A decomposition method to construct cubature formulae of degree 3

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Abstract

Numerical integration formulas in $n$-dimensional Euclidean space of degree three are discussed. For the integrals with permutation symmetry we present a method to construct its third-degree integration formulas with $2n$ real points. We present a decomposition method and only need to deal with $n$ one-dimensional moment problems independently.

Keywords: Numerical integration; Degree three; Cubature formulae; Decomposition method; One-dimensional moment problem

1 Introduction

Let $\Pi^n = \mathbb{R}[x_1, \ldots, x_n]$ be the space of polynomials in $n$ real variables and $\mathcal{L}$ be a square positive linear functional defined on $\Pi^n$ such as those given by $\mathcal{L}(f) = \int_{\mathbb{R}^n} f(x)W(x)dx$, where $W$ is a nonnegative weight function with finite moments of all order. Let $\Pi_d^n$ be the space of polynomials of degree at most $d$. Here we discuss numerical integration formulas of the form

$$\mathcal{L}(f) \approx \sum_k a_k f(u^{(k)}),$$

where $a_k$ are constants and $u^{(k)}$ are points in the spaces. The formulas are called degree of $d$ if they are exact for integrations of any polynomials of $x$ of degree at most $d$ but not $d + 1$.

In this paper, we only deal with the construction of third-degree cubature formula. It looks like a simple problem, however it remains to be solved. The well known result is due to Stroud [1–3]. He presented a method to construct numerical integration formulas of degree 3 for centrally symmetrical region and recently Xiu [5] also considered the similar numerical formulas for integrals as

$$\mathcal{L}(f) = \int_a^b \ldots \int_a^b w(x_1) \ldots, w(x_n) f(x_1, x_2, \ldots, x_n)dx_1 \ldots dx_n.$$

In [5], Xiu assumed that every single integral is symmetrical, which means his result naturally belongs to centrally symmetrical case. Recently, the authors [12] extended Stroud’s results and

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presented formulas of degree 3 of 2n points or 2n + 1 points for integrals as

\[ \mathcal{L}(f) = \int_{a_1}^{b_1} \ldots \int_{a_n}^{b_n} w_1(x_1) \ldots w_n(x_n) f(x_1, \ldots, x_n) \, dx_1 \ldots dx_n, \]

\[ w_i(x_i) \geq 0, \ x_i \in [a_i, b_i], \ i = 1, \ldots, n. \]

Besides, many scholars employed the invariant theory method to deal with symmetrical case and we can refer to [7–9] and the references therein. As far as we know, 2n is the minimum of the integration points except some special regions (see [11]), and for centrally symmetrical region Mysovskikh [6] had shown this point. However, for the general integration case, it remains unknown how to construct the formulas of degree 3 with 2n points. In the two-dimensional case, third degree integration formulas with 4 real points was given in [13, 14] for any regions. But it is difficult to extend it to higher dimension. For other related work, we can refer to [15–17] and the reference therein.

This paper will extend the results in [12] for the integrals of product regions to those with permutation symmetry. First we present a condition which is satisfied by the integral. And then we prove that under this condition, the construction problem of cubature formulae with degree three can be transformed into two smaller sub-cubature problems. Finally, for the construction of cubature rules of the integrals with permutation symmetry can be decomposed into n one-dimensional moment problems.

This paper is organized as follows. The construction of cubature formulas of degree 3 are presented in section 2. And section 3 will present two examples to illustrate the construction process. Finally, section 4 will make a conclusion.

## 2 The construction of third-degree formulas

Assume that \( \mathcal{L} \) has the following property:

\( \text{(P)} \)

*There exist \( n \) linearly independent polynomial \( l_i(x_1, \ldots, x_n), i = 1, 2, \ldots, n \) such that all \( l_i l_j (i = 1, \ldots, n − 1) \) are the orthogonal polynomials of degree two with respect to \( \mathcal{L} \).*

Let

\[ T : \ l_i(x_1, \ldots, x_n) \rightarrow t_i, \ i = 1, 2, \ldots, n \]  

(2)

be a linear transformation and \( \mathcal{L} \) be transformed into \( \mathcal{L}' \). Then by the assumption all \( t_i t_j (i = 1, 2, \ldots, n − 1) \) are the orthogonal polynomial of degree two with respect to \( \mathcal{L}' \). Here we do not require that all \( t_i t_j t_n (i = 1, 2, \ldots, n − 1) \) can constitute a basis of orthogonal polynomials of degree 2 with respect to \( \mathcal{L}' \). We can also assume that the third-degree formula of \( \mathcal{L}' \) has the following form

\[ \begin{array}{c}
\upsilon^{(1)} = (v_{1,1}, v_{1,2}, v_{1,3}, \ldots, v_{1,n-1}, 0) \ \omega_1 \\
\upsilon^{(2)} = (v_{2,1}, v_{2,2}, v_{2,3}, \ldots, v_{2,n-1}, 0) \ \omega_2 \\
\vdots \\
\upsilon^{(N)} = (v_{N,1}, v_{N,2}, v_{N,3}, \ldots, v_{N,n-1}, 0) \ \omega_N \\
\upsilon^{(N+j)} = (0, 0, 0, \ldots, 0, v_{N+j,n}) \ \omega_{N+j}, \ j = 1, 2.
\end{array} \]  

(3)

To enforce polynomial exactness of degree 3, it suffices to require (3) to be exact for

\[ 1, t_1, t_2, \ldots, t_n, t_it_j, t_it_jt_k \ \ i, j, k = 1, 2, \ldots, n. \]
Then we have

$$\omega_1 + \omega_2 + \ldots + \omega_{N+2} = L'(1)$$

(4a)

$$\begin{cases}
\omega_1 v_{1,i} + \omega_2 v_{2,i} + \ldots + \omega_N v_{N,i} = L'(t_i), & i = 1, 2, \ldots, n - 1 \\
\omega_{N+1} v_{1,i} + \omega_{N+2} v_{2,i} + \ldots + \omega_N v_{N,i} = L'(t_n)
\end{cases}$$

(4b)

$$\begin{cases}
\omega_1 v_{1,j} + \omega_2 v_{2,j} + \ldots + \omega_N v_{N,j} = L'(t_j), & i = 1, 2, \ldots, n - 1 \\
\omega_{N+1} v_{1,j} + \omega_{N+2} v_{2,j} + \ldots + \omega_N v_{N,j} = L'(t_n)
\end{cases}$$

