CLASSIFICATION OF GENERALIZED YAMABE SOLITONS

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Abstract. In this paper, we consider generalized Yamabe soliton version of the Perelman’s conjecture. We consider complete gradient conformal solitons and classify them. As a corollary, we recover the classification of three-dimensional complete gradient Yamabe solitons. Furthermore, we also classify complete gradient conformal solitons with vanishing condition on Weyl, Cotton and Cao-Chen.

1. Introduction

Let \((M, g)\) be an \(n\)-dimensional Riemannian manifold. For smooth functions \(F\) and \(\varphi\) on \(M\), \((M, g, F, \varphi)\) is called a gradient conformal soliton (cf. [16], [15] and [7]), if it satisfies
\[
\varphi g = \nabla \nabla F.
\]
If \(F\) is constant, \(M\) is called trivial. As is well known, Gradient conformal solitons were studied by Cheeger-Colding ([8], see also [15]). Recently, the special case of it has been studied. It is the gradient Yamabe soliton:
\[
(R - \rho)g = \nabla \nabla F,
\]
where, \(R\) is the scalar curvature on \(M\), and \(\rho \in \mathbb{R}\) is a constant. Yamabe solitons are special solutions of the Yamabe flow introduced by R. Hamilton [11]. In the last decade, Yamabe solitons have developed rapidly.

The Yamabe soliton is similar to the Ricci soliton. As is well known, S. Brendle [3] brought significant progress to 3-dimensional gradient

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Ricci solitons, that is, he showed that “any 3-dimensional complete noncompact $\kappa$-noncollapsed gradient steady Ricci soliton with positive curvature is rotationally symmetric” which is a famous conjecture of Perelman [14]. Therefore, it is interesting to consider the similar problem: “Is the Yamabe soliton rotationally symmetric under some natural assumption?” Daskalopoulos and Sesum are the first ones who consider the problem [9]. In the seminal paper, they showed that any complete locally conformally flat gradient Yamabe solitons with positive sectional curvature is rotationally symmetric. Cao, Sun and Zhang [6], and Catino, Mantegazza and Mazzieri [7] relaxed the assumption. Inspired by their works, we consider the conformal soliton version of Perelman’s conjecture, that is, rotational symmetry of gradient conformal solitons with nonnegative scalar curvature. More generally, we consider the following problem:

Problem 1. Classify complete gradient conformal solitons with nonnegative scalar curvature.

Conformal gradient solitons were studied by Cheeger and Colding [8]. They gave a characterization of warped product manifolds. Inspired by their work, we will drastically simplify the proof of the classification result of it given by Tashiro [15] (see also Catino-Mantegazza-Mazzieri’s work [7]).

Theorem 1.1. A nontrivial complete gradient conformal soliton $(M^n, g, F, \varphi)$ is either

(1) compact and rotationally symmetric, or

(2) rotationally symmetric and equal to the warped product

$([0, \infty), dr^2) \times |\nabla F| (\mathbb{S}^{n-1}, \bar{g}_S)$,

where, $\bar{g}_S$ is the round metric on $\mathbb{S}^{n-1}$, or

(3) the warped product

$(\mathbb{R}, dr^2) \times |\nabla F| (\mathbb{N}^{n-1}, \bar{g})$,

where, the scalar curvature $\bar{R}$ of $N$ satisfies

$|\nabla F|^2 R = \bar{R} - (n - 1)(n - 2)\varphi^2 - 2(n - 1)g(\nabla F, \nabla \varphi)$.

Remark 1.2. By Theorem 1.1, to consider rotational symmetry of gradient conformal solitons, we only have to consider (3) of Theorem 1.1.

Remark 1.3. To understand the Yamabe soliton, many generalizations of it have been introduced. For example, almost Yamabe solitons [2], gradient $k$-Yamabe solitons [7] (see also [1]), $h$-almost gradient Yamabe...
solitons have been introduced. Gradient conformal solitons include these notions. Therefore, we can apply all the result in this paper to these ones.

