Almost Quasi-linear Utilities in Disguise: Positive-representation
An Extension of Roberts’ Theorem

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Abstract. This work deals with the implementation of social choice rules using dominant strategies for unrestricted preferences. The seminal Gibbard-Satterthwaite theorem shows that only few unappealing social choice rules can be implemented unless we assume some restrictions on the preferences or allow monetary transfers. When monetary transfers are allowed and quasi-linear utilities w.r.t. money are assumed, Vickrey-Clarke-Groves (VCG) mechanisms were shown to implement any affine-maximizer, and by the work of Roberts, only affine-maximizers can be implemented whenever the type sets of the agents are rich enough.

In this work, we generalize these results and define a new class of preferences: Preferences which are positive-represented by a quasi-linear utility. That is, agents whose preference on a subspace of the outcomes, which is defined by a threshold, can be modeled using a quasi-linear utility. We show that the characterization of VCG mechanisms as the incentive-compatible mechanisms extends naturally to this domain. We show that the original characterization of VCG mechanism is an immediate corollary of our generalized characterization. Our result follows from a simple reduction to the characterization of VCG mechanisms. Hence, we see our result more as a fuller more correct version of the VCG characterization than a new non-quasi-linear domain extension.

This work also highlights a common misconception in the community attributing the VCG result to the usage of transferable utility. Our result shows that these results extend naturally to the non-transferable utility domain. That is, that the incentive-compatibility of the VCG mechanisms does not rely on money being a common denominator, but rather on the ability of the designer to fine the agents on a continuous (maybe agent-specific) scale.

We also provide simple characterizations of the types which are represented and pos-represented by quasi-linear utility functions. We show characterizations both in utility function terms and in preference terms, by that supplying a full comparison of the different classes.

We think these two insights, considering the utility as a representation and not as the preference itself (which is common in the economic community) and considering utilities which represent the preference only for the relevant domain, would turn out to fruitful in other domains as well.

Keywords: Mechanism Design Strategy-proofness Dominant Strategy Incentive Compatibility Non Quasi-linear Utilities Positive-representation Roberts’ Theorem

1 Introduction

Consider the problem of a designer who wishes to implement a given social choice rule. That is, consider a finite set of agents \( N \) and a finite set of possible social alternatives \( A \) s.t. each agent holds a preference (a total order) over \( A \); The designer needs to choose, as a function of the agents’ preferences, one social alternative out of \( A \), e.g., one that maximizes a social welfare or some other criterion, but the preferences are a-priori unknown to the designer. A naïve procedure for the designer would be to first query the preferences of the

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agents and choose the alternative accordingly. The problem with such solution is that an agent might report her preference untruthfully if she thinks it might result in a better alternative for her, by that preventing the designer from choosing correctly.

In this work, we study the fundamental question of which social choice rules are implementable by a principal, using the most basic implementation concept of dominant-strategies. That is, we aim to characterize which social choice rules can be implemented s.t. no agent has an incentive to misreport her preference, independent of the reports of the other agents, and to show a mechanism for any such implementable social choice rule. Furthermore, we analyze implementation using deterministic incentive-compatible direct revelation mechanisms. In a direct revelation mechanism each agent is asked to report her true preference and subsequently a decision is made according to a decision rule. Incentive-compatibility for these mechanisms means that an agent cannot benefit from reporting a preference different from her true preference. In the sequel, we assume the mechanisms are direct and deterministic and in Section 4 we discuss these restrictions and how to relax them. In particular, note that the assumption of the mechanism being direct is without loss of the generality, as according to the revelation principle [11] any implementable social choice rule is implementable using a direct revelation mechanism and the characterization of direct revelation implementations is easily lifted to the general characterization.

The seminal works of Gibbard [3] and Satterthwaite [14] show that without further assumptions, if all profiles of preferences are feasible, one cannot devise a mechanism for choosing a single alternative, besides trivial procedures which a-priori ignore most agents or most alternatives. Assuming the preferences of the agents are richer and include intensity of preferences (that is, a cardinal utility function $u: \mathcal{A} \rightarrow \mathbb{R}$ which assigns for every alternative its utility or desirability for the agent) does not circumvent this impossibility.

On the other hand, the seminal works of Vickrey [15], Clarke [2], and Groves [5] show that allowing the designer to induce monetary transfers (pay money to the agents or charge them) gives rise to a non-trivial family of implementable social choice rules, while still assuming that all profile preferences are feasible. That is, the outcome is both a chosen alternative and in addition a sequence of payments to the agents. In addition, the preferences of the agents are extended to preferences over $\mathcal{A} \times \mathbb{R}$, the tuples of an alternative and a payment of the agent. Note that (under some mild conditions) such preference also defines a monetary value for each alternative and hence defines a cardinal utility $v: \mathcal{A} \rightarrow \mathbb{R}$ over the alternatives.

When all agents’ preferences are defined by quasi-linear utility functions $u: \mathcal{A} \times \mathbb{R} \rightarrow \mathbb{R}$ of the form $u(a,z) = v(a) - z$ for some $v: \mathcal{A} \rightarrow \mathbb{R}$, the Vickrey-Clarke-Groves (VCG) mechanisms are incentive-compatible direct mechanisms which can implement any affine-maximizer of the agents’ valuation functions $v$ [15, 2, 5]. Moreover, Green and Laffont [4] and Roberts [13] later showed that these are the only incentive-compatible onto mechanisms when all profiles of quasi-linear utility functions are feasible.

However, quasi-linearity is a strong assumption. On the individual level, it is an assumption that in particular there is no income effect in the evaluations of the alternatives, which might be violated due to, for example, lack of liquidity, wealth effect, or risk-aversion. On the society level, it is an assumption of a transferable utility model, that is, an assumption that the ‘utility loss’ due to paying $1 for Agent $i$ equals to the ‘utility gain’ due to receiving this amount for Agent $j$.

Hence, it is of interest of other preference domains for which non-trivial implementable social choice rules exist, while still inducing the full unrestricted set of preferences over the alternatives. To the best of our knowledge, except the aforementioned works for quasi-linear utilities, very few papers considered social choice mechanisms with monetary transfers without restricting the possible profiles of preferences over the alternatives (In much more restricted domains, like single-peaked facility location [9] or auctions, just to name two, there are many truthful mechanisms that are not affine-maximizers.).

The work most relevant to this paper is the result by Ma et al. [7]. In this work, Ma et al. present a new family of utility functions, the parallel utility functions, which is a super-set of the family of the quasi-linear utility functions. Following the works of Vickrey, Clarke, and Groves and the work of Roberts, they present a family of mechanisms which are incentive-compatible whenever all agents have parallel utility functions, and prove that the induced social choice rules are the only implementable rules (under two additional mild conditions, No subsidy & Individual rationality) if all induced valuation profiles are feasible.
Our Results

We think that parallel utility functions can be seen as a special case of disguised almost quasi-linear utility functions, by that strengthening the knife-edginess of the quasi-linear scenario and the characterization of VCG mechanisms. We formalize this by two insights (which catches this intuitive notion of ‘disguised almost quasi-linear’). We discuss this connection we see in more details in Section 4.

First, we notice that VCG mechanisms do not depend on the quasi-linear representation an agent reports and can be stated (while still maintaining the characterization result) as mechanisms that get as input the preferences of the agents and not utility functions. In particular, VCG mechanisms can be extended to receive non-quasi-linear utility functions as long as there is some quasi-linear representation of it (quasi-linear in disguise). While this observation is technically trivial, we think it is conceptually important as being consistent with the 
*revealed preferences principle* and in particular with the fact that one cannot test whether an agent holds one utility representation or the other, so in general there is no base to assume that one utility function better represents the agent.

Second, our main insight is a weaker notion of representation, *positive-representation* (and shortly *pos-representation*). We say an agent’s preference is pos-represented by a utility function if they coincide whenever the utility of at least one of the two comparands is positive (and not for any two comparands as with regular representation by a utility function). This new simple notion allows us to extend the previous above works of Vickrey, Clarke, Groves, Green-Laffont, Roberts.

We extend the VCG mechanisms to mechanisms that get as input preferences which are pos-represented by quasi-linear functions. We show that these mechanisms are incentive compatible (Theorem 1) and show that when all induced valuation profiles are feasible, essentially these are the only incentive-compatible onto mechanisms (Theorem 2). This result extends the characterizations of Roberts [13] and Green-Laffont [4].

