A smooth variant of the Afriat-Varian theorem

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Abstract
We present a simple geometric construction for smoothing polyhedral utility functions.

Keywords: Afriat-Varian theorem, the utility function, the economic indices theory, convex analysis.

1 Introduction
The problem of the integrability of the demand functions is studied in mathematical economics for more than hundred years. (The functional dependencies between the volume of purchases and the corresponding prices are called the demand functions.) It seems that italorian economist J. Antonelli was one of the first to notice in 1886 (cf. 1) that under certain conditions the demand functions are rationalized, i.e. they can be obtained via maximization of the utility function under budget constraint. The attention to this problem was revived in 1905 in connection with the discourse between V. Volterra and V. Pareto about the interpretation of the demand functions’ rationalization conditions 2. V. Volterra pointed out that Pareto’s result on the rationalizability of the demand functions for two kinds of goods could not be generalized to the case of more than two kinds of goods. However, it is known as well that rationalization is an implicit assumption for the computation of the price indices and the volume of purchases in economic statistics. V. Pareto has tried to justify rationalization conditions and to show that they should be always valid, but in 1915 E. Slutsky 3 proved that Frobenius integrability conditions are necessary for the rationalization of the demand functions (and the inverse demand functions). On one hand, small perturbations of the demand functions should violate rationalization conditions. On the other hand, there was an allusion to the second thermodynamics principle which has been formulated by Caratheodory also in the form of the Frobenius integrability conditions. The original question about the interpretation and satisfiability of the integrability conditions of the differential form recovered from the demand functions has been recognized as a problem that was investigated by such economists as P. Samuelson, K. Arrow, H. Houthakker, L. Hurwitz and others. The Frobenius integrability conditions were reformulated in terms of the strong axiom of the revealed preference theory, a discrete analog of the Caratheodory- Caratheodory- Chow criterion. The axiom allows an explicit testing of the market statistics data (the list of values of the demand functions at the prescribed points). Computational experiments showed that violation of the strong axiom took place in the course of the large-scaled structural changes in the economy similar to the Great Depression of the thirties. A well-known Afriat- Varian theorem form the theoretical ground for these experiments. According to one variant of this theorem the market statistics can be extended to the demand functions that are rationalized in the class of positive homogeneous utility functions if and only if it satisfies the homogeneous strong axiom.

However, the extension figuring in the Afriat-Varian theorem is not smooth, although smoothness of the utility functions is an implicit informal assumption. In this paper we study is it possible to make the extension involved continuous and additionally to have continuous inverses.

This question in turn is equivalent to rationalization of the demand functions in the class of smooth economic indices (positively homogeneous utility functions and price indices). We note in passing that rationalization of market statistics in the class of smooth utility functions and in the class of positive homogeneous utility functions is considered in [3], and in [10], respectively.

2 Notation
Denote by $\mathcal{F}$ the set all nonnegative, positive homogeneous, concave functions in the nonnegative orthant $\mathbb{R}^n_+$, i.e. any $f \in \mathcal{F}$ is a map from $\mathbb{R}^n_+$ to $\mathbb{R}_+$ and moreover $f(\cdot)$ is concave and $f(\lambda x) = \lambda f(x)$ for any nonnegative $\lambda$ and for any $x \in \mathbb{R}^n_+$.

The Euclidean norm in $\mathbb{R}^n$ is denoted by $\| \cdot \|$. Let $\mathcal{B} = \{ x \in \mathbb{R}^n \mid \| x \| \leq 1 \}$ and $\mathcal{B}(x_0, r) = \{ x \in \mathbb{R}^n \mid \| x - x_0 \| \leq r \}$ be, respectively, the unit Euclidean ball and the Euclidean ball of the radius $r$ centered at $x_0$.

If $A$ and $B$ are sets in $\mathbb{R}^n$ their Minkowski sum (i.e. the set of all points of the form $c = a + b$, $a \in A, b \in B$) is denoted by $A \oplus B$.

A face is an intersection of a convex polyhedral set with its supporting hyperplane. A $(n-1)$-dimensional face of a $n$-dimensional convex polyhedral set is called a facet.
The boundary of a set $A \subseteq \mathbb{R}^n$ is denoted by $Bd(A)$.

