Principles and Dynamics of Quantum Mechanics

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We determine that the wavefunction of an interacting particle is the superposition of its virtual concurrent states. In many cases we can tell what concurrent states arise. Furthermore, we derive the Schrodinger and Dirac equations as the unique conditions the wavefunction must satisfy at each point in order to fulfill the corresponding averaged energy relation. In the balanced particle state and wave approach we follow, quantum dynamics becomes tangible and we obtain a logically connected and physically justified quantum theory.

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1. INTRODUCTION

Quantum mechanics has been based on the axiom that the Schrodinger equation “plays a role logically analogous to Newton’s second law” and that “is not derivable from any more basic principle, but is one of the laws of quantum physics.”\textsuperscript{1-2} Relying on the postulated Schrodinger equation, quantum mechanics becomes a space-time based theory, losing the dual, particle state and wave, description of quantum systems. Furthermore, the physical origin of the wavefunction is unspecified since it is the solution of a postulated equation.

In our approach, the wavefunction of a particle is the superposition of its virtual concurrent wave-states. The variation of the wavefunction in space and time arises from the interference of particle waves that have constant amplitudes.

In particle scatterings, we can tell what concurrent states arise and thus obtain the wavefunction. For example, in the electron double slit experiment the wavefunction at some point arises from the merging of two concurrent wave states at that point. On the other hand, the wavefunction of a bound particle fulfills the averaged energy relation from which we derive the Schrodinger equation as the unique condition the wavefunction must satisfy at each point.

We will first discuss particle wave-states and define the wavefunction as the superposition of the virtual concurrent particle states. Next, we derive the Schrodinger equation and then present the essential aspects of quantum dynamics by considering a series of scattering processes and bound particle states.

2. PRINCIPLES OF QUANTUM MECHANICS

atomic scale phenomena happen with specific probabilities that we obtain from the interference of concurrent particle waves. The fundamental principle of quantum mechanics is that the wavefunction of an interacting particle is the superposition of its virtual concurrent wave states.
A. Particle States and Particle Waves

Particles do not oscillate per se, since their intrinsic properties are measured to be always the same. On the other hand, an interacting particle has virtual concurrent states that interfere like waves and determine the probabilities of the various outcomes.

We address this situation—non-waving single particle state and wave interference of concurrent states—by associating each particle state of momentum $p$ and energy $E$ with two particle waves (one real and one imaginary) of the form

$$\psi(r,t) = a(p,E) e^{i(p \cdot r - E t)}$$

(1)

The waves in $\psi(r,t)$ have $\lambda = h/p$ wavelength and $T = h/E$ period. The intensity $|\psi(r,t)|^2 = |a(p,E)|^2$ of a single particle wave is constant in space and time—the amplitude does not depend on the phase—so a single particle state is not waving. On the other hand, two or more particle states interfere like waves.

In one dimension, we can visualize a particle wave by imagining at each point a clock with one hand rotating with period $T = h/E$, while the phase angle increases in the direction of the momentum attaining the same orientation $\lambda = h/p$ meters farther. In three dimensions, the plane wave fills the entire space and the “clock hands” rotate at all points with period $T$ while their phase angles increase, in the direction of the momentum, completing a full cycle every $\lambda$ meters.

B. Concurrent States and Wavefunction

The particles we are interested in have positions and momenta within certain ranges, interact with boundaries and with energy potentials at all points simultaneously and so are described by superpositions of concurrent wave-states.

The amplitude $a(p_i,E_i)$ of each concurrent state corresponds to the probability of finding the particle in that state, but the particle is not in any particular state before detection. The concurrent states are virtual; they do not specify where the particle is or how it moves. On the other hand, the concurrent states represent all possibilities of where a particle may be or how it may move and their interference yields the probabilities of the various outcomes.

The wavefunction $\psi(r,t)$ of a particle is the superposition of its virtual concurrent wave states [we use $\psi(r,t)$ for plane waves and for wavefunctions]

$$\psi(r,t) = \sum_i a(p_i,E_i) e^{i(p \cdot r - E t)}$$

$$a(p_i,E_i) = \frac{1}{V_0} \int_V |\psi(r,t)| e^{-i(p \cdot r - E t)} d^3rdt$$

(2)

where $V_0$ is the normalization volume. We consider that the particle is within a cube of side $L$ ($V_0 = h^2L^3)$ and so has discrete values of momentum ($p_i$) and energy ($E_i$). One benefit of this quantization scheme is that our expressions become clearer by involving space-time integrations and momentum-energy summations.

