Stable determination of coefficients in the dynamical Schrödinger equation in a magnetic field

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Abstract
In this paper we consider the inverse problem of determining, on a compact
Riemannian manifold, the electric potential or the magnetic field in a
Schrödinger equation via Dirichlet data from measured Neumann boundary
observations. This information is enclosed in the dynamical Dirichlet-
to-Neumann map associated to the magnetic Schrödinger equation. We prove
in dimension \( \dim \geq 2 \) that the knowledge of the Dirichlet-to-Neumann map for
the Schrödinger equation uniquely determines the magnetic field and the
electric potential and we establish Hölder-type stability.

Keywords: inverses problems, stability estimates, Dirichlet-to-Neumann
map, Schrödinger equation

1. Introduction and main results

This article is devoted to the study of the following inverse boundary value problem: given
a Riemannian manifold with boundary determine the magnetic potential in a dynamical
Schrödinger equation in a magnetic field from the observations made at the boundary. Let
\((\mathcal{M}, g)\) be a smooth and compact Riemannian manifold with boundary \(\partial \mathcal{M}\). We denote by \(\Delta\)
the Laplace–Beltrami operator associated to the Riemannian metric \(g\). In local coordinates,
g\(x = (g_{jk}), \) the Laplace operator \(\Delta\) is given by

\[
\Delta = \frac{1}{\sqrt{|g|}} \sum_{j,k=1}^{n} \frac{\partial}{\partial x_j} \left( \sqrt{|g|} g^{jk} \frac{\partial}{\partial x_k} \right)
\]

Here \((g^{jk})\) is the inverse of the metric \(g\) and \(|g| = \det(g_{jk})\). In this paper we study an inverse
problem for the dynamical Schrödinger equation in the presence of a magnetic potential.
Given \( T > 0 \), we denote \( Q = (0, T) \times \mathcal{M} \) and \( \Sigma = (0, T) \times \partial \mathcal{M} \). We consider the following initial boundary value problem for the magnetic Schrödinger equation with a magnetic potential \( A \) and electric potential \( V \),

\[
\begin{align*}
(i\partial_t + \mathcal{H}_{A,V})u &= 0 \quad \text{in } Q, \\
u(0, \cdot) &= 0 \quad \text{in } \mathcal{M}, \\
\begin{cases}
\end{cases} \\
u = f \quad \text{on } \Sigma,
\end{align*}
\]

where

\[
\mathcal{H}_{A,V} = \frac{1}{\sqrt{|\mathcal{M}|}} \sum_{j,k=1}^n \left( \frac{\partial}{\partial x_j} - i a_j \right) \sqrt{|\mathcal{M}|} g^{jk} \left( \frac{\partial}{\partial x_k} - i a_k \right) + V = \Delta - 2i A \cdot \nabla - i \delta A - |A|^2 + V.
\]

Here \( V : \mathcal{M} \to \mathbb{R} \) is the electric potential and \( A = a_j dx^j \) is a covector field (1-form) with real-valued coefficients \( a_j \in C^\infty(\mathcal{M}) \) is the magnetic potential and \( \delta \) is the coderivative (codifferential) operator sending 1-forms to a function by the formula

\[
\delta A = \frac{1}{\sqrt{|\mathcal{M}|}} \sum_{j,k=1}^n \frac{\partial}{\partial x^j} \left( g^{jk} \sqrt{|\mathcal{M}|} a_k \right).
\]

The norm of the covector field \( A \) and the product \( A \cdot \nabla \) are given by

\[
|A|^2 = \sum_{j,k=1}^n g^{jk} a_j a_k, \quad A \cdot \nabla = \sum_{j,k=1}^n g^{jk} a_j \frac{\partial}{\partial x^k}.
\]

We may define the Dirichlet-to-Neumann (DN) map associated with magnetic Schrödinger operator \( \mathcal{H}_{A,V} \) by

\[
\Lambda_{A,V}(f) = (\partial_{\nu} + iA(\nu))u, \quad f \in H^{2,1}_0(\Sigma),
\]

where \( \nu = \nu(x) \) denotes the unit outward normal to \( \partial \mathcal{M} \) at \( x \) and \( H^{2,1}_0(\Sigma) \) is anisotropic Sobolev space defined below. The solution of the Schrödinger equation is the wave function \( u \), which describes the state of a particle moving in the potential \( V \) and in the magnetic potential \( A \).

We consider the inverse problem to know whether the DN map \( \Lambda_{A,V} \) determines uniquely the magnetic potential \( A \) and the electric potential \( V \).

In the absence of magnetic potential, the identifiability problem of the electric potential \( V \) was solved by [13]. In the presence of a magnetic potential \( A \), let us observe that there is an obstruction to uniqueness. In fact as it was noted in [19], the DN map is invariant under the gauge transformation of the magnetic potential. Specifically, given \( \varphi \in C^\infty(\mathcal{M}) \) such that \( \varphi|_{\partial\mathcal{M}} = 0 \) one has

\[
e^{-i\varphi} \mathcal{H}_{A_0,V} e^{i\varphi} = \mathcal{H}_{A_0 + d\varphi,V}, \quad e^{-i\varphi} \Lambda_{A_0,V} e^{i\varphi} = \Lambda_{A_0 + d\varphi,V} = \Lambda_{A,V}, \quad d\varphi = \sum_{j=1}^n \frac{\partial \varphi}{\partial x^j} dx^j.
\]

Therefore, the magnetic potential \( A \) cannot be uniquely determined by the DN map \( \Lambda_{A,V} \). From a geometric view point this can be seen as follows. Since \( \mathcal{M} \) is a compact Riemannian manifold with boundary, for every covector \( A \in H^2(\mathcal{M}, T^*\mathcal{M}) \) there exist uniquely determined \( A' \in H^2(\mathcal{M}, T^*\mathcal{M}) \) and \( \varphi \in H^{2+1}(\mathcal{M}) \) such that:

\[
A = A' + d\varphi, \quad \delta A' = 0, \quad \varphi|_{\partial\mathcal{M}} = 0.
\]
We call the fields \( \mathbf{A}' \) and \( \varphi \) the solenoidal and potential parts of the covector \( \mathbf{A} \). The non-uniqueness manifested in (1.4) says that the best we could hope to reconstruct from the DN map \( \Lambda_{AV} \) is the solenoidal part \( \mathbf{A}' \) of the covector \( \mathbf{A} \).

Physically, our inverse problem consists in determining the magnetic field \( \mathbf{A}' \) induced by the magnetic potential \( \mathbf{A} \) of an anisotropic medium by probing it with disturbances generated on the boundary. The data are responses of the medium to these disturbances which are measured on the boundary and the goal is to recover the magnetic field \( \mathbf{A}' \) which describes the property of the medium. Here we assume that the medium is quiet initially and \( f \) is a disturbance which is used to probe the medium. Roughly speaking, the data is \( (\partial_\nu + i \nu \cdot \mathbf{A})u \) measured on the boundary for different choices of \( f \).

The uniqueness in the determination, from the DN map, of electromagnetic potential appearing in a Schrödinger equation in a domain with obstacles was proved by Eskin [19]. The main ingredient in his proof is the construction of geometric optics solutions. Using this geometric optics construction Salazar [39] shows that the boundary data allows us to recover integrals of the potentials along light rays and he establishes the uniqueness of these potentials modulo a gauge transform. Also, a logarithmic stability estimate is obtained and the presence of obstacles inside the domain is studied. In [2], Avdonin et al use the so-called BC (boundary control) method to prove that the DN map determines the electrical potential in a one dimensional Schrödinger equation.

In recent years significant progress has been made regarding the problem of identifying the electrical potential. In [36], Rakesh and Symes prove that the DN map uniquely determines the time-independent potential in a wave equation. Ramm and Sjöstrand [37] have extended the result in [36] to the case of time-dependent potentials. Isakov [23] has considered the simultaneous determination of a zeroth order coefficient and a damping coefficient. A key ingredient in the existing results is the construction of complex geometric optics solutions of the wave equation, concentrated along a line, and the relationship between the hyperbolic DN map and the x-ray transform play a crucial role. For the wave equation with a lower order term \( q(t, x) \), Waters [48] proves that we can recover the x-ray transform of time dependent potentials \( q(t, x) \) from the dynamical DN map in a stable way. He derives conditional Hölder stability estimates for the x-ray transform of \( q(t, x) \).

The uniqueness by a local DN map is well solved (e.g. Belishev [5], Eskin [19, 20], Katchlov et al [26], Kurylev and Lassas [28]). The stability estimates in the case where the DN map is considered on the whole lateral boundary were established in Stefanov and Uhlmann [41], Sun [44], Bellassoued and Dos Santos Ferreira [12]. In [33] Montalo proves Hölder type stability estimates near generic simple Riemannian metrics for the inverse problem of recovering simultaneously the metric, the magnetic field, and electric potential from the associated hyperbolic DN map modulo a class of gauge transformation.

In the case of the Schrödinger equation, Avdonin and Belishev gave an affirmative answer to this question for smooth metrics conformal to the Euclidean metric in [3]. Their approach is based on the BC method introduced by Belishev [5] and uses, in an essential way, a unique continuation property. Because of the use of this qualitative property, it seems unlikely that the BC method would provide accurate stability estimates. More precisely, when \( \mathcal{M} \) is a bounded domain of \( \mathbb{R}^n \), and \( \varrho, q \in C^2(\mathcal{M}) \) are real functions, Avdonin and Belishev [3] show that for any fixed \( T > 0 \) the response operator (or the Neumann-to-Dirichlet map) of the Schrödinger equation \( i\varrho \partial_t u + \Delta u - qu = 0 \) uniquely determines the coefficients \( \varrho \) and \( q \). The problem is reduced to recovering \( \varrho, q \) from the boundary spectral data. The spectral data are extracted from the response operator by the use of a variational principle.
The analogue problem for the wave equation has a long history. Unique determination of the metric goes back to Belishev and Kurylev [6] using the BC method and involves works of Katchalov et al [26], Kurylev and Lassas [28], Lassas and Oksanen [29] and Anderson et al [1]. In fact, Katchalov et al proved that the determination of the metric from the DN map was equivalent for the wave and Schrödinger equations (as well as other related inverse problems) in [27].

The importance of control theory for inverse problems was first understood by Belishev [5]. He used control theory to develop the first variant of the control (BC) method. This method gives an efficient way to reconstruct a Riemannian manifold via its response operator (dynamical DN map) or spectral data (a spectrum of the Beltrami–Laplace operator and traces of normal derivatives of the eigenfunctions), themselves, whereas the coefficients on these manifolds are recovered automatically. More precisely, let \( \mathcal{M} \) and \( \mathcal{M}' \) be two smooth compact manifolds with mutual boundary \( \partial \mathcal{M} = \partial \mathcal{M}' = \Gamma \) endowed with smooth potentials \( q \) and \( q' \) respectively, \( \Lambda_{q, \mathcal{M}} \) and \( \Lambda_{q', \mathcal{M}'} \) their DN map on \( (0, T) \times \partial \mathcal{M} \); if \( \Lambda_{q, \mathcal{M}} = \Lambda_{q', \mathcal{M}'} \) then there exists a diffeomorphism \( \Psi : \mathcal{M} \rightarrow \mathcal{M}' \times \Gamma \) such that \( \Psi \circ \partial \mathcal{M} = \partial \mathcal{M}' \) and \( q = q' \circ \Psi \).

As for the stability of the wave equation in the Euclidian case, we also refer to [44] and [24]; in those papers, the DN map was considered on the whole boundary. Isakov and Sun [24] proved that the difference in some subdomain of two coefficients is estimated by an operator norm of the difference of the corresponding local DN maps, and that the estimate is of Hölder type. Bellassoued et al [11] considered the inverse problem of recovering a time independent potential in the hyperbolic equation from the partial DN map. They proved a logarithm stability estimate. Moreover in [35] it is proved that if an unknown coefficient belongs to a given finite dimensional vector space, then the uniqueness follows by a finite number of measurements on the whole boundary. In [8], Bellassoued and Benjoud used complex geometrical optics solutions concentrating near lines in any direction to prove that the DN map determines uniquely the magnetic field induced by a magnetic potential in a magnetic wave equation.

In [7], the authors treat the inverse problem of determining two time-dependent coefficients appearing in a dissipative wave equation in the Euclidian case, from measured boundary observations. The authors establish, in dimension \( n \geq 2 \), stability estimates with respect to the DN map of these coefficients, provided that there are no outside cloaking regions. The generalization of this result to the Riemannian case seems very difficult and constitutes a major open problem, since the geodesic x-ray transform of a time dependent function is far from understood.

In the case of the anisotropic wave equation, the problem of establishing stability estimates in determining the metric was studied by Stefanov and Uhlmann in [41, 42] for metrics close to Euclidean and generic simple metrics. In [12], the author and Dos Santos Ferreira proved stability estimates for the wave equation in determining a conformal factor close to 1 and time independent potentials in simple geometries. We refer to this paper for a longer bibliography in the case of the wave equation. In [30] Liu and Oksanen consider the problem of reconstructing a wave speed \( c \) from acoustic boundary measurements modelled by the hyperbolic DN map. They introduced a reconstruction formula for \( c \) that is based on the BC method and also incorporates features from the complex geometric optics solutions approach.

For the DN map for an elliptic equation, the paper by Calderón [15] is a pioneering work. We also refer to Bukhgeim and Uhlmann [14], Hech-Wang [21], Salo [38] and Uhlmann [46] as a survey. In [18] Dos Santos Ferreira et al prove that the knowledge of the Cauchy data for the Schrödinger equation in the presence of magnetic potential, measured on a possibly very small subset of the boundary, uniquely determines the magnetic field. In [45], Tzou proves a log log-type estimate which show that the magnetic field and the electric potential of
the magnetic Schrödinger equation depends stably on the DN map even when the boundary measurement is taken only on a subset that is slightly larger than half of the boundary. In [17], Cheng and Yamamoto prove that the stability estimations imply the convergence rate of the Tikhonov regularized solutions.

The main goal of this paper is to study the stability of the inverse problem for the dynamical anisotropic Schrödinger equation with magnetic and electric potentials. We follow the same strategy as in [12] inspired by the works of Dos Santos Ferreira et al [18], Stefanov and Uhlmann [41, 42] and Bellassoued and Choulli [9].

