Strong immersions and maximum degree

Zdeněk Dvořák∗  Tereza Klímošová†

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Abstract

A graph $H$ is strongly immersed in $G$ if $G$ is obtained from $H$ by a sequence of vertex splittings (i.e., lifting some pairs of incident edges and removing the vertex) and edge removals. Equivalently, vertices of $H$ are mapped to distinct vertices of $G$ (branch vertices) and edges of $H$ are mapped to pairwise edge-disjoint paths in $G$, each of them joining the branch vertices corresponding to the ends of the edge and not containing any other branch vertices. We show that there exists a function $d: \mathbb{N} \to \mathbb{N}$ such that for all graphs $H$ and $G$, if $G$ contains a strong immersion of the star $K_{1,d(\Delta(H))}$ whose branch vertices are $\Delta(H)$-edge-connected to one another, then $H$ is strongly immersed in $G$. This has a number of structural consequences for graphs avoiding a strong immersion of $H$. In particular, a class $\mathcal{G}$ of simple 4-edge-connected graphs contains all graphs of maximum degree 4 as strong immersions if and only if $\mathcal{G}$ has either unbounded maximum degree or unbounded tree-width.

In this paper, graphs are allowed to have parallel edges and loops, where each loop contributes 2 to the degree of the incident vertex. A graph without parallel edges and loops is called simple.

Various containment relations have been studied in structural graph theory. The best known ones are minors and topological minors. A graph $H$ is a minor of $G$ if it can be obtained from $G$ by a sequence of edge and vertex removals and edge contractions. A graph $H$ is a topological minor of $G$ if a subdivision of $H$ is a subgraph of $G$, or equivalently, if $H$ can be obtained from $G$ by a sequence of edge and vertex removals and by suppressions of vertices of degree two. In a fundamental series of papers, Robertson and Seymour developed the theory of graphs avoiding a fixed minor, giving a description of their structure [14] and proving that every proper minor-closed class of graphs is characterized by a finite set of forbidden minors [15]. The topological minor relation is somewhat harder to deal with (and in particular, there exist proper topological minor-closed classes that are not characterized by a finite set of forbidden topological minors), but a description of their structure is also available [4, 5].

In this paper, we consider a related notion of a graph immersion. Let $H$ and $G$ be graphs. An immersion $\theta$ of $H$ in $G$ is a function $\theta$ from vertices and edges of $H$ such that

- $\theta(v)$ is a vertex of $G$ for each $v \in V(H)$, and $\theta \upharpoonright V(H)$ is injective.
- $\theta(e)$ is a connected subgraph of $G$ for each $e \in E(H)$, and if $f \in E(H)$ is distinct from $e$, then $\theta(e)$ and $\theta(f)$ are edge-disjoint.
- If $e \in E(H)$ is incident with $v \in V(H)$, then $\theta(v)$ is a vertex of $\theta(e)$, and if $e$ is a loop, then $\theta(e)$ contains a cycle passing through $\theta(v)$.

An immersion $\theta$ is strong if it additionally satisfies the following condition:

∗Computer Science Institute of Charles University, Prague, Czech Republic. E-mail: rakdver@iuuk.mff.cuni.cz. Supported the Center of Excellence – Inst. for Theor. Comp. Sci., Prague (project P202/12/G061 of Czech Science Foundation), and by project LH12095 (New combinatorial algorithms - decompositions, parameterization, efficient solutions) of Czech Ministry of Education.

†Institute of Mathematics and DIMAP, University of Warwick, Coventry, UK. E-mail: T.Klimosova@warwick.ac.uk. Her work leading to this invention has received funding from the European Research Council under the European Union’s Seventh Framework Programme (FP7/2007-2013)/ERC grant agreement no. 259385.
• If $e \in E(H)$ is not incident with $v \in V(H)$, then $\theta(e)$ does not contain $\theta(v)$.

When we want to emphasize that an immersion does not have to be strong, we call it weak. Let $E(\theta)$ denote $\bigcup_{e \in E(H)} E(\theta(e))$ and let $V(\theta)$ denote $\bigcup_{e \in E(H)} V(\theta(e))$. Let us note that by choosing the subgraphs $\theta(e)$ as small as possible, we can assume that $\theta(e)$ is a path with endvertices $\theta(u)$ and $\theta(v)$ if $e$ is a non-loop edge of $H$ joining $u$ and $v$, and that $\theta(e)$ is a cycle containing $\theta(v)$ if $e$ is a loop of $H$ incident with $v$; we call an immersion satisfying these constraints slim.

If $H$ is a topological minor of $G$, then $H$ is also strongly immersed in $G$. On the other hand, an appearance of $H$ as a minor does not imply an immersion of $H$, and conversely, an appearance of $H$ as a strong immersion does not imply the appearance as a minor or a topological minor. Nevertheless, many of the results for minors and topological minors have analogues for immersions and strong immersions. For example, any simple graph with minimum degree at least $200k$ contains a strong immersion of the complete graph $K_k$ (DeVos et al. [3]), as compared to similar sufficient minimum degree conditions for minors ($\Omega(k^2)$, Bollobás and Thomason [1], Komlós and Szemerédi [8]). A structure theorem for weak immersions appears in DeVos [4] and Wollan [18]. Note that unlike the grid theorem of Robertson and Seymour [13]. Chudnovsky et al. [2] proved the following variation on the grid theorem of Robertson and Seymour [13].

**Theorem 1.** For every $g \geq 1$, there exists $t \geq 0$ such that every $4$-edge-connected graph of tree-width at least $t$ contains the $g \times g$ grid as a strong immersion.

