Remarks on Kähler Ricci Flow

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Abstract

We show the convergence of Kähler Ricci flow directly if the $\alpha$-invariant of the canonical class is greater than $n/n+1$. Applying these convergence theorems, we can give a Kähler Ricci flow proof of Calabi conjecture on such Fano manifolds. In particular, the existence of KE metrics on a lot of Fano surfaces can be proved by flow method. Note that this geometric conclusion (based on the same assumption) was established earlier via elliptic method by G. Tian (cf. [Tian87], [Tian90] and [Tian97]). However, a new proof based on Kähler Ricci flow should be still interesting in its own right.

1 Introduction

On a Fano manifold, the Kähler Ricci flow was introduced as a possible means to search for Kähler Einstein (KE) metrics. Following Yau’s estimate ([Yau78]), H. D. Cao ([Cao85]) proved that the Kähler Ricci flow with smooth initial metric always exists globally. On a KE manifold, the first named author and Tian showed that Kähler Ricci flow converges exponentially fast toward the KE metric if the initial metric has positive bisectional curvature (cf. [CT1], [CT2]). One important feature of these two papers is that the authors introduced a family of functionals to obtain a uniform $C^0$-estimate of the evolved potential function $\varphi(t, \cdot)$. Once the uniform $C^0$-estimate is established, geometry of the evolved Kähler metrics is completely controlled and the flow will converge in the holomorphic category.

Using his famous $\mu$-functional, Perelman proved that scalar curvature, diameter and normalized Ricci potential are all uniformly bounded along Kähler Ricci flow (cf. [PST]). In 2002, he announced that the Kähler Ricci flow will always converge to the KE metric on any KE manifold. This result was generalized to manifolds with Kähler Ricci solitons by Tian and Zhu ([TZ]). Based on these fundamental estimates, if we assume that initial metric has positive bisectional curvature, the first named author, S. Sun and Tian ([CTS]) proved that the Kähler Ricci flow will converge to a KE metric automatically. Consequently, they give a Ricci flow proof of Frankel conjecture which was initially proved by Siu-Yau (cf. [SiY]) and Morri (cf. [Mor]) independently. Prior to [CTS], partial progress was made in various works, e.g., [Chen] and [PSSW]. The study of Kähler Ricci flow is

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very intense after G. Perelman’s fundamental estimates. In this short note, we don’t plan to analyze all these works in depth. However, we want to list a few references here for the convenience of readers: [Se1], [PSSW2], [PSS], [Hei], [Ru1], [CH], [TZs], [RZZ], [FS] and references therein.

On a Kähler manifold, Calabi conjectured that it admits a KE metric whenever its first Chern class $c_1$ has a definite sign or $c_1$ vanishes. He pointed out that the existence of KE metric is equivalent to the solvability of some Monge-Ampere equation which can be attacked by continuity method. This famous Calabi conjecture was proved by S. T. Yau in his celebrated work [Yau78] when $c_1 < 0$ or $c_1 = 0$. The case of $c_1 < 0$ was also obtained independently by T. Aubin (cf. [Au]). For the case $c_1 > 0$ (Fano manifolds), situation is much more delicate. In [Tian87], [TY] and [Siu], the existence of KE metric was proved on some special Fano manifolds. In the celebrated work [Tian90], Tian finally proved the Calabi conjecture in dimension 2. He showed that a Fano surface $M$ admits a KE metric if and only if its automorphism group is reductive. In higher dimensional Fano manifolds, few general results are known. In [Tian87], Tian introduced the $\alpha$-invariant of a Fano manifold $M^n$ and subsequently proved that $C^0$-estimate holds if $\alpha$-invariant is greater than $\frac{n}{n+1}$. In [Tian90] and [Tian97], he showed that $C^0$-estimate can be obtained from the properness of $F$-functional. Therefore, the Monge-Ampère equation is solvable whenever the $F$-functional is proper or the $\alpha$-invariant of $M^n$ is greater than $\frac{n}{n+1}$. Consequently, there is a KE metric in the canonical class of such Fano manifolds.

Inspired by these famous works in complex Monge-Ampere equation, we attempt to develop some estimates about the potential function $\varphi$ over Kähler Ricci flow. The core issue is to obtain the $C^0$-estimate along the flow. Like the way used in continuity method, one can reduce the $C^0$-estimate of potential function $\varphi$ to an integral estimate of $\varphi$. This point was already observed by Yanir Rubinstein in [Ru1]. As an easy application, we can prove directly that potential function is uniformly bounded along the flow if $\alpha$-invariant is greater than $\frac{n}{n+1}$ or $F$ functional is proper. However, the $C^0$-estimate of $\varphi$ under the condition $F$-functional being proper was proved by Tian and Zhu [TZp] (cf. Proposition3.1 of [TZ] also). We thank Tian for pointing this out to us. For the completeness of our presentation, we include a proof for the same statement.

After we obtain the uniform $C^0$-norm of $\varphi$, from the Kähler Ricci flow equation

$$\dot{\varphi} = \log \frac{\omega^n}{\omega_n^n} + \varphi - h_\omega.$$  

(1)

and the free boundedness of $\dot{\varphi}$ (Lemma 3.1), one can easily see that every metric $\omega_\varphi$ is uniformly equivalent to the initial metric $\omega$. Moreover, the method in section 6 and 7 of [CT2] applies directly on equation (1). It follows that all $k$-th derivatives of $\varphi$ in the fixed gauge are uniformly bounded. Therefore, for every sequence $t_i \to \infty$, there is a subsequence $t_{i_k}$ such that $\omega + \sqrt{-1} \partial\bar{\partial}\varphi(t_{i_k})$ smoothly converges to a limit metric $\omega + \partial\bar{\partial}\varphi(\infty)$ in the same gauge. This limit metric must be a KE metric since it is a Kähler Ricci Soliton metric with constant scalar curvature (See Section 4 for more details). Therefore, the existence of
KE metric is already proved. Furthermore, by considering the first eigenvalue of $\Delta_{\omega}$, one can even show that this flow converges to a unique KE metric exponentially fast (cf. [CT1]).

As applications of our theorems, we can prove the existence of KE metrics on a lot of Fano surfaces by flow method. By classification theory, every Fano surface with zero Futaki invariant is either $\mathbb{CP}^1 \times \mathbb{CP}^1$, $\mathbb{CP}^2$ or diffeomorphic to $\mathbb{CP}^2 \# k\mathbb{CP}^2$, $3 \leq k \leq 8$. As $\mathbb{CP}^1 \times \mathbb{CP}^1$, $\mathbb{CP}^2$ and $\mathbb{CP}^2 \# 3\mathbb{CP}^2$ are all toric surfaces, the Kähler Ricci flows on them are studied in [CW]. The only interesting cases are $4 \leq k \leq 8$. Starting from any metric in canonical class, we can show the $C^0$-norms of potential functions are uniformly bounded if $M \sim \mathbb{CP}^2 \# k\mathbb{CP}^2$, $k = 4, 5, 7$. For $M \sim \mathbb{CP}^2 \# 6\mathbb{CP}^2$, we can only show that $C^0$-estimate hold for some complex structure with nice symmetry. If the symmetry of the complex structure is bad, we need to develop other methods to obtain the $C^0$-estimate, which will be discussed together with $k = 8$ case in a subsequent paper.

