Group Analysis of Differential Equations and Generalized Functions

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Abstract. We present an extension of the methods of classical Lie group analysis of differential equations to equations involving generalized functions (in particular: distributions). A suitable framework for such a generalization is provided by Colombeau's theory of algebras of generalized functions. We show that under some mild conditions on the differential equations, symmetries of classical solutions remain symmetries for generalized solutions. Moreover, we introduce a generalization of the infinitesimal methods of group analysis that allows to compute symmetries of linear and nonlinear differential equations containing generalized function terms. Thereby, the group generators and group actions may be given by generalized functions themselves.

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1 Introduction

Symmetry properties of distributions and group invariant distributional solutions (in particular: fundamental solutions) to particular types of linear differential operators have been studied by Methée ([15]), Tengstrand ([27]), Szmydt and Ziemian ([24, 25, 26, 28]). A systematic investigation of the transfer of classical

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group analysis of differential equations into a distributional setting is due to Berest and Ibragimov ([2, 3, 4, 5], [11]), again with a view to determining fundamental solutions of certain linear partial differential equations. A survey of the lastnamed studies including a comprehensive bibliography can be found in the third volume of [12]. As these approaches use methods from classical distribution theory, their range is confined to linear equations and linear transformations of the dependent variables.

Algebras of generalized functions offer the possibility of going beyond these limitations towards a generalization of group analysis to genuinely nonlinear problems involving singular terms, like distributions or discontinuous nonlinearities. In the present paper we develop a theory of group analysis of differential equations in algebras of generalized functions that allows a satisfactory treatment of such problems. This line of research has been initiated in [20] and has been taken up in [14]. Applications to different types of algebras of generalized functions can be found in [22] and [23].

The plan of the paper is as follows: In section 2 we consider systems of partial differential equations together with a classical symmetry group $G$ that transforms smooth solutions into smooth solutions. Assuming polynomial bounds on the action of $G$, we can extend it to generalized functions belonging to Colombeau algebras and ask whether $G$ remains a symmetry group for generalized solutions. In section 2.1 we develop methods based on a factorization property of the transformed system of equations. Essentially, polynomial bounds on the factors suffice to give a positive answer. In the scalar case we show this to be automatically satisfied whenever the equation contains at least one of the derivatives of the solution as an isolated term. While the conditions of section 2.1 concern some mild assumptions on the algebraic structure of the equations, section 2.2 develops a topological criterion, applicable to systems of linear equations: the existence of a $C^\infty$-continuous homogeneous right inverse guarantees a positive answer as well. Along the way we give examples of nonlinear symmetry transformations of shock and delta wave solutions to linear and nonlinear systems.

The purpose of section 3 is to develop the general theory, allowing the equations and the group action (hence also its generators) to be given by generalized functions. Using the characterization of Colombeau generalized functions by their generalized pointvalues established in [19] as well as results on Colombeau solutions to ODEs, we show that the classical procedure for computing symmetries can be literally transferred to the generalized function situation. The defining equations are derived as usual, but their solutions are sought in generalized functions. This enlarges the reservoir of possible symmetries of classical equations and allows the study of symmetries of equations with singular terms. An example is provided by a conservation law with discontinuous flux function.

The remainder of the introduction is devoted to fixing the notation and recalling some basic definitions. Concerning symmetries of differential equations, we follow the notations and terminology of [21]. Thus for the action of a Lie group $G$
on some manifold $M$, assumed to be an open subset of some space $\mathcal{X} \times \mathcal{U}$ of independent and dependent variables (with $\dim(\mathcal{X}) = p$ and $\dim(\mathcal{U}) = q$) we write $g \cdot (x, u) = (\Xi_g(x, u), \Phi_g(x, u))$. Transformation groups are always supposed to act regularly on $M$. If $\Xi_g$ does not depend on $u$, the group action is called projectable. Elements of the Lie algebra $\mathfrak{g}$ of $G$ as well as the corresponding vector fields on $M$ will typically be denoted by $\mathbf{v}$ and the one-parameter subgroup generated by $\mathbf{v}$ by $\eta \rightarrow \exp(\eta \mathbf{v})$. $M^{(n)}$ denotes the $n$-jet space of $M$; the $n$-th prolongation of a group action $g$ or vector field $\mathbf{v}$ is written as $\text{pr}^{(n)}g$ or $\text{pr}^{(n)}\mathbf{v}$, respectively. Any system $S$ of $n$-th order differential equations in $p$ dependent and $q$ independent variables can be written in the form

$$\Delta_{\nu}(x, u^{(n)}) = 0, \quad 1 \leq \nu \leq l,$$

where the map

$$\Delta : \mathcal{X} \times \mathcal{U}^{(n)} \rightarrow \mathbb{R}^l$$

$$(x, u^{(n)}) \rightarrow (\Delta_1(x, u^{(n)}), \ldots, \Delta_l(x, u^{(n)}))$$

will be supposed to be smooth. Hence $S$ is identified with the subvariety

$$S_\Delta = \{(x, u^{(n)}): \Delta(x, u^{(n)}) = 0\}$$

of $\mathcal{X} \times \mathcal{U}^{(n)}$. For any $f : \Omega \subseteq \mathcal{X} \rightarrow \mathcal{U}$, $\Gamma_f$ is the graph of $f$ and $\Gamma_f^{(n)} := \{(x, \text{pr}^{(n)} f(x)): x \in \Omega\}$ is the graph of the $n$-jet of $f$.

Modelling of generalized functions will be carried out in the so-called 'special version' of Colombeau's algebras of generalized functions (cf. [1], [6], [7], [13]) whose definition we shortly recall: Let $\Omega \subseteq \mathbb{R}^n$ be open, $I = (0, \infty)$ and set $\mathcal{G}(\Omega) = \mathcal{E}_\mathcal{M}(\Omega)/\mathcal{N}(\Omega)$, where

$$\mathcal{E}_\mathcal{M}(\Omega) := \{(u_z)_z \in C^\infty(\Omega)^I : \forall K \subset \subset \Omega, \forall \alpha \in \mathbb{N}^n_0, \exists p \in \mathbb{N} \ \text{with} \ \sup_{x \in K} |\partial^\alpha u_z(x)| = O(\varepsilon^{-p}) \ \text{as} \ \varepsilon \rightarrow 0\}$$

$$\mathcal{N}(\Omega) := \{(u_z)_z \in C^\infty(\Omega)^I : \forall K \subset \subset \Omega, \forall \alpha \in \mathbb{N}^n_0, \forall q \in \mathbb{N} \ \sup_{x \in K} |\partial^\alpha u_z(x)| = O(\varepsilon^q) \ \text{as} \ \varepsilon \rightarrow 0\}.$$
\[ E_r(\Omega) = \{(u_x)_x \in (OM(\Omega))^t : \forall \alpha \in \mathbb{N}_0^n \exists p > 0 \]
\[ \sup_{x \in \Omega} (1 + |x|)^{-p} |\partial^\alpha u_x(x)| = O(\varepsilon^{-p}) \ (\varepsilon \to 0) \} \]

\[ N_r(\Omega) = \{(u_x)_x \in (OM(\Omega))^t : \forall \alpha \in \mathbb{N}_o^n \exists p > 0 \forall q > 0 \]
\[ \sup_{x \in \Omega} (1 + |x|)^{-p} |\partial^\alpha u_x(x)| = O(\varepsilon^q) \ (\varepsilon \to 0) \} \]

The map \( \iota \) defined above is a linear embedding of \( S'(\mathbb{R}^n) \) into \( G_r(\mathbb{R}^n) \) commuting with partial derivatives and making
\[ O_C(\mathbb{R}^n) = \{ f \in C^\infty(\mathbb{R}^n) : \exists p > 0 \forall \alpha \in \mathbb{N}_o^n \sup_{x \in \mathbb{R}^n} (1 + |x|)^{-p} |\partial^\alpha f(x)| < \infty \} \]
a faithful subalgebra. Elements of \( OM(\Omega) \) are called \emph{slowly increasing}. Componentwise insertion of elements of \( G \) into slowly increasing functions yields well defined elements of \G_r. The importance of \( G_r(\Omega) \) for our purposes stems from the fact that elements of this algebra can be composed (again by componentwise insertion, cf.\,[9, [13]], a necessary prerequisite for generalizing symmetry methods, see section 3. Especially in the theory of ODEs in the generalized function context it is often useful to consider the algebra \( G_r(\Omega \times \mathbb{R}') \) whose elements satisfy \( G \)-bounds in the \( \Omega \)-variables and \( G_o \)-bounds in the \( \mathbb{R}' \)-variables (cf. \[9 \] or \[13 \]). Elements of Colombeau algebras are usually denoted by capital letters with the understanding that \( (u_x)_x \) denotes an arbitrary representative of \( U \in G \). \( U \) is called associated with some distribution \( w \) if \( u_x \to w \) in \( \mathcal{D}' \). Generalized numbers (i.e. the ring of constants in case \( \Omega \) is connected) in any of the above algebras will be denoted by \( \mathcal{R} \). Componentwise insertion of points into representatives of generalized functions yields well defined elements of \( \mathcal{R} \). Our choice of the special variants of Colombeau algebras is aimed at notational simplicity. However, all results presented in the sequel carry over to the respective ‘full’ variants of the algebras (distinguished by the fact that they allow a canonical embedding of distributions) as well.

