Nonlinear Schrödinger equations with exceptional potentials

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Abstract
We consider the cubic nonlinear Schrödinger equation with an exceptional potential. We obtain a sharp time decay for the global in time solution and we get the large time asymptotic profile of small solutions. We prove the existence of modified scattering for this model, that is, linear scattering modulated by a phase. Our approach is based on the spectral theorem for the perturbed linear Schrödinger operator and a factorization technique, that allows us to control the resonant nonlinear term. We make some parity assumptions in order to control the small-energy behavior of the scattering coefficients and of the wave functions.

1 Introduction
In this article, we consider the cubic nonlinear Schrödinger equation
\[
\begin{cases}
  i\partial_t u = -\frac{1}{2} \partial_x^2 u + Vu + \lambda |u|^2 u, & x \in \mathbb{R}, \ t > 0, \\
  u(0, x) = u_0(x), & x \in \mathbb{R},
\end{cases}
\]  (1.1)
where the potential \( V(x) \) is a real-valued given function and the parameter \( \lambda \in \mathbb{R} \). This equation is related to the Ginzburg-Landau equation of superconductivity ([19]), it was used to describe one-dimensional self-modulation of a monochromatic wave ([45], [7]), stationary two-dimensional self-focusing of a plane wave ([11]), propagation of a heat pulse in a solid Langmuir waves in plasmas ([40]) and the self-trapping phenomena of nonlinear optics ([32]). For other applications of (1.1), we refer to [10], [39], [22], [18], [9], and [44].

In the case when the external potential \( V = 0 \), it is well known that the Cauchy problem
\[
\begin{cases}
  i\partial_t u = -\frac{1}{2} \partial_x^2 u + \lambda |u|^2 u, & x \in \mathbb{R}, \ t > 0, \\
  u(0, x) = u_0(x), & x \in \mathbb{R},
\end{cases}
\]  (1.2)
is globally well posed in a variety of spaces, for instance in \( \mathbf{H}^1 \) or \( \mathbf{L}^2 \). See e.g. [33]. Concerning the asymptotic behavior of the solutions the cubic nonlinearity in dimension 1 is a limiting case. Indeed, if the nonlinearity \( \lambda |u|^2 u \) in (1.2) is replaced by \( \lambda |u|^\alpha u \), with \( \alpha > 2 \), there is low energy scattering, i.e. a solution of (1.2) with a sufficiently small initial value (in some appropriate sense) behaves asymptotically free as \( |t| \to \infty \), i.e. similar to the solutions of the linear Schrödinger equation \( i\partial_t u = -\Delta u \) ([43, 24, 25, 14, 26, 36, 16]). On the other hand, as it shown in [42, Theorem 3.2 and Example 3.3, p. 68] and [8], if the power \( \alpha \leq 2 \), then low energy scattering for (1.2) cannot be expected. In the case of the cubic power \( \alpha = 2 \), the relevant notion for equation (1.2) is modified scattering, i.e. standard scattering modulated by a phase. The existence of modified wave operators was established in [38]. That is, for all sufficiently small asymptotic state \( u^+ \), there exists a solution of (1.2), which behaves like \( e^{i\phi(t, \cdot)} e^{i\Delta} u^+ \), as \( t \to \infty \), where the phase \( \phi \) is given explicitly in terms of \( u^+ \). (See also [13, 30, 41].)

Conversely, for small initial values, it was shown in [28] that there is such modification in the asymptotic behavior of the corresponding solution to (1.2). (See also [27, 29, 34, 17].)

The well-posedness of the perturbed NLS equation (1.1) is also known for a wide class of potentials \( V \) ([15]). Compared to the case of the free NLS equation, not so much is known however about the large time behaviour of the solutions to (1.1). The existence of standing wave solutions to (1.1) was investigated in [23]. Concerning the existence of low energy scattering, the first paper in this direction to our knowledge is [46]. It was established that for the NLS equation
\[
i\partial_t u = Hu + f(u)u,
\]  (1.3)
where the power of the nonlinearity $f(u)$ is $5 \leq p < \infty$, the nonlinear scattering operator $S_V$, associated to (1.3), is a homeomorphism from some neighborhood of $0$ in $L^2$ onto itself. For the power nonlinearities with $3 < p < 5$, the existence of low energy scattering was established in [6]. In the case of the NLS equation under a partial quadratic confinement, the existence of low energy scattering was proved in [5]. The long-time behavior of solutions to a focusing cubic one-dimensional NLS equation with a Dirac potential was considered in [21].

In our previous paper [37] we considered the cubic nonlinear Schrödinger equation (1.1) in the case of generic potentials $V$. That is, potentials $V$ such that the corresponding Jost solutions $f_{\pm}(x,k)$ at zero energy ($k = 0$) satisfy $[f_{+}(x,0), f_{-}(x,0)] \neq 0$, where $[f, g]$ denote the Wronskian of $f$ and $g$, i.e. $[f, g] = g\partial_x f - f\partial_x g$. See Section 2 for the definition of the Jost functions. In [37], we proved the existence of unique solutions $u \in C([0, \infty); H^1)$ of (1.1) which decay for long times as solutions to the linear free Schrödinger equation $i\partial_t u + \frac{1}{2}\partial_x^2 u = 0$. Namely, $u$ satisfies $\|u(t)\|_{L^\infty} \leq C(1 + t)^{-\frac{4}{5}}$. Moreover, we obtained the large time asymptotic representation for these solutions, which have a modified character. Thus, we proved that as in the case of the free NLS equation (1.2), the cubic nonlinearity in the problem (1.1) in the case of generic potentials $V$ is critical from the point of view of large time asymptotic behavior.

In the present paper we continue this study and consider the case of exceptional potentials. We say that a potential $V$ is exceptional if the Jost solutions at zero energy satisfy $[f_{+}(x,0), f_{-}(x,0)] = 0$. Equivalent definitions of the exceptional potentials, as well as a comprehensive study of their properties can be found in papers [2], [3], [4], and the references there in. Note that the trivial potential $V(x) = 0$ is exceptional. If a potential $V$ is non-trivial and positive, it is generic. As an example of an exceptional case consider the square-well potential in the depth when a bound state is added to the potential. Observe that at any other depth the square-well potential is generic. This example exhibits the unstable nature of the exceptional potentials: a small perturbation of the exceptional case usually makes the case generic ([2]). For the linear Schrödinger equation $i\partial_t u - \frac{1}{2}\partial_x^2 u + V u$, the transmission coefficient $T(k)$ defines the probability $|T(k)|^2$ that a particle of energy $k^2$ can tunnel through the potential $V$. (For the definition of the transmission and reflection coefficients $T(k), R_\pm(k)$ see Section 2.) In the case of generic potentials, the zero-energy transmission coefficient $T(0)$ is zero, meaning that a particle with no energy cannot tunnel through a non-trivial potential. This is not longer true for exceptional potentials, since in this case $T(0) \neq 0$. Also, we observe that in the exceptional case, the energy zero can be a resonant energy - an energy where the potential $V$ is "perfectly transparent", that is $R_\pm(0) = 0$. We note that the resonant energies are important for tunneling spectroscopy ([47]). These special properties of the exceptional potentials make them particularly interesting.

We now state our main result. First, we define the class of exceptional potentials we will consider in this paper. For some $N \geq 1$, let us partition the real axis $\mathbb{R}$ as $-\infty < x_1 < x_2 < \ldots < x_N < +\infty$. We denote $I_j = (x_{j-1}, x_j)$ for $j = 1, \ldots, N + 1$, with $x_0 = -\infty, x_{N+1} = +\infty$. We obtain a fragmentation of the potential $V \in L^{1,1}$ by setting

$$V(x) = \sum_{j=1}^{N+1} V_j(x), \quad (1.4)$$

where

$$V_j(x) = \begin{cases} V(x), & x \in I_j, \\ 0, & \text{elsewhere.} \end{cases}$$

We assume the following:

**Condition 1.1** The potential $V \in L^{1,3}$ is exceptional, symmetric and such that the linear Schrödinger operator $H = -\frac{1}{2}\partial_x^2 + V$ does not have negative eigenvalues. In addition, for some $N \geq 1$, there is a partition (1.4) such that each part $\langle \cdot \rangle^2 V_j \in W^{1,1}(I_j)$.

**Remark 1.2** Of course the trivial potential $V = 0$ satisfies Condition 1.1. An example of a non-trivial potential satisfying the above assumptions is

$$V(x) = V_1(x) + V_2(x), \quad (1.5)$$

where

$$V_1(x) = \begin{cases} A^2, & x \in (0,1), \\ -B^2, & x \in (1,2), \\ 0, & \text{elsewhere,} \end{cases}$$

$$V_2(x) = V_1(-x),$$

with $B = \frac{\pi}{2}$ and $A$ satisfying $A \tan h A = \frac{\pi}{2}$. Indeed, it follows from Section IV of [4] that $V_1$ is an exceptional potential without bound states. Then, the same is true for $V_2$. Thus, from Theorem 2.3 of [2] we see that $V$ is exceptional. Moreover, by (i) on page 2454 of [4] we conclude that $V$ does not have any eigenvalue.
Define
\[ M = e^{\frac{t}{2\pi}}, \]
and
\[ \mathcal{D}_t \phi = (it)^{-\frac{3}{4}} \phi (xt^{-1}). \]

**Theorem 1.3** Suppose that \( V \) satisfies Condition 1.1. Let the initial data \( u_0 \in H^1 \cap H^{0,1} \). In addition, we suppose that \( u_0 (x) \) are odd if \( T (0) = 1 \) and \( u_0 (x) \) are even in the case when \( T (0) = -1 \). Then, there exists \( \varepsilon_0 > 0 \) such that for any \( 0 < \varepsilon < \varepsilon_0 \) and \( \| u_0 \|_{H^1} + \| u_0 \|_{H^{0,1}} \leq \varepsilon \), there exists a unique solution \( u \in C ([0, \infty); H^1) \) of the Cauchy problem (1.1) satisfying the estimate
\[ \| u (t) \|_{L^\infty} \leq C (1 + t)^{-\frac{3}{4}}, \] (1.6)
for any \( t \geq 0 \). Moreover there exists a unique modified final state \( w_+ \in L^\infty \) such that the following asymptotics is valid
\[ u (t) = MD_t w_+ e^{-i|w_+|^2 \log t} + O \left( \varepsilon t^{-\beta / 2} \right), \] (1.7)
for \( t \to \infty \), uniformly with respect to \( x \in \mathbb{R} \), where \( \beta > 0 \).

**Remark 1.4** Here are some comments on Theorem 1.3.

(i) If the potential is symmetric, \( \lim_{x \to -\infty} f_+ (x, 0) = \pm 1 \) and then, \( T (0) = \pm 1 \) (see (3.2) and (4.22)).

(ii) The difference between the generic and exceptional cases becomes relevant when the small-energy behavior of the scattering coefficients and of the wave functions is considered ([2]). In the case of Theorem 1.3 we need to impose the parity assumptions of Theorem 1.3 since in this case the scattering coefficients, as well as the function \( w \) have some special symmetries. We also need to control the function \( w \) at zero energy. In the case of generic potentials, \( w (0) = 0 \) and \( (\mathcal{V} (t) w) (0) \) gain some extra decay, as \( t \to \infty \). This is not true for arbitrary exceptional potentials: in general \( w (0) \) may not tend to zero, as \( t \to \infty \). Fortunately, under the assumptions of Theorem 1.3 \( w (t, 0) = 0 \) (see (3.40)). Also, we prove that \( (\mathcal{V} (t) w) (0) \) has an extra decay, when \( t \to \infty \) (see (3.16)). As is pointed in Remark 3.4, if \( \lim_{t \to \infty} w (t, 0) \neq 0 \), the asymptotics of the solution \( u \) may differ from (1.7) because of an extra term of the form \( h (t, x) \), with \( h \in L^\infty \). Once, we control \( \| u \|_{X_T} \), via a local existence result, we extend the solution for all times \( t > 0 \).

The rest of the paper is organized as follows. In Section 2 we recall some known properties for the linear Schrödinger equations. The proof of Theorem 1.3 is given in Section 3. This proof depends on several intermediate results, which are stated in Section 3, but whose proofs are deferred until Sections 5 to 7. Finally, in Section 4 we present some properties of the Jost solutions in the case of exceptional potentials that are used in the proof.

Throughout this paper, we use the following notation. The usual Fourier transforms \( \mathcal{F}_0 \) and \( \mathcal{F}_0^{-1} \) are defined by
\[ \mathcal{F}_0 \phi = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ixk} \phi (x) \, dx \]
and
\[ \mathcal{F}_0^{-1} \phi = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ixk} \phi (k) \, dk. \]
respectively. We denote by $L^p = L^p (\Omega)$, $1 \leq p \leq \infty$, the Lebesgue space on a domain $\Omega \subset \mathbb{R}$ and the weighted Lebesgue spaces are given by

$$L^{p,s} = \{ \phi : \| \phi \|_{L^{p,s}} = \| \langle \cdot \rangle^s \phi (\cdot) \|_{L^p} < \infty \}, \quad s \in \mathbb{R},$$

where

$$\langle x \rangle = (1 + x^2)^{1/2}.$$

For any $k \in \mathbb{N} \cup \{0\}$ and $p \geq 1$, we denote by $W^{k,p} = \left\{ \phi : \left( \sum_{j=0}^{k} \| \phi^{(j)} \|_{L^p}^p \right)^{1/p} < \infty \right\}$, the Sobolev space of order $k$ based on $L^p$ (see e.g. [1] for the definitions and properties of these spaces.) For any $r \in \mathbb{R}$, we denote by $H^r = H^r (\mathbb{R})$ the Sobolev space based on $L^2$ consisting of the completion of the Schwartz class in the norm $\| \phi \|_{H^r} = \| F_0 \phi \|_{L^{2,r}}$. Moreover, for any $r, s \in \mathbb{R}$, we define the weighted Sobolev spaces by

$$H^{r,s} = \{ \phi : \| \phi \|_{H^{r,s}} = \| \langle \cdot \rangle^s \phi (\cdot) \|_{H^r} < \infty \}.$$

Finally, the same letter $C$ may denote different positive constants which particular value is irrelevant.

## 2 Basic notions.

### 2.1 Free Schrödinger equation.

We consider first the free linear Schrödinger equation

$$\begin{align*}
\begin{cases}
i \partial_t u = H_0 u, \\
u(0, x) = u_0(x)
\end{cases} \quad (2.1)
\end{align*}$$

where $H_0 = -\frac{1}{2} \frac{d^2}{dx^2}$ is the free Schrödinger operator. $H_0$ is a self-adjoint operator on $L^2$ with domain $D (H_0) = H^2$. The solution to (2.1) is given by $u(t, x) = U_0(t) u_0$, where

$$U_0(t) = e^{-i t H_0} = F_0^{-1} e^{-\frac{i}{2} k^2} F_0$$

is the free Schrödinger evolution group. The asymptotic representation formula for $U_0(t)$ is the relation (see [31])

$$U_0(t) F_0^{-1} = M D_t V_0(t),$$

where $M = e^{\frac{i t^2}{2}}$, $D_t \phi = (it)^{-\frac{1}{2}} \phi (xt^{-1})$ and

$$V_0(t) \phi = F_0 M F_0^{-1} \phi = \sqrt{\frac{it}{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{i}{4} (\xi - \xi')^2} \phi (\xi) d\xi.$$

Observe that

$$\| V_0(t) \phi \|_{L^2} = \| F_0 M F_0^{-1} \phi \|_{L^2} = \| \phi \|_{L^2}, \quad (2.4)$$

$$\| (V_0(t) - 1) \phi \|_{L^2} = \| F_0 (M - 1) F_0^{-1} \phi \|_{L^2} \leq C \left\| \left( \frac{x^2}{2t} \right)^{\frac{3}{4}} F_0^{-1} \phi \right\|_{L^2} \leq C t^{-\frac{1}{4}} \| \partial_k \phi \|_{L^2} \quad (2.5)$$

and

$$\| \partial_x (V_0(t) - 1) \phi \|_{L^2} \leq C \| \partial_k \phi \|_{L^2}. \quad (2.6)$$

### 2.2 Linear Schrödinger equation with a potential.

Consider now the linear Schrödinger equation with a potential

$$\begin{align*}
\begin{cases}
i \partial_t u = H u, \\
u(0, x) = u_0(x)
\end{cases} \quad (2.7)
\end{align*}$$
where $H = H_0 + V(x)$ and the potential $V(x)$ is a real valued function for all $x \in \mathbb{R}$. If $V \in L^{1,1}$, it is known that the operator $H$ is self-adjoint ([46]), the absolutely-continuous spectrum of $H$ is given by $\sigma_{ac}(H) = (0, \infty)$, $H$ has no singular-continuous spectrum, $H$ has no eigenvalues that are positive or equal to 0 and $H$ has a finite number of negative eigenvalues that are simple ([20]).

