SOME HOMOGENEOUS COORDINATE RINGS THAT ARE KOSZUL ALGEBRAS

Rikard Bögvad

Abstract. Using reduction to positive characteristic and the method of Frobenius splitting of diagonals, due to Mehta and Ramanathan, it is shown that homogeneous coordinate rings for either proper and smooth toric varieties or Schubert varieties are Koszul algebras.

1. Introduction. All varieties will be assumed to be defined and proper over an algebraically closed field $k$. Assume that $\mathcal{L}$ is a line bundle on an algebraic variety $X$, and define the graded $k$-algebra

$$R(\mathcal{L}) := \oplus_{n \geq 0} \Gamma(X, \mathcal{L}^\otimes n).$$

Since $X$ is proper this algebra is finite dimensional in each degree as a $k$-vector space, and it is the integral closure of the homogeneous coordinate ring of the image of $X$ under the map defined by $\mathcal{L}$. The striking result of Mumford that $R(\mathcal{L}^\otimes n)$ has quadratic relations if $n$ is large enough has been generalized in several ways, for example resulting in an analysis of the degrees of the syzygies of $R(\mathcal{L})$ considered as a quotient of a minimal polynomial algebra (cf. [EL]). Another generalization is the theorem of J. Backelin [B] that for $n$ large enough $R(\mathcal{L}^\otimes n)$ is a Koszul algebra (See Def 2.1.). G.R.Kempf has noted in [K2] that for certain types of varieties (e.g. abelian varieties, Grassman varieties, and curves) $R(\mathcal{L})$ is Koszul except for very few $\mathcal{L}$, and he suggests the problem of finding more examples. Our first result is the following.

**Theorem 1.** Let $X$ be a nonsingular proper toric variety, and suppose that $\mathcal{L}$ is an ample line bundle on $X$. Then $R(\mathcal{L})$ is a Koszul algebra.

In the case of homogeneous spaces we obtain the slightly stronger result

**Theorem 2.** Let $X$ be a Schubert variety in a homogeneous space $G/P$ of a reductive algebraic group modulo a parabolic subgroup, and suppose that $\mathcal{L}$ is an effective line bundle on $G/P$. Then $R(\mathcal{L})$ is a Koszul algebra.

It is known (cf.[B,R]) that $R(\mathcal{L})$ is generated by $\Gamma \mathcal{L}$ in either case so $R(\mathcal{L})$ is the homogeneous coordinate ring of $X$ in the embedding given by $\mathcal{L}$. A Koszul
algebra always has quadratic relations, so this is a generalization of the results of Ramanathan [R] – homogeneous spaces – and [B] – toric varieties. In [K], it is shown that algebras with a straightening law whose discrete algebra is defined by quadratic monomials are Koszul algebras. These algebras include, as stated there, at least embeddings of Grassmannians by Plücker coordinates, but I ignore whether they include all homogeneous varieties of the theorem.

There is also a relative version of Theorem 2, simplifying proofs, and generalizing the statement that Schubert varieties are linearly defined in an embedding of the corresponding Borel variety given by an effective line bundle $L$ (cf.[R],[KR]). A ring homomorphism is said to be 1-linear if it is relatively Koszul (see Def.2.1).

Theorem 3. Let $X$ be as in Theorem 2. Suppose that $Y \subset X$ is a Schubert variety and that $L$ is an effective line bundle on $X$. Let $S := R(Y, L)$ be the restriction to $Y$ of $R = R(X, L)$. Then the surjective map $s : R \to S$ is 1-linear (cf. Def 2.1).

The above results follow by generalizing the methods used by Ramanathan to prove that homogeneous coordinate rings of Schubert varieties have quadratic relations to higher syzygies. This generalization is contained in the following theorem. Assume that $Y \subset X$ is a closed subvariety, and define the partial diagonal $D_{s,i+1}^i \subset Y \times X \times \ldots \times X = Y \times X(s)$ (s factors $X$), as the set

$$\{(x_0, x_1, \ldots, x_s), x_0 \in Y, x_k \in X, 1 \leq k \leq s, x_i = x_{i+1}\}.$$ 

Theorem 4. Assume that the characteristic of $k$ is positive and that $Y \subset X$ is a closed subvariety of the proper variety $X$, possibly the empty set (this corresponds to the Koszul property, see section 2). Assume that each $Y \times X(n), n = 1, 2, \ldots$ is Frobenius split and that all diagonals $\Delta_{i+1}^n$ for $i = 0, \ldots, n-1$, are simultaneously compatibly split as closed subvarieties of $X(n)$. Let $S := R(Y, L)$ be the restriction to $Y$ of $R := R(X, L)$. Then the map $s : R \to S$ is 1-linear.

The idea of the proof of Theorem 4 is to find conditions for the existence of a (partial) homotopy of the identity on a certain bar complex with the zero map, showing that the homology is concentrated in the right degrees. These conditions, infinite in number, are formulated as the acyclicity in the right dimension of a kind of model, as is done in, for example, the proof of the Eilenberg-Zilber theorem in elementary algebraic topology [M, VIII,7-8]. (See Theorem 5.) Since we are working with the bar complex they are then easily rephrased in terms of restriction maps of cohomology of line bundles to partial diagonals. Then the results of Mehta and Ramanathan on compatible splitting show that the resulting conditions are satisfied if the partial diagonals are compatibly split.

Since our conditions for having a Koszul algebra unfortunately are infinite in number, Backelin’s theorem does not follow from them (examples have been constructed by J.-E Roos [Ro] showing that the right behaviour for an arbitrary graded ring of a finite number of torsion groups does not suffice to prove that it is a Koszul algebra, even if the number of generators of the algebra is bounded). Thus one might hope that a closer analysis of a resolution of a homogeneous algebra would give much sharper results, valid generally. In a sequel a generalization of the above results to multi-cones as in [KR] will be given.

