Closed Formulas of the Arithmetic Mean Component Competitive Ratio for the 3-Objective and 4-Objective Time Series Search Problems

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Abstract: For the multi-objective time series search problem, Hasegawa and Itoh [Proc. of WALCOM, LNCS 9627, 2016, pp. 201-212] presented the best possible online algorithm balanced price policy BPP for any monotone function $f : \mathbb{R}^k \rightarrow \mathbb{R}$ and derived the exact values of the competitive ratio for several monotone functions. Specifically for the monotone function $f(c_1, \ldots, c_k) = (c_1, \ldots, c_k)/k$, Hasegawa and Itoh derived the exact value of the arithmetic mean component competitive ratio for $k = 2$ but it is not known for any integer $k \geq 3$. In this paper, we derive the exact values of the arithmetic mean component competitive ratio for $k = 3$ and $k = 4$.

Key Words: multi-objective time series search problem, monotone functions, arithmetic mean component competitive ratio.

1 Introduction

There are many single-objective online optimization problems such as paging and caching (see [10] for a survey), metric task systems (see [6] for a survey), asset conversion problems (see [7] for a survey), buffer management of network switches (see [4] for a survey), etc., and Sleator and Tarjan [8] introduced a notion of competitive analysis to measure the efficiency of online algorithms for single-objective online optimization problems. For online problems of multi-objective nature, Tiedemann, et al. [9] presented a framework of multi-objective online problems as an online version of multi-objective optimization problems [2] and formulated a notion of the competitive ratio for multi-objective online problems as the extension of the competitive ratio for single-objective online problems. On defining the competitive ratio for $k$-objective online problems, Tiedemann, et al. [9] regarded multi-objective online problems as a family of (possibly dependent) single-objective online problems and applied a monotone function $f : \mathbb{R}^k \rightarrow \mathbb{R}$ to the family of the single-objective online problems. Let $A$ be an algorithm for a $k$-objective online problem. Then we regard the algorithm $A$ as a family of algorithms $A_i$ for the $i$th objective. For $c_1, \ldots, c_k$, where $c_i$ is the competitive ratio of the algorithm $A_i$, we say that the algorithm $A$ is $f(c_1, \ldots, c_k)$-competitive with respect to a monotone function $f : \mathbb{R}^k \rightarrow \mathbb{R}$. In fact, Tiedemann, et al. [9] defined the competitive ratio by several monotone (continuous) functions, e.g., the worst component competitive ratio by $f_1(c_1, \ldots, c_k) = \max(c_1, \ldots, c_k)$, the arithmetic mean component competitive ratio by $f_2(c_1, \ldots, c_k) = (c_1 + \cdots + c_k)/k$, and the geometric mean component competitive ratio by $f_3(c_1, \ldots, c_k) = (c_1 \times \cdots \times c_k)^{1/k}$.

1.1 Previous Work

A single-objective time series search problem (initially investigated by El-Yaniv, et al. [3]) is defined as follows: Let $\text{ALG}$ be a player that searches for the maximum price in a sequence of prices. At the beginning of each time period $t \in \{1, \ldots, T\}$, the player $\text{ALG}$ receives a price $p_t \in \mathbb{R}_+$ and must decide whether to accept or reject the price $p_t$. Assume that prices are chosen from the interval $\text{ITV} = [m, M]$, where $0 < m \leq M$. If the player $\text{ALG}$ accepts $p_t$, then the game ends and
the return for ALG is \( p_t \). If the player ALG rejects \( p_t \) for every \( t \in \{1, \ldots, T\} \), then the return for ALG is defined to be \( m \). Let \( r = M/m \) be the fluctuation ratio of possible prices. For the case that \( m \) and \( M \) are known to online algorithms, El-Yaniv et al. \[3\] presented a (best possible) deterministic algorithm reservation price policy \( \text{RPP} \), which is \( \sqrt{r} \)-competitive, and a randomized algorithm exponential threshold \( \text{EXPO} \), which is \( O(\log r) \)-competitive.\[3\]

As a natural extension of the single-objective time series search problem, a multi-objective (k-objective) time series search problem \[9\] can be defined as follows: At the beginning of each time period \( t \in \{1, \ldots, T\} \), the player \( \text{ALG}_k \) receives a price vector \( \vec{p}_t = (p^1_t, \ldots, p^k_t) \in \mathbb{R}^k \) and must decide whether to accept or reject the price vector \( \vec{p}_t \). As in the case of a single-objective time series search problem, assume that price \( p^i_t \) is chosen from the interval \( \text{ITV}_i = [m_i, M_i] \), where \( 0 < m_i \leq M_i \) for each \( i \in \{1, \ldots, k\} \), and that the player \( \text{ALG}_k \) knows \( m_i \) and \( M_i \) for each \( i \in \{1, \ldots, k\} \). If the player \( \text{ALG}_k \) accepts \( \vec{p}_t \), then the game ends and the return for \( \text{ALG}_k \) is \( \vec{p}_t \). If the player \( \text{ALG}_k \) rejects \( \vec{p}_t \) for every \( t \in \{1, \ldots, T\} \), then the return for \( \text{ALG}_k \) is defined to be the minimum price vector \( \vec{p}_{\text{min}} = (m_1, \ldots, m_k) \). Without loss of generality, assume that \( M_i/m_i \geq \cdots \geq M_k/m_k \). For the case that all of \( \text{ITV}_1 = [m_1, M_1], \ldots, \text{ITV}_k = [m_k, M_k] \) are real intervals, Tiedemann et al. \[9\] presented best possible online algorithms for the multi-objective time series search problem with respect to the monotone functions \( f_1, f_2, \) and \( f_3 \), i.e., the best possible online algorithm \( \text{RPP-HIGH} \) for the multi-objective time series search problem with respect to the monotone function \( f_1 \) \[9, \text{Theorems 1 and 2}\], the best possible online algorithm \( \text{RPP-MULT} \) for the bi-objective time series search problem with respect to the monotone function \( f_2 \) \[9, \text{Theorem 3 and 4}\] and the best possible online algorithm \( \text{RPP-MULT} \) for the bi-objective time series search problem with respect to the monotone function \( f_3 \) \[9, \S3.2\]. Recently, Hasegawa and Itoh \[5\] presented the deterministic online algorithm balanced price policy \( \text{BPP} \) and showed that \( \text{BPP} \) is best possible for any monotone function \( f : \mathbb{R}^k \to \mathbb{R} \) and for any integer \( k \geq 2 \).

1.2 Our Contribution

For \( k = 2 \), Hasegawa and Itoh \[5\] pointed out that with respect to the monotone (continuous) function \( f_2 \), the (arithmetic mean component) competitive ratio for the bi-objective time series search problem \[9, \text{Theorems 3 and 4}\] is incorrect and derived the exact one, however, no best possible value of the competitive ratio is known for any integer \( k \geq 3 \). In this paper, we derive best possible value of the (arithmetic mean component) competitive ratio for integers \( k = 3 \) and \( k = 4 \) with respect to the monotone function \( f_2 \), i.e., \( f_2(c_1, \ldots, c_k) = (c_1 + \cdots + c_k)/k \).

2 Preliminaries

2.1 Notations

Let \( k \geq 2 \) be an integer. For each \( 1 \leq i \leq k \), let \( \text{ITV}_i = [m_i, M_i] \) be the interval of the \( i \)-th component of price vectors for the \( k \)-objective time series search problem, and we use \( r_i = M_i/m_i \) to denote the fluctuation ratio of the interval \( \text{ITV}_i = [m_i, M_i] \). Without loss of generality, we assume that \( r_1 \geq \cdots \geq r_k \geq 1 \). For any monotone continuous function \( f : \mathbb{R}^k \to \mathbb{R} \), define

\[
\mathcal{S}^k_f = \left\{ (x_1, \ldots, x_k) \in \text{ITV}_1 \times \cdots \times \text{ITV}_k : f \left( \frac{x_1}{m_1}, \ldots, \frac{x_k}{m_k} \right) = f \left( \frac{M_1}{x_1}, \ldots, \frac{M_k}{x_k} \right) \right\},
\]

\[\text{We can show that if only the fluctuation ratio } r = M/m \text{ is known (but not } m \text{ or } M \text{) to the (deterministic) player ALG, then no better competitive ratio than the trivial one of } r \text{ is achievable.}\]
and Hasegawa and Itoh \[5\] Theorem 3.1] showed that with respect to any monotone function $f: \mathbb{R}^k \to \mathbb{R}$, the competitive ratio for the $k$-objective time series search problem is given by

$$z_f^k = \sup_{(x_1, \ldots, x_k) \in S_f^k} f \left( \frac{M_1}{x_1}, \ldots, \frac{M_k}{x_k} \right).$$

