The $r$-matrix structure of the Euler-Calogero-Moser model

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Abstract

We construct the $r$-matrix for the generalization of the Calogero-Moser system introduced by Gibbons and Hermsen. By reduction procedures we obtain the $r$-matrix for the $O(N)$ Euler-Calogero-Moser model and for the standard $A_N$ Calogero-Moser model.
1 Introduction

The Calogero-Moser model seems to occupy a particular place in the world of integrable systems. On one hand, its complete solution is known \[1, 2, 3, 4\] even at the quantum level, but on the other hand, the techniques which have now become standard — in particular the $r$-matrix formalism — did not seem to apply to it.

The beginning of an answer to this problem was given in \[5\] where the classical $r$-matrix was calculated and found to be a dynamical object, which is a non standard feature. This result was later extended to the elliptic case in \[6, 7\].

While the general theory of constant classical $r$-matrices is now well developped, this is not so for the dynamical ones and we still are in the phase of producing examples. Dynamical $r$-matrices appear when we consider higher Poisson structures \[8, 9\]. They also appear in non linear $\sigma$-models \[10, 11\], or in the theory of Sine Gordon solitons \[12\]. Finally one should mention an interesting example of dynamical $r$-matrix for the hyperbolic Gaudin model \[13\].

In this paper we will consider the generalization of the Calogero-Moser model introduced by Gibbons and Hermsen \[14\]. This class of models has recently received some attention in relation with the evolution of energy levels with respect to the parameter $t$ for a perturbed system of Hamiltonian $H = H_0 + tV$ \[15, 16, 17\].

Let us introduce a set of dynamical variables $(q_i, p_i)_{i=1\ldots N}$ and $(f_{ij})_{i,j=1\ldots N}$ together with the Poisson brackets

$$\{p_i, q_j\} = \delta_{ij} \quad (1)$$
$$\{f_{ij}, f_{kl}\} = \delta_{jk} f_{il} - \delta_{li} f_{kj} \quad (2)$$

and the Hamiltonian

$$H = \frac{1}{2} \sum_{i=1}^{N} p_i^2 - \frac{1}{2} \sum_{i,j=1\atop i \neq j}^{N} f_{ij} f_{ji} (q_i - q_j)^2. \quad (3)$$

In order to have a non degenerate Poisson bracket it is assumed that the $(f_{ij})_{i,j=1\ldots N}$ are restricted to a symplectic submanifold of $\mathbb{R}$. Notice furthermore that the quantities $(f_{ii})_{i=1\ldots N}$ Poisson commute with the Hamiltonian; we can therefore reduce the flows to the surfaces $(f_{ii} = \text{constant})_{i=1\ldots N}$. These models were shown to be integrable precisely on these surfaces and our purpose is to compute their $r$-matrices.

The plan of the paper is as follows : after a short review on the $r$-matrix formalism in section 2, we find in section 3 the Poisson structure of the Lax operator which implies the integrability property of the model on the manifolds $(f_{ii} = \text{constant})_{i=1\ldots N}$. In section 4 we use the previous result to obtain the $r$-matrix structure for the $O(N)$ Euler-Calogero-Moser model introduced by Wojciechowski \[18\]. This is achieved by using the mean procedure introduced in \[19, 20, 21\]. Finally in section 5 we again use the results of section 3 to present a new construction of the $r$-matrix of the standard Calogero-Moser model computed for the first time in \[3\] and obtained here by Hamiltonian reduction from the model \[2\].

2 Dynamical $r$-matrices

In the Lax representation of a dynamical system, the equations of motion are written in the form

$$\dot{L} = [L, M] \quad (4)$$

where $L$ and $M$ take values in a Lie algebra $\mathcal{G}$ with basis $\{X_\mu\}$.

Eq.\[4\] implies that the quantities $tr(L^n)$ are conserved. Liouville integrability requires their involution under the Poisson brackets. It was shown in \[24\] that this property $\{tr(L^n), tr(L^n)\} = 0$ is equivalent to the existence of an $r$-matrix.