We argue that preferences which are pos-represented by quasi-linear utility functions maintain the essential properties of quasi-linear utility functions (hence we call them ‘disguised almost quasi-linear utility functions’) and we see the simple proofs by reduction as a supporting evidence for that. This strengthens the knife-edginess of the VCG phenomenon by showing that the quasi-linear domain is essentially the only unrestricted preference domain for which an incentive-compatible non-trivial implementation is known, and we hope this gives a better research direction of exploring mechanism design beyond quasi-linear preferences.

Furthermore, we think that our work addresses a common misconception in the community of the reason why introducing money enables the design of general mechanisms and circumventing impossibility results à la Gibbard-Satterthwaite. Many works attribute the VCG result to the extended model being a transferable utility model. For example, in the Algorithmic Game Theory book [12], we found that VCG Mechanisms (and the driving force behind their incentive compatibility) are introduced using phrases like “Money can be transferred between players. ... (This) will allow us to do things that we could not do otherwise.” Our work shows that the main driving-force is the individual additive-separability and that the introduction of transferable utility does not play a role in the characterization. For instance, the principal might use different incomparable payment methods for different agents. We discuss the connection to the stronger assumption of quasi-linearity and transferable utility in more details in Section 4.

Identifying the preference of an agent with her cardinal utility is basic in the theory of modeling economic agents, and in most scenarios, unless additional strong assumptions are added, the utility is only a representation of the preference and does not hold any additional information of the decision behavior of the agent. Moreover, in many cases, one’s assumptions on the agents’ preferences are mostly irrelevant when considering outcomes below some threshold or an outside option. Hence, we think these two insights could be fruitful in other domains as well.

To ease the reading flow, some of the less significant proofs are postponed to the appendix.

2 Model

We denote the set of agents by $\mathcal{N}$, the set of alternatives by $\mathcal{A}$, and their respective size by $n = |\mathcal{N}|$ and $m = |\mathcal{A}|$. Each agent holds a preference, i.e. an order, over $\mathcal{A} \times \mathbb{R}$, the set of tuples of an alternative and
a payment of the agent. We also refer to the preference as the agent’s type. We say a cardinal function \( u : A \times \mathbb{R} \rightarrow \mathbb{R} \) represents \([8, \text{Def. 1.B.2}]\) a preference \( \succeq \) (and refer to \( u \) as the utility function) if for any two alternatives \( a, b \in A \) and two payments \( z_a, z_b \in \mathbb{R} \),

\[
\langle a, z_a \rangle \succ \langle b, z_b \rangle \iff u(a, z_a) \geq u(b, z_b).
\]

It is not hard to see that two utility functions \( u, w : A \times \mathbb{R} \rightarrow \mathbb{R} \) represent the same preference iff one is a monotone transformation of the other, that is, there exists a monotone bijection \( \varphi : \mathbb{R} \rightarrow \mathbb{R} \) s.t. \( u = \varphi \circ w \). In this work, we assume that the preferences of the agents can be represented by a utility function \( u \) s.t. \( u \) is strictly decreasing in the payment, i.e., an agent prefers to pay less. We denote the set of all utility functions which are strictly decreasing in the payment by \( U \) and the set of preferences (types) represented by utility functions from \( U \) by \( \mathcal{T} \). We use the notations of types and type sets in our results to emphasize that our results are invariant to the choice of representations. In places it is not important, in order to ease the readability, we sometimes use utility functions to characterize directly the agents.

A special family of utility functions are the quasi-linear utility functions.\(^1\) These are the utility functions of the form \( u(a, z) = v(a) - z \) for some function \( v : A \rightarrow \mathbb{R} \) which is referred to as the valuation. We denote the set of all quasi-linear utility functions by \( \mathcal{U}^{QL} \). We note that the quasi-linear representations of a preference are closed to shifts by a constant and in particular a preference can be represented by a continuum of quasi-linear utility functions.

**Claim 1.** Let \( u, w : A \times \mathbb{R} \rightarrow \mathbb{R} \) be two quasi-linear utility functions. Then the following three statements are equivalent:

1. Both \( u \) and \( w \) represent the same preference over \( A \times \mathbb{R} \).
2. There exists a constant \( C \in \mathbb{R} \) s.t. for any alternative \( a \in A \) \( u(a, 0) = w(a, 0) + C \).
3. There exists a constant \( C \in \mathbb{R} \) s.t. for any alternative \( a \in A \) and \( z \in \mathbb{R} \) \( u(a, z) = w(a, z) + C \).

In a recent work, Ma el al. defined a new class of utility function extending the quasi-linear family, which they named Parallel utility functions.

**Definition 1 (Parallel utility functions [7]).**

A utility function \( u : A \times \mathbb{R} \rightarrow \mathbb{R} \) is a parallel utility function if \( u(a, z) \) is continuous and strictly decreasing in \( z \), • For any alternative \( a \in A \), \( \lim_{z \to \infty} u(a, z) = \min_{a' \in A} u(a', 0) \), i.e., an agent prefers her worst alternative for free over paying a large enough payment for \( a \), and • For any two alternatives \( a, b \in A \) s.t. \( u(a, 0) \geq u(b, 0) \) and any payment \( z \geq 0 \) s.t. \( u(b, z) \geq \min_{a' \in A} u(a', 0) \) (i.e., any positive payment \( z \) s.t. the agent prefers paying \( z \) for \( b \) over receiving some alternative for free)

\[
u(a, z + (p^a - p^b)) = u(b, z),
\]

for \( p^a \) being the payment \( z \) s.t. \( u(a, z) = \min_{a' \in A} u(a', 0) \), i.e., \( p^a \) is the maximal payment \( z \) s.t. the agent prefers paying it for a over receiving for free her worst alternative, and \( p^b \) is defined similarly.

**Example i.** Let \( u \in \mathcal{U}^{QL} \) be a quasi linear utility function, i.e., \( u(a, z) = v(a) - z \) for some valuation function \( v : A \rightarrow \mathbb{R} \). Then, for any two alternatives \( a, b \in A \), \( p^a = v(a) - \min_{a' \in A} v(a') \) and

\[
u(a, z + (p^a - p^b)) = v(a) - [z + (v(a) - v(b))] = v(b) - z = u(b, z),
\]

i.e., \( u \) is a parallel utility function.

**Example ii.** Generalizing the former, let \( u \in \mathcal{U}^{QL} \) be a quasi-linear utility function and consider a utility function \( u' \in \mathcal{U} \) which coincides with \( u \) whenever \( u(a, z) \geq \min_{a' \in A} u(a', 0) \) (but still \( u' \) is continuous and downward monotone in the payment). For any two alternatives \( a, b \in A \) s.t. \( u'(a, 0) \geq u'(b, 0) \) and any payment \( z \geq 0 \) s.t. \( u'(b, z) \geq \min_{a' \in A} u'(a', 0) \), it holds that • \( u(a, 0) = u'(a, 0) \geq u'(b, 0) \), • \( p^a = u(a, 0) - \min_{a' \in A} u(a', 0) \) and similarly for \( b \). So \( p^a \geq p^b \), and hence

\[
u'(b, z) = u'(b, z) = u(a, z + (p^a - p^b)) = u'(a, z + (p^a - p^b)),
\]

i.e., \( u \) is a parallel utility function.

