Frame-like formulation for free mixed-symmetry bosonic massless higher-spin fields in $AdS_d$

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Abstract

In this paper we discuss in detail the frame-like formulation of free bosonic massless higher-spin fields of general symmetry type in $AdS_d$, announced recently in [1,2]. Properties of gauge invariant and $AdS$ covariant action functionals and their flat limits are carefully analyzed.

Contents

1 Introduction 2

2 Sketch of the frame-like formulation 4

3 Young tableaux and compensator formalism 10

3.1 Tensor modules of orthogonal algebras 10

3.1.1 Symmetric basis 11

3.1.2 Antisymmetric basis 12

3.2 Tracelessness conditions 13

3.3 Dimensional reduction and compensator 14

4 Higher-spin gauge fields in $AdS_d$ 16

4.1 Example of gravity 16

4.2 Symmetric higher-spin gauge fields 18

4.3 Mixed-symmetry higher-spin gauge fields in $AdS$ basis 21

4.4 Mixed-symmetry higher-spin gauge fields in Lorentz basis 22

4.5 Frame versus metric 25

5 General properties of a higher-spin action 27

5.1 Background 27

5.2 Degrees of freedom 28

5.3 Higher-spin action in the frame-like formalism 31

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1 Introduction

Free field dynamics of massless higher-spin (HS) gauge fields has been extensively studied within various approaches. The case of totally symmetric massless fields both on the flat and (anti)-de Sitter backgrounds of any dimensions has been fully investigated [3]-[18]. Non-symmetric (mixed-symmetry) massless fields were also studied for decades attracting some more attention in recent years [19]-[38].

There are two different approaches to HS massless fields. The metric-like approach generalizes the metric formulation of gravity. It was extensively elaborated both for totally symmetric and for mixed-symmetry HS fields starting from the original papers of Fronsdal [3] and de Wit and Freedman [5]. The frame-like approach, that generalizes the Cartan formulation of gravity, was suggested in Ref. [7] for $4d$ totally symmetric bosonic and fermionic HS fields and, independently, in [8] for HS fermions.

In this work we consider in detail the frame-like formulation of HS massless field dynamics presented recently in [1, 2] where manifestly gauge invariant Lagrangian formulation for HS massless bosonic fields of any symmetry type (i.e. of any spin) was announced. The basic idea of the frame-like formulation is that HS massless fields are described in terms of differential forms that take values in appropriate irreducible tensor representations of the $AdS_d$ algebra $o(d-1,2)$. The formalism of differential forms provides clear geometric realization of $AdS_d$ HS gauge symmetries and gauge-invariant HS field strengths. In particular, manifestly $AdS$ covariant and gauge-invariant action functionals for HS fields of any symmetry type are constructed as specific bilinear combinations of field strengths, thus generalizing the MacDowell-Mansouri action for gravity [39] and the previously known Lagrangian formulation for free totally symmetric HS fields [9, 10].
There are two degenerate cases of bosonic massless fields not considered in this paper. First is the case of totally antisymmetric fields described by \( p \)-forms including the case of a scalar field as a zero-form. Here the action does not have a form of the wedge product of the gauge invariant field strengths, requiring the Hodge star as is well-known, e.g., from the example of Maxwell theory. Although this case is also covered by the frame-like formalism we do not consider it here because the corresponding model and action is well known. The second special case is that of self-dual HS fields in \( AdS_{1+4k} \). It requires special consideration that will be given elsewhere.

The frame-like formulation can also be applied to description of HS massless fields in Minkowski and de Sitter backgrounds (for bosons). In the latter case, the \( dS_d \) algebra \( o(d, 1) \) admits, however, no lowest weight (i.e., bounded energy) unitary representations that implies the instability of the corresponding theories. The frame-like HS theory in Minkowski space can be obtained in the limit \( \Lambda \rightarrow 0 \) of the \( AdS_d \) HS theory, where \( \Lambda \) is the cosmological constant. For mixed-symmetry HS massless fields the flat limit is not completely trivial because, generically, a given \( AdS_d \) HS massless field has more degrees of freedom (i.e., less gauge symmetries) than its flat cousin \([24, 25]\). In other words, a generic irreducible \( AdS \) field decomposes into a set of massless Minkowski fields in the flat limit. Alternatively, one can take the flat limit so that it will exhibit an enhancement of additional gauge symmetries that gauge away all extra degrees of freedom of the \( AdS \) theory compared to the Minkowski one. As we will see this flat limit enhancement of gauge symmetries serves as the guiding principle that fixes the correct HS Lagrangian in \( AdS_d \).

