A LOWER BOUND FOR THE DETERMINANTAL COMPLEXITY OF A HYPERSURFACE

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Abstract. We prove that the determinantal complexity of a hypersurface of degree \( d > 2 \) is bounded below by one more than the codimension of the singular locus, provided that this codimension is at least 5. As a result, we obtain that the determinantal complexity of the 3 \( \times \) 3 permanent is 7. We also prove that for \( n > 3 \), there is no nonsingular hypersurface in \( \mathbb{P}^n \) of degree \( d \) that has an expression as a determinant of a \( d \times d \) matrix of linear forms while on the other hand for \( n \leq 3 \), a general determinantal expression is nonsingular. Finally, we answer a question of Ressayre by showing that the determinantal complexity of the unique (singular) cubic surface containing a single line is 5.

1. Introduction

Let \( k \) be a field. For a positive integer \( m \), let \( X = (x_{ij})_{1 \leq i,j \leq m} \) denote an \( m \times m \) matrix of linear forms. Let \( \det_m = \det(X) \in k[x_{ij}] \) be the determinant polynomial.

Definition 1.1. Let \( f(x) \in k[x_1, \ldots, x_n] \) be a polynomial. A determinantal expression of size \( m \) for \( f \) is an affine linear map \( L: k^n \to k^{m \times m} \) such that \( f(x) = \det_m(L(x)) \). The determinantal complexity of \( f \), denoted by \( dc(f) \), is the smallest \( m \) such that there exists a determinantal expression of size \( m \) for \( f \).

The main result of this paper is the following lower bound for the determinantal complexity of a homogeneous polynomial \( f \in k[x_1, \ldots, x_n] \). We denote by \( \text{Sing}(f) \) the singular locus of the hypersurface \( V(f) \subseteq k^n \).

Theorem 1.2. Let \( k \) be a field. Let \( f \in k[x_1, \ldots, x_n] \) be a homogeneous polynomial of degree \( d > 2 \). If \( \text{codim}(\text{Sing}(f)) > 4 \), then

\[
dc(f) \geq \text{codim}(\text{Sing}(f)) + 1.
\]

Our first application of this result is to give a new lower bound for the determinantal complexity of the permanent polynomial

\[
\text{perm}_n = \sum_{\sigma \in S_n} x_{1\sigma(1)}x_{2\sigma(2)}\cdots x_{n\sigma(n)}.
\]

In [Val79a] and [Val79b], Valiant conjectured that \( dc(\text{perm}_n) \) is not bounded above by any polynomial in \( n \) as long as \( \text{char}(k) \neq 2 \). Moreover, Valiant proposed an algebraic analogue of the P versus NP question by introducing the complexity classes \( \text{VP}_e \) and \( \text{VNP} \) consisting of sequences of polynomials which are “computable by
arithmetic trees in polynomial time” and “definable in polynomial time,” respectively. Valiant showed that if \( dc(\text{perm}_n) \) grows faster than any polynomial, then in fact \( VP \neq VNP \).

In characteristic 0, the best known bounds for the determinantal complexity of \( \text{perm}_n \) for \( n > 2 \) are:

\[
\frac{n^2}{2} \leq dc(\text{perm}_n) \leq 2^n - 1
\]

where the lower bound was established by Mignon and Ressayre [MR04] (whose argument was subsequently generalized in [CCL08] to characteristic \( p \neq 2 \) to provide the bound \( (n-2)(n-3)/2 \leq dc(\text{perm}_n) \)) and the upper bound was established by Grenet [Gre11] (which holds in any characteristic).

