Ergodicity and hydrodynamic projections in quantum spin lattices at all frequencies and wavelengths

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Obtaining rigorous and general results about the quantum dynamics of extended many-body systems is a difficult task. Given the panoply of phenomena that can be observed and are being investigated, it is crucial to delineate the boundaries of what is possible. In quantum lattice models, the Lieb-Robinson (LR) bound tells us that the spatial extent of operators grows at most linearly in time. But what happens within this light-cone? We obtain a universal form of ergodicity showing that operators get “thinner” almost everywhere within the light-cone. This includes what we believe is the first general result about decay of correlation functions within the LR light-cone, and is applicable to any locally interacting system in space-time translation invariant spatially mixing states. This weak notion of ergodicity is sufficient to prove a universal Boltzmann-Gibbs principle in all such systems: the projection of observables onto hydrodynamic modes at long times in correlation functions. In particular, we give an accurate formal definition of the complete space of hydrodynamic modes. This accounts for all types of dynamics, integrable or not. A surprising outcome, using Hilbert spaces of observables, is the realisation that all rigorous results, including hydrodynamic projections, are generalisable to arbitrary frequencies and wavelengths. This extends the concept of dynamical symmetries studied recently, and opens the door to the use of hydrodynamics to address oscillating behaviours at long times. We illustrate this by explaining the oscillatory algebraic decay of free fermion correlation functions using oscillatory hydrodynamic projections.

The dynamics of many-body, extended, isolated, interacting quantum systems has been the focus of much recent theoretical and experimental research [1–3]. Of particular interest is their universal, large-scale behaviours away from equilibrium. In this respect, the notions of ergodicity and hydrodynamics are natural avenues of investigation. But proving that, and how, ergodicity occurs and hydrodynamics emerges, whereby a large amount of information is lost despite the reversible, Hamiltonian dynamics, are some of the deepest problems in many-body physics. The appearance of new laws at large scales from microscopic descriptions is part of Hilbert’s sixth problem [4], and remains largely open in the context of extended quantum systems out of equilibrium.

On the one hand, ergodicity is the idea that, over long times, the system’s state covers uniformly enough the manifold of states, or at least the part of it that is dynamically accessible, such as the energy shell. In the language of Gibbs’ statistical mechanics, this translates into the equivalence of time averages with ensemble averages. In what situations does this happen in extended quantum systems? In particular, a natural idea is that this requires the presence of chaos – classically, the exponential separation, over time, of initially nearby trajectories; and quantum mechanically, certain conditions on the distribution of energies, or on out-of-time-ordered correlators (OTOC) [5, 6]. Is there such a link between short-time (chaotic) and long-time (ergodic) behaviours? Von Neumann’s work [7] offers plenty of insight, see [8, 9]. On the other hand, hydrodynamics purports that the dynamics of the system reduces to that of few emergent, slowly-decaying degrees of freedom. One stark realisation of this is the Boltzmann-Gibbs principle [10–12], a part of hydrodynamic linear response theory: correlations due to local perturbations are carried by long-lived modes that slowly propagate at hydrodynamic velocities. Indeed, the dominant correlations between a person’s vocal chords and another’s eardrum are carried by sound waves. The principle holds in extended Hamiltonian quantum models, including with integrability [13–15]. What are the conditions for the emergence of the Boltzmann-Gibbs principles? How is this related to ergodicity?

In this letter, we report on progress in these directions. Ergodicity and hydrodynamic modes are rather subtle notions, and standard approaches have many drawbacks. For instance, defining ergodicity as the covering of the energy shell is not sufficient: with few other conserved charges the shell may be subdivided and the system deemed ergodic, but what if there are an infinite number? What kind of charges should be admitted? A similar problem arises with establishing the full set of slowly decaying modes that contribute to large-scale correlations. What, precisely, qualifies as a slowly decaying mode?

Going beyond the conventional paradigm, we propose to separate the questions into two parts: general notions – such as the emergence of hydrodynamic modes – that hold for a wide family of systems; and specific structures – such as the type and number of modes – that require a model analysis. We obtain mathematically rigorous results pertaining to the former. The results are valid in all translation-invariant, infinitely-extended quantum spin lattices (restricting for simplicity to D-dimensional hypercubes), with short-range interactions (interaction strength decaying at least exponentially with distance).

Our main results are as follows. (1) Almost-everywhere ergodicity. Infinite-time averages of local observables along almost every velocity are non-fluctuating, and, as operators, are “thin”. In particular, this implies the vanishing of time-averaged OTOC, emphasising the lack of connection between (short-time) chaos and (long-time)
ergodicity. (2) The Boltzmann-Gibbs principle. The Euler scaling limit and the complete set of emergent hydrodynamic modes are formally defined in a model-independent way. (3) Almost-everywhere ergodicity and the Boltzmann-Gibbs principle remain valid with respect to every frequency and wavelength. This is a consequence of the general Hilbert space structures underlying the above two results. Thus hydrodynamics can be applied in the neighbourhood of any value \((\omega, k)\) in the frequency-wavelength plane. This shows that recent ideas on dynamical symmetries [16–20] can be extended to the hydrodynamic regime of correlation functions.

These results hold independently from many physical properties of the system, be it diffusive, integrable, constrained, localised, etc. They are consequences of the extensivity of the system, including the short enough range of the interaction, and of the small enough space of local degrees of freedom, which essentially forbid the build-up over large regions of strong correlations. They do not depend on any specific short-time or microscopic features. The idea that hydrodynamics is a useful concept beyond its conventional field of applications, for instance not requiring chaos, has come to the fore with the recent development of generalised hydrodynamics for integrable systems [21–24], and ties in with the idea that thermodynamic concepts should not rely on short-time behaviours, as emphasised recently in [25]. However, rigorous results about many-body dynamics, especially of such generality, are notoriously difficult to obtain.

The full proofs are provided in separate papers: in [26] a subset of almost-everywhere ergodicity results are established for arbitrary dimensions, frequencies and wavelengths, and for a wider space of interactions with exponentially decaying strength; and in the upcoming paper [28], the Boltzmann-Gibbs principle is shown in the same context. Our proofs introduce a number of new techniques, which we will however not discuss here.

We consider a quantum lattice model with local spaces \(\mathbb{C}^N\) on \(\mathbb{Z}^D\) \((N, D \geq 1)\). The Hamiltonian is homogeneous (space-translation invariant) and with short-range interaction (exponentially decaying or faster). Observables form a \(C^*\) algebra \(\mathfrak{U}\), the norm-completion of the algebra \(\mathfrak{U}_{\text{loc}}\) of local observables (supported on finite numbers of sites). We denote \(A(x, t), B(x, t), \ldots\) observables \(A \in \mathfrak{U}\) translated to space-time points \(x, t\), and sometimes omit the time argument if \(t = 0\). A state \(\langle \cdots \rangle\) gives ensemble averages, normalised to \(\langle 1 \rangle = 1\). It is assumed homogeneous and stationary, \(\langle A(x, t) \rangle = \langle A \rangle \forall x \in \mathbb{Z}^D, t \in \mathbb{R}\), and spatially mixing [29]:

\[
\lim_{x \to -\infty} \langle A(x)B \rangle = \langle A \rangle \langle B \rangle, \quad A, B \in \mathfrak{U}. \quad (1)
\]

Any single-phase thermal state – that is, spatially mixing Kubo-Martin-Schwinger (KMS) state – is an example: for \(D = 1\) this is every thermal state, for \(D > 1\) this includes every thermal state above a certain temperature [30, Section 5.3]. One associates a Hilbert space \(\mathcal{H}_{\text{GNS}}\) and a vector \(|\Omega\rangle\) to any state, \(\langle A \rangle = \langle \Omega | A | \Omega \rangle\): this is the Gelfand-Naimark-Segal (GNS) construction [31, Chapter 2.3] (we use the same notation for \(A \in \mathfrak{U}\) and its representation as a bounded operator on \(\mathcal{H}_{\text{GNS}}\). \(\mathcal{H}_{\text{GNS}}\) is the space of excitations above the mixed-state “vacuum”. See the Supplementary Material (SM) for a complete discussion of the mathematical setup and accurate expressions of the theorems.

