AN EXTREMAL EIGENVALUE PROBLEM IN KÄHLER GEOMETRY

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ABSTRACT. We study Laplace eigenvalues \( \lambda_k \) on Kähler manifolds as functionals on the space of Kähler metrics with cohomologous Kähler forms. We introduce a natural notion of a \( \lambda_k \)-extremal Kähler metric and obtain necessary and sufficient conditions for it. A particular attention is paid to the \( \lambda_1 \)-extremal properties of Kähler-Einstein metrics of positive scalar curvature on manifolds with non-trivial holomorphic vector fields.

1. INTRODUCTION

1.1. Motivation. Let \( M \) be a closed manifold. For a Riemannian metric \( g \) on \( M \) we denote by

\[
0 = \lambda_0(g) < \lambda_1(g) \leq \ldots \leq \lambda_k(g) \leq \ldots
\]

the eigenvalues of the Laplace-Beltrami operator \( \Delta_g \) repeated according to their multiplicity. In real dimension 2, by classical results of Hersch [22], Yang and Yau [42], and Li and Yau [31], the first eigenvalue \( \lambda_1(g) \) is bounded when the Riemannian metric \( g \) ranges over metrics of fixed volume. A basic question is: for a given conformal class \( c \) on \( M \), is there a metric that maximizes \( \lambda_1(g) \) among metrics \( g \in c \) with \( \text{vol}(M, g) = 1 \)? What are its properties? When \( M \) has a zero genus, the answers go back to classical results of Hersch [22] and Li and Yau [31]. For higher genus surfaces this circle of questions have been studied extensively in the last decades, see [13, 14, 15, 23, 24, 33] and the most recent papers [30, 34, 35].

In particular, Nadirashvili and Sire [34], developing earlier ideas by Nadirashvili [33], have stated an existence theorem for \( \lambda_1 \)-maximizers in conformal classes along with an outlined proof. Later, it has been improved by Petrides [35], who has also given a rigorous argument for the statement, using previous work [18, 30]. Mention that the \( \lambda_1 \)-maximizers given by the above existence theorems may have conical singularities, and can be described as metrics that admit harmonic maps into round spheres by their first eigenfunctions. The latter statement actually holds for arbitrary \( \lambda_1 \)-maximizers (and even \( \lambda_k \)-extremals for any \( k \geq 1 \)) with conical singularities, see [30], where much more general statements in this direction have been proven. Besides, the combination of [35, Theorem 1] and [30, Theorem E1] shows that the set of all conformal smooth metrics with conical singularities that maximize \( \lambda_1 \) is compact, see also [29].

1.2. Eigenvalue problems on Kähler manifolds. The purpose of this paper is to describe an extremal eigenvalue problem in higher dimensions on a Kähler manifold \( (M, J, g, \omega) \), which generalizes the described extremal problem on Riemannian surfaces. Here we view the eigenvalue \( \lambda_k(g) \) as a functional on the space \( \mathcal{K}_\Omega(M, J) \) that is formed by Kähler metrics \( g \) whose Kähler forms \( \omega \) represent a given de Rham cohomology class \( \Omega \). When \( (M, J, g) \) is a Riemannian surface with a volume form \( \omega \), the space \( \mathcal{K}_{\omega}(M, J) \) is precisely the set of metrics of a fixed area that are conformal to \( g \), see the discussion in Sect. 2. By a classical work due to Bourguignon, Li, and Yau [35], the first eigenvalue \( \lambda_1(g) \) is bounded on \( \mathcal{K}_\Omega(M, J) \) when \( (M, J) \) is projective and

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the de Rham class $\Omega$ belongs to $H^2(M,\mathbb{Q})$. The eigenvalue bound in [8] shows that the Fubini-Study metric on $\mathbb{C}P^n$ is a $\lambda_1$-maximizer in its Kähler class. Recently, these results have been generalized by Arezzo, Ghigi, and Loi [6] to more general Kähler manifolds that admit holomorphic stable vector bundles over $M$ with sufficiently many sections. In particular, they show that the symmetric Kähler-Einstein metrics on the Grassmannian spaces are also $\lambda_1$-maximizers in their Kähler classes. Moreover, as is shown in [7], so are symmetric Kähler-Einstein metrics on Hermitian symmetric spaces of compact type. These results motivate a further investigation of spectral geometry of Kähler-Einstein metrics of positive scalar curvature; see [9, 40] for the general existence theory of such metrics.

