On the radical of cluster tilted algebras

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Abstract

We determine the minimal lower bound \( n \), with \( n \geq 1 \), where the \( n \)-th power of the radical of the module category of a representation-finite cluster tilted algebra vanishes. We give such a bound in terms of the number of vertices of the underline quiver. Consequently, we get the nilpotency index of the radical of the module category for representation-finite self-injective cluster tilted algebras. We also study the non-zero composition of \( m \), \( m \geq 2 \), irreducible morphisms between indecomposable modules in representation-finite cluster tilted algebras lying in the \( (m + 1) \)-th power of the radical of their module category.

Keywords  Irreducible morphism · Radical · Projective cover ·Injective hull

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1 Introduction

Let \( A \) be a finite dimensional algebra over an algebraically closed field. The representation theory of an algebra \( A \) deals with the study of the module category of finitely generated \( A \)-modules, \( \text{mod} \, A \). A fundamental tool in the study of \( \text{mod} \, A \) is the Auslander–Reiten theory, based on irreducible morphisms and almost split sequences.

For \( X, Y \in \text{mod} \, A \), we denote by \( \mathfrak{m}(X, Y) \) the set of all morphisms \( f : X \to Y \) such that, for any indecomposable \( A \)-module \( M \), and any pair of morphisms \( h : M \to X \) and \( g : Y \to M \), there exists a morphism \( d : M \to \text{mod} \, A \) such that \( df = gh \).

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and \( h' : Y \rightarrow M \) the composition \( h'fh \) is not an isomorphism. Inductively, the powers of \( \mathfrak{H}(X, Y) \) are defined.

There is a close connection between irreducible morphisms and the powers of the radical, given by a well-known result proved by Bautista in [2], which states that if \( f : X \rightarrow Y \) is a morphism between indecomposable \( A \)-modules then \( f \) is irreducible if and only if \( dp(f) = 1 \), see Sect. 2.2.

In the case \( \mathfrak{H}^n(M, N) = 0 \) for some positive integer \( n \) and for all \( M \) and \( N \) in \( \text{mod} \ A \), we write this fact by the expression \( \mathfrak{H}^n(\text{mod} \ A) = 0 \). We recall that an algebra \( A \) is representation-finite (or of finite representation type) if and only if there is a positive integer \( n \) such that \( \mathfrak{H}^n(\text{mod} \ A) \) vanishes (see [1, p. 183]).

In [17], Liu defined the notion of degree of an irreducible morphism (see Sect. 2.4) which has been a powerful tool to study many problems of the representation of Artin algebras’s theory. In particular, it has been an important tool to solve the problem concerning the nilpotency of the radical of a module category of an algebra \( A \), in the case of finite dimensional \( k \)-algebras over an algebraically closed field of finite representation type.

If \( A \) is a finite dimensional basic algebra over an algebraically closed field then we know that \( A \simeq k \, Q_A/I_A \). In addition, if \( A \) is representation-finite then by [10] all irreducible epimorphisms and all irreducible monomorphisms are of finite left and right degree, respectively. In particular, the irreducible monomorphism \( r_a : \text{rad}(P_a) \hookrightarrow P_a \), where \( P_a \) is the projective module corresponding to the vertex \( a \) in \( Q_A \), has finite right degree. Dually, the irreducible epimorphism \( g_a : I_a \rightarrow I_a/S_a \), where \( I_a \) is the injective module corresponding to the vertex \( a \) in \( Q_A \), has finite left degree. We denote by \( S_a \) the simple \( A \)-module corresponding to the vertex \( a \) in \( Q_A \).

By [8], we know that for a finite dimensional algebra over an algebraically closed field \( A \simeq k \, Q_A/I_A \) where \( A \) is representation-finite we can compute the nilpotency index \( r_a \) of \( \mathfrak{H}(\text{mod} \ A) \) by \( \max \{ r_a \}_{a \in Q_A} + 1 \) where \( r_a \) is equal to the length of any path of irreducible morphisms between indecomposable modules from the projective \( P_a \) to the injective \( I_a \), going through the simple \( S_a \).

Applying the above mentioned result we give the minimal positive integer \( m \) such that \( \mathfrak{H}(\text{mod} \Gamma) \) vanishes, where \( \Gamma \) is a cluster tilted algebra of type \( \overline{\Delta} \), with \( \Delta \) a Dynkin quiver. More precisely, we prove the following result.

**Theorem A** Let \( \mathcal{C} \) be the cluster category of a representation-finite hereditary algebra \( H \). Let \( \mathcal{T} \) be an almost complete tilting object in \( \mathcal{C} \) with complements \( M \) and \( M^* \). Consider \( \Gamma = \text{End}_\mathcal{C}(\mathcal{T})^{\text{op}} \) and \( \Gamma' = \text{End}_\mathcal{C}(\mathcal{T}')^{\text{op}} \) the cluster tilted algebras with \( T = \mathcal{T} \oplus M \) and \( T' = \mathcal{T}' \oplus M^* \). Let \( r_\Gamma \) and \( r_{\Gamma'} \) be the nilpotency indices of \( \mathfrak{H}(\text{mod} \Gamma) \) and \( \mathfrak{H}(\text{mod} \Gamma') \), respectively. Then, \( r_\Gamma = r_{\Gamma'} \).

As a consequence of Theorem A and [20, Theorem 4.11] we get the following corollary, which is one of the main results of this paper.

**Corollary B** Let \( \Delta \) be a Dynkin quiver and let \( \Gamma \) be a cluster tilted algebra of type \( \overline{\Delta} \). Let \( r_\Gamma \) be the nilpotency index of \( \mathfrak{H}(\text{mod} \Gamma) \). Then the following conditions hold:

(a) If \( \overline{\Delta} = A_n \), then \( r_\Gamma = n \) for \( n \geq 1 \).
(b) If \( \overline{\Delta} = D_n \), then \( r_\Gamma = 2n - 3 \) for \( n \geq 4 \).
(c) If $\Delta = E_6$, then $r_{\Gamma} = 11$.
(d) If $\Delta = E_7$, then $r_{\Gamma} = 17$.
(e) If $\Delta = E_8$, then $r_{\Gamma} = 29$.