For the monotone function $f(c_1, \ldots, c_k) = (c_1 + \cdots + c_k)/k$, we refer to $z_f^k$ as the arithmetic mean component competitive ratio for the $k$-objective time series search problem. In the rest of this paper, we focus on the function $f(c_1, \ldots, c_k) = (c_1 + \cdots + c_k)/k$. For the bi-objective time series search problem, Tiedemann, et al. \[9, Theorem 3\] derived the arithmetic mean component competitive ratio $(r_1 r_2)^{1/4}$, which is disproved by Hasegawa and Itoh \[5, Theorem 4.1\], i.e.,

$$z_f^2 = \frac{1}{4} \left\{ \sqrt{4r_1^2 + (r_2 - 1)^2} + (r_2 + 1) \right\} > (r_1 r_2)^{1/4}. \tag{1}$$

In this paper, we will derive closed formulas of the arithmetic mean component competitive ratio for the $k$-objective time series search problem for an integer $k = 3$ (see Theorems \[3.1\] and \[3.2\] in Section 3) and for an integer $k = 4$ (see Theorems \[4.1\], \[4.2\] and \[4.3\] in Section 4).

For the function $f(c_1, \ldots, c_k) = (c_1 + \cdots + c_k)/k$, it is immediate that $S_f^k$ and $z_f^k$ are given by

$$S_f^k = \left\{ (x_1, \ldots, x_k) \in \text{ITV}_1 \times \cdots \times \text{ITV}_k : \frac{1}{k} \left( \frac{x_1}{m_1} + \cdots + \frac{x_k}{m_k} \right) = \frac{1}{k} \left( \frac{M_1}{x_1} + \cdots + \frac{M_k}{x_k} \right) \right\},$$

$$z_f^k = \sup_{(x_1, \ldots, x_k) \in S_f^k} \frac{1}{k} \left( \frac{M_1}{x_1} + \cdots + \frac{M_k}{x_k} \right) = \frac{1}{k} \sup_{(x_1, \ldots, x_k) \in S_f^k} \left\{ \frac{1}{2} \left( \frac{x_1}{m_1} + \cdots + \frac{x_k}{m_k} \right) + \frac{1}{2} \left( \frac{M_1}{x_1} + \cdots + \frac{M_k}{x_k} \right) \right\},$$

respectively. For each $i \in \{1, \ldots, k\}$, let $\alpha_i = \sqrt{M_i/m_i} = \sqrt{r_i}$. Since $r_1 \geq \cdots \geq r_k \geq 1$, we have that $\alpha_1 \geq \cdots \geq \alpha_k \geq 1$. For $x > 0$, let $\phi(x) = \frac{x^{\frac{1}{\alpha_i^2}}}{x^{\frac{1}{\alpha_i^2}}}$ and for each $i \in \{1, \ldots, k\}$, let

$$\xi_i = \alpha_i \phi \left( \frac{x_i}{\sqrt{m_i M_i}} \right). \tag{2}$$

Note that the function $\phi$ is monotonically increasing and it is immediate that $\phi(x^{-1}) = -\phi(x)$. So the correspondence $x_i \to \xi_i$ in (2) bijectively maps the interval $\text{ITV}_i = [m_i, M_i]$ to

$$\left[ \alpha_i \phi \left( \frac{m_i}{\sqrt{m_i M_i}} \right), \alpha_i \phi \left( \frac{M_i}{\sqrt{m_i M_i}} \right) \right] = [\alpha_i \phi(\alpha_i^{-1}), \alpha_i \phi(\alpha_i)] = [-\alpha_i \phi(\alpha_i), \alpha_i \phi(\alpha_i)].$$

For simplicity, set $\beta_i = \alpha_i \phi(\alpha_i)$. Then for each $1 \leq i \leq k$, we have that $2\beta_i = \alpha_i^2 - 1 = r_i - 1$ and let $\text{ITV}_i = [-\alpha_i \phi(\alpha_i), \alpha_i \phi(\alpha_i)] = [-\beta_i, \beta_i]$. Note that $\beta_1 \geq \cdots \geq \beta_k \geq 0$.

### 2.2 Formulations

In this subsection, we present several observations that are crucial in the subsequent discussions.

**Proposition 2.1:** Assume that the correspondence $x_i \to \xi_i$ is given by (2) for each $1 \leq i \leq k$. Then $(x_1, \ldots, x_k) \in S_f^k$ iff both of the following conditions hold: (i) $\xi_i \in [-\beta_i, \beta_i]$ for each $1 \leq i \leq k$ and (ii) $\xi_1 + \cdots + \xi_k = 0$. 3
Then it is easy to see that \((x_1, \ldots, x_k) \in S^k_f\) iff both of the conditions (i) and (ii) hold.

Proof: From the definition of the correspondence by (2), we have that

\[
\frac{1}{2} \sum_{i=1}^{k} \left( \frac{x_i}{m_i} - \frac{M_i}{x_i} \right) = \sum_{i=1}^{k} \frac{M_i}{m_i} \cdot \frac{1}{2} \left( \frac{x_i}{\sqrt{m_i M_i}} - \frac{\sqrt{m_i M_i}}{x_i} \right) = \sum_{i=1}^{k} \alpha_i \phi \left( \frac{x_i}{\sqrt{m_i M_i}} \right) = \sum_{i=1}^{k} \xi_i.
\]

Then the problem of maximizing the function \(H(x_1, \ldots, x_k)\) over \(S^k_f\) is equivalent to the problem of maximizing the function \(G(\xi_1, \ldots, \xi_k)\) over \(T^k_f\).

Proposition 2.2: Assume that the correspondence \(x_i \rightarrow \xi_i\) is given by (2) for each \(1 \leq i \leq k\). Then the problem of maximizing the function \(H(x_1, \ldots, x_k)\) over \(S^k_f\) is equivalent to the problem of maximizing the function \(G(\xi_1, \ldots, \xi_k)\) over \(T^k_f\).

Proof: By straightforward calculations, we have that for each \(1 \leq i \leq k\),

\[
\frac{x_i}{m_i} + \frac{M_i}{x_i} = 2 \sqrt{\frac{M_i}{m_i} \cdot \frac{x_i}{\sqrt{m_i M_i}} + \frac{\sqrt{m_i M_i}}{x_i}} = 2 \alpha_i \sqrt{\left( \frac{x_i}{\sqrt{m_i M_i}} + \frac{\sqrt{m_i M_i}}{x_i} \right)^2}
\]

\[
= 2 \alpha_i \left( 1 + \frac{x_i}{\sqrt{m_i M_i}} - \frac{\sqrt{m_i M_i}}{x_i} \right)^2 = 2 \alpha_i \left( 1 + \phi^2 \left( \frac{x_i}{\sqrt{m_i M_i}} \right) \right)
\]

\[
= 2 \alpha_i + \alpha_i^2 \phi^2 \left( \frac{x_i}{\sqrt{m_i M_i}} \right) = 2 \alpha_i^2 + \xi_i^2,
\]

where the last equality follows from the correspondence \(x_i \rightarrow \xi_i\) in (2). Thus it is immediate that the problem of maximizing the function \(H(x_1, \ldots, x_k)\) over \(S^k_f\) is equivalent to the problem of maximizing the function \(G(\xi_1, \ldots, \xi_k)\) over \(T^k_f\).

Lemma 2.1: If \(\xi^* = (\xi_1, \ldots, \xi_k) \in T^k_f\) maximizes the function \(G\), then (i) there exists at most a single unfilled variable \(\xi_j\) and (ii) \(-\xi^*\) \(\in T^k_f\) maximizes the function \(G\).

