ON EXPONENTIAL DECAY RATE OF SEMIGROUP ASSOCIATED WITH SECOND ORDER LINEAR DIFFERENTIAL EQUATION IN HILBERT SPACE WITH STRONG DAMPING OPERATOR

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Abstract. We obtain estimate of the exponential decay rate of semigroup associated with second order linear differential equation
\[ u'' + Du' + Au = 0 \]
in Hilbert space. We assume that \( A \) is a selfadjoint positive definite operator, \( D \) is an accretive sectorial operator and \( \text{Re} D \geq \delta A, \delta > 0 \). We obtain a location of the spectrum of a pencil associated with linear differential equation.

1. In Hilbert space \( H, (\cdot, \cdot) \) we consider a second order linear differential equation
\[ u''(t) + Du'(t) + Au(t) = 0, \tag{1} \]
here \( u(t) \) is a vector-value function on semi-axis \( \mathbb{R}_+ = [0, +\infty) \). Many evolution equations arising in mechanics can be reduced to the equation (1) in an appropriate space (see, for example, \([4, 6, 7]\)). In this case \( A \) represents potential energy and \( D \) represents dissipation (\( D \) is a damping operator). We will assume that

Condition 1. \( A \) is a selfadjoint positive definite operator with dense domain \( \mathcal{D}(A) \). Let
\[ a_0 = \inf_{x \in \mathcal{D}(A), \|x\|=1} (Ax, x) = \inf_{x \in \mathcal{D}(A^{1/2}), \|x\|=1} (A^{1/2}x, A^{1/2}x) > 0. \]

By \( H_s \) we denote a collection of Hilbert spaces generated by \( A^{1/2} \), i.e. for \( s \geq 0 \) the space \( H_s \) is the domain \( \mathcal{D}(A^{s/2}) \) endowed with the norm \( \|x\|_s = \|A^{s/2}x\| \), for \( s < 0 \) the space \( H_s \) is the completion of \( H \) with respect to the norm \( \| \cdot \|_s \). By definition \( H_0 = H, H_1 = \mathcal{D}(A^{1/2}), H_2 = \mathcal{D}(A) \) and \( H_s \leftrightarrow H_r \) for \( s > r \). Since \( |(x, y)| = |(A^{-s/2}x, A^{s/2}y)| \leq \|x\|_{-s} \cdot \|y\|_s \) for \( s > 0 \) and for all \( x \in H, y \in H_s \), then the sesquilinear form \( (x, y) \) can be extended by continuity to a sesquilinear form \( (x, y)_s \) on \( H_{-s} \times H_s \). Therefore we can regard the space \( H_{-s} \) as a

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dual of \( H_s \) \( (H_{-s} = H_s^*) \), the duality is determined by the sesquilinear form \((x, y)_{-s,s}\) (duality with respect to the pivot space \( H \)). By \( \mathcal{L}(X, Y) \) we will denote a space of bounded operators acting from a space \( X \) into a space \( Y \). Operator \( A \) can be regarded as a bounded operator acting in the collection of Hilbert spaces: \( A \in \mathcal{L}(H_s, H_{s-2}) \forall s \in \mathbb{R} \).

Following [7] we will assume that

**Condition 2.** \( D \in \mathcal{L}(H_1, H_{-1}) \) is an accretive sectorial operator, i.e. \( \text{Re}(Dx, x)_{-1,1} \geq 0 \) and \( |\text{Im}(Dx, x)_{-1,1}| \leq \nu \text{Re}(Dx, x)_{-1,1} \) for all \( x \in H_1 \) and some \( \nu > 0 \).

Denote (infimum with respect to \( x \in H_1, x \neq 0 \))

\[
\alpha = \inf \frac{\text{Re}(Dx, x)_{-1,1}}{||x||^2}, \quad \beta = \inf \frac{\text{Re}(Dx, x)_{-1,1}}{||x||^2}, \quad \delta = \inf \frac{\text{Re}(Dx, x)_{-1,1}}{||x||^2}.
\]

Inequality \( ||x||^2 \geq a_0 ||x||^2 \geq a_0^2 ||x||^2_1 \) \( (\forall x \in H_1) \) implies, that

\[
\alpha \geq a_0 \beta \geq a_0^2 \delta \geq 0.
\]

By \( ||D|| = \sup_{x \in H_1, ||x|| = 1} ||Dx||_1 \) we denote a norm of the operator \( D \). Note, that the operator \( D \in \mathcal{L}(H_1, H_{-1}) \) is accretive (sectorial) in the sense of the condition 2 iff the operator \( A^{-1/2}DA^{-1/2} \in \mathcal{L}(H) \) is accretive (sectorial).