The frame-like formulation is of particular importance for the study of the nonlinear HS theory \([40, 41]\) (see also Refs. \([42, 43]\) for reviews) because, in first place, it makes nonlinear symmetries manifest. Since frame-like HS fields are treated as gauge connections, they contain information on the structure of a global HS algebra, which is a specific infinite-dimensional extension of the \( AdS_d \) spacetime symmetry. The results of this paper are expected to give an important information on the structure of an extension of the nonlinear HS gauge theory of totally symmetric fields to a HS theory with mixed-symmetry gauge fields. Hopefully, this analysis will eventually shed light on a symmetric phase of string theory known to contain mixed-symmetry HS fields which are however massive in its standard formulation.

The layout of the present paper is as follows. In section 2 we review the general structure of the frame-like formulation of generic massless HS gauge fields in \( AdS_d \), summarizing the main results and ideas.

In section 3 we collect relevant facts about tensor representations of (pseudo)-orthogonal algebras and the compensator formalism.

In section 4 frame-like and metric-like fields are introduced and their relationship is established. Frame-like fields are described both in anti-de Sitter \( o(d−1, 2) \) and Lorentz \( o(d−1, 1) \) bases. In particular, the dynamical roles of different Lorentz frame-like fields is explained.

The discussion of general properties of \( AdS_d \) HS action functionals both in
metric-like and in frame-like forms is the content of section 5. A set of conditions on HS action functionals which single out a physically correct theory are formulated. The key role in this analysis is played by the analysis of the flat limit Λ = 0 of the AdSₜ HS field dynamics.

In section 6 AdSₜ HS action is reformulated in terms of an appropriate fermionic Fock space. The problem of finding a free field action is reduced to the analysis of a differential complex with the derivation Q associated with the variation of the action. The key property of Q is that it is equivalent to a certain de Rham differential. In subsection 6.4 it is shown how the action can be reconstructed from field equations in the frame-like formalism.

In section 7 the field equations are found that describe correctly the HS gauge fields of general symmetry type in AdSₜ and give rise to the gauge invariant HS action by virtue of the procedure elaborated in subsection 6.4.

Explicit check of the flat gauge symmetry enhancement for the constructed AdSₜ HS action functional is the content of section 8.

Conclusions and outlook are given in section 9. The Appendix contains some relations helpful in the analysis of the HS field equations.

Throughout the paper we use the mostly minus signature and adhere notations \( m, n = 0 \div d - 1 \) for world indices, \( a, b = 0 \div d - 1 \) for tangent Lorentz \( o(d - 1, 1) \) vector indices and \( A, B = 0 \div d \) for tangent AdSₜ \( o(d - 1, 2) \) vector indices. We also use condensed notations of \([9]\) for a set of antisymmetric or symmetric vector indices: \( a[k] \equiv [a_1 \ldots a_k] \) and \( a(k) \equiv (a_1 \ldots a_k) \). We use the convention that upper (lower) indices denoted by the same letter are assumed to be symmetrized or antisymmetrized as \( S^{a(2)} = \frac{1}{2}(S^{a_1a_2} + S^{a_2a_1}) \) or \( A^{a[2]} = \frac{1}{2}(A^{a_1a_2} - A^{a_2a_1}) \).

### 2 Sketch of the frame-like formulation

From the group-theoretical point of view a free particle propagating in the \( d \)-dimensional anti-de Sitter spacetime \( AdS_d \) corresponds to an irreducible highest weight unitary module \( D(E_0, s) \) of the AdSₜ isometry algebra \( o(d - 1, 2) \). The module is characterized by the energy \( E_0 \) and the spin \( s \) which are highest weights associated with the maximal compact subalgebra \( o(2) \oplus o(d - 1) \subset o(d - 1, 2) \). The energy \( E_0 \) is the weight of \( o(2) \) and the spins \( s \) are the weights of \( o(d - 1) \). The module \( D(E_0, s) \) is induced from the (vacuum) finite-dimensional weight \( E_0, s \) \( o(2) \oplus o(d - 1) \)-module. Note that, because we do not consider AdS (anti)self-dual fields in this paper, the corresponding vacuum module is not necessarily irreducible for \( d - 1 = 4k \) but decomposes into the sum of of two submodules with the positive and negative last weight (if it is different from zero). Correspondingly, we will assume that the spins \( s \) are positive.

In the bosonic case we discuss in this paper, where all spins \( s \) are integer, vacuum modules can be realized by \( o(d - 1) \) traceless tensors with the Young symmetry.
properties associated with the spins \( s \) being lengths of rows of the corresponding Young tableau. It is convenient to unify rows of equal lengths into horizontal blocks as follows

\[
\begin{array}{cccc}
  \cdots & s_2 & \cdots & \cdots \\
  \cdots & \cdots & \cdots & \cdots \\
\end{array}
\]

The uppermost block of length \( s \) and height \( p \) plays the distinguished role in the whole analysis.

