Entropy of Eigenfunctions on Quantum Graphs

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Distribution of eigenfunctions: Quantum ergodicity

\((M, g)\) compact Riemannian manifold:

\[-\Delta_g \psi_n = k_n^2 \psi_n, \quad \|\psi_n\|_{L^2} = 1\]

Shnirelman (74), Zelditch (87), Colin de Verdiere (85), Quantum Ergodicity: ergodic geodesic flow, then almost all eigenfunctions equidistribute for \(k_n \to \infty\):

\[
\lim_{j \to \infty} \int_M a(x)|\psi_{n_j}(x)|^2 \, d\nu = \int_M a(x) \, d\nu ,
\]

along a subsequence \(n_j\) of density 1, \(d\nu\) Riemannian measure.

- density 1: \(\lim_{N \to \infty} \frac{|\{n_j \leq N\}|}{N} = 1\)
- valid for \(\langle \psi_{n_j}, \text{Op}[a] \psi_{n_j} \rangle\) with \(a \in C^\infty(S^*M)\)
Quantum Unique Ergodicity?

Quantum Unique Ergodicity: Does

$$\lim_{n \to \infty} \int_M a(x)|\psi_n(x)|^2 \, d\nu = \int_M a(x) \, d\nu.$$ 

hold?

- Kurlberg Rudnick ’00, Marklof Rudnik ’00: **Yes** for Hecke eigenbasis of cat maps and for parabolic maps
- Faure, Nonnenmacher, De Bièvre ’03; Chang, Krueger, RS, Troubetzkoy ’08: **No** for cat maps and other quantised maps
- Lindenstrauss ’06: **Yes** Quantum Unique Ergodicity holds for Hecke eigenbasis on arithmetic surfaces.
- Hassell ’10: **No** for Stadium billiards
- Anantharaman et.al. ’07: lower bounds on the entropy of quantum limits on manifolds of negative curvature.
Quantum Graphs

- $G = (V, E)$, finite undirected connected graph. $V$-vertices, $E$-edges, $E \ni e = [i, j], i, j \in V$, bonds = oriented edges, $b = (i, j)$, then $\hat{b} := (j, i) \neq b$

- Length $L \in \mathbb{R}_+^{\mid E\mid}$: assign to each edge a length $L_e > 0$, identify $e$ with interval $[0, L_e]$.

$$L^2(G, L) := \bigoplus_{e \in E} L^2([0, L_e]) \quad H_s(G, L) := \bigoplus_{e \in E} H_s((0, L_e)) .$$

- Laplace operator: $\Delta : H_2(G, L) \to L^2(G, L)$, $f = (f_1, f_2, \cdots, f_{\mid E\mid}) \in H_2(G, L)$, then

$$\Delta f = (f_1'', f_2'', \cdots, f_{\mid E\mid}'') .$$

- need boundary conditions at vertices to define self-adjoint operator
S-matrix and Boundary conditions

describe boundary conditions on vertex $i$ of degree $d_i$ in terms of $S$-matrix $S^{(i)}$: unitary $d_i \times d_i$ matrix

- $[i,j], j \sim i$, edges adjacent to $i$, oriented away from $i$:
- Solutions to $-\Delta f = k^2 f$:
  \[
  f_{[i,j]}(x) = a_{(j,i)}e^{-ikx} + a_{(i,j)}e^{ikx}
  \]
- $a_{i}^{in} := (a_{(j_1,i)}, \cdots , a_{(j_{d_i},i)}), a_{i}^{out} := (a_{(i,j_1)}, \cdots , a_{(i,j_{d_i})}).$
  \[
  a_{i}^{out} = S^{(i)}(k)a_{i}^{in}
  \]
- Boundary conditions classified by Kostrykin Schrader ’99
Examples

- **Neumann conditions**: \( f_e = f_{e'} \) for all \( e, e' \) meeting at \( i \) and \( \sum f'_{e} = 0 \).

\[
S^{(i)}_{e,e'} = \frac{2}{d_i} - \delta_{e,e'}
\]

For large \( d_i \) backscattering dominates!

- **Equi-transmitting conditions** (Harrison, Smilansky, Winn 07):

\[
|S^{(i)}_{e,e'}|^2 = \begin{cases} 
0 & e = e' \\
\frac{1}{d_i - 1} & e \neq e'
\end{cases}
\]

No backscattering!