Proof: For the statement (i), assume that there exist two distinct unfilled variables \(\xi_{j_1}\) and \(\xi_{j_2}\), i.e., \(-\beta_{j_1} < \xi_{j_1} < \beta_{j_1}\) and \(-\beta_{j_2} < \xi_{j_2} < \beta_{j_2}\). So we have that

\[
G(\xi_1, \ldots, \xi_k) = 2 \left( \sqrt{\alpha_{j_1}^2 + \xi_{j_1}^2} + \sqrt{\alpha_{j_2}^2 + \xi_{j_2}^2} + \sum_{j \in \{1, \ldots, k\} \setminus \{j_1, j_2\}} \sqrt{\alpha_j^2 + \beta_j^2} \right)
\]

Then there exists \(\eta \neq 0\) such that \(\xi_{j_1} + \eta \in \text{ITV}'_{j_1}\) and \(\xi_{j_2} - \eta \in \text{ITV}'_{j_2}\). Let \(\xi'_{j_1} = \xi_{j_1} + \eta, \xi'_{j_2} = \xi_{j_2} - \eta, \text{ and } \xi'_j = \xi_j\) for each \(j \in \{1, \ldots, k\} \setminus \{j_1, j_2\}\). It is immediate that \((\xi'_1, \ldots, \xi'_k) \in T^k_f\). For the rest of the proof, we use the following claim (the proof of the claim is given in Appendix A).
Claim 2.1: For $a, b, c, d > 0$, let $h(x) = \sqrt{a^2 + (b + x)^2} + \sqrt{c^2 + (d - x)^2}$. Then $h'(x)$ and $h''(x)$ are continuous, and (i) $\text{sgn } h'(0) = \text{sgn } (cb - ad)$ and (ii) $h''(0) > 0$.

For Claim 2.1, set $a = \alpha_{j_1}$, $b = \xi_{j_1}$, $c = \alpha_{j_2}$, and $d = \xi_{j_2}$. If $h'(0) = \text{sgn}(\alpha_{j_2}\xi_{j_1} - \alpha_{j_1}\xi_{j_2}) \neq 0$, then by the continuity of $h'$, we can take a small $\eta \neq 0$ to satisfy the following conditions: $\text{sgn } \eta = \text{sgn } h'(0)$ and $\text{sgn } h'(x) = \text{sgn } h'(0)$ for all $x$ between 0 and $\eta$. On the other hand, if $h'(0) = \text{sgn}(\alpha_{j_2}\xi_{j_1} - \alpha_{j_1}\xi_{j_2}) = 0$, then by the continuity of $h''$, we can take a small $\eta \neq 0$ to satisfy the following conditions: $\text{sgn } \eta = \pm 1$ and $h''(x) = h''(0)$ for all $x$ between 0 and $\eta$. Then in either case, it follows from the mean value theorem that

$$\sqrt{\alpha_{j_1}^2 + \xi_{j_1}^2} + \sqrt{\alpha_{j_2}^2 + \xi_{j_2}^2} < \sqrt{\alpha_{j_1}^2 + (\xi_{j_1} + \eta)^2} + \sqrt{\alpha_{j_2}^2 + (\xi_{j_2} - \eta)^2},$$

where the inequality follows from Claim 2.1 (i) for the case that $\alpha_{j_2}\xi_{j_1} - \alpha_{j_1}\xi_{j_2} \neq 0$ and from Claim 2.1 (ii) for the case that $\alpha_{j_2}\xi_{j_1} - \alpha_{j_1}\xi_{j_2} = 0$. This implies that $G(\xi_1, \ldots, \xi_k) < G(\xi'_1, \ldots, \xi'_k)$, i.e., $\xi = (\xi_1, \ldots, \xi_k) \in \mathcal{T}_k^k$ is not a maximizer of $G$.

For the statement (ii), it is immediate from the fact that $G(\xi_1, \ldots, \xi_k) = G(-\xi_1, \ldots, -\xi_k)$ and the definition of $\mathcal{T}_k^k$, i.e., $-(\xi_1 + \cdots + \xi_k) = 0$ for any $\xi = (\xi_1, \ldots, \xi_k) \in \mathcal{T}_k^k$.

### 2.3 A Simplified Derivation of the Competitive Ratio for $k = 2$

As a consequence of Lemma 2.1 for $k = 2$, we can derive a closed formula of the arithmetic mean component competitive ratio for $k = 2$ given in (1). For $(\xi_1, \xi_2) \in \mathcal{T}_2^2$ that maximizes $G$, consider the following cases: (2.0) $\xi_1$ and $\xi_2$ are filled, (2.1) $\xi_1$ is unfilled, and (2.2) $\xi_2$ is unfilled.

The case (2.0) is possible only when $\beta_1 = \beta_2$. In this case, it is immediate that

$$G(\pm \beta_1, \mp \beta_2) = G(\pm \beta_2, \mp \beta_2) = 2\sqrt{4r_1 + (r_2 - 1)^2} + (r_2 + 1).$$

For the case (2.1), we have that $\xi_2$ is filled, i.e., $\xi_2 = \pm \beta_2$. Since $(\xi_1, \xi_2) \in \mathcal{T}_2^2$, it is obvious that $\xi_1 = -\xi_2 = \mp \beta_2$. Then from Lemma 2.1, it follows that

$$G(\pm \beta_2, \mp \beta_2) = 2\left(\sqrt{\alpha_1^2 + \beta_2^2} + \sqrt{\alpha_2^2 + \beta_2^2}\right) = 2\sqrt{4r_1 + (r_2 - 1)^2} + (r_2 + 1).$$

For the case (2.2), we have that $\xi_1$ is filled, i.e., $\xi_1 = \pm \beta_1$. Without loss of generality, assume by Lemma 2.1 (ii) that $\xi_1 = \beta_1$. Since $\xi_2$ is unfilled and $(\xi_1, \xi_2) \in \mathcal{T}_2^2$, it is immediate that $-\beta_2 < \xi_2 = -\xi_1 = -\beta_1$, which contradicts the assumption that $\beta_1 \geq \beta_2 \geq 1$. Thus the case (2.2) never occurs. Then the arithmetic mean component competitive ratio $z_f^2$ is given by

$$z_f^2 = \frac{1}{4} \cdot G(\pm \beta_2, \mp \beta_2) = \frac{1}{4} \cdot \left\{\sqrt{4r_1 + (r_2 - 1)^2} + (r_2 + 1)\right\}.$$

In the following sections, we extend the above argument for the case that $k = 2$ to the case that $k = 3$ (see Section 3) and the case that $k = 4$ (see Section 4).

### 3 Competitive Ratio for $k = 3$

In this section, we derive closed formulas of the arithmetic mean component competitive ratio for the 3-objective time series search problem.
Theorem 3.1: If \((r_1 - 1) \geq (r_2 - 1) + (r_3 - 1)\), then the arithmetic mean component competitive ratio for the 3-objective time series search problem is

\[
    z_f^3 = \frac{1}{6} \cdot \left[ \sqrt{4r_1 + \{(r_2 - 1) + (r_3 - 1)\}^2 + (r_2 + 1) + (r_3 + 1)} \right].
\]

Theorem 3.2: If \((r_1 - 1) < (r_2 - 1) + (r_3 - 1)\), then the arithmetic mean component competitive ratio for the 3-objective time series search problem is

\[
    z_f^3 = \frac{1}{6} \cdot \left\{ \sqrt{4r_1 + (r_2 - r_2)^2 + (r_1 + 1) + (r_2 + 1)} \right\}.
\]

Let \((\xi_1, \xi_2, \xi_3) \in \mathcal{T}_3^2\) be a maximizer of \(G\). By Lemma 2.1(i), there can exist a unfilled variable, consider the following cases: (3.1) none of the variables \(\xi_1, \xi_2,\) and \(\xi_3\) is unfilled or the variable \(\xi_1\) is unfilled, (3.2) the variable \(\xi_2\) is unfilled, and (3.3) the variable \(\xi_3\) is unfilled.

For the case (3.1), two cases (3.1.1) \(\xi_1 = \pm (\beta_2 + \beta_3)\) and (3.1.2) \(\xi_1 = \pm (\beta_2 - \beta_3)\) are possible, for the case (3.2), two cases (3.2.1) \(\xi_2 = \pm (\beta_1 + \beta_3)\) and (3.2.2) \(\xi_2 = \pm (\beta_1 - \beta_3)\) are possible, and for the case (3.3), two cases (3.3.1) \(\xi_3 = \pm (\beta_1 + \beta_2)\) and (3.3.2) \(\xi_3 = \pm (\beta_1 - \beta_2)\) are possible. Then it is immediate to observe that

- Case (3.1.1) Possible only if \(\beta_1 \geq \beta_2 + \beta_3\).
- Case (3.1.2) Always possible.
- Case (3.2.1) Impossible.
- Case (3.2.2) Possible only if \(\beta_1 < \beta_2 + \beta_3\).
- Case (3.3.1) Impossible.
- Case (3.3.2) Possible only if \(\beta_1 < \beta_2 + \beta_3\).

