Optimization mathematical models of the peaceful subordinating interactions of two States

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Abstract. Pioneer dynamic mathematical optimization models of the peaceful subordinating interactions between two States in favor of one of them are created on the platform of the dynamic mathematical optimization model of a State, earlier constructed and investigated by the author from both theoretical and applied points of view. These models appear to be the totally new ones, since their active agents are States and not firms or another smaller subjects of the economics. Consequently at the current time moment they are purely conceptual. Neither theoretical nor practical methods of solution of the formulated optimization problem are considered in the paper.

1. Introduction
Several mathematical models of competition and interaction between some subjects of economics were constructed and discussed in the articles and monographies [1 - 7]. Actual agents in all abovementioned models as well as in some others were not namely states. In the articles [8-10] and in the monographs [11 - 13] a general pioneer and all-encompassing optimization mathematical model of a state was introduced. Articles [9, 10] are devoted to the computational simulation task of a search for the optimized controls for a state based on the abovementioned model.

A breakthrough to creation on this platform of optimization generalized mathematical models of interstate interactions (between several states) has occurred in 2018, when the author managed to formalize mathematically some subordinating interactions of two states (both military and peaceful) [14]. Exactly these interaction types are the mostly noticeable ones in the modern history of 20th and 21st centuries. The main goal of the both kinds of subordinating interactions - military or peaceful - is the same. It is to achieve the greatest discrepancy (divergence) between the joint wealth of interactive states towards the end of a certain period of time (in favor of one of the interacting states). The ways (methods) of control, however, may be qualitatively different in these interactions.

In the present work we are going to consider namely a peaceful (trade and currency) subordinating interaction between two states with an optimal one-sided control in favor of a subordinating state (a non-coordinated control). This article, being the pioneer one, at the present time is the purely conceptual paper. No applied methods of solution of the considered optimization tasks are discussed in this article. All the used customary and special terms must be understood in the mathematical meaning only.

2. A State, its organization and functioning
We shall consider the mankind as the totality of all human beings ever lived (or living now) on the planet Earth. The united - material (and mental) - world (at the taken moment of time) is the totality containing the following two parts:
1) the native world - material and mental - of the Universe (and the Earth as its part);
2) the over-native world - material or mental - created by the mankind during its existence (in the process of its being).
Thus, the mankind itself must be considered as a part of the over-native world.

A society appears to be an object of the united world which necessarily includes a certain part of the mankind and all parts of the over-native world, directly connected with it. A State (in the broad sense, understood in the meaning of a so-called State-Country) is a society, which appears to be a three-level complicated structure (or a system) consisting of the (seven) basic subsystems. These seven subsystems depend on each other, and they are connected to each other that way, so the State cannot function without anyone of those subsystems. The basic (sub)systems of a State we shall call the providing, the existential, the accounting, the joint administrative (which consists of 3 parts itself), and the supreme (sub)system.

The existential (sub)system C performs the following methods of a vital activity within a State: the meta-agrarian method (that means a process of removing of some vital resources from the native environment), the industrial method (that means processing of those natural resources, or remaking of them), and servicing method. The providing (sub)system D is responsible for the order, laws and legislation, guarding, public morality, etc. The accounting (sub)system E issues money, carries out their storage and distribution, provides the collection and processing of statistical data, and so on. The providing, the existential, and the accounting (sub)systems are being controlled (ruled over) with administrative (sub)systems F, G, and H, correspondingly. Hence, they can be named as providing-administrative (sub)system, existential-administrative (sub)system, and accounting-administrative (sub)system, correspondingly. They are the three parts of the joint administrative system and form it altogether. The ruling (supreme) (sub)system P provides a control (rules) over the joint administrative system.

We consider a State as an open system which has the following three surroundings important for its being, referred to as a native surrounding (a native environment) A1, a foreign over-native surrounding A2, and an own shady surrounding of a State A3. They are collectively referred to as the surroundings of a State.

