Coordinate-choice independent expression for drift orbit flux and flux-force relation in neoclassical toroidal viscosity theory

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A coordinate-choice independent expression does not depend how the magnetic surface is parametrized by (θ, ζ). Flux-force relation in neoclassical toroidal viscosity (NTV) theory has been generalized in a coordinate-choice independent way. The expression for the surface averaged drift orbit flux in 1/ν regime is derived without the requirement of straight field line coordinates. The resulted formula is insensitive to how the magnetic surface is parametrized and broadens the cases where flux-force relation can be applied. Construction of straight field line coordinates is avoided when the formula is used for numerical computation.

I. INTRODUCTION

Plasma rotation is of great importance in plasma physics [1-5]. Neoclassical toroidal viscosity (NTV) theory is one of the candidates for explaining the effect on plasma rotation when a non-axisymmetric perturbation is applied to tokamak [6-10]. Such effect from NTV theory has been observed in many experiments [11-13]. A fundamental relation in NTV theory is the flux-force relation, which relates the neoclassical viscosity forces on plasma to the drift orbit flux through each magnetic surface [1, 7, 14, 15]. By solving the drift kinetic equation, the drift orbit flux can be further expressed with respect to the variation of B field on the magnetic surface [8, 16]. The drift orbit flux calculation gives a non-zero result once the axisymmetry is broken by a 3D perturbation. This whole chain of derivation shows a mechanism that how the variation of B introduced by perturbation will result in a surface averaged force that modifies the plasma rotation profile.

Despite the wide usage of such NTV theory, the existing formula for flux-force relation and the drift orbit flux is not coordinate-choice independent. By coordinate-choice independent [17], the authors of this paper mean that the formula does not depend on how the magnetic surface is parametrized by (θ, ζ) — the flux label ψ will still be used because the flux surfaces are determined once the B field is given, but the (θ, ζ) can be chosen even as non-straight field line coordinates. The flux-force relation was first derived under Hamada [14], and then is generalized to other straight field line coordinates (magnetic coordinates) [6]. The expression for drift orbit flux is only derived under Hamada [6]. The flux-force relation gives the change rate of the surface averaged quantity \( \langle \vec{U} \cdot \vec{Q} \rangle \), where \( \vec{U} \) is the macro velocity and \( \vec{Q} \) a vector field with certain constrain such that the relation to transport quantity holds.

In the case derived in Hamada [14], it is shown that \( \vec{Q} \) can be the vector field \( \vec{e}_\theta, \vec{e}_\phi \) or a linear combination of them, where \( \theta \) and \( \zeta \) are the Hamada coordinates. In the cases derived for other straight field line coordinates in [6], it has been shown that by including the pressure term \( p \) together with the viscous tensor \( \nabla \times \vec{e}_\phi \), one can have \( \vec{Q} = m \sqrt{\nabla \psi} \times \vec{e}_\phi - n \sqrt{\nabla \psi} \times \vec{e}_\phi = m \vec{e}_\phi + n \vec{e}_\phi \) and the flux-force relation still holds. Specifically, if \( \alpha = m \theta - n \zeta \) is the helical angle for symmetry, \( \vec{Q} \) becomes the symmetry vector. \( \vec{Q} \) thus only represents one vector field. In all these cases shown, the expression for \( \vec{Q} \) depends on how the magnetic surface is parametrized by (θ, ζ) and the coordinates has to be a straight field line coordinate.

The set of vector fields from which \( \vec{Q} \) can be chosen constrains how the theory can be applied. It is often regarded as a formula for the NTV toroidal torque if one chooses \( \vec{Q} = \vec{e}_\phi \) and \( \langle \vec{U} \cdot \vec{e}_\phi \rangle \) is treated as the toroidal angular momentum. Since \( \zeta \) has to be one of the parameters in straight field line coordinates, this equivalence is not exact for non-axisymmetric cases where \( \vec{e}_\phi \) wobbles and does not agree with the toroidal direction in the lab’s frame. The quasi-symmetry vector \( \vec{Q}_q \) defined in [17] has another definition \( \vec{Q}_q = (F \vec{B} + \vec{B} \times \nabla \psi) / 2 \pi B^2 \), where \( F(\psi) \) is the poloidal electric current flux. This definition does not depend on the choice of (θ, ζ), but the function \( F(\psi) \) is not a free parameter and \( \vec{Q} \) thus only represents one vector field. In this paper, the authors give a coordinate-choice independent expression for a family of vector fields: \( \vec{S} = h(\vec{r}) \vec{B} + C(\psi) \vec{B} \times \nabla \psi / B^2 \) which includes all the cases of \( \vec{Q} \) shown previously and more. For any vector fields described by such format, the flux-force relation will hold. Such format does not depend on how the magnetic surface is parametrized by (θ, ζ) and can be constructed even in coordinates using lab's toroidal and poloidal angle.

The expression for drift orbit flux, which is related to \( \vec{P} \) term in flux-force relation, will be non-zero when the toroidal symmetry is broken. This drift orbit flux can be expressed by the variation of B field on the magnetic surface. The analytic formula has so far been derived in Hamada [8, 16]. Using the vector field \( \vec{S} \), the authors generalize the expression of drift orbit flux for the 1/ν regime case and obtain a coordinate-choice independent formula that can be
evaluated without straight field line coordinates.

The results derived in this paper also serve as a potential speed up method for the numeric calculations since the computation work for straight field line coordinates construction is avoided. In the formula, one still needs the flux label $\psi$ for the magnetic field. The reconstruction codes usually provide such information \[18, 19\].

This paper is organized as follows. In section §II, we briefly introduce how we find the format for $\vec{S}$. In section §III, we prove the coordinate-choice independent flux-force relation using $\vec{S}$ where the derivation avoids using $\theta$ or $\zeta$ at all. In section §IV, the coordinate-choice independent expression for the drift orbit flux is derived. In section §V, the results are compared with existing theory and some new application using the generalized results is discussed. Conclusion is made in section §VI. In appendix, we discuss the usage of Fourier analysis on the formula to further speed up the computation.

II. MOTIVATION OF THE GENERALIZATION

In this section, we take a brief review of the big idea behind flux-force relation and explain the motivation of finding $\vec{S}$.

