Vector Mesons on the Light Front

K. Naito\textsuperscript{a,1}, S. Maedan\textsuperscript{b,2} and K. Itakura\textsuperscript{c,d,3}

\textsuperscript{a} Meme Media Laboratory, Hokkaido University, Sapporo 060-8628, Japan
\textsuperscript{b} Department of Physics, Tokyo National College of Technology, Tokyo 193-0997, Japan
\textsuperscript{c} RIKEN BNL Research Center, Brookhaven National Laboratory, Upton, NY 11973, USA
\textsuperscript{d} Service de Physique Théorique, CEA/Saclay, F91191 Gif-sur-Yvette Cedex, France

Abstract

We apply the light-front quantization to the Nambu–Jona-Lasinio model with the vector interaction, and compute vector meson’s mass and light-cone wavefunction in the large $N$ limit. Following the same procedure as in the previous analyses for scalar and pseudo-scalar mesons, we derive the bound-state equations of a $q\bar{q}$ system in the vector channel. We include the lowest order effects of the vector interaction. The resulting transverse and longitudinal components of the bound-state equation look different from each other. But eventually after imposing an appropriate cutoff, one finds these two are identical, giving the same mass and the same (spin-independent) light-cone wavefunction. Mass of the vector meson decreases as one increases the strength of the vector interaction.

\textsuperscript{1}knaito@nucl.sci.hokudai.ac.jp
\textsuperscript{2}maedan@tokyo-ct.ac.jp
\textsuperscript{3}itakura@quark.phy.bnl.gov, itakura@spht.saclay.cea.fr
\textsuperscript{4}present address
The light-cone (LC) wavefunction of a hadron is one of the most useful quantities for describing the hadron structure in terms of its underlying degrees of freedom [1]. In general, it contains information about soft dynamics among quarks, antiquarks and gluons, and one can compute various scattering processes involving hadrons in initial/final states, by combining it with the hard part of the diagrams. Diffractive vector meson production is one of the typical examples of such processes [2]. To compute the amplitude of diffractive electro/photoproduction of a vector meson for a wide range of kinematics, one needs to know the LC wavefunction of a vector meson with non-perturbative information. In this Letter, we are going to discuss this LC wavefunction of the vector meson in a simple model. As another interesting example, E791 experiment at Fermilab [3] has recently attempted to determine the LC wavefunction (squared) of pions through the diffractive pion dissociation process (dijets production) according to Ref. [4]. Although it is argued that determination of the pion LC wavefunction from the experimental data is actually quite hard [5], it is still true that one cannot compute the amplitude of this process without knowing the pion LC wavefunction. Similar experiments are possible in the dijets production from a virtual photon, where, according to the vector meson dominance model, the vector meson contribution forms the hadronic part of the photon wavefunction.

Perturbative calculations provide us with the so called ”asymptotic” form of the LC wavefunctions. For example, the asymptotic form of the pion LC wavefunction is known as well as the vector meson’s one [6, 7]. However, non-perturbative study is quite few. Lattice simulation can compute the first few moments of meson’s distribution function, but at present they are not sufficient to determine the LC wavefunction itself. Therefore, it is very important to develop a non-perturbative technique which allows us to directly obtain the LC wavefunction. Clearly, the most straightforward and natural framework is the Hamiltonian formalism in the light-front (LF) quantization [8, 1]. Before challenging the problem in the real QCD, one should be able to learn much from the analyses of simpler models such as the Nambu–Jona-Lasinio (NJL) model. Indeed, this model was recently studied by two of the authors within the LF quantization [9, 10] and we follow the same procedure to get the LC wavefunctions of vector mesons. As is well known, there is a paradoxical situation in the LF quantization. It has been asked how one can describe spontaneous symmetry breaking in a formalism having only a trivial Fock vacuum. This was answered in Ref. [9] within the NJL model with N component fermions: \[ \mathcal{L}_{NJL} = \bar{\Psi}(i\partial - m_0)\Psi + \frac{1}{2}G_1[(\bar{\Psi}\Psi)^2 + (\bar{\Psi}\gamma_5\Psi)^2] \]. Based on the analogy with the description of spontaneous symmetry breaking in a scalar theory on the LF, they found that, still with the trivial Fock vacuum, chiral symmetry breaking is described in such a way that one selects an appropriate Hamiltonian depending on the phases of the symmetry. In the NJL model, different Hamiltonians are originated from different solutions to the constraint equation, which exists only in the LF formalism. The ”bad” component of the spinor, \( \psi_- \) (where \( \psi_\pm = \frac{1}{2}\gamma^\mp\gamma^\pm\bar{\Psi} \)) is not a dynamical variable and is subject to a constraint equation, as we will see below. This ”fermionic constraint” is a nonlinear equation in the NJL model and leads to the ”gap equation” for the chiral condensate if one adopts an appropriate cutoff. Namely, using the parity invariant cutoff \( |p^\pm| < \Lambda \), one gets

