t = iC_2 \Phi_{xx} + C_1 \Phi_x + C_0 \Phi, \]
\[ (135) \]
\[ (136) \]
with
\[ B_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad B_0 = \begin{pmatrix} 0 & q \\ r & 0 \end{pmatrix}, \]
\[ C_2 = \begin{pmatrix} b + 1 & 0 \\ 0 & b \end{pmatrix}, \quad C_1 = \begin{pmatrix} 0 & iq \\ ir & 0 \end{pmatrix}, \quad C_0 = \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix}, \]
\[ c_{12} = i(2b + 1)q_x + i\alpha q_y, \quad c_{21} = -2ibr_x - i\alpha r_y. \]
\[ (137) \]
\[ (138) \]
Here \( c_{jj} \) satisfy the following system of equations
\[ c_{11x} - \alpha c_{11y} = iq_r x + rc_{12} - qc_{21}, \quad \alpha c_{22y} = -irq_x + rc_{12} - qc_{21}. \]
\[ (140) \]

The compatibility condition of equations(135)-(136) gives the following (2+1)-dimensional non-linear Schrödinger equation:
\[ iq_t + q_{\xi\xi} + uq = 0, \]
\[ ir_t + r_{\xi\xi} - ur = 0, \]
\[ v_r + 2(rq)_{\xi} = 0, \]
\[ (141) \]
\[ (142) \]
\[ (143) \]
or after \( \eta \to t \) we have
\[
\begin{align*}
iq_t + q_{xx} + vq &= 0, \\
ir_t - r_{xx} - vr &= 0, \\
v_t + 2(rq)_x &= 0.
\end{align*}
\]
(144)
(145)
(146)
It coincide with the ME (74)-(75). Thus we have presented the new LR for the ME or for the YOE. Consequently we found the new form of the gauge equivalent counterpart of the ME and/or the YOE, namely, the M-XXXIV equation.

10 The M-V equation

Our next example of integrable generalized HFE is the so-called Myrzakulov-V (M-V) equation. The M-V equation reads as
\[
i R_t + 2[R, R_y]_x + 3(R^2 R_y R)_x = 0
\]
(147)
or
\[
i R_t + \frac{1}{2}[R, R_y]_x + 3\frac{1}{2}[R^2, (R^2)_y]_x = 0,
\]
(148)
where the spin matrix \( R \) satisfies the following conditions
\[
R^3 = R, \quad \text{tr}(R) = 0, \quad \det(R) = 0.
\]
(149)
The M-V equation (148) is the (2+1)-dimensional integrable equation. Its LR looks like
\[
\begin{align*}
\Psi_x &= U_1 \Psi, \\
\Psi_t &= V_1 \Psi,
\end{align*}
\]
(150)
(151)
where
\[
\begin{align*}
U_1 &= -i\lambda R, \\
V_1 &= -2i\lambda^2 R + \frac{\lambda}{2} ([R, R_y] + 3[R^2, (R^2)_y]).
\end{align*}
\]
(152)
(153)
To find its gauge equivalent equation, let us consider the transformation
\[
R = \Phi^{-1} \Sigma \Phi,
\]
(154)
where \( \Sigma = \text{diag}(1, 0, -1) \). Let us assume that \( \Phi \) satisfies the following equations
\[
\begin{align*}
\Phi_x &= -i\lambda \Sigma + Q, \\
\Phi_t &= (\mu_2 \lambda^2 + \mu_1 \lambda + \mu_0) \Phi_y + V \Phi,
\end{align*}
\]
(155)
where the matrix \( Q \) is given, \( V \) is unknown matrix and \( \mu_j = \text{consts} \). The compatibility condition \( \Phi_{xt} = \Phi_{tx} \) gives the following two equations
\[
Q_t - V_x + [U, V] - (\mu_2 \lambda^2 + \mu_1 \lambda + \mu_0) Q_y = 0
\]
(156)
and
\[
\lambda_t - (\mu_2 \lambda^2 + \mu_1 \lambda + \mu_0) \lambda_y = 0.
\]
(157)
The equation (156) is the desired nonlinear Schrödinger type equation coupled with the equation for the potential \( v(x, t) \). At the same time, the equation (157) tells us that in this case, we have the nonisospectral problem, where \( \lambda = \lambda(y, t) \).
11 Magnon-phonon models as HFE with self-consistent potentials

In this section we want to present some HFE with self-consistent potentials (HFESCP), namely, the so-called magnon-phonon systems or spin-phonon systems. These models describe nonlinear interactions between the spin waves and the lattice waves. Some of these models have the different classes of nonlinear solutions: solitons, kinks, breathers and so on. We continue to study the key properties of integrable spin-phonon nonlinear systems serving to mimicry some dynamical features of physically motivated but non-integrable spin-phonon nonlinear models. Some of these models are integrable but others are nonintegrable. In particular, the Kuralay-I equation (166)-(167), the Shynaray-I equation (168)-(170), the Zhaidary-I equation (171)-(173) are integrable, admit Lax representations and so on. Here some of these spin-phonon or magnon-phonon models.

