GRADED CODIMENSIONS OF LIE SUPERALGEBRA $b(2)$

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Abstract. We study asymptotic behaviour of graded codimensions of Lie superalgebra $b(2)$. We prove that graded PI-exponent exists and is equal to $3 + 2\sqrt{3}$.

1. Introduction

We consider finite dimensional Lie superalgebras over a field of characteristic zero and study their $\mathbb{Z}_2$-graded identities. We pay main attention to numerical invariants of identities, in particular, to graded codimensions and their asymptotic behaviour.

It is well-known that in case $\dim L < \infty$ both graded and ordinary codimensions are exponentially bounded ([2]). One of the more important questions of the theory of numerical invariants of polynomial identities is: does a (graded) PI-exponent exist?

There are many papers where the existence of PI-exponent is proved for different classes of algebras. For example, if $A$ is an associative PI-algebra or a finite dimensional Lie, Jordan or alternative algebra then its PI-exponent exists and is a non-negative integer (see [4], [5], [6], [11]). The existence of PI-exponent for any finite dimensional simple algebra was proved in [8]. It is not difficult to show that if PI-exponent of $A$ exists then it is less than or equal to $d$ provided that $d = \dim A < \infty$ (see for example [2]). In many important classes of algebras over an algebraically closed field (associative, Lie, Jordan, alternative) the equality $\exp(A) = \dim A$ is equivalent to simplicity of $A$ ([4], [5], [11]). Recently [8] it was shown that $\exp(L) < \dim L$ for any finite dimensional simple Lie superalgebra $L$ of the type $b(t), t \geq 3$, (in the notation of [9]). The existence of PI-exponent and similar inequality $\exp(L) < \dim L$ for $b(2)$ was also proved in [8] although it is not simple superalgebra.

Graded codimensions of Lie superalgebras were studied much less. In particular, it is still unknown if $\exp^{gr}(L)$ exists even when $L$ is a finite dimensional simple Lie superalgebra. In recent paper [10] it was proved that an upper graded PI-exponent of Lie superalgebra $b(t), t \geq 2$, is less than or equal to $t^2 - 1 + t\sqrt{t^2 - 1}$. In particular, this gives an upper bound for ordinary PI-exponent of $b(t)$. In the present paper we prove the existence of graded PI-exponent of $b(2)$. We also prove that $\exp^{gr}(b(2)) = 3 + 2\sqrt{3}$. Note that it was recently announced that an ordinary PI-exponent does not exist in general non-associative case (see [12]).

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2. Main constructions and definitions

Let $L$ be a Lie superalgebra over a field $F$ of characteristic zero, that is $L = L_0 \oplus L_1$ is a non-associative $\mathbb{Z}_2$-graded algebra satisfying two identical relations

$$xy + (-1)^{|x||y|}yx = 0,$$

$$x(yz) = (xy)z + (-1)^{|x||y|}y(xz) = 0$$

where $x, y, z \in L_0 \cup L_1$ and $|x| = 0$ if $x \in L_0$ while $|x| = 1$ if $x \in L_1$.

Elements from $L_0 \cup L_1$ are called homogeneous and we say that $x$ is even if $x \in L_0$ or $x$ is odd if $x \in L_1$.

Denote by $\mathcal{L}(X,Y)$ a free Lie superalgebra with infinite sets of even generators $X$ and odd generators $Y$. A polynomial $f = f(x_1, \ldots, x_m, y_1, \ldots, y_n) \in \mathcal{L}(X,Y)$ is said to be a graded identity of Lie superalgebra $L = L_0 \oplus L_1$ if $f(a_1, \ldots, a_m, b_1, \ldots, b_n) = 0$ whenever $a_1, \ldots, a_m \in L_0, b_1, \ldots, b_n \in L_1$.