\[
\frac{M - m_0}{M} = \frac{G_1NA^2}{4\pi^2} \left\{ 2 - \frac{M^2}{\Lambda^2} \left( 1 + \ln \frac{2\Lambda^2}{M^2} \right) \right\},
\]

where \( M = m_0 - G_1\langle\bar{\Psi}\Psi\rangle \) is the dynamical mass of the fermion. When the coupling constant \( \bar{G}_1 = G_1NA^2/4\pi^2 \) is larger than the critical value \( \bar{G}_1^{\text{critical}} = 1/2 \), the gap equation has a
non-zero solution even in the chiral limit $m_0 \to 0$. This means that the fermionic constraint
allows for "symmetric" and "broken" solutions corresponding to those of the gap equation. If
one selects the "broken" solution, and substituting it to the canonical Hamiltonian, one obtains
the "broken" Hamiltonian. This governs the dynamics in the broken phase and is completely
different from the Hamiltonian with the "symmetric" solution. In Ref. [9], the fermionic con-
straint was solved by using the $1/N$ expansion [Indeed, Eq. (1) is the leading order result], and
they obtained the Hamiltonian in both symmetric and broken phases. They also solved the
bound-state equations for the scalar and pseudo-scalar mesons\(^6\), and obtained their LC wave-
functions and masses, as well as the PCAC and GOR relations.

One can of course apply the same procedure for vector states, but we know that the NJL
model $L_{NJL}$ does not allow for a bound state in the vector channel \([11, 12]\). Vector states start
to bind if one adds the vector interaction so that the attractive force between a quark and an
antiquark in the vector channel becomes stronger. Therefore, in this Letter, we include the
vector interaction minimally by adding the following interaction:

$$L_V = -\frac{G_2}{2} \left[ (\bar{\psi} \gamma_\mu \psi) :^2 + (\bar{\psi} \gamma_\mu \gamma_5 \psi :)^2 \right].\quad (2)$$

This interaction, however, makes the fermionic constraint tremendously complicated:

\[
i \partial_- \psi_- = \left( i \gamma^i_\perp \partial_\perp + m_0 \right) \frac{1}{2} \gamma^+ \psi_+ \\
- \frac{G_1}{2} \left[ \gamma^+ \psi_+ (\bar{\psi}_+ \psi_- + \bar{\psi}_- \psi_+) - i \gamma_5 \gamma^+ \psi_+ (\bar{\psi}_+ i \gamma_5 \psi_- + \bar{\psi}_- i \gamma_5 \psi_+) \right] \\
+ G_2 \left[ \psi_- (\bar{\psi}_+ \gamma^+ \psi_+) + \gamma_5 \psi_- (\bar{\psi}_+ \gamma^+ \gamma_5 \psi_+) \right] \\
+ \frac{G_2}{2} \left[ \gamma^+ \gamma_\perp \psi_+ : \left\{ \bar{\psi}_+ \gamma^i_\perp \psi_- + \bar{\psi}_- \gamma^i_\perp \psi_+ \right\} - \gamma^i_\perp \gamma_5 \gamma^+ \psi_+ : \left\{ \bar{\psi}_+ \gamma^i_\perp \gamma_5 \psi_- + \bar{\psi}_- \gamma^i_\perp \gamma_5 \psi_+ \right\} \right]. \quad (3)
\]

Here we have followed the same operator ordering as in the previous analysis without the vector
interaction. After rewriting this equation into a bilocal form, one can solve it in the quantum
level by using the $1/N$ expansion, which is systematically generated by the Holstein-Primakoff
technique \([13]\). It turns out that the leading order equation gives the same gap equation as
Eq. (1). This is natural because we have taken the normal order\(^7\) in the interaction (2).