In the present paper, we prove a Hölder-type estimate which shows that a magnetic field \( A' \) induced by a magnetic potential and the electric potential depends stably on the DN map \( \Lambda_{AV} \).
\[ T^*_x \mathcal{M} \to T^*_x \mathcal{M}, \quad \mathcal{A} \mapsto \mathcal{A}^l, \quad (1.5) \]
given in local coordinates by
\[ (a^i dx^l)^j = a^i \partial_j, \quad a^i = \sum_{k=1}^n g^{ik} a_k, \quad (1.6) \]
where \((dx^1, \ldots, dx^n)\) is the basis in the space \(T^*_x \mathcal{M}\) which is the dual to the basis \((\partial_1, \ldots, \partial_n)\). For the Riemannian manifold \((\mathcal{M}, g)\) we define the inner product of 1-forms in \(T^*_x \mathcal{M}\) by
\[ \langle A, B \rangle = \langle A^l, B^l \rangle = \sum_{j,k=1}^n g^{jk} a^l b^k, \quad (1.7) \]
The metric tensor \(g\) induces the Riemannian volume \(dv^n = |g|^{1/2} dx_1 \wedge \cdots \wedge dx_n\). We denote by \(L^2(\mathcal{M})\) the completion of \(C^\infty(\mathcal{M})\) endowed with the usual inner product
\[ \langle f_1, f_2 \rangle = \int_{\mathcal{M}} f_1(x) \overline{f_2(x)} dv^n, \quad f_1, f_2 \in C^\infty(\mathcal{M}). \quad (1.8) \]
A smooth section of vector bundle \(E\) over the Riemannian manifold \(\mathcal{M}\) is a smooth map \(s : \mathcal{M} \to E\) such that for each \(x \in \mathcal{M}\), \(s(x)\) belongs to the fiber over \(x\). We denote by \(C^\infty(\mathcal{M}, E)\) the space of smooth sections of the vector bundle \(E\). Using this, we denote by \(C^\infty(\mathcal{M}, T^* \mathcal{M})\) the space of smooth vector fields on \(\mathcal{M}\) and \(C^\infty(\mathcal{M}, T \mathcal{M})\) the space of smooth 1-forms on \(\mathcal{M}\). Similarly, we may define the spaces \(L^2(\mathcal{M}, T^* \mathcal{M})\) (resp. \(L^2(\mathcal{M}, T \mathcal{M})\)) of square integrable 1-forms (resp. vectors) by using the inner product
\[ \langle A, B \rangle = \int_{\mathcal{M}} (A^l B^l) dv^n, \quad A, B \in T^* \mathcal{M}. \]
Let \(T^k \mathcal{M}\) be the space of tensor fields of type \(k\) on \(T^*_x \mathcal{M}\). We denote by \(T^k \mathcal{M}\) the tensor bundle of type \(k\). In the local coordinate system a \(k\)-tensor field \(u\) can be written as
\[ t = t_{h_1} \cdots t_{h_k} dx^{h_1} \otimes \cdots \otimes dx^{h_k}. \]
For each \(x \in \mathcal{M}\), \(T^k \mathcal{M}\) is endowed with an inner product as follows
\[ \langle t_1, t_2 \rangle = \sum_{j_1, \ldots, j_k=1}^n t_1(\partial_{j_1}, \ldots, \partial_{j_k}) t_2(\partial_{j_1}, \ldots, \partial_{j_k}). \]
Let \(C^\infty(\mathcal{M}, T^k \mathcal{M})\) the space of the smooth \(k\)-tensor fields on \(\mathcal{M}\). In view of (1.8), we denote by \(L^2(\mathcal{M}, T^k \mathcal{M})\) the space of square integrable \(k\)-tensors fields on \(\mathcal{M}\) as the completion of \(C^\infty(\mathcal{M}, T^k \mathcal{M})\) endowed with the following inner product
\[ \langle t_1, t_2 \rangle = \int_{\mathcal{M}} (t_1, t_2) dv^n, \quad t_1, t_2 \in T^k \mathcal{M}. \]
The Sobolev space \(H^k(\mathcal{M})\) is the completion of \(C^\infty(\mathcal{M})\) with respect to the norm \(\| \cdot \|_{H^k(\mathcal{M})}\)
\[ \| f \|^2_{H^k(\mathcal{M})} = \| f \|^2_{L^2(\mathcal{M})} + \sum_{k=1}^n \| \nabla^k f \|^2_{L^2(\mathcal{M}, T^k \mathcal{M})}, \]
where \(\nabla^k\) is the covariant differential of \(f\) in the metric \(g\). If \(f\) is a \(C^\infty\) function on \(\mathcal{M}\), then \(\nabla f\) is the vector field such that
\[ X(f) = \langle \nabla f, X \rangle, \]
for all vector fields \(X\) on \(\mathcal{M}\). This reads in coordinates
The normal derivative is
\[ \partial_n u := \langle \nabla u, \nu \rangle = \sum_{j,k=1}^n g^{jk} \frac{\partial u}{\partial x_k}, \]
where \( \nu \) is the unit outward vector field to \( \partial M \).

Likewise, we say that 1-form \( A = a_i dx^i \) in \( H^k(M, T^*M) \) if each component \( a_j \) in \( H^k(M) \), which can be viewed as the Hilbert space with respect to the norm
\[ \| A \|_{H^k(M, T^*M)} = \sum_{j=1}^n \| a_j \|_{H^k(M)}. \]

Before stating our main results on the inverse problem, we give the following result concerning the wellposedness of the initial boundary problem (1.1), when \( u \) is a weak solution in the class \( C^{1,0}(\overline{M}, T^*M) \). The following theorem gives conditions on \( f, A \) and \( V \), which guarantee uniqueness and continuous dependence on the data of the solutions of the magnetic Schrödinger equation (1.1) with non-homogeneous Dirichlet boundary condition.

We denote \( \Omega^2(M) \) the vector space of smooth 2-forms on \( M \). In local coordinates 2-form \( \omega \) can be represented as
\[ \omega = \omega_{jk} dx^j \wedge dx^k, \]
where \( \omega_{jk} \) are smooth real-valued functions on \( M \). For smooth and compactly supported 2-form \( \omega \) in \( M \), we define the Sobolev norm \( H^s(M, \Omega^2(M)) \), \( s \in \mathbb{R} \), by
\[ \| \omega \|_{H^s(M, \Omega^2(M))} = \sum_{j,k} \| \omega_{jk} \|_{H^s(M)}. \]

Finally, we introduce the anisotropic Sobolev spaces
\[ H^{2,1}_0(\Sigma) = H^2(0, T; L^2(\partial M)) \cap L^2(0, T; H^1(\partial M)), \]
equipped with the norm
\[ \| f \|_{H^{2,1}_0(\Sigma)} = \| f \|_{H^2(0, T; L^2(\partial M))} + \| f \|_{L^2(0, T; H^1(\partial M))}. \]
Finally we set
\[ H^{2,1}(\Sigma) = \{ f \in H^{2,1}_0(\Sigma), f(0, \cdot) = \partial_t f(0, \cdot) \equiv 0 \}. \]

**Theorem 1.1.** Let \( T > 0 \) be given, \( A \in C^0(M, T^*M) \) and \( V \in W^{1,\infty}(M) \). Suppose that \( f \in H^{2,1}_0(\Sigma) \). Then the unique solution \( u \) of (1.1) satisfies
\[ u \in C^0(0, T; H^1(M)). \quad (1.11) \]
Furthermore we have \( \partial_t u \in L^2(\Sigma) \) and there is a constant \( C = C(T, M, \| A \|_{W^{1,\infty}}, \| V \|_{W^{1,\infty}}) > 0 \) such that
\[ \| \partial_t u \|_{L^2(\Sigma)} \leq C \| f \|_{H^{2,1}_0(\Sigma)}. \quad (1.12) \]
The Dirichlet-to-Neumann map \( \Lambda_{A,V} \) defined by (1.3) is therefore continuous and we denote by \( \| \Lambda_{A,V} \| \) its norm in \( \mathcal{L}(H^{2,1}_0(\Sigma), L^2(\Sigma)). \)
Theorem 1.1 gives a rather comprehensive treatment of the regularity problem for (1.1) with stronger boundary condition \( f \). Moreover, our treatment clearly shows that a regularity for \( f \in H^1_0(\Sigma) \) is sufficient to obtain the desired interior regularity of \( u \) on \( Q \) while the full strength of the assumption \( f \in H^1_0(\Sigma) \) is used to obtain the desired boundary regularity for \( \partial_{\nu}u \) and then the continuity of the DN map \( \Lambda_{A,V} \).

1.2. Stable determination

In this section we state the main stability results. Let us first introduce the admissible class of manifolds for which we can prove uniqueness and stability results in our inverse problem. For this we need the notion of simple manifolds [42].

Let \((M,g)\) be a Riemannian manifold with boundary \( \partial M \), we denote by \( D \) the Levi-Civita connection on \((M,g)\). For a point \( x \in \partial M \), the second quadratic form of the boundary 
\[ \Pi(\theta, \theta) = \langle D_{\theta \nu}, \theta \rangle, \quad \theta \in T_x(\partial M) \]
is defined on the space \( T_x(\partial M) \). We say that the boundary is strictly convex if the form is positive-definite for all \( x \in \partial M \) (see [40]).

**Definition 1.2.** We say that the Riemannian manifold \((M,g)\) (or that the metric \(g\)) is simple in \(M\), if \( \partial M \) is strictly convex with respect to \(g\), and for any \( x \in M \), the exponential map \( \exp : \exp^{-1}(M) \rightarrow M \) is a diffeomorphism. The latter means that every two points \( x, y \in M \) are joined by a unique geodesic smoothly depending on \( x \) and \( y \).

Manifolds are the central objects of differential geometry and play an important role in theoretical physics. Recently, theory of manifolds made an entrance into the world of inverse problems and image processing. Since the famous paper [34] by J. Radon in 1917, it has been agreed that integral geometry problems consist in determining some function or tensor, which is defined on a manifold, given its integrals over submanifolds of prescribed class. Moreover, in the Riemannian geometry of image processing, differential geometric methods are exploited for treating images as Riemannian manifolds in general Euclidian embedding spaces for applications in linear and nonlinear scale-space image processing, and for developing improved mathematical procedures for enhancement, smoothing, and segmentation of multi spectral and texture images. Also of relevance in image processing are the very useful class of manifold, called Hadamard–Cartan manifolds which are complete and simply-connected Riemannian manifolds \((M,g)\) and have everywhere non-positive sectional curvature. The manifold of symmetric positive definite matrices \( \mathbb{P}^n \) has been successfully used for data representation in image set classification. We note that \( \mathbb{P}(n) \) is of dimension \( n(n + 1)/2 \) (see [32]).

Note that if \((M,g)\) is simple, one can extend it to a simple manifold \( M_1 \) such that \( M_1^{\text{int}} \supset M \).

Let us now introduce the admissible sets of magnetic potentials \( A \) and electric potentials \( V \). Let \( m_1, m_2 > 0 \) and \( k \geq 1 \) be given, set
\[ \mathcal{A}^{k}(m_1, k) = \{ A \in C^{\infty}(M, T^* M), \| A \|_{H^k(M, T^* M)} \leq m_1 \}, \quad (1.13) \]
and
\[ \mathcal{V}(m_2) = \{ V \in W^{1, \infty}(M), \| V \|_{W^{1, \infty}(M)} \leq m_2 \}. \quad (1.14) \]
The main results of this paper are as follows.

**Theorem 1.3.** Let \((M,g)\) be a simple compact Riemannian manifold with boundary of dimension \( n \geq 2 \) and let \( T > 0 \). There exist \( k \geq 1, \epsilon > 0, C > 0 \) and \( \kappa \in (0, 1) \) such that for any \( A_1, A_2 \in \mathcal{A}^{k}(m_1, k) \) and \( V_1, V_2 \in \mathcal{V}(m_2) \) coincide near the boundary \( \partial M \) and any with
\[ \| A_1^I - A_2^I \|_{C^0} \leq \varepsilon, \text{ the following estimate holds true} \]
\[ \| A_1^I - A_2^I \|_{L^2(M,T^*M)} + \| V_1 - V_2 \|_{L^2(M)} \leq C \| \Lambda_{A_1,V_1} - \Lambda_{A_2,V_2} \|, \]  \hfill (1.15)

where \( C \) depends on \( M, m_m, m_n, \) and \( \varepsilon. \)

For 1-form \( A = a_j dx^j \) where \( a_j \) are smooth functions on \( M \). The exterior derivative of \( A \) is given by
\[ da = \sum_{j,k=1}^{n} \frac{1}{2} (\partial_j a_k - \partial_k a_j) dx^j \wedge dx^k, \]
where \( \wedge \) is the antisymmetric wedge product
\[ dx^j \wedge dx^k = - dx^k \wedge dx^j. \]
Since \( d^2 = 0 \) for all forms, we get
\[ dA = da. \]

By theorem 1.3, we can readily derive the following.

**Corollary 1.4.** Let \( (M,g) \) be a simple compact Riemannian manifold with boundary of dimension \( n \geq 2 \) and let \( T > 0 \). There exist \( k \geq 1, \varepsilon > 0, C > 0 \) and \( \kappa \in (0,1) \) such that for any \( A_1, A_2 \in \mathcal{A}(m, k) \) and \( V_1, V_2 \in \mathcal{V}(m, 2) \) coincide near the boundary \( \partial M \) and any with \( \| A_1^I - A_2^I \|_{L^2(M)} \leq \varepsilon, \) the following estimate holds true
\[ \| dA_1 - dA_2 \|_{H^k(M,T^*M)} + \| V_1 - V_2 \|_{L^2(M)} \leq C \| \Lambda_{A_1,V_1} - \Lambda_{A_2,V_2} \| \]  \hfill (1.16)

where \( C \) depends on \( M, m_m, m_n, \) and \( \varepsilon. \)

By theorem 1.3, we can readily derive the following uniqueness result.

**Corollary 1.5.** Let \( (M,g) \) be a simple compact Riemannian manifold with boundary of dimension \( n \geq 2 \) and let \( T > 0 \). There exist \( k \geq 1, \varepsilon > 0, \) such that for any \( A_1, A_2 \in \mathcal{A}(m, k) \) and \( V_1, V_2 \in \mathcal{V}(m, 2) \) coincide near the boundary \( \partial M \) and any with \( \| A_1^I - A_2^I \|_{L^2(M)} \leq \varepsilon, \) we have that \( \Lambda_{A_1,V_1} = \Lambda_{A_2,V_2} \) implies \( A_1 = A_2 \) and \( V_1 = V_2 \) everywhere in \( M. \)

Our proof is inspired by techniques used by Stefanov and Uhlmann [42], and Bellassoued and Dos Santos Ferreira [13], which prove uniqueness theorems for an inverse problem without magnetic potential.

The outline of the paper is as follows. In section 2 we study the geodesical ray transform for 1-forms and functions on a manifold. In section 3 we construct special geometrical optics solutions to magnetic Schrödinger equations. In sections 4 and 5, we establish stability estimates for the solenoidal part of the magnetic field and the electric potential. The appendix A is devoted to the study of the Cauchy problem for the Schrödinger equation and we prove theorem 1.1.