The bulk plasma satisfies the momentum equation

$$\rho \frac{d\vec{U}}{dt} = \vec{j} \times \vec{B} - \vec{\nabla} \cdot \vec{P}$$

One can dot this equation with any vector field $\vec{A}$ and obtain

$$\rho \vec{A} \cdot \frac{d\vec{U}}{dt} = \vec{A} \cdot (\vec{j} \times \vec{B}) - \vec{A} \cdot (\vec{\nabla} \cdot \vec{P})$$  \hspace{1cm} (1)

While $\vec{A}$ is arbitrary so far, only certain choices of $\vec{A}$ would make the terms on the R.H.S. of equation (1) meaningful in the sense that they could be related to transport flux quantities. It has been shown that in straight field line coordinates $(\psi, \theta, \zeta)$ \[7\], if one chooses $\vec{A} = \vec{Q} = m\sqrt{g} \vec{\nabla} \psi \times \vec{\nabla} \theta - n\sqrt{g} \vec{\nabla} \psi \times \vec{\nabla} \zeta$. Then $\vec{Q} \cdot (\vec{j} \times \vec{B}) \propto \vec{j} \cdot \vec{\nabla} \psi$ and $\langle \vec{A} \cdot (\vec{\nabla} \cdot \vec{P}) \rangle \propto \langle U_d \cdot \vec{\nabla} \psi \rangle$ where $U_d$ is the drift orbit velocity and $\langle X \rangle$ denotes the surface average of $X$.

$$\langle X \rangle = \iint X \sqrt{g} d\theta d\zeta$$

$$= \frac{d}{d\psi} \iiint_V X \sqrt{g} d^3x$$  \hspace{1cm} (2)

Here, it is not normalized by $V'(\psi)$, because each term in our final results will be a surface average of some quantity and keeping $V'(\psi)$ all the way along the derivation only makes it lengthy. In all, one has \[7\]:

$$\rho \frac{d}{dt} \langle \vec{Q} \cdot \vec{U} \rangle = \langle m - nq \rangle \langle \vec{j} \cdot \vec{\nabla} \psi \rangle - \langle \vec{Q} \cdot (\vec{\nabla} \cdot \vec{P}) \rangle$$

$$\langle \vec{Q} \cdot (\vec{\nabla} \cdot \vec{P}) \rangle = \langle m - nq \rangle e \langle \vec{U}_d \cdot \vec{\nabla} \psi \rangle_{na}$$

$$= \langle m - nq \rangle \langle eU_d \cdot \vec{\nabla} \psi \rangle$$

While all the other quantities are insensitive to how the surface is parametrized by $(\theta, \zeta)$, the vector field $\vec{Q} = me\vec{z} + na\vec{e}_\theta$ restricts the application to straight field line coordinates and the vector field $\vec{Q}$ also has to be straight under such coordinates.

The authors in this paper have relaxed the constrain for $\vec{A}$ in equation (1). As soon as the flux surfaces are defined with flux label $\psi$, one can define the perpendicular vector field:

$$\vec{S}_0 = \frac{\vec{\nabla} \psi \times \vec{B}}{B^2}$$

The authors then show in section §III that for any vector field $\vec{A} = \vec{S}$ which can be casted in the form:

$$\vec{S} = h(x)\vec{B} + a(\psi)\vec{S}_0$$  \hspace{1cm} (3)
where \( h(\vec{x}) \) is an arbitrary spatial function and \( a(\psi) \) is a flux surface function, the terms on the R.H.S. of equation (1) can be related to transport quantities. As is noted in [17], any vector field tangential to the magnetic surface \( \psi(r) = \text{const} \) can be represented by:

\[
Q^* = C_0(\vec{r}) \left[ \frac{\vec{B} \times \vec{\nabla} \psi}{B^2} + \lambda(\vec{r}) \frac{\vec{B}}{B^2} \right]
\]

where \( C_0(\vec{r}) \) and \( \lambda(\vec{r}) \) are free to choose. By specifying \( C_0(\vec{r}) = 1/2\pi \) and letting \( \lambda(\vec{r}) = F(\psi) \), the poloidal current flux, one obtains the quasi-symmetry vector \( \vec{Q}_a \). The expression for \( \bar{S} \) is obtained by only specifying \( C_0(\vec{r}) = C_0(\psi) \) while leaving the function \( \lambda \) undetermined. Thus, \( \bar{Q}_a \) is included in the set of \( \bar{S} \). In fact, equation (3) is equivalent as the condition \( \vec{B} \times \vec{S} = a(\psi) \vec{\nabla} \psi \). For \( \bar{Q} = m\vec{c}_s + n\vec{e}_\phi \) in straight field line cases where \( \vec{B} = q\vec{\nabla} \psi \times \vec{\nabla} \theta + \vec{\nabla} \zeta \times \vec{\nabla} \psi \), such condition is satisfied with \( a(\psi) = m - nq \). Thus, the cases of \( \bar{Q} \) described in [7, 14] are included in the set of \( \bar{S} \).

III. PROOF OF THE GENERAL FLUX-FORCE RELATION

In this section, the authors prove the coordinate-choice independent flux-force relation with the vector field \( \vec{A} \) taking the form in equation (3). We first show a few important identities related to \( \vec{S} \) and the guiding center drift velocity. Then we proceed to prove the relation.

A. An identity for vector field \( \vec{S} \).

Since \( \vec{\nabla} \times (a(\psi) \vec{\nabla} \psi) = 0 \), one has

\[
0 = \vec{\nabla} \times (\vec{B} \times \vec{S}) = (\vec{\nabla} \cdot \vec{S}) \vec{B} + \vec{S} \cdot (\vec{\nabla} \vec{B}) - \vec{B} \cdot (\vec{\nabla} \vec{S})
\]

Dotting it with \( \vec{B} \) one has

\[
(\vec{\nabla} \cdot \vec{S}) B^2 - \vec{B} \vec{B} : (\vec{\nabla} \vec{S}) = -\vec{S} \cdot \vec{\nabla} \left( \frac{B^2}{2} \right) = -B \vec{S} \cdot \vec{\nabla} B
\]

Since \( \vec{T} \cdot \vec{\nabla} \vec{S} = \vec{\nabla} \cdot \vec{S} \), above is equivalent as saying:

\[
(\vec{T} - \vec{bb}) : (\vec{\nabla} \vec{S}) = -\frac{\vec{S} \cdot \vec{\nabla} B}{B} \tag{4}
\]

where \( \vec{b} = \vec{B}/B \). This identity will be used many times in the following derivation.

B. Guiding center drift

The drift orbit flux \( \langle \vec{U}_d \cdot \vec{\nabla} \psi \rangle \) is the transport quantity we want to relate to. The guiding center drift velocity of a single particle is [20]:

\[
\vec{v}_D = \frac{E \times \vec{B}}{B^2} + \frac{m}{qB^2} \vec{B} \times \left( \frac{\mu}{m} \vec{\nabla} B + v_{\parallel}^2 \vec{b} \cdot \vec{\nabla} \vec{b} + v_{\parallel} \frac{\partial \vec{b}}{\partial t} \right)
\]

where \( q \) is the charge of the particle. In the following derivations in this paper, we will rarely use the safety factor, so \( q \) will mostly be used to denote the particle’s charge unless specified otherwise. Ignoring the polarization drift \( \frac{\partial \vec{b}}{\partial t} \) part with semi-steady assumption and noticing that \( \vec{E} = -\vec{\nabla} \Phi(\psi) \propto \vec{\nabla} \psi \), we have

\[
\vec{\nabla} \psi \cdot \vec{v}_D = \frac{m}{q} \frac{\vec{\nabla} \psi \cdot \vec{B}}{B^2} : \left( \frac{\mu}{m} \vec{\nabla} B + v_{\parallel}^2 \vec{b} \cdot \vec{\nabla} \vec{b} \right)
\]

\[
= \frac{m}{q} \left( \frac{v_{\parallel}^2}{2B} \vec{S}_0 \cdot \vec{\nabla} B + v_{\parallel}^2 \vec{b} \cdot (\vec{\nabla} \vec{b}) \cdot \vec{S}_0 \right)
\]
Since $\vec{S}_0 \cdot \vec{b} = 0$,
\[
\frac{a}{m} \vec{\nabla} \psi \cdot \vec{v}_D = \frac{v^2}{2B} \vec{S}_0 \cdot \vec{\nabla} B - v_\parallel^2 \vec{b} \cdot (\vec{\nabla} \vec{S}_0) \cdot \vec{b}
\]  
(5)

