String instanton in $AdS_4 \times CP^3$

Alessandra Cagnazzo $^{*\dagger}$, Dmitri Sorokin $^\dagger$ and Linus Wulff $^\dagger$

$^*$ Dipartimento di Fisica “Galileo Galilei”,
Universitá degli Studi di Padova

$^\dagger$ INFN, Sezione di Padova,
via F. Marzolo 8, 35131 Padova, Italia

Abstract

We study the string instanton wrapping a non–trivial two–cycle in $CP^3$ of the type IIA string theory compactified on $AdS_4 \times CP^3$ superspace and find that it has twelve fermionic zero modes associated with 1/2 of the supersymmetry of the background thus manifesting that this classical instanton configuration is 1/2 BPS.
1 Introduction

M–theory compactified on $AdS_4 \times S^7 / \mathbb{Z}_k$ and a corresponding Type IIA superstring theory compactified on $AdS_4 \times CP^3$ are on the bulk side of the $AdS_4 / CFT_3$ holography whose boundary superconformal Chern–Simons–matter theory is assumed to provide an effective worldvolume description of a stack of multiple M2–branes \([1, 2, 3, 4, 5, 6, 7]\). This new $AdS_4 / CFT_3$ correspondence shares some features with the well studied $AdS_5 / CFT_4$ correspondence whose bulk theory is Type IIB superstring theory compactified on $AdS_5 \times S^5$ and the boundary theory is the superconformal $\mathcal{N} = 4, D = 4$ super Yang–Mills theory.

However, the two examples of the $AdS/CFT$ correspondence are quite different. First of all, the Type IIB superstring in $AdS_5 \times S^5$ is maximally supersymmetric. Its 32 supersymmetries are part of the superisometry group $PSU(2, 2|4)$. On the other hand the $AdS_4 \times S^7 / \mathbb{Z}_k$ solution of $D = 11$ supergravity has 32 supersymmetries only for $k = 1, 2$, while for $k > 2$ the theory is invariant under 24 supersymmetries and so is its ten–dimensional counterpart, the Type IIA superstring theory in the $AdS_4 \times CP^3$ supergravity background whose superisometries form the supergroup $OSp(6|4)$ \([8, 9, 10]\). As a result, while the Green–Schwarz action for the $AdS_5 \times S^5$ superstring amounts to a worldsheet sigma–model on the supercoset space $PSU(2, 2|4)/SO(1, 4) \times SO(5)$ \([11, 12]\), the target superspace of the Green–Schwarz action of the $AdS_4 \times CP^3$ superstring is not a supercoset space, though it possesses the $OSp(6|4)$ isometry \([13, 14]\). Only when the superstring is extended in $CP^3$, its dynamics can be described by the $OSp(6|4)/U(3) \times SO(1, 3)$ supercoset sigma–model \([15, 16, 17, 18, 19]\). It can be obtained from the complete Green–Schwarz action \([13]\) by partially gauge fixing kappa–symmetry in a way which puts to zero the eight worldsheet fermionic modes corresponding to the eight broken supersymmetries of the $AdS_4 \times CP^3$ superbackground. Such a gauge fixing is admissible only when the string moves in $CP^3$. This gauge is, however not admissible when the string moves entirely in the $AdS_4$ part of the superbackground. One of the peculiar consequences of this peculiarity situation is that though the subsector of the $AdS_4 \times CP^3$ superstring theory described by the supercoset sigma–model is classically integrable \([15, 16]\), the explicit proof of the classical integrability of the complete $AdS_4 \times CP^3$ superstring still remains an open problem. This is because the complete $AdS_4 \times CP^3$ superspace is not a supercoset space, and the methods used to prove the integrability of the $AdS_5 \times S^5$ superstring \([20]\) and of the supercoset sector of the $AdS_4 \times CP^3$ superstring \([15, 16]\) do not apply.

Another interesting peculiarity of the $AdS_4 \times CP^3$ superstring, which the $AdS_5 \times S^5$ superstring does not have, is the existence on $CP^3$ of string instantons\(^1\). They are formed in the Wick rotated theory by the string worldsheet wrapping a topologically non–trivial two–cycle of $CP^3$. This two–cycle is a $CP^1 \simeq S^2$ corresponding to the closed Kähler two–form $J_2$ on $CP^3$. As we shall show, also in the case of the string instantons on $CP^3$ the consistent gauge fixing of kappa–symmetry does not allow reducing the string action to the supercoset sigma model, i.e. to eliminate, by using kappa–symmetry, the eight fermionic modes corresponding to the broken supersymmetries.

The main goal of this paper is to study the superstring instanton on $CP^3$ and analyze its fermionic zero modes. In the case of branes moving in a supersymmetric background, their fermionic equations have solutions which are associated with the Killing spinors of the back-\(^1\)

\(^1\)For a recent review and references on string and brane instantons and their effects see e.g. \([21, 22]\).
ground that guarantee its supersymmetries. The number of physical modes of the worldvolume Dirac operator associated with the Killing spinors is equal to the number of components of the Killing spinors which are not annihilated by the kappa–symmetry projector of the brane. If the background is maximally supersymmetric, one concludes that the number of dynamical zero modes on the brane are half the number of supersymmetries, since the rank of the kappa–symmetry projector is equal to half the number of the maximal supersymmetries of the background. When the background is not maximally supersymmetric, as the $AdS_4 \times CP^3$ one, the number of the brane fermion zero modes associated with unbroken supersymmetries depends on how many of them are not eliminated by the kappa–symmetry projector. The action of the kappa–symmetry projector and the corresponding number of fermionic zero modes depends on how the given brane configuration is embedded into target space. In the cases with less supersymmetries the worldvolume Dirac equation may, in general, also have solutions which are not associated with unbroken supersymmetries, but with the broken ones. Thus the analysis of the brane fermionic modes in less supersymmetric backgrounds should be made case by case (see e.g. [23] for a more detailed discussion of this point).

As we shall see, in the $AdS_4 \times CP^3$ case the string instanton on $CP^3$ has twelve zero modes all of which are associated with the supersymmetries of the background and there are no zero modes of the fermions corresponding to the supersymmetries broken by $AdS_4 \times CP^3$. So the instanton under consideration is 1/2 BPS. It is interesting that these twelve zero modes are divided into eight and four ones which have different geometrical and physical meaning. The eight massive fermionic zero modes are four copies of the two–component Killing spinor on $S^2$ and the four other fermionic modes are two copies of massless chiral and anti-chiral fermion on $S^2$ electrically coupled to the electromagnetic potential created on $S^2$ by a monopole placed in the center of $S^2$. The monopole potential arises as part of the $CP^3$ spin connection pulled–back on the instanton $S^2$. This, at least formally and remotely, reminds us the peculiarity of the presence of different, light and heavy, physical worldsheet degrees of freedom in a Penrose limit of the $AdS_4 \times CP^3$ superstring [24, 25, 26, 27].

In M–theory compactified on $AdS_4 \times S^7/Z_k$, the counterpart of the string instanton considered in this paper is an Euclidean M2–brane that wraps the non–trivial 3–cycle $S^3/Z_k$ (for $k > 1$) inside $S^7/Z_k$.

The presence of the string instanton and its fermionic zero modes may generate non–perturbative corrections to the string effective action, which may affect its properties and if so should be taken into account in studying, e.g. the $AdS_4/CFT_3$ correspondence. The instantons may, perhaps, contribute to the worldsheet S–matrix and/or to energies of a semiclassical string. To study these effects one needs to find a way of merging the instanton and Minkowski solutions, such as spinning strings or BMN geodesics. In addition, in the presence of the instanton fermionic zero modes, the worldsheet correlator, to be non–zero, should contain a number of fermion insertions $2$.

The paper is organized as follows. In Section 2 we review the form of the string action in the $AdS_4 \times CP^3$ superbackground and truncate it to the second order in fermions. In Section 3 we describe the bosonic part of the string instanton solution and in Section 4 we study its fermionic zero modes. Section 5 contains a summary and brief discussion of a possible relation of the string instanton to some features of the $AdS_4/CFT_3$ correspondence. Appendix

\footnote{We are thankful to Konstantin Zarembo for these comments.}
A contains a description of our conventions and notation and of the geometry of the \( AdS_4 \times CP^3 \) superspace. In Appendix B we give the explicit form of the \( CP^3 \) Fubini–Study metric, vielbeins and connection which were used for the analysis of the string fermion equations.

## 2 \( AdS_4 \times CP^3 \) superstring action up to the quadratic order in fermions

To simplify the study of the fermionic zero modes of the string instanton we reduce the complete superstring action of [13] to the quadratic order in fermions (though, the solutions we find satisfy the complete non–linear equations of motion to all orders in fermions). Alternatively, one can use the quadratic type IIA superstring action derived in [28, 29] for a generic superbackground and substitute into it the values of the supergravity fields corresponding to the \( AdS_4 \times CP^3 \) background. As a consistency check we have performed the reduction of both of the actions. We shall see that they give the same result upon a redefinition of bosonic \( CP^3 \) coordinates of the target–superspace of [13]. This redefinition is required due to a particular parametrization used in [13] to construct an explicit form of the \( AdS_4 \times CP^3 \) supergeometry.

The Green–Schwarz superstring action in a generic type IIA supergravity background has the well known form

\[
S = -\frac{1}{4\pi\alpha'} \int d^2\xi \sqrt{-h} h^{IJ} \mathcal{E}_I^A \mathcal{E}_J^B \eta_{AB} - \frac{1}{2\pi\alpha'} \int B_2, \tag{2.1}
\]

where \( \xi^I (I, J = 0, 1) \) are the worldsheet coordinates, \( h_{IJ}(\xi) \) is an intrinsic worldsheet metric, \( \mathcal{E}_I^A \) are worldsheet pullbacks of target superspace vector supervielbeins and \( B_2 \) is the pull–back of the NS–NS 2–form.

The kappa–symmetry transformations of the worldsheet fields \( Z^M(\xi) = (X^M(\xi), \Theta^A(\xi)) \) which leave the superstring action (2.1) invariant (provided the superbackground obeys the superspace supergravity constraints) are

\[
\begin{align*}
\delta_\kappa Z^M & : \mathcal{E}_M^\alpha = \frac{1}{2}(1 + \Gamma)\tilde{\alpha}_\beta^\alpha h_{\beta}(\xi), & \alpha = 1, \cdots, 32 \\
\delta_\kappa Z^M & : \mathcal{E}_M^A = 0, & A = 0, 1, \cdots, 9
\end{align*}
\tag{2.2}
\]

where \( \kappa^\alpha(\xi) \) is a 32–component spinor parameter, \( \frac{1}{2}(1 + \Gamma)\tilde{\alpha}_\beta^\alpha \) is a spinor projection matrix with

\[
\Gamma = \frac{1}{2\sqrt{-\det G_{IJ}}} \varepsilon^{IJ} \mathcal{E}_I^A \mathcal{E}_J^B \Gamma_{AB} \Gamma_{11}, \quad \Gamma^2 = 1, \tag{2.4}
\]

where \( G_{IJ} = \mathcal{E}_I^A \mathcal{E}_J^B \eta_{AB} \) is the induced metric on the worldsheet. The explicit form of the supervielbeins \( \mathcal{E}_A(Z) \) and the NS–NS 2–form \( B_2 \) which describe the geometry of the \( AdS_4 \times CP^3 \) superspace are given in Appendix A (see also [13, 14]).

\( ^3 \)The kappa–variations of \( Z^M \) should be accompanied by a kappa–variation of the worldsheet metric \( h_{IJ} \) whose explicit form the reader may find in [13], eq. (4.21).
Up to second order in fermions the supervielbeins and the $B$-field have the following form

$$
\mathcal{E}^a = e^{\frac{i}{4} \phi_0} \left( e^a (1 - \frac{1}{R} \nu \gamma^5 \nu) + i \Theta \gamma^a \mathcal{D} \Theta \right),
$$
$$
\mathcal{E}^{a'} = e^{\frac{i}{4} \phi_0} \left( e^{a'} (1 - \frac{1}{R} \nu \gamma^5 \nu) + i \delta \gamma^{a'} \gamma^5 \mathcal{D} \nu + 2i \nu \gamma^{a'} \gamma^5 \mathcal{D} \Theta \right),
$$
$$
\mathcal{E}^\mu = e^{\frac{i}{4} \phi_0} (\mathcal{D} \Theta)^\mu,
$$

where $e^{\phi_0} = \frac{R}{\mu}$ is the vacuum expectation value of the dilaton, $R$ is the radius of the $S^7$ sphere whose base is $CP^3$, $l_p$ is the eleven-dimensional Planck length related to the string tension as follows $l_p = e^{\frac{1}{4} \phi_0} \sqrt{\alpha'}$ and $k$ is the Chern–Simons level related to the units of $F_4$ and $F_2$ Ramond–Ramond flux which support the $AdS_4 \times CP^3$ solution of Type IIA supergravity.