2. Geodesical ray transform on a simple manifold

In this section we first collect some formulas needed in the rest of this paper and introduce the geodesical ray transform for 1-form. Denote by \( \text{div} X \) the divergence of a vector field \( X \in H^1(M, T^*M) \) on \( M, \) i.e. in local coordinates (see p 42, [26]),
\[ \text{div} X = \frac{1}{\sqrt{\text{det} g}} \sum_{i=1}^{n} \partial_i \left( \sqrt{\text{det} g} X^i \right), \quad X = \sum_{i=1}^{n} X^i \partial_i. \]  \hfill (2.1)

Using the inner product of 1-form, we can define the coderivative operator \( \delta \) as the adjoint of the exterior derivative via the relation
\[ (\delta A, \nu) = (A, \text{div} \nu), \quad A \in \mathcal{C}^\infty(M, T^*M), \quad \nu \in \mathcal{C}^\infty(M). \] (2.2)

Then \( \delta A \) is related to the divergence of vector fields by \( \delta A = -\text{div} (A^\sharp) \), where the divergence is given by (2.1). If \( X \in H^1(M, T^*M) \) the divergence formula reads
\[
\int_M \text{div} X \, \text{d}v^n = \int_{\partial M} \langle X, \nu \rangle \, \text{d}\sigma^{n-1},
\] (2.3)

and for a function \( f \in H^1(M) \) Green’s formula reads
\[
\int_M \text{div} X f \, \text{d}v^n = -\int_M \langle X, \nabla f \rangle \, \text{d}v^n + \int_{\partial M} \langle X, \nu \rangle f \, \text{d}\sigma^{n-1}.
\] (2.4)

Then if \( f \in H^1(M) \) and \( w \in H^2(M) \), the following identity holds
\[
\int_M \Delta w \, \text{d}v^n = -\int_M \langle \nabla w, \nabla f \rangle \, \text{d}v^n + \int_{\partial M} \partial_n w \, \text{d}\sigma^{n-1}.
\] (2.5)

For \( x \in M \) and \( \theta \in T_xM \) we denote by \( \gamma_{x,\theta} \) the unique geodesic starting at the point \( x \) in the direction \( \theta \). We consider
\[
SM = \{ (x, \theta) \in T^*M; \ |\theta| = 1 \}, \quad S^*M = \{ (x, p) \in T^*M; \ |p| = 1 \},
\]
the sphere bundle and co-sphere bundle of \( M \). The exponential map \( \exp_x : T_xM \rightarrow M \) is given by
\[
\exp_x(\nu) = \gamma_{x,\nu}[\nu], \quad \theta = \frac{\nu}{|\nu|}.
\] (2.6)

A compact Riemannian manifold \( (M, g) \) with boundary is called a convex non-trapping manifold, if it satisfies two conditions:

(i) the boundary \( \partial M \) is strictly convex, i.e. the second fundamental form of the boundary is positive definite at every boundary point,

(ii) for each \( (x, \theta) \in SM \), the maximal geodesic \( \gamma_{x,\theta}(t) \) satisfying the initial conditions \( \gamma_{x,\theta}(0) = x \) and \( \gamma_{x,\theta}(0) = \theta \) is defined on a finite segment \( [\tau_-(x, \theta), \tau_+(x, \theta)] \). We recall that a geodesic \( \gamma : [a, b] \rightarrow M \) is maximal if it cannot be extended to a segment \( [a - \varepsilon_1, b + \varepsilon_2] \), where \( \varepsilon_1 > 0 \) and \( \varepsilon_1 + \varepsilon_2 > 0 \).

The second condition is equivalent to all geodesics having finite length in \( M \).

An important subclass of convex non-trapping manifolds are simple manifolds. We say that a compact Riemannian manifold \( (M, g) \) is simple if it satisfies the following properties

(a) the boundary is strictly convex,

(b) there are no conjugate points on any geodesic.

A simple \( n \)-dimensional Riemannian manifold is diffeomorphic to a closed ball in \( \mathbb{R}^n \), and any pair of points in the manifold are joined by an unique geodesic.

Given \( (x, \theta) \in SM \), there exist a unique geodesic \( \gamma_{x,\theta} \) associated to \( (x, \theta) \) which is maximally defined on a finite interval \( [\tau_-(x, \theta), \tau_+(x, \theta)] \), with \( \gamma_{x,\theta}(\tau_-(x, \theta)) \in \partial M \). We define the geodesic flow \( \Phi_t \) as following
\[
\Phi_t : SM \rightarrow SM, \quad \Phi_t(x, \theta) = (\gamma_{x,\theta}(t), \dot{\gamma}_{x,\theta}(t)), \quad t \in [\tau_-(x, \theta), \tau_+(x, \theta)],
\] (2.7)

and \( \Phi_t \) is a flow, that is, \( \Phi_t \circ \Phi_s = \Phi_{t+s} \).
Now, we introduce the submanifolds of inner and outer vectors of $S\mathcal{M}$
\[ \partial_{\pm} S\mathcal{M} = \{(x, \theta) \in S\mathcal{M}, x \in \partial \mathcal{M}, \pm \langle \theta, \nu(x) \rangle < 0 \}, \]
(2.8)
where $\nu$ is the unit outer normal to the boundary. Note that $\partial_+ S\mathcal{M}$ and $\partial_- S\mathcal{M}$ are compact manifolds with the same boundary $S\mathcal{M}$, and $\partial S\mathcal{M} = \partial_+ S\mathcal{M} \cup \partial_- S\mathcal{M}$. We denote by $C^\infty(\partial S\mathcal{M})$ be the space of smooth functions on the manifold $\partial S\mathcal{M}$. Thus we can define two functions $\tau_\pm: \mathcal{M} \to \mathbb{R}$ which satisfy
\[ \tau_\pm(x, \theta) = -\tau_\pm(x, -\theta), \]
\[ \tau_\pm(x, \theta) = 0, \quad (x, \theta) \in \partial_\pm S\mathcal{M}, \]
\[ \tau_\pm(\Phi(x, \theta)) = \tau_\pm(x, \theta) - t, \quad \tau_\pm(\Phi(x, \theta)) = \tau_\pm(x, \theta) + t. \]
For $(x, \theta) \in \partial S\mathcal{M}$, we denote by $\gamma_{\theta, 0}: [0, \tau_\pm(x, \theta)] \to \mathcal{M}$ the maximal geodesic satisfying the initial conditions $\gamma_{\theta, 0}(0) = x$ and $\dot{\gamma}_{\theta, 0}(0) = \theta$. For each smooth 1-form $A \in C^\infty(\mathcal{M}, T^*\mathcal{M})$, $A = a_j \, dx^j$ we introduce the smooth symbol function $\sigma_\theta \in C^\infty(S\mathcal{M})$ given by
\[ \sigma_\theta(x, \theta) = \sum_{j=1}^n a_j(x) \theta^j = \langle A(x), \theta \rangle, \quad (x, \theta) \in S\mathcal{M}. \]
(2.9)

The Riemannian scalar product on $T_x \mathcal{M}$ induces the volume form on $S\mathcal{M}$, denoted by $d\omega_\theta$ and given by
\[ d\omega_\theta(x, \theta) = \sqrt{[\mathcal{E}]} \sum_{k=1}^n (-1)^k \theta^k \wedge \cdots \wedge \hat{d}\theta^k \wedge \cdots \wedge d\theta^n. \]
As usual, the notation $\wedge$ means that the corresponding factor has been dropped. We introduce the volume form $d\nu_{2n-1}$ on the manifold $S\mathcal{M}$ by
\[ d\nu_{2n-1}(x, \theta) = d\omega_\theta(x, \theta) \wedge d\nu^n, \]
where $d\nu^n$ is the Riemannian volume form on $\mathcal{M}$. By Liouville’s theorem, the form $d\nu_{2n-1}$ is preserved by the geodesic flow. The corresponding volume form on the boundary $\partial S\mathcal{M} = \{(x, \theta) \in S\mathcal{M}, x \in \partial \mathcal{M} \}$ is given by
\[ d\sigma^{2n-2} = d\omega_\theta(x, \theta) \wedge d\sigma^{n-1}, \]
where $d\sigma^{n-1}$ is the volume form of $\partial \mathcal{M}$.

We now recall the Santaló formula
\[ \int_{S\mathcal{M}} F(x, \theta) d\nu_{2n-1}(x, \theta) = \int_{\partial S\mathcal{M}} \left( \int_0^{\tau_\pm(x, \theta)} F(\Phi(x, \theta)) d\tau \right) \mu(x, \theta) d\sigma^{2n-2} \]
(2.10)
for any $F \in C(S\mathcal{M})$.

Let $L^2(\partial S\mathcal{M})$ be the space of square integrable functions with respect to the measure $\mu(x, \theta) d\sigma^{2n-2}$ with $\mu(x, \theta) = \{|\theta, \nu(x)\}|$. This Hilbert space is endowed with the scalar product
\[ (u, v)_\mu = \int_{\partial S\mathcal{M}} u(x, \theta) \overline{v}(x, \theta) \mu(x, \theta) d\sigma^{2n-2}. \]
(2.11)
2.1. Geodesical ray transform of 1-forms

The ray transform of 1-forms on a simple Riemannian manifold \( (M, g) \) is the linear operator:

\[
I_1 : C^\infty(M, T^*M) \longrightarrow C^\infty(\partial_+ SM)
\]

defined by

\[
I_1(A)(x, \theta) = \int_{\gamma_{x,\theta}} A = \sum_{j=1}^{n} \int_0^{\tau_{x,\theta}} a_j(\gamma_{x,\theta}(t)) \gamma_{x,\theta}^j(t) dt = \int_0^{\tau_{x,\theta}} \sigma_A(\Phi(x, \theta)) dt,
\]

where \( \gamma_{x,\theta} : [0, \tau(x, \theta)] \rightarrow M \) is a maximal geodesic satisfying the initial conditions \( \gamma_{x,\theta}(0) = x \) and \( \dot{\gamma}_{x,\theta}(0) = \theta \). It is easy to see that \( I_1(\varphi) = 0 \) for any smooth function \( \varphi \) in \( M \) with \( \varphi|_{\partial M} = 0 \). It is known that \( I_1 \) is injective on the space of solenoidal 1-forms satisfying \( \delta A = A_0 \) for simple metric \( g \). In other words, \( I_1(A) = 0 \) implies \( A = 0 \), i.e. \( A = \partial \varphi \) with some \( \varphi \) vanishing on \( \partial M \). So we have

\[
I_1(\varphi) = \int_{\partial_+ SM} \varphi(\gamma_{x,\theta}(\tau(x, \theta))) d\sigma = 0, \quad A \in C^0(M, T^*M).
\]  

We will now determine the adjoint \( I_1^* \) of \( I_1 \). The ray transform \( I_1 \) is a bounded operator from \( L^2(M, T^*M) \) into \( L^2(\partial_+ SM) \). For \( A \in L^2(M, T^*M) \) and \( \Psi \in L^2(\partial_+ SM) \), we get

\[
(I_1(A), \Psi)_0 = \int_{\partial_+ SM} \overline{A}(x, \theta) \Psi(x, \theta) d\sigma = \int_{\partial_+ SM} \left( \int_0^{\tau_{x,\theta}} \sigma_A(\Phi(x, \theta)) d\tau(x, \theta) \right) \Psi(x, \theta) d\sigma = \int_{\partial_+ SM} \sigma_A(\theta) \Psi(x, \theta) d\sigma = (A, I_1^*(\Psi)),
\]

where the adjoint \( I_1^* : L^2(\partial_+ SM) \longrightarrow L^2(M, T^*M) \) is given by

\[
(I_1^*(\Psi)(x))_j = \int_{S_1, M} \psi(x, \theta) d\omega_{\gamma_{x,\theta}}(\theta)
\]

where the extension of the function \( \Psi \) from \( \partial_+ SM \) to \( S \) constant on every orbit of the geodesic flow, i.e.

\[
\tilde{\Psi}(x, \theta) = \Psi(\gamma_{x,\theta}(\tau(x, \theta)), \dot{\gamma}_{x,\theta}(\tau(x, \theta))) = \Psi(\Phi(\tau(x, \theta), \theta)(x, \theta)), \quad (x, \theta) \in S \).
\]

The ray transform of 1-forms on a simple Riemannian manifold can be extended to the bounded operator

\[
I_1 : L^2(M, T^*M) \longrightarrow L^2(\partial_+ SM).
\]

Now, we recall some properties of the ray transform of 1-forms on a simple Riemannian manifold proved in [43]. Let \( (M, g) \) be a simple metric, we assume that \( g \) extends smoothly as a simple metric on \( M_{\text{int}} \supset M \) and let \( N_1 = I_1^* I_1 \). Then there exist \( C_1 > 0, C_2 > 0 \) such that

\[
C_1 \|A\|_{L^2(M)} \leq \|N_1(A)\|_{L^2(\partial_+ SM)} \leq C_2 \|A\|_{L^2(M)}
\]

for any \( A \in L^2(M, T^*M) \). If \( \mathcal{O} \) is an open set of the simple Riemannian manifold \( (M_{\text{int}}, g) \), the normal operator \( N_1 \) is an elliptic pseudodifferential operator of order \(-1\) on \( \mathcal{O} \) (see appendix B for more details) whose principal symbol is \( \sigma(x, \xi) = (\partial_{\mu}(\xi, \xi))_{i \in \text{L}
\]

where
\[ g_{ij}(x, \xi) = \frac{c_n}{|\xi|} \left( g_{jk} - \frac{\xi_j \xi_k}{|\xi|^2} \right). \]

Therefore for each \( k \geq 0 \) there exists a constant \( C_k > 0 \) such that for all \( A \in H^k(M, T^*M) \) compactly supported in \( \mathcal{O} \)

\[ \|N(A)\|_{H^{k+1}(M)} \leq C_k \|A\|_{H^k(\mathcal{O})}. \] (2.16)

2.2. Geodesical ray transform of function

The ray transform (also called geodesic x-ray transform) on a convex non trapping manifold \( M \) is the linear operator

\[ \mathcal{I}_0 : C^\infty(M) \longrightarrow C^\infty(\partial, S, M) \] (2.17)

defined by the equality

\[ \mathcal{I}_0 f(x, \theta) = \int_0^{\tau_\gamma(x, \theta)} f(\gamma_{x, \theta}(t)) \, dt. \] (2.18)

The right-hand side of (2.18) is a smooth function on \( \partial, S, M \) because the integration limit \( \tau_\gamma(x, \theta) \) is a smooth function on \( \partial, S, M \), see lemma 4.1.1 of [40]. The ray transform on a convex non trapping manifold \( M \) can be extended as a bounded operator

\[ \mathcal{I}_0 : H^k(M) \longrightarrow H^k(\partial, S, M) \] (2.19)

for every integer \( k \geq 0 \), see theorem 4.2.1 of [40].

The ray transform \( \mathcal{I}_0 \) is a bounded operator from \( L^2(M) \) into \( L^2(\partial, S, M) \). The adjoint \( \mathcal{I}_0^* : L^2(\partial, S, M) \rightarrow L^2(M) \) is given by

\[ \mathcal{I}_0^* \Psi(x) = \int_{\partial, S, M} \tilde{\Psi}(x, \theta) \, d\omega_\gamma(\theta) \] (2.20)

where \( \tilde{\Psi} \) is the extension of the function \( \Psi \) from \( \partial, S, M \) to \( S, M \) constant on every orbit of the geodesic flow, i.e.

\[ \tilde{\Psi}(x, \theta) = \Psi(\gamma_{x, \theta}(\tau_\gamma(x, \theta))). \]

Let \( (M, g) \) be a simple metric, we assume that \( g \) extends smoothly as a simple metric on \( M_1^\text{int} \supseteq M \) and let \( N_0 = T_0^* \mathcal{I}_0 \). Then there exist \( C_1 > 0, C_2 > 0 \) such that

\[ C_1 \| f \|_{H^k(M_1)} \leq \|N_0(f)\|_{H^{k+1}(M_1)} \leq C_2 \| f \|_{H^k(M_1)} \] (2.21)

for any \( f \in L^2(M) \). If \( \mathcal{O} \) is an open set of the simple Riemannian manifold \( (M_1, g) \), the normal operator \( N_0 \) is an elliptic pseudodifferential operator of order \(-1\) on \( \Omega \) whose principal symbol is a multiple of \( |\xi|^{-1} \) (see [42]). Therefore there exists a constant \( C_3 > 0 \) such that for all \( f \in H^k(\mathcal{O}) \) compactly supported in \( \mathcal{O} \)

\[ \|N_0(f)\|_{H^{k+1}(M_1)} \leq C_3 \| f \|_{H^k(\mathcal{O})}. \] (2.22)

3. Geometrical optics solutions of the magnetic Schrödinger equation

We now proceed to the construction of geometrical optics solutions to the magnetic Schrödinger equation. We extend the manifold \((M_1, g)\) into a simple manifold \( M_1^\text{int} \supseteq M \). The potentials \( A_1, A_2 \) may also be extended to \( M_1 \) and their \( H^k(M_1, T^*M_1) \) norms may be bounded by \( M_0 \).
Since $A_1 = A_2$ and $V_1 = V_2$ near the boundary, their extension outside $\mathcal{M}$ can be taken the same so that $A_1 = A_2$ and $V_1 = V_2$ in $\mathcal{M}_i \setminus \mathcal{M}$.