We classify the problem instances based on the magnitude of \(\beta_1\), i.e., \(\beta_1 \geq \beta_2 + \beta_3\) and \(\beta_1 < \beta_2 + \beta_3\). Then the possibilities for the cases (3.1.1), \ldots, (3.3.2) can be summarized in Table 1.

|               | \(\beta_1 \geq \beta_2 + \beta_3\) | \(\beta_1 < \beta_2 + \beta_3\) |
|---------------|----------------------------------|---------------------------------|
| Case (3.1.1)  | possible                         | —                               |
| Case (3.1.2)  | possible                         | possible                        |
| Case (3.2.1)  | —                               | possible                        |
| Case (3.2.2)  | —                               | possible                        |
| Case (3.3.1)  | —                               | —                               |
| Case (3.3.2)  | —                               | possible                        |

### 3.1 Proof of Theorem 3.1

Assume that \(\beta_1 \geq \beta_2 + \beta_3\), i.e., \((r_1 - 1) \geq (r_2 - 1) + (r_3 - 1)\). For the cases (3.1.1) and (3.1.2), let \(V_{(3.1.1)}\) and \(V_{(3.1.2)}\) be the potential maximum values of \(G\), respectively, i.e.,

\[
    V_{(3.1.1)} = G(\pm (\beta_2 + \beta_3), \mp \beta_2, \mp \beta_3) = 2 \left\{ \sqrt{\alpha_1^2 + (\beta_2 + \beta_3)^2} + \sqrt{\alpha_2^2 + \beta_2^2} + \sqrt{\alpha_3^2 + \beta_3^2} \right\}
\]

\[
    = \sqrt{4r_1 + \{(r_2 - 1) + (r_3 - 1)\}^2 + (r_2 + 1) + (r_3 + 1)};
\]

\[
    V_{(3.1.2)} = G(\pm (\beta_2 - \beta_3), \mp \beta_2, \pm \beta_3) = 2 \left\{ \sqrt{\alpha_1^2 + (\beta_2 - \beta_3)^2} + \sqrt{\alpha_2^2 + \beta_2^2} + \sqrt{\alpha_3^2 + \beta_3^2} \right\}
\]

\[
    = \sqrt{4r_1 + \{(r_2 - 1) - (r_3 - 1)\}^2 + (r_2 + 1) + (r_3 + 1)}.
\]
Since \( r_2 \geq r_3 \geq 1 \), it is immediate to see that \( V_{(3.1.1)} \geq V_{(3.1.2)} \). Thus for the case that \( (r_1 - 1) \geq (r_2 - 1) + (r_3 - 1) \), we can conclude that
\[
z_j^2 = \frac{V_{(3.1.1)}}{6} = \frac{1}{6} \left[ \sqrt{4r_1 + \{(r_2 - 1) + (r_3 - 1)\}^2} + (r_2 + 1) + (r_3 + 1) \right].
\]

### 3.2 Proof of Theorem 3.2

The following proposition is crucial (and its proof is given in Appendix B).

**Proposition 3.1:** Let \( F_3(x, y, z) = \sqrt{4y + (x - z)^2} + (x - y) - \sqrt{4x + (y - z)^2} \). For any \( x \geq y \),
(a) if \( z \leq x \) and \( y \geq x - z + 1 \), then \( F_3(x, y, z) \geq 0 \);
(b) if \( y \geq z - x + 1 \), then \( F_3(x, y, z) \geq 0 \).

Assume that \( \beta_1 < \beta_2 + \beta_3 \), i.e., \( (r_1 - 1) < (r_2 - 1) + (r_3 - 1) \). For the cases (3.1.2), (3.2.2), and (3.3.2), let \( V_{(3.1.2)} \), \( V_{(3.2.2)} \) and \( V_{(3.3.2)} \) be the potential maximum values of \( G \), respectively, i.e.,

\[
\begin{align*}
V_{(3.1.2)} &= G(\pm(\beta_2 - \beta_3), \mp\beta_2, \pm\beta_3) = 2\left\{ \sqrt{\alpha_1^2 + (\beta_2 - \beta_3)^2} + \sqrt{\alpha_2^2 + \beta_2^2} + \sqrt{\alpha_3^2 + \beta_3^2} \right\} \\
&= \sqrt{4r_1 + (r_2 - r_3)^2} + (r_2 + 1) + (r_3 + 1); \\
V_{(3.2.2)} &= G(\pm\beta_1, \mp(\beta_1 - \beta_3), \mp\beta_3) = 2\left\{ \sqrt{\alpha_2^2 + (\beta_1 - \beta_3)^2} + \sqrt{\alpha_1^2 + \beta_1^2} + \sqrt{\alpha_3^2 + \beta_3^2} \right\} \\
&= \sqrt{4r_2 + (r_1 - r_3)^2} + (r_1 + 1) + (r_3 + 1); \\
V_{(3.3.2)} &= G(\pm\beta_1, \mp\beta_2, \pm(\beta_1 - \beta_2)) = 2\left\{ \sqrt{\alpha_3^2 + (\beta_1 - \beta_2)^2} + \sqrt{\alpha_1^2 + \beta_1^2} + \sqrt{\alpha_2^2 + \beta_2^2} \right\} \\
&= \sqrt{4r_3 + (r_1 - r_2)^2} + (r_1 + 1) + (r_2 + 1). \\
\end{align*}
\]

In the following lemmas, we show that \( V_{(3.3.2)} \) is the maximum in \( V_{(3.1.2)} \), \( V_{(3.2.2)} \), and \( V_{(3.3.2)} \) for the case that \( (r_1 - 1) < (r_2 - 1) + (r_3 - 1) \).

**Lemma 3.1:** \( V_{(3.2.2)} \geq V_{(3.1.2)} \).

**Proof:** From (4) and (3), it is immediate that
\[
V_{(3.2.2)} - V_{(3.1.2)} = \sqrt{4r_2 + (r_1 - r_3)^2} + (r_1 - r_2) - \sqrt{4r_1 + (r_2 - r_3)^2}.
\]
Set \( x = r_1, \ y = r_2, \) and \( z = r_3 \). From the fact that \( r_1 \geq r_2 \geq r_3 \geq 1 \), we have that \( x \geq y \) and \( z \leq x \), and from the assumption that \( (r_1 - 1) < (r_2 - 1) + (r_3 - 1) \), it follows that \( y > x - z + 1 \). Thus from Proposition 3.1(a), the lemma follows. \( \blacksquare \)

**Lemma 3.2:** \( V_{(3.3.2)} \geq V_{(3.2.2)} \).

**Proof:** From (5) and (4), it is immediate that
\[
V_{(3.3.2)} - V_{(3.2.2)} = \sqrt{4r_3 + (r_1 - r_2)^2} + (r_2 - r_3) - \sqrt{4r_2 + (r_1 - r_3)^2} \\
= \sqrt{4r_3 + (r_1 - r_2)^2} + (r_2 - r_3) - \sqrt{4r_2 + (r_1 - r_3)^2}.
\]
Set \( x = r_2, \ y = r_3, \) and \( z = r_1 \). From the fact that \( r_1 \geq r_2 \geq r_3 \geq 1 \), we have that \( x \geq y \) and from the assumption that \( (r_1 - 1) < (r_2 - 1) + (r_3 - 1) \), it follows that \( y > z - x + 1 \). Thus from Proposition 3.1(b), the lemma follows. \( \blacksquare \)

From Lemmas 3.1 and 3.2, it follows that \( V_{(3.3.2)} \) is the maximum in \( V_{(3.1.2)}, V_{(3.2.2)} \) and \( V_{(3.3.2)} \). Thus Theorem 3.2 holds, i.e., if \( (r_1 - 1) < (r_2 - 1) + (r_3 - 1) \), then
\[
z_j^2 = \frac{V_{(3.3.2)}}{6} = \frac{1}{6} \left\{ \sqrt{4r_3 + (r_1 - r_2)^2} + (r_1 + 1) + (r_2 + 1) \right\}. 
\]
4 Competitive Ratio for $k = 4$

In this section, we derive closed formulas of the arithmetic mean component competitive ratio for the 4-objective time series search problem.