We want to derive an asymptotic representation formula for the evolution group defined by (2.7), similar to (2.3). For that purpose, we need the spectral theorem for the perturbed operator $H$, that follows from the Weyl-Kodaira-Titchmarsh theory (see, for example, [20]). Assume that $V \in L^{1,1}$. The Jost functions $f_{\pm}(x, k)$ are solutions to the stationary Schrödinger equation

\[
\begin{aligned}
-\frac{1}{2} \frac{d^2}{dx^2} f_\pm + V f_\pm &= \frac{k^2}{2} f_\pm, \\
\quad f_\pm(x, k) &= e^{\pm ikx}, \ x \to \pm \infty.
\end{aligned}
\]  

They are solutions to Volterra integral equations and are obtained by iteration as uniformly convergent series ([20], [46]). Since $f_{\pm}(x, k)$ are independent solutions to (2.8) for $k \neq 0$, there are unique coefficients $T(k)$ and $R_{\pm}(k)$ such that

\[
f_{\pm}(x, k) = \frac{R_{\pm}(k)}{T(k)} f_+(x, k) + \frac{1}{T(k)} f_{\pm}(x, -k),
\]

for $k \neq 0$ (see page 144 of [20]). The function $T(k)$ is the transmission coefficient, $R_-(k)$ is the reflection coefficient from left to right of the plane wave $e^{ikx}$ and $R_+(k)$ is the reflection coefficient from right to left of the plane wave $e^{-ikx}$. They satisfy the identity (the unitarity of the scattering matrix)

\[
|T(k)|^2 + |R_{\pm}(k)|^2 = 1, \ k \in \mathbb{R}.
\]

We now introduce the generalized Fourier transform $\mathcal{F}$. Let the Heaviside function $\theta(k)$ be $\theta(k) = 1$ for $k \geq 0$, and $\theta(k) = 0$ for $k < 0$. We define

\[
\Psi(x, k) = \theta(k) T(k) f_+(x, k) + \theta(-k) T(-k) f_-(x, -k).
\]

The generalized Fourier transform

\[
\mathcal{F} \phi = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \psi(x, k) \phi(x) \, dx
\]

is unitary from the continuous subspace of $H$ denoted by $\mathcal{H}_c$ onto $L^2$. Under the assumption that $V$ has no negative eigenvalues $\mathcal{F}$ is unitary on $L^2$ and $\mathcal{F}^{-1}$ is given by

\[
\mathcal{F}^{-1} \phi = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \psi(x, k) \phi(k) \, dk.
\]

Then, the operator $\mathcal{F} H \mathcal{F}^{-1}$ acts as multiplication by \( \frac{\omega^2}{2} \) and the solution to (2.7) is given by $u = \mathcal{U}(t) u_0$, where

\[
\mathcal{U}(t) = e^{-itH} = \mathcal{F}^{-1} e^{-it k^2/2} \mathcal{F}.
\]

We now deduce an asymptotic representation formula for $\mathcal{U}(t)$. We define

\[
\Phi(x, k) = e^{-ikx} \Psi(x, k).
\]

Then, we have

\[
\mathcal{U}(t) \mathcal{F}^{-1} \phi = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{i}{2} k^2} \Phi(x, k) \phi(k) \, dk = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{i}{2} k^2 + ikx} \Phi(x, k) \phi(k) \, dk,
\]

and thus

\[
\mathcal{U}(t) \mathcal{F}^{-1} \phi = \frac{1}{\sqrt{2\pi}} M D_i \sqrt{\frac{i}{2}} \int_{-\infty}^{\infty} e^{-\frac{i}{2} k^2 + ikx} \Phi(x, k) \phi(k) \, dk.
\]

Denoting

\[
\mathcal{V}(t) \phi = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{i}{2} (k-x)^2} \Phi(x, k) \phi(k) \, dk = D_i e^{-\frac{i}{2} k^2} \mathcal{F}^{-1} \phi,
\]

from (2.15) we obtain

\[
\mathcal{U}(t) \mathcal{F}^{-1} \phi = M D_i \mathcal{V}(t) \phi,
\]
which is an asymptotic representation formula for the evolution group \( U(t) \). We also calculate the inverse asymptotic representation formula

\[
    F^{-1}(t) \phi = \mathcal{V}^{-1}(t) D_{t}^{-1} \mathcal{M} \phi,
\]

where

\[
    \mathcal{V}^{-1}(t) \phi = \sqrt{\frac{t}{2\pi i}} \int_{-\infty}^{\infty} e^{-\frac{i}{4} (k-x)^2} \Phi(ik, k) \phi(x) dx = e^{\frac{i}{4} k^2} \mathcal{F} M D_{t} \phi.
\]

### 3 Proof of Theorem 1.3.

This section is devoted to the proof of the main result. Suppose that the potential \( V \in L^{1,1} \) is exceptional. Then, the Jost solutions at \( k = 0 \) satisfy \( [f_+(x,0), f_-(x,0)] = 0 \). In this case, \( a = \lim_{x \to -\infty} f_+(x,0) = 0 \) ([35], [2], [46]). We also assume that the potential \( V(x) \) is symmetric, i.e. \( V(-x) = V(x) \). This ensures that the function \( f_+(x,0) \) is either even or odd (see [12], page 160). Indeed, for symmetric potentials

\[
    f_+(x,k) = f_-(x,k).
\]

Then, since \( V \) is exceptional \( f_+(x,0) = \alpha f_+(x,0) \), for some \( \alpha \in \mathbb{R} \). Changing \( x \to -x \) in the last equation we get \( f_+(x,0) = \alpha f_+(x,0) \). The combination of these two equations yields \( f_+(x,0) = \alpha^2 f_+(x,0) \). Then, \( \alpha = \pm 1 \), and thus, \( f_+(x,0) \) is even or odd. Therefore,

\[
    a = \lim_{x \to -\infty} f_+(x,0) = \pm 1.
\]

Also, we observe that for symmetric potential \( V(x) \) the problem (1.1) conserves the parity condition, i.e. if the initial data \( u_0(x) \) are even (odd) then the solution \( u(t,x) \) is even (odd) for all time. From (3.1) it follows that \( \Psi(-x,-k) = \Psi(x,k) \), so that if \( \phi(x) \) is even (odd) and the potential \( V \) is symmetric, then \( \mathcal{F} \phi \) is also even (odd). Then, we suppose additionally that the initial data \( u_0(x) \) are odd if \( a = 1 \) and \( u_0(x) \) are even if \( a = -1 \). We also note that in the case of symmetric potentials \( R_+(k) = R_-(k) \). Then, we denote

\[
    R(k) = R_+(k) = R_-(k).
\]

The parity assumption on the potential and initial data is made in order to assure that (3.2) is true and \( \mathcal{F}(u(t)) \) is either even or odd. We need this extra information in order to control the small-energy behaviour of the scattering coefficients and the wave functions.

We begin by presenting the results of Sections 5, 6 and 7 that are involved in the proof of Theorem 1.3. We note that under Condition 1.1, all the relations of Section 4 are valid. Taking into account the commentaries about the properties of the symmetric potentials made at the beginning of this section, we see that for potentials satisfying Condition 1.1, the results of Sections 5, 6 and 7 are true. Therefore, we have the following.

**Lemma 3.1** Suppose that \( V \) satisfies Condition 1.1. If \( T(0) = 1 \), let \( \phi \in H^{0,1} \) be odd and if \( T(0) = -1 \), suppose that \( \phi \) is even. Then

\[
    (\mathcal{F} \phi)(0) = 0.
\]

Moreover

\[
    \| \mathcal{F} \phi \|_{H^1} \leq K_0 \| \phi \|_{H^{0,1}},
\]

for some \( K_0 > 0 \) is true.

**Lemma 3.2** Suppose that \( V \) satisfies Condition 1.1. Then, for any \( t \in \mathbb{R} \) and \( \phi \in H^1 \cap H^{0,1} \) the estimates

\[
    \| U(t) \phi \|_{H^1} \leq K_0 \| \phi \|_{H^1}
\]

and

\[
    \| U(t) \phi \|_{H^{0,1}} \leq K_0 \| \phi \|_{H^1} + \| \phi \|_{H^{0,1}}
\]

for some \( K_0 > 0 \) are valid.

**Lemma 3.3** Suppose that \( V \) satisfies Condition 1.1. Then, if \( T(0) = 1 \), the estimate

\[
    \| V(t) w - T(|x|) w(x) - R(|x|) w(-x) \|_{L^\infty} \leq C \| w(0) \| + C t^{-1/4} \| w \|_{H^1}
\]

is true for all \( t \geq 1 \). Moreover, in the case \( T(0) = -1 \),

\[
    \| V(t) w - T(|x|) w(x) - R(|x|) w(-x) - \sqrt{\frac{2}{\pi}} w(0) \int_{\sqrt{tx}}^{\infty} e^{-\frac{k^2}{2}} dk \|_{L^\infty} \leq C t^{-1/4} \| w \|_{H^1}
\]

for some \( K_0 > 0 \) is valid.
Furthermore, if $T(0) = 1$, let $w \in H^1$ be odd and if $T(0) = -1$, suppose that $w$ is even. Also suppose that $w$ can be represented as $w = F\psi$, for some $\psi \in H^{0,1}$. Then, the estimate
\[
\| \partial_x V(t) w \|_{L^2} \leq C \| w \|_{L^\infty} \log(t) + C \| w \|_{H^1},
\]
(3.10)
is true for all $t \geq 1$.

**Remark 3.4** It can be proved that in general the following asymptotic formula holds
\[
\| V(t) w - T(|x|) w(x) - R(|x|) w(-x) - h(t,x) w(0) \|_{L^\infty} \leq Ct^{-1/4} \| w \|_{H^1},
\]
for some function $h \in L^\infty((0, \infty) \times \mathbb{R})$. In principle $h$ might be not identically zero. Thus, if $\lim_{t \to \infty} w(t,0) \neq 0$, the above expansion together with (1.8) suggest a different behaviour for $u(t)$, as $t \to \infty$, compared with the asymptotics in the generic case obtained in [37] or (1.7).

**Lemma 3.5** If $V$ satisfies Condition 1.1, the estimates
\[
\left\| \nabla^{-1}(t) \phi - \frac{T(|k|)}{T(|k|)} \phi(-k) \right\|_{L^\infty} \leq C |\phi(0)| + C t^{-\frac{1}{4}} \| \phi \|_{H^1},
\]
(3.11)
and
\[
\| \partial_k V^{-1}(t) \phi \|_{L^2} \leq C t^{-\frac{1}{4}} |\phi(0)| + C \| \phi \|_{H^1},
\]
(3.12)
are valid for all $t \geq 1$.

We postpone the proof of Lemmas 3.1, 3.2, 3.3 and 3.5 to Sections 5, 6 and 7. We now use these results to prove some estimates.

**Lemma 3.6** Suppose that $V$ satisfies Condition 1.1. If $T(0) = 1$, let $w \in H^1$ be odd and if $T(0) = -1$, suppose that $w$ is even. Also suppose that $w$ can be represented as $w = F\psi$, for some $\psi \in H^{0,1}$. Then, for any $0 < \beta < \frac{1}{2}$, the asymptotics
\[
\nabla^{-1}(t) \left( |V(t)|^2 V(t) w \right) = |w(k)|^2 w(k) + O \left( \left( |w(0)| + t^{-\frac{1}{4}} \| w \|_{H^1} \right)^2 \left( \| w \|_{L^\infty} + t^{-\beta} \| w \|_{H^1} \right)^2 \right)
\]
(3.13)
holds for all $t \geq 1$.

**Proof.** It follows from (3.11) that
\[
V^{-1}(t) \left( |V(t)|^2 V(t) w \right) = \frac{T(|k|)}{T(|k|)} \left( |V(t)|^2 V(t) w \right)(k) + R(|k|) \left( |V(t)|^2 V(t) w \right)(-k) + R_1,
\]
(3.14)
where
\[
|R_1| \leq C |(V(t) w)(0)|^3 + C \left( \frac{1}{2} \right)^{-\frac{1}{4}} \| V(t) w \|_{L^\infty}^2 \| V(t) w \|_{H^1}.
\]
(3.15)
If $T(0) = 1$, $w$ is odd. In particular, $w(0) = 0$. If $T(0) = -1$, then
\[
T(0) w(0) + R(0) w(0) + \sqrt{\frac{2t}{\pi}} w(0) \int_0^\infty e^{-tk^2} dk = 0.
\]
Hence, for $T(0) = \pm 1$, by (3.8) and (3.9) we get
\[
|\nabla(t) w(0)| \leq Ct^{-\frac{1}{4}} \| w \|_{H^1},
\]
(3.16)
and via (2.10)
\[
\| V(t) w \|_{L^\infty} \leq C \| w \|_{L^\infty} + Ct^{-1/4} \| w \|_{H^1}.
\]
(3.17)
Using (3.16), (3.17) and (3.10) in (3.15), as $\| V(t) w \|_{L^2} = \| w \|_{L^2}$ we estimate
\[
|R_1| \leq Ct^{\beta - \frac{1}{4}} \left( \| w \|_{L^\infty} + t^{-\beta} \| w \|_{H^1} \right)^3.
\]
By (3.8) and (3.9) we get
\[
V(t) w = T(|x|) w(x) + R(|x|) w(-x) + O \left( |w(0)| + t^{-\frac{1}{4}} \| w \|_{H^1} \right).
\]
Then, using (2.10) and (3.17), from (3.14) we derive
\[
V^{-1} (t) \left( |V(t)|^2 V(t) w \right) = T(|k|) \left( |T(|k|) w(\pm k)| + 2 \right) \left( |T(|k|) w(\pm k)| + R(|k|) w(-k) \right)\\
+ O \left( \left( |w(0)| + t^{-\frac{1}{2}} \|w\|_{H^1} \right) \left( \|w\|_{L^\infty} + t^{-\beta} \|w\|_{H^1} \right)^2 \right) + O \left( t^{\beta - \frac{1}{2}} \left( \|w\|_{L^\infty} + t^{-\beta} \|w\|_{H^1} \right)^3 \right).
\] (3.18)

Note that (see (3.3) and (4.14) below)
\[
T(k)R(k) = 0.
\] (3.19)

Then, using that \(w\) is odd for \(T(0) = 1\) and even for \(T(0) = -1\), in view of (3.19) and (2.10) we obtain
\[
|T(|k|) w(\pm k)| = |T(|k|) w(\pm k)| = |T(|k|) w(\pm k)| = |T(|k|) w(\pm k)| = |T(|k|) w(\pm k)| = \left( |T(|k|)|^2 + |R(|k|)|^2 \right) \left( |T(|k|) R(|k|) + T(|k|) R(|k|)| \right) |w(\pm k)|^2 = |w(\pm k)|^2,
\]
for \(T(0) = \pm 1\). Hence, using again (3.19) and (2.10) from (3.18) we obtain (3.13).

Also, we present the estimate for the derivative of the nonlinear term.

**Lemma 3.7** Suppose that \(V\) satisfies Condition 1.1. If \(T(0) = 1\), let \(w \in H^1\) be odd and if \(T(0) = -1\), suppose that \(w\) is even. Also suppose that \(w\) can be represented as \(w = F\psi\), for some \(\psi \in H^0\). Then the estimate
\[
\left\| \partial_t V^{-1} (t) \left( |V(t)|^2 V(t) w \right) \right\|_{L^2} \leq C \|w\|_{L^\infty}^2 \log (t) + C t^{-1/4} \|w\|_{H^1}^3 + C \|w\|_{L^\infty} \|w\|_{H^1}.
\] (3.20)
is valid for all \(t \geq 1\).

**Proof.** (3.12) implies
\[
\left\| \partial_t V^{-1} (t) \left( |V(t)|^2 V(t) w \right) \right\|_{L^2} \leq C t^{\frac{1}{2}} \left( \|V(t) w|^2 V(t) w \right)(0) + C \|V(t) w|^2 V(t) w \right\|_{H^1}^2 + C \|V(t) w\|_{L^\infty}^2 \|V(t) w\|_{H^1}^2.
\]

By (3.10)
\[
\left\| \partial_t V(t) \phi \right\|_{L^2} \leq C \|w\|_{L^\infty} \log (t) + C \|w\|_{H^1}.
\]
Hence, using (3.16), (3.17) and \(\|V(t) w\|_{L^2} = \|w\|_{L^2}\) we deduce (3.20). ■

We are now in position to prove the main result. Using (2.17) we represent the solution to (1.1) as
\[
u(t) = U(t) F^{-1} w = MD_t V(t) w(t),
\] (3.21)

Then, by (3.8) and (3.9)
\[
u(t) = M D_t (\pm |x|) w(x) + R (|x|) w(-x) + O \left( \frac{|w(0)|}{\sqrt{t}} \right) + O \left( t^{-\frac{3}{4}} \|w\|_{H^1} \right),
\] (3.22)
as \(t \to \infty\), uniformly with respect to \(x \in \mathbb{R}\). This expression shows that in order to estimate the solution in the uniform norm \(\|u(t)\|_{L^\infty}\), we need to control \(\|w\|_{L^\infty}\) and \(\|w\|_{H^1}\). Therefore, given \(u(t)\), we introduce a new dependent variable by letting \(w(t) = F(\pm t) u(t)\) and define the following space
\[X_T = \{ u \in C \left( [0,T] ; H^1 \right); \|u\|_{X_T} < \infty \}, \ T > 0,\]
where
\[
\|u\|_{X_T} = \sup_{t \in [0,T]} \left( \|w\|_{L^\infty} + t^{-\beta} \|w\|_{H^1} \right),
\]
with \(\beta > 0\).

For the local well-posedness of equation (1.1) in \(H^1\) we refer to [15] (see Theorem 4.3.1 and Corollary 4.3.3). We now prove a local existence result which is adapted to our purposes. We have the following.
Theorem 3.8 Suppose that $V$ satisfies Condition 1.1. Let the initial data $u_0 \in H^1 \cap H^{0,1}$. In addition, suppose that $u_0(x)$ are odd if $T(0) = 1$ and $u_0(x)$ are even in the case when $T(0) = -1$. Then, for some $T > 0$, the Cauchy problem for (1.1) has a unique solution $u \in C \left([0,T];H^1 \cap H^{0,1}\right)$, such that

$$U(-t) u \in C \left([0,T];H^1 \cap H^{0,1}\right)$$

(3.23) with

$$\|u\|_{X_T} + \sup_{t \in (0,T)} \left(\langle t \rangle^{-1-\beta} \|u(t)\|_{H^{0,1}}\right) + \sup_{t \in (0,T)} \left(\langle t \rangle^{-\beta} \|u(t)\|_{H^1}\right) \leq C \left(\|u_0\|_{H^1} + \|u_0\|_{H^{0,1}}\right).$$

(3.24)

Moreover, if $\|u_0\|_{H^1} + \|u_0\|_{H^{0,1}} \leq \varepsilon$, there is $\varepsilon_0 > 0$, such that for every $0 < \varepsilon < \varepsilon_0$ the existence time $T > 1$. Furthermore, $u$ can be extended on a maximal existence interval $[0,T_{max})$, to a solution $u \in C \left([0,T_{max});H^1 \cap H^{0,1}\right)$, and if $T_{max} < \infty$, then

$$\sup_{\tau \in (0,t)} \left(\langle \tau \rangle^{-1-\beta} \|u(\tau)\|_{H^{0,1}}\right) + \sup_{\tau \in (0,t)} \left(\langle \tau \rangle^{-\beta} \|u(\tau)\|_{H^1}\right) \to \infty, \text{ as } t \uparrow T_{max}.$$  

(3.25)

Proof. We consider the space $\mathcal{X}_T$ with $u$ can be extended on a maximal existence interval $[0,T_{max})$, to a solution $u \in C \left([0,T_{max});H^1 \cap H^{0,1}\right)$, and if $T_{max} < \infty$, then

$$\sup_{\tau \in (0,t)} \left(\langle \tau \rangle^{-1-\beta} \|u(\tau)\|_{H^{0,1}}\right) + \sup_{\tau \in (0,t)} \left(\langle \tau \rangle^{-\beta} \|u(\tau)\|_{H^1}\right) \to \infty, \text{ as } t \uparrow T_{max}.$$  

(3.25)

Using (3.6) we estimate from (3.26)

$$\|u(t)\|_{H^1} \leq \|U(t) u_0\|_{H^1} + |\lambda| \int_0^t \left(\|U(t) \left(|u|^2 u\right)(\tau)\|_{H^1}\right) d\tau \leq K_0 \|u_0\|_{H^1} + K_0 |\lambda| \int_0^t \left(\|u(\tau)\|_{H^1}\right) d\tau.$$  

(3.27)

Then

$$\|u(t)\|_{H^{0,1}} \leq K_0 \|u_0\|_{H^{0,1}} + 3K_0 |\lambda| \int_0^t \|u(\tau)\|_{L^\infty} \|u(\tau)\|_{H^{0,1}} d\tau.$$  

(3.28)

Moreover, (3.26) and (3.7) imply

$$\|u(t)\|_{H^{0,1}} \leq \|U(t) u_0\|_{H^{0,1}} + |\lambda| \int_0^t \left(\|U(t) \left(|u|^2 u\right)(\tau)\|_{H^{0,1}}\right) d\tau \leq K_0 \|u_0\|_{H^1} + \|u_0\|_{H^{0,1}} + K_0 |\lambda| \int_0^t \left(\|u(\tau)\|_{H^1}\right) d\tau \leq K_0 \|u_0\|_{H^1} + \|u_0\|_{H^{0,1}}.$$  

(3.29)

The existence of a unique solution $u \in C \left([0,T];H^1 \cap H^{0,1}\right)$ follows from a contraction mapping argument on the space $\mathcal{E}_T = \{u \in C \left([0,T];H^1 \cap H^{0,1}\right): \|u(t)\|_{L^\infty((0,T),H^1)} \leq M \text{ and } \|u(t)\|_{L^\infty((0,T),H^{0,1})} \leq M\}$ where

$$M = (K_0 + 1) \left(\langle T \rangle \|u_0\|_{H^1} + \|u_0\|_{H^{0,1}}\right)$$

with $T > 0$ satisfying

$$12 |\lambda| (K_0 + 1)^2 M^2 \langle T \rangle < 1.$$  

(3.30)

The inclusion (3.23) is consequence of (3.6), (3.7) and $u \in C \left([0,T];H^1 \cap H^{0,1}\right)$. Using (3.5) and (3.7) we get

$$\|u(t)\|_{H^1} = \|\mathcal{F}U(-t) u(t)\|_{H^1} \leq K_0 \|U(-t) u(t)\|_{H^{0,1}} \leq K_0 \langle t \rangle \|u(t)\|_{H^1} + \|u(t)\|_{H^{0,1}}.$$  

