Infrared resummation for derivative interactions in de Sitter space

Hiroyuki Kitamoto

Physics Division, National Center for Theoretical Sciences
National Tsing-Hua University, Hsinchu 30013, Taiwan

Abstract

In de Sitter space, scale invariant fluctuations give rise to infrared logarithmic corrections to physical quantities, which spoil perturbation theories eventually. For non-derivative interactions, it has been known that the field equation reduces to a Langevin equation with a white noise in the leading logarithm approximation. The stochastic equation allows us to evaluate the infrared effects nonperturbatively. We extend the resummation formula as it is applicable also in models with derivative interactions. We first consider the nonlinear sigma model, and next consider a more general model consisting of a non-canonical kinetic term and a potential term. The stochastic equations derived from the infrared resummation in these models can be understood as generalizations of the standard one to curved target spaces.
1 Introduction

In de Sitter (dS) space, the propagator for a massless and minimally coupled scalar field has a dS symmetry breaking term. This term is a direct consequence of the scale invariant spectrum at the superhorizon scale and is expressed as a logarithm of the scale factor of the Universe \cite{1-3}. In the presence of the scalar field, physical quantities may acquire time dependences through the propagator. In tribute to their origin, we call them quantum infrared (IR) effects in dS space.

By employing the Schwinger–Keldysh formalism \cite{4,5}, we can investigate interacting field theories in dS space perturbatively. The IR effects at each loop level manifest as polynomials in the IR logarithm whose degrees increase with the loop level. At a late time, the leading IR effects come from the leading IR logarithms at each loop level. This fact indicates that the perturbative investigation breaks down after a large enough cosmic expansion. In order to understand such a situation, we need to evaluate the IR effects nonperturbatively.

For non-derivative interactions, A. A. Starobinsky and J. Yokoyama proposed that the IR dynamics can be described nonperturbatively by a Langevin equation with a white noise \cite{6,7}. N. C. Tsamis and R. P. Woodard proved that the stochastic approach is equivalent to the resummation of the leading IR logarithms to all-loop orders \cite{8}.

In this paper, we extend the resummation formula of the leading IR logarithms as it is applicable also in models with derivative interactions. As specific models, we consider the nonlinear sigma model and a more general model consisting of a non-canonical kinetic term and a potential term. Although there has been a study to derive stochastic equations in these models \cite{9}, the consistency with the resummation of the leading IR logarithms has not been verified completely.

We show that even though the kinetic term is non-canonical, the Yang–Feldman formalism reduces to a Langevin equation with a white noise in the leading logarithm approximation. The resulting stochastic equation is expressed as a covariant form with respect to the field coordinate transformation. As a specific feature of derivative interactions, we need to take into account the contribution from the subhorizon scale in evaluating the leading IR effects. This fact can be confirmed also in our previous studies about the energy-momentum tensor of the nonlinear sigma model \cite{10,11}.

The organization of this paper is as follows. In Sec. 2, we review the propagator for a massless and minimally coupled scalar field in dS space, in particular its IR behavior. In Sec. 3, we derive a Langevin equation for the nonlinear sigma model by applying the leading logarithm approximation to its Yang–Feldman formalism. In Sec. 4, we apply the same approach to a general scalar field theory consisting of a non-canonical kinetic term and a potential term. We also refer to an association with the Euclidean field theory on sphere. We conclude discussions in Sec. 5.
2 Free scalar field theory

Here we review a free scalar field which is massless and minimally coupled to the dS background. In particular, we focus on its IR dynamics which is a source of time dependent quantum effects.

In the Poincaré coordinate, the metric of dS space is given by

\[ ds^2 = -dt^2 + a^2(t)d\mathbf{x}^2, \quad a(t) = e^{Ht}, \]

where the dimension of the spacetime is taken as an arbitrary \( D \) and \( H \) is the Hubble constant. In the conformally flat coordinate,

\[ g_{\mu\nu} = a^2(\tau)\eta_{\mu\nu}, \quad a(\tau) = \frac{1}{-H\tau}. \]

The conformal time \( \tau \) is related to the cosmic time \( t \) as \( \tau = -\frac{1}{H}e^{-Ht} \). In this paper, \( D \)-dimensional vectors and tensors are expressed in the conformally flat coordinates.

The quadratic action for a massless and minimally coupled scalar field is

\[ S = \int \sqrt{-g}d^Dx \left[ -\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi \right]. \]

The free scalar field \( \phi_0 \) can be expanded by the annihilation and the creation operators \( a_p, a_p^\dagger \) as

\[ \phi_0(x) = \int \frac{d^{D-1}p}{(2\pi)^{D-1}} \left[ \phi_p(x)a_p + \phi_p^*(x)a_p^\dagger \right], \]

\[ \phi_p(x) = \sqrt{\frac{\pi}{2}}H^{\frac{D-2}{2}}(-\tau)^\frac{D-3}{2}H_{\frac{D-1}{2}}^{(1)}(-p\tau)e^{ip\cdot x}, \]

where \( p = |p| \) and \( H_{\frac{D-1}{2}}^{(1)} \) is the Hankel function of the first kind. We consider the Bunch–Davies vacuum \( a_p|0\rangle = 0 \) in this paper. It should be noted that \( p \) denotes the comoving momentum and the physical momentum is given by \( P = p/a(\tau) \).

