Thermodynamic Geometry and Type 0A Black Holes

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Abstract

In this note we study thermodynamic geometry of the type 0A black hole solution in string theory using a variety of different methods (Ruppeiner, Weinhold and Geometrothermodynamics). Our results indicate that the curvature invariants are finite for all physical solutions, suggesting that there is no phase transition. It is also found that the cutoff of the entropy, which is the singular limit of the theory, appears geometrically in the Weinhold picture as the thermodynamic cone itself.

1 Introduction

The study of black hole thermodynamics has been an intensely active subject for the past few decades. The fact that such objects emit Hawking radiation has been at the heart of recent developments concerning the paradox of information loss. Whilst Hawking’s calculations indicated that emitted radiation was thermal \cite{1}, recent arguments have suggested that there are in fact correlations between emitted quanta which can carry away information and therefore preserve unitarity. The tunneling method of Parikh and Wilczek \cite{2}, in particular, has been an important milestone on this particular journey. There are several puzzling issues which remain, even if the information paradox is resolved, which pertain to the nature of the entropy on the horizon - and its corresponding microstates. Such questions would hopefully be answered by a quantum theory of gravity, however even within string theory such questions are often difficult to pose unless we include supersymmetry. One particular class of stringy models that may be exactly solvable are those of dilaton gravity in 1 + 1 dimensions in the type 0 string \cite{3, 4, 5}. Such a string only contains bosonic degrees of freedom due to the non-chiral projection acting on the two-dimensional superstring. It has long been known that charged (asymptotically flat) black hole solutions exist in this theory, and its relative simplicity means that it is a good toy model for many of our current conjectures regarding black hole information loss. Because the theory arises from a truncation of the full superstring, and one can also use it to test matrix models \cite{6, 8}.

In this note we will be interested in the thermodynamic geometry of such a black hole configuration, in particular we wish to understand whether a phase transition should be expected.
at a critical value of the entropy as proposed in [12]. We begin by briefly recapitulating the black hole solution before moving on to a discussion of information metrics and different bases in which they are defined.

2 Black holes in type 0A string theory

It has long been known that the low energy effective action for the type 0A theory, with a single non-zero RR flux term and no tachyon, can be written as follows [3, 4, 5]

\[ S = - \int d^2\sigma \sqrt{-g} \left\{ e^{-2\Phi} \left( R + 4(\nabla \Phi)^2 + 4k^2 \right) + \Lambda \right\} \]  

(2.1)

where \( k^2 = 2/\alpha'^2 \) is a cosmological constant term depending on the string length, and \( \Lambda = -Q^2/(2\pi\alpha') \) is a constant coming from the gauge field via \( F_{\mu
u}^A = -Q/(2\pi\alpha') \). Denoting the coordinates by \( \sigma = (t, \phi) \) we find there is a black hole solution corresponding to the following:

\[ ds^2 = -f(\phi)dt^2 + \frac{d\phi^2}{f(\phi)}, \quad \Phi = -k\phi \]  

(2.2)

where the function \( f(\phi) \) has the following form

\[ f(\phi) = 1 - \frac{1}{2k} (M - \Lambda \phi) e^{-2k\phi} \]  

(2.3)

with \( M \) corresponding to the mass of the black hole. The event horizon exists as the solution to

\[ e^{-2k\phi_H} (M - \Lambda \phi_H) = 2k \]  

(2.4)

and the temperature is given by the usual expression

\[ T = \frac{f'(\phi)}{4\pi} \bigg|_{\phi_H} = \frac{1}{8k\pi} \left( \Lambda + 2k(M - \Lambda \phi_H) \right) e^{-2k\phi_H}. \]  

(2.5)

Using the WKB approximation, one can argue that Hawking radiation arises from quantum tunneling of particles through the event horizon\(^3\). A simple calculation then yields the following result for the temperature

\[ T = \frac{k}{2\pi} \left( 1 + \frac{\Lambda}{4k^2} e^{-2k\phi_H} \right) \]  

(2.6)

and therefore one can use the standard thermodynamic relation \( dM = TdS \) combined with (2.4) to determine the entropy, which becomes

\[ S = 4\pi e^{2k\phi_H}. \]  

(2.7)

Given the entropy of the system, one can then express the mass (in the fixed charge ensemble) as a function of the entropy and the charge

\[ M = \frac{kS}{2\pi} - kQ^2 \left( \frac{S}{8\pi} \ln \left( \frac{S}{4\pi} \right) \right) \]  

