Static Observers in Curved Spaces and Non-inertial Frames in Minkowski Spacetime

F. Dahia and P. J. Felix da Silva
Departamento de Física, Universidade Federal de Campina Grande,
58109-970, PB, Brazil

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Abstract

Static observers in curved spacetimes may interpret their proper acceleration as the opposite of a local gravitational field (in the Newtonian sense). Based on this interpretation and motivated by the equivalence principle, we are led to investigate congruences of timelike curves in Minkowski spacetime whose acceleration field coincides with the acceleration field of static observers of curved spaces. The congruences give rise to non-inertial frames that are examined. Specifically we find, based on the locality principle, the embedding of simultaneity hypersurfaces adapted to the non-inertial frame in an explicit form for arbitrary acceleration fields. We also determine, from the Einstein equations, a covariant field equation that regulates the behavior of the proper acceleration of static observers in curved spacetimes. It corresponds to an exact relativistic version of the Newtonian gravitational field equation. In the specific case in which the level surfaces of the norm of the acceleration field of the static observers are maximally symmetric two-dimensional spaces, the energy-momentum tensor of the source is analyzed.

1 Introduction

Based on the equivalence principle, it is expected that some physical features of gravity can be mimicked by accelerated frames in Minkowski spacetime. The Rindler frame, which is adapted to a family of uniformly accelerated observers, is a famous example of a non-inertial system that simulates some characteristics of a black hole's geometry[1, 2].

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This frame has been widely investigated in the literature and here we are going to start our discussion pointing out a peculiar aspect of the Rindler frame. It is related to the remarkable characteristic that the proper acceleration $a$ of Rindler observers, which is constant along their worldlines, varies according to the law $a = 1/\rho$ in relation to the observers, where $\rho$ corresponds to the initial distance of the observer with respect to the origin of an inertial frame. As it is well known, the reason of this behavior is related to the geometric properties of the timelike Killing field which generates the congruence.

On the other hand, from the physical point of view, it is very suggestive that this particular dependence of $a$ and $\rho$ is connected to the behavior of static observers in Schwarzschild geometry in the vicinity of the horizon. Indeed, if $\rho$ denotes the radial distance of an observer to the horizon, then, the proper acceleration the observers need in order to stay at rest in their position close to the horizon is proportional to $1/\rho$. Therefore the Rindler congruence and the static Schwarzschild observers have the same acceleration field $a(\rho)$. This equivalence reinforce the linkage between Rindler frame and the exterior of a black hole.

However, it happens that the inverse law holds only in the region where $\rho$ is small in comparison to the Schwarzschild radius of the black hole. Indeed, static observers at long distances are submitted to an acceleration field proportional to $1/\rho^2$, (i.e, they obey the inverse square law of Newtonian gravitation) and, in the intermediate domain, the dependence of $a$ in terms of $\rho$ is described by a much more complicated function.

We can say, then, that static observers in the Schwarzschild geometry are accelerated according to a cumbersome function $a(\rho)$ which reduces to the power law $1/\rho$ only in the region near the horizon. All these considerations led us to turn our attention to the study of a congruence of timelike curves whose acceleration field $a(\rho)$ coincides with the acceleration field of static observers in the Schwarzschild spacetime. In this case, only in the range of small values of $\rho$, the congruence defined in Minkowski spacetime will be equivalent to the worldlines of Rindler observers.

Of course this discussion is not limited to the Schwarzschild spacetime and can be extend to encompass any static spacetime. In this paper, motivated by these ideas, we are going to investigate congruences in Minkowski spacetime that obey a generic law $a(\rho)$, focusing on the problem of determining a coordinate system adapted to the non-inertial frame associated to these congruences.

A fundamental ingredient in this formalism is the determination of the simultaneity hypersurfaces relative to the non-inertial frame. As we shall see, the embedding functions of the simultaneity hypersurfaces into Minkowski
spacetime are explicitly obtained for any arbitrary function $a(\rho)$, by using
the locality principle [3, 4]. The formulation of this general scheme allows
us to discuss the correspondence between non-inertial frames in Minkowski
spacetime and static observers of any static spacetimes. The particular case
concerning the Schwarzschild spacetime, due to its relevance, is investigated
in detail.

Considering the importance of the acceleration of static observers in this
context, it is desirable to find a field equation that regulates the behavior
of $a$. By using the embedding formalism we obtain, from the Einstein equa-
tions, a covariant field equation for the proper acceleration that has a clear
physical interpretation. Indeed, from the perspective of static observers,
non-interacting bodies are accelerated as if they were under the influence of
a gravitational field whose intensity is precisely equal to $a$ but in the opposite
direction. Thus, the field equation for $a$ may be understood as that the exact
relativistic version of the gravitational field equation of Newtonian theory,
as we shall see.

By admitting additional symmetries, we can focus our attention in a
special class of static spacetimes characterized by the fact that the level
surfaces of the norm of the proper acceleration field are maximally symmet-
ric two-dimensional spaces. This assumption simplifies greatly the Einstein
equations and as a consequence the field equation can be totally written in
terms of the extrinsic and intrinsic curvature of the level surfaces and the
acceleration field.

The general form of the energy-momentum tensor of the possible sources
of this special kind of static spacetimes can be deduced from the Einstein
equation. As we shall see, it corresponds to non-viscous fluids that may
be anisotropic. Indeed, the pressure in the parallel direction ($p_\parallel$) to the
acceleration field may be different from the pressure the fluid exerts in the
orthogonal direction ($p_\perp$). In the particular case in which $p_\perp = 0$ and the
parallel pressure satisfies the state equation characteristic of a false vacuum
state, the produced spacetime possesses an intriguing property: the static
observers are accelerated precisely according to the Rindler law, although
they are in a curved spacetime.

This paper is organized as follows. In the first section we review the
Rindler frame in order to establish the notation and the techniques necessary
for the construction of a coordinate system adapted to a non-inertial frame.
In the second section, we study the induced geometry in the simultaneity
hypersurfaces relative to the accelerated congruence. Next, in the third sec-
tion, the idea of establishing a mapping between non-inertial frame and static
observers is elaborated and the particular case related to the Schwarzschild
spacetime is discussed in detail. The fourth section is dedicated to determine
the field equation that regulates the acceleration field of static observers in curved spacetimes and to study the energy-momentum tensor of the sources that generate a special class of static spacetimes.

2 Accelerated Observers in Minkowski spacetime

For the sake of simplicity, let us initially consider a two-dimensional Minkowski spacetime mapped by coordinates \((t, x)\) of an inertial system \(S\). Consider now an observer submitted to a constant proper acceleration \(a(\rho)\), where \(\rho\) corresponds to the initial distance of the observer with respect to the origin of \(S\). The worldline of this observer, that we will denote by \(O_\rho\) henceforth, is described by the following parametric curve:

\[
\begin{align*}
t(\tau_\rho) &= \frac{1}{a(\rho)} \sinh (a(\rho) \tau_\rho) \\
x(\tau_\rho) &= \frac{1}{a(\rho)} \cosh (a(\rho) \tau_\rho) - \frac{1}{a(\rho)} + \rho
\end{align*}
\]

where \(\tau_\rho\) is the proper time of \(O_\rho\). It can be directly checked that the proper acceleration of the observer is indeed \(a(\rho)\) and that in the initial position, \(x(0) = \rho\), the observer is instantaneously at rest.