In this paper, we focus on the contribution from the superhorizon scale where curved spacetime-specific effects are dominant. At the superhorizon scale: \( P \ll H \Leftrightarrow -p\tau \ll 1 \), the wave function behaves as

\[ \phi_p(x) \simeq -i\frac{2^{\frac{D-3}{2}}\Gamma(\frac{D-1}{2})}{\sqrt{\pi}} \frac{H^{\frac{D-2}{2}}}{p^{\frac{D-1}{2}}}e^{ip\cdot x}. \]

The quantum fluctuation of a massless and minimally coupled scalar field is frozen at the superhorizon scale as the leading IR term of the wave function has no time dependence.
From (2.6), the contribution from the superhorizon scale to the propagator at the coincident point is given by

\[ \langle \phi_0^2(x) \rangle \simeq \int_{p < H a(\tau)} \frac{d^{D-1}p}{(2\pi)^{D-1}} \frac{2^{D-3} \Gamma^2(D-1)}{\pi} \frac{H^{D-2}}{p^{D-1}}. \]  

The upper bound of the momentum integral is taken on the horizon because the $1/p^{D-1}$ spectrum is dominant at the superhorizon scale. The propagator has an IR logarithmic divergence due to the presence of the scale invariant spectrum.

In order to regularize the IR divergence, we introduce an IR cut-off $p_0$ which fixes the minimum value of the comoving momentum. Under the setting, the IR contribution to the propagator is expressed as the logarithm of the scale factor [1-3]:

\[ \langle \phi_0^2(x) \rangle \simeq \frac{2H^{D-2} \Gamma(D-1)}{(4\pi)^{D/2} \Gamma(D/2)} \int_{p_0}^{H a(\tau)} \frac{dp}{p} \int_{p_0/a(\tau)}^{a(a)} \frac{dP}{P} \log \left( \frac{a(a)}{a_0^2} \right), \]

where the initial time $a_0$ is related to the IR cut-off as $a_0 = p_0/H$. Physically speaking, $a(a)/p_0$ means a size of the Universe and it expands with the scale factor. Since the Hubble scale $1/H$ is constant, the degrees of freedom at the superhorizon scale increase with time. This increase gives rise to the secular growth of the propagator through the scale invariant spectrum.

We show the explicit form of the propagator for a massless and minimally coupled scalar field [12] as

\[ \langle \phi_0(x) \phi_0(x') \rangle = I(y) + \frac{H^{D-2} \Gamma(D-1)}{(4\pi)^{D/2} \Gamma(D/2)} \log \left( \frac{a(\tau)a(\tau')}{a_0} \right), \]

\[ I(y) = \frac{H^{D-2}}{(4\pi)^{D/2}} \left[ \frac{\Gamma(D/2 - 1)}{2} \left( \frac{y}{4} \right)^{1 - D/2} - \frac{\Gamma(D/2 + 1)}{2} \left( \frac{y}{4} \right)^{2 - D/2} + \frac{\Gamma(D - 1)}{\Gamma(D/2)} \delta \right. \]

\[ + \sum_{n=1}^{\infty} \left[ \frac{\Gamma(D - 1 + n)}{n \Gamma(D/2 + n)} \left( \frac{y}{4} \right)^n - \frac{\Gamma(D/2 + 1 + n)}{(2 - D/2 + n)(n + 1)!} \left( \frac{y}{4} \right)^{2 - D/2 + n} \right], \]

\[ \delta = -\psi \left( 1 - \frac{D}{2} \right) + \psi \left( D - 1 \right) + \psi \left( 0 \right), \quad \psi(z) \equiv \frac{1}{\Gamma(z)} \frac{d\Gamma(z)}{dz}, \]

where $y$ denotes the square of the dS invariant distance:

\[ y = H^2 a(\tau)a(\tau') \Delta x^2, \quad \Delta x^2 \equiv -(\tau - \tau')^2 + (x - x')^2. \]
In (2.9), the first term respects the dS symmetry while the second term breaks the dS symmetry as a consequence of the IR cut-off dependence.

For the subsequent discussion, we evaluate the twice-differentiated propagator at the coincident point: \( \langle \partial_\mu \varphi_0(x) \partial_\nu \varphi_0(x) \rangle \). In the dimensional regularization, the \( y^\alpha \) term disappears at the coincident point if \( \alpha \) becomes positive for a sufficiently low \( D \). Considering this fact, the twice-differentiated propagator at the coincident point is evaluated as

\[
\langle \partial_\mu \varphi_0(x) \partial_\nu \varphi_0(x) \rangle = \frac{H^D}{(4\pi)^{\frac{D}{2}}} \frac{\Gamma(D)}{4\Gamma(\frac{D}{2}+1)} \partial_\mu \partial'_\nu y^1 \big|_{x' \to x} = -\frac{H^D}{(4\pi)^{\frac{D}{2}}} \frac{\Gamma(D)}{D\Gamma(\frac{D}{2})} g_{\mu\nu}.
\]  

(2.13)

The twice-differentiated propagator respects the dS symmetry. That is because if differential operators act on both of \( x \) and \( x' \), the dS symmetry breaking term in (2.9) does not have any contribution as

\[
\partial_\mu \partial'_\nu \log(\frac{a(\tau)}{a_0}/a_0^2) = 0.
\]  

(2.14)

In interacting field theories with massless and minimally coupled scalar fields, physical quantities may acquire IR logarithmic corrections through the propagator. At the late time \( \log(\frac{a(\tau)}{a_0}) \gg 1 \), the leading IR effects come from the leading IR logarithms at each loop level. For example, for the contribution from the \( \lambda \varphi^i \varphi^j \varphi^i \varphi^j \) term, quantum corrections are multiplied by up to the following factor with each increase in the loop level:

\[
\lambda H^{D-4} \log^2(\frac{a(\tau)}{a_0}).
\]  

(2.15)