Our construction here is a modification of a similar result in [12], which dealt with the situation of the Schrödinger equation without magnetic potential.

We suppose, for a moment, that we are able to find a function $\psi \in C^2(\mathcal{M})$ which satisfies the eikonal equation

$$|\nabla \psi|^2 = \sum_{i,j=1}^n g^{ij} \frac{\partial \psi}{\partial x_i} \frac{\partial \psi}{\partial x_j} = 1, \quad \forall x \in \mathcal{M}, \quad (3.1)$$

and assume that there exists a function $\alpha \in H^1(\mathbb{R}, H^2(\mathcal{M}))$ which solves the transport equation

$$\partial_t \alpha + \langle d\psi, d\alpha \rangle + \frac{1}{2} (\Delta \psi) \alpha = 0, \quad \forall t \in \mathbb{R}, x \in \mathcal{M}, \quad (3.2)$$

which satisfies for some $T_0 > 0$

$$\alpha(t, x)|_{t=0} = \alpha(t, x)|_{t=0} = 0, \quad \forall x \in \mathcal{M}. \quad (3.3)$$

Moreover, we assume that there exists a function $\beta \in H^1(\mathbb{R}, H^2(\mathcal{M}))$ which solves the transport equation

$$\partial_t \beta + \langle d\psi, d\beta \rangle - i(A, d\psi) \beta = 0, \quad \forall t \in \mathbb{R}, x \in \mathcal{M}. \quad (3.4)$$

We also introduce the norm $\| \cdot \|$, given by

$$\| \alpha \| = \| \alpha \|_{H^1(\mathcal{M})}. \quad (3.5)$$

**Lemma 3.1.** Let $A \in C^0(\mathcal{M}, T^* \mathcal{M})$ and $V \in W^{1,\infty}(\mathcal{M})$. The magnetic Schrödinger equation

$$(i\partial_t + \mathcal{H}_{A,V})u = 0, \quad \text{in} \quad Q, \quad u(0, x) = 0, \quad \text{in} \quad \mathcal{M},$$

has a solution of the form

$$u(t, x) = \alpha(2\lambda t, x) \beta(2\lambda t, x) e^{i\lambda(\alpha - \lambda)} + v_0(t, x), \quad (3.6)$$

such that

$$u \in C^0(0, T; L^2(\mathcal{M})) \cap C(0, T; H^2(\mathcal{M})), \quad (3.7)$$

where $v_0(t, x)$ satisfies

$$v_0(t, x) = 0, \quad (t, x) \in \Sigma, \quad v_0(0, x) = 0, \quad x \in \mathcal{M}. \quad (3.8)$$

Furthermore, there exist $C > 0$ such that, for all $\lambda \geq T_0/2T$ the following estimates hold true.

$$\|v_0(t, \cdot)\|_{L^k(\mathcal{M})} \leq C \lambda^{k-1} \|\alpha\|_*, \quad k = 0, 1. \quad (3.9)$$

The constant $C$ depends only on $T$ and $\mathcal{M}$ (that is $C$ does not depend on $a$ and $\lambda$). The result remains true if the initial condition $u(0, x) = 0$ is replaced by the final condition $u(T, x) = 0$ provided $\lambda \geq T_0/2T$; in this case $v_0$ is such that $v_0(T, x) = 0$.\]
Proof. Let us consider

\[ R(t,x) = -(i\partial_t + \mathcal{H}_{A,V})(\sigma_\beta)(2\lambda, x) e^{i\lambda \psi - \lambda t}). \]  

(3.9)

Let \( \nu \) solve the following homogeneous boundary value problem

\[
\begin{aligned}
(i\partial_t + \mathcal{H}_{A,V})\nu(t,x) &= R(t,x) \quad \text{in} \ \Omega, \\
\nu(0,x) &= 0, \quad \text{in} \ \mathcal{M}, \\
\nu(t,x) &= 0 \quad \text{on} \ \Sigma.
\end{aligned}
\]

(3.10)

To prove our lemma it would be enough to show that \( \nu \) satisfies the estimates (3.8). The case where the condition \( u(T,x) = 0 \) is imposed rather than the initial condition may be handled in a similar fashion by imposing the corresponding condition \( \nu(T,x) = 0 \) on \( \nu \) since \( \alpha(2\lambda T, \cdot) = 0 \) if \( \lambda > T_0/2T \). By a simple computation, we have

\[
-R(t,x) = e^{i\mathcal{H}_{A,V} - \lambda t}\mathcal{H}_{A,V}(\sigma_\beta)(2\lambda, x))
+ \frac{2i}{1} e^{i\mathcal{H}_{A,V} - \lambda t}\beta(2\lambda, x) \left( \partial_t \sigma - (d\psi, d\sigma) + \frac{\alpha}{2} \Delta \psi \right) (2\lambda, x)
+ \frac{2i}{1} e^{i\mathcal{H}_{A,V} - \lambda t}\alpha(2\lambda, x) (\partial_t \beta + (d\psi, d\beta) - i(A, d\psi) \beta)(2\lambda, x)
+ \lambda^2 \alpha(2\lambda, x) e^{i\mathcal{H}_{A,V} - \lambda t} \left( 1 - |d\psi|^2 \right).
\]

(3.11)

Taking into account (3.1), (3.2) and (3.4), the right-hand side of (3.11) becomes

\[
R(t,x) = -e^{i\mathcal{H}_{A,V} - \lambda t}\mathcal{H}_{A,V}(\sigma_\beta)(2\lambda, x))
\equiv -e^{i\mathcal{H}_{A,V} - \lambda t}R_0(2\lambda, x).
\]

(3.12)

Since \( R_0 \in H^2_0(0,T; L^2(\mathcal{M})) \) for \( \lambda > T_0/2T \), by lemma A.1, we find

\[
\nu_\lambda \in C(0,T; L^2(\mathcal{M})) \cap C(0,T; H^2(\mathcal{M}) \cap H^1_{\partial}(\mathcal{M})).
\]

(3.13)

Furthermore, there is a constant \( C > 0 \), such that

\[
\|\nu_\lambda(t, \cdot)\|_{L^2(\mathcal{M})} \leq C \int_0^T \|R_0(2\lambda s, \cdot)\|_{L^2(\mathcal{M})} \, ds
\leq \frac{C}{\lambda} \int_0^T \|R_0(s, \cdot)\|_{L^2(\mathcal{M})} \, ds
\leq \frac{C}{\lambda} \|\sigma\|.
\]

(3.14)

Moreover, for any \( \eta > 0 \), we have

\[
\|\nabla \nu_\lambda(t, \cdot)\|_{L^2(\mathcal{M})} \leq C \eta \int_0^T \left( \lambda^2 \|R_0(2\lambda s, \cdot)\|_{L^2(\mathcal{M})} + \lambda \|\partial_t R_0(2\lambda s, \cdot)\|_{L^2(\mathcal{M})} \right) \, ds
+ \eta^{-1} \int_0^T \|R_0(2\lambda s, \cdot)\|_{L^2(\mathcal{M})} \, ds.
\]

(3.15)
Finally, choosing \( \eta = \lambda^{-1} \), we obtain
\[
\| \nabla v_3(t, \cdot) \|_{L^2(M)} \leq C \left( \int_{\mathbb{R}} \| R_0(s, \cdot) \|_{L^2(M)} \, ds + \int_{\mathbb{R}} \| \partial_s R_0(s, \cdot) \|_{L^2(M)} \, ds \right) \quad (3.16)
\]
Combining (3.16) and (3.14), we immediately deduce the estimate (3.8).

We will now construct the phase function \( \psi \) solution to the eikonal equation (3.1) and the amplitudes \( \alpha \) and \( \beta \) solutions to the transport equations (3.2)–(3.4).

Let \( y \in \partial M \). Denote points in \( M \) by \( (r, \theta) \) where \( (r, \theta) \) are polar normal coordinates in \( M \) with center \( y \). That is \( x = \exp_y(r \theta) \) where \( r > 0 \) and
\[
\theta \in S_r M = \{ \theta \in T_r M, \ |\theta| = 1 \}.
\]
In these coordinates (which depend on the choice of \( y \)) the metric takes the form
\[
g_0(r, \theta) = dr^2 + g_0(r, \theta),
\]
where \( g_0(r, \theta) \) is a smooth positive definite metric. For any function \( u \) compactly supported in \( M \), we set for \( r > 0 \) and \( \theta \in S_r M \)
\[
a(r, \theta) = u(\exp_y(r \theta)),
\]
where we have extended \( u \) by 0 outside \( M \). An explicit solution to the eikonal equation (3.1) is the geodesic distance function to \( y \in \partial M \)
\[
\psi(x) = d_y(x, y).
\] (3.17)
By the simplicity assumption, since \( y \in M \setminus \overline{M} \), we have \( \psi \in C^\infty(M) \) and
\[
\tilde{\psi}(r, \theta) = r = d_y(x, y).
\] (3.18)
The next step is to solve the transport equation (3.2). Recall that if \( f(r) \) is any function of the geodesic distance \( r \), then
\[
\Delta g f(r) = f''(r) + \frac{\rho^{-1}}{2} \frac{\partial}{\partial r} f'(r).
\] (3.19)
Here \( \rho = \rho(r, \theta) \) denotes the square of the volume element in geodesic polar coordinates. The transport equation (3.2) becomes
\[
\frac{\partial \tilde{\alpha}}{\partial t} + \frac{\tilde{\alpha}}{\partial r} \frac{\partial \tilde{\alpha}}{\partial r} + \frac{1}{4} \tilde{\alpha} \rho^{-1} \frac{\partial}{\partial r} \frac{\partial}{\partial r} \tilde{\alpha} = 0.
\] (3.20)
Thus \( \tilde{\alpha} \) satisfies
\[
\frac{\partial \tilde{\alpha}}{\partial t} + \frac{\tilde{\alpha}}{\partial r} \frac{\partial \tilde{\alpha}}{\partial r} + \frac{1}{4} \tilde{\alpha} \rho^{-1} \frac{\partial}{\partial r} \tilde{\alpha} = 0.
\] (3.21)
Let \( \phi \in C^\infty_0(\mathbb{R}) \) and \( \Psi \in H^2(\partial, S M) \). Let us write \( \tilde{\alpha} \) in the form
\[
\tilde{\alpha}(t, r, \theta) = \rho^{-1/4} \phi(t - r) \Psi(y, \theta).
\] (3.22)
Direct computations yield
\[
\frac{\partial \tilde{\alpha}}{\partial t}(t, r, \theta) = \rho^{-1/4} \phi'(t - r) \Psi(y, \theta)
\] (3.23)
and
\[
\frac{\partial \tilde{\alpha}}{\partial r}(t, r, \theta) = -\frac{1}{4} \rho^{-\frac{3}{4}} \frac{\partial}{\partial r} \phi(t - r) \Psi(y, \theta) - \rho^{-\frac{1}{4}} \phi(t - r) \Psi(y, \theta). \tag{3.24}
\]

Finally, (3.24) and (3.23) yield
\[
\frac{\partial \tilde{\alpha}}{\partial t}(t, r, \theta) + \frac{\partial \tilde{\alpha}}{\partial r}(t, r, \theta) = -\frac{1}{4} \rho^{-\frac{1}{4}} \phi(t, r, \theta) \frac{\partial}{\partial r}. \tag{3.25}
\]

Now if we assume that supp(\phi) \subseteq (0, 1), then for any \( x = \exp_\gamma(r\theta) \in \mathcal{M} \), it is easy to see that \( \tilde{\alpha}(t, r, \theta) = 0 \) if \( t \leq 0 \) and \( t \geq T_0 \) for some \( T_0 > 1 + \text{diam}(\mathcal{M}) \).

In geodesic polar coordinates the gradient vector \( \nabla\psi(x) \) is given by \( \gamma_{y, \theta}(r) \) we give the proof in appendix C (see also [22]), then
\[
(\tilde{A}(r, y, \theta), \nabla\psi) = \left( \tilde{A}(r, y, \theta), \nabla\psi \right) = \tilde{\sigma}(\Phi(y, \theta)).
\]

The transport equation (3.4) becomes
\[
\frac{\partial \tilde{\beta}}{\partial t} + \frac{\partial \tilde{\psi}}{\partial r} \frac{\partial \tilde{\beta}}{\partial r} - i \tilde{\sigma}(r, y, \theta) \tilde{\beta} = 0, \tag{3.26}
\]
where \( \tilde{\sigma}(r, y, \theta) := \sigma_\theta(\Phi(y, \theta)) = \{ \gamma_{y, \theta}(r), \tilde{A}(\gamma_{y, \theta}(r)) \} \). Thus \( \tilde{\beta} \) satisfies
\[
\frac{\partial \tilde{\beta}}{\partial t} + \frac{\partial \tilde{\beta}}{\partial r} - i \tilde{\sigma}(r, y, \theta) \tilde{\beta} = 0. \tag{3.27}
\]

Thus, we can choose \( \tilde{\beta} \) as following
\[
\tilde{\beta}(t, y, r, \theta) = \exp \left( i \int_0^t \tilde{\sigma}(r - s, y, \theta) ds \right).
\]

Hence (3.4) is solved.

### 4. Stable determination of the solenoidal part of the magnetic field

In this section, we prove the stability estimate of the solenoidal part \( A^s \) of the magnetic field \( A \). We are going to use the geometrical optics solutions constructed in the previous section; this will provide information on the geodesic ray transform of the difference of magnetic potentials.

#### 4.1. Preliminary estimates

The main purpose of this section is to present a preliminary estimate, which relates the difference of the potentials to the DN map. As before, we let \( A_1, A_2 \in \mathcal{S}(m_1, k) \) and \( V_1, V_2 \in \mathcal{H}(m_2) \) such that \( A_1 = A_2, V_1 = V_2 \) near the boundary \( \partial \mathcal{M} \). We set
\[
A(x) = (A_1 - A_2)(x), \quad V(x) = (V_1 - V_2)(x).
\]

Recall that we have extended \( A_1, A_2 \) as \( H^2(\mathcal{M}_\partial, \mathcal{M}) \) in such a way that \( A = 0 \) and \( V = 0 \) on \( \mathcal{M} \setminus \mathcal{M}_\partial \).

**Lemma 4.1.** Let \( T > 0 \). There exist \( C > 0 \) such that for any \( \alpha, \beta \in H^1(\mathbb{R}, H^2(\mathcal{M})) \) satisfying the transport equation (3.2) with (3.3), the following estimate holds true:
\[
\left| 2\lambda \int_0^T \int_M \langle A, d\psi \rangle (\alpha_2 R_1)(2\lambda, x)(\beta_2 R_1)(2\lambda, x) \, d\nu^\# \, dr \right| \\
\leq C \lambda^{-1} \lambda^2 \| A_{A_1, V_1} - A_{A_2, V_2} \| \| \alpha_1 \| \| \alpha_2 \|, 
\tag{4.1}
\]
for all \( \lambda > T_0/2T \).

