On the blow-up problem for the axisymmetric 3D Euler equations

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Abstract

In this paper we study the finite time blow-up problem for the axisymmetric 3D incompressible Euler equations with swirl. The evolution equations for the deformation tensor and the vorticity are reduced considerably in this case. Under the assumption of local minima for the pressure on the axis of symmetry with respect to the radial variations we show that the solution blows up in finite time. If we further assume that the second radial derivative vanishes on the axis, then the system reduces to the form of Constantin–Lax–Majda equations and can be integrated explicitly.

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1. The axisymmetric 3D Euler equations

We are concerned with the following Euler equations for the homogeneous incompressible fluid flows in a domain \( \Omega \subset \mathbb{R}^3 \),

\[
\frac{Dv}{Dt} = -\nabla p, \tag{1.1}
\]

\[
\text{div } v = 0, \tag{1.2}
\]

\[
v(x, 0) = v_0(x), \tag{1.3}
\]

where \( D/Dt \) is the material derivative defined by

\[
\frac{D}{Dt} = \frac{\partial}{\partial t} + (v \cdot \nabla).
\]

Here \( v = (v_1, v_2, v_3) \), \( v_j = v_j(x, t), j = 1, 2, 3 \), is the velocity of the flow, and \( p = p(x, t) \) is the scalar pressure and \( v_0 \) is the given initial velocity, satisfying \( \text{div } v_0 = 0 \). Since the classical result [12] on the local well-posedness for the 3D Euler equations in the standard Sobolev
space $H^m(\mathbb{R}^3)$, $m > 5/2$, the problem of finite time singularity for such a local smooth solution is still an outstanding open problem, although there is a celebrated result on the blow-up criterion [1] and its refinements [2, 7, 9] taking into account geometric considerations on the vorticity directions. By an axisymmetric solution of the Euler equations we mean the velocity field $v(r, x_3, t)$, solving the Euler equations and having the representation

$$v(r, x_3, t) = v'(r, x_3, t)e_r + v^\theta(r, x_3, t)e_\theta + v^3(r, x_3, t)e_3$$

in the cylindrical coordinate system, where

$$e_r = \left(\frac{x_1}{r}, \frac{x_2}{r}, 0\right), \quad e_\theta = \left(-\frac{x_2}{r}, \frac{x_1}{r}, 0\right), \quad e_3 = (0, 0, 1), \quad r = \sqrt{x_1^2 + x_2^2}.$$ 

In this case also the question of finite time blow-up of the solution is wide open (see, e.g. [3–5] for preliminary studies of the problem; see also [10] for the related recent result in the case of helical symmetry). The vorticity $\omega = \text{curl} \ v$ is computed as

$$\omega = \omega^r e_r + \omega^\theta e_\theta + \omega^3 e_3,$$

where

$$\omega^r = -\partial_3 v^\theta, \quad \omega^\theta = \partial_3 v^r - \partial_r v^3, \quad \omega^3 = \frac{v^\theta}{r} + \partial_r v^\theta.$$

We denote

$$\tilde{v} = v' e_r + v^3 e_3.$$

The Euler equations for the axisymmetric solution are

$$\begin{cases}
\partial_t v^r + (\tilde{v} \cdot \nabla) v^r = -\partial_r p + \frac{(v^\theta)^2}{r}, \\
\partial_t v^3 + (\tilde{v} \cdot \nabla) v^3 = -\partial_3 p, \\
\partial_t v^\theta + (\tilde{v} \cdot \nabla) v^\theta = -\frac{v^r v^\theta}{r}, \\
\text{div} \ \tilde{v} = 0, \\
v(r, x_3, 0) = v_0(r, x_3),
\end{cases}$$

where $\nabla = e_r \partial_r + e_3 \partial_3$. Note that the above representation of the Euler equations in the cylindrical coordinate system is valid off the axis of symmetry, which is chosen to be the $x_3$-axis. Hence, in order to analyse the equation on the $x_3$-axis, we mainly use the equations in the Cartesian coordinate system. Below the functional values of a cylindrically symmetric function $f(x_1, x_2, x_3) = f(r, x_3)$, $r = \sqrt{x_1^2 + x_2^2}$ on the $x_3$-axis should be understood as $\lim_{r \to 0^+} f(r, x_3) := f(x_3)$.

