THE KOSZUL PROPERTY FOR GRADED TWISTED TENSOR PRODUCTS

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Abstract. Let $K$ be a field. Let $A$ and $B$ be connected $\mathbb{N}$-graded $K$-algebras. Let $C$ denote a twisted tensor product of $A$ and $B$ in the category of connected $\mathbb{N}$-graded $K$-algebras. The purpose of this paper is to understand when $C$ possesses the Koszul property, and related questions. We prove that if $A$ and $B$ are quadratic, then $C$ is quadratic if and only if the associated graded twisting map has a property we call the unique extension property. We show that $A$ and $B$ being Koszul does not imply $C$ is Koszul (or even quadratic), and we establish sufficient conditions under which $C$ is Koszul whenever both $A$ and $B$ are. We analyze the unique extension property and the Koszul property in detail in the case where $A = K[x]$ and $B = K[y]$.

1. Introduction

Though the study of factorization structures in mathematics has a long history, there has been much recent interest in very general questions about the dual notions of product and factorization for associative algebras over a field. Čap, Schichl, and Vanžura introduced in [13] a very general notion of product for a pair of associative algebras, or equivalently, a notion of factorization, called the twisted tensor product. This product is analogous to the Zappa-Szép product for groups, see [4] for example, and the bicrossed product for Hopf algebras, [1]. Commutative tensor products, Ore extensions, and smash products of algebras can all be realized as particular cases of twisted tensor products. Homological and ring-theoretic properties of these particular cases are well-studied, and a number of recent papers establish such

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properties of twisted tensor products in the graded setting; see [9], [12], [14] for example.

One particularly important homological property of a graded algebra is the Koszul property, introduced by Priddy in [11]. Let $K$ be a field. Let $A$ be an $\mathbb{N}$-graded $K$-algebra. We further assume $A$ is connected ($\dim_K A_0 = 1$) and locally finite dimensional ($\dim_K A_i < \infty$ for all $i \geq 0$). For a $K$-vector space $V$, let $T(V)$ denote the tensor algebra generated by $V$.

The graded algebra $A$ is called one-generated if the canonical multiplication map $\pi : T(A_1) \to A$ is surjective. Let $I = \langle \ker \pi \cap A_1 \otimes A_1 \rangle$ be the ideal of $T(A_1)$ generated by the degree 2 elements of the kernel of $\pi$. If $A$ is one-generated, the quadratic part of $A$ is the algebra $q(A) = T(A_1)/I$. If $A \cong q(A)$ then $A$ is called quadratic.

The graded algebra $A$ is called Koszul if the trivial module $K = A_0 = A/A_+$ admits a resolution

$$\cdots \to P_3 \to P_2 \to P_1 \to P_0 \to K \to 0$$

such that each $P_i$ is a graded free left $A$-module generated in degree $i$.

Two well-known facts follow immediately from this definition:

1. every Koszul algebra is quadratic,
2. every one-generated, free algebra is Koszul.

Many important quadratic algebras arising naturally in mathematics are Koszul, and Koszul algebras satisfy a powerful duality property that sometimes underlies deep connections between apparently unrelated problems. One famous example is the duality between the symmetric algebra and the exterior algebra underlying the so-called BGG correspondence [2]. For additional motivation, background, and examples of Koszul algebras, we encourage the interested reader to consult the book [10], which remains the most comprehensive and useful treatment of the theory of quadratic and Koszul algebras we have read.

Determining whether or not a given quadratic algebra has the Koszul property can be quite difficult. Given the nice homological properties possessed by Koszul algebras, it is therefore natural to study the degree to which the Koszul property is preserved under notions of factorization and product. Our interest in this problem arose from questions about Koszulity of certain smash products of graded Hopf algebras in [5]. The purpose of the current paper is not to answer questions raised in [5], but rather to address some basic questions about the behavior of the Koszul property with regard to twisted tensor products in the category of connected graded algebras.

A twisted tensor product of $K$-algebras $A$ and $B$ can be fruitfully studied by considering an associated $K$-linear twisting map $\tau : B \otimes A \to A \otimes B$. The twisted tensor product algebra associated to $\tau$ is denoted $A \otimes_{\tau} B$. In the classical case of a commutative tensor product, $\tau(b \otimes a) = a \otimes b$. See Section 2 for more details. If $A$ and $B$ are graded, the $K$-linear map $\tau$ must be a map of graded vector spaces. This raises the question of how the space $A \otimes B$ is graded. If $A$ and $B$ are $\mathbb{N}$-graded, then $A \otimes B$ is $\mathbb{N} \times \mathbb{N}$ graded in the obvious way, but $A \otimes B$ is also $\mathbb{N}$-graded by the Künneth grading: $(A \otimes B)_i = \bigoplus_{p+q=i} A_p \otimes B_q$. The commutative tensor product obviously fits either choice of grading.

The existing literature on graded twisting maps all but exclusively deals with the very restrictive $\mathbb{N} \times \mathbb{N}$ grading, assuming $\tau(B_j \otimes A_i) = A_j \otimes B_i$, which further
implies \( \tau \) is invertible. This enables one to prove theorems that closely parallel those for commutative tensor products. However, this restriction also excludes many algebras of interest, including Ore extensions with nontrivial derivations and many smash products of algebras. Indeed, assuming \( A \) and \( B \) are augmented \( \mathbb{K} \)-algebras, a twisting map \( \tau : B \otimes A \to A \otimes B \) induces the structure of a left \( B \)-module on \( A \) via \((1_A \otimes \epsilon_B)\tau\) and that of a right \( A \)-module on \( B \) via \((\epsilon_A \otimes 1_B)\tau\), where \( \epsilon_A \) and \( \epsilon_B \) are the augmentation maps on \( A \) and \( B \), respectively. Requiring \( \tau \) to preserve the \( \mathbb{N} \times \mathbb{N} \) grading forces both of these induced actions to be trivial, leading readily to the following theorem.

**Theorem 1.1** ([14, Theorem 4.18], [14, Proposition 1.8]). Suppose that \( A \) and \( B \) are Koszul algebras. Let \( \tau : B \otimes A \to A \otimes B \) be an invertible graded twisting map such that \( \tau(B_i \otimes A_j) = A_j \otimes B_i \) for all \( i, j \). Then the twisted tensor product algebra \( A \otimes_{\tau} B \) is Koszul.

It is well known that graded Ore extensions of Koszul algebras are Koszul, indicating this theorem holds more generally. On the other hand, there are simple examples showing that a graded twisted tensor product of Koszul algebras need not be Koszul (see Example 5.3). Thus, in this paper, we study the most general type of graded twisting map \( \tau : B \otimes A \to A \otimes B \). We do not insist that \( \tau \) be invertible, and we only require the Künneth grading to be preserved: \( \tau(B \otimes A)_n \subseteq (A \otimes B)_n \) for all \( n \geq 0 \). We provide answers to the following questions.

1. When are graded twisted tensor products of quadratic algebras quadratic?
2. When are graded twisted tensor products of Koszul algebras Koszul?

Here is an outline of the paper. In Section 2 we recall background, make relevant definitions and extensions of the results in [13] and [3] to the graded setting, and discuss the use of filtrations for later use in proving the Koszul property.

Section 3 is concerned with existence and uniqueness of graded twisting maps. To define a graded twisting map, it is often useful to work inductively. In Lemma 3.1 we characterize when a graded map which is twisting to degree \( n \) may be extended (uniquely) to a graded map which is twisting to degree \( n + 1 \). Theorem 3.2 gives very useful and, in practice, checkable sufficient conditions for when a graded map defined to degree \( n + 1 \) which is twisting to degree \( n \) is also twisting to degree \( n + 1 \).

In the last part of Section 3 we construct a new large class of graded twisting maps \( \tau : B \otimes A \to A \otimes B \) where \( A \) and \( B \) are free algebras of arbitrary finite rank, then in Theorem 3.7 we determine conditions for when a graded twisting map defined on free algebras induces a graded twisting map on algebras with relations.

In Section 4 we determine when a twisted tensor product of quadratic algebras is quadratic. If \( A \) and \( B \) are \( \mathbb{N} \)-graded \( \mathbb{K} \)-algebras, we say a graded twisting map \( \tau \) has the unique extension property if for all graded twisting maps \( \tau' : B \otimes A \to A \otimes B \) and all \( n \in \mathbb{N} \) such that \( \tau_i' = \tau_i \) for \( i < n \), then \( \tau'_n = \tau_n \). Our main result in this section is Theorem 3.13 which has the following corollary.

**Theorem 1.2** (Corollary 3.14). If \( A \) and \( B \) are quadratic, then \( \tau \) has the unique extension property if and only if \( A \otimes_{\tau} B \) is quadratic.

In Section 5 we consider the Koszul property for graded twisted tensor products. We start by proving, Theorem 5.3 that the twisted tensor product of Koszul algebras is Koszul for one-sided twisting maps (if \( \tau \) is one-sided, then one of the component algebras of the twisted tensor product is normal). We also prove, Proposition 5.5 that for an arbitrary graded twisting map \( \tau \), that \( A \otimes_{\tau} B \) is Koszul if
A and B are free algebras, and $A \otimes_\tau B$ is quadratic. Example 6.4 shows that the graded twisted tensor product of Koszul algebras need not be Koszul. The remainder of Section 5 is concerned with the introduction of a large class of two-sided twisting maps which we call separable. We determine, in Theorem 5.10 sufficient conditions to ensure that $A \otimes_\tau B$ is Koszul when $\tau$ is separable.

**Theorem 1.3.** Let $A$ and $B$ be quadratic algebras. Let $\tau : B \otimes A \to A \otimes B$ be a separable graded twisting map. Assume $\tau$ satisfies

1. $\tau_{A_0 \otimes A_2 \otimes B_1} (1 \otimes \tau)(\tau_0 \otimes 1)(B_1 \otimes I_2) = 0$, and
2. $\tau_{A_1 \otimes B_2 \otimes B_0} (\tau \otimes 1)(1 \otimes \tau_0)(J_2 \otimes A_1) = 0$.

Let $F$ denote the filtration on $A \otimes_\tau B$ defined prior to Lemma 5.7 and let $F^B$ denote its restriction to the subalgebra $B$. Let $\tilde{B} = gr^F(B)$. If $A$ is Koszul, the quadratic part of $\tilde{B}$ is Koszul and $\tilde{B}$ has no defining relations in degree 3, then $A \otimes_\tau B$ is Koszul.

This theorem has the following immediate corollary.

**Corollary 1.4.** Let $A$ be a Koszul algebra with quadratic relation space $I_2$, let $B$ be a free algebra, and suppose $\tau : B \otimes A \to A \otimes B$ is a separable graded twisting map. Assume that $\tau_{A_0 \otimes A_2 \otimes B_1} (1 \otimes \tau)(\tau_0 \otimes 1)(B_1 \otimes I_2) = 0$. Then $A \otimes_\tau B$ is Koszul.

In Section 6 we analyze twisting maps $\tau : B \otimes A \to A \otimes B$ where $A = \mathbb{K}[x]$ and $B = \mathbb{K}[y]$. Our work here is complemented by the work of Guccione, Guccione and Valqui in [8]. We prove the following theorem.

**Theorem 1.5.** Suppose that $\tau : B \otimes A \to A \otimes B$ is a graded twisting map and write $\tau(y \otimes x) = ax^2 \otimes 1 + bx \otimes y + 1 \otimes cy^2$. If $1 - ac \neq 0$ and $b \neq 0$ or $c \neq 0$, then $\tau$ has the unique extension property and $A \otimes_\tau B$ is quadratic, hence Koszul.

We also consider when the definition $\tau(y \otimes x) = x^2 \otimes 1 + 1 \otimes cy^2$ extends to a graded twisting map, culminating in a complete answer in Theorem 6.5 and Proposition 6.6. Our characterization exhibits a curious connection to the Catalan numbers.

Finally in Section 7 we consider three examples that illustrate the main theorems of the paper. Most importantly Example 7.4 is an example of a twisted tensor product that arises from a separable twisting map that is not isomorphic to any twisted tensor product coming from a one-sided twisting map.

2. Preliminaries

As discussed above, this paper explores the transfer of the Koszul property between a twisted tensor product and its component subalgebras, which need not be normal. In this section we recall the relevant background.

2.1. Twisted products and twisting maps. Throughout the paper, let $\mathbb{K}$ denote a field. Tensor products taken with respect to $\mathbb{K}$ are denoted by $\otimes$. If $V$ is a $\mathbb{K}$-vector space, we write $1_V$ for the identity map on $V$; if $V$ is a unital $\mathbb{K}$-algebra we will abuse notation and also write $1_V \in V$ for the identity element.

Let $A$ and $B$ be (unital) $\mathbb{K}$-algebras with multiplication maps $\mu_A$ and $\mu_B$. Following [13], an internal twisted tensor product of $A$ and $B$ is a triple $(C, i_A, i_B)$ where $C$ is a $\mathbb{K}$-algebra and $i_A : A \to C$ and $i_B : B \to C$ are injective $\mathbb{K}$-algebra homomorphisms such that the $\mathbb{K}$-linear map $A \otimes B \to C$ given by $a \otimes b \mapsto i_A(a) i_B(b)$ is an isomorphism of $\mathbb{K}$-vector spaces. We say two internal twisted tensor products
(C, i_A, i_B) and (C', i'_A, i'_B) of (graded) K-algebras A and B are isomorphic if there exist (graded) algebra homomorphisms α : A → A', β : B → B', and γ : C → C' such that γi_A = i'_A α, γi_B = i'_B β, and γ is an isomorphism.

We call a K-linear map τ : B ⊗ A → A ⊗ B an algebra twisting map if τ(1_B ⊗ a) = a ⊗ 1_B and τ(b ⊗ 1_A) = 1_A ⊗ b for all a ∈ A and b ∈ B and if

τ(µ_B ⊗ µ_A) = (µ_A ⊗ µ_B)(1_A ⊗ τ ⊗ 1_B)(τ ⊗ τ)(1_B ⊗ τ ⊗ 1_A).

This definition was introduced in [13]. We will refer to the conditions τ(1_B ⊗ a) = a ⊗ 1_B and τ(b ⊗ 1_A) = 1_A ⊗ b for all a ∈ A and b ∈ B as the unital twisting conditions. For any K-linear map τ : B ⊗ A → A ⊗ B we adopt Sweedler-type notation and write τ(b ⊗ a) = aτ ⊗ bτ.

If A and B carry a grading by the semigroup N, the K-linear tensor product A ⊗ B admits an N-grading by the Künneth formula

\[(A ⊗ B)_m = \bigoplus_{k+l=m} A_k ⊗ B_l.\]

More generally, if V and W are N-graded K-vector spaces, we grade V ⊗ W by the Künneth formula. Throughout this paper the term graded will mean N-graded. A K-linear map f : V → W is called graded if it preserves the N grading: f(V_i) ⊆ W_i for all i ≥ 0. If an algebra twisting map τ : B ⊗ A → A ⊗ B is graded we call it a graded algebra twisting map, or just graded twisting map. If C is a graded algebra, we write C_+ for \(\bigoplus_{i>0} C_i\).

If V and W are graded K-vector spaces and f : V → W is a graded K-linear map, we denote the degree n component of f by f_n and define f_{≤n} = \(\bigoplus_{i=0}^n f_i\) and f_{>n} = \(\bigoplus_{i>n} f_i\).

We say a graded K-linear map t : (B ⊗ A)_{≤n} → (A ⊗ B)_{≤n} is graded twisting in degree n if t(1_B ⊗ a) = (a ⊗ 1_B) and t(b ⊗ 1_A) = (1_A ⊗ b) for all a ∈ A_n and b ∈ B_n and

\[t_n(µ_B ⊗ µ_A) = (µ_A ⊗ µ_B)(1_A ⊗ t_{≤n} ⊗ 1_B)(t_{≤n} ⊗ t_{≤n})(1_B ⊗ t_{≤n} ⊗ 1_A)\]
as maps defined on (B ⊗ B ⊗ A ⊗ A)_{n}.

