Discrete Spectrum of a Model Operator Related to Three-Particle Discrete Schrödinger Operators

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Abstract

A model operator $H_\mu$, $\mu > 0$ corresponding to a three-particle discrete Schrödinger operator on a lattice $\mathbb{Z}^3$ is considered. We study the case where the parameter function $w$ has a special form with the non-degenerate minimum at the $n, n > 1$ points of the six-dimensional torus $\mathbb{T}^6$. If the associated Friedrichs model has a zero energy resonance, then we prove that the operator $H_\mu$ has infinitely many negative eigenvalues accumulating at zero and we obtain an asymptotics for the number of eigenvalues of $H_\mu$ lying below $z, z < 0$ as $z \to -0$.

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1 INTRODUCTION

We are going to discuss the following remarkable phenomenon of the spectral theory of the three-body Schrödinger operators, known as the Efimov effect: if a system of three particles interacting through pair short-range potentials is such that none of the three two-particle subsystems has bound states with negative energy, but at least two of them have a zero energy resonance, then this three-particle system has an infinite number of three-particle bound states with negative energy accumulating at zero.

For the first time the Efimov effect has been discussed in [7]. An independent proof on a physical level of rigor has been also given in [5] and then many works devoted to this subject, see for example, [6, 11, 12, 13, 14]. A rigorous mathematical proof of the existence of Efimov’s effect was originally carried out in [16].

Denote by $N(z)$ the number of eigenvalues of the Hamiltonian lying below $z, z < 0$. The growth of $N(z)$ has been studied in [2] for the symmetric case. Namely, the authors of [2] have first found (without proofs) the exponential asymptotics of eigenvalues corresponding to spherically symmetric bound states. This result is consistent with the lower bound $\lim \inf_{z \to 0} N(z)|\log|z||^{-1} > 0$ established in [13] without any symmetry assumptions.

In [12] the asymptotics of the form $N(z) \sim U_0|\log|z||$ as $z \to -0$ for the number $N(z)$ of bound states of a three-particle Schrödinger operator below $z, z < 0$ was obtained, where the coefficient $U_0$ depends only on the ratio of the masses of the particles.

Recently in [15] the existence of the Efimov effect for $N$-body quantum systems with $N \geq 4$ has been proved and a lower bound on the number of eigenvalues was given.

In [1, 3, 8, 9, 10] the presence of Efimov’s effect for the three-particle discrete Schrödinger operators has been proved and in [11, 13] an asymptotics for the number of eigenvalues similarly to [12, 14] was obtained.
In the present paper, we study the model operator $H_{\mu}$, $\mu > 0$ corresponding to a three-particle discrete Schrödinger operator on a lattice $\mathbb{Z}^3$. Here we are interested to discuss the case where the parameter function $w$ has a special form with the non degenerate minimum at the $n, n > 1$ points of the six-dimensional torus $\mathbb{T}^6$. If the associated Friedrichs model has a zero energy resonance, then we prove that the operator $H_{\mu}$ has infinitely many negative eigenvalues accumulating at zero (in the considering case zero is the bottom of the essential spectrum of $H_{\mu}$). Moreover, we establish the asymptotic formula

$$\lim_{z \to -0} \frac{N_{\mu}(z)}{|\log|z||} = \frac{n\gamma_0}{4\pi}$$

for the number $N_{\mu}(z)$ of eigenvalues of $H_{\mu}$ lying below $z$, $z < 0$. Here the number $n \equiv n(n)$, $n > 1$ is defined in Remark 2.3 (see below) and the number $\gamma_0$ is a unique positive solution of the equation

$$\gamma \sqrt{3} \cos \frac{\pi \gamma}{2} = 8 \sin \frac{\pi \gamma}{6}. \quad (1.1)$$

The asymptotics obtained in this paper can be considered as a generalization of the asymptotics, which was obtained in [1, 3, 4, 12, 14]. In [4] the non symmetric version of the operator $H_{\mu}$ was considered and the spectrum of this operator was analyzed for an arbitrary function $w$ with $n = 1$.