For any $\vec{S}$ given by equation (3) (including $\vec{S}_0$), using equation (4) one has
\[
\frac{v^2}{2B} \vec{S} \cdot \vec{\nabla} B - v_\parallel^2 \vec{b} \cdot (\vec{\nabla} \vec{S}) \cdot \vec{b} = -\frac{|v_\parallel|}{B} \vec{S} \cdot (\vec{\nabla} (B|v_\parallel(\mu, E, B)|)) - v_\parallel^2 \vec{b} \cdot \vec{S}
\]  
(6)

To use above expression, $v_\parallel$ must be considered as a function $v_\parallel(\mu, E, B)$ when taking the derivative.

C. R.H.S. terms of equation (1)

Now, we show the R.H.S. terms of equation (1) can be related to transport flux quantities. The first term is easy to deal with:
\[
\vec{S} \cdot (\vec{j} \times \vec{B}) = \vec{j} \cdot (\vec{B} \times \vec{S}) = a(\psi)\vec{j} \cdot \vec{\nabla} \psi
\]

For the second term,
\[
\vec{S} \cdot (\vec{\nabla} \cdot \vec{P}) = \vec{\nabla} \cdot (\vec{S} \cdot \vec{P}) - (\vec{\nabla} \vec{S}) \cdot \vec{P}
\]
we notice that
\[
\vec{P} = \sum_s \int \int \int m_s f_s \vec{v}_s \vec{v}_s d^3v
\]
\[
= \sum_s [(\vec{\nabla} - \vec{b})P_{s,\perp} + \vec{b} : (\vec{\nabla} \vec{S}_0)]
\]
where the summation is over different species. The two terms for $\vec{S}$ in equation (3) are considered individually. Using equation (4) and equation (5) we have:
\[
(\vec{\nabla} \vec{S}_0) : \vec{P}_s = -\frac{\vec{S}_0 \cdot \vec{\nabla} B}{B} P_{s,\perp} + \vec{b} \cdot (\vec{\nabla} \vec{S}_0) P_{s,\parallel}
\]
\[
= -\int \int \int m_s \left[ \frac{1}{2} v_{s,\perp} \vec{S}_0 \cdot \vec{\nabla} B - v_\parallel^2 \vec{b} : (\vec{\nabla} \vec{S}_0) \right] f_s d^3v
\]
\[
= -q_s \int \int v_d \cdot \vec{\nabla} \psi f_s d^3v
\]
\[
= -q_s \vec{U}_{s,d} \cdot \vec{\nabla} \psi
\]
where $\vec{U}_{s,d}$ is the macro drift orbit velocity for species $s$. Similarly, using equation (4) and equation (6) we have:
\[
(\vec{\nabla} \vec{h} \vec{B}) : \vec{P}_s = \int \int \int \left[ -\frac{v_{s,\perp}^2 \vec{h} \vec{B} \cdot \vec{\nabla} B}{2B} + v_\parallel^2 \vec{b} \cdot (\vec{\nabla} \vec{h} \vec{B}) \cdot \vec{b} \right] m_s f_s d^3v
\]
\[
= \int \int \int \frac{|v_{s,\perp}|}{B} \vec{\nabla} \cdot (h \vec{B} B|v_{s,\perp}(\mu, E, B)|) m_s f_s d^3v
\]
\[
= \frac{2\pi}{m_s} \int \int \left\{ \vec{\nabla} \cdot [h \vec{B} B|v_{s,\perp}(\mu, E, B)| f_s] - |v_{s,\perp}| h \vec{B} B \cdot \vec{\nabla} f_s \right\} d\mu dE
\]
\[
= \vec{\nabla} \cdot (P_{s,\parallel} h \vec{B}) - \frac{2\pi}{m_s} \int |v_{s,\parallel}| h \vec{B} B \cdot \vec{\nabla} f_s(\mu, E) d\mu dE
\]
In all, since \( \vec{S} = h\vec{B} + a\vec{S}_0 \), we have:

\[
\rho \vec{S} \cdot \frac{d\vec{U}}{dt} = a\vec{j} \cdot \nabla \psi - \nabla \cdot (\vec{S} \cdot \vec{B}) - a \sum_s q_s \vec{U}_{s,d} \cdot \nabla \psi + \nabla \cdot (P_\parallel h\vec{B}) - \sum_s \frac{2\pi}{m_s} \int |v_{s,\parallel}| h\vec{B} \cdot \nabla f_s(\mu, E) d\mu dE
\]

Since \( \langle \nabla \cdot \vec{X} \rangle = \frac{d}{d\tau} \langle \vec{X} \cdot \nabla \psi \rangle \) and \( \vec{b} \cdot \nabla \psi = 0 \), \( \vec{S} \cdot \nabla \psi = 0 \), those total divergence terms should vanish after surface average. If we also take into account that under low collisionality regime, \( \vec{b} \cdot \nabla f_s(\mu, E) = 0 \) [14, 21], then we obtain:

\[
\langle \rho \vec{S} \cdot \frac{d\vec{U}}{dt} \rangle = a\langle \vec{j} \cdot \nabla \psi \rangle - a \sum_s \langle q_s \vec{U}_{s,D} \cdot \nabla \psi \rangle
\]  

(7)

Such flux-force relation, where \( \vec{S} \) is expressed by equation (43), is insensitive to how the magnetic surface is parametrized by \((\theta, \zeta)\).

IV. EVALUATION OF THE DRIFT ORBIT FLUX

The formula for evaluating the drift orbit flux \( \langle q_s \vec{U}_{s,D} \cdot \nabla \psi \rangle \) has been derived under Hamada coordinates [8] and this quantity should be a surface invariant that does not depend on the choice of \((\theta, \zeta)\) [8]. So technically, one already has a method to calculate \( \langle \rho \vec{S} \cdot \frac{d\vec{U}}{dt} \rangle \). However, the derivation in the previous section encourages the authors to find a way of representing the quantity \( \langle q_s \vec{U}_{s,D} \cdot \nabla \psi \rangle \) without Hamada. A coordinate-choice independent formula for this transport flux under 1/\(\nu\) regime will be derived in this section, which can be evaluated under much more general coordinates. The drift orbit flux under other regimes [16, 22, 23] are not covered in this paper.