Massless and singleton fields on \( \text{AdS}_d \) are described by UIRs with lowest energies saturating the unitarity bound \( E_0 = E_0(s) \). As shown in [24], for bosonic massless gauge fields

\[
E_0(s) = s - p + d - 2. \tag{2.2}
\]

The limiting module \( \lim_{E_0 \to E_0(s)} (D(E_0, s)) \) necessarily contains null states (that should become negative states for \( E_0 < E_0(s) \)) to be factored out to obtain the irreducible module \( D(E_0, s) \). Field-theoretically, this factorization manifests gauge symmetry. The irreducible module \( D(E_0(s), s) \) describes either a gauge massless field with local degrees of freedom in the \( \text{AdS}_d \) or a singleton field with local degrees on the boundary of \( \text{AdS}_d \) (i.e., with all bulk degrees of freedom gauged away [44]).

In the framework of the frame-like formulation the dynamics of massless (gauge) fields is described by a \( p \)-form field [1]

\[
\Omega^I_{(p)}(x) = dx^{n_1} \wedge \ldots \wedge dx^{n_p} \Omega^I_{n_1 \ldots n_p}(x) \tag{2.3}
\]

taking values in the finite-dimensional \( o(d-1,2) \)-module \( I \) described by the \( o(d-1,2) \) traceless Young tableau

\[
\begin{array}{cccc}
  \cdots & \cdots & \cdots & \cdots \\
  \cdots & \cdots & \cdots & \cdots \\
\end{array}
\]

which is obtained from (2.1) by adding the uppermost (i.e. the longest) row of the Young tableau (2.1) and then cutting off the rightmost (i.e. the shortest) column.

\(^4\)Massless gauge fields correspond to \( p < \frac{d-1}{2} \). The case \( p = \frac{d-1}{2} \) for odd \( d \) corresponds to singletons which are massless fields on the boundary of \( \text{AdS}_d \).
Other way around, this rule allows one to reconstruct the $o(d - 1)$ weights from a $o(d - 1, 2)$ Young tableau associated with a given $p$-form gauge field $(2.3)$. It is worth to note that since $p \geq 1$, the uppermost horizontal block in $(2.4)$ must have at least two rows for the $o(d - 1, 2)$ Young tableau to be associated with one or another massless field.

For example, to describe the spin two massless field, which corresponds to the $o(d - 1)$ tableau $\square$, one introduces the 1-form gauge field $\Omega^{AB}(x) = -\Omega^{BA}(x)$ that takes values in the representation $\mathbb{B}$ of $o(d-1,2)$. This gauge field can be interpreted as the gauge connection of $o(d-1,2)$. Its decomposition into representations of the Lorentz algebra $o(d - 1, 1) \subset o(d - 1, 2)$ gives rise to the conventional frame field and Lorentz spin connection 1-forms as explained in more detail below.

With $p$-form gauge fields one associates (linearized) curvatures which are $(p+1)$-forms taking values in the same $o(d - 1, 2)$-module $I$

$$R^I_{(p+1)} = D_0 \Omega^I_{(p)} ,$$

where $D_0 T^A = dT^A + \Omega_0^{A \alpha} T^\alpha$ is a $o(d - 1, 2)$ covariant derivative evaluated with respect to the background 1-form connection $\Omega_0^{AB}$ that satisfies the zero curvature equation

$$D_0 D_0 = d\Omega_0^{AB} + \Omega_0^{A \alpha} C \wedge \Omega_0^{B \beta} = 0$$

that can be taken as a definition of AdS space. Relation $(2.6)$ implies that curvatures are invariant under the gauge transformations

$$\delta \Omega^I_{(p)} = D_0 \xi^I_{(p-1)} ,$$

where the $(p - 1)$-form $\xi^I_{(p-1)}$ is a gauge parameter. The Bianchi identities take the form

$$D_0 R^I_{(p+1)} = 0 .$$

To elucidate the dynamical content of a theory formulated in terms of the $p$-form gauge field $\Omega^I_{(p)}$, let us decompose the $o(d - 1, 2)$ representation $I$ carried by the tangent indices into representations of the Lorentz subalgebra $o(d - 1, 1) \subset o(d - 1, 2)$. Schematically, the result is

$$\Omega^I_{(p)} \longrightarrow \left( e_{(p)} \oplus \omega_{(p)} \right) \oplus \sum \omega'_{(p)} \oplus \sum w_{(p)} ,$$

where $p$-form gauge fields on the right-hand-side have tangent indices corresponding to the all possible traceless $o(d - 1, 1)$ Young tableaux resulting from the $o(d - 1, 2)$ Young tableau $(2.3)$. There is a useful classification of Lorentz-covariant fields in the decomposition $(2.9)$ according to their different dynamical roles. So we distinguish between physical field $e_{(p)}$, relevant auxiliary field $\omega_{(p)}$, irrelevant auxiliary fields $\omega'_{(p)}$, and extra fields $w_{(p)}$.