Note: There is an additional term in the phases of the above expressions; instead of $[p_i \cdot r - E_it]$ we should had written $[p_i \cdot r - E_it + \theta_i]$, where the relative phase $\theta_i$ is the phase of
the particle wave at \( r=0 \). To see the role of \( \theta_i \)'s, consider the ground electron eigenfunction in a Hydrogen atom which is a superposition of particle waves having momenta \( p \) (from \(-\infty\) to \(+\infty\)) and, as we will see, amplitudes proportional to \([1+b^2p^2]^{-2}\), where \( b \) is a constant number. Now, it is the values of the relative phase angles [\( \theta(p)=0 \) in this case] that specify that the highest intensity of the electron eigenfunction happens in the vicinity of the nucleus (at \( r=0 \)). Nevertheless, the relative phases of particle waves can be omitted because in particle scatterings all states have common origin, therefore we can take \( \theta(p,E)=0 \); on the other hand, the wavefunctions of bound particles are obtained as solutions of the Schrödinger equation and incorporate the space-time effects of the relative phase angles.

Equations (2) interrelate the particle space-time description [the wavefunction \( \psi(r,t) \)] to its momentum-energy description [amplitudefunction \( a(p_i,E_i) \)]. A detection of the position of a particle will locate it at point \( (r,t) \) with probability \(|\psi(r,t)|^2\), while a measurement of momentum-energy will find the particle in some state \( (p_i,E_i) \) with probability \(|a(p_i,E_i)|^2\). Repeated measurements verify these probabilistic predictions.

Quantum mechanics is the ultimate non-local theory since particles are described by states that have constant amplitudes over all space. Given that quantum states are detected in different ways, seemingly puzzling results arise, especially when measuring electron spins and photon polarizations of entangled states. Quantum mechanics specifies, correctly, that measurement detects only one of the possible concurrent states.

Knowing the wavefunction \( \psi(r,t) \) and, consequently, the amplitudefunction \( a(p_i,E_i) \) of a particle, we can calculate the average value of quantities that depend on energy-momentum or on space-time. For example, the average kinetic and potential energies of an electron in the ground state of a Hydrogen atom are

\[
\frac{\overline{p^2}}{2m} = \int |a_i(p)|^2 \frac{p^2}{2m} d^3 p = +13.6eV
\]

\[
\overline{V(r)} = \int |\psi_i(r)|^2 - k\frac{e^2}{r} d^3 r = -27.2eV
\]

where \(|a_i(p)|^2\) and \(|\psi_i(r)|^2\) are shown in the next section.

When we have angular momentum eigenfuctions we can calculate the value of their total angular momentum as well.

**C. Calculation of Wavefunctions**

The characteristics of the interaction of a particle determine what concurrent states arise and, also, how to proceed and obtain the particle wavefunction and amplitudefunction.

In particle scatterings – i.e. electron passing through slits, photon reflecting off a mirror, alpha particles hitting a barrier, electron colliding with a nucleus, etc. – we can tell what virtual concurrent states arise and thus obtain the wavefunctions directly.

On the other hand, we often find the wavefunction of a bound particle, i.e. of an atomic electron, from the requirement that its energy is equal to its average kinetic plus potential energies. We will see that the averaged energy relation can be fulfilled by specific wavefunctions only, the solutions of the Schrödinger equation.
3. DERIVATION OF THE SCHRODINGER EQUATION

Since the particle wavefunction consists of wave states of constant amplitude, the interaction of a particle with an energy potential \( V(r) \) is properly expressed in global terms: the particle energy is equal to the averaged kinetic plus potential energies.

