The Capacity of Symmetric Private Information Retrieval under Arbitrary Collusion and Eavesdropping Patterns

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Abstract

We study the symmetric private information retrieval (SPIR) problem under arbitrary collusion and eavesdropping patterns for replicated databases. We find its capacity, which is the same as the capacity of the original SPIR problem with the number of databases $N$ replaced by a number $F^*$. The number $F^*$ is the optimal solution to a linear programming problem that is a function of the joint pattern, which is the union of the collusion and eavesdropping pattern. This is the first result that shows how two arbitrary patterns collectively affect the capacity of the PIR problem. We draw the conclusion that for SPIR problems, the collusion and eavesdropping constraints are interchangeable in terms of capacity. As special cases of our result, the capacity of the SPIR problem under arbitrary collusion patterns and the capacity of the PIR problem under arbitrary eavesdropping patterns are also found.

I. INTRODUCTION

In a computer network based on a client-server model, the behavior of information retrieval is often related to privacy leaks. A malicious server that monitors user queries will retrieve the server’s history and infer the user’s behavior in conjunction with other servers. Simultaneously, the privacy of the database often needs to be protected as well, i.e., the user cannot get more information than the desired message. For data queries, especially in the case of information

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retrieval that requires high confidentiality, it is necessary to ensure the privacy of the retrieval process in the sense of information theory.

The problem of private information retrieval (PIR) was first proposed in [1], where the user wants to retrieve a certain bit out of a database of $K$ bits, from $N$ servers each storing a replicated version of the database. The privacy of the user is protected when the queries of the user do not reveal any information about which bit is of interest to any single database. The PIR problem was reformulated in [2], where the user wants to retrieve a sufficiently large message from the database. The goal of query design considered in [2] is to maximize the download efficiency, which is defined as the ratio of the size of the desired message to the total number of downloaded symbols from the servers. The maximum download efficiency, termed the capacity, of the PIR problem was shown to be [2]

$$C_{\text{PIR}} = \left(1 + \frac{1}{N} + \frac{1}{N^2} + \cdots + \frac{1}{N^{K-1}}\right)^{-1}. \quad (1)$$

When considering the privacy of the database, in addition to the privacy of the user, the symmetric private information retrieval (SPIR) problem was proposed [3]. In SPIR problems, in addition to protecting the user’s message index from every single server, the servers must collectively prevent the user from gaining any information about the undesired messages of the database. The capacity of the SPIR problem was found to be [4]

$$C_{\text{SPIR}} = \begin{cases} 1 - \frac{1}{N}, & \text{if } \rho \geq \frac{1}{N - 1} \\ 0, & \text{otherwise} \end{cases} \quad (2)$$

where $\rho$ is the amount of common randomness the servers share relative to the message size. Noted that both $C_{\text{PIR}}$ and $C_{\text{SPIR}}$ increase with the number of servers $N$, since with the help of more servers, the privacy of the user can be hidden better from any single server. Further note that $C_{\text{SPIR}}$ is independent to the number of messages $K$, and we have $C_{\text{SPIR}} = \lim_{K \to \infty} C_{\text{PIR}}$.

In the scenario where some subsets of databases may communicate and collude to learn about the message index that is of interest to the user, the colluding PIR and SPIR problems were proposed and studied in [5] and [6], respectively. The collusion structure studied was a symmetric one, called $T$-colluding servers, where out of the $N$ servers, any up to $T$ number of servers may collude. To preserve the privacy of the user under possible collusion among databases, the number of downloaded symbols needs to be increased. It is shown in [5] and [7]
that under $T$-colluding servers, the capacities of the PIR and SPIR problems still take on the same form as (1) and (2), respectively, but with $N$ replaced by $\frac{N}{T}$. In other words, when any $T$ databases may collude, the number of effective databases has decreased from $N$ to $\frac{N}{T}$, where $\frac{N}{T}$ does not need to be an integer.

In addition to the database privacy considered in $T$-colluding SPIR, where database privacy is protected against the querying user, reference [7] further considers database privacy protection against an external eavesdropper. More specifically, the external eavesdropper has the ability to overhear the queries to and the answers from any $E$ out of $N$ servers. This problem has been termed $T$-ESPIR, and its capacity was found in [7] to take on the same form of (2), with $N$ replaced by $\frac{N}{\max(T,E)}$.

Many other variants of the PIR problem have been studied since [2], in addition to the scenarios mentioned above, i.e., that of colluding servers, database privacy against the querying user, and database privacy against external eavesdroppers. Due to limited space, we provide the references here without going into the detailed settings and results of each one [8]–[109].

The majority of existing PIR papers study the case of symmetric servers. Symmetry simplifies the problem setting and often lead to capacity results. In practice, however, heterogeneity among servers is prevalent. For example, servers belonging to the same company are more likely to collude, and servers who are located geographically close are more likely to eavesdrop at the same time. In this paper, we study the SPIR problem under the heterogeneous collusion and eavesdropping pattern. More specifically, a heterogeneous colluding pattern may be represented by its maximal colluding sets [110], [111] as $\mathcal{P}_c = \{\mathcal{T}_1, \mathcal{T}_2, \cdots, \mathcal{T}_{M_c}\}$, where the servers in set $\mathcal{T}_m$, $m \in [1 : M_c]$ may collude, and there are $M_c$ such colluding sets. Similarly, a heterogeneous eavesdropping pattern may be represented by its maximal eavesdropping sets as $\mathcal{P}_e = \{\mathcal{E}_1, \mathcal{E}_2, \cdots, \mathcal{E}_{M_e}\}$, where the servers in set $\mathcal{E}_m$, $m \in [1 : M_e]$ may be tapped by the passive eavesdropper simultaneously, and there are $M_e$ such eavesdropping sets.

While this is the first paper to study arbitrary eavesdropping patterns, the problem of arbitrary collusion patterns has been studied before. Tajeddine et. al in [110] first proposed PIR problems under arbitrary collusion patterns and studied it for MDS-coded databases, where the database messages are encoded using an $[N,J]$ MDS code, and the coded bits are stored in the $N$ servers. Several other works followed, including [112] for replicated databases, [111] Section VII for MDS-coded databases, and some discussions in [113] Appendix D], for both the replicated and MDS-coded databases scenarios. Though collusion patterns are diverse, and at first glance, the
problem requires a case-by-case analysis due to the property of each specific collusion pattern \cite{111}, reference \cite{114} found a general formula for the PIR capacity that holds for any collusion pattern $\mathcal{P}_c$. The capacity formula was shown to be of the form (1), with $N$ replaced by a number $S^*$, where $S^*$ is the optimal value of the following linear programming problem

$$\max_y \quad 1_N^T y$$

subject to

$$B_{\mathcal{P}_c}^T y \leq 1_M$$

$$y \geq 0_N,$$

where $B_{\mathcal{P}_c}$ is the incidence matrix, of size $N \times M_c$, of the collusion pattern $\mathcal{P}_c$, i.e., if Server $n$ is in the $m$-th colluding set $T_m$ in $\mathcal{P}_c$, we let the $(n,m)$-th element of $B_{\mathcal{P}_c}$ be 1, otherwise, it is zero. $1_k$ ($0_k$) is the column vector of size $k$ whose elements are all one (zero).