The physical field has tangent indices described by the traceless $o(d - 1, 1)$ Young tableau with the minimal possible number of cells in the decomposition $(2.9)$. This
is obtained from (2.4) by removing the row of length \( s - 1 \) from the uppermost horizontal block. Equivalently, the same tableau can be obtained by removing the column of height \( p \) of the Young tableau (2.1) so that the length of the uppermost horizontal block of (2.1) becomes \( s - 1 \). The auxiliary fields are described by various Young tableaux which differ from that of the physical field by one additional cell.

The relevant auxiliary field has an additional cell in the first column while irrelevant auxiliary fields have an additional cell in any other column. All other possible Young tableaux in the decomposition (2.9) with two or more additional cells correspond to extra fields.

For the spin two field example mentioned above, the decomposition has the form \( \Omega^{AB} \to e^a \oplus \omega^{ab} \), where \( e^a \) is the frame field (the physical 1-form field) and \( \omega^{ab} = -\omega^{ba} \) is the Lorentz spin connection (the relevant auxiliary 1-form field). Irrelevant and extra fields are absent in this case.

To extract Lorentz-covariant components of the \( o(d-1,2) \) field in a manifestly \( o(d-1,2) \) covariant manner it is convenient to introduce a compensator field which is an \( o(d-1,2) \) vector \( V^A(x) \) normalized as

\[
V^A V_A = 1. \tag{2.10}
\]

The Lorentz subalgebra \( o(d-1,1) \subset o(d-1,2) \) can be identified with the stability algebra of the compensator, while the Lorentz-irreducible components can be represented as \( o(d-1,2) \) tensors orthogonal to the compensator.

In the case of gravity the decomposition takes the form \( \Omega^{AB} = \omega^{AB} + \lambda(V^A E_B - V^B E_A) \), where \( o(d-1,2) \) covariantized versions of \( e^a \) and \( \omega^{ab} \) are defined as \( \lambda E^A = D(V^A) \) and \( D_L V^A = dV^A + \omega^{AB} V_B = 0 \) and \( \lambda^2 = -\Lambda \). From the condition (2.10) it follows that \( E^AV_A = 0 \). For the linearized gravity the condition \( \omega^{AB} V_A = 0 \) is also true what is most evident when \( V^A = \text{const} \). The decomposition procedure for generic free HS fields is analogous.

The more traditional metric-like formulation of the HS field dynamics results from a partial gauge fixing of the frame-like formulation considered in the present paper. In these terms, a mixed-symmetry massless field is described by a Lorentz-covariant tensor field \( \Phi(x) \) that carries Lorentz indices with the symmetry properties of the Young tableau (2.1). It is not traceless however, satisfying some relaxed tracelessness conditions that generalize the Fronsdal double tracelessness conditions for symmetric HS gauge fields [3]. The metric-like field \( \Phi(x) \) is a component of the physical \( p \)-form gauge field \( e_{(p)} \), i.e. it is contained in the tensor product of the \( p \) antisymmetric form (world) indices and the tangent Young tableau indices of \( e_{(p)} \).
Note that the Young tableaux on the left-hand-side of (2.11) are traceless while that one on the right-hand-side, which is identified with the metric-like gauge field, is not. One can see that the generalized Fronsdal double tracelessness conditions on the field $\Phi(x)$ that follow from this construction require that

- the double contraction of four indices of any row of the upper horizontal block is zero;
- contraction of any two indices that do not belong to the first horizontal block is zero.

For the case of the one-row tableau, that corresponds to a totally symmetric field, one recovers the usual Fronsdal double-tracelessness condition [3]. In the spin two case, the metric-like field becomes the traceful metric tensor while the “other components” consist of the antisymmetric part of the frame.

