Radiative heavy quark energy loss in a dynamical QCD medium

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The computation of radiative energy loss in a dynamically screened QCD medium is a key ingredient for obtaining reliable predictions for jet quenching in ultra-relativistic heavy ion collisions. We calculate, to first order in the opacity, the energy loss suffered by a heavy quark traveling through an infinite and time-independent QCD medium and show that the result for a dynamical medium is almost twice that obtained previously for a medium consisting of randomly distributed static scattering centers. A quantitative description of jet suppression in RHIC and LHC experiments thus must correctly account for the dynamics of the medium’s constituents.

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I. INTRODUCTION

Studying the suppression pattern of high transverse momentum hadrons is a powerful tool to map out the density of a QCD plasma created in ultra-relativistic heavy ion collisions [1–3]. This suppression (called jet quenching) results from the energy loss of high energy partons moving through the plasma [4–7]. Recent non-photonic single electron data [8, 9] (which present an indirect probe of heavy quark energy loss) showed that radiative energy loss alone can not explain the results as long as realistic parameter values are used [10]. Inclusion of collisional energy loss [11–14] improves the agreement with available data [15], but still does not yield a perfect description.

The currently available studies suffer from one crucial drawback: The medium induced radiative energy loss is computed in a QCD medium consisting of randomly distributed but static scattering centers (“static QCD medium”). In such a medium the collisional energy loss is exactly zero. This approximation was motivated by early estimates [16–21], which indicated that the typical collisional energy loss should be small compared to the radiative one. However, recent calculations [11–14] showed that the collisional contribution is important and comparable to the radiative energy loss. The static approximation is thus qualitatively wrong as far as the computation of collisional energy loss is concerned and should therefore also be revisited in the context of radiative energy loss.

In this paper, we report on a first important step, the calculation of heavy quark radiative energy loss in an infinite and time-independent QCD medium consisting of dynamical constituents. By comparing with the static medium calculation, this permits us to qualitatively assess the importance of dynamical effects on radiative energy loss. The more demanding problem of including finite medium size corrections, i.e. the Landau-Pomeranchuk-Migdal (LPM) effect, will be left to a future study.

Here is the outline of our paper: In Section II we compute, to first order in the opacity, the radiative energy loss in an infinite, dynamical QCD medium. In Section III we obtain the corresponding result in the static approximation. While the analytical results in both cases lead to formally very similar expressions, they give remarkably different numerical values for the energy loss. These are presented in Section IV. We will see that a dynamical medium leads to approximately twice the radiative energy loss obtained in the static approximation. In Section V we present a short summary and conclude that representing the QCD medium by a random ensemble of static scattering centers is not a good approximation for RHIC and LHC phenomenology. Some technical steps of our calculation are reproduced in the Appendix. We will use the following notation for 4-vectors: $k = (k_0, \vec{k}) = (k_0, k_z, k)$, i.e. $\vec{k}$ (with an explicit vector superscript) describes a 3-vector while $k$ (without a vector superscript) denotes the 2-vector transverse to the direction of motion $z$ of the heavy quark. Correspondingly $d^4k \equiv dk_z d^2k$.

II. RADIATIVE ENERGY LOSS IN A DYNAMICAL QCD MEDIUM

In this Section we compute the medium induced radiative energy loss for a heavy quark to first order in the opacity. For simplicity we consider an infinite QCD medium and assume that the on-shell heavy quark is produced at time $x_0 = -\infty$. In this medium we compute the radiative energy loss per unit length, $dE/dL$. For phenomenological applications in heavy-ion collisions one would, as a first approximation, use this result and simply multiply it with the effective thickness $L$ of the medium to calculate the total energy loss. A more rigorous derivation would have to start with a finite size medium from the beginning; we leave this for the future.

Medium induced radiative energy loss is caused by the radiation of one or more gluons induced by collisional interactions between the quark of interest and partons in the medium. The energy loss rate can be expanded in
the number of scattering events suffered by the heavy quark which is equivalent to an expansion in powers of the opacity. For a finite medium, the opacity is given by the product of the density of the medium with the scattering cross section, integrated along the path of the heavy quark. The lowest (first) order contribution corresponds to one collisional interaction with the medium, accompanied by emission of a single gluon. We adopt this as a definition of the "first order in opacity" also for the infinite medium.

For a medium consisting of dynamical quarks and gluons in thermal equilibrium, the corresponding energy loss contribution involves two cut Hard-Thermal Loop (HTL) gluon propagators. The associated Feynman diagrams are plotted in Figs. 10-12 and computed in Appendices B-D. The diagrams represent an on-shell heavy quark with momentum \( p' \) which (in arbitrary order) exchanges a virtual gluon of momentum \( q \) with a parton in the medium and radiates a gluon with momentum \( k \). The heavy quark emerges with (measured) momentum \( p \). Since the exchanged gluon momentum is space-like \([13, 18, 22]\), only the Landau damping contribution \((q_0 \leq |q|)\) to the cut HTL effective gluon propagator \( D(q) \) needs to be taken into account \([13, 17, 18]\).

The radiated gluon has timelike momentum \( k = (\omega, \vec{k}) \), so only the quasi-particle contribution at \( \omega > |\vec{k}| \) from the cut gluon propagator \( D(k) \) contributes \([23-25]\). Energy and momentum conservation requires \( p' = p + k + q \). Since our focus is on heavy quarks with mass \( M \gg gT \), we neglect the thermal shift of the heavy quark mass.

The effective gluon propagator has both transverse and longitudinal contributions \([26-32]\). The 1-HTL gluon propagator has the form

\[
\frac{iD^{\mu\nu}(l)}{l^2 - \Pi_T(l)} + \frac{Q_{\mu\nu}(l)}{l^2 - \Pi_L(l)},
\]

where \( l = (l_0, \vec{l}) \) is the 4-momentum of the gluon and \( P^{\mu\nu}(l) \) and \( Q_{\mu\nu}(l) \) are the transverse and longitudinal projectors, respectively. The transverse and longitudinal HTL gluon self energies \( \Pi_T \) and \( \Pi_L \) are given by \([29]\)

\[
\Pi_T(l) = \mu^2 \left[ \frac{y^2}{2} + \frac{y(1-y^2)}{4} \ln \left( \frac{y + 1}{y-1} \right) \right], \quad \Pi_L(l) = \mu^2 \left[ 1 - y^2 - \frac{y(1-y^2)}{2} \ln \left( \frac{y + 1}{y-1} \right) \right],
\]

where \( y \equiv l_0/|\vec{l}| \) and \( \mu = gT \sqrt{N_c/3 + N_f/6} \) is the Debye screening mass.

While the results obtained in this paper are gauge invariant \([22]\), we present the calculation for simplicity in Coulomb gauge. In this gauge the only nonzero terms in the transverse and longitudinal projectors are

\[
P^{ij}(l) = -\delta^{ij} + \frac{\Pi^{ij}}{l^2}, \quad Q^{00}(l) = -\frac{l^2}{l^2} = 1 - \frac{l_0^2}{l^2} = 1 - y^2.
\]

As in \([33-39]\), we assume validity of the soft gluon and soft rescattering approximations (see Appendix A for details). With these assumptions we compute in Appendices B-D the diagrams \( M_{1,0}, M_{1,1} \) and \( M_{1,2} \), which contribute to the first order radiative energy loss. For the interaction rate we find

\[
\Gamma(E) = \frac{1}{2E} 2 \text{Im} \, M_{\text{tot}} = \frac{1}{2E} (2 \text{Im} \, M_{1,0} + 2 \text{Im} \, M_{1,1} + 2 \text{Im} \, M_{1,2}),
\]

where (see Eqs. (B21), (C10), and (D6))

\[
2 \text{Im} \, M_{1,0} = 8E \, g^4 T \, [t_a, t_b] \left[ \{ t_c, t_a \} \right] \int \frac{d^3 k}{(2\pi)^3 2\omega} \frac{d^2 q}{(2\pi)^2} \frac{\mu^2}{q^2(q^2 + \mu^2)} \frac{k^2}{(k^2 + \chi)^2},
\]

\[
2 \text{Im} \, M_{1,1} = 8E \, g^4 T \, [t_a, t_b] \left[ \{ t_c, t_a \} \right] \int \frac{d^3 k}{(2\pi)^3 2\omega} \frac{d^2 q}{(2\pi)^2} \frac{\mu^2}{q^2(q^2 + \mu^2)} \frac{-2k \cdot (k+q)}{(k^2 + \chi) ((k+q)^2 + \chi)},
\]

\[
2 \text{Im} \, M_{1,2} = 8E \, g^4 T \, [t_a, t_b] \left[ \{ t_c, t_a \} \right] \int \frac{d^3 k}{(2\pi)^3 2\omega} \frac{d^2 q}{(2\pi)^2} \frac{\mu^2}{q^2(q^2 + \mu^2)} \frac{(k+q)^2}{((k+q)^2 + \chi)^2}.
\]

Here \([t_a, t_b]\) is a color commutator. \( m_g^2 = \mu^2/2 \) is the effective mass for gluons with hard momenta \( k \gg T \), and \( \chi \equiv M^2 x^2 + m_g^2 \) where \( x \) is the longitudinal momentum fraction of the heavy quark carried away by the emitted gluon. We assume constant coupling \( g \).

By using the above equations, the interaction rate becomes

\[
\Gamma(E) = 4 g^4 T \, [t_a, t_b] \left[ \{ t_c, t_a \} \right] \int \frac{d^3 k}{(2\pi)^3 2\omega} \frac{d^2 q}{(2\pi)^2} \frac{\mu^2}{q^2(q^2 + \mu^2)} \left( \frac{k}{k^2 + \chi} - \frac{k+q}{(k+q)^2 + \chi} \right)^2 \approx D_R \frac{C_R \alpha_s}{\pi} C_2(G) \alpha_s T \int \frac{dx}{x} \frac{d^2 k}{\pi} \frac{d^2 q}{\pi} \frac{\mu^2}{q^2(q^2 + \mu^2)} \left( \frac{k}{k^2 + \chi} - \frac{k+q}{(k+q)^2 + \chi} \right)^2,
\]
where we used \([t_a, t_c] [t_b, t_a] = C_2(G)C_R D_R\) (with \(C_2(G) = 3\), \(C_R = \frac{4}{3}\), and \(D_R = 3\)) and, in the second step, the soft 
rescattering approximation \(|k| \ll k_s \approx \omega\).

