Improved invariant polytope algorithm and applications

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In several papers of 2013 – 2016, Guglielmi and Protasov made a breakthrough in the problem of the joint spectral radius computation, developing the invariant polytope algorithm which for most matrix families finds the exact value of the joint spectral radius. This algorithm found many applications in problems of functional analysis, approximation theory, combinatorics, etc. In this paper we propose a modification of the invariant polytope algorithm making it roughly 3 times faster and suitable for higher dimensions. The modified version works for most matrix families of dimensions up to 25, for non-negative matrices the dimension is up to three thousand. Besides we introduce a new, fast algorithm for computing good lower bounds for the joint spectral radius. The corresponding examples and statistics of numerical results are provided. Several applications of our algorithms are presented. In particular, we find the exact values of the regularity exponents of Daubechies wavelets of high orders and the capacities of codes that avoid certain difference patterns.

1 INTRODUCTION AND NOTATION

The joint spectral radius (JSR) of a set of matrices is a quantity which describes the maximal asymptotic growth rate of the norms of products of matrices from that set (with repetitions permitted). Precisely, given a finite set $\mathcal{A} = \{A_j : j = 1, \ldots, J\} \subseteq \mathbb{R}^{s \times s}$, $s \in \mathbb{N}$, then

$$\text{JSR}(\mathcal{A}) := \lim_{n \to \infty} \max_{A_j \in \mathcal{A}} \left\| A_{j_n} \cdots A_{j_2} A_{j_1} \right\|^{1/n}. \quad (1)$$

In [11] it is proved that (for finite $\mathcal{A}$)

$$\text{JSR}(\mathcal{A}) = \lim_{n \to \infty} \max_{A_j \in \mathcal{A}} \rho(A_{j_n} \cdots A_{j_2} A_{j_1})^{1/n}, \quad (2)$$

where $\rho$ is the classical spectral radius of a matrix. With $\# \mathcal{A}$ we denote the number of elements of the set $\mathcal{A}$. If $\# \mathcal{A} = 1$, then the JSR reduces to the spectral radius of a matrix.

The JSR has been defined in [31] and since appeared in many (seemingly unrelated) mathematical applications, e.g. for computing the regularity of wavelets and subdivision schemes [15], the capacity of codes [27], the stability of linear switched systems [21] or in connection with the Euler partition function [30].

The computation of the JSR is a notoriously hard problem. Even for non-negative matrices with rational coefficients this problem is NP-hard [9]. Moreover, the question whether or not $\text{JSR}(\mathcal{A}) \leq 1$ for a given set $\mathcal{A}$ is algorithmically undecidable [10]. Most algorithms which try to compute or to approximate the JSR make use of the following inequality [15]

$$\max_{A_j \in \mathcal{A}} \rho(A_{j_k} \cdots A_{j_1})^{1/k} \leq \text{JSR}(\mathcal{A}) \leq \max_{A_j \in \mathcal{A}} \|A_{j_k} \cdots A_{j_1}\|^{1/k}, \quad (3)$$

which holds for any $k \in \mathbb{N}$. For a product $A_{j_k} \cdots A_{j_1}$ we say the number $\rho(A_{j_k} \cdots A_{j_1})^{1/k}$ is its normalized spectral radius. Equation (3) tells us that the normalized spectral radius of every product is a valid lower bound for the JSR, on the contrary, one has to compute the norms of all products of a fixed length $k \in \mathbb{N}$ to obtain a valid upper bound.

If there exists a product $\Pi = A_{j_n} \cdots A_{j_1}$, $A_j \in \mathcal{A}$, such that $\rho(\Pi)^{1/n} = \text{JSR}(\mathcal{A})$, we call the product a spectral maximizing product (s.m.p.). Not all sets of matrices possess an s.m.p. [22], or in other words, there exist sets of matrices such that the normalized spectral radius of every finite product is strictly less than the JSR. It is an open question whether pairs of binary matrices always possess an s.m.p. [5].

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There are three common strategies to exploit (3): (i) Compute all products up to a length $k \in \mathbb{N}$ [16, 27, 28]; (ii) Take a suitable family of norms and minimize the right hand side of (3) with respect to that family [1, 7, 8, 29]; (iii) Construct a norm which gives good estimates in (3) for short products, preferably for products of length 1 [17, 19, 20, 24, 25]. So far, there exists only two algorithms which can compute the exact value of the JSR for a large number of matrix families: the tree-based branch and bound approach [28] and the invariant polytope algorithm [17, 19]. In this paper we concentrate on the invariant polytope algorithm and thus follow strategy (iii).

We will call a norm $\| \cdot \|$ extremal for $\mathcal{A}$ if

$$\|A_j x\| \leq \text{JSR}(\mathcal{A}) \cdot \|x\| \text{ for all } x \in \mathbb{R}^s \text{ and for all } A_j \in \mathcal{A}. \quad (4)$$

In [2] it is shown that every irreducible family of matrices, i.e. a family of matrices which have no trivial common invariant subspaces, possesses an extremal norm. Its construction is easily described in terms of the set

$$P(v) = \text{co} \bigcup_{n \in \mathbb{N}_0, A_j \in \mathcal{A}} \left\{ \pm A_{j_n} \cdots A_{j_1} v \right\}, \quad (5)$$

where $\text{co}$ denotes the convex hull and $v \in \mathbb{R}^s$.

**Theorem 1.1.** [11, 20]. If $\mathcal{A}$ is irreducible, $\text{JSR}(\mathcal{A}) \geq 1$ and for a given $v \in \mathbb{R}^s \setminus \{0\}$ the set $P(v)$ is bounded and has non-empty interior, then $\text{JSR}(\mathcal{A}) = 1$ and $P(v)$ is the unit ball of an extremal norm $\| \cdot \|$ for $\mathcal{A}$.

Conversely, if $\mathcal{A}$ is irreducible and $\text{JSR}(\mathcal{A}) = 1$, then for any $v \in \mathbb{R}^s$, $P(v)$ is a bounded subset of $\mathbb{R}^s$.

Clearly, the unit ball completely describes the corresponding norm. Given $P \subseteq \mathbb{R}^s$, a closed, convex and balanced ($\alpha P \subseteq P$ for all $|\alpha| < 1$) body with non-empty interior, the so-called *Minkowski norm* $\| \cdot \|_P : \mathbb{R}^s \rightarrow \mathbb{R}$,

$$\| \cdot \|_P = \inf \{ r > 0 : x \in rP \} \quad (6)$$

fulfills $\{ x \in \mathbb{R}^s : \|x\|_P \leq 1 \} = P$.

The idea of the invariant polytope algorithm [17, 19] is to construct the set (5) in finitely many steps, whenever it is a polytope.

We will describe polytopes by the convex hull of its vertices. For finite $V \subseteq \mathbb{C}^s$ we define the complex convex hull of $V$ by

$$\text{absco } V = \{ x \in \mathbb{C}^s : x = \sum_{v \in V} t_v v \text{ with } \sum_{v \in V} |t_v| \leq 1, \ t_v \in \mathbb{C}^s \}. \quad (7)$$

For finite $V \subseteq \mathbb{R}^s$ we define the symmetrized convex hull of $V$ by

$$\text{co}_s V = \{ x \in \mathbb{R}^s : x = \sum_{v \in V} t_v v \text{ with } \sum_{v \in V} |t_v| \leq 1, \ t_v \in \mathbb{R}^s \} = \text{co}(V \cup -V). \quad (8)$$

For finite $V \subseteq \mathbb{R}^s_+$ we define the *cone* of $V$ with respect to the first orthant by

$$\text{co}_+ V = \{ x \in \mathbb{R}^s_+ : x = y - z, \ y \in \text{co}(V), \ z \in \mathbb{R}^s_+ \}. \quad (9)$$

For simplicity, we denote with $\text{co } V$ any of these convex hulls ($\text{co}$, absco, $\text{co}_s$, $\text{co}_+$) depending on the context.

In all cases we identify the (finite) set $V$ with the matrix whose columns are the coordinates of the points $v \in V$. 

2
1.1 Invariant polytope algorithm and outline for the paper

We present here the idea of the invariant polytope algorithm [17, 19]. The major goal of this paper is to modify this algorithm making it

- faster,
- more robust and
- more efficient for larger matrices.

These modifications are outlined in Section 2. The actual modified invariant polytope algorithm 4.1 is given in Section 4. In Section 3 we introduce the modified Gripenberg algorithm 3.1 which is capable of finding very long s.m.p.-candidates in short time. Section 5 is devoted to numerical examples showing where the algorithms from Sections 3 and 4 perform well and where they are not applicable any more. We finish the paper with a Matlab implementation of the modified Gripenberg algorithm 3.1 in a copy-pastable format.

Algorithm 1.2 (Invariant polytope algorithm [17, 19]). Given a finite set of matrices \( \mathcal{A} = \{ A_j : j = 1, \ldots, J \} \subseteq \mathbb{R}^{n \times n} \).

1. For some \( D \in \mathbb{N} \) look over all products of matrices in \( \mathcal{A} \) of length \( \leq D \) and choose the shortest products \( \Pi_1 \) such that \( \rho_c := \rho(\Pi_1)^{1/l} \) is maximal, where \( l \) is the length of the product and call \( \Pi_1 \) spectral maximizing product-candidate (s.m.p.-candidate). Set \( \mathcal{A} := \rho_c^{-1} \mathcal{A} \). Now we try to prove that \( \operatorname{JSR}(\widehat{\mathcal{A}}) \leq 1 \).
2. Let \( v_1 \) be the leading eigenvector of \( \Pi_1 \), i.e. the eigenvector with respect to the biggest eigenvalue in magnitude.
3. Construct the cyclic root \( \mathcal{H} \): Let \( v_1^{(i)} \), \( i = 1, \ldots, l - 1 \), be the leading eigenvectors of the cyclic permutations of \( \Pi_1 \), i.e. for \( \Pi_1 = \hat{A}_{j_1} \cdots \hat{A}_{j_l} \) we get \( v_1^{(i)} := \hat{A}_{j_i} \cdots \hat{A}_{j_l} v_1 \). Set \( \mathcal{H} := \{ v_1^{(0)}, \ldots, v_1^{(l-1)} \} \) and \( V := \mathcal{H} \).
4. For all \( v \in V \) and for all \( j = 1, \ldots, J \) do
   - If \( \| \hat{A}_j v \|_{\operatorname{co} V} > 1 \) set \( V := V \cup \hat{A}_j v \).
   - Depending on the matrices in \( \mathcal{A} \) and the leading eigenvector \( v_0 \) we use different convex hulls in the algorithm; if all entries of the matrices \( A_j \) are non-negative, we can take a non-negative leading eigenvalue and we use co_− (case (P)); if the matrices have positive and negative entries and the leading eigenvector is real we use co_ (case (R)); otherwise we use absco (case (C)).
5. Repeat step 4 until \( \widehat{\mathcal{A}} V \subseteq \operatorname{co} V \).
6. If the algorithm fulfils \( \widehat{\mathcal{A}} V \subseteq \operatorname{co} V \), then the algorithm terminates and we have found an invariant polytope \( V \), which implies that \( \| \hat{A}_j \|_{\operatorname{co} V} \leq 1 \) for all \( j = 1, \ldots, J \), or in other words \( \operatorname{JSR}(\widehat{\mathcal{A}}) \leq 1 \).

