Complexity of piecewise convex transformations in two dimensions, with applications to polygonal billiards

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Abstract

ABSTRACT We introduce the class of piecewise convex transformations, and study their complexity. We apply the results to the complexity of polygonal billiards on surfaces of constant curvature.

Introduction

The following situation frequently occurs in geometric dynamics. There is a phase space $X$, a transformation $T : X \to X$; and there is a finite decomposition $\mathcal{P} : X = X(a) \cup X(b) \cup \cdots$. Let $\mathcal{A} = \{a, b, \ldots\}$ be the corresponding alphabet. A phase point $x \in X$ is regular if every element of the orbit $x, Tx, T^2x, \ldots$ belongs to a unique atom of $\mathcal{P}$. Suppose that $x \in X(a), Tx \in X(b)$, etc. The corresponding word $ab \cdots$ is the code of $x$. Let $\Sigma(n)$ be the set of words in $\mathcal{A}$ of length $n$ obtained by coding points in $X$. The positive function $f(n) = |\Sigma(n)|$ is the associated complexity. Its

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behavior as \( n \to \infty \) is an important characteristic of the dynamical system in question. The following examples have motivated our study.

**Example A.** Let \( P \subset \mathbb{R}^2 \) be a polygon with sides \( a, b, \ldots \), and let \( X \) be the phase space of the billiard map \( T_b \) in \( P \). The coding generated by the corresponding decomposition \( \mathcal{P} : X = X(a) \cup X(b) \cup \cdots \) is the traditional coding of billiard orbits by the sides they hit \[16\]. Basic questions about its complexity are open \[9\].

**Example B.** Let \( P \subset \mathbb{R}^2 \) be a convex polygon with vertices \( a, b, \ldots \). The complement \( X = \mathbb{R}^2 - P \) is the phase space of the outer billiard \( T_o \) about \( P \). (It is also called the dual billiard. See \[16\]). The conical regions bounded the singular lines of \( T_o \) form the natural decomposition \( \mathcal{P} : X = X(a) \cup X(b) \cup \cdots \). In \( X(a), X(b), \ldots \) the mapping \( T_o \) is the symmetry about \( a, b, \ldots \). The decomposition \( \mathcal{P} \) yields the coding of outer billiard orbits by the vertices they hit.

We will study the complexity of (2-dimensional) piecewise convex transformations. This is a wide class of geometric dynamical systems; it contains the examples above. Our setting is as follows. (For simplicity of exposition, we restrict our attention to two dimensions.)

Let \( X \) be a geodesic surface, and let \( \Gamma \subset X \) be a finite geodesic graph. A subset \( Y \subset X \) is convex if for any \( x, y \in Y \) there is a unique geodesic in \( Y \) with endpoints \( x, y \in Y \). Suppose that the closed faces \( X(a), X(b), \ldots \) of \( \Gamma \) are convex, and let \( \mathcal{P} : X = X(a) \cup X(b) \cup \cdots \) be the corresponding decomposition. We say that \( \mathcal{P} \) is a convex partition of \( X \). A differentiable mapping \( T : Y \to X \) is convex if it sends geodesics to geodesics. Suppose that \( T : X \to X \) is a convex diffeomorphism on \( X(a), X(b), \ldots, \) and that \( T(X(a)), T(X(b)), \ldots \) also form a convex partition of \( X \). We say that the triple \((X, T, \mathcal{P})\) is a piecewise convex transformation with the defining partition \( \mathcal{P} \).

Making further assumptions on \((X, T, \mathcal{P})\), we obtain more specialized classes of transformations, e.g., piecewise isometries, piecewise affine mappings, etc. The case when the faces of \( \Gamma \) are convex euclidean polyhedra, and \( T \) is isometric on them arises in task scheduling problems \[1\].

In Section \[12\] we develop geometric and combinatorial techniques to study the complexity of piecewise convex transformations. In the rest of the paper we apply these results to the inner and outer polygonal billiards on (simply

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1The function \( f(\cdot) \) can be bounded, or grow polynomially, or grow exponentially, etc.
connected) surfaces of constant curvature $\chi = 0, 1, -1$. One of the goals of this work is to develop a uniform approach to these dynamical systems. While there is a vast literature on the parabolic case ($\chi = 0$), the elliptic ($\chi = 1$) and the hyperbolic ($\chi = -1$) cases have been studied only sporadically.

Let $M$ be a surface of constant curvature, and let $P \subset M$ be a polygon. In Section 2 we cast the (inner) billiard map in $P$ as a piecewise convex transformation $(X, T, \mathcal{P})$. See Theorem 1. We do it simultaneously for all curvatures, and without making additional assumptions on $P$. Thus, we don’t assume that $P$ is convex or simple, and have to pay a price for this. The partition $\mathcal{P}$ is finer than the “natural” one; the coding it generates is more refined than the standard coding by sides [15]. We develop a dictionary between the language of piecewise convex transformations and that of billiard orbits. In the remaining part of the paper we come back to Examples A, B and study the complexity of natural coding.

In Section 3 we investigate the complexity of inner billiard orbits in a polygon $P$ on a surface of constant curvature $\chi$. In section 3.1 $P, \chi$ are arbitrary; later on we specialize to convex $P$ but arbitrary $\chi$, to $\chi = 0$, and to $\chi = \pm 1$ respectively. Below we formulate the main results.

The side complexity of billiard orbits in any rational euclidean polygon grows at most cubically; see Theorem 3.
The side complexity of billiard orbits in any spherical polygon grows subexponentially; see Theorem 4.
The side complexity of billiard orbits in any hyperbolic (i.e., $\chi = -1$) polygon grows exponentially; the exponent in question is the topological entropy of the billiard map; see Theorem 5.

Section 4 is the outer billiard counterpart of Section 3. Here are some of its results.

For $\chi = 0$ and arbitrary polygon (resp. rational polygon) we obtain polynomial bounds from above and below (resp. quadratic asymptotics) for the complexity; see Theorem 6 and Theorem 7 respectively.
For $\chi = 1$ complexity grows subexponentially; see Theorem 8.
For $\chi = -1$ and arbitrary polygon, we obtain linear lower bound for complexity, which is sharp: for the so-called large polygons complexity grows linearly; see Theorem 9.

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1 Piecewise convex transformations

1.1 Convex geodesic surfaces

We will introduce a class of maps of geodesic spaces. In order to simplify our exposition, we will restrict it to two dimensions, i.e., to geodesic surfaces. Let $M$ be one. Then $M$ is a topological surface. It may have a boundary, $\partial M$, and a finite number of cone points. The surface $M$ is endowed with a collection of geodesics, satisfying the standard properties. In particular, $\partial M$ is a finite union of geodesics. Let $x, y \in M$ be any pair of points. Then $M$ is geodesically convex if there is a unique shortest geodesic $\gamma = \gamma(x, y) \subset M$ joining them.\(^3\)

Examples. i) The set $\mathbb{R}^2$ endowed with a projectively flat (e.g., Minkowski) metric is a geodesic surface. Its geodesics are straight lines. Let $M \subset \mathbb{R}^2$ be a closed, bounded convex region. Then it is a geodesic surface iff $M$ is a polygon. The geodesic $\gamma(x, y)$ is the segment with endpoints $x, y$.

ii) Let $S^2$ be the round sphere, and let $M \subset S^2$ be a convex spherical polygon. Then $M$ is a geodesic surface iff it does not contain antipodal points. This holds iff $M$ is contained in an open hemisphere.

In what follows we will often say “convex geodesic surface” instead of “geodesically convex geodesic surface”. Let $X$ be a geodesic surface. A finite graph $\Gamma$ drawn on $X$ is a geodesic graph if the edges of $\Gamma$ are geodesics. We will assume that the cone points of $X$ are contained in the set of vertices of $\Gamma$, and that the latter are nondegenerate. (A vertex is degenerate if it has two adjacent edges, and they are collinear.)

Let $X(a), X(b), \ldots$ be the (open) faces of $\Gamma$, and let $P_a, P_b, \ldots \subset X$ be their closures. We will say that $\Gamma$ is a convex geodesic graph if the surfaces $P_a, P_b, \ldots \subset X$ are geodesically convex. We will refer to the data $(X, \Gamma)$ as a piecewise (geodesically) convex geodesic surface. We will also say that\(^2\)

\(\text{See especially Remark 7 in [2].}\)

\(\text{Although the uniqueness condition may seem too restrictive, it is crucial for our study of iterations of piecewise convex transformations. See section 1.4 below.}\)
\[ P : X = P_a \cup P_b, \ldots \] is a \textit{convex geodesic partition} of the geodesic surface \( X \). The boundary \( \partial P \subset X \) is spanned by the edges and vertices of \( \Gamma \). We will also refer to it as the \textit{support} of \( \Gamma \) and denote by \( < \Gamma > \). There is a \( 1 - 1 \) correspondence between convex geodesic graphs drawn on \( X \) and convex geodesic partitions of \( X \). The latter concept provides an alternative approach to the material below \([10]\). However, the language of convex geodesic graphs is more suitable for our purposes, and we will use it in what follows.

Let \( X \) be a convex geodesic surface, and let \( Y \) be an arbitrary geodesic surface. A diffeomorphism (not necessarily surjective) \( T : X \to Y \) is a \textit{geodesically convex transformation} if it sends geodesics to geodesics. Then \( T(X) = Z \subset Y \) is a convex geodesic surface as well, and the inverse map \( T^{-1} : Z \to X \) is a geodesically convex transformation.

1.2 \textbf{Piecewise convex transformations: iterations, coding, and complexity}

We will now introduce a class of dynamical systems which, on one hand, is sufficiently general to include several interesting examples, and on the other, is special enough to allow a common geometric framework. Let \( X, \Gamma, P = P(\Gamma) \) be as above. Let \( p \) be the number of atoms of \( P \) and write \( P : X = \bigcup_{i=1}^p P_i \). Suppose that for \( 1 \leq i \leq p \) there is a convex transformation \( T_i : P_i \to X \). Set \( Q_i = T_i(P_i) \). Suppose that \( X = \bigcup_{i=1}^p Q_i \) and that the open sets \( \text{int}(Q_i) \) are pairwise disjoint. Then \( Q : X = \bigcup_{i=1}^p Q_i \) is a convex geodesic partition, and \( \partial Q \) is the support of a convex geodesic graph, \( \Gamma^{-1} \), drawn on \( X \).

These data determine a \textit{piecewise convex transformation} \( T : X \to X \) with the \textit{defining partition} \( P \), and we will use the notation \((X,T,P)\) for it. The inverse of \((X,T,P)\) is also a piecewise convex transformation. Its defining partition is \( Q = T(P) \). Thus in our notation, \((X,T,P)^{-1} = (X,T^{-1},Q)\).

Invertible piecewise isometric (as well as affine, or projective) maps are examples of piecewise convex transformations. Let \( M \) be a simply connected surface of constant curvature, and let \( P \subset M \) be a geodesic polygon. In Sections \([3]\) and \([4]\) we will put the inner and the outer billiard about \( P \) into the framework of piecewise convex transformations.

