THE CANONICALITY OF THE INTEGRAL MODELS OF RSZ SHIMURA VARIETIES

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Abstract. We show that the integral models of Shimura varieties of Rapoport, Smithling and Zhang in relation to variants of the arithmetic Gan–Gross–Prasad conjecture, the arithmetic fundamental lemma conjecture and the arithmetic transfer conjecture are canonical in the sense prescribed by Pappas. In particular, we prove that they are isomorphic to the models constructed by Kisin and Pappas.

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1. Introduction

The Shimura varieties have long been important in number theory because of their links to both arithmetic geometry and automorphic representation theory. The Langlands program is a famous example of such links, and another emerging subject is the intersection theory of special cycles on these varieties and its relation to automorphic $L$-functions, starting from the Gross–Zagier formula [4].

In this direction, Rapoport, Smithling and Zhang [17] have recently formulated a variant of the arithmetic Gan–Gross–Prasad conjecture and global analogues of the arithmetic fundamental lemma conjecture and the arithmetic transfer conjecture (cf. [3, §27], [22] and [16]). One of the main parts of the formulation is their explicit moduli theoretic construction of semi-global and global integral models of certain Shimura varieties. The sequel paper [18] has vastly generalized the construction, and is the subject of our paper. Their moduli problem

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is about pairs of abelian schemes with additional data unlike classical integral models in [8] and [19].

As the last two references show, the integral models of Shimura varieties are of arithmetic interest in their own right, and have been investigated by various people. Lately, Kisin and Pappas [7] constructed well-behaved semi-global integral models for a large class of Shimura varieties with parahoric level structure at $p$. These models apparently depend on various choices, but Pappas [13] has introduced the notion of canonical integral models in a broader context than [10] and its analogues in the ramified case, and has showed that the models of Shimura varieties of Hodge type in [7, 4.3.6] are independent of choices by the fact that they are canonical. However, such models are abstractly defined, not easy to study for some applications.

Our goal in this paper is to prove that models in [18, Theorem 5.4] coincide with those in [7, 4.3.6]. This is done by showing that the RSZ models are also canonical. This paper has potential effects on future research in two directions. First, since the appearance of [13], there have been several papers which refer to some integral models being canonical, e.g., [5, p. 501], [21, p. 3] and [14, p. 3], but there are still few research on this canonicality. Second, [17] has posed the question of formulating a variant of the arithmetic Gan–Gross–Prasad conjecture using the Kisin–Pappas integral models. Our paper suggests that this question is reasonable.

The condition for a system of models of Shimura varieties to be canonical consists mainly of two parts. One is the extension property close to [10, Definition 2.5], which we cope with in a similar way to the existing method. The other condition involves a notion relevant to a section rigid in the first order in [19, Definition 3.31]. As a consequence, we use the deformation theory of abelian varieties and $p$-divisible groups.

With the above background in mind, we state our main result. Let $F$ be a CM field, and $F_0$ be its maximal totally real subfield. Set $\Phi$ to be a CM type of $F/F_0$. Let $W$ be a nondegenerate $F/F_0$-hermitian space of positive dimension $n$. For each element $\varphi$ of $\Phi$, denote the signature of the hermitian space $W \otimes_{F,\varphi} \mathbb{C}$ by $(r_{\varphi}, r_{\overline{\varphi}})$. We then have the unitary group $G = U(W)$ over $F_0$ and the torus over $\mathbb{Q}$ as follows:

$$Z^\mathbb{Q}(R) = \{ z \in \text{Res}_{F/\mathbb{Q}} G_m(R) \mid \text{Nm}_{F/F_0}(z) \in R^\times \},$$

$R$ being a $\mathbb{Q}$-algebra. Set $\tilde{G}$ to be the direct product of $Z^\mathbb{Q}$ and $\text{Res}_{F_0/\mathbb{Q}} G$. The PEL type Shimura variety $\text{Sh}_{K_{\tilde{G}}}(\tilde{G}, h_{\tilde{G}})$ attached to this group is considered mainly in this paper. The reasons of adapting this $\tilde{G}$ is in [18, Remark 2.6].

In our case, we have a particular choice of connected parahoric group scheme $\tilde{G}$ over $\mathbb{Z}_p$ with generic fiber $\tilde{G}$ and choose some groups $K_{\tilde{G}}$. 
whose $p$ part is $\tilde{G}(\mathbb{Z}_p)$ and whose prime-to-$p$ part $K^p_{\tilde{G}}$ is sufficiently small. Then we have a flat semi-global integral model $\mathcal{M}_{K_{\tilde{G}}}$ of the above Shimura variety having the level $K_{\tilde{G}}$. [17] and [18] think of only one prime-to-$p$ level in terms of $\mathbb{Z}_Q$ in their integral models, but here we generalize their auxiliary model to allow various levels so that we can discuss the canonicality of the resulting RSZ models. We also have a local model $M^{\text{loc}}$ attached by [15] to $\tilde{G}$ and the minuscule cocharacter of $\tilde{G}$ coming from the Shimura datum.

As we vary $K^p_{\tilde{G}}$, we obtain by construction a system of models $\left(\mathcal{M}_{K_{\tilde{G}}(p)}\right)_{K_{\tilde{G}}^p}$ with transition morphisms for $K_{\tilde{G}}^p \subseteq K_{\tilde{G}}^p'$;

$$\pi_{K_{\tilde{G}}^p,K_{\tilde{G}}^p'}: \mathcal{M}_{\tilde{G}(\mathbb{Z}_p)K_{\tilde{G}}^p} \to \mathcal{M}_{\tilde{G}(\mathbb{Z}_p)K_{\tilde{G}}^p'}$$

extending the transition morphisms on generic fibers.

**Main Theorem** (Theorem 5.3). This system is canonical with respect to $(\tilde{G}, M^{\text{loc}})$. This means what follows among other things.

1. The morphisms $\pi_{K_{\tilde{G}}^p,K_{\tilde{G}}^p'}$ are finite etale.
2. For a discrete valuation ring $R$ of mixed characteristic $(0,p)$, the map

$$\lim_{\leftarrow K_{\tilde{G}}^p} \mathcal{M}_{K_{\tilde{G}}}(R) \to \lim_{\leftarrow K_{\tilde{G}}^p} \mathcal{M}_{K_{\tilde{G}}(p)} \left( R \left[ \frac{1}{p} \right] \right)$$

is bijective.
3. For each $K_{\tilde{G}}^p$, there is a locally universal $(\tilde{G}, M^{\text{loc}})$-display on the $p$-adic formal completion of $\mathcal{M}_{K_{\tilde{G}}}$ associated with the pro-etale local system on $\text{Sh}_{K_{\tilde{G}}}(\tilde{G}, h_{\tilde{G}})$ given by the covers

$$\text{Sh}_{K_{\tilde{G}}'(p),K_{\tilde{G}}^p}(\tilde{G}, h_{\tilde{G}}) \to \text{Sh}_{K_{\tilde{G}}}(\tilde{G}, h_{\tilde{G}}),$$

$K_{\tilde{G}}'$ ranging over open subgroups of $\tilde{G}(\mathbb{Z}_p)$. These $(\tilde{G}, M^{\text{loc}})$-displays are compatible with respect to the base change by $\pi_{K_{\tilde{G}}^p,K_{\tilde{G}}^p'}$.

As a result, the system is isomorphic to the one introduced in [7, 4.3.6].

We refer to the body of the paper and [13] for unexplained materials some of which are implicit here.

We make a remark about how the nature of RSZ models is reflected in our proof. The construction of integral models involves delicate conditions, for instance, the refined spin condition [17, §4.4], to make the models flat. However, such conditions does not explicitly appear in our proof. Rather, we use the resulting flatness almost everywhere. This flatness is necessary to obtain $(\tilde{G}, M^{\text{loc}})$-displays, for example.

Finally, we turn to the organization of the paper. In §2 we review the notion of canonical integral models of Shimura varieties following [13]. In §3 we explain more about the Shimura variety $\text{Sh}_{K_{\tilde{G}}}(\tilde{G}, h_{\tilde{G}})$. 
In §4, we discuss the auxiliary models that supplement the RSZ integral models. In §5, we write down the integral models of Shimura varieties whose canonicality is showed in the rest of the paper. We address the extension property of the models in §6 while we construct the locally universal \((\tilde{G}, \text{M}^\text{loc})\)-displays in §7.

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**Notation.** Set \(p\) to be an odd prime. If \(K\) is a usual or \(p\)-adic number field, then \(\mathcal{O}_K\) is its ring of integers, and \(\overline{\mathcal{O}}_K\) is the integral closure of \(\mathcal{O}_K\) in \(K\). We fix an emebdding \(\mathbb{Q} \rightarrow \mathbb{Q}_p\). For a number field \(K\), let \(\mathbb{A}_K\), \(\mathbb{A}_K, f\) and \(\mathbb{A}_K, f\) denote the adele of \(K\), finite adele of \(K\) and finite adele of \(K\) away from \(p\). In the three symbols, \(K\) is omitted if \(K = \mathbb{Q}\). For an abelian group \(\Lambda\), we put \(\hat{\Lambda} = \lim_{\substack{\leftarrow \to \n \in \mathbb{N} \setminus \{0\}} } \Lambda/n\Lambda\) and \(\hat{\Lambda}^p = \lim_{\substack{\leftarrow \to \n \in \mathbb{N} \setminus \{0\} \setminus \{p\}\}} \Lambda/n\Lambda\).

