EVALUATION OF THE CONVOLUTION SUM INVOLVING THE SUM OF
DIVISORS FUNCTION FOR 14, 22 AND 26

AYŞE ALACA, ŞABAN ALACA AND EBÉNÉZER NTIENJEM

ABSTRACT. For all natural numbers \( n \), we discuss the evaluation of the convolution sum,

\[
\sum_{\substack{(l,m)\in\mathbb{N}^2 \setminus \{(0,0)\} \atop \alpha l + \beta m = n}} \sigma(l)\sigma(m),
\]

where \( \alpha \beta = 14, 22, 26 \). We generalize the extraction of the convolution sum using Eisenstein forms of weight 4 for all pairs of positive integers \((\alpha, \beta)\). We also determine formulae for the number of representations of a positive integer by the octonary quadratic forms \( a(x_1^2 + x_2^2 + x_3^2 + x_4^2) + b(x_5^2 + x_6^2 + x_7^2 + x_8^2) \), where \((a, b) = (1, 1), (1, 3), (1, 9), (2, 3)\). These numbers of representations of a positive integer are applications of the evaluation of certain convolution sums by J. G. Huard et al. [12], A. Alaca et al. [13, 14] and D. Ye [29].

1. INTRODUCTION

Let \( \mathbb{N}, \mathbb{N}_0, \mathbb{Z}, \mathbb{Q}, \mathbb{R} \) and \( \mathbb{C} \) denote the sets of positive integers, non-negative integers, integers, rational numbers, real numbers and complex numbers, respectively.

Let \( k, n \in \mathbb{N} \). The sum of positive divisors of \( n \) to the power of \( k \), \( \sigma_k(n) \), is defined by

\[
\sigma_k(n) = \sum_{0 \leq d \mid n} d^k.
\]

We write \( \sigma(n) \) as a synonym for \( \sigma_1(n) \). For \( m \notin \mathbb{N} \) we set \( \sigma_k(m) = 0 \).

Let \( \alpha, \beta \in \mathbb{N} \) be such that \( \alpha \leq \beta \). The convolution sum, \( W_{(\alpha, \beta)}(n) \), is defined by

\[
W_{(\alpha, \beta)}(n) = \sum_{\substack{(l,m)\in\mathbb{N}^2 \setminus \{(0,0)\} \atop \alpha l + \beta m = n}} \sigma(l)\sigma(m).
\]

We write \( W_{\alpha}(n) \) as a synonym for \( W_{(1, \beta)}(n) \).

The values of \((\alpha, \beta)\) for those convolution sums \( W_{(\alpha, \beta)}(n) \) that have so far been evaluated are given in Table 4. We discuss the evaluation of the convolution sum for \( \alpha \beta = 14, 22, 26 \), i.e., \((\alpha, \beta) = (2, 7), (1, 22), (2, 11), (1, 26), (2, 13)\). These convolution sums have not been evaluated yet as one can notice from Table 4. We also discuss the generalization of the extraction of the convolution sum using Eisenstein forms of weight 4 for all pairs of positive integers \((\alpha, \beta)\).

As an application, convolution sums are used to determine explicit formulæ for the number of representations of a positive integer \( n \) by the octonary quadratic forms

\[
a(x_1^2 + x_2^2 + x_3^2 + x_4^2) + b(x_5^2 + x_6^2 + x_7^2 + x_8^2),
\]
and

\[
c(x_1^2 + x_1x_2 + x_2^2 + x_3x_4 + x_4^2) + d(x_5^2 + x_5x_6 + x_6^2 + x_7x_8 + x_8^2),
\]

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respectively, where \(a, b, c, d \in \mathbb{N}\).

So far known explicit formulae for the number of representations of \(n\) by the octonary \(\text{form}\) \[a \text{ and } b \geq 0, \quad c \geq 0, \quad d \geq 0, \quad a + b + c + d = n\] are referenced in \(\text{Table 5}\). We determine formulae for the number of representations of a positive integer \(n\) by the octonary \(\text{quadratic form}\) \[a \text{ and } b \geq 0, \quad c \geq 0, \quad d \geq 0, \quad a + b + c + d = n\] for which \((a, b) = (1, 1), (1, 3), (2, 3), (1, 9)\). These new results are applications of the evaluation of some convolution sums by J. G. Huard et al. \([12]\), A. Alaca et al. \([1, 3]\) and D. Ye \([29]\).

This paper is organized in the following way. In \(\text{Section 2}\) we discuss modular forms, briefly define eta functions and convolution sums, and prove the generalization of the extraction of the convolution sum. Our main results on the evaluation of the convolution sums are discussed in \(\text{Section 3}\). The determination of formulae for the number of representations of a positive integer \(n\) is discussed in \(\text{Section 4}\). A brief outlook is given in \(\text{Section 5}\).

Software for symbolic scientific computation is used to obtain the results of this paper. This software comprises the open source software packages \(\text{GiNaC, Maxima, REDUCE, SAGE}\) and the commercial software package \(\text{MAPLE}\).

2. MODULAR FORMS AND CONVOLUTION SUMS

Let \(\mathbb{H}\) be the upper half-plane, that is \(\mathbb{H} = \{z \in \mathbb{C} \mid \text{Im}(z) > 0\}\), and let \(G = \text{SL}_2(\mathbb{R})\) be the group of \(2 \times 2\)-matrices \(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\) such that \(a, b, c, d \in \mathbb{R}\) and \(ad - bc = 1\) hold. Let furthermore \(\Gamma = \text{SL}_2(\mathbb{Z})\) be a subset of \(\text{SL}_2(\mathbb{R})\). Let \(N \in \mathbb{N}\). Then \(\Gamma(N) = \{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{Z}) \mid \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \pmod{N} \}\) is a subgroup of \(G\) and is called a \(\text{principal congruence subgroup of level } N\). A subgroup \(H\) of \(G\) is called a \(\text{congruence subgroup of level } N\) if it contains \(\Gamma(N)\).