(3.31)

As $\|u(t)\|_{L^\infty((0,T),H^1)} + \|u(t)\|_{L^\infty((0,T),H^{0,1})} \leq 2 (K_0 + 1) \langle T \rangle \|u_0\|_{H^1} + \|u_0\|_{H^{0,1}}$, via Sobolev’s inequality we have

$$\|u\|_{X_T} \leq C \left(\|u_0\|_{H^1} + \|u_0\|_{H^{0,1}}\right).$$

Since $u \in \mathcal{E}_T$, $\|u(t)\|_{L^\infty((0,T),H^1)} \leq (K_0 + 1) \langle T \rangle \|u_0\|_{H^1}$. Thus, we deduce (3.24). If $\|u_0\|_{H^1} + \|u_0\|_{H^{0,1}} \leq \varepsilon$, then from (3.30) we conclude that there is $\varepsilon_0 > 0$, such that for every $0 < \varepsilon < \varepsilon_0$ the existence time $T > 1$. Finally, we prove the blowup alternative (3.25). Assume by contradiction that $T_{max} < \infty$, and that there exist $B > 0$ and a sequence $(t_n)_{n \geq 1}$ such that $t_n \uparrow T_{max}$ and

$$\sup_{\tau \in (0,t_n)} \left(\langle \tau \rangle^{-1-\beta} \|u(\tau)\|_{H^{0,1}}\right) + \sup_{\tau \in (0,t_n)} \left(\langle \tau \rangle^{-\beta} \|u(\tau)\|_{H^1}\right) \leq B.$$  

We consider the space $\mathcal{E}_T$ with

$$M = (2K_0 \langle T_{max} \rangle^{2+\beta} + 1) B.$$  

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We fix $T_{\text{max}} < \tau_1 < \infty$, then we choose $0 < T < \tau_1 - T_{\text{max}}$ sufficiently small so that
\[
12K_0 |\lambda| M^2 \left( 2K_0 \langle T_{\text{max}} \rangle^{2+\beta} + 1 \right) \langle T \rangle T < 1.
\]
Then, by a contraction mapping argument, for all $n \geq 1$ there exists $u_n \in C([0, T]; H^1 \cap H^{0,1})$ which is a solution to the equation
\[
u_n (t) = \mathcal{U} (t) u (t_n) - i \lambda \int_0^t \mathcal{U} (t - \tau) \left( |u_n|^2 u_n \right) (\tau) \, d\tau.
\]
Setting now $v_n (t) = \begin{cases} u (t) & 0 \leq t \leq t_n \\ u_n (t - t_n) & t_n \leq t \leq t_n + T \end{cases}$ we see that $v_n \in C([0, t_n + T]; H^1 \cap H^{0,1})$ is a solution of (1.1) on $[0, t_n + T]$. Since $t_n + T > T_{\text{max}}$, for $n$ big enough, this yields a contradiction. Hence, (3.25) holds. \hfill \blacksquare

We now prove an a priori estimate for the solutions of the Cauchy problem (1.1). Let us define
\[ K = 13 \left( K_0 + 1 \right)^3, \]
where $K_0 > 0$ is given by Lemma 3.1 and 3.2. We have the following.

**Lemma 3.9** Suppose that $V$ satisfies Condition 1.1. Let the initial data $u_0 \in H^1 \cap H^{0,1}$. In addition, we suppose that $u_0 (x)$ are odd if $T (0) = 1$ and $u_0 (x)$ are even in the case when $T (0) = -1$. Let $T > 0$ be given by Theorem 3.8. Then, there exists $\varepsilon_0 > 0$ such that for any $0 < \varepsilon < \varepsilon_0$ and $\|u_0\|_{H^1} + \|u_0\|_{H^{0,1}} \leq \varepsilon$, the estimate
\[
\|u\|_{X_T} + \sup_{\tau \in (0, T)} \left( \langle \tau \rangle^{-1-\beta} \|u (\tau)\|_{H^{0,1}} \right) + \sup_{\tau \in (0, T)} \left( \langle \tau \rangle^{-\beta} \|u (\tau)\|_{H^1} \right) < K \varepsilon
\]
holds for the solutions $u$ of the Cauchy problem (1.1).

**Proof.** We prove (3.32) by a contradiction argument. Suppose that the statement of the lemma is not true. Similarly to (3.31) from inclusion $u \in C([0, T]; H^1 \cap H^{0,1})$ and (3.23) it follows that the left-hand side in (3.32) is continuous as function of $T > 0$. Then, for any $\varepsilon > 0$ and initial data $\|u_0\|_{H^1} + \|u_0\|_{H^{0,1}} \leq \varepsilon$, we can find $T_1 \leq T$, with the property
\[
\|u\|_{X_{T_1}} + \sup_{\tau \in (0, T_1)} \left( \langle \tau \rangle^{-1-\beta} \|u (\tau)\|_{H^{0,1}} \right) + \sup_{\tau \in (0, T_1)} \left( \langle \tau \rangle^{-\beta} \|u (\tau)\|_{H^1} \right) = K \varepsilon.
\]
Let us show that this yields a contradiction. Theorem 3.8 guarantees that there is $\varepsilon_0 > 0$ such that for any $0 < \varepsilon \leq \varepsilon_0$ and $\|u_0\|_{H^1} + \|u_0\|_{H^{0,1}} \leq \varepsilon$ the existence time $T > 1$. In particular, as $u \in E_1$, it follows that
\[
\sup_{t \in (0,1)} \|u (t)\|_{H^1} \leq (K_0 + 1) \left( \sqrt{2} \|u_0\|_{H^1} + \|u_0\|_{H^{0,1}} \right) \leq \sqrt{2} (K_0 + 1) \varepsilon
\]
and
\[
\sup_{t \in (0,1)} \|u (t)\|_{H^{0,1}} \leq \sqrt{2} (K_0 + 1) \varepsilon.
\]
Then, using (3.31) we have
\[
\sup_{t \in (0,1)} \|w (t)\|_{H^1} \leq \sqrt{2} K_0 \left( \sup_{t \in (0,1)} \|u (t)\|_{H^1} + \sup_{t \in (0,1)} \|u (t)\|_{H^{0,1}} \right) \leq 4K_0^2 (K_0 + 1) \varepsilon.
\]
Moreover,
\[
\sup_{t \in (0,1)} \|w (t)\|_{L^\infty} \leq \sqrt{2} \sup_{t \in (0,1)} \|u (t)\|_{H^1} \leq 4\sqrt{2} K_0^2 (K_0 + 1) \varepsilon.
\]
Hence,
\[
\|u\|_{X_T} \leq \sup_{t \in (0,1)} \|w\|_{L^\infty} + \sup_{t \in (0,1)} \|w\|_{H^1} \leq 4 \left( 1 + \sqrt{2} \right) K_0^2 (K_0 + 1) \varepsilon.
\]
Suppose now that $t \geq 1$. We represent the solution $u (t) = \mathcal{U} (t) F^{-1} w = MD \mathcal{V} (t) w (t)$ and use (3.22) to reduce the problem to estimate $w$. As mentioned at the beginning of this Section, if the potential is symmetric and the initial data $u_0 (x)$ are even or odd, then $u (t, x)$ is even or odd, respectively, for all $t \in [0, T]$. Then, since $F$ preserves the parity, by the assumptions
of Theorem 1.3, \( w(k) \) is odd if \( T(0) = 1 \), and \( w(k) \) is even if \( T(0) = -1 \). To derive the equation for \( w(t) \) we apply the operator \( F \) to equation (1.1). We find

\[
i \partial_t (FU(-t) u) = \lambda FU(-t) \left( \|u\|^2 u \right).
\]

Hence, substituting \( u(t) = U(t) F^{-1} w \) we obtain the equation for \( w(t) \),

\[
i \partial_t w = \lambda FU(-t) \left( \|U(t) F^{-1} w\|^2 U(t) F^{-1} w \right).
\]

By virtue of operators \( V(t) \) and \( V^{-1}(t) \) we find

\[
l \lambda FU(-t) \left( \|U(t) F^{-1} w\|^2 U(t) F^{-1} w \right)
= \lambda V^{-1}(t) D_t^{-1} M \left( |M D_t V(t) w|^2 M D_t V(t) w \right)
= \lambda V^{-1}(t) D_t^{-1} \left( |D_t V(t) w|^2 D_t V(t) w \right)
= \lambda V^{-1}(t) \left( \|V(t) w\|^2 V(t) w \right).
\]

Thus we arrive to the equation

\[
i \partial_t w = \lambda V^{-1}(t) \left( \|V(t) w\|^2 V(t) w \right).
\] (3.38)

Applying Lemma 3.6 we get

\[
i \partial_t w = \lambda V^{-1}(t) \|w(k)\|^2 w(k) + O \left( t^{-1} \left( \|w(0)\| + t^{-\frac{1}{2}} \|w\|_{L^1} \right)^2 \right) + O \left( \varepsilon^3 t^{\beta - \frac{3}{2}} \left( \|w\|_{L^\infty} + t^{-\beta} \|w\|_{L^1} \right)^3 \right).
\] (3.39)

We claim that

\[
w(t,0) = 0.
\] (3.40)

Indeed, we have \( w(t,0) = (Fu(t))(0) \). Since \( u(t,x) \) is odd if \( T(0) = 1 \) and it is even when \( T(0) = -1 \), (3.40) follows from (3.4). Going back to (3.39) and using (3.33) we obtain

\[
i \partial_t w = \lambda V^{-1}(t) \|w(k)\|^2 w(k) + O \left( \varepsilon^3 t^{\beta - \frac{3}{2}} \right).
\] (3.41)

Multiplying (3.41) by \( w(k) \), taking the imaginary part and using (3.33) we find

\[
\partial_t \|w(k)\|^2 = O \left( \varepsilon^3 \langle t \rangle^{\beta - \frac{3}{2}} \|w\|_{L^\infty} \right) = O \left( \varepsilon^3 \langle t \rangle^{\beta - \frac{3}{2}} \right).
\] (3.42)

By (3.36)

\[
\|w(1)\|_{L^\infty} \leq 4\sqrt{2} K_0^2 (K_0 + 1) \varepsilon.
\]

Then, equation (3.42) shows that

\[
\sup_{t \in (1,T)} \|w(t)\|_{L^\infty} \leq 4\sqrt{2} K_0^2 (K_0 + 1) \varepsilon + C \varepsilon^{3/2}.
\] (3.43)

To estimate the derivative \( \|\partial_k w\|_{L^2} \), we differentiate equation (3.38). We get

\[
i \partial_t w_k = \lambda V^{-1}(t) \left( \|V(t) w\|^2 V(t) w \right).
\]

Taking the \( L^2 \)-norm in the last equation, using the estimate of Lemma 3.7 and relation (3.33) we get

\[
\frac{d}{dt} \|\partial_k w\|_{L^2} \leq C t^{-1} \log \langle t \rangle \|w\|_{L^\infty}^3 + C t^{-\frac{5}{2}} \|w\|_{H^1}^3 + C \varepsilon \|w\|_{H^1} \leq C \varepsilon t^{\beta - 1},
\]

for \( 0 < \beta \leq \frac{1}{8} \). Integrating the last inequality we obtain

\[
\|\partial_k w\|_{L^2} \leq \|\partial_k w(1)\|_{L^2} + C \varepsilon \langle t \rangle^\beta.
\]

By (3.35) \( \|\partial_k w(1)\|_{L^2} \leq 4 K_0^2 (K_0 + 1) \varepsilon \). Then,

\[
\|\partial_k w\|_{L^2} \leq \langle t \rangle^\beta \left( 4 K_0^2 (K_0 + 1) \varepsilon + C \varepsilon^3 \right).
\] (3.44)
Moreover, as \(\|u(t)\|_{L^2} = \|u_0\|_{L^2}\), for all \(t \in [0, T]\), and \(\mathcal{F}\) and \(\mathcal{U}(-t)\) are unitary on \(L^2\), we get

\[
\|w\|_{L^2} = \|\mathcal{F}\mathcal{U}(-t)u(t)\|_{L^2} = \|u(t)\|_{L^2} = \|u_0\|_{L^2} \leq \varepsilon \leq \varepsilon \langle t \rangle^\beta .
\] (3.45)

(3.44) and (3.45) imply

\[
\sup_{t \in (1,T_1)} \left( (t)^{-\beta} \|w(t)\|_{H^1} \right) \leq 4K_0^2 (K_0 + 1) \varepsilon + \varepsilon + C\varepsilon^3 .
\] (3.46)

Hence, by (3.43) and (3.46)

\[
\sup_{t \in (1,T_1)} \left( \|w\|_{L^\infty} + \langle t \rangle^{-\beta} \|w\|_{H^1} \right) \leq \left( 4 + 4\sqrt{2} \right) K_0^2 (K_0 + 1) \varepsilon + \varepsilon + C\varepsilon^3/2 \leq \left( 5 + 4\sqrt{2} \right) (K_0 + 1)^3 \varepsilon + C\varepsilon^3/2 .
\] (3.47)

From (3.37) and (3.47) we deduce

\[
\|u\|_{X_{T_1}} \leq \left( 5 + 4\sqrt{2} \right) (K_0 + 1)^3 \varepsilon + C\varepsilon^3/2 .
\] (3.48)

Using (2.10), (3.22), (3.33) we estimate

\[
\|u(t)\|_{L^\infty} \leq C\varepsilon \langle t \rangle^{-1/2} , \quad t \in [0,T_1].
\] (3.49)

Then, from (3.28) and (3.33)

\[
\|u(t)\|_{H^1} \leq K_0 \varepsilon + C\varepsilon^3 \langle t \rangle^{-1+\beta} dt \leq K_0 \varepsilon + C\varepsilon^3 \langle t \rangle^\beta .
\]

Thus,

\[
\sup_{\tau \in (0,T_1)} \left( \langle \tau \rangle^{-\beta} \|u(\tau)\|_{H^1} \right) \leq K_0 \varepsilon + C\varepsilon^3 .
\] (3.50)

Similarly, from (3.29), (3.49), (3.33) we get

\[
\|u(t)\|_{H^{0.1}} \leq K_0 (t) \varepsilon + C\varepsilon^3 (t) \int_0^t \langle \tau \rangle^{-1+\beta} d\tau \leq (K_0 + C\varepsilon^3) \langle t \rangle^{1+\beta} ,
\] (3.51)

for all \(t \in [0,T_1]\). By (3.48), (3.50) and (3.51) we see that

\[
\|u\|_{X_{T_1}} + \sup_{\tau \in (0,T_1)} \left( \langle \tau \rangle^{-1-\beta} \|u(\tau)\|_{H^{0.1}} \right) + \sup_{\tau \in (0,T_1)} \left( \langle \tau \rangle^{-\beta} \|u(\tau)\|_{H^1} \right) \leq \left( 7 + 4\sqrt{2} \right) (K_0 + 1)^3 \varepsilon + C\varepsilon^3/2 < K\varepsilon,
\] (3.52)

for all \(\varepsilon > 0\) small enough. This contradicts (3.33). Therefore, there is \(\varepsilon_1 > 0\), such that for all \(0 < \varepsilon \leq \varepsilon_1\), the estimate (3.32) is valid. \(\blacksquare\)

**Proof of Theorem 1.3.** From Theorem 3.8 and Lemma 3.9 it follows that there is a global solution \(u \in C([0,\infty);H^1)\) to (1.1) satisfying the estimate

\[
\|u\|_{X_{\infty}} < K\varepsilon.
\] (3.53)

In particular, from (3.22) via (2.10) we get the uniform estimate (1.6). We now study the asymptotics of \(w(t,k)\) as \(t \to \infty\). Integrating (3.42) in time we get

\[
|w(t,k)|^2 - |w(s,k)|^2 = O \left( \varepsilon^2 \langle s \rangle^{-\frac{1}{2}} \right)
\]

for all \(t > s > 1\). The last estimate shows that \(|w(t,k)|\) is a Cauchy sequence. Then, there exists a limit \(\Xi \in L^\infty\) with the property

\[
\Xi(k) - |w(t,k)|^2 = O \left( \varepsilon^2 \langle t \rangle^{-\frac{1}{2}} \right).
\]

Since \(w(t,k)\) is even or odd, \(\Xi(k)\) is an even function. Next, we rewrite equation (3.41) as

\[
i\partial_t w = \lambda t^{-1} \Xi(k) w(k) + O \left( \varepsilon^3 \|w(k)\|_{L^\infty} t^{\beta-\frac{1}{2}} \right) + O \left( \varepsilon^3 \langle t \rangle^{-\beta-\frac{1}{2}} \right)
\]

\[
= \lambda t^{-1} \Xi(k) w(k) + O \left( \varepsilon^3 \langle t \rangle^{-\beta-\frac{1}{2}} \right),
\]

where in the last equality we used (3.53) to control \(\|w(k)\|_{L^\infty}\). Putting \(w(t,k) = v(t,k) e^{-i\lambda \Xi(k) \log t}\), we exclude the resonant nonlinear term \(\lambda t^{-1} \Xi(k) w(k)\), and we are left with the equation for \(v\)

\[
i\partial_t v = O \left( \varepsilon^3 \langle t \rangle^{-\beta-\frac{1}{2}} \right).
\]
Integrating in time the last equation we get
\[ v(t, k) - v(s, k) = O \left( \varepsilon^3 s^{\beta - \frac{1}{2}} \right) \]
for all \( t > s > 1 \). Thus \( v(t, k) \) is a Cauchy sequence, and hence, there exists \( v_+ \in L^\infty \) such that \( v_+(k) - v(t, k) = O \left( \varepsilon^3 t^{\beta - \frac{1}{2}} \right) \). Therefore, we get the asymptotics for \( w \)
\[ w(t, k) = v_+(k) e^{-i\lambda k \log t} + O \left( \varepsilon t^{\beta - \frac{1}{2}} \right), \]
as \( t \to \infty \), uniformly on \( k \in \mathbb{R} \). Moreover, noting that \( \Xi(k) = \lim_{t \to \infty} |w(t, k)|^2 = \lim_{t \to \infty} |v(t, k)|^2 = |v_+|^2 \), we obtain
\[ w(t, k) = v_+(k) e^{-i\lambda |v_+|^2 \log t} + O \left( \varepsilon t^{\beta - \frac{1}{2}} \right), \]
Since \( \Xi(k) \) is even, \( v \) has the same parity as \( w \), and thus \( v_+ \) is either even or odd. We now conclude as follows. Relations (3.22), (3.40) and (3.53) imply
\[ u(t) = MD_t \left( T(|x|) w(x) + R(|x|) w(-x) \right) + O \left( \varepsilon t^{\beta - \frac{1}{2}} \right), \]
as \( t \to \infty \), uniformly with respect to \( x \in \mathbb{R} \). Hence
\[ u(t) = MD_t w_+ e^{-i\lambda |v_+|^2 \log t} + O \left( \varepsilon t^{\beta - \frac{1}{2}} \right), \]
with \( w_+ = (T(|x|) v_+(x) + R(|x|) v_+(-x)) \). Also we note that by (2.10) and (3.19) \( |v_+|^2 = |w_+|^2 \). Therefore, the asymptotic formula (1.7) follows. Theorem 1.3 is proved.

4 Jost solutions.

In this Section we expose some properties and estimates for the Jost solutions that are involved in the proof of the main result. Assume that \( V \in L^{1,1} \). Let
\[ m_\pm(x, k) = e^{\mp ikx} f_\pm(x, k). \]
By Lemma 1 of [20] (see page 130), the functions \( m_\pm(x, k) \) are the unique solutions of the Volterra integral equations
\[ m_+(x, k) = 1 + \int_x^\infty K(y - x, k) V(y) m_+(y, k) dy \]
and
\[ m_-(x, k) = 1 + \int_{-\infty}^x K(y - x, k) V(y) m_-(y, k) dy, \]
respectively, where \( K(x, k) = \int_0^x e^{ikx} dz \). We need the following estimates (see [20], [37]).

