Interior Structure of Supernova Remnants from Ejecta-Dominated to Sedov-Taylor Phase

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Received ________________; accepted ________________
ABSTRACT

Supernova remnants (SNRs) evolve through different phases, from an early Ejecta-Dominated phase to a middle-aged Sedov-Taylor phase, and to late-age radiative and dissipation phases. Here we consider spherically symmetric SNR evolution up to the onset of the radiative phase. Numerical calculations of interior structure are carried out for the self-similar phases. Hydrodynamic simulations are carried out for the full SNR evolution prior to onset of radiative losses. The SNR structure for the full evolution is analyzed to produce integrated emission measures and temperatures. Fitting formulae are presented which can be used in comparing model SNRs with observed SNR emission measures and temperatures. This allows determination of SNR model explosion energy, ejecta mass, ejecta density profile and circumstellar medium density from SNR observations. The new results are incorporated into the SNR modelling code SNRPy.

Subject headings: supernova remnants:
1. Introduction

Supernova remnants (SNRs) have a great impact on the evolution of galaxies and the interstellar medium (ISM) within galaxies (Vink 2012 and references therein). They do this via their energy input into the ISM, and return of elements.

SNRs are observed primarily in X-rays, by emission from hot interior gas with temperature \( \sim 1 \) keV, and in radio, by synchrotron emission from relativistic electrons accelerated by the SNR shockwave. Only a small number of the \( \sim 300 \) observed SNRs in our Galaxy have been well enough characterized to determine their evolutionary state, including supernova (SN) type, explosion energy and age. In order to expedite characterization of a significant number of SNRs, Leahy & Williams (2017) presented a set of SNR models and a software implementation in Python, called SNRPy. By fitting models to X-ray observations of SNRs (Leahy 2017, Leahy & Ranasinghe 2018), valuable information can be obtained on the nature of SN explosions. The SN population properties can be used as inputs to constrain the evolution of the Galaxy and its interstellar medium.

Section 2 of this paper presents an overview of SNR evolution and the quantities that are required for modelling X-ray observations of SNRs. Section 3 describes calculations of interior structure for the self-similar Ejecta-Dominated (ED) phase, and the self-similar Sedov-Taylor (ST) phase for uniform ISM and cloudy ISM cases. Section 4 describes hydrodynamic simulations which include the early ED phase and the late ST phase, and the transition between them. Section 5.1 discusses results from the self-similar solutions, including emission measures (\( EMs \)) and \( EM \)-weighted temperatures. Section 5.2 describes the results from the hydrodynamic simulations, and analytic fits to \( EMs \) and temperatures. Section 6 gives a summary and where to access the results and the updated SNR modelling code SNRPy.
2. Supernova Remnant Evolution and Structure

A SN explosion creates a SNR starting with the ejection of the SN progenitor envelope at high speed, typically $\sim 10000$ km/s. General descriptions of SNR evolution are given in numerous places (e.g., Cioffi et al. 1988, Truelove & McKee 1999—hereafter TM99, and Leahy & Williams 2017—hereafter LW17). The ejecta collides with the circumstellar medium (CSM) or ISM, causing a forward shock (FS) to propagate outward and a reverse shock (RS) to propagate back into the ejecta.

The general sequence of SNR evolution starts with the ED phase for which the effect of the ejected mass is important. This gradually evolves to the ST phase, for which the swept-up mass by the SN shock far exceeds the ejected mass. For ED, transition and ST phases, radiative energy losses are unimportant. Beyond the ST phase, radiative losses become important (e.g. Cioffi et al. 1988). In the current work, the phases prior to the radiative phases are considered.

The basic interior structure of a SNR, prior to the ST phase, has the following regions from outside to inside: i) the undisturbed CSM; ii) the FS moving into the CSM; a layer of shocked CSM; iii) the contact discontinuity (CD) separating the shocked CSM from the shocked ejecta; iv) the layer of shocked ejecta; v) the RS moving inward relative to the ejecta; and vi) the undisturbed ejecta. The unshocked ejecta has a homologous velocity profile ($v \propto r$ at fixed time).

After the reverse shock reaches the center of the SNR, the entire ejecta is fully shocked. Reflected shocks and sound waves are generated at this time, and die out slowly over time (Cioffi et al. 1988). The reflected shocks and sound waves are clearly seen in the numerical simulations presented below.

In order to calculate SNR evolution and structure, the following simplifying assumptions
are made. The SNR is spherically symmetric. The CSM has: i) constant density; or ii) \(1/r^2\) stellar wind density profile centered on the SN, i.e., \(\rho_{CSM} = \rho_s r^{-s}\) with \(s=0\) or \(s=2\). The unshocked ejecta has a constant density core for \(r \leq R_{\text{core}}\), and a power-law density envelope: \(\rho_{ej} \propto r^{-n}\) for \(r > R_{\text{core}}\).

2.1. Characteristic Scales

Non-radiative supernova remnants undergo a unified evolution (TM99). The characteristic radius and time for \(s = 0\) are given by \(R_{ch} = (M_{ej}/\rho_0)^{1/3}\) and \(t_{ch} = E_0^{-1/2} M_{ej}^{5/6} \rho_0^{-1/3}\) with \(M_{ej}\) the ejected mass and \(E_0\) the explosion energy. The characteristic velocity is \(V_{ch} = R_{ch}/t_{ch}\) and characteristic shock temperature is \(T_{ch} = 3/16 \mu n_e V_{ch}^2\), with \(\mu\) the mean mass per particle. For SNR in a CSM with \(s = 2\), the characteristic radius and time are given by \(R_{ch} = (M_{ej}/\rho_s)\) and \(t_{ch} = E_0^{-1/2} M_{ej}^{3/2} v_w/\dot{M}\), with \(\dot{M}\) and \(v_w\) the wind mass loss rate and velocity, and \(\rho_s = \dot{M}/4\pi v_w\).

2.2. Emission Measure (EM), \(EM\)-weighted Temperature and Column \(EM\)

Because the emission from the hot shocked gas in a SNR is dominated by two-body processes, it depends on the product of electron and ion densities (e.g., Raymond et al. 1976). \(EM\) is defined in terms of electron density \(n_e\) and hydrogen ion density \(n_H\) by \(EM = \int n_e(r)n_H(r)dV\). \(EM\) can be measured by X-ray observations, so the measured \(EM\) is critical to determining the evolution state of a SNR.

