We review the framework we and our collaborators have developed for the study of one-loop quantum corrections to extended field configurations in renormalizable quantum field theories. We work in the continuum, transforming the standard Casimir sum over modes into a sum over bound states and an integral over scattering states weighted by the density of states. We express the density of states in terms of phase shifts, allowing us to extract divergences by identifying Born approximations to the phase shifts with low order Feynman diagrams. Once isolated in Feynman diagrams, the divergences are canceled against standard counterterms. Thus regulated, the Casimir sum is highly convergent and amenable to numerical computation. Our methods have numerous applications to the theory of solitons, membranes, and quantum field theories in strong external fields or subject to boundary conditions.

1. Introduction

In these talks, we describe our project to develop reliable, accurate, and efficient techniques for a variety of calculations in renormalizable quantum field theories in the presence of background fields. These background field configurations need not be solutions of the classical equations of motion. Our calculations are exact to one loop, allowing us to proceed where perturbation theory or the derivative expansion would not be valid. For example, in a model with no classical soliton we can demonstrate the existence of a non-topological soliton stabilized at one loop order by quantum fluctuations. We renormalize divergences in the conventional way: by combining counterterms with low order Feynman diagrams and satisfying renormalization conditions in a fixed scheme. In this way we are certain that the theory is being held fixed as the background field is varied. Our methods are

*Based on talks presented by the authors at the 5th workshop ‘QFTEX’, Leipzig, Sept. 2001.
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also efficient and practical for numerical computation: the quantities entering the numerical calculation are cutoff independent and do not involve differences of large numbers. The numerical calculations themselves are highly convergent.

Our methods are limited to one loop and, except in special cases, to static field configurations. We also require the background field configuration to have enough symmetry that the associated scattering problem admits a partial wave expansion. The one-loop approximation includes all quantum effects at order $\hbar$. It is a good approximation for strong external fields or when the number of particles circulating in the loop becomes large. Even when it cannot be rigorously justified, the one-loop approximation can provide insight into novel structures in the same way that classical solutions to quantum field theories have done in the past. We can address a wide variety of problems, including

- The stabilization of solitons by quantum effects in theories that do not have classical soliton solutions.
- The direct calculation of induced charges, both fractional and integer, carried by background field configurations.
- The analysis of the divergences and physical significance of calculations of vacuum energies in the presence of boundaries — i.e. the traditional “Casimir effect”.
- The computation of quantum fluctuations in strong external fields.
- Quantum contributions to the properties of branes and domain walls.

This report presents an introduction to our methods and examples of applications. Sections 2–4 lay out the method and address renormalization and computational efficiency. Sections 5–9 introduce examples. In Section 2 we describe our method in general terms. In Section 3 we illustrate the method with the case of a charged boson field in a bosonic background in three spatial dimensions. We also explain how to compute phase shifts and their Born approximations efficiently. In Section 4 we turn to renormalization using methods adapted from dimensional regularization. We work in $n$ space dimensions and show that the leading terms in the Born expansion, which diverge for integer $n$, can be unambiguously identified with Feynman diagrams. This approach resolves several longstanding ambiguities in Casimir calculations. In Section 5 we show how to compute fractional and integer charges carried by background fields. We illustrate the importance of gauge invariant regularization of Casimir calculations in a study of the charge carried by an electrostatic “hole” in one space dimension. In Section 6 we consider a chiral model in one space dimension to show that quantum effects of a heavy fermion can stabilize a soliton that is not present in the classical theory. We compute corrections to the energy and central charge in $1 + 1$ dimensional supersymmetric models in Section 7. In Section 8 we extend our results to interfaces. Finally we conclude with a summary of some topics that are currently under investigation or the subject for future projects.
2. Overview

We start with an overview of our method. We treat the simplified example of a fluctuating boson or fermion field of mass $m$ in a static, spherically symmetric background potential $\chi(r)$ in three dimensions. Since we encounter divergences, we imagine that we have analytically continued to values of the space dimension $n$ where the integrals are convergent. In Section 4 we provide the rigorous justification for this procedure.

We take the interaction Lagrangian $L_I = g\bar{\psi}\chi\psi$ for fermions and $L_I = g\psi^\dagger\chi\psi$ for bosons where $\psi$ is the fluctuating field. We want to compute the one-loop “effective energy,” the effective action per unit time. It is given either by the sum of all one loop diagrams with all insertions of the background $\chi(r)$,

$$
\Delta E_{\text{bare}}[\chi] = \text{Diagram}
$$

or by the “Casimir sum” of the shifts in the zero-point energies of all the small oscillation modes in the background $\chi$,

$$
\Delta E_{\text{bare}}[\chi] = \pm \frac{1}{2} \sum_j |\epsilon_j| - |\epsilon_0|
$$

for bosons (+) and fermions (−) respectively. Both of these representations are divergent and require renormalization. We start from the second expression and work in the continuum. We rewrite the Casimir sum as a sum over bound states plus an integral over scattering states, weighted by the density of states $\rho(k)$. We subtract from the integral the contribution of the trivial background, which is given by the free density of states $\rho^0(k)$. Thus we have

$$
\Delta E_{\text{bare}}[\chi] = \pm \left( \frac{1}{2} \sum_j |\omega_j| + \frac{1}{2} \int_0^\infty \omega(k) \left( \rho(k) - \rho^0(k) \right) dk \right)
$$

where $\omega_j$ denotes the energy of the $j^{\text{th}}$ bound state, and $\omega(k) = \sqrt{k^2 + m^2}$.

The density of states is related to the $S$-matrix and the phase shifts by

$$
\rho(k) - \rho^0(k) = \frac{1}{2\pi i} \frac{d}{dk} \text{Tr} \ln S(k) = \sum_\ell D^\ell \frac{1}{\pi} \frac{d\delta_\ell(k)}{dk}
$$

where $\ell$ labels the basis of partial waves in which $S$ is diagonal. $D^\ell$ is the degeneracy factor. For example, $D^\ell = 2\ell + 1$ for a boson in three dimensions. It is convenient
to use Levinson’s theorem to express the contribution of the bound states to eq. (3) in terms of their binding energy. Levinson’s theorem relates the number of bound states to the difference of the phase shift at $k = 0$ and $\infty$,

$$n^{\text{bound}}_\ell = \frac{1}{\pi}(\delta_\ell(0) - \delta_\ell(\infty)) = -\int_0^\infty dk \frac{d\delta_\ell(k)}{dk}.$$  \hfill (5)

Subtracting $mn^{\text{bound}}_\ell$ from the sum over bound states in eq. (3) and using eqs. (5) and (4), we obtain

$$\Delta E_{\text{bare}}[\chi] = \pm \left( \frac{1}{2} \sum_{j,\ell} D^\ell (|\omega_{j,\ell}| - m) + \int_0^\infty \frac{dk}{2\pi} (\omega(k) - m) \sum_\ell D^\ell \frac{d\delta_\ell(k)}{dk} \right)$$  \hfill (6)

where the sum over partial waves is to be performed before the $k$ integration. While the phase shifts and bound state energies are finite, $\Delta E_{\text{bare}}[\chi]$ is divergent because the $k$–integral diverges in the ultraviolet. To better understand the origin and character of the divergences, we go back to the diagrammatic representation of the vacuum energy, eq. (1). Since we are working with a renormalizable theory, only the first few diagrams are divergent, and these divergences can be canceled by a finite number of counterterms. The series of diagrams gives an expansion of the effective energy in powers of the background field $\chi(r)$. Likewise, the phase shift calculation can be expanded in powers of $\chi(r)$ using the Born series,

$$\delta^N_\ell(k) = \sum_{i=1}^N \delta^{(i)}_\ell(k)$$  \hfill (7)

where $\delta^{(i)}_\ell(k)$ is the contribution to the phase shift at order $i$ in the potential $\chi(r)$. In general, the Born expansion is a poor approximation at small $k$, especially if the potential has bound states, when it typically does not converge. What is important for us, however, is that the contributions to $\Delta E_{\text{bare}}$ from successive terms in the Born series correspond exactly to the contributions from successive Feynman diagrams. That is, the $i$th term in the Born series generates a contribution to the vacuum energy which is exactly equal to the contribution of the Feynman diagram with $i$ external insertions of $\chi$.