The organization of the paper is as follows. In Section 2 the model operator $H_{\mu}$ is introduced as a bounded self-adjoint operator and the main result of the paper is formulated. In Section 3 some spectral properties of the associated Friedrichs model $h_{\mu}(p)$, $p \in (-\pi, \pi]^3$ are studied. In Section 4, we reduce the eigenvalue problem by the principle of Birman-Schwinger. Section 5 is devoted to the prove of the main result of the paper.

## 2 MODEL OPERATOR AND STATEMENT OF THE MAIN RESULT

Let us introduce some notations used in this work. Denote by $\mathbb{T}^3$ the three-dimensional torus, the cube $(-\pi, \pi]^3$ with appropriately identified sides. The torus $\mathbb{T}^3$ will always be considered as an abelian group with respect to the addition and multiplication by real numbers regarded as operations on the three-dimensional space $\mathbb{R}^3$ modulo $(2\pi\mathbb{Z})^3$. Let $(\mathbb{T}^3)^2 = \mathbb{T}^3 \times \mathbb{T}^3$ be a Cartesian product, $L_2(\mathbb{T}^3)$ be the Hilbert space of square-integrable (complex) functions defined on $\mathbb{T}^3$ and $L_2^s((\mathbb{T}^3)^2)$ be the Hilbert space of square-integrable symmetric (complex) functions defined on $(\mathbb{T}^3)^2$.

Let us consider a model operator $H_{\mu}$ acting on the Hilbert space $L_2^s((\mathbb{T}^3)^2)$ as

$$H_{\mu} = H_0 - \mu V_1 - \mu V_2,$$

where

$$(H_0f)(p, q) = w(p, q)f(p, q),$$

$$(V_1f)(p, q) = \varphi(p) \int_{\mathbb{T}^3} \varphi(s)f(s, q)ds,$$

$$(V_2f)(p, q) = \varphi(q) \int_{\mathbb{T}^3} \varphi(s)f(p, s)ds.$$
Here $\mu$ is a positive real number, the function $\varphi(\cdot)$ is a real-valued analytic even function on $\mathbb{T}^3$ and the function $w$ has form

$$w(p,q) = \varepsilon(p) + \varepsilon(p+q) + \varepsilon(q)$$

with

$$\varepsilon(p) = \sum_{j=1}^{3}(1 - \cos mp(j)), \ p = (p^{(1)}, p^{(2)}, p^{(3)}) \in \mathbb{T}^3,$$

where $m$ is the positive integer number.

Under these assumptions the operator $H_\mu$ is bounded and self-adjoint in $L^2_2(\mathbb{T}^3)^2$.

To formulate the main result of the paper we introduce the Friedrichs model $h_\mu(p), p \in \mathbb{T}^3$, which acts in $L^2_2(\mathbb{T}^3)$ as

$$h_\mu(p) = h_0(p) - \mu v,$$

where

$$(h_0(p)f_1)(q) = w(p,q)f(q),$$

$$(vf)(q) = \varphi(q) \int_{\mathbb{T}^3} \varphi(s)f(s)ds.$$

The perturbation $\mu v$ of the operator $h_0(p), p \in \mathbb{T}^3$ is a self-adjoint operator of rank one. Therefore in accordance with the invariance of the essential spectrum under finite rank perturbations the essential spectrum $\sigma_{\text{ess}}(h_\mu(p))$ of $h_\mu(p), p \in \mathbb{T}^3$ fills the following interval on the real axis:

$$\sigma_{\text{ess}}(h_\mu(p)) = [m(p); M(p)],$$

where the numbers $m(p)$ and $M(p)$ are defined by

$$m(p) = \varepsilon(p) + 2 \sum_{j=1}^{3}(1 - \cos \frac{mp^{(j)}}{2}), \ p = (p^{(1)}, p^{(2)}, p^{(3)}) \in \mathbb{T}^3,$$

$$M(p) = \varepsilon(p) + 2 \sum_{j=1}^{3}(1 + \cos \frac{mp^{(j)}}{2}), \ p = (p^{(1)}, p^{(2)}, p^{(3)}) \in \mathbb{T}^3.$$ 

The following Theorem describes the location of the essential spectrum of $H_\mu$.

**Theorem 2.1** For the essential spectrum $\sigma_{\text{ess}}(H_\mu)$ of the operator $H_\mu$ the equality

$$\sigma_{\text{ess}}(H_\mu) = \bigcup_{p \in \mathbb{T}^3} \sigma_{\text{disc}}(h_\mu(p)) \cup [0; \frac{27}{2}]$$

holds, where $\sigma_{\text{disc}}(h_\mu(p))$ is the discrete spectrum of $h_\mu(p), p \in \mathbb{T}^3$.