The remaining sections are organized as follows. Section 2 is devoted to the proof of Theorem 1.1. By using Theorem 1.1, we also classify three-dimensional complete gradient Yamabe solitons which was shown by Cao, Sun and Zhang [6]. In section 3, we give two classification results under divergence-free Cotton tensor, and vanishing Cao-Chen tensor introduced by Cao and Chen [5] (see also [4]). Classification of locally conformally flat gradient conformal solitons is given in section 4.

2. Proof of Theorem 1.1 and the classification of three-dimensional gradient Yamabe solitons

In this section, we prove Theorem 1.1. We first define some notions. The Riemannian curvature tensor is defined by

\[ R(X, Y)Z = -\nabla_X \nabla_Y Z + \nabla_Y \nabla_X Z + \nabla_{[X,Y]} Z, \]

for \( X, Y, Z \in \mathfrak{X}(M) \). The Ricci tensor \( R_{ij} \) is defined by \( R_{ij} = \overline{R}(\partial_i, \partial_j)\partial_k, \partial_\ell \).

**Proposition 2.1.** Let \( (M, g, F, \varphi) \) be a complete gradient conformal soliton. Assume that \( \Sigma_c = F^{-1}(c) \) is a regular level surface. Then, we have

1. \( |\nabla F| \) and \( \varphi \) is constant on \( \Sigma_c \),
2. the second fundamental form of \( \Sigma_c \) is \( B_{ab} = \frac{\varphi}{|\nabla F|^2} g_{ab} \),
3. the mean curvature \( H = (n-1)\frac{\varphi}{|\nabla F|^2} \) is constant on \( \Sigma_c \),
4. in any open neighborhood \( F^{-1}(\alpha, \beta) \) of \( \Sigma_c \) in which \( F \) has no critical points, the soliton metric \( g \) can be expressed as

\[ g = dr^2 + \frac{(F'(r))^2}{(F'(r_0))^2} \bar{g}_{r_0}, \]

where, \( \bar{g}_{r_0} = g_{ab}(r_0, x)dx^a dx^b \) is the induced metric on \( \Sigma_c \), and \((x^2, \ldots, x^n)\) is a local coordinate system on \( \Sigma_c \).

**Proof.** Let \( c_0 \) be a regular value of \( F \), and \( \Sigma_{c_0} = F^{-1}(c_0) \). Assume that \( I(\supset c_0) \) is an open interval, such that \( F \) has no critical point in an open neighborhood \( U_I = F^{-1}(I) \) of \( \Sigma_{c_0} \). Then, one has

\[ g = \frac{1}{|\nabla F|^2} dF^2 + \bar{g}_{\Sigma_{c_0}} = \frac{1}{|\nabla F|^2} dF^2 + g_{ab}(F, x)dx^a dx^b, \]
where, $\mathcal{g}_{\Sigma_{c_0}}$ is an induced metric, $x = (x^2, \ldots, x^n)$ is a local coordinate system on $\Sigma_{c_0}$, and $a, b = 2, 3, \ldots, n$.

Since

$$\nabla(|\nabla F|^2) = 2\nabla\nabla F \nabla F = 2 \varphi g(\nabla F, \cdot),$$

$|\nabla F|^2$ is constant on $\Sigma_c$ which is diffeomorphic to $\Sigma_{c_0}$.

On $U_I$, let $r = \int \frac{dF}{|\nabla F|}$. Then, one has

$$g = dr^2 + g_{ab}(r, x)dx^a dx^b.$$ 

Let $\nabla r := \partial_1 := \partial_r \left(= \frac{\partial}{\partial r}\right)$, then one has $|\nabla r| = 1$ and $\nabla F = F'(r)\partial_1$. Here we remark that without loss of generality, one can assume that $F' > 0$ on $U_I$. Assume that $I = (\alpha, \beta)$ with $F'(r) > 0$ for all $r \in I$. Since $\nabla \partial_1 \partial_1 = 0$, integral curves to $\nabla r$ are normal geodesics. By the soliton equation,

$$F''(r) = \varphi.$$

Thus, $\varphi$ is constant on $\Sigma_c$. The second fundamental form can be written by

$$B_{ab} = \frac{F''(r)}{F'(r)} g_{ab}.$$ 

Hence, the mean curvature can be written by $H = (n - 1)\frac{F''(r)}{F'(r)}$. By a direct computation,