**Proposition 4.1** Suppose that \( V \in L^{1,1} (\mathbb{R}) \). Then,
\[ |m_\pm(x, k) - 1| \leq C \langle k \rangle^{-1} \langle x \rangle, \]
for all \( x, k \in \mathbb{R} \). If \( V \in L^{1,2+\delta} (\mathbb{R}) \), for some \( \delta \geq 0 \), then, for all \( k \in \mathbb{R} \) and \( \pm x \geq 0 \), we have
\[ |m_\pm(x, k) - 1| \leq C \langle k \rangle^{-1} \langle x \rangle^{-1-\delta}, \]
\[ |\partial_k m_\pm(x, k)| \leq C \langle k \rangle^{-1} \langle x \rangle^{-\delta}. \]

We also need the following.

**Proposition 4.2** Assume that \( V \in L^{1,2+\delta} (\mathbb{R}) \). In addition, suppose that there is a partition (1.4) such that each part \( \langle \cdot \rangle^{2+\delta} V_j \in W^{1,1} (I_j) \). Then, for all \( k \in \mathbb{R} \) and \( \pm x \geq 0 \), estimates
\[ |\partial_x m_\pm(x, k)| \leq C \langle k \rangle^{-1} \langle x \rangle^{-2-\delta}, \]
and
\[ |\partial_k \partial_x m_\pm(x, k)| \leq C \langle k \rangle^{-1} \langle x \rangle^{-1-\delta}. \]
are valid.
Proof. First we prove (4.5) for the upper sign. The other case is considered similarly. From (4.1) we obtain
\[
\partial_x m_+ (x, k) = - \int_x^\infty e^{2ik(y-x)} V (y) m_+ (y, k) \, dy.
\] (4.7)

For \(|k| \leq 1\), by using (4.3) we get
\[
|\partial_x m_+ (x, k)| \leq C \langle x \rangle^{-2-\delta} \int_x^\infty \langle y \rangle^{2+\delta} |V (y)| \, dy \leq C \langle k \rangle^{-1} \langle x \rangle^{-2-\delta} \int_x^\infty \langle y \rangle^{2+\delta} |V (y)| \, dy.
\] (4.8)

and hence (4.5) follows. Let now \(|k| \geq 1\). From (4.7) we have
\[
\partial_x m_+ (x, k) = - \int_x^\infty e^{2ik(y-x)} V (y) \, dy - \int_x^\infty e^{2ik(y-x)} V (y) (m_+ (y, k) - 1) \, dy.
\] (4.9)

Using (4.3) we estimate
\[
\left| \int_x^\infty e^{2ik(y-x)} V (y) (m_+ (y, k) - 1) \, dy \right| \leq C \langle k \rangle^{-1} \langle x \rangle^{-2-\delta} \int_x^\infty \langle y \rangle |V (y)| \, dy \leq C \langle k \rangle^{-1} \langle x \rangle^{-2-\delta}.
\] (4.10)

To control the first term in the right-hand side of (4.9) we use (1.4). Suppose that \(x \in I_l\), for some \(l = 1, ..., N + 1\). Then
\[
\int_x^\infty e^{2ik(y-x)} V (y) \, dy = \int_x^{x_l} e^{2ik(y-x)} V_i (y) \, dy + \sum_{j=l+1}^{N+1} \int_{x_{j-1}}^{x_j} e^{2ik(y-x)} V_j (y) \, dy.
\]

Integrating by parts we get
\[
\int_x^\infty e^{2ik(y-x)} V (y) \, dy = \frac{1}{2ik} \left( e^{2ik(y-x)} V_i (y) \bigg|_{x}^{x_l} - \int_x^{x_l} e^{2ik(y-x)} \partial_y V_i (y) \, dy \right.
\]
\[
+ \sum_{j=l+1}^{N+1} \left( e^{2ik(y-x)} V_j (y) \bigg|_{x_{j-1}}^{x_j} - \int_{x_{j-1}}^{x_j} e^{2ik(y-x)} \partial_y V_j (y) \, dy \right) \bigg).
\]

Hence, by Sobolev embedding theorem,
\[
\left| \int_x^\infty e^{2ik(y-x)} V (y) \, dy \right| \leq C \langle k \rangle^{-1} \langle x \rangle^{-a} \sum_{j=l}^{N+1} \| \langle \cdot \rangle^a V_j \|_{L^\infty (I_j)} + C \langle k \rangle^{-1} \langle x \rangle^{-a} \sum_{j=l}^{N+1} \| \langle \cdot \rangle^a V_j \|_{W^{1,1} (I_j)}
\]
\[
\leq C \langle k \rangle^{-1} \langle x \rangle^{-a} \sum_{j=l}^{N+1} \| \langle \cdot \rangle^a V_j \|_{W^{1,1} (I_j)} \leq C \langle k \rangle^{-1} \langle x \rangle^{-a},
\] (4.11)

for \(0 \leq a \leq 2 + \delta\). Using (4.10) and (4.11) with \(a = 2 + \delta\) in (4.9) we get (4.5) for \(|k| \geq 1\).

Let us prove (4.6) for the upper sign. Differentiating (4.7) with respect to \(k\) we have
\[
\partial_k \partial_x m_+ (x, k) = -2i \int_x^\infty e^{2ik(y-x)} (y-x) V (y) (m_+ (y, k) - 1) \, dy - \int_x^\infty e^{2ik(y-x)} V (y) (\partial_k m_+ (y, k)) \, dy - 2i \int_x^\infty e^{2ik(y-x)} (y-x) V (y) \, dy.
\] (4.12)

Using (4.3) and (4.4) we estimate the first two terms in the right-hand side of (4.12) by \(C \langle k \rangle^{-1} \langle x \rangle^{-2-\delta}\). Using (4.11) with \(V (x)\) replaced with \((y-x) V (y)\), we control the last term in the right-hand side of (4.12) by \(C \langle k \rangle^{-1} \langle x \rangle^{-1-\delta}\). Hence, we deduce (4.6) in the case \(x \geq 0\). The case \(x \leq 0\) is treated similarly. □

Equation (2.9) can be rewritten in terms of \(m_\pm (x, k)\) as follows
\[
T (k) m_\pm (x, k) = R_\pm (k) e^{\pm 2ikx} m_\pm (x, k) + m_\pm (x, -k).
\] (4.13)

The coefficients \(T (k)\) and \(R_\pm (k)\) satisfy the following relations ([20], pages 144-146):
\[
T (k) = T (-k),
\]
\[ R_\pm (k) = R_\pm (-k) \]
\[ T(k) R_- (k) + T(k) R_+ (k) = 0. \]  
(4.14)

Moreover, the integral representations
\[ \frac{1}{T(k)} = 1 - \frac{1}{2ik} \int_{-\infty}^{\infty} V(y) m_+(y,k) dy \]  
(4.15)\n
and
\[ \frac{R_\pm (k)}{T(k)} = \frac{1}{2ik} \int_{-\infty}^{\infty} e^{\mp 2iky} V(y) m_\mp (y,k) dy \]  
(4.16)\n
hold. We observe that if \( V \in L^{1,1}(\mathbb{R}) \), by (4.2), (2.10), (4.15), (4.16)
\[ |T(k) - 1| \leq \frac{C}{|k|}, \text{ for } |k| \geq 1, \]  
(4.17)\n
and
\[ |R_\pm (k)| \leq \frac{C}{|k|}. \]  
(4.18)\n
Indeed, we have
\[ |T(k) - 1| \leq \frac{|T(k)|}{|k|} \int_{-\infty}^{\infty} \langle y \rangle |V(y)| dy \leq \frac{C}{|k|}, \text{ for } |k| \geq 1, \]  
and
\[ |R_\pm (k)| \leq \frac{|T(k)|}{|k|} \int_{-\infty}^{\infty} \langle y \rangle |V(y)| dy \leq \frac{C}{|k|}, \text{ for } |k| \geq 1. \]

In the following Proposition, we expose some estimates for the coefficients \( T(k) \) and \( R_\pm (k) \) (see Theorem 2.3 of [46]).

**Proposition 4.3** Suppose that \( V \in L^{1,3}(\mathbb{R}) \) is exceptional. Then, the estimates
\[ \left| \frac{d}{dk} T(k) \right| \leq C \langle k \rangle^{-1} \]  
(4.19)\n
\[ |T(k) - T(0)| \leq C |k|, \text{ as } k \to 0, \]  
(4.20)\n
\[ |R_\pm (k) - R_\pm (0)| \leq C |k|, \text{ as } k \to 0, \]  
(4.21)\n
are valid.

We also need to calculate the limit of \( T(k) \) and \( R_\pm (k) \), as \( k \to 0 \). Let
\[ a = \lim_{x \to -\infty} f_+(x,0). \]

We have the following result (see Theorem 2.1 of [35] or [46], page 52).

**Proposition 4.4** Suppose that the potential \( V \in L^{1,3}(\mathbb{R}) \) is exceptional. Then, we get
\[ T(k) = \frac{2a}{1 + a^2} + O(k), \text{ as } k \to 0, \]  
(4.22)\n
\[ R_\pm (k) = \pm \frac{1}{1 + a^2} + O(k), \text{ as } k \to 0. \]  
(4.23)\n
Finally, we present the following proposition.

**Proposition 4.5** Suppose that the potential \( V \in L^{1,3}(\mathbb{R}) \) is exceptional. Then,
\[ \left| \frac{d}{dk} R_\pm (k) \right| \leq C \langle k \rangle^{-1}. \]  
(4.24)
Proof. Similarly to the proof of (4.2) given in Lemma 1 of [20], we show
\[ |\partial_k m_+ (x, k)| \leq C \langle x \rangle^2 \] (4.25)
and
\[ |\partial_k^2 m_+ (x, k)| \leq C \langle x \rangle^3, \] (4.26)
for any \( x \in \mathbb{R} \). Using (4.2), (4.25) and (4.26) we see that
\[ \left| \frac{d^j}{dk^j} \int_{-\infty}^{\infty} e^{\mp 2iky} V (y) m_+ (y, k) dy \right| \leq C, \quad j = 0, 1, 2. \] (4.27)
In the case of exceptional potentials \( \int_{-\infty}^{\infty} V (y) m_+ (y, 0) dy = 0 \) (see [2]). Then,
\[ \left| \frac{d}{dk} \left( \frac{1}{k} \int_{-\infty}^{\infty} e^{\mp 2iky} V (y) m_+ (y, k) dy \right) \right| \leq \frac{C}{\langle k \rangle}, \quad k \in \mathbb{R}. \] (4.28)
Hence, multiplying equation (4.16) by \( T(k) \), derivating the resulting relation and using (2.10), (4.19), (4.27) and (4.28) we prove (4.24). □

5 Regularity and weighted estimates for \( \mathcal{F} \) and \( \mathcal{F}^{-1} \).

This section is dedicated to the proof of Lemmas 3.1 and 3.2. In all of the following results we only ask the properties of the Jost solutions that we use in the proof to be true. It is straightforward to check that under Condition 1.1 all the required properties (results of Section 4) are satisfied.

We need to control the \( L^2 \)-norm of the expressions \( \partial_k \mathcal{F} \phi, x \mathcal{U} (t) \phi \) and \( \partial_x (\mathcal{U} (t) \phi) \). We prove the following.

Lemma 5.1 Suppose that the estimates (2.10), (4.3) with \( \delta = 0 \), (4.4) with \( \delta > 1/2 \), (4.17), (4.18), (4.19), (4.22), (4.23) and (4.24) hold. Also, assume that \( m_+ (x, 0) = m_- (-x, 0) \). If \( a = 1 \), let \( \phi \in \mathcal{H}^{0, 1} \) be odd and if \( a = -1 \), suppose that \( \phi \) is even. Then, relation
\[ (\mathcal{F} \phi) (0) = 0. \] (5.1)
Furthermore
\[ \| \mathcal{F} \phi \|_{\mathcal{H}^1} \leq K_0 \| \phi \|_{\mathcal{H}^{0, 1}}, \] (5.2)
for some \( K_0 > 0 \) is true.

Proof. Recall that
\[ \mathcal{F} \phi = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx} \Phi (x, k) \phi (x) dx \] (5.3)
where
\[ \Phi (x, k) = \theta (k) T(k) m_+ (x, k) + \theta (-k) T(-k) m_- (x, -k). \]
In the domain \( x > 0 \) we express \( \Phi (x, k) \) in terms of \( m_+ \) using (4.13). We find
\[ \Phi (x, k) = \theta (k) T(k) m_+ (x, k) + \theta (-k) \left( e^{2ikx} R_+ (-k) m_+ (x, -k) + m_+ (x, k) \right). \] (5.4)
and in the domain \( x < 0 \) we express \( \Phi (x, k) \) in terms of \( m_- \)
\[ \Phi (x, k) = \theta (k) \left( e^{2ikx} R_- (k) m_- (x, k) + m_- (x, -k) \right) + \theta (-k) T(-k) m_- (x, -k). \] (5.5)
Using (2.12), (5.4) and (5.5) in (5.3) we have
\[ (\mathcal{F} \phi) (k) = \frac{\theta (k)}{\sqrt{2\pi}} \left( \int_0^\infty e^{-ikx} T(k) m_+ (x, k) \phi (x) dx + \int_{-\infty}^0 e^{-ikx} \left( e^{2ikx} R_- (k) m_- (x, k) + m_- (x, -k) \right) \phi (x) dx \right) \]
\[ + \frac{\theta (-k)}{\sqrt{2\pi}} \left( \int_{-\infty}^0 e^{-ikx} \left( T(-k) m_- (x, -k) \right) \phi (x) dx + \int_0^\infty e^{-ikx} \left( e^{2ikx} R_+ (-k) m_+ (x, -k) + m_+ (x, k) \right) \phi (x) dx \right). \] (5.6)
Note that if \( a = \pm 1 \), by (4.22) \( T(0) = \pm 1 \) and by (4.23) \( R_{\pm} (0) = 0 \). Also, by assumption \( m_+ (x, 0) = m_- (-x, 0) \). Then, using that \( \phi \) is odd when \( a = 1 \) and \( \phi \) is even if \( a = -1 \), we get
\[ \int_0^\infty T(0) m_+ (x, 0) \phi (x) dx + \int_{-\infty}^0 m_- (x, 0) \phi (x) dx = 0 \]
and
\[ \int_{-\infty}^{0} \left( \mathcal{T}(0)m_{-}(x,0) \right) \phi(x) \, dx + \int_{0}^{\infty} m_{+}(x,0)\phi(x) \, dx = 0. \]

In particular, these equalities prove (5.1). Moreover, we can commute the derivative \( \partial_{k} \) and the cut-off function \( \theta \) in (5.6). Since \( \mathcal{F} \) is unitary on \( L^{2} \), \( ||\mathcal{F}\phi||_{L^{2}} = ||\phi||_{L^{2}} \). We estimate \( \partial_{k}\mathcal{F}\phi \). Using (5.6) we decompose
\[ \mathcal{F}\phi = \sum_{i=1}^{6} J_{i} + \mathcal{F}_{0}\phi, \quad (5.7) \]
where
\begin{align*}
J_{1} &= \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} e^{-ikx} \theta(k) \left( \mathcal{T}(k)m_{+}(x,k) - 1 \right) \phi(x) \, dx, \\
J_{2} &= \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} e^{-ikx} \theta(-k) \left( m_{+}(x,k) - 1 \right) \phi(x) \, dx, \\
J_{3} &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{0} e^{-ikx} \theta(k) \left( \mathcal{T}(-k)m_{-}(x,-k) - 1 \right) \phi(x) \, dx, \\
J_{4} &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{0} e^{-ikx} \theta(k) \left( m_{-}(x,-k) - 1 \right) \phi(x) \, dx, \\
J_{5} &= \frac{\theta(-k) R_{+}(k)}{\sqrt{2\pi}} \int_{0}^{\infty} e^{ikx} m_{+}(x,-k)\phi(x) \, dx, \\
J_{6} &= \frac{\theta(k) R_{-}(k)}{\sqrt{2\pi}} \int_{-\infty}^{0} e^{ikx} m_{-}(x,k)\phi(x) \, dx.
\end{align*}

We estimate \( ||\partial_{k}J_{1}|| \). We have
\[ ||\partial_{k}J_{1}||_{L^{2}} \leq C \left( \left| \mathcal{T}(k) - 1 \right| \int_{0}^{\infty} e^{-ikx} \left( m_{+}(x,k) - 1 \right) x\phi(x) \, dx \right)_{L^{2}} + C \left( \mathcal{T}(k) - 1 \right) ||\mathcal{F}_{0}\theta(x)\phi(x)||_{L^{2}} + C \left| \mathcal{T}(k) - 1 \right| \left( \int_{0}^{\infty} e^{-ikx} \left( m_{+}(x,k) - 1 \right) x\phi(x) \, dx \right)_{L^{2}} + C \left( \int_{0}^{\infty} e^{-ikx} \left( m_{+}(x,k) - 1 \right) x\phi(x) \, dx \right)_{L^{2}} + C \left( \int_{0}^{\infty} e^{-ikx} \left( m_{+}(x,k) - 1 \right) x\phi(x) \, dx \right)_{L^{2}} + C \left( \int_{0}^{\infty} e^{-ikx} \left( m_{+}(x,k) - 1 \right) x\phi(x) \, dx \right)_{L^{2}}.\]

Then, using (2.10), (4.3) with \( \delta = 0, (4.4) \) with \( \delta > 1/2, (4.17), (4.19) \) and (4.24) we obtain \( ||\partial_{k}J_{1}||_{L^{2}} \leq ||\phi||_{H^{0.1}} \). Similarly, we show that \( ||\partial_{k}J_{l}||_{L^{2}} \leq ||\phi||_{H^{0.1}} \) for \( l = 2, 3, 4, 5, 6 \), and hence, (5.2) follows.

**Lemma 5.2** Suppose that the estimates (2.10), (4.3) with \( \delta = 0, (4.4) \) with \( \delta > 1/2, (4.5) \) with \( \delta = 0, (4.17), (4.18), (4.19) \) and (4.24) hold. Suppose that (5.2) is true. Then, for any \( t \in \mathbb{R} \), the estimates (3.6) and (3.7) are satisfied.