**Theorem 4.1:** If $(r_1 - 1) \geq (r_2 - 1) + (r_3 - 1) + (r_4 - 1)$, then the arithmetic mean component competitive ratio for the 4-objective time series search problem is

$$z_4^1 = \frac{1}{8} \left[ \sqrt{4r_1 + \{(r_2 - 1) + (r_3 - 1) + (r_4 - 1)\}^2} + (r_2 + 1) + (r_3 + 1) + (r_4 + 1) \right].$$

**Theorem 4.2:** If $(r_2 - 1) + (r_3 - 1) - (r_4 - 1) \leq (r_1 - 1) < (r_2 - 1) + (r_3 - 1) + (r_4 - 1)$, then the arithmetic mean component competitive ratio for the 4-objective time series search problem is

$$z_4^2 = \frac{1}{8} \left[ \sqrt{4r_4 + \{(r_1 - 1) - (r_2 - 1) - (r_3 - 1)\}^2} + (r_1 + 1) + (r_2 + 1) + (r_3 + 1) \right].$$

**Theorem 4.3:** If $(r_1 - 1) < (r_2 - 1) + (r_3 - 1) - (r_4 - 1)$, then the arithmetic mean component competitive ratio for the 4-objective time series search problem is

$$z_4^3 = \frac{1}{8} \left[ \sqrt{4r_3 + \{(r_1 - 1) - (r_2 - 1) + (r_4 - 1)\}^2} + (r_1 + 1) + (r_2 + 1) + (r_4 + 1) \right].$$

Let $(\xi_1, \xi_2, \xi_3, \xi_4) \in T^4_k$ be a maximizer of $G$. By Lemma 2.1(i), there can exist an unfilled variable $\xi_j \in \{\xi_1, \xi_2, \xi_3, \xi_4\}$. According to an unfilled variable, consider the following cases: (4.1) none of the variables $\xi_1, \xi_2, \xi_3$, and $\xi_4$ is unfilled or the variable $\xi_1$ is unfilled, (4.2) the variable $\xi_2$ is unfilled, (4.3) the variable $\xi_3$ is unfilled, and (4.4) the variable $\xi_4$ is unfilled.

For the case (4.1), we have four cases (4.1.1) $\xi_1 = \pm(\beta_2 + \beta_3 + \beta_4)$, (4.1.2) $\xi_1 = \pm(\beta_2 + \beta_3 - \beta_4)$, (4.1.3) $\xi_1 = \pm(\beta_2 - \beta_3 + \beta_4)$, and (4.1.4) $\xi_1 = \pm(-\beta_2 + \beta_3 + \beta_4)$. Then it is immediate that

Case (4.1.1) Possible only if $\beta_1 \geq \beta_2 + \beta_3 + \beta_4$. Case (4.1.3) Always possible.

Case (4.1.2) Possible only if $\beta_1 \geq \beta_2 + \beta_3 - \beta_4$. Case (4.1.4) Always possible.

For the case (4.2), we have four cases (4.2.1) $\xi_2 = \pm(\beta_1 + \beta_3 + \beta_4)$, (4.2.2) $\xi_2 = \pm(\beta_1 + \beta_3 - \beta_4)$, (4.2.3) $\xi_2 = \pm(-\beta_1 + \beta_3 + \beta_4)$, and (4.2.4) $\xi_2 = \pm(-\beta_1 + \beta_3 - \beta_4)$. Then it is immediate that

Case (4.2.1) Impossible. Case (4.2.3) Possible only if $\beta_1 < \beta_2 + \beta_3 - \beta_4$.

Case (4.2.2) Impossible. Case (4.2.4) Possible only if $\beta_1 < \beta_2 + \beta_3 + \beta_4$.

For the case (4.3), we have four cases (4.3.1) $\xi_3 = \pm(\beta_1 + \beta_2 + \beta_4)$, (4.3.2) $\xi_3 = \pm(\beta_1 + \beta_2 - \beta_4)$, (4.3.3) $\xi_3 = \pm(-\beta_1 + \beta_2 + \beta_4)$, and (4.3.4) $\xi_3 = \pm(-\beta_1 + \beta_2 - \beta_4)$. Then it is immediate that

Case (4.3.1) Impossible. Case (4.3.3) Possible only if $\beta_1 < \beta_2 + \beta_3 - \beta_4$.

Case (4.3.2) Impossible. Case (4.3.4) Possible only if $\beta_1 < \beta_2 + \beta_3 + \beta_4$.

For the case (4.4), we have four cases (4.4.1) $\xi_4 = \pm(\beta_1 + \beta_2 + \beta_3)$, (4.4.2) $\xi_4 = \pm(\beta_1 + \beta_2 - \beta_3)$, (4.4.3) $\xi_4 = \pm(-\beta_1 + \beta_2 + \beta_3)$, and (4.4.4) $\xi_4 = \pm(-\beta_1 + \beta_2 - \beta_3)$. Then it is immediate that

Case (4.4.1) Impossible. Case (4.4.3) Impossible.

Case (4.4.2) Impossible. Case (4.4.4) Possible only if $\beta_2 + \beta_3 - \beta_4 < \beta_1 < \beta_2 + \beta_3 + \beta_4$.

We classify the problem instances based on the magnitude of $\beta_1$, i.e., consider the cases $\beta_1 \geq \beta_2 + \beta_3 + \beta_4$, $\beta_2 + \beta_3 - \beta_4 \leq \beta_1 < \beta_2 + \beta_3 + \beta_4$, and $\beta_1 < \beta_2 + \beta_3 - \beta_4$. Then the possibilities for the cases (4.1.1), . . . , (4.4.4) can be summarized in Table 2.
4.1 Proof of Theorem 4.1

Assume that $\beta_1 \geq \beta_2 + \beta_3 + \beta_4$, i.e., $(r_1 - 1) \geq (r_2 - 1) + (r_3 - 1) + (r_4 - 1)$. For the cases (4.1.1), (4.1.2), (4.1.3), and (4.1.4), we use $V_{(4.1.1)}$, $V_{(4.1.2)}$, $V_{(4.1.3)}$, and $V_{(4.1.4)}$ to denote the potential maximum values of $G$, respectively, i.e.,

$$
V_{(4.1.1)} = G(\pm(\beta_2 + \beta_3 + \beta_4), \mp\beta_2, \mp\beta_3, \mp\beta_4)
$$

$$
= 2 \left\{ \sqrt{\alpha_2^2 + (\beta_2 + \beta_3 + \beta_4)^2} + \sqrt{\alpha_2^2 + \beta_2^2} + \sqrt{\alpha_3^2 + \beta_3^2} + \sqrt{\alpha_4^2 + \beta_4^2} \right\}
$$

$$
= \sqrt{4r_1 + \{(r_2 - 1) + (r_3 - 1) + (r_4 - 1)\}} + (r_2 + 1) + (r_3 + 1) + (r_4 + 1);
$$

$$
V_{(4.1.2)} = G(\pm(\beta_2 + \beta_3 - \beta_4), \mp\beta_2, \mp\beta_3, \mp\beta_4)
$$

$$
= 2 \left\{ \sqrt{\alpha_2^2 + (\beta_2 + \beta_3 - \beta_4)^2} + \sqrt{\alpha_2^2 + \beta_2^2} + \sqrt{\alpha_3^2 + \beta_3^2} + \sqrt{\alpha_4^2 + \beta_4^2} \right\}
$$

$$
= \sqrt{4r_1 + \{(r_2 - 1) + (r_3 - 1) - (r_4 - 1)\}} + (r_2 + 1) + (r_3 + 1) + (r_4 + 1);
$$

$$
V_{(4.1.3)} = G(\pm(\beta_2 - \beta_3 + \beta_4), \mp\beta_2, \pm\beta_3, \mp\beta_4)
$$

$$
= 2 \left\{ \sqrt{\alpha_2^2 + (\beta_2 - \beta_3 + \beta_4)^2} + \sqrt{\alpha_2^2 + \beta_2^2} + \sqrt{\alpha_3^2 + \beta_3^2} + \sqrt{\alpha_4^2 + \beta_4^2} \right\}
$$

$$
= \sqrt{4r_1 + \{(r_2 - 1) - (r_3 - 1) + (r_4 - 1)\}} + (r_2 + 1) + (r_3 + 1) + (r_4 + 1);
$$

$$
V_{(4.1.4)} = G(\pm(-\beta_2 + \beta_3 + \beta_4), \pm\beta_2, \mp\beta_3, \mp\beta_4)
$$

$$
= 2 \left\{ \sqrt{\alpha_2^2 + (-\beta_2 + \beta_3 + \beta_4)^2} + \sqrt{\alpha_2^2 + \beta_2^2} + \sqrt{\alpha_3^2 + \beta_3^2} + \sqrt{\alpha_4^2 + \beta_4^2} \right\}
$$

$$
= \sqrt{4r_1 + \{-(r_2 - 1) + (r_3 - 1) + (r_4 - 1)\}} + (r_2 + 1) + (r_3 + 1) + (r_4 + 1).
$$