Let us call “a wealth of a State” every mentally highlighted part of this State (including the State population), as well as a part of internal environment of this State. Wealths are preserved (contained) in the systems (institutions) of a State and in the surroundings of it. All the State’s wealthes we shall further divide into the following types: the existential wealth (marked with the code 1), the providing wealth (code 4), the accounting wealth (code 5), the administrative wealth (code 2), and the supreme (marked with the code 3) wealth. A totality of wealthes of a given type for all institutions of a given basic (sub)system compose wealthes of this basic (sub)system, while a totality for all basic (sub)systems of a State compose wealthes of this State of a given type. A totality of wealthes of a State of all types composes a joint wealth of a State.

Each and every basic (sub)system generates a wealth belonging to its special type. Generating this wealth, the (sub)system uses some of wealthes which were already existing inside of it. All basic (sub)systems are connected together (to the almost each other subsystem) via the streams of wealthes. The providing, the existential, and the accounting systems extract from the surroundings of a State and return into them (corresponding to their essence) their special types of wealthes.

3. General notations and definitions. Streams of wealthes of a State
Let us fix a certain State S and consider the main interval of time \([T_1, T_2]\). In this time interval we fix the moment \(t\) and an increment of time \(\Delta t\) such as \(T_1 \leq t - \Delta t \leq t \leq t + \Delta t \leq T_2\). A totality of all specific (over-native) wealthes of the type (or kind) \(m\) in the system \(M\) at the moment \(t\) we denote by \(V^M_m(t)\), and a totality of all specific wealthes of all types \(m = 1, 2, 3, 4, 5\) in all the (sub)systems \(M = C, D, E, F, G, H, P\) let’s denote by \(V(t)\). Now we consider the general (complete) totality \(V(T_1, T_2)\) of all specific wealthes \(x\), which belongs to all totalities \(V(t)\) for every \(t\) from the main (principal) interval of
time \([T_1, T_2]\). We denote with \(c(t)\) a real value of the specific (concrete) wealth \(x\) at the time moment \(t\), denominated in money of the State \(S\) (part 4.6 in monographs [11 - 13]). The number \(W^m(t)\) which equals to the sum of real values \(c(t)\) of all specific (concrete) wealths \(x\) from the totality \(V^m(t)\) will be called the evaluated joint wealth of the kind (type) \(m\) of the system \(M\) at the moment of time \(t\).

A totality \(R^m_{MN}(t - \Delta t, t + \Delta t)\) including all the specific (concrete) wealths \(x\) from the general totality \(V(T_1, T_2)\), such that \(x\) is included into the totality \(V^m(t - \Delta t)\), and \(x\) is not included into the totality \(V^m(t + \Delta t)\), and \(x\) is included into the totality \(V^m(t + \Delta t)\), we shall refer to as the stream on the interval of time \([t - \Delta t, t + \Delta t]\) coming from the totality of specific (concrete) wealths of the type (the kind) \(m\) of the system \(M\) into the totality of specific (concrete) wealths of the kind \(n\) of the system \(N\). The number \(S^m_{MN}(t - \Delta t, t + \Delta t)\), which is equal to the sum of real values \(c(t)\) of all specific (concrete) wealths \(x\) from the stream \(R^m_{MN}(t - \Delta t, t + \Delta t)\), we shall refer to as the evaluated stream on the interval of time \([t - \Delta t, t + \Delta t]\) from the totality of specific (concrete) wealths of the kind \(m\) of the system \(M\) into the totality of specific (concrete) wealths of the kind \(n\) of the system \(N\). The number \(S^m_{MN}(t) = \lim_{\Delta t \to 0} \frac{S^m_{MN}(t - \Delta t, t + \Delta t)}{2\Delta t}\) we shall call the evaluated stream at the time moment \(t\) from the totality of specific (concrete) wealths of the type (kind) \(m\) of the system \(M\) into the totality of specific (concrete) wealths of the type (kind) \(n\) of the system \(N\). We shall omit the words «evaluated» and «joint» as well as the indication of the time moment \(t\) in our further considerations and statements.