**Theorem 1.1.** In the axisymmetric 3D Euler equations with the symmetry of axis chosen to be the $x_3$-axis, we write

$$\tilde{\omega} := \partial_1 v^2 - \partial_2 v^1 = 2\partial_1 v^2 = 2\partial_r v^\theta, \quad \lambda := \partial_3 v^3,$$

which are defined on the $x_3$-axis. Suppose the initial data satisfy

$$S := \{x_3 \in \mathbb{R} | \tilde{\omega}(x_3) = 0, \lambda_0(x_3) > 0, \partial_3^2 p_0(x_3) \geq 0\} \neq \emptyset,$$

where we denoted

$$\tilde{\omega}_0(x_3) = \tilde{\omega}(x_3, 0), \quad \lambda_0(x_3) = \lambda(x_3, 0), \quad p_0(x_3) = p(x_3, 0).$$

We define $T_1 = T_1(a)$ as

$$T_1 = \inf\{t > 0 | \partial_r^2 p(X_3(a, t), t) < 0\},$$
where $X_3(a, t)$ is the particle trajectory defined by the local classical solution $v(x, t)$.

$$\frac{\partial X_3(a, t)}{\partial t} = v_3(X_3(a, t), t), \quad X_3(a, 0) = a.$$ 

Then there exists no global classical solution to the axisymmetric 3D Euler equations if there exists $a \in S$ such that

$$T_1(a) \geq \frac{1}{\lambda_0(a)}.$$

(1.4)

**Remark 1.1.** After this research was done, Professor Peter Constantin informed me of the preprint [6], where it was shown earlier than this paper that the positivity of the Hessian of the pressure leads to a singularity in the general case. In our case of restriction to the axis of symmetry, the positivity is assumed only for the second radial derivative of the pressure on the axis.

**Remark 1.2.** The assumption on the positivity of the second radial derivative of the pressure on the $x_3$-axis is physically natural in view of the following heuristic argument (originally from Professor Peter Constantin’s private comment). We consider a axisymmetric compressible ideal fluid with swirl. Due to the centrifugal force the density of fluid becomes a local minimum on the $x_3$-axis, which implies a local minimum of pressure on the $x_3$-axis, hence $\partial^2_r p \geq 0$ on the axis. Now we take a zero Mach number limit for the pressure to obtain the pressure of the original axisymmetric incompressible fluid (see [13] for a rigorous result for this singular limit problem). In this limiting procedure it is plausible to expect the preservation of the local minimum property of the pressure on the $x_3$-axis.

**Theorem 1.2.** In the axisymmetric 3D Euler equations with the symmetry of axis chosen to be the $x_3$-axis, let us assume that there exists $T > 0$ such that

$$\partial^2_t p(x_3, t) = 0 \quad \forall (x_3, t) \in \mathbb{R} \times [0, T].$$

(1.5)

Then the pair $(\bar{\omega}, \lambda)$, which is defined in theorem 1.1, can be explicitly given by

$$\bar{\omega}(X_3(a, t), t) = \frac{4\bar{\omega}_0(a)}{(2 - \lambda_0(a)t)^2 + \bar{\omega}_0(a)^2t},$$

(1.6)

$$\lambda(X_3(a, t), t) = \frac{4\lambda_0(a) - 2[\lambda_0(a)^2 + \bar{\omega}_0(a)^2]t}{(2 - \lambda_0(a)t)^2 + \bar{\omega}_0(a)^2t}$$

(1.7)

along the particle trajectory $\{X_3(a, t)\}$ for all $(a, t) \in \mathbb{R} \times [0, T]$. Let us assume $S_0 = \{x_3 \in [\lambda_0(x_3) > 0, \bar{\omega}_0(x_3) = 0] \neq \emptyset$. Then the form of solution (1.7) implies that there exists no global classical solution to the 3D axisymmetric Euler equations if

$$T \geq \inf_{a \in S_0} \frac{2}{\lambda_0(a)}.$$

2. Proof of the main theorems

We begin with the following elementary lemma.

