An eigenvalue problem for the anisotropic $\Phi$-Laplacian

A. Alberico ∗– G. di Blasio †– F. Feo ‡

Abstract

We study an eigenvalue problem involving a fully anisotropic elliptic differential operator in arbitrary Orlicz-Sobolev spaces. The relevant equations are associated with constrained minimization problems for integral functionals depending on the gradient of competing functions through general anisotropic $N$-functions. In particular, the latter need neither be radial, nor have a polynomial growth, and are not even assumed to satisfy the so called $\Delta_2$-condition. The resulting analysis requires the development of some new aspects of the theory of anisotropic Orlicz-Sobolev spaces.

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1 Introduction

In the present paper, we deal with the existence of solutions to fully anisotropic eigenvalue problems having the form

$$
\begin{cases}
-\text{div} (\Phi_\xi(\nabla u)) = \lambda b(|u|) \text{sign } u & \text{in } \Omega \\
u = 0 & \text{on } \partial \Omega,
\end{cases}
$$

where $\Omega$ is an open bounded subset in $\mathbb{R}^n$, with $n \geq 2$, $\lambda$ is a positive real parameter, $\Phi : \mathbb{R}^n \to [0, \infty)$ is an $N$-function (see Section 2.1) belonging to $C^1(\mathbb{R}^n)$, and $b : [0, \infty) \to [0, \infty)$ is an increasing, left-continuous function such that $b(t) = 0$ if and only if $t = 0$ and $\lim_{t \to \infty} b(t) = +\infty$. Here, $\Phi_\xi$ denotes the gradient of $\Phi$. Let us emphasize that $\Phi(\xi)$ neither necessarily depends on $\xi$ through its length $|\xi|$, nor necessarily has a power type behavior.

Formally, problem (1.1) represents the Euler-Lagrange equation associated with the following constrained minimization problem

$$
\inf \left\{ \int_\Omega \Phi(\nabla u) \, dx : u \in W^{1,L}_{\Phi}(\Omega), \int_\Omega B(u) \, dx = r \right\},
$$

Key words: Anisotropic Sobolev spaces, Constrained minimum problems, Eigenvalue problems

Mathematics Subject Classifications: 46E30, 35J25, 35P30

∗Istituto per le Applicazioni del Calcolo “M. Picone” (IAC), Consiglio Nazionale delle Ricerche (CNR), Via P. Castellino 111, 80131 Napoli, Italy. E-mail:a.alberico@iac.cnr.it
†Dipartimento di Matematica e Fisica, Università degli Studi della Campania “L. Vanvitelli”, Viale Lincoln, 5 - 81100 Caserta, Italy. E-mail: giuseppina.diblasio@unicampania.it
‡Dipartimento di Ingegneria, Università degli Studi di Napoli “Pathenope”, Centro Direzionale Isola C4 80143 Napoli, Italy. E-mail: filomena.feo@unipartheneope.it

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where \( r \) is any positive real constant, \( W_0^1L_B,\Phi(\Omega) \) is the anisotropic Orlicz-Sobolev space built upon \( \Phi \) and \( B \), where \( B \) is the 1-dimensional \( N \)-function defined as \( B(t) = \int_0^{|t|} b(\tau) \, d\tau \). We point out that neither \( \Phi \) nor \( B \) are required to fulfill the \( \Delta_2 \)-condition. Due to this fact, differentiability of the functionals appearing in (1.2) is not guaranteed. Hence, problem (1.1) cannot be derived via standard methods like constrained minimization or critical point techniques.

The function \( B \) will be subject to a sharp growth condition that follows from the anisotropic Sobolev inequality for \( W_0^1L_B,\Phi(\Omega) \) proved in [12]. For a comprehensive treatment of this matter, we refer the reader to Section 2.3 and Section 3.

Our aim is to show that for any \( r > 0 \) there exist \( \lambda_r > 0 \) and \( u_r \in W_0^1L_B,\Phi(\Omega) \cap L^\infty(\Omega) \) such that

\[
\int_\Omega B(u_r) \, dx = r \quad \text{and} \quad u_r \text{ solves problem } (1.1) \text{ with } \lambda = \lambda_r.
\]

Classical results in this line of investigations deal with the eigenvalue problem for \( p \)-Laplacian

\[
\begin{aligned}
-\text{div} \left( |\nabla u|^{p-2} \nabla u \right) &= \lambda |u|^{p^*-2} u \quad \text{in } \Omega \\
\quad u &= 0 \quad \text{on } \partial \Omega,
\end{aligned}
\]

with \( 1 < p < N \) and \( 1 < q < p^* \), where \( p^* \) stands for the Sobolev conjugate of \( p \). Problem (1.3) is the Euler-Lagrange equation associated with the minimization problem (1.2) corresponding to the choice \( \Phi(\xi) = \frac{1}{p} |\xi|^p \). Several results are available in the literature on existence and properties of eigenvalues and corresponding eigenfunctions to problem (1.3) (see, e.g., [16, 17, 21, 22, 28]).

Isotropic eigenvalue problems and associated constrained minimization problems in the spirit of (1.1) and (1.2), respectively, with \( \Phi(\xi) = \Phi(|\xi|) \) and \( B(t) = \Phi(|\xi|) \), have been investigated in [23]. Our contribution extends the results of [23], not only in allowing for completely fully anisotropic differential operators, but also in admitting more general growths on the right-hand side \( b(|u|) \) sign \( u \). In particular, the generality of the problems under consideration calls for the use and further development of the unconventional functional framework of anisotropic Orlicz and Orlicz-Sobolev spaces which are not necessarily reflexive (see, e.g., [7, 10, 12, 13, 14, 15, 25, 26, 27, 29]).

Let us mention that elliptic equations and variational problems, whose growth is governed by an \( n \)-dimensional \( N \)-function \( \Phi \), have been studied under different perspectives in [2, 5, 6, 4, 3, 7, 9, 12, 13, 14, 15, 20].

The paper is organized as follows. Section 2 contains a background, as well as some new results, on anisotropic Orlicz and Orlicz-Sobolev spaces. The statements of our main results and some special instances are given in Section 3. The proofs of main results are presented in Section 4.

### 2 Functional setting

#### 2.1 Young functions

Let \( n \geq 1 \). Let \( \Phi : \mathbb{R}^n \to [0, +\infty) \) be an \( n \)-dimensional Young function, namely an even, convex function such that \( \Phi(0) = 0 \) and, for every \( t > 0 \), the set \( \{ \xi \in \mathbb{R}^n : \Phi(\xi) < t \} \) is bounded and contains an open neighborhood of 0. An \( n \)-dimensional Young function \( \Phi \) is called an \( n \)-dimensional \( N \)-function if it is a finite valued function, vanishes only at 0 and the following additional conditions are in force

\[
\lim_{|\xi| \to +\infty} \frac{\Phi(\xi)}{|\xi|} = +\infty \tag{2.1}
\]

and

\[
\lim_{|\xi| \to 0} \frac{\Phi(\xi)}{|\xi|} = 0. \tag{2.2}
\]

For \( n = 1 \), any 1-dimensional \( N \)-function \( A : \mathbb{R} \to [0, +\infty) \) takes the form

\[
A(t) = \int_0^{|t|} a(\tau) \, d\tau \quad \text{for } t \in \mathbb{R}, \tag{2.3}
\]
where \( a : [0, \infty) \to [0, \infty) \) is an increasing, right-continuous function, which is positive for \( \tau > 0 \) and satisfies conditions \( a(0) = 0 \) and \( \lim_{\tau \to +\infty} a(\tau) = +\infty \).

If \( \Phi \) is an \( n \)-dimensional Young function, then
\[
(2.4) \quad \Phi(h\xi) \leq |h|\Phi(\xi) \quad \text{for } |h| \leq 1 \text{ and } \xi \in \mathbb{R}^n.
\]

The Young inequality tells us that
\[
(2.5) \quad \xi \cdot \xi' \leq \Phi(\xi) + \Phi_\bullet(\xi') \quad \text{for } \xi, \xi' \in \mathbb{R}^n,
\]
where \( \Phi_\bullet \) is the Young conjugate of \( \Phi \) given by
\[
(2.6) \quad \Phi_\bullet(\xi') = \sup \{ \xi \cdot \xi' - \Phi(\xi) : \xi \in \mathbb{R}^n \} \quad \text{for } \xi' \in \mathbb{R}^n.
\]

Here, "." stands for scalar product in \( \mathbb{R}^n \). We observe that if \( \Phi \) is finite-valued and assumption (2.1) holds, then the function \( \Phi_\bullet \) is an \( n \)-dimensional Young function and finite-valued (see [7, Corollary 6.3]). Note also that the Young conjugation is involutive, i.e. \( \Phi_{\bullet\bullet} = \Phi \). Moreover, \( \Phi_\bullet \) is an \( N \)-function, provided that \( \Phi \) is.

An \( n \)-dimensional Young function \( \Phi \) is said to satisfy the \( \Delta_2 \)-condition near infinity, briefly \( \Phi \in \Delta_2 \) near infinity, if it is finite-valued and there exist constants \( C > 2 \) and \( K \geq 0 \) such that \( \Phi(2\xi) \leq C \Phi(\xi) \) for \( |\xi| > K \).

Let us consider a case when the \( n \)-dimensional \( N \)-function \( \Phi \) is given by
\[
(2.7) \quad \Phi(\xi) = \sum_{i=1}^{n} A_i(\xi_i) \quad \text{for } \xi \in \mathbb{R}^n,
\]
where \( A_i, \) for \( i = 1, \ldots, n, \) are 1-dimensional \( N \)-functions. A standard choice in (2.7) is \( A_i(t) = |t|^{p_i} \) for some powers \( 1 < p_i < +\infty, \) for \( i = 1, \ldots, n. \) One can easily verify that in (2.7) every function \( A_i \in \Delta_2 \) near infinity if and only if \( \Phi(\xi) \) does. An example of a function which does not satisfy the \( \Delta_2 \)-condition is given by
\[
\Phi(\xi) = \sum_{i=1}^{n} (e^{(|\xi_i|^{\alpha_i})} - 1) \quad \text{for } \xi \in \mathbb{R}^n
\]
with \( \alpha_i > 1, \) for any \( i = 1, \ldots, n. \)

The following proposition is a special case of [26, Theorem 5.1].

**Proposition 2.1 (Equality cases in the Young inequality)** Let \( \Phi \) be a differentiable \( n \)-dimensional Young function. Then, for any \( \xi_0 \in \mathbb{R}^n \)
\[
\xi_0 \cdot \eta = \Phi(\xi_0) + \Phi_\bullet(\eta)
\]
if and only if \( \eta = \Phi_\xi(\xi_0). \)

Thanks to Proposition 2.1 in [7, Proposition 6.7] the authors proved the following lemma when \( \Phi \in C^1(\mathbb{R}^n) \), but their proof runs also under the weaker assumption that \( \Phi \) is differentiable.

**Lemma 2.2** Let \( \Phi \) be a differentiable \( n \)-dimensional Young function. Assume that (2.1) holds. Then
\[
(2.8) \quad \Phi_\bullet(\Phi_\xi(\xi)) \leq \Phi_\xi(\xi) \cdot \xi \leq \Phi(2\xi) \quad \text{for } \xi \in \mathbb{R}^n.
\]

Finally, we show a technical lemma which will be very useful in the sequel.

We say that two \( n \)-dimensional \( N \)-functions \( \Phi \) and \( \Psi \) are equivalent if there exist positive constants \( k_1 \) and \( k_2, \) depending only on \( n, \) such that
\[
\Phi(k_1\xi) \leq \Psi(\xi) \leq \Phi(k_2\xi) \quad \text{for } \xi \in \mathbb{R}^n.
\]

We emphasize that \( \Phi \) and \( \Psi \) are equivalent if and only if \( \Phi_\bullet \) and \( \Psi_\bullet \) are.
Lemma 2.3 Given any $n$-dimensional $N$-function $\Phi$, there exists another $n$-dimensional $N$-function $\Psi$ which is strictly convex and equivalent to $\Phi$. As a consequence, $\Psi$ is differentiable.

Proof. Theorem 26.3 in [24] states that the strict convexity of an $N$-function guarantees the differentiability of its conjugate. Thus, it is enough to prove the existence of a strictly convex $N$-function equivalent to $\Phi$. Let $\Phi_- : \mathbb{R}^n \to [0, \infty)$ be the radial function defined as

$$
\Phi_-(\xi) = \sup \left\{ \Theta(\xi) : \Theta : \mathbb{R}^n \to [0, \infty) \text{ N-function, radial and } \Theta(\xi) \leq \Phi(\xi) \right\} \quad \text{for } \xi \in \mathbb{R}^n.
$$

By construction, $\Phi_-$ is an $N$-function. Indeed, $\Phi_-$ is a convex function since it is a supremum of convex functions, and one can easily check that conditions (2.1) and (2.2) are verified.