Phrases like “over Spec \(A\)” and equations like \(X \times_{\text{Spec} A} \text{Spec} B\) are abbreviated as “over \(A\)” and \(X \otimes_A B\), or simply \(X_B\), for example. If \(A\) is an abelian scheme over a scheme \(S\), then its \(p\)-adic Tate module, prime-to-\(p\) Tate module and full Tate module are denoted respectively by \(T_p(A)\), \(T^p(A)\) and \(\hat{T}(A)\). The conventions for \(p\)-divisible groups are analogous. Their rational versions are \(V_p(A)\), \(V^p(A)\) and \(\hat{V}(A)\). The dual abelian scheme of \(A\) is written \(A^\vee\). If \(A\) is an \(\mathcal{O}_K\)-action (resp. an \(\mathcal{O}_{K,(p)}\)-action, a \(K\)-action) \(i\) on \(A\) for a CM field \(K\), the default action of these rings on \(A^\vee\) is through \(i(\pi)^\vee\) for each \(\pi\) in these rings.

If \(A\) is an abelian scheme or a \(p\)-divisible group over a scheme \(S\), then its Lie algebra is denoted by \(\text{Lie} A\). The notation \(\pi_1\) means the etale fundamental groups whereas \(\pi_1^{\text{alg}}\) stands for the algebraic fundamental groups of algebraic groups. The notation \(\text{Sh}\) is not for Shimura varieties over \(\mathbb{C}\) but for canonical models. The moduli schemes and the moduli functors are denoted by the same symbols. For a finite set \(X\), set \(#X\) to be the number of elements of it.

2. **Overview of canonical integral models**

The description below is specific to our case.

2.1. **Notation.** The notation added here is different from that of other subsequent sections. Let \((G, X)\) be a Shimura datum. Especially, \(G\) is connected. We assume

- that \(G\) splits over a tamely ramified extension of \(\mathbb{Q}_p\),
- that \(p \nmid \# \pi_1^{\text{alg}}(G)\), and
- that the center of \(G\) has the same \(\mathbb{Q}\)-split rank as the \(\mathbb{R}\)-split rank.
Set $\mu$ to be a minuscule cocharacter of $G_{\overline{\mathbb{Q}}_p}$ coming from an element of $X$. Let $E$ be the reflex field of $(G, X)$. Set $\nu$ to be its $p$-adic place induced by the fixed embedding $\overline{\mathbb{Q}} \rightarrow \mathbb{Q}_p$. Let $\mathfrak{p}$ be the maximal ideal of $\mathcal{O}_E$. Set $X_\mu$ to be the projective smooth variety over $E$ to which $G_{\overline{\mathbb{Q}}_p}/P_\mu$ descends. Here, $P_\mu$ means the parabolic subgroup of $G_{\overline{\mathbb{Q}}_p}$ corresponding to $\mu$ [12, Theorem 25.1].

We take a connected parahoric group scheme $G$ associated with a point in the extended Bruhat–Tits building of $G_{\mathbb{Q}_p}$. In [15], Papas and Zhu construct a local model $M^\text{loc}$ with respect to $G$ and the conjugacy class of $\mu$ under our assumption on the splitting of $G$.

We assume that $(G, M^\text{loc})$ is of strongly integral local Hodge type [13, 3.1.4], i.e., there is a closed immersion $\iota : G \rightarrow \text{GL}_{n, \mathbb{Z}_p}$ such that

1. $\iota(G)$ includes the center of $\text{GL}_{n, \mathbb{Z}_p}$,
2. $\iota\mu$ is conjugate over $\overline{\mathbb{Q}}_p$ to the standard minuscule cocharacter sending $a$ to $\text{diag}(a^d, 1^{(n-d)})$ for some $0 \leq d \leq n$, and
3. the Zariski closure in $\text{Gr}(d, n)_{\mathcal{O}_{E_\nu}}$ of the image of $\iota_* : X_\mu, \mathcal{O}_{E_\nu} \rightarrow \text{Gr}(d, n)_{E_\nu}$ induced by $\iota$ is isomorphic to $M^\text{loc}$ including the $G$-action.

The notation $\iota_*$ also stands for the embedding $M^\text{loc} \rightarrow \text{Gr}(d, n)_{\mathcal{O}_{E_\nu}}$ in the last condition and its other variants.

2.2. Dieudonné $(G, M^\text{loc})$-displays. Let $R$ be a complete noetherian local flat $\mathcal{O}_{E_\nu}$-algebra with perfect residue field of characteristic $p$. Recall from [23, §1, §2], etc., that in this case we have a variant $\hat{W}(R)$ of the ring of Witt vectors equipped with a Frobenius $\varphi$. We also recall the functor of [13 Proposition 4.1.2]. It is from the groupoid of pairs $(P, q)$ of a $G$-torsor $P$ over $\hat{W}(R)$ and a $G$-equivariant morphism $q : P \otimes_{\hat{W}(R)} R \rightarrow M^\text{loc}$ of $\mathcal{O}_{E_\nu}$-schemes. It lands in the groupoid of tuples $(Q', Q, \alpha)$ of two $G$-torsors $Q', Q$ over $\hat{W}(R)$ and a $G$-equivariant morphism $\alpha : Q[1/p] \rightarrow \varphi^* Q'[1/p]$. In addition, the functor sends a pair $(P, q)$ to a tuple of the form $(P, Q, \alpha)$.

**Definition 2.1** ([13 Definition 4.3.2]). A Dieudonné $(G, M^\text{loc})$-display over $R$ is a tuple of

- a $G$-torsor $P$ over $\hat{W}(R)$,
- a $G$-equivariant morphism $q : P \otimes_{\hat{W}(R)} R \rightarrow M^\text{loc}$ and
- a $G$-isomorphism $\Psi : Q \rightarrow P$, where $Q$ is the second constituent of the image of $(P, q)$ by the above functor.

We turn to $(G, M^\text{loc})$-displays of less importance to us. Let $R_0$ be a $p$-adically complete flat $\mathcal{O}_{E_\nu}$-algebra formally of finite type. The above functor sending $(P, q)$ to $(P, Q, \alpha)$ is analogously defined when $\hat{W}(R)$ is replaced by $W(R_0)$.

*Not to be confused with elements of the CM type.*
Definition 2.2 ([13 Definition 4.2.2]). A \((\mathcal{G}, M^{\text{loc}})\)-display over \(R_0\) is a tuple of

- a \(\mathcal{G}\)-torsor \(\mathcal{P}\) over \(W(R_0)\),
- a \(\mathcal{G}\)-equivariant morphism \(q: \mathcal{P} \otimes_{W(R_0)} R_0 \to M^{\text{loc}}\) and
- a \(\mathcal{G}\)-isomorphism \(\Psi: \mathcal{Q} \xrightarrow{\sim} \mathcal{P}\), where \(\mathcal{Q}\) is the second constituent of the image of \((\mathcal{P}, q)\) by the above functor.

This definition is generalized in [13 4.2.4] for a \(p\)-adic flat formal scheme \(\mathcal{X}\) over \(\text{Spf} \mathcal{O}_{E_\nu}\) formally of finite type by substituting a sheaf of rings \(W(\mathcal{O}_\mathcal{X})\) for \(W(R_0)\).

2.3. Rigidity and locally universal displays. Assume that the local ring \((R, \mathfrak{m}, k)\) in \((\mathcal{G}, M^{\text{loc}})\)-display over \(R\).

We refer to [13 Definition 4.5.8] for the definition of a section \(s \in \mathcal{P}(W(R))\) being rigid in the first order at \(\mathfrak{m}\). Such a section always exists if \(k\) is algebraically closed.

Definition 2.3 ([13 Definition 4.10]). The Dieudonné \((\mathcal{G}, M^{\text{loc}})\)-display \((\mathcal{P}, q, \Psi)\) is called locally universal if there exists a section \(s \in \mathcal{P}(W(R))\) rigid in the first order such that \(q \circ (s \otimes 1): \text{Spec} \, R \to \mathcal{P} \otimes_{W(R)} R \to M^{\text{loc}} \otimes_{\mathbb{Z}_p} W(k)\) induces an isomorphism from the completion of the local ring at the image of \(\mathfrak{m}\) to \(R\).

In [13 §4.4], a classical Dieudonné display \(\mathcal{D}(\iota)\) is attached to the tuple \(\mathcal{D} = (\mathcal{P}, q, \Psi)\), which in turn gives a \(p\)-divisible group \(\mathcal{G}\) by the main theorem of [23].

Proposition 2.4 ([13 Proposition 4.5.15(b)]). Take a section \(s\) in \(\mathcal{P}(W(R))\) rigid in the first order at \(\mathfrak{m}\). Set \(T\) to be the tangent space of the deformation functor of \(\mathcal{G}_k\) on the category of artinian local rings with residue field \(k\). Let \(z\) be the image of \(\mathfrak{m}\) in \(\text{Gr}(d,n)\) over \(\mathcal{O}_{E_\nu}\) by \(\iota_* \circ q \circ (s \otimes 1)\).

Then, there is an isomorphism between \(T\) and \(T_z \text{Gr}(d,n) \otimes_{\mathcal{O}_{E_\nu}} k\) with the following condition. The base change of \(\mathcal{G}\) over \(\text{Spec} \, R/(\mathfrak{m}^2 + pR)\) seen as a deformation of \(\mathcal{G}_k\) gives a map \((\mathfrak{m}/(\mathfrak{m}^2 + pR))^{\vee} \to T\) of tangent spaces. The condition is that the last map equals \(q \circ (s \otimes 1)\) composed with \(\text{Spec} \, R/(\mathfrak{m}^2 + pR) \to \text{Spec} \, R\).