In this paper, inspired by the proof techniques of \cite{114}, we extend the capacity results found for T-ESPIR problems in \cite{7} to arbitrary collusion and eavesdropping patterns. We find its capacity, which is the same as the capacity of the original SPIR problem in (2) with the number of databases $N$ replaced by a number $F^*$. The number $F^*$ is the optimal solution to a linear programming problem that is a function of the joint pattern, which is the union of the collusion and eavesdropping pattern. Hence, we draw the conclusion that for SPIR problems, the collusion and eavesdropping constraints are interchangeable in terms of capacity. The result shows that the collusion and eavesdropping pattern affects the capacity of the SPIR problem only through the number $F^*$. This is the first result that shows how two arbitrary patterns collectively affect the capacity of the PIR/SPIR problem. As a special case of our result, the capacity of the SPIR problem under arbitrary collusion patterns and the capacity of the PIR problem under arbitrary eavesdropping patterns are also found.

II. SYSTEM MODEL

The system model is as shown in Fig. 1. Consider the problem where $K$ messages are stored on $N$ replicated databases. The $K$ messages, denoted as $W_k = (W_k^1, W_k^2, \cdots, W_k^L)$, $k \in [1 : K]$ are independent and each message consists of $L$ symbols, which are independently and uniformly
distributed over a finite field \( F_q \), where \( q \) is the size of the field, i.e.,

\[
H(W_k) = L, \quad k = 1, ..., K, \tag{3}
\]

\[
W_{[1:K]} = H(W_1, ..., W_K) = H(W_1) + H(W_2) + \cdots + H(W_K).
\]

A user wants to retrieve message \( W_\theta, \theta \in [1 : K] \), by sending designed queries to the databases, where the query sent to the \( n \)-th database is denoted as \( Q_n^{[\theta]} \). Since the queries are designed by the user, who do not know the content of the messages, we have

\[
I(W_{[1:K]}; Q_n^{[\theta]}) = 0, \quad \forall \theta \in [1 : K].
\]

In order to protect the privacy of the database from the user and possible eavesdroppers, the servers share a common random variable \( S \), which is independent to the messages and queries,
and is not known to the users and the eavesdroppers, i.e.:

$$I(S; W_{1:K}, Q_{1:N}^{[\theta]}) = 0.$$  

Upon receiving the query $Q_{n}^{[\theta]}$, Database $n$ calculates the answer, denoted as $A_{n}^{[\theta]}$, based on the query received $Q_{n}^{[\theta]}$, the messages $W_{1:K}$ and the common randomness $S$, i.e.,

$$H(A_{n}^{[\theta]}|Q_{n}^{[\theta]}, W_{1:K}, S) = 0, \quad \forall n \in [1:N], \theta \in [1:K]. \quad (4)$$

The queries need to be designed to satisfy the following four conditions:

1) Correctness of decoding at the user: the user is able to reconstruct the desired message $W_{\theta}$ from all the answers received from the databases, i.e.,

$$H(W_{\theta}|A_{1:N}^{[\theta]}, Q_{1:N}^{[\theta]}) = 0, \quad \forall \theta \in [1:K]. \quad (5)$$

2) Database privacy against the user: the user learns nothing about the undesired messages in the database, i.e.,

$$I(W_{\theta}^{\bar{\theta}}; A_{1:N}^{[\theta]}, Q_{1:N}^{[\theta]}) = 0, \quad \forall \theta \in [1:K]. \quad (6)$$

Note that in the PIR problem, the above constraint does not need to be satisfied.

3) User privacy under arbitrary server collusion pattern: The collusion pattern considered takes on the general form $\mathcal{P}_{c} = \{T_{1}, T_{2}, \ldots, T_{M_{c}}\}$, where $M_{c}$ is the number of colluding sets and $T_{m} \subseteq [1:N]$, $\forall m \in [1:M_{c}]$ is the $m$-th colluding set in $\mathcal{P}_{c}$. The representation $\mathcal{P}_{c}$ means that the databases in set $T_{m}$ may collude, and there are $M_{c}$ such colluding sets. As an example, for $N = 4$ databases, the 2-colluding case considered by [5] is denoted as $\mathcal{P}_{c} = \{\{1,2\}, \{1,3\}, \{1,4\}, \{2,3\}, \{2,4\}, \{3,4\}\}$. Note that all databases must appear in at least one element of $\mathcal{P}_{c}$, because at the very least, the privacy of the user must be preserved at each single database, which is the requirement of the original PIR problem [2]. To protect the privacy of the user, we require that databases that are in a colluding set can not learn anything about the desired message index $\theta$, i.e.,

$$(Q_{T}^{[\theta]}, A_{T}^{[\theta]}, W_{1:K}, S) \sim (Q_{T}^{[\theta]}, A_{T}^{[\theta]}, W_{1:K}, S), \quad \forall \theta \in [1:K], \quad \forall T \in \mathcal{P}_{c}. \quad (7)$$

We restrict ourselves to $K \geq 2$ in this paper, because in the case of $K = 1$, with only 1 message, protecting the user’s privacy becomes trivial. We also restrict ourselves to the
case where $\mathcal{P}_c$ does not include the set of all servers, as in this case, no scheme can simultaneously achieve protecting both the user’s privacy and database privacy against the user [7].

4) Database privacy against a possible passive eavesdropper: assume that there is a passive eavesdropper who is interested in the content of the messages, and can eavesdrop on the queries to and the answers from one of the following sets of servers $\mathcal{E}_1, \cdots, \mathcal{E}_{M_e} \subseteq [1 : N]$. We denote the eavesdropping pattern as $\mathcal{P}_e = \{\mathcal{E}_1, \mathcal{E}_2, \cdots, \mathcal{E}_{M_e}\}$, where $M_e$ is the number of possible eavesdropping sets and $\mathcal{E}_m \subseteq [1 : N], \forall m \in [1 : M_e]$ is the $m$-th eavesdropping set in $\mathcal{P}_e$. As an example, for $N = 4$ servers, the symmetric eavesdropping of any 3 servers considered by [7] is denoted as $\mathcal{P}_e = \{\{1, 2, 3\}, \{1, 3, 4\}, \{2, 3, 4\}, \{1, 2, 4\}\}$. We require that the eavesdropper knows nothing about the messages in the database, no matter which set in $\mathcal{P}_e$ it taps, i.e.,

$$I(W_1; K; A_\theta^{[\mathcal{E}]}, Q_\theta^{[\mathcal{E}]}|\mathcal{E}) = 0, \quad \forall \mathcal{E} \in \mathcal{P}_e. \quad (8)$$

We restrict ourselves to the case where $\mathcal{P}_e$ does not include the set of all servers, since when the eavesdropper observes the queries to and the answers from all servers, it can definitely decode $W_\theta$, same as the user. Hence, there is no scheme that can achieve database privacy against the passive eavesdropper [7].

For ease of notation, the collusion pattern $\mathcal{P}_c$ and eavesdropping pattern $\mathcal{P}_e$ satisfy the following: we only include the maximal set as elements of $\mathcal{P}_c (\mathcal{P}_e)$. For example, if $\{1, 2, 3\} \in \mathcal{P}_c (\mathcal{P}_e)$, then by definition, $\{1, 2\}$ is a colluding (eavesdropping) set too. But we do not include $\{1, 2\}$ in $\mathcal{P}_c (\mathcal{P}_e)$ for ease of representation.

