ON THE SEMINORMAL BASES AND DUAL SEMINORMAL BASES OF THE CYCLOTOMIC HECKE ALGEBRAS OF TYPE $G(\ell, 1, n)$

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ABSTRACT. This paper studies the seminormal bases $\{f_{st}\}$, the dual seminormal bases $\{g_{st}\}$, and the non-degenerate and the degenerate cyclotomic Hecke algebras $\mathcal{H}_n$ of type $G(\ell, 1, n)$. We present some explicit formulae for the constants $a_{st} := g_{st}/f_{st} \in K^\times$, $a_{st} := g_{st}/f_{st} \in K^\times$ in terms of the $\gamma$-coefficients $\{\gamma_{st}, \gamma'_{st}\}$ and the $r$-coefficients $\{r_{st}, r'_{st}\}$ of $\mathcal{H}_n$. In particular, we answer a question [21, Remark 3.6] of Mathas on the rationality of square roots of some quotients of products of $\gamma$-coefficients.

We obtain some explicit formulae for the expansion of each seminormal bases of $\mathcal{H}_n$ under the natural inclusion $\mathcal{H}_{n-1} \hookrightarrow \mathcal{H}_n$.

1. Introduction

Let $\ell, n$ be two positive integers. The cyclotomic Hecke algebras $\mathcal{H}_n$ of type $G(\ell, 1, n)$, also known as Ariki-Koike algebras, can be viewed as some generalizations of the Iwahori-Hecke algebras of types $A$ and $B$. They were introduced by Broué and Malle [9], and independently by Ariki and Koike [2], [4], and they play an important role in the modular representation theory of finite groups of Lie type over fields of non-defining characteristic. These algebras have been studied extensively both because of their rich representation theory and because of their close relationships with the affine Hecke algebras of type $A$, KLR algebras, BGG category $\mathcal{O}$ and geometric representation theory, see [3], [6], [7], [8], [10], [14] and [18].

The cyclotomic Hecke algebras $\mathcal{H}_n$ is cellular in the sense of Graham and Lehrer [13]. Using the cellular bases of $\mathcal{H}_n$ constructed in [12] and [21], Mathas has constructed in [21] a seminormal basis $\{f_{st}\}$ and a dual seminormal basis $\{g_{st}\}$ for the non-degenerate cyclotomic Hecke algebras $\mathcal{H}_n(q, Q)$ when they are semisimple and $q \not= 1$. These seminormal bases are important not only for the semisimple representation theory of $\mathcal{H}_n$, but also for the modular representation theory of $\mathcal{H}_n$, see [22], [15] and [19]. For the degenerate cyclotomic Hecke algebra $\mathcal{H}_n(u)$, there are similar constructions and results (e.g., seminormal basis $\{f_{st}\}$, dual seminormal basis $\{g_{st}\}$) in [3] and [27]. Muchof the theory on the seminormal bases applied in the paper goes back to Murphy in the symmetric groups and associated Hecke algebras cases (i.e., $\ell = 1$), see [23], [24], [25] and [26]. By the semisimplicity criterion of $\mathcal{H}_n$ and some eigenvalue (w.r.t. the Jucys-Murphy operators of $\mathcal{H}_n$) consideration, we see that $a_{st} := g_{st}/f_{st} \in K^\times$ and $a_{st} := g_{st}/f_{st} \in K^\times$, where $K$ is the ground field. However, these constants $a_{st}, a_{st}$ were not explicitly known as rational functions in the literature at the moment. The purpose of this paper is to give some explicit formulae of $a_{st}, a_{st}$ for both the non-degenerate and the degenerate cyclotomic Hecke algebras. To state our main result, we need some definitions and notations.

Let $R$ be an integral domain and $q \in R^\times$. Let $Q = (Q_1, \cdots, Q_t)$, where $Q_1, \cdots, Q_t \in R$. The non-degenerate cyclotomic algebra $\mathcal{H}_n(q, Q)$ of type $G(\ell, 1, n)$ is the unital associative $R$-algebra with generators $T_0, T_1, \cdots, T_{n-1}$ and the following defining relations:

\[
(T_0 - Q_1) \cdots (T_0 - Q_t) = 0;
\]

\[
T_0 T_1 T_0 = T_1 T_0 T_1;
\]

\[
(T_i - q)(T_i + 1) = 0, \quad \forall 1 \leq i \leq n - 1;
\]

\[
T_i T_j = T_j T_i, \quad \forall 1 \leq i < j - 1 < n - 1,
\]

\[
T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1}, \quad \forall 1 \leq i < n - 1.
\]

Following [11, §2], we define

\[
\mathcal{L}_m := q^{1-m}T_{m-1} \cdots T_1 T_0 T_1 \cdots T_{m-1}, \quad m = 1, 2, \cdots, n,
\]

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and call them the Jucys-Murphy operators of $H_{\ell,n}(q, Q)$.

Let $P_n$ be the set of multipartitions of $n$. For each $\lambda \in P_n$, let $\text{Std}(\lambda)$ be the set of standard $\lambda$-tableaux. Let $\{m_{\lambda t}[s, t \in \text{Std}(\lambda), \lambda \in P_n]\}$ be the Dipper-James-Mathas cellular basis of $H_{\ell,n}(q, Q)$. The definition of $m_{\lambda t}$ makes use of a “trivial representation” of the Hecke algebra $H_q(\mathfrak{S}_\lambda)$ associated to a standardYoung subgroup $S$ of $\mathfrak{S}_n$. Suppose that $q \neq 1$ and $H_{\ell,n}(q, Q)$ is semisimple. Let $\{f_{\lambda t}[s, t \in \text{Std}(\lambda), \lambda \in P_n]\}$ be the seminormal basis of $H_{\ell,n}(q, Q)$ corresponding to the cellular basis $\{m_{\lambda t}[s, t \in \text{Std}(\lambda), \lambda \in P_n]\}$. Replacing the “trivial representation” of $H_q(\mathfrak{S}_\lambda)$ in the construction of $m_{\lambda t}$ by the “sign representation” of $H_q(\mathfrak{S}_\lambda)$, one can also get a second cellular basis $\{n_{\lambda t}[s, t \in \text{Std}(\lambda), \lambda \in P_n]\}$. We refer the readers to Section 2 and Section 3 for unexplained notations here. The following theorem is the first main result of this paper.

**Theorem 1.1.** Let $\lambda \in P_n$ and $s, t \in \text{Std}(\lambda)$. Suppose $q \neq 1$ and $H_{\ell,n}(q, Q)$ is semisimple. Then

$$\alpha_{st} := n_{st}/f_{st} = (-q)^{-\ell(d(s')) - \ell(d(t'))} \frac{\gamma_s \gamma_t}{\gamma_s \gamma_t} = (-q)^{\ell(d(s')) + \ell(d(t'))} \frac{\gamma_s' \gamma_t'}{\gamma_s' \gamma_t'},$$

where for each $u \in \text{Std}(\lambda)$, $\gamma_u$ is the $\gamma$-coefficient defined in Definition 2.7. $\gamma_u'$ is defined as in Definition 3.8.

There is a natural algebra embedding $\iota: H_{\ell,n-1}(q, Q) \hookrightarrow H_{\ell,n}(q, Q)$, which is defined on generators by $\iota(T_i) := T_i$ for $0 \leq i < n - 1$. In order to avoid the confusion between the notations for $H_{\ell,n-1}(q, Q)$ and $H_{\ell,n}(q, Q)$. We add a superscript $(n)$ to indicate that it is the notation for $H_{\ell,n}(q, Q)$. Let $\mu \in P_{n-1}$ and $s, t \in \text{Std}(\lambda)$. Under the embedding $\iota$, we have

$$\iota^{(n-1)}_{st} = \sum_{\lambda \in P_n} \sum_{u, v \in \text{Std}(\lambda)} \beta_{uv}^{st} f_{uv}^{(n)}.$$

where $\beta_{uv}^{st} \in K$ for each pair $(u, v)$. The following theorem is the second main result of this paper.

**Theorem 1.3.** Let $\mu \in P_{n-1}, \lambda \in P_n$, and $s, t \in \text{Std}(\mu), u, v \in \text{Std}(\lambda)$. Suppose $q \neq 1$ and $H_{\ell,n}(q, Q)$ is semisimple. Then $\beta_{uv}^{st} \neq 0$ if and only if $u \downarrow_{n-1} = s$ and $v \downarrow_{n-1} = t$. In that case we have

$$\beta_{uv}^{st} = \frac{\gamma_u^{(n-1)}}{\gamma_v^{(n)}} = \frac{\gamma_t^{(n-1)}}{\gamma_s^{(n)}}.$$

If we set $\ell := 1$ and $Q_1 := 1$, then the above two main results Theorems 1.1 and 1.3 give the corresponding results for the classical semisimple Iwahori-Hecke algebra $H_q(\mathfrak{S}_n)$ associated to the symmetric group $\mathfrak{S}_n$ with Hecke parameter $q \neq 1$.

The degenerate case is parallel to the non-degenerate case with slight modification. Let $u = (u_1, \ldots, u_t)$, where $u_1, \ldots, u_t \in R$. The degenerate cyclotomic Hecke algebra $H_{\ell,n}(u)$ of type $G(\ell, 1, n)$ is the unital associative $R$-algebra with generators $s_1, \ldots, s_{n-1}, L_1, \ldots, L_n$ and the following defining relations:

$$\begin{align*}
(L_1 - u_1) \cdots (L_1 - u_t) & = 0; \\
S_i^2 & = 1, \quad \forall 1 \leq i \leq n - 1; \\
S_i S_j & = S_j S_i, \quad \forall 1 \leq i < j - 1 < n - 1; \\
S_i S_{i+1} S_i & = S_{i+1} S_i S_{i+1}, \quad \forall 1 \leq i < n - 1; \\
L_i L_k & = L_k L_i, \quad 1 \leq i < k, l \leq n, l \neq i, i + 1; \\
L_{i+1} & = S_i L_i S_i + S_i, \quad 1 \leq i < n.
\end{align*}$$

The elements $L_1, \ldots, L_n$ are called the Jucys-Murphy elements of the degenerate cyclotomic Hecke algebra $H_{\ell,n}(u)$.

As in the non-degenerate case, we have a cellular basis $\{m_{\lambda t}[s, t \in \text{Std}(\lambda), \lambda \in P_n]\}$ as well as a dual cellular basis $\{n_{\lambda t}[s, t \in \text{Std}(\lambda), \lambda \in P_n]\}$ of $H_{\ell,n}(u)$. Suppose that $H_{\ell,n}(u)$ is semisimple. Let $\{f_{\lambda t}[s, t \in \text{Std}(\lambda), \lambda \in P_n]\}$ be the seminormal basis of $H_{\ell,n}(u)$ corresponding to the cellular basis $\{m_{\lambda t}[s, t \in \text{Std}(\lambda), \lambda \in P_n]\}$. Let $\{g_{\lambda t}[s, t \in \text{Std}(\lambda), \lambda \in P_n]\}$ be the dual seminormal basis of $H_{\ell,n}(u)$ corresponding to the dual cellular basis $\{n_{\lambda t}[s, t \in \text{Std}(\lambda), \lambda \in P_n]\}$. Then we have that $g_{st} := g_{st}/f_{st} \in K^\times$ for any $s, t \in \text{Std}(\lambda), \lambda \in P_n$. The following two theorems are the analogues of Theorems 1.1, 1.3 for the degenerate case.
Then $b_{st} := g_{st} / f_{st} = (-1)^{-\ell(d(s'))-\ell(d(t'))} r_{\lambda}^{t'} r_{\lambda'}^{s} / r_{\lambda}^{s} r_{\lambda'},$
where for each $u \in \text{Std}(\lambda)$, $r_u$ is the $r$-coefficient defined in Definition 4.2. $r'_u$ is defined as in Definition 4.2.