Relevant for our purposes is the congruence subgroup \(\Gamma_0(N) = \{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{Z}) \mid c \equiv 0 \pmod{N} \}\).

Let \(k, N \in \mathbb{N}\) and let \(\Gamma' \subseteq \Gamma\) be a congruence subgroup of level \(N \in \mathbb{N}\). Let \(k \in \mathbb{Z}, \gamma \in \text{SL}_2(\mathbb{Z})\) and \(f : \mathbb{H} \cup \mathbb{Q} \cup \{\infty\} \to \mathbb{C} \cup \{\infty\}\). We denote by \(f^{[\gamma]}\) the function whose value at \(z\) is \((cz+d)^{-k}f(\gamma(z))\), i.e., \(f^{[\gamma]}(z) = (cz+d)^{-k}f(\gamma(z))\). The following definition is according to N. Koblitz \([14, \text{p. 108}]\).

**Definition 2.1.** Let \(N \in \mathbb{N}, k \in \mathbb{Z}, f\) be a meromorphic function on \(\mathbb{H}\) and \(\Gamma' \subseteq \Gamma\) a congruence subgroup of level \(N\).

(a) \(f\) is called a \(\text{modular function of weight } k\) for \(\Gamma'\) if

(a1) for all \(\gamma \in \Gamma'\) it holds that \(f^{[\gamma]} = f\).

(a2) for any \(\delta \in \Gamma\) it holds that \(f^{[\delta]}(z)\) can be expressed in the form \(\sum_{n \in \mathbb{Z}} a_ne^{2\pi i nz}\),

wherein \(a_n \neq 0\) for finitely many \(n \in \mathbb{Z}\) such that \(n < 0\).

(b) \(f\) is called a \(\text{modular form of weight } k\) for \(\Gamma'\) if

(b1) \(f\) is a modular function of weight \(k\) for \(\Gamma'\),

(b2) \(f\) is holomorphic on \(\mathbb{H}\),

(b3) for all \(\delta \in \Gamma\) and for all \(n \in \mathbb{Z}\) such that \(n < 0\) it holds that \(a_n = 0\).

(c) \(f\) is called a \(\text{cusp form of weight } k\) for \(\Gamma'\) if

(c1) \(f\) is a modular form of weight \(k\) for \(\Gamma'\),

(c2) for all \(\delta \in \Gamma\) it holds that \(a_0 = 0\).

For \(k, N \in \mathbb{N}\), let \(M_k(\Gamma_0(N))\) be the space of modular forms of weight \(k\) for \(\Gamma_0(N)\), \(S_k(\Gamma_0(N))\) be the subspace of cusp forms of weight \(k\) for \(\Gamma_0(N)\), and \(E_k(\Gamma_0(N))\) be the
subspace of Eisenstein forms of weight $k$ for $\Gamma_0(N)$. Then it is proved in W. A. Stein’s book (online version) \[24, p. 81\] that $M_k(\Gamma_0(N)) = E_k(\Gamma_0(N)) \oplus S_k(\Gamma_0(N))$.

As noted in Section 5.3 of W. A. Stein’s book \[24, p. 86\] if the primitive Dirichlet characters are trivial and $2 \leq k$ is even, then $E_k(q) = 1 - \frac{2k}{\pi} \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n$, where $B_k$ are the Bernoulli numbers.

For the purpose of this paper we only consider trivial primitive Dirichlet characters and $k \geq 4$. Theorems 5.8 and 5.9 in Section 5.3 of \[24, p. 86\] also hold for this special case.

2.1. Eta Functions. The Dedekind eta function, $\eta(z)$, is defined on the upper half-plane $\mathbb{H}$ by $\eta(z) = e^{\frac{\pi iz}{24}} \prod_{n=1}^{\infty} (1 - e^{2\pi inz})$. We set $q = e^{2\pi iz}$. Then $\eta(z) = q^{1/24} \prod_{n=1}^{\infty} (1 - q^n) = q^{1/24} F(q)$, where $F(q) = \prod_{n=1}^{\infty} (1 - q^n)$.

The Dedekind eta function was systematically used by M. Newman \[19, 20\] to construct modular forms for $\Gamma_0(N)$. M. Newman then determined when a function $f$ is a modular form for $\Gamma_0(N)$ by providing conditions (i)-(iv) in the following theorem. The order of vanishing of an eta function at the cusps of $\Gamma_0(N)$, which is condition (v) or (v$'$) in the following theorem, was determined by G. Ligozat \[17\].

The following theorem is proved in L. J. P. Kilford’s book \[13, p. 99\] and G. Kohler’s book \[15, p. 37\]; we will apply that theorem to determine eta functions, $f(z)$, which belong to $M_k(\Gamma_0(N))$, and especially those eta functions which are in $S_k(\Gamma_0(N))$.

**Theorem 2.2 (M. Newman and G. Ligozat).** Let $N \in \mathbb{N}$ and $f(z) = \prod \eta^{s_\delta}(\delta z)$ be an eta function which satisfies the following conditions:

(i) $\sum_{1 \leq \delta \parallel N} s_\delta \equiv 0 \pmod{24}$,  
(ii) $\sum_{1 \leq \delta \parallel N} \delta s_\delta \equiv 0 \pmod{24}$,  
(iii) $\prod_{1 \leq \delta \parallel N} s_\delta \delta$ is a square in $\mathbb{Q}$,  
(iv) $k = \frac{1}{2} \sum_{1 \leq \delta \parallel N} s_\delta$ is an even integer,  
(v) for each positive divisor $d$ of $N$ it holds that $\sum_{1 \leq \delta \parallel N} \frac{\gcd(\delta, d)^2}{\delta} r_\delta \geq 0$.