This correspondence is not trivial in light of divergences. We have verified the identification for the lowest order diagram by direct comparison in $n$ space dimensions where both are finite. At this order the Born and Feynman contributions to $\Delta E_{\text{bare}}[\chi]$ are precisely equal as analytic functions of $n$ as we will show in Section 4. We have also performed various numerical checks to verify the identification in higher orders.

We then define the subtracted phase shift

$$\tilde{\delta}^N_\ell(k) = \delta_\ell(k) - \delta^N_\ell(k)$$  \hfill (8)
where we take \( N \) to be the number of divergent diagrams in the expansion of eq. (4).

The effect of the Born subtraction is illustrated in Fig. 1. Note that the subtracted phase shift is large at small \( k \), so the Born approximation is very different from the true phase shift in this region. However, the Born approximation becomes good at large \( k \), so that the subtracted phase shift vanishes quickly as \( k \to \infty \).

Having subtracted the potentially divergent contributions to \( \Delta E_{\text{bare}}[\chi] \) via the Born expansion, we add back in exactly the same quantities as Feynman diagrams, \( \sum_{i=1}^{N} \Gamma_{\text{FD}}^{(i)}[\chi] \). We combine the contributions of the diagrams with those from the counterterms, \( \Delta E_{\text{CT}}[\chi] \), and apply standard perturbative renormalization conditions. We have thus removed the divergences from the computationally difficult part of the calculation and re-expressed them as Feynman diagrams, where the regularization and renormalization have been carried out with conventional methods. This approach to renormalization in strong external fields was first introduced by Schwinger in his work on QED in strong fields. Combining the renormalized Feynman diagrams,

\[
\Gamma_{\text{FD}}^{(N)}[\chi] = \sum_{i=1}^{N} \Gamma_{\text{FD}}^{(i)}[\chi] + \Delta E_{\text{CT}}[\chi] \tag{9}
\]

with the subtracted phase shift calculation, we obtain the complete, renormalized, one loop effective energy,

\[
\Delta E[\chi] = \pm \frac{1}{2} \sum_{\ell} \int d^{3}\mathbf{k} \left( \sum_{j} (|\omega_{j,\ell}| - m) + \int_{0}^{\infty} \frac{dk}{\pi} (\omega(k) - m) d\delta_{\ell}^{N} (k) \right)
+ \Gamma_{\text{FD}}^{(N)}[\chi] \tag{10}
\]

where the two pieces are now separately finite. Since the \( k \) integral is now convergent, we are free to interchange it with the sum over partial waves or to integrate by parts. This expression is suitable for numerical computation, since it does not contain differences of large numbers. The massless limit is also smooth, except for the case of one spatial dimension, where we expect incurable infrared divergences.
3. An Example

In this section we illustrate this approach with a simple example. We consider a charged scalar field, $\phi$, in a classical scalar background $\chi(r)$ in three dimensions. We show how to carry out a variational search for quantum–stabilized nontopological solitons, which are local minima of the effective energy functional, $E[\phi]$. Although we do not find any solitons in this simple model, we develop computational tools that will be useful when we consider models with more complex structure.

3.1. The model

The model Lagrangian is given by

$$\mathcal{L} = \frac{1}{2} (\partial_\mu \chi)^2 - \frac{\lambda}{4!} \left( \chi^2 - v^2 \right)^2 + \partial_\mu \phi^* \partial^\mu \phi - G \phi^* \chi^2 \phi^* + a(\partial_\mu \chi)^2 - b(\chi^2 - v^2) - c(\chi^2 - v^2)^2. \quad (11)$$

Here we have coupled $\phi$ to the square of the background field $\chi$ so that the classical potential for $\chi$ is positive–definite. The potential for $\chi$ is of the symmetry–breaking form with a minimum at $\chi = \pm v$. By Derrick’s theorem, this model has no soliton at the classical level.

When we integrate out the dynamical $\phi$ field, we leave behind its Casimir energy as a functional of $\chi$. The divergences of the Casimir energy are then canceled by the $\chi$-dependent counterterms explicitly given in eq. (11). The coefficients $a$, $b$ and $c$ are determined by considering deviations $h = \chi - v$ from the classical vacuum $\langle \chi \rangle = v$ and imposing the following perturbative renormalization conditions:

1) the tadpole diagram with an external $h$ field vanishes,

2) the location and residue of the pole of the $h$ propagator remain unchanged.

Condition 1) implies that there are no quantum corrections to the vacuum expectation value $\langle \chi \rangle = v$. Condition 2) gives the standard on–shell renormalization conditions for $h$. The small oscillations potential for the $\phi$ fluctuations is

$$V(r) = G\chi^2(r) - M^2 = G(h^2(r) + 2vh(r)) \quad (12)$$

where $M = Gv^2$ is the mass of the $\phi$ field.

3.2. Phase shifts and the Born approximation

Next we compute the phase shifts in each partial wave. An adaptation of the variable phase method provides a numerically stable and efficient way to compute both the phase shifts and the first $N$ terms in the Born approximation for potentials $V(r)$ that fall faster than $1/r^2$ as $r \to \infty$.

We write the Jost solution to the radial wave equation with angular momentum $\ell$ as

$$\varphi_\ell(k, r) = e^{2i\beta_\ell(k, r)} r h_\ell^{(1)}(kr) \quad (13)$$
where \( h^{(1)}_\ell \) are Hankel functions describing outgoing spherical waves. The complex function \( \beta_\ell(k,r) \) vanishes when \( V(r) = 0 \) and obeys the non-linear inhomogeneous ordinary differential equation

\[
-i\beta''_\ell - 2ikp_\ell(kr)\beta'_\ell + 2(\beta'_\ell)^2 + \frac{1}{2}gV(r) = 0
\]  

(14)

where a prime denotes differentiation with respect to \( r \). The rational function

\[
p_\ell(x) = \frac{d}{dx}\ln(h^{(1)}_\ell(x))
\]

(15)

can easily be computed numerically. We have introduced a parameter \( g \) in eq. (14) to keep track of orders in the potential. Eventually we will set \( g = 1 \). With the boundary conditions \( \lim_{r \to \infty} \beta_\ell(k,r) = \beta'_\ell(k,r) = 0 \), the solution \( u_\ell(k,r) \) that is regular at \( r = 0 \) is a superposition of \( \varphi_\ell \) and \( \varphi^*_\ell \) weighted by the S-matrix \( e^{2i\delta_\ell(k)} \),

\[
u_\ell(k,r) = \varphi^*_\ell(k,r) + e^{2i\delta_\ell(k)}\varphi_\ell(k,r).
\]  

(16)

Demanding that \( u_\ell \) be regular at the origin determines the phase shift,

\[
\delta_\ell(k) = -2\text{Re} \; \beta_\ell(k,r)|_{r=0}.
\]  

(17)

Next we extend this approach to the Born series, which is the expansion of \( \delta_\ell(k) \) in a power series in \( g \). From eqs. (16) and (17) we see that

\[
\delta_\ell^{(i)}(k) = -2\text{Re} \; \beta_\ell^{(i)}(k,r)|_{r=0}
\]  

(18)

where \( \beta_\ell^{(i)}(k,r) \) are the terms in an expansion of \( \beta_\ell(k,r) \) in powers of \( g \),

\[
\beta_\ell(k,r) \sim \sum_{i=1}^{\infty} g^i \beta_\ell^{(i)}(k,r).
\]  

(19)

We obtain a series of differential equations for the \( \{\beta_\ell^{(i)}\} \) by substituting the series expansion of \( \beta_\ell \) into eq. (14) and identifying powers of \( g \),

\[
-i\beta_{\ell}^{(1)''} - 2ikp_{\ell}(kr)\beta_{\ell}^{(1)'} = -\frac{1}{2}V(r)
\]

\[
-i\beta_{\ell}^{(2)''} - 2ikp_{\ell}(kr)\beta_{\ell}^{(2)'} = -2(\beta_{\ell}^{(1)})^2
\]

\[
\vdots
\]  

(20)

with the same boundary conditions as before. The source term for each successive order in the expansion involves only lower order terms. Thus these equations can be integrated simultaneously by integrating the vector \( \{\beta_\ell, \beta_\ell^{(1)}, \beta_\ell^{(2)}, \ldots \} \) in from \( r = \infty \). The full phase shift and the Born approximations are then determined by the value of the resulting vector at \( r = 0 \).
At this point, we have collected all the ingredients for the continuum part of the energy calculation. From Levinson’s theorem, we know how many bound states to look for, and then the energies of the bound states are easily obtained by standard shooting methods.