$$B_{ab} = g(\partial_1, -\nabla_a \partial_b)$$

$$= - \Gamma^1_{ab}$$

$$= - \frac{1}{2} g^{\ell \ell} \{ \partial_a g_{\ell b} + \partial_b g_{a \ell} - \partial_{\ell} g_{ab} \}$$

$$= \frac{1}{2} \partial_1 g_{ab}.$$ 

Thus, we have

$$\partial_1 g_{ab} = 2B_{ab} = 2 \frac{F''(r)}{F'(r)} g_{ab}.$$ 

Hence, one has

$$g_{ab}(r, x) = \left( \frac{F'(r)}{F'(r_0)} \right)^2 g_{ab}(r_0, x).$$

We will show Theorem 1.1.
Proof of Theorem [1.1]. The above argument shows that $|\nabla F|$ is constant on a regular level surface. Set $N^{n-1} = F^{-1}(c_0)$ and $\overline{g} = (F'(r_0))^{-2} \bar{g}_{r_0}$ for regular value $c_0$ of $F$. By the above argument, $F$ has at most 2 critical values. Without loss of generality, one can assume that $I = (-\infty, \infty)$, or $I = [0, \infty)$ with $F'(0) = 0$, or $I = [\alpha_0, \beta_0]$ with $F'(\alpha_0) = F'(\beta_0) = 0$.

We consider the first case. By a direct calculation, we can get formulas of the warped product manifold of the warping function $|\nabla F| = F'(r) > 0$ (cf. [13]).

For $a, b, c, d = 2, 3, \cdots, n$,

\begin{align*}
R_{1ab} &= -F'^2 F''' \bar{g}_{ab}, \quad R_{1bc} = 0, \\
R_{abcd} &= (F')^2 \bar{R}_{abcd} + (F' F'')^2 (\bar{g}_{ad} \bar{g}_{bc} - \bar{g}_{ac} \bar{g}_{bd}),
\end{align*}

(2.1)

\begin{align*}
R_{11} &= - (n - 1) \frac{F'''}{F'}, \quad R_{1a} = 0, \\
R_{ab} &= \bar{R}_{ab} - ((n - 2)(F'')^2 + F'' F''') \bar{g}_{ab},
\end{align*}

(2.2)

\begin{align*}
R &= (F')^{-2} \bar{R} - (n - 1)(n - 2) \left( \frac{F''}{F'} \right)^2 - 2(n - 1) \frac{F'''}{F'},
\end{align*}

(2.3)

where, the curvature tensors with bar are the curvature tensors of $(N, \bar{g})$. We consider the second case. Since $F$ has a unique critical point $x_0$, $r(x) = \text{dist}(x, x_0)$. Therefore, $\Sigma_c = \{F(x) = c\}$ is diffeomorphic to a geodesic sphere centered at $x_0$. By the smoothness of the metric $g$ at $x_0$, the induced metric $\overline{g}$ on $N^{n-1}$ is round. By an elementary argument shows that the third case, that is it is compact and rotationally symmetric.

By Theorem [1.1] one can recover the classification of 3-dimensional complete gradient Yamabe solitons. In fact, by non-existence theorem of compact gradient Yamabe solitons (cf. [12]), (1) of Theorem [1.1] cannot happen. We will consider the case (3) of Theorem [1.1] By the soliton equation $R - \rho = \varphi = F''$ and

\[ R = (F')^{-2} \bar{R} - 2 \left( \frac{F''}{F'} \right)^2 - 4 \frac{F'''}{F'}, \]

one can get that the scaler curvature $\bar{R}$ is constant. Therefore, $N^2$ is a space form.

**Corollary 2.2.** A nontrivial three-dimensional complete gradient Yamabe soliton $(M^3, g, F)$ is either
(1) rotationally symmetric and equal to the warped product 
\(([0, \infty), dr^2) \times |\nabla F| (\mathbb{S}^2, \bar{g}_S)\),
where, \(\bar{g}_S\) is the round metric on \(\mathbb{S}^2\), or
(2) the warped product 
\((\mathbb{R}, dr^2) \times |\nabla F| (N^2(c), \bar{g})\),
where, \(N^2(c)\) is a space form with constant curvature \(c\).

