Dirac neutrino mixings from hidden $\mu - \tau$ symmetry

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(Dated: February 14, 2018)

We explore masses and mixings for Dirac neutrinos in models where lepton number is conserved, under the guidance of a hidden, but broken, $\mu - \tau$ exchange symmetry, that makes itself evident in the squared hermitian mass matrix. We study the parameter space in the most general theory as allowed by current neutrino oscillation experiment data. By using a general parameterization of the mass matrix which contains only observable parameters we establish that the amount of breaking of the symmetry is in the range of the atmospheric mass scale, without regard to the neutrino hierarchy, the absolute neutrino mass and the Dirac CP phase. An estimate of the invisible branching ratio for a Higgs boson decaying into Dirac neutrinos, $H \rightarrow \nu\bar{\nu}$, is given and compared to recent measurements in this context.

I. INTRODUCTION

Neutrino oscillations have been for some time under the scope of a large number of theoretical studies and many experimental efforts, since they imply that neutrinos have mass and leptons mix flavors (for a review see for instance [2]). As the Standard Model (SM), on the contrary, predicts that neutrinos should be rather massless and no flavor mixing should exist in the lepton sector, neutrino physics seem to point towards the need of new physics. Compelling evidence for neutrino oscillations has been provided by data obtained from the observation of neutrinos arriving from the sun, from upper atmosphere interactions of cosmic rays, from nuclear reactors, and from particle accelerators. Most of such data can be understood in a framework with three weak flavor neutrinos, $\nu_{\ell}$ for $\ell = e, \mu, \tau$. Those corresponding to the three SM charged leptons.

Oscillation phenomena is not sensible to the actual mass of the neutrinos but to their squared mass differences, $\Delta m^2_{ab} = m_a^2 - m_b^2$, for $a, b = 1, 2, 3$. It is so, however, to the mixing that connects the weak to the mass eigenstates, $\nu_a$. In the two neutrino flavor approximation, the oscillation probability is simply given as $P_{\ell'\ell} = \sin^2 2\theta \sin^2 (\Delta m^2_{ab} E/4L)$, for the mixing angle $\theta$, Global fits [1, 2], with all three neutrinos, find for solar neutrino oscillations the scale $\Delta m^2_{21} = m_2^2 - m_1^2 \sim 7.5 \times 10^{-5} eV^2$, whereas for atmospheric neutrinos they give $\Delta m^2_{\text{ATM}} = |\Delta m^2_{31}| \sim 2.5 \times 10^{-3} eV^2$. Note that the hierarchy among the first two mass eigenstates is well known due to the contribution of matter effects on solar neutrino oscillations. However, the sign in $\Delta m^2_{21} = m_2^2 - m_1^2$, and therefore the neutrino mass hierarchy pattern, is still unknown. Data is so far consistent with both normal ($\Delta m^2_{21} > 0$) and inverted ($\Delta m^2_{21} < 0$) hierarchies. As for the mixing angles, global fits indicate that $\sin^2 \theta_{\odot} \approx 0.308 \pm 0.017$ for solar, $\sin^2 \theta_{\text{ATM}} \approx 0.437^{+0.034}_{-0.023}$ (0.455$^{+0.039}_{-0.031}$) for atmospheric, and $\sin^2 \theta_{13} \approx 0.0234^{+0.020}_{-0.019}$ (0.0240$^{+0.0019}_{-0.0022}$) for reactor neutrino oscillations with normal (inverted) hierarchy.

Despite the incontrovertible evidence for neutrino masses and mixings, none can be said still about the actual nature of the neutrino. The question regards to whether neutrino is its own antiparticle, in which case it is called a Majorana particle, or not, in which case it would be a Dirac particle. Neutrino oscillations are consistent with both the possibilities. Although Majorana neutrinos have an ideal signature on neutrinoless double beta decay, the experimental evidence for such processes is still lacking (for a review see for instance Ref. [3]). On the theoretical side, Majorana neutrinos are considered to be easier to understand, as the seesaw mechanism [4] can generate very small masses for the standard left handed neutrinos, at the account of introducing large masses for right handed ones, without assuming any small value for the Yukawa couplings to the Higgs field. Nevertheless, apart from this naturalness argument, there is no other theoretical or experimental reason to believe that the other possibility can be ruled out, and so it remains. Yet, Dirac neutrino nature could be understood if total lepton number were a conserved quantity and some models oriented to account for the smallness of the neutrino mass in this case had already been explored (see for instance [4]).

On the other hand, there is the intriguing observation, of particular interest for model building, that the measured mixing angles comply with the

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emplyrical relation

\[ 1/2 - \sin^2 \theta_{ATM} \approx \sin \theta_{13}/\text{few}, \tag{1.1} \]

which suggests that the deviation of \( \theta_{ATM} \) from its maximal value, \( \Delta \theta = \pi/4 - |\theta_{ATM}| \), could be correlated to the non zero value of \( \theta_{13} \). This can be thought to indicate a possible common physical origin for both angles, since null values of \( \Delta \theta \) and \( \theta_{13} \) do increase the symmetry in the Majorana neutrino mass sector by exhibiting a discrete \( \mu - \tau \) exchange symmetry. This has suggested the idea that observed values could be understood as a result of the breaking of \( 13 \) symmetry. Many theoretical studies had been inspired by this observation in the last years, but few attention has been given to exploring the possibilities of this symmetry for the description of Dirac neutrino mixings. Such is the main goal of the present work.

The outcome of our study has some interesting results that we will discuss below. First of all, unlike to what happens in the Majorana neutrino case, \( \mu - \tau \) symmetry appears to be quite natural in the Dirac neutrino framework. It of course arises from the condition of zero \( \Delta \theta \) and \( \theta_{13} \), which implies that the symmetry is rather broken. However, unlike the Majorana case (see for instance Ref. [2]), current experimental results indicate that the breaking of the symmetry is always relatively small for Dirac neutrinos, when compared to the heaviest neutrino mass, regardless of neutrino hierarchy and the Dirac CP phase value. Therefore, \( \mu - \tau \) must be regarded as a good approximate symmetry. This suggest that any realistic model built to provide Dirac neutrino masses and mixings should contain \( \mu - \tau \) symmetry as an implicit, or explicit, flavor symmetry.