“Other components” in the tensor product (2.11) are compensated by Stueckelberg part of the gauge transformation law of the physical $p$-form field $e_{(p)}$

$$\delta e_{(p)} = D\varepsilon_{(p-1)} + \text{Stueckelberg part} \tag{2.12}$$

where $D$ is the Lorentz covariant derivative in the background $AdS$ gravitational field and the $(p-1)$-form gauge parameter $\varepsilon_{(p-1)}$ carries tangent indices of the same type as the physical $p$-form $e_{(p)}$. The $(p-1)$-form gauge parameters of the “Stueckelberg part” in (2.12) carry tangent indices of the same types as auxiliary fields, that is with one cell added to the Young tableau associated with the physical field. It turns out that all “other components” in (2.11) can be gauge fixed to zero by the Stueckelberg gauge transformation so that the remaining nonzero components belong to the metric-like gauge field $\Phi(x)$. The gauge symmetry of the $\Phi(x)$ inherited from the transformation law (2.12) is

$$\delta \Phi(x) = \Pi(D\varepsilon(x)) \tag{2.13},$$

where a gauge parameter $\varepsilon(x)$ carries the indices described by the Young tableau resulting from that of $\Phi(x)$ by cutting a cell of the $p$-th row, and $\Pi$ is the projector to the tensor space of $\Phi(x)$. The gauge transformation law (2.13) is in agreement with the group-theoretical analysis of Metsaev [24]. Note that not all of the components of the parameter $\varepsilon_{(p-1)}$ contribute to $\varepsilon(x)$ because some of them are Stueckelberg with respect to gauge transformations for gauge parameters

$$\delta \xi_{(p-1)}^I = D_0\eta_{(p-2)}^I \tag{2.14}.$$

The form of the action functional for free HS gauge fields [10, 11, 17, 1, 36, 2, 38]

$$S_2 = \int_{\mathcal{M}^d} H^{\cdots} E_0^{\cdots} \cdots E_0^{\cdots} \wedge R_{(p+1)} \wedge R_{(p+1)} \tag{2.15},$$
is analogous to the MacDowell-Mansouri-Stelle-West action for gravity with the cosmological term \[39, 45, 47\]. Here \(o(d-1, 2)\) covariant coefficients \(H^-\), constructed of the compensator \(V^A\), tangent metric \(\eta^{AB}\) and the \(o(d-1, 2)\) Levi-Civita tensor, parameterise various ways of index contractions, \(E_0^-\) is the 1-form frame field of the \(AdS_d\) background. Any action of this form is manifestly invariant under the gauge transformations \[2.7\] because the field strength \(R_{(p+1)}\) is gauge invariant.

The coefficients in the action functional are to be determined by imposing the decoupling conditions

\[
\frac{\delta S_2}{\delta \omega'(p)} \equiv 0 \quad \text{and} \quad \frac{\delta S_2}{\delta w(p)} \equiv 0.
\] (2.16)

The meaning of these conditions is different.

The decoupling condition with respect to the extra fields \(w(p)\) implies effectively that the action is free of higher derivatives of the physical field. Indeed, once all extra fields are decoupled, the action depends non-trivially on the physical and auxiliary fields only. The auxiliary fields can be expressed by virtue of their equations of motion in terms of first derivatives of the metric-like gauge field \(\Phi(x)\) modulo pure gauge parts, so that the bosonic HS equations of motion will be of second-order.

The decoupling condition for the irrelevant auxiliary fields \(\omega'(p)\) guarantees \[2\] the correctness of the flat limit \(\Lambda \to 0\) of the \(AdS_d\) theory characterized by the enhancement of the additional gauge symmetries in the flat limit \[25\]. The point is that it is not enough to require the \(AdS_d\) action to be invariant under \(AdS_d\) gauge symmetries and to contain first order derivatives in order to guarantee that it describes a correct HS dynamics. The correct choice is dictated by the structure of the kinetic terms in the action which, for the metric-like field \(\Phi(x)\), is given by \(S^\text{flat}_2 \sim \int dx^d \partial \Phi \partial \Phi\) that should result from the \(AdS_d\) action \(S^\text{AdS}_2 \sim \int dx^d (\mathcal{D}\Phi \mathcal{D}\Phi + \Lambda \Phi^2)\) in the flat limit \(\Lambda \to 0\). In the \(AdS_d\) space, the mass-like terms \(\Lambda \Phi^2\) break down all the gauge symmetries of the action \(S^\text{flat}_2\) except for \(\delta \Phi = \mathcal{D} \varepsilon\) associated with the \(AdS_d\) gauge parameter \(\varepsilon\). The part of the Lagrangian that contains two derivatives is not uniquely fixed by the \(AdS\) gauge symmetry and may describe unwanted degrees of freedom, if not fine tuned by requiring maximal gauge symmetries in the flat limit,

\[
\delta \Phi(x) = \partial \varepsilon(x) + \sum_{I>1} \partial S_I(x),
\] (2.17)

where gauge parameters \(S_I(x)\) are described by the Young tableaux resulting from that of the field \(\Phi(x)\) by cutting off a cell from the last row of any \(I\)-th, \((I > 1)\) horizontal block with the convention that \(S_1(x) \equiv \varepsilon(x)\). Note that the additional gauge symmetry parameters \(S_I, I > 1\) are absent for rectangular tableaux. For example, totally symmetric and totally antisymmetric fields belong to this class.