Let $A \subseteq \mathbb{R}^n$ be a closed convex set and let $x \in Bd(A)$. Recall that by definition the supporting cone $T_A(x)$ of $A$ at $x$ is the intersection of all closed halfspaces containing $A$ and $x$. Respectively, the conjugate (polar) cone $N_A(x) = (T_A(X))^* = \{ p \in \mathbb{R}^n | \langle p, v \rangle \leq 0 \ \forall v \in T_A(x) \}$ is called the normal cone of $A$ at $x$.

The boundary of any closed convex body is called a convex hypersurface (or a convex curve when restricted to the plane).

The superdifferential $\partial f$ of a concave function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ at a point $x_0$ consists of all vectors $p \in \mathbb{R}^n$ such that $p(x-x_0) \geq f(x) - f(x_0)$.

### 3 Afriat-Varian theorem

The market statistics $S = \{ p^t, q^t \}, p^t, q^t \in \mathbb{R}^n, t = 1, \ldots, T$, where $p^t$ and $q^t$ are, respectively, the prices and the volume of purchases in time $t$, is called rationalizable in the class of utility functions $F$, if there exists such function $Q \in F$ that $q^t = \text{Argmax} \{ Q(q) | (p^t q) \leq (p^t q^t), q \geq 0 \}$, $t = 1, \ldots, T$.

According to the Afriat-Varian theorem (see, e.g. [1, 2]) the following statements are equivalent.

1. The market statistics $S = \{ p^t, q^t \}$ is rationalizable in the class of the utility functions $F$.
2. The market statistics $S = \{ p^t, q^t \}$ satisfies the strong homogeneous axiom of the revealed preference theory, i.e. for any ordered tuple $\{t(1), \ldots, t(k)\} \subseteq \{0, \ldots, T\}$ the inequality holds:

$$
(p^{t(1)} q^{t(2)}) (p^{t(2)} q^{t(3)}) \cdots (p^{t(k)} q^{t(1)}) \geq (p^{t(1)} q^{t(1)}) (p^{t(2)} q^{t(2)}) \cdots (p^{t(k)} q^{t(k)}).
$$

3. The following linear system is consistent

$$
\lambda_t (p^t q^t) \leq \lambda_t (p^t q^t), \ \lambda_t > 0, t, \tau = 1, \ldots, T. \tag{1}
$$

One of the particular utility functions can be recovered from an arbitrary solution of (1) by the formula

$$
Q(q) = \min_{t = 1, \ldots, t} \lambda_t p^t q. \tag{2}
$$

The superdifferential $\partial Q(q)$ at any point $q \in \mathbb{R}^n$ consists of the convex hull of all (scaled) minimizing prices $\partial Q(q) = \text{conv}_{u \in U} \{ \lambda_u p^u \}$, where $u \in U \subseteq \{1, \ldots, t\} \iff \lambda_u p^u q = Q(q)$.

Note that the utility function is not uniquely specified by the consistency of the system (1).

We need the following definition.

**Definition 1** Let $Q(\cdot)$ be an arbitrary function from $F$. The Lebesque set $\chi_Q = \{ x \in \mathbb{R}^n | P(x) \geq 1 \}$ is called the characteristic set of $Q(\cdot)$.

Denote by $\mathcal{X}$ the set family of all characteristic sets of functions from the class $F$.

Next simple lemma summarizes some properties of the characteristic sets.

**Lemma 1 (Characteristic Set)**

1. $\chi \in \mathcal{X}$ if $\chi$ is a closed convex set in $\mathbb{R}^n$ satisfying hereditary intersection property with any positive ray, i.e. for any positive ray $r_x = \{ \lambda x | x \in \mathbb{R}^n, x > 0, \lambda \geq 0 \}$ there exists a point $x_0 \in r_x$ such that $r_x \cap \chi = \{ \lambda x_0, \lambda \geq 1 \}$.

2. Any $f \in F$ can be uniquely restored from $\chi_f$ and vice versa. In other words, there is a canonical bijection between $\mathcal{F}$ and $\mathcal{X}$.