**Ergodicity.** It is commonly believed that, in typical interacting systems, auto-correlation functions of local observables vanish at long times, \(\langle A(0, t)B(0, 0) \rangle - \langle A \rangle \langle B \rangle \to 0 \ (t \to \infty)\). This is the notion of mixing [32]: the effect of the small localised perturbation \(B(0, 0)\) mixes with its surrounding and, locally, decays in time. A weaker notion, implied by mixing, is ergodicity: \(T^{-1} \int_0^T dt \langle A(0, t)B(0, 0) \rangle \to \langle A \rangle \langle B \rangle \ (T \to \infty)\). This is equivalent to mean-square ergodicity: the vanishing of the variance of the observable \(A_T := T^{-1} \int_0^T dt A(0, t)\) at long times, \(\langle A_T^2 \rangle - \langle A \rangle^2 \to 0\). Mean-square ergodicity says that infinite-time averages of observables, in the state \(\langle \cdots \rangle\), do not fluctuate, and are simply equal to ensemble averages. (Integrals of observables are defined via the Bochner integral [33], see the SM.)

Mixing and ergodicity are extremely difficult to establish rigorously in interacting systems. However, some results are available along “rays” in space-time, at large enough velocities. Most prominent is the celebrated Lieb-Robinson (LR) bound [34] (see also [35, Chapter 4]): the commutators of local observables \(\lbrace A(x, t), B \rbrace\), \(A, B \in \mathfrak{U}_{\text{loc}}\) decay (in algebra norm) exponentially fast as \(x, t \to \infty, |x| > v_{LR}t\). This is everywhere outside a “light-cone”
bounded by the LR velocity \( v_{\text{LR}} > 0 \) (the LR bound has been extended in a number of ways, see for instance [36]). The support of the operator \( A(0,t) \) grows, beyond that of \( A \), at most linearly in time, and therefore correlations beyond the LR cone decay. But this says nothing about what happens within the LR cone, where little is known rigorously. Yet it is there that the interesting physics of ergodicity and hydrodynamics emerge.

Our ergodicity result is as follows. On the lattice, it is natural to restrict to the set of rational spatial directions \( S_{Q}^{D-1} = \{ x/|x| : x \in \mathbb{Z}^D \} \) (this is dense on the sphere). Given \( T > 0, v \in \mathbb{R} \) and \( n \in S_{Q}^{D-1} \), denote the average of \( A \in \mathfrak{A} \) along the ray with velocity vector \( v = vn \) as

\[
\check{A}^v_T = \frac{1}{T} \int_0^T dt \langle |vt|, t \rangle
\]

where \( [a] = (|a_i|) \) is the vector of the integer parts.

**Theorem 1** For every rational direction \( n \in S_{Q}^{D-1} \) and almost every \( v \in \mathbb{R} \) with respect to the Lebesgue measure:

\[
\lim_{T \to \infty} \left( \langle \check{A}^v_T \rangle \right)^m = \langle A \rangle^m \tag{3}
\]

for every \( A \in \mathfrak{A}, m \in \mathbb{N} \). If, further, \( \langle \cdot \cdot \cdot \rangle \) is a KMS state, then

\[
\lim_{T \to \infty} \check{A}^v_T = \langle A \rangle 1, \quad \lim_{T \to \infty} [\check{A}^v_T, B] = 0 \tag{4}
\]

for every \( A, B \in \mathfrak{A} \), in the strong operator topology (SOT).

Eq. (2) is with respect to a strong-operator integration measure on \( \mathcal{H}_{\text{GNS}} \); and \( C_T \to 0 \) in the SOT means \( ||C_T|\psi|| \to 0 \) for all \( |\psi| \in \mathcal{H}_{\text{GNS}} \) (see the SM).

Taking the language of probability, Eq. (3) means “\( \check{A}^v_T \to \langle A \rangle \) in law”. This is almost-everywhere ergodicity: along almost every velocity, \( \check{A}^v_T \) tends to a non-fluctuating quantity. As a consequence, by the Cauchy-Schwarz inequality, \( \langle \check{A}^v_T B \rangle \to \langle A \rangle \langle B \rangle \) and \( \langle [\check{A}^v_T, B] \rangle \to 0 \) for every \( A, B \in \mathfrak{A} \). The latter means, by standard perturbation theory, that the effect at time \( T \) of a local perturbation to the dynamics, on observations made at \( x = |vt| \), is non-extensive in time for almost every \( v \).

Eqs. (4) are starker statements. They hold in spatially mixing, space-time translation invariant KMS states, including Gibbs and generalised Gibbs states [37–39]. They imply that the limits in (4) hold within any expectation value, multiplied by any other observables in any order. We thus obtain an extension of (a weak version of) the LR bound to within the LR cone: the ray-averaged operator \( \check{A}^v_T \) becomes “thin” as \( T \to \infty \), being un-observable by any \( B \), at almost every velocity \( v \). See Fig. 1 for a pictorial representation.

In particular, the infinite-time-averaged OTOC vanishes, \( \langle [\check{A}^v_T, B]^2 \rangle \to 0 \). At short times, the OTOC is expected to grow exponentially in chaotic systems [5]. Clearly, by boundedness of the state, the OTOC cannot grow forever. We find that its time average in fact must decay for almost every velocity.

The results do not say anything about what happens in the pure time direction, \( v = 0 \). For instance, it is easy to construct a trivial Hamiltonian, without interactions between spins, where the velocity \( v = 0 \) is not ergodic (see the SM). However, all results hold for velocities as near as desired to 0.

Theorem 1 is based on the von Neumann ergodic theorem [40, Theorem II.11], which relates time averages to projections onto invariant spaces. As far as we know, before our work there were no nontrivial results concerning the invariant subspace of \( \mathcal{H}_{\text{GNS}} \) for unitary time evolution in quantum lattice models. We show it to be spanned by \( |0 \rangle \) at almost every velocity. This happens because finite-dimensional local spaces (in fact, we believe that countable-dimensionality is sufficient) are too small to allow for operators to extend thickly on large distances over time: the set of speeds where the operator remains supported under averaging should be “small enough”.

In order to gain intuition, take the following standard calculation: if a system’s energy differences are non-degenerate, then the energy eigenstate \( |E\rangle \) is mean-square ergodic: \( \lim_{T \to \infty} \langle E \check{A}^v_T |E\rangle = \sum_{E'} \lim_{T \to \infty} T^{-1} \int_0^T dt e^{i(E-E')t} \langle E \check{A}^v_T |E'\rangle \langle E'|B\rangle = \langle E \check{A}^v_T |E\rangle B \). This requires finite energy differences, hence a finite system size. Instead, in Theorem 1, the limit \( T \to \infty \) is taken after the limit on the system’s size, already infinite in \( \langle \cdot \cdot \cdot \rangle \), and there is no requirement on energy differences; the timescales are smaller than any timescales associated to the system’s size. Classically, ergodicity in finite size is associated to the (naive) picture that the full energy shell is eventually covered under time evolution. By contrast, here naively the energy shell is not covered. Our results instead address local relaxation, and thus a form of typicality (see e.g. [9]). In fact, we believe they are related to a notion of many-body ergodicity (which would require further investigation): under a metric based on nearness of local averages, the neighbourhood of every state in the energy shell is crossed in the local relaxation time, even in infinite systems.

— Boltzmann-Gibbs. Almost-everywhere ergodicity embodies the disparity between the extensivity of the system and locality of the observable. It turns out that it leads to (a version of) the Boltzmann-Gibbs principle: the reduction of the number of degrees of freedom at large space-time separations by projections over hydrodynamic modes. See [15] for a modern discussion of this principle in many-body quantum Hamiltonian systems.