A. Bounce average

Bounce integral \( \oint dl \) is one of the key elements of the derivations in this section. Operators like \( \partial / \partial \mu \) or \( \int d\mu \) will be moved in/out the bounce integral and the surface average will also be decomposed into \( \oint \) and the integral over the field line label. On a perturbed magnetic surface where the trapping region possibly contains holes in it, these tricks could be confusing and usage of the formula could raise errors in numerical computation if the definitions are not clear. The authors try to clarify such issues in this subsection. The derivation will use the Clebsch coordinate \((l, \beta, \psi)\), but the \( \oint dl \) integral we adapt in the end is an integral along field lines that does not depend on this coordinate as long as the \( \vec{B} \) is properly defined.

The concept of trapped/passing particles are introduced when exchanging the order of integral over velocity and space. The velocity space \( dv \) can also be represented by \( d\mu Ed\gamma \) [20] and the \( \mu \) component only spans over \([0, E/B(\vec{x})]\). Consider a function \( h(\theta, \zeta, E, \mu) \) that depends on both space and velocity. One can integrate this function over the velocity space first and then further integrate it over the whole surface.

\[
I = \sum_\sigma \int_0^{2\pi} \int_0^{2\pi} \int_0^{\infty} \int_0^{E/B(\vec{x})} h\mu Ed\theta d\zeta d\mu dE
\]

where \( \sigma = \pm 1 \) denotes the direction of \( v_\parallel \) for the given \((\mu, E)\) [20], which will be omitted for most of the derivation. If one integrates it over \( \theta - \zeta \) domain first, then the maximum \( \mu \) it can reach is \( E/B_{\min} \) and for a given \((E, \mu)\) the integration area for \( \theta - \zeta \) may not cover the whole surface:

\[
I = \int_0^{\theta_2(\mu, E, \zeta)} \int_0^{E/B_{\min}} \int_0^{2\pi} \int_0^{\theta_2(\mu, E, \zeta)} h\theta d\zeta d\mu dE
\]

where \( \theta_1 \) and \( \theta_2 \) will be the turning point where \( B(\theta, \zeta) = E/\mu \). If \( E/\mu > B_{\max} \) then \( \theta_1 = 0 \) and \( \theta_2 = 2\pi \) and the area is still the whole \( \theta - \zeta \) domain. Another way of saying this is, for a given \((\mu, E)\) it determines a trapping domain \( \Omega(E, \mu) \) on the surface where \( \mu B(\vec{x}) < E \) is satisfied within this domain. The domain \( \Omega(E, \mu) \) is the area on which we should integrate \( d\theta d\zeta \).
Bounce integral is along field lines. If we define an indicator function $1_{\Omega}$, the value of which is one within the domain $\Omega$ and zero outside it, then the integration over $\Omega$ could be expressed as:

$$\int_{\Omega} 1_{\Omega} h \frac{dl}{B} d\beta = \int_{\Omega} \frac{dl}{B} \sum_{j} \chi_{\beta}(j) h d\beta$$

$$= \int_{\Omega} \frac{dl}{B} \sum_{j} \chi_{\beta}(j) h d\beta = \int_{\Omega} 1_{\Omega} h \frac{dl}{B} d\beta$$

(8)

where $g$ is the Jacobian. If the integration $\int dl$ is intersected by $1_{\Omega}$, then for each continuous pieces along the line, it becomes a bounce integration $\hat{\phi} dl$ on the field line once $\sum_{j}$ is considered. Sometimes the $dl$ integration has to go several rounds over the torus before it meets the edge of $\Omega$. Yet the domain of $\int dl d\beta$ in equation (8) only covers the surface one time and we may not get a complete $\hat{\phi} dl$. Also, while intuition suggests $\hat{\phi} dl$ for a passing particle should sample the whole surface, the relation is still not clear.

To clarify this issue and obtain a more uniform expression, we need to increase the domain of $l$ to cover the surface multiple times. If $(l_0, \beta)$ is used to parametrize the surface, then each point on the surface can actually be represented by multiple $(l_0, \beta)$ points. Suppose $A_1$ is one simple connected domain for $(l_0, \beta)$ such that each point on the surface is exactly represented once in $A_1$ and the lower boundary for $l$ in $A_1$ is uniformly 0. Let the span for $\beta$ in $A_1$ be $(0, \beta_0)$. The upper boundary for $l$, would then be a function of $\beta$: $L_1(\beta)$, which indicates the length of each $B$ field line in $A_1$ before it meets the boundary. If we keep the span of $\beta$ in $A_1$ fixed, but increase the upper boundary for $l$, then we can construct $A_2$, which has every point on the surface being mapped exactly to two points in $A_2$. We thus have $L_2(\beta)$, which allows the field line to go around the surface exactly twice. Similarly, we can define $A_N$ and $L_N(\beta) \rightarrow \infty$ as $N \rightarrow \infty$. Then because of the periodicity of the surface, the surface integral has the following property

$$\int_{\Omega} h \sqrt{g} dl d\beta = \int_{A_1} h \sqrt{g} dl d\beta = \frac{1}{N} \int_{A_N} h \sqrt{g} dl d\beta$$

Since the domain for $\beta$ is $(0, \beta_0)$, the surface integration becomes:

$$\int_{\Omega} h \sqrt{g} dl d\beta = \int_{0}^{\beta_0} \frac{1}{N} \int_{0}^{L_N(\beta)} h \sqrt{g} dl d\beta$$

If one thinks it in this way for the integration in equation (8). Then for trapped particles, $\frac{1}{N} \int_{0}^{L_N(\beta)} 1_{\Omega} h dl$ is effectively cut into several pieces divided by the edge of $\Omega$ and each piece forms a $\frac{1}{N} h dl$ with the $\sum_{\sigma}$ taken into consideration. For some segments containing the end points $l = 0$ or $l = L_N(\beta)$, it may not form a complete bounce circle, but we can always add the missing parts. As $N \rightarrow \infty$, the difference, scaled by $1/N$, goes to zero and the final expression converges to the same value whether completing the circle or not on the boundary. For each bouncing circle on the surface, since the surface is repeated $N$ times in $A_N$, it will occur $N$ times in the integration once sums over $(0, \beta_0)$. Thus, the total effective value for each bouncing integration will be still be $N \times \frac{1}{N} \phi dl = \phi dl$.

For passing particles, $\frac{1}{N} \int_{0}^{L_N(\beta)} 1_{\Omega} h dl$ is still denoted this integration by $\phi dl$. This is especially important if $h$ has the form of $h = \sigma \partial H / \partial l$ where $H$ is a periodic function defined on the surface. In such cases, suppose the maximum value of $|H|$ on the surface is $H_M$, then $\frac{1}{N} \int_{0}^{L_N(\beta)} h dl < \frac{2}{N} H_M$ which goes to 0 as $N \rightarrow \infty$. Considering the trivial trapped particle case together, we have that $\phi \sigma [\partial H / \partial l] dl = 0$ for both passing and trapped particles.