The contribution of the RR fluxes manifests itself in the presence of projectors $P_6$ and $P_2$ in the string action (see Appendix A.5). They split the 32–component fermionic variable $\Theta$ into the 24–component spinors $\vartheta^{a\alpha}$ ($\alpha = 1, \ldots, 4$; $a' = 1, \ldots, 6$) which correspond to the 24 supersymmetries of the $AdS_4 \times CP^3$ solution and the 8–component spinors $\vartheta^{a\alpha}$ ($q = 1, 2$) which correspond to the broken supersymmetries. The index $\alpha$ is a spinor index of $AdS_4$ (see Appendix A for more details). The covariant derivative $\mathcal{D} \Theta$ is defined as follows

$$
\mathcal{D} \Theta = \left\{ \begin{array}{c}
\mathcal{D} \nu = (d + \frac{1}{R} e^a \gamma^5 \gamma_a - \frac{1}{4} \omega^{ab} \gamma_{ab} - 2i A \gamma_7) \nu \\
\mathcal{D} \vartheta = P_6 (d + \frac{1}{R} e^{a'} \gamma^5 \gamma_{a'} + \frac{1}{4} \omega^{a'b'} \gamma_{a'b'} - \frac{1}{4} \omega^{ab} \gamma_{ab} - \frac{1}{4} \gamma_{a'b'} \gamma_{a'b'}) \vartheta
\end{array} \right.,
$$

where $e^a(x)$, $\omega^{ab}(x)$ and $\gamma^a$, $\gamma^5$ are, respectively the vielbein, connection and Dirac–matrices of $AdS_4$ of radius $R/2$. $e^{a'}(y)$ and $\omega^{a'b'}(y)$ are, respectively, the vielbein and connection on $CP^3$ of radius $R$ and $\gamma_{a'}$, $\gamma_7$ are $8 \times 8$ gamma–matrices of $Spin(6)$. $A(y) = \frac{1}{8} \omega^{a'b'} J_{a'b'}$ is the RR one-form potential whose field strength is the Kähler form on $CP^3$, $dA = \frac{1}{R^2} e^a e^{b'} J_{a'b'}$. See Appendix A for more details regarding the notation and conventions.

Substituting the expressions for the vielbeins (2.5) and the NS–NS two–form (2.6) into the action (2.1) and keeping only terms up to quadratic order in fermions we get the following action

$$
S = -\frac{e^{\frac{i}{4} \phi_0}}{4 \pi \alpha'} \int d^2 \xi \sqrt{-h} h^{IJ} \left( e_I^a e_J^b \eta_{ab} + e_I^{a'} e_J^{b'} \delta_{a'b'} \right) \\
- \frac{e^{\frac{i}{4} \phi_0}}{2 \pi \alpha'} \int d^2 \xi \Theta (\sqrt{-h} h^{IJ} - \epsilon^{IJ} \Gamma_{11}) [i e_I^A \Gamma_A \nabla_j \Theta - \frac{1}{R} e_I^A e_J^B \Gamma_A \Gamma_{11} \Gamma_B \Theta] \\
+ \frac{e^{\frac{i}{4} \phi_0}}{2 \pi \alpha'} \int d^2 \xi \sqrt{-h} h^{IJ} e_I^{a'} \nabla_j (i \Theta P_6 \Gamma_{a'} \Theta),
$$

where $\nabla \Theta = (d - \frac{1}{4} \omega^{AB} \Gamma_{AB}) \Theta$ is the worldsheet pullback of the conventional $AdS_4 \times CP^3$ covariant derivative.

The first two lines of this action coincide with the action which one gets by reducing to $AdS_4 \times CP^3$ the quadratic Green–Schwarz action in a generic type IIA superbackground
The last term in the action (2.8) appeared because of our choice of parametrization of the $AdS_3 \times CP^3$ superspace which allowed us to write its geometry in the simplest form. It is not hard to see that the last term in (2.8) can be canceled (modulo higher order terms in fermions) by making the following shift of the bosonic coordinates $y^m$ of $CP^3$

$$y^m = \hat{y}^m + i\Theta P_6 \Gamma^m \mathcal{P}_2 e_a^m(\hat{y}).$$

(2.9)

After this field redefinition the two forms of the string action become equivalent.

To study the string instantons we should perform a Wick rotation of the worldsheet and the target space in the action (2.8) to Euclidean signature. The Wick rotation basically consists in replacing $\sqrt{-h}$ and $\sqrt{-G}$, respectively with $\sqrt{h}$ and $\sqrt{G}$, replacing $\varepsilon^{IJ}$ with $-i\varepsilon^{IJ}$ and taking into account that the fermions $\Theta$ become complex spinors, since there are no Majorana spinors in ten-dimensional Euclidean space. However, the complex conjugate spinors do not appear in the Wick rotated action and, hence, the number of the fermionic degrees of freedom formally remains the same as before the Wick rotation. Note also that the Euclidean $\gamma^5$ is defined as $\gamma^5 = \gamma^1 \gamma^2 \gamma^3 \gamma^4$, where $\gamma^4$ is the Wick rotated $\gamma^0$. So $(\gamma^5)^2 = 1$ as in the case of Minkowski signature.

Thus, after the redefinition (2.9) and the Wick rotation the action takes the following form

$$S_E = \frac{e^{\frac{2\phi}{\alpha'}}}{4\pi\alpha'} \int d^2\xi \sqrt{h} h^{IJ} \left( e_1^a e_1^b \delta_{ab} + e_1^a e_1^b \delta_{ab} \right)$$

$$+ \frac{e^{\frac{2\phi}{\alpha'}}}{2\pi\alpha'} \int d^2\xi \Theta(\sqrt{h} h^{IJ} + i\varepsilon^{IJ} \Gamma_{11}) \left[ i e_1^A \Gamma_A \nabla_J \Theta - \frac{1}{R} e_1^A e_1^B \Gamma_A P_6 \gamma^5 \Gamma_B \Theta \right]$$

and the kappa–symmetry matrix $\Gamma$ gets replaced by

$$\Gamma = -\frac{i}{2 \sqrt{\det G_{IJ}}} \varepsilon^{IJ} \epsilon_I^A \epsilon_J^B \Gamma_{AB} \Gamma_{11}, \quad \Gamma^2 = 1.$$ 

(2.11)

### 3 String instanton wrapping a two–sphere inside $CP^3$

We are interested in a string whose worldsheet wraps a topologically non–trivial two–cycle inside $CP^3$ and thus is a stringy counterpart of the instantons of two–dimensional $CP^N$ sigma–models\footnote{The instanton solution in the $O(3)$ (or $CP^1$) sigma–model was first found in [30] and then generalized to the case of the $CP^n$ sigma–models in [31] [32] [33]. The instanton solution in the supersymmetric $CP^1$ sigma–model was first discussed in [34]. See [35] [36] [37] for a review and references on this subject.}. To be topologically non–trivial this two–cycle should have a non–zero pull–back on its worldsheet of the Kähler two–form $J_2 = \frac{1}{\pi} e^{\psi} e^{\alpha'} J_{\alpha'\psi}$ of $CP^3$. Such a two–cycle is a $CP^1 \cong S^2$ subspace of $CP^3$. To identify it, it is convenient to consider the form of the Fubini–Study metric on $CP^3$ given in [35]

$$ds^2 = R^2 \left[ \frac{1}{4}(d\theta^2 + \sin^2 \theta (d\varphi + \frac{1}{2} \sin^2 \alpha \sigma_3)^2) + \sin^2 \varphi \frac{1}{2} d\alpha^2 + \frac{1}{4} \sin^2 \theta \sin^2 \alpha (\sigma_1^2 + \sigma_2^2 + \cos^2 \alpha \sigma_3^2) \right],$$

(3.12)

where $0 \leq \theta \leq \pi$, $0 \leq \varphi \leq 2\pi$ and $0 \leq \alpha \leq \frac{\pi}{2}$, and $\sigma_1, \sigma_2, \sigma_3$ are three left-invariant one-forms on $SU(2)$ obeying $d\sigma_1 = -\sigma_2 \sigma_3$ etc. (see Appendix B for more details). Notice that with
this choice of the $CP^3$ coordinates, $\theta$ and $\varphi$ parameterize a two–sphere of radius $\frac{R}{2}$. This two–sphere is topologically non–trivial and associated to the Kähler form on $CP^3$. The string instanton wraps this sphere. For instance, if it wraps the sphere once $\theta$ and $\varphi$ can be identified with the string worldsheet coordinates, while all other $CP^3$ as well as $AdS_4$ coordinates are worldsheet constants in this case. Thus the pullback on the string instanton of the metric (3.12) of $CP^3$ (of radius $R$) is the metric of the sphere of radius $R/2$

$$ds^2 = \frac{R^2}{4} (d\theta^2 + \sin^2 \theta \, d\varphi^2) .$$

In this coordinate system the $S^2$ vielbein $e^i$ and the spin connection $\omega^{ij}_{S^2} (i, j = 1, 2)$ can be chosen in the form

$$e^1 = \frac{R}{2} d\theta , \quad e^2 = \frac{R}{2} \sin \theta \, d\varphi , \quad \omega^{12}_{S^2} = \cos \theta d\varphi ,$$

and the $S^2$ curvature is

$$R^{ij} = d\omega^{ij}_{S^2} = \frac{4}{R^2} e^i e^j .$$

### 3.1 Bosonic part of the instanton solution

The bosonic part of the Wick rotated string action (2.10) is

$$S_E = \frac{T}{2} \int d^2 \xi \, \sqrt{h} h^{IJ} e^i_I e^j_J \delta_{ij} ,$$

where $T = \frac{\alpha'}{2\pi \alpha'}$ and $e^i$ are the vielbeins on $S^2$. To discuss the instanton solution of this $CP^1$ sigma model it is convenient to introduce complex coordinates both on the worldsheet and in target space (see [35] for a review of instantons in two–dimensional sigma models). In the conformal gauge $\sqrt{h} h^{IJ} = \delta^{IJ}$ and in the $(z, \bar{z})$ coordinate system on the worldsheet the action takes the form

$$S_E = \frac{T}{2} \int d^2 z \, e^i_z e^j_{\bar{z}} \delta_{ij} .$$

To introduce complex coordinates on the target sphere it is convenient to describe it as $CP^1$. The Fubini-Study metric on $CP^1$ is

$$ds^2_{CP^1} = \frac{d\zeta \, d\bar{\zeta}}{(1 + |\zeta|^2)^2} .$$

If we choose $\zeta$ to be

$$\zeta = \tan \frac{\theta}{2} e^{i\varphi} ,$$

eq. (3.18) takes the form of the metric on $S^2$ of radius $\frac{1}{2}$

$$ds^2 = \frac{1}{4} (d\theta^2 + \sin^2 \theta \, d\varphi^2) .$$
In the $\zeta, \bar{\zeta}$ coordinate system the string action takes the following form (which is similar to that of the $O(3)$–sigma model)

$$S_E = \frac{TR^2}{4} \int d^2z \frac{|\partial\zeta|^2 + |\bar{\partial}\zeta|^2}{(1 + |\zeta|^2)^2}. \quad (3.21)$$

It is now obvious that a local minimum is attained if $\bar{\partial}\zeta = 0$ or $\partial\zeta = 0$, i.e. the embedding is given by a holomorphic function $\zeta = \zeta(z)$ for the instanton or by an anti–holomorphic function $\zeta = \zeta(\bar{z})$ for the anti–instanton. The remaining part of the action can be shown to be a topological invariant, namely,

$$S_I = \pi n TR^2 = n \frac{R_{CP^3}^2}{2\alpha'}, \quad (3.22)$$

where $n$ is the topological charge of the instanton and $R_{CP^3} = e^{\frac{1}{2}\phi_0} R$ is the $CP^3$ radius in the string frame.

What we have just reproduced is the classical instanton solution of the two–dimensional $O(3)$ sigma–model [30] or rather its extension to $CP^3$ [32, 31, 33] which in terms of the Fubini–Studi coordinates $\zeta^a$ ($a = 1, 2, 3$) of $CP^3$ (see eq. (B.1) of Appendix B) has the form

$$\zeta^a = \zeta^a(z) \text{ or } \zeta^a = \zeta^a(\bar{z}).$$

The difference with the $CP^N$ models is that in our case the string action is also invariant under worldsheet reparametrization. This means that every classical string solution must satisfy the Virasoro constraints implying that the worldsheet bosonic physical fields are associated with the string oscillations transverse to the worldsheet. For the instanton solution the string excitations along $AdS_4$ are zero and the Virasoro constraints have the following form in the conformal gauge

$$\left(\delta^{ab}(1 + |\zeta|^2) - \zeta^b \bar{\zeta}^a\right) \frac{\partial\zeta^a}{(1 + |\zeta|^2)^2} \bar{\partial}\zeta^b = 0. \quad (3.23)$$

We see that the Virasoro constraints are identically satisfied by the (anti)instanton solution.