Let $\rho$ be the amount of common randomness the servers share relative to the message size, i.e.,

$$\rho = \frac{H(S)}{L} \quad (9)$$

For a given common randomness amount $\rho$, the rate of the PIR problem with collusion pattern $\mathcal{P}_c$ and eavesdropping pattern $\mathcal{P}_e$, denoted as $R_\rho(\mathcal{P}_c, \mathcal{P}_e)$, is defined as the ratio between the message size $L$ and the total number of downloaded information from the databases, i.e.,

$$R_\rho(\mathcal{P}_c, \mathcal{P}_e) = \frac{L}{\sum_{n=1}^{N} H(A_n^{[\theta]})}, \quad (10)$$
which is not a function of $\theta$ due to the privacy constraint in (7). The capacity of the PIR problem with collusion pattern $P_c$ and eavesdropping pattern $P_e$ given common randomness amount $\rho$ is $C(\rho)(P_c, P_e) = \sup R(\rho)(P_c, P_e)$, where the supremum is over all possible retrieval schemes.

For the collusion pattern $P_c$ and eavesdropping pattern $P_e$, we define a joint pattern $P_J$ as the maximal set representation of $P' = P_c \cup P_e$. For example, for $P_c = \{\{1, 2\}, \{3, 4\}\}$ and $P_e = \{\{1, 2, 3\}, \{4\}\}$, the joint pattern $P_J = \{\{1, 2, 3\}, \{3, 4\}\}$. We denote the number of sets in the joint pattern $P_J$ as $M_J$.

For pattern $P$ consisting of $M$ sets, we define its corresponding incidence matrix $B_P$, of size $N \times M$, as follows: if Server $n$ is in the $m$-th set in $P$, we let the $(n, m)$-th element of $B_P$ be 1, otherwise, it is zero. For example, pattern $P = \{\{1, 2\}, \{2, 3\}, \{2, 4\}, \{1, 3, 4\}\}$ would correspond to an incidence matrix of

$$B_P = \begin{bmatrix}
1 & 0 & 0 & 1 \\
1 & 1 & 1 & 0 \\
0 & 1 & 0 & 1 \\
0 & 0 & 1 & 1
\end{bmatrix}.$$

The above definition of incidence matrix is applicable to the collusion pattern $P_c$, the eavesdropping pattern $P_e$ and their joint pattern $P_J$. Throughout the paper, we will denote the $k \times 1$ column vector of all ones as $1_k$, and the $k \times 1$ column vector of all zeros as $0_k$. $I_k$ is the size $k \times k$ identity matrix and when the size is evident, we write it as $I$. Similarly, $0_{k \times i}$ is the size $k \times i$ matrix of all zeros, and when the size is evident, we write it as $0$.

III. MAIN RESULTS

The main result on the SPIR capacity under arbitrary collusion and eavesdropping pattern is given in the following theorem.

**Theorem 1** When $K \geq 2$, the capacity of the SPIR problem under arbitrary collusion pattern $P_c$ and eavesdropping pattern $P_e$ is

$$C(\rho)(P_c, P_e) = \begin{cases}
1 - \frac{1}{F^*} & \text{if } \rho \geq \frac{1}{F^* - 1}, \\
0 & \text{otherwise}
\end{cases},$$

(11)
where $F^*$ is the optimal value of the following linear programming problem (LP1):

\[
(LP1) \quad \max_y \quad 1_N^T y \\
\text{s. t.} \quad B_{P_J}^T y \leq 1_{M_J} \quad (12) \\
y \geq 0_N, \quad (13)
\]

where $B_{P_J}$ is the incidence matrix, of size $N \times M_J$, of the joint pattern $P_J$ of collusion pattern $P_c$ and eavesdropping pattern $P_e$.

Theorem 1 will be proved in the following section. We will first show that $1 - \frac{1}{F^*}$ is achievable as long as the amount of common randomness relative to the message size satisfies $\rho \geq \frac{1}{F^* - 1}$. This is achieved by distributing the amount of data queried to each database proportional to the optimal solution $y^*$ of (LP1). Next, we present a converse theorem that gives an upper bound on capacity as $1 - \frac{1}{F_2^*}$, where $F_2^*$ is the optimal value of another linear programming problem (LP2). We further show in the converse that an achievable scheme that protects the user’s privacy against the colluding servers in $P_c$ and protects the database’s privacy against the user and the passive eavesdropper in $P_e$ exists, if and only if the amount of common randomness relative to the message size satisfies $\rho \geq \frac{1}{F_2^* - 1}$, i.e., if $\rho < \frac{1}{F_2^* - 1}$, the capacity is zero. Finally, we show that (LP1) and (LP2) are dual problems, which means $F^* = F_2^*$. This concludes the proof that (12) is the capacity of the SPIR problem under arbitrary collusion patterns $P_c$ and eavesdropping patterns $P_e$.

We make a few remarks here regarding the main result.

**Remark 1** The result of Theorem 1 has a similar form as the SPIR problem in (2), with $N$ replaced by the $F^*$. So arbitrary eavesdropping and collusion patterns affect the capacity of the SPIR problem by reducing the number of effective servers from $N$ to $F^*$.

**Remark 2** Theorem 1 shows that the arbitrary collusion and eavesdropping patterns affect the capacity of the SPIR problem only through the optimal solution of linear programming problem (LP1). Hence, the effect of arbitrary collusion and eavesdropping patterns affect the capacity of the SPIR problem through one number $F^*$.

**Remark 3** Theorem 1 is the first result that shows how two arbitrary patterns collectively affect the capacity of the PIR/SPIR problem. In the SPIR problem of arbitrary collusion and
eavesdropping patterns, the capacity $C(\mathcal{P}_c, \mathcal{P}_e)$ is determined by the joint pattern $\mathcal{P}_J$. Hence, it does not rely on the individual collusion pattern $\mathcal{P}_c$ or eavesdropping pattern $\mathcal{P}_e$, as long as their joint pattern $\mathcal{P}_J$ is the same. As a special case, we may switch the two patterns and still get the same capacity.

**Remark 4** Our result coincides with known capacity results of SPIR problems:

1) The original SPIR problem [4], with no collusion among servers and no passive eavesdropper, corresponds to the collusion pattern $\mathcal{P}_c = \{\{1\}, \{2\}, \ldots, \{N\}\}$ and eavesdropping pattern $\mathcal{P}_e = \emptyset$, the empty set. Hence, the joint pattern $\mathcal{P}_J = \mathcal{P}_c$, and its incidence matrix is $B_{\mathcal{P}_J} = I_N$. The optimal solution to (LP1) is $y^* = 1_N$, and the corresponding optimal value is $F^* = N$. Hence, the capacity formula in (11) becomes (2), consistent with [4].