3.3. Feynman diagrams

We now turn to the calculation of the Feynman diagrams, which we add back into the energy functional to compensate for the Born subtractions. In this scalar model, only the diagrams with one and two insertions of $V$ are divergent, so we take $N = 2$ in eq. (10). $\Gamma_{FD}^{(1)}[h]$ is local and thus completely canceled due to condition 1). The divergences of $\Gamma_{FD}^{(2)}[h]$ are canceled by the $b$ and $c$ type counterterms in eq. (11) leaving a renormalized contribution, $\Gamma_{FD}^{(2)}[h]$. The result is the one–loop energy functional

$$\Delta E[h] = \Gamma_{FD}^{(2)}[\chi] + \sum_{j,\ell} (2\ell + 1)(\omega_{j,\ell} - M)$$

$$- \int_0^{\infty} \frac{dk}{\pi} \frac{k}{\sqrt{k^2 + M^2}} \sum_{\ell} (2\ell + 1) \left( \delta_\ell(k) - \delta_\ell^{(1)}(k) - \delta_\ell^{(2)}(k) \right)$$

where we have integrated by parts. There is no surface term at $k = \infty$ because the Born subtractions have removed the leading large $k$ behavior of $\delta_\ell(k)$, and the Levinson subtraction has eliminated the contribution from $k = 0$. There is also an overall factor of 2 because the complex field $\phi$ contains two real components. The renormalized two–point Feynman diagram reads

$$\Gamma_{FD}^{(2)}[\chi] = -\frac{4v^2G^2}{(4\pi)^2} \int_0^{\infty} \frac{q^2 dq}{(2\pi)^2} \tilde{\phi}^2(q) \int_0^{1} dx \frac{x(1-x)}{M^2 - x(1-x)m^2}$$

$$+ \frac{G^2}{(4\pi)^2} \int_0^{\infty} \frac{q^2 dq}{(2\pi)^2} \tilde{V}^2(q) \int_0^{1} dx \left[ \ln \frac{M^2 + x(1-x)q^2}{M^2 - x(1-x)m^2} - \frac{x(1-x)m^2}{M^2 - x(1-x)m^2} \right]$$

where $\tilde{f}(q)$ denotes the Fourier transform, $\tilde{f}(q) = \int d^3r \exp(iq \cdot r)f(r)$, and $m = \sqrt{\lambda v^2/3}$ is the mass of $h$. We obtain the total energy functional by adding the classical energy for $h$ to the one–loop quantum contribution,

$$E[h] = E_{cl}[h] + \Delta E[h].$$

To illustrate a typical variational search we introduce a two parameter ansatz

$$h(r) = -dv e^{-r^2v^2/2w^2},$$

and compute the energy, $E(d, w)$, as a function of these variational parameters. Figure 2 shows the energy as a function of $w$ at $d = 1$ for several choices of $G$. For fixed renormalized model parameters $G$ and $m$, we vary the ansatz parameters $d$
and \( w \). One of the primary advantages of our renormalization procedure is that it is manifestly independent of the background field \( \chi \), so we can confidently vary the ansatz while keeping the model parameters fixed. We can then see if there exists a configuration for which the energy, eq. (23), plus the energy required to explicitly occupy the most tightly bound level is less than the mass of a free particle in the trivial background. If we find such a configuration, we know that further variations can only lower the energy. Thus there must exist a stable soliton, which carries charge, but is too light to decay into free particles. In Ref. 3 we did find such configurations, but only for values of \( G \) and \( m \) where further increase of \( w \) yielded \( E[h] < 0 \), signaling instability of the vacuum. This effect is depicted in the case of \( G = 8 \) in Figure 2. Although we did not find a non–trivial solution in this simple model, we have demonstrated the practicality of the method in three dimensions.

4. Dimensional Regularization

The identification of terms in the Born series with Feynman diagrams is crucial to fixing the renormalization procedure precisely in our approach. No arbitrariness can be tolerated in the renormalization process: if the manipulations of formally divergent quantities introduce finite ambiguities, the method is useless. There have been controversies for many years concerning the proper renormalization procedure for Casimir calculations. Since we are studying renormalizable quantum field theories, we know that the effective energy can be calculated unambiguously. In this section, we apply the methods of dimensional regularization to scattering from a central potential and prove that the lowest order term in the Born series is equal to the lowest order Feynman diagram as an analytic function of \( n \), the number of space dimensions. Since this is the most divergent diagram — quadratically divergent for \( n = 3 \) — we are confident that the same method will regulate all other divergences in the effective energy unambiguously. For simplicity, we will consider the self–interactions of a single real boson, \( \phi(x) \). The generalization to fermions is discussed in Ref. [4]
We consider a static, spherically symmetric background potential \( V(r) \) in \( n \) dimensions. For \( n = 1 \), \( V(r) \) reduces to a symmetric potential with even and odd parity channels. For \( n \neq 1 \) the \( S \)-matrix is diagonal in the basis of the irreducible tensor representations of \( SO(n) \). These are the traceless symmetric tensors of rank \( \ell \), where \( \ell = 0, 1, 2, \ldots \). We write the change in the full density of states as a sum over partial waves

\[
\rho_n(k) - \rho_n^0(k) = \frac{1}{\pi} \frac{d}{dk} \sum_{\ell=0}^{\infty} D^\ell_n \delta_{n\ell}(k)
\]

where \( D^\ell_n \) is the degeneracy of the \( SO(n) \) representation labeled by \( \ell \). For integer \( n \) and \( \ell \), \( D^\ell_n \) is given by the dimension of the space of symmetric, traceless tensors with \( \ell \) indices. Replacing factorials by \( \Gamma \)-functions allows us to define the analytic continuation of \( D^\ell_n \),

\[
D^\ell_n = \frac{\Gamma(n + \ell - 2)}{\Gamma(n - 1)\Gamma(\ell + 1)} (n + 2\ell - 2).
\]

For \( n = 3 \), \( D^\ell_n \) reduces to \( 2\ell + 1 \) as expected. As \( n \to 1 \) all the \( D^\ell_n \to 0 \) except for \( \ell = 0 \) and \( \ell = 1 \), for which \( \lim_{n \to 1} D^0_n = \lim_{n \to 1} D^1_n = 1 \), corresponding to the symmetric and antisymmetric channels respectively.

The phase shifts are obtained by solving the radial Schrödinger equation generalized to \( n \) dimensions,

\[
-\psi'' - \frac{n-1}{r} \psi' + \frac{\ell(\ell+n-2)}{r^2} \psi + V(r)\psi = k^2 \psi
\]

which reduces to Bessel’s equation for \( V = 0 \). At the origin, the regular solution \( \psi_{n\ell} \) is proportional to \( r^\ell \) independent of \( n \).

Incoming and outgoing waves are given by generalized Hankel functions,

\[
h^{(1,2)}_{n\ell}(kr) = \frac{1}{(kr)^{\frac{\ell+1}{2}}} \left( J_{\frac{\ell+1}{2}-1}(kr) \pm i Y_{\frac{\ell+1}{2}-1}(kr) \right)
\]

and the phase shifts are defined in the usual way by writing the solution \( \psi_{n\ell} \) regular at the origin as

\[
\psi_{n\ell} \sim h^{(2)}_{n\ell}(kr) + e^{2i\delta_{n\ell}(k)} h^{(1)}_{n\ell}(kr)
\]

for large \( r \), where the potential vanishes. In \( n \) dimensions, the one–loop effective energy functional is

\[
\Delta E_n[\chi] = \frac{1}{2} \sum_{\ell=0}^{\infty} D^\ell_n \left( |\omega_{j,n\ell}| - m \right) + \int_0^{\infty} \sum_{j} \frac{dk}{2\pi} (\omega(k) - m) \sum_{\ell=0}^{\infty} D^\ell_n \frac{d}{dk} \delta_{n\ell}(k)
\]

where the \( \omega_{j,n\ell} \) are the energies of the bound states in each partial wave \( \ell \). Eq. (30) is well defined for \( n < 1 \), where the integration over \( k \) and the sum over \( \ell \) converge.