Fixed $c > 0$ and let $g : [0, \infty) \to [0, \infty)$ be a strictly increasing function such that $0 < g(s) \leq c$ for $s \geq 0$. Then,

$$
G(t) = \int_0^t g(s) \, ds
$$

is a strictly convex, increasing function and $0 < G(t) \leq ct$ for any $t > 0$. Set

$$
\Upsilon(\xi) = G(\Phi_-(\xi)) .
$$

Since $\Phi_-$ is radial and $G$ is strictly convex, it follows that also $\Upsilon$ is strictly convex. Then, $\Phi + \Upsilon$ is an $N$-function, strictly convex and equivalent to $\Phi$ because

$$
\Phi(\xi) \leq \Phi(\xi) + \Upsilon(\xi) \leq \Phi(\xi) + c\Phi_-(\xi) \leq (1 + c)\Phi(\xi) \leq \Phi(1 + c)\xi ,
$$

where the last inequality is due to (2.4).

\[ \square \]

2.2 Anisotropic Orlicz spaces

In this section we present Orlicz spaces built upon both a 1-dimensional Young function (see, e.g., [1]) and $n$-dimensional Young functions (see, e.g., [7, 26, 27, 25]). For the convenience of the reader we give a briefly background.

Let $\Omega$ be a bounded measurable subset in $\mathbb{R}^n$, with $n \geq 2$. The Orlicz space $L_A(\Omega)$, associated with a 1-dimensional Young function $A$, is the set of all measurable functions $g : \Omega \to \mathbb{R}$ such that the Luxemburg norm

$$
\|g\|_{L_A(\Omega)} = \inf \left\{ k > 0 : \int_{\Omega} A \left( \frac{g(x)}{k} \right) \, dx \leq 1 \right\}
$$

is finite. The functional $\| \cdot \|_{L_A(\Omega)}$ is a norm on $L_A(\Omega)$, which makes the latter a Banach space.

Given two finite-valued 1-dimensional Young functions $A$ and $D$, we say that $A \ll D$, namely $A$ increases essentially more slowly than $D$ near infinity, if

$$
\lim_{t \to +\infty} \frac{A(\gamma t)}{D(t)} = 0 \quad \text{for every } \gamma > 0.
$$

Note that if $A \ll D$, then

$$
L_D(\Omega) \hookrightarrow L_A(\Omega) ,
$$

where the arrow “$\hookrightarrow$” stands for continuous embedding.
Let \( \Phi \) be an \( n \)-dimensional Young function. The anisotropic Orlicz class \( L_\Phi(\Omega; \mathbb{R}^n) \) is defined as

\[
L_\Phi(\Omega; \mathbb{R}^n) = \left\{ U : \Omega \rightarrow \mathbb{R}^n \text{ measurable s.t. } \int_\Omega \Phi(U) \, dx < +\infty \right\}.
\]

Note that \( L_\Phi(\Omega; \mathbb{R}^n) \) is a convex set of functions and it needs not be a linear space in general, unless \( \Phi \) satisfies the \( \Delta_2 \)-condition near infinity. The Orlicz space \( L_\Phi(\Omega; \mathbb{R}^n) \) is the linear hull of \( L_\Phi(\Omega; \mathbb{R}^n) \) and it is a Banach space with respect to the following Luxemburg norm

\[
\|U\|_\Phi = \inf \left\{ k > 0 : \int_\Omega \Phi \left( \frac{U}{k} \right) \, dx \leq 1 \right\}.
\]

We emphasize that \( L_\Phi(\Omega; \mathbb{R}^n) \subset L^1(\Omega) \) for any \( n \)-dimensional Young function \( \Phi \). We stress that if two \( n \)-dimensional Young functions \( \Phi \) and \( \Psi \) are equivalent, then \( \| \cdot \|_\Phi \) and \( \| \cdot \|_\Psi \) are equivalent and then \( L_\Phi \) and \( L_\Psi \) are the same space.

Let us denote by \( E_\Phi(\Omega; \mathbb{R}^n) \) the closure in \( L_\Phi(\Omega; \mathbb{R}^n) \) of the bounded measurable functions with compact support in \( \Omega \). In general

\[
E_\Phi(\Omega; \mathbb{R}^n) \subset L_\Phi(\Omega; \mathbb{R}^n) \subset L_\Phi(\Omega; \mathbb{R}^n).
\]

Both inclusions hold as equalities in (2.10) if and only if \( \Phi \) satisfies the \( \Delta_2 \)-condition near infinity (see [25, Corollary 5.1]).

From now on, let \( \Phi \) be an \( n \)-dimensional \( N \)-function. The following generalized Hölder inequality holds

\[
\int_\Omega U(x) \cdot V(x) \, dx \leq 2 \|U\|_\Phi \|V\|_\Phi^*.
\]

for every \( U \in L_\Phi(\Omega; \mathbb{R}^n) \) and \( V \in L_{\Phi^*}(\Omega; \mathbb{R}^n) \) (see [27, Theorem 4.1]). Fixed \( V \in L_{\Phi^*}(\Omega; \mathbb{R}^n) \), the integral in (2.11) defines a linear and continuous functional on \( L_\Phi(\Omega; \mathbb{R}^n) \). The space \( L_\Phi(\Omega; \mathbb{R}^n) \) can be also endowed with the following Orlicz norm

\[
\|U\|_{(\Phi)} = \sup_{\int_\Omega \Phi^*(V) \leq 1} \left| \int_\Omega U(x) \cdot V(x) \, dx \right|.
\]

Thanks to Lemma [2,3] we can assume that \( \Phi \) is differentiable (up to an equivalent \( N \)-function), and then Luxemburg norm (2.9) and Orlicz norm (2.12) are equivalent, i.e. \( \|U\|_\Phi \leq \|U\|_{(\Phi)} \leq 2\|U\|_\Phi \) (see [27, Theorem 4.5]).

Combining the Orlicz norm and the Luxemburg norm together it is possible to get this sharp form of generalized Hölder inequality

\[
\int_\Omega U(x) \cdot V(x) \, dx \leq \|U\|_{(\Phi)} \|V\|_{\Phi^*}
\]

for every \( U \in L_\Phi(\Omega; \mathbb{R}^n) \) and \( V \in L_{\Phi^*}(\Omega; \mathbb{R}^n) \).

If \( A \) is a 1-dimensional \( N \)-function, it is well-known that the dual space of \( E_A(\Omega) \) is isomorphic and homeomorphic to \( L_{A^*}(\Omega) \) (see [1, Theorem 8.18]). The analogue result holds for the anisotropic spaces.

**Proposition 2.4** Let \( \Phi \) be an \( n \)-dimensional \( N \)-function. The dual space of \( E_\Phi(\Omega; \mathbb{R}^n) \), denoted by \((E_\Phi(\Omega; \mathbb{R}^n))'\), is isomorphic and homeomorphic to \( L_{\Phi^*}(\Omega; \mathbb{R}^n) \) and the duality pairing is given by

\[
<V, U> = \int_\Omega V(x) \cdot U(x) \, dx
\]

for \( V \in L_{\Phi^*}(\Omega; \mathbb{R}^n) \) and \( U \in E_\Phi(\Omega; \mathbb{R}^n) \).
Remark 2.5 Note that if $\Phi \in \Delta_2$, then $(L_\Phi(\Omega; \mathbb{R}^n))' = L_{\Phi^*}(\Omega; \mathbb{R}^n)$.

**Proof of Proposition 2.2.** We proceed by steps. First we show that any element $V \in L_{\Phi^*}(\Omega; \mathbb{R}^n)$ determines a bounded linear functional defined as

\[
(2.13) \quad \langle l_V, U \rangle = \int_\Omega U(x) \cdot V(x) \, dx
\]

for every $U \in E_\Phi(\Omega; \mathbb{R}^n)$. Then, it remains to show that every bounded linear functional on $E_\Phi(\Omega; \mathbb{R}^n)$ can be written uniquely in the form $l_V$ for some $V \in L_{\Phi^*}(\Omega; \mathbb{R}^n)$. In order to do this, we prove that any bounded linear functional $l$ on $E_\Phi(\Omega; \mathbb{R}^n)$ has the form $(2.13)$ when we restrict ourselves to the set of vector-valued simple functions, i.e., vector-valued functions such that each component is a simple function (functions that assume a finite number of values). The density of this set in $E_\Phi$ allows us to conclude the proof.

**Step 1.** $l_V$ restricted on $E_\Phi(\Omega; \mathbb{R}^n)$ belongs to $(E_\Phi(\Omega; \mathbb{R}^n))'$.

It follows by $(2.11)$.

**Step 2.** The set of vector-valued simple functions is dense in $E_\Phi$.

Let us consider $U \in L^{\infty}(\Omega; \mathbb{R}^n)$. By standard measure theory, there exists a sequence $\{U_h\}_h$ of vector-valued simple functions such that

1. $U_h \to U$ a.e. in $\Omega$,
2. $\|U_h\|_{L^{\infty}(\Omega; \mathbb{R}^n)} \leq C_0 \|U\|_{L^{\infty}(\Omega; \mathbb{R}^n)}$ for all $h \in \mathbb{N}$ and for a given positive constant $C_0$.

We claim that $\|U_h - U\|_\Phi \to 0$. Indeed, by ii) we have

\[
\|U_h - U\|_{L^{\infty}(\Omega; \mathbb{R}^n)} \leq (C_0 + 1)\|U\|_{L^{\infty}(\Omega; \mathbb{R}^n)} := C_1.
\]

Given $\epsilon > 0$, set

\[
C_2(\epsilon) = \sup_{|x| \leq C_1} \Phi \left( \frac{\epsilon}{\xi} \right)
\]

and

\[
S_h = \left\{ x \in \Omega : \Phi \left( \frac{U_h(x) - U(x)}{\epsilon} \right) > \frac{1}{2|\Omega|} \right\}.
\]

By i) and continuity of $\Phi$, the measure $|S_h| \to 0$ as $h \to \infty$. Then, we can choose $h$ sufficiently large such that $|S_h| < \frac{1}{2C_2(\epsilon)}$. It follows that

\[
\int_\Omega \Phi \left( \frac{U_h - U}{\epsilon} \right) \, dx = \int_{S_h} \Phi \left( \frac{U_h - U}{\epsilon} \right) \, dx + \int_{\Omega \setminus S_h} \Phi \left( \frac{U_h - U}{\epsilon} \right) \, dx
\]

\[
\leq C_2(\epsilon) \frac{1}{2C_2(\epsilon)} + |\Omega| \frac{1}{2|\Omega|} = 1,
\]

that implies $\|U_h - U\|_\Phi \leq \epsilon$. By definition of $E_\Phi(\Omega; \mathbb{R}^n)$, the result holds for $U \in E_\Phi(\Omega; \mathbb{R}^n)$.

**Step 3.** Representation formula for $l \in (E_\Phi(\Omega; \mathbb{R}^n))'$ on vector-valued simple functions.

Let $l$ be any continuous linear functional on $E_\Phi(\Omega; \mathbb{R}^n)$ and let $G \subset \Omega$ be a measurable set. Let us consider $\chi_G(x)$, the characteristic function of the set $G$, and let us set

\[
\overrightarrow{\chi}_G(x) = (\chi_G(x), \ldots, \chi_G(x)) \quad \text{and} \quad \overrightarrow{\chi}_{G,h}(x) = (0, \ldots, 0, \chi_{G,h}(x), 0, \ldots, 0).
\]

Then $\overrightarrow{\chi}_G(x) = \sum_{h=1}^n \chi_G(x)e_h = \sum_{h=1}^n \chi_{G,h}(x)$, where $\{e_1, \ldots, e_n\}$ is the standard base of $\mathbb{R}^n$. For every $h \in \{1, \ldots, n\}$, by Step 2 we have that $\overrightarrow{\chi}_{G,h} \in E_\Phi(\Omega; \mathbb{R}^n)$. Moreover $\mu_h(G) = \langle l, \overrightarrow{\chi}_{G,h}(x) \rangle$ is an absolutely continuous measure. Indeed,

\[
(2.14) \quad | \langle l, \overrightarrow{\chi}_{G,h}(x) \rangle | \leq \|l\| \cdot \|\overrightarrow{\chi}_{G,h}\|_\Phi
\]
\[ \| \chi_{G,h} \|_{\Phi} = \inf \left\{ k : \int_{\Omega} \Phi \left( \frac{\chi_{G,h}(x)}{k} \right) dx \leq 1 \right\} = \inf \left\{ k : |G\Phi \left( \frac{\varepsilon h}{k} \right) | \leq 1 \right\}. \]

Let us define the 1-dimensional Young function \( A_h : \mathbb{R} \to [0, \infty) \) as \( A_h(t) = \Phi(te_h) \) for every \( t \in \mathbb{R} \). Then,

\[ (2.15) \quad \| \chi_{G,h} \|_{\Phi} = \frac{1}{A_h^{-1} \left( \frac{1}{|G|} \right)}. \]

The absolute continuity of measure \( \mu_h \) follows by combining (2.14) and (2.15). By virtue of the Radon-Nikodym’s Theorem, there exists a real valued function \( V_h \) belonging to \( L^1(\Omega) \) such that

\[ (2.16) \quad < l, \chi_{G,h} > = \int_{\Omega} V_h(x) \chi_G(x) dx \quad \forall h = 1, \ldots, n. \]

By (2.16),

\[ (2.17) \quad < l, \chi_G > = \sum_{h=1}^{n} < l, \chi_{G,h} > = \int_{\Omega} V(x) \cdot \chi_G(x) dx, \]

where \( V(x) = (V_1(x), \ldots, V_n(x)) \). Moreover, if \( U \) is a vector-valued simple function defined as

\[ U(x) = \sum_{j=1}^{n} \alpha_j \chi_{G_j}(x), \]

where \( \alpha_j \in \mathbb{R} \) and \( G_j \) are disjoint measurable subsets of \( \Omega \), by the linearity of \( l \) and (2.17), we get

\[ (2.18) \quad < l, U > = \sum_{j=1}^{n} \alpha_j < l, \chi_{G_j}(x) > = \sum_{j=1}^{n} \alpha_j \int_{\Omega} V(x) \cdot \chi_{G_j}(x) dx = \int_{\Omega} U \cdot V dx = < l_V, U >, \]

where \( l_V \) is defined by (2.13).