2) The T-ESPIR problem [6], where any up to $T$ servers many collude and any up to $E$ servers may be eavesdropped, corresponds to the collusion pattern $\mathcal{P}_c = \{\mathcal{S} \subseteq [1 : N] : |\mathcal{S}| = T\}$ and the eavesdropping pattern $\mathcal{P}_e = \{\mathcal{E} \subseteq [1 : N] : |\mathcal{E}| = E\}$. Thus, the joint pattern is $\mathcal{P}_J = \{\mathcal{D} \subseteq [1 : N] : |\mathcal{D}| = \max\{T, E\}\}$. Hence, the incidence matrix of the joint pattern is consists of $M_J = \left(\begin{array}{c} \frac{N}{\max\{T, E\}} \end{array}\right)$ columns, each with $\max\{T, E\}$ number of 1s and $N - \max\{T, E\}$ number of 0s. It is straightforward to see that the optimal solution to (LP1) is $y^* = \frac{1}{\max\{T, E\}} 1_N$, and the corresponding optimal value $F^* = \frac{N}{\max\{T, E\}}$. Hence, the capacity formula in (11) becomes

$$C_{T-ESPIR} = \begin{cases} 1 - \frac{\max\{T, E\}}{N} & \text{if } \rho \geq \frac{\max\{T, E\}}{N - \max\{T, E\}} \\ 0 & \text{otherwise} \end{cases}$$

consistent with [7].

**Remark 5** The special case of Theorem 1 provides us with the following new PIR capacity results:

1) The capacity of SPIR under arbitrary collusion pattern $\mathcal{P}_c$: By setting $\mathcal{P}_e = \emptyset$, we obtain the capacity of SPIR under arbitrary collusion pattern $\mathcal{P}_c$, which is previously unknown,

$$C_{\mathcal{P}_c} = \begin{cases} 1 - \frac{1}{S^*} & \text{if } \rho \geq \frac{1}{S^* - 1} \\ 0, & \text{otherwise} \end{cases}$$
where $S^*$ is the optimal value of the following linear programming problem:

$$\max_y \quad 1_N^T y$$

s. t. \quad \mathbf{B}_{P_c}^T y \leq 1_{M_c}$$

$$y \geq 0_N,$$

where $\mathbf{B}_{P_c}$ is the incidence matrix, of size $N \times M_c$, of the collusion pattern $P_c$. This is the SPIR version of [114, Theorem 1], which characterizes the PIR capacity under arbitrary collusion $P_c$. Comparing the two results, we see that again, the PIR capacity under arbitrary collusion pattern $P_c$ converges to the SPIR capacity under the same collusion pattern, as the number of messages $K$ goes to infinity.

2) The capacity of PIR and SPIR under arbitrary eavesdropping pattern $P_e$: By setting $P_c = \{\{1\}, \cdots, \{N\}\}$, we obtain the capacity of SPIR under arbitrary eavesdropping pattern $P_e$, which is previously unknown,

$$C_{P_e} = \begin{cases} 
1 - \frac{1}{Q^*} & \text{if } \rho \geq \frac{1}{Q^* - 1} \\
0 & \text{otherwise}
\end{cases}$$

where $Q^*$ is the optimal value of the following linear programming problem:

$$\max_y \quad 1_N^T y$$

s. t. \quad \mathbf{B}_{P_e}^T y \leq 1_{M_e}$$

$$y \geq 0_N,$$

where $\mathbf{B}_{P_e}$ is the incidence matrix, of size $N \times M_e$, of the eavesdropping pattern $P_e$. Interestingly, the corresponding PIR problem is simultaneously solved, as the capacities of the PIR and SPIR problems with arbitrary eavesdropping pattern $P_e$ are the same, i.e., the protection of database’s privacy against the user is free and achieved without increasing the total download from the servers. The detailed proof of its capacity is in Section IV-D.
IV. Proofs

A. Achievability

Recall that for each joint pattern $\mathcal{P}_J$ of collusion pattern $\mathcal{P}_c$ and eavesdropping pattern $\mathcal{P}_e$, there is a corresponding incidence matrix $B_{\mathcal{P}_J}$, as defined in Section II. Let $\mathbf{y} = [y_1 \ y_2 \ \cdots \ y_N]^T$ be a feasible and rational solution of (LP1), i.e., $\mathbf{y}$ consists of rational elements, and it satisfies the constraints (12) and (13). Let the value of the objective function in (LP1) corresponding to $\mathbf{y}$ be $F$, i.e., $F = \sum_{n=1}^{N} y_n$. Then, we have the following achievability theorem.

**Theorem 2** Consider the SPIR problem with collusion pattern $\mathcal{P}_c$ and eavesdropping pattern $\mathcal{P}_e$, Suppose $\mathbf{y}$ is a rational and feasible solution of (LP1) and $F = \mathbf{1}_{N}^T \mathbf{y} > 1$. Then the following rate is achievable, i.e.,

$$C_{\rho}(\mathcal{P}_c, \mathcal{P}_e) \geq 1 - \frac{1}{F}$$

(14)

when the amount of common randomness $\rho$ satisfies

$$\rho \geq \frac{1}{F - 1}.$$  

(15)

**Proof:** The details of the proof of Theorem 2 is provided in Appendix A, along with an illustrating example. The proof follows similarly to [7, Section V.A], and we note the difference here: 1) In place of $N$ in [7, Section V.A], we have $\bar{L}$, which will be chosen such that the number of symbols downloaded from each of the servers is an integer. Such an $\bar{L}$ can be found since $\mathbf{y}$ is rational. 2) In place of $\max\{T, E\}$ in [7, Section V.A], we have $\bar{L}$. 3) Rather than generating $N$ queries with each server receiving 1 query, the user generates $\bar{L}$ number of queries, and distribute the queries among the databases proportionally according to $(\mathbf{y}, F)$, more specifically, the number of queries to Server $n$ is based on the proportion $\frac{y_n}{F}$, $n \in [1 : N]$.

**Remark 6** The achievable scheme proposed is based on recognizing that in both colluding and eavesdropping sets, the number of queries/answers the servers or the eavesdropper see is crucial. When each colluding set of servers collectively see too many queries, the index of the user’s desired message can be deduced, and therefore, the user’s privacy is leaked. When the eavesdropper sees too many answers, the number of common randomnesses is not enough to protect all the messages, and the database privacy is leaked. Thus, in the design of the achievable scheme, we need to balance the amount of queries each server receives, and correspondingly,
the number of answers it sends. This balance is represented by (12), where by considering the joint pattern of collusion and eavesdropping, it is assured that both colluding servers and the passive eavesdropper do not see too many queries/answers. Considering the joint pattern, i.e., (12), may seem like a condition that is too strong, i.e., sufficient but not necessary, but as will be shown in the converse, this is in fact optimal.

Remark 7 It is easy to see that \( y = \frac{1}{N}1_N \) is a feasible and rational solution of (LP1). The corresponding cost function \( F = 1^T_N y = 1 \). Hence, the optimal cost function of (LP1) satisfies \( F^* \geq 1 \). The equality happens if and only if the collusion pattern or the eavesdropping pattern includes the set of all servers, which is not under consideration in this paper, as explained in Section II. Hence, for the cases of \( \mathcal{P}_c \) and \( \mathcal{P}_e \) considered in this paper, we always have \( F^* > 1 \).

Note that the right-hand side (RHS) of (14) is an increasing function of \( F \) and the RHS of (15) is a decreasing function of \( F \). Based on the result of Theorem 2 to find the largest possible achievable rate using the smallest amount of common randomness, we should find the maximum \( F = \sum_{n=1}^{N} y_n \) achievable over all \( y \) satisfying (12) and (13). Applying Theorem 2 for the optimal solution of (LP1), denoted as \( (y^*, S^*) \), and noting that \( y^* \) is rational due to the fact that the objective function and the linear constraints in (LP1) are both with integer coefficients, the rate of Theorem 1 is achievable.

