FOURIER COEFFICIENTS OF HALF-INTEGRAL WEIGHT CUSP FORMS AND WARRING’S PROBLEM

FABIAN WAIBEL

Abstract. Extending the approach of Iwaniec and Duke, we present strong uniform bounds for Fourier coefficients of half-integral weight cusp forms of level $N$. As an application, we consider a Waring-type problem with sums of mixed powers.

1. Introduction

A positive, integral, symmetric $k \times k$ matrix $A$ with even diagonal elements gives rise to a quadratic form $q(x) := \frac{1}{2}x^tAx$. It is a central problem of number theory to study the representation function $r(q,n) := \#\{x \in \mathbb{Z}^k | q(x) = n\}$.

One way to do so is by examining the theta series $\theta(q,z) := \sum_{x \in \mathbb{Z}^k} e(q(x)z) = \sum_{n=0}^{\infty} r(q,n)e(nz)$

which is a modular form of (generally) half-integral weight of level $N$, where $N$ is the level of $q$. To understand $r(q,n)$, decompose $\theta(q,z)$ into an Eisenstein series and a cusp form. To treat the cusp form contribution, one may apply the results from the late eighties from Iwaniec [6], Duke [2], and Duke-Schulze-Pillot [3]. Let $f(z) = \sum_{n \geq 1} a(n)e(nz)$ be a holomorphic cusp form of half-integral weight $k/2$, $k \geq 3$ for the group $\Gamma = \Gamma_0(N)$, normalized with respect to

\begin{equation}
\langle f, g \rangle = \int_{\Gamma \backslash \mathbb{H}} f(z)\overline{g(z)}y^k \frac{dx dy}{y^2}.
\end{equation}

Then, it was shown in [2, 6] that, for squarefree $n$,

\begin{equation}
a(n) \ll n^{k/4 - 2/7 + \epsilon}
\end{equation}

provided that $f \in U^\perp$ for $k = 3$, where $U$ is the subspace of theta functions of $S_{3/2}(N, \chi)$ of type $\sum_{n \geq 1} \psi(n)ne(tn^2z)$ for some real character $\psi$ and $4t \mid N$.

The aim of this paper is threefold: we extend the bound (2) to arbitrary $n$, we include forms of level $N$ with arbitrary nebentypus and improve the bound with respect to $N$. For the second point we need to bear in mind that the Weil-Estermann bound does not necessarily hold for twisted Kloosterman sums for prime power moduli (cf. [8, Example 9.9]). The main strategy follows the work of Duke and Iwaniec [2, 6] with the extensions of Blomer [1].

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If $d$ divides a power of $x$, we write $d \mid x^\infty$, and we denote the squarefree kernel by $\text{rad}(n)$.

**Theorem 1.** Fix an orthonormal basis $\{\varphi_j = \sum_{n \geq 1} a_j(n)e(nz)\}^d_{j=1}$ of $S_{k/2}(N,\chi)$ for odd $k \geq 5$ and of $U^+$ for $k = 3$. Then it holds for $n = tv^2w^2$ with $t$ squarefree, $v \mid N^\infty$, $(w,N) = 1$ and quadratic $\chi$ that

$$\sum_{j=1}^{d} |a_j(n)|^2 \ll n^{k/2-1} \left( \frac{t^{3/7}v^{6/7}}{N^{2/7}(n,N)^{1/7}} + \frac{t^{3/8}v^{3/4}}{N^{1/8}(n,N)^{1/4}} + \frac{v(n,N)}{N} + 1 \right) (nN)^{\epsilon}.$$ 

For arbitrary $\chi$, the last term within the bracket changes to $\frac{v(n,N)}{N} (c_\chi \text{rad}(c_\chi))^{1/4}$, where $c_\chi$ is the conductor of $\chi$.

We singled out the case of quadratic $\chi$ because this is the relevant case for quadratic forms and the main application that we proceed to present. It has been investigated by Wooley [19] under which conditions on the exponents $k_j$, $j = 1, \ldots, t$ the Diophantine equation

$$x_1^2 + x_2^2 + x_3^3 + x_4^3 + \sum_{j=1}^{t} y_j^k = n$$

has solutions for all sufficiently large $n$. His proof is based, among other things, on a result of Golubeva [4] Theorem 2 which we can improve by Theorem 1 as follows.

**Theorem 2.** Let $P \neq 3$ be an odd prime, $(n,6P) = 1$ and $n = tv^2$ with $t$ squarefree. Then

$$n = x^2 + y^2 + 6Pz^2$$

is solvable for $(x,y,z) \in \mathbb{N}^3$ if $P^{1+\epsilon} \leq \min(n^{1/17}\sqrt{12/17}, n^{1/11}\sqrt{6/11}, n^{1/3})$. This holds, in particular, if $nv^{28/3} > P^{17+\epsilon}$.

In [4], the bound is $nv^{12} > P^{21+\epsilon}$. For $k_i \in \mathbb{N}$ and $2 \leq k_1 \leq \ldots \leq k_t$ set

$$\gamma(k) = \prod_{i=1}^{t} \left(1 - \frac{1}{k_i}\right) \quad \text{and} \quad \bar{\gamma}(k) = \left(1 - \frac{1}{k_i}\right) \prod_{i=1}^{t-2} \left(1 - \frac{1}{k_i}\right).$$

**Theorem 3.** Assume the Riemann hypothesis for all $L$-functions associated with Dirichlet characters. Then, provided that $\gamma(k) < \frac{58}{81}$, all sufficiently large numbers $n$ are represented in the form of (3). The same conclusions hold without the assumption of the Riemann hypothesis if

(i) $t \geq 2$ and $\bar{\gamma}(k) < \frac{58}{81}$ or
(ii) $\gamma(k) < \frac{58}{81}$ and the exponents $k_1, \ldots, k_t$ are not all even.