In the leading order of the $1/N$ expansion, mesonic states are written as constituent states
with a (dynamical) quark and a (dynamical) antiquark, as was discussed in Ref. [9] for the
scalar and pseudo-scalar states. Thus, a generic vector state with total momentum $P^\mu$ and
helicity $\lambda$ can be represented as follows:

\[
|\text{vector}; \lambda, P \rangle = P^+ \int_0^{P^+} dk^+ \int d^2 k_\perp \phi_\lambda(x, k_\perp) \times \epsilon_\mu(P, \lambda) \left\{ \Gamma^\mu(-k, -P + k) + \tilde{\Gamma}^\mu(-P + k, -k) \right\}_{\alpha\beta} B^\dagger_{\alpha\beta}(k, P - k) |0\rangle,
\]

where $k^+$ and $k_\perp$ are the longitudinal and transverse momentum of the quark ($x = k^+/P^+$),
$\phi_\lambda(x, k_\perp)$ is the spin independent part\(^8\) of the LC wavefunction which should be determined

\(^6\)Though there was only one flavor in Ref. [9], these physically correspond to the pion and sigma mesons,
and also the vector meson to be discussed in the present Letter corresponds to the rho meson. Generalization
to multi flavors should be straightforward.

\(^7\)Normal order is defined with respect to the Fourier modes of the fermionic field.

\(^8\)By definition, $\phi_\lambda(x, k_\perp)$ should be independent of $\lambda$, but we retain $\lambda$ because, as we will see below, the
bound-state equations look different for different $\lambda$.\]
by the dynamics, \( e^\mu(P, \lambda) \) is a polarization vector, and \( B^\dagger_{\alpha\beta}(\mathbf{p}, \mathbf{q}) \) is a bosonic operator which was introduced to solve the fermionic constraint \([9]\) and corresponds to quark and antiquark creation operators \( \sim b^\dagger_{\alpha} d^\dagger_{\beta} \) in the leading order of the \( 1/N \) expansion. Spin dependent part of the LC wavefunction, \( \Gamma^\mu = (\Gamma^+, \Gamma^-, \Gamma^i) \), is determined by the interpolating field of the vector meson:

\[
\Gamma^+(\mathbf{p}, \mathbf{q}) = \Gamma^+(\mathbf{p}, \mathbf{q}) = \frac{1}{2} \cdot 1, \\
\Gamma^-(\mathbf{p}, \mathbf{q}) = \bar{\Gamma}^-(\mathbf{q}, \mathbf{p}) = -\frac{1}{4 p^+ q^+} \left\{ \gamma_\perp^i p_\perp^i \gamma_\perp^j q_\perp^j + M \gamma_\perp^i (p_\perp + q_\perp)^i + M^2 \right\}, \\
\Gamma^i(\mathbf{p}, \mathbf{q}) = -\frac{1}{2 q^+} \gamma_\perp^i (\gamma_\perp^j q_\perp^j + M), \quad \bar{\Gamma}^i(\mathbf{p}, \mathbf{q}) = -\frac{1}{2 q^+} (\gamma_\perp^j q_\perp^j + M) \gamma_\perp^i.
\]

The polarization vector is written in the rest frame of the meson \( P^\mu = (m_\nu/\sqrt{2}, m_\nu/\sqrt{2}, 0_\perp) \) as (see also \([14]\))

\[
e^\mu(P, \lambda = \pm 1) = \left( 0, 0, \pm \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right), \quad e^\mu(P, \lambda = 0) = \left( \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0_\perp \right).
\]

As we explained before, after solving the fermionic constraint (3) with the nontrivial solution \( M \neq 0 \) to the gap equation (1) and substituting its solution into the canonical Hamiltonian, one gets the “broken” Hamiltonian \( H_{\text{LF}} \). Its explicit form and the detailed derivation of the Hamiltonian are very complicated and will be reported elsewhere \([15]\). Here only the final results are shown. The Hamiltonian is derived order by order of the \( 1/N \) expansion,

\[
H_{\text{LF}} = N \sum_{n=0}^{\infty} \left( \frac{1}{\sqrt{N}} \right)^n h^{(n)},
\]

and \( h^{(2)} \) turns out to be the lowest non-trivial Hamiltonian since \( h^{(0)} \) is just a constant and \( h^{(1)} = 0 \). Therefore, keeping this non-trivial order, one can write the eigenvalue equation for a vector state as

\[
h^{(2)}|\text{vector}; P\rangle = \frac{m_\nu^2 + P^2}{2 P^+} |\text{vector}; P\rangle.
\]

Solving this equation yields both the spin independent part of the LC wavefunction \( \phi_\lambda(x, k_\perp) \) and mass of the vector meson \( m_\nu \) simultaneously.