Given positive integers $0 \leq k \leq n$, denote by $P_{k,n-k}$ the subspace of all multilinear polynomials $f = f(x_1, \ldots, x_k, y_1, \ldots, y_{n-k}) \in \mathcal{L}(X,Y)$ of degree $k$ on even variables and of degree $n - k$ on odd variables. Denote by $Id^{gr}(L)$ the ideal of $\mathcal{L}(X,Y)$ of all graded identities of $L$. Then $P_{k,n-k} \cap Id^{gr}(L)$ is the subspace of all multilinear graded identities of $L$ of total degree $n$ depending on $k$ even variables and $n - k$ odd variables. Also denote by $P_{k,n-k}(L)$ the quotient

$$P_{k,n-k}(L) = \frac{P_{k,n-k}}{P_{k,n-k} \cap Id^{gr}(L)}.$$  

Then the graded $(k, n-k)$-codimension of $L$ is

$$c_{k,n-k}(L) = \dim P_{k,n-k}(L)$$

and the total graded codimension of $L$ is

$$c_n^{gr}(L) = \sum_{k=0}^{n} \binom{n}{k} c_{k,n-k}(L).$$

It is known (see [2]) that if $\dim L < \infty$ then the sequence $\{c_n^{gr}(L)\}_{n \geq 1}$ is exponentially bounded and one can consider the related sequence $\sum c_n^{gr}(L)$. The latter sequence has the lower and upper limits

$$\exp^{gr}(L) = \liminf_{n \to \infty} \sqrt[n]{c_n^{gr}(L)}, \quad \exp^{gr}(L) = \limsup_{n \to \infty} \sqrt[n]{c_n^{gr}(L)}$$

called the lower and upper graded PI-exponents of $L$, respectively. If an ordinary limit exists, it is called an (ordinary) graded PI-exponent of $L$,

$$\exp^{gr}(L) = \lim_{n \to \infty} \sqrt[n]{c_n^{gr}(L)}.$$  

Symmetric groups and their representations play an important role in the theory of codimensions. One can find all details concerning application of representation theory of symmetric groups to study of polynomial identities in [1], [3], [7]. In case of graded identities one can consider $(S_k \times S_{n-k})$-action on multilinear graded polynomials. Namely, the subspace $P_{k,n-k} \subseteq \mathcal{L}(X,Y)$ has a natural structure of $(S_k \times S_{n-k})$-module where $S_k$ acts on even variables $x_1, \ldots, x_k$ while $S_{n-k}$ acts on odd variables $y_1, \ldots, y_{n-k}$. Clearly, $P_{k,n-k} \cap Id^{gr}(L)$ is the submodule under this
action and we get an induced \( S_k \times S_{n-k} \)-action on \( P_{k,n-k}(L) \). If \( G \) is a subgroup of \( S_k \times S_{n-k} \) then \( G \) also acts naturally on \( P_{k,n-k}(L) \). In particular,

(1) \[ c_{k,n-k}(L) \geq \dim M \]

for any subgroup \( G \subseteq S_k \times S_{n-k} \) and for any \( G \)-submodule \( M \) of \( P_{k,n-k}(L) \). We will use the relation (1) for getting a lower bound of \( c_{k,n-k}(L) \).

3. Graded PI-exponent of \( b(2) \)

Recall the construction of Lie superalgebra \( L = L_0 \oplus L_1 = b(2) \). Even component \( L_0 \) consists of all \( 4 \times 4 \) matrices of the type

\[ L_0 = \left\{ \begin{pmatrix} A & 0 \\ 0 & -A^t \end{pmatrix} | A \in M_2(F), tr(A) = 0 \right\}, \]

where \( A \) is a traceless matrix and \( t : A \to A^t \) is the usual transpose involution.

Odd component \( L_1 \) consists of \( 4 \times 4 \) matrices

\[ L_1 = \left\{ \begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix} | \text{ where } B^t = B, C^t = -C, B, C \in M_2(F) \right\}. \]

Also denote

\[ L_1^- = \left\{ \begin{pmatrix} 0 & 0 \\ C & 0 \end{pmatrix} | C^t = -C \in M_2(F) \right\}, \]

\[ L_1^+ = \left\{ \begin{pmatrix} 0 & B \\ 0 & 0 \end{pmatrix} | B^t = B \in M_2(F) \right\}. \]

Then \( \dim L_1^- = 1, \dim L_1^+ = 3. \)

As a vector space \( L \) is embedded into \( M_4(F) \). Using ordinary associative matrix multiplication we can define super-Lie product on \( L \) as

\[ \{x, y\} = xy - (-1)^{|x||y|}yx \]

for any homogeneous \( x, y \in L_0 \cup L_1 \) where \( |x| = 0 \) if \( x \in L_0 \) while \( |x| = 1 \), if \( x \in L_1 \).