\(^1\) In the economics literature (e.g., [4]) these functions are also referred to as separable or additively separable.
Example iii. On the other hand, consider the following utility function.

\[ u(a, z) = \begin{cases} 
  z \leq -3 & \rightarrow -3 - z \\
  z \geq -3 & \rightarrow -1 - z/3 
\end{cases} ; 
\]

\[ u(b, z) = -2 - z ; 
\]

\[ u(c, z) = \begin{cases} 
  z \leq -1 & \rightarrow -1 - z \\
  z \geq -1 & \rightarrow -3 - 3z. 
\end{cases} 
\]

Then, \( u \) coincides with the quasi-linear utility function

\[ \begin{cases} 
  u^{QL}(a, z) = -3 - z \\
  u^{QL}(b, z) = -2 - z \\
  u^{QL}(c, z) = -1 - z 
\end{cases} \]

whenever \( u^{QL}(x, z) \geq 0 = 3 + \min_{a' \in A} u^{QL}(a', 0) \) but it is not a parallel utility function. One could see that by noticing that \( p^a = 6, p^b = 1, \) and \( p^c = 0, \) but for \( z = 0: \)

\[ u(a, z + (p^a - p^b)) = u(a, 5) = -\frac{8}{3} \]

\[ u(b, z) = u(b, 0) = -2. \]

Given type domains \( T_1, \ldots, T_n \subseteq T \) for the agents, a direct mechanism with monetary transfers (or shortly a mechanism) for \( \times_{i \in N} T_i \) is defined by two functions \( x: \times_{i \in N} T_i \rightarrow A \) and \( p: \times_{i \in N} T_i \rightarrow \mathbb{R}^N \) (and we notate by \( p_i \) the \( i \)-th coordinate of \( p \)) in the following way: At the first stage each agent reports her type, and based on the reports \( t_1, \ldots, t_n \), the alternative \( x(t_1, \ldots, t_n) \) is chosen and Agent \( i \) pays \( p_i(t_1, \ldots, t_n) \). We say that a mechanism \( (x, p) \) is onto if the function \( x \) is onto, i.e., any alternative is the outcome of some report vector. \(^2\) We use the notation \( T_i \) for \( \times_{j \neq i} T_j \), and say that a mechanism \( (x, p) \) is incentive compatible if it is in an agent best interest to always report truthfully her type. Formally, for any type \( t_i \in T_i \) and any sequence of reports \( t_{-i} \in T_{-i} \),

\[ t_i \in \operatorname{argmax}_{t_i' \in T_i} \left( x(t_i', t_{-i}), p_i(t_i', t_{-i}) \right), \]

where the maximum is taken according to the preference \( t_i \).

An interesting class of incentive-compatible mechanisms are the affine VCG mechanisms which get as reports utility functions and not general preferences.

Definition 2 (Affine VCG mechanism).

We say that a mechanism \( (x, p) \) is an affine VCG mechanism if there exist an agent weight vector \( w \in \Delta(N) \) (i.e., \( w \in [0, 1]^N \) and \( \sum_{i \in N} w_i = 1 \)), an alternative cost vector \( c \in \mathbb{R}^A \), and a non-empty set of alternatives \( A' \subseteq A \) s.t. for any report vector \( (u_1, \ldots, u_n) \in \times_{i \in N} U_i, \)

\[ x(u_1, \ldots, u_n) \in \operatorname{argmax}_{a \in A'} \left( c_a + \sum_{i \in N} w_i \cdot u_i(a, 0) \right) \]

and there exist functions \( h_i: U_{-i} \rightarrow \mathbb{R} \) for \( i = 1, \ldots, n \) s.t. for any report vector \( (u_1, \ldots, u_n) \) the payment of Agent \( i \) is

\[ p_i(u_i, u_{-i}) = h_i(u_{-i}) - 1/w_i \cdot \left( \sum_{j \neq i} w_j \cdot u_j(a^*, 0) + c_a^* \right) \quad \text{for } a^* = x(u_i, u_{-i}). \]

if \( w_i > 0 \) and \( p_i(u_i, u_{-i}) = h_i(u_{-i}) \) if \( w_i = 0 \) (i.e., if Agent \( i \) has no influence of the chosen alternative \( x(u_i, u_{-i}) \)).

In their works, Vickrey [15], Clarke [2], and Groves [5] proved that when for all agents \( U_i \subseteq U^{QL} \), i.e., all agents hold quasi-linear utility functions, these mechanisms are incentive-compatible. In particular, an agent is indifferent between reporting different representations of her type. Furthermore, recalling that two (quasi-linear) utility functions represent the same preference iff they differ by a constant, we note that \( x \) and \( p_i \) are actually invariant to the representation an Agent \( i \) reports. Later, Green and Laffont [4] and Roberts [13] proved that if for all agents \( U_i = U^{QL} \), these are the only direct onto incentive-compatible mechanisms.

\(^2\) In the social choice and economics literature [1, 10], this property is also referred to as Citizen sovereignty and as Non-imposition.
Theorem (Roberts [13, Thm. 3.1] & Green and Laffont [4, Thm. 3]).

Let \((x, p)\) be an onto incentive-compatible mechanism for the case \([\forall i \ U_i = U^{QL}]\). Then, it is an affine VCG mechanism.

3 Main Result

In this work, we introduce a weaker notion of representation we call positive-representation.

Definition 3 (Positive-representation).

We say a utility function \(u: A \times \mathbb{R} \to \mathbb{R}\) pos-represents a preference \(\succ\) if for any two alternatives \(a, b \in A\) and payments \(z_a, z_b \in \mathbb{R}\) s.t. either \(u(a, z_a) \geq 0\) or \(u(b, z_b) \geq 0\),
\[
\langle a, z_a \rangle \succ \langle b, z_b \rangle \iff u(a, z_a) \geq u(b, z_b).
\]

That is, we require \(u\) to represent the preference only above some threshold (The choice of zero as the threshold is for normalization). E.g., in Example iii we saw a preference which coincides with a quasi-linear utility function \(u^{QL}\) whenever the utility is positive (but the threshold was not \(\min_{a \in A} u^{QL}(a, 0)\) and the utility was not a parallel utility). One could think of this threshold as representing an outside option so we are not interested in modeling the agent’s preferences between two unacceptable alternatives. This allows one to represent the agent using a simpler utility function (quasi-linear in our case). For example, a common practice is to model a buyer in a combinatorial auction using a valuation function \(v: 2^X \to \mathbb{R}\) which assigns a monetary value to each subset of items, and a quasi-linear utility function \(u(S, z) = v(S) - z\) for \(S \subseteq X\) and \(z \in \mathbb{R}\). In particular, such representation entails a preference between two outcomes in which the buyer over-paid for the bundle she received. But in most cases, it is assumed a buyer can always refuse to over-pay, and it might be a better modeling to assume indifference. Moreover, commonly this preference pays no role in the analysis, so the analysis should be applicable also if the buyer has a different preference between such outcomes.

In this work we are interested in agents that can be pos-represented by quasi-linear utility functions, and show that the characterization of incentive-compatible mechanisms is naturally extended for these preferences.

It is straight-forward from the definition that if a utility function \(u\) represents a preference then in particular \(u\) pos-represents the preference. Moreover, as a corollary of Claim 1 we get the following.

Corollary 1. Let \(u \in U^{QL}\) be a quasi-linear utility function and \(\succ\) a preference s.t. \(u\) represents \(\succ\). Then, for any constant \(C \in \mathbb{R}\) the utility function \(u + C\) pos-represents \(\succ\).

Note that only a weaker version of Claim 1 holds for pos-representations. One can find two quasi-linear utility functions \(u, v \in U^{QL}\) which differ by a constant but \(u\) pos-represents a preference which \(v\) does not. For instance, take \(A = \{a, b, c\}\) and the preference which is defined by the following utility function \(w: A \times \mathbb{R} \to \mathbb{R}\):
\[
w(a, z) = 1 - z; \quad w(b, z) = \begin{cases} z \leq 2 & 2 - z \\ z \geq 2 & 4 - 2z \end{cases}; \quad w(c, z) = \begin{cases} z \leq 3 & 3 - z \\ z \geq 3 & 9 - 3z.\end{cases}
\]
Then, this preference is pos-represented by the quasi-linear utility function \( u(a, z) = 1 - z \)
\( u(b, z) = 2 - z \) but not by \( u(c, z) = 3 - z \).

\[
\begin{align*}
w(a, z) &= 1 - z \\
w(b, z) &= 2 - z \\
w(c, z) &= 3 - z
\end{align*}
\]

\[
\begin{align*}
u'(a, 6) &= 7 - z \\
u'(b, 6) &= 8 - z \\
u'(c, 6) &= 9 - z
\end{align*}
\]

For instance, \( w(a, 6) = -5 \), \( w(b, 6) = -8 \), and \( w(c, 6) = -9 \).

Claim 2. Let \( u, w : A \times \mathbb{R} \rightarrow \mathbb{R} \) be two quasi-linear utility functions.

1. If both \( u \) and \( w \) pos-represent the same preference, then there exists a constant \( C \in \mathbb{R} \) s.t.
   \[
   \forall a \in A, \ z \in \mathbb{R} \quad w(a, z) = u(a, z) + C.
   \]

2. If there exists a positive constant \( C \geq 0 \) s.t.
   \[
   \forall a \in A, \ z \in \mathbb{R} \quad w(a, z) = u(a, z) - C,
   \]
   then \( w \) pos-represents any preference which is pos-represented by \( u \).