Before going into details, let us briefly discuss the \( q\bar{q} \) states to clarify the procedure we perform. In general, the LF energy of the two body state (4) may be schematically written as

\[
P^{-}_{q\bar{q}} = \frac{k^2_\perp + M^2}{2 k^+} + \frac{(P_\perp - k_\perp)^2 + M^2}{2 (P^+ - k^+)} + V(k, P),
\]

where the first two terms are the “kinetic” energies of the quark and the antiquark, and \( V \) is the potential which allows for a bound state. This form of the energy leads to the following bound-state equation:

\[
\left\{ m_\nu^2 - \frac{k^2_\perp + M^2}{x (1 - x)} \right\} \phi(x, k_\perp) = \int_0^1 dy \int d^2 p_\perp V(x, k_\perp; y, p_\perp) \phi(y, p_\perp),
\]

where we have chosen the vector meson’s rest frame, \( P = (P^+, P^\perp_\perp) = (m_\nu/\sqrt{2}, 0_\perp) \) for simplicity, and redefined \( V \) with some factors included. In the following, since we are interested
in seeing how the vector interaction (2) affects the vector channel, we will derive the potential $V$ up to the leading order of the vector interaction $G_2$. We will see that in this leading order the potential term $V(x, k_\perp; y, p_\perp)$ is separable with respect to the internal $(y, p_\perp)$ and external $(x, k_\perp)$ variables, and actually depends only on $y$ and $p_\perp$.

Now let us explicitly show the bound-state equations of the transverse and longitudinal components derived from the leading nontrivial Hamiltonian $h^{(2)}$. First, for a transversely polarized vector meson, a lengthy calculation yields the following potential $V_T (\epsilon(x) = x/|x|)$

$$V_T = -\frac{G_2 N}{(2\pi)^3} \left[ 1 + \frac{G_2 N}{(2\pi)^3} \int_{-\infty}^{\infty} dq^+ \int d^2 q_\perp \frac{\epsilon(q^+)}{P^+ - q^+} \right]^{-1} \frac{p_\perp^2 + M^2 - 2y(1-y)p_\perp^2}{y^2(1-y)^2}. \quad (12)$$

Notice that this is already independent of the external variables $x, k_\perp$. Taking the leading contribution of $G_2$, one arrives at an equation for the LC wavefunction $\phi_T(x, k_\perp) = \phi_{\lambda=\pm 1}(x, k_\perp)$:

$$\left\{ m_T^2 - \frac{k_\perp^2 + M^2}{x(1-x)} \right\} \phi_T(x, k_\perp) = -\frac{G_2 N}{(2\pi)^3} \int dy \, d^2 p_\perp \frac{p_\perp^2 + M^2 - 2y(1-y)p_\perp^2}{y^2(1-y)^2} \phi_T(y, p_\perp). \quad (13)$$

Next, the longitudinal component is much more involved. A longer, but straightforward calculation leads to a more complicated potential $V_L$:

$$V_L = -\frac{G_2 N}{(2\pi)^3} \left\{ 1 - \frac{G_2 N}{(2\pi)^3} \int_{0}^{1} dz \int d^2 q_\perp \right\}^{-1} \left\{ \frac{m_L^2 - \frac{k_\perp^2 + M^2}{x(1-x)}}{m_L^2 + \frac{k_\perp^2 + M^2}{x(1-x)}} \right\} \left\{ \frac{m_L^2 + \frac{k_\perp^2 + M^2}{x(1-x)}}{m_L^2 + \frac{p_\perp^2 + M^2}{y(1-y)}} \right\}^{-1} \times \left[ 2 + \frac{4(k_\perp^2 + M^2)}{m_L^2 x(1-x) - (k_\perp^2 + M^2)} \left\{ 1 - \frac{G_2 N}{(2\pi)^3} \int_{0}^{1} dz \int d^2 q_\perp \right\}^{1} \right] \left\{ m_L^2 + \frac{p_\perp^2 + M^2}{y(1-y)} \right\}. \quad (14)$$

Again, taking the leading term with respect to $G_2$ after careful modification of the bound-state equation, we eventually obtain the following simpler equation for the longitudinal mode $\phi_L(x, k_\perp) = \phi_{\lambda=0}(x, k_\perp)$:

$$\left\{ m_L^2 - \frac{k_\perp^2 + M^2}{x(1-x)} \right\} \phi_L(x, k_\perp) = -\frac{G_2 N}{(2\pi)^3} \int dy \, d^2 p_\perp \frac{4(p_\perp^2 + M^2)}{y(1-y)} \phi_L(y, p_\perp). \quad (15)$$