Denote by \( \mathcal{L} \) the \textit{full shift space} on the alphabet \( \mathcal{A} = \{a,b,c,\ldots\} \) of the faces of \( \Gamma \). A point \( x \in X \) is \textit{regular} if \( x,Tx,T^2x,\ldots \) belong to open faces of \( \Gamma \). Let \( X_\infty \subset X \) be the set of regular points. Assigning to \( x \in X_\infty \) the
sequence of faces of $\Gamma$ containing the consecutive elements $x, Tx, T^2x, \ldots$, we obtain the coding map $\sigma : X_\infty \to L$. Set $\Sigma = \sigma(X_\infty)$, and let $\Sigma(n)$ be the set of words of length $n$ that occur in $\Sigma$. The function $f(n) = |\Sigma(n)|$ is the complexity of $(X, T, P)$ with respect to the defining partition.

Let $\Gamma', \Gamma''$ be two geodesic graphs on $X$. Their join $\Gamma' \cup \Gamma''$ is the (unique) geodesic graph such that $< \Gamma' \cup \Gamma'' > = < \Gamma' > \cup < \Gamma'' >$. We write $\Gamma'' \prec \Gamma'$ if $\Gamma' \cup \Gamma'' = \Gamma''$. Recall that $\mathcal{P}' \cup \mathcal{P}''$ denotes the join of partitions, and that $\mathcal{P}'' \prec \mathcal{P}'$ holds if $\mathcal{P}' \cup \mathcal{P}'' = \mathcal{P}''$. Let $\mathcal{P}' = \mathcal{P}(\Gamma'), \mathcal{P}'' = \mathcal{P}(\Gamma'')$. If $\Gamma', \Gamma''$ are convex geodesic graphs, then so is $\Gamma' \cup \Gamma''$, and $\mathcal{P}(\Gamma' \cup \Gamma'') = \mathcal{P}(\Gamma') \cup \mathcal{P}(\Gamma'')$. Moreover, $\Gamma'' \prec \Gamma'$ iff $\mathcal{P}'' \prec \mathcal{P}'$.

Set $\Gamma_1 = \Gamma, \Gamma_2 = \Gamma_1 \cup T^{-1}(\Gamma), \ldots, \Gamma_{n+1} = \Gamma_n \cup T^{-n}(\Gamma), \ldots$. The convex geodesic graphs $\Gamma_k, k = 1, 2, \ldots$ form an increasing tower with respect to the relation $\prec$, and we set $S_k = \prec \Gamma_k >, \mathcal{P}_k = \mathcal{P}(\Gamma_k)$. The singular set $S_\infty = \bigcup_{k=1}^\infty S_k = X \setminus X_\infty$ is a countable (at most) union of geodesics. Note that $(X, T^n, \mathcal{P}_n)$ is a piecewise convex transformation for $n \geq 1$.

Let $x = x_1, Tx = x_2, \ldots, T^{n-1}x = x_n$ be a finite orbit such that the points $x_1, x_2, \ldots, x_n$ belong to open faces of $\Gamma$. We say that $x_1, x_2, \ldots, x_n$ is a regular orbit of length $n$ and denote by $\sigma(x_1, x_2, \ldots, x_n)$ the corresponding word on the alphabet $A$; it is the code of $x_1, x_2, \ldots, x_n$. Then $\Sigma(n)$ is the set of codes of regular orbits of length $n$; the orbit $x = x_1, Tx = x_2, \ldots, T^{n-1}x = x_n$ is regular iff $x$ belongs to an open face of the graph $\Gamma_n$.

The proposition below summarizes the discussion.

**Proposition 1** Let $(X, T, \mathcal{P})$ be a piecewise convex transformation, and let $< \Gamma > = \partial \mathcal{P}$. Then:

1. There is a sequence $\Gamma_k, k \geq 1$ of convex geodesic graphs in $X$ such that
   \[
   \cdots \prec \Gamma_{n+1} \prec \Gamma_n \prec \cdots \prec \Gamma_1 = \Gamma;
   \]

2. Set $\mathcal{P}_n = \mathcal{P}(\Gamma_n), n \geq 1$. Then $\mathcal{P}_n$ is a convex geodesic partition, and the $n$th iteration of $(X, T, \mathcal{P})$ is a piecewise convex transformation with the defining partition $\mathcal{P}_n$, i.e., $(X, T, \mathcal{P})^n = (X, T^n, \mathcal{P}_n)$.

3. There is a natural bijection between the set $\Sigma(n)$ and the set of atoms of the partition $\mathcal{P}_n$. Thus, the complexity $f(n)$ is the number of faces of the graph $\Gamma_n$.

### 1.3 A combinatorial lemma

We will need a general proposition that concerns the combinatorics of graphs. Let $X$ be a piecewise geodesic surface. Let $\Gamma', \Gamma''$ be geodesic graphs drawn
on $X$ and set $\Gamma = \Gamma' \lor \Gamma''$. (We do not assume that the faces of $\Gamma', \Gamma''$ are convex.)

Denote by $F', F'', E', E'', V', V''$ the respective sets of faces, edges and vertices. Let $E, F, V$ be the sets of faces, edges and vertices of $\Gamma$. If $e' \in E', e'' \in E''$ intersect non-transversally, then they (partially) overlap. There are 4 ways in which this can happen. See figure. Denote by $c(\Gamma', \Gamma'')$ the number of the overlapping pairs of edges.

\begin{figure}[h]
\centering
\includegraphics[width=0.8\textwidth]{overlap.png}
\caption{Overlapping of edges of two graphs}
\end{figure}

**Lemma 1** Let $\Gamma', \Gamma''$ be geodesic graphs drawn on a piecewise geodesic surface $X$. Denote by $\chi$ the Euler characteristic of $X$. Let $V_1', V_2''$ be the sets of vertices of $\Gamma', \Gamma''$ that are disjoint from the other graph, and set $V_{ess} = V - V_1' - V_2''$. Then

$$|F| - |F'| - |F''| + \chi = |V_{ess}| - c(\Gamma', \Gamma'').$$ (1)

**Proof.** Any graph $A$ drawn on $X$ satisfies

$$|F(A)| - |E(A)| + |V(A)| = \chi(X).$$ (2)

Applying (2) to graphs $\Gamma', \Gamma'', \Gamma$, we obtain for the left hand side of (1)

$$|F| - |F'| - |F''| + \chi = (|E| - |E'| - |E''|) + (|V'| + |V''| - |V|).$$ (3)

Denote by $e_i', e_j''$ the edges of $\Gamma', \Gamma''$ respectively. Let $a_i', a_j''$ be the number of vertices of $\Gamma''', \Gamma''$ that are located in the interior of $e_i', e_j''$ respectively. Let $b_i', b_j''$ be the number of times that the interior of $e_i', e_j''$ transversally
intersects the interior of an edge of $\Gamma''$, $\Gamma'$ respectively. Then $e'_i$, $e''_j$ contribute $a'_i + b'_i + 1, a''_j + b''_j + 1$ edges to $E$ respectively. Taking the overlapping into account, we obtain

$$|E| = \sum_i a'_i + \sum_i b'_i + |E'| + \sum_j a''_j + \sum_j b''_j + |E''| - c(\Gamma', \Gamma'').$$

Let $V_c$ be the set of common vertices of $\Gamma', \Gamma''$. Let $V'_e, V''_e$ be the sets of vertices of $\Gamma', \Gamma''$ that are in the interior of edges of the other graph. Then

$$|V'| = |V'_e| + |V'_d| + |V_c|, \quad |V''| = |V''_e| + |V''_d| + |V_c|. \quad (5)$$

Besides

$$\sum_i a'_i = |V''_e|, \quad \sum_j a''_j = |V'_e|. \quad (6)$$

Let $V_n = V \setminus (V' \cup V'')$ be the set of “new” vertices of $\Gamma$. Then

$$\sum_i b'_i = \sum_j b''_j = |V_n|. \quad (7)$$

We also have

$$|V| = |V''_e| + |V''_d| + |V_c| + |V_n|. \quad (8)$$

From equations (4), (5), (6) and (7), we obtain

$$|E| - |E'| - |E''| = |V'_e| + |V''_e| + 2|V_n| - c(\Gamma', \Gamma''). \quad (9)$$

By equation (9),

$$|V'| + |V''| - |V| = |V_c| - |V_n|. \quad (10)$$

Substituting (9), (11) into (8), and using (8) again, we obtain the claim.

The expression for $|F(\Gamma' \vee \Gamma'')| - |F(\Gamma')| - |F(\Gamma'')|$ in Lemma 1 involves information about vertices of $\Gamma', \Gamma''$ which is sometimes not available. The following corollary of Lemma 1 is useful.

**Proposition 2** Let $\Gamma', \Gamma''$ be geodesic graphs drawn on a piecewise geodesic surface $X$ of Euler characteristic $\chi$. Let $\Gamma = \Gamma' \vee \Gamma''$, and let the notation be as above. Then

$$|V_n| \leq |F| - |F'| - |F''| + \chi + c(\Gamma', \Gamma'') \leq |V|. \quad (11)$$
Proof. By Lemma 1 and equation (8), the quantity to estimate in (11) is equal to 

\[ |V_e'| + |V_e''| + |V_c| + |V_n|. \]

The following special cases of Lemma 1 and Proposition 2 will be especially useful.

**Corollary 1.** Let \( X \) be a geodesic surface homeomorphic to the open disc. Let \( \Gamma', \Gamma'' \) be geodesic graphs drawn on \( X \) such that their edges intersect only transversally. Let \( \Gamma = \Gamma' \lor \Gamma'' \) and let the sets \( F, F', F'', V, V_n, V'_d, V''_d, V_{\text{ess}} \) be as above. Then

\[ |F| - |F'| - |F''| + 1 = |V_{\text{ess}}| \]  

(12) (see figure 2), and

\[ |V_n| \leq |F| - |F'| - |F''| + 1 \leq |V|. \]  

(13)

If \( V'_d = V''_d = \emptyset \) then

\[ |F| - |F'| - |F''| + 1 = |V|. \]  

(14)

**Proof.** By our assumptions, \( c(\Gamma', \Gamma'') = 0 \) and \( \chi = 1 \). Hence Lemma 1 and Proposition 2 yield (12) and (13) respectively. If \( V'_d = V''_d = \emptyset \) then \( V = V_{\text{ess}} \) and (12) becomes (14).

1.4 Geometric formula for complexity of piecewise convex transformations

Let \( \Sigma \) be a language on a finite alphabet \( \mathcal{A} \), and let \( \Sigma(n) \) be the set of words of length \( n \) in \( \Sigma \). The complexity of \( \Sigma \) is the function \( f(n) = |\Sigma(n)| \). Set \( \varphi(n) = f(n+1) - f(n) \) and \( \psi(n) = \varphi(n+1) - \varphi(n) \). We will refer to the functions \( \varphi(\cdot), \psi(\cdot) \) as the first, second differences of complexity, respectively.