If t is graded twisting in degree i for all i ≤ n we say t is graded twisting to degree n. Evidently, a K-linear map τ : B ⊗ A → A ⊗ B is a graded twisting map if and only if the restriction of τ to (B ⊗ A)_{≤n} is graded twisting to degree n for all n ≥ 0 (which is equivalent to graded twisting in degree n for all n ≥ 0). Throughout the paper, we adopt the convention of using τ to describe potential graded twisting maps, that is, graded K-linear maps of the form B ⊗ A → A ⊗ B. We use t to indicate a graded K-linear map defined only on (B ⊗ A)_{≤n} → (A ⊗ B)_{≤n}, which is potentially graded twisting to degree n.

**Remark 2.1.** Let A and B be K-algebras and τ : B ⊗ A → A ⊗ B a K-linear map. As noted in [13] (Remark 2.4(1)) τ is a twisting map if and only if the following identities hold.

\[τ(1_B ⊗ µ_A) = (µ_A ⊗ 1_B)(1_A ⊗ τ)(τ ⊗ 1_A)\]
\[τ(µ_B ⊗ 1_A) = (1_A ⊗ µ_B)(τ ⊗ 1_B)(1_B ⊗ τ)\]

The reasoning in [13] applied to homogeneous components of τ in the graded setting shows that t : (B ⊗ A)_{≤n} → (A ⊗ B)_{≤n} is graded twisting to degree n if and only
if \( t(1_B \otimes a) = (a \otimes 1_B) \) and \( t(b \otimes 1_A) = (1_A \otimes b) \) for all \( a \in A_{\leq n} \) and \( b \in B_{\leq n} \), and for all \( j \leq n \),
\[
  t_j(1_B \otimes \mu_A) = (\mu_A \otimes 1_B)(1_A \otimes t_j)(t_{\leq j} \otimes 1_A) \\
  t_j(\mu_B \otimes 1_A) = (1_A \otimes \mu_B)(t_{\leq j} \otimes 1_B)(1_B \otimes t_j).
\]

For later use we record the following proposition. The easy proof is left to the reader.

**Proposition 2.2.** Let \( A \) and \( B \) be graded \( \mathbb{K} \)-algebras, and let \( \alpha : A \to A \) and \( \beta : B \to B \) be graded automorphisms.

Suppose that \( t : (B \otimes A)_{\leq n} \to (A \otimes B)_{\leq n} \) is a \( \mathbb{K} \)-linear map that is graded twisting to degree \( n \). Define \( t' : (B \otimes A)_{\leq n} \to (A \otimes B)_{\leq n} \) by \( t' = (\alpha \otimes \beta)t(\beta^{-1} \otimes \alpha^{-1})|_{(B \otimes A)_{\leq n}} \). Then \( t' \) is graded twisting to degree \( n \). Moreover, there exists a unique \( \mathbb{K} \)-linear extension of \( t, t : (B \otimes A)_{\leq n+1} \to (A \otimes B)_{\leq n+1} \) that is graded twisting to degree \( n+1 \) if and only if there exists a unique \( \mathbb{K} \)-linear extension of \( t', \tilde{t} : (B \otimes A)_{\leq n+1} \to (A \otimes B)_{\leq n+1} \) that is graded twisting to degree \( n+1 \).

If \( \tau : B \otimes A \to A \otimes B \) is a graded twisting map, then \( \tau' : B \otimes A \to A \otimes B \) defined by \( \tau' = (\alpha \otimes \beta)t(\beta^{-1} \otimes \alpha^{-1}) \) is a graded twisting map. Furthermore, the algebras \( A \otimes_{\tau} B \) and \( A \otimes_{\tau'} B \) are isomorphic as twisted tensor products of \( A \) and \( B \).

The relationship between twisting maps and twisted products was established for \( \mathbb{K} \)-algebras in [13]. Here we add graded and truncated algebra versions of the theorem as well.

**Proposition 2.3.** Let \( A \) and \( B \) be (graded) algebras. Let \( \tau : B \otimes A \to A \otimes B \) be a (graded) \( \mathbb{K} \)-linear map. Define a map \( \mu_\tau : A \otimes B \otimes A \otimes B \to A \otimes B \) by \( \mu_\tau = (\mu_A \otimes \mu_B)|_{(A \otimes B)_{\leq n}} \).

1. The map \( \mu : B \otimes A \to A \otimes B \) is a (graded) algebra twisting map if and only if \( \mu_\tau \) defines an associative multiplication giving \( A \otimes B \) the structure of a (graded) algebra.
2. Assume that \( A \) and \( B \) are graded and \( \tau : B \otimes A \to A \otimes B \) is graded. Then \( \tau \) is twisting to degree \( n \) if and only if \( \mu_\tau \) induces an associative multiplication giving \( (A \otimes B)/(A \otimes B)_{>n} \) the structure of a graded algebra.

**Proof.** Statement (1) in the not necessarily graded case is Proposition 2.3 of [13]; for the graded case of (1), one follows the proof of Proposition 2.3 of [13] and makes the appropriate changes.

For (2) we set \( C = A \otimes B/(A \otimes B)_{>n} \). The map \( \mu_\tau \) is graded, so, in particular,
\[
  \mu_\tau((A \otimes B)_{>n} \otimes (A \otimes B)_{>n}) \subseteq (A \otimes B)_{>n},
\]
hence we get a canonically defined graded \( \mathbb{K} \)-linear map \( \mu_\tau : C \otimes C \to C \). We identify \( C \) with \( (A \otimes B)_{\leq n} \) in the obvious way. Then note that under this identification \( \mu_\tau_{>n} = 0 \) and
\[
  \mu_\tau_{\leq n} = (\mu_A \otimes \mu_B)(1_A \otimes \tau_{\leq n} \otimes 1_B).
\]
Now, to prove that \( \mu_\tau \) is associative if and only if \( \tau \) is twisting to degree \( n \), one simply follows the proof of Proposition 2.3 in [13]. \( \square \)

We denote the algebra determined by \( \tau \) in Proposition 2.3 (1) by \( A \otimes_\tau B \). We call \( A \otimes_\tau B \) the **external twisted tensor product** of \( A \) and \( B \).
A given pair of $\mathbb{K}$-algebras may have non-isomorphic external twisted products, and a given $\mathbb{K}$-algebra may be expressed as an internal twisted product in more than one way.

**Proposition 2.4.** Let $(C, i_A, i_B)$ be a (graded) twisted tensor product of (graded) $\mathbb{K}$-algebras $A$ and $B$. Then there is a unique (graded) twisting map $\tau$ such that $C$ is isomorphic to $A \otimes B$ as a (graded) twisted tensor product.

**Proof.** The ungraded version of this statement is Proposition 2.7 in [13]. The proof in the graded case is entirely analogous. $\square$

If $A$ and $B$ are graded, a graded twisting map $\tau$ satisfies

$$\tau(B_+ \otimes A_+) \subseteq (A_+ \otimes B_0) \oplus (A_+ \otimes B_+) \oplus (A_0 \otimes B_+).$$

We will see in Section 5 that a few special cases are important to distinguish. If

$$\tau(B_+ \otimes A_+) \subseteq (A_+ \otimes B_0) \oplus (A_+ \otimes B_+)$$

or

$$\tau(B_+ \otimes A_+) \subseteq (A_+ \otimes B_+) \oplus (A_0 \otimes B_+)$$

we call the graded twisting map $\tau$ one-sided; if $\tau(B_+ \otimes A_+) \subseteq (A_+ \otimes B_+)$ we call $\tau$ pure; if $\tau(B_i \otimes A_j) \subseteq A_j \otimes B_i$ for all $i, j \geq 0$, then we call $\tau$ strongly graded.

In our experience, graded twisting maps may behave rather badly in general. For example, from a homological point of view, in general, one cannot naturally form a projective resolution of $A \otimes B$ from projective resolutions of $A$ and $B$, unless one uses bar complexes, see [7]. The class of one-sided twisting maps is much nicer; and pure and strongly graded twisting maps are nicer still. It is the last case which is most often encountered in the literature; [9], [12], [14]. We implore the reader to bear in mind:

*Many references to graded twisting maps in the literature in fact refer to the more restrictive cases of one-sided, pure, or strongly graded twisting maps.*

In light of this potential for confusion, when we wish to emphasize that a statement about graded twisting maps applies in full generality, we refer to the twisting map as two-sided.

In [3], the authors identify the twisted tensor product $A \otimes \tau B$ with a certain quotient of the free product algebra, $A \ast B$. As a vector space

$$A \ast B = \bigoplus_{i \geq 0; x_1, x_2 \in \{0, 1\}} A_{x_1}^i \otimes (B_+ \otimes A_+)^{\otimes i} \otimes B^*_+.$$  

We note that $A \ast B$ is $\mathbb{N}$-graded with the usual Künneth grading.

We will need a slight generalization of the graded version of their result. We refer to an element $b \otimes a - \tau(b \otimes a) \in A \ast B$ as a $\tau$-relation.

For a graded $\mathbb{K}$-linear map $\tau : B \otimes A \rightarrow A \otimes B$ and $n \in \mathbb{N}$, we let $I^n_\tau$ be the ideal of $A \ast B$ generated by $\tau$-relations of degree at most $n$. That is,

$$I^n_\tau = \langle b \otimes a - \tau(a \otimes b) \mid a \in A_i, b \in B_j, i + j \leq n \rangle.$$

The ideal generated by all $\tau$-relations is

$$I_\tau = \langle b \otimes a - \tau(a \otimes b) \mid a \in A, b \in B \rangle.$$
Proposition 2.5. Let $A$ and $B$ be graded, unital $\mathbb{K}$-algebras. If the $\mathbb{K}$-linear map $\tau: B \otimes A \to A \otimes B$ is graded twisting to degree $n$, then there is a $\mathbb{K}$-algebra homomorphism $\varphi: (A \ast B)/I^\tau_n \to (A \otimes B)/(A \otimes B)_{>n}$, where the algebra structure on $(A \otimes B)/(A \otimes B)_{>n}$ is given by Proposition 2.3(2). The map $\varphi$ is an isomorphism in degrees $\leq n$.

If $\tau$ is a graded twisting map, then $A \otimes_\tau B \cong (A \ast B)/I_\tau$.

Proof. First assume that $\tau$ is a graded twisting map. Using the universal property of the free product there exists a surjective graded algebra homomorphism

$$\pi: A \ast B \to A \otimes_\tau B.$$ 

It is clear that the ideal $I_\tau$ is contained in ker $\pi$. Moreover, it is evident from the defining generators of $I_\tau$ that in each graded component

$$\dim((A \ast B)/I_\tau)_i \leq \dim(A \otimes_\tau B)_i.$$ 

Therefore we have ker $\pi = I_\tau$ and $A \otimes_\tau B \cong (A \ast B)/I_\tau$.

Now suppose that $\tau: B \otimes A \to A \otimes B$ is graded twisting to degree $n$. From Proposition 2.3(2), $(A \otimes B)/(A \otimes B)_{>n}$ is canonically an associative algebra. There are obvious algebra maps $A \to (A \otimes B)/(A \otimes B)_{>n}$ and $B \to (A \otimes B)/(A \otimes B)_{>n}$ so the universal property of $A \ast B$ affords a surjective algebra homomorphism $A \ast B \to (A \otimes B)/(A \otimes B)_{>n}$ whose kernel contains $I^\tau_n$. Comparing dimensions of graded components in degrees $\leq n$ we see there is an induced map

$$\varphi: (A \ast B)/I^\tau_n \to (A \otimes B)/(A \otimes B)_{>n}$$

which is an isomorphism in degrees $\leq n$. \hfill $\square$

2.2. Quadratic and Koszul algebras. Recall that we assume $A$ is an $N$-graded $\mathbb{K}$-algebra that is connected and locally finite dimensional. As mentioned above, the graded algebra $A$ is called Koszul if the trivial module $\mathbb{K} = A_0 = A/A_+$ admits a resolution

$$\cdots \to P_3 \to P_2 \to P_1 \to P_0 \to \mathbb{K} \to 0$$

such that each $P_i$ is a graded free left $A$-module generated in degree $i$.

There are several formulations of the notion of Koszul algebra that are equivalent to the definition above, and these lead to a multitude of techniques for proving an algebra is Koszul. One very useful approach makes use of filtrations.

Let $\Gamma$ be a graded ordered semigroup. By definition, $\Gamma$ is a semigroup with unit $e$ endowed with a semigroup map $g: \Gamma \to N$ such that the fibers $\Gamma_n = g^{-1}(n)$ are totally ordered with their orders satisfying:

for all $\alpha, \beta \in \Gamma_k$ and $\gamma \in \Gamma_l$, $\alpha < \beta \implies \alpha \gamma < \beta \gamma$ and $\gamma \alpha < \gamma \beta$.

We also assume $g^{-1}(0) = \{e\}$.

A $\Gamma$-valued filtration on a graded algebra $A$ is a collection of subspaces $F_\alpha A_n \subseteq A_n$ for all $n \geq 0$ and $\alpha \in \Gamma_n$ such that

- $F_\alpha A_n \subseteq F_\beta A_m$ whenever $\alpha \leq \beta$,
- if $\gamma \in \Gamma_n$ is maximal, then $F_\gamma A_n = A_n$,
- $F_\alpha A_n \cdot F_\beta A_m \subseteq F_{\alpha \beta} A_{n+m}$.
The associated $\Gamma$-graded algebra is $\text{gr}^F A = \bigoplus_{\alpha \in \Gamma} (F_\alpha A_n / (\sum_{\alpha' < \alpha} F_{\alpha'} A_n))$. The filtration $F$ is called one-generated if

$$F_\alpha A_n = \sum_{i_1 + \cdots + i_k \leq \alpha} F_{i_1} A_1 \cdot F_{i_2} A_1 \cdots F_{i_k} A_1$$

Observe that $\text{gr}^F A$ is one-generated if and only if $F$ is. The next theorem illustrates the usefulness of filtrations for studying Koszul algebras.

**Theorem 2.6** ([10], Theorem 7.1, p. 89). Let $A$ be a quadratic algebra equipped with a one-generated filtration $F$ with values in a graded ordered semigroup $\Gamma$. Assume the associated $\Gamma$-graded algebra $\text{gr}^F A$ satisfies the following conditions

1. $\text{gr}^F A$ has no defining relations of degree 3;
2. $q(\text{gr}^F A)$ is Koszul

Then $\text{gr}^F A$ is quadratic (hence Koszul) and $A$ is Koszul.

The theorem is particularly useful in the case of a normal subalgebra.

**Corollary 2.7** ([10], Example 2, p. 90). Let $f : A \to C$ be a homomorphism of quadratic algebras such that $C_{1} f(A_1) \subset f(A) C_{1}$. Assume that $A$ and $C/f(A)C$ are Koszul algebras and the left action of $A$ on $C$ is free in degrees $\leq 3$. Then $C$ is a Koszul algebra.

### 3. Existence of Graded Twisting Maps

In the graded setting, it is often convenient to define graded twisting maps inductively. We now describe the inductive step in this process, which we refer to as the **twisting map extension problem**. Throughout this section let $A$ and $B$ denote connected, graded $\mathbb{K}$-algebras.

Let $t : (B \otimes A)_{\leq n} \to (A \otimes B)_{\leq n}$ be a $\mathbb{K}$-linear map that is graded twisting to degree $n$. Let $t^A = \pi^A \circ t$, where $\pi^A$ is the composition of canonical maps

$$(A \otimes B)_{\leq n} \to \bigoplus_{i \leq n} A_i \otimes B_0 \to A_{\leq n}.$$

Analogously, let $\pi^B$ be the composition

$$(A \otimes B)_{\leq n} \to \bigoplus_{i \leq n} A_0 \otimes B_i \to B_{\leq n}$$

and put $t^B = \pi^B \circ t$. Let $i_A : A \to A \otimes B$ and $i_B : B \to A \otimes B$ be the canonical inclusions.