As another example, for the contribution from the \( f^2 \varphi^i \varphi^j g^{\mu\nu} \partial_\mu \varphi^i \partial_\nu \varphi^j \) term, quantum corrections are multiplied by up to the following factor with each increase in the loop level:

\[
f^2 H^{D-2} \log(\frac{a(\tau)}{a_0}).
\]  

(2.16)

These secular growths indicate that even though the dimensionless couplings are small as \( \lambda H^{D-4}, f^2 H^{D-2} \ll 1 \), the perturbative investigation breaks down after the long enough time:

\[
\lambda H^{D-4} \log^2(\frac{a(\tau)}{a_0}) \sim 1 \quad \text{or} \quad f^2 H^{D-2} \log(\frac{a(\tau)}{a_0}) \sim 1.
\]  

(2.17)

A rational approach for evaluating such nonperturbative effects is to resum the leading IR logarithms to all-loop orders. The resummation formula for non-derivative interactions has been constructed by N. C. Tsamis and R. P. Woodard [8]. In subsequent sections, we construct the resummation formula for derivative interactions.
3 Nonlinear sigma model

As a specific model with derivative interactions, we consider the nonlinear sigma model:

\[
S = \int \sqrt{-g} d^Dx \left[ -\frac{1}{2} G_{ij}(\varphi) g^{\mu\nu} \partial_\mu \varphi^i \partial_\nu \varphi^j \right],
\]

where \( G_{ij} \) is the metric of the target space. The action is invariant under the general coordinate transformation which identifies \( \varphi^i \) as coordinates. The global symmetry guarantees that this model consists of massless and minimally coupled scalar fields without fine-tuning the quadratic action. For the subsequent discussion, we introduce the vielbein:

\[
G_{ij}(\varphi) = E^a_i(\varphi) E^b_j(\varphi) \delta_{ab},
\]

where \( a \) denotes the index of the flat tangent space \( E^N \).

The field equation for the nonlinear sigma model is given by

\[
\frac{1}{\sqrt{-g}} \partial_\mu \left\{ \sqrt{-g} G_{ij}(\varphi) g^{\mu\nu} \partial_\nu \varphi^j \right\} - \frac{1}{2} \partial_\varphi^i G_{jk}(\varphi) g^{\mu\nu} \partial_\mu \varphi^i \partial_\nu \varphi^k = 0.
\]

As we know, the equation is rewritten as the following form:

\[
\nabla^\mu \nabla_\mu \varphi^i + \Gamma^i_{jk}(\varphi) g^{\mu\nu} \partial_\mu \varphi^j \partial_\nu \varphi^k = 0,
\]

where \( \nabla_\mu \) is the covariant derivative for the spacetime and \( \Gamma^i_{jk} \) is the Levi–Civita connection for the target space. However, in order to find a covariant Yang–Feldman formalism, it is more useful to rewrite (3.3) as

\[
\frac{1}{\sqrt{-g}} \partial_\mu \left\{ \sqrt{-g} E^a_i(\varphi) g^{\mu\nu} \partial_\nu \varphi^i \right\} - D_i E^a_j(\varphi) g^{\mu\nu} \partial_\mu \varphi^i \partial_\nu \varphi^j = 0,
\]

where \( D_i \) is the covariant derivative for the target space.

The Yang–Feldman formalism for the nonlinear sigma model is given by

\[
- i \int d^Dx' \partial_\mu \left\{ \sqrt{-g(x')} g^{\mu\nu}(x') \partial_\nu G^R(x, x') E^a_i(\varphi(x')) \right\} \varphi^i(x')
= \varphi^a_0(x) - i \int \sqrt{-g(x')} d^Dx' G^R(x, x') D_i E^a_j(\varphi(x')) g^{\mu\nu}(x') \partial_\mu \varphi^i(x') \partial_\nu \varphi^j(x'),
\]

where \( G^R \) denotes the retarded propagator:

\[
G^R(x, x') = \theta(\tau - \tau') \langle [\varphi_0(x), \varphi_0(x')] \rangle,
\]

and it satisfies

\[
\nabla^\mu \nabla_\mu G^R(x, x') = \frac{i \delta^{(D)}(x - x')}{\sqrt{-g(x)}},
\]

5
We can derive (3.5) by applying the d’Alembert operator to the both sides of (3.6). It should be noted that the homogeneous solution, i.e. the free field, has the index $a$ rather than $i$, and the both sides of (3.6) have the index $a$.

The Yang–Feldman formalism describes quantum effects from all regions of momentum scale, and it is not exactly solvable in general. We show that in the leading logarithm approximation, the Yang–Feldman formalism reduces to a Langevin equation with a white noise, which is a more suitable tool for investigating the IR dynamics.

For the first term in the right side of (3.6), we extract its leading IR behavior as

$$\varphi_0^a(x) \simeq \bar{\varphi}_0^a(x) \equiv \int \frac{dD-1p}{(2\pi)^{D-1}} \theta(Ha(\tau) - p) \left[ -i \frac{2^{\frac{D-2}{2}} \Gamma(\frac{D-1}{2})}{\sqrt{\pi}} \frac{H^{\frac{D-2}{2}}}{p^{\frac{D-1}{2}}} e^{ip \cdot x} a_p^a + \text{(h.c.)} \right], \quad (3.9)$$

where $[a_p^a, a_{p'}^b] = (2\pi)^{D-1} \delta^{(D-1)}(p - p') \delta^{ab}$ and the others are zero. Since the $1/p^{D-1}$ term is dominant at the superhorizon scale, we introduce the step function which is nonzero at the IR region. By iterative substitution of (3.9), the Wightman functions left intact by differential operators $\langle \varphi_0(x) \varphi_0(x') \rangle, \langle \varphi_0(x') \varphi_0(x) \rangle$ induce the IR logarithms.

