Critical structure factors of bilinear fields in $O(N)$-vector models.

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Abstract

We compute the two-point correlation functions of general quadratic operators in the high-temperature phase of the three-dimensional $O(N)$ vector model by using field-theoretical methods. In particular, we study the small- and large-momentum behavior of the corresponding scaling functions, and give general interpolation formulae based on a dispersive approach. Moreover, we determine the crossover exponent $\phi_T$ associated with the traceless tensorial quadratic field, by computing and analyzing its six-loop perturbative expansion in fixed dimension. We find: $\phi_T = 1.184(12)$, $\phi_T = 1.271(21)$, and $\phi_T = 1.40(4)$ for $N = 2, 3, 5$ respectively.

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I. INTRODUCTION

In nature many physical systems undergo phase transitions belonging to universality classes of the O(N) vector models. Their universal critical properties can be determined theoretically by considering the \( \phi^4 \) Hamiltonian

\[
\mathcal{H} = \int d^d x \left[ \frac{1}{2} \partial_\mu \vec{\phi} \cdot \partial_\mu \vec{\phi} + \frac{1}{2} r \vec{\phi} \cdot \vec{\phi} + \frac{1}{4!} u(\vec{\phi} \cdot \vec{\phi})^2 \right],
\]

where \( \vec{\phi}(x) \) is an \( N \)-component real field. Various computational methods, supported by renormalization-group (RG) theory, have provided accurate determinations of several universal quantities, see, e.g., Ref. [1] for a recent comprehensive review. Among others, we should mention the critical exponents, the equation of state, and the correlation functions of the order parameter \( \vec{\phi}(x) \). However, for some experimental systems one is also interested in the behavior of correlation functions describing the critical fluctuations of secondary, quadratic local fields. Due to the symmetry of the theory, there are two independent quantities that are quadratic in the fundamental field \( \vec{\phi}(x) \): one is the local energy density

\[
E(x) = \vec{\phi}(x) \cdot \vec{\phi}(x),
\]

which is O(N) invariant; the other one is the anisotropic second-order traceless tensor

\[
T_{ij}(x) = \phi_i(x)\phi_j(x) - \delta_{ij} \frac{1}{N} \vec{\phi}(x) \cdot \vec{\phi}(x).
\]

The crossover exponent \( \phi_T \) associated with the traceless tensor field \( T_{ij}(x) \) describes the instability of the O(N)-symmetric theory against anisotropy [2–5]. It is thus relevant for the description of multicritical phenomena, for instance the critical behavior near a bicritical point where two critical lines with \( O(N) \) and \( O(M) \) symmetry meet, giving rise to a critical theory with enlarged \( O(N + M) \) symmetry, see, e.g., Refs. [6–8]. This bicritical behavior has been the object of new studies quite recently, since it appears in the \( SO(5) \) theory of superconductivity [3], and has been observed experimentally in organic conductors [10]. As discussed in Ref. [11], the correlation functions \( G_E(x - y) \equiv \langle E(x)E(y) \rangle \) and \( G_T(x - y) \equiv \langle T_{ij}(x)T_{ij}(y) \rangle \) are relevant in the description of strain-strain correlations in certain liquids and solids, where an effective coupling between the order parameter and the elastic deformations occurs. Moreover, in the special case \( N = 2 \), the traceless tensor field \( T_{ij}(x) \) is related to the second-harmonic order parameter in density-wave systems, whose critical behavior belongs to the XY universality class, see, e.g., Refs. [12,13,11]. Experimentally, such behavior is observed at the nematic-smectic A transition in liquid crystals [12,14,19,11]. In these systems the structure factor of the secondary order parameter \( T_{ij} \) has been measured using x-ray scattering techniques [18,19]. The crossover exponent \( \phi_T \) is also relevant [20] in the description of crossover effects in diluted Ising antiferromagnets with \( n \)-fold degenerate ground state [21], for instance in some diluted magnetic semiconductors such as \( \text{Cd}_{1-x}\text{Mn}_x\text{Te} \).

In this paper we determine the crossover exponent \( \phi_T \). Such a quantity has already been obtained in the framework of the \( \epsilon \)-expansion to three loops [22], from the analysis of
high-temperature expansions [6] for \( N = 2, 3, 5 \), and by means of a Monte Carlo simulation [23] for \( N = 5 \). Here, we consider the alternative field-theoretical (FT) method based on a fixed-dimension expansion in powers of the zero-momentum quartic coupling [24], and perform a six-loop calculation of \( \phi_T \). For the physically interesting cases \( N = 2, 3, 5 \) we obtain

\[
\begin{align*}
\phi_T &= 1.184(12) \quad (N = 2) , \\
\phi_T &= 1.271(21) \quad (N = 3) , \\
\phi_T &= 1.40(4) \quad (N = 5) .
\end{align*}
\] (4)

We also consider the correlation functions \( G_E(x) \) and \( G_T(x) \) in the high-temperature phase. In the critical limit, the Fourier transform \( \tilde{G}_T(q) \) obeys a scaling law that is analogous to that of the fundamental correlation function, i.e.

\[
\tilde{G}_T(q, t) = A_T^+ t^{-\gamma_T} f_T(q^2 \xi^2) ,
\] (5)

where \( t \equiv (T - T_c)/T_c \) is the reduced temperature, \( \gamma_T = 2\phi_T - 2 + \alpha \) is the tensor susceptibility exponent and \( \xi \) is the second-moment correlation length computed from the two-point function of the order parameter. The same scaling behavior holds for the correlation functions \( \tilde{G}_E(q, t) \) of systems in the Ising universality class, with \( \alpha \) replacing \( \gamma_T \), i.e. \( \tilde{G}_E(q, t) = A_E^+ t^{-\alpha} f_E(q^2 \xi^2) \). For \( N \geq 2 \), however, \( \alpha \) is negative and an additional background term should be taken into account. In this case, in the critical limit, we have

\[
\tilde{G}_E(q, t) = B_E + \tilde{G}_{E,\text{sing}}(q, t) = B_E + A_E^+ t^{-\alpha} f_E(q^2 \xi^2) .
\] (6)

The background term \( B_E \) is the dominant one and the singular part vanishes at criticality. In this case, by using positivity (unitarity in FT language) arguments, one may also show that \( A_E^+ < 0 \), as observed in experiments.

In this paper we extend the two-loop \( \epsilon \)-expansion computation of Refs. [11,19]. We compute the universal scaling functions \( f_E(q^2 \xi^2) \) and \( f_T(q^2 \xi^2) \) using the \( \epsilon \) expansion and the expansion in fixed dimension \( d = 3 \). First, we determine the small-momentum behavior to four loops in the fixed-dimension expansion and to three loops in \( \epsilon \)-expansion. In particular, we obtain accurate estimates of the experimentally relevant ratios \( X_{E,T} \equiv \xi_{E,T}^2 / \xi^2 \), where \( \xi_{E,T} \) is the second-moment correlation length computed from \( G_{E,T}(x) \) or from its singular part if \( \alpha \) is negative. For instance, for \( N = 1 \) we find

\[
X_E = 0.0140(5) ,
\] (7)

and for \( N = 2 \)

\[
X_E = -0.0017(1) , \quad X_T = 0.041(2) .
\] (8)

Moreover, we study the large-momentum behavior of the structure factors and construct interpolations valid for all momenta by using the dispersive approach applied to \( \langle \phi(0)\phi(x) \rangle \) by Bray [25].

The paper is organized as follows. In Sec. II we report the computation of the crossover exponent \( \phi_T \) to six loops in the fixed-dimension expansion and compare our results with
the existing theoretical and experimental estimates (Sec. II D). In Sec. III we report the computation of the structure factors. In Sec. III A we briefly summarize the expected behavior of the structure factors in the critical region and set our notations. In Sec. III B we explain our FT calculation, whose results are presented in Sec. III C. In Sec. III D we finally give approximate expressions for the structure factors by using a dispersive approach. Appendix A discusses the large-momentum behavior of the structure factors. Details of the perturbative calculation are reported in App. B.

II. THE CROSSOVER EXPONENT ASSOCIATED WITH THE TENSOR COMPOSITE FIELD

A. Zero-momentum scaling behavior

The zero-momentum behavior of correlation functions involving generic local operators \( O(x) \), such as \( E(x) \) and \( T_{ij}(x) \), can be obtained from the free energy in the presence of an external field \( h_O \) coupled with \( O(x) \). Indeed, the singular part of the free energy scales as

\[
F_{\text{sing}} \propto t^{2-\alpha} f \left( h/t^{\beta}, h_O/t^{\phi_O} \right),
\]

where \( h \) is the magnetic field, and \( \phi_O \) is the crossover exponent. Then, by differentiating with respect to \( h_O \), one obtains the zero-momentum correlations and the RG relations

\[
\beta_O = 2 - \alpha - \phi_O, \\
\gamma_O = -2 + \alpha + 2\phi_O,
\]

where the exponents \( \beta_O \) and \( \gamma_O \) describe respectively the critical (singular) behavior of the average \( \langle O(x) \rangle \sim |t|^{\beta_O} \) and of the susceptibility \( \chi_O \equiv \sum_x \langle O(0)O(x) \rangle_c \sim t^{-\gamma_O} \).

