FORWARD PERFORMANCE PROCESSES IN INCOMPLETE MARKETS AND ILL-POSED HJB EQUATIONS

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Abstract. We consider the problem of optimal portfolio selection under forward investment performance criteria in an incomplete market. The dynamics of the prices of the traded assets depend on a pair of stochastic factors, namely, a slow factor (e.g. a macroeconomic indicator) and a fast factor (e.g. stochastic volatility). We analyze the associated forward performance SPDE and provide explicit formulae for the leading order and first order correction terms for the forward investment process and the optimal feedback portfolios. They both depend on the investors initial preferences and the dynamically changing investment opportunities. The leading order terms resemble their time-monotone counterparts, but with the appropriate stochastic time changes resulting from averaging phenomena. The first-order terms compile the reaction of the investor to both the changes in the market input and his recent performance. Our analysis is based on an expansion of the underlying ill-posed HJB equation, and it is justified by means of an appropriate remainder estimate.

1. Introduction

This paper analyzes the optimal portfolio selection problem under forward investment criteria in incomplete markets. Incompleteness stems from the presence of imperfectly correlated stochastic factors that affect the dynamics of the traded assets. Such factors have been widely used in the literature and model an array of market inputs, like, among others, stochastic volatility, stochastic interest rates, predictability of asset returns, and various macroeconomic indicators. Herein, we consider a pair of such factors, which are taken, however, to move at different time scales.

The mathematical formulation of the problem of optimal investment in continuous time was pioneered by Merton in [Mc1], [Mc2] and is usually referred to as the Merton problem. In the classical Merton problem the investor faces a complete market and seeks an investment portfolio that optimizes her expected utility from wealth acquired in the investment process. Hereby, the investor’s utility function (or, equivalently, her preferences) is determined ex ante and does not change over time. The Merton problem has been studied in a variety of frameworks and we refer to the books [Du], [KS] for excellent accounts of the classical results. However, the setup of the Merton problem has two inherent drawbacks: 1) the investor has to decide about her utility function ex ante and cannot adapt it to market observations; 2) the investments over different time horizons are inconsistent with each other, that is: for $0 < T_1 < T_2$ the solution to the investment problem for the time period $[0, T_1]$ is not the restriction of the solution to the investment problem for the time period $[0, T_2]$ to $[0, T_1]$.

Forward investment performance criteria were introduced and developed in [MZ1] and [MZ2], and provide a complementary setting to the traditional expected utility framework. They allow for dynamic adaptation of the investor’s risk preferences given how the market conditions change and, also, take into account the updated performance of the implemented strategies. The forward performance process $U(t, \cdot), t \geq 0$ is a stochastic process adapted to the filtration of the investor with the properties that with probability one all functions $x \mapsto U(t, x)$ are increasing and concave (and, thus, can serve as utility functions); for every self-financing strategy $\pi$ and the corresponding portfolio value process $X^\pi$ the process $U(t, X^\pi(t)), t \geq 0$ is a supermartingale in the filtration of the investor; and there exists a self-financing strategy $\pi^*$ such that the process $U(t, X^{\pi^*}(t)), t \geq 0$
is a martingale in the filtration of the investor. The pair \((U, \pi^*)\) encodes how the preferences of
the investor and her optimal investment decisions jointly evolve in time from the given \(U(0, \cdot)\). For a more
detailed description of these criteria, further motivation and construction of concrete
examples, we refer the reader, among others, to [ElM], [KOZ], [MZ5], [MZ6], [NT] and [NZ1].

In practice, one can think of a client presenting a fund manager with the desired investment target
(e.g. 5% above the S&P 500 performance) and a band around the investment target (e.g. 4-6% above the S&P 500 performance), which give an indication about the client’s initial utility function
\(U(0, \cdot)\). Then, the fund manager’s problem is to find a pair \((U, \pi^*)\) with the given \(U(0, \cdot)\). Therefore,
the question of finding large classes of forward performance processes \(U\) and the corresponding
optimal portfolios \(\pi^*\) is of great importance.

Assuming the filtration of the investor to be generated by a Brownian motion and her forward
performance process \(U(t, x)\) to be an Itô process in \(t\) and twice continuously differentiable in \(x\), one can show (see [MZ5] and [NT] for more details) that \(U\) is a solution of the fully non-linear
stochastic partial differential equation (SPDE)

\[
\frac{dU(t, x)}{dt} = \frac{1}{2} \frac{||U_x(t, x) \lambda(t) + \sigma(t) \sigma^{-1}(t) a^W_x(t, x)||^2}{U_{xx}(t, x)} dt + a(t, x)^T dW(t).
\]

Here \(W = (W, \tilde{B})\) is a standard Brownian motion that generates the filtration of the investor;
\(W\) is a Brownian motion to whose filtration the asset prices are adapted; \(\sigma\) is the corresponding
volatility matrix of the asset prices and \(\sigma^{-1}\) is its Moore-Penrose pseudoinverse; \(\lambda\) is the market
price of risk; \(a = (a^W, a^{\tilde{B}})\) is a suitable stochastic process adapted to the filtration of the investor;
and the superscript \(T\) denotes transpose.

The forward SPDE (1.1) provides the analogue of the Hamilton-Jacobi-Bellman (HJB) equation
that is associated with the classical optimization problems of expected utility from terminal wealth.
As in the traditional setting, it is fully nonlinear and possibly degenerate. There are, however,
fundamental differences between (1.1) and its classical counterpart. Firstly, (1.1) is posed forward
in time, which makes the problem ill-posed. Secondly, the forward volatility process \(a(t, x)\) is up
to the investor to choose, in contrast to the classical case, where it is the mere outcome of the
Itô decomposition of the value function process. The specification of the correct class of forward
volatility processes is a very challenging problem, which remains open.

So far three classes of forward performance processes have been exhibited in the literature: 1) time-monotone forward performance processes, that is: forward performance processes which are of
finite variation in the time variable (see [MZ4] for more details); 2) homothetic forward performance
processes, that is: forward performance processes whose dependence on the investor’s wealth \(x\) is
of power form (see [NZ1] and [NT] for more details); 3) forward performance processes of factor
form in complete markets (see [NT] for more details). These three types of forward performance
processes result from significant simplifications of the SPDE (1.1) in certain special cases: time-
monotone forward performance processes are obtained by setting \(a \equiv 0\) and solving the resulting
partial differential equation (PDE); homothetic forward performance processes result from choosing
\(U\) to be of product form with power function dependence on \(x\); and forward performance processes
of factor form in complete markets are derived using a reduction of the SPDE (1.1) to a Hamilton-
Jacobi-Bellman (HJB) equation that can be linearized in the complete market framework using the
Fenchel-Legendre transform.

As in [NT] we consider forward performance processes of factor form (see Subsection 1.1 for
the exact details), so that (1.1) can be reduced to an HJB equation. However, we consider the
incomplete market case in which the HJB equation cannot be linearized by a simple transformation.
Nonetheless, we are able to find explicit formulas for the leading order and first-order correction
terms of the solution to such an HJB equation. These yield the leading order and first-order correction terms of the corresponding pair \((U, \pi^*)\). We complement these results by an appropriate estimate of the remainder term. A similar expansion for the classical Merton problem in an incomplete market was given in [FSZ]. In contrast to the Merton problem setup in [FSZ], we face the additional difficulty of the HJB equation being ill-posed. In addition, no general estimates of the remainder were given in [FSZ], so that our approach provides new insights in the Merton problem setting as well.

The following subsection describes our framework.

1.1. Setting. We consider \(n\) tradeable securities whose prices follow the stochastic differential equations

\[
dS_i(t) = S_i(t) \mu_i(Y^\delta(t), Y^\epsilon(t)) \, dt + S_i(t) \sigma_i(Y^\delta(t), Y^\epsilon(t))^T \, dW(t), \quad i = 1, 2, \ldots, n
\]

and where \(Y^\delta, Y^\epsilon\) are two observable real-valued stochastic factors. The factors are modelled by one-dimensional diffusion processes

\[
dY^\delta(t) = \delta b(Y^\delta(t)) \, dt + \sqrt{\delta} \kappa(Y^\delta(t)) \, dB_1(t),
\]

\[
dY^\epsilon(t) = \frac{1}{\epsilon} \gamma(Y^\epsilon(t)) \, dt + \frac{1}{\sqrt{\epsilon}} \alpha(Y^\epsilon(t)) \, dB_2(t).
\]

We think of \(\delta, \epsilon\) as being small positive numbers, so that \(Y^\delta\) should be thought of as a slow factor (e.g. a macroeconomic indicator) and \(Y^\epsilon\) as a fast factor (e.g. a fast mean-reverting stochastic volatility). Hereby, the noise \(W = (W, B_1, B_2)\) is jointly a \((d+2)\)-dimensional Brownian motion, \(W\) is a \(d\)-dimensional standard Brownian motion, \(B_1, B_2\) are one-dimensional standard Brownian motions, and the covariance structure is given by

\[
\rho^s_{jt} := \langle W_j, B_1 \rangle(t), \quad j = 1, 2, \ldots, d,
\]

\[
\rho^f_{jt} := \langle W_j, B_2 \rangle(t), \quad j = 1, 2, \ldots, d,
\]

\[
\rho^{sf}_{jt} := \langle B_1, B_2 \rangle(t).
\]

Since we allow for non-perfect correlation between the asset price processes and the stochastic factors, the market is in general incomplete.

In our setting the forward performance SPDE (1.1) reads

\[
dU(t, x) = \frac{1}{2} \left\| \frac{U_x(t, x) \lambda(Y^\delta(t), Y^\epsilon(t)) + \sigma(Y^\delta(t), Y^\epsilon(t)) \sigma(Y^\delta(t), Y^\epsilon(t))^{-1} a^W(t, x)}{U_{xx}(t, x)} \right\|^2 \, dt + a(t, x)^T \, d\tilde{W}(t).
\]

Here \(\sigma(Y^\delta(t), Y^\epsilon(t)) = (\sigma_1(Y^\delta(t), Y^\epsilon(t)), \ldots, \sigma_n(Y^\delta(t), Y^\epsilon(t)))\) is the volatility matrix of the stock price processes (1.2), and

\[
\lambda(Y^\delta(t), Y^\epsilon(t)) = (\sigma(Y^\delta(t), Y^\epsilon(t))^T)^{-1} \mu(Y^\delta(t), Y^\epsilon(t))
\]

is the market price of risk. The superscripts \(T\) and \(-1\) denote transpose and Moore-Penrose pseudoinverse as before; and \(\tilde{W}\) is the standard Brownian motion obtained from the Brownian motion \(W\) by left-multiplication with a suitable constant matrix.

In this paper we focus on solutions of (1.8) of factor form

\[
U(t, x) = V(t, x, Y^\delta(t), Y^\epsilon(t)),
\]

for some deterministic function \(V = V(t, x, y_1, y_2)\). As discussed in the introduction of [NT], such solutions are particularly natural from the economic point of view. Indeed, thinking of the forward performance process \(U\) as encoding the preferences of the investor on a set of trading strategies
given the state of the world she observes and assuming that there are only \textit{finitely} many quantities the investor keeps track of, it is natural to assume that the state enters her preferences through the corresponding finite number of factor processes. With a slight abuse of notation, we write $V(0, x)$ for the initial condition.