Then $f(z) \in M_k(\Gamma_0(N))$.

The eta quotient $f(z)$ belongs to $S_k(\Gamma_0(N))$ if (v) is replaced by

(v$'$) for each positive divisor $d$ of $N$ it holds that $\sum_{1 \leq \delta \parallel N} \frac{\gcd(\delta, d)^2}{\delta} r_\delta > 0$.

2.2. Convolution Sums $W_{(\alpha, \beta)}(n)$. Recall that for $\alpha, \beta \in \mathbb{N}$ such that $\alpha \leq \beta$, the convolution sum, $W_{(\alpha, \beta)}(n)$, is defined by $W_{(\alpha, \beta)}(n) = \sum_{(l, m) \in \mathbb{N}_0^2, \alpha l + \beta m = n} \sigma(l) \sigma(m)$.

As observed by A. Alaca et al. \[\Pi\], we can assume that $\gcd(\alpha, \beta) = 1$. Let $q \in \mathbb{C}$ be such that $|q| < 1$. Then the Eisenstein series $L(q)$ and $M(q)$ are defined as follows:

(2.1) $L(q) = E_2(q) = 1 - 24 \sum_{n=1}^{\infty} \sigma(n) q^n,$

(2.2) $M(q) = E_4(q) = 1 + 240 \sum_{n=1}^{\infty} \sigma_3(n) q^n.$

The following two relevant results are essential for the sequel of this work and are a generalization of the extraction of the convolution sum using Eisenstein forms of weight 4 for all pairs $(\alpha, \beta) \in \mathbb{N}^2$. 

Lemma 2.3. Let \( \alpha, \beta \in \mathbb{N} \). Then
\[
(\alpha L(q^\alpha) - \beta L(q^\beta))^2 \in M_4(\Gamma_0(\alpha\beta)).
\]
Proof. If \( \alpha = \beta \), then trivially \( 0 = (\alpha L(q^\alpha) - \alpha L(q^\alpha))^2 \in M_4(\Gamma_0(\alpha)) \) and there is nothing to prove. Therefore, we may suppose that \( \alpha \neq \beta > 1 \) in the sequel. We apply the result proved by W. A. Stein [24, Thms 5.8, 5.9, p. 86] to deduce \( L(q) - \alpha L(q^\alpha) \in M_2(\Gamma_0(\alpha)) \subseteq M_2(\Gamma_0(\alpha\beta)) \) and \( L(q) - \beta L(q^\beta) \in M_2(\Gamma_0(\beta)) \subseteq M_2(\Gamma_0(\alpha\beta)) \). Therefore,
\[
\alpha L(q^\alpha) - \beta L(q^\beta) = (L(q) - \beta L(q^\beta)) - (L(q) - \alpha L(q^\alpha)) \in M_2(\Gamma_0(\alpha\beta))
\]
and so \( (\alpha L(q^\alpha) - \beta L(q^\beta))^2 \in M_4(\Gamma_0(\alpha\beta)) \). \( \square \)

Theorem 2.4. Let \( \alpha, \beta \in \mathbb{N} \) be such that \( \alpha \) and \( \beta \) are relatively prime and \( \alpha < \beta \). Then
\[
(2.3) \quad (\alpha L(q^\alpha) - \beta L(q^\beta))^2 = (\alpha - \beta)^2 + \sum_{n=1}^{\infty} \left( 240 \alpha^2 \sigma_3(n) + 240 \beta^2 \sigma_3(n) \right.
\]
\[
+ 48 \alpha (\beta - 6n) \sigma(n) + \alpha \beta (\alpha - 6n) \sigma(n) - 1152 \alpha \beta W(\alpha, \beta)(n) \left. \right) q^n.
\]
Proof. We observe that
\[
(2.4) \quad (\alpha L(q^\alpha) - \beta L(q^\beta))^2 = \alpha^2 L^2(q^\alpha) + \beta^2 L^2(q^\beta) - 2 \alpha \beta L(q^\alpha)L(q^\beta).
\]
J. W. L. Glaisher [11] has proved the following identity
\[
(2.5) \quad L^2(q) = 1 + \sum_{n=1}^{\infty} \left( 240 \sigma_3(n) - 288 n \sigma(n) \right) q^n
\]
which we apply to deduce
\[
(2.6) \quad L^2(q^\alpha) = 1 + \sum_{n=1}^{\infty} \left( 240 \sigma_3(\frac{n}{\alpha}) - 288 \frac{n}{\alpha} \sigma(\frac{n}{\alpha}) \right) q^n
\]
and
\[
(2.7) \quad L^2(q^\beta) = 1 + \sum_{n=1}^{\infty} \left( 240 \sigma_3(\frac{n}{\beta}) - 288 \frac{n}{\beta} \sigma(\frac{n}{\beta}) \right) q^n.
\]
Since
\[
(\sum_{n=1}^{\infty} \sigma(\frac{n}{\alpha}) q^n)(\sum_{n=1}^{\infty} \sigma(\frac{n}{\beta}) q^n) = \sum_{n=1}^{\infty} (\sum_{\alpha k + \beta l = n} \sigma(k) \sigma(l)) q^n = \sum_{n=1}^{\infty} W(\alpha, \beta)(n) q^n,
\]
we conclude, when using the accordingly modified Equation 2.1 that
\[
(2.8) \quad L(q^\alpha)L(q^\beta) = 1 - 24 \sum_{n=1}^{\infty} \sigma(\frac{n}{\alpha}) q^n - 24 \sum_{n=1}^{\infty} \sigma(\frac{n}{\beta}) q^n + 576 \sum_{n=1}^{\infty} W(\alpha, \beta)(n) q^n.
\]
Therefore,
\[
(\alpha L(q^\alpha) - \beta L(q^\beta))^2 = (\alpha - \beta)^2 + \sum_{n=1}^{\infty} \left( 240 \alpha^2 \sigma_3(\frac{n}{\alpha}) + 240 \beta^2 \sigma_3(\frac{n}{\beta}) \right.
\]
\[
+ 48 \alpha (\beta - 6n) \sigma(\frac{n}{\alpha}) + 48 \beta (\alpha - 6n) \sigma(\frac{n}{\beta}) - 1152 \alpha \beta W(\alpha, \beta)(n) \left. \right) q^n
\]
as asserted. \( \square \)
3. Evaluation of the Convolution Sums $W_{(\alpha,\beta)}(n)$ for $\alpha \beta = 14, 22, 26$