B. Converse

Recall that for the joint pattern \( \mathcal{P}_J \) of collusion pattern \( \mathcal{P}_c \) and eavesdropping pattern \( \mathcal{P}_e \), there is a corresponding incidence matrix \( B_{P_J} \), as defined in Section II. Consider the following linear programming problem, which will be called (LP2),

\[
\text{(LP2)} \quad \min_x \quad 1^T_M x \\
\text{subject to} \quad B_{P_J} x \geq 1_N \\
x \geq 0_M. \quad (16)
\]

Let \( x = [x_1 \ x_2 \ \cdots \ x_M]^T \) be a feasible and rational solution of (LP2), i.e., \( x \) consists of rational elements, and it satisfies the constraints (16) and (17). Let the value of the objective function in (LP2) corresponding to \( x \) be \( F_2 \), i.e., \( F_2 = \sum_{m=1}^{M} x_m \).

We have the following converse theorem.
Theorem 3  Consider the SPIR problem with collusion pattern \( P_c \) and eavesdropping pattern \( P_e \). Let \( P_J \) be the joint pattern, whose incidence matrix is \( B_{P_J} \). Suppose \( x \) is a rational and feasible solution of (LP2) and \( F_2 = 1^T x \). Then, the capacity of the SPIR problem is upper bounded by

\[
C(P_c, P_e) \leq 1 - \frac{1}{F_2}. \tag{18}
\]

Furthermore, for an achievable scheme to exist, the amount of common randomness shared by the servers relative to the message size must satisfy

\[
\rho \geq \frac{1}{F_2 - 1}. \tag{19}
\]

i.e., if (19) is not satisfied, then the capacity of the SPIR problem is zero.

Proof: The proof follows by combining the converse results of SPIR with \( T \)-colluding and \( E \)-eavesdropping [7] with the converse results of PIR with arbitrary collusion pattern [114]. The details are provided in Appendix B.

The reason why only the joint pattern matters is because, for each eavesdropping set \( N \in P_e \), we have

\[
H(A_N^k | W_k, Q_N^k) = H(A_N^k | Q_N^k). \tag{20}
\]

Furthermore, in SPIR problems, for each colluding set, i.e., \( \forall N \in P_c \), (20) is still true [7]. Hence, in SPIR problems, colluding and eavesdropping sets play the same role in the converse proof. This is why only the joint pattern, i.e., the union of the colluding and eavesdropping pattern matters.

Note that the RHS of (18) is an increasing function of \( F_2 \) and the RHS of (19) is a decreasing function of \( F_2 \). Based on the result of Theorem 3 to find the tightest upper bound on capacity and the tightest lower bound on the common randomness, we should find the minimum \( F_2 = \sum_{m=1}^M x_m \) achievable over all \( x \) satisfying (16) and (17). Applying Theorem 3 for the optimal solution of (LP2), denoted as \((x^*, S_2^*)\), and noting that \( x^* \) is rational due to the fact that the objective function and linear constraints in (LP2) are both with integer coefficients, we have

\[
C(P_c, P_e) \leq 1 - \frac{1}{F_2^*}
\]

and the amount of common randomness shared by the servers relative to the message size must
satisfy
\[
\rho \geq \frac{1}{F_2^* - 1}.
\] (21)

C. Capacity

In Section IV-A, we have shown that when the amount of common randomness relative to the message size satisfies \( \rho \geq \frac{1}{F_2^* - 1} \), the SPIR capacity satisfies \( C(\mathcal{P}_c, \mathcal{P}_e) \geq 1 - \frac{1}{F^*} \). In Section IV-B, we have shown that when \( \rho \geq \frac{1}{F_2^* - 1} \), the SPIR capacity satisfies \( C(\mathcal{P}_c, \mathcal{P}_e) \leq 1 - \frac{1}{F_2^*} \), furthermore, when \( \rho < \frac{1}{F_2^* - 1} \), the SPIR capacity is zero. Recall that \( F^* \) and \( F_2^* \) are the optimal solutions to (LP1) and (LP2), respectively.

If \( F^* = F_2^* \), then the achievability and converse results meet, yielding the capacity result of Theorem I. This is indeed true, as (LP1) and (LP2) are actually dual problems of each other, which means \( F^* = F_2^* \).

Hence, we have found the capacity of the SPIR problem under arbitrary collusion pattern \( \mathcal{P}_c \) and eavesdropping pattern \( \mathcal{P}_e \), as described in Theorem I.

D. PIR problem with arbitrary eavesdropping pattern \( \mathcal{P}_e \)

Following the proof of our main result, we may get the following theorem on the PIR capacity with arbitrary eavesdropping pattern \( \mathcal{P}_e \).

**Theorem 4** When \( K \geq 2 \), the capacity of the PIR problem under arbitrary eavesdropping pattern \( \mathcal{P}_e \) is

\[
C^\text{PIR}_{\rho}(\mathcal{P}_e) = \begin{cases} 
1 - \frac{1}{F^*}, & \rho \geq \frac{1}{F^* - 1}, \\
0, & \text{otherwise}
\end{cases}
\]

where \( F^* \) is the optimal value of the following linear programming problem (LP1):

\[
(LP1) \quad \max_y \quad 1^T_N y \\
\text{s. t.} \quad B_{\mathcal{P}_e}^T y \leq 1_{M_e} \\
y \geq 0_N,
\]

where \( B_{\mathcal{P}_e} \) is the incidence matrix, of size \( N \times M_e \), of the eavesdropping pattern \( \mathcal{P}_e \).
The above theorem studies the PIR problem, which does not protect the databases’ privacy against the user, i.e., (6) does not need to be satisfied. In this case, (20) still holds for all the eavesdropping sets, and therefore, the converse holds for the eavesdropping pattern \(P_e\). In terms of achievability, we still employ the same achievability used for SPIR, i.e., even though (6) does not need to be satisfied, the proposed achievability still satisfies (6). Coupled with the converse results, we see that in PIR problems with arbitrary eavesdropping pattern \(P_e\), databases’ privacy against the user can be protected for free, i.e., without incurring an increase in the download cost.

PIR, and not SPIR, problems with arbitrary colluding and eavesdropping patterns are still open. There, the colluding pattern and eavesdropping patterns will play different roles, and how they interact with each other is part of ongoing research.

\section{V. Conclusions}

We have found the capacity of the SPIR problem under arbitrary collusion patterns \(P_c\) and eavesdropping pattern \(P_e\). We first link the achievable SPIR rate and its converse to the solutions of two linear programming problems. Then, we show that the two different linear programming problems have the same optimal value. As a result, the achievable SPIR rate and its converse meet, yielding the capacity. From our results, it can be seen that the collusion pattern and eavesdropping pattern do not matter individually, it is their joint pattern that determines the SPIR capacity of the problem.

