ON THE LIOUVILLE COUPLING CONSTANTS

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Abstract

For the general operator product algebra coefficients derived by Cremmer Roussel Schnittger and the present author with (non negative) integer screening numbers, the coupling constants determine the factors additional to the quantum group $6j$ symbols. They are given by path independent products over a two dimensional lattice in the zero mode space. It is shown that the ansatz for the three point function of Dorn-Otto and Zamolodchikov-Zamolodchikov precisely defines the corresponding flat lattice connection, so that it does give a natural generalization of these coupling constants to continuous screening numbers. The consistency of the restriction to integer screening charges is reviewed, and shown to be linked with the orthogonality of the (generalized) $6j$ symbols. Thus extending this last relation is the key to general screening numbers.
1 Introduction

One outcome of refs.[1, 2, 3, 4, 5] was the general expression for the fusion and braiding matrices of the general chiral operators noted \( V(J^e)(z) \), which are chiral components of the quantum Liouville exponentials. We follow the notational conventions of these works. Calling \( h \) the quantum group deformation parameter, the central charge is \( C_L = 1 + 6\frac{h}{\pi} + \frac{\pi}{h} + 2 \). The notation \( J^e \) characterizes a primary field with associated (rescaled) Liouville momentum \( \varpi_{J^e} = \varpi_0 + 2J^e \), where \( \varpi_0 = 1 + \pi/h \) is one of the \( sl(2) \) invariant vacua. Its weight is \( \Delta(J^e) = (\varpi_0^2 - \varpi_j^2)h/4\pi \). The discussion was carried out for arbitrary continuous \( J^e \), but for the degenerate fields, one has \( J^e = J + \hat{J}\pi/h \), such that \( 2J + 1 \) and \( 2\hat{J} + 1 \) are positive integers which characterize the finite dimensional quantum group representations. Thus \( J^e \) is thought of as the effective spin. We denote by \( \mathcal{P}_{J^e} \), the projector over the corresponding Verma module. The full fusion-equation reads\[5\]

\[
\mathcal{P}_{J^e_{123}} V(J^e_1)(z_1) \mathcal{P}_{J^e_{23}} V(J^e_2)(z_2) \mathcal{P}_{J^e_3} = \sum_{J_{12}} \frac{g^e_{J_{123}}}{g^e_{J_{123}J_{123}}} \left\{ \left\{ J^e_{123} J^e_{123} J^e_{123} \right\} q \right\} \left\{ \left\{ \hat{J}^e_{123} \hat{J}^e_{123} \hat{J}^e_{123} \right\} \hat{q} \right\}
\]

\[
\sum_{\nu_{123}} \mathcal{P}_{J^e_{123}} V(J^e_{12}, \nu_{12}) \left( z_2 \right) \mathcal{P}_{J^e_3} \left< \varpi_{J^e_{12}}, \nu_{12} \right| V(J^e_1)(z_1 - 2\pi/h) | \varpi_{J^e_2} > . \quad (1.1)
\]

Note that the last term is a \( c \) number (a matrix element) which is a book keeping device to handle all descendents at once. They are characterized, abstractly by a multi-index noted \( \nu \). The \( V \) fields are normalized such that

\[
< \varpi_L | V(J^e_1)(0) | \varpi_{K^e} > = 1 \quad \text{if} \quad J^e + K^e - L^e = p + \hat{p}\pi/h, \quad p \in \mathbb{Z}, \hat{p} \in \mathbb{Z}, \quad (1.2)
\]

\[
< \varpi_L | V(J^e_1)(0) | \varpi_{K^e} > = 0 \quad \text{otherwise} \quad (1.3)
\]

where \( \mathbb{Z} \) is the set of non negative integers. The symbol \( g^e_{J^e_{12}K^e} \) stands for the coupling constants which are the central point of the present note. They involve the contributions which are not solely determined by the quantum group symmetry, in contrast with the 6j symbols. The sum over \( \{\nu_{12} \} \) represents the summation over arbitrary states\[6\] of the Verma module with momentum \( \varpi_{J^e_{12}} \). This equation was derived for the most general case where the \( J^e \)'s are arbitrary continuous variables with the restriction that condition Eq.\[1.3\] be obeyed by each of the four \( V \) operators which appear in the fusion equation\[1.1\], so that