Note that this produces the correct wavefunction symmetry for \( \ell = 0 \) and \( \ell = 1 \) when \( n = 1 \).
We want to demonstrate explicitly that the contribution to the energy from the first Born approximation to the phase shift is precisely equal to the tadpole graph, which we calculate in ordinary Feynman perturbation theory. We show that these two quantities are equal by computing both as analytic functions of \( n \).

The first Born approximation to the phase shift is

\[
\delta^{(1)}_{nl}(k) = -\frac{\pi}{2} \int_{0}^{\infty} J_{\frac{n}{2}+\ell-1}(kr)^2 V(r) r dr
\]

and its contribution to the Casimir energy is

\[
\Delta E_n^{(1)}[\chi] = \int_{0}^{\infty} \frac{dk}{2\pi} (\omega(k) - m) \sum_{\ell=0}^{\infty} D_\ell^n d_\ell^{(1)}(k). \tag{32}
\]

Using the Bessel function identity

\[
\sum_{\ell=0}^{\infty} \frac{(2q+2\ell)\Gamma(2q+\ell)}{\Gamma(\ell+1)} J_{q+\ell}(z)^2 = \frac{\Gamma(2q+1)}{\Gamma(q+1)^2} \left( \frac{z}{2} \right)^{2q} \tag{33}
\]

with \( q = \frac{n}{2} - 1 \), we sum over \( \ell \) in eq. (32) and obtain

\[
\Delta E_n^{(1)}[\chi] = \left( \frac{V}{4\pi} \right)^2 \frac{(2-n)}{\Gamma\left( \frac{n}{2} \right)} \int_{0}^{\infty} (\omega(k) - m) k^{n-3} dk = \left( \frac{V}{4\pi} \right)^2 m^{n-1} \Gamma\left( \frac{1-n}{2} \right) \tag{34}
\]

which converges for \( 0 < n < 1 \). Here \( \langle V \rangle \) is the \( n \)-dimensional spatial average of \( V(r) \),

\[
\langle V \rangle = \int V(x) d^n x = \frac{2\pi^{\frac{n}{2}}}{\Gamma\left( \frac{n}{2} \right)} \int_{0}^{\infty} V(r) r^{n-1} dr. \tag{35}
\]

The tadpole diagram is easily computed using dimensional regularization, and the result agrees precisely with eq. (34). Thus we can be certain that our method of subtracting the first Born approximation and adding back the corresponding Feynman diagram is correct.

5. Fractional Induced Charges on Background Fields

A scalar background configuration can carry the charge associated with a boson or fermion field that fluctuates around it. The charge can be related to a trace of the Green’s function and, in turn, to a sum over bound states and an integral over phase shifts much like the zero–point energy. Our methods can be used to compute these charges. This approach was first developed and applied to simple cases by Blankenbecler and Boyanovsky\(^\text{6}\). Our methods provide generalizations and applications to more complex systems. Here we derive the general result and present two non-trivial examples, the electrostatic potential hole and the chiral bag in one dimension.
We consider a fermion field $\psi$ in a static background $V(r)$ in $n$ dimensions. Let $\omega$ denote the single-particle energies and $\alpha$ the remaining quantum numbers. The corresponding wave functions are normalized so that

$$\int \psi_\alpha^\dagger(x, \omega) \psi_{\alpha'}(x, \omega') d^n x = \delta(\omega - \omega') \delta_{\alpha, \alpha'}.$$  

For bound states the Dirac delta is replaced by a Kronecker delta. Then the charge density $j^0(x)$ is given by

$$j^0(x) = -\frac{1}{2} \sum_\alpha \int_\infty^{-\infty} d\omega \text{sign}(\omega) |\psi_\alpha(x, \omega)|^2$$

where $\int d\omega (\ldots)$ denotes a sum over bound states and integral over the continuum.

The Green’s function for the fermion field is

$$G(x, y, E) = \sum_\alpha \int_\infty^{-\infty} d\omega \frac{\psi_\alpha(x, \omega) \psi_\alpha^\dagger(y, \omega)}{E - \omega + i \text{sign}(\omega) \epsilon}$$

and its imaginary part is the density of states,

$$\rho(\omega) = \frac{dN}{d\omega} = \text{Im Tr} \frac{2}{\pi} \int G(x, x, \omega) d^n x = \frac{\text{sign}(\omega)}{\pi} \sum_\alpha \int d^n x |\psi_\alpha(x, \omega)|^2.$$  

Thus the charge $Q[V] = \int d^n x j^0(x)$ can be expressed via the density of states. Combining eqs. (37) and (39) and substituting eq. (4) for the density of states yields

$$Q[V] = -\sum_\alpha \left( \int_m^{\infty} \frac{d\omega}{2\pi} \frac{d\delta_\alpha(\omega)}{d\omega} + \sum_{\omega_{\alpha,j} > 0} \frac{1}{2} - \int_{-\infty}^{-m} \frac{d\omega}{2\pi} \frac{d\delta_\alpha(\omega)}{d\omega} - \sum_{\omega_{\alpha,j} < 0} \frac{1}{2} \right)$$

$$= \frac{1}{2\pi} \sum_\alpha (\delta_\alpha(m) - \delta_\alpha(\infty) - \pi n^\alpha_> + \pi n^\alpha_< - \delta_\alpha(-m) + \delta_\alpha(-\infty))$$

where we have defined the charge such that the configuration $V \equiv 0$ has zero charge. Here $n^\alpha_>$ and $n^\alpha<$ give the number of bound states with positive and negative energy respectively in each channel.

This result has an intuitive interpretation. If a state leaves the positive continuum but appears as a positive energy bound state, the spectral asymmetry remains unchanged and the charge does not change. However, if this state crosses $\omega = 0$ and becomes a negative energy bound state, the spectral asymmetry changes and the charge of the configuration increases by one. Likewise, if a negative energy level moves up through zero energy, the charge decreases by one. Levinson’s theorem tracks all states that enter and leave the two continua at $\omega = \pm m$. Thus

$$\delta_\alpha(m) - \delta_\alpha(\infty) + \delta_\alpha(-m) - \delta_\alpha(-\infty) - \pi n^\alpha_> - \pi n^\alpha_< = 0,$$

where

$$n^\alpha_> = |\{ \omega : \omega > 0, \psi_\alpha(x, \omega) \neq 0 \}|,$$

$$n^\alpha_< = |\{ \omega : \omega < 0, \psi_\alpha(x, \omega) \neq 0 \}|.$$
so that\(^d\)

\[ Q = \frac{1}{\pi} \sum_{\alpha} \delta_\alpha(m) - \delta_\alpha(\infty) - \pi n_\alpha^- = \frac{1}{\pi} \sum_{\alpha} \pi n_\alpha^- - \delta_\alpha(-m) + \delta_\alpha(-\infty). \]  

(42)

5.1. **QED and the need for regularization**

Conserved charges are not renormalized. That is, they do not receive any contributions from the counterterms of the theory. However, if the theory is not regularized in a manner consistent with the symmetry responsible for charge conservation, *i.e.* gauge invariance, then spurious, finite renormalization of the charge can occur. The case of QED in one dimension provides a clear illustration of this problem. Although the calculation of the charge is free of divergences, proper attention to regularization is essential.

We consider an electrostatic potential hole with depth \( A^0 \equiv \varphi \) and width \( 2L \). The phase shifts in the two parity channels are given by

\[
\frac{k}{m + \omega} \tan(kL + \delta^+(\omega)) = \frac{q}{m + \omega + e\varphi} \tan qL \\
\frac{m + \omega}{k} \tan(kL + \delta^-(\omega)) = \frac{m + \omega + e\varphi}{q} \tan qL, \tag{43}
\]

where \( e \) is the elementary charge, \( k = \sqrt{\omega^2 - m^2} \) and \( q = \sqrt{(\omega + e\varphi)^2 - m^2} \). As \( \omega \to \pm \infty \), the total phase shift, \( \delta^+ + \delta^- \), approaches \( \pm 2e\varphi L \) since \( q \to k \pm e\varphi \).

Substitution into eq. (42) would indicate a fractional induced charge on the potential hole. Although fractional charges are possible in other problems, as we will see below, they do not occur in one dimensional electrostatics. The reason for this misleading result is that we have implicitly used a regulator that is not gauge invariant.