The original bounds in [19] Theorem 1,2] are $\gamma(k) < 12/17$ with the assumption of the Riemann hypothesis, $\bar{\gamma}(k) < 74/105$ for (i) and $\gamma(k) < 74/105$ for (ii). As a consequence, it follows that every sufficiently large number $n$ is represented in the form

$$x_1^2 + x_2^2 + x_3^3 + x_4^3 + \sum_{j=1}^{t} x_j^3 = n,$$

with odd $t \leq 81$, or in the form

$$x_1^2 + x_2^2 + x_3^3 + x_4^3 + x_5^3 + x_6^{12} + x_7^{16} + x_8^{20} = n.$$
if the truth of the Riemann hypothesis is assumed.

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2. Shimura’s Lift and Maass Forms

We follow the exposition of [1]. For \( 0 \neq z \in \mathbb{C} \) and \( v \in \mathbb{R} \) define \( z^v \) by

\[
    z^v = |z|^v \exp(iv \arg(z)), \quad \text{where } \arg(z) \in (-\pi, \pi].
\]

For a holomorphic function on the upper half plane \( f : \mathbb{H} \to \mathbb{C} \) and \( \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(4) \) set

\[
    f[\gamma]_{k/2}(z) = \left( \epsilon_d^{-1} \left( \frac{c}{d} \right) \right)^{-k} (cz + d)^{-k/2} f(\gamma z),
\]

where \( \left( \frac{a}{b} \right) \) is the extended Kronecker symbol (cf. [17, p. 442]) and \( \epsilon_d = \left( \frac{-1}{d} \right)^{1/2} \).

From now on, \( \chi \) will always denote a character mod \( N \) and \( 4 \mid N \). For odd \( k \), we denote the spaces of modular forms and cusp forms of half-integral weight \( k/2 \) for \( \Gamma_0(N) \) and transformation behavior \( f[\gamma]_{k/2}(z) = \chi(d)f(z) \) by \( M_{k/2}(N, \chi) \) and \( S_{k/2}(N, \chi) \). For \( f, g \in S_{k/2}(\Gamma, \chi) \), the inner product is defined by [1]. For \( (n, N) = 1 \), let \( T(n) : M_k(N, \chi) \to M_k(N, \chi) \) be the Hecke operator (cf. [9, Chapter 4.3]).

For \( f = \sum_{n \geq 1} c(n)e(nz) \in S_{k/2}(N, \chi), \ k \geq 3 \text{ odd}, \ v = (-1)^{(k-1)/2} \text{ and } t \) without square factors (other than 1) prime to \( N \), define \( C_t(n) \) by the formal identity

\[
    \sum_{n=1}^{\infty} C_t(n)n^{-s} = L(s - k/2 + 3/2, \chi_4t\chi) \sum_{n=1}^{\infty} c(tn^2)n^{-s}.
\]

Then \( F_t(z) = \sum_{n=1}^{\infty} C_t(n)e(nz) \in M_{k-1}(N/2, \chi^2) \) is called the \( t \)-Shimura lift. If \( f \) is an eigenform for all Hecke operators \( T(p^2), \ p \nmid N \) with eigenvalues \( \lambda_p \), then \( F_t \), if it is not equal to 0, is an eigenform for all \( T_p, p \nmid N \) with the same eigenvalues, and it holds for \( (n, N) = 1 \) that [17 Corollary 1.8]

\[
    C_t(n) = c(t) \cdot \lambda_n.
\]

There exists an orthonormal basis of \( U^\perp \) and of \( S_{k/2}(N, \chi), k \geq 5 \), of simultaneous eigenforms for all \( T(p^2), p \nmid N \). Consequently, if the \( t \)-Shimura lift of \( f \) is cuspidal, it follows by Deligne’s bound for integral-weight modular forms for \( (w, N) = 1 \) that

\[
    |c(tw^2)| = |c(t)| \sum_{d \mid w} \mu(d) \chi_4t\chi(\overline{d})d^{k/2-3/2} \lambda_{w/m} \leq |c(t)| w^{k/2-1} \tau(w)^2.
\]

For \( k \geq 5 \) the Shimura lift is always cuspidal. However, for \( k = 3 \) the \( t \)-Shimura lift is cuspidal for all squarefree \( t \) if and only if if \( f \in U^\perp \), i.e. \( f \) does not live in the subspace of theta functions.

The theory of Maass forms with general weights was introduced by Selberg [16]. For \( \gamma \in \Gamma_0(4) \) and \( k \in \mathbb{Z} \) set

\[
    f[\gamma]_{k/2}(z) = \left( \epsilon_d^{-1} \left( \frac{c}{d} \right) \right)^{-k} e^{-ik/2 arg(cz+d)} f(\gamma z).
\]

We call a function \( f : \mathbb{H} \to \mathbb{C} \) an automorphic form of weight \( k/2 \) if it satisfies for all \( \gamma \in \Gamma \) the transformation rule

\[
    f[\gamma]_{k/2}(z) = \chi(d)f(z),
\]
and \( f(z) \ll y^\sigma + y^{1-\sigma} \) for some \( \sigma > 0 \). A Maaß form is an automorphic form that is an eigenfunction of

\[
\Delta_{k/2} = -y^2 \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) + i(k/2)y \frac{\partial}{\partial x},
\]

with eigenvalue \( \lambda = s(1-s) \). We denote the space of such forms by \( \mathcal{A}_s(\Gamma \backslash \mathbb{H}, k/2, \chi) \). Their inner product is defined by

\[
\langle f, g \rangle = \int_{\Gamma \backslash \mathbb{H}} f(z) \overline{g(z)} \frac{dx dy}{y^2}.
\]

Every form \( f \) in \( \mathcal{A}_s(\Gamma \backslash \mathbb{H}, k/2, \chi) \) has a Fourier expansion at the cusp \( \infty \) given by

\[
f(z) = \rho^+ y^s + \rho^- y^{s-1} + \sum_{\nu \in \mathbb{Z}, \nu \neq 0} \rho(n)W_{\text{sgn}(n)k/4, s-1/2}(4\pi|n|y)e(nz),
\]

where \( W_{\alpha,\beta}(z) \) denotes the standard Whittaker function \([10, \text{p. 295}]\). If the zero coefficient of \( f \in \mathcal{A}_s(\Gamma \backslash \mathbb{H}, k/2, \chi) \) vanishes at every cusp, then it is called a Maaß cusp form and the space of such forms is denoted by \( \mathcal{C}_s(\Gamma \backslash \mathbb{H}, k/2, \chi) \).