\section{Appendix A}

\textbf{Proof of Theorem 2}

\textbf{A. An Illustrative Example:} \(K = 3, N = 5, P_c = \{\{1, 2\}, \{1, 4\}, \{2, 4\}, \{3, 4\}, \{5\}\}, P_e = \{\{1, 2, 3\}, \{2, 4\}, \{5\}\}\)

The joint pattern is \(P_J = \{\{1, 2, 3\}, \{1, 4\}, \{2, 4\}, \{3, 4\}, \{5\}\}\), and its corresponding incidence matrix is:

\[
B_{P_J} = \begin{bmatrix}
1 & 1 & 0 & 0 & 0 \\
1 & 0 & 1 & 0 & 0 \\
1 & 0 & 0 & 1 & 0 \\
0 & 1 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 1
\end{bmatrix}
\]
A feasible solution to (LP1) is \( y = \begin{bmatrix} 1 & 1 & 1 & 2 & 1 \end{bmatrix}^T \), and the corresponding cost is \( F = \frac{8}{3} \). The solution \( y \) is in fact the optimal solution of (LP1) but its optimality is not used here.

Pick \( \bar{L} \) such that the following numbers are integers: \( \bar{L}, \frac{L}{F}, \frac{\bar{L}}{F} y_n, n \in [1 : N] \). The smallest \( \bar{L} \) that satisfies this is 8.

Suppose each message has length \( L = \bar{L} \left( 1 - \frac{1}{F} \right) = 5 \). We stack the symbols of all \( K = 3 \) messages into a \( KL = 15 \) column vector, i.e.,

\[
W = \begin{bmatrix} W^1_1 & W^2_1 & \ldots & W^5_1 & \ldots & W^1_3 & \ldots & W^5_3 \end{bmatrix}^T
\]

The queries are generated at the user in the following way. First, the user generates \( \frac{\bar{L}}{F} = 3 \) many independent and random column vectors \( U_1, U_2, U_3 \), each with length \( KL = 15 \). The queries generated by the users are given as

\[
\left[ \bar{Q}_1 \quad \bar{Q}_2 \quad \ldots \quad \bar{Q}_8 \right] \triangleq \left[ U_1 \quad U_2 \quad U_3 \right] \cdot G_{(8,3)} + \left[ 0 \quad \ldots \quad 0 \quad e^{[\theta]}_1 \quad \ldots \quad e^{[\theta]}_5 \right]
\]  

(22)

where \( G_{(8,3)} \) is the generating matrix of an \((8,3)\)-GRS code, \( e^{[\theta]}_k \) is a column vector of length 15 where only the \((15(\theta - 1) + k)\)-th element is 1, and all other elements are 0. The number of zero vectors in the second term of the RHS of (22) is \( \bar{L} - L = 3 \). The total number of query column vectors in (22) is \( \bar{L} = 8 \), and these will be distributed to the servers according to \((y, F)\), more specifically, the number of queries to server \( n \) is \( \frac{y_n}{F} \bar{L} \), i.e., Servers 1, 2, 3 receives \( \bar{Q}_1, \bar{Q}_2, \bar{Q}_3 \), respectively, Server 4 receives \( (\bar{Q}_4, \bar{Q}_5) \) and Server 5 receives \( (\bar{Q}_6, \bar{Q}_7, \bar{Q}_8) \).

We now show that the queries generated above protects user privacy, i.e., it satisfies (7). Recall that the colusion pattern is \( \mathcal{P}_c = \{\{1, 2\}, \{1, 4\}, \{2, 4\}, \{3, 4\}, \{5\}\} \). The number of query vectors colluding set \( \{1, 2\} \) sees is 2, and the number of query vectors colluding sets \( \{1, 4\}, \{2, 4\}, \{3, 4\}, \{5\} \) see is 3 each. Thus, we may conclude that the number of query vectors each colluding set sees is no more than 3. Due to the MDS property of the \((8,3)\)-GRS code, any 3 out of \( \bar{Q}_1, \ldots, \bar{Q}_8 \) are independent and uniformly distributed. Hence, user privacy is preserved against any colluding set of servers in \( \mathcal{P}_c = \{\{1, 2\}, \{1, 4\}, \{2, 4\}, \{3, 4\}, \{5\}\} \).

Next, we describe server answers based on the query vector it receives. The \( N = 5 \) servers share a common randomness vector \( S = \begin{bmatrix} S_1 & S_2 & S_3 \end{bmatrix} \). Thus, the common randomness shared relative to the message size is \( \rho = \frac{3}{5} \), which is equal to \( \frac{1}{F - 1} \). Each server calculates the following row vector \( \bar{S} = \begin{bmatrix} \bar{S}_1 & \bar{S}_2 & \ldots & \bar{S}_8 \end{bmatrix} \triangleq S \cdot G_{(8,3)} \). Server \( n \) calculates the inner product of each query vector it receives with the message vector \( W \), and then adds the corresponding elements
from $\bar{S}$ and sends it to the user, i.e., let
\[ A_n \triangleq Q_n^T W + S_n, \quad n \in [1 : 8] \] (23)

Servers 1, 2, 3 sends $\bar{A}_1$, $\bar{A}_2$, $\bar{A}_3$, respectively, Server 4 sends $(\bar{A}_4, \bar{A}_5)$, and Server 5 sends $(\bar{A}_6, \bar{A}_7, \bar{A}_8)$.

Now, we verify that the proposed scheme satisfies the decoding constraint at the user, i.e., (5), the database privacy constraint at the user, i.e., (6) and the database privacy constraint at the passive eavesdropper, i.e., (8). To do this, it is useful to write the server answers in a different form by noticing that both the random vectors $U_1, U_2, U_3$ and the common randomness $S_1, S_2, S_3$ are encoded by the same $(8, 3)$-GRS code with generating matrix $G_{(8,3)}$, i.e., the answers sent by the servers may be written in the following matrix form
\[
\begin{bmatrix}
\bar{A}_1 & \bar{A}_2 & \cdots & \bar{A}_8 \\
\end{bmatrix}
= \begin{bmatrix}
W^TU_1 + S_1 & W^TU_2 + S_2 & W^TU_3 + S_3 \\
\end{bmatrix} \cdot G_{(8,3)} + \begin{bmatrix}
0 & \cdots & 0 & W^1_\theta & \cdots & W^5_\theta \\
\end{bmatrix}
= \begin{bmatrix}
X_1 & X_2 & X_3 & W^1_\theta & \cdots & W^5_\theta \\
\end{bmatrix} \cdot \begin{bmatrix}
G_{(8,3)} \\
0_{5 \times 3} & I_5 \\
\end{bmatrix}
\]

where we have defined $X_l = W^TU_l + S_l$, $l \in [1 : 3]$.

