Self-dual Hamiltonians as Deformations of Free Systems

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Abstract

We formulate the problem of finding self-dual Hamiltonians (associated with integrable systems) as deformations of free systems given on various symplectic manifolds and discuss several known explicit examples, including recently found double elliptic Hamiltonians. We consider as basic the notion of self-duality, while the duality in integrable systems (of the Toda/Calogero/Ruijsenaars type) comes as a derivative notion (degenerations of self-dual systems).

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1. The new energy that theory of integrable many-systems obtained during last several years is mostly due to the unexpected discovery of relations between the $N = 2$ supersymmetric gauge theories \[1\] and integrable systems \[2\] (see also \[3\] and references therein). In fact, this latter still gets new motivations from the physical base.

For instance, the needs in description of $6d$ gauge theories have led to the notion of double-elliptic systems \[4\], \[5\]. These systems have been constructed using the notion of duality \[4\], which is also obliged to a physical set-up \[6\].

In fact, the duality in integrable systems was first observed by S.Ruijsenaars \[7\] in rational and trigonometric Calogero and Ruijsenaars many-body systems. Since then, it was extended to the elliptic systems \[4\], \[5\], \[8\]. However, the very notion of duality was not unambiguously formulated so far. Moreover, it looks so that there are plenty of dual systems, constructing them does not look a problem at all!

In this short note we discuss that one can start from a more restrictive notion of self-duality. It can be easier defined, naturally leads to duality and, being more restrictive, immediately put a problem of finding more/all self-dual systems. Here we restrict ourselves with just several manifest examples of self-dual systems. We also do not discuss the symplectic geometry behind the notion of self-duality.

The standard facts and known examples of dual systems, as well as a proper technique to deal with them can be found in reviews \[9\].

2. Let us consider a free Hamiltonian system with $N$ degrees of freedom whose phase space (symplectic manifold) is just $\mathbb{R}^{2N}$. In Darboux coordinates the symplectic form is $\sum_i dp_i \wedge dq_i$. Then, there are $N$ independent functions on the phase space, integrals of motion in involution whichever one of them chosen to be the Hamiltonian of the system. The choice of these integrals is quite free, say, they can be just symmetric powers of $p_i$: $H_k(p, q) = \sum_i p_i^k$.

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There are no more independent integrals in involution, say, similar functions of coordinates. Therefore, this choice in a sense breaks the symmetry between permutations of \( \{ p \} \) and \( \{ q \} \) (i.e. fixes the polarization). In order to restore this symmetry, one should perform an (anti)canonical transformation \( p_i \rightarrow Q_i, q_i \rightarrow P_i \). In the new coordinates \( \hat{H}_k(P, Q) = H_k(p, q) = \sum_i Q_i^k \), while \( \sum_i q_i^k = H_k(P, Q) \). Here \( \hat{H}_k(P, Q) \) are integrals (Hamiltonians) after the canonical transformation is done.

This means that the action of the symplectic group on the Hamiltonians should be provided with a proper (canonical) change of variables on the symplectic manifold (a kind of covariance property). It can be also given by the pair of relations

\[
H_k(p, q) = \sum_i Q_i^k \\
H_k(P, Q) = \sum_i q_i^k
\]

(1)

These relations manifestly realize the symmetry between \( \{ Q \} \) (essentially, old \( \{ p \} \)) and \( \{ q \} \).

Now let us consider a less trivial, interacting system. We ask it to be integrable in the Liouville sense, i.e. still to possess \( N \) independent integrals of motion in involution. Then, the Hamiltonians can be quite tricky functions on the phase space. One may ask how to generalize the symmetry between coordinates and momenta to this case. In order to do this, one can just use the same relations (1).

This means in interacting system that, at the first stage, one makes a canonical change to the free system (essentially, action variables) and then makes the anticanonical transformation just permuting coordinates and momenta. Note that the relations (1) already contains \( 2N \) equations, and there is additionally the condition of canonicity of the transformation \( p_i \rightarrow Q_i, q_i \rightarrow P_i \). This restricts the Hamiltonians.