3. Proof of Theorem 1

Let \( \{\varphi_j = \sum_{n \geq 1} a_j(n)e(nz)\}_{j=1}^d \) be an orthonormal basis of \( S_{k/2}(N, \chi) \) for odd \( k \geq 5 \) and of \( U^\perp \) for \( k = 3 \). Set \( n = tv^2w^2 \) with \( \mu^2(t) = 1, v \mid N^{\infty} \) and \( (w, N) = 1 \). The square part of \( n \) coprime to \( N, w \), can be easily handled by \([11]\) since \( |a_j(n)|^2 \leq w^{k/2-1}|a_j(tv^2)|^2 \). Therefore, it is sufficient to prove that

\[
\sum_{j=1}^d |a_j(n)|^2 \ll n^{k/2-1} \left( \frac{t^{3/7}v^{6/7}}{N^{2/7}(n, N)^{1/7}} + \frac{t^{3/8}v^{3/4}}{N^{1/8}(n, N)^{1/4}} + \frac{v(n, N)}{N} + 1 \right) (nN)^\varepsilon
\]

for \( n = tv^2 \), with \( \mu^2(t) = 1 \) and \( v \) arbitrary.

The proof follows the Iwaniec-Duke approach very closely and we assume some familiarity with the article \([6]\). For \( k \geq 5 \), we directly apply the Petersson formula while for \( k = 3 \), we first embed the weight 3/2 cusp forms into the space of Maaß cusp forms of weight 3/2 via \( f(x+iy) \mapsto y^{3/4}f(x+iy) \) and then apply the Kuznetsov formula. The Petersson formula for half-integral weights states that \([14, \text{p. 89}]\)

\[
\frac{\Gamma(k/2-1)}{(4\pi n)^{k/2-1}} \sum_{j=1}^d |a_j(n)|^2 = 1 + 2\pi i^{-k/2} \sum_{N \mid c} c^{-1} J_{k-1} \left( \frac{4\pi n}{c} \right) K^k_{\chi}(n, n; c),
\]

where \( J_{k-1} \) is the Bessel function of order \( k/2 - 1 \) and

\[
K_{\chi}(m, n; c) = \sum_{d \mid (m, c)} e \left( \frac{md + \overline{m\chi}(d)}{c} \right) e \left( \frac{md + \overline{m\chi}(d)}{c} \right)
\]

is a twisted Kloosterman sum. If \( f(z) \) is a normalized cusp form for \( \Gamma_0(N) \) with respect to \([14]\), then \( \left[ \Gamma_0(Q) : \Gamma_0(N) \right]^{-1/2} f(z) \) is a normalized cusp form for \( \Gamma_0(Q) \) provided that \( N \mid Q \). Instead of applying the Petersson formula for the level \( N \), we use it for higher levels \( Q = pN \) with primes \( p \in P = \{ p \mid P < p \leq 2P \} \). Since \( |\Gamma_0(pN) : \Gamma_0(N)| \leq p + 1 \), this yields (cf. \([12, \text{p. 400}]\))

\[
\sum_{j=1}^d |a_j(n)|^2 \ll n^{k/2-1} \left( \sum_{p \in \mathbb{P}} \left( \sum_{(pN) \mid c} c^{-1} K^k_{\chi}(n, n; c) J_{k/2-1} \left( \frac{4\pi n}{c} \right) \right) \log P, \right)
\]
where we choose $P > 1 + (\log 2nN)^2$ to ensure that $\# \mathcal{P} = P(\log P)^{-1}$. After expressing the Bessel function by means of its asymptotic formula and applying partial summation, it remains to find a bound for sums of the type $\sum_{Q \in \mathcal{Q}} |K_Q(x)|$, where

$$K_Q(x) := \sum_{c \in \mathcal{Z}, Q \mid c} c^{-1/2} K^k(x, n; c) e \left( \frac{2np}{c} \right)$$

with $-1 \leq \nu \leq 1$ and $Q \in \mathcal{Q} = \{ pN \mid p \in \mathcal{P} \}$.

First, we factor the modulus $c$ into $qr$, where $q$ is coprime to $2nN$ and $r \mid (2nN)^\nu$. This way, (9) decomposes into a Kloosterman sum of modulus $r$ and a Salié sum of modulus $q$ which is explicitly computable. Very similar to [6, Lemma 6], we obtain

$$K^k(x, n; c) = q^{1/2} \sum_{s \equiv r/2 \pmod{2s}} \epsilon_s^{-2k} f_r(2s, \chi) \left( (1 + i^s) \left( \frac{nr}{q} \right) + (1 - i^s) \left( \frac{-nr}{q} \right) \right)$$

$$\sum_{ab = q} e \left( 2n \left( \frac{ar}{b} - \frac{br}{a} + \frac{ab}{r} \right) \right).$$

The main difference is that $f_r(2s, \chi) = \sum_{d \equiv r \pmod{r}} \left( \frac{r}{d} \right) \chi(d)$.

**Lemma 4.** For quadratic $\chi$, one has the following bound

$$|K^k(x, n; c)| \leq \tau(c)(n, c)^{1/2} c^{1/2},$$

while, for arbitrary $\chi$ one gets an additional factor of $(c_r \text{rad}(c_r))^{1/4}$ on the right-hand side.