As explained in section \[6\] the correct action does exist being fixed up to an overall factor and total derivatives by the decoupling conditions \[2.16\] \[2\]. Technically, the problem of finding coefficients satisfying the decoupling conditions gets complicated if operating in terms of multi-index tensors. To simplify the problem we reformulate it in terms of an appropriate fermionic Fock space where HS fields
are described as Fock vectors. In this setup, a Fock space version of the action functional (2.15) reads as
\[
S_2 = \int_{\mathcal{M}} \langle 0 \mid H (\wedge E_0)^{d-2p-2} \wedge R_{(p+1)} \wedge R_{(p+1)} | 0 \rangle , \tag{2.18}
\]
where \(E_0\) and \(R_{(p+1)}\) denote the Fock space realizations of the background frame field and HS curvatures, respectively. A nice feature of this formulation is that the variation of the action (2.18) has the form
\[
\delta S_2 = \int_{\mathcal{M}} \langle 0 \mid Q H (\wedge E_0)^{d-2p-1} \wedge R_{(p+1)} \wedge \delta \Omega_{(p)} | 0 \rangle , \tag{2.19}
\]
where the operator \(Q\) satisfies
\[
Q^2 = 0 . \tag{2.20}
\]
It can be shown that, in a certain basis, the operator \(Q\) has the simple form of the de Rham operator. This observation suggests the following strategy of finding action function \(H\). Firstly, we find the equations of motion in the \(Q\)-closed form consistent with the decoupling conditions and then reconstruct the action function \(H\) that leads to these equations of motion by a homotopy based on the Poincare lemma. (Note that the ambiguity in adding \(Q\)-exact terms \(H \sim QT\), that do not contribute to the field equations, manifests the ambiguity of the Lagrangian up to total derivatives.) The idea of this approach is due to the observation that it is easier to find correct equations of motion than to analyze the decoupling conditions directly in terms of the action (2.18).

3 Young tableaux and compensator formalism

In this section we summarize relevant facts about tensor modules of the orthogonal algebra which are used in our analysis of tensor modules of the \(AdS_d\) algebra \(o(d-1,2)\) its Lorentz subalgebra \(o(d-1,1)\) and massive Wigner little algebra \(o(d-1)\). We also consider the decomposition of a \(o(d-1,2)\)-module into \(o(d-1,1)\)-modules.

3.1 Tensor modules of orthogonal algebras

Any irreducible tensor module of the complex Lie algebra \(o(M|\mathbb{C})\) is defined by a highest weight vector \(l = (l_1, l_2, ..., l_\nu)\), where components \(l_i\) are integers that satisfy the conditions (see, e.g., [48])

\[
M = 2\nu : \quad l_1 \geq l_2 \geq \cdots \geq l_{\nu-1} \geq |l_\nu| , \tag{3.1}
\]
\[
M = 2\nu + 1 : \quad l_1 \geq l_2 \geq \cdots \geq l_{\nu-1} \geq l_{\nu} \geq 0 . \tag{3.2}
\]

The weights \(l\) (modulo a sign) can be depicted as a Young tableau
where the \( i \)-th row consists of \( s_i = |l_i| \) cells and \( j \)-th column consists of \( h_j \) cells. Proper realization of the irreducible module corresponding to the highest weight \( \mathbf{l} \) can be given in terms of the complex rank-\( P \) \( o(M) \)-tensors \( T^{ab...} \), \( a, b = 1, \ldots, M \) where \( P = s_1 + \ldots + s_\nu \) is the total number of cells in (3.3). The irreducibility conditions on \( T \) are

- Young symmetry conditions discussed in subsections 3.1.1 and 3.1.2
- tracelessness conditions
  \[
  \eta_{ab} T^{...a...b...} = 0 ;
  \]  
  \[
  (3.4)
  \]
- (anti-)selfduality conditions
  \[
  *T = \pm T ,
  \]  
  \[
  (3.5)
  \]
  where * is the Hodge automorphism, for the case of even \( M \) and \( l_\nu \neq 0 \).