In this section we compute the crossover exponent \( \phi_T \) associated with the tensor field \( T_{ij}(x) \) in the fixed-dimension FT framework, by performing a six-loop perturbative expansion. Of course, the crossover exponent associated with the energy density \( E(x) \) is trivial, i.e., \( \phi_E = 1 \) and \( \gamma_E = \alpha \).

B. The fixed-dimension expansion: generalities

In the fixed-dimension FT approach, one renormalizes the theory by introducing a set of zero-momentum conditions for the two-point and four-point one-particle irreducible correlation functions

\[
\Gamma_{ij}^{(2)}(p) = \delta_{ij} Z^{-1}_\phi [m^2 + p^2 + O(p^4)],
\]

\[
\Gamma_{ijkl}^{(4)}(0) = m^\epsilon Z^{-2}_\phi g^{\frac{1}{3}} \left( \delta_{ij} \delta_{kl} + \delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk} \right),
\]

where \( \epsilon \equiv 4 - d \) and \( d \) is the space dimension. They relate the mass \( m \) and the zero-moment renormalized coupling \( g \) to the corresponding Hamiltonian parameters \( r \) and \( u \).
\[ u = m^e g Z_u(g) Z_{\phi}(g)^{-2}. \] (13)

In addition, one introduces the function \( Z_t \) that is defined by the relation
\[
\Gamma_{ij}^{(1,2)}(0) = \delta_{ij} Z_t(g)^{-1},
\] (14)

where \( \Gamma^{(1,2)}(p) \) is the one-particle irreducible two-point function with an insertion of \( \frac{1}{2} \phi^2 \).

The critical theory is obtained by setting \( g = g^* \), where \( g^* \) is the nontrivial zero of the \( \beta \)-function
\[
\beta(g) = \frac{\partial g}{\partial m} \bigg|_u.
\] (15)

The standard critical exponents are then obtained by evaluating the RG functions
\[
\eta_\phi(g) = \left. \frac{\partial \ln Z_\phi}{\partial \ln m} \right|_u,
\]
\[
\eta_t(g) = \left. \frac{\partial \ln Z_t}{\partial \ln m} \right|_u
\] (16)

at the fixed point \( g^* \), i.e.
\[
\eta = \eta_\phi(g^*),
\]
\[
\frac{1}{\nu} = 2 + \eta_t(g^*) - \eta_\phi(g^*).
\] (17)

In three dimensions these RG functions are known to six loops for generic values of \( N \) [26,27]. For \( N = 0, 1, 2, 3 \), seven-loop series for \( \eta_\phi \) and \( \eta_t \) were computed in Ref. [28].

In order to evaluate the crossover exponent \( \phi_T \) associated with the operator \( T_{ij}(x) \), we define the renormalization function \( Z_T(g) \) from the one-particle irreducible two-point function \( \Gamma_T^{(2)}(p) \) with an insertion of the operator \( T_{ij} \), i.e. we set
\[
\Gamma_T^{(2)}(0)_{ij;kl} = Z_T^{-1}(g) A_{ijkl},
\] (18)

where
\[
A_{ijkl} = \delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk} - \frac{2}{N} \delta_{ij} \delta_{kl},
\] (19)

so that \( Z_T(0) = 1 \). Then, we compute the RG function
\[
\eta_T(g) = \left. \frac{\partial \ln Z_T}{\partial \ln m} \right|_u = \beta(g) \frac{d \ln Z_T}{dg},
\] (20)

and \( \eta_T = \eta_T(g^*) \). Finally, the RG scaling relation
\[
\phi_T = (2 + \eta_T - \eta) \nu
\] (21)

allows us to determine \( \phi_T \).
C. The fixed-dimension expansion: six-loop results

We computed $\Gamma^{(2)}_T(0)$ to six loops. The calculation is rather cumbersome, since it requires the evaluation of 563 Feynman diagrams. We handled it with a symbolic manipulation program, which generates the diagrams and computes the symmetry and group factors of each of them. We used the numerical results compiled in Ref. [29] for the integrals associated with each diagram. We obtained

$$\eta_T(\bar{g}) = -\bar{g} \frac{2}{8 + N} + \bar{g}^2 \frac{2(6 + N)}{3(8 + N)^2} - \bar{g}^3 \frac{18.312844 + 3.433275N - 0.21674589N^2}{(8 + N)^3}$$

$$+ \bar{g}^4 \frac{140.79937 + 37.573408N + 1.0362736N^2 + 0.09432565N^3}{(8 + N)^4}$$

$$- \bar{g}^5 \frac{1340.075 + 416.71657N + 17.622623N^2 - 0.91128056N^3 - 0.050833747N^4}{(8 + N)^5}$$

$$+ \bar{g}^6 \frac{15651.266 + 5665.6519N + 433.68712N^2 + 1.0675503N^3 + 0.67910559N^4 + 0.031393004N^5}{(8 + N)^6}$$

$$+ O(\bar{g}^7),$$

where, as usual, we have introduced the rescaled coupling $\bar{g}$ defined by

$$g = \frac{48\pi}{8 + N} \bar{g}.$$  

Field-theoretical perturbative expansions are divergent, and thus, in order to obtain accurate results, an appropriate resummation is required. We use the method of Ref. [30] that takes into account the large-order behavior of the perturbative expansion, see, e.g., Ref. [31]. Mean values and error bars are computed using the algorithm of Ref. [32].

Given the expansion of $\eta_T(\bar{g})$, we determine the perturbative expansion of $\phi_T(\bar{g})$, $\beta_T(\bar{g})$, and $\gamma_T(\bar{g})$, using the relations (10) and (21). For $N = 2$ we obtain $[33] \phi_T = 1.176(4)$, $1.178(3)$, $\beta_T = 0.821(6)$, $0.825(5)$, and $\gamma_T = 0.355(2)$, $0.358(3)$, where, for each exponent, we report the estimate obtained from the direct analysis and from the analysis of the series of the inverse, i.e. from $1/\phi_T(g)$, etc. The two estimates obtained for each exponent agree within error bars, but, with the quoted errors, the scaling relations (10) are not well satisfied. For instance, using $\nu = 0.67155(27)$ (Ref. [34]) and $\beta_T = 0.823(6)$ we obtain $\phi_T = 1.192(6)$, while using the same value of $\nu$ and $\gamma_T = 0.3565(30)$ we have $\phi_T = 1.1855(15)$. These two estimates are slightly higher than those obtained from the analysis of $\phi_T(g)$ and $1/\phi_T(g)$. Clearly, the errors are somewhat underestimated, a phenomenon that is probably connected with the nonanalyticity [35–37] of the RG functions at the fixed point $\bar{g}^*$. 

In order to obtain a conservative estimate, we have thus decided to take as estimate of $\phi_T$ the weighted average of the direct estimates and of the estimates obtained using $\beta_T$ and $\gamma_T$ together with the scaling relations [38]. The (very conservative) error is such to include all estimates. The other exponents are dealt with analogously. The final results for several values of $N$ are reported in Table I.
TABLE I. Critical exponents associated with the tensor field $T_{ij}(x)$.

| $N$ | $\phi_T$ | $\beta_T$ | $\gamma_T$ |
|-----|-----------|------------|------------|
| 2   | 1.184(12) | 0.830(12)  | 0.354(25)  |
| 3   | 1.271(21) | 0.863(21)  | 0.41(4)    |
| 4   | 1.35(4)   | 0.90(4)    | 0.45(8)    |
| 5   | 1.40(4)   | 0.90(4)    | 0.50(8)    |
| 8   | 1.55(4)   | 0.94(4)    | 0.61(8)    |
| 16  | 1.75(6)   | 0.98(6)    | 0.77(12)   |

D. Comparison with previous results

The exponent $\phi_T$ can also be computed in the $\epsilon$ expansion. Three-loop series were derived in Ref. [22]:

$$
\phi_T = 1 + \epsilon + \frac{N}{2(N+8)} + \epsilon^2 \frac{N^3 + 24N^2 + 68N}{4(N+8)^3} + \epsilon^3 \frac{N^5 + 48N^4 + 788N^3 + 3472N^2 + 5024N - 48N(5N + 22)(N+8)\zeta(3)}{8(N+8)^5} + O(\epsilon^4).
$$

(24)

The coefficients of this series decrease rapidly; for instance, we have

$$
\phi_T(N = 2) = 1 + 0.1\epsilon + 0.06\epsilon^2 - 0.00735899\epsilon^3 + O(\epsilon^4),
$$

(25)

$$
\phi_T(N = 3) = 1 + 0.136364\epsilon + 0.083959\epsilon^2 + 0.000991\epsilon^3 + O(\epsilon^4),
$$

(26)

for $N = 2$ and 3 respectively. Thus, any resummation gives estimates that do not differ significantly from those obtained by simply setting $\epsilon = 1$. For $N = 2, 3$ we obtain $\phi_T \approx 1.15$, $\phi_T \approx 1.22$, in reasonable agreement—keeping into account that these are three-loop results—with the estimates of Table I. They are also in agreement with the estimate of Ref. [14] that reports $\phi_T = 1.16(7)$ for $N = 2$, which has been obtained by analyzing the same $O(\epsilon^3)$ series and the two-loop series calculated in the framework of the fixed-dimension expansion.