A. Ground State of Hydrogen Atom

We consider the ground eigenstate of the Hydrogen atom in which we can detect the concurrent states of the electron and measure their probabilities. Hitting the atom with an energetic photon and measuring the momenta of the scattered photon and the ejected electron, we deduce the initial electron momentum. Such “photoelectric” experiments have been carried out with Hydrogen and Helium atoms. From these measurements we determine that the probability of finding the electron in a momentum state \( p \) is

\[
\text{momentum probability} = |a_i(p)|^2 = \frac{1}{\pi^2} \left( \frac{2r_0}{\hbar} \right)^3 \frac{1}{\left[ 1 + \frac{r_0^2}{\hbar^2 p^2} \right]^4}
\]

(4)

where \( r_0 = 0.053 \text{ nm} \).

Next, we can either calculate the position probability amplitude from (2) or measure it since it is possible, in principle, to locate the electron by taking very fast photographs. Such measurements would show that the probability of finding the electron at a distance \( r \) from the nucleus is

\[
\text{position probability} = |\psi_i(r)|^2 = \left[ \frac{1}{\sqrt{\pi r_0^3}} e^{-\frac{r}{r_0}} \right]^2
\]

(5)

Thus, we have obtained, experimentally, the electron amplitude function and eigenfunction of the ground state of a Hydrogen atom.

All concurrent electron states on the ground level have the same energy equal to its average kinetic plus potential energies

\[
E_i = \frac{p^2}{2m} + V(r)
\]

\[
E_i = \int |a_i(p)|^2 \frac{p^2}{2m} d^3p + \int |\psi_i(r)|^2 \frac{k e^2}{r} d^3r = +13.6 - 27.2 = -13.6 \text{ eV}
\]

(6)

Now, we ask the question: Does the averaged energy relation determine the particle eigenfunction? Ordinarily, an average does not specify the particular combination of values that have produced it. However, in the present case the eigenfunction consists of particle wave-states that extend over all space and in which the space-time variables are interrelated to the momentum-energy ones. We will find then that the averaged energy relation does specify the eigenfunctions that fulfill it.
B. Averaged Energy Relation and Schrodinger Equation

We consider first that the interaction potential $V(r)$ depends only on space. Then the particle energy is constant, the time-energy part of all particle waves factors out, and the space part of the eigenfunction has the form

$$\psi(r) = \sum_i a(p_i) e^{i p_i r}$$  \hspace{1cm} (7)

The averaged energy relation is

$$E = \frac{p^2}{2m} + V(r) \rightarrow \frac{1}{V_0} \int |\psi|^2 E d^3r = \sum_i |a(p_i)|^2 \frac{p_i^2}{2m} + \frac{1}{V_0} \int |\psi|^2 V(r) d^3r$$  \hspace{1cm} (8)

and since from (7) we have

$$\sum_i |a(p_i)|^2 \frac{p_i^2}{2m} = \frac{1}{V_0} \int \psi^* \left[ -\frac{\hbar^2}{2m} \frac{\partial^2 \psi}{\partial r^2} \right] d^3r$$  \hspace{1cm} (9)

we put the averaged energy relation in the following form

$$E - \frac{p^2}{2m} - V(r) = 0 \rightarrow \frac{1}{V_0} \int \psi^* \left[ E\psi + \frac{\hbar^2}{2m} \frac{\partial^2 \psi}{\partial r^2} - V(r)\psi \right] d^3r = 0$$  \hspace{1cm} (10)

The above equation states that the average value of the integrand is zero. But, as we will justify below, the integrand itself is zero at each point $(r)$. Dividing the integrand by $\psi^*$ we get

$$-\frac{\hbar^2}{2m} \frac{\partial^2 \psi}{\partial r^2} + V(r)\psi = E\psi \hspace{1cm} at \hspace{1cm} each \hspace{1cm} r$$  \hspace{1cm} (11)

This is the Schrodinger equation, derived to be the unique condition the wavefunction must satisfy at each point in order to fulfill the averaged energy relation.

We realize that the integrand $I(r)$ in (10) is equal to zero everywhere because if it differs from zero at some point that would alter the value of the potential. For example, suppose that the integrand at point $r$ is $I(r) = N(r) = \psi^* n(r) \psi$. Then, we would have

$$I(r) = \psi^* \left[ E\psi + \frac{\hbar^2}{2m} \frac{\partial^2 \psi}{\partial r^2} - V(r)\psi \right] = \psi^* n(r) \psi = \psi^* \left[ E\psi + \frac{\hbar^2}{2m} \frac{\partial^2 \psi}{\partial r^2} - \{V(r) + n(r)\} \psi \right] = 0$$  \hspace{1cm} (12)
If $n(r)$ is not zero, the value of the classical potential changes from $V(r)$ to $V(r) + n(r)$ and we are dealing with a different potential and wavefunction. Simply put, since the potential is the only quantity that depends explicitly on space, its value will be affected if the integrand is not zero at some point.