Assuming $V \in C^{1,2,2}$, applying Itô’s formula to $V(t, x, Y^\delta(t), Y^\epsilon(t))$, equating first the resulting martingale part with the martingale part on the right-hand side of (1.8), and then the two bounded variation parts, one concludes that the function $V(t, x, y_1, y_2)$ is a classical solution of the HJB equation

$$
(1.10) \quad V_t + A^\delta_y V - \frac{1}{2} \| V_x \lambda + \sigma \sigma^{-1}(V_{xy1} \sqrt{\delta} \kappa \rho^* + V_{xy2} \frac{1}{\sqrt{\epsilon}} \alpha \rho^f) \|_x^2 = 0.
$$

Here $A^\delta_y$ is the generator of the diffusion process $(Y^\delta, Y^\epsilon)$. We also note that the initial condition $U(0, \cdot)$ for the SPDE (1.8) translates into an initial condition for the HJB equation (1.10), so that the latter is posed in the “wrong” time direction and, in particular, one does not expect solutions to exist for all initial conditions or to depend continuously on them. In general, this ill-posedness is the main mathematical difficulty in dealing with forward performance processes.

The main results of the paper (Proposition 4.1, Theorem 4.3) identify explicitly the leading order and first order correction terms of the solution $V$ of (1.10) in the limit regime $\delta \downarrow 0$, $\epsilon \downarrow 0$. This allows to identify the leading order and first order correction terms of the corresponding pair $(U, \pi^*)$ explicitly as well (see Proposition 5.1). All our results are obtained under the following two assumptions.

\textbf{Assumption 1.1.} (i) The range of left-multiplication by the matrix $\sigma$ is all of $\mathbb{R}^d$, so that $\sigma \sigma^{-1}$ is the $d \times d$ identity matrix. In particular, this implies $n \geq d$.

(ii) The function $\xi \mapsto V(0, (V_x)^{-1}(0, e^{-\xi}))$ is a Laplace transform of a non-negative Borel measure on $\mathbb{R}$, where $(V_x)^{-1}(0, \cdot)$ is the inverse of the function $V_x(0, \cdot)$.

The latter condition is related to the ill-posedness of the initial value problem for the HJB equation (1.10), and will turn out to be necessary for the leading order term of $V$ to be well-defined. A class of possible initial conditions is given by $V(0, x) = c_1 x^{c_2}$, $c_1 > 0$, $c_2 \in (0, 1)$.

\textbf{Assumption 1.2.} The process $Y^\epsilon$ is positive recurrent with a unique invariant distribution $\mu$. Clearly, the latter does not depend on the value of $\epsilon$ (since a change in $\epsilon$ corresponds to a multiplication of the generator of $Y^\epsilon$ by a constant).

\textbf{1.2. Outline.} To ensure that the main ideas are not obscured by cumbersome notation we first consider the cases where only the slow factor $Y^\delta$ is present (“slow factor case”, Section 2) or only the fast factor $Y^\epsilon$ is present (“fast factor case”, Section 3).

In the slow factor case we provide explicit formulas for the leading order and first order correction terms of $V$ in Propositions 2.1 and 2.2 and justify the approximation of $V$ by such in Theorem 2.5. The corresponding results in the fast factor case can be found in Propositions 3.1 and 3.2 and Theorem 3.3. In Section 4 we consider the general case and give explicit formulas for the leading order and first order correction terms of $V$ in Proposition 4.1. The corresponding remainder estimate can be found in Theorem 4.3. Finally, in Section 5 we give explicit formulas for the portfolios associated with our approximation (Proposition 5.1) and explain in which sense they are approximately optimal (Remark 5.3).

\textbf{2. Forward investment problem with a slow factor}

The first situation we consider is the slow factor case, that is when $\mu_i$ and $\sigma_i$ in (1.2) depend only on $Y^\delta$, and so $V(t, x, y_1, y_2)$ in (1.9) does not depend on $y_2$. Moreover, to simplify the notation
we write $\rho$ for $\rho^s$ and $y$ for $y_1$ throughout the present section. In view of Assumption 1.1 and with these notations the HJB equation (1.10) becomes

$$V_t + \frac{1}{2} \delta \kappa(y)^2 V_{yy} + \delta b(y) V_y - \frac{1}{2} \frac{\|V_x \lambda(y) + V_{xy} \sqrt{\delta} \kappa(y) \rho\|^2}{V_{xx}} = 0.$$  

Here we aim to find an expansion of $V$ of the form

$$V = V^{(0)} + \sqrt{\delta} V^{(1)} + O(\delta)$$

in the limit regime $\delta \downarrow 0$. To this end, we will first derive expressions for $V^{(0)}$ and $V^{(1)}$ informally and then justify the resulting expansion in Theorem 2.5 below.

To obtain the leading order term $V^{(0)}$ we set $\delta = 0$ in (2.1):

$$V^{(0)} = - \frac{1}{2} \|\lambda(y)\|^2 \frac{V^{(0)}_{xx}}{V^{(0)}_x} = 0.$$  

In addition, we endow the latter equation with the initial condition $V^{(0)}(0, x, y) = V(0, x)$. The resulting problem corresponds to taking the volatility coefficient in the forward performance SPDE (1.8) to be zero. This is precisely the case of time-monotone forward performance processes studied in [MZ4]. The formula for the solution $V^{(0)}$ of (2.3) can be therefore recovered directly from [MZ4, Theorems 4 and 8].

Proposition 2.1 (Leading order term, slow factor). The solution $V^{(0)}$ of the HJB equation (2.3) with the initial condition $V^{(0)}(0, x, y) = V(0, x)$ admits the following representation in terms of a non-negative Borel measure $\nu$ on $\mathbb{R}$:

$$V^{(0)}(t, x, y) = u\left(\|\lambda(y)\|^2 t, x\right)$$

where $u$ is given by

$$u(t, x) = - \frac{1}{2} \int_0^t e^{-h^{(-1)}(s, x) + \frac{1}{2} \kappa_2^{(0)}(s, x)} h_x(s, h^{(-1)}(s, x)) \, ds + V(0, x),$$

$$h(t, x) = \int_{\mathbb{R}} e^{z x - \frac{1}{2} z^2 t} \nu(dz),$$

and $h^{(-1)}$ denotes the inverse of $h$ in the variable $x$.

This follows from a transformation of (2.3) to the ill-posed heat equation and Widder’s Representation Theorem of positive solutions to this equation ([Wi, Theorem 8.1]). Both this transformation and Widder’s Theorem will be used to construct higher order terms of the expansion.

Next, we turn to the correction term $V^{(1)}$ in (2.2). To obtain an equation for $V^{(1)}$ we plug $V^{(0)} + \sqrt{\delta} V^{(1)}$ into (2.1) and collect the terms on the order of $\sqrt{\delta}$. To this end, we note the expansions

$$\| (V^{(0)} + \sqrt{\delta} V^{(1)}) \lambda \|^2 + \sqrt{\delta} (2 V^{(0)} V^{(1)} \lambda \lambda^T \rho + O(\delta) ,$$

$$\frac{1}{V^{(0)}_{xx} + \sqrt{\delta} V^{(1)}_{xx}} = \frac{1}{V^{(0)}_{xx} V^{(0)}_{xx}} - \sqrt{\delta} \frac{V^{(1)}_{xx}}{V^{(0)}_{xx}} + O(\delta).$$

The resulting equation for $V^{(1)}$ reads

$$V^{(1)}_t + \frac{1}{2} \frac{\|\lambda(y)\|^2 (V^{(0)}_x)^2}{(V^{(0)}_{xx})^2} V^{(1)}_{xx} - \frac{\|\lambda(y)\|^2 V^{(0)}_x}{V^{(0)}_{xx}} V^{(1)}_x = \kappa(y) \lambda(y)^T \rho \frac{V^{(0)}_x V^{(0)}_y}{V^{(0)}_{xx}} \frac{V^{(0)}_y}{V^{(0)}_{xx}}$$
and it is endowed with the initial condition $V^{(1)}(0, x, y) = 0$, since the zeroth order term $V^{(0)}$ has already satisfied the initial condition for $V$.

**Proposition 2.2** (Correction term, slow factor). The solution $V^{(1)}$ of the PDE\(^{(2.8)}\) endowed with the initial condition $V^{(1)}(0, x, y) = 0$ is given by

$$V^{(1)}(t, x, y) = \frac{t}{2} \kappa(y) \lambda(y)^T \rho \frac{V_x^{(0)}(t, x, y)V_{xx}^{(0)}(t, x, y)}{V_{xx}^{(0)}(t, x, y)}.$$

**Proof.** We start by introducing the change of variables

$$\tag{2.9} (t, \xi, y) := \left(t, -\log V_x^{(0)} - \frac{\|\lambda(y)\|^2}{2} t, y\right)$$

and set $w^{(0)}(t, \xi, y) = V_x^{(0)}(t, x, y)$, $w^{(1)}(t, \xi, y) = V^{(1)}(t, x, y)$. The latter functions are well-defined, since $V^{(0)}$ is strictly increasing and strictly concave in $x$, so that $\xi$ is a strictly increasing function of $x$. By viewing the equation \((2.3)\) as the “linear” equation

$$V^{(0)}_t + \frac{\|\lambda(y)\|^2}{2} \left(\frac{V_x^{(0)}}{V^{(0)}_{xx}}\right)^2 V_{xx}^{(0)} - \|\lambda(y)\|^2 \left(\frac{V_x^{(0)}}{V^{(0)}_{xx}}\right) V_{xx}^{(0)} = 0$$

with coefficients depending on $V^{(0)}$ and computing

$$V^{(0)}_t = w^{(0)}_t + \frac{\|\lambda(y)\|^2}{2} w^{(0)}_\xi \left(-\left(\frac{V_x^{(0)}}{V^{(0)}_{xx}}\right)_x - 2\right),$$

$$V^{(0)}_x = w^{(0)}_\xi \left(-\frac{V_{xx}^{(0)}}{V_x^{(0)}}\right),$$

$$V^{(0)}_{xx} = w^{(0)}_{\xi\xi} \left(\frac{V_{xx}^{(0)}}{V_x^{(0)}}\right)^2 + w^{(0)}_\xi \left(\frac{V_x^{(0)}}{V^{(0)}_{xx}}\right)^2 \left(\frac{V_x^{(0)}}{V^{(0)}_{xx}}\right)_x,$$

we deduce that \((2.3)\) becomes

$$\tag{2.13} w^{(0)}_t + \frac{\|\lambda(y)\|^2}{2} w^{(0)}_{\xi\xi} = 0$$

in the coordinates $(t, \xi, y)$.