I have been told that results similar to those in the present note on homogeneous varieties have been obtained by M.S.Ravi.

I would like to thank J.Backelin, T.Ekedahl, R.Fröberg, C.Löfwall and J.-E Roos for helpful conversations on these topics.
2.1 The bar complex. Let $k$ be a field (of arbitrary characteristic). We will only consider finitely generated commutative graded $k$-algebras

$$R = \oplus_{i \geq 0} R_i$$

who are connected, i.e. $R_0 = k$. Homomorphisms between such graded rings will always be homogeneous of degree 0. Let $R^+ := \oplus_{i > 0} R_i$ be the irrelevant ideal. The normalized standard complex (often called the bar complex) of $R$ has underlying vector space

$$N_\ast = \bigoplus_{n \geq 0} R \otimes_k (R^+)^n.$$ 

Here $(R^+)^n$ denotes the tensor product over $k$ of $R^+$ with itself $n$ times (letting $(R^+)^0 = k$) (Cf. [CE, X.2]). A typical element of the complex is denoted by

$$\lambda_0[\lambda_1, \lambda_2, \ldots, \lambda_n] := \lambda_0 \otimes \lambda_1 \otimes \lambda_2 \otimes \ldots \otimes \lambda_n \in N_n.$$ 

The $R$-linear differential $d_n : N_n \to N_{n-1}$ of the complex is given by

$$d_n[\lambda_1, \lambda_2, \ldots, \lambda_n] := \lambda_1[\lambda_2, \ldots, \lambda_n] + \sum_{0 < i < n} (-1)^i [\lambda_1, \lambda_2, \ldots, \lambda_i \lambda_{i+1}, \ldots, \lambda_n] \quad (1)$$

With this differential the complex is an $R$-free resolution of $k$. Observe that it is bigraded; in addition to its grading as a complex — its homological degree — it has a second grading induced from the grading of $R$, called the internal degree. The homogeneous piece of $N_\ast$ of bidegree $(i, j)$ will be denoted by $N_{ij}$. The differential has bidegree $(-1, 0)$ as a homogeneous map in this bigrading. If $M$ is a graded $R$-module, the induced differential of $M \otimes_R N_\ast$ is still homogeneous, and hence the bigrading passes to the homology

$$H(M \otimes_R N_\ast) = Tor^R_\ast(M, k) = \bigoplus_{i,j \geq 0} Tor^R_{ij}(M, k).$$

We now introduce more notation. Assume that $S := \oplus_{i \geq 0} S_i$, is a graded connected $R$-algebra. Note that it is immediately clear from the construction of the bar complex that $(S \otimes_R N_{ij}) = (S \otimes_k (R^+)^i) = 0$, if $j < i$ and that hence in this case also $Tor^R_{ij}(S, k) = 0$. The diagonal part $D(S) := \bigoplus_{n \geq 0} S_n \otimes_k (R_1)^n$, is the subspace of $S \otimes_R N_\ast$ consisting in each homological degree of the part of minimal internal degree. The graded complement $V(S) := \bigoplus_{n=0}^\infty S^+ \otimes_k (R^+)^n$, will be called the off-diagonal part of the complex $S \otimes_R N_\ast$. It is a subcomplex and contains all boundaries, by reason of degree. Hence there is an injection

$$Z(S \otimes_R N_\ast) \cap D(S) \cong \bigoplus_{i \geq 0} Tor^R_{ii}(S, k) \hookrightarrow H(S \otimes_R N_\ast). \quad (2)$$

Denote the free $R$-module of rank $d$ with generators in degree $j$ by $R[j]^d$. The dimensions of the torsion groups $d_{ij} = dim_k Tor^R_{ij}(S, k)$ may also be interpreted in terms of ranks of the minimal free graded resolution $F_i$ of $S$. Namely it is true that $F_i = \oplus_j R[j]^{d_{ij}}$. This motivates the first part of the following (fairly standard) terminology.
Definition 2.1. (Cf. [BF], [EG]). If $S = R/I$ is a quotient of $R$, such that $Tor_{ij}^R(S, k) = 0$ if $i \neq j$, then $S$ is said to have a 1-linear $R$-resolution, or the map $R \to S$ is said to be 1-linear. If $k = R/R^+$ has a 1-linear resolution then $R$ is called a Koszul algebra.

Using the previous terminology it is clear that $R \to S$ is 1-linear iff the inclusion $V(S) \subset S \otimes_R N.$ induces the zero map on homology. In the more enthusiastic terminology of Kempf [K] one says that $R$ is wonderful if $R$ is a Koszul algebra, and that $S$ is an awesome module if it has a 1-linear resolution. Note that a Koszul algebra $R$ is generated by $R_1$ (since $Tor_{ij}^R(k, k) = 0$ if $j \neq 1$, see e.g. [Le]) and if it is presented as a graded quotient of the polynomial algebra $k[R_1]$, it will have relations in degree 2 (since $Tor_{2j}^R(k, k) = 0$ if $j \neq 2$, loc.cit.). It is also clear that a ring $S$ with an 1-linear resolution will be a quotient of $R$ by an ideal generated by elements of degree 1.

We have use for another concept, due to G. Levin [L], from the theory of the homology of commutative rings.

Definition 2.2. A homomorphism $\eta : R \to S$ that induces a surjection $Tor_R^i(k, k) \to Tor_S^i(k, k)$ is called a large homomorphism.

Some examples are given by the following lemma.