The non-zero composition of $n$ irreducible morphisms between indecomposable modules could belong to $\Re^{n+1}$. In the last years, there have been many works done in this direction. The first to give a partial solution to that problem were Igusa and Todorov in [16], where they proved that if

$$X_0 \xrightarrow{f_1} X_1 \to \cdots \to X_{n-1} \xrightarrow{f_n} X_n$$

is a sectional path then $f_n \ldots f_1$ lies in $\Re^*(X_0, X_n)$ but not in $\Re^{n+1}(X_0, X_n)$.

In [9], Coelho, Trepode and the first named author characterized when the composition of two irreducible morphisms is non-zero and lies in $\Re^*(\mod A)$ for $A$ an Artin algebra. In [11], Le Meur, Trepode and the first named author solved the problem of when the composition of $n$ irreducible morphisms between indecomposable modules is non-zero and belongs to $\Re^{n+1}(\mod A)$ for finite dimensional $k$-algebras over a perfect field $k$.

As a consequence of the results of this work, we obtain when the composition of $n$ irreducible morphisms between indecomposable $A$-modules belongs to the $n + 1$ power of the radical of their module category, for a representation-finite cluster tilted algebra $A$. More precisely, we prove the following result.

**Theorem C** Let $\Gamma$ be a representation-finite cluster tilted algebra. Consider the irreducible morphisms $h_i : X_i \to X_{i+1}$, with $X_i \in \text{ind } \Gamma$ for $1 \leq i \leq m$. Then $h_m \ldots h_1 \in \Re^{m+1}(X_1, X_{m+1})$ if and only if $h_m \ldots h_1 = 0$.

**2 Preliminaries**

Throughout this work, by an algebra we mean a finite dimensional basic $k$-algebra over an algebraically closed field, $k$.

**2.1 Notions on quivers and algebras**

A quiver $Q$ is given by a set of vertices $Q_0$ and a set of arrows $Q_1$, together with two maps $s, e : Q_1 \to Q_0$. Given an arrow $\alpha \in Q_1$, we write $s(\alpha)$ for the starting vertex of $\alpha$ and $e(\alpha)$ for the ending vertex of $\alpha$. We denote by $\overline{Q}$ the underlying graph of $Q$. For each algebra $A$ there is a quiver $Q_A$, called the ordinary quiver of $A$, such that $A$ is the quotient of the path algebra $k Q_A$ by an admissible ideal.

Let $A$ be an algebra. We denote by $\mod A$ the category of finitely generated left $A$-modules and by $\text{ind } A$ the full subcategory of $\mod A$ which consists of one representative of each isomorphism class of indecomposable $A$-modules.

We say that $A$ is a representation-finite algebra if there is only a finite number of isomorphism classes of indecomposable $A$-modules.
We denote by $\Gamma_A$ the Auslander–Reiten quiver of $\text{mod } A$, and by $\tau$ the Auslander–Reiten translation $D\text{Tr}$ with inverse $\text{Tr}D$ denoted by $\tau^{-1}$.

### 2.2 On the radical of a module category

A morphism $f : X \to Y$, with $X, Y \in \text{mod } A$, is called irreducible provided it does not split and whenever $f = gh$, then either $h$ is a split monomorphism or $g$ is a split epimorphism.

If $X, Y \in \text{mod } A$, the ideal $\mathfrak{R}(X, Y)$ of $\text{Hom}(X, Y)$ is the set of all the morphisms $f : X \to Y$ such that, for each $M \in \text{ind } A$, each $h : M \to X$ and each $h' : Y \to M$ the composition $h'fh$ is not an isomorphism. For $n \geq 2$, the powers of $\mathfrak{R}(X, Y)$ are inductively defined. By $\mathfrak{R}^\infty(X, Y)$ we denote the intersection of all powers $\mathfrak{R}^i(X, Y)$, with $i \geq 1$.

We say that the depth of a morphism $f : M \to N$ in $\text{mod } A$ is infinite if $f \in \mathfrak{R}^\infty(M, N)$; otherwise, the depth of $f$ is the integer $n \geq 0$ for which $f \in \mathfrak{R}^n(M, N)$ but $f \notin \mathfrak{R}^{n+1}(M, N)$. We denote the depth of $f$ by $\text{dp}(f)$.

By [2], it is known that for $X, Y \in \text{ind } A$, a morphism $f : X \to Y$ is irreducible if and only if $\text{dp}(f) = 1$.

We recall the next proposition fundamental for our results.

**Proposition 2.1** ([1, V, Proposition 7.4]) Let $M$ and $N$ be indecomposable modules in $\text{mod } A$ and let $f$ be a morphism in $\mathfrak{R}^n(M, N)$, with $n \geq 2$. Then, the following conditions hold:

(i) There exist a natural number $s$, indecomposables $A$-modules $X_1, \ldots, X_s$, morphisms $f_i \in \mathfrak{R}(M, X_i)$, and morphisms $g_i : X_i \to N$, with each $g_i$ a sum of compositions of $n - 1$ irreducible morphisms between indecomposable modules such that $f = \sum_{i=1}^s g_i f_i$.

(ii) If $\text{dp}(f) = n$, then at least one of the $f_i$ in (i) is irreducible.

It is well known by a result of Auslander that an algebra $A$ is representation-finite if and only if $\mathfrak{R}^\infty(\text{mod } A) = 0$. That is, there is a positive integer $n$ such that $\mathfrak{R}^n(X, Y) = 0$ for all $X, Y$ $A$-modules. The minimal positive integer $m$ such that $\mathfrak{R}^m(\text{mod } A) = 0$ is called the nilpotency index of $\mathfrak{R}(\text{mod } A)$. We denote such an index by $r_A$.

### 2.3 Basic definitions of paths

A path in $\text{mod } A$ is a sequence $M_0 \xrightarrow{f_1} M_1 \xrightarrow{f_2} M_2 \to \cdots \to M_{n-1} \xrightarrow{f_n} M_n$ of non-zero non-isomorphisms $f_1, \ldots, f_n$ between indecomposable $A$-modules with $n \geq 1$. In case that $f_1, \ldots, f_n$ are irreducible morphisms, we say that the path is in $\Gamma_A$ or equivalently that it is a path in $\Gamma_A$. The length of a path in $\Gamma_A$ is defined as the number of irreducible morphisms (not necessarily different) involved in the path.

