Entanglement properties of the two-dimensional SU(3) AKLT state

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Two-dimensional (spin-2) Affleck-Kennedy-Lieb-Tasaki (AKLT) type valence bond solids on the square lattice are known to be symmetry protected topological (SPT) gapped spin liquids [Shintaro Takayoshi, Pierre Pujol, and Akihiro Tanaka Phys. Rev. B 94, 235159 (2016)]. Using the projected entangled pair state (PEPS) framework, we extend the construction of the AKLT state to the case of SU(3), relevant for cold atom systems. The entanglement spectrum is shown to be described by an alternating SU(3) chain of “quarks” and “antiquarks”, subject to exponentially decaying (with distance) Heisenberg interactions, in close similarity with its SU(2) analog. We discuss the SPT feature of the state.

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properties. In addition, the PEPS framework enables to compute entanglement properties [21, 38] – entanglement spectrum (ES), entanglement Hamiltonian (EH), etc... – in a very efficient way. It turns out that the 1D or 2D SU(2) AKLT states have extremely simple representations in terms of MPS [23] and PEPS [21], respectively, which make the analysis of their bulk and edge properties accurately computable.

Although AKLT parent Hamiltonians are fine-tuned, the AKLT states provide in fact simple paradigms for the simplest (non-topological) gapped spin liquid phases, which can occupy a rather extended region in the parameter space of realistic Hamiltonians. For example, the 1D spin-1 AKLT state corresponds to a special point of the well-known extended Haldane phase describing several experimental spin-1 chains. Since localized SU(N) spin systems can now be realized on optical 1D and 2D lattices, SU(N) AKLT states are expected to describe generic spin liquid phases in such systems and are therefore of high interest. In the case of a SPT phase, the edge modes of the AKLT wave function will also be generic of the whole phase, being protected by symmetry. In this rapid communication, we extend the 2D AKLT state to the case of SU(3) symmetry. We show that it can be represented as a simple tensor network, allowing for extensive studies. We explore its bulk properties on an infinite cylinder, using transfer matrix methods. The edge physics is investigated by computing the entanglement spectrum and the related entanglement Hamiltonian. We show that the latter can be very well approximated by a simple SU(3) Heisenberg Hamiltonian with exponentially decaying interactions.

**SU(3) AKLT wavefunction.** We now extend the recipe for the construction of SU(2) AKLT states to SU(3), in a straightforward way. In that case, we use standard Young tableau notations to label the SU(3) irreps or “spins” (also denoted by their dimension in bold). First, in order to realize SU(3) singlets on all NN bonds of the square lattice, four “quarks” in the fundamental [1] = 3 irrep (“ antiquarks” in the anti-fundamental [1] = 3 irrep) are attached on each even (odd) site. This way, neighboring virtual spins on every NN bond belong to 3 and 3 irreps and can then be projected onto SU(3) [1, 1, 1] = 1 singlets. Then, in order to entangle this simple product of singlets, one projects the group of four quarks on each even (odd) site onto the most symmetric [4] = 15 ([4, 4] = 15) irrep corresponding to the actual physical degrees of freedom, as seen in figure 2(a). Note that the assignment as fundamental or anti-fundamental is arbitrary, the same tensor being placed on every site. As for SU(2), a simple parent Hamiltonian can be built from bond projectors on the largest, most-symmetric [8, 4] (self-conjugate) irrep obtainable from the tensor-product 15 ⊗ 15.

\[
H_{SU(3)}^{2D} = \sum_{(i,j)} P_{i,j}^{[8,4]},
\]

where the sum runs over all NN bonds.

**Description of the PEPS formalism.** For simplicity, let us first start with a periodic (L-site) 1D chain with d onsite physical degrees of freedom labeled by α (e.g. the components of the physical spin). By definition, the amplitudes \( c_{\alpha_1, \alpha_2, \ldots, \alpha_L} \) of a (translationally-invariant) MPS of virtual dimension D are given solely in terms of d \( \times \) D matrices \( A^\alpha \) as \( c_{\alpha_1, \alpha_2, \ldots, \alpha_L} = \text{Tr}\{A^{\alpha_1}A^{\alpha_2}\cdots A^{\alpha_L}\} \). It is easy to see that the 1D SU(2) AKLT state of Fig. 1(a) is in fact a MPS defined from a set of three \( 2 \times 2 \) matrices labelled by the physical spin (i.e. \( d = 3 \) and \( D = 2 \)). This construction can easily be generalized in 2D by replacing the d matrices by d rank-z tensors, where z is the lattice coordination number (z = 4 in our case). The amplitudes of the PEPS are then obtained from the tensor network defined by attaching a tensor on each lattice site and by contracting the site tensors over the virtual indices [32–36]. The \( S = 2 \) AKLT state of Fig. 1(b) can then be viewed as a simple PEPS with \( D = 2 \) virtual degrees of freedom (corresponding to the attached virtual spin-1/2) and \( d = 2S + 1 = 5 \) physical spin components [21]. Similarly, the SU(3) AKLT state of Fig. 2(b) can be interpreted as a PEPS of virtual dimension \( D = 3 \) (for the three colors of the quarks) and \( d = 15 \) physical dimension, as depicted in Fig. 2(b).