The second term in the right side of (3.6) includes two differentiated fields $\partial'_\mu \varphi^i(x') \partial'_\nu \varphi^j(x')$. Each diagram with the leading IR logarithms includes a loop of the twice-differentiated propagators, which starts at $x'$ and ends at $x'$. The other diagrams, which are obtained by transferring any of the differential operators from inside to outside the loop, have reduced powers of the IR logarithms. After partial integrations, the diagrams with the leading IR logarithms can be summarized as

$$\begin{align*}
\begin{array}{c}
\text{Diagram 1:} \\
D_i E_i \\
\text{Diagram 2:} \\
D_i E_j \\
\text{Diagram 3:} \\
D_i E_j G_{ij} \\
\text{...} \\
\end{array}
\end{align*}
\simeq \begin{array}{c}
\text{Diagram 1:} \\
D_i E_i \\
\text{Diagram 2:} \\
D_i E_j \\
\text{Diagram 3:} \\
D_i E_j G_{ij} \\
\text{...} \\
\end{array}, \quad (3.10)
$$

where the horizontal line segment denotes $G^R(x, x')$ and $\delta G_{ij} = G_{ij} - \delta_{ij}$. It should be noted that the diagrams with differentiated $\delta G_{ij}$ are negligible up to the leading logarithm accuracy. In the Yang–Feldman formalism, the approximation is expressed as

$$\begin{align*}
\partial'_\mu \varphi^i(x') \partial'_\nu \varphi^j(x') &\simeq -G^{ij}(\varphi(x')) \langle \partial'_\mu \varphi_0(x') \partial'_\nu \varphi_0(x') \rangle \\
&= -G^{ij}(\varphi(x')) \frac{H^D}{(4\pi)^{\frac{D}{2}}} \frac{\Gamma(D)}{D!} g_{\mu\nu}(x'), \quad (3.11)
\end{align*}
$$

where we substituted the value of the twice-differentiated propagator (2.13). It should be recalled that the IR logarithm does not have any contribution in evaluating the twice-differentiated propagator. In other words, even in the leading logarithm approximation, the statement holds true only if the retarded propagator starting at $x'$ is not differentiated. So, it does not apply to the left side of (3.6) with $\partial'_\mu G^R(x, x')$. Please refer to [11] for details of the power counting of the IR logarithms.
we need to take into account the contribution from the subhorizon scale. In fact, the contribution from (3.11) is overlooked (and is added later by hand) in [9].

The second term in the right side of (3.6) also includes the retarded propagator. In evaluating it, we need to know the real and the imaginary parts of the wave function. So, the wave function (2.5) should be expanded further than done in (2.6) as

\[
\phi_p(x) \simeq -i \left\{ \frac{2^{-\frac{D-2}{2}} \Gamma(\frac{D-1}{2}) \sqrt{\pi}}{p^{\frac{D-2}{2}}} + i \frac{\sqrt{\pi}}{2^{\frac{D+1}{2}} \Gamma(\frac{D+1}{2})} H_{\frac{D-2}{2}} p^{\frac{D-1}{2}} (-\tau)^{D-1} \right\} e^{ip \cdot x}. \tag{3.12}
\]

Substituting it, the retarded propagator is evaluated as

\[
G^R(x, x') \simeq \theta(\tau - \tau') \int \frac{d^{D-1}p}{(2\pi)^{D-1}} \frac{-i H^{D-2}}{D-1} \left[ (-\tau')^{D-1} - (-\tau)^{D-1} \right] e^{ip \cdot (x - x')} = \frac{-i}{(D - 1) H} \theta(\tau - \tau') \left[ a^{-(D-1)}(\tau') - a^{-(D-1)}(\tau) \right] \delta^{(D-1)}(x - x'). \tag{3.13}
\]

The retarded propagator itself does not have the IR logarithm in contrast to the Wightman functions, while the vertex integral with it induces the IR logarithm as

\[
\int \sqrt{-g(x')} d^{D}x' \ G^R(x, x') \simeq -i \int_{\tau}^{\tau'} a(\tau') d\tau' \left[ 1 - (a(\tau')/a(\tau))^{D-1} \right] \simeq -i \frac{1}{(D - 1) H} \int_{t}^{t'} dt', \tag{3.14}
\]

where the trivial spatial integral is not shown. In the last line, we neglected the second term because it does not induce the IR logarithm. Seeing that \(dt' = d(\log a(t'))/H\), the vertex integral provides an IR logarithm in addition to the contribution from the integrand \(\int D_{\mu}E^a_{\mu}(\varphi(x')) g^{\mu \nu}(x') \partial_{\mu} \varphi^i(x') \partial_{\nu} \varphi^j(x') \). It should be noted that the higher-order terms neglected in (3.13) reduce the powers of the IR logarithms from the integrand.