Our approach is based on Cassaigne’s formula for \( \psi(\cdot) \) [3]. Denote by \( m_l(w), m_r(w) \) the number of left, right one-letter extensions of the word \( w \) respectively. We will assume, following [3], that \( m_l(w), m_r(w) \geq 1 \) for any \( w \in \Sigma \). A word is bispecial if \( m_l(w), m_r(w) > 1 \). Let \( m_b(w) \) be the number of extensions of the type \( awb \) where \( a, b \in \mathcal{A} \). Let \( \mathcal{B} \subset \Sigma \) be the set of bispecial words, and set \( \mathcal{B}(n) = \mathcal{B} \cap \Sigma(n) \). The Cassaigne index is defined by

\[ \mu(w) = m_b(w) - m_l(w) - m_r(w) + 1. \]  

(15)
Figure 2: Illustrating formula (12): $|F| = 12, |F'| = 4, |F''| = 5, |\text{Vess}| = 4$

Note that if $w$ is not bispecial then $\mu(w) = 0$. The Cassaigne formula says

$$\psi(n) = \sum_{w \in B(n)} \mu(w) = \sum_{w \in \Sigma(n)} \mu(w).$$

(16)

We define the cumulative index by $\mu(n) = \sum_{w \in \Sigma(n)} \mu(w)$ for $n \geq 1$, and $\mu(0) = 0$. Set

$$M(n) = \sum_{k \leq n} \mu(k).$$

(17)

Lemma 2 The complexity of a piecewise convex transformation satisfies

$$f(n) = f(1) + (n - 1)(f(2) - f(1)) + \sum_{k \leq n - 2} M(k).$$

(18)

Proof. Denote by $g(n)$ the right hand side of (18). Since, by (16,17), the second differences of $f(\cdot), g(\cdot)$ are equal, $f(n) - g(n)$ is linear in $n$. But $f(1) - g(1) = f(2) - g(2) = 0$. $lacksquare$

Let $(X, T, \mathcal{P})$ be a piecewise convex transformation, and let $\Sigma$ be its coding language. For $w \in \Sigma(n)$ let $X(w) \subset X$ be the corresponding open face of the graph $\Gamma_n$. We denote by $\Gamma(w)$ the restriction of $\Gamma_{-1} \vee \Gamma_{n+1}$ to $X(w)$. Let $\text{Vess}(w)$ (resp. $\text{OE}(w)$) be the set of essential vertices (resp. pairs of overlappings edges) for $\Gamma(w)$. 

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Lemma 3 For any $w \in \Sigma$ we have
\[ \mu(w) = |V_{\text{ess}}(w)| - |\text{OE}(w)|. \] (19)

Proof. Denote by $\Gamma', \Gamma''$ the restrictions of $\Gamma_{n+1}, \Gamma_{-1}$ to $X(w)$ respectively. Then in the notation of Lemma 1
\[ m_b(w) = |F|, m_r(w) = |F'|, m_l(w) = |F''|. \]
Since $X(w)$ is contractible, $\chi = 1$ in equation (1). Thus, the left hand side of equation (1) is $\mu(w)$. But its right hand side is $|V_{\text{ess}}(w)| - |\text{OE}(w)|$.

We will use the notation $c(w) = |\text{OE}(w)|$. For $n \geq 1$ set
\[ V(n) = \sum_{k \leq n} v(k), \] (20)
and
\[ C(n) = \sum_{k \leq n} c(k). \] (21)
Thus, $v(n)$ (resp. $c(n)$) is the number of “new” essential vertices (resp. “new” edge overlappings) of the graph $\Gamma_{-1} \lor \Gamma_{n+1}$, while $V(n)$ (resp. $C(n)$) is the total number of essential vertices (resp. edge overlappings) of $\Gamma_{-1} \lor \Gamma_{n+1}$. Note that only bispecial words contribute to these numbers.

Proposition 3 Let $(X, T, P)$ be a piecewise convex transformation and let $P_1 = P, P_2, \ldots$ be the corresponding sequence of convex partitions. Then the complexity of $(X, T, P)$ satisfies
\[ f(n) = |P_1| + (n - 1)(|P_2| - |P_1|) + \sum_{k \leq n-2} V(k) - \sum_{k \leq n-2} C(k). \] (22)

Proof. We have $f(1) = |P_1|$, $f(2) = |P_2|$. By Lemma 3 $\mu(n) = v(n) - c(n)$, hence $M(n) = V(n) - C(n)$. The claim now follows from equation (18).

2 Billiard map as a piecewise convex transformation: the dictionary

Let $M$ be a complete simply connected surface of constant curvature,\(^4\) and let $P \subset M$ be a geodesic polygon. In order to cast the billiard map in $P$ as a\(^4\)We normalize the curvature to 0 or −1 or 1. Thus $M$ is either the euclidean plane or the hyperbolic plane or the unit sphere. We will refer to these geometries as parabolic, hyperbolic and elliptic respectively.
piecewise convex transformation simultaneously for all three cases at hand, we will use the models of the hyperbolic and elliptic geometries where the geodesics are straight lines in the euclidean plane. (We will refer to them as the \textit{projective models}.)

For $\chi = -1$, this is the Klein-Beltrami model of hyperbolic geometry. The hyperbolic plane is represented by the open unit disc, the unit circle $S$ is the “circle at infinity”, and geodesics are the chords of this disc. Let $x, y$ be distinct points in the disc, and let $a, b$ be the intersection points of the line $xy$ with $S$. We identify the oriented line $xy$ with $\mathbb{R}$, and hence $x, y, a, b$ with real numbers. The cross-ratio of the four points is given by

$$[a, x, y, b] = \frac{(y-a)(b-x)}{(x-a)(b-y)}.$$  

The distance between points satisfies $2d(x, y) = \ln[a, x, y, b]$. In this model, isometries of the hyperbolic geometry are the projective transformations of the euclidean plane, preserving the unit disc.

Projective model of the elliptic geometry (i.e., $\chi = +1$) is as follows. We restrict our attention to an open hemisphere; consider for concreteness the northern hemisphere. Its central projection to the tangent plane at $(0, 0, 1)$ is a surjective diffeomorphism that takes spherical geodesics to euclidean straight lines.

In order to endow the \textit{billiard map phase space} $X = X(P)$ with the structure of a geodesic surface in a uniform fashion, we will impose a slight restriction on $P$. Let $P \subset M$ be a geodesic polygon, and let $a$ be a side of $P$. We denote by $s_a$ the geodesic reflection in $M$ about $a$, and set $P_a = P \cup s_a(P)$.

\textbf{Definition 1} Let $P \subset M$ be a spherical polygon. Then $P$ is \textit{admissible} if for any side $a$ the polygon $P_a$ is contained in an open hemisphere.

Note that the admissibility is a restriction only in the elliptic case.\textsuperscript{5} Unless we state otherwise, we will consider only admissible polygons, suppressing the qualifier.

The space $X = X(P)$ consists of directed geodesic segments inside $P$ both of whose endpoints belong to $\partial P$. Since we are using projective models, $P \subset \mathbb{R}^2$ is a euclidean polygon in any of the three cases at hand. Geodesic segments in $P$ are straight; we refer to their endpoints as the \textit{beginning

\textsuperscript{5}We think that all of our results remain valid without this restriction.
The mapping \( x \mapsto (b(x), e(x)) \) is an embedding \( X \subset \partial P \times \partial P \); it induces a topology on \( X \).

Let \( L \) be the space of oriented straight lines (rays) in the euclidean plane. Endowed with the natural topology, it is a cylinder. Besides, \( L \) is a geodesic surface: geodesics in \( L \) are the pencils of rays passing through a point or the pencils of parallel rays. Equivalently, we define this structure via the canonical embedding \( \mathbb{R}^2 \subset \mathbb{RP}^2 \). Lines in \( \mathbb{R}^2 \) become points of the dual projective plane \( (\mathbb{RP}^2)^* \simeq \mathbb{RP}^2 \). The geodesic surface structure of \( L \) is thus induced by that of the real projective plane.

Denote by \( l(x) \in L \) the ray containing the chord \( x \). The mapping \( X \to L \) given by \( x \mapsto l(x) \) is finite-to-one; if \( P \) is convex then it is one-to-one. It induces the structure of a geodesic surface on \( X \).

Let \( \Delta^+ \subset X \) (resp. \( \Delta^- \subset X \)) be given by the condition that \( e(x) \) (resp. \( b(x) \)) is a corner of \( P \). Let \( \Delta_0 \subset X \) be the set of chords that contain a corner of \( P \) in the interior. Set

\[
\Delta = \Delta^+ \cup \Delta^- \cup \Delta_0.
\]

By definition, \( \Delta \) is a geodesic graph in \( X \). The following lemma is crucial.

**Lemma 4** The faces of the graph \( \Delta \) are geodesically convex.

**Proof.** Let \( x_0 = A_0B_0, x_1 = A_1B_1 \in X \setminus \Delta \), and let \( x_t = A_tB_t, t \in [0,1] \), be a path in \( X \setminus \Delta \) connecting them. Then the points \( A_t, t \in [0,1] \) (resp. \( B_t, t \in [0,1] \)) belong to the interior of a side, \( a \subset \partial P \) (resp. \( b \subset \partial P \)).

Up to relabeling and orientation reversal, the points in question may form four a priori possible configurations; the four cases are shown in figure 3.

Cases 3, 4 contradict to our assumptions. Indeed, in case 4 the sides \( a, b \) intersect at their interior points. In case 3 let \( V_0, V_1 \) be the endpoints of the side \( b \). The two frames \( A_0B_0, V_0V_1 \) and \( A_1B_1, V_0V_1 \) have opposite orientations, thus for some \( 0 < t < 1 \) the vectors \( A_tB_t, V_0V_1 \) are colinear, in contradiction to the assumption that \( A_tB_t \in X \setminus \Delta \).

In cases 1, 2 let \( O \in \mathbb{R}^2 \) be the intersection point of the straight lines \( A_0B_0, A_1B_1 \).\(^6\) Let \( \gamma \) be the curve in \( X \) obtained by rotating the oriented line \( A_0B_0 \) about \( O \) towards \( A_1B_1 \) by an angle less than \( \pi \). It is a geodesic. To show that \( \gamma \) does not intersect \( \Delta \), it suffices to prove that the quadrilateral \( A_0A_1B_1B_0 \) contains no vertices of \( P \). Assume that it does, and let \( V \) be such

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\(^6\)In case 1 the two lines may be parallel, but the same argument works.
Figure 3: Convexity of $\Delta$
a vertex. The frames \( A_0 V, A_0 B_0 \) and \( A_1 V, A_1 B_1 \) have opposite orientations. Hence, for some \( 0 < \tau < 1 \) the vectors \( A_\tau V, A_\tau B_\tau \) are colinear, and thus the segment \( x_\tau = A_\tau B_\tau \) contains a corner of \( P \), in contradiction to the assumptions.

We denote by \( T_b : X \to X \) the billiard map and by \( T_b^{-1} : X \to X \) its inverse. They are not defined on all of \( X \), and there are choices in defining them on \( \Delta \). To avoid the ambiguities, we will now precisely define \( T_b, T_b^{-1} \) on their respective domains of definition.

Let \( x = [b(x), e(x)] \) be a phase point. Then \( x_1 = T_b(x) \) (resp. \( x_{-1} = T_b^{-1}(x) \)) is not defined iff \( e(x) \) (resp. \( b(x) \)) is a corner of \( P \). Thus, the natural domain of definition for \( T_b \) (resp. \( T_b^{-1} \)) is \( X \setminus \Delta_+ \) (resp. \( X \setminus \Delta_- \)). Suppose now that \( e(x) \) is not a corner, and let \( a \) be the unique side of \( P \) that contains \( e(x) \). Let \( l_1(x) \) be the reflection of \( l(x) \) about \( a \). Let \( x_1 = T_b(x) = [e(x), e(x_1)] \subset l_1(x) \) be the longest chord such that \([e(x), e(x_1)] \in X\). This defines \( T_b \) on \( X \setminus \Delta_+ \). The definition of \( T_b^{-1} \) on \( X \setminus \Delta_- \) is analogous, and we leave it to the reader. Set

\[
\Gamma = \Delta \cup T_b^{-1}(\Delta \setminus \Delta_-), \quad \Theta = (\Delta \setminus \Delta_-) \cup T_b^{-1}(\Delta_0).
\]

**Proposition 4** The following holds:

1. The subset \( \Theta \subset X \) is the set of discontinuities of the billiard map.
2. The faces of \( \Gamma \) are geodesically convex.