Now let us consider a more general situation when the phase space is not flat and may be compact. Then, even a free system possesses less trivial Hamiltonians \( h_k(p) \), since they are to be good functions on the phase space. Then, the system (1) should be substituted by

\[
H_k(p, q) = h_k(Q) \\
H_k(P, Q) = h_k(q)
\]

(2)

**Definition.** The system with Hamiltonians that satisfy (2) with the variables \( (p, q) \) and \( (P, Q) \) related by a canonical transformation we call self-dual.

We do not know any constructive way to build such systems, and, therefore, just restrict ourselves here with several explicit examples.

Note that (2) enjoys a kind of covariance property: instead of a set \( h_k \) of free Hamiltonians, one can use any other set of proper functions on the phase space. All of them are some functions of the originally fixed \( h_k \), while the new dual Hamiltonians are the same functions of the old ones.

3. We shall consider the complexified version of the phase space so that Hamiltonians should celebrate good analytic properties on the phase space. However, the typical situation is when the phase space is very asymmetric under the change of polarization, i.e. under interchanging the coordinates and momenta. In this case one can not use the equations (2).

Our strategy below in such a case will be to construct a self-dual system on the non-trivial phase space that depends on moduli and then to go to the boundary of the moduli space producing different
degenerate situations, where say, only $q$ but not $Q$ gets into degeneration region. In the vicinity of the boundary the two lines in (2) look different

\begin{align*}
H_k^{(1)}(p, q) &= h_k^{(1)}(Q) \\
H_k^{(2)}(P, Q) &= h_k^{(2)}(q)
\end{align*}  \tag{3}

and give rise to two different sets of Hamiltonians. We call them dual. The dual Hamiltonians are just the Hamiltonian of the self-dual system taken in two different regions of the moduli space. In other words, this means that one should use just the same equations (3) but in their two lines properly interchange points of the moduli space: $H_k^{(1,2)}(p, q) = H_k(p, q; m_{1,2})$, $h_k^{(1,2)}(q) = h_k(q; m_{1,2})$.

Therefore, one can get the counterpart of self-duality for more complicated phase-spaces via degeneration. In fact, the inverse is also correct: the dual system proved to be a powerful technical tool for constructing the self-dual Hamiltonians to start with the dual ones, since these latter are often constructed simpler.

Let us start with a system with one degree of freedom (it also includes systems with two degrees of freedom with decoupling the centre of mass). In this case, the new variables $P$ and $Q$ are given by the two relations (2) and one should impose the requirement of canonicity of the transformation:

\[
\frac{\partial H(p, q)}{\partial p} h'(q) = \frac{\partial H(P, Q)}{P} h'(Q) \tag{4}
\]

Using again (2) one now rewrites the r.h.s. of (4) as a function of $p$ and $q$ and obtains a complicated functional differential equation that defines the Hamiltonian of the self-dual system. Certainly, we are not able to solve this equation and even to say how many solutions it has. However, this is one equation for one function.

In a system with $N$ degrees of freedom, the relation (2) again defines the new coordinates and momenta, while the canonicity condition gives a system of equations for the Hamiltonian.

4. Now we turn to the concrete examples. We start with the simplest case of the system with one degree of freedom. The free Hamiltonians below are chosen in the form that leads to some standard Hamiltonian systems.

- The phase space $\mathcal{M} = \mathbb{R}^2$. The free system is given by the Hamiltonian $H = p^2/2$. A solution to the self-duality equations (1) is $H = p^2/2 + g^2/q^2$, where $g$ is an arbitrary parameter. This is the rational Calogero Hamiltonian with coupling constant $g$.