A similar computation for $w^{(1)}$ shows that \((2.8)\) transforms into

$$\tag{2.14} w^{(1)}_t + \frac{\|\lambda(y)\|^2}{2} w^{(1)}_{\xi\xi} = \kappa \lambda(y)^T \rho t \lambda(y)^T \lambda'(y) w^{(0)}_{\xi\xi}$$

in the new variables. Hereby, to obtain the right-hand side of \((2.14)\) we have relied on the following two considerations:

(a) By differentiating \((2.13)\) in $y$ and rearranging one obtains

$$(w^{(0)}_y)_t + \frac{\|\lambda(y)\|^2}{2} (w^{(0)}_y)_{\xi\xi} = -\lambda(y)^T \lambda'(y) w^{(0)}_{\xi\xi}.$$

Moreover, the latter equation, endowed with the initial condition $w^{(0)}_y(0, \xi, y) = 0$, has the unique solution $w^{(0)}_y = -t \lambda(y)^T \lambda'(y) w^{(0)}_{\xi\xi}$ (note that the uniqueness of the solution is a consequence of Widder’s Theorem). In the original coordinates this solution reads

$$\tag{2.15} V^{(0)}_y = -t \lambda(y)^T \lambda'(y) \left(\frac{V_x^{(0)}}{V^{(0)}_{xx}}\right)_x \frac{V_x^{(0)}}{V^{(0)}_{xx}}.$$
(b) In addition, by using (2.11), (2.12) one finds
\[ w^{(0)}_{\xi\xi\xi} = -\frac{V^{(0)}_x}{V^{(0)}_{xx}} \left( \frac{\left( \frac{V^{(0)}_x}{V^{(0)}_{xx}} \right)^2 \left( \frac{V^{(0)_x}}{V^{(0)}_{xx}} \right)}{x} \right). \]

A combination of (a) and (b) gives the right-hand side of (2.14). At this point, one can check that the unique solution of (2.14), endowed with the initial condition \( w^{(1)}(0, \xi, y) = 0 \) is given by
\[ w^{(1)}(t, \xi, y) = \frac{t^2}{2} \kappa(y) \lambda(y)^T \rho \lambda(y)^T \lambda'(y) w^{(0)}_{\xi\xi\xi}. \]

Hereby, the uniqueness part of the statement follows by another application of Widder’s Theorem. To obtain the proposition, it remains to change the coordinates back to \((t, x, y)\) and use that
\[ V^{(0)}_{xy} = -t \lambda(y)^T \lambda'(y) \left( \frac{\left( \frac{V^{(0)}_x}{V^{(0)}_{xx}} \right)^2 \left( \frac{V^{(0)}_x}{V^{(0)}_{xx}} \right)}{x} \right), \]
which can be obtained from (2.15) by a differentiation in \( x \).

We now give another representation of the correction term \( V^{(1)} \) which has a natural interpretation in terms of the original forward performance problem.

**Proposition 2.3** (Natural parametrization of correction term, slow factor). The solution \( V^{(1)} \) of the PDE (2.8) endowed with the initial condition \( V^{(1)}(0, x, y) = 0 \), written as \( w^{(1)}(t, \xi, y) \) in the coordinates \((t, \xi, y)\) defined in (2.9), admits the representation
\[ w^{(1)}(t, \xi, y) = \int_0^t w^{(1),s}(t, \xi, y) \, ds \]
where each \( w^{(1),s} \) is the solution of the initial value problem
\[ w^{(1),s}_t + \frac{\|\lambda(y)\|^2}{2} w^{(1),s}_{\xi\xi} = 0, \quad t \geq s, \]
\[ w^{(1),s}(s, \xi, y) = s \kappa(y) \lambda(y)^T \rho \|\lambda(y)\|^2 w^{(0)}_{\xi\xi}(s, \xi, y). \]

In particular, each \( w^{(1),s} \) can be represented as
\[ w^{(1),s}(t, \xi, y) = \int_{\mathbb{R}} e^{z\xi - z^2(t-s)} \nu^{(s)}(dz) \]
with \( \nu^{(s)} \) being a signed Borel measure on \( \mathbb{R} \).

In the original coordinates, the same representation reads
\[ V^{(1)}(t, x, y) = \int_0^t V^{(1),s}(t, x, y) \, ds \]
where each \( V^{(1),s} \) is the solution of the initial value problem
\[ V^{(1),s}_t + \frac{\|\lambda(y)\|^2}{2} \left( \frac{V^{(0)}_x}{V^{(0)}_{xx}} \right)^2 V^{(1),s}_{xx} - \|\lambda(y)\|^2 \frac{V^{(0)}_x}{V^{(0)}_{xx}} V^{(1),s}_x = 0, \]
\[ V^{(1),s}(s, x, y) = \kappa(y) \lambda(y)^T \rho \frac{V^{(0)}_x(s, x, y) V^{(0)}_{xy}(s, x, y)}{V^{(0)}_{xx}(s, x, y)}. \]
Remark 2.4. We remark that each of the processes $V^{(1),s}$ (or, equivalently, $w^{(1),s}$) can be viewed as an auxiliary forward performance process. These should be interpreted as the first order corrections that the investor makes at any given time $s$ in reaction to the market conditions she observes. Furthermore, when making a projection of her future preferences from a time $t$ on, the investor corrects her leading order forward performance criterion $V^{(0)}$ (or, equivalently, $w^{(0)}$) by aggregating all her previous first order corrections $V^{(1),s}$, $s \in [0,t]$ (or, equivalently, $w^{(1),s}$).

Proof of Proposition 2.3. We first recall that $w^{(0)}$ is a classical solution of the forward heat equation (2.13). Hence, the same is true for $w^{(0)}_{\xi \xi}$. At this point, an application of Widder’s Theorem shows that the initial value problem (2.17), (2.18) has a solution that exists for all $t \geq s$. In other words, each function $w^{(1),s}$, $s \geq 0$ is well-defined. Since the forward heat equation with source (2.14) has a unique classical solution starting from the zero initial condition by Widder’s Theorem, the representation (2.19) will follow once we establish that the right-hand side of (2.19) is a classical solution of (2.14). This is the result of the following computation:

\[
\partial_t \left( \int_0^t w^{(1),s}(t, \xi, y) \, ds \right) = w^{(1),t}(t, \xi, y) + \int_0^t w^{(1),s}(t, \xi, y) \, ds
\]

\[
= t \kappa(y) \lambda(y)^T \rho \lVert \lambda(y) \rVert^2 w^{(0)}(t, \xi, y) - \int_0^t \frac{\|\lambda(y)\|^2}{2} w^{(1),s}(t, \xi, y) \, ds
\]

\[
= t \kappa(y) \lambda(y)^T \rho \lVert \lambda(y) \rVert^2 \frac{w^{(0)}_{\xi \xi}(t, \xi, y) - \|\lambda(y)\|^2}{2} \partial_{\xi \xi} \left( \int_0^t w^{(1),s}(t, \xi, y) \, ds \right).
\]

Finally, (2.19) follows from another application of Widder’s Theorem [W1] Theorem 8.1], and (2.20) is the result of writing (2.19) in the original coordinates $(t, x, y)$.

The next theorem shows that, under appropriate assumptions, the error in the approximation of the true value function $V$ by $V^{(0)} + \sqrt{\delta} V^{(1)}$ is indeed on the order of $\delta$ as one would expect. To this end, we define the non-linear functional

\[
\eta := \frac{\|\lambda\|^2}{2\delta} \frac{V_{xx}}{V_{xx}} - \frac{V_x^{(0)}}{V_{xx}^{(0)}}^2 + \frac{\kappa \lambda^T \rho}{\sqrt{\delta}} \frac{V_x V_{xy} - V_{xx}^{(0)} V_{xy}^{(0)}}{V_{xx}^{(0)}} + \frac{1}{2} \kappa^2 \|\rho\|^2 \frac{V_{xy}^2}{V_{xx}^2} - \frac{1}{2} \kappa^2 V_{yy} - b V_y,
\]

recall the change of coordinates $(t, x, y) \mapsto (t, \xi, y)$ of (2.20), and set

\[
\tilde{\eta}(t, \xi, y) = \eta(t, x, y).
\]

The bound on the approximation error can then be stated as follows.

Theorem 2.5 (Remainder estimate, slow factor). Suppose that there exist $\delta_0 > 0$ and $T \leq \infty$ such that for all $\delta \in (0, \delta_0)$ the HJB equation (2.1) has a solution $V \in C^{1,2,2}([0, T) \times (0, \infty) \times \mathbb{R})$ which is increasing and strictly concave in the second argument. Then the quantity

\[
(2.24) \quad \int_{\mathbb{R}} e^{-\frac{\delta}{2 \|\lambda\|^2 s}} \sum_{k=0}^{\infty} \frac{(-1)^k z^{2k}}{(2k)!} \|\lambda\|^{2k} k^k \left( \frac{d}{d\xi} \right)^{2k} \int_0^t \int_{\mathbb{R}} \tilde{\eta}(s, \xi - \chi, y) s^{-1/2} e^{-\frac{\delta}{2 \|\lambda\|^2 s}} d\chi \, ds \, dz,
\]

with $\tilde{\eta}$ as in (2.23), is well-defined and finite for all $\delta \in (0, \delta_0)$. Moreover, for every $(t, x, y) \in [0, T) \times (0, \infty) \times \mathbb{R}$ for which the limit superior

\[
\lim_{\delta \downarrow 0} \int_{\mathbb{R}} e^{-\frac{\delta}{2 \|\lambda\|^2 s}} \sum_{k=0}^{\infty} \frac{(-1)^k z^{2k}}{(2k)!} \|\lambda\|^{2k} k^k \left( \frac{d}{d\xi} \right)^{2k} \int_0^t \int_{\mathbb{R}} \tilde{\eta}(s, \xi - \chi, y) s^{-1/2} e^{-\frac{\delta}{2 \|\lambda\|^2 s}} d\chi \, ds \, dz
\]

is finite, the error bound

\[
\lim_{\delta \downarrow 0} \delta^{-1} \left| V(t, x, y) - V^{(0)}(t, x, y) - \sqrt{\delta} V^{(1)}(t, x, y) \right| < \infty
\]
Proof of Theorem 2.5. We start by expressing the HJB equation (2.1) as

\[ V_t - \frac{||\lambda||^2}{2} V_{xx} - \sqrt{\delta} \kappa \lambda^T \rho V_x V_{yy} + \frac{\delta}{2} \kappa^2 ||\rho||^2 V_{xy}^2 - \frac{\delta}{2} \kappa^2 V_{yy} - \delta b V_y. \]  

Next, we write \( V = V^{(0)} + \sqrt{\delta} V^{(1)} + \sqrt{\delta} Q \), insert the latter expression into the left-hand side of (2.27), and expand in \( \sqrt{\delta} \) using the elementary identity

\[ \frac{1}{a + \sqrt{\delta} b} = \frac{1}{a} - \sqrt{\delta} \frac{b}{a^2 + \sqrt{\delta} ab}, \quad a < 0, \quad b < -\frac{a}{\sqrt{\delta}}. \]

Recalling that the functions \( V^{(0)} \) and \( V^{(1)} \) were constructed in such a way that all terms in (2.27) on the orders of 1 and \( \sqrt{\delta} \) cancel out, and collecting the remaining terms we obtain after a lengthy but straightforward computation that

\[ \sqrt{\delta} Q_t - \sqrt{\delta} ||\lambda||^2 \frac{V^{(0)}_{xx}}{V^{(0)}_x} Q_x + \sqrt{\delta} \frac{||\lambda||^2}{2} \left( \frac{V^{(0)}_x}{V^{(0)}_{xx}} \right)^2 Q_{xx} = \delta \eta. \]