With the linear differential equation (1) we associate a quadratic operator pencil [2] [7]

\[
L(\lambda) = \lambda^2 \mathcal{J} + \lambda D + A,
\]

here \( \mathcal{J} : H_1 \hookrightarrow H_{-1} \) is an embedding operator, \( \lambda \in \mathbb{C} \) is a spectral parameter. We regard the pencil as an operator-function \( L(\lambda) \in \mathcal{L}(H_1, H_{-1}) \). As usual one can define a resolvent set

\[
\rho(L) = \{ \lambda \in \mathbb{C} : \exists L^{-1}(\lambda) \in \mathcal{L}(H_{-1}, H_1) \}
\]

and a spectrum \( \sigma(L) = \mathbb{C}\setminus \rho(L) \) of the pencil \( L(\lambda) \).

The second order differential equation (1) can be linearized as a first order differential equation [2] [6] [7]

\[
w'(t) = \mathcal{T}w(t), \quad w(t) = (u' \quad u)^T
\]

in "energy" space \( \mathfrak{H} = H \times H_1 \) with the matrix operator

\[
\mathcal{T} = \begin{pmatrix} -D & -A \\ I & 0 \end{pmatrix}, \quad \mathcal{D}(\mathcal{T}) = \left\{ \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \in H_1 \times H_1 : Dw_1 + Aw_2 \in H \right\}
\]

Since \( \text{Re}(\mathcal{T}w, w)_{\mathfrak{H}} = -\text{Re}(Dw_1, w_1)_{-1,1} \leq 0 \) and \( 0 \in \rho(\mathcal{T}) \), then \( -\mathcal{T} \) is a maximal accretive operator and, therefore, \( \mathcal{T} \) is a generator of \( C_0 \)-semigroup \( \exp(t\mathcal{T}) \) of contractions [5]. In [2] it is shown that if the conditions 1 and 2 hold and \( \beta > 0 \), then the operator \( \mathcal{T} \) is a generator.
of exponentially decaying semigroup. In [1], in particular, it is proved that, under the conditions [1, 2] and $\delta > 0$, the operator $T$ is a generator of analytic semigroup.

In papers [3, 6] for the case $\beta > 0$ was obtained results on the exponential decay rate of the semigroup $\exp(tT)$ and on location of the spectrum of the pencil $L(\lambda)$. In paper [7] was obtained results on analyticity of the semigroup $\exp(tT)$ and on the location of the spectrum of the pencil $L(\lambda)$. In present paper using another technique for the cases $\delta > 0$ we obtain an estimate for the exponential decay rate of the semigroup generated by $T$.

2. In the space $H$ with respect to the given inner product the operator $(-T)$ is neither uniformly accretive nor sectorial. For $\theta \geq 0$ introduce a collection of sesquilinear forms

$$[w, v]_\theta = (w_1, v_1) + \theta(w_1, v_1)_- + (w_2, v_2)_+ + \theta(w_2, v_2)_- + \theta(Dw_2, Dw_2)_- +$$

$$\theta(Dw_2, v_1)_-, \ w = (w_1, w_2), \ v = (v_1, v_2) \in H,$$

here $(\cdot, \cdot)_s = (A^{s/2}, A^{s/2})$ is an inner product in the space $H_s$. Since

$$|w|^2_{\theta} = [w, w]_\theta = \|w_1\|^2 + \|w_2\|_1^2 + \theta\|w_2\|^2 + \|w_1 + Dw_2\|_{-1}^2,$$

then $[\cdot, \cdot]_\theta$ is an inner product in $H$ topologically equivalent to the given one. Obviously $[\cdot, \cdot]_0 = (\cdot, \cdot)_s$.

**Proposition 1.** Let the conditions [7] and [2] are satisfied and $\delta > 0$. Then for arbitrary $\theta > 0$ and $0 \leq b \leq \sqrt{\theta}$ for all $w = (w_1, w_2)^\top \in D(T)$ the following inequalities

$$\text{Re}[T w, w]_\theta \leq -\omega_\theta |w|_{\theta}^2, \quad |\text{Im}[T w, w]_\theta| \leq M_{\theta, b} |\text{Re}[T w, w]_\theta| + b |w|_{\theta}^2,$$

hold, where

$$\frac{1}{\omega_\theta} = \frac{1}{\beta} + \frac{\|D\|^2}{2\delta} + \frac{1}{2} \left( \frac{1}{\alpha} + \frac{1}{\theta\delta} \right) + \frac{1}{2} \sqrt{\left( \frac{1}{\alpha} + \frac{1}{\theta\delta} + \frac{\|D\|^2}{\delta} \right)}^2 - \frac{4}{\alpha\delta} > 0,$$

$$M_{\theta, b} = \nu + \frac{2}{\delta(b + \sqrt{b^2 + 4\theta})} + \frac{\sqrt{\theta - b}}{\beta}.$$