The abovementioned (sub)systems, wealths, and streams of wealths of a State \(S\) are indicated below on the Figure 1.

**Fig. 1.** The representation of (sub)systems, surroundings and streams of wealths of a State
the articles [9, 10]. The non-linearity of control members in the corresponding equations makes this task the somehow peculiar one. Despite of this difficulties, the explicit analytical optimal solution of the system of equations for the State $S$ was found with the help of the maximal principle of L.S. Pontryagin.

4. Peaceful interactions of two States. Additional wealths and streams of wealths

Now we are going to consider some two States: the already described State $S$ and another State $S(I)$. All values corresponding to this second state will be equipped with the Roman digit $I$ situated in round brackets. We set a goal to create mathematical models of the optimal peaceful interactions between these two States, which are implemented by trade and currency controls stemming from the subordination of the State $S$ to the State $S(I)$.

For this aim we need to introduce the additional existential wealth $1(I)$ of the system $C(I)$ belonging to the State $S(I)$ into the existential system $C$ of the State $S$, and introduce also the additional existential wealth $1$ of the system $C$ (of the State $S$) into the existential system $C(I)$ (of the State $S(I)$). Additionally, we introduce the accounting wealth having the code $w$ (and named the fixed (generalized) world currency) into the accounting (sub)systems of the both States – the accounting (sub)system $E$ of the State $S$ as well as into the accounting (sub)system $E(I)$ of the State $S(I)$. This generalized world currency allows to carry out trade operations between the States $S$ and $S(I)$.

Also it is necessary to add some new streams $Z^{1(I)}_{CC}$, $Z^{I(I)}_{CC(I)}$, $Z^{11}_{C(I)C}$, and $Z^{I1}_{C(I)C(I)}$ for existential (sub)systems of both States: $C$ and $C(I)$.

The first new stream $Z^{11}_{CC}$ will be further named as “the selling stream in $S$ for $S(I)$”. Its meaning is the follows: the existential wealth $1$ of the system $C$ was sold by the State $S$ to the State $S(I)$ and was bought by the State $S(I)$ in the State $S$ (with the use of the world generalized currency), hence after that it has become the existential wealth $1(I)$ of the existential (sub)system $C(I)$ of the State $S(I)$.

The second new stream $Z^{I1(I)}_{CC(I)}$ will be further named as “the export from $S$ into $S(I)$ and the import into $S(I)$ from $S$”. Its meaning is: the existential wealth $1(I)$ which is acquired (bought) by the State $S(I)$ was dropped into the existential (sub)system $C(I)$.

The third new stream $Z^{11}_{C(I)C}$ will be further named as “the selling in $S(I)$ for $S$”. Its meaning is the follows: the existential wealth $1(I)$ of the existential (sub)system $C(I)$ was sold by the State $S(I)$ to the State $S$ and was bought by the State $S$ in the State $S(I)$ with the help of the world generalized currency, and thus it has become the existential wealth $1$ of the existential (sub)system $C$ (of the State $S$).

The fourth new stream $Z^{I1}_{C(I)C}$ will be further named as “the export from $S(I)$ into $S$ and the import into $S$ from $S(I)$”. Its meaning is that the existential wealth $1$ acquired (bought) by the State $S$ has now been conveyed into the existential (sub)system $C$ (of the State $S$).

Additionally, we have to introduce also the new streams $Y^{ww}_{E(I)E}$ and $Y^{ww}_{EE(I)}$. The stream $Y^{ww}_{E(I)E}$ will be further called “the exportation-paying by the State $S(I)$ to the State $S$“. It reflects the fact that the world currency $w$ is transferred from the accounting (sub)system $E(I)$ into the accounting (sub)system $E$ for the sale by the State $S$ to the State $S(I)$ of the existential wealth $1$ of the system $C$ of the State $S$.