3. If $f \in F$ is k-smooth (i.e. $f \in C^k(\mathbb{R}^n_+)$) iff $\chi_f$ has k-smooth field of tangential supporting hyperplanes.

Let define the gauge transform of $Q(\cdot)$:

$$
\mathcal{P}(p) = \inf_{q \geq 0} \frac{qp}{Q(q)}.
$$

As $Q(\cdot) \in \mathcal{F}$ then the gauge transform $\mathcal{P}(\cdot)$ also belongs to class $F$ and forms some dual index of prices. By (1, 2)

$$
Q(q) = \inf_{p \geq 0} \frac{qp}{\mathcal{P}(p)}.
$$

Moreover, next lemma shows the correspondence between the level sets of the dual gauges $Q(q)$ and $\mathcal{P}(p)$.

**Lemma 2** Let $Q(q) = 1$ (i.e. $q \in Bd(\chi_Q)$). If $p \in \partial Q(q)$ then $\mathcal{P}(p) = 1$ (i.e. $p \in Bd(\chi_P)$ and $q \in \partial \mathcal{P}(p)$). In terminology of [3] the sets $\chi_Q$ and $\chi_P$ form a blocking pair.

**Proof.** For any $p \in \partial Q(q)$ and every $\tilde{q}$ the inequality holds $p(q - \tilde{q}) \geq Q(q) - Q(\tilde{q})$. Let $\tilde{q} = \lambda q$, where $\lambda > 0$. Then $(\lambda - 1)p q \geq (\lambda - 1)Q(q)$ for every $\lambda > 0$. So we have $p q = Q(q) = 1$. (The Euler identity for homogeneous but non-smooth $Q(\cdot)$.) It follows from Kuhn-Tukker theorem that if $p \in \partial Q(q)$ then $q \in \text{Argmax} \{ Q(q) | p q \leq p q \}$. Thus $\mathcal{P}(p) = \inf_{q \geq 0} \frac{qp}{Q(q)} = pq = 1$. We have from duality of the gauge transform that $Q(q) = \inf_{p \geq 0} \frac{pq}{\mathcal{P}(p)} = 1$. So for every $\tilde{p} \in R^+\_n$ the inequality $q \tilde{p} \geq \mathcal{P}(\tilde{p})$ holds. Then for every $\tilde{p} \in R^+\_n$ we obtain $q(p - \tilde{p}) \geq \mathcal{P}(\tilde{p}) - \mathcal{P}(p)$, i.e. $q \in \partial \mathcal{P}(p)$.

It is shown in [3] that the list of the statements equivalent to the Afriat-Varian theorem can be enlarged by the following propositions.

4. Such $\mathcal{P} \in \mathcal{F}$ exists that $p^t \in \text{Argmax} \{ \mathcal{P}(p) | (p^t q) \leq (p^t q^t), p \geq 0 \}$ $t = 1, \ldots, T$.

5. The following linear system is consistent

$$
\mu_t (p^t q^t) \leq \mu_t (p^t q^t), \ \mu_t > 0, t, \tau = 1, \ldots, T. \tag{3}
$$

**Definition 2** Set $Q \in \mathcal{F}_k$ (k ≥ 1) if Q ∈ F and additionally both Q and the dual to Q gauge $\mathcal{P}$ belong to $C^k(\mathbb{R}^n_+)$ (i.e. are k-smooth).

**Theorem 1 (Smooth Afriat-Varian theorem)** The following statements are equivalent.
1. The market statistics $S = \{p^t, q^t\}$ is rationalizable in the class of the utility functions $\mathcal{F}_k$.