Let $\delta : (B_+ \otimes A_+)_{\leq n+1} + (B_+ \otimes B_+ \otimes A_+)_{\leq n+1} \to (B_+ \otimes A_+)_{\leq n+1}$ be the $\mathbb{K}$-linear map defined by

$$\delta = (1_{B} \otimes \mu_{A} - t^B \otimes 1_{A}) + (\mu_{B} \otimes 1_{A} - 1_{B} \otimes t^A)$$

and let $R : (B_+ \otimes A_+ \otimes A_+)_{\leq n+1} + (B_+ \otimes B_+ \otimes A_+)_{\leq n+1} \to (A \otimes B)_{\leq n+1}$ be given by

$$R = (\mu_{A} \otimes 1_B)(1_A \otimes t)((t - i_B t^B) \otimes 1_A)$$

$$+ (1_A \otimes \mu_{B})(t \otimes 1_B)(1_B \otimes (t - i_A t^A))$$

It is important to note that in this context the compositions $(1_A \otimes t)((t - i_B t^B) \otimes 1_A)$ and $(t \otimes 1_B)(1_B \otimes (t - i_A t^A))$ are well-defined, but if $t$ is two-sided, $(1_A \otimes t)(t \otimes 1_A)$ and $(t \otimes 1_B)(1_B \otimes t)$ may not be.
Lemma 3.1 (Twisting Map Extension Problem). Let \( t : (B \otimes A)_{\leq n} \rightarrow (A \otimes B)_{\leq n} \) be a \( \mathbb{K} \)-linear map that is graded twisting to degree \( n \). There exists a \( \mathbb{K} \)-linear extension \( t' : (B \otimes A)_{\leq n+1} \rightarrow (A \otimes B)_{\leq n+1} \) of \( t \) that is graded twisting to degree \( n+1 \) if and only if there exists a \( \mathbb{K} \)-linear map \( f : (B_+ \otimes A_+)_{n+1} \rightarrow (A \otimes B)_{n+1} \) satisfying \( f_{n+1} = R_{n+1} \). If such an extension \( t' \) exists, it is unique if and only if \( \delta_{n+1} \) is surjective.

If \( \tau \) is a graded twisting map, then the twisting map extension problem has a solution for every \( n \). We will show shortly that non-uniqueness of these extensions is captured precisely by the minimal generators of \( I_\tau \).

Proof. First, suppose \( t' \) is any \( \mathbb{K} \)-linear extension of \( t \). Note that since \( i_B t_B \subseteq A_0 \otimes B \),

\[
(\mu_A \otimes 1_B)(1_A \otimes t')(i_B t_B \otimes 1_A) = t'(t_B \otimes 1_A)
\] (1).

Now assume that \( t' \) is graded twisting to degree \( n+1 \) and put \( f = t'|_{(B_+ \otimes A_+)_{n+1}} \).

By Remark 2.1

\[
f(1_B \otimes \mu_A - t^B \otimes 1_A)_{n+1} = (\mu_A \otimes 1_B)(1_A \otimes t')(i_B t_B \otimes 1_A)_{n+1} - t'(t_B \otimes 1_A)_{n+1}
\]

\[
= (\mu_A \otimes 1_B)(1_A \otimes t')(i_B t_B \otimes 1_A)_{n+1}
\]

On elements of \( (B_+ \otimes A_+ \otimes A_+)_{n+1} \), we have \( t' \otimes 1_A = t \otimes 1_A \). Furthermore,

\[
((t - i_B t^B) \otimes 1_A)(B_+ \otimes A_+ \otimes A_+)_{n+1} \subseteq (A_+ \otimes B \otimes A_+)_{n+1}.
\]

Thus,

\[
f(1_B \otimes \mu_A - t^B \otimes 1_A)_{n+1} = (\mu_A \otimes 1_B)(1_A \otimes t')(i_B t_B \otimes 1_A)_{n+1}
\]

\[
= (\mu_A \otimes 1_B)(1_A \otimes t)(i_B t_B \otimes 1_A)_{n+1}
\]

on \( (B_+ \otimes A_+ \otimes A_+)_{n+1} \). An analogous calculation shows

\[
f(\mu_B \otimes 1 - 1 \otimes t^A)_{n+1} = (1_A \otimes \mu_B)(t \otimes 1_B)(1_B \otimes (t - i_A t^A))_{n+1}
\]

on \( (B_+ \otimes B_+ \otimes A_+)_{n+1} \). Hence \( f_{n+1} = R_{n+1} \).

Now suppose there exists a \( \mathbb{K} \)-linear map \( f : (B_+ \otimes A_+)_{n+1} \rightarrow (A \otimes B)_{n+1} \) such that \( f_{n+1} = R_{n+1} \). Extend \( f \) to \( f' : (B \otimes A)_{n+1} \rightarrow (A \otimes B)_{n+1} \) by \( f'(1_B \otimes a) = a \otimes 1_B \) and \( f'(b \otimes 1_A) = 1_B \otimes b \) for all \( a \in A_{n+1} \) and \( b \in B_{n+1} \). Define \( t' = t \otimes f' \).

We must show \( t' \) is graded twisting to degree \( n+1 \). It suffices to show \( t' \) is graded twisting in degree \( n+1 \). We have,

\[
t'(1_B \otimes \mu_A)_{n+1} = f'(1_B \otimes \mu_A)_{n+1}
\]

\[
= t'(t_B \otimes 1_A) + (\mu_A \otimes 1_B)(1_A \otimes t)(t - i_B t_B) \otimes 1_A)_{n+1}
\]

\[
= (\mu_A \otimes 1_B)(1_A \otimes t')(i_B t_B \otimes 1_A) + (\mu_A \otimes 1_B)(1_A \otimes t')(t - i_B t_B) \otimes 1_A)_{n+1}
\]

\[
= (\mu_A \otimes 1_B)(1_A \otimes t')(t' \otimes 1_A)_{n+1}.
\]

The second equality follows from \( f_{n+1} = R_{n+1} \) and the third is given by (1) above. An analogous calculation shows

\[
t'(\mu_B \otimes 1_A)_{n+1} = (1_A \otimes \mu_B)(t' \otimes 1_B)(1_B \otimes t')_{n+1}.
\]

By Remark 2.1 this completes the proof of the existence part of the Lemma.

If \( \delta_{n+1} \) is surjective, let \( f' \) be another linear map satisfying \( f'_{n+1} = R_{n+1} \). Then for any \( x \in (B \otimes A)_{n+1} \), we have \( (f - f')(x) = (f - f')\delta(y) = (R - R)(y) = 0 \). So \( f \) is unique.
If $\delta_{n+1}$ is not surjective, let $W$ be a (nontrivial) complementary subspace of im $\delta_{n+1}$ in $(B \otimes A)_{n+1}$. Define $f' = f$ on im $\delta$ and freely define $f'$ on $W$ such that $f'|_W \neq f|_W$. Then $f'\delta_{n+1} = R$ but $f' \neq f$. \hfill \Box

The following simpler condition for extending a linear map that is graded twisting to some degree is often useful in practice.

**Theorem 3.2.** Let $A$ and $B$ be one-generated $\mathbb{K}$-algebras. Let $n$ be a positive integer. Suppose that $t : (B \otimes A)_{\leq n+1} \to (A \otimes B)_{\leq n+1}$ is a graded $\mathbb{K}$-linear map such that $t$ satisfies the unital conditions: $t(1 \otimes a) = a \otimes 1$ and $t(b \otimes 1) = 1 \otimes b$, and

1. $t|_{(B \otimes A)_{\leq n}}$ is a twisting map to degree $n$;
2. $t(1_B \otimes \mu_A) = (\mu_A \otimes 1_B)(1_A \otimes t)(t \otimes 1_A)$ on $(B \otimes A \otimes A)_{n+1}$;
3. $t(\mu_B \otimes 1_A) = (1_A \otimes \mu_B)(t \otimes 1_B)(1_B \otimes t)$ on $(B_1 \otimes B \otimes A)_{n+1}$.

Then $t$ is a twisting map to degree $n+1$.

**Proof.** Let $f = f|_{(B_+ \otimes A_+)_{n+1}}$. We must show $f\delta_{n+1} = R_{n+1}$. We first prove that $f(1_B \otimes \mu_A) = R_{n+1} + f(\mu_B \otimes 1_A)$.

Let $b \otimes a_1 \otimes a_2 \in (B_+ \otimes A_+ \otimes A_+)_{n+1}$, where $\deg(a_2) \geq 2$. Write $t(b \otimes a_1) = \sum_i A \otimes b_i$, where $l(b \otimes a_1) = (t - i_B t^B)(b \otimes a_1) \in A_+ \otimes B$ and $\sum_i A \otimes b_i = i_B t^B(b \otimes a_1)$.

Since $A$ is one-generated, we have $a_2 = \mu_A(\sum_j a_j' \otimes x_j)$ for some $x_j \in A_1$.

We compute

$$t(b \otimes a_1 a_2) = t(b \otimes a_1 \mu_A(\sum_j a_j' \otimes x_j))$$

$$= \sum_j t(b \otimes \mu_A(a_1 a_j' \otimes x_j))$$

$$= \sum_j t(1_B \otimes \mu_A)(b \otimes a_1 a_j' \otimes x_j)$$

$$= \sum_j (\mu_A \otimes 1_B)(1_A \otimes t)(t \otimes 1_A)(b \otimes a_1 a_j' \otimes x_j) \text{ using condition (2),}$$

$$= \sum_j (\mu_A \otimes 1_B)(1_A \otimes t)(t(b \otimes a_1 a_j') \otimes x_j)$$

$$= \sum_j (\mu_A \otimes 1_B)(1_A \otimes t)[(\mu_A \otimes 1_B)(1_A \otimes t)(t \otimes 1_A)(b \otimes a_1 a_j') \otimes x_j] \text{ using condition (1),}$$

$$= \sum_j (\mu_A \otimes 1_B)(1_A \otimes t)[(\mu_A \otimes 1_B)(1_A \otimes t)((l(b \otimes a_1) + \sum_1 b_i) \otimes a_j') \otimes x_j]$$

$$= \sum_j (\mu_A \otimes 1_B)(1_A \otimes t)[(\mu_A \otimes 1_B)(1_A \otimes t)((l(b \otimes a_1) \otimes a_j') \otimes x_j)]$$

$$+ \sum_j (\mu_A \otimes 1_B)(1_A \otimes t)(\sum_i (\mu_A \otimes 1_B)(1_A \otimes t)(1 \otimes b_i \otimes a_j') \otimes x_j).$$

We next make two claims.

$$\sum_j (\mu_A \otimes 1_B)(1_A \otimes t)[(\mu_A \otimes 1_B)(1_A \otimes t)((l(b \otimes a_1) \otimes a_j') \otimes x_j)] = R_{n+1}(b \otimes a_1 \otimes a_2) \text{ (i),}$$
\[
\sum_j (\mu_A \otimes 1_B)(1_A \otimes t)[\sum_i (\mu_A \otimes 1_B)(1_A \otimes t)(1 \otimes b_i \otimes a'_j) \otimes x_j] = f(t^B \otimes 1_A)(b \otimes a_1 \otimes a_2) \] (ii).

Before starting the proof of (i) we note that
\[
(1_A \otimes t)[(\mu_A \otimes 1_B)(1_A \otimes t) \otimes 1_A] = (\mu_A \otimes 1_A \otimes 1_B)(1_A \otimes (1_A \otimes t)(t \otimes 1_A)).
\]
Now we prove (i).

\[
R_{n+1}(b \otimes a_1 \otimes a_2)
= (\mu_A \otimes 1_B)(1_A \otimes t)(l(b \otimes a_1) \otimes a_2)
= (\mu_A \otimes 1_B)(1_A \otimes t(1_B \otimes \mu_A))(l(b \otimes a_1) \otimes \sum_j a'_j \otimes x_j)
= (\mu_A \otimes 1_B)(1_A \otimes (\mu_A \otimes 1_B)(1_A \otimes t)(t \otimes 1_A))(\sum_j l(b \otimes a_1) \otimes a'_j \otimes x_j) \text{ using condition (1),}
= (\mu_A \otimes 1_B)(1_A \otimes (1_A \otimes t)[(\mu_A \otimes 1_B)(1_A \otimes t) \otimes 1_A])(\sum_j l(b \otimes a_1) \otimes a'_j \otimes x_j) \text{ since } \mu_A \text{ is associative,}
= \sum_j (\mu_A \otimes 1_B)(1_A \otimes t)(\mu_A \otimes 1_B)(1_A \otimes t \otimes 1_A)(l(b \otimes a_1) \otimes a'_j \otimes x_j), \text{ as desired.}
\]

The proof of (ii) is more transparent.

\[
\sum_j (\mu_A \otimes 1_B)(1_A \otimes t)[\sum_i (\mu_A \otimes 1_B)(1_A \otimes t)(1 \otimes b_i \otimes a'_j) \otimes x_j]
= \sum_j (\mu_A \otimes 1_B)(1_A \otimes t)[(\mu_A \otimes 1_B)(1_A \otimes t) \otimes 1_A](\sum_i 1 \otimes b_i \otimes a'_j \otimes x_j)
= \sum_j (\mu_A \otimes 1_B)(1_A \otimes t)(t \otimes 1_A)(\sum_i b_i \otimes a'_j \otimes x_j)
= \sum_i t(b_i \otimes a_2) \text{ using condition (2),}
= f(t^B \otimes 1_A)(b \otimes a_1 \otimes a_2).
\]

We leave it to the reader to check that \( f(\mu_B \otimes 1_A) = R_{n+1} + f(1_B \otimes t^A). \)

In some cases one can construct graded twisting maps on algebras with relations by first defining a twisting map on free algebras and then checking that the ideals of relations are preserved. We first address the case of free algebras, in particular, we examine the question of when defining a linear map on free generators extends to a graded twisting map on free algebras. (We consider the uniqueness of such extensions in the next section.)

We start with some background material following [13]. If \( V \) and \( W \) are \( K \)-vector spaces, let \( L(V, W) \) denote the space of all linear maps from \( V \) to \( W \). Let \( A \) and \( B \) be associative, unital, \( K \)-algebras and let \( V \) be a \( K \)-vector space. Then \( L(V, V \otimes B) \) and \( L(V, A \otimes V) \) are associative, unital, \( K \)-algebras. Their unit elements
are $v \mapsto v \otimes 1_B$ and $v \mapsto 1_A \otimes v$, respectively. The multiplication in $L(V, V \otimes B)$ is given by
\[
\phi \star \psi = (1_V \otimes \mu_B)(\phi \otimes 1_B)\psi;
\]
the multiplication in $L(V, A \otimes V)$ is given by
\[
\phi \star \psi = (\mu_A \otimes 1_V)(1_A \otimes \psi)\phi.
\]

**Proposition 3.3** ([13], Proposition 2.6). Let $\tau : B \otimes A \to A \otimes B$ be a $\mathbb{K}$-linear map. Define linear maps $\tau^B : B \to L(A, A \otimes B)$ and $\tau^A : A \to L(B, A \otimes B)$ by, for all $a \in A$ and $b \in B$,
\[
\tau^B(b)(a) = (\mu_A \otimes 1_B)(1_A \otimes ev_{a_2} \tau^B(\tau^B(b)(a_1))),
\]
where $ev_a : L(A, A \otimes B) \to A \otimes B$ is the map determined by evaluating at $a \in A$.

**Proposition 3.4.** Suppose that $\tau^B : B \to L(A, A \otimes B)$ is an algebra homomorphism that respects multiplication in $A$. Define $\tau^A : A \to L(B, A \otimes B)$ by $\tau^A(a)(b) = \tau^B(b)(a)$ for all $a \in A$ and $b \in B$. Then $\tau^A$ is an algebra homomorphism.

**Proof.** Let $a_1, a_2 \in A$ and $b \in B$. Then
\[
\tau^A(a_1a_2)(b) = \tau^B(b)(a_1a_2)
\]
\[
= (\mu_A \otimes 1_B)(1_A \otimes ev_{a_2} \tau^B(\tau^B(b)(a_1)))
\]
\[
= (\mu_A \otimes 1_B)(1_A \otimes \tau^A(a_2))((\tau^A(a_1))(b))
\]
\[
= (\tau^A(a_1) \star \tau^A(a_2))(b).
\]

Therefore $\tau^A(a_1a_2) = \tau^A(a_1) \star \tau^A(a_2)$, as desired. \qed

**Corollary 3.5.** Given an algebra homomorphism $\tau^B : B \to L(A, A \otimes B)$ which respects multiplication in $A$, define a map $\tau : B \otimes A \to A \otimes B$ via $\tau(b \otimes a) = \tau^B(b)(a)$, for all $a \in A$ and $b \in B$. Then $\tau$ is a twisting map.