**Proof.** If we split the sum for $c = rq, r \mid 2^\nu$, $(2, q) = 1$ we obtain

$$K^k(x, n; c) = K^k(x, n; q) S(x, n, q),$$

where $\chi_r$ and $\chi_q$ are characters modulo $r$ and $q$ respectively and the latter sum

$$S(x, n, q) = \sum_{d \equiv q \pmod{q}} \chi(d) \left( \frac{d}{q} \right) e \left( \frac{n(d + \bar{d})}{q} \right)$$

is a Kloosterman sum twisted by a character. For arbitrary $\chi$, we apply [8, Theorem 9.3] and get $|S(x, n, q)| \leq \tau(q)(n, q)^{1/2} q^{1/2} (q \chi \text{rad}(q \chi))^{1/4}$. Since the conductor of a real character with odd modulus is always squarefree, we obtain the Weil bound for real $\chi$ by applying [8, Proposition 9.4, 9.7 and 9.8], i.e. $|S(x, n, q)| \leq \tau(q)(n, q)^{1/2} q^{1/2}$. To bound the first term on the right-hand side of (11), we modify [7, Lemma 12.2 and Lemma 12.3]. Therefore, we set $r = 2^\nu$ and assume that $\alpha \geq 4$ to ensure that $\epsilon_r = \epsilon_n$ for $r = a + b 2^\beta$, where $N \equiv \beta = \frac{a}{2}$ or $\frac{2a+b}{2}$ respectively. By following the argument of Iwaniec very closely, we obtain

$$|K^k(x, n; 2^n)| \leq 2^\beta M,$$

where $M$ is the number of solutions modulo $2^\beta$ of $-na^2 + Ba + n \equiv 0 \pmod{2^\beta}$ for $B$ defined as in [7, Lemma 12.2 and Lemma 12.3]. To bound $M$, we proceed as
in [8] Lemma 9.6, Proposition 9.7 and Proposition 9.8 obtaining $|K^k_{\chi}(n, n; r)| \leq \tau(r)(n, r)^{1/2} r^{1/2}$.

We split $K_Q(x)$ according to whether $t \mid c$. By applying Lemma 4 we get, for quadratic $\chi$, that

$$|K_{[t, Q]}(x)| \leq \frac{x(t, Q)(v, Q/(Q, t))^{1/2}}{tQ} \tau(tQ)(xn)^e \leq \frac{x(t, Q)(v^2, Q/(Q, t))^{1/2}}{t^2} \tau(tQ)(xn)^e \leq \frac{xv(n, Q)}{n^{1/2}Q} \tau(tQ)(xn)^e$$

since $(t, Q)^2(v, Q/(Q, t))$ divides both $Q^2$ and $n^2$. In particular, one has

$$\sum_{Q \leq t} |K_{[t, Q]}(x)| \leq xv(n, N)n^{-1/2}N^{-1}(xnN)^e.$$  \hspace{1cm} (12)

For general $\chi$, we get an additional factor of $(c_\chi \text{rad}(c_\chi))^{1/4}$ on the right-hand side. The remaining part of $K_Q(x)$ can be reduced to partial sums of the type

$$K_q^\ast(y) = \sum_{y < c \leq 2y} c^{-1/2} K^k_{\chi}(n, n; c) e \left( \frac{2\nu n}{c} \right)$$

with $4 \leq y \leq x$. There are $O(\log(x))$ such partial sums. For even $t$, we trivially estimate $|K_{[t/2, Q]}(x)|$ and assume that $K_q^\ast(y)$ runs with $\frac{t}{2} \mid c$ to ensure that $n/(n, r)$ is not a perfect square. By (10) we conclude that

$$K_q^\ast(y) = \sum_{r \in \mathfrak{R}} r^{-1/2} \sum_{s \equiv r/2 \mod r/2} \epsilon_s^{-k} f_r(2s) \left[ (1 + i^2)F_{r,s}^+(p) + (1 - i^2)F_{r,s}^-(p) \right],$$

where $\mathfrak{R} = \{ r : N \mid r \mid (2nN)\mathcal{O}, t \nmid r \}$ and

$$F_{r,s}^\pm(p) = \sum_{y < c \leq 2y} \sum_{(a, b) = 1, p \mid ab} \left( \frac{\pm nr}{ab} \right) e \left( \frac{2n}{r} \left( \frac{\overline{a}}{b} - \frac{br}{a} + \frac{sab}{r} + \frac{\nu}{abr} \right) \right)$$

with $(ab, 2nN) = 1$. We treat $F_{r,s}^\pm(p)$ according to the values of $a$ and $b$ and split it into dyadic ranges $A < a \leq 2A$ and $B < b \leq 2B$ with $y < rAB \leq 2y$ and $A, B \geq \frac{1}{2}$ which we denote by $F(A, B; p)$.

For either $A$ or $B$ small, we apply the Weil bound for the Kloosterman sum and estimate trivially. Following [6, p.396] word by word, we get

$$F(A, B; p) \ll \left( 1 + \frac{n}{y} \right) \sum_{B < b \leq 2B \atop (b, 2nN) = 1} \sum_{A_1 < a \leq A_2 \atop (a, b) = 1} \left( \frac{\pm nr}{a} \right) e \left( \frac{2nm}{b} \frac{\overline{a}}{br} \right),$$

with $m$ defined by $mp_b = r^2 + 1 + sbb(\text{mod} b)$ and $A_1, A_2$ such that $Ap_b = A_1 < A_2 \leq 2Ap_b$, where $p_b := p/(b, p)$. Set $\delta_1 = \frac{n}{(n, r)}$ and $\delta_2 = \frac{r}{(n, r)}$. At this point, we cannot proceed as in Iwaniec [6 Section 5] because $8 \mid \delta_2$ is generally not satisfied. To solve this, we distinguish three cases:

1. $2 \nmid \delta_1$. Set $\Delta_1 = \delta_1$ and $\Delta_2 = 16\delta_2$.
2. $\text{ord}_2(\delta_1) = 1$ or 2. Set $\Delta_1 = 2^{-\text{ord}_2(\delta_1)}\delta_1$ and $\Delta_2 = 2^{2+\text{ord}_2(\delta_1)}\delta_2$.
3. $8 \mid \delta_1$. Set $\Delta_1 = \delta_1$ and $\Delta_2 = \delta_2$. 