The variant of Theorem 1 for weak immersions was also proved by Wollan [18]. Note that unlike the grid theorem for minors [13], Theorem 1 does not admit a weak converse—there exist graphs of bounded tree-width containing arbitrarily large grids as strong immersions. A connected graph $H$ with at least three vertices is a multistar if it has no loops and contains a vertex $c$ incident with all its edges. The vertex $c$ is the center of the multistar and all its other vertices are rays. We write $c(H)$ for the center of the multistar and $R(H)$ for the set of its rays. Let the multistar with $n$ rays of degree $k$ be denoted by $S_{n,k}$. Note that every graph with at most $n$ vertices and with maximum degree at most $k$ is contained as a strong immersion in $S_{n,k}$, which has tree-width $1$. Furthermore, subdividing each edge of $S_{n,k}$ results in a simple graph of tree-width $2$ containing every graph with at most $n$ vertices and with maximum degree at most $k$ as a strong immersion.

Consequently, to turn Theorem 1 into an approximate characterization, we need to deal with the star-like graphs. The main result of this paper essentially states that if the maximum degree of $H$ is $k$ and a $k$-edge-connected graph $G$ contains a sufficiently large star as a strong immersion, then $G$ also contains $H$ as a strong immersion. Let us now state the result more precisely.

A $k$-system of magnitude $d$ in a graph $G$ is a pair $(H, \sigma)$, where $H$ is a multistar and $\sigma$ is a strong immersion of $H$ in $G$ satisfying the following conditions:

1. $H$ has at least $d$ edges,
2. rays of $H$ have degree at most $k$ in $H$, and
3. for each $v \in R(H)$, there exists no edge cut in $G$ of size less than $k$ separating $\sigma(c(H))$ from $\sigma(v)$.

Let us remark that by Menger’s theorem, (s3) implies that $G$ contains $k$ pairwise edge-disjoint paths from $\sigma(c(H))$ to $\sigma(v)$ (not necessarily belonging to or disjoint with the immersion).

A strong immersion $\theta$ of $S_{n,k}$ in a graph $G$ respects a strong immersion $\sigma$ of a multistar $H$ in $G$ if $\theta(c(S_{n,k})) = \sigma(c(H))$ and $\theta(R(S_{n,k})) \subseteq \sigma(R(H))$. Let us define $d(k) = (2k + 1)^{3k+1}k^2(k+1)$.

**Theorem 2.** If $k \geq 3$ and $n \geq 2$ are integers and $(H, \sigma)$ is a $k$-system of magnitude at least $d(k)n$ in a graph $G$, then $G$ contains $S_{n,k}$ as a strong immersion respecting $(H, \sigma)$. 

If $G$ is $k$-edge-connected and contains a vertex $c$ with at least $d$ distinct neighbors, then the neighborhood of $c$ contains a $k$-system of magnitude $d$. Let us recall that if a graph $F$ has $n$ vertices and maximum degree at most $k$, then $F$ is strongly immersed in $S_{n,k}$.

Furthermore, the relation of strong immersion is transitive. If $H_1$ has an immersion $\theta_1$ in $H$ and $H$ has an immersion $\theta$ in $G$, then let $\theta \circ \theta_1$ be defined as follows: $(\theta \circ \theta_1)(v) = \theta(\theta_1(v))$ for each $v \in V(H_1)$ and $(\theta \circ \theta_1)(e) = \bigcup_{f \in E(\theta_1(e))} \theta(f)$ for each $e \in E(H_1)$. Note that $\theta \circ \theta_1$ is an immersion of $H_1$ in $G$, and if $\theta_1$ and $\theta$ are strong, then $\theta \circ \theta_1$ is strong. Consequently, Theorem 2 has the following corollary.

**Corollary 3.** For every integer $k \geq 3$ and a graph $F$ of maximum degree at most $k$, if a $k$-edge-connected graph $G$ contains a vertex with at least $d(k)|V(F)|$ distinct neighbors, then $F$ appears in $G$ as a strong immersion.

The version of Corollary 3 for weak immersions was previously obtained by a different method by Marx and Wollan [12]. As a consequence of Corollary 3, we obtain the following strengthening of Theorem 1.

**Theorem 4.** Let $\mathcal{G}$ be a class of 4-edge-connected simple graphs. The following propositions are equivalent:

(i) There exists a graph $F$ of maximum degree 4 that does not appear as a weak immersion in any graph in $\mathcal{G}$.

(ii) There exists a graph $F$ of maximum degree 4 that does not appear as a strong immersion in any graph in $\mathcal{G}$.

(iii) There exists an integer $t \geq 4$ such that every graph in $\mathcal{G}$ has tree-width at most $t$ and maximum degree at most $t$.

Let us remark that the assumption that graphs in $\mathcal{G}$ are simple is important—if $\mathcal{G}$ is the class of all graphs that can be obtained from paths by replacing each edge by at least four parallel edges, then $\mathcal{G}$ satisfies (ii), but not (i) and (iii). Furthermore, if $\mathcal{G}$ is the class of graphs obtained from simple 4-edge-connected 4-regular graphs of bounded tree-width (say at most 10) by replacing one edge by any positive number of parallel edges, then $\mathcal{G}$ satisfies (i) and (ii), but not (iii). The implications $(iii) \Rightarrow (i) \Rightarrow (ii)$ hold even if $\mathcal{G}$ contains non-simple graphs, though. Obviously, in propositions (i) and (ii), we could restrict $F$ to be a square grid.

Flows in networks can be used to determine whether a $k$-system of large magnitude with a given center and rays exists. This enables us to restate Theorem 2 in the following form.

**Theorem 5.** Let $G$ be a graph and $c$ a vertex of $G$. Let $X \subseteq V(G) \setminus \{c\}$ be any set of vertices such that $G$ contains no edge cut of size less than $k$ separating $c$ from a vertex in $X$. If a graph $F$ of maximum degree at most $k \geq 3$ does not appear in a graph $G$ as a strong immersion, then there exist sets $Y \subseteq X$ and $K \subseteq E(G)$ such that $k|Y| + |K| < d(k)|V(F)|$ and the component of $G - Y - K$ that contains $c$ does not contain any vertex of $X$.