Since $r_1 \geq r_2 \geq r_3 \geq r_4 \geq 1$, it is easy to see that $V_{(4.1.1)} \geq V_{(4.1.2)}$, $V_{(4.1.1)} \geq V_{(4.1.3)}$, and $V_{(4.1.1)} \geq V_{(4.1.4)}$, i.e., $V_{(4.1.1)}$ is the maximum in $V_{(4.1.1)}$, $V_{(4.1.2)}$, $V_{(4.1.3)}$, and $V_{(4.1.4)}$. Thus for the case that

| Case     | $\beta_1 \geq \beta_2 + \beta_3 + \beta_4$ | $\beta_2 + \beta_3 - \beta_4 \leq \beta_1 < \beta_2 + \beta_3 + \beta_4$ | $\beta_1 < \beta_2 + \beta_3 - \beta_4$ |
|----------|---------------------------------|-------------------------------------------------|---------------------------------|
| Case (4.1.1) | possible | — | — |
| Case (4.1.2) | possible | possible | — |
| Case (4.1.3) | possible | possible | possible |
| Case (4.1.4) | possible | possible | possible |
| Case (4.2.1) | — | — | — |
| Case (4.2.2) | — | — | — |
| Case (4.2.3) | — | — | possible |
| Case (4.2.4) | — | possible | possible |
| Case (4.3.1) | — | — | — |
| Case (4.3.2) | — | — | — |
| Case (4.3.3) | — | — | possible |
| Case (4.3.4) | — | possible | possible |
| Case (4.4.1) | — | — | — |
| Case (4.4.2) | — | — | — |
| Case (4.4.3) | — | — | — |
| Case (4.4.4) | — | possible | — |
Proposition 4.1: Let $G = \max \{x, y, z, p\}$, then

$$z^4 = \frac{V_{(4,1,1)}}{8} = \frac{1}{8} \cdot \left[ \sqrt{4r_1 + \{ (r_2 - 1) + (r_3 - 1) + (r_4 - 1) \}}^2 + (r_2 + 1) + (r_3 + 1) + (r_4 + 1) \right].$$

4.2 Proof of Theorem 4.2

The following proposition is crucial (and its proof is given in Appendix C).

Proposition 4.1: Let $F_4(x, y, z, p) = \sqrt{4y + (z - x - p)^2 + (x - y) - \sqrt{4x + (z - y - p)^2}}$. For any $x \geq y$, if $y \geq z - x - p + 1$, then $F_4(x, y, z, p) \geq 0$.

Assume that $\beta_2 + \beta_3 - \beta_4 \leq \beta_1 < \beta_2 + \beta_3 + \beta_4$, i.e.,

$$(r_2 - 1) + (r_3 - 1) - (r_4 - 1) \leq (r_1 - 1) < (r_2 - 1) + (r_3 - 1) + (r_4 - 1).$$

For the cases (4.1.2), (4.1.3), and (4.1.4), let $V_{(4,1,2)}$, $V_{(4,1,3)}$, and $V_{(4,1,4)}$ be the potential maximum values of $G$, respectively, i.e.,

$$V_{(4,1,2)} = G(\pm (\beta_2 + \beta_3 - \beta_4), \mp \beta_2, \mp \beta_3, \pm \beta_4)$$

$$= 2 \left\{ \sqrt{\alpha_1^2 + (\beta_2 + \beta_3 - \beta_4)^2} + \sqrt{\alpha_2^2 + \beta_2^2} + \sqrt{\alpha_3^2 + \beta_3^2} + \sqrt{\alpha_4^2 + \beta_4^2} \right\}$$

$$= \sqrt{4r_1 + \{(r_2 - 1) + (r_3 - 1) - (r_4 - 1)\}^2 + (r_2 + 1) + (r_3 + 1) + (r_4 + 1)}; \quad (6)$$

$$V_{(4,1,3)} = G(\pm (\beta_2 - \beta_3 + \beta_4), \mp \beta_2, \pm \beta_3, \mp \beta_4)$$

$$= 2 \left\{ \sqrt{\alpha_1^2 + (\beta_2 - \beta_3 + \beta_4)^2} + \sqrt{\alpha_2^2 + \beta_2^2} + \sqrt{\alpha_3^2 + \beta_3^2} + \sqrt{\alpha_4^2 + \beta_4^2} \right\}$$

$$= \sqrt{4r_1 + \{(r_2 - 1) - (r_3 - 1) + (r_4 - 1)\}^2 + (r_2 + 1) + (r_3 + 1) + (r_4 + 1)};$$

$$V_{(4,1,4)} = G(\pm (\beta_2 + \beta_3 + \beta_4), \pm \beta_2, \pm \beta_3, \mp \beta_4)$$

$$= 2 \left\{ \sqrt{\alpha_1^2 + (\beta_2 + \beta_3 + \beta_4)^2} + \sqrt{\alpha_2^2 + \beta_2^2} + \sqrt{\alpha_3^2 + \beta_3^2} + \sqrt{\alpha_4^2 + \beta_4^2} \right\}$$

$$= \sqrt{4r_1 + \{-(r_2 - 1) + (r_3 - 1) + (r_4 - 1)\}^2 + (r_2 + 1) + (r_3 + 1) + (r_4 + 1)}.$$

Since $r_1 \geq r_2 \geq r_3 \geq r_4 \geq 1$, it is obvious that $V_{(4,1,2)} \geq V_{(4,1,3)}$ and $V_{(4,1,2)} \geq V_{(4,1,4)}$, i.e., $V_{(4,1,2)}$ is the maximum in $V_{(4,1,2)}$, $V_{(4,1,3)}$, and $V_{(4,1,4)}$. For the cases (4.2.4), (4.3.4), and (4.4.4), we use $V_{(4,2,4)}$, $V_{(4,3,4)}$, and $V_{(4,4,4)}$ to denote the potential maximum values of $G$, respectively, i.e.,

$$V_{(4,2,4)} = G(\pm \beta_1, \mp (\beta_2 + \beta_3 + \beta_4), \mp \beta_3, \mp \beta_4)$$

$$= 2 \left\{ \sqrt{\alpha_2^2 + (-\beta_1 + \beta_3 + \beta_4)^2} + \sqrt{\alpha_1^2 + \beta_1^2} + \sqrt{\alpha_3^2 + \beta_3^2} + \sqrt{\alpha_4^2 + \beta_4^2} \right\}$$

$$= 2 \left\{ \sqrt{\alpha_2^2 + (-\beta_1 - \beta_3 - \beta_4)^2} + \sqrt{\alpha_1^2 + \beta_1^2} + \sqrt{\alpha_3^2 + \beta_3^2} + \sqrt{\alpha_4^2 + \beta_4^2} \right\}$$

$$= \sqrt{4r_2 + \{(r_1 - 1) - (r_3 - 1) - (r_4 - 1)\}^2 + (r_1 + 1) + (r_3 + 1) + (r_4 + 1)}; \quad (7)$$

$$V_{(4,3,4)} = G(\pm \beta_1, \mp \beta_2, \pm (\beta_1 + \beta_2 + \beta_4), \mp \beta_4)$$

$$= 2 \left\{ \sqrt{\alpha_3^2 + (-\beta_1 + \beta_2 + \beta_4)^2} + \sqrt{\alpha_1^2 + \beta_1^2} + \sqrt{\alpha_2^2 + \beta_2^2} + \sqrt{\alpha_4^2 + \beta_4^2} \right\}$$

$$= 2 \left\{ \sqrt{\alpha_3^2 + (\beta_1 - \beta_2 - \beta_4)^2} + \sqrt{\alpha_1^2 + \beta_1^2} + \sqrt{\alpha_2^2 + \beta_2^2} + \sqrt{\alpha_4^2 + \beta_4^2} \right\}$$

$$= \sqrt{4r_3 + \{(r_1 - 1) - (r_2 - 1) - (r_4 - 1)\}^2 + (r_1 + 1) + (r_2 + 1) + (r_4 + 1)}; \quad (8)$$
\[ V_{(4.4.4)} = G(\pm \beta_1, \mp \beta_2, \mp \beta_3, \pm (-\beta_1 + \beta_2 + \beta_3)) \]
\[ = 2 \left\{ \sqrt{\alpha_1^2 + (-\beta_1 + \beta_2 + \beta_3)^2} + \sqrt{\alpha_2^2 + \beta_1^2 + \sqrt{\alpha_3^2 + \beta_2^2}} \right\} \]
\[ = 2 \left\{ \sqrt{\alpha_1^2 + (\beta_1 - \beta_2 - \beta_3)^2} + \sqrt{\alpha_2^2 + \beta_1^2 + \sqrt{\alpha_3^2 + \beta_2^2}} \right\} \]
\[ = \sqrt{4r_4 + \{(r_1 - 1) - (r_2 - 1) - (r_3 - 1)\}^2 + (r_1 + 1) + (r_2 + 1) + (r_3 + 1)}. \quad (9) \]