Following the ideas in [33] [16] for other extremal eigenvalue problems, we introduce the notion of a $\lambda_k$-extremal Kähler metric under the deformations in its Kähler class. The class of such extremal metrics contains $\lambda_k$-maximizers in $X_M^{\Omega}(M,J)$. We assume that the metrics under consideration are always smooth, and show that for a $\lambda_k$-extremal Kähler metric there exists a collection of non-trivial $\lambda_k$-eigenfunctions $f_1, \ldots, f_\ell$ such that

$$
\lambda_k^2 \left( \sum_{i=1}^{\ell} f_i^2 \right) - 2\lambda_k \left( \sum_{i=1}^{\ell} |\nabla f_i|^2 \right) + \sum_{i=1}^{\ell} |dd^c f_i|^2 = 0.
$$

For the first eigenvalue the latter hypothesis is also sufficient for a metric to be $\lambda_1$-extremal. This statement shows that for a $\lambda_k$-extremal Kähler metric $g$ the eigenvalue $\lambda_k(g)$ is always multiple (Corollary 2.2), and allows to produce examples of $\lambda_1$-extremal metrics as products.

We proceed with considering in more detail the case when $(M,J)$ is a complex Fano manifold with non-trivial holomorphic vector fields. Recall that, by a classical result of Matsushima [32], for a Kähler-Einstein metric $g$ on $(M,J)$ the $\lambda_1$-eigenfunctions are potentials of Killing vector fields. Using this fact, we show that a Kähler-Einstein metric $g$ is $\lambda_1$-extremal if and only if there exist non-trivial Killing potentials $f_1, \ldots, f_\ell$ such that the function $\sum f_i^2$ is also a Killing potential. As an application, we conclude that a toric Fano Kähler-Einstein manifold whose connected group of automorphisms is a torus is not $\lambda_1$-extremal. For example, so is a Kähler-Einstein metric on $\mathbb{C}P^2$ blown up at three points in a generic position, see [38, 39]. As another example, we show that Kähler-Einstein manifolds different from $\mathbb{C}P^n$ that admit hamiltonian 2-forms of order $\geq 1$ are also never $\lambda_1$-extremal. This conclusion applies to the non-homogeneous Kähler-Einstein metrics from [26, 27, 28].

2. AN EXTREMEIGENVALUE PROBLEM

2.1. Statement of the problem. Let $(M,g,J,\omega)$ be a compact Kähler manifold of real dimension $n = 2m$. Recall that due to the $\bar{\partial}\partial$-lemma any Kähler metric $\check{g}$ whose Kähler form $\check{\omega}(\cdot,\cdot) = \check{g}(J\cdot,\cdot)$ is co-homologous to $\omega$ has the form $\omega + dd^c \varphi$, where $d^c = JdJ^{-1} = i(\partial - \bar{\partial})$ and the action of $J$ on the cotangent bundle is defined via the duality with respect to $g$. The smooth function $\varphi$ above is determined uniquely by the condition

$$
\int_M \varphi v_g = 0, \quad \text{where } v_g = \omega^m/m!
$$

is the Riemannian volume form on $M$. By $X^{\Omega}(M,J)$ we denote the space of Kähler metrics on $(M,J)$ whose Kähler forms represent a given de Rham cohomology class $\Omega$. For a representative Kähler form $\omega$ it can be identified with the space of functions

$$
\{ \varphi \in C^\infty(M) : \omega + dd^c \varphi > 0, \int_M \varphi v_g = 0 \}.
$$
Besides, for any function \( \varphi \) with a zero mean-value there exists \( \varepsilon > 0 \) such that \( \omega + tdd^c \varphi \in \mathcal{K}_\Omega(M, J) \) for all \( |t| < \varepsilon \), and the space \( C^0_\varepsilon(M) \), formed by such functions \( \varphi \), can be thought as the tangent space at \( \tilde{g} \in \mathcal{K}_\Omega(M, J) \).

For a Kähler metric \( g \) on \((M, J)\) we denote by

\[
0 = \lambda_0(g) < \lambda_1(g) \leq \lambda_2(g) \leq \ldots \leq \lambda_k(g) \leq \ldots
\]

the eigenvalues of the Laplace–Beltrami operator \( \Delta_g = \delta d \) acting on functions, where \( \delta \) is the \( L_2 \)-adjoint of the exterior derivative \( d \) with respect to \( g \). Recall that due to the standard Kato–Rellich perturbation theory the functions \( t \mapsto \lambda_k(g_t) \) have left and right derivatives for analytic deformations \( g_t \). We view the eigenvalues \( \lambda_k(g) \) as functionals on the space \( \mathcal{K}_\Omega(M, J) \) and introduce the following definition.