\[
J^e_1 + J^e_2 - J^e_{123} = p_{1,23} + \frac{\pi}{h} \hat{p}_{1,23}, \quad p_{1,23}, \hat{p}_{1,23} \in \mathbb{Z}
\]

\[
J^e_2 + J^e_3 - J^e_{23} = p_{2,3} + \frac{\pi}{h} \hat{p}_{2,3}, \quad p_{2,3}, \hat{p}_{2,3} \in \mathbb{Z}
\]

\[
J^e_{12} + J^e_3 - J^e_{123} = p_{12,3} + \frac{\pi}{h} \hat{p}_{12,3}, \quad p_{12,3}, \hat{p}_{12,3} \in \mathbb{Z}
\]

\[
J^e_1 + J^e_2 - J^e_{12} = p_{1,2} + \frac{\pi}{h} \hat{p}_{1,2}, \quad p_{1,2}, \hat{p}_{1,2} \in \mathbb{Z}. \quad (1.4)
\]

From the viewpoint of Coulomb gas, the \( p \)'s are the screening numbers. Thus, there is a consistent operator product algebra where all these numbers are non negative integers. The symbols between double braces are the corresponding generalized 6j symbols associated with the two quantum group parameters \( h, \tilde{h} (q = e^{ih}, \tilde{q} = e^{\tilde{h}}) \), themselves related to the two screening charges \( \alpha_\pm \) by \( h = \pi \alpha_\pm^2 / \pi, \tilde{h} = \pi \alpha_\pm^2 / \pi \). In

\[2\] to simplify the formulae we assume that they are orthonormalized.
the $6j$ symbols associated with $\hat{q}$, we have introduced the convenient notation $J^e = J^e h/\pi$ which makes the symmetry between $h$ and $\hat{h}$ more explicit. The sum over $J_{12}^e$ runs over all the values of $J_{12}^e$ allowed by the four screening number conditions. It can be viewed as a double sum on $p_{1,2}, \hat{p}_{1,2}$, such that $p_{1,2} + \hat{p}_{1,2} \pi/h = J^e_1 + J^e_2 - J^e_{12}$. Thus it is a summation over non negative integers. By construction, the fusion formula only involves the coupling constants $g_{J^e_{12}}^{J^e_{13}}$, with the restriction that $J^e + K^e - L^e = p + \hat{p}_n/h$, where $p$, and $\hat{p}$ are non negative integers.

We recall the braiding equation as well. It was derived in ref.[3] for one half of the algebra and in ref.[4] for the full algebra. It can be deduced from the fusion by the three leg symmetry of the vertices[3, 4, 3]:

$$< \varpi_{12} | V(J^e_1) | \varpi_2 > = e^{i\pi(\Delta(J^e_1) + \Delta(J^e_2) - \Delta(J^e_{12}))} < \varpi_{12} | V(J^e_2) | \varpi_1 > . \quad (1.5)$$

This gives

$$\mathcal{P}_{J^e_{123}} V(J^e_1)(z_1) \mathcal{P}_{J^e_{23}} V(J^e_2)(z_2) \mathcal{P}_{J^e_3} = \sum_{J^e_{13}} e^{i\pi(\Delta(J^e_{13}) + \Delta(J^e_3) - \Delta(J^e_{23}) - \Delta(J^e_{12}))} \times$$

$$\frac{g_{J^e_{123}}^{J^e_{13}} g_{J^e_{123}}^{J^e_{23}}}{g_{J^e_{23}}^{J^e_{13}} g_{J^e_{13}}^{J^e_{23}}} \left\{ \left\{ J^e_{12}, J^e_{13} | J^e_{23} \right\} \right\}_{q} \left\{ \left\{ J^e_{13}, J^e_{23} | J^e_{12} \right\} \right\}_{q} \mathcal{P}_{J^e_{123}} V(J^e_2)(z_2) \mathcal{P}_{J^e_{13}} V(J^e_1)(z_1) \mathcal{P}_{J^e_3} \quad (1.6)$$

where again the sum over $J^e_{13}$ is to be understood as a double sum.