To ensure that we maintain gauge invariance, we use dimensional regularization and work in \( n \) space dimensions, taking \( n \to 1 \) at the end of the calculation. The large \( k \) behavior of the phase shifts is given by the first Born approximation. Using eqs. (31) and (33), we find that in \( n \) dimensions, the sum over all channels of the first Born approximation to the phase shift is

\[
\delta_n^{(1)}(\omega) = \omega k^{n-2} \frac{N_D e\varphi \pi}{2^{n-2} \Gamma\left(\frac{n}{2}\right)^2} \int_0^\infty A^0(r) r^{n-1} dr = \omega k^{n-2} \frac{N_D L^n e\varphi \pi}{2^{n-2} n \Gamma\left(\frac{n}{2}\right)^2} \tag{44}
\]

where \( 2N_D \) is the dimension of the Dirac algebra for spacetime dimension \( D = n + 1 \). If we take the limit \( n \to 1 \) before taking the limit \( \omega \to \pm \infty \), we recover the previous fractional result \( \pm 2e\varphi L \) with \( N_D = 1 \). However, the proper prescription is to compute in \( n < 1 \) dimensions and only take \( n \to 1 \) at the end. In that case we see that \( \delta_n^{(1)}(\omega) \) vanishes as \( \omega \to \pm \infty \), so that the square well does not carry fractional

\(^d\)There are peculiarities in the symmetric channel in one dimension. See Ref.\[5\]
charge. We note that other common schemes, such as zeta-function regularization, would yield the spurious fractional result.

Since this result involves only the first Born approximation, it can be checked by considering the corresponding Feynman diagram. The lowest diagram contributing to the charge in an external field has two insertions: one of the charge operator, $\gamma^0$, and the other of the external field. Thus we must consider the vacuum polarization diagram, which in $D$ spacetime dimensions becomes

$$\Pi_{\mu \nu}(p) = -2e^2 N_D \int_0^1 d\xi \int \frac{d^D k}{(2\pi)^D}$$

$$\times 2\xi(1-\xi)(g_{\mu\nu} p^2 - p\mu p\nu) + g_{\mu\nu} \left[ m^2 - p^2 \xi(1-\xi) + k^2(\frac{2}{D} - 1) \right].$$

If we had not regulated the theory by analytically continuing the space-time dimension, we would not have found the last term, which vanishes if we set $D = 2$ from the outset. Taking the $D \to 2$ limit carefully shows that this term exactly cancels the two terms that precede it, leaving the transverse form of the vacuum polarization required by gauge invariance. When these terms are not canceled, they lead to the same fractional fermion number we obtained from the phase shifts with the naive phase shift calculation. Thus we must include in our definition of the field theory the additional information that the theory is regulated in order to preserve gauge invariance at the quantum level. Dimensional regularization provides a way to implement this requirement in terms of both phase shifts and Feynman diagrams.

5.2. Fractional charge in a chiral model

As an example where genuine fractional charges occur, we consider a chiral bag in $D = 1 + 1$ dimensions. The generalization to the three dimensional bag model is discussed in Ref. 5.

We consider a free fermion on the half-line $x > 0$ satisfying the boundary condition

$$ie^{i\gamma^5 \theta} \Psi = \gamma^1 \Psi$$

at $x = 0$, with $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$ parameterizing the model. We break the scattering into generalized parity channels. The corresponding phase shifts are given by

$$\cot \delta^+(\omega) = -\frac{k}{\omega - m} \tan \beta \quad \text{and} \quad \tan \delta^-(\omega) = \frac{k}{\omega - m} \tan \beta$$

where $\beta = \frac{\pi}{4} - \frac{\theta}{2}$. At the thresholds $\omega \to \pm m$, the phase shifts are either 0 or $\pi$. At large $\omega$, we find $\delta^+(\pm \omega) + \delta^-(\pm \omega) \to -\frac{\pi}{2} \pm (\frac{\pi}{2} - \theta)$. There is a single bound state determined by the condition

$$\sqrt{m^2 - \omega^2} = (m + \omega) \tan \beta.$$

Collecting these results into eq. (42), we find that the fermion number is $\frac{\theta}{\pi} - \text{sign}(\theta)$. We have fixed the overall integer constant by noting that the jump in fermion
number as a function of $\theta$ should occur where there is a Jackiw-Rebbi\footnote{15} zero mode, namely at $\theta = 0$. One can also obtain this constant by considering the fermion number as a function of temperature\footnote{16}.

6. Chiral Model in One Dimension

As an application of our method, we show how a quantum soliton can appear in a theory with a heavy fermion. We consider a one-dimensional chiral model in which the fermion gets its mass from its coupling to a scalar condensate. It is easy to find a spatially varying scalar background which has a tightly bound fermion level. If the classical energy of the background field plus the energy of the tightly bound fermion is less than the free fermion mass $m$, this configuration would appear to be a stable soliton, since it is unable to decay into free fermions. However, the energy of the lowest fermion level enters at the same order in $\hbar$ as the full one-loop fermion effective energy, since the latter simply corresponds to the shift of the zero-point energies, eq. (2), of all the fermion modes. The question of stability can therefore only be addressed by computing the full one-loop effective energy. Here we summarize our analysis of this system and show that it supports stable solitons. We also illustrate how our scattering theory methods generalize to fermions. More details of this calculation can be found in Ref.\footnote{17}.

6.1. The model

We consider a chiral model in one dimension with a symmetry-breaking scalar potential. We couple a two-component real boson field $\vec{\phi} = (\phi_1, \phi_2)$ chirally to a fermion $\Psi$

$$\mathcal{L} = \frac{1}{2} \partial_\mu \vec{\phi} \cdot \partial^\mu \vec{\phi} - V(\vec{\phi}) + \bar{\Psi} \{ i \gamma \cdot \partial - G (\phi_1 + i \gamma_5 \phi_2) \} \Psi$$

(49)

where the potential for the boson field is given by

$$V(\vec{\phi}) = \frac{\lambda}{8} \left( \vec{\phi} \cdot \vec{\phi} - v^2 \right)^2 - \alpha v^3 (\phi_1 - v) + \text{const.}$$

(50)

$V(\vec{\phi})$ has its minimum at $\vec{\phi} = (v, 0)$. Terms proportional to $\alpha$ break the chiral symmetry explicitly. If $\alpha$ we set to zero, the chiral symmetry appears to break spontaneously, but quantum fluctuations in one dimension restore the symmetry\footnote{18}. For large enough $\alpha$, the classical vacuum $\vec{\phi} = (v, 0)$ is stable against quantum corrections and $m = Gv$ is the fermion mass. The coefficient $c$ in the counterterm Lagrangian

$$\mathcal{L}_{c.t.} = c \left( \vec{\phi} \cdot \vec{\phi} - v^2 \right)$$

(51)

is fixed by the condition that the quantum corrections do not change the VEV of $\vec{\phi}$. This model has no stable soliton solutions at the classical level.

We are interested in the mass of the lightest state carrying unit fermion number. If its mass is less than $m$, this state is a stable soliton. We neglect boson loops,
so that the effective energy is given by the sum of the classical and the fermion loop contributions, \( E_{\text{tot}}[\vec{\phi}] = E_{\text{cl}}[\vec{\phi}] + E_{\text{f}}[\vec{\phi}] \). This approximation is exact in the limit where the number of independent fermion species becomes large. The fermion contribution to the effective energy is \( E_{\text{f}} = E_{\text{Cas}} + E_{\text{val}} \) where \( E_{\text{Cas}} \) is the sum over zero–point energies, calculated with the methods we have developed. \( E_{\text{val}} \) is the energy required for the soliton to have unit charge. Using the methods of the previous section, we can calculate the fermion number of the background field. If a level has crossed zero, then the background field will already carry the required fermion number and \( E_{\text{val}} = 0 \). If the background field has zero charge, we must explicitly fill the most tightly bound level, giving \( E_{\text{val}} = \epsilon_0 \), where \( \epsilon_0 \) is the energy of that level.