The organization of this paper is as follows. In section 2, we setup the notations and basic properties of Kähler geometry. In section 3, we write down some $C^0$-estimates of potential function $\varphi$ along Kähler Ricci flow. As applications of these estimates, we obtain some convergence theorems of Kähler Ricci flow and apply them on Fano surfaces to prove Calabi conjecture in section 4.

The authors would like to remark that the results of this paper have important overlaps with the results of [Ru1]. Many theorems of this paper are actually implied in [Ru1]. All of them are developed by the authors without being aware of the results of [Ru1]. We thank Y.A. Rubinstein for pointing these overlaps out to us.

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2 Basic Kähler Geometry

Let $M$ be an $n$-dimensional compact Kähler manifold. A Kähler metric can be given by its Kähler form $\omega$ on $M$. In local coordinates $z_1, \cdots, z_n$, this $\omega$ is of the form

$$\omega = \sqrt{-1} \sum_{i,j=1}^{n} g_{\bar{j}j} dz^i \wedge d\bar{z}^j > 0,$$

where $\{g_{\bar{j}j}\}$ is a positive definite Hermitian matrix function. The Kähler condition requires that $\omega$ is a closed positive (1,1)-form. Given a Kähler metric $\omega$, its volume form is

$$\omega^n = \frac{1}{n!} \left( \sqrt{-1} \right)^n \det \left( g_{\bar{j}j} \right) dz^1 \wedge d\bar{z}^1 \wedge \cdots \wedge dz^n \wedge d\bar{z}^n.$$
The curvature tensor is
\[ R_{ijkl} = -\frac{\partial^2 g_{ij}}{\partial z^k \partial z^l} + \sum_{p,q=1}^n g^{pq} \frac{\partial g_{ij}}{\partial z^k} \frac{\partial g_{pl}}{\partial z^q}, \quad \forall i,j,k,l = 1,2,\ldots,n. \]

The Ricci curvature form is
\[ \text{Ric}(\omega) = \sqrt{-1} \sum_{i,j=1}^n R_{ij} \omega^i \wedge \omega^j = -\sqrt{-1} \partial \bar{\partial} \log \det(g_{ij}). \]

It is a real, closed (1,1)-form and \[ [\text{Ric}] = 2\pi c_1(M). \]

From now on we assume \( M \) has positive first Chern class, i.e., \( c_1(M) > 0 \). We call \([\omega]\) as a canonical Kähler class if \([\omega] = [\text{Ric}] = 2\pi c_1(M)\). If we require the initial metric is in canonical class, then the normalized Ricci flow (c.f. (author?) [Cao85]) on \( M \) is
\[ \frac{\partial g_{ij}}{\partial t} = g_{ij} - R_{ij}, \quad \forall i,j = 1,2,\ldots,n. \]

Denote \( \omega = \omega_{g(0)}, \omega_{g(t)} = \omega + \sqrt{-1} \partial \bar{\partial} \varphi_t \). \( \varphi_t \) is called the Kähler potential and sometime it is denoted as \( \varphi \) for simplicity. On the level of Kähler potentials, Kähler Ricci flow becomes
\[ \frac{\partial \varphi}{\partial t} = \log \frac{\omega^n}{\omega_{n-1}^n} + \varphi - h_\omega, \]
where \( h_\omega \) is defined by
\[ \text{Ric}(\omega) - \omega = \sqrt{-1} \partial \bar{\partial} h_\omega, \text{ and } \int_M (e^{h_\omega} - 1) \omega^n = 0. \]

As usual, the flow equation (2) or (3) is referred as the Kähler Ricci flow in canonical class of \( M \). It is proved by Cao (author?) [Cao85], who followed Yau’s celebrated work (author?) [Yau78], that this flow exists globally for any smooth initial Kähler metric in the canonical class.

In this note, we only study Kähler Ricci flow in the canonical class. For the simplicity of notation, we may not mention that the flow is in canonical class every time.

Let \( \mathcal{P}(M,\omega) = \{ \varphi | \omega + \sqrt{-1} \partial \bar{\partial} \varphi > 0 \} \). It is shown in [Tian87] that there is a small constant \( \delta > 0 \) such that
\[ \sup_{\varphi \in \mathcal{P}(M,\omega)} \frac{1}{V} \int_M e^{-\delta(\varphi - \sup_{\mathcal{P}(M,\omega)} \varphi)} \omega^n < \infty. \]

The supreme of such \( \delta \) is called the \( \alpha \)-invariant of \((M,\omega)\) and it is denoted as \( \alpha(M,\omega) \). Let \( G \) be a compact subgroup of \( \text{Aut}(M) \) and \( \omega \) is a \( G \)-invariant form. We denote
\[ \mathcal{P}_G(M,\omega) = \{ \varphi | \omega + \sqrt{-1} \partial \bar{\partial} \varphi > 0, \varphi \text{ is invariant under } G \}. \]

Similarly, we can define \( \alpha_G(M,\omega) \).
For every $\varphi \in \mathcal{P}(M, \omega)$, we denote $\omega_\varphi = \omega + \sqrt{-1} \partial \bar{\partial} \varphi$. There are some well known functionals defined in this space.

\[
I_\omega(\varphi) \triangleq \frac{1}{V} \int_M \varphi (\omega^n - \omega_\varphi^n) = \frac{1}{V} \sum_{i=0}^{n-1} \int_M \sqrt{-1} \partial \varphi \wedge \bar{\partial} \varphi \wedge \omega_i \wedge \omega_\varphi^{n-1-i}.
\]

(4)

\[
J_\omega(\varphi) \triangleq \frac{1}{V} \sum_{i=0}^{n-1} \frac{i+1}{n+1} \int_M \sqrt{-1} \partial \varphi \wedge \bar{\partial} \varphi \wedge \omega_i \wedge \omega_\varphi^{n-1-i}.
\]

(5)

\[
F_0^\omega(\varphi) \triangleq J_\omega(\varphi) - \frac{1}{V} \int_M \varphi \omega^n.
\]

(6)

\[
F_\omega(\varphi) \triangleq F_0^\omega(\varphi) - \log \left( \frac{1}{V} \int_M e^{h_\omega - \varphi} \omega^n \right).
\]

(7)

\[
\nu_\omega(\varphi) \triangleq F_\omega(\varphi) + \frac{1}{V} \int_M h_\omega \omega^n - \frac{1}{V} \int_M h_\omega \omega_\varphi^n.
\]

(8)

The last functional is the well known Mabuchi K-energy. It is generally defined by its derivative. The formula here is proved in [DT].

