An Empirical Bayes Approach to Frequency Estimation

Giorgio Picci and Bin Zhu

Abstract—In this paper we show that the classical problem of frequency estimation can be formulated and solved efficiently in an empirical Bayesian framework by assigning a uniform a priori probability distribution to the unknown frequency. We discover that the covariance matrix of the signal model is the discrete-time counterpart of the operator whose eigenfunctions are the famous prolate spheroidal wave functions, introduced by Slepian and coworkers in the 1960’s and widely studied in the signal processing literature although motivated by a different class of problems. The special structure of the covariance matrix is exploited to design an estimator for the hyperparameters of the prior distribution which is essentially linear, based on subspace identification. This is in contrast to standard parametric estimation methods which are based on iterative optimization algorithms of local nature. Simulations show that the approach is quite promising and seems to compare very favorably with classical methods from the literature.

Index Terms—Frequency estimation, empirical Bayes, bandpass energy concentration, asymptotic eigenvalue distribution, subspace method.

I. INTRODUCTION

Frequency estimation is an old nonlinear problem which is encountered in many branches of science and engineering and has generated a huge literature. Since the literature on this problem is so large it is impossible to summarize it in this introduction. We shall limit to referring the readers to the books [1], [2], to the references therein and just point to some recent interesting contributions such as [3], [4]. Besides spectral analysis, which tends to produce nonconsistent estimates and must rely on ad hoc reorganizations of partial spectral estimates (see e.g. [5] and the comments in the introduction of Thomson’s paper [6]), accurate frequency estimation has been mostly approached in the literature by nonlinear optimization techniques (typically variants of Maximum Likelihood) which because of non convexity are generally local and not guaranteed to yield a unique minimum. In this paper we propose a new Bayesian approach by imposing a prior distribution on the unknown frequency. This prior is a uniform distribution on a small frequency range which can be interpreted as a confidence interval centered about the unknown nominal frequency. Its width and the relative center frequency are estimated from data, which in the simplest case could be an harmonic oscillation signal corrupted by additive noise, whose frequency is modeled as a randomly varying parameter. It turns out that by this approach one ends up describing the samples of the observed signal as those of a particular class of stationary processes which could well be named Bandlimited white noise processes first studied in the conference papers [7], [8]. The covariance operator of these processes has remarkable properties which have been uncovered in the 60’s and 70’s by D. Slepian and coworkers in a famous series of papers concerning the energy concentration problems of time- and band- limited signals [9]–[13].

Because of the uniform frequency prior, the covariance of the resulting process turns out to be a function of the modulated sinc-type, which in the special case when the nominal center frequency is zero, has been well studied in the afore-cited literature. The key property of the covariance operator in question is that its eigenvalues decay extremely fast to zero for indices greater than an a priori computable number (the so-called Slepian frequency). This means that the eigenfunction expansion of the kernel involves essentially only a finite number of terms. This key feature is also evident from simulations when the center frequency is non-zero but a thorough understanding of the behavior of these modulated sinc operators was posed as an open problem in [10] p. 63. Later it was shown to hold for continuous-time modulated sinc kernels [14], [15] but the discrete-time case was still left open. In this paper, we provide a proof that discrete-time kernels behave in a completely analogous way. This allows a direct and rather simple estimation of the hyperparameters of the prior. The resulting center frequency estimate is then computed by a simple averaging process which seems to yield very accurate and robust results, at least for a large enough sample size. This new estimation method is also expounded for signals with multiple unknown frequencies.

Empirical Bayes hyperparameter estimation of stationary a posteriori descriptions of harmonic signals is normally dealt with by conditional likelihood maximization as considered for example in [16], but our work uses more deeply the structure of the process and does not need to involve optimization, going well beyond this approach.

A well-known difficulty in frequency estimation is that stationary random processes with periodic components, even when the frequencies are exactly known, are not ergodic. Nonergodicity means in particular that, when the sample size goes to infinity, the limit of the process sample covariance is sample dependent, that is, the limit sample covariance depends on the random amplitudes of its elementary oscillatory components (see e.g. [17] pp. 105-109). This lack of ergodicity is even more serious when the frequency is random. For this reason, one-sample-path estimation runs into difficulty.

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and a reasonable approach for many practical situations is to consider estimation from cross-sectional or panel data, as described in e.g., [18]. This extends the scope of the paper [8]. Note that cross-sectional data of oscillatory nature can be the result of intermittent observations and are quite common for example in mechanical, astrophysical, biomedical, and econometric applications. Data may be often collected from several trajectories of a harmonic oscillation process whose frequency may deviate randomly about an unknown nominal value. It is then reasonable to model them as realizations of a random variable, that is, as noisy versions of the same nominal frequency parameter.

We will first deal with signals with one hidden sinusoidal component but the techniques and results are then extended to treat signals with multiple harmonic components of unknown frequencies by assigning to them non-overlapping rectangular (uniform) prior distributions. In this way the overall covariance kernel becomes the sum of the individual covariances of uncorrelated harmonic components. Our technique can still be applied somehow reminiscent of Multiple Kernel methods as in [19], [20].

The paper is organized as follows: In Section II, we formulate the Bayesian framework for the frequency estimation problem. Then in Section III we discuss the special structure of the signal covariance which is a discrete-time counterpart of the modulated sinc kernel class discussed in the literature. We prove the sharp decay property of the eigenvalues using techniques that lead to the continuous-time results from the literature. Then we illustrate our findings through a numerical example.

In Section IV we first exploit the covariance structure to propose an extremely simple frequency estimate for signals with only one unknown frequency, which is only based on spectral data of the covariance. Note that because of non-ergodicity, consistent estimation of the covariance data is a non-trivial issue. This issue is discussed at the end of the section with an excursus on exchangeability which may in practice be a reasonable assumption on the data structure. Section V attacks the main theme of the paper, namely estimation of multiple frequencies using a subspace method by introducing a natural approximate state-space model of the data. In the following Section VI the method is applied to several test examples. As can be seen, the results are very encouraging.

Section VII concludes the paper.