**Proof.** It is elementary to see that \( \Delta_+ \) belongs to the set of discontinuities. (The billiard map is not even defined on \( \Delta_- \).) Let \( x \in \Delta_0 \). Then arbitrarily close to \( x \) there are phase points \( x', x'' \) such that \( e(x'), e(x'') \) belong to distinct, and non-adjacent, sides of \( P \). Hence \( b(T_b(x')) \) and \( b(T_b(x'')) \) are not close to each other, and \( T_b \) is discontinuous at \( x \). Let \( x \notin \Delta_+ \). Then \( x_1 = T_b(x) \) is defined. If however \( x_1 \in \Delta_0 \), then arguing as above, we conclude that \( e(T_b(x')) \) and \( e(T_b(x'')) \) are not close to each other, and \( T_b \) is discontinuous at \( x \). See Figure 4. On the other hand, if \( x_1 \notin \Delta_0 \), then \( x \) is a point of continuity of \( T_b \). This proves the first claim.

Let \( x', x'' \) belong to a face of \( \Gamma \). Arguing as in the proof of Lemma 4 we see that \( e(x'), e(x'') \) belong to the same side, \( a \), of \( P \). Set \( Q = P_a \), and let \( Y = X(Q) \). The “reflection” trick associates with any phase point \( x \in X \) such that \( e(x) \in a \) a phase point \( y \in Y \). The correspondence \( x \mapsto y \) maps a curve \( x(t), 0 \leq t \leq 1 \), connecting \( x', x'' \) in \( X \) into a curve \( y(t), 0 \leq t \leq 1 \),
connecting $y', y''$ in $Y$. Let $\Delta(Q) \subset Y$ be the corresponding set of phase points containing corners. The condition $x(t) \in X \setminus \Gamma, 0 \leq t \leq 1$, implies that $y(t) \in Y \setminus \Delta(Q), 0 \leq t \leq 1$. By Lemma 4, phase points $y', y''$ are connected by a geodesic, $\beta$, in $Y \setminus \Delta(Q)$. Folding $\beta$ back into $X$ by the geodesic reflection $s_a$, we obtain the geodesic $\alpha$ connecting $x', x''$ in $X \setminus \Gamma$. This establishes the second claim.

We will now cast the billiard mapping as a piecewise convex transformation.

**Theorem 1** Let $M$ be a simply connected surface of constant curvature, and let $P \subset M$ be a geodesic polygon. Let $X = X(P)$ be the phase space of the billiard map endowed with the structure of a geodesic surface. Let $T_b : X \rightarrow X$ be the billiard map.

Let $\Gamma$ be the graph defined by (23), and let $P : X = X_\alpha \cup X_\beta \cup \cdots$ be the corresponding convex geodesic partition.

Then $\Gamma$ is a convex geodesic graph, and $T_b$ is continuous on the faces of $\Gamma$. Let $T$ be the map that coincides with $T_b$ on the open faces of $\Gamma$ and extends to their closures by continuity. Then $(X, T, P)$ is a piecewise convex transformation; the billiard map $T_b$ and the transformation $(X, T, P)$ coincide on $X \setminus \Gamma$.  

![Figure 4: Discontinuity of the billiard map at a point in $T_b^{-1}(\Delta_0)$](image-url)
Proof. Most of the statements have been established in the course of the preceding discussion. For instance, Proposition 4 asserts that $\Gamma$ is a convex geodesic graph. It remains to show that the restriction of $T_b$ to any open face of $\Gamma$ is a geodesically convex transformation.

The geodesic structure on $X$ has been defined via ray focusing. Geodesic reflections send focusing beams into focusing beams. Let $Y \subset X$ be a face of $\Gamma$. As we have seen in the proof of Proposition 4, the endpoints $b(x)$ of all phase points $x \in Y$ belong to the same side, $a$, of $P$. Hence, for all $x \in Y$ the new phase point $x_1 = T_b(x)$ is obtained via the geodesic reflection $s_a$. Hence $T_b$ sends geodesic segments of $Y$ into geodesic segments.

It has been traditional to study the complexity of billiard orbits in a polygon $P$ using the coding by sides of $P$ [16]. Let $(X, T, P)$ be as in Theorem 1. The complexity of billiard orbits with respect to the coding by atoms of $P$ coincides with the complexity of $(X, T, P)$. We will study it, and then apply our results to the traditional billiard complexity.

First, we will establish a dictionary between the language of billiard orbits [9, 15, 16] and that of piecewise convex transformations. A billiard orbit of (combinatorial) length $n$ is a sequence $\gamma = x_1, \ldots, x_n$ of phase points such that $x_{i+1} = T_b(x_i)$. Geometrically, $\gamma$ is a sequence of $n$ consecutive chords of $P$, where each chord is the reflection of the preceding one. In particular, the points $e(x_1), \ldots, e(x_{n-1})$ are not corners. We will also say that $\gamma$ is an $n$-segment orbit. If $e(x_n)$ is not a corner, then the 1-step forward extension is $x_1, \ldots, x_n, x_{n+1}$, where $x_{n+1} = T_b(x_n)$. If $b(x_1)$ is not a corner, then the 1-step backward extension is $x_{-1}, x_1, \ldots, x_n$, where $x_{-1} = T_b^{-1}(x_1)$. We can iterate these extentsions in obvious ways.

A billiard orbit is regular if it does not contain corners. Otherwise, $\gamma$ is singular. If $\gamma$ does not contain corners of $P$ in its interior, but the endpoints $b(\gamma) = b(x_1), e(\gamma) = e(x_n)$ are corners, we say, following [13], that $\gamma$ is a generalized diagonal of length $n$.

The notion of generalized diagonals works well for the billiard in a convex polygon. We introduce an extension of this notion which works for arbitrary polygons.

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[7] We use the self-explanatory language of geometric optics in discussing billiard dynamics [16].
Definition 2. An \( n \)-segment billiard orbit \( \gamma = x_1, \ldots, x_n \) is strongly singular if \( x_2, \ldots, x_{n-1} \) do not contain corners but \( x_1 \) and \( x_n \) do.

A family of billiard orbits is a one-parameter family \( \gamma(t) = x_1(t), \ldots, x_n(t) \) where the mapping \( t \mapsto \gamma(t) \) is injective and continuously differentiable. In particular, we will consider families of strongly singular billiard orbits. Any such is contained in a unique maximal family, and we will consider only them, suppressing the qualifier maximal. A strongly singular billiard orbit is isolated if it is not contained in a family of such orbits.

Proposition 5. Let \( P \subset M \) be a geodesic polygon, and let \( (X, T, \mathcal{P}) \) be the corresponding piecewise convex transformation. Let \( n \geq 1 \). Then:

1. There is a bijection between the set \( V_{\text{Ess}}(n) \) and the union of the sets of isolated \((n+2)\)-segment and \((n+3)\)-segment strongly singular billiard orbits;
2. There is a bijection between the set \( O\text{E}(n) \) and the union of the sets of families of \((n+2)\)-segment and \((n+3)\)-segment strongly singular billiard orbits.

Proof. Let \( x \in X \setminus \Gamma_n \). Iterating the billiard map, we obtain an \((n+1)\)-segment billiard orbit \( x = x_1, \ldots, x_n, x_{n+1} \), where \( x_{n+1} = T_b^n(x) \). Let \( x_{n+2} = T_b(x_{n+1}) \). Then \( x \in \Gamma_{n+1} \) iff either \( x_{n+1} \in \Delta_0 \cup \Delta_+ \) or \( x_{n+1} \in X \setminus \Delta \) but \( T_b(x_{n+1}) = x_{n+2} \in \Delta \).

Set \( x_{-1} = T_b^{-1}(x) \). Then \( x \in \Gamma_{-1} \) iff \( x_{-1} \in \Delta \setminus \Delta_+ \). Thus, we have obtained a surjective map from \( < \Gamma_{-1} \vee (\Gamma_{n+1} \setminus \Gamma_n) > \subset X \) to the union of set of \((n+2)\)-segment and \((n+3)\)-segment strongly singular billiard orbits. (The orbit corresponding to \( x \) is \( x_{-1}, x_1, \ldots, x_n, x_{n+1} \) in the former case, and \( x_{-1}, x_1, \ldots, x_n, x_{n+1}, x_{n+2} \) in the latter.)

The map above is a bijection. Moreover, isolated orbits correspond to the essential vertices of \( \Gamma_{-1} \vee (\Gamma_{n+1} \setminus \Gamma_n) \), and families of orbits correspond to overlapping edges of \( \Gamma_{-1} \) and \( \Gamma_{n+1} \setminus \Gamma_n \).

3 Complexity of the billiard in a polygon

We will now apply the preceding material to the complexity of billiards in geodesic polygons on surfaces of constant curvature. In the beginning of this section we consider the three cases at hand simultaneously, emphasizing their similarities.
3.1 Arbitrary curvature, any polygon

Let $M$ be a surface of constant curvature $\chi = 0, \pm 1$, and let $P \subset M$ be a geodesic polygon. The operation of unfolding sends billiard orbits in $P$ into geodesics in $M$ \[11\], see figure 5. If $\gamma = x_1, x_2, \ldots, x_n$ is a billiard orbit, then its unfolding is the geodesic $\tilde{\gamma} = x_1, \tilde{x}_2, \ldots, \tilde{x}_n$ where the segments $x_i, \tilde{x}_i$ differ by an isometry of $M$.

![Figure 5: Unfolding a billiard trajectory](image)

Let $G(P) \subset Iso(M)$ be the group generated by the geodesic reflections in the sides of $P$. Denote by $G^{(n)}(P) \subset G(P)$ the set of elements obtained from products of at most $n$ reflections. (These are the elements of length at most $n$.) Then $G^{(n)}(P), 0 \leq n < \infty$, is an increasing tower of finite sets; their union is $G(P)$.

**Lemma 5** 1. Let $\chi \leq 0$. Then all strongly singular billiard orbits in $P$ are isolated.

2. Let $\chi > 0$. Let $\gamma = x_1, x_2, \ldots, x_n$ be a strongly singular billiard orbit in $P$ and let $\tilde{\gamma} = x_1, \tilde{x}_2, \ldots, \tilde{x}_n$ be its unfolding. Suppose that $\gamma$ extends to a family of strongly singular billiard orbits. Then there are corner points $p_1 \in x_1, p_n \in x_n$ and an isometry $g$ of $M$ such that $g(x_n) = \tilde{x}_n$ and the points $p_1, g(p_n)$ either coincide or are antipodal.

**Proof.** Let $\gamma(t) = x_1(t), x_2(t), \ldots, x_n(t), 0 \leq t \leq 1$, be a family of strongly singular billiard orbits. The segments $x_1(t)$ (resp. $x_n(t)$) pass through the same corner point $p_1$ (resp. $p_n$). We view $\gamma(t)$ as a beam of trajectories that emanate from the focusing point $p_1$ and refocus at $p_n$.

