A one-parameter model for turbine wakes from the entrainment hypothesis

Paolo Luzzatto-Fegiz
Department of Mechanical Engineering, University of California, Santa Barbara, CA 93106, USA
E-mail: fegiz@engineering.ucsb.edu

Abstract. The entrainment hypothesis consists of a concise turbulence model, which has been widely used to model geophysical flows including plumes, gravity currents, and breaking internal waves. This paper employs the entrainment hypothesis to obtain a model for the wake of a single wind turbine, without having to linearize the governing equations, or needing to assume a wake radius that grows linearly (or in any other prescribed manner) with distance downstream. Assuming a top-hat velocity deficit and a zero pressure gradient, the analysis presented here re-discovers a result first due to Morton (B.R. Morton. J. Fluid Mech. 10:101–112, 1961), which however appears to have been completely overlooked until now. The predictions from this entrainment wake model are compared with published laboratory data and field observations, finding good agreement. Since the entrainment model does not require assuming a specific relationship between wake diameter and distance downstream, it may also enable simple extensions to account for complex terrain and for effects of yaw.

1. Introduction
Simple models for turbine wakes have been used extensively in the wind energy community, both as independent tools for engineering calculations, as well as to complement more refined and computationally-intensive techniques. One of the most popular approaches was introduced by Jensen [1], who developed a model assuming that the wake radius grows linearly with downstream distance \( x \), and approximating the velocity deficit with a top-hat profile, as sketched in figure 1a. This model is often labeled the ‘park’ model; it is given by [1]

\[
\frac{D_w(x)}{D} = \frac{D_i}{D} + \frac{2k(x-x_i)}{D}, \quad \frac{U_w(x)}{U_o} = 1 - \frac{2a}{1 + 2k(x-x_i)/D_i^2},
\]

where \( U_w \) and \( U_o \) are the velocities inside and outside the wake, \( a \) is the axial induction coefficient (related to \( C_t = 4a(1-a) \), such that \( a = \frac{1}{2}(1 - \sqrt{1-C_t}) \)), \( D \) is the turbine diameter and \( D_i \) is the initial wake diameter, which starts at a downstream location \( x_i \) from the turbine. Following the approach of Frandsen [2], \( D_i \) is often set as the diameter after initial wake expansion, such that actuator disc theory gives \( D_i = D[(1-a)/(1-2a)]^{1/2} \). The value of \( x_i \) is often set to zero [2]. The park model constitutes a pioneering and practically useful result, and has been widely implemented in the wind energy community [3, 4]. More recently, several authors [5, 6] noted that the park model does not conserve momentum. This fact can be verified by first
noting that, for a general axisymmetric wake with a top-hat profile, the $x$-momentum deficit (relative to a uniform stream $U_o$) is given by (e.g. [7])

$$\dot{M} = \rho \frac{\pi}{4} D^2 w \left( U_o - U_w \right),$$

(2)

where $\rho$ is the density. Conservation of linear momentum implies that $\dot{M}$ should be constant and equal to the thrust on the turbine. If one evaluates $\dot{M}$ for the park model using (1), and differentiates it with respect to $x$, the result is

$$\frac{d\dot{M}_{\text{park}}}{dx} = \rho 4\pi a^2 k U_o^2 \frac{D_i^4}{[D_w(x)]^3} > 0,$$

(3)

such that $\dot{M}_{\text{park}}$ increases with $x$. The momentum deficit of the park model is plotted in figure 1b, using values of $C_t = 0.8$ and $k = 0.06$, which are typical in applications (values of $k$ range between approximately 0.05 and 0.075 depending on terrain; see [6] and references therein). By $x/D = 10$, $\dot{M}_{\text{park}}$ has grown to approximately twice the thrust actually exerted by the turbine.

