Time Dependent Behavior of Granular Material in a Vibrating Box

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Abstract

Using numerical and analytic methods, we study the time dependent behavior of granular material in a vibrating box. We find, by molecular dynamics simulation, that the temporal fluctuations of the pressure and the height expansion scale in $A f$, where $A$ ($f$) is the amplitude (frequency) of the vibration. On the other hand, the fluctuations of the velocity and the granular temperature do not scale in any simple combination of $A$ and $f$. Using the kinetic theory of Haff, we study the temporal behaviors of the hydrodynamic quantities by perturbing about their time averaged values in the quasi-incompressible limit. The results of the kinetic theory disagree with the numerical simulations. The kinetic theory predicts that the whole material oscillates roughly as a single block. However, the numerical simulations show that the region of active particle movement is localized and moves with time, behavior very similar to the propagation of a sound wave.

Keywords: Granular media; Vibration; Wave; Scaling; Kinetic theory
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1 Introduction

Systems of granular particles (e.g. sand) exhibit many interesting phenomena, such as segregation under vibration or shear, density waves in the outflow through a hopper and a tube, and the formation of heaps and convection cells under vibration [1-5]. These phenomena are consequences of the unusual dynamical response of the systems, and are for the most part still poorly understood.

We focus on the vertical vibration of a box containing granular particles. There are many interesting phenomena associated with this system, such as convection cells [6-11], heap formation [12-17], sub-harmonic instability [18], surface waves [19, 20] and even turbulent flows [21]. The basis for understanding these diverse phenomena is the state of granular media under vibration. The state is characterized by the hydrodynamic fields of the system, such as the density, velocity and granular temperature fields.

There have been several studies on the state of granular particles under vibration. Thomas et al. studied the system in three dimensions, mainly focusing on the behavior of shallow beds [22]. Clément and Rajchenbach experimentally measured the density, velocity and temperature fields of a two-dimensional vertical packing of beads [23]. They found that the temperature increases monotonically with the distance from the bottom plate. The same system was studied by molecular dynamics (MD) simulation with similar results [24]. In a series of simulations and experiments, Luding et
al. studied the behavior of the one and two-dimensional systems [25-27]. They found that the height expansion, which is the rise of the center of mass due to the vibration, scales in the variable $x = Af$. Here, $A$ and $f$ are the amplitude and the frequency of the vibration. Warr et al. experimentally confirmed the scaling, and they also gave an argument for its origin [28]. In recent MD simulations of the three-dimensional system, Lan and Rosato measured the density and temperature fields [29]. They compared the results with the theoretical predictions by Richman and Martin [30], and found good agreement. Also, an approximate theory was developed for the system in one dimension, which agrees with simulations in the weak and the strong dissipative regimes [31].

In a previous paper, we studied the time averaged behavior of the two-dimensional system using numerical and analytic methods [32]. Using MD simulation, we found that the time averaged value of not only the expansion but also the density and the granular temperature fields scale in $x$. We also used the kinetic theory of Haff [33] to determine the time averaged hydrodynamic fields in the quasi-incompressible limit. The results are, in general, consistent with the numerical data, and in particular show scaling behavior in the variable $x$. We found that the origin of the scaling can be understood within the framework of the theory.

In the present paper, we extend our study to the time dependent behavior of the system, whose understanding is not only essential in studying
various time dependent phenomena, but also necessary to understand the mechanisms of certain steady state phenomena. For example, many of the arguments for the mechanism of the convection involve the variations of certain hydrodynamic quantities over a vibration cycle [7, 8, 10, 11, 16].

We find, by MD simulation, that the temporal fluctuations of the pressure and the height expansion scale in $x$, while the velocity and the granular temperature do not scale in any simple combination of $A$ and $f$. We also study the system using the kinetic theory of Haff, where the temporal behavior is studied by perturbing about the time average in the quasi-incompressible limit. The results of the kinetic theory disagree with the numerical data. The kinetic theory predicts that the whole system of particles moves as one effective “block.” The numerical simulations, on the other hand, show that the region of active particle movement is localized, and moves with time. The presence of the wave is partly responsible for the discrepancy.

The paper is organized as follows. In Sec. 2, we specify the interaction of the particles used in the MD simulations. We then present the temporal fluctuations of the expansion and the fields obtained by the simulations. Analytic results will be discussed in Sec. 3. The continuum equations for granular material will be given, and perturbation equations are derived. We present the solution of the equations, and compare with the numerical data. In Sec. 4, we check the assumptions used in obtaining the solution. We also discuss various properties of the waves. Conclusions are given in Sec. 5.
2 Numerical Simulation

We start by describing the interactions used in the MD simulations. The simulations are done in two dimensions with disk shaped particles. The interaction between the particles is that of Cundall and Strack [34], which allows the particles to rotate as well as translate. Particles interact only by contact, and the force between two such particles $i$ and $j$ is the following.