It is evident that the right-hand side is again independent of the variables $x, k_\perp$. Since the step from Eq. (14) to Eq. (15) is a bit nontrivial, let us show the easiest way to derive Eq. (15), which is however less systematic. First of all, if one ignores the $G_2$ dependent term in the second line of Eq. (14) that gives the higher order in $G_2$ and thus can be ignored anyway, one immediately finds

$$\left\{ m_L^2 - \frac{k_\perp^2 + M^2}{x(1-x)} \right\} \left\{ m_L^2 + \frac{k_\perp^2 + M^2}{x(1-x)} \right\}^{-1} \left[ 2 + \frac{4(k_\perp^2 + M^2)}{m_L^2 x(1-x) - (k_\perp^2 + M^2)} \right] = 2.$$ 

Then one integrates the resulting bound-state equation over $x$ and $k_\perp$, obtaining the following:

$$\int dx \, d^2 k_\perp \left\{ m_L^2 - \frac{k_\perp^2 + M^2}{x(1-x)} \right\} \phi_L(x, k_\perp) = -\frac{G_2 N}{(2\pi)^3} \left( \int dx \, d^2 k_\perp \right) \int dy \, d^2 p_\perp \frac{p_\perp^2 + M^2}{y(1-y)} \phi_L(y, p_\perp).$$

One can modify the bound-state equation by using this integral in the right-hand side of it. Finally, taking the leading term with respect to $G_2$, one obtains Eq. (15). It should be noted
that the above equation is consistent with Eq. (15) since it is the integration of Eq. (15) over $x$ and $k_\perp$.

At first glance, the above two eigenvalue equations (13) and (15) look different and thus seem to give different masses for the transverse and longitudinal vector mesons. This is of course physically unacceptable, and as we will verify soon, these equations are essentially the same and give the same mass $m_T = m_L$. This equivalence will be achieved after one specifies cutoff scheme. It is not hard to identify the origin of this (fake) difference with the lack of Lorentz covariance in the LF formalism.

Nevertheless, even at this stage, one can see that solutions to the above equations are an identical function of $x$ and $k_\perp$. To this end, it should be noted again that these two equations have a very simple structure: Their right-hand sides depend on neither $x$ nor $k_\perp$.

This immediately implies that the solutions should be

$$\phi_{T/L}(x, k_\perp) = \frac{C_{T/L}}{m_T^2 - \frac{k_\perp^2 + M^2}{x(1-x)}}.$$  \hspace{1cm} (16)

Thus, if the masses of transverse and longitudinal vector mesons coincide with each other, then so do the LC wavefunctions ($C_{T/L}$ are determined by the normalization). The LC wavefunction (16) has a peak at $x = 1/2$ as we will see explicitly below. Since our description of the vector states is with respect to the quark (antiquark) having a dynamical mass $M \neq 0$, this shape of the LC wavefunction implies the constituent picture.

Now let us verify that both the two equations (13) and (15) derive the same equation for a vector meson mass $m_V$. Inserting the solution (16) into these equations, one arrives at integral equations for $m_V$. We evaluate the integrals by introducing the "extended parity invariant cutoff" [9] which is actually equivalent to the Lepage-Brodsky cutoff [6]:

$$\frac{p_\perp^2 + M^2}{y(1-y)} < 2 \Lambda^2.$$  \hspace{1cm} (17)

Indeed, this is a natural extension of the parity invariant cutoff in the two body sector, $K^+K^- < \Lambda^2$ where $K^\pm$ are the sum of (on-shell) quark and antiquark longitudinal momenta and energies $[K^+ = p^+ + (P^+ - p^+)] = P^+, \ K^- = (p^2 + M^2)/2p^+ + (p^2 + M^2)/2(P^+ - p^+)]$. This cutoff apparently preserves transverse rotation and parity symmetry separately, but in fact it does work better. First, it also respects the usual three dimensional space rotation [16]. Thus one can relate the above cutoff $\Lambda$ to the 3-momentum cutoff $\sum_{i=1,2,3}(p^i)^2 < \Lambda_{3M}^2$ through $2(\Lambda_{3M}^2 + M^2) = \Lambda^2$. Next, the cutoff $K^+K^- < \Lambda^2$ is invariant under the boost transformation $K^\pm \rightarrow e^{\pm \beta}K^\pm$, which is necessary for the relativistic formulation.

In Ref. [9], the parity invariant cutoff was specified as $|K^\pm| < \Lambda$ which contains two independent conditions. In actual calculations, however, the authors of Ref. [9] utilized only the Lepage-Brodsky cutoff which is obtained by combining the two conditions\footnote{Therefore, body of the calculations in Ref. [9] is correct, while the derivation of the Lepage-Brodsky cutoff was not appropriate.}. Namely, the condition $|K^\pm| < \Lambda$ was introduced only to derive the Lepage-Brodsky cutoff. However, putting the cutoff on the longitudinal momentum $K^+ < \Lambda$ for the total momentum is not preferable from the viewpoint of boost symmetry. Thus, in the present paper, we redefined the parity invariant cutoff in the two body sector by $K^+K^- < \Lambda^2$. On the other hand, there is no problem in putting $|p^\pm| < \Lambda$ in the gap equation (1) because the momentum is not the external
momentum but the internal one to be integrated out. Indeed, the gap equation comes from the zero longitudinal momentum of the fermionic constraint written in the bilocal form.

