Generalized Snell Envelope as a Minimal Solution of BSDE With Lower Barriers

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Abstract

The aim of this paper is to characterize the snell envelope of a given \( \mathcal{P} \)-measurable process \( l := (l_t)_{0 \leq t \leq T} \) as the minimal solution of some backward stochastic differential equation with lower general reflecting barriers and to prove that this minimal solution exists.

Keys Words: Backward stochastic differential equation; comparison theorem; Snell envelope.

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1 Introduction and notations

Let \( (\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \leq T}, \mathcal{P}) \) be a stochastic basis on which is defined a Brownian motion \( (B_t)_{t \leq T} \) such that \( (\mathcal{F}_t)_{t \leq T} \) is the natural filtration of \( (B_t)_{t \leq T} \) and \( \mathcal{F}_0 \) contains all \( \mathcal{P} \)-null sets of \( \mathcal{F} \). Note that \( (\mathcal{F}_t)_{t \leq T} \) satisfies the usual conditions, i.e. it is right continuous and complete.

Let us first introduce the following notations:

- \( \mathcal{P} \) is the sigma algebra of \( \mathcal{F}_t \)-predictable sets on \( \Omega \times [0, T] \).
- \( \mathcal{D} \) is the set of \( \mathcal{P} \)-measurable and right continuous with left limits (\( \text{rell} \) for short) processes \( (Y_t)_{t \leq T} \) with values in \( \mathbb{R} \).

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• For a given process \( Y \in \mathcal{D} \), we denote : \( Y_{t-} = \lim_{s \uparrow t} Y_s, t \leq T \) \( (Y_{0-} = Y_0) \), and \( \Delta_s Y = Y_s - Y_{s-} \) the size of its jump at time \( s \).

• \( \mathcal{K} := \{ K \in \mathcal{D} : K \) is nondecreasing and \( K_0 = 0 \} \).

• \( \mathcal{L}^{2,d} \) the set of \( \mathbb{R}^d \)-valued and \( \mathcal{P} \)-measurable processes \((Z_t)_{t \leq T}\) such that

\[
\int_0^T |Z_s|^2 ds < \infty, P - a.s.
\]

The aim of this paper is to characterize the snell envelope of a given \( \mathcal{P} \)-measurable process \( l := (l_t)_{0 \leq t \leq T} \) as the minimal solution of some reflected BSDE with lower barriers (RBSDE for short).

Let \( l := (l_t)_{0 \leq t \leq T} \) be an \( \mathcal{F}_t \)-adapted right continuous with left limits (rccl for short) process with values in \( \mathbb{R} \) of class \( D([0, T]) \), that is the family \((l_t)_{t \in T} \) is uniformly integrable, where \( T \) is the set of all \( \mathcal{F}_t \)-stopping times \( \nu \), such that \( 0 \leq \nu \leq T \). The Snell envelope \( \mathcal{S}_t(l) \) of \( l := (l_t)_{0 \leq t \leq T} \) is defined as

\[
\mathcal{S}_t(l) = \text{ess sup}_{\nu \in \mathcal{T}_t} \mathbb{E} [l_\nu | \mathcal{F}_t],
\]

where \( \mathcal{T}_t \) is the set of all stopping times valued between \( t \) and \( T \). According to the work of Mertens (see [4]), \( \mathcal{S} \) is the smallest rccl-supramartingale of class \( D([0, T]) \) which dominates the process \( l \), i.e., \( \mathcal{P} \)-a.s., \( \forall t \leq T, l_t \leq \mathcal{S}_t(l) \).