The interaction rate sums over all initial and final colors of the heavy quark. The heavy quark radiative energy 
loss per unit length is obtained from the above expression for the interaction rate by weighting it with the energy \(\omega\) of 
the emitted gluon and averaging over the initial color of the heavy quark [18, 22, 25]:

\[
\frac{dE_{\text{dyn}}}{dL} = \frac{1}{D_R} \int d\omega \omega \frac{d\Gamma(E)}{d\omega} \approx \frac{E}{D_R} \int dx \frac{d\Gamma(E)}{dx},
\]

(2.7)

This leads to

\[
\frac{\Delta E_{\text{dyn}}}{E} = \frac{C_R \alpha_s \pi}{\lambda_{\text{dyn}}} \int dx \frac{d^2 k}{\pi} \frac{d^2 q}{\pi} \frac{\mu^2}{q^2 (q^2 + \mu^2)} \left( \frac{k}{k^2 + \chi} - \frac{k+q}{(k+q)^2 + \chi} \right)^2
\]

\[
= \frac{C_R \alpha_s \pi}{\lambda_{\text{dyn}}} \int dx \frac{dk^2}{2 \pi} \frac{dq^2}{2 \pi} \frac{\mu^2}{q^2 (q^2 + \mu^2)} \frac{1}{k^2 + \chi}
\times \left[ \frac{q^2 + \chi}{\sqrt{\chi^2 + 2 \chi (k^2+q^2) + (k^2-q^2)^2}} - \frac{\chi}{k^2 + \chi} + \frac{q^2 \chi (\chi - 3k^2 + q^2)}{[\chi^2 + 2 \chi (k^2+q^2) + (k^2-q^2)^2]^2} \right],
\]

(2.9)

where the second step is obtained after angular integration and we defined a “dynamical mean free path” \(\lambda_{\text{dyn}}\) through

\[
\lambda_{\text{dyn}}^{-1} \equiv C_2(G) \alpha_s T = 3 \alpha_s T.
\]

(2.10)

Under the assumption that \(\alpha_s\) is not running, Eq. (2.9) can be further analytically integrated over \(0 \leq |k| \leq k_{\text{max}}\) 
where \(k_{\text{max}} = 2E \sqrt{1-x}\) [37]. We obtain

\[
\frac{\Delta E_{\text{dyn}}}{E} = \frac{C_R \alpha_s \pi}{\lambda_{\text{dyn}}} \int dx \frac{d^2 q}{\pi} J_{\text{dyn}}(q, x),
\]

(2.11)

where

\[
J_{\text{dyn}}(q, x) = \frac{\mu^2}{2 q^2 (q^2 + \mu^2)} \left[ -1 - \frac{2 k_{\text{max}}^2}{k_{\text{max}}^2 + \chi} + \frac{q^2 - k_{\text{max}}^2 + \chi}{\sqrt{\chi^4 + 2 q^2 (\chi - k_{\text{max}}^2) + (k_{\text{max}}^2 + \chi)^2}} \right]
\]

\[
+ \frac{2 (q^2 + 2 \chi)}{q^2 \sqrt{1 + \frac{4 \chi}{q^2}}} \ln \left( \frac{k_{\text{max}}^2 + \chi}{\chi} \right) \left( q^2 + 3 \chi + \sqrt{1 + \frac{4 \chi}{q^2}} \left( q^2 + \chi \right) \sqrt{q^4 + 2 q^2 (\chi - k_{\text{max}}^2) + (k_{\text{max}}^2 + \chi)^2} \right) \right],
\]

(2.12)

Alternatively, Eq. (2.9) can be integrated over \(0 \leq |q| \leq q_{\text{max}}\) where \(q_{\text{max}} = \sqrt{4ET}\) [14]. This leads to an analytical 
expression for radiated gluon spectrum \(\tilde{J}_{\text{dyn}}(k, x)\). We find

\[
\frac{\Delta E_{\text{dyn}}}{E} = \frac{C_R \alpha_s \pi}{\lambda_{\text{dyn}}} \int dx \frac{d^2 k}{\pi} \tilde{J}_{\text{dyn}}(k, x),
\]

(2.13)

where

\[
\tilde{J}_{\text{dyn}}(k, x) = \frac{2 \chi \mu^2}{(k^2 + \chi) G(\chi, k, -\mu^2)} \left( 1 - \frac{k^2 + \chi}{\sqrt{G(\chi, k, q_{\text{max}}^2)}} \right)
\]

\[
- \frac{\mu^2 (k^2+\mu^2-3 \chi)}{2 (k^2+\chi) G(\chi, k, -\mu^2)} \left( \frac{k^2-3 \chi}{k^2+\chi} - \frac{k^2-q_{\text{max}}^2-3 \chi}{k^2+\chi} \right)
\]

\[
- \left( \frac{\chi}{(k^2+\chi)^2} - \frac{2 \chi - \mu^2}{(k^2+\chi) \sqrt{G(\chi, k, -\mu^2) + \chi (k^2+\chi-\mu^2)^2}} \right) \ln \left( \frac{\mu^2}{q_{\text{max}}+\mu^2} \right)
\]

\[
- \frac{\chi}{(k^2+\chi)^2} \ln \left[ \frac{(k^2+\chi)^2 - (k^2-\chi) q_{\text{max}}^2 + (k^2+\chi) G(\chi, k, q_{\text{max}}^2)}{2 (k^2+\chi)^2} \right]
\]

\[
+ \frac{1}{\sqrt{G(\chi, k, -\mu^2)}} \left( \frac{2 \chi - \mu^2}{k^2+\chi} - \frac{\chi (k^2-\mu^2)}{G(\chi, k, -\mu^2)} \right)
\]

\[
\times \ln \left[ \frac{(k^2+\chi)^2 - (k^2+\mu^2-\chi) q_{\text{max}}^2 + (k^2-\chi) G(\chi, k, q_{\text{max}}^2) G(\chi, k, -\mu^2)}{(k^2+\chi)^2 + (k^2-\chi) G(\chi, k, -\mu^2)} \right],
\]

(2.14)
with

\[
G(\chi, \mathbf{k}, \kappa^2) \equiv (k^2 + \chi)^2 - 2\kappa^2(k^2 - \chi) + \kappa^4
\] (2.15)

It is worth noting that each of the three contributions in Eq. (2.5) diverges logarithmically in the limit of zero transverse momentum of the exchanged gluon, \(q \to 0\). The reason is that in a dynamical QCD medium both transverse and longitudinal gluon exchange contribute to the radiative energy loss. While Debye screening renders the longitudinal gluon exchange infrared finite, transverse gluon exchange causes a well-known logarithmic singularity [25, 40], due to the absence of a magnetic screening [41]. However, the infrared divergences cancel in the sum (2.4), giving rise to a finite energy loss rate. (This can be seen from Eq. (2.14), where analytical integration over \(q\) is performed.) This is a nontrivial result since it was previously believed that in a dynamical medium a magnetic cutoff must be artificially introduced in order to avoid divergent results [25, 40].

III. RADIATIVE ENERGY LOSS IN A STATIC QCD MEDIUM

Let us now briefly revisit for comparison the heavy quark radiative energy loss in a static QCD medium, using a derivation which parallels that of the previous section and thus clearly exhibits the differences between the two situations. We again consider an on-shell heavy quark produced in the remote past propagating through an infinite QCD medium that now consists of randomly distributed static scattering centers. The static interactions are modeled as Gyuulassy-Wang [33] color-screened Yukawa potentials

\[
V_n = \frac{2\pi}{\bar{q}_n \cdot \bar{x}_n} T_n(R) \otimes T_n(n)
\] (3.1)

here \(\bar{x}_n\) is the location of the \(n^{th}\) scattering center, the two \(T\) symbols (with \(a_n\) being summed over) denote the color matrices of the heavy quark and the \(n^{th}\) scattering center, and \(v(\bar{q}_n) \equiv 4\pi\alpha_s,/q_n^2 + \mu^2\).

The diagrams contributing to the radiative energy loss at first order in opacity are shown in Fig. 1. As seen in the figure, the quark scatters of one of the color centers with momentum \(q = (0, q_z, \mathbf{q})\) and radiates a gluon with momentum \(k = (\omega, k_z, \mathbf{k})\). As in the previous section, energy-momentum conservation requires \(p' = p + k + q\).

FIG. 1: Feynman diagrams \(M_1^{\text{stat}}, M_2^{\text{stat}}\) and \(M_3^{\text{stat}}\) contributing to the soft gluon radiation amplitude in a static medium to first order in opacity. The static color charge has color \(a\) and exchanges momentum \(q = (0, q_z, \mathbf{q})\) with the heavy quark. The radiated gluon has color \(c\) and carries momentum \(k = (\omega, k_z, \mathbf{k})\).