As in the non-degenerate case, we also have an algebra embedding $\iota_1 : H_{\ell,n-1}(u) \rightarrow H_{\ell,n}(u)$, which is defined on generators by $\iota_1(s_i) := s_i, \iota_1(L_k) = L_k$, for $1 \leq i < n - 1, 1 \leq k \leq n - 1$. In order to avoid the confusion between the notations for $H_{\ell,n-1}(u)$ and $H_{\ell,n}(u)$. We add a superscript $(n)$ to indicate that it is the notation for $H_{\ell,n}(u)$. Let $\mu \in \mathcal{P}_{n-1}, \lambda \in \mathcal{P}_n$. Let $\Phi_s$ be the symmetric group on $n$. A partition of $\lambda$ is a weakly decreasing sequence $\lambda = \lambda_1, \lambda_2, \ldots$ of non-negative integers such that $|\lambda| := \sum_{i \geq 1} \lambda_i = a$. Let $\lambda = (\lambda_1, \lambda_2, \ldots) + a$ be a partition of $a$. We define $\lambda' = (\lambda'_1, \lambda'_2, \ldots)$, where for each $i$, $\lambda'_i := \# \{ j | \lambda_j \geq i \}$. Then $\lambda'$ is again a partition of $a$ and is called the conjugate of $\lambda$. A multipartition of $n$ is an $\ell$-tuple $\Lambda = (\lambda(1), \ldots, \lambda(\ell))$ of partitions such that $|\lambda(1)| + \cdots + |\lambda(\ell)| = n$. We define the Young diagram of $\Lambda$ to be $|\Lambda| := \{(i,j,c)| 1 \leq j \leq \lambda_j^{(c)}, 1 \leq c \leq \ell \}$. A $\Lambda$-tableau $t$ is a bijective map $t : |\Lambda| \rightarrow [1, 2, \ldots, n]$. If the $\Lambda$-tableau $t$ satisfies that $t(i,j,l) \leq t(a,b,l)$ for any $i \leq a$ and $j \leq b$ and $1 \leq l \leq \ell$, then we say $t$ is standard. We use $\text{Std}(\Lambda)$ to denote the set of standard $\Lambda$-tableaux. If $t \in \text{Std}(\lambda)$, then we set Shape$(t) := \lambda$, and we can write $t = (t^{(1)}, \ldots, t^{(\ell)})$, where each $t^{(\ell)}$ is a standard $\lambda^{(\ell)}$-tableaux.

Theorem 1.4. Let $\lambda \in \mathcal{P}_n$ and $s, t \in \text{Std}(\lambda)$. Suppose $H_{\ell,n}(u)$ is semisimple. Then

$$a_{st} := g_{st} / f_{st} = (-1)^{-\ell(d(s'))-\ell(d(t'))} r_{\lambda}^{t'} r_{\lambda'}^{s} / r_{\lambda}^{s} r_{\lambda'},$$

where for each $u \in \text{Std}(\lambda)$, $r_u$ is the $r$-coefficient defined in Definition 4.2.

Theorem 1.6. Let $\mu \in \mathcal{P}_{n-1}, \lambda \in \mathcal{P}_n$. Suppose $H_{\ell,n}(u)$ is semisimple. Then $b_{st}^{(\mu)} \neq 0$ if and only if $u \downarrow_{n-1} = s$ and $v \downarrow_{n-1} = t$. In that case we have

$$b_{st}^{(\mu)} = \frac{r^{(\mu)}_{s}}{r^{(\mu)}_{u}} \cdot \frac{r^{(\mu)}_{t}}{r^{(\mu)}_{v}}.$$

If we set $\ell := 1$ and $u_1 := 0$, then the above two main results Theorems 1.4 and 1.6 give the corresponding results for the semisimple symmetric group algebra $K[\mathcal{S}_n]$.

The content of the paper is organised as follows. In Section 2 we give some preliminary results on the structure and representation theory of the cyclotomic Hecke algebras $\mathcal{H}_{\ell,n}(q, Q)$. In particular, we shall recall the construction of cellular bases and seminormal bases of $\mathcal{H}_{\ell,n}(q, Q)$. Then we reveal some hidden relationship between various $\gamma$-coefficients in Lemma 5.2.2. Combining this with the use of certain remarkable invertible elements $\Phi_s$ introduced in Mathas’s work [21], we finally give the proof of the main results Theorem 1.4 and Theorem 1.6. In Section 4 we deal with the degenerate cyclotomic Hecke algebra $H_{\ell,n}(u)$. The argument is similar to the non-degenerate case. In particular, we give the proof of the main results Theorem 1.4 and Theorem 1.6.

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2. Preliminary

Let $\mathcal{H}_{\ell,n} \in \{\mathcal{H}_{\ell,n}(q, Q), H_{\ell,n}(u)\}$. Let $\mathcal{S}_n$ be the symmetric group on $\{1, 2, \ldots, n\}$. For each $1 \leq i < n$, we set $s_i := (i, i + 1)$. A word $w = s_{i_1} s_{i_2} \ldots s_{i_k}$ for $w \in \mathcal{S}_n$ is called a reduced expression of $w$ if $k$ is minimal; in this case we say $w$ has length $k$ and we write $\ell(w) = k$. Given a reduced expression $s_{i_1} \cdots s_{i_k}$ of $w \in \mathcal{S}_n$, we define $T_w = T_{i_1} \cdots T_{i_k}$, which is independent of the choice of the reduced expression of $w$ because the braid relations hold in $\mathcal{H}_{\ell,n}(q, Q)$. Let “$\ast$” be the unique anti-involution of $\mathcal{H}_{\ell,n}$ which fixes its defining generators.

Dipper, James and Mathas have shown in [12] that the algebra $\mathcal{H}_{\ell,n}$ is cellular in the sense of [13]. To recall the cellular structure given in [12], we need some combinatorial notions and notations. Let $a$ be a positive integer. A partition of $a$ is a weakly decreasing sequence $\lambda = (\lambda_1, \lambda_2, \ldots)$ of non-negative integers such that $|\lambda| := \sum_{i \geq 1} \lambda_i = a$. Let $\lambda = (\lambda_1, \lambda_2, \ldots) + a$ be a partition of $a$. We define $\lambda' = (\lambda'_1, \lambda'_2, \ldots)$, where for each $i$, $\lambda'_i := \# \{ j | \lambda_j \geq i \}$. Then $\lambda'$ is again a partition of $a$ and is called the conjugate of $\lambda$. A multipartition of $a$ is an $\ell$-tuple $\Lambda = (\lambda(1), \ldots, \lambda(\ell))$ of partitions such that $|\lambda(1)| + \cdots + |\lambda(\ell)| = n$. We define the Young diagram of $\Lambda$ to be $|\Lambda| := \{(i,j,c)| 1 \leq j \leq \lambda_j^{(c)}, 1 \leq c \leq \ell \}$. A $\Lambda$-tableau $t$ is a bijective map $t : |\Lambda| \rightarrow [1, 2, \ldots, n]$. If the $\Lambda$-tableau $t$ satisfies that $t(i,j,l) \leq t(a,b,l)$ for any $i \leq a$ and $j \leq b$ and $1 \leq l \leq \ell$, then we say $t$ is standard. We use $\text{Std}(\Lambda)$ to denote the set of standard $\Lambda$-tableaux. If $t \in \text{Std}(\Lambda)$, then we set Shape$(t) := \lambda$, and we can write $t = (t^{(1)}, \ldots, t^{(\ell)})$, where each $t^{(\ell)}$ is a standard $\lambda^{(\ell)}$-tableaux.
Let \( \mathcal{P}_n \) be the set of multipartitions of \( n \). For each multipartition \( \lambda = (\lambda^{(1)}, \ldots, \lambda^{(t)}) \in \mathcal{P}_n \), let \( \mathfrak{S}_\lambda \) be the corresponding standard Young subgroup of \( \mathfrak{S}_n \). That is,
\[
\mathfrak{S}_\lambda := \mathfrak{S}_{(\lambda^{(1)}, \ldots, \lambda^{(1)})} \times \mathfrak{S}_{(\lambda^{(2)}, \ldots, \lambda^{(1)})} \times \cdots \times \mathfrak{S}_{(\lambda^{(t)}, \ldots, \lambda^{(1)})},
\]
where \( b_i := (\lambda^{(i)})_1 \) for \( i = 1, 2, \ldots, \ell \). For each \( \lambda \in \mathcal{P}_n \), we define
\[
\lambda' := (\lambda^{(t)}), \ldots, \lambda^{(1)}),
\]
and call it the conjugate of \( \lambda \). For each \( t \in \text{Std}(\lambda) \), we define
\[
t' = (t^{(1)}'), \ldots, t^{(1)}).
\]
Then \( t' \in \text{Std}(\lambda') \).

Let \( t^\lambda \) be the initial standard \( \lambda \)-tableaux in which the numbers \( 1, 2, \ldots, n \) are entered in order first along the rows of \( t^{(1)} \) and then the rows of \( t^{(2)} \) and so on. We define \( t_\lambda := (t^\lambda)' \). In particular, \( t_\lambda \) is the standard \( \lambda \)-tableaux in which the numbers \( 1, 2, \ldots, n \) are entered in order first along the columns of \( t^{(t)} \) and then the columns of \( t^{(t-1)} \) and so on. For each \( t \in \text{Std}(\lambda) \), let \( d(t) \in \mathfrak{S}_n \) be the unique element in \( \mathfrak{S}_n \) such that \( \lambda d(t) = t \), and we set \( w_\lambda := d(t_\lambda) \).

For any \( \lambda, \mu \in \mathcal{P}_n \), we write \( \lambda \gg \mu \) if for all \( 1 \leq s \leq \ell \) and all \( i \geq 1 \),
\[
\sum_{i=1}^{s-1} |\lambda^{(i)}| + \sum_{j=1}^{i} \lambda_j^{(s)} \geq \sum_{i=1}^{s-1} |\mu^{(i)}| + \sum_{j=1}^{i} \mu_j^{(s)}.
\]
Clearly \( \mathcal{P}_n \) is a poset with respect to the partial order \( \gg \).