(4c)

$$\begin{cases}
\omega_1 v_{1,k} + \omega_2 v_{2,k} + \ldots + \omega_N v_{N,k} = L'(t_k), & i = 1, 2, \ldots, n - 1 \\
\omega_{N+1} v_{1,k} + \omega_{N+2} v_{2,k} + \ldots + \omega_N v_{N,k} = L'(t_n)
\end{cases}$$

(4d)

and the equation (4a) can be rewritten as

$$\omega_1 + \omega_2 + \ldots + \omega_N = \xi_1,$$

$$\omega_{N+1} + \omega_{N+2} = \xi_2,$$

$$\xi_1 + \xi_2 = L'(1).$$

Hence we can rewrite the equations (4) as

$$\begin{cases}
\omega_1 + \omega_2 + \ldots + \omega_N = \xi_1, \\
\omega_1 v_{1,i} + \omega_2 v_{2,i} + \ldots + \omega_N v_{N,i} = L'(t_i), & i = 1, 2, \ldots, n - 1 \\
\omega_1 v_{1,j} + \omega_2 v_{2,j} + \ldots + \omega_N v_{N,j} = L'(t_j), & i = 1, 2, \ldots, n - 1 \\
\omega_{N+1} v_{1,j} + \omega_{N+2} v_{2,j} + \ldots + \omega_N v_{N,j} = L'(t_n)
\end{cases}$$

(5a)

$$\begin{cases}
\omega_{N+1} v_{1,j} + \omega_{N+2} v_{2,j} + \ldots + \omega_N v_{N,j} = L'(t_j), & i = 1, 2, \ldots, n - 1 \\
\omega_{N+1} v_{1,k} + \omega_{N+2} v_{2,k} + \ldots + \omega_N v_{N,k} = L'(t_k), & i = 1, 2, \ldots, n - 1 \\
\omega_{N+1} v_{1,k} + \omega_{N+2} v_{2,k} + \ldots + \omega_N v_{N,k} = L'(t_n)
\end{cases}$$

(5b)

$$\omega_{N+1} v_{1,k} + \omega_{N+2} v_{2,k} + \ldots + \omega_N v_{N,k} = L'(t_n)$$

(5c)

$$\xi_1 + \xi_2 = L'(1).$$

Once $\xi_i$ is determined by (5c), then (5a) and (5b) become one $n - 1$ dimensional and the other one-dimensional moment problems respectively. If these two lower dimensional moment problems can be solved, then we can get a cubature formula of degree 3 with respect to the original integration problem. Generally speaking, the one-dimensional moment problem can be easily solved, but it is difficult to be solved for the $n - 1$ dimensional moment problem. However, if the $n - 1$ dimensional problem can be divided into one $n - 2$ dimensional moment problem and the other one-dimensional moment problem and further the $n - 2$ dimensional moment problem can continue this process, then the original integration problem can be turned into $n$ one-dimensional moment problems.

In what follows, we will prove that if $L$ is an integral functional with permutation symmetry, then the construction problem of third-degree cubature formulae can be turned into $n$ one-dimensional moment problems. In fact, we usually encounter this kind of integral functional, for example, the integration over the simplex, the square, the ball or the positive sector of the ball, that is $\{ (x_1, x_2, \ldots, x_n) | x_i \geq 0, i = 1, 2, \ldots, n; x_1^2 + x_2^2 + \ldots + x_n^2 \leq 1 \}$.

We first prove that the integral functional with permutation symmetry must meet the property (P). Thus the original cubature problem can be turned into two sub-cubature problems.

**Theorem 1.** Let $n \geq 2$ and

$$l_i(x_1, \ldots, x_n) = x_i - x_n, \quad i = 1, 2, \ldots, n - 1$$

$$l_n(x_1, \ldots, x_n) = x_1 + x_2 + \ldots + x_n + c_n$$

where

$$c_n = - \frac{L(x_1^3 + (n - 3)x_1^2x_2 - (n - 2)x_1x_2x_3)}{L(x_1^2 - x_1x_2)}.$$
If $\mathcal{L}$ is permutation symmetrical, then $l_1, l_n$ ($i = 1, 2, \ldots, n - 1$) is orthogonal to the polynomials of degree $\leq 1$.

**Proof.** Take $l_1, l_n$ as an example. We first prove $\mathcal{L}(x_1^2 - x_1 x_2) \neq 0$ to confirm the existence of $l_n$. In fact, by the symmetry and the positivity,
\[
\mathcal{L}(x_1^2 - x_1 x_2) = \frac{1}{2} \mathcal{L}(x_1^2 - x_1 x_2 - x_1 x_2 + x_2^2) = \frac{1}{2} \mathcal{L}((x_1 - x_2)^2) > 0.
\]

Let us exam the orthogonality of $l_1 l_n$. Assume $n \geq 3$. By the symmetry, we have
\[
\mathcal{L}(l_1 l_n) = \mathcal{L}(x_1^2 - x_n^2 + (x_1 - x_n)(x_2 + \ldots + x_{n-1} + c_n)) = 0,
\]
\[
\mathcal{L}(x_1 l_1 l_n) = \mathcal{L}(x_1^2 - x_n^2 x_1 + (x_1 - x_n)(x_2 x_1 + \ldots + x_{n-1} x_1 + c_n x_1)) = 0, \quad 2 \leq i \leq n - 1
\]
and for $i = 1$ (the same to $i = n$)
\[
\mathcal{L}(x_1 l_1 l_n) = \mathcal{L}(x_1^3 - x_n^2 x_1 + (x_1 - x_n)(x_2 x_1 + \ldots + x_{n-1} x_1 + c_n x_1)) = 0.
\]
It is easy to verify that the result holds when $n = 2$. This completes the proof. \qed