Notation and convention

Boldface symbols denote random quantities. For a square summable sequence $y$ of complex numbers, we take the definition of the discrete-time Fourier transform (DTFT) as the following

$$\mathcal{F} : \ell^2 \rightarrow L^2[-\pi, \pi]$$

$$y \mapsto \hat{y}(\omega) := \sum_{t \in \mathbb{Z}} y(t) e^{-it\omega},$$

where the convergence of the Fourier series is understood in $L^2$ norm. The inverse transform is given by

$$\mathcal{F}^{-1} : \hat{y} \mapsto y(t) := \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{it\omega} \hat{y}(\omega) d\omega.$$

The $\ell^2$ norm of $y$ is known as their energy of the signal. The indicator function on a set $S \subset \Omega$ is defined as

$$\chi_S(\omega) = \begin{cases} 1 & \text{for } \omega \in S, \\ 0 & \text{for } \omega \in \Omega \setminus S. \end{cases}$$

II. The Signal Model

Consider the following signal model

$$y(t) = x(t) + w(t); \quad t \in \mathbb{Z} \quad (1)$$

where $t$ represents time, $x$ is the sum of random oscillatory components (a quasi periodic process), that is

$$x(t) := \sum_{\ell=1}^{\nu} a_\ell \cos(\omega_\ell t) + b_\ell \sin(\omega_\ell t), \quad (2)$$

and $w$ is additive white noise. The angular frequencies $\omega_\ell$ are unknown but their number $\nu$ is fixed. In addition we shall require that:

- the amplitude pairs $a_k, b_k$ are zero-mean pairwise and mutually uncorrelated for all $k$ and the two components $a_k, b_k$ have equal variance: $\sigma_k^2 = \text{var}[a_k] = \text{var}[b_k], k = 1, \ldots, \nu$;
- each angular frequency $\omega_\ell$ is a random variable taking independent of the amplitudes;
- The noise $w(t)$ is white, zero-mean, stationary of variance $\sigma_w^2$, independent of everything else.

We shall let $\omega := [\omega_1, \ldots, \omega_\nu]^\top$ and denote by $a, b$ two similarly arranged amplitude vectors. Note that the model is linear in $a, b$, and hence estimation of the amplitudes and their variance is just a standard linear estimation problem when the frequencies are known. For this reason, in this paper we shall just concentrate on the problem of frequency estimation.

Let us now introduce an empirical Bayesian framework. We shall impose that each component $\omega_\ell$ of the random vector $\omega$ follows a uniform distribution on the frequency band $[\theta_\ell - W_\ell, \theta_\ell + W_\ell]$ such that the symmetrized sets w.r.t. the origin $S_\ell := [\theta_\ell - W_\ell, \theta_\ell + W_\ell] \cup [-\theta_\ell - W_\ell, -\theta_\ell + W_\ell], \quad \ell = 1, \ldots, \nu$ do not overlap. For simplicity we shall assume that the assigned bandwidth is the same for different frequencies, i.e., $W_1 = \cdots = W_\nu = W$. Here $-\pi \leq \theta_\ell \leq \pi$ is called a center-frequency and $0 \leq W \leq \pi$ the bandwidth. In the literature, both $\theta$ and $W$ are called hyperparameters of the a priori distribution for the frequency $\omega$.

The frequency estimation problem is posed as estimation of the hyperparameters. For a critical survey and a bibliography on empirical Bayes see [21], [22].

The stated assumptions imply that for each fixed frequency value $\omega$ the $\nu$ components, say $x_\ell, \ell = 1, \ldots, \nu$ of the signal (2) are stationary uncorrelated processes. Hence the covariance
function of the process $y$ for a fixed deterministic $\omega$ has the form
\[
\Sigma(t, s \mid \omega) := \mathbb{E} \{y(t)y(s) \mid \omega\} = K(t, s \mid \omega) + \sigma^2 \delta(t, s),
\]
where $\delta(t, s)$ is the Kronecker symbol, and
\[
K(t, s \mid \omega) := \sum_{\ell=1}^{\nu} \mathbb{E} \{x_\ell(t)x_\ell(s) \mid \omega\} = \sum_{\ell=1}^{\nu} K_\ell(t, s \mid \omega)
\]
is the a priori conditional covariance of the signal $x$ given $\omega = \omega$. To lighten the notation, we shall temporarily suppress the subscripts. The formulas below should be interpreted as holding for a generic index $\ell$.

By the model assumptions, the following computation is straightforward:
\[
K(t, s \mid \omega) = \mathbb{E} \{a^2 \cos(\omega t) \cos(\omega s) + ab \cos(\omega t) \sin(\omega s) + ab \sin(\omega t) \cos(\omega s) + b^2 \sin(\omega t) \sin(\omega s)\}
= \sigma^2 \cos(\omega \tau)
(4)
\]
where $\tau := t - s$, and then computing the a posteriori covariance by integrating the function w.r.t. the uniform prior density, one gets
\[
K(t, s) = \sigma^2 \mathbb{E} \{\cos(\omega \tau)\} = \sigma^2 \int_{-\omega}^{\omega} \cos(\omega \tau) \frac{1}{2\omega} d\omega = \sigma^2 \cos(\omega \tau) W_\omega.
(5)
\]
Since the covariance function depends only on $\tau$, the signal $x$ is stationary, and so is $y$. In the following, we will write $K(\tau)$ in place of $K(t, s)$.

For $\theta = 0$, the covariance function $K$ is the well-known sinc function, which is the inverse Fourier transform of a rectangular function, namely
\[
\sigma^2 \sin(\omega \tau) W_\omega = \sigma^2 \int_{-\omega}^{\omega} e^{i\omega \tau} d\omega.
(6)
\]
It follows that a zero-frequency component of the process $x$ must have a uniform spectral density $\frac{\sigma^2}{2\omega} \chi_S[-W, W](\omega)$. When $W = \pi$, the process is just a usual stationary white noise of variance $\sigma^2$. For $W < \pi$, the process $x$ is nontrivial, called a bandlimited white noise within the frequency band $[-W, W]$. In this case, it is a purely deterministic process with an absolutely continuous spectral distribution, since the logarithm of the density is obviously not integrable (see e.g., [23, p. 144]).

In this paper, we are mostly interested in the case $\theta_\ell \neq 0$, for which we make the assumption that $|\theta_\ell| > W$, so that each support set
\[
S := [\theta - W, \theta + W] \cup [-\theta - W, -\theta + W]
(7)
is composed of two disjoint intervals symmetric with respect to the origin. Then the last expression in (4) can be rewritten as
\[
\sigma^2 \cos(\omega \tau) W_\omega = \sigma^2 \int_{-\omega}^{\omega} e^{i\omega \tau} \chi_S(\omega) d\omega = \frac{\pi \sigma^2}{2W} \int_{-\pi}^{\pi} e^{i\omega \tau} \chi_S(\omega) d\omega.
(8)
\]
where $\chi_S$ is the indicator function on $S$, and the second equality holds due to the symmetry of the integrand. From the above relation, we see that the spectral density of the process $x$ is now the sum of $\nu$ disjoint spectral terms, each of the form
\[
\phi_{x_\ell}(\omega) = \frac{\pi \sigma^2}{2W} \left(\chi_{[\theta_\ell-W, \theta_\ell+W]} + \chi_{[-\theta_\ell-W, -\theta_\ell+W]}\right).
\]
The signal $x$ can therefore be described as a sum of independent deterministic carriers, each of angular frequency $\theta_\ell$, amplitude-modulated by a bandlimited white noise process described before. For the same reason, the covariance function $[\mathbb{H}]$ has been called a modulated sinc kernel in [24], where it arises in a somewhat different context.