The newly introduced stream $Y^{ww}_{EE(I)}$ will be named as “the importation-paying by the State $S$ to the State $S(I)$“. Its meaning is that the world currency $w$ must be transferred from the accounting (sub)system $E$ into the accounting (sub)system $E(I)$ with the aim of the bought by the State $S$ in the State $S(I)$ of the existential wealth $1(I)$ of the (sub)system $C(I)$.

Due to the fact that some streams are calculated (evaluated) in money 5 of the State $S$, while the others are expressed (evaluated) in money 5(I) of the State $S(I)$, we ought to describe connections of
those streams, which were originally represented (calculated) in different money, resorting to conversion of those money into some theoretically generalized (fixed) world currency \( w \). Let us consider the equality \( q(t) = s(t)q_w(t) \). Here \( q(t) \) be a quantity of internal money \( 5 \) of the State \( S \), while \( q_w(t) \) be a quantity of currency (at the time moment \( t \), when the actual exchange of internal money and the currency takes place). The number \( s(t) \) in the abovementioned equality symbolizes the exchange rate for internal money \( 5 \) of the State \( S \) into the fixed (generalized) world currency \( w \) at the time moment \( t \). In the same vein, the number \( s(I)(t) \) in the equality \( q(I)(t) = s(I)q_w(t) \), where \( q(I)(t) \) be a quantity of internal money \( 5(I) \) of the State \( S(I) \), while \( q_w(t) \) be a quantity of currency (at the time moment \( t \) of the actual exchange operation), will be called the exchange rate for internal money \( 5(I) \) of the State \( S(I) \) into the fixed (generalized) world currency \( w \) at the time moment \( t \).

The equalities considered below appear from the definitions mentioned earlier: \( Z_{1}^{11}(I) = sY_{1}^{ww}(E_{1})(I) \), \( Z_{C(I) C}^{11} = sY_{C(E)}^{ww}(I) \), \( Z_{C(I) C}^{11} = s(I)Y_{C(E)}^{ww}(I) \), and also \( Z_{C(I) C}^{11} = s(I)Y_{C(E)}^{ww}(I) \). Further we accept the assumption, that all the transmitted and transferred streams will be considered as «equal» with respect to the generalized currency \( w \). Thus, we can conclude: they are interconnected through the equalities \( (1/s)Z_{C(I) C}^{11} = (1/s)(s(I))Z_{C(I) C}^{11} \) as well as \( (1/s)Z_{C(I) C}^{11} = (1/s)(s(I))Z_{C(I) C}^{11} \).

5. Evolutionary equations for the State \( S(I) \)

The evolutionary equations system for the State and its functioning is written in accordance with the main principle of preservation. This preservation principle provides: the velocity of change of the valued joint wealth of a given type (or kind) in a given (sub)system at a given moment \( t \) equals to the sum of all entering valued streams of this kind of wealth into that (sub)system at this moment of time \( t \) minus the sum of all outgoing valued streams (of the same kind of wealth) from that given (sub)system (at the same time moment \( t \) under consideration). According to such a main principle, we obtain the system of evolutionary and functioning equations for the State \( S(I) \) given below:

\[
\begin{aligned}
W_{C(I)}^{11}(I) &= L_{E(I) C(I)}^{11}(I) - (p_{1}(I)B_{E(I) D(I)}^{11}(I) + p_{2}(I)B_{E(I) E(I)}^{11}(I)) + p_{3}(I)B_{E(I) F(I)}^{11}(I) + p_{4}(I)B_{E(I) G(I)}^{11}(I) + p_{5}(I)B_{E(I) H(I)}^{11}(I) \\
1(I) + p_{6}(I)B_{E(I) P(I)}^{11}(I) - e_{0}(I)W_{C(I)}^{11}(I) + s(I)Y_{E(I)}^{ww} - s(I)Y_{E(E)}^{ww},
\end{aligned}
\]