In this section, we give explicit formulae for the convolution sums $W_{(2,7)}(n)$, $W_{(1,22)}(n)$, $W_{(2,11)}(n)$, $W_{(1,26)}(n)$ and $W_{(2,13)}(n)$. Note that an explicit formula for the convolution sum $W_{(1,14)}(n)$ is proved by E. Royer [23].

3.1. Bases for $E_4(\Gamma_0(\alpha \beta))$ and $S_4(\Gamma_0(\alpha \beta))$ for $\alpha \beta = 14, 22, 26$. We use the dimension formulae for the space of Eisenstein forms and the space of cusp forms in T. Miyake’s book [18] Thm 2.5.2, p. 60] or W. A. Stein’s book [24] Prop. 6.1, p. 91] to deduce that $\dim(E_4(\Gamma_0(14))) = \dim(E_4(\Gamma_0(22))) = \dim(E_4(\Gamma_0(26))) = 4$, $\dim(S_4(\Gamma_0(14))) = 4$, $\dim(S_4(\Gamma_0(22))) = 7$ and $\dim(S_4(\Gamma_0(26))) = 9$. By Theorem 2.2, the following eta functions

- $A_i(q), 1 \leq i \leq 4$, are elements of $S_4(\Gamma_0(14))$.
  
  $A_1(q) = \frac{\eta(z)^3 \eta(7z)}{\eta(14z)}$  
  $A_2(q) = \frac{\eta^2(z) \eta^2(2z) \eta^2(7z)}{\eta^2(14z)}$  
  $A_3(q) = \frac{\eta^6(z) \eta(14z)}{\eta^3(2z) \eta^2(7z)}$  
  $A_4(q) = \frac{\eta^8(z) \eta^6(11z)}{\eta^5(2z) \eta^4(11z)}$

- $B_i(q), 1 \leq i \leq 7$ are elements of $S_4(\Gamma_0(22))$.
  
  $B_1(q) = \frac{\eta^3(z) \eta^6(11z)}{\eta^2(2z) \eta^3(11z)}$  
  $B_2(q) = \frac{\eta^4(z) \eta^4(11z)}{\eta^2(2z) \eta^2(11z)}$  
  $B_3(q) = \frac{\eta^2(z) \eta^3(2z) \eta^2(11z)}{\eta(z)}$  
  $B_4(q) = \frac{\eta^4(2z) \eta^4(22z)}{\eta^2(z) \eta^2(11z)}$  
  $B_5(q) = \frac{\eta^6(2z) \eta^6(22z)}{\eta^2(z) \eta^4(11z)}$  
  $B_6(q) = \frac{\eta^2(z) \eta^3(11z) \eta^2(22z)}{\eta(z)}$  
  $B_7(q) = \frac{\eta^4(2z) \eta^4(11z)}{\eta^2(z) \eta^2(22z)}$

- $C_i(q), 1 \leq i \leq 9$, are elements of $S_4(\Gamma_0(26))$.
  
  $C_1(q) = \frac{\eta(z)^3 \eta(2z) \eta^3(13z)}{\eta(26z)}$  
  $C_2(q) = \frac{\eta^3(z) \eta^3(2z) \eta(13z) \eta(26z)}{\eta(z)}$  
  $C_3(q) = \frac{\eta(z) \eta^3(2z) \eta^3(13z) \eta(26z)}{\eta(z)}$  
  $C_4(q) = \frac{\eta^3(z) \eta(2z) \eta(13z) \eta^3(26z)}{\eta(z)}$  
  $C_5(q) = \frac{\eta(z) \eta(2z) \eta^3(13z) \eta(26z)}{\eta(z)}$  
  $C_6(q) = \frac{\eta^3(z) \eta(13z) \eta^3(26z)}{\eta(2z)}$  
  $C_7(q) = \frac{\eta^3(2z) \eta^3(13z) \eta(26z)}{\eta(z)}$  
  $C_8(q) = \frac{\eta^3(2z) \eta^3(13z)}{\eta(z)}$  
  $C_9(q) = \frac{\eta^2(z) \eta^2(26z)}{\eta(z) \eta^2(13z)}$

The eta functions

- $A_i(q), 1 \leq i \leq 4$, can be expressed in the form $\sum_{n=1}^{\infty} a_i(n)q^n$;

- $B_i(q), 1 \leq i \leq 7$, can be expressed in the form $\sum_{n=1}^{\infty} b_i(n)q^n$; and

- $C_i(q), 1 \leq i \leq 9$, can be expressed in the form $\sum_{n=1}^{\infty} c_i(n)q^n$. 