2 Transfer of Classical Symmetry Groups

2.1 Factorization Properties

The first question to be answered in trying to extend the applicability of classical group analysis to generalized solutions concerns permanence properties of classical symmetries: Let \( G \) be the symmetry group of some system \( S \) of PDEs and consider \( S \) within the framework of \( G(\Omega) \). Under which conditions do elements of \( G \) also transform generalized solutions into other generalized solutions? It is the aim of this and the following section to answer this question. To begin with, let us fix some terminology:

\begin{definition}
Let \( G \) be a projectable local group of transformations acting on some open set \( M \subseteq X \times U \) according to \( g \cdot (x, u) = (\Xi_g(x), \Phi_g(x, u)) \). \( g \) is called
\end{definition}
slowly increasing if the map \( u \rightarrow \Phi_g(x, u) \) is slowly increasing, uniformly for \( x \) in compact sets. \( g \) is strictly slowly increasing if \( \Phi_g \in \mathcal{O}_M(M) \). If \( \Omega \subseteq X, U \in \mathcal{G}(\Omega) \) and \( g \) is (strictly) slowly increasing, the action of \( g \) on \( U \) is defined as the element

\[
gU := \text{cl}[(\Phi_g \circ (id \times u_x)) \circ \Xi_g^{-1}] \tag{1}
\]

of \( \mathcal{G}(\Xi_g(\Omega)) \).

If \( U \) is a smooth function, (1) reproduces the classical notion of group action on functions. Henceforth we make the tacit assumption that the differential equations under consideration are of a form that allows for an insertion of elements of Colombeau generalized functions (i.e. the function \( \Delta \) representing the equations on the prolongation space is slowly increasing). Also, slowly increasing group actions are always understood to be projectable. Analogous to the classical setting we give the following

\[\textbf{2.2 Definition} \] Let \( S \) be some system of differential equations with \( p \) variables and \( q \) unknown functions. A solution of \( S \) in \( \mathcal{G} \) is an element \( U \in (\mathcal{G}(\Omega))^q \), with \( \Omega \subseteq X \) open, which solves the system with equality in \( (\mathcal{G}(\Omega))^q \). A symmetry group of \( S \) in \( \mathcal{G} \) is a local transformation group acting on \( X \times U \) such that if \( U \) is a solution of the system in \( \mathcal{G} \), \( g \in G \) and \( g \cdot U \) is defined, then also \( g \cdot U \) is a solution of \( S \) in \( \mathcal{G} \).

Let us take a look at the transition problem from classical to generalized symmetry groups on the level of representatives. Thus, let \( G \) be a slowly increasing symmetry group of some differential equation

\[
\Delta(x, u^{(n)}) = 0. \tag{2}
\]

This means that if \( f \) is a classical solution, i.e. if \( \Delta(x, \text{pr}^{(n)} f(x)) = 0 \) for all \( x \) then also \( \Delta(x, \text{pr}^{(n)} (g \cdot f)(x)) = 0 \). Now let \( U \in \mathcal{G}(\Omega) \) be a generalized solution to (2). Then for any representative \( (u_x)_\varepsilon \) of \( U \) there exists some \( (n_x)_\varepsilon \in \mathcal{N}(\Omega) \) such that for all \( x \) and all \( \varepsilon \) we have

\[
\Delta(x, \text{pr}^{(n)} u_x(x)) = n_x(x). \tag{3}
\]

In particular, the differential equation (2) need not be satisfied for even one single value of \( \varepsilon \). This basic observation displays quite fundamental obstacles to a direct utilization of the classical symmetry group properties of \( G \) in order to obtain statements on the status of \( G \) in the Colombeau-setting. Therefore we have to derive properties of symmetry groups that are better suited to allow a transfer to differential algebras. The starting point for our considerations is a slight modification of a well known factorization property of smooth maps (cf. [21], Proposition 2.10):

\[\text{5}\]
2.3 Proposition Let $F$ be a smooth mapping from some manifold $M$ to $\mathbb{R}^k$ ($k \leq n = \dim(M)$), let $f : (-\eta_c,\eta_c) \times M \to \mathbb{R}$ be smooth and suppose that $f(\eta, \cdot)$ vanishes on the zero set $S_F$ of $F$, identically in $\eta$. If $F$ is of maximal rank (= $k$) on $S_F$ then there exist smooth functions $Q_1, \ldots, Q_k : (-\eta_c,\eta_c) \times M \to \mathbb{R}$ such that

$$f(\eta, m) = Q_1(\eta, m)F_1(m) + \cdots + Q_k(\eta, m)F_k(m)$$

for all $(\eta, m) \in (-\eta_c,\eta_c) \times M$.

We are mainly interested in the following application of Proposition 2.3:

2.4 Theorem Let

$$\Delta_\nu(x, u^{(n)}) = 0, \quad 1 \leq \nu \leq l$$

be a nondegenerate system of PDEs. Let $G = \{g_0 : \eta \in (-\eta_c,\eta_c)\}$ be a one parameter symmetry group of (4) and set $g_0 : (x, u) = (\Xi(x, u), \Phi(x, u))$. Then there exist $C^\infty$-functions $Q_{\mu\nu} : (-\eta_c,\eta_c) \times \mathbb{V} \to \mathbb{R}$ ($1 \leq \mu, \nu \leq l$, $\mathbb{V}$ an open subset of $M^{(n)}$) such that if $u : \Omega \subseteq \mathbb{R}^p \to \mathbb{R}$ is smooth and $g_0u$ exists we have

$$\Delta_\nu(\Xi(x, u(x)), \text{pr}^{(n)}(g_0u)(\Xi(x, u(x)))) = \sum_{\mu=1}^l Q_{\mu\nu}(x, \text{pr}^{(n)}u(x))\Delta_\mu(x, \text{pr}^{(n)}u(x))$$

(5)

on the domain of $g_0u$ for $1 \leq \nu \leq l$.

Proof. Denote by $z$ the coordinates on $M^{(n)}$. That $g_0$ is an element of the symmetry group of the system is equivalent with

$$\Delta(z) = 0 \Rightarrow \Delta_\nu(\text{pr}^{(n)}g_0(z)) = 0 \quad (1 \leq \nu \leq l)$$

for all $\eta$ and $z$ such that this is defined. $\Delta$ is of maximal rank because (4) is nondegenerate. Hence, by Proposition 2.3 there exist $C^\infty$-functions $P_{\mu\nu} : (-\eta_c,\eta_c) \times \mathbb{V} \to \mathbb{R}$ ($1 \leq \mu \leq l$, $\mathbb{V}$ an open subset of $M^{(n)}$) such that

$$\Delta_\nu(\text{pr}^{(n)}g_0(z)) = \sum_{\mu=1}^l Q_{\mu\nu}(\eta, z)\Delta_\mu(z).$$

(6)

Now for a smooth function $u : \Omega \subseteq \mathbb{R}^p \to \mathbb{R}$ as in our assumption and $x \in \Omega$ we set

$$z_u(x) := (x, \text{pr}^{(n)}u(x)) \in M^{(n)}.$$  

(7)

Then by definition $\text{pr}^{(n)}g_0(z_u(x)) = (\Xi(x, u(x)), \text{pr}^{(n)}(g_0u)(\Xi(x, u(x))))$, so the result follows.

For a single PDE $\Delta(x, \text{pr}^{(n)}u) = 0$, equation (5) takes the simpler form

$$\Delta(\Xi(x, u(x)), \text{pr}^{(n)}(g_0u)(\Xi(x, u(x)))) = \sum_{\mu=1}^l Q_{\mu\nu}(\eta, x, \text{pr}^{(n)}u(x))\Delta(x, \text{pr}^{(n)}u(x)).$$

(8)

Theorem 2.4 will be one of our main tools in transferring classical symmetry groups of (systems of) PDEs into the setting of algebras of generalized functions.
2.5 Proposition Let \( \eta \to g_\eta \) be a slowly increasing one parameter symmetry group of (4). If \( P_{\mu \nu} := (Q_{\mu \nu}(\eta, \Xi_{-\eta}(\cdot)), \text{pr}^{(n)}u(\Xi_{-\eta}(\cdot))) \) belongs to \( E_M(\Omega) \) for \( 1 \leq \mu, \nu \leq l \) and every \( (u_\cdot)_x \in E_M(\Omega) \), then \( \eta \to g_\eta \) is a symmetry group of (4) in \( \mathcal{G} \) as well. This condition is satisfied if

\[
(x, u^{(n)}) \to Q_{\mu \nu}(\eta, x, u^{(n)})
\]

is slowly increasing in the \( u^{(n)} \)-variables, uniformly in \( x \) on compact sets for \( 1 \leq \mu, \nu \leq l \) and every \( \eta \).

