Conformal invariance without referring to metric

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Abstract. Field theories invariant to conformal transformations are a very important class of models. Besides their theoretical significance due to their large symmetry group, they are important also from the practical point of view. For instance the kinematic part of the Standard Model Lagrangian also shows conformal invariance. In the usual approach, a field theory is called conformal invariant whenever its field equations or its action is invariant to the conformal transformations of the spacetime metric tensor along with corresponding transformation of the field quantities. For this, an action of the conformal transformations on the fields needs to be specified a priori, and conformal invariance only makes sense along with this group action. In this paper we introduce a simple new method of generating field theories in terms of their Lagrangian, without a priori specifying the action of the conformal group on the fields. The interesting aspect of this method is that it does not to refer to a spacetime metric tensor a priori, and therefore becomes particularly useful when searching for theories where the spacetime metric tensor is an emergent quantity, not a fundamental field.

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1. Introduction

An important class of spacetime transformations are the conformal diffeomorphisms. Given a general relativistic spacetime model \((M, g)\), \(M\) being a four dimensional real smooth manifold and \(g\) being a smooth Lorentz signature metric tensor field over it, a conformal transformation is a pair \((\phi, \Omega)\), where \(\phi\) is an \(M \rightarrow M\) diffeomorphism and \(\Omega\) is an \(M \rightarrow \mathbb{R}^+\) positive valued smooth scalar field, for which \(\phi^*g = \Omega^2g\) holds, \(\phi^*\) denoting the pullback operation of the diffeomorphism \(\phi\). An important subgroup of the conformal transformations are the conformal rescalings where \(\phi\) is the identity of \(M\) and \(\Omega\) is kept to be an arbitrary positive valued smooth scalar field. In a field theory, a group action of the conformal transformations on the fundamental fields may be specified, and then the conformal group can act on all the fields in the theory simultaneously. Whenever the field equations or the action functional of the model is invariant to such an group action on the metric tensor and the fields, the theory is called conformally invariant [1, 2, 3].

Conformally invariant field theories are very important class of models. Some of them are rather artificial, and are more interesting from the mathematical point of view. However, one may observe that in all physical field theories the kinematic terms in the Lagrangian are always conformally invariant. The simplest example is the classical Standard Model Lagrangian, which if considered over a generic curved spacetime, its kinematic term is seen to be indeed conformally invariant: the only term which breaks the conformal symmetry is the Higgs self-interaction term due to the prescribed constant nonzero Higgs vacuum expectation value.

From the above, widely used definition of conformally invariant field theories it is clear that first one needs a field equation or Lagrangian, and that the spacetime metric must be one of the fundamental fields. Then, a group action of the conformal transformations on all the fields must be given. Given all these, one may or may not observe that the field equations (or the action functional) is invariant to such a group action. The disadvantage of such approach is that it is relatively difficult to systematically generate conformally invariant field theories e.g. through their Lagrangians.

This paper provides a simple methodology on how to generate such field theory Lagrangians which shall be guaranteed to be conformally invariant with appropriately chosen group action of the conformal transformations on the fields. In fact, this principle can be in particular useful when searching for theories where the spacetime metric tensor field is not a fundamental quantity, but a derived one. The approach is based on the notion of measure lines introduced in [4]. This basically gives precise mathematical formalization to dimensional analysis, quite well-known in physics. Our approach can be summarized as doing dimensional analysis in each point of spacetime, independently, using the well-known technique of vector bundles. The interesting property of this approach is that no reference to the spacetime metric tensor is needed in first place, only the physical dimensions of the fields are necessary.

First, we recall the Lagrangian formulation of classical field theories in a way which does not rely on a spacetime metric tensor field.