Frandsen et al. [5] proposed a momentum-conserving model, given by

$$\frac{D_w(x)}{D} = \frac{D_i}{D} + 2k^* \left( \frac{x - x_i}{D} \right), \quad \frac{U_w(x)}{U_o} = \frac{1}{2} \left[ 1 + \sqrt{1 - \frac{8a(1 - 2a)}{\left[ 1 + 2k^*(x - x_i)/D_i \right]^2}} \right],$$

(4)

where the linear wake spreading parameter $k^*$ is in general different from $k$. More recently, linear-spread wake models that improve on the top-hat assumption have been introduced. These include FLORIS [8], which nests several linearly-spread velocity deficit regions, as well as the model of Bastankhah & Porté-Agel [6], which employs a Gaussian velocity deficit. These models significantly improve agreement with experiments and Large Eddy Simulations of turbine wakes.

While the linear spreading assumption facilitates conceptual modeling, it requires empirical estimates of the spreading rate, which have not been related to more general turbulence parameterizations. In this paper, the assumption of a specific global relationship between $D_w$
and $x$ is removed, by instead employing a simple, local turbulence closure that has seen wide usage in geophysical fluid dynamics, namely the entrainment hypothesis [9], which is outlined in section 2. In section 3, the entrainment hypothesis is used to obtain a model for wind turbine wakes. Section 4 compares the entrainment wake model to other approaches and to data for high Reynolds number wakes. Section 5 discusses the growth of the wake, its approximation by linear or other powers of $x$, as well as its asymptotic behavior. Concluding remarks follow in section 6.

2. The entrainment hypothesis

The ‘entrainment hypothesis’ consists of a simple turbulence parameterization, developed for flows where a strongly turbulent region advances into a relatively lower-turbulence background fluid. As noted earlier, this parameterization appears to have been first introduced by G.I. Taylor, in the context of modeling turbulent plumes [9]. This turbulence closure has had a vast impact on the environmental fluid dynamics literature; to date, reference [9] has accumulated 1,356 citations according to Web of Science. As previously mentioned, the entrainment hypothesis has been applied extensively in geophysical and industrial flow problems including plumes, jets, natural ventilation and gravity currents [9, 10, 11, 12, 13, 14, 15].

To describe this turbulence parameterization, consider an interface between a high-turbulence region and relatively low-turbulence ambient fluid, as sketched in figure 2a. Consistently with the definitions for a turbulent wake used so far in this paper, let the streamwise velocities in the high- and low-turbulence regions be $U_w$ and $U_o$, respectively. The entrainment hypothesis assumes that fluid outside the turbulent region crosses the turbulent interface with an entrainment velocity of magnitude $v_e(x)$, which is proportional to the streamwise velocity difference across the interface, such that (see figure 2b)

$$v_e = E |U_o - U_w|,$$

where $E$ is known as the entrainment coefficient (typically of the order of 0.1), which is obtained empirically [9]; see also section 4.

Similarly to the approach taken in the park and Frandsen models, here the axial velocity across the wake is approximated by a top-hat profile, such that $U_o$ and $U_w$ are the flow velocities...
inside and outside the wake, and $D_w$ defines the location of the wake boundary.

It has been observed that if $E$ can be assumed to depend mainly on local processes, then its value should not change significantly even when one considers qualitatively different large-scale flows. Extensive oceanographic datasets exist to parameterize $E$ as a function of local Reynolds number and Froude number (or, equivalently, Richardson number) ([14] and references therein). It is worth noting that these data were collected in field environments, where ambient turbulence is inevitably present. Introducing the entrainment hypothesis to turbine wake modeling could enable wind energy practitioners to leverage this wealth of established data on describing turbulent interfaces.

3. Entrainment model for wind turbine wakes

With reference to the wake sketched in figure 2c, consider a control volume of infinitesimal length $\Delta x$ in the axial direction, whose diameter matches the diameter $D_w(x)$ of the wake. Assuming the fluid has uniform density, conservation of mass implies

$$\frac{\pi}{4} D_w^2 U_w + v_e \pi D_w \Delta x = \frac{\pi}{4} D_w^2 U_w + \frac{d}{dx} \left( \frac{\pi}{4} D_w^2 U_w \right) \Delta x. \quad (6)$$