Note that \( L_0 \) is a Lie algebra isomorphic to \( sl_2(F) \). We will identify \( L_0 \) with \( sl_2(F) \) and use the standard basis of \( sl_2(F) \)

\[ e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \]

Furthermore we will not use associative multiplication. This will allow us to omit super-Lie brackets, i.e. to write \( ab \) instead of \( \{a, b\} \). We will also use the notation \( ab \cdots c \) for the left-normed product \( \{\ldots\{a, b\},\ldots, c\} \).

We will also use the following agreement for denoting alternating sets of variables. If \( f = f(x_1, \ldots, x_k, y_1, \ldots, y_n) \) is a multilinear polynomial and we apply to \( f \) the operator of alternation on variables \( x_1, \ldots, x_k \), then we will write the same symbol (bar, double bar, tilde, double tilde, etc.) over the variables \( x_1, \ldots, x_k \), that is

\[ f(\bar{x}_1, \ldots, \bar{x}_k, y_1, \ldots, y_n) = \sum_{\sigma \in S_k} (\text{sgn } \sigma) f(x_{\sigma(1)}, \ldots, x_{\sigma(k)}, y_1, \ldots, y_n). \]

For example \( \bar{x}a\bar{y} = xay - yax \), or

\[ \bar{x}_1\bar{a}\bar{x}_2\bar{b}\bar{x}_3 = \sum_{\sigma \in S_3} (\text{sgn } \sigma)x_{\sigma(1)}ax_{\sigma(2)}bx_{\sigma(3)}. \]
We will also use this notation for non-multilinear polynomials with repeating variables as follows
\[
\bar{x}_1 \bar{x}_2 a \bar{x}_1 \bar{x}_2 = x_1 x_2 a x_1 x_2 - x_2 x_1 a x_1 x_2 =
\]
\[
x_1 x_2 a x_1 x_2 - x_1 x_2 a x_2 x_1 - x_2 x_1 a x_1 x_2 + x_2 x_1 a x_2 x_1.
\]

Following this agreement we consider an alternating expression \( h \vec{e} \vec{f} \vec{h} \) in the Lie algebra \( L_0 = sl_2(F) \). Since \( he = 2e, hf = -2f, ef = h \), it easily follows that
\[
h \vec{e} \vec{f} \vec{h} = 8h
\]
and
\[
(2) \quad h \vec{e} \vec{f} \vec{h} \cdots \vec{e} \vec{f} \vec{h} = 8^t h.
\]

Consider a multilinear polynomial
\[
g = g(x_0, x_1^1, x_2^1, x_3, \ldots, x_i^j, x_2^i, x_3^i) = \text{Alt}_1 \text{Alt}_2 \ldots \text{Alt}_t (x_0 x_1^1 \cdots x_3^i)
\]
where \( \text{Alt}_j \) is the operator of alternation on \( x_1^j, x_2^j, x_3^j, 1 \leq j \leq t \), that is
\[
g = x_0 x_1^1 x_2^1 \cdots x_1^t x_2^t x_3^t.
\]
The evaluation \( \varphi : x_1^1, \ldots, x_1^i \to e, x_2^1, \ldots, x_2^i \to f, x_0, x_3, \ldots, x_3^i \to h \) gives us
\[
(3) \quad \varphi(g) = 8^t h
\]
in \( L_0 \) by (2). Moreover, if we denote the symmetrization on variables \( x_1^1, \ldots, x_i^j \), \( i = 1, 2, 3 \) by \( \text{Sym}_i \), then it follows from (3) and the definition of \( \varphi \) that
\[
(4) \quad \varphi(\text{Sym}_1 \text{Sym}_2 \text{Sym}_3(g)) = (t!)^3 8^t h
\]
in \( L_0 \) by virtue of (2). An element
\[
g' = \text{Sym}_1 \text{Sym}_2 \text{Sym}_3(g)
\]
with the fixed \( x_0 \) generates in \( P_{3t+1,0} \) an irreducible \( S_{3t} \)-submodule with the character \( \chi_\lambda, \lambda = (t, t, t) \) where the permutation group \( S_{3t} \) acts on \( x_1^1, x_2^1, x_3, \ldots, x_1^t, x_2^t, x_3^t \).