2. The market statistics $S = \{p^t, q^t\}$ satisfies the strong homogeneous axiom of the revealed preference theory, i.e. for any ordered tuple $\{t(1), \ldots, t(k)\} \subseteq \{0, \ldots, T\}$ the inequality holds:

$$
\left(p^{t(1)} q^{t(2)} (p^{t(2)} q^{t(3)}) \ldots (p^{t(k)} q^{t(1)}) > \right. 
\left( p^{t(1)} q^{t(2)} p^{t(2)} q^{t(3)} \ldots (p^{t(k)} q^{t(k)}) \right).
$$

3. The following linear system is (strictly) consistent

$$
\lambda_t (p^t q^t), \lambda_t > 0, t, \tau = 1, \ldots, T. \quad (3)
$$

4. Such $P \in F$ exists that

$$p^t \in \text{Argmax} \{P(p) \mid (q^t p) \leq (q^t p^t), p \geq 0 \} \quad t = 1, \ldots, T.
$$

5. The following linear system is (strictly) consistent

$$
\mu_t (p^t q^t) < \mu_t (p^t q^t), \mu_t > 0, t, \tau = 1, \ldots, T.
$$

**Proof.** In fact, it is enough to prove equivalence of the first and the third statement (all other implications are established by more or less standard arguments in the framework of the ordinary Afriat-Varian theorem).

1. $\Rightarrow$ (3). If market statistics is rationalizable in the class $\mathcal{F}_k$ then both dual gauges $\mathcal{Q}(\cdot)$ and $\mathcal{P}(\cdot)$ are smooth and their supergradients at any points are ordinary gradients (i.e. each supergradient consists of a single vector only). Therefore, by Lemma 2, the strict solution of the system (3) is given by $\lambda_t = \|\text{grad}(\mathcal{Q}(q^t))\|$, $t = 1, \ldots, T$.

3. $\Rightarrow$ (1). Technically, the proof consists in smoothing of the piecewise-linear utility function that is obtained from some solution of the system (3) via suitable convolutions.

Next Lemma is the main technical tool for smoothing.

**Definition 3** Let $K \subseteq \mathbb{R}^n$ be a closed convex set and let $n(y), \|n(y)\| = 1$ be any unit normal vector at point $y \in \text{Bd}(K)$, i.e. the inclusion $K \subseteq \{x \in \mathbb{R}^n \mid \langle n(y), y - x \rangle \leq 0\}$ holds. Define the quasi indicator function of $K$ as follows:

$$
\text{q-ind}_{K}(x) = \left\{ 
\begin{array}{ll}
1, & x \in K \\
\inf_{y \in \text{Bd}(K)} \{1 - \langle n(y), x - y \rangle\}, & \text{otherwise}
\end{array}
\right.
$$

By construction $\text{q-ind}_{K}(\cdot)$ is concave.

**Lemma 3 (Smoothing)** Let $h(\cdot) \in C^k_{\text{loc}}(\mathbb{R}^n)$ be any nonnegative spherically symmetric with respect to the origin “cap”-function that vanishes outside some $\varepsilon$-ball $B(0, \varepsilon)$ and such that $\int_\mathbb{R}^n h(x)dx = 1$ (there are plenty of such “caps” even in the $C^\infty(\mathbb{R}^n)$-class). Let $H = \{x \in \mathbb{R}^n \mid \langle a, x \rangle \leq a, \alpha \neq 0\}$ be any halfspace through the origin.

Set $\alpha = \int_\mathbb{R}^n \text{q-ind}_{H}(x-c)h(c)dc$. Then

(i) The convolution $f = \text{q-ind}_{K} * h = \int_{\mathbb{R}^n} \text{q-ind}_{K}(x-c)h(c)dc$ is a concave function of the class $C^k(\mathbb{R}^n)$.

(ii) The following inclusion holds for the Lebesque set $\chi_{fa} = \{x \in \mathbb{R}^n \mid f(x) \geq \alpha\} \subseteq K$, in particular, $\chi_{fa} = K$ if $K$ is any halfspace.

(iii) Let $x \in \text{Bd}(K)$ be any boundary point of $K$ and let the boundary $\text{Bd}(K)$ be sufficiently locally flat at $x$, i.e. $B(x, \varepsilon) \cap K = H_x \cap K$ (here $H_x$ is a supporting halfspace to $K$ at $x$) then $f(x) = \alpha$.

(iv) If $K \subseteq \mathcal{X}$ is a characteristic set (of some function from $\mathcal{F}$) then $\chi_{fa}$ is also a characteristic set from $\mathcal{X}$ (that corresponds to some function from $\mathcal{F}$).