To state our case, we organize the present discussion as follows. We start by revisiting the origin of Dirac masses in a lepton conserving extension of the SM, and introducing a phenomenological parameterization for the Yukawa couplings that uses only all known experimental observables. We argue that those are the only physical parameters that are relevant to reconstruct the most general couplings in the lepton sector. Next we use this results to study \( \mu - \tau \) symmetry and to parameterize its breaking in the context of Dirac neutrinos. Furthermore, we use the above mentioned experimental results on neutrino masses and mixings to explore the breaking parameter space in order to establish the amount of breaking allowed for the experimental data. We also calculate explicit expressions for \( \Delta \theta \) and \( \theta_{13} \), in the limit of a small breaking of the symmetry to evidence their correlation under the \( \mu - \tau \) symmetric approach. Some comments about the implications of our Yukawa couplings parameterization on the invisible width of the Higgs are also made. Some further discussion and our conclusions are finally presented.

## II. A PHENOMENOLOGICAL PARAMETERIZATION FOR YUKAWA COUPLINGS

Extending the SM with the simple addition of right handed sijet neutrinos, \( N_\ell \), introduces an anomaly free global symmetry in the theory, which is associated to the combination of baryon and total lepton numbers, \( B - L \). As Majorana mass terms violate the conservation of lepton number by two units, they are not possible if \( L \) (that is, \( B - L \)) is assumed to be conserved. Under such an assumption, the most general Yukawa couplings are written as

\[ f_\ell L_\ell H \ell_R + y_{\ell\ell} L_\ell \tilde{H} N_\ell + h.c. \tag{2.1} \]

They would be responsible for lepton masses through the Higgs mechanism. Here \( L \) stands for the standard left handed lepton doublets, \( H \) for the Higgs, and \( \ell_R \) for the standard right handed charged leptons. Notice that, without loss of generality, we have chosen to work in the basis where lepton charge couplings are already diagonal. This can always be made. If a specific model for lepton masses were to provide non diagonal Yukawa couplings in that sector, one can always trasform it into the expression given above by picking up the specific \( U(3) \times U(3) \) flavor transformations, \( L_\ell \rightarrow O_{LL} L_\ell \) and \( \ell_R \rightarrow O_{RR} \ell_R \), that diagonalize the corresponding Yukawa matrix, under which all other SM terms are invariant, but for the Yukawa couplings that involve the right handed neutrinos. These last would just be properly redefined by the \( O^L \) transformation. We work on such a basis on what follows. After that electroweak symmetry breaking is introduced, one gets the most general Dirac mass terms as

\[ m_\ell \tilde{\ell}_L \ell_R + (M_\ell)_{H\ell} \nu_R N_\ell \ell + h.c., \tag{2.2} \]

where, clearly, \( m_\ell = f_\ell v \) and \( (M_\ell)_{H\ell} = v y_{\ell\ell} \), with \( v = \langle H \rangle \) the Higgs vacuum expectation value. Note that Dirac neutrino mass matrix, \( M_\ell \), is non diagonal and complex in general. Its diagonalization is done through a bi-unitary transformation, that requires the simultaneous transformation of left and right handed neutrino flavor spaces. Thus, we take

\[ \nu_\ell = U_{\ell a} \nu_a \quad \text{and} \quad N_\ell = V_{\ell a} N_a, \tag{2.3} \]

such that in the new basis the neutrino mass terms become \( (M_\ell)_{ab} \nu_a N_b + h.c., \) where

\[ M_d = U^\dagger M_\nu V = \text{diag}\{m_1, m_2, m_3\}. \tag{2.4} \]
Notice that the above mass eigenvalues can always be taken to be real and positive, and we will do so hereafter. Indeed, since right handed neutrinos have no further interactions in any other sector of the SM, one can always rephase right handed neutrino wave functions to absorb the mass phases within the $V$ matrix. In other words, such phases have not any physical meaning.

Determination of the required mixings, $U$ and $V$, can be done by considering the hermitian squared matrices

$$M_L = M_v M_u^\dagger \quad \text{and} \quad M_R = M_v^\dagger M_v, \quad (2.5)$$

By setting $\nu (V)$ in above expressions, it is easy to see that $U (V)$ is actually the unitary matrix that diagonalizes $M_L (M_R)$, since

$$U^\dagger M_L U = M_L^2 = \text{diag} \{ m_1^2, m_2^2, m_3^2 \}, \quad (2.6)$$

and similarly for $M_R$, since $V^\dagger M_R V = M_R^2$.

The transformation of the left handed neutrino sector, on the other hand, does affect other SM sector. Specifically, in the neutrino mass basis, charged current interactions, the $W$ boson connects neutrinos to charged leptons, are now given by the coupling term $W^\mu \ell_L^\gamma V_{\ell a}^R$. Note that this is just as in the quark sector where the mixing is expressed by the CKM matrix. In the standard parameterization the $U$ mixing matrix is given by the Pontecorvo-Maki-Nakagawa-Sakata matrix [11-12],

$$\left( \begin{array}{ccc} c_{12} c_{13} & s_{12} c_{13} & z \\ -s_{12} c_{23} - c_{12} s_{23} \tilde{\zeta} & c_{12} c_{23} - s_{12} s_{23} \tilde{\zeta} & s_{23} c_{13} \\ s_{12} s_{23} - c_{12} c_{23} \tilde{\zeta} & -c_{12} c_{23} - c_{23} s_{12} \tilde{\zeta} & c_{13} \end{array} \right),$$

where $z = s_{13} \exp -i \delta_{CP}$, with $\delta_{CP}$ the still undetermined Dirac CP phase. Here, $\tilde{\zeta}$ stands for the complex conjugated, whereas $c_{ij}$ and $s_{ij}$ stand for $\cos \theta_{ij}$ and $\sin \theta_{ij}$, respectively, of the mixing angles with the proper identification of $\theta_{12} = \theta_{\odot}$, and $\theta_{23} = \theta_{ATM}$.