Given a partition \( \mu = (\mu_1, \ldots, \mu_d) + n \), we define the function
\[
\Phi(\mu) = \frac{1}{z_1 \ldots z_d},
\]
where
\[
z_1 = \frac{\mu_1}{n}, \ldots, z_d = \frac{\mu_d}{n}.
\]
The value of \( \Phi(\mu) \) is closely connected with \( \text{deg} \chi(\mu) \).

**Lemma 1.** [8, Lemma 1] Let \( n \geq 100 \). Then
\[
\frac{\Phi(\mu)^n}{n^{d^2+d}} \leq d_\mu \leq n\Phi(\mu)^n.
\]

In particular, if \( m = 3t \) and \( \mu = (t, t, t) \) then
\[
(6) \quad \text{deg} \chi_\mu \geq m^{-12} 3^m.
\]
In the next step we will construct an irreducible $S_{6k}$-submodule in $P_{1,6k+1} \not\subset Id^{2(r)}(L)$ where $S_{6k}$ acts on some $6k$ odd variables. Denote

$$d = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} \in L_1^-.$$ 

Then $L_1^- = \text{Span} < d >$. It is not difficult to check that $L_1^+L_1^- = L_0$. Hence there exist $a, b, c \in L_1^+$ such that

$$ad = e, \quad bd = f, \quad cd = h$$

where $\{e, f, h\}$ is the fixed basis of $L_0$. It follows that

$$h(ad)(bd)(cd) = 8h$$

in $L$ and

$$h(ad)(bd)(cd)(\tilde{a}d)(\tilde{b}d)(\tilde{c}d) = h(ad)(bd)(cd)(\tilde{a}d)(\tilde{b}d)(\tilde{c}d) = 64h.$$ 

Repeating this procedure and using (3) we obtain a multialternating expression (7)

$$H = h(ad)(bd)(cd)(\tilde{a}d)(\tilde{b}d)(\tilde{c}d) \cdots (\tilde{a}d)(\tilde{b}d)(\tilde{c}d)(ad)(bd)(cd) = 8^{k+1}h$$

depending on one $h$, $k + 1$ elements $a, b, c$ and $3(k + 1)$ elements $d$. The element $H$ on the left hand side of (7) contains $k$ alternating sets $\{a, b, c, d\}$. The first set consists of $1st$, $1st$, $1st$ and $4th$ $d$. The second set consists of $2nd$ $a$, $2nd$ $b$, $2nd c$ and $7th d$, and so on. The element $H$ also contains $2(k + 1) + 1$ non-alternating entries $d$ and four extra factors $a, b, c, d$ out of alternating sets. This $H$ is a value of the following multilinear polynomial: denote by

$$w = w(x_0, y_1^1, y_2^1, z_1^1, z_2^1, z_3^1, \ldots, y_1^k, y_2^k, y_3^k, z_1^{k+1}, z_2^{k+1}, z_3^{k+1}) =$$

$$\text{Alt}_1 \ldots \text{Alt}_k(x_0(y_1^1 z_1^1)(y_2^1 z_2^1)(y_3^1 z_3^1)(y_1^2 z_1^2)(y_2^2 z_2^2)(y_3^2 z_3^2) \cdots (y_1^{k+1} z_1^{k+1})(y_2^{k+1} z_2^{k+1})(y_3^{k+1} z_3^{k+1}))$$

where $x_0$ is an even variable, all $y_j^i$, $z_j^i$ are odd and $\text{Alt}_j$ is the operator of alteration on $y_j^i, y_j^i, z_j^{i+1}, j = 1, \ldots, k$. Then $\varphi(w) = H$ where $\varphi$ is an evaluation of the form

$$\varphi(x_0) = h, \quad \varphi(y_j^i) = a, \quad \varphi(y_j^i) = b, \quad \varphi(z_j^i) = c, \quad \varphi(z_j^i) = d, \quad j = 1, \ldots, k + 1, i = 1, 2, 3.$$ 