**Proof.** (i) follows directly from the properties of convolution (actually it is valid for an arbitrary locally integrable function).

(ii) and (iii) are reduced to an easy check.

(iv) follows from (ii) and from the observation that for $K \subseteq \mathcal{X}$ all far enough from the origin points of any positive ray ultimately fall inside $K$ and are sufficiently far from the boundary of $K$.

It is worth restating here the aforementioned condition for the existence of the utility function in terms of the characteristic sets. Namely, the utility function for a given market statistics $S = \{p^t, q^t\}, p^t, q^t \in \mathbb{R}^+_{i} \cup \{i = 1, \ldots, t\}$ exists if there exists a set $\chi \subseteq \mathcal{X}$, such that

$$p^t \in N_{\chi}(q^t), i = 1, \ldots, t. \quad (4)$$

Here $\hat{q}^t$ is a (unique) point of intersection of the set $\text{Bd}(\chi)$ and the ray $r_{q^t} = \{\lambda q^t, \lambda \geq 0\}, i = 1, \ldots, t$ and $N_{\chi}(q^t)$ is a corresponding normal cone to $\chi$ at point $\hat{q}^t$. In particular, the consistent system (3) defines a polyhedral characteristic set $\chi$ (actually, the consistency of (3) is a restatement of the condition (4)). Thus, in order to smooth a polyhedral utility function it is sufficient to smooth the corresponding polyhedral characteristic set preserving the inclusions (4).

Now recall that $\mathcal{P}(\cdot)$ is smooth if and only if the characteristic set $\chi_{\mathcal{P}}$ has smooth boundary without flat regions, i.e. the set $\chi_{\mathcal{P}}$ should have strictly convex boundary.

As above let $\mathcal{Q}(\cdot)$ be the polyhedral utility function corresponding to the strict solution of (3) given by Lemma 3. Let $\chi_{\mathcal{Q}}$ be the polyhedral characteristic set of $\mathcal{Q}(\cdot)$. Smoothing by convolution is not enough for our purposes as it preserves some flat regions of $\chi_{\mathcal{Q}}$. To overcome this difficulty we will slightly change polyhedral $\chi_{\mathcal{P}}$ replacing its “flat” facets by curved “spherical” facets preserving the supergradient inclusions (4).

By construction any facet of $\chi_{\mathcal{Q}}$ is intersected in its relative interior by a unique ray $r_{q^t} = \{x \in \mathbb{R}^+_i \mid x = \lambda q^t, \lambda \geq 0\}, i = 1, \ldots, t$. Now take the system of balls with equal radii $B_i = B(s_i + \rho p_i, \rho), i = 1, \ldots, t$. The radius $\rho$ should be chosen large enough to satisfy the following conditions:

1. any ball $B_i$ is intersected by any nonnegative ray;

2. for all $i = 1, \ldots, t$ all points $s_j, i \neq j, j = 1, \ldots, t$ should fall inside the ball $B_i$.

Set $\chi_{\mathcal{Q}} = (\cap_{i=1, \ldots, t} B_i \cap \chi_{\mathcal{Q}}) \oplus \mathbb{R}^+_{i}$. 

3
Namely, to choose \( \varepsilon \) we require that for all \( i = 1, \ldots, t \), \( B(s_i, \varepsilon) \cap B_i = B(s_i, \varepsilon) \cap \tilde{\chi}_Q \).

Generally, call a boundary point \( x \) of \( \tilde{\chi}_Q \) belonging to the boundary of some \( B_i \) \( \varepsilon \)-round if \( B(x, \varepsilon) \cap B_i = B(x, \varepsilon) \cap \tilde{\chi}_Q \) (thus our requirement for the smallness of \( \varepsilon \) means that all points \( s_i \) should be \( \varepsilon \)-round).

Let \( \beta = q \text{ind}_{B_i} * h(s_i + x) \). By our assumptions \( \beta \) is a constant for all \( i \). Moreover, by construction this equality holds if we take any point from the set \( U_{s_i} \) of all boundary points of \( \tilde{\chi}_Q \) in some neighborhood of \( s_i \) (as all points from \( U_{s_i} \) are \( \varepsilon \)-round).