Proof. It suffices to observe that (5) gives

\[
\Delta_{\eta}(x, \text{pr}^{(n)}(g_\eta u)(x)) = \\
= \sum_{\mu=1}^{l} Q_{\mu \nu}(\eta, \Xi_{-\eta}(x), \text{pr}^{(n)}u(\Xi_{-\eta}(x))) \Delta_{\mu}(\Xi_{-\eta}(x), \text{pr}^{(n)}u(\Xi_{-\eta}(x))).
\]

For any solution \( U \in \mathcal{G}(\Omega) \) with representative \( u = (u_\cdot)_x \), this expression is in \( N(\Omega) \) since \( P_{\mu \nu} \in E_M(\Omega) \) for each \( \mu, \nu \), and every \( \Delta_{\mu}(\Xi_{-\eta}(\cdot), \text{pr}^{(n)}u(\Xi_{-\eta}(\cdot))) \) is in \( N(\Omega) \) because \( U \) is a solution and \( \Xi_{-\eta} \) is a diffeomorphism. \( \square \)

2.6 Example In [18], the system

\[
\begin{align*}
U_t + UU_x &= 0 \\
V_t + UV_x &= 0 \\
U \big|_{t=0} &= U_0, \quad V \big|_{t=0} &= V_0
\end{align*}
\]

(9)

where \( U, V \in \mathcal{G}_{s,0}(\mathbb{R} \times [0, \infty)), U_0, V_0 \in \mathcal{G}_{s,0}(\mathbb{R}) \) is analyzed (\( \mathcal{G}_{s,0} \) is a variant of the Colombeau algebra with global instead of local bounds). In the following we present some applications of the above results to this system (for a more detailed study, see [14]). For \( U'_o \geq 0 \) (9) has a unique solution \((U, V)\) in \( \mathcal{G}_{s,0}(\mathbb{R} \times [0, \infty)) \) with \( \partial_x U \geq 0 \). We consider solutions in \( \mathcal{G}_{s,0}(\mathbb{R} \times [0, \infty)) \) with initial data \( U_o(x) = u_L + (u_R - u_L)H(x) \) and \( V_o(x) = v_L + (v_R - v_L)H(x) \), where \( H \) is a generalized Heaviside function with \( H' \geq 0 \), i.e. \( H \) is a member of \( \mathcal{G}_{s,0}(\mathbb{R}) \) with a representative \( (h_\cdot)_x \) coinciding with the classical Heaviside function \( Y \) off the interval \([-\varepsilon, \varepsilon]\). For \( u_L < u_R \) the solution \((U, V)\) is associated with the rarefaction wave

\[
u(x, t) = \begin{cases} 
\frac{u_L}{t}, & x \leq u_L t \\
\frac{v_L}{t}, & u_L t \leq x \leq u_R t \\
u_R, & u_R t \leq x
\end{cases}
\]

(10)

\[
v(x, t) = \begin{cases} 
\frac{v_L}{t} + \frac{u_L v_L - u_R v_R}{u_R - u_L}, & x \leq u_L t \\
\frac{v_R}{t} + \frac{v_L u_R - v_R u_L}{u_R - u_L}, & u_L t \leq x \leq u_R t \\
v_R, & u_R t \leq x
\end{cases}
\]

(11)
However, choosing different generalized Heaviside functions for modelling the initial data \( U_\sigma \), respectively \( V_\sigma \) we may obtain a superposition of the rarefaction wave (10) in \( u \) with a shock wave

\[
v(x, t) = v_L + (v_R - v_L)Y(x - ct)
\]

with arbitrary shock speed \( c, u_L \leq c \leq u_R \). We are going to construct a one parameter symmetry \( \eta \rightarrow g_\eta \) of (9) which transforms any of the solutions (11), (12) into a shock wave solution as \( \eta \rightarrow \pm \infty \). For this we employ the two-dimensional Lorentz-transformation \((\eta, (x, t)) \rightarrow (x \cosh(\eta) - t \sinh(\eta), -x \sinh(\eta) + t \cosh(\eta))\) with infinitesimal generator \( X_\zeta = -t \partial_x - x \partial_t \). Then \( X := X_\zeta + (u^2 - 1) \partial_u \) generates a projectable one-parameter symmetry group of (9). Assuming that \(-1 < u_L < u_R < 1\), we can extend the solution \((U, V)\) to the region \( \Omega = \mathbb{R}^2 \setminus \{(x, t) : t \leq 0, u_R t \leq x \leq u_L t\} \) by the method of characteristics applied to representatives. Then the Lorentz-transformed solutions

\[
\tilde{u}_\zeta(x, t) = -\tanh(\eta - \text{Artanh}(u_\zeta(x \cosh(\eta) + t \sinh(\eta))), \\
x \sinh(\eta) + t \cosh(\eta))) \\
\tilde{v}_\zeta(x, t) = v_\zeta(x \cosh(\eta) + t \sinh(\eta), x \sinh(\eta) + t \cosh(\eta))
\]

are well defined at least on \( \mathbb{R} \times (0, \infty) \). The factorization property (5) in this case reads

\[
(\partial_t \tilde{u}_\zeta + \tilde{u}_\zeta \partial_x \tilde{u}_\zeta)(x, t) = \\
(\partial_t u_\zeta + u_\zeta \partial_x u_\zeta)/(\cosh^3(\text{Artanh}(u_\zeta - \eta)) \cosh(\text{Artanh}(u_\zeta))) (\Xi^{-1}(x, t))
\]

and similarly for the second line in (9), demonstrating that \((\tilde{U}, \tilde{V})\) is again a solution. For each \( \eta, \tilde{U} \) is associated with a piecewise smooth function which converges to \( \mp 1 \) as \( \eta \rightarrow \pm \infty \). Observing that the coordinate transformations in (13), (14) approach boosts in the directions \((\mp 1, 1)\) as \( \eta \rightarrow \pm \infty \), we see that the functions associated with \( \tilde{V} \) converge to the shock wave \( v_L + (v_R - v_L)Y(x \pm t) \) as \( \eta \rightarrow \pm \infty \), for whatever solution \( \tilde{V} \) given in (11) or (12).

Although Proposition 2.5 provides a manageable algorithm to determine if classical symmetry groups carry over to generalized solutions it would certainly be preferable to have criteria at hand that allow to judge directly from the given PDE if the factors \( P_{\mu \nu} \) behave nicely (given slowly increasing group actions). The first step in this direction is gaining control over the behaviour of the map \( z \rightarrow \text{pr}^{(n)}g_\eta(z) \), defined on \( \mathcal{M}^{(n)} \).

### 2.7 Proposition

If \( \eta \rightarrow g_\eta \) is a (strictly) slowly increasing group action on \( \mathcal{M} \) then \( z \rightarrow \text{pr}^{(n)}g_\eta(z) \) is (strictly) slowly increasing as well.

**Proof.** Let \( N := \dim(\mathcal{M}^{(n)}) \). For \( z = (z_1, \ldots, z_p, z_{p+1}, \ldots, z_N) \in \mathcal{M}^{(n)} \) we choose some smooth function \( h : X \rightarrow \mathcal{U} \) satisfying \( z = z_h(z_1, \ldots, z_p) \), with
with the aid of Proposition 2.7 we still need some information about the explicit form of the factors $P_{\mu \nu}$. By the definition of prolonged group actions we have to find estimates for every

$$A_z := ((\Phi_\eta \circ (id \times h)) \circ \Xi_\eta)^{(s)}(\bar{x})$$

(16)

(where (s) denotes the derivative of order s) in terms of $z$. The above formula contains the components of $pr^{(n)}g(z)$ of order s $(s \leq n)$. Note that the particular choice of $h$ has no influence on (16), i.e. $A_z$ depends exclusively on $z$. To compute $A_z$ explicitly we use the formula for higher order derivatives of composite functions (see [8]). Denoting by $\mathcal{Y}_m$ the group of permutations of $\{1, \ldots, m\}$ we have:

$$A_z(r_1, \ldots, r_s) = \sum_{i=1}^{s} \sum_{\sigma \in \mathcal{Y}_s} \sum_{l_k \in \mathcal{Y}_s} \frac{1}{i!m!} ([\Phi_\eta \circ (id \times h)]^{(i)}((\bar{x}))(t_1, \ldots, t_i),$$

(17)

where

$$t_1 = \Xi_\eta^{(k_1)}(\bar{x})(r_{\sigma(1)}, \ldots, r_{\sigma(k_1)}), \ldots, t_i = \Xi_\eta^{(k_i)}(\bar{x})(r_{\sigma(s-k_i+1)}, \ldots, r_{\sigma(s)}).$$

and

$$((\Phi_\eta \circ (id \times h))^{(i)}(x)(t_1, \ldots, t_i)) = \sum_{j=1}^{s} \sum_{\tau \in \mathcal{Y}_s} \sum_{l_j \in \mathcal{Y}_s} \frac{1}{j!n!} \Phi_\eta^{(j)}(x, u)(s_1, \ldots, s_j),$$

(18)

where

$$s_1 = (id \times h)^{(l_1)}(x)(t_{\tau(1)}, \ldots, t_{\tau(l_1)}), \ldots, s_j = (id \times h)^{(l_j)}(x)(t_{\tau(i)-l_j+1}, \ldots, t_{\tau(i)}).$$

Each $s_m$ consists of sums of products of certain $t_{\tau(k)}$ with certain $z_i$ and an analogous assertion holds for the $\Phi_\eta^{(j)}(x, u)(s_1, \ldots, s_j)$. Hence from (17) and (18) the result follows. \hfill \Box

Returning to our original task of finding a priori estimates for the factors $P_{\mu \nu}$, even with the aid of Proposition 2.7 we still need some information about the explicit form of the $Q_{\mu \nu}$ to go on. In general this seems quite difficult to achieve. However, there is a large and important class of PDEs that allow a priori statements on the concrete form of the factorization. Namely, we are going to show that each scalar PDE in which at least $a$ or one of its derivatives appears as a single term with constant coefficient belongs to this class.