Suppose now that \( l \) is neither of class \( D([0, T]) \) nor a rccl process but just \( \mathcal{P} \)-measurable, it is natural to ask whether we can define the smallest local supermartingale which dominates the process \( l \)? In order to give a positive answer to this question, let \( L \in \mathcal{D} \) and \( \delta \in \mathcal{K} \) and assume that there exists a local martingale \( M_t = M_0 + \int_0^t \kappa_s dB_s \) such that \( P \)-a.s.,

\[
L_t \leq M_t \text{ on } [0, T] \quad \text{and} \quad l_t \leq M_t \quad d\delta_t - a.e. \text{ on } [0, T] \quad \text{and} \quad L_T \leq M_T.
\]

Theorem 3.1 states that \( Y \) the minimal solution of the following RBSDE with lower barriers \( L \) and \( l \),

\[
\begin{cases}
(i) & Y_t = L_T + \int_t^T dB_s - \int_t^T Z_s dB_s, t \leq T, \\
(ii) & \forall t \in [0, T], L_t \leq Y_t, \\
(iii) & \text{on } [0, T], l_t \leq Y_{t-}, d\delta_t - a.e. \\
(iv) & \forall L^* \in \mathcal{D} \quad \text{satisfying} \quad \forall t < T, L_t \leq L^*_t \leq Y_t \quad \text{and} \quad \text{on } [0, T], l_t \leq L^*_t, \quad d\delta_t - a.e. \\
& \quad \text{we have} \int_0^T (Y_{t-} - L^*_t) dB_t = 0, \quad a.s., \\
(v) & Y \in \mathcal{D}, \quad K^+ \in \mathcal{K}, \quad Z \in \mathcal{L}^{2,d},
\end{cases}
\]

is the smallest rccl local supermartingale satisfying

\[
\forall t \in [0, T], L_t \leq Y_t, \quad l_t \leq Y_{t-} - d\delta - a.e., \text{ on } [0, T] \quad \text{and} \quad L_T \leq Y_T.
\]

The process \( Y \) will be called later the generalized Snell envelope associated to \( L, l \) and \( \delta \) and it will be denoted by \( \mathcal{S} (L, l, \delta, L_T) \). It is worth mention here that when the process \( l \) is bounded and progressively measurable and \( \delta \) is the Lebesgue measure, L. Stettner and J. Zabczyk characterize the strong Snell envelope \( V \), which is the smallest right continuous non-negative supermartingale such that \( V \geq l, dtdP \)-a.s., as the limit of some non-linear equation.
As by product, if we suppose that there exist $L \in \mathcal{D}$ and $M$ a local martingale such that $L_t \leq l_t \leq M_t$, $dt$–a.e. and $l_T \leq M_T$. We prove that $Y$ the minimal solution of the following reflected BSDE

\[
\begin{cases}
(i) & Y_t = L_T + \int_t^T dK^+_s - \int_t^T Z_sdB_s, t \leq T, \\
(ii) & \text{on } [0,T], L_t \leq Y_t, dt$–a.e \\
(iii) & \forall L^* \in \mathcal{D} \text{ satisfying } L_t \leq L_t^* \leq Y_t \ dt$–a.e. we have \\
& \int_0^T (Y_{t^-} - L^*_{t-})dK^+_s = 0, \ a.s., \\
(iv) & Y \in \mathcal{D}, \quad K^+ \in \mathcal{K}, \quad Z \in \mathcal{L}^{2,d},
\end{cases}
\]

is the smallest rcll local supermartingale bounding the given process $l := (l_t)_{0 \leq t \leq T}$, i.e.

\[
l_t \leq Y_t, \ dt$–a.e \ and \ l_T \leq Y_T.
\]

We shall prove later that equation (1.2) has a minimal solution. We shall also characterize the solution $Y$ as the generalized snell envelope $S(L) = \mathcal{S}(L, l, \delta, L_T)$ and we shall show that the generalized snell envelope $S(L, 0, 0, L_T)$ coincides with the usual snell envelope defined by equality (1.1) if the process $L$ is of class $D[0,T]$.

We need also the following notations:

- For a set $B$, we denote by $B^c$ the complement of $B$ and $1_B$ denotes the indicator of $B$.
- For each $(a, b) \in \mathbb{R}^2$, $a \wedge b = \min(a, b)$ and $a \vee b = \max(a, b)$.
- For all $(a, b, c) \in \mathbb{R}^3$ such that $a \leq c$, $a \vee b \wedge c = \min(\max(a, b), c) = \max(a, \min(c, b))$.