The procedure for the calculation of the diagrams shown in Fig. 1 was already presented in [37] (see in particular appendices A and B there), so we will not repeat it here. We obtain

\[
M_1^{\text{stat}}(p, k) \approx 4igE \int \frac{d^4q_1}{(2\pi)^4} V(q_1) e^{iq_1 \cdot x_1} \frac{p \cdot \epsilon}{(p + k)^2 - M^2} t_c t_{a_1},
\]

\[
M_2^{\text{stat}}(p, k) \approx 4igE \int \frac{d^4q_1}{(2\pi)^4} V(q_1) e^{iq_1 \cdot x_1} \frac{p' \cdot \epsilon}{(p' - k)^2 - M^2} t_{a_1} t_c,
\]

\[
M_3^{\text{stat}}(p, k) \approx 4igE \int \frac{d^4q_1}{(2\pi)^4} V(q_1) e^{iq_1 \cdot x_1} \frac{\epsilon \cdot (k + q_1)}{(k + q_1)^2 - m_g^2} \left[ t_c, t_{a_1}\right],
\] (3.2)

where \(\epsilon \equiv \epsilon(k) = \left(0, \frac{2\epsilon \cdot k}{k_z + \omega}, \epsilon\right)\) (with \(\epsilon = (1, 0)\) or \((0,1)\)) is the polarization vector of the emitted gluon. By using
Eqs. (3.1), (A14), and (C7), together with \( p \cdot \epsilon \approx p' \cdot \epsilon \approx \epsilon \cdot k/x \) [34, 37], Eq. (3.2) becomes

\[
M_1^{\text{stat}} \approx 4igE \int \frac{d^3q_1}{(2\pi)^3} v(q_1^a) e^{-i\epsilon q_1 \cdot \xi_1} \frac{\epsilon \cdot k}{k^2 + \chi} t_c t_a t_{a_1} T_{a_1},
\]

\[
M_2^{\text{stat}} \approx -4igE \int \frac{d^3q_1}{(2\pi)^3} v(q_1^a) e^{-i\epsilon q_1 \cdot \xi_1} \frac{\epsilon \cdot k}{k^2 + \chi} t_{a_1} t_c T_{a_1},
\]

\[
M_3^{\text{stat}} \approx -4igE \int \frac{d^3q_1}{(2\pi)^3} v(q_1^a) e^{-i\epsilon q_1 \cdot \xi_1} \frac{\epsilon \cdot (k+q)}{(k+q)^2 + \chi} [t_c, t_{a_1}] T_{a_1}. \tag{3.3}
\]

Therefore

\[
M_{\text{tot}}^{\text{stat}}(p, k) = M_1^{\text{stat}}(p, k) + M_2^{\text{stat}}(p, k) + M_3^{\text{stat}}(p, k)
\approx 4igE [t_c, t_{a_1}] T_{a_1} \int \frac{d^3q_1}{(2\pi)^3} v(q_1^a) e^{-i\epsilon q_1 \cdot \xi_1} \left[ \frac{\epsilon \cdot k}{k^2 + \chi} - \frac{\epsilon \cdot (k+q)}{(k+q)^2 + \chi} \right]. \tag{3.4}
\]

Squaring this and ensemble-averaging the result over the positions \( \xi_1 \) of the scattering centers gives

\[
\langle |M_{\text{tot}}^{\text{stat}}|^2 \rangle(p, k) \approx 16g^2E^2 [t_c, t_{a_1}] [t_{a_1}, t_c] T_{a_1} T_{a_2} \int \frac{d^3q_1}{(2\pi)^3} \frac{d^3q_2}{(2\pi)^3} \langle e^{-i(\xi_1 - \bar{\xi}_2) \cdot \epsilon_1} \rangle \langle v(q_1) \rangle \langle v(q_2) \rangle 
\times \sum_{\epsilon} \left( \frac{\epsilon \cdot k}{k^2 + \chi} - \frac{\epsilon \cdot (k+q)}{(k+q)^2 + \chi} \right) \left( \frac{\epsilon \cdot k}{k^2 + \chi} - \frac{\epsilon \cdot (k+q)}{(k+q)^2 + \chi} \right)
\approx 16g^2E^2 [t_c, t_{a_1}] [t_{a_1}, t_c] C_2(T) \frac{d_T}{d_g} \frac{1}{V} \int \frac{d^3q}{(2\pi)^3} |v(\bar{q})|^2 \left( \frac{k}{k^2 + \chi} - \frac{k+q}{(k+q)^2 + \chi} \right)^2, \tag{3.5}
\]

In the last step we used [34] \( \text{Tr}(T_{a_1} T_{a_2}) = \delta_{a_1 a_2} C_2(T) d_T/d_g \) (we assume that all scattering centers (“target partons”) have the same \( d_T \)-dimensional color representation with Casimir \( C_2(T) \)). Furthermore, we assumed (similarly to [34]) that the ensemble average over the phase factor gives

\[
\langle e^{-i(\xi_1 - \bar{\xi}_2) \cdot \epsilon_1} \rangle = \frac{(2\pi)^3}{V} \delta^{(3)}(\xi_1 - \bar{\xi}_2), \tag{3.6}
\]

where \( V = LA_1 \), with \( L \) and \( A_1 \) being the depth (along the heavy quark’s path) and transverse area of the medium.

We can now compute the interaction rate for the heavy quark in a static medium of scatterers with color representation \( T \):

\[
\Gamma_{\text{stat}}^T(E) = \int \frac{d^3p}{(2\pi)^32E} \frac{d^3k}{(2\pi)^32\omega} \left( 2\pi \right)^4 \delta^{(4)}(p' - p - k - q) \frac{1}{2E} |\langle M_{\text{tot}}^{\text{stat}}|^2 \rangle(p, k)|
\approx \int \frac{d^3k}{(2\pi)^32\omega} 2\pi \delta(q_z - \frac{k^2 + M^2x^2 + m_q^2}{2x E}) \frac{1}{4E^2} |\langle M_{\text{tot}}^{\text{stat}}|^2 \rangle|
\approx 4\alpha_s[t_c, t_{a_1}] [t_{a_1}, t_c] C_2(T) \frac{d_T}{d_g} \frac{1}{V} \int \frac{dx}{x} \frac{d^2k}{(2\pi)^2} \frac{d^2q}{(2\pi)^2} |v(0, q)|^2 \left( \frac{k}{k^2 + \chi} - \frac{k+q}{(k+q)^2 + \chi} \right)^2
\approx D_R d_T \frac{C_R \alpha_s}{\pi} \frac{1}{\lambda_T} \int \frac{dx}{x} \frac{d^2k}{\pi} \frac{d^2q}{\pi} \frac{\mu^2}{g^2 + \mu^2} \left( \frac{k}{k^2 + \chi} - \frac{k+q}{(k+q)^2 + \chi} \right)^2, \tag{3.7}
\]

where we defined the mean free path for a heavy quark scattering off quark-type (”q”) or gluon-type (”g”) scattering centers through [15, 33, 34]

\[
\frac{1}{\lambda_T} = \frac{3}{8} C_2(T) \frac{1}{V} \int \frac{d^2q}{(2\pi)^2} |v(0, q)|^2 \quad \Rightarrow \quad \frac{1}{\lambda_T} = \frac{2}{3} \cdot 2\pi \frac{g^2}{\pi^2} \cdot \rho_g = \frac{2\pi^2 g^2}{2} \cdot \frac{1.205}{\pi^2} \cdot 16T^3
\]

\[
\frac{1}{\lambda_T} = \frac{2\pi \alpha_s^2}{\pi^2} \cdot \rho_q = \frac{2\pi \alpha_s^2}{2} \cdot \frac{1.205}{\pi^2} \cdot 9n_f T^3. \tag{3.8}
\]

After averaging over the initial colors of the heavy quark and the scattering centers and weighting the rate with the energy \( \omega \) of the emitted gluon, the heavy quark energy loss in an infinite static QCD medium is then given by

\[
\frac{dE_{\text{stat}}}{dL} = \frac{1}{D_R} \int d\omega \omega \left( \frac{1}{d_g} \frac{d\Gamma_{\text{stat}}^T(E)}{d\omega} + \frac{1}{d_g} \frac{d\Gamma_{\text{stat}}^T(E)}{d\omega} \right). \tag{3.9}
\]
This leads to

$$\frac{\Delta E_{\text{stat}}}{E} = \frac{C_R \alpha_s}{\pi} \frac{L}{\lambda_{\text{stat}}} \int dx \frac{d^2 k}{\pi} \frac{d^2 q}{\pi} \frac{\mu^2}{(q^2 + \mu^2)^2} \left( \frac{k}{k^2 + \chi} - \frac{k + q}{(k + q)^2 + \chi} \right)^2,$$  \hspace{1cm} (3.10)

with

$$\frac{1}{\lambda_{\text{stat}}} = \frac{1}{\lambda_q} + \frac{1}{\lambda_g} = \frac{6}{\pi^2} \frac{1 + n_f/6}{1 + n_f/4} \alpha_s T = c(n_f) \frac{1}{\lambda_{\text{dyn}}},$$  \hspace{1cm} (3.11)

where $c(n_f) \equiv \frac{6\sqrt{\pi} + n_f/4}{\pi^2 + n_f/6}$ is a slowly increasing function of $n_f$ that varies between $c(0) \approx 0.73$ and $c(\infty) \approx 1.09$. For a typical value $n_f = 2.5$ (which we use in this paper), $c(2.5) \approx 0.84 \simeq 1$.

As in the previous section, under the assumption that $\alpha_s$ is not running, Eq. (3.10) can be analytically integrated over $k$ or $q$. Integration over $k$ yields

$$\frac{\Delta E_{\text{stat}}}{E} = \frac{C_R \alpha_s}{\pi} \frac{L}{\lambda_{\text{stat}}} \int dx \frac{d^2 q}{\pi} J_{\text{stat}}(q, x),$$

with the simple relationship

$$J_{\text{stat}}(q, x) = J_{\text{dyn}}(q, x) \frac{q^2}{q^2 + \mu^2} \frac{\mu^2}{\lambda_g},$$  \hspace{1cm} (3.13)

where $J_{\text{dyn}}(q)$ is given by Eq. (2.12). Integrating Eq. (3.10) instead over $q$ gives

$$\frac{\Delta E_{\text{stat}}}{E} = \frac{C_R \alpha_s}{\pi^2} \frac{L}{\lambda_{\text{stat}}} \int dx d^2 k J_{\text{stat}}(k, x),$$

where

$$J_{\text{stat}}(k, x) = -\frac{\chi q_{\text{max}}^2}{(q_{\text{max}}^2 + \mu^2)(k^2 + \chi)^2} + \frac{2\chi - 3\mu^2}{2G(\chi, k, -\mu^2)} + \frac{3\chi\mu^2(3k^2 + \mu^2 - \chi)}{2G(\chi, k, -\mu^2)^2}$$

$$+ \frac{\mu^2\sqrt{\mathcal{G}(\chi, k, q_{\text{max}})}}{(q_{\text{max}}^2 + \mu^2) \mathcal{G}(\chi, k, -\mu^2)} \left( \frac{\mu^2 - 2\chi}{k^2 + \chi} + \frac{3\chi(k^2 + \chi - \mu^2)}{\mathcal{G}(\chi, k, -\mu^2)} - 2\chi(k^2 + \chi + q_{\text{max}}^2) / \mathcal{G}(\chi, k, q_{\text{max}}) \right)$$

$$+ \frac{\mu^2(k^2 + \chi - q_{\text{max}}^2)}{2\mathcal{G}(\chi, k, -\mu^2) \sqrt{\mathcal{G}(\chi, k, q_{\text{max}})}}$$

$$+ \frac{\mu^2}{\mathcal{G}(\chi, k, -\mu^2)^2} \left( -\frac{4\chi k^2}{k^2 + \chi} + 2(k^2 + \chi + \mu^2) \left( \frac{6\chi k^2}{\mathcal{G}(\chi, k, -\mu^2)} - 1 \right) \right)$$

$$\times \ln \left[ \frac{\mu^2}{q_{\text{max}}^2 + \mu^2} \left( \frac{k^2 + \chi^2}{(k^2 + \chi)^2} - \frac{(k^2 + \mu^2 - \chi)q_{\text{max}}^2}{(k^2 + \mu^2 - \chi)^2} + \frac{(k^2 - \chi)k^2}{(k^2 - \chi)k^2 + (k^2 + \chi)^2 + (k^2 + \chi)^2} \sqrt{\mathcal{G}(\chi, k, -\mu^2)} \right) \right],$$

with $\mathcal{G}$ given by Eq. (2.15). Note that the emitted gluon spectra, Eq. (2.14) for $J_{\text{dyn}}(k, x)$ and Eq. (3.15) for $J_{\text{stat}}(k, x)$, while showing some similarities, don’t exhibit a similarly simple analytical relationship as was the case for $J_{\text{dyn}}(q, x)$ and $J_{\text{stat}}(q, x)$ (see Eq. (3.13)).