In all, for an integration over the surface $\int \frac{dl}{B} d\beta$, it can be reformed as an integration over $A_N$ and take the limit of $N \rightarrow \infty$.

$$\int_{\Omega} 1_{\Omega} h \frac{dl}{B} d\beta = \frac{1}{N} \int_{A_N} 1_{\Omega} h \frac{dl}{B} d\beta$$

$$= \int_{0}^{\beta_0} \frac{1}{N} \int_{0}^{L_N(\beta)} 1_{\Omega} h \frac{dl}{B} d\beta$$

$$= \lim_{N \rightarrow \infty} \int_{0}^{\beta_0} \sum_{k} \left( \frac{1}{N} \int_{0}^{L_N(\beta)} h \frac{dl}{B} d\beta \right)$$

(9)

where $\hat{\phi}_k dl$ is the $kth$ continuous piece on $[0, L_N(\beta)]$ that has been cut by the boundary of $\Omega$. As $N$ increases, the total number of $k$ for each $\beta$ also increases. The total weight for each bouncing circle on the surface is still one. We can omit the summation and $1/N$ notation together if no ambiguity would arise. And we have $\int 1_{\Omega} h \frac{dl}{B} d\beta = \int (\phi \frac{dl}{B}) d\beta.$
It is not wise to numerically evaluate the surface average as a limit shown in equation (9). This relation, as will be seen later, serves as a transient step for the derivation. In the final expression shown in equation (17), the surface average can be evaluated using equation (2). In the circumstances shown in section VA, \( A_1 \) is well contained in \( A_1 \) and one can just use equation (9) for the surface average and take \( N = 1 \).

**B. Drift orbit flux**

The expression for the drift orbit flux is obtained by solving the drift kinetic equation. The procedure of obtaining the quadratic \( \partial f/\partial \mu \) expression is very similar to the work in [8] which uses Hamada. We present it here in a modified way to show that it is actually coordinate-choice independent. Then we proceed to solve for \( \partial f/\partial \mu \) and express that with vector field \( \vec{S} \). The final coordinate-choice independent expression provides a method for numeric calculation that does not need straight field line coordinates.

The 0th order drift kinetic equation gives a Maxwellian solution \( f_M \) for \( f_0 \). The first order equation is [8]:

\[
v_{||} |\vec{b}| \cdot \nabla f_{1,0} = 0
\]

Further expansion of this equation with \( 1/\nu \) ordering suggests:

\[
v_{||} |\vec{b}| \cdot \nabla f_{1,1} + |\vec{v}| \cdot \nabla |\vec{\psi}| \frac{df_0}{d\psi} = \frac{v_{||}}{B} \frac{\nu M |\vec{v}|}{v_{||}} \frac{\partial f_{1,0}}{\partial \mu}
\]

The first equation implies \( f_{1,0} = f_{1,0}(E, \mu, |\vec{\psi}|, \beta) \) does not vary along the field line. Although two different lines are essentially one field line on irrational surface, \( f_{1,0} \) can have different value on two field lines for trapped particles with the same \( (E, \mu) \). This is because those two field lines are disconnected by the boundaries of trapping region \( \Omega(E, \mu) \). \( v_{||} \) is not properly defined outside the trapping region and equation (10) is not valid outside the trapping region. So, the value of \( f_{1,0} \) on this two pieces of lines does not talk to each other.

The second equation is on both the spatial and velocity domain \( (\beta, l, E, \mu) \). For a given \( (E, \mu) \), one knows whether it is a passing particle or trapped particle. If it is a trapped particle, then the spatial point \( (\beta, l) \) corresponds to a specific \( f_0 dl \) whose integration domain includes this point. If it is a passing particle, then it corresponds to the \( f_0 dl \) with \( l \) goes from 0 to \( L_N(\beta) \), which essentially samples the whole surface as \( N \to \infty \). In either case, one can multiply equation (11) with \( 1/|v_{||}| \) and perform the \( f_0 dl \) integration. The first term on the L.H.S takes the form \( \int f_0 \sigma|\partial H/\partial \mu|dl \) and will vanish for both passing and trapped cases, as is discussed in section VA. Thus we have:

\[
\int \frac{B}{|v_{||}|} \vec{v}_d \cdot \nabla |\vec{\psi}| \frac{df_0}{d\psi} = \int \frac{\partial}{\partial \mu} \left( \frac{\nu M |\vec{v}|}{|v_{||}|} \frac{\partial f_{1,0}}{\partial \mu} \right) dl
\]

Moving the operator \( \partial/\partial \mu \) outside \( f_0 dl \) in the last step of deriving needs more explanation because the domain for \( dl \) depends on \( \mu \). Suppose the equation is used for a given \( \mu_1 \), then moving \( f_0 dl \) inside requires a proper definition for the integral domain for at least \( \mu \in [\mu_1, \mu_1 + \epsilon) \), where \( \epsilon \) is just a small quantity. For example, if the integral of \( f_0 dl \) is on \([l_1, l_2]\) for \( \mu_1 \), while for \( \mu_1 + \epsilon \), \( |B| \) has reached some critical point within the region and the domain \([l_1, l_2]\) has to be broken into several pieces, then the \( f_0 dl \) for \( \mu_1 + \epsilon \) has to be actually a summation \( \sum_k f_0 dl \) so that all the pieces in \([l_1, l_2]\) are added to make the last step in equation (12) valid.

The drift orbit flux from \( f_0 \) does not contribute when surface averaged. The first non-zero contribution comes from the \( f_{1,0} \) part. As has been noted in the previous section, the integration over the surface can also be represented as the limit of integration over \( A_N \) and divided by \( N \). This makes sure that when \( f_0 dl \) is performed, it will be a complete circle. By exchanging the order of integration, we have:
\[ q\Gamma = \left\langle d^3v q\vec{v}_d \cdot \nabla \psi f_{1,0} \right\rangle \]
\[ = \left\langle \frac{2\pi q}{M^2} \int \int B \left[ f_{1,0} \vec{v}_d \cdot \nabla \psi d\mu dE \right] \right\rangle \]
\[ = \frac{2\pi q}{M^2} \int \int_{\mathbb{R}^N} \frac{1}{N} \int \int B \left[ \vec{v}_d \cdot \nabla \psi f_{1,0} d\mu dE \right] \frac{dl}{B} d\beta \]
\[ = \frac{2\pi q}{M^2} \int \left( \int \int B \vec{v}_d \cdot \nabla \psi \frac{dl}{B} f_{1,0} d\beta \right) d\mu dE \]

Then substitute in equation (12) and exchange the order of integration again, we obtain the expression with respect to \( \frac{\partial f_{1,0}}{\partial \mu} \):

\[ q\Gamma = -\frac{2\pi q}{M^2} \int \int \left( \int \nu |v|| \frac{df_{1,0}}{d\psi} \right) \frac{\partial f_{1,0}}{\partial \mu} \frac{dl}{B} d\beta d\mu dE \]
\[ = -\frac{2\pi q}{M} \int \nu |v| \left( \frac{\partial f_{1,0}}{\partial \mu} \right)^2 \frac{df_{0}}{d\psi} \]
\[ \times \frac{dl}{B} d\beta d\mu dE \]  \hspace{1cm} (13)

where the surface integral has been treated the same as that in deriving equation (9) and we have used equation (10) to assume that \( f_{1,0} \) and \( \frac{\partial f_{1,0}}{\partial \mu} \) can be moved inside/outside the integral \( \frac{dl}{B} \).