Next, we let \( \tilde{Q} := \sqrt{\delta} Q \), recall the change of coordinates

\[ (t, \xi, y) := \left( t, -\log V^{(0)}_x(t, x, y) - \frac{||\lambda(y)||^2}{2} t, y \right) \]

of (2.24), and define \( q(t, \xi, y) := \tilde{Q}(t, x, y) \). By the same computation as in the proof of Proposition 2.2, the partial differential equation (2.28) can be rewritten as

\[ q_t + \frac{||\lambda||^2}{2} q_{\xi \xi} = \delta \tilde{\eta} \]

where \( \tilde{\eta}(t, \xi, y) = \eta(t, x, y) \) as before. Then, Duhamel’s principle for the backward equation (2.29) implies that

\[ \frac{1}{\sqrt{2\pi t}} \int_{\mathbb{R}} q(t, \xi - \chi, y) e^{-\frac{\chi^2}{2||\lambda||^2 t}} d\chi = \delta \int_{\mathbb{R}} \left( \frac{1}{\sqrt{2\pi s}} \right) \int_{\mathbb{R}} \tilde{\eta}(s, \xi - \chi, y) e^{-\frac{\chi^2}{2||\lambda||^2 s}} d\chi ds. \]

It follows that the right-hand side of the latter equation is in the domain of the inverse Weierstrass transform in the sense of \( \text{[Wi2, equations (5), (6)]} \). This, in turn, implies that the quantity in (2.24) is well-defined and finite for all \( \delta \in (0, \delta_0) \). Moreover, applying the inverse Weierstrass transform to both sides of (2.30), we see that, up to a multiplicative constant, the function \( \xi \mapsto \delta^{-1} q(t, \xi, y) \) is given by the quantity inside the absolute value in (2.25). The statement of the theorem is now immediate. □
3. Forward investment problem with a fast factor

The second situation we consider is the fast factor case, that is when \( \mu_i \) and \( \sigma_i \) in (1.2) depend only on \( Y^i \), and so \( V(t, x, y_1, y_2) \) in (1.9) does not depend on \( y_1 \). To simplify the notation, we write \( \rho \) for \( \rho^T \) and \( y \) for \( y_2 \) throughout this section. With these notations and in view of Assumption 1.1, the HJB equation (1.10) becomes

\[
V_t + \frac{\alpha(y)^2}{2\epsilon} V_{yy} + \frac{\gamma(y)}{\epsilon} V_y - \frac{1}{2} \frac{||V_x \lambda(y) + V_{xy} \frac{1}{\sqrt{\epsilon}} \alpha(y) \rho||^2}{V_{xx}} = 0.
\]

Our goal here is to find an explicit expansion of the solution \( V \) to (3.1) of the form

\[
V = V^{(0)} + \sqrt{\epsilon} V^{(1)} + O(\epsilon)
\]

in the limit regime \( \epsilon \downarrow 0 \). As in the previous section, we will first derive formulas for \( V^{(0)} \) and \( V^{(1)} \) informally, and then justify the resulting expansion by means of a suitable remainder estimate.

To find \( V^{(0)} \) we plug (3.2) into (3.1) and collect the leading order terms (namely, those on the order of \( \epsilon^{-1} \)) to get

\[
\frac{\alpha(y)^2}{2} V_{yy}^{(0)} + \gamma(y) V_y^{(0)} - \frac{1}{2} \alpha(y)^2 \|\rho\|^2 \frac{(V_{xy}^{(0)})^2}{V_{xx}^{(0)}} = 0.
\]

Note that we can satisfy (3.3) by choosing \( V^{(0)} \) as a function of \( t \) and \( x \) only. As we explain below, the exact choice of \( V^{(0)} \) will be pinned down by the lower order terms in the expansion of (3.1).

To proceed, we plug (3.2) into (3.1), and collect the terms of order \( \epsilon^{-1/2} \). We obtain

\[
-\alpha(y) \lambda(y)^T \rho \frac{V_x^{(0)} V_{xy}^{(0)}}{V_{xx}^{(0)}} - \alpha(y)^2 \|\rho\|^2 \frac{V_{xy}^{(0)} V_{yy}^{(1)}}{V_{xx}^{(0)}} + \alpha(y)^2 \|\rho\|^2 \frac{2 V_{xy}^{(0)} V_{xx}^{(1)}}{V_{xx}^{(0)} V_{xx}^{(0)}} + \alpha(y)^2 \frac{V_{yy}^{(1)} + \gamma(y) V_y^{(1)}}{2} = 0.
\]

Choosing \( V^{(0)} \) to be independent of \( y \) as we noted earlier, the latter equation simplifies to

\[
\frac{\alpha(y)^2}{2} V_{yy}^{(1)} + \gamma(y) V_y^{(1)} = 0.
\]

Clearly, the above equation can be satisfied by choosing \( V^{(1)} \) to be a function of \( (t, x) \) only. As with \( V^{(0)} \), the exact choice of \( V^{(1)} \) will result from considering lower order terms in the expansion of (3.1).

Next, we insert the extended expansion

\[
V = V^{(0)} + \sqrt{\epsilon} V^{(1)} + \epsilon V^{(2)} + \epsilon^{3/2} V^{(3)} + O(\epsilon^2)
\]

into (3.1) in order to find the terms of order 1. This results in the equation

\[
V_t^{(0)} - \frac{||\lambda(y)||^2}{2} \frac{(V_{xx}^{(0)})^2}{V_{xx}^{(0)}} + \frac{\alpha(y)^2}{2} V_{yy}^{(2)} + \gamma(y) V_y^{(2)} = 0.
\]

Hereby, we have used the fact that \( V^{(0)} \) and \( V^{(1)} \) do not depend on \( y \). The next proposition shows that there is a unique choice of \( V^{(0)} \) such that \( V^{(0)}(0, x, y) = V(0, x) \) and the equation (3.4), viewed as an elliptic partial differential equation for \( V^{(2)} \), has a solution.

Proposition 3.1 (Leading order term, fast factor). There is a unique function \( V^{(0)} \) such that \( V^{(0)}(0, x, y) = V(0, x) \) and the equation (3.4) possesses a solution \( V^{(2)} \). With

\[
\tilde{\lambda} = \left( \int_{\mathbb{R}} \|\lambda(y)\|^2 \mu(dy) \right)^{1/2},
\]

\[ \frac{\tilde{\lambda}^2}{2} = \frac{\lambda^T \lambda}{2}, \]

where \( \lambda \) is given by (3.2) and \( \mu(dy) \) is the measure that appears in the statement of Assumption 1.1.
such a function $V^{(0)}$ admits the representation

$$ V^{(0)}(t, x, y) = V^{(0)}(t, x) = u(\bar{\lambda}^2 t, x) $$

where $u$ is given by

$$ u(t, x) = -\frac{1}{2} \int_0^t e^{-h^{-1}(s, x)} h_x(s, h^{-1}(s, x)) \, ds + V(0, x), $$

$$ h(t, x) = \int_{\mathbb{R}} e^{x - \frac{1}{2} z^2 t} \nu(dz). $$

Here $h^{-1}$ is the inverse of $h$ in the variable $x$; $\nu$ is a non-negative Borel measure on $\mathbb{R}$; and $C$ is a real constant.

Proof. We start by integrating \textbf{[3.1]} with respect to the invariant distribution $\mu$ of Assumption \textbf{1.2}. Since $V^{(0)}$ does not depend on $y$ and

$$ \int_{\mathbb{R}} \left( \frac{\alpha(y)^2}{2} Y_{yy}^{(2)} + \gamma(y) V_y^{(2)} \right) \mu(dy) \equiv 0 $$

(due to the invariance of $\mu$), we obtain

$$ V_t^{(0)} - \frac{\bar{\lambda}^2}{2} \frac{V_x^{(0)}(0)^2}{V_{xx}^{(0)}} = 0. $$

We easily conclude using \textbf{[MZ4] Theorems 4 and 8].}

To obtain $V^{(1)}$ we will expand \textbf{[3.1]} up to order $\epsilon^{1/2}$, which however requires further information on $V^{(2)}$. As a first step, we subtract from the equation \textbf{[3.1]} its averaged version \textbf{[3.7]} to get

$$ \frac{\alpha(y)^2}{2} V_{yy}^{(2)} + \gamma(y) V_y^{(2)} = \frac{\|\lambda(y)\|^2 - \bar{\lambda}^2}{2} \frac{V_x^{(0)}(0)^2}{V_{xx}^{(0)}}. $$

We introduce the notation

$$ \phi(y) = \int_0^\infty \mathbb{E}^y \left[ \|\lambda(Y^1(s))\|^2 - \bar{\lambda}^2 \right] \, ds, $$

where $Y^1$ denotes the fast factor process, solution of the SDE \textbf{[1.4]}, but with $\epsilon = 1$. Then (see e.g. \textbf{FTSS Section 3.2, p. 94]), the solution $V^{(2)}$ of \textbf{[3.8]} admits the stochastic representation

$$ V^{(2)}(t, x, y) = -\frac{1}{2} \frac{(V_x^{(0)}(t, x))^2}{V_{xx}^{(0)}(t, x)} \phi(y) + C(t, x), $$

where $C(t, x)$ is a function that does not depend on $y$.

We can now expand the HJB equation \textbf{[3.1]} up to order $\epsilon^{1/2}$ to obtain

$$ V_t^{(1)} + \frac{\|\lambda(y)\|^2}{2} \left( \frac{V_x^{(0)}(0)}{V_{xx}^{(0)}} \right)^2 V_{xx}^{(1)} - \frac{V_x^{(0)}(0)}{V_{xx}^{(0)}} \lambda(y)^T \left( \frac{(V_x^{(0)}(0))^2}{V_{xx}^{(0)}} \right) x \alpha(y) \rho $$

$$ + \frac{\alpha(y)^2}{2} V_{yy}^{(3)} + \gamma(y) V_y^{(3)} = 0. $$

Averaging this equation with respect to the invariant distribution $\mu$ of Assumption \textbf{1.2} we obtain further

$$ V_t^{(1)} + \frac{\bar{\lambda}^2}{2} \left( \frac{V_x^{(0)}(0)}{V_{xx}^{(0)}} \right)^2 V_{xx}^{(1)} - \bar{\lambda} \frac{V_x^{(0)}(0)}{V_{xx}^{(0)}} V_{xx}^{(1)} - \frac{V_x^{(0)}(0)}{V_{xx}^{(0)}} \left( \frac{(V_x^{(0)}(0))^2}{V_{xx}^{(0)}} \right) x \left( \int_{\mathbb{R}} \phi'(y) \alpha(y) \lambda(y)^T \mu(dy) \right) \rho = 0, $$

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which is the desired partial differential equation for \( V^{(1)} \). Since \( V^{(0)} \) satisfies the initial condition for \( V \), we endow (3.11) with the initial condition \( V^{(1)}(0, x, y) = 0 \).

**Proposition 3.2** (Correction term, fast factor). The unique classical solution of the partial differential equation (3.11) with the initial condition \( V^{(1)}(0, x, y) = 0 \) is given by

\[
V^{(1)}(t, x, y) = t \frac{V^{(0)}_x}{V^{(0)}_{xx}} \left( \frac{V^{(0)}_x}{V^{(0)}_{xx}} \right)^2 \left( \int_{\mathbb{R}} \phi'(y) \alpha(y) \lambda(y)^T \mu(dy) \right) \rho,
\]

with \( \phi \) as in (3.9).