**Definition 2.1.** A Kähler metric \( g \in \mathcal{K}_\Omega(M, J) \) is called \( \lambda_k \)-extremal, if for any analytical deformation \( g_t \in \mathcal{K}_\Omega(M, J) \) with \( g_0 = g \) the following relation holds

\[
\frac{d}{dt}\bigg|_{t=0^-} \lambda_k(g_t) \leq \frac{d}{dt}\bigg|_{t=0^+} \lambda_k(g_t) < 0. \tag{2.1}
\]

It is straightforward to see that a Kähler metric is \( \lambda_k \)-extremal if and only if either the inequality

\[
\lambda_k(g_t) \leq \lambda_k(g) + o(t) \quad \text{as} \quad t \to 0,
\]

or the inequality

\[
\lambda_k(g_t) \geq \lambda_k(g) + o(t) \quad \text{as} \quad t \to 0
\]

occurs. In particular, we see that any \( \lambda_k \)-maximizer in \( \mathcal{K}_\Omega(M, J) \) is \( \lambda_k \)-extremal. The following two remarks are consequences of the results in [16]. First, for the first eigenvalue only the first of the above inequalities may occur. Second, for a deformation \( \omega_t = \omega + dd^c \varphi_t \) the validity or the failure of relation \((\text{2.1})\) depends only on the function \( \dot{\varphi} = (d/dt)|_{t=0} \varphi_t \), and hence, in Definition \((\text{2.1})\) we may consider only deformations with \( \varphi_t = t \varphi \), where \( \varphi \in C^0_\varepsilon(M) \).

In the sequel we use the fourth order differential operator \( L(f) \) defined as \( \delta^c \sigma (\sigma d \sigma f) \), where \( \delta \) and \( \delta^c \) stand for the \( L_2 \)-adjoints of \( d \) and \( d^c \) respectively. Recall that they satisfy the relations

\[
\delta \psi = - \sum_{i=1}^{2m} t_{e_i} \langle D_{e_i} \psi \rangle \quad \text{and} \quad \delta^c \psi = - \sum_{i=1}^{2m} t_{Je_i} \langle D_{Je_i} \psi \rangle,
\]

where \( D \) is the Levi-Civita connection of \( g \), and \( \{e_1, Je_1, \ldots, e_m, Je_m\} \) is a \( J \)-adapted orthonormal frame. We then calculate

\[
L(f) = \delta^c \left( - (dd^c f)(df^2, \cdot) + fd^c \delta df \right)
\]

\[
= \sum_{i=1}^{2m} t_{e_i} \langle D_{e_i}(dd^c f)(df^2, \cdot) \rangle - \sum_{i=1}^{2m} t_{Je_i}(dd^c f)(D_{Je_i} df^2, \cdot) + f \delta^c d^c \delta df - (df, d\Delta f)
\]

\[
= (\delta^c dd^c f, df) + (dd^c f, dd^c f) + f \Delta^2_g f - (\Delta f, df, df)
\]

where we used the facts that \( dd^c f \) is \( J \)-invariant and that \( J \) commutes with \( D \) and \( \Delta_g \) (when acting on 1-forms), as well as the standard identities between the operators \( d, d^c, \delta, \delta^c \), and \( \Delta_g \).

The proof of the following statement is close in the spirit to the arguments in [33] [16] and is given at the end of the section.
Theorem 2.1. Let $g \in \mathcal{K}_\Omega(M,J)$ be a $\lambda_k$-extremal Kähler metric. Then there exists a finite collection $f_1, \ldots, f_l$ of non-trivial eigenfunctions corresponding to $\lambda_k(g)$ such that

$$
\sum_{i=1}^l L(f_i) = \lambda_k^2 \left( \sum_{i=1}^l f_i^2 \right) - 2\lambda_k \left( \sum_{i=1}^l |\nabla f_i|^2 \right) + \sum_{i=1}^l |\ddbar f_i|^2 = 0.
$$

(2.3)

For $k = 1$ the existence of such a collection of eigenfunctions is sufficient for a Kähler metric $g$ to be $\lambda_1$-extremal.

Considering maximal and minimal values of an eigenfunction $f$, it is straightforward to see that the operator

$$
L(f) = \lambda_k^2 f^2 - 2\lambda_k |\nabla f|^2 + |\ddbar f|^2
$$

has a trivial kernel on $\lambda_k$-eigenspaces for $k \geq 1$. Thus, we obtain the following corollary.

Corollary 2.2. Let $g \in \mathcal{K}_\Omega(M,J)$ be a $\lambda_k$-extremal Kähler metric. Then the eigenvalue $\lambda_k(g)$ is multiple.

As another consequence of Theorem 2.1 we see that the notion of $\lambda_1$-extremality behaves well under products.

Corollary 2.3. Let $g \in \mathcal{K}_\Omega(M)$ be a $\lambda_1$-extremal Kähler metric on $(M,J)$. Then for any Kähler metric $g'$ on $(M',J')$ such that $\lambda_1(g') \geq \lambda_1(g)$ the product metric $g \times g'$ is $\lambda_1$-extremal along deformations in its Kähler class on $(M,J) \times (M',J')$.