Also denote by $\text{Sym}_1, \text{Sym}_2, \text{Sym}_3$ the symmetrization on the sets $\{y_1^1, \ldots, y_1^k\}$, $\{y_2^1, \ldots, y_2^k\}$, $\{y_3^1, \ldots, y_3^k\}$, respectively, and by $\text{Sym}_4$ the symmetrization on $\{z_1^2, z_2^2, z_3^2, \ldots, z_1^{k+1}, z_2^{k+1}, z_3^{k+1}\}$. If

$$w' = \text{Sym}_1\text{Sym}_2\text{Sym}_3\text{Sym}_4(w)$$

then

$$\varphi(w') = (3k)!(k!)^3\varphi(w) = (3k)!(k!)^38^{k+1}h$$

in $L$. In particular, $w'$ is not an identity of $L$.

Now let the permutation group $S_{6k}$ act on the set $\{y_j^i, z_j^{i+1} | 1 \leq j \leq k, i = 1, 2, 3\}$ and let $x_0, z_1^1, z_2^1, z_3^1, y_1^1, y_2^1, y_3^1, y_4^1, y_5^1, y_6^1$ be fixed. Then $w'$ generates an irreducible $S_{6k}$-submodule in $P_{1,6(k+1)}$ corresponding to the partition $\mu = (3k, k, k, k) \vdash r = 6k$ with

$$\Phi(\mu) = 2\sqrt{3}.$$
Hence
\[ \deg \chi_{\mu} \geq r^{-20(2\sqrt{3})^{y}} \]
by Lemma 1.

Now let
\[ u = w'(g', y_{11}^{1}, y_{21}^{1}, y_{31}^{1}, \ldots, z_{1}^{k+1}, z_{2}^{k+1}, z_{3}^{k+1}) \]
where \( g' \) is taken from (5). If \( m = 3t, r = 6k \), group \( S_{m} \) acts on even variables \( x_{i}^{j} \) from \( g' \) whereas \( S_{r} \) acts on odd variables \( \{y_{i}^{j}, z_{i}^{j}\} \) (except \( z_{1}^{1}, z_{2}^{1}, y_{1}^{k+1}, y_{2}^{k+1}, y_{3}^{k+1} \)) then \( u \) is not a graded identity of \( L \) as follows from (4), (6) and (7) and it generates an irreducible \( S_{m} \times S_{r} \)-submodule \( M \) in \( P_{m+1, r+6} \) with the character \( \chi_{\lambda, \mu} \) where \( \lambda = (t, t, t) \), \( \mu = (3k, k, k, k) \). Hence by (6), (8)
\[ \dim M = \deg \chi_{\lambda} \deg \chi_{\mu} \geq \frac{1}{(m + r)^{32}} 3^{m(2\sqrt{3})^{y}} \]
and then by (1)
\[ c_{m+1, r+6}(L) = \dim P_{m+1, r+6}(L) \geq \dim M \geq \frac{3^{m(2\sqrt{3})^{y}}}{(m + r)^{32}}. \]

The inequality (9) means that we have proved the following lemma.
\section*{Lemma 2.} Let \( t, r \geq 1 \) be arbitrary integers and \( m = 3t, r = 6k \). Then
\[ c_{m+1, r+6}(L) \geq \frac{3^{m(2\sqrt{3})^{y}}}{(m + r)^{32}}. \]

The proof of Lemma 2 is straightforward.