- **non-Robin boundary conditions**: \( S \) independent of \( k \). Equivalent to \( S^* = S \). Then \( S = P_+ - P_- \) where \( P_\pm \) orthogonal projections with \( P_+ + P_- = I \), \( P_+ P_- = 0 \) and boundary conditions are

\[
P_- f = 0 \quad P_+ f' = 0
\]
Bond S-matrix and quantisation conditions

- Quantum Graph: \((G, L, \{S^{(i)}\}_{i \in V})\)
- Bond S-matrix \(\mathcal{U}(k) = (u_{b,b'})\): \(2|E| \times 2|E|\) matrix defined by
  \[
  u_{(i,j),(k,l)} = \delta_{jk} S^{(j)}_{(i,j),(j,l)} e^{ikL_{[i,j]}} , \quad \mathcal{U}(k) = e^{ikL} S
  \]
- Quantisation conditions:
  \[
  \mathcal{U}(k)a = a , \quad a \in \mathbb{C}^{2|E|} \setminus \{0\} ,
  \]
  if and only if \(f\) defined by
  \[
  f_{[i,j]} = a_{(i,j)} e^{ikx_{i,j}} + a_{(j,i)} e^{ikx_{j,i}}
  \]
  is eigenfunction.
- Eigenvalues determined by secular equation
  \[
  \det(\mathcal{U}(k) - I) = 0
  \]
Paths and classical dynamics

- Path of length $t \in \mathbb{N}$: $\gamma = (b_1, b_2, \cdots, b_{t-1}, b_t)$ where if $b_s = (i, j)$ and $b_s+1 = (k, l)$ then $j = k$.
- $\Gamma_t(b, b')$-set of paths connecting $b$ and $b'$ in $t$ steps
- $\Gamma'_t(b, b')$-set of paths without backtracking: $b_{s+1} \neq \hat{b}_s$.

Set $L_\gamma = \sum_{b \in \gamma} L_b$, $s_\gamma = \prod_{s=1}^t S_{b_s, b_{s+1}}$, then

$$U(k)^t = (u^{(t)}_{b', b}) \quad u^{(t)}_{b', b} = \sum_{\gamma \in \Gamma_t(b, b')} s_\gamma e^{ikL_\gamma},$$

if no backscattering: $u^{(t)}_{b', b} = \sum_{\gamma \in \Gamma'_t(b, b')} s_\gamma e^{ikL_\gamma}$

**Classical dynamics:** Set $M = (m_{b, b'})$ with $m_{b, b'} := |u_{b, b'}|^2$. $M$ is doubly stochastic and defines a Markov chain with

$$M^t x = \frac{x \cdot e}{2|E|} e + O_G, x(e^{-\gamma G t})$$

for some $\gamma_G > 0$ and $e = (1, 1, \cdots, 1)$. 
Quantum Graphs: History

• introduced independently in different areas: Chemistry, Physics, Mathematics
• Quantum ergodicity on quantum graphs is open! Partial results:
  • Berkolaiko, Keating, Winn (04): No quantum ergodicity on star graphs
  • Berkolaiko, Keating, Smilanski (07): Quantum ergodicity for graphs related to interval maps.
  • Gnutzman, Keating, Piotet (10): quantum ergodicity under gap condition, non-rigorous.
  • Anantharaman, LeMasson (13): quantum ergodicity on d-regular combinatorial graphs.
  • Jakobson, Strohmaier, Safarov (13): quantum ergodicity with ray-splitting
  • Winn (14, in preparation): quantum ergodicity on d-regular quantum graphs which large girth.
  • Colin de Verdière (14): classification of quantum limits on finite graphs with Neumann bc: no quantum ergodicity
Entropy

Let \( \mathbf{a} \in \mathbb{C}^N \) with \( \| \mathbf{a} \| = 1 \). Entropy:

\[
S(\mathbf{a}) := \frac{1}{\ln N} \sum_{n=1}^{N} -|a_n|^2 \ln |a_n|^2
\]

- \( 0 \leq S(\mathbf{a}) \leq 1 \)
- \( S(\mathbf{a}) = 0 \) iff \( \mathbf{a} = \mathbf{e}_m = (\delta_{m,n}) \) and \( S(\mathbf{a}) = 1 \) iff \( \mathbf{a} = \frac{1}{\sqrt{N}} \mathbf{e} \)
- if \( \mathbf{a} = (a_n) \), \( a_n = 0 \) for \( n \in K \subset \{1, 2, \cdots, N\} \) then

\[
S(\mathbf{a}) \leq \frac{\ln(N - |K|)}{\ln N}
\]