Theorem 5 forms a basis for a structure theorem for strong immersions analogous to the one for weak immersions [4, 18], which we develop in a future paper. Here, let us state just the first step towards this structure.

**Theorem 6.** For every graph $F$ and an integer $m \geq 0$, there exists a constant $M$ such that the following holds. Let $G$ be a graph and $X \subseteq V(G)$ a set of its vertices such that no two vertices of $X$ are separated by an edge cut of size less than $M$ in $G$. Let $G_X$ be the graph with vertex set $X$ in that two vertices $u, v \in X$ are adjacent if $G$ contains $m$ pairwise edge-disjoint paths joining $u$ with $v$ and otherwise disjoint with $X$. If $G$ does not contain $F$ as a strong immersion, then $G_X$ is connected.
Let $K_F$ be a multistar with rays $V(F)$, such that each vertex $v \in V(F)$ has the same degree in $F$ and in $K_F$. Observe that if $m \geq 2|E(F)|$, then the graph $G_X$ cannot contain the star $K_{t_1,|V(F)|}$ as a minor, as otherwise Menger's theorem would imply that $G$ contains $K_F$ as a strong immersion, and consequently that $G$ contains $F$ as a strong immersion. This restricts the structure of $G_X$ significantly, and to obtain a structure theorem, it remains to argue how the rest of the graph can attach to this well-structured part of $G$.

In Section 1 we prove Theorems 4, 5 and 6 under assumption that Theorem 2 holds. Section 2 is devoted to the proof of Theorem 2.

1 Corollaries

Proof of Theorem 4. The implication (i) $\Rightarrow$ (ii) is trivial.

Note that for every graph $F$ of maximum degree 4, there exists an integer $g$ such that $F$ is strongly immersed in the $g \times g$ grid. Let $t_1$ be the constant of Theorem 1 for this $g$. Let $t_2 = d(4)|V(F)|$, and let $t = \max(t_1, t_2)$. If a graph $F$ of maximum degree 4 does not appear as a strong immersion in a graph $G \in G$, then by Theorem 1 $G$ has tree-width at most $t_1 \leq t$, and by Corollary 3 $G$ has maximum degree at most $t_2 \leq t$. Therefore, (ii) $\Rightarrow$ (iii) holds.

Suppose now that (iii) holds, i.e., there exists an integer $t \geq 4$ such that every graph $G \in G$ has tree-width at most $t$ and maximum degree at most $t$. Let $F$ be a sufficiently large 4-regular expander (see [10] for a construction); say, for some $\varepsilon > 0$, $|V(F)| \geq 5t + 5 + \frac{3(t+1)^2}{\varepsilon}$ and for each $S \subseteq V(F)$ of size at most $|V(F)|/2$, there are at least $\varepsilon|S|$ edges in $F$ between $S$ and $V(F) \setminus S$. We claim that $F$ does not appear as a weak immersion in $G$.

Suppose on the contrary that $\theta$ is an immersion of $F$ in $G$, and let $T = \theta(V(F))$. Since $G$ has tree-width at most $t$, there exist sets $A, B \subseteq V(G)$ such that $A \cup B = V(G)$, no edge of $G$ joins a vertex in $A \setminus B$ with a vertex in $B \setminus A$, $|A \cap B| \leq t+1$, $|A \cap T| \geq \frac{1}{3}(|T|-2t-2)$, and $|B \cap T| \geq \frac{2}{3}(|T|-2t-2)$. Without loss of generality, assume that $|T \setminus B| \leq |T \setminus A|$. Let $S_A = \theta^{-1}(T \setminus B)$ and $S_B = \theta^{-1}(T \cap B) = V(F) \setminus S_A$. Note that $\frac{1}{3}(|V(F)|-5t-5) \leq |S_A| \leq |V(F)|/2$. Let $Z$ be the set of edges of $F$ between $S_A$ and $S_B$; we have $|Z| \geq \varepsilon|S_A|$. Consequently, the subgraph $Q = \bigcup_{e \in Z} \theta(e)$ contains at least $\varepsilon|S_A|$ edges incident with vertices of $A \cap B$, and at least one vertex of $A \cap B$ has degree at least $\varepsilon|S_A|/(t+1) \geq t+1$. This contradicts the assumption that the maximum degree of $G$ is at most $t$, showing that (iii) $\Rightarrow$ (i) holds.

Proof of Theorem 5. Since $G$ does not contain $F$ as a strong immersion, Theorem 2 implies that $G$ does not contain a $k$-system of magnitude $d(k)|V(F)|$. Let $G'$ be the network with the vertex set $V(G) \cup \{z\}$, where $z$ is a new vertex not appearing in $V(G)$, and the edge set defined as follows: For each edge $e \in E(G)$ not incident with a vertex in $X$, add a pair of edges in opposite directions joining the endvertices of $e$. For each edge $e \in E(G)$ joining a vertex $u \notin X$ with a vertex $v \in X$, add an edge directed from $u$ to $v$. For each vertex $x \in X$, add an edge directed from $x$ to $z$. The edges incident with $z$ have capacity $k$, while all other edges of the network have capacity 1. If $G'$ contained a flow of size $d(k)|V(F)|$ from $e$ to $z$, then the corresponding pairwise edge-disjoint paths in $G$ would form a $k$-system of magnitude $d(k)|V(F)|$, as each vertex of $X$ is contained in at most $k$ such paths. Consequently, no such flow exists.