Remark 1.3. In step 1.2 (4) we actually add a vertex \( \hat{A}_j v \notin \mathcal{H} \) even if it lies slightly inside of the polytope, i.e. if \( \| \hat{A}_j v \|_{\operatorname{co} V} > 1 - \epsilon \) depending on the accuracy \( \epsilon > 0 \) in which we can compute the norm.

2 SUMMARY OF THE MAIN MODIFICATIONS

In this section we present the modifications of algorithm 1.2 and explain their importance. For more details see Sections 3 and 4.

2.1 Multiple trees

If there are multiple s.m.p.-candidates \( \Pi_1, \ldots, \Pi_R, R \in \mathbb{N} \) in (1), then one has to construct multiple cyclic roots \( \mathcal{H}_r, r = 1, \ldots, R \), in (3) and balance the sizes of the corresponding polytopes using
the dual leading eigenvectors $v_r^*$ [19, Section 2.3]. In particular, for the s.m.p.-candidate $\tilde{\Pi}_r$ define $\tilde{\Pi}_r^* v_r^* = v_r^*$ with $(v_r^{(0)}, v_r^*) = 1$, $r = 1, \ldots, R$, where $\tilde{\Pi}_r^*$ is the conjugate transpose of $\tilde{\Pi}_r$ and $(\cdot, \cdot)$ is the usual inner product. One then has to find factors $\alpha_i > 0$ such that, for some $h \in \mathbb{N}$,

$$\alpha_i \sup_{z \in \mathbb{R}^h \{v_r^{(0)}, \ldots, v_r^{(l(l+1))}\}} |(v_r^*, z)| < \alpha_j, \text{ for all } i, j = 1, \ldots, R.$$  

(10)

Afterwards set $\mathcal{H}_r := \{\alpha_1 v_r^{(0)}, \ldots, \alpha_r v_r^{(l(r-1))}\}$, $v_r^* := \alpha_r^{-1} v_r^*$, and $V := \bigcup_{r=1}^R \mathcal{H}_r$.

Example 4.3 is a set of matrices, where it was wrongly assumed that all balancing factors $\alpha_i \equiv 1$

2.2 Nearly s.m.p.s

The invariant polytope algorithm [17, 19] for cases (P) and (R), terminates if and only if (a) the s.m.p.-candidates $\Pi_r$, $r = 1, \ldots, R$, are dominant, i.e. there exists $\gamma$ such that $0 < \gamma < JSR(\mathcal{A})$ and $\rho(A_{j_1} \cdots A_{j_l})^{1/l} < \gamma$ whenever $A_{j_1} \cdots A_{j_l}$ is not an s.m.p.-candidate (or a cyclic permutation of one), and (b) the leading eigenvectors $v_0^*$ are simple.

If the spectral gap is small, i.e. $\gamma \approx JSR(\mathcal{A})$ then the generated polytope very slowly absorbs new vertices. In this case it is helpful to consider also the polytopes generated by the so-called nearly-s.m.p.s, products whose normalized spectral radius is nearly that of the s.m.p.-candidates [19]. The suggested balancing procedure of the nearly-s.m.p.s in [19, Remark 3.7] does not work always, as Example 4.4 shows. Section 4.5 presents a new method to solve this problem.

2.3 Extra-vertices

Also in [19], the authors introduce the so-called extra-vertices [19, Section 4], which can speed up the algorithm in cases where the constructed polytope is very flat. In Section 4.4 we suggest a method which automatically chooses a good set of extra-vertices – a task which was done by hand so far.

2.4 Finding s.m.p. candidates

algorithm 1.2 only terminates, if the s.m.p.-candidates $\Pi_r$ are indeed s.m.p.s. Thus, algorithm 1.2 heavily relies on a correct initial guess for the s.m.p.-candidates. A plain brute-force search in (1) will fail, if the s.m.p.s length is large. Our numerical tests have shown that even for random pairs of matrices, s.m.p.s of length greater than 30 are not uncommon. A particular easy example of two matrices with a very long s.m.p. is given in Example 5.2. We present two new methods that search for s.m.p.s efficiently in Sections 3 and 4.10.

2.5 Bounds for the JSR

If algorithm 1.2 does not find an invariant polytope in reasonable time, algorithm 1.2 can still give upper bounds for the JSR, if one keeps track of the norms $||A_j v||_{\infty}$. In Lemma 4.2 we show that our modified algorithm also returns these bounds. Nevertheless, these bounds are usually quite rough. A simple modification, presented in Remark 4.3 on the other hand, increases the accuracy of these intermediate bounds on the drawback that the exact value of the JSR becomes uncomputable.

2.6 Parallization and natural selection of vertices

A disadvantage of algorithm 1.2 in its current form is that the polytope is changed inside of the main loop in (4), which implies that the norm of $A_j v$ has to be computed with respect to a different polytope for each vertex. Therefore, the linear programming problem is different for each norm and the so-called warm start of linear programming problems cannot be used. Furthermore, the main loop cannot be parallelised. We eliminate these two problems and additionally speed up the algorithm in Section 4.8.
With the same technique we can solve problems arising when the number of matrices in $\mathcal{A}$ is large. In such cases algorithm 1.2 will stall, simply due to the fact, that the number of vertices to test, increases (in the worst case) by a factor of $\#\mathcal{A}$ in each iteration, e.g. if $\#\mathcal{A} = 256$, the original algorithm is likely never to reach the third level.

2.7 Estimating the Minkowski norm
To reduce the number of norms one has to compute, and, thus, to speed up the algorithm, we use the estimates for the Minkowski norm in Lemma 4.5.

3 MODIFIED GRIPENBERG ALGORITHM
From (3) we know that the normalized spectral radius of any matrix product is a lower bound for the JSR. Thus, by a clever guess of a matrix product one easily obtains good (maybe sharp) lower bounds. algorithm 3.1 finds in nearly all of our numerical tests an s.m.p.

algorithm 3.1 is a modification of the well-known Gripenberg algorithm [16], one of the first algorithms which gave reasonable estimates for the JSR. We briefly describe how it works: Given some accuracy $0 < \delta \leq 1$ we iteratively compute the sets $C_k$, $k \in \mathbb{N}$, $C_1 := \mathcal{A}$ and $C_{k+1}$ consists of all matrices $C \in \mathcal{A}C_k$ with $\|C\|^{1/(k+1)} \geq \delta b_-$, where $b_- = \max\{\rho(C)^{1/n} : C \in C_n, n = 1, \ldots, k\}$ is the current lower bound for the JSR. For each $k$ the JSR lies in the interval $[b_-, b_+]$ with $b_+ = \min_{n=1, \ldots, k} \max\{\|C\|^{1/n} : C \in C_n\}$. If $\delta < 1$ the algorithm terminates, i.e. there exists $K \in \mathbb{N}$ such that $C_K = \emptyset$ and the algorithm returns the JSR up to an accuracy of $\delta$, i.e. $b_-/b_+ \leq \delta$. For real-world applications the algorithm works well for $\delta \leq 0.95$. For larger $\delta$ the number of products to compute is usually too large.

The modified Gripenberg algorithm 3.1 uses a different selection mechanism. Instead of dismissing products with norms less than some threshold, it keeps the products with highest and lowest norms, products with norms in between are thrown away. This way, the algorithm cannot determine upper bounds for the JSR anymore, but it still works for finding s.m.p.s of considerably length.

Theorem 3.2. The modified Gripenberg algorithm 3.1 has linear complexity in the number $J = \#\mathcal{A}$ of matrices, in the number $N \in \mathbb{N}$ of kept products in each level and in the maximal length $D \in \mathbb{N}$ of the products.

Proof. In every iteration, in total $D$ many, the algorithm computes at most $2 \cdot N \cdot J$ norms and spectral radii.

Remark 3.3. The modified Gripenberg algorithm 3.1 with parameters $N = D = \infty$ is exactly the Gripenberg algorithm with accuracy $\delta = 1$.

Remark 3.4. Clearly one can pursue other selection strategies in step (11). The straightforward choice of taking the $2 \cdot N$ products with highest normalized norm performs very badly. Taking an arbitrary subset of $\mathcal{M}_d$ of size $2 \cdot N$ in step (11) performs mostly similarly to the proposed algorithm 3.1, but in some cases worse, see Table 3. Furthermore. Algorithm 1.2 in the given form is deterministic, so we prefer it over the non-deterministic version.

Remark 3.5. The modified invariant polytope algorithm 4.1 can also be used to search for s.m.p.-candidates. Thus, we present the numerical examples showing the performance of the modified Gripenberg algorithm 3.1 only after Section 4.

4 MODIFIED INVARIANT POLYTOPE ALGORITHM
In this section, we present the modifications for the algorithm 1.2. Its current implementation [26, December 2018] only handles the case $(P)$ (as explained in 1.2 (4)) with non-negative matrices and
Algorithm 3.1 (modified Gripenberg algorithm).