\(EM\) can be calculated from a model SNR density profile. During self-similar phases of a SNR, the density profile has a constant functional form with normalization and scaling with radius dependent on time. The dimensionless \(EM\), \(dEM\), was defined by LW17 as \(dEM = EM/(n_{e,s} n_{H,s} R_{FS}^3)\) with \(n_{e,s}\) and \(n_{H,s}\) are \(n_e\) and \(n_H\) immediately inside the
forward shock (FS). We extend this to define $dEM_{FS}$ and $dEM_{RS}$ for the gas heated by the FS and for gas heated by the RS, respectively.

\[
dEM_{FS} = \frac{1}{n_{e,s}n_{H,s}R_{FS}^3} \int_{R_{CD}}^{R_{FS}} n_e(r)n_H(r) dV
\]

\[
dEM_{RS} = \frac{1}{n_{e,s}n_{H,s}R_{FS}^3} \int_{R_{RS}}^{R_{CD}} n_e(r)n_H(r) dV
\]

The observed temperature of a SNR, derived from the X-ray spectrum, depends on the state of the SNR and on the adopted X-ray spectrum model. Most commonly a single electron-temperature non-equilibrium ionization model is used. The X-ray temperature measures the $EM$-weighted temperature of the shocked gas. LW17 defined the dimensionless temperature $dT = \frac{1}{T_{FS}} \frac{1}{EM} \int n_e(r)n_H(r)T(r) dV$, with $T_{FS}$ the forward shock temperature. We extend this to define $dT_{FS}$ and $dT_{RS}$ for the gas heated by the FS and by the RS, respectively.

\[
dT_{FS} = \frac{1}{T_{FS}} \frac{1}{EM} \int_{R_{CD}}^{R_{FS}} n_e(r)n_H(r)T(r) dV
\]

\[
dT_{RS} = \frac{1}{T_{FS}} \frac{1}{EM} \int_{R_{RS}}^{R_{CD}} n_e(r)n_H(r)T(r) dV
\]

The surface brightness of a SNR depends on the line-of-sight integral of the emission coefficient $j(\nu) = n_e n_H \epsilon(\nu)$, with emissivity $\epsilon(\nu)$. The column emission measure ($CEM$) is often used as a proxy for surface brightness, valid when the emission coefficient is only weakly dependent on the temperature history of the parcel of gas (e.g. see White & Long 1991, hereafter WL91). $CEM$ is given by $CEM(B) = \int n_e n_H dS$, where the integral is along the line of sight through the SNR at impact parameter $B$ from center.

We define the dimensionless $CEM$ using the scaled densities and dimensionless impact parameter, $b = B/R_{FS}$, by

\[
cEM(b) = \int \frac{n_e(x(s))}{n_{e,s}} \frac{n_H(x(s))}{n_{H,s}} ds
\]
with \( s = S/R_{FS} \) and \( x(s) = \sqrt{b^2 + s^2} \). More generally, we define the dimensionless \( cEM(b) \) separately for gas heated by the forward shock and gas heated by the reverse shock, dimensionless \( cEM_{FS}(b) \) and \( cEM_{RS}(b) \). For \( cEM_{FS}(b) \), \( x(s) \) is limited to values between \( R_{CD}/R_{FS} \) and 1, while for \( cEM_{RS}(b) \), \( x(s) \) is limited to values between \( R_{RS}/R_{FS} \) and \( R_{CD}/R_{FS} \).

3. Calculations of SNR Structure for Self-similar Phases

3.1. Pure ST and SNR in cloudy ISM

Following LW17, we refer to the standard ST solution, with zero ejected mass, as "pure ST" to differentiate it from the ST phases of TM99, with non-zero ejected mass. WL91 present self-similar models for SNR evolution in a cloudy ISM, assuming zero ejected mass.

For simplicity we only consider their one parameter models which depend on \( C/\tau \). Here \( C = \rho_c/\rho_0 \), with \( \rho_c \) is the ISM density if the clouds were uniformly dispersed in the ISM and \( \rho_0 \) is the intercloud density prior to cloud evaporation. The evaporation timescale parameter is \( \tau = t_{evap}/t \), with \( t_{evap} \) the evaporation timescale and \( t \) the age of the SNR. The WL91 case \( C/\tau = 0 \) is the same as the pure ST solution.

The WL91 models were recalculated by solving the self-similar differential equations given in WL91, but to higher accuracy using a variety of differential equation solvers. The equations were solved within both MathCad and Mathematica software packages, using fourth-order Runge-Kutta with adaptive step size, Burlisch-Stoer method, and a hybrid solver which uses a combination of Adams and BDF (backwards differentiation formula). The results were compared and all agreed to 5 digits or better. The solutions agree with the figures shown in WL91.

Results are presented here for \( C/\tau=0 \) (pure ST), 1, 2 and 4. The solutions are available
as described at the end of this paper. Fig. 1 shows the interior structure (pressure, density, gas velocity and gas temperature) vs. scaled radius, $r/R_{\text{shock}}$. The dimensionless $cEM$ is shown vs. dimensionless impact parameter $b$. The integrated quantities $dEM$ and $dT$ for the WL91 solutions are given in Table 1.

### 3.2. Early ED Phase

For SNR with non-zero ejected mass, the evolution starts with the early ED phase. The ejecta has a constant density core and power-law density envelope. The self-similar evolution starts at $t=0$ and ends when the reverse shock approaches the ejecta core (TM99). The self-similar solutions exist for $n > 5$ and are discussed by [Chevalier (1982)](Chevalier1982).

Here we calculate the self-similar solutions, labelled CP (Chevalier-Parker) using the methods outlined in [Chevalier (1982)](Chevalier1982) and [Parker (1963)](Parker1963). The equations were solved within both MathCad and Mathematica software packages, using different differential equation solvers, and comparing the results to ensure consistency. The cases $s=0$ and $s=2$, for $n=6, 7, 8, 9, 10, 11, 12, 13$ and $14$ are computed, and are made available as data tables as described at the end of the paper.

The $s=0$ and $s=2$ solutions for $n=7$ and $n=12$ were consistent with those given in [Chevalier (1982)](Chevalier1982), but are of higher accuracy. Interior solutions for $s=0$, $n=6, 8, 10$ and $12$ are shown in Fig. 2 for the regions from the reverse shock to the forward shock. Interior solutions for $s=2$, $n=6, 8, 10$ and $12$ are shown in Fig. 3.
4. Hydrodynamic Calculations of SNR Structure

After the early self-similar evolution, the evolution is calculated using hydrodynamic equations. The evolution follows a unified evolution as shown by TM99, before radiative losses become important. Unified evolution means that solutions have the same dependence on $t/t_{ch}$ if radius is scaled by $R_{ch}$, velocity is scaled by $V_{ch}$ and temperature is scaled by $T_{ch}$. Because there is no smooth transition for $s=2$ from ED to post-ED (e.g. TM99), we calculate the post-ED phases only for the $s=0$ case. That evolution is the subject of this section.