In each case $\Delta_1$ and $\Delta_2$ satisfy that \( \frac{\Delta_1 \Delta_2}{a} \) is either \( 8 \mid \Delta_1 \) or \( 8 \mid \Delta_2 \) and \( \Delta_1, \Delta_2 \) and \( b \) are pairwise coprime. Set \( 2\Delta = 2\Delta_2 \), where \( j = 5, j = 3 + 2 \text{ord}_2(\delta_1) \) or \( j = 1 \) according to the corresponding case. Thus, the innermost sum of (15) is equal to
\[
\sum_{a} := \sum_{A_1 < a \leq A_2} \left( \frac{\pm \Delta_1 \Delta_2}{a} \right) e \left( 2jm \frac{\Delta_1 a}{\Delta_2 b} \right).
\]
By applying [6, (3.14)] it follows for \( D = \Delta_1 \Delta_2 b \) that
\[
\left| \sum_a \right| \leq \sum_{1 \leq |d| \leq D/2} \frac{1}{|d|} \left| \sum_{x \pmod{D}} \left( \frac{\pm \Delta_1 \Delta_2}{x} \right) e \left( 2jm \frac{\Delta_1 \overline{x}}{\Delta_2 b} + \frac{dx}{D} \right) \right|.
\]
The sum modulo \( D \) can be factored into three sums in the same manner as in [6, p.396]. Note that \( \Delta_1 \) is not a perfect square because there exists an odd prime divisor of \( t \) which, by definition of \( \mathfrak{M} \), does not divide \( r \). Therefore, \( x \mapsto (\overline{\Delta_1 x}) \) is not the trivial character. By following Iwaniec step by step and making use of \( \mathfrak{M} \), we get
\[
F(A, B; p) \ll B^{3/2} \left( 1 + \frac{n}{y} \right) (nr)^{1/2}(n, N)^{-1}r^2(r) \log(ny)
\]
and
\[
F(A, B; p) \ll A^{3/2} \left( 1 + \frac{n}{y} \right) (nr)^{1/2}(n, N)^{-1}r^2(r) \log(ny).
\]
If both \( A \) and \( B \) are large, we make use of the flexibility gained through the averaging over the levels. We want to estimate
\[
F_p(A, B) = \sum_{p \leq P} |F(A, B; p)|.
\]
Setting \( \lambda_p := \text{sgn}F(A, B; p) \) we get
\[
F_p(A, B) = \sum_{A < a \leq 2A} \sum_{B < b \leq 2B} \sum_{P < p \leq 2P} \sum_{y < abr \leq 2y} (a, b) = 1 \lambda_p \left( \frac{\pm nr}{ab} \right) e \left( 2n \left( \frac{anr}{b} - \frac{br}{a} + \frac{sab}{r} + \frac{\nu}{abr} \right) \right).
\]
To bound this, we follow [6, Section 6] step by step. First, we split the sum according to whether \( p \mid a \) or \( p \mid b \). In each case we interchange the sums, apply Cauchy-Schwarz to the square and change the sums back. Hence, we have two \( p \)-sums. If the summands of both \( p \)-sums coincide, we trivially estimate, otherwise we apply the Weil bound. Since [6, Lemma 7] does not hold, we cannot use \( (n, r) \leq r^{1/2} \) for [6, (6.1)]. Instead, we use \( (n, N) \leq (n, r) \leq r \) and (6.3) from Iwaniec changes to
\[
F(A, B) \ll yr^{-1}P^{-1/2} + \left( 1 + \frac{n}{y} \right)^{1/2} (s^2 - 1, r)^{1/2}(r) \log y
\]
\[
\left( y^{7/8}r^{-5/8}P^{3/8}(n, N)^{-1/4} + (A^{-1/2} + B^{-1/2})y^{-1} \right).
\]
In particular, we lose a factor of \( r^{-1/4} \) in the second term within the bracket. To bound \( K_Q(y) \), we modify [6, Section 7] accordingly and apply (16) and (17) in case that either \( A \) or \( B \) is
\[
\leq \left( 1 + \frac{n}{y} \right)^{-1/4} n^{-1/4}r^{-3/4}y^{1/2}P^{-1/2}(n, N)^{1/2}
\]
holds, by Proskurin’s variant \[12\] of the Kuznetzov formula, that
\[\sum_{p \in \mathcal{P}} |F_{r,s}^{\pm}(A, B; p)| \ll \gamma r^{-1} P^{-1/2} + (y + n)^{5/8} r^{-5/8} (n, N)^{-1/4} \quad (s^2 - 1, r)^{1/2} \tau^2(r) (\log ny) \left(y^{1/4} P^{3/8} + n^{1/8} y^{1/8} P^{1/4}\right).\]

According to \[13\], it remains to sum this inequality over \(s \not\equiv r/2 \pmod{2}\) and \(r \in \mathfrak{R}\). The more general form of \(f_r(2s, \chi)\) does not affect \[6\] (7.2) and (7.3)]. Hence,
\[\sum_{s \not\equiv r/2} |f_r(2s, \chi)|(s^2 - 1, r)^{1/2} \ll r^{1/2} \tau^2(r) \quad \text{and} \quad \sum_{r \in \mathfrak{R}} r^{-1/8} \tau^4(r) \ll \tau(nN)^{-1/8}.\]

Combining this with \[12\], we conclude, for quadratic \(\chi\), that
\[\sum_{q \leq Q} |K_Q(x)| \ll \left(x^2(n, N)^{-1/2} N^{-1} + xP^{-1/2} N^{-1/2}\right) + (x + n)^{5/8} \left(x^{1/4} P^{3/8} + n^{1/8} x^{1/8} P^{1/4}\right) N^{-1/8} (n, N)^{-1/4} \left(n x N\right)^{\epsilon} \]
which is an improvement of \[5\] Theorem 3]. By \[8\], we infer
\[\sum_{j} \left|a_j(n)\right|^2 \ll k^{k/2-1} \left(\frac{n^{1/2}}{P^{1/2} N^{1/2}} + \frac{n^{3/8} P^{3/8}}{(n, N)^{1/4}}\right) (nNP)^{\epsilon}.\]

Choosing \(P = n^{1/7} (n, N)^{2/7} / N^{3/7} + (nN)^{\epsilon}\) yields, for real \(\chi\), that
\[\sum_{j} \left|a_j(n)\right|^2 \ll k^{k/2-1} \left(\frac{n^{1/2}}{P^{1/2} N^{1/2}} + \frac{n^{3/8} P^{3/8}}{(n, N)^{1/4}}\right) (nN)^{\epsilon},\]
while, for an arbitrary character \(\chi\), the first term changes to \(\frac{v(n, N)}{N} (c_{\chi} \text{rad}(c_{\chi}))^{1/4}\).