By the flow-cut duality, it follows that $G'$ contains an edge cut $K'$ of capacity less than $d(k)|V(F)|$ separating $c$ from $z$. Let $Y$ be the set of vertices $y \in X$ such that the edge $xz$ belongs to $K'$. Let $K$ be the set of edges of $G$ corresponding to the edges of $K'$ not incident with $z$. Clearly, $G - Y - K$ contains no path from $c$ to a vertex of $X$. Furthermore, $k|Y| + |K|$ is equal to the capacity of $K'$, and thus $k|Y| + |K| < d(k)|V(F)|$.

Proof of Theorem 6. Let $k = \max(\Delta(F), 3)$, $s = d(k)|V(F)|$ and $M = ms^3 + s^2$. Suppose that a graph $G$ and a set $X \subseteq V(G)$ satisfy the assumptions of Theorem 6. Consider any nonempty disjoint sets $A, B \subseteq X$ such that $A \cup B = X$. Let $c_A$ be an arbitrary vertex of $A$ and apply Theorem 5 for $c_A$ and $B$, obtaining sets $Y_B \subseteq B$ and $K_B \subseteq E(G)$, where $k|Y_B| + |K_B| < s$, such that the component of
G – K_B – Y_B that contains c_A does not contain any vertex of B. For each y ∈ Y_B, apply Theorem\textsuperscript{5} for y and A, obtaining sets Y^y_A ⊆ A and K^y_A ⊆ E(G), where k|Y^y_A| + |K^y_A| < s, such that the component of G – K^y_A – Y^y_A that contains y does not contain any vertex of A. Let K = K_B ∪ \bigcup_{y ∈ Y_B} K^y_A and let Y_A = \bigcup_{y ∈ Y_B} Y^y_A, and note that |K| ≤ s^2 and |Y_A| ≤ s^2.

Let c_B be an arbitrary vertex of B. By Menger’s theorem, there exists a set \mathcal{P}_0 of M pairwise edge-disjoint paths from c_A to c_B in G. Let \mathcal{P} ⊆ \mathcal{P}_0 consist of the paths that do not contain edges of K; we have |\mathcal{P}| ≥ M – s^2. Consider a path P ∈ \mathcal{P}. Let v_0, v_1, ..., v_p be the vertices of P in order, where v_0 = c_A and v_p = c_B. Let j > 0 be the smallest index such that v_j belongs to B. As the component of G – K_B – Y_B that contains c_A does not contain any vertex of B, the vertex v_j belongs to Y_B. Let i be the largest index such that i < j and v_i belongs to A. As the component of G – K^v_j – Y^v_j that contains v_j does not contain any vertex of A, it follows that v_i belongs to Y^v_j ⊆ Y_A. Consequently, G contains a set of |\mathcal{P}| pairwise edge-disjoint paths joining vertices of Y_A with vertices of Y_B and otherwise disjoint from X. By the pigeonhole principle, there exist vertices a ∈ A and b ∈ B incident with at least \frac{|\mathcal{P}|}{|Y_A|} ≥ m of these paths, and thus ab is an edge of G_X.

Therefore, for all nonempty disjoint sets A, B ⊆ X = V(G_X) such that A ∪ B = V(G_X), there exists an edge between a vertex of A and a vertex of B in G_X. It follows that G_X is connected. \hfill \Box

2 Proof of Theorem\textsuperscript{2}

We need the following variation on the Mader’s splitting theorem [11]. Let G be a graph, x a vertex of G and for each s, t ∈ V(G) \setminus \{x\}, let \lambda(s, t) denote the maximum number of pairwise edge-disjoint paths between s and t in G. Let e and f be edges joining x to vertices u and v, respectively, and let G’ be the graph obtained from G – \{e, f\} by adding a new edge joining u with v. We say that the pair of edges e and f is splittable if for every s, t ∈ V(G’) \setminus \{x\}, the graph G’ contains \lambda(s, t) pairwise edge-disjoint paths between s and t. We say that G’ is obtained by lifting the edges e and f. Note that G’ is immersed in G.

Theorem 7 (Frank [8]). Let G be a graph and let x be a vertex of G not incident with any 1-edge cut. If x has degree m ≠ 3, then there are \lfloor m/2 \rfloor pairwise disjoint splittable pairs of edges incident with x.

If (H, σ) is a k-system and σ is slim, we consider the paths in σ(E(H)) to be directed away from \sigma(c), where c = c(H). That is, if e = cv is an edge of H, then \sigma(c) is the first vertex of \sigma(e) and \sigma(v) is the last vertex of \sigma(e).

Definition 1. We say that a triple (G, H, σ) is well-behaved if (H, σ) is a k-system of magnitude d in a graph G, such that σ is slim and the following conditions are satisfied:

\begin{itemize}
  \item (w1) G is 3-edge connected,
  \item (w2) vertices in V(G) \setminus V(σ) have degree exactly 3,
  \item (w3) for every v ∈ R(H), if K is an k-edge cut in G separating σ(c(H)) from σ(v), then K consists of the edges incident with \sigma(v),
  \item (w4) every edge of E(G) \setminus E(σ) is incident with a vertex in σ(R(H)) of degree exactly k,
  \item (w5) for each vertex v ∈ V(σ) \setminus σ(V(H)), at most one edge incident with v does not belong to E(σ).
\end{itemize}

Furthermore, if there is such an edge and v belongs to the path \sigma(e) for an edge e ∈ E(H), then v is the next-to-last vertex of \sigma(e) and the last vertex of \sigma(e) has degree exactly k.