Lemma 2.3. a) A retraction $\eta : R \to S$ onto a graded subalgebra $S \subset R$ is a large homomorphism.

b) Suppose that $R \to S = R/I$ is 1-linear. Then it is large.

c) If $R$ is a Koszul algebra and $\eta : R \to S$ is a large map, then $S$ is also a Koszul algebra, and furthermore $\eta : R \to S$ is then 1-linear.

Part a) is wellknown [L, Theorem 2.3] and immediate (see the similar proof in Lemma 3.1.2 in [B].) The remaining statements are a generalization of [Ba, Lemma 2.3] (Backelin assumes that both rings involved are Koszul algebras). The proof uses the following characterization of large homomorphisms $\eta : R \to S$, given in [L, Theorem 1.1]. Namely, $\eta$ is large iff the map

$$p_* : Tor^R_i(S, k) \to Tor^R_i(k, k),$$

induced by the natural surjection $p : S \to S/S^+ = k$ is an injection (Levin studies local rings but his arguments are true in our graded situation). Let us prove that this criterion is true if $\eta : R \to S$ is 1-linear, i.e. that $Tor^R_i(k, k) = 0$ if $i \neq j$. Consider the bar complex $N.$ of $R$. Then $p$ induces $p_*:

$$S \otimes_R N. \to k \otimes_R N.$$

This map has bidegree $(0, 0)$, so it maps in particular the diagonal part

$$D(S) := \bigoplus_{n \geq 0} S_0 \otimes_k (R^+)^n,$$

isomorphically to $D(k) := \bigoplus_{n \geq 0} k \otimes_k (R^+)^n$, and since, as mentioned above, by reason of degree, $D(k)$ contains no boundaries it follows that $p_*$ is injective on the diagonal part of the homology:

$$p_* : Z(S \otimes_R N.) \cap D(S) \cong \bigoplus Tor^R_i(S, k) \hookrightarrow Z(k \otimes_R N.) \cap D(k) \cong \bigoplus Tor^R_i(k, k).$$
(Note that this known result is valid for arbitrary rings). If $R \to S$ is 1-linear the only non-zero homology is the one on the diagonal, and thus the map is large. Finally the last part of the lemma is an immediate consequence of the definitions, and the previously used characterization of large homomorphisms.

2.2 The bar complex of a homogeneous coordinate ring. The following situation will be our main object of study. Suppose that $L$ is a line bundle on the proper variety $X$. Form, as in the introduction the graded connected algebra

$$R = R(X, L) = k \oplus R^+.$$  

Let $Y$ be a subvariety of $X$, restrict $L$ to $Y$ and consider in the same way the $k$-algebra $S := S(L)$. There is a canonical restriction homomorphism $R \to S$. Consider the bar complex $N$ of $R$ tensored with $S$.

$$S \otimes_k N = \bigoplus_{n \geq 0} S \otimes_k (R^+)^n \cong \bigoplus_{1 \leq n, 0 \leq i_0, 1 \leq i_1, \ldots, i_n} \Gamma(Y \times X(n), L^{i_0} \times \ldots \times L^{i_n}),$$

Here $Y \times X(n)$ denotes the product of $Y$ with $n$ copies of $X$, and we have used the canonical identification

$$\Gamma(Y \times X(n), L^{i_0} \times \ldots \times L^{i_n}) \cong \Gamma(Y, L^{\otimes i_0}) \otimes_k \cdots \otimes_k \Gamma(X, L^{\otimes i_n}).$$

The differential $d_n : N_n \to N_{n-1}$ of the complex may be given as the sum

$$(1)$$

$$d_n[\lambda_1, \lambda_2, \ldots, \lambda_n] := \lambda_1[\lambda_2, \ldots, \lambda_n] + \sum_{0 < i < n} (-1)^i[\lambda_1, \lambda_2, \ldots, \lambda_i\lambda_{i+1}, \ldots, \lambda_n]$$

$$= d^n_{01}[\lambda_1, \lambda_2, \ldots, \lambda_n] + \sum_{0 < i < n} (-1)^i d^n_{i+1}[\lambda_1, \lambda_2, \ldots, \lambda_n].$$

of restriction maps

$$d^n_{i+1} : S \otimes_k (R^+)^n \supset \Gamma(Y \times X(n), L^{i_0} \times \ldots \times L^{i_n}) \to \Gamma(\Delta_{ii+1}^n, L^{i_0} \times \ldots \times L^{i_n})$$

$$\cong \oplus \Gamma(Y \times X(n-1), L^{i_0} \times \ldots \times L^{i_0} \otimes L^{i+i-1} \times L^{i_n}) \subset S \otimes_k (R^+)^{n-1}$$

for $i = 0, \ldots, n-1$, on cohomology from $Y \times X(n)$ to partial diagonals $\Delta_{ii+1}^n$ consisting of those points in $Y \times X(n)$ that have coordinate $i$ equal to coordinate $i + 1$. These maps will be called partial differentials.

With joyfully Bourbakistic nostalgia we extend the definition of homogeneous coordinate rings to the empty subset, and defining $R(\emptyset, L) := k$ if $L$ is the restriction of a line bundle $L$ on $X$, we obtain the quotient $k$ as the restriction to a closed subvariety. This gives us the added comfort to be able to treat uniformly possibly 1-linear restrictions of $R$ as well as the question whether $R$ itself is a Koszul algebra.