Throughout the paper we introduce the following data:

- $\xi$ is an $\mathcal{F}_T$-measurable one dimensional random variable.
- $L := \{L_t, 0 \leq t \leq T\}$ is a barrier which belongs to $\mathcal{D}$.
- $l := \{l_t, 0 \leq t \leq T\}$ is a $\mathcal{P}$–measurable process.
- $\delta \in \mathcal{K}$.
- $\mathcal{M} = \mathcal{M}(L, l, \delta, \xi)$ is the set of rcll local supermartingale $V_t = V_0 - A_t + \int_0^t \chi_s dB_s$, where $A \in \mathcal{K}$ and $\chi \in \mathcal{L}^{2,d}$ such that

\[
L_t \leq V_t, \ l_t \leq V_{t-} - \delta_t \ dt$–a.e. and $\xi \leq V_T.
\]

We should note here that if $V_t = V_0 - A_t + \int_0^t \chi_s dB_s \in \mathcal{M}$, then we have

1. $V_t + 1 \in \mathcal{M}$.
2. $V_t + A_t = V_0 + \int_0^t \chi_s dB_s \in \mathcal{M}$.

## 2 Preliminaries

In view of clarifying this issue, we recall some results concerning generalized reflected BSDEs (GRBSDE for short) with two rcll obstacles. We present both the existence and comparison theorem for minimal
solutions of this kind of equations. Those results will play a crucial role in our proofs (see [2] for more details). We should note here that the notion of reflected BSDE with two obstacles has been first introduced by Civitanic and Karatzas [1].

2.1 Existence of a minimal solutions for GRBSDE

Let us recall first the following definition of two singular measures.

**Definition 2.1.** Let $K^1$ and $K^2$ be two processes in $\mathcal{K}$. We say that $K^1$ and $K^2$ are singular if and only if there exists a set $D \in \mathcal{P}$ such that

$$\mathbb{E} \int_0^T 1_D(s, \omega) dK^1_s(\omega) = \mathbb{E} \int_0^T 1_D(s, \omega) dK^2_s(\omega) = 0.$$  

This is denoted by $dK^1 \perp dK^2$.

Let us now define the notion of solution of the GRBSDE with two obstacles $L$ and $U$. For this reason, let:

- $g : [0, T] \times \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ be a function such that
  
  $\forall y \in \mathbb{R}, \ (t, \omega) \mapsto g(t, \omega, L_t(\omega) \vee y \wedge U_t(\omega))$ is $\mathcal{P}$ - measurable.
- $U := \{U_t, 0 \leq t \leq T \}$ be a barrier such that $L_t \leq U_t, \forall t \in [0, T]$.

**Definition 2.2.**

1. We say that $(Y, Z, K^+, K^-) := (Y_t, Z_t, K^+_t, K^-_t)_{t \leq T}$ is a solution of the generalized reflected BSDE, associated with the data $(\xi, g, \delta, L, U)$, if the following hold:

   $$
   \left\{
   \begin{array}{l}
   (i) \quad Y_t = \xi + \int_t^T g(s, Y_s) d\delta_s + \int_t^T dK^+_s - \int_t^T dK^-_s - \int_t^T Z_s dB_s, t \leq T, \\
   (ii) \quad \forall t \in [0, T], \ L_t \leq Y_t \leq U_t, \\
   (iii) \quad \int_0^T (Y_{t-} - L_{t-}) dK^+_t = \int_0^T (U_{t-} - Y_{t-}) dK^-_t = 0, \ a.s., \\
   (iv) \quad Y \in \mathcal{D}, \quad K^+, K^- \in \mathcal{K}, \quad Z \in \mathcal{L}^{2, d}, \\
   (v) \quad dK^+ \perp dK^-.
   \end{array}
   \right.
   $$

2. We say that the GRBSDE (2.4) has a minimal solution $(Y_t, Z_t, K^+_t, K^-_t)_{t \leq T}$ if for any other solution $(Y'_t, Z'_t, K'^+_t, K'^-_t)_{t \leq T}$ of (2.4) we have for all $t \leq T, Y_t \leq Y'_t$, $P$-a.s.