Dividing by $\pi \Delta x/4$, canceling the first term on each side, and re-arranging terms so the derivative is on the left-hand side,

$$\frac{d}{dx} \left( D_w^2 U_w \right) = 4v_e D_w. \quad (7)$$

Using the definition of the entrainment velocity given by (5), and assuming $U_o > U_w$ to remove the absolute value,

$$\frac{d}{dx} \left( D_w^2 U_w \right) = 4E(U_o - U_w) D_w. \quad (8)$$

In considering conservation of axial momentum, the pressure gradient is assumed to be negligible over the majority of the wake. This is typically the case after the wake has undergone the initial pressure adjustment shortly after the turbine. If the wake model is initialized after this initial pressure adjustment has already taken place, conservation of axial momentum yields

$$\frac{\pi}{4} D_w^2 U_o^2 + v_e \pi D_w \Delta x U_o = \frac{\pi}{4} D_w^2 U_w^2 + \frac{d}{dx} \left( \frac{\pi}{4} D_w^2 U_w^2 \right) \Delta x. \quad (9)$$

Similarly to the procedure used to obtain (8) from (6), the momentum equation (9) is divided by $\pi \Delta x/4$ and the first terms on each side cancel. Using the definition of $v_e$ and moving the derivative to the left-hand-side, one obtains

$$\frac{d}{dx} \left( D_w^2 U_w^2 \right) = 4E(U_o - U_w) D_w U_o. \quad (10)$$

Equations (8) and (10) admit an analytical solution. Multiplying (8) by $U_o$ and subtracting (10), the right-hand sides cancel out, leaving

$$U_o \frac{d}{dx} (D_w^2 U_w) - \frac{d}{dx} (D_w^2 U_w^2) = 0. \quad (11)$$

This can be integrated to give

$$D_w^2 U_w (U_o - U_w) = C_1, \quad (12)$$

where $C_1$ is a constant. The left hand-side of (12) is the momentum deficit (2), divided by $\rho \pi^2$. Equation (12) therefore proves that the momentum deficit associated with the entrainment wake model is indeed constant. The value of $C_1$ can be found by requiring that the momentum deficit
must equate the thrust \( T = C_t \frac{1}{2} \rho U_o^2 \frac{x}{4} D^2 \), such that (12) becomes, after dividing both sides by \( \rho U_o^2 \frac{x}{4} D^2 \)

\[
\left( \frac{D_w}{D} \right)^2 \frac{U_w}{U_o} \left( \frac{U_o - U_w}{U_o} \right) = \frac{C_t}{2}.
\]  

Equation (13) can be used to eliminate \( D_w \) from the momentum equation (2), by first solving (13) for \( D_w \)

\[
D_w = \left( \frac{C_t}{2} \right)^{1/2} \left[ \frac{U_o}{U_w} \left( \frac{U_o}{U_o - U_w} \right) \right]^{1/2}.
\]  

Substituting (14) into (10) and simplifying,

\[
\frac{d}{dx} \left( \frac{U_w}{U_o - U_w} \right) = \frac{4E}{D} \left( \frac{2}{C_t} \right)^{1/2} \left( \frac{U_o - U_w}{U_w} \right)^{1/2}.
\]  

Introducing the substitution

\[
f = \frac{U_w}{U_o - U_w},
\]  

equation (15) becomes

\[
\frac{df}{dx} = \frac{4E}{D} \left( \frac{2}{C_t} \right)^{1/2} f^{-1/2},
\]  

which can be integrated to give

\[
\frac{2}{3} f^{3/2} = 4E \left( \frac{2}{C_t} \right)^{1/2} \frac{x}{D} + C_2 \equiv 4E \left( \frac{2}{C_t} \right)^{1/2} \left( \frac{x - x_v}{D} \right),
\]  

where the constant of integration \( C_2 \) has been re-written in terms of a ‘virtual origin’ \( x_v \), which is a hypothetical location upstream where the wake would attain \( U_w(x_v) = 0 \). (This approach is common, for example, in models of plumes and jets [9].) To verify \( U_w(x_v) = 0 \), use the definition for \( f \) from (16) into (18), and multiply by 3/2

\[
\left( \frac{U_w}{U_o - U_w} \right)^{3/2} = 6E \left( \frac{2}{C_t} \right)^{1/2} \frac{(x - x_v)}{D}.
\]  