In practice we can only observe a finite sample of size $N$ from the process $y$. For clarity of exposition, we shall now assume that $\nu = 1$ and neglect the subscript $\ell$ altogether. The generalization to multiple sinusoids, i.e., $\nu > 1$, will be obvious. Collect the observed random variables into a column vector, and in particular, let $X_N := [x(t), x(t+1), \ldots, x(t+N-1)]^T$. Then consider the $N \times N$ covariance matrix
\[
K_N := E\{XX_N^T\} = \begin{bmatrix} K(0) & K(1) & \cdots & K(N-1) \\ K(1) & K(0) & \cdots & K(N-2) \\ \vdots & \vdots & \ddots & \vdots \\ K(N-1) & K(N-2) & \cdots & K(0) \end{bmatrix}.
(9)
\]
This symmetric Toeplitz structure of the covariance matrix comes from the fact that the process is stationary and real-valued. Similarly, we can define the $N \times N$ covariance matrix of the process $y$, say $\Sigma_N$, and we have the relation
\[
\Sigma_N = K_N + \sigma^2 I_N.
(10)
\]
The eigen-structure of $K_N$ will be of great importance to our frequency estimation problem, and that will be the content of the next section.

### III. Properties of the Covariance Matrix

In this section, we show that the covariance matrix (9) also arises in a quadratic form when considering an energy concentration problem for discrete-time deterministic signals. Here we make contact with classical works of David Slepian and colleagues which started in the early sixties at Bell Labs [10]–[13]. We also mention the monograph [25] for a good reference on this topic. In particular, here we make special reference to the bandpass analogues of Prolate Spheroidal Wave Functions mentioned in a concluding remark in the paper [10].

In order to state the problem, we need to set up some notations first.

Let $J$ be a set that is a union of a finite number of pairwise disjoint closed subintervals of $[-\pi, \pi]$, e.g., a union of sets like $S$ in (7). Define the band-limiting operator
\[
\mathcal{B} : \ell^2 \rightarrow \ell^2, \quad y \mapsto F^{-1}[\chi_J F(y)]
(11)
\]
that corresponds to a bandpass filter with prescribed bandwidth $\{\omega \in J\}$. Fix a positive integer $N$ and let
\[
I := \{0, 1, \ldots, N-1\}.
(12)
Define similarly the time-limiting operator
\[ \mathcal{T} : \ell^2 \to \ell^2, \quad y \mapsto \chi_I y, \]
where \( \chi_I \) is the indicator function in the time domain \( \mathbb{Z} \).

The energy concentration problem that will be discussed in this section is
\[ \sup_{y \in l^2} \frac{\| \mathcal{B} \mathcal{T} y \|^2_2}{\| y \|^2_2}. \]
(14)

Notice that the supremum can only be attained at a time-limited \( y \), because the objective value of \( \hat{y} := \mathcal{T} y \) is equal to \( \| \mathcal{B} \hat{y} \|^2_2 / \| \hat{y} \|^2_2 \) which is not less than that of \( y \). Therefore, it is equivalent to consider the problem
\[ \sup_{y \in l^2} \frac{\| \mathcal{B} y \|^2_2}{\| y \|^2_2}, \]
(15)

where \( \text{supp}(\cdot) \) denotes the support of a function. In other words, the aim is to find a time-limited signal whose energy is most concentrated in the frequency band \( J \).

A. The eigenvalue problem

The impulse response of the ideal bandpass filter \( \chi_J(\omega) \) is just the inverse Fourier transform
\[ \rho(t) := \frac{1}{2\pi} \int_J e^{it\omega} d\omega, \quad t \in \mathbb{Z}. \]
(16)

Observe that the function \( \rho \) has the symmetry \( \rho(-t) = \rho(t)^* \). Here \( z^* \) means the complex conjugate (transpose) of \( z \in \mathbb{C} \).

According to the definitions \( \mathcal{B} \mathcal{T} \) and \( \mathcal{T} \), we have
\[ \mathcal{B} \mathcal{T} y = \mathcal{F}^{-1} \left[ \chi_J(\omega) \mathcal{F}(\mathcal{T} y) \right] \]
\[ = \mathcal{F}^{-1} \left[ \chi_J(\omega) \sum_{t=0}^{N-1} y(t)e^{-it\omega} \right] \]
\[ = \rho \ast \mathcal{T} y \]
where \( \ast \) denotes convolution. It follows that
\[ \| \mathcal{B} \mathcal{T} y \|^2_2 = \sum_{t} \left| \sum_{k=0}^{N-1} \rho(t-k)y(k) \right|^2 \]
(18)
\[ = \sum_{j=0}^{N-1} y(j)^* \sum_{k=0}^{N-1} y(k) \rho(t-j)^* \rho(t-k). \]

The last summation can be rewritten
\[ \sum_{t \in \mathbb{Z}} \rho(t-j)^* \rho(t-k) = \sum_{t \in \mathbb{Z}} \rho(j-t)\rho(t-k) \]
\[ = (\rho \ast x)(j), \]
(19)

where the sequence \( x(t) := \rho(t-k) \) has Fourier transform \( \hat{x}(\omega) = e^{-ik\omega} \chi_J(\omega) \). The Fourier transform of \( \rho \ast x \) is simply again \( e^{-ik\omega} \chi_J(\omega) \). Hence the above sum is equal to \( \rho(j-k) \), and we arrive at
\[ \| \mathcal{B} \mathcal{T} y \|^2_2 = \sum_{j=0}^{N-1} y(j)^* \sum_{k=0}^{N-1} y(k) \rho(j-k) \]
\[ = y^* R y, \]
(20)

where \( y = [y(0), y(1), \ldots, y(N-1)]^T \) is a slight abuse of notation, and
\[ R = \begin{bmatrix}
\rho(0) & \rho(-1) & \cdots & \rho(-N+1) \\
\rho(1) & \rho(0) & \cdots & \rho(-N+2) \\
\vdots & \vdots & \ddots & \vdots \\
\rho(N-1) & \rho(N-2) & \cdots & \rho(0)
\end{bmatrix}. \]
(21)

The matrix \( R \) has a Hermitian Toeplitz structure, and it is also positive definite because the quadratic form determines the energy of \( \mathcal{B} \mathcal{T} y \). Notice that when the set \( J \) is symmetric w.r.t. the origin such as \( S \) in \( (7) \), then the integral in \( (16) \) reduces to \( \int J \cos(\omega) d\omega \). In that case, \( \rho \) is an even function of time, and the matrix \( R \) is real symmetric.

Now the objective functional in the energy concentration problem \( (14) \) is in fact equal to the Rayleigh quotient associated to \( R \). By the min-max theorem, the maximum of the objective is equal to the largest eigenvalue of \( R \), and it is attained when \( y \) is taken to be the corresponding eigenvector. It is obvious that the eigenvalues of \( R \) do not exceed 1, simply because both \( \mathcal{B} \) and \( \mathcal{T} \) are projection operators.