2. Non-metric formulation of classical field theories

In this section the precise mathematical definition of classical field theories is recalled in terms of the Lagrangian and variational principles: for a comprehensive overview,
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see e.g. [2, 6, 7]. The used definition deliberately does not refer to an a priori known spacetime metric tensor field, and thus resembles basically to a Palatini type formulation [1]. In the followings, we shall denote the tangent bundle of a manifold \( M \) by \( T(M) \), and by \( T^*(M) \) the corresponding cotangent bundle. The vector bundle of maximal forms (also called volume forms) is denoted by \( \wedge^n T^*(M) \) where \( n = \dim(M) \). We shall use the elementary fact that the sections of the volume form bundle may be integrated over an oriented manifold without any further assumption. In the followings every differential geometrical object is assumed to be smooth for simplicity of presentation (strict differentiability counting is performed in [5]). The vector space of smooth sections of some vector bundle \( V(M) \) over \( M \) is denoted by \( \Gamma(V(M)) \). We shall use the terms section and field interchangeably. The affine space of covariant derivations over \( V(M) \) shall be denoted by \( D(V(M)) \). The corresponding dual vector bundle of \( V(M) \) is denoted by \( V^*(M) \). These notations are the usual ones in differential geometry literature. In addition, we shall use Penrose abstract indices \([2, 1]\) for denoting complicated tensor traces and expressions concerning the tensor powers of \( T(M) \) and \( T^*(M) \). The abstract indices of \( T(M) \) shall be denoted by superscripted lower case latin letters \( (a b c d . . .) \), whereas for \( T^*(M) \) subscripted lower case latin letters \( (a b c d . . .) \) shall be used. The index symmetrization operation shall be denoted by curly brackets, e.g. \( \tilde{t}_{(abc)} \), whereas the antisymmetrization operation shall be denoted by square brackets, e.g. \( t_{[abc]} \), furthermore their normalization convention shall be set as in e.g. [1]. Namely, the normalization is chosen in such a way that symmetrization and antisymmetrization shall become a projection operator.

Recall that the space of smooth sections \( \Gamma(V(M)) \) of some vector bundle \( V(M) \) admits a natural \( \mathcal{E} \) topology [3]: without any further assumptions it is meaningful to define convergence of a sequence \( (\varphi_n)_{n \in \mathbb{N}} \) in \( \Gamma(V(M)) \) to a limit \( \varphi \) in \( \Gamma(V(M)) \) with requiring that the field \( (\varphi - \varphi_n)_{n \in \mathbb{N}} \) and all of its derivatives uniformly converge to zero on any compact region of \( M \). Whenever the manifold \( M \) is compact, or a fixed compact region \( K \subset M \) is considered, the \( \mathcal{E} \) topology naturally gives rise to a norm equivalence class on the fields over the pertinent region [5]. Because of this, ordinary (Fréchet) derivatives of functions of such local fields can be naturally defined without any further mathematical structures.

As usual in the differential geometry literature [1], a covariant derivation on a vector bundle \( V(M) \) may be uniquely extended to all the tensor powers of \( V(M) \) and its dual bundle \( V^*(M) \) by requiring Leibniz rule over tensor product, commutativity with tensor contraction, and correspondence to the exterior derivation over the scalar line bundle \( M \times \mathbb{R} \). Similarly, given two different vector bundles along with covariant derivation on each, then they naturally give rise to a joint covariant derivation, which uniquely extends to all tensor powers of the pertinent vector bundles and their duals, by requiring the very same properties.

**Remark 1.** Let \( J^a_{c_1 \ldots c_n} \) be a smooth section of \( T(M) \otimes \wedge^n T^*(M) \), i.e. a volume form valued tangent vector field. Then, given any covariant derivation \( \nabla \) on \( T(M) \), one has that the expression \( \nabla_a J^a_{c_1 \ldots c_n} \) is independent of the choice of the covariant derivation, where \( \tilde{\nabla} \) denotes the torsion-free part of \( \nabla \). I.e. the divergence of a volume form valued vector field is naturally defined without further assumptions. Similarly, for a smooth section \( K_{c_1 \ldots c_n}^{a_1 \ldots a_n} \) of \( T(M) \otimes T(M) \otimes \wedge^n T^*(M) \) one has that \( \tilde{\nabla}_a K^{a_1 \ldots a_n}_{c_1 \ldots c_n} \) is independent of the choice of the covariant derivation and thus the divergence of such field is naturally defined without further assumptions.
Given the above notions and observations, a classical field may be defined as a quartet 