The mechanism

Given a non-empty set of alternatives \( A' \subseteq A \), an agent weight vector \( w \in \Delta(N) \), and an alternative cost vector \( c \in \mathbb{R}^A \), we define the following mechanism for the scenario that for all \( i \in \mathcal{N} \), any type of Agent \( t \), \( t \in T_i \), is pos-represented by (at least one) quasi-linear utility function \( u \in U_{QL} \).

Mechanism 1. Given a vector of reports \((t_1, \ldots, t_n)\) for \( t_i \in T_i \),
- Define \( u_i \) to be an arbitrary quasi-linear utility function which pos-represents \( t_i \).
- Choose \( a^* \in \arg\max_{a \in A'} (c_a + \sum_{i \in \mathcal{N}} w_i \cdot u_i(a, 0)) \) and return
  - The chosen alternative: \( x(t_1, \ldots, t_n) = a^* \).
  - The payment of Agent \( i \): If \( w_i > 0 \),
    \[
    p_i(t_1, \ldots, t_n) = h_i(t_{-i}) - 1/w_i \cdot \left( c_{a^*} + \sum_{j \neq i} w_j \cdot u_j(a^*, 0) \right),
    \]
    for some function \( h_i \) which depends only on the reports of the other agents, and in case \( w_i = 0 \),
    \[
    p_i(t_1, \ldots, t_n) = h_i(t_{-i}).
    \]

There is no constraint on the choice of the pos-representations \( u_i \) in the first step of the mechanism. Nevertheless, we note that, similarly to our note for the VCG mechanism, the choice of pos-representation for Agent \( i \) does not influence the chosen alternative \( x \) or her payment \( p_i \). Based on Claim 2, the set \( \arg\max_{a \in A'} (c_a + \sum_{i \in \mathcal{N}} w_i \cdot u_i(a, 0)) \) does not depend on the chosen pos-representations, and the payment of
Agent \( i \) is influenced only by the choice of pos-representations for other agents. Hence, one can derive a mechanism with a more natural input language which gets (a compact representation of) some pos-representation of the agent’s preference. We show that Mechanism 1 is incentive-compatible under an additional bound on the payment functions \( h_i \).

**Theorem 1.**

Let \( \emptyset \neq \mathcal{A}' \subseteq \mathcal{A} \), \( w \in \Delta(\mathcal{N}) \) be an agent weight vector, \( c \in \mathbb{R}^A \) an alternative cost vector, and \( (x, p) \) Mechanism 1 defined by \( \langle \mathcal{A}', w, c \rangle \). Then \( (x, p) \) is an incentive-compatible mechanism whenever the following hold:

- For all \( i \in \mathcal{N} \), any type of Agent \( i \) is pos-represented by a quasi-linear utility function \( u \in U^{QL} \).
- For all \( i \in \mathcal{N} \), the function \( h_{-i} \) in the definition of Mechanism 1 satisfies that for any profile of types \( t_1 \in T_1, \ldots, t_n \in T_n \), there exist quasi-linear utility functions \( u_1, \ldots, u_n \in U^{QL} \) s.t. \( u_i \) pos-represents \( t_i \) and

\[
h_i(t_{-i}) \leq \frac{1}{w_i} \cdot \max_{a \in \mathcal{A}'} \left[ c_a + \sum_{j \neq i} w_j \cdot u_j(a, 0) \right].
\]

Before proving the theorem, it is worthwhile to understand the bound on \( h_i \) for the following two scenarios: If all types of all agents are represented by some quasi-linear utility function, then for any function \( h_i \) we can find pos-representations of the types s.t. the bound will hold (since for any constant \( C > 0 (u + C) \) represents, and hence also pos-represent, the same preference as \( u \)). If all types of all agents are pos-represented by some quasi-linear utility function \( u_i \in U^{QL} \) which satisfies \( \min_{a \in \mathcal{A}} u_i(a, 0) \geq 0 \), then the Clarke pivot rule [12, Def. 9.19]

\[
h_i(t_{-i}) = \frac{1}{w_i} \cdot \max_{a \in \mathcal{A}'} \left[ c_a + \sum_{j \neq i} w_j \cdot u_j(a, 0) \right]
\]

satisfies this bound. Last, we note that such function might not exist when the type sets are too rich. E.g., when \( c_a \equiv 0 \), \( w_i \equiv 1 \), and \( T_i = \{u_\alpha\}_{\alpha \in \mathbb{R}} \) for \( v(x) = \begin{cases} x = a & 1 \\ x = b & 2 \\ x = c & 3 \end{cases} \) and \( u_\alpha(x, z) = \begin{cases} z \leq v(x) - \alpha & v(x) - \alpha - z \\ O/W & (v(x) - \alpha - z)^3 \end{cases} \), \( u_\alpha \) (for large enough \( \alpha \)) is pos-represented only by quasi-linear utility functions satisfying \( u^{QL}(x, 0) \leq -\alpha \) for all alternatives \( x \in \mathcal{A} \), so the bound is not feasible.

**Proof of Theorem 1.**

We will show that Agent \( i \) maximizes her preference by reporting truthfully. First, we notice that if \( w_i = 0 \) this claim is trivial since the agent has no influence on the chosen alternative or on her payment. Henceforth, we assume that \( w_i > 0 \).

Let \( t_i \in T_i \) be the preference of Agent \( i \), and let \( t_{-i} \in \times_{j \neq i} T_j \) be the reports of the other agents. Next, for \( j = 1, \ldots, n \), let \( u_j \) be the quasi-linear pos-representation of \( t_j \) which was chosen by the mechanism. First, we show that Agent \( i \) maximizes the utility \( u_i \) by reporting truthfully. If an alternative \( a \) is chosen, the utility is

\[
u_i(a, 0) - h_i(t_{-i}) + \frac{1}{w_i} \cdot \left[ c_a + \sum_{j \neq i} w_j \cdot u_j(a, 0) \right],
\]

and this expression is maximized when

\[
c_a + \sum_j w_j \cdot u_j(a, 0)
\]

is maximized which is what happens when Agent \( i \) reports truthfully.

\(^3\) Our proof follows the steps of the incentive-compatibility proof of the VCG mechanism of Nisan [12, Prop. 9.31] with few modifications.
Based on Claim 2, the set of maximizers, \( \text{argmax}_{a \in A} \left( c_a + \sum_{j \in N} w_j \cdot u_j(a, 0) \right) \), is invariant to the choice of pos-representations of the agents’ preferences. By the assumption on \( h_{-i} \), there exist pos-representations \( u_1, \ldots, u_n \in U^{QL} \) of the agents preferences s.t.

\[
\begin{align*}
  u_i(x(t_1, \ldots, t_n), p_i(t_1, \ldots, t_n)) &= \frac{1}{w_i} \cdot \left[ c_{x(t_1, \ldots, t_n)} + \sum_{j} w_j \cdot u_j(a^*, 0) \right] - h_i(t_{-i}) \geq 0,
\end{align*}
\]

so maximizing \( u_i \) also maximizes the preference of Agent \( i \).

Next, we show that when the type sets \( T_i \) are rich enough, there is no other onto incentive-compatible mechanisms.

**Theorem 2.**

*If there are at least three alternatives (\( |A| \geq 3 \)) and for any \( i \in N \) there exist a bijection \( \varphi_i \) between \( T_i \) and \( \{ u \in U^{QL} \mid \min_{a \in A} u(a, 0) = 0 \} \) s.t. for any type \( t \in T_i \), \( t \) is pos-represented by \( \varphi_i(t) \), then any incentive-compatible onto mechanism \( (x, p) \) s.t. for any profile of types \( t_1 \in T_1, \ldots, t_n \in T_n \), the payment of the \( i \)-th agent satisfies

\[
p_i(t_1, \ldots, T_n) \leq [\varphi_i(t)](x(t_1, \ldots, t_n), 0),
\]

can be defined as Mechanism 1 w.r.t. \( \mathcal{A}' = A \), an agent weight vector \( w \in \Delta(N) \), and an alternative cost vector \( c \in \mathbb{R}^d \).*

Following the steps of Roberts [13, Thm. 3.2], we get as a corollary that without monetary transfers the only incentive-compatible mechanisms are dictatorships.