**Input:**
- Set of square matrices $\mathcal{A} = \{A_j : j = 1, \ldots, J\} \subseteq \mathbb{R}^{s \times s}$
- Number of products kept in each step $N \in \mathbb{N}$
- Maximal length of products $D \in \mathbb{N}$

**Output:**
- s.m.p.-candidates $C$
- Lower bound $\rho_c$ for JSR($\mathcal{A}$)

**Initialization:**
- Start with the product of length 0, $M_0 = \{I\}$, where $I$ is the identity matrix
- Set current lower bound for JSR, $\rho_c := 0$

**Algorithm:**

for $d = 1, \ldots, D$ do

Compute all possible new products $M_d := \mathcal{A}M_{d-1}$

Update lower bound $\rho_c := \max\{\rho_c, \rho(M_d)^{1/d} : M_d \in M_d\}$

Remove products whose norms are less than $\rho_c$, $M_d := \{M_d \in M_d : \|M_d\|^{1/d} \geq \rho_c\}$

Keep only products with highest and lowest norms:

Sort $M_d$ w.r.t $\|M_d\|$ and disregard the matrices with indices $N + 1, \ldots, \#M_d - N - 1$

Thus $M_d = \{M_1, \ldots, M_N, M_{#M_d-N}, \ldots, M_{#M_d} : M_i \in M_d\}$

Choose products $C = \{M_{d_1} \in M_d : \rho(M_{d_1})^{1/d} = \rho_c, d = 1, \ldots, D\}$

Remove cyclic permutations and powers of products from $C$.

return $C$, $\rho_c$

the case (R) with real leading eigenvalues. Case (C) with complex leading eigenvalues is not yet implemented. Thus, we also restrict the discussion of the modified invariant polytope algorithm 4.1 to the cases (P) and (R). Nevertheless, most of the results are valid in the case (C) as well.

For some vertex $w = A_jv \in V_{k+1}$, $A_j \in \mathcal{A}$, $v \in V_k$, $k \in \mathbb{N}$, we say that $w$ is a child of $v$, and that $v$ is the parent of $w$.

**Theorem 4.2.** Let $\mathcal{A} = \{A_j, j = 1, \ldots, J\} \subseteq \mathbb{R}^{s \times s}$ be a finite set of square matrices.
- For $\delta = 1$, the modified invariant polytope algorithm 4.1 terminates if and only if the original algorithm [17, 19] terminates, i.e. $\mathcal{A}$ is irreducible and $\Pi_1, \ldots, \Pi_R$ are dominant s.m.p.s.
- For $0 < \delta < 1$ the modified invariant polytope algorithm 4.1 terminates if JSR(\mathcal{A}) < $\delta^{-1} \cdot \rho_c$.
- Moreover, for any iteration $k \in \mathbb{N}$, JSR(\mathcal{A}) $\in [\rho_c, b_k \cdot \rho_c]$, where $\rho_c$ and and $b_k$ are defined in algorithm 4.1.

Before presenting the proof of Theorem 4.2 in 4.11, we describe all modifications and extensions of algorithm 1.2. These are numbered (12)-(21) in algorithm 4.1.

**4.1 Irreducibility of input matrices (12)**

The input matrices $\mathcal{A}$ should be irreducible, i.e. do not have a trivial common invariant subspace, because otherwise the algorithm (1.2) may not be able to terminate. If the matrices are reducible,
Algorithm 4.1 (Modified invariant polytope algorithm). The constant $\epsilon > 0$ appearing in this algorithm is the accuracy in which we can compute the norms $N(v)$. All modifications to algorithm 1.2 are marked with *. Lines with numbers are subroutines, which are described in detail in Sections 4.1-4.10.

**Input:**
Set of irreducible square matrices $\mathcal{A} = \{A_j : j = 1, \ldots, J\} \subseteq \mathbb{R}^{s \times s}$

Accuracy $0 < \delta \leq 1$ ($\delta \simeq 1$)

**Output:**
Exact value $\rho_c$ of JSR or bound $[\rho_c, b \cdot \rho_c]$ for JSR

Invariant polytope $V$

Spectral maximizing products $\Pi_i$

**Initialization:**
* Search for s.m.p.-cand. and nearly-s.m.p.s $\Pi_r = A_{j_{r_1}} \cdots A_{j_{r_l}}, r = 1, \ldots, R$

Set $\rho_r := \rho(\Pi_r)^{1/l(r)}, \rho_c := \max \rho_r, \tilde{A} := \delta \rho_c^{-1} A$

Compute the leading eigenvectors $v_r$ of $\tilde{A}$

Compute the root vectors $v_r^{(i)} := (\rho_c/\delta \rho_r)^i A_{j_{r_1}} \cdots A_{j_{r_l}} v_r, i = 0, \ldots, l(r) - 1$

* Compute extra-vertices $v_{R+1}, \ldots, v_S \in \mathbb{R}^s$

Provide the balancing factors $a_1, \ldots, a_S \in \mathbb{R}$

Set $\mathcal{H} := \{a_1 v_1^{(0)}, a_1 v_1^{(1)}, a_1 v_1^{(2)}, \ldots, a_S v_S^{(l(r)-1)}\}, V_0 := \mathcal{H} \cup \{a_{R+1} v_{R+1}, \ldots, a_S v_S\}$

Set $N(v) := \infty$ for all $v \in V_0$, $b_0 := \infty, k := 0$

**Main Loop:**

while $\tilde{A} V_k \setminus \mathcal{H} \not\subseteq (1 - \epsilon) \text{co} V_k$

* Select new children $E_{k+1} \subseteq \tilde{A} V_k \setminus \mathcal{H}$

* Choose subset of vertices $W_k \subseteq V_k$

* Compute/estimate norm $N(v) := \|v\|_{\text{co} W_k}$ for all $v \in E_{k+1}$

$V_{k+1} := V_k \cup \{v \in E_{k+1} : N(v) > 1 - \epsilon\}$

$b_{k+1} := \delta^{-1} \cdot \min\{b_k, \max\{1, N(v)(1 - \epsilon)^{-1} : v \in V_n \land \tilde{A} v \not\subseteq \text{co} V_n, \text{ where } n \text{ is the biggest integer such that } \tilde{A} V_n \subseteq \text{co} V_{k+1}\}\}$

* Test spectral radii and restart algorithm if a better s.m.p. candidate is found.

Increase $k := k + 1$

if $\delta = 1$

return $V, \{\Pi_i\}_i, \rho_c$ and exit

else

print $\text{JSR} \in [\rho_c, b_{k+1} \cdot \rho_c]$
then there exists a basis in which all of the matrices $A_j$ have block upper triangular form. The JSR of the matrices then equals to the maximum of the JSR of the diagonal blocks. In our implementation [26], we therefore automatically search for trivial common invariant subspaces prior to starting the algorithm. Here we make use of the functions permTriangul and jointTriangul from [23], as well as a new method invariantSubspace [26] which searches for trivial common invariant difference subspaces as described in [13].

### 4.2 Search for s.m.p.-candidates (13)

In our implementation [26], we use the modified Gripenberg algorithm 3.1 to search for s.m.p.-candidates.

Every product, which is shorter than the s.m.p.-candidate and having normalized spectral radius greater or equal to $\tau \cdot \rho_c$ is considered to be a nearly-s.m.p.. In our implementation we use a heuristic default value of $\tau = 0.9999$.

### 4.3 Approximate computation (14)

If we multiply the set of matrices $\tilde{A}$ by a factor $0 < \delta < 1$, algorithm 4.1 cannot return exact values for the JSR anymore, but only up to a relative accuracy of $\delta$. Indeed, if algorithm 4.1 terminates, then $\|\tilde{A}_j v\|_\infty V \leq 1 \Leftrightarrow \|A_j v\|_\infty V \leq \delta^{-1} \cdot \rho_c \Leftrightarrow \text{JSR}(A) \leq \delta^{-1} \cdot \rho_c$. Nevertheless, there are cases where this procedure is of high importance.

(a) If the dimension $s$ of matrices is large, algorithm 4.1 will probably not terminate anyway, and thus only give bounds for the JSR. A factor $\delta \approx 0.97$ will speed up the computation tremendously and the returned bounds are mostly better (at least in our numerical examples) than for $\delta = 1$.

(b) If the s.m.p.s are not dominant, or there is an infinite number of dominant s.m.p.s, or $\mathcal{A}$ is not irreducible, algorithm 4.1 will not terminate. In these cases, choosing $\delta \approx 1 - 10^{-9}$ ensures that algorithm 4.1 terminates and the obtained bounds will be nearly the same as when $\delta = 1$.

(c) If one is interested only whether $\text{JSR}(A) < C$ for some $C > 0$, one can choose $1 > \delta > \rho_c/C$ and algorithm 4.1 will terminate much faster.

In our implementation, we use the Matlab function eig to compute the leading eigenvalue. This may not be the fastest available procedure, but it is fast enough in comparison to the time the main loop needs to terminate.

### 4.4 Adding extra-vertices automatically (15)

Given some threshold $T > 0$, we compute the singular value decomposition of $\mathcal{H}$. We then add all singular vectors (extra-vertices) $v_{R+1}, \ldots, v_S$ corresponding to singular values which are in modulus less than $T$ to the cyclic root $v_1^{(0)}, v_1^{(1)}, \ldots, v_{\ell(R)-1}$. This strategy yielded a good behaviour of the algorithm in most of our examples. Note that with this procedure the polytope $co V_0$ always has non-empty interior.

In our implementation we use a heuristic default value of $T \approx 0.1$.