Let us recall that paths in $\Gamma_A$ having the same starting vertex and the same ending vertex are called parallel paths.
Let Γ be a component of Γ_A. We say that Γ is a component with length if parallel paths in Γ have the same length. Otherwise, it is called a component without length, see [12].

By a directed component we mean a component Γ that there is no sequence

\[ M_0 \xrightarrow{f_1} M_1 \xrightarrow{f_2} M_2 \rightarrow \cdots \xrightarrow{f_n} M_n \]

of non-zero non-isomorphisms \( f_1, \ldots, f_n \) between indecomposable \( A \)-modules with \( M_0 = M_n \).

Given a directed component \( \Gamma \) of \( \Gamma_A \), its orbit graph \( O(\Gamma) \) is a graph defined as follows: the points of \( O(\Gamma) \) are the \( \tau \)-orbits \( \tau(M) \) of the indecomposable modules \( M \) in \( \Gamma \). There is an edge between \( O(M) \) and \( O(N) \) in \( O(\Gamma) \) if there are positive integers \( n, m \) and either an irreducible morphism from \( \tau^m M \) to \( \tau^n N \) or from \( \tau^n N \) to \( \tau^m M \) in \( \text{mod } A \).

Note that if the orbit graph \( O(\Gamma) \) is of tree-type, then \( \Gamma \) is a simply connected translation quiver, and by [3] we know that \( \Gamma \) is a component with length.

2.4 On the nilpotency index of the radical of a module category

Next, we recall the definition of degree of an irreducible morphism given by Liu in [17].

Let \( f : X \rightarrow Y \) be an irreducible morphism in \( \text{mod } A \), with \( X \) or \( Y \) indecomposable. The left degree \( d_l(f) \) of \( f \) is infinite, if for each integer \( n \geq 1 \), each module \( Z \in \text{ind } A \), and each morphism \( g : Z \rightarrow X \) with \( dp(g) = n \) we have that \( dp(fg) = n + 1 \). Otherwise, the left degree of \( f \) is the least natural \( m \) such that there are an \( A \)-module \( Z \) and a morphism \( g : Z \rightarrow X \) with \( dp(g) = m \) such that \( fg \in \mathcal{R}^{m+2}(Z, Y) \).

The right degree \( d_r(f) \) of an irreducible morphism \( f \) is dually defined.

In order to compute the nilpotency index of the radical of any module category we shall strongly use [8, Theorem A]. For the convenience of the reader, we recall it below.

Let \( A = kQ_A/I_A \) be a representation-finite algebra. Let \( a \in (Q_A)_0 \) and \( P_a, I_a \) and \( S_a \) be the projective, injective and simple \( A \)-modules, respectively, corresponding to the vertex \( a \).

For each \( a \in (Q_A)_0 \), let \( n_a \) be the number defined as follows:

- If \( P_a = S_a \), then \( n_a = 0 \).
- If \( P_a \not\cong S_a \), then \( n_a = d_r(\iota_a) \), where \( \iota_a \) is the irreducible morphism \( \iota_a : \text{rad}(P_a) \rightarrow P_a \).

Dually, for each \( a \in (Q_A)_0 \), let \( m_a \) be the number defined as follows:

- If \( I_a = S_a \), then \( m_a = 0 \).
- If \( I_a \not\cong S_a \), then \( m_a = d_r(\theta_a) \), where \( \theta_a \) is the irreducible morphism \( \theta_a : I_a \rightarrow I_a/S_a \).

We write \( r_a = m_a + n_a \).
Theorem 2.2 ([8, Theorem A])

Let \( A \simeq k Q_A / I_A \) be a finite dimensional algebra over an algebraically closed field and assume that \( A \) is a representation-finite algebra. Then the nilpotency index \( r_A \) of \( \mathfrak{N}(\text{mod } A) \) is \( r_A = \max \{ r(a) \mid a \in (Q_A)_0 \} + 1 \).

Following [8, Remark 1], \( r(a) \) is equal to the length of any path of irreducible morphisms between indecomposable modules from the projective \( P_a \) to the injective \( I_a \), going through the simple \( S_a \).

Finally, we recall a result of [8] that shall be useful in this work.

Lemma 2.3 ([8, Lemma 2.4]) Let \( A \simeq k Q / I \) be a representation-finite algebra. Given \( a \in Q_0 \), consider \( r(a) \) the number defined as above. Then, the following conditions hold:

(a) Every non-zero morphism \( f : P_a \to I_a \) that factors through the simple \( A \)-module \( S_a \) is such that \( \text{dp}(f) = r(a) \).

(b) Every non-zero morphism \( f : P_a \to I_a \) which does not factor through the simple \( A \)-module \( S_a \) is such that \( \text{dp}(f) < r(a) \).

2.5 The cluster category

Let \( H \) be a hereditary algebra. We denote by \( \mathcal{D} = \mathcal{D}^b(\text{mod } H) \) the bounded derived category of \( \text{mod } H \). The cluster category, \( \mathcal{C} \), is defined as the quotient \( \mathcal{D} / F \), where \( F \) is the composition \( \tau^{-1}[1] \) of the suspension functor and the Auslander–Reiten translation in \( \mathcal{D} \). The objects of \( \mathcal{C} \) are the \( F \)-orbits of the objects in \( \mathcal{D} \), and the morphisms in \( \mathcal{C} \) are defined as

\[
\text{Hom}_{\mathcal{C}}(\tilde{X}, \tilde{Y}) = \bigsqcup_{i \in \mathbb{Z}} \text{Hom}_{\mathcal{D}}(F^i X, Y),
\]

where \( X, Y \) are objects in \( \mathcal{D} \) and \( \tilde{X}, \tilde{Y} \) are the corresponding objects in \( \mathcal{C} \). By [4, Proposition 1.5], the summands of (1) are almost all zero.

We recall some basic and useful properties of \( \mathcal{C} \).

(i) \( \mathcal{C} \) is a Krull–Schmidt category.