2.2. Discussion and basic examples. We proceed with considering the case when $m = 1$, that is when $M$ is an oriented Riemann surface. Let $g$ be a Riemannian metric and $\omega = \nu_g$ be its volume form. It is straightforward to see that for any smooth function $\phi \in C^\infty(M)$ the hypothesis $\omega + \ddbar \phi > 0$ holds if and only if $1 - \Delta_g \phi > 0$. Thus, the space $\mathcal{K}_{\omega}(M,J)$ can be defined as

$$
\mathcal{K}_{\omega}(M,J) = \{ \phi \in C^\infty(M) : 1 - \Delta_g \phi > 0, \int_M \phi \nu_g = 0 \}.
$$

The following lemma is elementary; we state it for the convenience of references.

Lemma 2.4. Let $(M,g)$ be a Riemannian surface, and $\omega$ be its volume form. Then the space of Kähler metrics in $\mathcal{K}_{\omega}(M,J)$ coincides with the space of Riemannian metrics $\tilde{g}$ that are conformal to $g$ and such that $\text{vol}(M,\tilde{g}) = \text{vol}(M,g)$.

Proof. Let $\varphi$ be a function from the above space $\mathcal{K}_{\omega}(M,J)$. Then the metric $\tilde{g} = (1 - \Delta_g \varphi)g$ is clearly has the same volume as the metric $g$. Conversely, for a given conformal metric $\tilde{g} = e^\sigma g$ of the same volume as $g$ the equation

$$
e^\sigma = 1 - \Delta_g \varphi$$

has a unique solution $\varphi \in C^\infty(M)$ with zero mean-value, see [20] for standard existence results for solutions of elliptic equations.

As a consequence of the lemma above, we see that a Riemannian metric $g$ on $M$ is $\lambda_k$-extremal in the sense of Definition 2.1 if and only if it is $\lambda_k$-extremal under volume preserving conformal deformations. There has been constructed a variety of examples of $\lambda_1$-extremal and $\lambda_1$-maximal metrics, and as is known [34] [35], every conformal class on a closed surface contains a $\lambda_1$-maximizer, which may have conical singularities. Thus, using Corollary 2.3, we obtain a variety of examples of $\lambda_1$-extremal Kähler metrics on the products of Riemann surfaces and Kähler manifolds.
Recall that by [14, 16] for a metric $g$ that is $\lambda_k$-extremal under the volume preserving conformal deformations there exists a collection $f_1, \ldots, f_t$ of non-trivial $\lambda_k$-eigenfunctions such that $\sum f_i^2 = 1$. When $m = 1$, we see that the latter condition coincides with the necessary condition given by Theorem 2.1. Indeed, in this case the operator $L(f)$ takes the form

$$L(f) = 2(\lambda_k^2 f^2 - \lambda_k |\nabla f|^2),$$

where we used the identities $(\Delta_g f) = -dd^c f$ and $|\omega|^2 = 1$. Now the relation

$$\Delta_g(\sum f_i^2) = 2(\lambda_k \sum f_i^2 - \sum |\nabla f_i|^2) = \lambda_k^{-1} \sum L(f_i)$$

implies the claim. More generally, in higher dimensions $m \geq 2$ the hypothesis $\Sigma L(f_i) = 0$ in Theorem 2.1 is equivalent to the relation

$$\Delta_g(\sum f_i^2) = \lambda_k^{-1} m(m - 1)(\sum dd^c f_i \wedge dd^c f_i) \wedge \omega_1^{m-2}/\omega^m.$$

Here we used the fact that $(\Delta_g f)^2 - |dd^c f|^2 = m(m - 1)dd^c f \wedge dd^c f \wedge \omega_1^{m-2}/\omega^m$, see identity (2.8) in the proof of Theorem 2.1 below.

2.3. Proof of Theorem 2.1. We start with the following lemma.

**Lemma 2.5.** Let $g \in \mathcal{K}(M, J)$ be a $\lambda_k$-extremal Kähler metric, and $E_k$ be an eigenspace for $\lambda_k(g)$. Then for any function $\varphi \in C_0^\infty(M)$ the quadratic form

$$Q_\varphi(f) = \int_M \varphi L(f)v_g$$

is indefinite on $E_k$. For $k = 1$ the hypothesis that the form $Q_\varphi$ is indefinite for any $\varphi \in C_0^\infty(M)$ is also sufficient for a Kähler metric $g$ to be $\lambda_k$-extremal.