Let H be a multistar with a strong immersion σ in a graph G and let H’ be a multistar with a strong immersion σ’ in a graph G’. We say that (G’, H’, σ’) is a reduction of (G, H, σ) if there exists a weak immersion \theta of G’ in G satisfying the following conditions:

\begin{itemize}
  \item \theta(σ’(c(H’))) = σ(c(H)),
\end{itemize}
Suppose that

\[ \theta(\sigma'(R(H'))) \subseteq \sigma(R(H)), \]

if \( e \in E(G') \) is not incident with a vertex \( v \in \sigma'(V(H')) \), then \( \theta(e) \) does not contain \( \theta(v) \).

Note that if a strong immersion of \( S_{n,k} \) in \( G' \) respects \( (H', \sigma') \), then \( \theta \circ \alpha \) is a strong immersion of \( S_{n,k} \) in \( G \) respecting \((H, \sigma)\). Furthermore, the reduction relation is transitive.

**Lemma 8.** If \((H_0, \alpha_0)\) is a \( k \)-system of magnitude \( d \) in a graph \( G_0 \), then there exists a reduction \((G, H, \sigma)\) of \((G_0, H_0, \alpha_0)\) such that \((H, \sigma)\) is a \( k \)-system of magnitude \( d \) and \((G, H, \sigma)\) is well-behaved.

**Proof.** Let \( G \) with a \( k \)-system \((H, \sigma)\) of magnitude \( d \) be chosen so that \((G, H, \sigma)\) is a reduction of \((G_0, H_0, \alpha_0)\), and subject to that \(|V(G)| + |E(G)| + |E(\sigma)|\) is minimal. Such triple \((G, H, \sigma)\) exists, since \((G_0, H_0, \alpha_0)\) is a reduction of itself. Clearly, \( \sigma \) is slim and \( G \) has no loops. We claim that \((G, H, \sigma)\) is well-behaved. Let us discuss the conditions \((w1), \ldots, (w5)\) separately.

\( (w1) \) Suppose that there is an edge cut \( K \) of size at most 2 in \( G \); we can assume that \( K \) is minimal. Since \( k \geq 3 \), \((s3)\) implies that all vertices of \( \sigma(V(H)) \) are in the same component \( C \) of \( G - K \). Let \( G' = C \) if \(|K| \leq 1 \); let \( G' \) be the graph obtained from \( C \) by adding an edge \( e \) (possibly a loop) between the vertices in \( C \) incident with \( K \) if \(|K| = 2 \). Let \( \sigma'(v) = \sigma(v) \) for \( v \in V(H) \). Consider \( f \in E(H) \). If \( \sigma(f) \) contains two edges of \( K \), then let \( \sigma'(f) = (\sigma(f) \cap C) + e \), otherwise let \( \sigma'(f) = \sigma(f) \cap C \). Observe that \((G', H', \sigma')\) is a reduction of \((G, H, \sigma)\). This is a contradiction, as \((G, H, \sigma)\) was chosen so that \(|V(G)| + |E(G)| + |E(\sigma)|\) is minimal.

\( (w2) \) Suppose \( G \) satisfies \((w1)\) and contains a vertex \( v \in V(G) \setminus V(\sigma) \) of degree greater than 3. By \( \text{Theorem } 7 \) we can lift a pair of edges incident to \( v \) without violating the condition \((s3)\). This way, we obtain a reduction \((G', H, \sigma')\) contradicting the minimality of \((G, H, \sigma)\).

\( (w3) \) Suppose \( G \) contains an edge cut \( K \) of size \( k \) separating \( \sigma(c(H)) \) from \( \sigma(v) \) for some \( v \in R(H) \), such that \( K \) does not consist of the edges incident with \( \sigma(v) \). By \((s3)\), \( G - K \) has only two components. Let \( G' \) be the graph obtained from \( G \) by replacing the component \( C \) of \( G - K \) that contains \( \sigma(v) \) by a single vertex \( w \) of degree \( k \), incident with the edges of \( K \). Let \( Z \subseteq E(H) \) be the set of edges \( e \in E(H) \) such that \( \sigma(e) \) contains an edge of \( K \). Let \( H' \) be the multistar obtained from \( H \) by making all edges of \( Z \) incident with \( v \) instead of their original incident ray and removing all resulting isolated vertices. Let \( \sigma' \) be the strong immersion of \( H' \) in \( G' \) such that \( \sigma'(v) = w, \sigma'(x) = \sigma(x) \) for \( x \in V(H') \setminus \{v\}, \sigma'(e) = \sigma(e) \) for \( e \in E(H) \setminus Z \), and \( \sigma'(e) \) is the segment of the path \( \sigma(e) \) between \( \sigma'(c(H')) \) and the first edge of \( K \) appearing in the path (inclusive). Note that \( w \) has degree exactly \( k \) in \( H' \), and thus \( H' \) satisfies \((s2)\). Furthermore, if \( x \in R(H') \), then \( \sigma'(x) \) is not separated from \( \sigma(c(H')) \) by an edge cut \( K' \) of size less than \( k \), as otherwise \( K' \) (with the edges incident with \( w \) replaced by the corresponding edges of \( K \)) separates \( \sigma(x) \) from \( \sigma(c(H)) \); hence, \((s3)\) holds for \((H', \sigma')\). Also, by \((s3)\) for \((H, \sigma)\), there exist \( k \) pairwise edge-disjoint paths in \( G \) joining \( \sigma(v) \) with the edges of \( K \). Consequently, there exists a strong immersion of \( G' \) in \( G \), showing that \((G', H', \sigma')\) is a reduction of \((G, H, \sigma)\). This contradicts the minimality of the latter.

\( (w4) \) Suppose that \( G \) satisfies \((w3)\) and \( e \in E(G) \setminus E(\sigma) \) is not incident with a vertex of \( \sigma(R(H)) \) of degree exactly \( k \). Consequently, \( e \) is not contained in any \( k \)-edge cut separating \( \sigma(c(H)) \) from a vertex in \( \sigma(R(H)) \), and we conclude that \((G - e, H, \sigma)\) is a reduction contradicting the minimality of \((G, H, \sigma)\).