The exponent $\phi_T$ has been also computed in the $1/N$ expansion [33] for $d = 3$:

$$
\phi_T = 2 - \frac{32}{\pi^2N} + O\left(\frac{1}{N^2}\right).
$$

(27)

For $N = 16$ it gives $\phi_T = 1.80$, which agrees with the FT result of Table I.

The exponent $\phi_T$ has been estimated by high-temperature expansion techniques in Ref. [3], obtaining $\phi_T = 1.175(15)$ for $N = 2$ and $\phi_T = 1.250(15)$ for $N = 3$, in agreement with the FT estimates. For $N = 5$, the exponent $\phi_T$ has also been determined by means of a Monte Carlo simulation [23]: $\phi_T = 1.387(30)$.

Experimental estimates of $\phi_T$ are reported in Ref. [11]. We mention the experimental result $\phi_T = 1.17(2)$ for the $(2 \rightarrow 1 + 1)$ bicritical point in GdAlO$_3$ [11]. The $(3 \rightarrow 2 + 1)$ bicritical behavior has been studied in MnF$_2$ [12], obtaining $\phi_T = 1.279(31)$. The experimental results obtained for a nematic–smectic-A transition reported in Ref. [18] are $\beta_T = 0.76(4)$ and $\gamma_T = 0.41(9)$.  

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III. THE STRUCTURE FACTOR OF THE BILINEAR FIELDS IN THE HIGH-TEMPERATURE PHASE

A. Scaling behavior

The two-point correlation function of the fundamental field, i.e. $G(x) = \langle \hat{\phi}(0) \cdot \hat{\phi}(x) \rangle$, is of central importance because its Fourier transform $\tilde{G}(q)$ is directly related to the scattering intensity in scattering experiments. For $t \to 0^+$, its asymptotic behavior is given by

$$\tilde{G}(q) = C^+ t^{-\gamma} f(q^2\xi^2),$$

(28)

where $C^+$ is the amplitude of the magnetic susceptibility and the function $f(y)$ is universal. Taking the second-moment correlation length

$$\xi^2 \equiv \frac{1}{2d} \sum_x |x|^2 G(x) = -\tilde{G}(0)^{-1} \frac{\partial \tilde{G}(q)}{\partial q^2} \bigg|_{q^2=0}$$

(29)

as length scale, the small-momentum behavior of $f(y)$ is $f(y) = 1/(1+y) + O(y^2)$, with very small $O(y^2)$ corrections. Theoretical results for the correlation function $\tilde{G}(q)$ are reviewed, e.g., in Ref. [1].

In this section we study the scaling behavior of the two-point correlation functions of the bilinear fields $E(x)$ and $T_{ij}(x)$. Like the specific heat, which is given by the zero-momentum component of the two-point function $\tilde{G}_E(q,t)$, the asymptotic behavior of $\tilde{G}_{E,T}(q,t)$ for $t \to 0^+$ is not as simple as that of the fundamental two-point function. Indeed, in the scaling limit $t \to 0^+$, $q^2 \to 0$ with $q^2\xi^2$ fixed, RG theory predicts

$$\tilde{G}_{E,T}(q,t) = B_{E,T} [1 + O(t)] + A^{+}_{E,T} t^{-\gamma_{E,T}} f_{E,T}(q^2\xi^2) \left[ 1 + O(t^{\Delta}) \right],$$

(30)

where $B_{E,T}$ and $A^{+}_{E,T}$ are nonuniversal constants, $f_{E,T}(y)$ is a universal function satisfying $f_{E,T}(0) = 1$, and $\Delta$ is the exponent related to the leading irrelevant operator. As amply discussed in textbooks—see, e.g., Ref. [31]—the presence of the background term $B_E$ in the asymptotic behavior of $\tilde{G}_E(q,t)$ is related to the need of an additive renormalization. One may easily see that the same argument applies to the two-point function $\tilde{G}_{T}(q,t)$ of $T_{ij}$.

Since $\gamma_T > 0$ for all $N \geq 2$, the leading behavior of the tensor two-point function is determined by the singular term depending on the scaling function $f_T(q^2\xi^2)$:

$$\tilde{G}_T(q,t) = A^+_T t^{-\gamma_T} f_T(q^2\xi^2) \left[ 1 + O(t^\Delta) + O(t^{\gamma_T}) \right].$$

(31)

The background term $B_T$ gives subleading corrections of order $t^{\gamma_T}$, that turn out to be more relevant than the standard scaling corrections of order $t^\Delta$. Indeed, for the physically relevant cases $N = 2,3$, one finds that $\gamma_T < \Delta$ ($\Delta \approx 0.53$ for $N = 2$ and $\Delta \approx 0.55$ for $N = 3$, see, e.g., the results reviewed in Ref. [1]). The difference decreases as $N \to \infty$, since both $\gamma_T$ and $\Delta$ converge to 1 with the same $1/N$ correction.

The same thing holds for the energy two-point function in the case of the Ising universality class for which $\alpha$ is positive, $\alpha = 0.1199(7)$ (Ref. [15]), i.e.
\[ \tilde{G}_E(q,t) = A_E^+ t^{-\alpha} f_E(q^2 \xi^2) \left[ 1 + O(t^\Delta) + O(t^\alpha) \right], \]  
(32)

where \( \Delta \approx 0.53 \), see e.g. Ref. [46]. On the other hand, for the \( O(N) \) vector models with \( N \geq 2 \), since \( \alpha < 0 \), the background term \( B_E \) gives the leading behavior of the energy two-point function \( \tilde{G}_E(q,t) \):

\[ \tilde{G}_E(q,t) = B_E + A_E^+ t^{-\alpha} f_E(q^2 \xi^2) \left[ 1 + O(t^\Delta) \right] + O(t). \]  
(33)

In these cases, the singular part vanishes for \( t = 0 \) and is usually responsible for a cusp-like finite maximum in the specific heat at the critical point, as it is observed in experiments and in lattice models. This requires the nonuniversal constant \( A_E^+ \) to be negative (see the discussion in Sec. III C 1).

In order to single out the singular behavior, one may consider the derivative with respect to the reduced temperature \( t \)

\[ W_{E,T}(q,t) \equiv \frac{\partial \tilde{G}_{E,T}}{\partial t} = -\gamma_{E,T} A_E^+ t^{-1-\gamma_{E,T}} w_{E,T}(q^2 \xi^2) \left[ 1 + O(t^\Delta, t^{1+\gamma_{E,T}}) \right], \]  
(34)

where

\[ w_{E,T}(y) = f_{E,T}(y) + \frac{2\nu}{\gamma_{E,T}} y f_{E,T}'(y) = 1 + O(y) \]  
(35)

is another universal function.

1. Small-momentum behavior

At small momentum, i.e. for \( y \equiv q^2 \xi^2 \ll 1 \), the scaling functions \( f_{E,T}(y) \) behave as

\[ f_E(y) = 1 + \sum_{n=1}^{\infty} e_n y^n, \]  
(36)

\[ f_T(y) = 1 + \sum_{n=1}^{\infty} a_n y^n. \]  
(37)

Using Eq. (35), these expansions can be related to those of the scaling functions \( w_{E,T}(y) \),

\[ w_E(y) = 1 + \sum_{n=1}^{\infty} \bar{e}_n y^n, \]  
(38)

\[ w_T(y) = 1 + \sum_{n=1}^{\infty} \bar{a}_n y^n. \]  
(39)

Indeed, it is immediate to obtain

\[ \bar{e}_n = e_n \left( 1 + \frac{2n\nu}{\alpha} \right), \]  
(40)

\[ \bar{a}_n = a_n \left( 1 + \frac{2n\nu}{\gamma_T} \right). \]  
(41)
Simple arguments based on perturbation theory suggest that the convergence radius \( R_c \) of the small-momentum expansions is determined by the two-particle cut. The singularity in the complex plane closest to the origin is expected to be \( y_s = -4S_M^+ \), where \( S_M^+ = \xi^2/\xi_{\text{gap}}^2 \) and \( \xi_{\text{gap}} \) is the exponential correlation length that determines the large-distance exponential behavior of the fundamental two-point function. Therefore, \( R_c = 4S_M^+ \). For the \( O(N) \) vector models, \( S_M^+ \) is very close to one, so that \( R_c \approx 4 \). For example, \( S_M^+ = 0.999634(4) \) for the Ising universality class \([45]\), \( S_M^+ = 0.999592(6) \) for the XY universality class \([34]\), \( S_M^+ = 0.99959(4) \) for the Heisenberg universality class \([47]\), and \( S_M^+ = 1-0.004590/N + O(1/N^2) \) in the large-\( N \) limit \([47]\). As a consequence, for \( n \to \infty \),

\[
\frac{e_{n+1}}{e_n} \approx \frac{a_{n+1}}{a_n} \approx \frac{\bar{e}_{n+1}}{\bar{e}_n} \approx \frac{\bar{a}_{n+1}}{\bar{a}_n} \approx -\frac{1}{4}. \tag{42}
\]