2.4. Canonical integral models. Let \(K^p\) be a sufficiently small compact open subgroup of \(G(\mathbb{A}_f^p)\). Put \(K^p = \mathcal{G}(\mathbb{Z}_p)\) and \(K = K_pK^p \subseteq G(\mathbb{A}_f)\).

Let \(\mathcal{S}_K\) be a normal flat separated \(\mathcal{O}_{E,(\nu)}\)-schemes of finite type with generic fiber \(\text{Sh}_K(G,X)\). Let \(\mathcal{L}_K\) be the pro-etale \(\mathcal{G}(\mathbb{Z}_p)\)-local system on \(\text{Sh}_K(G,X)\) that is given by the covers \((\text{Sh}_{K_pK^p}(G,X) \to\)
Sh_{G(z_p) \subseteq G} (G, X))_{K \subseteq G(z_p)}. Suppose that we are given a Dieudonné $(G, M^{loc})$-display $D_x$ over the strict completion $R_x$ of the local ring of $\mathcal{O}_K$ at each $x \in \mathcal{O}_K$. As in §2.3, we associate a $p$-divisible group $G_x$ over $R_x$ with $D_x$. We cite [13, Definition 6.1.5] for the notion that $L_{K}$ and $D_x$ are associated, and for the definition of $(L_{K}, (D_x, \alpha_x)_{x \in \mathcal{O}_K})_{x \in \mathcal{S}_K}$ being an associated system. Here, $\alpha_x$ is an isomorphism $T_p \left( G_x \otimes_{R_x} R_x \left[ \frac{1}{p} \right] \right) \cong L_{K,R_x} \left[ \frac{1}{p} \right] \times \mathbb{A}_Z^p$ of pro-etale $\mathbb{Z}_p$-local systems on $R_x[1/p]$, where the push-out on the right hand side is via $\iota$. These definitions do not depend on $\iota$ by [13, Proposition 6.5.1].

**Theorem 2.5** ([13, Theorem 6.4.1]). Suppose that there exists a $p$-divisible group on $\mathcal{S}_K$ whose pull-back over $\mathcal{O}_K$ has the $p$-adic Tate module isomorphic to $L_{K} \times G_{A^n \mathbb{Z}_p}$. Then, there exists a unique associated system $(L_{K}, (D_x, \alpha_x)_{x \in \mathcal{O}_K})_{x \in \mathcal{S}_K}$ up to unique isomorphism.

We also consider a situation where we have one $(G, M^{loc})$-display instead of pointwise Dieudonné $(G, M^{loc})$-displays $D_x$.

**Definition 2.6** ([13, Definition 6.2.1, Definition 6.2.2]). Assume that $D_K$ is a $(G, M^{loc})$-display over the $p$-adic formal completion $\mathcal{O}_{\mathcal{S}_K}$ of $\mathcal{S}_K$ if for each $x \in \mathcal{O}_K$, there is a locally universal Dieudonné $(G, M^{loc})$-display $D_x$ over $R_x$ associated with $L_{K}$ such that the $(G, M^{loc})$-display $D_x \otimes_{W(R_x)} W(R_x)$ is isomorphic to $D_K \otimes_{W(\mathcal{O}_{\mathcal{S}_K})} W(R_x)$.

We slightly modify [13, Definition 7.1.3] in response to our moduli problems.

**Definition 2.7.** Let $(\mathcal{S}_K)_{K^p}$ be a system of $\mathcal{O}_{E, \nu}$-schemes as above for $K^p$ running over a cofinal subset of the set of compact open subgroups of $G_{A^n \mathbb{Z}_p}$, together with morphisms $\pi_{K^p, K^p} : \mathcal{I}_{K_{K^p}} \rightarrow \mathcal{I}_{K_{K^p}}$ for $K^p \subseteq K^p$ extending $\mathcal{O}_{\mathcal{S}_K} \rightarrow \mathcal{O}_{\mathcal{S}_K} (G, X) \rightarrow \mathcal{O}_{\mathcal{S}_K} (G, X)$. Then the system is called canonical if

1. for each $K^p \subseteq K^p$, the morphism $\pi_{K^p, K^p}$ is finite etale,
2. for any discrete valuation ring $R$ of mixed characteristic $(0, p)$, the map

$$\lim_{\longrightarrow} \mathcal{I}_K (R) \rightarrow \lim_{\longrightarrow} \mathcal{I}_K \left( R \left[ \frac{1}{p} \right] \right)$$

is bijective, and
(3) for each $K^p$, there is a locally universal $(G, M^{\text{loc}})$-display $\mathcal{D}_K$ on the $p$-adic completion $\mathfrak{G}_K$ associated with $L_K$. These $(G, M^{\text{loc}})$-displays are compatible with respect to the base change by $\pi_{K^p, K^p}$.

As an example, if $(G, X)$ is of Hodge type, then the integral models of $\text{Sh}_K(G, X)$ by Kisin and Papas \cite[4.3.6]{7} are canonical \cite[Theorem 8.2.1]{13}.

The theorem below is stronger than the uniqueness result of canonical integral models. We take advantage of that difference later.

**Theorem 2.8** (cf. \cite[Theorem 7.1.7]{13}). Suppose that systems $(\mathcal{S}_K)_{K^p}$ and $(\mathcal{S}'_K)_{K^p}$ satisfy the conditions of Definition 2.7, except that the following condition takes the place of the condition (3):

\[ (3^*) \text{ the } \mathcal{O}_{E,v} \text{-schemes } \mathcal{S}_K \otimes_{\mathcal{O}_{E,(v)}} \mathcal{O}_{E,v} \text{ (and } \mathcal{S}'_K \otimes_{\mathcal{O}_{E,(v)}} \mathcal{O}_{E,v} \text{) support locally universal associated systems.} \]

Then for each $K^p$, there exists an isomorphism $\mathcal{S}_K \sim \mathcal{S}'_K$ that is the identity on the generic fibers and compatible with transition morphisms.

### 3. The Shimura variety

We review some of \cite[§2, §3]{18}.

**3.1. Notation.** The convention here is the same as in §1 and valid through the rest of the paper. The same is true for the symbols in the upcoming sections. Let $F$ be a CM field with $F_0$ its maximal totally real subfield. Set $\Phi$ to be a CM type of $F/F_0$. We fix a totally negative element $\Delta$ of $F_0$ and its square root $\sqrt{\Delta}$ such that an embedding $\varphi$ of $F$ in $\mathbb{C}$ is in $\Phi$ if and only if $\varphi(\sqrt{\Delta})\sqrt{-1}^{-1} > 0$.

Let $W$ be a nondegenerate $F/F_0$-hermitian space of positive dimension $n$. For each $\varphi \in \Phi$, denote the signature of the hermitian space $W \otimes_{F, \varphi} \mathbb{C}$ by $(r_\varphi, r_{\bar{\varphi}})$. Set $G$ to be the unitary group $U(W)$ over $F_0$.

We also define a group scheme $G^Q$ over $\mathbb{Z}$, $G^Q$ over $\mathbb{Q}$ and $\tilde{G}$ over $\mathbb{Q}$ as

\[
Z^Q(R) := \{ z \in \text{Res}_{O_F/\mathbb{Z}} \mathbb{G}_m(R) \mid \text{Nm}_{O_F/O_{F_0}}(z) \in R^x \},
\]
\[
G^Q(R) := \{ g \in \text{Res}_{F_0/\mathbb{Q}} \text{GU}(W)(R) \mid c(g) \in R^x \},
\]
\[
\tilde{G} := Z^Q \times_{\mathbb{G}_m} G^Q,
\]

where $R$ is a ring (resp. a $\mathbb{Q}$-algebra) and $c$ means the similitude.

The groups $\tilde{G}$ and $Z^Q \times \text{Res}_{F_0/\mathbb{Q}} G$ are isomorphic by sending $(z, g)$ to $(z, z^{-1}g)$. We also note that $\tilde{G}$ satisfies the three assumptions imposed on $G$ in §2.1.

Let $S = \text{Res}_{\mathbb{C}/\mathbb{R}} \mathbb{G}_m$ be the Deligne torus. If $h : S \to H$ is a morphism of algebraic groups over $\mathbb{R}$, then $\{h\}$ denotes its conjugacy class.
3.2. The Shimura data. For \( Z^Q \), we identify \( R \otimes_Q F \) with \( \prod_{\varphi \in \Phi} C \) through elements of \( \Phi \), getting an isomorphism between \( (Z^Q)_R \) and a subgroup of \( \prod_{\varphi \in \Phi} S \). We let \( h_{Z^O}: S \to (Z^Q)_R \) send \( z \) in \( C^\infty \) to \((z)_{\varphi \in \Phi}\). The equivalence relation is denoted \( \sim \).

The Shimura datum \( \{h_{Z^O}\} \) is a Shimura datum. Let \( E_\Phi \) be its reflex field, or the reflex field of \( \Phi \).

We next turn to \( G^Q \). Identify \( W \otimes_{F, \varphi} C \) with \( C^n \) so that the hermitian form on \( W \otimes_{F, \varphi} C \) is represented by \( \text{diag}(1^{(r_\varphi)}, -1^{(s_\varphi)}) \). Then, we have \( h_{G^O}: S \to (G^Q)_R \), where \( h_{G^O}(R) \) is induced by an \( R \)-algebra homomorphism \( C \to \prod_{\varphi \in \Phi} \text{End}(C^n) \) sending \( \sqrt{-1} \) to \( (\sqrt{-1}\text{diag}(1^{(r_\varphi)}, -1^{(s_\varphi)}))_{\varphi} \).