We say $F_\omega$ is proper on $\mathcal{P}(M, \omega)$ if there exists an increasing function $\mu : \mathbb{R} \to [c, \infty)$ satisfying

\[
\lim_{t \to \infty} \mu(t) = \infty; \quad F_\omega(\varphi) \geq \mu(I_\omega(\varphi)), \quad \forall \varphi \in \mathcal{P}(M, \omega).
\]

Here $c$ is some number. In particular, $F_\omega$ is proper implies that $F_\omega$ is bounded from below.

We list some basic properties of these functionals without giving proofs. Interested readers are referred to [Di], [Tb], [CT1] and references therein for more details.

**Proposition 2.1.** Suppose $M$ to be a Fano manifold without nontrivial holomorphic vector field. If $M$ admits a KE metric in its canonical class $[\omega]$, then $F_\omega$ is proper in $\mathcal{P}(M, \omega)$.

**Proposition 2.2.** Along Kähler Ricci flow, $\frac{\partial}{\partial t} F_\omega(\varphi_t) \leq 0$, $\frac{\partial}{\partial t} \nu_\omega(\varphi_t) \leq 0$.

### 3 Estimates along Kähler Ricci flow

As Kähler Ricci flow on Riemannian surface is very clear, we only consider Kähler Ricci flow on Fano manifolds with complex dimension $n \geq 2$.

We first list some well-known estimates.

In his unpublished work, Perelman got some deep estimates along Kähler Ricci flow. We list his estimates below. The detailed proof can be found in Sesum and Tian’s note [PST].

**Proposition 3.1** (Perelman’s Estimates). Suppose $\{(M^n, g(t)), 0 \leq t < \infty\}$ to be a Kähler Ricci flow solution. There are two positive constants $D, \kappa$ depending only on this flow such that the following two estimates hold.
1. Let $R_{g_t}$ be the scalar curvature under metric $g_t$, $h_{\omega(t)}$ be the Ricci potential of form $\omega(t)$ satisfying $\frac{1}{V} \int_M e^{h_{\omega(t)}} \omega^n \omega = 1$. Then we have

$$\|R_{g_t}\|_{C^0} + \text{diam}_{g_t} M + \|h_{\omega(t)}\|_{C^0} + \|\nabla h_{\omega(t)}\|_{C^0} < D.$$ 

2. $\text{Vol}(B_{g_t}(x,r)) r^2 > \kappa$ for every $r \in (0,1)$, $(x,t) \in M \times [0,\infty)$.

In [Zhang] and [Ye], Zhang and Ye obtained independently that Sobolev constant is uniformly bounded along every Kähler Ricci flow solution on a Fano manifold.

**Proposition 3.2** (Sobolev Constant Estimate). There is a uniform Sobolev constant $C_S$ along the Kähler Ricci flow solution $\{ (M^n, g(t)), 0 \leq t < \infty \}$. In other words, for every $f \in C^\infty(M)$, we have

$$\left( \int_M |f|^{\frac{2n}{n-1}} \omega^n \right)^{\frac{n-1}{n}} \leq C_S \left( \int_M |\nabla f|^2 \omega^n + \frac{1}{V^n} \int_M |f|^2 \omega^n \right).$$

As an application of Perelman's estimate, weak Poincaré constant is also bounded along Kähler Ricci flow.

**Proposition 3.3** (Weak Poincaré Constant Estimate). There is a uniform weak Poincaré constant $C_P$ along the Kähler Ricci flow solution $\{ (M^n, g(t)), 0 \leq t < \infty \}$. In other words, for every nonnegative function $f \in C^\infty(M)$, we have

$$\frac{1}{V} \int_M f^2 \omega^n \leq C_P \left\{ \frac{1}{V} \int_M |\nabla f|^2 \omega^n + (\frac{1}{V} \int_M f \omega^n)^2 \right\}.$$

More details about this Proposition can be found in [TZ].

**Lemma 3.1** (c.f. [PSS]). By properly choosing initial condition, we have

$$\|\dot{\varphi}\|_{C^0} + \|\nabla \dot{\varphi}\|_{C^0} < C$$

for some constant $C$ independent of time $t$.

From now on, we always choose initial condition properly such that $\dot{\varphi}$ is uniformly bounded. Remember $\varphi$ satisfies the equation $\dot{\varphi} = \log \frac{\omega^n}{\omega_n} + \varphi - h_\omega$. This equation gives us a lot information.

**Remark 3.1.** When Sobolev constants, Poincaré constants and $\dot{\varphi}$ are all uniformly bounded, we can follow directly from continuity method to estimate $|\varphi|_{C^0}$ from integrations of $\varphi$. This process is described from Lemma 3.2 to Theorem 3.1. However, from a conversation with Yanir Rubinstein, we know that this similarity was already observed by Yanir. Rubinstein [Ru1]. So these estimates were already implied in [Ru1] (See [Ru2] also). Just for the convenience of the readers, we include a complete proof here.
Lemma 3.2. There is a constant $C$ such that
\[
\frac{1}{V} \int_M (-\varphi)\omega^n_{\varphi} \leq n \sup_M \varphi - \sum_{i=0}^{n-1} \frac{i}{n+1} \int_M \sqrt{-1} \partial \varphi \wedge \bar{\partial} \varphi \wedge \omega^i \wedge \omega^{n-1-i} + C. \tag{9}
\]

In particular, we have
\[
\frac{1}{V} \int_M (-\varphi)\omega^n_{\varphi} \leq n \sup_M \varphi + C. \tag{10}
\]

Proof. According to Proposition 2.2, $F_{\omega}(\varphi)$ is non-increasing along Kähler Ricci flow. Therefore, we have
\[
0 = F_{\omega}(0) \geq F_{\omega}(\varphi) = J_{\omega}(\varphi) - \frac{1}{V} \int_M \varphi \omega^n - \log\left(\frac{1}{V} \int_M e^{\varphi \omega^n}\right) = J_{\omega}(\varphi) - I_{\omega}(\varphi) + \frac{1}{V} \int_M (-\varphi)\omega^n_{\varphi} - \log\left(\frac{1}{V} \int_M e^{-\varphi} \omega^n_{\varphi}\right).
\]