It will be convenient to study these maps in a more general context. Let $T_{ii+1}^n$ be the ideal of

$$\Delta_{ii+1}^n \subset Y \times X(n),$$

for $i = 0, \ldots, n-1$.
and denote the inclusion by \( i \). Assume that \( \mathcal{L}_\alpha := \mathcal{L}_0 \times \ldots \times \mathcal{L}_n \), is an invertible sheaf on \( Y \times X(n) \), which is the product of the invertible sheaves \( \mathcal{L}_i \). There is then a short sequence.

\[
\begin{align*}
\Gamma(Y \times X(n), T^n_{ii+1} \mathcal{L}_\alpha) &\hookrightarrow \Gamma(Y \times X(n), \mathcal{L}_\alpha) \\
&\rightarrow \Gamma(\Delta^n_{ii+1}, i^* \mathcal{L}_\alpha) \cong \Gamma(Y \times X(n-1), \mathcal{L}_0 \times \ldots \times \mathcal{L}_i \otimes \mathcal{L}_{i+1} \times \ldots \times \mathcal{L}_n).
\end{align*}
\]

If \( \mathcal{L}_j = \mathcal{L}^i_j \), the map composed of the last two maps in the short sequence is precisely the differential \( d_{ii+1}^n \), in the complex \( S \otimes_R N \), described above. For this reason, we will continue to call the composite map \( d_{ii+1}^n \) in the general case, suppressing any reference to the line bundles are involved. The first part of the following lemma is clear. The second part is a consequence of the fact that taking global sections commutes with finite inverse limits.

**Lemma 2.2.** If \( X \) is a proper variety and \( \mathcal{L}_\alpha \) and \( d_{ii+1}^n \) are as above then

\[
\ker d_{ii+1}^n = \Gamma(Y \times X(n), T^n_{ii+1} \mathcal{L}_\alpha).
\]

Furthermore

\[
\bigcap_{i \in I} \ker d_{ii+1}^n = \Gamma(Y \times X(n), \bigcap_{i \in I} T^n_{ii+1} \mathcal{L}_\alpha), \text{ if } I \subset \{0, \ldots , n\}.
\]

3. **Models for constructing a homotopy.** Let \( \mathcal{L}_\alpha = \mathcal{L}_0 \times \ldots \times \mathcal{L}_{n+1} \), for \( n \geq 0 \), be a line bundle on \( Y \times X(n+1) \). We will now construct an associated *model complex* \( K = K(n, \mathcal{L}_\alpha) \). It is zero in all (homological) degrees except possibly \( n-1, n, n+1 : \)

\[
K = K_{n-1} \oplus K_n \oplus K_{n+1}.
\]

and

\[
\begin{align*}
K_{n+1} &:= \Gamma(Y \times X(n+1), \mathcal{L}_\alpha) \\
K_n &:= \bigoplus_{0 \leq i \leq n} \Gamma(Y \times X(n), \mathcal{L}_0 \times \ldots \times \mathcal{L}_i \otimes \mathcal{L}_{i+1} \times \ldots \times \mathcal{L}_{n+1}) \\
K_{n-1} &:= \bigoplus_{0 \leq k < i \leq n} \Gamma(Y \times X(n-1), \mathcal{L}_0 \times \ldots \times \mathcal{L}_k \otimes \mathcal{L}_{k+1} \times \ldots \times \mathcal{L}_i \otimes \mathcal{L}_{i+1} \times \ldots \times \mathcal{L}_{n+1}).
\end{align*}
\]

The differential restricted to \( K_{n-1} \) is zero, while it is the same as in the standard complex acting on \( K_n \) or \( K_{n+1} \). Thus, if a typical element of \( K_{n+1} \) or \( K_n \) is denoted by the symbol \([\lambda_0, \ldots , \lambda_s] \), for \( s = n-1 \), or \( n \), where each \( \lambda_i \in \Gamma(X, M) \) for some \( M = \mathcal{L}_j \) or \( M = \mathcal{L}_j \otimes \mathcal{L}_{j+1} \), then formula 2.(1) precisely describes the differential. (Using the partial differentials described as a preliminary of lemma 2.1). For the definition to be intelligible for \( n = 0 \) we consider \( X(0) \) to be \( \text{spec} \ k \), and \( X(-1) := \emptyset \). Then \( K(0, \mathcal{L}_0 \times \mathcal{L}_1) \) is the complex

\[
\begin{align*}
(1) \quad \Gamma(Y \times X, \mathcal{L}_0 \times \mathcal{L}_1) &\cong \Gamma(Y, \mathcal{L}_0) \otimes \Gamma(X, \mathcal{L}_1) \rightarrow \Gamma(Y, \mathcal{L}_0 \otimes \mathcal{L}_1) \rightarrow 0,
\end{align*}
\]

and the only non-zero differential \( d_1 = d_{01}^0 \) corresponds to multiplication of global sections.

We will now inductively describe the homology of the models.
Lemma 3.1. Assume that $\mathcal{L}_\alpha = \mathcal{L}_0 \times \ldots \times \mathcal{L}_{n+1}$ is an invertible sheaf on $Y \times X(n+1)$ and let $\bar{\mathcal{L}}_\alpha$ be the sheaf $\mathcal{L}_0 \times \ldots \times \mathcal{L}_n \otimes \mathcal{L}_{n+1}$ on $Y \times X(n)$. The partial differential

$$d_{n+1}^{m+1} : A := \Gamma(Y \times X(n+1), \mathcal{L}_\alpha) \rightarrow B := \Gamma(Y \times X(n), \mathcal{L}_0 \times \ldots \times \mathcal{L}_n \otimes \mathcal{L}_{n+1})$$

induces a map $\delta$ of subspaces:

$$\delta : A \supset \bigcap_{0 \leq i \leq n-1} \ker d_{n+1}^{ii+1} \rightarrow \bigcap_{0 \leq i \leq n-1} \ker d_{n+1}^{ii+1} \subset B,$$

and there is an exact sequence

$$cok\delta \hookrightarrow H_n(K(n, \mathcal{L}_\alpha)) \rightarrow H_{n-1}(K(n-1, \bar{\mathcal{L}}_\alpha)) \otimes \Gamma(X, \mathcal{L}_{n+1}).$$

If $n = 0$ there is an isomorphism

$$H_0(K(0, \mathcal{L}_\alpha) \cong cok\delta = cok d_{01}^1.$$

Proof. The case $n = 0$ is immediately clear from the description (1) of $H_0(K(0, \mathcal{L}_0 \times \mathcal{L}_1)$.