(2.8)

\(^3\)We refer the reader to [12] for further details.
allowing us to derive the following thermodynamic variables

\[
T = \frac{\partial M}{\partial S} = \frac{k}{2\pi} \left( 1 - \frac{Q^2}{4S} \right),
\]

\[
\Phi = \frac{\partial M}{\partial Q} = \frac{kQ}{4\pi} \ln \left( \frac{S}{4\pi} \right),
\]

\[
C = \frac{\partial M}{\partial T} = 4S^2 Q^2 - S,
\]

\[
\chi = \frac{\partial Q}{\partial \Phi} = -\frac{4\pi}{\ln(S/4\pi)}. \tag{2.9}
\]

The expression for the specific heat is interesting because it implies that (in physically acceptable regions) the specific heat is positive which is in stark contrast to most black hole solutions. This suggests that the black hole is thermodynamically stable, and suggests that the canonical ensemble is valid. The temperature of the solution is also bounded from above by \(T_{\text{max}} = 1/\sqrt{2\pi^2 \alpha'}\) which corresponds to the Hagedorn temperature \(T_H\). It is believed that one can analytically continue to temperatures greater than \(T_H\) using T-duality, in which case one simply studies the black hole solution in type 0B at \(T < T_H\) \[6\]. The expectation is that there will then be a phase transition occurring at the Hagedorn temperature, which separates these two distinct phases. However it is not clear whether one can study fluctuations through this point.

Note that the non-zero charge allows for the existence of an extremal black hole solution, which can be found by demanding both \(f(\phi_{\text{ex}}) = f'(\phi_{\text{ex}}) = 0\), which has a mass

\[
M_{\text{ex}} = \frac{kQ^2}{8\pi} \left( 1 - \ln \left( \frac{Q^2}{16\pi} \right) \right). \tag{2.10}
\]

and zero-temperature (as expected for an extremal solution). Positivity of the extremal mass therefore imposes an upper bound on the flux such that \(Q^2 \leq 16\pi e \approx 136.64\).

### 2.1 The Phase Transition Puzzle

In Davies paradigm \[7\], one can see that the specific heat \(2.9\) does not diverge, therefore a phase transition is not expected. However if one makes a Legendre transformation to work in the fixed potential ensemble, then the specific heat can be written in the form

\[
C_\Phi \sim \frac{S}{4\pi^2 \Phi^2} \ln \left( \frac{S}{4\pi} \right) \frac{4\pi^2 \Phi^2 - k^2 S \ln(S/4\pi)^2}{2 + \ln(S/4\pi)} \tag{2.11}
\]

which diverges at \(S = S_c = 4\pi e^{-2}\), therefore one may expect a phase transition at this value of the entropy. Such phase transitions are commonly studied in the condensed matter community, but less so in the black hole community. Since this value is independent of any other variables, one may expect something unusual occur at \(S = S_c\) in the fixed charge ensemble but this is actually not the case. All the thermodynamic variables are continuous through the curve \(S = S_c\). One may also be concerned that we are not in Einstein frame, therefore the concept of a gravitational phase transition may be ill defined. However because our black hole solution is asymptotically flat, we can use the results of \[9\] to show that this is not the case.
Figure 1: Specific heat, $C_{\phi}$, of the 0A black hole. The singular limit of the theory is the entropy cutoff, $S = S_c = 4\pi e^2$, which corresponds to the divergence of $C_{\phi}$.

In the following section we will study the black hole thermodynamics using information geometry, which involves the construction of a metric using the thermodynamic variables. One can then construct invariants from the metric, which should be insensitive to the thermodynamic ensemble under consideration.

3 Information Geometry

Information geometry [10] is the study of probability and information by way of differential geometry. In this note we focus in particular on the Fisher type [11] of information geometry in that the Hessian matrix of the second derivatives of the energy, or alternatively the entropy, can be regarded as a Riemannian metric on the space of thermodynamical states. When energy is used as a potential this metric is called the Weinhold metric [13], when the entropy is used it is called the Ruppeiner metric [14].