In the spacetime diagram of \(S\), this worldline is a hyperbola given by the equation

\[
(x - b(\rho))^2 - t^2 = \frac{1}{a(\rho)}
\]

where \(a^{-1}(\rho)\) is the distance from the vertex of the hyperbola to its center \(b(\rho) = \rho - \frac{1}{a(\rho)}\), located at the \(x\)-axis.

The motion equations and may also be understood according to a different perspective that is more appropriate to our discussion. They can be considered as a set of equations which describe a family of accelerated observers, where for each value of \(\rho\) corresponds a different observer. Thus, with respect to the congruence, the coordinate \(\rho\) can be viewed as a parameter that identify each member of the family. It is important to emphasize that each observer suffers an uniform acceleration along its worldline. However, distinct observers are submitted to different accelerations according to a generic law \(a(\rho)\).

We want now to construct a coordinate system adapted to this non-inertial frame. A crucial element in this discussion is the notion of simultaneity associated to the frame. There is no privileged or natural way to
establish it and as a consequence there are a multitude of options that can be found in the literature. Every possible way leads to a different kind of coordinate system [5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20].

Here we will follow a very reasonable idea known as the locality principle. According to this principle the accelerated observer is instantaneously equivalent to the co-moving inertial observer. The reason is that they share the same position and velocity instantaneously and therefore, as both observers have the same physical state according to non-quantum mechanics, they should be considered physically equivalent at that moment.

If we admit this hypothesis as being valid then it is natural to assume that the set of simultaneous events for the accelerated observer coincides locally with the set of simultaneous events relative to the co-moving inertial observer.

In Minkowski spacetime this set could be identified as follows. Consider $S^\prime(\rho, \tau)$ the co-moving inertial frame relative to the observer $O_\rho$ at a certain instant of time $\tau$. Obviously (by definition), the accelerated observer will be find in $O^\prime$, the origin of $S^\prime(\rho, \tau)$, with null velocity instantaneously. Now, with respect to the inertial frame $S$, let $R^\mu$ be the components of the position vector of a generic event $E$ and $R^\mu_0$ the vector that localize the origin of $S^\prime(\rho, \tau)$ - which corresponds also to the spacetime position of $O_\rho$ at instant $\tau$. The event $E$ will be considered simultaneous to $O^\prime$, in the inertial frame $S^\prime(\rho, \tau)$, if the relative position of $E$ (which is given by $\Delta R^\mu = R^\mu - R^\mu_0$) has no timelike component when decomposed in the coordinate basis associated to $S^\prime$, since, in this case, there will be no timelike separation between the events $E$ and $O^\prime$ as viewed from $S^\prime$ frame.

Considering that the direction of the timelike axis of the frame $S^\prime$ coincides with the direction of the vector $U^\mu$ (the proper velocity of $O_\rho$), then, it follows that the condition of simultaneity is equivalent to the orthogonality condition between the relative position and $U^\mu$ in the Minkowski metric:

$$U_\mu (R^\mu - R^\mu_0) = 0$$

In the two-dimensional case, the solution of the above equation, obviously, corresponds exactly to the $x^\prime$-axis of $S^\prime$.

When we are dealing with an observer $O_\rho$ of the accelerated congruence, then, based on the principle of locality, we should be aware that the validity of equation (4) is local, since $U^\mu$ changes in time (the observer is accelerated) and with respect to the space (different observers are submitted to different accelerations). So the equation that defines simultaneous events relative to accelerated frame is an infinitesimal version of the above equation, i.e.,

$$U_\mu dR^\mu = 0$$

In the two-dimensional case, the solution of the above equation, obviously, corresponds exactly to the $x^\prime$-axis of $S^\prime$. 
This means that the proper velocity of each observer must be orthogonal to any infinitesimal displacement in the simultaneity section, $dR^\mu$, in every point of the hypersurface. In other words, proper velocity is the normal vector of the simultaneity sections.

Let us assume that the simultaneity sections can be described by a function $\varphi$ defined in Minkowski spacetime. More precisely, let us admit that each equation $\varphi = \text{constant}$ determines a hypersurface in spacetime which corresponds to a simultaneity section adapted to the accelerated observers. We know that the normal vector of the hypersurface is proportional to the gradient of the function $\varphi$. Therefore, in order to satisfy equation (5), we must have:

$$\frac{1}{|\nabla \varphi|} \frac{\partial \varphi}{\partial x^\mu} = U_\mu$$

(6)

where $|\nabla \varphi|$ is the modulus of the norm of the gradient in Minkowski metric.

For the sake of simplicity let us use coordinates $\rho$ and $\tau$ to localize events.\footnote{Henceforth we will write $\tau$ in the place of $\tau_\rho$ in order to make the notation simpler.}

Of course, whenever it is necessary, we can, from equations (1) and (2), obtain the corresponding $(t, x)$-coordinates in the frame $S$.

Thus, assuming we have the function $\varphi (\rho, \tau)$, equation (6) can be written, by using a well-known relation involving partial derivatives, in the following way

$$- \left( \frac{\partial \tau}{\partial \rho} \right) \varphi = \frac{(\frac{\partial \varphi}{\partial \rho})_\tau}{(\frac{\partial \varphi}{\partial \tau})_\rho} = \frac{U_\rho}{U_\tau}$$

(7)

On the other hand, by using equations (1) and (2) as transformation equations, $U_\rho$ and $U_\tau$ can be determined from the components of the proper velocity $U_t = - \cosh a \tau$ and $U_x = \sinh a \tau$. We find

$$U_\tau = \left( \frac{\partial t}{\partial \tau} \right)_\rho U_t + \left( \frac{\partial x}{\partial \tau} \right)_\rho U_x = -1$$

(8)

$$U_\rho = \left( \frac{\partial t}{\partial \rho} \right)_\tau U_t + \left( \frac{\partial x}{\partial \rho} \right)_\tau U_x = - \frac{a^' \tau}{a} + \left( \frac{a^'}{a^2} + 1 \right) \sinh (a \tau)$$

(9)

Therefore, the equation for simultaneity hypersurface takes the form:

$$\left( \frac{\partial \tau}{\partial \rho} \right) \varphi = - \frac{a^' \tau}{a} + \left( \frac{a^'}{a^2} + 1 \right) \sinh (a \tau)$$

(10)

In order to solve this equation it is convenient to introduce a new coordinate

$$\eta = a (\rho) \tau$$

(11)
in terms of which the above equation reduces to

\[ \left( \frac{\partial \eta}{\partial \varphi} \right) = \left( \frac{a'}{a} + a \right) \sinh(\eta) \] (12)

This equation can be directly integrated and the solution is

\[ \tau = \frac{1}{a} \ln \left[ \frac{1 + f(\varphi) a \exp \left( \int a \, d\rho \right)}{1 - f(\varphi) a \exp \left( \int a \, d\rho \right)} \right] \] (13)

where \( f(\varphi) \) is some arbitrary function. For each specified value \( \varphi = \text{const} \), equation (13) gives us the coordinates \((\tau, \rho)\) of the simultaneous events relative to the non-inertial frame.