(ii) \( \mathcal{C} \) is a triangulated category, whose suspension functor over \( \mathcal{C} \) is denoted by \( [1] \).

(iii) \( \mathcal{C} \) has Auslander–Reiten triangles, which are induced by the Auslander–Reiten triangles of \( \mathcal{D} \). We also denote the Auslander–Reiten translation of \( \mathcal{C} \) by \( \tau \).

Remark 2.4 We deduce by (iii) that the irreducible morphisms in \( \mathcal{C} \) are induced by the irreducible morphisms in \( \mathcal{D} \). Moreover, the non-zero paths of irreducible morphisms between indecomposable objects in \( \mathcal{C} \) are induced by non-zero paths of irreducible morphisms between indecomposable objects in \( \mathcal{D} \), and both paths have the same length.

We denote by \( \mathcal{S} \) the set \( \text{ind}(\text{mod } H \vee H[1]) \) consisting of the indecomposable \( H \)-modules together with the objects \( P[1] \), where \( P \) is an indecomposable projective \( H \)-module. We can see the set \( \mathcal{S} \) as the fundamental domain of \( \mathcal{C} \) for the action of \( F \) in \( \mathcal{D} \), containing exactly one representative object from each \( F \)-orbit in \( \text{ind } \mathcal{D} \).

It is known that given \( X \) and \( Y \) objects in \( \mathcal{S} \), \( \text{Hom}_{\mathcal{D}}(F^i X, Y) = 0 \) for all \( i \neq -1, 0 \). Moreover, if \( H \) is a representation-finite algebra, then at least one, \( \text{Hom}_{\mathcal{D}}(F^{-1} X, Y) \) or \( \text{Hom}_{\mathcal{D}}(X, Y) \) vanishes.
2.6 On tilting objects

An object $T$ in $\mathcal{C}$ is said to be a tilting object if $\text{Ext}^1_{\mathcal{C}}(T, T) = 0$ and $T$ is maximal with that property, that is, if $\text{Ext}^1_{\mathcal{C}}(T \oplus X, T \oplus X) = 0$ then $X \in \text{add} T$.

We say that an object $\overline{T}$ in $\mathcal{C}$ is an almost complete tilting object if $\text{Ext}^1_{\mathcal{C}}(\overline{T}, \overline{T}) = 0$ and there is an indecomposable object $X$, which is called complement of $\overline{T}$, such that $\overline{T} \oplus X$ is a tilting object. It is known that an almost complete tilting object $\overline{T}$ in $\mathcal{C}$ has exactly two non-isomorphic complements. We denote them by $M$ and $M^*$.

The algebra $\text{End}_{\mathcal{C}}(T)^{\text{op}}$, where $T$ is a tilting object in $\mathcal{C}$, is called a cluster tilted algebra of type $\overline{Q}$, where $Q$ is the quiver whose path algebra is the hereditary algebra $H$, that is, $H = kQ$.

We denote by $\Gamma$ the cluster tilted algebra $\text{End}_{\mathcal{C}}(T)^{\text{op}}$, and by $\Gamma'$ the cluster tilted algebra $\text{End}_{\mathcal{C}}(T')^{\text{op}}$, with $T = \overline{T} \oplus M$ and $T' = \overline{T} \oplus M^*$ where $\overline{T}$ is an almost complete tilting object in $\mathcal{C}$ with complements $M$ and $M^*$, respectively. In [6, Theorem 1.3], the authors proved that we can pass from one algebra to the other by using mutation.

The next theorem shall be fundamental to develop some results of this paper.

**Theorem 2.5** ([5]) Let $T$ be a tilting object in $\mathcal{C}$ and we denote by $G$ the functor $\text{Hom}_{\mathcal{C}}(T, -) : \mathcal{C} \to \text{mod } \Gamma$. Then, the functor $\overline{G} : \mathcal{C}/\text{add}(\tau T) \to \text{mod } \Gamma$ (induced by $G$) is an equivalence.

It follows from the above theorem that an indecomposable projective $\Gamma$-module $P_u$ is of the form $\text{Hom}_{\mathcal{C}}(T, T_u)$, where $T_u$ is an indecomposable summand of $T$. Moreover, it is known that the indecomposable injective $\Gamma$-module $I_u$, which is the injective cover of the simple $S_u = \text{top } P_u$, is of the form $\text{Hom}_{\mathcal{C}}(T, \tau^2 T_u)$.

Furthermore, the Auslander–Reiten sequences in $\text{mod } \Gamma \simeq \mathcal{C}/\text{add}(\tau T)$ are induced by the Auslander–Reiten triangles in $\mathcal{C}$. We can deduce that the irreducible morphisms in $\text{mod } \Gamma$ are induced by irreducible morphisms in $\mathcal{C}$ which do not factor through $\text{add}(\tau T)$.

Consequently, a path of irreducible morphisms between indecomposable modules in $\text{mod } \Gamma$ is induced by a path of irreducible morphisms between indecomposable objects in $\mathcal{C}$, and both have the same length.

Finally, we recall the following important results useful for our further considerations.

**Proposition 2.6** Let $\overline{T}$ be a cluster tilted object in $\mathcal{C}$ with complements $M$ and $M^*$. We consider $\Gamma = \text{End}_{\mathcal{C}}(T)^{\text{op}}$ and $\Gamma' = \text{End}_{\mathcal{C}}(T')^{\text{op}}$ with $T = \overline{T} \oplus M$ and $T' = \overline{T} \oplus M^*$. Then,

(a) The $\Gamma$-module $\text{Hom}_{\mathcal{C}}(T, \tau M^*)$ is simple. Moreover, $\text{Hom}_{\mathcal{C}}(T, \tau M^*) \simeq \text{top } P_x$, where $P_x = \text{Hom}_{\mathcal{C}}(T, M)$.

(b) The $\Gamma'$-module $\text{Hom}_{\mathcal{C}}(T', \tau M)$ is simple. Moreover, $\text{Hom}_{\mathcal{C}}(T', \tau M) \simeq \text{top } P'_y$, where $P'_y = \text{Hom}_{\mathcal{C}}(T', M^*)$.