Now the integral in the equations is replaced as follows:

\[
\int dy \int d^2 p_\perp \rightarrow \int_{y_-}^{y_+} dy \int_0^{2\Lambda^2 y(1-y) - M^2} \pi d(p_\perp^2), \tag{18}
\]

with

\[
y_\pm = \frac{1 \pm \beta}{2}, \quad \beta \equiv \sqrt{1 - \frac{2M^2}{\Lambda^2}}. \tag{19}
\]

Then one can explicitly prove that the two equations from Eqs. (13) and (15) do give the same equation that determines the mass \( m_V = m_T = m_L \). It is very important to recognize that we can derive this equation simply by inserting the LC wavefunction \((16)\) into the bound-state equations with the above cutoff. We have just evaluated the right-hand sides of the equations. Since the LC wavefunction \((16)\) is a direct consequence of the bound-state equations, to obtain the same equation for \( m_V \) means that the original equations are also equivalent to each other.

The explicit form of the equation for the vector meson mass is given by

\[
\frac{1}{\tilde{G}^2} = \frac{2}{3} \left[ \beta + (1 - \beta^2) \left\{ r \ln \left( \frac{1+\beta}{1-\beta} \right) - (2r + 1) \sqrt{\frac{1-r}{r}} \arctan \frac{\beta}{\sqrt{1-r}} \right\} \right], \tag{20}
\]

where we have defined a dimensionless coupling constant \( \tilde{G}_2 = G_2 N \Lambda^2 / 4\pi^2 \) and \( r \) is (square of) the ratio of the vector meson mass to the threshold mass \( 2M \):\

\[
r \equiv \left( \frac{m_V}{2M} \right)^2. \tag{21}
\]

A physical bound-state appears only when the ratio \( r \) is in the range \( 0 < r < 1 \). Equation (20) has a solution in this region when the strength of the coupling constant \( \tilde{G}_2 \) is in the range \( \tilde{G}_2^{(\text{min})} < \tilde{G}_2 < \tilde{G}_2^{(\text{max})} \) defined by

\[
\tilde{G}_2^{(\text{min})} \equiv \frac{3}{2} \left\{ \beta + (1 - \beta^2) \ln \left( \frac{1+\beta}{1-\beta} \right) \right\}^{-1}, \quad \tilde{G}_2^{(\text{max})} \equiv \frac{3}{2} \cdot \frac{1}{\beta^2}. \tag{22}
\]

Two limiting cases \( \tilde{G}_2 = \tilde{G}_2^{(\text{min})} \) and \( \tilde{G}_2 = \tilde{G}_2^{(\text{max})} \) correspond to \( r = 1 \) (loose binding limit), and \( r = 0 \) (tight binding limit), respectively. When \( M/\Lambda \to 0 \ (\beta \to 1) \), the physical bound-state region shrinks \( \tilde{G}_2^{(\text{min})} \to \tilde{G}_2^{(\text{max})} \), while it becomes wider as \( M/\Lambda \) grows large. The existence of \( \tilde{G}_2^{(\text{min})} \) is consistent with the observation that there is no bound state in the NJL model without the vector interaction. Similar behaviors have been found in Ref. [17].

In Figure 1, a numerical solution to Eq. (20) is shown as a function of \( \tilde{G}_2 \), where the constituent quark mass is taken to be \( M/\Lambda = 0.4 \ (\beta = 0.82) \) as an example. As we expect, the bound state appears for \( \tilde{G}_2 \) larger than some critical value and the mass starts to decrease from the threshold value \( 2M \) as one increases the strength of the vector interaction. The value of critical coupling constants are exactly the same as the values predicted by analytic calculation. When \( \beta = 0.82 \), they are \( \tilde{G}_2^{(\text{min})} = 0.95, \tilde{G}_2^{(\text{max})} = 2.72.\)

10Alternatively, one can regard that the LC wavefunction \((16)\) has support defined by the Lepage-Brodsky cutoff.
Figure 1: $m_V/2M$ as a function of $\tilde{G}_2$. Constituent quark mass is $M/\Lambda = 0.4$. A bound state appears only in the regime $\tilde{G}_2^{(\text{min})} = 0.95 < \tilde{G}_2 < \tilde{G}_2^{(\text{max})} = 2.72$.