Historically Ruppeiner proposed in 1979 a geometrical way of studying thermodynamics of equilibrium systems. In his theory certain aspects of thermodynamics and statistical mechanics of the system under consideration are encoded in a single geometrical object, namely the metric describing its thermodynamic state space. The distance on this space is given by

$$ds^2_R = g_{ij}^R dX^i dX^j,$$

where $g_{ij}^R$ is the so-called Ruppeiner metric defined as

$$g_{ij}^R = -\partial_i \partial_j S(X), \quad X = (U, N^a); \quad a = 1, 2, \ldots, n$$

(3.1)
where $U$ is the system’s internal energy (mass in our study), $S(X)$ is an entropy function of the thermodynamic system one wishes to consider and $N^a$ stand for other extensive variables, or mechanically conserved charges of the system in our application. The minus sign in the definition is due to concavity of the entropy function. It was observed by Ruppeiner that in thermodynamic fluctuation theory the Riemannian curvature of the Ruppeiner metric measures the complexity of the underlying statistical mechanical model, i.e. it is flat for the ideal gas whereas any curvature singularities are a signal of critical behavior. The Ruppeiner theory has been applied to numerous systems and yielded significant results, for details see [15]. Ruppeiner originally developed his theory in the context of thermodynamic fluctuation theory, for systems in canonical ensembles. The Ruppeiner metric is conformally related to the Weinhold metric through

$$g_{ij}^R = \frac{1}{T} g_{ij}^W$$

where $T$ is thermodynamic temperature of the system. It is worth noting that some flat Ruppeiner geometries can be accounted for by the flatness theorem [16]. More precisely, whenever the entropy function takes the form

$$S = M^a f \left( \frac{M}{Q} \right)$$

where $a \neq 1$ otherwise the Ruppeiner metric is degenerate. This theorem is, however, not applicable to our study because of the form of the entropy. The Ruppeiner method is not only usable for determining the underlying statistical mechanical model of the system, but also detecting its critical behavior. Incidentally it is consistent with the so-called Poincaré’s linear series method for analyzing stability in non-extensive systems. This method is simple owing to the fact that it utilizes only a few thermodynamic functions such as the fundamental relations in order to study/analyze (in)stabilities. This method can thus be applied to BHs although they are non-extensive systems. There have been some compelling results [17] but we will not dwell on this as it is outside the scope of our present work.

Applications of this type of information geometry to black holes can be found in e.g. [18, 20, 21, 22, 23, 24, 25, 26]. This method has also been applied to exotic systems such as the hot QCD model [27]. An alternate approach known as Geometrothermodynamics (GTD) [28, 29] has been devoted to constructing Legendre invariant information metrics, which also appears to yield useful information about the black hole state space. This approach ensures that the form of the thermodynamic potential does not dictate the phase space structure of the black hole. In this paper we will apply it to the study of the 0A black hole thermodynamics.

Majority of black holes have negative specific heats (which are characteristics of self-gravitating systems) and are described microcanonically. In spite of this we have found that the Ruppeiner geometry of black holes is often surprisingly simple and elegant. Furthermore some findings are physically suggestive in particular for ultra-spinning Myers-Perry black holes, in which the onset of ultra-spinning instability is detected by the singularity of the Ruppeiner curvature [19].

\footnote{Henceforth we refer to it as the Ruppeiner curvature.}
3.1 Weinhold analysis

It is natural to work in the fixed charge ensemble, since this implies that the black hole solution is in thermal and electrical equilibrium with the surrounding heat bath.

\[ dM = TdS + \Phi dQ, \]  
(3.5)

which ensures that we obtain the correct temperature for the black hole system. The Weinhold analysis can then be determined by identifying the mass of the black hole with the internal energy. The Weinhold analysis then follows by determining the fluctuations of the mass due to changes in the entropy-charge phase space, allowing one to construct a thermodynamic metric from the corresponding Hessian. Note that when \( S = 4\pi \), the term proportional to \( Q^2 \) in the definition of the mass will vanish \( (2.9) \) and therefore our two-parameter system reduces to a one-parameter system. Physically one sees that \( \Phi \) vanishes at \( S = 4\pi \), therefore there is no source for the electric charge. For physical solutions we must then ensure \( S \neq 4\pi \).