Let the coordinate of the center of particle $i$ ($j$) be $\vec{R}_i$ ($\vec{R}_j$), and $\vec{r} = \vec{R}_i - \vec{R}_j$. We use a new coordinate system defined by two vectors $\hat{n}$ (normal) and $\hat{s}$ (shear). Here, $\hat{n} = \vec{r}/|\vec{r}|$, and $\hat{s}$ is obtained by rotating $\hat{n}$ clockwise by $\pi/2$.

The normal component $F_{j\rightarrow i}^n$ of the force acting on particle $i$ from particle $j$ is

$$F_{j\rightarrow i}^n = k_n(a_i + a_j - |\vec{r}|) - \gamma_n m_e (\vec{v} \cdot \hat{n}),$$  

(1)

where $a_i$ ($a_j$) is the radius of particle $i$ ($j$), and $\vec{v} = d\vec{r}/dt$. The first term is the linear elastic force, where $k_n$ is the elastic constant of the material. The constant $\gamma_n$ of the second term is the friction coefficient of the velocity dependent damping force, and $m_e$ is the effective mass, $m_i m_j/(m_i + m_j)$.

The shear component $F_{j\rightarrow i}^s$ is given by

$$F_{j\rightarrow i}^s = -\text{sign}(\delta s) \min(k_s |\delta s|, \mu |F_{j\rightarrow i}^n|).$$  

(2)

The term represents static friction, which requires a finite amount of force ($\mu F_{j\rightarrow i}^n$) to break a contact. Here, $\mu$ is the friction coefficient, $\delta s$ the total shear displacement during a contact, and $k_s$ the elastic constant of a virtual
The shear force also affects the rotation of the particles. The torque acting on particle $i$ due to particle $j$ is

$$T_{j\rightarrow i} = \vec{r}_c \times \hat{s} F_{j\rightarrow i}^s,$$  \hspace{1cm} (3)

where $\vec{r}_c$ is the vector from the center of particle $i$ to the point where particles $i$ and $j$ overlap. Since the particles used in the simulations are very stiff (large $k_n$), the area of the overlap is very small. It is thus a good approximation to use $-a_i \hat{n}$ as $\vec{r}_c$.

A particle can also interact with a wall. The force and torque on particle $i$, in contact with a wall, are given by (1) - (3) with $a_j = 0$ and $m_e = m_i$. A wall is assumed to be rigid, i.e., it is not moved by collisions with particles. Also, the system is under a gravitational field $\vec{g}$. A more detailed explanation of the interaction is given elsewhere \cite{35}.

The movements of the particles are calculated using a fifth order predictor-corrector method. We use two Verlet tables. One is a usual table with finite skin thickness. The other table is a list of pairs of actually interacting particles, which is needed to calculate the shear force. The interaction parameters used in this study are fixed as follows, unless otherwise specified: $k_n = 10^5, k_s = 10^5, \gamma_n = 2 \times 10^2$ and $\mu = 0.2$. The timestep is taken to be $5 \times 10^{-6}$. This small timestep is necessary for the large elastic constant used in the simulations. For too small values of the elastic constant, the system loses the character of a system of distinct particles, and behaves like
a viscous material. In order to avoid artifacts of a monodisperse system (e.g., hexagonal packing), we choose the radius of the particles from a Gaussian distribution with the mean 0.1 and the width 0.02. The density of the particles is 0.1. Throughout this paper, CGS units are implied.

We put the particles in a two-dimensional rectangular box. The box consists of two horizontal (top and bottom) plates which oscillate sinusoidally along the vertical direction with given amplitude $A$ and frequency $f$. The separation between the two plates $H$ is chosen to be much larger ($10^5$ times) than the average radius of the particles, so the particles do not interact with the top plate for all cases studied here. We apply a periodic boundary condition in the horizontal direction. The width of the box is $W = 1$. We also try different values of $W$, and find no essential difference in the following results.

We start the simulation by inserting the particles at random positions in the box. We let them fall by gravity and wait while they lose energy by collisions. We wait for $10^5$ iterations for the particles to relax, and during this period we keep the plates fixed. The typical velocity at the end of the relaxation is of order $10^{-2}$. After the relaxation, we vibrate the plates for 50 cycles before taking measurements in order to eliminate any transient effect. Measurements are made during the next 200 cycles.