Note that, forgetting differentials, $K(n, \mathcal{L}_\alpha)$ is the direct sum of vectors paces of global sections of the form $\Gamma(Y \times X(s), M_0 \times \ldots \times M_s)$, where $M_i$, $0 \leq i \leq s$, are invertible sheaves on $Y$ or $X$ and $s = n - 1, n$ or $n + 1$. Let $V$ and $W$ be the sub vectors paces of $K(n, \mathcal{L}_\alpha)$ consisting of the direct sums of those $\Gamma(Y \times X(s), M_1 \times \ldots \times M_s)$ with $M_s = \mathcal{L}_{n+1}$, respectively $M_s \neq \mathcal{L}_{n+1}$. Then $K(n, \mathcal{L}_\alpha) = V \oplus W$, and $W$ is clearly a subcomplex. It is clear that there are isomorphisms of vectors paces

$$V \cong K(n, \mathcal{L}_\alpha)/W \cong K(n-1, \bar{\mathcal{L}}_\alpha) \otimes \Gamma(X, \mathcal{L}_{n+1}).$$

The last vectors pace may be considered as a complex, namely as the tensor product (over $k$) of $K(n-1, \bar{\mathcal{L}}_\alpha)$ with the complex $\Gamma(X, \mathcal{L}_{n+1})[-1]$, which has differential zero and is concentrated in degree 1. (Here, as usual, $M[-1]$ of a complex $(M, d)$ denotes the complex, which in (homological) degree $m$ has $M[-1]_m = M_{m-1}$ and differential $-d$.) Then the final vector space isomorphism of (2) is in fact an isomorphism of complexes. Now take the long homology sequence belonging to the short exact sequence of complexes

$$W \hookrightarrow K(n, \mathcal{L}_\alpha) \xrightarrow{\delta} K(n-1, \bar{\mathcal{L}}_\alpha) \otimes \Gamma(X, \mathcal{L}_{n+1})[-1].$$

We get the exact sequence

$$H_{n+1}(K(n-1, \bar{\mathcal{L}}_\alpha) \otimes \Gamma(X, \mathcal{L}_{n+1})[-1] \rightarrow H_n(W) \rightarrow$$

$$\rightarrow H_n(K(n-1, \bar{\mathcal{L}}_\alpha)) \rightarrow H_n(K(n-1, \bar{\mathcal{L}}_\alpha) \otimes \Gamma(X, \mathcal{L}_{n+1})[-1]).$$
Since $W_{n+1} = 0$ and $W_n = \Gamma(Y \times X(n), \mathcal{L}_0 \times \ldots \times \mathcal{L}_{n-1} \times \mathcal{L}_n \otimes \mathcal{L}_{n+1})$, it is clear that

$$H_n(W) = \ker d_n \cap W_n = \ker(\Sigma_{i=0}^{n-1} \ker d_n^{i+1}) \cap W_n = \bigcap_{0 \leq i \leq n-1} \ker d_n^{i+1} \cap W_n.$$  

Similarly

$$(4) \quad H_{n+1}(K(n-1, \tilde{\mathcal{L}}_\alpha) \otimes \Gamma(X, \mathcal{L}_{n+1})[-1]) = \bigcap_{0 \leq i \leq n-1} \ker d_{n+1}^{i+1},$$

(the intersection taken in $A$ of the lemma), and

$$H_n(K(n-1, \tilde{\mathcal{L}}_\alpha) \otimes \Gamma(X, \mathcal{L}_{n+1})[-1]) = H_{n-1}(K(n-1, \tilde{\mathcal{L}}_\alpha)) \otimes \Gamma(X, \mathcal{L}_{n+1}),$$

so it suffices to check that the connecting map in the long exact sequence (3) is given by $d_{n+1}^{n+1}$. But this follows, from (4), since, using the canonical lifting $K(n, \mathcal{L}_\alpha)_{n+1}/W_{n+1} \cong V_{n+1} \subset K(n, \mathcal{L}_\alpha)_{n+1}$, the connecting map is $d_{n+1}$ restricted to $V$, and since the maps $d_{n+1} = \Sigma_{i=0}^{n} d_{n+1}^{i+1}$ and $d_{n+1}^{n+1}$ evidently coincide on $\cap_{0 \leq i \leq n-1} \ker d_{n+1}^{i+1}$.

By induction it is now possible to give reasonable geometric conditions for the models to have middle degree homology zero.

**Lemma 3.2.** Assume that the following conditions are satisfied by an invertible sheaf $\mathcal{L}_\alpha$ on $Y \times X(n+1)$:

i) $H^1(Y \times X(n+1), \mathcal{L}_\alpha) = 0$.

ii) $\Gamma(Y \times X(n+1), \mathcal{L}_\alpha) \rightarrow \Gamma(\bigcup_{i=0}^{n-1} \Delta_{i+1}^{n+1}, \mathcal{L}_\alpha)$.

iii) $\Gamma(Y \times X(n+1), \mathcal{L}_\alpha) \rightarrow \Gamma(\Delta_{n+1}^{n+1}, \mathcal{L}_\alpha)$.

Then $\text{cok} \delta$ (see the preceding lemma) is zero. If the conditions i) to iii) are satisfied for all $\mathcal{L}_0 \times \ldots \times \mathcal{L}_{m+1}$ where $0 \leq m \leq n$ then $H_n(K(n, \mathcal{L}_\alpha)) = 0$.