To calculate the constant of integration \( x_v \), use (18) and require that at a location \( x_i \) shortly downstream of the rotor (which is at \( x = 0 \)) we have \( U_w(x = x_i) = U_o(1 - 2a) \), consistently with actuator disc theory. The value of \( x_i \) may be selected from a model for wake expansion, or may be set to zero consistently with typical implementations of the park and Frandsen models. Therefore, setting \( U_w(x = x_i) = U_o(1 - 2a) \) in (18) and solving for \( x_v \),

\[
\frac{x_v}{D} = \frac{x_i}{D} - \frac{(1 - 2a)^{3/2}(1 - a)^{1/2}}{12aE}.
\]  

To use \( C_t \) throughout the solution instead of \( a \), employ \( C_t = 4a(1 - a) \) to write \((1 - a) = C_t/(2a) - 1\), as well as \((1 - a) = C_t/(4a) \) and \( a = \frac{1}{2}(1 - \sqrt{1 - C_t}) \)

\[
\frac{x_v}{D} = \frac{x_i}{D} - \frac{(1 - C_t)^{3/4}(2C_t)^{1/2}}{12E [1 - (1 - C_t)^{1/2}]^{3/2}}.
\]
For brevity, we define a rescaled $x$-variable

$$X = 6E \left( \frac{2}{C_t} \right)^{1/2} \frac{(x - x_v)}{D},$$

(22)

$$= 6E \left( \frac{2}{C_t} \right)^{1/2} \frac{(x - x_i)}{D} + \frac{(1 - C_t)^{3/4}}{[1 - (1 - C_t)^{1/2}]^{3/2}},$$

(23)

such that solving (19) for $U_w/U_o$ yields

$$\frac{U_w}{U_o} = \frac{X^{2/3}}{X^{2/3} + 1}.$$  

(24)

The wake diameter is found by substituting $U_w/U_o$ from (24) into (14). After simplifying, and summarizing also the expressions for $U_w$ and $X$, for convenience,

$$\frac{D_w}{D} = \left( \frac{C_t}{2} \right)^{1/2} \frac{X^{2/3} + 1}{X^{1/3}}, \quad U_w \frac{U_o}{U_o} = \frac{X^{2/3}}{X^{2/3} + 1},$$

(25)

$$X = 6E \left( \frac{2}{C_t} \right)^{1/2} \frac{(x - x_i)}{D} + \frac{(1 - C_t)^{3/4}}{[1 - (1 - C_t)^{1/2}]^{3/2}},$$

where, once again, $x_i$ is the location where streamtube expansion associated with canonical actuator disc theory is assumed complete.

After obtaining (25) independently, the author found a paper of Morton [11], which provided the same result, through a different derivation, in the context of a broader discussion of entrainment models. Morton’s paper appears to have been overlooked by the fluid dynamics and wind energy communities; according to Web of Science, it has accumulated only 16 citations over 57 years. The next section provides the first validation of this model.

4. Comparison to data and to other one-parameter wake models

To provide a consistent comparison between the three models, set $x_i = 0$ for all wake theories. As noted earlier, this practice is consistent with common implementations of the park model [2]. In order to test performance in realistic conditions, data is extracted from the literature, for field observations from full-scale turbines. To ensure a robust comparison between models and field data, $U(x, r, \theta)$ is averaged over the surface of a ‘virtual’ rotor at each $x$-location, as sketched in the inset of figure 3,

$$\langle U \rangle_{\text{ rotor}}(x) = \frac{4}{\pi D^2} \int_0^{2\pi} \int_0^{D/2} U(x, r, \theta) r \, dr \, d\theta.$$  

(26)

The requirement that $U(x, r, \theta)$ be available over a transverse cross-section of the flow (rather than, say, only centerline measurements or velocity profiles) somewhat limits the data that can be used for this purpose. We employ field data collected by [17] (from their figure 5) as well as by [18] (based on power reduction between their turbines T41 and T42). The turbine-scale Reynolds number $Re = U_o D/\nu$ (with $\nu$ the kinematic viscosity) is approximately $6 \times 10^7$. In addition, the laboratory results of [18] are added, which have $Re \simeq 2 \times 10^4$ (specifically from their figure 7, which reports $U$ over a transverse cross section at $x = 5D$).