M-XXXIII equation:

\[ 2iS_t = [S, S_{xx}] + 2iuS_x, \]  
\[ u_t + u_x + \alpha(u^2)_x + \beta u_{xxx} + \lambda tr(S^2)_x = 0. \]

M-XXXIV equation:

\[ S_t = S \wedge S_{xx} + uS_x, \]  
\[ u_t + 4\eta(S^2)_x = 0. \]

M-XXXV equation:

\[ 2iS_t = [S, S_{xx}] + 2iuS_x, \]  
\[ \rho u_{tt} = \nu_0^2 u_{xx} + \alpha(u^2)_{xx} + \beta u_{xxxx} + \frac{\lambda}{4} tr(S^2)_x. \]

M-XXXVI equation:

\[ 2iS_t = [S, S_{xx}] + 2iuS_x, \]  
\[ \rho u_{tt} = \nu_0^2 u_{xx} + \frac{\lambda}{4} tr(S^2)_x. \]

Kuralay equation:

\[ S_t - S \wedge S_{xt} - uS_x = 0, \]  
\[ u_x + \frac{1}{2}(S^2)_t = 0. \]

Shynaray equation:

\[ S_t - S \wedge S_{xt} - uS_x - 2d^2 S_t - 4c(uS)_x = 0, \]  
\[ u_x + \frac{1}{2}(S^2)_t = 0. \]
\[ w_x + \frac{1}{16l^2c^2}(S^2)_t = 0. \]

Zhaidary equation:

\[ S_t - S \wedge S_{xt} - uS_x - 2(l + d) S_t - 4c(uS)_x = 0, \]  
\[ u_x + \frac{1}{2}(S^2)_t = 0. \]
\[ w_x + \frac{1}{4(2l + d)c^2}(S^2)_t = 0. \]
M-XXXVII equation:

\[ 2iS_t = [S, S_{xxxx}] + 2 \left\{((1 + \mu)S^2_x - u + m)[S, S_x]\right\}_x, \]
\[ u_t + u_x + \alpha(u^2)_x + \beta u_{xxx} + \lambda(S^2_x)_x = 0. \]  

M-XXXVIII equation:

\[ 2iS_t = [S, S_{xxxx}] + 2 \left\{((1 + \mu)S^2_x - u + m)[S, S_x]\right\}_x, \]
\[ u_t + u_x + \lambda(S^2_x)_x = 0. \]  

M-XXXIX equation:

\[ 2iS_t = [S, S_{xxxx}] + 2 \left\{((1 + \mu)S^2_x - u + m)[S, S_x]\right\}_x, \]
\[ \rho u_{tt} = \nu_0^2 u_{xx} + \alpha(u^2)_{xx} + \beta u_{xxxx} + \lambda(S^2_x)_{xx}. \]  

M-XXXX equation:

\[ 2iS_t = [S, S_{xxxx}] + 2 \left\{((1 + \mu)S^2_x - u + m)[S, S_x]\right\}_x, \]
\[ \rho u_{tt} = \nu_0^2 u_{xx} + \lambda(S^2_x)_{xx}. \]  

M-XXXXI equation:

\[ 2iS_t = \{(\mu S^2_x - u + m)[S, S_x]\}_x, \]
\[ u_t + u_x + \alpha(u^2)_x + \beta u_{xxx} + \lambda(S^2_x)_x = 0. \]  

M-XXXXII equation:

\[ 2iS_t = \{(\mu S^2_x - u + m)[S, S_x]\}_x, \]
\[ u_t + u_x + \lambda(S^2_x)_x = 0. \]  

M-XXXXIII equation:

\[ 2iS_t = \{(\mu S^2_x - u + m)[S, S_x]\}_x, \]
\[ \rho u_{tt} = \nu_0^2 u_{xx} + \alpha(u^2)_{xx} + \beta u_{xxxx} + \lambda(S^2_x)_{xx}. \]  