In the following lemmas, we show that \( V_{(4.4.4)} \) is the maximum in \( V_{(4.1.2)}, V_{(4.2.4)}, V_{(4.3.4)}, \) and \( V_{(4.4.4)} \) for the case that \((r_2 - 1) + (r_3 - 1) - (r_4 - 1) \leq (r_1 - 1) < (r_2 - 1) + (r_3 - 1) + (r_4 - 1).\)

**Lemma 4.1:** \( V_{(4.4.4)} \geq V_{(4.1.2)}. \)

**Proof:** From (9) and (6), it is immediate that
\[ V_{(4.4.4)} - V_{(4.1.2)} = \sqrt{4r_4 + (r_1 - r_2 - r_3 + 1)^2 + (r_1 - r_4)} - \sqrt{4r_4 + (r_2 + r_3 - r_4 - 1)^2}. \]

Set \( x = r_1, y = r_4, z = r_2 + r_3, \) and \( p = 1. \) From the fact that \( r_1 \geq r_2 \geq r_3 \geq r_4 \geq 1, \) we have that \( x \geq y, \) and from the assumption that \((r_2 - 1) + (r_3 - 1) - (r_4 - 1) \leq (r_1 - 1), \) it follows that \( y \geq z - x - p + 1. \) Thus from Proposition 4.1, the lemma follows.

**Lemma 4.2:** \( V_{(4.3.4)} \geq V_{(4.2.4)}. \)

**Proof:** From (8) and (7), it is immediate that
\[ V_{(4.3.4)} - V_{(4.2.4)} = \sqrt{4r_3 + (r_1 - r_2 - r_4 + 1)^2 + (r_2 - r_3)} - \sqrt{4r_2 + (r_1 - r_3 - r_4 + 1)^2}. \]

Set \( x = r_2, y = r_3, z = r_1 - r_4, \) and \( p = -1. \) From the fact that \( r_1 \geq r_2 \geq r_3 \geq r_4 \geq 1, \) we have that \( x \geq y, \) and from the assumption that \((r_1 - 1) < (r_2-1)+(r_3-1)+(r_4-1), \) it follows that \( y > z - x - p + 1. \) Thus from Proposition 4.1, the lemma follows.

**Lemma 4.3:** \( V_{(4.4.4)} \geq V_{(4.3.4)}. \)

**Proof:** From (9) and (8), it is immediate that
\[ V_{(4.4.4)} - V_{(4.3.4)} = \sqrt{4r_4 + (r_1 - r_2 - r_3 + 1)^2 + (r_3 - r_4)} - \sqrt{4r_3 + (r_1 - r_2 - r_4 + 1)^2}. \]

Set \( x = r_3, y = r_4, z = r_1 - r_2, \) and \( p = -1. \) From the fact that \( r_1 \geq r_2 \geq r_3 \geq r_4 \geq 1, \) we have that \( x \geq y, \) and from the assumption that \((r_1 - 1) < (r_2-1)+(r_3-1)+(r_4-1), \) it follows that \( y > z - x - p + 1. \) Thus from Proposition 4.1, the lemma follows.

From Lemmas 4.1, 4.2, and 4.3 it is immediate that \( V_{(4.4.4)} \) is the maximum in \( V_{(4.1.2)}, V_{(4.2.4)}, V_{(4.3.4)}, \) and \( V_{(4.4.4)}. \) Thus Theorem 4.2 holds, i.e., if \((r_2 - 1) + (r_3 - 1) - (r_4 - 1) \leq (r_1 - 1) < (r_2 - 1) + (r_3 - 1) + (r_4 - 1), \) then it follows that
\[ z^4 = \frac{V_{(4.4.4)}}{8} = \frac{1}{8} \left[ \sqrt{4r_4 + \{(r_1 - 1) - (r_2 - 1) - (r_3 - 1)\}^2 + (r_1 + 1) + (r_2 + 1) + (r_3 + 1)} \right]. \]

**4.3 Proof of Theorem 4.3**

Assume that \( \beta_1 < \beta_2 + \beta_3 - \beta_4, \) i.e., \((r_1 - 1) < (r_2 - 1) + (r_3 - 1) - (r_4 - 1). \) For the cases (4.1.3), (4.1.4), (4.2.3), (4.2.4), (4.3.3), and (4.3.4), let \( V_{(4.1.3)}, V_{(4.1.4)}, V_{(4.2.3)}, V_{(4.2.4)}, V_{(4.3.3)}, \) and \( V_{(4.3.4)} \) be the potential maximum values of \( G, \) respectively, i.e.,
\[ V_{(4,1.3)} = G(\pm (\beta_2 - \beta_3 + \beta_4), \mp \beta_2, \pm \beta_3, \mp \beta_4) \]
\[ = \sqrt{\alpha_1^2 + (\beta_2 - \beta_3 + \beta_4)^2 + \sqrt{\alpha_2^2 + \beta_2^2 + \sqrt{\alpha_3^2 + \beta_3^2 + \sqrt{\alpha_4^2 + \beta_4^2}}} \]
\[ = \sqrt{4r_1 + \{(r_2 - 1) - (r_3 - 1) + (r_4 - 1)\}^2 + (r_2 + 1) + (r_3 + 1) + (r_4 + 1); \quad (10) \]
\[ V_{(4,1.4)} = G(\pm (\beta_2 + \beta_3 + \beta_4), \pm \beta_2, \pm \beta_3, \pm \beta_4) \]
\[ = \sqrt{\alpha_1^2 + (-\beta_2 + \beta_3 + \beta_4)^2 + \sqrt{\alpha_2^2 + \beta_2^2 + \sqrt{\alpha_3^2 + \beta_3^2 + \sqrt{\alpha_4^2 + \beta_4^2}}} \]
\[ = \sqrt{4r_1 + \{(r_2 - 1) - (r_3 - 1) + (r_4 - 1)\}^2 + (r_2 + 1) + (r_3 + 1) + (r_4 + 1); \quad (11) \]
\[ V_{(4,2.3)} = G(\mp \beta_1, \pm (\beta_1 - \beta_3 + \beta_4), \pm \beta_3, \mp \beta_4) \]
\[ = \sqrt{\alpha_1^2 + (\beta_1 - \beta_3 + \beta_4)^2 + \sqrt{\alpha_2^2 + \beta_2^2 + \sqrt{\alpha_3^2 + \beta_3^2 + \sqrt{\alpha_4^2 + \beta_4^2}}} \]
\[ = \sqrt{4r_2 + \{(r_1 - 1) - (r_3 - 1) + (r_4 - 1)\}^2 + (r_1 + 1) + (r_3 + 1) + (r_4 + 1); \quad (12) \]
\[ V_{(4,2.4)} = G(\pm \beta_1, \pm (-\beta_1 + \beta_3 + \beta_4), \mp \beta_3, \mp \beta_4) \]
\[ = \sqrt{\alpha_1^2 + (-\beta_1 + \beta_3 + \beta_4)^2 + \sqrt{\alpha_2^2 + \beta_2^2 + \sqrt{\alpha_3^2 + \beta_3^2 + \sqrt{\alpha_4^2 + \beta_4^2}}} \]
\[ = \sqrt{4r_2 + \{(r_1 - 1) - (r_3 - 1) + (r_4 - 1)\}^2 + (r_1 + 1) + (r_3 + 1) + (r_4 + 1); \quad (11) \]
\[ V_{(4,3.3)} = G(\mp \beta_1, \pm \beta_2, \pm (\beta_1 - \beta_2 + \beta_4), \mp \beta_4) \]
\[ = \sqrt{\alpha_1^2 + (\beta_1 - \beta_2 + \beta_4)^2 + \sqrt{\alpha_2^2 + \beta_2^2 + \sqrt{\alpha_3^2 + \beta_3^2 + \sqrt{\alpha_4^2 + \beta_4^2}}} \]
\[ = \sqrt{4r_3 + \{(r_1 - 1) - (r_2 - 1) + (r_4 - 1)\}^2 + (r_1 + 1) + (r_2 + 1) + (r_4 + 1); \quad (12) \]
\[ V_{(4,3.4)} = G(\pm \beta_1, \pm \beta_2, \pm (-\beta_1 + \beta_2 + \beta_4), \mp \beta_4) \]
\[ = \sqrt{\alpha_1^2 + (-\beta_1 + \beta_2 + \beta_4)^2 + \sqrt{\alpha_2^2 + \beta_2^2 + \sqrt{\alpha_3^2 + \beta_3^2 + \sqrt{\alpha_4^2 + \beta_4^2}}} \]
\[ = \sqrt{4r_3 + \{(r_1 - 1) - (r_2 - 1) + (r_4 - 1)\}^2 + (r_1 + 1) + (r_2 + 1) + (r_4 + 1). \]