**Proof.** Let us prove (3.6). Due to the unitarity of \( \mathcal{F} \) on \( L^{2} \), \( ||\mathcal{U}(t)\phi||_{L^{2}} = ||\phi||_{L^{2}} \). Since \( \mathcal{U}(t)\phi = \mathcal{F}^{-1}e^{-\frac{it}{2}\mathcal{H}}\mathcal{F}\phi \), to estimate \( ||\partial_{k}\mathcal{U}(t)\phi||_{L^{2}} \) it suffices to prove that
\[ ||\partial_{x}\mathcal{F}^{-1}\phi||_{L^{2}} \leq C ||\phi||_{H^{0.1}} \quad (5.8) \]
and
\[ ||\mathcal{F}\phi||_{H^{0.1}} \leq C ||\phi||_{H^{1}}. \quad (5.9) \]

First, we consider \( \partial_{x}\mathcal{F}^{-1}\phi \). By definition
\[ \partial_{x}\mathcal{F}^{-1}\phi = i\mathcal{F}^{-1}(k\phi(k)) + \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ikx} (\partial_{x}\Phi(x,k)) \phi(k) \, dk. \quad (5.10) \]

By the unitarity of \( \mathcal{F}^{-1} \), the first term in the right-hand side of (5.10) is estimated by \( K ||\phi||_{H^{0.1}} \). So, we turn to the second term. Using (2.11) and (2.14) we get
\[ \Phi(x,k) = e^{-ikx} (\theta(k) T(k) f_{+}(x,k) + \theta(-k) T(-k) f_{-}(x,-k)) \].
Substituting $m_\pm (x, k) = e^{\mp ikx} f_\pm (x, k)$ we get
\[ \Phi (x, k) = \theta (k) T (k) m_+ (x, k) + \theta (-k) T (-k) m_- (x, -k). \] (5.11)

Moreover, using (4.13) for the upper sign we have
\[ \Phi (x, k) = \theta (k) T (k) m_+ (x, k) + \theta (-k) m_+ (x, k) + \theta (-k) R_+ (-k) e^{-2ikx} m_+(x, -k), \] (5.12)
for $x \geq 0$. Using (4.13) for the lower sign in (5.11) we get
\[ \Phi (x, k) = \theta (k) R_- (k) e^{-2ikx} m_- (x, k) + \theta (k) m_- (x, -k) + \theta (-k) T (-k) m_- (x, -k) \] (5.13)
in the case $x \leq 0$. Using (5.12) and (5.13) we have
\[ \partial_x \Phi (x, k) = (\theta (k) T (k) + \theta (-k)) \partial_x m_+ (x, k) + \theta (-k) R_+ (-k) \partial_x (e^{-2ikx} m_+ (x, -k)) \]
for $x \geq 0$ and
\[ \partial_x \Phi (x, k) = \theta (k) R_- (k) \partial_x (e^{-2ikx} m_- (x, k)) + (\theta (k) + \theta (-k) T (-k)) \partial_x m_- (x, -k) \]
for $x \leq 0$. Then,
\[ \int_{-\infty}^{\infty} e^{ikx} (\partial_x \Phi (x, k)) \phi (k) dk = \sum_{j=1}^{6} I_j, \] (5.14)
where
\[ I_1 = \theta (x) \int_{-\infty}^{\infty} e^{ikx} \theta (k) T (k) + \theta (-k) \partial_x m_+ (x, k) \phi (k) dk \]
\[ I_2 = \theta (x) \int_{-\infty}^{\infty} e^{-ikx} \theta (-k) R_+ (-k) \partial_x m_+ (x, -k) \phi (k) dk \]
\[ I_3 = \theta (-x) \int_{-\infty}^{\infty} e^{-ikx} \theta (k) R_- (k) \partial_x m_- (x, k) \phi (k) dk \]
\[ I_4 = \theta (-x) \int_{-\infty}^{\infty} e^{ikx} \theta (k) + \theta (-k) T (-k) \partial_x m_- (x, -k) \phi (k) dk \]
\[ I_5 = -2i\theta (x) \int_{-\infty}^{\infty} e^{-ikx} \theta (-k) R_+ (-k) m_+ (x, -k) k \phi (k) dk \]
\[ I_6 = -2i\theta (-x) \int_{-\infty}^{\infty} e^{-ikx} \theta (k) R_- (k) m_- (x, k) k \phi (k) dk. \]

Using (2.10) and (4.5) with $\delta = 0$ we estimate
\[ \int_{-\infty}^{\infty} \left| (\theta (k) T (k) + \theta (-k)) \partial_x m_+ (x, k) \right| \phi (k) dk \leq C \langle x \rangle^{-2} \int_{-\infty}^{\infty} \langle k \rangle^{-1} | \phi (k) | dk. \]

Then, by Cauchy-Schwarz inequality we get
\[ \| I_1 \|_{L^2} \leq C \| \langle x \rangle^{-2} \|_{L^2} \| \langle k \rangle^{-1} \|_{L^2} \| \phi \|_{L^2} \leq C \| \phi \|_{L^2}. \]

Similarly we show
\[ \| I_j \|_{L^2} \leq C \| \phi \|_{L^2} \]
for $j = 2, 3, 4$. We write $I_5$ as
\[ I_5 = -2i\theta (x) \int_{-\infty}^{\infty} e^{-ikx} \theta (-k) R_+ (-k) (m_+ (x, -k) - 1) k \phi (k) dk \]
\[ -2\sqrt{\pi} i \theta (x) \mathcal{F}_0 (\theta (-k) R_+ (-k) k \phi (k)). \]

By (2.10) and (4.3) with $\delta = 0$ we obtain
\[ \int_{-\infty}^{\infty} \theta (-k) R_+ (-k) | \theta (x) (m_+ (x, -k) - 1) | k \phi (k) dk \leq C \langle x \rangle^{-1} \int_{-\infty}^{\infty} \langle k \rangle^{-1} | k \phi (k) | dk. \]
Then, via Cauchy-Schwartz inequality
\[
\left\| \int_{-\infty}^{\infty} e^{-ikx} \theta (-k) R_+ (-k) \theta (x) (m_+ (x, -k) - 1) k \phi (k) \, dk \right\|_{L^2} \leq C \left\| \langle x \rangle^{-1} \right\|_{L^2} \left\| (k) \right\|_{L^2} \left\| k \phi \right\|_{L^2} \leq C \left\| \phi \right\|_{H^{0.1}}.
\]
Since by \((2.10)\) and Parseval’s identity \(\left\| \theta (x) \mathcal{F}_0 (-k) R_+ (-k) k \phi (k) \right\|_{L^2} \leq \left\| k \phi (k) \right\|_{L^2} \leq \left\| \phi \right\|_{H^{0.1}}\), we deduce
\[
\left\| I_j \right\|_{L^2} \leq C \left\| \phi \right\|_{H^{0.1}}.
\]
Similarly, we show that \(\left\| I_0 \right\|_{L^2} \leq C \left\| \phi \right\|_{H^{0.1}}\). Gathering together the estimates for \(I_j, j = 1, 2, 3, 4, 5, 6\), from \((5.14)\) we prove \((5.8)\). We now consider \((5.9)\). We use \((5.7)\). We write \(k J_1\) as
\[
k J_1 = \theta (k) \left\{ k \left( \frac{\mathcal{F}_0 (\theta (x) \phi (x))}{\sqrt{2\pi}} \right) k \int_{-\infty}^{\infty} e^{-ikx} \left( m_+ (x, k) - 1 \right) \phi (x) \, dx. \tag{5.15}\]
Using \((4.17)\) we estimate
\[
\left\| \theta (k) \left\{ k \left( \frac{\mathcal{F}_0 (\theta (x) \phi (x))}{\sqrt{2\pi}} \right) k \int_{-\infty}^{\infty} e^{-ikx} \left( m_+ (x, k) - 1 \right) \phi (x) \, dx. \right\|_{L^2} \leq C \left\| \mathcal{F}_0 (\theta (x) \phi (x)) \right\|_{L^2} \leq C \left\| \phi \right\|_{L^2}. \tag{5.16}\]
Integrating by parts in the second term of the right-hand side of \((5.15)\) we get
\[
k \int_{0}^{\infty} e^{-ikx} \left( m_+ (x, k) - 1 \right) \phi (x) \, dx = -i \left( m_+ (x, k) - 1 \right) \phi (x) \bigg|_{x=0} - i \int_{0}^{\infty} e^{-ikx} \partial_x \left( \left( m_+ (x, k) - 1 \right) \phi (x) \right) \, dx.
\]
Then, using \((2.10), (4.3)\) and \((4.5)\) with \(\delta = 0\) we estimate
\[
\left\| \theta (k) \left\{ k \left( \frac{\mathcal{F}_0 (\theta (x) \phi (x))}{\sqrt{2\pi}} \right) k \int_{-\infty}^{\infty} e^{-ikx} \left( m_+ (x, k) - 1 \right) \phi (x) \, dx. \right\|_{L^2} \leq C \left\| \mathcal{F}_0 (\theta (x) \phi (x)) \right\|_{L^2} \leq C \left\| \phi \right\|_{L^2}. \tag{5.17}\]
Using \((5.16)\) and \((5.17)\) in \((5.15)\) we show
\[
\left\| J_1 \right\|_{H^{0.1}} \leq C \left\| \phi \right\|_{H^{1}}.
\]
Similarly we prove that \(k J_2, k J_3\) and \(k J_4\) in \((5.7)\) are controlled by \(C \left\| \phi \right\|_{H^{1}}\). We write \(k J_5\) as
\[
k J_5 = k \frac{\theta (-k) R_+ (-k)}{\sqrt{2\pi}} \int_{0}^{\infty} e^{ikx} \left( m_+ (x, -k) - 1 \right) \phi (x) \, dx - i \frac{\theta (-k) R_+ (-k)}{\sqrt{2\pi}} \int_{0}^{\infty} \partial_x e^{ikx} \phi (x) \, dx
\]
\[
= k \frac{\theta (-k) R_+ (-k)}{\sqrt{2\pi}} \int_{0}^{\infty} e^{ikx} \left( m_+ (x, -k) - 1 \right) \phi (x) \, dx + i \frac{\theta (-k) R_+ (-k)}{\sqrt{2\pi}} \phi (0) + i \frac{\theta (-k) R_+ (-k)}{\sqrt{2\pi}} \int_{0}^{\infty} e^{ikx} \partial_x \phi (x) \, dx.
\]
Then, using \((4.3)\) with \(\delta = 0\) and \((4.18)\) we show that
\[
\left\| J_5 \right\|_{H^{0.1}} \leq C \left\| \phi \right\|_{H^{1}}.
\]
Similarly, we estimate \(J_6\). Therefore, as \(k \mathcal{F}_0 \phi = i \mathcal{F}_0 (\partial_x \phi)\), from \((5.7)\) we attain \((5.9)\).

We now turn to \((3.7)\). Observe that
\[
x \mathcal{U} (t) \phi = x \mathcal{F}^{-1} \left( e^{-\frac{ik}{2} k^2} - 1 \right) \mathcal{F} \phi + x \phi = L + x \phi \tag{5.18}\]
with
\[
L = \frac{x}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ikx} \left( \left( e^{-\frac{ik}{2} k^2} - 1 \right) \Phi (x, k) \right) (\mathcal{F} \phi) (k) \, dk.
\]
Using \((5.12)\) and \((5.13)\) we get
\[
L = \theta (x) (L_1 + L_2) + \theta (-x) (L_3 + L_4),
\]
where
\[
L_1 = \frac{x}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ikx} \left( e^{-\frac{ik}{2} k^2} - 1 \right) \left( \theta (k) T (k) + \theta (-k) \right) m_+ (x, k) \left( \mathcal{F} \phi \right) (k) \, dk,
\]
\[
L_2 = \frac{x}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx} \left( e^{-\frac{ik}{2} k^2} - 1 \right) \theta (-k) R_+ (-k) m_+ (x, -k) \left( \mathcal{F} \phi \right) (k) \, dk,
\]
\[
L_3 = \frac{x}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx} \left( e^{-\frac{ik}{2} k^2} - 1 \right) \theta (k) R_+ (k) m_- (x, k) \left( \mathcal{F} \phi \right) (k) \, dk.
\]
\[ L_4 = \frac{x}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ikx} \left( e^{-\frac{ik^2}{2}} - 1 \right) (\theta (k) + \theta (-k)) T (-k)) m_- (x, -k) (F\phi) (k) dk. \]

Integrating by parts in \( L_1 \) we have

\[ L_1 = L_{11} + L_{12} + L_{13} + L_{14}, \quad (5.19) \]

where

\[ L_{11} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ikx} e^{-\frac{ik^2}{2}} (\theta (k) T (k) + \theta (-k)) m_+ (x, k) (F\phi) (k) dk, \]

\[ L_{12} = i \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ikx} \left( e^{-\frac{ik^2}{2}} - 1 \right) \theta (k) \partial_k T (k) m_+ (x, k) (F\phi) (k) dk, \]

\[ L_{13} = i \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ikx} \left( e^{-\frac{ik^2}{2}} - 1 \right) (\theta (k) T (k) + \theta (-k)) \partial_k m_+ (x, k) (F\phi) (k) dk \]

and

\[ L_{14} = i \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ikx} \left( e^{-\frac{ik^2}{2}} - 1 \right) (\theta (k) T (k) + \theta (-k)) m_+ (x, k) \partial_k (F\phi) (k) dk. \]

We estimate \( L_{11} \) by using (2.10) and (4.3) with \( \delta = 0 \)

\[ \left| L_{11} \right| \leq \left| \frac{t}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ikx} e^{-\frac{ik^2}{2}} (\theta (k) T (k) + \theta (-k)) m_+ (x, k) (F\phi) (k) dk \right| \]

\[ + \left| tF_0^{-1} \left( e^{-\frac{ik^2}{2}} (\theta (k) T (k) + \theta (-k)) k (F\phi) (k) \right) dk \right| \]

\[ \leq C \left| \frac{t}{\sqrt{2\pi}} (x)^{-1} \int_{-\infty}^{\infty} \langle k \rangle^{-1} k (F\phi) (k) dk \right| + C \left| tF_0^{-1} \left( e^{-\frac{ik^2}{2}} (\theta (k) T (k) + \theta (-k)) k (F\phi) (k) \right) dk \right|. \]

Then, via Cauchy-Schwarz inequality and (5.9) we get

\[ \| L_{11} \|_{L^2} \leq C t \left( \| (x)^{-1} \|_{L^2} + 1 \right) \| F\phi \|_{H^{0.1}} \leq C \| \phi \|_{H^1}. \quad (5.20) \]

Similarly, by using the estimates (2.10), (4.3) with \( \delta = 0, (4.4) \) with \( \delta > 1/2 \), and (4.19) we prove that

\[ \| L_{12} \|_{L^2} + \| L_{13} \|_{L^2} \leq C \| \phi \|_{H^1}. \quad (5.21) \]

We now study \( L_{14} \). We write

\[ L_{14} = iF_0^{-1} \left( \left( e^{-\frac{ik^2}{2}} - 1 \right) (\theta (k) T (k) + \theta (-k)) \partial_k (F\phi) (k) \right) \]

\[ + i \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ikx} \left( e^{-\frac{ik^2}{2}} - 1 \right) (\theta (k) T (k) + \theta (-k)) (m_+ (x, k) - 1) \partial_k (F\phi) (k) dk. \]

Then,

\[ \left| L_{14} \right| \leq \left| F_0^{-1} \left( \left( e^{-\frac{ik^2}{2}} - 1 \right) (\theta (k) T (k) + \theta (-k)) \partial_k (F\phi) (k) \right) \right| \]

\[ + C \int_{-\infty}^{\infty} \left| (\partial_k (F\phi) (k) \right| \left| \partial_k (F\phi) (k) \right| \right| \right| \]

\[ \leq \left| \partial_k (F\phi) (k) \right| \| L^2 \| + C \int_{-\infty}^{\infty} \left| \partial_k (F\phi) (k) \right| \right| \right| \]

\[ \leq \left| \partial_k (F\phi) (k) \right| \| L^2 \| + C \left| \partial_k (F\phi) (k) \right| \right| \right| \]

Hence, using (5.2) we estimate

\[ \| L_{14} \|_{L^2} \leq C \| \phi \|_{H^{0.1}}. \quad (5.22) \]

Relations (5.19), (5.20), (5.21) and (5.22) imply

\[ \| L_1 \|_{L^2} \leq C \left( \| \phi \|_{H^1} + \| \phi \|_{H^{0.1}} \right) \]

In the same spirit we estimate \( L_2, L_3 \) and \( L_4 \). Hence, from (5.18) we see that

\[ \| u(t) \phi \|_{L^2} \leq C \left( \| \phi \|_{H^1} + \| \phi \|_{H^{0.1}} \right). \]

Using that \( \| u(t) \|_{L^2} = \| \phi \|_{L^2} \), we attain (3.7).
6 Estimates for dilatation $\mathcal{V}$.

This section is devoted to the proof of Lemma 3.3. In the following lemma we study the large-time asymptotics for $\mathcal{V}(t)$. We denote by

$$
\Lambda(x) = \theta(x) R_+(x) + \theta(-x) R_-(x). 
$$

(6.1)

We prove the following:

**Lemma 6.1** Let $a = \pm 1$. Suppose that (2.10), (4.3) and (4.4) with $\delta = 0$, (4.19), (4.22), (4.23) and (4.24) hold. Then, if $a = 1$, the estimate

$$
\| \mathcal{V}(t) \phi - T(|x|) \phi(x) - \Lambda(x) \phi(-x) \|_{L^\infty} \leq C |\phi(0)| + Ct^{-1/4} \|\phi\|_{H^1},
$$

(6.2)

is true for all $t \geq 1$. Moreover, in the case $a = -1$,

$$
\left\| \mathcal{V}(t) \phi - T(|x|) \phi(x) - \Lambda(x) \phi(-x) - \sqrt{\frac{2i}{\pi}} \phi(0) \int_{\sqrt{tx}} e^{-\frac{ik^2}{4}} dk \right\|_{L^\infty} \leq Ct^{-1/4} \|\phi\|_{H^1},
$$

(6.3)

holds for all $t \geq 1$.

