A VARIANT OF SOME CYCLOTONOMIC MATRICES INVOLVING TRINOMIAL COEFFICIENTS

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Abstract. In this paper, by using the theory of circulant matrices we study some matrices over finite fields which involve the quadratic character and trinomial coefficients.

1. Introduction

Let $p$ be an odd prime and let $(\cdot/p)$ be the Legendre symbol. The study of the matrices involving Legendre symbols can trace back to the works of Lehmer [6] and Carlitz [1]. For example, Carlitz [1] initiated the study of the matrix

$$C_p = \left[ \left( \frac{j-i}{p} \right) \right]_{1 \leq i,j \leq p-1}.$$  

Carlitz [1, Thm. 4 (4.9)] proved that the characteristic polynomial of $C_p$ is

$$f_p(t) = \left( t^2 - (-1)^{\frac{p-1}{2}} \right) \left( t^2 - (-1)^{\frac{p-1}{2}} \right).$$

Later Chapman [2, 3] and Vsemirnov [8, 9] investigated many variants of Carlitz’s matrix $C_p$. In particular, using sophisticated matrix decompositions Vsemirnov [8, 9] confirmed Chapman’s “evil” determinant conjecture which says that

$$\det \left[ \left( \frac{j-i}{p} \right) \right]_{1 \leq i,j \leq \frac{p+1}{2}} = \begin{cases} -a_p & \text{if } p \equiv 1 \pmod{4}, \\ 1 & \text{if } p \equiv 3 \pmod{4}. \end{cases}$$

The number $a_p$ is defined by the following equality

$$\varepsilon_p^{2-(\frac{2}{p})} h_p = a_p + b_p \sqrt{p}, \quad a_p, b_p \in \mathbb{Q},$$

where $\varepsilon_p > 1$ and $h_p$ denote the fundamental unit and class number of the real quadratic field $\mathbb{Q}(\sqrt{p})$.

Recently, Sun [7] further studied some variants of Carlitz’s matrix $C_p$. For example, Sun [7, Thm. 1.2] showed that

$$- \det \left[ \left( \frac{i^2 + j^2}{p} \right) \right]_{1 \leq i,j \leq p-1}$$

Key words and phrases. Central Trinomial Coefficients, Finite Fields, Determinants.

2020 Mathematics Subject Classification. Primary 05A19, 11C20; Secondary 15A18, 15B57, 33B10.
is always a quadratic residue modulo $p$. Along this line, for any integers $c, d$, the arithmetic properties of the matrix
\[
\begin{pmatrix}
\frac{i^2 + cij + dj^2}{p}
\end{pmatrix}_{1\leq i,j\leq p-1}
\]
were extensively studied. Readers may refer to [5, 7, 10, 11, 13] for details on this topic.

On the other hand, in the same paper Sun posed a conjecture (see [7, Remark 1.3]) which states that
\[
2 \det \begin{pmatrix}
\frac{1}{i^2 - ij + j^2}
\end{pmatrix}_{1\leq i,j\leq p-1}
\]
is a quadratic residue modulo $p$ whenever $p \equiv 2 \pmod{3}$ is an odd prime. This conjecture was later proved by Wu, She and Ni [12].

Also, Let
\[
D_p = \begin{pmatrix}
(i^2 + j^2) \left( \frac{i^2 + j^2}{p} \right)
\end{pmatrix}_{1\leq i,j\leq (p-1)/2}.
\]
Recently, Wu, She and Wang [13] proved a conjecture posed by Sun which states that
\[
\frac{D_p}{p} = \begin{cases}
1 & \text{if } p \equiv 1 \pmod{4}, \\
(-1)^{h(-p)-1} & \text{if } p \equiv 3 \pmod{4},
\end{cases}
\]
where $h(-p)$ is the class number of $\mathbb{Q}(\sqrt{-p})$.