If we set

$$L_1 = L \otimes 1$$
\[ L_2 = 1 \otimes L \]

this condition reads

\[
\{L_1, L_2\} = [r_{12}, L_1] - [r_{21}, L_2]
\]

\[ (5) \]

where

\[
r_{12} = \sum r_{\mu\nu} X_\mu \otimes X_\nu \quad \text{and} \quad r_{21} = \sum r_{\mu\nu} X_\nu \otimes X_\mu.
\]

In general the \( r \)-matrix is a dynamical object and possesses no special symmetry. Let us recall here some of its properties.

1. The Lax operator \( L \) together with the \( r \)-matrix completely describe the dynamical system: if we choose \( \frac{1}{n} tr(L^n) \) as a Hamiltonian, the equations of motion take the Lax form with \( M = tr_2(r_{12}L_2^{-1}) \).

2. The left-hand side of eq. (5) being a Poisson bracket, the Jacobi identity implies a constraint on \( r \). It is easy to check that it takes the form

\[
[L_1, [r_{12}, L_1] + [r_{32}, r_{13}] + [L_2, r_{13}] - \{L_3, r_{12}\}] + \text{cyclic permutations} = 0.
\]

\[ (6) \]

It is interesting to note that the display of indices in (6) is not the one which appears in the usual Yang-Baxter equation. We recover it when the \( r \)-matrix is antisymmetric.

3. Since the existence of an \( r \)-matrix is equivalent to the involution property of the eigenvalues of \( L \), the form of eq. (5) is not modified by a conjugation of the Lax matrix. Indeed if \( L' = g^{-1}Lg \) then

\[
\{L'_1, L'_2\} = [r'_{12}, L'_1] - [r'_{21}, L'_2]
\]

with

\[
r'_{12} = g^{-1}_1 g^{-1}_2 \left( r_{12} - \{g_1, L_2\} g^{-1}_1 + \frac{1}{2} [u_{12}, L_2] \right) g_1 g_2
\]

\[ (7) \]

where \( u_{12} = \{g_1, g_2\} g^{-1}_1 g^{-1}_2 \).

4. Finally we recall that the \( r \)-matrix of a given Lax matrix is not unique. In particular if we introduce a symmetric matrix \( \tau_{12} = \tau_{21} \) of \( G \otimes G \), the change

\[
r_{12} \rightarrow r_{12} + [\tau_{12}, L_2]
\]

does not affect the equation (5).

3 The Poisson structure for the \( sl(N) \) model

The Lax matrix of the system (3) is

\[
L(\lambda) = \sum_{i=1}^{N} \left( p_i - \frac{f_{ii}}{\lambda} \right) e_{ii} - \sum_{i,j=1}^{N} \left( \frac{1}{q_i - q_j} + \frac{1}{\lambda} \right) f_{ij} e_{ij}.
\]

\[ (9) \]

where \( (e_{ij})_{kl} = \delta_{ik} \delta_{jl} \).

The Poisson brackets of the elements of the Lax matrix (3) can be recast in the form

\[
\{L_1(\lambda), L_2(\mu)\} = [r_{12}(\lambda, \mu), L_1(\lambda)] - [r_{21}(\mu, \lambda), L_2(\mu)] - \sum_{i,j=1}^{N} \frac{f_{ii} - f_{jj}}{(q_i - q_j)^2} e_{ij} \otimes e_{ji}
\]

\[ (10) \]

where

\[
r_{12}(\lambda, \mu) = \frac{C}{\lambda - \mu} - \sum_{i,j=1}^{N} \frac{1}{q_i - q_j} e_{ij} \otimes e_{ji}
\]

\[ (11) \]
and \( C \) is the Casimir element of \( sl(N) \)

\[
C = \sum_{i,j=1}^{N} e_{ij} \otimes e_{ji}.
\]

Some comments are in order:

1. Eq.(10) holds for the non reduced dynamical system (3) and it is known that \( tr L^n(\lambda) \) are not in involution for this system. This fact is responsible for the additional term in eq.(10) compared to eq.(5). Indeed we have

\[
\{ tr L^n(\lambda), tr L^m(\mu) \} = n m \sum_{i,j=1}^{N} \frac{f_{ii} - f_{jj}}{(q_i - q_j)^2} [L^{n-1}(\lambda)]_{ij} [L^{m-1}(\mu)]_{ji}.
\]

2. The constraints \( f_{ii} \) generate on \( L(\lambda) \) a conjugation by a diagonal matrix. Hence \( tr L^n(\lambda) \) Poisson commute with \( f_{ii} \). It follows that one can compute their reduced Poisson bracket on the manifold \( (f_{ii} = \text{constant})_{i=1...N} \). This amounts to setting \( f_{ii} = \text{constant} \) in eq.(10) and therefore \( tr L^n(\lambda) \) are in involution.

3. Since the quantities \( tr L^n(\lambda) \) commute on the reduced phase space, \( L(\lambda) \) has an \( r \)-matrix. However its explicit computation requires some care since \( L(\lambda) \) is not a function on the reduced phase space.

In the following two sections we give two examples of reductions where we can explicitely obtain the \( r \)-matrix, starting from the initial structure (10).

4 The \( r \)-matrix for the \( O(N) \) model

We now consider the following model, introduced by Wojciechowski in [18]. The dynamical variables are \((q_i, p_i)_{i=1...N}\) and antisymmetric \((h_{ij} = -h_{ji})_{i,j=1...N}\). The Poisson bracket is now:

\[
\{ p_i, q_j \} = \delta_{ij}
\]

(12)

\[
\{ h_{ij}, h_{kl} \} = \frac{1}{2} (\delta_{il} h_{jk} + \delta_{ki} h_{lj} + \delta_{jk} h_{il} + \delta_{lj} h_{ki}) .
\]

(13)

The Hamiltonian of the system reads

\[
H = \frac{1}{2} \sum_{i=1}^{N} p_i^2 + \frac{1}{2} \sum_{i,j=1, i \neq j}^{N} \frac{h_{ij}^2}{(q_i - q_j)^2} .
\]

As shown in [18] the equations of motion can be written in the form

\[
\dot{L}(\lambda) = [L(\lambda), M(\lambda)]
\]

where the Lax pair \((L, M)\) is given by

\[
L(\lambda) = \sum_{i=1}^{N} p_i e_{ii} - \sum_{i,j=1, i \neq j}^{N} \left( \frac{1}{q_i - q_j} + \frac{1}{\lambda} \right) h_{ij} e_{ij}
\]

(14)

\[
M(\lambda) = \sum_{i,j=1, i \neq j}^{N} \frac{h_{ij}}{(q_i - q_j)^2} e_{ij} ,
\]

(15)

and the Hamiltonian is \( H = \frac{1}{2} \int \frac{d\lambda}{2\pi i} tr L(\lambda)^2 \).

In [18] this system was proved to have Poisson commuting Hamiltonians \( tr L^n(\lambda) \). It follows that this model admits an \( r \)-matrix, which we now calculate.
First of all we remark that the Lax matrix of the $O(N)$ model is obtained from the Lax matrix of the $sl(N)$ model as

$$L^{O(N)}(\lambda) = \frac{1}{2} (1 - \sigma) L^{sl(N)}(\lambda)$$  \hspace{1cm} (16) $$

where $\sigma$ is the following involutive automorphism of the loop algebra:

$$\sigma : \lambda^n e_{ij} \longrightarrow -(-\lambda)^n e_{ji}.$$  \hspace{1cm} (17) $$

In this average procedure

$$h_{ij} = \frac{1}{2} (f_{ij} - f_{ji})$$  \hspace{1cm} (18) $$

and the Poisson structure (3) of the $sl(N)$ model becomes the structure (13).