Remark 1. Although it does not particularly interest us here, it is worth mentioning that the energy concentration problem \( (14) \) has a “dual” problem obtained by interchanging the two operators \( \mathcal{B} \) and \( \mathcal{T} \), namely
\[ \sup_{y \in l^2} \frac{\| \mathcal{B} \mathcal{T} y \|^2_2}{\| y \|^2_2}. \]
(22)

The problem \( (22) \) is equivalent to determining the supremum of \( \| \mathcal{B} \mathcal{T} y \|^2_2 \) over all band-limited signals subject to the constraint \( \| y \|^2_2 = 1 \). By a standard variational argument using the Lagrange multiplier, one can conclude that the maximum of the dual objective is equal to the largest eigenvalue of a linear integral operator with a (modified) Dirichlet kernel. Moreover, following the lines in \( (26) \) Section 5, it is not difficult to show that the eigenvalues of such an integral operator are identical to those of \( R \), and the corresponding eigenfunctions are related via the Fourier transform.

B. Asymptotic distribution of the eigenvalues

We shall allow the dimension of \( R \) to increase. In other words, the integer \( N \) introduced by the set \( I \) in \( (12) \) is now a variable. Let \( \lambda_j(N) \) be the \( j \)-th eigenvalue (arranged in nonincreasing order) of \( R \). We know from the previous subsection that \( 0 < \lambda_j(N) \leq 1 \) for all \( j = 1, \ldots, N \). It also follows easily that
\[ \sum_{j=1}^{N} \lambda_j(N) = \text{tr} R = N \rho(0) = \frac{m(J)}{2\pi N}, \]
(23)

where the notation \( m(\cdot) \) denotes the Lebesgue measure of a set. Now for a real number \( 0 < \gamma < 1 \), define \( M(\gamma, N) \) to be the number of eigenvalues of \( R \) that are no less than \( \gamma \). Again we have included the explicit dependence on the dimensional variable \( N \). The next result is a first-order description of the asymptotic eigenvalue distribution of the matrix \( R \). The proof borrows techniques from \( (14) \) and can be found in the appendix.
Theorem 1. It holds that
\[ \lim_{N \to \infty} \frac{M(\gamma, N)}{N} = \frac{m(J)}{2\pi} \] (24)

independent of \( \gamma \).

A more precise formula for the asymptotic expansion of the quantity \( M(\gamma, N) \) is given and justified in [13] for the continuous-time case. The second term in the asymptotic expansion is shown to be proportional to \( \log N \). Slepian’s asymptotic expressions for the eigenvalues, valid for \( \theta = 0 \), are also reported in [7] p. 1059. Although we believe that analogous discrete-time estimates should hold, a formal proof is yet to be worked out. For our problem of frequency estimation, Corollary 1 below is anyway sufficient.

By choosing \( \gamma \) arbitrarily close to 1, an immediate consequence of the above theorem and formula (23) is the following.

Corollary 1. For \( N \to \infty \), the matrix \( R \) has rank
\[ n = Nm(J)/2\pi, \] (25)

and all the nonzero eigenvalues tend to 1.

The convergence is very fast since, as it is shown in the proof of the Theorem, the matrix \( R \) has only \( o(N) \) eigenvalues that are between 0 and 1, and for large sample size they can be reasonably neglected.

For \( \nu = 1 \) the covariance matrix in (9) is just a scalar multiple of \( R \) via \( K_N = \frac{\pi W}{2\pi} R_N \) (here for notational consistency we have added the subscript \( N \) to \( R \)). Clearly, the constant factor only rescales the eigenvalues. In particular, the assertion on the rank in Corollary 1 holds for \( K \). Below we show some simulations of how the eigenvalues decay.

Fig. 1 shows the behavior of the eigenvalues \( \mu_k \) of the sinc kernel for \( N = 1000, W/2\pi = 0.02, \sigma^2 = 1 \) which yields a rank approximately equal to 40. We can clearly see that for \( n < 40 \) the eigenvalues are all equal to the same constant while for \( n > 40 \) the \( \mu_k \)'s very quickly decrease to zero.

The behavior of the eigenvalues of \( R \) is the same except that the normalization makes the \( \mu_k \) all practically equal to one for \( k < n \). In order to get the same normalization we just need to substitute \( \mu_k \) with \( 2W\mu_k/2\pi \).

As for the modulated sinc kernel, Fig. 2 shows the eigenvalues of a matrix \( K \) with the same values of \( N, W, \) and \( \sigma^2 \). One sees that the eigenvalues have exactly the same behavior as those of the sinc kernel. Only the value of \( n \) such that for \( k > n \), \( \mu_k \approx 0 \) is now \( 4NW/2\pi \approx 80 \), i.e., twice the value of \( n \) for the sinc kernel. Moreover, the amplitudes of the eigenvalues for \( k < n \) are half of those of the sinc kernel, for equal values of \( W \). This follows the from the symmetry of the spectrum and matches also the experimental findings of [24].

In order to get the largest eigenvalues of the modulated sinc kernel equal to one, a different normalization should be made by substituting \( \mu_k \) with \( 4W\mu_k/2\pi \). This agrees with the matrix rescaling described above.

IV. SOLUTION TO THE PROBLEM WITH ONE HIDDEN FREQUENCY

In the case of one hidden frequency, we have \( \text{rank } K_N \approx \frac{2W}{\pi} N \) according to Corollary 1. We can see that the bandwidth \( W \) can be inferred from the rank information of the signal covariance matrix \( K \). Since our measurements come from the process \( y \), we start by estimating its covariance matrix \( \Sigma \).

It is well-known that a sinusoidal signal is not second-order ergodic. Even more so if its amplitude and frequency are both random (see Proposition 2 in the Appendix). Therefore, one sample path of the process \( y \) is not enough to estimate the covariance matrix. For this reason, we shall assume that our observed data consist of \( L \) strings of sample observations, assumed for simplicity all of length \( N \):
\[ y_k(t) = a_k \cos(\omega_k t) + b_k \sin(\omega_k t) + w_k(t), \] (26)

where \( k = 1, \ldots, L, \) \( t = 1, \ldots, N \), \( (a_k, b_k) \) are sample determinations of the random variables \( (a, b) \), and the frequencies \( \omega_k \) are sample determinations of the random variable \( \omega \) which is uniformly distributed on the fixed interval \( [\theta - W, \theta + W] \). We assume that noises of different cross sections are independent. Furthermore, we assume that the random samples \( (a_k, b_k, \omega_k) \) come from i.i.d. copies of \( (a, b, \omega) \), then apparently the covariance matrix can be estimated by doing a
cross-sectional average

\[ \hat{\Sigma}_N := \frac{1}{L} \sum_{k=1}^L \mathcal{Y}_k \mathcal{Y}_k^\top, \]  

where \( \mathcal{Y}_k = [y_k(1) \cdots y_k(N)]^\top \) is a column vector. By the strong law of large numbers, we have

\[ \hat{\Sigma}_N \to \Sigma_N \text{ as } L \to \infty \]  

almost surely. Let \( \hat{\lambda}_N \) be the smallest eigenvalue of \( \hat{\Sigma}_N \). Then given (10) and Theorem 1, we have

\[ \lim_{L,N \to \infty} \hat{\lambda}_N = \sigma_w^2. \]