**Definition 2.2** The set $\bigcup_{p \in \mathbb{T}^3} \sigma_{\text{disc}}(h_\mu(p))$ resp. $[0; \frac{27}{2}]$ is called two- resp. three-particle branch of the essential spectrum $\sigma_{\text{ess}}(H_\mu)$ of the operator $H_\mu$, which will be denoted by $\sigma_{\text{two}}(H_\mu)$ resp. $\sigma_{\text{three}}(H_\mu)$. 

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Denote by $n \equiv n(m)$ the number of the all points of the form $(p_i, q_j) \in (\mathbb{T}^3)^2$ with $p_i = (p_i^{(1)}, p_i^{(2)}, p_i^{(3)})$ and $q_j = (q_j^{(1)}, q_j^{(2)}, q_j^{(3)})$ such that $p_i^{(k)}, q_j^{(k)} \in \{0, \pm \frac{2\pi}{m}; \pm \frac{4\pi}{m}; \cdots; \pm \frac{2\pi'}{m} \}$, $k = 1, 2, 3$ and $p_s \neq p_l, q_s \neq q_l$ for $s \neq l$, where

$$m' = \begin{cases} m - 2, & \text{if the number } m \text{ is even} \\ m - 1, & \text{if the number } m \text{ is odd}. \end{cases}$$

It is easy to check that the function $w(\cdot, \cdot)$ has the non-degenerate minimum at that points $(p_i, q_j) \in (\mathbb{T}^3)^2$ and $n = (m' + 1)^6$.

Now we additionally assume that $m \geq 3$. Because, it is easy to show that, if $m = 1, 2$, then $n = 1$. In this paper we are interested to study the case where $n > 1$.

We denote that $\mathbf{1}_n = \{1, 2, \cdots, n\}$.

**Remark 2.3** In our analysis of the discrete spectrum of $H_\mu$, crucial role is played by the zeroes of the function $\varphi(\cdot)$ at the points $q_j \in \mathbb{T}^3, j = 1, \sqrt{n}$ (see, for example [3]). Suppose that at only $n, 1 < n \leq n$ points of the set $\{q_j\}_{j=1}^n$ the value of the function $\varphi(\cdot)$ is nonzero. We consider the set $\{(p_{i}, q_{s}) \in (\mathbb{T}^3)^2 : i = \mathbf{1}_n, q_{s} = \overline{1, n}, \text{such that } \varphi(q_{s}) \neq 0, i = \mathbf{1}_n, \text{and } \varphi(q_{s}) = 0, i = \mathbf{n}+1, \mathbf{n} \}$. Throughout this paper we shall use this notation without further comments.

**Remark 2.4** Note that the equality $h_\mu(p_{s_1}) \equiv h_\mu(p_{s_2}), i = \mathbf{1}_n$ holds.

Let $C(\mathbb{T}^3)$ (resp. $L_4(\mathbb{T}^3)$) be the Banach space of continuous (resp. integrable) functions on $\mathbb{T}^3$.

**Definition 2.5** The operator $h_\mu(p_{s_1})$ is said to have a zero energy resonance if the number 1 is an eigenvalue of the integral operator

$$(G\psi_\alpha)(q) = \frac{\mu \varphi(q)}{2} \int_{\mathbb{T}^3} \frac{\varphi(s)\psi(s)ds}{\varepsilon(s)}, \quad \psi \in C(\mathbb{T}^3)$$

and at least one (up to normalization constant) of the associated eigenfunctions $\psi$ satisfies the condition $\psi(q_{s}) \neq 0, i = \mathbf{1}_n$.

**Remark 2.6** We notice that if the operator $h_\mu(p_{s_1})$ has a zero energy resonance, then the function

$$f(q) = \frac{\mu \varphi(q)}{2\varepsilon(q)} \in L_4(\mathbb{T}^3) \setminus L_2(\mathbb{T}^3), \quad (2.1)$$

obeys the equation $h_\mu(p_{s_1})f = 0$ (see Lemma 3.7).