It is worth noticing that $V$ actually contains no further physical information. As a matter of fact, the definition of the right handed neutrino flavor basis is ambiguous. Although broken, the $U(3)_N$ flavor symmetry of the sector allows an arbitrary redefinition of the Yukawa couplings given in Eq. (2.1) through the transformation $N_L \rightarrow O_{\ell \nu}^N N_L$. This is unlike the mass eigenstate basis, which is rather well defined. We make use of this fact to introduce a simple bottom up reconstruction of the Dirac neutrino mass matrix, based only on physical and measurable parameters. This truly phenomenological approximation would have the advantage of providing a general and simple criteria to fix the weak interaction flavor basis as the one connected to the mass basis only through the physical rotation associated to the left handed neutrinos, given by the PMNS mixing matrix. In such a basis, we simply write $M_v = U \cdot M_d$. Moreover, in this basis weak interactions are diagonal and the Yukawa couplings are expressed without lost of generality as

$$\frac{m_\nu}{v} \bar{L}_\ell H R + \frac{m_\nu}{v} U_{\ell a} \bar{L}_\ell \tilde{H} N_a + h.c., \quad (2.7)$$

where all unphysical parameters had been explicitly removed. These terms provide a well defined extension to the SM that contains no further parameters than those already known or which can be determined in the future. The hardest to probe might be the absolute mass scale. Current bound is at the 2 eV range [13], but if it were high enough we could expect to have some positive results from tritium beta decay experiments [14-15], with no positive signal from neutrinoless double beta decay experiments. Of course, the immediate form of the couplings derived from any specific flavor model would in general differ from above expression, but as we have argued, it can always be converted into that. Furthermore, above approach has another clear advantage since it also allows to identify some symmetries that are hidden in the general expression (2.1), as we discuss next.

III. $\mu - \tau$ SYMMETRY WITH DIRAC NEUTRINOS

By using the phenomenological parameterization given in Eq. (2.7), it is straightforward to see that in the limit where $\theta_{13}$ is null and $\theta_{23} = -\pi/4$, i.e. $\Delta \theta = 0$, the neutrino mass matrix exhibits a $\mu - \tau$ exchange symmetric structure,

$$M^\circ_{\nu} = \left( \begin{array}{ccc} m^0_{\mu_1} & m^0_{\mu_2} & 0 \\ m^0_{\mu_2} & m^0_{\mu_3} & m^0_{\mu_3} \\ 0 & m^0_{\mu_3} & -m^0_{\mu_3} \end{array} \right). \quad (3.1)$$

Notice the odd behaviour of the last column. In terms of the observables, in our approximation the above mass terms are given as $m^0_{\mu_1} = m_1 c_{12} s_{13}$; $m^0_{\mu_2} = m_2 s_{12}$; $m^0_{\mu_3} = -\sqrt{2} m_1 s_{12}$; $m^0_{\mu_3} = -\sqrt{2} m_2 s_{12}$; and $m^0_{\mu_3} = -\sqrt{2} m_3$. We should stress that this $\mu - \tau$ realization is actually close related to the parametric form of the PMNS mixing matrix.

Here we also note that the alternative choice for the quadrant of the atmospheric angle, that is taking its maximal value at $\theta_{23} = \pi/4$, only changes by a global sign the third row on $M^\circ_{\nu}$, which still reflects a symmetric relation associated to $\mu - \tau$ exchange. This alternative can also be interpreted as
a change in the flavor and mass state basis, where tau flavours and third neutrino mass eigenstate are replaced by $\pi$ (a simple sign change in the wave functions). This does not affect mass eigenvalues (they remain positive), neither our main conclusion about the size of $\mu - \tau$ breaking, as we will explicitly show later on for matter of completeness. In what follows we shall first concentrate in analyzing the previously given case.

A less parametric dependant way of realizing the existence of $\mu - \tau$ symmetry arises when one rather looks at the more generic form of the hermitian squared matrix, $\tilde{M}_L$, in the diagonal charged lepton basis used for Eq. (3.1), and directly from Eq. (3.2), the result of calculating the hermitian squared mass matrix elements, for $\theta_{13} = \Delta \theta = 0$, shows that $\tilde{M}_L$ is symmetric and exhibits a perfect $\mu - \tau$ exchange symmetry,

$$\tilde{M}_L^0 = \begin{pmatrix}
\tilde{m}_{e\mu}^0 & \tilde{m}_{e\tau}^0 & \tilde{m}_{e\mu}^0 \\
\tilde{m}_{\mu\mu}^0 & \tilde{m}_{\mu\tau}^0 & \tilde{m}_{\mu\mu}^0 \\
\tilde{m}_{\tau\mu}^0 & \tilde{m}_{\tau\tau}^0 & \tilde{m}_{\mu\mu}^0 \\
\end{pmatrix},$$

(3.2)

where the only four relevant terms are

$$\tilde{m}_{e\mu}^0 = \left\{ \begin{array}{ll}
\Delta m_{21}^2 s_{12}^2 + m_0^2 & \text{(NH)} \\
\Delta m_{21}^2 s_{12}^2 + |\Delta m_{31}^2| + m_0^2 & \text{(IH)}
\end{array} \right.$$  

(3.3)

for normal (NH) and inverted hierarchy (IH), respectively, and

$$\tilde{m}_{e\mu}^0 = \frac{1}{\sqrt{2}} C_{12} s_{12} \Delta m_{21}^2;$$

$$\tilde{m}_{\mu\tau}^0 = \frac{1}{2} \left( \Delta m_{21}^2 c_{12}^2 + |\Delta m_{31}^2| \right);$$

$$\tilde{m}_{\mu\mu}^0 = \frac{1}{2} \left( |\Delta m_{31}^2| + \Delta m_{21}^2 c_{12}^2 + 2m_0^2 \right);$$

(3.4)

where the minus (plus) sign in $\tilde{m}_{e\mu}^0$ corresponds to NH (IH) hierarchy, and $m_0$ stands for the lighter neutrino mass. Notice that in this limit there is no CP violation implied, because all matrix elements are real in the reconstructed matrix in Eq. (3.2). Also note that above expressions imply that, in the current approximation, neutrino oscillation scales are given straightforwardly by the off diagonal terms of $\tilde{M}_L^0$, with the proper identifications, such that $\Delta m_{21}^2 = \sqrt{8} \tilde{m}_{e\mu}^0 / \sin 2\theta_{12}$, whereas $\Delta m_{31}^2 \approx \mp 2\tilde{m}_{\mu\mu}^0$. However, as this is just a naive approximation that neglects the contributions of $\theta_{13}$ and $\Delta \theta$, we need to keep in mind that it is likely to provide wrong predictions for the scales if the corrections from the breaking of $\mu - \tau$ symmetry were not negligible. We will address this issue below.