The left side of (3.6) is expressed as

\[
E_i^a(\varphi(x)) \varphi^i(x) - i \int \sqrt{-g(x')} d^{D}x' \ g^{\mu \nu}(x') \partial_{\mu} G^R(x, x') \partial_{\nu} E^a_{\mu}(\varphi(x')) \varphi^i(x'). \tag{3.15}
\]

For the first term, we can extract the leading IR logarithms just by iterative substitution of (3.9). In order to evaluate the leading IR effects from the second term, we approximate the differentiated retarded propagator as

\[
\partial_{\mu} G^R(x, x') \simeq i \theta(\tau - \tau') \delta^{00} a^{-(D-2)}(\tau') \delta^{(D-1)}(x - x'). \tag{3.16}
\]

As we did in (3.13), we expanded the differentiated retarded propagator in powers of \((-p\tau)\) and \((-p\tau')\) and kept the zeroth-order term. Substituting (3.16), the vertex integral with
the differentiated propagator is given by

\[
\int \sqrt{-g(x')} d^D x' \ g^{\mu\nu}(x') \partial_\mu G(x, x') \simeq -i \int^t d\tau \ \delta_0^{\nu}
= -i \int^t dt' \ a^{-1}(t') \delta_0^{\nu}, \quad (3.17)
\]

where the trivial spatial integral is not shown. For the integrand \( \partial_0^a E_\nu^a(\varphi(x')) \varphi^i(x') \), the differential operator \( \partial_0^a = a(t') \partial_0^a \) removes an IR logarithm from \( E_\nu^a(\varphi(x')) \) and adds one scale factor \( a(t') \). The scale factor from the differential operator is canceled by \( a^{-1}(t') \) in (3.17), and then the vertex integral provides an IR logarithm as \( dt' = d(\log a(t'))/H \). That is how the second term in (3.15) induces the leading IR logarithms.

Applying the leading logarithm approximation: (3.9), (3.10), (3.12) and (3.17), the Yang–Feldman formalism (3.6) reduces to

\[
E_\nu^a(\varphi(x)) \varphi^i(x) = \varphi_0^a(x) + \frac{H^{D-1} \Gamma(D-1)}{(4\pi)^{D/2} \Gamma(D/2)} \int^t dt' \ D_i E^{ia}(\varphi(t', x)).
\]

(3.18)

Differentiating its both sides with respect to \( t \), we obtain the local equation:

\[
E_\nu^a(\varphi(x)) \varphi^i(x) = \ddot{\varphi}_0^a(x) + \frac{H^{D-1} \Gamma(D-1)}{(4\pi)^{D/2} \Gamma(D/2)} D_i E^{ia}(\varphi(x)),
\]

(3.19)

where \( \dot{\varphi}^i = \partial_t \varphi^i \) and the correlation function of \( \ddot{\varphi}_0^a \) is given by

\[
\langle \ddot{\varphi}_0^a(t, x) \ddot{\varphi}_0^a(t', x) \rangle = \delta^{ab} \int \frac{dD-1 p}{(2\pi)^{D-1}} \left( H^2 a(t) \right)^2 \delta(Ha(t) - p) \delta(Ha(t) - Ha(t')) \times \frac{2^{D-3} \Gamma^2(D-1)}{\pi} \frac{H^{D-2}}{p^{D-1}} \delta^{ab}(t - t').
\]

(3.20)

It should be recalled that the time dependence of \( \varphi_0^a \) appears only through the step function which denotes the superhorizon scale. As a consequence of this fact, the correlation function of \( \ddot{\varphi}_0^a \) is proportional to the temporal delta function. We call this type of fluctuation a white noise. The equation (3.19) + (3.20) is known as a Langevin equation with a white noise [13].

From the Langevin equation, we can derive the equation for the probability density \( \rho \):

\[
\dot{\rho}(t, \phi) = \frac{H^{D-1} \Gamma(D-1)}{(4\pi)^{D/2} \Gamma(D/2)} \frac{\partial^2}{\partial \phi^i \partial \phi^j} \left\{ G^{ij}(\phi) \rho(t, \phi) \right\} - \frac{H^{D-1} \Gamma(D-1)}{(4\pi)^{D/2} \Gamma(D/2)} \frac{\partial}{\partial \phi^i} \left[ \left[ \int \frac{\partial}{\partial \phi^j} E^i_a(\phi) E^{ja}(\phi) + E^i_a(\phi) D_j E^{ja}(\phi) \right] \rho(t, \phi) \right].
\]
\[ H^{D-1} \frac{\Gamma(D-1)}{(4\pi)^{D/2}} \frac{\partial^2}{\partial \phi^i \partial \phi^j} \{ G^{ij}(\phi) \rho(t, \phi) \} \]
\[ + H^{D-1} \frac{\Gamma(D-1)}{(4\pi)^{D/2}} \frac{\partial}{\partial \phi^i} \{ \Gamma^i_{jk}(\phi) G^{jk}(\phi) \rho(t, \phi) \}, \quad (3.21) \]
which is known as a Fokker–Planck equation. By using its solution, we can evaluate the vacuum expectation value (vev) of each operator as
\[ \langle F^{i_1 \cdots i_n}(\phi(x)) \rangle = \int d^N \phi \ F^{i_1 \cdots i_n}(\phi) \rho(t, \phi), \quad (3.22) \]
where \( F^{i_1 \cdots i_n} \) denotes an arbitrary function. It should be noted that \( \phi^i \) are operators while \( \phi \) are \( \gamma \)-numbers. For a general case, we show the relation between the Langevin equation and the Fokker–Planck equation in Appendix A. We can rewrite (3.21) as the following covariant form:
\[ \dot{\rho}(t, \phi) = \frac{H^{D-1} \Gamma(D-1)}{(4\pi)^{D/2}} \frac{\partial}{\partial \phi^i} \{ \rho(t, \phi) \sqrt{G(\phi)} \}. \quad (3.23) \]
It should be noted that as seen in (3.22), \( \rho \) itself is not a scalar quantity but \( \rho / \sqrt{G} \) is a scalar quantity.