Now we will find a lower bound for \( n \)th graded codimension of \( L \) for the special case of \( n \).
\section*{Lemma 3.} Let \( n - 7 \) be a multiple of 6. Then
\[ c_{n}^{gr}(L) \geq \frac{1}{3^{18} n^{38}} (3 + 2\sqrt{3})^{n}. \]

\textbf{Proof.} Let \( q = n - 7 \). Then applying Lemma 2 we obtain
\[ c_{n}^{gr}(L) = \sum_{i} \binom{n}{i} c_{i, n-i}(L) \geq \sum_{j=0}^{q/6} \binom{n}{6j+1} c_{1+6j, q+6-6j}(L) \geq \]
\[ \frac{1}{q^{32}} \sum_{j=0}^{q/6} \binom{n}{6j+1} 3^{6j}(2\sqrt{3})^{q-6j} \geq A \frac{1}{n^{33}} \]
where
\[ A = \sum_{j=0}^{q/6} \binom{n}{6j} 3^{6j}(2\sqrt{3})^{q-6j} \]

since
\[ \binom{n}{i} \leq n \binom{n}{i+1}. \]

Now, since \( 0 \leq j < \frac{q}{6} \), we have
\[ \binom{q}{6j+i} 3^{6j+i}(2\sqrt{3})^{q-6j-i} < (3q)^{5} \binom{q}{6j} 3^{6j}(2\sqrt{3})^{q-6j} \]
for all $1 \leq i \leq 5$. It follows that

$$A > \frac{6}{(3n)^5} \sum_{i=0}^{q} \binom{q}{i} 3^i (2\sqrt{3})^{q-i} = \frac{6}{(3n)^5} (3 + 2\sqrt{3})^q > \frac{2}{3^{18} n^5} (3 + 2\sqrt{3})^n.$$ 

Hence

$$c^n_{gr}(L) > \frac{(3 + 2\sqrt{3})^n}{3^{18} n^{38}}.$$ 

□

Now we consider the case when $n - 7$ is not a multiple of 6, that is $n - 7 \equiv i (mod 6)$ with $1 \leq i \leq 5$.

**Lemma 4.** Let $t, r, i \geq 1$ be arbitrary integers, $m = 3t$, $r = 6k$ and $i \leq 5$. Then

$$c_{m+1+i,r+6}(L) \geq \frac{3^m (2\sqrt{3})^r}{(m+r)^{32}}.$$ 

**Proof.** The proof is similar to the proof of Lemma 2. We only need to change the polynomial $u = w(g', y_1^1, \ldots, z_{k+1}^1)$ to $u' = ux_1 \cdots x_i$ and consider an evaluation $\varphi$ with the same values on $x_0, x_1^2, y_1^j, z_1^j$ as in Lemma 2 and $\varphi(x_1) = e, \varphi(x_2) = \ldots = \varphi(x_i) = h$ if $i \geq 2$. Then $\varphi(u') = \pm 2^i \varphi(u) \neq 0$ and we are done. □

Slightly modifying arguments of Lemma 3 and using Lemma 4, we get the following result for arbitrary $1 \leq i \leq 5$.

**Lemma 5.** Let $n - 7 \equiv i (mod 6), 1 \leq i \leq 5$. Then

$$c^n_{gr}(L) \geq \frac{1}{3^{18} n^{43}} (3 + 2\sqrt{3})^n.$$ 

□

Now we are ready to prove the main result of the paper.

**Theorem 1.** Graded PI-exponent of Lie superalgebra $\mathcal{L} = b(2)$ exists and is equal to

$$\exp^{gr}(L) = 3 + 2\sqrt{3}.$$ 

**Proof.** By Lemmas 3 and 5

$$\exp^{gr}(L) = \liminf_{n \to \infty} \sqrt[n]{c^n_{gr}(L)} \geq 3 + 2\sqrt{3}.$$ 

On the other hand, $\exp^{gr}(b(t)) \leq t^2 - 1 + t\sqrt{t^2 - 1}$ for all $t \geq 2$ as proved in [10]. Hence the limit

$$\exp^{gr}(b(2)) = \lim_{n \to \infty} \sqrt[n]{c^n_{gr}(b(2))}$$

exists and

$$\exp^{gr}(L) = 3 + 2\sqrt{3}.$$ 

□

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