**Proof.** We introduce a new space variable

\[
\xi := - \log V^{(0)}_x - \frac{\bar{\lambda}^2}{2} t.
\]

Since \( V^{(0)}_x \) is strictly increasing and strictly concave in \( x \) for any given \( t \), \( \xi \) is a strictly increasing function of \( x \) and we may define

\[
w^{(0)}(t, \xi) := V^{(0)}(t, x) \quad \text{and} \quad w^{(1)}(t, \xi) := V^{(1)}(t, x).
\]

A straightforward application of the chain rule together with the partial differential equation for \( V^{(0)}_x \) corresponding to (3.7) now gives

\[
V^{(0)}_t = w^{(0)}_t + \frac{\bar{\lambda}^2}{2} w^{(0)}_\xi \left( - \left( \frac{V^{(0)}_x}{V^{(0)}_{xx}} \right)_x - 2 \right),
\]

\[
V^{(0)}_x = w^{(0)}_\xi \left( - \frac{V^{(0)}_{xx}}{V^{(0)}_x} \right),
\]

\[
V^{(0)}_{xx} = w^{(0)}_{\xi \xi} \left( \frac{V^{(0)}_{xx}}{V^{(0)}_x} \right)^2 + w^{(0)}_\xi \left( - \frac{V^{(0)}_{xx}}{V^{(0)}_x} \right)_x.
\]

In addition, we note that (3.7) can be viewed as the “linear” equation

\[
(3.16) \quad V^{(0)}_t + \frac{1}{2} \bar{\lambda}^2 \left( \frac{V^{(0)}_x}{V^{(0)}_{xx}} \right)^2 \frac{V^{(0)}_{xx}}{V^{(0)}_x} - \bar{\lambda}^2 \left( \frac{V^{(0)}_x}{V^{(0)}_{xx}} \right) V^{(0)}_x = 0
\]

(with the coefficients depending on the solution). Plugging (3.13), (3.14) and (3.15) into (3.16) we obtain

\[
w^{(0)}_t + \frac{1}{2} \bar{\lambda}^2 \left( \frac{V^{(0)}_x}{V^{(0)}_{xx}} \right)^2 \left( w^{(0)}_\xi \left( \frac{V^{(0)}_{xx}}{V^{(0)}_x} \right)^2 + w^{(0)}_\xi \left( - \frac{V^{(0)}_{xx}}{V^{(0)}_x} \right)_x \right) - \bar{\lambda}^2 \left( \frac{V^{(0)}_x}{V^{(0)}_{xx}} \right) w^{(0)}_\xi \left( - \frac{V^{(0)}_{xx}}{V^{(0)}_x} \right) = 0
\]

which readily simplifies to

\[
(3.17) \quad w^{(0)}_t + \frac{1}{2} \bar{\lambda}^2 w^{(0)}_{\xi \xi} = 0.
\]

A similar computation shows that the transformed correction term \( w^{(1)} \) satisfies the forward heat equation

\[
(3.18) \quad w^{(1)}_t + \frac{1}{2} \bar{\lambda}^2 w^{(1)}_{\xi \xi} = w^{(0)}_t \left( \int_{\mathbb{R}} \phi'(y) \alpha(y) \lambda(y)^T \mu(dy) \right) \rho,
\]

with the initial condition \( w^{(1)}(0, \xi) = 0 \). At this point, it is easy to check (using (3.17)) that

\[
(3.19) \quad w^{(1)} = t w^{(0)}_\xi \left( \int_{\mathbb{R}} \phi'(y) \alpha(y) \lambda(y)^T \mu(dy) \right) \rho
\]
satisfies (3.18) with the desired initial condition. Changing back to the original coordinates we easily obtain (3.12).

The uniqueness part of the proposition follows from the uniqueness of the solution of the Cauchy problem for the forward heat equation (see [Wi, Theorem 8.1]). □

Next, we give another representation for the correction term \( V^{(1)} \) which has a natural interpretation in terms of the original portfolio optimization problem.

**Proposition 3.3** (Natural parametrization of correction term, fast factor). Let \( w^{(0)}, w^{(1)} \) be the functions \( V^{(0)}, V^{(1)} \) from Propositions 3.1, 3.2 written in the coordinates

\[
(t, \xi) := \left( t, -\log V_x^{(0)} - \frac{\lambda^2}{2} t \right).
\]

Then:

(a) \( w^{(1)} \) admits the representation

\[
w^{(1)}(t, \xi) = \int_0^t w^{(1),s}(t, \xi) \, ds
\]

where each \( w^{(1),s} \) is the solution of the initial-value problem

\[
w^{(1),s}_t + \frac{\lambda^2}{2} w^{(1),s}_{\xi\xi} = 0, \quad t \geq s,
\]

\[
w^{(1),s}(s, \xi) = w^{(0)}_{\xi\xi}(s, \xi) \left( \int_{\mathbb{R}} \phi'(y) \alpha(y) \lambda(y)^T \mu(dy) \right) \rho.
\]

and can be therefore represented as

\[
w^{(1),s}(t, \xi) = \int_{\mathbb{R}} e^{s\xi - z^2(t-s)} \nu(s)(dz),
\]

with \( \nu(s) \) being a suitable signed Borel measure on \( \mathbb{R} \).

(b) In the original coordinates, the same representation reads

\[
V^{(1)}(t, x) = \int_0^t V^{(1),s}(t, x) \, ds
\]

where each \( V^{(1),s} \) is the solution of the initial-value problem

\[
V^{(1),s}_t + \frac{\lambda^2}{2} \left( \frac{V_x^{(0)}}{V_{xx}^{(0)}} \right)^2 V^{(1),s}_{xx} - \frac{\lambda^2}{2} \frac{V_x^{(0)}}{V_{xx}^{(0)}} V^{(1),s}_x = 0, \quad t \geq s,
\]

\[
V^{(1),s}(s, x) = \frac{V_x^{(0)}(s, x)}{V_{xx}^{(0)}(s, x)} \left( \frac{V_x^{(0)}(s, x)}{V_{xx}^{(0)}(s, x)} \right)^2 \left( \int_{\mathbb{R}} \phi'(y) \alpha(y) \lambda(y)^T \mu(dy) \right) \rho.
\]

**Remark 3.4.** The quantities \( V^{(1),s} \) (or, equivalently, \( w^{(1),s} \)) of Proposition 3.3 should be interpreted in the same way as their analogues in the slow factor case. We refer to Remark 2.4 above for more details.

**Proof of Proposition 3.3.** We first recall from the proof of Proposition 3.2 that \( w^{(0)} \) is a classical solution of the forward heat equation (3.17). Hence, \( w^{(0)}_{\xi\xi} \) is a solution of the same equation and, therefore, the solutions \( w^{(1),s}, s \geq 0 \) of (3.21), (3.22) are well-defined and given by

\[
w^{(1),s}(t, x) = w^{(0)}_{\xi\xi}(t, \xi) \left( \int_{\mathbb{R}} \phi'(y) \alpha(y) \lambda(y)^T \mu(dy) \right) \rho.
\]
Therefore the right-hand side of the representation (3.20) is equal to the right-hand side of (3.19), and, thus, (3.20) immediately follows. Moreover, for every \( x, x' \in C \), the error in the approximation of the true value function \( V \) by \( V^{(0)} + \sqrt{\varepsilon} V^{(1)} \) is of the order \( \varepsilon \). To this end, we introduce the non-linear functional\(^{(3.26)}\)

\[
\eta := \frac{1}{2} \varepsilon^{1/2} \| \lambda(y) \|^2 \left( \frac{V_x^{(0)}}{V_{xx}^{(0)}} \right)^2 \frac{1}{V_{xx}} (V_{xx} - V_{xx}^{(0)})
+ \frac{1}{2\varepsilon} \| \lambda(y) \|^2 \left( \frac{V_x^{(0)}}{V_{xx}^{(0)}} \right)^2 \frac{1}{V_{xx}} (V_{xx} - V_{xx}^{(0)} - \varepsilon^{1/2} V_{xx}^{(1)}) (V_{xx} - V_{xx}^{(0)})
+ \lambda(y)^T V_x^{(0)} (\lambda(y) V_x^{(1)} + \alpha(y) \rho V_{xy}^{(2)}) \left( \frac{1}{V_{xx}} - \frac{1}{V_{xx}^{(0)}} \right)
- \frac{1}{\varepsilon} \| \lambda(y) \|^2 V_x^{(0)} (V_x - V_x^{(0)} - \varepsilon^{1/2} V_x^{(1)}) \left( \frac{1}{V_{xx}} - \frac{1}{V_{xx}^{(0)}} \right)
+ \frac{1}{\varepsilon^{3/2}} \lambda(y)^T \alpha(y) \rho V_{xy}^{(2)} (V_{xy} - \varepsilon V_x^{(2)}) \left( \frac{1}{V_{xx}} \right) + \frac{1}{2\varepsilon} \frac{1}{V_{xx}} \| \lambda(y) V_x - \lambda(y) V_x^{(0)} + \varepsilon^{-1/2} \alpha(y) \rho V_{xy} \|^2
+ \frac{1}{\varepsilon^{2}} \left( \left( V_t - \frac{1}{2} \frac{\| V_x \lambda(y) + V_{xy} \sqrt{\varepsilon} \alpha(y) \rho \|^2}{V_{xx}} \right) - \varepsilon \left( V_t^{(0)} - \| \lambda(y) \|^2 \frac{V_x^{(0)}}{V_{xx}^{(0)}} \right) \right)
- \frac{\varepsilon}{2} \frac{V^{(1)}_t}{V_{xx}^{(0)}} + \frac{\| \lambda(y) \|^2}{2} \left( \frac{V^{(0)}_x}{V_{xx}^{(0)}} \right)^2 \left( V_{xx}^{(0)} - \left( \frac{V^{(0)}_x}{V_{xx}^{(0)}} \right) \lambda(y)^T (V_x^{(1)} \lambda(y) + \phi'(y) \left( \frac{(V^{(0)}_x)}{V_{xx}^{(0)}} \right) \alpha(y) \rho) \right) \right).
\]

Here \( V^{(2)} \) is defined through \( (3.10) \), and we note that the value of \( \eta \) does not depend on the choice of the constant \( C(t, x) \) in \( (3.10) \). We also set \( (t, \xi, y) := (t, -\log V_x^{(0)} - \| \lambda(y) \|^2 t, y) \), and let \( \tilde{\eta}(t, \xi, y) = \eta(t, x, y) \).