### 4.5 Balancing of cyclic trees (16)

The original balancing procedure is described in [19, Section 3]. We present its improved version for the case of extra-vertices and nearly-s.m.p.s. If $\delta < 1$ no balancing is necessary as can be seen in the proof of Theorem 4.2. If $\delta = 1$, for $v_j^*, j = 1, \ldots, R$, i.e. the dual eigenvectors of the s.m.p.-candidates, and $i = 1, \ldots, S$ we define for $h \in \mathbb{N}$

$$q_{l,j} = \sup_{z \in \mathcal{A}^h ((\rho_c/\rho_r)^j, v_1^{(0)}, \ldots, v_{\ell(i)-1})} |(v_j^*, z)|,$$  \hspace{1cm} (22)
if \( v_1 \) is an extra-vertex the supremum clearly is only over the set \( z \in \mathcal{A}^h\{v_1\} \). The factor \((\rho_c/\rho_r)^j\) ensures that all vertices of the cyclic root of nearly-s.m.p.s get the same weight in the computation. If \( v_1 \) is the leading eigenvector of an s.m.p.-candidate, we have \( \rho_c/\rho_r = 1 \). Now one has to find numbers \( \alpha_1, \ldots, \alpha_S > 0 \) such that
\[
\begin{align*}
\alpha_i q_{i,j} &< \alpha_j \quad \text{whenever } v_1 \text{ is the leading eigenvector of an s.m.p.-candidate} \\
\alpha_i q_{i,j} &< 1 \quad \text{otherwise}
\end{align*}
\]
and multiply all vertices \( v_i^{(j)}, i = 1, \ldots, R, j(i) = 0, \ldots, l(i)-1 \), and extra-vertices \( v_i, i = R+1, \ldots, S \), from the root \( \mathcal{H} \) with the corresponding balancing factor \( \alpha_i \). In our implementation we distinguish between extra-vertices and vertices from nearly-s.m.p.s., precisely we solve the following system
\[
\begin{align*}
\alpha_i q_{i,j} &< \alpha_j \quad \text{whenever } v_1 \text{ is the leading eigenvector of an s.m.p.-candidate} \\
\alpha_i q_{i,j} &= .999 \cdot \rho_i \quad \text{whenever } v_1 \text{ is the leading eigenvector of a nearly-s.m.p.} \\
\alpha_i q_{i,j} &= 1/100 \quad \text{whenever } v_1 \text{ is an extra-vertex.}
\end{align*}
\]
Now [19, Theorem 3.3] ensures that the modified invariant polytope algorithm 4.1 terminates when started with both the s.m.p.-candidates, the nearly-s.m.p.s and the extra-vertices if and only if it terminates when started solely with the s.m.p.-candidates.

It was assumed (personal communication), at least for dimension \( s = 1 \), that the balancing factors for transition matrices occurring in subdivision theory are always equal to 1. While it is not hard to find counterexamples in dimensions \( s > 1 \), the claim is also not valid in the univariate case, as the next example will show.

**Example 4.3.** Let \( S \) be the univariate subdivision scheme\(^1\) defined by the mask \( a \) and the dilation matrix \( M \) given by
\[
a = \frac{1}{12} \begin{bmatrix} 3 & 3 & 4 & 3 & 4 & 3 & 4 & 3 & 3 \end{bmatrix}^T, \quad M = [-3].
\]

The basic limit function can be seen in Figure 1. Taking the digit set \( D = \{-2, -1, 0\} = M[0,1] \cap \mathbb{Z} \), we construct the set \( \Omega_C = \{-4, -3, -2, -1, 0, 1\} \) (using [12, Lemma 3.8]) and the corresponding transition matrices \( T_d = (a(\alpha - M\beta))_{\alpha, \beta \in \Omega_C}, d \in D \). The restriction of the transition matrices to the space \( V \) of first order differences with basis
\[
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 \\
-1 & 1 & 0 & 0 & 0 \\
0 & -1 & 1 & 0 & 0 \\
0 & 0 & -1 & 1 & 0 \\
0 & 0 & 0 & -1 & 1 \\
0 & 0 & 0 & 0 & -1
\end{bmatrix}
\]
yields the set of matrices \( T|_V = \{ T_{-2}|_V, T_{-1}|_V, T_0|_V \} \) with
\[
T_{-2}|_V = \frac{-1}{12} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
3 & 0 & 1 & 2 & 0 & 0 & 0 & 0 & 0 \\
2 & 0 & 2 & 1 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 3 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}, \quad T_{-1}|_V = \frac{-1}{12} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 3 & 0 & 1 & 2 & 0 & 0 & 0 & 0 \\
1 & 2 & 0 & 2 & 1 & 0 & 0 & 0 & 0 \\
2 & 1 & 0 & 3 & 0 & 0 & 0 & 0 & 0 \\
3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}, \quad T_0|_V = \frac{-1}{12} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 3 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 1 & 2 & 0 & 2 & 0 & 0 & 0 & 0 \\
0 & 2 & 1 & 0 & 3 & 0 & 0 & 0 & 0 \\
0 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}.
\]

\(^1\) Subdivision schemes are computational means for generating finer and finer meshes in \( \mathbb{R}^s \), usually in dimension \( s = 1, 2, 3 \). At each step of the subdivision recursion, the topology of the finer mesh is inherited from the coarser mesh and the coordinates \( c^{(n+1)} \) of the finer vertices are computed by local averages of the coarser ones \( c^{(n)} \) by \( c^{(n+1)} = S c^{(n)} = \sum_{\alpha \in \mathbb{Z}^s} a(\cdot - M\alpha) c^{(n)}(\alpha) \). See [12] for a more thorough explanation.
For the s.m.p.s $\Pi_1 = T_2T_1T_{-1}|V$ and $\Pi_2 = T_1T_0T_{-1}|V$ with balancing vector $[1 \quad 9/10]$, the invariant polytope algorithm terminates after 4 iterations. With the balancing-vector $[1 \quad 1]$ the invariant polytope algorithm does not terminate.

The basic limit function of the scheme can be seen in Figure 1, its Hölder regularity is $0.9413 \ldots$.

Example 4.4 shows the advantage of the new balancing procedure in connection with nearly-s.m.p.s.

**Example 4.4.** Given $E_1 = \begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix},$ $E_2 = \begin{bmatrix} 2 & 0 \\ 2 & 1 \end{bmatrix}$, the irreducible set $E = \{E_1, E_2\}$ has $E_2E_1$ as an s.m.p. and $\rho(E) = 2.5396 \ldots$ Assuming we start algorithm 4.1 with that candidate and the nearly-s.m.p. $E_2$, with corresponding leading eigenvectors $\tilde{v}_1^{(0)} = [0.9121 \ldots \ 0.4100 \ldots]^T$, $\tilde{v}_2^{(0)} = [0.4472 \ldots \ 0.8944 \ldots]^T$ and leading dual eigenvectors $\tilde{v}_1^* = [0.9958 \ldots \ 0.2238 \ldots]^T$, $\tilde{v}_2^* = [2.2361 \ldots \ 0.0000 \ldots]^T$. For the balancing procedure as described in [19, Remark 3.7] we need to find numbers $\alpha_1, \alpha_2 > 0$ such that for some $h \in \mathbb{N}$, say $h = 10$, $q_{1,2} = \sup_{z \in \tilde{E}^h} |(\tilde{v}_1^*, z)| = 2.0395 \ldots$ and $q_{2,1} = \sup_{z \in \tilde{E}^h} |(\tilde{v}_2^*, z)| = 0.8196 \ldots$ the following two inequalities hold

$$\alpha_1 \cdot 2.0395 \ldots = \alpha_1 q_{1,2} < \alpha_2$$
$$\alpha_2 \cdot 0.8196 \ldots = \alpha_2 q_{2,1} < \alpha_1.$$

This is clearly impossible. Since there are no admissible balancing factors for $h = 10$, there are no admissible balancing factors for $h > 10$ [19, Section 3].

Since $E_1E_2$ is a dominant s.m.p., algorithm 4.1 terminates if it is started only with that candidate, and thus there exists balancing factors such that the algorithm terminates when started with $E_2E_1$ and $E_1$, e.g. $\alpha_1 = 1$, $\alpha_2 = 1/2$.

4.6 Natural selection of vertices (17)

In the original algorithm 1.2, in every iteration all vertices generated in the last iteration, which were not mapped inside the polytope, were used to construct new vertices. In the modified algorithm 4.1 we only take a subset of those. We choose the vertices under the mild condition that for every $n \in \mathbb{N}$, every vertex of $\{\tilde{A}_j\}^nV_0$ eventually will be selected, given that it is not absorbed already. In other words, we do not forget any vertex to select. Two selection strategies turned out to work well:

(a) Choose those vertices that have the largest (e.g. highest decile) norm $\|V^+ \cdot \|_2$, where $V^+$ denotes any pseudo-inverse of $V$, and

(b) Choose those vertices whose parent vertex has largest norm with respect to the norm $\| \cdot \|_V$. 

Fig. 1. The basic limit function for the subdivision scheme from Example 4.3.
Strategy (a) reduces the number of vertices in $V$ by roughly 20%, strategy (b) by roughly 10%. Since the intermediate bounds $b_k$ for the JSR decreases very slowly when we use strategy (a) only, we use three times (a) and one time (b) in our implementation.

The natural selection of new vertices also makes algorithm 4.1 applicable for problems with a large number of matrices, since it ensures that the number of norms to be computed in each iteration is reasonably small. Of course, it does not substantially decrease the total number of norms we need to compute.

4.7 Simplified polytope (18)

In some examples the vertices constructed by algorithm 4.1 are very near to each other, i.e. are at distances in the order of the machine epsilon. Those vertices are irrelevant for the size of the polytope so we disregard them. This also protects against stability problems in the LP-programming part, since for simplices with vertices very near to each other, LP-solvers perform very badly. This phenomenon happens frequently when there are multiple s.m.p.s. In our implementation we use a variable threshold in (18) when determining which vertices of the polytope we use in the computation of the norm.

It would also be possible to choose a polytope $W(v)$ for each norm $\|v\|_{co W(v)}$ we need to compute, since for each $v \in \mathcal{A}E$ we only need $s + 1$ vectors from $V$ to compute the norm $\|v\|_{co V}$ exactly. Unfortunately we have no idea so far, how to select a good subset of $V_k$ in a reasonable amount of time, i.e. faster than the computation of the norm would take.

4.8 Parallelisation (19) & (20)

This is one of the main differences to the original implementation.

Instead of testing each vertex one after another, and adding it immediately to the set of vertices $V_k$ if it is outside of the polytope, we compute all selected vertices from step (17) with respect to the same polytope. Afterwards we add all vertices (which are outside of the polytope) at once to the set $V_k$.

This clearly leads to larger polytopes, in our examples the number of vertices increases by 10%, but this is compensated by the fact that we can parallelise the computations of the norms. The speed-up is nearly linear in the number of available threads. Since the linear programming model does not change, we can speed up this part further by warm starting the linear programming problems, i.e. we reuse the solutions obtained from the computations of the other vertices. If there are no suitable candidates to warm start with, we still can speed up the LP-problem by starting the search for the solution at the nearest vertex point of the polytope $W$. The speed-up from warm starting is roughly 50-70%.