It follows that
\[
\frac{1}{V} \int_M (-\varphi)\omega^n_{\varphi} \leq I_{\omega}(\varphi) - J_{\omega}(\varphi) + \log\left(\frac{1}{V} \int_M e^{-\varphi} \omega^n_{\varphi}\right).
\]

Plugging the expression of functional $I$ and $J$, we have
\[
\frac{1}{V} \int_M (-\varphi)\omega^n_{\varphi} \leq \sum_{i=0}^{n-1} \frac{n-i}{n+1} \frac{1}{V} \int_M \sqrt{-1} \partial \varphi \wedge \bar{\partial} \varphi \wedge \omega^i \wedge \omega^{n-1-i} + \log\left(\frac{1}{V} \int_M e^{-\varphi} \omega^n_{\varphi}\right)
\]
\[
\leq \frac{n}{n+1} I_{\omega}(\varphi) - \sum_{i=0}^{n-1} \frac{i}{n+1} \frac{1}{V} \int_M \sqrt{-1} \partial \varphi \wedge \bar{\partial} \varphi \wedge \omega^i \wedge \omega^{n-1-i} + \log\left(\frac{1}{V} \int_M e^{-\varphi} \omega^n_{\varphi}\right)
\]
\[
= \frac{n}{n+1} \frac{1}{V} \int_M \varphi \omega^n + \frac{n}{n+1} \frac{1}{V} \int_M (-\varphi)\omega^n_{\varphi} + \log\left(\frac{1}{V} \int_M e^{-\varphi} \omega^n_{\varphi}\right)
\]
\[
- \sum_{i=0}^{n-1} \frac{i}{n+1} \frac{1}{V} \int_M \sqrt{-1} \partial \varphi \wedge \bar{\partial} \varphi \wedge \omega^i \wedge \omega^{n-1-i}.
\]

Note that $\varphi$ is bounded, so we have
\[
\frac{1}{V} \int_M (-\varphi)\omega^n_{\varphi} \leq \frac{n}{V} \frac{1}{V} \int_M \varphi \omega^n - \sum_{i=0}^{n-1} \frac{i}{V} \int_M \sqrt{-1} \partial \varphi \wedge \bar{\partial} \varphi \wedge \omega^i \wedge \omega^{n-1-i} + C
\]
\[
\leq n \sup_M \varphi - \sum_{i=0}^{n-1} \frac{i}{V} \int_M \sqrt{-1} \partial \varphi \wedge \bar{\partial} \varphi \wedge \omega^i \wedge \omega^{n-1-i} + C.
\]

\[\Box\]
Lemma 3.3. There is a small constant $\delta$ and a big constant $C$ depending only on this flow such that

$$\sup_M \varphi < \frac{1 - \delta}{\delta} \int_M (-\varphi) \omega^n + C. \quad (11)$$

Proof. As $M$ is a Fano manifold, we know there are two constants $\delta, C_\delta$ depending only on the initial metric $\omega$ such that

$$\sup_{\varphi \in \mathcal{P}(M, \omega)} \frac{1}{V} \int_M e^{-\delta(\varphi - \sup_M \varphi)} \omega^n < C_\delta. \quad (12)$$

In particular, for every time $t$, $\varphi_t$ satisfies this inequality. Along Kähler Ricci flow, we have $\frac{\omega^n}{\omega^n} = e^{\varphi - h}$. Plugging this into the previous inequality yields that

$$\frac{1}{V} \int_M e^{(1 - \delta)\varphi + \delta \sup_M \varphi - h} \omega^n < C_\delta. \quad (13)$$

The convexity of exponential function tells us

$$\frac{1}{V} \int_M \left\{ \delta \sup_M \varphi + (1 - \delta)\varphi - h \right\} \omega^n < \log C_\delta.$$

Therefore there is a constant $C$ depending only on this flow such that

$$\sup_M \varphi < \frac{1 - \delta}{\delta} \frac{1}{V} \int_M (-\varphi) \omega^n + C. \quad (14)$$

\[ \square \]

Lemma 3.4. There is a constant $C$ depending only on the flow such that

$$\|\varphi\|_{C^0(M)} < C(\max \{0, \sup_M \varphi\} + 1).$$

Proof. Lemma 3.2 tells us $\frac{1}{V} \int_M (-\varphi) \omega^n \leq n \sup_M \varphi + C_1$. So we can choose constant $C_2 > n$ such that

$$\frac{1}{V} \int_M (-\varphi) \omega^n \leq C_2(\max \{0, \sup_M \varphi\} + 1). \quad (15)$$

Clearly, $\sup_M \varphi \leq \max \{0, \sup_M \varphi\} < C_2(\max \{0, \sup_M \varphi\} + 1)$. Define

$$\tilde{\varphi} \triangleq \varphi - 2C_2(\max \{0, \sup_M \varphi\} + 1).$$

\[ 1 \]

\[ \begin{align*}
\frac{1}{V} & \int_M (-\varphi) \omega^n \\
& \leq C_2(\max \{0, \sup_M \varphi\} + 1).
\end{align*} \]
We have
\[
\sup_M \hat{\varphi} < -1, \quad \frac{1}{V} \int_M |\hat{\varphi}| \omega^n_\varphi = \frac{1}{V} \int_M (-\hat{\varphi}) \omega^n_\varphi < 3C_2(\max_M \{0, \sup_M \varphi\} + 1).
\]

Direct computation shows that
\[
\int_M |\nabla| \hat{\varphi}^{p+1} \omega^n_\varphi = \frac{(p+1)^2}{4} \int_M \sqrt{-1} \partial \bar{\partial} \hat{\varphi}^{p+1} \omega^n_\varphi
\]
\[
= \frac{(p+1)^2}{4p} \int_M \sqrt{-1} \partial |\hat{\varphi}|^{p-1} \partial |\hat{\varphi}| \wedge \omega^n_\varphi
\]
\[
= -\frac{(p+1)^2}{4p} \int_M |\hat{\varphi}|^p \sqrt{-1} \partial \bar{\partial} |\hat{\varphi}| \wedge \omega^n_\varphi
\]
\[
= \frac{(p+1)^2}{4p} \int_M |\hat{\varphi}|^p \sqrt{-1} \partial \bar{\partial} |\hat{\varphi}| \wedge \omega^n_\varphi
\]
\[
\leq \frac{(p+1)^2}{4p} \int_M |\hat{\varphi}|^p \omega^n_\varphi. \tag{14}
\]

