Time reversing procedure of SLE and 2d gravity

Yoshiki Fukusumi

Institute for Solid State Physics, University of Tokyo
Kashiwa, Chiba 277-8581, Japan

November 2, 2017

Abstract

We analyse the time reversing procedure of Schramm Loewner evolutions and its relation to Liouville field theory or 2d pure gravity. We can get martingale observables by the calculation of LFT correlation functions without matter.

1 Introduction

There are a wide variety of connections between Schramm Loewner evolutions and conformal field theories. One of the most direct and physical methods to construct SLE and CFT correspondence is the construction of martingale observables whose expectation values coincide with vertex operator in CFT. \[1\] \[2\] \[3\] This martingale condition ensures CFT calculations of SLE probabilities like left passive probability or connectivity weight. Beginning from the work by D. Bernard and M. Bauer, W. Wernar and R. Friedrich, there are a lot of variant of that connection. Recently generalized SLEs are constructed by field theoretic argument and a lot of them are consistent with Monte Carlo simulation of 2d lattice models \[4\].

Moreover by the series of works by Duplantier and Sheffield, it was widely recognized that SLE martingale is closely related to the random measure of 2d gravity. \[5\]. They constructed random measure of 2d gravity by SLE on free boundary condition. It is different from usual SLE/CFT correspondence in that SLE describes interface or boundary condition changing operator. And series of their papers, they showed conformal invariance of such one point functions.

In this paper, we show Liouville field theory multipoint functions are constructed by the martingale distribution function of the time reversed SLE. Our formulation is based on previous work of Duplantier and Sheffield, but we assume reverse flow of SLE should be treated as chordal version of full plane SLE. If our formulation are valid, we can get boundary Liouville field theory as martingale distribution functions of generalized SLE.

\[1\] E-mail: y.fukusumi@issp.u-tokyo.ac.jp
2 Time reversed SLE

What we call time reversed SLE is the same as the reverse SLE considered in 2d gravity coupled with matter. It is defined by time reversal of SLE and related to reversability of SLE. First we introduce the chordal SLE \( g_t(z) \).

\[
dg_t(z) = \frac{2dt}{g_t(z) - \xi_t}, \quad \xi_t = \sqrt{\kappa} B_t \tag{2.1}
\]

and it satisfies the initial conditions \( g_0(z) = z \). For the later discussion, the property of tip of SLE curve is important i.e. \( g_t(\gamma_t) = \xi_t \).

For introducing reverse time SLE, we think about the SLE process from \( t = 0 \) to \( t = T \). Then we can obtain the data of \( (B_t)_{t=0}^T \). In this situation, one can define time reversed Loewner map,

\[
dg_{T-t}(z) = \frac{-2dt}{g_{T-t}(z) - \xi_{T-t}}. \tag{2.2}
\]

For convenience I omitted the time \( T \) and transform \( t \rightarrow -t \). This process is defined by the condition \( (B_t)_{t=0}^T \). This \( B_t \) is not stochastic object in this stage but the consequence of Brownian motion and it is fractal object. So we can assume,

\[
(dB_t)^2 = dt, \text{ a.e.} \tag{2.3}
\]

Therefore we define time reversed SLE,

\[
dg_t(z) = -\frac{-2dt}{g_t(z) - \xi_t}. \tag{2.4}
\]

with

In the previous paper by Duplantier, the radial stochastic version of this theory are confirmed to become a uniformizing map \([6]\). However the chordal version of this SLE is a map \( H \rightarrow H \setminus K_t \). In the section 4, we show this map has the Liouville theory correlation function as the martingale observable.

3 Very short review of the generalalized Liouville Field theory

In this section, we introduce the generalized Liouville field theory which includes minimal matter CFT and Liouville field theory. In general we can define these CFT by Hilbert space and OPE or commutation relation of operators. (For complete discussion, please see the paper by Ribault \([8]\) and its reference) In SLE sense, it naturally generates martingale observables.

For convenience the generalized minimal model is given by the energy momentum tensor in coulomb gas form,

\[
T(z) = \partial^2 \phi + Q\phi \tag{3.1}
\]

\( Q \) is a complex number. The theory is called Liouville field theory when \( Q \) is a real number, and the theory is called (minimal) matter CFT when \( Q \) is a pure imaginary. The central charge and conformal dimension of the theory is given by,

\[
c = 1 + 6Q^2, \quad Q = b + 1/b. \tag{3.2}
\]
This parameter $b$ is a complex number which determines the theory. The theory becomes matter CFT if $b$ is pure imaginary, and becomes pure gravity or Liouville field Theory if $b$ is real. For calculation of correlation functions, degenerate fields are useful. In bosonisation or Wakimoto free field representation scheme, each contour of screening field corresponds to the solution of the differential equation derived from degenerate conditions. Hence the conformal dimensions of degenerate fields are given by

$$h_{(r,s)} = \alpha_{(r,s)}(Q - \alpha_{(r,s)}), \quad \alpha_{(r,s)} = \frac{Q}{2} - \frac{1}{2}(br + b^{-1}s)$$  \hspace{1cm} (3.3)