**Proof.** This follows immediately from Proposition 2.6 in [13]. \qed

The reason for the Corollary is its usefulness for constructing twisting maps: writing down an algebra homomorphism $B \to L(A, A \otimes B)$ which respects multiplication in $A$ can be easier than verifying that some linear map $\tau : B \otimes A \to A \otimes B$ is a twisting map. We now illustrate this point.

Let $A_1$ and $B_1$ be finite-dimensional $\mathbb{K}$-vector spaces. Let $A = T(A_1)$ and $B = T(B_1)$ be the associated tensor algebras with the usual weight grading. Let \{\(x_1, \ldots, x_l\)\} be a basis of $A_1$; let \{\(u_1, \ldots, u_m, d_1, \ldots, d_n\)\} be a basis of $B_1$. Let $B'_1 = \text{span}\{u_1, \ldots, u_m\}$, and $B''_1 = \text{span}\{d_1, \ldots, d_n\}$. Suppose that $t : B_1 \otimes A_1 \to A \otimes B$ is a linear map such that
\[
t(B'_1 \otimes A_1) \subseteq A_1 \otimes B'_1 \oplus A_2 \otimes B_0
\]
and
\[
t(B''_1 \otimes A_1) \subseteq A_1 \otimes B''_1 \oplus A_0 \otimes \mu_B(B_1 \otimes B''_1).
\]

**Theorem 3.6.** There exists a graded twisting map $\tau : B \otimes A \to A \otimes B$ such that $\tau|_{B_1 \otimes A_1} = t$. 

\[\boxed{}
\]
Proof. By Corollary 3.3 it suffices to construct an algebra homomorphism \( \tau^B : B \to L(A, A \otimes B) \) that respects multiplication in \( A \).

We start by defining a linear map \( \tau^B : B_1 \to L(A, A \otimes B) \). Fix a basis element \( u_i \in B_1 \). In order to define \( \tau^B(u_i) \in L(A, A \otimes B) \) it suffices to define \( \tau^B(u_i) \) on the canonical monomial basis of \( A \) coming from the algebra generating set \( \{x_1, \ldots, x_l\} \).

First set \( \tau^B(u_i)(1_A) = 1_A \otimes u_i \) and \( \tau^B(u_i)(x_j) = t(u_i \otimes x_j) \). Then extend this definition linearly to get a linear map

\[
\tau^B : B_0 \oplus B_1' \to L(A_{\leq 1}, A_{\leq 2} \otimes (B_0 \oplus B_1')).
\]

With this definition the right hand side of the following equation is well-defined. Set

\[
\tau^B(u_i)(x_{j_1} x_{j_2}) = (\mu_A \otimes 1_B)(1_A \otimes \text{ev}_{x_{j_2}} \tau^B)(\tau^B(u_i)(x_{j_1})).
\]

Then, for \( k \geq 3 \), define, inductively,

\[
\tau^B(u_i)(x_{j_1} \cdots x_{j_k}) = (\mu_A \otimes 1_B)(1_A \otimes \text{ev}_{x_{j_k}} \tau^B)(\tau^B(u_i)(x_{j_1} \cdots x_{j_{k-1}})).
\]

Thus we have defined a linear map \( \tau^B(u_i) : A \to A \otimes B \).

Next we fix a basis element \( d_i \in B_1'' \). Set \( \tau^B(d_i)(1_A) = 1_A \otimes d_i \) and \( \tau^B(d_i)(x_j) = t(d_i \otimes x_j) \). The last definition may be extended linearly to obtain a map

\[
\tau^B : B''_1 \to L(A_{\leq 1}, A_{\leq 1} \otimes B);
\]

then since, as noted above, \( L(A_{\leq 1}, A_{\leq 1} \otimes B) \) is an associative unital algebra we obtain an algebra homomorphism

\[
\tau^B : T(B''_1) \to L(A_{\leq 1}, A_{\leq 1} \otimes B).
\]

Next define

\[
\tau^B(u_k d_i)(x_j) = (1_A \otimes \mu_B)(\tau^B(u_k) \otimes 1_B)(\tau^B(d_i)(x_j)).
\]

Now we extend the definition of \( \tau^B(d_i) \) to the space \( A_{\leq 2} \) by setting

\[
\tau^B(d_i)(x_{j_1} x_{j_2}) = (\mu_A \otimes 1_B)(\mu_A \otimes \text{ev}_{x_{j_2}} \tau^B)(\tau^B(d_i)(x_{j_1})).
\]

Please note that the right hand side of the last equation makes sense as we have defined \( \tau^B : \mu_B(B_1 \otimes B_1) \to L(A_{\geq 2}, A_{\geq 3} \otimes B) \). Suppose \( k \geq 3 \). Let us assume inductively that for any monomial \( w \in T(B_1)_{k-1} \) that \( \tau^B(w) : A_{\leq 1} \to A_{\leq k} \otimes B \) has been defined by multiplicative extension using the fact that \( B \) is a free algebra and the algebra structure of \( L(A_{\leq k}, A_{\leq k} \otimes B) \). (There is a natural inclusion \( L(A_{\leq 1}, A_{\leq k} \otimes B) \to L(A_{\leq k}, A_{\leq k} \otimes B) \) given by extension by 0.) Then define the map \( \tau^B(wd_j) \) on the space \( A_{\leq 1} \) by

\[
\tau^B(wd_j) = (1_A \otimes \mu_B)(\tau^B(w) \otimes 1_B)\tau^B(w).
\]

Assume that \( \tau^B(d_i)(x_{j_1} \cdots x_{j_{k-1}}) \) has been defined, then set

\[
\tau^B(d_i)(x_{j_1} \cdots x_{j_k}) = (\mu_A \otimes 1_B)(1_A \otimes \text{ev}_{x_{j_k}} \tau^B)(\tau^B(d_i)(x_{j_1} \cdots x_{j_{k-1}})).
\]

To be clear about why the right hand side makes sense write

\[
\tau^B(d_i)(x_{j_1} \cdots x_{j_{k-1}}) = (x_{j_1} \cdots x_{j_{k-1}}) \tau(d_i) \tau,
\]

where \( (d_i)_\tau \) is a sum of monomials in \( T(B_1) \) of degree \( \leq k \) and each such monomial ends in an element of \( \{d_1, \ldots, d_n\} \). By construction, \( \tau^B((d_i)_\tau)(x_{j_k}) \) makes sense. Therefore we have defined \( \tau^B(d_i) \in L(A, A \otimes B) \).
We have defined a linear map $\tau^B : B_1 \to L(A, A \otimes B)$, so the universal property of the tensor algebra affords an algebra homomorphism

$$\tau^B : B \to L(A, A \otimes B).$$

Note that the preliminary definitions, for example $\tau^B(d_i)_{d_j}^B$ or $\tau^B(u_i)_{d_j}^B$ as linear maps on $A \simeq B$, are compatible with this definition.

Now we check that $\tau^B$ respects multiplication in $A$. Notice that for any element $b_1 \in B_1$ we have defined

$$\tau^B(b_1)(x_{j_1} \cdots x_{j_k}) = (\mu_A \otimes 1_B)(1_A \otimes \ev_{x_{j_k}} \tau^B)(\tau^B(b_1)(x_{j_1} \cdots x_{j_{k-1}})).$$

It follows easily from the definitions that in fact, for all $b \in B$,

$$\tau^B(b)(x_{j_1} \cdots x_{j_k}) = (\mu_A \otimes 1_B)(1_A \otimes \ev_{x_{j_k}} \tau^B)(\tau^B(b)(x_{j_1} \cdots x_{j_{k-1}})).$$

Let $b \in B$ be arbitrary. We claim that for all integers $l \geq 2$ and $1 \leq k \leq l$,

$$\tau^B(b)(x_{n_1} \cdots x_{n_k} x_{n_{k+1}} \cdots x_{n_l}) = (\mu_A \otimes 1_B)(1_A \otimes \ev_{x_{n_k+1}} \cdots x_{n_{k+1}} \cdots x_{n_{l-1}} \tau^B)(\ev_{x_{n_1} \cdots x_{n_k}} \tau^B(b)).$$

As observed above, the case $l = 2$ is true. Suppose that $l \geq 3$ and that the equation holds for all $1 \leq j < l$ and all $1 \leq k \leq l - 1$.

On the left hand side of the claimed equation:

$$\tau^B(b)(x_{n_1} \cdots x_{n_k} x_{n_{k+1}} \cdots x_{n_l})
= (\mu_A \otimes 1_B)(1_A \otimes \ev_{x_{n_1}} \tau^B)(\tau^B(b)(x_{n_1} \cdots x_{n_k} x_{n_{k+1}} \cdots x_{n_{l-1}}))
= (\mu_A \otimes 1_B)(1_A \otimes \ev_{x_{n_1}} \tau^B)(1_A \otimes \ev_{x_{n_k+1} \cdots x_{n_{l-1}}} \tau^B)(\ev_{x_{n_1} \cdots x_{n_k}} \tau^B(b)).$$

$$= (\mu_A(\mu_A \otimes 1_A) \otimes 1_B)(1_A \otimes \ev_{x_{n_1}} \tau^B)(1_A \otimes \ev_{x_{n_k+1} \cdots x_{n_{l-1}}} \tau^B)(\ev_{x_{n_1} \cdots x_{n_k}} \tau^B(b)).$$

On the right hand side of the claimed equation:

$$(\mu_A \otimes 1_B)(1_A \otimes \ev_{x_{n_k+1} \cdots x_{n_l}} \tau^B)(\ev_{x_{n_1} \cdots x_{n_k}} \tau^B(b))
= (\mu_A \otimes 1_B)(1_A \otimes (\mu_A \otimes 1_B)(1_A \otimes \ev_{x_{n_1}} \tau^B)(\ev_{x_{n_k+1} \cdots x_{n_{l-1}}} \tau^B)))(\ev_{x_{n_1} \cdots x_{n_k}} \tau^B(b))
= (\mu_A \otimes 1_B)(1_A \otimes (\mu_A \otimes 1_B)(1_A \otimes \ev_{x_{n_1}} \tau^B)))(\ev_{x_{n_k+1} \cdots x_{n_{l-1}}} \tau^B)(\ev_{x_{n_1} \cdots x_{n_k}} \tau^B(b))
= (\mu_A(\mu_A \otimes 1_A) \otimes 1_B)(1_A \otimes (\mu_A \otimes 1_B)(1_A \otimes \ev_{x_{n_1}} \tau^B))(\ev_{x_{n_k+1} \cdots x_{n_{l-1}}} \tau^B)(\ev_{x_{n_1} \cdots x_{n_k}} \tau^B(b))
= (\mu_A(\mu_A \otimes 1_A) \otimes 1_B)(1_A \otimes 1_A \otimes \ev_{x_{n_1}} \tau^B)(\ev_{x_{n_k+1} \cdots x_{n_{l-1}}} \tau^B)(\ev_{x_{n_1} \cdots x_{n_k}} \tau^B(b)).$$

Since $\mu_A$ is associative, the claim follows.

We conclude that $\tau^B$ respects multiplication in $A$. Hence there exists a twisting map $\tau : B \otimes A \to A \otimes B$ which extends $t$.

We conclude with conditions on a graded twisting map that imply the map induces a graded twisting map on graded quotient algebras. The non-graded version appears in [3] p. 13.

Let $A$ and $B$ be graded $K$-algebras. Let $I$ and $J$ be homogeneous ideals in $A$ and $B$ respectively, and let $A' = A/I$ and $B' = B/J$. Let $\pi_A : A \to A'$ and $\pi_B : B \to B'$ denote the canonical projections, which are graded $K$-algebra homomorphisms. If $X \in \{A, B, A', B'\}$ we denote the multiplication map on $X$ by $\mu_X : X \otimes X \to X$. Choose graded $K$-linear splittings $\eta_A : A' \to A$ and $\eta_B : B' \to B$ such that
\[ \pi_A \eta_A = 1_A \text{ and } \pi_B \eta_B = 1_B. \] Assume that \( \tau : B \otimes A \to A \otimes B \) is a graded twisting map such that
\[ \tau(B \otimes I + J \otimes A) \subset I \otimes B + A \otimes J. \]

**Theorem 3.7.** Make all of the assumptions of the last paragraph. Then the linear map \( \tau' : B' \otimes A' \to A' \otimes B' \) given by
\[ \tau' = (\pi_A \otimes \pi_B) \circ \tau \circ (\eta_B \otimes \eta_A) \]
is also a graded twisting map. Moreover, \( \pi_A \otimes \pi_B : A \otimes\tau B \to A' \otimes\tau B' \) is a graded algebra homomorphism.

We omit the proof and refer the reader to [3, Theorem, p. 13]. We also remark that it is easy to check that the map \( \tau' \) does not depend on the choice of the splitting maps \( \eta_A \) and \( \eta_B \).

## 4. Uniqueness of Extensions and Quadratic Relations

Koszul algebras must be quadratic, so we begin by characterizing quadratic twisted tensor products of quadratic algebras. We show that a twisted tensor product is quadratic if and only if the corresponding twisting map is determined by its lowest-degree component. Initially, it is useful to consider two different notions of “low-degree determination.”

Throughout this section, let \( A \) and \( B \) be connected, graded, unital \( K \)-algebras. Let \( \tau : B \otimes A \to A \otimes B \) be a graded twisting map.

**Definition 4.1.** The graded twisting map \( \tau \) is one-determined if for any graded twisting map \( \tau' : B \otimes A \to A \otimes B \) such that \( \tau_2 = \tau'_2 \) we have \( \tau = \tau' \).

The unital condition on graded twisting maps implies that degree 2 is the lowest degree in which a difference could occur between \( \tau \) and \( \tau' \).

**Definition 4.2.** The graded twisting map \( \tau \) has the unique extension property to degree \( n \) if, for any linear map \( \tau' : B \otimes A \to A \otimes B \) that is graded twisting to degree \( n \) such that \( \tau_i = \tau'_i \) for all \( i < n \), we have \( \tau_n = \tau'_n \).

The graded twisting map \( \tau \) has the unique extension property if \( \tau \) has the unique extension property to degree \( n \) for all \( n \geq 3 \).

It is obvious that a graded twisting map \( \tau \) with the unique extension property is one-determined.

Example [7,2] shows that the converse is false, though we do not know of a counterexample where both \( A \) and \( B \) are quadratic.

When \( \tau \) is a graded twisting map such that the ideal \( I_\tau \) is generated by (homogeneous) quadratic elements, we say that \( A \otimes\tau B \) is \( \tau \)-quadratic. (The definition of \( I_\tau \) occurs immediately preceding Proposition [2.3].) Now we characterize graded twisting maps which have the unique extension property as follows.

**Theorem 4.3.** A graded twisting map \( \tau : B \otimes A \to A \otimes B \) has the unique extension property if and only if \( A \otimes\tau B \) is \( \tau \)-quadratic.

**Proof.** Suppose \( \tau \) has the unique extension property. Assume inductively that \( I_\tau^n = I_\tau^2 \). Let \( b \otimes a \in (B \otimes A)_{n+1} \) where \( a \in A_+ \) and \( b \in B_+ \). Let \( r = b \otimes a - \tau(b \otimes a) \in A_+ B \).