Let \( p = 1 \), we have \( \int_M |\nabla| \hat{\varphi}|^2 \omega^n_\varphi \leq \int_M |\hat{\varphi}| \omega^n_\varphi \). Applying Poincaré inequality (Proposition 3.3) to \(-\hat{\varphi}\) yields
\[
\int_M |\hat{\varphi}| \omega^n_\varphi \leq C_P \left( \int_M |\nabla \hat{\varphi}|^2 \omega^n_\varphi + \frac{1}{V} \left( \int_M |\hat{\varphi}| \omega^n_\varphi \right)^2 \right)
\]
\[
\leq C_P \left( \int_M |\hat{\varphi}| \omega^n_\varphi + \frac{1}{V} \left( \int_M |\hat{\varphi}| \omega^n_\varphi \right)^2 \right).
\]

Using inequality (13), we obtain
\[
\left( \int_M |\hat{\varphi}|^2 \omega^n_\varphi \right)\frac{1}{2} \leq C_3(\max_M \{0, \sup_M \varphi\} + 1). \tag{15}
\]

For general \( p \geq 1 \), inequality (14) can be rewritten as
\[
\int_M |\nabla| \hat{\varphi}^{p+1} \omega^n_\varphi \leq \frac{(p+1)^2}{4p} \int_M |\hat{\varphi}|^p \omega^n_\varphi < \frac{(p+1)^2}{4p} \int_M |\hat{\varphi}|^{p+1} \omega^n_\varphi.
\]

Since Sobolev constants are uniformly bounded along Kähler Ricci flow, standard Moser iteration yields that
\[
\|\hat{\varphi}\|_{C^0} \leq C_4 \left( \int_M |\hat{\varphi}|^2 \omega^n_\varphi \right)^{\frac{1}{2}}.
\]

Combining this with inequality (15), we obtain
\[
\|\hat{\varphi}\|_{C^0} \leq C_3 C_4(\max_M \{0, \sup_M \varphi\} + 1).
\]
\[
\int_M (\varphi) \omega^n < C \quad \Longleftrightarrow \quad \sup_M \varphi < C \quad \Longleftrightarrow \quad \int_M \varphi \omega^n < C
\]
\[
\inf_M \varphi > -C \quad \Longleftrightarrow \quad \sup_M \varphi < C \quad \Longleftrightarrow \quad \int_M \varphi \omega^n < C
\]
\[
|\varphi|_{C^0(M)} < C \quad \Longleftrightarrow \quad \text{Osc}_M \varphi < C
\]

Table 1: The relations among bounds

Remember \( \varphi = \bar{\varphi} + 2C_2(\max\{0, \sup_M \varphi\} + 1) \). Let \( C = 2C_2 + C_3C_4 + 1 \), we have

\[ \| \varphi \|_{C^0(M)} < C(\max\{0, \sup_M \varphi\} + 1). \]

**Theorem 3.1.** Along Kähler Ricci flow \( \{(M^n, g(t)), 0 \leq t < \infty\} \) in the canonical class of Fano manifold \( M \), the following conditions are equivalent.

- \( \varphi \) is uniformly bounded.
- \( \sup_M \varphi \) is uniformly bounded from above.
- \( \inf_M \varphi \) is uniformly bounded from below.
- \( \int_M \varphi \omega^n \) is uniformly bounded from above.
- \( \int_M (\varphi) \omega^n \) is uniformly bounded from above.
- \( I_\omega(\varphi) \) is uniformly bounded.
- \( \text{Osc}_M \varphi \) is uniformly bounded.

**Proof.** Look at Table 1, it contains three circles: (1234), (256) and (12789). In order to prove this theorem, we only need to show the induction go through in every circle. However, step 1 is nothing but Lemma 3.3, step 2 is just Lemma 3.4, steps 3, 4, 5, 7 are trivial. So only steps 6, 8, 9 need proof.

**Step 6.** \( \int_M \varphi \omega^n \) bounded from above \( \Rightarrow \sup_M \varphi \) bounded from above.

Since \( \omega + \sqrt{-1} \partial \bar{\partial} \varphi > 0 \), take trace under metric \( \omega \), we have \( -\Delta \varphi < n \). Plugging it
into Green’s function formula implies

\[
\varphi(p) = \frac{1}{V} \int_M \varphi \omega^n - \frac{1}{V} \int_M G(p, q) \Delta \varphi \omega^n(q)
\]

\[
\leq \frac{1}{V} \int_M \varphi \omega^n + \frac{n}{V} \int_M G(p, q) \omega^n(q)
\]

\[
\leq \frac{1}{V} \int_M \varphi \omega^n + C_1.
\]

Here \(G\) is the nonnegative Green function of \((M, \omega)\), \(p\) is any point in \(M\). It follows that

\[
\sup_M \varphi \leq \frac{1}{V} \int_M \varphi \omega^n + C_1.
\]

Therefore \(\sup_M \varphi\) is bounded from above whenever \(\int_M \varphi \omega^n\) is bounded from above.

**Step 8.** \(\text{Osc}_M \varphi\) bounded \(\Rightarrow I_\omega(\varphi)\) bounded.

Note that \(I_\omega(\varphi) = \frac{1}{V} \int_M \varphi \omega^n - \frac{1}{V} \int_M \varphi \omega^n\) \(\leq \sup_M \varphi - \inf_M \varphi = \text{Osc}_M \varphi\).

**Step 9.** \(I_\omega(\varphi)\) bounded \(\Rightarrow \int_M (-\varphi) \omega^n\) bounded from above.

The Kähler Ricci flow equation yields

\[
\frac{1}{V} \int_M (-\varphi) \omega^n = \frac{1}{V} \int_M \{ \log \frac{\omega^n}{\omega^n} - \dot{\varphi} - h_\omega \} \omega^n
\]

\[
\leq \frac{1}{V} \int_M \{ -\dot{\varphi} - h_\omega \} \omega^n + \log \left\{ \frac{1}{V} \int_M \omega^n \right\}
\]

\[
= \frac{1}{V} \int_M \{ -\dot{\varphi} - h_\omega \} \omega^n
\]

\[
< C_2. \tag{16}
\]

Here we used the fact that both \(\dot{\varphi}\) and \(h_\omega\) are uniformly bounded. Therefore, we have

\[
\frac{1}{V} \int_M (-\varphi) \omega^n = I_\omega(\varphi) + \frac{1}{V} \int_M (-\varphi) \omega^n < I_\omega(\varphi) + C_2. \tag{17}
\]

This means that \(I_\omega(\varphi)\) is bounded implies \(\int_M (-\varphi) \omega^n\) is bounded from above. \(\square\)

