Linearized warp drive and the energy conditions

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“Warp drive” spacetimes are useful as “gedanken-experiments” and as a theoretician’s probe of the foundations of general relativity. Applying linearized gravity to the weak-field warp drive, i.e., for non-relativistic warp-bubble velocities, we find that the occurrence of energy condition violations in this class of spacetimes is generic to the form of the geometry under consideration and is not simply a side-effect of the “superluminal” properties. Using the linearized construction it is now possible to compare the warp field energy with the mass-energy of the spaceship, and applying the “volume integral quantifier”, extremely stringent conditions on the warp drive spacetime are found.

1 Introduction

“Warp drive” spacetimes [1, 2] are specific examples of solving the Einstein field equation in the reverse direction, in which one engineers an interesting spacetime metric, then finds the matter distribution responsible for the respective geometry. The analysis of wormhole geometries is treated in a similar manner [3, 4]. Following this philosophy, it was found that these spacetimes violate the energy conditions of general relativity. Although most (but not all) classical forms of matter are thought to obey the energy conditions, they are definitely violated by certain quantum fields [5]. Another interesting feature of these spacetimes is that they allow “effective” superluminal travel, although, locally, the speed of light is not surpassed [6]. However, to provide a general global definition of superluminal travel is no trivial matter [7]. Nevertheless, it was found that negative energy densities and superluminal travel are intimately related [8], and that certain classical systems, such as non-minimally coupled scalar fields, violate the null and the weak energy conditions [9, 10]. Another severe drawback is that by using the “quantum inequality” [11] it can be argued that truly enormous amounts of energy are needed to sustain superluminal spacetimes [12, 13, 14].

In this work we will not focus our attention on the superluminal features of the “warp drive” [15], but rather on the weak-field limit, considering the bubble velocity to be non-relativistic, $v \ll 1$. We shall be interested in applying linearized gravity to warp drive spacetimes, testing the energy conditions at first and second order of the warp-bubble velocity. A particularly interesting aspect of this construction is that one may now place a finite mass spaceship at the origin and consequently compare the mass-energy of the warp field with the mass-energy of the spaceship. This is not possible in the usual finite-strength warp field, since in the usual formalism the spaceship is always treated as a massless test particle, moving along a geodesic. It is interesting to note that if it is possible to realise even a weak-field warp drive in nature, such a spacetime appears to be an example of a “reaction-less drive”. That is, the warp bubble moves by interacting with the geometry of
spacetime instead of expending reaction mass, and the spaceship is simply carried along with it.

We shall also apply the “volume integral quantifier”, as defined in [16, 17], to the weak-field limit, at first and second order of the warp-bubble velocity, and thus, find extremely stringent conditions on the warp drive spacetime. If the construction of a “strong-field” warp drive starts from an approximately Minkowski spacetime, and inexorably passes through a weak-field regime, these conditions are so stringent that it appears unlikely that the “warp drive” will ever prove technologically useful. However, “warp drive” spacetimes are likely to retain their status as useful “gedanken-experiments”, as they are useful primarily as a theoretician’s probe of the foundations of general relativity.

2 Alcubierre warp drive

Alcubierre demonstrated, within the framework of general relativity, that it is in principle possible to warp spacetime in a small bubble-like region, in such a way that the bubble may attain arbitrarily large velocities [1]. The enormous speed of separation arises from the expansion of spacetime itself. The Alcubierre model for hyper-fast travel resides in creating a local distortion of spacetime, producing an expansion behind the bubble, and an opposite contraction ahead of it.

The warp drive spacetime metric, in cartesian coordinates, is given by (with $G = c = 1$

$$ds^2 = -dt^2 + [dx - \beta(x, y, z - z_0(t)) \, dt] \cdot [dx - \beta(x, y, z - z_0(t)) \, dt]. \tag{1}$$

In terms of the ADM formalism, this corresponds to a spacetime wherein space is flat, while the “lapse function” is identically unity, and the only non-trivial structure lies in the “shift vector” $\beta(t, x)$. The Alcubierre warp drive corresponds to taking the shift vector to lie in the direction of motion, i.e., $\beta(x, y, z - z_0(t)) = v(t) \hat{z} f(x, y, z - z_0(t))$, in which $v(t) = dz_0(t)/dt$ is the velocity of the warp bubble, moving along the positive $z$-axis. An alternative to the Alcubierre spacetime, is the Natário warp drive [2], where the shift vector is constrained by being divergence-free, $\nabla \cdot \beta(x, y, z) = 0$.