Consider a scalar PDE $\Delta(x, u^{(n)}) = 0$ together with a symmetry group $\eta \rightarrow g_\eta$. Then we have

$$\Delta(z) = 0 \Rightarrow \Delta(pr^{(n)}g_\eta(z)) = 0$$

Set $F(z) := \Delta(z)$, $f(z) := \Delta(pr^{(n)}g_\eta(z))$ and $N = \dim(\mathcal{M}^{(n)})$. Suppose that in a neighborhood of some $z$ with $F(z) = 0$ we have $\frac{\partial F}{\partial z_k} > 0$ for some $1 \leq k \leq N$. Then we have

$$\Delta(z) = 0 \Rightarrow \Delta(pr^{(n)}g_\eta(z)) = 0$$

Set $F(z) := \Delta(z)$, $f(z) := \Delta(pr^{(n)}g_\eta(z))$ and $N = \dim(\mathcal{M}^{(n)})$. Suppose that in a neighborhood of some $z$ with $F(z) = 0$ we have $\frac{\partial F}{\partial z_k} > 0$ for some $1 \leq k \leq N$. Then we have
Then by the implicit function theorem, locally there exists a smooth function \( \psi : \mathbb{R}^{N-1} \to \mathbb{R} \) such that in a suitable neighborhood of \( z \) we have

\[
F(z) = 0 \quad \iff \quad z_k = \psi(z'),
\]
where \( z' = (z_1, \ldots, \hat{z}_k, \ldots, z_N) \) (meaning that the component \( z_k \) is missing from \( z' \)). It follows that

\[
F(z) = (z_k - \psi(z')) \int_0^1 \frac{\partial F}{\partial z_k}(z_1, \ldots, z_{k-1}, \tau z_k + (1 - \tau)\psi(z'), \ldots, z_N) \, d\tau,
\]
and on the other hand

\[
f(z) = (z_k - \psi(z')) \int_0^1 \frac{\partial f}{\partial z_k}(z_1, \ldots, z_{k-1}, \tau z_k + (1 - \tau)\psi(z'), \ldots, z_N) \, d\tau.
\]
Thus in the said neighborhood we have

\[
f(z) = F(z) \int_0^1 \frac{\partial f}{\partial z_k}(z_1, \ldots, z_{k-1}, \tau z_k + (1 - \tau)\psi(z'), \ldots, z_N) \, d\tau
\quad \text{(19)}
\]
provided the denominator of this expression is \( \neq 0 \). In particular, if for some constant \( c \neq 0 \) we have \( \frac{\partial F}{\partial z_k} \equiv c \) in a neighborhood of \( z \) then (19) simplifies to

\[
f(z) = \frac{1}{c} F(z) \int_0^1 \frac{\partial f}{\partial z_k}(z_1, \ldots, z_{k-1}, \tau z_k + (1 - \tau)\psi(z'), \ldots, z_N) \, d\tau \quad \text{(20)}
\]
After these preparations we can state

**2.8 Theorem** Let \( \eta \to g_\eta \) be a slowly increasing symmetry group of the equation \( \Delta(x, u^{(n)}) = 0 \). Set \( N = \dim(\mathcal{M}^{(n)}) \) and suppose that \( \frac{\partial \Delta}{\partial \zeta_k} \equiv c \neq 0 \) for some \( p + 1 \leq k \leq N \). Then \( \eta \to g_\eta \) is a symmetry group of \( \Delta(x, u^{(n)}) = 0 \) in \( \mathcal{G} \).

**Proof.** Without loss of generality we may assume \( c = 1 \). Using the above notations we have \( F(z) = z_k - \psi(z') \), so (20) implies

\[
f(z) = F(z) \int_0^1 \frac{\partial f}{\partial z_k}(z_1, \ldots, z_{k-1}, \tau z_k + (1 - \tau)(z_k - F(z)), \ldots, z_N) \, d\tau =: F(z)Q(\eta, z).
\]

From Proposition 2.7 we know that \( z \to f(z) \) is slowly increasing in the \( u^{(n)} \)-variables (i.e. in those \( z_i \) with \( i > p \)), uniformly in \( x = (z_1, \ldots, z_p) \) on compact sets. Since \( F \) is slowly increasing we infer that \( Q(\eta, z_n(x)) \in \mathcal{E}_M(\Omega) \) for any \( u \in \mathcal{E}_M(\Omega) \) (with \( z_n \) as in (7)). Finally,

\[
\Delta(x, pr^{(n)}(g_\eta u)(x)) = \Delta(\overline{z}_\eta(x), pr^{(n)}(\overline{z}_\eta(x)))Q(\eta, \overline{z}_\eta(x), pr^{(n)}u(\overline{z}_\eta(x))).
\]
Since $\Xi$ is a diffeomorphism, it follows that if $U = cl[u]$ solves the equation, so does $g \circ U$. 

As the proof shows, we can drop the assumption $p + 1 \leq k$ if we require the group action to be strictly slowly increasing. It is clear that many PDEs satisfy the requirements of Theorem 2.8. For example, in the Hopf equation $\Delta(x, t, u, u_x, u_t) = u_t + uu_x$ or $\Delta(z_1, \ldots, z_k) = z_k + z_3z_4$ one can take $k = 5$. Note however that not every symmetry group of this equation is automatically slowly increasing. Theorem 2.8 constitutes a useful tool for transferring classical symmetry groups to Colombeau algebras.

2.9 Example The initial value problem

$$U_t + \lambda U_x = f(U)$$
$$U|_{t=0} = U_0$$

has unique solutions in $\mathcal{G}(\mathbb{R}^\nu)$, given $U_0 \in \mathcal{G}(\mathbb{R})$, provided $f \in \mathcal{O}_M$ is globally Lipschitz (see [17]). If in addition $f$ is bounded and the initial data are distributions with discrete support, say $U_0(x) = \sum_{i,j} a_{ij} \delta^{(i)}(x - \xi_j)$ then the generalized solution is associated with the delta wave $v + w$ where

$$v(x, t) = \sum_{i,j} a_{ij} \delta^{(i)}(x - \lambda t - \xi_j)$$

and $w$ is the smooth solution to $w_t + \lambda w_x = f(w)$, $w(0) = 0$.

The vector field $X = cf(u)\partial_u$ generates an infinitesimal symmetry of (21) for arbitrary $c \in \mathbb{R}$. With $F(u) := \int du / f(u)$, the corresponding Lie point transformation is

$$(x, t, u) \rightarrow (\tilde{x}, \tilde{t}, \tilde{u}) = (x, t, F^{-1}(c \eta + F(u))).$$

This provides a well-defined nonlinear transformation of the generalized solution $U \in \mathcal{G}(\mathbb{R}^\nu)$, provided that the right hand side in (23) is slowly increasing.

In the example

$$U_t + \lambda U_x = \tanh(U)$$

the generalized solution is associated with $v(x, t)$ and $w$ vanishes identically. Applying (23) we obtain (due to Theorem 2.8) the new generalized solution

$$\tilde{U}(x, t) = \text{Arsinh}(e^{c_0} \sinh(U(x, t)))$$

We are going to show that $\tilde{U}$ is still associated with the delta wave $v$ in (22). To simplify the argument we assume $\lambda = 0$ and $U_0(x) = \delta^{(i)}(x)$. Representatives of $U$ resp. $\tilde{U}$ are $u_x(x, t) = \text{Arsinh}(e^{c_0} \sinh(\rho^{(i)}(x)))$ and $\tilde{u}_x(x, t) = \text{Arsinh}(e^{c_0+1} \sinh(\rho^{(i)}(x)))$. For $\psi \in \mathcal{D}(\mathbb{R}^\nu)$ we have

$$I^i := \int \int_{\mathbb{R}^\nu} \tilde{u}_x(x, t) \psi(x, t) dxdt = \int \int_{\mathbb{R}^\nu} \theta(e^{c_0+1}, \sigma^{c_0+1} \rho^{(i)}(x)) d\sigma \rho^{(i)}(x) \psi(\sigma x, t) dxdtdt$$

11
where \( \theta(\alpha, y) := \frac{d}{dy} \text{Ar sinh}(\text{asinh}(y)) \) for \( \alpha > 0, y \in \mathbb{R} \). Since \( \theta \) is bounded by \( \max(1, \alpha) \) and \( \lim_{|\sigma| \to \infty} \theta(\alpha, y) = 1 \) it follows that \( I^\sigma \to \int \psi(0, t) dt \), so \( \tilde{U} \) is associated with the delta function on the \( t \)-axis, as desired. For \( i \geq 1 \) we write

\[
I^\sigma_i = \iint_0^1 (\theta(e^{\sigma+i}, \sigma e^{-1} \rho^{(i)}(x)) - 1) \, d\sigma e^{-i} \rho^{(i)}(x) \psi(\varepsilon x, t) \, dxdt + \left(-1)^i \iint \rho(x) \partial_x^i \psi(\varepsilon x, t) \, dxdt
\]

Here the second term converges to \((-1)^i \int \partial_x^i \psi(0, t)\) and the first term goes to zero since \( \int_0^1 |\theta(\alpha, \sigma y) - 1| \, d\sigma \leq \frac{2\sqrt{\varepsilon - 1}}{|\sigma|} |1 - e^{-|x|}| \) for \( y \neq 0 \). This proves the claim for \( \rho \in D(\mathbb{R}) \). For \( \rho \in S(\mathbb{R}) \) splitting the \( x \)-integral into one from \(-\frac{1}{\sqrt{\varepsilon}}\) to \( \frac{1}{\sqrt{\varepsilon}} \) and one over \( |x| \geq \frac{1}{\sqrt{\varepsilon}} \) gives the same result.