### 4 Application of the Estimates

If \(\varphi\) is uniformly bounded along the Kähler Ricci flow, then there must be a KE metric in the canonical class. Actually, as we discussed in the introduction, there is a limit metric form \(\omega_\infty = \omega + \sqrt{-1} \partial \bar{\partial} \varphi_\infty\) in the canonical class. As both \(\varphi\) and \(h_\omega\) (Perelman’s estimate) are uniformly bounded along the flow, it’s easy to see from the definition of the functionals that \(I_\omega(\varphi), J_\omega(\varphi), F_\omega(\varphi)\) and \(\nu_\omega(\varphi)\) are all bounded. In particular, the \(K\)-energy \(\nu_\omega(\varphi)\) is bounded from below. Therefore \(\omega_\infty\) must be a metric with constant scalar curvature (cf. section 7 of \textit{[CT2]}, or \textit{[PSSW1]}). On the other hand, as a critical metric of Perelman’s
W-functional, $\omega_\infty$ must be a Kähler Ricci soliton. So there is a smooth function $f$ such that

$$R_{ij} + f_{ij} - g_{ij} = 0, \quad f_{ij} = f_{i\bar{j}} = 0.$$  

Taking trace yields $R + \Delta_{\omega_\infty} f - 2 = 0$. Since $R$ is constant, $\Delta_{\omega_\infty} f$ has to be a constant and consequently zero. It follows that $f$ is a constant and we have $R_{ij} = g_{ij}$ under the metric $\omega_\infty$. This means $\omega_\infty$ is a KE metric in the canonical class. Notice our convergence is in a fixed gauge, every limit KE metric form $\omega_\infty$ is compatible with the original complex structure. There is no “jump” of complex structure at all in this limit process. Using the method in [CT1] and [CT2], we are able to show that the Kähler Ricci flow converges exponentially fast to a KE metric.

Therefore, as corollary of Theorem 3.1, we have the following theorems.

**Theorem 4.1.** \{$(M^n, g(t)), 0 \leq t < \infty$\} is a Kähler Ricci flow solution initiating from a $G$-invariant metric $\omega$. $\alpha_G(M, \omega) > \frac{n}{n+1}$. Then this flow converges exponentially fast to a KE metric.

**Proof.** We only need to show $\varphi$ is uniformly bounded along the flow.

Recall Lemma 3.2 and Lemma 3.3 Combining inequality (10) and (11), we have

$$\frac{1}{V} \int_M (-\varphi) \omega^n \varphi \leq n \sup_M \varphi + C_1 \leq n \cdot \frac{1 - \delta}{\delta} \frac{1}{V} \int_M (-\varphi) \omega^n \varphi + C_2.$$  

Since $\alpha_G > \frac{n}{n+1}$, we can choose $\delta > \frac{n}{n+1}$ such that $n \cdot \frac{1 - \delta}{\delta} < 1$. Therefore,

$$(1 - n \cdot \frac{1 - \delta}{\delta}) \frac{1}{V} \int_M (-\varphi) \omega^n \varphi < C_2.$$  

It follows that both $\frac{1}{V} \int_M (-\varphi) \omega^n \varphi$ and $\sup_M \varphi$ are uniformly bounded. So Theorem 3.1 implies that $\varphi$ is uniformly bounded.

**Remark 4.1.** This theorem implies the existence of KE metric on a lot of Fano manifolds. For example, every Fano manifold $M$ without $G$-invariant multiplier ideal sheaf (See [Na]) has $\alpha_G(M, \omega) \geq 1 > \frac{n}{n+1}$. Therefore KE metric exists in its canonical class.

On a Mukai-Umemura 3-fold $M$, Donaldson ([Don]) showed that $\alpha_{SO(3)}(M, \omega) = \frac{5}{6} > \frac{3}{4}$. Therefore Calabi conjecture holds on this manifold.

**Theorem 4.2** ([TZ1]). \{$(M^n, g(t)), 0 \leq t < \infty$\} is a Kähler Ricci flow solution initiating from $\omega$. $F_\omega$ is proper on the space $\mathcal{P}(M^n, \omega)$. Then this flow converges exponentially fast to a KE metric.

**Proof.** As $F$ is proper and $F_\omega(\varphi) \leq F_\omega(0)$ along Kähler Ricci flow, we see that $I_\omega(\varphi)$ is uniformly bounded. So $\varphi$ is uniformly bounded.
Perelman has claimed that the Kähler Ricci flow will converge to the KE metric if KE metric exists in the canonical class (A generalization of this claim is proved in [TZ]). These two theorems can be achieved directly from Tian’s existence theorem of KE metrics and Perelman’s claim. However, we obtain these theorems from Ricci flow without assuming the existence of KE metric.

**Lemma 4.1.** Suppose \((M^n, \omega)\) is a Fano manifold, \((N^n, \omega_{KE})\) is a Kähler Einstein manifold, \(\pi : M \to N\) is a branched covering map satisfying \([\pi^*\omega_{KE}] = \lambda c_1(M)\) for some number \(\lambda > 1\), \(G\) is the deck transformation group and \(\omega\) is \(G\)-invariant.

If \(\int_M (\pi^*\omega^n_{KE})^{-\gamma} \omega^n < \infty\) for some \(\gamma > \frac{1}{\lambda - 1}\), then the Kähler Ricci flow initiating from \((M, \omega)\) will converge exponentially fast to a KE metric. In particular, there exists a KE metric on \(M\).

**Proof.** For simplicity, we suppose this covering is \(p\)-sheeted, i.e., \(|G| = p\). We denote \(f\) as the smooth function \(\pi^* \omega_{KE}\). Let \(\varphi = \varphi_t, \pi^* (\frac{1}{\lambda} \omega_{KE}) = \omega + \sqrt{-1} \partial \bar{\partial} u\), we have

\[
F_\omega(\varphi) = F^0_\omega(\varphi) - \log \left( \frac{1}{V} \int_M e^{\varphi - \varphi_0} \omega^n \right)
\]

\[
= F^0_\omega(u) + F^0_{\pi^* (\frac{1}{\lambda} \omega_{KE})} (\varphi - u) - \log \left( \frac{1}{V} \int_M e^{-\varphi} \omega^n \right).
\]

Note that \(u\) is a fixed function and \(F^0_\omega(u)\) is a fixed number. So we have

\[
F_\omega(\varphi) \geq F^0_{\pi^* (\frac{1}{\lambda} \omega_{KE})} (\varphi - u) - C
\]

\[
= \frac{1}{\lambda} F^0_{\pi^* (\omega_{KE})} (\lambda (\varphi - u)) - C
\]

\[
= \frac{1}{\lambda} F^0_{\omega_{KE}} (\lambda (\pi^* (\varphi - u))) - C. \quad (18)
\]

The last step is well defined since \(\varphi\) is \(G\)-invariant. Notice that \(F_{\omega_{KE}} (\lambda (\pi^* (\varphi - u)))\) is bounded from below. It follows from inequality (18) that

\[
F_\omega(\varphi) \geq \frac{1}{\lambda} \left\{ F_{\omega_{KE}} (\lambda (\pi^* (\varphi - u))) \right\} + \log \left( \frac{1}{\text{Vol}(N)} \int_N e^{\lambda (\pi^* (u - \varphi))} \omega^n \right) - C
\]

\[
\geq \frac{1}{\lambda} \log \left( \frac{1}{p \text{Vol}(N)} \int_M e^{\lambda (u - \varphi)} \pi^* \omega_{KE} \right) - C
\]

\[
\geq \frac{1}{\lambda} \log \left( \int_M e^{-\lambda \varphi} \pi^* \omega_{KE} \right) - C. \quad (19)
\]

In the last step, we used the property that \(u\) is bounded on \(M\).