The classical potential $V(r)$ is an effective description of the interaction. Even though quantum energy fluctuations do happen, the classical potential is the average over these background fluctuations and this precludes additional variations.

When the potential $V(r,t)$ depends on space and time we follow similar steps and derive from the averaged energy relation the space-time Schrödinger equation

$$\frac{p^2}{2m} + V(r,t) = E \quad \rightarrow \quad -\frac{\hbar^2}{2m} \frac{\partial^2 \psi}{\partial r^2} + V(r,t)\psi = i\hbar \frac{\partial \psi}{\partial t}$$

(13)

Only wavefunctions that are solutions of the Schrödinger equation can fulfill the averaged energy relation.

Furthermore, by enlarging the quantization volume (taking the edge $L$ of the quantization cube to infinity) the above equation can be used to describe the interaction of free particles.

### 4. PARTICLE SCATTERINGS

In this and the next section, we illustrate our approach by considering a series of particle scatterings and bound states. When particles encounter fixed boundaries or in particle collisions, we can tell what concurrent states arise and obtain directly the particle wavefunction.

#### A. Electron Double Slit Experiment

In the electron double slit experiment (considered in two dimensions), electrons are fired at a certain horizontal velocity toward a screen with two slits. Each electron that passes through leaves a dot on a film plate behind the slits and we observe that dots accumulate at specific areas on the film separated by blank zones. Each electron that lands on the film has passed through one of the slits. However, the probability that an electron may land on some point depends on its two virtual states merging at that point.

The electron wavefunction at any point on the film arises from the interference of two virtual concurrent states emerging from the slits. The intensity of the wavefunction at some point corresponds to the probability that the electron may land on that point, and is high at the center between the slits and varies periodically from zero to high at points farther from the center. By firing many electrons, the probabilistic prediction is verified by the distribution of dots on the film.

#### B. Light Reflection and Refraction

The wavefunction of a photon reflecting off a plane mirror is the superposition of infinite concurrent paths (states) from the light source to any point of the mirror and then to the eye of an observer. However, considering a strand of neighboring photon paths bouncing off some tiny region of the mirror to the eye, the difference in travel time within this strand is continuous and so the virtual photon states, merging with all kinds of
phases, cancel out. Only the photon states neighboring the fastest (bouncing at equal angles) route have equally fast travel times, propagate with same phases and interfere constructively creating a strong wavefunction along this route that photons do follow.

Likewise, in light refraction the interference of the infinite virtual concurrent states (paths) explains why photons propagate along the fastest route specified by Snell’s law.

It sounds paradoxical, but a photon travels along the fastest path because it may travel equally well along any other path.

C. Electron-Nucleus Scattering

In low energy electron-nucleus collisions, we can consider that the nucleus, being thousands of times more massive than the electron, is the almost unperturbed center of scattering. The energy of the scattered electron is equal to that of the incoming electron, since the recoil of the nucleus is infinitesimal.

The outgoing electron state in direction $\theta$ (relative to the initial electron momentum) arises from the contributions of all concurrent scattering events that produce electron states of momentum $p_\theta$.

Let us consider such an event at point $r$ from the nucleus. The incoming electron state $\psi_0(r)$ interacts with the electric potential $V(r) (= kZe^2/r)$ and a scattered wave $\psi_{sc}(r)$ is radiated from point $r$ in all directions. The scattered wave is proportional to the incoming wave and to the strength of the electric potential of the nucleus at $r$

$$\psi_{sc}(r) = V(r)\psi_0(r) \quad (14)$$

Summing over all concurrent scattering events, we find the amplitude $A_\theta$ of the electron state $p_\theta$ by taking the “projection” of the scattered waves $\psi_{sc}$ to $\psi_0$.