Finally, we can compare the radiative energy loss rates to first order in opacity in dynamic and static QCD media, Eqs. (2.8) and (3.10). Both expressions yield an energy loss that increases linearly with the path length $L$ through the medium. This reflects our neglect of LPM interference effects [4, 5] – our result corresponds to the Bethe-Heitler limit. In spite of this and many other similarities between Eqs. (2.8) and (3.10), they feature two important differences: The first is an $O(15\%)$ decrease in the mean free path

$$\lambda_{\text{dyn}} \Leftrightarrow \lambda_{\text{stat}} = \lambda_{\text{dyn}} / c(n_f),$$

which increases the energy loss rate in the dynamical medium by $O(20\%)$. The second difference is a change in the shape and normalization of the emitted gluon spectrum, which in the energy loss rate is reflected by the replacement

$$\left[ \frac{\mu^2}{(q^2 + \mu^2)^2} \right]_{\text{dyn}} \Leftrightarrow \left[ \frac{\mu^2}{(q^2 + \mu^2)^2} \right]_{\text{stat}}.$$

As we will see in the next section, this second difference leads to an additional significant increase of the heavy quark energy loss rate and of the emitted gluon radiation spectrum by about 50% for the dynamical QCD medium.
IV. NUMERICAL RESULTS

In this section we present numerical results for the total radiative energy loss, as well as its differential rate with respect to the energy fraction and transverse momentum carried by the exchanged and emitted gluons, to first order in opacity, using the expressions for infinite QCD media derived in the two preceding sections. Specifically, we discuss a static quark-gluon plasma of temperature $T = 225$ MeV, with $n_f = 2.5$ effective light quark flavors and strong interaction strength $\alpha_s = 0.3$, as representative of average conditions encountered in Au+Au collisions at RHIC. Further below we will raise the temperature of the medium to $T = 400$ MeV to simulate (average) conditions in Pb+Pb collisions at the LHC. For the light quarks we assume that their mass is dominated by the thermal mass $\mu$, while for the bottom mass we use $M = 4.75$ GeV. As noted before, the radiative energy loss in the Bethe-Heitler limit considered here increases linearly with the path length $L$ traveled by the fast quark – for normalization purposes we will set this path length to a standard value of $L = 5$ fm throughout.

![Graphs showing fractional radiative energy loss](image)

**FIG. 2:** Fractional radiative energy loss for an assumed path length $L = 5$ fm as a function of momentum for light, charm and bottom quarks (left, center, and right panels, respectively). Full and dashed curves correspond to the energy loss in a dynamical and a static QCD medium, respectively, and are obtained from Eqs. (2.8) and Eq. (3.10).

In Figure 2 we compare the momentum dependence of the radiative energy loss over an assumed path length $L = 5$ fm for a dynamical and a static QCD medium (as given by Eqs. (2.8) and (3.10)). For all three types of quarks, the dynamical medium is seen to cause almost 70% higher energy loss than the static medium. The left panel in Fig. 4 below shows that $\sim 50\%$ of this increase arises from the larger values of the function $\tilde{J}_{\alpha}(k, x)$ which describes the shape of the emitted gluon spectrum, with an additional $\sim 20\%$ increase stemming from the shorter mean free path in the dynamical medium.

To better understand the kinematic distribution of the dynamical medium effects we will now study the energy loss differentially as a function of the gluon energy fraction $x = \omega/E$ and the transverse momenta $k$ and $q$ of the emitted and exchanged gluons. We define the double-differential energy loss spectra

$$S(q, x) = \frac{1}{E} \frac{d(E)}{dx \, dq} = \frac{C_R \alpha_s}{\pi^2} \frac{L}{\lambda} \tilde{J}(q, x),$$

and their single-differential analogues

$$S(q) = \frac{1}{E} \frac{d(E)}{dq} = \frac{C_R \alpha_s}{\pi^2} \frac{L}{\lambda} \int dx \tilde{J}(q, x),$$

$$S(k) = \frac{1}{E} \frac{d(E)}{dk} = \frac{C_R \alpha_s}{\pi^2} \frac{L}{\lambda} \int dx \tilde{J}(k, x),$$

$$S(x) = \frac{1}{E} \frac{d(E)}{dx} = \frac{C_R \alpha_s}{\pi^2} \frac{L}{\lambda} \int dq \tilde{J}(q, x) = \frac{C_R \alpha_s}{\pi^2} \frac{L}{\lambda} \int d^2 k \tilde{J}(k, x).$$

In Figure 3 we plot the transverse momentum dependence of the exchanged ($S(q)$, left panel) and emitted gluon spectrum ($S(k)$, right panel). The largest differences between static (dashed) and dynamical media (solid) are observed at low transverse momenta $|q|, |k| \lesssim 1$ GeV, especially for the exchanged momentum spectrum $S(q)$. The latter is
seen to exhibit qualitatively different low-$q$ behavior in static and dynamical media, but the difference is seen to rapidly disappear at $|q| > 1$ GeV (middle panel in Fig. 4), as expected from Eq. (3.17).

While the dynamical medium effects on the exchanged gluon spectrum are mostly concentrated at low transverse momenta, the right panels in Fig. 3 and (more clearly) Fig. 4 show that the effect on the emitted gluon spectrum extends over the entire transverse momentum region, causing enhancements by more than a factor 2.5 at low $|k|$, and settling down at high $|k| \gtrsim 1$ GeV to an approximately $|k|$-independent enhancement by a factor $\sim 1.3$.

The $k$-integrated effect on the radiative charm energy loss is shown in the left panel of Fig. 4. The energy loss ratio between dynamical and static media is almost independent of the momentum $p$ of the fast charm quark, saturating at $\sim 1.7$ above $p \gtrsim 20$ GeV and being even somewhat larger at smaller momenta. (This includes the $\sim 20\%$ effect arising from the shorter mean free path in the dynamical medium.) We checked that the dynamical enhancement persists at constant level to the largest possible charm quark energies. This is shown explicitly in the left panel of Fig. 5 where we plot the charm quark energy loss ratio for an otherwise identical medium of higher temperature $T = 400$ MeV (“LHC conditions”). There is no quark energy domain where the assumption of static scatterers in the medium becomes a valid approximation.

The mass of the fast quark plays only a minor role for its energy loss. The right panel in Fig. 5 shows (for LHC conditions, but similar statements apply at lower medium temperature) the asymptotic energy loss ratio for very high energy quarks as a function of the quark mass. While the dynamical enhancement is largest for light quarks, the
difference between light and bottom quarks is only about 15%, and $b$ quarks still suffer about 70% more energy loss in a dynamical medium than in one with static scattering centers.

Stronger quark mass effects are seen in the shapes of the exchanged and emitted gluon spectra themselves. Figure 6 shows the fractional energy loss as a function of the energy fraction $x = \omega/E$ of the radiated gluon relative to the initial quark energy. For both static and dynamic media the fractional energy loss for bottom quarks (i.e. the emitted gluon energy spectrum) is seen to be significantly softer for bottom than for charm quarks. No such strong mass effect is visible in the shapes of the transverse momentum spectra of exchanged and emitted gluons: Figs. 7 and 8 show that, at fixed $x$, the exchanged and emitted gluon transverse momentum spectra have very similar shapes for charm and bottom quarks, and that the main difference shows up in the $x$-dependence of these spectra. This strong quark mass effect on the emitted gluon energy spectrum is a consequence of the well known “dead-cone effect” [37, 43].

Except for very low $x$-values, the differential energy loss $S(x)$ is a decreasing function of $\chi = M^2 x^2 + m_g^2$, and $\chi$ grows significantly faster with $x$ for bottom than for charm quarks. Thus, as $x$ increases, the contribution to the energy loss decreases more rapidly for bottom than for charm quarks. This is borne out by Fig. 6.

In the present study, large-$x$ contributions to the total energy loss are not strongly suppressed, and even for bottom quarks the contribution from $x$ regions where the soft gluon approximation $\omega \ll E$ becomes doubtful could be as large as 30%. This is mainly a deficiency of our approximations – for a medium of static scatterers it is known that the LPM effect strongly suppresses the emission of large-$x$ gluons (see Eq. (11) in the second paper of Ref. [37]). Including
such effects should improve the applicability of our approximations, by reducing large-$x$ gluon emission. That is, the
total radiative energy loss should be reduced, without qualitatively affecting the energy loss ratio between static and dynamical media (since this ratio is seen in Fig. 6 to be largely independent of $x$).

V. CONCLUSION

Static scattering center approximation was used in all previous calculations of heavy quark radiative energy loss. An important consequence of this approximation is that it results in exactly zero collisional energy loss. However, it was recently found [11–14] that, under RHIC conditions, heavy quark collisional energy loss is significant and comparable to the previously calculated radiative energy loss. Since the static approximation is evidently inadequate in the computation of collisional energy loss, there arises a question whether such approximation is appropriate in the radiative energy loss case.