## 2 The coupling constant from a flat connection

The general expression of the coupling constants was given in refs.[1, 3] under the form

$$g_{J^e_{13}}^{J^e_{12}} = g_p g_0 \frac{\gamma H_p(\varpi, J^e_1) H_p(\varpi, J^e_2) H_p(\varpi, J^e_3)}{H_p(\varpi, J^e_{12}, J^e_{13})} \quad (2.1)$$

with $\varpi_{p/2, \hat{p}/2} = \varpi_0 + p + \hat{p}_n/h$, $J^e_1 + J^e_2 - J^e_{12} = p + \hat{p}_n/h$, $p, \hat{p} \in \mathbb{Z}$, and

$$H_p(\varpi) = \frac{\Pi_{r=1}^{\hat{p}} \sqrt{F((\varpi - r)h/\pi)}}{\Pi_{r=1}^{\hat{p}} \sqrt{F((\varpi - \hat{p}_n/h)h/\pi)}} \left( \varpi \sqrt{h/\pi} - r \sqrt{h/\pi - \hat{p}_n/h} \right). \quad (2.2)$$

The constants $g_0$ and $\hat{g}_0$ are arbitrary. The function $F$ is defined by

$$F(z) = \frac{\Gamma(z)}{\Gamma(1 - z)} \quad (2.3)$$

By absorbing the denominator factors, an equivalent form was given:

$$H_p(\varpi) = \prod_{i=1}^{n-1} \left\{ F \left[ \varpi - N_i - (\tilde{N}_i + 1 + \epsilon_i) \frac{\pi}{h} \right] \left( \frac{-\pi}{h} \right)^{N_i} \right\}^{\epsilon_i/2}$$

$$\left\{ F \left[ \frac{h}{\pi} \varpi - \tilde{N}_i - (N_i + 1 + \epsilon_i) \frac{h}{\pi} \right] \left( \frac{-h}{\pi} \right)^{\tilde{N}_i} \right\}^{\epsilon_i/2} \quad (2.4)$$
where the \((N_i, \hat{N}_i), i = 1...n\) describe an arbitrary planar path going from \((0,0) = (N_1, \hat{N}_1)\) to \((p, \hat{p}) = (N_n, \hat{N}_n)\). The allowed elementary steps \((\epsilon_i, \hat{\epsilon}_i) \equiv (N_{i+1} - N_i, \hat{N}_{i+1} - \hat{N}_i)\) are \((0, \pm 1)\) and \((\pm 1, 0)\). Of course, this expression only makes sense if \(p\) and \(\hat{p}\) are integers. However, it is not restricted to the case where they are positive (more on this below). One sees that it is given by a sort of Wilson product of a discrete flat connection in a two dimensional square lattice with spacings \(1 \) and \(\pi/h\).

The point of the present note is to show that the function \(\Upsilon\) introduced in refs.\[7, 8\] precisely allows us to integrate explicitly the above Wilson integral. This will give a natural interpolation for the connection away from the discrete lattice, which will define \(g\) for arbitrary continuous \(p\) and \(\hat{p}\).

The derivation goes as follows. Recall the definition of refs.\[7, 8\]:

\[F(bx) = \frac{\Upsilon(x + b)}{\Upsilon(x)} b^{2bx-1}, \quad F\left(\frac{1}{b}x\right) = \frac{\Upsilon\left(x + \frac{1}{b}\right)}{\Upsilon(x)} \left(\frac{1}{b}\right)^{2x/b-1} \tag{2.5}\]