Therefore, we can answer to the Yamabe soliton version of Perelman’s conjecture.

**Corollary 2.3.** Any 3-dimensional complete nontrivial nonflat gradient Yamabe soliton \((M^3, g, F)\) with \(R \geq 0\) is rotationally symmetric.

### 3. Gradient conformal solitons with Vanishing condition on Cotton and Cao-Chen

In this section, we give two classification results. We first recall the Cotton tensor \(C\) and the Weyl tensor \(W\).

\[ C_{ijk} = \nabla_i S_{jk} - \nabla_j S_{ik} \]
\[ = \nabla_i R_{jk} - \nabla_j R_{ik} - \frac{1}{2(n-1)}(g_{jk} \nabla_i R - g_{ik} \nabla_j R), \]

where, \(S = \text{Ric} - \frac{1}{2(n-1)} R g\) is the Schouten tensor. The Cotton tensor is skew-symmetric in the first two indices and totally trace free, that is,

\[ C_{ijk} = -C_{jik} \quad \text{and} \quad g^{jk} C_{ijk} = g^{ik} C_{ijk} = 0. \]

\[ W_{ijk\ell} = R_{ijk\ell} - \frac{1}{n-2}(R_{ikg_{j\ell}} + R_{j\ell g_{ik}} - R_{i\ell g_{jk}} - R_{jkg_{i\ell}}) \]
\[ + \frac{R}{(n-1)(n-2)}(g_{ik} g_{j\ell} - g_{i\ell} g_{jk}). \]

As is well known, a Riemannian manifold \((M^n, g)\) is locally conformally flat if and only if (1) for \(n \geq 4\), the Weyl tensor vanishes; (2) for \(n = 3\), the Cotton tensor vanishes. Moreover, for \(n \geq 4\), if the Weyl tensor vanishes, then the Cotton tensor vanishes. We also see that for \(n = 3\), the Weyl tensor always vanishes, but the Cotton tensor does not vanish in general.

In Section 4, we will classify locally conformally flat gradient conformal solitons. Therefore, we consider gradient conformal solitons under
weaker assumption, that is, divergence-free Cotton tensor. Here we remark that $C \equiv 0$ and $\text{div } C \equiv 0$ are equivalent.

**Lemma 3.1.** On any Riemannian manifold, the following are equivalent.

1. $C \equiv 0$,
2. $\text{div } C \equiv 0$.

**Proof.**

\[
\nabla_i \nabla_k C_{kij} = \nabla_i \nabla_k (\nabla_k S_{ij} - \nabla_i S_{kj}) \\
= \nabla_i \nabla_k S_{ij} - \nabla_k \nabla_i S_{kj} \\
= R_{ikkp} \nabla_p S_{ij} + R_{ikjp} \nabla_k S_{pj} + R_{ikjp} \nabla_k S_{ip} \\
= - R_{ip} \nabla_p S_{ij} + R_{kp} \nabla_k S_{pj} \\
+ (S_{ij} g_{kp} + S_{kp} g_{ij} - S_{ip} g_{kj} - S_{kj} g_{ip}) \nabla_k S_{ip} \\
= S_{ij} C_{kik} + S_{ip} C_{ijp} \\
= - C_{jip} R_{ip}.
\]

Hence, one has

\[
\nabla_i \nabla_j \nabla_k C_{kji} = - \nabla_i C_{ijk} R_{jk} - C_{ijk} \nabla_i R_{jk}.
\]

By the definition and a property of the Cotton tensor,

\[
C_{ijk} \nabla_i R_{jk} = C_{ijk}(C_{ijk} + \nabla_j R_{ik} + \frac{1}{4}(g_{jk} \nabla_i R - g_{ik} \nabla_j R)) \\
= |C_{ijk}|^2 - C_{ijk} \nabla_j R_{ik}.
\]

Thus, we have

\[
C_{ijk} \nabla_i R_{jk} = \frac{1}{2} |C_{ijk}|^2.
\]