Let us note that though in the $AdS_4 \times CP^3$ background the purely bosonic components of the NS–NS 3–form field strength $H_3$ are zero, the NS–NS 2–form may have non–zero expectation values proportional to the Kähler two–form on $CP^3$, $B_2 = \frac{\alpha'}{R^2} a J_{a'b'} e^{a'} e^{b'}$, where $a$ plays the role of a constant axion. For such a two–form, $H_3 = dB_2$ is zero since $J_2$ is the closed (but not exact) form, $dJ_2 = 0$. In this case also the Wess–Zumino part of the (Wick rotated) string action (2.1) will contribute to the instanton action, which becomes

$$S_I = n (\pi R^2 T - ia) = n \left(\frac{R_{CP^3}^2}{2\alpha'} - ia\right). \quad (3.24)$$

A similar situation one has in the case of string instantons on Calabi–Yau spaces [39, 40]. In the context of the $AdS_4/CFT_3$ correspondence, the co–homologically non–trivial $B_2$ field appears from the string side in the generalization of the ABJM model to include gauge groups of a different rank proposed in [41] (see the Summary below for more discussion of this point).

Finally, we note that the bosonic string instanton has twelve zero modes. Four of them correspond to the directions along $AdS_4$ and eight are the instanton zero modes on $CP^3$ [36]. We are now in a position to proceed with the study of the fermionic zero modes carried by the string instanton. We shall see that their number is also twelve.

---

5We are thankful to Massimo Bianchi for bringing our attention to this fact.
4 Fermionic equations of motion and the fermionic zero modes of the string instanton on $CP^3$

In a general supergravity background the equation of motion for the fermions following from the Green–Schwarz action (2.1) (with the choice of superspace constraints given in Appendix A.4) is

\[
0 = -\sqrt{-h^{IJ}} \mathcal{E}_I^A T_{A\alpha} \mathcal{E}_J^A - \frac{1}{2} \varepsilon^{IJ} \mathcal{E}_J^B \mathcal{E}_I^A H_{AB\alpha} = 2i\sqrt{-h^{IJ}} \mathcal{E}_J^A (\Gamma_A \mathcal{E}_I^A) + 2i\varepsilon^{IJ} \mathcal{E}_J^A (\Gamma_A \Gamma_{11} \mathcal{E}_I^A) + i\sqrt{-h^{IJ}} G_{IJ} \lambda^\alpha - i\varepsilon^{IJ} \mathcal{E}_J^A \mathcal{E}_I^B (\Gamma_{AB} \Gamma_{11} \lambda^\alpha). \tag{4.25}
\]

Taking into account that on the mass shell the auxiliary metric $h^{IJ}$ and the induced metric $G_{IJ} = \frac{1}{2} h^{IJ} (h^{KL} G_{KL}) \Rightarrow \sqrt{-h^{IJ}} G_{IJ} = \sqrt{-h^{IJ}} = \sqrt{-G} \Rightarrow \sqrt{-h^{IJ}} G_{IJ} = 2\sqrt{-G}$ the fermionic equations of motion take the following form which reflects the kappa–symmetry of the theory

\[(1 - \Gamma) [G^{IJ} \mathcal{E}_I^A \Gamma_A \mathcal{E}_J^A + \lambda] = 0, \tag{4.26}\]

where $\Gamma$ is the matrix which appears in the kappa–symmetry projector (2.2) and $\lambda^\alpha$ is the dilatino superfield.

For completeness, let us also present the equations of motion of the string bosonic modes

\[\nabla_I (\sqrt{-G} G^{IJ} \mathcal{E}_J^A) + \sqrt{-G} G^{IJ} \mathcal{E}_J^B T_{BA}^D \mathcal{E}_I^D + \frac{1}{2} \varepsilon^{IJ} \mathcal{E}_J^B \mathcal{E}_I^C H_{CBA} = 0, \tag{4.27}\]

where $T_{BA}^D$ are torsion components and $H_{CBA}$ are components of the NS–NS superfield strength with vector indices $A, D$ and with the indices $C$ and $B$ standing for both the vector and the spinor indices (see Appendix A.4).

At the linearized level in the $AdS_4 \times CP^3$ superspace the equation of motion for the fermions (4.26) reduces to

\[0 = (1 - \Gamma) (g^{IJ} e_I^A \Gamma_A \mathcal{D}_J \Theta + \frac{2i}{R} \gamma^5 \nu) \tag{4.28}\]

where $g^{IJ}$ is inverse of

\[g_{IJ} = e_I^A e_J^B \eta_{AB} = e^{-\frac{2}{3} \phi_0} G_{IJ}\big|_{\Theta = 0} \tag{4.29}\]

and

\[\nabla_I \Theta = \left\{ \begin{array}{l}
\nabla_I \nu = (\partial_I - \frac{1}{4} \omega^{ab}_{I}(\partial_{ab} - 2i A_J \gamma_{11})\nu)
\n\nabla_I \vartheta = (\partial_I - \frac{1}{4} \omega^{ab}_{I}(\partial_{ab} - \frac{1}{4} \omega_{a'b'} \gamma_{a'b'})\vartheta
\end{array} \right., \tag{4.30}\]

where remember that $A_J = \frac{1}{8} \omega_{a'b'} J_a b'$.

Note that one can alternatively derive eq. (4.28) by varying the quadratic action (2.8) or its Wick rotated counterpart (2.10).
4.1 Restriction to the instanton solution

As we have discussed in Section 3, the instanton solution is supported on the \( CP^3 \) two-dimensional subspace whose tangent space is characterized e.g. by the first two values of the \( CP^3 \) tangent space index \( a'=1,2,\ldots,6 \). Restricting to this solution we have

\[
e_f^a = 0, \quad e_f^{a'} = (e_f^i, e_f^\bar{a}) = 0, \quad J_{ij} = \varepsilon_{ij}, \quad i = 1,2 \quad \text{and} \quad \bar{a} = 3, 4, 5, 6. \tag{4.31}
\]

It will be convenient to choose the \( CP^3 \) gamma matrices as follows

\[
\gamma^a' = (\rho^i \otimes 1, \rho^3 \otimes \gamma^\bar{a}), \quad \gamma_7 = -\rho^3 \otimes \gamma_5, \quad \gamma_5 = \frac{1}{4!} \varepsilon_{\bar{a}\bar{b}\bar{c}\bar{d}} \gamma_{\bar{a}\bar{b}\bar{c}\bar{d}}, \tag{4.32}
\]

where \((\rho^1, \rho^2, \rho^3) = (\sigma^1, \sigma^3, -\sigma^2 = i\varepsilon)\) are the (re-labeled) Pauli matrices so that \( \rho^1 \rho^2 = i\rho^3 \), and \( \gamma^\bar{a} \) are \( 4 \times 4 \) Dirac gamma matrices corresponding to the four-dimensional subspace of \( CP^3 \) orthogonal to the instanton \( CP^1 \) and \( \gamma_5^2 = 1 \).

The Wick rotated kappa–symmetry projection matrix \((2.11)\) then reduces to

\[
\Gamma = i \frac{e_i^A}{2\sqrt{G}} \varepsilon^{IJ} e_i^I e_j^J \rho_{ij} \rho^3 \gamma_5 \gamma_5 = -\frac{\det e_i^I}{\sqrt{\det e_i^I e_j^J \delta_{ij}}} \gamma_5 \gamma_5 = -\gamma_5 \gamma_5
\]

and the fermionic part of the Euclidean action \((2.10)\) becomes

\[
S_F = T \int d^2 \xi \sqrt{g} g^{IJ} \Theta (1 - \Gamma) \gamma_5 \left[ i e_i^I \rho_i \nabla_I \Theta - \frac{1}{R} e_i^I e_j^J \rho_i \rho_j \Theta \right], \tag{4.34}
\]

where the metric \( g_{IJ} \) was defined in \((4.29)\). Note that in our case the fermionic terms of this two–dimensional theory differ from those of the conventional \( 2d \) supersymmetric \( O(3) \sim CP^1 \) (or in general \( CP^N \)) sigma–model (see \([35, 37]\) for a review and references). For comparison, the \( CP^N \) sigma–model Lagrangian is

\[
L_{CP^N} = G_{ab}(\zeta, \bar{\zeta}) \left( \partial_I \zeta^b \partial_I \zeta^a + i \Psi^b \rho^I D_I \Psi^a \right) - \frac{1}{2} R_{abcd}(\Psi^b \Psi^c) (\Psi^d \Psi^e), \tag{4.35}
\]

where now \( \zeta^a(\xi) \) \((a, \bar{a} = 1, \ldots, N)\) are the complex \( CP^N \) coordinates and \( \Psi^a \) and \( \Psi^{\bar{a}} \) are independent complex \( 2N \)-component spinor fields, \( G_{ab}(\zeta, \bar{\zeta}) \) is the Kähler (Fubini-Study) metric on \( CP^N \) (see eq. \((B.1)\) of Appendix B for the \( CP^3 \) case), \( D_I \Psi^a = \partial_I \Psi^a + \Gamma_{bc}^a \partial_I \zeta^b \Psi^c \) and \( \Gamma_{bc}^a \) and \( R_{abcd} \) are the \( CP^N \) Christoffel symbol and curvature, respectively.

In view of the form of the quadratic action \((4.34)\) and of the fermionic equation \((4.28)\) it is natural to impose on the fermionic fields the conventional kappa–symmetry gauge–fixing condition

\[
\frac{1}{2} (1 + \Gamma) \Theta = \frac{1}{2} (1 - \gamma_5 \gamma_5) \Theta = 0, \tag{4.36}
\]

which means that the fermions split into two sectors according to their chiralities in \( AdS_4 \) and in the four-dimensional subspace of \( CP^3 \) orthogonal to the instanton \( CP^1 \)

\[
\Theta_+ : \quad \gamma_5 \Theta_+ = \gamma_5 \Theta_+ = \Theta_+, \quad \Theta_- : \quad \gamma_5 \Theta_- = \gamma_5 \Theta_- = -\Theta_-. \tag{4.37}
\]
Using the form of the $CP^3$ gamma–matrices (4.32) we find that

$$J = -i J_{a' b'} \gamma^{a' b'} \gamma_7 = -2 \gamma_5 + i J_{\tilde{a} \tilde{b}} \gamma_5 \rho^3 = -2 \gamma_5 + 2 \rho^3 \tilde{J} (1 - \gamma_5),$$

where

$$\tilde{J} = -\frac{i}{4} J_{\tilde{a} \tilde{b}} \gamma_5 = -\frac{i}{8} J_{\tilde{a} \tilde{b}} (1 - \gamma_5) \quad \tilde{J}^2 = \frac{1}{2} (1 - \gamma_5).$$

So, the supersymmetry projection matrices $\mathcal{P}_2$ and $\mathcal{P}_6$ become

$$\mathcal{P}_2 = \frac{1}{8} (2 + J) = \frac{1}{4} (1 + \rho^3 \tilde{J}) (1 - \gamma_5)$$

$$\mathcal{P}_6 = \frac{1}{8} (6 - J) = \frac{1}{4} (3 + \gamma_5 - \rho^3 \tilde{J} (1 - \gamma_5)).$$

Their action on the two sets of the chiral fermions is

$$\mathcal{P}_6 \Theta^+ = \Theta^+ = \vartheta^+$$

$$\mathcal{P}_2 \Theta^+ = \vartheta^+ = 0,$$

$$\mathcal{P}_6 \Theta^- = \frac{1}{2} (1 - \rho^3 \tilde{J}) \Theta^- = \vartheta^-$$

$$\mathcal{P}_2 \Theta^- = \frac{1}{2} (1 + \rho^3 \tilde{J}) \Theta^- = \vartheta^-.$$

Note that from eqs. (4.41) and (4.42) it follows that all the eight $\vartheta^+$ are fermions which correspond to unbroken supersymmetries of the $AdS_4 \times CP^3$ superbackground, while in the $\Theta^-$ sector four fermions ($\vartheta^-$) correspond to unbroken supersymmetries and other four ($\vartheta^-$) to the broken ones. Note also that since for the instanton configuration the kappa–symmetry projector (4.33) commutes with the ‘supersymmetry’ projectors (4.40), it is not possible to choose the kappa–symmetry gauge–fixing condition which would put to zero all the eight ‘broken–supersymmetry’ fermions. In terms of the fields $\vartheta^+, \vartheta^-$ and $\vartheta$ the fermionic action (4.34) takes the form

$$S_F = 2T \int d^2 \xi \det e \left[ i \vartheta^+ e^I_i \rho^I I \vartheta^+ + \frac{2i}{R} \vartheta^+ \vartheta^+ - 2 (i \vartheta^+ e^I_i \rho^I I \vartheta^- - \frac{1}{R} \vartheta^+ \vartheta^-) \right],$$

where $e^I_i$ is the inverse vielbein on $S^2$.

For the instanton configuration the fermionic equation (4.28) reduces to the following ones

$$e^I_i \rho^I I \vartheta^+ + \frac{2i}{R} \vartheta^+ = 0,$$

$$e^I_i \rho^I I \vartheta^- + \frac{2i}{R} \vartheta^- = 0,$$

$$e^I_i \rho^I I \vartheta = 0.$$

From the form of the action (4.43) and the equation of motion (4.45) it follows that the field $\vartheta$ can be regarded as an auxiliary one, which can be expressed in terms of a derivative of $\vartheta^-$. However, for the analysis of the solutions of eqs. (4.41)–(4.46) it is more convenient to consider it as an independent variable satisfying the Dirac equation (4.46).