We here revisited the problem of heavy quark radiative energy loss, but now in dynamical medium of thermally distributed massless quarks and gluons. Our work has two goals: 1) To address the applicability of static approximation in radiative energy loss computations, and 2) To compute collisional and radiative energy losses within a consistent theoretical framework. In this paper we report the first step in this direction, where we compute the 1st order in opacity contribution to the radiative energy loss in a dynamical QCD medium of infinite size.

We have shown that each individual contribution in the diagrammatic expansion of the energy loss in a dynamical medium is infrared divergent, due to the absence of magnetic screening [25]. However, it is interesting that the sum of these contributions lead to an infrared safe result. The magnetic infrared divergence is thus naturally regulated, eliminating the need for introducing an artificial magnetic gluon mass when computing the radiative energy loss in a dynamical QCD medium.

The analytic expression for the radiative energy loss in a dynamical QCD medium was found to be remarkably similar to the one obtained in the static approximation. Still, the seemingly small differences, observed in the analytical expressions, were found to have important quantitative consequences: At the same (first) order in opacity, fast quarks that propagate through a dynamical QCD medium lose energy at almost twice the rate computed for a medium of static scatterers. Recoil of the (massless) quarks and gluons in the medium is thus a phenomenologically important effect, which can not be neglected. We found no corners of the kinematic phase-space where the static scattering approximation is valid, neither for light nor for heavy fast quarks. Hence, the constituents of QCD medium can not be approximated as static scattering centers, and for reliable predictions of radiative energy loss, dynamical effects have to be included.

High precision heavy flavor measurements are expected to emerge from the upcoming high luminosity RHIC and LHC experiments. An important goal of heavy flavor energy loss measurements is to provide a tomographic diagnostic tool for the hot QCD matter created in these collisions. Therefore, reliable quantitative predictions for these experiments are essential. The results presented in this paper lead to the important qualitative conclusion that the observed quark energy loss could be significantly larger than previously thought. Turning this qualitative insight into a quantitative comparison with existing data, and reliable predictions for upcoming data requires, however, a significant additional work. Most importantly, the present study does not take into account coherent interference (LPM) effects and their modification by the finite size of the medium created in heavy-ion collision fireballs. The computation of heavy quark energy loss in a finite size dynamical QCD medium is therefore our next important goal.

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APPENDIX A: ASSUMPTIONS AND APPROXIMATIONS

1. Kinematics

In this paper we consider a heavy quark of mass $M$ which is produced in the remote past on its mass shell, with large spatial momentum $p' \gg M$. We choose coordinates such that the momentum of the initial quark is along the $z$
functions defined by emitted gluon can be simply approximated by \[23\] expressions for the transverse and longitudinal self energies \(\Pi^{\text{ex}}\) exchanges (in arbitrary sequence) one virtual gluon with space-like momentum

\[ q = (q_0, \mathbf{q}) = (q_0, q_z, \mathbf{q}), \quad q_0 \leq |\mathbf{q}| \]

with a parton in the medium and radiates one (medium-modified) real gluon with time-like momentum

\[ k = (k_0, \mathbf{k}) = (\omega, k_z, \mathbf{k}), \quad k_0 \geq |\mathbf{k}| \]

into the medium. The quark emerges with 4-momentum \(p'\).

We are interested in the radiative energy loss to first order in the opacity, so we study the case in which the quark exchanges (in arbitrary sequence) one virtual gluon with space-like momentum

\[ q = (q_0, \mathbf{q}) = (q_0, q_z, \mathbf{q}), \quad q_0 \leq |\mathbf{q}| \]

with a parton in the medium and radiates one (medium-modified) real gluon with time-like momentum

\[ k = (k_0, \mathbf{k}) = (\omega, k_z, \mathbf{k}), \quad k_0 \geq |\mathbf{k}| \]

into the medium. The quark emerges with 4-momentum \(p'\).

For the computation of the Feynmann diagrams given in Appendices B-D we will need cut propagators for the heavy quark \(p\) \((D^>(p))\), the radiated gluon \(k\) \((D^>(k))\), and the exchanged gluon \(q\) \((D^>(q))\).

The effective 1-HTL gluon propagators for the exchanged and emitted gluons have the form given in Eq. (2.1). By following the procedure outlined in [25], we obtain for the cut full gluon propagator

\[ D^>(\mu\nu)(l) = -(1 + f(l_0)) \left( P_{\mu\nu}(l) \rho_T(l) + Q_{\mu\nu}(l) \rho_L(l) \right), \]

where \(l\) is gluon momentum, \(f(l_0) = (e^{l_0/T} - 1)^{-1}\), and \(T\) is the temperature of the medium. \(\rho_{L,T}(l)\) are spectral functions defined by

\[ \rho_{L,T}(l) = 2\pi \delta(l^2 - \Pi_{T,L}(l)) - 2 \text{Im} \left( \frac{1}{l^2 - \Pi_{T,L}(l)} \right) \theta(1 - \frac{l^2}{f^2}). \]

It was shown in [23] that for the radiated gluon with momentum \(k\) the longitudinal contribution can be neglected relative to the transverse one, and that for the exchanged gluon the self energy \(\Pi_T(k)\) can be approximated by \(m_g^2\), where \(m_g \approx \mu/\sqrt{2} = gT \sqrt{(6 + n_f)/12}\) is the asymptotic mass. These approximations are true in the soft rescattering limit \(\omega \gg |\mathbf{q}| \sim |\mathbf{k}| \sim gT\) which we use in this paper. With these approximations the HTL gluon propagator for the emitted gluon can be simply approximated by [23]

\[ D_{\mu\nu}(k) \approx -i \frac{P_{\mu\nu}(k)}{k^2 - m_g^2 + i\epsilon}, \]

where \(P_{\mu\nu}\) is the transverse projector. The cut propagator for the radiated gluon is then given by [23, 25]

\[ D^>(\mu\nu)(k) \approx -2\pi(1 + f(\omega)) \frac{P_{\mu\nu}(k)}{2\omega} \delta(k_0 - \omega), \]

where \(\omega \approx \sqrt{k^2 + m_g^2}\).

By using Eqs. (A11, A13) defined below, we obtain \(f(\omega) = (e^{x_0/E} - 1)^{-1} \ll 1\) for highly energetic jets and \(x > T/E\). Eq. (A7) can then be simplified to

\[ D^>(\mu\nu)(k) \approx -2\pi \frac{P_{\mu\nu}(k)}{2\omega} \delta(k_0 - \omega). \]

Similarly, the cut propagator for the heavy quark (with \(D(p) = \frac{i}{p^2 - M^2 + i\epsilon}\)) is given by

\[ D^>(p) = 2\pi \frac{1}{2E} \delta(p_0 - E). \]

Unfortunately, the above approximations cannot be used for the virtual gluon mediating the collisional interaction. Both transverse and longitudinal contributions have to be kept in the gluon propagator \(D(q)\), and it can be shown numerically that both contributions are equally important. Furthermore, no further simplifications can be made in the expressions for the transverse and longitudinal self energies \(\Pi_T(q)\) and \(\Pi_L(q)\) (see Eq. (2.2)), beyond the restriction
energy loss using our approximation in the region $|q| \approx |k| \ll k_z$) and the full curve in the right panel of Fig. 9. We additionally tested that the error made by computing the contribution to the energy loss comes from the difference between longitudinal and transverse integrands (see Eqs. (B11) and (B12)). Together with conservation of energy and momentum $(p' = p + k + q)$ they yield

$$\begin{align*}
k &= (\omega \approx k_z + \frac{k^2 + m^2}{2k_z}, k_z, k), \\
p &= (E \approx p_z + \frac{(k+q)^2 + M^2}{2p_z}, p_z, -(k+q)),
\end{align*}$$

and

$$p' = \left( E' \approx p_z + k_z + q_z + \frac{M^2}{2(p_z + k_z + q_z)}, p_z + k_z + q_z, 0 \right).$$

In the next subsection we will show that it is reasonable to assume that $q_z$ has the same order of magnitude as $|q|$. Since $|k| \ll k_z$ and $q_z \sim |q| \sim |k|$, we then also have $q_z \ll k_z$. Thus $k_z + q_z \approx k_z$ and $p_z + k_z + q_z \approx p_z + k_z \approx p_z + q_z \approx p_z$. Defining

$$x \equiv \frac{k_z}{p_z},$$

we can further rewrite

$$(p + k)^2 - M^2 \approx \frac{k^2 + M^2 x^2 + m^2}{x} \approx M^2 - (p' - k)^2$$

and show that

$$p'' P_{\mu \nu}(k)p'' \approx p'' P_{\mu \nu}(k)p'' = p'' P_{\mu \nu}(k)p'' \approx p'' P_{\mu \nu}(k)p'' \approx - \frac{k^2}{x^2},$$

where $P_{\mu \nu}(k)$ is a transverse projector of radiated gluon, defined by Eq. (2.3).

Finally, by using Eqs. (A2) and (A11)–(A12), we obtain

$$E' - E - \omega - q_0 \approx q_z - q_0 - \frac{k^2 + M^2 x^2 + m^2}{2x E} \approx q_z - q_0.\quad (A16)$$

2. $q_z$ vs. $|q|$ comparison

Equation (A16) together with energy conservation implies $q_0 \sim q_z$. Introducing the variable $y = \frac{q_0}{\sqrt{q_z^2 + q^2}}$ (with $-1 \leq y \leq 1$), we can further express $q_z$ in terms of $q^2$:

$$q_z^2 = \frac{q^2}{1 - y^2}.\quad (A17)$$

The left panel in Fig. 9 shows the ratio $q_z/q_L = q_z/\sqrt{q^2}$ over the entire $y$ range. We see that, except for the region $y \rightarrow 1$, $q_z$ and $|q|$ are comparable, and that for $|y| < 0.95$ the ratio $\frac{q_z}{|q|}$ remains below 3. Hence, for $|y| < 0.95$, $q_z$ and $|q|$ have the same order of magnitude.

We next want to test how important the region $|y| > 0.95$ is for the energy loss. To do this, we start from the following equation:

$$F_{T,L}(y) = \frac{1}{2\pi y} \frac{1}{y} \left( \frac{2 \text{Im} \Pi_{T,L}(y)}{q^2 + \text{Re} \Pi_{T,L}(y)} \right) \left( \frac{\text{Im} \Pi_{T,L}(y)}{2} \right)^2,\quad (A18)$$

which gives $y$-integrand of the transverse and longitudinal contributions to the energy loss (see Eqs. (B11) and (B12)). By using Eq. (A18) we obtain $F_{T,L}(y)$ for a typical transverse momentum $|q| = 0.5 \text{ GeV}$ of the exchanged gluon, which is shown in the right panel of Fig. 9.