The pair \( (G^Q, \{h_{G^O}\}) \) makes up a Shimura datum.

We finally come to \( \tilde{G} \). Let \( h_{\tilde{G}} \) denote the map \( h_{Z^O} \times_{G^O} h_{G^O}: S \to \tilde{G}_R \).

The pair \( (\tilde{G}, \{h_{\tilde{G}}\}) \) is a PEL type Shimura datum. Set \( E \subseteq C \) to be its reflex field.

4. Auxiliary models

Set \( \nu \) to be the \( p \)-adic place of \( E \) induced by the fixed embedding \( \mathcal{O} \to \mathcal{O}_p \). Let \( a \neq 0 \) be an ideal of \( O_F \) prime to \( p \) such that the following category \( \mathcal{M}^S_\Phi \) fibered in groupoids over the category of locally noetherian \( O_E \)-schemes is nontrivial. Such \( a \) exists by [18, Remark 3.7(iii)].

Viewed as a pseudo-functor, \( \mathcal{M}^S_\Phi \) associates with an object \( S \) of the source category the groupoid of tuples \( (A_0, i_0, \lambda_0) \) of

- an abelian scheme \( A_0 \) over \( S \),
- \( i_0: O_F \to \text{End}_S(A_0) \) with the Kottwitz condition that for every \( a \in O_F \), the characteristic polynomial of \( i_0(a) \) acting on \( \text{Lie} A_0 \) is \( \prod_{\varphi \in \Phi} (T - \varphi(a)) \), and
- \( \lambda_0: A_0 \to A_0^\vee \) is an \( O_F \)-linear polarization with kernel \( A_0[a] \).

Arrows of this groupoid are isomorphisms of abelian schemes preserving the other structures. This moduli problem is represented by a finite etale \( O_E \)-scheme by applying the argument of [6, Proposition 3.1.2].

We briefly recall \( \mathcal{M}^S_\Phi \) in [18, (4.2)]. Via the first homology group, \( \mathcal{M}^S_\Phi(\mathbb{C}) \) bijects to the set \( \mathcal{L}_\Phi^\text{a} \) of the isomorphism classes of the following pairs \( (\Xi_0, (\ast, \ast)_0) \). This pair comprises an invertible \( O_F \)-module \( \Xi_0 \) and an alternating \( O_F/O_{F_0} \)-balanced form \( (\ast, \ast)_0: \Xi_0 \times \Xi_0 \to \mathbb{Z} \) such that \( \langle \sqrt{\Delta \ast}, \ast \rangle_0 \) is positive definite and the dual of \( \Xi_0 \) with respect to \( \langle \ast, \ast \rangle_{0, \mathbb{Q}} \) is \( a^{-1} \Xi_0 \). Elements \( \Xi_0, \Xi_0^\vee \in \mathcal{L}_\Phi^\text{a} \) are called equivalent if there are \( O_F \)-linear isomorphisms \( \Xi_0 \otimes_{\mathbb{Z}} \mathbb{Z} \iso \Xi_0^\vee \otimes_{\mathbb{Z}} \mathbb{Z} \) and \( \Xi_0 \otimes_{\mathbb{Z}} \mathbb{Q} \iso \Xi_0^\vee \otimes_{\mathbb{Z}} \mathbb{Q} \) by which alternating forms correspond up to a constant in \( \mathbb{Z}^\times \) (resp. \( \mathbb{Q}^\times \)). The equivalence relation is denoted \( \sim \). If \( \xi \) is in \( \mathcal{L}_\Phi^\text{a}/\sim \), then \( \mathcal{M}^S_\Phi(\xi) \) is
the Zariski closure in $\mathcal{M}_0^a$ of the set of the image of $x$ in $\mathcal{M}_0^a$ where $x \in \mathcal{M}_0^a(\mathbb{C})$ corresponds to an element of $\xi$. This $\mathcal{M}_0^{a,\xi}$ is clopen in $\mathcal{M}_0^a$.

Let $(\Lambda_0, \psi_0)$ be an element of $\mathcal{L}_0^a$. Put $K_{\mathbb{Z}^Q} = Z^Q(\mathbb{Z}_p)K_{\mathbb{Z}^Q}$ for an open subgroup $K_{\mathbb{Z}^Q}$ of $Z^Q(\hat{\mathbb{Z}}^p)$. If $S$ is a locally noetherian $\mathcal{O}_{E,(p)}$-scheme and $(A_0, i_0, \lambda_0)$ is an object of $\mathcal{M}_0^a(S)$, then let $\text{Isom}((\hat{\Lambda}_0^p)_S, T^p(A_0))$ be the pro-etale sheaf of an $\hat{\mathcal{O}}^p$-linear isomorphism from the constant sheaf $(\hat{\Lambda}_0^p)_S$ over $S$ to the pro-etale sheaf $T^p(A_0)$ transferring $\psi_0$ to the Riemann form of $\lambda_0$ up to a constant in $\hat{\mathbb{Z}}^{p,\infty}$.

We define the following category $\mathcal{M}_{0,K_{\mathbb{Z}^Q}}$ fibered in groupoids over the category of locally noetherian $\mathcal{O}_{E,(p)}$-schemes. As a pseudo-functor, a locally noetherian $\mathcal{O}_{E,(p)}$-scheme $S$ is sent to the groupoid of tuples $(A_0, i_0, \lambda_0, e^p)$ composed of

- an object $(A_0, i_0, \lambda_0)$ of $\mathcal{M}_0^a(S)$ and
- a section $e^p$ over $S$ of the pro-etale sheaf

$$\text{Isom}((\hat{\Lambda}_0^p)_S, T^p(A_0))/K_{\mathbb{Z}^Q}. $$

Arrows of this groupoid are the ones of $\mathcal{M}_0^a$ by which the level structures coincide. Note that $K_{\mathbb{Z}^Q}$ acts on $\text{Isom}((\hat{\Lambda}_0^p)_S, T^p(A))$ by the action on $\hat{\Lambda}_0^p$. Also, if $S$ is connected with a geometric point $s \to S$, then $e^p$ can be identified with a $K_{\mathbb{Z}^Q}$-class of $\hat{\mathcal{O}}^p$-linear isomorphisms $\hat{\Lambda}_0^p \sim T^p(A_s)$ transferring $\psi_0$ to the Riemann form of $\lambda_0$ up to a constant in $\hat{\mathbb{Z}}^{p,\infty}$ and the $\pi_1(S, s)$-action on the target to actions on the source by elements of $K_{\mathbb{Z}^Q}$. In addition, we may replace the condition on $\text{Ker} \lambda_0$ in the moduli problem by the one that $p \not| \deg \lambda_0$ by the isometry property of $e^p$.

Toward the representability of the auxiliary models, we first observe in the sequel that $\mathcal{M}_{0,\mathbb{Z}^Q(\hat{\mathbb{Z}})}^{\Lambda_0}$ gives a fibered full subcategory of $\mathcal{M}_0^a \otimes_{\mathbb{Z}} \mathbb{Z}(p)$. Indeed, the level structure with respect to $\mathbb{Z}^Q(\hat{\mathbb{Z}})$ attached to $(A_0, i_0, \lambda_0)$ in $\mathcal{M}_0^a(S)$ is unique if exists since the group of automorphisms of $\hat{\Lambda}_0^p$ preserving the $\hat{\mathcal{O}}^p$-action and $\psi_0$ up to a scalar is $\mathbb{Z}^Q(\hat{\mathbb{Z}}^p)$. We identify $\mathcal{M}_{0,\mathbb{Z}^Q(\hat{\mathbb{Z}})}^{\Lambda_0}$ with its essential image in $\mathcal{M}_0^a \otimes_{\mathbb{Z}} \mathbb{Z}(p)$.

We also relate $\mathcal{M}_{0,\mathbb{Z}^Q(\hat{\mathbb{Z}})}^{\Lambda_0}$ to $\mathcal{M}_0^{a,\xi}$. Set $\xi$ to be the element $[(\Lambda_0, \psi_0)]$ of $\mathcal{L}_0^a/\sim$. Then we claim that $\mathcal{M}_{0,\mathbb{Z}^Q(\hat{\mathbb{Z}})}^{\Lambda_0,\xi}$ contains $\mathcal{M}_0^{a,\xi} \otimes_{\mathbb{Z}} \mathbb{Z}(p)$. In fact, we have only to construct a level structure on $(A_0, i_0, \lambda_0) \in \mathcal{M}_0^{a,\xi}(S)$ for each connected locally noetherian $\mathcal{O}_{E,(p)}$-scheme $S$. If $S$ has a point with residue field of characteristic 0, then $S$ has a geometric point $s$ such that the corresponding element of $\mathcal{M}_0^{a,\xi}(k(s))$ factors through a $\mathbb{C}$-valued point $t$ of $\mathcal{M}_0^{a,\xi}$. We can think of a complex torus $A_0(\mathbb{C})$ via $t$. Its first homology group over $\hat{\mathbb{Z}}$ is isomorphic to $\hat{\Lambda}_0^p$ up to a similitude in $\hat{\mathbb{Z}}^{\infty}$, and that translates into the level structure $\hat{\Lambda}_0^p \sim T^p(A_{0,t})$. We
can deal with the other case by taking a lift to characteristic 0, relying on the finite etaleness of $\mathcal{M}_0^{\Lambda_0}$. We obtain a decomposition

$$\mathcal{M}_0^{\Lambda_0} \times_{\mathcal{M}_0^{\Lambda_0}} \mathcal{M}_0^{h_0}$$

where the disjoint sum ranges over $[\Xi_0] \in \mathcal{L}_{\Phi}$. Such that $\widehat{\mathbb{Z}}_0^p$ is isometric to $\Lambda_0^p$ up to a similitude in $(\widehat{\mathbb{Z}}^p)^\times$. In particular, the left hand side is a finite etale scheme over $\text{Spec} \mathcal{O}_{F,(p)}$.