M-XXXXIV equation:

\[ 2iS_t = \{(\mu S^2_x - u + m)[S, S_x]\}_x, \]
\[ \rho u_{tt} = \nu_0^2 u_{xx} + \lambda(S^2_x)_{xx}. \]  

M-XXXXV equation:

\[ 2iS_t = [S, S_{xx}] + (uS_3 + h)[S, \sigma_3], \]
\[ u_t + u_x + \alpha(u^2)_x + \beta u_{xxx} + \lambda(S^2_x)_x = 0. \]  

M-XXXXVI equation:

\[ 2iS_t = [S, S_{xx}] + (uS_3 + h)[S, \sigma_3], \]
\[ u_t + u_x + \lambda(S^2_3)_x = 0. \]
M-XXXXVII equation:

\[ 2iS_t = [S, S_{xx}] + (uS_3 + h)[S, \sigma_3], \]
\[ \rho u_{tt} = \nu_0^2 u_{xx} + \alpha (u^2)_{xx} + \beta u_{xxxx} + \lambda (S^2_{3})_{xx}. \]

M-XXXXVIII equation:

\[ 2iS_t = [S, S_{xx}] + (uS_3 + h)[S, \sigma_3], \]
\[ \rho u_{tt} = \nu_0^2 u_{xx} + \lambda (S^2_{3})_{xx}. \]

M-XXXXIX equation:

\[ 2iS_t = [S, S_{xx}] + (u + h)[S, \sigma_3], \]
\[ u_t + u_x + \alpha (u^2)_x + \beta u_{xxx} + \lambda (S^2_{3})_x = 0. \]

M-XL equation:

\[ 2iS_t = [S, S_{xxxx}] + 2 \left\{ ((1 + \mu)\bar{S}^2_x - u + m)[S, S_x] \right\}_x, \]
\[ \rho u_{tt} = \nu_0^2 u_{xx} + \lambda (\bar{S}^2_x)_{xx}. \]

M-XLI equation:

\[ 2iS_t = \left\{ (\mu \bar{S}^2_x - u + m)[S, S_x] \right\}_x, \]
\[ u_t + u_x + \alpha (u^2)_x + \beta u_{xxx} + \lambda (\bar{S}^2_x)_x = 0. \]

M-XLII equation:

\[ 2iS_t = \left\{ (\mu \bar{S}^2_x - u + m)[S, S_x] \right\}_x, \]
\[ u_t + u_x + \lambda (\bar{S}^2_x)_x = 0. \]

M-XLIII equation:

\[ 2iS_t = \left\{ (\mu \bar{S}^2_x - u + m)[S, S_x] \right\}_x, \]
\[ \rho u_{tt} = \nu_0^2 u_{xx} + \alpha (u^2)_{xx} + \beta u_{xxxx} + \lambda (S^2_{3})_{xx}. \]

M-XLIV equation:

\[ 2iS_t = \left\{ (\mu \bar{S}^2_x - u + m)[S, S_x] \right\}_x, \]
\[ \rho u_{tt} = \nu_0^2 u_{xx} + \lambda (\bar{S}^2_x)_{xx}. \]

M-XLV equation:

\[ 2iS_t = [S, S_{xx}] + (uS_3 + h)[S, \sigma_3], \]
\[ u_t + u_x + \alpha (u^2)_x + \beta u_{xxx} + \lambda (S^2_{3})_x = 0. \]

M-XLVI equation:

\[ 2iS_t = [S, S_{xx}] + (uS_3 + h)[S, \sigma_3], \]
\[ u_t + u_x + \lambda (S^2_{3})_x = 0. \]
Nonlinear models describing interactions of long and short (LS) waves are given by the Yajima-Oikawa type equations. These long wave - short wave interaction models have been derived and proposed with various motivations, which mainly come from fluid and plasma physics. It is well known that in these long wave-short wave equations is that a long wave always arises as generated by short waves. In this paper, we study some of integrable LS models, namely, the Yajima-Oikawa equation, the Newell equation, the Ma equation, the Geng-Li equation and etc. Any integrable equations admitting the Lax representations, generally speaking, are gauge equivalent to some integrable generalized Heisenberg ferromagnet equations with self-consistent potentials (HFESCP). The associated Lax representations of these HFESCP are given. We also presented several spin-phonon equations which describe nonlinear interactions of spin and lattice subsystems in ferromagnetic materials.
Acknowledgements

This work was supported by the Ministry of Education and Science of Kazakhstan, Grant AP08856912.

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