**Proof.** We consider first the case $x \geq 0$. Using (5.12) in (2.16) we have

$$
\mathcal{V}(t) \phi = \sqrt{\frac{it}{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{t^2(k-x)^2}{4}} \left( \theta(k) T(k) m_+(tx,k) + \theta(-k) m_+(tx,k) + \theta(-k) R_+(k) e^{-2ikt} m_+(tx,-k) \right) \phi(k) dk.
$$

(6.4)

Let us denote by $\mathcal{V}^+(t)$ to $\mathcal{V}(t)$ in the case when $a = 1$ and by $\mathcal{V}^-(t)$ to $\mathcal{V}(t)$ in the case when $a = -1$. We decompose

$$
\mathcal{V}^\pm(t) \phi = \sum_{j=1}^{3} \mathcal{V}^\pm_j(t),
$$

(6.5)

where

$$
\mathcal{V}^1_1(t) = \sqrt{\frac{it}{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{t^2(k-x)^2}{4}} \theta(k) (T(k) \mp 1) m_+(tx,k) \phi(k) dk,
$$

$$
\mathcal{V}^2_2(t) = \sqrt{\frac{it}{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{t^2(k-x)^2}{4}} \theta(k) R_+(k) m_+(tx,k) \phi(k) dk,
$$

$$
\mathcal{V}^3_3(t) = \sqrt{\frac{it}{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{t^2(k-x)^2}{4}} (\theta(-k) \pm \theta(k)) m_+(tx,k) \phi(k) dk.
$$

We first study $\mathcal{V}^1_1(t)$. Using that $\int_{-\infty}^{\infty} e^{-\frac{t^2(k-x)^2}{4}} dk = \sqrt{\frac{2\pi}{it}}$, we decompose $\mathcal{V}^1_1(t)$ as

$$
\mathcal{V}^1_1(t) = (T(x) \mp 1) \phi(x) + \mathcal{V}^1_{11}(t) + \mathcal{V}^1_{12}(t),
$$

(6.6)

with

$$
\mathcal{V}^1_{11}(t) = (T(x) \mp 1) (m_+(tx,x) - 1) \phi(x),
$$

$$
\mathcal{V}^1_{12}(t) = \sqrt{\frac{it}{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{t^2(k-x)^2}{4}} \Phi_1(x,k) dk
$$

and

$$
\Phi_1(x,k) = \theta(k) (T(k) \mp 1) m_+(tx,k) \phi(k) - (T(x) \mp 1) m_+(tx,x) \phi(x).
$$

From (4.3) $\delta = 0$ and (4.22) it follows

$$
|\mathcal{V}^1_{11}(t)| \leq C |x| (tx)^{-1} \|\phi\|_{L^\infty} \leq \frac{C}{t} \|\phi\|_{H^1}.
$$

(6.7)
Now, we estimate $\mathcal{V}_{12}^\pm (t)$. Using the identity

$$e^{-\frac{i}{2} (k-x)^2} = B \partial_k \left( (k-x) e^{-\frac{i}{2} (k-x)^2} \right)$$

with $B := \left( 1 - it (k-x)^2 \right)^{-1}$, integrating by parts in the definition of $\mathcal{V}_{12} (t)$ and using $(k-x) \partial_k B = 2it (k-x)^2 B^2$ we have

$$\mathcal{V}_{12}^\pm (t) = \sqrt{\frac{it}{2\pi}} \int_\infty^{-\infty} e^{-\frac{i}{2} (k-x)^2} (k-x) B \partial_k \Phi (x,k) \, dk + 2it \sqrt{\frac{it}{2\pi}} \int_\infty^{-\infty} e^{-\frac{i}{2} (k-x)^2} (k-x)^2 B^2 \Phi (x,k) \, dk.$$

It follows from (2.10), (4.3), (4.4), (4.19) and

$$|\phi (k) - \phi (x)| \leq C |k-x|^\frac{3}{2} \| \partial_k \phi \|_{L^2}$$

that

$$|\Phi (x,k)| \leq C |k-x| |\phi (k)| + C |k-x|^{\frac{3}{2}} \| \partial_k \phi \|_{L^2}$$

and

$$|\partial_k \Phi (x,k)| \leq C (|\phi (k)| + |\partial_k \phi (k)|).$$

Then, we estimate

$$|\mathcal{V}_{12}^\pm (t)| \leq C \sqrt{\frac{t}{\pi}} \int_\infty^{-\infty} \frac{|k-x||\phi (k)| + |\partial_k \phi (k)|}{1 + it(k-x)^2} \, dk + C t \sqrt{\frac{t}{\pi}} \int_\infty^{-\infty} \frac{|(k-x)^2 |\phi (k)| + |k-x|^{\frac{3}{2}} \| \partial_k \phi \|_{L^2}}{(1 + it(k-x)^2)^2} \, dk$$

$$\leq C \sqrt{\frac{t}{\pi}} \int_\infty^{-\infty} \frac{|k-x||\phi (k)| + |\partial_k \phi (k)|}{1 + it(k-x)^2} \, dk + C \sqrt{\frac{t}{\pi}} \| \phi \|_{H^1} \int_\infty^{-\infty} \frac{|k-x|^{\frac{3}{2}}}{1 + it(k-x)^2} \, dk + C \int_\infty^{-\infty} \frac{|\phi (k)|}{(1 + it(k-x)^2)^{\frac{3}{4}}} \, dk.$$

Hence, via Cauchy-Schwarz inequality we obtain

$$|\mathcal{V}_{12}^\pm (t)| \leq C \sqrt{\frac{t}{\pi}} \left( \int_\infty^{-\infty} \frac{(k-x)^2}{(1 + it(k-x)^2)^2} \, dk \right)^{1/2} \left( \int_\infty^{-\infty} \left( |\phi (k)|^2 + |\partial_k \phi (k)|^2 \right) \, dk \right)^{1/2}$$

$$+ C \| \phi \|_{H^1} \sqrt{\frac{t}{\pi}} \int_\infty^{-\infty} \frac{|k|^{\frac{3}{2}}}{1 + itk^2} \, dk + C \left( \int_\infty^{-\infty} \frac{dk}{1 + itk^2} \right)^{\frac{3}{4}} \| \phi \|_{L^2}$$

$$\leq C \| \phi \|_{H^1} \sqrt{\frac{t}{\pi}} \left( \int_\infty^{-\infty} \frac{k^2}{(1 + itk^2)^2} \, dk \right)^{1/2} + C \int_\infty^{-\infty} \frac{|k|^{\frac{3}{2}}}{1 + itk^2} \, dk + C t^{-1/4} \| \phi \|_{L^2} \leq C t^{-1/4} \| \phi \|_{H^1}.$$
with
\[
\mathcal{V}_{31}^+ = \sqrt{\frac{it}{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{it}{4t}(k-x)^2} (m_+ (tx,k) \phi(k) - m_+ (tx,x) \phi(x)) \, dk.
\]

For \( \mathcal{V}_{3}^- \) we have
\[
\mathcal{V}_{3}^- (t) = \phi(x) \sqrt{\frac{it}{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{it}{4t}(k-x)^2} \left( \theta(-k) - \theta(k) \right) \, dk + \mathcal{V}_{31}^+ (t) + \mathcal{V}_{32}^- (t) + \mathcal{V}_{33}^- (t),
\]  

and thus,
\[
\mathcal{V}_{31}^- (t) = (m_+ (tx,x) - 1) \phi(x) \sqrt{\frac{it}{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{it}{4t}(k-x)^2} \left( \theta(-k) - \theta(k) \right) \, dk
\]

and
\[
\mathcal{V}_{32}^- (t) = -\sqrt{\frac{it}{2\pi}} \int_{0}^{\infty} e^{-\frac{it}{4t}(k-x)^2} (m_+ (tx,k) \phi(k) - m_+ (tx,x) \phi(x)) \, dk.
\]

Similarly to (6.10) we show that
\[
|\mathcal{V}_{31}^+ (t)| \leq Ct^{-1/4} \|\phi\|_{H^1},
\]

and then,
\[
|\mathcal{V}_{3}^- (t) - m_+ (tx,x) \phi(x)| \leq Ct^{-1/4} \|\phi\|_{H^1}.
\]  

Using (4.3) we get
\[
|\mathcal{V}_{31}^- (t)| \leq C \left| (m_+ (tx,x) - 1) \int_{-\sqrt{t\pi}}^{\sqrt{t\pi}} e^{-\frac{1}{4t}k^2} \, dk \right| \|\phi\|_{H^1} \leq \frac{C \|\phi\|_{H^1}}{\sqrt{t}}.
\]  

Taking into account identity (6.8) and integrating by parts in \( \mathcal{V}_{32}^- (t) \) we have
\[
\mathcal{V}_{32}^- (t) = -\sqrt{\frac{it}{2\pi}} \int_{0}^{\infty} e^{-\frac{it}{4t}(k-x)^2} \left( m_+ (tx,0) \phi(0) - m_+ (tx,x) \phi(x) \right)
\]

\[
+ \sqrt{\frac{it}{2\pi}} \int_{0}^{\infty} \left( k-x \right) e^{-\frac{it}{4t}(k-x)^2} \partial_k \left( B(m_+ (tx,k) \phi(k) - m_+ (tx,x) \phi(x)) \right) \, dk.
\]

Using (4.4) and (6.9), with \( k = 0 \), we estimate
\[
|m_+ (tx,0) \phi(0) - m_+ (tx,x) \phi(x)| \leq C \|\phi\|_{H^1}.
\]

Then,
\[
\left| \sqrt{\frac{it}{2\pi}} \int_{0}^{\infty} e^{-\frac{it}{4t}(k-x)^2} \left( m_+ (tx,0) \phi(0) - m_+ (tx,x) \phi(x) \right) \right| \leq C \|\phi\|_{H^1} \sqrt{\frac{|x|^3}{(1 + t x^2)^{3/2}}} \leq Ct^{-1/4} \|\phi\|_{H^1}.
\]

Moreover, similarly to (6.10) we control
\[
\left| \sqrt{\frac{it}{2\pi}} \int_{0}^{\infty} \left( k-x \right) e^{-\frac{it}{4t}(k-x)^2} \partial_k \left( B(m_+ (tx,k) \phi(k) - m_+ (tx,x) \phi(x)) \right) \, dk \right| \leq Ct^{-1/4} \|\phi\|_{H^1},
\]

and thus,
\[
|\mathcal{V}_{32}^- (t)| \leq Ct^{-1/4} \|\phi\|_{H^1}.
\]  

Similarly we get
\[
|\mathcal{V}_{33}^- (t)| \leq Ct^{-1/4} \|\phi\|_{H^1}.
\]
Using (6.15), (6.16) and (6.17) in (6.13) we get
\[
\left| \mathcal{V}_3^- (t) - \phi (x) \right| \leq Ct^{-1/4} \| \phi \|_{H^1}.
\] (6.18)

Introducing (6.11), (6.12), (6.14) and (6.18) into (6.5) we obtain
\[
\left| \mathcal{V}^+ (t) \phi - T (x) \phi (x) - R_+ (x) \phi (-x) - (m_+ (tx, x) - 1) \phi (x) \right| \leq Ct^{-1/4} \| \phi \|_{H^1}.
\] (6.19)

and
\[
\left| \mathcal{V}^- (t) \phi - T (x) \phi (x) - R_- (x) \phi (-x) - \phi (x) \right| \leq Ct^{-1/4} \| \phi \|_{H^1}.
\] (6.20)

Using (4.3) and (6.9) with \( k = 0 \), we get
\[
\left| (m_+ (tx, x) - 1) (\phi (x) - \phi (0)) \right| \leq C \sqrt{t} \| \partial_k \phi \|_{L^2}.
\]

Also, by (4.3) \( |m_+ (tx, x) - 1| \leq C \). Using the last two inequalities in (6.19), we arrive to (6.2) in the case \( x \geq 0 \). Using (6.9), with \( k = 0 \), to estimate
\[
\left| \phi (x) - \phi (0) \right| + \sqrt{t} \int_{-\infty}^{\infty} \sqrt{t} e^{-4(k-x)^2} \theta (-k) - \theta (k) \, dk \leq Ct^{-1/2} \| \phi \|_{H^1},
\]

and noting that
\[
1 + \sqrt{t} \int_{-\infty}^{\infty} e^{-4(k-x)^2} \theta (-k) - \theta (k) \, dk = \sqrt{\frac{2t}{\pi}} \int_{\sqrt{tx}}^{\infty} e^{-\frac{4}{k^2}} \, dk,
\]

we deduce (6.3) in the case \( x \geq 0 \).

We consider now the case \( x \leq 0 \). Introducing (5.13) into (2.16) we have
\[
\mathcal{V} (t) \phi = \sqrt{\frac{2i}{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{4}{k^2} (k-x)^2} \theta (k) T (k) m_- (tx, k) \phi (-k) \, dk
+ \sqrt{\frac{2i}{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{4}{k^2} (k-x)^2} \theta (k) R_+ (k) m_+ (tx, k) \phi (k) \, dk
+ \sqrt{\frac{2i}{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{4}{k^2} (k-x)^2} \theta (-k) m_- (tx, k) \phi (-k) \, dk
\]

Therefore, proceeding similarly to the case of \( x \geq 0 \), we obtain the result for \( x \leq 0 \).  \[ \Box \]

Next we estimate the derivative \( \partial_x \mathcal{V} (t) \).

**Lemma 6.2** Suppose that the estimates (4.4) with \( \delta > \frac{1}{2} \), (4.5) and (4.6) with \( \delta = 0 \) are true. Also, assume that (2.10), (4.3), (4.19), (4.22), (4.23) and (4.24), are satisfied. Moreover, if \( a = 1 \), let \( \phi \in H^1 \) be odd and if \( a = -1 \), suppose that \( \phi \) is even. Moreover, suppose that \( \phi \) can be represented as \( \phi = F \psi \), for some \( \psi \in H^{0,1} \). Then, the estimate
\[
\| \partial_x \mathcal{V} (t) \phi \|_{L^2} \leq C \| \phi \|_{L^\infty} \log (t) + C \| \phi \|_{H^1}
\] (6.21)
is true for all \( t \geq 1 \).

**Proof.** We consider the case of \( x \geq 0 \). We depart from relation (6.4). We denote
\[
\Theta (tx, k) = \theta (k) m_+ (tx, k) (T (k) \phi (k) + R_+ (k) \phi (-k)) + \theta (-k) m_- (tx, k) \phi (k).
\]

Then,
\[
\mathcal{V} (t) \phi = \sqrt{\frac{it}{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{4}{k^2} (k-x)^2} \Theta (tx, k) \, dk.
\] (6.22)
Taking the derivative with respect to $x$ in the last relation we obtain
\[
\partial_x V(t) \phi = -\sqrt{\frac{it}{2\pi}} \int_{-\infty}^{\infty} \partial_k e^{-\frac{it}{4}(k-x)^2} \Theta (tx, k) \, dk + \sqrt{\frac{it}{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{it}{4}(k-x)^2} \partial_x (\Theta (tx, k)) \, dk. \tag{6.23}
\]
Integrating by parts in the first term of the right-hand side of (6.23) and using (5.1) we get
\[
\partial_x V(t) \phi = I_1 + I_2, \tag{6.24}
\]
where
\[
I_1 = \sqrt{\frac{it}{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{it}{4}(x-k)^2} \partial_k \Theta (tx, k) \, dk
\]
and
\[
I_2 = \sqrt{\frac{it}{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{it}{4}(x-k)^2} \partial_x (\Theta (tx, k)) \, dk.
\]
We begin by estimating $I_1$. We split $I_1$ as follows
\[
I_1 = \sum_{j=1}^{4} I_{1j}, \tag{6.25}
\]
where
\[
I_{1j} = \sqrt{\frac{it}{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{it}{4}(x-k)^2} \Theta_j (k) \, dk = \mathcal{V}_0 (t) \Theta_j,
\]
$j = 1, 2,$ and
\[
I_{1j} = \sqrt{\frac{it}{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{it}{4}(x-k)^2} \Theta_j (tx, k) \, dk,
\]
$j = 3, 4,$ with
\[
\Theta_1 (k) = \theta (k) ((\partial_k T (k)) \phi (k) + \partial_k R^+ (k) \phi (-k)),
\]
\[
\Theta_2 (k) = \theta (k) T (k) \partial_k \phi (k) + \theta (k) R^+ (k) \partial_k \phi (-k) + \theta (-k) \partial_k \phi (k),
\]
\[
\Theta_3 (tx, k) = (\partial_k m^+ (tx, k)) \theta (k) T (k) \phi (k) + \theta (k) R^+ (k) \phi (-k) + \theta (-k) \phi (k) + (m^+ (tx, k) - 1) \Theta_1 (k),
\]
and
\[
\Theta_4 (tx, k) = (m^+ (tx, k) - 1) \Theta_2 (k).
\]
By using (2.5) we estimate
\[
\|I_{1j}\|_{L^2(\mathbb{R}^+)} = \|\mathcal{V}_0 (t) \Theta_j\|_{L^2(\mathbb{R}^+)} \leq C \|\Theta_j\|_{L^2}, \tag{6.26}
\]
for $j = 1, 2$. Relations (4.19) and (4.24) imply
\[
\|\Theta_1\|_{L^2} \leq C (\|\partial_k T\|_{L^\infty} + \|\partial_k R\|_{L^\infty}) \|\phi\|_{L^2} \leq C \|\phi\|_{L^2}, \tag{6.27}
\]
Moreover, using (2.10) we estimate
\[
\|\Theta_2\|_{L^2} \leq C (1 + \|T\|_{L^\infty} + \|R\|_{L^\infty}) \|\partial_k \phi\|_{L^2} \leq C \|\partial_k \phi\|_{L^2}. \tag{6.28}
\]
Hence, it follows from (6.26), (6.27), (6.28) that
\[
\|I_{1j}\|_{L^2(\mathbb{R}^+)} \leq C \|\phi\|_{H^1}, \tag{6.29}
\]
for $j = 1, 2$. We use (2.10), (4.3), (4.4) with $\delta > \frac{1}{2}$, (4.19) and (4.24) to control $\Theta_3$. We obtain
\[
|\Theta_3 (tx, k)| \leq C |\partial_k m^+ (tx, k)| \left( \|T\|_{L^\infty} + \|R\|_{L^\infty} + 1 \right) (|\phi (k)| + |\phi (-k)|)
\]
\[
+ C |m^+ (tx, k) - 1| \left( \|\partial_k T\|_{L^\infty} + \|\partial_k R\|_{L^\infty} \right) (|\phi (k)| + |\phi (-k)|)
\]
\[
\leq C \langle tx \rangle^{-\delta} \langle k \rangle^{-1} (|\phi (k)| + |\phi (-k)|).
\]
Hence, via Cauchy-Schwartz inequality we derive
\[
\|I_{13}\|_{L^2(\mathbb{R}^+)} = \left\| \sqrt{\frac{it}{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{it}{4}(x-k)^2} \Theta_3 (tx, k) \, dk \right\|_{L^2(\mathbb{R}^+)} \leq C t^{\frac{\delta}{2}} \left\| \langle tx \rangle^{-\delta} \int_{-\infty}^{\infty} \langle k \rangle^{-1} (|\phi (k)| + |\phi (-k)|) \, dk \right\|_{L^2(\mathbb{R}^+)}
\]
\[
\leq C t^{\frac{\delta}{2}} \left\| \langle tx \rangle^{-\delta} \right\|_{L^2(\mathbb{R}^+)} \left\| \langle k \rangle^{-1} \right\|_{L^2} \|\phi\|_{L^2} \leq C \|\phi\|_{L^2}. \tag{6.30}
\]
In the same manner, by using (2.10) and (4.3) we find

\[ |\Theta_4 (x, k)| \leq C |m_+ (tx, k) - 1| (\|T\|_{L^\infty} + \|R\|_{L^\infty} + 1) (|\partial_k \phi (k)| + |\partial_k \phi (-k)|) \]

\[ \leq C \langle tx \rangle^{-1} \langle k \rangle^{-1} (|\partial_k \phi (k)| + |\partial_k \phi (-k)|) , \]

and thus,

\[ \|I_{14}\|_{L^2 (\mathbb{R}^+)} = C t \frac{i}{2\pi} \int_0^{\infty} e^{-\frac{it}{2} (x-k)^2} \Theta_4 (tx, k) \, dk \]

\[ \leq C t \frac{i}{2\pi} \int \langle tx \rangle^{-1} \langle k \rangle^{-1} (|\partial_k \phi (k)| + |\partial_k \phi (-k)|) \, dk \]

\[ \leq C t \frac{i}{2\pi} \int_0^{\infty} \langle tx \rangle^{-1} \|\partial_k \phi\|_{L^2} \leq C \|\partial_k \phi\|_{L^2} . \]

(6.31)

Therefore, from (6.25), (6.29), (6.30) and (6.31) we conclude that

\[ \|I_1\|_{L^2 (\mathbb{R}^+)} \leq C \|\phi\|_{H^4} . \]

(6.32)

Next, we turn to the term \( I_2 \). We have

\[ I_2 = \sqrt{\frac{i}{2\pi}} t^{\frac{3}{2}} \int_0^{2\pi} e^{-\frac{it}{2} (x-k)^2} (T (k) \phi (k) + R_+ (k) \phi (-k)) (\partial_x m_+) (tx, k) \, dk \]

\[ + \sqrt{\frac{i}{2\pi}} t^{\frac{3}{2}} \int_{2\pi}^{\infty} e^{-\frac{it}{2} (x-k)^2} (T (k) \phi (k) + R_+ (k) \phi (-k)) (\partial_x m_+) (tx, k) \, dk \]

\[ + \sqrt{\frac{i}{2\pi}} t^{\frac{3}{2}} \int_{-\infty}^{0} e^{-\frac{it}{2} (x-k)^2} (\partial_x m_+) (tx, k) \phi (k) \, dk . \]

Making the change of variables \( k = 2x - k' \) in the third integral of the right-hand side of the last relation we decompose

\[ I_2 = I_{21} + I_{22} + I_{23} , \]