6.2. Phase shifts

We consider configurations with \( \phi_1(x) = \phi_1(-x) \) and \( \phi_2(x) = -\phi_2(-x) \), so that parity is a good quantum number. Hence the solutions \( \psi(x) \) to the Dirac equation in the background \( \vec{\phi} \) can be decomposed into parity channels,

\[
\psi_{\pm}(x) = \pm \psi_{\pm}(x).
\]

Then the solution to the Dirac equation is

\[
\varphi_{+k}(x) = \left( \frac{f(x)}{i \frac{f'(x) + G\phi_1(x) f(x)}{\omega + G\phi_2(x)}} \right)
\]

\[
\varphi_{-k}(x) = \left( \frac{f^*(x)}{i \frac{f^*(x) + G\phi_1(x) f^*(x)}{\omega + G\phi_2(x)}} \right)
\]

where \( f(x) \) obeys a real second–order differential equation, which is amenable to the variable phase method. We write \( f(x) = e^{i\beta(x,\omega)} e^{ikx} \) with boundary conditions \( \beta(\infty, \omega) = \beta'(\infty, \omega) = 0 \), so that \( \varphi_{+k} \) and \( \varphi_{-k} \) asymptotically describe outgoing and incoming plane wave spinors, respectively. We substitute into the Dirac equation and solve the resulting differential equation for \( \beta(x, \omega) \) numerically,

\[
-ik'\frac{\beta'(x, \omega)}{\beta'(x, \omega)} + 2k\beta'(x, \omega) + \beta''(x, \omega) - m^2 + G^2 \phi_1^2(x) + G^2 \phi_2^2(x) - G\phi_1'(x) \omega + G\phi_2(x) \left[ G\phi_1(x) + i(k + \beta'(x, \omega)) \right] = 0.
\]

We define the phase shifts \( \delta_{\pm}(\omega) \) by writing scattering wavefunctions in the basis of parity eigenstates,

\[
\psi_{\pm}(x) = \varphi_{-k}(x) \pm \frac{m - ik}{\omega} e^{2i\delta_{\pm}(\omega)} \varphi_{+k}(x)
\]

where we have introduced the factor \( (m - ik)/\omega \) to guarantee that \( \delta_{\pm} = 0 \) when \( \vec{\phi} = (v, 0) \). Imposing the boundary conditions eq. (52) onto the scattering solution
\(\delta_\pm(\omega) = - \text{Re} \beta(0, \omega) - \text{arg} \left[ 1 + \frac{i \beta'(0, \omega) + G(\phi_1(0) - v)}{\mp \omega + m + ik} \right]. \quad (56)\)

Finally we sum the contributions from positive and negative energies and both parities,

\[ \delta_F(k) = \delta_+(\omega(k)) + \delta_+(-\omega(k)) + \delta_-(\omega(k)) + \delta_-(\omega(k)). \quad (57)\]

Renormalization is particularly simple in this model. The first and second Born approximations corresponding to the Feynman diagrams with one and two insertions of \(\vec{\phi} - (v, 0)\) diverge. However, the divergences are related by chiral symmetry. Both are canceled by a counterterm proportional to \(\vec{\phi}^2 - v^2\). It suffices to subtract the first Born approximation to \(\delta(k)\) and the part of the second related to it by chiral symmetry,

\[ \delta^{(1)}(k) = \frac{2G^2}{k} \int_0^\infty dx \left( v^2 - \vec{\phi}^2(x) \right). \quad (58)\]

The condition that the VEV of \(\vec{\phi}\) does not get renormalized requires that the counterterm exactly cancel the Feynman diagrams that are added back in compensation for the Born subtractions. Thus we have

\[ E_{\text{Cas}} = -\frac{1}{2} \sum_j (|\omega_j| - m) - \int_0^\infty \frac{dk}{2\pi} \left( \omega(k) - m \right) \frac{d}{dk} \left( \delta v(k) - \delta^{(1)}(k) \right). \quad (59)\]

### 6.3. Numerical studies

We consider variational ansätze for the background field. As \(x \to \pm \infty\), \(\vec{\phi}\) must go to its vacuum value, \((v, 0)\). We find that energetically favored configurations execute a loop in the \((\phi_1, \phi_2)\) with radius \(R > v\) so that they enclose the origin. A simple ansatz with these properties is

\[ \phi_1 + i\phi_2 = v \left\{ 1 - R + R \exp \left[ i\pi \left( 1 + \tanh(Gvx/w) \right) \right] \right\} \quad (60)\]

with the width \((w)\) and amplitude \((R)\) as variational parameters. For particular model parameters \(G, \alpha, \lambda\) and \(v\), we compute \(B = E_{\text{tot}}/m - 1\) as a function of the variational parameters \(w\) and \(R\). We show the resulting binding energy surface in Figure 3 for one set of model parameters. The contour \(B = 0\) separates the region in which the effective energy of background configuration is less than \(m\) from the region in which it is larger than \(m\). The maximal binding is indicated by a star. In Figure 4 we present the profiles \(\phi_1\) and \(\phi_2\) corresponding to this variational minimum as functions of the dimensionless coordinate \(\xi = x/m\). This background field configuration does not carry fermion number in this case, so the most strongly bound level must explicitly be occupied. The total charge density is shown in Figure 4. It receives contributions from the polarized fermion vacuum, given by eq. (37), and from the explicitly occupied valence level, given by \(\psi^\dagger_0(x)\psi_0(x)\) where \(\psi_0(x)\) is the bound state wavefunction of the valence level.
**Fig. 3.** $\mathcal{B}$ as a function of the *ansatz* parameters for the class of *model* parameters characterized by the relations $\alpha = 0.5G^2$, $\lambda = G^2$, and $v = 0.375$. A solid curve marks the contour $\mathcal{B} = 0$. The star indicates the minimum at $w = 2.808$ and $R = 0.586$.

**Fig. 4.** $\phi_1$, $\phi_2$, and the fermion number density $j_0$ at the variational minimum. The left panel shows $\phi_1(\xi)$ and $\phi_2(\xi)$, and the right panel shows the charge density $j_0(\xi)$, which gets contributions from both the polarized fermion vacuum eq. (37) and the filled valence level. The model parameters are as in Figure 3.

Figure 3 shows the result of repeating the binding energy calculation for various sets of model parameters. When $\mathcal{B}$ is negative, the configuration is a fermion with lower energy than a fermion propagating in the trivial background. Since the true minimum of the energy will have even lower energy, we know that a soliton exists.

We have extended this analysis to a chiral Yukawa model with $SU(2)$ symmetry in three dimensions. The analysis is more complicated: rotational symmetry is replaced by grand spin, the sum of rotations in spatial $SU(2)$ and isospin $SU(2)$, and diagrams up to fourth order in the external field are divergent. Nevertheless, the program can still be carried out. However, we do not find evidence for a bound fermionic soliton in this theory. In general, binding is weaker than in one dimension and it occurs in regions of parameter space where the model or the restriction to one fermion loop is internally inconsistent. It is possible that expanding the model to also include gauge fields may change this result, and work is underway to consider this possibility.
7. Quantum Corrections to the Energy and Central Charge for Supersymmetric Solitons in 1+1 Dimensions

$N = 1$ supersymmetric models in 1 + 1 dimension provide a simple example of our methods. The ability to study configurations that are not solutions to the equations of motion (and therefore not supersymmetric) and to handle renormalization unambiguously allows us to resolve long-standing questions regarding saturation of the BPS bound$^{11,12}$. Here we present only a brief introduction to the results of Ref.$^{13}$ and refer the reader there for a more complete presentation.