**Proposition 4.1.** $\mathcal{M}_0^{\Lambda_0}$ is represented by a scheme finite etale over $\mathcal{M}_0^{\Lambda_0, K_{\mathbb{Z}}}$.  

**Proof.** Let $S$ be a connected locally noetherian $\mathcal{O}_{F,(p)}$ with an object $(A_0, i_0, \lambda_0, \epsilon_0)$ of $\mathcal{M}_0^{\Lambda_0, K_{\mathbb{Z}}}$. We fix a representative $\Lambda_0^p \to T^p(A_0, \lambda_0, \epsilon_0)$ of $\epsilon_0^p$ denoted by the same symbol. Set $N$ to be the fiber product of $\mathcal{M}_0^{\Lambda_0, K_{\mathbb{Z}}}$. Then, $N$ can be seen as a set-valued functor from the category of locally noetherian $S$-schemes, sending an $S$-scheme $T$ to the inverse image of $\epsilon_0^p$ by the map

$$(\text{Isom}((\Lambda_0^p)_S, T^p(A_0))/K_{\mathbb{Z}}^p)(T) \to (\text{Isom}((h_0)_S, T^p(A_0))/K_{\mathbb{Z}}^p)(T).$$

For a given morphism $f: T \to S$ of connected locally noetherian schemes and a geometric point $t$ of $T$, we would like to find out the set of $\epsilon \in Z_{\mathbb{Q}}(\widehat{\mathbb{Z}}^p)/K_{\mathbb{Z}}^p$ with the condition below. Let $g: \pi_1(S, t) \to Z_{\mathbb{Q}}(\widehat{\mathbb{Z}}^p)$ be induced by $\epsilon_0^p$. The condition is that $f_*(\pi_1(T, t) \to \pi_1(S, t))$ composed with $\epsilon^{-1}g\epsilon$ factors through $K_{\mathbb{Z}}^p$. Although $\epsilon^{-1}g\epsilon = g$ holds since $Z_{\mathbb{Q}}$ is commutative, writing $\epsilon$ explicitly clarifies the proof.

Set $G_{\epsilon}$ to be the inverse image of $K_{\mathbb{Z}}^p$ by $\epsilon^{-1}g\epsilon$. Then $\pi_1(S, t)/G_{\epsilon}$ is a $\pi_1(S, t)$-set in correspondence with a finite etale cover $S' \to S$. Lifts of $f$ to $T \to S'$ are in bijection with $zG_{\epsilon} \in \pi_1(S, t)/G_{\epsilon}$ fixed by $f_*(\pi_1(T, t))$. For $z \in \pi_1(S, t)$, the stabilizer of $zG_{\epsilon} \in \pi_1(S, t)/G_{\epsilon}$ in $\pi_1(S, t)$ is $zG_{\epsilon}z^{-1} = G_{g(z)\epsilon}$. Thus, $N$ is represented by the disjoint sum of $S'$ over $\epsilon \in g(\pi_1(S, t))/Z_{\mathbb{Q}}(\widehat{\mathbb{Z}}^p)/K_{\mathbb{Z}}^p$. \hfill \qed

Finally, the following description \[\text{Sh}_{K_{\mathbb{Z}}}^F(\mathcal{M}_{\mathbb{Z}}, \{h_{\mathbb{Z}}\})\] and the comparison of $C$-valued points [11, Theorem 8.17] show that the generic fiber of

$$\mathcal{M}_0 = \mathcal{M}_0^{\Lambda_0, K_{\mathbb{Z}}} \times_{\mathcal{M}_0^{\Lambda_0}} \mathcal{M}_0^{h_0}$$

is $\text{Sh}_{K_{\mathbb{Z}}}^F(\mathcal{M}_{\mathbb{Z}}, \{h_{\mathbb{Z}}\}) \otimes \mathbb{E}_k$. The scheme $\text{Sh}_{K_{\mathbb{Z}}}^F(\mathcal{M}_{\mathbb{Z}}, \{h_{\mathbb{Z}}\})$ is a clopen subscheme of the $\mathbb{E}_k$-scheme representing a functor from the category of locally noetherian $\mathbb{E}_k$-schemes to that of groupoids. The functor sends such a scheme $S$ to the groupoid of tuples $(A_0, i_0, \lambda_0, \epsilon)$, where

- $A_0$ is an abelian scheme over $S$,
- $i_0: F \to \text{End}_S(A_0) \otimes \mathbb{Q}$ satisfies the Kottwitz condition with respect to $\Phi$ as in \[\text{[8, \S5]}\].
\begin{itemize}
  \item $\lambda_0 \in \text{Hom}(A_0, A_0^\vee) \otimes_{\mathbb{Z}} \mathbb{Q}$ is an $F$-linear $\mathbb{Q}$-polarization, and
  \item $\epsilon \in (\text{Isom}(\Lambda_0 \otimes_{\mathbb{Z}} \mathbb{A}_f, \hat{V}(A_0))/K_{Z^0})(S)$,
\end{itemize}

with arrows comprising $F$-linear $\mathbb{Q}$-isogenies of abelian schemes preserving the level structures and sending polarizations with each other up to a factor in $\mathbb{Q}_S^2(S)$. This description of $\text{Sh}_{K_{Z^0}}(Z^0, \{h_{Z^0}\})$ works even when the level at $p$ is not $Z^0(\mathbb{Z}_p)$.

5. RSZ integral models

Our formulation is based on [18, §4.1, §5.2].

5.1. Notation. For each $p$-adic place $w$ of $F$, we fix a uniformizer $\pi_w$ of $F_w$. We take them so that if a place $v$ of $F_0$ splits into $w$ and $\overline{w}$ in $F$, then $\pi_w = \pi_{\overline{w}}$ as elements of $F_{0,v}$. For a $p$-adic place $v$ of $F_0$, we let $\pi_v$ be $\pi_w$ if $w$ is the unique place of $F$ above $v$, and an element of $F_v$ sent to $(\pi_w, \pi_{\overline{w}})$ in $F_w \times F_{\overline{w}}$ if $v$ splits into $w$ and $\overline{w}$ in $F$.

Let $(\ast, \ast)_0: A_0 \times \Lambda_0 \to \mathcal{O}_F$ be the hermitian form such that $\psi_0 = \text{Tr}_{F/\mathbb{Q}} \sqrt{\Delta}^{-1}(\ast, \ast)_0$. Likewise, we view $W$ also as a symplectic space via $\text{Tr}_F/\mathbb{Q} \sqrt{\Delta}^{-1}(\ast, \ast)_0$, the form $(\ast, \ast)$ being the hermitian one of $W$. We take an $\mathcal{O}_F$-lattice $\Lambda$ of $W$ such that for each $p$-adic place $v$ of $F_0$, the inclusion $\Lambda_v \subseteq \Lambda^\vee_v \subseteq \pi_v^{-1}\Lambda_v$ holds, the dual being the symplectic one. Regarding the levels, set $K_{G,v}$ to be a sufficiently small open compact subgroup of $G(\mathbb{A}_f^p)$. We introduce the following groups:

$$K_{G,v} := \text{Stab}_{G(F_0,v)}(\Lambda_v),$$

$$K_{G,p} := \prod_{v \mid p} K_{G,v},$$

$$K_G := K_{G,p} K_{G,v},$$

$$K_{\tilde{G}} := K_{Z^0} G \subseteq \tilde{G}(\mathbb{A}_f).$$

We set $\tilde{G}$ to be the subgroup $K_{Z^0, p} K_{G,p}$ of $\tilde{G}(\mathbb{Q}_p)$, seen as a connected parahoric group scheme associated with a point in the extended Bruhat–Tits building of $G(\mathbb{Q}_p)$. Finally, we define $L = \text{Hom}_{\mathcal{O}_F}(\Lambda_0, \Lambda)$. The $F$-vector space $V = L_Q$ admits a hermitian form $V \times V \to F$ such that for each $m, m'$ in $L_0$ and $x, x'$ in $L$,

$$(x, x')(m, m')_0 = (xm, x'm'),$$

where all the forms are hermitian.

5.2. Naive models. For two abelian schemes $A_0$ and $A$ over a scheme $S$ with $\mathcal{O}_{F,(p)}$-actions up to prime-to-$p$ isogeny, we define $T^p(A_0, A)$ and $V^p(A_0, A)$ to be the pro-etale sheaf $\text{Hom}_{\mathcal{O}_{F,(p)}}(T^p(A_0), T^p(A))$ and $\text{Hom}_{\mathcal{O}_{F,(p)}}(V^p(A_0), V^p(A))$, respectively. If $A_0$ and $A$ are endowed with $\mathcal{O}_{F,(p)}$-linear $\mathbb{Q}$-polarizations $\lambda_0$ and $\lambda$ respectively, then we have a
hermitian form on the sheaf $V^p(A_0, A)$ sending $x, x' \in V^p(A_0, A)$ to 
\[ \lambda_0^{-1} \circ x' \circ \lambda \circ x \in \text{End}_S(V^p(A_0)) \isom (A^p_{F,f})_S(S). \]

We define the category $\mathcal{M}^\text{naive}_{\tilde{G}}$ fibered in groupoids over the category of locally noetherian $\mathcal{O}_{E,(p)}$-schemes. It associates with each such scheme $S$ the groupoid of tuples $(A_0, i_0, \lambda_0, e^p, A, i, \lambda, \eta^p)$, where

- $(A_0, i_0, \lambda_0, e^p) \in \mathcal{M}_0(S)$,
- $A$ is an abelian scheme over $S$,
- $i : O_{F} \to \text{End}_S(A)$ satisfies another Kottwitz condition that for every $a \in O_F$, the characteristic polynomial of $i(a)$ acting on $\text{Lie} A$ is
  \[ \prod_{\varphi \in \text{Hom}(F, \mathbb{Q})} (T - \varphi(a))^{r_{\varphi}}, \]
- $\lambda \in \text{Hom}_{O_F}(A, A^\vee) \otimes \mathbb{Z}_{(p)}$ is a $\mathbb{Q}$-polarization, and
- $\eta^p$ is a section over $S$ of the pro-etale sheaf $\text{HermIsom}((\hat{L}^p)_S, T^p(A_0, A))/K^p_{\tilde{G}}$.