**Corollary 2.**

*If there are at least three alternatives and the type sets \( \{ T_i \}_{i \in N} \) satisfy the conditions of Thm. 2, then for any incentive-compatible onto mechanism \( (x, p) \) without transfers (i.e., \( p_i(t_1, \ldots, t_n) = 0 \) for all \( i \in N \)), there exists a unique agent \( d \in N \) (a dictator) s.t. for any type profile \( (t_1, \ldots, t_n) \)

\[
x(t_1, \ldots, t_n) \in \text{argmax}_{a \in A} u_d(a, 0).
\]

**Proof of Theorem 2.**

Let \( (x, p) \) be an incentive-compatible onto mechanism as stated in the theorem. We define the following auxiliary mechanism for the case in which the type sets of all agents are \( U^{QL} \), i.e., all quasi-linear utility functions. (Since \( \varphi_i \) is a bijection we use the notation \( \varphi_i \) for \( \varphi_i^{-1} \) as well)

**Mechanism.** Given a vector of reports \( (u_1, \ldots, u_n) \in (U^{QL})^N \),

\begin{itemize}
  \item For \( i = 1, \ldots, n \)
    \begin{itemize}
      \item Define \( \tilde{u}_i \in U^{QL} \) by \( \tilde{u}_i(a, 0) = u_i(a, 0) - \min_{a' \in A} u_i(a', 0) \).
      \item Define \( t_i \in T_i \) to be \( \varphi_i(\tilde{u}) \in T_i \).
    \end{itemize}
  \item Apply the mechanism \( (x, p) \) on the type vector \( (t_1, \ldots, t_n) \) and return the same.
\end{itemize}

Then the following two properties hold:

- The mechanism is onto. Let \( a \in A \). Since \( (x, p) \) is an onto mechanism, there exists a type profile \( (t_1 \in T_1, \ldots, t_n \in T_n) \) which is mapped by the mechanism \( (x, p) \) to the alternative \( a \). For \( i = 1, \ldots, n \), let \( u_i \) be a quasi-linear utility function s.t. \( \varphi_i(u_i) = t_i \). For this profile, in the first step of the mechanism \( u_i = \tilde{u}_i \) and it is mapped to \( t_i \) and hence the chosen alternative is \( a \).

- The mechanism is incentive compatible. We will show that Agent \( i \) maximizes her utility by reporting truthfully. Let \( u_i \in U^{QL} \) be the utility of Agent \( i \), \( u_{-i} \in \times_{j \neq i} U^{QL} \) be the reports of the other agents, and \( (a, z) \) be the outcome of the mechanism when Agent \( i \) reports truthfully, i.e., for the profile \( (u_i, u_{-i}) \). Next, consider a report for Agent \( i \), \( \tilde{u}_i \), and let \( (\tilde{a}, \tilde{z}) \) be the outcome for the profile \( (\tilde{u}_i, u_{-i}) \). We define
Mechanism 2. Given a vector of reports \((Vickrey, Clarke, Groves, Green-Laffont, and Roberts [15, 2, 5, 13, 4])\) of linear utility functions, we get the classic incentive-compatibility characterization for quasi-linear utilities of the agents' types. For the special case in which all types of all agents can be represented by quasi-linear utility functions we get as corollaries the following mechanism and theorem. In particular, if the agents are reporting their quasi-linear utility functions, so also the mechanism, so also the type \(h\) that pos-represents \(t\) and our assumption on the payments, we get that \(\tilde{u}_i(a, z_i) \geq \tilde{u}_i(\tilde{a}, \tilde{z}_i)\) and hence

\[
 u_i(a, z_i) = \tilde{u}_i(a, z_i) + \min_{a' \in A} u_i(a', 0) \geq \tilde{u}_i(\tilde{a}, \tilde{z}_i) + \min_{a' \in A} u_i(a', 0) = u_i(\tilde{a}, \tilde{z}_i).
\]

Hence, by the theorems of Roberts [13, Thm. 3.1] and Green-Laffont [4], there exist an agent weight vector \(w \in \Delta(N)\) and an alternative cost vector \(c \in \mathbb{R}^A\) s.t. for any report vector \((u_1, \ldots, u_n) \in \left(U^{QL}\right)^N\) the alternative returned by the mechanism \(a^*\) satisfies

\[
a^* \in \arg \max_{a \in A} \left( c_a + \sum_{i \in N} w_i \cdot u_i(a, 0) \right),
\]

and for \(i = 1, \ldots, n\), the payment of Agent \(i\) equals to

\[
p_i(u_i, u_{-i}) = h_i(u_{-i}) - 1/w_i \cdot \left( \sum_{j \neq i} w_j \cdot u_j(a^*, 0) + c_{a^*} \right)
\]

if \(w_i > 0\) and to \(h_i(u_{-i})\) otherwise, for some function \(h_i : U_{-i} \rightarrow \mathbb{R}\).

By Claim 2, if two quasi-linear utility functions \(u, w \in U^{QL}\) pos-represent a preference \(t_i \in \mathcal{T}_i\), then \(u\) and \(w\) differ by a constant. Since the set of maximizers, \(\arg \max_{a \in A} \left( c_a + \sum_{i \in N} w_i \cdot u_i(a, 0) \right)\), is invariant to such constant shifts, \(a^* \in \arg \max_{a \in A} \left( c_a + \sum_{i \in N} w_i \cdot u_i(a, 0) \right)\) for any choice of quasi-linear pos-representations of the agents' types. Hence, we get the required characterization of the allocation rule \(x\).

Next, notice that for \(u, w \in U^{QL}\) as above, \(u\) and \(w\) are mapped to the same type at the first stage of the mechanism, so also \(h_i\) does not depend on the pos-representations. I.e., it can be defined as a function of the types \(h_i : \mathcal{T}_{-i} \rightarrow \mathbb{R}\), which gives us the required characterization of the payment rule \(p\).

Types represented by quasi-linear utility functions

For the special case in which all types of all agents can be represented by quasi-linear utility functions we get as corollaries the following mechanism and theorem. In particular, if the agents are reporting their quasi-linear utility functions, we get the classic incentive-compatibility characterization for quasi-linear utilities of Vickrey, Clarke, Groves, Green-Laffont, and Roberts [15, 2, 5, 13, 4].

Mechanism 2. Given a vector of reports \((t_1, \ldots, t_n)\) for \(t_i \in \mathcal{T}_i\),

- Define \(u_i\) to be an arbitrary quasi-linear utility function which represents \(t_i\).
- Choose \(a^* \in \arg \max_{a \in A'} \left( c_a + \sum_{i \in N} w_i \cdot u_i(a, 0) \right)\) and return
  - The chosen alternative: \(x(t_1, \ldots, t_n) = a^*\).
  - The payment of Agent \(i\): If \(w_i > 0\),

\[
p_i(t_1, \ldots, t_n) = h_i(t_{-i}) - 1/w_i \cdot \left( c_{a^*} + \sum_{j \neq i} w_j \cdot u_j(a^*, 0) \right),
\]

for some function \(h_i\) which depends only on the reports of the other agents, and in case \(w_i = 0\),

\[
p_i(t_1, \ldots, t_n) = h_i(t_{-i}).
\]

Theorem 3. If all types of all agents are represented by a quasi-linear utility functions, then
(a) Let \( \emptyset \neq A' \subseteq A \), \( w \in \Delta(N) \) be an agent weight vector, \( c \in \mathbb{R}^A \) an alternative cost vector, and \( \langle x, p \rangle \) Mechanism 2 defined by \( (A', w, c) \). Then \( \langle x, p \rangle \) is an incentive-compatible mechanism.

(b) If there are at least three alternatives \( |A| \geq 3 \) and for all agents \( i \in N \), any quasi-linear utility function \( u \in \mathcal{U}^{QL} \) represents some type \( t \in T_i \), then any incentive-compatible onto mechanism \( \langle x, p \rangle \) can be defined as Mechanism 2 w.r.t. \( A' = A \), an agent weight vector \( w \in \Delta(N) \), and an alternative cost vector \( c \in \mathbb{R}^A \).

Characterizations

For completeness, we also show two characterizations, one in utility terms and one in preference terms, of the types which are represented by a quasi-linear function and of the types which are pos-represented by a quasi-linear function.

**Theorem 4.**

Let \( u \) be a utility function which is strongly downward monotone in money. Then,

- \( u \) is represented by a quasi-linear utility function iff there exist a function \( v: A \to \mathbb{R} \) and a strongly monotone function \( \varphi: \mathbb{R} \to \mathbb{R} \) s.t. \( u(a, z) = \varphi(v(a) - z) \).