**Proof.** Recall that $\mathcal{T}_{i+1}^{n+1}$ is the ideal in $Y \times X(n+1)$ of the partial diagonal $\Delta_{i+1}^{n+1}$ (and that $n+1$ is here not a power but an index). Observe that conditions i) and ii) of the lemma, together with the exact sequence

$$\Gamma(Y \times X(n+1), \mathcal{L}_\alpha) \rightarrow \bigcap_{i=0}^{n-1} \Delta_{i+1}^{n+1}, \mathcal{L}_\alpha) \rightarrow$$

$$\rightarrow H^1(Y \times X(n+1), \bigcap_{i=0}^{n-1} \mathcal{T}_{i+1}^{n+1} \mathcal{L}_\alpha) \rightarrow H^1(Y \times X(n+1), \mathcal{L}_\alpha)$$

imply that $H^1(Y \times X(n+1), \bigcap_{i=0}^{n-1} \mathcal{T}_{i+1}^{n+1} \mathcal{L}_\alpha) = 0$. (Note that $\bigcap_{i=0}^{n-1} \mathcal{T}_{i+1}^{n+1} \mathcal{L}_\alpha$ is the ideal of $\bigcup_{i=0}^{n-1} \Delta_{i+1}^{n+1}$).

There is a short exact sequence

$$(5) \quad \bigcap_{i=0}^{n} \mathcal{T}_{i+1}^{n+1} \ightarrow \bigcap_{i=0}^{n} \mathcal{T}_{i+1}^{n+1} \oplus \mathcal{T}_{n+1}^{n+1} \rightarrow J := \bigcap_{i=0}^{n} \mathcal{T}_{i+1}^{n+1} + \mathcal{T}_{n+1}^{n+1} \subset \mathcal{O}_{Y \times X(n+1)}.$$

Taking the tensor product of (5) with $\mathcal{L}_\alpha$ (over $\mathcal{O} = \mathcal{O}_{Y \times X(n+1)}$) and taking the long cohomology sequence shows that

$$
\Gamma(Y \times X(n+1), (\bigcap_{i=0}^{n-1} T_{ii+1}^{n+1} \oplus T_{nn+1}^{n+1}) \mathcal{L}_\alpha) \rightarrow \Gamma(Y \times X(n+1), \mathcal{J} \mathcal{L}_\alpha)
$$

(6) is a surjection. Let $\mathcal{I}_1 := \bigcap_{i=0}^{n-1} T_{ii+1}^{n+1}, \mathcal{I}_2 := T_{nn+1}^{n+1}$ and $S := Y \times X(n+1)$, and consider the commutative diagram:

$$
\begin{array}{ccc}
\Gamma(S, \mathcal{I}_1 \mathcal{L}_\alpha \oplus \mathcal{I}_2 \mathcal{L}_\alpha) & \rightarrow & \Gamma(S, \mathcal{J} \mathcal{L}_\alpha) \\
\downarrow & & \downarrow \epsilon \\
\Gamma(S, \mathcal{J} \mathcal{L}_\alpha) & \rightarrow & \Gamma(S, \mathcal{J} \mathcal{L}_\alpha) \\
\downarrow \cong & & \downarrow \eta \\
\Gamma(Y \times X(n), \bigcap_{i=0}^{n-1} T_{ii+1}^{n+1} \mathcal{L}_\alpha) & \rightarrow & \Gamma(Y \times X(n), \mathcal{J} \mathcal{L}_\alpha)
\end{array}
$$

The map composed of the middle vertical maps is precisely the partial differential $d_{n+1}$. The composition of the left hand vertical morphisms kills $\Gamma(Y \times X(n+1), \mathcal{I}_2 \mathcal{L}_\alpha)$, and then by lemma 2.2, the cokernel of this composed map equals $\text{cok}\delta$. In view of (5), this implies that $\text{cok}\delta \cong \text{cok}\epsilon$. By a diagram chase, $\text{cok}\epsilon = 0$ if $\text{cok}\eta = 0$, since the right hand vertical map is injective. This, however, is condition iii) of the lemma, which hence is proved. (The last statement follows from the preceding lemma by an obvious induction).

### 4.1. A criterion for a restriction map of homogeneous coordinate rings to be 1-linear.

We will now study the homology of homogeneous coordinate rings, using the model complexes. The object of this paragraph is the following result.

**Theorem 5.** Let $X$ be a proper variety, $\mathcal{L}$ a line bundle on $X$ and $R := R(X, \mathcal{L})$. Suppose further that $S$ is the restriction of $R$ to the closed subvariety $Y$ of $X$, and that the conditions i-iii) of Lemma 3.2 are fulfilled for all line bundles of the type $\mathcal{L}^{\alpha} := \mathcal{L}^{\otimes a_0} \times \ldots \times \mathcal{L}^{\otimes a_{n+1}}$ where $a_0 \geq 0$, and $a_i > 0$ if $i > 0$. Then $R \rightarrow S$ is 1-linear. (In particular note that putting $Y = \emptyset$ in Lemma 3.2 gives conditions for $R$ to be Koszul.)