**$\mu - \tau$ symmetry predictions**

It is not difficult to see that Eq. (3.2) does correspond to the most general structure of the left handed hermitian squared matrix allowed by $\mu - \tau$ symmetry. Indeed, in the top-down approximation where one starts by imposing the symmetry on the otherwise general hermitian $M_L$, it is required that its elements satisfy the conditions $\tilde{m}_{e\mu} = \tilde{m}_{e\tau}$, $\tilde{m}_{\mu e} = \tilde{m}_{\tau e}$, $\tilde{m}_{\mu\mu} = \tilde{m}_{\mu\tau}$ and $\tilde{m}_{\mu\tau} = \tilde{m}_{\tau\mu}$. Hermiticity, on the other hand, implies that all matrix elements in general obey the condition $\tilde{m}_{a\beta} = \tilde{m}_{\beta\alpha}$. As a consequence of the last, $\tilde{m}_{\mu\tau}$, as well as the diagonal components, must be real numbers. Therefore, only the off diagonal terms on first row and column could be complex. That is only $\tilde{m}_{e\mu}$ does it. The single phase of this term, however, is non physical. As it can be easily checked, the last can be rephased away by a global redefinition of the electron neutrino, and electron wave function phases, which finally renders $\tilde{m}_{e\mu}$ to be a real number. This procedure shows that, indeed, no CP violation is implied in the $\mu - \tau$ symmetric case.

A straightforward calculation in the top-down approximation, shows that the squared neutrino masses predicted from a $\mu - \tau$ symmetric $\tilde{M}_L$ are given as

$$m_1^2 = \frac{1}{2} \left[ \tilde{m}_{e\mu}^0 + \tilde{m}_{\mu\mu}^0 + \tilde{m}_{\mu\tau}^0 - \frac{\sqrt{8}}{\sin 2\theta_{12}} \tilde{m}_{e\mu}^0 \right];$$

$$m_2^2 = \frac{1}{2} \left[ \tilde{m}_{e\tau}^0 + \tilde{m}_{\mu\mu}^0 + \tilde{m}_{\mu\tau}^0 + \frac{\sqrt{8}}{\sin 2\theta_{12}} \tilde{m}_{e\mu}^0 \right];$$

$$m_3^2 = \tilde{m}_{\mu\mu}^0 - \tilde{m}_{\mu\tau}^0$$

(3.5)

where the $\theta_{12}$ mixing angle goes as

$$\tan 2\theta_{12} = \frac{\sqrt{8} \tilde{m}_{e\mu}^0}{\tilde{m}_{e\mu}^0 - (\tilde{m}_{\mu\mu}^0 + \tilde{m}_{\mu\tau}^0)}.$$

(3.6)

**IV. $\mu - \tau$ SYMMETRY BREAKING**

The symmetry under consideration, however, is not in any way an exact one and previous predictions would result to be inexact. The observed non zero values for $\theta_{13}$ and $\Delta \theta$ are a clear indication of that. Nevertheless, the fact that these last are actually small suggest that $\mu - \tau$ symmetry could still be treated as an approximated flavor symmetry. Exploring how good that approximation actually is, is the question we address next.
In order to study the effects of the breaking of \(\mu - \tau\) symmetry we will focus in the hermitian squared matrix \(\tilde{M}_L\), which, as we have already argued, has a general form that is independent of the chosen right handed neutrino basis. In its more general form any such a matrix can always be rewritten as

\[
\tilde{M}_L = \tilde{M}_L^S + \delta \tilde{M}_L ,
\]

where \(\tilde{M}_L^S\) is an explicitly \(\mu - \tau\) exchange invariant matrix, and \(\delta \tilde{M}_L\) stands for the non invariant parts. In terms of its components, the symmetric part of the hermitian matrix is given as

\[
\tilde{M}_L^S = \begin{pmatrix}
\tilde{m}_{ee} & \tilde{m}_{e\mu} & \tilde{m}_{e\tau} \\
\tilde{m}_{\mu e} & \tilde{m}_{\mu\mu} & \text{Re}(\tilde{m}_{\mu\tau}) \\
\tilde{m}_{\tau e} & \text{Re}(\tilde{m}_{\tau\mu}) & \tilde{m}_{\tau\tau}
\end{pmatrix} .
\]

As we have discussed in previous section above matrix can be made all real by a global rephasing on the electron sector. However, with a broken symmetry this operation can not completely remove the phase anymore and CP violation should arise. Indeed, the rephasing of electron leptons only moves the \(\tilde{m}_{\mu e}\) phase into \(\delta \tilde{M}_L\) matrix elements. In order to keep our discussion simple and as general as possible we assume hereafter only the natural conditions implied from the symmetry and hermiticity. As such, we take \(\tilde{m}_{e\mu}\) as the only possible complex matrix element in \(\tilde{M}_L^S\). Notice that this conditions also requires that \((\tilde{M}_L^S)_{\mu\tau}\) be the real part of the in general complex \(\tilde{m}_{\mu\tau}\).

The nonsymmetric part, and the source of the breaking of the symmetry, is then expressed in general by the hermitic matrix

\[
\delta \tilde{M}_L = \begin{pmatrix}
0 & 0 & \alpha \\
0 & 0 & \zeta \\
\alpha^* & \zeta^* & \beta
\end{pmatrix} ,
\]

where the involved symmetry breaking parameters are exactly defined as

\[
\alpha = \tilde{m}_{e\tau} - \tilde{m}_{e\mu} ,
\beta = \tilde{m}_{\mu\tau} - \tilde{m}_{\mu\mu} ,
\zeta = \text{Im}(\tilde{m}_{\mu\tau}) .
\]

We note that hermiticity implies the existence of only one arbitrary phase which is contained in the \(\alpha\) parameter. \(\beta\) is a real number, whereas \(\zeta\) is purely imaginary by definition. From here, it is easy to see that a possible removal of the \(\tilde{m}_{e\mu}\) phase in \(\tilde{M}_L^S\) does only change the phase of \(\alpha\), without affecting the other parameters. In that basis, the rephasing process explicitly shows that only one \(CP\) phase does become physical. The relation of the final phase with the Dirac \(CP\) phase in the PMNS matrix, however, is not straightforward, as our next results show. Thus, we do not find any further advantage on explicitly using such a rephasing.