The resolvent set of the operator $T$ is non-empty, therefore

**Corollary 1.** Under the conditions of the proposition [7]

$$\sigma(T) \subset \{ \lambda \in \mathbb{C} : \text{Re} \lambda \leq -\omega_\theta, |\text{Im} \lambda| \leq M_{\theta, b} |\text{Re} \lambda| + b \}.$$

Putting $b = 0$ we obtain
Colorary 2. Under the conditions of the proposition \[7\] for all \(\theta > 0\)

\[\sigma(T) \subset \left\{ \lambda \in \mathbb{C} : \text{Re} \lambda \leq -\omega, |\text{Im} \lambda| \leq \left( \nu + \frac{1}{\delta \sqrt{\theta}} + \frac{\sqrt{\theta}}{\beta} \right) |\text{Re} \lambda| \right\}\]

It’s easy to prove \[7\], that \(\sigma(L) = \sigma(T)\).

**Theorem 1.** Let the conditions \[4\] and \[5\] are satisfied and \(\delta > 0\). Then

the operator \(T\) is a generator of the (analytic) semigroup \(\exp(tT)\) in

the space \(\mathcal{H}\) with exponential decay rate

\[\omega = \left( \frac{1}{\beta} + \frac{2}{\sqrt{\alpha \delta}} + \frac{\|D\|^2}{2\delta} + \sqrt{\frac{4\|D\|^2}{\delta \sqrt{\alpha \delta}} + \frac{\|D\|^4}{\delta^4}} \right)^{-1} > 0,\]

i.e. for all \(t \geq 0\) the inequality \(\|\exp(tT)\|_{\mathcal{H}} \leq \text{const} \cdot \exp(-\omega t)\) holds.

For all \(b \geq 0\)

\[\sigma(L) = \sigma(T) \subset \{ \lambda \in \mathbb{C} | \text{Re} \lambda \leq -\omega, |\text{Im} \lambda| \leq M_b |\text{Re} \lambda| + b \}\]

where \(M_b = \min_{\theta \geq b^2} M_{\theta, b}\).

Colorary 3. Under the conditions of the theorem \[7\] for all \((u_1, u_0)^T \in D(T)\) there exists a unique solution \(u(t)\) of the Cauchy problem for the
differential equation \[11\] with initial conditions \(u(0) = u_0, u'(0) = u_1\)

and

\[\|u(t)\|_1^2 + \|u'(t)\|_1^2 \leq \text{const} \cdot \exp(-2\omega t)(\|u_0\|_1^2 + \|u_1\|_1^2).\]

Putting \(b = 0\) we have

Colorary 4. Under the conditions of the theorem \[7\]

\[\sigma(L) = \sigma(T) \subset \{ \lambda \in \mathbb{C} | \text{Re} \lambda \leq -\omega, |\text{Im} \lambda| \leq \left( \nu + \frac{2}{\sqrt{\delta \beta}} \right) |\text{Re} \lambda| \}.\]

**Remark 1.** In \[7\] under the condition of the theorem \[4\] was obtained
the following location of the pencil’s spectrum

\[\sigma(L) = \sigma(T) \subset \{ \lambda \in \mathbb{C} | \text{Re} \lambda \leq 0, |\text{Im} \lambda| \leq \nu |\text{Re} \lambda| + \delta^{-1} \}\]

\[\sigma(L) = \sigma(T) \subset \left\{ \lambda \in \mathbb{C} | \delta \leq \frac{|\text{Re} \lambda|}{a_0^{-2} + |\lambda|^{-2}} \right\}\]

3. With the operator \(T\) in the space \(\mathcal{H}\) endowed with the inner
product \[\langle \cdot, \cdot \rangle_\theta\] we can associate a linearization of the pencil \(L(\lambda)\) under
the form \(L(\lambda) = \lambda Q - T\), where

\[Q = \begin{pmatrix} I + \theta A^{-1} & \theta A^{-1} D \\ \theta D^* A^{-1} & A + \theta I + \theta D^* A^{-1} D \end{pmatrix} \quad T = \begin{pmatrix} -D & -A - \theta I \\ A + \theta I & -\theta D^* \end{pmatrix} .\]
The linearization $L(\lambda)$ can be regarded as an operator-function $L(\lambda) \in L(H_1 \times H_1, H_{-1} \times H_{-1})$.

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