To clarify the gauge analogy, consider an abelian flat connection in two dimensions \(\vec{A} = \vec{\nabla} \Lambda\). We put it on a square lattice with spacings \(b\) and \(1/b\) by introducing the Wilson link variables

\[U_{\vec{r}, \vec{\mu}} = e^{\int_{\vec{r}}^{\vec{r}+\vec{\mu}} \tilde{\nabla} d\vec{x}} = e^{\Lambda(\vec{r}+\vec{\mu}) - \Lambda(\vec{r})}, \tag{2.6}\]

where \(\vec{r}\) is a point on the lattice (with coordinates \((nb, m/b)\) \(n, m\) integers), and \(\vec{\mu}\) has coordinates \((b, 0)\) or \((0, 1/b)\). Of course, by construction, the Wilson line product along any path on the lattice only depends upon the initial and final points:

\[\prod_{i=0}^{N} U_{\vec{r}_i, \vec{\mu}_i} = e^{\Lambda(\vec{r}_N + \vec{\mu}_N) - \Lambda(\vec{r}_0)} \tag{2.7}\]

To make contact with the previous quantum group formulae, we let

\[b = \sqrt{\frac{\hbar}{\pi}}, \quad \varpi = \frac{1}{b} x, \tag{2.8}\]

and rewrite Eq.\[2.4\] as

\[H_{pp}(\varpi) = \prod_{i=1}^{n-1} \left\{ F \left[ \frac{1}{b} \left( x - bN_i - \frac{1}{b} (\hat{N}_i + \frac{1 + \epsilon_i}{2}) \right) \right] \left( \frac{-1}{b^2} \right)^{N_i} \hat{\epsilon}_i/2 \right\} \left\{ F \left[ b \left( x - \frac{1}{b} \hat{N}_i - (N_i + \frac{1 + \epsilon_i}{2}) b \right) \right] \left( -b^2 \right)^{\hat{N}_i} \hat{\epsilon}_i/2 \right\}. \tag{2.9}\]

Clearly, the values of \(x\) which appear are of the form \(x = x_0 + nb + m/b, n m\) integers, \(x_0\) fixed continuous constant. If \(b\) is irrational, this uniquely defines a point in the two dimensional square lattice introduced above. Thus we have a discrete flat connection of the general type mentioned above. The basic point is that Eqs.\[2.4\] simply define the corresponding gauge function \(\Lambda\). This is easily seen by using Eqs.\[2.3\] to integrate the product \[2.3\]. Since the result does not depend upon the path, one may take the simplest rectangular one. Then one finds

\[H_{p, \hat{p}}(\varpi) = \frac{\sqrt{\Upsilon(x)}}{\sqrt{\Upsilon(x - pb - \hat{p}^2)}} b^{2bx - 1 - b^2(p+1)/2} \left( \frac{1}{b} \right)^{(2x/b - 1 - (\hat{p}+1)/b^2)\hat{p}/2} \tag{2.10}\]
One sees that up to the last two terms we have obtained an equation of the type Eq.(2.7) with \( \Lambda = \ln(\Upsilon(x))/2 \). Of course, once it is derived it give a natural generalisation of the flat connection to the continuum – reversing the discretization performed by writing Eq.(2.3). The next question concerns possible zeros or poles of \( \Upsilon \). Their positions are easily seen by looking at possible singularities in the path formulation Eq.(2.9). It follows from Eqs.(2.3, 2.5) that

\[
\frac{\Upsilon(x + b)}{\Upsilon(x)} \begin{cases}
\infty & \text{if } x = -n/b, \ n \in \mathbb{Z}; \\
0 & \text{if } x = (n + 1)/b, \ n \in \mathbb{Z}; \\
is regular otherwise
\end{cases} \tag{2.11}
\]

\[
\frac{\Upsilon(x + \frac{1}{b})}{\Upsilon(x)} \begin{cases}
\infty & \text{if } x = -nb, \ n \in \mathbb{Z}; \\
0 & \text{if } x = (n + 1)b, \ n \in \mathbb{Z}; \\
is regular otherwise;
\end{cases} \tag{2.12}
\]

This agrees with the fact that \( \Upsilon(x) \) vanishes iff \( x = -nb - m/b \), and \( x = (n + 1)b + (m + 1)/b \) with \( n \) and \( m \) both non-negative integers.