The decoding constraint at the user is satisfied, as upon receiving $\bar{A}_l$, $l \in [1 : 8]$, the user may decode $\begin{bmatrix}
X_1 & X_2 & X_3 & W^1_\theta & \cdots & W^5_\theta \\
\end{bmatrix}$ because $\begin{bmatrix}
G_{(8,3)} \\
0_{5 \times 3} & I_5 \\
\end{bmatrix}$ is invertible due to the MDS property of the generating matrix $G_{(8,3)}$. The database privacy against the user is satisfied because apart from its decoded message $W_\theta$, it can further decode $X_1, X_2, X_3$, each of which includes a common randomness $S_1, S_2$ or $S_3$ to protect the other $K - 1 = 2$ messages. Finally, we check that the proposed scheme protects database privacy against the passive eavesdropper. Recall that the eavesdropping pattern is $\mathcal{P}_e = \{\{1, 2, 3\}, \{2, 4\}, \{5\}\}$. Hence, the number of answers each eavesdropping set $\{1, 2, 3\}$, $\{2, 4\}$ and $\{5\}$ sees is equal to 3. From (23), we see that each answer $\bar{A}_k$ is protected by a randomness $\bar{S}_k$, and as long as the eavesdropper sees no more than 3 answers, the corresponding 3 $\bar{S}$s it sees are linearly independent, due to the MDS property of the generating matrix $G_{(8,3)}$. Hence, the inner product of the corresponding query vector and message vector is fully protected against the passive eavesdropper who observes answers from servers in each eavesdropping set in $\mathcal{P}_e = \{\{1, 2, 3\}, \{2, 4\}, \{5\}\}$. 

October 19, 2020 DRAFT
B. General Achievability Scheme for arbitrary number of messages $K$, arbitrary number of databases $N$ and arbitrary joint pattern $\mathcal{P}_J$ of collusion pattern $\mathcal{P}_c$ and eavesdropping pattern $\mathcal{P}_e$

Let $y = \begin{bmatrix} y_1 & y_2 & \cdots & y_N \end{bmatrix}^T$ be a feasible and rational solution of (LP1), i.e., $y$ consists of rational elements, and it satisfies the constraints (12) and (13). Let the value of the objective function in (LP1) corresponding to $y$ be $F$, i.e., $F = \sum_{n=1}^{N} y_n$. Furthermore, $y$ satisfies $F > 1$.

The encoding of the messages follows the scheme in [7, Section V.A] closely with $N$ replaced by $\bar{L}$, and $\max\{T, E\}$ replaced with $\bar{L} F$. For completeness, we state the scheme here.

Pick $\bar{L}$ such that the following numbers are integers: $\bar{L}$, $\bar{L} F$, $\bar{L} F y_n$, $n \in [1 : N]$. Note that the above involves $N + 2$ numbers which is finite. Such an $\bar{L}$ can be found because $F$ and $y_n$, $n \in [1 : N]$ are rational numbers.

Suppose each message has length $L = \bar{L} \left( 1 - \frac{1}{F} \right)$. Note that $L$ thus defined is positive, because $y$ satisfies $F > 1$. We stack the symbols of all $K$ messages into a $KL$ column vector, i.e.,

$$W = \begin{bmatrix} W_1^1 & W_1^2 & \cdots & W_1^K & \cdots & W_L^1 & \cdots & W_L^K \end{bmatrix}^T$$

The queries are generated at the user in the following way. First, the user generates $\bar{L} F$ many independent and random column vectors $U_1, U_2, \cdots, U_{\bar{L} F}$, each with length $KL$. The queries generated by the users are given as

$$\bar{Q}_1 \bar{Q}_2 \cdots \bar{Q}_L \triangleq \begin{bmatrix} U_1 & U_2 & \cdots & U_{\bar{L} F} \end{bmatrix} \cdot G(L, \bar{L} F) + \begin{bmatrix} 0 & \cdots & 0 & e_1^{[\theta]} & \cdots & e_{\bar{L}}^{[\theta]} \end{bmatrix}$$

(24)

where $G(L, \bar{L} F)$ is the generating matrix of an $(L, \bar{L} F)$-GRS code, $e_k^{[\theta]}$ is a column vector of length $KL$ where only the $((\theta - 1)L + k)$-th element is 1, and all other elements are 0. The number of zero vectors in the second term of the RHS of (24) is $\bar{L} - L$. The total number of query column vectors in (24) is $\bar{L}$, and these will be distributed to the servers according to $(y, F)$, more specifically, the number of queries to server $n$ is $\frac{y_n \bar{L}}{F}$, i.e., server $n$ receives $\bar{Q}_n$, $n \in \left[ \sum_{i=1}^{n-1} \frac{y_i \bar{L}}{F} + 1 : \sum_{i=1}^{n} \frac{y_i \bar{L}}{F} \right]$ as query vectors.

We now show that the queries generated above protects the user’s privacy, i.e., it satisfies (7). The number of query vectors each colluding set $\mathcal{T}_m$, $m \in [1 : M_c]$ sees is

$$\sum_{n \in \mathcal{T}_m} \frac{y_n \bar{L}}{F} = \frac{L}{F} \sum_{n \in \mathcal{T}_m} y_n \leq \frac{L}{F}$$

(25)

The last inequality of (25) is true because either $\mathcal{T}_m$ is in the joint pattern $\mathcal{P}_J$, or it is a subset of
some eavesdropping set, which is in $\mathcal{P}_J$. Eitherway, since $y$ satisfies (12), we have $\sum_{n \in T_m} y_n \leq 1$. Hence, we conclude that the number of query vectors each colluding set of server sees is no more than $\frac{L}{F}$. Due to the MDS property of the $\left( L, \frac{L}{F} \right)$-GRS code, any $\frac{L}{F}$ number of column vectors in the LHS of (24) are independent and uniformly distributed. Hence, user privacy is preserved against any colluding set of servers $T_m, m \in [1 : M_c]$.

We note here that by a similar argument, the passive eavesdropper who observes an eavesdropping set $\mathcal{E}_m, m \in [1 : M_e]$, from the eavesdropping pattern $\mathcal{P}_e$ does not know anything about the user’s index of interest, and hence, the scheme preserves the user’s privacy against the passive eavesdropper as well.

Next, we describe server answers based on the query vector it receives. The $N$ servers share a common randomness vector $S = [S_1 \ S_2 \ \cdots \ S_{\frac{L}{F}}]$. Thus, according to (9), the amount of common randomness shared relative to the message size is

$$\rho = \frac{H(S)}{L} = \frac{\frac{L}{F}}{L \left( 1 - \frac{1}{F} \right)} = \frac{1}{F - 1}$$

Each server calculates the following row vector $\bar{S} = [\bar{S}_1 \ \bar{S}_2 \ \cdots \ \bar{S}_{\frac{L}{F}}] \triangleq S \cdot G(L, \frac{L}{F})$ of size $1 \times \bar{L}$. Server $n$ calculates the inner product of each query vector it receives with the message vector $W$, and then adds the corresponding elements from $\bar{S}$ and sends it to the user, i.e., let

$$\bar{A}_k \triangleq Q_k^T W + \bar{S}_k, \quad k \in [1 : \bar{L}] , \quad (26)$$

Server $n$ sends $\bar{A}_k, k \in [\sum_{i=1}^{n-1} \frac{y_i}{F} \bar{L} + 1 : \sum_{i=1}^{n} \frac{y_i}{F} \bar{L}]$ back to the user.