We measure hydrodynamic quantities—density, velocity and granular temperature—which characterize the state of the system. The most detailed
information is contained in the time series of their fields (e.g., density field), which will be discussed later. Here, we want to start with something simple and representative of the system.

The center of mass of the particles can be loosely related to the density. Let \( y(t) \) be the vertical coordinate of the center of mass at time \( t \). The mean density is related to the mean interparticle distance, which is also related to \( y(t) \). Since \( y(t) \) is a scalar which also can be easily measured, we study it as a representative of the density. In the same spirit, we study the space averaged vertical velocity \( V_y(t) \) and granular temperature \( \tau(t) \) instead of the complete local fields. We define the spatial average of \( A(x, y, t) \) as

\[
\langle A(t) \rangle = \frac{\int \int A(x, y, t) \rho(x, y, t) dxdy}{\int \int \rho(x, y, t) dxdy},
\]

where \( \rho(x, y, t) \) is the density field at position \((x, y)\). It thus follows that \( V_y(t) = \langle v_y(t) \rangle \) and \( \tau(t) = \langle T(t) \rangle \), where \( v_y(x, y, t) \) and \( T(x, y, t) \) is the vertical velocity and the granular temperature field, respectively. Another quantity of interest is the pressure field \( p(x, y, t) \). For information on this quantity, we will study the total pressure at the bottom

\[
p_o(t) = \frac{\int p(x, y = 0, t) \rho(x, y = 0, t) dx}{\int \rho(x, y = 0, t) dx}.
\]

We studied the time averaged behavior of the system in our previous paper [32]. Here, we study the temporal behavior, especially focusing on the variations of the fields within a vibration cycle. We first measure the temporal fluctuation, which is defined as the standard deviation of a temporal
sequence. Let $\langle A \rangle_t$ be the time average of $A(t)$; then, we use

\begin{align*}
\bar{y}_{\text{exp}} &= \langle y_{\text{exp}} \rangle_t, \quad \delta y_{\text{exp}} = \sqrt{\langle y_{\text{exp}}^2 \rangle_t - \langle y_{\text{exp}} \rangle_t^2} \\
\bar{V}_y &= \langle V_y \rangle_t, \quad \delta V_y = \sqrt{\langle V_y^2 \rangle_t - \langle V_y \rangle_t^2} \\
\bar{\tau} &= \langle \tau \rangle_t, \quad \delta \tau = \sqrt{\langle \tau^2 \rangle_t - \langle \tau \rangle_t^2} \\
\bar{p}_o &= \langle p_o \rangle_t, \quad \delta p_o = \sqrt{\langle p_o^2 \rangle_t - \langle p_o \rangle_t^2},
\end{align*}

(6)

where the expansion $y_{\text{exp}}(t)$ is defined as the difference between $y(t)$ during and before the vibration. We measure these quantities at every $1/100$ of a period for 200 cycles.

In our previous paper, it was shown that $\bar{y}_{\text{exp}}$ scales in $Af$, in agreement with earlier simulations and experiments [25-28]. We find that the rotation of the particles included in the present simulation does not change this scaling, but does significantly decrease the value of $\bar{y}_{\text{exp}}$ [30]. The decrease is probably due to the fact that the average translational energy becomes smaller, since some of the energy is transferred to the rotation. We then consider the fluctuation of the expansion $\delta y_{\text{exp}}$. In Fig. 1(a), we show $\delta y_{\text{exp}}$ for several values of $A$ and $f$. The quality of the scaling is not very good, but the data is still consistent with $Af$ scaling, especially without the persistent deviation at low $A$ part of the $f = 20$ data. This deviation is also present in the scaling of $\bar{y}_{\text{exp}}$.

The behavior of $\tau(t)$ is very similar. It was shown that $\bar{\tau}$ scales in $Af$ [32]. Again, we find that the rotation does not change the scaling of $\bar{\tau}$.
but does change its value. The situation becomes a little different when we consider the fluctuation $\delta \tau$ as shown in Fig. 1(b). It is clear from the figure that $\delta \tau$ does not scale in $Af$. In fact, it does not scale in any of the simple combinations of $A$ and $f$ we have tried.