**Step 4. Function \( V \) that appears in (2.18) belongs to \( L_{\Phi^*}(\Omega; \mathbb{R}^n) \).**

Let \( U \in E_\Phi(\Omega; \mathbb{R}^n) \). By Step 2, we know that there exists a sequence \( U_h \) of simple functions such that \( U_h \to U \) in \( L_\Phi(\Omega; \mathbb{R}^n) \). This means that \( U_h \to U \) almost everywhere and also the sequence \( |U_h \cdot V| \to |U \cdot V| \) almost everywhere. Moreover, fixed some positive constant \( K \), one can choose \( h \) sufficiently large such that

\[ \|U_h\|_{\Phi} \leq \|U\|_{\Phi} + \|U_h - U\|_{\Phi} \leq \|U\|_{\Phi} + K. \]

Now, if \( U_h \cdot V \geq 0 \), on applying Fatou’s Lemma, we get

\[ (2.19) \quad \int_{\Omega} U \cdot V dx \leq \liminf_{h} \int_{\Omega} U_h \cdot V dx \leq \liminf_{h} \left\| U_h \cdot V \right\| dx \leq \liminf_{h} \|l\| \|U_h\|_{\Phi} \leq \|l\| (\|U\|_{\Phi} + K). \]

On the other hand, if \( U_h \cdot V \leq 0 \), on applying Fatou’s Lemma again, we get

\[ \int_{\Omega} -U \cdot V dx \leq \liminf_{h} \int_{\Omega} -U_h \cdot V dx \leq \liminf_{h} \left\| -U_h \cdot V \right\| dx \leq \liminf_{h} \|l\| \|U_h\|_{\Phi} \leq \|l\| (\|U\|_{\Phi} + K). \]

By (2.19) and (2.20), we deduce

\[ (2.21) \quad \left| \int_{\Omega} U \cdot V dx \right| < +\infty \]
for any $U \in E_{\Phi}(\Omega; \mathbb{R}^n)$. This means that if we choose $U(x) = (U_1(x), \ldots, U_n(x))$ such that

$$U_i(x) = \frac{\partial \Phi}{\partial e_i}(x) \quad \text{for} \quad i = 1, \ldots, n$$

then, by Proposition 2.1, we get

$$\int_{\Omega} \Phi(V) \, dx \leq \int_{\Omega} \Phi(V) \, dx + \int_{\Omega} \Phi(U) \, dx = \int_{\Omega} U \cdot V \, dx.$$ 

We stress that the extra assumption on differentiability of $\Phi$ required in Proposition 2.1 can be dropped thanks to Lemma 2.3. Finally, since (2.21), it follows that $V \in L_{\Phi^*}(\Omega; \mathbb{R}^n)$ and $l_V$ is linear bounded functional on $E_{\Phi}(\Omega, \mathbb{R}^n)$.

**Step 5. Identification between $l$ and $l_V$.**

We note that both the functionals $l_V$ defined as in (2.13) and $l$ assume the same values on the set of vector-valued simple functions. Since the last set is dense in $E_{\Phi}(\Omega, \mathbb{R}^n)$, they agree with $E_{\Phi}(\Omega, \mathbb{R}^n)$ and the proof is complete.

For the convenience of the reader, let us recall a few definitions concerning the convergence and boundedness. We say that a sequence $\{U_h\}_h \subset L_{\Phi}(\Omega; \mathbb{R}^n)$ converges in mean to $U \in L_{\Phi}(\Omega; \mathbb{R}^n)$ if

$$\int_{\Omega} \Phi(U_h - U) \, dx \rightarrow 0.$$ 

Note that the convergence in norm implies the convergence in mean, and they are equivalent if and only if $\Phi$ satisfies the $\Delta_2$-condition. Moreover, if a function $U \in L_{\Phi}(\Omega; \mathbb{R}^n)$ is bounded in mean, namely

$$\int_{\Omega} \Phi(U) \, dx < C$$

for some constant $C > 0$, then it is bounded in norm as well. The converse is not true unless $\Phi$ satisfies the $\Delta_2$-condition.

Finally, we extend [19, Lemma 1] to vector-valued functions that it will be useful in what follows.

**Lemma 2.6** Let $\Phi$ be a differentiable $n$-dimensional $N$-function. For all $V \in L_{\Phi}(\Omega; \mathbb{R}^n)$

$$\sup \left\{ \int_{\Omega} U \cdot V \, dx - \int_{\Omega} \Phi(U) \, dx : U \in L_{\Phi^*}(\Omega; \mathbb{R}^n) \right\} = \int_{\Omega} \Phi(V) \, dx$$

$$= \sup \left\{ \int_{\Omega} U \cdot V \, dx - \int_{\Omega} \Phi(U) \, dx : U \in E_{\Phi^*}(\Omega; \mathbb{R}^n) \right\}$$

**Proof.** Since $E_{\Phi^*}(\Omega; \mathbb{R}^n) \subset L_{\Phi^*}(\Omega; \mathbb{R}^n)$, we have only to prove that

$$\sup \left\{ \int_{\Omega} U \cdot V \, dx - \int_{\Omega} \Phi(U) \, dx : U \in L_{\Phi^*}(\Omega; \mathbb{R}^n) \right\} \leq \int_{\Omega} \Phi(V) \, dx$$

$$\leq \sup \left\{ \int_{\Omega} U \cdot V \, dx - \int_{\Omega} \Phi(U) \, dx : U \in E_{\Phi^*}(\Omega; \mathbb{R}^n) \right\}.$$ 

The left-hand side in (2.22) follows by applying Young inequality.
Now let us prove the right-hand side in (2.22). Let $V \in L_\Phi(\Omega; \mathbb{R}^n)$. We define $U_h = \Phi_\xi (V_h)$ by

$$V_h = \begin{cases} 
V(x) & \text{for } x \text{ s.t. } |V(x)| \leq h \\
0 & \text{otherwise},
\end{cases}$$

where $|\cdot|$ denotes the modulus in $\mathbb{R}^n$. We claim that $U_h \in L_\infty(\Omega; \mathbb{R}^n) \subset E_{\Phi}(\Omega; \mathbb{R}^n)$. Indeed, by (2.8) we get

$$0 \leq \Phi_\xi (U_h) \leq \Phi(2V_h).$$

By definition of $N$-function and (2.23), it easily follows that if $V \equiv 0$ then $U_h \equiv 0$ and the claim is obvious. Let $V \not\equiv 0$ and suppose by contradiction that $\sup_{x \in \Omega} |U_h(x)| = +\infty$. Then, there exists a sequence $\{x_j\} \subset \Omega$ such that $|U_h(x_j)| \to +\infty$. Thus, by (2.23), we get

$$0 \leq \frac{\Phi_\xi (U_h(x_j))}{|U_h(x_j)|} \leq \frac{\Phi(2V_h(x_j))}{|U_h(x_j)|}.$$

By (2.1), the term in the center blows up, while the right-hand side goes to zero, because $\Phi$ is continuous and $V_h(x)$ is bounded. This proves that $U_h \in L_\infty(\Omega; \mathbb{R}^n)$.

By Proposition 2.1, we obtain $\Phi(V_h) = U_h \cdot V_h - \Phi_\xi (U_h)$. So, by integrating on $\Omega$, it follows that

$$\int_{\Omega} \Phi(V_h) \, dx = \int_{\Omega} U_h \cdot V_h \, dx - \int_{\Omega} \Phi_\xi (U_h) \, dx \leq \int_{\Omega} U_h \cdot V \, dx - \int_{\Omega} \Phi_\xi (U_h) \, dx$$

and then

$$\int_{\Omega} \Phi(V_h) \, dx \leq \sup \left\{ \int_{\Omega} U \cdot V \, dx - \int_{\Omega} \Phi_\xi (U) \, dx : U \in E_{\Phi}(\Omega; \mathbb{R}^n) \right\}.$$  

By Fatou’s Lemma, the left-hand side in (2.24) converges to $\int_{\Omega} \Phi(V) \, dx < +\infty$ and (2.22) follows.

\[ \square \]

### 2.3 Anisotropic Orlicz-Sobolev spaces

Let $\Phi$ be an $n$-dimensional $N$-function. Let us define the Banach space $W_{0}^{1}L_{\Phi}(\Omega)$ (see [7]) as

$$W_{0}^{1}L_{\Phi}(\Omega) = \{ u : \Omega \to \mathbb{R} : \text{ the continuation of } u \text{ by } 0 \text{ outside } \Omega \text{ is weakly differentiable and } \nabla u \in L_{\Phi}(\Omega; \mathbb{R}^n) \}$$

equipped with the norm

$$\|u\|_{W_{0}^{1}L_{\Phi}(\Omega)} = \|\nabla u\|_{L_{\Phi}(\Omega; \mathbb{R}^n)}.$$

We emphasize that the following anisotropic Sobolev type inequality holds for any function in $W_{0}^{1}L_{\Phi}(\Omega)$ (see [12]). Assume that $\Phi$ fulfils

$$(2.25) \quad \int_{0}^{1} \left( \frac{\tau}{\Phi_\xi (\tau)} \right)^{\frac{1}{n-1}} \, d\tau < \infty,$$

where $\Phi_\xi : [0, \infty) \to [0, \infty)$ is an $N$-function satisfying

$$|\{\xi \in \mathbb{R}^n : \Phi(\xi) \leq t\}| = |\{\xi \in \mathbb{R}^n : \Phi_\xi (|\xi|) \leq t\}| \quad \text{for } t \geq 0.$$
Note that the function $\xi \mapsto \Phi_\circ (|\xi|)$ agrees with the spherically increasing symmetral of $\Phi$. We denote by $\Phi_n : [0, \infty) \to [0, \infty]$ the optimal Sobolev conjugate of $\Phi$ defined as

$$(2.26) \quad \Phi_n(t) = \Phi_\circ (H^{-1}(t)) \quad \text{for} \ t \geq 0,$$

where $H : [0, \infty) \to [0, \infty)$ is given by

$$H(t) = \left( \int_0^t \left( \frac{\tau}{\Phi_\circ(\tau)} \right) \frac{1}{n-1} \ d\tau \right)^{\frac{n-1}{n}} \quad \text{for} \ t \geq 0,$$

provided that the integral is convergent. Here, $H^{-1}$ denotes the generalized left-continuous inverse of $H$.

If

$$(2.27) \int_0^\infty \left( \frac{\tau}{\Phi_\circ(\tau)} \right)^{\frac{1}{n-1}} d\tau = \infty,$$

then there exists a constant $C_1 = C_1(n)$ such that

$$(2.28) \quad \|u\|_{L^{\Phi_n}(\Omega)} \leq C_1 \|u\|_{W_0^1 L^{\Phi}(\Omega)}$$

for every $u \in W_0^1 L^{\Phi}(\Omega)$ (see [12, Theorem 1 and Remark 1]).

If

$$(2.29) \int_0^\infty \left( \frac{\tau}{\Phi_\circ(\tau)} \right)^{\frac{1}{n-1}} d\tau < \infty,$$

then there exists a constant $C_2 = C_2(\Phi, n, |\Omega|)$ such that

$$(2.30) \quad \|u\|_{L^{\infty}(\Omega)} \leq C_2 \|u\|_{W_0^1 L^{\Phi}(\Omega)}$$

for every $u \in W_0^1 L^{\Phi}(\Omega)$ (see e.g. [15, Theorem 1.2]).