As it was already emphasized in the discussion on previous section, up to non physical phases, the general squared mass matrix \(M_L\) can be reconstructed purely from neutrino observables using Eq. \((4.4)\), according to which \(\tilde{m}_{a\beta} = \sum_a U_{a\alpha} U_{a\beta}^* m_a^2\). By comparing this reconstruction with our above parameterization we find that the general symmetry breaking parameters, written in terms of the neutrino observables without any approximation, are given by

\[
\alpha = \pm a z \Delta m_{ATM}^2 + (a s_{12}^2 z - c_{12} s_{12} a') \Delta m_{\odot}^2 ,
\beta = \pm b c_{13}^2 \Delta m_{\odot}^2 + [b' + d \cos{\delta_{CP}}] \Delta m_{\odot}^2 ,
\zeta = - i c_{12} s_{12} s_{13} \sin{\delta_{CP}} \Delta m_{\odot}^2 ,
\]

where a short hand notation has been introduced to account for the following combinations among the mixings, \(a = c_{13} (s_{23} - c_{23})\), \(a' = c_{13} (s_{23} + c_{23})\), \(b = c_{23}^2 - s_{23}^2\), \(b' = s_{13}^2 s_{12}^2 - c_{12}^2\), \(d = 4 s_{12} c_{23} c_{13} s_{13} s_{12}\). In above, the sign on top (bottom) corresponds to \(NH\) (IH) from now on. We stress that above expresions do cancell in the limit where \(\theta_{13} = -\pi/4\) and \(\theta_{13} = 0\). Furthermore, none of the breaking parameters do depend on the absolute scale of the neutrino mass, which seems remarkeable.

It is also worth noticing that \(\alpha\) and \(\beta\) corrections are dominated by the atmospheric scale, whereas \(\zeta\) is just proportional to the solar scale. This represents a relevant correction to the symmetric condition, \(\tilde{m}_{a\beta}^0 = \tilde{m}_{a\mu}^0\) as presented in Eq. \((3.4)\). In contrast, the most general expression of such a mass term goes as \(\tilde{m}_{\mu\tau} = \pm c_{13} s_{23} z \Delta m_{ATM} + c_{13} s_{12} (c_{12} c_{23} - s_{12} z_{23}) \Delta m_{\odot}\), which also has the atmospheric scale as the leading contribution to it. This means that the predicted \(\Delta m_{21}^2\) scale in the symmetric limit would be larger than the actual value of the solar scale, but not as large as the atmospheric scale itself (due to the \(z\) factor).

In order to get a quantitative estimate of how good the \(\mu - \tau\) approximation is, we introduce a set of dimensionless parameters that compare the breaking parameters against the correspond-
ing matrix elements, such that we define

\[
\hat{\alpha} = \frac{\alpha}{m_{e\mu}} = \frac{\sum_a U_{ea} [U^*_{\tau a} - U^*_{\mu a}] m_a^2}{\sum_a U_{ea} U_{\mu a} m_a^2},
\]

\[
\hat{\beta} = \frac{\beta}{m_{\mu\mu}} = \frac{\sum_a [U_{\tau a}^2 - |U_{\mu a}|^2] m_a^2}{\sum_a |U_{\mu a}|^2 m_a^2},\tag{4.6}
\]

\[
\hat{\gamma} = \frac{\gamma}{R_{e}(m_{\mu\tau})} = \frac{i \Delta m_{\mu\tau}^2}{\sum_a U_{ea} U_{\mu a} m_a^2}.
\]

We should notice that this new set of parameters is invariant under the rephasing of \(m_{e\mu}\) discussed above.

It is straightforward to use above definitions to make an estimate of the leading order values of the dimensionless parameters, up to corrections of the order of \(x = \Delta m_{\mu\tau}^2/\Delta m_{ATM}^2\). After some algebra we get

\[
\hat{\alpha} \approx \cot \theta_{23} - 1 \pm \left( \frac{a_1}{a_3} + \frac{a_2}{a_3} \right) x + O(x^2)\tag{4.7}
\]

and

\[
\hat{\beta} \approx \frac{\pm b c_{13}^2}{y + a_4} \left[ \frac{b d + d \cos \delta_{CP}}{y + a_4} \right] + \frac{b |a_3|^2}{s_{12}^2(y + a_4)^2} x + O(x^2)\tag{4.8}
\]

where our short hand notation now stands for

\[
y = m_0^2/\Delta m_{ATM}^2, \quad a_1 = a s_{12}^2 c_{12} c_{13} - c_{12} s_{12} a', \quad a_2 = c_{13} s_{12}(c_{12} c_{23} - z s_{12} s_{23}), \quad a_3 = c_{13} s_{23}
\]

\[
a_4 = \begin{cases} 
2 s_{12}^2 c_{13}^2 \
1 - s_{23}^2 c_{13}^2 
\end{cases} \quad \text{(NH)}
\]

\[
1 - s_{23}^2 c_{13}^2 \quad \text{(IH)}
\]

(4.9)

Considering best fit values for the observed mixings we get that

\[
\hat{\alpha} \approx -2.13 + O(x).\tag{4.10}
\]

for any hierarchy and any value of the absolute neutrino mass, which represents a non small number, although \(\alpha\) itself is always smaller than the atmospheric scale. This is, as a matter of fact, the largest of the corrections to the symmetric mass matrix elements that are required to account for the observed data. Indeed, for \(\hat{\beta}\) we get, at a first order evaluation on the best fit values, that

\[
|\hat{\beta}| < \begin{cases} 
0.288 \quad \text{(NH)} \\
0.215 \quad \text{(IH)}
\end{cases}\tag{4.11}
\]

It is also straightforward to show that

\[
\hat{\gamma} \approx \mp i \frac{c_{12} s_{12} s_{14} \sin \delta_{CP}}{c_{13}^2 c_{23}^2} x + O(x^2).\tag{4.12}
\]

and therefore, that \(|\hat{\gamma}|\) has a best fit value of order \(10^{-2}\) at the highest.