where

\[
\begin{aligned}
L_{E(I) C(I)}^{11}(I) &= a(I)W_{C(I)}^{11}(I)(K(I) - W_{C(I)}^{11}(I)) + d(I)r(I) + \\
(B_{E(I) C(I)}^{11}(I) + B_{E(I) E(I)}^{11}(I) + B_{E(I) F(I)}^{11}(I)) + B_{E(I) G(I)}^{11}(I) + B_{E(I) H(I)}^{11}(I),
\end{aligned}
\]

\[
\begin{aligned}
2(I) &= W_{D(I)}^{11}(I) - B_{E(I) D(I)}^{11}(I) - e_{1}(I)W_{D(I)}^{11}(I) \\
3(I) &= W_{E(I)}^{11}(I) - B_{E(I) E(I)}^{11}(I) - e_{2}(I)W_{E(I)}^{11}(I) \\
4(I) &= W_{F(I)}^{11}(I) - B_{E(I) F(I)}^{11}(I) - e_{3}(I)W_{F(I)}^{11}(I) \\
5(I) &= W_{G(I)}^{11}(I) - B_{E(I) G(I)}^{11}(I) - e_{4}(I)W_{G(I)}^{11}(I) \\
6(I) &= W_{H(I)}^{11}(I) - B_{E(I) H(I)}^{11}(I) - e_{5}(I)W_{H(I)}^{11}(I) \\
7(I) &= W_{P(I)}^{11}(I) - B_{E(I) P(I)}^{11}(I) - e_{6}(I)W_{P(I)}^{11}(I)
\end{aligned}
\]
\[ W^w_{E(I)} = Y^w_{EE(I)} - Y^w_{E(I)E} - e^w_I W^w_{E(I)}. \]

In this system \( K(I) > 0, a(I) > 0, c(I) > 0, d(I) > 0, 0 < e(I) < 1, 0 < p(I) < 1 \) are numeric parameters for the State \( S(I) \), \( C(I) \) denotes a budget (sub)system of the existential system of this State, \( r(I) \) means a key rate of this State’s emission center, while \( B^S(I)_{E(I)C_{E(I)}}(I), B^S(I)_{E(I)D(I)}, B^S(I)_{E(I)E(I)}(I), B^S(I)_{E(I)F(I)}, B^S(I)_{E(I)G(I)}(I), B^S(I)_{E(I)H(I)}(I), B^S(I)_{E(I)P(I)}(I) \) marks the budget streams directed from the State \( S(I) \) accounting (sub)system into all the indicated (see low indices) - thoroughly described earlier in this paper - (sub)systems of the State \( S(I) \). While the equations with the numbers from (1) to (7) are written in money \( S(I) \) which are in use in the State \( S(I) \), the equation (8) is written in the generalized world currency \( w \).

Let us consider for the system of evolutionary and functioning equations (1)-(8) for the State \( S(I) \) the initial totality \( \sigma(I)[t_0, t] \) of all \( S(I) \)-internal controls \( r(I), B^S(I)_{E(I)C_{E(I)}}(I), B^S(I)_{E(I)D(I)}, B^S(I)_{E(I)E(I)}(I), B^S(I)_{E(I)F(I)}, B^S(I)_{E(I)G(I)}(I), B^S(I)_{E(I)H(I)}(I), B^S(I)_{E(I)P(I)}(I) \) which are assigned (introduced) as the functions of a time moment \( t' \) on the interval of time \([t_0, t]\).