If \( \lambda \gg \mu \) and \( \lambda \neq \mu \), then we write \( \lambda \gg \mu \). Let \( s \in \text{Std}(\lambda) \), \( t \in \text{Std}(\mu) \). We write \( s \gg t \) if for any \( 1 \leq k \leq n \), \( \text{Shape}(s \downarrow_{(1,2,\ldots,k)}) \gg \text{Shape}(t \downarrow_{(1,2,\ldots,k)}) \). If \( s \gg t \) and \( s \neq t \) then we write \( s \gg t \). Clearly, \( t^\lambda \gg s \gg t \lambda \) for any \( s \in \text{Std}(\lambda) \).

**Definition 2.1.** \([11, 5]\) Let \( \lambda \in \mathcal{P}_n \) and \( s, t \in \text{Std}(\lambda) \). We define
\[
m_{st} := T_{d(s)} \left( \sum_{w \in \mathfrak{S}_\lambda} T_w \left( \prod_{s=2}^{\ell} \prod_{k=1}^{\lambda^{(s-1)}} (L_k - Q_s) \right) d(t),
\]
\[
m_{st} := d(s)^{-1} \left( \sum_{w \in \mathfrak{S}_\lambda} w \left( \prod_{s=2}^{\ell} \prod_{k=1}^{\lambda^{(s-1)}} (L_k - u_s) \right) d(t),
\]

**Theorem 2.2.** \([11, 5, 27]\) With respect to the poset \( (\mathcal{P}_n, \gg) \) and the anti-involution \( * \), the set \( \{m_{st} | s, t \in \text{Std}(\lambda), \lambda \in \mathcal{P}_n \} \) forms a cellular basis of \( \mathcal{H}_{\ell,n}(q, Q) \), while the set \( \{m_{st} | s, t \in \text{Std}(\lambda), \lambda \in \mathcal{P}_n \} \) forms a cellular basis of \( H_{\ell,n}(u) \).

One of the remarkable properties of the basis \( \{m_{st}\} \) is that it can be defined over an arbitrary ground ring, though the computation of the product of these bases can be rather complicated. When the Hecke algebra \( \mathcal{H}_{\ell,n}(q, Q) \) is semisimple, there is another basis (called seminormal basis) of \( \mathcal{H}_{\ell,n}(q, Q) \) which is much easier for calculation. Henceforth we assume that \( q \neq 1 \). Let us recall the following criteria of semisimplicity for \( \mathcal{H}_{\ell,n}(q, Q) \).

**Lemma 2.3.** \([1]\) Let \( R = K \) be a field. Suppose \( 1 \neq q \in K^\times \). Then \( \mathcal{H}_{\ell,n}(q, Q) \) is semisimple if and only if
\[
(2.4) \quad \prod_{i=1}^{n} (1 + q + q^2 + \cdots + q^{i-1}) \prod_{1 \leq i < j \leq \ell} (q^d Q_i - Q_j) \in K^\times.
\]

For any \( t = (t^{(1)}, \ldots, t^{(t)}) \in \text{Std}(\lambda) \) and any \( 1 \leq k \leq n \), we define
\[
\text{res}_t(k) = q^{i-k} Q_c, \quad \text{if } k \text{ appears in row } i \text{ and column } j \text{ of } t^{(c)}
\]
We also define \( R(k) := \{\text{res}_t(k) | t \in \text{Std}(\lambda), \lambda \vdash n \} \).

The condition (2.3) is actually equivalent to the following statement:
\[
\text{for any } \lambda, \mu \in \mathcal{P}_n, \ s \in \text{Std}(\lambda), t \in \text{Std}(\mu), \text{ if } s \neq t, \text{ then there exists } \ 1 \leq k \leq n \text{ such that } \text{res}_s(k) - \text{res}_t(k) \in K^\times.
\]
Definition 2.7. ([25, 21 Definition 2.4]) Suppose \( q \neq 1 \) and (2.4) holds. Let \( \lambda \in \mathcal{P}_n \) and \( t \in \text{Std}(\lambda) \). We define
\[
F_t = \prod_{k=1}^{n} \prod_{c \in R(k), c \neq \text{res}_t(k)} \frac{L_k - c}{\text{res}_t(k) - c}.
\]

For any \( \lambda \in \mathcal{P}_n \) and \( s, t \in \text{Std}(\lambda) \), we define
\[
f^{(s)}_{t} := F_s m_{st} F_t.
\]
(2.8)

When the context is clear, we shall omit the superscript “\((n)\)” and write \( f_{st} \) instead of \( f^{(n)}_{st} \).

For any \( k \in \mathbb{Z}_{\geq 0} \), we define \( |k|_q = \sum_{i=0}^{k-1} q^i \). For any \( m \in \mathbb{Z}_{\geq 0} \), we set \( [m]_q! = [1]_q [2]_q \cdots [m]_q \). If \( \lambda = (\lambda^{(1)}, \ldots, \lambda^{(t)}) \in \mathcal{P}_n \), then we define \( [\lambda]_{q}! = \prod_{c=1, i \geq 1} [\lambda^{(c)}]_{q}! \).

Definition 2.9. ([25, 19 (3.17)-(3.19)], [21, 2.9]) Suppose \( q \neq 1 \) and (2.4) holds. Let \( \lambda \in \mathcal{P}_n \). The \( \gamma \)-coefficients \( \{ \gamma^{(n)}_{t} \mid t \in \text{Std}(\lambda), \lambda \in \mathcal{P}_n \} \) are defined to be a multiset of invertible scalars in \( K^\times \) which are uniquely determined by:

\[ \gamma^{(n)}_{t} = |\lambda|_{q}! \prod_{1 \leq c < t \leq \ell, 1 \leq j \leq |\lambda|_{c}} (q^{j-i} Q_{c} - Q_{t}); \]
and

\[ \gamma^{(n)}_{S} = (q \text{res}_s(i) - \text{res}_t(i))(\text{res}_s(i) - q \text{res}_t(i)) \]
(2.11)

When the context is clear, we shall omit the superscript “\((n)\)” and write \( \gamma_{t} \) instead of \( \gamma^{(n)}_{t} \).

Lemma 2.10. ([21 Theorems 2.11, 2.15, Corollary 2.13]) Suppose \( q \neq 1 \), (2.4) holds and \( R = K \) is a field. Then
\[ \{ f_{st} \mid s, t \in \text{Std}(\lambda), \lambda \in \mathcal{P}_n \} \]
is a basis of \( \mathcal{H}_{\ell,n}(q, Q) \). Moreover,

1) if \( s, t, u \) and \( v \) are standard tableaux, then \( f_{st} f_{uv} = \delta_{uv} \gamma_{s} f_{st} \);
2) if \( \lambda \in \mathcal{P}_n \), \( s, t \in \text{Std}(\lambda) \) and \( 1 \leq k \leq n \), then \( f_{st} L_k = \text{res}_k f_{st} \), \( L_k f_{st} = \text{res}_k f_{st} \);
3) for each \( \lambda \in \mathcal{P}_n \) and \( t \in \text{Std}(\lambda) \), \( F_t = \frac{1}{t} f_{tt} \) and \( F_t \) is a primitive idempotent;
4) \( \{ F_t \mid t \in \text{Std}(\lambda), \lambda \in \mathcal{P}_n \} \) is a complete set of pairwise orthogonal primitive idempotents in \( \mathcal{H}_{\ell,n}(q, Q) \).

We call \( \{ f_{st} \mid s, t \in \text{Std}(\lambda), \lambda \in \mathcal{P}_n \} \) the seminormal basis of \( \mathcal{H}_{\ell,n}(q, Q) \) corresponding to the cellular basis \( \{ m_{st} \mid s, t \in \text{Std}(\lambda), \lambda \in \mathcal{P}_n \} \) of \( \mathcal{H}_{\ell,n}(q, Q) \).

In the rest of this section, we consider the degenerate cyclotomic Hecke algebra \( H_{\ell,n}(u) \). First, let’s recall the following criteria of semisimplicity for \( H_{\ell,n}(u) \).

Lemma 2.11. ([5 Theorem 6.11]) Let \( R = K \) be a field. Then \( H_{\ell,n}(u) \) is semisimple if and only if
\[ (n)! \prod_{1 \leq i < j \leq \ell} (d \cdot 1_K + u_i - u_j) \in K^\times. \]
(2.12)

For any \( t = (t^{(1)}, \ldots, t^{(\ell)}) \in \text{Std}(\lambda) \) and any \( 1 \leq k \leq n \), we define
\[ c_i(k) = j - i + u_c, \quad \text{if } k \text{ appears in row } i \text{ and column } j \text{ of } t^{(c)} \]
(2.13)

We also define \( C(k) := \{ c_i(k) \mid t \in \text{Std}(\lambda), \lambda \vdash n \} \).

The condition (2.12) is actually equivalent to the following statement:
\[ \text{for any } \lambda, \mu \in \mathcal{P}_n, s \in \text{Std}(\lambda), t \in \text{Std}(\mu), \text{if } s \neq t, \text{ then there exists } 1 \leq k \leq n \text{ such that } c_s(k) - c_t(k) \in K^\times. \]
(2.14)

Definition 2.15. ([24, 5 Definition 6.7]) Suppose (2.4) holds. Let \( \lambda \in \mathcal{P}_n \) and \( t \in \text{Std}(\lambda) \). We define
\[ F_t = \prod_{k=1}^{n} \prod_{c \in C(k), c \neq \text{res}_t(k)} \frac{L_k - c}{c_t(k) - c}. \]
For any $\lambda \in \mathcal{P}_n$ and $s, t \in \text{Std}(\lambda)$, we define
\begin{equation}
\ell^{(n)}_{st} := F_s m_{st} F_t.
\end{equation}

When the context is clear, we shall omit the superscript and write $f_{st}$ instead of $f^{(n)}_{st}$.

**Definition 2.17.** ([23, [5] Lemma 6.10]) Suppose (2.12) holds. Let $\lambda \in \mathcal{P}_n$. We define a multiset of elements $\{r^{(n)}_t\} \in K^{\times} \mid t \in \text{Std}(\lambda), \lambda \in \mathcal{P}_n \}$ in $K^{\times}$ as follows:

1. $r^{(n)}_t = \left( \prod_{s=1}^{l} \prod_{i \geq 1} \lambda_i^{(i)} \right) \prod_{1 \leq s < t \leq l} (j - i + u_s - u_t)$; and
2. if $s = (i, i + 1) \triangleright t$ then
\[
\frac{r^{(n)}_t}{r^{(n)}_s} = \frac{(1 + c_t(i) - c_t(i))(c_s(i) - c_t(i) - 1)}{(c_s(i) - c_t(i))^2}.
\]

When the context is clear, we shall omit the superscript “(n)” and write $r_t$ instead of $r^{(n)}_t$.

**Lemma 2.18.** ([22 Proposition 3.4]) Suppose (2.13) holds and $R = K$ is a field. Then
\[
\{f_{st} \mid s, t \in \text{Std}(\lambda), \lambda \in \mathcal{P}_n \}
\]
is a basis of $H_{\ell,n}(u)$.