The Fokker–Planck equation is exactly solvable for an equilibrium state which is eventually established: \( \rho(t, \phi) \rightarrow \rho_\infty(\phi) \) at \( t \rightarrow \infty \) if it exists. Since \(-D^i D_i^j = D^i D_i^j\) is nonnegative-definite, the solution for the equilibrium state is given by
\[ D_i \left\{ \frac{\rho_\infty(t, \phi)}{\sqrt{G(\phi)}} \right\} = 0 \quad \Rightarrow \quad \rho_\infty(\phi) = Z^{-1} \sqrt{G(\phi)}. \quad (3.24) \]
The overall coefficient is fixed as the total integral of the probability density is kept unity as
\[ Z = \int dN \phi \ \sqrt{G(\phi)}. \quad (3.25) \]
As seen in (3.24)-(3.25), the convergence of \( \sqrt{G(\phi)} \) at \( |\phi| \rightarrow \infty \) determines whether an equilibrium state is eventually established or not. If the asymptotic behavior of \( \sqrt{G(\phi)} \) is at most \( |\phi|^{-\beta}, \beta > 0 \), any equilibrium state is not established. We show a specific example which induces an equilibrium state as
\[ G_{ij}(\phi) = \exp \left( -\frac{f^2}{2N} \phi^k \phi^k \right) \delta_{ij}, \quad (3.26) \]
where \( f \) is a coupling constant. In this case, the saturation values of \( \langle (\phi^i(x) \phi^i(x))^n \rangle \) are given by
\[ \rho_\infty(\phi) = Z^{-1} \exp \left( -\frac{f^2}{4} \phi^k \phi^k \right) \Rightarrow \langle (\phi^i(x) \phi^i(x))^n \rangle |_{t \rightarrow \infty} = \frac{\Gamma \left( \frac{N}{2} + n \right)}{\Gamma \left( \frac{N}{2} \right)} \left( \frac{2}{f} \right)^{2n}. \quad (3.27) \]
As seen in the \( 1/f^{2n} \) dependences of the saturation values, the equilibrium solution describes the leading IR effects nonperturbatively.
As a more general model, we consider the hybrid model consisting of a non-canonical kinetic term and a potential term:

\[ S = \int \sqrt{-g} d^D x \left[ -\frac{1}{2} G_{ij}(\varphi) g^{\mu\nu} \partial_\mu \varphi^i \partial_\nu \varphi^j - V(\varphi) \right]. \tag{4.1} \]

Each coupling in the potential term, which is made dimensionless by the Hubble constant, is kept small, so that the scalar fields can be identified as pseudo Nambu–Goldstone bosons. The covariant Yang–Feldman formalism for the hybrid model is given by

\[ -i \int d^D x' \partial'_\mu \left\{ \sqrt{-g(x')} g^{\mu\nu}(x') \partial'_\nu G^R(x, x') E_i^a(\varphi(x')) \right\} \varphi^i(x') = \varphi^a_0(x) - i \int \sqrt{-g(x')} d^D x' \ G^R(x, x') \left\{ D_i E_j^a(\varphi(x')) g^{\mu\nu}(x') \partial'_\mu \varphi^i(x') \partial'_\nu \varphi^j(x') \right. \\
\left. + E^{ia}(\varphi(x')) D_i V(\varphi(x')) \right\}. \tag{4.2} \]

Applying the leading logarithm approximation: (3.9), (3.11), (3.14) and (3.17), and differentiating its both sides with respect to \( t \), the Yang–Feldman equation reduces to the Langevin equation with the white noise:

\[ \dot{\varphi}^a_0(x) = \varphi^a_0(x) + \frac{H^{D-1}}{(4\pi)^{D/2} \Gamma(D/2)} D_i E_i^a(\varphi(x)) \\
- \frac{1}{(D-1)H} E^{ia}(\varphi(x)) D_i V(\varphi(x)), \tag{4.3} \]

\[ \langle \dot{\varphi}^a_0(t, x) \dot{\varphi}^b_0(t', x) \rangle = \frac{2H^{D-1}}{(4\pi)^D \Gamma(D/2)} \delta^{ab} \delta(t - t'). \tag{4.4} \]

It should be emphasized that (3.9) holds true even though the potential term is present in the action. Since the differential operators reduce the number of the IR logarithms from non-derivative interactions, (3.9) does not explicitly depend on the potential term up the leading logarithm accuracy.

From the Langevin equation (4.3)-(4.4), we can derive the Fokker–Planck equation:

\[ \frac{\dot{\rho}(t, \phi)}{\sqrt{G(\phi)}} = \frac{H^{D-1}}{(4\pi)^{D/2} \Gamma(D/2)} D^i D_i \left\{ \rho(t, \phi) \right\} + \frac{1}{(D-1)H} D^i \left\{ D_i V(\phi) \rho(t, \phi) \sqrt{G(\phi)} \right\}. \tag{4.5} \]

In order to solve the Fokker–Planck equation for an equilibrium state, it is useful to introduce

\[ \Psi(t, \phi) = \exp \left\{ -\frac{1}{2} \frac{2\pi^{D+1}}{\Gamma(D+1) H D} V(\phi) \right\} \rho(t, \phi) \sqrt{G(\phi)}. \tag{4.6} \]
We can rewrite (1.5) as the equation for the rescaled probability density:

\[ \dot{\Psi}(t, \phi) = -\frac{H D^{-1} \Gamma(D - 1)}{4\pi^{D/2}} \mathcal{A}^i \mathcal{A}_i \Psi(t, \phi), \]  

(4.7)

\[ \mathcal{A}_i = D_i + \frac{1}{2} \frac{2 \pi^{D+1}}{\Gamma(D+1) H^D} D_i V(\phi). \]  

(4.8)