**Theorem 2.7** Let $\overline{T}$ be an almost complete tilting object in $\mathcal{C}$ with complements $M$ and $M^*$. We consider $\Gamma = \text{End}_{\mathcal{C}}(T)^{\text{op}}$ and $\Gamma' = \text{End}_{\mathcal{C}}(T')^{\text{op}}$ as above. Let $S_x$ and $
$S'_y$ be the simple modules top$(\text{Hom}_\mathcal{E}(T, M))$ and top$(\text{Hom}_\mathcal{E}(T', M^*))$, respectively. Then, there is an equivalence

$$\theta: \text{mod } \Gamma/\text{add } S_x \rightarrow \text{mod } \Gamma'/\text{add } S'_y.$$
Consider a morphism $f : X \to Y$ with $X, Y \in \text{ind } \Gamma$. Then, $f$ is an irreducible morphism in $\text{mod } \Gamma$ which does not factor through $\text{add } S_x$ if and only if $\theta(f)$ is an irreducible morphism in $\text{mod } \Gamma'$ which does not factor through $\text{add } S'_y$.

**Proof** Let $\Gamma$ and $\Gamma'$ be the cluster tilted algebras as above. Let $X$ and $Y$ be indecomposable modules in $\text{mod } \Gamma$ and let $f : X \to Y$ be a non-zero morphism such that $f$ does not factor through $\text{add } S_x$. By the equivalence of Theorem 2.7, we have that $\theta(f) : \theta(X) \to \theta(Y)$ is a non-zero morphism and moreover $\theta(f)$ does not factor through $\text{add } S'_y$.

Assume that $f$ is irreducible. We prove that $\theta(f)$ so is. In fact, assume that $\theta(f)$ is a section. Then there exists a morphism $\tilde{f} : \theta(Y) \to \theta(X)$ such that $\tilde{f} \theta(f) = 1_{\theta(X)}$. Moreover, $\tilde{f}$ does not factor through $\text{add } S'_y$ because $\theta(f)$ neither does. Then, there is a morphism $f' : Y \to X$ such that $\tilde{f} \theta(f') = 1_{\theta(X)}$.

Therefore,

$$\theta(1_X) = 1_{\theta(X)} = \theta(f') \theta(f) = \theta(f' f)$$

and since $\theta$ is a faithful functor, $1_X = f' f$, which is a contradiction to the fact that $f$ is not a section. Thus, we prove that $\theta(f)$ is not a section.

Analogously, we can prove that $\theta(f)$ is not a retraction.

Now, assume that there is a $\Gamma'$-module $\tilde{Z}$ and that there are morphisms $\tilde{g} : \theta(X) \to \tilde{Z}$ and $\tilde{h} : \tilde{Z} \to \theta(Y)$, such that $\tilde{g} \theta(f) = \tilde{h} \tilde{g}$. Since $\theta(f)$ does not factor though $\text{add } S'_y$, we infer that neither do the morphisms $\tilde{g}$ and $\tilde{h}$. By Theorem 2.7, there exist $Z \in \text{mod } \Gamma$ and morphisms $g : X \to Z$ and $h : Z \to Y$ which do not factor trough $\text{add } S_x$ such that $\tilde{g} \approx \theta(g)$ and $\tilde{h} \approx \theta(h)$. Then, $\theta(f) = \theta(h) \theta(g) = \theta(h g)$ and consequently $f$ is an irreducible morphism, $g$ is a section (and therefore $\theta(g)$ also does) or $h$ is a retraction (and therefore $\theta(h)$ also does). Thus, $\theta(f)$ is an irreducible morphism.

The converse follows by considering $\theta'$ the quasi-inverse equivalence of $\theta$. 

Our next goal is to prove that the nilpotency index is invariant under mutation.

Following the above notation, we denote by $a$ the vertex of $Q_{\Gamma}$ and of $Q_{\Gamma'}$, which comes from $T_a$, a direct indecomposable summand of $\overline{T}$, and we denote by $x$ ($y$, respectively) the vertex of $Q_{\Gamma'}$ ($Q_{\Gamma'}$, respectively) which comes from $M$, the summand of $T$ ($M^*$, the summand of $T'$, respectively).

We start with some lemmas in order to prove one of the main theorems of this section.

**Lemma 3.3** Let $\mathcal{C}$ be a cluster category of a representation-finite hereditary algebra $H$, and let $\overline{T}$ be an almost complete tilting object in $\mathcal{C}$ with complements $M$ and $M^*$. Consider $\Gamma = \text{End}_\mathcal{C}(\overline{T} \oplus M)^{\text{op}} \simeq k Q_{\Gamma} / I_{\Gamma}$ and $\Gamma' = \text{End}_\mathcal{C}(\overline{T} \oplus M^*)^{\text{op}} \simeq k Q_{\Gamma'} / I_{\Gamma'}$ the cluster tilted algebras. Then, for all indecomposable summands $T_a$ of $\overline{T}$, we have that $r_a = r'_a$.

**Proof** Let $\Gamma$ and $\Gamma'$ be cluster tilted algebras as in the statement, and let $T_a$ be an indecomposable summand of $\overline{T}$. Consider $P_a$, $S_a$ and $I_a$ the projective, simple and injective $\Gamma$-modules, respectively, corresponding to the vertex $a \in Q_{\Gamma}$, and $P'_a$, $S'_a$ and $I'_a$ the projective, simple and injective $\Gamma'$-modules, respectively, corresponding to
Let \( \mathcal{A} \) be the cluster tilted algebra of a representation-finite hereditary algebra \( H \). Let \( \Gamma \) be an almost complete tilting object in \( \mathcal{A} \). Let \( r_a = \tau_i \). Let \( f_a : P_a \rightarrow I_a \) be a non-zero morphism in \( \text{mod} \Gamma \) that factors through \( S_a \). By Lemma 2.3, we have that \( \text{dp}(f_a) = r_a \). Therefore, by Proposition 2.1, we can write the morphism \( f_a \) as follows:

\[
f_a = \sum_{i=1}^{s} g_i f_i
\]

for some \( s \geq 1 \), where \( f_i \in \mathfrak{M}_\Gamma(P_a, X_i) \), with \( X_i \in \text{ind} \Gamma \), and \( g_i \) is a finite sum of composition of \( r_a - 1 \) irreducible morphisms between indecomposable modules, for \( i = 1, \ldots, s \).