The simultaneity hypersurfaces can also be described by parametric equations in the coordinate system of the inertial frame \( S \). Indeed, by using equations (1), (2) and (13), we find:

\[ t = F_{\varphi}(\rho) \equiv \frac{2}{a} \left[ \frac{f(\varphi) a \exp \left( \int a \, d\rho \right)}{1 - \left( f(\varphi) a \exp \left( \int a \, d\rho \right) \right)^2} \right] \] (14)

\[ x = G_{\varphi}(\rho) \equiv \frac{2}{a} \left[ \frac{(f(\varphi) a \exp \left( \int a \, d\rho \right))^2}{1 - \left( f(\varphi) a \exp \left( \int a \, d\rho \right) \right)^2} \right] + \rho \] (15)

As we have mentioned, the coordinate \( \varphi \) can be used to label the simultaneity hypersurfaces. For each value \( \varphi = \text{const} \) corresponds a simultaneity hypersurface and the equations above give the embedding functions of the hypersurface in the Minkowski spacetime. By varying the coordinate \( \rho \), we can find the image of the hypersurfaces embedded in the spacetime diagram of \( S \) frame.

It is also possible to connect \( \varphi \) with the proper time that is measured by observers. Choosing a certain particular observer to serve as a reference, let us say \( \rho_0 \), then, from equation (13), we find the following relation

\[ f(\varphi) = \frac{\tanh \left( \frac{a(\rho_0) \tau_0}{2} \right)}{a(\rho_0) \exp \left( \int_{\rho_0}^{\rho} a d\rho \right)} \] (16)

Thus, the simultaneity section may be labelled, in an equivalent way, by the proper time \( \tau_0 \) of the observer \( \rho_0 \).

The equations (14) and (15) are very general in the sense that they are applicable to any function \( a(\rho) \). In the appendix, we analyze several simple cases of physical interest. Figure (1) shows that the general formulas (14) and
reproduce the Rindler congruence when we take $a = 1/\rho$. In the second figure, we have considered a congruence that follows the inverse square law. And at last, the uniform acceleration field $a = \text{cte}$, which was also considered with a different notion of simultaneity in the literature [18, 21], is examined in Figure (3).

2.1 Geometry of the simultaneity hypersurfaces

A generalization to (3+1)-dimensions can be immediately obtained as soon as we have established the symmetry of the spatial distribution of the observers. This is directly connected with the interpretation of $\rho$ as a spatial coordinate. For instance, if the spherical symmetry is admitted, then, it is implicitly assumed that the observers are performing a radial motion and consequently $\rho$ plays the role of a radial coordinate. It follows then that the simultaneity hypersurfaces are described by the equations:

$$t = F_\varphi (\rho) \quad (17)$$
$$x = G_\varphi (\rho) \sin \theta \cos \phi \quad (18)$$
$$y = G_\varphi (\rho) \sin \theta \sin \phi \quad (19)$$
$$z = G_\varphi (\rho) \cos \theta \quad (20)$$

However, in the case of cylindrical symmetry, the embedding functions are

$$t = F_\varphi (\rho) \quad (21)$$
$$x = G_\varphi (\rho) \cos \phi \quad (22)$$
$$y = G_\varphi (\rho) \sin \phi \quad (23)$$
$$z = z' \quad (24)$$

while the plane symmetry leads to the following embedding map

$$t = F_\varphi (\rho) \quad (25)$$
$$x = G_\varphi (\rho) \quad (26)$$
$$y = y' \quad (27)$$
$$z = z' \quad (28)$$

At this point, once we have already described the hypersurface by the embedding functions, we are able now to study the induced geometry in this hypersurface. With this purpose, let us first recall some elements of the embedding formalism. Let $\psi : \Sigma \rightarrow M$ be an embedding map of a
hypersurface $\Sigma$ in a manifold $M$. Given a coordinate system $\{\xi^i\}$ on $\Sigma$ and $\{x^a\}$ on $M$, the embedding functions can be explicitly written as

$$x^a = \psi^a(\xi^1, \xi^2, \xi^3)$$

(29)

The differential $df$, which is injective by definition, maps vectors of the tangent space of $\Sigma$ into vectors of the tangent space of $M$. In particular, the basis vectors can be written as

$$\frac{\partial}{\partial \xi^a} = e^a_\alpha \frac{\partial}{\partial x^\alpha}$$

(30)

where

$$e^a_\alpha = \frac{\partial \psi^\alpha}{\partial \xi^a}$$

(31)

If the manifold $M$ is equipped with a metric, $g_{\alpha\beta}$, then, the embedding induces a metric $h_{ab}$ in the hypersurface, which, in terms of the coordinate basis, is given by

$$h_{ab} = e^a_\alpha e^b_\beta g_{\alpha\beta}$$

(32)

The normal vector of the hypersurface $\Sigma$ relative to $M$ is, in the case of our congruence, the timelike vector $U^\alpha$. The orthogonality condition with respect to the tangent vector $\frac{\partial}{\partial \xi^a}$ can be expressed as $e^a_\alpha U_\alpha = 0$.

In the present formalism an important concept is that of a projection tensor $\Pi_{\alpha\beta}$, which maps vectors of the tangent space of $M$ onto the tangent space of the hypersurface $\Sigma$. Its components are defined as

$$\Pi_{\alpha\beta} = g_{\alpha\beta} + U_\alpha U_\beta$$

(33)

It is clear that, for any vector $V^\alpha$, the projection $\Pi^\alpha_\beta V^\beta$ can be considered as a vector of the tangent space of $\Sigma$. Therefore it can be written in terms of the basis $\left\{ \frac{\partial}{\partial \xi^a} \right\}$. In particular, the projection of $\partial_\alpha$ can be written as $e^a_\alpha \partial_\alpha$, for some 'vielbein' $e^a_\alpha$. Thus, we have

$$\Pi^\gamma_\alpha \frac{\partial}{\partial x^\gamma} = e^a_\alpha \frac{\partial}{\partial \xi^a}$$

(34)

Considering this relation and taking the inner product with vectors of the basis $\left\{ \partial_\beta \right\}$, it is possible to show that $e^a_\alpha$ is related to $e^a_\alpha$ according to the following formula:

$$e^a_\alpha = g_{\alpha\beta} h^{ab} e^b_\beta$$

(35)

The embedding also induces a covariant derivative in $\Sigma$. If $\nabla$ is the covariant derivative defined in $M$, compatible with $g_{\alpha\beta}$, then, by using the
projection tensor, we can define a covariant derivative in $\Sigma$ in the following way

\[ (3) \nabla_\beta v^\alpha = e^\alpha_A e_\beta B \nabla_B v^A \]

where $v^\alpha$ is some vector which belongs to the tangent space of $\Sigma$ (more precisely, an extension of the vector). It can be checked that the induced covariant derivative is also compatible with the induced metric $h_{ab}$ \[22\].

Another important concept in the context of the embedding formalism is that of extrinsic curvature $K_{ab}$. Roughly speaking, we can say that it measures the variation of the normal vector along tangent directions of the hypersurface $\Sigma$. In terms of its components, we have the following definition:

\[ K_{ab} = e_\alpha^a e_\beta^b \nabla_\beta U_\alpha \]

From the induced covariant derivative, the intrinsic Riemann tensor \( R^{abcd} \) of the hypersurface $\Sigma$ can be naturally constructed. As it is well known, from relation \( (36) \), the tensor \( R^{abcd} \) can be put in connection with the components of the Riemann tensor $R_{\alpha\beta\gamma\delta}$ defined in $M$, according to the Gauss equation \[22\]:

\[ (3) R^{abcd} = e^\alpha_a e^\beta_b e^\gamma_c e^\delta_d R_{\alpha\beta\gamma\delta} + (K_{ad}K_{bc} - K_{ac}K_{bd}) \]

There is also the Codazzi equation that gives the variation of extrinsic curvature on the hypersurface $\Sigma$ \[22\]:

\[ (3) \nabla_c K_{ab} - (3) \nabla_b K_{ac} = U^\mu e_\alpha^\mu e_\beta^\gamma e_\gamma^\delta R_{\mu\alpha\beta\gamma} \]

In our case the ambient space $M$ is the Minkowski space, so these equations reduce to:

\[ (3) R^{abcd} = K_{ad}K_{bc} - K_{ac}K_{bd} \]

\[ (3) \nabla_c K_{ab} = (3) \nabla_b K_{ac} \]

After we have briefly reviewed these general concepts regarding the embedding formalism, let us now turn our attention to our particular embedding maps in order to study the induced geometry in $\Sigma$. By using the embedding functions, given by the sets of equations \[17\]-\[20\], \[21\]-\[24\] and \[25\]-\[28\], and identifying explicitly the intrinsic coordinates for each embedding map, we can directly calculate $e_\alpha^a$ and $K_{ab}$.