Set

$$\mu_0 = 2 \left( \int_{\mathbb{T}^3} \frac{\varphi^2(s)ds}{\varepsilon(s)} \right)^{-1}.$$ 

**Remark 2.7** We remark that the operator $h_\mu(p_{s_1})$ has a zero energy resonance if and only if $\mu = \mu_0$ (see Lemma 3.2).

Let us denote by $\tau_{ess}(H_\mu)$ the bottom of the essential spectrum $\sigma_{ess}(H_\mu)$ of $H_\mu$ and by $N_\mu(z)$ the number of eigenvalues of $H_\mu$ lying below $z, z < \tau_{ess}(H_\mu)$. 

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Remark 2.8 We note that $\tau_{\text{ess}}(H_{\mu_0}) = 0$ (see Lemma 3.6).

The main result of this paper is the following

**Theorem 2.9** The operator $H_{\mu_0}$ has an infinitely many negative eigenvalues accumulating at zero and for the function $N_{\mu_0}(\cdot)$ the relation

$$
\lim_{z \to -0} \frac{N_{\mu_0}(z)}{\log|z|} = \frac{n \gamma_0}{4\pi}
$$

(2.2)

holds, where the number $n$ is defined in Remark 2.3 and the number $\gamma_0$ is a positive solution of the equation (1.1).

Remark 2.10 Clearly, the infinite cardinality of the negative discrete spectrum of $H_{\mu_0}$ follows automatically from the positivity of the number $\gamma_0$.

Remark 2.11 We point out that the asymptotics (2.2) is new and similar asymptotics have not yet been obtained for the three-particle Schrödinger operators on $\mathbb{R}^3$ and $\mathbb{Z}^3$.

3 SPECTRAL PROPERTIES OF THE OPERATOR $h_\mu(p)$

In this section we study some spectral properties of the Friedrichs model $h_\mu(p), p \in \mathbb{T}^3$, which plays a crucial role in our analysis of the discrete spectrum of the operator $H_\mu$.

Let $\mathbb{C}$ be the field of complex numbers. For any $p \in \mathbb{T}^3$ we define an analytic function $\Delta_\mu(p; \cdot)$ (the Fredholm determinant associated with the operator $h_\mu(p), p \in \mathbb{T}^3$) in $\mathbb{C} \setminus \sigma_{\text{ess}}(h_\mu(p))$ by

$$
\Delta_\mu(p; z) = 1 - \mu \int_{\mathbb{T}^3} \frac{\varphi^2(q) dq}{w(p, q) - z}.
$$

The following statement (see [4]) establishes a connection between of eigenvalues of $h_\mu(p), p \in \mathbb{T}^3$ and zeroes of the function $\Delta_\mu(p; \cdot)$, $p \in \mathbb{T}^3$.

**Lemma 3.1** For any $p \in \mathbb{T}^3$ the operator $h_\mu(p)$ has an eigenvalue $z \in \mathbb{C} \setminus \sigma_{\text{ess}}(h_\mu(p))$ if and only if $\Delta_\mu(p; z) = 0$.

Since the function $w(\cdot, \cdot)$ has the non-degenerate minimum at the points $(p_{s_i}, q_{s_i}) \in (\mathbb{T}^3)^2, i = \overline{1, n}$ and the function $\varphi(\cdot)$ is an analytic function on $\mathbb{T}^3$, the integral

$$
\int_{\mathbb{T}^3} \frac{\varphi^2(q) dq}{w(p, q)}, p \in \mathbb{T}^3
$$

is finite.

By Lebesgue’s dominated convergence theorem and the equality $\Delta_\mu(p_{s_i}; 0) = \Delta_\mu(p_{s_i}; 0)$, $i = \overline{1, n}$ it follows that

$$
\Delta_\mu(p_{s_i}; 0) = \lim_{p \to p_{s_i}} \Delta_\mu(p; 0), i = \overline{1, n}.
$$

We remark that the following three statements, which are useful for the proof of main result can be proven similarly to corresponding statements of [1, 4] and hence here for completeness we only reproduce these statements without proofs.
Lemma 3.2 The operator $h_{p,q}(p_{s_1})$ has a zero energy resonance if and only if $\mu = \mu_0$.