Moreover,
1. if $s, t, u$ and $v$ are standard tableaux, then $f_{st}f_{uv} = \delta_{uv}f_{st}$;
2. if $\lambda \in \mathcal{P}_n, s, t \in \text{Std}(\lambda)$ and $1 \leq k \leq n$, then $f_{st}L_k = c_1(k)f_{st}, L_kf_{st} = c_s(k)f_{st}$;
3. for each $\lambda \in \mathcal{P}_n$ and $t \in \text{Std}(\lambda)$, $F_t = \frac{1}{r_t} f_t$ is a primitive idempotent;
4. $\{F_t \mid t \in \text{Std}(\lambda), \lambda \in \mathcal{P}_n \}$ is a complete set of pairwise orthogonal primitive idempotents in $H_{\ell,n}(u)$.

We call $\{f_{st} \mid s, t \in \text{Std}(\lambda), \lambda \in \mathcal{P}_n \}$ the seminormal basis of $H_{\ell,n}(u)$ corresponding to the cellular basis $\{m_{st} \mid s, t \in \text{Std}(\lambda), \lambda \in \mathcal{P}_n \}$ of $H_{\ell,n}(u)$.

## 3. The non-degenerate case

In this section we shall only consider the non-degenerate cyclotomic Hecke algebra $\mathcal{H}_{\ell,n}(q, \mathbb{Q})$. Our purpose is to give the proof of the main results Theorem 1.4 and Theorem 1.3. Throughout this section, we assume that $R = K$ is a field, $q \neq 1$ and (2.12) holds. In particular, this implies that $\mathcal{H}_{\ell,n}(q, \mathbb{Q})$ is (split) semisimple over $K$.

Let $\{m_{st} \mid s, t \in \text{Std}(\lambda), \lambda \in \mathcal{P}_n \}$ be the Dipper-James-Mathas cellular basis, and $\{f_{st} \mid s, t \in \text{Std}(\lambda), \lambda \in \mathcal{P}_n \}$ be the corresponding seminormal basis of $H_{\ell,n}(q, \mathbb{Q})$. For each $\lambda \in \mathcal{P}_n$, we define
\[
\mathcal{H}_{\ell,n}^{\lambda} := K\text{-Span}\{m_{st} \mid s, t \in \text{Std}(\mu), \lambda \subset \mu \in \mathcal{P}_n \},
\]
which is a cell ideal of $\mathcal{H}_{\ell,n}(q, \mathbb{Q})$ with respect to the cellular basis. For any $1 \leq k \leq n$ and $s, t \in \text{Std}(\lambda)$, we have that ([21, (2.3)])
\begin{equation}
m_{st}L_k = \text{res}_t(k)m_{st} + \sum_{v \in \text{Std}(\lambda) \atop v \triangleright t} a_v m_{sv} \pmod{\mathcal{H}_{\ell,n}^{\lambda}},
\end{equation}
where $a_v \in K$ for each $t < v \in \text{Std}(\lambda)$.

Mathas has yet introduced in [21 §3] another cellular basis which will be called the dual cellular basis of $\mathcal{H}_{\ell,n}(q, \mathbb{Q})$. We now recall his construction.

For each $\lambda \in \mathcal{P}_n$, we define
\begin{equation}
\ell^{(n)}_{\lambda} := \left( \sum_{w \in \mathfrak{S}_\lambda} (-q)^{-f(w)}T_w \right) \prod_{s=2}^{l} \prod_{k=1}^{[\lambda^{(s)}+1]} (L_k - Q_{t-s+1}).
\end{equation}

If $t \in \text{Std}(\lambda)$, then we define $d'(t) \in \mathcal{S}_n$ by $t^{\lambda}d'(t) = t$. For any $s, t \in \text{Std}(\lambda)$, we set
\begin{equation}
\ell^{(n)}_{st} = (-q)^{-f(c_t(t))}T_{d'}(s)\ell^{(n)}_{t^{\lambda}a\lambda}T_{d'}(t).
\end{equation}

**Theorem 3.4.** ([21]) With respect to the opposite poset $(\mathcal{P}_n, \preceq)$ and the anti-involution $\ast$, the set $\{\ell^{(n)}_{st} \mid s, t \in \text{Std}(\lambda), \lambda \in \mathcal{P}_n \}$ forms a cellular basis of $\mathcal{H}_{\ell,n}(q, \mathbb{Q})$. 


We call it the dual cellular basis of $\mathcal{H}_{\ell,n}(q, Q)$. For each $\lambda \in \mathcal{P}_n$, we define

$$\mathcal{H}_{\ell,n}^{\lambda} := K - \text{Span}\{n_{st}| s, t \in \text{Std}(\mu), \lambda \triangleright \mu \in \mathcal{P}_n\},$$

which is a cell ideal of $\mathcal{H}_{\ell,n}(q, Q)$ with respect to the dual cellular basis. For any $1 \leq k \leq n$ and $s, t \in \text{Std}(\lambda)$, we have that (21 Proposition 3.3)

$$(3.5) \quad n_{st}E_k = \text{res}_s(k)n_{st} + \sum_{v \in \text{Std}(\lambda), v < t} b_v n_{sv} \pmod{\mathcal{H}_{\ell,n}^{\lambda}},$$

where $b_v \in K$ for each $t \triangleright v \in \text{Std}(\lambda)$.

**Definition 3.6.** Let $\lambda \in \mathcal{P}_n$. For any $s, t \in \text{Std}(\lambda)$, we define

$$g_{st} := F_s n_{st} F_t.$$

**Remark 3.7.** Note that our notations $n_{st}, g_{st}$ differ with the corresponding notations in 21 by a conjugation and an invertible scalar. The elements $n_{st}, g_{st}$ in the current paper should be identified with the elements $n_{st}, g_{st}$ in 21 up to some invertible scalar. In particular, our dual cellular basis $\{n_{st}\}$ use the partial order “$\leq$” while 21 use the partial order “$\geq$” for the dual cellular basis. Our convention for the notations $n_{st}$ in this paper agrees with the one used in 17 Section 3.

**Definition 3.8.** (21 §3) Suppose $\hat{q}, \hat{Q}_1, \ldots, \hat{Q}_\ell$ are indeterminates over $\mathbb{Z}$. Set $\mathcal{A} := \mathbb{Z}[\hat{q}^{\pm 1}, \hat{Q}_1, \ldots, \hat{Q}_\ell]$. Let $H := \mathbb{Q}(\hat{q}, \hat{Q}_1, \ldots, \hat{Q}_\ell)$ be the rational functional field on $\hat{q}, \hat{Q}_1, \ldots, \hat{Q}_\ell$. Let $\mathcal{H}_{\ell,n}(\hat{q}, \hat{Q})$ be the non-degenerate cyclotomic Hecke algebra of type $G(\ell, 1, n)$ over $\mathcal{A}$ with Hecke parameter $\hat{q}$ and cyclotomic parameters $\hat{Q} := (\hat{Q}_1, \ldots, \hat{Q}_\ell)$. Set $\mathcal{H}_{\ell,n}^\mathcal{A}(\hat{q}, \hat{Q}) := \mathcal{A} \otimes_{\mathcal{A}} \mathcal{H}_{\ell,n}(\hat{q}, \hat{Q})$. Then $\mathcal{H}_{\ell,n}^\mathcal{A}(\hat{q}, \hat{Q})$ is split semisimple. We set $'$ to be the unique ring involution of $\mathcal{H}_{\ell,n}(\hat{q}, \hat{Q})$ (21 §3) which is defined on generators by

$$T'_0 := T_0, \quad T'_i := -\hat{q}^{-1}T_i, \quad \hat{q}' := \hat{q}^{-1}, \quad \hat{Q}'_i := \hat{Q}_{i+1} - 1, \quad 1 \leq i < n, \quad 1 \leq j \leq \ell.$$

Clearly, $'$ naturally extends to a ring involution of $\mathcal{H}_{\ell,n}^\mathcal{A}(\hat{q}, \hat{Q})$. We have $L'_m = L_m$ for any $1 \leq m \leq n$, and $m_{st}' = n_{sv}'$, $(\text{res}_s(k))' = \text{res}_t(k)$ for any $1 \leq k \leq n$ by 21 (3.2)). It follows from Definition 2.7 that

$$(3.9) \quad F'_t = F_t, \quad f'_{st} = (F_s n_{st} F_t)' = F'_s n_{st} F'_t = F'_s n_{sv} F_v = g_{sv}. $$

For any rational function $f$ on $\hat{q}, \hat{Q}_1, \ldots, \hat{Q}_\ell$, we use $f'$ to denote the rational function obtained from $f$ by substituting $\hat{q}$ and $\hat{Q}$ (for $1 \leq i \leq \ell$) with $-\hat{q}^{-1}$ and $\hat{Q}_{i+1}$ respectively. By Definition 2.9 for each $t \in \text{Std}(\lambda)$, the scalar $\gamma_t$ is given by the evaluation of a rational function $\gamma_t(\hat{q}, \hat{Q}_1, \ldots, \hat{Q}_\ell)$ at $\hat{q} := q, \hat{Q}_i := Q_i, 1 \leq i \leq \ell$. Thus the notation

$$\gamma'_t := 1_K \otimes_{\mathcal{A}} \gamma_t(\hat{q}, \hat{Q}_1, \ldots, \hat{Q}_\ell) \in K^\mathcal{A}$$

does make sense.

Note that in general we have $\gamma'_t \neq \gamma_t$. For example, if $\ell = 1 = Q_1, \lambda = (2, 1)$, $t = t^1s_2$, then

$$\gamma_t = \frac{(q^2 - q^{-1})(q - 1)(1 + q)}{(q - q^{-1})^2}, \quad \gamma'_t = \frac{(q^2 - q)(q^{-1} - 1)(1 + q^{-1})}{(q - q^{-1})^2} \neq \gamma_t = 1 + q.$$

**Corollary 3.10.** Suppose $q \neq 1$, (2.3) holds and $R = K$ is a field. Then

$$(3.11) \quad \{g_{st} \mid s, t \in \text{Std}(\lambda), \lambda \in \mathcal{P}_n\}$$
is a basis of $\mathcal{H}_{\ell,n}(q, Q)$. Moreover,

1) if $s, t, u$ and $v$ are standard tableaux, then $g_{st}g_{uv} = \delta_{tu} \gamma'_t g_{sv}$;

2) if $\lambda \in \mathcal{P}_n$, $s, t \in \text{Std}(\lambda)$ and $1 \leq k \leq n$, then $g_{st}E_k = \text{res}_s(k)g_{st}, \quad L_k g_{st} = \text{res}_s(k)g_{st}$;

3) for each $\lambda \in \mathcal{P}_n$ and $t \in \text{Std}(\lambda)$, $F_t = g_{st}/\gamma'_t$.

**Proof.** This follows from (3.9) and Lemma 2.11. □

We call (3.11) the dual seminormal basis of $\mathcal{H}_{\ell,n}(q, Q)$ corresponding to the dual cellular basis $\{n_{st}| s, t \in \text{Std}(\lambda), \lambda \in \mathcal{P}_n\}$.