For the real form \( o(M - t, t) \) of \( o(M|\mathbb{C}) \), tensor representations are also characterized by different Young tableaux (3.3). The \( o(M - t, t) \) irreducible module corresponding to (3.3) can be realized as the space of rank-\( P \) real tensors satisfying the irreducibility conditions described above with (anti-)selfduality conditions to be imposed if \( l_\nu \neq 0 \) and \( M - 2t \equiv 0 \mod 4 \).

### 3.1.1 Symmetric basis

Let \( l_q, q \leq \nu \) be the last nonzero component in the weight vector \( \mathbf{l} \). We group indices of a rank-\( P \) tensor \( T \) into the sets corresponding to the rows of the Young tableau (3.3),

\[
T^{a_1(s_1), \ldots, a_q(s_q)} , \quad s_i = |l_i| \]

and require \( T \) to be symmetric in each group of indices \( a_i(s_i) \) and to satisfy the conditions that symmetrization of all indices of a \( i \)-th group with any index from a \( j \)-th group gives zero for \( i < j \).

In the symmetric basis, it may be convenient to characterize a Young tableaux by horizontal blocks. Namely, combining rows of equal length into horisontal blocks, the Young tableau (3.3) can be described by a set of pairs of positive integers \((\tilde{s}_I, p_I)\), \( I = 1, \ldots, k \) with \( \tilde{s}_1 > \tilde{s}_2 > \cdots > \tilde{s}_k > 0 \) and \( p_I \) such that \( p_1 + \ldots + p_k = q \). The
result can be depicted as

\[(\tilde{s}_1, p_1)
\]

\[(\tilde{s}_2, p_2)
\]

\[
\vdots
\]

\[
\vdots
\]

where an \(I\)-th horizontal block has length \(\tilde{s}_I\) and height \(p_I\). The exact identification of rows of equal length in (3.3) and horizontal blocks in (3.7) reads as

\[
\tilde{s}_1 = s_1 = \ldots = s_{p_1} > \tilde{s}_2 = s_{p_1+1} = \ldots = s_{p_1+p_2} > \ldots > \tilde{s}_k = s_{p_1+\ldots+p_{k-1}+1} = \ldots = s_q.
\]

(3.8)

It is worth to note that, as a consequence of its Young symmetry properties, a horizontal block \((\tilde{s}_I, p_I)\) is invariant with respect to exchange of its rows up to a sign factor \((-1)^{\tilde{s}_I}\).

### 3.1.2 Antisymmetric basis

Let us group the indices of a rank-\(P\) tensor \(T\) into the sets corresponding to the columns of the Young tableau (3.3)

\[
T^{a_1[h_1], a_2[h_2], \ldots, a_{s_1}[h_{s_1}]},
\]

(3.9)

where the length of the first row \(s_1\) is the number of columns in (3.3). In the antisymmetric basis, \(T\) is required to be antisymmetric in each group of indices \(a_i[h_i]\) and to satisfy the conditions that antisymmetrization of all indices of a \(i\)-th group with any index of a \(j\)-th group gives zero if \(i < j\). One should note that tensors (3.6) and (3.9) corresponding to the same Young tableau form isomorphic modules of the orthogonal algebra defined in different (namely, symmetric and antisymmetric) Young bases. Let \(Y_{M-t,t}(s_1, \ldots, s_\nu)\) denote the space of \(o(M-t, t)\) tensors satisfying the Young symmetry conditions either in symmetric or in antisymmetric basis.

In the antisymmetric basis, it may be convenient to characterize Young tableaux by vertical blocks. Namely, combining columns of equal height into vertical blocks, the Young tableau (3.3) can be described by a set of pairs of positive integers \((m_1, \tilde{h}_1)\) with \(\tilde{h}_1 > \tilde{h}_2 > \ldots > \tilde{h}_k > 0\) and \(m_I\) such that \(m_1 + \ldots + m_k = s_1\). The result can be depicted as

\[
\vdots
\]

\[
\vdots
\]

\[
(m_k, \tilde{h}_k)
\]

(3.10)
where $I$-th vertical block has length $m_I$ and height $\tilde{h}_I$. The exact identification of columns of equal height in (3.3) and vertical blocks in (3.10) reads as

$$
\tilde{h}_1 = h_1 = \ldots = h_{m_1} > \tilde{h}_2 = h_{m_1+1} = \ldots = h_{m_1+m_2} > \ldots
$$

$$
\ldots > \tilde{h}_k = h_{m_1+\ldots+m_k-1+1} = \ldots = h_{s_1}.
$$

(3.11)

Note that a number of vertical blocks of any Young tableau equals to the number of its horizontal blocks. Also note that, as a consequence of its Young symmetry properties, a vertical block is invariant under exchange of its columns.