\[(M, V(M), dL, S),\]  

where \(M\) is some finite dimensional differentiable manifold possibly with boundary (this is called the base manifold — it models the spacetime or a compactified spacetime with or without a boundary), \(V(M)\) is some finite dimensional smooth vector bundle over it (this is called the vector bundle of matter fields). The Lagrange form \(dL\) is then a smooth volume form valued vector bundle homomorphism \(V(M) \times T^*(M) \otimes V(M) \times T^*(M) \otimes V(M) \otimes V^*(M) \rightarrow \wedge^n T^*(M)\). In particular, it acts on the sections as

\[
dL : \Gamma(V(M) \times T^*(M) \otimes V(M) \times T^*(M) \otimes V(M) \otimes V^*(M)) \rightarrow \Gamma(\wedge^n T^*(M)),
\]

\[
(v, Dv, F) \mapsto dL(v, Dv, F).
\]

Then, the action functional \(S(K)\) is defined on a compact region \(K \subset M\) as:

\[
S(K) : \Gamma(V(M)) \times D(V(M)) \rightarrow \mathbb{R},
\]

\[
(v, \nabla) \mapsto S_{v, \nabla}(K) := \int_K dL(v, \nabla v, F(\nabla)).
\]

Here, \(\nabla v\) is the covariant derivative of the field \(v\) by the covariant derivation \(\nabla\), and \(F(\nabla)\) is the curvature tensor of \(\nabla\). As usually, the solutions of the field equation of the field theory shall be the stationary points of the action functional with the fields having fixed boundary value. Namely the field \((v, \nabla) \in \Gamma(V(M)) \times D(V(M))\) is said to be a solution of the field theory whenever for all compact regions \(K \subset M\) one has

\[
D^a S_{v, \nabla}(K) = 0,
\]

where \(D^a S(K)\) denotes the Fréchet derivative \(DS(K)\) of \(S(K)\) projected along the closed sub-affine space of \(\Gamma(V(M)) \times D(V(M))\) which consists of all fields equal to \((v, \nabla)\) along the boundary set \(\partial K\). In the end, as quite expected [5], this is equivalent to the Euler-Lagrange equations

\[
\begin{align*}
D_1 dL(v, \nabla v, F(\nabla)) - \tilde{\nabla}_a D_2^a dL(v, \nabla v, F(\nabla)) &= 0, \\
D_2 dL(v, \nabla v, F(\nabla)) \left(\cdot \right) v - \tilde{\nabla}_a 2 D_2^a dL(v, \nabla v, F(\nabla)) \left(\cdot \right) &= 0
\end{align*}
\]

throughout the interior of any compact region \(K \subset M\) and thus throughout \(M\). Here \(D_1 dL, D_2 dL, D_3 dL\) means the partial derivative of \(dL\) with respect to its first, second and third argument, respectively, i.e. the derivative of the Lagrange form along the matter fields, the matter field gradients, and the curvature tensor. One should note that because of Remark [4], the covariant derivation may be chosen arbitrarily over \(T(M)\) in the divergence expressions of Eq. [4].

**Remark 2.** Note that whenever a model is considered in which \(M\) is compact (possibly with boundary), then the field equations can be written in a simpler form

\[
D S_{v, \nabla}(M) = 0.
\]

This is quite similar to as in Eq. [4], but variation on the boundary does not need to be excluded. Because of the boundary term of \(\partial M\), along with the Euler-Lagrange equations Eq. [4], one gets additional boundary field equations

\[
D_2^a dL(v, \nabla v, F(\nabla)) = 0,
\]
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\[ 2D^b_\alpha \text{d}L(v, \nabla v, F(\nabla))(\cdot) = 0 \]  

which can eventually be used to impose dynamical constraints to the fields on the boundary \( \partial M \).