The limit here and those similar ones in the following are understood as first letting \( L \to \infty \) and then \( N \to \infty \). In this sense we are able to build a consistent estimator of the signal covariance matrix \( K_N \):

\[ K_N := \hat{\Sigma}_N - \hat{\lambda}_N I_N. \]

Next, for \( \varepsilon > 0 \) close to zero, the numerical rank of \( K_N \) can be described using Theorem \[1\] as

\[ \text{rank}(K_N) \approx M(\varepsilon, N) = \frac{2W}{\pi} N \]

with an approximation error which roughly grows as \( \log N \). From this estimate we can then define an estimator of \( \hat{W} \) as follows

\[ \hat{W} := \frac{\pi \text{rank}(K_N)}{2}. \]

From the relation for the scalar covariance of lag 1

\[ \Sigma(1) = \sigma_w^2 \cos\theta \frac{\sin W}{W}, \]

we can write down an estimator of the center frequency:

\[ \hat{\theta} := \arccos \left( \frac{\Sigma(1) \hat{W}}{\sigma_w^2 \sin W} \right), \]

where \( \Sigma(1) \) is an estimator of \( \Sigma(1) \). An individual estimator of the covariance lag \( \Sigma(\tau) \) may be obtained by doing a second average along the \( \tau \)-th subdiagonal of \( \hat{\Sigma}_N \). The procedure is equivalent to first computing the standard biased covariance estimator within one sample path \[1\] Chapter 2,

\[ \hat{\Sigma}_k(\tau) := \frac{1}{N} \sum_{t=1}^{N-\tau} y_k(t+\tau)y_k(t), \]

and then doing cross sectional average w.r.t. \( k \). A consistent estimator of the signal variance is given by

\[ \hat{\sigma}^2 := \hat{\Sigma}(0) - \hat{\lambda}_N, \]

since we have \( \Sigma(0) = \sigma^2 + \sigma_w^2 \) by \[3\]. Under the cross-sectional independence assumption, the estimators \[32\] and \[34\] for \( W \) and \( \theta \) are also consistent. In particular we have

\[ \hat{W} \to \frac{\pi \text{rank}(K_N)}{2} \approx W \]

when \( N \) is large.

In the next section we shall describe a different and more general estimator based on the subspace philosophy.

Remark 2. The independence assumption for the cross sections may seem quite strong. A more natural assumption could be to require that the strings \( \{y_k\} \) are sample observations of length \( N \) from an exchangeable sequence of \( N \)-dimensional random vectors, \( \{y_k\}_{k=1}^L \). This means that the joint probability distribution of the sequence \( \{y_k\}_{k=1}^L \) is independent of the order in which the random elements \( y_k \) are listed \[27\]. Roughly speaking, this implies that the order in which the \( N \)-dimensional samples \( \{y_k\} \) are observed is not important. Since the additive noise process is i.i.d. (and hence exchangeable) this implies that the sequence of random vectors \( \{x_k\}_{k=1}^L \) in the sequence

\[ y_k = x_k + w_k, \quad k = 1, \ldots, L, \]

should also be exchangeable. By assumption the amplitudes \( \{a_k, b_k\} \) are uncorrelated, with the same variance \( \sigma^2 \) and independent of \( w_k(t) \) so that each parent observation process is wide sense stationary. The sample frequencies \( \omega_k \) are thought of as being determinations of a fixed random variable \( \omega \).

Assume now that \( L \) is very large, i.e. \( L \to \infty \). It follows from De Finetti representation theorem \[27\]-\[30\] that the \( L \) exchangeable random observations \( \{y_k\}_{k=1}^L \) must be conditionally independent given a certain fixed random variable. In other words their joint distribution must be a mixture of the (equal) distributions of each cross-sectional process \( y_k \), given some invariant probability measure called the De Finetti measure \[27\] of the family. Here by imposing our Bayesian prior assumption, i.e., assuming that the frequencies \( \omega_k \) are picked at random from a common uniform distribution on the interval \( [\theta - W, \theta + W] \), we impose this prior distribution to be equal to the De Finetti measure of the exchangeable collection of random vectors \( \{y_k\} \). This measure of course depends on the hyperparameters \( (\theta, W) \). What is important in this setting is that on one hand, once a fixed sample frequency \( \omega_k \) is randomly selected to produce one \( N \)-dimensional sample \( y_k \) and this sample is averaged over \( N \) yielding a function of \( \omega_k \) (see Proposition \[2\] in the appendix), these sample averages are still independent (even if not identically distributed) as their probability distributions \( P(y_k | \omega_k) \) combine to form the joint distribution in a multiplicative fashion by conditional independence. Therefore, the Law of Large Numbers applies also to our exchangeable sequence of observations \( \{y_k\} \) and leads via a second averaging with respect to \( k \), to a probabilistic average (expectation) with respect to the measure governing the \( \omega_k \)’s. In particular a consistent covariance estimate can result by first averaging each \( N \)-dimensional sample \( \{y_k\} \) with respect to \( t \) and successively computing a second cross-sectional average with respect to \( k \).

This identification is actually rather standard and goes back to the foundations of Bayesian Statistics, see for example the classroom notes of the course Stat260: Bayesian Modeling and Inference by M. Jordan \[31\].

A. Simulation Part I

In this simulation part, we test the performance of the rank estimator for \( W \) \[32\] and the \( \arccos \) estimator for the center.
frequency \( \theta \) using finite data. Since the processes considered here are real-valued, their power spectra are symmetric w.r.t. the origin, and one may think that the power content is concentrated on the half interval \([0, \pi]\). In the first example, the measurements \((\mathbf{x}_t)\) are generated with \(a, b\) standard normal and \(\omega\) drawn from the uniform distribution in \([\theta - W, \theta + W]\) with the hyperparameters \(\theta = 2\pi \times 0.3\) and \(W = 2\pi \times 0.02\). The additive noise is i.i.d. Gaussian with variance 1. The length of one sample path is \(N = 10^3\) and the number of sample paths is \(L = 10^4\).

In Fig. 3 we report the eigenvalues of the estimated covariance matrix \((\hat{\Sigma})\). By comparison with the eigenvalues of the theoretical covariance matrix (blue line), we can see some distortions in the large and small eigenvalues due to the slow convergence of the estimator \((\hat{\Sigma})\). However, from Fig. 3 we also see that one unique feature retained by the estimated eigenvalues is their fast decay. Inspired by the nice overlap of the two vertical lines in the L-shape part, we propose an ad-hoc solution: replace \(\text{rank}(\hat{\mathbf{K}}_N)\) given by \((32)\) with the index corresponding to the corner of the L-shape of the estimated eigenvalues, namely

\[
\text{rank}(\hat{\mathbf{K}}_N) \leftarrow \arg\max \Delta^2 \lambda(\hat{\Sigma}_N)
\]

where \(\Delta^2 \lambda(\hat{\Sigma}_N)\) is the second-order difference of the eigenvalue sequence which is already in decreasing order. The maximum should be obtained at the corner since there the first-order difference changes significantly.