C. Solution for \( \frac{\partial f_{1,0}}{\partial \mu} \)

Now, we have obtained the quadratic \( \frac{\partial f_{1,0}}{\partial \mu} \) expression for the drift orbit flux. It remains to solve for \( \frac{\partial f_{1,0}}{\partial \mu} \). This can be done by substituting equation (9) and equation (10) into equation (12):

\[ q \frac{\partial}{\partial \mu} \left( \int \nu |v|| \frac{df_{1,0}}{d\psi} \right) \frac{dl}{B} \frac{\partial f_{1,0}}{\partial \mu} \]
\[ = \left( \int \nabla \cdot (\vec{S}_0 B |v||\mu, E, B|) \right) \frac{dl}{B} \frac{df_{0}}{d\psi} \]  \hspace{1cm} (14)

Integration by \( \mu \),

\[ q \int \nu |v|| \frac{df_{1,0}}{d\psi} \frac{dl}{B} \frac{\partial f_{1,0}}{\partial \mu} \]
\[ = -\int_{\mu_{\max}}^{\mu} \left( \sum_k \vec{v} \cdot (\vec{N} \cdot \vec{S}_0 B |v||\mu, E, B|) \right) \frac{dl}{B} \frac{df_{0}}{d\psi} d\bar{\mu} \]  \hspace{1cm} (15)

The discussion about the last step in equation (12) makes sure that the \( \frac{dl}{B} \) we obtain on the L.H.S. is consistent with our previous definitions for \( \frac{dl}{B} \). There’s no constant term because L.H.S also equals 0 at \( \mu = \mu_{\max} \). The integration limit of \( \frac{dl}{B} \) on the R.H.S. is given by \( (E, \tilde{\mu}) \). As has been noted in previous section, when integrating equation (14), one has to connect all the locally defined equation. Suppose equation (15) is for \( \mu = \mu_1 \) and \([l_1, l_2] \) is the corresponding domain for \( \frac{dl}{B} \). Then for \( \tilde{\mu} > \mu_1 \) on the R.H.S. of equation (15), it has to include all the pieces of \( \frac{dl}{B} \) within \([l_1, l_2] \).

Moving the \( \frac{dl}{B} \) outside \( \int \frac{dl}{B} \) integration will affect the domain of \( \frac{dl}{B} \), which will become the largest for all \( \tilde{\mu} \in [\mu, \mu_{\max}] \). The integration domain for \( d\mu \) will also be affected. At each spatial point \( x \), the integration domain for \( d\mu \) will be \([\mu, \tilde{\mu}_{\max}] \) where \( \tilde{\mu}_{\max} = E/B(x) \).

\[ -\int_{\mu_{\max}}^{\mu} \left( \frac{\nabla \cdot (\vec{N} \cdot \vec{S}_0 B |v||\mu, E, B|) \right) \frac{dl}{B} \frac{df_{0}}{d\psi} d\bar{\mu} \]
\[ = \int \nabla \cdot (\vec{S}_0 B \int_{\mu}^{\tilde{\mu}_{\max}} |v||\mu, E, B|) \frac{dl}{B} \frac{df_{0}}{d\psi} \]
\[ = \frac{m}{3} \int \nabla \cdot (\vec{N} \cdot |v|) \frac{dl}{B} \frac{df_{0}}{d\psi} \]
This procedure has no problem of including the passing particle case, where \( \frac{\partial}{\partial \mu} \frac{\partial f_{1,0}}{\partial \mu} \), and thus we have:

\[
\frac{\partial f_{1,0}}{\partial \mu} = \frac{m \phi \cdot \nabla \cdot (S_0 |v_0|^3) \frac{df_0}{d\psi} - \frac{m|v_0|^3}{3q\mu} \nabla \cdot S_0 \frac{df_0}{d\psi}}{3q\mu \frac{df_0}{d\psi}}
\]

In the case of \( \vec{S} = a\vec{S}_0 + \vec{h}\vec{B} \), for trapped particles \( |v_0| = 0 \) at the turning point and

\[
\int \nabla \cdot (h\vec{B}|v|^3) \frac{dl}{B} = \int \vec{b} \cdot \nabla (h|v|^3) dl = 0
\]

Thus

\[
\frac{\partial f_{1,0}}{\partial \mu} = \frac{m \phi \cdot \nabla \cdot (S|v|^3) \frac{df_0}{d\psi}}{3aq\mu \frac{df_0}{d\psi}} \tag{16}
\]

This is also valid for passing particles because \( \phi \vec{b} \cdot \nabla (h|v|^3) dl \) is bounded by \( Max(2h|v|^3)/N \), which goes to zero as \( N, L_N(\beta) \to \infty \), while \( \phi |v|^3 \frac{df_0}{d\psi} \) stays finite. Thus the \( h\vec{B} \) part still does not contribute in equation (16). Finally, we obtain the formula for the surface integrated drift orbit flux.

\[
\langle qU_d \cdot \nabla \psi \rangle = -\frac{2\pi m}{9a^2q} \int \frac{1}{\nu} |v_0| \left((\frac{\phi \cdot \nabla \cdot (\vec{S}|v|^3) \frac{df_0}{d\psi}}{\phi |v|^3 \frac{df_0}{d\psi}})^2 \frac{df_0}{d\psi} \right) d\mu dE \tag{17}
\]

\[
= -\frac{2\pi m}{aqm} \int \frac{1}{\nu} |v_0| \left((\frac{\phi(\vec{S} \cdot \nabla B - \frac{m|v|^3}{3q\mu} \nabla \cdot \vec{S}) \frac{df_0}{d\psi}}{\phi |v|^3 \frac{df_0}{d\psi}})^2 \frac{df_0}{d\psi} \right) d\mu dE \tag{18}
\]

This expression is coordinate-choice independent and can be evaluated even in non-straight field line coordinates. For trapped particles, in order to perform \( \phi \frac{df_0}{d\psi} \), one needs to calculate the boundaries of trapping region, which are the contours given by \( |B| = E/\mu \). For passing particles, \( \phi \frac{df_0}{d\psi} \) effectively becomes the surface integration. Since \( \langle \nabla \cdot (\vec{S}|v|^3) \rangle = 0 \), it means that the passing particles do not contribute.

This expression, combined with the flux-force relation in equation (4), gives a way to calculate the effect of a 3D magnetic field perturbation on plasma rotation. It should be noted that the \( \vec{S} \) in equation (17) does not need to be the same \( \vec{S} \) as that in equation (4). One can use \( \vec{S}_1 \) to calculate the drift orbit flux and use \( \vec{S}_2 \) in equation (7) to calculate the change rate of \( \langle \vec{U} \cdot \vec{S}_2 \rangle \), as long as they all follow the general expression for \( \vec{S} \) in equation (3).