### A. Mixings near the symmetric limit

Above results indicate that \(\mu - \tau\) symmetry can only be considered as a good approximate flavor symmetry in a weak sense, that is when the breaking parameters are compared to the heaviest neutrino mass in the spectrum, which is given as \(m_\nu^2 = \Delta m_{ATM}^2 + m_0^2 (+\Delta m_\odot^2)\) for the NH (IH) case. This claim becomes transparent if we consider the initial observation that \(\sin \theta_{13}\) and \(\sin \Delta \theta\) are, as a matter of fact, small numbers. Thus, \(\alpha\) and \(\beta\) breaking parameters can be expressed by the following approximated first order formulae

\[
\alpha \approx \sqrt{2}\left[ \sin \left( \pm \Delta m_{ATM}^2 - s_{12}^2 \Delta m_\odot^2 \right) - c_{12} s_{12} s_{\Delta \theta} \Delta m_\odot^2 \right]
\]

\[
\beta \approx \mp 2 s_{\Delta \theta} \Delta m_{ATM}^2 + [2 b' s_{\Delta \theta} + g_{s13}] \Delta m_\odot^2,\tag{4.13}
\]

with \(g = 2 s_{12} c_{12}(s_{\Delta \theta}^2 - 1)\cos \delta_{CP}\) and \(s_{\Delta \theta}\) standing for \(\sin \Delta \theta\), which for the present case is defined by the relation \(\theta_{23} = -\pi/4 + \Delta \theta\). Above expressions do stress that indeed \(|\alpha|, |\beta|, |\gamma| < < m_\nu^2\). At the leading order (when \(x \approx 0\)) and taking neutrino oscillation scales as known inputs, we get the following predictions

\[
\sin \theta_{13} \approx \frac{|\alpha|}{\sqrt{2} \Delta m_{ATM}^2},\tag{4.14}
\]

and

\[
\sin \Delta \theta \approx \frac{\beta}{2 \Delta m_{ATM}^2},\tag{4.15}
\]

that should be valid for any flavor model that results consistent with neutrino data.

From last expressions, the phenomenological relation given in Eq. (4.11) gets justified, since our \(\mu - \tau\) parametrization now suggests that

\[
\sin \Delta \theta \approx \text{sign} \theta_{13} \times \beta/\sqrt{2} |\alpha|,\tag{4.16}
\]

which is an expression given just in terms of the mass matrix elements of \(M_L\).

Including solar scale contribution in our calculations provides somewhat more complicated expressions for the small mixings that cannot be easily resolved analytically in terms only of the mass matrix elements. However, one can read Eq. (4.13) as a constraint on the small neutrino mixings, when all other neutrino oscillation parameters are taken as known, within experimental uncertainties. Following this line of thought, and after some lengthy algebra we can express the relation among the deviation of the atmospheric mixing from its maximal value in terms of the predicted value for \(\theta_{13}\), as

\[
\sin \Delta \theta \approx \text{sign} \theta_{13} \frac{g s_{13}(\sqrt{2}B g_{s12} + A s_{12} c_{12} \Delta m_\odot^2)}{2 A g_{sc} + \sqrt{2} B s_{12} c_{12} \Delta m_\odot^2},\tag{4.17}
\]
where we have defined $A = \text{Re}(\alpha) + \text{Im}(\alpha)$, $B = \beta + |c_1|$, and also $g_6 = \pm \Delta m^2_{\text{ATM}} - \Delta m^2_{\text{sol}}$, $g_c = \pm \Delta m^2_{\text{ATM}} - \Delta m^2_{\text{nu}},$ and $g_6 = \cos \delta_{\text{CP}} - \sin \delta_{\text{CP}}$. In the same footing, the predicted value for $\sin \theta_{13}$ mixing is given as
\[
\sin \theta_{13} \approx \frac{2Ag_c + \sqrt{2}Bg_{12}s_{12}\Delta m^2_{\text{sol}}}{2\sqrt{2}g_6[g_c - (s_{12}^2c_{12}\Delta m^2_{\text{sol}})]}. \tag{4.18}
\]

V. $\mu - \tau$ ANTI-SYMMETRY CASE.

Let us next add some comments regarding the case where $\theta_{23}$ is chosen to lay in the first quadrant, such that its maximal value corresponds to $\theta_{23} = \pi/4$. As we have stated before this alternative should corresponds to the one we have already discussed, up to wave function phase redefinitions in both flavor and mass neutrino basis. Therefore one would not expect any fundamental conclusion to change. Nevertheless, as the given choice effectively affects the way $\mu - \tau$ symmetry realizes, and, hence, general formulae may also change accordingly, we believe that considering in some detail the changes introduced in the analysis for this choice can be of interest for model building.

First of all, for a strictly positive value of $\theta_{23} = \pi/4$, joint to a zero value for $\theta_{13}$, our phenomenological reconstruction of the mass matrix, in the diagonal charged lepton basis, now leads to
\[
M^0_{\mu\tau} = \begin{pmatrix}
\tilde{m}_{e\tau} & \tilde{m}_{\mu\tau} & -\tilde{m}_{e\mu} \\
\tilde{m}_{\mu\tau} & \tilde{m}_{\mu\mu} & -\tilde{m}_{e\mu} \\
-\tilde{m}_{e\mu} & -\tilde{m}_{e\mu} & \tilde{m}_{\mu\mu}
\end{pmatrix}, \tag{5.1}
\]
Comparing with Eq. 3.3, one can notice that only the third row of the above mass matrix has changed by global sign. However, this does indeed change the way $\mu - \tau$ manifests itself in the hermitian squared mass matrix, as usually defined by $\tilde{M}^0_{\mu\tau} = M^0_{\mu\tau}M^0_{\mu\tau}$, which now becomes
\[
\tilde{M}^0_{\mu\tau} = \begin{pmatrix}
\tilde{m}_{e\tau} & \tilde{m}_{e\tau} & -\tilde{m}_{e\mu} \\
\tilde{m}_{e\tau} & \tilde{m}_{e\tau} & -\tilde{m}_{e\mu} \\
-\tilde{m}_{e\mu} & -\tilde{m}_{e\mu} & \tilde{m}_{\mu\mu}
\end{pmatrix}, \tag{5.2}
\]
where its entries are given by equations 3.5 and 3.6 just as in the symmetric case. Above matrix now exhibits a $\mu - \tau$ antisymmetry, where $\tilde{M}^0_{AL}$ remains invariant under the exchange $\nu_{\mu} \leftrightarrow -\nu_{\tau}$.