The form function $f(x, y, z)$ possesses the general features of having the value $f = 0$ in the exterior and $f = 1$ in the interior of the bubble. The general class of form functions chosen by Alcubierre was spherically symmetric: $f(r(t)) = f(x, y, z - z_0(t))$ with $r(t) = \{(z - z_0(t))^2 + x^2 + y^2\}^{1/2}$. Whenever a more specific example is required we adopt

$$f(r) = \frac{\tanh[\sigma(r + R)] - \tanh[\sigma(r - R)]}{2 \tanh(\sigma R)}, \tag{2}$$

in which $R > 0$ and $\sigma > 0$ are two arbitrary parameters. $R$ is the “radius” of the warp-bubble, and $\sigma$ can be interpreted as being inversely proportional to the bubble wall thickness.

Note that observers with the four velocity $U^\mu = (1, 0, 0, v f)$ move along geodesics, as their 4-acceleration is zero, i.e., $a^\mu = U^\nu U^\mu_{\;\nu} = 0$. The hypothetical spaceship, which in the original formulation is treated as a test particle and placed within the Alcubierre warp bubble, moves along a timelike curve $z = z_0(t)$ regardless of the value of $v(t)$. Proper time along this curve equals the coordinate time [1] and the centre of the perturbation corresponds to the spaceship’s position $z_0(t)$. The expansion of the volume elements, $\theta = U^\mu_{\;\mu}$, is given by $\theta = v \partial f / \partial z$. Using equation (2), one may prove that the volume elements are expanding behind the spaceship, and contracting in front of it [1, 15].

If we treat the spaceship as more than a test particle, we must confront the fact that by construction we have forced $f = 0$ outside the warp bubble. This implies that the spacetime geometry is asymptotically Minkowski space, and in particular the ADM mass is zero. That
is, the ADM mass of the spaceship and the warp field generators must be exactly compensated by the ADM mass due to the stress-energy of the warp-field itself. Viewed in this light it is now patently obvious that there must be massive violations of the classical energy conditions. One of our tasks in the current article will be to see if we can make qualitative and quantitative statements concerning the localization and “total amount” of energy condition violations. A similar attempt at quantification of the “total amount” of energy condition violation in traversable wormholes was recently presented in [16, 17], by introducing the notion of the “volume integral quantifier”. This notion amounts to calculating the definite integrals \( \int T_{\mu\nu}U^\mu U^\nu \, dV \) and \( \int T_{\mu\nu}k^\mu k^\nu \, dV \), and the amount of violation is defined as the extent to which these integrals become negative, where \( U^\mu \) and \( k^\mu \) are any timelike and null vectors, respectively, and \( T_{\mu\nu} \) is the stress-energy tensor.

The weak energy condition (WEC) states \( T_{\mu\nu}U^\mu U^\nu \geq 0 \). Its physical interpretation is that the local energy density is positive. By continuity it implies the null energy condition (NEC). We verify that for the warp drive metric, the WEC is violated, i.e.,

\[
T_{\mu\nu} U^\mu U^\nu = -\frac{v^2}{32\pi} \left[ \left( \frac{\partial f}{\partial x} \right)^2 + \left( \frac{\partial f}{\partial y} \right)^2 \right] = -\frac{1}{32\pi} \frac{v^2(x^2 + y^2)}{r^2} \left( \frac{df}{dr} \right)^2 < 0. \tag{3}
\]

Using an orthonormal basis, the energy density of the warp drive spacetime is given by \( T_{ij} = T_{\mu\nu} U^\mu U^\nu \), i.e., eq. (3), which is distributed in a toroidal region around the \( z \)-axis, in the direction of travel of the warp bubble [13]. Note that the energy density for this class of spacetimes is nowhere positive, and the fact that the total ADM mass can nevertheless be zero is due to the intrinsic nonlinearity of the Einstein equations.