It is also interesting to see $m_V$ as a function of $G_1$ and $G_2$. The dependence on $G_1$ enters only through the constituent mass $M$ [see Eq. (1)]. In Figure 2, we show the vector meson mass $m_V$ as a function of $G_1 = G_1 N \Lambda^2 / 4\pi^2$ and $G_2$ in the broken phase $G_1 > G_1^{(\text{critical})} = 1/2$ and in the chiral limit. As the coupling constant $G_1$ becomes large, the constituent mass (namely, the chiral condensate) becomes large. For fixed cutoff $\Lambda$, this means to increase the value $M/\Lambda$ and thus enlarges the bound-state region. This can be seen clearly in the figure.

In Figure 3, we compare the LC wavefunction of the vector meson (16) with that of the pseudo-scalar meson $\phi_{PS}(x, k_\perp) \propto (m_{PS}^2 - (k_\perp^2 + M^2)/x(1-x))^{-1}$ which is the result of the previous analysis where the vector interaction was not included [9]. The transverse momentum $k_\perp$ is set to be zero for simplicity and the wavefunctions are normalized at $x = 1/2$ for comparison$^{11}$. We have chosen the masses to be $(m_{PS}/2M)^2 = 0.01$ and $(m_V/2M)^2 = 0.95$ as a typical case in the chirally broken phase with non-zero current quark mass $m_0 \neq 0$. In spite of the absence of the vector interaction in the previous analysis, this comparison makes sense because the effects of the vector interaction on the mass of a pseudo-scalar meson is small. Indeed, after the Fierz transformation, the vector interaction will generate terms proportional to the original $G_1$ interaction, but of the higher order in $1/N$. Notice that the LC wavefunction of the pseudo-scalar meson has the same functional form of $x$ and $k_\perp$ as that of Eq. (16). Therefore, difference of the shape is due to different values of the mass. The constituent picture works better in the vector meson than in the pseudo-scalar meson.

Before concluding the paper, let us show one more evidence for the equivalence between the transverse and longitudinal equations. As we already mentioned, the superficial difference comes from the fact that the Lorentz covariance (in particular, the 3 dimensional rotational invariance) is not manifest in the LF coordinates. If one works in a framework with obvious rotational invariance, then there should be no difference between transverse and longitudinal components. Indeed, in the leading order of the $1/N$ expansion, it is possible to derive the same equations from the covariant Bethe-Salpeter (BS) equation:

$$\hat{\phi}_{BS}^\text{BS}(q; P) = \frac{G_2 N}{(2\pi)^4 i} \int d^4k \text{ tr} \left[ \gamma^i \frac{1}{k - M + i\epsilon} \gamma_i \hat{\phi}_{BS}(k; P) \frac{1}{k - P - M - i\epsilon} \right], \quad (23)$$

$^{11}$ If one includes the Lepage-Brodsky cutoff into the definition of the LC wavefunction, support of $x$ for $k_\perp = 0$ is given by $y_- < x < y_+$ with $y_{\pm}$ given by eq. (19).
Figure 2: Vector meson mass $m_V/\Lambda$ as a function of $\tilde{G}_1$ and $\tilde{G}_2$.

Figure 3: LC wavefunctions $\phi(x, k_\perp = 0)$ of pseudo-scalar (dashed) and vector (solid) mesons with typical masses $(m_{PS}/2M)^2 = 0.01$, $(m_V/2M)^2 = 0.95$. Normalized at $x = 1/2$ for comparison.
where $\hat{\phi}^{\text{BS}}(q; P)$ is the amputated BS amplitude for the vector channel ($P$ and $q$ are the total momentum and relative momentum of a quark and an antiquark system, respectively). The LC wavefunction is obtained by the LF energy integral of the BS amplitude. Here, however, we have to be much careful because we must specify the cutoff scheme to make the equations well-defined. When we impose the cutoff, it is of course (and again) desirable to maintain the symmetries such as the rotational invariance. For example, in Eq. (23), we can use the so called 3-momentum cutoff scheme which respects 3-dimensional rotational invariance. Then, it turns out that the resulting two equations are equivalent to Eqs. (13) and (15) respectively, with the parity invariant cutoff (17) if one uses the following variable transformation $y \leftrightarrow (E_k + k_z)/2E_k$, with $E_k = \sqrt{k^2 + M^2}$ and $k$ being the three momentum here. Also, one needs to replace the 3-momentum cutoff $\Lambda_{3M}$ by the extended parity invariant cutoff $\Lambda$ using the relation shown before.