4.9 Norm estimation (19)

Before computing the exact norm of a vector $A_j v$, we try to determine the relative position (inside or outside of the polytope) using the estimates in Lemma 4.5. If a vertex is proven to be inside or outside of the polytope, we do not have to compute its exact norm anymore. Unfortunately, these estimates are quite rough and fail to determine the position for most vertices, except in the positive case ($P$) where Lemma 4.5 (5) gives very good estimates.

**Lemma 4.5.** Let $V \subseteq \mathbb{R}^s$ and $x \in \mathbb{R}^s$. Then

(1) $\|x\|_{co V} \leq \|x\|_1$, where $V \lambda = x$.

(2) $\|x\|_{co V} \geq \|V^+ x\|_2$, where $V^+$ is the Moore-Penrose pseudo-inverse of $V$.

(3) If there exists $w \in \mathbb{R}^s$ such that $|\langle w, v \rangle| < |\langle w, x \rangle|$ for all $v \in V$, then $x \notin co V$. 


(4) Let $W$ be the vertices of another central symmetric polytope with non-empty interior such that $\text{co } W \subseteq \text{co } V$, then $\| \cdot \|_{\text{co } V} \leq \| \cdot \|_{\text{co } W}$.

(5) If $V \subseteq \mathbb{R}^+_n$, $x \in \mathbb{R}^+_n$ and there exists $v \in V$ such that $x_i \leq v_i$ for all $i = 1, \ldots, s$, then $x \in \text{co } V$.

**Proof.**  
(1) For every $x \in \partial \text{co } V$ there exists a unique $\hat{\lambda} \in \mathbb{R}^n$ such that $x = \sum_{i=1}^{\# V} \hat{\lambda}_i v_i$, $\| \hat{\lambda} \|_1 = 1$. Any other $\lambda$ which satisfies $x = \sum_{i=1}^{\# V} \lambda_i v_i$ has norm $\| \lambda \|_1 \geq 1$, because if otherwise, this implies $x \in (\text{co } V)^\circ$. Therefore, all $x \in \partial \text{co } V$ fulfill $\| x \|_{\text{co } V} \leq \| \lambda \|_1$.

(2) By construction of the Moore-Penrose inverse, $\hat{\lambda} = V^+ x$ is the unique solution to $V \hat{\lambda} = x$ with minimum 2-norm, from which the claim follows.

(3) If $| \langle w, v \rangle | < | \langle w, x \rangle |$ for all $v \in V$, then there exists a hyperplane which separates the point $x$ and the polytope $V$. From this the claim directly follows.

(4) This immediately follows from the definition of the Minkowski norm.

(5) Defining $z := v - x$ we see that $z \in \text{co } V$ which implies $x = z + v \in \text{co } V$.

**Remark 4.6.** Estimate 4.5 (5) uses the fact that the norms $\| \cdot \|_{\text{co } V}$ are orthant monotonic. It would be interesting to know whether and when Minkowski norms $\| \cdot \|_{\text{co } V}$ are orthant monotonic. This would allow to transfer the estimate 4.5 (5) to the case (R).

### 4.10 New stopping criterion (21)

In our implementation we added a routine which computes the spectral radii of all interchangeably occurring matrix products. If the candidates $\Pi_r$ are s.m.p.s, then all interchangeably occurring matrix products will have spectral radius less than 1. On the other hand, if the candidates are not s.m.p., then it still can happen that all interchangeably occurring matrix products have spectral radius less than 1, but algorithm 4.1 never stops, see Example 4.8.

Nevertheless, this never happened in any non-artificial example. Furthermore, products with larger normalized spectral radius always occurred very fast. Thus, from a practical point of view, this is a better way to check whether the candidates are s.m.p. or not, than the method described in [17, Proposition 2]. The method [17, Proposition 2] on the other hand, is fail-proof and eventually always strikes when an s.m.p.-candidate is not an s.m.p.. In our implementation of the algorithm [26] we thus implemented both stopping criteria.

We now illustrate how the stopping criterion (21) may fail. For that purpose, we introduce for given $\delta > 0$ the set

$$
\mathcal{M}_\delta = \{(i_n)_n \in \{1, \ldots, J\}^\mathbb{N} : \rho(\tilde{A}_{i_m} \cdots \tilde{A}_{i_1})^{1/m} < \delta, \; \forall m \in \mathbb{N}\}.
$$

For $\delta = 1$, the products $\tilde{A}_{i_n} \cdots \tilde{A}_{i_1}$, $(i_n)_n \in \mathcal{M}_1$, are exactly the products occurring in algorithm 4.1 until the stopping criterion (21) strikes. The hope would be, that the norms of the products in that sequence stay bounded, i.e. $\exists C > 0$ such that $\| \tilde{A}_{i_n} \cdots \tilde{A}_{i_1} \| < C$ for all $n \in \mathbb{N}$. This is wrong, even for arbitrary small $\delta$ the norms of the products may go to infinity.

**Example 4.7.** $A = \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$. Clearly, $\text{JSR}(\{A, B\}) = 2$. The matrices of the (left-infinite) sequence $(A^n B)_n$, $n \in \mathbb{N}$, have spectral radius zero, yet the sequence $\| A^n B \|$ goes to infinity as $n \to \infty$.

Unfortunately Example 4.8 consists of reducible matrices. Thus, we need another example to proof that the stopping criterion may fail.
Example 4.8. Let $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ and $B = \frac{3}{4} A$. Then $\text{JSR}(\{A, B\}) = 1$, $\rho(A^n B) < 1$ for all $n \in \mathbb{N}$, but $\|A^n B\| \to \infty$. Indeed, $A^n B = \begin{bmatrix} 3/4 & (3n + 3)/4 \\ 0 & 3/4 \end{bmatrix}$, thus it follows that $\rho(A^n B) = \frac{3}{4}$ and $\|A^n B\|_1 = 6 + 3n$. 

Now define the irreducible set $\mathcal{A} = \{A, A^T, B, B^T\}$. Then algorithm 4.1, together with an (unlucky) version of the natural selection of vertices procedure and started with the (wrong) s.m.p.-candidates $\{A, A^T\}$ and root vectors $v^{(0)}_0 = [1 \ 0]^T$, $v^{(0)}_1 = [0 \ 1]^T$, can construct an infinitely big polytope, solely with products whose spectral radius is strictly less than one. Indeed, for $n \in \mathbb{N}$, applying the sequence of products $A^n B$ to the starting vector $v_1$ we get the sequence of vectors $A^n B v_1 = [3n/4 \ 3n/4]^T$. The same calculation shows that $(A^T)^n B^T v_0 = [3/4 \ 3n/4]^T$. These two sequences generate an unbounded polytope.

4.11 Proof for Theorem 4.2

Proof. (i) Assume the original algorithm [17, 19] terminates at depth $n$ with vertices $V^n_0$, then $\tilde{A}^0 \cap \text{co} V^n_0 \subseteq \text{co} V^n_0$ for all $m \in \mathbb{N}$ [17, Lemma 13]. Assume further that the modified invariant polytope algorithm 4.1 does not terminate. Due to Section 4.6 (17) there exists $k \in \mathbb{N}$ such that $V^n_0 \subseteq \text{co} V_{k+1}$, where $\text{co} V_{k+1}$ is the invariant polytope constructed by the modified algorithm after the $k$th iteration. Note that the set $V_{k+1}$ consists only of vertices from $V^n_0$ and vertices from $\tilde{A}^k V_0$. Both sets are subsets of $\text{co} V^n_0$. Thus, $\tilde{A}^k \cap \text{co} V_{k+1} \subseteq \tilde{A}^k \cap \text{co} V_n \cup \tilde{A}^{k+1} V_0 = \text{co} V^n_0$, implying that the modified algorithm terminated at latest in the $k$th iteration, contradicting our assumption. 

The other direction follows in exactly the same way.

(ii) Assume that $\text{JSR}(\tilde{A}) < 1$, or equivalently, $\text{JSR}(\tilde{A}) < \eta < 1$ for some $\eta > 0$. [11, Theorem I (b)] implies that $\|\tilde{A}_k \cdots \tilde{A}_i\| \to 0$ for any product $\tilde{A}_k \cdots \tilde{A}_i \in \tilde{A}^n$ as $k \to \infty$. Thus, the algorithm eventually terminates.

(iii) Assume in step $k$ we found an upper bound $b_{k+1}$ such that $b_{k+1} < b_k$ and the algorithm will not terminate after that iteration. Define $n \in \mathbb{N}$ as the biggest integer such that $\mathcal{A} V_n \subseteq V_{k+1}$. We then have $b_{k+1} = \delta^{-1} \cdot \max\{N(v)(1 - \epsilon)^{-1} : v \in V_n \land \mathcal{A} v \not\subseteq \text{co} V_n\}$. Let $j \in \{1, \ldots, J\}$ and $v \in V_n$. If $\mathcal{A} v \not\subseteq \text{co} V_n$ we trivially get $\|A_j v\|_{\text{co} V_n} \leq 1 \leq b_{k+1}$. If $\mathcal{A} v \not\subseteq \text{co} V_n$ we directly get from the definition of $b_{k+1}$ that $\|A_j v\|_{\text{co} V_n} \leq b_{k+1}$.

5 APPLICATIONS AND NUMERICAL RESULTS

In this section we illustrate algorithm 3.1 and 4.1 with numerical examples. For our tests we use matrices from standard applications, as well as random matrices. We also try to repeat tests which were performed in similar papers [3, 4, 6, 17, 19, 27].

The parameters for the various algorithms are chosen such that they terminate after a reasonably short time. For the modified invariant polytope algorithm 4.1 the parameters are chosen such that the algorithm terminates at all, hopefully in shortest time. We do not report the exact parameters, since we believe they are of no value for the reader. The tests are performed using an Intel Core i5-4670S@3.8GHz, 8GB RAM with the software Matlab R2017a, JSR-Toolbox v1.2b [23], SeDuMi Toolbox v1.32 and Gurobi solver v8.0.

For the tests we report • the dimension $\text{dim}$ of the matrices, • the duration $\text{time}$ needed for the computation (this value is only to be understood in magnitudes), • the number of matrices $J$ in the test set $\mathcal{A}$, • the number of vertices $\#V$ of the invariant polytope, • spectral maximizing product(s) $\text{s.m.p.}$, and • the number $\#\text{tests}$ of test runs.
5.1 Main results for the modifications.