Returning to Eq.(2.1), we next deduce the expression for the coupling constant, namely,

\[
\frac{\mathcal{J}_i^3}{\mathcal{J}_i} = \left( g_{0}b^{-(1+1/2)} \right)^p \left( (g_{0}/b)^{-1+1/b} \right)^{1/\beta} \sqrt{\Upsilon(x_1)} \sqrt{\Upsilon(x_2)} \sqrt{\Upsilon(x_3)} \sqrt{\Upsilon(x_4)} \sqrt{\Upsilon((p + 1)b + (\beta + 1)/b)}
\]

where we have let \( x_i = b\alpha\pi_i \), so that

\[
x_1 + x_2 - x_3 = (2p + 1)b + (2\beta + 1)/b. \tag{2.14}
\]

Several comments are in order. Let us begin by discussing the properties of the expression just derived as it is (we shall connect with the earlier expressions of refs.[4, 8] last). First, in the most restricted case, namely when \( J_i^c = J_i + \tilde{J}_i \pi/h \), \( 2J_i \in \mathbb{Z}, 2\tilde{J}_i \in \mathbb{Z} \). All the terms is the numerator and in denominator vanish, so that this formula is not very useful. The expression is perfectly finite however, and is better derived from the previous product relation 2.2 or 2.4. Second, consider the case where the spins are continuous, but \( p \) and \( \beta \) are still integers. If \( p \) and \( \beta \) are both positive or both non-positive, the vanishing of \( \Upsilon((p + 1)b + (\beta + 1)/b) \) in the denominator compensates the vanishing of \( \Upsilon(b + 1/b) \) in the numerator and the ratio is still finite. Third, on the contrary, in the mixed situation, Eq.(2.13) gives zero since it contains \( \sqrt{\Upsilon(b + 1/b)} \) as the only vanishing term in the numerator. Fourth, if everything is continuous, Eq.(2.13) gives zero, for the same reason.

In the earlier discussions[4, 8], one modifies the above expression, and replaces the vanishing term \( \Upsilon(b + 1/b) \) by its derivative which is non zero. As a result, the new expression blows up when Eq.(2.13) is finite, and is finite when Eq.(2.13) gives a vanishing result. This explains why the cases discussed in refs.[1–5] appear as poles in the work of [4, 8]. A similar situation is encountered in the heuristic expressions proposed in ref.[1, 4] of amplitudes with continuous screening which are proportional to gamma functions\(^3\) whose arguments are the opposite of the screening numbers, which one may factor out at will.

\(^3\) This is really the case only if the two screening charges are dealt with independently contrary to ref.[1], as was done for instance in ref.[2].
3 The Liouville n-point functions.

Since this question has attracted much attention lately, it is worthwhile summarizing how they arise in the present context. We shall deal with the conformal bootstrap solution based on the fusing and braiding relations recalled above where the screening numbers are consistently kept integers. On the sphere, the n-point functions of chiral operators may be calculated from the matrix element

\[ G_{\omega_1, \omega_2, \ldots, \omega_n}^{(n,V)} (z_1, z_2, \ldots, z_n) = -\omega_0 |V'(J^n_0) (z_n) V'(J^n_{n-1}) (z_{n-1}) \ldots V'(J^n_1) (z_1)| \omega_0 >, \]

where

\[ \omega_i = \omega_0 + 2J^i. \]  

(3.1)

Inserting complete sums over intermediate states, one sees that we may rewrite

\[ G_{\omega_1, \omega_2, \ldots, \omega_n}^{(n,V)} (z_1, z_2, \ldots, z_n) = \sum_{K_1^n} \ldots \sum_{K_{n-1}^n} < -\omega_0 |V'(J^n_0) (z_n) \mathcal{P}_{K_1} V'(J^n_{n-1}) (z_{n-1}) \mathcal{P}_{K_2} \ldots \mathcal{P}_{K_{n-1}} V'(J^n_1) (z_1)| \omega_0 >. \]

(3.3)