The evolution of $R_{FS}$ and $R_{RS}$ were calculated using an analytic approximation for ED, ED to ST and ST phases by TM99. The reverse shock slows its outward motion (relative to the ISM) about the time that it reaches the ejecta core, at time $t_{core}$ (TM99). Then it propagates inward, reaching the center of the SNR at time $t_{rev}$ (TM99). The evolution is continuous, but it is useful to label the phases as ‘ED’ for $0 < t < t_{core}$, ‘ED to ST’ for $t_{core} < t < t_{rev}$, and ‘ST’ for $t_{rev} < t < t_{PDS}$, where $t_{PDS}$ is the time where radiative losses affect the evolution (Cioffi et al. 1988, TM99 and LW17). However the ‘ST’ phase can be quite different that the ‘pure ST’ evolution, as pointed out by LW17.

Here we calculate the evolution for $s=0$ and $n=6$ to 14 using the hydrodynamics code PLUTO (Mignone et al. 2007, Mignone et al. 2012). A core-envelope structure for the ejecta is assumed. For the simulations the fundamental code units were set to $\rho_u = 10^{-18}$ gm/cm$^{-3}$ (density), $r_u = 10^{16}$ cm (distance) and $v_u = 10^7$ cm/s (velocity). The resulting code units for time, pressure, mass and energy are $t_u = 10^9$ s, $P_u = 10^{-4}$ dyne cm$^{-2}$, $M_u = 10^{30}$ gm and $E_u = 10^{44}$ erg.

We tested different values for the ISM density, ejecta mass and explosion energy to verify that SNR evolution in scaled variables (density scaled by $\rho_{ISM}$, time scaled by $t_{ch}$, radius by $R_{ch}$, velocity by $V_{ch}$ and pressure by $P_{ch} = \rho_{ISM}V_{ch}^2$) was independent of those
initial quantities. Then we set the ISM density $= 10^{-22}\text{gm/cm}^3$, ejected mass $= 1\ M_\odot$, and explosion energy $= 10^{51}\text{ erg}$ for the remaining calculations. This yields characteristic scales of $t_{ch} = 3.839 \times 10^9\ s = 121.7\ yr$, $R_{ch} = 2.714 \times 10^{18}\ cm = 0.8797\ pc$, $V_{ch} = 7.071 \times 10^{3}\ km/s$ and $P_{ch} = 5.000 \times 10^{-5}\ \text{dyne/cm}^2$.

The SNR initial conditions consisted of unshocked core and envelope plus shocked and unshocked ISM. The resulting time-dependent solutions showed large transient fluctuations in the hydrodynamic variables. Use of more accurate initial conditions should result in smaller transient fluctuations. Thus a more accurate second case of initial conditions was constructed from the self-similar CP solutions, consisting of unshocked and shocked ejecta, and shocked and unshocked ISM.

For the first case initial conditions, a small outer ejecta radius $R_{ej} = 5 \times 10^{12}\ cm$ was chosen. The core radius was taken as $10^{-1.5}$ of $R_{ej}$. The core density was set so that the integrated mass from $r = 0$ to $r = R_{ej}$ was $1\ M_\odot$. The velocity increases linearly with radius for the unshocked ejecta, so the velocity profile is specified by the velocity $v_{ej}$ at $r = R_{ej}$. $v_{ej}$ was determined by requiring the total ejecta kinetic energy to be the explosion energy. For example, $v_{ej} = 2.888 \times 10^5\ km/s$ and $v_{core} = 9.132 \times 10^3\ km/s$ is found for $n=7$. The ejecta pressure was set to a low value ($10^{-8}\ \text{dyne cm}^{-2}$).

A layer of shocked ISM with density $4\ \rho_{ISM}$ was added outside the ejecta from $R_{ej}$ to $R_{FS,0}$. The outer initial forward shock radius is $R_{FS,0} = (13/12)R_{ej}$, determined from the requirement that the mass of shocked ISM equals the total ISM mass swept up between $r = 0$ and $R_{FS,0}$. The velocity of the layer is $v_{ej}$, because the high density at early times of the ejecta makes it act like a rigid piston. This yields a shock velocity at $R_{FS,0}$ of $v_{FS,0} = (4/3)v_{ej}$ and an interior pressure of the ejecta layer of $(3/4)\rho_{ISM}v_{FS,0}^2$. Including the time for the outer edge of the ejecta to expand to $R_{ej}$ from $r = 0$, the initial solution has $t/t_{ch} = 4.5 \times 10^{-8}$ and $R_{FS}/R_{ch} = 1.99 \times 10^{-6}$ for $n=7$. The initial conditions for the
n=7 simulation are shown in the left panel of Fig. 6.

The second case for the initial conditions utilizes the CP self-similar solutions. The initial density profile is determined by matching both the unshocked ISM and the unshocked ejecta to the CP solution. The outer boundary of the CP solution at $R_{FS,0}$ is set to $4\rho_{ISM}$, and the outer boundary of the unshocked ejecta at $R_{RS,0}$ is set to $(1/4)\rho_{CP}(R_{RS,0})$ from the CP solution.

The ejecta mass includes the core, $r < R_{core}$, the unshocked powerlaw envelope, $R_{core}$ to $R_{RS}$, and the shocked ejecta, $R_{RS}$ to $R_{CD}$. The shocked ejecta contains a pileup of the shocked envelope mass. We define $w_{core} = R_{core}/R_{RS}$, which is different than TM99 who don’t have a layer of shocked ejecta in their initial density profile. We set $w_{core} = 10^{-1.5}$. For small $w_{core}$, the contribution from the shocked ejecta is small. Thus we use the analytic value of mass from the unshocked ejecta, $M_{ej} \simeq M_{core} + M_{env} = M_{core} \frac{n-3}{n-5} w_{core}^{n-3}$ to obtain the initial estimate of $M_{core}$ from $M_{ej}$. We integrate to obtain an accurate value of all three contributions to the ejecta mass. Then we set the total to the desired ejecta mass, $1M_\odot$, to determine a more accurate value of $M_{core}$ . $\rho_{core}$ is found from $M_{core} = (4/3)\pi R_{core}^3 \rho_{core}$ for given $R_{core}$. $R_{core,0}$ is chosen large enough to be resolved with enough grid cells in the hydrocode. It is small enough to give $R_{FS,0} << R_{ch}$, to include enough of the early ED phase prior to the post-ED evolution. E.g., for n=7 we chose $R_{core,0} = 4.0 \times 10^{14}$ cm, yielding $R_{FS,0} = 1.60 \times 10^{16}$ cm = $3.31 \times 10^{-3} R_{ch}$.