As above, set \( \tilde{\chi}_Q = \chi_{\phi_{\beta}} = \{ x \in \mathbb{R}^n \mid \varphi(x) \geq \beta \} \), where \( \varphi(\cdot) \) is a convolution of the quasi indicator function of \( \tilde{\chi}_Q \) and \( h(\cdot) \). By the reasoning analogous to the smoothing lemma \( \tilde{\chi}_Q \subset \tilde{\chi}_Q \) and \( \tilde{\chi}_Q \in X \). Moreover, for all \( i = 1, \ldots, t \) all \( \varepsilon \)-round points from \( U_{s_i} \) (including \( s_i \)) belong to the boundary of \( \tilde{\chi}_Q \) so that supergradient inclusions (1) hold for \( \tilde{\chi}_Q \).

Thus smooth by lemma 1 index of goods \( \hat{Q}(\cdot) \) corresponding to \( \tilde{\chi}_Q \) has a smooth dual index of prices \( P(\cdot) \) both consistent with the given statistics.

Note that we have proved only \( C^1 \)-smoothness of the indices involved. \( C^k \)-smoothness is established by the following argument.

Denote by \( \tilde{\chi}_P \) the characteristic set of the smooth by construction dual index of prices \( P(\cdot) \). Lemma 2 states that the image of \( C^k \)-smooth boundary of the characteristic set \( B(d(\tilde{\chi}_Q)) \) is mapped under the gradient map \( q \to \partial Q \) into the boundary \( B(d(\tilde{\chi}_P)) \) of the characteristic set of the corresponding dual (and \( C^1 \)-smooth by construction) index of prices. (And moreover this map is one-to-one.) Let prove that the map \( q \to \partial Q \) has nonzero Jacobian on the boundary surface \( B(d(\tilde{\chi}_Q)) \). Then \( C^k \)-smoothness of \( P(\cdot) \) immediately follows from the implicit function theorem. At first, note that the corresponding Weingarten map is invertible on the convex surface \( B(d(\tilde{\chi}_Q)) \) (Weingarten map surely is invertible on the spherical patches and hence on the surface obtained after applying convolution with nonnegative function, i.e. on \( B(d(\tilde{\chi}_Q)) \) itself.) Moreover, it follows from the Euler identity \( q \partial Q = 1 \) that restricted to the surface \( B(d(\tilde{\chi}_Q)) \) our map \( q \to \partial Q \) can be obtained by Weingarten map by some smooth scaling of the normal vector.

**Remark 1** \( C^1 \)-smoothness of the resulting utility function could be obtained by a simple geometric construction. Set \( \tilde{\chi} = (\chi_Q \oplus \varepsilon B) \cap \mathbb{R}_+^n \), where positive \( \varepsilon \) is small enough. By construction \( \tilde{\chi} \in X \) and moreover \( \tilde{\chi} \) has \( C^1 \)-smooth boundary.

The last assertion could be proved as follows. The set \( K \oplus \varepsilon B \) is usually called the outer parallel set of a convex set \( K \). It consists of all points within Euclidean distance \( \varepsilon \) from the set \( K \). Now the fact that \( \varepsilon \)-neighborhood of any closed convex set has \( C^1 \)-smooth boundary surely belongs to the public mind but we were unable to find it in the literature. **We’ll be grateful for pointing any relevant references.** For completeness sake we sketch a short geometric proof of the statement, which was independently communicated to us by M.Arslanov and S.Chukanov. Namely, any point \( x \in Bd(K) \oplus \varepsilon B \) is also a boundary point of a ball \( B(y, \varepsilon) \subseteq K \) centered at some (boundary) point \( y \in K \). The supporting hyperplane to \( K \oplus \varepsilon B \) at \( x \) coincides with the supporting hyperplane to \( B(y, \varepsilon) \) at \( x \) and is thus unique. Hence, the boundary of \( K \oplus \varepsilon B \) is \( C^1 \)-smooth.

**Acknowledgements.**

We are grateful to A.Kitaev for proposing to use convolutions in smoothing procedures and to M.Arslanov and S.Chukanov for communicating a short proof of the smoothness of an outer parallel convex body.

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