Here, $\text{HermIsom}((\hat{L}^p)_S, T^p(A_0, A))$ is the pro-etale sheaf of $\hat{O}_F$-linear isomorphisms between the two sheaves preserving the hermitian forms on $L \otimes \mathbb{Z}_{(p)} A^p_{f}$ and $V^p(A_0, A)$.

The tuple is further required to satisfy the two conditions below.

- We have a decomposition with respect to $p$-adic places of $F_0$
  \[ A[p^\infty] = \prod_{v|p} A[v^\infty], \]
  where the quotient $O_{F_0,v}$ of $O_{F_0} \otimes \mathbb{Z}_p$ acts on $A[v^\infty]$ through $i$. For each $p$-adic place $v$ of $F_0$, the kernel of $\lambda_v : A[v^\infty] \to A^\vee[v^\infty]$ induced by $\lambda$ should have the rank $\#(\Lambda^\vee_v/\Lambda_v)$ and should be included in $A[\pi_v] = \text{Ker}(i(\pi_v) : A \to A)$.
- For every $p$-adic place $v$ of $F_0$ nonsplit in $F$ and each geometric point $s \to S$, \[ \text{[17] Appendix A} \] associates the sign invariant
  \[ \text{inv}^r_v(A_{0,s}, i_{0,s}, \lambda_{0,s}, A_s, i_s, \lambda_s) \]
  in $\{\pm 1\}$. This should equal
  \[ (-1)^{n(n-1)/2} \text{det } V_v \]
  in $F_{0,\varepsilon}/\text{Nm } F_v^\times \simeq \{\pm 1\}$. Here, the determinant is the one of a matrix representation of the hermitian form of $V_v$.

The latter condition is called the sign condition. Arrows of this groupoid are pairs consisting of an arrow in $\mathcal{M}_0$ and an isomorphism of the other abelian schemes preserving the other data. We note that $\eta^p$ admits a similar identification to $e^p$ described in terms of stalks.

We compare this moduli with its analogue described by isogenies. The naive model in [18] means this isogeny version. Let $\mathcal{R}^\text{naive}_{\tilde{G}}$ be the category fibered in groupoids over the category of locally noetherian
The tuple is requested to satisfy the two conditions as before. Here, is constant with respect to depends on the isogeny class of (the involved actions are up to isogeny. Moreover, the invariant only with essentially the same definition of Hermsom as its counterpart.

The tuple is requested to satisfy the two conditions as before. Here, the sign invariant at a geometric point \( s \to S \) is also defined when the involved actions are up to isogeny. Moreover, the invariant only depends on the isogeny class of \( (A_{0,s}, i_{0,s}, \lambda_{0,s}, A_s, i_s, \lambda_s) \), and is locally constant with respect to \( s \).

Arrows of this groupoid are composed of pairs of an arrow in \( \mathcal{M}_0(S) \) and an \( \mathcal{O}_{F,(p)} \)-linear \( \mathbb{Z}^x \)-isogeny between other abelian schemes which preserves the other data.

**Lemma 5.1** (cf. [3] Corollary 1.3.5.4). Set \( S \) to be a connected locally noetherian scheme of residual characteristic 0 or \( p \) with a geometric point \( s \to S \). Set \( A \) to be an abelian scheme over \( S \) with an action \( i : \mathcal{O}_{F,(p)} \to \text{End}_S(A) \otimes_{\mathbb{Z}} \mathbb{Z}(p) \).

Then the isomorphism classes of \( \mathcal{O}_{F,(p)} \)-linear \( \mathbb{Z}^x \)-isogenies from \( A \) to different targets consisting of an abelian scheme over \( S \) with an \( \mathcal{O}_{F,(p)} \)-action are in bijection with \( \pi_1(S, s) \)-invariant open compact subgroups of \( V^p(A_s) \). This is by sending the class of \( f \in \text{Hom}_{\mathcal{O}_{F,(p)}}(A, A') \otimes_{\mathbb{Z}} \mathbb{Z}(p) \) to \( V^p(f)^{-1}(T^p(A'_s)) \).

Moreover, the \( \mathcal{O}_{F,(p)} \)-action of \( A' \) is derived from an action \( \mathcal{O}_F \to \text{End}_S(A') \) if and only if the corresponding subgroup of \( V^p(A_s) \) is \( \mathcal{O}_F \)-invariant.

**Proposition 5.2.** The forgetful functor \( \mathcal{M}^\text{naive}_K \to \mathcal{R}^\text{naive}_K \) is an equivalence of categories fibered in groupoids.

**Proof.** We show the equivalence of the \( S \)-valued points, mimicking [3] Proposition 1.4.3.4]. We may assume that \( S \) is connected. We take its geometric point \( s \). We first address the essential surjectivity. Take an object \( (A_0, i_0, \lambda_0, e^p, A, i, \lambda, \eta^p) \) of \( \mathcal{R}^\text{naive}_K \). By Lemma 5.1, there is an abelian scheme \( A' \) over \( S \), an action \( \mathcal{O}_F \to \text{End}_S(A') \) and an \( \mathcal{O}_{F,(p)} \)-linear \( \mathbb{Z}^x \)-isogeny \( f \in \text{Hom}(A, A') \otimes_{\mathbb{Z}} \mathbb{Z}(p) \) such that

\[
V^p(f)^{-1}(T^p(A'_s)) = \eta^p(\hat{T}^p)(T^p(A_{0,s})).
\]

We also obtain a \( \mathbb{Q} \)-polarization \( \lambda' \) of \( A' \) from \( \lambda \). The Kottwitz condition and the sign condition is still valid with \( A' \) in place of \( A \) and the condition on \( \text{Ker} \lambda'_e \) holds again. We are left with \( \eta^p \). We fix a
representative \( \mathbb{L} \otimes_{\mathbb{Z}} A^p_f \to V^p(A_{0,s}, A_s) \) of \( \eta^p \) at \( s \) denoted by the same symbol. The homomorphism \( V^p(f) : V^p(A_{0,s}, A_s) \to V^p(A_{0,s}', A_s') \) is isometric due to the compatible \( \lambda \) and \( \lambda' \). This leads to the isometry of \( V^p(f) \circ \eta^p \). The image \( V^p(f) \circ \eta^p(\tilde{L}^p) \) equals \( T^p(A_0, A') \) as \( \eta^p \) induces the isomorphism \((L \otimes_{\mathbb{Z}} A^p_f) \otimes_{\mathbb{K}^F,f} V^p(A_{0,s}) \to V^p(A_s)\), through which \( \tilde{L}^p \otimes_{\mathbb{K}^F} T^p(A_{0,s}) \) and \( V^p(f)^{-1}(T^p(A'_s)) \) coincide.

Next we focus on the fully-faithfulness. The faithfulness is trivial. For the other part, take two objects \((A_0, i_0, \lambda_0, e^p, A, i, \lambda, \eta^p)\) and \((A'_0, i'_0, \lambda'_0, e'^p, A', i', \lambda', \eta'^p)\) of \( \mathcal{M}^{\text{naive}}_{K_G}(S) \), and a morphism of \( \mathcal{R}^{\text{naive}}_{K_G}(S) \) from the first to the second. Let \( f \in \text{Hom}_{\mathcal{O}_E}(A, A') \otimes_{\mathbb{Z}} \mathbb{Z}(p) \) be its constituent. Then, using the level structures, we have \( V^p(f)(T^p(A_s)) = T^p(A'_s') \), implying that \( f \) is an actual morphism.

The functors \( \mathcal{M}^{\text{naive}}_{K_G} \) and \( \mathcal{R}^{\text{naive}}_{K_G} \) are representable by \( \mathcal{O}_{E,(p)} \)-schemes of finite type by a standard argument (cf. \cite{8} §5). These schemes are separated by the valuative criterion. Besides, their generic fibers are \( \text{Sh}_{K_G}(\tilde{G}, h_G) \) by \cite{18} Theorem 4.4].

5.3. Flat models. When \( K^p_{\mathbb{Z}^\mathbb{Q}} = \mathbb{Z}^\mathbb{Q}(\mathbb{Z}(p)) \), we refer to \cite{18} Theorem 5.4 for the definition of \( \mathcal{M}_{K_G} \); except that it is a scheme over \( \mathcal{O}_{E,(0)} \) as in \cite{17} §4, and that in (d) of the theorem, we suppose that \( \{r_{\varphi}, r_{\varphi'}\} = \{1, n-1\} \). The latter is because we do not know if the local model in \cite{20} related to (d) is normal, affecting in turn the proof of the normality of our models. The definition of \cite{18} Theorem 5.4] uses \( \mathcal{R}^{\text{naive}}_{K_G} \), but we can describe it in terms of \( \mathcal{M}^{\text{naive}}_{K_G} \). In general, we put \( \mathcal{M}_{K_G} = \mathcal{M}_{Z^\mathbb{Q}(\mathbb{Z})K_G} \times_{\mathcal{M}_{K_G}} \mathcal{M}_0 \).