Since it is known that \( \text{codim}(\text{Sing}(\text{perm}_n)) > 4 \) for \( n > 2 \) (c.f. [vzG87, Lem. 2.3]) if \( \text{char}(k) \neq 2 \), we obtain:

**Corollary 1.3.** Let \( k \) be a field with \( \text{char}(k) \neq 2 \). If \( n > 2 \), then

\[
dc(\text{perm}_n) \geq \text{codim}(\text{Sing}(\text{perm}_n)) + 1.
\]

For \( n = 3 \), the inequalities (1.1) imply that \( 5 \leq dc(\text{perm}_3) \leq 7 \). It has been an open question to determine the actual value of \( dc(\text{perm}_3) \). Similarly, for \( n = 4 \), the inequalities (1.1) imply that \( 8 \leq dc(\text{perm}_4) \leq 15 \). Since it can be readily computed that \( \text{codim}(\text{Sing}(\text{perm}_3)) = 6 \) and \( \text{codim}(\text{Sing}(\text{perm}_4)) = 8 \) (c.f. [vzG87]), we obtain:

**Corollary 1.4.** Let \( k \) be a field with \( \text{char}(k) \neq 2 \). Then \( dc(\text{perm}_3) = 7 \) and \( dc(\text{perm}_4) \geq 9 \).

It was recently shown in [HI15] that if \( \text{char}(k) = 0 \), the smallest determinantal expression for \( \text{perm}_3 \) such that every entry is 0, 1 or a variable has size 7.

**Remark 1.5.** Since any matrix where the first two columns are zero is a singular point of \( V(\text{perm}_n) \), it follows that \( \text{codim}(\text{Sing}(\text{perm}_n)) \leq 2n \). Therefore, even if the codimension of \( \text{Sing}(\text{perm}_n) \) achieves the maximum value \( 2n \), Corollary 1.3 would only give the linear bound \( dc(\text{perm}_n) \geq 2n + 1 \).

Our second application is toward the determinantal complexity of homogeneous forms of low degree. It is a classical problem to determine which homogeneous forms of degree \( d \) have a determinantal expression of size \( d \); see [Dic21] (or [Bea00] for a modern treatment). For this discussion, we assume that \( k \) is algebraically closed. As any binary form \( f(x, y) \) of degree \( d \) factors into a product of \( d \) linear forms, it is clear that \( dc(f) = d \). It is also known that any (possibly singular) plane curve \( f(x, y, z) \) of degree \( d \) has \( dc(f) = d \) [Bea00] Rmk. 4.4 and any quadratic form \( f \) in \( n \leq 4 \) variables has \( dc(f) = 2 \). It is a classical fact that a general cubic surface \( (n = 4, d = 3) \) admits a determinantal expression of size 3 [Sch63, Cre68]. However, in all other cases \( (n = 4, d > 3 \text{ or } n > 4) \), a dimension count yields that a general homogeneous form \( f(x_1, \ldots, x_n) \) of degree \( d \) has \( dc(f) > d \) [Dic21, Thm. 1]. As a direct consequence of Theorem 1.2 we obtain the following bound for the determinantal complexity for nonsingular hypersurfaces in the case that \( n > 4 \) and \( d \leq n \):

**Corollary 1.6.** Let \( k \) be a field. Assume that \( n > 4 \) and \( d \leq n \). If \( f(x_1, \ldots, x_n) \) is a non-degenerate homogeneous form (i.e. the hypersurface \( V(f) \subseteq \mathbb{P}^{n-1} \) is nonsingular) of degree \( d > 2 \), then \( dc(f) \geq n + 1 \).
In particular, Corollary 1.6 implies that for $n > 4$ and $d \leq n$, there does not exist a nonsingular hypersurface of degree $d$ in $\mathbb{P}^{n-1}$ having a determinantal expression of size $d$.

Using Bertini’s theorem, we can obtain the even stronger result:

**Theorem 1.7.** Let $k$ be an algebraically closed field with $\text{char}(k) = 0$. Let $L: k^n \to k^{m \times m}$ be a linear map with $m \geq 2$. Let $f(x) := \det_m(L(x))$. Then $\text{codim}(\text{Sing}(f)) \leq \min(4, n)$ and equality holds if $L$ is a general linear map.

In particular, if we let $V(f) \subseteq \mathbb{P}^{n-1}$ be the projective hypersurface defined by $f$, then the following statements hold:

1. if $n \leq 4$ then $V(f)$ is nonsingular for a general linear map $L$; and
2. if $n > 4$ then $V(f)$ is singular for every linear map $L$.