Lemma 3.3 The following decomposition holds
\[
\Delta_{\mu_0}(p; z) = 2\pi^2 \mu_0 \sum_{j=1}^{n} \varphi^2(q_{s_j}) \sqrt{\frac{3}{4}p - p_{s_j}}| - z + O(|p - p_{s_j}|^2) + O(|z|)
\]
as $|p - p_{s_j}| \to 0, i = \overline{1,n}$ and $z \to -0$.

Set
\[
U_{\delta}(p_0) = \{ p \in T^3 : |p - p_0| < \delta \}, \quad p_0 \in T^3, \quad \delta > 0.
\]

Lemma 3.4 There exist positive numbers $C_1$, $C_2$, $C_3$ and $\delta$ such that
\[
C_1|p - p_{s_i}|^2 \leq |\Delta_{\mu_0}(p; 0)| \leq C_2|p - p_{s_i}|^2, \quad p \in U_{\delta}(p_{s_i}), \quad i = \overline{1,n};
\]
\[
|\Delta_{\mu_0}(p; 0)| \geq C_3, \quad p \in T^3 \setminus \bigcup_{i=1}^{n} U_{\delta}(p_{s_i}).
\]

From the representation
\[
w(p, q) = |p - p_{s_i}|^2 + (p - p_{s_i}, q - q_{s_i}) + |q - q_{s_i}|^2 + O(|p - p_{s_i}|^4) + O(|q - q_{s_i}|^4)
\]
as $|p - p_{s_i}|, |q - q_{s_i}| \to 0, i = \overline{1,n}$ it follows the following

Lemma 3.5 There exist the numbers $C_1, C_2, C_3 > 0$ and $\delta > 0$ such that
1) $C_1(|p - p_{s_i}|^2 + |q - q_{s_i}|^2) \leq w(p, q) \leq C_2(|p - p_{s_i}|^2 + |q - q_{s_i}|^2)$ for $(p, q) \in U_{\delta}(p_{s_i}) \times U_{\delta}(q_{s_i}), \quad i = \overline{1,n};$
2) $w(p, q) \geq C_3$ for all $p, q$, which at least one of the conditions $p \notin \bigcup_{i=1}^{n} U_{\delta}(p_{s_i})$ and $q \notin \bigcup_{i=1}^{n} U_{\delta}(q_{s_i})$ is fulfilled.

Lemma 3.6 The operator $h_{\mu_0}(p), p \in T^3$ has no negative eigenvalues.

Proof. First we show that for any $p \in T^3 \setminus \{p_{s_1}, p_{s_2}, \ldots, p_{s_n}\}$ the inequality $\Delta_{\mu_0}(p; 0) > \Delta_{\mu_0}(p_{s_1}; 0)$ holds. Denote
\[
\Lambda(p) = \int_{T^3} \frac{\varphi^2(q)dq}{w(p, q)}.
\]
Since the functions $\varphi(\cdot)$ and $w(\cdot, \cdot)$ are even, the function $\Lambda(\cdot)$ is also even. Then
\[
\Lambda(p) - \Lambda(p_{s_1}) = \frac{1}{4} \int_{T^3} \frac{2w(p_{s_1}, q) - (w(p, q) + w(-p, q))}{w(p, q)w(-p, q)w(p_{s_1}, q)} [w(p, q) + w(-p, q)]^2 \varphi^2(q) dq -
\]
\[
- \frac{1}{4} \int_{T^3} \frac{[w(p, q) + w(-p, q)]^2}{w(p, q)w(-p, q)w(p_{s_1}, q)} \varphi^2(q) dq.
\]

By the equalities
\[
w(p_{s_1}, q) - \frac{w(p, q) + w(-p, q)}{2} = \sum_{j=1}^{3} \left( \cos m p^{(j)} - 1 \right) \left( 1 + \cos m q^{(j)} \right)
\]
and (3.1) we have the inequality \( \Lambda(p) - \Lambda(p_{s1}) < 0 \) for any \( p \in \mathbb{T}^3 \setminus \{p_{s1}, p_{s2}, \ldots, p_{sn}\} \).