We assume that clustering (1) is faster than \( |x|^{-D} \) for all local observables. The notions needed for a full expression of the Boltzmann-Gibbs principles are as follows. We define the positive-semidefinite sesquilinear form using the connected correlator \( \langle AB \rangle^c = \langle AB \rangle - \langle A \rangle \langle B \rangle \),

\[
(A, B) = \sum_{x \in \mathbb{Z}^D} \langle A^\dagger(x) B \rangle^c, \tag{5}
\]
and the equivalence $A \simeq A'$ iff $(A' - A, A' - A) = 0$. Clearly $A(x) \simeq A$. It is natural to see the equivalence class $\Sigma A := \{ A' : A' \simeq A \}$ as the “extensive observable” associated to the density $A$, formally $\Sigma A = \sum_x A(x)$.

We Cauchy complete to the Hilbert space of extensive observables $H = \{ \Sigma A : A \in H_{\text{loc}} \}$. One can show that time evolution acts as a unitary operator $\tau_t$ on $H$, and we form the space of extensive conserved quantities $Q = \{ a \in H : \tau_t a = a \forall t \in \mathbb{R} \}$; formally $[H, a] = 0$. This is closed, hence there is an orthogonal projection $P : H \rightarrow Q$. Elements of $Q$ that are local are those $a = \Sigma A$ such that $A \in H_{\text{loc}}$ is a conserved density with a constant $B$:

$$\frac{d}{dt} A(x, t) + \sum_i (B(x + e_i, t) - B(x, t)) = 0$$

(6)

(where $e_i$ are unit vectors in directions $i$). But $Q$ may contain more elements, not necessarily local (but still, by definition, extensive), for instance the quasi-local charges known to exist in integrable systems [41].

The objects of interest are the time-averaged, long-wavelength Fourier transform of two-point functions:

$$S_{\Sigma A, \Sigma B}(\kappa) := \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T dt \sum_{x \in \mathbb{Z}^D} e^{i\kappa \cdot x / t} (A^\dagger(x, t)B)^c.$$  

The limit on $T$ is in general a Banach limit, see [42, Chapter III.7] and [26]. As the notation implies, the result only depends on the equivalence classes $\Sigma A, \Sigma B$; in fact, it gives a continuous sesquilinear form on $H$. Then:

**Theorem 2** For every $a, b \in H$ and $\kappa \in \mathbb{R}^D$,

$$S_{a, b}(\kappa) = S_{P_{a, b}c}(\kappa).$$

(8)

That is, the correlation of $A$ and $B$ at large space-time separation, under the appropriate space-time fluid-cell averaging expressed in (7), is obtained by projecting the extensive observables $a = \Sigma A$ and $b = \Sigma B$ onto the extensive conserved quantities $Q$. The physical idea is that the initial, dynamically complicated disturbance quickly relaxes and projects, at the Euler scale of long time and large distances, onto the extensive conserved quantities, that then carry correlations. The proof of Theorem 2 [26, 28] shows that the notion of relaxation given by almost-everywhere ergodicity is sufficient for this projection to occur, and $Q$ is the space of remaining dynamical degrees of freedom (see Fig. 1).

We believe this is the first general, rigorous result concerning the Boltzmann-Gibbs principle in interacting systems. The principle is therefore applicable to the large class of quantum lattices, going beyond interacting particle systems conventionally studied in statistical physics. It provides further support to the idea that the basic principles of hydrodynamics hold independently from the details of the microscopic dynamics.

Hydrodynamic projections were originally considered in the celebrated works by Mori [43] and Zwanzig [44], where integration over “fast modes” gave rise to a projection mechanism over “slow modes”, see also [10–12]. It is a difficult problem to correctly identify the full space of slow modes, and the importance of conserved quantities was noted early on [45]. In conventional particle systems they are assumed to be the energy, momentum and number of particles, while in integrable systems, the set of conserved charges is infinite [13, 24], see the review [15].

Our result shows that, in every spatially mixing state of short-range quantum lattices: (1) there is a specific fluid-cell mean under which projection occurs (see [14, 15, 46] for discussions of fluid-cell means); (2) the space of extensive conserved quantities $Q$ always exists and is universally defined; (3) $Q$ may in general be state-dependent. It remains an important problem to show that $Q$ is spanned by the conventional extensive conserved quantities in chaotic systems.

— All wavelengths and frequencies. As usual in the context of hydrodynamics, the results above pertain to what happens around the point $(\omega, \kappa) = (0, 0)$ in the frequency-wavenumber plane. Surprisingly, they in fact generalise to arbitrary wavelengths and frequencies:

**Theorem 3** The results of Theorems 1 and 2 stay valid under the replacement $A(x, t) \rightarrow e^{i\omega t - i\kappa \cdot x} A(x, t)$ for any value of $\kappa \in \mathbb{R}^D$ and $\omega \in \mathbb{R}$.

In (2) this gives the oscillatory ray averages $A_{\omega, \kappa}^c$; in (3) and (4) one-point functions are replaced by $\lim_{T \rightarrow \infty} (A_{\omega, \kappa}^c) = 0$ for generic $(\omega, \kappa)$. Instead of (6) we have the $k$-inner product $(A, B)_k = \sum_{x \in \mathbb{Z}^D} e^{i\kappa \cdot x} (A(x)B)_c$ and likewise there is an extra factor $e^{i\kappa \cdot x - i\omega t}$ in (7) for $S_{\Sigma A, \Sigma B}(\kappa)$, representing $(\omega, \kappa)$-fluid-cell means (see Fig. 1, and the SM for other expressions of fluid-cell means). The completion of the equivalence classes $\Sigma A$ induced by $(\cdot, \cdot)_k$ gives the space $H_\kappa$ of $k$-extensive observables, formally the quantities $\sum e^{-i\kappa \cdot x} A(x)$; and the $\omega, k$-conserved charges $Q_{\omega, \kappa}$ are the $k$-extensive observables $a$ satisfying $e^{i\omega t + \tau_t a} = a$, formally $[H, a] + \omega a = 0$. Theorem 3 is a consequence of the fact that oscillatory space-time translations, although not algebra automorphisms, still are unitary operators on the Hilbert space of observables (the space $H_{\text{GNS}}$).

Almost-everywhere ergodicity with nonzero frequency and/or wavelength rigorously forbids sustained oscillatory behaviours, over extended regions, for correlations of local operators and OTOC such as $(A(x, t)B)_c$ and $\langle A(x, t), B \rangle^2$. Recent results have established lower bounds on localised persistent oscillations (on the ray $\psi = 0$) using “strictly local dynamical symmetries” [17, 19, 20]. We find that such symmetries may only exist on a set of rays $\psi$ of measure zero: for infinitely-many rays as near as desired to $0$, there is no such oscillation. ($\omega, \kappa$)-conserved densities and currents satisfy

$$0 = \frac{d}{dt} A(x, t) + i\omega A(x, t)$$

$$+ \sum_i (B(x + e_i, t) - B(x, t) + (e^{-i\kappa \cdot e_i} - 1)B(x + e_i, t),$$

(9)
and if they exist, correlation functions may present oscillatory behaviours at large scales of space and time: \((\omega, k)\)-fluid-cell means project onto \(Q_{\omega, k}\), as per Theorem 2. One expects only certain values of \((\omega, k)\) giving \(Q_{\omega, k} \neq \{0\}\). A simple example is the free fermionic lattice: creation and annihilation operators at momentum \(k\) are \((\omega, k)\)-extensive conserved quantities for \(\omega = E(k)\) given by the dispersion relation, and \((\omega, k)\)-hydrodynamic projection explains the oscillatory algebraic decay of fermion two-point functions conventionally obtained by a saddle-point analysis; see the SM. \((\omega, k)\)-hydrodynamic projection is a hydrodynamic expansion, of linear-response type, in the neighbourhood of arbitrary \((\omega, k)\), which extends the paradigm of hydrodynamics to oscillatory behaviours. It generalises ideas in recent works on time crystals, where dynamical symmetries are used to explain persistent oscillations in Drude weights [16, 18].