**Proof.** For any $\varphi \in C_0^\infty(M)$ consider the Kähler deformation $\omega_t = \omega + tdd^c \varphi$, defined for a sufficiently small $|t|$. By [16, Theorem 2.1] for a proof of the lemma it is sufficient to show that the form $Q_\varphi$ satisfies the relation

$$Q_\varphi(f) = \int_M f(\Delta_g f)v_g,$$

where $\Delta_g f$ stands for the value $(d/dt)|_{t=0}(\Delta_g f)$. First, we claim that the operator $\Delta_\varphi$ satisfies the identity

$$\Delta_\varphi f = (dd^c f, dd^c \varphi).$$

Indeed, differentiating the relation

$$m \cdot dd^c f \wedge \omega_1^{m-1} = (-\Delta_g f) \omega_1^m,$$

we obtain

$$dd^c f \wedge dd^c \varphi \wedge \omega_1^{m-2}/(m - 2)! = (\Delta_g f)(\Delta_g \varphi) - \Delta_\varphi f) \omega_1^m/m!.$$

Denote by $\wedge^{1,1}(M)$ the bundle of real $(1, 1)$-forms on the manifold $(M, J)$, that is 2-forms $\psi$ such that $\psi(J \cdot, J \cdot) = \psi(\cdot, \cdot)$. For a given Kähler metric $(g, \omega)$ on $(M, J)$, it decomposes as a direct $g$-orthogonal sum

$$\wedge^{1,1}(M) = \mathbb{R} \omega \oplus \wedge_0^{1,1}(M)$$

(2.7)
of the irreducible $U(m)$-invariant subspaces of $(1,1)$-forms proportional to $\omega$ and primitive (trace-free) $(1,1)$-forms, respectively. Let us consider the symmetric $U(m)$-invariant bilinear form on $\Lambda^{1,1}(M)$ defined as

$$q(\phi, \psi) = (\text{tr}_\omega \phi)(\text{tr}_\omega \psi) - \frac{\phi \wedge \psi \wedge (\omega^{m-2}/(m-2)!)}{\omega^m/m!},$$

where

$$\text{tr}_\omega \psi = (\psi, \omega)_g = m \frac{\psi \wedge (\omega^{m-1}/(m-1)!)}{\omega^m/m!}.$$  

Since the form $q(\cdot, \cdot)$ leaves the two irreducible factors in the decomposition (2.7) orthogonal, by the Schur lemma we conclude that it is proportional to the induced Euclidean product $(\cdot, \cdot)_g$ on each factor in (2.7). Evaluating $q(\omega, \omega)$ and $q(\psi_0, \psi_0)$ with $\psi = \alpha \wedge J\alpha$ for a unitary 1-form $\alpha$, we see that in fact the form $q(\cdot, \cdot)$ coincides with $(\cdot, \cdot)_g$, that is

$$(\text{tr}_\omega \phi)(\text{tr}_\omega \psi) - \frac{\phi \wedge \psi \wedge (\omega^{m-2}/(m-2)!)}{\omega^m/m!} = (\phi, \psi)_g.$$  

Now combining the last relation with (2.6), we obtain identity (2.5). Using the latter we have

$$Q_\phi(f) = \int_M \phi L(f)v_g = \int_M f(dd^c f, dd^c \phi)v_g = \int_M f(\Delta_\phi f)v_g,$$

and thus, obtain relation (2.4). \hfill \Box

By Lemma 2.5 in order to prove the theorem it is sufficient to show that the quadratic form $Q_\phi(f)$ is indefinite on $E_k$ if and only if there exists a collection of eigenfunctions $f_1, \ldots, f_\ell \in E_k$ such that

$$(2.9) \quad \sum_{i=1}^\ell L(f_i) = 0 \quad \text{and} \quad \sum_{i=1}^\ell \int_M f_i^2 v_g = 1.$$  

Consider the convex subset

$$K = \left\{ \sum_{i=1}^\ell L(f_i) : f_i \in E_k, \sum_{i=1}^\ell \int_M f_i^2 v_g = 1 \right\}$$

in the space $L^2(M)$. We are going to show that the form $Q_\phi(f)$ is indefinite if and only if $0 \notin K$. Suppose that $Q_\phi(f)$ is indefinite for any $\phi \in C_0^\infty(M)$ and $0 \notin K$. Then by the Hahn–Banach separation theorem there exists a function $\psi \in L^2(M)$ and $\varepsilon > 0$ such that

$$\int_M \psi u v_g \geq \varepsilon > 0 \quad \text{for any} \ u \in K.$$  

Since the set $K$ lies in a finite-dimensional subspace, then choosing $\varepsilon > 0$ smaller, if necessary, by approximation we may assume that the function $\psi$ belongs to $C^\infty(M)$. Define $\psi_0$ as the zero mean-value part of $\psi$, that is

$$\psi_0 = \psi - \frac{1}{\text{vol}(M,g)} \int_M \psi v_g.$$  

Since the operator $L(f)$ takes values among zero mean-value functions, we obtain

$$Q_{\psi_0}(f) = \int_M \psi_0 L(f)v_g = \int_M \psi L(f)v_g > 0$$

for any non-trivial $f \in E_k$. Thus, we arrive at a contradiction with the assumption that the form $Q_\phi$ is indefinite for any $\phi \in C_0^\infty(M)$. 
Conversely, given a collection $f_1, \ldots, f_\ell \in E_k$ that satisfy relationships (2.9), we have
\[
\sum_{i=1}^{\ell} Q_\varphi(f_i) = \int_M \varphi \left( \sum_{i=1}^{\ell} L(f_i) \right) v_g = 0.
\]
Thus, the quadratic form $Q_\varphi(f)$ is indeed indefinite for any $\varphi \in C^0_c(M)$. \hfill \Box