The constants \( e_1 \) and \( a_1 \) are related to the universal ratios \( X_{E,T} \equiv \xi_{E,T}^2/\xi^2 \) introduced in Refs. \([19,14]\), where \( \xi_{E,T} \) are the second-moment correlation lengths associated with the singular part of the energy and of the tensor two-point functions respectively. More precisely, if \( \gamma_{T,E} > 0 \), the correlation length is defined by Eq. \((29)\), replacing \( \tilde{G}(q) \) with \( \tilde{G}_{E,T}(q) \). If the exponent is negative, then

\[
\xi_{E}^2 = -(\tilde{G}_{E}(0) - B_{E})^{-1} \frac{\partial \tilde{G}_{E}(q)}{\partial q^2} \bigg|_{q^2=0}. \tag{43}
\]

The universal ratios \( X_E \) and \( X_T \) are given by \( X_E = -e_1 \) and \( X_T = -a_1 \).

2. Large-momentum behavior

The large-momentum behavior of the fundamental correlation function is given by the Fisher-Langer formula \([48]\):

\[
f(y) \approx \frac{A_1}{y^{1-\eta/2}} \left( 1 + \frac{A_2}{y^{(1-\alpha)/(2\nu)}} + \frac{A_3}{y^{1/(2\nu)}} \right). \tag{44}
\]

One may derive a similar expression for the correlation functions of the bilinear fields. The large-momentum behavior of the structure factors can be studied by performing a short-distance expansion of the two-point functions \( G_E(x) \) and \( G_T(x) \). Following the method outlined in Ref. \([48]\), we obtain the corresponding asymptotic expansions for \( y \to \infty \):

\[
f_E(y) \approx E_1 y^{-\alpha/(2\nu)} \left( 1 + \frac{E_2}{y^{(1-\alpha)/(2\nu)}} + \frac{E_3}{y^{1/(2\nu)}} \right), \tag{45}
\]

\[
f_T(y) \approx T_1 y^{-\gamma_T/(2\nu)} \left( 1 + \frac{T_2}{y^{(1-\alpha)/(2\nu)}} + \frac{T_3}{y^{1/(2\nu)}} \right). \tag{46}
\]

The derivation of these formulae is reported in App. \([4]\). Notice that for the \( O(N) \) vector models with \( N \geq 2 \), since \( \alpha < 0 \), \( f_E(y) \) increases as \( y \to \infty \).
B. Field-theory calculations: generalities

Because of the presence of the background term, the FT calculation of the scaling functions \( f_E(y) \) and \( f_T(y) \) requires some care. First, we define the dimensionless functions

\[
G_{E,T}(g, y) \equiv u \tilde{G}_{E,T}(q, t, u),
\]

where \( g \) is the four-point renormalized coupling. Then, in order to eliminate the constant additive renormalization term, we consider the derivative with respect to \( m \) of \( G_{E,T}(g, y) \):

\[
W_{E,T}(g, y) = m \frac{\partial}{\partial m} G_{E,T}(g, y) \bigg|_{u = \beta(g)} = \beta(g) \frac{\partial G_{E,T}(g, y)}{\partial g} - 2y \frac{\partial G_{E,T}(g, y)}{\partial y}.
\]

(48)

At the fixed point \( g^* \), the functions \( W_{E,T}(g, y) \) differ from \( W_{E,T}(q, t) \), defined in Eq. (34), by a multiplicative factor independent of \( q \). Therefore, the scaling functions \( w_{E,T}(g, y) \), defined in Eq. (35), are given by

\[
w_{E,T}(g, y) = \frac{W_{E,T}(g, y)}{W_{E,T}(g, 0)}.
\]

(49)

Note that the zero-momentum functions \( W_{E,T}(g, 0) \) are related to the exponents \( \gamma_{E,T} \) by the relation

\[
-\frac{\gamma_{E,T}}{\nu} = \lim_{g \to g^*} \beta(g) \frac{d \ln W_{E,T}(g, 0)}{dg}.
\]

(50)

C. Field-theoretical results

1. Small-momentum expansion

We compute the small-momentum expansion of the structure factors to four loops in the fixed-dimension approach and to three loops in the \( \epsilon \) expansion.

In the fixed-dimension approach, we first determine the expansion in powers of \( g \) of the coefficients \( \bar{e}_i \) and \( \bar{a}_i \) defined in Eqs. (38) and (39). The explicit expressions are reported in App. [B]. In order to obtain numerical estimates we use the same resummation procedure outlined in the previous section. Our numerical results are presented in Tables II and III. Note that, as expected, the ratios \( \bar{e}_{i+1}/\bar{e}_i \) and \( \bar{a}_{i+1}/\bar{a}_i \) quickly approach \(-1/4\). The corresponding coefficients \( e_i \) and \( a_i \) are obtained by using the relations (40) and (41). For the exponent \( \nu \) we use the same values reported before [38], while for \( \gamma_T \) we use the results of Table I. In the case of \( a_i \) a large part of the uncertainty is due to the error in the exponent \( \gamma_T \) that enters the relation between \( \bar{a}_i \) and \( a_i \). The results are reported in Table IV. We also performed direct analyses of the coefficients \( e_i \), \( a_i \), considering the \( g \)-series that can be obtained from Eqs. (10) and (11). The results are substantially consistent with those obtained by first estimating \( \bar{e}_i \) and \( \bar{a}_i \). In the case of \( a_i \) they turn out to be more precise; we show also them in Table IV (third column of results).
TABLE II. Estimates of the coefficients $\bar{e}_i$ for several values of $N$.

| $N$ | $\bar{e}_1$ | $\bar{e}_2/\bar{e}_1$ | $\bar{e}_3/\bar{e}_2$ | $\bar{e}_4/\bar{e}_3$ | $\bar{e}_5/\bar{e}_4$ |
|-----|-------------|-----------------------|-----------------------|-----------------------|-----------------------|
| 1   | -0.170(5)   | -0.206(1)             | -0.221(1)             | -0.229(1)             | -0.234(1)             |
| 2   | -0.155(5)   | -0.199(1)             | -0.216(2)             | -0.226(2)             | -0.232(2)             |
| 3   | -0.142(5)   | -0.193(2)             | -0.213(2)             | -0.222(2)             | -0.230(3)             |
| 4   | -0.133(6)   | -0.189(3)             | -0.211(2)             | -0.222(3)             | -0.228(3)             |
| 5   | -0.126(6)   | -0.186(3)             | -0.209(3)             | -0.221(3)             | -0.228(4)             |
| 8   | -0.111(5)   | -0.180(3)             | -0.206(3)             | -0.219(4)             | -0.227(4)             |

TABLE III. Estimates of the coefficients $\bar{a}_i$ for several values of $N$.

| $N$ | $\bar{a}_1$ | $\bar{a}_2/\bar{a}_1$ | $\bar{a}_3/\bar{a}_2$ | $\bar{a}_4/\bar{a}_3$ | $\bar{a}_5/\bar{a}_4$ |
|-----|-------------|-----------------------|-----------------------|-----------------------|-----------------------|
| 2   | -0.203(2)   | -0.224(2)             | -0.232(1)             | -0.236(1)             | -0.239(1)             |
| 3   | -0.208(2)   | -0.226(1)             | -0.234(1)             | -0.238(1)             | -0.240(1)             |
| 4   | -0.213(2)   | -0.228(1)             | -0.235(1)             | -0.239(1)             | -0.241(1)             |
| 5   | -0.216(1)   | -0.230(1)             | -0.236(1)             | -0.240(1)             | -0.242(1)             |
| 8   | -0.224(1)   | -0.235(1)             | -0.239(1)             | -0.242(1)             | -0.244(1)             |

TABLE IV. Results of the coefficients $e_i$ and $a_i$ for several values of $N$ and from various analyses: (a) ($d = 3$) by using the fixed-dimension results for $\bar{e}_i$ and $\bar{a}_i$, and by directly analyzing the series for $a_i$; (b) ($\epsilon$-exp) by resummation of the three-loop $\epsilon$-expansion.