We introduce also the following assumption:

(H) The function $g$ and the barrier $U$ satisfy the following:

- (a) There exists $\beta \in L^0(\Omega, L^1([0, T], \mathcal{D}(dt), \mathbb{R}_+))$ such that
  $$\forall y \in \mathbb{R}, \ |g(t, \omega, L_t(\omega) \vee y \wedge U_t(\omega))| \leq \beta_t(\omega), \quad \delta(dt)P(d\omega) - a.e. \quad \text{and} \quad \delta(dt)P(d\omega) - a.e.,$$

(b) the function $y \mapsto g(t, \omega, L_t(\omega) \vee y \wedge U_t(\omega))$ is continuous.

(c) The barrier $U$ is a rcll local supermartingale, i.e. there exist $\alpha \in \mathcal{K}$ and $\gamma \in \mathcal{L}^{2, d}$ such that

$$U_t = U_0 - \alpha_t + \int_0^t \gamma_s dB_s.$$

The following theorem has already been proved in [2]. We should note here that the barriers $L$ and $U$ are rcll, the continuous case has been studied in [3].

**Theorem 2.1.** If assumption (H) holds then the GRBSDE (2.4) has a minimal solution.
2.2 Comparison theorem for minimal solutions

Let us now recall the following comparison theorem which plays a crucial rule in the proof of the existence of solutions for RBSDE. The proof of this comparison theorem is based on an exponential change and an approximation scheme, see [2]. Let \((Y, Z, K^+, K^-)\) be the minimal solution for the following GRBSDE

\[
\begin{align*}
(i) \quad & Y_t = \xi + \int_t^T g(s, Y_s^-)d\delta_s + \int_t^T dK^+_s - \int_t^T dK^-_s - \int_t^T Z_sdB_s, t \leq T, \\
(ii) \quad & \forall t \in [0, T], \; L_t \leq Y_t \leq U_t, \\
(iii) \quad & \int_0^T (Y_{t-} - L_{t-})dK^+_t = \int_0^T (U_{t-} - Y_{t-})dK^-_t = 0, \; \text{a.s.,} \\
(iv) \quad & Y \in \mathcal{D}, \; K^+, K^- \in \mathcal{K}, \; Z \in \mathcal{L}^{2,d}, \\
(v) \quad & dK^+ \perp dK^-.
\end{align*}
\]

Let \((Y', Z', K'^+, K'^-)\) be a solution for the following GRBSDE

\[
\begin{align*}
(i) \quad & Y'_t = \xi' + \int_t^T dA'_s + \int_t^T dK'^+_s - \int_t^T dK'^-_s - \int_t^T Z'_s dB_s, t \leq T, \\
(ii) \quad & \forall t \in [0, T], \; L'_t \leq Y'_t \leq U'_t, \\
(iii) \quad & \int_0^T (Y'_{t-} - L'_{t-})dK'^+_t = \int_0^T (U'_{t-} - Y'_{t-})dK'^-_t = 0, \; \text{a.s.,} \\
(iv) \quad & Y' \in \mathcal{D}, \; K'^+, K'^- \in \mathcal{K}, \; Z' \in \mathcal{L}^{2,d}, \\
(v) \quad & dK'^+ \perp dK'^-,
\end{align*}
\]

where \(A'\) is a process in \(\mathcal{K}, \; L'\) and \(U'\) are two barriers which belong to \(\mathcal{D}\) such that \(L'_t \leq U'_t, \; \forall t \in [0, T]\). Assume moreover that for every \(t \in [0, T]\)

(a) \(\xi \leq \xi'\).

(b) \(Y'_t \leq U_t, \; L'_t \leq Y_t, \; \forall t \in [0, T]\).

(c) \(g(s, Y'_s^-)d\delta_s \leq dA'_s\) on \([0, T]\).

Theorem 2.2. (Comparison theorem for minimal solutions, see [2]) Assume that the above assumptions hold then we have:

1. \(Y_t \leq Y'_t\), for every \(t \in [0, T]\), \(P\)-a.s.

2. \(1_{\{U'_t = U_t\}}dK^-_t \leq dK^-_t\) and \(1_{\{U'_t = L_t\}}dK'^+_t \leq dK'^+_t\).