The proof will take the rest of the section. By Lemma 3.2 all higher homology is zero for all model complexes:

$$
H_n R(n+1, \mathcal{L}^{\otimes a_0} \times \ldots \times \mathcal{L}^{\otimes a_{n+1}}) = 0, \text{ if } n \geq 1,
$$

and $a_0 \geq 0$, $a_i > 0$ if $i > 0$. Let $D = D(S)$ be the diagonal vector subspace of $K := S \otimes_R N$, which contains everything on the diagonal,

$$
D = \bigoplus_{n \geq 0} \Gamma(Y \times X(n), \mathcal{O}_Y \times \mathcal{L} \times \ldots \times \mathcal{L}).
$$

Consider $D$ as a complex with the zero differential. There is a surjective map of complexes $K. \rightarrow D$ (with kernel the off-diagonal subcomplex $V := V(S)$) (see section 2.1). The theorem asserts that this map gives an injection after taking homology. To prove this it suffices to prove that the injection $V. \subset K.$ is homotopic to 0.
4.2 Homotopy.

First we will define a function which will give the target of the homotopy. The function \( \sigma \) is a map from sequences \((\alpha) := (a^0, \ldots, a^n)\) of length \(n + 1\) consisting of nonnegative integers with \(a^0 \geq 0\) and \(a^i > 0\) if \(i > 0\), and such that either \(a^0 \neq 0\) or \(a^i \neq 1\) for some \(i \neq 0\), (i.e. those indices for which \(M(n, \alpha)\) occurs in \(V\)) to sequences of length \(n + 2\), defined in the following way: If \(a^0 \neq 0\) then

\[
\sigma(\alpha) = (0, a^0, \ldots, a^n).
\]

Otherwise, let \(a^i\) be the first element in the sequence, different from 1. Then

\[
\sigma(\alpha) := (a^0, \ldots, a^{i-1}, 1, a^i - 1, a^{i+1}, \ldots, a^n) = (0, 1, \ldots, 1, 1, a^i - 1, a^{i+1}, \ldots, a^n).
\]

Let

\[
M(n, \alpha) := \Gamma(Y \times X(n), \mathcal{L}^{\otimes a_0} \times \ldots \times \mathcal{L}^{\otimes a_n})
\]

and recall that the partial differential \(a_n^{i+1}\) is a map

\[
M(n, a^0, \ldots, a^n) \to M(n - 1, a^0, \ldots, a^i + a^{i+1}, \ldots, a^n).
\]

Call temporarily the sequence \((a^0, \ldots, a^i + a^{i+1}, \ldots, a^n)\) a partial differential of the sequence \((a^0, \ldots, a^n)\). The following easy lemma is crucial in the construction of the homotopy.

**Lemma 4.1.** Suppose that \(\beta = (b^0, b^1, \ldots, b^{n-1}) = (a^0, \ldots, a^i + a^{i+1}, \ldots, a^n)\) is a partial differential of \(\alpha = (a^0, \ldots, a^n)\). Then \(\sigma(\beta)\) is a partial differential of \(\sigma(\alpha)\). Also \(\alpha\) itself is a partial differential of \(\sigma(\alpha)\).

**Proof.** The last statement is immediate from the definition of \(\sigma\). The remaining proof falls into several cases. If \(a^0 \neq 0\) then clearly \(b^0 \neq 0\) and \(\sigma(\beta) = (0, b^0, b^1, \ldots, b^{n-1})\) is a partial differential of \(\sigma(\alpha) = (0, a^0, \ldots, a^n)\). If \(a^0 = 0\), but \(b^0 \neq 0\) then \(i = 0\) and \(\sigma(\beta) = (0, b^0, b^1, \ldots, b^{n-1}) = \alpha\), which is a partial differential of \(\sigma(\alpha)\), by the last part of the lemma. If finally \(a^0 = b^0 = 0\), let \(a^i\) be the first non-zero element in the sequence different from 1. There are now three cases. If \(i < l\) then \(1 = a^i\) and \(a^i + a^{i+1} - 1 = a^{i+1}\) and hence \(\sigma(\beta) = \alpha\). If \(i = l\) then \(\sigma(\beta) = (0, 1, \ldots, 1, a^l + a^{l+1} - 1, \ldots, a^n)\) is clearly a partial differential of \(\sigma(\alpha) = (0, 1, \ldots, 1, a^l - 1, a^{l+1}, \ldots, a^n)\). If \(i > l\) then \(\sigma(\beta) = (0, 1, \ldots, 1, a^l - 1, \ldots, a^i + a^{i+1}, \ldots, a^n)\) and this is a partial differential of \(\sigma(\alpha) = (0, 1, \ldots, 1, a^l - 1, \ldots, a^n)\).

We will now construct a degree +1 map \(s : V. \to N.\), such that the inclusion map \(V. \to N.\) is homotopic to 0 through \(s\), i.e. \(v = dsv + sdrv, v \in V.\) This map will be constructed inductively. Assume the condition on models of the theorem . Note that a model complex \(K(n, \mathcal{L}^{\otimes a_0} \times \ldots \times \mathcal{L}^{\otimes a_{n+1}})\) is a subcomplex of \(K.\), and that it contains in the middle degree precisely those \(M(n, \beta)\) for which \(\beta\) is a partial differential of \(\alpha\).

**First step.** To construct

\[
s_0 : V_0 = \bigoplus \Gamma(Y, \mathcal{L}^i) \to K.,
\]
consider the model complex $K := K(0, L \times L^{i-1})$ (note that $i \geq 2$)

$$K : \quad \Gamma(Y \times X, L \times L^{i-1}) \xrightarrow{d_1^{i-1}} \Gamma(Y, L^i) \to 0$$

By assumption this complex is acyclic in degree 0, hence $d_1^{i-1}$ is surjective and there is a splitting $s_0$. Choose such a splitting $s_0$ for each $i \geq 2$. Clearly $d_{12}s_0a = a$, and hence $s_0$ is acceptable as the first part of a homotopy of the inclusion of $V \subset K$ with the zero map.