3. Kähler–Einstein Manifolds with a Non-Trivial Automorphism Group

We now specialize the considerations to the case when $(g, J, \omega)$ is a Kähler–Einstein manifold with positive scalar curvature, that is the Ricci form $\rho$ is a positive constant multiple of the Kähler form $\omega$. Rescaling the metric, we may assume that $\rho = \omega$, or equivalently, the Kähler class is $\Omega = 2\pi c_1(M, J)$. Under this assumption, the scalar curvature $\text{Scal}_g$ equals $2m$.

As a Fano manifold, $(M, J)$ is simply-connected \cite{25, 43}, and hence, any real holomorphic vector field $X$ can be uniquely written in the form
\[
X = \text{grad}_g h_X + J\text{grad}_g f_X,
\]
where $h_X$ and $f_X$ are smooth functions with zero mean-values, see \cite{19} for details. The complex-valued function $h_X + if_X$ is called the holomorphy potential of $X$. For a Killing vector field $Y$, the above decomposition reduces to
\[
Y = J\text{grad}_g f_Y,
\]
and the corresponding function $f_Y \in C^0_c(M)$ is called the Killing potential of $Y$. The following statement is due to Matsushima \cite{32}, see also \cite{19} Ch. 3.

**Proposition 3.1.** Let $(M, g, J, \omega)$ be a compact Kähler–Einstein manifold with scalar curvature $\text{Scal}_g = 2m$. Then the Lie algebra $\mathfrak{h}(M)$ of real holomorphic vector fields on $(M, J)$ decomposes as the direct sum
\[
\mathfrak{h}(M) = \mathfrak{t}(M, g) \oplus J\mathfrak{t}(M, g),
\]
where $\mathfrak{t}(M, g)$ is the sub-algebra of Killing vector fields for $g$. Moreover, the algebra $\mathfrak{t}(M, g)$ is Lie algebra isomorphic to the space
\[
E_1(g) = \{ f \in C^0_c(M) : \Delta_g(f) = 2f \}
\]
equipped with the Poisson bracket of functions with respect to $\omega$, via the map $f \to J\text{grad}_g f$. Furthermore, the first eigenvalue satisfies the inequality $\lambda_1(g) \geq 2$.

As a consequence of the proposition above, we see that $\lambda_1(g) = 2$ if and only if the connected component of the identity $\text{Aut}_0(M, J)$ of the group of biholomorphic automorphisms of $(M, J)$ is non-trivial. In this case, we have the following statement.

**Theorem 3.2.** Let $(M, g, J, \omega)$ be a compact Kähler–Einstein manifold with scalar curvature $\text{Scal}_g = 2m$, and suppose that the connected component $\text{Aut}_0(M, J)$ of the automorphism group is non-trivial. Then the metric $g$ is $\lambda_1$-extremal in $\Omega = 2\pi c_1(M, J)$ if and only if there exist non-trivial $\lambda_1$-eigenfunctions $f_1, \ldots, f_\ell$ such that the zero mean-value part of the sum $\sum_{i=1}^{\ell} f_i^2$ is a (possibly trivial) $\lambda_1$-eigenfunction, that is
\[
(3.1) \quad \sum_{i=1}^{\ell} f_i^2 - \frac{1}{\text{vol}(M, g)} \int_M \left( \sum_{i=1}^{\ell} f_i^2 \right) v_g \in E_1(g).
\]
Proof. We show that the necessary and sufficient condition in Theorem 2.1 is equivalent to relation (3.1). First, note that by Proposition 3.1, we have \( \lambda_1(g) = 2 \), and any eigenfunction \( f \in E_1(g) \) is a Killing potential. The last fact implies that such an eigenfunction \( f \) satisfies \( Dd^c f = (1/2)dd^c f \), and thus, we obtain

\[
\Delta_{g} |df|^2 = 2 \left( (D^*D(d^c f), d^c f) - (D(d^c f), D(d^c f)) \right)
\]

\[
= \left( \delta d d^c f, d^c f \right) - (dd^c f, dd^c f) = (\Delta_g d^c f, d^c f) - (dd^c f, dd^c f)
\]

\[
= 2|df|^2 - |dd^c f|^2.
\]