We now consider how the vector space $\text{Sym}^d(k^n)^\vee$ of homogeneous forms in $n$ variables of degree $d$ decomposes by the determinantal complexity. For this discussion, we assume that $k$ is algebraically closed of characteristic 0. For $n = 2$ or 3, this decomposition of $\text{Sym}^d(k^n)^\vee$ is trivial as all non-zero forms have determinantal complexity $d$. The first interesting case is quadratic forms. For a quadratic form $f$ in $n > 4$ variables of rank $r$, the determinantal complexity of $f$ is $\lceil (r+1)/2 \rceil$ for $r \geq 4$ and 2 otherwise [MR04, Thm. 1.4]. Thus, the decomposition of $\text{Sym}^2(k^2)^\vee$ by the determinantal complexity is the same as the stratification by the above function of the rank $r$.

The next interesting case to consider is cubic surfaces. It is a classical fact that any nonsingular cubic surface has a determinantal expression of size 3; see [Gra55] and [Bea00]. More generally, it was shown in [BL98] that any cubic surface not projectively equivalent to $f = xy^2 + yt^2 + z^3$ has determinantal complexity 3 and moreover that $\text{dc}(f) > 3$. This form $f$ is also the unique cubic surface (up to projective equivalence) in $\mathbb{P}^3$ containing a single line. While it is possible to write down a determinantal expression of size 5, it has been an open question to determine whether $\text{dc}(f)$ is 4 or 5. Using a similar idea to the proof of Theorem 1.2, we establish:

**Theorem 1.8.** Let $k$ be a field with $\text{char}(k) = 0$. Then $\text{dc}(xy^2 + yt^2 + z^3) = 5$.

**Remark 1.9.** It is worthwhile to include the following well-known observation: since $\text{dc}(xy^2 + yt^2 + z^3) > 3$, one sees that the determinantal complexity function $f \mapsto \text{dc}(f)$ is not in general upper semicontinuous (i.e. the locus of forms $f$ with $\text{dc}(f) \geq m$ for a fixed $m$ is not necessarily Zariski-closed). Indeed, the cubic surface $xy^2 + yt^2 + z^3$ degenerates to singular cubic surfaces (e.g. $z^3 = \lim_{r \to 0} \epsilon xy^2 + \epsilon yt^2 + z^3$) with determinantal complexity 3. Nevertheless, for each $m$, the locus of forms $f$ with $\text{dc}(f) = m$ is constructible.

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2. **Proofs**

We begin with an easy generalization of [vzG87] Thm. 3.1.
Proposition 2.1. Let $k$ be any field. Let $f \in k[x_1, \ldots, x_n]$ be a homogeneous polynomial $f \in k[x_1, \ldots, x_n]$ satisfying $\text{codim}(\text{Sing}(f)) > 4$. If $L : k^n \to k^{m \times m}$ is a determinantal expression for $f$, then $\text{im}(L) \cap \text{Sing}(\det_m) = \emptyset$. In particular, $\text{rank}(L(0)) = m - 1$.

Proof. Let $y_{kl}$ for $1 \leq k, l \leq m$ be coordinates on $k^{m \times m}$ and write $L = (L_{kl})$ where each $L_{kl}$ is an affine linear form in $x_1, \ldots, x_n$. By the chain rule,

$$\frac{\partial f}{\partial x_i}(x) = \sum_{1 \leq k, l \leq m} \frac{\partial \det_m(L(x))}{\partial y_{kl}} \frac{\partial L_{kl}}{\partial x_i}(x)$$

and it follows that $L^{-1}(\text{Sing}(\det_m)) \subseteq \text{Sing}(f)$. If $L^{-1}(\text{Sing}(\det_m)) \neq \emptyset$, then

$$\text{codim}(\text{Sing}(f)) \leq \text{codim}(L^{-1}(\text{Sing}(\det_m))) \leq \text{codim}(\text{Sing}(\det_m)) = 4$$

where the second inequality is the standard bound for the codimension of an inverse image (c.f. [Har95, Thm. 17.24]) and the last equality follows from the fact that the singular locus of $\det_m$ consists of matrices $A$ with $\text{rank}(A) \leq m - 2$. The above inequalities contradict our hypothesis that $\text{codim}(\text{Sing}(f)) > 4$.