For clarity, we note that in the general relativity terminology the above approach resembles to the Palatini action principle: the covariant derivation is varied independently from the field quantities (e.g. from the metric tensor).

3. Non-metric formulation of conformal invariance

An advantage of the formalism revised in the previous section is that the notion of conformal invariance can be formulated without referring to an explicit group action of the conformal group, which usually would assume some a priori knowledge on the relation of the field equations and the metric tensor. The key idea is motivated by \([4]\). In that work of T. Matolcsi, the mathematical model of special relativistic spacetime is considered to be a triplet \((M, L, \eta)\), where \(M\) is a four dimensional real affine space (modelling the flat spacetime), \(L\) is a one dimensional vector space (modelling the one dimensional vector space of length values), and \(\eta : \sqrt{2} M \to L \otimes L\) is the flat Lorentz signature metric (constant throughout the space time), where \(M\) is the underlying vector space of \(M\) (can be considered as the tangent space of \(M\)). The important idea in that construction is that the field quantities, such as the metric tensor, are not simply real valued, but they take their values in the tensor powers of the measure line \(L\), which formalizes the physical expectation that quantities actually have physical dimensions. This is indeed nothing but the precise mathematical formulation of dimensional analysis, because then fields taking their values in different tensor powers of the measure line \(L\) reside in different vector spaces, and therefore cannot be added for instance. In terms of dimensional analysis this simply formalizes our physical intuition that one should not add physical quantities of different physical dimensions.

Such mathematically precise formulation of dimensional analysis, although may seem to be a relatively innocent idea at a first glance, becomes quite powerful tool when carried over to a general relativistic framework. Namely, let our base manifold \(M\) be some four dimensional real manifold (with or without boundary), and let \(L(M)\) be a real vector bundle over \(M\), with one dimensional fiber. The fiber of \(L(M)\) over each point of \(M\) shall model the vector space of length values, and the pertinent line bundle shall be called the measure line bundle. Just like proposed in \([4]\), the field quantities shall carry certain tensor powers of \(L(M)\) or \(L^*(M)\). For simplicity, the notation \(L^n(M) := \otimes^n L(M)\) and \(L^{-n}(M) := \otimes^n L^*(M)\) shall be used, for all \(n \geq 0\) integers, conforming to the conventions of \([4]\), and also to our physical intuition of dimensional analysis. Our idea can be physically formulated as: the field quantities are tagged with a spacetime point dependent physical dimension, i.e. that the physical dimensions in different spacetime points are not necessarily comparable, a priori (a covariant derivation over \(L(M)\) needs to be explicitly specified for that).

The first observation which can be made is that whenever the fields are tagged with such point dependent physical dimensions, this poses a restriction on possible expressions for Lagrangians. That is because only pure dimensionless volume form field may be integrated throughout a manifold, and therefore the physical dimensions of the field quantities need to cancel when evaluated by the Lagrangian. Given that the vector bundle of field quantities are properly equipped with known physical
dimensions, this consistency principle rules out several, otherwise possible Lagrangian expressions.

Given a classical field theory model \((M, V(M), dL, S)\), where the vector bundle of fields \(V(M)\) is properly equipped with physical dimensions, i.e. the tensor powers of the measure line \(L(M)\), we can formulate the metric independent definition of conformal invariance. The model shall be called \textit{conformally invariant} whenever the action functional does not depend on the choice of the covariant derivation on the measure line bundle \(L(M)\). Physically, this would mean that the model is insensitive to the relation of physical dimensions in different spacetime points.