Theorem 3.1.  
(a) The sets 
\[ B_{E,14} = \{ M(q^t) \mid t \text{ is a positive divisor of } 14 \}, \]
\[ B_{E,22} = \{ M(q^t) \mid t \text{ is a positive divisor of } 22 \}, \]
\[ B_{E,26} = \{ M(q^t) \mid t \text{ is a positive divisor of } 26 \} \]
constitute bases of \( E_4(\Gamma_0(14)) \), \( E_4(\Gamma_0(22)) \) and \( E_4(\Gamma_0(26)) \), respectively.
(b) The sets \( B_{S,14} = \{ A_i(q) \mid 1 \leq i \leq 4 \} \), \( B_{S,22} = \{ B_i(q) \mid 1 \leq i \leq 7 \} \) and \( B_{S,26} = \{ C_i(q) \mid 1 \leq i \leq 9 \} \) are bases of \( S_4(\Gamma_0(14)) \), \( S_4(\Gamma_0(22)) \) and \( S_4(\Gamma_0(26)) \), respectively.
(c) The sets \( B_{M,14} = B_{E,14} \cup B_{S,14} \), \( B_{M,22} = B_{E,22} \cup B_{S,22} \) and \( B_{M,26} = B_{E,26} \cup B_{S,26} \) constitute bases of \( M_4(\Gamma_0(14)) \), \( M_4(\Gamma_0(22)) \) and \( M_4(\Gamma_0(26)) \), respectively.

Proof. We only give the proof for the case related to 14 since the other two cases are proved similarly.

(a) By Theorem 5.8 in Section 5.3 of [24] p. 86 each \( M(q^t) \) is in \( M_4(\Gamma_0(t)) \), where \( t \) is 
a positive divisor of 14. Since the dimension of \( E_4(\Gamma_0(14)) \) is finite, it suffices to show that \( M(q^t) \) with \( t \mid 14 \) are linearly independent. Suppose that \( x_1, x_2, x_7, x_{14} \in \mathbb{C} \) and \( \sum_{q \mid 14} x_q M(q^5) = 0 \). That is,
\[ \sum_{q \mid 14} x_q + 240 \sum_{q \mid 14} x_q \left( \sum_{q \mid 14} q^5 q^n \right) = 0. \]

We then equate the coefficients of \( q^n \) for \( n = 1, 2, 7, 14 \) to obtain the following system of linear equations
\[ x_1 = 0 \]
\[ 9x_1 + x_2 = 0 \]
\[ 344x_1 + x_7 = 0 \]
\[ 3096x_1 + 344x_2 + 9x_7 + x_{14} = 0 \]
whose unique solution is \( x_1 = x_2 = x_7 = x_{14} = 0 \). So, the set \( B_{E,14} \) is linearly independent. Hence, the set \( B_{E,14} \) is a basis of \( E_4(\Gamma_0(14)) \).

(b) We first show that each \( A_i(q) \), where \( 1 \leq i \leq 4 \), is in the space \( S_4(\Gamma_0(14)) \). That is implicit since \( A_i(q) \) with \( 1 \leq i \leq 4 \) are obtained from an exhaustive search using [Theorem 2.2] (i) \( \rightarrow \) (iv'). Since the dimension of \( S_4(\Gamma_0(14)) \) is 4, it suffices to show that the set \( \{ A_i(q) \mid 1 \leq i \leq 4 \} \) is linearly independent. Suppose that \( x_1, x_2, x_3, x_4 \in \mathbb{C} \) and
\[ x_1 A_1(q) + x_2 A_2(q) + x_3 A_3(q) + x_4 A_4(q) = 0. \]
Then
\[ \sum_{n=1}^{\infty} \left( x_1 a_1(n) + x_2 a_2(n) + x_3 a_3(n) + x_4 a_4(n) \right) q^n = 0. \]
So, when we equate the coefficients of \( q^n \) for \( n = 1, 2, 3, 4 \), we obtain the following system of linear equations
\[ x_1 - 5x_2 + 6x_3 + 5x_4 = 0 \]
\[ x_2 - 2x_3 - 3x_4 = 0 \]
\[ x_3 + x_4 = 0 \]
\[ x_3 - 6x_4 = 0 \]
whose unique solution is $x_1 = x_2 = x_3 = x_4 = 0$. So, the set $B_{5,14}$ is linearly independent. Hence, the set $B_{5,14}$ is a basis of $S_4(\Gamma_0(14))$.

(c) Since $M_4(\Gamma_0(14)) = E_4(\Gamma_0(14)) \oplus S_4(\Gamma_0(14))$, the result follows from (a) and (b).

\[ \Box \]

3.2. Evaluation of $W_{(\alpha, \beta)}(n)$ for $(\alpha, \beta) = (2,7), (1,22), (2,11), (1,26), (2,13)$.

**Lemma 3.2.** We have

\[
(3.1) \quad (2L(q^2) - 7L(q^7))^2 = 25 + \sum_{n=1}^{\infty} \left( \frac{-672}{25} \sigma_3(n) + \frac{21312}{25} \sigma_3\left(\frac{n}{2}\right) + \frac{261072}{25} \sigma_3\left(\frac{n}{7}\right) - \frac{131712}{25} \sigma_3\left(\frac{n}{14}\right) + \frac{672}{25} a_1(n) + \frac{96}{25} a_2(n) + \frac{5376}{25} a_3(n) + 384 a_4(n) \right) q^n, \\
(3.2) \quad (2L(q^2) - 11L(q^{11}))^2 = 81 + \sum_{n=1}^{\infty} \left( \frac{15840}{61} \sigma_3(n) + \frac{37440}{61} \sigma_3\left(\frac{n}{2}\right) + \frac{494208}{61} \sigma_3\left(\frac{n}{11}\right) + \frac{36864}{61} b_1(n) + \frac{357408}{61} b_2(n) + \frac{1160352}{61} b_3(n) + \frac{1539072}{61} b_4(n) + \frac{834048}{61} b_5(n) - 22176 b_6(n) - 864 b_7(n) \right) q^n, \\
(3.3) \quad (L(q) - 26L(q^{26}))^2 = 625 + \sum_{n=1}^{\infty} \left( \frac{19152}{85} \sigma_3(n) - \frac{4992}{85} \sigma_3\left(\frac{n}{2}\right) - \frac{120912}{85} \sigma_3\left(\frac{n}{13}\right) + \frac{12946752}{85} \sigma_3\left(\frac{n}{13}\right) + \frac{82848}{85} c_1(n) - \frac{4128}{17} c_2(n) + \frac{61920}{17} c_3(n) - \frac{177216}{85} c_4(n) - \frac{53664}{17} c_5(n) + \frac{1077024}{85} c_7(n) + \frac{291072}{85} c_8(n) - \frac{1248}{85} c_9(n) \right) q^n, \\
(3.4) \quad (2L(q^2) - 13L(q^{13}))^2 = 121 + \sum_{n=1}^{\infty} \left( \frac{-1248}{85} \sigma_3(n) + \frac{76608}{85} \sigma_3\left(\frac{n}{2}\right) + \frac{3236688}{85} \sigma_3\left(\frac{n}{26}\right) - \frac{843648}{85} \sigma_3\left(\frac{n}{26}\right) + \frac{1248}{85} c_1(n) + \frac{12192}{17} c_2(n) + \frac{52128}{17} c_3(n) + \frac{181824}{85} c_4(n) + \frac{158496}{17} c_5(n) + \frac{16224}{85} c_7(n) - \frac{35328}{85} c_8(n) - \frac{82848}{85} c_9(n) \right) q^n.
\]