We can, in analogy with the definitions in [16, 17], quantify the “total amount” of energy condition violating matter in the warp bubble by defining

\[
M_{\text{warp}} = \int \rho_{\text{warp}} \, d^3x = \int T_{\mu\nu} U^\mu U^\nu \, d^3x = -\frac{v^2}{12} \int \left( \frac{df}{dr} \right)^2 \, r^2 \, dr. \tag{4}
\]

This is emphatically not the total mass of the spacetime, but it characterizes the amount of negative energy that one needs to localize in the walls of the warp bubble. For the specific shape function (2) we can estimate \( M_{\text{warp}} \approx -v^2 R^2 \sigma \).

The NEC states that \( T_{\mu\nu} k^\mu k^\nu \geq 0 \). Considering the NEC for a null vector oriented along the \( \pm \hat{z} \) directions, and in particular, if we average over the \( \pm \hat{z} \) directions we have [15]

\[
\frac{1}{2} \{ T_{\mu\nu} k_{\pm \hat{z}}^\mu k_{\pm \hat{z}}^\nu + T_{\mu\nu} k_{\hat{z} \pm \hat{z}}^\mu k_{\hat{z} \pm \hat{z}}^\nu \} = -\frac{v^2}{8\pi} \left[ \left( \frac{\partial f}{\partial x} \right)^2 + \left( \frac{\partial f}{\partial y} \right)^2 \right], \tag{5}
\]

which is manifestly negative, and so the NEC is violated for all \( v \).

Using the “volume integral quantifier”, we may estimate the “total amount” of averaged null energy condition violating matter in this spacetime, given by \( \int T_{\mu\nu} k_{\pm \hat{z}}^\mu k_{\pm \hat{z}}^\nu \, d^3x \approx -v^2 R^2 \sigma \approx M_{\text{warp}} \). The key aspects to note here are that the net volume integral of the \( O(v) \) term is zero, and that the net volume average of the NEC violations is approximately the same as the net volume average of the WEC violations, which are \( O(v^2) \) [15].

### 3 Linearized warp drive

Our goal now is to try to build a more realistic model of a warp drive spacetime where the warp bubble is interacting with a finite mass spaceship. To do so we first consider the linearized theory applied to warp drive spacetimes, for non-relativistic velocities, \( v \ll 1 \). In linearized theory, the spacetime metric is given by \( ds^2 = (\eta_{\mu\nu} + h_{\mu\nu}) \, dx^\mu \, dx^\nu \), with \( h_{\mu\nu} \ll 1 \).
and $\eta_{\mu\nu} = \text{diag}(-1, 1, 1, 1)$. The analysis can be simplified by defining the trace reverse of $h_{\alpha\beta}$, given by $\overline{h}_{\alpha\beta} = h_{\alpha\beta} - \frac{1}{2} \eta_{\alpha\beta} h$, with $h = \overline{h}^{\beta}_{\alpha} = -h$. In terms of $\overline{h}_{\alpha\beta}$, the linearized Einstein tensor reads

$$G_{\alpha\beta} = -\frac{1}{2} \left[ \overline{h}_{\alpha\beta,\mu}^\mu + \eta_{\alpha\beta} \overline{h}_{\mu\nu,\mu}^\nu - \overline{h}_{\alpha\mu,\beta}^\mu - \overline{h}_{\beta\mu,\alpha}^\mu + O\left(\overline{h}_{\alpha\beta}^2\right) \right]. \quad (6)$$

Now the results deduced from applying linearized theory are only accurate to first order in $v$. This is equivalent to neglecting the $v^2 f^2$ from the metric (1), retaining only the first order terms in $v$. That is, we are making the following approximation

$$(\overline{h}_{\mu\nu}) = \begin{pmatrix} 0 & 0 & 0 & -v f \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -v f & 0 & 0 & 0 \end{pmatrix}. \quad (7)$$

The trace of $\overline{h}_{\mu\nu}$ is identically null, i.e., $h = h^\mu_{\mu} = 0$. Therefore, the trace reverse of $\overline{h}_{\mu\nu}$, defined in eq. (3), is given by $\overline{h}_{\mu\nu} = h_{\mu\nu}$, i.e., eq. (7) itself.