| $N$ | $e_i$ ($d = 3$) | $a_i$ ($d = 3$) | $a_i$ ($d = 3$) | $e_i$ ($\epsilon$-exp) | $a_i$ ($\epsilon$-exp) |
|-----|----------------|----------------|----------------|------------------------|------------------------|
| 1   |                |                |                | -0.014(5)              | -0.014(5)              |
| 2   | -0.147(3)×10^{-2} | -0.147(3)×10^{-2} | -0.147(3)×10^{-2} | -0.014(5)              | -0.014(5)              |
| 3   | -0.219(4)×10^{-3} | -0.219(4)×10^{-3} | -0.219(4)×10^{-3} | -0.014(5)              | -0.014(5)              |
| 4   | -0.38(1)×10^{-4}  | -0.38(1)×10^{-4}  | -0.38(1)×10^{-4}  | -0.014(5)              | -0.014(5)              |
| 5   | -0.77(1)×10^{-5}  | -0.77(1)×10^{-5}  | -0.77(1)×10^{-5}  | -0.014(5)              | -0.014(5)              |

| $N$ | $e_i$ ($d = 3$) | $a_i$ ($d = 3$) | $a_i$ ($d = 3$) | $e_i$ ($\epsilon$-exp) | $a_i$ ($\epsilon$-exp) |
|-----|----------------|----------------|----------------|------------------------|------------------------|
| 2   |                |                |                | -0.014(5)              | -0.014(5)              |
| 3   |                |                |                | -0.014(5)              | -0.014(5)              |
| 4   |                |                |                | -0.014(5)              | -0.014(5)              |
| 5   |                |                |                | -0.014(5)              | -0.014(5)              |

| $N$ | $e_i$ ($d = 3$) | $a_i$ ($d = 3$) | $a_i$ ($d = 3$) | $e_i$ ($\epsilon$-exp) | $a_i$ ($\epsilon$-exp) |
|-----|----------------|----------------|----------------|------------------------|------------------------|
| 1   |                |                |                | -0.014(5)              | -0.014(5)              |
| 2   |                |                |                | -0.014(5)              | -0.014(5)              |
| 3   |                |                |                | -0.014(5)              | -0.014(5)              |
| 4   |                |                |                | -0.014(5)              | -0.014(5)              |
| 5   |                |                |                | -0.014(5)              | -0.014(5)              |

| $N$ | $e_i$ ($d = 3$) | $a_i$ ($d = 3$) | $a_i$ ($d = 3$) | $e_i$ ($\epsilon$-exp) | $a_i$ ($\epsilon$-exp) |
|-----|----------------|----------------|----------------|------------------------|------------------------|
| 1   |                |                |                | -0.014(5)              | -0.014(5)              |
| 2   |                |                |                | -0.014(5)              | -0.014(5)              |
| 3   |                |                |                | -0.014(5)              | -0.014(5)              |
| 4   |                |                |                | -0.014(5)              | -0.014(5)              |
| 5   |                |                |                | -0.014(5)              | -0.014(5)              |

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TABLE V. Final estimates of the coefficients $e_1$ and $a_1$.

| $N$ | $e_1$     | $a_1$       |
|-----|-----------|-------------|
| 1   | $-0.0140(5)$ | $-0.041(2)$ |
| 2   | $0.0017(1)$    | $-0.046(1)$ |
| 3   | $0.014(3)$     | $-0.051(2)$ |
| 4   | $0.024(3)$     | $-0.0533(5)$|
| 5   | $0.030(1)$     | $-0.0533(5)$|
| 8   | $0.047(1)$     | $-0.062(2)$ |

In $\epsilon$ expansion we directly resum the expansions of $e_i$ and $a_i$ reported in App. [3]. The results are also reported in Table [V] and are in substantial agreement with the fixed-dimension results. For $a_1$, we also perform a constrained analysis that makes use of the available results for $a_1$ in two and one dimensions. Such a method was introduced in Ref. [50] and generalized in Refs. [35,51]. In many instances it has provided quite accurate results for critical quantities. We use the estimates of $a_1$ in two dimensions reported in Refs. [52,53]: $a_1 = -0.0812(5), -0.1014(6),$ and $-0.1313(9)$ for $N = 3, 4, 8$ respectively. We also make use of the one-dimensional result [54], $a_1 = -(N-1)^2/(4N^2)$. From the analysis constrained in one dimension, in three dimensions we obtain: $a_1 = -0.0397(2)$ for $N = 2$ and $a_1 = -0.0533(3)$ for $N = 5$, while from the analysis constrained in two dimensions we obtain $a_1 = -0.0460(3)$ for $N = 3$, $a_1 = -0.0507(5)$ for $N = 4$, $a_1 = -0.0621(2)$ for $N = 8$. Constraining the analysis both in two and one dimension, we obtain: $a_1 = -0.0458(1)$ for $N = 3$, $a_1 = -0.0514(6)$ for $N = 4$, $a_1 = -0.0625(2)$ for $N = 8$. These results are compatible with those of Table [V].

Taking into account the above results for $e_1$ and $a_1$, we consider as our final estimates the numbers reported in Table [V], where we have been rather conservative in giving the errors, which include all the results we have obtained.

As already noted in Ref. [1], $e_1 = -\alpha/(6\gamma) + O(\epsilon^3)$ and $a_1 = -\gamma_T/(6\gamma) + O(\epsilon^3)$. These relations are not satisfied to order $\epsilon^3$, see App. B. Nonetheless, they still provide very good approximations to $e_1$ and $a_1$. For instance (see Ref. [4] for the estimates of the critical exponents): $-\alpha/(6\gamma) = -0.01377(8), 0.00185(10), 0.0159(8)$ respectively for $N = 1, 2, 3$, where the error is related to the uncertainty on the estimates of $\alpha$ and $\gamma$.

The coefficient $e_1$ has also been computed for $N = 1$ by Monte Carlo simulations [53]. The numerical data are well described by $-\alpha_{\text{eff}}(t)/(6\gamma_{\text{eff}}(t))$, where $\alpha_{\text{eff}}(t)$ and $\gamma_{\text{eff}}(t)$ are effective exponents determined from the specific heat and the susceptibility.

It is interesting to note that the signs of $a_i$ and $e_i$ are strictly related to the signs of the amplitudes $A_{E,T}^+$ and of the exponents $\alpha$ and $\gamma_T$. First, we observe that in the critical limit the correlation functions are nonnegative, i.e. $G_{E,T}(x) \geq 0$. Indeed, the lattice $\phi^4$ model with nearest-neighbor couplings is exactly reflection positive and therefore, the above-reported inequalities are rigorously true for any value of the couplings. At criticality, they should hold for any model in the same universality class. Therefore, all moments are positive, i.e. $\sum_x |x|^{2n} G_{E,T}(x) \geq 0$. If the correlation functions have the scaling forms (31) and (32), this implies

$$A_{E,T}^+ \geq 0, \quad (-1)^n a_n \geq 0. \quad (51)$$
For $N \geq 2$, using Eq. (33), we obtain
\[ B_E \geq 0, \quad (-1)^n A_{E_n}^+ \geq 0. \] (52)

Relations (51) are satisfied by our results, while Eq. (52) and our result $e_1 > 0$ imply $A_{E,T}^- < 0$. Thus, although $A_{E,T}^+$ is nonuniversal, the positivity (unitarity in FT language) of the theory fixes its sign.

As a final remark, note that $a_1$ and $e_1$ are very small. The unexpectedly small value of $a_1$ was crucial to provide an explanation [19,11], consistently with RG theory, of the experimental results of Ref. [18] for density-wave systems, at the nematic–smectic-A transition of liquid crystals. The values of $a_1$ are quite smaller than what would be naively expected.

The nearest singularity in the complex $y$ plane corresponds to the two-particle cut, thus at large distance $G_T(x) \sim |x|^q \exp(-|x|/\xi_T^*)$ with $\xi_T^* = \xi_{\text{gap}}/2$, where $\xi_{\text{gap}}$ is the exponential correlation length that determines the large-distance exponential behavior of the fundamental two-point function. Positivity of $G_T(x)$ would then require that the second-moment correlation length $\xi_T$ be smaller than $\xi_T^*$. As a consequence, since $\xi_{\text{gap}} \simeq \xi$ (see Sec. III A 1), $a_1 \lesssim 1/4$. But this bound turns out to be much larger than the actual value of $a_1$.