In the previous paper, the behaviors of $V_y(t)$ and $p_o(t)$ were not discussed in detail, since their time averaged values are trivial. Since the system is in a steady state, $\bar{V}_y$ is zero, which is also confirmed by the simulations. Since the pressure $p_o(t)$ is caused by the weight of the particles, one might guess $\bar{p}_o$ is simply the total weight of the particles divided by the area of the bottom ($W = 1$). We find that $\bar{p}_o$ is indeed a constant independent of $A$ and $f$, whose value is consistent with the total weight. On the other hand, the behavior of their fluctuations is far from trivial. The fluctuation of the velocity $\delta V_y$ for several values of $A$ and $f$ is shown in Fig. 1(c). It is apparent that $\delta V_y$ does not scale in $Af$. Also, $\delta V_y$ does not scale in any simple combination of $A$ and $f$. The data for the fluctuation of the pressure $\delta p_o$ is consistent with scaling in $Af$ as shown in Fig. 1(d). The quality of collapse is again not very good, especially for the low $A$ part of the $f = 20$ data.

The behavior of the time averaged quantities is either trivial ($\bar{V}_y$ and $\bar{p}_o$), or scales in $Af$ ($\bar{y}_{\text{exp}}$ and $\bar{\tau}$). The reason for the trivial behavior has been discussed, and the scaling in $Af$ can be understood from a kinetic theory of granular particles [32]. The behavior of the fluctuations is, however, not easy to understand. For example, one might naively expect that $\delta V_y$
behaves as $Af$, the velocity fluctuation of the bottom plate; but the numerical data suggests this is not so. Also, one might guess $\delta p_o$ behaves as $Af^2$, the variation of the effective gravity; but the numerical simulations suggest scaling in $Af$. Thus, the behaviors of the temporal fluctuations seem to be inconsistent with an intuitive picture. In the next section, we discuss an attempt to understand this behavior.

3 Kinetic Theory

In this section, we study the time dependent behavior of the system using a kinetic theory of granular material. We use the formalism by Haff [33], which was successfully applied to the time averaged behavior of the system [32]. For more details on the formalism and other kinetic theories of granular material, see Ref. [32] and references therein.

Haff’s formulation consists of equations of motion for mass, momentum and energy conservation. The mass conservation equation is

$$\frac{\partial}{\partial t} \rho + \nabla \cdot (\rho \vec{v}) = 0,$$

where $\rho$ and $\vec{v}$ are the density and the velocity fields, respectively. Next is the $i$-th component of the momentum conservation equation,

$$\rho \frac{\partial}{\partial t} v_i + \rho (\vec{v} \cdot \nabla) v_i = \frac{\partial}{\partial x_i} [-p + \lambda (\nabla \cdot \vec{v})] + \frac{\partial}{\partial x_j} [\eta (\frac{\partial v_j}{\partial x_i} + \frac{\partial v_i}{\partial x_j})] + \rho g_i,$$

where summation over index $j$ is implied. The coefficients $\lambda$ and $\eta$ are viscosities which will be determined later. Also, $p$ is the internal pressure, and
$g_i$ is the $i$-th component of the gravitational field. Although (8) resembles the Navier-Stokes equation, the coefficients as well as the internal pressure are now functions of the fields instead of being constant. The last of the equations of motion is energy conservation,

$$\frac{\partial}{\partial t} \left( \frac{1}{2} \rho v^2 + \frac{1}{2} \rho T \right) + \frac{\partial}{\partial x_i} \left[ \left( \frac{1}{2} \rho v^2 + \frac{1}{2} \rho T \right) v_i \right]$$

$$= - \frac{\partial}{\partial x_i} (pv_i)$$

$$+ \frac{\partial}{\partial x_i} [\lambda (\nabla \cdot \vec{v}) v_i] + \frac{\partial}{\partial x_i} [\eta (\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}) v_j]$$

$$+ \rho v_i g_i$$

$$+ \frac{\partial}{\partial x_i} \left[ K \frac{\partial}{\partial x_i} \left( \frac{1}{2} \rho T \right) \right] - I.$$

Here, $T$ is the granular temperature field, $K$ is the “thermal conductivity,” $I$ is the rate of the dissipation due to inelastic collisions, and summations over indices $i$ and $j$ are implied. Although the form of (9) is somewhat different from that of the Navier-Stokes equations, the equation can still be easily understood. The left hand side of (9) is simply the material derivative of the total kinetic energy, where the total kinetic energy is divided into the convective part (involving $\vec{v}$) and the fluctuating part (involving $T$). On the right hand side of the equation, the first three lines are simply the rate of work done by the internal pressure, viscosity and gravity, respectively. The term involving $K$ is the rate of energy transported by “thermal conduction.” The dissipation term $I$, which is a consequence of the inelasticity of the particles, is responsible for many of the unique properties of granular material.
We now discuss the coefficients which are yet to be determined. Derivation of the relations of these coefficients to the fields is based on intuitive arguments \[33\]. Also, the derivation assumes that the density is not significantly smaller than the close-packed density, i.e., the system is almost incompressible. The relation for the internal pressure is

\[ p = t \rho \frac{T}{s}, \]