### 2.2 Continuity Properties

In this section we work out a different strategy for transferring classical point symmetries into the \( G \)-setting. This approach, suggested in [20], consists in a more topological way of looking at the transfer problem by using continuity properties of differential operators. As we have pointed out in the discussion following (3), the main obstacle against directly applying classical symmetry groups componentwise to representatives of generalized solutions is that the differential equations need not be satisfied componentwise. However, there are certain classes of partial differential operators that do allow such a direct application. Consider a linear partial differential operator \( P \) giving rise to an equation

\[
PU = 0
\]

in \( G \) and let \( G \) be a classical slowly increasing symmetry group of (26). Furthermore, suppose that \( P \) possesses a continuous homogeneous (but not necessarily linear) right inverse \( Q \). If \( U = \text{cl}[u] \) is a solution to (26) in \( G(\Omega) \) then there exists some \( n \in \mathcal{N}(\Omega) \) such that

\[
Pn = n.
\]

Since \( Q \) is a right inverse of \( P \) this implies

\[
P(u_x - Qn_x) = 0 \quad \forall \varepsilon \in I.
\]

Also, \( Qn \in \mathcal{N}(\Omega) \) due to the continuity and homogeneity assumption on \( Q \). If \( g \in G \), (27) implies

\[
P(g(u_x - Qn_x)) = 0 \quad \forall \varepsilon \in I.
\]

By definition,

\[
P(gU) = \text{cl}[P(gu)] = \text{cl}[P(g(u - Qn))],
\]

so \( gU \) is a solution as well. Summing up, \( G \) is a symmetry group in \( G \). The following result will serve to secure the existence of a right inverse as above for a large class of linear differential operators.
2.10 Proposition Let $E$, $F$ be Fréchet spaces and $A$ a continuous linear map from $E$ onto $F$. Then $A$ has a continuous homogeneous right inverse $B : F \to E$.

Proof. See [16], p. 364. \hfill \Box

From these preparations we conclude

2.11 Theorem Let

$$\Delta_{\nu}(x, u^{(n)}) = 0, \quad \nu = 1, \ldots, l$$

be a system of linear PDEs with slowly increasing $\Delta_{\nu}$ and let $\eta \to g_\eta$ be a slowly increasing symmetry group of this system. Assume that the operator defined by the left hand side is surjective $(C^\infty(\Omega))^l \to (C^\infty(\Omega))^l$. Then $\eta \to g_\eta$ is a symmetry group for the system in $G(\Omega)$ as well. \hfill \Box

The assumptions of Theorem 2.11 are automatically satisfied for any linear partial differential operator with constant coefficients on an arbitrary convex open domain (see [10], 10.6).

2.12 Example The system of one-dimensional linear acoustics

$$\begin{align*}
P_t + U_x &= 0 \\
U_t + P_x &= 0.
\end{align*}$$

is transformed via $U = V - W$, $P = V + W$ into

$$\begin{align*}
V_t + V_x &= 0 \\
W_t - W_x &= 0.
\end{align*}$$

Using the infinitesimal generators $\Phi(v)\partial_v + \Psi(w)\partial_w$ ($\Phi, \Psi$ arbitrary smooth functions) of (29) we obtain symmetry transformations for (28) of the form

$$\begin{align*}
\bar{U} &= F^{-1} \left( \eta + F\left(\frac{1}{2}(P + U)\right) \right) - G^{-1} \left( \theta + G\left(\frac{1}{2}(P - U)\right) \right) \\
\bar{P} &= F^{-1} \left( \eta + F\left(\frac{1}{2}(P + U)\right) \right) + G^{-1} \left( \theta + G\left(\frac{1}{2}(P - U)\right) \right)
\end{align*}$$

with arbitrary diffeomorphisms $F, G$. Since (28) satisfies the assumptions of Theorem 2.11 on $\Omega = \mathbb{R}^2$ it follows that any slowly increasing transformation of this form is a symmetry of (28). In particular, this includes nonlinear transformations of distributional solutions, cf. Example 2.13.

In the remainder of this section we discuss the interplay between symmetry groups and solutions of PDEs in the sense of association. Consider

$$\Delta_{\nu}(x, u^{(n)}) \approx 0, \quad 1 \leq \nu \leq l$$

in $\mathcal{G}$. A slowly increasing symmetry group of the corresponding system

$$\Delta(x, u^{(n)}) = 0, \quad 1 \leq \nu \leq l$$

13
is called a symmetry group in the sense of association if it transforms solutions of (30) into other such solutions. The first question to be answered in this context is whether one can derive conditions on the form of the factorization (8) that will yield symmetry groups in the sense of association. It is clear that a sufficient condition is to suppose that \( Q \) depends exclusively on \( \eta \) and \( x \). Distributional solutions to linear PDEs arise as a special case of (8) and have been treated in [4]. There, the validity of equation (8) with \( Q \) depending on \( \eta \) and \( x \) only is actually used to define symmetry groups in \( \mathcal{D}' \). In order to remain within the classical distributional framework, the admissible group transformations in [4] are restricted to projectable ones acting linearly in the dependent variables. On the other hand, the method developed there is even applicable to linear equations containing distributional terms which allows to use invariance methods to compute fundamental solutions.

Second, if \( u \) is a solution to \( \Delta(x, u^{[n]}) = 0 \) in \( \mathcal{G}(\Omega) \) possessing an associated distribution, one may ask for which group actions \( g \) this implies that \( gu \) as well possesses an associated distribution. This is certainly the case for admissible transformations in the above sense. On the other hand, we have already seen in Example 2.9 that even genuinely nonlinear symmetry transformations may preserve association properties.

The next example shows that nonlinear group actions may transform distributional solutions in Examples 2.9 and 2.12 into more complicated distributional solutions or into generalized solutions in \( \mathcal{G}(\mathbb{R}^2) \) not admitting associated distributions.

**2.13 Example** We consider the equation \( U_t + \lambda U_x = 0 \) arising in (21) or (29). We have already observed that \( U = F^{-1}(\eta + F(U)) \) defines a symmetry transformation for arbitrary diffeomorphisms \( F \). Here we take \( F \in \mathcal{C}^\infty(\mathbb{R}), F' > 0, F(y) = \text{sign}(y)\sqrt{|y|} \) for \( |y| \geq 1 \). We wish to compute \( \bar{U} \) when \( U \in \mathcal{G}(\mathbb{R}^2) \) is a delta wave solution \( U(x, t) \approx \delta(x) \approx \delta(x - \lambda t) \). We take \( U \) as the class of \( \rho \in \mathcal{D}([-1, 1]) \). We have when \( \eta \geq 0 \):

(i) If \( i = 0 \), that is \( U \approx \delta(x - \lambda t) \), then \( \bar{U} \approx \delta(x) - \lambda \delta'(x - \lambda t) \);

(ii) If \( i = 1 \), that is \( U \approx \delta'(x - \lambda t) \), then \( \bar{U} \approx \delta''(x) + \eta \delta'(x) \approx \delta''(x - \lambda t) + \delta'(x - \lambda t) \);

(iii) If \( i \geq 2 \) then \( \bar{U} \) does not admit an associated distribution.

To see this, we may assume that \( \lambda = 0 \) and write \( a_\pm(x) := \frac{\lambda}{\eta} \) for brevity. Note that \( F^{-1}(\eta) = \text{sign}(\eta)\sqrt{\eta} \) for \( |\eta| \geq 1 \). Let \( A_\pm = \{ x \in [\pm \varepsilon, \varepsilon] : |a_\pm(x)| \leq (\eta + 1)^2 \} \). If \( x \in A_\pm \) and \( a_\pm(x) \geq 0 \) then \( \eta + F(a_\pm(x)) \geq 1 \) and \( F^\prime(\eta + F(a_\pm(x))) = \eta + 2\eta \sqrt{a_\pm(x)} + a_\pm(x) \). Also, if \( x \in A_\pm \) and \( a_\pm(x) < 0 \) then \( \eta + F(a_\pm(x)) \leq -1 \) and \( F^\prime(\eta + F(a_\pm(x))) = -\eta^2 + 2\eta \sqrt{|a_\pm(x)|} + a_\pm(x) \). The functions \( F^{-1}(\eta + F(a_\pm(x))), |a_\pm(x)|, \text{and } \sqrt{|a_\pm(x)|} \) are bounded on the complement of \( A_\pm \). Thus

\[
\int_{-\varepsilon}^{\varepsilon} F^{-1}(\eta + F(a_\pm(x)))dxdt = \]
\[ \int_{A_{\alpha}} \left( \pm \eta^2 + 2\eta \sqrt{a_z(x) + a_x(x)} \right) \psi(x,t) dx dt + O(\varepsilon) = \int_{A_{\alpha}} \sqrt{a_z(x) + a_x(x)} \psi(x,t) dx dt + O(\varepsilon) \]

while

\[ \int_{|\xi| \geq \varepsilon} F^{-1}(\eta + F(a_z(x))) \psi(x,t) dx dt \rightarrow F^{-1}(\eta + F(0)) \int_{\mathbb{R}^d} \psi(x,t) dx dt \]

It follows that \( F^{-1}(\eta + F(a_z(x))) \) converges in \( D'(\mathbb{R}^2) \) if and only if \( 2\eta \sqrt{a_z} + a_z \) admits an associated distribution. A simple computation yields the particular results (i), (ii), (iii).