First we are going to consider the case of spherical symmetry. Of course, the intrinsic coordinates of the simultaneity hypersurface are: $\xi^1 = \rho, \xi^2 = \cdots$
\[ \theta, \xi^3 = \phi. \] In this coordinate system the induced metric in \( \Sigma \), according to the equation (32), is given by
\[
dl^2 = \left[ \frac{a'}{a^2} - \left( 1 + \frac{a'}{a^2} \right) \cosh (\eta) \right]^2 d\rho^2 + G^2_\varphi (\rho) \left[ d\theta^2 + \sin^2 \theta d\phi^2 \right]
\]
where \( \eta = a \tau \) is given by equation (13).

We can also verify that the extrinsic curvature is diagonal and its non-null components are the following
\[
K_{\rho \rho} = \left( a' + a \right) \left[ \frac{a'}{a^2} - \left( 1 + \frac{a'}{a^2} \right) \cosh \eta \right] \sinh \eta \quad (43)
\]
\[
K_{\theta \theta} = \left[ \frac{1}{a} \left( \cosh \eta - 1 \right) + \rho \right] \sinh \eta \quad (44)
\]
\[
K_{\phi \phi} = (\sin^2 \theta) K_{\theta \theta}^2 \quad (45)
\]

As it is well known, in a 3-dimensional space, the Riemann tensor has only 6 algebraically independent components. In this present case, all non-null components of \((3) R_{abcd}\) can be determined by this following set of components:
\[
(3) R_{\rho \theta \rho \theta} = -K_{\rho \rho} K_{\theta \theta} \quad (46)
\]
\[
(3) R_{\rho \phi \rho \phi} = -K_{\rho \rho} K_{\theta \theta} (\sin^2 \theta) \quad (47)
\]
\[
(3) R_{\phi \phi \phi \phi} = - (\sin^2 \theta) K_{\theta \theta}^2 \quad (48)
\]
It is interesting to note that the component \( R_{\phi \phi \phi \phi} \) is non-null unless the observers are not accelerated \((a = 0)\). This means that the simultaneity hypersurfaces adapted to accelerated observers are curved regardless to the form of the function \( a (\rho) \).

Concerning the cylindrical symmetry, the induced metric is given by
\[
dl^2 = \left[ \frac{a'}{a^2} - \left( 1 + \frac{a'}{a^2} \right) \cosh (\eta) \right]^2 d\rho^2 + G^2_\varphi (\rho) \left[ d\theta^2 + \sin^2 \theta d\phi^2 + dz^2 \right]
\]
and the extrinsic curvature has the following non-null components:
\[
K_{\rho \rho} = \left( a' + a \right) \left[ \frac{a'}{a^2} - \left( 1 + \frac{a'}{a^2} \right) \cosh \eta \right] \sinh \eta \quad (50)
\]
\[
K_{\phi \phi} = \left[ \frac{1}{a} \cosh \eta - \frac{1}{a} + \rho \right] \sinh \eta \quad (51)
\]
Now, the only non-null component of the Riemann tensor are \((3) R_{\phi \rho \rho \theta} = -K_{\rho \rho} K_{\phi \phi}\) and the algebraically equivalent components. Note that, besides
the case of null acceleration, $R_{\rho\theta\rho\theta}$ is zero for a congruence whose acceleration is $a = \frac{1}{\rho}$. This means that the Rindler congruence is the unique non-inertial frame in which the adapted simultaneity hypersurfaces are flat.

For the plane symmetry, we have

$$dl^2 = \left[ \frac{a'}{a^2} - \left( 1 + \frac{a'}{a^2} \right) \cosh (\eta) \right]^2 d\rho^2 + dy^2 + dz^2$$ (52)

and the hypersurfaces are flat for any function $a(\rho)$.

3 Static observers in curved spacetime

Intuitively it is expected that, with respect to static observers, free-falling bodies seem to be accelerated. Evoking a Newtonian picture, this acceleration can be interpreted as the effect of a local ‘gravitational field’ [23, 24]. Thus, motivated by the equivalence principle, we are led to conjecture that accelerated observers in Minkowski spacetime with the same proper acceleration of the static observers could simulate aspects of the gravitational field in their non-inertial frame. This establishes a connection between static observers of curved spaces and accelerated observers in Minkowski spacetime. Here we want to examine this connection in detail considering the Schwarzschild spacetime.

Let us start our discussion recalling some important definitions. A spacetime is static if it admits a timelike Killing field $T^\mu$, which is also hypersurface-orthogonal [25]. In a spacetime that possess this symmetry, a static observer can be defined as a particle which follows a worldline whose proper velocity $U^\mu = \frac{dx^\mu}{d\tau}$ is proportional to the Killing field:

$$U^\mu = \frac{T^\mu}{V}$$ (53)

where $V(x)$ is the normalization function which satisfies $V^2 = -T^\mu T_\mu$.

The reason for this definition is very clear if we consider coordinates adapted to this Killing field. It can be shown that there exist coordinates in which the metric can be put in the following form [25]:

$$ds^2 = g_{00} dt^2 + g_{ij} dx^i dx^j$$ (54)

where the components $g_{00}$ and $g_{ij}$ do not depend on $t$. In this coordinates, the Killing field assumes the following simple form: $T^\mu = (1, 0, 0, 0)$. Hence, static observers, according to (53), will be characterized by worldlines along...
which the spatial coordinates do not change. Note also that \( g_{00} = -V^2 \), in these coordinates.

In a curved spacetime static observers do not follow geodesics. Obviously they must be accelerated against the 'gravitational attraction' in order to keep their spatial position unchanged. By using the definition and the Killing equation, we can show that the observer's acceleration, \( a^\mu = U^\nu \nabla_\nu U^\mu \), can be written as [25]:

\[
a_\mu = \nabla_\mu \ln V
\]

(55)

Usually the norm of the proper acceleration of an observer is interpreted as the acceleration that is measured by an instantaneous co-moving free-falling frame. Indeed, if we denote the quantities in the free-falling frame by a hat sign and \( \Pi^{\mu\nu} \) as the projection operator orthogonal to \( U^\mu \) (see equation (33)) then, the proper acceleration norm can written as

\[
a_\mu = \sqrt{g_{\mu\nu} a^\mu a^\nu}
\]

(56)

note that we have used the orthogonality condition \( U_\mu a_\mu = 0 \) and the fact that in the free-falling frame, the projection tensor assume locally the diagonal form \( \tilde{\Pi}^{\mu\nu} = \text{diag}(0, 1, 1, 1) \).