For practical evaluation, a simple observation can be quite useful. Given two covariant derivations \(\nabla_a\) and \(\nabla'_a\) over any vector bundle with one real dimensional fiber (i.e. over any real line bundle), then one has \(\nabla'_a = \nabla_a + C_a\), where \(C_a\) is a smooth real covector field. Because of that, conformal invariance of a classical field theory model \((M, V(M), dL, S)\) can be easily verified: conformal invariance holds if and only if for any field \(v \in \Gamma(V(M))\) and covariant derivation \(\nabla \in \mathcal{D}(V(M))\) the action functional \(S_v, \nabla(K)\) is invariant to the transformation \(\nabla_a \mapsto \nabla_a + C_a\) of the covariant derivation over the measure line bundle \(L(M)\) with any smooth real covector field \(C_a\). In practice, this is a condition which is very simple to verify.

It was seen that a definition of conformal invariance can be given which neither refers to spacetime metric tensor, nor to an a priori known group action of the conformal group on the fields. In the following section an example shall be provided which shows that indeed, whenever a metric tensor field is present in the model, the construction shall be conformally invariant in the usual sense, defined by the metric rescaling. The proposed approach can, however, come especially useful in constructing models where the metric tensor is an emergent quantity, not a fundamental field.

4. Example: conformal invariant version of vacuum general relativity

For illustrative purpose, we present the formulation of the conformally invariant version of vacuum general relativity. The model is specified via a slightly generalized form of the Einstein-Hilbert Lagrangian, namely let us take the Lagrange form

\[
dL : \Gamma(V(M) \times T^*(M) \otimes V(M) \times T^*(M) \wedge T^*(M) \otimes V(M) \otimes V^*(M)) \to \Gamma(\wedge^4 T^*(M)),
\]

\[
((\varphi, g_{ab}), (D\varphi, Dg_{ab}), (r_{gh}, R_{gh})^{ij}) \mapsto d\textbf{v}(g)\varphi^2 g^{km} g^{l'n} R_{klmn}^{ij}.
\]

(8)

where the base manifold \(M\) is assumed to be 4 dimensional and oriented, whereas the vector bundle of fields is defined to be \(V(M) := L^{-1}(M) \times L^2(M) \otimes \sqrt{2} T^*(M)\), where \(L(M)\) is called the line bundle of lengths. The symbol \(d\textbf{v}(g)\) denotes the canonical volume form field generated by \(g \in \Gamma(L^2(M) \otimes \sqrt{2} T^*(M))\). Note that by construction one has \(d\textbf{v}(g) \in \Gamma(L^4(M) \otimes \wedge^4 T^*(M))\), i.e. the canonical volume form has dimension length to the four, as physically expected. As already mentioned in Section 2, in our variational scheme the quantities are varied independently, i.e. no a priori relation is assumed between the metric and covariant derivation, furthermore, also the torsion of the covariant derivation is not restricted initially. It is seen that Eq.(8) simply corresponds to the standard Einstein-Hilbert Lagrangian with a slight generalization: the inverse Planck length (here denoted by \(\varphi\)) is not assumed to be constant, but can (must) have location dependence, i.e. it is rather a field than a
constant in this model, as it is set to be a section of the vector bundle $L^{-1}(M)$. With this simple generalization, the theory becomes conformally invariant in terms of our definition in Section 3, as the action functional does not depend on the covariant derivation over the line bundle of lengths. After direct substitution of $dL$ in Eq. (5) into Eq. (8), along with subsequent usage of the identities
\begin{align*}
\frac{\partial}{\partial g_{ab}} g^{cd} &= -\frac{1}{2} \left( g^{ca} g^{bd} + g^{cb} g^{ad} \right),
\frac{\partial}{\partial g_{ab}} (\varphi^2 g^{cd}) &= -\frac{1}{2} (\varphi^2 g^{ab}) (\varphi^2 - 2 g^{ef}) (\varphi^2 - 2 g^{ef})
\end{align*}

It is straightforward to show (see also [5]), that the field equations read as
\begin{align*}
\Delta_a (\varphi^2 g_{bc}) &= 0, \\
\varphi^2 E (\nabla, \varphi^2 g)_{ab} &= 0
\end{align*}

which is equivalent to
\begin{align*}
\Delta_a (\varphi^2 g_{bc}) &= 0, \\
\varphi^2 E (\nabla, \varphi^2 g)_{ab} &= \varphi^2 T (\nabla, \varphi^2 g)_{ab},
\end{align*}