**Proof.** First, if \( \alpha_2 \) satisfies (3.2), \( \beta_2 \) satisfies (3.3), and \( \lambda > T_0/2T \), lemma 3.1 guarantees the existence of a geometrical optics solution \( u_2 \)

\[
u_2(t, x) = (\alpha_2) (2\lambda, x)e^{-\lambda(x-y)} + v_2, \lambda(t, x),
\tag{4.2}
\]
to the Schrödinger equation corresponding to the potentials \( A_2 \) and \( V_2 \),

\[
\left( i\partial_t + H_{A_2, V_2} \right) u(t, x) = 0 \quad \text{in } Q, \quad u(0, x) = 0 \quad \text{in } M,
\]
where \( v_2, \lambda \) satisfies

\[
\lambda \| v_2, \lambda(t, \cdot) \|_{L^2(M)} + \| \nabla v_2, \lambda(t, \cdot) \|_{L^2(M)} \leq C \| \alpha_2 \|, \quad v_2, \lambda(t, x) = 0, \quad \forall (t, x) \in \Sigma. \tag{4.3}
\]

Moreover

\[
u_2 \in \mathcal{C}(0, T; L^2(M)) \cap \mathcal{C}(0, T; H^2(M)).
\]

Let us denote by \( f_\lambda \) the function

\[
f_\lambda(t, x) = (\alpha_2, \beta_2)(2\lambda, x)e^{-\lambda(x-y)}, \quad (t, x) \in \Sigma.
\]

Let us consider \( v \) the solution of the following non-homogeneous boundary value problem

\[
\begin{aligned}
(i\partial_t + H_{A_1, V_1})v &= 0, & \quad (t, x) & \in Q, \\
v(0, x) &= 0, & \quad x & \in M, \\
v(t, x) &= u_2(t, x) := f_\lambda(t, x), & \quad (t, x) & \in \Sigma.
\end{aligned}
\tag{4.4}
\]

Denote \( w = v - u_2 \). Therefore, \( w \) solves the following homogeneous boundary value problem for the magnetic Schrödinger equation

\[
\begin{aligned}
(i\partial_t + H_{A_1, V_1})w(t, x) &= 2i\langle A, d\nu_2 \rangle + W(x)u_2(t, x) & \quad (t, x) & \in Q, \\
w(0, x) &= 0, & \quad x & \in M, \\
w(t, x) &= 0, & \quad (t, x) & \in \Sigma,
\end{aligned}
\]

where

\[
W(x) = i\delta(A) - |A_2| + |A_1| + V \equiv W_A + V.
\]

Using the fact that \( W(x)u_2 \in W^{1,1}(0, T; L^2(M)) \) with \( u_2(0, \cdot) \equiv 0 \), by lemma 1, we deduce that

\[
w \in \mathcal{C}(0, T; L^2(M)) \cap \mathcal{C}(0, T; H^2(M) \cap H^2(M)).
\]

Therefore, we have constructed a special solution

\[
u_1 \in \mathcal{C}(0, T; L^2(M)) \cap \mathcal{C}(0, T; H^2(M))
\]
to the backward magnetic Schrödinger equation

\[(i\partial_t + \mathcal{H}_{A_\lambda}) u_t(t,x) = 0, \quad (t,x) \in Q,\]

\[u_T(t,x) = 0, \quad x \in \mathcal{M},\]

having the special form

\[u_t(t,x) = (\alpha_1\beta_1)(2\lambda t,x)e^{i\lambda(x(t)-\lambda t)} + v_{1,\lambda}(t,x),\quad (t,x) \in \Sigma,\]  

(4.5)

which corresponds to the potentials \(A_1\) and \(V_1\), where \(v_{1,\lambda}\) satisfies for \(\lambda > T_0/2T\)

\[
\lambda \norm{v_{1,\lambda}(t,\cdot)}_{L^2(A_\lambda)} + \norm{\nabla v_{1,\lambda}(t,\cdot)}_{L^2(A_\lambda)} \leq C \norm{\alpha_1}_\ast.
\]  

(4.6)

Integrating by parts and using Green’s formula (2.5), we find

\[
\int_0^T \int_{\mathcal{M}} (i\partial_t + \mathcal{H}_{A_\lambda}) u_t(t,x) \varpi dx \, dt = \int_0^T \int_{\mathcal{M}} 2i \langle A, du_T \rangle \varpi dx \, dt + \int_0^T \int_{\mathcal{M}} (W_\lambda + V)(x) u_T \varpi dx \, dt
\]

\[= - \int_0^T \int_{\partial\mathcal{M}} (\partial_\nu + \nu A_1) \cdot v_T \varpi d\sigma^{n-1} \, dt.
\]  

(4.7)

Taking (4.7), (4.4) into account, we deduce

\[- \int_0^T \int_{\mathcal{M}} 2i \langle A, du_T \rangle \varpi(t,x) \varpi dx \, dt = \int_0^T \int_{\partial\mathcal{M}} (A_{\lambda;\lambda_1} - A_{\lambda_1;\lambda}) f_\lambda(t,x) \varpi dx \, dt
\]

\[+ \int_0^T \int_{\mathcal{M}} (W_\lambda + V)(x) u_T \varpi dx \, dt
\]  

(4.8)

where \(h_\lambda\) is given by

\[h_\lambda(t,x) = (\alpha_1\beta_1)(2\lambda t,x)e^{i\lambda(x(t)-\lambda t)}, \quad (t,x) \in \Sigma.
\]

It follows from (4.8), (4.5) and (4.2) that

\[2\lambda \int_0^T \int_{\mathcal{M}} \langle A, du_\lambda \rangle (2\lambda t,x)(\beta_1\lambda_1 \varpi) \varpi dx \, dt
\]

\[= \int_0^T \int_{\mathcal{M}} \varpi (A_{\lambda;\lambda_1} - A_{\lambda_1;\lambda}) f_\lambda dx \, dt - 2\lambda \int_0^T \int_{\mathcal{M}} (A, du_\lambda)(2\lambda t,x) \varpi \lambda_1 e^{i\lambda(x(t)-\lambda t)} \varpi dx \, dt
\]

\[+ 2i \int_0^T \int_{\mathcal{M}} (A, du_\lambda)(2\lambda t,x) \varpi \lambda_1 e^{i\lambda(x(t)-\lambda t)} \varpi dx \, dt + 2i \int_0^T \int_{\mathcal{M}} (A, d\varphi_\lambda)(2\lambda t,x) \varpi \lambda_1 e^{i\lambda(x(t)-\lambda t)} \varpi dx \, dt
\]

\[+ 2i \int_0^T \int_{\mathcal{M}} (A, d\varphi_\lambda)(2\lambda t,x) e^{-i\lambda(x(t)-\lambda t)} \varpi dx \, dt + 2i \int_0^T \int_{\mathcal{M}} (A, du_\lambda)(t,x) \varpi \lambda_1 \varpi dx \, dt
\]

\[+ \int_0^T \int_{\partial\mathcal{M}} (W_\lambda + V)(x) u_T \varpi dx \, dt
\]

\[= \int_0^T \int_{\partial\mathcal{M}} (A_{\lambda;\lambda_1} - A_{\lambda_1;\lambda}) f_\lambda dx \, dt + \mathcal{R}_\lambda.
\]  

(4.9)

In view of (4.6) and (4.3), we have

\[\mathcal{R}_\lambda \leq C \frac{\lambda}{\lambda} \norm{\alpha_1}_\ast \norm{\alpha_2}_\ast.
\]  

(4.10)
On the other hand, by the trace theorem, we find
\[
\left| \int_0^T \int_{\partial M} (A_{\alpha}, v_1 - A_{\alpha}, v_2)(f_x) \nabla \alpha \, d\sigma \, dt \right| \leq \| A_{\alpha}, v_1 - A_{\alpha}, v_2 \|_{L^2(S^2)} \| b\|_{L^2(S^2)} \leq C_\lambda \| \alpha_1 \| \| \alpha_2 \| \| A_{\alpha}, v_1 - A_{\alpha}, v_2 \|. \tag{4.11}
\]

The estimate (4.1) follows easily from (4.9)–(4.11). This completes the proof of the lemma.

Lemma 4.2. There exists \( C > 0 \) such that for any \( \Psi \in H^2(\partial, S, M) \), the following estimate
\[
\left| \int_{S_y M} \int_{0}^{\tau_0 (y, \theta)} \tilde{\partial}_2 s (y, \theta) \Psi(y, \theta) \Psi(y, \theta) \, ds \, d\omega_{\beta}(\theta) \right| \leq C \| A_{\alpha}, v_1 - A_{\alpha}, v_2 \|^{1/2} \| \Psi(y, \cdot) \|_{H^2(S_y M)}
\]+ \| A_{\alpha}, v_1 - A_{\alpha}, v_2 \|^{1/2} \| \Psi(y, \cdot) \|_{L^2(S_y M)} \tag{4.12}
\]
holds for any \( y \in \partial M \).

We use the notation
\[
S_y^+ M_\theta = \{ \theta \in S_y M : \langle \nu, \theta \rangle < 0 \}.
\]

Proof. Following (3.22), we pick \( T_0 > 1 + \text{diam} M \) and take two solutions to (3.2) and (3.3) of the form
\[
\tilde{\alpha}_1(t, r, \theta) = \rho^{-1/4} \dot{\phi}(t - r) \Psi(y, \theta),
\]
\[
\tilde{\alpha}_2(t, r, \theta) = \rho^{-1/4} \dot{\phi}(t - r) \mu(y, \theta).
\]

We recall that \( \mu(y, \theta) = \langle \nu(y), \theta \rangle \) is the density of the \( L^2 \) space where the image of the geodesic ray transform lies. Now we change variable in the left term of (4.1), \( x = \exp((t, \theta), r > 0 \) and \( \theta \in S_y M \), we have
\[
2\lambda \int_0^T \int_{S_y M} \langle A, \dot{d}(\nu) (2 \lambda, t, x) (2 \lambda, t, x) \rangle \, dv \, dt = 2\lambda \int_0^T \int_{S_y M} \int_{0}^{\tau_0 (y, \theta)} \tilde{\partial}_2 s (y, \theta) \dot{\phi}^2 (2 \lambda, t, \theta) \Psi(y, \theta) \, ds \, d\omega_{\beta}(\theta) \, dt
\]
\[
= 2\lambda \int_0^T \int_{S_y M} \int_{0}^{\tau_0 (y, \theta)} \tilde{\partial}_2 s (y, \theta) \dot{\phi}^2 (2 \lambda, t, \theta) \Psi(y, \theta) \, ds \, d\omega_{\beta}(\theta) \, dt
\]
\[
= 2\lambda \int_0^T \int_{S_y M} \int_{0}^{\tau_0 (y, \theta)} \tilde{\partial}_2 s (2 \lambda, t, \theta) \Psi(y, \theta) \, ds \, d\omega_{\beta}(\theta) \, dt
\]
\[
= 2\lambda \int_0^T \int_{S_y M} \int_{0}^{\tau_0 (y, \theta)} \tilde{\partial}_2 s (2 \lambda, t, \theta) \Psi(y, \theta) \, ds \, d\omega_{\beta}(\theta) \, dt
\]
\[
= 2\lambda \int_0^T \int_{S_y M} \int_{0}^{\tau_0 (y, \theta)} \tilde{\partial}_2 s (2 \lambda, t, \theta) \Psi(y, \theta) \, ds \, d\omega_{\beta}(\theta) \, dt
\]
\[
= 2\lambda \int_0^T \int_{S_y M} \int_{0}^{\tau_0 (y, \theta)} \tilde{\partial}_2 s (2 \lambda, t, \theta) \Psi(y, \theta) \, ds \, d\omega_{\beta}(\theta) \, dt
\]
\[
= 2\lambda \int_0^T \int_{S_y M} \int_{0}^{\tau_0 (y, \theta)} \tilde{\partial}_2 s (2 \lambda, t, \theta) \Psi(y, \theta) \, ds \, d\omega_{\beta}(\theta) \, dt
\]
\[
= \int_{S_y M} \hat{\phi} (\tau) \Psi(y, \theta) \, d\omega_{\beta}(\theta) \tag{4.13}
\]
By the support properties of the function $\phi$, we get that the left-hand side term in the previous inequality reads

$$
\int_{\mathbb{R}} \phi^2(\tau) \int_{S, \mathcal{M}_i} \left[ \exp \left( \int_0^{2\pi \tau} \tilde{\alpha}(s - \tau, y, \theta) ds \right) - 1 \right] \Psi(y, \theta) \mu(y, \theta) d\tau \ d\omega_j(\theta) = \int_{S, \mathcal{M}_i} \left[ \exp \left( \int_0^{2\pi (y, \theta)} \tilde{\alpha}(s, y, \theta) ds \right) - 1 \right] \Psi(y, \theta) \mu(y, \theta) d\omega_j(\theta).
$$

Then, by (4.13) and (4.1) we get

$$
\left| \int_{S, \mathcal{M}_i} \left( \exp(i \mathcal{I}_0(A)(y, \theta)) - 1 \right) \Psi(y, \theta) \mu(y, \theta) d\omega_j(\theta) \right| \leq C \left( \lambda^{-1} + \lambda^2 \| \Lambda_{A_0, \mathcal{M}_i} - \Lambda_{A_1, \mathcal{M}_i} \| \right) \left\| \Psi(y, \cdot) \right\|_{H^1(S^{*}, \mathcal{M}_i)}.
$$

Finally, minimizing in $\lambda$ in the right hand-side of (4.14) we obtain

$$
\left| \int_{S, \mathcal{M}_i} \left( \exp(i \mathcal{I}_0(A)(y, \theta)) - 1 \right) \Psi(y, \theta) \mu(y, \theta) d\omega_j(\theta) \right| \leq C \left\| \Lambda_{A_0, \mathcal{M}_i} - \Lambda_{A_1, \mathcal{M}_i} \right\|^{1/4} \left\| \Psi(y, \cdot) \right\|_{H^1(S^{*}, \mathcal{M}_i)}.
$$

Using the fact that

$$
\exp(i \mathcal{I}_0(A)(y, \theta)) - 1 = i \mathcal{I}_0(A)(y, \theta) - (\mathcal{I}_0(A)(y, \theta))^2 \int_0^1 \exp(i \mathcal{I}_0(A)(y, \theta))(1 - t) dt,
$$

we deduce from (2.12)

$$
\left| \int_{S, \mathcal{M}_i} \mathcal{I}_0(A)(y, \theta) \Psi(y, \theta) \mu(y, \theta) d\tau \ d\omega_j(\theta) \right| \leq C \left\| \Lambda_{A_0, \mathcal{M}_i} - \Lambda_{A_1, \mathcal{M}_i} \right\|^{1/4} \left\| \Psi(y, \cdot) \right\|_{H^1(S^{*}, \mathcal{M}_i)} + \left\| \Psi(y, \cdot) \right\|_{L^2(S^{*}, \mathcal{M}_i)} \left\| A' \right\|_{C^2}.
$$