Since \( \mathcal{A}^i \mathcal{A}_i \) is nonnegative-definite, the solution for the equilibrium state: \( \Psi(t, \phi) \rightarrow \Psi_\infty(\phi) \) at \( t \rightarrow \infty \) is given by

\[ \mathcal{A}_i \Psi_\infty(\phi) = 0 \quad \Rightarrow \quad \rho_\infty(\phi) = Z^{-1} \sqrt{G(\phi)} \exp \left\{ -\frac{2 \pi^{D+1}}{\Gamma(D+1) H^D} V(\phi) \right\}, \]  

(4.9)

where the normalization factor is

\[ Z = \int d^N \phi \sqrt{G(\phi)} \exp \left\{ -\frac{2 \pi^{D+1}}{\Gamma(D+1) H^D} V(\phi) \right\}. \]  

(4.10)

If \( V(\phi) \) behaves like \( \lambda |\phi|^\alpha \), \( \lambda, \alpha > 0 \) at \( |\phi| \rightarrow \infty \), even though the asymptotic behavior of \( \sqrt{G(\phi)} \) is at most \( |\phi|^{-\beta} \), \( \beta > 0 \), an equilibrium state is eventually established.

The equilibrium solution (4.9)-(4.10) shows a strong association with the Euclidean field theory on \( S_D \). The IR effects discussed in this paper are interpreted as nonequilibrium phenomena as they break the dS symmetry and evolve with time. However, if an equilibrium state is eventually established, it may be described by the Euclidean field theory on \( S_D \). For non-derivative interactions, it has been known that the equilibrium solution of the stochastic equation can be reproduced by considering the zero mode dynamics in the Euclidean field theory on \( S_D \) [14,16]. As shown below, the discussion holds true even though the kinetic term is non-canonical.

The scalar fields can be expanded by the spherical harmonics \( Y_{l_1, \ldots, l_D} \) on \( S_D \) as

\[ \varphi^i(x_E) = \sum_{\{l_1, \ldots, l_D\}} \varphi^i_{\{l_1, \ldots, l_D\}} Y_{\{l_1, \ldots, l_D\}}(x_E), \]  

(4.11)

where \( x_E \) are the spherical coordinates and \( l_1 \geq |l_2| \geq \cdots \geq |l_D| \geq 0 \) are the angular momentums. In order to evaluate the IR effects, we extract the zero mode as

\[ \varphi^i(x_E) \simeq \varphi^i_{\{0, \ldots, 0\}} Y_{\{0, \ldots, 0\}}. \]  

(4.12)

It should be recalled that the zero mode has no coordinate dependence.
In the zero mode approximation, only the potential term contributes to the Euclidean action as

\[ S_E = \sqrt{g_E} d^D x_E \left[ \frac{1}{2} G_{ij}(\varphi) g_{E}^{\mu
u} \partial_{\mu} \varphi^i \partial_{\nu} \varphi^j + V(\varphi) \right] \]

\[ \simeq V(\varphi_{0,\ldots,0}) Y_{0,\ldots,0} \int \sqrt{g_E} d^D x_E \]

\[ = V(\varphi_{0,\ldots,0}) Y_{0,\ldots,0} \cdot \frac{2\pi^{D+1}}{\Gamma\left(\frac{D+1}{2}\right) H^D}, \quad (4.13) \]

where the coefficient of the potential term is nothing but the volume of \( S_D \) of radius 1/H.

In the path integral formalism with (4.13), the vev of each operator is evaluated as

\[ \langle F_{i_1,\ldots,i_n}(\varphi(x)) \rangle \]

\[ = \frac{\int \sqrt{G} D\varphi \, F_{i_1,\ldots,i_n}(\varphi(x)) e^{-S_E}}{\int \sqrt{G} D\varphi \, e^{-S_E}} \]

\[ \simeq \frac{\int \sqrt{G} d^N(\varphi_{0,\ldots,0}) Y_{0,\ldots,0}) \cdot F_{i_1,\ldots,i_n}(\varphi_{0,\ldots,0}) Y_{0,\ldots,0}) \cdot \exp \left\{ -\frac{2\pi^{D+1}}{\Gamma\left(\frac{D+1}{2}\right) H^D} V(\varphi_{0,\ldots,0}) Y_{0,\ldots,0}) \right\}}{\int \sqrt{G} d^N(\varphi_{0,\ldots,0}) Y_{0,\ldots,0}) \cdot \exp \left\{ -\frac{2\pi^{D+1}}{\Gamma\left(\frac{D+1}{2}\right) H^D} V(\varphi_{0,\ldots,0}) Y_{0,\ldots,0}) \right\}} \]

The covariant functional integral \( \sqrt{G} D\varphi \) reduces to \( \sqrt{G} d^N(\varphi_{0,\ldots,0}) Y_{0,\ldots,0}) \) in the zero mode approximation. Identifying \( \varphi_{0,\ldots,0}) Y_{0,\ldots,0}) \) as \( \phi \), (4.14) is equal to the stochastic evaluation with the equilibrium solution (4.9)-(4.10).

5 Conclusion

We extended the resummation formula of the leading IR logarithms as it is applicable to a general scalar field theory whose kinetic term is non-canonical. As seen in (3.11), the contribution from the subhorizon scale should be taken into account if such derivative interactions are present. The subhorizon dynamics can be treated as in the free field theory except for the nontrivial overall factor \( G^{ij}(\varphi) \).