The initial estimate of $v_{core}$ is obtained from the energy of core and unshocked envelope. This is given by $E_{ej} \simeq E_{core} + E_{env} = E_{core} \frac{n-5}{n-3} w_{core}^{n-5}$ and $E_{core} = (2/5)\pi R_{core}^3 \rho_{core} v_{core}^2$. The error in $v_{core}$ is small for small $w_{core}$. A more accurate $v_{core}$ is obtained by integrating the energy in the core, unshocked envelope and the shocked envelope and setting to the explosion energy, $10^{51}$ erg. The velocity at the outer edge of the unshocked envelope is $v_{env} = v_{core} \times R_{RS,0}/R_{core,0}$. The time since explosion for the initial solution is given by
\( t_0 = R_{\text{core},0}/v_{\text{core}} \). For \( n=7 \), \( v_{\text{core}} = 9.13 \times 10^3 \text{ km/s} \) and \( t_0/t_{\text{ch}} = 4.14 \times 10^{-5} \).

To match velocities with the CP solution, we apply shock jump conditions at both forward and reverse shocks. The postshock pressure at \( R_{\text{FS}} \) is \( P_{\text{FS}} = (3/4)\rho_{\text{ISM}}V_{\text{FS}}^2 \). The pressure ratio \( x_{RF} = P_{RS}/P_{FS} \) is an \( n \)-dependent constant given by the CP solution. The reverse shock velocity, relative to the envelope gas, is \( P_{RS} = (3/4)\rho_{\text{env}}V_{RS}^2 \) with the reverse shock velocity in the envelope frame \( V_{RS} = v_{\text{env}} - V_{RS,\text{obs}} \) with \( V_{RS,\text{obs}} \) the reverse shock velocity in the observer frame. The gas velocity relative to the post-shock gas \( v_{sh,\text{rel}} \) is \( 1/4 \) of the pre-shock gas \( v_{un,\text{rel}} \): \( v_{sh,\text{rel}} = (1/4)v_{un,\text{rel}} \). After a bit of algebra we find:

\[
V_{FS} = \frac{4 v_{env}}{3(\sqrt{y_{RF}} \rho_{\text{ISM}}/\rho_{\text{env}} + y_{RF})}
\]

where the ratio of post-shock gas velocities from the CP solution is given by \( y_{RF} = v_{sh,RS}/v_{sh,FS} \).

The above procedure fully determines the initial conditions which satisfy the shock jump conditions at both shocks and have the correct total energy and ejecta mass. The initial CP solution for \( n=7 \) is shown in the right panel of Fig. 6. This has \( v_{\text{env},0} = 2.89 \times 10^5 \text{ km/s} \) and \( V_{FS,0} = 2.09 \times 10^5 \text{ km/s} \).

The initial conditions for case 1 were computed analytically using a modified init.c program in PLUTO. The initial conditions for case 2 consist of a binary file which includes the CP numerical solutions matched to the unshocked ejecta and and the ISM. For both cases, we added a passive scalar tracer field to track the contact discontinuity and the ejecta core-envelope boundary.

The SNR evolution includes a large range in time and spatial scales. The typical initial time is \( \sim 10^{-8}t_{\text{ch}} \) (case 1) to \( 10^{-4}t_{\text{ch}} \) (case 2) and initial radius is \( \sim 10^{-6}R_{\text{ch}} \) (case 1) to \( 10^{-3}R_{\text{ch}} \) (case 2). For case 1, we started the simulation at very early time and small radius in order to allow the approximate initial conditions to relax to a more accurate solution.
The late stage time is $\sim 10^4 t_{ch}$ and late stage radius is $\sim 10^2 R_{ch}$, for both cases. Thus it is not possible to compute the SNR structure in a single run of PLUTO. Instead we ran the code successively in stages, with the output of each stage used as input for the next stage. The computational grid was chosen so that the SNR initial outer shock radius was $\simeq 1/5$ of the grid size which allowed room for the SNR to expand to the edge of the grid before initiating a new stage. The spatial grid size was chosen 5000 points, so the SNR was resolved by a minimum of 1000 points at any time. Typically, 7 to 10 stages were computed for each evolution, allowing a $\simeq 10^5 - 10^8$ factor in radial expansion. The time steps were adjusted by PLUTO to satisfy the Courant condition, yielding $\sim 80,000$ timesteps per stage. The times for saving structure files (or snapshots) of the evolution were chosen manually, resulting in $\sim 250$ snapshots per evolution.

5. Results and Discussion

5.1. Self-similar SNR Solutions

During the self-similar phases of evolution of an SNR, $dEM_{FS}$, $dEM_{RS}$, $dT_{FS}$ and $dT_{RS}$ are constants; $cEM_{FS}(b)$ and $cEM_{RS}(b)$ are functions independent of time. The integrated quantities $dEM$ and $dT$ for the WL solutions for $C/\tau = 0, 1, 2$ and $4$ are given in Table 1. The dimensionless $cEM$ for the WL solutions are shown as a function of impact parameter $b = B/R_{shock}$ in Fig. [1]

$cEM$ for the CP solutions is shown as a function of $b = B/R_{shock}$ for select $s=0$ cases in Fig. [4]. The $cEM$ for gas between the CD and the FS is shown in the left panel. It varies smoothly with $b$ peaking approximately midway between the CD and the FS because of projection effects. The $cEM$ for gas between the RS and the CD is shown in the right panel. Because the RS-heated gas forms a thinner and much denser shell than FS-heated
gas (Fig. 2), the $cEM$ is much more peaked at $b$ between the RS and the CD. Fig. 5 shows $cEM$ vs. $b$ for the $s=2$ cases. For $s=2$, both FS-heated gas and RS-heated gas are concentrated in thin and dense shells close to the CD (Fig. 3). In projection, this explains the sharp peak in $cEM$ for both FS-heated gas (left panel) and RS-heated gas (right panel). The extended tail in $cEM$ for $b$ from CD and FS is caused by projection of the low density part of the FS-heated gas.

The integrated quantities $dEM$ and $dT$ from the CP solutions for FS-heated gas and RS-heated gas are given in Table 1. $dEM$ and $dT$ for FS-heated gas varies slowly with $n$ for $s=0$, whereas for RS-heated gas $dEM$ increases by 2 orders of magnitude and $dT$ decreases by 1 order of magnitude. For $s=2$ FS-heated gas, $dEM$ increases by a factor of 5 from $n=6$ to 14 and $dT$ decreases by a factor or 3.5. For $s=2$ RS-heated gas, $dEM$ increases by 2 orders of magnitude for $n=6$ to 14, and $dT$ decreases 1 order of magnitude. In summary, for both $s=0$ and $s=2$ as $n$ increases from 6 to 14, the RS heated gas is brighter and of lower temperature relative to FS heated gas.