For later convenience we introduce the level 2 null vectors of generalized minimal models.

$$\left(b^2L^2_{-1} + L_{-2}\right)\phi_{(1,2)} = \left(\frac{1}{b^2}L^2_{-1} + L_{-2}\right)\phi_{(2,1)} = 0$$ \hspace{1cm} (3.4)

What is important for later discussion is that the sign of $b^2$ changes by the transformation $b' = ib$ and it changes central charge $c' = 26 - c$. This dual transformation relates pure gravity and matter CFT, which can couple.

If we assume mode expansion of energy momentum tensor $T(z) = \sum_n L_n z^{n+2}$, and operator expansion of fields,

$$\left(b^2\partial^2_z + \sum_i \left(\frac{\delta_{z_i}}{(z_i - z)^2} - \frac{\partial_{z_i}}{z_i - z}\right)\right)\langle \Pi_i X(z_i)\phi_{(1,2)}(z) \rangle = 0,$$ \hspace{1cm} (3.5)

with $\Pi_i X(z_i) = \Pi_i \phi_{\delta_i}(z_i)$.

### 4 Martingale observable of reverse time SLE

In this section, I show the correlation functions of Liouville field theory are consistent with the time reversed SLE.

First we define observables. We define the primary fields with conformal dimension $h$,

$$\phi_h(z) = (g'(z))^{h} \phi_h(g_t(z)).$$ \hspace{1cm} (4.1)

Moreover we assume interfaces are described by boundary condition changing operator $\psi$. In this stage, we don’t assume its conformal dimension of this operator. What should be done is detecting the conformal dimension of this b.c.c. operator on each case. We consider following observable on $H$ in analogy with usual SLE,

$$F(y_\alpha, \xi, t) = \prod_{\alpha} \langle \Pi_{\alpha} \phi_{h_\alpha}(g_t(y_\alpha))\psi(\infty)\psi(\xi) \rangle_H / \langle \psi(\infty)\psi(\xi) \rangle_H.$$ \hspace{1cm} (4.2)

For this curve goes to $\infty$, we took boundary condition changing operator at $\infty$. Unfortunately, $g_t$ maps $H$ to $H \setminus K_t$, usual SLE CFT correspondence is not valid for chordal SLE case. However, in radial SLE, such difficulty does not appear.

Then we derive derivative of primary fields,

$$d\phi_{h_\alpha}(y_\alpha) = -2dt(g'_t(y_\alpha))^{h_\alpha} \left(\frac{\partial_{y_\alpha}}{g_t(y_\alpha) - \xi_t} - \frac{\delta_{y_\alpha}}{(g_t(y_\alpha) - \xi_t)^2}\right)\phi_{h_\alpha}(g_t(y_\alpha)).$$ \hspace{1cm} (4.3)
And derivative of boundary field is,
\[ d\psi(\xi_t) = d\xi_t \psi'(\xi_t) + \frac{\kappa}{2} dt \psi''(\xi_t). \] (4.4)

Therefore the total derivative of the observable is,
\[
\prod_\alpha g_t(y_\alpha)^{-h_\alpha} dF(y_\alpha, \xi_t, t) = \left( dt \left( \frac{\kappa}{2} \frac{\partial^2}{\partial \xi_t^2} - 2 \sum_\alpha \left( \frac{\partial g_t(y_\alpha)}{g_t(y_\alpha) - \xi_t} - \frac{\delta g_t(y_\alpha)}{(g_t(y_\alpha) - \xi_t)^2} \right) \right) + d\xi_t \partial \xi_t \right) \\
\left( \prod_\alpha \phi_{h_\alpha}(g_t(y_\alpha)) \psi(\xi_t) \psi(-\infty) \right).
\] (4.5)

Therefore the martingale condition is nothing but null vector equation of Liouville gravity in the previous section with \( b^2 = \kappa/4, \psi = \phi_{(1,2)}. \) If there is no other freedom of Brownian motion like ones on Lie group manifold [7], we can get the time reversed SLE observables by calculating Liouville field theory correlation function.

5 An explicit form of maritingale observable

As we saw in the previous sections we can get the martingale observables in the form of Liouville Field theory correlation functions. In this section, we give the most simplest observables,

\[
\langle \phi_{(1,3)}(z) \rangle_H = (g_t'(z))^{-1 - \frac{\kappa}{4}} (g_t(z) - \xi_t)^{-1 - \frac{\kappa}{4}}
\] (5.1)

It is the almost same form of observable which is in the paper by Duplantier and Sheffield.

What Duplantier and Sheffield was proved was following,

\[
(D, \phi)_H = (\psi(D), \phi)_H_{K_t}.
\] (5.2)

Their right hand side should be calculated by free boundary condition on \( H. \)

On the other hand our formulation is consistent with BCFT picture.