We see that the main contribution to the energy loss comes from the region $|y| < 0.95$, especially after accounting that the contribution to the energy loss comes from the difference between longitudinal and transverse integrands (see Eq. (B11) and the full curve in the right panel of Fig. 9). We additionally tested that the error made by computing the energy loss using our approximation in the region $|y| > 0.95$ (where the approximation breaks down) is less than 2%.
FIG. 9: Left: The ratio $q_z/q_\perp \equiv q_z/|q_\perp|$ as a function of $y$. Right: Transverse (dot-dashed) and longitudinal (dashed) contributions to the energy loss from virtual gluons with typical transverse momenta $|q| = 0.5$ GeV, as functions of $y$. Full curve shows the difference between longitudinal and transverse contributions (see Eq. (B11)). The vertical arrow indicates the $y$-region above which $q_z$ exceeds $|q|$ by more than a factor 3.

APPENDIX B: COMPUTATION OF DIAGRAMS $M_{1,0,1} - M_{1,0,4}$

In this appendix we present in some detail the calculation of the diagrams shown in Fig. 10. These diagrams present contributions where both ends of the exchanged gluon $q$ are attached to the heavy quark, i.e. none is attached to the radiated gluon $k$ and no 3-gluon vertex is involved.

Here and later the diagrams are labeled as follows: In $M_{1,i,j}$, $1$ denotes that these diagrams contribute to the energy loss to first order in opacity; $i$ denotes how many ends of the virtual gluon $q$ are attached to the radiated gluon $k$; and $j$ labels the specific diagram in that class.

FIG. 10: Feynman diagrams $M_{1,0,1}$, $M_{1,0,2}$, $M_{1,0,3}$ and $M_{1,0,4}$ contributing to the radiative energy loss to first order in opacity. The large dashed circles (“blob”) represent effective HTL gluon propagators [23]. A cut gluon propagator with momentum $k$ and color $c$ corresponds to the radiated gluon ($\omega > |\vec{k}|$). A cut gluon propagator with momentum $q$ and color $a$ corresponds to a collisional interaction with a parton in the medium ($q_0 \leq |q|$).
1. We will first calculate the cut diagram \( M_{1,0,1}^{\gamma} = 2 \text{Im} M_{1,0,1} \) [25]:

\[
M_{1,0,1}^{\gamma} = \int (\text{Im} (p' - q)^\mu) D_{\rho \sigma}(q)(i g(2p' - q)^\nu) \frac{i}{(p+k)^2 - M^2 + i\epsilon} \frac{-i}{(p+k)^2 - M^2 - i\epsilon} (-i g(2p+k)^\rho) D_{\rho \sigma}^\gamma(k)(i g(2p+k)^\sigma) \\
\times D^\gamma(p) t_a t_c t_a (2\pi)^4 \delta^{(4)}(p' - p - k - q) \frac{d^4 p}{(2\pi)^4} \frac{d^4 q}{(2\pi)^4} \frac{d^4 k}{(2\pi)^4},
\]

where \( D_{\rho \sigma}^\gamma(k) \), \( D^\gamma(p) \) and \( D_{\rho \sigma}(q) \) (given by Eqs. (A8), (A9) and (A10)) are the cut propagators for the radiated gluon, the heavy quark, and the exchanged gluon, respectively. \( M_{1,0,1}^{\gamma} \) then becomes

\[
M_{1,0,1}^{\gamma} = \int g^4 \frac{1}{(p+k)^2 - M^2 + i\epsilon} \frac{1}{(p+k)^2 - M^2 - i\epsilon} (2p+k)^\rho P_{\rho \sigma}(2p+k)^\sigma (-2\pi) \frac{\delta(k_0 - \omega)}{2\omega} \\
\times \theta(1 - \frac{q_0^2}{q^2}) (1 + f(q_0)) (2p' - q)^\mu \text{Im} \left( \frac{P_{\rho \sigma}(q)}{q^2 - \Pi_T(q)} + \frac{Q_{\rho \sigma}(q)}{q^2 - \Pi_L(q)} \right) (2p' - q)^\nu \\
\times t_a t_c t_a (2\pi) \frac{\delta(\mu_0 - k_0 - q_0)}{2\pi} \frac{d^4 q}{(2\pi)^4} \frac{d^4 k}{(2\pi)^4},
\]

where we have used three of the \( \delta \)-functions to do the integral over \( d^3 p \). Correspondingly, it should be kept in mind that in (B2) spatial components of \( p \) should be replaced by the corresponding components of \( p' - k - q \) which can be then further simplified with the approximations discussed in Appendix A.1. The same will be understood when evaluating the other three diagrams further below.

From Eq. (A15) and the fact that \( k^\rho P_{\rho \sigma}(k) = 0 \) we obtain

\[
(2p+k)^\rho P_{\rho \sigma}(k)(2p+k)^\sigma \approx -4 \frac{k^2}{q^2}.
\]

For highly energetic jets, and by using Eqs. (2.3) and (A16), we obtain \( (2p' - q)^\mu P_{\rho \sigma}(q)(2p' - q)^\nu \approx -(2p' - q)^\mu Q_{\rho \sigma}(q)(2p' - q)^\nu \approx -4E'^2 q^2 / q^2 \), which leads to

\[
(2p' - q)^\mu \text{Im} \left( \frac{P_{\rho \sigma}(q)}{q^2 - \Pi_T(q)} + \frac{Q_{\rho \sigma}(q)}{q^2 - \Pi_L(q)} \right) (2p' - q)^\nu \approx 4E'^2 \frac{q^2}{q^2} \text{Im} \left( \frac{1}{q^2 - \Pi_L(q)} - \frac{1}{q^2 - \Pi_T(q)} \right)
\]

By also using Eqs. (B3) and (B4), and after performing integrations over \( p_0 \) and \( k_0 \), Eq. (B2) reduces to

\[
M_{1,0,1}^{\gamma} = g^4 t_a t_c t_a \int \frac{1}{(p+k)^2 - M^2} \frac{4k^2}{x^2} \theta(1 - \frac{q_0^2}{q^2}) (4E'^2) \frac{q^2}{q^2} (1 + f(q_0)) 2 \text{Im} \left( \frac{1}{q^2 - \Pi_L(q)} - \frac{1}{q^2 - \Pi_T(q)} \right) \\
\times \frac{1}{2E} \frac{d^4 q}{(2\pi)^4} \frac{d^4 k}{(2\pi)^4} \frac{d^4 \mu_0}{(2\pi)^4}.
\]

We now use Eqs. (A14), (A16), as well as \( \omega = \sqrt{k^2 + m^2} \approx x E \) and \( 4E'^2 / 2E \approx 2E \), to obtain:

\[
M_{1,0,1}^{\gamma} = 8E g^4 t_a t_c t_a \int \frac{d^4 k}{(2\pi)^4} \frac{2}{2\omega} \frac{k^2}{(k^2 + M^2 + x^2 m^2)^2} \int \frac{d^4 q}{(2\pi)^4} \frac{2\mu_0}{2\omega} \frac{2}{(k^2 - \Pi_L(q))^2} \frac{2\text{Im} \Pi_T(q)}{(q^2 - \Pi_T(q))^2 + (\text{Im} \Pi_T(q))^2},
\]

where \( f(q_0) = (e^{q_0/T} - 1)^{-1} \) and \( T \) is the temperature of the medium. For small \( q_0 \) we can expand

\[
1 + f(q_0) \approx \frac{T}{q_0} + \frac{1}{2} + O \left( \frac{q_0}{T} \right).
\]

With this approximation Eq. (B6) becomes

\[
M_{1,0,1}^{\gamma} = 2E g^4 t_a t_c t_a \int \frac{d^4 k}{(2\pi)^4} \frac{4k^2}{(k^2 + M^2 + m^2)^2} L_q
\]
where \( I_q \) is given by

\[
I_q = \int \frac{d^4q}{(2\pi)^4} 2\pi \delta(q_0-q_z) \frac{q^2}{q_z^2} \left( \frac{1}{2} + \frac{T}{q_0} \right) \frac{2 \text{Im} \Pi_L(q)}{(q^2+\text{Re} \Pi_L(q))^2 + (\text{Im} \Pi_L(q))^2} - \frac{2 \text{Im} \Pi_T(q)}{(q^2-\text{Re} \Pi_T(q))^2 + (\text{Im} \Pi_T(q))^2}.
\]

With the help of Eq. (A17) and noting that \( 2 \text{Im} \Pi_{T,L} \) depend only on \( y \) and that \( 2 \text{Im} \Pi_{T,L}(y)/([q^2+\text{Re} \Pi_{T,L}(y)]^2 + (\text{Im} \Pi_{T,L}(y))^2) \) is an odd function of this variable, we can rewrite Eq. (B9) as

\[
I_q = T \int \frac{d^2q}{(2\pi)^2} \frac{1}{2\pi} \int_{-1}^{1} \frac{dy}{y} \left( \frac{2 \text{Im} \Pi_L(y)}{(q^2+\text{Re} \Pi_L(y))^2 + (\text{Im} \Pi_L(y))^2} - \frac{2 \text{Im} \Pi_T(y)}{(q^2+\text{Re} \Pi_T(y))^2 + (\text{Im} \Pi_T(y))^2} \right)
\]

Here we used the sum rules [44]

\[
\int_{-1}^{1} \frac{dy}{y} \frac{2 \text{Im} \Pi_{T,L}(y)}{(q^2+\text{Re} \Pi_{T,L}(y))^2 + (\text{Im} \Pi_{T,L}(y))^2} = \left( \frac{1}{q^2+\text{Re} \Pi_{T,L}(y=\infty)} - \frac{1}{q^2+\text{Re} \Pi_{T,L}(y=0)} \right)
\]

with

\[
\text{Re} \Pi_{T,L}(y=\infty) = \frac{\mu^2}{3}, \quad \text{Re} \Pi_T(y=0) = 0, \quad \text{Re} \Pi_L(y=0) = \mu^2.
\]