5.2. Hydrodynamic SNR Solutions

The evolution of the interior density, velocity and pressure is captured in the snapshots from PLUTO vs. time. Because of the homologous velocity profile of unshocked ejecta ($v \propto r$), the density interior to the RS drops as $1/t^3$ and the core-envelope boundary expands linearly with time. These properties are reproduced by the hydro simulations, with both cases of initial conditions. Numerical errors are visible in the results, e.g. the top two panels of Fig. 7. The relative errors are largest in unshocked ejecta pressure because the initial pressure is small in that region. Errors in density are small except at the origin, and velocity errors are small everywhere. Case 1 hydro solutions have larger errors than case 2.
Evidence that the solutions are reliable comes from comparison of the solutions with different initial conditions. Despite large differences in the initial conditions, both case 1 and case 2 evolve to the same structure after time of \( t \approx 0.01t_{ch} \). The \( t < 0.01t_{ch} \) differences between the case 1 hydro solution and the case 2 hydro solution can be attributed to the inaccuracy of the case 1 initial conditions. Because the simulations for case 1 and case 2 agree after \( t \approx 0.01t_{ch} \), and the fluctuations for case 2 are smaller than for case 1, hereafter we use the results from case 2.

The self similar evolution of the interior structure from early times \( (t << t_{ch}) \) was verified. Deviations from self-similar evolution as time increases are expected. These deviations are apparent starting at \( t \approx 0.3t_{ch} \) when reverse shock propagates inward to reach the the core boundary. After this time \( R_{RS}/R_{FS} \) decreases. This can be seen in the animations of the structure files provided with this paper. At \( t/t_{ch} = 1 \), \( R_{RS} \) is well inside the core (top right panel of Fig. 7), and a sawtooth shape density forms just inside the CD. This sawtooth density at the CD persists for the remainder of the SNR evolution.

After \( t/t_{ch} = 1 \), the reverse shock accelerates inward, reaching the SNR center at \( t/t_{ch} \approx 2.5 \), in agreement (to \( \sim 10\% \)) with the results of TM99. After the RS hits the center, a reflected shock slowly propagates outward. In the bottom left panel of Fig. 7 for \( t/t_{ch} \approx 3 \), the reflected shock is propagating outward and is visible as the pressure, velocity and density jump at \( r/R_{FS} = 0.2 \). The reflected shock reaches \( r/R_{FS} = 0.7 \) at \( t/t_{ch} \approx 10 \) (bottom right panel of Fig. 7), and finally reaches the forward shock at \( t/t_{ch} \approx 80 \).

The evolution for \( s=0, n=8 \) of \( R_{FS}, V_{FS} \) and \( R_{RS} \) is shown in Fig. 8. The deviation from self-similar behaviour is seen at \( t/t_{ch} \approx 0.3 \). The reverse shock moves inward after \( t/t_{ch} \approx 1 \) and hits the SNR center at \( t/t_{ch} \approx 2.5 \). A perturbation in \( V_{FS} \) is seen when the reflected shock hits the FS at \( t/t_{ch} \approx 80 \).

Fig. 9 (left) shows \( dEM \) and \( dT \) for FS shocked gas for \( n=8 \). \( dEM_{FS} \) and \( dT_{FS} \) vary
weakly with time, with maximum range of \( \simeq 0.8 - 0.5 \) for \( dE_{M_{FS}} \), and \( \simeq 1.1 - 1.3 \) for \( dT_{FS} \). By comparing different runs with different \( n \), case 1 and case 2 initial conditions, and with other varied parameters, we determined that the decrease near \( t/t_{ch} \simeq 0.6 \) and the peak and drop near \( t/t_{ch} \simeq 40 \) in \( dT_{FS} \) are real. The snapshots show that these are caused by the outward moving reflected shock. Fig. 9 (right) shows \( dE_{M} \) and \( dT \) for RS shocked gas for \( n=8 \). For RS shocked gas, \( dE_{M_{RS}} \) and \( dT_{RS} \) change rapidly with time after the early self-similar phase. \( dE_{M_{RS}} \) and \( dT_{RS} \) both show an increase around \( t/t_{ch} \simeq 5 \), which is real and caused by the reflected shock.

5.2.1. Fits to \( dE_{M} \) and \( dT \) from the hydrodynamic SNR solutions

In order to facilitate usage of \( dE_{M} \) and \( dT \) for modelling SNRs, we provide fitting functions to \( dE_{M} \) and \( dT \) for both FS and RS gas. \( dE_{M_{FS}} \), \( dT_{FS} \), \( dE_{M_{RS}} \) and \( dT_{RS} \) were extracted from the hydro simulations for \( n=6 \) to 14, as functions of scaled time \( t_{s} = t/t_{ch} \). We found that piecewise powerlaw functions provide good approximations to these quantities. The minimum number of segments was chosen to give a fit to the data using least squares minimization.

Fig. 9 shows the extracted \( dE_{M} \) and \( dT \) for \( n=8 \) and the \( n=8 \) fitting functions. Because \( dT_{FS} \) has the most complex behaviour, we show three different fits: one with 6 segments; a second with 5 segments; and a third with 3 segments. The latter one fits a smoothed version of \( dT_{FS} \). Because real SNRs are not completely spherically symmetric, the RS from different directions is expected to hit the center at different times, thus resulting in smoothing of the peaks in \( dT_{FS} \) compared to our hydro simulations.

We thus chose to use the 3 segment fit to the smoothed \( dT_{FS} \) here, for all \( n \) values,
given by:

\[ dT_{FS}(t_s) = dT_{FS,0} \quad \text{if } t_s < t_1 \]
\[ dT_{FS,0}(t_s/t_1)^{p_1} \quad \text{if } t_1 < t_s < t_2 \]
\[ dT_{FS,0}(t_2/t_1)^{p_1}(t_s/t_2)^{p_2} \quad \text{if } t_2 < t_s \]

(7)

For \( dEM_{FS} \), a model with 5 segments fits the simulation results:

\[ dEM_{FS}(t_s) = dEM_{FS,0} \quad \text{if } t_s < t_1 \]
\[ dEM_{FS,0}(t_s/t_1)^{p_1} \quad \text{if } t_1 < t_s < t_2 \]
\[ dEM_{FS,0}(t_2/t_1)^{p_1}(t_s/t_2)^{p_2} \quad \text{if } t_2 < t_s < t_3 \]
\[ dEM_{FS,0}(t_2/t_1)^{p_1}(t_3/t_2)^{p_2}(t_s/t_3)^{p_3} \quad \text{if } t_3 < t_s < t_4 \]
\[ dEM_{FS,0}(t_2/t_1)^{p_1}(t_3/t_2)^{p_2}(t_4/t_3)^{p_3}(t_s/t_4)^{p_4} \quad \text{if } t_4 < t_s \]

(8)

The right panel of Fig. 9 shows \( dEM \) and \( dT \) for RS shocked gas. After \( t/t_{ch} \sim 0.3 \), \( dEM_{RS} \) decreases and \( dT_{RS} \) increases. The main cause of the decrease of \( dEM_{RS} \) is the increase of volume of FS gas (via \( R_{FS} \)) relative to volume of RS gas (see equation 2). The main cause of the increase of \( dT_{RS} \) is the decrease of \( T_{FS} \) relative to \( T \) of RS gas (see equation 4).