We define the anisotropic Orlicz-Sobolev space $W^{1} L^{B,\Phi}(\Omega)$ as

$$W^{1} L^{B,\Phi}(\Omega) = \{u \in L^{B}(\Omega) : u \text{ is weakly differentiable in } \Omega \text{ and } \nabla u \in L^{\Phi}(\Omega; \mathbb{R}^n)\}.$$

The space $W^{1} E^{B,\Phi}(\Omega)$ is defined accordingly by replacing $L^{B}(\Omega)$ and $L^{\Phi}(\Omega; \mathbb{R}^n)$ by $E^{B}(\Omega)$ and $E^{\Phi}(\Omega; \mathbb{R}^n)$, respectively. Both $W^{1} L^{B,\Phi}(\Omega)$ and $W^{1} E^{B,\Phi}(\Omega)$ can be identified to subspaces of the product $L^{B}(\Omega) \times L^{\Phi}(\Omega; \mathbb{R}^n)$. The spaces $W^{1} L^{B,\Phi}(\Omega)$ and $W^{1} E^{B,\Phi}(\Omega)$ equipped with the norm

$$(2.31) \quad \|u\|_{W^{1} L^{B,\Phi}(\Omega)} = \|u\|_{L^{B}(\Omega)} + \|\nabla u\|_{L^{\Phi}(\Omega; \mathbb{R}^n)}$$

are Banach spaces (see [1, Theorem 3.2]).

Let us denote by $W_0^1 L^{B,\Phi}(\Omega)$ the $\sigma(L^{B} \times L^{\Phi}, E^{B} \times E^{\Phi})$-closure of $D(\Omega)$ in $W^{1} L^{B,\Phi}(\Omega)$. Analogously, $W_0^1 E^{B,\Phi}(\Omega)$ stands for the closure of $D(\Omega)$ in $W^{1} E^{B,\Phi}(\Omega)$ with respect to the norm (2.31).

Let us emphasize that, given a function $u \in W_0^1 L^{B,\Phi}(\Omega)$, the function obtained by extending $u$ outside $\Omega$ by zero belongs to $W^{1} L^{B,\Phi}(\mathbb{R}^n)$. Thus,

$$(2.32) \quad W_0^1 L^{B,\Phi}(\Omega) \subset W_0^1 L^{\Phi}(\Omega).$$

Both spaces, $W_0^1 L^{B,\Phi}(\Omega)$ and $W_0^1 L^{\Phi}(\Omega)$, are reflexive if and only if $\Phi \in \Delta_2$ near infinity. Embedding (2.32) yields directly that Sobolev type inequalities (2.28) and (2.30) hold for $W_0^1 L^{B,\Phi}(\Omega)$.

Moreover, the following compact embedding holds.
Proposition 2.7 Let $\Phi$ be an $N$-function. Assume that either (2.27) holds and $B \prec\prec \Phi_n$ or (2.29) holds and $B$ is anything. Then

$$W_0^1L_{B,\Phi}(\Omega) \hookrightarrow E_B(\Omega),$$

where the arrows $\hookrightarrow$ stand for compact embedding.

Proof. Arguing as in the proof of [8, Theorem 2.1], we deduce that

$$\left\{ u : \int_{\Omega} \Phi_n \left( \frac{|u(x)|}{\lambda} \right) \, dx < \infty \text{ for every } \lambda > 0 \right\} \subset \text{closure of } L^\infty(\Omega) \text{ in } L_{\Phi_n}(\Omega).$$

Finally, observing that $\int_{\Omega} \Phi_n \left( \frac{|u(x)|}{\lambda} \right) \, dx < \infty$ for every $\lambda > 0$ whenever $u \in W_0^1L_{B,\Phi}(\Omega)$ (see also [11, Remark 7]), we conclude that, if $B \prec\prec \Phi_n$,

$$W_0^1L_{B,\Phi}(\Omega) \subset E_{\Phi_n}(\Omega) \subset E_B(\Omega).$$

Let $\{u_h\}_h$ be a bounded sequence in $W_0^1L_{B,\Phi}(\Omega)$. Since the compact embedding (see [7])

$$W_0^1L_{B,\Phi}(\Omega) \hookrightarrow W_0^{1,1}(\Omega) \hookrightarrow L^1(\Omega),$$

it follows that (up a subsequence) $\{u_h\}_h$ converges in $L^1(\Omega)$ and then in measure in $\Omega$. If $B \prec\prec \Phi_n$, the convergence in measure and the boundedness in $L_{\Phi_n}(\Omega)$ of $\{u_h\}_h$ (that follows by (2.28)) yield that $\{u_h\}_h$ converges in $L_B(\Omega)$ (see [11, Theorem 8.22]). The embedding (2.34) and the closure of $E_B(\Omega)$ conclude the proof.

If (2.29) holds, we have that for any $N$-function $B$,

$$W_0^1L_{B,\Phi}(\Omega) \subset L^\infty(\Omega) \subset E_{B_1}(\Omega) \subset E_B(\Omega),$$

where $B_1$ is an $N$-function such that $B \prec\prec B_1$. Thus, one can use the same argument as in the previous case on replacing $\Phi_n$ by $B_1$ in order to conclude the proof.

\[ \square \]

2.4 Complementary systems

Let $X$ and $K$ be real Banach spaces in duality with respect to continuous pairing $\langle \cdot, \cdot \rangle$, and let $X_0$ and $K_0$ be subspaces of $X$ and $K$, respectively. Then, $(X, X_0; K, K_0)$ represents a so-called complementary system if, by means of $\langle \cdot, \cdot \rangle$, the dual of $X_0$ can be identified to $K$ and that of $K_0$ to $K$.

Given a complementary system $(X, X_0; K, K_0)$ and a closed subspace $Y$ of $X$, it is possible to construct a new complementary system imposing some restrictions on $Y$. More precisely, set $Y_0 = Y \cap X_0$, $Z = K/Y_0^\perp$ and $Z_0 = \{ z + Y_0^\perp : z \in K_0 \} \subset Z$, where $Y_0^\perp = \{ z \in K : \langle y, z \rangle = 0 \text{ for every } y \in Y_0 \}$.

The theory on complementary system has been investigated e.g. in [18], and, for the convenience of the reader, we recall Lemma 1.2 contained in it. The relevant lemma provides conditions so that $(Y, Y_0; Z, Z_0)$ is a complementary system generated by $Y$ in $(X, X_0; K, K_0)$.

Lemma 2.8 The pairing $\langle \cdot, \cdot \rangle$ between $X$ and $K$ induces a pairing between $Y$ and $Z$ if and only if $Y_0$ is $\sigma(X, K)$ dense in $Y$. In this case, $(Y, Y_0; Z, Z_0)$ is a complementary system if $Y$ is $\sigma(X, K_0)$ closed, and conversely, when $K_0$ is complete, $Y$ is $\sigma(X, K_0)$ closed if $(Y, Y_0; Z, Z_0)$ is a complementary system.
The topologies \( \sigma(Y,Z) \) and \( \sigma(Y,Z_0) \) are the weak topologies induced on \( Y \) by \( \sigma(X,K) \) and \( \sigma(X,K_0) \), respectively, and \( Z_0 \) is the subspace of the dual space of \( Y_0 \) equals the set of those linear functionals on \( Y_0 \) which are \( \sigma(X_0,K_0) \) continuous.

Here, our aim is to prove that \( Y = W^1_0L_{B,\phi}(\Omega) \) generates a new complementary system in \( (X,X_0,K,K_0) = (L_B \times L_\phi, E_B \times E_\phi; L_{B,*} \times L_{\phi,*}, E_{B,*} \times E_{\phi,*}) \).

In order to do this, we assume that \( \Omega \) enjoys the segment property, namely there exist a locally finite open covering \( \{\Omega_j\}_j \) of \( \partial\Omega \) and corresponding vectors \( \{y_j\}_j \) such that \( x + ty_j \in \Omega \) with \( x \in \overline{\Omega} \cap \Omega_j \) and \( 0 < t < 1 \). This condition is essential in Lemma 2.9 below.

Let us verify that the conditions in Lemma 2.8 are fulfilled. First, \( W^1_0L_{B,\phi}(\Omega) \) is \( \sigma(L_B \times L_\phi,L_{B,*} \times L_{\phi,*}) \) closed thanks to the very definition of \( W^1_0L_{B,\phi}(\Omega) \). Moreover, we have to verify that \( W^1_0L_{B,\phi}(\Omega) \cap (E_B \times E_\phi) \) agrees with \( W^1_0E_{B,\phi}(\Omega) \) and it is \( \sigma(L_B \times L_\phi,L_{B,*} \times L_{\phi,*}) \) dense in \( W^1_0L_{B,\phi}(\Omega) \).

**Lemma 2.9** If \( \Omega \) enjoys the segment property, then

(a) \( W^1_0E_{B,\phi}(\Omega) = W^1_0L_{B,\phi}(\Omega) \cap (E_B \times E_\phi) \),

(b) \( W^1_0E_{B,\phi}(\Omega) \) is \( \sigma(L_B \times L_\phi,L_{B,*} \times L_{\phi,*}) \) dense in \( W^1_0L_{B,\phi}(\Omega) \).

**Proof.** (a) To prove the assert, one can be reduced to prove that \( \mathcal{D}(\Omega) \) is norm dense in \( W^1_0L_{B,\phi}(\Omega) \cap (E_B \times E_\phi) \). By [18, Theorem 1.3 and Corollary 1.10], it is enough to verify that \( \mathcal{D}(\Omega) \) is \( \sigma(L_B \times L_\phi,L_{B,*} \times L_{\phi,*}) \) dense in \( W^1_0L_{B,\phi}(\Omega) \). It follows by an appropriate version of Lemmas 1.4 - 1.7 in [18] applied to the \( n \)-dimensional Young function \( \Phi \). In fact, one can easily verify that those lemmas hold for vectorial functions, as well.

(b) Let us recall that, by definition, \( \mathcal{D}(\Omega) \) is dense in \( W^1_0E_{B,\phi}(\Omega) \) with respect to the norm (2.9) and is \( \sigma(L_B \times L_\phi,L_{B,*} \times L_{\phi,*}) \) dense in \( W^1_0L_{B,\phi}(\Omega) \). Our goal is to prove that \( W^1_0E_{B,\phi}(\Omega) \) is \( \sigma(L_B \times L_\phi,L_{B,*} \times L_{\phi,*}) \) dense in \( W^1_0L_{B,\phi}(\Omega) \), namely that for every \( u \in W^1_0L_{B,\phi}(\Omega) \) there exists a sequence \( \{u_h\}_h \subset W^1_0E_{B,\phi}(\Omega) \) such that

\[
u_h \to u \quad \text{in} \quad \sigma(L_B \times L_\phi,L_{B,*} \times L_{\phi,*}),
\]
i.e.

\[
\int_{\Omega} u_h \psi_1 dx + \int_{\Omega} \nabla u_h \cdot \psi_2 dx \to \int_{\Omega} u \psi_1 dx + \int_{\Omega} \nabla u \cdot \psi_2 dx \quad \forall (\psi_1,\psi_2) \in L_{B,*} \times L_{\phi,*}.
\]

Let us suppose by contradiction that there exists a function \( \bar{u} \) in \( W^1_0L_{B,\phi}(\Omega) \) such that, for every sequence \( \{u_h\}_h \subset W^1_0E_{B,\phi}(\Omega) \),

\[
(2.36) \quad \lim_{h \to \infty} u_h \neq \bar{u} \quad \text{in} \quad \sigma(L_B \times L_\phi,L_{B,*} \times L_{\phi,*}).
\]

On the other hand, by the very definition of \( W^1_0E_{B,\phi}(\Omega) \), for every \( u_h \in W^1_0E_{B,\phi}(\Omega) \) there exists a sequence \( \{v^k_h\}_{k \in \mathbb{N}} \subset \mathcal{D}(\Omega) \) such that \( v^k_h \to u_h \) in norm, and then

\[
(2.37) \quad \lim_{k \to \infty} v^k_h = u_h \quad \text{in} \quad \sigma(L_B \times L_\phi,L_{B,*} \times L_{\phi,*}).
\]

The statement follows by observing that (2.37) does not agree with (2.36) and by recalling that \( W^1_0L_{B,\phi}(\Omega) \) is the \( \sigma(L_B \times L_\phi,E_{B,*} \times E_{\phi,*}) \)-closure of \( \mathcal{D}(\Omega) \) in \( W^1L_{B,\phi}(\Omega) \).\[\square\]
Lemma 2.8 and Lemma 2.9 assure that \((W_0^1L_B,\Phi(\Omega), W_0^1E_B,\Phi; W^{-1}L_{B^*},\Phi(\Omega), W^{-1}E_{B^*},\Phi(\Omega))\) is the complementary system generated by \(W_0^1L_B,\Phi(\Omega)\) in \((L_B \times L_\Phi, E_B \times E_\Phi; L_{B^*} \times L_{\Phi^*}, E_{B^*} \times E_{\Phi^*})\), where

\[
W^{-1}L_{B^*},\Phi(\Omega) = \left\{ f \in \mathcal{D}'(\Omega) : f = f_0 - \sum_{i=1}^{n} \frac{\partial}{\partial x_i} f_i, \quad f_0 \in L_{B^*}(\Omega), (f_1, \ldots, f_n) \in L_{\Phi^*}(\Omega; \mathbb{R}^n) \right\}
\]

and

\[
W^{-1}E_{B^*},\Phi(\Omega) = \left\{ f \in \mathcal{D}'(\Omega) : f = f_0 - \sum_{i=1}^{n} \frac{\partial}{\partial x_i} f_i, \quad f_0 \in E_{B^*}(\Omega), (f_1, \ldots, f_n) \in E_{\Phi^*}(\Omega; \mathbb{R}^n) \right\}.
\]

3 Main results

Assume that \(\Omega\) is an open bounded subset of \(\mathbb{R}^n\), with \(n \geq 2\), satisfying the segment property, and \(\Phi \in C^1(\mathbb{R}^n)\) is an \(N\)-function fulfilling (2.25).