Let $\mathcal{L}_1$ and $\mathcal{L}^1$ be two linear functionals defined on $\Pi^{n-1}$ and $\Pi^1$, whose moments are determined by
\[
\mathcal{L}_1(t_1 t_2 t_3) = \mathcal{L}(l_1 l_2 l_3), \quad \mathcal{L}_1(t_1 t_3) = \mathcal{L}(l_1 l_3), \quad \mathcal{L}_1(t_1) = \mathcal{L}(l_1), \quad \mathcal{L}_1(1) = \xi^{(0)}_1 \tag{6}
\]
\[
\mathcal{L}^1(t_n^3) = \mathcal{L}(t_n^3), \quad \mathcal{L}^1(t_n^2) = \mathcal{L}(t_n^2), \quad \mathcal{L}^1(t_n) = \mathcal{L}(l_n), \quad \mathcal{L}^1(1) = \xi^{(0)} \tag{7}
\]
\[
\xi^{(0)}_1 + \xi^{(0)} = \mathcal{L}(1) \quad \text{and} \quad 1 \leq i, j, k \leq n - 1
\]
respectively. Thus the construction problem of third-degree formulas with respect to $\mathcal{L}$ is turned into two smaller problems, one of which is the construction with respect to $\mathcal{L}_1$ and the other of which is the construction with respect to $\mathcal{L}^1$. It is easy to compute
\[
\mathcal{L}^1(t_n) = \mathcal{L}(l_n) = n \cdot \mathcal{L}(x_1) + c_n \mathcal{L}(1),
\]
\[
\mathcal{L}^1(t_n^2) = n \mathcal{L}(x_1^2) + n(n-1) \mathcal{L}(x_1 x_2) + 2 n c_n \mathcal{L}(x_1) + c_n^2 \mathcal{L}(1)
\]
\[
\mathcal{L}^1(t_n^3) = n \mathcal{L}(x_1^3) + 6 \binom{n}{2} \mathcal{L}(x_1^2 x_2) + 6 \binom{n}{3} \mathcal{L}(x_1 x_2 x_3)
\]
\[+ 3 c_n (n \mathcal{L}(x_1^2) + n(n-1) \mathcal{L}(x_1 x_2)) + 3 n c_n^2 \mathcal{L}(x_1) + c_n^3 \mathcal{L}(1)
\]
\[= n \mathcal{L}(x_1^3) + 3 n(n-1) \mathcal{L}(x_1^2 x_2) + n(n-1)(n-2) \mathcal{L}(x_1 x_2 x_3)
\]
\[+ 3 c_n (n \mathcal{L}(x_1^2) + n(n-1) \mathcal{L}(x_1 x_2)) + 3 n c_n^2 \mathcal{L}(x_1) + c_n^3 \mathcal{L}(1).
\]

Next we will show this decomposition process can continue.

Let us consider the cubature formula with respect to $\mathcal{L}_1$. Obviously, $\mathcal{L}_1$ is also permutation symmetrical, which allows us to employ theorem \[\square\] continuously. Define
\[
l^{(1)}_i(t_1, \ldots, t_{n-1}) = t_i - t_{n-1}, \quad i = 1, 2, \ldots, n - 2,
\]
\[
l^{(1)}_{n-1}(t_1, \ldots, t_n) = t_1 + t_2 + \ldots + t_{n-1} + c_{n-1},
\]
where
\[
c_{n-1} = \frac{-\mathcal{L}_1(t_1^3 + (n-4)t_1^2 t_2 - (n-3)t_1 t_2 t_3)}{\mathcal{L}_1(t_1^2 - t_1 t_2)}.
\]
then \( l^{(1)}_i l^{(1)}_{n-1}, n = 1, 2, \ldots, n - 2 \) are the orthogonal polynomials of degree two with respect to \( L_1 \). Noticing \( t_i = l_i(x_1, x_2, \ldots, x_n) \), we have

\[
\begin{align*}
l^{(1)}_i(t_1, \ldots, t_{n-1}) &= t_i - t_{n-1} = x_i - x_{n-1}, \quad i = 1, 2, \ldots, n - 2, \\
l^{(1)}_{n-1}(t_1, \ldots, t_n) &= t_1 + t_2 + \ldots + t_{n-1} + c_{n-1} \\
&= x_1 + x_2 + \ldots + x_{n-1} - (n-1)x_n + c_{n-1}.
\end{align*}
\]

Again let \( L_2 \) and \( L^2 \) be two linear functionals defined on \( \Pi^{n-2} \) and \( \Pi^1 \), whose moments are determined by

\[
\begin{align*}
L_2(\tau_i \tau_j \tau_k) &= L_1(l^{(1)}_i l^{(1)}_j l^{(1)}_k), \quad L_2(\tau_i \tau_j) = L_1(l^{(1)}_i l^{(1)}_j), \quad L_2(\tau_i) = L_1(l^{(1)}_i), \quad L_2(1) = \xi^{(1)}_1, \\
L^2(\tau^3_{n-1}) &= L_1((l^{(1)}_{n-1})^3), \quad L^2(\tau^2_{n-1}) = L_1((l^{(1)}_{n-1})^2), \quad L^2(\tau_{n-1}) = L_1(l^{(1)}_{n-1}), \quad L^2(1) = \xi^{(1)}_2 \\
\xi^{(1)}_1 + \xi^{(1)}_2 &= \xi^{(0)}_i \quad \text{and} \quad 1 \leq i, j, k \leq n - 2
\end{align*}
\]

respectively. It is easy to compute

\[
\begin{align*}
L^2(\tau_{n-1}) &= L_1(l^{(1)}_{n-1}) = L((l^{(1)}_{n-1})^2) + c_{n-1}(L_1(1) - L(1)), \\
L^2(\tau_{n-2}) &= L((l^{(1)}_{n-1})^2) = L_1([(t_1 + t_2 + \ldots + t_{n-1}) + c_{n-1}]^2) \\
&= L((l^{(1)}_{n-1})^2) + c^2_{n-1}(L_1(1) - L(1)), \\
L^2(\tau_{n-3}) &= L((l^{(1)}_{n-1})^3) + c^3_{n-1}(L_1(1) - L(1)).
\end{align*}
\]

Assume that \( L_k \) is a linear functional defined on \( \Pi^{n-k} \) for every \( k(0 \leq k < n) \) and satisfies property (P). Then a cubature problem of degree 3 with respect to \( L_k \) can be divided into two smaller cubature problems—one with respect to \( L_{k+1} \) and the other with respect to \( L^{k+1} \). Moreover \( L_{k+1} \) also satisfies the property (P) and then this process can continue and will end when \( k = n - 1 \). Finally, an \( n \)-dimensional cubature problem can be transformed into \( n \) one-dimensional cubature problems.