**Lemma 3.12.** (21 Remark 3.6) Let $\lambda$ be a multipartition of $n$ and $s, t \in \text{Std}(\lambda)$. Suppose $q \neq 1$, (2.4) holds and $R = K$ is a field. Then

1) For any standard tableau $t$, we have

$$g_{tv} = f'_{st} = \gamma'_t F_v = \frac{\gamma'_t}{\gamma'_t} f_{tv}. \quad \tag{7}$$
2) There exists a unique scalar $\alpha_{st} \in K^\times$ such that $g_{st} = \alpha_{st} f_{st}$. Moreover, $\alpha_{st}^2 = \gamma_t^{\alpha_{st}}/\gamma_t^{\gamma_t}$.

**Proof.** Part 1) follows from (3.9, Corollary 3.10 3) and Definition 3.6. For Part 2), on the one hand, combining Lemma 2.10 2), Corollary 2.11 2) with (2.6), we can deduce that $g_{st} = \alpha_{st} f_{st} \in K^\times$. On the other hand, applying the anti-involution “$^\ast$”, we can get that $g_{st} = \alpha_{st} f_{st}$ and hence $\alpha_{st} = \alpha_{st}$. Therefore,

$$
\gamma_t^{\alpha_{st}} g_{st} = g_{st} f_{st} = \alpha_{st}^2 \gamma_t^{\gamma_t} f_{st} = \alpha_{st}^2 \gamma_t^{\gamma_t} f_{st}^{\gamma_t}.
$$

By 1) we have $g_{st} f_{st}^{\gamma_t} = \gamma_t^{\gamma_t} / \gamma_t^{\gamma_t}$. Hence Part 2) of the lemma follows. 

**Remark 3.13.** Suppose that $q, Q_1, \cdots, Q_\ell$ are indeterminates over $\mathbb{Z}$. Then by (3.14)

$$
f_{st} = g_{st} = \alpha_{st} f_{st},
$$

for any $s, t \in \text{Std}(\lambda)$ and $\lambda \in \mathcal{P}_n$. Note that the scalar $\alpha_{st}$ in our paper should be identified with the scalar $\alpha_{st}$ in the notation of [21]. In view of our convention of notations, we have that $g_{st} = \alpha_{st} f_{st}$, while in view of the convention of notations in [21], we have $g_{st} = \alpha_{st} f_{st}$. It follows from Lemma 3.12 that $\gamma_t^{\alpha_{st}}/\gamma_t$ always has a square root in $K^\times$ which is a rational function on $q, Q_1, \cdots, Q_\ell$. In [21] Remark 3.6] Mathas has asked whether one can give an intrinsic explanation of this fact and in particular determine the sign of each scalar $\alpha_{st}$. In this paper we present some explicit combinatorial formulae for these scalars $\alpha_{st}$ as some rational functions on $q, Q_1, \cdots, Q_\ell$ and affirmatively answer Mathas’s above question.

For the reader’s convenience, we include below a lemma which gives a recursive formula for the $\gamma_t$-coefficients associated to the dual seminormal bases.

**Lemma 3.15.** Suppose $q \neq 1$ and (2.4) holds. Let $\lambda \in \mathcal{P}_n$. The coefficients of the dual seminormal basis $\{g_{st} | s, t \in \text{Std}(\lambda), \lambda \in \mathcal{P}_n\}$ can be uniquely determined by:

a) $\gamma_t^{\alpha_{st}} = \gamma_t^{\alpha_{st}} = q^{c^\lambda} \prod_{1 \leq c < \ell} \prod_{1 \leq j < \lambda_i} (q^{-1} Q_s - Q_t)$, where $C = - \sum_{c=1}^{\ell} \gamma_t^{\alpha_{st}} (\lambda_i^{(\gamma_t^{-1})} - 1)$; and

b) if $s = (i, i + 1) < t$ then

$$
\gamma_t^{\alpha_{st}} = q^{c^\lambda} \frac{(q res_a(i) - res_d(i)) (res_a(i) - q res_d(i)) (res_a(i) - q res_d(i))}{(res_a(i) - res_d(i))^2}.
$$

**Proof.** This follows from Definition 3.8 and the equality $(res_a(k)) = res_a(k)$. 

**Lemma 3.16.** ([21 Proposition 2.7]) Suppose $q \neq 1$, (2.4) holds and $R = K$ is a field. Let $\lambda \in \mathcal{P}_n$ and $s, u \in \text{Std}(\lambda)$. Let $i$ be an integer with $1 \leq i < n$ and $t := s(i, i + 1)$. If $t$ is standard then

$$
f_{us} T_i = \begin{cases} A_i(s) f_{us} + f_{ut}, & \text{if } t < s, \\ A_i(s) f_{us} + B_i(s) f_{ut}, & \text{if } s < t, \end{cases}
$$

where

$$
A_i(s) = \frac{(q - 1) res_a(i + 1)}{res_a(i + 1) - res_a(i)}, \quad B_i(s) := \frac{q res_d(i) - res_d(i + 1)}{res_d(i + 1) - res_d(i)^2}.
$$

If $t$ is not standard then

$$
f_{us} T_i = \begin{cases} q f_{us}, & \text{if } i \text{ and } i + 1 \text{ are in the same row of } s, \\ -f_{us}, & \text{if } i \text{ and } i + 1 \text{ are in the same column of } s. \end{cases}
$$

Let $\mathcal{H}_q(\mathcal{S}_n)$ be the Iwahori-Hecke algebra of the symmetric group $\mathcal{S}_n$, which can be identified with the $K$-subalgebra of $H_{t_i, n}(q, Q)$ generated by $T_1, \cdots, T_{n-1}$.

**Lemma 3.17.** ([23, 21 Proposition 4.1, Lemma 4.3]) Suppose $q \neq 1$, (2.4) holds and $R = K$ is a field. Let $\lambda \in \mathcal{P}_n$ and $i$ be an integer with $1 \leq i < n$. Then there exist invertible elements $\{\Phi_t | t \in \text{Std}(\lambda)\}$ in $\mathcal{H}_q(\mathcal{S}_n)$ such that

(i) for any $s, t \in \text{Std}(\lambda)$, $f_{st} = \Phi_s f_{t s} \Phi_t$;

(ii) $\Phi_{i t} = 1$ if $s := t(i, i + 1) < t$,

$$
\Phi_s = \Phi_t (T_i - A_i(t)).
$$

**Lemma 3.18.** Let $\lambda \in \mathcal{P}_n$ and $t \in \text{Std}(\lambda)$. Let $i$ be an integer with $1 \leq i < n$. Suppose $q, Q_1, \cdots, Q_\ell$ are indeterminates over $\mathbb{Z}$. If $s := t(i, i + 1) \in \text{Std}(\lambda)$ with $s < t$, then $\Phi_s = (-q)^{-1} \Phi_t (T_i - A_i(t))$. 

Proof. Recall the ring involution $'$ introduced in Definition 3.8 which is defined on generators by
\[ T_i' := T_i, \quad T_i' := -q^{-1}T_i, \quad q' := q^{-1}, \quad Q_j' := Q_{\ell-j+1}, \quad 1 \leq i < n, 1 \leq j \leq \ell. \]
It follows from Lemma 3.17 that $\Phi_a' = \Phi_a'(-q^{-1}T_i - A_i(t'))$. Thus it suffices to show that $A_i(t') = -q^{-1}A_i(t')$.

Since $q' = q^{-1}$, we have that
\[
A_i(t') = \frac{(q-1) \text{res}_t(i+1)}{\text{res}_t(i+1) - \text{res}_t(i)}(q^{-1} - 1) \text{res}_t(i) = \frac{(q^{-1} - 1) \text{res}_t(i+1)'}{\text{res}_t(i+1)' - \text{res}_t(i)'} = (-q^{-1})(q-1) \text{res}_t(i+1)' \frac{\text{res}_t(i+1) - \text{res}_t(i)}{\text{res}_t(i+1)' - \text{res}_t(i)'} = (-q)^{-1}A_i(t').
\]
This completes the proof of the lemma. □

Lemma 3.19. Let $\lambda \in \mathcal{P}_n$ and $t \in \text{Std}(\lambda)$. Suppose $q, Q_1, \ldots, Q_\ell$ are indeterminates over $\mathbb{Z}$. Then we have
\[
\Phi_\lambda \Phi_t = (-q)^{-\ell(d(t))} \frac{\gamma_{\lambda^t} \gamma_t^\lambda}{\gamma_t \gamma_{\lambda^t}} f_{\lambda^t} f_t.
\]

Proof. Recall that $d(t) \in \mathfrak{S}_n$ such that $t^d d(t) = t$. Fix a reduced expression $d(t) = s_{i_1} \cdots s_{i_l}$, where $1 \leq i_j < n$ for each $j$. For each $1 \leq k \leq \ell$, we define $w_k = s_{i_1} s_{i_2} \cdots s_{i_k}$ and set $t_k := t^k w_k$, $t_0 := t$. Then $t_l = t$. We get the following sequence of standard $\lambda$-tableaux:
\[
t^\lambda = t_0 \triangleright t_1 \triangleright t_2 \triangleright \cdots \triangleright t_l = t.
\]
Combining this with Lemmas 3.16 and 3.17 we get that
\[ (3.20) \quad \Phi_t \Phi_\lambda = (-q)^{-\ell(d(t))} f_{\lambda^t} f_t.
\]

Note that $t_{k-1} < t_k = t_{k-1}s_{i_k}$ for each $1 \leq k \leq l$. We get the following sequence of standard $\lambda'$-tableaux:
\[
t_{\lambda'} = t'_0 \triangleright t'_1 \triangleright t'_2 \triangleright \cdots \triangleright t'_l = t'.
\]
Applying Lemma 3.16, Lemma 3.17 and (3.20), we get that
\[
\Phi_t \Phi_\lambda = (-q)^{-\ell(d(t))} \frac{\gamma_{t_{\lambda'}} \gamma_{t_{\lambda}^t}}{\gamma_t \gamma_{t_{\lambda}}^t} f_{t_{\lambda'}} f_{t_{\lambda}}.
\]

Applying the anti-automorphism $*$ and noting that $*$ commutes with $'$, we get that
\[
\Phi_\lambda \Phi_t \Phi_\lambda^* = (-q)^{-\ell(d(t))} \frac{\gamma_{t_{\lambda'}} \gamma_{t_{\lambda}^t}}{\gamma_t \gamma_{t_{\lambda}}^t} f_{t_{\lambda'}} f_{t_{\lambda}}.
\]
This completes the proof of the lemma. □

The following result reveals some hidden relationship between $\gamma_t, \gamma_{t_{\lambda}}, \gamma_{t_{\lambda}^t}$ and $\gamma_{t_{\lambda}'}$.