As usual, Möbius invariance is such that it is enough to compute in the limit \( z_n \to 0, \ z_1 \to \infty \). One gets\(^4\)

\[ G_{\omega_1, \omega_2, \ldots, \omega_n}^{(n,V)} (0, z_2, \ldots, z_{n-1}, \infty) = \sum_{K_2^n} \ldots \sum_{K_{n-2}^n} < -\omega_0 |V'(J^n_{n-1}) (z_{n-1}) \mathcal{P}_{K_2} \ldots \mathcal{P}_{K_{n-2}} V'(J^2_2) (z_2)| \omega_1 >. \]

(3.4)

This fact was actually derived and shown to be consistent with the operator algebra for half integer spins only. There may be subtleties in the general case which we shall leave out. In any case, the restriction to non negative integer values of the screening operators is such that the summations over the intermediate spins is discrete. This is easily seen by recursion since \( K_j^e \) must be of the form \( K_j^e = K_j^{e-1} + J_j^e \pm | \hat{p}_j | \pi / h \) with \( p_j \) and \( \hat{p}_j \) integers. Concerning the four-point function, Eq. [1] allows us to re-express it in terms of three-point functions as follows

\[
G_{\omega_1, \omega_2, \omega_3, -\omega_4}^{(4,V)} (0, z_2, z_3, \infty) = \sum_{J_{23}^e} \sum_{J_{12}^e} \left( \frac{g_{J_{23}^e} g_{J_{12}^e}}{g_{J_{23}^e} g_{J_{12}^e}} \right) \left\{ \left\{ \{ J_{23}^e \} | J_{12}^e \} \right\} \left\{ \left\{ \{ J_{23}^e \} | J_{12}^e \} \right\} \right\} \right)
\]

\[
\sum_{\{ \nu_3 \}} G_{\omega_1, \omega_2, \omega_3, -\omega_4}^{(3,V)} (0, z_1, \infty) G_{\omega_2, \omega_3, -\omega_4}^{(3,V)} (0, z_3 - z_1, \infty)
\]

(3.5)

This involves of course three point functions with descendents defined, in general, according to

\[ G_{\omega_1, \{ \nu_1 \}}^{(3,V)} (0, z_2, \infty) = -\omega_3, \{ \nu_3 \} |V'(J_{\{ \nu_2 \}}^2) (z_2)| \omega_1, \{ \nu_1 \} > \quad (3.6) \]

In this way all n point functions are expressed as discrete sums of three point functions. The braiding equation 1.6 leads to similar relations, albeit with quantum numbers put differently:

\[ G_{\omega_1, \omega_2, \omega_3, -\omega_4}^{(4,V)} (0, z_2, z_3, \infty) = \]

\(^4\) up to a divergent factor in \( z_n \) which we drop
\[
\sum_{J_{12}^{(c)}} \left( \sum_{J_{13}^{(c)}} \left( \frac{g_{J_{13}^{(c)} J_{12}^{(c)}}}{g_{J_{12}^{(c)} J_{13}^{(c)}}} \left\{ \left\{ J_{12}^{(c)} J_{13}^{(c)} | J_{12}^{(c)} \right\}_q \left\{ \left\{ J_{12}^{(c)} J_{13}^{(c)} | J_{12}^{(c)} \right\}_q \right\} e^{\pm i \pi (\Delta(J_{13}^{(c)}) + \Delta(J_{12}^{(c)}) - \Delta(J_{13}^{(c)}) - \Delta(J_{12}^{(c)}))} \right) \right) \right) \right)
\]

\[
\sum_{\{\nu_{13}\}} G_{\nu_{13}}^{(3, V)}(0, z_2, \infty) G_{\nu_{13}}^{(3, V)}(0, z_3, \infty) \right].
\]