\( (w5) \) Suppose that \( G \) satisfies \((w1)\) and \((w4)\) and consider a vertex \( v \in V(\sigma) \setminus V(H(\sigma)) \). If at least two edges incident with \( v \) do not belong to \( E(\sigma) \), then by \( \text{Theorem } 7 \) there exists a splittable pair of edges \( e \) and \( f \) incident with \( v \) such that \( e \notin E(\sigma) \). Let \( u \) and \( w \) be the endvertices of \( e \) and \( f \), respectively, distinct from \( v \). Let \( e' \) be an edge incident with \( v \) and not belonging to \( \{e\} \cup E(\sigma) \), and let \( z \) be the vertex incident with \( e' \) distinct from \( v \). By \((w4)\), \( u \) and \( z \) are vertices of \( \sigma(R(H)) \) of degree exactly \( k \). Let \( G' \) be the graph obtained from \( G \) by lifting the
edges \( e \) and \( f \), creating a new edge \( h \). If \( f \not\in E(\sigma) \), then \( H' = H \) and \( \sigma' = \sigma \). Otherwise, consider the edge \( f_0 \in E(H) \) such that \( f \in E(\sigma(f_0)) \). If \( w \) appears before \( v \) in \( \sigma(f_0) \), then let \( H' \) be the graph obtained from \( H \) by making \( f_0 \) incident with \( \sigma^{-1}(w) \) instead of its original incident ray (and possibly removing the resulting isolated vertex) and let \( \sigma' \) be obtained from \( \sigma \) by letting \( \sigma'(f_0) \) consist of \( h \) and the subpath of \( \sigma(f_0) \) between \( \sigma(c(H)) \) and \( w \). If \( w \) appears after \( v \) in \( \sigma(f_0) \), then let \( H' \) be the graph obtained from \( H \) by making \( f_0 \) incident with \( \sigma^{-1}(z) \) instead of its original incident ray (and possibly removing the resulting isolated vertex) and let \( \sigma' \) be obtained from \( \sigma \) by letting \( \sigma'(f_0) \) consist of \( e' \) and the subpath of \( \sigma(f_0) \) between \( \sigma(c(H)) \) and \( v \). Note that \((G', H', \sigma')\) is a reduction contradicting the minimality of \((G, H, \sigma)\).

Let us now consider the case that \( v \in V(\sigma(V(H))) \) is incident with exactly one edge \( e \) not belonging to \( E(\sigma) \), where \( e \) joins \( v \) with a vertex \( u \). Note that \( u \) belongs to \( \sigma(R(H)) \) and has degree exactly \( k \). Let \( f_0 \) be an edge of \( H \) such that \( \sigma(f_0) \) contains \( v \). Let \( H' \) be obtained from \( H \) by making \( f_0 \) incident with \( \sigma^{-1}(u) \) instead of its original incident ray (and possibly removing the resulting isolated vertex) and let \( \sigma' \) be obtained from \( \sigma \) by letting \( \sigma'(f_0) \) consist of \( e \) and the subpath of \( \sigma(f_0) \) between \( \sigma(c(H)) \) and \( v \). Note that \((G', H', \sigma')\) is a reduction of \((G, H, \sigma)\). By the minimality of \((G, H, \sigma)\), we have that \( v \) is the next-to-last vertex of the path \( \sigma(f_0) \). Furthermore, if the last vertex \( x \) of \( \sigma(f_0) \) had degree greater than \( k \), then we could find a reduction of \((G', H', \sigma')\) contradicting the minimality of \((G, H, \sigma)\) in the same way as in the proof of (w3) or (w4).

\[\square\]

**Lemma 9.** If \((H, \sigma)\) is a \( k \)-system of magnitude \( d \) in a graph \( G \), then there exists a reduction \((G', H', \sigma')\) of \((G, H, \sigma)\) such that \((H', \sigma')\) is a \( k \)-system of magnitude \( \frac{d}{k(k+1)} \) in \( G' \), \((G', H', \sigma')\) is well-behaved and each vertex of \( \sigma'(R(H')) \) has degree exactly \( k \) in \( G' \).

**Proof.** By Lemma we can assume that \((G, H, \sigma)\) is well-behaved. Let \( S \) be the set of all vertices \( s \in R(H) \) such that \( \sigma(s) \) has degree exactly \( k \) in \( G \) and let \( B = R(H) \setminus S \). Since \( \sigma \) is slim, (s2) and (s3) imply that for each \( v \in B \), there exists an edge \( e \in E(G \setminus E(\sigma)) \) incident with \( \sigma(v) \). By (w4), \( e \) is incident with a vertex in \( \sigma(S) \). We conclude that \(|B| \leq k|S|\). Since \(|B| + |S| = |R(H)|\), we have \(|S| \geq |R(H)|/(k + 1) \). Let \( H_0 = H - B \). Since \( H \) is connected and \( \sigma \) satisfies (s2), we have \(|E(H_0)| \geq |S| \geq |R(H)|/(k + 1) \geq \frac{d}{k(k+1)} \). Let \( \sigma_0 = \sigma \restriction (V(H_0) \cup E(H_0)) \). Clearly, \((H_0, \sigma_0)\) is a \( k \)-system of magnitude \( \frac{d}{k(k+1)} \) in \( G \), \((G, H_0, \sigma_0)\) is a reduction of \((G, H, \sigma)\) and each vertex of \( \sigma_0(R(H_0)) \) has degree exactly \( k \) in \( G \). Finally, we obtain a well-behaved reduction \((G', H', \sigma')\) by Lemma since no vertices of degree greater than \( k \) belonging to \( \sigma'(R(H')) \) are created in its proof.