Lemma 3.21. Suppose $q \neq 1$, (2.4) holds and $R = K$ is a field. Let $\lambda \in \mathcal{P}_n$ be a multipartition of $n$ and $t \in \text{Std}(\lambda)$. Then we have that
\[
\gamma_t \gamma_{t_{\lambda}'} = q^{-2\ell(d(t))} \gamma_{t_{\lambda}^t} \gamma_{t_{\lambda}}.
\]

Proof. Without loss of generality we can assume that $q, Q_1, \ldots, Q_\ell$ are indeterminates over $\mathbb{Z}$. As in the proof of Lemma 3.19 we fix a reduced expression $d(t) = s_{i_1} \cdots s_{i_l}$, where $1 \leq i_j < n$ for each $j$. For each $1 \leq k \leq l$, we define $w_k = s_{i_1} s_{i_2} \cdots s_{i_k}$ and set $t_k := t^k w_k$, $t_0 := t$. Then we get the following two sequences of standard tableaux:
\[
t^\lambda = t_0 \triangleright t_1 \triangleright t_2 \triangleright \cdots \triangleright t_l = t, \quad t_{\lambda'} = t'_0 \triangleright t'_1 \triangleright t'_2 \triangleright \cdots \triangleright t'_l = t'.
\]
By definition, we have
\[
\gamma_t = \frac{\gamma_{t_{\lambda}} \gamma_{t_{\lambda}^t}}{\gamma_{t_{\lambda}} \gamma_{t_{\lambda}^t}} \cdots \frac{\gamma_{t_{\lambda}} \gamma_{t_{\lambda}^t}}{\gamma_{t_{\lambda}} \gamma_{t_{\lambda}^t}}.
\]
Applying the ring involution $'$, we get that
\[
\gamma'_t = \gamma'_{t_{k}} \left( \frac{\gamma_{t_{k+1}}}{\gamma_{t_{k}}} \right)' \left( \frac{\gamma_{t_{k+2}}}{\gamma_{t_{k}}} \right)' \cdots \left( \frac{\gamma_{t_{l}}}{\gamma_{t_{l-1}}} \right)'
\]
For each $1 \leq k \leq l$, by Definition 2.9 and Definition 3.8 we have that
\[
\left( \frac{\gamma_{t_{k}}}{\gamma_{t_{k-1}}} \right)' = \left( \frac{q \text{res}_{t_{k-1}}(i_k) - \text{res}_{t_{k}}(i_k)}{\text{res}_{t_{k-1}}(i_k) - \text{res}_{t_{k}}(i_k)} \right)^{-1}
\]
Hence, we can get that
\[
\text{Combining this with (3.22), we can deduce that}
\]
\[
\alpha
\]
\[
\gamma_t = \gamma_t x \left( \frac{\gamma_{t_{k+1}}}{\gamma_{t_{k}}} \right)' \left( \frac{\gamma_{t_{k+2}}}{\gamma_{t_{k}}} \right)' \cdots \left( \frac{\gamma_{t_{l}}}{\gamma_{t_{l-1}}} \right)'
\]
It follows that $\gamma_t \gamma'_t = q^{-2\ell(d(t))} \gamma_{t_{k}} \gamma'_{t_{k}}$. This completes the proof of the lemma.

Let $\lambda \in \mathcal{P}$ and $s, t \in \text{Std}(\lambda)$. Recall that $\alpha_{st} \in K^*$ is an invertible scalar introduced in Lemma 3.12 such that $\alpha_{st} = \alpha_{st} f_{st}$. Now we can give the proof of the first main result of this paper which presents some explicit formulae for the scalar $\alpha_{st}$.

**Proof of Theorem 1.1** To prove the theorem, we can assume without loss of generality that $q, Q_1, \cdots, Q_\ell$ are indeterminate over $\mathbb{Z}$. In this case, we can use the ring involution $'$ of $\mathcal{H} q(\lambda) Q$ introduced in Definition 3.11. By Lemma 3.17, we have that $f_{s't'} = \Phi_{s'}^{f_{t'a}} \Phi_{t'}$. Applying the involution $'$ and using Lemma 3.12, we can get that
\[
(3.22) \quad g_{st} = f_{s't'} = (\Phi_{s'}^{f_{t'a}} \Phi_{t'})' = (\Phi_{s'}')^{f_{t'a}} \Phi_{t'} = \frac{\gamma_t x}{\gamma_t} (\Phi_{s'}')^{f_{t'a}} \Phi_{t'}
\]
Applying Lemma 3.19, we can deduce that
\[
(\Phi_{s'}')^{f_{t'a}} \Phi_{t'} = (-q)^{-\ell(d(s'))} \gamma_{s'} \gamma_{t'} f_{t'a} \Phi_{t'}
\]
\[
= (-q)^{-\ell(d(s'))} \frac{1}{\gamma_{s'}} f_{t'a} \gamma_{t'} \Phi_{t'}
\]
\[
= (-q)^{-\ell(d(s')) - \ell(d(t'))} \gamma_{t'} \gamma_{t'} f_{t'a} \Phi_{t'} = (-q)^{-\ell(d(s')) - \ell(d(t'))} \frac{\gamma_t x}{\gamma_t} f_{st}
\]
Combining this with (3.22), we can deduce that
\[
\theta_{st} = (-q)^{-\ell(d(s')) - \ell(d(t'))} \frac{\gamma_t x}{\gamma_t} f_{st}
\]
Hence $\alpha_{st} = (-q)^{-\ell(d(s')) - \ell(d(t'))} \gamma_{t_{k}} \gamma'_{t_{k}}$. This proves the first equality of the theorem.

Finally, by Lemma 3.21, we have that
\[
\frac{\gamma_{t_{k}}}{\gamma_{s}} = q^{2\ell(d(s'))} \gamma'_{t_{k}} \gamma_{t_{k}}, \quad \frac{\gamma'_{t_{k}}}{\gamma_{t_{k}}} = q^{2\ell(d(t))} \gamma'_{t_{k}} \gamma_{t_{k}}
\]
It follows that
\[
(-q)^{-\ell(d(s')) - \ell(d(t'))} \frac{\gamma_{t_{k}}}{\gamma_{s}} = (-q)^{-\ell(d(s')) + \ell(d(t'))} \frac{\gamma'_{t_{k}}}{\gamma_{t_{k}}}
\]
which proves the second equality of the theorem.

Let $i : \mathcal{H}_{t-1}(q, Q) \hookrightarrow \mathcal{H}_{t}(q, Q)$ be the natural inclusion which is defined on generators by $i(T_i) := T_i$ for $0 \leq i < n - 1$. In order to avoid the confusion between the notations for $\mathcal{H}_{t-1}(q, Q)$ and
\( \mathcal{H}_n(q, Q) \). We add a superscript \((n)\) to indicate that it is the notation for \( \mathcal{H}_n(q, Q) \). Let \( \mu \in \mathcal{P}_{n-1} \) and \( s, t \in \text{Std}(\lambda) \). Then we have

\[
\beta_{st}^{(n-1)} = \sum_{\lambda \in \mathcal{P}_{n-1}} \sum_{u, v \in \text{Std}(\lambda)} \beta_{uv} \beta_{uv}^{(n-1)},
\]

where \( \beta_{uv}^{(n-1)} \in K \) for each pair \((u, v)\). In the rest of this section, we shall give some explicit formulae for these scalars \( \beta_{uv}^{(n)} \).

**Lemma 3.28.** Suppose \( q \neq 1 \), \( (2.4) \) holds and \( R = K \) is a field. Let \( \mu \in \mathcal{P}_{n-1}, \lambda \in \mathcal{P}_n \), and \( s, t \in \text{Std}(\mu) \). Then

1. \( \beta_{st}^{(n)} \neq 0 \) only if \( u \downarrow_{n-1} = s \) and \( v \downarrow_{n-1} = t \);
2. \( \beta_{ss}^{(n)} \neq 0 \) if and only if \( u \downarrow_{n-1} = s \). In that case, \( \beta_{ss}^{(n)} = \gamma_{ss}^{(n-1)} / \gamma_s^{(n)} \).

**Proof.** Without loss of generality we can assume \( n \geq 2 \). Suppose that \( \beta_{uv}^{(n)} \neq 0 \). Then the equalities \( u \downarrow_{n-1} = s \) and \( v \downarrow_{n-1} = t \) follows from \((2.4)\) and Lemma \( 2.10 \) by considering the left and the right actions of \( L \) for \( 1 \leq m \leq n \). This proves the part a) of the lemma.

Let \( \mu \in \mathcal{P}_{n-1} \) and \( s \in \text{Std}(\mu) \). By the part a) of the lemma, we can write

\[
\beta_{ss}^{(n-1)} = \sum_{\lambda \in \mathcal{P}_{n-1}} \sum_{u, v \in \text{Std}(\lambda)} \beta_{uv} \beta_{uv}^{(n-1)}.
\]

Suppose \( \beta_{ss}^{(n)} \neq 0 \). Then by Lemma \( 3.24 \), we see that \( u \downarrow_{n-1} = s = v \downarrow_{n-1} \). Since \( \text{Shape}(u) = \text{Shape}(v) \), it follows that \( u = v \). Therefore, we can get that

\[
\beta_{ss}^{(n-1)} = \sum_{\lambda \in \mathcal{P}_{n-1}} \sum_{u, v \in \text{Std}(\lambda)} \beta_{uv} \beta_{uv}^{(n-1)}.
\]

Since \( \beta_{ss}^{(n-1)} \) is a primitive idempotent and \( \{\beta_{uv}^{(n-1)} / \gamma_s^{(n)}\} \) is a complete set of pairwise orthogonal primitive idempotents in \( \mathcal{H}_n(q, Q) \), it follows that \( \sum_{u \in \text{Std}(\lambda)} \beta_{uv} \beta_{uv}^{(n-1)} = \beta_{ss}^{(n-1)} \) if and only if \( u \downarrow_{n-1} = s \). Thus

\[
\beta_{ss}^{(n-1)} = \gamma_s^{(n-1)} / \gamma_s^{(n)}.
\]

On the other hand, we have that

\[
\sum_{\lambda \in \mathcal{P}_{n-1}} \sum_{u, v \in \text{Std}(\lambda)} \beta_{uv} \beta_{uv}^{(n-1)} = 1.
\]

Combining this with \((2.4)\), \( (2.27) \), and the equality \( \beta_{ss}^{(n-1)} = \beta_{ss}^{(n-1)} / \gamma_s^{(n)} \), we can deduce that \( \beta_{ss}^{(n)} \) is a removable node of \( \lambda \).

Let \( \lambda \in \mathcal{P}_n \) and \( \alpha \in [\lambda] \). If \( [\lambda] \setminus \{\alpha\} \) is again the Young diagram of a multipartition, then we say that \( \alpha \) is a removable node of \( \lambda \).

**Lemma 3.28.** Let \( \mu \in \mathcal{P}_{n-1}, \lambda \in \mathcal{P}_n \) such that \( \mu = \lambda \setminus \{\alpha\} \) for some removable node \( \alpha \) of \( \lambda \). Let \( s, t \in \text{Std}(\mu) \), \( u, v \in \text{Std}(\lambda) \). If \( u \downarrow_{n-1} = s \), then we have

\[
\gamma_s^{(n-1)} / \gamma_u^{(n)} = \gamma_t^{(n-1)} / \gamma_v^{(n)}.
\]
By the inductive definition of the $\gamma$-coefficients given in Definition 2.10, we can deduce that for all $1 \leq k \leq m$,
\[
\frac{\gamma_k^{(n-1)}}{\gamma_k^{(n-1)}} = \frac{\gamma_k^{(n)}}{\gamma_k^{(n)}}.
\]
It follows that
\[
\frac{\gamma_k^{(n-1)}}{\gamma_k^{(1)}} = \frac{\gamma_k^{(n-1)}}{\gamma_k^{(1)}} = \frac{\gamma_k^{(n)}}{\gamma_k^{(1)}}.
\]
This completes the proof of the lemma.