Our first main result concerns the existence of solutions to the following Dirichlet problem

\[
\begin{align*}
\Omega & \quad \begin{cases}
\text{−div} (\Phi \xi (\nabla u)) = \lambda \ b(|u|) \ \text{sign} \ u \\
u = 0
\end{cases} \\
\partial \Omega & \quad \text{in} \ \Omega
\end{align*}
\]

where \(\Phi \xi\) denotes the gradient of \(\Phi\), \(\lambda > 0\) and for \(t > 0\) function \(b(t)\) is the derivative (see (2.23)) of a 1-dimensional \(N\)-function \(B\) fulfilling some suitable assumptions.

**Definition 3.1** A function \(u \in W_0^1L_\Phi(\Omega)\) is called a weak solution to problem (3.1) if \(\Phi \xi (\nabla u) \in L_{\Phi^*}(\Omega; \mathbb{R}^n)\), \(b(|u|) \in L_{B^*}(\Omega)\) and

\[
\int_{\Omega} \Phi \xi (\nabla u) \cdot \nabla \varphi \ dx = \lambda \int_{\Omega} \frac{b(|u|)}{|u|} u \varphi \ dx
\]

for any \(\varphi \in W_0^1L_\Phi(\Omega) \cap L^\infty(\Omega)\).

The existence result for solutions to (3.1) reads as follows.

**Theorem 3.2** Let \(\Omega\) be a bounded Lipschitz domain in \(\mathbb{R}^n\). Let \(\Phi \in C^1(\mathbb{R}^n)\) be an \(n\)-dimensional \(N\)-function fulfilling (2.25). Assume that \(B\) is a 1-dimensional \(N\)-function such that \(B \prec \prec \Phi \) if (2.27) holds or \(B\) is any if (2.29) is in force. Then, for any \(r > 0\) there exists a weak solution \(u_r \in W_0^1L_\Phi(\Omega) \cap L^\infty(\Omega)\) to problem (3.1), with \(\lambda = \lambda_r\), such that \(\int_{\Omega} B(u_r) \ dx = r\).

**Remark 3.3** A close inspection of the proof of Theorem 3.2 reveals that the solution \(u_r\) actually belongs to the space \(W_0^1L_B,\Phi(\Omega) \subset W_0^1L_\Phi(\Omega)\) (see (2.32)).

**Remark 3.4** Note that any bounded Lipschitz domain in \(\mathbb{R}^n\) satisfies the segment property, also.

In order to establish our main result we consider the following constrained minimization problem

\[
c_r = \inf \left\{ \int_{\Omega} \Phi (\nabla u) \ dx : u \in W_0^1L_B,\Phi(\Omega), \int_{\Omega} B(u) \ dx = r \right\}
\]

for any \(r > 0\), where \(B\) is as above.

As already observed in the Introduction, since \(\Phi \notin \Delta_2\) and \(B \notin \Delta_2\), differentiability of the functionals appearing in (3.3) is not guaranteed. Then we cannot apply the standard method of Lagrange multipliers to obtain Theorem 3.2. However, problem (3.1) can be still regarded as the Euler-Lagrange equation associated with problem (3.3).

Our next result guarantees the existence of a minimizer of problem (3.3).
Theorem 3.5 Under the same assumptions as in Theorem 3.2, for any $r > 0$, minimization problem (3.3) has at least one minimizer $u_r \in W_0^1 L_B(\Omega)$.

We observe that since no $\Delta_2$-condition is required on $\Phi$ and on $B$, conditions $\Phi_\xi(\nabla u_r) \in L_{\Phi_*}(\Omega; \mathbb{R}^N)$ and $b(|u_r|) \in L_{B_*}(\Omega)$ does not necessarily occur, then in general (3.1) is not well-defined. Nevertheless, we are able to prove the following result.

Proposition 3.6 Under the same assumptions as in Theorem 3.2, if $u_r \in W_0^1 L_B(\Omega)$ is a minimizer of problem (3.3), then

(i) $\Phi_\xi(\nabla u_r) \in L_{\Phi_*}(\Omega; \mathbb{R}^n)$;

(ii) $b(|u_r|) \in L_{B_*}(\Omega)$.

3.1 Examples

In this Subsection, we specialize Theorem 3.2 to some classes of $N$-functions $\Phi$, which govern the differential operator in the equation in (3.1), with a distinctive structure.

If $\Phi$ is defined as in (2.7), problem (3.1) takes the form

$$
\begin{cases}
-\sum_{i=1}^n \left( A_i'(u_{x_i}) \right) x_i = \lambda b(|u|) \text{sign} u & \text{in } \Omega \\
u = 0 & \text{on } \partial \Omega,
\end{cases}
$$

(3.4)

where $A_i$, for $i = 1, \ldots, n$, are 1-dimensional $N$-functions. One has that (see [12])

$$
\Phi_o(t) \approx \overline{A}(t) \quad \text{near infinity},
$$

(3.5)

where $\overline{A}$ is the 1-dimensional $N$-function obeying

$$
\overline{A}^{-1}(\tau) = \left( \prod_{i=1}^n A_i^{-1}(\tau) \right)^{\frac{1}{n}}.
$$

(3.6)

Thus, Theorem 3.2 holds for problem (3.4) where $\Phi_o$ is replaced by $\overline{A}$ in the definition of $\Phi_n$ (see (2.26)).

Example 1. Let

$$
A_i(t) = \frac{1}{p_i} t^{p_i} \log^{\alpha_i}(c + t) \quad \text{for } t > 0,
$$

(3.7)

where $p_i > 1$, $\alpha_i \in \mathbb{R}$, $i = 1, \ldots, n$, and $c$ positive constant sufficiently large for all functions $A_i(t)$ be convex. Let $\overline{p}$ and $\overline{\alpha}$ be defined as

$$
\frac{1}{\overline{p}} = \frac{1}{n} \sum_{i=1}^n \frac{1}{p_i} \quad \text{and} \quad \overline{\alpha} = \frac{\overline{p}}{n} \sum_{i=1}^n \alpha_i.
$$

With this choice of $A_i$ in problem (3.4) one has that function $\Phi_n(x)$ defined as in (2.26) has the following behavior

$$
\Phi_n(x) \approx \begin{cases}
    t^{\overline{p}} (\log(c + t))^{\overline{\alpha} - \overline{p}} & \text{if } \overline{p} < n \\
    e^{n - \overline{\alpha} - 1} & \text{if } \overline{p} = n, \overline{\alpha} < n - 1 \\
    e^{n - \overline{\alpha}} & \text{if } \overline{p} = n, \overline{\alpha} = n - 1,
\end{cases}
$$

(3.8)
near infinity. When \( \overline{p} > n \), or \( \overline{p} = n \) and \( \overline{\alpha} > n - 1 \), condition (2.29) holds. Assume that
\[
B(t) \leq \begin{cases} 
\left( t^p \log(c + t) \right)^{\frac{\tau}{n-p}} & \text{if } \overline{p} < n \\
\left( e^{\frac{n}{n-p}} \right)^{\frac{\tau}{n-p}} & \text{if } \overline{p} = n, \overline{\alpha} < n - 1 \\
\left( e^{\frac{n}{n-p}} \right)^{\frac{\tau}{n-p}} & \text{if } \overline{p} = n, \overline{\alpha} = n - 1 ,
\end{cases}
\]
and \( B(t) \) is any if \( \overline{p} > n \) or \( \overline{p} = n, \overline{\alpha} > n - 1 \).
Hence, thanks to Theorem 3.2 for any \( r > 0 \) there exist a constant \( \lambda_r > 0 \) and \( u_r \in W_0^1 L^\overline{p} \log^\overline{\alpha} L(\Omega) \cap L^\infty(\Omega) \) such that \( \int_\Omega B(u_r) \, dx = r \) and \( u_r \) solves problem (3.4) with \( A_i \) as in (3.7), and with \( \lambda = \lambda_r \).

Example 2. Now we show a special instance with
\[
\Phi(\xi) = \sum_{i=1}^n \left( e^{\alpha_i |\xi_i|} - 1 \right) \quad \text{for } \xi \in \mathbb{R}^n ,
\]
where \( \alpha_i > 1 \).
The corresponding problems read
\[
\begin{cases} 
- \sum_{i=1}^n \left( \alpha_i |u_{xi_i}|^{\alpha_i} |u_{x_i}|^{\alpha_i-2} u_{x_i} \right) = \lambda b(|u|) \text{sign } u & \text{in } \Omega \\
u = 0 & \text{on } \partial \Omega .
\end{cases}
\]
By (3.5) and (3.6) again, we have
\[
\Phi^{-1}(s) \approx \left( \prod_{i=1}^n \left( \log(1 + s) \right)^{\frac{1}{\alpha_i}} \right)^{\frac{1}{n \overline{\alpha}}} = (\log(1 + s))^{\frac{1}{n} \sum_{i=1}^n \frac{1}{\alpha_i}} = (\log(1 + s))^{\frac{1}{n \overline{\alpha}}} \quad \text{near infinity} ,
\]
where \( \overline{\alpha} \) is the harmonic average of \( \alpha_i \), for \( i = 1, \ldots, n \). Then,
\[
\Phi(t) \approx e^{t^\overline{p}} - 1 \quad \text{near infinity} ,
\]
and condition (2.29) is always verified. Thus, Theorem 3.2 holds for any \( N \)-function \( B \).

Example 3. Let us consider now another particular case of the function (2.7) given by
\[
\Phi(\xi) = \sum_{i=1}^{n-1} \frac{1}{p_i} |\xi_i|^{p_i} + \left( e^{\alpha |\xi_n|^{\alpha}} - 1 \right) \quad \text{for } \xi \in \mathbb{R}^n ,
\]
where \( p_i > 1 \), for \( i = 1, \ldots, n - 1 \), and \( \alpha > 1 \). Note the \( \Phi \notin \Delta_2 \). Now, problem (3.4) agrees with
\[
\begin{cases} 
- \sum_{i=1}^{n-1} \left( |u_{xi_i}|^{p_i-2} u_{xi_i} \right) + \left( \alpha e^{u_{x_n}^{\alpha}} |u_{x_n}|^{\alpha-2} u_{x_n} \right) = \lambda b(|u|) \text{sign } u & \text{in } \Omega \\
u = 0 & \text{on } \partial \Omega .
\end{cases}
\]
One can verify via (3.5) and (3.6) that
\[
\Phi^{-1}(s) \approx \left( \prod_{i=1}^n A_i^{-1}(s) \right)^{\frac{1}{n}} \approx \left( s^{\sum_{i=1}^n \frac{1}{p_i} \left( \log(1 + s) \right)^{\frac{1}{p_i}}} \right)^{\frac{1}{n}}
\]
\[
= s^{\frac{1}{n} \sum_{i=1}^n \frac{1}{p_i} \left( \log(1 + s) \right)^{\frac{1}{p_i}}} = s^{\frac{1}{n} \left( \log(1 + s) \right)^{\frac{1}{\overline{p}}}} \quad \text{near infinity} ,
\]
where \( \frac{1}{\beta} = \frac{1}{n} \sum_{i=1}^{n-1} \frac{1}{p_i} \). Then

\[
\Phi(t) \approx t^\beta (\log(1 + t))^{-\frac{\beta}{\alpha}} \quad \text{near infinity}.
\]

If \( \Phi \) verifies condition (2.27), namely if \( \frac{1-\beta}{n-1} > -1 \), i.e. if \( \sum_{i=1}^{n-1} \frac{1}{p_i} > 1 \), then

\[
\Phi_n(t) \approx t^\beta (\log(1 + t))^{-\frac{\beta}{\alpha(n-\beta)}} \quad \text{near infinity}.
\]

Whereas, if \( \Phi \) fulfills condition (2.29), i.e. if \( \sum_{i=1}^{n-1} \frac{1}{p_i} \leq 1 \), then \( \Phi_n \) agrees with \(+\infty\) near infinity.

By assuming that

\[
B(t) \ll s^\beta (\log(1 + t))^{-\frac{\beta}{\alpha(n-\beta)}} \quad \text{if } \sum_{i=1}^{n-1} \frac{1}{p_i} > 1
\]

and

\[
B(t) \quad \text{is any} \quad \text{if } \sum_{i=1}^{n-1} \frac{1}{p_i} \leq 1,
\]

Theorem 3.2 holds.