**Lemma 2.1.** Let $v = (v^1, v^2, v^3) = v(x_1, x_2, x_3)$ be an axially symmetric $C^1$-vector field on $\mathbb{R}^3$ with the axis of symmetry chosen as the $x_3$-axis, satisfying $\text{div} v = 0$, and let $p = p(x_1, x_2, x_3)$ be an axially symmetric $C^2$-scalar function on $\mathbb{R}^3$. Then on the axis of
symmetry we have
\[ v^1 = v^2 = \partial_3 v^1 = \partial_3 v^2 = \partial_1 v^3 = \partial_2 v^3 = 0, \]  
\[ v^r = v^\theta = \partial_3 v^r = \partial_3 v^\theta = \partial_r v^3 = 0, \]  
\[ \partial_3 v^1 = \partial_2 v^2 = -\frac{\partial_3 v^3}{2} = \partial_r v^r = \lim_{r \to 0} \frac{v^r}{r}, \]  
\[ \partial_3 v^2 = -\partial_2 v^1 = \partial_r v^\theta = \lim_{r \to 0} \frac{v^\theta}{r}, \]
\[ \partial_1 v^3 = \partial_2 p = \partial_1 \partial_2 p = \partial_1 \partial_3 p = \partial_2 \partial_3 p = \partial_r \partial_3 p = 0, \]  
\[ (2.1) \quad (2.2) \quad (2.3) \quad (2.4) \quad (2.5) \quad (2.6) \]

**Proof.** Here we use the notation
\[ (x_1', x_2') := (-x_2, x_1) \quad \text{and} \quad (\bar{x}_1, \bar{x}_2) := \left( \frac{r}{\sqrt{2}}, \frac{r}{\sqrt{2}} \right), \quad r = \sqrt{x_1^2 + x_2^2}. \]
Let us observe first
\[ v^1(x_1, x_2, x_3) = \frac{x_1}{r} v^r - \frac{x_2}{r} v^\theta \quad \text{and} \quad v^3(x_1, x_2, x_3) = \frac{x_2}{r} v^r + \frac{x_1}{r} v^\theta, \quad r > 0, \]
and thus
\[ v^1(x_1', x_2', x_3) = -\frac{x_2}{r} v^r - \frac{x_1}{r} v^\theta = -v^2(x_1, x_2, x_3), \]  
\[ v^3(x_1', x_2', x_3) = \frac{x_1}{r} v^r - \frac{x_2}{r} v^\theta = v^1(x_1, x_2, x_3). \]

Passing \( r \to 0 \) in (2.7) and (2.8), we find that \( v_1 = v_2 = 0 \) on the \( x_3 \)-axis. On the other hand,
\[ v^1(\bar{x}_1, \bar{x}_2, x_3) + v^2(\bar{x}_1, \bar{x}_2, x_3) = \sqrt{2} v^r(\bar{x}_1, \bar{x}_2, x_3), \]
\[ v^1(\bar{x}_1, \bar{x}_2, x_3) - v^3(\bar{x}_1, \bar{x}_2, x_3) = -\sqrt{2} v^\theta(\bar{x}_1, \bar{x}_2, x_3), \]
and passing \( r \to 0 \) in (2.9) and (2.10), we also find that \( v^\theta = v^r = 0 \) on the \( x_3 \)-axis. Replacing \( v_1, v_2, v^r, v^\theta \) by \( \partial_3 v^1, \partial_3 v^2, \partial_3 v^r, \partial_3 v^\theta \), respectively, in the above argument we also deduce that \( \partial_3 v^1 = \partial_3 v^2 = \partial_3 v^r = \partial_3 v^\theta = 0 \). Next we note that
\[ \partial_1 v^3(x_1, x_2, x_3) = \frac{x_1}{r} \partial_x v^3, \quad \partial_2 v^3(x_1, x_2, x_3) = \frac{x_2}{r} \partial_x v^3, \quad r > 0, \]
and therefore
\[ \partial_1 v^3(x_1', x_2', x_3) = -\frac{x_2}{r} \partial_x v^3 = -\partial_2 v^3(x_1, x_2, x_3), \]  
\[ (2.11) \quad (2.12) \]
for all \( r > 0 \). Similarly to the above, passing \( r \to 0 \) in (2.11) and (2.12), we deduce \( \partial_1 v^3 = \partial_2 v^3 = 0 \) on the \( x_3 \)-axis. Since
\[ \partial_1 v^3(\bar{x}_1, \bar{x}_2, x_3) - \partial_2 v^3(\bar{x}_1, \bar{x}_2, x_3) = \sqrt{2} \partial_r v^3(x_1, x_2, x_3), \]
we are led to \( \partial_1 v^3 = 0 \) on the \( x_3 \)-axis by passing \( r \to 0 \). In order to verify (2.3) we compute
\[ \partial_1 v^1(x_1, x_2, x_3) = \frac{v^r}{r} - \frac{x_2^2}{r^3} v^r + \frac{x_2^2}{r^3} \partial_x v^r + \frac{x_1 x_2}{r^2} \partial_x v^\theta, \]  
\[ \partial_2 v^2(x_1, x_2, x_3) = \frac{v^r}{r} - \frac{x_1^2}{r^3} v^r + \frac{x_1^2}{r^3} \partial_x v^r - \frac{x_1 x_2}{r^2} \partial_x v^\theta. \]
for $r > 0$. Hence,