2. Large-momentum expansion

We also compute the constants $E_i$ and $T_i$ of the large-momentum behavior of $f_{E,T}(y)$. Matching the large-momentum expansion of the two-loop expression of $\tilde{G}_{E,T}(q,t)$ with Eqs. (15) and (16), we obtain
\[ E_1 = 1 + \frac{4 - N}{8 + N} \epsilon + O(\epsilon^2), \] (53)
\[ E_2 = -2 + \frac{2(5 + N)}{8 + N} \epsilon + E_{22}\epsilon^2 + O(\epsilon^3), \]
\[ E_3 = 2 - \frac{14 + N}{8 + N} \epsilon + E_{32}\epsilon^2 + O(\epsilon^3), \]
and
\[ T_1 = 1 + \frac{4 + N}{8 + N} \epsilon + O(\epsilon^2), \] (54)
\[ T_2 = -\frac{2(4 + N)}{4 - N} - \frac{2(4 + N)(20 - 13N - N^2)}{(4 - N)^2(8 + N)} \epsilon + T_{22}\epsilon^2 + O(\epsilon^3), \]
\[ T_3 = \frac{2(4 + N)}{4 - N} - \frac{(4 + N)(56 - 34N - N^2)}{(4 - N)^2(8 + N)} \epsilon + T_{32}\epsilon^2 + O(\epsilon^3). \]

Moreover,
\[ E_{22} + E_{32} = -\frac{(N^3 - 14N^2 - 140N - 432)}{2(N + 8)^3} + \frac{N - 4}{12(N + 8)^2} \pi^2, \]
\[ T_{22} + T_{32} = -\frac{(N + 4)(N^2 + 14N + 108)}{2(N + 8)^3} - \frac{N + 4}{12(N + 8)^2} \pi^2. \] (55)
The constants $E_1$ and $T_1$ are in agreement with the results of Ref. [11]. The divergence of the coefficients $T_2$ and $T_3$ for $N \to 4$ is related to the vanishing of the $O(\epsilon)$ term in the expansion of $\alpha[56]$. The large size of the coefficients makes it difficult to resum the perturbative series. For the physically interesting case of $N = 2$ we report the result obtained by setting $\epsilon = 1$ and give as error the size of the last coefficient. In this way we obtain: $E_1 = 1.3(3)$, $E_2 = 0.7(1.3)$, and $E_3 = 0.3(1.7)$ for $N = 1$, and $E_1 = 1.2(2)$, $E_2 = 0.6(1.4)$, $E_3 = 0.4(1.6)$, $T_1 = 1.6(6)$, $T_2 = -9(3)$, $T_3 = 8.4(2.4)$ for $N = 2$. Moreover, $E_2 + E_3 = 0.0(2)$ and $T_2 + T_3 = -0.7(1)$ for $N = 2$.

D. Interpolations of the structure factors

In Ref. [11] the authors discuss several approximate forms for $f_{E,T}(y)$. They present generalizations of the Fisher-Burford [43] approximant for $\langle \phi \phi \rangle$. These approximations are quite crude and do not reproduce the full Fisher-Langer behavior for large $y$. A better approach based on dispersion theory was put forward by Bray [25]. Here, we will apply the same method to the universal functions $f_E(y)$ and $f_T(y)$.

A generalization of the arguments presented in Ref. [25] gives the following representation for $f_T(y)$

$$f_T(y) = 1 - \frac{y T_1}{\pi} \sin \left( \frac{\pi T_1}{2\nu} \right) \int_{4 S_M^+}^{\infty} dx \frac{x^{-1-\gamma_T/(2\nu)}}{x+y} F_T(x), \quad (56)$$

where $F_T(x)$ is the spectral function satisfying $F_T(\infty) = 1$. We assume here that the only singularities of $f_T(y)$ in the complex plane are branch cuts on the negative real axis and that the leading one corresponds to the two-particle state, so that the disk $|y| < 4 S_M^+$ is free of singularities. Under this assumption, the representation (56) is exact.

For generic $F_T(x)$, Eq. (56) does not give the correct Fisher-Langer behavior (16). Indeed, for $y \to \infty$ we obtain $f_T(y) \approx \text{constant} + T_1 y^{-\gamma_T/(2\nu)}$. We must thus require the constant to be zero. This gives the sum rule

$$\frac{T_1}{\pi} \sin \left( \frac{\pi T_1}{2\nu} \right) \int_{4 S_M^+}^{\infty} dx \frac{x^{-1-\gamma_T/(2\nu)}}{x} F_T(x) = 1, \quad (57)$$

which allows the determination of $T_1$ once $F_T(x)$ is given.

Eq. (56) applies also to $f_E(y)$ with the obvious replacements. However, the sum rule (57) requires $\alpha > 0$ and can thus be used only in the Ising case. For $\alpha < 0$, Eq. (57) is replaced by

$$\frac{E_1}{\pi} \sin \left( \frac{\pi \alpha}{2\nu} \right) \left[ \frac{2\nu}{\alpha} (4 S_M^+)^{-\alpha/(2\nu)} + \int_{4 S_M^+}^{\infty} dx \frac{x^{-1-\alpha/(2\nu)}}{x} (F_E(x) - 1) \right] = 1. \quad (58)$$

In order to obtain approximate expressions for the structure factors, we must assume a specific form for the spectral function. For this purpose, we assume, as in Ref. [24], that $F_T(x)$ gives the exact Fisher-Langer behavior on the cut. Explicitly, we consider
We also report the large-$y$ behavior, $f_E(y) \approx 1.199y^{-0.08725}$ and the small-$y$ behavior, $f_E(y) \approx 1 - 0.01366y + 0.001467y^2 - 0.000219y^3$.

To completely determine the spectral function, we must specify the constants $T_2$ and $T_3$. We use here the $\epsilon$-expansion results of Sec. III C 2. These estimates are not very precise, but the interpolation is quite insensitive on $T_2$ and $T_3$ separately. Indeed, what really matters is their sum $T_2 + T_3$ that is more accurately determined. In order to test these interpolations, we can compare the estimates of $T_1$ and $a_i$—and, analogously, of $E_1$ and $e_i$—with those of the preceding sections. For $N = 2$, using $T_2 = -9$ and $T_3 = 8.4$, we obtain $T_1 \approx 1.56$, $a_1 \approx -0.055$, $a_2 \approx 0.008$, which are reasonably close to the estimates reported before. Analogously, using $E_2 = -0.6$ and $E_3 = 0.4$, we obtain $E_1 \approx 1.00$, $e_1 \approx 0.005$ and $e_2 \approx -0.0007$, again in reasonable agreement with previous results. In particular, the fact that $|e_1| \ll |a_1|$ is correctly predicted by the approximation. For $N = 1$, using $E_2 = -2/3$, $E_3 = 1/3$, we obtain $E_1 \approx 1.20$, $e_1 \approx -0.016$, $e_2 \approx 0.0019$, in reasonable agreement with what reported above.

In Fig. 1 we report $f_E(y)$ for $N = 1$ and in Figs. 2 and 3 a graph of $f_E(y)$ and of $f_T(y)$ for $N = 2$. It is interesting to note that for $N = 2$ the function $f_E(y)$ varies slowly and differs from one only for quite large values of $y$. Taking also into account that the prefactor vanishes as $t \to 0$, the $q^2$ dependence of $G_E(q,t)$ should be hardly visible in experiments and in numerical Monte Carlo simulations. Moreover, in this case $f_E(y) \geq 1$ for all $y$, so that, because of the inequalities (52), there is an attenuation of the singular behavior for increasing $q$, as generally expected.
FIG. 2. Universal function $f_E(y)$ obtained using Eqs. (56) and (59), for $N = 2$. We also report the large-$y$ behavior, $f_E(y) \approx 1.00y^{0.010908}$ and the small-$y$ behavior, $f_E(y) \approx 1 + 0.00171y - 0.000169y^2 + 0.0000243y^3$.

FIG. 3. Universal function $f_T(y)$ obtained using Eqs. (56) and (59), for $N = 2$. We also report the large-$y$ behavior, $f_T(y) \approx 1.559y^{-0.263569}$ and the small-$y$ behavior, $f_T(y) \approx 1 - 0.0397y + 0.0053y^2 - 0.000852y^3$. 
APPENDIX A: LARGE-MOMENTUM BEHAVIOR FOR THE BILINEAR CORRELATION FUNCTIONS

In this appendix we compute the large-momentum behavior of the correlation function. We follow closely the discussion of Refs. [57,49] for the correlation function of the field \( \phi \).