**Example 4.** We present now a possible instance of examples which generalize one from [Tr] provided by \( N \)-functions \( \Phi \) of the form

\[
\Phi(\xi) = \sum_{k=1}^{K} A_k \left( \left| \sum_{i=1}^{n} \alpha_i^k \xi_i \right| \right) \quad \text{for } \xi \in \mathbb{R}^n,
\]

where \( A_k \) are \( N \)-functions of one variable, \( K \in \mathbb{N} \) and coefficients \( \alpha_i^k \in \mathbb{R} \) are arbitrary.

When \( n = 2 \), we consider, for example, the \( N \)-function given by (see [3, Example 5])

\[
\Phi(\xi) = |\xi_1 - \xi_2|^p + |\xi_1|^q (\log(c + |\xi_1|))^\alpha \quad \text{for } \xi \in \mathbb{R}^2,
\]

where \( c \) is a sufficiently large constant for \( \Phi \) to be convex, \( p > 1 \) and either \( q \geq 1 \) and \( \alpha > 0 \), or \( q = 1 \) and \( \alpha > 0 \).

Hence, problem (3.1) becomes

\[
\begin{cases}
- \left[ (\Phi x_1(u_{x_1}, u_{x_2})) x_1 + (\Phi x_2(u_{x_1}, u_{x_2})) x_2 \right] = \lambda b(|u|) \sign u & \text{in } \Omega \\
u = 0 & \text{on } \partial\Omega,
\end{cases}
\]

where

\[
\Phi x_1(u_{x_1}, u_{x_2}) = p|u_{x_1} - u_{x_2}|^{p-2}(u_{x_1} - u_{x_2}) + q|u_{x_1}|^{q-2}u_{x_1} (\log(c + |u_{x_1}|))^\alpha \\
+ \alpha|u_{x_1}|^{q-1} \frac{u_{x_1}}{c + |u_{x_1}|} (\log(c + |u_{x_1}|))^\alpha - 1,
\]

and

\[
\Phi x_2(u_{x_1}, u_{x_2}) = -p|u_{x_1} - u_{x_2}|^{p-2}(u_{x_1} - u_{x_2}).
\]

The Sobolev conjugate of \( \Phi \) takes the following values

\[
\Phi_2(s) = \begin{cases}
\frac{2 s^{2 q}}{s^{p+q} - p q} \log \frac{p q}{s^{p+q} - p q} (t) & \text{if } p q < p + q \\
\exp \left( t^{\frac{2 q}{p+q}} \right) & \text{if } p q = p + q \text{ and } p q < p + q \\
\exp(\exp(t^{2})) & \text{if } p q = p q = p + q
\end{cases}
\]

(3.8)
near infinity.
If \( pq = p + q \) and \( pq > p + q \), or \( pq > p + q \), then condition (2.29) holds.
By assuming that
\[
B(t) \preceq \Phi_2(t),
\]
where \( \Phi_2 \) is as in (3.3) and
\[
B(t) \text{ is any if } pq = p + q \text{ and } pq > p + q, \text{ or } pq > p + q,
\]
Theorem 3.2 holds.

4 Proofs of main results
In this section, we provide the proof of our main results stated in section § 3. In order to prove Theorem 3.2, we first show the existence of a minimizer of constrained minimization problem (3.3).
We only focus on the case \( B \preceq \Phi_n \), since the other case runs easily. In both cases, the key tool is the compact embedding of \( W_0^1 L_{B, \Phi}(\Omega) \) into \( E_B \) which is guaranteed by Proposition 2.7.

**Proof of Theorem 3.5.** Let us introduce the following functionals \( F : W_0^1 L_{B, \Phi}(\Omega) \to [0, \infty) \) defined as
\[
F(u) = \int_\Omega \Phi(\nabla u) \, dx,
\]
and
\[
G(u) = \int_\Omega B(u) \, dx,
\]
respectively. We observe that \( F \) is a finite-valued functional on \( W_0^1 L_{B, \Phi}(\Omega) \) if and only if \( \Phi \) fulfils the \( \Delta_2 \)-condition. Whereas, \( G(u) \) is always finite for every \( u \in W_0^1 L_{B, \Phi}(\Omega) \) because the compact embedding stated in Proposition 2.7.
In order to prove the existence of a minimizer, we have to show first the continuity of \( G \) and the lower semicontinuity of \( F \) with respect to the topology \( \sigma(W_0^1 L_{B, \Phi}(\Omega), W^{-1} E_{B_\Phi, \Phi^*}(\Omega)) \), where \( W_0^1 L_{B, \Phi}(\Omega) \) has to be understood as the dual space of \( W^{-1} E_{B_\Phi, \Phi^*}(\Omega) \).

**Step 1.** \( G \) is \( \sigma(W_0^1 L_{B, \Phi}(\Omega), W^{-1} E_{B_\Phi, \Phi^*}(\Omega)) \) continuous.
It is enough to prove that if
\[
(4.1) \quad u_h \to u \quad \text{in } \sigma(W_0^1 L_{B, \Phi}(\Omega), W^{-1} E_{B_\Phi, \Phi^*}(\Omega)),
\]
then \( G(u_h) \to G(u) \).
By (4.1), it follows that \( u_h \) is bounded in \( W_0^1 L_{B, \Phi}(\Omega) \). By the compact embedding of \( W_0^1 L_{B, \Phi}(\Omega) \) in \( E_B(\Omega) \) (see Proposition 2.7), we have that \( u_h \) converges to \( u \) in norm in \( E_B(\Omega) \). Since convergence in norm implies the mean convergence (see § 2.2), we get \( B(2(u_h - u)) \to 0 \) in \( L^1(\Omega) \). It follows, (up a subsequence), that \( u_h \to u \) a.e. in \( \Omega \) and there exists (up a subsequence) a function \( w \in L^1(\Omega) \) such that
\[
B(2|u_h - u|) \leq w(x) \quad \text{a.e. in } \Omega.
\]
Owing to the strictly monotonicity and the convexity of function \( B \), we obtain
\[
B(u_h) \leq \frac{1}{2} B(2u) + \frac{1}{2} w \quad \text{a.e. in } \Omega,
\]
and then the statement of Step 1 follows thanks to Lebesgue’s dominate convergence Theorem.

**Step 2.** \( F \) is \( \sigma(W_0^1 L_{B, \Phi}(\Omega), W^{-1} E_{B_\Phi, \Phi^*}(\Omega)) \) lower semicontinuous.
By definition, it is enough to prove that $F(u) \leq \lim \inf F(u_h)$ if (4.1) holds.

Let us fix $\varepsilon > 0$. Since $\nabla u_h \in L_\Phi(\Omega; \mathbb{R}^n)$ for all $h$, by Lemma 2.6 there exists a function $W \in E_{\Phi^*}(\Omega; \mathbb{R}^n)$ such that

$$F(u_h) = \int_\Omega \Phi(\nabla u_h) \, dx \geq \int_\Omega \nabla u_h \cdot W \, dx - \int_\Omega \Phi^*(W) \, dx \quad \forall h \in \mathbb{N}$$

and

$$F(u) = \int_\Omega \Phi(\nabla u) \, dx \leq \int_\Omega \nabla u \cdot W \, dx - \int_\Omega \Phi^*(W) \, dx + \varepsilon,$$

namely

$$(4.2) \quad F(u_h) - F(u) \geq \int_\Omega \nabla u_h \cdot W \, dx - \int_\Omega \nabla u \cdot W \, dx - \varepsilon \quad \forall h \in \mathbb{N}.$$ 

By (4.2) and (4.1), we get

$$\lim \inf F(u_h) \geq F(u) - \varepsilon,$$

and the proof of Step 2 follows by the arbitrariness of $\varepsilon$.

**Step 3. Existence of a minimizer of (3.3).**

Let $\{u_h\}_h \subset W^1_0 L_B(\Omega)$ be a minimizing sequence of (3.3), i.e.

$$G(u_h) = \int_\Omega B(u_h) \, dx = r \quad \forall h \in \mathbb{N}$$

and

$$F(u_h) = \int_\Omega \Phi(\nabla u_h) \, dx \to c_r \quad \text{as } h \to \infty.$$ 

This means that $\{\nabla u_h\}_h$ is bounded in mean and then in norm in $L_\Phi(\Omega; \mathbb{R}^n)$ (see § 2.2). By Banach-Alaoglu’s Theorem, there exists (up a subsequence) $u_r \in W^1_0 L_B(\Omega)$ such that $u_h \to u_r$ in $\sigma(W^1_0 L_B, \Phi(\Omega), W^{-1} E_{\Phi^*}(\Omega))$. By Step 1 and Step 2, it follows

$$G(u_r) = r \quad \text{and} \quad F(u_r) \leq \lim \inf F(u_h) = c_r.$$ 

By definition of $c_r$, we conclude that $F(u_r) = c_r$.

Our next aim is to prove Proposition 3.6. To do this, the next auxiliary lemmas will be critical.

**Lemma 4.1** Let $U \in L_\Phi(\Omega; \mathbb{R}^n)$. Then the following statements hold

(a) $\Phi \xi((1 - \varepsilon)U) \in L_{\Phi^*}(\Omega; \mathbb{R}^n)$ for all $\varepsilon \in (0, 1]$;

(b) $\Phi \xi((1 - \varepsilon)U + V) \in L_{\Phi^*}(\Omega; \mathbb{R}^n)$ for all $V \in E_\Phi(\Omega; \mathbb{R}^n)$ and for all $\varepsilon \in (0, 1]$.

**Proof.** Let $U \in L_\Phi(\Omega; \mathbb{R}^n)$ and $V \in E_\Phi(\Omega; \mathbb{R}^n)$. The case $\varepsilon = 1$ is trivial, so let $\varepsilon \in (0, 1)$.

(a) By (2.8) and the convexity of $\Phi$, we get

$$\frac{\varepsilon}{1 - \varepsilon} \Phi^* \left( \Phi \xi((1 - \varepsilon)U) \right) \leq \varepsilon U \cdot \Phi \xi((1 - \varepsilon)U) \leq \varepsilon U \cdot \Phi \xi((1 - \varepsilon)U) + \Phi((1 - \varepsilon)U) \leq \Phi(U).$$
Then, since \( U \in \mathcal{L}_{\Phi}(\Omega; \mathbb{R}^n) \), it follows \( \Phi_* (\Phi_{\xi} ((1 - \varepsilon) U)) \in L^1(\Omega) \), namely \( \Phi_{\xi} ((1 - \varepsilon) U) \in \mathcal{L}_{\Phi_*}(\Omega; \mathbb{R}^n) \).

(b) Thanks to the convexity of \( \Phi \), we have that

\[
(4.3) \quad \Phi \left( \frac{1}{1 - \varepsilon/2} ((1 - \varepsilon) U + V) \right) \leq \frac{1 - \varepsilon}{1 - \varepsilon/2} \Phi (U) + \left( 1 - \frac{1}{1 - \varepsilon/2} \right) \Phi \left( \frac{2V}{\varepsilon} \right).
\]

Inequality (4.3) gives \( \Phi \left( \frac{1}{1 - \varepsilon/2} ((1 - \varepsilon) U + V) \right) \in L^1(\Omega, \mathbb{R}^n) \). Owing to (a), the statement (b) follows.

\[\square\]

For convenience of the reader, we state Lemma 4.2 in [23].

**Lemma 4.2** Let \( u, v \in E_B(\Omega), u \neq 0 \) and \( \int_{\Omega} B(u)v \, dx \neq 0 \). Then the condition

\[
\int_{\Omega} B((1 - \varepsilon) u + \delta v) \, dx = \int_{\Omega} B(u) \, dx
\]

defines a continuously differentiable function \( \delta = \delta(\varepsilon) \) in some interval \((-\varepsilon_0, \varepsilon_0)\) with \( \varepsilon_0 > 0 \).
Moreover, \( \delta(0) = 0 \) and

\[
(4.4) \quad \delta'(0) = \frac{\int_{\Omega} b(u)u \, dx}{\int_{\Omega} b(u)v \, dx},
\]

where \( b \) is the derivative of \( B \).