### 3.1 Energy condition violations

In linearized theory the 4-velocity can be approximated by $U^\mu = (1, 0, 0, 0)$, and by using equation (6) we verify that the WEC is identically “saturated”, i.e., $T_{\mu\nu} U^\mu U^\nu = T_{00} = O(v^2)$. Although in this approximation, at least to first order in $v$, the WEC is not violated, it is on the verge of being so.

Despite the fact that the observers, with $U^\mu = (1, 0, 0, 0)$, measure zero energy density, it can be shown that observers which move with any other arbitrary velocity, $\beta$, along the positive $z$ axis measure a negative energy density, to first order in $v$. Consider a Lorentz transformation, and using equation (6) the energy density measured by these observers is given by

$$T_{00} = \frac{\gamma^2 \beta^2 v}{8\pi} \left[ \left( \frac{x^2 + y^2}{r^2} \right) \frac{d^2 f}{dr^2} + \left( \frac{x^2 + y^2 + 2(z - z_0(t))^2}{r^3} \right) \frac{df}{dr} \right] + O(v^2). \quad (8)$$

with $\gamma = (1 - \beta^2)^{-1/2}$. A number of general features can be extracted from the terms in square brackets, without specifying an explicit form of $f$. In particular, $f$ decreases monotonically from its value at $r = 0$, $f = 1$, to $f \approx 0$ at $r \geq R$, so that $df/dr$ is negative in this domain. The form function attains its maximum in the interior of the bubble wall, so that $d^2 f/dr^2$ is also negative in this region. Therefore there is a range of $r$ in the immediate interior neighbourhood of the bubble wall where one necessarily encounters negative energy density, as measured by the observers considered above. Again we find that WEC violations persist to arbitrarily low warp bubble velocities.

One can show that the NEC is proportional to the energy density, $T_{00}$, of equation (8). Thus, we verify that the NEC is also violated in the immediate interior vicinity of the bubble wall (see [15] for details).

### 3.2 Spaceship immersed in the warp bubble

Consider now a spaceship in the interior of an Alcubierre warp bubble, which is moving along the positive $z$ axis with a non-relativistic constant velocity, i.e., $v \ll 1$. The metric is given by

$$ds^2 = -dt^2 + dx^2 + dy^2 + \left[ dz - v f(x, y, z - vt) dt \right]^2$$

$$-2\Phi(x, y, z - vt) \left[ dt^2 + dx^2 + dy^2 + (dz - v f(x, y, z - vt) dt)^2 \right]. \quad (9)$$
If $\Phi = 0$, the metric (9) reduces to the warp drive spacetime of eq. (1). If $v = 0$, we have the metric representing the gravitational field of a static source, in particular, that of a spaceship. Note that the mass density of the spaceship, $\rho$, is related to the gravitational potential $\Phi$ by Poisson’s equation, $\nabla^2 \Phi = 4\pi \rho$.

**First order approximation**

Applying the linearized theory, keeping terms linear in $v$ and $\Phi$ but neglecting all superior order terms, the matrix elements, $h_{\mu\nu}$, and the respective trace-reversed elements, $\overline{h}_{\mu\nu}$, of the metric (9) are given by the following approximations

$$
(h_{\mu\nu}) = \begin{bmatrix} -2\Phi & 0 & 0 & -vf \\ 0 & -2\Phi & 0 & 0 \\ 0 & 0 & -2\Phi & 0 \\ -vf & 0 & 0 & -2\Phi \end{bmatrix} \quad \text{and} \quad (\overline{h}_{\mu\nu}) = \begin{bmatrix} -4\Phi & 0 & 0 & -vf \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -vf & 0 & 0 & 0 \end{bmatrix},
$$

(10)

where the trace of $h_{\mu\nu}$ is given by $h = h^{\mu\mu} = -4\Phi$.