We now remark that the Lax matrix of the $O$ model as

$$\{L_1^{O(N)}(\lambda), L_2^{O(N)}(\mu)\} = \frac{1}{4} (1 - \sigma \otimes 1 - 1 \otimes \sigma + \sigma \otimes \sigma) \cdot$$

\[\left( [r_{12}(\lambda, \mu), L_1^{sl(N)}(\lambda)] - [r_{21}(\mu, \lambda), L_2^{sl(N)}(\mu)] - \sum_{i,j=1 \atop i \neq j}^N \frac{f_{ii} - f_{jj}}{(q_i - q_j)^2} e_{ij} \otimes e_{ji} \right). \]  \hspace{1cm} (19) $$

We now remark that

$$(1 - \sigma \otimes 1 - 1 \otimes \sigma + \sigma \otimes \sigma) \sum_{i,j=1 \atop i \neq j}^N \frac{f_{ii} - f_{jj}}{(q_i - q_j)^2} e_{ij} \otimes e_{ji} = 0.$$  \hspace{1cm} (20) $$

Moreover since

$$\sigma \otimes \sigma r_{12}(\lambda, \mu) = -r_{12}(\lambda, \mu)$$  \hspace{1cm} (21) $$

we finally get an explicit $r$-matrix structure for $L^{O(N)}(\lambda)$:

$$\{L_1^{O(N)}(\lambda), L_2^{O(N)}(\mu)\} = [r_{12}^{O(N)}(\lambda, \mu), L_1^{O(N)}(\lambda)] - [r_{21}^{O(N)}(\mu, \lambda), L_2^{O(N)}(\mu)]$$  \hspace{1cm} (22) $$

with

$$r_{12}^{O(N)}(\lambda, \mu) = \frac{1}{2} (1 + \sigma \otimes 1) r_{12}^{sl(N)}(\lambda, \mu)$$  \hspace{1cm} (23) $$

or explicitly

$$r_{12}^{O(N)}(\lambda, \mu) = - \frac{\lambda}{\lambda^2 - \mu} \sum_{i=1}^N e_{ii} \otimes e_{ii} - \frac{1}{2} \sum_{i,j=1 \atop i \neq j}^N \left( \frac{1}{q_i - q_j} + \frac{1}{\lambda + \mu} \right) e_{ij} \otimes e_{ij}$$

$$- \frac{1}{2} \sum_{i,j=1 \atop i \neq j}^N \left( \frac{1}{q_i - q_j} + \frac{1}{\lambda - \mu} \right) e_{ij} \otimes e_{ji}.$$  \hspace{1cm} (24) $$

This is an example of application of the well known mean procedure [1, 21, 21]. It is interesting to see how the Jacobi identity (3) is fulfilled for this dynamical $r$-matrix (24). We find that

$$[r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{32}, r_{13}] = \frac{1}{2} \sum_{i,j=1 \atop i \neq j}^N \frac{1}{(q_i - q_j)^2} e_{ij} \otimes \left[ (e_{ij} + e_{ji}) \otimes (e_{ii} - e_{jj}) - (e_{ii} - e_{jj}) \otimes (e_{ij} + e_{ji}) \right].$$

It is then easy to check that

$$[r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{32}, r_{13}] + \{L_2, r_{13}\} - \{L_3, r_{12}\} = 0.$$  \hspace{1cm} (25) $$

The remarkable feature of this equation is the vanishing of its right-hand side. This is in contrast with other cases [2, 13] where eq. (23) has a non vanishing right-hand side, albeit of a special form.
5 A new construction of the $r$-matrix for the Calogero model

We now show how the dynamical $r$-matrix for the usual Calogero model \cite{5, 6, 7} stems from eq.(10) through a procedure of Hamiltonian reduction.

Let us restrict ourselves to the symplectic manifold

$$ f_{ij} = \xi_i \eta_j $$

with $\{\xi_i, \xi_j\} = 0$, $\{\eta_i, \eta_j\} = 0$, $\{\xi_i, \eta_j\} = -\delta_{ij}$. The Poisson brackets of $f_{ij}$ are indeed given by (2).