By the definition of \( \mu_0 \) we have \( \Delta_{\mu_0}(p_{s1}; 0) = 0 \). Hence the inequality

\[
\Delta_{\mu_0}(p; z) > \Delta_{\mu_0}(p_{s1}; 0) = 0
\]

holds for any \( p \in \mathbb{T}^3 \) and \( z < 0 \). By Lemma [3.1] it means that, the operator \( h_{\mu_0}(p), p \in \mathbb{T}^3 \)

has no negative eigenvalues. \( \square \)

**Lemma 3.7** The function \( f \), which is defined by (2.1), obeys the equation \( h_{\mu_0}(p_{s1})f = 0 \).

**Proof.** First we show that \( f \in L_1(\mathbb{T}^3) \setminus L_2(\mathbb{T}^3) \), that is,

\[
\int_{\mathbb{T}^3} |f(q)|dq < \infty \quad \text{and} \quad \int_{\mathbb{T}^3} |f(q)|^2dq = \infty.
\]

From the definition of \( \mu_0 \) it follows that \( \Delta_{\mu_0}(p_{s1}; 0) = 0 \). By the construction of the set \( \{(p_{s1}, q_{s1}) \in (\mathbb{T}^3)^2 : i = \overline{1,n}\} \) we have that \( \varphi(q_{s1}) \neq 0, i = \overline{1,n} \) and \( \varphi(q_{s1}) = 0, i = \overline{n+1,n} \).

Using these facts and the definition of the function \( \varepsilon(\cdot) \) we obtain that there exist the numbers \( C_1, C_2, C_3 > 0 \) and \( \delta > 0 \) such that

\[
C_1|q - q_{s1}|^2 \leq \varepsilon(q) \leq C_2|q - q_{s1}|^2, \quad q \in U_{\delta}(q_{s1}), \quad i = \overline{1,n},
\]

\[
\varepsilon(q) \geq C_3, \quad q \in \mathbb{T}^3 \setminus \bigcup_{i=1}^{n} U_{\delta}(q_{s1}),
\]

\[
|\varphi(q)| \geq C_3, \quad q \in U_{\delta}(q_{s1}), \quad i = \overline{1,n}
\]

and in the case where \( n < n \) we have that

\[
C_1|q - q_{s1}|^2 \leq |\varphi(q)| \leq C_2|q - q_{s1}|^2, \quad q \in U_{\delta}(q_{s1}), \quad i = \overline{n+1,n}.
\]

Applying latter inequalities we obtain that

\[
\int_{\mathbb{T}^3} |f(q)|dq \leq C_1 \sum_{j=1}^{n} \int_{U_{\delta}(q_{s1})} \frac{dq}{|q - q_{s1}|^2} + C_2 < \infty,
\]

\[
\int_{\mathbb{T}^3} |f(q)|^2dq \geq C_1 \sum_{j=1}^{n} \int_{U_{\delta}(q_{s1})} \frac{dq}{|q - q_{s1}|^4} + C_2 = \infty.
\]

It is easy to check that the function \( f \) obeys the equation \( h_{\mu_0}(p_{s1})f = 0 \). \( \square \)

4 **THE BIRMAN-SCHWINGER PRINCIPLE**

For a bounded self-adjoint operator \( A \), acting in Hilbert space \( \mathcal{R} \), we define \( d(\lambda, A) \) as

\[
d(\lambda, A) = \sup\{dimF : (Au, u) > \lambda, u \in F \subset \mathcal{R}, ||u|| = 1\}.
\]

\( d(\lambda, A) \) is equal to the infinity, if \( \lambda \) is in the essential spectrum and if \( d(\lambda, A) \) is finite, it is equal to the number of the eigenvalues of \( A \) bigger than \( \lambda \).

By the definition of \( N_\mu(z) \) we have

\[
N_\mu(z) = d(-z, -H_\mu), \quad -z > -\tau_{ess}(H_\mu).
\]

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Since the function $\Delta_\mu(\cdot, \cdot)$ is positive on $\mathbb{T}^3 \times (-\infty, \tau_{ess}(H_\mu))$, the positive square root of $\Delta_\mu(p; z)$ exists for any $p \in \mathbb{T}^3$ and $z < \tau_{ess}(H_\mu)$.