To summarize, we have applied the LF quantization to the NJL model with the vector interaction, and obtained the eigenvalue equations for vector meson’s LC wavefunctions. Due to the addition of the vector interaction, the vector state becomes a bound state. At first glance, transverse and longitudinal components of the bound-state equations look different from each other, but eventually after imposing an appropriate cutoff scheme, one finds these two coincide with each other. Mass of the vector meson decreases as one increases the strength of the vector interaction. This behavior is consistent with Refs.[17, 18] which also treated the vector meson within the NJL model with the vector interaction.

Once we obtain the LC wavefunctions, we can compute various physical quantities. One of such important quantities is the physical form factors. For the pseudo-scalar case, this was done [19] in the NJL model with two flavor quarks. Similar analysis can be done in the vector meson case and will be reported in the future publications [15].

Our non-perturbative approach plays a complementary role to the perturbative calculation of the asymptotic form [7], because our LC wavefunction is expected to describe that of low energy scale (Of course we have to include flavor degrees of freedom, which is straightforward. See for example, [16]). It is interesting to find a way to interpolate these two different approaches. One of the possible ways for this problem is to include the effects of gluon propagation between a quark and an antiquark. This is easily incorporated by using the nonlocal current current interaction $j^\mu D_{\mu \nu}j^\nu$ instead of the point interaction in the NJL model. Such kind of interpolation between low and high momentum regimes is used in various situations. It is also interesting to apply our approach to the heavy quark system such as $J/\psi$ or $\Upsilon$. The use of the nonlocal interaction is also convenient from the technical point of view. In the presence of the gauge field, one needs to consider the longitudinal zero modes of the gauge field. In spite of the favorable aspect that the nontrivial vacuum structure such as the theta vacua may be attributed to the dynamics of gauge zero modes, inclusion of the gauge zero modes makes the canonical structure terribly complicated [20]. Thus, for the problem of quarkoniums where one can ignore the vacuum physics, it is easier to replace the gauge field effects by the non-local current-current interaction.

References

[1] For example, S.J. Brodsky, “QCD Phenomenology and Light-Front Wavefunctions”, hep-ph/0111340.
[2] S.J. Brodsky, et al. Phys. Rev. D50 (1994) 3134.

[3] E. M. Aitala, et al. (E791 Collaboration), Phys. Rev. Lett. 86 (2001) 4768.

[4] L. Frankfurt, G.A. Miller, and M. Strikman, Phys. Lett. B304 (1993) 1; ibid. Found. Phys. 30 (2000) 5333.

[5] V.M. Braun, et al. Phys. Lett. B509 (2001) 43; Nucl. Phys. B638 (2002) 111, V. Chernyak, Phys. Lett. B516 (2001) 116.

[6] G.P. Lepage, and S.J. Brodsky, Phys. Rev. D22 (1980) 2157.

[7] P. Ball, et al. Nucl. Phys. B529 (1998) 323.

[8] S.J. Brodsky, H.C. Pauli, and S.S. Pinsky, Phys. Rept. 301 (1998) 299.

[9] K. Itakura, S. Maedan, Phys. Rev. D61 (2000) 045009; ibid. D62 (2000) 105016.

[10] K. Itakura, S. Maedan, Prog. Theor. Phys. 105 (2001) 537.

[11] S.P. Klevansky, Rev. Mod. Phys. 64 (1992) 649, U. Vogl and W. Weise, Prog. Part. Nucl. Phys. 27 (1991) 195.

[12] S. Klimt et al. Nucl. Phys. A516 (1990) 429.

[13] K. Itakura, Phys. Rev. D54 (1996) 2853.

[14] C.R. Ji, P.L. Chung, and S.R. Cotanch, Phys. Rev. D45 (1992) 4214.

[15] K. Naito, S. Maedan and K. Itakura, in preparation.

[16] W. Bentz, T. Hama, T. Matsuki and K. Yazaki, Nucl. Phys. A651 (1999) 143.

[17] V. Dmitrašinović, Phys. Lett. B451 (1999) 170.

[18] T. Kugo, Prog. Theor. Phys. 55 (1976) 2032.

[19] T. Heinzl, Light-Cone Dynamics of Particles and Fields, hep-th/9812190.

[20] V.A. Franke, Yu. V. Novozhilov and E.V. Prokhvatilov, Lett. Math. Phys. 5 (1981) 239, ibid 5 (1981) 437, K. Itakura, S. Maedan and M. Tachibana, Phys. Lett. B442 (1998) 217, K. Itakura and S. Maedan, Nucl. Phys. A670 (2000) 76c.