Let \(\beta = \frac{\lambda \gamma}{\gamma + 1} > 1\). Hölder inequality implies

\[
\int_M e^{-\beta \varphi} \omega^n = \int_M e^{-\beta \varphi} \cdot f^\frac{\beta}{\gamma} \cdot f^\frac{\beta}{\gamma} \omega^n
\]

\[
\leq \left( \int_M e^{-\lambda \varphi} f^n \right)^{\frac{\gamma}{\gamma + 1}} \left( \int_M f^{-\gamma} \omega^n \right)^{\frac{1}{\gamma + 1}}.
\]

13
As $\int_M f^{-\gamma} \omega^n$ is a finite number, we have

$$\int_M e^{-\beta \varphi} \omega^n \leq C \left( \int_M e^{-\lambda \varphi} \pi^n \omega_{KE}^n \right)^{\frac{\gamma}{\gamma+1}}.$$  

It follows that

$$\frac{\gamma}{\gamma+1} \log\left( \int_M e^{-\lambda \varphi} \pi^n \omega_{KE}^n \right) \geq -C + \log\left( \int_M e^{-\beta \varphi} \omega^n \right).$$

Putting this inequality into (19) gives us

$$F_\omega(\varphi) \geq \left( \frac{\gamma}{\gamma+1} \right) \frac{\beta - 1}{\beta} \int_M (-\varphi) \omega_{\varphi}^n - C.$$  

The convexity of exponential map together with monotonicity of $F_\omega(\varphi)$ implies that

$$0 = F_\omega(0) \geq F_\omega(\varphi) \geq \left( \frac{\beta - 1}{\beta} \right) \int_M (-\varphi) \omega_{\varphi}^n - C.$$  

As $\beta - 1 > 0$, the previous inequality implies $\int_M (-\varphi) \omega_{\varphi}^n$ is uniformly bounded from above along the flow. By Theorem 3.1 we know $\varphi$ is uniformly bounded along the flow. Therefore this flow converges exponentially fast to a KE metric.

If we denote $R(\pi) \subset M$ as the ramification divisor of $\pi$. Choose $x \in R(\pi)$, let $s$ be the defining holomorphic function (locally defined) of $R(\pi)$ at $x$. Define

$$\alpha_x(R(\pi)) \triangleq \sup\{ \lambda \geq 0 : |s|^{-2\lambda} \text{ is } L^1 \text{ on a neighborhood of } x \}.$$  

$$\alpha(R(\pi)) \triangleq \inf_{x \in R(\pi)} \alpha_x(R(\pi)).$$

Note that $\alpha(R(\pi)) = 1$ if $R(\pi)$ is a reduced smooth divisor. Denote

$$c \triangleq \sup\{ \lambda \geq 0 : \int_M \left( \frac{\pi^n}{\omega_{KE}^n} \right)^{-\lambda} \omega^n < \infty \}.$$  

It’s shown in [Ab] (Lemma 2.8) that $c = \alpha(R(\pi))$. Therefore, we have the following theorem.

**Theorem 4.3.** Suppose $(M^n, \omega)$ is a Fano manifold, $(N^n, \omega_{KE})$ is a Kähler Einstein manifold, $\pi : M \to N$ is a branched covering map satisfying $[\pi^* \omega_{KE}] = \lambda c_1(M)$ for some number $\lambda > 1$. $R(\pi)$ is the ramification divisor of $\pi$, $G$ is the deck transformation group and $\omega$ is $G$-invariant.

If $\alpha(R(\pi)) > \frac{1}{\chi - 1}$, then the Kähler Ricci flow initiating from $(M, \omega)$ will converge exponentially fast to a KE metric. In particular, there exists a KE metric on $M$.  

14
As in \[An\], we can generalize this theorem to multiple covers. We first fix some notations.

Let \( D_0, \ldots, D_l \) be divisors of \( M \). Fix \( x \in \bigcup_{i=0}^l D_i \). Let \( f_i \) be the local defining holomorphic functions of \( D_i \). Define
\[
\alpha_x(D_0, \ldots, D_l) \triangleq \sup \{ \delta \geq 0 : (|f_0| + \cdots |f_l|)^{-2\delta} \text{ is } L^1 \text{ on a neighborhood of } x \}. 
\]
\[
\alpha(D_0, \ldots, D_l) \triangleq \inf_{x \in \bigcup_{i=0}^l D_i} \alpha_x(D_0, \ldots, D_l).
\]

Then Theorem 4.3 can be generalized as the following.

**Theorem 4.4.** Suppose \((M^n, \omega)\) is a Fano manifold, \((N^n_i, \omega_{KE}), 0 \leq i \leq l\) is a Kähler Einstein manifold, \(\pi_i : M \rightarrow N_i\) is a branched covering map satisfying \([\pi^* \omega_{KE}] = \lambda_i c_1(M)\) for some number \(\lambda_i > 1\). \(R(\pi_i)\) is the ramification divisor of \(\pi_i\), \(G_i\) is the deck transformation group and \(\omega\) is \(G_i\)-invariant for every \(0 \leq i \leq l\).

If \(\alpha(R(\pi_0), \ldots, R(\pi_l)) > \max_{0 \leq i \leq l} \frac{1}{\lambda_i - 1}\), then the Kähler Ricci flow initiating from \((M, \omega)\) will converge exponentially fast to a KE metric. In particular, if \(\bigcap_{i=0}^l R(\pi_i) = \emptyset\), then the Kähler Ricci flow initiating from \((M, \omega)\) will converge exponentially fast to a KE metric.

Then let’s apply previous theorems on Fano surfaces.

**Corollary 4.1** ([Hei]). Let \(M\) be a Fano surface and \(M \sim \mathbb{CP}^2 \# 4 \mathbb{CP}^2\). \(\omega\) is a metric form in canonical class. Then the Kähler Ricci flow initiating from \(\omega\) converges exponentially fast to a KE metric.