**Proof.** Since the other cases are proved similarly, we only give the proof for $(2L(q^2) - 7L(q^7))^2$.

We apply Lemma 2.3 with $\alpha = 2$ and $\beta = 7$ and we use Theorem 3.1 (c) to infer that there exist $x_1, x_2, x_7, x_{14}, y_1, y_2, y_3, y_4 \in \mathbb{C}$ such that

\[
(2L(q^2) - 7L(q^7))^2 = \sum_{\delta|14} x_\delta M(q^\delta) + \sum_{j=1}^{4} y_j A_j(q) \\
= \sum_{\delta|14} x_\delta + \sum_{i=1}^{\infty} \left( \frac{240}{\delta} \sum_{\delta|14} x_\delta \sigma_3\left(\frac{n}{\delta}\right) + \sum_{j=1}^{4} y_j a_j(n) \right) q^n.
\]
Now, when we equate the right hand side of Equation 3.5 with that of Equation 2.3, and when we take the coefficients of $q^n$ for which $n = 1, 2, 3, 4, 5, 7, 9, 14$ for example, we obtain a system of linear equations whose solution is unique. Hence, we obtain the stated result.

Now we state and prove our main result of this Subsection.

**Theorem 3.3.** Let $n$ be a positive integer. Then

$$W_{(2,7)}(n) = \frac{1}{600} \sigma_3(n) + \frac{1}{150} \sigma_3\left(\frac{n}{2}\right) + \frac{49}{600} \sigma_3\left(\frac{n}{7}\right) + \frac{49}{150} \sigma_3\left(\frac{n}{14}\right) + \left(\frac{1}{24} - \frac{1}{28} n\right) \sigma_1(n)$$

$$+ \left(\frac{1}{24} - \frac{1}{8} n\right) \sigma\left(\frac{n}{2}\right) - \frac{1}{600} a_1(n) - \frac{1}{4200} a_2(n) - \frac{1}{75} a_3(n) - \frac{1}{42} a_4(n),$$

$$W_{(1,22)}(n) = \frac{17}{1464} \sigma_3(n) - \frac{1}{122} \sigma_3\left(\frac{n}{2}\right) + \frac{35}{488} \sigma_3\left(\frac{n}{11}\right) + \frac{125}{366} \sigma_3\left(\frac{n}{22}\right)$$

$$+ \left(\frac{1}{24} - \frac{1}{88} n\right) \sigma(n) + \left(\frac{1}{24} - \frac{1}{4} n\right) \sigma\left(\frac{n}{22}\right) - \frac{71}{2684} b_1(n) - \frac{159}{5368} b_2(n)$$

$$- \frac{69}{3315} b_3(n) - \frac{32}{671} b_4(n) + \frac{2}{61} b_5(n) - \frac{7}{8} b_6(n) - \frac{3}{88} b_7(n),$$

$$W_{(2,11)}(n) = -\frac{5}{488} \sigma_3(n) + \frac{5}{366} \sigma_3\left(\frac{n}{2}\right) + \frac{137}{1464} \sigma_3\left(\frac{n}{11}\right) + \frac{39}{122} \sigma_3\left(\frac{n}{22}\right)$$

$$+ \left(\frac{1}{24} - \frac{1}{4} n\right) \sigma\left(\frac{n}{2}\right) + \left(\frac{1}{24} - \frac{1}{8} n\right) \sigma\left(\frac{n}{11}\right) + \frac{16}{671} b_1(n) - \frac{1241}{5368} b_2(n)$$

$$- \frac{4029}{5368} b_3(n) - \frac{668}{671} b_4(n) + \frac{362}{671} b_5(n) + \frac{7}{8} b_6(n) + \frac{3}{88} b_7(n),$$

$$W_{(1,26)}(n) = \frac{1}{2040} \sigma_3(n) + \frac{1}{510} \sigma_3\left(\frac{n}{2}\right) + \frac{169}{2040} \sigma_3\left(\frac{n}{13}\right) + \frac{169}{510} \sigma_3\left(\frac{n}{26}\right)$$

$$+ \left(\frac{1}{24} - \frac{1}{104} n\right) \sigma(n) + \left(\frac{1}{24} - \frac{1}{4} n\right) \sigma\left(\frac{n}{26}\right) - \frac{863}{26520} c_1(n) + \frac{43}{3315} c_2(n)$$

$$- \frac{215}{1768} c_3(n) + \frac{71}{1020} c_4(n) + \frac{1}{408} c_5(n) - \frac{863}{2040} c_7(n) - \frac{379}{3315} c_8(n)$$