5.1.1 Modified invariant polytope algorithm. To summarize, we can say that the single-threaded modified invariant polytope algorithm 4.1 is roughly three times faster than the original algorithm [17, 19]. If the dimension of the matrices is sufficiently large, the parallelised algorithm 4.1 scales nearly linearly with the number of available threads. More precisely

- for pairs of random matrices the modified invariant polytope algorithm 4.1 reports the exact value of the JSR in reasonable time up to dimension 25,
- for Daubechies matrices the modified invariant polytope algorithm reports the exact value of the JSR in reasonable time up to dimension 42,
- for non-negative matrices it strongly depends on the problem. For random, sparse, non-negative matrices the algorithm works up to dimension 3000 or higher. For the (sparse) matrices arising in the context of code capacities (Section 5.4) the algorithm works well only up to dimension 16.

5.1.2 Modified Gripenberg algorithm. For the modified Gripenberg algorithm 3.1 we can say, that it finds in almost all cases an s.m.p.. Thus, for fast estimates of the JSR, the modified Gripenberg algorithm 3.1 may be used independently, e.g. in applications where the parameters where a matrix family has highest/lowest JSR need to be determined. In a second step one then may compute the exact JSR for the found parameters using the modified invariant polytope algorithm.

Clearly, since the computation of the JSR is NP-hard, there must be sets of matrices for which algorithm 3.1 fails and we report mostly these cases together with a comparison with other known algorithms. These are (a) the random modified Gripenberg algorithm 3.1 described in Remark 3.4, (b) the Gripenberg algorithm itself, (c) the modified invariant polytope algorithm 4.1 and (d) the genetic algorithm [3]. The latter one being a Monte Carlo algorithm and its current implementation (September 2018) has a minor bug, and thus the results may be better than documented here. At least in our test runs, the modified Gripenberg algorithm 3.1 performed best, in the sense that in most cases it returned a correct s.m.p. in fastest time. More precisely,

- for long s.m.p.s the modified Gripenberg algorithm 3.1 performed best,
- for large sets of matrices the genetic algorithm and the modified invariant polytope algorithm 4.1 performed best.

5.2 Randomly generated matrices

We first present the behaviour of the modified invariant polytope algorithm 4.1 for pairs of matrices of dimensions 2 to 20 with normally distributed values whose (a) matrices have the same 2-norm, (b) matrices have the same spectral radius, and (c) matrices have the same spectral radius and \( \delta = 0.99 \) (where \( \delta \) was the parameter controlling the accuracy of the algorithm, see Section 4.3 (14)). We print the median values for these three examples, since there are always some outliers in tests if \( \delta = 1 \). The average value is roughly 100 times bigger. We see in Table 1 that the algorithm is applicable for pairs of random matrices up to dimension 25, for which it takes roughly one weekend to complete. For \( \delta = 0.95 \) the algorithm is comparable to Gripenbergs algorithm.

Compared to the same test with the original invariant polytope algorithm in [17, Table 2] we see that the modified invariant polytope algorithm 4.1 produces polytopes with roughly twice as much vertices, but still works for matrices of dimension 20.

Random matrices with non-negative entries are an worthy test case, since the computation of the invariant polytope (i.e. the loop (17)-(20)) always finishes after a few seconds, nearly regardless of the dimension. Since the algorithm is not optimized for such high dimensions, the algorithm still needs some minutes to terminate, mostly due to the preprocessing steps (13)-(16). For sparse
Table 1. Computation of the JSR for random pairs of matrices. \( \delta \): accuracy parameter \( \delta \) for the modified invariant polytope algorithm (14), dim: dimension of the matrices, \#V: number of vertices of the invariant polytope, time: time needed to compute the invariant polytope, \( J \): number of matrices, \#test: number of test runs.

| \( J = 2, \#test = 20 \), median values | \( J = 2, \#test = 20 \), median values | \( J = 2, \#test = 20 \), median values |
|-----------------------------------------|-----------------------------------------|-----------------------------------------|
| \( \delta = 1 \) | \( \delta = 1 \) | \( \delta = 0.99 \) |
| \( \delta = 1 \) | equal spectral radius | equal spectral radius | equal spectral radius |
| \( \delta = 1 \) | time | \#V | time | \#V | time | \#V |
| 2 | 1.1 s | 5-2 | 1.2 s | 6-2 | 0.2 s | 5-2 |
| 4 | 1.4 s | 17-2 | 1.8 s | 77-2 | 0.8 s | 19-2 |
| 6 | 2.0 s | 47-2 | 2.5 s | 130-2 | 1.5 s | 47-2 |
| 8 | 2.5 s | 100-2 | 3.9 s | 220-2 | 2.1 s | 98-2 |
| 10 | 4.9 s | 270-2 | 5.1 s | 320-2 | 3.3 s | 220-2 |
| 12 | 4.7 s | 280-2 | 11 s | 770-2 | 6.6 s | 570-2 |
| 14 | 8.4 s | 510-2 | 21 s | 1100-2 | 12 s | 800-2 |
| 16 | 25 s | 1100-2 | 33 s | 1400-2 | 25 s | 1000-2 |
| 18 | 90 s | 2100-2 | 200 s | 2500-2 | 44 s | 1600-2 |
| 20 | 295 s | 3100-2 | 5000 s | 6200-2 | 800 s | 3900-2 |

Table 2. Computation of the JSR for random pairs of matrices with non-negative entries. dim: dimension of the matrices, \( J \): number of matrices, \#test: number of test runs. time: time needed to compute the invariant polytope, \#V: number of vertices of the invariant polytope. † Most cones have 8 or 16 vertices, because the algorithm terminates after 3 or 4 iterations. The algorithm does not check whether all of these vertices are really outside of the polytope. †† Since the matrices are random, most of the sparse matrices have non-trivial invariant subspaces which reduces the effective dimension of the matrices by roughly 10%.

| \( J = 2, \#test = 20 \), non-negative entries, equal spectral radius, median values | \( J = 2, \#test = 20 \), non-negative entries, equal spectral radius, median values | \( J = 2, \#test = 20 \), non-negative entries, equal spectral radius, median values |
|-----------------------------------------|-----------------------------------------|-----------------------------------------|
| \( \delta = 1 \) | \( \delta = 1 \) | \( \delta = 0.99 \) |
| \( \delta = 1 \) | 0% sparsity | 90% sparsity | 98% sparsity | 99% sparsity |
| \( \delta = 1 \) | time | \#V | time | \#V | time | \#V | time | \#V | time | \#V |
| 20 | 0.3 s | 7 | 1.7 s | 42 | 2.2 s | 50 | 2.2 s | 50 | 2.2 s | 50 |
| 50 | 0.3 s | 8 | 1.6 s | 50 | 2.2 s | 50 | 2.2 s | 50 | 2.2 s | 50 |
| 100 | 0.4 s | 8 | 0.8 s | 25 | 17 s | 1300 | 17 s | 1300 | 17 s | 1300 |
| 200 | 0.5 s | 8 | 1.0 s | 23 | 5.0 s | 220 | 110 s | 2600 | 110 s | 2600 |
| 500 | 1.2 s | 8 | 1.8 s | 16 | 7.7 s | 90 | 26 s | 310 | 26 s | 310 |
| 1000 | 6.3 s | 8 | 11 s | 16 | 30 s | 45 | 72 s | 110 | 72 s | 110 |
| 2000 | 35 s | 8 | 72 s | 16 | 35 s | 8 | 290 s | 64 | 290 s | 64 |

matrices with non-negative entries, algorithm performs slightly worse, but is still applicable up to dimension 2000 or higher. Again, it is very likely that it stills work for even larger matrices if the algorithm were optimized for such matrices, see Table 2 for the results. We again give the median values. The average values for these cases are roughly 10% higher. A better benchmark for non-negative matrices is presented in Section 5.4.

In Table 3 we see how the modified Gripenberg algorithm performs on random matrices. These random matrices have equally distributed values in \([-5, 5]\) to mimic the test in [3, Section 4.2]. Interestingly the genetic algorithm performs very bad, as does the random modified Gripenberg algorithm. We report the success-rate, i.e. how often the algorithms did find an s.m.p. in percent.
Table 3. Performance of the modified Gripenberg algorithm 3.1 for sets of random matrices. dim: dimension of the matrices, J: number of matrices, success: percentage of how often a correct s.m.p. is found. #test: number of test runs. time: time needed by the algorithm.

| Algorithm                | J = 2, dim = 2 | J = 4, dim = 4 | J = 8, dim = 8 |
|-------------------------|----------------|----------------|----------------|
|                          | success | time  | success | time  | success | time  |
| mod. invariant polytope | 100%    | 1.1 s | 100%    | 4.3 s | 100%    | 40.0 s|
| mod. Gripenberg         | 100%    | 1.9 s | 100%    | 4.1 s | 100%    | 5.4 s |
| random Gripenberg       | 100%    | 1.8 s | 99%     | 3.8 s | 82%     | 4.3 s |
| Gripenberg              | 100%    | 3.8 s | 100%    | 20.3 s| 100%    | 82.1 s|
| brute force             | 100%    | 180 s | 98%     | 180.0 s| 74%     | 180.0 s|
| genetic                 | 100%    | 7.1 s | 97%     | 9.3 s | 87%     | 12.0 s|

Table 4. For the test set $X$ (Example 5.1) all fast algorithms fail. dim: dimension of the matrices, lower bd.: computed lower bound for the JSR, J: number of matrices, time: time needed by the algorithm.

| Testset       | Algorithm            | lower bd.   | time |
|---------------|----------------------|-------------|------|
| $X$           | mod. invariant polytope | 1.01179... | 40 s |
| $J = 2$       | mod. Gripenberg      | 1.01130... | 4 s  |
| dim = 2       | random Gripenberg    | 1.01172... | 10 s |
|                | Gripenberg           | 1.01179... | 580 s|
|                | genetic              | 1.01130... | 8 s  |

5.3 Handpicked generic matrices

Example 5.1. Let

$$X_1 = \begin{bmatrix} 15 & -73 \\ 92 & 79 \\ 56 & 89 \\ 59 & 118 \end{bmatrix}, \quad X_2 = \begin{bmatrix} -231 & -143 \\ 241 & 219 \\ 103 & -38 \\ 153 & 65 \end{bmatrix}.$$ 

The set $X = \{X_1, X_2\}$ has an s.m.p. of length 119 with normalized spectral radius $\text{JSR}(X) = 1.01179...$. Gripenberg’s algorithm finds the s.m.p. after an evaluation of ~630k products, taking roughly ten minutes. Both the modified Gripenberg algorithm, as well as the genetic algorithm fail. The modified invariant polytope algorithm 4.1 finds an s.m.p. after less than one minute. The test results are in Table 4.