By Lemma [3.1] there exists \( y \in (B \otimes A) \otimes (B \otimes B \otimes A) \) such that \( \delta(y) = b \otimes a \) and \( R(y) = \tau(b \otimes a) \). We can assume \( y = \sum b_i \otimes a_i \otimes a_i' + \sum b_j' \otimes b_j'' \otimes a_j'' \) where \( a_i, a_i', a_j'' \in A_+ \) and \( b_i, b_j', b_j'' \in B_+ \).
Letting $\tau' = \tau - i_A\tau^A - i_B\tau^B$, we perform the following calculation in the free product $A * B$.

$$r = b \otimes a - R(y) = \delta(y) - R(y)$$

$$= \sum b_i \otimes a_i a_i' - \tau^B (b_i \otimes a_i) \otimes a_i' - R(b_i \otimes a_i) \otimes a_i'$$

$$+ \sum b_j' b_j' \otimes a_j' - b_j' \otimes \tau^A (b_j' \otimes a_j') - R(b_j' \otimes b_j' \otimes a_j')$$

$$= \sum (b_i \otimes a_i - \tau^B (b_i \otimes a_i)) \otimes a_i' - (\mu_A \otimes 1)(1_A \otimes \tau)(\tau'(b_i \otimes a_i) \otimes a_i')$$

$$+ \sum b_j' \otimes (b_j' \otimes a_j' - \tau^A (b_j' \otimes a_j') - \tau^B (b_j' \otimes a_j')) - (1_A \otimes \mu_B)(\tau \otimes 1_B)(b_j' \otimes \tau'(b_j' \otimes a_j'))$$

Now, $\tau'(b_i \otimes a_i) \otimes a_i' - (1_A \otimes \tau)(\tau'(b_i \otimes a_i) \otimes a_i')$

and $b_j' \otimes \tau' (b_j' \otimes a_j') - (\tau \otimes 1_B)(b_j' \otimes \tau'(b_j' \otimes a_j'))$

are both elements of $I_n^\tau = I_2^\tau$. Adding these relations to $r$, we see that

$$r - \sum (b_i \otimes a_i - \tau^B (b_i \otimes a_i)) \otimes a_i' - \sum b_j' \otimes (b_j' \otimes a_j' - \tau^B (b_j' \otimes a_j')) \in I_2^\tau.$$
Choose graded $K$-linear sections $\eta_A : A \to T(A_1)$ and $\eta_B : B \to T(B_1)$ of the canonical algebra maps $\pi_A : T(A_1) \to A$ and $\pi_B : T(B_1) \to B$, respectively. Define ideals of $T(A_1 \oplus B_1)$ by

$$J_\tau = \langle (\eta_B \otimes \eta_A)(b \otimes a - a_\tau \otimes b_\tau) : a \in A, b \in B \rangle$$

and

$$J_\tau^2 = \langle (\eta_B \otimes \eta_A)(b \otimes a - a_\tau \otimes b_\tau) : a \in A_i, b \in B_j, i + j \leq 2 \rangle.$$ 

By Proposition 2.3, we know that $A \otimes \tau B \cong (A \ast B)/I_\tau$. Then we have the following isomorphisms

$$(A \ast B)/I_\tau \cong A \otimes \tau B$$

$$\cong \frac{T(A_1 \oplus B_1)}{J_A + J_B + J_\tau}$$

$$\cong \frac{T(A_1 \oplus B_1)}{(J_A + J_B + J_\tau)_2}$$

$$\cong \frac{T(A_1 \oplus B_1)}{J_A^2 + J_B^2 + J_\tau^2}$$

$$\cong (A \ast B)/I_\tau^2.$$ 

We have used the hypothesis that $A \otimes \tau B$ is quadratic on the third isomorphism, and the assumption that $A$ and $B$ are quadratic on the last isomorphism. The fourth isomorphism is actually an equality since it is easy to check that $(J_A + J_B + J_\tau)_2 = J_A^2 + J_B^2 + J_\tau^2$. Now, since $I_\tau^2 \subseteq I_\tau$, we conclude that $I_\tau^2 = I_\tau$. Hence $A \otimes \tau B$ is $\tau$-quadratic, from which it follows, by Theorem 4.3, that $\tau$ has the unique extension property. \hfill $\square$

To conclude this section we will show that a certain large class of two-sided graded twisting maps has the unique extension property. As promised, this will show that the graded twisting map constructed in Theorem 3.6 is unique. Furthermore, we will study the Koszul property for a subclass of this class in the next section.

For the rest of this section assume that $A$ and $B$ are one-generated $K$-algebras. Suppose that $\tau : B \otimes A \to A \otimes B$ is a graded twisting map and that there is a vector space decomposition $B_1 = B'_1 \oplus B''_1$ such that

$$\tau(B'_1 \otimes A_1) \subseteq A_2 \otimes B_0 \oplus A_1 \otimes B_1,$$

$$\tau(B''_1 \otimes A_1) \subseteq A_1 \otimes B_1 \oplus A_0 \otimes \mu_B(B_1 \otimes B''_1 + B'_1 \otimes B'_1).$$

**Theorem 4.6.** Assume the hypotheses of the last paragraph. Then $\tau$ has the unique extension property.

**Proof.** Let $n \geq 2$ and let $\tau' : B \otimes A \to A \otimes B$ be a linear map that is graded twisting to degree $n$ such that $\tau_{\leq 2} = \tau'_{\leq 2}$. We first show that $\tau_{(B_1 \otimes A)_n} = \tau'_{(B_1 \otimes A)_n}$. The case $n = 2$ is true by assumption. Assume that $n \geq 3$ and that $\tau_{(B_1 \otimes A)_{<n}} = \tau'_{(B_1 \otimes A)_{<n}}$.

First, let $u \otimes x \in (B'_1 \otimes A)_n$. Write $x = \mu_A(x_1 \otimes x_2)$, where $\deg(x_1) = 1$ (note that we are using Sweedler-type notation here since actually $x$ may be a sum of
terms of the form $\mu_A(x_1 \otimes x_2)$. Then

$$\tau(u \otimes x) = \tau(1_B \otimes \mu_A)(u \otimes x_1 \otimes x_2)$$

$$= (\mu_A \otimes 1_B)(1_A \otimes \tau)(\tau_2 \otimes 1_A)(u \otimes x_1 \otimes x_2)$$

$$= (\mu_A \otimes 1_B)(1_A \otimes \tau)(\tau'_2 \otimes 1_A)(u \otimes x_1 \otimes x_2).$$

By assumption $\tau'_2(u \otimes x_1) = (x_1)_r \otimes u \in A_2 \otimes B_0 \oplus A_1 \otimes B_1$. Since $\deg(x_2) < \deg(x)$ it follows by induction that $\tau(u \otimes x_2) = \tau'(u \otimes x_2)$. Therefore $\tau(u \otimes x) = \tau'(u \otimes x)$.

Second, let $d \otimes x \in (B'_1 \otimes A)_n$. We start by noting that if we write $\tau(d \otimes x) = x_r \otimes d_r \in A \otimes B$, then $\deg(x_r) \leq \deg(x)$. Write $x = \mu_A(x_1 \otimes x_2)$, where $\deg(x_1) = 1$. Then

$$\tau(d \otimes x) = \tau(1_B \otimes \mu_A)(d \otimes x_1 \otimes x_2)$$

$$= (\mu_A \otimes 1_B)(1_A \otimes \tau)(\tau_2 \otimes 1_A)(d \otimes x_1 \otimes x_2)$$

$$= (\mu_A \otimes 1_B)(1_A \otimes \tau)(\tau'_2 \otimes 1_A)(d \otimes x_1 \otimes x_2).$$

By assumption $\tau'_2(d \otimes x_1) = (x_1)_r \otimes d_r \in A_1 \otimes B_1 \otimes A_0 \otimes \mu_B(B_1 \otimes B'_1 \otimes B'_1 \otimes B'_1)$. Therefore we may write $d_r$ as a sum of terms in $B_1$, $\mu_B(B_1 \otimes B'_1)$, or $\mu_B(B'_1 \otimes B'_1)$. We address each of these three possibilities. If $b \in B_1$ is a term in $d_r$, then using the inductive assumption, $\tau(b \otimes x_2) = \tau'(b \otimes x_2)$. If $\mu_B(b \otimes d') \in \mu_B(B_1 \otimes B'_1)$ is a term in $d_r$, then, using the inductive assumption and the result of the previous paragraph, we see that $\tau(\mu_B(b \otimes d') \otimes x_2) = \tau'(\mu_B(b \otimes d') \otimes x_2)$. Finally if $\mu_B(u_1 \otimes u_2) \in \mu_B(B'_1 \otimes B'_1)$ is a term in $d_r$, then two applications of the result in the previous paragraph yields $\tau(\mu_B(u_1 \otimes u_2) \otimes x_2) = \tau'(\mu_B(u_1 \otimes u_2) \otimes x_2)$. Therefore it follows that $\tau(d \otimes x) = \tau'(d \otimes x)$.

Finally, let $n \geq 3$. Fix a tensor $b \otimes a \in (B_+ \otimes A_+)_n$; also assume, without loss of generality, that $b$ is a monomial (with respect to the algebra generating set $B_1$). We induct on $k = \deg(b)$. The base of the induction, where $k = 1$, is taken care of above.

Let $k \geq 2$ and write $b = b_1 b_2$, where $\deg(b_2) = 1$. Assume inductively that $\tau|_{(B_+ \otimes A)_\leq n} = \tau'|_{(B_+ \otimes A)_\leq n}$. Then

$$\tau(b \otimes a) = \tau(\mu_B \otimes 1_A)(b_1 \otimes b_2 \otimes a)$$

$$= (1_A \otimes \mu_B)(\tau \otimes 1_B)(\tau_2 \otimes 1_A)(b_1 \otimes b_2 \otimes a)$$

$$= (1_A \otimes \mu_B)(\tau' \otimes 1_B)(\tau'_2 \otimes 1_A)(b_1 \otimes b_2 \otimes a)$$

$$= \tau'(b \otimes a).$$

5. Koszul twisted tensor products

Recall that a graded twisting map $\tau : B \otimes A \rightarrow A \otimes B$ is called one-sided if

$$\tau(B_+ \otimes A_+) \subseteq (A_+ \otimes B_0) \oplus (A_+ \otimes B_+)$$

or

$$\tau(B_+ \otimes A_+) \subseteq (A_+ \otimes B_+) \oplus (A_0 \otimes B_+).$$

The most common examples of algebras constructed from graded twisting maps: Ore extensions, crossed product algebras, or $H$-module algebra smash products, where $H$ is a graded Hopf algebra, come from one-sided twisting maps.
The following simple observation reveals why one-sided twisting maps more readily permit transfer of structure from \(A\) and \(B\) to \(A \otimes \tau B\) than (two-sided) graded twisting maps in general.

**Proposition 5.1.** If \(\tau : B \otimes A \to A \otimes B\) is one-sided, then either \(A\) or \(B\) is a normal subalgebra of \(A \otimes \tau B\).

**Proof.** This follows immediately from Proposition 2.5.

Proposition 5.1 suggests that twisted tensor products arising from one-sided twisting maps can be fruitfully studied using standard techniques concerning normal subalgebras. Such is the case with Theorem 5.3 below. Before addressing the Koszul property, we examine one-sided twisting maps in the framework of Section 4.

**Proposition 5.2.** Let \(A\) and \(B\) be one-generated graded \(K\)-algebras. If \(\tau : B \otimes A \to A \otimes B\) is a one-sided graded twisting map, then \(\tau\) has the unique extension property.

**Proof.** This result follows immediately from Theorem 4.6.

**Theorem 5.3.** Let \(\tau : B \otimes A \to A \otimes B\) be a one-sided graded twisting map.

1. If \(A\) and \(B\) are quadratic \(K\)-algebras, then \(A \otimes \tau B\) is quadratic.
2. If \(A\) and \(B\) are Koszul \(K\)-algebras, then \(A \otimes \tau B\) is Koszul.

**Proof.** Statement (1) is an immediate consequence of Proposition 5.2 and the last statement of Proposition 2.5.

Assume that \(A\) and \(B\) are Koszul algebras and

\[
\tau(B_+ \otimes A_+) \subseteq (A_+ \otimes B_+) \oplus (A_0 \otimes B_+).
\]

Let \(C = A \otimes \tau B\). Then the map \(i_B : B \to C\) given by \(i_B(b) = 1 \otimes b\) for all \(b \in B\) is an injective homomorphism of algebras. Moreover, \(C\) is a free right \(B\)-module. The twisting map condition ensures that \(i_B(B_1)C_1 \subseteq C_1i_B(B_1)\). Finally note that \(C/(i_B(B_1)C) \cong A\) as algebras. The case where \(\tau(B_+ \otimes A_+) \subseteq (A_+ \otimes B_0) \oplus (A_+ \otimes B_+)\) is analogous. Now Statement (2) follows from Statement (1) and Corollary 2.7.

We assume that (2) of Theorem 5.3 is well-known, but we were not able to find the statement in this generality in the literature.

Absent the requirement that at least one of \(A\) and \(B\) is a normal subalgebra of \(A \otimes \tau B\), questions about the structure of \(A \otimes \tau B\) become much more difficult to answer. Even constructing graded twisting maps that are not one-sided is far from straightforward. We begin with an example that shows that a twisted tensor product of Koszul algebras need not be Koszul, nor even quadratic. The example also shows: the twisted tensor product of two algebras of finite global dimension need not have finite global dimension, and the class of Artin-Schelter regular algebras is not closed under taking twisted tensor products.

**Example 5.4.** Let \(A = K[x]\), \(B = K[y]\). Define a \(K\)-linear map \(\tau : B \otimes A \to A \otimes B\) by

\[
\tau(y^i \otimes x^j) = \begin{cases} 
  x^j \otimes y^i & \text{if } i \text{ or } j \text{ is even} \\
  x^{j+1} \otimes y^{i-1} - x^j \otimes y^i + x^{j-1} \otimes y^{i+1} & \text{if } i \text{ and } j \text{ are both odd.}
\end{cases}
\]
is defined arbitrarily. It is straightforward to check that $\tau_{B,3}$. Consequently, by Theorem 4.3, $\tau$ is a twisting map.

Similiarly, (1) twisting to degree $n$.

For (1) we use Theorem 3.2. First of all, in degree 3, we have

$$(\mu_A \otimes 1_B)(1_A \otimes \tau)(\tau \otimes 1_A)(y \otimes x \otimes x) = (\mu_A \otimes 1_B)(1_A \otimes \tau)((x^2 \otimes 1_B - x \otimes y + 1_A \otimes y^2) \otimes x)$$

$$= (\mu_A \otimes 1_B)(x^2 \otimes x \otimes 1_B - x \otimes x^2 \otimes 1_B + x \otimes x \otimes y$$

$$- x \otimes 1_A \otimes y^2 + 1_A \otimes x \otimes y^2)$$

$$= x^2 \otimes y$$

$$= \tau(y \otimes x^2).$$

Similiarly, $(1_A \otimes \mu_B)(\tau \otimes 1_B)(1_B \otimes \tau)(y \otimes y \otimes x) = \tau(y^2 \otimes x)$. Now assume $\tau_{n,3}$ is twisting to degree $n$, for $n \geq 3$. We check condition (2) of Theorem 3.2 and leave the check of condition (3) of Theorem 3.2 to the reader.

Suppose that $i + j = n + 1$ and $i, j \geq 1$.

**Case 1**: $i$ is even.

Then

$$(\mu_A \otimes 1_B)(1_A \otimes \tau)(\tau \otimes 1_A)(y^i \otimes x^j \otimes x) = (\mu_A \otimes 1_B)(1_A \otimes \tau)(x^j \otimes y^i \otimes x)$$

$$= (\mu_A \otimes 1_B)(x^j \otimes x \otimes y^i)$$

$$= x^{j+1} \otimes y^i$$

$$= \tau(y^i \otimes x^{j+1}).$$

**Case 2**: $i$ and $j$ are both odd.

Then

$$(\mu_A \otimes 1_B)(1_A \otimes \tau)(\tau \otimes 1_A)(y^i \otimes x^j \otimes x)$$

$$= (\mu_A \otimes 1_B)(1_A \otimes \tau)((x^{j+1} \otimes y^{i-1} - x^j \otimes y^i + x^{j-1} \otimes y^{i+1}) \otimes x)$$

$$= (\mu_A \otimes 1_B)(x^{j+1} \otimes x \otimes y^{i-1} - x^j \otimes (x^2 \otimes y^{i-1} - x \otimes y^i + 1_A \otimes y^{i+1}) + x^{j-1} \otimes x \otimes y^{i+1}))$$

$$= x^{j+1} \otimes y^i$$

$$= \tau(y^i \otimes x^{j+1}).$$

The case where $i$ is odd and $j$ is even is left to the reader. We conclude that $\tau$ is a twisting map.