On this manifold we have a symplectic action of an Abelian Lie group

$$ \xi_i \rightarrow \lambda_i \xi_i, \ \eta_i \rightarrow \lambda_i^{-1} \eta_i. $$

The Hamiltonian (3) is invariant under this action and we can apply the method of Hamiltonian reduction. The infinitesimal generator of this action is

$$ H_\varepsilon = \sum_{i=1}^{N} \varepsilon_i f_{ii}. $$

The reduction is performed by first fixing the momentum. We choose:

$$ f_{ii} = \alpha. $$

The isotropy group $G_\alpha$ of $\alpha$ is the whole group itself since it is Abelian and we still have to quotient out the action of this group. At the end of the procedure, the $2N$ degrees of freedom $(\xi_i, \eta_i)_{i=1..N}$ are eliminated, leaving as reduced Hamiltonian (3) the Hamiltonian of the usual Calogero-Moser model

$$ H_{\text{Cal}} = \frac{1}{2} \sum_{i=1}^{N} p_i^2 - \frac{1}{2} \alpha^2 \sum_{i,j=1 \atop i \neq j}^{N} \frac{1}{(q_i - q_j)^2}. $$

In order to perform the reduction at the level of the Lax matrix we remark that the group acts on $L$ as conjugation by the matrix $\text{diag}(\lambda_i)_{i=1..N}$. We now observe that the matrix

$$ L_{\text{Cal}} = \text{diag}(\xi_i^{-1}) L \text{ diag}(\xi_i) $$

is invariant under $G_\alpha$ and we can therefore compute the Poisson brackets of its matrix elements safely.

$$ L_{\text{Cal}} = \sum_{i=1}^{N} \left( p_i - \frac{\alpha}{\lambda} \right) e_{ii} - \alpha \sum_{i,j=1 \atop i \neq j}^{N} \left( \frac{1}{q_i - q_j} + \frac{1}{\lambda} \right) e_{ij}. $$

The $r$-matrix of this $L$ matrix is now obtained from eq.(11) by applying the conjugation formula (7).

When setting $f_{ii} = \alpha$ the extra term in eq.(10) vanishes, leaving us with the $r$-matrix

$$ r_{12}^{\text{Cal}}(\lambda, \mu) = r_{12}(\lambda, \mu) - \frac{1}{\mu} \sum_{i=1}^{N} e_{ii} \otimes e_{ii} + \sum_{i,j=1 \atop i \neq j}^{N} \left( \frac{1}{q_i - q_j} - \frac{1}{\mu} \right) e_{ii} \otimes e_{ji}. $$

which gives

$$ r_{12}^{\text{Cal}}(\lambda, \mu) = -\frac{\lambda}{\mu(\lambda - \mu)} \sum_{i=1}^{N} e_{ii} \otimes e_{ii} - \sum_{i,j=1 \atop i \neq j}^{N} \left( \frac{1}{q_i - q_j} + \frac{1}{\lambda - \mu} \right) e_{ij} \otimes e_{ji} + \sum_{i,j=1 \atop i \neq j}^{N} \left( \frac{1}{q_i - q_j} - \frac{1}{\mu} \right) e_{ii} \otimes e_{ji}. $$

This $r$-matrix does not coincide with the $r$-matrix obtained in \cite{3, 4, 8}, however it is easily related to it by a transformation (8) with $\tau_{12} = \frac{1}{2} \sum_{i=1}^{N} e_{ii} \otimes e_{ii}$. 

\[ \text{Page 5} \]
6 Conclusion

The class of models introduced by Gibbons and Hermsen [14] turns out to be a particularly interesting generalization of the Calogero model. In particular their $r$-matrix structure is rather elegant and is general enough to allow the construction of a host of new models by various reduction procedures. Generalization to trigonometric and elliptic potentials is possible and will be developed further.

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