*Proof.* First we show that there exists a KE metric in the canonical class. Start Kähler Ricci flow from a metric \(\omega_0\) which is invariant under the finite automorphism group \(G\). Remember \(\alpha_G(M, \omega_0) > \frac{2}{3}\) (See \([TY]\), Theorem 4.1) implies the existence of KE metric in the canonical class.

Now we consider the Kähler Ricci flow initiating from any metric \(\omega\) in the canonical class. Note that \(M\) has no nontrivial holomorphic vector field. By Proposition 2.1 \(F_\omega\) is proper since the existence of a KE metric. By Theorem 4.2 we obtain the result we need.

**Corollary 4.2.** Let \(M\) be a Fano surface and \(M \sim \mathbb{CP}^2 \# k \mathbb{CP}^2, k = 5, 7\). \(\omega\) is a metric form in canonical class. Then the Kähler Ricci flow initiating from \(\omega\) converges exponentially fast to a KE metric.

*Proof.* As Aut\((M)\) is discrete for each such \(M\), the existence of KE metric will imply the properness of \(F_\omega\) by Proposition 2.1. Therefore, by theorem 4.2 we only need to show the existence of a KE metric. We will use Kähler Ricci flow with symmetry to find the KE
metric. The main tools are Theorem 4.3 and Theorem 4.4. For the construction of the branched coverings, see [De] for details.

**Case 1.** $M \sim \mathbb{CP}^2 \# 5 \mathbb{CP}^2$.

Let $N_0 = N_1 = N_2 = N_3 = N_4 = \mathbb{CP}^1 \times \mathbb{CP}^1$, we can construct $\pi_i : M \to N_i$, $0 \leq i \leq 4$ such that

$$\bigcap_{i=0}^{4} R(\pi_i) = \emptyset.$$

Moreover, let $G_i$ be the deck transformation for covering $\pi_i$, we can find a metric $\omega$ which is invariant under every $G_i$. Theorem 4.4 applies and Kähler Ricci flow tends to a KE metric in $\bigcap_{i=0}^{4} \mathcal{P}_{G_i}(M, \omega)$.

Actually, $M \sim \mathbb{CP}^2 \# 5 \mathbb{CP}^2$ can be embedded into $\mathbb{CP}^4$ as the complete intersection of two quadrics $Q_1$ and $Q_2$. By the result of Miles Reid ([Re]), we can find a coordinate system of $\mathbb{CP}^4$ such that

$$Q_1 = \{x_0^2 + x_1^2 + x_2^2 + x_3^2 + x_4^2 = 0\}$$

$$Q_2 = \{a_0 x_0^2 + \cdots + a_4 x_4^2 = 0\}$$

where $a_i \neq a_j$ when $i \neq j$. As $M = Q_1 \cap Q_2$, by “forgetting” $x_0$, we obtain a projection map $\pi_0 : M \to N_0 \subset \mathbb{CP}^3$ where

$$N_0 = \{(x_1 : x_2 : x_3 : x_4) | (a_1 - a_0)x_1^2 + \cdots (a_4 - a_0)x_4^2 = 0\}.$$

So $N_0$ is a smooth quadratic surface in $\mathbb{CP}^3$. It is biholomorphic to $\mathbb{CP}^1 \times \mathbb{CP}^1$ and it admits a KE metric. Similarly, we can define $\pi_i, 1 \leq i \leq 4$. We have

$$\bigcap_{i=0}^{4} R(\pi_i) \subset \bigcap_{i=0}^{4} \{x_i = 0\} = \emptyset.$$

$G_i = \mathbb{Z}_2$ and it acts on $M$ by multiplying $\pm 1$ on the $i$-th coordinate. Clearly, by taking average, we can find a metric $\omega$ which is invariant under $\oplus_{i=0}^{4} G_i$.

**Case 2.** $M \sim \mathbb{CP}^2 \# 7 \mathbb{CP}^2$.

In this case, $M$ is a branched double cover of $N = \mathbb{CP}^2$. $R(\pi)$ is a smooth curve, so $\alpha(R(\pi)) = 1$. $[\pi_*(\omega_{KE})] = 3c_1(M)$, so $\lambda = 3$. Clearly, $\alpha(R(\pi)) = 1 > \frac{1}{\lambda - 1} = \frac{1}{2}$. So Theorem 4.3 applies and there exists a KE metric in $\mathcal{P}_G(M, \omega)$.

**Corollary 4.3.** Let $M$ be a Fano surface and $M \sim \mathbb{CP}^2 \# 6 \mathbb{CP}^2$. It is well known that $M$ is a cubic surface in $\mathbb{CP}^3$. Suppose that

$$M = \{x_0^3 + \cdots + x_3^3 + f(x_1, \cdots, x_3) = 0\} \subset \mathbb{CP}^3$$

for some $l \geq 1$. $\omega$ is a metric form in canonical class. Then the Kähler Ricci flow initiating from $\omega$ converges exponentially fast to a KE metric.
Proof. As in the previous two corollaries, we only need to prove the existence of KE metric.

In this case, $M$ admits $l+1$ branched covering $\pi_i : M \to N_i = \mathbb{CP}^2$ obtained by

$$\pi(x_0, \cdots, x_3) = (x_0, \cdots, \hat{x}_i, \cdots, x_3), \quad 0 \leq i \leq l.$$ $G_i = \mathbb{Z}_3$ acts by multiplication of roots of $z^3 = 1$ on the $i$-th coordinate of $\mathbb{CP}^3$. Direct computation shows $[\pi_i^*(\omega_{FS})] = 3c_1(M)$ for every $0 \leq i \leq l$. As argued in Proposition 3.1 of [Ar], we can obtain

$$\alpha(R(\pi_0), \cdots, R(\pi_l)) > \frac{1}{3-1} = \frac{1}{2}.$$ So Theorem 4.4 applies. Starting from an $\omega$ which is invariant under $\oplus_{i=0}^l G_i$, Kähler Ricci flow will converge to a KE metric.

For those cubic surfaces $(\mathbb{CP}^2 \# 6\mathbb{CP}^2)$ with bad symmetry, i.e., those cubic surfaces whose equations cannot be written as in Corollary 4.3, we have other methods to study the behavior of Kähler Ricci flow on it, which will be discussed together with Kähler Ricci flow on $\mathbb{CP}^2 \# 8\mathbb{CP}^2$ in a subsequent paper.

Remark 4.2. On $\mathbb{CP}^2 \# 4\mathbb{CP}^2$ and $\mathbb{CP}^2 \# 5\mathbb{CP}^2$ with special complex structure, Gordon Heier ([Hei]) proved the convergence of Kähler Ricci flow by multiplier ideal sheaf method. This method is first studied in [PSS] for general Kähler Ricci flow on Fano manifolds. It is improved by Yanir A. Rubinstein in [Ru1].

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