The covariant derivative $\nabla_I$ (defined in (4.30)) contains the pullback on the instanton two–sphere of the $CP^3$ spin connection whose explicit form is given in Appendix B

$$\nabla_I \vartheta^\pm = \left( \partial_I - \frac{1}{4} \omega^i_{\alpha' \beta'} \gamma_{a' a'} \right) \vartheta^\pm.$$
Computing the pullback of the $CP^3$ connection, substituting it into the Dirac equations and taking into account the projection properties of the spinors we get the fermionic equations in the following form

\[ e_i^I \rho^i (\nabla^{S^2}_{I} + i \tilde{A}_I \rho^3) \vartheta_+ = 0, \]

\[ e_i^I \rho^i (\nabla^{S^2}_{I} \vartheta_+ + \frac{2i}{R} e_i^j \rho_j \vartheta_+) = 0, \]

\[ e_i^I \rho^i (\nabla^{S^2}_{I} - i \tilde{A}_I \rho^3) \vartheta_+ = 0, \]

where $\nabla_{S^2} = d - \frac{1}{4} \omega^{ij}_{S^2} \rho_{ij}$ is the intrinsic covariant derivative on the sphere of radius $R_{S^2} = R/2$ with curvature $R^{ij}_{S^2} = d \omega^{ij}_{S^2} = \frac{4}{R^2} e^i e^j$ and $\tilde{A}$ can be interpreted as the electromagnetic potential induced by a magnetic monopole of charge $g = -1/2$ placed at the center of the sphere. This is due to the fact that

\[ F = d \tilde{A} = \frac{1}{R^2} e^i e^j \epsilon_{ij} = \frac{1}{2} e^i e^j F_{ji} \Rightarrow F_{ij} = -\frac{2}{R^2} \epsilon_{ij} = \frac{g}{R^2} \epsilon_{ij}. \]

Note that $\frac{1}{4} \omega^{ij}_{S^2} \epsilon_{ij}$ and $\tilde{A}$ are equivalent up to a total derivative term

\[ \tilde{A} = \frac{1}{4} \omega^{ij}_{S^2} \epsilon_{ij} + d \lambda. \]

In our parametrization of $CP^3$ (see Appendix B) and for a given embedding of $S^2$ in $CP^3$, $\omega_{S^2}$ and $\tilde{A}$ have the following form in terms of the angular coordinates on $S^2$

\[ \omega^{12}_{S^2} = \cos \theta d \varphi, \quad \tilde{A} = \frac{1}{2} (1 + \cos \theta) d \varphi. \]

We are now in a position to analyze the solutions of the fermionic equations (4.48)–(4.50). Eq. (4.48) has the form of the Dirac equation for a fermion of mass $\frac{2}{R}$. It is the product of $e_i^I \rho^i$ with the Killing spinor equation on the sphere

\[ (\nabla^{S^2}_{I} + \frac{i}{R} e_i^j \rho_j) \vartheta_+ = 0. \]

The Killing spinor equation on $S^2$ for a two–component spinor has two non–trivial solutions [42]. Our $\vartheta_+$ spinors carry four (independent) external indices in addition to the $S^2$–spinor index. Therefore, eq. (4.53) has eight solutions which are obviously solutions of the Dirac equation (4.48). These are actually the only regular eigenspinors of the Dirac operator on the sphere with the eigenvalue $-2i/R$ [43]. Thus, in the $\Theta_+$ sector the string instanton has eight fermionic zero modes which are the solutions of the Killing spinor equation (4.53). In spherical coordinates they have the explicit form [44]

\[ \vartheta_+ = e^{-\frac{i}{2} \theta^1} e^{i \phi^3} \epsilon_+ = \left( \cos \frac{\theta}{2} - i \rho^1 \sin \frac{\theta}{2} \right) \left( \cos \frac{\phi}{2} + i \rho^3 \sin \frac{\phi}{2} \right) \epsilon_+, \]

where $\epsilon_+$ is an arbitrary constant spinor satisfying the chirality conditions $\gamma_5 \epsilon_+ = \gamma_5 \epsilon_+ = \epsilon_+$.  

11
Let us now proceed with the analysis of the third fermionic equation (4.50). As we have already mentioned, this equation describes the electric coupling of the fermionic field $\vartheta$ to the monopole potential on the sphere. The electric charge of $\vartheta$ is $e = \pm 1$ for $\vartheta = \pm \rho_3 \vartheta$, i.e. when $\vartheta$ is a chiral/anti–chiral two–dimensional spinor, respectively. The analysis in [45] then tells us that there are non–trivial solutions of the charged Dirac equation (4.50) of positive chirality when $ge \geq 1/2$ and of negative chirality when $ge \leq -1/2$. Since we are in the opposite situation, there are no non–trivial solutions in our case and hence $\vartheta = 0$.

If $\vartheta = 0$, eq. (4.49) implies that $\vartheta_-$ should satisfy the massless Dirac equation

$$e_i^I \rho^I (\nabla_I S^2 + i \ddot{A}_I \rho^3) \vartheta_- = e_i^I \rho^I (\partial_I + \frac{i}{2} \rho_3 i \partial I \varphi) \vartheta_- = 0.$$  
(4.55)

We observe that the electric charge of $\vartheta_-$ is opposite to that of $\vartheta$, i.e. it is $e = \mp 1$ depending on whether $\vartheta_-$ is chiral or anti–chiral two–dimensional spinor, i.e. whether $\vartheta_- = \pm \rho_3 \vartheta_-$. Now we are in the situation in which the requirement of [45] for the Dirac equation (4.55) to have non–trivial solutions is saturated, i.e. in our case for $\vartheta_-$ of positive $\rho^3$–chirality $ge = 1/2$ and for $\vartheta_-$ of negative $\rho^3$–chirality $ge = -1/2$. By the Atiyah–Singer index theorem there is one solution for each $\rho^3$–chirality of $\vartheta_-$. The general solution of (4.55) has actually a very simple form

$$\vartheta_- = \frac{1}{2} e^{-\frac{i}{2} \rho_3 i \varphi} [(1 + \rho^3) \lambda_- (\zeta) + (1 - \rho^3) \mu_- (\bar{\zeta})],$$  
(4.56)

where $\lambda_- (\zeta)$ and $\mu_- (\bar{\zeta})$ are holomorphic and anti–holomorphic spinors in the projective coordinates $\zeta$ and $\bar{\zeta}$ of $S^2 \simeq CP^1$ which are anti–chiral in the directions transverse to the instanton, i.e. $\lambda_- = - \gamma_5 \lambda_-, \mu_- = - \gamma_5 \mu_-$ and $\mu_- = - \gamma_5 \mu_-$. For the anti–instanton the solution takes the same form but with anti–holomorphic $\lambda_- (\bar{\zeta})$ and holomorphic $\mu_- (\zeta)$.

In [45] it has been shown that the only normalizable solutions of the Dirac equation (4.55) are those with constant $\lambda_-$ and $\mu_-$ in (4.56). This allows us to conclude that in the $\vartheta_-$ sector the string instanton has four zero modes characterized by eq. (4.56) with constant $\lambda_-$ and $\mu_-$. Note that for $\lambda_-=const$ and $\mu_-=const$ the spinor (4.56) is the solution of the stronger equation

$$(\partial_I + \frac{i}{2} \rho_3 i \partial I \varphi) \vartheta_- = 0.$$  
(4.57)

This equation is the projection on the instantonic sphere of the $AdS_4 \times CP^3$ Killing spinor equation for $\vartheta_-$. To summarize, when $\vartheta = 0$ and in view of the form of the fermionic supervielbeins $E^{a'}_{a'} (a' = 1, \ldots, 6)$ of the supercoset $OSp(6|4)/U(3) \times SO(1,3)$ (see Appendix A.7), the non–linear fermionic equation of motion (4.20) as well as the linear one (4.28) involve the pull–back on the string worldsheet of the $AdS_4 \times CP^3$ Killing spinor operator

$$D \vartheta = D_{24} \vartheta = \mathcal{P}_6 (d + \frac{i}{R} e^a \gamma^5 \gamma_a + \frac{i}{R} e^{a'} \gamma_{a'} - \frac{1}{4} \omega^{ab} \gamma_{ab} - \frac{1}{4} \omega^{a'b'} \gamma_{a'b'}) \vartheta,$$  
(4.58)

which acts on the 24 fermions $\vartheta$ associated with the supersymmetry of $AdS_4 \times CP^3$ (see Appendix A.7). Therefore, if $\vartheta$ are the 24 Killing spinors on $AdS_4 \times CP^3$ they solve not only

\footnote{Remember that the eight–component spinor $\vartheta_-$ satisfies the additional projection condition (4.42) which reduces the number of its components to four.}

\footnote{To avoid confusion, let us note that the index $a'$ on spinors is different from the same index on bosonic quantities. See the end of Appendix A.5 for a more detailed explanation.}
the linearized equations (4.28) but also the complete fermionic equations (4.26). In the case of the string instanton considered above, the kappa–symmetry projector reduces the number of solutions of the pulled–back Killing spinor equation by half, leaving us with the twelve physical fermionic zero modes. It should also be noted that these fermionic zero modes do not contribute to the bosonic equations (4.27). This guarantees that the bosonic instanton solution does not have a back reaction from the fermionic modes.

We should note that the Dirac equations (4.48)–(4.50) may have (non–normalizable) solutions which are not the Killing spinors (as e.g. eq. (4.56) with non–constant λ and µ). However, these other fermionic modes would modify the string field equations at higher order in fermions. In particular, they would produce a non–trivial contribution to the bosonic field equations (4.27), i.e. back–react on the form of the purely bosonic instanton and, hence, should be discarded.

Let us stress once again that, as we have shown, for the instanton solution considered above the kappa–symmetry cannot eliminate all the eight fermions \(\upsilon\) associated with the supersymmetries broken in the \(AdS_4 \times CP^3\) background. Therefore, even if among the instanton fermionic zero modes there is no \(\upsilon\)–modes, the fluctuations around the instanton solution will have four physical fermionic degrees of freedom corresponding to the target–space supersymmetries broken by the \(AdS_4 \times CP^3\) background.

4.2 Fermionic zero modes and supersymmetry

Let us discuss in more detail how the fermionic zero modes are related to supersymmetry of the \(AdS_4 \times CP^3\) superbackground and, correspondingly, of the superstring action. At the linearized level in fermions the supersymmetry part of the \(OSp(6|4)\) transformations acts as follows

\[
\begin{align*}
\delta \vartheta &= \epsilon, \\
\delta \upsilon &= 0, \\
\delta X^M e_A^A(X) &= -i \epsilon \Gamma^A \vartheta,
\end{align*}
\]

where \(\epsilon \equiv P_6 \epsilon(X)\) are 24 supersymmetry parameters of \(OSp(6|4)\) satisfying the \(AdS_4 \times CP^3\) Killing spinor equation

\[
D\epsilon = \nabla \epsilon + \frac{i}{R} e^A \mathcal{P}_6 \gamma^5 \Gamma_A \epsilon = 0
\]

with the explicit form of \(D\) given in eq. (4.58). Note that, at the leading order in fermions, the eight fermions \(\upsilon\) are not subject to the supersymmetry transformations. The action of the isometry group \(OSp(6|4)\) on these fermions is such that it takes the form of induced \(SO(1,3) \times U(1)\) rotations with parameters depending on \(X, \vartheta\) and the \(OSp(6|4)\) parameters

\[
\delta \upsilon = \frac{1}{4} \Lambda_{AB}(\epsilon, X, \vartheta) \Gamma^{AB} \upsilon.
\]

Thus the first nontrivial term in the supersymmetry variation of \(\upsilon\) is quadratic in fermionic fields which is beyond the linear approximation we are interested in.

It is not hard to see that the quadratic string action (2.28) is invariant under the supersymmetry transformations (4.59) (up to quadratic order in fermions). At the same time, the action
(2.10), which is obtained from (2.8) by the shift (2.9) of the \( CP^3 \) coordinates, is invariant under the supersymmetry with the transformations of the \( \text{shifted} \) bosonic coordinates being

\[
\delta \hat{X}^M e_M^A(\hat{X}) = -i\epsilon \Gamma^A \partial - i\epsilon \Gamma^A \nu = -i\epsilon \Gamma^A \Theta.
\] (4.62)

Let us now briefly recall how the target–space supersymmetry gets converted into worldsheet supersymmetry upon elimination of the un–physical fermionic degrees of freedom by gauge fixing kappa–symmetry. A more detailed discussion of such a “transmutation” of supersymmetry and its partial breaking in the Green–Schwarz formulation of superstrings and superbranes the reader may find \( e.g. \) in \([46, 47, 48, 49]\).