Now let $\mathbb{F}_q = \{0, a_1, \cdots, a_{q-1}\}$ be the finite field with $q$ elements, where $q$ is an odd prime power. Also, let $\chi$ be the unique quadratic multiplicative character of $\mathbb{F}_q$, i.e.,
\[
\chi(x) = \begin{cases}
0 & \text{if } x = 0, \\
1 & \text{if } x \text{ is a nonzero square}, \\
-1 & \text{otherwise}.
\end{cases}
\]

Motivated by the above results, in this paper, we shall study the following matrix over $\mathbb{F}_q$:
\[
S_q := \begin{pmatrix}
(a_i^2 + a_ia_j + a_j^2)\chi(a_i^2 + a_ia_j + a_j^2)
\end{pmatrix}_{1\leq i,j\leq q-1}.
\]

Let $n$ be a non-negative integer. The central trinomial coefficient $T_n$ is defined to be the coefficient of $x^n$ in the polynomial $(x^2 + x + 1)^n$. Equivalently, $T_n$ is the constant term of $(x + 1 + x^{-1})^n$. Now we state our main result.

**Theorem 1.1.** Let $q$ be an odd prime power. Then there exists an element $u \in \mathbb{F}_q$ such that
\[
\det S_q = T_{\frac{q+1}{2}} \cdot u^2.
\]

As a consequence of Theorem 1.1, we have the following result.

**Corollary 1.1.** Let $p$ be an odd prime. Suppose $p \nmid \det S_p$. Then
\[
\left( \frac{\det S_p}{p} \right) = \left( \frac{T_{\frac{p+1}{2}}}{p} \right).
\]
Next we shall give a necessary and sufficient condition for $S_q$ to be singular. Let $n$ be a non-negative integer. Then the trinomial coefficients $\left(\begin{array}{c} n \\ k \end{array}\right)_2$ is defined by

$$\left(x + \frac{1}{x} + 1\right)^n = \sum_{k=-n}^{n} \left(\begin{array}{c} n \\ k \end{array}\right)_2 x^k.$$  

Clearly $\left(\begin{array}{c} n \\ 0 \end{array}\right)_2 = T_n$. Now we state our last result.

**Theorem 1.2.** Let $\mathbb{F}_q$ be the finite field with $q > 5$ and $(q, 22) = 1$. Then

$$\det S_q = \frac{121}{64} \cdot T_{q+1} \cdot \prod_{k=1}^{(p-5)/2} \left(\begin{array}{c} q+1 \\ 2 \end{array}\right)_2 \in \mathbb{F}_p,$$

where $p$ is the characteristic of $\mathbb{F}_q$. Also, $S_q$ is a singular matrix over $\mathbb{F}_q$ if and only if

$$\left(\begin{array}{c} q+1 \\ 2 \end{array}\right)_2 \equiv 0 \pmod{p}$$

for some $0 \leq k \leq (q - 5)/2$.

We will prove our main results in Section 2 and Section 3 respectively.

## 2. Proof of Theorem 1.1

We first introduce the definition of the circulant matrices. Let $R$ be a commutative ring. Let $m$ be a positive integer and $t_0, t_1, \ldots, t_{m-1} \in R$. We define the circulant matrix $C(t_0, \ldots, t_{m-1})$ to be an $m \times m$ matrix whose $(i-j)$-entry is $t_{j-i}$ where the indices are cyclic module $m$. Wu [11, Lemma 3.4] obtained the following result.

**Lemma 2.1.** Let $R$ be a commutative ring. Let $m$ be a positive even integer. Let $t_0, t_1, \ldots, t_{m-1} \in R$ such that

$$t_i = t_{m-i} \text{ for each } 1 \leq i \leq m-1.$$  

Then there exists an element $u \in R$ such that

$$\det C(t_0, \ldots, t_{m-1}) = \left(\sum_{i=0}^{m-1} t_i\right) \left(\sum_{i=0}^{m-1} (-1)^i t_i\right) \cdot u^2.$$  

We also need the following known result.

**Lemma 2.2.** Let $k$ be an integer. Then

$$\sum_{x \in \mathbb{F}_q \setminus \{0\}} x^k = \begin{cases} -1 & \text{if } p - 1 \mid k, \\ 0 & \text{otherwise}. \end{cases}$$
Now we are in a position to prove our main results. For simplicity, the summations \( \sum_{x \in \mathbb{F}_q} \) and \( \sum_{x \in \mathbb{F}_q \setminus \{0\}} \) are abbreviated as \( \sum_{x} \) and \( \sum_{x \neq 0} \) respectively.