(10)

where \( t \) is an undetermined constant, and \( d \) is the average diameter of the particles. The variable \( s \), which is roughly the gap between the particles, is related to the density by

\[ \rho \equiv \frac{m}{(d + s)^3}, \]

(11)

where \( m \) is the average mass of the particles. Then, the viscosity \( \eta \) is given as

\[ \eta = q d^2 \rho \frac{\sqrt{T}}{s}, \]

(12)

where \( q \) is an undetermined constant. In a similar way, the thermal conductivity is found to be

\[ K = r d^2 \frac{\sqrt{T}}{s}. \]

(13)

Here again, \( r \) is an undetermined constant. Finally, the rate of dissipation is

\[ I = \gamma \rho \frac{T^{3/2}}{s}, \]

(14)

where \( \gamma \) is an undetermined constant. The viscosity \( \lambda \) is left undetermined, due to the fact that, in the range where these relations are valid, the term containing \( \lambda \) is negligible and is dropped from the calculation.
We impose two constraints in order to make the equations analytically tractable. The first is the horizontal periodic boundary condition. Due to the boundary condition, there are no significant variations of the fields along the horizontal direction. Thus, we only have to deal with a one-dimensional equation, instead of a two or three-dimensional one. The other constraint is incompressibility, which is a little tricky. Incompressibility implies, strictly speaking, that the density $\rho$ is constant. Due to the relation between $\rho$ and $s$ \(^{(11)}\), $s$ also has to be constant. Here, we are interested in the situation where $s$ is much smaller than $d$, but still non-zero. In such a case, the variation of the density can be ignored, but not the variation of a variable that depends directly on $s$. We call this condition quasi-incompressibility.

Under these conditions, we solved the equations for the time averaged fields,

\begin{align*}
T^{(0)}(y) &\simeq B^2 v_w^2 \frac{y}{y_o - y} \exp(-2y/\ell) \\
\bar{s}^{(0)}(y) &\simeq \frac{t^2 \rho_o g}{g} \frac{d^2 y}{(y_o - y)^2} \exp(-2y/\ell) \\
v_y^{(0)}(y) &= 0 \\
p^{(0)}(y) &= \rho_o g (y_o - y).
\end{align*}

(15)

Here, $v_w = 2\pi Af$ the maximum velocity of the bottom, $y_o$ the height of the free surface, and $\ell = \sqrt{r/\gamma} d$ the dissipation length. Also, $\rho_o$ is the density of the maximum packing, and

\begin{equation}
B^2 = \frac{(1 + e_w)^2}{1 - e_w^2 - (2rd/\alpha \ell) (1 - \ell/2y_o)},
\end{equation}

(16)
where \( e_w \) is the coefficient of restitution of collisions between a particle and a wall [12].

We study the time dependent behavior of a quantity by perturbing it from its time averaged value. We assume that the perturbation term oscillates with \( f \)—the frequency of the vibration. In general, the assumption is not valid, since one has to consider all the modes with different frequencies. However, when the amplitude of the vibration is small enough, in many cases, the mode with frequency \( f \) dominates the time dependent behavior. We expect there is a range of \( A \) in which the assumption is valid, which will be determined later by the numerical simulations. We thus use

\[
T(y, t) = T^{(0)}(y) + T^{(1)}(y) \cdot \exp(i\omega t)
\]
\[
s(y, t) = s^{(0)}(y) + s^{(1)}(y) \cdot \exp(i\omega t)
\]
\[
v_y(y, t) = v_y^{(0)}(y) + v_y^{(1)}(y) \cdot \exp(i\omega t)
\]
\[
p(y, t) = p^{(0)}(y) + p^{(1)}(y) \cdot \exp(i\omega t).
\] (17)