V. COMPARISON WITH EXISTING THEORIES AND APPLICATIONS

A. Flux-force relation

One can choose \( \vec{S} \) to be many vector fields of interest, as long as it can be cast into the form expressed by equation (3), which is equivalent as requiring \( \vec{B} \times \vec{S} = a(\psi) \nabla \psi \). The vector field \( \vec{e}_\theta, \vec{e}_\zeta \) in (11, 12, 13) all satisfy such requirements.

For example, in straight field line cases: \( \vec{B} = \frac{1}{\sqrt{g}}(\vec{e}_\zeta + q \vec{e}_\theta) \), where \( q \) is the safety factor. If one takes \( \vec{S} = m\sqrt{g} \nabla \psi \times \nabla \theta - n\sqrt{g} \nabla \psi \times \nabla \zeta \), then \( \vec{B} \times \vec{S} = (m - nq) \nabla \psi \). This is sufficient to say that \( \vec{S} = h\vec{B} + (m - nq)\vec{S}_0 \), where the explicit expression for \( h(\vec{x}) \) is yet to be calculated. Using such \( \vec{S} \) in equation (7), one recovers the results obtained in (11, 12).

The relation is equally useful in non-straight field line coordinates \( (\theta, \zeta, \psi) \), where \( \psi \) is still the flux label and

\[
\vec{B} = \frac{1}{\sqrt{g}} \left[B_4(\theta, \zeta, \psi)\vec{e}_\zeta + B_p(\theta, \zeta, \psi)\vec{e}_\theta \right]
\]
where $B_t$ and $B_p$ are no longer required to be constant on the magnetic surface. One can construct a subset of $\vec{S}$ with:

$$\vec{S} = \frac{a_1(\psi)}{B_p} \vec{e}_\zeta + \frac{a_2(\psi)}{B_t} \vec{e}_\theta$$

such that $\vec{B} \times \vec{S} = (a_1 - a_2)\vec{\nabla}\psi$ is always satisfied. Especially, if one further specifies $a_2(\psi) = 0$, then $\vec{S} = a_1\vec{e}_\zeta/B_p$.

Since there is no requirement on $(\theta, \zeta)$, one can choose them to be the natural toroidal and poloidal angles. $\vec{S}$ obtained with such choice is close to the lab’s toroidal direction, but still not exactly the same. This explicitly shows that in non-axisymmetric cases, $\langle \rho\vec{U} \cdot \vec{S} \rangle$ is not really the toroidal angular momentum, but with some modulation by $B_p$ and the wobbliness on magnetic surface.

### B. drift orbit flux

The drift orbit flux formula has previously been derived in Hamada $(\theta, \zeta, V)$. If one chooses $\vec{S} = \vec{e}_\zeta$, then $\vec{\nabla} \cdot \vec{S} = 0$ and the corresponding term in equation (13) is annihilated. One gets the result consistent with that obtained in [8].

The surface average is performed on the flux surface after perturbation. In deriving equation (17), we didn’t make the assumption about the normalized perturbation amplitude being much smaller than $\epsilon$, the inverse aspect ratio. Without this assumption, the bounce integral on different field lines could not be factored out since the turning point is modified by the 3D perturbation once $\delta B/B$ is comparable to $\epsilon$. If we further adapt this assumption as that in [8, 16], then equation (17) could be further simplified by using Fourier analysis and performing the integration with respect to the field line label $\beta$ separately from the bounce integral. This is shown in section VII A. The authors want to keep the formula general to include stellerator-like perturbed cases as well.

The formula given by equation (17) or equation (18) can be used under other straight field line coordinates and even non-straight field line coordinates. One still needs to determine the trapping region $\Omega(E, \mu)$ for each given $(E, \mu)$, the boundaries of which are defined by the contours: $|B| = E/\mu$. This is a necessary work for calculating $\oint d\ell$ related integrals.

### VI. SUMMARY

The authors show that the flux-force relation expressed by equation (7) is valid as long as the vector field $\vec{S}$ can be cast into the coordinate-choice independent form expressed by equation (3). This broadens the application of flux-force relation to a bigger set of $\vec{S}$ and drops the requirement of straight field line coordinates.

The surface averaged drift orbit flux $\langle q_s U_{s,D} \cdot \vec{\nabla}\psi \rangle$, which is directly related to the flux-force formula, is also expressed in a coordinate-choice independent way with respect to the variation of $|B|$ under $1/\nu$ regime. The evaluation of equation (17) or equation (18) drops the requirement of Hamada coordinate.

The formula being coordinate-choice independent are intuitive because the flux-force relation and drift orbit flux are physical quantities that are invariant to how the magnetic surface is parametrized. These formula could also be preferred in numerical computation because the construction of straight field line coordinates for a 3D perturbed plasma is avoided. One can choose the natural poloidal and toroidal angle $(\theta, \phi)$ as parameters and the flux label $\psi$ can be obtained from the magnetic field reconstruction code [18, 19].

### VII. APPENDIX

#### A. Fast calculation by Fourier transformation

Evaluation for expressions taking the form

$$I = \langle \int \int [\oint h(\vec{x}, E, \mu) \frac{d\ell}{B^2} dEd\mu] \rangle$$

(19)

can be accelerated with Fourier analysis if certain assumptions are made. The basic idea is that for a given $(E, \mu)$, the value of $\oint h d\ell$ is the same for each points on a bounce circle and may only different by a ‘phase’ between different
field lines. Thus, if we can factor this $\hat{f} \frac{h}{B}$ out and do the integration with respect to other parameters separately, the calculation can be simplified a lot.

To make this improvement, certain assumptions have to be made. We only consider the trapped particles in this case. The assumptions are:

1. The magnetic surface comes from a perturbation to an axisymmetric $B$ field, and the magnitude of the perturbation is much smaller to the $|B|$ variation on the original field.

2. Following the 1st assumption, we further assume that the turning point for a trapped particle is not changed by the perturbation.

With such assumptions, if we use the natural poloidal angle $\theta$ as one parameter, and the field line label $\beta$ as another parameter, then $\hat{f} \frac{h}{B} = \int_{\theta_{12}}^{\theta_{12}} h \sqrt{\mathbf{g}} d\theta$ has the same $\theta_{11}, \theta_{12}$ on different field lines with a given $(E, \mu)$.