This completes the proof of the lemma. \qed

### 4.2. End of the proof of the stability estimate of the magnetic field

Let us now complete the proof of the stability estimate of the solenoidal part of the magnetic field. Using lemma 4.2, for any $y \in \partial \mathcal{M}_i$ and $\Psi \in H^2(\partial_i, S \mathcal{M}_i)$ we have

$$
\left| \int_{S, \mathcal{M}_i} \mathcal{I}_0(A)(y, \theta) \Psi(y, \theta) \mu(y, \theta) d\omega_j(\theta) \right| \leq C \left\| \Lambda_{A_0, \mathcal{M}_i} - \Lambda_{A_1, \mathcal{M}_i} \right\|^{1/4} \left\| \Psi(y, \cdot) \right\|_{H^1(S^{*}, \mathcal{M}_i)} + \left\| \Psi(y, \cdot) \right\|_{L^2(S^{*}, \mathcal{M}_i)} \left\| A' \right\|_{C^2(\mathcal{M}_i,T^*; \mathcal{M}_i)}.
$$

Integrating with respect to $y \in \partial \mathcal{M}_i$ we obtain
\[ \left| \int_{\Omega} I_0(A) (y, \theta) \Psi(y, \theta) \mu(y, \theta) d\sigma \right| \leq C \| \Lambda_{A_0, V_1} - \Lambda_{A_2, V_2} \|^{1/4} \| \Psi \|_{H^1(\partial, S_{\Omega, M})} \]
\[ + \| \Psi \|_{L^2(\partial, S_{\Omega, M})} \| A' \|_{C^0(M, \mathbb{R}^3)}^2. \]  

(4.15)

Now we choose
\[ \Psi(y, \theta) = I_0(N_1(A))(y, \theta). \]

Taking into account (2.16) and (4.15), we obtain
\[ \| N_1(A) \|_{L^2(M)} \leq C \| \Lambda_{A_0, V_1} - \Lambda_{A_2, V_2} \|^{1/4} \| A' \|_{H^1} + \| A' \|_{L^2} \| A' \|_{L^2}^2. \]

By interpolation, it follows that for any \( a \in (0, 1) \) there exists \( k > 0 \) such that
\[ \| N_1(A) \|_{L^2(M)} \leq C \| N_1(A) \|_{L^2(M)}^{2(1-a)} \| N_1(A) \|_{L^2(M)}^{2a} \leq C \| N_1(A) \|_{L^2(M)}^{2a} \leq C \| \Lambda_{A_0, V_1} - \Lambda_{A_2, V_2} \|^{1/4} \| A' \|_{H^1}^{1-a} \| A' \|_{L^2}^a. \]

Moreover, for any \( b \in (0, 1) \) there exists \( k' > 0 \) such that
\[ \| A' \|_{L^2} \leq C \| A' \|_{H^{2-\epsilon}} \leq C \| A' \|_{L^2} \| A' \|_{H^{1-\epsilon}} \leq C \| A' \|_{L^2}. \]

(4.16)

Using (2.15), we deduce that
\[ \| A' \|_{L^2}^2 \leq C \| \Lambda_{A_0, V_1} - \Lambda_{A_2, V_2} \|^{1/4} \| A' \|_{L^2}^{1+2b}. \]

Selecting \( a, b \in (0, 1) \) such that \( a(1 + 2b) > 2 \), we deduce that
\[ \| A' \|_{L^2}^2 \leq C \| \Lambda_{A_0, V_1} - \Lambda_{A_2, V_2} \|^{1/4} + C \epsilon^{a(1+2b)-2} \| A' \|_{L^2}^2. \]

So, for \( \epsilon \) small, we deduce
\[ \| A' \|_{L^2(M)} \leq C \| \Lambda_{A_0, V_1} - \Lambda_{A_2, V_2} \|^{1/4}. \]

(4.18)

Furthermore by (4.17) and (4.18) we get
\[ \| A' \|_{L^2} \leq C \| \Lambda_{A_0, V_1} - \Lambda_{A_2, V_2} \|^{1/2}, \quad \kappa_1 = \frac{ab}{8}. \]

(4.19)

This completes the proof of the Hölder stability estimate of the solenoidal part of the magnetic potential.

5. Stable determination of the electric potential

The goal of this section is to prove a stability estimate for the electric potential. The proof of that stability estimate involves using the stability result we have already obtained for the magnetic field. Apply the Hodge decomposition to \( A = A_1 - A_2 = A' + d\varphi \). Define \( A'_j = A_1 - \frac{1}{2} d\varphi \) and \( A'_2 = A_2 + \frac{1}{2} d\varphi \) so that \( A' = A'_1 - A'_2 = A' \). First we replace the magnetic potential \( A_j \) by \( A'_j, j = 1, 2 \). Since the DN map is invariant under gauge transformation we have
\[ \Lambda_{A_0, V_1} = \Lambda_{A'_j, V_j}, \quad j = 1, 2. \]
Define $\alpha_j, \beta_j$ and $u_j$ as in section 4 with $A_j$ replaced by $A'_j, j = 1, 2$.

**Lemma 5.1.** Let $T > 0$. There exist $C > 0$ such that for any $\alpha_j, \beta_j \in H^1(\mathbb{R}, H^2(\mathcal{M}))$ satisfying the transport equation (3.2) with (3.4), the following estimate holds true:

$$
\left| \int_0^T \int_\mathcal{M} V(x)(\alpha_1(t)) (2\lambda t, x) ) (\beta_2(t), 2\lambda, x) \, dv \, dt \right| \\
\leq C(\lambda^{-2} + \lambda \|A'_j\|_{\infty} + \lambda^3 \|\Lambda_{A'j, v}\|) \|\alpha_1\|_s \|\alpha_2\|_s, 
$$

for all $\lambda > T_0/2T$.

**Proof.** We start with identity (4.8), except this time we will isolate the electric potential term on the LHS:

$$
- \int_0^T \int_\mathcal{M} V(x)u_2 \bar{\eta}_0 \, dv \, dt = \int_0^T \int_\mathcal{M} (\Lambda_{A'_j, v_1} - \Lambda_{A'_j, v_2}) f_j(t, x) \bar{\eta}_0(t, x) \, d\sigma^{-1} \, dt \\
+ \int_0^T \int_\mathcal{M} 2i(A'_j, du_2) \bar{\eta}_0(t, x) \, dv \, dt + \int_0^T \int_\mathcal{M} W_A(x) u_2 \bar{\eta}_0 \, dv \, dt, 
$$

where $h_0$ is given by

$$
h_0(t, x) = (\alpha_1(t))(2\lambda t, x)e^{i\lambda(x(\lambda x))}, \quad (t, x) \in \Sigma.
$$

It follows from (5.2), (4.5) and (4.2) that

$$
\int_0^T \int_\mathcal{M} V(x)(\alpha_2 \bar{\eta}_0)(2\lambda t, x)(\beta_2 \bar{\eta}_0)(2\lambda t, x) \, dv \, dt \\
= \int_0^T \int_\mathcal{M} \bar{f}_0(\Lambda_{A'_j, v_1} - \Lambda_{A'_j, v_2}) f_j \, d\sigma^{-1} \, dt + \int_0^T \int_\mathcal{M} V(x)(\alpha_2(2\lambda t, x)) \bar{\eta}_0(t, x) \, d\sigma^{-1} \, dv \, dt \\
+ \int_0^T \int_\mathcal{M} V(x) \nu_{2, \lambda}(\bar{\eta}_0)(2\lambda t, x) e^{-i\lambda(x(\lambda x))} \, dv \, dt + \int_0^T \int_\mathcal{M} V(x) \nu_{2, \lambda}(t, x) \bar{\eta}_0(t, x) \, d\sigma^{-1} \, dv \, dt \\
+ \int_0^T \int_\mathcal{M} W_A(x) u_2(t, x) \nu_{2, \lambda} \bar{\eta}_0(t, x) \, dv \, dt + \int_0^T \int_\mathcal{M} (\Lambda'_{A'_j, du_2}) \bar{\eta}_0(t, x) \, dv \, dt \\
= \int_0^T \int_\mathcal{M} \bar{f}_0(\Lambda_{A'_j, v_1} - \Lambda_{A'_j, v_2}) f_j \, d\sigma^{-1} \, dt + \mathcal{B}'_{A'_j}, 
$$

(5.3)

In view of (4.6) and (4.3), we have

$$
\left| \mathcal{B}'_{A'_j} \right| \leq \left( \frac{1}{\lambda^2} + \lambda \|A'_j\|_{\infty} \right) \|\alpha_1\|_s \|\alpha_2\|_s. 
$$

(5.4)

On the other hand, by the trace theorem, we find

$$
\left| \int_0^T \int_\mathcal{M} (\Lambda_{A'_j, v_1} - \Lambda_{A'_j, v_2})(f_j) h_0 \, d\sigma^{-1} \, dt \right| \leq \|\Lambda_{A'_j, v_1} - \Lambda_{A'_j, v_2}\| \|f_j\|_{L^1(\Sigma)} \|h_0\|_{L^2(\Sigma)} \\
\leq C \lambda^3 \|\alpha_1\|_s \|\alpha_2\|_s \|\Lambda_{A'_j, v_1} - \Lambda_{A'_j, v_2}\|. 
$$

(5.5)

The estimate (5.1) follows easily from (5.3), (5.4). This completes the proof of the lemma. □
Lemma 5.2. There exists $C > 0$ and $\kappa_2 \in (0, 1)$ such that for any $b \in H^2(\partial_s \mathcal{M}_h)$, the following estimate
\[
\left| \int_{S_{\mathcal{M}_h}} \int_0^{\gamma_2(\cdot, \theta)} \nabla(s, \theta) b(y, \theta) \mu(y, \theta) \, ds \, d\omega_j(\theta) \right| \leq C \| \Lambda_{\mathcal{A}_h, \mathcal{V}_h} - \Lambda_{\mathcal{A}_h, \mathcal{V}_h} \|^2 \| b(\cdot, \cdot) \|_{H^p(S_{\mathcal{M}_h}^*)}.
\]
(5.6)
holds for any $y \in \partial \mathcal{M}_h$.

Proof. Following (3.22), we pick $T_0 > 1 + \text{diam} \mathcal{M}_h$ and take two solutions to (3.2) and (3.3) of the form
\[
\tilde{a}_1(t, r, \theta) = \rho^{-1/4} \phi(t - r) b(y, \theta),
\]
\[
\tilde{a}_2(t, r, \theta) = \rho^{-1/4} \phi(t - r) \mu(y, \theta).
\]
Now we change variable in (5.1), $x = \exp_y(r\theta)$, $r > 0$ and $\theta \in S_{\mathcal{M}_h}$, we have
\[
\int_0^T \int_{S_{\mathcal{M}_h}} V(x) \alpha_{1}(r\theta; 2\lambda, x) \beta_{1}(2\lambda, x) d\nu_{\theta} \, dt
\]
\[
= \int_0^T \int_{S_{\mathcal{M}_h}} \int_{0}^{\gamma_2(\cdot, \theta)} \nabla(r, \theta) \tilde{a}_1(r\theta; 2\lambda, r, \theta) \beta_{1}(2\lambda, r, \theta) d\nu_{\theta} \, d\omega_j(\theta) \, dt
\]
\[
+ \int_0^T \int_{S_{\mathcal{M}_h}} \int_{0}^{\gamma_2(\cdot, \theta)} \nabla(r, \theta) \tilde{a}_2(r\theta; 2\lambda, r, \theta) \beta_{1}(2\lambda, r, \theta) d\nu_{\theta} \, d\omega_j(\theta) \, dt
\]
\[
= \frac{1}{2\lambda} \int_{0}^{\gamma_2(\cdot, \theta)} \nabla(r, \theta) \tilde{a}_1(r\theta; 2\lambda, r, \theta) \beta_{1}(2\lambda, r, \theta) d\nu_{\theta} \, d\omega_j(\theta) \, dt
\]
By virtue of lemma 5.1, we conclude that
\[
\left| \int_0^\infty \int_{S_{\mathcal{M}_h}} \int_0^{\gamma_2(\cdot, \theta)} \nabla(r, \theta) \tilde{a}_1(r\theta; 2\lambda, r, \theta) \beta_{1}(2\lambda, r, \theta) d\nu_{\theta} \, d\omega_j(\theta) \, dt \right|
\]
\[
\leq C \lambda^{-1} + \lambda^2 \| \Lambda_{\mathcal{A}_h, \mathcal{V}_h} - \Lambda_{\mathcal{A}_h, \mathcal{V}_h} \| + \lambda^2 \| \Lambda_{\mathcal{A}_h, \mathcal{V}_h} \|_{H^p(S_{\mathcal{M}_h}^*)} \| b(\cdot, \cdot) \|_{H^p(S_{\mathcal{M}_h}^*)}.
\]
(5.7)
By the support properties of the function $\phi$, we get that the left-hand side term in the previous inequality reads
\[
\int_0^\infty \int_{S_{\mathcal{M}_h}} \int_0^{\gamma_2(\cdot, \theta)} \nabla(r, \theta) \tilde{a}_1(r\theta; 2\lambda, r, \theta) \beta_{1}(2\lambda, r, \theta) d\nu_{\theta} \, d\omega_j(\theta) \, dt
\]
\[
= \left( \int_{-\infty}^{\infty} \phi^2(t) \, dt \right) \int_{S_{\mathcal{M}_h}} \int_0^{\gamma_2(\cdot, \theta)} \nabla(r, \theta) \tilde{a}_1(r\theta; 2\lambda, r, \theta) \beta_{1}(2\lambda, r, \theta) d\nu_{\theta} \, d\omega_j(\theta) \, dt.
\]
Then taking account (4.19) we obtain
\[ \int_{S,M_0} \int_0^{\tau_1(y,\theta)} \tilde{V}(r,\theta)b(y,\theta)\mu(y,\theta)\,dr\,d\omega_y(\theta) \leq C \left( \lambda^{-1} + \lambda^4 \left\| \Lambda_{\mathcal{A},V_1} - \Lambda_{\mathcal{A},V_2} \right\|_{r} \right) \times \left\| \phi \right\|_{H^2(S,M_0)} \|b(y,\cdot)\|_{H^2(\Sigma^+,M_0)}. \]

Finally, minimizing in \( \lambda \) in the right-hand side of the last inequality we obtain
\[ \int_{S,M_0} \int_0^{\tau_1(y,\theta)} \tilde{V}(s,\theta)b(y,\theta)\mu(y,\theta)\,ds\,d\omega_y(\theta) \leq C \|\Lambda_{\mathcal{A},V_1} - \Lambda_{\mathcal{A},V_2}\| \|b(y,\cdot)\|_{H^2(\Sigma^+,M_0)}. \]

This completes the proof of the lemma. \( \square \)

### 5.1. End of the proof of the stability estimate

Let us now complete the proof of the stability estimate in theorem 1.3. Using lemma 5.2, for any \( y \in \partial M_1 \) and \( b \in H^2(\partial_1,SM_1) \) we have
\[ \left\| \int_{\partial_1,SM_1} \mathcal{I}_0(V)(y,\theta)b(y,\theta)\mu(y,\theta)\,d\omega_y(\theta) \right\| \leq C \|\Lambda_{\mathcal{A},V_1} - \Lambda_{\mathcal{A},V_2}\| \|b(y,\cdot)\|_{H^2(\Sigma^+,M_0)}. \]

Integrating with respect to \( y \in \partial M_1 \) we obtain
\[ \int_{\partial,SM_1} \mathcal{I}_0(V)(y,\theta)b(y,\theta)\mu(y,\theta)\,d\sigma^{2n - 2}(y,\theta) \leq C \|\Lambda_{\mathcal{A},V_1} - \Lambda_{\mathcal{A},V_2}\| \|b\|_{H^2(\partial,SM_1)}. \quad (5.8) \]

Now we choose
\[ b(y,\theta) = \mathcal{I}_0(N_0(V))(y,\theta). \]

Taking into account (2.19) and (2.21), we obtain
\[ \|N_0(V)\|_{L^2} \leq C \|\Lambda_{\mathcal{A},V_1} - \Lambda_{\mathcal{A},V_2}\| \|V\|_{H^2}. \]

By interpolation, it follows that
\[ \left\| N_0(V) \right\|_{H^2} \leq C \left\| N_0(V) \right\|_{L^2} \|N_0(V)\|_{H^2} \leq C \left\| N_0(V) \right\|_{L^2} \|V\|_{H^2} \leq C \|\Lambda_{\mathcal{A},V_1} - \Lambda_{\mathcal{A},V_2}\| \|V\|_{H^2}. \quad (5.9) \]

Using (2.22), we deduce that
\[ \|V\|_{L^2(S,M_1)}^2 \leq C \|\Lambda_{\mathcal{A},V_1} - \Lambda_{\mathcal{A},V_2}\|^2. \]

This completes the proof of theorem 1.3.