**Proof of Proposition 3.6** (i). Let \( r > 0 \) and \( u_r \in W^1_0 L_{B, \Phi}(\Omega) \) be a minimizer of (3.3). Suppose, by contradiction, that \( \Phi_{\xi} (\nabla u_r) \notin L_{\Phi}^* (\Omega, \mathbb{R}^N) \). By Proposition 2.1 we get

\[
(4.5) \quad \int_{\Omega} \Phi_{\xi} (\nabla u_r) \cdot \nabla u_r \, dx = \int_{\Omega} \Phi (\nabla u_r) \, dx + \int_{\Omega} \Phi_* (\Phi_{\xi} (\nabla u_r)) \, dx = +\infty.
\]

Now, we choose \( v \in W^1_0 E_{B, \Phi}(\Omega) \) such that \( \int_{\Omega} b(u_r)u_r \, dx = \int_{\Omega} b(u_r)v \, dx \). By Lemma 4.2 and by (3.3), there exist \( \varepsilon_0 \in (0, 1) \) and a function \( \delta \in C^1(\mathbb{R}) \) fulfilling

\[
(4.6) \quad \int_{\Omega} B((1 - \varepsilon) u_r + \delta(\varepsilon)v) \, dx = \int_{\Omega} B(u_r) \, dx = r \quad \text{for all } \varepsilon \in (-\varepsilon_0, \varepsilon_0).
\]

Moreover, \( \delta(0) = 0 \) and \( \delta'(0) = 1 \). Then, there exists \( \varepsilon_1 \in (0, \varepsilon_0) \) such that \( \delta(\varepsilon) \geq 0 \) for all \( \varepsilon \in [0, \varepsilon_1] \) and

\[
(4.7) \quad |\delta'(\varepsilon)| \leq \frac{3}{2} \quad \text{for all } \varepsilon \in [-\varepsilon_1, \varepsilon_1].
\]

By (4.7),

\[
(4.8) \quad |\delta(\varepsilon)| \leq \frac{3}{2} |\varepsilon| \quad \text{for all } \varepsilon \in [-\varepsilon_1, \varepsilon_1].
\]

Let us define the function \( \Psi : [0, \varepsilon_1] \to \mathbb{R} \) by

\[
\Psi(\varepsilon) = \int_{\Omega} \Phi(W_\varepsilon(x)) \, dx,
\]

where

\[
W_\varepsilon(x) = (1 - \varepsilon) \nabla u_r(x) + \delta(\varepsilon) \nabla v(x) \quad \text{for } x \in \Omega.
\]

\[19\]
Since, by (2.8), $\Phi_\varepsilon(W_\varepsilon) \cdot W_\varepsilon \geq 0$ a.e. in $\Omega$ and $\Phi_\varepsilon(W_\varepsilon) \cdot W_\varepsilon \rightarrow \Phi_\varepsilon(\nabla u_r) \cdot \nabla u_r$ a.e. in $\Omega$, then by Fatou’s Lemma and (4.5) we have

(4.9) \[ \int_\Omega \Phi_\varepsilon(W_\varepsilon) \cdot W_\varepsilon \, dx \rightarrow +\infty \quad \text{for } \varepsilon \rightarrow 0. \]

Let $\varepsilon \in (0, 1)$. By the convexity of $\Phi$, estimates (4.8) and (2.4), it follows that

\[
\Phi(W_\varepsilon) \leq (1 - \varepsilon)\Phi(\nabla u_r) + \varepsilon \Phi\left(\frac{\delta(\varepsilon)}{\varepsilon} \nabla v\right) \leq \Phi(\nabla u_r) + \frac{2\delta(\varepsilon)}{3} \Phi\left(\frac{3}{2} \nabla v\right)
\]

\[
\leq \Phi(\nabla u_r) + \Phi\left(\frac{3}{2} \nabla v\right) \in L^1(\Omega)
\]

for all $\varepsilon \in (0, \varepsilon_1)$, because $v \in W^{1}_0 E_{B, \Phi}(\Omega)$ and $u_r$ is a minimizer of problem (3.3). Thanks to Lebesgue’s dominated convergence Theorem, $\Psi(\varepsilon)$ is continuous in $(0, \varepsilon_1]$. It is easily to check the continuity in $\varepsilon = 0$, as well.

Let $\varepsilon_2 \in (0, \varepsilon_1)$ be arbitrary and set $U = \frac{2}{2 - \varepsilon} \left[(1 - \varepsilon)\nabla u_r + \delta(\varepsilon)\nabla v\right]$. First, by (2.8) and convexity of $\Phi$ we get

(4.10) \[ \Phi_\varepsilon((1 - \varepsilon)\nabla u_r + \delta(\varepsilon)\nabla v) = \Phi_\varepsilon\left(\frac{1}{2} U\right) \leq \Phi_\varepsilon\left(\frac{1}{2} U\right) \cdot \frac{1}{2} U \]

\[
\leq \frac{2 - \varepsilon}{\varepsilon} \left[\Phi_\varepsilon\left(\frac{1}{2} U\right) \cdot \frac{\varepsilon}{2} U + \Phi\left(\frac{1}{2} U\right) \cdot \frac{\varepsilon}{2} U\right] \leq \frac{2 - \varepsilon}{\varepsilon} \Phi(U).
\]

Moreover, Young inequality (2.5), (4.10), (2.4), (4.7) and (4.8) yield

\[
\left|\frac{\partial}{\partial \varepsilon} \Phi_\varepsilon(W_\varepsilon)\right| = \frac{1}{\varepsilon} \left|\Phi_{\varepsilon}(\frac{1}{2} U) \cdot (\varepsilon \delta(\varepsilon)\nabla v - \varepsilon \nabla u_r)\right|
\]

\[
\leq \frac{1}{\varepsilon} \left[\Phi_{\varepsilon}\left(\frac{1}{2} U\right) + \Phi(\varepsilon \delta(\varepsilon)\nabla v - \varepsilon \nabla u_r)\right]
\]

\[
\leq \frac{1}{\varepsilon} \left[\frac{2 - \varepsilon}{\varepsilon} \Phi(\frac{2 - \varepsilon}{2} \nabla u_r + \frac{2\delta(\varepsilon)}{2 - \varepsilon} \nabla v) + \frac{1}{\varepsilon} \Phi\left(\frac{\varepsilon \delta(\varepsilon)}{1 - \varepsilon} \nabla v\right)\right]
\]

\[
\leq \frac{1}{\varepsilon} \left[\frac{2 - \varepsilon}{\varepsilon} \left(\frac{2 - \varepsilon}{2 - \varepsilon} \Phi(\nabla u_r) + \left(\frac{\varepsilon}{2 - \varepsilon}\right) \Phi\left(\frac{1}{1 - \varepsilon} \nabla v\right)\right)\right]
\]

\[
+ \varepsilon \delta'(\varepsilon) \Phi(\nabla v) + \varepsilon \Phi(\nabla u_r)
\]

\[
\leq \frac{1}{\varepsilon_2} \left[\frac{2 - \varepsilon}{\varepsilon} \Phi(\nabla u_r) + \frac{\delta(\varepsilon)}{\varepsilon} \Phi(4\nabla v) + \frac{3}{2} \Phi(\nabla v) + \Phi(\nabla u_r)\right]
\]

\[
\leq \frac{1}{\varepsilon_2} \left[\frac{2 - \varepsilon}{\varepsilon} \Phi(\nabla u_r) + \frac{\delta(\varepsilon)}{2} \Phi(4\nabla v) + \frac{3}{2} \Phi(\nabla v) + \Phi(\nabla u_r)\right] \in L^1(\Omega)
\]

for all $\varepsilon \in (\varepsilon_2, \varepsilon_1)$, because $v \in W^{1}_0 E_{B, \Phi}(\Omega)$ and $u_r$ is a minimizer of problem (3.3). Then we conclude that

(4.11) \[ \Psi'(\varepsilon) = \int_\Omega \Phi_\varepsilon(W_\varepsilon) \cdot (\delta'(\varepsilon)\nabla v - \nabla u_r) \, dx \]

for all $\varepsilon \in (\varepsilon_2, \varepsilon_1)$. By the arbitrariness of $\varepsilon_2$, equality (4.11) holds for all $\varepsilon \in (0, \varepsilon_1)$.

We note that

(4.12) \[ \Psi'(\varepsilon) = \int_\Omega \Phi_\varepsilon(W_\varepsilon) \left(\delta'(\varepsilon)\nabla v - \frac{W_\varepsilon - \delta(\varepsilon)\nabla v}{1 - \varepsilon}\right) \, dx \]

\[= -\frac{1}{1 - \varepsilon} \int_\Omega \Phi_\varepsilon(W_\varepsilon) \cdot W_\varepsilon \, dx + \left(\delta'(\varepsilon) + \frac{\delta(\varepsilon)}{1 - \varepsilon}\right) \int_\Omega \Phi_\varepsilon(W_\varepsilon) \cdot \nabla v \, dx. \]
Owing to Young inequality \((2.5)\) and inequality \((2.8)\), we have

\[
(4.13) \quad \Phi_\xi(\xi) \cdot 2\eta \leq \Phi_\bullet (\Phi_\xi(\xi)) + \Phi (2\eta) \leq \xi \cdot \Phi_\xi(\xi) + \Phi (2\eta) \quad \forall \eta, \xi \in \mathbb{R}^n.
\]

Since \(v \in W_0^1 E_B, \Phi (\Omega)\), by \((4.12)\) and \((4.13)\), we can deduce that

\[
(4.14) \quad \Psi'(\varepsilon) \leq \left( \frac{\delta'(\varepsilon)}{2} + \frac{\delta(\varepsilon)/2 - 1}{1 - \varepsilon} \right) \int_\Omega \Phi_\xi(W_\varepsilon) \cdot W_\varepsilon \, dx + \left( \frac{\delta'(\varepsilon)}{2} + \frac{\delta(\varepsilon)}{2(1 - \varepsilon)} \right) \int_\Omega \Phi (2\nabla v) \, dx
\]

\[
\leq \left( \frac{3}{4} + \frac{\delta(\varepsilon)/2 - 1}{1 - \varepsilon} \right) \int_\Omega \Phi_\xi(W_\varepsilon) \cdot W_\varepsilon \, dx + C \quad \text{for } \varepsilon \in (0, \varepsilon_1),
\]

where \(C\) is a positive constant independent of \(\varepsilon\). The last estimate \((4.14)\) and limit \((4.9)\) imply \(\lim_{\varepsilon \to 0} \Psi'(\varepsilon) = -\infty\). Then, there exists \(\varepsilon_3 > 0\) such that \(\Psi(\varepsilon_3) < \Psi(0)\). On setting \(\hat{u}_r = (1 - \varepsilon_3) u_r + \delta(\varepsilon_3)v\), we have

\[
\int_\Omega \Phi(\nabla \hat{u}_r) \, dx = \Psi(\varepsilon_3) < \Psi(0) = \int_\Omega \Phi(\nabla u_r) \, dx
\]

and

\[
\int_\Omega B(\hat{u}_r) \, dx = \varepsilon,
\]

which is a contradiction recalling that \(u_r\) is a minimizer. Hence, \(\Phi_\xi (\nabla u_r) \in \mathcal{L}_\bullet (\Omega; \mathbb{R}^n)\) and the proof of \((i)\) is complete.

\((ii)\). The idea of the proof is similar to that of \((i)\). For convenience of the reader, we give all details. Let \(r > 0\) and let \(u_r \in W_0^1 L_B, \Phi (\Omega)\) be a minimizer of problem \((3.3)\). Thanks to embedding \((2.33)\), \(u_r \in E_B(\Omega)\). Let \(v \in E_B(\Omega)\) such that \(\int_\Omega b(u_r)u_r \, dx = \int_\Omega b(u_r)v \, dx\), and Lemma \(4.2\) guarantees that there exist \(\varepsilon_0 \in (0, 1)\) and a function \(\delta \in C^1(-\varepsilon_0, \varepsilon_0)\) satisfying \((4.6)\). Moreover, \(\delta(0) = 0\), \(\delta(\varepsilon) \geq 0\) for all \(\varepsilon \in [0, \varepsilon_1]\), and \((4.7)\) and \((4.8)\) hold. On setting

\[
\Lambda(\varepsilon) = \int_\Omega B(\omega_\varepsilon(x)) \, dx \quad \text{with } \varepsilon \in [0, \varepsilon_1],
\]

where \(\omega_\varepsilon(x) = (1 - \varepsilon)u_r(x) + \delta(\varepsilon)v(x)\) for \(x \in \Omega\), by \((4.3)\), it follows that \(\Lambda(\varepsilon) = \varepsilon\) and then \(\Lambda'(\varepsilon) = 0\) for every \(\varepsilon \in [0, \varepsilon_1]\). Now, we assume by absurdum that \(b(u_r) \notin L_B(\Omega)\), i.e.

\[
(4.15) \quad \int_\Omega B(\omega_\varepsilon(x)) \, dx = +\infty.
\]

Let \(\varepsilon_2 \in (0, \varepsilon_1)\) be arbitrary. The monotonicity of \(b\) and Lemma \(4.1\) for 1-dimensional Young function give

\[
\left| \frac{\partial}{\partial \varepsilon} B(\omega_\varepsilon(x)) \right| = \left| b((1 - \varepsilon)u_r(x) + \delta(\varepsilon)v(x)) (\delta'(\varepsilon)v(x) - u_r(x)) \right|
\]

\[
\leq \left| b((1 - \varepsilon)u_r(x) + \delta(\varepsilon)v(x)) \right| \left( \left| \delta'(\varepsilon) \right| |v(x)| + |u_r(x)| \right)
\]

\[
\leq \left| b\left( (1 - \varepsilon_2) u_r(x) + \frac{3}{2}v(x) \right) \right| \left( \frac{3}{2} |v(x)| + |u_r(x)| \right) \in L^1(\Omega)
\]

for any \(\varepsilon \in (\varepsilon_2, \varepsilon_1)\). For the arbitrariness of \(\varepsilon_2\), it follows

\[
\Lambda'(\varepsilon) = \int_\Omega b(\omega_\varepsilon(\delta'(\varepsilon)v(x) - u_r(x)) \, dx
\]
for every $\varepsilon \in (0, \varepsilon_1)$.