**Theorem 2.** Let

\[
l^{(k)}_{n-k} = x_1 + x_2 + \ldots + x_{n-k} - (n-k)x_{n-k+1} + c_{n-k}
\]

and \( L^k \) be a linear functional defined on \( \Pi^1 \) according to the above process, then the corresponding moments are

\[
\begin{align*}
L^{k+1}(1) &= \xi^{(k)}_2, \\
L^{k+1}(\tau) &= L((l^{(k)}_{n-k})^2) + c_{n-k}(L_k(1) - L(1)) \\
&= c_{n-k}L_k(1), \\
L^{k+1}(\tau^2) &= L((l^{(k)}_{n-k})^2) + c^2_{n-k}(L_k(1) - L(1)) \\
&= (n-k)(n-k+1)L(x^2_1 - x_1 x_2) + c^2_{n-k}L_k(1), \\
L^{k+1}(\tau^3) &= L((l^{(k)}_{n-k})^3) + c^3_{n-k}(L_k(1) - L(1)) \\
&= (n-k)(n-k+1)(n-k+2)L(-x^3_1 + 3x^2_1 x_2 - 2x_1 x_2 x_3) + c^3_{n-k}L_k(1)
\end{align*}
\]

where

\[
c_{n-k} = -\frac{L(x^3_1 - 3x^2_1 x_2 + 2x_1 x_2 x_3)}{L(x^2_1 - x_1 x_2)}, \quad k = 1, 2, \ldots, n - 2.
\]
Proof. It remains to prove Eq. (8). It follows from theorem 1 that
\[ c_{n-k} = \frac{\mathcal{L}_k(t_1^3 + (n - 3 - k)t_1^2 t_2 - (n - 2 - k)t_1 t_2 t_3)}{\mathcal{L}_k(t_1^3 - t_1 t_2)}. \]
According to the definition of \( \mathcal{L}_k \), we have
\[
\mathcal{L}_k(t_1^3 + (n - 3 - k)t_1^2 t_2 - (n - 2 - k)t_1 t_2 t_3) \\
= \mathcal{L}_k((x_1 - x_{n-k})^3 - (n - 3 - k)(x_1 - x_{n-k})^2(x_2 - x_{n-k}) \\
- (n - 2 - k)(x_1 - x_{n-k})(x_2 - x_{n-k})(x_3 - x_{n-k})) \\
= \mathcal{L}_k(x_1^3 - 3x_1^2 x_2 + 2x_1 x_2 x_3),
\]
where the permutation symmetry is used. This completes the proof. \( \square \)

Remark 1. In fact \( \mathcal{L}_{n-1} \) is also a one-dimensional integration functional. The corresponding moment can be calculated by
\[
\mathcal{L}_{n-1}(1) = \xi_1^{(n-2)} \\
\mathcal{L}_{n-1}(\tau) = \mathcal{L}(x_1 - x_2) = 0, \\
\mathcal{L}_{n-1}(\tau^2) = \mathcal{L}((x_1 - x_2)^2) = 2\mathcal{L}(x_1^2 - x_1 x_2), \\
\mathcal{L}_{n-1}(\tau^3) = \mathcal{L}((x_1 - x_2)^3) = 0.
\]
For convenience, in what follows let \( \mathcal{L}^n = \mathcal{L}_{n-1} \).

Suppose that
\[
\mathcal{L}^k(g) \approx \sum_{i=1}^{n_k} w_{i,k} g(t_{i,k}), k = 1, 2, \ldots, n \]
is exact for any \( g \in \Pi_3^1 \). And let \( v^{i,k} = (v_{i,k}^{(1)}, v_{i,k}^{(2)}, \ldots, v_{i,k}^{(n)}) \) be the solution of
\[
\begin{cases}
x_1 + x_2 + \ldots + x_n + c_n = t_{i,1} \\
x_{n-1} - x_n = 0 \\
\vdots \\
x_2 - x_n = 0 \\
x_1 - x_n = 0 \\
\end{cases}
\text{for } k = 1, \tag{11}
\]
\[
\begin{cases}
x_1 + x_2 + \ldots + x_n + c_n = 0 \\
x_1 + x_2 + \ldots + x_{n-1} - (n - 1)x_n + c_{n-1} = 0 \\
\vdots \\
x_1 + x_2 - 2x_3 + c_2 = 0 \\
x_1 - x_2 = t_{i,n} \\
\end{cases}
\text{for } k = n, \tag{12}
\]
and
\[
\begin{align*}
x_1 + x_2 + \ldots + x_n + c_n &= 0 \\
x_1 + x_2 + \ldots + x_{n-1} - (n-1)x_n + c_{n-1} &= 0 \\
\cdots \cdots \cdots \\
x_1 + x_2 + \ldots + x_{n-k+1} - (n-k+1)x_{n-k+2} + c_{n-k+1} &= t_{i,k} \quad \text{for } k = 2, 3, \ldots, n-1, \\
x_{n-k} - x_{n-k+1} &= 0 \\
\cdots \cdots \cdots \\
x_1 - x_{n-k+1} &= 0
\end{align*}
\]

(13)

Hence the final cubature formula can be written as
\[
\mathcal{L}(f) \approx \sum_{k=1}^{n} \sum_{i=1}^{n_k} w_{i,k} f(v^{i,k})
\]

which is exact for any \( f \in \Pi_3^n \). It is clear that the solution of Eq. (11) is
\[
(\eta_1, \eta_2, \ldots, \eta_i), \quad \eta_i = \frac{t_{i,1} - c_n}{n}
\]

and the solution of Eq. (13) is
\[
\begin{align*}
x_n &= \frac{c_{n-1} - c_n - \delta_{2,k}t_{i,2}}{n} \\
x_{n-1} &= x_n + \frac{c_{n-2} - c_{n-1}}{n-1} = x_n \\
\cdots \cdots \\
x_{n-k+3} &= x_{n-k+4} \\
x_{n-k+2} &= x_{n-k+3} + \frac{c_{n-k+2} - c_{n-k+3} - t_{i,k}}{n-k+2} = x_{n-k+3} - \frac{t_{i,k}}{n-k+2} \\
x_1 = x_2 = \ldots = x_{n-k+1} &= x_{n-k+2} + \frac{t_{i,k} - c_{n-k+1}}{n-k+1}
\end{align*}
\]

where \( \delta_{2,k} = 1 \) if \( k = 2 \) and \( \delta_{2,k} = 0 \) if \( k \neq 2 \), and the solution of Eq. (12) is
\[
\begin{align*}
x_n &= x_{n-1} = \ldots = x_3 = \frac{c_{n-1} - c_n}{n} \\
x_2 &= x_3 - \frac{c_{n-1} + t_{i,n}}{2} \\
x_1 &= x_2 + t_{i,n} = x_3 + \frac{t_{i,n} - c_{n-1}}{2}
\end{align*}
\]