Finally, Eq. (B8) becomes

\[
2 \text{Im} M_{1,0,1} = M_{1,0,1}^2 = 8E g^4 T t_a t_c t_a t_c \int \frac{d^3k}{(2\pi)^3} \frac{d^2q}{(2\pi)^2} \frac{k^2}{(k^2+M^2)^2} \frac{\mu^2}{q^2(q^2+\mu^2)}.
\]

2. Next we consider the diagram \( M_{1,0,2}^2 = 2 \text{Im} M_{1,0,2} \):

\[
M_{1,0,2}^2 = \int (-ig(2p'-k)\mu) D_{\mu\nu}(k) (ig(2p'-k)\nu) \frac{i}{(p'-k)^2 - M^2 + i\epsilon} \frac{-i}{(p'-k)^2 - M^2 - i\epsilon} \times (-ig(2p+q)\mu) D_{\rho\sigma}(q) (ig(2p+q)\sigma) D^>(p) t_a t_a t_c t_c (2\pi)^4 \delta^{(4)}(p' - p - k - q) \frac{d^4p}{(2\pi)^4} \frac{d^4q}{(2\pi)^4} \frac{d^4k}{(2\pi)^4}.
\]

By applying the same techniques as above and using Eq. (A14) we obtain

\[
2 \text{Im} M_{1,0,2} = 8E g^4 T t_a t_c t_a t_c \int \frac{d^3k}{(2\pi)^3} \frac{d^2q}{(2\pi)^2} \frac{k^2}{(k^2+M^2)^2} \frac{\mu^2}{q^2(q^2+\mu^2)}.
\]

3. Let us now compute the diagram \( M_{1,0,3}^2 = 2 \text{Im} M_{1,0,3} \):

\[
M_{1,0,3}^2 = \int (-ig(2p'-q)\mu) D_{\mu\nu}(q) (ig(2p+q)\nu) \frac{i}{(p+k)^2 - M^2 + i\epsilon} \frac{-i}{(p+k)^2 - M^2 - i\epsilon} \times \frac{-i}{(p'-k)^2 - M^2 + i\epsilon} t_a t_c t_a t_c D^>(p) (2\pi)^4 \delta^{(4)}(p' - p - k - q) \frac{d^4p}{(2\pi)^4} \frac{d^4q}{(2\pi)^4} \frac{d^4k}{(2\pi)^4} \times \theta(1 - \frac{q_0^2}{q^2}) (1 + f(q_0)) (2p'-q)\mu \int \left( \frac{P_{\mu\nu}(q)}{q^2 - \text{Re} \Pi_T(q)} + \frac{Q_{\mu\nu}(q)}{q^2 - \text{Im} \Pi_T(q)} \right) (2p+q)\nu \times 2\pi \int \frac{dp_0 dp_0}{(2\pi)^4} \frac{d^4q}{2\pi} \frac{d^4k}{2\pi} \delta^{(p_0 - E)} \frac{dp_0}{2\pi} \frac{d^4q}{(2\pi)^4} \frac{d^4k}{(2\pi)^4}.
\]
where we have used Eqs. (A8), (A9) and (A10). We also used three of the $\delta$-functions to do the integral over $d^4p$. By using Eqs. (A14)–(A16), the cut amplitude of diagram $M_{1,0,3}$ becomes

$$2 \text{Im } M_{1,0,3} = 2E g^4 t_a t_c t_a t_c \int \frac{d^3k}{(2\pi)^32\omega} \frac{4k^2}{(k^2+M^2x^2+m_g^2)^2} \int \frac{d^4q}{(2\pi)^4} 2\pi \delta(q_0-q_z) \left(1 + f(q_0)\right) \frac{q^2}{q^4} \times \left(\frac{2\text{Im } \Pi_T(q)}{(q^2-\text{Re } \Pi_T(q))^2 + (\text{Im } \Pi_T(q))^2} - \frac{2\text{Im } \Pi_L(q)}{(q^2-\text{Re } \Pi_L(q))^2 + (\text{Im } \Pi_L(q))^2}\right)$$

$$= -8E g^4 t_a t_c t_a t_c \int \frac{d^3k}{(2\pi)^32\omega} \frac{k^2}{(k^2+M^2x^2+m_g^2)^2} I_q,$$

where $I_q$ is given by Eq. (B11), giving finally

$$2 \text{Im } M_{1,0,3} = -8E g^4 T t_a t_c t_a t_c \int \frac{d^3k}{(2\pi)^32\omega} \frac{d^2q}{(2\pi)^2} \frac{k^2}{(k^2+M^2x^2+m_g^2)^2} \frac{\mu^2}{q^2(q^2+\mu^2)}.$$  

(B18)

4. In the same way we obtain

$$2 \text{Im } M_{1,0,4} = -8E g^4 T t_a t_c t_a t_c \int \frac{d^3k}{(2\pi)^32\omega} \frac{d^2q}{(2\pi)^2} \frac{k^2}{(k^2+M^2x^2+m_g^2)^2} \frac{\mu^2}{q^2(q^2+\mu^2)}.$$  

(B19)

5. The sum of all four diagrams (B14), (B16), (B19), and (B20) thus becomes

$$2 \text{Im } M_{1,0} = 2 \text{Im } M_{1,0,1} + 2 \text{Im } M_{1,0,2} + 2 \text{Im } M_{1,0,3} + 2 \text{Im } M_{1,0,4}$$

$$= 8E g^4 T [t_a, t_c] [t_c, t_a] \int \frac{d^3k}{(2\pi)^32\omega} \frac{d^2q}{(2\pi)^2} \frac{k^2}{(k^2+M^2x^2+m_g^2)^2} \frac{\mu^2}{q^2(q^2+\mu^2)},$$

where $[t_a, t_c]$ is a color commutator.

**APPENDIX C: COMPUTATION OF DIAGRAMS $M_{1,1,1} - M_{1,1,4}$**

In this Appendix we calculate the diagrams shown in Fig. 11 where one of the ends of the exchanged gluon $q$ is attached to the radiated gluon $k$.

We start off with $M^-_{1,1,1} = 2 \text{Im } M_{1,1,1}$:

$$M^-_{1,1,1} = \int \left(-i(g(2p' - k')^\mu t_b) D_{\mu \nu} (k') g \bar{f}^{c b a} \left(g^{\nu \tau} (k' + q) \lambda + g^{\lambda \tau} (k - q) \rho - g^{\lambda \rho} (k' + k)^\tau\right) D_{\lambda \nu}^\tau (i g(2p + k)^\tau t_c) \right. \right.$$  

$$\times D_{\sigma \tau}^\gamma (i g(2p - q)^\sigma t_a) \left. \right] \frac{1}{(p+k)^2 - M^2 - i\epsilon} D_{\sigma \tau}^\gamma (p)(2\pi)^4 \delta(4)(p' - p - k - q) \int \frac{d^4p}{(2\pi)^4} \frac{d^4q}{(2\pi)^4} \frac{d^4k}{(2\pi)^4}$$

$$= g^4 \frac{2E t_b t_a t_c t_a}{2E} \int \frac{d^4q}{(2\pi)^4} \frac{d^4k}{(2\pi)^4} \left. \frac{1}{(p+k)^2 - M^2 - i\epsilon} \right. 2\pi \delta(q_0 - E - k_0 - q_0) G ,$$

(C1)

where we used Eq. (A9), and performed the integral over $d^4p$. As in the previous section, $\vec{p} = \vec{k} - \vec{q}$ should be substituted and we define $G$ as

$$G = (2p' - k')^\mu (2p + k)^\nu (2p - q)^\rho D_{\mu \rho} (k') D_{\nu \lambda}^\tau (k) D_{\sigma \tau}^\gamma (q) \left(g^{\nu \tau} (k' + q) \lambda + g^{\lambda \tau} (k - q) \rho - g^{\lambda \rho} (k' + k)^\tau\right)$$

$$= G_1 + G_2 - G_3$$

(C2)

with

$$G_1 = \left[(2p' - k')^\mu D_{\mu \rho} (k') D_{\rho \sigma}^\gamma (q) (2p - q)^\sigma\right] \left[(k' + q)^\lambda D^\tau_{\lambda \nu} (k) (2p + k)^\nu\right],$$

$$G_2 = \left[(2p' - k')^\mu D_{\mu \rho} (k') (k - q)^\nu\right] \left[(2p + k)^\gamma D^\tau_{\nu \lambda} (k) D^\rho_{\lambda \sigma} (q) (2p' - q)^\sigma\right],$$

$$G_3 = \left[(2p' - k')^\mu D_{\mu \rho} (k') D_{\rho \sigma}^\gamma (k) (2p + k)^\nu\right] \left[(k + k')^\tau D^\gamma_{\tau \sigma} (q) (2p - q)^\sigma\right].$$

(C3)

Recalling that for the radiated gluon only the transverse part of the HTL propagator contributes, it is straightforward to show that (under the assumptions listed in Appendix A) the dominant contribution to Eq. (C2) comes from $G_3$.
FIG. 11: Feynman diagrams $M_{1,1,1}$, $M_{1,1,2}$, $M_{1,1,3}$ and $M_{1,1,4}$ contributing to the radiative energy loss to first order in opacity, labeled in the same way as Fig. 10.