1. The energy correlation function

The basic ingredient of the calculation is the short-distance expansion of the product of operators \( E(x + y/2)E(x - y/2) \). For \( y \to 0 \), see, e.g., Ref. [31], this product is equal to the sum of all the operators that are allowed by symmetries, multiplied by \( C \)-number coefficients, that take into account the short-distance behavior. The most singular contribution comes from the operators of smallest dimension. In this case, neglecting the contribution related to the identity operator, it implies

\[
E(x + y/2)E(x - y/2) = C(y)E(x) + \text{less singular contributions}.
\] (A1)

Now, let us consider the connected correlation function of \( l \) composite operators \( E(x), G^{(l)}(p_1, \ldots, p_l) \), and its renormalized counterpart \( G_R^{(l)}(p_1, \ldots, p_l) = Z_LZ_\phi^{-1}G^{(l)}(p_1, \ldots, p_l) \). Then, Eq. (A1) implies for \( p \gg m \)

\[
G_R^{(l)}(p, -p, 0, \ldots, 0) \approx \tilde{C}(p; m)G_R^{(l-1)}(0, \ldots, 0),
\] (A2)

where we have explicitly written the mass dependence of the short-distance coefficient. Since renormalized correlation functions scale canonically, i.e.

\[
G_R^{(l)}(p, -p, 0, \ldots, 0) = m^{d-2l}f(p/m),
\] (A3)

we have

\[
\tilde{C}(p; m) = m^{-2}\tilde{C}(p/m)
\] (A4)

Renormalized correlation functions satisfy the Callan-Symanzik equation

\[
\left[ m\frac{\partial}{\partial m} + \beta(g)\frac{\partial}{\partial g} - l\eta_2(g) \right] G_R^{(l)}(p_1, \ldots, p_l) = m^2\sigma(g)G_R^{(l+1)}(0, p_1, \ldots, p_l),
\] (A5)

where \( \sigma(g) \) is a RG function satisfying \( \sigma(g^*) = 2 - \eta \), and \( \eta_2(g) = \eta_1(g) - \eta_\phi(g) \). Applying the Callan-Symanzik equation to the relation (A2) we obtain, setting \( g = g^* \),

\[
\left[ m\frac{\partial}{\partial m} - \eta_2(g^*) \right] \tilde{C}(p; m) = 0,
\] (A6)

and therefore, using Eq. (A4), we have

\[
\tilde{C}(p; m) \sim m^{-2}(p/m)^{-2 - \eta_2} = m^{-2}(p/m)^{-1/\nu}.
\] (A7)
Now, using the above-reported results and $Z_t/Z_\phi \sim m^{\eta_t-\eta_\phi} \sim m^{1/\nu-2}$ for $g = g^*$, see Eq. (13), we obtain
\[
\frac{\partial^2}{\partial t^2} G^{(2)}(p, -p) = G^{(4)}(0, 0, p, -p) \sim m^{8-4/\nu} G^{(4)}_R(0, 0, p, -p) \\
\approx m^{8-4/\nu} \tilde{C}(p; m) G^{(3)}_R(0, 0, 0) \sim (p/m)^{-1/\nu} m^{d-4/\nu} \sim t^{-1-\alpha} p^{-1/\nu}.
\] (A8)

Integrating this equation twice with respect to $t$, we have
\[
G^{(2)}(p, -p) = a(p) + b(p)t + ct^{-\alpha} p^{-1/\nu} + o(t^{-\alpha}),
\] (A9)
where $a(p)$ and $b(p)$ are unknown functions of $p$. Comparing this result with the scaling equations (32) and (33), we obtain finally Eq. (45).

2. The tensor correlation function

The calculation is analogous. The short-distance expansion of the product $T_{ij}(x)T_{ij}(y)$ is given by
\[
T_{ij}(x+y/2)T_{ij}(x-y/2) = C_T(y) E(x) + \text{less singular contributions}.
\] (A10)

Now, we consider the connected correlation function with $l$ fields $E(x)$ and two fields $T_{ij}(x)$ with the indices summed over, $G^{(l)}_{T,R}(p_1, p_2; q_1, \ldots, q_l)$, and its renormalized counterpart $G^{(l)}_{T,R}(p_1, p_2; q_1, \ldots, q_l) = Z_T^2 Z_l^2 Z_{\phi}^{-1/2} G^{(l)}_{T}(p_1, p_2; q_1, \ldots, q_l)$. For $p \gg m$ we have
\[
G^{(l)}_{T,R}(p, -p; 0, \ldots, 0) \approx \tilde{C}_{T}(p; m) G^{(l+1)}_R(0, \ldots, 0),
\] (A11)

The coefficient $\tilde{C}_{T}(p; m)$ scales as in Eq. (A4) and satisfies the RG equation
\[
\left[ m \frac{\partial}{\partial m} - 2\eta'_2 + \eta_2 \right] \tilde{C}_{T}(p; m) = 0,
\] (A12)
where $\eta'_2 = \eta_T - \eta_\phi$. Therefore
\[
\tilde{C}_{T}(p; m) \sim m^{-2}(p/m)^{-2-2\eta'_2+\eta_2} = m^{-2}(p/m)^{-(1+\gamma_T-\alpha)/\nu}.
\] (A13)

As in the energy case, we consider the second derivative of $G^{(0)}_{T}(p, -p)$ with respect to $t$. For $p \gg m$ we have
\[
\frac{\partial^2}{\partial t^2} G^{(0)}_{T}(p, -p) = G^{(2)}_{T}(p, -p; 0, 0) \sim m^{8-2/\nu-2\phi_T/\nu} G^{(2)}_{T,R}(p, -p; 0, 0) \\
\approx m^{8-2/\nu-2\phi_T/\nu} \tilde{C}(p; m) G^{(3)}_R(0, 0, 0) \sim p^{-(1+\gamma_T-\alpha)/\nu} t^{-1-\alpha},
\] (A14)

where we have used the fact that, for $g = g^*$, $Z_T/Z_\phi \sim m^{\eta_T-\eta_\phi} \sim m^{\phi_T/\nu-2}$, see Eqs. (16), (20). Integrating this equation twice with respect to $t$ and using the scaling equation (31), we obtain the large-momentum behavior (46).
APPENDIX B: PERTURBATIVE EXPANSION OF THE TWO-POINT FUNCTIONS \( G_{E,T} \)

In order to compute the structure factor of the bilinear fields, we determine the one-particle-irreducible diagrams with insertions of two operators \( E \) or \( T_{ij} \) and zero external legs. We use the susceptibility \( \chi \) as inverse mass square, so that tadpole diagrams can be neglected. Also, subdiagrams that correspond to diagrams of the two-point function \( \langle \phi \phi \rangle \) are subtracted at zero momentum. The diagrams contributing up to four loops are drawn in Fig. 4. The structure factors of the bilinear fields can be expanded as

\[
G(\tilde{g}, y) = uG_{E,T}(u, q) = \sum_{j=1} \left(-1\right)^{l-1} u^l \chi^{l/2} S_j C_{j}^{E,T} I_j(q^2\chi) \tag{B1}
\]

where the sum is over the graphs without tadpoles, \( l \) is the number of loops of the graph, \( S_j \) the graph symmetry factor, \( C_{j}^{E,T} \) the group factor, and \( I_j(q^2) \) the loop integral with unit mass. In Table VI we report \( S_j, C_{j}^{E,T} \).

We computed the coefficients \( \tilde{e}_i \) and \( \tilde{a}_i \) to four loops in the fixed-dimensions expansion. The expansion of the loop integrals \( I_j(q^2) \) is reported in Table VI. In the calculation we used the results of Refs. [58, 59]. We also used the expression of the bare coupling \( u \) as a function of the renormalized coupling \( \tilde{g} \),
TABLE VI. For each diagram $j$ contributing to the energy and tensor two-point function we report: the number of loops $l$, the symmetry factor $S_j$, the group factors $C_j^{E,T}$, and the expansion of the integral $I_j(y)$ in fixed dimension $d = 3$.

| $j$ | $l$ | $S_j$ | $\frac{C_j^E}{N}$ | $\frac{4C_j^T}{N(N-1)}$ | $(8\pi)^l I_j(y)$ |
|-----|-----|------|----------------|----------------|------------------|
| 1   | 1   | 2    | 2              | $\frac{2}{\sqrt{y}} \arctan \frac{\sqrt{y}}{2}$ |                      |
| 2   | 2   | 1    | $\frac{2+4N}{9}$ | $\frac{4}{3}$    | $I_1^2$          |
| 3   | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{(2+N)^2}{9}$ | $\frac{8}{5}$    | $I_1^3$          |
| 4   | 3   | $\frac{2}{3}$ | $\frac{2+4N}{3}$ | $\frac{2(2+N)}{3}$ | $-0.0376821 + 0.00160802y + 0.000209975y^2 - 0.00012236y^3$ + $0.0000404108y^4 - 0.0000116689y^5 + O(y^6)$ |
| 5   | 3   | 1    | $\frac{2+4N}{3}$ | $\frac{2(6+N)}{9}$ | $0.5 - 0.105903y + 0.0222193y^2 - 0.00477681y^3$ + $0.0104957y^4 - 0.00234591y^5 + O(y^6)$ |
| 6   | 4   | $\frac{1}{4}$ | $\frac{(2+N)^3}{27}$ | $\frac{16}{27}$ | $I_1^4$          |
| 7   | 4   | $\frac{2}{3}$ | $\frac{(2+N)^2}{9}$ | $\frac{4(2+N)}{9}$ | $I_1 I_4$        |
| 8   | 4   | 1    | $\frac{(2+N)^2}{9}$ | $\frac{4(6+N)}{27}$ | $I_1 I_5$        |
| 9   | 4   | 1    | $\frac{(2+N)(8+N)}{27}$ | $\frac{2(2+N)(8+N)}{27}$ | $-0.0266277 + 0.0012789y + 0.0001577474y^2 - 0.000095736y^3$ + $0.000031929y^4 - 9.2602210^{-6}y^5 + O(y^6)$ |
| 10  | 4   | 2    | $\frac{(2+N)(8+N)}{27}$ | $\frac{8(4+N)}{27}$ | $0.25 - 0.0601852y + 0.0132661y^2 - 0.00292294y^3$ + $0.000651627y^4 - 0.000147057y^5 + O(y^6)$ |
| 11  | 4   | $\frac{1}{2}$ | $\frac{(2+N)(8+N)}{27}$ | $\frac{2(16+6N+N^2)}{27}$ | $0.322467 - 0.0786798y + 0.0175558y^2 - 0.0039039y^3$ + $0.0008763y^4 - 0.000198793y^5 + O(y^6)$ |
\[ u = m \frac{48\pi \bar{g}}{(8 + N)} \left[ 1 + \bar{g} + \frac{27N^2 + 350N + 1348}{27(8 + N)^2} \bar{g}^2 + \frac{N^3 + 17.3632N^2 + 120.783N + 315.831}{(8 + N)^3} \bar{g}^3 + O(\bar{g}^4) \right] \]  