For (2), let $\alpha, \beta, \gamma, \delta \in \mathbb{K}$ be arbitrary. Define a graded $\mathbb{K}$-linear map $\tau' : B \otimes A \to A \otimes B$ by insisting that $\tau'$ satisfy the unital twisting conditions, $\tau'(y \otimes x) = x^2 \otimes 1_B - x \otimes y + 1_A \otimes y^2$, $\tau'(y \otimes x^2) = \alpha x^3 \otimes 1_B + \beta x^2 \otimes y + \gamma x \otimes y^2 + \delta 1_A \otimes y^3$, $\tau'(y^2 \otimes x) = \alpha x^3 \otimes 1_B + (\beta - 1)x^2 \otimes y + (\gamma + 1)x \otimes y^2 + \delta 1_A \otimes y^3$, and $\tau'$ on $(B_+ \otimes A_+ \otimes A_+ \otimes B_+ \otimes A_+ \otimes A_+ \otimes B_+ \otimes A_+) \geq 4$ is defined arbitrarily. It is straightforward to check that $\tau'$ is a twisting map to degree 3. We conclude that $\tau$ does not have the unique extension property to degree 3. Consequently, by Theorem 3.3, $A \otimes B$ is not quadratic.

For (3), assume that $\tau' : B \otimes A \to A \otimes B$ is graded twisting to degree $n$, $n \geq 4$, and $\tau'_i = \tau_i$ for all $i < n$. We have to show that $\tau'_n = \tau_n$. Let $i, j \geq 1$ and $i + j = n$. For (1) we use Theorem 3.2. First of all, in degree 3, we have
If \( i = 1 \), then \( j \geq 3 \), so
\[
\tau'(y \otimes x^i) = \tau'(1_B \otimes \mu_A)(y \otimes x^2 \otimes x^{j-2}) = (\mu_A \otimes 1_B)(1_A \otimes \tau')(\tau' \otimes 1_A)(y \otimes x^2 \otimes x^{j-2}) = \tau(y \otimes x^j).
\]

If \( i \geq 2 \), then
\[
\tau'(y^i \otimes x^j) = \tau'(\mu_B \otimes 1_A)(y^{i-2} \otimes y^2 \otimes x^3) = (1_A \otimes \mu_B)\tau' \otimes 1_A)(1_A \otimes \tau')(1_A \otimes \tau')(y^{i-2} \otimes y^2 \otimes x^3) = \tau(y \otimes x^j).
\]

We conclude that \( \tau \) has the unique extension property to degree \( n \).

Finally, it is easy to prove that \( A \otimes \tau B \) is isomorphic to the algebra \( \mathbb{K}\langle x, y \rangle / (x^2, y^2x - xy^2) \). This algebra has infinite global dimension, so \( A \otimes \tau B \) is not Artin-Schelter regular.

In Example 5.4 the twisted tensor product of free algebras is not Koszul because the graded twisting map \( \tau \) fails to have the unique extension property. When \( A \) and \( B \) are free algebras, the unique extension property completely determines Koszulity.

**Proposition 5.5.** If \( A \) and \( B \) are one-generated free algebras and \( \tau : B \otimes A \to A \otimes B \) is a graded twisting map, then \( A \otimes \tau B \) is a Koszul algebra whenever it is quadratic.

**Proof.** Suppose that \( A = \mathbb{K}\langle x_1, \ldots, x_m \rangle \) and \( B = \mathbb{K}\langle y_1, \ldots, y_n \rangle \) are one-generated free algebras. Assume that \( \tau : B \otimes A \to A \otimes B \) is a graded twisting map. Let \( C = A \otimes \tau B \) and suppose that \( C \) is quadratic. The Hilbert series of \( A \) and \( B \) are respectively, \( h_A(t) = (1 - mt)^{-1} \) and \( h_B(t) = (1 - nt)^{-1} \), so
\[
h_C(t) = (1 - (m + n)t + mnt^2)^{-1}.
\]
That \( C \) is Koszul follows immediately from [10] Chapter 2, Proposition 2.3. \( \Box \)

We note that under the assumption that \( A \) and \( B \) are free, Theorem 4.3 implies \( A \otimes \tau B \) is quadratic if and only if \( \tau \) has the unique extension property. In the very simplest case where \( A \) and \( B \) are free on a single generator, almost all graded twisting maps have the unique extension property; see Section 4.

We do not know of an example where \( A \) and \( B \) are Koszul, and \( A \otimes \tau B \) is quadratic but not Koszul. We would not be surprised if such examples exist. The graded twisting map \( \tau \) for such an example would necessarily be two-sided.

In the remainder of this section we introduce a large class of graded two-sided twisting maps whose associated twisted tensor products are Koszul algebras. This class contains algebras whose Koszulity cannot be explained by Theorem 5.3; see Example 7.3.

Let \( A \) and \( B \) be one-generated \( \mathbb{K} \)-algebras. Recall that for a vector space \( V \), the tensor algebra generated by \( V \) is denoted \( T(V) \). Since \( A \) and \( B \) are one-generated, there are canonical projections \( \pi_A : T(A_1) \to A \) and \( \pi_B : T(B_1) \to B \).

**Definition 5.6.** A graded twisting map \( \tau : B \otimes A \to A \otimes B \) is called *separable* if there exists a decomposition \( B_1 = B_1' \oplus B_1'' \) such that
\[
\tau(B_1' \otimes A_1) \subset A_2 \otimes B_0 \oplus A_1 \otimes B_1'
\]
\[
\tau(B_1'' \otimes A_1) \subset A_1 \otimes B_1'' \oplus A_0 \otimes \mu_B(B_1 \otimes B_1'').
\]
We remark that the class of separable twisting maps is a subclass of the twisting maps introduced in the paragraph prior to Theorem 4.4. Hence if \( \tau : B \otimes A \to A \otimes B \) is separable, then \( \tau \) has the unique extension property, and so \( A \otimes \tau B \) is quadratic when \( A \) and \( B \) are quadratic.

One approach to proving an algebra is Koszul that is useful in many situations makes use of semigroup filtrations. In the context of separable graded twisting maps, it is natural to consider the following \( \mathbb{N}^3 \) filtration - indeed this filtration motivates the definition of separable graded twisting map.

Let \( \Gamma = \mathbb{N}^3 \). Then \( \Gamma \) is a commutative monoid under componentwise addition, and we equip \( \Gamma \) with a grading by total degree; that is, we let \( \tau : \Gamma \to \mathbb{N} \) be the grading homomorphism defined by \( g(a, b, c) = a + b + c \). Ordering the fibers \( g^{-1}(n) \) lexicographically, with

\[
(0, 0, 1) < (0, 1, 0) < (1, 0, 0) \quad \text{in } g^{-1}(1)
\]
determines the structure of a graded, ordered monoid on \( \Gamma \). In particular, if \( \alpha, \beta \in g^{-1}(n) \) and \( \gamma \in g^{-1}(m) \), then \( \alpha < \beta \) implies \( \alpha + \gamma < \beta + \gamma \).

We define a one-generated, \( \Gamma \)-valued filtration \( F_{\alpha} \) on \( A \otimes \tau B \) as follows:

\[
F_{(0,0,0)} = A_0 \otimes B_0 \quad F_{(0,0,1)} = A_0 \otimes B_0' + F_{(0,0,0)}
\]

\[
F_{(0,1,0)} = A_1 \otimes B_0 + F_{(0,0,1)} \quad F_{(1,0,0)} = A_0 \otimes B_1' + F_{(0,1,0)}
\]

and for all \( \alpha \in \Gamma \) such that \( g(\alpha) > 1 \),

\[
F_{\alpha} = \sum_{i_1 + \cdots + i_k = \alpha} F_{i_1} \cdot F_{i_2} \cdots F_{i_k}
\]

Note that by restricting our attention to the subalgebra \( B \) we obtain a filtration by the commutative monoid \( \mathbb{N}^2 \), viewed as the submonoid of \( \Gamma \) generated by \( (1, 0, 0) \) and \( (0, 0, 1) \). We denote this filtration by \( F^B \).

With Theorem 5.3 in mind, we address the question of when the associated graded algebra of \( A \otimes \tau B \) with respect to the filtration \( F \) is a graded twisted tensor product.

We begin with several lemmas. In what follows we will use 1 to denote either of the identity maps \( I_A \) and \( I_B \). It is clear from context which one we mean.

**Lemma 5.7.** Let \( A \) and \( B \) be quadratic algebras with quadratic relation spaces \( I_2 \) and \( J_2 \), respectively. If \( R : B_1 \otimes A_1 \to A_1 \otimes B_1 \) is a linear map such that

\[
(1 \otimes R)(R \otimes 1)(B_1 \otimes I_2) \subseteq I_2 \otimes B_1
\]

and

\[
(R \otimes 1)(1 \otimes R)(J_2 \otimes A_1) \subseteq A_1 \otimes J_2
\]

then \( R \) extends uniquely to a graded twisting map \( \tilde{R} : B \otimes A \to A \otimes B \).

**Proof.** By Theorem 5.4 \( R \) uniquely determines a graded twisting map between free algebras \( \tilde{R} : T(B_1) \otimes T(A_1) \to T(A_1) \otimes T(B_1) \). Since \( R(B_1 \otimes A_1) \subseteq A_1 \otimes B_1 \), it follows by induction on total degree that \( \tilde{R}(T_n(B_1) \otimes T_m(A_1)) \subseteq T_m(A_1) \otimes T_n(B_1) \) for all \( m, n \in \mathbb{N} \). Furthermore, the assumption

\[
(1 \otimes R)(R \otimes 1)(B_1 \otimes I_2) \subseteq I_2 \otimes B_1
\]

imply

\[
(1 \otimes \tilde{R})(\tilde{R} \otimes 1)(T(B_1) \otimes (I_2)) \subseteq (I_2) \otimes T(B_1)
\]
and similarly
\[(R \otimes 1)(1 \otimes R)(J_2 \otimes A_1) \subseteq A_1 \otimes J_2\]
implies
\[(\hat{R} \otimes 1)(1 \otimes \hat{R})(\langle J_2 \rangle \otimes T(A_1)) \subseteq T(A_1) \otimes \langle J_2 \rangle.\]
By Theorem 3.7, \(\hat{R}\) induces a graded twisting map \(\hat{R} : B \otimes A \to A \otimes B\).

Let \(I_2 = \ker (\mu_A|_{A_1 \otimes A_1})\) and \(J_2 = \ker (\mu_B|_{B_1 \otimes B_1})\), so if \(A\) and \(B\) are quadratic, then \(A \cong T(A_1)/\langle I_2 \rangle\) and \(B \cong T(B_1)/\langle J_2 \rangle\). Define \(R : B_1 \otimes A_1 \to A_1 \otimes B_1\) by \(R = \pi_{A_1 \otimes B_1} \circ \tau|_{B_1 \otimes A_1}\). Also define \(\tau_A : B_1 \otimes A_1 \to A_2 \otimes B_0\) by \(\tau_A = \pi_{A_2 \otimes B_0} \circ \tau|_{B_1 \otimes A_1}\) and \(\tau_B : B_1 \otimes A_1 \to A_0 \otimes B_2\) by \(\tau_B = \pi_{A_0 \otimes B_2} \circ \tau|_{B_1 \otimes A_1}\).

Lemma 5.8. If \(\tau : B \otimes A \to A \otimes B\) is a separable graded twisting map, then
\[(1 \otimes R)(R \otimes 1)(B_1' \otimes I_2) \subseteq I_2 \otimes B_1'.\]
Furthermore, if
\[\pi_{A_0 \otimes A_2 \otimes B_1}(1 \otimes \tau)(\tau_B \otimes 1)(B_1 \otimes I_2) = 0\]
then
\[(1 \otimes R)(R \otimes 1)(B_1'' \otimes I_2) \subseteq I_2 \otimes B_1''.\]

It is obvious from the definitions that the additional condition is equivalent to
\[\pi_{A_0 \otimes A_2 \otimes B_1}(1 \otimes \tau)(\tau_B \otimes 1)(B_1'' \otimes I_2) = 0.\]

Proof. For all \(u \in B_1'\) and \(x, y \in A_1\) we have
\[(1 \otimes \tau)(\tau \otimes 1)(u \otimes x \otimes y) = (1 \otimes \tau)(\tau_A(u \otimes x) \otimes y + R(u \otimes x) \otimes y) = (1 \otimes \tau)(\tau_A(u \otimes x) \otimes y) + x_R \otimes \tau_A(u_R \otimes y) + x_R \otimes R(u_R \otimes y).\]

Since the first two terms are in \(A_+ \otimes A_+ \otimes B_0\), we have
\[(1 \otimes R)(R \otimes 1)(B_1' \otimes I_2) = \pi_{A_1 \otimes A_1 \otimes B_1}(1 \otimes \tau)(\tau \otimes 1)(B_1' \otimes I_2).\]

Since \((1 \otimes \tau)(\tau \otimes 1)(B_1' \otimes I_2) \subseteq I_2 \otimes B_0\), it follows that
\[(1 \otimes R)(R \otimes 1)(B_1' \otimes I_2) \subseteq I_2 \otimes B_1'.\]

Now, for all \(d \in B_1''\) and \(x, y \in A_1\) we have
\[(1 \otimes \tau)(\tau \otimes 1)(d \otimes x \otimes y) = (1 \otimes \tau)(\tau_B(d \otimes x) \otimes y + R(d \otimes x) \otimes y)
= (1 \otimes \tau)(\tau_B(d \otimes x) \otimes y) + x_R \otimes \tau_B(d_R \otimes y) + x_R \otimes R(d_R \otimes y).\]

Assume
\[\pi_{A_0 \otimes A_2 \otimes B_1}(1 \otimes \tau)(\tau_B \otimes 1)(B_1 \otimes I_2) = 0.\]
Then for any element \(d \otimes \sigma \in B_1'' \otimes I_2\), the first two terms in the last line above are in \((A \otimes A)_{\leq 1} \otimes B_{\geq 2}\) and hence
\[(1 \otimes R)(R \otimes 1)(B_1'' \otimes I_2) = \pi_{A_1 \otimes A_1 \otimes B_1''}(1 \otimes \tau)(\tau \otimes 1)(B_1'' \otimes I_2).\]

As before, since \((1 \otimes \tau)(\tau \otimes 1)(B_1'' \otimes I_2) \subseteq I_2 \otimes B_0\), we conclude that
\[(1 \otimes R)(R \otimes 1)(B_1'' \otimes I_2) \subseteq I_2 \otimes B_1''.\]
When \((1 \otimes R)(R \otimes 1)(B_1 \otimes I_2) \subseteq I_2 \otimes B_1\), the extension of \(R\) to \(B_1 \otimes A_2\) given by
\[
R(b \otimes a) = (\mu_A \otimes 1)(1 \otimes R)(R \otimes 1)(b \otimes \hat{a}),
\]
where \(\hat{a} \in \mu_A^{-1}(a)\) is any preimage of \(a\) in \(A_1 \otimes A_1\), is well-defined. Next, we establish analogous conditions under which \(R\) extends to \(B_2 \otimes A_1\). This will be enough to ensure \(R\) extends to a graded twisting map on all of \(B \otimes A\).

**Lemma 5.9.** If \(\tau : B \otimes A \to A \otimes B\) is a separable graded twisting map, then for \((X,Y) \in \{(B'_1,B'_1'),(B'_1',B''_1'),(B'_1',B'_1''),(B''_1',B'_1'')\}\) we have
\[
(R \otimes 1)(1 \otimes R)(X \otimes Y \otimes A_1) \subseteq A_1 \otimes X \otimes Y.
\]
Furthermore, if
\[
\pi_{A_1 \otimes B_2 \otimes B_0}(\tau \otimes 1)(1 \otimes \tau_A)(J_2 \otimes A_1) = 0,
\]
then
\[
(R \otimes 1)(1 \otimes R)(J_2 \otimes A_1) \subseteq A_1 \otimes J_2.
\]

In applications, and in the proof, it can be helpful to note that the additional hypothesis is satisfied if and only if
\[
\pi_{A_1 \otimes B_2 \otimes B_0}(\tau \otimes 1)(1 \otimes \tau_A)(r(J_2) \otimes A_1) = 0,
\]
where \(r : B_1 \otimes B_1 \to B''_1 \otimes B'_1\) is the canonical projection.