The unfolding $\gamma \mapsto \tilde{\gamma}$ transforms directed billiard orbits into directed geodesics in $M$, preserving the length and sending focusing beams of billiard
orbits into focusing beams of geodesics. The beam \( \tilde{\gamma}(t) \) emanates from \( p_1 \in M \) and refocuses at \( g(p_n) \in M \) where \( g \in G(P) \). The distance between \( p_1, g(p_n) \) along any geodesic \( \tilde{\gamma}(t) \) is the same as the distance between \( p_1, p_n \) along the orbit \( \gamma(t) \), hence positive.

If \( \chi \leq 0 \), this implies \( p_1 \neq g(p_n) \); thus the geodesic beam \( \tilde{\gamma}(t) \) has two different focusing points. This is impossible, which proves the first claim. If \( \chi > 0 \), i.e., \( M \) is the sphere, the focusing points in question are either antipodal or they coincide.

We denote by \( s(k) \) (resp. \( fs(k) \)) the number of isolated (resp. families of) strongly singular \( k \)-segment billiard orbits in \( P \). Set

\[
S(n) = \sum_{3 \leq k \leq n} s(k), \quad FS(n) = \sum_{3 \leq k \leq n} sf(k). \tag{24}
\]

**Proposition 6** Let \( P \) be a geodesic polygon on a surface \( M \) of constant curvature \( \chi \). Let \( (X, T, \mathcal{P}) \) be the associated piecewise convex transformation, and let \( f(\cdot) \) be the billiard complexity corresponding to the partition \( \mathcal{P} \). Then there are positive integers \( q_1, q_2 \) depending on \( P \), so that:

1. If \( \chi = 1 \), then

\[
f(n) = q_1 + q_2 n + S(n+1) - FS(n+1) + 2 \left( \sum_{3 \leq k \leq n} S(k) - \sum_{3 \leq k \leq n} FS(k) \right). \tag{25}
\]

2. If \( \chi \leq 0 \), then

\[
f(n) = q_1 + q_2 n + S(n+1) + 2 \sum_{3 \leq k \leq n} S(k). \tag{26}
\]

**Proof.** We use Proposition \ref{prop5} to express the right hand side of (22) via the numbers of (families of) strongly singular billiard orbits. This yields the first claim. Taking into account Lemma \ref{lem5}, we obtain the second claim. The factor of 2 in (25) (26) is due to the fact that the contribution of \( k \)-segment strongly singular orbits with \( k \leq n \) is counted twice. See Proposition \ref{prop4}.

\[\text{\footnotesize 8Although [16] assumes that } \chi = 0, \text{ the argument applies to arbitrary } \chi.\]
There is a correspondence between strongly singular billiard orbits and generalized diagonals. Let $\gamma = x_1, x_2, \ldots, x_{n-1}, x_n$, $n \geq 3$, be a strongly singular orbit. Let $p \in x_1$ (resp. $q \in x_n$) be the last (resp. the first) corner point. Note that $p \neq e(x_1), q \neq b(x_n)$. Set $\tilde{x}_1 = [p, e(x_1)], \tilde{x}_n = [b(x_n), q]$. Then $\tilde{\gamma} = \tilde{x}_1, x_2, \ldots, x_{n-1}, \tilde{x}_n$ is a generalized diagonal. The mapping $\gamma \mapsto \tilde{\gamma}$ is a bijection, and it preserves the combinatorial length.\footnote{However, it does not preserve the property to be isolated.}

Let $gd(n)$ (resp. $fgd(n)$) be the number of $n$-segment isolated (resp. families of) generalized diagonals, $n \geq 3$. Set
\[
GD(n) = \sum_{3 \leq k \leq n} gd(k), \quad FGD(n) = \sum_{3 \leq k \leq n} fgd(k).
\]

**Theorem 2** Let $P$ be a geodesic polygon on a surface $M$ of constant non-positive curvature. Let $(X, T, P)$ be the associated piecewise convex transformation, and let $f(\cdot)$ be the billiard complexity corresponding to the partition $P$. Then there are integers $c_1, c_2$ depending on $P$, so that:
\[
f(n) = c_1 + c_2 n + GD(n + 1) + 2 \sum_{3 \leq k \leq n} GD(k).
\]

**Proof.** By Lemma\footnote{The same argument shows that generalized diagonals are isolated, as well. The claim now follows from (26) and the correspondence between strongly singular billiard orbits and generalized diagonals.} strongly singular billiard orbits are isolated. The same argument shows that generalized diagonals are isolated, as well. The claim now follows from (26) and the correspondence between strongly singular billiard orbits and generalized diagonals.

We will now apply the preceding material to the complexity, $F(\cdot)$, of the coding billiard orbits by the sides of $P$. To avoid ambiguities, we will define this coding now. Let $P$ have $p$ sides $a, b, \ldots$, and denote by $X(a, b) \subset X$ the set of phase points $x$ such that $b(x) \in a, e(x) \in b$. As $a \neq b$ run through the pairs of sides such that $X(a, b)$ has a nonempty interior, the sets $X(a, b)$ form the partition $P_{\text{side}}$ of the phase space. Then the side complexity, $F(\cdot)$, is the complexity of the billiard map with respect to $P_{\text{side}}$.

It will be useful to have a direct definition of the side complexity. We denote by $L_{\text{side}}$ the full language on the alphabet of sides of $P$. Let now $\gamma = x_1, \ldots, x_n$ be a regular $n$-segment billiard orbit. Then $b(x_1), \ldots, b(x_n), e(x_n)$ are interior points of sides $s_1, \ldots, s_n, s_{n+1}$, and we set
\[
\sigma_{\text{side}}(\gamma) = s_1, \ldots, s_n, s_{n+1} \in L_{\text{side}}(n + 1).
\]
The language $\Sigma_{\text{side}}$ is the range of the mapping, and we set $\Sigma_{\text{side}}(n) = \Sigma_{\text{side}} \cap L_{\text{side}}(n)$. Then the side complexity satisfies

$$F(n) = |\Sigma_{\text{side}}(n+1)|.$$  \hfill (29)

To compare complexities, we will use an elementary lemma.

**Lemma 6** Let $(X,T,\mathcal{P})$ be a piecewise convex transformation, and let $f(\cdot)$ be the corresponding complexity. Let $\mathcal{P}_1$ be a partition, such that $\mathcal{P} \prec \mathcal{P}_1$, and let $f_1(\cdot)$ be the complexity of $(X,T,\mathcal{P})$ with respect to $\mathcal{P}_1$. Then $f_1(n) \leq f(n)$.

**Corollary 2** Let $P$ be a geodesic polygon on a surface $M$ of constant curvature $\chi$, and let $F(\cdot)$ be the complexity of coding billiard orbits by the sides of $P$. Then the following inequalities hold:

1. If $\chi = 1$, then there are constants $q_1, q_2$ such that

$$F(n) \leq q_1 + q_2 n + S(n+1) - FS(n+1) + 2 \left( \sum_{3 \leq k \leq n} S(k) - \sum_{3 \leq k \leq n} FS(k) \right).$$ \hfill (30)

2. If $\chi \leq 0$, then there are constants $c_1, c_2$ such that

$$F(n) \leq c_1 + c_2 n + GD(n+1) + 2 \sum_{3 \leq k \leq n} GD(k).$$ \hfill (31)

**Proof.** Let $\mathcal{P}$ be the defining partition of the associated piecewise convex transformation. By Proposition 4 the partitions $\mathcal{P}, \mathcal{P}_{\text{side}}$ satisfy $\mathcal{P} \prec \mathcal{P}_{\text{side}}$. Now the first (resp. second) claim is immediate from Proposition 6 (resp. Theorem 2) and Lemma 6. \hfill \blacksquare

### 3.2 Arbitrary curvature, convex polygon

The preceding considerations drastically simplify for convex polygonal billiard tables.

**Proposition 7** Let $P \subset M$ be a convex geodesic polygon on a surface of constant curvature $\chi$. Let $F(\cdot)$ be the traditional complexity of billiard orbits in $P$ coming from the coding of billiard orbits in $P$ by the sides they hit. Then there exist constants $c_1, c_2$ depending only on $P$ such that the following holds:
1. Let $\chi = 1$. Then
\[
F(n) = c_1 + c_2 n + \sum_{3 \leq k \leq n} GD(k) - \sum_{3 \leq k \leq n} FGD(k); \quad (32)
\]

2. Let $\chi \leq 0$. Then
\[
F(n) = c_1 + c_2 n + \sum_{3 \leq k \leq n} GD(k). \quad (33)
\]

Proof. In the notation of Section 2, we have
\[
\Delta_0 = \emptyset, \quad \Theta = \Delta_+, \quad \Gamma = \Delta = \Delta_+ \cup \Delta_-.
\]
The standard defining partition is $\mathcal{P} = \mathcal{P}(\Gamma)$, and we have $\mathcal{P}_{sides} = \mathcal{P}(\Delta)$. In view of the above, $\mathcal{P}_{sides} = \mathcal{P}$, and hence the two complexities coincide: $F(n) = f(n)$.

For a convex polygon, the notion of strongly singular billiard orbits and the notion of generalized diagonals coincide. The argument of Proposition 5 works. Due to the coincidences we just pointed out, it establishes a bijection between $\text{Vess}(n)$ (resp. $\text{OE}(n)$) and the set of isolated (resp. families of) $n$-segment generalized diagonals in the convex polygon $P$.

From this bijection and (22), we obtain the first claim. Now the first statement of Lemma 5 yields the second claim.

Remark 1 The preceding proof yields expressions for the constants $c_1, c_2$. From (22), we have $c_1 = 2|P_1| - |P_2|, c_2 = |P_2|$. Let the polygon $P$ have $p$ sides. The integer $|P_1|$ (resp. $|P_2|$) is the number of types of 1-segment (resp. 2-segment) billiard orbits in $P$. By convexity, $|P_1| = p(p - 1)$. However, $|P_2|$ is not determined by $p$ alone.

We will now apply the preceding material to the complexity of polygonal billiards. We will consider the three cases separately.

### 3.3 The Euclidean Case

There is a considerable literature on the billiard dynamics in euclidean polygons. See [16] for references. Many basic questions remain open [9]. One of them is whether the complexity (of the coding by sides) of billiard orbits grows (at most) polynomially. It is known that the growth is subexponential [13, 7, 11], implying that the billiard in a euclidean polygon has zero...

\[\text{[10]}\text{The discussion in [13, 7] is restricted to simply connected polygons.}\]
topological entropy. By Corollary 2, the complexity is bounded from above by the counting function for generalized diagonals. The latter also grows subexponentially [10], and is believed to grow at most polynomially [9].

A euclidean polygon is rational if all of its angles are rational multiples of $\pi$. Rational polygons play an important role in the subject [15, 16].

**Theorem 3**  Let $P$ be a rational euclidean polygon, and let $F(\cdot)$ be the side complexity of billiard orbits in $P$. Then there exists $c > 0$ such that

$$F(n) < cn^3.$$  \hfill (34)

If $P$ is convex, then there exist positive constants $c_1, c_2$ such that

$$c_1n^3 < F(n) < c_2n^3.$$  \hfill (35)

**Proof.** By a theorem of H. Masur [15], $GD(n)$ grows quadratically for rational polygons. More precisely, there exist positive constants $c'_1, c'_2$ such that

$$c'_1n^2 < GD(n) < c'_2n^2.$$  \hfill (36)

The first claim follows from this and the bound (31). The second claim follows the same way from the formula above and the identity (33).