In practice, one needs to compute the PEPS wave function norm \( \langle \Psi | \Psi \rangle \) or expectation values \( \langle \Psi | O | \Psi \rangle \) of local operators \( O \). For such purpose, one first defines a two-layer tensor network, each layer representing the ket and bra wave functions. By contracting two identical tensors on their physical indices one gets a new tensor \( z \) of dimension \( D^z = 9 \).

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**FIG. 2.** (Color online) (a,b) The AKLT SU(3) wave function is defined similarly to the SU(2) case: four virtual states in the fundamental (anti-fundamental) irrep of SU(3) of dimension \( D = 3 \), are attached on even (odd) sites and projected onto the fully symmetric 15 (T5) irrep. Virtual states of all neighboring sites are projected on SU(3) singlets to form a tensor network. (c) By contracting two identical site tensors on their physical indices one gets a new tensor \( z \) of dimension \( D^z = 9 \).
and right boundary states are identical.

![Fig. 3](image)

**FIG. 3.** The fixed-point boundary state is defined as the leading eigenvector of the transfer matrix. The latter is defined by contracting the local $E$ tensor along a circle, leaving the left and right legs open.

**FIG. 4.** (Color online) (a) Bulk gap of an infinite AKLT SU(N) cylinder vs circumference $N_v$. The extrapolated $N_v \to \infty$ values of $\xi = 1/\Delta$ are shown on the plot. (b) Coefficients of the effective entanglement Hamiltonian (decomposed in term of Heisenberg-like operators) for SU(2) and SU(3) AKLT wavefunctions vs site separation (semi-log plot). Straight lines are fits according to an exponential behavior $J(r) = J_0 \exp(-r/\lambda)$. Data for SU(2) are taken from reference [21].

**Bulk properties.** The gap $\Delta$ in the bulk can easily be computed from the two largest eigenvalues of the transfer matrix, $\Delta = \ln(E_1/E_2)$, with $E_1 > E_2$, the correlation length $\xi$ being defined as the inverse of the gap. We have computed $\Delta$ for cylinders of perimeter $N_v = 2, 4, 6, 8$ and extrapolated the result in the limit $N_v \to \infty$, as shown in figure 4(a). We find that the extrapolation of $\xi$ for the SU(3) case is very short ($\xi_3 \approx 1.2$), even shorter than the SU(2) value ($\xi_2 \approx 2.1$). Note that the extrapolation is very accurate, the scaling being exponential and the system size being large compared to $\xi$.

**Entanglement Hamiltonian and entanglement spectrum.** In order to construct the entanglement Hamiltonian (EH), the fixed-point state (see above) is reshaped as a $D^{N_v} \times D^{N_v}$ boundary density matrix $\Sigma_b$, acting on virtual variables. It has previously been shown [21] that this matrix can be mapped onto the reduced density matrix of the half cylinder $\rho$ via an isometry, $\rho = U^\dagger(\Sigma_b)U$. The entanglement Hamiltonian $\mathcal{H}$ acting on virtual boundary configurations is defined via $(\Sigma_b)^2 = \exp(-\mathcal{H})$.

The spectrum of $\mathcal{H} - $ the entanglement spectrum (ES) – has been conjectured by Li and Haldane [39], to be in one-to-one correspondence with the physical edge modes of the system. We compare the ES of SU(2) and SU(3) AKLT wavefunctions in figure 5. We observe they are very much similar at low energy: (i) the ground state is a singlet with momentum $k = 0$ (when $N_v = 4n$), (ii) low-energy excitations follow a sinusoidal dispersion typical of the lower edge of a 2-spinon continuum, shown in figure 5(c). This can be explained from the simple (approximate) analytical form of the EH (derived next).