It is important to point out two things in here. First, the squared mass matrix is also hermitian in the $\mu - \tau$ anti-symmetric case, and second, that $\tilde{M}^0_{AL}$ can also be constructed by a topdown method akin to the $\mu - \tau$ symmetric case using the PMNS mixing matrix elements by considering $\theta_{23} = \pi/4$ instead of $\theta_{23} = -\pi/4$. Following our own previous steps, in the most general case $\mu - \tau$ anti-symmetry breaking can be explicitly parameterized as
\[
\tilde{M}_{\mu\tau} = \tilde{M}^0_{\mu\tau} + \delta\tilde{M}_{\mu\tau}, \tag{5.3}
\]
where the $\tilde{M}^0_{\mu\tau}$ is the $\mu - \tau$ antisymmetric part generaly written as
\[
\tilde{M}^0_{\mu\tau} = \begin{pmatrix}
\tilde{m}_{e\tau} & \tilde{m}_{e\tau} & -\tilde{m}_{e\mu} \\
-\tilde{m}_{e\mu} & \text{Re}(\tilde{m}_{\mu\tau}) & \tilde{m}_{\mu\mu}
\end{pmatrix}, \tag{5.4}
\]
As for the anti-symmetry breaking matrix, $\delta\tilde{M}_{\mu\tau}$, it is clear that the only entry which is different from the symmetric case, is the one associated to $\tilde{m}_{e\tau}$, and thus it is now written as
\[
\delta\tilde{M}_{\mu\tau} = \begin{pmatrix}
0 & 0 & \alpha_A \\
0 & 0 & \zeta \\
\alpha_A & \zeta & \beta
\end{pmatrix}, \tag{5.5}
\]
for which we now define
\[
\alpha_A = \tilde{m}_{e\tau} + \tilde{m}_{\mu\tau}, \tag{5.6}
\]
whereas $\beta$ and $\zeta$ are given as before. In terms of the neutrino oscillation parameters we now get
\[
\alpha_A = \pm a'z\Delta m^2_{\text{ATM}} - (ac_{12}s_{12} + a's_{12}^2)\Delta m^2_{\text{sol}}. \tag{5.7}
\]
Thus, the corresponding dimensionless parameter, defined as earlier by $\hat{\alpha}_A = \alpha_A/\tilde{m}_{e\tau}$, is given up to first order in the neutrino oscillation scales ratio, as
\[
\hat{\alpha}_A \approx 1 + \cot \theta_{23} - \left(\alpha_A' a_{13} + \alpha_A a_{13}'\right)x + \mathcal{O}(x^2) \approx 2.13 + \mathcal{O}(x) \tag{5.8}
\]
where $a_{13}' = a's_{12}^2 + ac_{12}s_{12}$. Last row gives the leading order contribution with central values in neutrino mixings, which is, up to a sign, as large as the one obtained in the symmetric realization of $\mu - \tau$ symmetry and, as before, $\alpha_A$ remains smaller than the largest neutrino mass.

By considering the near to anti-symmetric case, where we parameterize the atmospheric mixing as $\theta_{23} = \pi/4 + \Delta \theta$ for $|\Delta \theta| \ll 1$, we get the following expression for the predicted relationship among $\theta_{13}$ and $\Delta \theta$ mixings,
\[
\sin \Delta \theta \approx \sin \theta_{13} \frac{\sqrt{2}B g_{12}s_{12}c_{12}\Delta m^2_{\text{sol}}}{\sqrt{2}B g_{12}s_{12}\Delta m^2_{\text{sol}} - 2A'g_6}, \tag{5.9}
\]
where $A' = \text{Re}(\alpha_A) + \text{Im}(\alpha_A)$ and $g_6 = \cos \delta_{\text{CP}} + \sin \delta_{\text{CP}}$. Notice that this expression mimics the corresponding one that we obtained in the previous case [Eq. 4.17].
A. Reconstructed mass matrix elements

By using the general parameterization we have given for the hermitian squared mass matrix, it is easy to get a numerical idea of the order of magnitude of the off-diagonal matrix elements. This is because it turns out that they do not depend (as we already emphasized along our previous discussions) on the absolute mass scale of the neutrino. For this we can just set in the best fit values of the oscillation neutrino parameters, assuming no CP violation. Thus, the following are the numerically reconstructed off-diagonal mass matrix elements, in the most general case, for NH (IH),

\[
\begin{align*}
\tilde{m}_{\mu e} & \approx 2.65 \times 10^{-4} (2.16 \times 10^{-4}) \text{eV}^2, \\
\tilde{m}_{\mu \tau} & \approx 1.19 \times 10^{-3} (1.24 \times 10^{-3}) \text{eV}^2, \\
\tilde{m}_{\tau e} & \approx 2.46 \times 10^{-4} (2.96 \times 10^{-4}) \text{eV}^2. 
\end{align*}
\]

On the other hand, the reconstruction of the diagonal terms in the CP conserving case gives the following expressions,

\[
\begin{align*}
\tilde{m}_{ee} & = m_0^2 + \Delta m_{ATM}^2 \left[ \frac{1}{2} \pm \frac{1}{2} \right] + s_{23} c_{13} \Delta m_{23}^2, \\
\tilde{m}_{\mu \mu} & = m_0^2 + \Delta m_{ATM}^2 \left[ \frac{1}{2} \pm \frac{1}{2} \right] + s_{23} c_{13} \Delta m_{23}^2, \\
\tilde{m}_{\tau \tau} & = m_0^2 + \Delta m_{ATM}^2 \left[ \frac{1}{2} \pm \frac{1}{2} \right] + c_{23} c_{13} \Delta m_{23}^2,
\end{align*}
\]

where, to simplify, we have used \( h = c_{12}^2 c_{23} + s_{12}^2 s_{23} s_{13} - 2 c_{12} c_{23} s_{12} s_{23} s_{13} \cos \delta_{CP} \), and \( h' = c_{12}^2 s_{23} + s_{12}^2 c_{23} s_{13} + 2 c_{12} c_{23} s_{12} s_{23} s_{13} \cos \delta_{CP} \), and where the top(bottom) sign corresponds to normal (inverted) hierarchy and \( m_0 \) stands for the lightest neutrino mass, as usual.