Substituting (3.3) into (3.2), we have

\[
\nabla_i \nabla_j \nabla_k C_{kji} = - \nabla_i C_{ijk} R_{jk} - \frac{1}{2} |C_{ijk}|^2.
\]

By the assumption, the Cotton tensor vanishes. \qed

**Proposition 3.2.** A nontrivial complete gradient conformal soliton $(M^n, g, F, \varphi)$ with $\text{div } C \equiv 0$ $(C \equiv 0)$ is either

1. compact and rotationally symmetric, or
2. rotationally symmetric and equal to the warped product
   \[(\mathbb{R}, dr^2) \times |\nabla F| (S^{n-1}, \bar{g}_S),\]
where, $\bar{g}_S$ is the round metric on $S^{n-1}$, or
3. the warped product
   \[(\mathbb{R}, dr^2) \times |\nabla F| (N^{n-1}, \bar{g}),\]
where, $N$ has constant scalar curvature $\bar{R}$. Furthermore, if $R \geq 0$, then either $\bar{R} > 0$, or $R = \bar{R} = 0$ and $(M, g)$ is isometric to the Riemannian product $(\mathbb{R}, dr^2) \times (N^{n-1}, \bar{g})$.

**Proof.** We only have to consider the case (3) of Theorem 1.1. By the same argument as in the proof of Theorem 1.1, one has

\[(3.4) \quad R_{11} = - (n - 1) \frac{F'''}{F'}, \quad R_{1a} = 0, \]

and

\[(3.5) \quad R = (F')^{-2} \bar{R} - (n - 1)(n - 2) \left( \frac{F''}{F'} \right)^2 - 2(n - 1) \frac{F'''}{F'},\]

for $a, b = 2, 3, \ldots, n$. By the definition of the Cotton tensor, we have

\[C_{a1} = \nabla_a R_{11} - \frac{1}{n} (g_{a1} \nabla_1 R - g_{1a} \nabla_1 R) \]

\[= \frac{1}{n} \nabla_a R.\]

From this and (3.5), the scalar curvature $\bar{R}$ of $N$ is constant. Assume that $R \geq 0$. If $\bar{R} \leq 0$, then by (3.5), $F'' \leq 0$. Therefore, $F'(> 0)$ is concave, which means that $F'$ is constant. By (3.5) again, $R = \bar{R} = 0$ and $(M, g)$ is the Riemannian product $(\mathbb{R}, dr^2) \times (N^{n-1}, \bar{g})$. \qed

For a gradient Ricci soliton $(\text{Ric} - \lambda g = \nabla \nabla F)$, Cao and Chen introduced a new tensor $D$ (cf. [5], [4]). We call it the Cao-Chen tensor:

\[D_{ijk} = \frac{1}{n-2} (R_{kj} \nabla_i F - R_{ki} \nabla_j F) \]

\[+ \frac{1}{(n-1)(n-2)} (R_{dt} g_{jk} \nabla_i F - R_{jt} g_{ik} \nabla_i F) \]

\[= \frac{R}{(n-1)(n-2)} (g_{kj} \nabla_i F - g_{ki} \nabla_j F).\]

Roughly speaking, on gradient Ricci solitons, the Cao-Chen tensor estimates the difference between the Weyl tensor and the Cotton tensor, but, on gradient Yamabe solitons, it doesn’t. Therefore it is interesting to consider the gradient conformal solitons with $D \equiv 0$.

**Theorem 3.3.** A nontrivial complete gradient conformal soliton $(M^n, g, F, \varphi)$ with vanishing Cao-Chen tensor is either

1. compact and rotationally symmetric, or
2. rotationally symmetric and equal to the warped product

\[((0, \infty), dr^2) \times |\nabla F| (S^{n-1}, \bar{g}_S),\]
where, \( \bar{g}_S \) is the round metric on \( S^{n-1} \), or

\( (3) \text{ the warped product} \)

\[ (\mathbb{R}, dr^2) \times [\nabla F] (N^{n-1}_{Ein}, \bar{g}), \]

where, \( N_{Ein} \) is an Einstein manifold. Furthermore, if \( R \geq 0 \), then either \( \bar{R} > 0 \), or \( R = \bar{R} = 0 \) and \( (M, g) \) is isometric to the Riemannian product \( (\mathbb{R}, dr^2) \times (N^{n-1}, \bar{g}) \), where, \( N \) is Ricci flat.