$$\partial_1 v^1(x_1', x_2', x_3) = \partial_2 v^2(x_1, x_2, x_3) \quad \forall r > 0. \quad (2.13)$$

Passing $r \to 0$ in (2.13), we have $\partial_1 v^1 = \partial_2 v^2$ on the $x_3$-axis. The condition $v = 0$ implies $\partial_1 v^1 = \partial_2 v^2 = -\frac{1}{2} \partial_3 v^3$. We note

$$\partial_1 v^1(\bar{x}_1, \bar{x}_2, x_3) - \partial_2 v^2(\bar{x}_1, \bar{x}_2, x_3) = \frac{v^\theta(x_1, x_2, x_3)}{r} - \partial_r v^\theta(x_1, x_2, x_3) \quad (2.14)$$

and

$$\partial_1 v^1(\bar{x}_1, \bar{x}_2, x_3) + \partial_2 v^2(\bar{x}_1, \bar{x}_2, x_3) = \frac{v^r(x_1, x_2, x_3)}{r} + \partial_r v^r(x_1, x_2, x_3). \quad (2.15)$$

From (2.14) we have

$$\lim_{r \to 0} \frac{v^\theta}{r} = \lim_{r \to 0} \partial_r v^\theta. \quad (2.16)$$

Let us compute

$$\partial_2 v^1(x_1, x_2, x_3) = -\frac{x_1 x_2}{r^3} v^r + \frac{x_1 x_2}{r^2} \partial_r v^r - \frac{v^\theta}{r} + \frac{x_1^2}{r^3} v^\theta - \frac{x_2^2}{r^2} \partial_r v^\theta,$$

$$\partial_1 v^2(x_1, x_2, x_3) = -\frac{x_1 x_2}{r^3} v^r + \frac{x_1 x_2}{r^2} \partial_r v^r + \frac{v^\theta}{r} - \frac{x_1^2}{r^3} v^\theta + \frac{x_2^2}{r^2} \partial_r v^\theta,$$

and, hence

$$\partial_1 v^2(x_1', x_2', x_3) = -\partial_2 v^1(x_1, x_2, x_3) \quad (2.17)$$

for all $r > 0$. Passing $r \to 0$ in (2.17), we obtain $\partial_1 v^2 = -\partial_2 v^1$ on the $x_3$-axis. Let us compute

$$\partial_2 v^1(\bar{x}_1, \bar{x}_2, x_3) + \partial_1 v^2(\bar{x}_1, \bar{x}_2, x_3) = -\frac{v^r(x_1, x_2, x_3)}{r} + \partial_r v^r(x_1, x_2, x_3) \quad (2.18)$$

and

$$\partial_2 v^1(\bar{x}_1, \bar{x}_2, x_3) - \partial_1 v^2(\bar{x}_1, \bar{x}_2, x_3) = -\frac{v^\theta(x_1, x_2, x_3)}{r} - \partial_r v^\theta(x_1, x_2, x_3). \quad (2.19)$$

Equation (2.18) provides us with

$$\lim_{r \to 0} \frac{v^r}{r} = \lim_{r \to 0} \partial_r v^r, \quad (2.20)$$

while (2.19), combined with (2.16), shows $\partial_1 v^2 = \partial_r v^\theta$, respectively, on the $x_3$-axis. Using (2.20), passing $r \to 0$ in (2.15), we deduce $\partial_1 v_1 = \partial_r v_r$ on the $x_3$-axis. As for (2.5) the proof of $\partial_1 p = \partial_2 p = \partial_3 p = \partial_3 p = \partial_r p = \partial_r p = 0$ is exactly the same as the above and we omit it. We note

$$\partial_1 \partial_2 p(x_1, x_2, x_3) = -\frac{x_1 x_2}{r^3} \partial_r p + \frac{x_1 x_2}{r^2} \partial_r^2 p,$$