3 Generalized Snell envelope as a solution of some RBSDE

In this section, we prove an existence result of a minimal solution for some reflected BSDE with lower barriers. We shall also characterize this minimal solution \(Y\) as the smallest \(\text{rcll}\) local supermartingale satisfying

\(\forall t \in [0, T], \; L_t \leq Y_t, \; l_t \leq Y_{t-} \; d\delta_t - a.e., \; \text{on } [0, T]\) and \(\xi \leq Y_T\).

Let us now introduce the definition of our RBSDE with lower obstacles.
Definition 3.1. 1. We call \( (Y, Z, K^+):= (Y_t, Z_t, K^+_t)_{t \leq T} \) a solution of the RBSDE, associated with the data \((\xi, L, l, \delta)\), if the following hold:

\[
\begin{align*}
(i) & \quad Y_t = \xi + \int_t^T dK^+_s - \int_t^T Z_s dB_s, \ t \leq T, \\
(ii) & \quad \forall t \in [0, T], \ L_t \leq Y_t, \\
(iii) & \quad \text{on } [0, T], \ l_t \leq Y^+_{t-}, \ d\delta_t \ - \ a.e. \\
(iv) & \quad \forall L^* \in D \text{ satisfying } \forall t < T, \ L_t \leq L^+_t \leq Y_t \quad \text{and} \\
& \quad \text{on } [0, T], \ l_t \leq L^-_{t-}, \ d\delta_t \ - \ a.e. \\
& \quad \text{we have } \int_0^T (Y^-_t - L^-_{t-}) dK^+_t = 0, \ a.s., \\
(v) & \quad Y \in D, \ K^+ \in K, \ Z \in L^{2,d}.
\end{align*}
\]

2. We say that the RBSDE \((7)\) has a minimal solution \((Y_t, Z_t, K^+_t)_{t \leq T}\) if for any other solution \((Y'_t, Z'_t, K^+_t)_{t \leq T}\) of \((7)\) we have for all \(t \leq T\), \(Y_t \leq Y'_t\), \(P\)-a.s.

3.1 Main result

Let \(L \in D, \ \xi \in L^0(\Omega), \ l \in L^0(\Omega \times [0, T])\) and \(\delta \in K\). We assume the following hypothesis:

(A) There exists a local martingale \(M_t = M_0 + \int_0^t \kappa_s dB_s\) such that \(P\)-a.s., \(L_t \leq M_t\) on \([0, T]\) and \(l_t \leq M_t, d\delta_t \ - \ a.e.\ on \ [0, T]\) and \(\xi \leq M_T\). This is equivalent to \(M \neq \emptyset\).

The main result of this paper is the following.

Theorem 3.1. If assumption (A) hold then the RBSDE \((7)\) has a minimal solution \((Y_t, Z_t, K^+_t)_{t \leq T}\). Moreover \(Y\) is the smallest \(rcll\) local supermartingale satisfying

\[
\forall t \in [0, T], \ L_t \leq Y_t, \ l_t \leq Y^+_{t-} - \delta \ - \ a.e., \ on \ [0, T] \ and \ \xi \leq Y_T.
\]

We say that \(Y\) is the generalized Snell envelope associated to \(L, l, \delta\) and \(\xi\). We denote it by \(S(L, l, \delta, \xi)\).

3.1.1 Auxiliary penalized equation

Let \(V_t = V_0 - A_t + \int_0^t \chi_s dB_s \in M\). Let also \((Y^{(n,V)}, Z^{(n,V)}, K^{(n,V)}+), K^{(n,V)}-\) be the minimal solution of the following penalized RBSDE with two \(rcll\) barriers:

\[
\begin{align*}
(i) & \quad Y^{(n,V)}_t = \xi + n \int_t^T (l_s - Y^{(n,V)}_{s-}) d\delta_s + \int_t^T dK^{(n,V)}_s + \\
& \quad - \int_t^T dK^{(n,V)}_- - \int_t^T Z^{(n,V)}_s dB_s, \ t \leq T, \\
(ii) & \quad \forall t \in [0, T], \ L_t \leq Y^{(n,V)}_t \leq V_t, \\
(iii) & \quad \int_0^T (Y^{(n,V)}_{t-} - L^-_{t-}) dK^{(n,V)}_t = \int_0^T (Y^{(n,V)}_{t-} - L^-_{t-}) dK^{(n,V)}_- = 0, \ P\ - \ a.s., \\
(iv) & \quad Y^{(n,V)} \in D, \ K^{(n,V)}+, K^{(n,V)}- \in K, \ Z^{(n,V)} \in L^{2,d}, \\
(v) & \quad dK^{(n,V)}+ \perp dK^{(n,V)}-.
\end{align*}
\]
It follows from comparison theorem 2.2, applied to $Y$ satisfying (3.9) to the process $Y$. Since $Y$ is a rcll local supermartingale satisfying