**Proof.** Note that for \(a \in A_2, u \in B'_1, \tau(u \otimes a) \in A_{\geq 2} \otimes B_{\leq 1}\). With this in mind, the following are straightforward to check, as in the proof of the preceding lemma. For all \(u,u_1,u_2 \in B'_1, d,d_1,d_2 \in B''_1, \) and all \(x \in A_1,\)
\[
(R \otimes 1)(1 \otimes R)(u_1 \otimes u_2 \otimes x) = \pi_{A_1 \otimes B_2 \otimes B_0}(\tau \otimes 1)(1 \otimes \tau)(u_1 \otimes u_2 \otimes x) \in A_1 \otimes B'_1 \otimes B'_1
\]
\[
(R \otimes 1)(1 \otimes R)(d_1 \otimes d_2 \otimes x) = \pi_{A_1 \otimes B_2 \otimes B_0}(\tau \otimes 1)(1 \otimes \tau)(d_1 \otimes d_2 \otimes x) \in A_1 \otimes B''_1 \otimes B'_1
\]
\[
(R \otimes 1)(1 \otimes R)(u \otimes d \otimes x) = \pi_{A_1 \otimes B_2 \otimes B_0}(\tau \otimes 1)(1 \otimes \tau)(u \otimes d \otimes x) \in A_1 \otimes B''_1 \otimes B''_1.
\]
These calculations prove the first part of the lemma. It is also straightforward to check that
\[
(R \otimes 1)(1 \otimes R)(d \otimes u \otimes x) = \pi_{A_1 \otimes B_2 \otimes B_0}(\tau \otimes 1)(1 \otimes \tau)(d \otimes u \otimes x) \in A_1 \otimes B''_1 \otimes B'_1
\]
if and only if
\[
\pi_{A_1 \otimes B_2 \otimes B_0}(\tau \otimes 1)(1 \otimes \tau_A)(d \otimes u \otimes x) = 0. \quad (*)
\]
Now, assume that
\[
\pi_{A_1 \otimes B_2 \otimes B_0}(\tau \otimes 1)(1 \otimes \tau_A)(J_2 \otimes A_1) = 0.
\]
Then (*) holds for all \(\rho \otimes x \in J_2 \otimes A_1\). Since \((\tau \otimes 1)(1 \otimes \tau)(J_2 \otimes A_1) \subseteq A \otimes J_2\), it follows that
\[
(R \otimes 1)(1 \otimes R)(J_2 \otimes A_1) = \pi_{A_1 \otimes B_2 \otimes B_0}(\tau \otimes 1)(1 \otimes \tau_A)(J_2 \otimes A_1) \subseteq A_1 \otimes J_2.
\]

**Theorem 5.10.** Let \(A\) and \(B\) be quadratic algebras. Let \(\tau : B \otimes A \to A \otimes B\) be a separable graded twisting map. Assume \(\tau\) satisfies
\begin{align*}
(1) \quad & \pi_{A_0 \otimes A_2 \otimes B_0}(\tau_B \otimes 1)(B_1 \otimes J_2) = 0, \quad \text{and} \\
(2) \quad & \pi_{A_1 \otimes B_2 \otimes B_0}(\tau \otimes 1)(1 \otimes \tau_A)(J_2 \otimes A_1) = 0.
\end{align*}
Let $F$ denote the filtration on $A \otimes_B A$ defined above, and let $F^B$ denote its restriction to the subalgebra $B$. Let $\overline{B} = \text{gr}^{F^B}(B)$. If $A$ is Koszul, the quadratic part of $\overline{B}$ is Koszul and $\overline{B}$ has no defining relations in degree 3, then $A \otimes_B A$ is Koszul.

**Proof.** Assume that the quadratic part of $\overline{B}$ is Koszul and $\overline{B}$ has no defining relations in degree 3. Then Theorem 2.6 implies that $\overline{B}$ is Koszul (and $B$ is Koszul).

Let $R = \pi_{A_1 \otimes B_1} \circ \tau|_{B_1 \otimes A_1}$. First we show that $R$ extends uniquely to a graded twisting map $\overline{R} : \overline{B} \otimes A \rightarrow A \otimes \overline{B}$. Since $B_1 = B_1$, we have $(1 \otimes R)(R \otimes 1)(B_1 \otimes A) \subseteq A \otimes B_1$.

by Lemma 5.8 We define

\[
J_{(0,0,2)} = J_2 \cap (F_{(0,0,1)} \otimes F_{(0,0,1)}) = J_2 \cap (B_1'' \otimes B_1''),
\]

\[
J_{(1,0,1)} = J_2 \cap (F_{(0,0,1)} \otimes F_{(1,0,0)} \oplus F_{(1,0,0)} \otimes F_{(0,0,1)}),
\]

\[
J_{(2,0,0)} = J_2 \cap (F_{(1,0,0)} \otimes F_{(1,0,0)}) = J_2.
\]

The relation space of $\overline{B}$ is

\[
J_{(2,0,0)}/J_{(1,0,1)} \oplus J_{(0,0,2)} \oplus J_{(1,0,2)}.
\]

Since $J_{(0,0,2)} ⊂ B_1'' \otimes B_1''$, Lemma 5.9 shows

\[
(R \otimes 1)(1 \otimes R)(J_{(0,0,2)} \otimes A_1) \subseteq A_1 \otimes J_{(0,0,2)},
\]

\[
(R \otimes 1)(1 \otimes R)(J_{(1,0,1)} \otimes A_1) \subseteq A_1 \otimes J_{(1,0,1)},
\]

\[
(R \otimes 1)(1 \otimes R)(J_{(2,0,0)} \otimes A_1) \subseteq A_1 \otimes J_{(2,0,0)},
\]

from which it follows that

\[
(R \otimes 1)(1 \otimes R)((J_{(1,0,1)}/J_{(0,0,2)}) \otimes A_1) \subseteq A_1 \otimes (J_{(1,0,1)}/J_{(0,0,2)}),
\]

and

\[
(R \otimes 1)(1 \otimes R)((J_{(2,0,0)}/J_{(1,0,1)}) \otimes A_1) \subseteq A_1 \otimes (J_{(2,0,0)}/J_{(1,0,1)}).
\]

By Lemma 5.7 $R$ extends uniquely to $\overline{R}$ as desired.

Next, we show that $C = A \otimes_B \overline{B}$ is isomorphic to $Q = q(\text{gr}^F(A \otimes_B B))$. To see this, observe that $Q$ is quadratic and its space of relations is

\[
\text{gr}^F(I_2 + J_2 + I_+) = \text{gr}^F(I_2) + \text{gr}^F(J_2) + \text{gr}^F(I_+)
\]

where

\[
\text{gr}^F(I_2) = I_2,
\]

\[
\text{gr}^F(J_2) = J_{(2,0,0)}/J_{(1,0,1)} \oplus J_{(0,0,2)} \oplus J_{(0,0,2)},
\]

\[
\text{gr}^F(I_+) = \text{span} \{ b \otimes a - R(b \otimes a) : a \in A_1, b \in B_1 \}.
\]

By Proposition 2.5 $C \cong Q$.

Finally, since the Hilbert series of $A \otimes_B B$ is the same as that of $C$, and hence that of $Q$, we conclude that $Q \cong \text{gr}^F(A \otimes_B B)$. If $A$ is Koszul, then Theorem 1.1 implies that $C$ is Koszul. The result now follows from Theorem 2.6.

\[\square\]

**Corollary 5.11.** Let $A$ be a Koszul algebra with quadratic relation space $I_2$, let $B$ be a free algebra, and suppose $\tau : B \otimes A \rightarrow A \otimes B$ is a separable graded twisting map. Assume that $\pi_{A_1 \otimes A_1} \otimes B_1(1 \otimes \tau)(\tau_B \otimes 1)(B_1 \otimes I_2) = 0$. Then $A \otimes_B B$ is Koszul.
6. Graded twisted tensor products on two generators

Let $A = \mathbb{K}[x]$ and $B = \mathbb{K}[y]$ be free $\mathbb{K}$-algebras and let $\tau : B \otimes A \to A \otimes B$ be a graded twisting map. Define $\tau_B : B \otimes A \to A_0 \otimes B$ by $\tau_B = \pi_{A_0 \otimes B} \tau$. Suppressing the tensors, write $\tau(yx) = ax^2 + bxy + cy^2$ for $a, b, c \in \mathbb{K}$.

Let $s_i^d$ denote the coefficient of $y^d$ when $\tau(y^i x^{d-i})$ is written in terms of monomials $x^p y^q$. Then for all $1 \leq i \leq d-1$, applying the first identity from Remark 2.1 to the factorization $y^i x^{d-i+1} = y^i x^{d-i} x$ shows that

\[
\tau(y^i x^{d-i+1}) - s_i^d \tau(y^d x) = (\mu_A \otimes 1_B)(1_A \otimes \tau)((\tau - \tau_B)(y^i x^{d-i}) \otimes x) \in A_+ \otimes B.
\]

Applying the second identity from Remark 2.1 to the factorization $y^d x = y^{d-1} y x$ shows

\[
\tau(y^d x) - a \tau(y^{d-1} x^2) - (bs_{d-1}^d + c)y^{d+1} \in A_+ \otimes B.
\]

Thus

\[
(1 - ac) \tau(y^k x) - (bc + c)y^{k+1} \in A_+ \otimes B
\]

is an element of $A_+ \otimes B$.

**Lemma 6.1.** If $1 - ac \neq 0$, then $1 - as_{d-1}^d \neq 0$ for all $d \geq 2$.

**Proof.** Suppose, to the contrary, that $1 - as_{d-1}^d = 0$ for some $d \geq 3$. Then the calculation preceding the lemma implies that $bs_{d-1}^d + c = 0$. Multiplying both sides of this equation by $a$ gives $b + ac = 0$. The assumption $1 - ac \neq 0$ implies $b + 1 \neq 0$. We claim that $s_{k-1}^k = c$ for all $k \geq 2$.

By definition, $s_1^2 = c$. Suppose $s_{k-1}^k = c$. Then we have

\[
(1 - ac) \tau(y^k x) - (bc + c)y^{k+1} = 0
\]

Since $1 - ac \neq 0$, it follows that $s_{k+1}^k = c(b+1)/(1-ac) = c(b+1)(1+b) = c$, and our claim follows by induction. We have reached a contradiction since $s_{d-1}^d = c$, so $1 - ac = 0$, contrary to assumption. □

We are ready to prove that under generic conditions on $a$, $b$, and $c$, $A \otimes \tau B \cong \mathbb{K}[x,y]/(yx - ax^2 - bxy - cy^2)$.

**Theorem 6.2.** If $1 - ac \neq 0$, then $\tau$ has the unique extension property and $A \otimes \tau B$ is $\tau$-quadratic.

**Proof.** By Theorem 1.3 it suffices to prove $\tau$ has the unique extension property. Fix $d \geq 2$ and assume $\tau' : B \otimes A \to A \otimes B$ is graded twisting to degree $d+1$ and that $\tau_i = \tau_i'$ for all $i \leq d$. Then for every $k \leq d$, the coefficient of $y^k$ when $\tau'(y^i x^{k-1})$ is written in terms of monomials $x^p y^q$ is $s_i^k$, and for all $1 \leq i \leq d-1$,

\[
\tau(y^i x^{d-i+1}) - s_i^d \tau(y^d x) = (\mu_A \otimes 1_B)(1_A \otimes \tau)((\tau - \tau_B)(y^i x^{d-i}) \otimes x)
\]

\[
= (\mu_A \otimes 1_B)(1_A \otimes \tau')(((\tau - \tau_B)(y^i x^{d-i}) \otimes x)
\]

\[
= (\mu_A \otimes 1_B)(1_A \otimes \tau')((\tau' - \tau_B')(y^i x^{d-i}) \otimes x)
\]

\[
= \tau'(y^i x^{d-i+1}) - s_i^d \tau'(y^d x).
\]

The second and third equalities follow from the assumption that $\tau_d = \tau_d'$, and the fourth follows from the assumption that $\tau'$ is graded twisting to degree $d + 1$. Similarly,
Proof. It is straightforward to check, using Theorem 3.2 and Lemma 6.4, that \( \tau \) is a graded twisting map. Moreover, it is a linear map as defined by Definition 6.3.

Certain polynomials arise which we define first. Assuming that there exists a basis for \( (A \otimes B)_{\leq 1} \), \( b \), and \( c \) satisfy the unital twisting conditions and

\[
\tau(y^d x) - a \tau(y^{d-1} x^2) - (bs^d_{d-1} + c)y^{d+1} = \tau'(y^d x) - a \tau'(y^{d-1} x^2) - (bs^d_{d-1} + c)y^{d+1}
\]

hence

\[
(1 - as^d_{d-1})\tau(y^d x) - (bs^d_{d-1} + c)y^{d+1} = (1 - as^d_{d-1})\tau'(y^d x) - (bs^d_{d-1} + c)y^{d+1}
\]

and

\[
(1 - as^d_{d-1})\tau(y^d x) = (1 - as^d_{d-1})\tau'(y^d x).
\]

By Lemma 6.1, \( 1 - as^d_{d-1} \neq 0 \), so \( \tau(y^d x) = \tau'(y^d x) \), and hence by (*), \( \tau'(y^d x^{d-i+1}) = \tau(y^d x^{d-i+1}) \) for all \( 1 \leq i \leq d \). Together with the unital conditions, this shows \( \tau = \tau' \) on a basis for \( (B \otimes A)_{d+1} \).

Now we investigate the existence of twisting maps for which \( b = 0 \) and \( ac \neq 0 \). Assuming that there exists \( \lambda \in K \) such that \( \lambda^2 = ac^{-1} \) we use Proposition 2.2 to assume that, without loss of generality, \( \tau(y \otimes x) = x^2 \otimes 1_B + c1_A \otimes y^2 \), where \( c \in K \). Certain polynomials arise which we define first.

Definition 6.3. Define a sequence of polynomials \( S_n(t) \in K[t] \) by \( S_1(t) = 1 \), \( S_2(t) = 1 \), and \( S_n(t) = S_{n-1}(t) - tS_{n-2}(t) \) for all \( n \geq 1 \).

It is straightforward to prove that \( S_n(t) \) has an explicit form:

\[
S_n(t) = \sum_{j=0}^{r} (-1)^j \binom{n - 1 - j}{j} t^j, \quad \text{where } r = \begin{cases} \frac{n-2}{2} & \text{if } n \text{ is even} \\ \frac{n-1}{2} & \text{if } n \text{ is odd} \end{cases}
\]

Moreover, these polynomials are interestingly related to the Catalan numbers [6]. It would be remarkable to uncover any deeper connection here.

Lemma 6.4. For all \( n \geq 2 \) and \( i, j \geq 1 \),

1. \( S_{i+1}(t)S_{i+j}(t) = t^i S_j(t) + S_{i+j+1}(t)S_i(t) \)
2. \( S_n^2(t) - t^{n-1} = S_{n+1}(t)S_{n-1}(t) \)

The proof of (1) is a straightforward double induction using the recursive definition of \( S_n(t) \) and is left to the reader. Notice that (2) is a special case of (1).

For \( c \in K \), let \( S_n = S_n(c) \).

Theorem 6.5. Suppose that \( c \in K \) is not a root of \( S_n(t) \) for all \( n \geq 1 \). Then the linear map \( \tau : B \otimes A \to A \otimes B \) defined by insisting that \( \tau \) satisfy the unital twisting conditions and

\[
\tau(y^{i} \otimes x^j) = S_{i+j}^{-1}(S_j x^{i+j} \otimes 1_B + c^j S_1 A \otimes y^{i+j}) \text{ for all } i, j \geq 1
\]

is a graded twisting map. Moreover, \( \tau \) has the unique extension property.

Proof. It is straightforward to check, using Theorem 3.2 and Lemma 6.4 (1), that \( \tau \) is a graded twisting map.

We show that \( \tau \) has the unique extension property. Let \( n \geq 2 \) and suppose that \( \tau' : B \otimes A \to A \otimes B \) is a graded linear map that is twisting to degree \( n+1 \) and \( \tau'_n = \tau_n \). We must show that \( \tau' = \tau \).