We consider the Lagrangian

$$\mathcal{L} = \frac{m^2}{2\lambda} \left( (\partial_\mu \phi)(\partial^\mu \phi) - U(\phi)^2 + i\bar{\Psi}\gamma^\mu\partial_\mu \Psi - U'(\phi)\bar{\Psi}\Psi \right)$$

(61)

where $\phi$ is a real scalar, $\Psi$ is a Majorana fermion, and $U(\phi) = W'(\phi)$ where $W(\phi)$ is the superpotential. If $U(\phi)^2$ is of the symmetry breaking form with equal minima at $\phi = \pm 1$, then a soliton is a solution to

$$\frac{d\phi_0(x)}{dx} = -U(\phi_0(x))$$

(62)

where $\phi_0 \to \pm 1$ as $x \to \pm \infty$. An antisoliton is obtained by sending $x$ to $-x$. The boson and fermion small oscillation modes are given by

$$\left( -\frac{d^2}{dx^2} + U'(\phi_0)^2 + U(\phi_0)U''(\phi_0) \right) \eta_k(x) = \omega^2 \eta_k(x)$$

(63)

$$\gamma^0 \left( -i\gamma^1 \frac{d}{dx} + U'(\phi_0) \right) \psi_k(x) = \omega \psi_k(x).$$

(64)

Defining

$$V(x) = U'(\phi_0)^2 + U(\phi_0)U''(\phi_0) - m^2$$
\[ \tilde{V}(x) = U'(\phi_0)^2 - U(\phi_0)U''(\phi_0) - m^2 \]  \hspace{1cm} (65)

and squaring the Dirac equation, we obtain

\[
\begin{pmatrix}
\frac{d^2}{dx^2} + V(x) & 0 \\
0 & -\frac{d^2}{dx^2} + \tilde{V}(x)
\end{pmatrix}
\begin{pmatrix}
\eta_k(x) \\
\psi_k(x)
\end{pmatrix}
= \begin{pmatrix}
\omega^2 \eta_k(x) \\
0
\end{pmatrix}
\]

(66)

It is easy to show that the bound state spectra of the effective scalar potentials \(V(x)\) and \(\tilde{V}(x)\) will always coincide, except possibly for zero modes.

7.1. Supersymmetric Spectrum

To be specific, we will consider the special case of \(U(\phi) = \frac{m^2}{2}(\phi^2 - 1)\), where the soliton is the standard “kink,” \(\phi_0(x) = \tanh \frac{mx}{2}\). Our ability to consider configurations that are not solutions to the classical equations of motion allows us to study a sequence of background fields, \(\phi_0(x, x_0)\), which interpolate between the trivial background at \(x_0 = 0\) and a widely separated kink-antikink pair as \(x_0 \to \infty\),

\[
\phi_0(x, x_0) = \tanh \frac{m}{2}(x + x_0) - \tanh \frac{m}{2}(x - x_0) - 1.
\]  \hspace{1cm} (67)

This procedure enables us to avoid potential ambiguities regarding the choice of boundary conditions when \(\phi_0\) tends toward different vacua as \(x \to \pm \infty\) or at boundaries introduced to discretize the problem. To stabilize an arbitrary background, \(\phi_0(x, x_0)\), we must insert a source term, \(J(x) = \frac{d^2 \phi_0}{dx^2} - \frac{1}{2}m^2(\phi_0^3 - \phi_0)\) into the SUSY lagrangian, eq. (61). With this choice, \(\phi_0(x, x_0)\) is a stationary point of the action (though not necessarily a global minimum). The source breaks supersymmetry except when \(x_0 = 0\) and as \(x_0 \to \infty\), but it allows us to track the properties of the system continuously from the trivial case \((x_0 = 0)\) to the case of interest \((x_0 \to \infty)\), both of which are supersymmetric. Technical issues associated with the source, including restoration of translation invariance and the appearance of modes with imaginary frequencies, are discussed in Ref. 14. They do not complicate the picture presented here. To understand the subtleties of the renormalized energy calculation, it is instructive to compare the bosonic and fermionic spectra as functions of separation \(x_0\) for kink-antikink pair. For large separation, the boson and fermion modes match, as required by supersymmetry, except we have two boson zero modes but only one fermion zero mode. Figure 6 illustrates this discrepancy.

Note that for each state in the spectrum, there is another equivalent state with the opposite sign of the energy. Since we are considering a real scalar and a Majorana fermion, in both cases we only consider one of these two states.

The zero modes matter even though they do not contribute directly to the vacuum polarization energy because they are related to the continuum through Levinson’s theorem. Since the number of bound states is different for bosons and fermions, by Levinson’s theorem the phase shifts at \(k = 0\) must differ. Since the
Fig. 6. Bosonic (left) and fermionic (right) bound state spectra for kink/antikink background with separation $2x_0$. We display $\omega_B^2$ for the bosonic modes and $\omega_F$ for the fermionic modes as functions of $x_0$. In limit of infinite separation, we have supersymmetry, so the modes match except for the zero modes.

Phase shifts are continuous, they must also differ as functions of $k$. Indeed, for the widely separated soliton/antisoliton pair, we find

$$\delta_B(k) - \delta_F(k) = 2 \arctan \frac{m}{k}$$

and so the renormalized one-loop quantum correction to the energy as $x_0 \to \infty$ is

$$\Delta E = \frac{1}{2} \sum_j (\omega_j^B - m) - \frac{1}{2} \sum_j (\omega_j^F - m)$$

$$+ \int_0^\infty \frac{dk}{2\pi} \left( \sqrt{k^2 + m^2 - m} \right) \frac{d}{dk} \left( \delta_B(k) - \delta_F(k) - \delta^{(1)}(k) \right)$$

$$= -\frac{m}{2} + \int_0^\infty \frac{dk}{2\pi} \left( \sqrt{k^2 + m^2 - m} \right) \frac{d}{dk} \left( 2 \arctan \frac{m}{k} - 2m \right)$$

where we have fixed the coefficient of the counterterm

$$\mathcal{L}_{ct} = -CU'''(\phi)U(\phi) - CU'''(\phi)\bar{\Psi}\Psi$$

by requiring that the tadpole graph vanish, with no further finite renormalizations.

We assign half this energy shift to the soliton and half to the antisoliton, so the result for the soliton is $\Delta E = -\frac{m}{\pi^2}$. This assignment is supported by a careful consideration of the zero modes: for a single soliton, we find one bosonic zero mode, and “one-half” of a fermionic zero mode. Just as the bosonic zero mode reflects the breaking of translation invariance, the fermionic mode reflects the breaking of one of the two supersymmetry generators. This mode is the Majorana fermion analog of the Jackiw-Rebbi mode. For large $x_0$, only one mode appears near $\omega = 0$ in the spectrum of positive energy solutions the Dirac equation. When we reduce to the
case of a single soliton, still keeping only the positive energy states, it is weighted by one half. Indeed one can verify that the residue of the pole in the fermionic Green’s function at $\kappa^2 = -m^2$ is half the usual result for a bound state. Our result has since been confirmed using the generalized effective action approach\(^3\).

### 7.2. BPS bound

In the supersymmetric system, there are additional restrictions on the quantum Hamiltonian $H$. The central charge

$$Z = \frac{m^2}{\lambda} \int dx \, U(\phi) \frac{d\phi}{dx}$$  

obeys the BPS bound

$$\langle H \rangle \geq |\langle Z \rangle| .$$

Classically, the bound is saturated,

$$E_{\text{cl}} = \frac{m^2}{2\lambda} \int dx \, \left[ \left( \frac{d\phi_0}{dx} \right)^2 + U^2(\phi_0) \right] = -\frac{m^2}{\lambda} \int dx \, U(\phi_0) \frac{d\phi_0}{dx} = -Z_{\text{cl}}$$

and the contribution from the counterterm is equal and opposite as well,

$$\Delta E_{\text{ct}} = C \int dx \, U''(\phi_0)U(\phi_0) = -C \int dx \, U''(\phi_0)\phi'_0 = -\Delta Z_{\text{ct}} .$$

The negative quantum correction to the energy we found in the previous section would appear to lead to a violation of the bound, which cannot be correct. The resolution is that a similar analysis, taking careful account of renormalization by using the scattering data, yields a compensating correction to the central charge. Expanding the field $\phi$ around the classical solution $\phi_0(x)$ gives

$$\Delta Z = \langle Z \rangle_\phi - Z_{\text{cl}} = \Delta Z_{\text{ct}} + \frac{m^2}{2\lambda} \int dx \, \left\langle \left( \frac{d}{dx} + U' \right) \eta \right\rangle^2 + \langle U'' \eta^2 \rangle_{\phi_0}$$

where $\phi(x) = \phi_0(x) + \eta(x)$ and

$$\eta(x) = \sqrt{\frac{\lambda}{m^2}} \left( \int \frac{dk}{\sqrt{4\pi \omega_k}} \left( a_k \eta_k(x) e^{-i \omega_k t} + a_k^\dagger \eta_k^*(x) e^{i \omega_k t} \right) + \eta_{\omega=0}(x) a_{\omega=0} \right)$$

where $\omega_k = \sqrt{k^2 + m^2}$; the creation and annihilation operators obey the usual commutation relations, and we have explicitly separated the contribution of the zero mode. The other bound states are understood to give discrete contributions to the integral.
We can compute this expectation value and connect it to our scattering theory formalism using the relationship between the wavefunction and the density of states,

\[ \rho(k) - \rho_0(k) = \frac{1}{\pi} \int dx \left( |\eta_k(x)|^2 - 1 \right) \]  

yielding as a result

\[ \Delta Z = \frac{1}{4} \sum_j (|\tilde{\omega}_j| - m) - \frac{1}{4} \sum_j (|\omega_j| - m) + \int_0^\infty \frac{dk}{4\pi} \left( \sqrt{k^2 + m^2} \right) \frac{d}{dk} \left( \hat{\delta}(k) - \delta(k) + 2\delta^{(1)}(k) \right) \]

\[ = \frac{m}{4} - \int_0^\infty \frac{dk}{2\pi} \left( \sqrt{k^2 + m^2} - m \right) \frac{d}{dk} \left( \arctan \frac{m}{k} - \frac{m}{k} \right) = \frac{m}{2\pi}, \]  