$$
\forall t \in [0, T], \ L_t \leq Y_t, \ l_t \leq Y_{t-} \ \delta_t - a.e., \ \text{on} \ [0, T] \ \text{and} \ \xi \leq Y_T.
$$

Moreover, for every $V \in \mathcal{M}$ and all $(n, t) \in \mathbb{N} \times [0, T], \ Y_t^{(n, V)} \leq V_t$. Since $Y^{(n, M)}$ is also the minimal solution of (3.9), then for every $V$, $Y^{(n, V)} = Y^{(n, M)}$. From now on we denote the solution of (3.9) by $(Y^{n, V}, Z^{n, V}, K^{(n, V)})$.

Now by using comparison theorem 2.2 we get, for every $V \in \mathcal{M}$, that

$$
L_t \leq Y_t^n \leq Y_t^{n+1} \leq V_t. \quad (3.10)
$$

Now let us set

$$
Y_t = \sup_n Y_t^n \ \text{and} \ Y_t^- = \sup_n Y_{t-}^n. \quad (3.11)
$$

The following results guarantee that the process $Y$ is the smallest rcll local supermartingale satisfying

$$
\forall t \in [0, T], \ L_t \leq Y_t, \ l_t \leq Y_{t-} \ \delta_t - a.e., \ \text{on} \ [0, T] \ \text{and} \ \xi \leq Y_T.
$$

By letting $n$ to infinity in (3.10) and using assumption (A) we have the following.

**Lemma 3.1.** For every $t \in [0, T]$ we have for every $V \in \mathcal{M}$,

$$
L_t \leq Y_t \leq V_t \ \text{on} \ [0, T] \ \text{and} \ L_{t-} \leq Y_{t-} \leq V_{t-} \ \text{on} \ [0, T].
$$

**Proposition 3.1.** The process $Y$ defined by (3.11) satisfy the following properties:

1. $Y$ is a rcll local supermartingale and $Y_t^- \leq Y_t$, for every $t \in [0, T]$.

2. $l_t \leq Y_t^-, \ \delta_t - a.e., \ \text{on} \ [0, T]$.

In particular it follows that $Y$ belongs to $\mathcal{M}$. 

Proof. 1. Recall that \( M_t = M_0 + \int_0^t \kappa_s dB_s \in \mathcal{M} \). We have

\[
Y_t^n - M_t = \xi - M_T + n \int_t^T (l_s - Y_{s-}^n)^+ d\delta_s + \int_t^T dK_s^n + \int_t^T (Z_s^n - \kappa_s) dB_s.
\]

Let \((\tau_i)_{i \geq 1}\) be the family of stopping times defined by

\[
\tau_i = \inf\{s \geq 0 : M_s - L_s \geq i + M_0 - L_0\} \wedge T.
\]

Note that \( \tau_i > 0 \), \( P \)-a.s., for every \( i \geq 1 \). By using a localization procedure we have for every \( i \geq 1 \) and \( n \geq 0 \)

\[
\mathbb{E}(M_0 - Y_0^n) + n \mathbb{E} \int_0^{\tau_i^-} (l_s - Y_{s-}^n)^+ d\delta_s + \mathbb{E} K_s^n \leq i + \mathbb{E}(M_0 - L_0).
\]

Put

\[
M^n_t = Y^n_t - M_t,
\]

\[
i M^n_t = M^n_t 1_{\{t < \tau_i\}} + M^n_{t-} 1_{\{t \geq \tau_i\}},
\]

we have

\[-i - \mathbb{E}(M_0 - L_0) \leq i M^n_t \leq 0 \quad \text{and} \quad i M^n_t \leq i M^n_{t+} \quad \text{and} \quad t \to i M^n_t \text{ is a rcll supermartingale.}\]

It follows then from Dellacherie and Meyer [4] that \( \sup_n i M^n_t \) is also a rcll process supermartingale process. Since \( P \left( \bigcup_{n=1}^{\infty} (\tau_n = T) \right) = 1 \), it follows that \( Y_t \) is a rcll local supermartingale on \([0, T]\).