The line element for the Weinhold metric can be written in the form

\[ ds^2_w = \frac{k}{8\pi} \left( \frac{Q^2}{S^2} dS^2 - 4 \frac{Q}{S} dS dQ - 2 \ln \left( \frac{S}{4\pi} \right) dQ^2 \right) \]  
(3.6)

which is conformally flat (as expected for a two-dimensional metric). To see this in detail first define the new variable \( u = Q^2 / S \) to eliminate the cross-terms in the metric, allowing us to work in the \( u, Q \) basis. Then define a new variable

\[ \tau = \mp \sqrt{2} Q \sqrt{2 + \ln \left( \frac{Q^2}{4\pi u} \right)} \]  
(3.7)

which immediately implies

\[ ds^2_w = \frac{k}{8\pi \tau^2} (d\tau^2 - dQ^2). \]  
(3.8)

Note that the entire line element vanishes when \( \tau \to 0 \) which corresponds to \( S = S_c = 4\pi e^{-2} \). This will give rise to a singularity in the curvature invariant, however this does not imply a phase transition. This singular point marks the break-down of the classical theory of thermodynamic fluctuations, in much the same way as we expect the scale factor singularity in FRW cosmology to correspond to a break-down of the classical description of space-time. We propose that \( S_c \) is actually the minimum (classical) entropy one can associate to this black hole system, and therefore physical entropy satisfies \( S > S_c \). Furthermore note that the Weinhold metric has Lorentzian signature with \( SO(1,1) \) symmetry. The metrics in Eq. \( 3.6 \) allow us to construct the Weinhold curvature invariants

\[ R_w = - \frac{4\pi}{kQ^2} \left( 2 + \ln \left( \frac{S}{4\pi} \right) \right)^{-2} \]

\[ R^w_{ab} R^{ab} = \frac{8\pi^2}{k^2 Q^4} \left( 2 + \ln \left( \frac{S}{4\pi} \right) \right)^{-4} \]  
(3.9)

which are both singular at the solution \( S_c \) indicating that this is indeed a singular point of the thermodynamic metric.
Figure 2: A Weinhold state space of the 0A black hole. The null cone represents a singular limit which is the entropy cutoff in this case, $S = S_c = 4\pi e^2$.

**Weinhold state space**

We can transform the metric in Eq. (3.8) into a manifestly flat form (in this case Minkowskian metric) in order to study the state space of this BH and compare it to that of other BHs studied elsewhere. We first write it in Rindler form ($ds^2 = -d\alpha^2 + \alpha^2 d\beta^2$) where we obtain

$$\alpha = \sqrt{-k} \frac{\tau^2}{2\pi Q}, \quad \beta = \frac{2Q}{\tau}.$$  \hspace{1cm} (3.10)

and then we transform this to Minkowski form using the hyperbolic transformations

$$t = \alpha \cosh \beta, \quad x = \alpha \sinh \beta.$$  \hspace{1cm} (3.11)

The opening of the wedge embedded in this Minkowskian null cone is given by

$$\Delta = \frac{t}{x} = \coth \beta = \coth \left( \frac{2Q}{\tau} \right).$$  \hspace{1cm} (3.12)

With the definition of $\tau$ in Eq. (3.7) we write this as a function of the entropy

$$\Delta(S) = \coth \left( \frac{\sqrt{2}}{\sqrt{2 + \ln(S/4\pi)}} \right).$$  \hspace{1cm} (3.13)

Because the entropy has a cutoff limit at $S = S_c$ we see that $\beta$ diverges which implies that $\Delta = 1$. This becomes the singular limit for the 0A BH instead of $S = 0$ for black holes we have studied previously. Indeed one can see that the coth function is imaginary for $S < S_c$, and only becomes real (and smoothly increasing) for physical entropy satisfying $S > S_c$.

To study the physical fluctuations, however, one must work in the Ruppeiner geometry since the Ruppeiner scalar curvature is a measure of the correlation length of the system. The metric can be calculated to give

$$ds_r^2 = \frac{\tau^2}{4Q^2} \left( 1 - \frac{Q^2}{4\pi} \exp \left( \frac{4Q^2 - \tau^2}{2Q^2} \right) \right)^{-1} (d\tau^2 - dQ^2).$$  \hspace{1cm} (3.14)
which again has an apparent singularity as $\tau \to 0$ where the thermodynamic description breaks down. The corresponding Ruppeiner curvature invariant becomes
\[
R_r = \frac{3Q^4 + 16Q^2S + 16S^2 + (Q^4 + 28Q^2S)\ln(S/4\pi) + 8Q^2S\ln(S/4\pi)^2}{2Q^2(Q^2 - 4S)(2 + \ln(S/4\pi))^2},
\]
which blows up at $S = S_c$ and also $S = Q^2/4$, which is the extremal limit of the black hole.