(78)
in terms of the phase shifts \( \delta(k) \) and \( \tilde{\delta}(k) \) and the bound states energies \( \omega_j \) and \( \tilde{\omega}_j \) in the potentials \( V \) and \( \tilde{V} \) respectively. Comparison with eq. (69) shows that the correction to central charge for a single soliton or antisoliton satisfies \( |\Delta E| = |\Delta Z| \), so the BPS bound remains saturated. This result was subsequently confirmed by SUSY methods to be the matrix element of an anomalous correction to the central charge operator.

8. Quantum Energies of Interfaces

As a final application, we follow Ref. 17 and extend our formalism to the case of interfaces, background fields that are independent of some of the spatial coordinates, and symmetric in the remaining ones. Our goal will be to compute the energy per unit transverse area, which corresponds to an induced quantum surface tension or cosmological constant. For an application to the case with only one nontrivial dimension, see Ref. 18 which applies functional methods. The general problem was considered in Ref. 19, though this approach does not make contact with perturbative renormalization conditions. It also requires that the fluctuating field have an explicit mass term, so that massless fields or fields that get their masses through spontaneous symmetry breaking cannot be considered.

The interface is described as a background field in \( n + s \) dimensions. The background is independent of the coordinates in \( s \) “transverse” dimensions, but varies as a function of the \( n \) “nontrivial” coordinates. Examples include a domain wall in three space, where \( s = 2 \) and \( n = 1 \), a magnetic vortex also in three space, where \( s = 1 \) and \( n = 2 \), and branes in general, where \( s \) is the brane dimension and \( n \) is the codimension. Since the background is independent of the \( s \) coordinates, the scattering phase shifts are independent of the momentum \( \vec{p} \) along the interface. The energy per unit transverse area can be written as a natural generalization of
eq. (10),

\[ \mathcal{E}_{n,s}[\phi] = \pm \int \frac{d^p p}{(2\pi)^s} \sum_{\ell} D_n^\ell \left[ \int_0^\infty \frac{d k}{2\pi} (\omega(k,p) - m(p)) \frac{d}{d k} \delta_{n\ell}^N(k) \right. \\
\left. + \frac{1}{2} \sum_{j} (|\omega_{j,\ell}(p)| - m(p)) \right] + \mathcal{C}^N_{n,s}[\phi] \quad (79) \]

where \( m(p) = (p^2 + m^2)^{1/2} \), \( \omega(p,k) = (k^2 + m(p)^2)^{1/2} \), \( \omega_j(p) = (m(p)^2 - \kappa_j^2)^{1/2} \), and \( p = |\vec{p}| \). The bound states and Born–subtracted phase shifts, \( \delta_{n\ell}^N(k) \), are computed in the scattering theory ignoring the transverse dimension \( s \).

Next we integrate over the \( s \) coordinates of \( p \) and obtain

\[ \mathcal{E}_{n,s}[\phi] = \pm \frac{\Gamma(-\frac{s+1}{2})}{(4\pi)^{\frac{s+1}{2}}} \sum_{\ell} D_n^\ell \left[ \int_0^\infty \frac{d k}{2\pi} \omega(k)^{s+1} - m^{s+1} \frac{d}{d k} \delta_{n\ell}^N(k) \right. \\
\left. + \frac{1}{2} \sum_{j} (|\omega_{j,\ell}|^{s+1} - m^{s+1}) \right] + \mathcal{C}^N_{n,s}[\phi]. \quad (80) \]

In eq. (80), all of the divergences have been handled with our standard prescription. Nonetheless, eq. (80) appears to diverge when \( s \) approaches an odd integer because of the pole in the \( \Gamma \) function. Since we know that for any smooth background all potential divergences have been treated by Born subtraction and subsequent renormalization, the quantity in brackets must vanish in each channel. Thus we are led to scattering theory sum rules, such as

\[ \int_0^\infty \frac{d k}{\pi} k^2 \frac{d}{d k} \delta^{(1)}(k) - \sum_j \kappa_j^2 = 0 \quad (81) \]

which can be proved using Jost–function techniques. Using these identities, we find that the limit where \( s \) approaches an odd integer is smooth. For the case of \( s \to 1 \), for example, we obtain

\[ \mathcal{E}_{n,1}[\phi] = \pm \frac{1}{(4\pi)} \sum_{\ell} D_n^\ell \left[ \int_0^\infty \frac{d k}{2\pi} \omega(k)^2 \frac{d}{d k} \delta_{n\ell}^N(k) \right. \\
\left. + \frac{1}{2} \sum_{j} \omega_{j,\ell}^2 \log \frac{\omega_{j,\ell}^2}{m^2} \right] + \mathcal{C}^N_{n,1}[\phi] \quad (82) \]

which is free of divergences. The result is independent of the scale of the logarithms by eq. (81) and Levinson’s theorem. In Figure 7 we consider a specific potential for \( n = 1 \) and study the behavior of \( \mathcal{E}_{1,s} \) as a function of \( s \). It is smooth at \( s = 1 \) and \( s = 3 \), showing that the naive divergences indeed vanish.
Figure 7: $E_{1,s}[^\phi]/m^{s+1}$ as a function of $s$ for a bosonic field in the background $V(x) = -\frac{\ell + 1}{\ell}m^2\text{sech}^2\frac{mx}{\ell}$ with $\ell = 1.5$. For the particular cases $s = 1$ and $s = 3$, the limits have been taken analytically using the sum rule eq. (81) and its $s = 3$ analogue.

As a concrete example, we can apply this formula to the kink domain wall in $2 + 1$ dimensions. The Lagrangian is

$$\mathcal{L} = \frac{m^2}{2\lambda} \left( \partial_\mu \phi \partial^\mu \phi - \frac{m^2}{4} (\phi^2 - 1)^2 \right) + C(\phi^2 - 1) \quad (83)$$

where $\phi$ is a real scalar field with mass $m$, and the counterterm $C$ is fixed by the renormalization condition that $\langle \phi \rangle = 1$ is not renormalized, with no further finite renormalization. The kink solution is

$$\phi_0(x) = \tanh \frac{mx}{2} \quad (84)$$

and the corresponding small oscillations potential is

$$V(x) = -\frac{3}{2}\text{sech}^2 mx \quad (85)$$

which is an exactly solvable Posch-Teller reflectionless potential. The total phase shift and its first Born approximation are

$$\delta(k) = 2 \arctan \frac{3m}{2k}$$
$$\delta^{(1)}(k) = \frac{3m}{k} \quad (86)$$

and there are bound states at $\omega = 0$, $\omega = m\sqrt{3}/2$ and a "half-bound" threshold state at $\omega = m$. Substituting these data into eq. (82) gives

$$E_{1,1}[\phi_0] = \frac{3m^2}{16\pi} \left( \arccoth(2) - 2 \right) \quad . \quad (87)$$
9. Conclusions

We have presented a general procedure that is applicable to a variety of problems in quantum field theory. It gives a concrete prescription for handling field theory divergences in a concrete way. Ref. 23 extends this approach to systems with simple time dependence and Ref. 22 applies this formalism to nonzero temperature. Work is underway to apply these techniques to Higgs solitons in the Standard Model of the weak interactions, to field theories constrained to obey boundary conditions (such as the original Casimir problem), and to local densities. It could also be applied to compute the determinants associated with instantons and bounces.

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