- \( u \) is pos-represented by a quasi-linear utility function iff there exist a function \( v: A \to \mathbb{R} \) and a strongly monotone function \( \varphi: \mathbb{R} \to \mathbb{R} \) s.t. \( u(a, z) = \varphi(v(a) - z) \) whenever \( z \preceq v(a) \).

**Theorem 5.**

Let \( \succeq \) be a strongly downward monotone in money preference. Then,

- \( \succeq \) is represented by a quasi-linear utility function iff there exist an alternative \( a^* \in A \) and a payment \( z^* \in \mathbb{R} \) s.t.

1. The preference \( \succeq \) is continuous [8, Def. 3.C.1] in money.
2. For any \( x \in A \) there exist \( z_x, z'_x \in \mathbb{R} \) s.t. \( \langle x, z_x \rangle \prec \langle a^*, z^* \rangle \prec \langle x, z'_x \rangle \).
3. For any two alternatives \( x, y \in A \) and payments \( z_x, z_y \in \mathbb{R} \) s.t. \( \langle x, z_x \rangle \succeq \langle y, z_y \rangle \) it holds that

\[ \forall \alpha \in \mathbb{R} \quad \langle x, z_x - \alpha \rangle \succeq \langle y, z_y - \alpha \rangle \, . \]

- \( \succeq \) is pos-represented by a quasi-linear utility function iff there exist an alternative \( a^* \in A \) and a payment \( z^* \in \mathbb{R} \) s.t.

1. For any \( x \in A \) there exist \( z_x, z'_x \in \mathbb{R} \) s.t. \( \langle x, z_x \rangle \prec \langle a^*, z^* \rangle \prec \langle x, z'_x \rangle \).
2. For any two alternatives \( x, y \in A \) and payments \( z_x, z_y \in \mathbb{R} \) s.t.

\[ \forall \alpha \in \mathbb{R}^+ \quad \langle x, z_x - \alpha \rangle \succeq \langle y, z_y - \alpha \rangle \, . \]

4 Discussion

Identifying the preference of an agent with her cardinal utility is basic in the theory of modeling economic agents, and in most scenarios, unless additional strong assumptions are added, the utility is only a representation of the preference and does not hold any additional information of the decision behavior of the agent. This rather simple insight allows one to extend incentive-compatibility results (or similar results regarding decisions of individuals) beyond a-priori restrictive domains of utility functions. Moreover, in many cases, one’s assumptions on the agents’ preferences are mostly irrelevant when considering outcomes below some threshold or an outside option, which might allow us to extend such results even further.

In this work, we formalized this insight by defining a new representation notion, pos-representation, and showed that the characterization of VCG mechanisms extends naturally and easily to type sets which are pos-represented by quasi-linear utility functions. The proof technique of a simple reduction to the works of Roberts [13] and Green-Laffont [4] also hints that one could see our characterization as a fuller version of the VCG characterization. We expect that also the characterization of Holmstrom [6] can extended in a similar fashion to characterization of type domains. Holmstrom [6] shows that one can replace the assumption of
unrestrictiveness of the valuation domains by a weaker assumption that the domains are smoothly-connected. We also suspect that finding the counterpart of smooth-connectiveness in preferences terms would be interesting by its own.

We expect this insight to be applicable in many more scenarios in which results deal with quasi-linear utilities while essentially not assuming the cardinal utility is an exact unique representation of the agents’ preferences, and moreover when assuming it represents the preferences only for a well-defined subdomain of the outcomes.

### 4.1 The quasi-linear model (The transferable utility assumption)

As we argued in Section 1, our result shows that the main driving-force of the VCG characterization is not the ability to compare the intensity of preferences of the different agents and in particular we did not assume a transferable utility model. Our mechanism is given the ordinal preferences of the agents and outputs payments but in no place in the characterization we assumed that ‘money’ is the same commodity for all agents and one could think of scenarios is which the designer would fine different agents using different commodities (a trivial example would be fining using different currencies). The main message we see in our work is that the main driving-force of the VCG characterization is the separability of the agents’ preferences and the ability of the designer to fine the agents on a continuous scale, by that leaving the transferable utility domain of quasi-linear utilities and returning to the non-transferable utility domain of Gibbard-Satterthwaite.

On the other hand, it is important to note that in our general model, (a) Exactly because lacking the transferable utility assumption, *budget-balancedness* becomes more complex to define and it no longer implies Pareto-optimality of the mechanism; (b) A randomized mechanism is not necessarily equivalent to a deterministic one (See also below).

### 5 Other ordinal generalization

Common to the pos-representation notion we presented here and the Parallel utilities notion of Ma et al. [7] is that both can be defined as representation by a quasi-linear utility function over a subset of the domain: In the pos-representation case, whenever the utility is above a threshold; And in the Parallel utilities domain, whenever the utility and the payment are positive. In a subsequent work, we follow this common trait and show that our work extends easily to deal with quadrant domains in which both the utility and the payment are bounded from below by some threshold.

Moreover, we can generalize the pos-representation assumption of the preferences to assume that a type is pos-represented by a utility function of the form \( u_i(a, z) = u_i(a, 0) - \phi_i(z) \) for some continuous monotone bijection \( \phi_i : \mathbb{R} \to \mathbb{R} \) (i.e., an individual utility of money), and our characterization easily extends to this case as well under an additional assumption that the designer knows the functions \( \phi_i \) for all agents.\(^4\)

Last, we note that some of our initial assumptions on the mechanisms can be easily relaxed.

#### 5.1 Non-direct revelation mechanisms

One could consider more general mechanisms in which the agents, instead of reporting utility functions or valuation vectors, use more abstract actions in a sequential manner or a one-round one, and instead of incentive-compatibility require implementation using dominant strategies. Applying a simple direct revelation principle [11] shows that any such general dominant strategy implementation is equivalent to a direct revelation incentive-compatible mechanism. The two mechanisms implement the same mapping of the agents’ private preferences to a chosen alternative and a payment. Hence, more general mechanisms cannot implement different allocation rules than those implemented by direct-revelation mechanisms. Still, one might prefer using non-direct mechanisms for other reasons, like a more natural input language or lower complexity of the mechanism.

\(^4\) This knowledge assumption might seems very strong, but note that this is the common assumption in the quasi-linear model.
5.2 Non-Deterministic mechanisms

The characterization problem of random incentive compatible mechanisms remains open. We did not model the preferences of the agents over lotteries of outcomes, and in particular did not assume they have von Neumann-Morgenstern preferences over lotteries [8, Def. 6.B.5]. Hence we can not derive results regarding truthful mechanisms except the following easy claim.

**Corollary 3.** Given the conditions of Theorem 2, the only universally truthful mechanisms [12, Def. 9.38] (i.e., truth-telling is a dominant strategy for all coin flips by the mechanism) are lotteries over mechanisms of the type of Mechanism 1.

Extending our work to random mechanisms, even when assuming von Neumann-Morgenstern preferences, should include also extending the notion of pos-representation. But we conjecture that in many cases such extensions, especially of pos-representation using quasi-linear utility functions, will lose a lot of the bite of the pos-representation notion.
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A Proof of Claim 1

Claim. Let \( u, w : A \times \mathbb{R} \rightarrow \mathbb{R} \) be two quasi-linear utility functions. Then the following three statements are equivalent:

1. Both \( u \) and \( w \) represent the same preference over \( A \times \mathbb{R} \).
2. There exists a constant \( C \in \mathbb{R} \) s.t. for any alternative \( a \in A \) \( u(a, 0) = w(a, 0) + C \).
3. There exists a constant \( C \in \mathbb{R} \) s.t. for any alternative \( a \in A \) and \( z \in \mathbb{R} \) \( u(a, z) = w(a, z) + C \).

Proof.

(1) \( \Rightarrow \) (2): If both \( u \) and \( w \) represent the same preference, then

\[
\forall a, b \in A, z_a, z_b \in \mathbb{R} \quad u(a, z_a) \geq u(b, z_b) \iff w(a, z_a) \geq w(b, z_b).
\]

Both \( u \) and \( w \) are quasi-linear so we get that

\[
\forall a, b \in A, z_a, z_b \in \mathbb{R} \quad u(a, 0) - u(b, 0) \geq (z_a - z_b) \iff w(a, 0) - w(b, 0) \geq (z_a - z_b),
\]

and hence

\[
\forall a, b \in A \quad u(a, 0) - u(b, 0) = w(a, 0) - w(b, 0)
\]

and there exists a constant \( C \in \mathbb{R} \) s.t.

\[
\forall a \in A \quad u(a, 0) = w(a, 0) + C.
\]

(2) \( \Rightarrow \) (3): Since \( u(a, z) = u(a, 0) - z \) and \( w(a, z) = w(a, 0) - z \), this is a trivial derivation.