Let \( S_x \) be the simple top of the projective \( \Gamma \)-module \( P_x = \text{Hom}_\mathcal{A}(T, M) \). Since \( S_a \neq S_x \), neither \( f_i \) nor \( g_i \) factor through add \( S_x \), because \( \text{Hom}_\Gamma(P_a, S_x) = 0 = \text{Hom}_\Gamma(S_x, I_a) \). Then, by the equivalence \( \theta : \text{mod} \Gamma/\text{add} S_x \rightarrow \text{mod} \Gamma'/\text{add} S_y \) defined above, we have that \( \theta(f_a) = \sum_{i=1}^{s} \theta(g_i) \theta(f_i) \) is a non-zero morphism, where each \( \theta(g_i) \in \mathfrak{M}_\Gamma(\theta(P_a), \theta(X_i)) \). Moreover, by Proposition 3.2, each \( \theta(g_i) \) is a finite sum of composition of \( r_a - 1 \) irreducible morphisms between indecomposable modules. Then, \( \theta(f_a) \in \mathfrak{M}_\Gamma(\theta(P_a), \theta(I_a)) \), that is, \( \theta(f_a) \in \mathfrak{M}_\Gamma(P_a, I_a) \). By Lemma 2.3 we have that \( r_a \leq r'_a \).

Similarly, we can prove that \( r'_a \leq r_a \). Hence, \( r_a = r'_a \) as we wish to prove. \( \square \)

**Lemma 3.4** Let \( \mathcal{C} \) be the cluster category of a representation-finite hereditary algebra \( H \). Let \( \overline{T} \) be an almost complete tilting object in \( \mathcal{C} \) with complements \( M \) and \( M^\ast \). Consider \( \Gamma = \text{End}_\mathcal{C}(\overline{T} \oplus M)^{op} \simeq kQ_\Gamma/I_\Gamma \) and \( \Gamma' = \text{End}_\mathcal{C}(\overline{T} \oplus M^\ast)^{op} \simeq kQ_{\Gamma'}/I_{\Gamma'} \), the cluster tilted algebras. Let \( x \) and \( y \) be the vertices of \( Q_\Gamma \) and \( Q_{\Gamma'} \), respectively which come from the summands \( M \) of \( T \) and \( M^\ast \) of \( T' \), respectively. Then \( r_x = r'_y \).

**Proof** Let \( \Gamma \simeq kQ_\Gamma/I_\Gamma \) and \( \Gamma' \simeq kQ_{\Gamma'}/I_{\Gamma'} \) be the cluster tilted algebras defined as above.

To prove that \( r_x = r'_y \), we shall prove that \( n_x = m'_y \) and \( m_x = n'_y \), where \( r_x = n_x + m_x \) and \( r'_y = n'_y + m'_y \) are the bounds defined in Notation 2.9.

Consider a non-zero morphism \( f_x : P_x \rightarrow S_x \) and the irreducible morphism \( i_x : \text{rad} P_x \rightarrow P_x \). Since the cluster tilted algebra \( \Gamma \) is representation-finite, by [10, Theorem A] the right degree of \( i_x \) is finite and more precisely it is \( n_x \). Therefore, by the dual result of [10, Proposition 3.4], we have that \( \text{dp}(f_x) = n_x \). Hence, by Proposition 2.1 we know that there is a non-zero path of irreducible morphisms between indecomposable modules of length \( n_x \) in mod \( \Gamma \) as follows:

\[
\varphi_x : P_x \xrightarrow{h_1} X_1 \xrightarrow{h_2} X_2 \rightarrow \cdots \rightarrow X_{n_x-1} \xrightarrow{h_{n_x}} S_x.
\]

By the equivalence defined in Theorem 2.5, it is induced by a non-zero path of also \( n_x \) irreducible morphisms between indecomposable objects in the cluster category \( \mathcal{C} \), such that it does not factor through add \( \tau T \)

\[
\tilde{\varphi}_x : M \xrightarrow{\tilde{h}_1} \tilde{X}_1 \xrightarrow{\tilde{h}_2} \tilde{X}_2 \rightarrow \cdots \rightarrow \tilde{X}_{n_x-1} \xrightarrow{\tilde{h}_{n_x}} \tau M^\ast
\]
where \( P_x = \text{Hom}_C(T, M) \), \( S_x = \text{Hom}_C(T, \tau M^*) \) and \( X_i = \text{Hom}_C(T, \tilde{X}_i) \) for \( 1 \leq i \leq n_x - 1 \).

On the other hand, if we consider a non-zero morphism \( g'_y : S'_y \rightarrow I'_y \) in mod \( \Gamma' \), by Theorem A and [10, Proposition 3.4], we have that \( dp(g'_y) = m'_y \). Hence, with an analogous analysis to the previous one, there exists a non-zero path \( \psi'_y \) of \( m'_y \) irreducible morphisms between indecomposable modules from \( S'_y \) to \( I'_y \) in mod \( \Gamma' \). Moreover, such a path is induced by a non-zero path \( \tilde{\psi}'_y \), from \( \tau M \) to \( \tau^2 M^* \), of \( m'_y \) irreducible morphisms between indecomposable modules in the cluster category \( \mathcal{C} \) and such that it does not factor through add \( T \).

\[
\tilde{\psi}'_y : \tau M \rightarrow \tilde{Y}'_1 \rightarrow \tilde{Y}'_2 \rightarrow \cdots \rightarrow \tilde{Y}'_{m_y-1} \rightarrow \tau^2 M^* \tag{3}
\]

because \( S'_y = \text{Hom}_C(T', \tau M) \) and \( I'_y = \text{Hom}_C(T', \tau^2 M^*) \).