In general, there is no way to reexpress the right hand side in terms of a 4 point function, since the braiding matrix explicitly depends upon \(J_{13}^{(c)}\). Thus the n point functions of the V fields do not have well defined monodromy properties. This will not be the case for the expectation values of the Liouville exponentials as we next show. According to the operator expression \(4\) of the Liouville exponentials, they are defined such that

\[
G^{(n, \Phi)}_{\nu_1, \nu_3, \cdots, \nu_n}(0, 0, z_2, z_2, z_3, \infty, \infty) = \sum_{J_{12}^{(c)}} \left( \frac{g_{J_{12}^{(c)} J_{13}^{(c)}}}{g_{J_{12}^{(c)} J_{13}^{(c)}}} \right)^2
\]

\[
< \nu_4 | V^{(J_{13}^{(c)})}(z_3) \mathcal{P}_{J_{12}^{(c)}} V^{(J_{12}^{(c)})}(z_2) | \nu_1 > < \nu_4 | V^{(J_{12}^{(c)})}(z_2) \mathcal{P}_{J_{12}^{(c)}} V^{(J_{13}^{(c)})}(z_2) | \nu_1 > .
\]

Making use of Eq.\(3.7\) for the two chiralities one gets

\[
G^{(4, \Phi)}_{\nu_1, \nu_2, \nu_3, \cdots, \nu_4}(0, 0, z_2, z_2, z_3, z_3, \infty, \infty) = \sum_{J_{13}^{(c)}, J_{13}^{(c)}} g_{J_{13}^{(c)} J_{12}^{(c)}} g_{J_{13}^{(c)} J_{12}^{(c)}} g_{J_{13}^{(c)} J_{12}^{(c)}} g_{J_{13}^{(c)} J_{12}^{(c)}}
\]

\[
\sum_{J_{12}^{(c)}} \left\{ \left\{ J_{12}^{(c)} J_{13}^{(c)} | J_{12}^{(c)} \right\}_q \left\{ \left\{ J_{12}^{(c)} J_{13}^{(c)} | J_{12}^{(c)} \right\}_q \right\} \right) \left\{ \left\{ J_{12}^{(c)} J_{13}^{(c)} | J_{12}^{(c)} \right\}_q \left\{ \left\{ J_{12}^{(c)} J_{13}^{(c)} | J_{12}^{(c)} \right\}_q \right\} \right) \right) \right)
\]

\[
\sum_{\{\nu_{13}\}} G_{\nu_{13}}^{(3, V)}(0, z_2, \infty) G_{\nu_{13}}^{(3, V)}(0, z_3, \infty) \right).
\]

The basic difference with Eq.\(3.9\) is that the summation over the intermediate \(13\) quantum numbers is done independently for the two chiralities, since it only appears on the right hand side of Eq.\(3.7\). The symbol \(\nu_{13}\) is defined as equal to \(\nu_0 + 2 \mathcal{T}_{13}\). However, it was verified in ref.\(4\) that, as long as the screening numbers are non negative integers, the generalized 6j are orthonormal polynomials such that

\[
\sum_{J_{12}^{(c)}} \left( \left\{ \left\{ J_{12}^{(c)} J_{13}^{(c)} | J_{12}^{(c)} \right\}_q \left\{ \left\{ J_{12}^{(c)} J_{13}^{(c)} | J_{12}^{(c)} \right\}_q \right\} \right) \left\{ \left\{ J_{12}^{(c)} J_{13}^{(c)} | J_{12}^{(c)} \right\}_q \left\{ \left\{ J_{12}^{(c)} J_{13}^{(c)} | J_{12}^{(c)} \right\}_q \right\} \right) \right) = \delta_{J_{13}^{(c)}} \mathcal{T}_{13}.
\]

(3.11)
Since this point is very important, let us review how the range of summation is specified, following e.g. ref. [14]. Each generalised 6j symbol is defined so that the corresponding four pairs of screening numbers are non-negative integers. Their entries, however only differ by the spin with index 13, so that the screening numbers are

\[ J_1^c + J_5^c - J_{13}^c = p_{1,3} + \frac{\pi}{\hbar} \hat{p}_{1,3}, \quad J_2^c + J_6^c - J_{12}^c = p_{1,2} + \frac{\pi}{\hbar} \hat{p}_{1,2}; \]

\[ J_{13}^c + J_{23}^c = p_{13,2} + \frac{\pi}{\hbar} \hat{p}_{13,2}, \quad J_{12}^c + J_{32}^c = p_{12,3} + \frac{\pi}{\hbar} \hat{p}_{12,3}; \]

\[ J_1^c + J_3^c - J_{13}^c = \overline{p}_{1,3} + \frac{\pi}{\hbar} \hat{\overline{p}}_{1,3}, \quad J_{13}^c + J_2^c - J_{12}^c = \overline{p}_{13,2} + \frac{\pi}{\hbar} \hat{\overline{p}}_{13,2}. \] (3.12)