Entropy large \( \rightarrow \) \( \mathbf{a} \) can’t be concentrated on small set

Entropy is a measure for the distribution of \( \mathbf{a} \)
Entropic Uncertainty Principle

Maassen Uffink '88: Let $U = (u_{n,m}) \in \mathbb{C}^{N \times N}$ be unitary, then

$$S(a) + S(Ua) \geq -\frac{\ln \left( \max_{n,m} |u_{n,m}|^2 \right)}{\ln N}$$

$\sum_n |u_{n,m}|^2 = 1$: optimal case $|u_{n,m}|^2 = 1/N$, $S(a) + S(Ua) \geq 1$

Example: Fourier transform $F = (f_{n,m})$, $f_{n,m} = \frac{1}{\sqrt{N}} e^{2\pi i \frac{nm}{N}}$

$$S(a) + S(Fa) \geq 1$$

Application to eigenvectors: If $Ua = a$ then

$$S(a) \geq -\frac{1}{2 \ln N} \ln \left( \max_{n,m} |u_{n,m}|^2 \right)$$

and

$$S(a) \geq -\frac{1}{2 \ln N} \ln \left( \max_{n,m} |u_{n,m}^{(t)}|^2 \right), \quad \text{where} \quad U^t = (u_{n,m}^{(t)})$$
Star Graphs, equi-transmitting

Theorem (Kameni, RS 13/14)

Let \((G, E)\) be a star graph with equi-transmitting boundary conditions, then for any eigenfunction

\[
S(a) \geq \frac{1}{2} \frac{\ln(|E| - 1) + 2 \ln 2}{\ln|E| + \ln 2} > \frac{1}{2}
\]

• eigenfunctions: \(f_e(x) = A_e \cos (k(x - L_e))\),

\[
S(A) := \frac{1}{\ln|E|} \sum_{e=1}^{\lvert E \rvert} -|A_e|^2 \ln|A_e|^2 , \ l\|A\| = 1
\]

• \(e^{ikL} Se^{ikL} A = A, \ |S_{e,e'}|^2 = (1 - \delta_{e,e'}) \frac{1}{|E|-1}\)

\[
S(A) \geq \frac{1}{2} \frac{\ln(|E| - 1)}{\ln|E|}
\]

• \(S(a) = \frac{\ln|E|}{\ln(2|E|)} S(A) + \frac{\ln 2}{\ln(2|E|)}\)
Star Graphs, Neuman

Theorem (Kameni, RS 13/14)

Let \((G, E)\) be a star graph with Neumann boundary conditions, \(L\) rationally independent, and \(a^{(n)}\) is the \(n\)'th eigenfunction, then the average entropy \(\langle S \rangle := \lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} S(a^{(n)})\) satisfies

\[
\langle S(a) \rangle = \frac{\alpha}{\ln |E|} + O(|E|\Delta L)
\]

where \(\Delta L = \max_e L_e - \min_e L_e\) and \(\alpha = 1.2692\ldots\).

- proof based on Barra & Gaspard (99), further developed in Keating, Marklof and Winn (03), Colin de Verdière (14)
- quantisation condition \(\det(I - \mathcal{U}(k)) = F(kL \mod 2\pi)\), function on torus \(T^{|E|}\) evaluated on trajectory \(kL \mod 2\pi\), as are \(\mathcal{U}(k)\) and eigenvectors \(a\).
- use Weyl’s Theorem (unique ergodicity of \(\phi^t(x) = x + tL \mod 2\pi\)) to transform energy average in average over torus \(T^{|E|}\).
Equi-transmitting versus Neumann, Star Graphs

Figure: 6235 eigenfunctions on Star graph with 338 edges.
Left: Entropy of eigenfunctions with equi-transmitting boundary conditions.
Right: Entropy of eigenfunctions with Neumann boundary conditions.
Regular graphs

- **G** $d + 1$ regular graph if every vertex has degree $d + 1$
  - $2|E| = (d + 1)|V|$
  - equi-transmitting boundary conditions: $|S_{e,e'}|^2 = \frac{1}{d}(1 - \delta_{e,e'})$
- $G_N = (V_N, E_N)$, $N \in \mathbb{N}$, graphs with $\lim_{N \to \infty} |V_N| = \infty$
  - $G_N$ expander if there exits $\gamma > 0$ such that
    \[
    M_N^t a = \frac{a \cdot e}{2|E_N|} e + O(e^{-\gamma t})
    \]
    Expansion rate uniform in $N$!
  - $G_N$ has large girth if there exist a $\delta > 0$ such that the length $T_N$ of the shortest cycle satisfies
    \[
    T_N \geq 2\delta \ln(2|E_N|)
    \]
  - If $b, b'$ have distance $t$ less then $\delta \ln(2|E|)$ then there exist only one path of length $t$ connecting them.
  - Any ball of radius less then $\delta \ln(2|E_N|)$ is a tree.
Regular graphs: Large girth