Remark 3. The presented variational problem may be reformulated on the closed affine subspace of torsion-free covariant derivations, in which case the torsion tensor $T(\nabla)_{ab}$ automatically vanishes and thus the source term $T(\nabla, \varphi^2 g)_{ab}$ vanishes on the right hand side of Eq. (12) along with having automatically $\Delta_a = \nabla_a$. The field equations Eq. (12) may be re-expressed also in terms of the original metric $g_{ab}$ which is not rescaled to be dimensionless. More specifically, Eq. (12) is seen to be equivalent to
\begin{align*}
\tilde{D}_a (g_{bc}) &= 0,
\end{align*}
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$$E(\tilde{D}, g)_{ab} = T(\nabla, \varphi^2 g)_{ab}$$

$$+ \varphi^{-1} \tilde{D}_a \tilde{D}_b \varphi + \varphi^{-1} \tilde{D}_b \tilde{D}_a \varphi - 2g_{ab} g^{ef} \varphi^{-1} \tilde{D}_e \tilde{D}_f (\varphi)$$

$$- 4 \varphi^{-1} \tilde{D}_a (\varphi) \varphi^{-1} \tilde{D}_b (\varphi) + g_{ab} g^{ef} \varphi^{-1} \tilde{D}_e (\varphi) \varphi^{-1} \tilde{D}_f (\varphi),$$

$$g^{ab} \tilde{D}_a \tilde{D}_b \varphi - \frac{1}{6} \mathcal{R}(\tilde{D}, g) \varphi = \frac{1}{6} g^{ab} T(\nabla, \varphi^2 g)_{ab} \varphi, \quad (13)$$

where in this case $\tilde{D}_a$ is a torsion-free covariant derivation over $L^{-1}(M) \otimes T(M)$ such that it is metric compatible ($\tilde{D}_a (g_{bc}) = 0$), furthermore $E(\tilde{D}, g)_{ab}$ is the Einstein tensor of $\tilde{D}_a$ and $g_{bc}$, whereas $\mathcal{R}(\tilde{D}, g)$ is the Ricci scalar of $\tilde{D}_a$ and $g_{bc}$. The obtained field equation is seen to be nothing but the coupled conformally invariant Einstein-Klein-Gordon equation for $g_{ab}$ and $\varphi$, along with some source term coming from a possible torsion contribution (which may be zeroed out by means of Remark 4). Again, when further matter fields are present, they contribute to the right hand side in terms of an energy-momentum tensor. As the field equations Eq. (13) are known to be conformally invariant in the usual sense, it is clear that our definition of conformal invariance in Section 3 is consistent with the conventional definition using metric rescaling group action.

**Remark 4.** Whenever the base manifold $M$ has a boundary $\partial M$ and the variation on the manifold boundary is allowed as in Remark 2, the boundary field equations read as

$$\varphi^2 g_{ab} = 0 \quad \text{(throughout } \partial M). \quad (14)$$

The field equations Eq. (13) and Eq. (14) mean together that the rescaled metric $\varphi^2 g_{ab}$ and its Levi-Civita covariant derivation $\nabla_a$ obey vacuum Einstein equations with a possible additional source term originating from the torsion of $\nabla_a$. Furthermore, the rescaled metric $\varphi^2 g_{ab}$ is pressed to zero as approaching the boundary with a conformal boundary condition (just like the asymptotical behavior in the case of Friedman-Robertson-Walker cosmological solutions).