— Conclusion. We have shown almost-everywhere ergodicity and the Boltzmann–Gibbs principle in every short-range quantum spin model on \(\mathbb{Z}^D\), and at every frequency \(\omega\) and wavenumber \(k\). This underlines the large universality of general ergodic and hydrodynamic principles: they emerge solely from the separation of scales, the large gap between extensivity and locality. The mathematical construction based on spaces of local observables showed the flexibility in the notions of extensivity (or homogeneity) and stationarity, allowing for oscillations. The results are rigorous. Applying them to explicit examples, including specific structures inside the LR light-cone and oscillatory hydrodynamic projections in interacting systems generalising the free-fermion example, would be very interesting. The \((\omega, k)\)-hydrodynamic projection result paves the way for a full \((\omega, k)\)-hydrodynamics, a subject which should help uncover new universal dynamics and which we hope to investigate in the near future.

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[1] J. Eisert, M. Friesdorf, and C. Gogolin, Nature Physics 11, 124 (2015).
[2] C. Gogolin and J. Eisert, Reports on Progress in Physics 79, 056001 (2016).
[3] L. D’Alessio, Y. Kafri, A. Polkovnikov, and M. Rigol, Advances in Physics 65, 239 (2016).
[4] D. R. Hilbert (Bull. Amer. Math. Soc. 8 (1902), 437–479, 1902).
[5] J. Maldacena, S. H. Shenker, and D. Stanford, Journal of High Energy Physics 2016, 106 (2016).
[6] K. Hashimoto, K. Murata, and R. Yoshii, Journal of High Energy Physics 2017, 135 (2017).
[7] J. v. Neumann, Zeitschrift für Physik 57, 30 (1929).
[8] S. Goldstein, J. L. Lebowitz, R. Tumulka, and N. Zhang, The European Physical Journal H 35, 173 (2010).
[9] S. Goldstein, J. Lebowitz, C. Maistrodonato, R. Tumulka, and N. Zanghì, Proceedings of The Royal Society A Mathematical Physical and Engineering Sciences 466 (2009), 10.1098/rspa.2009.0635.
[10] H. Spohn. Large scale dynamics of interacting particles (Springer-Verlag, 1991).
[11] A. DeMasi and E. Presutti. Mathematical methods for hydrodynamic limits (Springer Berlin Heidelberg, 2006).
[12] C. Kipnis and C. Landim. Scaling limits of interacting particle systems, Vol. 320 (Springer Science & Business Media, 1998).
[13] B. Doyon and H. Spohn, SciPost Physics 3, 039 (2017).
[14] B. Doyon, SciPost Phys. 5, 54 (2018).
[15] J. Nardis, B. Doyon, M. Medenjak, and M. Panfil, Correlation functions and transport coefficients in generalised hydrodynamics (to appear in J. Stat. Mech, 2021).
[16] B. Buča, J. Tindall, and D. Jaksch, Nature Communications 10, 1730 (2019).
[17] B. Buca, A. Purlavastha, G. Guarnieri, M. T. Mitcheson, D. Jaksch, and J. Goold, “Quantum many-body attractors,” (2020), arXiv: 2008.11166 [quant-ph].
[18] M. Medenjak, T. Prosen, and L. Zadnik, SciPost Phys. 9, 3 (2020), publisher: SciPost.
[19] B. Buča, “Local hilbert space fragmentation and out-of-time-ordered crystals,” (2021), arXiv: 2108.13411 [cond-mat.stat-mech].
[20] T. Gunawardana and B. Buča, “Dynamical l-bits in Stark many-body localization,” (2021), arXiv: 2110.13135 [cond-mat.dis-nn].
[21] O. A. Castro-Alvaredo, B. Doyon, and T. Yoshimura, Physical Review X 6, 041065 (2016).
[22] B. Bertini, M. Collura, J. De Nardis, and M. Fagotti, Physical Review Letters 117, 207201 (2016).
[23] V. B. Bulchandani, R. Vasseur, C. Karrasch, and J. E. Moore, Physical Review B 97, 045407 (2018).
[24] B. Doyon, SciPost Phys. Lect. Notes (2020), 10.21468/SciPostPhysLectNotes.18.
[25] S. Chakraborti, A. Dhar, S. Goldstein, A. Kundu, and J. L. Lebowitz, arXiv:2109.07742 [cond-mat] (2021).
[26] B. Doyon, “Hydrodynamic projections and the emergence of linearised Euler equations in one-dimensional isolated systems,” (2020), arXiv: 2011.00611 [math-ph].
[27] D. Ampelogiannis and B. Doyon, “Almost everywhere ergodicity in quantum lattice models,” (2021), in preparation.
[28] D. Ampelogiannis and B. Doyon, “The Boltzmann–Gibbs principle in quantum lattice models,” (2021), in preparation.
[29] In [27] the state is assumed to be “factorial”, which implies mixing and is a wide enough set of states; but in a separate work we will show, using the LR bound, that mixing is indeed sufficient.
[30] O. Bratteli and D. Robinson, Operator Algebras and
Quantum Statistical Mechanics 2: Equilibrium States
Models in Quantum Statistical Mechanics, Operator Algebras and Quantum Statistical Mechanics (Springer, 1997).

[31] O. Bratteli and D. Robinson, Operator Algebras and Quantum Statistical Mechanics I: C*- and W*-Algebras. Symmetry Groups. Decomposition of States, Operator Algebras and Quantum Statistical Mechanics (Springer, 1987).

[32] In quantum statistical mechanics, “mixing” is sometimes used to describe what we call here spatial mixing, Eq. (1), and likewise “ergodicity” is used to describe a property related to spatial integrals. Here we reserve these words to the more physical notions involving time.

[33] E. Hille and R. Phillips, Functional Analysis and Semigroups, American Mathematical Society: Colloquium publications (American Mathematical Society, 1996) issue: v. 31, pt. 1.

[34] E. H. Lieb and D. W. Robinson, Communications in Mathematical Physics 28, 251 (1972).

[35] P. Naaijkens, Quantum Spin Systems on Infinite Lattices: A Concise Introduction, Lecture Notes in Physics, Vol. 933 (Springer International Publishing, Cham, 2017).

[36] M. C. Tran, A. Y. Guo, C. L. Baldwin, A. Ehrenberg, A. V. Gorshkov, and A. Lucas, Physical review letters 127, 160401 (2021).

[37] M. Rigol, V. Dunjko, V. Yurovsky, and M. Olshanii, Physical Review Letters 98, 050405 (2007).

[38] E. Ilievski, J. De Nardis, B. Wouters, J.-S. Caux, F. H. L. Essler, and T. Prosen, Physical Review Letters 115, 157201 (2015).

[39] F. H. L. Essler and M. Fagotti, Journal of Statistical Mechanics: Theory and Experiment 2016, 064002 (2016).

[40] M. Reed and B. Simon, I: Functional Analysis, Methods of Modern Mathematical Physics (Elsevier Science, 1981).

[41] E. Ilievski, M. Medenjak, T. Prosen, and L. Zadnik, Journal of Statistical Mechanics: Theory and Experiment 2016, 064008 (2016), publisher: IOP Publishing.

[42] J. B. Conway, A Course in Functional Analysis, Graduate Texts in Mathematics, Vol. 96 (Springer New York, New York, NY, 2007).

[43] H. Mori, Progress of Theoretical Physics 33, 423 (1965).

[44] R. Zwanzig, Lectures in theoretical physics 3, 106 (1961).

[45] Th. Brøx and H. Rost, The Annals of Probability 12, 742 (1984).