The measurements above describe $U$ in the wake. Since the flow of kinetic energy is proportional to $\langle U \rangle_{\text{ rotor}}^3$, the normalized wake power content $(\langle U \rangle_{\text{ rotor}}^3/U_o)^3$ is plotted versus $x/D$, shown by the symbols in figure 3. Each wake model is fitted with respect to their respective free parameter, finding the values $k = 0.067$, $k^* = 0.027$, $E = 0.15$.
Figure 3. Power available in a turbine wake: comparison between one-parameter wake models (showing the original park model [1], the momentum-conserving model of reference [5], as well as the entrainment model) and data from field measurements [18, 17] and from laboratory experiments [18]. Each model has been fitted with respect to one parameter, which determines the spreading rate, yielding $k = 0.067$, $k^* = 0.027$ and $E = 0.15$.

5. Discussion: linearization, asymptotic behavior, and relation between $k$ and $E$

It is interesting to examine the trends exhibited by the entrainment model in the light of the literature on axisymmetric turbulent wakes. Uberoi & Freymuth [19] examined the wake of a sphere at $Re \sim 8,600$, where they observed self-similarity for $x/D \gtrsim 50$, with $D_w \sim x^{1/3}$, consistently with canonical theory for turbulent axisymmetric wakes; this scaling was confirmed by subsequent experiments using discs and porous obstacles [7]. More recently, Eames et al. [20] proposed that wakes developing with a strongly turbulent inflow will scale as $D_w \sim x$. This scaling was supported by sphere wake data from [21] for $x/D \lesssim 6$, with $220 \lesssim Re \lesssim 1,100$ and turbulence intensity between 15% and 26%. Subsequently, [22] proposed that the $D_w \sim x$ scaling should also hold in wind turbine wakes. They presented experimental measurements behind a porous obstacle for $1 \lesssim x/D \lesssim 8$, at $Re \sim 34,000$ and with turbulence intensity of 18%. Rind & Castro [23] noted that the experiments supporting $D_w \sim x$ were performed for $x/D$ values smaller than traditionally considered ‘far wake’ behavior in the free shear flow literature. In [23], they performed Direct Numerical Simulations (DNS, at $Re \sim 10,000$) with ambient turbulence between 2.2% and 8.7%, finding a scaling for $D_w$ between $x^{1/3}$ and $x^{1/2}$, which was confirmed by subsequent DNS studies [24]. Nedić et al. [25] proposed that turbulent axisymmetric wakes behind complex objects could scale as $D_w \sim x^{1/2}$, due to the nonequilibrium scaling followed by the dissipation in the lee of such obstacles [26]. The scaling $D_w \sim x^{1/2}$ was supported by experiments at $Re \sim 80,000$, turbulence intensity around 5%, over the range $10 \lesssim x/D \lesssim 50$. In all of these studies, the turbulence lengthscale was of the order of the size of the obstacle.

In light of the findings of [20], and of the fact that several wake models assume a linearly-spreading wake [6], it is interesting to examine the result of linearizing the wake expansion in the entrainment model (25). Setting $x_i = 0$, noting $D_w$ has no singularities for $x > 0$, a Taylor
Define $D_{w,lin}$ as the linearly-spreading wake that results from neglecting the higher-order terms in (27). To choose a value of $x_r$, require that the first term on the right-hand side of (27) must equal the initial wake diameter $D_i/D = \sqrt{(1-a)/(1-2a)}$. This sets $x_r = x_{lin}$. Then

$$\frac{D_{w,lin}}{D} = \frac{D_i}{D} + 2k_E x_{lin},$$

where $k_E = E\frac{\zeta^{2/3} - 1}{\zeta^{4/3}}$, (29)

where $x_{lin}$ is found by solving (note that $\zeta$ also depends on $x_{lin}$, according to (28))

$$\left(\frac{C_l}{2}\right) \frac{1}{\zeta^{2/3} + 1} - 2E x_{lin} \frac{\zeta^{2/3} - 1}{\zeta^{4/3}} = \frac{D_i}{D}.$$  