and find that $\partial_1 \partial_2 p(x_1', x_2', x_3) = -\partial_1 \partial_2 p(x_1, x_2, x_3)$. Hence, passing $r \to 0$, we have $\partial_1 \partial_2 p = 0$ on the $x_3$-axis. We also compute

$$\partial_2^2 p(x_1, x_2, x_3) = \frac{1}{r} \partial_r p + \frac{x_1^2}{r^3} \partial_r p + \frac{x_2^2}{r^2} \partial_r^2 p = \partial_2^2 p(x_1', x_2', x_3)$$

and deduce that

$$\partial_2^2 p = \partial_2^2 p \text{ on the } x_3\text{-axis by passing } r \to 0. \quad (2.21)$$
Note that
\[ \partial_1^2 p + \partial_2^2 p = \frac{1}{r} \partial_r p + \partial_r^2 p \]  
and
\[ \partial_1 \partial_2 p(\bar{x}_1, \bar{x}_2, x_3) = -\frac{1}{2r} \partial_r p + \frac{1}{2} \partial_r^2 p \rightarrow 0 \quad \text{as } r \rightarrow 0. \]  
(2.23)

From (2.21), (2.22) and (2.23) we have \( \partial_1^2 p = \partial_2^2 p = \partial_r^2 p = \lim_{r \to 0} \frac{1}{r} \partial_r p. \) \( \square \)

Next we recall the matrix representation of the Euler equations (see, e.g. [14]). Given velocity \( v(x, t) \) and pressure \( p(x, t) \), we introduce the \( 3 \times 3 \) matrices,
\[ V_{ij} = \frac{\partial v_j}{\partial x_i}, \quad S_{ij} = V_{ij} + V_{ji}, \quad A_{ij} = V_{ij} - V_{ji}, \quad P_{ij} = \partial^2 p / \partial x_i \partial x_j, \]
with \( i, j = 1, 2, 3 \). Then we have the decomposition \( V = (V_{ij}) = S + A \), where \( S = (S_{ij}) \) represents the deformation tensor of the fluid, and \( A = (A_{ij}) \) is related to the vorticity \( \omega \) by the formula,
\[ A_{ij} = \frac{3}{2} \sum_{k=1}^{3} \epsilon_{ijk} \omega_k, \quad \omega_i = 3 \sum_{j,k=1}^{3} \epsilon_{ijk} A_{jk}, \]  
(2.24)

where \( \epsilon_{ijk} \) is the skewsymmetric tensor with the normalization \( \epsilon_{123} = 1 \). Note that \( P = (P_{ij}) \) is the Hessian of the pressure. Let \( \{\lambda_1, \lambda_2, \lambda_3\} \) be the set of eigenvalues of \( S \). Computing partial derivatives \( \partial / \partial x_k \) of (1.1) yields
\[ \frac{DV}{Dt} = -V^2 - P, \]  
(2.25)

Taking the symmetric part of (2.25), we have
\[ \frac{DS}{Dt} = -S^2 - A^2 - P, \]  
(2.26)

from which, using formula (2.24), we derive
\[ \frac{DS_{ij}}{Dt} = -\sum_{k=1}^{3} S_{ik} S_{kj} + \frac{1}{4}(|\omega|^2 \delta_{ij} - \omega_i \omega_j) - P_{ij}, \]  
(2.27)

where \( \delta_{ij} \) is the Kronecker delta defined by \( \delta_{ij} = 1 \) if \( i = j \) and \( \delta_{ij} = 0 \) otherwise. The antisymmetric part of (2.25) is
\[ \frac{DA}{Dt} = -SA - AS, \]  
(2.28)

which, using formula (2.24) again, we obtain easily
\[ \frac{D\omega}{Dt} = S\omega, \]  
(2.29)

which is the well-known vorticity evolution equation that could be derived also by taking the curl of (1.1). Taking the trace of (2.27), we have the identity
\[ - (\lambda_1^2 + \lambda_2^2 + \lambda_3^2) + \frac{1}{2} |\omega|^2 = \Delta p. \]  
(2.30)