We should emphasize that this description so far developed here is valid for all static spacetimes. Now let us concentrate our discussion on the special case of the Schwarzschild spacetime. Considering the Schwarzschild metric

\[
\begin{align*}
\text{ds}^2 &= -\left(1 - \frac{2GM}{r}\right) dt^2 + \left(1 - \frac{2GM}{r}\right)^{-1} dr^2 + r^2 d\Omega^2,
\end{align*}
\]

(57)

where \( d\Omega = d\theta^2 + \sin^2 \theta d\phi^2 \), it is easy to verify, by using (55), that the proper acceleration field depends on the position of the static observers according to the formula:

\[
a = \frac{GM}{r^2} \left(1 - \frac{2GM}{r}\right)^{-1/2}
\]

(58)

Here the acceleration is explicitly written in terms of the coordinate \( r \). In order to make a comparison with the Rindler congruence, it is convenient to find \( a \) as a function of \( \rho \) - the distance of the observer's position to the event horizon located at Schwarzschild radius \( (R_s = 2GM) \). Integrating the line element in the radial direction from \( R_s \) to \( r \), we find

\[
\rho = r \left(1 - \frac{R_s}{r}\right)^{1/2} + \frac{1}{2} R_s \ln \left[ \frac{2r}{R_s} - 1 + \frac{2r}{R_s} \left(1 - \frac{R_s}{r}\right)^{1/2}\right]
\]

(59)
It would be very convenient to write $r$ in terms of $\rho$, but the inverse function cannot be found in an exact form. However, employing the perturbation method, approximate expressions can be obtained. For instance, in a domain close to the horizon, the following expansion is valid

$$ r = R_s + \frac{\delta^2(\rho)}{R_s} \quad (60) $$

where $\delta(\rho)$ is some function of $\rho$ which is small compared to $R_s$. Substituting (60) in the equation (59) we obtain, up to order $\delta^3$, the result $\rho = 2\delta(\rho) + \frac{1}{3R_s^2}\delta^3(\rho)$. Now considering the expansion of the function $\delta(\rho)$ in power series, this last equation gives

$$ \delta(\rho) = \frac{1}{2}\rho - \frac{1}{48R_s^2}\rho^3 \quad (61) $$

which, combined with equation (58) and (60) yields the proper acceleration of static observers that lie in the vicinity the horizon

$$ a = \frac{1}{\rho} - \frac{1}{3R_s^2}\rho^3 \quad (62) $$

This result clearly shows that the dependence $1/\rho$, which holds for Rindler observers, is valid only approximately for those observers which are very close to the horizon.

In the opposite case, $r >> R_s$, the expression (59) gives us, in the first order approximation, $\rho = r$ and, then, equation (58) now yields

$$ a = \frac{GM}{\rho^2} \quad (63) $$

which, obviously, reproduces the inverse square law predicted by Newtonian theory of gravity.

The equation (58) gives us the proper acceleration that the static observers in Schwarzschild spacetime are submitted to. In the context of the present of discussion, we are now led to consider a congruence of observers in Minkowski spacetime whose acceleration field $a(\rho)$ obeys the same law (58).

The general scheme was already developed in the previous section. The analysis relative to the Schwarzschild static observers can be promptly done as a particular case, just by using the appropriate acceleration law (58). Below we show explicitly the embedding functions in terms of the coordinate...
Field equation for the proper acceleration
of static observers

The connection between static observers and accelerated frames in Minkowski spacetime depends crucially on the acceleration of the static observers. In this section we want to deduce the field equation that dictates the behavior of proper acceleration field in curved spaces. This can be achieved by rewriting Einstein equations conveniently. With this purpose, we are going to use the embedding formalism once again. But now we should keep in mind that the ambient space $M$ is a static curved manifold and the hypersurface are the 3-dimensional space orthogonal to the timelike Killing field $T^\mu$.

An important characteristic of this embedding is that the extrinsic curvature is null. In order to check this, consider the covariant derivative of $U^\mu$:

$$\nabla_\mu U_\nu = \frac{1}{V} (\nabla_\mu T_\nu - T_\nu \nabla_\mu \ln V)$$  (66)

In Fig.(4), we have plotted on the Minkowski spacetime diagram several simultaneity sections related to this accelerated congruence. It is interesting to note, comparing with Fig.(1), that for short distances the behavior is very similar to the Rindler congruence while that asymptotically the inverse square law is recovered (see Fig.(2)).

As we have already seen in the previous section, once the embedding functions are determined, the induced geometry can be directly analyzed by the techniques presented in the section (2). Considering the spherical symmetry of Schwarzschild spacetime, equations (17)-(20) are the natural choice for the embedding functions. This means that the induced metric is given by (42) and the curvature of the simultaneity hypersurfaces are described by the Riemann tensor calculated in (46), (47) and (48), taking into account the extrinsic curvature which is evaluated in equations (43), (44) and (45).
It happens that, for a hypersurface-orthogonal Killing field, we may write
\[ \nabla_\mu T_\nu = T_\nu \nabla_\mu \ln V - T_\mu \nabla_\nu \ln V \]  
(67)

It follows immediately that
\[ \nabla_\mu U_\nu = -U_\mu a_\nu \]  
(68)

which, according to the definition (67), leads to \( K_{ab} = 0 \).

As a consequence, the Gauss-Codazzi equation reduces to
\[ ^{(3)} R_{abcd} = e^\alpha_a e^\beta_b e^\gamma_c e^\delta_d R_{\alpha\beta\gamma\delta} \]  
(69)

\[ U^\mu e^\alpha_a e^\beta_b R_{\mu\alpha\beta\gamma} = 0 \]  
(70)

Now in order to obtain the Einstein equations, let us first determine the Ricci tensor. Here we are going to follow closely the notation of reference [27]. Contracting the first and third indices of the Riemann tensor using the spacetime metric conveniently written as \( g_\alpha^\gamma = e^\alpha_a e^\gamma_c h_{ac} - U^\alpha U^\gamma \), we obtain
\[ R_{\beta\nu} = (h_{ac} e^\alpha_a e^\gamma_c R_{\alpha\beta\gamma\nu} - U^\alpha U^\gamma R_{\alpha\beta\gamma\nu}) \]  
(71)

Each index of the Ricci tensor can be projected either in the tangential direction of \( \Sigma \), given by \( e^\beta_b \), or along the orthogonal direction \( U_\mu \). Taking into account the Gauss-Codazzi equations, these projections yield [27]
\[ e^\mu_i e^\nu_j R_{\beta\mu} = ^{(3)} R_{ij} - ^{(3)} E_{ij} \]  
(72)

\[ R_{\mu\nu} U^\mu e^\nu_j = 0 \]  
(73)

\[ R_{\mu\nu} U^\mu U^\nu = ^{(3)} E_{ij} h^{ij} \]  
(74)

where \( ^{(3)} R_{ij} \) is the intrinsic Ricci tensor of \( \Sigma \) and \( ^{(3)} E_{ij} \) is a symmetric tensor defined as
\[ ^{(3)} E_{ij} = U^\mu U^\nu e^\alpha_i e^\beta_j R_{\alpha\mu\beta\nu} \]  
(75)

This tensor inhabits the hypersurface \( \Sigma \), but it is not an intrinsic geometric quantity of the hypersurface since it depends on the orthogonal components of Riemann tensor defined in the ambient space \( M \). As we shall see, \( ^{(3)} E_{ij} \) can be written in terms of the components of the proper acceleration. Indeed, from the definition of the Riemann tensor, we have
\[ R_{\alpha\beta\gamma\nu} U^\mu = D_\beta D_\nu U_\alpha - D_\nu D_\beta U_\alpha \]  
(76)

By using (68) in the above equation and contracting it with \( U^\nu e^\alpha_a e^\beta_b \), we find
\[ ^{(3)} E_{ij} = a_i a_j + e^\mu_i e^\nu_j D_\mu a_\nu \]  
(77)
where, for the sake of simplicity, we are using the notation: \( a_i = e_i^\alpha a_\alpha \).