In the general scalar field theory, the leading logarithm approximation of the Yang–Feldman formalism leads to a Langevin equation with a white noise. As seen in (4.13), the resulting stochastic equation can be understood as a generalization of the standard one to curved target spaces. From the generalized one, we can conclude that if the target space is compact sufficiently, even though the potential term is absent, an equilibrium state is eventually established. The equilibrium state can be reproduced by considering the zero mode dynamics in the Euclidean field theory on \( S_D \). It should be emphasized that if the target space is not so compact, any equilibrium state is not established.
This study can be interpreted as a necessary first step to the nonperturbative investigation of the IR effects from gravity. The gravitational field includes massless and minimally coupled modes [17], and so induces the IR effects which spoil perturbation theories eventually. The Einstein gravity consists of derivative interactions, gauge degrees of freedom and tensor fields. This study shows how to resum the leading IR logarithms from derivative interactions of the scalar field. The resummation formula in the presence of gauge degrees of freedom and tensor fields is a future subject.

The contribution from gravity should be considered also in evaluating the IR effects in inflation theories. If only the inflaton induced the IR effects, we could evaluate them non-perturbatively by using the current resummation formula. However, there is no reason to neglect the IR effects from gravity. In particular, if the pseudo shift symmetry is respected, the IR effects from the gravitational interaction are more dominant compared with those from the self-interaction of the inflaton [18].

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A Langevin and Fokker–Planck eqs.

Here we show the relation between the Langevin equation and the Fokker–Planck equation. Please refer to [13] for how to derive the relation.

If the Langevin equation is expressed as

\[ \dot{\xi}^i(t) = A^i_a(t, \xi(t))\eta^a(t) + B^i(t, \xi(t)), \]  
\[ \langle \eta^a(t) \rangle = 0, \quad \langle \eta^a(t)\eta^b(t') \rangle = \delta^{ab}\delta(t-t'), \]  

the corresponding Fokker–Planck equation is given by

\[ \dot{\rho}(t, x) = \frac{1}{2} \frac{\partial^2}{\partial x^i\partial x^j} \left\{ A^i_a(t, x)A^{ja}(t, x)\rho(t, x) \right\} \]
\[ - \frac{\partial}{\partial x^i} \left\{ \left[ \frac{1}{2} \frac{\partial}{\partial x^j} A^i_a(t, x)A^{ja}(t, x) + B^i(t, x) \right] \rho(t, x) \right\}. \]

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Using its solution, the vev of each operator can be evaluated as

\[
\langle F_{i_1\cdots i_n}(\xi(t)) \rangle = \int d^N x \ F_{i_1\cdots i_n}(x) \rho(t, x),
\]

where \( F_{i_1\cdots i_n} \) denotes an arbitrary function. In the derivation of (3.21) from (3.19)–(3.20), we made the following replacement:

\[
\xi^i(t) \rightarrow \varphi^i(t, x),
\]

\[
A^i_a \rightarrow \left\{ \frac{H^{D-1} \Gamma(D-1)}{(4\pi)^{D/2} \Gamma(D/2)} \right\}^{1/2} E^i_a,
\]

\[
B^i \rightarrow \frac{H^{D-1} \Gamma(D-1)}{(4\pi)^{D/2} \Gamma(D/2)} E^i_a D^j E^j_a.
\]  

(A.5)

In the main text, the probability density for one-point functions is discussed. Here we refer to the probability density for spatially-separated two-point functions. In investigating it, we need to know the correlation function of \( \dot{\varphi}_0^a \) at spatially separated points:

\[
\langle \dot{\varphi}_0^a(t, x) \dot{\varphi}_0^b(t', x') \rangle = \frac{2H^{D-1} \Gamma(D-1)}{(4\pi)^{D/2} \Gamma(D/2)} \delta^{ab} \delta(t - t') \theta(1 - Ha(t)r),
\]  

(A.6)

where \( r = |x - x'| \). In the derivation of (A.6), we approximated the Bessel function of the first kind by the step function as

\[
\frac{2H^{D-1} \Gamma(D-1)}{(4\pi)^{D/2} \Gamma(D/2)} J_{D-4}(Ha(t)r) \simeq \theta(1 - Ha(t)r).
\]

(A.7)

From (3.19) and (A.6), the Fokker–Planck equation for spatially-separated two-point functions is given by

\[
\frac{\dot{\rho}(t, \phi, \phi')}{\sqrt{G(\phi)G(\phi')}} = \frac{H^{D-1} \Gamma(D-1)}{(4\pi)^{D/2} \Gamma(D/2)} D^i D_i \left\{ \frac{\rho(t, \phi, \phi')}{\sqrt{G(\phi)G(\phi')}} \right\}
\]

\[
+ \frac{H^{D-1} \Gamma(D-1)}{(4\pi)^{D/2} \Gamma(D/2)} D^i D_i \left\{ \frac{\rho(t, \phi, \phi')}{\sqrt{G(\phi)G(\phi')}} \right\}
\]

\[
+ \theta(1 - Ha(t)r) \frac{2H^{D-1} \Gamma(D-1)}{(4\pi)^{D/2} \Gamma(D/2)} D_i \left\{ E^i_a(\phi) E^j_a(\phi') \frac{\rho(t, \phi, \phi')}{\sqrt{G(\phi)G(\phi')}} \right\}
\]

(A.8)

where \( \phi, \phi' \) denote \( \phi(x) \), \( \phi(x') \). Using its solution, each spatially-separated two-point function can be evaluated as

\[
\langle P_{i_1\cdots i_m}(\varphi(t, x)) Q_{j_1\cdots j_n}(\varphi(t, x')) \rangle = \int d^N \phi d^N \phi' \ P_{i_1\cdots i_m}(\phi) Q_{j_1\cdots j_n}(\phi') \rho(t, \phi, \phi'),
\]

(A.9)

where \( P_{i_1\cdots i_m}, Q_{j_1\cdots j_n} \) denote arbitrary functions.
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