Now, we verify that the proposed scheme satisfies the decoding constraint at the user, i.e., (5), the database privacy constraint at the user, i.e., (6) and the database privacy constraint at the passive eavesdropper, i.e., (8). To do this, it is useful to write the server answers in a different form by noticing that both the random vectors $U_1, U_2, \cdots, U_{\frac{L}{F}}$ and the common randomness $S_1, S_2, \cdots, S_{\frac{L}{F}}$ are encoded by the same $\left( L, \frac{L}{F} \right)$-GRS code with generating matrix $G(L, \frac{L}{F})$, i.e.,
the answers sent by the servers may be written in the following matrix form

\[
\begin{bmatrix}
\bar{A}_1 & \bar{A}_2 & \cdots & \bar{A}_L
\end{bmatrix}
= \begin{bmatrix}
W^T U_1 + S_1 & W^T U_2 + S_2 & \cdots & W^T U_L + S_L
\end{bmatrix} \cdot 
\begin{bmatrix}
G_{(L, \frac{L}{p})} & \begin{bmatrix}
0 & \cdots & 0 & W^1_\theta & \cdots & W^L_\theta
\end{bmatrix}
\end{bmatrix}
\]

where we have defined \(X_l = W^T U_l + S_l\), \(l \in \left[1 : \bar{L}\right]\).

The decoding constraint at the user is satisfied, as upon receiving \(\bar{A}_l\), \(l \in \left[1 : \bar{L}\right]\), the user may decode \(X_l = W^T U_l + S_l\) because \(G_{(L, \frac{L}{p})} \cdot \begin{bmatrix} 0_{L \times \frac{L}{p}} & I_L \end{bmatrix}\) is invertible due to the MDS property of the generating matrix \(G_{(L, \frac{L}{p})}\). The database privacy against the user is satisfied because apart from its decoded message \(W_\theta\), it can further decode \(X_l\), \(l \in \left[1 : \bar{L}\right]\), which includes a common randomness \(S_l\) to protect the other \(K - 1\) messages. Finally, the proposed scheme protects database privacy against the passive eavesdropper because the number of answers each eavesdropping set \(E_m\), \(m \in \left[1 : M_e\right]\) sees is

\[
\sum_{n \in E_m} \frac{y_n \bar{L}}{F} = \frac{\bar{L}}{F} \sum_{n \in E_m} y_n \leq \frac{\bar{L}}{F}
\]

which follows similar reasoning as (25). From (26), we see that each answer \(\bar{A}_k\) is protected by a randomness \(\bar{S}_k\), and as long as the eavesdropper sees no more than \(\frac{\bar{L}}{F}\) many answers, the corresponding \(\frac{\bar{L}}{F}\) number of \(\bar{S}_s\) are linearly independent, due to the MDS property of the generating matrix \(G_{(L, \frac{L}{p})}\). Hence, the inner product of the corresponding query vector and message vector is fully protected against the passive eavesdropper who observes answers from servers in \(E_m\).

**Appendix B**

**Proof of Theorem 3**

Define \(Q\) as the complete set of queries, i.e., \(Q = \{Q_n^{[k]} | n \in [1 : N], k \in [1 : K]\}\). Using standard SPIR converse techniques in [7], we can obtain

\[
H(S) \geq H(A^{[k]}_N|Q)
\]  

(27)
and

\[ H(W_k) \leq H(A^{[k]}_{[1:N]}|Q) - H(A^{[k]}_N|Q) \]  

(28)

for any \( \mathcal{N} \subset [1 : N] \) that is a collusion set in \( \mathcal{P}_c \) or an eavesdropping set in \( \mathcal{P}_e \), i.e., in SPIR problems, the following equality holds true for \( \mathcal{N} \) whether it is a colluding set or an eavesdropping set [7] :

\[ H(A^{[k]}_N|W_k, Q^{[k]}_N) = H(A^{[k]}_N|Q^{[k]}_N). \]  

(29)

The fact that (29) holds for all eavesdropping sets \( \mathcal{N} \in \mathcal{P}_e \) is obvious, as for any passive eavesdropper, observing the queries to and answers from \( \mathcal{N} \) servers should tell nothing about \( W_k \), i.e., \( (Q^{[k]}_N, A^{[k]}_N) \) is independent to \( W_k \), \( \forall k \in [1 : K] \). As for a colluding set \( \mathcal{N} \in \mathcal{P}_c \), we have

\[ H(A^{[k]}_N|W_k, Q^{[k]}_N) = H(A^{[k]}_N'|W_k, Q^{[k]}_N') = H(A^{[k]}_N'|Q^{[k]}_N') = H(A^{[k]}_N|Q^{[k]}_N) \]

the first and third equality follows from protecting user’s privacy against the server, i.e., (7), while the second equality follows from protecting the database’s privacy against the user, i.e., (6). Hence, we conclude that (27) and (28) both hold when \( \mathcal{N} \in \mathcal{P}_J \).

Similar to [114], for each server set \( \mathcal{N}_m \in \mathcal{P}_J, m \in [1 : M_J] \), multiply both sides of (27) by \( x_m \), which is the \( m \)-th element of \( x \). Note that \( x \) satisfies (17), which means that we are multiplying non-negative numbers and the sign of the inequality does not need to be changed. Then, adding all these \( M_J \) inequalities together, we obtain

\[ F_2 \cdot H(S) \geq \sum_{m=1}^{M_J} x_m H(A^{[k]}_{N_m}|Q), \]

where we have used the definition of \( F_2 \), i.e., \( F_2 = \sum_{m=1}^{M} x_m \). Based on the results in [114], we have

\[ \sum_{m=1}^{M_J} x_m H \left( A^{[k]}_{N_m}|Q \right) \geq H \left( A^{[k]}_{[1:N]}|Q \right). \]  

(30)

Hence, we have shown that

\[ H(S) \geq \frac{1}{F_2} H \left( A^{[k]}_{[1:N]}|Q \right). \]  

(31)
which tells us that in order to protect the user’s privacy against colluding server sets in \( \mathcal{P}_c \), the database’s privacy against the user, and the database’s privacy against the passive eavesdropping on the sets in \( \mathcal{P}_e \), the amount of common randomness must be no smaller than the ratio of the downloaded amount and \( F_2 \).

Similarly, for each server set \( N_m \in \mathcal{P}_J \), multiply both sides of (28) by \( x_m \), which is the \( m \)-th element of \( x \), and adding all these \( M_J \) inequalities together, we obtain

\[
F_2 \cdot H(W_k) \leq F_2 \cdot H(A_{1:N}^{[k]}|Q) - \sum_{m=1}^{M_J} x_m H(A_N^{[k]}|Q)
\]
\[
\leq F_2 \cdot H(A_{1:N}^{[k]}|Q) - H(A_N^{[k]}|Q)
\]
(32)

where (32) follows from (30). Thus, we have

\[
H(A_{1:N}^{[k]}|Q) \geq \frac{F_2}{F_2 - 1} H(W_k)
\]
(33)

which combined with (31) gives us a lower bound on the common randomness relative to the message size, i.e.,

\[
\rho = \frac{H(S)}{H(W_k)} \geq \frac{1}{F_2 - 1}
\]
(34)

Using (33), we obtain an upper bound on the capacity of the SPIR problem under collusion pattern \( \mathcal{P}_c \) and \( \mathcal{P}_e \), i.e.,

\[
C(\mathcal{P}_c, \mathcal{P}_e) = \frac{H(W_k)}{\sum_{n=1}^{N} H(A_n^{[k]}|Q)} \leq \frac{H(W_k)}{\sum_{n=1}^{N} H(A_{1:N}^{[k]}|Q)} \leq \frac{H(W_k)}{H(A_{1:N}^{[k]}|Q)} \leq 1 - \frac{1}{F_2}
\]
(35)

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