Collecting the above discussion, we have

**Theorem 3.** Assume that \( \mathcal{L} \) is permutation symmetrical. Then there must exist a cubature formula
\[
\mathcal{L}(f) \approx \sum_{k=2}^{n} \sum_{i=1}^{m_k} w_{i,k} f(\beta_{i,k}, \gamma_{i,k}, \ldots, \gamma_{i,k}) + \sum_{i=1}^{m_1} w_{i,1} f(\alpha_{i,1}, \ldots, \alpha_{i,1}) := C(f)
\]

which is exact for every polynomial of degree \( \leq 3 \). In the formula, \( \alpha \)'s and \( \beta \)'s can be computed
by

\[
\begin{aligned}
\gamma &= \frac{c_{n-1} - c_n}{n} \\
\beta_{i,k} &= \gamma - \frac{t_{i,k}}{n-k+2}, \quad 2 \leq k \leq n-1; \\
\alpha_{i,k} &= \beta_{i,k} + \frac{t_{i,k} - c_{n-k}}{n-k+1}, \quad 2 \leq k \leq n-1; \\
\alpha_{i,1} &= \frac{t_{i,1} - c_n}{n}, \\
\beta_{i,n} &= \gamma - \frac{t_{i,n} + c_2}{2}, \\
\alpha_{i,n} &= \beta_{i,n} + t_{i,n}
\end{aligned}
\]

where \( t_{i,k} \)'s and \( w_{i,k} \)'s are the nodes and weights of the quadrature formula \((10)\) with respect to \( L^k \).

The proof is a direct result of the computation and is omitted.

Remark 2. For the quadrature problem of the one-dimensional moment, it is well known that the number of the nodes \( n_k = 2 \) in the general case. Hence the total number of the nodes of the cubature formula with respect to \( L \) is \( 2n \) generally and \( 2n \) is usually the minimum among all the cubature formula of degree 3 except one case of the integration over the \( n \)-dimensional simplex \([11]\). For more knowledge of the problem of the one-dimensional moment, we can refer to the appendix of \([4]\).

Remark 3. For convenience, we present the relations of \( \xi \)'s as follows

\[
\begin{aligned}
L(1) &\rightarrow \xi_1^{(0)} \rightarrow \xi_1^{(1)} \rightarrow \cdots \rightarrow \xi_1^{(n-2)} \\
&+ \xi_2^{(0)} + \cdots + \xi_2^{(n-2)}
\end{aligned}
\]

According to the previous discussion, it is clear that

\[
L_1^k(1) = \xi_2^{(k-1)} \quad \text{for} \quad 1 \leq k \leq n-1 \quad \text{and} \quad L_1^n(1) = L_{n-1}(1) = \xi_1^{(n-2)}.
\]

3 Numerical Examples

- Firstly take the integration over the \( n \)-dimensional simplex as an example. Define

\[
L(f) = \int_{T_n} f(x_1, x_2, \ldots, x_n) dx_1 dx_2 \ldots dx_n
\]

where

\[
T_n = \{(x_1, x_2, \ldots, x_n) | x_1 + x_2 + \ldots + x_n \leq 1, \quad x_i \geq 0, \quad i = 1, 2, \ldots, n\}.
\]

It is well known that

\[
L(x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}) = \frac{\alpha_1! \alpha_2! \cdots \alpha_n!}{(n + \alpha_1 + \alpha_2 + \ldots + \alpha_n)!}.
\]

Then by a simple computation, we have

\[
c_i = -\frac{2}{n+3}, \quad i = 2, 3, \ldots, n-1, \quad c_n = -\frac{n+2}{n+3}, \quad \gamma = \frac{1}{n+3}.
\]
Here if we take
\[ \xi^{(i)}_2 = t_{i+1} \cdot \frac{1}{n \cdot n!}, \quad \xi^{(n-2)}_1 = t_n \cdot \frac{1}{n \cdot n!}, \quad i = 0, 1, \ldots, n-2, \]
and \( \sum_{i=1}^n t_i = n \), then the moments of \( \mathcal{L}^{k+1}(1 \leq k \leq n-2) \) are
\[
\begin{align*}
\mathcal{L}^{k+1}(1) &= t_{k+1} \cdot \frac{1}{n \cdot n!}, \\
\mathcal{L}^{k+1}(\tau) &= c_{n-k} \mathcal{L}_k(1) = -\frac{2(n - \sum_{i=1}^k t_i)}{(n+3)n \cdot n!}, \\
\mathcal{L}^{k+1}(\tau^2) &= \mathcal{L}(t^{(k)}_{n-k})^2 + c_{n-k}^2 (\mathcal{L}_k(1) - \mathcal{L}(1)) = \frac{(n-k)(n-k+1)}{(n+2)!} + \frac{4(n - \sum_{i=1}^k t_i)}{n(n+3)^2 \cdot n!}, \\
\mathcal{L}^{k+1}(\tau^3) &= (n-k)(n-k+1)(n-k+2)\mathcal{L}( -x_1^3 + 3x_1^2 x_2 - 2x_1 x_2 x_3 ) + c_{n-k}^3 \mathcal{L}_k(1) \\
&= \frac{-2(n-k)(n-k+1)(n-k+2)}{(n+3)!} - \frac{8(n - \sum_{i=1}^k t_i)}{n(n+3)^3 \cdot n!}
\end{align*}
\]
and
\[
\begin{align*}
\mathcal{L}^1(1) &= \frac{t_1}{n \cdot n!}, \\
\mathcal{L}^1(\tau) &= -\frac{2}{(n+1)!(n+3)}, \\
\mathcal{L}^1(\tau^2) &= \frac{n^2 + 5n + 8}{(n+3)!(n+3)}, \\
\mathcal{L}^1(\tau^3) &= -\frac{2(n+2)(n+4)}{(n+3)^2(n+3)!}
\end{align*}
\]
and
\[
\begin{align*}
\mathcal{L}^n(1) &= \frac{t_n}{n \cdot n!}, \\
\mathcal{L}^n(\tau) &= 0, \\
\mathcal{L}^n(\tau^2) &= \frac{2}{(n+2)!}, \\
\mathcal{L}^n(\tau^3) &= 0.
\end{align*}
\]

By taking different values for \( \xi_s \), we can get different cubature formulae. For \( n = 3 \) and \( n = 4 \), if we take all \( t_i = 1 \), then we can get formulas as showed in Tables 1 and 2.