**Remark 2** We believe that the cubic lower bound on complexity (35) is valid for arbitrary rational polygons. However, the geometry of the billiard map is much simpler in the convex case, as we saw in section 3.2. Figure 6 shows that the singular graph of the billiard map in a quadrilateral is much more complicated in the nonconvex case. The bounds (35) were obtained in [4].

### 3.4 The elliptic and the hyperbolic geometries

The billiard in a spherical polygon has two salient features. First, the billiard map is a piecewise isometry. Second, a spherical polygon may have non-isolated strongly singular orbits, in particular, non-isolated generalised diagonals.

We will use the notation $Geo(M)$ for the space of oriented geodesics on $M$. Let $\phi : Geo(S^2) \rightarrow S^2$ be the standard diffeomorphism. (See Section 4.2)
Figure 6: Singular set for the billiard in a euclidean quadrilateral: i) convex; ii) nonconvex

for a discussion of spherical duality.) Pulling back by $\phi$ the round metric on $S^2$, we obtain an invariant metric; the distance between two great circles is the angle between them. Metric geodesics and geodesics of the projective structure coincide.

Let now $P \subset S^2$ be a polygon. Let $X = X(P)$ be the billiard map phase space, and let $\varphi : X \to Geo(S^2)$ be the map introduced in Section 2. Pulling back by $\varphi$ induces not only the structure of a piecewise convex surface on $X$ but also a metric on convex pieces of the phase space. Locally, the billiard map $T_b : X \to X$ is a geodesic reflection, i.e., an isometry. We summarize this discussion as a proposition.

**Proposition 8** Let $P$ be an arbitrary spherical polygon. Then the billiard map phase space is a piecewise metrically convex Riemannian surface. The billiard map is a piecewise isometry.

This observation has consequences for complexity of billiard orbits.

**Theorem 4** Let $P$ be any spherical polygon.\(^{11}\) Then the complexity of the coding of billiard orbits by sides of $P$ grows subexponentially.

\(^{11}\)We do not assume that $P$ is an admissible polygon.
Proof. Let \((X,T,P)\) be the associated piecewise convex transformation. Denote by \(f(\cdot)\) its complexity, and let \(F(\cdot)\) be the standard complexity of billiard orbits. By the proof of Corollary \(\text{\ref{cor:main}}\), \(F(n) \leq f(n)\). Hence, it suffices to show that \(f(n)\) grows subexponentially. By Proposition \(\text{\ref{prop:iso}}\), \((X,T,P)\) is a piecewise isometry on a convex partition. By Theorem 4.2 of \(\text{\cite{10}}\), its complexity has subexponential growth.

The examples below illustrate peculiarities of spherical polygonal billiards.

**Example 1.** Let \(P \subset S^2\) be a polygon such that \(G(P)\) is a finite group. For instance, \(P\) may be the fundamental domain of a finite, generated by reflections group of isometries.\(^{12}\) Then every billiard orbit in \(P\) is periodic. Moreover, there is a finite number of symbolic codes corresponding to prime periodic orbits, hence complexity is bounded.

**Example 2.** Let \(P\) be a “bigon”; it is bounded by two geodesics, \(a,b\) connecting the North and the South poles. Let \(\alpha\) be the angle between them.\(^{13}\) If \(\alpha\) is \(\pi\)-rational, then we are in the situation of Example 1. We will now discuss the case when \(\alpha\) is \(\pi\)-irrational.

First, we point out that in any case the set \(\Sigma_{\text{side}}(n)\) consists of 2 elements: \(a,b,a,b,\ldots\) and \(b,a,b,a,\ldots\). Thus, \(F(n) = 2\).

There is an obvious periodic orbit. It corresponds to the intersection of \(P\) with the equator, and it has 2 segments, perpendicular to \(a,b\). We will denote this orbit by \(\gamma_0\).

**Claim.** Let \(\alpha\) be \(\pi\)-irrational. Then \(\gamma_0\) is the only prime periodic orbit in \(P\).

We will show that any periodic orbit \(\gamma\) is a multiple of \(\gamma_0\). We can assume that \(\gamma\) has an even number, \(2m\), segments, and that its symbolic code is \(b,a,\ldots,b,a\). Then the element, \(g(\gamma)\), of the group \(G(P)\) that we obtain by tracing \(\gamma\) is \((\rho)^m\) where \(\rho\) is the rotation about the vertical axis by the angle \(2\alpha\).

Let \(\ell(\gamma)\) be the spherical geodesic corresponding to \(\gamma\). (Note that \(\ell(\gamma)\) differs, in general, from the unfolding \(\tilde{\gamma}\), which is a geodesic segment along \(\ell(\gamma)\).) The periodicity of \(\gamma\) implies that \(\ell(\gamma)\) is invariant under \(g(\gamma)\), which rotates the sphere about the vertical axis by \(2m\alpha\). The only geodesic invariant under this (nontrivial!) rotation is the equator, which implies the

\(^{12}\)These polygons are well known. See, e.g., \(\text{\cite{11}}\).

\(^{13}\)Note that \(P\) is not an admissible polygon.
By convention, a periodic billiard orbit in $P$ does not pass through its corners. In particular, it cannot trace the boundary of $P$. It is not known if every euclidean polygon has a periodic orbit \[9\]. Below we present a spherical polygon without periodic orbits.

**Example 3.** For $0 < \alpha < 2\pi$ let $Q = Q(\alpha)$ be the isosceles spherical triangle with two right angles, and whose third angle is $\alpha$. If $\alpha$ is $\pi$-rational then every billiard orbit in $Q$ is periodic. If $\alpha$ is $\pi$-irrational, then $Q$ has no periodic billiard orbits. We outline a proof below.

The triangle $Q$ is obtained from the bigon $P$ of Example 2 by folding it about the equator. In this situation, every billiard orbit, $\gamma$, in $Q$ uniquely lifts to a billiard orbit $\tilde{\gamma}$ in $P$; the orbit $\gamma$ is periodic iff so is $\tilde{\gamma}$. If $\alpha$ is $\pi$-rational, then $Q$ satisfies the conditions of Example 1. Let $\alpha$ be $\pi$-irrational, and let $\gamma$ be a periodic orbit in $Q$. By preceding remark and Example 2, $\tilde{\gamma}$ runs along the equator. Thus, $\gamma$ traces the boundary of $Q$.

We will now discuss the hyperbolic case. A positive function, $s(\cdot)$, of natural argument is *subexponential* if $s(n) < e^{hn}$, $s^{-1}(n) < e^{hn}$ for any $h > 0$ and all sufficiently great $n$.

**Theorem 5** Let $P \subset H^2$ be a geodesic polygon, and let $f(\cdot)$ be the complexity of the coding of billiard orbits by sides of $P$. Denote by $h_{\text{top}}$ the topological entropy of the billiard map in $P$. Then $h_{\text{top}} > 0$; there exists a subexponential function $s(\cdot)$ such that $f(n) = s(n)e^{h_{\text{top}}n}$.

**Proof.** The billiard flow of $P$ is (uniformly) hyperbolic \[8\]. Thus, the metric entropy of the billiard flow with respect to the Liouville measure is positive. By Abramov’s formula, the metric entropy of the billiard map in $P$ is positive, as well. The metric entropy is a lower bound on the topological entropy, hence our first claim.

Let $\mathcal{P}, \mathcal{P}_{\text{side}}$ be our defining partition and the partition by sides of the phase space of the billiard map. Let $\alpha(t), \beta(t)$ be infinite billiard orbits that bounce of the same sides of $P$ as $-\infty < t < \infty$. Let $\tilde{\alpha}(t), \tilde{\beta}(t)$ be their unfoldings. Then $\tilde{\alpha}(t), \tilde{\beta}(t)$ are infinite geodesics in $H^2$ such that the distance...
between them is bounded for \(-\infty < t < \infty\). Hence \(\alpha(-\infty) = \beta(-\infty)\) and \(\alpha(\infty) = \beta(\infty)\) implying \(\alpha = \beta\). Thus, \(\alpha = \beta\), and hence \(\mathcal{P}_{\text{side}}\) is a generating partition.

Since \(\mathcal{P} \prec \mathcal{P}_{\text{side}}\), the convex partition \(\mathcal{P}\) is generating as well. By [10], the complexity of \((X, T, \mathcal{P})\) has the form \(t(n)e^{h_{\text{top}}n}\), where \(t(\cdot)\) is a subexponential function. The relation \(\mathcal{P} \prec \mathcal{P}_{\text{side}}\) implies \(f(n) \leq t(n)e^{h_{\text{top}}n}\). The cardinality of the number of atoms of \(\mathcal{P}(n)\) that partition an atom of \(\mathcal{P}_{\text{side}}(n)\) grows at most polynomially; thus \(f(n) \geq t(n)n^{-d}e^{h_{\text{top}}n}\) for some positive integer \(d\).

**Remark 3** Unlike \(Geo(S^2)\), the spaces \(Geo(H^2)\) and \(Geo(R^2)\) do not have Riemannian metrics, invariant under the natural actions of the groups of isometries of \(H^2\) and \(R^2\), respectively. However, there exists an invariant Lorentz metric on \(Geo(H^2)\). We describe it below.

Let \(H\) be the upper sheet of the hyperboloid \(z^2 - x^2 - y^2 = 1\) in \(R^3\) equipped with the pseudo-Riemannian metric \(g = dx^2 + dy^2 - dz^2\). The induced metric on \(H\) is the metric of constant negative curvature. Isometries of \(H\) are the restrictions of \(g\)-orthogonal transformations of the ambient space. Geodesics in \(H\) are its intersections with planes through the origin.

Given such a plane, its \(g\)-orthogonal complement is a line that intersects the hyperboloid of one sheet \(H_1 = \{z^2 - x^2 - y^2 = -1\}\) at two antipodal points; if the plane is oriented then so is the line, and one can canonically choose one of these intersection points. This construction identifies \(Geo(H^2)\) with \(H_1\). The metric \(g\) induces a pseudo-Riemannian metric of signature \((1, 1)\) on \(H_1\). The identification above yields a Lorentz metric on \(Geo(H^2)\).

Let now \(P \subset H^2\) be a geodesic polygon, and let \(X\) be the phase space of the billiard map in \(P\). The canonical geodesic space structure in \(X\) is induced by the natural mapping \(p : X \to Geo(H^2)\). Pulling back the Lorentz metric on \(Geo(H^2)\), we obtain a metric \(h\) on \(X\). By construction, \(h\) is locally invariant under the billiard map.\(^{15}\) Thus, the billiard map in a hyperbolic polygon is a piecewise Lorentz isometry. We do not know of any applications of this observation.

\(^{15}\)In general, \(h\) is a Lorentz metric with singularities. If \(P\) is a convex polygon, then \(h\) is regular.
4 Complexity of polygonal outer billiards

Let \((X, T, \mathcal{P})\) be a piecewise convex transformation. If the mappings \(T_i : P_i \to X\) are isometries, then \((X, T, \mathcal{P})\) is a piecewise convex isometry. Piecewise isometries in one dimension are the interval exchange maps. They arise, in particular, from the billiard in rational polygons \([15]\), and have been much studied. We will investigate the complexity of a particular class of piecewise convex isometries – the outer billiard transformations.