For \( dT_{RS} \) and for \( dEM_{RS} \), 4 segments fit the simulation results:

\[ dT_{RS}(t_s) = dT_{RS,0} \quad \text{if } t_s < t_1 \]
\[ dT_{RS,0}(t_s/t_1)^{p_1} \quad \text{if } t_1 < t_s < t_2 \]
\[ dT_{RS,0}(t_2/t_1)^{p_1}(t_s/t_2)^{p_2} \quad \text{if } t_2 < t_s < t_3 \]
\[ dT_{RS,0}(t_2/t_1)^{p_1}(t_3/t_2)^{p_2}(t_s/t_3)^{p_3} \quad \text{if } t_3 < t_s \]

(9)
\[ dEM_{RS}(t_s) = \begin{cases} 
  dEM_{RS,0} & \text{if } t_s < t_1 \\
  dEM_{RS,0}(t_s/t_1)^{p_1} & \text{if } t_1 < t_s < t_2 \\
  dEM_{RS,0}(t_2/t_1)^{p_1}(t_s/t_2)^{p_2} & \text{if } t_2 < t_s < t_3 \\
  dEM_{RS,0}(t_2/t_1)^{p_1}(t_3/t_2)^{p_2}(t_s/t_3)^{p_3} & \text{if } t_3 < t_s 
\end{cases} \] (10)

The best fit coefficients for \(dT_{FS}(t_s), dEM_{FS}(t_s), dT_{RS}(t_s)\) and \(dEM_{RS}(t_s)\) are given in Table 2 for the different values of n. \(dT_{FS,0}, dEM_{RS,0}, dT_{FS,0}\) and \(dEM_{RS,0}\) were fixed at the values for the initial CP self-similar phase for s=0, given in Table 1.

With the calculated time-dependent \(dEM\) and \(dT\), we can compare how the properties of the shocked ISM and shocked ejecta change with time. The FS EM and EM-weighted T, in dimensionless form, remain remarkably constant over the whole SNR evolution: From Fig. 9 (left), we see that \(dT_{FS}\) only rises a small amount (\(\sim 10\%\)) between \(t/t_{ch} = 1\) and 10, where the significant changes in \(dT\) occur. Those changes are caused by the reverse shock propagating rapidly through the ejecta core, reflecting from convergence at the SNR center, and then propagating outward through the shocked ejecta, the CD and finally the shocked ISM. The very slow decrease of \(dT_{FS}\) after \(t/t_{ch} \sim 100\) occurs in all the simulations, providing evidence it is a real effect. This decrease is probably caused by the effect of the reflected shock on expanding the forward shock more than it does in the self-similar solution. \(dEM_{FS}\) exhibits small changes, with a sharp decrease of \(\sim 40\%\) between \(t/t_{ch} = 0.4\) and 2, during the time the reverse shock propagate through the ejecta core, until convergence at the SNR center.

For shocked ejecta, \(dT_{RS}\) rises steadily with time after the self-similar evolution ends, near \(t/t_{ch} = 0.3\), and has a bump around \(t/t_{ch} \sim = 6\), when the reflected shock passes
through the dense shell of ejecta near the CD. We find that the shocked ejecta is hotter than the shocked ISM \((dT_{RS} > dT_{FS})\) for \(t/t_{ch} \gtrsim 2\) for \(n=6\). This transition of \(T_{RS} > T_{FS}\) (noting that the scaling is the same for both values to obtain dimensionless values) gradually increases from \(t/t_{ch} \gtrsim 2\) for \(n=6\) to \(t/t_{ch} \gtrsim 4\) for \(n=14\).

\(dEM_{RS}\) (right panel of Fig. 9) decreases steadily with time after the self-similar evolution ends, near \(t/t_{ch} = 0.3\), and has a bump around \(t/t_{ch} \sim 6\), caused by the reflected shock. The \(EM\) of shocked ejecta is smaller than that of shocked ISM \((dEM_{RS} < dEM_{FS})\) at all times for \(n=6\) and 7. For \(n=8\) to 14, \(dEM_{RS} > dEM_{FS}\) at early times and \(dEM_{RS} << dEM_{FS}\) at late times. The transition time for the \(EM\) of shocked ISM to exceed that of shocked ejecta increases from \(t/t_{ch} = 0.4\) for \(n=8\) to \(t/t_{ch} = 0.6\) for \(n=14\).

To illustrate the above changes, an animation of \(dT_{FS}(t_s), dEM_{FS}(t_s), dT_{RS}(t_s)\) and \(dEM_{RS}(t_s)\) vs. time with animation parameter \(n\) is provided as an online attachment.

6. Summary and Conclusion

The unified evolution of a SNR (TM99), from explosion to the onset of significant radiative losses, is considered here. The early ED evolution is self-similar and the CP solutions are recalculated here, with higher accuracy. Tables of the solutions are provided as online attachments, for \(s=0\) and \(s=2\) and all \(n\) from 6 to 14, with resolution of 500 points between RS and FS. The dimensionless column \(EM\), \(cEM\), for these solutions are presented in Fig. 4 and Fig. 5 and included in the CP solution tables. The summary quantities \(dT_{FS}\), \(dEM_{FS}\), \(dT_{RS}\) and \(dEM_{RS}\) are given in Table 1.

For non-radiative SNRs with the ejecta mass much smaller than the swept-up ISM mass, the evolution is self-similar. The WL91 solutions are recalculated here for the cases of uniform ISM \((C/\tau=0)\) and cloudy ISM \((C/\tau=1, 2\) and 4\). The \(C/\tau=0\) case is the same
as the pure Sedov-Taylor solution. Tables of the solutions and $cEMs$ are provided as online attachments. The summary quantities $dT_{FS}$ and $dEM_{FS}$ are given in Table 1.

We calculate the unified phase evolution using the publicly available hydrodynamic code PLUTO \cite{Mignone2007} for $s=0$ and $n$ values from 6 to 14. The final results presented here use the CP solutions as initial conditions. The evolution is calculated from early times ($t/t_{ch} \sim 10^{-4}$) to late times ($t/t_{ch} \sim 10^{4}$), while maintaining a minimum resolution of the SNR of 1000 grid points center to FS. We verify that the early-time hydro solutions ($t/t_{ch} \lesssim 0.2$) exhibit self-similar evolution agreeing with the CP solutions.