If we impose on the fermionic fields \( \Theta = (\vartheta, \nu) \) a kappa–symmetry gauge condition as \( e.g. \) the one we have used for studying the instanton solution, eq. (4.36),

\[
\frac{1}{2}(1 + \Gamma_0) \Theta = 0,
\] (4.63)

the kappa–symmetry gauge–fixing condition will not be invariant under all the twenty–four supersymmetries (4.59) but only under half of them satisfying the condition

\[
\epsilon_{br} = \frac{1}{2}(1 - \Gamma_0) \epsilon.
\] (4.64)

In eqs. (4.63) and (4.64) we denoted the gauge–fixing projector by \( \Gamma_0 \) to distinguish it from the more general projector matrix \( \Gamma \) that appears in the kappa–symmetry transformations (2.2)–(2.4).

The target–space supersymmetries with the parameter \( \epsilon_{br} \) are those which are spontaneously broken by the presence of the string. The reason is that the remaining twelve fermionic fields \( \vartheta = \frac{1}{2}(-1 - \Gamma_0) \vartheta \) get shifted by these transformations and hence behave as Volkov–Akulov goldstinos \([50, 51]\).

The supersymmetries which remain unbroken and which become worldsheet supersymmetries are identified as follows. The gauge fixing condition (4.63) is not invariant under the supersymmetry transformations (4.59) with the parameter \( \epsilon_w = \frac{1}{2}(1 - \Gamma_0) \epsilon \). However, this can be cured by an appropriate compensating kappa–symmetry transformation that (at the leading order in fermions) satisfies the condition

\[
-\frac{1}{4} \mathcal{P}_6 (1 + \Gamma_0) (1 + \Gamma) \kappa = \epsilon_w \equiv \frac{1}{2}(1 + \Gamma_0) \epsilon.
\] (4.65)

This condition relates the components of the \( \kappa \)–symmetry parameter appearing in the transformation of \( \vartheta \) to the supersymmetry parameter \( \epsilon_w \). Since kappa–symmetry is the worldsheet fermionic symmetry which can actually be identified with the conventional local worldsheet supersymmetry \([52]\), eq. (4.65) thus converts the unbroken target–space supersymmetries into worldsheet supersymmetry. Note that eq. (4.65) does not involve the part of the kappa–symmetry transformation acting on the \( \nu \)–fermions since they are singled out with the complementary projector \( \mathcal{P}_2 \). This part of kappa–symmetry is fixed by the gauge condition

\[
\frac{1}{2}(1 + \Gamma_0) \nu = 0 \quad (\text{see eq. (4.63)}).
\]

As a result, \( e.g. \) at a leading order in fermions and bosons) under the broken and unbroken supersymmetries the worldsheet fermionic fields remaining after the gauge-fixing (4.63) \( \Theta \equiv
$\frac{1}{2} (1 - \Gamma_0) \Theta = (\vartheta, \nu)$ and the bosonic fields $\hat{X}^M$ transform as follows

\[
\delta \vartheta = 0, \quad \delta \nu = \epsilon_{br} + \frac{1}{4} P_6 (1 - \Gamma_0)(1 + \Gamma) \kappa, \quad (4.66)
\]

\[
\delta \hat{X}^M e_M^i (\hat{X}) = -i \epsilon_w \Gamma^i \Theta - \delta \kappa \mathcal{E}^i (\hat{X}, \Theta) + O(\epsilon, \Theta, \hat{X}), \quad (4.67)
\]

\[
\delta \hat{X}^M e_M^\perp (\hat{X}) = -2i \epsilon_w \Gamma^\perp \Theta + \delta \kappa \mathcal{E}^\perp (\hat{X}, \Theta) + O(\epsilon, \Theta, \hat{X}), \quad (4.68)
\]

where $i = 0, 1$ and $\perp = 2, \ldots, 9$ indicate the directions parallel and orthogonal to the string worldsheet, respectively, the second terms in (4.67) and (4.68) come from the compensating kappa–symmetry transformation (2.2) that at the linearized level is just $-i \epsilon \Gamma^A \Theta$, and $O(\epsilon, \Theta, \hat{X})$ stand for terms which are non–linear in fields (and their derivatives).

It is instructive to notice that the leading (linear) term in the supersymmetry transformations of $\hat{X}^M$ along the directions transverse to the string worldsheet contains the parameter of the unbroken supersymmetries, while along the worldsheet the linear term contains the broken supersymmetry parameter. This reflects the fact that the bosonic excitations transversal to the classical string configuration and kappa–gauge fixed fermionic fields are associated with worldsheet physical fields forming supermultiplets under the unbroken supersymmetry. At the same time the supersymmetry transformations along the string worldsheet can be compensated by an appropriated worldsheet reparametrization.

For the instanton solution under consideration we have $\Gamma = \Gamma_0 = -\gamma_5 \tilde{\gamma}_5$ and $\nu = 0$. So the supersymmetry transformations (at the leading order) become

\[
\delta \vartheta = 0 + \ldots, \quad \delta \nu = \epsilon_{br} + \ldots, \quad (4.69)
\]

\[
\delta \hat{X}^M e_M^\perp (\hat{X}) = -2i \epsilon_w \Gamma^\perp \Theta + \ldots, \quad (4.70)
\]

where the dots stand for higher order terms in fermions.

Under the unbroken supersymmetry transformations the fermionic zero modes induce an (isometry) transformation of the string coordinates in the transverse directions which results in a shift of the bosonic parameters characterizing the string instanton. This is analogous to the supersymmetry transformations of the ‘collective coordinates’ of the $CP^N$ sigma–model instanton [35].

From eqs. (4.69) and (4.70) we also see that if we start from the purely bosonic instanton solution discussed in Section 3 we can find at least part of the instanton fermionic zero modes by looking at the variation of the fermionic fields under supersymmetry. The form of the supersymmetry transformations implies that the bosonic instanton configuration is 1/2 BPS. Namely, the string instanton solution with $\Theta = 0$ is invariant under the twelve supersymmetries $\epsilon_w$. Fermionic zero modes are generated by the target–space supersymmetries (with the parameter $\epsilon_{br}$) which are broken by the string configuration, as we have already discussed in the end of Section 4.1 where we have also shown that the instanton does not have other fermionic zero modes associated with the fields $\nu$. Note that the latter cannot be obtained from the purely bosonic solution by a supersymmetry transformation since the corresponding variation of $\nu$ is proportional to $\nu$ itself (see eq. (4.61)).

Let us now compare our $AdS_4 \times CP^3$ superstring worldsheet theory (which has 12 unbroken worldsheet supersymmetries) with the supersymmetry properties of the conventional
\[ n = (2, 2) \] supersymmetric \( CP^N \) sigma–model (described by the Lagrangian (4.35)) and with its instanton solutions [35, 37].

The supersymmetry transformations in the \( CP^N \) sigma–model have the following form

\[ \delta \zeta^a = \epsilon \Psi^a, \quad \delta \Psi^a = i \rho^I \partial_I \zeta^a + \cdots, \quad (a = 1, \ldots, N) \] (4.71)

where \( \epsilon \) is now a constant complex two–component spinor parameter of \( n = (2, 2) \) supersymmetry and the dots stand for the terms non–linear in the fields. The \( CP^N \) sigma–model is also invariant under superconformal transformations [53]

\[ \delta \zeta^a = \bar{\eta}(z, \bar{z}) \Psi^a, \quad \delta \Psi^a = i \rho^I \eta(z, \bar{z}) \partial_I \zeta^a + \cdots, \quad (a = 1, \ldots, N) \] (4.72)

The superconformal transformations are similar to the rigid supersymmetries (4.71) but with the complex two–component spinor parameters whose chiral and anti–chiral components are, respectively, holomorphic and anti–holomorphic \( \eta(z, \bar{z}) = (\eta_+(z), \eta_-(\bar{z})) \).

The superconformal symmetry of the \( CP^N \) sigma–model (which is broken by quantum anomalies [35]) is in a sense a counterpart of the spontaneously broken part of the target–space supersymmetry of the superstring action.

If one starts from the purely bosonic instanton solution of the \( CP^N \) sigma–model

\[ \partial \zeta^a = 0 \quad \text{or} \quad \partial \zeta^a = 0 \quad \text{and} \quad \Psi = 0 \] (4.73)

one can then generate solutions of the fermionic field equations and find the corresponding fermionic zero modes by considering the supersymmetry transformations (4.71) and (4.72) of \( \Psi \). In this way, taking into account that for the instanton the fields \( \zeta^a \) are either holomorphic or anti–holomorphic, one finds that only half of the supersymmetry transformations (4.71) and of the superconformal transformations (4.72) are non–trivial, those with the parameters \( \epsilon \) and \( \eta \) being (anti)chiral 2d spinors. The fermionic zero modes obtained in this way are (anti)holomorphic (anti)chiral 2d spinor fields. We observe that in contrast to the case of the string instanton whose fermionic zero modes are generated by the spontaneously broken supersymmetry transformations, in the case of the \( CP^N \) sigma–model half of the fermionic zero modes are generated by the rigid supersymmetry transformations and another half by the superconformal symmetry.

## 5 Summary and Discussion

We have thus found that the string instanton wrapping the non–trivial two–cycle inside \( CP^3 \) has twelve fermionic zero modes. As we have already mentioned, the eight string fermionic fields \( \vartheta_+ \) which have an effective mass \( \frac{2}{R} \) and four massless \( \vartheta_- \) electrically coupled to the \( S^2 \) monopole field, correspond to twelve (of the twenty four) supersymmetries of the \( AdS_3 \times CP^3 \) background. The fermionic zero modes thus play the role similar to Volkov–Akulov goldstinos [50, 51] and manifest partial breaking of supersymmetry. Note that in \( AdS_3 \times CP^3 \) there also exists an NS5–brane instanton wrapping the entire \( CP^3 \). It would be of interest to analyze possible effects of these instantons in \( AdS_4 \times CP^3 \) superstring theory and to understand their counterparts in the boundary \( CFT_3 \) theory.

\[ ^8n \] labels the real number of left– and right–handed worldsheet supersymmetries.
As was mentioned briefly in Section 3.1, the instanton solution can be generalized by switching on a NS–NS field $B_2 \sim J_2$ of a non–trivial co–homology on $CP^1 \simeq S^2$. The coupling of the string to the $B$–field then results in the instanton action being shifted by a constant imaginary piece. In fact, the co–homologically non–trivial $B$–field arises on the string side of the $AdS_4/CFT_3$ correspondence when one considers the ABJ–model [41] which generalizes the ABJM construction to gauge groups of different rank, i.e. $U(N + l)_k \times U(N)_{-k}$ with $0 < l < k$ instead of $U(N)_k \times U(N)_{-k}$. The appearance of the $B$–field in the string action thus results in breaking the parity–invariance of the theory. In [41] it has been shown that the integral of $B_2$ on the $CP^1$ cycle inside $CP^3$ takes a fractional value

$$\frac{1}{2\pi} \int_{CP^1} B_2 = \frac{l}{k}.$$  

(5.74)

In eleven dimensions this corresponds to a co–homologically nontrivial three–form potential on the 3–cycle $S^3/Z_k \subset S^7/Z_k$. From the point of view of M2–branes probing a $\mathbb{C}^4/Z_k$ singularity this is associated to $l$ fractional M2–branes sitting at the singularity. The fractional M2-branes can be thought of as M5–branes wrapping the corresponding vanishing 3-cycle at the orbifold point, see [41].

This picture suggests that the string instanton considered in this paper should correspond in M–theory to an instantonic M2–brane, i.e. an M2–brane whose worldvolume wraps a 3–cycle $S^3/Z_k \subset S^7/Z_k$. There is, however, a subtlety here. Namely, while in the case of the $D = 10$ type IIA string instanton there are an infinite number ($|n| = 1, 2, \ldots, \infty$) of topologically different configurations, the number of non–equivalent M2–brane configurations wrapping the 3–cycle in $S^7/Z_k$ of the $D = 11$ theory is $k$, since $H_3(S^7/Z_k, \mathbb{Z}) = \mathbb{Z}_k$. The reason is that the string instanton solution has been considered in the $AdS_4 \times CP^3$ background of the pure type IIA $D = 10$ supergravity, i.e. in the limit in which (from the $D = 11$ perspective) the radius of the $S^1$ fiber of $S^7$ tends to zero ($k \to \infty$). The consideration at finite $k$ would require taking into account Kaluza–Klein modes and D–brane effects.

It would be interesting to find out what these instantonic strings, M2–branes and NS5–branes correspond to in the ABJ/ABJM gauge–theory picture.

### Acknowledgements

The authors are grateful to Konstantin Zarembo for the suggestion to look at this stringy instanton problem and for valuable discussions and comments. They are also thankful to Massimo Bianchi, Marco Matone, Nikita Nekrasov and Mario Tonin for useful comments and discussion. This work was partially supported by the INFN Special Initiative TV12. D.S. was also partially supported by an Excellence Grant of Fondazione Cariparo (Padova) and the grant FIS2008-1980 of the Spanish Ministry of Science and Innovation.

### Appendix A. Main notation and conventions

The convention for the ten and eleven dimensional metrics is the ‘almost plus’ signature ($-, +, \cdots, +$). Generically, the tangent space vector indices are labeled by letters from the
beginning of the Latin alphabet, while letters from the middle of the Latin alphabet stand for curved (world) indices. The spinor indices are labeled by Greek letters.