We also have that \( \tilde{\varphi}_x \in \text{Hom}_C(M, \tau M^*) \), with \( \tilde{\varphi}_x \neq 0 \), where

\[
\text{Hom}_C(M, \tau M^*) = \text{Hom}_D(F^{-1} M, \tau M^*) \oplus \text{Hom}_D(M, \tau M^*);
\]

and \( \tilde{\psi}'_y \in \text{Hom}_C(\tau M, \tau^2 M^*) \), with \( \tilde{\psi}'_y \neq 0 \), where

\[
\text{Hom}_D(\tau M, \tau^2 M^*) = \text{Hom}_D(F^{-1} \tau M, \tau^2 M^*) \oplus \text{Hom}_D(\tau M, \tau^2 M^*).
\]

In both cases, only one of the summands is non-zero since \( H \) is representation-finite. Hence, if \( \text{Hom}_D(F^{-1} M, \tau M^*) \neq 0 \), then \( \text{Hom}_D(F^{-1} \tau M, \tau^2 M^*) \neq 0 \) because

\[
\text{Hom}_D(F^{-1} M, \tau M^*) = \text{Hom}_D(\tau M[-1], \tau M^*) \cong \text{Hom}_D(\tau^2 M[-1], \tau^2 M^*) \cong \text{Hom}_D(F^{-1} \tau M, \tau^2 M^*).
\]

Therefore, the path in (2) is induced by a path of irreducible morphisms between indecomposable modules of length \( n_x \) from \( \tau M[-1] \) to \( \tau M^* \) in \( \mathcal{D} \), and the path in (3) is induced by a path of \( m'_y \) irreducible morphisms between indecomposable modules from \( \tau^2 M[-1] \) to \( \tau^2 M^* \) in \( \mathcal{D} \). Moreover, since \( \text{Hom}_D(\tau M[-1], \tau M^*) \cong \text{Hom}_D(\tau^2 M[-1], \tau^2 M^*) \) and \( \Gamma(\mathcal{D}) \) is a translation quiver with length, then \( n_x = m'_y \).

Now, if \( \text{Hom}_D(M, \tau M^*) \neq 0 \), with the same argument as before we can conclude that \( n_x = m'_y \).

Analogously, considering a non-zero morphism \( g_x : S_x \rightarrow I_x \) in mod \( \Gamma \) and a non-zero morphism \( f'_y : P'_y \rightarrow S'_y \) in mod \( \Gamma' \), with a similar analysis as above, we conclude that \( m_x = n'_y \). Thus, \( r_x = m_x + n_x = n'_y + m'_y = r'_y \).

\[ \square \]

Now, we are in position to show Theorem A.

**Proof of Theorem A** Let \( \Gamma \cong kQ_{\Gamma}/I_{\Gamma} \) and \( \Gamma' \cong kQ_{\Gamma'}/I_{\Gamma'} \) be the cluster tilted algebras defined as above. Since \( H \) is a representation-finite algebra, so are \( \Gamma \) and \( \Gamma' \). We denote...
by \( r_\Gamma \) and \( r_\Gamma' \), the nilpotency indices of \( \mathcal{N}(\text{mod } \Gamma) \) and \( \mathcal{N}(\text{mod } \Gamma') \), respectively. We prove that \( r_\Gamma = r_\Gamma' \). In fact, we know that

\[
\begin{align*}
    r_\Gamma &= \max \{ r_u | u \in (Q_\Gamma)_0 \} + 1 = \max \{ r_u | T_u \in \text{ind}(\text{add } T) \} + 1 \quad \text{and} \\
    r_\Gamma' &= \max \{ r'_v | v \in (Q_\Gamma')_0 \} + 1 = \max \{ r'_v | T_v \in \text{ind}(\text{add } T') \} + 1.
\end{align*}
\]

By Lemmas 3.3 and 3.4, we have that

\[
\begin{align*}
    r_\Gamma &= \max \{ r_u | T_u \in \text{ind}(\text{add } T) \} + 1 \\
    &= \max \{ r_u | T_u \in \text{ind}(\overline{T}), r_x \} + 1 \\
    &= \max \{ r'_u | T_u \in \text{ind}(\overline{T}), r'_x \} + 1 \\
    &= \max \{ r_v | T_v \in \text{ind}(\overline{T'}) \} + 1 \\
    &= r_\Gamma',
\end{align*}
\]

proving the result. \( \square \)

For the convenience of the reader we state [20, Theorem 4.11]. For a different proof of this result, involving the concept of degree of an irreducible morphism, we refer the reader to [14].

**Theorem 3.5** Let \( H = k\Delta \) be a representation-finite hereditary algebra and let \( r_H \) be the nilpotency index of \( \mathcal{N}(\text{mod } H) \). Then the following conditions hold:

(a) If \( \Delta = A_n \), then \( r_H = n \), for \( n \geq 1 \).
(b) If \( \Delta = D_n \), then \( r_H = 2n - 3 \), for \( n \geq 4 \).
(c) If \( \Delta = E_6 \), then \( r_H = 11 \).
(d) If \( \Delta = E_7 \), then \( r_H = 17 \).
(e) If \( \Delta = E_8 \), then \( r_H = 29 \).

The next result shall be important to prove the corollary stated in the introduction, and follows from [5] and [6].

**Corollary 3.6** Let \( \Delta \) be a connected and acyclic quiver. The classes of quivers obtained of \( \Delta \) by mutations coincide with the classes of quivers of the cluster tilted algebras of type \( \Delta \). Moreover, if \( \Delta \) is of Dynkin type then there is a finite number of the mentioned classes.

Now, we are in condition to present the corollary stated in the introduction, which is a consequence of Theorem A and [20, Theorem 4.11].

**Proof of Corollary B** Let \( \Gamma \simeq kQ_\Gamma / I_\Gamma \) be a cluster tilted algebra of type \( \overline{\Delta} \), where \( \Delta \) is a Dynkin quiver and let \( H \) be the hereditary algebra \( H = k\Delta \). Since \( H \) is representation-finite, then so is \( \Gamma \).
Let \( r_H \) and \( r_\Gamma \) the nilpotency indices of \( \mathcal{H}(\text{mod } H) \) and \( \mathcal{H}(\text{mod } \Gamma) \), respectively. We claim that \( r_\Gamma = r_H \). In fact, by Corollary 3.6, we can change the algebra \( \Gamma \) into the algebra \( H \) by a finite sequence of mutations of the quiver \( Q_\Gamma \). By Theorem A, we have that \( r_\Gamma = r_H \) where \( r_H \) takes the values given in Theorem 3.5. \( \square \)

In [18], Ringel proved that all self-injective cluster tilted algebras are representation-finite. Furthermore, the author showed that this particular algebras are cluster tilted algebras of type \( D_n \), with \( n \geq 3 \) (considering \( D_3 = A_3 \)).