Let \(M\) be a simply connected surface of constant curvature \(\chi = 0, \pm 1\). For \(x \in M\) let \(T_x : M \to M\) be the geodesic symmetry about \(x\). Let \(P \subset M\) be a convex polygon with \(p\) vertices \(a, b, c, \ldots\) listed counterclockwise. If \(\chi \neq 1\), set \(X = X(P) = M - P\). If \(\chi = 1\), i.e., \(M\) is the sphere, we assume that \(P\) is contained in a hemisphere.\(^{16}\) Let \(P'\) be the antipodal polygon, and set \(X = X(P) = M - P - P'\).

\[\begin{array}{c}
R_b \\
\hline
T(x) \\
\hline
b \\
a \\
\hline
\end{array}\]

\[\begin{array}{ccc}
& A_b & \\
\hline
R_a & & \\
\hline
P & x & \\
\hline
& A_a & \\
\end{array}\]

Figure 7: Definition of the outer billiard map

For a vertex, say \(a\), of \(P\), let \(R_a \subset X\) be the geodesic ray extending the side \(ab\) in the direction of \(a\). The rays \(R_a, R_b, \ldots\) partition \(X\) into convex

\(^{16}\)This will be our standing assumption; we will not restate it.
polygons $X_a, X_b, \ldots$. See figure [4] We denote this partition by $\mathcal{P}$. The statement below defines the protagonist of this section.

**Definition 3** Let $M$ be a simply connected surface of constant curvature $\chi$, and let $P \subset M$ be a convex polygon. Set $X = M - P$ if $\chi \neq 1$, and $X = M - P - P'$ if $\chi = 1$.

Let $\mathcal{P} : X = X_a \cup X_b \cup \cdots$ be the partition of $X$ defined above. Then $\mathcal{P}$ is a convex geodesic partition. The piecewise convex isometry $(X, T, \mathcal{P})$ defined by the geodesic symmetries $T_a : X_a \to X, T_b : X_b \to X, \ldots$ is the outer billiard about $P$, and the geodesic surface $X = X(P) \subset M$ (it is a topological annulus) is the phase space of the outer billiard.

We will use the notation $T_o : X \to X$ for the outer billiard, suppressing the subscript if no confusion arises. By complexity of the outer billiard we mean the complexity of $(X, T, \mathcal{P})$ with respect to the partition $\mathcal{P}$. Now we introduce notation and terminology that will be used throughout this section. If $g(n), h(n)$ are two positive sequences then we write $g \prec h$ if there is a constant $C$ such that for all $n$ sufficiently large $g(n) \leq Ch(n)$. If $g \prec h$ and $h \prec g$, then we write $g \sim h$; we will say that the sequences have the same growth or are in the same (growth) class. If $g \prec n^d$ then we say that $g$ grows at most polynomially with degree $d$, or that $g$ is bounded by $n^d$.

If $G$ is a group with a finite set $S = \{s_1, \ldots, s_p\}$ of generators, we denote by $G_S^{(n)} \subset G$ the set of elements that can be represented by products of at most $n$ elements of $S$ and their inverses. The growth class of the sequence $g_S(n) = |G_S^{(n)}|$ does not depend on the choice of $S$. If $g_S(n) \sim n^d$, then we say that the group $G$ grows polynomially, with degree $d$.

Let $G = G(P) \subset \text{Iso}(M)$ be the group generated by the geodesic reflections in the vertices of the polygon $P$. We will relate the growth of $G$ and the complexity of the outer billiard about $P$. We proceed to study separately the three cases at hand.

### 4.1 The euclidean case

We will obtain polynomial bounds on the complexity of outer billiard.

**Theorem 6** Let $P$ be a convex euclidean $p$-gon, and let $f(\cdot)$ be the complexity of the outer billiard about $P$. Then $n \prec f(n) < n^{p+1}$.  

30
Proof. The edges of the graph $\Gamma_n$ are parallel to the sides of $P$; each edge is a segment or a half-line. Assume, for simplicity of exposition, that $P$ has no parallel sides. Then there are $p$ directions. For each direction there are $n$ parallel half-lines, hence their total number is $pn$. Since they partition $X$ into $pn$ components, the number of faces of $\Gamma_n$ is at least $pn$. This yields the linear lower bound on complexity. Now, for the upper bound.

Let $G = G(P)$, and let $S = \{T_1, \ldots, T_p\}$ be its natural set of generators. We will need a few lemmas.

Lemma 7 The growth of $G$ is bounded by $n^{p-1}$.

Proof. The subgroup $H \subset G$ generated by $T_1 T_p, T_2 T_p, \ldots, T_{p-1} T_p$ is a quotient group of $\mathbb{Z}^{p-1}$, hence its growth is bounded by $n^{p-1}$. Since $H$ is a normal subgroup of $G$ of index 2, the two groups have the same growth. 

Let $\Gamma = \partial P$, and let $\Gamma_1, \Gamma_2, \ldots$ be the canonical sequence of graphs; see Section 1. Let $\gamma_n$ be the set of edges of $\Gamma_n \setminus \Gamma_{n-1}$

Lemma 8 The first difference of the sequence $|\gamma_n|$ is bounded by $n^{p-1}$.

Proof. The edges of $\gamma_{n+1}$ are obtained from the edges of $\gamma_n$ by applying the inverse map $T^{-1}$. Each time a singularity half-line of $T^{-1}$ intersects an edge of $\gamma_n$, this edge splits into two, and thus contributes 1 to $|\gamma_{n+1}| - |\gamma_n|$.

Let $L_n$ be the set of straight lines obtained by reflecting at most $n$ times in the vertices of $P$ the extensions of the sides of $P$. By Lemma 7, $|L_n| \ll n^{p-1}$. Each of these lines intersects a singularity half-line of $T^{-1}$ at most once, therefore the total number of intersections of the lines in $L_n$ with the singularity half-lines of $T^{-1}$ is bounded above by $n^{p-1}$. The edges of $\gamma_n$ belong to the lines from $L_n$, therefore the total number of intersections of these edges with the singularity half-lines of $T^{-1}$ is bounded above by $n^{p-1}$. (Note that the number of the edges of $\gamma_n$ could be bigger.)

We will now obtain the desired bound on complexity, i.e., we will estimate the number of faces of $\Gamma_n$. Denote by $|F_n|, |E_n|, |V_n|$ the number of faces, edges, vertices of the graph $\Gamma_n$ respectively. By Lemma 8, growth of the second difference of the sequence $|E_n|$ is at most polynomial of degree

\footnote{We conjecture that there is a universal quadratic lower bound.}
$p - 1$, hence $|E_n| \prec n^{p+1}$. The edges of $\Gamma_n$ are parallel to the sides of $P$, thus may have at most $p$ possible directions. Therefore, each face of $\Gamma_n$ is at most a $2p$-gon, and the valence of each vertex of $\Gamma_n$ is at most $2p$. Thus, $|E_n| \leq p|F_n|$, $|E_n| \leq p|V_n|$. Euler’s formula $|V_n| - |E_n| + |F_n| = 0$ implies $p|F_n| \leq (p - 1)|E_n|$, hence $|F_n| \prec |E_n|$.

We have obtained our bound, assuming that the rank of the abelian group generated by the sides of $P$ is $p - 1$, i.e., maximal possible. Although generically this is the case, the rank may drop. Our argument proves, in fact, the statement below.

**Corollary 3** Let $P$ be a convex euclidean $p$-gon, and let $r \leq p - 1$ be the rank of the abelian group generated by translations in the sides of $P$. Then the complexity of the outer billiard about $P$ is bounded by $n^{r+2}$.

A polygon is *rational* if the rank above is 2. Rational polygons are dense in the space of all polygons. We will study complexity of the outer billiard about a rational polygon. First, we recall preliminaries.

![Examples of polygons](image.png)

**Figure 8:** Outer billiard; examples of polygons $P$ and $Q$

We regard the plane as a vector space, with the center in the interior of the convex $p$-gon $P$. A well known construction [16] associates with $P$ a homothetic family of centrally symmetric convex polygons with at most (resp. exactly) $2p$ sides (resp. if $P$ is a generic $p$-gon). Let $Q$ be a particular polygon in this family. Each of its sides is parallel to a diagonal of $P$. See figure [8]. We endow the plane with a Minkowski norm such that $Q$ is the unit
The vector norm $|\cdot|$, radius, etc, will be understood with respect to it. We set $Q(r) = r \cdot Q$.

The polygon $Q$ determines the geometry of orbits of $T^2$ “at infinity” \[^{16}\]. We will elaborate. Let $x$ be a point in the plane which is sufficiently far from the origin. Let $Q_x$ be the circle centered at the origin and passing through $x$. Let $a \subset Q_x$ be the side containing $x$, and let $d$ be the corresponding diagonal of $P$ (parallel to $a$). Then $T^2$ translates $x$ along $a$ by $2|d|$; this continues until the orbit of $x$ overshoots $a$. Let $y = T^{2m}x$ be the corresponding point. Then the recipe above is applied to $y$, etc. See figure 9.

Figure 9: Second iteration of the outer billiard map “at infinity”

Let $a$ be an arbitrary side of $Q$, and let $d$ be the corresponding diagonal of $P$. The polygon $P$ is quasirational if, up to a common factor, the $p$ numbers $|a|/|d|$ are rational.

**Theorem 7** Let $P$ be a rational polygon, and let $f(\cdot)$ be the complexity of the outer billiard about $P$. Then $f(n) \sim n^2$.

**Proof.** Every rational polygon is quasirational. By a construction of R. Kolodziej \[^{14}\], there is a nested sequence of $T$-invariant polygonal simply connected domains $\cdots \subset U_i \subset U_{i+1} \subset \cdots$ exhausting the plane.\[^{18}\] By \[^{14}\], this construction was used in \[^{14}\] to prove the boundedness of all outer billiard orbits about quasirational polygons. If this is the case for arbitrary $p$-gons remains an open question for $p \geq 4$.\[^{16}\]

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\[^{16}\] This construction was used in \[^{14}\] to prove the boundedness of all outer billiard orbits about quasirational polygons. If this is the case for arbitrary $p$-gons remains an open question for $p \geq 4$.\[^{16}\]
there exists a constant $C = C(P) > 0$ such that the Kolodziej domains satisfy $Q(Ci) \subset U_i \subset Q(C(i+1))$. We will need a general lemma.

**Lemma 9** Let $P$ be an arbitrary convex polygon, and let $f(\cdot)$ be the complexity of the outer billiard about $P$. There exists $C_1 > 0$ such that the contribution to $f(n)$ of the exterior of the disc of radius $C_1 n$ grows linearly.

**Proof.** We will use the preceding notation and terminology. For any side $a$ of $Q$ set $r_a = |a|/|d|$. Let $C_1 > 2/r_a$ for all sides of $P$.

Consider the $T^2$-orbit of length $n$ of an arbitrary point $x$ outside of $Q(C_1 n)$. It follows a side, $a$, of $Q$ for $k \leq n$ iterations, then it “jumps” to the adjacent side, $a'$, and follows it for $n-k$ iterations. Counting the possibilities (and assuming that $Q$ is a $2p$-gon, i.e., that we are in the generic situation) we obtain $2p(n+1)$ types of $T^2$-orbits of length $n$. But different types mean different contributions to $f(2n)$, and vice versa.