The screening numbers are not all independent since one has

\[ p_{1,3} + p_{13,2} = p_{1,2} + p_{12,3} = \overline{p}_{1,3} + \overline{p}_{13,2}, \]

with similar relations for the hatted counterparts. It is convenient to consider \( p_{1,3}, \overline{p}_{1,3}, p_{1,2}, \overline{p}_{1,2}, \) and their hatted counterparts as independent. Then clearly, only \( p_{1,2}, \overline{p}_{1,2} \) vary in the summation. Since \( J_1^c, J_2^c \) are fixed, we may replace the summation over \( J_{12}^c \), by the summation over \( p_{1,2}, \overline{p}_{1,2} \), which is thus a discrete sum, even for continuous spins.

Now, returning to Eq. (3.10), we may resum over intermediate states on the right hand side, using Eq. (3.11)—so that only contributions with \( J_{13}^c = \overline{J}_{13}^c \) remain—and re-obtain a four point functions of Liouville exponentials with 2 and 3 exchanged. One gets

\[ G_{\omega_1, \omega_2, \omega_3, \omega_4}(0, 0, z_2, \bar{z}_2, z_3, \bar{z}_3, \infty, \infty) = G_{\omega_1, \omega_3, \omega_2, \omega_4}(0, 0, z_3, \bar{z}_3, z_2, \bar{z}_2, \infty, \infty), \] (3.13)

in agreement with locality. Performing a similar calculation for the fusion, one gets a very simple result

\[ G_{\omega_1, \omega_2, \omega_3, \omega_4}(0, 0, z_2, \bar{z}_2, z_3, \bar{z}_3, \infty, \infty) = \sum_{J_{23}^c, \{\nu_{23}\}} G_{\omega_1, \omega_2, \omega_3, \omega_4}(0, 0, \nu_{23}, \infty, \infty)G_{\omega_1, \omega_3, -\omega_2, -\omega_4}(0, 0, \nu_{23}, \infty, \infty). \] (3.14)

Thus the fusing and braiding matrices of the Liouville exponentials are equal to one.

4 Comments

The summary of the operator algebra presented here shows that we have a perfectly consistent theory, if we restrict ourselves to screening numbers which are non-negative integers, so that condition [3] is fulfilled by each vertex operator. The definition of coupling constant displayed by Eq. (2.13) is well suited for this restricted scheme, which is consistent, since the orthogonality relation Eq. (3.11) is obtained by a discrete sum over integer screening charges. Assuming a symmetry between spins \( J \) and \(-J - 1\), this is sufficient to recover the results of matrix models [13]. On the other hand, this symmetry leads to screening charges which are negative integers, and in general there are arguments to include screening charges which are negative integers or continuous. In this latter case recently discussed in ref. [8], by change of
normalisation, one obtains the case discussed here as residues of poles, which one may regard as on-shell contributions. Besides refs.[7], and [8], there are at present interesting progress in understanding the role of screening charges which are non positive integers: see refs.[10]. Another interesting paper [9] determines the coupling constant directly from the conformal bootstrap of the Liouville exponential. It seems closely related to the earlier discussions of refs.[1]–[5] which do so from the OPE of the chiral components of the Liouville exponentials. From the viewpoint presented here, the fundamental step to include continuous screening charges will be to generalize the orthogonality condition Eq.3.11. If 6j symbols may be extended to the case where conditions Eqs.1.4 are violated, so that a generalization of Eq.3.11 holds, then the machinery summarized here will be at work. For continuous screening numbers, Eq.3.11 would include a continuous summation instead of a discrete one. Note that the generalized 6j symbols with discrete screening numbers already correspond[4][5] to the most general Ashkey-Wilson orthogonal polynomials presently known, so that this extension is highly non trivial.

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