(B2)

and the relation between \( \chi \) and \( m \):

\[ m^2 \chi = Z_\phi(g) = 1 - \frac{4(N + 2)}{27(N + 8)^2} \bar{g}^2 - 0.106993 \frac{N + 2}{(N + 8)^2} \bar{g}^3 + O(\bar{g}^4). \]  

(B3)

Writing

\[ \bar{e}_i = \sum_{j=0} \bar{e}_{i,j} \bar{g}^j, \]  

(B4)

\[ \bar{a}_i = \sum_{j=0} \bar{a}_{i,j} \bar{g}^j, \]  

(B5)

we computed the coefficients \( \bar{e}_{i,j} \) and \( \bar{a}_{i,j} \) up to \( j = 3 \). They are reported in Table VII.

We computed the coefficients \( e_i \) and \( a_i \) in \( \epsilon \) expansion to three loops, i.e. to order \( \epsilon^3 \). The expansion of the integrals \( I_4(y) \) and \( I_5(y) \) was obtained by using the algebraic algorithm of Ref. [60].

We write

\[ e_i = \frac{N - 4}{N + 8} x_1 \epsilon + \frac{1}{(N + 8)^3} x_2 \epsilon^2 + \left[ \frac{(N + 2)(N - 4)}{(N + 8)^3} x_3 \lambda + \frac{(N + 2)}{(N + 8)^4} x_4 \zeta(3) + \frac{1}{(N + 8)^5} x_5 \right] \epsilon^3 + O(\epsilon^4), \]  

(B6)

\[ a_i = \frac{N + 4}{N + 8} x_1 \epsilon + \frac{N + 4}{(N + 8)^3} x_2 \epsilon^2 + \left[ \frac{(N + 2)(N + 4)}{(N + 8)^3} x_3 \lambda + \frac{1}{(N + 8)^4} x_4 \zeta(3) + \frac{1}{(N + 8)^5} x_5 \right] \epsilon^3 + O(\epsilon^4), \]  

(B7)

where \( \lambda = 1.171953619344729445 \). The coefficients \( x_i \) are reported in Tables VIII and IX.

Note that [11] to order \( \epsilon^2 \), \( e_1 \approx -a/(6\gamma) \) and \( a_1 \approx -\gamma T/(6\gamma) \). These relations do not hold at order \( \epsilon^3 \).
### TABLE VII. Coefficients $\bar{e}_{i,j}$ and $\bar{a}_{i,j}$, cf. Eqs. (B4) and (B5).

| $i$ | $j$ | $\bar{e}_{i,j}(8 + N)^j/(2 + N)^{(1-\delta)\bar{N}}$ | $\bar{a}_{i,j}(N + 8)^j$ |
|-----|-----|---------------------------------|-----------------------------|
| 1   | 0   | $-\frac{1}{7}$                  | $-\frac{1}{7}$              |
| 1   | 1   | $\frac{1}{7}$                   | $\frac{1}{7}$               |
| 2   | 0   | 0.0280008                       | 0.0560016 + 0.00585959 $N$  |
| 2   | 1   | $-0.046127 + 0.0223387 N$       | 0.0560016 + 0.00585959 $N$  |
| 3   | 0   | $-\frac{1}{64}$                | $\frac{1}{64}$              |
| 3   | 1   | $\frac{1}{64}$                | $\frac{1}{64}$              |
| 3   | 2   | $-0.0179798 - 0.0114583 N$     | $-0.0359598 - 0.00137167 N$ |
| 3   | 3   | $0.00517395 - 0.00217669 N + 0.00231481 N^2$ | $0.0103479 - 0.03337373 N - 0.0056229 N^2$ |
| 4   | 0   | $\frac{1}{256}$               | $\frac{1}{256}$             |
| 4   | 1   | $\frac{1}{256}$               | $\frac{1}{256}$             |
| 4   | 2   | $0.00733893 + 0.00444944 N$    | $0.0146779 + 0.000299779 N$ |
| 4   | 3   | $-0.000842908 - 0.00212265 N + 0.00162037 N^2$ | $-0.00168582 + 0.0117321 N + 0.00170808 N^2$ |
| 5   | 0   | $-\frac{1}{1024}$             | $\frac{1}{1024}$            |
| 5   | 1   | $\frac{1}{1024}$             | $\frac{1}{1024}$            |
| 5   | 2   | $-0.00253604 - 0.00149678 N$  | $-0.00507208 - 0.0000641563 N$ |
| 5   | 3   | $-0.000390898 - 0.0015352 N - 0.000747354 N^2$ | $-0.000781794 - 0.0037964 N - 0.000501577 N^2$ |

### TABLE VIII. Expansion coefficients $x_1$, $x_2$ and $x_3$ for $e_i$ and $a_i$.

| $x_1$ | $x_2$ | $x_3$ |
|-------|-------|-------|
| $e_1$  |       |       |
| $e_2$  |       |       |
| $e_3$  |       |       |
| $e_4$  |       |       |
| $e_5$  |       |       |
| $a_1$  |       |       |
| $a_2$  |       |       |
| $a_3$  |       |       |
| $a_4$  |       |       |
| $a_5$  |       |       |
### TABLE IX. Expansion coefficients $x_4$ and $x_5$ for $e_i$ and $a_i$.

|   | $x_4$                                                                 | $x_5$                                                                 |
|---|----------------------------------------------------------------------|----------------------------------------------------------------------|
| $e_1$ | $-1184 - 348 \frac{N}{7} N^2$                                      | $-(2+N) \left( -4112 - 2506 N - 460 N^2 + N^3 \right)$              |
|     |                                                                     | $4512 + 1420 N + 35 N^2$                                               |
| $x_5$ |                                                                      | $(2+N) \left( -170624 - 78048 N - 8148 N^2 + 599 N^3 \right)$          |
| $e_2$ |                                                                     | $7600$                                                                |
|     |                                                                     | $43876270$                                                            |
| $e_3$ | $586167808 + 604936960 N + 160069792 N^2 + 8862854 N^3 - 1077019 N^4 + 2048 N^5$ |
|     | $-2311264 - 720000 N - 17885 N^2$                                    |                                                                     |
| $e_4$ |                                                                     | $106349019200$                                                        |
|     |                                                                     | $3084090980864 + 377841482880 N + 961368439624 N^2 + 41380542442 N^3 - 69875168685 N^4 + 2097152 N^5$ |
| $a_1$ | $1376 + 764 N - 301 N^2 + 14 N^3$                                    |                                                                     |
| $a_2$ |                                                                     | $7552 - 2784 N - 5692 N^2 - 1667 N^3 - 100 N^4$                        |
|     | $-3104 + 13000 N + 3185 N^2 + 140 N^3$                                |                                                                     |
| $a_3$ |                                                                     | $145632 + 196600 N + 144000 N^2 + 17885 N^3$                           |
| $a_4$ |                                                                     | $568281088 + 262815488 N - 1452256 N^2 + 13472924 N^3 - 1114341 N^4 - 9216 N^5$ |
|     |                                                                     | $-3381888 + 3454972 N + 7018809 N^2 + 25638 N^3$                      |
| $a_5$ |                                                                     | $440401920$                                                           |
|     |                                                                     | $-17032726784 - 7719736448 N + 90484812 N^2 + 49916471 N^3 + 36197264 N^4 + 491520 N^5$ |
|     |                                                                     | $5244438496 + 4565788340 N + 853347495 N^2 + 26696670 N^3$             |
|     |                                                                     | $310012961680$                                                        |
|     |                                                                     | $34179580035328 + 14923763131008 N - 51376540188 N^2 - 895119755607 N^3 - 8146159226 N^4 - 137625600 N^5$ |
|     |                                                                     | $104639496192000$                                                     |
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