### 3 Generalized Group Actions

Although the methods introduced in the previous sections enable an application of large classes of classical symmetry groups to elements of Colombeau algebras, they are but the first step in a theory of generalized group analysis of differential equations. In this section we develop an extension of the methods of group analysis that will allow to consider symmetry groups of differential equations whose actions are generalized functions themselves.

#### 3.1 Generalized Transformation Groups

Simple examples indicate the necessity of extending the methods of group analysis of PDEs to equations involving generalized functions themselves:

**3.1 Example** Considering (21) in \( G_{\tau} \) with a generalized function \( f = \text{cl}[(f_x)_x] \in G_{\tau} \) we can apply the classical algorithm for calculating symmetry groups componentwise to the equations

\[ \partial_t u_x + \lambda \partial_x u_x = f_x(u_x) \]

thereby obtaining infinitesimal generators with generalized coefficient functions. Thus the question arises in which sense such generators induce symmetries of the differential equation. More generally, one can consider differential equations in \( G_{\tau} \) of the form

\[ P(x,U^{(n)}) = 0 \]

where \( P \) is a generalized function.

As is indicated by Example 3.1, composition of generalized functions will inevitably occur in a generalization of group analysis. For this purpose, we shall apply suitable variants of Colombeau algebras for the following considerations, namely \( G_{\tau}(\mathbb{R}^n) \) and \( \tilde{G}_{\tau}(\mathbb{R} \times \mathbb{R}^n) = \tilde{G}_{\tau}(\mathbb{R}^{1+n}) \).
3.2 Definition A generalized group action on \( \mathbb{R}^n \) is an element \( \Phi \) of \( (\mathcal{G}_r(\mathbb{R}^{1+n}))^n \) such that:

(i) \( \Phi(0, \cdot) = \text{id} \) in \( (\mathcal{G}_r(\mathbb{R}^n))^n \).

(ii) \( \Phi(\eta_1 + \eta_2, \cdot) = \Phi(\eta_1, \Phi(\eta_2, \cdot)) \) in \( (\mathcal{G}_r(\mathbb{R}^n))^n \), \( \forall \eta_1, \eta_2 \in \mathbb{R} \).

Before we turn to an infinitesimal description of generalized group actions let us shortly recall some basic definitions from [19] that are needed for a pointvalue characterization of generalized functions which in turn plays a fundamental role in the following considerations. Thus for any open set \( \Omega \subseteq \mathbb{R}^n \) we set

\[
\Omega_M := \{(x_e)_e \in \Omega^I : \exists \rho > 0 \ \exists \eta > 0 \ \forall x | x_e | \leq \varepsilon^{-\rho} \ (0 < \varepsilon < \eta)\}.
\]

On \( \Omega_M \) we define an equivalence relation by

\[
(x_e)_e \sim (y_e)_e \iff \forall q > 0 \ \exists \eta > 0 \ \forall x_e - y_e | \leq \varepsilon^q \ (0 < \varepsilon < \eta)
\]

and set \( \hat{\Omega} := \Omega_M / \sim \). \( \hat{\Omega} \) is called the set of generalized points corresponding to \( \Omega \). The set of compactly supported points is defined as

\[
\hat{\Omega}_e = \{ \hat{x} \in \hat{\Omega} : \exists \text{ representative } (x_e)_e \exists K \subset \subset \Omega \ \exists \eta > 0 : x_e \in K, \varepsilon \in (0, \eta)\}.
\]

Note that for \( \Omega = \mathbb{R} \) we have \( \hat{\Omega} = \mathbb{R} \). Theorems 2.4, 2.7 and 2.10 of [19] establish that elements of \( \mathcal{G}(\Omega), \mathcal{G}_e(\Omega) \) or \( \mathcal{G}_r(\Omega \times \Omega') \) are uniquely determined by their pointvalues in \( \hat{\Omega}_e, \hat{\Omega}, \) or \( \hat{\Omega}_e \times \hat{\Omega}' \), respectively. For the theory of ODEs in the Colombeau framework we refer to [9].

3.3 Definition Let \( \xi = (\xi_1, \ldots, \xi_n) \in (\mathcal{G}_r(\mathbb{R}^n))^n \). The generalized vector field

\[X = \sum_{i=1}^n \xi_i(x) \partial_{x_i}\]

is called \( \mathcal{G} \)-complete if the initial value problem

\[
\dot{x}(t) = \xi(x(t)) \\
x(t_o) = \bar{x}_o
\]

is uniquely solvable in \( \mathcal{G}(\mathbb{R})^n \) for any \( \bar{x}_o \in \mathbb{R}^n \) and any \( t_o \in \mathbb{R} \).

3.4 Definition Let \( \Phi \) be a generalized group action on \( \mathbb{R}^n \) and set

\[
\xi := \frac{d}{d \eta_0} \bigg| \Phi(\eta, \cdot) \in (\mathcal{G}_r(\mathbb{R}^n))^n.
\]

If the generalized vector field \( X = \sum_{i=1}^n \xi_i(x) \partial_{x_i} \) is \( \mathcal{G} \)-complete, then \( X \) is called the infinitesimal generator of \( \Phi \). In this case, \( \Phi \) is also called \( \mathcal{G} \)-complete.

By [9], every generalized vector field with \( \mathcal{G}_r \)-components whose gradient is of \( L^\infty \)-log-type is \( \mathcal{G} \)-complete. The notion of infinitesimal generator is well-defined due to
3.5 Proposition Every $G$-complete generalized group action is uniquely determined by its infinitesimal generator.

Proof. Let $\Phi', \Phi''$ be two $G$-complete generalized group actions with the same infinitesimal generator $X = \sum_{i=1}^{n} \xi_i(x) \partial_{x_i}$. Then both functions satisfy
\[
\frac{d}{d\eta} \Phi(\eta, x) = \frac{d}{d\mu} \Phi(\eta + \mu, x) = \frac{d}{d\mu} \Phi(\mu, \Phi(\eta, x)) = \xi(\Phi(\eta, x)).
\]
Now given any $\bar{x} \in \mathbb{R}^n$, it follows that both $\eta \to \Phi'(\eta, \bar{x})$ and $\eta \to \Phi''(\eta, \bar{x})$ solve the initial value problem
\[
\dot{x}(\eta) = \xi(x(\eta)) \\
x(0) = \bar{x}
\]
By assumption this entails that $\Phi'(\cdot, \bar{x}) = \Phi''(\cdot, \bar{x})$ in $(\mathcal{G}(\mathbb{R}))^n$. Consequently,
\[
\Phi'(\eta, \bar{x}) = \Phi''(\eta, \bar{x})
\]
for all $\eta \in \mathcal{R}$ and all $\bar{x} \in \mathbb{R}^n$. The claim now follows from [19], Theorem 2.10. □

As in the classical theory, we are first going to investigate symmetry groups of algebraic equations:

3.6 Definition Let $F \in \mathcal{G}_r(\mathbb{R}^n)$ and let $\Phi$ be a generalized group action on $\mathbb{R}^n$. $\Phi$ is called a symmetry group of the equation
\[
F(x) = 0
\]
in $\mathcal{G}_r(\mathbb{R}^n)$ if for any $\bar{x} \in \mathbb{R}^n$ with $F(\bar{x}) = 0 \in \mathcal{R}$ it follows that $\eta \to F(\Phi(\eta, \bar{x})) = 0$ in $\mathcal{G}(\mathbb{R})$ (or, equivalently, $F(\Phi(\eta, \bar{x})) = 0$ in $\mathcal{R}$ for every $\eta \in \mathcal{R}$).

A characterization of symmetry groups of (generalized) algebraic equations in terms of infinitesimal generators is provided by

3.7 Theorem Let $F \in \mathcal{G}_r(\mathbb{R}^n)$ be of the form
\[
F(x_1, \ldots, x_n) = x_i - f(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)
\]
for some $1 \leq i \leq n$ and $f \in \mathcal{G}_r(\mathbb{R}^{n-1})$. Let $\Phi$ be a $G$-complete generalized group action with infinitesimal generator $X = \sum_{i=1}^{n} \xi_i(x) \partial_{x_i}$ and suppose that $x' \to \xi(x', f(x'))$ defines a generalized vector field on $\mathbb{R}^{n-1}$ such that the corresponding system of ODEs possesses a flow in $(\mathcal{G}_r(\mathbb{R}^{1+(n-1)}))^{n-1}$. The following conditions are equivalent:

(i) $\Phi$ is a symmetry group of $F(x) = 0$.