Animations of the SNR with $\sim 250$ timesteps each are made available as online attachments. These illustrate the changes in interior structure as the SNR evolves. Summary quantities $dT_{FS}$, $dEM_{FS}$, $dT_{RS}$ and $dEM_{RS}$ were calculated as a function of time. Those for $n=8$ are presented in Fig. 9 as an example, and an online attachment shows these functions for all values of $n$. The other values of $n$ show similar behaviour to that seen for $n=8$, including the significant changes caused by the RS accelerating toward the center, the RS reflecting off of the center, and the RS passing through the material concentrated near the CD and the FS. The latter change occurs at the late time of $t/t_{ch} \sim 80$.

Model emission measures and temperatures are required for comparison with X-ray observations of SNRs. Piecewise powerlaws were least-squares fit to the dimensionless emission measures and temperatures, $dT_{FS}(t/t_{ch})$, $dEM_{FS}(t/t_{ch})$, $dT_{RS}(t/t_{ch})$ and $dEM_{RS}(t/t_{ch})$. The powerlaws are given by equations (7) to (10) with coefficients given in Table 2 for the different values of $n$. The emission measure and temperature fitting functions can be used in modelling SNRs, without the need to run the hydrodynamic simulations.

A Python code SNR modelling software (SNRPy) was presented by LW17. This included positions of FS and RS vs. time for several values of $s$ and $n$ from the TM99
solutions, and for some other models. The temperatures, emission measures, interior structures and surface brightness profiles were calculated for $s=0$, $n=7$ and 12 using the low-resolution published CP solutions, and for low resolution WL91 solutions. SNRPy has been updated to include the new high resolution CP and WL91 solutions. It now provides emission measures, interior structure and surface brightness profiles for all values of $s=0$ and $s=2$ for all $n$ from 7 to 14. SNRPy now provide $EM$ and $EM$-weighted $T$ for both shocked ISM and shocked ejecta as functions of time. A number of other corrections and updates were made. The new SNRPy code is available for download from the website quarknova.ca in the software section, and from GitHub in repository denisleahy/SNRmodels. The new CP and WL91 structure files are included in the Data directory of SNRPy. The animations of the PLUTO hydro simulations are provided as zip files with SNRPy.

This work was supported by a grant from the Natural Sciences and Engineering Research Council of Canada. DL thanks Robert Bell, who validated the CP and WL91 solutions.
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This manuscript was prepared with the AAS I$\TeX$ macros v5.2.
Fig. 1.— The interior structure of the WL91 self-similar solutions for $C/\tau=0$ (top left), 1 (top right), 2 (bottom left) and 4 (bottom right). The pressure, density, velocity and temperature are plotted vs. radius and are scaled to their values at the forward shock. Further scaling factors are applied to temperature, as noted in the figure legend. The dimensionless column emission measure (long dash line) is plotted vs impact parameter.
Fig. 2.— The interior structure of the Chevalier-Parker self-similar solutions for \( s=0 \) and \( n=6 \) (top left), 8 (top right), 10 (bottom left), 12 (bottom right). The values are scaled to the post-forward-shock values of pressure, density and temperature and the forward shock velocity. Further scaling factors are applied to density and temperature, as noted in the figure legend. The reverse shock is the point at smallest radius, and the contact discontinuity is where the density goes to zero.
Fig. 3.— The interior structure of the Chevalier-Parker self-similar solutions for $s=2$ and $n=6$ (top left), 8 (top right), 10 (bottom left), 12 (bottom right). The values are scaled to the post-forward-shock values of pressure, density and temperature and the forward shock velocity. Further scaling factors are applied to density and temperature, as noted in the figure legend. The reverse shock is the point at smallest radius, and the contact discontinuity is where the density goes to infinity.
Fig. 4.— The column emission measures vs. impact parameter $b$ for the Chevalier-Parker self-similar solutions for $s=0$ and $n=6, 7, 8, 10, 12$ and $14$. The left panel is for material heated by the forward shock, plotted with linear scale. The right panel is for material heated by the reverse shock, plotted with log scale.
Fig. 5.— The column emission measures vs. impact parameter $b$ for the Chevalier-Parker self-similar solutions for $s=2$ and $n=6,7,8,10,12$ and $14$. The left panel is for material heated by the forward shock and the right panel is for material heated by the reverse shock.
Fig. 6.— PLUTO hydrodynamic simulation of an SNR with s=0, n=7: Left panel: case 1 initial conditions of unshocked ejecta with shocked ISM. Right panel: case 2 initial conditions using the CP self-similar solution with shocked ejecta and shocked ISM. Density, velocity, pressure, time and forward shock radius $R_{FS}$ are in characteristic units, the x-axis is in units of $r/R_{FS}$. 


Fig. 7.— Snapshots of the interior structure for s=0, n=8 from hydrodynamic simulations at 4 characteristic times: $t/t_{ch} \simeq 0.1$ (top left), $t/t_{ch} \simeq 1$ (top right), $t/t_{ch} \simeq 3$ (bottom left) and $t/t_{ch} \simeq 10$ (bottom right). The time $t/t_{ch}$ and the forward shock radius in units of characteristic radius ($R_{FS}/R_{ch}$) are given at the top of each panel. The density ($\rho \rho_{ch}$), velocity ($v/v_{ch}$) and pressure ($p/p_{ch}$) are scaled to their characteristic values, and are plotted vs. radius in units of the forward shock radius ($r/R_{FS}$). Negative (inward) gas velocities are plotted in green.
Fig. 8.— Forward shock radius $R_{FS}$ and velocity $V_{FS}$, and reverse shock radius $R_{RS}$ extracted from the hydrodynamic simulations for $s=0$, $n=8$. Quantities are plotted as in units of characteristic radius or velocity as a function of characteristic time, $t/t_{ch}$. 
Fig. 9.— Extracted quantities from the hydrodynamic simulations for $s=0$, $n=8$ as a function of characteristic time, $t/t_{ch}$. Left: dimensionless temperature $dT_{FS}$ and dimensionless emission measure $dEM_{FS}$ of forward-shocked gas. Right: dimensionless temperature $dT_{RS}$ and dimensionless emission measure $dEM_{RS}$ of reverse-shocked gas. The functions fit to $dT_{FS}$, $dEM_{FS}$, $dT_{RS}$ and $dEM_{RS}$ vs. $t/t_{ch}$ are shown by the lines labelled model.
Table 1. Dimensionless Emission Measures and Temperatures for Self-similar Phases