One can set the $\theta = 0$ line at the most inner board position, where $|B|$ reaches its maximum. As $\theta$ goes to $2\pi$ after a poloidal circle, it reaches to the same line. This makes sure that for any trapped particle trajectory, $\theta$ will always stay within $[0, 2\pi)$ and we don’t have to worry about jumps or consistency issues of $\beta$, as $\theta$ goes beyond $2\pi$. What remains is to do Fourier transform on $h \sqrt{\mathbf{g}}$ w.r.t $\beta$. We only need the definition of $h \sqrt{\mathbf{g}}$ on $(\theta, \beta) \in [0, 2\pi] \times [0, 2\pi)$ and then make a periodic expansion.

$$h \sqrt{\mathbf{g}} = \sum_n (a_n(\theta) \sin(n\beta) + b_n(\theta) \cos(n\beta))$$

After changing the order of integration on $d\mu dE$ and the integration on surface:

$$I = \left\langle \int \left[ \int h(\bar{x}, E, \mu) \frac{d\bar{x}}{\sqrt{B}} \right]^2 dE d\mu \right\rangle$$

$$= \int \int \int \left[ \int \left[ \int h(\bar{x}, E, \mu) \sqrt{\mathbf{g}} d\theta \right] \sqrt{\mathbf{g}} d\beta d\theta d\mu \right]$$

$$= \int \int \int \sum_n \int \left[ \tilde{a}_n \sin(n\beta) + \tilde{b}_n \cos(n\beta) \right]^2 \sqrt{\mathbf{g}} d\theta d\beta dE d\mu$$

where

$$\tilde{a}_n = \int_{\theta_{11}}^{\theta_{12}} a_n(\theta) d\theta$$

$$\tilde{b}_n = \int_{\theta_{11}}^{\theta_{12}} b_n(\theta) d\theta$$

$\tilde{a}_n, \tilde{b}_n$ are independent of $\theta$ and $\beta$. If we further assume that the small perturbation to the axisymmetric equilibrium will result in a $\sqrt{\mathbf{g}} = \sqrt{\mathbf{g}}(\theta)$ which does not vary on $\beta$, then we can integrate on $\beta$ first and get

$$I = \int \int \sum_n (\tilde{a}_n^2 + \tilde{b}_n^2) dE d\mu$$

This expression gets rid of the dimension on $\beta$ and is faster in computation than equation (11). As for equation (12), we have

$$I = \left\langle \int \int \frac{1}{\mu E} |v| \left( \frac{\hat{\nabla} \cdot (\hat{S}|v|^3)}{\hat{\nabla} |v|^3} \right) \left( \frac{df}{d\psi} \right) d\mu dE \right\rangle$$

$$= \int \int \frac{1}{\mu E} \int \left( \left( \frac{\hat{\nabla} \cdot (\hat{S}|v|^3)}{\hat{\nabla} |v|^3} \right) \sqrt{\mathbf{g}} d\theta \right) d\beta \left( \frac{df}{d\psi} \right) d\mu dE$$

We assume that $|v| \sqrt{\mathbf{g}}$ has no $\beta$ dependence and perform Fourier transform of $\hat{\nabla} \cdot (\hat{S}|v|^3) \sqrt{\mathbf{g}}$ w.r.t $\beta$ where the dependence of $\beta$ comes from $|B|$ variation. Then

$$I = \int \int \sum_n \left( \frac{\hat{a}_n^2 + \hat{b}_n^2}{\hat{\nabla} |v|^3} \right) \frac{df}{d\psi} d\mu dE$$

where $\hat{a}_n$ and $\hat{b}_n$ are the integration on $\theta$ of the corresponding Fourier coefficients.

If one wants to do the Fourier transform without the involvement of $v$ (so that FFT will only need to be done once), then one can use equation (13) and evaluate.
\[
I = \left\langle \iint \frac{\mu}{\nu} \left( \left| v_\parallel \right|^3 \hat{\mathbf{S}} \cdot \mathbf{\nabla} B - \frac{m |v_\parallel|^3}{3\mu} \mathbf{\nabla} \cdot \mathbf{S} \right) \frac{d\Omega_\parallel}{d\psi} \right\rangle \left| v_\parallel \right|^2 \left( \frac{df_0}{d\psi} \right) d\mu dE \right. \\
= \iint \frac{\mu}{\nu} \left\langle \left( \left| v_\parallel \right|^3 \hat{\mathbf{S}} \cdot \mathbf{\nabla} B - \frac{m |v_\parallel|^3}{3\mu} \mathbf{\nabla} \cdot \mathbf{S} \right) \frac{d\Omega_\parallel}{d\psi} \right\rangle \left| v_\parallel \right|^2 \left( \frac{df_0}{d\psi} \right) d\mu dE 
\]

Suppose the Fourier coefficients of $\sqrt{g} \hat{\mathbf{S}} \cdot \mathbf{\nabla} B$ and $\sqrt{g} \mathbf{\nabla} \cdot \mathbf{S}$ are:

\[
\sqrt{g} \hat{\mathbf{S}} \cdot \mathbf{\nabla} B = \sum_n \left( a_n(\theta) \sin(n\beta) + b_n(\theta) \cos(n\beta) \right) \\
\sqrt{g} \mathbf{\nabla} \cdot \mathbf{S} = \sum_n \left( c_n(\theta) \sin(n\beta) + d_n(\theta) \cos(n\beta) \right)
\]

Then we have:

\[
I = \iint \frac{\pi\mu}{\nu} \left( \sum_n \left( \tilde{k}_\sin^2 + \tilde{k}_\cos^2 \right) \right) \left( \frac{df_0}{d\psi} \right) d\mu dE 
\]

where the coefficients are defined as:

\[
\tilde{k}_\sin = \int_{\theta_1}^{\theta_2} (\left| v_\parallel \right| a_n(\theta) - \frac{m |v_\parallel|^3}{3\mu} c_n(\theta)) d\theta \\

\tilde{k}_\cos = \int_{\theta_1}^{\theta_2} (\left| v_\parallel \right| b_n(\theta) - \frac{m |v_\parallel|^3}{3\mu} d_n(\theta)) d\theta
\]

This gives a boost in computation for the case where the ratio of perturbation over the axisymmetric equilibrium is much smaller compared to the aspect ratio $\epsilon$.

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[17] M. Y. Issev, M. I. Mikhailov, and V. D. Pustovitov, Plasma physics Reports 20, 319 (1994).
[18] P. Zanca and S. Martini, Control fusion 1251 (1999).
[19] J. D. Hanson, S. P. Hirshman, S. F. Knowlton, L. L. Lao, E. a. Lazarus, and J. M. Shields, Nuclear Fusion 49, 075031 (2009), ISSN 0029-5515,
[20] R. D. Hazeltine and J. D. Meiss, Plasma Confinement (Dover, New York, 2003), dover ed., ISBN 0-486-43242-4.
[21] K. C. Shaing and J. Callen, Physics of Fluids 26, 3315 (1983), ISSN 00319171,
[22] K. C. Shaing, M. S. Chu, C. T. Hsu, S. a. Sabbagh, J. C. Seol, and Y. Sun, Plasma Physics and Controlled Fusion 54, 124033 (2012), ISSN 0741-3335,
[23] Y. Sun, Y. Liang, K. Shaing, H. Koslowski, C. Wiegmann, and T. Zhang, Nuclear Fusion 51, 053015 (2011), ISSN 0029-5515,