**Proof.** We only have to consider (3) of Theorem 1.1. By the same argument as in the proof of Theorem 1.1, one has

\[ R_{11} = -(n-1)\frac{F'''}{F'}, \quad R_{1a} = 0, \]

\[ R_{ab} = \bar{R}_{ab} - ((n-2)(F'')^2 + F'F''')\bar{g}_{ab}, \]

\[ R = (F')^{-2}\bar{R} - (n-1)(n-2)\left(\frac{F''}{F'}\right)^2 - 2(n-1)\frac{F'''}{F'}, \]

for \( a, b = 2, 3, \cdots, n \). By the definition of the Cao-Chen tensor \( D \), we have

\[ D_{1ab} = \frac{1}{n-2}(R_{ba}\nabla_1 F - R_{ba} g_a F) \]

\[ + \frac{1}{(n-1)(n-2)}(R_{1t} g_{ab} \nabla_t F - R_{at} g_{tb} \nabla_t F) \]

\[ - \frac{R}{(n-1)(n-2)}(g_{ba} \nabla_1 F - g_{ba} \nabla_a F) \]

\[ = \frac{1}{n-2}R_{ba} \nabla_1 F + \frac{1}{(n-1)(n-2)}R_{11} g_{ab} \nabla_1 F \]

\[ - \frac{R}{(n-1)(n-2)}g_{ba} \nabla_1 F. \]

Since the Cao-Chen tensor vanishes, and \( F' > 0 \), one has

\[ R_{ab} = \frac{\bar{R} - R_{11}}{n-1} g_{ab}. \]

From this, (3.6) and (3.7), we have

\[ \bar{R}_{ab} = \frac{\bar{R}}{n-1} \bar{g}_{ab}. \]

Hence, \( N \) is an Einstein manifold. Thus, the scalar curvature \( \bar{R} \) of \( N \) is constant. Assume that \( R \geq 0 \). If \( \bar{R} \leq 0 \), then by (3.7), \( F'''' \leq 0 \). Therefore, \( F' > 0 \) is concave, which means that \( F' \) is constant. By (3.7) again, \( R = \bar{R} = 0 \) and \( \bar{Ric} = 0 \) on \( N \). Therefore, \( (M, g) \) is the Riemannian product \( (\mathbb{R}, dr^2) \times (N^{n-1}, \bar{g}) \), where, \( N \) is Ricci flat.  \( \Box \)
As is well known, Einstein manifolds of dimension $n \leq 3$ are space forms, hence one can get the following.

**Corollary 3.4.** A nontrivial complete gradient conformal soliton $(M^n, g, F, \varphi)$ $(n \leq 4)$ with vanishing Cao-Chen tensor is either

1. compact and rotationally symmetric, or
2. rotationally symmetric and equal to the warped product

$$([0, \infty), dr^2) \times_{|\nabla F|} (S_{n-1}^{n-1}, \bar{g}_S),$$

where, $\bar{g}_S$ is the round metric on $S^{n-1}$, or

3. the warped product

$$(\mathbb{R}, dr^2) \times_{|\nabla F|} (N^{n-1}(c), \bar{g}),$$

where, $(N^{n-1}(c), \bar{g})$ is a space form.

Therefore, we have the following.

**Corollary 3.5.** Any complete nontrivial nonflat gradient conformal soliton $(M^n, g, F, \varphi)$ $(n \leq 4)$ with $D \equiv 0$ and $R \geq 0$ is rotationally symmetric.

4. **Classification of locally conformally flat gradient conformal solitons**

In this section, inspired by [4], [6], [7] and [9], we classify locally conformally flat gradient conformal solitons.

**Lemma 4.1.** Let $(M, g, F, \varphi)$ be a nontrivial complete locally conformally flat gradient conformal soliton. Assume that $F$ has no critical point. Then, $(M, g, F, \varphi)$ is warped product

$$(\mathbb{R}, dr^2) \times_{|\nabla F|} (N^{n-1}(c), \bar{g}),$$

where, $(N^{n-1}(c), \bar{g})$ is a space form.