Since \( \tau' \) is twisting in degree \( n+1 \) we have
\[
\tau'(y \otimes x^n) = S_n^{-1}(S_{n-1}x^{n+1} \otimes 1_B + c^{n-1}\tau'(y^n \otimes x))
\]
\[
\tau'(y^n \otimes x) = S_n^{-1}(\tau'(y \otimes x^n) + cS_{n-1}1_A \otimes y^{n+1}.
\]

Multiplying the first equation by \(S_n^{-1}\) and adding the result to the second equation yields

\[
(1 - c^{n-1}S_n^{-2})\tau'(y^n \otimes x) = S_n^{-2}S_{n-1}(x^{n+1} \otimes 1_B + cS_{n-1}1_A \otimes y^{n+1}),
\]
then multiplying through by \(S_n^2\) and using Lemma \ref{lem:lem1} (2) we get:

\[
S_{n+1}S_{n-1}\tau'(y^n \otimes x) = S_{n-1}(x^{n+1} \otimes 1_B + cS_{n-1}1_A \otimes y^{n+1}).
\]
It follows that \(\tau'(y^n \otimes x) = \tau(y^n \otimes x)\).

Next suppose that \(i + j = n + 1\) and \(j \geq 2\). Then
\[
\tau'(y^i \otimes x^j) = (\mu_A \otimes 1_B)(1_A \otimes \tau')(\tau' \otimes 1_A)(y^i \otimes x^{j-1} \otimes x)
= (\mu_A \otimes 1_B)(1_A \otimes \tau')(S_n^{-1}(S_{n-1}x^n \otimes 1_B + c^{n-1}S_{n-1}1_A \otimes y^n) \otimes x)
= \tau(y^i \otimes x^j) \text{ since } \tau'(y^n \otimes x) = \tau(y^n \otimes x).
\]
We conclude that \(\tau\) has the unique extension property. \qed

**Proposition 6.6.** Let \(c \in K\) be a root of \(S_n(t)\) for some \(n \geq 3\). Then there is no graded twisting map \(\tau : B \otimes A \to A \otimes B\) such that \(\tau(y \otimes x) = x^2 \otimes 1_B + c1_A \otimes y^2\).

**Proof.** Let \(n \geq 3\) be minimal such that \(S_n = 0\). Suppose, to the contrary, that \(\tau : B \otimes A \to A \otimes B\) is a graded twisting map such that \(\tau(y \otimes x) = x^2 \otimes 1_B + c1_A \otimes y^2\).

Assume, inductively, that
\[
\tau(y \otimes x^{n-2}) = S_{n-1}^{-1}(S_{n-2}x^{n-1} \otimes 1_B + c^{n-2}1_A \otimes y^{n-1})
\]
\[
\tau(y^n \otimes x) = S_{n-1}^{-1}(x^{n-1} \otimes 1_B + cS_{n-2}1_A \otimes y^{n-1}).
\]

Then using the assumption that \(\tau\) is a twisting map we have
\[
\tau(y \otimes x^{n-1}) = S_{n-1}^{-1}(S_{n-2}x^n \otimes 1_B + c^{n-2}\tau(y^{n-1} \otimes x))
\]
\[
\tau(y^{n-1} \otimes x) = S_{n-1}^{-1}(\tau(y \otimes x^{n-1}) + cS_{n-2}1_A \otimes y^n).
\]
Isolating the term \(\tau(y^{n-1} \otimes x)\) exactly as in the proof of the last theorem yields
\[
S_nS_{n-2}\tau(y^{n-1} \otimes x) = S_{n-2}(x^n \otimes 1_B + S_{n-1}1_A \otimes y^n).
\]
Since \(S_n = 0\) and \(S_{n-2} \neq 0\) this is the desired contradiction. \qed

We conclude this section by remarking that we do not have a complete picture of necessary and sufficient conditions for the existence of twisting maps in the case \(b \neq 0, ac \neq 0\). If \(K\) is closed under taking square roots, one may assume, using Proposition \ref{prop:prop2} that the putative twisting map satisfies \(\tau(y \otimes x) = cx^2 \otimes 1_B + bx \otimes y + c1_A \otimes y^2\), but even with this simplification the analysis seems complicated.
7. Additional Examples

In this section we give examples. Our first example uses many of the main theorems of this paper.

Example 7.1. Let $A = \mathbb{K} \langle x, y \rangle$, $B = \mathbb{K} \langle d, u \rangle$. Use Theorem 3.3 to define a separable graded twisting map $\tau$ by

$$\tau(d \otimes x) = x \otimes d + 1 \otimes d^2, \quad \tau(d \otimes y) = y \otimes d + 1 \otimes d^2,$$

$$\tau(u \otimes x) = x \otimes u + x^2 \otimes 1, \quad \tau(u \otimes y) = y \otimes u + y^2 \otimes 1.$$  

Also assume that $\tau$ satisfies the unital twisting conditions.

Then Proposition 5.5 implies that $A \otimes \tau B$ is Koszul and moreover it is easy to check that this algebra has global dimension equal to 2. Another nice feature of this example is that $\tau$ preserves $xy - yx$ so we get via Theorem 3.7 an induced twisting map $\tau' : B \otimes A' \to A' \otimes B$, where $A' = \mathbb{K} \langle x, y \rangle$. To see this, let $I$ denote the two-sided ideal of $A$ generated by $xy - yx$. The condition $\tau(B \otimes I) \subset I \otimes B$ follows from the easily verified formulas:

$$\tau(d \otimes (xy - yx)) = (xy - yx) \otimes d$$

$$\tau(u \otimes (xy - yx)) = [(xy - yx)(x + y) + (x + y)(xy - yx)] \otimes 1_B + (xy - yx) \otimes u.$$  

Finally, Corollary 5.11 ensures that the algebra $A' \otimes \tau B$ is Koszul and it is easy to check that this algebra has global dimension equal to 3.

Example 7.2. Here we give an example of a one-determined, graded twisting map that does not have the unique extension property. Let $A = \mathbb{K} \langle x \rangle / \langle x^3 \rangle$ and $B = \mathbb{K} \langle y \rangle / \langle y^3 \rangle$. We claim that there is a unique graded twisting map $\tau : B \otimes A \to A \otimes B$ such that $\tau(y \otimes x) = x^2 \otimes 1_B - x \otimes y + 1_A \otimes y^2$.

Consider an arbitrary graded linear map $t : B \otimes A \to A \otimes B$ that satisfies the unital twisting conditions and $t(y \otimes x) = x^2 \otimes 1_B - x \otimes y + 1_A \otimes y^2$. There exist scalars $\lambda, \mu, \alpha, \beta, \gamma \in \mathbb{K}$ such that

$$t(y^2 \otimes x) = \lambda x^2 \otimes y + \mu x \otimes y^2,$$

$$t(y \otimes x^2) = \alpha x^2 \otimes y + \beta x \otimes y^2,$$

$$t(y^2 \otimes x^2) = \gamma x^2 \otimes y^2.$$  

It is straightforward to check that in order for $t$ to be graded twisting to degree 3, we must have $\alpha = \lambda + 1$ and $\beta = \mu - 1$. Considering the factorization $y^2 \otimes x^2 = (1 \otimes \mu_A)(y^2 \otimes x \otimes x)$ shows that for $t$ to be twisting to degree 4 it is necessary that $\gamma = \lambda + \mu^2$.

Define, for all $\lambda, \mu \in \mathbb{K}$, a two-parameter family of linear maps $\tau_{\lambda, \mu} : B \otimes A \to A \otimes B$ by

$$\tau_{\lambda, \mu}(y \otimes x) = x^2 \otimes 1_B - x \otimes y + 1_A \otimes y^2,$$

$$\tau_{\lambda, \mu}(y^2 \otimes x) = \lambda x^2 \otimes y + \mu x \otimes y^2,$$

$$\tau_{\lambda, \mu}(y \otimes x^2) = (\lambda + 1)x^2 \otimes y + (\mu - 1)x \otimes y^2,$$

$$\tau_{\lambda, \mu}(y^2 \otimes x^2) = (\lambda + \mu^2)x^2y^2.$$  

Also assume that $\tau_{\lambda, \mu}$ satisfies the unital twisting conditions. Then $\tau_{\lambda, \mu}$ is a graded twisting map if and only if $\lambda = -1$ and $\mu = 0$. To see this, consider

$$
0 = \tau((1 \otimes \mu_A)(y \otimes x \otimes x^2))
\quad = (\mu_A \otimes 1_B)((1 \otimes \tau)(y \otimes x) \otimes x^2)
\quad = -(\mu - 1)x^2 \otimes y^2 + y \otimes x^2
\quad = (-\mu + 1)x^2y^2 + (1_A \otimes \mu_B)(\tau \otimes 1)(y \otimes y \otimes x^2))
\quad = (\lambda + 1)^2x^2y^2.
$$

So $\lambda = -1$. Then

$$
0 = \tau((\mu_B \otimes 1)(y \otimes y^2 \otimes x))
\quad = (1 \otimes \mu_B)(\tau \otimes 1)(y \otimes y \otimes x)\quad = (1 \otimes \mu_B)(\lambda \tau(x \otimes y)^2 \otimes y + \mu \tau(y \otimes x) \otimes y^2)
\quad = \lambda x^2y^2
$$

so $\mu = 0$. We leave it to the reader to complete the verification that $\tau = \tau_{-1,0}$ is indeed a graded twisting map; it remains to show that $\tau(\mu_B \otimes 1_A)(y \otimes y \otimes x^2) = (1_A \otimes \mu_B)(\tau \otimes 1_B))(1_B \otimes \tau)(y \otimes y \otimes x^2)$.

Since $\tau_{-1,0}$ is the only graded twisting map $B \otimes A \rightarrow A \otimes B$ such that $y \otimes x \mapsto x^2 \otimes 1 - x \otimes y + 1 \otimes y^2$, it follows that $\tau$ is one-determined. However, $\tau_{-1,0}$ does not have the unique extension property. Recall that the unique extension property for a graded twisting map $\tau$ fails in degree $n$ whenever another linear map $\tau'$ is graded twisting to degree $n$ and satisfies $(\tau')_{<n} = \tau_{<n}$ and $\tau'_n \neq \tau_n$. *Note that $\tau'$ need not be a graded twisting map.*

Since all of the linear maps $\tau_{\lambda, \mu}$ are graded twisting to degree 3, all are identical on $(\tau_{\lambda, \mu})_{\leq 2}$, and all differ in degree 3, we see that $\tau_{-1,0}$ does not have the unique extension property to degree 3. Hence $\tau_{-1,0}$ does not have the unique extension property.

**Example 7.3.** Here is an example that shows that $A \otimes_k B$ Koszul does not imply that $A$ and $B$ are quadratic. In [5] the algebra $U(g_3)$ presented as the quotient of the free algebra $\mathbb{Q}[x_{ij} \mid 1 \leq i \neq j \leq 3]$ by the homogeneous ideal generated by

$$
[x_{ij}, x_{ik} + x_{jk}] \quad i, j, k \text{ distinct}
$$

$$
[x_{ik}, x_{jk}] \quad i, j, k \text{ distinct},
$$

where $[a, b] = ab - ba$, was shown to be Koszul. Furthermore in [5] it was proved that $U(g_3)$ is isomorphic to $A \otimes \mathbb{Q}[x_{12}, x_{21}]$ where $A$ is isomorphic to the subalgebra of $U(g_3)$ generated by $\{x_{13}, x_{23}, x_{31}, x_{32}\}$; the twisting map $\tau$ in this case is one-sided so the subalgebra $A$ is normal. Finally, it was proved in [5] that the algebra $A$ is not finitely presented.

**Example 7.4.** Let $A = \mathbb{K}[x]$, $B = \mathbb{K}(d, u)$, and let $\tau : B \otimes A \rightarrow A \otimes B$ be the separable, graded twisting map uniquely determined by

$$
\tau(d \otimes x) = x \otimes d + 1 \otimes d^2, \quad \tau(u \otimes x) = x^2 \otimes 1 + x \otimes u.
$$

For the sake of readability, we will suppress the tensor symbol for the rest of the example. We claim that there do not exist non-trivial one-generated, graded $\mathbb{K}$-algebras $C$ and $D$ and a one-sided graded twisting map $\tau' : D \otimes C \rightarrow C \otimes D$ such
that $A \otimes_r B \cong C \otimes_r D$ as graded algebras. This shows that the class of twisted tensor products arising from two-sided graded twisting maps strictly contains the class arising from one-sided twisting maps. Furthermore, since $A$ and $B$ are free algebras, $A \otimes_r B$ is Koszul by Proposition 5.5.

To the contrary, suppose such $C$, $D$, and $\tau'$ exist. Then there are vector space isomorphisms

$$A_1 \oplus B_1 \cong (A \otimes_r B)_1 \cong (C \otimes_r D)_1 \cong C_1 + D_1$$

and, since neither $C$ nor $D$ is the trivial algebra, either $\dim K C_1 = 1$ or $\dim K D_1 = 1$. A dimension count in homogeneous degree 2 shows that $C$ and $D$ must be free.

There is no loss of generality in assuming $\dim K C_1 = 1$. To see why, first note that $A \otimes_r B$ is isomorphic to its opposite algebra via $x \mapsto x, d \mapsto -d$, and $u \mapsto -u$. Thus the same is true for $C \otimes_r D$. Let $s : C \otimes D \to D \otimes C$ be the “trivial” twisting map $s(c \otimes d) = d \otimes c$. It is straightforward to check that $\tau'' = s\tau's$ is a graded twisting map $C^{op} \otimes D^{op} \to D^{op} \otimes C^{op}$. Moreover, $s$ is an algebra isomorphism $C \otimes_r D \to (D^{op} \otimes_r C^{op})^{op}$.

Let $C = K(a)$ and $D = K(b, c)$. Let $\varphi : A \otimes_r B \to C \otimes_r D$ be an isomorphism. Let $\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3, \gamma_1, \gamma_2, \gamma_3 \in K$ such that

$$\varphi(x) = \alpha_1 a + \beta_1 b + \gamma_1 c$$
$$\varphi(u) = \alpha_2 a + \beta_2 b + \gamma_2 c$$
$$\varphi(d) = \alpha_3 a + \beta_3 b + \gamma_3 c.$$  

Since $\tau'$ is one-sided, either $\tau'(D_1 \otimes C_1) \subseteq C \otimes D_+$ or $\tau'(D_1 \otimes C_1) \subseteq C_+ \otimes D$. We examine the latter case, leaving the former case to the reader.

Expanding the difference $\varphi(x^2 + xu) - \varphi(u)\varphi(x)$ gives

$$0 = \varphi(x^2 + xu) - \varphi(u)\varphi(x)$$
$$= \alpha_1^2 a^2 + (\alpha_1 \beta_1 + \alpha_1 \beta_2 - \alpha_2 \beta_1)ab + (\alpha_1 \gamma_1 + \alpha_1 \gamma_2 - \alpha_2 \gamma_1)ac$$
$$+ \beta_1^2 b^2 + (\beta_1 \gamma_1 + \beta_1 \gamma_2 - \beta_2 \gamma_1)bc + (\beta_1 \gamma_1 + \beta_2 \gamma_1 - \beta_1 \gamma_2) + \gamma_1^2 c^2$$
$$+ (\alpha_1 \beta_1 + \alpha_2 \beta_1 - \alpha_1 \beta_2)\tau'(b \otimes a) + (\alpha_1 \gamma_1 + \alpha_2 \gamma_1 - \alpha_1 \gamma_2)\tau'(c \otimes a).$$

By assumption, $\tau'(D_1 \otimes C_1) \subseteq C_+ \otimes D$, so the last two terms do not contain $b^2$ or $c^2$. It follows that $\beta_1 = \gamma_1 = 0$. The analogous calculation for $\varphi(xd + d^2) - \varphi(d)\varphi(x)$ shows that $\beta_3 = \gamma_3 = 0$, leaving us with

$$0 = \varphi(xd + d^2) - \varphi(d)\varphi(x) = \alpha_3^2 a^2$$
so $\alpha_3 = 0$, and hence $\varphi(d) = 0$, a contradiction.

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