| phase | s  | n  | C/τ | dEM<sub>FS</sub> | dT<sub>FS</sub> | dEM<sub>RS</sub> | dT<sub>RS</sub> |
|-------|----|----|-----|----------------|---------------|----------------|---------------|
| WL91  | n/a| n/a| 0   | 0.5164         | 1.2896        | n/a            | n/a           |
|       | n/a| n/a| 1   | 0.7741         | 1.3703        | n/a            | n/a           |
|       | n/a| n/a| 2   | 1.6088         | 1.3693        | n/a            | n/a           |
|       | n/a| n/a| 4   | 6.9322         | 1.3833        | n/a            | n/a           |
| CP    | 0  | 6  | n/a | 0.6746         | 1.2614        | 0.1604         | 0.8868        |
|       | 0  | 7  | n/a | 0.7542         | 1.2227        | 0.6250         | 0.5204        |
|       | 0  | 8  | n/a | 0.8081         | 1.1922        | 1.549          | 0.3368        |
|       | 0  | 9  | n/a | 0.8471         | 1.1694        | 3.087          | 0.2347        |
|       | 0  | 10 | n/a | 0.8767         | 1.1518        | 5.400          | 0.1723        |
|       | 0  | 11 | n/a | 0.8998         | 1.1380        | 8.634          | 0.1318        |
|       | 0  | 12 | n/a | 0.9184         | 1.1270        | 12.98          | 0.1039        |
|       | 0  | 13 | n/a | 0.9337         | 1.1179        | 18.53          | 0.0840        |
|       | 0  | 14 | n/a | 0.9465         | 1.1103        | 25.50          | 0.0693        |
| CP    | 1  | 6  | n/a | 17.60          | 0.1417        | 12.95          | 0.0413        |
|       | 1  | 7  | n/a | 99.35          | 0.0298        | 47.88          | 0.0254        |
|       | 1  | 8  | n/a | 56.92          | 0.0533        | 116.0          | 0.0169        |
|       | 1  | 9  | n/a | 45.84          | 0.0677        | 227.3          | 0.0119        |
|       | 1  | 10 | n/a | 62.73          | 0.0515        | 391.5          | 0.00887       |
|       | 1  | 11 | n/a | 79.16          | 0.0420        | 619.3          | 0.00684       |
|       | 1  | 12 | n/a | 94.66          | 0.0360        | 920.8          | 0.00644       |
|       | 1  | 13 | n/a | 75.18          | 0.0451        | 1308           | 0.00442       |
|       | 1  | 14 | n/a | 83.76          | 0.0411        | 1788           | 0.00367       |
Table 2. Coefficients for Fits to $dT_{FS}(t_s)$, $dEM_{FS}(t_s)$, $dT_{RS}(t_s)$ and $dEM_{RS}(t_s)$

|       | n=6  | n=7  | n=8  | n=9  | n=10 | n=11 | n=12 | n=13 | n=14 |
|-------|------|------|------|------|------|------|------|------|------|
| $dT_{FS}(t_s)$ |      |      |      |      |      |      |      |      |      |
| $t_1$ | 0.100 | 3.011 | 2.567 | 0.8303 | 0.4707 | 0.4149 | 0.3158 | 0.3999 | 0.2428 |
| $t_2$ | 1.197 | 19.43 | 17.85 | 11.95 | 11.24 | 13.72 | 11.86 | 16.48 | 8.654 |
| $p_1(\times 10^{-2})$ | -0.741 | 2.601 | 3.346 | 2.713 | 2.739 | 2.805 | 3.019 | 3.592 | 3.594 |
| $p_2(\times 10^{-3})$ | 0.919 | -5.29 | -2.43 | -0.352 | -0.479 | -0.285 | -0.328 | -4.50 | -0.322 |
| $dEM_{FS}(t_s)$ |      |      |      |      |      |      |      |      |      |
| $t_1$ | 0.8058 | 0.5343 | 0.4037 | 0.3352 | 0.2562 | 0.2400 | 0.2083 | 0.2031 | 0.1928 |
| $t_2$ | 2.270 | 1.762 | 1.122 | 1.189 | 1.247 | 1.070 | 1.187 | 1.194 | 1.149 |
| $t_3$ | 9.438 | 9.785 | 3.960 | 8.933 | 7.256 | 4.364 | 6.020 | 7.464 | 7.480 |
| $t_4$ | 45.69 | 47.79 | 240.7 | 40.98 | 41.69 | 55.84 | 51.34 | 53.16 | 42.56 |
| $p_1(\times 10^{-2})$ | -0.2099 | -0.2713 | -0.3197 | -0.3235 | -0.2810 | -0.2937 | -0.2772 | -0.2932 | -0.2882 |
| $p_2(\times 10^{-2})$ | -4.777 | -4.595 | -11.31 | -6.303 | -6.753 | -10.27 | -8.984 | -5.229 | -7.038 |
| $p_3(\times 10^{-2})$ | 4.576 | 5.211 | 1.795 | 5.983 | 4.489 | 2.715 | 5.765 | 3.410 | 5.166 |
| $p_4(\times 10^{-3})$ | -4.81 | -6.31 | -7.08 | -1.35 | -0.056 | -0.572 | -9.58 | -0.562 | -2.12 |
| $dT_{RS}(t_s)$ |      |      |      |      |      |      |      |      |      |
| $t_1$ | 0.1001 | 0.2290 | 0.4398 | 0.4016 | 0.3075 | 0.2618 | 0.2382 | 0.2162 | 0.1966 |
| $t_2$ | 1.376 | 1.229 | 2.037 | 2.523 | 2.686 | 2.492 | 2.766 | 2.801 | 2.925 |
| $t_3$ | 5.923 | 6.995 | 4.631 | 4.504 | 4.520 | 4.582 | 4.558 | 4.508 | 4.457 |
| $p_1$ | -0.0392 | -0.2713 | -0.3197 | -0.3235 | -0.2810 | -0.2937 | -0.2772 | -0.2932 | -0.2882 |
| $p_2$ | 2.376 | 1.166 | 1.562 | 1.842 | 1.996 | 1.841 | 2.094 | 2.079 | 2.247 |
| $p_3$ | 0.7157 | 0.7137 | 0.7131 | 0.7109 | 0.7159 | 0.7204 | 0.7209 | 0.7211 | 0.7242 |
| $dEM_{RS}(t_s)$ |      |      |      |      |      |      |      |      |      |
| $t_1$ | 0.6146 | 0.4272 | 0.3382 | 0.3221 | 0.2357 | 0.2209 | 0.2104 | 0.1939 | 0.1747 |
| $t_2$ | 2.833 | 2.657 | 2.953 | 2.462 | 2.799 | 2.832 | 2.665 | 2.730 | 2.681 |
| $t_3$ | 4.882 | 4.888 | 4.513 | 4.984 | 4.819 | 4.861 | 4.922 | 4.821 | 4.848 |
| $p_1$ | 2.465 | 2.652 | 2.733 | 2.955 | 2.718 | 2.781 | 2.884 | 2.891 | 2.882 |
| $p_2$ | 1.247 | 1.384 | 0.781 | 1.335 | 1.008 | 1.023 | 0.989 | 1.009 | 1.006 |
| $p_3$ | 1.919 | 1.919 | 1.919 | 1.920 | 1.926 | 1.927 | 1.941 | 1.928 | 1.933 |