| \( x_1 \) | \( x_2 \) | \( x_3 \) | weight |
|-----------|-----------|-----------|--------|
| 0.34240723692377 | 0.34240723692377 | 0.34240723692377 | 0.01469064053612 |
| 0.14125289379518 | 0.14125289379518 | 0.14125289379518 | 0.04086491501944 |
| 0.41353088165296 | 0.41353088165296 | 0.00627157002742 | 0.0188711233337 |
| 0.12380973765487 | 0.12380973765487 | 0.58571385802358 | 0.03668444322218 |
| 0.05947205458075 | 0.05947205458075 | 0.16666666666667 | 0.02777777777778 |
| 0.5947205458075 | 0.5947205458075 | 0.16666666666667 | 0.02777777777778 |
| 0.60719461208592 | 0.05947205458075 | 0.16666666666667 | 0.02777777777778 |
| 0.05947205458075 | 0.60719461208592 | 0.16666666666667 | 0.02777777777778 |

Table 1: Nodes and weights for \( T_3 \)

In Table 1 the first point is outside of the region. To avoid it, we can take
\[ t_1 = \frac{93}{85}, \quad t_2 = \frac{378}{391}, \quad t_3 = \frac{108}{115}, \]
\[
\begin{array}{cccc}
\text{x}_1 & \text{x}_2 & \text{x}_3 & \text{x}_4 \\
0.27145760185760 & 0.27145760185760 & 0.27145760185760 & 0.27145760185760 \\
0.12024746726682 & 0.12024746726682 & 0.12024746726682 & 0.12024746726682 \\
0.30652570925957 & 0.30652570925957 & 0.30652570925957 & -0.06243427063585 \\
0.11154151763119 & 0.11154151763119 & 0.11154151763119 & 0.52251830424930 \\
0.37131176827505 & 0.37131176827505 & -0.02833782226438 & 0.14285714285714 \\
0.09266869570542 & 0.09266869570542 & 0.09266869570542 & 0.52251830424930 \\
0.54391317546145 & 0.54391317546145 & 0.54391317546145 & 0.14285714285714 \\
0.02751539596712 & 0.02751539596712 & 0.02751539596712 & 0.14285714285714 \\
\end{array}
\]

\[w_1 = 0.00254167472911, \quad w_2 = 0.00787499193755, \quad w_3 = 0.00294495824332, \quad w_4 = 0.00747170842335, \quad w_5 = 0.00365639117145, \quad w_6 = 0.00676027549522, \quad w_7 = 0.00520833333333, \quad w_8 = 0.00520833333333.\]

Table 2: Nodes and weights for \(T_4\)

and the corresponding cubature formula is listed in Table 3. In the formula, the first and third nodes are on the boundary of the region \(T_3\).

If we take
\[t_1 = \frac{94}{85}, \quad t_2 = 1, \quad t_3 = \frac{76}{85},\]
then all the nodes are inside the region, see Table 4.

| \text{x}_1 | \text{x}_2 | \text{x}_3 | \text{x}_4 | \text{weight} |
| --- | --- | --- | --- | --- |
| 0.33333333333333 | 0.33333333333333 | 0.33333333333333 | 0.0187500000000 | 
| 0.14285714285714 | 0.14285714285714 | 0.14285714285714 | 0.04203431372549 | 
| 0.41666666666667 | 0.41666666666667 | 0.00000000000000 | 0.0187500000000 | 
| 0.12037037037037 | 0.12037037037037 | 0.59259259259259 | 0.03495843989970 | 
| 0.61593041596355 | 0.05073625070311 | 0.16666666666667 | 0.02608695652174 | 
| 0.05073625070311 | 0.61593041596355 | 0.16666666666667 | 0.02608695652174 | 

Table 3: Nodes and weights for \(T_3\)

In Table 2 all the weights are positive. However, we find there exist three points outside of the region. If we take
\[t_1 = \frac{104}{75}, \quad t_2 = \frac{3577}{2775}, \quad t_3 = \frac{9947}{8880}, \quad t_4 = \frac{49}{60},\]
and add one more point with weight \(-\frac{618391}{961698}\), then the corresponding formula is shown in Table 5.

In Table 5 there are 5 points on the boundary. To make all the nodes inside the region, we can take
\[t_1 = \frac{7}{5}, \quad t_2 = \frac{187}{145}, \quad t_3 = \frac{179522}{160283}, \quad t_4 = \frac{5}{6},\]
and add one more node with weight \(-\frac{618391}{961698}\) and the corresponding formula is shown in Table 6.
\[ w_1 = 0.00555555555556, w_2 = 0.00888888888889, w_3 = 0.00600490196078, \]
\[ w_4 = 0.00742227521640, w_5 = 0.00633074935401, w_6 = 0.00533755957993, \]
\[ w_7 = 0.00425347222222, w_8 = 0.00425347222222, w_9 = -49/80. \]

Table 5: Nodes and weights for \( T_4 \)

| \( x_1 \) | \( x_2 \) | \( x_3 \) | \( x_4 \) |
|-----------------|-----------------|-----------------|-----------------|
| 0.24955035825246 | 0.24955035825246 | 0.24955035825246 | 0.24955035825246 |
| 0.12511271452382 | 0.12511271452382 | 0.12511271452382 | 0.12511271452382 |
| 0.28567845223177 | 0.28567845223177 | 0.28567845223177 | 0.00000000000000 |
| 0.11210697374487 | 0.11210697374487 | 0.11210697374487 | 0.52082193590823 |
| 0.35714285714286 | 0.35714285714286 | 0.00000000000000 | 0.14285714285714 |
| 0.28571428571429 | 0.28571428571429 | 0.14285714285714 | 0.14285714285714 |

\[ w_1 = 0.00566710734383, w_2 = 0.0089162259895080, w_3 = 0.006035977209200, \]
\[ w_4 = 0.00739793083678, w_5 = 0.0063627387083462, w_6 = 0.005780572945942, \]
\[ w_7 = 0.00434027777778, w_8 = 0.00434027777778, w_9 = -0.643019950129875. \]

Table 6: Nodes and weights for \( T_4 \)

The integration problem on the \( n \)-simplex was studied very extensively. According to the collection of R. Cools in the website (http://nines.cs.kuleuven.be/research/ecf/ecf.html), the minimum number of nodes in the third-degree formulas is \( n + 2 \), in which there is a negative weight. Except the \((n + 2)\)-point formula, the minimum number is 8 and 10 for \( n = 3 \) and \( n = 4 \), respectively. If we only consider the formulas with positive weight, the minimum of the points is 8 and 11 for \( n = 3 \) and \( n = 4 \) respectively. Therefore our formula for \( n = 3 \) and \( n = 4 \) have the fewest numbers among all the formulas with positive weights.