(3) \( \Rightarrow \) (1): If there exists a constant \( C \in \mathbb{R} \) s.t.

\[
\forall a \in A, z \in \mathbb{R} \quad u(a, z) = w(a, z) + C,
\]

then

\[
\forall a, b \in A, z_a, z_b \in \mathbb{R} \quad u(a, z_a) \geq u(b, z_b) \iff w(a, z_a) \geq w(b, z_b).
\]
B Proof of Claim 2

Claim. Let \( u, w : A \times \mathbb{R} \to \mathbb{R} \) be two quasi-linear utility functions.
1. If both \( u \) and \( w \) pos-represent the same preference, then there exists a constant \( C \in \mathbb{R} \) s.t.
   \[
   \forall a \in A, z \in \mathbb{R} \quad w(a, z) = u(a, z) + C.
   \]
2. If there exists a positive constant \( C \geq 0 \) s.t.
   \[
   \forall a \in A, z \in \mathbb{R} \quad w(a, z) = u(a, z) - C,
   \]
   then \( w \) pos-represents any preference which is pos-represented by \( u \).

Proof.
1. If both \( u \) and \( v \) pos-represent the same preference, then for any two alternatives \( a, b \in A \) and \( z_a, z_b \in \mathbb{R} \) s.t. \( z_a \leq \min(u(a, 0), w(a, 0)) \) we get that \( u(a, z_a) \geq 0 \) and \( w(a, z_a) \geq 0 \) so
   \[
   u(a, z_a) \geq u(b, z_b) \iff w(a, z_a) \geq w(b, z_b).
   \]
   Both \( u \) and \( w \) are quasi-linear so we get that
   \[
   u(a, 0) - u(b, 0) \geq (z_a - z_b) \iff w(a, 0) - w(b, 0) \geq (z_a - z_b),
   \]
   and hence
   \[
   \forall a, b \in A \quad u(a, 0) - u(b, 0) = w(a, 0) - w(b, 0)
   \]
   and there exists a constant \( C \in \mathbb{R} \) s.t.
   \[
   \forall a \in A \quad u(a, 0) = w(a, 0) + C, \quad \text{and} \quad \forall a \in A, z \in \mathbb{R} \quad u(a, z) = w(a, z) + C.
   \]
2. Let \( \succeq \) be a preference which is pos-represented by \( u \). Let \( a, b \in A \) be two alternatives and \( z_a, z_b \in \mathbb{R} \) two payments s.t. either \( w(a, z_a) \geq 0 \) or \( w(b, z_b) \geq 0 \). Then, either \( u(a, z_a) \geq w(a, z_a) \geq 0 \) or \( u(b, z_b) \geq w(b, z_b) \geq 0 \), and
   \[
   w(a, z_a) \geq w(b, z_b) \iff u(a, z_a) \geq u(b, z_b) \iff \langle a, z_a \rangle \succeq \langle b, z_b \rangle.
   \]
\[\square\]
C  Proof of Corollary 2

Corollary.

If there are at least three alternatives and the type sets \( \{ T_i \}_{i \in \mathcal{N}} \) satisfy the conditions of Thm. 2, then for any incentive-compatible onto mechanism \( (x, p) \) without transfers (i.e., \( p_i (t_1, \ldots , t_n) \equiv 0 \) for all \( i \in \mathcal{N} \)), there exists a unique agent \( d \in \mathcal{N} \) (a dictator) s.t. for any type profile \( (t_1, \ldots , t_n) \)

\[
x(t_1, \ldots , t_n) \in \arg\max_{a \in \mathcal{A}} u_d(a, 0).
\]

Proof.

By Thm. 2, there exists an agent weight vector \( w \in \Delta(\mathcal{N}) \) and an alternative cost vector \( c \in \mathbb{R}^A \) s.t. for any report vector,

\[
x(t_1, \ldots , t_n) \in \arg\max_{a \in \mathcal{A}} \left( c_a + \sum_{i \in \mathcal{N}} w_i \cdot u_i(a, 0) \right).
\]

Let \( d \in \mathcal{N} \) be an agent s.t. \( w_d > 0 \) and notice that if Agent \( d \) knows the actions of the others she can enforce any outcome in \( \mathcal{A} \). Since there are no monetary transfers and the mechanism is incentive compatible we get that for any type profile \( (t_1, \ldots , t_n) \)

\[
x(t_1, \ldots , t_n) \in \arg\max_{a \in \mathcal{A}} u_d(a, 0).
\]

Since this property cannot hold for two different agents the dictator must be unique. \( \square \)
D Proof of Theorem 4

Claim 3. Let \( u \) be a utility function which is strongly downward monotone in money. Then, \( u \) is represented by a quasi-linear utility function iff there exist a function \( v : A \to \mathbb{R} \) and a strongly monotone function \( \varphi : \mathbb{R} \to \mathbb{R} \) s.t.

\[
u(a, z) = \varphi (v(a) - z).
\]

Proof.
\( \Leftarrow \): Trivial since \( \varphi (v(a) - z) \) and \( (v(a) - z) \) represent the same preference.

\( \Rightarrow \): By definition, there exists a function \( v : A \to \mathbb{R} \) s.t. for any \( x, y \in A \) and \( z_x, z_y \in \mathbb{R} \)

\[
u(x, z_x) \geq \nu(y, z_y) \iff (x) - z_x \geq (y) - z_y,
\]

and in particular

\[
u(x, z_x) = \nu(y, z_y) \iff (x) - z_x = (y) - z_y,
\]

and \( u(x, z_x) = \nu(y, z_x + (y) - (x)) \).

Let \( a^* \) be an arbitrary alternative and define a mapping \( \varphi : \mathbb{R} \to \mathbb{R} \) by \( \varphi (\alpha) = u(a^*, v(a^*) - \alpha) \). Then, \( \varphi \) is monotone: If \( \alpha < \beta \) then \( v(a^*) - \alpha > v(a^*) - \beta \) and hence

\[
\varphi (\alpha) = u(a^*, v(a^*) - \alpha) < u(a^*, v(a^*) - \beta) = \varphi (\beta).
\]

\( u(a, z) = \varphi (v(a) - z) \): For any \( a \in A \) and \( z \in \mathbb{R} \): \( u(a, z) = u(a^*, z + (a^*) - v(a)) = \varphi (v(a) - z). \) \( \Box \)
Claim 4. Let $u$ be a utility function which is strongly downward monotone in money. Then $u$ is positionally represented by a quasi-linear utility function iff there exist a function $v: \mathcal{A} \to \mathbb{R}$ and a strongly monotone function $\varphi: \mathbb{R} \to \mathbb{R}$ s.t.

$$u(a, z) = \varphi(v(a) - z) \quad \text{whenever } z \leq v(a).$$

Proof.

$\Leftarrow$: Trivial since $\varphi(v(a) - z)$ and $(v(a) - z)$ represent the same preference and hence pos-represented by the same utility functions.

$\Rightarrow$: By definition, there exists a function $v: \mathcal{A} \to \mathbb{R}$ s.t. for any two alternatives $x, y \in \mathcal{A}$ and payments $z_x, z_y \in \mathbb{R}$, if $v(x) \geq z_x$,

$$u(x, z_x) \geq u(y, z_y) \iff v(x) - z_x \geq v(y) - z_y,$$

and in particular

$$u(x, z_x) = u(y, z_y) \iff v(x) - z_x = v(y) - z_y,$$

and $u(x, z_x) = u(y, z_x + v(y) - v(x)).$

Let $a^*$ be an arbitrary alternative and define a mapping $\varphi: \mathbb{R} \to \mathbb{R}$ by

$$\varphi(\alpha) = \begin{cases} 
\alpha \geq 0 & u(a^*, v(a^*) - \alpha) \\
\alpha \leq 0 & u(a^*, v(a^*)) + \alpha.
\end{cases}$$

Then, $\varphi$ is monotone:

- If $\alpha < \beta \leq 0$ then $\varphi(\beta) - \varphi(\alpha) = \beta - \alpha > 0$.
- If $0 \leq \alpha < \beta$ then $\varphi(\beta) - \varphi(\alpha) = u(a^*, v(a^*) - \beta) - u(a^*, v(a^*) - \alpha) > 0$.
- If $\alpha \leq 0 < \beta$ then $\varphi(\alpha) = u(a^*, v(a^*)) + \alpha \leq u(a^*, v(a^*)) < u(a^*, v(a^*) - \beta) = \varphi(\beta)$.

$$u(a, z) = \varphi(v(a) - z) \quad \text{whenever } z \leq v(a); \quad \text{For any } a \in \mathcal{A} \text{ and } z \leq v(a):$$

$$u(a, z) = u(a^*, z + v(a^*) - v(a)) = \varphi(v(a) - z). \quad \square$$
\section*{E \hspace{1cm} Proof of Theorem 5}

\textbf{Claim 5.} A preference $\succ$ is represented by a quasi-linear utility function iff there exist an alternative $a^* \in A$ and a payment $z^* \in \mathbb{R}$ s.t.