This concludes the proof for \(k \geq 5\).

To prove the case \(k = 3\) we follow \[2\] Section 3 & 5], but include an arbitrary nebentypus \(\chi\). The map \(f(z) \mapsto y^{3/4} f(z)\) induces an injective mapping \(S_{3/2}(N, \chi) \rightarrow C_{3/4}(N, 3/2, \chi)\) and one has \(a(n) = (4\pi n)^{3/4} \rho(n)\), where \(a(n)\) denote the Fourier coefficients of \(f\) and \(\rho(n)\) the coefficients, see \[6\], of the corresponding Maaß cusp form. Let \(u_i(z)\) be an orthonormal basis of Maaß cusp forms of weight 3/2 with eigenvalues \(\lambda_j\) and Fourier coefficients \(\rho_j(n)\) and let \(\{f_{ij} = \sum_{n \geq 1} a_{ij}(n) e(nz)\}_{i=1}^{d_j}\) be an orthonormal basis of \(S_{3/2+j}(N, \chi)\). Then it holds, by Proskurin’s variant \[12\] p. 3888 of the Kuznetzov formula, that
\[\sum_{N \mid c} \frac{K_{1}(n, n; c)}{c} \varphi(4\pi n/c) = 4n \sum_{j} \left|\frac{\rho_j(n)}{\cosh(\pi t_j)}\right|^2 \hat{\varphi}(t_j) \quad + \sum_{a} \int_{-\infty}^{\infty} \frac{|\phi_{a, n}(1/2 + it)|^2}{\cosh(\pi t)|\Gamma(1/2 + 3/4 + it)|^2} \hat{\varphi}(t) dt \quad + 4 \sum_{j \geq 1} \frac{\Gamma(3/2 + 2j)e(3/8 + j/2)\hat{\varphi}(3/2 + 2j)}{(4\pi)^{3/2 + 2j} n^{1/2 + 2j}} \sum_{i=1}^{d_j} |a_{ij}(n)|^2.\]

Here, \(\varphi(x)\) is a suitable test function, \(\sum_{a}\) refers to the summation over the nonequivalent non-singular cusps of \(\Gamma_0(N)\), \(t_j\) is defined by \(s_j = 1/2 + it_j\) and \(\phi_{a, n}\) are the coefficients of an Eisenstein series (cf. \[12\] p. 3876]). Similar to the choice in \[8\] p. 51], we set \(\varphi(x) = c_0 x^{-7/2} J_{13/2}(x)\) for \(c_0 = -2^4 e(-3/8) \pi^{-2} \Gamma(9/2)^{-1}\) and \(J_k(z)\)
by direct computation, one can show that

\[ \text{Theorem 1.2}. \]

For this purpose, let \( Z \) be the Kuznetsov formula and by means of the Weber-Schafheitlin integral \([5, (6.574.2)]\) it is straightforward to calculate

\[ \hat{\varphi}(t) = \frac{t^2 + 1/4}{\cosh(2\pi t)\Gamma(-1/4 + it)\Gamma(-1/4 - it)\Gamma(6 + it)\Gamma(6 - it)}. \]

Observe that \( \hat{\varphi}(t) > 0 \) for \( t \in \mathbb{R} \) and for \( t \in [-i/4, i/4] \), the value at \( it = 1/4 \) defined by

\[ \lim_{t \to \pm i/4} \hat{\varphi}(t) = \frac{3}{64\pi^{3/2}\Gamma(23/4)\Gamma(25/4)}. \]

Thus, we may drop all terms of the first sum on the right-hand side of \([20]\) which represent eigenvalues distinct to 3/16 as well as the contribution from the continuous spectrum (the integral over the Eisenstein coefficients). Since the weights of \( f_{ij} \) are greater than or equal to 5/2, we can use our previous results to bound the last term of \([20]\). As before, we apply Iwaniec’s method of averaging over the levels. If \( u(z) \) is a normalized Maaß cusp form for \( \Gamma_0(N) \), then \([\Gamma_0(Q): \Gamma_0(N)]^{-1/2}u(z) \) is a normalized Maaß cusp form for \( \Gamma_0(Q), Q \in \mathbb{Q} \). Hence, by applying the Kuznetsov formula for every level \( Q \in \mathbb{Q} \), it follows

\[
\sum_{\lambda_i = 3/16} n \left| p_j(n) \right|^2 \ll \log P \sum_{Q \in \mathbb{Q}} \left| \sum_{Q \in c} \frac{K^2((n, n, c)}{c} \left( \frac{c}{n} \right)^{7/2} J_{13/2} \left( \frac{4\pi n}{c} \right) \right|
\]

\[ + \left( P + \frac{n^{1/2}}{P^{1/2}N^{1/2}} + \frac{v(n, N)}{N} + \frac{n^{3/8}P^{3/8}}{N^{1/8}(n, N)^{1/4}} \right) (nNP)^{\varepsilon}. \]

Since 13/2 is half integral and since for \( x > n \)

\[ n^{-7/2} \left( e^{x} J_{13/2} \left( \frac{4\pi n}{x} \right) \right) \ll n x^{-5/2}, \]

the right-hand side of \([21]\) can be treated exactly as in \([10] \) Section 8 \) taking into account \([19]\) and our choice of \( P \). This concludes the proof of Theorem 1.

4. An Application

Finally, we give an application of Theorem 1, particularly an improvement of \([19] \) Theorem 1.2.\) For this purpose, let \( A \) be a positive, integral, symmetric \( k \times k \) matrix with even diagonal elements, let \( q(x) := \frac{1}{2}x^tAx \) be the corresponding quadratic form and let \( N \) be the level of \( A \), i.e., the smallest integer such that \( NA^{-1} \) is integral with even diagonal. This section aims at finding a lower bound for the Fourier coefficients \( r(q, n) = \#\{ x \in \mathbb{Z}^k | q(x) = n \} \) of \( \theta(q, z) \) to conclude that \( n \) is represented by \( q \). By direct computation, one can show that \( \theta(q, z) \in M_{k/2}(N, \chi_{-1})^{\text{det}(A)} \) \([17] \) p. 456].