From the fact that \( r_1 \geq r_2 \geq r_3 \geq r_4 \geq 1 \), it is immediate that \( V_{(4,1.3)} \geq V_{(4,1.4)}, V_{(4,2.3)} \geq V_{(4,2.4)}, \) and \( V_{(4,3.3)} \geq V_{(4,3.4)}. \) In the following lemmas, we show that \( V_{(4,3.3)} \) is the maximum in \( V_{(4,1.3)}, V_{(4,2.3)}, \) and \( V_{(4,3.3)} \) for the case that \( (r_1 - 1) < (r_2 - 1) + (r_3 - 1) - (r_4 - 1). \)

**Lemma 4.4:** \( V_{(4,2.3)} \geq V_{(4,1.3)}. \)

**Proof:** From (11) and (10), it is immediate that
\[ V_{(4,2.3)} - V_{(4,1.3)} = \sqrt{4r_2 + (r_1 - r_2 + r_4 - 1)^2 + (r_1 - r_2) - \sqrt{4r_1 + (r_2 - r_3 + r_4 - 1)^2 - \sqrt{4r_2 + (r_3 - r_4 - r_1 + 1)^2 + (r_1 - r_2) - \sqrt{4r_1 + (r_3 - r_4 - r_1 + 1)^2}}. \]

Set \( x = r_1, y = r_2, z = r_3 - r_4, \) and \( p = -1. \) Since \( r_1 \geq r_2 \geq r_3 \geq r_4 \geq 1, \) we have that \( x \geq y \) and \( y - z + x + p - 1 = r_2 - r_3 + r_4 + r_1 - 2 = (r_2 - r_3) + (r_4 - 1) + (r_1 - 1) \geq 0, \)
i.e., \( y \geq z - x - p + 1. \) Thus from Proposition 4.1, the lemma follows. ■

**Lemma 4.5:** \( V_{(4,3.3)} \geq V_{(4,2.3)}. \)

**Proof:** From (12) and (11), it is immediate that
\[ V_{(4,3.3)} - V_{(4,2.3)} = \sqrt{4r_3 + (r_1 - r_2 + r_4 - 1)^2 + (r_2 - r_3) - \sqrt{4r_2 + (r_1 - r_2 + r_4 - 1)^2} - \sqrt{4r_3 + (r_1 + r_4 - r_2 - 1)^2 + (r_2 - r_3) - \sqrt{4r_2 + (r_1 + r_4 - r_3 - 1)^2. \]} \]
Set \( x = r_2, y = r_3, z = r_1 + r_4, \) and \( p = 1. \) From the fact that \( r_1 \geq r_2 \geq r_3 \geq r_4 \geq 1, \) we have that \( x \geq y, \) and from the assumption that \( (r_1 - 1) < (r_2 - 1) + (r_3 - 1) - (r_4 - 1), \) it follows that \( y > z - x - p + 1. \) Thus from Proposition 4.1, the lemma follows.

From Lemmas 4.4 and 4.5, it follows that \( V_{(4,3,3)} \) is the maximum in \( V_{(4,1,3)}, V_{(4,2,3)}, \) and \( V_{(4,3,3)}. \) Thus Theorem 4.3 holds, i.e., if \( (r_1 - 1) < (r_2 - 1) + (r_3 - 1) - (r_4 - 1), \) then

\[
\frac{z^4}{V_{(4,3,3)}} = \frac{1}{8} \left[ \sqrt{4r_3 + \{(r_1 - 1) - (r_2 - 1) + (r_4 - 1)\}^2} + (r_1 + 1) + (r_2 + 1) + (r_4 + 1) \right].
\]

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A Proof of Claim 2.1

By straightforward calculations, we have that

\[
h'(x) = \frac{x + b}{\sqrt{a^2 + (x + b)^2}} + \frac{x - d}{\sqrt{c^2 + (d - x)^2}}; \\
h''(x) = \frac{a^2}{\{a^2 + (x + b)^2\}^{3/2}} + \frac{c^2}{\{c^2 + (d - x)^2\}^{3/2}}.
\]
It is easy to show that \( h' \) and \( h'' \) are continuous. For the statement (ii), it is obvious that \( h''(0) > 0 \). For the statement (i), it is also immediate that

\[
\text{sgn } h'(0) = \text{sgn} \left( \frac{b}{\sqrt{a^2 + b^2}} - \frac{d}{\sqrt{c^2 + d^2}} \right) = \text{sgn}(b\sqrt{c^2 + d^2} - d\sqrt{a^2 + b^2}) = \text{sgn}(b^2(c^2 + d^2) - d^2(a^2 + b^2)) = \text{sgn}(c^2b^2 - a^2d^2) = \text{sgn}(cb - ad).
\]

**B Proof of Proposition 3.1**

By straightforward calculations, we have that

\[
F_3(x, y, z) = \sqrt{4y + (x - z)^2 + (x - y) - \sqrt{4x + (y - z)^2}}
\]

\[
= \frac{\left\{ \sqrt{4y + (x - z)^2 + (x - y)} \right\}^2 - \left\{ \sqrt{4x + (y - z)^2} \right\}^2}{\sqrt{4y + (x - z)^2 + (x - y) + \sqrt{4x + (y - z)^2}}}
\]

\[
= \frac{2(x - y) \left\{ \sqrt{4y + (x - z)^2 + (x - z - 2)} \right\}}{\sqrt{4y + (x - z)^2 + (x - y) + \sqrt{4x + (y - z)^2}}}
\]

For the statement (a), it is immediate that

\[
\sqrt{4y + (x - z)^2 + (x - z - 2)} \geq \sqrt{4(x - z + 1) + (x - z)^2 + (x - z - 2)} = \sqrt{(x - z + 2)^2 + (x - z - 2)} = 2(x - z) \geq 0,
\]

where the 1st inequality follows from the condition that \( y \geq x - z + 1 \), and the last equality and the 2nd inequality follow from the condition that \( z \leq x \). Thus the statement (a) holds.

For the statement (b), it is immediate that

\[
\sqrt{4y + (x - z)^2 + (x - z - 2)} \geq \sqrt{4(z - x + 1) + (x - z)^2 + (x - z - 2)}
\]

\[
= \sqrt{(z - x + 2)^2 + (x - z - 2)} = \sqrt{(z - x + 2)^2 - (z - x + 2)}
\]

\[
= |z - x + 2| - (z - x + 2) \geq (z - x + 2) - (z - x + 2) = 0,
\]

where the 1st inequality is due to the condition that \( y \geq z-x+1 \). Thus the statement (b) holds.

**C Proof of Proposition 4.1**

Substitute \( z - p \) for \( z \) in \( F_3(x, y, z) \) of Proposition 3.1 Then we have that

\[
F_4(x, y, z, p) = F_3(x, y, z - p) = \sqrt{4y + (x - z + p)^2 + (x - y) - \sqrt{4x + (y - z + p)^2}}
\]

\[
= \sqrt{4y + (z - x - p)^2 + (x - y) - \sqrt{4x + (z - y - p)^2}}.
\]

Then the proposition immediately follows from Proposition 3.1(b).