The acceleration vector \( a^\mu \) is orthogonal to the proper velocity \( U^\mu \), this means that it belongs to the tangent space of \( \Sigma \). Thus, by using the definition of induced covariant derivative (see equation (36)), it possible to write

\[
(3) \ E_{ij} = a_i a_j + (3)^{\;} \nabla_i a_j \tag{78}
\]

Thus, the equations (72) and (74) assume the form

\[
e_i^\mu e_j^\nu R_{\mu\nu} = (3)^{\;} \ R_{ij} - a_i a_j - (3)^{\;} \nabla_i a_j \tag{79}
\]

\[
U^\mu U^\nu R_{\mu\nu} = a^2 + (3)^{\;} \nabla_i a^i \tag{80}
\]

Taking into account the Einstein equations, \( R_{\mu\nu} = 8\pi G \left( T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T \right) \), we can say that the above equations represent the field equations for the proper acceleration of static observers. In particular the equation (80) has a very interesting form that resembles the Newtonian gravitational field equation. However, we have to highlight three important modifications due to the relativistic effects: i) The energy density \( T_{\mu\nu} U^\mu U^\nu \) is no longer the exclusive source of gravity, since there is now contributions that come from other components of the energy-momentum tensor; ii) There is the non-linear term \( a^2 \), whose presence in (80) demonstrates that gravity is a self-interacting field in the relativistic regime; and iii) there is the fact that the divergence of \( a^i \) depends on the intrinsic geometry of the hypersurface \( \Sigma \), which can be curved.

We should also point out that equation (73) does not constrain neither the acceleration nor the metric of \( \Sigma \), but, on the other hand, it imposes the natural condition according to which there is no energy flux in static spacetimes as seen from static observers.

Now, in order to proceed further, it is important to find a relation between the scalar curvature of \( M \) and the intrinsic scalar curvature of \( \Sigma \). With the help of equations (72), (73), (74), it can be shown that

\[
R = (3)^{\;} \ R - 2 \left( a^2 + \nabla_i a^i \right) \tag{81}
\]

Now, equations (72), (73), (74) and (81) permit us to examine the decomposition of the Einstein tensor. It is not difficult to check that the components of the Einstein tensor can be written as

\[
G_{\mu\nu} U^\mu U^\nu = \frac{1}{2} (3)^{\;} R \tag{82}
\]

\[
G_{\mu\nu} e_i^\nu = 0 \tag{83}
\]

\[
G_{ij} = (3)^{\;} G_{ij} - a_i a_j - \nabla_i a_j + h_{ij} \left( a^2 + \nabla_i a^i \right) \tag{84}
\]
where we are using again the notation $G_{ij} = \epsilon_i^\alpha \epsilon_j^\beta G_{\alpha\beta}$.

In the previous sections, we deal with acceleration fields described by functions of a unique coordinate, by virtue of the symmetries of the congruence. Thus, in order to make contact with the previous discussion, let us admit that the static spacetime exhibit additional symmetries.

So suppose that $M$ possesses a spacelike Killing vector field $X^\mu$ orthogonal to the vector field $T^\mu$. Furthermore, let us admit that it commutates with the time-translation generator, $T^\mu$. Of course, this means that $X^\mu$ satisfies the equation:

$$T^\alpha \nabla_\alpha X_\mu = X^\alpha \nabla_\alpha T_\mu$$  \hspace{1cm} (85)

We shall show that $X^\mu$ is also orthogonal to $a^\mu$. Indeed, taking the inner product with $T^\mu$ and using the Killing equation for $X^\mu$, the above equation gives $X^\alpha \nabla_\alpha V^2 = 0$, which, recalling the equation (55), implies the orthogonality condition:

$$X^\alpha a_\alpha = 0$$  \hspace{1cm} (86)

As consequence, we shall see that $X^\mu$ is a vector that belongs to the tangent space of the level surfaces of the function $V(x)$. Indeed, in any one of the hypersurfaces of $M$ that are orthogonal to the timelike Killing vector field $T^\mu$, (remember that these hypersurfaces are isometric 3-manifolds, since they do not evolve in a static spacetime), the equation $V(x) = c$, where $c$ is some positive constant, defines a level surface of the function $V(x)$, which we denote by $S_c$. Of course equation (85) implies that $a^\mu$ is perpendicular to this level surface. Thus, considering that $X^\mu$ is also orthogonal to $T^\mu$, the equation (86) leads us to conclude that $X^\mu$ lies in the tangent space of $S_c$ for some $c$. This holds in every point of $M$.

In general, the norm of acceleration vector $a^\mu$ may vary in the surface $S_c$. However, it is easy to show that, in the direction of Killing field $X^\mu$, the derivative of the norm $a = \sqrt{a_\mu a^\mu}$ is null:

$$X^\mu \nabla_\mu a = 0$$  \hspace{1cm} (87)

If there exist at least two linearly independent Killing field like this vector $X^\mu$, then, as they will span the two-dimensional tangent space of $S_c$, it follows that derivative of $a$ is zero along any tangent direction of $S_c$. In other words, this means that $S_c$ is also a level surface for the acceleration $a$. Therefore, we can write

$$a = Y(V)$$  \hspace{1cm} (88)

for some function $Y$.

Now consider the normal vector of $S_c$, i.e., $\sigma^\mu = \frac{a^\mu}{a}$. Since $S_c$ is a level surface of $a$, the gradient of $a$ is co-linear to $\sigma^\mu$. This fact allows us to write

$$\nabla_\mu a = (\sigma^\nu \nabla_\nu a) \sigma_\mu$$  \hspace{1cm} (89)
Based on this and also on the condition $\nabla_\mu a_\nu = \nabla_\nu a_\mu$ (that follows directly from equation (55)), we can verify that $\sigma^\mu$ satisfies the geodesic equation

$$\sigma^\mu \nabla_\mu \sigma^\nu = 0$$

As the embedding of $\Sigma$ into $M$ has null extrinsic curvature then the above equation is equivalent to $\sigma^i (\nabla_i \sigma^j) = 0$, i.e., the vector $\sigma^i = e_\mu^i \sigma^\mu$ satisfies the geodesic equation with respect to the induced geometry of the hypersurface $\Sigma$.

Let us now explore the consequences of this condition. Pick a certain value for $c$. As we know, associated to this value corresponds a particular level surface $S_c$. Let $x_1^{(1)}$ and $x_1^{(2)}$ be intrinsic coordinates of $S_c$. Solving the equation (91) for geodesics that cross $S_c$ perpendicularly in every point, we can use the affine parameter of these curves, $\rho$, together with coordinates $\{x_1\}$, to build a Gaussian normal coordinate system adapted to $S_c$ in its neighborhood. It follows that, in these new coordinate system, $V$ depends exclusively on $\rho$.