Two positive quadratic forms are in the same genus if they are equivalent over all \( \mathbb{Z}_p \). Define the theta series of the genus \( \theta(\text{gen } q, z) = \sum_{n=0}^{\infty} r(q, n)e(nz) \) by

\[
r(q, n) = \sum_{\tilde{q} \in \text{gen } q} w(\tilde{q})r(\tilde{q}, n) \quad \text{with} \quad w(\tilde{q}) = \left( \frac{1}{\#O_Z(\tilde{q})} \right)^{-1} \left( \frac{1}{\#O_Z(\tilde{q})} \right).
\]

where the summation is taken over a set of representative classes in the genus. Let \( S(z) = \theta(q, z) - \theta(\text{gen } q, z) \). Then \( S(z) \) is the orthogonal projection of \( \theta(q, z) \) onto
the subspace of cusp forms and \( \theta(\text{gen } q, z) \) is an Eisenstein series \footnote{\cite{15} Korollar 1}. Consequently, write

\[
\theta(q, z) = \theta(\text{gen } q, z) + S(z) =: \sum_{n=0}^{\infty} r(\text{gen } q, n)e(nz) + \sum_{n=1}^{\infty} a(q, n)e(nz).
\]

We would like to treat \( r(\text{gen } q, n) \) as the main term for \( r(q, n) \) and \( a(q, n) \) as the error term. To compute the Eisenstein coefficients \( r(\text{gen } q, n) \), we use Siegel’s formula \footnote{\cite{18}. From now on, let \( k = 3 \). Then

\[
(23) \quad r(\text{gen } q, n) = \frac{2\pi}{\sqrt{\Delta/8}} \prod_{p} r_p(q, n),
\]

where \( \Delta \) is the determinant of \( A \) and \( r_p(q, n) \) are the \( p \)-adic densities defined by

\[
r_p(q, n) := \lim_{\nu \to \infty} \frac{1}{p^{2\nu}} \# \{ x \in (\mathbb{Z}/p^{\nu} \mathbb{Z})^3 \mid q(x) \equiv n \pmod{p^{\nu}} \}.
\]

Apart from a finite number of cases, \( (p, Nn) \neq 1 \), the densities are easy to compute \footnote{\cite{18} Hilfssatz 12}

\[
r_p(q, n) = 1 + \frac{\chi - 2n\Delta}{p}, \quad p \nmid nN.
\]

The space of theta functions \( U \) poses a problem since their Fourier coefficients grow like \( n^{1/2} \) which is roughly the same size as \( r(\text{gen } q, n) \). Thus, to show that \( n \) can be represented by a quadratic form \( q \) using Theorem 1, it is necessary that the \( n \)-th coefficient of the projection of \( \theta(q, z) \) onto \( U \) vanishes.

For a ring \( R \) let \( O_R(q) := \{ S \in GL_{2k}(R) \mid S^tAS = A \} \) be the finite set of \( R \)-automorphs of \( q \). Two quadratic forms \( A_1, A_2 \) in the same genus with \( A_1 = S^tA_2S \) for \( S \in GL_k(\mathbb{Z}) \) belong to the same spinor genus, if \( S \in O_Q(A_2) \cap O_{Q,a}(A_2)GL_k(\mathbb{Z}_p) \), where \( O_{Q,a}(A) \) is the subgroup of \( p \)-adic automorps \( O_Q(A) \) of determinant and spinor norm 1 (cf. \footnote{\cite{11} Section 55}). Define the theta series of the spinor genus \( \theta(\text{spin } q, z) = \sum_{m=0}^{\infty} r(\text{spin } q, m)e(mz) \) by

\[
(24) \quad r(\text{spin } q, n) = \sum_{\overline{q} \in \text{spin } q} w(\overline{q})r(\overline{q}, n) \text{ with } w(\overline{q}) = \left( \sum_{\overline{q} \in \text{spin } q} \frac{1}{\#O_Z(\overline{q})} \right)^{-1} \frac{1}{\#O_Z(\overline{q})},
\]

where the summation is taken over a set of representative classes in the spinor genus of \( q \). Schulze-Pillot \footnote{\cite{15} has shown that the orthogonal projection of \( \theta(q, z) \) onto the subspace of \( U^\perp \) is \( \theta(q, z) - \theta(\text{spin } q, z) \). Therefore, write

\[
\theta(q, z) = \theta(\text{gen } q, z) + H(z) + f(z),
\]

with \( H(z) = \theta(\text{spin } q, z) - \theta(\text{gen } q, z) \in U \) and \( f \in U^\perp \). The contribution from the Fourier coefficients of \( f \) is easy to handle by Theorem \footnote{If \( r(\text{gen } q, n) = r(\text{spin } q, n) \), then the \( n \)-the Fourier coefficient of \( H(z) \) vanishes. This obviously holds when \( n \neq \{tm^2 : 4t \mid N,m \in \mathbb{N} \} \) since the coefficients of the theta functions vanish. According to the definitions \footnote{\cite{22} and \cite{24} it follows that \( r(\text{spin } q, n) = r(\text{gen } q, n) \) is satisfied if

\[
r(\text{spin } q, n) = r(\text{spin } q', n)
\]

for all \( q' \) in the same genus as \( q \). According to Schulze-Pillot \footnote{\cite{15} Korollar 2.3 (ii)} it holds for any \( q, q' \) in the same genus and squarefree \( t \) that

\[
r(\text{spin } q, tm^2) = r(\text{spin } q', tm^2)
\]
if \( N = 4t' t^2 \) with squarefree \( t' \) and \( h \mid m \). In particular, if \( N/4 \) is squarefree, one has \( \theta(q, z) = \theta(spz, z) \).