**Proof of Theorem 1.3** Recall the invertible elements \{\Phi_t|t \in \text{Std}(\mu)\} of $\mathcal{H}_{q}(\mathfrak{S}_{m-1})$ defined in Lemma 3.17. By Lemma 3.17, for $s, t \in \text{Std}(\mu)$, \[\Phi_s^{(n)} = \Phi_s^{(n-1)} \Phi_t.\] Applying Lemma 3.24 we get that
\[
f_{s,t}^{(n-1)} = \Phi_s^{(n-1)} \Phi_t.\]

Let $\lambda \in \mathfrak{P}_n$. Note that $d(s), d(t) \in \mathfrak{S}_{m-1}$. For any $a \in \text{Std}(\lambda)$ satisfying $a \downarrow_{m-1} = t^a$, it is clear that $ad(s), ad(t) \in \text{Std}(\lambda)$ and $ad(s) \downarrow_{m-1} = s, ad(t) \downarrow_{m-1} = t$. Therefore, it follows from the definitions of $\Phi_s, \Phi_t$ and Lemma 3.16 that $\Phi_s^{(n)} \Phi_t = f_{s,t}^{(n)}$. Thus we have that
\[
f_{s,t}^{(n-1)} = \Phi_s^{(n-1)} \Phi_t = \Phi_s^{(n-1)} \left( \sum_{\lambda \in \mathfrak{P}_n} \sum_{a \in \text{Std}(\lambda)} \gamma_{a}^{(n)} f_{a,s}^{(n-1)} \right) \Phi_t.\]
(B by Lemma 3.21)
\[
= \sum_{\lambda \in \mathfrak{P}_n} \sum_{a \in \text{Std}(\lambda)} \gamma_{a}^{(n)} f_{a,s}^{(n-1)} f_{a,t}^{(n-1)}.
\]
\[
= \sum_{\lambda \in \mathfrak{P}_n} \sum_{u, v \in \text{Std}(\lambda)} \gamma_{u}^{(n)} f_{u,v}^{(n-1)} f_{v,u}^{(n-1)}.
\]
where the last equality follows because $ad(s)$ is the unique $u \in \text{Std}(\lambda)$ satisfying $u \downarrow_{m-1} = s$, and $ad(t)$ is the unique $v \in \text{Std}(\lambda)$ satisfying $v \downarrow_{m-1} = t$. Since $\frac{\gamma_{a}^{(n)}}{\gamma_{a}^{(n)}} \in K^\times$, the above equality also implies that $\beta_{s,t}^{(n)} \neq 0$ if and only if $u \downarrow_{m-1} = s, v \downarrow_{m-1} = t$.

Finally, combining the above equality and Lemma 3.25, we can deduce that
\[
\beta_{s,t}^{(n)} = \frac{\gamma_{u}^{(n)}}{\gamma_{u}^{(n)}} = \frac{\gamma_{v}^{(n)}}{\gamma_{v}^{(n)}} = \frac{\gamma_{t}^{(n)}}{\gamma_{t}^{(n)}}.
\]
This completes the proof of the theorem.

4. **The degenerate case**

Let $u = (u_1, \ldots, u_r)$, where $u_1, \ldots, u_r \in K$. Let $H_{e,n}(u)$ be the degenerate cyclotomic Hecke algebra over $R$ with cyclotomic parameters $u_1, \ldots, u_r$. The purpose of this section is to give a proof of Theorem 1.3 and Theorem 1.0. The argument of the proof is similar to the non-degenerate case. Throughout this section, we shall assume \[\text{2.12}\] holds. In particular, \[\text{2.13}\] holds and $H_{e,n}(u)$ is semisimple over $K$.

Let $\lambda \in \mathfrak{P}_n$. For any $t = (t^{(1)}, \ldots, t^{(r)}) \in \text{Std}(\lambda)$ and any $1 \leq k \leq n, u$ define
\[
c_t(k) = j - i + u c_i, \quad \text{if } k \text{ appears in row } i \text{ and column } j \text{ of } t^{(c)}
\]
We also define $C(k) := \{c_t(k) \mid t \in \text{Std}(\lambda), \lambda \in \mathfrak{P}_n\}$. 

Definition 4.1. ([5 Definition 6.7]) Suppose (2.12) holds and \( R = K \) is a field. Let \( \lambda \in \mathcal{P}_n \) and \( t \in \text{Std}(\lambda) \). We define

\[
F_t = \prod_{k=1}^{n} \prod_{s \in C(k)} \frac{L_k - c}{c_t(k) - c}
\]

Definition 4.2. ([5 Lemma 6.10]) Suppose (2.12) holds and \( R = K \) is a field. Let \( \lambda \in \mathcal{P}_n \). The \( r \)-coefficients \( \{r_t | t \in \text{Std}(\lambda), \lambda \in \mathcal{P}_n \} \) are defined to be a multiset of invertible scalars in \( K^* \) which are uniquely determined by:

a) \( r_{t\lambda} = \left( \prod_{s=1}^{t} \prod_{i=1}^{\lambda^r(s)} (c_t(i) - c_s(i) + 1)(c_t(i) - c_s(i) - 1) \right) \prod_{1 \leq s \leq t \leq n} (j - i + u_s - u_t) \); and

b) if \( s = t(i, i + 1) \rightarrow t \) then

\[
\frac{r_t}{r_s} = \frac{(c_t(i) - c_s(i) + 1)(c_t(i) - c_s(i) - 1)}{(c_t(i) - c_s(i))^2}
\]

Let \( \{m_{st} | s, t \in \text{Std}(\lambda), \lambda \in \mathcal{P}_n \} \) be the cellular basis of \( H_{t,n}(u) \) introduced in Section 2. Let \( \{f_{st} | s, t \in \text{Std}(\lambda), \lambda \in \mathcal{P}_n \} \) be the corresponding seminormal basis of \( H_{t,n}(u) \). For each \( \lambda \in \mathcal{P}_n \), we define

\[
H_{t,n}^{\lambda} := \text{Span}_R \{m_{st} | s, t \in \text{Std}(\mu), \lambda \rightarrow \mu \in \mathcal{P}_n \}
\]

which is a cell ideal of \( H_{t,n}(u) \) with respect to the cellular basis. For any \( 1 \leq k \leq n \) and \( s, t \in \text{Std}(\lambda) \), we have that

\[
m_{st}L_k = c_t(k)m_{st} + \sum_{v \in \text{Std}(\lambda) \atop v < t} a_v m_{sv} \pmod{H_{t,n}^{\lambda}}
\]

where \( a_v \in K \) for each \( t \leq v \in \text{Std}(\lambda) \).

The dual seminormal basis of \( H_{t,n}(u) \) can be constructed in the same manner as that of the non-degenerate cyclotomic Hecke algebras \( \mathcal{H}_{t,n}(q, Q) \). First, we recall the construction of the dual cellular basis for \( H_{t,n}(u) \). Let \( \lambda \in \mathcal{P}_n \). We define

\[
m_{\lambda t \lambda} := (-1)^{n(\lambda)} \left( \prod_{w \in \mathcal{S}_\lambda} (-1)^{f'(w)} w \right) \left( \prod_{k=1}^{n} \prod_{s=2}^{t} \prod_{1 \leq j \leq n} (L_k - u_{t-s+1}) \right)
\]

where \( n(\lambda) := \sum_{i=1}^{t} (i - 1)|\lambda^r(i)| \).

Recall that for any \( t \in \text{Std}(\lambda) \), \( d'(t) \in \mathcal{G}_n \) is such that \( t \lambda d'(t) = t \). For any \( s, t \in \text{Std}(\lambda) \), we define

\[
n_{st} := (-1)^{(d'(s) - t)(d'(t))} d'(s)^{-1} m_{\lambda t \lambda} d'(t).
\]

Then, with respect to the poset \( (\mathcal{P}_n, \leq) \) and the anti-involution \( \ast^* \), \( \{n_{st} | s, t \in \text{Std}(\lambda), \lambda \in \mathcal{P}_n \} \) forms another cellular basis of \( H_{t,n}(u) \). We call it the dual cellular basis of \( H_{t,n}(u) \).

For each \( \lambda \in \mathcal{P}_n \), we define

\[
\tilde{H}_{t,n}^{\lambda} := K \text{-Span}\{n_{st} | s, t \in \text{Std}(\mu), \lambda \rightarrow \mu \in \mathcal{P}_n \}
\]

which is a cell ideal of \( H_{t,n}(u) \) related to the dual cellular basis. For any \( 1 \leq k \leq n \) and \( s, t \in \text{Std}(\lambda) \), we have that

\[
n_{st}L_k = c_t(k)n_{st} + \sum_{v \in \text{Std}(\lambda) \atop v < t} b_v n_{sv} \pmod{\tilde{H}_{t,n}^{\lambda}}
\]

where \( b_v \in K \) for each \( t \rightarrow v \in \text{Std}(\lambda) \).

Definition 4.7. Suppose (2.12) holds and \( R = K \) is a field. Let \( \lambda \in \mathcal{P}_n \). For any \( s, t \in \text{Std}(\lambda) \), we define

\[
g_{st} := F_t n_{st} F_t
\]

Definition 4.8. ([27 Definition 2.9]) Suppose that \( \hat{u}_1, \ldots, \hat{u}_\ell \) are indeterminates over \( Z \). We set \( \hat{\mathcal{H}} := \mathbb{Z}[\hat{u}_1, \ldots, \hat{u}_\ell] \) and \( \hat{\mathcal{H}}_d := \mathbb{Q}(\hat{u}_1, \ldots, \hat{u}_\ell) \). Let \( H_{t,n}(u) \) be the degenerate cyclotomic Hecke algebra of type \( G(\ell, 1, n) \) over \( \hat{\mathcal{H}}_d \) with cyclotomic parameters \( \hat{u} := (\hat{u}_1, \ldots, \hat{u}_\ell) \). Set \( H_{t,n}^{\hat{\mathcal{H}}_d}(u) := H_{t,n}(u) \otimes_{\hat{\mathcal{H}}_d} H_{t,n}(u) \). It is clear that \( H_{t,n}^{\hat{\mathcal{H}}_d}(u) \) is semisimple. In this case, we set \( \ell' \) to be the unique ring involution of \( H_{t,n}(u) \) which is defined on generators by

\[
\delta_i := -\delta_i, \quad L_m := -L_m, \quad \delta_i' := -\delta_{i+j+1}, \quad 1 \leq i < n, 1 \leq m \leq n, 1 \leq j \leq \ell.
\]
Clearly, ′ naturally extends to a ring involution of $H^{\mathcal{F}}$. In particular, in this case $\mathcal{F}' = \mathcal{F}$, $(c_1(k))' = -c_1(k)$ for any $1 \leq k \leq n$. It follows from Definition 4.13 that

\begin{equation}
F_i' = F_i', \quad F_k' = (F_k^{'\mathcal{F}})^{\mathcal{F}'} = F_k^{\mathcal{F}'} = F_k', \quad F_{s'} = F_{s'}, \quad F_{\mathcal{F}'}. \quad \text{(4.9)}
\end{equation}

For any rational function $f$ on $\hat{u}_1, \ldots, \hat{u}_\ell$, $f'$ is the rational function obtained from $f$ by substituting $\hat{u}_i$ with $-\hat{u}_{i-1}$ for each $1 \leq i \leq \ell$

By Definition 2.9, for each $\lambda \in \mathcal{P}_n$, $f_\lambda$ is given by the evaluation of a rational function $r_\lambda(\hat{u}_1, \ldots, \hat{u}_\ell)$ at $\hat{u}_i := u_i$, $1 \leq i \leq \ell$. Thus the notation $r_\lambda(\hat{u}_1, \ldots, \hat{u}_\ell)$ makes sense.