5.4. Statement of the main theorem. We think of \( \text{GL}(\Lambda_0 \oplus \Lambda) \) as an algebraic group over \( \mathbb{Z}_p \) identified with \( \text{GL}(\mathbb{F}_{\mathbb{Q}(n+1)}, \mathbb{Z}_p) \). Set \( \iota : \tilde{G} \to \text{GL}(\Lambda_0 \oplus \Lambda) \) to be the inclusion. The pair \((\tilde{G}, M^{\text{loc}})\) is of strongly integral local Hodge type by \( \iota \), as we now explain. The first two conditions on \( \iota \) in \cite{2.1} are straightforward. The last one is by \cite{7} Proposition 2.3.7]

**Theorem 5.3.** The system of schemes \( \mathcal{M}_{K_G} \) for varying \( K^p_{\mathbb{Z}^\mathbb{Q}} \) and \( K^p_{\mathbb{Z}^\mathbb{Q}} \) and the transition morphisms is a canonical integral model with respect to \((\tilde{G}, M^{\text{loc}})\). In particular, the system is isomorphic to the system of models in \cite{7} 4.3.6] with the corresponding levels.

Here is the strategy for the proof of Theorem 5.3. The construction of \((\tilde{G}, M^{\text{loc}})\)-displays appears tough. However, there are pointwise Dieudonné \((\tilde{G}, M^{\text{loc}})\)-displays constructed in a general situation in Theorem 2.5. The local universality of these local objects in our case follows from Proposition 2.4], which translates the sections rigid in the first order into the language of deformation theory as mentioned in §4]. Then the strong uniqueness result in Theorem 2.8 shows that RSZ models are...
isomorphic to the ones by Kisin and Pappas, whose \((\tilde{G}, M^\text{loc})\)-displays are already in \cite[Theorem 8.2.1]{15}.

The rest of this section concisely handles the fact that the models satisfy properties in Definition \ref{def:models} other than (2) and (3). The \(\mathcal{O}_{E,\nu}\)-scheme \(\mathcal{M}_{K,\tilde{G}}\) is flat with generic fiber \(\text{Sh}_{K,\tilde{G}}(\tilde{G}, h_{\tilde{G}})\) (cf. \cite[Theorem 5.4]{18}). It is separated of finite type since so is the naive model. We claim the normality of \(\mathcal{M}_{K,\tilde{G}}\). The local model diagram in the proof of \cite[Theorem 5.4]{18} attributes this to the normality of local models. Local models cited in the same proof are the Pappas–Zhu local models of \cite[Theorem 5.4]{18}. The finite etaleness of the transition morphisms is similar to the finite etale group scheme \(\tilde{A}\) by those citations and \cite[§8.2.5(c)]{13}. This implies that their base change over \(\mathcal{O}_{E,\nu}\) is normal by \cite[Theorem 9.1]{15} (cf. \cite[Proposition 9.2]{15}). The finite etaleness of the transition morphisms is similar to Proposition \ref{prop:extension}.

6. The extension property

This section is about the extension property, that is, \(\psi\) of Definition \ref{def:models}. First we show the extension property for \(\mathcal{M}_{K,\tilde{G}}^\text{naive}\). If \(S\) is a locally noetherian \(\mathcal{O}_{E,(p)}\)-scheme, then as a set, \(\varprojlim_{K^p \subseteq K_{G}} \mathcal{M}_{K,\tilde{G}}^\text{naive}(S)\) is the set of isomorphism classes of tuples \((A_0, i_0, \lambda_0, \epsilon^p, A, i, \lambda, \eta_p)\) of

- an abelian scheme \(A_0\) over \(S\),
- \(i_0: \mathcal{O}_F \to \text{End}_S(A_0)\) with the Kottwitz condition,
- an \(\mathcal{O}_F\)-linear polarization \(\lambda_0: A_0 \to A_0^\vee\) of degree prime to \(p\) such that \((A_0, i_0, \lambda_0) \in M^\text{loc}_0(S)\),
- an \(\mathcal{O}_F^p\)-linear isomorphism \(\epsilon^p: (\hat{A}_0)^p_S \to T^p(A_0)\) transferring \(\psi_0\) to the Riemann form of \(\lambda_0\) up to \((\mathbb{Z}/p)^\times\),
- \(A\) is an abelian scheme over \(S\),
- \(i: \mathcal{O}_F \to \text{End}_S(A)\) with the other Kottwitz condition,
- a \(\mathbb{Q}\)-polarization \(\lambda \in \text{Hom}_{\mathcal{O}_E}(A, A^\vee) \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)}\), and
- a hermitian isomorphism \(\eta_p: (\hat{L}^p)_S \to T^p(A_0, A)\),

further satisfying two conditions in the definition of \(\mathcal{M}_{K,\tilde{G}}^\text{naive}\).

Let \(R\) be a discrete valuation ring of mixed characteristic \((0, \nu)\) and let \((A_0, i_0, \lambda_0, \epsilon^p, A, i, \lambda, \eta_p)\) give an element of \(\varprojlim_{K^p \subseteq K_{G}} \mathcal{M}_{K,\tilde{G}}^\text{naive}(R[1/p])\).

Then by the Nerón–Ogg–Shafarevich criterion, \(A_0\) uniquely extends to an abelian scheme \(\hat{A}_0\) over \(R\). The data \(i_0\) and \(\lambda_0\) uniquely extend to data on \(\hat{A}_0\) by, e.g., \cite[Lemma 1]{2} and \cite[Proposition 2.14]{10}. For \(\epsilon^p\), the finite etale group scheme \(A_0[N]\) for a positive integer \(N\) prime to \(p\) is constant as \(A_0[N]\) is, so that \(\epsilon^p\) also uniquely extends to a datum over \(R\). The rest of the argument is analogous except that we use \((\hat{L}^p)_S \otimes_{\mathcal{O}_F^p} T^p(A_0) \sim T^p(A)\) induced by \(\eta_p\).
The extension property for \( \mathcal{M}_{K_{\tilde{G}}} \) holds by that for the naive model and the valuative criterion for the closed immersion \( \mathcal{M}_{K_{\tilde{G}}} \to \mathcal{M}^{\text{naive}}_{K_{\tilde{G}}} \).

7. The locally universal display

7.1. The local system. To use Theorem [25], we investigate \( \mathcal{L}_{K_{\tilde{G}}} \), the pro-etale \( \tilde{G}(\mathbb{Z}_p) \)-local system on \( \text{Sh}_{K_{\tilde{G}}}((\tilde{G}, \{ h_{\tilde{G}} \})) \) given by the covers \( \text{Sh}_{K_{\tilde{G}}(\text{pro-etale})}((\tilde{G}, \{ h_{\tilde{G}} \})) \to \text{Sh}_{K_{\tilde{G}}}((\tilde{G}, \{ h_{\tilde{G}} \})) \) for some pro-etale \( K_{\tilde{G}} \)-local system \( \tilde{G}((\tilde{G}, \{ h_{\tilde{G}} \})) \). For a positive integer \( n \), we set \( K_{\tilde{G}}(p^n) \) (resp. \( K_{G}(p^n) \)) to be the kernel of \( G/\mathbb{Z}_p \to GL(\Lambda_0/p^n\Lambda_0) \) (resp. \( K_{G}(p^n) \to GL(\Lambda/p^n\Lambda) \)). We also put \( K_{\tilde{G}}(p^n) = K_{\tilde{G}}(p^n) \). Denote by \( (A_0, i_0, \lambda, A, i, \eta, \Lambda, \Lambda_p, \Lambda_0, \lambda_0, \eta_p, \Lambda_p) \) the universal object over \( \mathcal{M}_{K_{\tilde{G}}} \) only in this subsection. Over the generic fiber, the tuple is also seen as the universal object of the moduli in [18, Definition 3.8] with the auxiliary model replaced by a clopen part of the moduli at the end of §4.

Lemma 7.1. The scheme \( \text{Sh}_{K_{\tilde{G}}(p^n)}(\text{pro-etale})((\tilde{G}, \{ h_{\tilde{G}} \})) \) represents a set-valued functor from the category of locally noetherian \( \text{Sh}_{K_{\tilde{G}}}((\tilde{G}, \{ h_{\tilde{G}} \})) \)-schemes, sending such a connected scheme \( S \) having some geometric point \( s \to S \) to the following set.