**Proof of Theorem 1.1.** Fix a primitive element \( g \) of \( \mathbb{F}_q \). Then one can verify that

\[
\det S_q = \prod_{i=1}^{q-1} a_i^2 \cdot \det \left[ \left( \left( \frac{a_j}{a_i} \right)^2 + \frac{a_j}{a_i} + 1 \right) \chi \left( \left( \frac{a_j}{a_i} \right)^2 + \frac{a_j}{a_i} + 1 \right) \right]_{1 \leq i, j \leq q-1}
\]

\[
= \det \left[ \frac{1}{g^{j-i}} (g^{2(j-i)} + g^{j-i} + 1) \frac{q+1}{2} \right]_{0 \leq i, j \leq q-2}.
\]

Let \( t_i = g^{-i}(g^{2i} + g^i + 1) \frac{q+1}{2} \) for \( 0 \leq i \leq q-2 \). Then

\[
\det S_q = \det C(t_0, t_1, \cdots, t_{q-2})
\]

and \( t_i = t_{q-1-i} \) for \( 1 \leq i \leq q-3 \). Applying Lemma 2.1 there is an element \( u \in \mathbb{F}_q \) such that

\[
\det S_q = \left( \sum_{i=0}^{q-2} t_i \right) \left( \sum_{i=0}^{q-2} (-1)^i t_i \right) u^2. \tag{2.2}
\]

We first evaluate \( \sum_{i=0}^{q-2} t_i \).

\[
\sum_{i=0}^{q-2} t_i = \sum_{x \neq 0} \frac{1}{x} (x^2 + x + 1) \frac{q+1}{2}
\]

\[
= \sum_{x \neq 0} \left( x + \frac{1}{x} + 1 \right) \cdot (x^2 + x + 1) \frac{q+1}{2}
\]

\[
= 2 \sum_{x \neq 0} x \cdot (x^2 + x + 1) \frac{q+1}{2} + \sum_{x \neq 0} (x^2 + x + 1) \frac{q+1}{2}
\]

\[
= -1 + 2 \sum_{x} x \left( \left( x + \frac{1}{2} \right)^2 + \frac{3}{4} \right) \frac{q-1}{2} + \sum_{x} \left( \left( x + \frac{1}{2} \right)^2 + \frac{3}{4} \right) \frac{q-1}{2}
\]

\[
= -1 + 2 \sum_{x} \left( x - \frac{1}{2} \right) \left( x^2 + \frac{3}{4} \right) \frac{q-1}{2} + \sum_{x} \left( x^2 + \frac{3}{4} \right) \frac{q-1}{2} = -1.
\]

Hence we obtain

\[
\sum_{i=0}^{q-2} t_i = -1. \tag{2.3}
\]

Next we turn to \( \sum_{i=0}^{q-2} (-1)^i t_i \).

\[
\sum_{i=0}^{q-2} (-1)^i t_i = \sum_{x \neq 0} \frac{1}{x} \cdot (x^2 + x + 1) \cdot \chi \left( \frac{1}{x} \right) \chi (x^2 + x + 1)
\]
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\[ = \sum_{x \neq 0} \left( x + \frac{1}{x} + 1 \right)^{\frac{q+1}{2}} = -T_{\frac{q+1}{2}}. \]

The last equality follows from Lemma 2.2. We therefore obtain

\[ \sum_{i=0}^{q-2} (-1)^i t_i = -T_{\frac{q+1}{2}}. \] (2.4)

Combining (2.3) and (2.4) with (2.2), we see that \( \det S_q = T_{\frac{q+1}{2}} \cdot u^2 \) for some \( u \in \mathbb{F}_q \).

This completes the proof. \( \square \)

3. PROOF OF THEOREM 1.2

We begin with the following known result (see [4, Lemma 10]).

**Lemma 3.3.** Let \( R \) be a commutative ring and let \( n \) be a positive integer. For any polynomial \( P(T) = p_{n-1}T^{n-1} + \cdots + p_1T + p_0 \in R[T] \) we have

\[ \det [P(X_iY_j)]_{1 \leq i,j \leq n} = \prod_{i=0}^{n-1} p_i \prod_{1 \leq i<j \leq n} (X_j - X_i) (Y_j - Y_i). \]

We also need the following lemma.

**Lemma 3.4.** Let \( q \) be an odd prime. Then for any non-zero element \( a \in \mathbb{F}_q \) we have

\[ (a^2 + a + 1)^{\frac{q+1}{2}} = f(a), \]

where

\[ f(T) = \frac{11}{8} + T + \frac{11}{8} T^2 + \sum_{k=-(q-5)/2}^{(q-5)/2} \left( \frac{q+1}{2} \right)_2 T^{k+\frac{q+1}{2}} \] (3.5)

is a polynomial over \( \mathbb{F}_q \).