### A.1 AdS4 space

AdS4 is parametrized by the coordinates \( x^m \) and its vielbeins are \( e^a = dx^m e^a_m(x) \), \( m = 0, 1, 2, 3; a = 0, 1, 2, 3 \). The \( D = 4 \) gamma–matrices satisfy:

\[
\{\gamma^a, \gamma^b\} = 2 \eta^{ab}, \quad \eta^{ab} = \text{diag}(-, +, +, +),
\]

(A.1)

\[
\gamma^5 = i\gamma^0 \gamma^1 \gamma^2 \gamma^3, \quad \gamma^5 \gamma^5 = 1.
\]

(A.2)

The charge conjugation matrix \( C \) is antisymmetric, the matrices \((\gamma^a)_{\alpha\beta} \equiv (C\gamma^a)_{\alpha\beta}\) and \((\gamma^{ab})_{\alpha\beta} \equiv (C\gamma^{ab})_{\alpha\beta}\) are symmetric and \(\gamma^5_{\alpha\beta} \equiv (C\gamma^5)_{\alpha\beta}\) is antisymmetric, with \(\alpha, \beta = 1, 2, 3, 4\) being the indices of a 4–dimensional spinor representation of \(SO(1, 3)\) or \(SO(2, 3)\).

### A.2 CP3 space

CP3 is parametrized by the coordinates \( y^{m'} \) and its vielbeins are \( e^{a'} = dy^{m'} e^{a'}_{m'}(y) \), \( m' = 1, \cdots , 6; a' = 1, \cdots , 6 \). The \( D = 6 \) gamma–matrices satisfy:

\[
\{\gamma^{a'}, \gamma^{b'}\} = 2 \delta^{a'b'}, \quad \delta^{a'b'} = \text{diag}(+, +, +, +, +, +),
\]

(A.3)

\[
\gamma^7 = \frac{i}{6!} \varepsilon_{a'_1 a'_2 a'_3 a'_4 a'_5 a'_6} \gamma^{a'_1} \cdots \gamma^{a'_6} \gamma^7 \gamma^7 = 1.
\]

(A.4)

The charge conjugation matrix \( C' \) is symmetric and the matrices \((\gamma^{a'})_{\alpha'\beta'} \equiv (C'\gamma^{a'})_{\alpha'\beta'}\) and \((\gamma^{a'b'})_{\alpha'\beta'} \equiv (C'\gamma^{a'b'})_{\alpha'\beta'}\) are antisymmetric, with \(\alpha', \beta' = 1, \cdots , 8\) being the indices of an 8–dimensional spinor representation of \(SO(6)\).

### A.3 Type IIA AdS4 × CP3 superspace

The type IIA superspace whose bosonic body is \(AdS_4 \times CP^3\) is parametrized by 10 bosonic coordinates \( X^M = (x^m, y^{m'}) \) and 32-fermionic coordinates \( \Theta^\mu = (\Theta^{\mu'}) \) \( (\mu = 1, 2, 3, 4; \mu' = 1, \cdots , 8) \). These combine into the superspace supercoordinates \( Z^\mathcal{M} = (x^m, y^{m'}, \Theta^{\mu'}) \). The type IIA supervielbeins are

\[
\mathcal{E}^A = dZ^\mathcal{M} \mathcal{E}_\mathcal{M} A(Z) = (\mathcal{E}^A, \mathcal{E}^{a'}), \quad \mathcal{E}^A(Z) = (\mathcal{E}^a, \mathcal{E}^{a'}), \quad \mathcal{E}^{a'}(Z) = \mathcal{E}^{a'a'}. \quad (A.5)
\]

### A.4 Superspace constraints

In our conventions the superspace constraint on the bosonic part of the torsion is

\[
T^A = -i\mathcal{E} \Gamma^A \mathcal{E} + i\mathcal{E}^A \mathcal{E} \lambda + \frac{1}{3} \mathcal{E}^A \mathcal{E}^B \nabla_B \phi, \quad (A.6)
\]
while the constraints on the RR and NS–NS field strengths are

\[
F_2 = -ie^{-\phi} \xi_{11} \epsilon + 2ie^{-\phi} \xi^A \xi \Gamma a \Gamma 11 \lambda + \frac{1}{2} \xi^B \xi^A F_{AB}, \quad (A.7)
\]
\[
F_4 = -\frac{i}{2} e^{-\phi} \xi^B \xi^A \xi \Gamma_{AB} \epsilon + \frac{1}{4!} \xi^B \xi^C \xi^B \xi^A F_{ABCD}, \quad (A.8)
\]
\[
H_3 = -ie^A \xi \Gamma a \Gamma_{11} \epsilon + ie^B \xi^A \xi \Gamma_{AB} \Gamma^{11} \lambda + \frac{1}{3!} \xi^C \xi^B \xi^A H_{ABC}. \quad (A.9)
\]

These differ from the conventional string frame constraints by the dilatino \( \lambda \)–term in \( T^A \) and related terms in \( F_2, F_4 \) and \( H_3 \). This is a consequence of the dimensional reduction from eleven dimensions. The constraints can be brought to a more conventional string–frame form by shifting the fermionic supervielbein \( E^\alpha \) by \( -\frac{1}{2} E^A (\Gamma^A \lambda) \) accompanied by a related shift in the connection. Note also that the purely bosonic part of the torsion, the last term in (A.6), can be eliminated by a proper redefinition of the spin connection.

The \( D = 10 \) gamma–matrices \( \Gamma^A \) are given by

\[
\{ \Gamma^A, \Gamma^B \} = 2 \eta^{AB}, \quad \Gamma^A = (\Gamma^a, \Gamma^a'), \quad (A.10)
\]

\[
\Gamma^a = \gamma^a \otimes 1, \quad \Gamma^{a'} = \gamma^5 \otimes \gamma^{a'}, \quad \Gamma^{11} = \gamma^5 \otimes \gamma^7, \quad a = 0, 1, 2, 3; \quad a' = 1, \ldots, 6.
\]

The charge conjugation matrix is \( C = C \otimes C' \).

The fermionic variables \( \Theta^a \) of IIA supergravity carrying 32–component spinor indices of \( \text{Spin}(1,9) \), in the \( AdS_4 \times CP^3 \) background and for the above choice of the \( D = 10 \) gamma–matrices, naturally split into 4–dimensional \( \text{Spin}(1,3) \) indices and 8–dimensional spinor indices of \( \text{Spin}(6) \), i.e. \( \Theta^a = \Theta^a \alpha' \) (\( \alpha = 1, 2, 3, 4; \quad \alpha' = 1, \ldots, 8 \)).

A.5 24 + 8 splitting of 32 \( \Theta \)

24 of \( \Theta^a = \Theta^a \alpha' \) correspond to the unbroken supersymmetries of the \( AdS_4 \times CP^3 \) background. They are singled out by a projector introduced in [8] which is constructed using the \( CP^3 \) Kähler form \( J_{ab} \) and seven \( 8 \times 8 \) antisymmetric gamma–matrices (A.3). The \( 8 \times 8 \) projector matrix has the following form

\[
P_6 = \frac{1}{8}(6 - J), \quad (A.11)
\]

where the \( 8 \times 8 \) matrix

\[
J = -iJ_{ab} \gamma^a \gamma^b \gamma^7 \quad \text{such that} \quad J^2 = 4J + 12 \quad (A.12)
\]

has six eigenvalues \(-2\) and two eigenvalues \(6\), \( i.e. \) its diagonalization results in

\[
J = \text{diag}(-2, -2, -2, -2, -2, -2, -2, 6, 6). \quad (A.13)
\]

Therefore, the projector (A.11) when acting on an 8–dimensional spinor annihilates 2 and leaves 6 of its components, while the complementary projector

\[
P_2 = \frac{1}{8}(2 + J), \quad P_2 + P_6 = 1 \quad (A.14)
\]
annihilates 6 and leaves 2 spinor components.

Thus the spinor
\[ \vartheta^{a\alpha'} = (\mathcal{P}_6 \Theta)^{a\alpha'} \quad \iff \quad \vartheta^{a\alpha'} \quad a' = 1, \ldots, 6 \] (A.15)
has 24 non-zero components and the spinor
\[ \upsilon^{a\alpha'} = (\mathcal{P}_2 \Theta)^{a\alpha'} \quad \iff \quad \upsilon^{a\alpha'} \quad i = 1, 2 \] (A.16)
has 8 non-zero components. The latter corresponds to the eight supersymmetries broken by the \( AdS_4 \times \mathbb{C}P^3 \) background.

To avoid confusion, let us note that the index \( a' \) on spinors is different from the same index on bosonic quantities. They are related by the usual relation between vector and spinor representations, i.e. given two \( \text{Spin}(6) \) spinors \( \psi_1^{\alpha'} \) and \( \psi_2^{\alpha'} \), projected as in (A.15), their bilinear combination \( v^{a'} = \psi_1 \mathcal{P}_6 \gamma^{a'} \mathcal{P}_6 \psi_2 = \psi_1 (\mathcal{P}_6 \gamma^{a'} \mathcal{P}_6) \upsilon^{b'} \psi_2^{c'} \) transforms as a 6-dimensional 'vector'.

### A.6 The explicit form of the geometry of the \( AdS_4 \times \mathbb{C}P^3 \) superspace

The supervielbeins have the following form

\[
egin{align*}
\mathcal{E}^{a'}(x, y, \vartheta, \upsilon) &= e^{\frac{1}{4} \phi(\upsilon)} \left( E^{a'}(x, y, \vartheta) + 2iv \frac{\sinh m}{m} \gamma^a \gamma^5 E(x, y, \vartheta) \right), \\
\mathcal{E}^a(x, y, \vartheta, \upsilon) &= e^{\frac{1}{4} \phi(\upsilon)} \left( E^a(x, y, \vartheta) + 4iv \frac{\sinh^2 \mathcal{M}/2 \mathcal{M}^2}{\mathcal{M}^2} Dv \right) \Lambda^{\alpha}_a(v) \\
&\quad - e^{-\frac{1}{4} \phi(\upsilon)} \frac{R^2}{k_l p} \left( A(x, y, \vartheta) - \frac{4}{R} v \varepsilon^\beta \frac{\sinh^2 \mathcal{M}/2 \mathcal{M}^2}{\mathcal{M}^2} Dv \right) E^a_7(v), \\
\mathcal{E}^{\alpha i}(x, y, \vartheta, \upsilon) &= e^{\frac{1}{4} \phi(\upsilon)} \left( \frac{\sinh \mathcal{M}}{\mathcal{M}} Dv \right)^{\beta j} \gamma^{a'} \gamma^{5} \epsilon_{ji}^\alpha \gamma_5^a \upsilon^{\alpha i}, \\
\mathcal{E}^{a\alpha'}(x, y, \vartheta, \upsilon) &= e^{\frac{1}{4} \phi(\upsilon)} E^{\gamma b'}(x, y, \vartheta) \left( \delta_5^\beta - \frac{8}{R} \frac{\sinh^2 m/2 m^2}{m^2} \right) v^{\beta i} \upsilon^{\alpha i}. 
\end{align*}
\]

The new objects appearing in these expressions, \( m, \mathcal{M}, \Lambda_{a}^{b}, E^{7}_{\alpha}, \) and \( S^{\beta}_{\alpha} \), are functions of \( \upsilon \) and their explicit forms are given in Appendix A.8 while the dilaton \( \phi \), dilatino \( \lambda \) and RR one–form \( \mathcal{A}_1 \) are given below. Contracted spinor indices have been suppressed, e. g. \( (\upsilon \varepsilon^5 \gamma^a)_{\alpha i} = \upsilon^{\beta i} \varepsilon_{ji} \gamma^5_{\beta a} \), where \( \varepsilon_{ij} = -\varepsilon_{ji}, \varepsilon_{12} = 1 \) is the \( SO(2) \) invariant tensor. Note that \( \varepsilon = -i \mathcal{P}_2 \gamma_7 \mathcal{P}_2 \).