Theorem (Kameni, RS 13/14)

Let $G$ be a $d + 1$ regular graph with girth $T_G = 2R_G + 1$, then for equi-transmitting boundary conditions

$$S(a) \geq \frac{1}{2} \frac{R_G \ln d}{\ln(2|E|)}.$$ 

Corollary

Assume $G_N$ has large girth, $T_G = 2\delta \ln(2|E|)$, then $S(a) \geq \frac{\delta \ln d}{2}$.

Main idea: for $t \leq R_G$, we have $|\Gamma'_t(b, b')| \leq 1$ hence for $t = R_G$

$$|u^{(t)}_{b, b'}|^2 = \left| \sum_{\gamma \in \Gamma'_t(b, b')} s_\gamma e^{ikL_\gamma} \right|^2 \leq |s_\gamma|^2 = \frac{1}{d^t}.$$
Regular graphs: Large girth and expanding

t large: $\Gamma'_t(b, b')$ contains exponentially many elements, turn

$$u^{(t)}_{b, b'} = \sum_{\gamma \in \Gamma'_t(b, b')} s_\gamma e^{i k L_\gamma}$$

into a sum over random variables by making length $L$ random.

Assumption: $L_e$ independent and
- $\mathbb{P}(L_e \leq \delta) = 0$ with $\delta > 0$ independent of $e$ and $G_N$.
- there exits an $f(k) \in C(\mathbb{R})$ with $\lim_{k \to \pm\infty} f(k) = 0$ such that $|\mathbb{E}(e^{i k L_e})| \leq f(k)$ independent of $e \in E_N$ and $G_N$.

Theorem (Kameni, RS 13/14)

$G_N = (V_N, E_N)$ expanding, large girth, random length and equi-transmitting. Then for any $\varepsilon > 0$ there exist a $k_0 > 0$ such that if $k \geq k_0$ and $a$ is an eigenvector of $\mathcal{U}(k)$ we have

$$\mathbb{P}\left( S(a) \geq \frac{1 - \varepsilon}{2} \right) \geq 1 - \frac{4d}{|V_N|^\varepsilon}$$
Regular graphs: Large girth and expanding

Proof strategy:

- **Chebychev’s inequality:** \( |u_{b,b'}^{(t)}| \sim \sqrt{\mathbb{E}(|u_{b,b'}^{(t)}|^2)} \)

- \( \mathbb{E}(|u_{b,b'}^{(t)}|^2) = \sum_{\gamma,\gamma' \in \Gamma'(b,b')} s_{\gamma}s_{\gamma'} \mathbb{E}(e^{ik(L_{\gamma}-L_{\gamma'})}) \)

large girth: if \( \gamma \neq \gamma' \)

\[ \mathbb{E}(e^{ik(L_{\gamma}-L_{\gamma'})}) \leq [f(k)]^{(2RG)} \]

- \( N_t(b, b') := |\Gamma'_t(b, b')|, |s_{\gamma}|^2 = d^{-t}, \) then

\[ \mathbb{E}(|u_{b,b'}^{(t)}|^2) \leq \frac{N_t(b, b')}{d^t} (1 + N_t(b, b')[f(k)]^{(2RG)}) \]

- **expander:** there exist \( \mu < 1 \), independent of \( G_N \), such that

\[ \frac{N_t(b, b')}{d^t} \leq \frac{1}{2|E|} + \mu^t. \]
Equi-transmitting versus Neumann, Regular Graphs

Figure: 2708 eigenfunctions on a 6-regular graph with 450 edges.
Left: Entropy of eigenfunctions with equi-transmitting boundary conditions.
Right: Entropy of eigenfunctions with Neumann boundary conditions.
Summary

• Entropy of eigenfunctions on graphs gives a measure for their localisation or delocalisation.
• We derive lower bounds on the entropy by using the Entropic Uncertainty Principle.
• Main assumptions are large girth and expansion, which allow to explore the Entropic Uncertainty Principle.
• For regular graphs with large girth, expanding, and with random bond-length, we obtain a bound similar to the Anantharaman bound on manifolds of negative curvature.