To understand its nature we decompose the EH on the canonical basis of SU(3) operators acting on the virtual degrees of freedom at the boundary. The latter are being defined in a fermionic representation as

$$S^\alpha_\beta(i) = \left\{ \begin{array}{ll} c^\dagger_{\alpha,i} c_{\beta,i} - \delta_{\alpha,\beta}/3 & \text{if } i \text{ is even} \\ c^\dagger_{\alpha,i} c_{\beta,i} - \delta_{\alpha,\beta}/3 & \text{if } i \text{ is odd} \end{array} \right. \quad (2)$$

where $\alpha, \beta$ label the three SU(3) colors. Note that the definition takes into account the anti-fundamental representation on odd sites [9], which in the fermion language is obtained via a particle-hole transformation. Since the Hamiltonian is SU(3) invariant, there is a limited number of combination of operators that can appear, in particular no linear term can appear. The only second order SU(3) invariant terms are Heisenberg-like terms, $S_i \cdot S_j = \sum_{\alpha,\beta} S^\alpha_\beta(i)S^\beta_\alpha(j)$. Hence,

$$\mathcal{H} = E_0 + \sum_{i \neq j} J(|i-j|) S_i \cdot S_j + \mathcal{H}_{\text{rest}} \quad (3)$$

where $E_0 = \text{Tr}(\mathcal{H})$. The higher order terms $\mathcal{H}_{\text{rest}}$ are corrections of much lower weights – only 5% (6%) of the euclidean norm of $\mathcal{H} - E_0$ for $N_v = 8$ ($N_v = 2$) – and are expected to be irrelevant. We show in figure 4(b) that the weights $J(r)$ follow an exponential decay with distance, from with we can extract a typical decay length $\lambda$. By comparing SU(3) and SU(2), we see that $\lambda_3 < \lambda_2$, fulfilling the same inequality than the bulk correlation length $\xi_3 < \xi_2$. This is in agreement with a general argument based on PEPS that the range $\lambda$ of the EH tracks the bulk correlation length $\xi$ [21].

**Discussion and outlook.** Interestingly, the EH of the SU(3) AKLT state is adiabatically connected to the nearest neighbor 3–3 Heisenberg chain [9]. The latter can be mapped to a spin-1 chain with a purely negative bi-quadratic coupling and was shown to exhibit a small spontaneous dimerization [40–43]. It is however plausible that the extra $J(2) \sim 0.3 J(1)$ coupling will close the gap and lead to a gapless spectrum. Indeed, the numerical ES shown in Figs. 5(b,c) does not show any hint of spontaneous translation symmetry breaking (implying GS two-fold degeneracy in the $N_v \to \infty$ limit). The conformal field theory (CFT) description of our EH is an
open problem which would require the numerical treat-
ment of very long chains. Interestingly, the parent Ham-
iltonian [44, 45] for a CFT wave function constructed from
the $SU(3)$ Wess-Zumino-Witten (WZW) models [46] is,
onece truncated, quite similar to our quasi-local EH, al-
though with a larger ratio $J(2)/J(1) \simeq 0.56$ and a 3-body
term of significant amplitude. Hence a description of the
EH in terms of a $SU(3)$ WZW theory seems natural and,
at least, agrees with our low-energy ES shown in figure
5(c). Tu et al. [44] report critical properties deviating
from the expected behaviors of the $SU(3)$ WZW model.
We note however that the two (local) models may sit in
different critical phases.

Another interesting question is the possible correspon-
dence between the ES and the edge physics [39]. As for
the $SU(2)$ AKLT state, one can construct a local $SU(3)$-
variant parent Hamiltonian or “PEPS model” [47, 48]
for which, any region with an open 1D boundary $\partial R$
will have a degenerate manifold of (at most) $D^{3\text{D}}$ AS.
For any PEPS models in a trivial (i.e. short-ranged en-
tangled) phase, any Hamiltonian can be realized on the
edge [48] by slightly perturbing the (fine-tuned) $SU(3)$
PEPS model. However, it is still possible to protect edge
properties by symmetries in the bulk [24]. For example,
similarly to the $SU(2)$ AKLT model, $SU(3)$ symmetry
and translation invariance rule out a gapped edge which
does not break any symmetry [28]. This is in direct cor-
respondence with the properties of the (infinite size) ES
discussed above.

Lastly, we comment on the relevance of this work to
cold atoms physics. Constructing bipartite lattices of
localized $SU(3)$ atoms in staggered conjugate irrep is
possible experimentally although challenging [49]. It is
also of interest to enforce the same irrep on every site.
For this goal, a different AKLT construction exits, involv-
ing virtual states belonging to the smallest self-conjugate
irrep. For this goal, a different AKLT construction exits, involv-
ing virtual states belonging to the smallest self-conjugate
irrep. For $SU(3)$ it corresponds to the adjoint $[2, 1]$ (8-
dimensional) irrep. The case of $SU(4)$ would be sim-
pler using the self-conjugate (antisymmetric) $[1, 1]$ (6-
dimensional) [15] irrep for the virtual states. The phys-
ical site degrees of freedom on a 2D square lattice cor-
respond to atoms in the $[4, 4]$ (105-dimensional) irrep of
$SU(4)$.

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