\[\square\]

Let \((G, H, \sigma)\) be well-behaved, where \((H, \sigma)\) is a \( k \)-system of magnitude \( d \) in a graph \( G \) such that each vertex of \( \sigma(R(H)) \) has degree exactly \( k \). We can assume that all edges of \( G \) between \( \sigma(e) \) and \( \sigma(R(H)) \) belong to \( E(\sigma) \), as otherwise we can add more edges to \( H \). Let \( N(\sigma) \) consist of \( \sigma(R(H)) \) and of all vertices incident with edges of \( E(G) \setminus E(\sigma) \). Let \( M(\sigma) = N(\sigma) \cap V(\sigma) \setminus \sigma(R(H)) \). Let us note that by (w5), \( M(\sigma) \) is an independent set in \( G \). Consequently, \( \sigma(e) \) intersects \( N(\sigma) \) in at most two vertices for each \( e \in E(H) \). Let \( G' \) be the graph with vertex set \( \sigma(c(H)) \cup N(\sigma) \) and the edge set defined as follows: the subgraphs of \( G \) and \( G' \) induced by \( N(\sigma) \) are equal; and, the edges incident with \( \sigma(c(H)) \) are \( \{f_e : e \in E(H)\} \), where \( f_e \) joins \( \sigma(c(H)) \) with the first vertex of \( \sigma(e) \) that belongs to \( N(\sigma) \). Let us define a strong immersion \( \sigma' \) of \( H \) in \( G' \) as follows: for \( v \in V(H) \), we set \( \sigma'(v) = \sigma(v) \). For \( e \in E(H) \), let \( \sigma'(e) \) consist of \( f_e \) and of \( \sigma(e) \cap G[N(\sigma)] \). Note that \( \sigma'(e) \) has length at most two. Clearly, \((G', H, \sigma')\) is a reduction of \((G, H, \sigma)\). We say that \((G', H, \sigma')\) is a core of \((G, H, \sigma)\).

Let us define a function \( g : E(G') \to E(G) \) as follows: if \( f \in E(G') \) is not incident with \( \sigma(c(H)) \), then let \( g(f) = f \). Otherwise, \( f = f_e \) for some \( e \in E(H) \), and we let \( g(f) \) be equal to the last edge of \( \sigma(e) \) that does not belong to \( G[N(\sigma)] \). We say that \( g \) is the origin function of the core.
Lemma 10. Let $\langle G, H, \sigma \rangle$ be well-behaved, where $(H, \sigma)$ is a $k$-system of magnitude $d$ in a graph $G$ such that each vertex of $\sigma(R(H))$ has degree exactly $k$. If $(G', H, \sigma')$ is the core of $(G, H, \sigma)$, then $(H, \sigma')$ is a $k$-system of magnitude $d$ in $G'$.

Proof. It suffices to check that $(G', H, \sigma')$ satisfies the condition (s3). Let $g$ be the origin function of $(G', H, \sigma')$. Consider an edge cut $K$ in $G'$ separating $\sigma'(c(H))$ from $\sigma'(v)$ for some $v \in R(H)$. Observe that $g(K)$ is an edge cut in $G$ separating $\sigma(c(H))$ from $\sigma(v)$, and thus $|K| \geq |g(K)| \geq k$ by (s3) for $(G, H, \sigma)$.

Let $(H, \sigma)$ be a $k$-system of magnitude $d$ in a graph $G$. We say that $(G, H, \sigma)$ is peeled if it is well-behaved, each vertex of $\sigma(R(H))$ has degree exactly $k$, all edges incident with $\sigma(c(H))$ belong to $E(\sigma)$ and $V(G) = \{\sigma(c(H))\} \cup N(\sigma)$. Note that every core is peeled.

Lemma 11. Let $(H_0, \sigma_0)$ be a $k$-system of magnitude $d$ in a graph $G_0$. Then there exists a reduction $(G, H, \sigma)$ of $(G_0, H_0, \sigma_0)$ that is peeled, the $k$-system $(H, \sigma)$ has magnitude $\frac{d}{k(k+1)}$ and no vertex of $G$ other than $\sigma(c(H))$ has degree greater than $2k + 1$.

Proof. Let $(G, H, \sigma)$ be a peeled reduction of $(G_0, H_0, \sigma_0)$, where the $k$-system $(H, \sigma)$ has magnitude $\frac{d}{k(k+1)}$, with $|E(G)|$ as small as possible (which exists by Lemmas 9 and 10). Suppose that a vertex $v \in V(G)$ has degree at least $2k + 2$ and $v \neq \sigma(c(H))$. By (w2) and the assumption that $(G, H, \sigma)$ is peeled, we have $v \in M(\sigma)$. Note that $v$ is joined with $\sigma(c(H))$ by at least $k + 1$ edges. Select an arbitrary edge $e \in E(H)$ such that $\sigma(e)$ contains $v$ and let $f_1$ and $f_2$ be the edges of $\sigma(e)$, where $f_1$ is incident with $\sigma(c(H))$. Let $G'$ be the graph obtained from $G$ by lifting $f_1$ and $f_2$, creating a new edge $f$. Let $\sigma'$ be obtained from $\sigma$ by letting $\sigma'(e)$ be the path consisting only of $f$. Note that $(G', H, \sigma')$ is a reduction of $(G, H, \sigma)$.