To avoid confusion, let us stress that the indices \( i, j \) carried by the \( \mathcal{P}_2 \)–projected spinors \( \upsilon^{a\alpha} \) and by the associated quantities, are different from the indices \( i, j \) used in the main text of the paper to define the tangent–space indices of \( S^2 \). The indices carried by \( \upsilon \) never appear in the main text. Using the same indices but for labeling different quantities in some instances which do not cause the confusion we thus avoid redundant proliferation of different labels.
The covariant derivative is defined as

\[ D\upsilon = \left( d + \frac{i}{R} E^a(x, y, \vartheta) \gamma^5 \gamma_a - \frac{1}{4} \Omega^{ab}(x, y, \vartheta) \gamma_{ab} \right) \upsilon. \]  

(A.18)

The type IIA RR one–form gauge superfield is

\[ \mathcal{A}_1(x, y, \vartheta, \upsilon) = Re^{-\frac{4}{3} \phi(\upsilon)} \left[ \left( A(x, y, \vartheta) - \frac{4}{R} \upsilon \vartheta^5 \frac{\sinh^2 \mathcal{M}/2}{\mathcal{M}^2} D\upsilon \right) \frac{R}{kl_p} \Phi(\upsilon) 
+ \frac{1}{kl_p} \left( E^a(x, y, \vartheta) + 4i \upsilon \gamma^a \frac{\sinh^2 \mathcal{M}/2}{\mathcal{M}^2} D\upsilon \right) E_7a(\upsilon) \right]. \]  

(A.19)

The RR four-form and the NS–NS three-form superfield strengths are given by

\[ F_4 = d\mathcal{A}_3 - \mathcal{A}_1 H_3 = -\frac{1}{4!} \varepsilon^d e^c e^b e^a \left( \frac{6}{kl_p} e^{-2\phi} \varepsilon_{abcd} \right) - \frac{i}{2} \varepsilon^B \varepsilon^A \varepsilon^1 \varepsilon^2 e^{-\phi} (\Gamma_{AB})_{a\beta}, \]  

\[ H_3 = dB_2 = -\frac{1}{3!} \varepsilon^c \varepsilon^b \varepsilon^a \left( \frac{6}{kl_p} e^{-\phi} \varepsilon_{abcd} E_7^d \right) - i \varepsilon^A \varepsilon^\beta \varepsilon^\alpha (\Gamma_{A} \Gamma_{11})_{a\beta} + i \varepsilon^B \varepsilon^A \varepsilon^{\alpha} (\Gamma_{AB} \Gamma_{11}) \lambda, \]  

(A.20)

and the corresponding gauge potentials are

\[ B_2 = b_2 + \int_0^1 dt i_\Theta H_3(x, y, t\Theta), \quad \Theta = (\vartheta, \upsilon) \]  

(A.21)

\[ \mathcal{A}_3 = a_3 + \int_0^1 dt i_\Theta (F_4 + \mathcal{A}_1 H_3) (x, y, t\Theta), \]  

(A.22)

where \( b_2 \) and \( a_3 \) are the purely bosonic parts of the gauge potentials and \( i_\Theta \) means the inner product with \( \Theta^\alpha \). Note that \( b_2 \) is pure gauge in the \( AdS_4 \times CP^3 \) solution while \( a_3 \) is the RR three-form potential of the bosonic background.

The dilaton superfield \( \phi(\upsilon) \), which depends only on the eight fermionic coordinates corresponding to the broken supersymmetries, has the following form in terms of \( E_7^a(\upsilon) \) and \( \Phi(\upsilon) \)

\[ e^{\frac{4}{3} \phi(\upsilon)} = \frac{R}{kl_p} \sqrt{\Phi^2 + E_7^a E_7^b \eta_{ab}}. \]  

(A.23)

The value of the dilaton at \( \upsilon = 0 \) is

\[ e^{\frac{4}{3} \phi(\upsilon)} |_{\upsilon=0} = e^{\frac{2}{3} \phi_0} = \frac{R}{kl_p}. \]  

(A.24)

The fermionic field \( \lambda^{ai}(\upsilon) \) describes the non–zero components of the dilatino superfield and is given by the equation \[54\]

\[ \lambda_{ai} = -\frac{i}{3} D_{ai} \phi(\upsilon). \]  

(A.25)

In the above expressions \( E^a(x, y, \vartheta) \), \( E^a(\upsilon) \) and \( \Omega^{ab}(x, y, \vartheta) \) are the supervielbeins and the \( AdS_4 \) part of the spin connection of the supercoset \( OSp(6|4)/U(3) \times SO(1, 3) \) and \( A(x, y, \vartheta) \) is the corresponding type IIA RR one–form gauge superfield whose explicit form is given below.
A.7 \( OSp(6|4)/U(3) \times SO(1,3) \) supercoset realization and other ingredients of the \((10|32)\)-dimensional \( AdS_4 \times CP^3 \) superspace

The supervielbeins and the superconnections of the \( OSp(6|4)/U(3) \times SO(1,3) \) supercoset which appear in the definition of the geometric and gauge quantities of the \( AdS_4 \times CP^3 \) superspace can be parametrized in the following way

\[
E^a = e^a(x) + 4i \theta \gamma^a \frac{\sinh^2 M_{24}/2}{M_{24}^2} D_{24} \theta, \\
E^{a'} = e^{a'}(y) + 4i \theta \gamma^{a'} \gamma^5 \frac{\sinh^2 M_{24}/2}{M_{24}^2} D_{24} \theta, \\
E^{aa'} = \left( \frac{\sinh M_{24}}{M_{24}} D_{24} \theta \right)^{aa'}, \\
\Omega^{ab} = \omega^{ab}(x) + \frac{8}{R} \theta \gamma_{ab} \gamma^5 \frac{\sinh^2 M_{24}/2}{M_{24}^2} D_{24} \theta, \\
\Omega^{a'b'} = \omega^{a'b'}(y) - \frac{4}{R} \theta (\gamma^{a'b'} - i J^{a'b'} \gamma^5) \gamma^5 \frac{\sinh^2 M_{24}/2}{M_{24}^2} D_{24} \theta, \\
A = \frac{1}{8} J^{a'b'} \Omega^{a'b'} = A(y) + \frac{4i}{R} \theta \gamma^5 \frac{\sinh^2 M_{24}/2}{M_{24}^2} D_{24} \theta,
\]

where

\[
R(M_{24}^2)^{aa'}_{\beta \beta'} = 4 \theta^{a'}_\beta \left( \delta^{a'}_\gamma \gamma_\beta \right) - 4 \delta^{a'}_\beta \theta \gamma^{a'} \left( \delta^c_\beta \gamma^5 \right) - 2 (\gamma^{a'} \theta) \delta^{a'}_\beta \gamma^5_\beta - (\gamma^{ab}) \delta^{a'}_\beta \gamma^5_\beta
\]

The derivative appearing in the above equations is defined as

\[
D_{24} \theta = \mathcal{D}_6 (d + i \frac{1}{R} e^a \gamma^5 \gamma_a + i \frac{1}{R} e^{a'} \gamma^{a'} - \frac{1}{4} \omega^{a'b} \gamma_\beta - \frac{1}{4} \omega^{a'b'} \gamma_\beta) \theta,
\]

where \( e^a(x) \), \( e^{a'}(y) \), \( \omega^{ab}(x) \), \( \omega^{a'b'}(y) \) and \( A(y) \) are the vielbeins and connections of the bosonic solution. The \( U(3) \) connection \( \Omega^{a'b'} \) satisfies the condition

\[
(P^-)^{a'}_{a'b'} \omega^{d'd'} \Omega^{c'd'} = \frac{1}{2} \left( \delta^c_{[a'} \delta^d_{b']} - J^c_{a'} J^d_{b'} \right) \Omega^{c'd'} = 0,
\]

where \( J^{a'b'} \) is the Kähler form on \( CP^3 \).

Let us stress once again that the index \( a' \) on spinors is different from the same index on bosonic quantities. See the end of Appendix A.5 for more explanation.

A.8 Other quantities appearing in the definition of the \( AdS_4 \times CP^3 \) superspace

\[
R(M^2)^{\alpha_i}_{\beta_j} = 4 (\varepsilon \varepsilon^5)^{\alpha_i}(\varepsilon \varepsilon^5)_{\beta_j} - 2 (\gamma^5 \gamma^a \varepsilon^i (\varepsilon \gamma_\beta \gamma_a)_{\beta_j} - (\gamma^{ab} \varepsilon^i (\varepsilon \gamma_{ab} \gamma^5)_{\beta_j},
\]

\[
(m^2)^{ij} = - \frac{4}{R} \varepsilon^i \gamma^5 \varepsilon^j,
\]
\[ \Lambda_a^b = \delta_a^b - \frac{R^2}{k l_p^2} \cdot \frac{e^{-\frac{2}{3} \phi}}{e^{\frac{2}{3} \phi} + \frac{R}{k l_p} \Phi} E_7 a E_7^b, \]

\[ S_{2 \alpha} = \frac{e^{-\frac{1}{3} \phi}}{\sqrt{2}} \left( \sqrt{e^{\frac{2}{3} \phi} + \frac{R}{k l_p} \Phi} - \frac{R}{k l_p} \frac{E_7^a \Gamma a_{11}}{\sqrt{e^{\frac{2}{3} \phi} + \frac{R}{k l_p} \Phi}} \right) \]

\[ E_7^a (\nu) = -\frac{8i}{R} \nu^a \frac{\sin^2 \mathcal{M}/2}{\mathcal{M}^2} \varepsilon \nu, \]

\[ \Phi (\nu) = 1 + \frac{8}{R} \nu^a \varepsilon^a \frac{\sin^2 \mathcal{M}/2}{\mathcal{M}^2} \varepsilon \nu. \]

Let us emphasise that the \( SO(2) \) indices \( i, j = 1, 2 \) are raised and lowered with the unit matrices \( \delta_{ij} \) and \( \delta^{ij} \) so that there is actually no difference between the upper and the lower \( SO(2) \) indices, \( \varepsilon_{ij} = -\varepsilon_{ji} \), \( \varepsilon^{ij} = -\varepsilon^{ji} \) and \( \varepsilon^{12} = \varepsilon_{12} = 1 \).

**Appendix B. \( CP^3 \) Geometry**

The Fubini-Study metric on \( CP^3 \) is

\[ ds^2 = \rho^{-2} d\tilde{\zeta}_a d\zeta^a - \rho^{-4} \zeta^a d\tilde{\zeta}_a \tilde{\zeta}_b d\zeta^b, \]

where \( \zeta^a \) are three complex numbers and \( \rho^2 = 1 + \zeta^a \tilde{\zeta}_a \). Real coordinates adapted to the \( U(3) \) isotropy group can be introduced as follows [38]

\[ \zeta^1 = \tan \frac{\theta}{2} \sin \alpha \sin \frac{\vartheta}{2} e^{i(\psi - \chi)/2} e^{i\varphi}, \]

\[ \zeta^2 = \tan \frac{\theta}{2} \cos \alpha e^{i\varphi}, \]

\[ \zeta^3 = \tan \frac{\theta}{2} \sin \alpha \cos \frac{\vartheta}{2} e^{i(\psi + \chi)/2} e^{i\varphi}, \]

where \( 0 \leq \theta, \vartheta \leq \pi, 0 \leq \varphi, \chi \leq 2\pi, 0 \leq \alpha \leq \frac{\pi}{2} \) and \( 0 \leq \psi \leq 4\pi \). In these coordinates the metric becomes

\[ ds^2 = \frac{1}{4} \left( d\theta^2 + \sin^2 \theta (d\varphi + \frac{\sin^2 \alpha \sigma_3}{2})^2 + \sin^2 \frac{\theta}{2} d\alpha^2 + \frac{1}{4} \sin^2 \frac{\theta}{2} \sin^2 \alpha (\sigma_1^2 + \sigma_2^2 + \cos^2 \alpha \sigma_3^2) \right), \]

where

\[ \sigma_1 = \sin \psi d\vartheta - \cos \psi \sin \vartheta d\chi \]

\[ \sigma_2 = \cos \psi d\vartheta + \sin \psi \sin \vartheta d\chi \]

\[ \sigma_3 = d\psi + \cos \vartheta d\chi \]

are three left-invariant one-forms on \( SU(2) \) obeying \( d\sigma_1 = -\sigma_2 \sigma_3 \) etc. Notice that with this choice of coordinates \( \theta \) and \( \varphi \) parameterize a two-sphere of radius \( \frac{1}{2} \). This two-sphere is
topologically non-trivial and associated to the Kähler form on $CP^3$. We choose the $CP^3$ vielbeins as follows

\[ e^1 = \frac{1}{2}d\theta \]
\[ e^2 = \frac{1}{2}\sin(\theta d\varphi + \frac{1}{2}\sin^2 \alpha \sigma_3) \]
\[ e^3 = -\frac{1}{2}\sin \frac{\theta}{2}\sin \alpha \sigma_2 \]
\[ e^4 = \frac{1}{2}\sin \frac{\theta}{2}\sin \alpha \sigma_1 \]
\[ e^5 = \sin \frac{\theta}{2}d\alpha \]
\[ e^6 = \frac{1}{4}\sin \frac{\theta}{2}\sin(2\alpha) \sigma_3. \]  

(B.5)

Using the fact that

\[ de^1 = 0 \]
\[ de^2 = 2\cot \theta e^2 e^1 + 2\cot \frac{\theta}{2} e^6 e^5 + 2\cot \frac{\theta}{2} e^4 e^3 \]
\[ de^3 = \cot \frac{\theta}{2} e^3 e^1 + \frac{\cot \alpha}{\sin \frac{\theta}{2}} e^3 e^5 + \frac{4}{\sin \frac{\theta}{2}\sin(2\alpha)} e^6 e^4 \]
\[ de^4 = \cot \frac{\theta}{2} e^4 e^1 + \frac{\cot \alpha}{\sin \frac{\theta}{2}} e^4 e^5 + \frac{4}{\sin \frac{\theta}{2}\sin(2\alpha)} e^3 e^6 \]
\[ de^5 = \cot \frac{\theta}{2} e^5 e^1 \]
\[ de^6 = \frac{\theta}{2} e^6 e^1 + \frac{2\cot(2\alpha)}{\sin \frac{\theta}{2}} e^6 e^5 + \frac{2\cot \alpha}{\sin \frac{\theta}{2}} e^4 e^3 \]  