Example 5.2 is of interest because it is a rather simple family of two matrices with an arbitrary long s.m.p.

Example 5.2. Let $n \in \mathbb{N}$, $C_0 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ and $C_n = \begin{bmatrix} 0 & e^{1/n} \\ \frac{1}{n} e^{1/n} & 0 \end{bmatrix}$. Then $C_n^0 C_n$ is an s.m.p. for the set $C_n = \{C_0, C_n\}$ with $\text{JSR}(C_n) = e^{1/n}$.

The genetic algorithm fails for most matrices of that family. All other algorithms report the correct s.m.p. in less than 5 s. The test results are in Table 5.

Proof for Example 5.2. Define $\tilde{C}_n = \begin{bmatrix} 0 & 0 \\ n & 0 \end{bmatrix}$, $n \in \mathbb{N}$. A product of $C_0$ and $\tilde{C}_n$ is non-zero if and only if it is of the form $C_0^{i_1} \tilde{C}_n C_0^{i_2} \tilde{C}_n \cdots \tilde{C}_n C_0^{i_m}$. Since the spectral radius does not change under cyclic permutation, we can assume that the product is of the form $C_0^{i_1} \tilde{C}_n C_0^{i_2} \tilde{C}_n \cdots C_0^{i_m} \tilde{C}_n$. A
Table 5. For the test sets $C_n$ (Example 5.2) the genetic algorithm mostly fails. $\text{dim}$: dimension of the matrices, $\text{lower bd.}$: computed lower bound for the JSR, $J$: number of matrices, $\text{s.m.p.}$: an s.m.p., $\text{time}$: time needed to compute the invariant polytope.

| Testset  | Algorithm           | lower bd. | time |
|----------|---------------------|-----------|------|
| $C_{15}$ | mod. invariant polytope | 1.0689... | 1.7 s |
| $J = 2$  | mod. Gripenberg      | 1.0689... | 3.3 s |
| $\text{dim} = 2$ | random Gripenberg | 1.0689... | 3.2 s |
| $\text{s.m.p.} = C_{15}^{15} C_{15}$ | Gripenberg | 1.0689... | 0.1 s |
| genetic | 1.0689... | 7.0 s |
| $C_{30}$ | mod. invariant polytope | 1.0338... | 2.5 s |
| $J = 2$  | mod. Gripenberg      | 1.0338... | 4.0 s |
| $\text{dim} = 2$ | random Gripenberg | 1.0338... | 4.3 s |
| $\text{s.m.p.} = C_{30}^{30} C_{30}$ | Gripenberg | 1.0338... | 0.1 s |
| genetic | 1.0215... | 6.6 s |
| $C_{60}$ | mod. invariant polytope | 1.0168... | 4.0 s |
| $J = 2$  | mod. Gripenberg      | 1.0168... | 3.1 s |
| $\text{dim} = 2$ | random Gripenberg | 1.0168... | 4.3 s |
| $\text{s.m.p.} = C_{60}^{60} C_{60}$ | Gripenberg | 1.0168... | 0.1 s |
| genetic | 1.0000... | 6.3 s |

A (lengthy) straightforward computation shows that the normalized spectral radius of this product is $(n^m \prod_{i=1}^m i_j)\frac{1}{(m + \sum_{i=1}^m i_j)}$. Taking the gradient with respect to $i$ and setting it to zero, we immediately get that all $i_j$ must be equal. Thus, the normalized spectral radius of all finite products is maximized with a product of the form $C_m^n \tilde{C}_n$ whose normalized spectral radius equals $mn^{1/(1+m)}$. For fixed $m \in \mathbb{N}$ this term has its maximum at $n = \frac{m}{m} e^{1+1/m}$. Thus, $C_m^n C_n$ is the product with largest normalized spectral radius under all finite products. Using (2) we conclude that $\text{JSR}(C) = \rho(C_m^n C_n)^{1/(n+1)} = (e^{(n+1)/n})^{1/(n+1)} = e^{1/n}$. \hfill $\square$

### 5.4 Capacity of codes with forbidden difference sets

In some electromagnetic recording systems, the bit error rate is often dominated by a small set of certain forbidden difference patterns $D$. Thus, one needs to construct sets of allowed words with values in $\{0, 1\}$, all of whose possible differences do not yield such a forbidden pattern. Clearly, one wants codes which constrain the number of all possible patterns as least as possible. We are interested in how constraining a given forbidden difference pattern is, which we denote as the capacity $\cap D \in [0, 1]$. The larger the capacity, the better. This problem can be expressed in terms of the JSR of a finite set of matrices. See [27] for a more details.

The occurring matrices in this application only have entries in $\{0, 1\}$, but their dimension, as well as the number of matrices increases exponentially with the length of the forbidden difference patterns.

We use the modified polytope algorithm to compute the capacities for the forbidden difference patterns $D$ taken from [27, p. 10], [3, Table 1], [6, p. 6] and for difference sets with the additional symbol $\pm$, denoting $+1$ and $-1$, discussed in [6, Section v]. Nearly all of these capacities were not known exactly before.

For most difference sets $D$, there are several s.m.p.s., that not only share the same leading eigenvalue but also the same eigenvector. Due to this reason, algorithm 4.1 sometimes only gives a
Fig. 2. Number of added vertices to the invariant polytope in each iteration for the example computing the capacity of codes with forbidden difference $+ + + + + -$.

$D = [1 1 1 1 -1 0]$; $C =$ codecapacity$(D)$; $[\text{JSR,info}] =$ tjsr$(C)$;

```matlab
figure('units', 'centimeters', 'position', [5,5,12,9]); L = sum(vertcat(info.cyclictree.L:));
plot(L,'o-','Color',[0.6,0.6,0.6],'MarkerSize',5,'MarkerFaceColor',[0.6,0.6,0.6], 'LineWidth',1)
axis([0, 12, 0, 3000]); xticks([0 10 20 30 40]); yticks([0 1000 2000 3000 4000]);
xlabel('k', 'Interpreter', 'Latex'); ylabel('\Delta \# V', 'Interpreter', 'Latex');
set(findall(gcf, '-property', 'FontSize'), 'FontSize',15);
set(findall(gcf, '-property', 'YMinorTick'), 'YMinorTick','off')
print('numdeltaV', '-depsc', '-r600');
```

bound for the JSR up to the accuracy in which we can compute the norms $\|\tilde{A}_j\|_{co W}$. We implemented the Matlab routine codecapacity [26] which computes the set of matrices needed for the JSR computation for a given difference set $D$. It works for reasonably small difference sets, and theoretically also for difference words with entries in $\{-K, \ldots, K\}$, $K \in \mathbb{N}$.

The exact computation of the capacity using the modified invariant polytope algorithm 4.1 was only possible if we used the estimates for the Minkowski norm in Lemma 5 (4.5), which reduced the norms to be computed by a factor of 100.

The difference set $D_4 = \{00\circ + \circ -\}$, taken from [3, Table 1], is a good test case for the modified Gripenberg algorithm, since the computation of the capacity translates to the JSR of a set with 256 matrices of dimension 16. As one can expect, Gripenberg’s algorithm fails to find an s.m.p., also the modified Gripenberg algorithm 3.1 fails. The genetic algorithm in most cases finds a better product than the one found by Gripenberg’s algorithm. The modified invariant polytope algorithm 4.1 also finds that better product after a while, but it did not terminate in reasonable time. Thus, the exact capacity, and whether an s.m.p. exists or not, is still unknown. The test results are in Table 6 and 7.

In Figure 2 we see the number of vertices added to the polytope $V$ in each iteration $k$ of the modified invariant polytope algorithm 4.1.

Remark 5.3. From our examples, it seems that the capacity for $D = \pm \pm \cdots \pm$ is $\text{cap}(D) = \frac{n-1}{n}$.

5.5 Hölder exponents of Daubechies wavelets

An important application of the JSR is the computation of the regularity of refinable functions. These are functions $\phi \in C_0(\mathbb{R}^s)$ which fulfill a functional equation of the form $\phi(x) = \sum_{a \in \mathbb{Z}^s} a(\alpha)\phi(2x - \alpha)$, $x \in \mathbb{R}$, with $a \in \ell_0(\mathbb{Z}^s)$. We use algorithm 4.1 to compute the H"older regularity of the Daubechies wavelets $D_n$ [14]. The regularity of $D_2$, $D_3$, and $D_4$ was computed by Daubechies and Lagarias [15], Gripenberg [16] computed it for $D_5, \ldots, D_8$, then Guglielmi and Protasov [19], as a demonstration of the invariant polytope algorithm, computed the regularity of $D_9, \ldots, D_{20}$. Now with the modified algorithm, we can compute the Hölder regularity for Daubechies wavelets up to $D_{42}$.
Table 6. Capacity of various difference sets $D$. $\epsilon = 10^{-10}$: computational accuracy, $\text{cap}(D)$: capacity, $D$: set of forbidden differences, dim: dimension of the matrices, $J$: number of matrices, $#V$: number of vertices of the invariant polytope, s.m.p.: an s.m.p.