Now since for every \( s \in [0, T] \) and \( n \in \mathbb{N} \), \( Y^n_{s-} \leq Y_{s-} \), it follows that \( Y_{s-} \leq Y_{s-} \).

2. On another hand, by letting \( n \) to infinity in inequality (3.13) and using Fatou’s lemma it follows that

\[
\mathbb{E} \int_0^{\tau_i^-} (l_s - Y_{s-}^-)^+ d\delta_s = 0.
\]

Hence

\[(l_s - Y_{s-}^-)^+ = 0 \text{ d}\delta_s - a.e. \text{ on } [0, T].\]

Assume now that \( Y_T^- < l_T \text{ and } \Delta_T \delta > 0 \). It follows from [2], that for every \( V \in \mathcal{M} \)

\[
Y_{T-}^n = L_{T-} \vee [\xi + n(l_T - Y_{T-}^n)^+ \Delta_T \delta] \wedge V_{T-} \geq [\xi + n(l_T - Y_{T-}^-)^+ \Delta_T \delta] \wedge V_{T-}.
\]

We get \( Y_{T-}^- = V_T \), which is absurd since \( V_t + 1 \in \mathcal{M} \). Consequently

\[l_s \leq Y_{s-}^- \text{ d}\delta_s - a.e. \text{ on } [0, T].\]

The proof of Proposition 3.1 is finished. 

\[\Box\]
3.1.2 Proof of the main result

Proof of Theorem 3.1 Let $L^* \in \mathcal{D}$ be such that for every $t \in [0,T]$, $L_t \leq L^*_t \leq Y_t$ and $l_t \leq L^*_t - d\delta_t - a.e.$ Let also $(Y^*, Z, K^+, K^-)$, which is exists according to Theorem 3.1, the minimal solution of the following RBSDE

\[
\begin{align*}
(i) & \quad Y_t^* = \xi + \int_t^T dK^+_s - \int_t^T dK^-_s - \int_t^T Z_s dB_s, \ t \leq T, \\
(ii) & \quad \forall t \in [0,T], \ L^*_t \leq Y^*_t \leq Y_t, \\
(iii) & \quad \int_0^T (Y^-_t - L^*_t) dK^+_t = \int_0^T (Y^*_t - L^*_t) dK^+_t = 0, \ a.s., \ \\
(v) & \quad Y^* \in \mathcal{D}, \ K^+, K^- \in \mathcal{K}, \ Z \in \mathcal{L}^{2, d}, \\
(vi) & \quad dK^+ \perp dK^-.
\end{align*}
\]

By the same argument as before with $V = Y$ (where $Y$ is the process defined in the previous subsection), one can see that $dK^- = 0$, hence $Y^* \in \mathcal{M}$. By Lemma 3.1 and (ii) of Equation (3.15) we get

\[ Y^*_t = Y_t. \]

Henceforth

\[ (Y^-_t - L^*_t) dK^+_t = 0. \]

Consequently, for every $V \in \mathcal{M}$, $(Y, Z, K^+)$ is a solution of (3.7). Moreover the process $Y$ is the smallest rcll local supermartingale satisfying

\[ \forall t \in [0,T], \ L_t \leq Y_t, \ l_t \leq Y_t - d\delta_t - a.e., \text{ on } [0,T] \text{ and } \xi \leq Y_T. \]

As by product we obtain the following theorem.