[46] G. Del Vecchio Del Vecchio and B. Doyon, “The hydrodynamic theory of dynamical correlation functions in the XX chain,” (2021), arXiv: 2111.08420 [math-ph].
Supplemental Material for
“Ergodicity and the Boltzmann-Gibbs principle in quantum spin lattices
at all frequencies and wavelengths”

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I. THE SETUP

We consider a lattice $\mathbb{Z}^D$, with each site $x \in \mathbb{Z}^D$ admitting a quantum spin described by the (matrix) algebra of $\mathfrak{U} := \mathbb{C}^{N_x} \otimes \mathbb{C}^{N_x}$, $N_x, D \in \mathbb{N}$. We require that the dimension of the spin algebra is uniformly bounded, $N_x \leq N$, for some $N \in \mathbb{N}$. To each finite $\Lambda \subset \mathbb{Z}^D$ we associate the algebra $U_\Lambda := \bigotimes_{x \in \Lambda} U_x$, and the algebra of local observables is defined as the inductive limit of the increasing net of algebras $\{ U_\Lambda \}_{\Lambda \in P_f(\mathbb{Z}^D)}$, where $P_f(\mathbb{Z}^D)$ denotes the set of finite subsets of $\mathbb{Z}^D$, $U_{loc} := \bigcup_{\Lambda \in P_f(\mathbb{Z}^D)} U_\Lambda$. The norm completion of $U_{loc}$ defines the quasi-local algebra of observables of the quantum spin lattice: $\mathfrak{U} := \overline{U_{loc}}$. For detailed definition of quasi-local algebras, see [1, Chapter 3.2.3].

A quantum spin lattice model is described by a triple $(\mathfrak{U}, \iota, \tau)$: the quasi-local $C^*$-algebra and the group actions of space and time translation on this algebra.

**Definition 1 (Dynamical System).** A dynamical system of a quantum spin lattice is a triple $(\mathfrak{U}, \iota, \tau)$ where $\mathfrak{U}$ is a quasi-local $C^*$-algebra, $\tau$ is a strongly continuous representation of the group $\mathbb{R}$ by $^*$-automorphisms $\{ \tau_t : \mathfrak{U} \to \mathfrak{U} \}_{t \in \mathbb{R}}$, and $\iota$ is a representation of the translation group $\mathbb{Z}^D$ by $^*$-automorphisms $\{ \iota_x : \mathfrak{U} \to \mathfrak{U} \}_{x \in \mathbb{Z}^D}$, such that for any $\Lambda \in P_f(\mathbb{Z}^D)$: $A \in U_\Lambda \implies \iota_x(A) \in U_{\Lambda + x}$ for all $x \in \mathbb{Z}^D$.

We further assume that $\tau$ is such that $\tau_t \iota_x = \iota_x \tau_t$, $\forall t \in \mathbb{R}$, $x \in \mathbb{Z}^D$, i.e. time evolution is homogeneous.

We denote these group actions on $A \in \mathfrak{U}$ as $\iota_x \tau_t(A) := A(x, t)$.

The group action of time translations is defined from the Hamiltonian. On any $\Lambda \in P_f(\mathbb{Z}^D)$ the local Hamiltonian is defined as $H_\Lambda := \sum_{X \subset \Lambda} \Phi(X)$, where $\Phi(X) \in U_X$, with $\Phi^*(X) = \Phi(X)$, is the interaction of the spins in the sublattice $X$. Then, time evolution is defined when the limit

$$\tau_t(A) = \lim_{\Lambda \to \mathbb{Z}^D} e^{iH_\Lambda t} A e^{-iH_\Lambda t}$$

in the norm, for all $A \in U_{loc}$, and is a strongly continuous $^*$-automorphism. We consider space-translation invariant interactions, $\Phi(\Lambda + x) = \Phi(\Lambda)$ for all $x \in \mathbb{Z}^D$, $\Lambda \in P_f(\mathbb{Z}^D)$, that decay fast enough, specifically we require:

$$\| \Phi \|_\lambda := \sum_{\Lambda \geq 0} \| \Phi(\Lambda) \| \ |\Lambda| N^{2|\Lambda|} e^{\lambda \text{diam}(\Lambda)} < \infty$$

for some $\lambda > 0$ and for $N$ the uniform bound on the spin algebra dimension.

The state of the system is described a normalised linear functional on the quasi-local algebra, interpreted as the ensemble average:
Definition 2 (States, Invariance and Clustering). A state of a dynamical system $(\mathcal{U}, \iota, \tau)$ is a positive linear functional $\langle \cdot \cdot \cdot \rangle : \mathcal{U} \rightarrow \mathbb{C}$ such that $\| \langle 1 \rangle \| = 1$. The set of states is denoted by $E_\mathcal{U}$.

A state $\langle \cdot \cdot \cdot \rangle \in E_\mathcal{U}$ is called space and time invariant if $\langle A(x, t) \rangle = \langle A \rangle$, $\forall A \in \mathcal{U}, t \in \mathbb{R}, x \in \mathbb{Z}^D$. It is called clustering, or spatially mixing, when

$$\lim_{x \to \infty} \langle \iota(x)B \rangle = \langle A \rangle \langle B \rangle, \quad A, B \in \mathcal{U}_{\text{loc}}. \quad (3)$$

Of importance is the GNS representation of a C*-algebra.

Proposition 3 (GNS representation). Given a state $\langle \cdot \cdot \cdot \rangle \in E_\mathcal{U}$ of a unital C*-algebra $\mathcal{U}$ there exists a (unique, up to unitary equivalence) triple $(H, \pi, \Omega)$ where $H$ is a Hilbert space with inner product $\langle \cdot, \cdot \rangle$, $\pi$ is a representation of the C*-algebra by bounded operators acting on $H$ and $\Omega$ is a cyclic vector for $\pi$, i.e. the span$\{\pi(A)\Omega : A \in \mathcal{U}\}$ is dense in $H$, such that

$$\langle A \rangle = \langle \Omega, \pi(A)\Omega \rangle, \quad A \in \mathcal{U} \quad (4)$$

If additionally we have a group $G$ of automorphisms $\{\tau_g\}_{g \in G}$ of $\mathcal{U}$ and $\langle \cdot \cdot \cdot \rangle$ is $\tau$-invariant, then there exists a representation of $G$ by unitary operators $U(g)$ acting on $H$. This representation is uniquely determined by

$$U(g)\pi(A)U(g)^* = \pi(\tau_g(A)), \quad A \in \mathcal{U}, g \in G \quad (5)$$

and invariance of the cyclic vector

$$U(g)\Omega = \Omega, \forall g \in G \quad (6)$$

We will use the same notation for $A \in \mathcal{U}$ and its representation $\pi(A)$, i.e. we suppress $\pi(\cdot \cdot \cdot)$. We now give the precise statements of the Theorems.