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Appendix A. The Cauchy problem for the magnetic Schrödinger equation

In this section we will establish existence, uniqueness and continuous dependence on the data of solutions to the magnetic Schrödinger equation (1.1) with non-homogeneous Dirichlet boundary condition \( f \in H^{-1}_0(\Sigma) \). We will use the method of transposition, or adjoint isomorphism of equations, and we shall solve the case of non-homogeneous Dirichlet boundary conditions under stronger assumptions on the data than those in [4] and [10].

Let \( v \in C^0(\mathcal{M}) \) and \( N \) be a smooth real vector field. The following identity holds true (see [47])

\[
\left( \nabla v, \nabla \langle N, \nabla f \rangle \right) = DN(\nabla v, \nabla f) + \frac{1}{2} \text{div} (|\nabla v|^2 N) - \frac{1}{2} |\nabla v|^2 \text{div} N \tag{A.1}
\]

where \( D \) is the Levi-Civita connection and \( DN \) is the bilinear form on \( T_x\mathcal{M} \times T_x\mathcal{M} \) given by

\[
DN(X, Y) = \langle D_X N, Y \rangle, \quad X, Y \in T_x\mathcal{M}.
\]

Here \( D_X N \) is the covariant derivative of vector field \( N \) with respect to \( X \).

Let us first review the classical well-posedness results for the Schrödinger equation with homogeneous boundary conditions. After applying the transposition method, we establish theorem 1.1.

A.1. Homogeneous boundary condition

Let us consider the following initial and homogeneous boundary value problem for the Schrödinger equation

\[
\begin{aligned}
\left( i\partial_t + \mathcal{H}_{A,V} \right) v(t, x) &= F(t, x) \quad \text{in } Q, \\
v(0, x) &= 0 \quad \text{in } \mathcal{M}, \\
v(t, x) &= 0 \quad \text{on } \Sigma. 
\end{aligned} \tag{A.2}
\]

Firstly, it is well known that if \( F \in L^1(0, T; L^2(\mathcal{M})) \) then (A.2) admits an unique weak solution \( v \in C(0, T; L^2(\mathcal{M})) \).

Multiplying the first equation of (A.2) by \( v \) and using Green’s formula and Grönwall’s lemma, we obtain the following estimate

\[
\|v(t, \cdot)\|_{L^2(\mathcal{M})} \leq C \|F\|_{L^1(0, T; L^1(\mathcal{M}))}, \quad \forall t \in (0, T). \tag{A.4}
\]

Now assume that \( F \in L^1(0, T; H^1_0(\mathcal{M})) \). Using the classical result of existence and uniqueness of weak solutions in Cazenave and Haraux [16] (set for abstract evolution equations), we obtain that the system (A.2) has a unique solution \( v \) such that

\[
v \in C(0, T; H^1_0(\mathcal{M})). \tag{A.5}
\]

Multiplying the first equation of (A.2) by \( \Delta_A v \) and using Green’s formula and Grönwall’s lemma, we get

\[
\|v(t, \cdot)\|_{H^1_0(\mathcal{M})} \leq C \|F\|_{L^1(0, T; H^1_0(\mathcal{M}))}, \quad \forall t \in (0, T). \tag{A.6}
\]

**Lemma A.1.** Let \( T > 0 \). Suppose that \( F \in W^{1,1}(0, T; L^2(\mathcal{M})) \) is such that \( F(0, \cdot) \equiv 0 \). Then the unique solution \( v \) of (A.2) satisfies

\[
v \in C^0(0, T; L^2(\mathcal{M})) \cap \mathcal{C}(0, T; H^1_0(\mathcal{M}) \cap H^1_0(\mathcal{M})). \tag{A.7}
\]
Furthermore there is a constant $C > 0$ such that for any $0 < \eta \leq 1$, we have
\[
\|v(t, \cdot)\|_{L^2([0, T]; L^2(M))} \leq C(\eta \|\partial_t F\|_{L^2(0, T; L^2(M))} + \eta^{-1} \|F\|_{L^2(0, T; L^2(M))}).
\] (A.8)

**Proof.** If we consider the equation satisfied by $\partial_t v$, (A.3) provides the following regularity
\[v \in C^0(0, T; L^2(M)).\]
Furthermore, since $F(0, \cdot) = 0$, by (A.4), there is a constant $C > 0$ such that the following estimate holds true
\[
\|\partial_t v(t, \cdot)\|_{L^2(M)} \leq C \int_0^T \|\partial_t F(s, \cdot)\|_{L^2(M)} ds, \quad \forall t \in (0, T).
\] (A.9)
Then, by (A.2), we see that $\mathcal{H}_{A, v} = -i\partial_s v + F \in C(0, T; L^2(M))$ and therefore $v \in C(0, T; H^2(M))$. This completes the proof of (A.7).

Next, multiplying the first equation (A.2) by $\sigma$ and integrating by parts, we obtain
\[
\begin{aligned}
\text{Re} \left[\int_M (i\partial_t v)(t)\sigma(t) - |\nabla v(t)|^2 + V(x)|v(t)|^2 \right] dt
&= \text{Re} \left[\int_M \int_0^T \partial_t F(s, x) ds \right] \sigma(t, x) dt.
\end{aligned}
\] (A.10)
Then there exists a constant $C > 0$ such that the following estimate holds true
\[
\|\nabla v(t)\|_{L^2(M)}^2 \leq C \left[\|\partial_t v(t)\|_{L^2}^2 + \|v(t)\|_{L^2}^2 + \int_0^T \int_M |v(t, x)\partial_t F(s, x)| dt dx \right].
\] (A.11)
Using (A.9) and (A.4), we get
\[
\|\nabla v(t)\|_{L^2(M)}^2 \leq C \left[\|\partial_t F\|_{L^2(0, T; L^2(M))}^2 + \|F\|_{L^2(0, T; L^2(M))}^2 \right].
\] (A.12)
Thus, we deduce (A.8), and this concludes the proof of lemma A.1.

**Lemma A.2.** Let $T > 0$, be given and let $\mathcal{H} = L^1(0, T; H^2(M))$ or $\mathcal{H} = H^2(0, T; L^2(M))$. Then the mapping $F \mapsto \partial_v v$ where $v$ is the unique solution to (A.2) is linear and continuous from $\mathcal{H}$ to $L^2(\Sigma)$. Furthermore, there is a constant $C > 0$ such that
\[
\|\partial_v v\|_{L^2(\Sigma)} \leq C \|F\|_{\mathcal{H}}.
\] (A.13)

**Proof.** Let $N$ be a $C^2$ vector field on $\mathcal{M}$ such that
\[
N(x) = \nu(x), \quad x \in \partial \mathcal{M}; \quad |N(x)| \leq 1, \quad x \in \mathcal{M}.
\] (A.14)
Multiply both sides of the first equation in (A.2) by $(N, \nabla \sigma)$ and integrate over $(0, T) \times \mathcal{M}$, this gives
\[
\begin{aligned}
&\int_0^T \int_{\mathcal{M}} F(t, x)(N, \nabla \sigma) \, dv^a \, dt
= \int_0^T \int_{\mathcal{M}} i\partial_s N(N, \nabla \sigma) \, dv^a \, dt
+ \int_0^T \int_{\mathcal{M}} \Delta v(N, \nabla \sigma) \, dv^a \, dt
+ \int_0^T \int_{\mathcal{M}} (-2i(A, \nabla v) - i(A \nabla v) + |A|^2 v + V(x)v)(N, \nabla \sigma) \, dv^a = I_1 + I_2 + I_3.
\end{aligned}
\] (A.15)
Consider the first term on the right-hand side of (A.15); integrating by parts with respect \( t \), we get

\[
I_1 = i \left[ \int_M \nu(N, \nabla \tau) \, dv^n \right]_{t=0}^{t=T} - i \int_0^T \int_M \nu \langle N, \nabla \partial_t \tau \rangle \, dv^n \, dt \\
= i \int_0^T \int_M \nu(T, x) \langle N, \nabla \tau(T, x) \rangle \, dv^n - i \int_0^T \int_M \langle N, \nabla (\nu \partial_t \tau) \rangle \, dv^n \, dt - I_t. 
\]

(A.16)

Then, by (2.3), we obtain

\[
2 \text{Re} \, I_1 = i \int_M \nu(T, x) \langle N, \nabla \tau(T, x) \rangle \, dv^n + i \int_0^T \int_M \text{div} \, N \nu \partial_t \nu \, dv^n \, dt \\
- i \left[ \int_0^T \int_{\partial M} \nu \partial_t \nu \, d\sigma^{n-1} \, dt \right] \\
= i \int_M \nu(T, x) \langle N, \nabla \tau(T, x) \rangle \, dv^n + \int_0^T \int_M \langle \nabla \nu, \nabla (\text{div} \, N \nu) \rangle \, dv^n \, dt \\
+ \int_0^T \int_M \text{F} \, \text{div} \, N \nu \, dv^n \, dt - \int_0^T \int_M \tilde{q} \, \text{div} \, N \nu \, dv^n \, dt \\
- \left[ i \int_0^T \int_{\partial M} \nu \partial_t \nu \, d\sigma^{n-1} \, dt + \int_0^T \int_{\partial M} \partial_\nu \nu \, \text{div} \, N \, d\sigma^{n-1} \, dt \right].
\]

The last term vanishes, using (A.8) or (A.6), we conclude that

\[
|\text{Re} \, I_1| \leq C \|F\|^2_H. 
\]

(A.17)

On the other hand, by Green’s theorem, we get

\[
I_2 = - \int_0^T \int_M \langle \nabla \tau, \nabla \nu \rangle \, dv^n \, dt + \int_0^T \int_{\partial M} |\partial_\nu \nu|^2 \, d\sigma^{n-1} \, dt.
\]

Thus by (A.1), we deduce

\[
I_2 = \int_0^T \int_M |\partial_\nu \nu|^2 \, d\sigma^{n-1} \, dt - \frac{1}{2} \int_0^T \int_{\partial M} |\nabla \nu|^2 \, d\sigma^{n-1} \, dt \\
+ \int_0^T \int_M \text{D}N(\nabla \nu, \nabla \tau) \, dv^n \, dt - \frac{1}{2} \int_0^T \int_M |\nabla \nu|^2 \, \text{div} \, N \, dv^n \, dt.
\]

Using the fact \( |\nabla \nu|^2 = |\partial_\nu \nu|^2 + |\nabla_\nu \nu|^2 = |\partial_\nu \nu|^2 \), \( x \in \partial M \), where \( \nabla_\nu \) is the tangential gradient on \( \partial M \), we get

\[
\text{Re} \, I_2 = \frac{1}{2} \int_0^T \int_{\partial M} |\partial_\nu \nu|^2 \, d\sigma^{n-1} \, dt + \int_0^T \int_M \text{D}N(\nabla \nu, \nabla \tau) \, dv^n \, dt - \frac{1}{2} \int_0^T \int_M |\nabla \nu|^2 \, \text{div} \, N \, dv^n \, dt. 
\]

(A.18)

Finally by (A.6) and (A.8), we have
Collecting (A.20), (A.18), (A.17) and (A.15), we obtain

\[ \int_0^T \int_{\partial \omega} |\partial_\nu v|^2 \, d\sigma \, dt \leq C \| F \|_{H^1_0}. \]  

This completes the proof of (A.13).

A.2. Non-homogeneous boundary condition

We now turn to the non-homogeneous Schrödinger problem (1.1).

Let \( H = L^1(0, T; H_0^1(\omega)) \) or \( H = H_0^1(0, T; L^2(\omega)) \). By \( (\cdot, \cdot)_{H', H} \), we denote the dual pairing between \( H' \) and \( H \).

**Definition A.3.** Let \( T > 0 \) and \( f \in L^2(\Sigma) \), we say that \( \psi \in H' \) is a solution of (1.1) in the transposition sense if for any \( F \in H \) we have

\[ (\psi, F)_{H', H} = \int_0^T \int_{\partial \omega} f(t, x) \partial_\nu \psi(t, x) \, d\sigma \, dt, \]

where \( \psi = \psi(t, x) \) is the solution of the homogeneous boundary value problem

\[ \begin{aligned}
(i \partial_t + \mathcal{H}_{\lambda, \nu}) \psi(t, x) &= F(t, x) \quad \text{in } Q, \\
\psi(T, x) &= 0 \quad \text{in } \omega, \\
\psi(t, x) &= 0 \quad \text{on } \Sigma.
\end{aligned} \]

One gets the following lemma.

**Lemma A.4.** Let \( f \in L^2(\Sigma) \). There exists a unique solution

\[ \psi \in C(0, T; H^{-1}(\omega)) \cap H^{-1}(0, T; L^2(\omega)) \]

defined by transposition, of the problem

\[ \begin{aligned}
(i \partial_t + \Delta_\lambda) \psi(t, x) &= 0 \quad \text{in } Q, \\
\psi(0, x) &= 0 \quad \text{in } \omega, \\
\psi(t, x) &= f(t, x) \quad \text{on } \Sigma.
\end{aligned} \]

Furthermore, there is a constant \( C > 0 \) such that

\[ \| \psi \|_{C(0, T; H^{-1}(\omega))} + \| \psi \|_{H^{-1}(0, T; L^2(\omega))} \leq C \| f \|_{L^2(\Sigma)}. \]

**Proof.** Let \( F \in H = L^1(0, T; H_0^1(\omega)) \) or \( H = H_0^1(0, T; L^2(\omega)) \). Let \( \psi \in C(0, T; H_0^1(\omega)) \) solution of the backward boundary value problem for the Schrödinger equation (A.23). By lemma A.2 the mapping \( F \mapsto \partial_\nu \psi \) is linear and continuous from \( H \) to \( L^2(\Sigma) \) and there exists \( C > 0 \) such that

\[ \| \psi \|_{C(0, T; H_0^1(\omega))} \leq C \| F \|_H \]

(A.27)
We define a linear functional $\ell$ on the linear space $H$ as follows:

$$\ell(F) = \int_0^T \int_{\partial\Omega} f(t,x) \partial_\nu \varphi(t,x) d\sigma dt$$

where $\varphi$ solves (A.23). By (A.28), we obtain

$$|\ell(F)| \leq C \|f\|_{L^2(\Sigma)} \|F\|_{H^2}.$$ 

It is known that any linear bounded functional on the space $H$ can be written as

$$\ell(F) = (u, F)_{H', H},$$

where $u$ is some element from the space $H'$. Thus the system (A.25) admits a solution $u \in H'$ in the transposition sense, which satisfies

$$\|u\|_{H'} \leq C \|f\|_{L^2(\Sigma)}.$$

This completes the proof of the lemma.