The induction step. Assume that $s_k$, $k < n$ has already been constructed, such that $s_k(M(k, \gamma)) \subset M(k+1, \sigma(\gamma))$. If $m \in K_n$, the relation

$$d_nm = s_{n-2}d_{n-1}d_nm + d_n s_{n-1}d_nm = d_n s_{n-2}d_n m$$

is valid. Hence in particular $d_n(m - s_{n-1}d_nm) = d_n m - d_n s_{n-1}d_nm = 0$, so $m - s_{n-1}d_nm$ is a cycle. To define $s_n$, it is enough to consider $m \in M(n, \alpha) \subset V_n$. Consider the model complex $K(n, \alpha(n)) \subset K$. By the induction hypothesis and Lemma 4.1 it is clear that $s_{n-1}d_nm$ is in $K(n, \alpha(n))$ and hence also the cycle $m - s_{n-1}d_nm$. By the acyclicity of $K(n, \alpha(n))$ in degree $n$ there is a splitting $\theta : Z_n(K(n, \alpha(n))) \to K(n, \alpha(n)))_{n+1} = M(n, \sigma(\alpha))$ of $d_n$. Define

$$s_n m := \theta(m - s_{n-1}d_nm) \in M(n, \sigma(\alpha))$$

Clearly $m = d_{n+1}s_nm + s_{n-1}d_nm$, so $s_n$ is the next step of the homotopy as desired.

5. Remaining proofs.

We will now prove the results stated in the introduction. To get Theorem 4, just recall that the arguments of Mehta-Ramanathan (see e.g. [R,1.13]), show that restriction of ample invertible sheaves to compatibly Frobenius split subvarieties induces surjections on global sections, so that ii-iii) of Lemma 4.3.2 are satisfied, for ample $L_\alpha$, if the corresponding unions of partial diagonals are compatibly split in $X(n + 1)$. By the same argument, if $Y \times X(n)$ is Frobenius split then i) of the lemma is true. Hence it suffices to note that by elementary properties of ampleness $L_\alpha$ is ample when $L$ is ample. Thus Theorem 5 gives Theorem 4. Now note that it is possible to restrict oneself to positive characteristics as described e.g.in [R,3.11] and [B,1.3] in order to show that the hypothesis of Theorem 5 is fulfilled in arbitrary characteristics.

From Theorem 4, (taking $Y = \emptyset$) Theorem 1 follows directly for all homogeneous coordinate rings belonging to ample line bundles of those toric varieties, for which the requisite splitting properties of partial diagonals are known. It is proven in [B, 3.1] that all homogeneous coordinate rings of the type in the theorem are algebra retracts of such rings and hence Theorem 1 follows in complete generality by Lemma 2.3 a).

Theorem 2 on the Koszul property of homogeneous coordinate rings of Schubert varieties clearly follows from the relative version given in Theorem 3, by applying this theorem first to $Y = \emptyset$ to obtain that $R = R(X, L)$ is a Koszul algebra and then another time to get that the map $R \to S$ is 1-linear and finally using Lemma 2.3 c) to see that $S$ is a Koszul algebra.

Finally Theorem 3 for an ample line bundle $L$ in positive characteristic, is a consequence of Theorem 5 using the fact that all the partial diagonals are compatibly split in degree 0 as desired.
split. That this is true is a straightforward generalization, which we omit, of the ingenious proof of this result for $n = 3$ contained in Theorem 3.5 iii) of [R]. The extension of this case of Theorem 3 to a merely effective line bundle $L$, is immediate from the fact, proven e.g. in [R,3.8, 3.11], that there is a parabolic subgroup $Q$ such that, denoting the map $G/P \to G/Q$ by $\pi$, the following is true. First $L = \pi^*(L')$ where $L'$ is an ample line bundle on $G/Q$, and secondly $R(G/P, L) \cong R(G/Q, L')$ and $R(Y, L) \cong R(\pi(Y), L')$.

References

[Ba] J. Backelin, Some homological properties of “high” Veronese subrings, J. Algebra 146(1992)1-17.

[BF] J. Backelin, R. Fröberg, Koszul algebras, Veronese subrings and rings with linear resolutions, Rev. Roumaine Math. Pures Appl. 30 (1985), 85-97.

[B] R. Bögvad, On the homogeneous ideal of a projective nonsingular toric variety, University of Stockholm preprint, 1994 (to appear in J. Tohoku Math.).

[CE] H. Cartan and S. Eilenberg, Homological algebra, Princ. UP, 1960.

[EG] D. Eisenbud and S. Goto, Linear free resolutions and minimal multiplicity, J. Algebra 88(1984), 89-133.

[K1] G. R. Kempf, Some wonderful rings in algebraic geometry, J. Algebra, vol 134, 1990.

[K2] G. R. Kempf, Wonderful rings and awesome modules, Free resolutions in commutative algebra and algebraic geometry, Sundance 90; Jones and Bartlett Publishers, Boston London 1992.

[KR] G. R. Kempf and A. Ramanathan, Multi-cones over Schubert varieties, Inventiones Math 1987.

[Le] J.-M. Lemaire, Algèbres connexes et homologie des espaces de lacets, (Lect. Notes Math., vol. 422), Berlin Heidelberg New York: Springer 1974.

[L] G. Levin, Large homomorphisms of local rings, Math. Scand 46(1980), 209-215.

[M] S. MacLane, Homology, Berlin Heidelberg New York: Springer, 1963.
[R] A. Ramanathan, *Equations defining Schubert varieties and Frobenius splitting of diagonals*, Publ.Math., Inst.Hautes Étud.Sci. 65 (1987), 61-90.

[Ro] J.-E.Roos, *Commutative non-Koszul algebras having a linear resolution of arbitrarily high order. Applications to torsion in loop space homology*, C.R.Acad.Sci.Paris Se'r.I Math.316 (1993), 1123-1128.