To check this, first note that, in these coordinates, $\sigma^\mu = \left( \frac{\partial x_\mu}{\partial \rho} \right)_{x_1}$. Now, from the normalization condition for $\sigma^\mu$, conveniently written as

$$a_\mu \left( \frac{\partial x_\mu}{\partial \rho} \right)_{x_1} = 1$$

and recalling equation (55) and (88), we find the equation

$$\frac{1}{Y(V)} \left( \frac{\partial \ln V}{\partial \rho} \right)_{x_1} = 1$$

If we assume, as initial condition, that $V$ is a uniform function in that chosen surface $S_c$, then, it follows, from the solution of equation (92), that, in some neighborhood of $S_c$, $V$ is a function of coordinate $\rho$ only. Thus, we can write $V = V(\rho)$.

Therefore the spacetime metric, written in these coordinates, assume the following simple form

$$ds^2 = V^2(\rho) dt^2 + d\rho^2 + \gamma_{AB} dx_1^A dx_1^B$$

where the capital indices have the following range $A, B = 1, 2$. We should note that, depending on the commutation relation between the Killing fields that lies in $S_c$, the induced metric $\gamma_{AB}$ in the level surfaces may admit extra simplifications. For instance, Schwarzschild metric is a special case of (93) in which level surfaces has spherical symmetry.
Now considering the metric in this special form (93), we are led to examine the question of classifying the possible sources of this kind of space-time. In other words, we have to faced the task of characterizing the energy-momentum of matter distributions that could generate such metrics. In particular we are interested in determining the conditions the energy-momentum tensor should satisfy in order to static observers in a curved space have the same acceleration, \( a(\rho) = 1/\rho \), as the Rindler observers in Minkowski space-time.

The better approach to deal with this question is to express (3)\( G_{ij} \) and (3)\( R \), that appears in the equations (82), (83) and (84), in terms of intrinsic and extrinsic curvatures of the level surfaces \( S_c \), applying the embedding formalism again. Making some minor adjustment, the same procedure previously described can be repeated and now it yields the following equations [28]:

\[
(3)\ G_{ij} \sigma^i \sigma^j = -\frac{1}{2} (2)\ R - \frac{1}{2} \Omega^{AC} \Omega_{AC} + \frac{1}{2} \Omega^2 \tag{94}
\]

\[
(3)\ g_{ij} L^i_A \sigma^j = (2)\nabla_C \Omega^C_A - (2)\nabla_A \Omega \tag{95}
\]

\[
(3)\ g_{ij} L^i_A L^j_B = (2)\ G_{AB} - \sigma^c \nabla_c (\Omega_{AB} - \gamma_{AB} \Omega) +
+ (2\Omega^C_A \Omega_{BC} - 3\Omega \Omega_{AB}) + \frac{1}{2} \gamma_{AB} \left( \Omega^{CD} \Omega_{CD} + \Omega^2 \right) \tag{96}
\]

where \( \sigma^i \) is the spacelike normal vector of the level surfaces \( S_c \), \( (2)\nabla \) is the induced covariant derivative compatible with the induced metric \( \gamma_{AB} \) of \( S_c \), \( (2)\ G_{AB} \) and \( (2)\ R \) are the intrinsic Einstein tensor and scalar curvature of \( S_c \) surfaces respectively, the symmetric tensor \( \Omega_{AB} \) is the extrinsic curvature of the level surfaces embedded in \( \Sigma \) and finally \( L^i_A \) is the analogue of \( e^a_i \) (see equation (31)) relative to the embedding of the level surfaces \( S_c \) into the hypersurface \( \Sigma \).

It is important to emphasize here that, in order to deduce these relations, we had to handle the tensor \( (2)E_{AB} \) whose definition

\[
(2)E_{BD} = (3)\ R_{abcd} \sigma^a \sigma^c L^b_B L^d_D \tag{97}
\]

is the two-dimensional analogue of (75). By using the fact that \( \sigma^i \) satisfies the geodesic equation, we could show, from the definition of the Riemann tensor, that

\[
(2)E_{BD} = -\sigma^c \nabla_c \Omega_{BD} + \Omega^C_B \Omega_{CD} \tag{98}
\]

Another useful relation we have employed too was the following formula (valid in the Gaussian coordinate system):

\[
\Omega_{AB} = \frac{1}{2} \sigma^c \nabla_c \gamma_{AB} \tag{99}
\]
Now, before we introduce the energy-momentum tensor into the Einstein equations, let us make one additional assumption concerning the geometry of the spacetime $M$. For the sake of simplicity, henceforth we shall admit that the level surfaces $S_c$ are maximally symmetric spaces. As a consequence, the Riemann tensor and the extrinsic curvature reduce to these simple forms

$$R_{ABCD} = \frac{(2)R}{2} (\gamma_{AC} \gamma_{BD} - \gamma_{AD} \gamma_{BC})$$

$$\Omega_{AB} = \frac{\Omega}{2 \gamma_{AB}}$$

where $(2)R$ and $\Omega$ (trace of extrinsic curvature) do not depend on the intrinsic coordinates of $S_c$, but may depend on $\rho$. The value of $(2)R$ determines completely the intrinsic geometry of the level surfaces. It is well known that if $(3)R$ is positive, null or negative then, the geometry of $S_c$ is spherical, plane or hyperbolic, respectively.

A direct consequence of this assumption is that the Einstein tensor can be totally written as a function of the proper acceleration, the scalar curvature and extrinsic curvature of $S_c$ surfaces in the following form:

$$G_{\mu\nu} U^\mu U^\nu = \frac{1}{2} (2)R - \frac{3}{4} \Omega^2 - \Omega'$$

$$G_{\mu\nu} e_\nu = 0$$

$$G_{ij} \sigma^i \sigma^j = -\frac{1}{2} (2)R + \frac{1}{4} \Omega^2 + a \Omega$$

$$G_{ij} \sigma^j L_i^A = 0$$

$$G_{ij} L_i^A L_j^B = \left[ \frac{1}{2} \Omega' + \frac{1}{4} \Omega^2 + (a^2 + a') + \frac{1}{2} a \Omega \right] \gamma_{AB}$$

where the prime denote derivative with respect to $\rho$.

We should also note that, due to the Bianchi identities, the scalar curvature and the extrinsic curvature of the maximally symmetric surfaces $S_c$ must satisfy the equation:

$$(2)R' + (2)R \Omega = 0$$

Based on the form of the Einstein tensor, we can conclude that the most general source of this special class of static spacetimes (in which the level surfaces of the norm of static observers’ acceleration are maximally symmetric two-dimensional spaces) is a non-viscous fluid described by the following energy-momentum tensor

$$T_{\mu\nu} = (\epsilon + p_\perp) U_\mu U_\nu + (p_\parallel - p_\perp) \sigma_\mu \sigma_\nu + p_\perp g_{\mu\nu}$$
where $\epsilon$ is the energy density, $p_\parallel$ is the pressure in the parallel direction to acceleration of static observers and $p_\perp$ is the pressure in the orthogonal direction relative to $\sigma^i$. All the quantities are measured by the static observers.