- Secondly take the integration over the positive sector of a ball as an example. Define

\[ \mathcal{L}(f) = \int_{S_n} f(x_1, x_2, \ldots, x_n) \, dx_1 \, dx_2 \ldots \, dx_n \]

where \( S_n = \{(x_1, x_2, \ldots, x_n) | x_1^2 + x_2^2 + \ldots + x_n^2 \leq 1, x_1 \geq 0, \ldots, x_n \geq 0 \} \). It is easy to get that

\[ \mathcal{L}(x_1^{\alpha_1}x_2^{\alpha_2}\ldots x_n^{\alpha_n}) = \frac{(\alpha_1 - 1)!!(\alpha_2 - 1)!!\ldots(\alpha_n - 1)!!}{(n + \alpha_1 + \alpha_2 + \ldots + \alpha_n)!!} \cdot \left( \frac{\pi}{2} \right)^{\frac{n-1}{2}} \]

where \( n_o \) denotes the number of the odd number among \( \alpha_1, \alpha_2, \ldots, \alpha_n \) and \( m!! \) denotes the
double factorial of $m$ and $m!! = 1$ if $m \leq 0$. Then by a simple computation, we have
\[
c_i = \frac{(n+2)!!}{(n+3)!!} \cdot \left(\frac{\pi}{2}\right)^{\left[\frac{n-1}{2}\right]} \cdot \frac{4 - \pi}{\pi - 2}, \quad i = 2, 3, \ldots, n - 1,
\]
\[
c_n = \frac{(n+2)!!}{(n+3)!!} \cdot \left(\frac{\pi}{2}\right)^{\left[\frac{n-1}{2}\right]} \cdot \frac{(n-1)\pi - (2n - 4)}{\pi - 2},
\]
\[
\gamma = \frac{(n+2)!!}{(n+3)!!} \cdot \left(\frac{\pi}{2}\right)^{\left[\frac{n-1}{2}\right]}.
\]

Here if we take
\[
\xi^{(i)}_2 = \frac{t_{i+1}}{n \cdot n!!} \cdot \left(\frac{\pi}{2}\right)^{\left[\frac{i}{2}\right]} \cdot \xi^{(n-2)}_2 = \frac{t_n}{n \cdot n!!} \cdot \left(\frac{\pi}{2}\right)^{\left[\frac{n-2}{2}\right]}, \quad i = 0, 1, 2, \ldots, n - 2,
\]
then the moments of $\mathcal{L}^{k+1}(1 \leq k \leq n - 2)$ are
\[
\mathcal{L}^{k+1}(1) = \frac{t_{k+1}}{n \cdot n!!} \cdot \left(\frac{\pi}{2}\right)^{\left[\frac{k}{2}\right]},
\]
\[
\mathcal{L}^{k+1}(\tau) = c_{n-k} \mathcal{L}_k(1) = -\frac{(n+2) \cdot (n - \sum_{i=1}^{k} t_i)}{n \cdot (n+3)!!} \cdot \left(\frac{\pi}{2}\right)^{\left[\frac{n-k}{2}\right]} \cdot \frac{4 - \pi}{\pi - 2},
\]
\[
\mathcal{L}^{k+1}(\tau^2) = \frac{(n-k)(n-k+1)}{(n+3)!!} \cdot \left(\frac{\pi}{2}\right)^{\left[\frac{n-k}{2}\right]} + \frac{(n-k)(n-k+1)(n-k+2)}{(n+3)!!} \cdot \left(\frac{\pi}{2}\right)^{\left[\frac{n-k}{2}\right]} \cdot \frac{2^{\left[\frac{n-k}{2}\right]} - 2\left[\frac{n-k}{2}\right]}{\pi - 2},
\]
\[
\mathcal{L}^{k+1}(\tau^3) = \frac{(n-k)(n-k+1)(n-k+2)}{(n+3)!!} \cdot \left(\frac{\pi}{2}\right)^{\left[\frac{n-k}{2}\right]} \cdot \frac{4 - \pi}{\pi - 2} \cdot \frac{\pi}{2} - 2 \cdot \left(\frac{\pi}{2}\right)^{\left[\frac{n-k}{2}\right]} \cdot \frac{4 - \pi}{\pi - 2} \cdot \frac{\pi}{2} - 2 \cdot \left(\frac{\pi}{2}\right)^{\left[\frac{n-k}{2}\right]} \cdot \frac{4 - \pi}{\pi - 2} \cdot \frac{\pi}{2} - 2 \cdot \left(\frac{\pi}{2}\right)^{\left[\frac{n-k}{2}\right]} \cdot \frac{4 - \pi}{\pi - 2}
\]
\]
and
\[
\mathcal{L}^1(1) = \frac{t_1}{n \cdot n!!} \cdot \left(\frac{\pi}{2}\right)^{\left[\frac{1}{2}\right]},
\]
\[
\mathcal{L}^1(\tau) = \frac{2(\pi - 3n + \pi - 4)}{(n+3)!! \cdot (\pi - 2)} \cdot \left(\frac{\pi}{2}\right)^{\left[\frac{1}{2}\right]},
\]
\[
\mathcal{L}^1(\tau^2) = \frac{n(\frac{3}{2} + n - 1)}{(n+2)!!} \cdot \left(\frac{\pi}{2}\right)^{\left[\frac{1}{2}\right]} - \frac{2n \cdot (n+2)!!}{(n+1)!! \cdot (n+3)!!} \cdot \left(\frac{\pi}{2}\right)^{\left[\frac{1}{2}\right]} - \frac{4 - \pi}{\pi - 2} \cdot \left(\frac{\pi}{2}\right)^{\left[\frac{1}{2}\right]} \cdot \frac{(n-1)\pi - (2n - 4)}{\pi - 2},
\]
\[
\mathcal{L}^1(\tau^3) = \frac{3n^2 - n}{(n+3)!!} \cdot \left(\frac{\pi}{2}\right)^{\left[\frac{1}{2}\right]} + \frac{n(1-n)(n-2)!}{(n+3)!!} \cdot \left(\frac{\pi}{2}\right)^{\left[\frac{1}{2}\right]} + \frac{2^{\left[\frac{1}{2}\right]} - 2\left[\frac{1}{2}\right]}{\pi - 2} \cdot \left(\frac{\pi}{2}\right)^{\left[\frac{1}{2}\right]} \cdot \frac{(n-1)\pi - (2n - 4)}{\pi - 2},
\]
\[
- \frac{3n(\frac{3}{2} + n - 1)}{(n+3)!!} \cdot \left(\frac{\pi}{2}\right)^{\left[\frac{1}{2}\right]} \cdot \frac{(n-1)\pi - (2n - 4)}{\pi - 2} + \frac{3n}{(n+1)!!} \cdot \left(\frac{\pi}{2}\right)^{\left[\frac{1}{2}\right]} + \frac{2^{\left[\frac{1}{2}\right]} - 2\left[\frac{1}{2}\right]}{\pi - 2} \cdot \left(\frac{\pi}{2}\right)^{\left[\frac{1}{2}\right]} + \frac{2^{\left[\frac{1}{2}\right]} - 2\left[\frac{1}{2}\right]}{\pi - 2} \cdot \left(\frac{\pi}{2}\right)^{\left[\frac{1}{2}\right]} + \frac{2^{\left[\frac{1}{2}\right]} - 2\left[\frac{1}{2}\right]}{\pi - 2} \cdot \left(\frac{\pi}{2}\right)^{\left[\frac{1}{2}\right]} + \frac{2^{\left[\frac{1}{2}\right]} - 2\left[\frac{1}{2}\right]}{\pi - 2} \cdot \left(\frac{\pi}{2}\right)^{\left[\frac{1}{2}\right]} + \frac{2^{\left[\frac{1}{2}\right]} - 2\left[\frac{1}{2}\right]}{\pi - 2} \cdot \left(\frac{\pi}{2}\right)^{\left[\frac{1}{2}\right]} + \frac{2^{\left[\frac{1}{2}\right]} - 2\left[\frac{1}{2}\right]}{\pi - 2} \cdot \left(\frac{\pi}{2}\right)^{\left[\frac{1}{2}\right]},
\]
\]
12
and
\[ L_n(1) = \frac{t_n}{n \cdot n!} \left( \frac{\pi}{2} \right)^{\left\lceil \frac{n}{2} \right\rceil}, \]
\[ L_n(\tau) = 0, \]
\[ L_n(\tau^2) = \frac{2}{(n + 2)!} \left( \frac{\pi}{2} \right)^{\left\lceil \frac{n-2}{2} \right\rceil} \left( \frac{\pi}{2} - 1 \right), \]
\[ L_n(\tau^3) = 0. \]