In our analysis of the spectrum of $H_\mu$, the crucial role is played by the compact integral operator $T_\mu(z)$, $z < \tau_{ess}(H_\mu)$, which acts in $L_2(\mathbb{T}^3)$ with the kernel

$$
\frac{\mu \varphi(p)\varphi(q)}{\sqrt{\Delta_\mu(p; z)}\sqrt{\Delta_\mu(q; z)}(w(p,q) - z)}.
$$

The following lemma is a realization of the well known Birman-Schwinger principle for the operator $H_\mu$ (see [1,3,4,10,12,14,15]).

**Lemma 4.1** For $z < \tau_{ess}(H_\mu)$ the operator $T_\mu(z)$ is compact and continuous in $z$ and one has

$$
N_\mu(z) = d(1, T_\mu(z)).
$$

This lemma has been proven in [4] for the non symmetric case.

## 5 THE PROOF OF THE MAIN RESULT

In this section we shall derive the asymptotics (2.2) for the number $N_{\mu_0}(z)$ of eigenvalues of the operator $H_{\mu_0}$ lying below $z$, $z < 0$, that is, we shall prove Theorem 2.9.

We shall first establish the asymptotics for $d(1, T_{\mu_0}(z))$ as $z \to -0$. Then Theorem 2.9 will be deduced by a perturbation argument based on the following lemma.

**Lemma 5.1** Let $A(z) = A_0(z) + A_1(z)$, where $A_0(z)$ (resp. $A_1(z)$) is compact and continuous in $z < 0$ (resp. $z \leq 0$). Assume that for some function $f(\cdot)$, $f(z) \to 0$, $z \to 0$ one has

$$
\lim_{z \to -0} f(z) d(\gamma, A_0(z)) = l(\gamma),
$$

and is continuous in $\gamma > 0$. Then the same limit exists for $A(z)$ and

$$
\lim_{z \to -0} f(z) d(\gamma, A(z)) = l(\gamma),
$$

For the proof of Lemma 5.1 see Lemma 4.9 of [12].

Let $T(\delta; |z|)$ be the integral operator which acts in $L_2(\mathbb{T}^3)$ with the kernel

$$
\frac{1}{2\pi^2} \sum_{i=1}^n \frac{\chi_\delta(p - p_{si})\chi_\delta(q - q_{si})(\frac{3}{4}|p - p_{si}|^2 + |z|)^{-\frac{1}{4}}(\frac{3}{4}|q - q_{si}|^2 + |z|)^{-\frac{1}{4}}}{|p - p_{si}|^2 + (p - p_{si}, q - q_{si}) + |q - q_{si}|^2 + |z|}.
$$

Here $\chi_\delta(\cdot)$ is the characteristic function of $U_\delta(0)$.

The following lemma can be proven using Lemmas 5.3–8.3.

**Lemma 5.2** For any $z \leq 0$ and small $\delta > 0$ the error $T_{\mu_0}(z) - T(\delta; |z|)$ is Hilbert-Schmidt operator and is continuous in the uniform operator topology at the point $z = 0$.

The space of the functions $f$ having support in $\bigcup_{i=1}^n U_\delta(p_{si})$, is an invariant subspace for the operator $T(\delta; |z|)$. Let $T_0(\delta; |z|)$ be the restriction of the operator $T(\delta; |z|)$ to this subspace, that is, the integral operator acting in $L_2(\bigcup_{i=1}^n U_\delta(p_{si}))$ with the kernel

$$
\frac{1}{2\pi^2} \sum_{i=1}^n \frac{(\frac{3}{4}|p - p_{si}|^2 + |z|)^{-\frac{1}{4}}(\frac{3}{4}|q - q_{si}|^2 + |z|)^{-\frac{1}{4}}}{|p - p_{si}|^2 + (p - p_{si}, q - q_{si}) + |q - q_{si}|^2 + |z|}.
$$

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Denote by $\text{diag}\{A_1, A_2, \cdots, A_n\}$ the $n \times n$ diagonal matrix with operators $A_1, A_2, \cdots, A_n$ as diagonal entries.