Theorem 4. Consider a dynamical system $(\mathcal{U}, \iota, \tau)$ in a state $\langle \cdot \cdot \cdot \rangle$ that is invariant under space-time translations and spatially mixing. For every rational direction $\mathbf{n} \in \mathbb{Z}^{D-1}$ of the velocity $\mathbf{v} = v\mathbf{n}$ and every $A, B \in \mathcal{U}$, we have that

$$\lim_{T \to \infty} \frac{1}{T} \int_0^T \langle A([vt], t) B \rangle \, dt = \langle A \rangle \langle B \rangle \quad (7)$$

for almost every speed $v \in \mathbb{R}$. We also have for every non-zero frequency-wavenumber $(\omega, \mathbf{k}) \in \mathbb{R}^{D+1}$ that

$$\lim_{T \to \infty} \frac{1}{T} \int_0^T \langle \exp(\mathbf{k} \cdot \mathbf{v} - \omega) t \ A([vt], t) B \rangle \, dt = I_{\omega, \mathbf{k}} \langle A \rangle \langle B \rangle \quad (8)$$

for almost all $v \in \mathbb{R}$, where

$$I_{\omega, \mathbf{k}} = \lim_{T \to \infty} \frac{1}{T} \int_0^T \exp(i \mathbf{k} \cdot \mathbf{v} - \omega) t \, dt = \begin{cases} 1, & \text{if } \mathbf{k} \cdot \mathbf{v} - \omega = 0 \\ 0, & \text{otherwise} \end{cases} \quad (9)$$
Additionally, for any $m \in \mathbb{N}$ we have,
\[
\lim_{T \to \infty} \left\langle \left( \frac{1}{T} \int_0^T A(\lfloor vt \rfloor, t) \, dt \right)^m \rightangle = \langle A \rangle^m
\] (10)
and for any $(\omega, k) \in \mathbb{R}^{D+1}$,
\[
\lim_{T \to \infty} \left\langle \left( \frac{1}{T} \int_0^T e^{i(k \cdot v - \omega)t} A(\lfloor vt \rfloor, t) \, dt \right)^m \rightangle = I_{\omega, k} \langle A \rangle^m.
\] (11)
for almost every $v \in \mathbb{R}$. The integral $\int_0^T A(\lfloor vt \rfloor, t) \, dt$ is to be understood as Bochner integral, see below.

In particular, every extremal KMS state satisfies the assumptions, which includes every KMS state above a certain temperature. In the GNS representation of KMS states, we also have some stronger results:

**Theorem 5.** Consider a dynamical system $(\mathfrak{U}, \iota, \tau)$, a space-time invariant, spatialle mixing KMS state $\langle \cdots \rangle$ (which may be KMS with respect to another dynamics $\tau'$) and its GNS representation $(H, \pi, \Omega)$. For every rational direction $n \in \mathbb{S}_Q^{D-1}$ of the velocity $v = v n$ and every $A, B \in \mathfrak{U}$, we have that in the GNS representation (we suppress $\pi$) for almost every $v \in \mathbb{R}$:

\[
\lim_{T \to \infty} \frac{1}{T} \int_0^T A(\lfloor vt \rfloor, t) \, dt \Psi = \langle A \rangle \Psi, \quad \forall \Psi \in H
\] (12)
and
\[
\lim_{T \to \infty} \frac{1}{T} \int_0^T [A(\lfloor vt \rfloor, t), B] \, dt \Psi = 0, \quad \forall \Psi \in H
\] (13)
where the integrals are to be understood a Bochner integrals, and the results express convergence of the time average in the strong operator topology.

Note that for an operator valued function $A(T)$, strong operator convergence of $A(T)$ to an operator $A$ means that $\|A(T) \Psi - A \Psi\| \to 0$ for all $\Psi \in H$.

**II. BOCHNER INTEGRALS**

Throughout this work we deal with ray averaged observables $\frac{1}{T} \int_0^T A(\lfloor vt \rfloor, t) \, dt$, where $A \in \mathfrak{U}$ and $(\mathfrak{U}, \iota, \tau)$ is a dynamical system. This integral is to be understood in the Bochner sense [2]. It is easy to see that the function $A(\lfloor vt \rfloor, t)$ is Bochner integrable [2, Definition 3.7.3], as it can be approximated by the countable valued functions (simple functions):

\[
A_n(t) = A(\lfloor vt \rfloor, \frac{\lfloor nt \rfloor}{n})
\] (14)
Indeed, $A(\lfloor vt \rfloor, t) = \tau_t \lfloor vt \rfloor A$ where $\tau_t$ is strongly continuous and $\lim_n \frac{\lfloor nt \rfloor}{n} = t$. Hence,

\[
\lim_n \|A_n(t) - A(\lfloor vt \rfloor, t)\| = 0
\]
for all $t$, for any $A \in \mathcal{U}$. The integral is defined in any interval $[0,T]$ of $\mathbb{R}$, uniquely and independently of the choice of $A_n(t)$ as

$$
\int_0^T A(\{vt\},t) \, dt := \lim_{n \to \infty} \int_0^T A_n(t) \, dt
$$

(15)

where the integral of $A_n(t)$ is defined as a sum over its (countable) values, in the usual way. The Bochner integral calculated inside a linear functional can be pulled outside, by [2, Theorem 3.7.12], in particular for any state $\langle \cdots \rangle$ we have $\langle \int_0^T A(\{vt\},t) \, dt \rangle = \int_0^T \langle A(\{vt\},t) \rangle \, dt$.

Likewise, in the GNS representation $\int_0^T \pi(A(\{vt\},t)) \, dt$ is defined, and the two definitions are compatible in the sense that the integral commutes with $\pi$, i.e. $\pi(\int_0^T A(\{vt\},t) \, dt) = \int_0^T \pi(A(\{vt\},t)) \, dt$, by [2, Theorem 3.7.12].

### III. EXAMPLE: QUANTUM SPIN CHAIN WITH TRIVIAL HAMILTONIAN

Consider the quantum spin chain $\mathbb{Z}$ with the trivial interaction containing only one-body terms $\Phi(\{n\}) = \sigma^z_n$, $n \in \mathbb{Z}$, which act non-trivially only on the local algebra $\mathcal{U}_{\{n\}}$ (measuring $n$-th the spin in the $z$ direction). As such, the local Hamiltonian for finite $\Lambda \subset \mathbb{Z}$ is a sum of commuting terms $H_\Lambda = \sum_{n \in \Lambda} \sigma^z_n$. Despite the interaction being trivial, it demonstrates the “almost everywhere” aspect of ergodicity, as we will show that in this case ergodicity holds in every ray, except for the $v = 0$ one. Consider the local Gibbs state

$$
\omega_\Lambda(A) := \frac{\text{Tr}(e^{-\beta H_\Lambda} A)}{\text{Tr}(e^{-\beta H_\Lambda})}, \quad A \in \mathcal{U}_\Lambda
$$

(16)

It is well-established that every thermodynamic limit point $\Lambda \to \infty$ of $\omega_\Lambda$ is a $\beta$-KMS state, and at high temperatures this will also be the unique $\mathcal{Z}$-invariant KMS state [3, Section 6.2]. Hence the Ergodicity Theorem holds and we can directly verify this. Indeed, consider any local observables $A, B \in \mathcal{U}_{\text{loc}}$, a velocity $v \neq 0$ and take a finite $\Lambda \subset \mathbb{Z}$ and $T > 0$ sufficiently large,

$$
\frac{1}{T} \int_0^T \omega_\Lambda(A(\{vt\},t)B) \, dt = \frac{1}{T} \int_0^T \frac{\text{Tr}(e^{-\beta H_\Lambda} e^{itH_\Lambda} A(\{vt\},0) e^{-itH_\Lambda} B)}{\text{Tr}(e^{-\beta H_\Lambda})} \, dt
$$

(17)

We can split the integral as $\int_0^{T_0} + \int_{T_0}^T$ with $T_0$ large enough so that the support of $A(\{vt\},0)$ is disjoint with the support of $B$ for all $T > T_0$. At the long time limit the first term will go to zero. In the second integral we can separate the terms of the Hamiltonian as $H_\Lambda = H_{\text{supp}(B)} + H_{\Lambda \setminus \text{supp}(B)}$, so that $e^{-itH_{\text{supp}(B)}}$ commutes with $A(\{vt\},0)$ and cancels out. We get:

$$
\frac{1}{T} \int_{T_0}^T \frac{\text{Tr}(e^{-\beta H_{\Lambda \setminus \text{supp}(B)}} e^{itH_{\Lambda \setminus \text{supp}(B)}} A(\{vt\},0) e^{-itH_{\Lambda \setminus \text{supp}(B)}} e^{-\beta H_{\text{supp}(B)}} B)}{\text{Tr}(e^{-\beta H_{\text{supp}(B)}} e^{-\beta H_{\Lambda \setminus \text{supp}(B)}})} \, dt
$$