**Theorem 3.5** (Remainder estimate, fast factor). Suppose that there exist \( \varepsilon_0 > 0 \) and \( T \leq \infty \) such that for all \( \varepsilon \in (0, \varepsilon_0) \) the HJB equation \( (3.1) \) has a solution \( V \in C^{1,2,2}(0, T) \times (0, \infty) \times \mathbb{R} \) which is increasing and strictly concave in the second argument. Then the quantity

\[
\int_{\mathbb{R}} e^{-\frac{x^2}{2T}} \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)! 2k^k} \left( \frac{d}{d\xi} \right)^{2k} \int_{0}^{t} \int_{\mathbb{R}} \tilde{\eta}(s, \xi - \chi, y) s^{-1/2} e^{-\frac{\chi^2}{2k}} d\chi d\xi dz
\]

(defined via \( (3.26) \) and the paragraph following it) is well-defined and finite for all \( \varepsilon \in (0, \varepsilon_0) \). Moreover, for every \( (t, x, y) \in (0, T) \times (0, \infty) \times \mathbb{R} \) for which the limit superior

\[
\lim_{\varepsilon \downarrow 0} \left| \int_{\mathbb{R}} e^{-\frac{x^2}{2T}} \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)! 2k^k} \left( \frac{d}{d\xi} \right)^{2k} \int_{0}^{t} \int_{\mathbb{R}} \tilde{\eta}(s, \xi - \chi, y) s^{-1/2} e^{-\frac{\chi^2}{2k}} d\chi d\xi dz \right|
\]

is finite, the error bound

\[
\lim_{\varepsilon \downarrow 0} \varepsilon^{-1} \left| V(t, x, y) - V^{(0)}(t, x) - \sqrt{\varepsilon} V^{(1)}(t, x) \right| < \infty
\]

applies. If the limit superior \( (3.27) \) is bounded above uniformly on a subset of \( [0, T) \times (0, \infty) \times \mathbb{R} \), then the convergence in \( (3.28) \) is uniform on the same subset of \( [0, T) \times (0, \infty) \times \mathbb{R} \).
Remark 3.6. Condition (3.27) is of the same form as condition (2.25), and the detailed interpretation of the latter given in Remark 2.6 applies here as well.

Proof of Theorem 3.3. We proceed as in the proof of Theorem 2.5. More specifically, we insert the ansatz \( V = V^{(0)} + \epsilon V^{(1)} + \epsilon^2 V^{(2)} + \epsilon^3 V^{(3)} + \epsilon^2 Q \) into the HJB equation (3.11) and expand the resulting equation in the powers of \( \epsilon^{1/2} \). The terms \( V^{(0)}, V^{(1)}, V^{(2)} \) and \( V^{(3)} \) were chosen in such a way that all terms on the orders of \( \frac{1}{\epsilon}, \frac{1}{\epsilon^2}, 1 \) and \( \epsilon^{1/2} \) cancel out. At this point, a tedious but straightforward computation relying on the elementary identity

\[
\frac{1}{a + \epsilon^{1/2} b} = \frac{1}{a} - \epsilon^{1/2} \frac{b}{a^2 + \epsilon^{1/2} ab}, \quad a < 0, \quad b < -\epsilon^{-1/2} a
\]

allows to compute the terms of orders \( \epsilon \) and leads to

\[
(3.29) \quad \epsilon \dot{Q}_t + \frac{\epsilon}{2} \frac{\|\lambda(y)\|^2 (V_x^{(0)})^2}{(V_{xx}^{(0)})^2} \dot{Q}_{xx} - \epsilon \frac{\|\lambda(y)\|^2 V_x^{(0)}}{V_{xx}^{(0)}} \dot{Q}_x = \epsilon \eta
\]

where \( \dot{Q} := \epsilon^{-1} (V - V^{(0)} - \epsilon^{1/2} V^{(1)}) = V^{(2)} + \epsilon^{1/2} V^{(3)} + \epsilon Q \) and \( \eta \) is defined according to (3.26). One can now conclude the argument by repeating the steps in the proof of Theorem 2.5 making the change of coordinates

\[
(t, \xi, y) := \left(t, -\log V_x^{(0)} - \frac{\|\lambda(y)\|^2}{2} t, y\right)
\]

in (3.29), and combining Duhamel’s principle for the resulting equation with the formula for the inverse Weierstrass transform given in [W12]. \( \square \)

4. Multiscale forward investment problem

We combine our approaches to the forward investment problems with slow and fast factors to analyze the multiscale forward investment problem described in Section 1.1. We consider an expansion for \( V(t, x, y_1, y_2) \) in equation (1.10) of the form

\[
V = V^{(0)} + \sqrt{\delta} V^{(1,0)} + \sqrt{\epsilon} V^{(0,1)} + O(\delta + \epsilon)
\]

in the limit regime \( \delta \downarrow 0, \epsilon \downarrow 0 \). We first give the general results, and then explicit formulas for the case of power utilities in Section 4.2.

4.1. First Order Approximations. It is convenient to define:

\[
\tilde{\lambda}(y_1) = \left(\int_{\mathbb{R}} \|\lambda(y_1, y_2)\|^2 \mu(dy_2)\right)^{1/2},
\]

\[
C_{1,0}(y_1) = (\rho^s)^T \left(\int_{\mathbb{R}} \lambda(y_1, y_2) \mu(dy_2)\right) \kappa(y_1),
\]

\[
C_{0,1}(y_1) = (\rho^f)^T \left(\int_{\mathbb{R}} \lambda(y_1, y_2) \phi_{y_2}(y_1, y_2) \alpha(y_2) \mu(dy_2)\right),
\]

where

\[
\phi(y_1, y_2) = \int_0^\infty \mathbb{E}\left[\|\lambda(y_1, Y^1(s))\|^2 - \tilde{\lambda}_2^2(y_1) | Y^1(0) = y_2\right] ds,
\]

and \( Y^1 \) denotes the fast factor process, solution of the SDE (1.4), but with \( \epsilon = 1 \).

The following proposition gives explicit formulas for the leading order term \( V^{(0)} \) and the first order correction terms \( V^{(1,0)} \) and \( V^{(0,1)} \).
Proposition 4.1 (Explicit formulas, general case). (a) The leading order term \( V^{(0)} \) admits the representation

\[
V^{(0)}(t, x, y_1, y_2) = V^{(0)}(t, x, y_1) = u(\tilde{\lambda}^2(y_1)t, x)
\]

where \( u \) is given by

\[
u(t, x) = -\frac{1}{2} \int_0^t e^{-(\tilde{h}^{-1})(s, x) + \frac{\bar{x}^2}{2}} h_x(s, h^{-1}(s, x)) \, ds + V(0, x),
\]

\[
h(t, x) = \int_{\mathbb{R}} \frac{e^{zx^2 - \frac{1}{2}z^2t}}{z} \nu(dz).
\]

Here \( h^{-1} \) is the inverse of \( h \) in the variable \( x \), and \( \nu \) is a non-negative Borel measure on \( \mathbb{R} \).

(b) The slow scale correction term \( V^{(1,0)} \) is given by

\[
V^{(1,0)}(t, x, y_1) = \frac{t}{2} C_{1,0}(y_1) \frac{V^{(0)}_{x x}(0)}{V^{(0)}_x(0)},
\]

and admits the natural parametrization

\[
v^{(1,0)}(t, x, y_1) = \int_0^t v^{(1,0)}(s, x, y_1) \, ds,
\]

where each \( v^{(1,0)}(s, x, y_1) \) is a solution of the initial value problem

\[
v^{(1,0)}_t(s, x, y_1) + \frac{\tilde{\lambda}^2(y_1)}{2} \left( \frac{V^{(0)}_x(s, x, y_1)}{V^{(0)}_{xx}(s, x, y_1)} \right)^2 v^{(1,0)}(s, x, y_1) - \tilde{\lambda}^2(y_1) \left( \frac{V^{(0)}_x(s, x, y_1)}{V^{(0)}_{xx}(s, x, y_1)} \right) v^{(1,0)}(s, x, y_1) = 0, \quad t \geq s,
\]

\[
v^{(1,0)}(s, x, y_1) = C_{1,0}(y_1) \frac{V^{(0)}_{xx}(s, x, y_1) V^{(0)}_x(s, x, y_1)}{V^{(0)}_{xx}(0, x, y_1)}.
\]

(c) The fast scale correction term \( V^{(0,1)} \) is given by

\[
v^{(0,1)}(t, x, y_1) = t C_{0,1}(y_1) \left( \frac{V^{(0)}_x(s, x, y_1)}{V^{(0)}_{xx}(s, x, y_1)} \right) \left( \frac{V^{(0)}_x(s, x, y_1)}{V^{(0)}_{xx}(s, x, y_1)} \right)^2
\]

The function \( V^{(0,1)} \) admits the natural parametrization

\[
v^{(0,1)}(t, x, y_1) = \int_0^t v^{(0,1)}(s, x, y_1) \, ds,
\]

where each \( v^{(0,1)}(s, x, y_1) \) is a solution of the initial value problem

\[
v^{(0,1)}_t(s, x, y_1) + \frac{\tilde{\lambda}^2(y_1)}{2} \left( \frac{V^{(0)}_x(s, x, y_1)}{V^{(0)}_{xx}(s, x, y_1)} \right)^2 v^{(0,1)}(s, x, y_1) - \tilde{\lambda}^2(y_1) \left( \frac{V^{(0)}_x(s, x, y_1)}{V^{(0)}_{xx}(s, x, y_1)} \right) v^{(0,1)}(s, x, y_1) = 0, \quad t \geq s,
\]

\[
v^{(0,1)}(s, x, y_1) = C_{0,1}(y_1) \frac{V^{(0)}_x(s, x, y_1)}{V^{(0)}_{xx}(s, x, y_1)} \left( \frac{V^{(0)}_x(s, x, y_1)}{V^{(0)}_{xx}(s, x, y_1)} \right)^2.
\]

Remark 4.2. The quantities \( V^{(1,0)}_x, V^{(1,0)}_{xx} \) of Proposition 4.1 should be interpreted in the same way as their analogues in the single factor cases. We refer to Remark 2.4 for more details.

Proof of Proposition 4.1. We start with the proofs of parts (a) and (c). To this end, we insert \( \delta = 0 \) into the HJB equation (1.10), employ Assumption 1.1, and then proceed as in the proofs of
The arguments from there can be repeated directly by replacing $\lambda(y)$ by $\lambda(y_1, y_2)$. In particular, $V^{(0)}$ and $V^{(0,1)}$ are determined via an averaging of the equations

$$V^{(0)}_t - \frac{||\lambda||^2}{2} \frac{(V^{(0)}_{xx})^2}{V^{(0)}_{xx}} + \mathcal{L}_{y_2} V^{(0,2)} = 0,$$

$$V^{(0,1)}_t + \frac{||\lambda||^2}{2} \left( \frac{V^{(0)}_{xx}}{V^{(0)}_{xx}} \right)^2 V^{(0,1)}_{xx} - V^{(0)}_{xx} \lambda^T \left( V^{(0)} (x, y) \right)_x + \frac{\sqrt{\epsilon}}{2} \alpha \rho + \mathcal{L}_{y_2} V^{(0,3)} = 0$$

with respect to $\mu(dy_2)$. Here $\epsilon^{-1}\mathcal{L}_{y_2}$ is the generator of the fast factor $Y^\epsilon$, that is

$$\mathcal{L}_{y_2} = \frac{1}{2} \alpha(y_2)^2 \partial_{y_2} y_2 + \gamma(y_2) \partial_{y_2} y_2,$$

and the terms $V^{(0)}$, $V^{(0,1)}$, $V^{(0,2)}$, $V^{(0,3)}$ are the ones appearing in the expansion of the solution to (1.10) with $\delta = 0$. In particular, subtracting from (4.13) its averaged version, we obtain the expression

$$(4.15) \quad V^{(0,2)}(t, x, y_1, y_2) = -\frac{1}{2} \phi(y_1, y_2) \left( \frac{V^{(0)}_{xx}}{V^{(0)}_{xx}} \right)^2 + C(t, x, y_1)$$

where $C(t, x, y_1)$ is a function that does not depend on $y_2$, and $\phi$ was defined in (4.14).