(6.33)

where

\[ I_{21} = \sqrt{\frac{i}{2\pi}} t^{\frac{3}{2}} \int_0^{2\pi} e^{-\frac{it}{2} (x-k)^2} \Theta_5 (t, x, k) \, dk \]

with

\[ \Theta_5 (t, x, k) = (T (k) \phi (k) + R_+ (k) \phi (-k)) (\partial_x m_+) (tx, k) , \]

and

\[ I_{22} = \sqrt{\frac{i}{2\pi}} t^{\frac{3}{2}} \int_{2\pi}^{\infty} e^{-\frac{it}{2} (x-k)^2} \Theta_6 (t, x, k) \, dk , \]

with

\[ \Theta_6 (t, x, k) = (\partial_x m_+) (tx, 2x-k) (\phi (2x-k) - \phi (-k)) \]

and

\[ I_{23} = \sqrt{\frac{i}{2\pi}} t^{\frac{3}{2}} \int_{-\infty}^{0} e^{-\frac{it}{2} (x-k)^2} \Theta_7 (t, x, k) \, dk , \]

with

\[ \Theta_7 (t, x, k) = (T (k) \phi (k) + R_+ (k) \phi (-k)) (\partial_x m_+) (tx, k) + (\partial_x m_+) (tx, 2x-k) \phi (-k) . \]

Using (2.10) and (4.5) with \( \delta = 0 \) we get

\[ |\Theta_5 (t, x, k)| \leq C |(\partial_x m_+) (tx, k)| (\|T\|_{L^\infty} + \|R\|_{L^\infty} (|\phi (k)| + |\phi (-k)|)) \leq C \langle tx \rangle^{-2} \|\phi\|_{L^\infty} . \]

Thus, we obtain

\[ |I_{21}| \leq C t^{\frac{3}{2}} \int_0^{2\pi} |\Theta_5 (t, x, k)| \, dk \leq C t^{\frac{3}{2}} |x| \langle tx \rangle^{-2} \|\phi\|_{L^\infty} , \]

and therefore

\[ \|I_{21}\|_{L^2 (\mathbb{R}^+)} \leq C t^{\frac{3}{2}} \|\phi\|_{L^\infty} \|x\| \langle tx \rangle^{-2} \|\|_{L^2 (\mathbb{R}^+)} \leq C \|\phi\|_{L^\infty} . \]

(6.34)

By Plancherel’s theorem we have

\[ \|\phi (2x-k) - \phi (-k)\|_{L^2} = \| (e^{2ixy} - 1) (F_0 \phi) (y) \|_{L^2} \leq C \|x\| \|y (F_0 \phi) (y)\|_{L^2} = C \|x\| \|\partial_k \phi\|_{L^2} . \]
Thus, from (6.37) and (6.38)

\[ |I_{22}| \leq Ct^\frac{7}{2} \int_{2x}^{\infty} |\Theta_{0}(t,x,k)| \, dk \leq Ct^\frac{7}{2} \int_{2x}^{\infty} |(\partial_x m_+)(tx,2x-k)| \, d|\phi(2x-k) - \phi(-k)| \, dk \]

\[ \leq Ct^\frac{7}{2} \langle tx \rangle^{-2} \int_{2x}^{\infty} (k-2x)^{-1} |\phi(2x-k) - \phi(-k)| \, dk \]

\[ \leq Ct^\frac{7}{2} \langle tx \rangle^{-2} \|k|^{-1} \|\phi(2x-k) - \phi(-k)\|_{L^2} \leq Ct^\frac{7}{2} \langle tx \rangle^{-1} \|\partial_x\phi\|_{L^2} . \]

Hence

\[ \|I_{22}\|_{L^2(R^+)} \leq Ct^\frac{7}{2} \|\partial_x\phi\|_{L^2} \|\langle tx \rangle^{-1} \|_{L^2(R^+)} \leq C \|\partial_x\phi\|_{L^2} . \] (6.35)

Integrating by parts in \( I_{23} \) we find

\[ I_{23} = -\frac{7}{\sqrt{2\pi}} \int_{2x}^{\infty} \partial_x e^{-\frac{7}{2}(x-k)^2} (k-x)^{-1} \Theta_{0}(t,x,k) \, dk = \frac{7}{\sqrt{2\pi}} \frac{e^{-\frac{7}{2}(x-k)^2}}{x} \Theta_{0}(t,x,2x) \]

\[ -\frac{7}{\sqrt{2\pi}} \int_{2x}^{\infty} e^{-\frac{7}{2}(x-k)^2} (k-x)^{-2} \Theta_{0}(t,x,k) \, dk + \frac{7}{\sqrt{2\pi}} \int_{2x}^{\infty} e^{-\frac{7}{2}(x-k)^2} (k-x)^{-1} \partial_x \Theta_{0}(t,x,k) \, dk, \] (6.36)

Let us now estimate \( \Theta_{0}(t,x,k) \). First, observe that by (4.6) with \( \delta = 0 \)

\[ |(\partial_x m_+)(tx,2x-k) - (\partial_x m_+)(tx,k)| \leq \int_{k}^{2x-k} (\partial_x \partial_x m_+)(tx,k') \, dk' \leq C |k-x| \langle tx \rangle^{-1} \leq C \frac{|k-x|}{|k-x|} \langle tx \rangle^{-1} , \] (6.37)

for \(|k-x| \leq 1\). Moreover, for \(|k-x| \geq 1\), by (4.5) with \( \delta = 0 \) we have

\[ |(\partial_x m_+)(tx,2x-k) - (\partial_x m_+)(tx,k)| \leq |(\partial_x m_+)(tx,2x-k)| + |(\partial_x m_+)(tx,k)| \leq C \frac{|k-x|}{|k-x|} \langle tx \rangle^{-2} . \] (6.38)

Thus, from (6.37) and (6.38)

\[ |(\partial_x m_+)(tx,2x-k) - (\partial_x m_+)(tx,k)| \leq C \frac{|k-x|}{|k-x|} \langle tx \rangle^{-1} . \] (6.39)

If \( a = 1 \), \( \phi \) is odd. When \( a = -1 \), \( \phi \) is supposed to be even. Then,

\[ \Theta_{0}(t,x,k) = ((T(k) \mp 1) \phi(k) + R_{+}(k) \phi(-k)) (\partial_x m_+)(tx,k) + ((\partial_x m_+)(tx,2x-k) - (\partial_x m_+)(tx,k)) \phi(-k) , \] (6.40)

for \( a = \pm 1 \). By (2.10), (4.5) with \( \delta = 0 \), (4.22) and (4.23) we estimate

\[ |(\partial_x m_+)(tx,k)| (|T(k) \mp 1| \phi(k) + |R_{+}(k)| \phi(-k)|) \leq C \langle tx \rangle^{-1} |k| (|\phi(k)| + |\phi(-k)|) . \] (6.41)

From (6.39) we see that

\[ |(\partial_x m_+)(tx,2x-k) - \partial_x m_+)(tx,k)| \phi(k) \leq C |k-x| \langle tx \rangle^{-1} |\phi(k)| . \] (6.42)

Thus, by (6.40), (6.41) and (6.42) we deduce

\[ |\Theta_{0}(t,x,k)| \leq |(\partial_x m_+)(tx,k)| (|T(k) \mp 1| \phi(k) + |R_{+}(k)| \phi(-k)|) + |(\partial_x m_+)(tx,2x-k) - \partial_x m_+)(tx,k)| \phi(-k)| \]

\[ \leq C \langle tx \rangle^{-1} (|k| + |k-x|) (|\phi(k)| + |\phi(-k)|) , \]

for \( a = \pm 1 \). Therefore, we get

\[ \frac{7}{\sqrt{2\pi}} \frac{e^{-\frac{7}{2}(x-k)^2}}{x} \Theta_{0}(t,x,2x) + \frac{7}{\sqrt{2\pi}} \int_{2x}^{\infty} e^{-\frac{7}{2}(x-k)^2} (k-x)^{-2} \Theta_{0}(t,x,k) \, dk \]

\[ \leq Ct^\frac{1}{2} \langle t \rangle \Theta_{0}(t,x,2x) + Ct^\frac{1}{2} \int_{2x}^{\infty} (k-x)^{-2} \Theta_{0}(t,x,k) \, dk \]

\[ \leq Ct^\frac{1}{2} \langle tx \rangle^{-1} \|\phi\|_{L^\infty} + Ct^\frac{1}{2} \langle tx \rangle^{-1} \int_{2x}^{\infty} (k-x)^{-1} (|\phi(k)| + |\phi(-k)|) \, dk \]

\[ \leq Ct^\frac{1}{2} \langle tx \rangle^{-1} (\|\phi\|_{L^\infty} + \|\phi\|_{L^2}) + Ct^\frac{1}{2} \langle tx \rangle^{-1} \|\phi\|_{L^\infty} \log (1 + |k|^{-1}) . \]
Hence,
\[
\left\| \frac{t^\frac{1}{2}}{\sqrt{2\pi}} \int \frac{e^{-\frac{t^2}{2}x^2} \Theta_k (t, x, 2x)}{x} \right\|_{L^2(\mathbb{R}^+)} + \left\| \frac{t^\frac{1}{2}}{\sqrt{2\pi}} \int e^{-\frac{t}{2}(x-k)^2} (k-x)^{-2} \Theta_k (t, x, k) \text{d}k \right\|_{L^2(\mathbb{R}^+)} \\
\leq C t^\frac{1}{2} \left( \|\phi\|_{L^\infty} + \|\phi\|_{L^2} \right) \left\| \langle tx \rangle^{-1} \right\|_{L^2(\mathbb{R}^+)} + C t^\frac{1}{2} \|\phi\|_{L^\infty} \left\| \langle tx \rangle^{-1} \log \left( 1 + |x|^{-1} \right) \right\|_{L^2(\mathbb{R}^+)} \\
\leq C \|\phi\|_{L^\infty} \log (t) + C \|\phi\|_{L^2}.
\]

(6.43)

Derivating (6.40) with respect to \(k\) we get
\[
\partial_k \Theta_k (t, x, k) = ((T (k) \mp 1) \phi (k) + R_+ (k) \phi (-k)) (\partial_k T x_m) (t, x, k) \\
+ (\partial_k T (k) \phi (k) + \partial_k R_+ (k) \phi (-k)) (\partial_k T x_m) (t, x, k) \\
+ ((T (k) \mp 1) \partial_k \phi (k) - R_+ (k) \partial_k \phi (-k)) (\partial_k T x_m) (t, x, k) \\
- ((\partial_k T x_m) (t, 2x - k) - (\partial_k T x_m) (t, k)) (\partial_k \phi) (-k) \\
- ((\partial_k \partial_k T x_m) (t, 2x - k) + (\partial_k \partial_k T x_m) (t, k)) (\partial_k \phi) (-k).
\]

(6.44)

Using (2.10) and (4.6) with \(\delta = 0\) we get
\[
\left| [(T (k) \mp 1) \phi (k) + R_+ (k) \phi (-k)) (\partial_k \partial_k T x_m) (t, x, k)] \leq C \langle tx \rangle^{-1} (|\phi (k)| + |\phi (-k)|) \right.
\]

(6.45)

From (4.5) with \(\delta = 0\), (4.19) and (4.24) we derive
\[
\left| [(\partial_k T (k) \phi (k) + \partial_k R_+ (k) \phi (-k)) (\partial_k T x_m) (t, x, k)] \leq C \langle tx \rangle^{-1} (|\phi (k)| + |\phi (-k)|) \right.
\]

(6.46)

By (2.10), (4.5) with \(\delta = 0\), (4.22) and (4.23)
\[
\left| [(T (k) \mp 1) \partial_k \phi (k) - R_+ (k) \partial_k \phi (-k)) (\partial_k T x_m) (t, x, k)] \leq C \langle k \rangle \langle tx \rangle^{-1} (|\partial_k \phi (k)| + |\partial_k \phi (-k)|) \right.
\]

(6.47)

From (6.39) we get
\[
\left| [(\partial_k T x_m) (t, 2x - k) - (\partial_k T x_m) (t, x, k)] \partial_k \phi (-k) \right| \leq C \frac{[k - x]}{(k - x)} \langle tx \rangle^{-1} |\partial_k \phi (k)|.
\]

(6.48)

Finally, by (4.6) with \(\delta = 0\) we estimate
\[
\left| [(\partial_k \partial_k T x_m) (t, 2x - k) + (\partial_k \partial_k T x_m) (t, x, k)] \phi (-k)] \leq C \langle tx \rangle^{-1} |\phi (-k)|.
\]

(6.49)

Using (6.45)-(6.49) in (6.44) we deduce
\[
\left| \partial_k \Theta_k (t, x, k) \right| \leq C \langle tx \rangle^{-1} (|\phi (k)| + |\phi (-k)|) + C \left( \frac{|k|}{(k - x)} \right) \langle tx \rangle^{-1} (|\partial_k \phi (k)| + |\partial_k \phi (-k)|).
\]

Hence,
\[
\left\| \frac{t^\frac{1}{2}}{\sqrt{2\pi}} \int \frac{e^{-\frac{t^2}{2}x^2} \Theta_k (t, x, 2x)}{x} \right\|_{L^2(\mathbb{R}^+)} + \left\| \frac{t^\frac{1}{2}}{\sqrt{2\pi}} \int e^{-\frac{t}{2}(x-k)^2} (k-x)^{-2} \Theta_k (t, x, k) \text{d}k \right\|_{L^2(\mathbb{R}^+)} \\
\leq C t^\frac{1}{2} \langle tx \rangle^{-1} \int \left| \left( \frac{|k|}{(k - x)} \right) \langle tx \rangle^{-1} (|\partial_k \phi (k)| + |\partial_k \phi (-k)|) \right| \text{d}k
\]

and then
\[
\left\| \frac{t^\frac{1}{2}}{\sqrt{2\pi}} \int \frac{e^{-\frac{t^2}{2}x^2} \Theta_k (t, x, 2x)}{x} \right\|_{L^2(\mathbb{R}^+)} + \left\| \frac{t^\frac{1}{2}}{\sqrt{2\pi}} \int e^{-\frac{t}{2}(x-k)^2} (k-x)^{-2} \Theta_k (t, x, k) \text{d}k \right\|_{L^2(\mathbb{R}^+)} \\
\leq C t^\frac{1}{2} \|\phi\|_{H^1} \left\| \langle tx \rangle^{-1} \right\|_{L^2(\mathbb{R}^+)} + C t^\frac{1}{2} \|\phi\|_{L^\infty} \left\| \langle tx \rangle^{-1} \log \left( 1 + |x|^{-1} \right) \right\|_{L^2(\mathbb{R}^+)} \\
\leq C \|\phi\|_{L^\infty} \log (t) + C \|\phi\|_{H^1}.
\]

(6.50)

Using (6.43) and (6.50) in (6.36) we arrive to
\[
\|I_{23}\|_{L^2(\mathbb{R}^+)} \leq C \left( \|\phi\|_{L^\infty} \log (t) + \|\phi\|_{H^1} \right).
\]

(6.51)
Introducing the estimates (6.34), (6.35) and (6.51) into (6.33) we obtain
\[ \|I_2\|_{L^2(R^+)} \leq C (\|\phi\|_{L^\infty} \log(t) + \|\phi\|_{H^1}), \] (6.52)

Therefore, from (6.24), (6.32) and (6.52) we control \( \|\partial_x V(t) \phi\|_{L^2(R^+)} \) by \( C (\|\phi\|_{L^\infty} \log(t) + \|\phi\|_{H^1}) \). Proceeding similarly we estimate
\[ \|\partial_x V(t) \phi\|_{L^2(R^-)} \leq C (\|\phi\|_{L^\infty} \log(t) + \|\phi\|_{H^1}). \]

Hence, we attain (6.21).

7 Estimates for the inverse operator \( V^{-1} \)

In this section we prove Lemma 3.5. We want to obtain an asymptotic expansion for \( V^{-1}(t) \), as \( t \to \infty \), as well as a control of the \( L^2 \)-norm of the derivative \( \partial_x V^{-1}(t) \). These results are presented in Lemmas 7.1 and 7.2 below. Recall that
\[ \Lambda(k) = \theta(k) R_+ (k) + \theta(-k) R_-(k). \]

**Lemma 7.1** Suppose that (2.10) and (4.3) with \( \delta > 0 \) are verified. Then, the estimate
\[ \left\| V^{-1}(t) \phi - \frac{1}{2\pi i} \int_{-\infty}^{\infty} e^{\frac{i}{2}(k-x)^2} \Phi(t, x, k) \phi(x) \, dx \right\|_{L^\infty} \leq C |\phi(0)| + C t^{-\frac{\delta}{2}} \|\phi\|_{H^1}, \]
is true for all \( t \geq 1 \).

**Proof.** Recall that
\[ V^{-1}(t) \phi = \frac{1}{2\pi i} \int_{-\infty}^{\infty} e^{\frac{i}{2}(k-x)^2} \Phi(t, x, k) \phi(x) \, dx, \] (7.1)
where
\[ \Phi(t, x, k) = \theta(k) T(k) m_+(x, k) + \theta(-k) T(-k) m_-(x, -k). \]

Using (5.4) and (5.5) in (7.1), we have
\[ V^{-1}(t) \phi = \frac{t}{2\pi i} \theta(k) T(k) \int_{0}^{\infty} e^{\frac{i}{2}(k-x)^2} m_+(tx, k) \phi(x) \, dx 
+ \frac{t}{2\pi i} \theta(-k) \int_{0}^{\infty} e^{\frac{i}{2}(k-x)^2} m_+(-tx, k) \phi(-x) \, dx 
+ \frac{t}{2\pi i} \theta(k) \int_{0}^{\infty} e^{\frac{i}{2}(k-x)^2} m_-(tx, -k) \phi(x) \, dx 
+ \frac{t}{2\pi i} \theta(-k) \int_{0}^{\infty} e^{\frac{i}{2}(k-x)^2} m_-(tx, -k) \phi(-x) \, dx \] (7.2)

Using the notation
\[ V_0(t) \phi = \frac{1}{2\pi i} \int_{-\infty}^{\infty} e^{\frac{i}{2}(k-x)^2} \phi(x) \, dx \]
we write
\[ V^{-1}(t) \phi = \theta(k) T(k) V_0(-t) \left( \theta(x) (\phi(x) - \phi(0)) \right) 
+ \theta(-k) R_+(k) V_0(-t) \left( \theta(-x) (\phi(-x) - \phi(0)) \right) 
+ \theta(-k) V_0(-t) \left( \theta(x) (\phi(x) - \phi(0)) \right) 
+ \theta(k) R_-(k) V_0(-t) \left( \theta(-x) (\phi(-x) - \phi(0)) \right) 
+ \theta(k) V_0(-t) \left( \theta(-x) (\phi(x) - \phi(0)) \right) 
+ \theta(-k) T(-k) V_0(-t) \left( \theta(-x) (\phi(x) - \phi(0)) \right) + R_1 + R_2, \]
where
\[ R_1 = \phi (0) \theta (k) \overline{T(k)} V_0 (-t) \theta (x) + \phi (0) \theta (-k) \overline{R_+ (-k)} V_0 (-t) \theta (-x) \]
\[ + \phi (0) \theta (-k) V_0 (-t) \theta (x) + \phi (0) \theta (k) \overline{R_- (k)} V_0 (-t) \theta (x) \]
\[ + \phi (0) \theta (k) V_0 (-t) \theta (-x) + \phi (0) \theta (-k) \overline{T(-k)} V_0 (-t) \theta (-x) \]

and
\[ R_2 = \sqrt{\frac{t}{2\pi i}} \theta (k) \overline{T(k)} \int_0^\infty e^{\frac{x^2}{2}} \left( \frac{m_+ (tx,k) - 1}{m_+ (tx,k) - 1} \right) \phi (x) \, dx \]
\[ + \sqrt{\frac{t}{2\pi i}} \theta (-k) \overline{R_+ (-k)} \int_0^\infty e^{\frac{x^2}{2}} \left( \frac{m_+ (-tx,-k) - 1}{m_+ (-tx,k) - 1} \right) \phi (-x) \, dx \]
\[ + \sqrt{\frac{t}{2\pi i}} \theta (-k) \overline{R_- (k)} \int_0^\infty e^{\frac{x^2}{2}} \left( \frac{m_- (-tx,k) - 1}{m_- (-tx,k) - 1} \right) \phi (-x) \, dx \]
\[ + \sqrt{\frac{t}{2\pi i}} \theta (k) \overline{T(-k)} \int_0^\infty e^{\frac{x^2}{2}} \left( \frac{m_- (tx,-k) - 1}{m_- (tx,k) - 1} \right) \phi (x) \, dx \]

Using (2.6), (2.10) and \( \theta (k) \theta (-k) = 0 \) for \( k \neq 0 \), we find
\[ \left| V^{-1} (t) \phi - \overline{T(|k|)} \phi (k) - \overline{\Lambda (k)} \phi (-k) \right| \leq C |\phi (0)| + Ct^{-\frac{k}{2}} \| \phi \|_{H^1} + R_1 + R_2, \tag{7.3} \]

It follows from (2.10) that
\[ \| R_1 \|_{L^\infty} \leq C |\phi (0)|. \tag{7.4} \]

In order to estimate \( R_2 \) we use (2.10) and (4.3) with \( \delta > 0 \) to obtain
\[ \| R_2 \|_{L^\infty} \leq Ct^{\frac{k}{2}} \| \phi \|_{L^\infty} \int_0^\infty \langle tx \rangle^{-1-\delta} \, dx \leq Ct^{\frac{k}{2}} \| \phi \|_{L^\infty}. \tag{7.5} \]

Therefore the result of the lemma follows from (7.3), (7.4) and (7.5). \[ \blacksquare \]

In the next lemma we estimate the derivative \( \partial_k V^{-1} (t) \).