(18)

Now, since the trace of a tensor product is the product of the traces, we get

$$
\frac{1}{T} \int_{T_0}^T \omega_\Lambda(A(\{vt\},t)) \, dt \omega_\Lambda(B)
$$

(19)
Hence, we’ve shown for sufficiently large $\Lambda$ and $T$

$$\frac{1}{T} \int_0^T \omega_\Lambda(A([vt],t)B) \, dt = \frac{1}{T} \int_0^{T_0} \omega_\Lambda(A([vt],t)B) \, dt + \frac{1}{T} \int_{T_0}^T \omega_\Lambda(A([vt],t)) \, dt \omega_\Lambda(B) \quad (20)$$

for all $\Lambda$ sufficiently large and $T > T_0$. Omitting the details, we can first take the thermodynamic limit $\Lambda \to \mathbb{Z}$ and then perform the $T \to \infty$ limit. We thus have

$$\lim_{T \to \infty} \frac{1}{T} \int_0^T \omega(A([vt],t)B) \, dt = \omega(A)\omega(B), \quad \forall \nu \neq 0 \quad (21)$$

for all local observables, which easily extends to the whole quasi-local algebra, since local elements are dense.

In the case $\nu = 0$ the above arguments fail, since the observable is moved purely in time and its support does not change. Hence, one can find examples where the result will not be $\omega(A)\omega(B)$.

IV. EXAMPLE: OSCILLATORY ASYMPTOTICS OF FREE FERMION CORRELATION FUNCTIONS

In this section, we provide an example of application of the oscillatory Boltzmann-Gibbs principle. In order to make the example as clear as possible, we do not attempt to express rigorous results here.

For simplicity, and in order to verify the ideas, we consider a one-dimensional quadratic model, where asymptotics of correlation functions can be obtained by an elementary saddle-point analysis. Such models are integrable, and the usual Boltzmann-Gibbs principle has been studied widely in the context of generalised hydrodynamics [4–7], and verified to agree with a saddle-point analysis [8].

The mathematical results reported in the main text apply to quantum lattice models with finite local spaces, and are based on the bosonic version of the C$^*$ algebra formulation of quantum statistical mechanics, where local operators commute with each other. Hence, they do not cover free bosonic chains (infinite-dimensional local space), nor free fermionic chains (fermionic formulation). Nevertheless, all results are expected to hold in both cases. In this section, we give the example of the free fermionic chain, with Hamiltonian

$$H = -\frac{J}{2} \sum_{x \in \mathbb{Z}} (c_{x+1}^\dagger c_x + c_x^\dagger c_{x+1}), \quad \{c_x^\dagger, c_y\} = \delta_{x,y}. \quad (22)$$

A. Hydrodynamic projections

In one-dimensional models, the results of [9] are stronger than those reported in the main text, as the linearised Euler equation is also proven. In combination with the Boltzmann-Gibbs principle, this makes the asymptotic of correlation functions more explicit. At a non-rigorous level, the formulae obtained are reviewed in [7] in the usual (non-oscillatory) case. For our
purpose, it is sufficient to recall that if $A_i, B_i$ are pairs of conserved densities and currents, such that the extensive charges $a_i = \Sigma A_i$ form a basis for the conserved quantities $Q$, then

$$\left\langle A_i(x, t) A_j(0, 0) \right\rangle_c \sim \ell^{-1} \left[ \delta(\bar{x} - A\ell) C_{ij} \right]_{ij} \quad (x = \ell \bar{x}, \ t = \ell \bar{t}, \ \ell \to \infty) \quad (23)$$

where the flux Jacobian is (see eq. (5) in the main text for the inner product $(B_i, A_l)$)

$$A^j_i = \sum_l (B_i, A_l) C_{lj}, \quad (24)$$

the statistic susceptibility matrix is

$$C_{ij} = (A_i, A_j) \quad \text{(with } C^{ij} \text{ the inverse matrix, } \sum_j C^{ij} C_{jl} = \delta^{il}) \quad (25)$$

and the fluid-cell mean can be taken as

$$\overline{A(x, t)} = \frac{1}{L^2} \sum_{y = -\frac{L}{2}}^{\frac{L}{2}} \int_{x - \frac{L}{2}}^{x + \frac{L}{2}} A(x + y, t + s) \, ds. \quad (26)$$

Here the mesoscopic length can be taken as $L = L(\ell)$ with $L \to \infty$ fast enough as $\ell \to \infty$, and $L/\ell \to 0$ (or as $L = \epsilon \ell$, then taking $\epsilon \to 0^+$ on the asymptotic large-$\ell$ result).

Three remarks are in order, see the explanations in [7]:

(i) The right-hand side of eq. (23) is obtained from the right-hand side of eq. (7) in the main text by (1) expressing the continuity equation relating $\left\langle A_i(x, t) A_j(0, 0) \right\rangle_c$ and $\left\langle B_i(x, t) A_j(0, 0) \right\rangle_c$; (2) using the projection formula, eq. (7) in the main text, expressing $S_{\Sigma B_i, \Sigma A_j}(\kappa)$ in terms of $S_{\Sigma A_l, \Sigma A_j}(\kappa)$, Fourier transforming back to real space to get $\left\langle B_i(x, t) A_j(0, 0) \right\rangle_c$ in terms of $\left\langle A_l(x, t) A_j(0, 0) \right\rangle_c$; and (3) solving the resulting continuity equation for the matrix of correlators $\left\langle A_i(x, t) A_j(0, 0) \right\rangle_c$, with appropriate initial condition. The solution is (23).

(ii) The fluid-cell mean (26) is different from that in eq. (7) in the main text. Eq. (7) expresses a rigorous result, but the result is expected to hold for a variety of possible definitions of fluid-cell means. The fluid-cell mean chosen above is more physically transparent and more convenient for our purposes.

(iii) If there is a finite set of basis charges (a finite set of indices $i$), the right-hand side of (23) is a “generalised function”. Its meaning is that for $\bar{x}/\ell$ equal to an eigenvalue of $A$, the large-$\ell$ asymptotic decays more slowly than $1/\ell$, while for other velocities, it decays more rapidly. The delta-function comes about from the normalisation condition:

$$C_{ij} = \sum_{x \in \mathbb{Z}} \left\langle A_i(x, t) A_j(0, 0) \right\rangle_c = \ell \int d\bar{x} \left\langle A_i(\bar{x}, t) A_j(0, 0) \right\rangle_c \quad (27)$$

where we use $1 = d\bar{x} = \ell d\bar{x}$ and the fact that $\sum_{x \in \mathbb{Z}} A_i(x, t)$ is independent of time. If there is a continuous set of basis charges, such as it typical in integrable models, then the large-$\ell$ asymptotics is exactly $1/\ell$, and the right-hand side of (23) is an ordinary function. See [5, 7].
The observation made in [9] and, for higher-dimensional lattices, in the main text, is that all formulae apply equally well in the oscillatory case: with the space-time translation group defined as
\[ \eta_{x,t}^{\omega,k} : A \mapsto e^{i\omega t - ikx} A(x, t). \] (28)
This is because \( \eta_{x,t}^{\omega,k} \) still is a unitary operator on the GNS space. Thus we have a space of \( k \)-extensive observables, the equivalence classes \( \Sigma^k A \) built using the positive semidefinite sesquilinear form
\[ (A, B)_k = \sum_{k \in \mathbb{Z}} e^{ikx} \langle A(x) B(0) \rangle^c, \] (29)
and from it we may construct, formally, the \( \omega, k \)-conserved charges, \( Q_{\omega,k} \).