The next result follows immediately from Corollary B.

**Corollary 3.7** Let \( \Gamma \) be a self-injective cluster tilted algebra. Then, the nilpotency index of \( \mathcal{H}(\text{mod } \Gamma) \) is \( 2n - 3 \), where \( n \) is the number of vertices of the quiver \( Q_\Gamma \).

We end up this section with a remark on Coxeter numbers. We refer the reader to [13] for a detailed account on root systems and Coxeter groups.

**Remark 3.8** It is known that the theory of cluster algebras has many connections with different areas in mathematics. In particular, there exists a connection with root systems and with Coxeter groups.

An element in a Coxeter group \( W \) is called a Coxeter element if it is the product of all simple reflections and moreover its order is called the Coxeter number of \( W \). On the other hand, the Coxeter number is related with the highest root in its corresponding root system.

For a finite irreducible Coxeter group \( W \), there is a corresponding root system \( \Phi \) of Dynkin type \( \Delta \). Now, if \( \Gamma \) is a cluster tilted algebra of \( \Delta \) type then it is known that the cardinal of the set of positive roots of \( \Phi \) coincides with the cardinal of ind \( \Gamma \). Moreover, the Coxeter number of \( W \) is exactly one more than the nilpotency index of \( \mathcal{H}(\text{mod } \Gamma) \).

## 4 On composition of irreducible morphisms

In this section, we establish the relationship between the composition of irreducible morphisms between indecomposable modules and the power of the radical where it belongs. We start with the following proposition.

**Proposition 4.1** Let \( \Gamma \) be a representation-finite cluster tilted algebra. Let \( M \) and \( N \) be indecomposable \( \Gamma \)-modules such that \( \text{Irr}_\Gamma(M, N) \neq 0 \). Then, \( \dim_k(\text{Hom}_\Gamma(M, N)) = 1 \). In particular, \( \mathcal{H}^2(M, N) = 0 \).

**Proof** Let \( \Gamma \) be a representation-finite cluster tilted algebra. Then \( \Gamma = \text{End}_C(T)^{\text{op}} \), where \( C = \mathcal{D}/F \) is the cluster category of a representation-finite hereditary algebra \( H \) and \( T \) a tilting object in \( C \).

Since \( \text{Irr}_\Gamma(M, N) \neq 0 \), there exists an irreducible morphism \( f : M \to N \). We claim that all the morphism \( g : M \to N \) in \( \text{mod } \Gamma \) are \( k \)-linearly dependent with \( f \). In fact, suppose that there exists a non-zero morphisms \( g : M \to N \) \( k \)-linearly independent with \( f \). Since \( \Gamma \) is representation-finite, we know that \( \dim_k(\text{Irr}_\Gamma(M, N)) = 1 \). Hence, \( g \) is not irreducible. Then \( g \in \mathcal{H}^2(M, N) \). Moreover, there is an integer \( n \geq 2 \) such
that \( g \in \mathcal{R}^n(M, N)\backslash \mathcal{R}^{n+1}(M, N) \). Therefore, there exist morphisms \( \tilde{f}, \tilde{g} : \tilde{M} \to \tilde{N} \) in the cluster category \( \mathcal{C} \) such that they do not factor through \( \text{add}(\tau T) \). Furthermore, these morphisms are induced by morphisms in the derived category. Moreover, since \( \text{Hom}_\mathcal{C}(\tilde{M}, \tilde{N}) = \text{Hom}_\mathcal{D}(F^{-1}M, N) \oplus \text{Hom}_\mathcal{D}(M, N) \), and only one of these summands is non-zero, we can deduce the existence of an irreducible morphism in \( \Gamma(\mathcal{D}) \) and a path of length \( n \), with \( n \geq 2 \), between the same objects, contradicting the fact that \( \Gamma(\mathcal{D}) \) is a quiver with length.

Therefore, there is no morphism \( g : M \to N \) in \( \operatorname{mod}\Gamma \) linearly independent with \( f \). Then, \( \operatorname{dim}_k(\text{Hom}_\Gamma(M, N)) = 1 \). Moreover, since \( f \) is irreducible, we conclude that \( \mathcal{R}_\Gamma^2(M, N) = 0 \).

\( \square \)

**Proof of Theorem C** If \( h_m \ldots h_1 = 0 \), then clearly we have that \( h_m \ldots h_1 \in \mathcal{R}_1^n(X_1, X_{m+1}) \).

Conversely. Assume \( h_m \ldots h_1 \in \mathcal{R}_1^{m+1}(X_1, X_{m+1}) \) and \( h_m \ldots h_1 \neq 0 \). Then, by [10, Theorem 5.1] there are irreducible morphisms \( f_i : X_i \to X_{i+1} \), for \( 1 \leq i \leq m \), such that \( f_m \ldots f_1 = 0 \). By Proposition 4.1, we have that \( \operatorname{dim}_k(\text{Hom}_A(X_i, X_{i+1})) = 1 \), for each \( i \). Hence, \( f_i \) and \( h_i \) are \( k \)-linearly dependent, that is, \( h_i = \lambda_i f_i \) where \( \lambda_i \) is a non-zero element of \( k \). Thus, \( h_m \ldots h_1 = \lambda f_m \ldots f_1 = 0 \), which contradicts our assumption. Therefore, \( h_m \ldots h_1 = 0 \) proving the result.

\( \square \)

**Remark 4.2** We observe that if we consider a cluster tilted algebra of type \( A_n \) or \( D_n \), then the results of this article can be proven with the geometric approach developed for cluster categories and cluster tilted algebras of type \( A_n \) and \( D_n \) given in [7,19], respectively. For a detail account on this approach see [14].

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**Declaration**

**Conflict of interest** Data sharing not applicable to this article as no datasets were generated or analysed during the current study. This article describes entirely theoretical research.

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