Proof of theorem 1.1. Thanks to lemma 2.1 we have the following reduced representation for the deformation tensor, the vorticity and the Hessian of the pressure on the \( x_3 \)-axis.
\[ S = \text{diag} \left( -\frac{\lambda}{2}, -\frac{\lambda}{2}, \lambda \right), \quad \omega = (0, 0, \bar{\omega}), \quad P = \text{diag}(\partial_r^2 p, \partial_r^2 p, \partial_r^2 p). \]  
(2.31)
where
\[ \lambda = \partial_3 v_3 = -2\partial_1 v_1 = -2\partial_2 v_2 = -2\partial_r v_r, \quad \tilde{\omega} = \partial_1 v_2 - \partial_2 v_1 = 2\partial_r v_\theta. \]
on the \( x_3 \)-axis. The (11) and (22) components of the matrix equation (2.27) reduce to
\[ \frac{D\lambda}{Dt} = \frac{\lambda^2}{2} - \frac{\tilde{\omega}^2}{2} + 2\partial_r^2 p, \tag{2.32} \]
where we set
\[ \frac{D}{Dt} = \partial_t + v_3 \partial_3, \]
while the (33) component becomes
\[ \frac{D\lambda}{Dt} = -\lambda^2 - \partial_r^2 p. \tag{2.33} \]
We note that (2.30) reduces to
\[ \Delta p = -\frac{3}{2}\lambda^2 + \frac{\tilde{\omega}^2}{2}, \tag{2.34} \]
which is also obtained by taking the subtraction (2.32) and (2.33). The vorticity equation is written as
\[ \frac{D\tilde{\omega}}{Dt} = \lambda \tilde{\omega}, \tag{2.35} \]
which can be solved as
\[ \tilde{\omega}(X_3(a, t), t) = \tilde{\omega}_0(a) \exp \left[ \int_0^t \lambda(X_3(a, s), s) \, ds \right] \]
along the trajectory. This implies that \( \tilde{\omega}(X_3(a, t), t) = 0 \) for \( a \in S \) as long as the classical solution persists. Hence, for \( a \in S \) (2.33) can be written as
\[ \frac{\partial \lambda(X_3(a, t), t)}{\partial t} \geq \frac{\lambda^2(X_3(a, t), t)}{2} + 2\partial_r^2 p(X_3(a, t), t) \geq \frac{\lambda^2(X_3(a, t), t)}{2} \quad \forall t \in (0, T_1(a)). \tag{2.36} \]
The differential inequality (2.36) can be solved immediately to yield
\[ \lambda(X_3(a, t), t) \geq \frac{2\lambda_0(a)}{2 - \lambda_0(a)} \quad \forall t \in (0, T_1(a)) \quad \text{with} \quad T_s(a) := \min \left\{ T_1(a), \frac{2}{\lambda_0(a)} \right\}, \]
which shows that \( T_1(a) \geq 2/\lambda_0(a) \) is not consistent with the fact that the classical solution persists until \( T(a) \).

**Proof of theorem 1.2.** By the hypothesis equation (2.32) together with (2.35) reduces to
\[ \begin{cases} \frac{D\lambda}{Dt} = \frac{\lambda^2}{2} - \frac{\tilde{\omega}^2}{2}, \\ \frac{D\tilde{\omega}}{Dt} = \lambda \tilde{\omega}. \end{cases} \tag{2.37} \]
This is exactly the same system studied by Constantin–Lax–Majda in [8] with the material derivative replacing the partial derivative in time, which was proposed as a one-dimensional
model equation for the 3D Euler equations in the vorticity formulation. Similarly to [8] we set \( \Theta = \lambda + i\omega \). Then (2.37) becomes the following complex Riccati equation along the trajectory,
\[
\frac{d\Theta}{dt} = \frac{\Theta^2}{2},
\]
which can be solved explicitly as
\[
\Theta(X_3(a, t), t) = \frac{2\Theta_0(a)}{2 - \Theta_0(a)t} = \frac{2\lambda_0(a) + 2i\hat{\omega}_0(a)}{2 - [\lambda_0(a) + i\hat{\omega}_0(a)]t}.
\]
(2.38)

Taking the imaginary and real parts of (2.38) we obtain (1.6) and (1.7).

□

Remark after the proof. In [11] Hou-Li also obtained a system of equations similar in form to (2.37), but for a different pair of unknown functions under completely different assumptions. In our case the system is derived rigorously from the axisymmetric 3D Euler equation by taking the limit \( r \to 0 \) and assuming only \( \frac{\partial^2 r p}{\partial r} = 0 \) on the \( x_3 \)-axis.

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