Next, we do a Monte-Carlo simulation to test our idea. In each trial, the hyperparameters are generated randomly. More precisely, first the bandwidth \(W\) is drawn from the uniform distribution in \(2\pi \times [0.01, 0.05]\), and then the center frequency \(\theta\) is drawn from \([W, \pi - W]\). The other parameters are the same as above. Given \(L\) independent measurement sequences of length \(N\), the covariance matrix is estimated through \((27)\), and then the rank is computed via \((33)\), which gives an estimate of \(W\) by \((32)\). At last, an estimate of \(\theta\) is computed via \((34)\). The estimation error of \(W\) is defined by the absolute value \(|\hat{W} - W|\), and the relative error is \(|\hat{W} - W|/W\). The error of \(\hat{\theta}\) is defined accordingly. In Fig. 4 the relative errors of \(\hat{W}\) and \(\hat{\theta}\) in 1000 trials are depicted using the boxplot. One can see that the estimates are quite accurate, although, as it will be shown in the next section they can be improved considerably using subspace methods.

![Fig. 3. Eigenvalues of the theoretical and estimated covariance matrices of \(y\)](image)

V. MULTIPLE HIDDEN FREQUENCIES: A SUBSPACE METHOD

Consider now the general measurement model \((\mathbf{1})\), with the signal \(x\) consisting of multiple sinusoids as in \((\mathbf{2})\) satisfying all assumptions listed in Sec. \((\mathbf{I})\). For simplicity, we shall assume that the amplitude covariances are the same, \(\sigma_1^2 = \cdots = \sigma_{\nu}^2 = \sigma^2\). The covariance of \(y\) can then be computed similarly to that in Section \((\mathbf{II})\). We have

\[
\Sigma(\tau) = K(\tau) + \sigma_\nu^2 \delta(\tau, 0) = \sum_{\ell=1}^\nu \sigma^2 \cos \omega_\ell \tau + \sigma_\nu^2 \delta(\tau, 0)
\]

\[
= \sigma^2 \sin W \tau W \tau \sum_{\ell=1}^\nu \cos \theta_\ell \tau + \sigma_\nu^2 \delta(\tau, 0)
\]

\[
= \pi \sigma^2 \int_{-\pi}^\pi e^{i\omega \tau} \sum_{\ell=1}^\nu \chi_{S_\ell}(\omega) d\omega + \sigma_\nu^2 \delta(\tau, 0).
\]

Under the assumptions listed in Sec. \((\mathbf{II})\) the sum \(\sum_{\ell=1}^\nu \chi_{S_\ell}(\omega)\) is the indicator function on the set \(S := \bigcup_{\ell=1}^\nu S_\ell\). From the integral expression for the covariance function, we see immediately that Corollary \((\mathbf{II})\) is applicable, and the asymptotic rank of \(\hat{\mathbf{K}}_N\) is now \(2\pi W N/\pi\). A rank estimator for the bandwidth \(W\) similar to \((32)\) can be obtained. Notice that this is possible since we have assumed that the supporting intervals for different frequencies have a common bandwidth. Next, we will concentrate on the estimation of the center frequencies \(\theta := (\theta_1, \ldots, \theta_{\nu})\).

Remark 3. When the amplitudes \(\sigma_1^2, \ldots, \sigma_{\nu}^2\) are different, the spectral density of our signal is a sum of nonoverlapping rectangular functions and can always be written as a weighted
sum of indicator functions. The assertion on the rank in Corollary [1] must still hold, although the proof seems now to be complicated and we shall have to postpone it to a future publication.

Efficient estimation of the hyperparameters can be based on maximum likelihood, assuming Gaussian additive noise. See [32, p. 429] for a general discussion of this point. The Gaussian likelihood function based on the k-th string of N data can be written as (cf. [33])

\[ l_k(\theta, W) = -\frac{N}{2} \log 2\pi - \frac{1}{2} \log \det \Sigma(\theta, W) - \frac{1}{2} \Sigma(\theta, W)^{-1} \gamma_k, \]

(40)

where \( \gamma_k \) is the vector introduced in (27), \( \Sigma(\theta, W) \) is the theoretical covariance matrix of \( \gamma_k \), with entries given in (39) which do not depend on the index \( k \). The first constant can be dropped from the objective function. By the independence assumption on the sample paths, the log-likelihoods add to each other so that we end up with maximization of the function

\[ l(\theta, W) = -\frac{L}{2} \log \det \Sigma(\theta, W) - \sum_{k=1}^{L} \frac{1}{2} \Sigma(\theta, W)^{-1} \gamma_k \]

(41)

with respect to \( \theta, W \). This leads to the well-known unique maximizer, see e.g. [17, pp. 202–203], for the covariance matrix

\[ \Sigma(\theta, W) = \hat{\Sigma}_N, \]

(42)

where \( \hat{\Sigma}_N \) is given in (27). Such an equation should be solved for the unknown hyperparameters \( \theta, W \) appearing in the known structure (39). Note that this equation can be interpreted as resulting from the well-known method of moments which is the theoretical basis of Subspace Methods [23, Chapt. 13]. Since the equation is nonlinear, one may think of setting up at the outset an iterative solution scheme. However, these numerical algorithms very often converge only locally. In fact, the likelihood function is nonconvex and contains many flat regions. Therefore, brute-force optimization seems to be a hard task.

We shall instead take advantage of the structure of the equation (42) to propose a subspace-based approach. For a fixed and large enough \( N \), we may and shall here assume that the \( N \times N \) covariance matrix \( K_N \) of the process \( x \) has exactly rank \( n := 2w\pi N \). As discussed in Subsection III-B for \( N \) large this is a quite accurate approximation. In other words, we do a truncation in the spectral decomposition of the matrix \( K_N \), retaining the largest \( n \) eigenvalues, namely

\[ K_N = \frac{\pi \sigma^2}{2W} R_N = \frac{\pi \sigma^2}{2W} Q_N D_N Q_N^T \approx \frac{\pi \sigma^2}{2W} Q_N \text{diag}(I_n, O_{N-n}) Q_N^T, \]

(43)

where \( O_m \) denotes the square all-zero matrix of size \( m \). As before, the eigenvalues in the diagonal matrix \( D_N \) are arranged in nonincreasing order.