**Remark 5.** It is worth to note that as a consequence of our variational principle scheme, a dynamical torsion theory arises, like the Einstein-Cartan-Sciama-Kibble theory [8], however there is an essential difference from that: not the original covariant derivation $\nabla_a$ is compatible with the rescaled metric $\varphi^2 g_{ab}$, but the torsion free part of it. I.e., in our case, if the torsion is not required to be zero a priori, the field equations are a simple Einstein theory for $\nabla_a$ and $\varphi^2 g_{ab}$, but the torsion $T(\nabla)^c_{ab}$ also contributes to the energy-momentum tensor. Also, in consequence, one obtains the constraint equation of

$$\varphi^{-2} g^{ah} \nabla_a T(\nabla, \varphi^2 g)_{bc} = 0 \quad (15)$$

for the torsion tensor due to the automatic vanishing of the divergence of the Einstein tensor because of the Bianchi identities. It is seen that Eq. (13) along with Eq. (15) is different than that of ECSK field equations [8].

The proposed metric independent definition of conformal invariance becomes particularly useful when dealing with non-metric theories, i.e. with models in which the spacetime metric tensor is a derived quantity, not a fundamental one.
Remark 6. A simple example for a model in which the metric tensor is not a fundamental quantity can be readily given with spinorial formulation [1, 2] of general relativity. In that approach, one has a spinor bundle $S(M)$ with two complex dimensional fibers over the real four manifold $M$. The Lagrange form is the spinorial representation of the Einstein-Hilbert Lagrangian:

$$dL : \Gamma (V(M) \times T^*(M) \otimes V(M) \times T^*(M) \wedge T^*(M) \otimes V(M) \otimes V^*(M)) \rightarrow \Gamma (\Lambda^4 T^*(M)),$$

$$((\varphi, \epsilon_{AB}, \sigma^A, \chi^B), (D\varphi_b, D\epsilon_{AB}b, D\sigma^A_{ab}, D\chi^B_b),$$

$$(\tau_{ab}, \rho_{ab}AB, C^AB_{B^C}, P_{abA^B})) \mapsto dv(g(\sigma, \epsilon)) \varphi^2 g(\sigma, \epsilon)^{ac} \left( \sigma^A_{ac} P_{abA^B} \sigma^b_{cB} + \sigma^A_{bB} P_{abB^D} \sigma^b_{A^D} \right),$$

(16)

where

$$V(M) := L^{-1}(M) \times L(M) \otimes \Lambda^2 S^*(M) \times T^*(M) \otimes S(M) \otimes S(M) \times L^{-1}(S(M)).$$

Here $g(\sigma, \epsilon)_{ab} := \sigma^A_{ac} \sigma^b_{cB} \epsilon_{AB} \epsilon_{AB}$ denotes the canonical Lorentz metric tensor generated by an $\epsilon_{AB} \in \Gamma (L(M) \otimes \Lambda^2 S^*(M))$ and $\sigma^A_{ab} \in \Gamma (T^*(M) \otimes S(M) \otimes S(M))$, furthermore $dv(g(\sigma, \epsilon)) \in \Gamma (L^4(M) \otimes \Lambda^4 T^*(M))$ denotes the canonical volume form generated by $g(\sigma, \epsilon)_{ab} \in \Gamma (L^2(M) \otimes \Lambda^2 T^*(M))$. In the notation, Penrose abstract indices were used according to the conventions of [1, 2]. It is seen that the model defined by this Lagrange form is conformally invariant in the sense of Section 5, as the action functional is invariant to the change of the covariant derivation over the line bundle of lengths $L(M)$.

5. Concluding remarks

In this paper we presented a metric independent formulation for the property of conformal invariance for classical field theories. The method is basically dimensional analysis performed in each point of spacetime, independently. This is mathematically realized via the notion of measure line bundles. With this notion, a field theory is conformally invariant whenever its field equations or action functional is invariant to the choice of the connexion over the measure line bundle, i.e. whenever the physical dimensions in each spacetime point are independent. An advantage of this methodology of generating conformally invariant theories is that it does not require an a priori metric and corresponding action of the conformal group over all the fields. Thus the spacetime metric can eventually be allowed to be a derived quantity, not a fundamental one, and the method can be also used to systematically generate such conformally invariant models.

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