There are many other approaches to information geometry which one could consider. Let us therefore examine the GTD approach \[29\] where the metric is given by $g = M\partial_{\alpha\beta}MdE^\alpha dE^\beta$ where $E^\alpha = S, Q$ which is Legendre invariant. The corresponding Weinhold invariant for this metric becomes
\[
R_w = -\left(\frac{64\pi^2}{k^2Q^2}\right)N^4S\left(4S - 2Q\ln(S/4\pi)\right)^2
\]
where we have defined
\[
A = -(4\pi(4S(128Q^2S^2 + 320S^3 + Q^6(12 + 5\ln(4\pi)) + 4Q^4S(-17 + 8\ln(4\pi)) - 4Q^4S(5Q^2 + 32S)\ln(S) + \ln(S/4\pi)(256S^3(Q^2 + S) + Q^2\ln(S/4\pi)4(Q^6 - 5Q^4S - 8Q^2S^2 - 48S^3) + \ln(S/4\pi)(Q^6 + 20Q^4S - 128S^3 + 8Q^4S\ln(S/4\pi))))))
\]
\[
B = (kQ^2(Q^2 - 4S)S(2 + \ln(S/4\pi))^2(-4S + Q^2\ln(S/4\pi)^3).
\]
which, like before, has divergences only at $S = S_{\text{ex}}, S = S_c$.

There are, however, a large class of Legendre invariant metrics available (in this basis) and we should consider that they may lead to different results. Two common metrics can be parametrized as follows
\[
g^\alpha = (S\partial_S M + \epsilon Q\partial_Q M)\text{diag}(-M_{SS}, M_{QQ})
\]
where the first metric corresponds to $\epsilon = 1$ and the second one to $\epsilon = 0$. The Ruppeiner curvature invariant can be computed in both cases with the result
\[
R_r^{\epsilon=1} = \frac{32\pi^2}{k^2Q^2\ln(S/4\pi)^2(Q^2 - 4S + 2Q^2\ln(S/4\pi))^2}(Q^2 - 4S)^2 - 16S^2\ln(S/4\pi))
\]
\[
+ \frac{64\pi^2}{k^2\ln(S/4\pi)(Q^2 - 4S + 2Q^2\ln(S/4\pi))^3}(Q^2 + 6S + 4\ln(S/4\pi)(Q^2 + 2S + 2S\ln(S/4\pi)))
\]
\[
R_r^{\epsilon=0} = \frac{32\pi^2}{k^2Q^2}(Q^2 - 4S)^2 + 16S^2\ln(4\pi) + 4S(-4S\ln(S) + Q^2\ln(S/4\pi)(S + 2\ln(S/4\pi))]
\]
\[
(Q^2 - 4S)^3\ln(S/4\pi)^2
\]
The solution is invertible, but its precise form is not relevant for our purposes.
which only has singularities at $S = S_{ex}$. This is interesting because the singularity at $S_c$ has been removed in this picture.

### 3.2 The Ruppeiner analysis

The Weinhold method is useful because the Hessian is easily calculated, but does it contain all the information about the theory? Let us consider this question by working directly with the Ruppeiner analysis, where the information metric can be considered as the pull-back of the entropy function to a manifold spanned by physical observables $(M, Q)$. Indeed this is the form of the metric as first proposed by Ruppeiner from thermodynamic fluctuation theory. The entropy clearly takes the form

$$ S(M, Q) = -\frac{Q^2}{4} W \left[ -\frac{16\pi}{Q^2} e^{-8\pi M/kQ^2} \right] $$

where $W[z]$ is the Lambert-W function, which we will often employ as short-hand notation in what follows. It is straightforward to show that the extremal solution occurs when $S = Q^2/4$, corresponding to the condition

$$ M_{ex} = -\frac{kQ^2}{8\pi} \ln \left( \frac{Q^2}{16\pi e} \right) $$

which is identical to (2.10). The Ruppeiner metric in this case is determined via the Hessian

$$ g_{ij} = -\partial_i \partial_j S(M, Q), $$

and the scalar curvatures are shown to be of the form

$$ R_r = \frac{2W[z](16\pi M - 5kQ^2)(8\pi M - kQ^2) + 2kQ^2W[z](16\pi M - 4kQ^2 + kQ^2W[z])}{W[z](1 + W[z])(8\pi MQ - 2kQ^3 + kQ^3W[z])^2} $$

$$ + \frac{2kQ^2[8\pi M - 3kQ^2]}{W[z](1 + W[z])(8\pi MQ - 2kQ^3 + kQ^3W[z])^2} $$

$$ R_w = -\frac{4\pi Q^2}{(8\pi M - 2kQ^2 + kQ^2W[z])^2} $$

The Ruppeiner invariant has a single divergence at $M_{ex}$ as expected. The Weinhold invariant appears to diverge at

$$ M_c = \frac{8k\pi + e^2 kQ^2}{4\pi e^2} $$

but the Ruppeiner invariant remains finite at this value. One can actually identify $M_c$ with $S_c$ in the Weinhold picture, which we believe to be a cut-off for the entropy, therefore the Ruppeiner invariant is insensitive to this value. This suggests that the black hole must have mass greater than $M_C$ for the classical theory to be valid.