Remark 4.10. Note that our notations $n_{st}, g_{st}$ differ with the corresponding notations in [27] by a conjugation. Namely, the readers should identify the elements $n_{st}, g_{st}$ in the current paper with the elements $n_{st'}, g_{st'}$ in [27]. In particular, our dual cellular basis $\{n_{st}\}$ use the partial order $\leq$, while $\{n_{st}\}$ use the opposite partial order $\geq$ for the dual cellular basis.

The following corollary and lemma can be proved in a similar way as in the non-degenerate case.

Corollary 4.11. Suppose (2.12) holds and $R = K$ is a field. Then

\begin{equation}
\{g_{st} \mid s, t \in \text{Std}(\lambda), \lambda \in \mathcal{P}_n\}
\end{equation}

is a basis of $H_{\mathcal{F}}(\mathcal{U})$. Moreover,

1) if $s, t, u$ and $v$ are standard tableaux, then $g_{st}g_{uv} = \delta_{uv}r'_t g_{sv}$;

2) if $\lambda \in \mathcal{P}_n$, $s, t \in \text{Std}(\lambda)$ and $1 \leq k \leq n$, then $g_{st}L_k = c_k(k)g_{st}$, $L_kg_{st} = c_k(k)g_{st}$;

3) for each $\lambda \in \mathcal{P}_n$ and $t \in \text{Std}(\lambda)$, $F_t = g_{st}/r'_t$.

We call (4.12) the dual seminormal basis of $H_{\mathcal{F}}(\mathcal{U})$ corresponding to the dual cellular basis $\{n_{st} \mid s, t \in \text{Std}(\lambda), \lambda \in \mathcal{P}_n\}$.

Note that in general we have $r'_t \neq r_t$. For example, if $\ell = 1, u_1 = 0, \lambda = (2, 1), t = t^3 s_2$, then

\begin{equation}
r_1 = \frac{3 \cdot 1 \cdot 2}{2}, \quad r'_1 = \frac{3 \cdot 1 \cdot 2}{2} \neq r_1 = 2.
\end{equation}

Lemma 4.13. Suppose (2.12) holds and $R = K$ is a field. Suppose that $\lambda$ is a multipartition of $n$ and $s, t \in \text{Std}(\lambda)$.

1) For any standard tableau $t$, we have

\begin{equation}
g_{st} = f_{st}^{\mathcal{F}'} = r'_t F_t = \frac{r'_t}{r_t} F_t.
\end{equation}

2) There exists $a_{st} \in K^\times$ such that $g_{st} = a_{st}f_{st}$. Moreover, $a_{st}^2 = r'_t r'_t / r_t r_t$.

For the reader’s convenience, we include below a lemma which gives a recursive formula for the $r^t$-coefficients associated to the dual seminormal bases.

Lemma 4.14. Suppose (2.12) holds. Let $\lambda \in \mathcal{P}_n$. We define a multiset of elements $\{r_t^{(n)} \in K^\times \mid t \in \text{Std}(\lambda), \lambda \in \mathcal{P}_n\}$ in $K^\times$ as follows:

a) $r_t^{(n)} = r_t^{(\lambda)} = (-1)^\frac{1}{2} (\prod_{i \geq 1} (\prod_{1 \leq c < f, j \leq \lambda^{(n)}_c}) (j - i + u_s - u_t))$, where

\begin{equation}
C = \sum_{1 < s, t \geq 1} (s - 1) \lambda^{(n)}_c;
\end{equation}

and

b) if $s = t(i, i + 1) \triangleleft t$ then

\begin{equation}
r_t^{(n)} = \frac{(1 + c_t(i) - c_s(i))(c_t(i) - c_s(i) - 1)}{(c_t(i) - c_s(i))^2} = \frac{r_s}{r_t}.
\end{equation}

Proof. This follows from Definition 4.13 and the equality $(c_t(k))' = c_v(k)$. \qed

Lemma 4.15. ([16] Lemma 3.8) Suppose (2.12) holds and $R = K$ is a field. Let $\lambda \in \mathcal{P}_n$ and $s, u \in \text{Std}(\lambda)$. Let $i, m$ be integers with $1 \leq i < n, 1 \leq m < n$ and $t := s(i, i + 1)$. If $t$ is standard then

\begin{equation}
f_{us} s_i = \begin{cases} a_t(s) f_{us} + f_{ut}, & \text{if } t \lessdot s, \\ a_t(s) f_{us} + b_t(s) f_{ut}, & \text{if } s \lessdot t, \end{cases}
\end{equation}

\begin{equation}
f_{us} L_m = c_t(m) f_{us},
\end{equation}

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Then there exist a family of invertible elements $s, a, b, c, t$ where $s$ are in the same row of $a$, if $i$ and $i + 1$ are in the same column of $a$.

The following lemma can be proved in the same way as the proof of [31, Proposition 4.1, Lemma 4.3].

**Lemma 4.16.** Suppose (2.12) holds and $R = K$ is a field. Let $\lambda \in \mathcal{P}_n$ and $i$ an integer with $1 \leq i < n$. Then there exist a family of invertible elements $\{\phi_i | t \in \text{Std}(\lambda)\}$ in $K[\mathcal{G}_n]$ such that

(i) for any $s, t \in \text{Std}(\lambda)$, $f_{st} = \phi_s^t f_{t,s} \phi_t$;

(ii) $\phi_{t^\alpha} = 1$, and if $s := (i, i + 1) < t$, then $\phi_s = \phi_t(s_i - a_i(t))$.

**Lemma 4.17.** Suppose (2.7.2) holds and $R = K$ is a field. Let $\lambda \in \mathcal{P}_n$ and $t \in \text{Std}(\lambda)$. Let $i$ be an integer with $1 \leq i < n$. Suppose that $u_1, \ldots, u_\ell$ are indeterminates over $Z$. If $s := t(i, i + 1) \in \text{Std}(\lambda)$ with $s \triangleleft t$, then

$$\phi'_s = -\phi'_t(s_i - a_i(t)).$$

**Proof.** To prove the lemma, we can assume without loss of generality that $u_1, \ldots, u_\ell$ are indeterminates over $Z$. In this case, we can use the ring involution $\mathcal{I}$ introduced in [31, §3] which is defined on generators by

$s'_i := -s_i, L'_m := -L_m, u'_j := -u_{\ell - j + 1}, \forall 1 \leq i < n, 1 \leq m \leq n, 1 \leq j \leq \ell.$

By definition, $\phi'_s = \phi'_t(s_i - a_i(t'))$. Thus it suffices to show that $a_i(t') = -a_i(t')$. By definition, we have that $(c_i(k))' = -c_i(k)$. It follows that

$$a_i(t') = \left(\frac{1}{c_i(i + 1) - c_i(i)}\right)' = -\frac{1}{c_i(i) - c_i(i + 1)}$$

for $1 \leq k \leq n$. Hence, we can get that $a_i(t') = \frac{1}{c_{i + 1}(i + 1) - c_i(i)} = a_i(t')$. This completes the proof of the lemma.

The following lemma can be proved by using Lemma 4.17 and a similar argument used in the proof of Lemma 3.18.

**Lemma 4.18.** Suppose (2.7.2) holds and $R = K$ is a field. Let $\lambda \in \mathcal{P}_n$ and $t \in \text{Std}(\lambda)$. Suppose that $u_1, \ldots, u_\ell$ are indeterminates over $Z$. Then we have

$$f_{t^\alpha} = (-1)^{\ell(d(t))}\frac{r_{t^\alpha}}{r_t} f_{t, t^\alpha}.$$  

Let $\mu \in \mathcal{P}_{n-1}$ and $s, t \in \text{Std}(\mu)$. By (3.1), we have

$$r_{st} = \sum_{\lambda \in \mathcal{P}_n} \sum_{u, v \in \text{Std}(\lambda)} b_{uv}^{st} f_{uv}^{(n)}$$

where $b_{uv}^{st} \in K$ for each pair $(u, v)$.

Replacing the $\gamma$-coefficients and the element $\Phi_t$ of $\mathcal{H}_{\ell,n}(\mathbb{Q})$ with the $r$-coefficients and the element $\phi_t$ of $H_{t,n}(\mathbb{Q})$, the following lemmas can be proved in the same way as the proof of Lemmas 3.21, 3.23.

**Lemma 4.19.** Suppose (2.7.2) holds and $R = K$ is a field. Let $\lambda \in \mathcal{P}_n$ be a multipartition of $n$ and $t \in \text{Std}(\lambda)$. Then we have that

$$r_t r'_t = r_{t^\alpha} r'_{t^\alpha}.$$  

**Lemma 4.20.** Suppose (2.7.2) holds and $R = K$ is a field. Let $\mu \in \mathcal{P}_{n-1}, \lambda \in \mathcal{P}_n$, and $s, t \in \text{Std}(\mu), u, v \in \text{Std}(\lambda)$. Then

$$a) \quad b_{uv}^{st} \neq 0 \text{ only if } u \downarrow_{n-1} = s \text{ and } v \downarrow_{n-1} = t;$$

$$b) \quad b_{uv}^{st} \neq 0 \text{ if and only if } u \downarrow_{n-1} = s. \text{ In that case, } b_{uv}^{st} = r_{uv}^{(n-1)} r_{tu}^{(n)}. $$

**Lemma 4.21.** Let $\mu \in \mathcal{P}_{n-1}, \lambda \in \mathcal{P}_n$ such that $\mu = \lambda \setminus \{\gamma\}$ for some removable node $\gamma$ of $[\lambda]$. Let $s, t \in \text{Std}(\mu), u, v \in \text{Std}(\lambda)$. If $u \downarrow_{n-1} = s$, then we have

$$\frac{r_{s}^{(n-1)}}{r_{u}^{(n)}} = \frac{r_{s}^{(n)}}{r_{a}^{(n)}}.$$  

where $a \in \text{Std}(\lambda)$ is the unique standard $\lambda$-tableau such that $a \downarrow_{n-1} = t^\mu$. 

15
Proof of Theorems 1.4, 1.6: Theorem 1.4 follows from Lemmas 4.18, 4.18 and a similar argument used in the proof of Theorem 1.1. Theorem 1.6 follows from Lemmas 4.20, 4.21 and a similar argument used in the proof of Theorem 1.3. □

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