Above we used the relation \( \Delta_g d^c f = 2d^c f \) and the fact that the tensor norm of \( dd^c f \) is twice its norm as a differential 2-form. On the other hand, we clearly have

\[
(3.2) \quad \Delta_g f^2 = 2f\Delta_g f - 2|df|^2 = 4f^2 - 2|df|^2.
\]

Combining the last two relations, for any eigenfunction \( f \in E_1(g) \) we obtain

\[
(3.3) \quad \Delta_g \left( f^2 - |df|^2 \right) = 4f^2 - 4|df|^2 + |dd^c f|^2.
\]

Now comparing (3.3) with the relation \( \sum L(f_i) = 0 \) in Theorem 2.1 we conclude that the metric \( g \) is \( \lambda_1 \)-extremal if and only if there exist \( \{ f_1, \ldots, f_\ell \} \in E_1(g) \) such that

\[
(3.4) \quad \sum_{i=1}^\ell (f_i^2 - |df_i|^2) = c
\]

for some constant \( c \). Integrating the last relation, we see that the constant \( c \) is the negated mean-value of the sum \( \sum_{i=1}^\ell f_i^2 \). Setting \( f_0 := \sum_{i=1}^\ell f_i^2 + c \), and using (3.2), we further obtain

\[
(3.5) \quad \Delta_g f_0 = \sum_{i=1}^\ell \left( 4f_i^2 - 2|df_i|^2 \right)
\]

\[
= 2f_0 + 2 \left( \sum_{i=1}^\ell (f_i^2 - |df_i|^2) - c \right) = 2f_0.
\]

Thus, relation (3.4) is in turn equivalent to the hypothesis that there exist non-trivial \( f_1, \ldots, f_\ell \in E_1(g) \) such that the zero mean-value part \( f_0 \) of the sum \( \sum_{i=1}^\ell f_i^2 \) is a \( \lambda_1 \)-eigenfunction itself. \( \square \)

Theorem 3.2 is a useful criterion for verifying whether the Kähler–Einstein metric on \((M,J)\) is \( \lambda_1 \)-extremal in \( 2\pi c_1(M,J) \). We demonstrate this in the corollaries below.

**Corollary 3.3.** Let \((M,g,J,\omega)\) be a compact homogeneous Kähler–Einstein manifold. Then the metric \( g \) is \( \lambda_1 \)-extremal metric within its Kähler class.

**Proof.** Let \( \{ f_1, \ldots, f_\ell \} \) be an orthonormal basis of \( E_1(g) \) with respect to the \( L_2 \)-global product \( \langle \cdot, \cdot \rangle_g \) induced by \( g \). The group of Kähler isometries \( G \) of \((M,g,J,\omega)\) acts isometrically on the space \((E_1(g),\langle \cdot, \cdot \rangle_g)\). It follows that the function

\[
f = \sum_{i=1}^\ell f_i^2
\]

is \( G \)-invariant, and since \( G \) acts transitively on \( M \), is constant. \( \square \)

As another application, we consider toric Kähler–Einstein manifolds which have been studied in many places, see [11, 12, 21, 41], and for which the existence theory takes a fairly concrete shape.
Corollary 3.4. Let $(M, g, J, \omega)$ be a compact Kähler–Einstein manifold of real dimension $2m$ whose connected identity component $\text{Aut}_0(M, J)$ of the automorphism group is the complexification of an $m$-dimensional real torus. Then the metric $g$ is not $\lambda_1$-extremal.

Proof. The assumption on $\text{Aut}_0(M, J)$ implies that $(M, g, J, \omega)$ is a toric Kähler–Einstein metric in the sense of [11][12][21][41]. Indeed, by Proposition 3.1, the connected component of the isometry group of the Kähler–Einstein metric is a maximal connected compact subgroup of $\text{Aut}_0(M, J)$. In our case it must be a real $m$-dimensional torus $T$. The latter acts in a hamiltonian way (as any induced vector field is individually hamiltonian), and by Delzant theorem [11], the momentum map $\mu : M \to t^*$ sends $M$ onto a compact convex polytope in the dual vector space $t^*$ of the Lie algebra $t = \text{Lie}(T)$. The pull back $f = (u, \mu) + \lambda$ to $M$ of an affine function $(u, x) + \lambda$ on $t^*$ (with $u \in t$ and $\lambda \in \mathbb{R}$) defines an element in $E_1(g) \oplus \mathbb{R}$. Conversely, all elements of $E_1(g) \oplus \mathbb{R}$ are of this form by Proposition 3.1. Suppose that the metric $g$ is $\lambda_1$-extremal. Then, by Theorem 3.2 there exist non-trivial eigenfunctions $f_i = (u_i, \mu) + \lambda_i$ such that the sum $\sum_{i=1}^n f_i$ is the pull-back of an affine function on $t^*$. It is then straightforward to see that $u_i = 0$ for any $i$, and hence, $f_i = 0$ for any $i$. Thus, we arrive at a contradiction with the hypothesis that the $f_i$'s are non-trivial.