We claim that $(H, \sigma')$ is a $k$-system in $G'$. It suffices to check that it satisfies the condition (s3). Let $K'$ be a minimal edge cut in $G'$ separating $\sigma'(c(H))$ from $\sigma'(x)$ for some $x \in R(H)$. Let $C$ and $X$ be the vertex sets of the components of $G' - K'$, where $C$ contains $\sigma'(c(H))$. Let $K$ be the set of edges between $C$ and $X$ in $G$. If $K$ does not contain $f_1$, then $|K'| = |K|$, and thus $|K'| \geq k$ by (s3) for $(G, H, \sigma)$. If $K$ contains $f_1$, then $K$ also contains all edges parallel to $f_1$, and these edges belong to $K'$ as well. Since $v$ and $\sigma(c(H))$ are joined by at least $k + 1$ edges, we have $|K'| \geq k$. We conclude that $(G', H, \sigma')$ satisfies the condition (s3).

Note that $(G', H, \sigma')$ is peeled, and thus it contradicts the minimality of $(G, H, \sigma)$.

Lemma 12. Let $k \geq 3$ be an integer and let $(H, \sigma)$ be a $k$-system in a graph $G$, where $(G, H, \sigma)$ is peeled. For each $v \in \sigma(R(H))$, there exists a set of $k$ pairwise edge-disjoint paths in $G$, each of length at most $4k + 2$, joining $\sigma(v)$ with $\sigma(c(H))$.

Proof. By (s3), there exist $k$ pairwise edge-disjoint paths $Q_1, \ldots, Q_k$ between $\sigma(v)$ and $\sigma(c(H))$; let $Q = Q_1 \cup \ldots \cup Q_k$ and let $S$ be the set of edges $e \in E(H)$ such $\sigma(e) \subseteq Q_i$ for some $i \in \{1, \ldots, k\}$. Let us choose these paths so that $|E(Q)|$ is as small as possible, and subject to that $|S|$ is as large as possible. Clearly, $Q$ contains exactly $k$ edges incident with $\sigma(c(H))$.

Let $e$ be an edge of $H$ such that $\sigma(e)$ shares at least one edge $f$ with $Q$, say $f \in E(Q_1)$. Suppose that $f$ is not incident with $\sigma(c(H))$, and let $f'$ be the other edge of $\sigma(e)$. If $f'$ were not in $E(Q)$, then we could change $Q_1$ to use $f'$ to enter $\sigma(c(H))$, thus either decreasing $|E(Q)|$, or adding $e$ to $S$, contrary to the choice of the paths $Q_1, \ldots, Q_k$. We conclude that if $\sigma(e)$ contains an edge of $Q$, then $Q \cap \sigma(e)$ contains an edge incident with $\sigma(c(H))$. Consequently, there are at most $k$ such edges $e \in E(H)$. Let

$$W = \bigcup_{e \in E(H), E(\sigma(e)) \cap E(Q) \neq \emptyset} V(\sigma(e)) \setminus \{\sigma(c(H))\}.$$ 

Since $\sigma(e)$ has length at most two for each $e \in E(H)$, we have $|W| \leq 2k$.

Consider the path $Q_i$ for some $i \in \{1, \ldots, k\}$, and let $v_0v_1\ldots v_\ell$ be its vertices in order, where $v_0 = \sigma(c(H))$ and $v_\ell = v$. Let $Z$ be the set of vertices of $Q_i$ belonging to $\sigma(R(H)) \cup M(\sigma)$. Suppose
that there exists a vertex \( z \in Z \setminus W \). Note that there exists an edge \( e \in E(H) \) such that \( z \) belongs to \( \sigma(e) \), and by the definition of \( W \), we have \( E(\sigma(e)) \cap E(Q) = \emptyset \). Therefore, we can change the path \( Q_i \) to follow \( \sigma(e) \) from \( z \) to \( \sigma(e(H)) \). Since we have chosen the paths with \( |E(Q)| \) as small as possible, we conclude that the distance from \( z \) to \( \sigma(e(H)) \) in \( Q_i \) is at most two. Consequently, \( (Z \setminus W) \cap V(Q_i) \subseteq \{v_1, v_2\} \). Since \((G, H, \sigma)\) is peeled and satisfies \((w4)\), every edge of \( Q_i \) is incident with a vertex of \( Z \), and thus each edge of \( Q_i - \{v_0, v_1, v_2\} \) is incident with a vertex of \( Z \cap W \). Since \( |W| \leq 2k \) and \( v_\ell = v \in Z \), this implies that \( Q_i \) has length at most \( 4k + 2 \).

**Proof of Theorem 2.** Let \( N = (2k + 1)^{8k^4} \) and \( d = d(k)n = k^2(k + 1)Nn \). By Lemma 11, there exists a peeled reduction \((G', H', \sigma')\) of \((G, H, \sigma)\), where \((H', \sigma')\) is a \( k \)-system of magnitude \( d' = \frac{d}{k(k+1)} \) in \( G' \) and no vertex of \( G' \) other than \( \sigma'(c(H')) \) has degree greater than \( 2k + 1 \). Consequently, for each \( v \in \sigma'(R(H')) \), there exist at most \( N \) vertices at distance at most \( 8k + 4 \) from \( v \) in \( G' - \sigma'(c(H')) \). Since \( |R(H')| \geq d'/k \geq Nn \), we can greedily choose a set \( U \subseteq \sigma'(R(H')) \) of size \( n \) such the distance in \( G' - \sigma'(c(H')) \) between any two vertices of \( U \) is at least \( 8k + 5 \). By Lemma 12, we can, for each \( u \in U \), find a set \( S_u \) of \( k \) pairwise edge-disjoint paths in \( G' \) joining \( u \) with \( \sigma'(c(H')) \), each of length at most \( 4k + 2 \). By the choice of \( U \), the paths \( \bigcup_{u \in U} S_u \) are pairwise edge-disjoint and intersect only in their endvertices. This set of paths corresponds to a strong immersion of \( S_{n,k} \) in \( G' \) respecting \((H', \sigma')\). We conclude that \( G \) contains a strong immersion of \( S_{n,k} \) respecting \((H, \sigma)\).

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