**Proof.** As \( a \neq 0 \), we have \( a^{q+k} = a^{k+1} \) for any integer \( k \). Using this and \( \left( \begin{array}{c} n \\ k \end{array} \right)_2 = \left( \begin{array}{c} n \\ -k \end{array} \right)_2 \), one can verify that \( (a^2 + a + 1)^{\frac{q+1}{2}} \) is equal to

\[ \sum_{k=-(q-5)/2}^{(q-5)/2} \left( \frac{q+1}{2} \right)_2 a^{k+\frac{q+1}{2}} + \left( \frac{q+1}{2} \right)_2 \left( \frac{q+1}{2} \right)_2 \left( \frac{q+1}{2} \right)_2 (1 + a^2) + \left( \frac{q+1}{2} \right)_2 \left( \frac{q+1}{2} \right)_2 \left( \frac{q+1}{2} \right)_2 \right) a \]

\[ = \frac{11}{8} + a + \frac{11}{8} a^2 + \sum_{k=-(q-5)/2}^{(q-5)/2} \left( \frac{q+1}{2} \right)_2 a^{k+\frac{q+1}{2}}. \]

The last equality follows from (below the trinomial coefficient \( \left( \begin{array}{c} n \\ k \end{array} \right)_2 \) is viewed as an element of \( \mathbb{F}_q \)).

\[ \left( \frac{q+1}{2} \right)_2 = 1, \left( \frac{q+1}{2} \right)_2 = q + 1 = \frac{1}{2}, \left( \frac{q+1}{2} \right)_2 = \frac{1}{2}, q + \frac{1}{2}, q + \frac{3}{2} = \frac{3}{8}. \]

This completes the proof. \( \square \)
Now we are in a position to prove our last result.

**Proof of Theorem 1.2** By Lemma 3.4 one can verify that

\[
\det S_q = \prod_{i=1}^{q-1} a_i^{q+1} \cdot \det \left[ \left( \left( \frac{a_j}{a_i} \right)^2 + \frac{a_j}{a_i} + 1 \right)^{\frac{q+1}{2}} \right]_{1 \leq i, j \leq q-1}
\]

\[
= \det \left[ \left( \left( \frac{a_j}{a_i} \right)^2 + \frac{a_j}{a_i} + 1 \right)^{\frac{q+1}{2}} \right]_{1 \leq i, j \leq q-1}
\]

\[
= \det \left[ f \left( \frac{a_j}{a_i} \right) \right]_{1 \leq i, j \leq q-1},
\]

where \( f \) is defined by (3.5).

Now applying Lemma 3.3 we obtain that \( \det S_q \) is equal to

\[
\det \left[ f \left( \frac{a_j}{a_i} \right) \right]_{1 \leq i, j \leq q-1} = 12 \frac{61}{64} \cdot \left( \frac{q+1}{2} \right)^2 \cdot \prod_{k=1}^{(p-5)/2} \left( \frac{q+1}{k} \right)^2 \cdot \prod_{1 \leq i < j \leq q-1} (a_j - a_i) \left( \frac{1}{a_j} - \frac{1}{a_i} \right). \tag{3.6}
\]

By [12, Eq. (3.3)] we further have

\[
\prod_{1 \leq i < j \leq q-1} (a_j - a_i) \left( \frac{1}{a_j} - \frac{1}{a_i} \right) = 1. \tag{3.7}
\]

Hence by (3.6) and (3.7) we obtain

\[
\det S_q = 12 \frac{61}{64} \cdot \left( \frac{q+1}{2} \right)^2 \cdot \prod_{k=1}^{(p-5)/2} \left( \frac{q+1}{k} \right)^2. \tag{3.8}
\]

As \( q > 5 \) and \((q, 22) = 1\), by (3.8) we see that

\[
\det S_q = 0 \iff \left( \frac{q+1}{k} \right)_2 \equiv 0 \pmod{p} \text{ for some } 0 \leq k \leq (p-5)/2,
\]

where \( p \) is the characteristic of \( \mathbb{F}_q \). This completes the proof. \( \square \)

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