On Hardness of the Joint Crossing Number

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1 Two Planar Graphs in the Plane

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- Silly question → of course, no crossings are needed!
Two Embedded Graphs in one Surface

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→ Hence, indeed, some mutual crossings are needed even if each one of the two graphs (itself) embeds there.
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An easy solution?

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- NO; this tempting toroidal example is very misleading!
2 Joint Embedding: a Brief History

To minimize the number of mutual edge crossings in a joint embedding of two graphs (say, red and blue) in one common surface.

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  A general upper bound of $< 4g \cdot \beta(G_1)\beta(G_2)$.  

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- [Richter–Salazar, 2005]: Disproving the A.–B. conjecture in the double-torus, a replacement conjecture given. Improved Negami’s upper bound wrt. representativity.
- And more. . .?
Joint Embedding: Formal Definitions

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    and so it makes better sense to prove hardness without assuming artificial restrictions, but make the construction working with all the restrictions (e.g., homeomorphism).
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- **Face-anchored joint embedding** problem = prescribed faces of the blue graph must hold assigned vertices of the red graph.

- Need to show that **face-anchors** can be enforced in a joint embedding.
Highly Entwined Drawings, I

Getting to the plane

How?
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Getting to the plane

How? Use the following *gadget* for each face-anchor (the anchor is *thick red*):
The gadget and the construction

- Make the original blue and red edges medium thick.
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- Playing slightly with the weights of the red $K_{3,3}$ s and the blue grids, we can enforce a precise *one-to-one assignment* (and no other permutations).
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More entwined with less handles

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The high level idea of anchor multiplication – a multi-anchor gadget:

- Only *four* face-anchors are used to tie down two long vertex sequences.
The multi-anchor gadget

1. Make the base **blue frame** very thick to thick:
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3. Adjust weights on the horizontal red and on new (med.-light) vertical blue bars to enforce unique even distribution of the red ladder vertices.
Gadget details

How thick the edges are? \( T \gg k \gg 1 \)
5 Anchored Hardness Reduction

[Cabello–Mohar] (2012): Anchored planar joint crossing number is NP-hard:
Using our multi-anchor gadget

• **Anchored planar drawing** (by [Cabello–Mohar]):
  a drawing of $G$ in the unit disc such that selected vertices $A \subseteq V(G)$
  appear in the prescribed order on the disc boundary.
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- Two copies of the gadget to emulate the four sides of the C.–M. constr.:
Putting all together

Double multi-anchor + Cabello–Mohar
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= hardness of the joint crossing number with 6 face-anchors in the plane.
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Double multi-anchor + Cabello–Mohar

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**Theorem.** Joint Crossing Number, Joint Homeomorphic Crossing Number, and Joint OP-Homeomorphic Crossing Number are NP-hard problems in any orientable surface of genus 6 or higher. This remains true even if the inputs are restricted to simple 3-connected graphs.
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2. We can improve down to genus 4 (both orientable and non-orientable) – this uses a differently shaped multi-anchor gadget, though based on the same ideas as above.
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4. Consequently, from (3.) we get the following new result (almost-planar):

**Theorem.** Let \( G \) be a planar graph with only 16 vertices of degree \( > 3 \), and \( x, y \in V(G) \). Then it is NP-hard to decide the crossing number of \( G + xy \).
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(Previously, [Cabello–Mohar] required an unlimited number of degrees $> 3$.)
The improved multi-anchor gadget

Just a simple sketch...
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Thank you for your attention.