In what follows, we will need the following estimate for the non-homogeneous elliptic boundary value problem.

Let $\psi \in H^{-1}(\Omega)$ and $\phi \in H^1(\partial\Omega)$. Let $w \in H^1(\Omega)$ the unique solution of the following boundary value problem

$$\begin{cases}
\Delta w = \psi & \text{in } \Omega, \\
w = \phi & \text{on } \partial\Omega,
\end{cases}$$

(A.29)

then, by the elliptic regularity (see [31]), the following estimate holds true

$$\|w\|_{H^1(\Omega)} \leq C \left( \|\psi\|_{H^{-1}(\Omega)} + \|\phi\|_{H^{\frac{1}{2}}(\partial\Omega)} \right).$$

(A.30)

**A.3. Proof of theorem 1.1**

We proceed to prove theorem 1.1. Let $f \in H^{-1}_0(\Sigma)$ such that $f(0, \cdot) = \partial_0 f(0, \cdot) = 0$ and $u$ solve (1.1). Put $w = \partial_0^2 u$, then

$$\begin{cases}
(i \partial_t + \mathcal{A}_0)w(t,x) = 0 & \text{in } Q, \\
w(0, x) = 0 & \text{in } \Omega, \\
w(t,x) = \partial_0^2 f(t,x) & \text{on } \Sigma.
\end{cases}$$

(A.31)

Since $\partial_0^2 f \in L^2(\Sigma)$, by lemma A.4, we get

$$w \in C(0,T; H^{-1} (\partial\Omega)) \cap H^{-1} (0, T; L^2 (\Omega)).$$

(A.32)

Furthermore there is a constant $C > 0$ such that

$$\|w\|_{C(0,T; H^{-1}(\partial\Omega))} + \|u\|_{H^{-1}(0,T; L^2(\Omega))} \leq C \|f\|_{L^2(\Sigma)}.$$
Thus (A.32) implies the following regularity for $v := \partial_t u$

$$v \in C^0(0, T; H^{-1}(\mathcal{M})) \cap C(0, T; L^2(\mathcal{M})),$$

$$\Delta_M v \in C^0(0, T; H^{-1}(\mathcal{M})) \cap H^{-1}(0, T; L^2(\mathcal{M})).$$

Since $\partial f(t, \cdot) \in H^i(\partial \mathcal{M})$, by the elliptic regularity, we get

$$v \in C^0(0, T; H^i(\mathcal{M})) \cap C^0(0, T; H^{-1}(\mathcal{M})).$$

Moreover there exists $C > 0$ such that the following estimates hold true

$$\|v\|_{C^0(0, T; H^{-1}(\mathcal{M}))} + \|\Delta v\|_{C^0(0, T; H^{-1}(\mathcal{M}))} \leq C \|f\|_{H^i(\Sigma)}. \quad (A.34)$$

Using (A.30), we find

$$\|v\|_{C^0(0, T; H^{-1}(\mathcal{M}))} + \|\Delta v\|_{C^0(0, T; H^{-1}(\mathcal{M}))} \leq C \|f\|_{H^i(\Sigma)}. \quad (A.35)$$

We deduce the following regularity of the solution $u$

$$u \in C^0(0, T; H^i(\mathcal{M})).$$

Moreover there exists $C > 0$ such that the following estimates hold true

$$\|u\|_{C^0(0, T; H^i(\mathcal{M}))} \leq C \|f\|_{H^i(\Sigma)}. \quad (A.36)$$

The proof of (1.8) is as in lemma A.2. If one multiplies (1.1) by $\langle N, \nabla \pi \rangle$, the arguments leading to (A.15) give now

$$0 = \int_0^T \int_{\mathcal{M}} i \partial_t u(N, \nabla \pi) dv^a \, dt + \int_0^T \int_{\mathcal{M}} \Delta u(N, \nabla \pi) dv^a \, dt$$

$$+ \int_0^T \int_{\mathcal{M}} (-2i(A, \partial_t u) - i(\delta A)u + |A|^2 u + V(x)u)(N, \nabla \pi) dv^a \, dt = I_1' + I_2' + I_3'. \quad (A.37)$$

with

$$|\text{Re } I_1'| \leq C \|f\|_{H^i(\Sigma)}^2 + \|\partial_t u\|_{L^2(\Sigma)}, \quad (A.38)$$

where we have used (A.35) instead of (A.8)–(A.6). Furthermore, we derive from Green’s formula

$$\text{Re } I_2' = \frac{1}{2} \int_0^T \int_{\partial \mathcal{M}} \partial_t u^2 \, d\sigma^{n-1} \, dt + \int_0^T \int_{\mathcal{M}} D(N u, \nabla \pi) dv^a \, dt$$

$$- \frac{1}{2} \int_0^T \int_{\mathcal{M}} \nabla u^2 \, \text{div } N \, dv^a \, dt \quad (A.39)$$

This together with

$$|\text{Re } I_3'| \leq \|f\|_{H^i(\Sigma)}^2 \quad (A.40)$$

and (A.40), (A.39) and (A.38) imply

$$\|\partial_t u\|_{L^2(\Sigma)} \leq C \|f\|_{H^i(\Sigma)}, \quad (A.41)$$

where we have used (A.35) again. The proof of theorem 1.1 is now complete.
Appendix B. Some technical results

Definition B.1. Suppose that \((\mathcal{M}, g)\) is a Riemannian manifold. Given a path \(\gamma : [a, b] \rightarrow \mathcal{M}\), the parallel transport

\[ J_{\gamma(a), \gamma(b)} : T_{\gamma(a)}\mathcal{M} \rightarrow T_{\gamma(b)}\mathcal{M}, \]

along \(\gamma\) of the tangent vector \(X \in T_{\gamma(a)}\mathcal{M}\) is defined as

\[ J_{\gamma(a), \gamma(b)}(X) = V(b), \]

where the vector field \(V(t) \in T_{\gamma(t)}\mathcal{M}\) is such that

\[
\begin{align*}
\nabla_\gamma V(t) &= 0 \quad t \in [a, b] \\
V(a) &= X,
\end{align*}
\]

that is

\[ V(t) = J_{\gamma(a), \gamma(t)}(X) \]

and \(\nabla_\gamma\) is the covariant derivative along \(\gamma\). The parallel transport is a linear isometry between \(T_{\gamma(a)}\mathcal{M}\) and \(T_{\gamma(b)}\mathcal{M}\).

Lemma B.2. Parallel transport is linear, orthogonal, and respects the operations of reparametrization, inversion and composition:

\[ J_{\gamma(a), \gamma(b)} \in \mathcal{L}(T_{\gamma(a)}\mathcal{M}, T_{\gamma(b)}\mathcal{M}). \] (B.1)

\[ \{ J_{\gamma(a), \gamma(b)}(X), J_{\gamma(a), \gamma(b)}(Y) \} = \{ X, Y \}, \quad X, Y \in T_{\gamma(a)}\mathcal{M}. \] (B.2)

\[ J_{\gamma(a), \gamma(b)}^{-1} = J_{\gamma(b), \gamma(a)} \] (B.3)

\[ J_{\gamma(a), \gamma(c)} \circ J_{\gamma(c), \gamma(b)} = J_{\gamma(a), \gamma(b)}. \] (B.4)

For a fixed \(x \in \mathcal{M}\) let \(v \in T_x\mathcal{M}\). Let \(J_{x, \exp, v} : T_x\mathcal{M} \rightarrow T_{\exp, v}\mathcal{M}\) the parallel transport along the geodesic \(\gamma : t \rightarrow \exp t v, \ t \in [0, 1]\). We define the Fourier transform on \(T_x\mathcal{M}\) as the linear operator \(\mathcal{F} : \mathcal{S}'(T_x\mathcal{M}) \rightarrow \mathcal{S}'(T^*\mathcal{M})\) on the space of temporary distribution by

\[ \mathcal{F}(f)(\xi) = \frac{1}{(2\pi)^{\frac{d}{2}}} \int_{T_x\mathcal{M}} e^{-i\xi(v)}f(\exp v)dv. \]

Now, we compute the composition \(T'_x T_x\). Let \(A \in L^2(\mathcal{M}, T\mathcal{M})\), by (2.14) we have

\[ (T'_x T_x(A))(x) = \int_{S_x\mathcal{M}} \theta T_x(A)(x, \theta) d\omega_\theta, \]

\[ = \int_{S_x\mathcal{M}} \theta T_x(A)(\Phi_{x, (x, \theta)}(x, \theta)) d\omega_\theta \] (B.5)

Since, for \(\sigma(x, \theta) = \langle A, \theta \rangle\), we get
\[ I_j(A)(\Phi_{t,x}(x, \theta)) = \int_0^\tau \sigma_A(\Phi_t(\Phi_{t,x}(x, \theta)))dt \]
\[ = \int_{\tau(x, \theta)}^\tau \sigma_A(\Phi_{t+,x}(x, \theta)))dt \]
\[ = \int_{\tau(x, \theta)} \sigma_A(\Phi_t(x, \theta)))dt. \quad (B.6) \]

Then
\[ (I_j^a I_j^a(A)(x) = \int_{S_xM} \theta^j \int_{\tau(x, \theta)}^{\tau(x, \theta) - \tau(x, \theta)} \sigma_A(\Phi_t(x, \theta)))d\omega(x)(\theta) \]
\[ = 2 \int_{S_xM} \theta^j \int_0^{\tau(x, \theta)} \sigma_A(\Phi_t(x, \theta)))d\omega(x)(\theta). \quad (B.7) \]

We denote by \( dv_\theta(\xi) \) the volume form on \( T_xM \) for a fixed \( x \in M \), we consider the following change integration variables in \( T_xM \) as follows \( \xi = \theta^j \). Then \( dv_\theta(\xi) = |\xi|^{n-1} |d\tau \wedge d\omega(x)(\theta)| \)

\[ (I_j^a I_j^a(A)(x) = 2 \int_{T_xM} \frac{\nu^j}{|\nu|+1} \sigma_A(\exp_vJ\exp_v)(v))dv_\xi. \]

Since
\[ \sigma_A(\exp_vJ\exp_v)(v)) = \langle A(\exp_v)^j, J\exp_v(v) \rangle \]
\[ = \langle J\exp_vA(\exp_v)^j, v \rangle = \sum_{k=1}^n (J\exp_vA(\exp_v))_j v^k. \quad (B.8) \]

Thus
\[ (I_j^a I_j^a(A)(x) = 2 \sum_{k=1}^n \int_{T_xM} \frac{\nu^j}{|\nu|+1} (J\exp_vA(\exp_v))_j dv_\xi. \quad (B.9) \]

We denote by
\[ \varrho_k(x, \xi) = 2\mathcal{F} \left( \frac{\nu^j}{|\nu|+2} \right)(\xi) = 2\mathcal{F} \left( \frac{\nu^j}{|\nu|+2} \right) \xi; \]
the last equality holds because \( \mathcal{F} \) is applied to an even function. Thus by the inversion formula for the Fourier transform
\[ 2\frac{\nu^j}{|\nu|+2} = \frac{1}{(2\pi)^n} \int_{T_xM} e^{-i\nu(x)} \varrho_k(x, \xi)d\xi. \]
By (B.9) we deduce that
\[ (I_j^a I_j^a(A)(x) = \frac{1}{(2\pi)^n} \sum_{k=1}^n \int_{T_xM} \int_{T_xM} e^{-i\nu(x)} \varrho_k(x, \xi)(J\exp_vA(\exp_v))_k dv_\xi d\xi. \quad (B.10) \]
Appendix C. Smoothness of distance function

Now let \( y \in \partial \mathcal{M} \) and consider the distance function
\[
\psi : \mathcal{M} \rightarrow \mathbb{R}; \quad \psi(x) = d_{\mathcal{R}}(y, x).
\]
As we have already seen, \( \psi \) is a continuous function. However, it is not hard to see that \( \psi \) is not smooth on \( \mathcal{M} \). In fact, \( \psi \) is never smooth at \( y \).

**Theorem C.1.** The function \( \psi \) is smooth on \( \mathcal{M} \setminus \text{Cut}(y) \cup \{ y \} \). Moreover, for each \( x \in \mathcal{M} \setminus \text{Cut}(y) \cup \{ y \} \), if we let \( \gamma_{y, \theta} \) be the unique normal minimizing geodesic from \( y \) to \( x \), then the gradient of \( \nabla \psi(x) \) at \( x \) is
\[
\nabla \psi(x) = \frac{d}{ds} \gamma_{y, \theta}(r), \quad r = d_{\mathcal{R}}(y, x).
\]

**Proof.** For each \( x \in \mathcal{M} \setminus \text{Cut}(y) \cup \{ y \} \), let \( \gamma_{y, \theta} \) be the unique normal minimizing geodesic from \( y \) to \( x \), \( \theta \in S_y \mathcal{M} \). Let
\[
A = \left\{ \ell(\gamma_{y, \theta}) \theta, \quad x \in \mathcal{M} \setminus \text{Cut}(y) \cup \{ y \} \right\}.
\]
Then \( A \subset T_x \mathcal{M} \setminus \{ 0 \} \) is an open set and \( \exp_{\gamma} : A \rightarrow \mathcal{M} \setminus \text{Cut}(y) \cup \{ y \} \) is smooth. Moreover, at each vector in \( A \), \( \exp_{\gamma} \) is nonsingular and thus a local diffeomorphism. Since \( \exp_{\gamma} \) is globally one-to-one on \( A \), it is a diffeomorphism from \( A \) to \( \mathcal{M} \setminus \text{Cut}(y) \cup \{ y \} \). It follows that \( \exp_{\gamma}^{-1} : \mathcal{M} \setminus \text{Cut}(y) \cup \{ y \} \rightarrow A \subset T_x \mathcal{M} \setminus \{ 0 \} \) is smooth. Thus \( \psi(x) = \left| \exp_{\gamma}^{-1}(x) \right| \) is smooth on \( \mathcal{M} \setminus \text{Cut}(y) \cup \{ y \} \). To calculate its gradient at \( x \), we choose any \( X \in T_x \mathcal{M} \) and let \( \sigma(s) \) be a smooth curve in \( \mathcal{M} \setminus \text{Cut}(y) \cup \{ y \} \) tangent to \( X \) at \( x = \sigma(0) \).

Now we consider the variation of \( \gamma_{y, \theta} \) so that \( \nabla \psi(x) \) is the unique minimizing geodesic from \( y \) to \( \sigma(s) \). Observe that the variation field vector of this variation at the point \( x \) is exactly \( X \). So according to the first variation formula,
\[
X(\psi) = \frac{d}{ds} \psi(\sigma(s))|_{s=0} = \frac{d}{ds} \ell(V(r, s))|_{s=0} = \langle X, \gamma_{y, \theta}(r) \rangle.
\]
It follows that \( \nabla \psi(x) = \gamma_{y, \theta}(r) \).

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