For the final statement, we know that $\text{rank}(L(0)) \geq m - 1$. But since $f$ is homogeneous, we have $\det(L(0)) = f(0) = 0$ which implies that $\text{rank}(L(0)) = m - 1$. \hfill $\square$

Remark 2.2. Proposition 2.1 is true more generally (with the same proof) for any morphism $L : k^n \to k^{m \times m}$ of varieties such that $f(x) = \det_m(L(x))$.

Proof of Theorem 1.2. Let $L : k^n \to k^{m \times m}$ be a determinantal expression for $f$. Proposition 2.1 implies that $\text{rank}(L(0)) = n - 1$. By multiplying $L$ by matrices on the left and right, we may assume that $J = L(0)$ is the $m \times m$ matrix $(J_{ij})$ with $J_{ii} = 1$ for $2 \leq i \leq m$ and $J_{ij} = 0$ otherwise. Therefore, $L(x) = J + Z(x)$ where $Z = (Z_{ij})$ is an $m \times m$ matrix where each $Z_{ij}$ is a linear form in $x = (x_1, \ldots, x_n)$. Since $f(x) = \det_m(J + Z(x))$ and the left hand side is homogeneous of degree $d > 2$, we conclude that the equations $Z_{11} = 0$ and $\sum_{j=2}^m Z_{ij}Z_{j1} = 0$ hold.

Let $I \subseteq k[x_1, \ldots, x_n]$ be the ideal generated by the first row and column of $Z$. Since $Z_{11} = 0$, every summand of $\det(J + Z(x))$ is divisible by a product of two elements of $I$. In particular, $f \in I^2$ and all partial derivatives $\partial f/\partial x_i$ are in $I$. Thus, $V(I) \subseteq \text{Sing}(f)$.

We now obtain an upper bound on the codimension of $V(I)$ as follows. We introduce the linear map

$$G : k^n \to k^{2(m-1)}$$

$$(x_1, \ldots, x_n) \mapsto (Z_{12}, Z_{13}, \ldots, Z_{1m}, Z_{21}, Z_{31}, \ldots, Z_{m1}).$$

Since $V(I) = \ker(G)$, we have that $\text{codim}(V(I)) = \dim(\text{im}(G))$. Now $\text{im}(G)$ is a linear subspace which is entirely contained in the non-degenerate quadric $\sum_{j=2}^m w_{1j}w_{j1} = 0$ in $k^{2(n-1)}$ where $w_{12}, \ldots, w_{1m}, w_{21}, \ldots, w_{m1}$ are the coordinates on $k^{2(n-1)}$. Such subspaces have dimension at most $m - 1$ (c.f. [Har95, pg. 289]).

We conclude that

$$\text{codim}(\text{Sing}(f)) \leq \text{codim}(V(I)) \leq m - 1$$

so that $m \geq \text{codim}(\text{Sing}(f)) + 1$ as claimed. \hfill $\square$

\footnote{If $\text{char}(k) = 2$, one checks that the argument of [Har95, pg. 289] applies to $Q(w) = \sum_{j=2}^m w_{1j}w_{j1}$ using the bilinear form $Q_0(w, w') = Q(w + w') - Q(w) - Q(w')$.}
Proof of Theorem 1.7. We begin by making the following observation. If $L: k^n \to k^{m \times m}$ is not injective then $V(f) \subseteq k^n$ is a cone over the hypersurface $V(f') \subseteq k^s$ where $f'$ is the determinant of the linear map $k^s = k^n / \ker(L) \to k^{m \times m}$ (i.e. there is a choice of basis $x_1, \ldots, x_s, x_{s+1}, \ldots, x_n$ for $\mathbb{P}^{n-1}$ such that the hypersurface defined by $L$ does not involve the variables $x_{s+1}, \ldots, x_n$). It follows that the codimension of $\text{Sing}(f)$ in $k^n$ is equal to the codimension of $\text{Sing}(f')$ in $k^s$.