If take all \( t_i = 1 \) for \( n = 3 \) and \( n = 4 \), then we can get formulae as showed in Tables 7 and 8. If we take \( t_1 = 0.8, t_2 = 1.31, t_3 = 1.11 \) and \( t_4 = 0.78 \) for \( n = 4 \), then we can get a formula with all the nodes inside the region, see Table 9.

| \( x_1 \) | \( x_2 \) | \( x_3 \) | weight |
|----------|----------|----------|--------|
| 0.53887049476004 | 0.53887049476004 | 0.53887049476004 | 0.07852747507104 |
| 0.18341741723402 | 0.18341741723402 | 0.18341741723402 | 0.09600545012840 |
| 0.57520979290336 | 0.57520979290336 | 0.220616228206 | 0.06975676243570 |
| 0.20283315000517 | 0.7681444807844 | 0.1047761276373 | |
| 0.76016315955181 | 0.09981758853698 | 0.31250000000000 | 0.08726646259972 |
| 0.09981758853698 | 0.76016315955181 | 0.31250000000000 | 0.08726646259972 |

Table 7: Nodes and weights for \( S_3 \)

| \( x_1 \) | \( x_2 \) | \( x_3 \) | \( x_4 \) |
|----------|----------|----------|----------|
| 0.47721483105875 | 0.47721483105875 | 0.47721483105875 | 0.47721483105875 |
| 0.17126237887529 | 0.17126237887529 | 0.17126237887529 | 0.17126237887529 |
| 0.4842041705925 | 0.4842041705925 | 0.4842041705925 | -0.06966276495181 |
| 0.20526869095372 | 0.20526869095372 | 0.20526869095372 | 0.76713241336478 |
| 0.560049954954835 | 0.560049954954835 | -0.02818760532449 | 0.29102618165375 |
| 0.16531879543241 | 0.16531879543241 | 0.76127471370739 | 0.29102618165375 |
| 0.7487573599445 | 0.0524103869240 | 0.29102618165375 | 0.29102618165375 |
| 0.0524103869240 | 0.7487573599445 | 0.29102618165375 | 0.29102618165375 |

| \( w_1 = 0.03771636146294, w_2 = 0.03938992292057, w_3 = 0.02874304082, w_4 = 0.04835888504269, w_5 = 0.03167997303102, w_6 = 0.04542631135249, w_7 = 0.03855314219176, 0.03855314219176 |

Table 8: Nodes and weights for \( S_4 \)

4 Conclusion

In this paper, we present a method to construct third-degree formulae for integrals with permutation symmetry. Our method is a decomposition method, which is easy to compute. At the end of the paper, we present some numerical results, which seem to be new. Compared with the existing method, we focus on the case of permutation symmetry, which seldom is considered. The numerical results show that the number of the points attain of close to the minimum. Besides, in most cases, the weights in our formulas are all positive or at most one negative weight.

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Table 9: Nodes and weights for $S_4$

| $x_1$ | $x_2$ | $x_3$ | $x_4$ |
|-------|-------|-------|-------|
| 0.49819378497585 | 0.49819378497585 | 0.49819378497585 | 0.49819378497585 |
| 0.15640817934597 | 0.15640817934597 | 0.15640817934597 | 0.15640817934597 |
| 0.46048445804733 | 0.46048445804733 | 0.46048445804733 | 0.00148511203764 |
| 0.21848509706654 | 0.21848509706654 | 0.21848509706654 | 0.72748319509617 |
| 0.54494615070832 | 0.54494615070832 | 0.00202000298217 | 0.29102618165375 |
| 0.16998718727254 | 0.16998718727254 | 0.75193793030368 | 0.29102618165375 |
| 0.79451246595922 | 0.0063765695922 | 0.29102618165375 | 0.29102618165375 |
| 0.0063765695922 | 0.79451246595922 | 0.29102618165375 | 0.29102618165375 |

$w_1 = 0.02857087469598$, $w_2 = 0.0311415281083$, $w_3 = 0.04213154579078$, $w_4 = 0.05887768675432$, $w_5 = 0.03842850515254$, $w_6 = 0.04715947052132$, $w_7 = 0.03855314219176$, $w_8 = 0.03855314219176$. 

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