It remains to prove part (b). To this end, we again employ Assumption 1.1 and insert the ansatz $V^{(0)} + \sqrt{\delta} V^{(1)}$ into the HJB equation (1.10). Collecting the terms of order $\sqrt{\delta}$ in the resulting equation we get

$$(4.16) \quad V^{(1)}_t - \left( \lambda \frac{V^{(0)}_{xx}}{V^{(0)}_{xx}} + \frac{\sqrt{\epsilon}}{2} \alpha \frac{V^{(1)}_{xx}}{V^{(0)}_{xx}} \right) \left( \frac{\rho \alpha}{\sqrt{\epsilon}} \frac{V^{(1)}_{xx}}{V^{(0)}_{xx}} + \frac{\epsilon \alpha}{\sqrt{\epsilon}} V^{(1)}_{y_2} \right) + \frac{1}{2} \frac{\gamma V^{(1)}_{y_2} + \frac{\sqrt{\epsilon}}{2} \alpha V^{(1)}_{y_2} + \frac{1}{2} \kappa \alpha V^{(1)}_{y_2} + \frac{1}{2} \kappa \alpha \rho^{2} V^{(0)}_{y_2} = 0.$$

We can now write $V^{(1)} = V^{(1,0)} + \sqrt{\epsilon} V^{(1,1)} + \epsilon V^{(1,2)}$ and expand equation (4.16) in powers of $\epsilon$ as in the proof of Proposition 3.2. By doing so, we conclude that $V^{(1,0)}$ and $V^{(1,1)}$ can be chosen as functions independent of $y_2$. Moreover, $V^{(1,0)}$ can be determined by averaging the equation

$$(4.17) \quad V^{(1,0)}_t + \frac{1}{2} \left( \lambda \frac{V^{(0)}_{xx}}{V^{(0)}_{xx}} \right) \frac{V^{(1,0)}_{xx}}{V^{(0)}_{xx}} - \frac{1}{2} \lambda \frac{V^{(0)}_{xx}}{V^{(0)}_{xx}} = 0$$

with respect to $\mu(dy_2)$. The averaged equation is endowed with the initial condition $V^{(1,0)}(0, x, y) = 0$ and can be solved explicitly by means of a transformation to an ill-posed backward heat equation as in the proofs of Propositions 2.2 and 2.3. This gives the explicit formula for $V^{(1,0)}$ and its natural parametrization.

We now complement the explicit formulas for the leading order terms $V^{(0)}$, $V^{(1,0)}$, $V^{(0,1)}$ by a convergence theorem justifying the approximation of the true value function $V$ by the function $V^{(0)} + \sqrt{\delta} V^{(1,0)} + \sqrt{\epsilon} V^{(0,1)}$. We will need the non-linear functional $\eta$, whose lengthy formula we give in Appendix A. We also set $(t, \xi, y_1, y_2) := (t, \log \frac{V^{(0)}_{xx}}{2} t, y_1, y_2)$, and let

$$(4.18) \quad \tilde{\eta}(t, \xi, y_1, y_2) := \eta(t, x, y_1, y_2).$$

**Theorem 4.3** (Remainder estimate, general case). Suppose that there are $\delta_0 > 0$, $\epsilon_0 > 0$ and $T \leq \infty$ such that, for all $(\delta, \epsilon) \in (0, \delta_0) \times (0, \epsilon_0)$, the HJB equation (1.10) has a solution $V \in$
\[ C^{1,2,2}([0, T] \times (0, \infty) \times \mathbb{R}^2) \text{ which is increasing and strictly concave in the second argument. Then, the quantity} \]
\[ (\delta + \epsilon)^{-1} \int_{\mathbb{R}} e^{-\frac{x^2}{2\delta}} \sum_{k=0}^{\infty} \frac{(-1)^k z^{2k}}{(2k)! 2^k k^k} \left( \frac{d}{d\xi} \right)^{2k} \int_0^t \int_{\mathbb{R}} \tilde{\eta}(s, \xi - \chi, y_1, y_2) s^{-1/2} e^{-\frac{x^2}{2\epsilon}} d\chi ds dz, \]

with \( \tilde{\eta} \) defined in (4.18), is well-defined and finite for all \((\delta, \epsilon) \in (0, \delta_0) \times (0, \epsilon_0)\). Moreover, for every \((t, x, y_1, y_2) \in [0, T] \times (0, \infty) \times \mathbb{R}^2\) for which the limit superior
\[ \lim_{\delta, \epsilon \to 0} (\delta + \epsilon)^{-1} \left| \int_{\mathbb{R}} e^{-\frac{x^2}{2\delta}} \sum_{k=0}^{\infty} \frac{(-1)^k z^{2k}}{(2k)! 2^k k^k} \left( \frac{d}{d\xi} \right)^{2k} \int_0^t \int_{\mathbb{R}} \tilde{\eta}(s, \xi - \chi, y_1, y_2) s^{-1/2} e^{-\frac{x^2}{2\epsilon}} d\chi ds dz \right| \]
is finite, the error bound
\[ \lim_{\delta, \epsilon \to 0} (\delta + \epsilon)^{-1} \left| V(t, x, y_1, y_2) - V(0)(t, x, y_1) - \sqrt{\delta} V^{(1,0)}(t, x, y_1) - \sqrt{\epsilon} V^{(0,1)}(t, x, y_1) \right| < \infty \]
applies. If the limit superior (4.19) is bounded above uniformly on a subset of \([0, T] \times (0, \infty) \times \mathbb{R}^2\), then the convergence in (4.20) is uniform on the same subset of \([0, T] \times (0, \infty) \times \mathbb{R}^2\).

**Remark 4.4.** Condition (4.19) is of the same form as condition (2.25), and the detailed interpretation of the latter given in Remark 2.6 applies here as well.

**Proof of Theorem 4.3.** We proceed as in the proof of Theorem 2.5. More specifically, we plug \( V = V(0) + \sqrt{\delta} V^{(1,0)} + \sqrt{\epsilon} V^{(0,1)} + Q \) into the HJB equation (1.10) and Taylor expand the resulting equation in \( \sqrt{\delta} \) and \( \sqrt{\epsilon} \). Hereby, we use the elementary identity
\[ \frac{1}{a + b} = \frac{1}{a} - \frac{b}{a^2 + ab} \]
and the definitions of \( V(0), V^{(1,0)}, V^{(0,1)}, V^{(2)}, V^{(3)}, V^{(1,1)} \) and \( V^{(1,2)} \) to eliminate the terms of orders 1, \( \sqrt{\delta} \) and \( \sqrt{\epsilon} \). The remaining equation then reads
\[ Q_t + \frac{||\lambda||^2}{2} \left( \frac{V_x^{(0)}}{V^{(0)}} \right)^2 Q_{xx} - ||\lambda||^2 \left( \frac{V_x^{(0)}}{V^{(0)}} \right) Q_x = \eta, \]
with \( \eta \) is defined prior to the statement of the theorem. One can now conclude by repeating the steps in the proof of Theorem 2.5, that is, by making the change of coordinates
\[ (t, \xi, y_1, y_2) := (t, -\log V_x^{(0)} - \frac{||\lambda||^2}{2} t, y_1, y_2) \]
in (4.21), and combining Duhamel’s principle for the resulting equation with the formula for the inverse Weierstrass transform given in [Wi2].

**4.2. Power utility example.** We illustrate the results with the family of power utility forward performance processes. For a constant risk aversion coefficient \( \gamma < 1 \), we impose the initial condition of the HJB equation (1.10) to be
\[ V(0, x) = \gamma \frac{x^{1-\gamma}}{1-\gamma}. \]
This is Example 16 in [MZ3] and, as shown there, leads to the following explicit solution for the constant parameter value function \( V^{(0)}(t, x, y_1) \) in part (a) of Proposition 4.1:
\[ V^{(0)}(t, x, y_1) = u(\lambda^2(y_1)t, x), \quad \text{where} \quad u(t, x) = \gamma \frac{x^{1-\gamma}}{1-\gamma} e^{-\frac{1}{2} t \Gamma}, \quad \text{and} \quad \Gamma = \frac{1-\gamma}{\gamma}, \]
which can be verified by taking the measure \( \nu \) to be a Dirac delta centered at \( \gamma^{-1} \).
Then, from \((4.7)\) in Proposition 4.1, we compute
\[
V^{(1,0)}(t, x, y_1) = \frac{1}{2} t^2 C_{1,0}(y_1) \Gamma^2 \bar{\lambda}''(y_1) V^{(0)}(t, x, y_1),
\]
and from \((4.10)\) we obtain
\[
V^{(0,1)}(t, x, y_1) = t C_{0,1}(y_1) \Gamma^2 V^{(0)}(t, x, y_1).
\]
Then the three-term approximation to the forward performance value function is given by
\[
(4.23)
\]
\[
V(t, x, y_1, y_2) = \left(1 + \frac{1}{2} \sqrt{\delta} t C_{1,0}(y_1) \Gamma^2 \bar{\lambda}''(y_1) + \sqrt{\epsilon} t C_{0,1}(y_1) \Gamma^2 \right) V^{(0)}(t, x, y_1) + O(\delta + \epsilon),
\]
where \(V^{(0)}\) is given explicitly in \((4.22)\).

5. Approximately optimal portfolio

In this last section we give an explicit formula for the approximately optimal feedback portfolio function associated with the approximation \(V \approx V^{(0)} + \sqrt{\delta} V^{(1,0)} + \sqrt{\epsilon} V^{(0,1)}\).

Proposition 5.1. The approximately optimal feedback portfolio function \(\pi^*\) associated with the approximation \(V \approx V^{(0)} + \sqrt{\delta} V^{(1,0)} + \sqrt{\epsilon} V^{(0,1)}\) is given by
\[
\pi^* = - (\sigma^T)^{-1} \lambda \left( \frac{V^{(0)}}{V^{(0)}_{xx}} + \sqrt{\delta} \frac{V^{(0)}_{xx} V^{(1,0)} - V^{(0)}_{x} V^{(1,0)}_{x}}{(V^{(0)}_{xx})^2} + \sqrt{\epsilon} \frac{V^{(0)}_{xx} V^{(0,1)} - V^{(0)}_{x} V^{(0,1)}_{x}}{(V^{(0)}_{xx})^2} \right)
\]
(5.1)
\[
- \sqrt{\delta} \kappa (\sigma^T)^{-1} \rho^s \frac{V^{(0)}_{xy}}{V^{(0)}_{xx}} + \sqrt{\epsilon} (\sigma^T)^{-1} \rho^f \phi y \frac{(V^{(0)}_{xx})^2}{V^{(0)}_{xx}}
\]
where \(V^{(0)}\), \(V^{(1,0)}\) and \(V^{(0,1)}\) are as in Proposition 4.1 and \(\phi\) is given in \((4.4)\).