**Lemma 7.2** Suppose that (4.2), (4.3) with \( \delta = 0 \), (4.4) with \( \delta > \frac{1}{2} \), and (4.5) with \( \delta = 0 \) are true. Also, assume that \( m_+ (x,0) = m_- (-x,0) \) and (4.18), (4.19), (4.22), (4.23), (4.24). If \( a = 1 \), let \( \phi \in H^1 \) be odd and if \( a = -1 \), suppose that \( \phi \) is even. Then the estimate
\[ \| \partial_k V^{-1} (t) \phi \|_{L^2} \leq Ct^{\frac{k}{2}} |\phi (0)| + C \| \phi \|_{H^1}, \tag{7.6} \]
is valid for all \( t \geq 1 \).

**Proof.** Taking into account (7.2), we split \( V^{-1} (t) \) in two parts:
\[ V^{-1} (t) \phi = \theta (k) I_+ (k) + \theta (-k) I_- (k), \tag{7.7} \]

where
\[ I_+ (k) = \sqrt{\frac{t}{2\pi i}} \int_0^\infty e^{\frac{x^2}{2}} \overline{T(k)} m_+ (tx,k) \phi (x) \, dx \]
\[ + \sqrt{\frac{t}{2\pi i}} \int_0^\infty e^{\frac{x^2}{2}} \overline{R_+ (-k)} m_- (-tx,k) \phi (-x) \, dx \]
\[ + \sqrt{\frac{t}{2\pi i}} \int_0^\infty e^{\frac{x^2}{2}} m_- (tx,-k) \phi (x) \, dx \]
and

\[
I_-(k) = \sqrt{\frac{t}{2\pi i}} \int_{-\infty}^{0} e^{\frac{z^2}{2}} T(-k)m_{-}(tx, -k)\phi(x) \, dx \\
+ \sqrt{\frac{t}{2\pi i}} \int_{-\infty}^{0} e^{\frac{z^2}{2}} R_+(-k)m_{+}(-tx, -k)\phi(-x) \, dx \\
+ \sqrt{\frac{t}{2\pi i}} \int_{0}^{\infty} e^{\frac{z^2}{2}} m_{+}(tx, k)\phi(x) \, dx.
\]

Note that if \(a = \pm 1\), by (4.22) \(T(0) = \pm 1\) and by (4.23) \(R_{\pm}(0) = 0\). Also, by assumption \(m_{+}(x, 0) = m_{-}(-x, 0)\). Then, using that \(\phi\) is odd when \(a = 1\) and \(\phi\) is even if \(a = -1\), we get

\[
I_+(0) = \sqrt{\frac{t}{2\pi i}} \int_{0}^{\infty} e^{\frac{z^2}{2}} m_{+}(tx, 0)(a\phi(x) + \phi(-x)) \, dx = 0
\]

(7.8)

and

\[
I_-(0) = \sqrt{\frac{t}{2\pi i}} \int_{0}^{\infty} e^{\frac{z^2}{2}} m_{+}(tx, 0)(a\phi(-x) + \phi(x)) \, dx = 0.
\]

(7.9)

Thus, derivating (7.7) we get

\[
\partial_t V^{-1}(t) \phi = \theta(k) \partial_t I_+(k) + \theta(-k) \partial_t I_-(k),
\]

and hence,

\[
\|\partial_t V^{-1}(t) \phi\|_{L^2} \leq \|\partial_t I_+(k)\|_{L^2(\mathbb{R}^+)} + \|\partial_t I_+(k)\|_{L^2(\mathbb{R}^-)}.
\]

(7.10)

Making the change \(z = k - x\) we get

\[
I_+(k) = \sqrt{\frac{t}{2\pi i}} \int_{-\infty}^{k} e^{\frac{z^2}{2}} T(k)m_{+}(t(k-z), k)\phi(k-z) \, dz \\
+ \sqrt{\frac{t}{2\pi i}} \int_{-\infty}^{k} e^{\frac{z^2}{2}} R_-(k)m_{-}(-t(k-z), k)\phi(z-k) \, dz \\
+ \sqrt{\frac{t}{2\pi i}} \int_{k}^{\infty} e^{\frac{z^2}{2}} m_{-}(t(k-z), -k)\phi(k-z) \, dz
\]

and

\[
I_-(k) = \sqrt{\frac{t}{2\pi i}} \int_{k}^{\infty} e^{\frac{z^2}{2}} T(-k)m_{-}(t(k-z), -k)\phi(k-z) \, dz \\
+ \sqrt{\frac{t}{2\pi i}} \int_{k}^{\infty} e^{\frac{z^2}{2}} R_+(k)m_{+}(-t(k-z), -k)\phi(z-k) \, dz \\
+ \sqrt{\frac{t}{2\pi i}} \int_{-\infty}^{k} e^{\frac{z^2}{2}} m_{+}(t(k-z), k)\phi(k-z) \, dz
\]

For \(\partial_t I_+(k)\) we have

\[
\partial_t I_+(k) = \sqrt{\frac{t}{2\pi i}} \int_{-\infty}^{k} e^{\frac{z^2}{2}} \partial_t \left( T(k)m_{+}(t(k-z), k)\phi(k-z) \right) \, dz \\
+ \sqrt{\frac{t}{2\pi i}} \int_{-\infty}^{k} e^{\frac{z^2}{2}} \partial_t \left( R_-(k)m_{-}(-t(k-z), k)\phi(z-k) \right) \, dz \\
+ \sqrt{\frac{t}{2\pi i}} \int_{k}^{\infty} e^{\frac{z^2}{2}} \partial_t \left( m_{-}(t(k-z), -k)\phi(k-z) \right) \, dz \\
+ \sqrt{\frac{t}{2\pi i}} e^{\frac{k^2}{2}} \phi(0) \left( T(k)m_{+}(0, k) + R_+(k)m_{+}(0, k) - m_{-}(0, -k) \right).
\]

Note that by (4.13)

\[
T(k)m_{+}(0, k) - m_{-}(0, -k) = R_-(k)m_{-}(0, k).
\]
Then, returning to the old variable of integration $x = k - z$ we get

$$
\partial_k I_+ (k) = \sqrt{\frac{t}{2\pi i}} \int_{-\infty}^{\infty} e^{\frac{i}{2}(k-x)^2} \Theta_1^{(+)} (t, x, k) \, dx + \sqrt{\frac{t}{2\pi i}} \int_{-\infty}^{\infty} e^{\frac{i}{2}(k-x)^2} \Theta_2^{(+)} (t, x, k) \, dx
$$

$$
+ \sqrt{\frac{t}{2\pi i}} \int_{-\infty}^{\infty} e^{\frac{i}{2}(k-x)^2} \Theta_3^{(+)} (t, x, k) \, dx + 2 \sqrt{\frac{t}{2\pi i}} e^{\frac{i}{2}k^2} \phi (0) \overline{R_- (k)m_- (0, k)},
$$

(7.11)

with

$$
\Theta_1^{(+)} (t, x, k) = \theta (x) \partial_k \left( T (k)m_+ (tx, k) \phi (x) + \overline{R_- (k)m_- (tx, k)} \phi (-x) \right) + \theta (-x) \partial_k m_- (tx, -k) \phi (x)
$$

$$
\Theta_2^{(+)} (t, x, k) = \theta (x) \left( T (k)m_+ (tx, k) \partial_x \phi (x) - \overline{R_- (k)m_- (tx, k)} \partial_x \phi (-x) \right) + \theta (-x) m_- (tx, -k) \partial_x \phi (x)
$$

$$
\Theta_3^{(+)} (t, x, k) = \theta (x) \left( T (k) \partial_x m_+ (tx, k) \phi (x) + \overline{R_- (k)} \partial_x m_- (tx, k) \phi (-x) \right) + \theta (-x) m_- (tx, -k) \phi (x)
$$

We estimate the last term in (7.11) by using (4.2) and (4.18)

$$
\left\| \sqrt{\frac{t}{2\pi i}} e^{\frac{i}{2}k^2} \phi (0) \overline{R_- (k)m_- (0, k)} \right\|_{L^2} \leq Ct \bigl| \phi (0) \bigr| \left\| \overline{R_- (k)m_- (0, k)} \right\|_{L^2} \leq Ct \bigl| \phi (0) \bigr| \left\| \langle k \rangle^{-1} \right\|_{L^2} \leq Ct \bigl| \phi (0) \bigr|.
$$

(7.12)

To estimate the term in (7.11) containing $\Theta_1^{(+)} (t, x, k)$ we decompose

$$
\Theta_1^{(+)} (t, x, k) = \Theta_{11}^{(+)} (x, k) + \Theta_{12}^{(+)} (t, x, k),
$$

(7.13)

where

$$
\Theta_{11}^{(+)} (x, k) = \theta (x) \phi (x) \partial_k T (k) + \theta (x) \phi (-x) \partial_k R_- (k)
$$

and

$$
\Theta_{12}^{(+)} (t, x, k) = \left( m_+ (tx, k) - 1 \right) \theta (x) \phi (x) \partial_k T (k) + \left( m_- (tx, k) - 1 \right) \theta (x) \phi (-x) \partial_k R_- (k)
$$

$$
+ T (k) \theta (x) \phi (x) \partial_k m_+ (tx, k) + R_- (k) \theta (x) \phi (-x) \partial_k m_- (tx, k)
$$

$$
+ \theta (-x) \phi (x) \partial_k m_- (tx, -k).
$$

Observe that

$$
\sqrt{\frac{t}{2\pi i}} \int_{-\infty}^{\infty} e^{\frac{i}{2}(k-x)^2} \Theta_{11}^{(+)} (x, k) \, dx = \partial_k T (k) \nu_0 (t) (\theta (x) \phi (x)) + \partial_k R_- (k) \nu_0 (t) (\theta (x) \phi (-x)) .
$$

Then, it follows from (2.5), (4.19) and (4.24) that

$$
\left\| \sqrt{\frac{t}{2\pi i}} \int_{-\infty}^{\infty} e^{\frac{i}{2}(k-x)^2} \Theta_{11}^{(+)} (x, k) \, dx \right\|_{L^2 (\mathbb{R}^+)} \leq \left\| \partial_k T (k) \nu_0 (t) (\theta (x) \phi (x)) \right\|_{L^2 (\mathbb{R}^+)} + \left\| \partial_k R_- (k) \nu_0 (t) (\theta (x) \phi (-x)) \right\|_{L^2 (\mathbb{R}^+)} \leq C \| \phi \|_{L^2}.
$$

(7.14)

Moreover, using (2.10), (4.3) with $\delta = 0$, (4.4) with $\delta > \frac{1}{4}$, (4.19) and (4.24) we estimate

$$
\left| \Theta_{12}^{(+)} (t, x, k) \right| \leq C \theta (x) \left( \left| m_+ (tx, k) - 1 \right| + \left| m_- (tx, -k) - 1 \right| \right) \left( | \phi (x) | + | \phi (-x) | \right) \left( \| \partial_k T \|_{L^\infty} + \| \partial_k R_- \|_{L^\infty} \right)
$$

$$
+ C \theta (x) \left( | \phi (x) | + | \phi (-x) | \right) \left( \| T \|_{L^\infty} + \| R_- \|_{L^\infty} \right) \left( | \partial_k m_+ (tx, k) | + | \partial_k m_- (tx, -k) | \right)
$$

$$
+ C \theta (-x) \left| \phi (x) \right| \left| \partial_k m_- (tx, -k) \right| \leq C \langle k \rangle^{-1} \langle tx \rangle^{-\delta} \left( | \phi (x) | + | \phi (-x) | \right).
$$

Then,

$$
\left\| \sqrt{\frac{t}{2\pi i}} \int_{-\infty}^{\infty} e^{\frac{i}{2}(k-x)^2} \Theta_{12}^{(+)} (t, x, k) \, dx \right\|_{L^2 (\mathbb{R}^+)} \leq Ct \left\| \langle k \rangle^{-1} \int_{-\infty}^{\infty} \langle tx \rangle^{-\delta} | \phi (x) | \, dx \right\|_{L^2 (\mathbb{R}^+)} \leq C \left\| \phi \right\|_{L^2}.
$$

(7.15)
Combining (7.13), (7.14) and (7.15) we get
\[
\left\| \sqrt{\frac{t}{2\pi i}} \int_{-\infty}^{\infty} e^{\frac{it}{2} (k-x)^2} \Theta_1^{(+)} (t, x, k) \, dx \right\|_{L^2(\mathbb{R}^+)} \leq C \| \phi \|_{L^2} .
\] (7.16)

Next we consider the term in (7.11) containing \( \Theta_2^{(+)} (t, x, k) \). We split
\[
\Theta_2^{(+)} (t, x, k) = \Theta_{21}^{(+)} (x, k) + \Theta_{22}^{(+)} (t, x, k) ,
\] (7.17)
where
\[
\Theta_{21}^{(+)} (x, k) = \theta (x) \overline{T (k)} \partial_x \phi (x) - \theta (x) \overline{R_- (k)} (\partial_x \phi) (-x) + \theta (-x) \partial_x \phi (x)
\]
and
\[
\Theta_{22}^{(+)} (t, x, k) = \theta (x) \overline{T (k)} \left( \overline{m_+ (tx, k)} - 1 \right) \partial_x \phi (x)
- \theta (x) \overline{R_- (k)} \left( \overline{m_- (-tx, k)} - 1 \right) (\partial_x \phi) (-x)
+ \theta (-x) \left( \overline{m_- (tx, -k)} - 1 \right) \partial_x \phi (x)
\]

By using (2.5) and (2.10) we estimate
\[
\left\| \sqrt{\frac{t}{2\pi i}} \int_{-\infty}^{\infty} e^{\frac{it}{2} (k-x)^2} \Theta_{21}^{(+)} (x, k) \, dx \right\|_{L^2(\mathbb{R}^+)} \leq \left\| \overline{T (k)} \overline{V}_0 (-t) (\theta (x) \partial_x \phi (x)) \right\|_{L^2(\mathbb{R}^+)}
+ \left\| \overline{R_- (k)} \overline{V}_0 (-t) (\theta (x) \partial_x \phi (x)) \right\|_{L^2(\mathbb{R}^+)} \leq C \| \partial_x \phi \|_{L^2} ,
\] (7.18)

Moreover, from relation (2.10) and (4.3) with \( \delta = 0 \) we get
\[
\left| \Theta_{22}^{(+)} (t, x, k) \right| \leq \theta (x) \left| \overline{T (k)} \right| \left| \overline{m_+ (tx, k)} - 1 \right| \left| \partial_x \phi (x) \right| + \theta (x) \left| \overline{R_- (k)} \right| \left| \overline{m_- (-tx, k)} - 1 \right| \left| (\partial_x \phi) (-x) \right|
+ \theta (-x) \left| \overline{m_- (tx, -k)} - 1 \right| \left| \partial_x \phi (x) \right| \leq C \langle k \rangle^{-1} \langle tx \rangle^{-1} \left| \partial_x \phi (x) \right| + \left| \partial_x \phi (x) \right| ,
\]
and hence,
\[
\left\| \sqrt{\frac{t}{2\pi i}} \int_{-\infty}^{\infty} e^{\frac{it}{2} (k-x)^2} \Theta_{22}^{(+)} (t, x, k) \, dx \right\|_{L^2(\mathbb{R}^+)} \leq C t^{\frac{1}{4}} \langle k \rangle^{-\frac{1}{2}} \langle tx \rangle^{-\frac{1}{2}} \left\| \partial_x \phi \right\|_{L^2} \leq C \| \partial_x \phi \|_{L^2} .
\] (7.19)

Using (7.17), (7.18) and (7.19) we obtain
\[
\left\| \sqrt{\frac{t}{2\pi i}} \int_{-\infty}^{\infty} e^{\frac{it}{2} (k-x)^2} \Theta_2^{(+)} (t, x, k) \, dx \right\|_{L^2(\mathbb{R}^+)} \leq C \| \partial_x \phi \|_{L^2} .
\] (7.20)

Finally by (2.10) and (4.5) with \( \delta = 0 \) we get
\[
\left| \Theta_3^{(+)} (t, x, k) \right| \leq C \theta (x) \left| \overline{R_- (k)} \right| \left| \overline{m_+ (tx, k)} \right| \left| \partial_x \phi (x) \right| + C \theta (x) \left| \overline{R_- (k)} \right| \left| \overline{m_- (-tx, k)} \right| \left| \partial_x \phi (x) \right|
+ C \theta (-x) \left| \overline{m_- (tx, -k)} \right| \left| \partial_x \phi (x) \right| \leq C t \langle k \rangle^{-1} \langle tx \rangle^{-2} \left( \left| \partial_x \phi (x) \right| + \left| \partial_x \phi (x) \right| \right).
\]

Therefore, using that \( \left| \phi (x) - \phi (0) \right| \leq \left| x \right|^{\frac{1}{2}} \| \partial_x \phi \|_{L^2} \), we conclude
\[
\left\| \sqrt{\frac{t}{2\pi i}} \int_{-\infty}^{\infty} e^{\frac{it}{2} (k-x)^2} \Theta_3^{(+)} (t, x, k) \, dx \right\|_{L^2(\mathbb{R}^+)} \leq C t^{\frac{1}{4}} \| \partial_x \phi \|_{L^2} \langle tx \rangle^{-\frac{3}{2}} \left\| \partial_x \phi \right\|_{L^2} + C t^{\frac{1}{4}} \| \partial_x \phi \|_{L^2} \langle tx \rangle^{-\frac{3}{2}} \left\| \partial_x \phi \right\|_{L^2} \leq C t^{\frac{1}{4}} \| \partial_x \phi \|_{L^2} + C \| \partial_x \phi \|_{L^2} .
\] (7.21)

Introducing (7.12), (7.16), (7.20) and (7.21) into (7.11) we obtain
\[
\| \partial_k I_+ (k) \|_{L^2(\mathbb{R}^+)} \leq C t^{\frac{1}{4}} \| \phi (0) \| + C \| \phi \|_{H^1} .
\]

Proceeding similarly to estimate \( \partial_k I_- \), from (7.10) we attain (7.6).
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