Substituting (17) into the mass conservation condition (7) and using the quasi-incompressibility condition, we obtain \( dv_y^{(1)}(y)/dy = 0 \). Here and in the rest of the calculation, we consider terms up to the first order in the expansion. Since \( v_y(y = 0, t) \) should be the velocity of the bottom plate,

\[
v_y^{(1)}(y) = iv_w,
\] (18)

which is a consequence of quasi-incompressibility and the one-dimensional nature of the system. Also, momentum conservation equation (8), combined
with (17), becomes
\[ p^{(1)}(y) = \rho_0 A \omega^2 (y_o - y), \] (19)
which can be easily understood. The pressure at the bottom is proportional
to the total weight of the particles. Thus, the fluctuation of \( p_o(t) \) is the
total mass \( \rho_0 y_o \) times the fluctuation of the effective gravity \( A \omega^2 \). Finally,
we consider the energy conservation equation (9). Combined with (17), it
becomes
\[
i \omega \rho_o U^{(0)} U^{(1)} + i \rho_o v_y^{(1)} U^{(0)} \frac{d}{dy} U^{(0)} = \frac{r d}{t d y} \left( p^{(0)} \frac{d}{dy} U^{(1)} + p^{(1)} \frac{d}{dy} U^{(0)} \right)
- \frac{\gamma}{t d} (U^{(1)} p^{(0)} + U^{(0)} p^{(1)}), \] (20)
where we introduce the variable \( U(y, t) = \sqrt{T(y, t)} \). One has to solve (20)
for \( T(y, t) \). We can not, unfortunately, get an analytic solution of the resulting
nonlinear differential equation. Since \( s(y, t) \) has to be calculated from
the relation (10) between \( T(y, t), s(y, t) \) and \( p(y, t) \), we also can not get an
expression for \( s^{(1)}(y) \).

We compare the results from the kinetic theory with the numerical simu-
lations (Fig. 1). First, we consider \( v_y(y, t) \). Since the temporal fluctuation of
\( v_y(y, t) \) is proportional to \( v_y^{(1)}(y) \), the kinetic theory predicts \( \delta V_y \sim A f \). The
data from the simulations, however, is inconsistent with this scaling. The
situation is similar for \( p(y, t) \). The kinetic theory predicts \( \delta p_o \sim A f^2 \), while
the numerical data scales in \( A f \). We do not have analytic expressions for
$T(y, t)$ and $s(y, t)$ to compare with the simulation data.

The failure of the kinetic theory, when applied to the time dependent behavior, is in sharp contrast with its success in studying the time averaged behavior. The kinetic theory correctly predicts the scaling behavior of all the time averaged hydrodynamic quantities. We suspect the failure is associated with the breakdown of a key assumption(s) used in the theory. Three key assumptions, besides the ones employed in the formulation of the kinetic theory, are made to obtain a simple analytic solution for the time dependent behavior. The first is quasi-incompressibility, which is shown to be valid for $Af \ll 1.5$ from the study of the time averaged behavior \cite{32}. However, the scaling behavior of the time averaged quantities remains unchanged even in the compressible regime. The second assumption is that the time dependence of a quantity is a sinusoidal oscillation with frequency $f$, which we expect to be valid for small $A$. The last assumption is that the time dependent term in (17) is much smaller than its time averaged value in order for the perturbation to be valid. In the next section, we check the validity of these assumptions by comparing to the numerical simulations.

4 Validity of Assumptions

First, we check the assumption that the mode with the driving frequency dominates the time dependent behavior of the hydrodynamic quantities. We obtain a time series of $y_{\text{exp}}(t)$ by measuring it at every $1/100$ of a period for
200 cycles. We then calculate the power spectrum of $y_{\text{exp}}(t) - \bar{y}_{\text{exp}}$ using a FFT routine in the NAG library (c06gbf). The results with $f = 100$ are shown in Fig. 2. The mode with $f = 100$ is dominant for $A = 0.01$ (Fig. 2(a)). When $A$ is increased further to 0.03, however, the $f \simeq 20$ mode becomes dominant (Fig. 2(b)). The results with $f = 20$ are entirely similar. The $f = 20$ mode loses its dominance when $A$ is increased to about 0.5. In both cases, the mode with frequency $f$ is dominant until $\Gamma \sim 10$, where $\Gamma = A\omega^2/g$.

The measurements of $V_y(t)$ and $p_o(t)$ also support the above observations. In Fig. 3(a), we show the variation of $V_y(t)$ in one cycle, where $f = 100$ and the data are averaged over 200 periods. The curve with $A = 0.01$ is nearly sinusoidal, while clear deviation is seen for $A = 0.03$. An important point to note is the behavior of the maximum value of $V_y(t)$. The kinetic theory predicts, in (18), the maximum value to be proportional to $A$, which is clearly not consistent with the data. The measured value is quite smaller than what is predicted. For example, for $A = 0.03$, the predicted value is $6\pi$, while the measured value is about 2.4. In Fig. 3(b), we show $p_o(t)$ in one period, where again $f = 100$ and the data is averaged over 200 cycles. The curve seems to deviate from the sinusoidal even for small $A$, and it is difficult to determine whether the mode with $f = 100$ dominates. Again, the predicted behavior of the maximum value of $p_o(t)$ is not consistent with the measurement. The maximum value of $p_o(t)$ is predicted to increase linearly with $A$, as in (19), which is clearly not consistent with the data. The observed maximum value
\( \sim 400 \) is quite smaller than the predicted value \( \sim 4,000 \).