- The phase space is $\mathcal{M} = S^1 \times S^1$, the product of two circles of radii $R_1$ and $R_2$. The free system is given by the Hamiltonian $\cos(p/R_1)$. A self-dual Hamiltonian is the Ruijsenaars Hamiltonian $H = \sqrt{1 - 2g^2/\sin^2(q/R_2)} \cos(p/R_1)$. Here the radii of the circles give a moduli space of the theory. The self-duality relation is

\begin{align*}
H(p, q; R_1, R_2) &= \cos(Q/R_1) \\
H(p, q; R_2, R_1) &= \cos(q/R_2)
\end{align*}  \tag{5}

By degenerating the momentum circle ($R_1 \to \infty$), one gets the trigonometric Calogero Hamiltonian $H_{tC}$, while degenerating the coordinate one, one obtains the rational Ruijsenaars Hamiltonian $H_{rR}$ so that the relation (3) becomes the duality relation:
\[ \begin{align*}
(H(p, q; R_1 = \infty, R_2) & \equiv H_{IC}(p, q) = Q^2/2 \\
(H(p, q; R_2, R_1 = \infty) & \equiv H_{IR}(P, Q) = \cos(q/R_2)
\end{align*} \] (6)

- One can consider the complexified phase space of the previous problem, i.e. the coordinate and momentum living on a cylinder. Then, there is another solution to the self-dual equations, that with trigonometric functions substituted with the hyperbolic ones.

- Generalizing the previous item, one can consider the coordinate and momentum living on torii. The free system is given by the Hamiltonian \( cn(p|k) \), the elliptic (Jacobi) cosine with the elliptic modulus \( k \). In order to find a solution to the self-dual equations (2) in this case, one can first solve them in the degenerate case, when, say, momentum torus becomes sphere and \( h^{(1)}(Q) = Q^2/2 \) in (3) (while \( h^{(2)}(q) \) is still \( cn(q|k) \)). Then, one can naturally assume that \( H^{(1)}(p, q) \) is the elliptic Calogero-Moser Hamiltonian, \( H(p, q) = p^2/2 + g^2/sn^2(q|k) \) and obtain solving (3)

\[ H(P, Q) = cn(q|k) = \alpha(Q) \cdot \frac{\sqrt{k^2 + k^2\alpha^2(Q)}}{\sqrt{k^2 + \alpha^2}} \] (7)

with

\[ \alpha^2(Q) = 1 - \frac{2g^2}{Q^2}, \quad k^2 \equiv 1 - k^2 \] (8)

This result prompts an anzats for the self-dual case \( H(P, Q|k, \tilde{k}) = \alpha(Q|k, \tilde{k}) \cdot \frac{\sqrt{k^2 + \alpha^2}}{\sqrt{k^2 + \alpha^2}} \)

\[ \alpha^2(q|\tilde{k}, k) = 1 - \frac{2\nu^2}{sn^2(q|k)}, \quad \beta^2(\tilde{q}|\tilde{k}, k) = k^2 + \tilde{k}^2\alpha^2(q|k), \quad \gamma^2(q|\tilde{k}, k) = \frac{\tilde{k}^2\alpha^2(q|k)}{k^2 + \alpha^2} \] (10)

This system is called double-elliptic system \( H \). Further details can be found in \( H \).

One can similarly find multiparticle generalizations. Some technique for doing this was developed in \( H \). However, the solutions found so far are only the multi-particle rational Calogero system, the multi-particle trigonometric Ruijsenaars system and the multi-particle double-elliptic system constructed not quite manifestly.

5. In conclusion, we introduced the notion of self-duality (duality) and constructed several explicit examples. We discussed only classical Hamiltonian systems, the generalization to the quantum case is quite immediate \( H \). Indeed, one just needs to consider a pair of sets of Shrödinger equations

\[ \begin{align*}
\hat{H}_{k}(\partial_x, x)\psi(x; \lambda) = h_k(\lambda)\psi(x; \lambda) \\
\hat{H}_{\tilde{k}}(\partial_{\tilde{x}}, \tilde{x})\psi(x; \lambda) = h_{\tilde{k}}(x)\psi(x; \lambda)
\end{align*} \] (11)

where the variable \( \lambda \) plays the role of (function of) energy in the first set of equations and \( x \) does in the second one.
There are also two other issues left beyond the scope of this note. First, note that the notion of duality can be generalized to the deformations of non-free systems. To this end, one just can allow on the r.h.s. of (2) to be arbitrary Hamiltonians.

The other important issue missed is the geometrical meaning of the self-duality. As far as it provides a symmetry between the old coordinates $q$ and the new ones $Q$, one would expect a manifest symplectic group invariance on these coordinates. It is really observed in examples, although the realization does not look simple, see [11].

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