Thus, for frequency-wavenumber \( \omega, k \), if \( A_i, B_i \) are pairs of oscillatory conserved densities and currents (eq. (9) in the main text), such that \( a_i = \Sigma^k A_i \) form a basis for \( \mathcal{Q}_{\omega,k} \), then
\[ \langle \left( A_i(x, t)^{\omega,k} \right)^\dagger A_j(0, 0) \rangle^c \sim e^{i\omega t - ikx} \ell^{-1} [\delta(\bar{x} - A^{\omega,k} \bar{t}) C_{\omega,k}]_{ij} \] (30)
where the oscillatory flux Jacobian is
\[ [A^{\omega,k}]_{ij}^k = \sum_l (B_i, A_l)_k [C_{\omega,k}]_{lj}^k, \] (31)
the statistic susceptibility matrix is
\[ [C^{\omega,k}]_{ij} = (A_i, A_j)_k \] (32)
and the fluid-cell mean is
\[ A^{\omega,k}(x, t) = \frac{1}{L^2} \sum_{y = -L^2}^{L^2} \int_{-L^2}^{L^2} e^{i\omega s - iky} A(x + y, t + s) \, ds, \] (33)

B. Saddle point analysis

We are looking to evaluate the asymptotic form of the thermal correlation function
\[ \langle c_x^\dagger(t) c_0(0) \rangle = \frac{\text{Tr}(e^{-\beta H} c_0^\dagger(t) c_0(0))}{\text{Tr}(e^{-\beta H})}, \quad c_x(t) = e^{iHt} c_x e^{-iHt} \] (34)
as \( x, t \to \infty \). The Hamiltonian (22) is diagonalised by the Fourier transform
\[ c_x = \int_{-\pi}^{\pi} \frac{dk}{\sqrt{2\pi}} e^{ikx} a(k), \quad \{a(\bar{k}), a(k')\} = \delta(k - k') \] (35)
as
\[
H = -\int_{-\pi}^{\pi} dk J \cos k \ a^\dagger(k) a(k)
\] (36)

with energy spectrum \(E(k) = -J \cos k\) and
\[
a(k, t) = e^{iHt} a(k) e^{-iHt} = e^{-iE(k)t} a(k).
\] (37)

Therefore we obtain the usual expression for the correlator
\[
\langle c^\dagger \ x(t) c_0(0) \rangle = \int_{-\pi}^{\pi} \frac{dk}{2\pi} n(k) e^{iE(k)t - ikx}, \quad n(k) = \frac{1}{1 + e^{\beta E(k)}}.
\] (38)

A similar analysis as that below can be performed in a generalised Gibbs ensemble (GGE), where \(n(k)\) takes an arbitrary form that characterises the GGE.

The asymptotic behaviour is easily obtained by a saddle point analysis. With \(k_\pm\) solving \(v(k_\pm) = x/t\) where \(v(k) = E'(k) = J \sin k\) and \(k_+ \in [-\pi/2, \pi/2], k_- = \text{sgn}(k_+)(\pi - k_+)\), we obtain
\[
\langle c^\dagger \ x(t) c_0(0) \rangle \sim \sum_{\pm} n(k_\pm) \sqrt{\frac{2\pi}{iE'(k_\pm)}} e^{iE(k_\pm)t - ik_\pm x}
\] (39)
as \(t \to \infty\) with \(x/t = \xi\) fixed. The decay in \(1/\sqrt{t}\), which is slower than \(1/t\), indicates that the hydrodynamic projection formula should give a generalised function. As all factors explicitly written as functions of \(k_\pm\) are slowly varying, they may be assumed to be constant within the fluid cell. Then we see that the support of the \((\omega, k)\)-fluid-cell mean (33) is on \(k = k_\pm\) and \(\omega = E(k_\pm)\), which is
\[
\xi = v(k), \quad \omega = E(k).
\] (40)

This saddle point analysis, however, does not provide the full shape of the correlation function around the velocity \(\xi = v(k)\). However, because we know the support, we may write
\[
\langle (c_t(x)^{\omega,k})^\dagger c_0(0) \rangle \sim e^{i\omega t - ikx} \ell^{-1} \delta(\bar{x} - v(k) \ell) R(k).
\] (41)

The normalisation \(R(k)\) is obtained by evaluating
\[
\frac{1}{L} \int_{-\frac{L}{2}}^{\frac{L}{2}} ds \sum_{x \in \mathbb{Z}} e^{ikx - i\omega s} \langle c^\dagger_t(c(t + s)c_0(0)) \rangle \sim \ell \int d\bar{x} e^{ikx} \langle (c_t(x)^{\omega,k})^\dagger c_0(0) \rangle
\] (42)
where again we use \(1 = dx = \ell d\bar{x}\). The left-hand side is found to be \(e^{i\omega t} n(k)\) from (38), while the right-hand side is found to be \(e^{i\omega t} R(k)\) from (41), giving
\[
R(k) = n(k).
\] (43)
C. Oscillatory hydrodynamic projection

We note that

\[ e^{iE(k)t}a(k,t) \]  

(44)

is independent of time. Further,

\[ a(k) = \frac{1}{\sqrt{2\pi}} \sum_x e^{-ikx}c_x. \]  

(45)

Therefore, \( a(k) \) is a \( k \)-extensive, \( \omega \)-conserved charge, \( a(k) \in Q_{\omega,k} \), for \( \omega = E(k) \), with local density

\[ A = \frac{c_0}{\sqrt{2\pi}}. \]  

(46)

It is a simple matter to verify that the associated current has the form

\[ B = \frac{iJ}{2\sqrt{2\pi}}(c_{-1} - e^{ik}c_0) \]  

(47)

in that \( A(x,t) \) and \( B(x,t) \) satisfy eq. (9) in the main text. Then we evaluate

\[ C^{\omega,k} = \frac{1}{\sqrt{2\pi}}\langle a(t)c_0 \rangle = \frac{n(k)}{2\pi}, \quad (B,A)_k = \langle B^a(k) \rangle = \frac{n(k)v(k)}{2\pi} \]  

(48)

and thus

\[ A^{\omega,k} = v(k). \]  

(49)

Hence, the prediction from hydrodynamic projections is

\[ \langle \left( \frac{c_x(t)^{\omega,k}}{c_0(0)} \right)^\dagger c_0(0) \rangle \sim e^{i\omega t - ikx}e^{-1}\delta(x - v(k)t) n(k) \]  

(50)

in agreement with (41) and (43).

\[ \begin{align*}
[1] & \text{O. Bratteli and D. Robinson, Operator Algebras and Quantum Statistical Mechanics 1: C*- and W*- Algebras. Symmetry Groups. Decomposition of States, Operator Algebras and Quantum Statistical Mechanics (Springer, 1987).} \\
[2] & \text{E. Hille and R. Phillips, Functional Analysis and Semi-groups, American Mathematical Society: Colloquium publications (American Mathematical Society, 1996) issue: v. 31, pt. 1.} \\
[3] & \text{O. Bratteli and D. Robinson, Operator Algebras and Quantum Statistical Mechanics 2: Equilibrium States Models in Quantum Statistical Mechanics, Operator Algebras and Quantum Statistical Mechanics (Springer, 1997).} \\
[4] & \text{B. Doyon and H. Spohn, SciPost Physics 3, 039 (2017).} \\
[5] & \text{B. Doyon, SciPost Phys. 5, 54 (2018).} 
\end{align*} \]
[6] B. Doyon, SciPost Phys. Lect. Notes (2020), 10.21468/SciPostPhysLectNotes.18.
[7] J. Nardis, B. Doyon, M. Medenjak, and M. Panfil, Correlation functions and transport coefficients in generalised hydrodynamics (to appear in J. Stat. Mech, 2021).
[8] G. Del Vecchio Del Vecchio and B. Doyon, “The hydrodynamic theory of dynamical correlation functions in the XX chain,” (2021), arXiv: 2111.08420 [math-ph].
[9] B. Doyon, “Hydrodynamic projections and the emergence of linearised Euler equations in one-dimensional isolated systems,” (2020), arXiv: 2011.00611 [math-ph].