By estimate $b(s) t \leq \frac{1}{2} b(s) s + b(2t) t$ for all $s, t \in \mathbb{R}$, we have

$$
(4.16) \quad \Lambda'(\varepsilon) = \int_{\Omega} b(\omega_{\varepsilon}) \left( \delta'(\varepsilon) v(x) + \frac{\delta(\varepsilon) v(x) - \omega_{\varepsilon}}{1 - \varepsilon} \right) dx
$$

$$
= -\frac{1}{1 - \varepsilon} \int_{\Omega} b(\omega_{\varepsilon}) \omega_{\varepsilon} dx + \int_{\Omega} \left( \delta'(\varepsilon) + \frac{\delta(\varepsilon)}{1 - \varepsilon} \right) b(\omega_{\varepsilon}) v(x) dx
$$

$$
\leq \left( \frac{\delta'(\varepsilon)}{2} + \frac{1}{1 - \varepsilon} \left( \frac{\delta(\varepsilon)}{2} - 1 \right) \right) \int_{\Omega} b(\omega_{\varepsilon}) \omega_{\varepsilon} dx + \left( \delta'(\varepsilon) + \frac{\delta(\varepsilon)}{1 - \varepsilon} \right) \int_{\Omega} b(2v) v dx
$$

for every $\varepsilon \in (0, \varepsilon_1)$. Young inequality and (2.8) yield

$$
(4.17) \quad \int_{\Omega} b(2v) v dx \leq \int_{\Omega} B(v) dx + \int_{\Omega} B_*(b(2v)) dx \leq \int_{\Omega} B(v) dx + \int_{\Omega} B(4v) dx < +\infty.
$$

By Proposition 2.1 and (4.15), we get

$$
(4.18) \quad \int_{\Omega} b(u_r) u_r dx = \int_{\Omega} B(u_r) dx + \int_{\Omega} B_*(b(u_r)) dx = +\infty.
$$

By the continuity of $b$, it follows that $b(\omega_{\varepsilon}) \omega_{\varepsilon} \to b(u_r) u_r$ a.e. in $\Omega$. Since $b(\omega_{\varepsilon}) \omega_{\varepsilon} \geq 0$ a.e. in $\Omega$, Fatou’s Lemma and (4.18) yield

$$
(4.19) \quad \lim_{\varepsilon \to 0^+} \int_{\Omega} b(\omega_{\varepsilon}) u_r dx = \int_{\Omega} b(u_r) u_r dx = +\infty.
$$

Now, we pass to the limit in the right-hand side of (4.16). By (4.17) and (4.19), we have $\lim_{\varepsilon \to 0^+} \Lambda'(\varepsilon) = -\infty$ that is in contradiction with the fact that $\Lambda'(\varepsilon) = 0$.

This implies that $b(u_r) \in L_{B_*}(\Omega)$ and complete the proof of (ii).

\[\square\]

We are now in a position to accomplish the proof of Theorem 3.2. In what follows it is important to underline that $(W^{1}_0 L_{B,\Phi}(\Omega), W^{1}_0 E_{B,\Phi}(\Omega); W^{-1} L_{B_*\Phi_*}(\Omega), W^{-1} E_{B_*\Phi_*}(\Omega))$ is a complementary system (see Subsection 2.4) under our assumption on $\Omega$.

**Proof of Theorem 3.2.** Let us define the functionals $dF$ ad $dG$ by

$$
\langle dF, v \rangle = \int_{\Omega} \Phi(\nabla u_r) \cdot \nabla v dx
$$

and

$$
\langle dG, v \rangle = \int_{\Omega} \frac{b(|u_r|)}{|u_r|} u_r v dx
$$

for any $v \in W^{1}_0 E_{B,\Phi}(\Omega)$, where $u_r$ is a minimizer of problem 3.3. By Proposition 3.6 the previous functionals are well-defined. Set

$$
\ker dF = \{ v \in W^{1}_0 E_{B,\Phi}(\Omega) : \langle dF, v \rangle = 0 \}
$$

and

$$
\ker dG = \{ v \in W^{1}_0 E_{B,\Phi}(\Omega) : \langle dG, v \rangle = 0 \}.
$$

If we prove that

$$
(4.20) \quad \ker dF \subset \ker dG,
$$

then...
then Proposition 43.1 in [30] assures the existence of \( \lambda_r \in \mathbb{R} \), associated with the minimizer \( u_r \), such that equality (3.2), for \( u = u_r \), holds for any test function \( \varphi \) in \( W_0^1 E_{B, \Phi}(\Omega) \). Finally, the \( \sigma(W_0^1 L_{B, \Phi}(\Omega), W^{-1} L_{B, \Phi^*}(\Omega)) \)-density of \( W_0^1 E_{B, \Phi}(\Omega) \) in \( W_0^1 L_{B, \Phi}(\Omega) \) (see Lemma 2.9 above) guarantees that (4.20) it is enough to conclude.

Then our goal is to prove (4.20), which will follow by the inclusion

\[
V_G := \left\{ v \in W_0^1 E_{B, \Phi}(\Omega) : \int_{\Omega} \frac{b(|u_r|)}{|u_r|} u_r v \, dx > 0 \right\} 
\subset \left\{ v \in W_0^1 E_{B, \Phi}(\Omega) : \int_{\Omega} \Phi(\nabla u_r) \cdot \nabla v > 0 \right\} := V_F.
\]

In order to verify the last inclusion, let us consider an arbitrary \( v \in V_G \). By Lemma 4.2 there exist \( \varepsilon_0 \in (0, 1) \) and \( \delta \in C^1(-\varepsilon_0, \varepsilon_0) \) such that

\[
\int_{\Omega} B((1-\varepsilon)u_r + \delta(\varepsilon)v) \, dx = \int_{\Omega} B(u_r) \, dx \quad \forall \varepsilon \in (-\varepsilon_0, \varepsilon_0).
\]

On setting \( w_\varepsilon = (1-\varepsilon)u_r + \delta(\varepsilon)v \), the definition of \( u_r \) and (4.21) assure that

\[
\int_{\Omega} \frac{\Phi(\nabla w_\varepsilon) - \Phi(\nabla u_r)}{\delta(\varepsilon)} \, dx \geq 0 \quad \forall \varepsilon \in (0, \varepsilon_1).
\]

By (4.4), we have \( \delta'(0) > 0 \) and there exists \( \varepsilon_1 \in (0, \varepsilon_0) \) such that \( \frac{\delta'(0)}{2} < \delta'(\varepsilon) < 2\delta'(0) \) for all \( \varepsilon \in (-\varepsilon_1, \varepsilon_1) \). By integrating with respect to \( \varepsilon \), we obtain

\[
\frac{\delta'(0)}{2} < \frac{\delta'(\varepsilon)}{\varepsilon} < 2\delta'(0) \quad \forall \varepsilon \in (0, \varepsilon_1).
\]

Since \( \Phi \) is locally Lipschitz, we get in the ball \( B(0, R) \) of radius \( R = |\nabla u_r| + \frac{3}{2}|\nabla v| \) that

\[
\left| \frac{\Phi(\nabla w_\varepsilon) - \Phi(\nabla u_r)}{\delta(\varepsilon)} \right| \leq \frac{L}{\delta(\varepsilon)} |\nabla w_\varepsilon - \nabla u_r| \leq L \frac{\varepsilon}{\delta(\varepsilon)} |\nabla u_r| + L |\nabla v| \leq L \frac{2}{\delta'(0)} |\nabla u_r| + L |\nabla v|,
\]

where \( L \) is the Lipschitz constant and the last inequality follow by (4.23). The rightmost side of (4.24) belongs to \( L^1(\Omega) \) because \( v \in W_0^1 E_{B, \Phi}(\Omega) \) and \( u_r \) is the solution to (3.3).

On recalling that \( \Phi \) is differentiable, easily computation gives

\[
\lim_{\varepsilon \to 0^+} \frac{\Phi(\nabla w_\varepsilon) - \Phi(\nabla u_r)}{\delta(\varepsilon)} = \Phi(\nabla u_r) \cdot \nabla v - \Phi(\nabla u_r) \cdot \frac{\nabla u_r}{\delta'(0)} \quad a.e. \text{ in } \Omega.
\]

Then, by Lebesgue’s dominate convergence Theorem, it follows

\[
\lim_{\varepsilon \to 0^+} \int_{\Omega} \frac{\Phi(\nabla w_\varepsilon) - \Phi(\nabla u_r)}{\delta(\varepsilon)} \, dx = \int_{\Omega} \left[ \Phi(\nabla u_r) \cdot \nabla v - \Phi(\nabla u_r) \cdot \frac{\nabla u_r}{\delta'(0)} \right] \, dx.
\]

Combining (4.22) and (4.25) we have

\[
\int_{\Omega} \Phi(\nabla u_r) \cdot \nabla v \, dx \geq \frac{1}{\delta'(0)} \int_{\Omega} \Phi(\nabla u_r) \cdot \nabla u_r \, dx > 0,
\]

namely \( v \in V_F \). Then \( V_G \subset V_F \) follows by arbitrariness of \( v \in V_F \).

At this point of the proof, we have found a function \( u_r \in W_0^1 L_{B, \Phi}(\Omega) \) such that \( \int_{\Omega} B(u_r) \, dx = r \), and fulfils the equation

\[
\int_{\Omega} \Phi(\nabla u_r) \cdot \nabla \varphi \, dx = \lambda_r \int_{\Omega} \frac{b(|u_r|)}{|u_r|} u_r \varphi \, dx
\]
for any $\varphi \in W^1_0 L_{B, \Phi}(\Omega)$.

Our aim is now to prove that this function $u_r$ is actually a weak solution to problem (3.1) as stated in Definition 3.1.

To do this, we first observe that, by inclusion (2.32), $u_r \in W^1_0 L_{\Phi}(\Omega)$.

Next, by [3, Proposition 2.4], one has that, given any function $\varphi \in W^1_0 L_{\Phi}(\Omega) \cap L^\infty(\Omega)$, there exist a constant $C = C(\Omega)$ and a sequence $\{\varphi_k\}_k \subset C^\infty_0(\Omega)$ such that

$$\varphi_k \to \varphi \quad \text{a.e. in } \Omega, \tag{4.27}$$

$$\|\varphi_k\|_{L^\infty(\Omega)} \leq C\|\varphi\|_{L^\infty(\Omega)} \quad \text{for every } k \in \mathbb{N}, \tag{4.28}$$

$$\nabla \varphi_k \to \nabla \varphi \quad \text{modularly in } L_{\Phi}(\Omega; \mathbb{R}^n). \tag{4.29}$$

Then, we have

$$\int_\Omega \Phi_\xi(\nabla u_r) \cdot \nabla \varphi_k \, dx = \lambda_r \int_\Omega \frac{b(|u_r|)}{|u_r|} u_r \varphi_k \, dx \tag{4.30}$$

for any $\varphi_k \in C^\infty_0(\Omega)$.

Condition (4.29) means that there exists a constant $l > 0$ such that

$$\lim_{k \to \infty} \int_\Omega \Phi_\xi \left( \frac{\nabla \varphi_k - \nabla \varphi}{l} \right) \, dx = 0.$$

Moreover, [3, Proposition 2.2] yields that, owing to conditions (4.29) and $\Phi_\xi(\nabla u_r) \in L_{B_*}(\Omega; \mathbb{R}^n)$, there exists a subsequence of $\{\nabla \varphi_k\}_k$, still indexed by $k$, such that

$$\lim_{k \to \infty} \int_\Omega \Phi_\xi(\nabla u_r) \cdot \nabla \varphi_k \, dx = \int_\Omega \Phi_\xi(\nabla u_r) \cdot \nabla \varphi \, dx.$$

Therefore, by the dominated convergence theorem coupling with (4.27), (4.28) and the fact that $b(|u_r|) \in L_{B_*}(\Omega)$, and hence in $L^1(\Omega)$, we have that

$$\lim_{k \to \infty} \int_\Omega \frac{b(|u_r|)}{|u_r|} u_r \varphi_k \, dx = \int_\Omega \frac{b(|u_r|)}{|u_r|} u_r \varphi \, dx.$$

Then, $u_r$ is a weak solution to (3.1) in $W^1_0 L_{\Phi}(\Omega)$ in the sense of Definition 3.1. Concerning the boundedness of $u_r$, it follows directly by [2, Theorem 4.1].

\[\square\]

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