1. The preference $\succ$ is continuous in money.
2. For any $x \in \mathcal{A}$ there exists $z_x \in \mathbb{R}$ s.t. $\langle x, z_x \rangle \succ (a^*, z^*)$.
3. For any $x, y \in \mathcal{A}$ and payments $z_x, z_y \in \mathbb{R}$ s.t. $\langle x, z_x \rangle \succ (y, z_y)$ it holds that
   \[ \forall \alpha \in \mathbb{R} \quad \langle x, z_x - \alpha \rangle \succ (y, z_y - \alpha). \]

\textbf{Proof.}

$\Rightarrow$: Trivial by the properties of quasi-linear functions.

$\Leftarrow$: First, we show that given any alternative $a \in \mathcal{A}$, $\succ$ is weakly monotone in money. Assume for contradiction that there exist $z_1 < z_2 < z_3$ s.t. \[
\begin{cases}
\langle a, z_1 \rangle \prec \langle a, z_2 \rangle \\
\langle a, z_2 \rangle \succ \langle a, z_3 \rangle
\end{cases}
\]
and \[
\langle a, z_3 \rangle \succ \langle a, z_1 \rangle
\]
are symmetric. By (1) there exists $z'_3 \in (z_1, z_3]$ s.t. $\langle a, z'_3 \rangle \sim \langle a, z_3 \rangle$. W.l.o.g., $z'_3 = z_3$. By (3), for any $0 \leq k \leq n$

\[ (a, z_1) \sim \langle a, z_1 + k/n (z_3 - z_1) \rangle \sim \langle a, z_1 \rangle. \]

On the other hand, by (1) there exists a $\varepsilon > 0$ s.t.

\[ \forall z'_2 \text{ s.t. } |z_2 - z'_2| < \varepsilon \quad \langle a, z'_2 \rangle \succ (a, z_1), \]

so we get a contradiction.

Moreover, we show that $\succ$ is strongly monotone in money. Assume for contradiction that there exist $z_1 < z_2$ s.t. $\langle a, z_1 \rangle \sim \langle a, z_2 \rangle$. Then, by weak monotonicity, for any $z \in [z_1, z_2]$

\[ \langle a, z_1 \rangle \sim \langle a, z \rangle \sim \langle a, z_2 \rangle, \]

and by (3) $\langle a, z_1 \rangle \sim \langle a, z \rangle$ for any $z \in \mathbb{R}$, in contradiction to (2).

Note that it cannot be that $\succ$ is upward monotone in money for some $x \in \mathcal{A}$ and downward monotone for a different $y \in \mathcal{A}$. Otherwise, by (2), there exist $z_x, z_y \in \mathbb{R}$ s.t. $\langle x, z_x \rangle \succ (a^*, z^*) \succ \langle y, z_y \rangle$, and by the monotonicity and (2), there exists $\delta > 0$ large enough s.t.

\[ \langle x, z_x - \delta \rangle \prec (a^*, z^*) \quad \text{and} \quad \langle y, z_y - \delta \rangle \succ (a^*, z^*), \]

in contradiction to (3).

W.l.o.g, assume that $\succ$ is strongly downward monotone in money (Otherwise, we'll show it is represented by a utility function quasi-linear in $(-z)$). Next, we define $v : \mathcal{A} \to \mathbb{R}$ by

\[ v(x) = \max \{ z \mid \langle x, z \rangle \succ (a^*, z^*) \}, \]

and note that

$-$ By (1) the set $\{ z \mid \langle x, z \rangle \succ (a^*, z^*) \}$ is closed.

$-$ By (2) the set $\{ z \mid \langle x, z \rangle \succ (a^*, z^*) \}$ is not empty and bounded from above.

Hence, the maximum exists. Moreover, because of (1), it holds that $\langle x, v(x) \rangle \sim (a^*, z^*)$ and

\[ \forall x, y \in \mathcal{A} \quad \langle x, v(x) \rangle \sim (y, v(y)). \]

Now, for any $x, y \in \mathcal{A}$ and $z_x, z_y \in \mathbb{R}$

\[ \langle x, z_x \rangle \succ (y, z_y) \iff (3) \langle x, v(x) \rangle \succ (y, z_y + v(x) - z_x) \]
\[ \iff \langle y, v(y) \rangle \succ (y, z_y + v(x) - z_x) \]
\[ \iff (\text{MON}) v(y) \leq z_y + v(x) - z_x \]
\[ \iff v(x) - z_x \geq v(y) - z_y. \quad \square \]
Claim 6. A strongly downward monotone in money preference $\succsim$ is pos-represented by a quasi-linear utility function iff there exist an alternative $a^* \in A$ and a payment $z^* \in \mathbb{R}$ s.t.

1. For any $x \in A$ there exists a payment $z_x \in \mathbb{R}$ s.t. $(x, z_x) \succ (a^*, z^*)$.

For any $x \in A$ there exists a payment $z_x \in \mathbb{R}$ s.t. $(x, z_x) \prec (a^*, z^*)$.

2. For any two alternatives $x, y \in A$ and payments $z_x, z_y \in \mathbb{R}$ s.t.

\[
\forall \alpha \in \mathbb{R}_+ \quad (x, z_x - \alpha) \succsim (y, z_y - \alpha).
\]

Proof.

$\Rightarrow$: Trivial by the properties of quasi-linear functions and choosing $a^*$ and $z^*$ s.t. the quasi-linear function satisfies $u(a^*, z^*) = 0$.

$\Leftarrow$: First, we note that since $\succsim$ is strongly monotone, it is continuous in money.\(^5\) Let $a \in A$ and $\{z_t\}, \{w_t\}$ be two sequences of monetary transfers s.t. $\forall t \quad (a, z_t) \trianglerighteq (a, w_t)$, $\lim_{t \to \infty} z_t = z$, and $\lim_{t \to \infty} w_t = p$. Then, $\forall t \quad w_t \leq z_t$ and hence $w \leq z$ and $(a, w) \succsim (a, z)$.

Next, we define $v: A \to \mathbb{R}$ by

\[
v(x) = \max \{ z \mid (x, z) \succsim (a^*, z^*) \},
\]

and note that

- By continuity the set $\{ z \mid (x, z) \succsim (a^*, z^*) \}$ is closed.

- By (1) the set $\{ z \mid (x, z) \succsim (a^*, z^*) \}$ is not empty and bounded from above.

Hence, the maximum exists. Moreover, because of the continuity, it holds that $(x, v(x)) \sim (a^*, z^*)$ and

\[
\forall x, y \in A \quad (x, v(x)) \sim (y, v(y)).
\]

Now, for any $x, y \in A$ and $z_x, z_y \in \mathbb{R}$ s.t. $v(x) - z_x \geq 0$

\[
(x, z_x) \succsim (y, z_y) \iff (2) (x, v(x)) \succsim (y, z_y + v(x) - z_x) \\
\iff (y, v(y)) \succsim (y, z_y + v(x) - z_x) \\
\iff v(y) \leq z_y + v(x) - z_x \\
\iff v(x) - z_x \geq v(y) - z_y.
\]

\(\square\)

---

\(^5\) We note that this claim does not hold for weakly downward monotone preferences. For example, the preference which is defined by $u(a, z) = \begin{cases} 0 & z \leq 0 \\ 1 & z > 0 \end{cases}$ is weakly downward monotone in money but is not continuous. The preference is not continuous since $\forall n \quad (a, 1/n) \preceq (a, 1/2)$ but $(a, 0) \succ (a, 1/2)$.  

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21