Theorem 3.2. Let $(T_i)_{i \geq 1}$ be a sequence of stopping times such that $\|T_i\| \cap \|T_j\| = \emptyset, \forall i \neq j$ and $\bigcup_{i \geq 1} \|T_i\| = \{(t, \omega) \in [0, T] \times \Omega : \Delta_\delta(\omega) > 0\}$. Under assumption (A), $Y$ the minimal solution of (3.7) is the smallest rcll local supermartingale satisfying $P$-a.s.

\[ \forall t \in [0, T], \ L_t \leq Y_t, \ l_t \leq Y_t - d\delta_t - a.e., \text{ on } [0, T], \forall i \geq 1, \ l_{T_i} \leq Y_{T_i} \text{ and } \xi \leq Y_T. \]

3.1.3 Some properties of the generalized Snell envelope

The generalized Snell envelope $Y = S(L, l, \delta, \xi)$ solution of RBSDE (3.7) has the following properties whose proofs are immediate.

Corollary 3.1. 1. $S(L, l, \delta, \xi) = S(L, \overline{\delta}, \delta, \xi)$, with $\overline{\delta} = l_s \lor L_{s-}$. 

2. If $L' \leq L$, $d\delta' \ll d\delta, l' \leq l$, $d\delta' - a.e., \xi' \leq \xi$ then $(L', l', \delta', \xi')$ satisfies condition (A) and $S(L', l', \delta', \xi') \leq S(L, l, \delta, \xi)$. 

3. $S(L, l, \delta, \xi) \geq S(L^f)$ (with equality if $l_t \leq L_t - d\delta_t - a.e., \text{ on } [0, T]$) where $S(L^f) = S(L, 0, 0, \xi)$ and $L^f_t = L_t \mathbf{1}_{\{t < T\}} + \xi \mathbf{1}_{\{t = T\}}$. 

4. Put $Y = S(L, l, \delta, \xi)$. If $l_t \leq Y_t', d\delta - a.e., \text{ on } [0, T]$ and $L_t \leq L_t' \leq Y_t, \forall t \in [0, T]$, and $d\delta \sim d\delta'$.
then \( S(L, l, \delta, \xi) = S(L', l', \delta', \xi) \).

In particular for every \( L^* \in \mathcal{D} \) such that \( P - a.s. \),

\[
L_t \leq L_t^* \leq Y_t, \quad \forall t \in [0, T], \quad \text{and} \quad l_t \leq L_t^* - Y_t, \quad dt - a.e. \quad \text{on} \quad [0, T] \quad \text{and} \quad L_T^* = \xi
\]

we have \( S(L, l, \delta, \xi) = S(L^*) \).

**Remark 3.1.** We know that if \( L \) is of class \( D \) then \( L \) satisfies assumption \( (A) \) (see Dellacherie-Meyer [4]). In this case our generalized snell envelope \( S(L) = S(L, 0, 0, L_T) \) coincides with the usual snell envelope \( \text{esssup}_{\tau \in \mathcal{T}} \mathbb{E}[L_\tau | F_t] \), where \( \mathcal{T} \) is the set of all stopping times valued between \( t \) and \( T \), as presented in Dellacherie-Meyer [4] and studied by several authors.

**Example 3.1.** If \( \delta_t = t \) and there exist \( L \in \mathcal{D} \) and \( M \) a local martingale such that \( L_t \leq l_t \leq M_t \) and \( \xi \leq M_T \). Let \((Y, Z, K^+)\) be the minimal solution of the following RBSDE

\[
\begin{align*}
(i) & \quad Y_t = \xi + \int_t^T dK_t^+ - \int_t^T Z_s dB_s, t \leq T, \\
(ii) & \quad \text{on} \quad [0, T], \quad l_t \leq Y_t, \quad dt - a.e. \\
(iii) & \quad \forall L^* \in \mathcal{D} \quad \text{satisfying} \quad l_t \leq L_t^* \leq Y_t \quad dt - a.e. \quad \text{we have} \\
& \quad \int_0^T (Y_{t-} - L_{t-}) dK_t^+ = 0, \quad a.s., \\
(v) & \quad Y \in \mathcal{D}, \quad K^+ \in \mathcal{K}, \quad Z \in \mathcal{L}^{2,d},
\end{align*}
\]

Then \( Y \) is the smallest local supermartingale such that

\[
l_t \leq Y_t, \quad dt - a.e \quad \text{and} \quad \xi \leq Y_T.
\]

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