6 Some conjectures

At least radial SLE case, there exists whole plane SLE which corresponds to our reverse SLE interpretation, in which the curve goes to outside of domain[3] and map the tip to boundary Brownian motion. Hence we expect we can get martingale observable of whole plane SLE as the correlation functions of Liouville field theory.

Based on this observation, we propose two processes related to 2d gravity coupled with matter. One may correspond to the matter part and the other may correspond to the gravity part, after the coupling of the theory. The matter part is described by following map \( h_t \circ g_t : H \setminus K_{t,2} \to H \setminus K_{t,1}. \)

\[
dh_t(z) = \frac{-2 dt}{h_t(z) - \xi_{t,1}}, \quad dh_t(z) = \frac{-2 dt}{h_t(z) - \xi_{t,1}}.
\] (6.1)

\[
dg_t(z) = \frac{2 dt}{g_t(z) - \xi_{t,2}}, \quad dg_t(z) = \frac{2 dt}{g_t(z) - \xi_{t,2}}.
\] (6.2)
with $h_t : H \to H \setminus K_{t,1}$, $g_t : H \setminus K_{t,2} \to H$, and initial condition $g_0 = h_0 = id$ and the tips are given by $g_t(\gamma_{t,1}) = \xi_{t,1}$, $\xi_{t,1}$ and $\xi_{t,2}$ are Brownian motions.

Moreover, we can get multiple time reversed SLE by the almost same procedure for usual multiple SLE. In the proceeding paper, we will treat that generalization.

Finally, if we assume this time reversing procedure changes the matter to the corresponding gravity, we can get corresponding gravity for usual Wess-Zumino-Witten models. For example, it may be interesting to check coupling condition for $SU(2)_k$ WZW model\[9\]. It is known this model can couple to $SL(2,R)$ WZW model with $c_{SU(2)} + c_{SL(2)} = 6$\[10\]. Moreover, although SLE for $Z_n$ parafermion is constructed by R. Santachiara and M. Picco \[12, 13, 14\], the gravity which can couple to $Z_n$ parafermion is not known. Therefore it may be interesting to check the null vector condition for $SL(2,R)_k/U(1)_k$ gravity.

7 Conclusion

Surprisingly, we can get the martingale observable of time reversed SLE by calculating the correlation functions of Liouville field theory or 2d pure gravity. That means we can get pure 2d gravity CFT by calculating distribution function of SLE. Inversely, Liouville field theory is consistent with martingale or heat equation of SLE.

We hope our formulation fulfills the gap between SLE on minimal and reverse SLE on 2d gravity coupled to minimal matter. In the forth coming paper we will try to construct multiple reverse SLE on Liouville field theory\[3\].

Just before the end of this work, I noticed the work which is confirming my observation for radial case \[17\].

8 Acknowledgement

First I would like to thank Sylvain Ribault for the beautiful lecture on ”Quantum integrable systems conformal field theories and stochastic processes” at Cargese. I also thank Kazumitsu Sakai for helpful comment and Yuji Tachikawa and Masaki Oshikawa for discussion and comments.

A Radial case

In this section, we will show whole plane SLE $g_t$ satisfies martingale for Liouville field theory. The monotonicity of the map corresponding to $g_t^{-1}$ has already shown. Therefore we assume $g_t$ as a map to unit disc. The same procedure was already done by O. Alekseev in the context of Laplacian growth and Liouville gravity.

First we transform the whole plane SLE as a map to upper half plane. The same procedure was done in radial SLE. The stochastic equation becomes,

$$\frac{dg_t(z)}{g_t(z)} = \frac{1}{2} \frac{2 + \eta g_t(z)}{g_t(z) - \eta_t} dt, \quad \eta_t = \tan \xi_t \tag{A.1}$$

We heavily use the group theroretic discussion of M. Bauer and D. Bernard \[15\].
\[ H_t^{-1}dH_t = dt \left( 2W_{-2} + \frac{\kappa}{2}W_{-1}^2 \right) + d\xi_t W_{-1}, \quad \text{(A.2)} \]

\[ W_{-1} = \frac{1}{2}(L_{-1} + L_1), \quad W_{-2} = \frac{1}{4}(L_0 + L_{-2}). \quad \text{(A.3)} \]

Then by the stochastic Ito calculus, we get the martingale observable,

\[ e^{-\hbar_r t}H_t\phi_{(1,2)}, h_r = \frac{(2 + \kappa)(6 + \kappa)}{8\kappa}. \quad \text{(A.4)} \]

This condition can be calculated by

\[ \left( 2W_{-2} + \frac{\kappa}{2}W_{-1}^2 \right) \phi_{(1,2)} = \frac{(2 + \kappa)(6 + \kappa)}{8\kappa} \phi_{(1,2)}. \quad \text{(A.5)} \]

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