We verify that the WEC is given by $T_{\mu\nu}U^\mu U^\nu = \rho + O(v^2, v\Phi, \Phi^2)$, where $\rho$ is the energy density of the spaceship which is manifestly positive. In linearized theory, the total ADM mass of the space-time simply reduces to the mass of the space-ship, i.e.,

$$M_{\text{ADM}} = \int_{T_00} d^3x = \int \rho d^3x + O(v^2, v\Phi, \Phi^2) = M_{\text{ship}} + O(v^2, v\Phi, \Phi^2).$$

(11)

The NEC, with $k^\mu \equiv (1, 0, 0, \pm 1)$, to first order in $v$ and $\Phi$, and neglecting the crossed terms $v\Phi$, takes the form

$$T_{\mu\nu} k^\mu k^\nu = \rho \pm \frac{v}{8\pi} \left( \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} \right) + O(v^2, v\Phi, \Phi^2).$$

(12)

From this, one can deduce the existence of localized NEC violations even in the presence of a finite mass spaceship, and can also make deductions about the net volume-averaged NEC violations. First, note that for reasons of structural integrity one wants the spaceship itself to lie well inside the warp bubble, and not overlap with the walls of the warp bubble. But this means that the region where $\rho \neq 0$ does not overlap with the region where the $O(v)$ contribution due to the warp field is non-zero. So regardless of how massive the spaceship itself is, there will be regions in the wall of the warp bubble where localized violations of NEC certainly occur. If we now look at the volume integral of the NEC, we have

$$\int T_{\mu\nu} k^\mu_\pm k^\nu_\pm d^3x = \int \rho d^3x + O(v^2, v\Phi, \Phi^2) = M_{\text{ship}} + O(v^2, v\Phi, \Phi^2).$$

(13)

The net result of this $O(v)$ calculation is that irrespective of the mass of the spaceship there will always be localized NEC violations in the wall of the warp bubble, and these localized NEC violations persist to arbitrarily low warp velocity. However at $O(v)$ the volume integral of the NEC violations is zero, and so we must look at higher order in $v$ if we wish to deduce anything from the consideration of volume integrals to probe “net” violations of the NEC.

**Second order approximation**

Consider the approximation in which we keep the exact $v$ dependence but linearize in the gravitational field of the spaceship $\Phi$. The WEC is given by (see [15] for details)

$$T_{\tilde{\mu}\tilde{\nu}} U^{\tilde{\mu}} U^{\tilde{\nu}} = \rho - \frac{v^2}{32\pi} \left( \left( \frac{\partial f}{\partial x} \right)^2 + \left( \frac{\partial f}{\partial y} \right)^2 \right) + O(\Phi^2).$$

(14)

Once again, using the “volume integral quantifier”, we find the following estimate
\[ \int T_{\mu \nu} U^\mu U^\nu \, d^3x = M_{\text{ship}} - v^2 R^2 \sigma + \int O(\phi^2) \, d^3x. \]  

(15)

Now suppose we demand that the volume integral of the WEC at least be positive, then \( v^2 R^2 \sigma \leq M_{\text{ship}} \). This inequality is the reasonable condition that the net total energy stored in the warp field be less than the total mass-energy of the spaceship itself, which places a powerful constraint on the velocity of the warp bubble. Re-writing this in terms of the size of the spaceship \( R_{\text{ship}} \) and the thickness of the warp bubble walls \( \Delta = 1/\sigma \), we have

\[ v^2 \leq \frac{M_{\text{ship}}}{R_{\text{ship}}} \frac{R_{\text{ship}} \Delta}{R^2}. \]  

(16)

For any reasonable spaceship this gives extremely low bounds on the warp bubble velocity. In a similar manner, we find the same restriction as eq. (16) whilst analyzing the NEC [15].

4 Conclusion

We have verified that the warp drive spacetimes necessarily violate the classical energy conditions, and continue to do so for arbitrarily low warp bubble velocity. Thus the energy condition violations in this class of spacetimes is generic to the form of the geometry under consideration and is not simply a side-effect of the “superluminal” properties.

Using linearized theory, we have built a more realistic model of the warp drive spacetime where the warp bubble interacts with a finite mass spaceship. We have applied the “volume integral quantifier” to the weak-field limit, and found that this places an extremely stringent condition on the warp drive spacetime, namely, that for all conceivably interesting situations the bubble velocity should be absurdly low. In view of this analysis, it therefore appears unlikely that the warp drive will ever prove to be technologically useful.

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