The mode with the driving frequency dominates the time dependent behavior for small values of \( A \), where a rough criterion for the dominance is \( \Gamma \ll 10 \). Also, it was shown that quasi-incompressibility is valid for \( Af < 1.5 \). The validity of the linear perturbation approximation is a little tricky. We require that the perturbation terms are smaller than the time averaged terms, which is valid for the expansion and the temperature. The two terms are, however, comparable for the pressure even at small value of \( A = 0.05 \). The consequence of the large pressure fluctuation on the validity of the perturbation is unclear. For large \( A \), all of the above assumptions are not valid, which complicates the analysis of the system. For example, in order to study the time dependent behavior, one has to consider additional modes with different frequencies.

The surprise is that the predictions for the maximum values of the vertical velocity and the pressure are not correct even when all the assumptions seem to be valid (e.g., \( A = 0.01 \) and \( f = 100 \)). We inspect again the predictions of the kinetic theory. As given in (18), the velocity field is uniform, and its value is \( v_w \) the velocity of the bottom. Therefore, the solution of the perturbation expansion suggests that the whole system of particles is moving as a single “block” attached to the bottom. The spatial and temporal variations of the other fields do not change the single block picture, but rather describe the structure of the block. A consequence of the picture is that the pressure at
the bottom is proportional to the effective gravity which reaches its maximum 
\((\Gamma + 1)g\) at phase \(3\pi/2\) of the vibration, which is exactly (19).

However, the measurements of \(V_y(t)\) and \(p_o(t)\) are not consistent with the picture. The measured maximum value of \(V_y(t)\) is much smaller than what is predicted, which suggests that only a small fraction of the particles move together at a given time. Also, the fact that the measured maximum value of \(p_o(t)\) is smaller than the prediction also supports this observation. Furthermore, the phase at which \(p_o(t)\) reaches the maximum is about 0, in contrast to \(3\pi/2\) suggested by the single block picture. The discrepancy can be understood as follows. Since the maximum acceleration of the bottom is larger than that of gravity, particles initially lying on the bottom will be “launched” at a certain phase of the vibration. The maximum pressure at the bottom will occur when most of the launched particles come back and collide with the bottom, which occurs around \(\phi = 0\).

The direct evidence against the single block picture is the time evolution of the whole velocity field shown in Fig. 4(a). Here, we use \(f = 100, A = 0.01\), at which the assumptions used to derive the predictions of the kinetic theory seem to be valid. It is clear from the figure that the region of significant motion is localized, and travels like a wave. The propagation of the disturbance seems to be very similar to that of sound waves in a gas. Also, the maximum velocity is about 6, close to the prediction \(2\pi\). The localization of the particle motion can also be seen in the time evolution of the granular temperature
field shown in Fig. 4(b). Again, the region of high temperature is localized, and travels upwards [37, 38]. Furthermore, the location of the high temperature region coincides with that of the large velocity region. The density field, on the other hand, does not vary significantly as shown in Fig. 4(c), which agrees with the previous experiment [23]. It is clear that the presence of the “waves” changes the behaviors of the fields, and possibly their scaling properties. The absence of the waves in the single block solution is, at least, partly responsible for the failure of the kinetic theory. The absence is due to quasi-incompressibility. In fact, it is easy to derive the single block picture only from the quasi-incompressibility condition and one-dimensional nature of the equation. It is thus necessary to consider the general case of the kinetic theory of a compressible gas, which unfortunately is quite complicated.