Combining equations (102)-(106) and (108), the Einstein equations reduces to

\[
\frac{1}{2} (2) R - \frac{3}{4} \Omega^2 - \Omega' = 8\pi G \epsilon \tag{109}
\]

\[
-\frac{1}{2} (2) R + \frac{1}{4} \Omega^2 + a\Omega = 8\pi G p_\parallel \tag{110}
\]

\[
\frac{1}{2} \Omega' + \frac{1}{4} \Omega^2 + (a^2 + a') + \frac{1}{2} a\Omega = 8\pi G p_\perp \tag{111}
\]

Besides, the fluid must satisfies, as a consequence of the energy-momentum conservation, the following equation

\[
p_\parallel' + (\epsilon + p_\parallel) a + (p_\parallel - p_\perp) \Omega = 0 \tag{112}
\]

Once these equations have been established, we are now in an appropriate position to attack the problem concerning the existence of a curved static spacetime in which the 'Rindler law' is reproduced exactly. As we have seen, the inverse law is satisfied in the Schwarzschild spacetime only approximately by those static observes who are near the horizon. Now, we want to investigate whether there is any source capable of generating a spacetime where the acceleration of static observer is precisely given by $a = \frac{1}{\rho}$. It is easy to see that $a^2 + a' = 0$ for a Rindler acceleration and, then, the equations (109), (110) and (111) allow us to write the extrinsic and intrinsic curvature directly in terms of the energy density and the pressures of the fluid as

\[
\Omega = 4\pi G \rho (\epsilon + p_\parallel + 2p_\perp) \tag{113}
\]

\[
(2) R = 8\pi G (\epsilon - p_\parallel + 2p_\perp) + \frac{1}{2} \left[4\pi G \rho (\epsilon + p_\parallel + 2p_\perp)\right]^2 \tag{114}
\]

where the energy density and the pressures are not totally independent since they must satisfy the conservation equation. A simple solution of equation (112) is obtained by taking the following state equation

\[
p_\parallel = -\epsilon_0 \tag{115}
\]

\[
p_\perp = 0 \tag{116}
\]

where $\epsilon_0$ is a uniform energy density that should be positive in order to satisfy the weak energy condition. These polytropic state equations imply, according to equations (113) and (114), that the level surfaces have null
extrinsic curvature ($\Omega = 0$) and a positive intrinsic curvature given by $(2) R = 16\pi G \epsilon_0$.

It is interesting to note that these are sufficient conditions to completely determine the curvature of this particular spacetime $M$. Indeed, by using the embedding formalism, we can show that the non-null components of the Riemann tensor are reduced to these following components

$$R_{ijkl} L^i_A L^j_B L^k_C L^l_D = 8\pi G \epsilon_0 (\gamma_{AC} \gamma_{BD} - \gamma_{AD} \gamma_{BC})$$

where $R_{ijkl} = R_{\mu\nu\alpha\beta} \epsilon^\mu_i \epsilon^\nu_j \epsilon^\alpha_k \epsilon^\beta_l$. Therefore, $M$ is flat only in the empty space. If $\epsilon_0$ is non-null, then, $M$ is curved and it is such that static observers follow the Rindler acceleration law.

Finally, let us make some comments concerning the source of this special spacetime. As it is well known, the state equation $p = -\epsilon$ is characteristic of a cosmological term or of an energy-momentum tensor related to a field that is found in a false vacuum state. The state equation (115) shows a similar dependence between the parallel pressure and energy density, however because of (116) it is clear that the fluid is not isotropic. Hence, we may say that the source is a kind of anisotropic false vacuum state. It is worthy of mention that some inflationary models speculate about the possibility that the Universe has passed through such rather exotic phase at the early stages of the cosmic evolution [29].

5 Summary

In curved static spacetime, static observers have a non-zero proper acceleration. From their perspective, free-falling bodies are accelerated due to the attraction of a local gravitational field. Because of this interpretation and motivated by the equivalence principle, we have studied congruences, defined in Minkowski spacetime, composed of timelike curves whose acceleration field coincides to the acceleration field of the static observers. Our purpose is to simulate in the non-inertial frame defined in Minkowski spacetime some aspects of the gravitational field that is experienced by static observers in curved spaces.

The embedding of the simultaneity hypersurfaces adapted to the non-inertial frame associated to the congruences, which is an important element in this context, was determined explicitly for any acceleration field, based on the locality principle. We have also investigated the induced geometry of the simultaneity hypersurfaces, determining explicitly the induced metric and the intrinsic Riemann tensor.
The particular case of the Schwarzschild spacetime was investigated in detail. The simultaneity sections were determined and illustrated in Fig. (4). This figure shows clearly that the congruence is equivalent to the Rindler frame only near the event horizon Fig. (1). While that, at large distances, it behaves like a congruence whose acceleration field obeys the inverse square law (see Fig.(2)).

We have also obtained the field equation that regulates the behavior of accelerated field of static observers from the Einstein equations. We have shown that this equation can be understood as the relativistic version of the Newtonian gravitational field equation. As we have seen, according to this equation the divergence (calculated with respect to the induced geometry of the simultaneity hypersurface) of the acceleration field depends on the effective energy density (energy density plus the trace of energy-momentum tensor) and on the norm squared of the acceleration field. The presence of this non-linear term is clearly a consequence of the self-interacting property of gravity in the relativistic regime.

When the spacetime is characterized by the fact that level surfaces of the proper acceleration are maximally symmetric spaces, the field equations are simplified and can be totally expressed in terms of the intrinsic and extrinsic curvature of the level surfaces and the acceleration field. The general form of tensor-energy momentum tensor of the possible sources for this kind of static spacetime was determined. We have seen that it corresponds to a non-viscous fluid totally characterized by the energy density, the parallel pressure and orthogonal pressure, which, in principle, may be different.

When the state equation of the fluid is that of an anisotropic false vacuum state, the produced spacetime has curvature and it is such that static observers are accelerated exactly according to the inverse law characteristic of the Rindler congruence.
6 Appendix

Here we have a list of figures illustrating the embedding of the simultaneity hypersurfaces into Minkowski spacetime adapted to different congruences of accelerated observers.

Figure 1: Non-inertial frame associated to the Rindler congruence which satisfies the inverse law \( a = \frac{1}{\rho} \). The dashed lines are hyperbolas that represent the worldlines of some observers. The full lines are simultaneity hypersurfaces adapted to the non-inertial frame corresponding to different instants of time (\( f(\varphi) = 0.1 \) and \( f(\varphi) = 0.3 \), respectively)

Figure 2: Non-inertial frame associated to a congruence that obey the inverse square law \( a = \frac{1}{\rho^2} \). The dashed lines are hyperbolas that represent the wordlines of some observers. The full lines are simultaneity hypersurfaces adapted to the non-inertial frame corresponding to different instants of time (\( f(\varphi) = 0.5 \) and \( f(\varphi) = 1.2 \), respectively)
Figure 3: Non-inertial frame associated to a congruence of observers submitted to the same acceleration $a = 1$. The dashed lines are hyperbolas that represent the worldlines of some observers. The full lines are simultaneity hypersurfaces adapted to the non-inertial frame corresponding to different instants of time ($f(\varphi) = 0.1$ and $f(\varphi) = 0.3$, respectively).

Figure 4: Non-inertial frame associated to a congruence of timelike curves whose acceleration field coincides with the acceleration of static observers in the Schwarzschild spacetime ($a = \frac{GM}{r^2} \left(1 - \frac{2GM}{r} \right)^{-1/2}$). The dashed lines are hyperbolas that represent the worldlines of some observers. We have assumed $GM = 1$. The full lines are simultaneity hypersurfaces adapted to the non-inertial frame corresponding to different instants of time ($f(\varphi) = 1$ and $f(\varphi) = 2.8$, respectively). Note the equivalence with the Rindler congruence, for small $\rho$. At large distances, the behavior is that of a congruence whose acceleration field obeys the inverse square law.
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