The set comprises pairs of two isomorphisms \( (\Lambda_0/p^n\Lambda_0)_{S} \to A_{0,S}[p^n] \) and \( (\Lambda/p^n\Lambda)_{S} \to A_{S}[p^n] \) lifting to \( \mathcal{O}_F \)-linear isomorphisms \( \Lambda_0 \otimes_{\mathbb{Z}} \mathbb{Z}_p \sim \to T_p(\Lambda_0,S) \) and \( \Lambda \otimes_{\mathbb{Z}} \mathbb{Z}_p \sim \to T_p(\Lambda,S) \) in the commutative diagram below for some \( \mathbb{Z}_p \sim \to \mathbb{Z}_p(1) \):

\[
\begin{array}{ccc}
\Lambda_0 \otimes_{\mathbb{Z}} \mathbb{Z}_p \times \Lambda_0 \otimes_{\mathbb{Z}} \mathbb{Z}_p & \xrightarrow{\psi_0} & T_p(\Lambda_0,S) \times T_p(\Lambda_0,S) \\
\downarrow & & \downarrow \\
\mathbb{Z}_p & \sim \to & \mathbb{Z}_p(1) \\
\downarrow \psi & & \downarrow \\
\Lambda \otimes_{\mathbb{Z}} \mathbb{Z}_p \times \Lambda \otimes_{\mathbb{Z}} \mathbb{Z}_p & \xrightarrow{\chi} & T_p(\Lambda,S) \times T_p(\Lambda,S).
\end{array}
\]

Proof. First, a morphism \( g_0 : (\Lambda_0/p^n\Lambda_0)_S \sim \to A_{0,S}[p^n] \) having a lift \( \tilde{g}_0 : \Lambda_0 \otimes_{\mathbb{Z}} \mathbb{Z}_p \to T_p(\Lambda_0,S) \) such that the upper half of the diagram commutes. The lower half of the diagram for \( \psi_0 : \mathbb{Z}_p \sim \to \mathbb{Z}_p(1) \) corresponds to specifying the \( K_{\tilde{G}}(p^n) \)-level structure for \( A_{0,S} \) since \( K_{\tilde{G}} \) is the stabilizer of \( \Lambda_0 \otimes_{\mathbb{Z}} \mathbb{Z}_p \). The set of \( \mathcal{O}_F \)-linear isomorphisms \( (L/p^nL)_{S} \to \text{Hom}_{\mathcal{O}_F}(A_{0,S}[p^n], A_{S}[p^n]) \) bijects via \( g_0 \) to the set of \( \mathcal{O}_F \)-linear isomorphisms \( (\Lambda/p^n\Lambda)_{S} \sim \to A_{S}[p^n] \) in the commutative diagram below for \( \psi \)

Since \( K_{G,p} \) is the stabilizer of \( L \otimes_{\mathbb{Z}} \mathbb{Z}_p \), for any \( g_H : (L/p^nL)_{S} \sim \to \text{Hom}_{\mathcal{O}_F}(A_{0,S}[p^n], A_{S}[p^n]) \), it is derived from specifying the \( K_{\tilde{G}}(p^n) \)-level structure for \( \tilde{g}_0 \) and only if there exists the \( \mathcal{O}_F \)-linear isomorphism \( \tilde{g} : \Lambda \otimes_{\mathbb{Z}} \mathbb{Z}_p \sim \to T_p(\Lambda) \) such that for every \( x, x' \in \text{Hom}_{\mathcal{O}_F}(T_p(\Lambda_0,S), T_p(\Lambda,S)) \)
and every \( m, m' \in \Lambda_0 \),

\[
(\lambda_0^{-1} x' \lambda x)(m, m')_0 = (\tilde{g}^{-1} x\tilde{g} m, \tilde{g}^{-1} x'\tilde{g} m').
\]

Here, the first factor of the left side is an element of \( \text{End}_{O_F}(T_p(A_0, s)) \cong O_F \otimes \mathbb{Z} \). The last equation is equivalent to the lower half of the commutative diagram by calculating both sides of the equality via applying \( \text{Tr} \sqrt{\Delta - 1} \).

□

This description implies that the push-out of \( L_{K_0} \tilde{G} \) via \( \iota \) is isomorphic to \( T_p(A_0) \times T_p(A) \) pulled back over \( \text{Sh}_{K_0}(G, \{ h_G \}) \). By Theorem 2.5, there exists a unique associated system \( (L_{K_0} \tilde{G}, (D_x, \alpha_x)_{x \in M_{K_0}(F_p)}) \) up to unique isomorphism.

7.2. The locally universality. In view of (3*) of Theorem 2.8, we show the local universality of the obtained system, mimicking the proof of [13, Proposition 8.1.5]. Let \( p \) be the prime ideal of \( E_\nu \). Set \( \tilde{E}_\nu \) be the maximal unramified extension of \( E_\nu \). Denote by \( C \) (resp. \( C_0 \)) the category of artinian local \( O_{\tilde{E}_\nu} \)-algebras (resp. \( W(F_p) \)-algebras) whose structure homomorphism is local and induces the equality of residue fields. For each \( x \in M_{K_0}(F_p) \), let \( (R_x, m_x) \) denote the strict completion of the local ring of \( M_{K_0}(F_p) \) at \( x \). Let \( (A_0, i_0, \lambda_0, \varpi, \overline{A}, \overline{i}, \overline{\lambda}, \overline{\varpi}) \) be the object of \( M_{K_0}(F_p) \) in correspondence with \( x \). We fix a positive integer \( N \) prime to \( p \) such that \( N\lambda \) is an actual polarization. The complete noetherian ring \( R_x \) pre-represents the following set-valued deformation functor Def from \( C \). For any object \( S \) of \( C \), \( \text{Def}(S) \) is the set of the isomorphism classes of the objects of \( M_{K_0}(S) \) reduced to \( (A_0, i_0, \lambda_0, \varpi, \overline{A}, \overline{i}, \overline{\lambda}, \overline{\varpi}) \) over \( \overline{F}_p \).

**Lemma 7.2.** The morphism \( \text{Def} = \text{Spf} R_x \rightarrow \text{Def}_{W(F_p)}(A_0[p^\infty] \times \overline{A}[p^\infty]) \) given by \( \mathcal{D}_x \) as in the statement of Proposition 2.4 is a closed immersion, the target being the deformation functor on \( C_0 \).

**Proof.** First, we have a closed immersion

\[
\text{Def} \rightarrow \text{Def}^{\text{naive}}
\]

to a deformation functor for \( M_{K_0}^{\text{naive}} \) similarly defined to the functor Def. Next, we may forget about the level structures, since they are described in terms of etale fundamental groups and stalks of etale sheaves, always admitting a unique deformation when the other data are deformed. Similarly, we may forget the locally constant sign invariant, the condition on \( \deg \lambda_0 \) and the condition on Ker \( \lambda_v \). Then, forgetting the Kottwitz conditions and sending the quasi-polarization \( \lambda \) deforming \( \lambda \) to \( N\lambda \) yields a closed immersion

\[
\text{Def}^{\text{naive}} \rightarrow \text{Def}_{O_{E_\nu}}(A_0, \overline{\lambda}_0, \overline{i}_0) \times \text{Def}_{O_{E_\nu}}(\overline{A}, N\overline{\lambda}, \overline{i}),
\]
the target consisting of the deformation functors on $\mathcal{C}$ of the data in the parentheses. Moreover, we have a closed immersion \cite[Theorem 1.4.4.5]{11}

$$\text{Def}_{\mathcal{C}}(\overline{A}_0, \lambda_0, t_0) \times \text{Def}_{\mathcal{C}}(\overline{N}, \lambda, t) \to \text{Def}_{\mathcal{C}}(\overline{A}_0) \times \text{Def}_{\mathcal{C}}(\overline{A}).$$

The target is a restriction to $\mathcal{C}$ of a similar functor $\text{Def}_W([F_p])_{\text{st}}(\mathcal{A}_0, \lambda_0, \lambda_0, \xi_0) \times \text{Def}_W([F_p])_{\text{st}}(\mathcal{A}_0) \to \text{Def}_W([F_p])_{\text{st}}(\mathcal{A}_0) \times \text{Def}_W([F_p])_{\text{st}}(\mathcal{A}).$

Finally, the latter is isomorphic to $\text{Def}_W([F_p])_{\text{st}}(\mathcal{A}_0[p^\infty] \times \mathcal{A}[p^\infty])$ by the Serre–Tate theorem. The composition of all these is what we have claimed is a closed immersion by construction in the proof of Theorem 2.5.

\begin{lemma}
\label{7.3}
The system $(\mathcal{L}_{K^G}, (\mathcal{D}_x, \alpha_x)_{x \in \mathcal{M}_{K^G}([F_p]))}$ is locally universal.
\end{lemma}

\begin{proof}
Put $\mathcal{D}_x = (\mathcal{P}_x, q_x, \Psi_x).$ We take an arbitrary section $s \in \mathcal{P}_x(R_x)$ rigid in the first order. By Proposition 2.4 and Lemma 7.2, $\iota_s \circ q_x \circ (s \otimes 1): \text{Spec } R_x \to \text{Gr}(d, n)_{\mathcal{O}_{\mathcal{E}^\nu}}$ restricted to $\text{Spec } R_x/(m_x^2 + pR_x)$ induces an injection from $(m_x/(m_x^2 + pR_x))$ to a tangent space of the Grassmannian. If $y$ is the image of the closed point of $R_x$ by $q_x \circ (s \otimes 1), then this implies that $\mathcal{O}_{\mathcal{M}_{K^G}^{\text{loc}}[y]} \to R_x$ is surjective on the cotangent spaces. In turn, the map from the completion of the source to $R_x$ is surjective as can be seen by taking the associated graded algebras. Finally, by comparing dimensions via the local model diagram in \cite[Theorem 5.4]{13}, recalling that various local models cited in the proof there are the Pappas–Zhu local models, we see that in fact the map from the completion to $R_x$ is an isomorphism. 

By the above Lemma 7.3, Theorem 2.8 and \cite[Theorem 8.1.4]{13}, the system made of $\mathcal{M}_{K^G}$ is isomorphic to the one made of integral models in \cite[4.3.6]{7}. By this and \cite[Theorem 8.2.1]{13}, the former system is canonical. This completes the proof of Theorem 5.3.

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