**Proposition 1.** For \( N \) large enough, there are an \( n \times n \) matrix \( A \) and an \( n \)-dimensional row vector \( c \) such that the random oscillatory signal \( x \) can be represented by the system

\[ \xi(t + 1) = A\xi(t) \]

(44)

\[ x(t) = c\xi(t) \]

(45)

where \( \xi(t) = [\xi_1(t), \xi_2(t), \ldots, \xi_n(t)]^T \) is an \( n \)-dimensional basis vector spanning the Hilbert space \( H(x) \) linearly generated by the \( N \) random variables of the set \( \{x(s) : t \geq s \geq t - N + 1\} \).

**Proof.** It is well-known that a rank-deficient covariance matrix (of rank \( n \)) must necessarily be the covariance of a purely deterministic process [23, p. 138, 276]. When the total support of the spectrum \( S = \bigcup_{\ell=1}^{\ell=N} \mathcal{S}_\ell \) is a proper subset of \([-\pi, \pi]\), \( x \) in (2) is a purely deterministic process which can be represented by a deterministic linear recursion of order \( n \) or equivalently, by a \( n \)-dimensional state-space model. Any such state-space representation for the process \( x \) is of the form (44), (45) where \( A \) can be chosen orthogonal so that \( A^\top = A^{-1} \).

The output in (45) admits an expression \( x(t) = c^\top A^t \xi(0) \), from which we can compute the covariance function of the process as

\[ \sigma(t - s) = c^\top \mathbb{E}(\xi(0)\xi(0)^\top)(A^s)^t c^\top = c^\top A^t P A^{-s} c^\top. \]

The matrix \( P := \mathbb{E}(\xi(0)\xi(0)^\top) \) satisfies a degenerate Lyapunov equation and commutes with \( A \). Therefore, we have \( \sigma(\tau) = c P \sigma^\tau c^\top \). The spectral density of \( x \) is a sum of Dirac deltas. To see this, we first notice that since \( A \) is orthogonal, its spectral decomposition can be written \( A = T \Lambda T^\top \) where \( T \) is unitary and \( \Lambda = \text{diag}(\delta(\omega - \varphi_1), \ldots, \delta(\omega - \varphi_n)) \) is a diagonal matrix of eigenvalues all having modulus 1. Moreover, the eigenvalues should come in conjugate pairs \( e^{\pm i \varphi} \) if \( \varphi \neq 0, \pi \) due to the reality of \( A \). The spectrum of the output process now follows:

\[ \Phi_x(\omega) = c P F(A^\top)^t c^\top = c P T F(A^\top)^t T^\top c^\top \]

\[ = 2\pi c P T \text{diag}(\delta(\omega - \varphi_1), \ldots, \delta(\omega - \varphi_n)) T^\top c^\top, \]

(46)

where the weights for the Dirac deltas are determined by the vectors \( c P T \) and \( T^\top c^\top \). See also [23, Eq. (8.129)].

Since the state-space realization will be constructed from the truncated covariance matrix (43), its spectrum should approximate the true one, i.e., the indicator function on \( S \) times a constant factor, in the sense that the supports of the Dirac deltas should be clustered in \( S \). The center of each cluster, namely the average of the arguments \( \varphi_k \) inside one cluster, is an estimate of the center frequency. Such an idea is also justified by the fact that the (approximate) eigenvalues of \( K_N \) do not depend on the center frequencies \( \theta \). Hence the whole dependence on \( \theta \) must be in \( c \) and \( A \).

Now the remaining point is how to obtain the parameters \( c, A \) in the realization from the measurements of \( y \). First, we estimate the rank of \( K_N \) using the technique in Subsection IV-A. Secondly, one can easily verify that the finite covari-
In the last step of the subspace algorithm, the center of each cluster may be obtained by simply taking the average of all the points in the cluster. This yields the estimate

\[ \hat{\theta}_\ell = \frac{1}{n_\ell} \sum_{k=1}^{n_\ell} \varphi_k \quad \ell = 1, \ldots, \nu \]  

where \( n_\ell \) is the number of phase points in each cluster of positive phases.

Again we do a Monte-Carlo simulation of 1000 trials in the style of Subsection IV-A. For the data generation, we first draw \( W \) from the uniform distribution in \( 2\pi \times [0.01, 0.05] \). Then we take the center frequencies \( \theta_1, \theta_2 \) from \( U [W, \pi - W] \), making sure that \( |\theta_2 - \theta_1| > 2W \) so that the supporting intervals for the two frequencies do not overlap. Given the hyperparameters, we generate \( L = 10^4 \) independent measurement sequences of length \( N = 10^3 \) according to (26). Other parameters are the same as those in Subsection IV-A. We can now run the subspace algorithm given in the previous section.

Fig. 5 shows the discrete spectrum of the output process (45) in the case of one hidden frequency. The horizontal axis is scaled to represent the frequency in Hz, and the true hyperparameters are \( [\theta, W] = 2\pi \times [0.3, 0.02] \). One can see that the Dirac deltas indeed cluster around the nominal frequency inside the supporting interval. Similarly, Fig. 6 shows the discrete spectrum in the case two hidden frequencies where \( [\theta_1, \theta_2, W] = 2\pi \times [0.1, 0.3, 0.02] \). The same observation can be made as before.

![Fig. 5. Discrete spectrum estimate with one hidden frequency](image)

The relative errors in Monte-Carlo simulations are plotted in Fig. 7. The two boxes on the left concern the case of one hidden frequency, where the relative error is defined by \( |\hat{\theta} - \theta| / \theta \). The first box shows the performance of the \( \text{arccos} \) estimator in Subsection IV-A and the second one is about the subspace-based estimator which clearly is much better. The error of \( \hat{W} \) is not included here since both methods use the same rank-based estimator for it. The two boxes on the right concern the subspace algorithm for the case of two hidden frequencies, where the labels “sub. mult. 1” and “sub. mult. 2” correspond to the number of sample paths \( L = 10^4 \) and...
processing community. In this setting the solution can be based on essentially linear techniques in subspace identification. The simulation results are very encouraging.

APPENDIX

In the proof of Theorem 1 we shall need two auxiliary lemmas. The first is just a simple technical fact.

**Lemma 1.** If two sequences of bounded real numbers \( \{a_n\}, \{b_n\} \) are such that

\[
\lim_{n \to \infty} (a_n - b_n) = 0, \tag{49}
\]

then

\[
\limsup_{n \to \infty} a_n = \limsup_{n \to \infty} b_n,
\]

and

\[
\liminf_{n \to \infty} a_n = \liminf_{n \to \infty} b_n.
\]

**Proof.** The argument is quite standard. Let \( \bar{a} := \limsup_{n \to \infty} a_n \). Then there exists a subsequence \( \{a_{n_k}\} \) converging to \( \bar{a} \). Define \( \bar{b} := \limsup_{k \to \infty} b_{n_k} \). Then there exists a sub-subsequence \( \{b_{n_k}\} \) converging to \( \bar{b} \). The condition (49) holds for the subsequence indexed by \( n_{k_j} \), which implies that \( \bar{a} = \bar{b} \). It then follows that \( \bar{b} := \limsup_{n \to \infty} b_n \geq \bar{b} = \bar{a} \). A symmetric argument leads to \( \bar{a} \geq \bar{b} \), and therefore \( \bar{a} = \bar{b} \). The proof for the limit inferior is similar and hence omitted. \( \square \)

The next lemma concerns the sum of squared eigenvalues of \( \mathbf{R} \).