In the case that only a slow factor is present (that is, in the setting of Section 2) the approximately optimal portfolio corresponding to the approximation \(V \approx V^{(0)} + \sqrt{\delta} V^{(1)}\) reads
\[
(5.2)
\]
\[
\pi^* = - (\sigma^T)^{-1} \lambda \left( \frac{V^{(0)}}{V^{(0)}_{xx}} + \sqrt{\delta} \frac{V^{(0)}_{xx} V^{(1)} - V^{(0)}_{x} V^{(1)}_{x}}{(V^{(0)}_{xx})^2} \right) - \sqrt{\delta} \kappa (\sigma^T)^{-1} \rho \frac{V^{(0)}_{xy}}{V^{(0)}_{xx}}
\]
where \(V^{(0)}\) and \(V^{(1)}\) are as in Propositions 2.1 and 2.3 respectively.

Finally, in the case that only a fast factor is present (that is, in the setting of Section 3) the approximately optimal portfolio corresponding to the approximation \(V \approx V^{(0)} + \sqrt{\epsilon} V^{(1)}\) is given by
\[
(5.3)
\]
\[
\pi^* = - (\sigma^T)^{-1} \lambda \left( \frac{V^{(0)}}{V^{(0)}_{xx}} + \sqrt{\epsilon} \frac{V^{(0)}_{xx} V^{(1)} - V^{(0)}_{x} V^{(1)}_{x}}{(V^{(0)}_{xx})^2} \right) + \sqrt{\epsilon} (\sigma^T)^{-1} \rho \phi y \frac{(V^{(0)}_{xx})^2}{V^{(0)}_{xx}}
\]
where \(V^{(0)}\) and \(V^{(1)}\) are as in Propositions 3.1 and 3.3 respectively, and \(\phi(y)\) is given in \((3.9)\).

Proof. We recall that the non-linearity in \((1.10)\) results from the optimization problem
\[
\max_{\pi} \left( (\lambda V_x + \sqrt{\delta} \kappa \rho^s V_{xy} + \frac{1}{\sqrt{\epsilon}} \alpha \rho^f V_{xy2})^T (\sigma \pi) + \frac{1}{2} V_{xx} (\sigma \pi)^T (\sigma \pi) \right)
\]
(see for example equation (10) in \([NT]\)). Here we have used the fact that \(\sigma \sigma^{-1} = I\) the \(d \times d\) identity matrix by Assumption 1.1. It then follows that optimal portfolios \(\pi^*\) are characterized by the equation
\[
\sigma^T \pi^* = \lambda \frac{V_x}{V_{xx}} + \sqrt{\delta} \kappa \rho^s \frac{V_{xy}}{V_{xx}} + \frac{1}{\sqrt{\epsilon}} \alpha \rho^f \frac{V_{xy2}}{V_{xx}}.
\]
To obtain (5.1), it now remains to multiply both sides of the latter equation by $(\sigma^T)^{-1}$, to use the fact that $\sigma \sigma^{-1}$ is the $d \times d$ identity matrix, to replace $V$ by $V^{(0)} + \sqrt{\delta} V^{(1,0)} + \sqrt{\epsilon} V^{(0,1)} + \epsilon V^{(0,2)}$, to use (4.15) for the last term, and to compute the terms of orders 1, $\sqrt{\delta}$ and $\sqrt{\epsilon}$. The identities (5.2) and (5.3) can be established in the same manner.

Remark 5.2. One can use the formula (4.23) for the value function in the case of the power utility forward performance to compute the approximation (5.1) for the optimal portfolio. We omit the lengthy expression here.

Remark 5.3. Consider the case when only the slow factor is present, that is, the setting of Section 2. Then by its definition (see e.g. [MZ3, Definition 1]) the forward performance process $U(t, x) = V(t, x, Y^\delta(t)), t \geq 0$ has the property that the process $U(t, X^{\pi}(t)) = V(t, X^{\pi}(t), Y^\delta(t)), t \geq 0$ is a supermartingale for any self-financing portfolio $\pi$ where $X^{\pi}(t)$ is the value of the portfolio $\pi$ at time $t$. Optimal portfolios correspond hereby to the case that $V(t, X^{\pi}(t), Y^\delta(t)), t \geq 0$ is a martingale.

Note further that for self-financing portfolios the portfolio value satisfies

$$
dX^{\pi}(t) = \mu(Y^\delta(t))^T \pi(t) \, dt + \sigma(Y^\delta(t))^T \pi(t) \, dW(t).
$$

Therefore, we can apply Itô’s formula to $V(t, X^{\pi}(t), Y^\delta(t))$ (dropping the arguments $t, X^{\pi}(t), Y^\delta(t)$ to simplify the notation):

(5.4)

$$
dV = \left( V_t + \frac{1}{2} \delta \kappa^2 V_{yy} + \delta b V_y + \mu^T \pi V_x + \pi^T \sigma^T \pi V_{xx} + \pi^T \sigma \rho \sqrt{\delta} \kappa V_{xy} \right) \, dt + V_y \sqrt{\delta} \kappa \, dB_1(t) + V_x \sigma^T \pi \, dW(t).
$$

Replacing $V$ by $V^{(0)} + \sqrt{\delta} V^{(1)} + \delta Q$ and inserting the portfolio $\pi^*$ of Proposition 5.1 into the drift term we see that all summands of orders 1 and $\sqrt{\delta}$ cancel out. We conclude that the process $V$ of (5.4) fails to be a martingale only by a bounded variation term of order $\delta$. In this sense the portfolio $\pi^*$ of Proposition 5.1 is “$\delta$-optimal”. A similar statement holds in the cases when only a fast factor is present and when both factors are present.

6. Conclusion

We have provided a convergent approximation for forward performance processes in a multifactor incomplete markets model, as well as for the corresponding optimal portfolio. Our approach is based on a perturbation analysis of the corresponding ill-posed HJB equation. The principal term in the approximation results from the appropriately averaged problem, whose solution is known from Widder’s Theorem. The correction terms for fast and slow volatility factors can be computed explicitly in terms of this leading order term.

We have given explicit calculations in the case of power utility. The ease of the formulas provided in more general cases, as well as conditions for convergence, should allow future work to develop the financial implications of forward performance processes in realistic market environments.
Appendix A. Expression for $\eta$ in Section 4

$$
\eta := -\frac{1}{V_{xx}} \left( \frac{\delta}{2} \left\| \lambda \right\|^2 (V_x^{(1,0)})^2 + \frac{\epsilon}{2} \left\| \lambda \right\|^2 (V_x^{(0,1)})^2 + \frac{\left\| \lambda \right\|^2}{2} (V_x - V_x^{(0)}) - \sqrt{\delta} V_x^{(1,0)} - \sqrt{\epsilon} V_x^{(0,1)} \right)^2 \\
+ \frac{\kappa^2}{2} \left\| \rho^s \right\|^2 \delta (V_{xy}^{(0)})^2 + \frac{\delta^2}{2} \kappa^2 \left\| \rho^s \right\|^2 (V_{xy}^{(1,0)})^2 + \frac{\delta}{2} \epsilon \kappa^2 \left\| \rho^s \right\|^2 (V_{xy}^{(0,1)})^2 + \frac{1}{2 \epsilon} \alpha^2 \left\| \rho^f \right\|^2 (V_{xy}^{(0,1)})^2 \\
+ \frac{\delta}{2} \kappa^2 \left\| \rho^s \right\|^2 (V_{xy} - V_{xy}^{(0)}) - \sqrt{\delta} V_{xy}^{(1,0)} - \sqrt{\epsilon} V_{xy}^{(0,1)} \right)^2 + \delta \kappa \lambda^T \rho^s V_x^{(0)} \left( V_{xy}^{(0)} \right) \\
+ \kappa \lambda^T \rho^s \sqrt{\delta} \varepsilon V_x^{(0,1)} + \sqrt{\delta} \epsilon \left\| \lambda \right\|^2 \left( V_{xy}^{(0,1)} \right)^2 + \kappa \lambda^T \rho^s \delta V_x^{(0)} \left( V_{xy}^{(0,1)} \right) \\
+ \delta^{3/2} \kappa \lambda^T \rho^s V_x^{(1,0)} \left( V_{xy}^{(0,1)} \right) + \delta \sqrt{\epsilon} \left( V_{xy}^{(0,1)} \right) V_{xy}^{(1,0)} \left( V_{xy}^{(0,1)} \right) + \frac{\sqrt{\delta}}{\sqrt{\epsilon}} \alpha \lambda^T \rho^f V_x^{(1,0)} \left( V_{xy}^{(0,1)} \right) \\
+ \frac{\sqrt{\delta}}{\sqrt{\epsilon}} \alpha \left( \rho^s \right)^T \rho^f V_{xy}^{(0,1)} \left( V_{xy}^{(0,1)} \right) V_{xy}^{(1,0)} + \frac{\sqrt{\delta}}{\sqrt{\epsilon}} \alpha \kappa \left( \rho^s \right)^T \rho^f V_{xy}^{(0,1)} \left( V_{xy}^{(0,1)} \right) \\
+ \lambda \left( V_x - V_x^{(0)} - \sqrt{\delta} V_x^{(1,0)} \right) \\
+ \frac{1}{V_{xx}} \left( \frac{\epsilon}{2} \left\| \lambda \right\|^2 (V_x^{(1,0)})^2 + \frac{\left\| \lambda \right\|^2}{2} (V_x - V_x^{(0)}) - \sqrt{\delta} V_x^{(1,0)} - \sqrt{\epsilon} V_x^{(0,1)} \right)^2 \left( V_{xy} - V_{xy}^{(0)} - \sqrt{\delta} V_{xy}^{(1,0)} - \sqrt{\epsilon} V_{xy}^{(0,1)} \right) \\
- \left( \frac{1}{V_{xx}} - \frac{1}{V_{xxy}} \right) \left( \left\| \lambda \right\|^2 \sqrt{\delta} V_x^{(0,1)} \left( V_{xy}^{(0)} \right)^{\delta} + \left\| \lambda \right\|^2 \sqrt{\epsilon} V_x^{(0,1)} \left( V_{xy}^{(0)} \right)^{\epsilon} + \sqrt{\delta} \kappa \lambda^T \rho^s V_x^{(0)} \left( V_{xy}^{(0,1)} \right) \\
+ \frac{1}{\sqrt{\epsilon}} \alpha \lambda^T \rho^f V_x^{(0)} \left( V_{xy}^{(0,1)} \right) + \left\| \lambda \right\|^2 \left( V_x - V_x^{(0)} - \sqrt{\delta} V_x^{(1,0)} - \sqrt{\epsilon} V_x^{(0,1)} \right) \left( V_{xy} - V_{xy}^{(0)} - \sqrt{\delta} V_{xy}^{(1,0)} - \sqrt{\epsilon} V_{xy}^{(0,1)} \right) \right) \\
- \frac{\left\| \lambda \right\|^2}{2} \left( \frac{V_{xxy}}{V_{xx}} \right)^2 \left( V_{xx} - V_{xxy} \right)^2 + A_{y_1}^\delta V + \frac{A_{y_2}^\epsilon}{\epsilon} \left( V - \epsilon V_2^{(2)} - \epsilon^{3/2} V_3^{(3)} - \sqrt{\delta} \epsilon^{3/2} V_1^{(1,2)} \right).$$

Here $A_{y_2}^\delta V_2^{(2)}$, $A_{y_2}^\epsilon V_3^{(3)}$ and $A_{y_2}^\epsilon V_1^{(1,2)}$ are defined through (4.13), (4.14) and (4.17), respectively.
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