(B.6)

one can show that the connection can be taken to be

\[ \omega^{12} = 2\cot \theta e^2 \]
\[ \omega^{1\tilde{a}} = \cot \frac{\theta}{2} \tilde{a} \]
\[ \omega^{23} = -\cot \frac{\theta}{2} e^4 \]
\[ \omega^{24} = \cot \frac{\theta}{2} e^3 \]
\[ \omega^{25} = -\cot \frac{\theta}{2} e^6 \]
\[ \omega^{26} = \cot \frac{\theta}{2} e^5 \]
\[ \omega^{34} = \cot \frac{\theta}{2} e^2 - 2\cot \frac{\theta}{2} e^2 + 2\cot \frac{\theta}{2} e^2 - 2\cot \frac{\theta}{2} e^2 \]
\[ \omega^{35} = \cot \frac{\theta}{2} e^3 \]
\[ \omega^{36} = \cot \frac{\theta}{2} e^4 \]
\[ \omega^{45} = \cot \frac{\theta}{2} e^4 \]
\[ \omega^{46} = \cot \frac{\theta}{2} e^3 \]
\[ \omega^{56} = \cot \frac{\theta}{2} e^5 + 2\cot(2\alpha) \]  

(B.7)

where $\tilde{a} = 3, 4, 5, 6$. The curvature of $CP^3$ is

\[ R^{ab} = d\omega^{ab} + \omega^c a \omega^{bc} = (\delta^c_a \delta^d_b + J_c a^d J_d^b) e^c e^d + J_a^{a'b} J_{d'}^{b'} e^c e^{d'} \]  

(B.8)

where $J_a^{a'b}$ are the components of the Kähler form with $J_{12} = J_{34} = J_{56} = 1$.

The $U(1)$ part of the connection is

\[ A = \frac{1}{8} J_a^{a'b} \omega^{a'b} = \frac{1}{8}( \cot \theta e^2 + \cot \frac{\theta}{2} e^2 - \frac{3\tan \alpha}{2\sin^2 \frac{\theta}{2}} e^6) = \cot \theta e^2 + \frac{1}{4} d\varphi - \frac{\tan \alpha}{2\sin \frac{\theta}{2}} e^6, \]  

(B.9)
and it is easy to verify that its derivative is proportional to the Kähler form

\[ dA = 2e^1 e^2 + 2e^3 e^4 + 2e^5 e^6. \]  

(B.10)

References

[1] J. H. Schwarz, “Superconformal Chern-Simons theories,” *JHEP* **11** (2004) 078, arXiv:hep-th/0411077.

[2] J. Bagger and N. Lambert, “Modeling multiple M2’s,” *Phys. Rev. D75* (2007) 045020, arXiv:hep-th/0611108.

[3] J. Bagger and N. Lambert, “Gauge Symmetry and Supersymmetry of Multiple M2-Branes,” *Phys. Rev. D77* (2008) 065008, arXiv:0711.0955 [hep-th].

[4] J. Bagger and N. Lambert, “Comments On Multiple M2-branes,” *JHEP* **02** (2008) 105, arXiv:0712.3738 [hep-th].

[5] A. Gustavsson, “Algebraic structures on parallel M2-branes,” *Nucl. Phys. B811* (2009) 66–76, arXiv:0709.1260 [hep-th].

[6] O. Aharony, O. Bergman, D. L. Jafferis, and J. Maldacena, “N=6 superconformal Chern-Simons-matter theories, M2-branes and their gravity duals,” *JHEP* **10** (2008) 091, arXiv:0806.1218 [hep-th].

[7] M. Benna, I. Klebanov, T. Klose, and M. Smedback, “Superconformal Chern-Simons Theories and AdS_4/CFT_3 Correspondence,” *JHEP* **09** (2008) 072, arXiv:0806.1519 [hep-th].

[8] B. E. W. Nilsson and C. N. Pope, “Hopf fibration of eleven-dimensional supergravity,” *Class. Quant. Grav.* 1 (1984) 499.

[9] D. P. Sorokin, V. I. Tkach, and D. V. Volkov, “Kaluza-Klein theories and spontaneous compactification mechanisms of extra space dimensions,” In *Moscow 1984, Proceedings, Quantum Gravity*, 376-392.

[10] D. P. Sorokin, V. I. Tkach, and D. V. Volkov, “On the relationship between compactified vacua of D = 11 and D = 10 supergravities;” *Phys. Lett. B161* (1985) 301–306.

[11] R. R. Metsaev and A. A. Tseytlin, “Supersymmetric D3 brane action in AdS(5) x S(5),” *Phys. Lett. B436* (1998) 281–288, arXiv:hep-th/9806095.

[12] I. A. Bandos, “Superembedding approach to superstring in AdS(5)xS^5 superspace,” arXiv:0812.0257 [hep-th].

[13] J. Gomis, D. Sorokin, and L. Wulff, “The complete AdS(4) x CP(3) superspace for the type IIA superstring and D-branes,” *JHEP* **03** (2009) 015, arXiv:0811.1566 [hep-th].
[14] P. A. Grassi, D. Sorokin, and L. Wulff, “Simplifying superstring and D-brane actions in $AdS(4) \times CP(3)$ superbackground,” arXiv:0903.5407 [hep-th].

[15] G. Arutyunov and S. Frolov, “Superstrings on $AdS_4 \times CP^3$ as a Coset Sigma-model,” JHEP 09 (2008) 129, arXiv:0806.4940 [hep-th].

[16] J. Stefanski, B., “Green-Schwarz action for Type IIA strings on $AdS_4 \times CP^3$,” Nucl. Phys. B808 (2009) 80–87, arXiv:0806.4948 [hep-th].

[17] P. Fré and P. A. Grassi, “Pure Spinor Formalism for $Osp(N|4)$ backgrounds,” arXiv:0807.0044 [hep-th].

[18] G. Bonelli, P. A. Grassi, and H. Safaai, “Exploring Pure Spinor String Theory on $AdS_4 \times CP^3$,” JHEP 10 (2008) 085, arXiv:0808.1051 [hep-th].

[19] R. D’Auria, P. Fre, P. A. Grassi, and M. Trigiante, “Superstrings on $AdS_4 \times CP^3$ from Supergravity,” Phys. Rev. D79 (2009) 086001, arXiv:0808.1282 [hep-th].

[20] I. Bena, J. Polchinski, and R. Roiban, “Hidden symmetries of the AdS(5) x S**5 superstring,” Phys. Rev. D69 (2004) 046002 arXiv:hep-th/0305116.

[21] R. Blumenhagen, M. Cvetic, S. Kachru, and T. Weigand, “D-brane Instantons in Type II String Theory,” arXiv:0902.3251 [hep-th].

[22] M. Bianchi and M. Samsonyan, “Notes on unoriented D-brane instantons,” arXiv:0909.2173 [hep-th].

[23] E. Bergshoeff, R. Kallosh, A.-K. Kashani-Poor, D. Sorokin, and A. Tomasiello, “An index for the Dirac operator on D3 branes with background fluxes,” JHEP 10 (2005) 102, arXiv:hep-th/0507069.

[24] T. Nishioka and T. Takayanagi, “On Type IIA Penrose Limit and N=6 Chern-Simons Theories,” JHEP 08 (2008) 001 arXiv:0806.3391 [hep-th].

[25] G. Grignani, T. Harmark, and M. Orselli, “The SU(2) x SU(2) sector in the string dual of N=6 superconformal Chern-Simons theory,” Nucl. Phys. B810 (2009) 115–134 arXiv:0806.4959 [hep-th].

[26] K. Zarembo, “Worldsheet spectrum in AdS(4)/CFT(3) correspondence,” arXiv:0903.1747 [hep-th].

[27] P. Sundin, “On the worldsheet theory of the type IIA AdS(4) x CP(3) superstring,” arXiv:0909.0697 [hep-th].

[28] A. A. Tseytlin, “On dilaton dependence of type II superstring action,” Class. Quant. Grav. 13 (1996) L81–L85, arXiv:hep-th/9601109.

[29] M. Cvetic, H. Lu, C. N. Pope, and K. S. Stelle, “T-Duality in the Green-Schwarz Formalism, and the Massless/Massive IIA Duality Map,” Nucl. Phys. B573 (2000) 149–176, arXiv:hep-th/9907202.
[30] A. M. Polyakov and A. A. Belavin, “Metastable States of Two-Dimensional Isotropic Ferromagnets,” *JETP Lett.* **22** (1975) 245–248.

[31] V. L. Golo and A. M. Perelomov, “Solution of the Duality Equations for the Two-Dimensional SU(N) Invariant Chiral Model,” *Phys. Lett.* **B79** (1978) 112.

[32] V. L. Golo and A. M. Perelomov, “Few remarks on chiral theories with sophisticated topology,” *Lett. Math. Phys.* **2** (1978) 477–482.

[33] A. D’Adda, M. Luscher, and P. Di Vecchia, “A 1/n Expandable Series of Nonlinear Sigma Models with Instantons,” *Nucl. Phys.* **B146** (1978) 63–76.

[34] P. Di Vecchia and S. Ferrara, “Classical Solutions in Two-Dimensional Supersymmetric Field Theories,” *Nucl. Phys.* **B130** (1977) 93.

[35] V. A. Novikov, M. A. Shifman, A. I. Vainshtein, and V. I. Zakharov, “Two-Dimensional Sigma Models: Modeling Nonperturbative Effects of Quantum Chromodynamics,” *Phys. Rept.* **116** (1984) 103.

[36] A. M. Perelomov, “Chiral Models: Geometrical Aspects,” *Phys. Rept.* **146** (1987) 135–213.

[37] A. M. Perelomov, “Supersymmetric Chiral Models: Geometrical Aspects,” *Phys. Rept.* **174** (1989) 229–282.

[38] C. N. Pope and N. P. Warner, “An SU(4) invariant compactification of d = 11 supergravity on a stretched seven sphere,” *Phys. Lett.* **B150** (1985) 352.

[39] X. G. Wen and E. Witten, “World sheet instantons and the Peccei-Quinn symmetry,” *Phys. Lett.* **B166** (1986) 397.

[40] M. Dine, N. Seiberg, X. G. Wen, and E. Witten, “Nonperturbative Effects on the String World Sheet,” *Nucl. Phys.* **B278** (1986) 769.

[41] O. Aharony, O. Bergman, and D. L. Jafferis, “Fractional M2-branes,” *JHEP* **11** (2008) 043, arXiv:0807.4924 [hep-th].

[42] Y. Fujii and K. Yamagishi, “Killing spinors on spheres and hyperbolic manifolds,” *J. Math. Phys.* **27** (1986) 979.

[43] A. A. Abrikosov, Jr., “Dirac operator on the Riemann sphere,” arXiv:hep-th/0212134.

[44] H. Lu, C. N. Pope, and J. Rahmfeld, “A construction of Killing spinors on $S^n$,” *J. Math. Phys.* **40** (1999) 4518–4526, arXiv:hep-th/9805151.

[45] S. Deguchi and K. Kitsukawa, “Charge quantization conditions based on the Atiyah-Singer index theorem,” *Prog. Theor. Phys.* **115** (2006) 1137–1149, arXiv:hep-th/0512063.
[46] J. Hughes and J. Polchinski, “Partially Broken Global Supersymmetry and the Superstring,” Nucl. Phys. B278 (1986) 147.

[47] J. Hughes, J. Liu, and J. Polchinski, “Supermembranes,” Phys. Lett. B180 (1986) 370.

[48] E. Bergshoeff, R. Kallosh, T. Ortin, and G. Papadopoulos, “Kappa-symmetry, supersymmetry and intersecting branes,” Nucl. Phys. B502 (1997) 149–169, arXiv:hep-th/9705040.

[49] P. Claus and R. Kallosh, “Superisometries of the AdS x S superspace,” JHEP 03 (1999) 014, arXiv:hep-th/9812087.

[50] D. V. Volkov and V. P. Akulov, “Possible universal neutrino interaction,” JETP Lett. 16 (1972) 438–440.

[51] D. V. Volkov and V. P. Akulov, “Is the Neutrino a Goldstone Particle?,” Phys. Lett. B46 (1973) 109–110.

[52] D. P. Sorokin, “Superbranes and superembeddings,” Phys. Rept. 329 (2000) 1–101, arXiv:hep-th/9906142.

[53] A. D’Adda, A. C. Davis, P. Di Vecchia, and P. Salomonson, “An Effective Action For The Supersymmetric $\mathbb{C}P^{N-1}$ Model,” Nucl. Phys. B222 (1983) 45.

[54] P. S. Howe and E. Sezgin, “The supermembrane revisited,” Class. Quant. Grav. 22 (2005) 2167–2200, arXiv:hep-th/0412245.