$$A_\theta \propto \int \psi^*_0(r)V(r)\psi_0(r)d^3r = \int e^{i\frac{p_\theta \cdot r}{\hbar}} \frac{kZe^2}{r} e^{i\frac{p_\theta \cdot r}{\hbar}} d^3r \quad (15)$$

Carrying out the space integration, we obtain the well known result

$$\text{probability} = |A_\theta|^2 = \frac{k^2Z^2e^4}{16E^2\sin^4(\theta/2)} \quad (16)$$

In addition, the incoming electron may be scattered twice, thrice, etc. by the potential (the probability amplitude between scatterings is given by the electron propagator) but these higher order contributions are tiny. The general theory of scattering is based on the time-depended Schrodinger equation (13), while for high-energy collisions, where additional particles may be created, we use the quantum interactions theory.

Here we can connect with the quantum interactions theory and consider that the electron scattering happens through concurrent processes in which one intermediate photon is exchanged between nucleus and electron. A virtual photon of momentum $q$ absorbed or emitted by the electron at point $r$ contributes a factor $1/q^2$ to the scattering amplitude; that is, a factor that is inversely proportional to the deviation from its real state. Summing first over $q$ we get the integral expression in (15). In conclusion, the exchange of low energy intermediate photons gives rise to the classical electric potential.
5. BOUND STATES

Bound particle states are superpositions of (detectable) virtual concurrent states and can serve as showcases of quantum mechanical concepts and insights. We present, mainly, the aspects characteristic to our approach.

A. Particle in a Box

Consider an electron inside a solid, one-dimensional box. Since the electron has zero average momentum, each energy eigenfunction is the superposition of two concurrent states having equal in magnitude amplitudes and opposite momenta. In addition, the eigenfunction must vanish at the impenetrable walls of the box, at \( x = 0 \) and \( x = L \), and this condition is satisfied for wavelengths \( \lambda_n = 2L/n \) (\( n=1, 2, \ldots \)) corresponding to discrete momentum values \( p_n = \pm \hbar/2L \).

The eigenfunctions of an electron in a one-dimensional box of size \( L \) have the form

\[
\psi_n(x,t) = a(+p_n) e^{i(p_n x - E_n t)\hbar}\sqrt{\frac{2}{L}} \sin \frac{p_n x}{\hbar} e^{\frac{-iE_n t}{\hbar}} + a(-p_n) e^{i(-p_n x - E_n t)\hbar}\sqrt{\frac{2}{L}} \sin \frac{p_n x}{\hbar} e^{\frac{iE_n t}{\hbar}}
\]

where \( p_n = n\hbar/\lambda_n = n\hbar/2L \) and \( a(+p_n) = -a(-p_n) = \frac{-i}{\sqrt{2L}} \).

The signs of the amplitudes \( a(+p_n) \) and \( a(-p_n) \) are determined by the boundary conditions \( \psi(0,t) = \psi(L,t) = 0 \), while their magnitudes are fixed by the normalization requirement that the total probability is 1.

On the lowest momentum-energy level \( (p_1 = \hbar/\lambda_1, E_1 = p_1^2/(2m)) \), the probability that the particle may be found at point \( x \) is proportional to \( \sin^2(\pi x/L) \), maximum at the middle \( (x=L/2) \) and decreasing toward the edges. If we could take a relatively low speed photograph (scores of bouncing photons) of an electron on the ground level inside an “atomic size glass sphere”, the photograph would show a spherical electron cloud thickest at the center of the sphere. This distribution resembles the spread of the electron in the ground state of a Hydrogen atom.

Textbooks apply the Schrodinger equation to obtain the eigenfunctions of a particle in a box and get the solution without its physical explanation. In our approach, the origin of each eigenfunction is physically justified since measurements would yield that the electron moves with momentum \( p_n \) inside the box half the times to the right and the other half to the left.

B. Time Variation

A wavefunction that is a superposition of concurrent states having different energies varies in time. For example, wavefunctions that are linear combinations of two or more energy eigenfunctions of a particle in a box, a harmonic oscillator, an atomic electron, etc. vary in space and time.

C. Hydrogen Atom

We have already analyzed experimentally the electron ground state of a Hydrogen atom. Below we show the theoretical calculation side.