**Proof of Theorem 2.** Let \( \theta(q, z) \) be the theta series of the quadratic form \( q = x^2 + y^2 + 6Pz^2 \). Then, \( \theta(q, z) \in \text{M}_{3/2}(24P, \chi) \) for a quadratic character \( \chi \) and since \( 6P \) is squarefree, it holds that \( \theta(q, z) = \theta(spz, z) \). Thus, the orthogonal projection of \( \theta(q, z) \) onto the subspace of cusp forms is in \( U^\perp \). Let \( \{\varphi_j(z) = \sum_{n \geq 1} a_j(n)e(nz)\}_{j=1}^d \) be an orthonormal basis of \( U^\perp \). Then

\[
\rho(q, n) = r(\text{gen } q, n) + \sum_{j=1}^d c_ja_j(n) = r(\text{gen } q, n) + O\left(\frac{d}{\sum_{j=1}^d c_j^2 \sum_{j=1}^d |a_j(n)|^2}\right).
\]

From \( \sqrt{\sum_{j=1}^d c_j^2} = O(P^{1/4+\epsilon}) \) (cf. [4 Theorem 3]) and Theorem 1, we conclude that

\[
(25) \quad \rho(q, n) = r(\text{gen } q, n) + O\left(\left(1^{1/2}v^{13/28}P^{3/28} + v^{7/16}P^{3/16} + v^{1/4}P^{1/4}\right)(Pn)^\epsilon\right)
\]

To bound \( r(\text{gen } q, n) \) from below, we apply (23), Siegel’s formula. If \( p \nmid 6P \), it holds by [13 Hilfssatz 16] that

\[
1 - \frac{1}{p} \leq \rho_p(n, q) \leq 1 + \frac{1}{p}.
\]

To treat the remaining densities, \( r_2(n, q), r_3(n, q) \) and \( r_P(n, q) \), we rely on Hensel’s lemma

**Lemma 5.** Assume that \( P \in \mathbb{Z}[x_1, \ldots, x_d] \) and \( \alpha \in \mathbb{Z}^d \) satisfy \( P(\alpha) \equiv 0 \mod p^k \). If it holds for at least one \( x_j \) that

\[
\frac{\partial f}{\partial x_j}(\alpha) \not\equiv 0 \mod p^l \quad \text{for some } l \leq \frac{k+1}{2},
\]

then \( P(x) \equiv 0 \mod p^{k+m} \) has \( p^{(d-1)} \) integer solutions. Each of these solutions \( \beta \) satisfies that \( \beta_j \equiv \alpha_j \mod p^{k-l+1} \) and \( \beta_i \equiv \alpha_i \mod p^k \) for all \( i \neq j \).

**Proof.** The case \( d = 1 \) is proven in [13 p. 48]. Assume \( j = 1 \). For each choice \( \beta_2, \ldots, \beta_d \) mod \( p^{k+m} \) with \( \beta_j \equiv \alpha_j \mod p^k \), we can apply the one-variable case to find \( \beta_1 \) such that \( P(\beta) \equiv 0 \mod p^{k+m} \).

For \( p = 2 \), consider the congruence

\[
(26) \quad x^2 + y^2 + 6Pz^2 \equiv n \mod 8
\]

for arbitrary odd \( n \). For each \( x \equiv 1, 3 \mod 4 \) (\( y \equiv 1, 3 \mod 4 \)), there are two possible choices for \( y \mod 8 \) (\( x \mod 8 \)) and four possibilities for \( z \mod 8 \) to solve (26). It follows by Lemma 5 that

\[
\rho_2(n, q) \geq \lim_{\nu \to \infty} \frac{32 \cdot 2^{2(\nu-3)}}{2^{2\nu}} = 1/2.
\]

If \( p \) is a prime, then \( \mathbb{Z}/p\mathbb{Z} \) is a finite field. In a finite field of odd order \( q \), every element unequal to zero can be expressed as the sum of two squares in \( q - 1 \) ways. Hence, for \( n \not\equiv 0 \mod P \), there exist \( P^2 - P \) solutions of

\[
(27) \quad x^2 + y^2 + 6Pz^2 \equiv n \mod P,
\]
with \((x, y) \neq 0 \mod P\). By Lemma \[5\] we infer \(r_3(n, q) \geq 2/3\) and \(r_P(n, q) \geq 1 - \frac{1}{P}\). It follows \(r_3(n, q) \geq 2/3\) and \(r_P(n, q) \geq 1 - \frac{1}{P}\). Thus, the main term of \(\text{(25)}\) dominates the error term as soon as

\[
P \leq \min\left(\varepsilon^{14/17} t^{1/17}, \varepsilon^{8/11} t^{1/11}, \varepsilon^{2/3} t^{1/3}\right)^{1-\epsilon}.
\]

If this holds true, it follows that \(x^2 + y^2 + 6Pz^2 = n\) has a solution in \(\mathbb{Z}_3\). Furthermore, we may assume that \(x, y, z\) are natural numbers since the number of integer solutions of \(x^2 + y^2 = n\) is \(O(n^\epsilon)\).

**Proof of Theorem 3**. We keep the notation from Wooley [19, Section 3] and modify only the parts concerning the bound of Golubeva’s theorem. The necessary requirements to apply Theorem [2] (i) \(NM^{12} > p^{17}\), (ii) \(NM^6 > p^{11}\) and (iii) \(N > p^3\), are fulfilled provided that (cf. [19, p. 14])

(i) \(\gamma_0 (6/c + 1) - 4/c - \epsilon > 17\gamma_0 - 34/3 + \epsilon\),
(ii) \(\gamma_0 (3/c + 1) - 2/c - \epsilon > 11\gamma_0 - 22/3 + \epsilon\), and
(iii) \(\gamma_0 - \epsilon > 3\gamma_0 - 2 + \epsilon\).

These inequalities yield the following conditions

(i) \(\gamma_0 < \frac{34c - 12 - 6\epsilon}{48c - 18}\),
(ii) \(\gamma_0 < \frac{22c - 6 - 6\epsilon}{30c - 9}\), and
(iii) \(\gamma_0 < 1 - 2\epsilon\).

Assuming the Riemann hypothesis, Wooley chooses \(c = 2 + 2\epsilon\) (cf. [19, p.15]). With this choice and \(\epsilon\) sufficiently small, the conditions are satisfied as long as \(\gamma_0 < 28/39 = \min(28/39, 38/51, 1)\). Otherwise, without assuming the Riemann hypothesis, the choice is \(c = 12/5 + 2\epsilon\), and it follows \(\gamma_0 < 58/81 = \min(58/81, 26/35, 1)\). The rest of the proof can be conducted exactly as in [19, Section 3].

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