| $D$          | s.m.p. | $\text{cap}(D)$ | $#V$ | $J$ | dim |
|--------------|--------|-----------------|------|-----|-----|
| $\pm \pm$    | $B_2B_3$ | 0.6942...       | 45.2 | 4   | 2   |
| $\ominus \ominus$ | $B_3$     | 1/2             | 3.2  | 4   | 2   |
| $\ominus + -$ | $B_4B_1$ | 0.9468...       | 86.2 | 2   | 8   |
| $\ominus + +$ | $B_1B_2$ | 0.8791...       | 43.2 | 4   | 8   |
| $\ominus + -$ | $B_3$     | 0.8791...       | 46.2 | 4   | 8   |
| $\ominus + +$ | $B_1B_2$ | 0.7396...       | 244.2| 16  | 8   |
| $\ominus + +$ | $B_1^2B_2^2$ | 0.7298...     | 804.2| 16  | 8   |
| $\ominus + +$ | $B_3$     | 0.9163...       | 1721.2| 16 | 16  |
| $\ominus + +$ | $B_6$     | 0.9163...       | 4559.2| 16 | 16  |
| $\ominus + + + -$ | $B_3$     | 0.9761...       | 992.2| 16 | 32  |
| $\ominus + + + +$ | $B_1B_2$ | 0.9614... + [0, $\epsilon$] | 54457.2| 16 | 32  |
| $\ominus + + + -$ | $B_3$     | 0.9761...       | 992.2| 16 | 64  |

Table 7. The modified Gripenberg algorithm 3.1 fails for the set of matrices corresponding to the forbidden difference set $D_4$ in Section 5.4. dim: dimension of the matrices, lower bd: computed lower bound for the JSR, $J$: number of matrices, time: time needed by the algorithm.

| Testset | Algorithm               | lower bd. | time |
|---------|-------------------------|-----------|------|
| $D_4 = \{\ominus \ominus \ominus \ominus \}$ | mod. invariant polytope | 1.6736... | 40 s |
| $J = 265$ | mod. Gripenberg         | 1.6663... | 2 s  |
| dim = 16 | random Gripenberg      | 1.6663... | 2 s  |
|         | Gripenberg              | 1.6663... | 60 s |
|         | genetic                 | 1.6736... | 10 s |

As noted in [19, Section 6.2], the polytopes generated by these matrices are very flat and the introduction of nearly-candidates and extra-vertices tremendously increases the performance of algorithm 4.1. Respectively, using the wrong set of nearly-candidates, the algorithm did not terminate at all. These cases are marked with $\hat{*}$ in Table 8. The right nearly-candidates and extra-vertices were merely found by trial and error. We report the number of extra-vertices and the vertices of the roots from the nearly-s.m.p.s together under $#Extra-V$. The number of the invariant polytopes vertices is depicted in Figure 3 (left side).

Remark 5.4. With the new values for $D_{21}$ to $D_{42}$ we can refine the observation in [18], that the differences of Hölder regularities $\alpha_n - \alpha_{n-1}$ seem to converge towards a value of 0.21 or maybe even 0.2, see Figure 3 (right side).
Table 8. Hölder regularity of Daubechies wavelets. $\alpha$: Hölder regularity of Daubechies wavelet, $D_n$: index of Daubechies wavelet, #V: number of vertices of the invariant polytope, #Extra-V: number of extra-vertices including those from nearly-s.m.p.s., s.m.p.: an s.m.p., time: time needed to compute the invariant polytope.

| $D_n$ | s.m.p. | #Extra-V | #V | time | $\alpha$ |
|-------|--------|----------|----|------|---------|
| 2     | $B_0^2$ | 0        | 0  | $<5\;s$ | 0.55001… |
| 3     | $B_0^2$ | 0        | 3  | $<5\;s$ | 1.08783… |
| 4     | $B_0^2$ | 2        | 9  | $<5\;s$ | 1.61793… |
| 5     | $B_0$ and $B_1$ | 2 | 14 | $<5\;s$ | 1.96896… |
| 6     | $B_0$ and $B_1$ | 3 | 18 | $<5\;s$ | 2.18914… |
| 7     | $B_0$ and $B_1$ | 4 | 27 | $<5\;s$ | 2.46041… |
| 8     | $B_0$ and $B_1$ | 5 | 40 | $<5\;s$ | 2.76082… |
| 9     | $B_0$ and $B_1$ | 6 | 55 | $<5\;s$ | 3.07361… |
| 10    | $B_0^2 B_1^2$ | 5 | 147 | $<5\;s$ | 3.36139… |
| 11    | $B_0$ and $B_1$ | 8 | 123 | 7 | 3.60347… |
| 12    | $B_0$ and $B_1$ | 9 | 91 | 7 | 3.83348… |
| 13    | $B_0$ and $B_1$ | 10 | 105 | 6 | 4.07348… |
| 14    | $B_0$ and $B_1$ | 11 | 134 | 8 | 4.31676… |
| 15    | $B_0^2 B_1$ | 11 | 386 | 6 | 4.55612… |
| 16    | $B_0^2 B_1$ | 12 | 346 | 7 | 4.78644… |
| 17    | $B_0$ and $B_1$ | 14 | 324 | 5 | 5.01380… |
| 18    | $B_0$ and $B_1$ | 15 | 282 | 8 | 5.23917… |
| 19    | $B_0$ and $B_1$ | 16 | 346 | 9 | 5.46532… |
| 20    | $B_0$ and $B_1$ | 17 | 529 | 12 | 5.69108… |
| 21    | $B_0^2 B_1$ | 17 | 868 | 15 | 5.91500… |
| 22    | $B_0^2 B_1$ | 22 | 433 | 9 | 6.13779… |
| 23    | $B_0$ and $B_1$ | 20 | 707 | 18 | 6.35958… |
| 24    | $B_0$ and $B_1$ | 21 | 701 | 16 | 6.58096… |
| 25    | $B_0$ and $B_1$ | 22 | 861 | 20 | 6.80198… |
| 26    | $B_0^2 B_1$ | 22 | 2471 | 73 | 7.02250… |
| 27    | $B_0^2 B_1$ | 29 | 2952 | 60 | 7.24241… |
| 28    | $B_0^2 B_1$ | 105 | 777 | 24 | 7.46187… |
| 29    | $B_0$ and $B_1$ | 26 | 1545 | 39 | 7.68091… |
| 30    | $B_0$ and $B_1$ | 27 | 2078 | 64 | 7.89962… |
| 31    | $B_0$ and $B_1$ | 29 | 2898 | 190 | 8.11801… |
| 32    | $B_0^2 B_1^2$ | 29 | 3791 | 760 | 8.33605… |
| 33    | $B_0^2 B_1^2$ | 30 | 4692 | 1330 | 8.55379… |
| 34    | $B_0$ and $B_1$ | 32 | 3047 | 628 | 8.77123… |
| 35    | $B_0$ and $B_1$ | 33 | 3191 | 727 | 8.98841… |
| 36    | $B_0$ and $B_1$ | 34 | 3887 | 881 | 9.20533… |
| 37    | $B_0^2 B_1^2$ | 70 | 8529 | 6503 | 9.42202… |
| 38    | $B_0^2 B_1^2$ | 38 | 6035 | 3540 | 9.63847… |
| 39    | $B_0^2 B_1^2$ | 40 | 7142 | 3900 | 9.85474… |
| 40    | $B_0$ and $B_1$ | 38 | 6909 | 5550 | 10.07073… |
| 41    | $B_0$ and $B_1$ | 39 | 8343 | 8743 | 10.28656… |
| 42    | $B_0$ and $B_1$ | 40 | 9508 | 16373 | 10.50220… |
6 CONCLUSION AND FURTHER WORK

6.1 Conclusion

The modified Gripenberg algorithm 3.1 together with the modified invariant polytope algorithm 4.1 can compute the exact value of the JSR in a short time (less than 30 minutes) for most matrix families up to dimension 22, in some cases even up to dimension 40. For matrices with non-negative entries, the algorithm may work up to a dimension of 3000. Since the modified invariant polytope algorithm 4.1 finds in almost all cases a correct s.m.p., it may be used independent to estimate the JSR.

6.2 Further work

From the mathematical point of view, the question why the modified Gripenberg algorithm 3.1 works so well is of interest, in particular why it works mostly better than the random modified Gripenberg algorithm. It also may be useful to search for better estimates for the Minkowski norms, e.g. with orthant-monotonic norms, which would lead to a considerable speed up of the modified invariant polytope algorithm.

From the algorithmic point of view, the algorithm could be made faster by using approximate solutions to the LP-problem when computing the Minkowski-norms, since the exact value of the norms is of minor interest — for the algorithm it is enough to know whether a point is inside or outside of the polytope.

We plan to implement the (seldom occurring) case (C) of complex leading eigenvalue in the near future and optimize the algorithm for a large number of parallel threads.

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APPENDIX

Algorithm 6.1 (Matlab implementation of the modified invariant polytope algorithm). For completeness, we provide a simple Matlab-implementation of the modified Gripenberg algorithm 3.1 in copy-paste-able format. The code is also available at http://tommsch.com/science.php for download, together with all other functions mentioned in that paper.

function [c] = gripenberg_modified(M,N,D)
%Tries to find smp-candidates in a fast way.
%Ex: gripenberg_modified({[2 1; 0 -2],[2 1; -1 -2]}, 4, 10)
J = length(M); %number of matrices
o = 1:J; %the orderings of the products to be checked
c = {}; %list of candidates
r = 0; %lower bound for JSR
for d = 1:D %do D iterations
    NR = zeros(2,size(o,2)); %norm and rho of candidates
    for i = 1:size(o,2) %can be parallelised using parfor!
        P = buildProduct(M,o(:,i)); %construct matrices
        NR(:,i) = [norm(P); max(abs(eig(P)))]; %compute norm and rho
    end
    NR = NR.^(1/d); %normalize norm and rho
    if r < max(NR(2,:)) %test if new bound was found
        c = {}; %delete candidates
        r = max(NR(2,:)); %update lower bound for JSR
    end
    c = [c num2cell(o(:,NR(2,:) >= r),1)]; %add candidates to c
    idx = NR(1,:) < r; %remove everything with norm less than JSR
    NR(:,idx) = []; o(:,idx) = [];
    %sort corresponding to norm
    NR = sortrows(NR');
    nNR = size(NR,2);
    if nNR > 2*N %keep highest and lowest norms
        o = o(:,[idx(1:N) idx(nNR-N+1:nNR)]);
    else %keep everything if N is too big
        o = o(:,idx);
    end
    o = repmat(o,[1 J]); %make new orderings of products
    reshape(repmat(1:J,[size(o,2) 1]),1,[]);
end

function M = buildProduct(A,prod)
%M constructs the product of matrices of A corresponding to prod.
M = eye(size(A{1},1));
for t = 1:length(prod); M = A{prod(t)}*M; end
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