(30)
Equation (30) is solved numerically for various values of $E$ and $C_t$. For $C_t = 0.8$ and $E = 0.15$, this yields $k_E = 0.031$, which is close to $k^* = 0.027$ found in section 4. The fact that $k_E$ is weakly dependent on $C_t$ is consistent with published data on turbulent wakes [7, 20, 25]. This relation between $k_E$ and $E$ enables relating the spreading parameter from the Frandsen model to a more general turbulence parameterization.

The value of $U_{w,lin}$ is found from $D_{w,lin}$ using momentum conservation (13). Figure 4 compares $U_w$ and $D_w$ from all models considered so far, for the range $0 < x/D < 8$ (as considered by [22]). The entrainment model is consistent with the linear wake scaling in this region.

Since the term ‘far wake’ has been used to imply widely different $x/D$ ranges, the limit $x/D \to \infty$ here is denoted as the ‘asymptotic wake’. The entrainment model exhibits canonical scalings

$$\lim_{x/D \to \infty} \frac{D_w}{D} = \left(\frac{C_t}{2}\right)^{1/2} X^{1/3}, \quad \lim_{x/D \to \infty} \frac{U_w}{U_o} = 1 - X^{-2/3}. \quad (31)$$

For the full solution (25) for $D_w$ to be within 5% of the asymptotic prediction (31), one needs $x/D \gtrsim 55$ (with $C_t = 0.8$ and $E = 0.15$), implying that this asymptotic scaling emerges at very large $x/D$. Overall, it is not the purpose of this paper to make an argument for the applicability of one asymptotic wake scaling over another, for wind turbine wakes. For wind energy purposes, it is possible that concerns about asymptotic wake limits may be largely academic, and that the choice of a wind turbine wake model over another may ultimately be reasonably guided by ease of use and agreement with data.

6. Outlook
This paper shows that the entrainment hypothesis yields an axisymmetric wake model, which does not assume a globally valid relation between wake diameter and downstream distance. The entrainment wake model has an analytical solution, which is in good agreement with field measurements. The derivation shown here assumes the coefficient $E$ does not change with $x$. This is consistent with the typical assumption, in the park and Frandsen wake models, that $k$ and $k^*$ are independent of $x$. It is in principle possible to let $E$ depend on $x$, integrating (8) and (10) numerically, albeit at the cost of additional complexity.

Often $k$ and $k^*$ are assumed to depend on ambient turbulence levels or effective roughness $z_0$ of the terrain, and empirical parameterizations have been constructed to express this dependence [6]. One can seek similar parameterizations for $E$. It is worthwhile noting that in section 4 we found $E = 0.15$ according to field measurements, corresponding to a situation with high-$Re$ and finite ambient turbulence. The value $E = 0.15$ therefore is likely to be close to an upper bound. Interestingly, measurements of entrainment in gravity currents and canopies also find that $E$ is at most approximately 0.15 [14, 27]. Laboratory experiments may expect smaller $E$s, especially in cases involving laminar inflow or stable stratification. It may be interesting to use wind turbine wake data to systematically test, and possibly refine, existing parameterizations for $E$ from geophysical fluid dynamics.

It is worthwhile noting that it is, in principle, possible to retain pressure gradient terms in the derivation of the momentum equation. The resulting set of governing equations may not admit an analytical solution, but may be integrated numerically, potentially enabling modeling wakes over complex terrain. It would also be valuable to introduce an equation for lateral momentum, as this may enable considering wake evolution behind a yawed rotor.

Acknowledgments
This work was partially supported by Churchill College, Cambridge through a Junior Research Fellowship, as well as by a Faculty Research Grant from the Academic Senate of the University of California, Santa Barbara. The author gratefully acknowledges interesting conversations on
the subject of entrainment with C.P. Caulfield and C. Cenedese, and thanks P. Rajborirug, N. Rommelfanger and one anonymous referee for helpful comments.

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