6. The peaceful subordinating interaction between two states and its mathematical model

Let us consider the secondary totality \( \tau(I)[t_0, t] \) of all \( S(I) \)-external controls \( Y^w_{EE(I)}, Y^w_{E(I)E} \) for the system of evolutionary and functioning equations (1)-(8) for the State \( S(I) \). The joint one-sided non-coordinated control \( \sigma(I)[t_0, t], \tau(I)[t_0, t] \) we shall denote by \( u[t_0, t] \) and call the peaceful joint one-sided control in the system of evolutionary equations (1)-(8) for the State \( S(I) \).

In a similar way we further shall introduce the joint wealth \( W_S(I)(t, u[t_0, t]) = (W^C_{E(I)}, W^D_{E(I)}, W^E_{E(I)}, W^F_{E(I)}, W^G_{E(I)}, W^H_{E(I)}, W^P_{E(I)}) \) of the State \( S(I) \) under the control \( u[t_0, t] \) at the moment of time \( t \) on the time interval \([t_0, T]\).

Let us consider the initial joint wealths of the interacting States: firstly the initial joint wealth \( W_S(t_0) = (W^1_C + W^4_D + W^5_E + W^2_F + W^2_G + W^2_P)(t_0) \) of the State \( S \) at the time moment \( t_0 \) and also the initial joint wealth \( W_S(I)(t_0) = (W^1_C(I) + W^4_D(I) + W^5_E(I) + W^2_F(I) + W^2_G(I) + W^2_H(I) + W^2_P(I))(t_0) \) of the State \( S(I) \) at the same moment of time \( t_0 \).

Now we introduce the target (objective) function \( \Psi(t, u[t_0, t]) = W_S(I)(t, u[t_0, t]) - W_S(t_0) \) of the discrepancy (divergence) between the joint wealth of the State \( S(I) \) at the current time moment \( t \) under the peaceful control \( u[t_0, t] \) in relation to the initial joint wealth of the State \( S \) at the initial moment of time \( t_0 \).

The interaction between the States \( S \) and \( S(I) \) under the introduced peaceful control \( u[t_0, T] \) in the system of evolutionary and functioning equations (1)-(8) for the State \( S(I) \) we shall name the peaceful interaction and denote by \( A(S, S(I), u[t_0, T]) \). Thus introduced peaceful interaction \( A(S, S(I), u[t_0, T]) \) we call the \((S(I), \alpha, \beta)\)-subordinating interaction (for the State \( S(I) \), and characterized with the numeric subordination levels \( 0 < \alpha, \beta, \gamma < 1 \), if:

1) \( W_S(I)(T, u[t_0, T]) - W_S(t_0) \geq \alpha(W_S(S(I), t_0) - W_S(t_0)) \), i.e. \( \Psi(T, u[t_0, T]) \geq \alpha\Psi(t_0) \) (the final discrepancy, or divergence);
2) \( W_S(I)(T, u[t_0, T]) \geq \beta W_S(S(I), t_0) \) (the final enrichment, or the profit obtaining).

A taken peaceful \((S(I), \alpha, \beta)\)-subordinating interaction \( A(S, S(I), u^*_{[t_0, T]} \) it is naturally to call the optimal interaction on the interval of time \([t_0, T]\) with respect to the chosen target (or objective) function \( \Psi(t, u[t_0, t]) \), if for any another peaceful \((S(I), \alpha, \beta)\)-subordinating interaction
\[ A(S, S(l), u[t_0, T]) \text{ the inequality } \Psi(T, u^*[t_0, T]) \geq \Psi(T, u[t_0, T]) \text{ holds. Formally (and traditionally) this relation - and this aim - may be formulated in the form } \Psi(T, u[t_0, T]) \to \max \text{ in the system of evolutionary and functioning equations (1)-(8) for the State } S(l). \]

In the considerably more complicated way we can investigate the case of peaceful interaction between the two States, where exists the consent, or agreement, to subordinate to the State \( S(l) \) from the side of the supreme (ruling) subsystem of the State \( S \). Therefore, under such conditions the coordinated two-sided control of the \( S(l) \)-subordination appears.

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