**Proof.** We consider (3) of Theorem 1.1. By the same argument as in the proof of Theorem 1.1, one can get formulas of the warped product manifold of the warping function $(0 <)|\nabla F| = F'(r)$. For $a, b, c, d = 2, 3, \cdots, n$.

$$R_{1a1b} = -F'F''\bar{g}_{ab}, \quad R_{1abc} = 0,$$

$$R_{abcd} = (F')^2 \bar{R}_{abcd} + (F'F'')^2(\bar{g}_{ad}\bar{g}_{bc} - \bar{g}_{ac}\bar{g}_{bd}),$$

(4.1)
(4.2) \[ R_{11} = - (n - 1) \frac{F'''}{F'}, \quad R_{1a} = 0, \]
\[ R_{ab} = \bar{R}_{ab} - ((n - 2)(F'')^2 + F'F''')\bar{g}_{ab}, \]

(4.3) \[ R = (F')^{-2} \bar{R} - (n - 1)(n - 2) \left( \frac{F''}{F'} \right)^2 - 2(n - 1) \frac{F'''}{F'}. \]

Case 1. \( \dim M = 3 \): By Proposition 3.2, \( N \) is a space form.

Case 2. \( \dim M \geq 4 \): By (3.1), (4.1), (4.2) and (4.3), one has

\[ W_{1ab} = - \frac{\bar{R}_{ab}}{n - 2} + \frac{\bar{R}}{(n - 1)(n - 2)} \bar{g}_{ab}, \]
\[ W_{1abc} = 0, \]
\[ W_{abcd} = (F')^2 \left( \tilde{W}_{abcd} + \frac{1}{(n - 2)(n - 3)} \left\{ \frac{2}{n - 1} \bar{R} (\bar{g}_{ad}\bar{g}_{bc} - \bar{g}_{ac}\bar{g}_{bd}) \right. \right. \]
\[ - \left( \bar{R}_{ad}\bar{g}_{bc} + \bar{R}_{bc}\bar{g}_{ad} - \bar{R}_{ac}\bar{g}_{bd} - \bar{R}_{bd}\bar{g}_{ac} \right) \right\} \right]. \]

Since \( M \) is locally conformally flat, one has

(4.4) \[ \bar{R}_{ab} = \frac{\bar{R}}{n - 1} \bar{g}_{ab}, \]

and

(4.5) \[ \tilde{W}_{abcd} = - \frac{1}{(n - 2)(n - 3)} \left\{ \frac{2}{n - 1} \bar{R} (\bar{g}_{ad}\bar{g}_{bc} - \bar{g}_{ac}\bar{g}_{bd}) \right. \]
\[ - \left( \bar{R}_{ad}\bar{g}_{bc} + \bar{R}_{bc}\bar{g}_{ad} - \bar{R}_{ac}\bar{g}_{bd} - \bar{R}_{bd}\bar{g}_{ac} \right) \right\}. \]

Substituting (4.4) into (4.5), one has \( \tilde{W}_{abcd} = 0 \). Therefore, \( N \) is Einstein and locally conformally flat, which means that \( N \) is a space form. \( \square \)

**Corollary 4.2.** A nontrivial complete locally conformally flat gradient conformal soliton \( (M^n, g, F, \varphi) \) is either

1. compact and rotationally symmetric, or
2. rotationally symmetric and equal to the warped product
   \[ ([0, \infty), dr^2) \times |\nabla F| (S^{n-1}, \bar{g}_S), \]
   where, \( \bar{g}_S \) is the round metric on \( S^{n-1} \), or
3. the warped product
   \[ (\mathbb{R}, dr^2) \times |\nabla F| (N^{n-1}(c), \bar{g}), \]
where, \((N^{n-1}(c), \bar{g})\) is a space form.

Therefore, we have the following.

**Corollary 4.3.** Any complete nontrivial nonflat locally conformally flat gradient conformal soliton with \(R \geq 0\) is rotationally symmetric.

**Data availability statement**

Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

**Conflict of interest**

There is no conflict of interest in the manuscript.

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