We want to finish this section by discussing some properties of the waves. We first consider the motion of the maximum disturbance. In Fig. 5(a), we show the phase $\phi_{\text{max}}(y)$ at which $v_y(y,t)$ reaches a maximum with $f = 100$ and several values of $A$. In other words, we plot the position of the maximum velocity in the $y - \phi$ plane. The velocity of the wave, inversely proportional to the slope of $\phi_{\text{max}}(y)$ curve, is a bit small. The time needed for the wave to propagate from the bottom to the top of the pile is of the order of the period of the vibration. The simulations with $f = 50$ show that, for the same values of $A$, the velocity of the wave does not change significantly, suggesting that there is a fixed time scale for the wave propagation. Also, it can be seen from the
figure that the velocity decreases when either $y$ or $A$ increases with the other parameters fixed. The decrease probably results from the decrease of the collision frequency between particles, due to the decrease of the density. Also, the location of the maximum granular temperature in the $y-\phi$ plane is shown in Fig. 5(b), where one can see the close correlation with Fig. 5(a). In fact, the maximum temperature always occurs just above the maximum velocity at a given phase, which probably is the point of the largest velocity gradient. We now discuss the values of the maximum disturbances. In Fig. 6(a), we show the maximum value of $v_y(y, t)$ for given $y-v_y(y, \phi_{\text{max}}(y))$. The maximum value decreases with $y$ roughly as an exponential, and the rate of the decrease is larger for larger $A$ with fixed $f$. The decrease is due to two causes: (1) some of the vertical velocity component is transferred to the horizontal one by interparticle collisions; (2) the kinetic energy is lost by inelastic collisions. The results with $f = 20$ are essentially the same. Also, the behavior of the value of the maximum temperature $T(y, \phi_{\text{max}}(y))$, as shown in Fig. 6(b), is very similar to that of $v_y(y, \phi_{\text{max}}(t))$. It decreases roughly as an exponential, and it decays faster with larger $A$ with fixed $f$, where the origin of the decrease is the same as the velocity field.

5 Conclusion

We have studied the temporal behavior of granular material in a vibrating box. We find that the temporal fluctuations of $y_{\exp}(t)$ and $p_o(t)$ scale in
$Af$, while no scaling is found for the fluctuations of $\tau(t)$ and $V_y(t)$. We study the behavior using the kinetic theory of Haff, where we perturb the hydrodynamic quantities from their time averaged values. The results of the kinetic theory are not consistent with the numerical data. We argue that the failure of the theory is, at least, partly due to the waves found in the simulations.

We discuss some possible ways to study the time dependent behavior of the system. Since the main problem of the present theory, as discussed above, is the condition of quasi-incompressibility, it sounds reasonable to study the system of equations in a fully compressible regime. However, the system of equations becomes too complicated. The three conservation equations (7)-(9) are written in general form, and do not need any modification. The relations of $p, \eta, K, I$ to the fields (10)-(14) as well as the boundary conditions have to be modified. It is not the modifications themselves, but the complexity of the resulting equations, that makes an analytic solution too difficult to obtain. However, we can still gain some information about the system by a perturbative or an approximate method, as well as the numerical solution of the kinetic equations.

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Figure Captions

Fig. 1: Scaling behaviors of the hydrodynamic quantities. Each datum is averaged over at least 3 samples, where 20000 measurements are made in a sample. (a) The fluctuation of the expansion $\delta y_{\text{exp}}$ seems to scale in $Af$, where (b) the fluctuation of the temperature $\delta \tau$ does not scale. (c) The fluctuation of the velocity $\delta V_y$ does not scale, where (d) the fluctuation of the pressure $\delta p_o$ seems to scale in $Af$.

Fig. 2: Power spectrum of $y_{\text{exp}}(t)$ with $f = 100$ and (a) $A = 0.01$, (b) $A = 0.03$. The expansion is measured at every $10^{-4}$ second for 200 cycles. The mode with $f = 100$ is dominant at $A = 0.01$, but loses its dominance at $A = 0.03$.

Fig. 3: Time evolution of (a) $V_y(t)$, (b) $p_o(t)$ in one cycle, where $f = 100$ and the data are averaged over 200 cycles. Deviation from a sinusoidal is apparent for $A = 0.03$.

Fig. 4: Time evolution of (a) $v_y(y,t)$, (b) $T(y,t)$ and (c) $\rho(y,t)$ fields, where $f = 100, A = 0.01$ and the data are averaged over 200 cycles. The density field is normalized to be the volume fraction.

Fig. 5: The position of (a) the maximum velocity and (b) the maximum temperature in the $y - \phi$ plane with $f = 100$ and several values of $A$. The data are averaged over 200 cycles. Note that the two sets of the
curves are almost identical.

**Fig. 6:** The maximum value of (a) the vertical velocity $v_y(y, \phi_{\text{max}}(y))$ and (b) the temperature $T(y, \phi_{\text{max}}(y))$ at height $y$ with $f = 200$. The data are averaged over 200 cycles.
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