An electron eigenstate in a Hydrogen atom is a superposition of concurrent states that have the same energy \( E_n \). Since there are no boundaries, virtual states of all momenta...
(from \(-\infty\) to \(+\infty\)) arise in every direction. The averaged electron energy relation is fulfilled only by eigenfunctions that satisfy the Schrodinger equation

\[
-\frac{\hbar^2}{2m} \frac{\partial^2 \psi}{\partial r^2} - \frac{ke^2}{r} \psi = E_n \psi
\]  

(18)

The solutions of the above equation yield all electron eigenfunctions and from these we can calculate the corresponding amplitudes functions.

For example, for the ground electron state we have

\[
\psi_1(r,t) = \psi_1(r) \otimes e^{\frac{-iE_1t}{\hbar}} = \int_{-\infty}^{\infty} a_1(p)e^{\frac{iE_1}{\hbar} p^2} d^3p \otimes e^{\frac{-iE_1t}{\hbar}}
\]  

(19)

Solving equation (18) we obtain \(\psi_1(r)\) and then we calculate \(a_1(p)\)

\[
\psi_1(r) = \frac{1}{\sqrt{\pi r_0^3}} e^{-\frac{r}{r_0}} \quad \text{and} \quad a_1(p) = \frac{1}{\pi} \left( \frac{2r_0}{\hbar} \right)^{3/2} \frac{1}{\sqrt{1 + \frac{r_0^2}{\hbar^2} p^2}}
\]  

(20)

where \(r_0 = \hbar^2/(mk^2) = 0.053\) nm, and we get \(E_1 = -ke^2/(2r_0) = -13.6\ eV\).

We note that there is no waving in \(\psi_1(r)\) which is a static function; particle waves are not detectable only their interference effects are.

In the traditional approach followed in textbooks, the electron eigenfunctions are obtained by the postulated Schrodinger equation. In our approach, experimentally justified by the analysis in section 3, each eigenfunction is a superposition of infinite concurrent states that fulfills the averaged energy relation.

6. DISCUSSION

We have based quantum mechanics on the experimentally justified and theoretically consistent principle that the wavefunction of a particle is the superposition of its virtual concurrent wave states. Given the characteristics of the interaction, either we can tell what concurrent states arise or obtain the wavefunction from the requirement that it fulfills the averaged energy relation. Following this approach, quantum dynamics becomes transparent, as we have demonstrated in discussing a series of particle scatterings and bound particle states.

When the interaction is described by a classical potential, the wavefunction must fulfill the averaged energy relation from which we have derived that the wavefunction satisfies the Schrodinger equation at each space-time point. Likewise, we can show that the Dirac equation of an electron in an electromagnetic field is the condition the wavefunction must satisfy at each point in order to fulfill the averaged relativistic energy relation. The Pauli equation, which also includes the interaction of the electron spin with the magnetic field, is obtained either as a low energy approximation of the Dirac equation or by “linearizing” the Schrodinger equation.\(^{(4)}\)
Here, a brief discussion of the history and status of the Schrodinger equation in quantum mechanics is in order. Erwin Schrodinger obtained his equation in January 1926, before it was realized that particle waves are probability waves (Born, July 1926). Schrodinger’s original derivation has been considered an “invention” because no solid and physically justified derivation was found. As a consequence, current textbooks regard the Schrodinger equation an axiom of quantum mechanics, introducing it with statements of the form “We cannot derive the Schrodinger equation from more basic principles, it is the basic principle”.\(^{(5)}\)

Multiplying the Schrodinger equation by \(\psi^*\) and integrating over space and time we obtain the averaged energy relation but it seems incomprehensible that the reverse holds. However, this is what we have shown.

There is a conceptual difference between the traditional approach to quantum mechanics and the one we follow. The traditional attitude is to find an “equation of motion” that would yield the particle wavefunction.\(^{(1)}\) Thus, the Schrodinger equation is postulated as the fundamental dynamical principle, analogous to Newton’s second law of motion in classical physics.

Our approach is that quantum dynamics has global character since particle wave-states have constant amplitudes. When a particle interacts with a classical potential, the wavefunctions that fulfill the averaged energy relation satisfy the Schrodinger equation at each point and this creates the local appearance of quantum dynamics.

The balanced particle state and wave approach we have followed yields a clear, experimentally justified and theoretically consistent quantum theory.

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