(ii) If $\bar{x} \in \mathbb{R}^n$ with $F(\bar{x}) = 0 \in \mathcal{R}$ it follows that $X(F)(\bar{x}) = 0$ in $\mathcal{R}$.
Proof. (i) ⇒ (ii): Consider the function \((\eta, x) \to F(\Phi(\eta, x)) \in \mathcal{G}_r(\mathbb{R}^{1+n})\). We have
\[
\frac{d}{d\eta} F(\Phi(\eta, x)) = \sum_{i=1}^n \frac{\partial F}{\partial x_i}(\Phi(\eta, x)) \xi_i(\Phi(\eta, x)) = X(F)(\Phi(\eta, x)),
\]
so that \(\frac{d}{d\eta} F(\Phi(\eta, x)) = X(F)(x)\) in \(\mathcal{G}_r(\mathbb{R}^n)\). Let \(\bar{x} \in \mathbb{R}^n\) such that \(F(\bar{x}) = 0\). Then \(F(\Phi(\eta, \bar{x})) = 0\) in \(\mathcal{G}(\mathbb{R})\). Thus \(\frac{d}{d\eta} F(\Phi(\eta, \bar{x})) = 0\) in \(\mathcal{R}\) which means that \(X(F)(\bar{x}) = 0\) in \(\mathcal{R}\).

(ii) ⇒ (i): We assume \(F(x_1, \ldots, x_n) = x_n - f(x_1, \ldots, x_{n-1})\) and abbreviate \((x_1, \ldots, x_{n-1})\) by \(x'\). Our first claim is that
\[
\xi_n(x', f(x')) = \sum_{j=1}^{n-1} \xi_j(x', f(x')) \partial_j f(x') \text{ in } \mathcal{G}_r(\mathbb{R}^{n-1})
\]
Indeed, if \(\bar{x}' \in \mathbb{R}^{n-1}\) then \(F(x', f(x')) = 0\) in \(\mathcal{R}\). Hence \(X(F)(\bar{x}', f(\bar{x}')) = 0\) in \(\mathcal{R}\) for all \(\bar{x}'\) by our assumption. Our claim now follows from [19], Theorem 2.7.

Consider the following system of ODEs in \(\mathcal{G}_r:\)
\[
\begin{align*}
\dot{x}_j(t) &= \xi_j(x', f(x')) \quad (j = 1, \ldots, n-1) \\
x'(0) &= \bar{a}' \in \mathbb{R}^{n-1}
\end{align*}
\]
By our assumption, this system has a flow \((\eta, a') \to (h_1(\eta, a'), \ldots, h_{n-1}(\eta, a'))\)
in \((\mathcal{G}_r(\mathbb{R}^{1+(n-1)}))^{n-1}\). Set \(g_n(\eta, a) := f(h_1(\eta, a'), \ldots, h_{n-1}(\eta, a'))\). Then \(g_n(0, a) = f(a')\) and
\[
g(\eta, a) = (g_1(\eta, a), \ldots, g_n(\eta, a)) := (h_1(\eta, a'), \ldots, h_{n-1}(\eta, a'), g_n(\eta, a))
\]
is in \((\mathcal{G}_r(\mathbb{R}^{1+n}))^n\). If \(\bar{a} \in \mathbb{R}^n\) then \(F(\eta, \bar{a}) = 0\) in \(\mathcal{R}\) for all \(\eta\). Therefore, if we can show that \(g(\cdot, a) = \Phi(\cdot, \bar{a})\) in \((\mathcal{G}_r(\mathbb{R}))^n\) for all \(\bar{a}\) with \(F(\bar{a}) = 0\), the proof is completed. Now we have \(\dot{g}_j(\eta, a) = \xi_j(g_1(\eta, a), \ldots, g_n(\eta, a))\) for \(1 \leq j \leq n-1\) and
\[
\dot{g}_n(\eta, a) = \left\{ \sum_{i=1}^{n-1} \frac{\partial j}{\partial x_i}(g_1(\eta, a), \ldots, g_{n-1}(\eta, a)) \right\} \dot{g}(\eta, a) = \xi_n(g_1(\eta, a), \ldots, g_n(\eta, a))
\]
If \(F(\bar{a}) = 0\) in \(\mathcal{R}\) then \(\bar{a}_n = f(\bar{a}')\), so that \(g(0, \bar{a}) = (\bar{a}', f(\bar{a}')) = \bar{a} = \Phi(0, \bar{a})\). Thus \(g(\cdot, \bar{a})\) and \(\Phi(\cdot, \bar{a})\) solve the same initial value problem. Since \(X\) is \(\mathcal{G}\)-complete, the claim follows.

\[\Box\]

3.2 Symmetries of Differential Equations

In this section we are going to apply the above results to symmetry groups of differential equations involving generalized functions. To this end, we will first have to define generalized group actions on generalized functions. Once we have
done this, by a symmetry group of a differential equation we will again mean a
group action that transforms solutions into other solutions. Thus, from now on
we will exclusively consider group actions on some space \( \mathbb{R}^p \times \mathbb{R}^q \) of independent
and dependent variables.

**3.8 Definition** A generalized group action \( \Phi \in (\mathcal{G}_r(\mathbb{R} \times \mathbb{R}^{p+q}))^{p+q} \) is called projectable if it is of the form

\[
\Phi(\eta, (x, u)) = (\Xi_\eta(x), \Psi_\eta(x, u)),
\]

where \( \Xi \in (\mathcal{G}_r(\mathbb{R} \times \mathbb{R}^p))^p \) and \( \Psi \in (\mathcal{G}_r(\mathbb{R} \times \mathbb{R}^{p+q}))^q \).

The group properties in this case read:

\[
\Xi_{\eta_1 + \eta_2} = \Xi_{\eta_1} \circ \Xi_{\eta_2} \quad \text{in } \mathcal{G}_r(\mathbb{R}^p) \forall \eta_1, \eta_2 \in \mathbb{R} \tag{31}
\]

\[
\Psi_{\eta_1 + \eta_2}(x, u) = \Psi_{\eta_1}(\Xi_{\eta_2}(x), \Psi_{\eta_2}(x, u)) \quad \text{in } \mathcal{G}_r(\mathbb{R}^{p+q}) \forall \eta_1, \eta_2 \in \mathbb{R} \tag{32}
\]

In particular, we have

\[
\Xi_\eta \circ \Xi_{-\eta} = \text{id} \quad \text{in } \mathcal{G}_r(\mathbb{R}^p) \forall \eta \in \mathbb{R}. \tag{33}
\]

An adaptation of Lie group analysis to spaces of distributions faces the fundamental problem that while the methods of classical Lie group analysis of differential
equations are geometric in the sense that group action on functions is defined via
graphs, in classical distribution theory there is no means of defining graphs of dis-
tributions. However, due to the pointvalue characterization obtained in [19] this
problem can be dealt with in a satisfactory manner within Colombeau algebras:

**3.9 Definition** Let \( U \in (\mathcal{G}(\mathbb{R}^p))^q \) and \( V \in (\mathcal{G}_r(\mathbb{R}^p))^q \). The graphs of \( U \) and \( V \) are defined as

\[
\Gamma_U := \{(\bar{x}, U(\bar{x})) : \bar{x} \in \mathbb{R}^p\}
\]

\[
\Gamma_V := \{(\bar{x}, U(\bar{x})) : \bar{x} \in \mathbb{R}^p\}.
\]

It follows directly from [19], Theorems 2.4 and 2.7 that any generalized function
is uniquely determined by its graph. Our next aim is to define generalized group
actions on generalized functions. As in the classical case this is done geometri-
cally, i.e. by transformation of graphs. The following result is immediate from the
definitions:

**3.10 Proposition** Let \( U \in (\mathcal{G}_r(\mathbb{R}^p))^q \) and let \( \Phi \) be a projectable generalized group action on \( \mathbb{R}^p \times \mathbb{R}^q \). Then \( \Phi_\eta(\Gamma_U) = \Gamma_{\Phi_\eta(U)} \) in \( \mathbb{R}^{p+q} \) for each \( \eta \), where \( \Phi_\eta(U) \)
denotes the element

\[
x \to \Psi_\eta(\Xi_{-\eta}(x), U \circ \Xi_{-\eta}(x))
\]
of \( (\mathcal{G}_r(\mathbb{R}^p))^q \). \qed
We are now able to give a geometric characterization of solutions of PDEs in $\mathcal{G}_r$.

3.11 Proposition Consider the system of PDEs

$$\Delta_\nu(x, U^{[n]}) = 0 \quad 1 \leq \nu \leq l$$

in $\mathcal{G}_r(\mathbb{R}^p)^q$ (where $\Delta \in (\mathcal{G}_r((\mathbb{R}^p \times \mathbb{R}^q)^{[n]}))^l$). Set

$$S_\Delta := \{ \tilde{z} \in \mathcal{R}^{[n]} : \Delta_\nu(\tilde{z}) = 0 \} \quad (1 \leq \nu \leq l).$$

Then $U \in (\mathcal{G}_r(\mathbb{R}^p))^q$ is a solution of the system iff $\Gamma_{pr^{[n]}U} \subseteq S_\Delta$.