The Corollary 3.4 shows in particular that the Kähler–Einstein metric on $\mathbb{C}P^2$ blown-up at three points in general position (see [38][39]) is not $\lambda_1$-extremal, and therefore is not a maximizer for $\lambda_1$ in its Kähler class.

As a final example, we consider non-homogeneous Kähler–Einstein metrics on projective bundles over the product of compact Kähler–Einstein manifolds with positive scalar curvatures that have been found and studied by Koiso and Sakane [26][27][28][37], see also [4][10][36] for alternative treatments. It is convenient to use the characterization from [4] saying that these are Kähler–Einstein metrics admitting a hamiltonian 2-form of order 1, in the sense of the theory in [2][3]. We can then prove the following statement.

Corollary 3.5. Let $(M, g, J, \omega)$ be a compact Kähler–Einstein manifold that is different from $\mathbb{C}P^n$ and admits a hamiltonian 2-form of order $\geq 1$. Then the metric $g$ is not $\lambda_1$-extremal.

Proof. As follows from the theory in [2][3], a hamiltonian 2-form $\phi$ on $(M, g, J, \omega)$ gives rise to an $\ell$-dimensional (real) torus $T$ in the isometry group of $(g, J, \omega)$, where $\ell \geq 1$ is the order of $\phi$. By the general classification [3] Thm. 6) and [2] Prop. 16], the corresponding moment map $\mu : M \to t^*$ sends $M$ to a Delzant simplex in the dual vector space of $t = \text{Lie}(T)$. Since $M$ is simply connected by the Kähler–Einstein assumption (see [4] Sec. 2.1) for the refinement in this case) and $(M, J) \neq \mathbb{C}P^n$, we conclude that it is the total space of a holomorphic projective bundle $P(\mathcal{E}_0 \oplus \mathcal{E}_1 \oplus \cdots \oplus \mathcal{E}_l) \to S$ over the product $S = \prod_j S_j$ of Kähler–Einstein manifolds $(S_j, g_j, J_j, \omega_j)$, where $\mathcal{E}_i$ are projectively-flat holomorphic vector bundles over $S$, satisfying certain topological conditions (in particular, $c_1(\mathcal{E}_i)/\text{rk}(\mathcal{E}_i)$ cannot all be equal when the order of $\phi$ is $\geq 1$). Moreover, the metric $g$ on $M$ is obtained by the generalized Calabi construction associated to this bundle, see [5] for a detailed treatment of this class of metrics.

As is shown in [17], when $(M, J) \neq \mathbb{C}P^n$, any other hamiltonian 2-form $\tilde{\phi}$ on $(M, g, J, \omega)$ must be a linear combination of $\phi$ and $\omega$. As the trace $\text{tr}_\omega \phi$ is the pull-back by $\mu$ of a non-constant affine function on $t^*$, see [2] Prop. 13], and is also invariant under the action of the isometry group $I(M, g)$ of $g$, the form $\phi$ is preserved by $I(M, g)$. Denote by $i(M, g)$ the Lie algebra of $I(M, g)$, which we identify with the space of Killing potentials of mean-value zero (using again that $(M, J)$ is Fano). It follows that the centralizer of $\mathfrak{t}$ in $i(M, g)$ equals $i(M, g)$. This allows us to use the description of $i(M, g)$ obtained in [5] Lemma 5]: it is shown in the proof of that
statement that any Killing potential \( f \in E_1(g) \) of \( (g, J, \omega) \) has the form

\[
f = \sum_j \left((u_j, \mu) + \lambda_j\right) f_j + \left((u, \mu) + \lambda\right),
\]

where \( f_j \) is a Killing potential of \((S_j, g_j, J_j, \omega_j)\), \( u, u_j \in \mathbb{T}, \lambda, \lambda_j \in \mathbb{R} \) and \( u_j, \lambda_j \) are determined by \((M, J)\). Furthermore, the \( u_j \)'s are expressed in terms of the degrees of \( \delta_j \) over \( S_j \) while the \( \lambda_j \)'s are determined by the Kähler class of \( \omega \). One easily deduces as in the proof of Corollary 3.4 that relation (3.1) is satisfied only if all \( u_j = 0 \), which is impossible unless the deRham classes \( c_1(\delta_j)/\text{rk}(\delta_j) \) are all equal, see [5]. As already observed, this is not the case when the projectively-flat bundles \( \delta_j \) are associated to a hamiltonian 2-form of order \( \ell \geq 1 \).

\[\Box\]

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