Let us now consider what happens to the GTD theory in the Ruppeiner analysis. In this case the metric takes the form

$$ g = -M \left( \frac{\partial S}{\partial M} \right)^{-1} \frac{\partial^2 S}{\partial X^a \partial X^b} dX^a dX^b $$

(3.25)
which results in the Ruppeiner curvature invariant

\[
R_r = \frac{1}{8M^3} \left( -4M + \frac{kQ^2}{\pi} + \frac{2}{\pi} (4M\pi + kQ^2)W[z] + \frac{kQ^2}{\pi} W[z]^2 + \frac{8kMQ^2 (kQ^2 - 12M\pi)}{(8M\pi - 2kQ^2 + kQ^2W[z])^2} \right) + \frac{1}{4M^3\pi} \frac{(16M^2\pi^2 - 6Mk\pi Q^2 + k^2Q^4)}{(8\pi M - 2kQ^2 + kQ^2W[z])}
\]

(3.26)

where we again employ the short handed notation for the Lambert function. This invariant does not have a non-trivial divergence\(^6\), which is somewhat troubling because we expect the solution to diverge at extremality.

Let us summarize the results in the following table, where we list all the divergences of the Ruppeiner invariant. Without a detailed analysis one would simply conclude that in the

| Formalism | Weinhold | Ruppeiner |
|-----------|----------|-----------|
| Standard  | \(S_c, S_{ex}\) | \(M_{ex}\) |
| GTD       | \(S_c, S_{ex}\) | none |
| \(\epsilon\) GTD | \(S_{ex}\) | - |

Table 1: Divergences of Ruppeiner invariant

Weinhold analysis there was a possible phase transition at \(S = S_c\), however we believe that the classical thermodynamic picture breaks down at this value, rather than suggesting the appearance of a phase transition. The underlying physics behind such a cut-off is not apparently clear because the thermodynamic observables are smoothly varying functions. However the fact that there is no additional dependence on the charge suggests that \(S_c\) is a fundamental scale of the theory, and the notion of equilibrium fluctuations breaks down at this point.

In the Ruppeiner analysis only the physical divergences seem to appear ie there are no signs of divergent behaviour at \(S_c\). However the GTD approach does not include the extremal case, suggesting that only the Ruppeiner invariant in the Ruppeiner analysis is capable of computing all the relevant physical information. The \(\epsilon\) GTD description (in the Weinhold analysis) also only selects this singular point.

4 Discussion

In this note we have considered the case of possible phase transitions in the OA black hole solution using information geometry. The potential phase transition in the fixed potential ensemble turns out to be associated with a singularity in the description of the thermodynamic geometry, where the line element vanishes. We interpret this as a (hard) cut-off for the entropy of the black hole, which must satisfy \(S > S_c\) in the physical domain - although we do not have a compelling thermodynamic explanation for its existence. We believe that this singularity represents the break-down of equilibrium thermodynamics, rather than indicating a phase transition.

\(^6\)Trivial in this sense refers to setting one of the parameters to zero.
The distance between equilibrium fluctuations is squeezed as we approach the $S_c$ curve, until it becomes ill defined precisely at $S = S_c$ indicating that all fluctuations exist at the same singular point.

Moreover we compared and contrasted several approaches to the problem in an attempt to understand which curvature invariants contain the physical information about the thermodynamic ensemble. Our results suggest that one should work predominantly with the Ruppeiner analysis, since the invariants in the Weinhold analysis tend to pick out all divergences - which may not all correspond to phase transitions. Indeed this is natural from the perspective of equilibrium thermodynamic fluctuation theory.

Furthermore the Ruppeiner invariant, in the Ruppeiner analysis, captures the physically relevant divergences (including the extremal point). Our results are preliminary and more detailed comparison of the two approaches is likely to yield better understanding of the power of contact geometry. It would also be useful to understand the physical relevance of the sign of the curvature invariant, since this should also tell us something useful about the microstates of the black hole theory.

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