Proof. This follows immediately from [19], Theorem 2.7. \hfill \Box

Prolongation of generalized group actions can be handled in a similar fashion as in the classical theory. Thus, let $\Phi$ be a projectable generalized group action on $\mathbb{R}^p \times \mathbb{R}^q$. We want to define the $n$-th prolongation $pr^{[n]}\Phi$ as a projectable generalized group action on $(\mathbb{R}^p \times \mathbb{R}^q)^{(n)}$. Let $z \in (\mathbb{R}^p \times \mathbb{R}^q)^{(n)}$ and choose $h \in \mathcal{O}_M(\mathbb{R}^p)^q$ such that $(z_1, \ldots, z_p, pr^{[n]}h(z_1, \ldots, z_p)) = z$. Now set

$$pr^{[n]}\Phi(\eta, z) := (\Xi_\eta(z_1, \ldots, z_p), pr^{[n]}(\Phi_\eta(\langle \Xi_\eta(z_1, \ldots, z_p)\rangle)).$$

(35)

Using for $h$ a suitable Taylor polynomial, it follows that $pr^{[n]}\Phi \in (\mathcal{G}_r(\mathbb{R} \times (\mathbb{R}^p \times \mathbb{R}^q)^{(n)}))^N$ (where $N = \dim((\mathbb{R}^{p+n})^{(n)})$). Moreover, the definition does not depend on the particular choice of $h$, which follows exactly as in the classical case.

3.12 Lemma Let $\tilde{z} \in (\mathbb{R}^p \times \mathbb{R}^q)^{(n)}$ and assume that $U \in (\mathcal{G}_r(\mathbb{R}^p))^q$ satisfies $(\tilde{z}_1, \ldots, \tilde{z}_p, pr^{[n]}U(\tilde{z}_1, \ldots, \tilde{z}_p)) = \tilde{z}$. Then

$$pr^{[n]}\Phi(\eta, \tilde{z}) = (\Xi_\eta(\tilde{z}_1, \ldots, \tilde{z}_p), pr^{[n]}(\Phi_\eta(U))(\Xi_\eta(\tilde{z}_1, \ldots, \tilde{z}_p))) \quad \forall \eta \in \mathbb{R}. \quad (36)$$

Proof. Let $U = cl[[e_x]]_z$ and choose a representative $(z_x)_z$ of $\tilde{z}$ such that

$$(z_1, \ldots, z_p, pr^{[n]}u_x(z_1, \ldots, z_p)) = z_x \quad \forall x.$$ 

Using the chain rule as in Proposition 2.7, it follows that the right hand sides of (35) (with $z$ replaced by $\tilde{z}$) and of (36) have the same representative (depending exclusively on $(z_x)_z$). \hfill \Box

3.13 Proposition $pr^{[n]}\Phi$ is a generalized group action on $(\mathbb{R}^p \times \mathbb{R}^q)^{(n)}$.

Proof. Property 3.2 (i) is clearly satisfied. Concerning (ii), according to [19], Theorem 2.7 it suffices to show that

$$pr^{[n]}\Phi(\eta_1 + \eta_2, \tilde{z}) = pr^{[n]}\Phi(\eta_1, pr^{[n]}\Phi(\eta_2, \tilde{z})) \quad \forall \eta_1, \eta_2 \in \mathbb{R}, \forall \tilde{z} \in (\mathbb{R}^p \times \mathbb{R}^q)^{(n)}.$$ 

Choose some $U \in (\mathcal{G}_r(\mathbb{R}^p))^q$ with $(\tilde{z}_1, \ldots, \tilde{z}_p, pr^{[n]}U(\tilde{z}_1, \ldots, \tilde{z}_p)) = \tilde{z}$. Then due to Lemma 3.12 we have

$$pr^{[n]}\Phi(\eta_2, \tilde{z}) = (\Xi_{\eta_2}(\tilde{z}_1, \ldots, \tilde{z}_p), pr^{[n]}(\Phi_{\eta_2}(U))(\Xi_{\eta_2}(\tilde{z}_1, \ldots, \tilde{z}_p))).$$

By (36) this implies $pr^{[n]}\Phi(\eta_1, pr^{[n]}\Phi(\eta_2, \tilde{z})) = pr^{[n]}\Phi(\eta_1 + \eta_2, \tilde{z})$. \hfill \Box

As in the classical case we therefore have (using the notations from Proposition 3.11):

20
3.14 Proposition Let $\Phi$ be a projectable generalized group action on $\mathbb{R}^p \times \mathbb{R}^q$ such that $\text{pr}^{(n)}\Phi$ is a symmetry group of the algebraic equation $\Delta(z) = 0$. Then $\Phi$ is a symmetry group of (34).

Proof. If $U \in \mathcal{G}_r(\mathbb{R}^p)$ is a solution of (34) then $\Gamma_{\text{pr}^{(n)}U} \subseteq \mathcal{S}_\Delta$ by Proposition 3.11. Thus

$$\Gamma_{\text{pr}^{(n)}(\Phi[U])} = \text{pr}^{(n)}\Phi_U(\Gamma_{\text{pr}^{(n)}U}) \subseteq \mathcal{S}_\Delta,$$

so that, again from Proposition 3.11, the claim follows.

3.15 Definition Let $X$ be a $\mathcal{G}$-complete generalized vector field. The $n$-th prolongation of $X$ is defined as the infinitesimal generator of the $n$-th prolongation of the generalized group action $\Phi$ corresponding to $X$:

$$\text{pr}^{(n)}X \big|_z = \left. \frac{d}{d\eta} \right|_0 \text{pr}^{(n)}\Phi_{\eta}(z),$$

provided that $\text{pr}^{(n)}\Phi$ is $\mathcal{G}$-complete as well. In this case, both $X$ and $\Phi$ are called $\mathcal{G}$-n-complete.

From Theorem 3.7 and Proposition 3.14 we immediately conclude

3.16 Theorem Under the assumptions of Proposition 3.11, let $\Phi$ be a $\mathcal{G}$-n-complete generalized group action on $\mathbb{R}^p \times \mathbb{R}^q$ with infinitesimal generator $X$ such that the conditions of Theorem 3.7 are satisfied for $\Delta$ and $\text{pr}^{(n)}\Phi$. If

$$\text{pr}^{(n)}X(\Delta) \big(\bar{z}\big) = 0 \quad \forall \bar{z} \in (\mathbb{R}^p \times \mathbb{R}^q)^{(n)} \text{ with } \Delta(\bar{z}) = 0,$$

then $\Phi$ is a symmetry group of (34).

In order to be able to apply the same algorithm as in classical Lie theory for the determination of the symmetry group of a generalized PDE, the final step is to verify that the formulas for prolongation of vector fields carry over to generalized vector fields.

3.17 Theorem Let

$$X = (x, u) \rightarrow \sum_{i=1}^{p} \xi_i(x) \partial_{x_i} + \sum_{\alpha=1}^{q} \psi_\alpha(x, u) \partial_{u_\alpha}$$

be a $\mathcal{G}$-n-complete generalized vector field with corresponding projectable group action $\Phi$ on $(\mathbb{R}^p \times \mathbb{R}^q)$. Then

$$\text{pr}^{(n)}X = X + \sum_{\alpha=1}^{q} \sum_{J} \psi_\alpha^J(x, u^{(n)}) \partial_{u_J}$$

where $J = (j_1, ..., j_k)$, $1 \leq j_k \leq p$ for $1 \leq k \leq n$ and

$$\psi_\alpha^J(x, u^{(n)}) = D_J(\psi_\alpha) - \sum_{i=1}^{p} \xi_i u_{i}^J + \sum_{i=1}^{p} \xi_i u_{i}^J,$$
Proof. Using the machinery developed so far, this is an easy modification of the proof of the classical result (see [21], Theorem 2.36).

We may summarize the results of this section as follows: In order to determine the symmetries of a differential equation involving generalized functions, the algorithm (as in the classical case) is to make an ansatz for the infinitesimal generators, calculate their prolongations according to Theorem 3.17 and then use Theorem 3.16 to determine the defining equations for the coefficient functions of the infinitesimal generators. The defining equations now yield PDEs in \( G \). Any solution of these equations that defines a \( G \)-\( n \)-complete generator will upon integration yield a symmetry group in \( G_r \).

3.18 Example As a simple example of a genuinely generalized group action we consider symmetries of the scalar conservation law

\[
    u_t + F(u)u_x = 0
\]

To illustrate the method we take \( f \) as the discontinuous function \( f(x) = x + \text{sgn}(x) \), suppose \( \rho \geq 0 \) and denote by \( F \) the (invertible) element \( \text{cl}[(f + \rho x)_x] \) of \( G_r(\mathbb{R}) \). The determining equations in this case read

\[
    \Phi_t + F\Phi_x = 0
\]

\[
    -\xi_x + F\tau_t + \tau F_t + \Phi F_u - F\xi_x + F^2\tau_x + \xi F_x = 0
\]

with infinitesimal generator \( v = \xi(x,t)\partial_x + \tau(x,t)\partial_t + \Phi(x,t,u)\partial_u \). As a particular solution we obtain \( v = xt\partial_x + t^2\partial_t + (F'(u))^{-1}(x - tF(u))\partial_u \). The corresponding generalized group action can be calculated explicitly in \( G_r \) showing that if \( u \) is a \( G_r \)-solution to (37) then so is

\[
    (x,t) \rightarrow F^{-1}(\eta x(1 + \eta t)^{-1} + F(u(x(1 + \eta t)^{-1}, t(1 + \eta t)^{-1})(1 + \eta t)^{-1})
\]

Note that (37) with \( F \) as above always possesses solutions in \( G_r(\mathbb{R} \times [0, \infty)) \) for initial data \( U_0 \in G_r(\mathbb{R}) \) with \( U_0' \geq 0 \).

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