We first show that $\text{codim}(\text{Sing}(f)) \leq 4$. By the above observation, we may assume that $L$ is injective and that $n \leq m^2$. Therefore, $V(f)$ is the intersection of the determinantal hypersurface $V(\det_m) \subseteq k^{m \times m}$ with $m^2 - n$ hyperplanes. As the intersection of a variety with a singular point $p$ with a hypersurface passing through $p$ also has a singularity at $p$, the codimension of the singular locus can only decrease after intersecting. As $\text{codim}(\text{Sing}(\det_m)) = 4$, we see that $\text{codim}(\text{Sing}(f)) \leq 4$.

We now show that $\text{codim}(\text{Sing}(f)) = \min(4, n)$ for general linear maps $L$. By the above observation, we may assume that $L$ is injective and $n \leq m^2$. Then the projective hypersurface $V(f) \subseteq \mathbb{P}^{m^2-1}$ is the intersection of $V(\det_m) \subseteq \mathbb{P}^{m^2-1}$ with $m^2 - n$ general hyperplanes. We recall Bertini’s Theorem (c.f. [Har95 Thm. 17.16]): if $X \subseteq \mathbb{P}^N$ is any variety over $k$ and $H$ is a general hyperplane, then $\text{Sing}(H \cap X) = H \cap \text{Sing}(X)$, and moreover $\dim(\text{Sing}(H \cap X)) = \dim(\text{Sing}(X)) - 1$ provided that $\dim(\text{Sing}(X)) > 0$.

We apply Bertini’s Theorem $m^2 - n$ times. Since the singular locus of $V(\det_m)$ in $\mathbb{P}^{m^2-1}$ has dimension $m^2 - 5$, we can conclude that the singular locus of a hypersurface $V(f) \subseteq \mathbb{P}^{m^2-1}$ defined by a general linear map $L$ has dimension $m^2 - 5 - (m^2 - n) = n - 5$ if $n \geq 5$ and is empty if $n < 5$, which was our intended goal.

For the final statements, if $n \leq 4$ (resp. $n > 4$), then we have shown that $\text{codim}(\text{Sing}(f)) = n$ for general linear maps $L$ (resp. $\text{codim}(\text{Sing}(f)) \leq 4$ for every linear map $L$) which implies that $\mathbf{V}(f) \subseteq \mathbb{P}^{n-1}$ is nonsingular for general $L$ (resp. singular for every $L$).

□

Proof of Theorem 1.8. By [BL98 Prop. 4.3], we know that $\text{dc}(f) > 3$. If $\text{dc}(f) = 4$, then we can assume that there is a determinantal expression $L = J + Z: k^4 \to k^{4 \times 4}$, where $Z = (Z_{ij})$ is a matrix of linear forms and $J = (J_{ij})$ is the matrix of rank $r$ with $J_{ii} = 1$ for $5 - r \leq i \leq 4$ and 0 otherwise. We will show that each possibility for the rank $r$ yields a contradiction. First, the rank $r$ cannot be 0 or 4 as $f$ is homogeneous of degree 3. If $r = 1$, the degree 3 component of $\det(L)$, namely $\det(Z_{ij})_{1 \leq i,j \leq 3}$, gives a determinantal expression of $f$ of size 3, contradicting the fact that $\text{dc}(f) > 3$.

If $r = 2$, then $Z_{11}Z_{22} - Z_{12}Z_{21} = 0$. We will argue that we can reduce to the case that $Z_{11} = Z_{22} = 0$. Indeed, if $Z_{11} = 0$, then either $Z_{21} = 0$ or $Z_{12} = 0$, and in the latter case we replace $L$ with its transpose. If $Z_{11} \neq 0$, then either $Z_{12}$ or $Z_{21}$ is a multiple of $Z_{11}$, and after potentially replacing $L$ with its transpose, we may assume that $Z_{12} \in (Z_{11})$. We may replace $L$ by $P^{-1}LP$ where $P$ is an invertible matrix of constants, so that $Z_{12} = 0$. But then $Z_{22} = 0$ and the claim is established by interchanging the first and second column (and negating one column). Since $y = z = 0$ is the unique line contained in this cubic surface, we can replace $L$ with
after replacing $L$ anti-symmetric, we can write $Z$ $Z$ $I$ $I$ $I$