Since $P$ is a rational polygon, the group $G \subset Iso(R^2)$ is discrete. The graphs $\Gamma_n$ are obtained from a finite collection of half-lines by $G$-action, hence $\Gamma_\infty = \cup_{n \geq 1} \Gamma_n$ belongs to a discrete collection of lines. Therefore $\Gamma_\infty$ is a graph, and the sequence $\Gamma_1 \subset \cdots \subset \Gamma_n \subset \cdots$ stabilizes on compacta. Moreover, there is a finite collection of convex polygons, such that every face of $\Gamma_\infty$ is congruent to a polygon in this collection. Hence the areas of the faces of $\Gamma_\infty$ are bounded away from zero and infinity.

Note that the constant $C_1$ in Lemma 9 can be chosen arbitrarily large. We choose it so that $C_1^2 = \tau \in \mathbb{N}$. Then for all $n$ sufficiently large

$$Q(C_1 n) \subset U_\tau \subset Q(C_1 n + C). \quad (37)$$

By Lemma 9 up to a linear term, $f(n)$ is the number of faces of $\Gamma_n$ intersecting $Q(C_1 n)$. By the left inclusion in (37), this is less than or equal to the number of faces of $\Gamma_\infty$ in $U_\tau$. By preceding remarks, there is $C_2 > 0$ such that that number is bounded by $C_2 \text{area}(U_\tau)$. By the right inclusion in (37), $\text{area}(U_\tau)$ is quadratic in $n$. We have obtained the upper bound $f(n) \prec n^2$.

Now for the lower bound. All regular points in $X$ are periodic [14, 12]. A face $F \subset X$ of $\Gamma_k$ is stable if $F$ is a face of $\Gamma_\infty$. Let $V_n \subset X$ be the set of points with period at most $n$. Each connected component of $V_n$ is an open, stable face of $\Gamma_n$. By remarks above, the number of connected components of $V_n$ has the same growth as the area of $V_n$, thus $\text{area}(V_n) \prec f(n)$. By
The following proposition is used in the proof of Theorem 7. It is also of independent interest. If $g, h$ are positive functions on $Y \subset \mathbb{R}^2$, the notation $g \prec h$ means that $g(x)/h(x)$ is bounded as $|x| \to \infty$. The notation $g \sim h$ means that $g \prec h, h \prec g$.

Proposition 9 Let $P$ be a convex polygon and let $X_{\text{per}} \subset X$ be the set of periodic points of the outer billiard. For $x \in X_{\text{per}}$ let $p(x)$ be the period.

1. We have $|x| \prec p(x)$.
2. Let $P$ be a rational polygon. Then for all regular points $p(x) \sim |x|$. Let $V_n$ be the set of points such that $p(x) \leq n$. Then $\text{area}(V_n) \sim n^2$.

Proof. We assume without loss of generality that $p(x) = 2m$. Let $Q_x$ be the circle through point $x$. The sequence $x, T_2(x), \ldots$ roughly follows $Q_x$. To come back to $x$, the sequence has to go around $Q_x$ at least once. Let $\delta$ be the “largest step” of $T_2$. Then we need at least $\text{perimeter}(Q_x)/\delta$ steps to return. Since $\text{perimeter}(Q_x) \sim |x|$, the first claim follows.

Let now $P$ be rational, and hence quasirational polygon, and let $U_k, k \geq 1$, be the Kolodziej domains. Let $k = k(x)$ be such that $x \in U_k \setminus U_{k-1}$. The relations $Q(Ck) \subset U_k \subset Q(C(k+1))$ imply that the function $k(x)$ satisfies $k(x) \sim |x|$. By inclusion $U_k \setminus U_{k-1} \subset Q(C(k+1)) \setminus Q(C(k-1))$, we have $\text{area}(U_k \setminus U_{k-1}) \sim |x|$. The point $x$ belongs to a unique face, $F = F(x)$, of $\Gamma_\infty$, hence $p(x)\text{area}(F) \leq \text{area}(U_k \setminus U_{k-1})$. By preceding remarks, $p(x) \prec \text{area}(U_k \setminus U_{k-1})$, implying $p(x) \prec |x|$, and hence the equivalence $p(x) \sim |x|$.

By this relation, there are constants $C_3, C_4 > 0$ such that for $n$ sufficiently large $Q(C_3n) \subset V_n \subset Q(C_4n)$, proving the last claim.

4.2 The elliptic and the hyperbolic cases

We will first study the outer billiard in elliptic geometry.

Theorem 8 Let $P \subset S^2$ be a convex spherical polygon. The complexity of outer billiards about $P$ grows subexponentially.

Proof. For $x \in S^2$ let $l = l^*$ be the appropriately oriented great circle centered at $x$. This diffeomorphism $S^2 \to \text{Geo}(S^2)$ is the spherical duality, and we denote by $x = l^*$ the inverse diffeomorphism.
Let $A, B, \ldots$ be the vertices of $P$. The geodesics $a = A^*, b = B^*, \ldots$ bound the convex polygon $P^*$. The correspondence $P \mapsto P^*$ is an automorphism of the space of convex spherical polygons. The proof of the following lemma is contained in [17]. (See also [16].)

**Lemma 10** Let $P, P^* \subset S^2$ be as above. Let $X_o, X_b$ be the phase spaces of the outer billiard about $P$, inner billiard about $P^*$, and let $T_o : X_o \rightarrow X_o, T_b : X_b \rightarrow X_b$ be the outer billiard, inner billiard maps respectively.

The spherical duality induces a diffeomorphism $X_o \rightarrow X_b$; it conjugates $T_o : X_o \rightarrow X_o$ and $T_b : X_b \rightarrow X_b$; it induces an isomorphism of the coding of $T_o$-orbits by corners of $P$ and the coding of $T_b$-orbits by sides of $P^*$.

Figure 10 illustrates Lemma 10. Let $f_o(n)$ (resp. $f_b(n)$) be the corner complexity of the outer billiard about $P$ (resp. billiard in $P^*$). By Lemma 10 $f_o(n) = f_b(n + 1)$. The claim now follows from Theorem 4.

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We refer to [6] for the background on polygonal outer billiards in the hyperbolic plane. Let $P \subset H^2$ be a $p$-gon, let $X = H^2 \setminus P$, and let $T : X \rightarrow X$ be the outer billiard map. It extends to a homeomorphism of the circle at infinity, $\tau : S \rightarrow S$. Its rotation number satisfies $\rho(P) \geq 1/p$ [6]. The polygon $P$ is large if $\rho(P) = 1/p$ and $\tau$ has a hyperbolic $p$-periodic orbit. See figure 11. The set of large polygons is open in the natural topology [6].

**Theorem 9** Let $P \subset H^2$ be an arbitrary convex polygon, and let $f(\cdot)$ be the complexity of the outer billiard about $P$. Then $n \prec f(n)$. If $P$ is a large polygon, then $f(n) \sim n$.

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Figure 11: A large quadrilateral

**Proof.** The bound $n < f(n)$ fails iff the sequence $\Gamma_k$, $k \geq 1$, stabilizes. Assume this to be the case, and let $\Gamma_m = \Gamma_{m+1} = \cdots = \Gamma_\infty$. The outer billiard map $T : X \rightarrow X$ preserves $\Gamma_\infty$; the restriction of $T$ to a closed face of $\Gamma_\infty$ is a diffeomorphism onto another one. Since $\Gamma_\infty$ is a finite graph, we find $n \in \mathbb{N}$ such that every face of $\Gamma_\infty$ is invariant under $T^n$.

Let $F$ be a closed face of $\Gamma_\infty$. Then $\partial F \cap S$ is either empty, or a vertex, or an edge of $F$. We will study the latter. Let $v_1, \ldots, v_N \in S$ be the consecutive endpoints of these edges, let $e_i \subset S$ (resp. $\alpha_i \subset \mathbb{H}^2$) be the circular arc (resp. the geodesic) with endpoints $v_i, v_{i+1}$ (we set $N + 1 = 1$), and let $F_i$ be the corresponding face of $\Gamma_\infty$. The restriction $T^n|_{F_i}$ is induced by an isometry, $g_i \in Iso(\mathbb{H}^2)$. The elements $g_1, \ldots, g_N$ are all equal to the identity iff $\tau^n = 1$.

**Lemma 11** The map $\tau : S \rightarrow S$ is not periodic.

**Proof.** Let $z$ be a vertex of $P$. For close points $x_1, y_1 \in S$ let $x_2, y_2 \in S$ be their reflections about $z$. Let $\lambda_1 = |x_2z|/|x_1z|$ and let $2\alpha_i$ be the angular measure of the arc $x_iy_i$, $i = 1, 2$; see figure 12. The triangles $x_1z y_1$ and $x_2z y_2$ are similar, therefore

$$\sin \alpha_2 = \lambda_1 \sin \alpha_1. \quad (38)$$

Let $x_1, \ldots, x_N$ be a periodic trajectory of the map $\tau$ consisting of smooth points, and let $\lambda_1 \ldots, \lambda_N$ be the respective ratios. Set $\Lambda = \prod_{i=1}^N \lambda_i$. Let
Figure 12: Computing the distortion of the map \( \tau \)

Let \( y_1 \) be a point sufficiently close to \( x_1 \), and let \( y_1, \ldots, y_N \) be its \( \tau \)-orbit; we assume that for both orbits the reflections occur in the same vertices of \( P \).

It follows from equation 38 that \( y_1, \ldots, y_N \) is a periodic trajectory iff \( \Lambda = 1 \).

In particular, if \( \tau \) has a periodic interval, then \( \Lambda = 1 \) there.

Let now \( x_1 \) cross counter-clockwise a singularity half-line of \( T \). In the notation of figure 13, \( \lambda_1 = (b + c)/a \) (resp. \( \lambda_1 = c/(a + b) \)) right before (resp. after) this. By \((b + c)/a > c/(a + b)\), the equality \( \Lambda = 1 \) before a singularity half-line implies that \( \Lambda < 1 \) immediately after it.

By Lemma 11 we can assume without loss of generality that \( g_1 \neq 1 \). Then \( g_1 \) is a (hyperbolic) parallel translation with the axis \( \alpha_1 \), and \( F_1 \) is the domain bounded by \( \alpha_1 \) and \( e_1 \). We will say that \( F_1 \) is a lunar face of \( \Gamma_\infty \). The union of lunar faces of \( \Gamma_\infty \) is invariant under \( T \). Therefore for any \( k > 0 \) there is \( l = l(k) \) such that \( T^{-k}(\alpha_1) = \alpha_l \). A geodesic \( \alpha_i \), \( 1 \leq i \leq N \), cannot contain a side of \( P \). If it does, then \( F_i \) contains a singular line of \( T \) in its interior, contrary to the definition of \( F_i \). See figure 13 where \( x_1x_2 \) represents now the geodesic \( \alpha_1 \). Thus, \( \alpha_1 \) is not an edge of \( \Gamma_m \) for any \( m \).
This contradiction proves our first claim.

Let now $P$ be a large $p$-gon. Then $\Gamma_n$ is a disjoint union of $p$ binary trees (see figure 14), and hence $|\Gamma_n|$ grows linearly.

Remark 4 Note that the function $f(\cdot)$ is bounded below by the complexity of the induced map $\tau : S \to S$ with respect to the natural partition. However, the latter may be finite. See figure 15.

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Figure 14: The graph $\Gamma_2$ for a large triangle

Figure 15: Finite complexity of the outer billiard map at infinity
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