$$+ \frac{1}{2040} c_9(n),$$

$$W_{(2,13)}(n) = \frac{1}{2040} \sigma_3(n) + \frac{1}{510} \sigma_3\left(\frac{n}{2}\right) + \frac{169}{2040} \sigma_3\left(\frac{n}{13}\right) + \frac{169}{510} \sigma_3\left(\frac{n}{26}\right)$$

$$+ \left(\frac{1}{24} - \frac{1}{52} n\right) \sigma\left(\frac{n}{2}\right) + \left(\frac{1}{24} - \frac{1}{8} n\right) \sigma\left(\frac{n}{13}\right) - \frac{1}{2040} c_1(n) - \frac{127}{5304} c_2(n)$$

$$- \frac{181}{1768} c_3(n) - \frac{947}{13260} c_4(n) - \frac{127}{408} c_5(n) - \frac{13}{2040} c_7(n) + \frac{46}{3315} c_8(n)$$

$$+ \frac{863}{26520} c_9(n).$$
Proof. We give the proof for the $W_{(2,7)}(n)$ since the other cases are proved similarly.

It follows immediately when we set $\alpha = 2$ and $\beta = 7$ in the right hand side of Equation 2.3, equate the so-obtained result with the right hand side of Equation 3.1 and solve for $W_{(2,7)}(n)$.

\[ \square \]

4. Number of Representations of a Positive Integer $n$ by the Octonary Quadratic Form using $W_{(1,4)}(n), W_{(3,4)}(n), W_{(3,8)}(n)$ and $W_{(4,9)}(n)$

The following number of representations of a positive integer $n$ are applications of the results of the evaluation of some convolution sums by J. G. Huard et al. [12], A. Alaca et al. [13] and D. Ye [29].

Let $n \in \mathbb{N}_0$ and the number of representations of $n$ by the quaternary quadratic form $x_1^2 + x_2^2 + x_3^2 + x_4^2$ be denoted by $r_4(n)$. That means,

\[ r_4(n) = \text{card}(\{(x_1, x_2, x_3, x_4) \in \mathbb{Z}^4 \mid m = x_1^2 + x_2^2 + x_3^2 + x_4^2\}). \]

We set $r_4(0) = 1$. For all $n \in \mathbb{N}$, the following Jacobi’s identity is proved in K. S. Williams’ book [27] Thrm 9.5, p. 83]

\[ (4.1) \quad r_4(n) = 8\sigma(n) - 32\sigma\left(\frac{n}{4}\right). \]

Let furthermore the number of representations of $n$ by the octonary quadratic form

\[ a(x_1^2 + x_2^2 + x_3^2 + x_4^2) + b(x_5^2 + x_6^2 + x_7^2 + x_8^2) \]

be denoted by $N_{(a,b)}(n)$. That means,

\[ N_{(a,b)}(n) = \text{card}(\{(x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8) \in \mathbb{Z}^8 \mid n = a(x_1^2 + x_2^2 + x_3^2 + x_4^2) + b(x_5^2 + x_6^2 + x_7^2 + x_8^2)\}). \]

We then infer the following result:

**Theorem 4.1.** Let $n \in \mathbb{N}$ and $(a, b) = (1, 1), (1, 3), (2, 3), (1, 9)$. Then

\[ N_{(1,1)}(n) = 16\sigma(n) - 64\sigma\left(\frac{n}{4}\right) + 64W_{(1,1)}(n) + 1024W_{(1,1)}\left(\frac{n}{4}\right) - 512W_{(1,4)}(n) \]

\[ = 16\sigma_3(n) - 32\sigma_3\left(\frac{n}{2}\right) + 256\sigma_3\left(\frac{n}{4}\right), \]

\[ N_{(1,3)}(n) = 8\sigma(n) - 32\sigma\left(\frac{n}{2}\right) + 8\sigma\left(\frac{n}{3}\right) - 32\sigma\left(\frac{n}{12}\right) \]

\[ + 64W_{(1,3)}(n) + 1024W_{(1,3)}\left(\frac{n}{4}\right) - 256\left(W_{(3,4)}(n) + W_{(1,12)}(n)\right), \]

\[ N_{(2,3)}(n) = 8\sigma\left(\frac{n}{2}\right) - 32\sigma\left(\frac{n}{8}\right) + 8\sigma\left(\frac{n}{3}\right) - 32\sigma\left(\frac{n}{12}\right) \]

\[ + 64W_{(1,3)}(n) + 1024W_{(1,3)}\left(\frac{n}{4}\right) - 256\left(W_{(3,8)}(n) + W_{(1,12)}(n)\right), \]

\[ N_{(1,9)}(n) = 8\sigma(n) - 32\sigma\left(\frac{n}{4}\right) + 8\sigma\left(\frac{n}{9}\right) - 32\sigma\left(\frac{n}{36}\right) \]

\[ + 64W_{(1,9)}(n) + 1024W_{(1,9)}\left(\frac{n}{4}\right) - 256\left(W_{(4,9)}(n) + W_{(1,36)}(n)\right). \]

Proof. We only prove the case $N_{(1,3)}(n)$ since the other cases are proved similarly.
It holds that
\[ N_{(1,3)}(n) = \sum_{\substack{(l,m) \in \mathbb{N}^2 \setminus \mathbb{N}_0^2 \\mid \, l+3m=n}} r_4(l)r_4(m) = r_4(n)r_4(0) + r_4(0)r_4\left(\frac{n}{3}\right) + \sum_{\substack{(l,m) \in \mathbb{N}^2 \\mid \, l+3m=n}} r_4(l)r_4(m) \]

We make use of Equation 4.1 to derive
\[
N_{(1,3)}(n) = 8\sigma(n) - 32\sigma\left(\frac{n}{4}\right) + 8\sigma\left(\frac{n}{3}\right) - 32\sigma\left(\frac{n}{12}\right) + \sum_{\substack{(l,m) \in \mathbb{N}^2 \\mid \, l+3m=n}} (8\sigma(l) - 32\sigma\left(\frac{l}{4}\right))(8\sigma(m) - 32\sigma\left(\frac{m}{4}\right)).
\]