Since the space $L_2(\bigoplus_{i=1}^{n} U_\delta(p_{si}))$ is an isomorphism to $\bigoplus_{i=1}^{n} L_2(U_\delta(p_{si}))$, the operator $T_0(\delta; |z|)$ can be written as diagonal operator

$$T_0(\delta; |z|) = \text{diag}\{T_0^{(1)}(\delta; |z|), T_0^{(2)}(\delta; |z|), \cdots, T_0^{(n)}(\delta; |z|)\},$$

where $T_0^{(i)}(\delta; |z|)$, $i = \overline{1, n}$ is the integral operator acting in $\bigoplus_{i=1}^{n} L_2(U_\delta(p_{si}))$ with the kernel

$$\frac{1}{2\pi^2} \frac{1}{|z|^{\frac{3}{2}}(\frac{3}{4}|p - p_{si}|^2 + |z|)^{-\frac{1}{2}}(\frac{3}{4}|q - q_{si}|^2 + |z|)^{-\frac{1}{2}}}.$$ 

One verifies that the operator $T_0(\delta; |z|)$ is unitary equivalent to the operator $T_1(r)$, $r = |z|^{-\frac{1}{2}}$ acting in $\bigoplus_{i=1}^{n} L_2(U_r(0))$ as

$$T_1(r) = \text{diag}\{T_1^{(1)}(r), T_1^{(2)}(r), \cdots, T_1^{(n)}(r)\},$$

where $T_1^{(i)}(r)$, $i = \overline{1, n}$ is the integral operator acting in the $L_2(U_r(0))$ with the kernel

$$\frac{1}{2\pi^2} \frac{1}{(|p|^2 + 1)^{\frac{5}{4}}(|q|^2 + 1)^{\frac{5}{4}}(|p|^2 + p, q) + |q|^2 + 1}}.$$ 

We note that the equivalence of these operators is performed by the unitary dilation

$$B_r = \text{diag}\{B_r^{(1)}, B_r^{(2)}, \cdots, B_r^{(n)}\} : \bigoplus_{i=1}^{n} L_2(U_\delta(p_{si})) \to \bigoplus_{i=1}^{n} L_2(U_r(0)).$$

Here the operator $B_r^{(i)} : L_2(U_\delta(p_{si})) \to L_2(U_r(0))$, $i = \overline{1, n}$ acting by

$$(B_r^{(i)} f)(p) = r^{-\frac{3}{2}} f \left( \frac{1}{r} (p - p_{si}) \right).$$

Since the space $\bigoplus_{i=1}^{n} L_2(U_r(0))$ is an isomorphism to $L_2(U_r(0))$, we rewrite the operator $T_1(r)$ as integral operator acting in $L_2(U_r(0))$ with the kernel

$$\frac{1}{2\pi^2} \frac{1}{(|p|^2 + 1)^{\frac{5}{4}}(|q|^2 + 1)^{\frac{5}{4}}(|p|^2 + p, q) + |q|^2 + 1}}.$$ 

Further, we may replace $(\frac{3}{4}|p|^2 + 1)^{\frac{3}{4}}, (\frac{3}{4}|q|^2 + 1)^{\frac{3}{4}}$ by $(\frac{1}{4}|p|^2 + 1 - \chi_1(p)), (\frac{1}{4}|q|^2 + 1 - \chi_1(q))$ and $|p|^2 + (p, q) + |q|^2 + 1$ by $(\frac{1}{4}|p|^2 + 1)(1 - \chi_1(p)), (\frac{1}{4}|q|^2 + 1)(1 - \chi_1(q))$ and $|p|^2 + (p, q) + |q|^2$, respectively, we have the operator $T_2(r)$. The error $T_1(r) - T_2(r)$ will be a Hilbert-Schmidt operator and continuous up to $z = 0$.

The space of functions having support in $L_2(U_r(0) \setminus U_1(0))$ is an invariant subspace for the operator $T_2(r)$. The kernel of this operator has form

$$K_n(p, q) = \frac{1}{\sqrt{3\pi^2} |p|^\frac{1}{2} |q|^\frac{1}{2} (|p|^2 + (p, q) + |q|^2)}.$$ 

Let $T'(r)$ be the integral operator acting on $L_2(U_r(0) \setminus U_1(0))$ with the kernel $K_2(p, q)$. The following lemma was proven in [1].
Lemma 5.3 The equality
\[ \lim_{z \to 0} \frac{d(1, T(z))}{|\log|z||} = \frac{\gamma_0}{2\pi} \]
is satisfied, where \( \gamma_0 \) is a positive solution of the equation (1.1).

Now Theorem 2.9 follows from Lemmas 4.1, 5.1–5.3.

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