To summarize, we have $P \in \text{LP}$ both spans would be equal to $P \in \text{LP}$ $L$ $Z$ $Z$ $L$ $Z$ $Z$

Since $f = 0$ along the subspace $Z_{12} = Z_{22} = 0$, we can use the fact again that $y = z = 0$ is the unique line in this cubic surface to replace $L$ with $P^{-1}LP$ so that $Z_{12} = y$ and $Z_{22} = z$. But then $f \in (y, z)^2$, a contradiction.

Finally, suppose $r = 3$. By the argument in the proof of Theorem 12, we know that $Z_{11} = 0$ and $Z_{12}Z_{21} + Z_{13}Z_{31} + Z_{14}Z_{41} = 0$. Moreover, we know that the dimension of the subspace $I_1 := \langle Z_{12}, Z_{13}, Z_{14}, Z_{21}, Z_{31}, Z_{41} \rangle$ is at most 3 and that if $I$ denotes the ideal generated by $I_1$, then $\text{Jac}(f) := (f_x, f_y, f_z, f_t) \subseteq I$. The equation $V(f) \subseteq k^4$ is singular along $y = z = t = 0$ which yields that $(y, z, t) = \sqrt{\text{Jac}(f)} \subseteq I$ and that $I_1 = \langle y, z, t \rangle$. Either the span of the first row or the first column must be equal to $I_1$; otherwise, as $y = z = 0$ is the unique line in the cubic surface, both spans would be equal to $(y, z)$ contradicting that $I_1 = \langle y, z, t \rangle$. Therefore, after replacing $L$ by $P^{-1}LP$ or $P^{-1}LT$, we can assume that $Z_{21} = z$, $Z_{31} = y$ and $Z_{41} = t$. As the matrix expressing $Z_{12}, Z_{13}, Z_{14}$ in terms of $z, y, t$ is necessarily anti-symmetric, we can write $Z_{12} = \alpha t + \beta y$, $Z_{13} = -\beta z + \gamma t$, and $Z_{14} = -\gamma y - \alpha z$.

To summarize, we have

$$f = \det \begin{pmatrix}
0 & Z_{12} & Z_{13} & Z_{14} \\
0 & Z_{22} & Z_{23} & Z_{24} \\
y & Z_{32} & 1 + Z_{33} & Z_{34} \\
z & Z_{42} & Z_{43} & 1 + Z_{44}
\end{pmatrix}.$$}

Comparing the coefficients in the above expression of the 6 monomials of degree 3 whose $x$-exponent is 1, one obtains six equations that the coefficients $X_{ij}$ of $x$ in $Z_{ij}$ must satisfy:

- $xy^2$: $\beta X_{23} - \gamma X_{43} = 1$
- $xz^2$: $-\beta X_{23} - \alpha X_{42} = 0$
- $xt^2$: $\alpha X_{24} + \gamma X_{43} = 0$
- $xyz$: $\beta(X_{22} - X_{33}) - \gamma X_{42} - \alpha X_{43} = 0$
- $xyz$: $\beta(X_{33} - X_{44}) + \alpha X_{23} + \beta X_{24} = 0$
- $xzt$: $\alpha(X_{22} - X_{44}) + \gamma X_{32} - \beta X_{34} = 0$

One can check that these equations are inconsistent unless $\alpha = 0$ and $\gamma \neq 0$. In this latter case, the top row of $L$ is 0 if $y = -\beta z + \gamma t = 0$ which in turn implies that $f$ vanishes on this subspace, a contradiction. We have therefore established that $\text{dc}(f) > 4$. On the other hand, one can check that

$$f = \det \begin{pmatrix}
-y & z & 0 & 0 & 0 \\
0 & 0 & z & t & x \\
z & 0 & 1 & 0 & 0 \\
0 & t & 0 & 1 & 0 \\
0 & y & 0 & 0 & 1
\end{pmatrix}$$

which implies that $\text{dc}(f) = 5$. □

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