We observe that
\[
(8\sigma(l) - 32\sigma\left(\frac{l}{4}\right))(8\sigma(m) - 32\sigma\left(\frac{m}{4}\right)) = 64\sigma(l)\sigma(m) - 256\sigma\left(\frac{l}{4}\right)\sigma\left(\frac{m}{4}\right) - 256\sigma(l)\sigma\left(\frac{m}{4}\right) + 1024\sigma\left(\frac{l}{4}\right)\sigma\left(\frac{m}{4}\right).
\]

The evaluation of
\[ W_{(1,3)}(n) = \sum_{\substack{(l,m) \in \mathbb{N}^2 \\mid \, l+3m=n}} \sigma(l)\sigma(m) \]

is shown by J. G. Huard et al. [12]. We map \( l \) to \( 4l \) to infer
\[ W_{(4,3)}(n) = \sum_{\substack{(l,m) \in \mathbb{N}^2 \\mid \, l+3m=n}} \sigma\left(\frac{l}{4}\right)\sigma(m) = \sum_{\substack{(l,m) \in \mathbb{N}^2 \\mid \, 4l+3m=n}} \sigma(l)\sigma(m). \]

The evaluation of \( W_{(4,3)}(n) = W_{(3,4)}(n) \) is proved by A. Alaca et al. [11]. We next map \( m \) to \( 4m \) to conclude
\[ W_{(1,12)}(n) = \sum_{\substack{(l,m) \in \mathbb{N}^2 \\mid \, l+3m=n \\text{and} \, m=4n}} \sigma(l)\sigma\left(\frac{m}{4}\right) = \sum_{\substack{(l,m) \in \mathbb{N}^2 \\mid \, l+12m=n}} \sigma(l)\sigma(m). \]

A. Alaca et al. [11] have shown the evaluation of \( W_{(1,12)}(n) \). We simultaneously map \( l \) to \( 4l \) and \( m \) to \( 4m \) to deduce
\[ \sum_{\substack{(l,m) \in \mathbb{N}^2 \\mid \, l+3m=n}} \sigma\left(\frac{l}{4}\right)\sigma\left(\frac{m}{4}\right) = \sum_{\substack{(l,m) \in \mathbb{N}^2 \\mid \, l+3m=n \\text{and} \, m=4n}} \sigma(l)\sigma(m) = W_{(1,3)}\left(\frac{n}{4}\right) \]

J. G. Huard et al. [12] have proved the evaluation of \( W_{(1,3)}(n) \).

We then put these evaluations together to obtain the stated result for \( N_{(1,3)}(n) \).

\[ \square \]

5. Concluding Remark and Future Work

As displayed on Table 4, convolution sums are so far evaluated individually, i.e., there is no evaluation of the convolution sums for a class of positive integers. Since convolution sums are used to determine explicit formulae for the number of representations of a positive integer \( n \) by the octonary quadratic forms in Equation 1.3 and Equation 1.4, respectively, there is no explicit formulae for the number of representations for a class of positive integers by the octonary quadratic forms as well. This is a work in progress.
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### Tables

**Table 4**: Known convolution sums \( W_{(\alpha, \beta)}(n) \)

| \((\alpha, \beta)\) | Authors | References |
|-------------------|---------|------------|
| (1,1)             | M. Besge, J. W. L. Glaisher, S. Ramanujan | [7][11][22] |
| (1,2), (1,3), (1,4) | J. G. Huard & Z. M. Ou & B. K. Spearman & K. S. Williams | [12] |
| (1,5), (1,7)      | M. Lemire & K. S. Williams, S. Cooper & P. C. Toh | [16][9] |
| (1,6), (2,3)      | S. Alaca & K. S. Williams | [6] |
| (1,8), (1,9)      | K. S. Williams | [26][25] |
| (1,10), (1,11), (1,13), (1,14) | E. Royer | [23] |
| (1,12), (1,16), (1,18), (1,24), (2,9), (3,4), (3,8) | A. Alaca & S. Alaca & K. S. Williams | [1][2][3][4] |
| (1,15), (3,5)     | B. Ramakrishnan & B. Sahu | [21] |
| (1,20), (2,5), (4,5) | S. Cooper & D. Ye | [10] |
| (1,23)            | H. H. Chan & S. Cooper | [8] |
| (1,25)            | E. X. W. Xia & X. L. Tian & O. X. M. Yao | [28] |
| (1,27), (1,32)    | S. Alaca & Y. Kesicioğlu | [5] |
| (1,36), (4,9)     | D. Ye | [29] |

**Table 5**: Known representations of \( n \) by the form Equation 1.3

| \((a, b)\) | Authors | References |
|-----------|---------|------------|
| (1,2)     | K. S. Williams | [26] |
| (1,4)     | A. Alaca & S. Alaca & K. S. Williams | [2] |
| (1,5)     | S. Cooper & D. Ye | [10] |
| (1,6)     | B. Ramakrishnan & B. Sahu | [21] |
| (1,8)     | S. Alaca & Y. Kesicioğlu | [5] |

Table 4: Known convolution sums \( W_{(\alpha, \beta)}(n) \)

Table 5: Known representations of \( n \) by the form Equation 1.3
EVALUATION OF THE CONVOLUTION SUM FOR 14, 22 AND 26

CENTRE FOR RESEARCH IN ALGEBRA AND NUMBER THEORY, SCHOOL OF MATHEMATICS AND STATISTICS, CARLETON UNIVERSITY, 1125 COLONEL BY DRIVE, OTTAWA, ONTARIO, K1S 5B6, CANADA

E-mail address: aalaca@math.carleton.ca
E-mail address: salaca@math.carleton.ca
E-mail address: ebenezer.ntienjem@carleton.ca; ntienjem@gmail.com