(i.e. $G_1$ and $G_2$ present small corrections which can be neglected). By using Eqs. (A8), (A10) and Eq. (A16), we obtain

\[
G_3 \approx \left[ 4p_\mu D^{\mu\nu}(k^') D_{\rho
\nu}^<(k) p^\rho \right] [(k+k^')^4 D^{\lambda}_{X\sigma}(q)(2p'-q)^7]
\]

\[
= \left[ -\frac{4p_\mu p'_\nu k_z k_z'}{(k_z'^2 + k^2_z)(k_z^2 + k^2_z)} \frac{i k \cdot (k+q)}{2\pi} \frac{2\pi}{\delta(k_0 - \omega)} \right]
\]

\[
\times \left[ (1 + f(q_0))(4k_z p'_z) \frac{q^2}{q^2 + q^2} 2 \text{Im} \left( \frac{1}{q^2 - \Pi_L(q)} - \frac{1}{q^2 - \Pi_T(q)} \right) \right]
\]

\[
\approx -16E^2 \frac{2\pi}{2\omega} \delta(k_0 - \omega) \frac{i}{(k+q)^2 - m_\sigma^2 + i\epsilon} \frac{k \cdot (k+q)}{x} (1 + f(q_0)) \frac{q^2}{q^2 + q^2} 2 \text{Im} \left( \frac{1}{q^2 - \Pi_L(q)} - \frac{1}{q^2 - \Pi_T(q)} \right).
\]

Hence, Eq. (C1) becomes

\[
M_{1,1,1} \approx 8E g^4 (if^{cba} t_{b\tau} t_a) \int \frac{d^4q}{(2\pi)^4} \frac{d^3k}{2\omega} \frac{1}{(p+k)^2 - M^2 (k+q)^2 - m_\sigma^2} \frac{k \cdot (k+q)}{x}
\]

\[
\times 2\pi \delta(p_0 - E - \omega - q_0) \frac{q^2}{q^2 + q^2} 2 \text{Im} \left( \frac{1}{q^2 - \Pi_L(q)} - \frac{1}{q^2 - \Pi_T(q)} \right).
\]

Noting that $if^{cba} t_{b\tau} t_a = \frac{1}{2} [t_a, t_c][t_c, t_a]$, the cut amplitude $M_{1,1,1}$ then reads

\[
M_{1,1,1} \approx 4E g^4 [t_a, t_c][t_c, t_a] \int \frac{d^4k}{(2\pi)^4} \frac{d^3q dq_0}{2\omega} \frac{1}{(p+k)^2 - M^2 (k+q)^2 - m_\sigma^2} \frac{k \cdot (k+q)}{x}
\]

\[
\times \left( q_0 - q_z + \frac{k^2 + M^2x^2 + m_\sigma^2}{2k_z} \right) \left[ (1 + f(q_0)) \frac{q^2}{q^2 + q^2} 2 \text{Im} \left( \frac{1}{q^2 - \Pi_L(q)} - \frac{1}{q^2 - \Pi_T(q)} \right) \right).
\]

The $\delta$-function implies that

\[
(k+q)^2 - m_\sigma^2 \approx -((k+q)^2 + M^2x^2 + m_\sigma^2).
\]
With the help of this and Eqs. (A14), (A16) we further obtain

\[
M_{1,1,1} = -4E g^4 [t_a, t_b] [t_c, t_d] \int \frac{d^3k}{(2\pi)^3} \frac{d^3q}{(2\pi)^3} \frac{d\rho_{q0q_c}}{2\pi^2} \delta(q_0 - q_c) \left(1 + f(q_0)\right) \frac{k \cdot (k + q)}{(k^2 + M^2x^2 + m_g^2)(k^2 + M^2x^2 + m_g^2)}
\times \frac{q^2}{q^2 + q^2} \left(\frac{2 \text{Im} \Pi_{L}(q)}{(q^2 - \text{Re} \Pi_{L}(q))^2 + (\text{Im} \Pi_{L}(q))^2} - \frac{2 \text{Im} \Pi_{T}(q)}{(q^2 - \text{Re} \Pi_{T}(q))^2 + (\text{Im} \Pi_{T}(q))^2}\right). \tag{C8}
\]

Finally, by applying the same procedure as in Eqs. (B7)–(B13), we obtain

\[
2 \text{Im} M_{1,1,1} = -4E g^4 T [t_a, t_c] [t_e, t_d] \int \frac{d^3k}{(2\pi)^3} \frac{d^3q}{(2\pi)^3} \frac{d^2\rho_{q0q_c}}{(2\pi)^2} \frac{d\rho_{q0q_c}}{(2\pi)^2} \frac{k \cdot (k + q)}{(k^2 + M^2x^2 + m_g^2)(k^2 + M^2x^2 + m_g^2)} \frac{\mu^2}{q^2(q^2 + \mu^2)}. \tag{C9}
\]

It is straightforward to show that the cut amplitudes of diagrams $M_{1,1,2}$, $M_{1,1,3}$, and $M_{1,1,4}$ each lead to the same result. The sum of all four diagrams computed in this section thus gives

\[
2 \text{Im} M_{1,1} = 2 \text{Im} M_{1,1,1} + 2 \text{Im} M_{1,1,2} + 2 \text{Im} M_{1,1,3} + 2 \text{Im} M_{1,1,4} = 8E g^4 T [t_a, t_c] [t_e, t_d] \int \frac{d^3k}{(2\pi)^3} \frac{d^3q}{(2\pi)^3} \frac{d^2\rho_{q0q_c}}{(2\pi)^2} \frac{d^2\rho_{q0q_c}}{(2\pi)^2} \frac{2k \cdot (k + q)}{(k^2 + M^2x^2 + m_g^2)(k^2 + M^2x^2 + m_g^2)} \frac{\mu^2}{q^2(q^2 + \mu^2)}. \tag{C10}
\]

**APPENDIX D: COMPUTATION OF DIAGRAM $M_{1,2}$**

In this Appendix we calculate the diagram shown in Fig. 12 where both ends of the exchanged gluon $q$ are attached to the radiated gluon $k$:

![Feynman diagram](image)

**FIG. 12:** Feynman diagram $M_{1,2}$ contributing to the radiative energy loss to first order in opacity, labeled in the same way as Fig. 10.

\[
M_{1,2} = \int (-ig(2p' - k')^\mu t_b) D_{\mu\rho}(k') g f^{bac}(g^{\rho\alpha}(k' + q)^\lambda + g^{\lambda\tau}(k - q)^\rho - g^{\rho\mu}(k' + k)^\tau) D_{\chi\alpha}(k) D_{\gamma\rho}(q)
\times g f^{bac}(g^{\sigma\beta}(k' + q)^\alpha + g^{\alpha\beta}(k - q)^\sigma - g^{\beta\sigma}(k' + k)^\beta) D_{\sigma\nu}(k')(ig(2p' - q)^\tau t_d) D_{\gamma}(p)
\times (2\pi)^4 \delta(4)(p' - p - k - q) \frac{d^4p}{(2\pi)^4} \frac{d^4q}{(2\pi)^4} \frac{d^4k}{(2\pi)^4}
\frac{2E}{g^4} \int f^{bac} t_b f^{dac} t_d \int \frac{d^4q}{(2\pi)^4} \frac{d^4k}{(2\pi)^4} 2\pi \delta(p_0' - E - k_0 - q_0) H, \tag{D1}
\]

where we used Eq. (A9), and performed the integral over $d^4p$. Again, $\vec{p} = \vec{p}' - \vec{k} - \vec{q}$ should be substituted and we define $H$ as

\[
H = (2p' - k')^\mu (2p' - k')^\nu D_{\mu\rho}(k') D_{\chi\alpha}(k) D_{\gamma\rho}(q) D_{\sigma\nu}(k')
\times \left(g^{\rho\alpha}(k' + q)^\lambda + g^{\lambda\tau}(k - q)^\rho - g^{\rho\mu}(k' + k)^\tau\right) \left(g^{\sigma\beta}(k' + q)^\alpha + g^{\alpha\beta}(k - q)^\sigma - g^{\beta\sigma}(k' + k)^\beta\right). \tag{D2}
\]

Note that the left and right parts of the $M_{1,2}$ diagram are complex conjugates (i.e. mirror images) of each other. Therefore, for the three-gluon vertices on the left and right side, we go in counter-clockwise and clockwise direction, respectively.
As in the previous sections, for the radiated gluon we only consider transverse polarization. Under the assumptions described in Appendix A, it is straightforward to show that the dominant contribution to Eq. (D2) is given by

\[
H \approx \left[ 4 \rho_\mu \tilde{D}^\mu(p') D_{\rho\sigma}(k) (D^{\rho\sigma}(k')) p'_\nu \right] \left[ (k'+k)^\nu D_{\gamma\eta}(q) (k'+k)^\eta \right]
\]

\[
\approx \left[ 4 \rho_\mu \frac{K^2}{k^2} \frac{1}{((k+q)^2 - m_g^2 + i\epsilon)((k+q)^2 - m_g^2 - i\epsilon)} \right] \times \left[ 4k^2(1 + f(q_0)) \frac{q^2}{q_2^2 + q^2} \right] \frac{2\text{Im}(\frac{1}{q_2^2 - \Pi_L(q)} - \frac{1}{q_2^2 - \Pi_T(q)})}{2\omega}.
\]

where we have used Eqs. (A8), (A10) and Eq. (A16).

Therefore, Eq. (D1) becomes

\[
M_{1,2}^> \approx 8E g^4 f^{bac t_0} f^{daci} t_d \int \frac{d^4 q}{(2\pi)^4} \frac{d^3 k}{(2\pi)^3 2\omega} 2\pi\delta(k_0 - E - \omega - q_0)
\]

\[
\times \frac{(k+q)^2}{((k+q)^2 - m_g^2)^2} (1 + f(q_0)) \frac{q^2}{q_2^2 + q^2} \frac{2\text{Im}(\frac{1}{q_2^2 - \Pi_L(q)} - \frac{1}{q_2^2 - \Pi_T(q)})}{2\omega}.
\]

By using \( i f^{bac t_0} = [t_a, t_c] \), Eq. (A16) and Eq. (C7), we obtain

\[
2\text{Im} M_{1,2} \approx 8E g^4 [t_a, t_c] [t_c, t_a] \int \frac{d^4 q}{(2\pi)^4} \frac{d^3 k}{(2\pi)^3 2\omega} \frac{(k+q)^2}{((k+q)^2 + M^2 x^2 + m_g^2)^2} q^2 (1 + f(q_0))
\]

\[
\times 2\pi\delta(q_0 - q_z) \frac{2\text{Im}\Pi_L(q)}{(q_2 - \text{Re}\Pi_L(q))^2 + (\text{Im}\Pi_L(q))^2} - \frac{2\text{Im}\Pi_T(q)}{(q_2 - \text{Re}\Pi_T(q))^2 + (\text{Im}\Pi_T(q))^2}).
\]

Finally, by applying the same procedure as in Eqs. (B7)–(B13), we obtain

\[
2\text{Im} M_{1,2} = 8E g^4 T [t_a, t_c] [t_c, t_a] \int \frac{d^4 k}{(2\pi)^4 2\omega} \frac{d^2 q}{(2\pi)^2} \frac{(k+q)^2}{((k+q)^2 + M^2 x^2 + m_g^2)^2} q^2 (q_2^2 + \mu^2).
\]
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