**Lemma 2.**

\[
\lim_{N \to \infty} \frac{1}{N} \sum_{j=1}^{N} \lambda_j^2(N) = \frac{m(J)}{2\pi}. \tag{50}
\]

**Proof.** Since \( \mathbf{R} \) is Hermitian, we have

\[
\sum_{j=1}^{N} \lambda_j^2(N) = \text{tr} \mathbf{R}^2 = \text{tr}(\mathbf{R}\mathbf{R}^*) = \sum_{j=-N+1}^{N-1} |\rho(j)|^2(N - |j|). \tag{51}
\]

It follows that

\[
\frac{1}{N} \sum_{j=1}^{N} \lambda_j^2(N) = \sum_{j=-N+1}^{N-1} |\rho(j)|^2 \left( 1 - \frac{|j|}{N} \right). \tag{52}
\]

We can view the latter summation over \( \mathbb{Z} \) by adding zeros. Apparently, each term in the infinite sum is dominated by \( |\rho(j)|^2 \). Moreover, for each fixed \( j \) the term-wise limit as \( N \to \infty \) is also \( |\rho(j)|^2 \). Applying Lebesgue’s dominated convergence theorem for the counting measure on \( \mathbb{Z} \), we can conclude that

\[
\lim_{N \to \infty} \frac{1}{N} \sum_{j=1}^{N} \lambda_j^2(N) = \sum_{j \in \mathbb{Z}} |\rho(j)|^2 \int_{-\pi}^{\pi} |\chi_j(\omega)|^2 d\omega = \frac{m(J)}{2\pi}. \tag{53}
\]
where the second equality is the Parseval identity.

**Proof of Theorem 1**

**Proof.** We first show that the number of eigenvalues not close to 0 or 1 is $o(N)$. To this end, define the function

$$
\mathcal{J}(N) := \sum_{j=1}^{N} \lambda_j(N)(1 - \lambda_j(N))
$$

(54a)

$$
= \sum_{j=1}^{N} \lambda_j(N) - \sum_{j=1}^{N} \lambda_j^2(N),
$$

(54b)

where each summand in (54a) is nonnegative. Then according to (23) and Lemma 2, we have

$$
\lim_{N \to \infty} \frac{\mathcal{J}(N)}{N} = 0.
$$

(55)

In other words, the function $\mathcal{J}(N)$ is $o(N)$. Fix $0 < \delta < \gamma < 1$, and the number of eigenvalues $\delta \leq \lambda_j(N) < \gamma$ is $M(\delta, n) - M(\gamma, N)$. Clearly, for these eigenvalues we have

$$
\lambda_j(N)(1 - \lambda_j(N)) > \delta(1 - \gamma) := \nu > 0,
$$

(56)

which implies that

$$
\mathcal{J}(N) \geq \sum_{\delta \leq \lambda_j(N) < \gamma} \lambda_j(N)(1 - \lambda_j(N)) \geq \nu \left[ M(\delta, N) - M(\gamma, N) \right] \geq 0.
$$

(57)

It follows that

$$
\lim_{N \to \infty} \frac{M(\delta, N) - M(\gamma, N)}{N} = 0,
$$

(58)

which means that the quantity $M(\delta, N) - M(\gamma, N)$ is also $o(N)$.

Next, define the quantities

$$
M_+ := \limsup_{N \to \infty} \frac{M(\gamma, N)}{N},
$$

$$
M_- := \liminf_{N \to \infty} \frac{M(\gamma, N)}{N}.
$$

(59)

Applying Lemma 1 in this appendix to the relation (58), we know that both $M_+$ and $M_-$ do not depend on $0 < \gamma < 1$. We want to establish that the two quantities coincide so that the ordinary limit in (24) exists and is equal to the common value. Observe that

$$
\text{tr} R = \sum_{j=1}^{N} \lambda_j(N) + \sum_{M(\gamma, N) + 1}^{N} \lambda_j(N) \geq \gamma M(\gamma, N),
$$

(60)

and similarly

$$
\sum_{j=1}^{N} \lambda_j^2(N) = \sum_{j=1}^{M(\gamma, N)} \lambda_j^2(N) + \sum_{M(\gamma, N) + 1}^{N} \lambda_j^2(N) \geq \gamma M(\gamma, N) + \gamma \text{tr} R,
$$

(61)

and furthermore, we have

$$
M_+ \leq \limsup_{N \to \infty} \frac{\text{tr} R}{\gamma N} = \frac{m(J)}{2\pi},
$$

(63a)

$$
M_- \geq \liminf_{N \to \infty} \frac{1}{N} \left( \sum_{j=1}^{N} \lambda_j^2(N) - \gamma \text{tr} R \right) = (1 - \gamma)^2 \frac{m(J)}{2\pi},
$$

(63b)

where we have used Lemma 2 again in (63b). Letting $\gamma \to 1$ in (63a) and $\gamma \to 0$ in (63b), we obtain

$$
m(J) \leq M_- \leq M_+ \leq \frac{m(J)}{2\pi},
$$

(64)

and the claim of the theorem follows.

The next proposition concerns the time average of one sample path of the noisy sinusoidal signal.

**Proposition 2.** Let

$$
y(t) = x(t) + w(t) = a \cos(\omega t) + b \sin(\omega t) + w(t)
$$

(65)

be a sample path of the process $\{1\}$, where $t = 1, 2, \ldots$. Then for each fixed $\omega$ with $|\omega| < \pi$,

$$
\frac{1}{N} \sum_{t=1}^{N} y(t + \tau)y(t) \to \frac{a^2 + b^2}{2} \cos \omega \tau + \sigma_w^2 \delta(\tau, 0)
$$

as $N \to \infty$ with probability one.

**Proof.** We have

$$
\frac{1}{N} \sum_{t=1}^{N} y(t + \tau)y(t) = \frac{1}{N} \sum_{t=1}^{N} [x(t + \tau)x(t) + x(t + \tau)w(t) + w(t + \tau)x(t) + w(t + \tau)w(t)]
$$

and that the first time average converges to $\frac{a^2 + b^2}{2} \cos \omega \tau$ is shown in [7] pp. 105-109] or [11] pp. 171-172]. That the average of each cross term in the middle tends to 0, follows since the process $\tilde{w}(t) := e^{i\omega t}w(t)$ is (complex) zero-mean i.i.d. and by the assumed uncorrelation so is also $\alpha e^{i\omega \tau}\tilde{w}(t)$ and hence so is its real part, so that the law of large numbers holds for each cross term. The time average of the last term tends to $\sigma_w^2 \delta(\tau, 0)$ again by the law of large numbers.

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