VI. HIGGS BOSON DECAY TO NEUTRINOS.

From the Yukawa interaction in the flavor basis, as expressed by equation (27), it is straightforward to compute the invisible Higgs boson decay width, \( \Gamma^\ell_\nu (H \rightarrow \nu \ell \nu) \). Also, considering the current observed value of the total Higgs’ decay width \( \Gamma_H < 1.7 \text{ GeV} \), the branching ratio for the invisible decay \( B(H \rightarrow \nu \ell \nu) \), to first order, can be estimated within the current framework. For the decay width, considering that \( m_a \ll m_H \), we get

\[
\Gamma^\ell_a \approx \frac{m_a^2}{2(v)^2} \frac{m_H}{8\pi} |U_{\ell a}|^2. \quad (6.1)
\]

The branching ratio is then given by the equation

\[
B(H \rightarrow \nu \ell \nu) = \frac{\sum_{\alpha \ell} \Gamma_{H}^{\ell_{\alpha}}}{\Gamma_{H}}. \quad (6.2)
\]

Furthermore, regarding the Higgs mass at a value of \( m_H = 125 \text{ GeV} \), its vev \( \langle v \rangle \approx 246 \text{ eV} \), and taking the neutrino mass spectrum at a high limit of \( m_\nu \approx 2 \text{ eV} \), a degenerated hierarchy as a means of estimating an upper limit value, the resulting invisible branching ratio comes out to be

\[
B(H \rightarrow \nu \ell \nu) \approx 5.5 \times 10^{-13}, \quad (6.3)
\]

at its larger possible value in the theory.

The upper bound experimentally set for the branching fraction of the invisible Higgs decay is currently at 0.28 \cite{15}, which places our current estimation at a much lower value.

VII. CONCLUSIONS

As our present analysis for Dirac neutrinos has shown, \( \mu-\tau \) symmetry arises as a slightly broken symmetry that makes itself evident in the diagonal charged lepton mass basis. This symmetry is actually already encoded by the observed mixings in the PMNS mixing matrix. An appropriate selection of the right handed neutrino basis then allows to remove all non physical parameters in the Yukawa sector. In such a parameterization, only the absolute neutrino mass scale and the CP phase remain unknown.

Furthermore, \( \mu-\tau \) breaking becomes easier to study when using the hermitian squared neutrino mass matrix \( \tilde{M} \). Such a matrix can either be symmetric or antisymmetric under the exchange of \( \mu \) and \( \tau \) labels. Both the realizations, however, as we have already argue, are actually connected with a simple rephasing in \( \tau \) neutrino flavor and third neutrino mass eigenstates and, therefore, main conclusions regarding symmetry breaking are alike. \( \tilde{M} \) allows for a natural and easy parameterization of the breaking of the symmetry that requires only three free parameters \((\alpha, \beta \text{ and } \zeta)\), with only one of them being a complex number \(\alpha\). Observed atmospheric and reactor mixings indicate that the symmetry breaking parameters are at most below the range of the atmospheric squared mass difference, without any dependance on the absolute scale of the neutrino mass and regardless of the actual value of the CP phase.

The largest of the symmetry breaking corrections, in any realization of the symmetry, corresponds to the \( \tilde{m}_{\ell \nu} \) mass term, parameterized by \( \alpha \), which in any case is always smaller than the heavier mass scale in the neutrino sector. This could indicate the presence of an unknown (perhaps broken) symmetry in the \( e-\tau \) coupling sector. It is also interesting to note that a perturbative behavior in the symmetry breaking sector only happens
when the neutrino mass spectrum becomes almost degenerate and, thus, when the absolute mass scale turns out to be larger than the atmospheric scale. Indeed, upon comparison of the matrix elements in $\delta M_{\alpha\beta}$ with the neutrino mass scale, it is only possible for the almost degenerate case to truely fulfill the relation $\delta M_{\alpha\beta} \ll m_{\nu_\alpha}^2$ with $m_{\nu_\alpha}^2$ being the squared minimum value of the neutrino mass scale. Therefore, the only general conclusion one can draw is that $\mu - \tau$ symmetry is indeed a good approximated symmetry only in a weak sense. Nevertheless, it is still interesting to point out the approximated relation that arises for the observed mass mixing data, according to which $2\Delta M_{\alpha\beta} \approx \sqrt{2} \alpha / \sin \theta_{13} \approx |\beta| / \sin \Delta \theta$. We believe this observations may be valuable for model building in the Dirac neutrino framework.

For further study, an analysis of the $\mu - \tau$ symmetry breaking can be performed using perturbation theory considering the results mentioned earlier.

Another noteworthy result, is the fact that for any realization of the symmetry (that is for a symmetric or anti-symmetric $M$), we always have that $\alpha, \hat{\alpha}_A >> \beta, \hat{\beta}$, which might also indicate an underlying flavor symmetry involving the $e - \mu$ coupling in the theory. Thus, in general, even though $\mu - \tau$ symmetry seems to be an underlying symmetry in the neutrino sector a complete understanding of neutrino oscillation parameters stills seems to need additional (extended ) flavor symmetries.

Finally, the estimation or the invisible Higgs decay to neutrinos could grant a good test for the nature of neutrinos as well as their mass spectrum, mass hierarchy and CP violation phase, provided that such quantity could be measured accurately. Unfortunately, however, it still lays far below the current experimental sensitivity.

Acknowledgments

This work was partially supported by CONACyT, México, under Grant No. 237004.

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