### 3.2 Tracelessness conditions

In this section we define subspaces of the spaces of traceful tensors that, apart from Young symmetry conditions, satisfy specific tracelessness conditions which extend the Fronsdal double tracelessness condition for massless symmetric fields to mixed-symmetry massless fields of general type.

Let $B^{d-1,1}(s_1, \ldots, s_q, 0, \ldots, 0)$ be the linear space of tensors which have the Young properties of the type $Y_{d-1,1}(s_1, \ldots, s_q, 0, \ldots, 0)$ and satisfy the tracelessness conditions

$$
\eta_{a_i a_j} T^{a_1(s_1), \ldots, a_q(s_q)} = 0, \quad 0 < i \leq m
$$

and

$$
\eta_{a_i a_j} T^{a_1(s_1), \ldots, a_q(s_q)} = 0, \quad m < i \leq q.
$$

(3.12)

(3.13)

Note that $B^{d-1,1}_m(s_1, \ldots, s_q, 0, \ldots, 0) \subset B^{d-1,1}_n(s_1, \ldots, s_q, 0, \ldots, 0)$ for $m < n$.

$B^{d-1,1}_0(s_1, \ldots, s_q, 0, \ldots, 0)$ is the space of traceless tensors with the $Y_{d-1,1}(s_1, \ldots, s_q, 0, \ldots, 0)$ Young properties.

The following lemmas are simple consequences of the definition of $B^{d-1,1}_m(s_1, \ldots, s_q, 0, \ldots, 0)$.

**Lemma 1**

Contraction of $\eta_{(a_i a_j a_k a_l)}$ with any four symmetrized indices of a tensor from $B^{d-1,1}_m(s_1, \ldots, s_q, 0, \ldots, 0)$ gives zero.

Lemma 1 is a corollary of (3.12) and the Young symmetry properties, which guarantee that any group of symmetrized indices can be placed in the first row.

**Lemma 2**

From Lemma 1 it follows that

$$
\eta_{(a_i a_j a_k a_l)} T^{a_1(s_1), \ldots, a_q(s_q)} = 0, \quad \forall \ i, j, k, l,
$$

(3.14)

i.e. any double trace gives zero provided that any three of the contracted indices are symmetrized.

This is because $\eta_{ab} \eta_{cd}$ belongs to the symmetric part of the tensor product

$$
\left(\otimes \otimes \right)_{\text{sym}} = \oplus \oplus
$$

(3.15)
so that the symmetrization of any three indices of $\eta_{ab}\eta_{cd}$ implies the total symmetrization. Therefore nonzero traces in $B_{m}^{d-1,1}(s_1, \ldots, s_q, 0, \ldots, 0)$ can only appear when all elementary contractions hit different rows.

**Lemma 3**

The condition (3.13) along with Lemma 2 mean that contraction of any $m + 1$ pairs of indices of $T^{a_1(s_1), \ldots, a_q(s_q)} \in B_{m}^{d-1,1}(s_1, \ldots, s_q, 0, \ldots, 0)$ gives zero.

Recall that a rectangular block is invariant (up to a sign) under exchange of its rows. As a result it follows

**Lemma 4**

Once (3.13) is true for one of the rows of a rectangular block it is true for the entire block, i.e. $B_{m}^{d-1,1}(s_1, \ldots, s_q, 0, \ldots, 0) = B_{n}^{d-1,1}(s_1, \ldots, s_q, 0, \ldots, 0)$ if $s_{m+1} = s_{n+1}$.

Therefore, it is sufficient to impose the trace condition (3.13) for any row inside a horizontal block (e.g., the upper row).

### 3.3 Dimensional reduction and compensator

Let us now address the question what is a pattern of the decomposition of a given $o(d-1,2)$-module described by a Young tableau into modules of the subalgebra $o(d-1,1) \subset o(d-1,2)$. For our purposes it is convenient to describe this decomposition in a manifestly $o(d-1,2)$ covariant manner. To this end, let us introduce a compensator $V^A$, which is an $o(d-1,2)$ vector normalized as $V_A V^A = 1$. (Note that when discussing gauged orthogonal algebras the compensator vector becomes an $x$-dependent field $V^A(x)$). The role of the compensator in the decomposition procedure has clear geometrical interpretation since the Lorentz algebra $o(d-1,1) \subset o(d-1,2)$ can be identified as the stability algebra of the compensator. This is most evident in the standard form of the compensator $V^A = (0, \ldots, 0, 1) = \delta^A_d$.

Let a traceless $o(d-1,2)$ tensor representation of some symmetry type $Y_{d-1,2}(s_1, \ldots, s_q, 0, \ldots, 0)$ be considered either in symmetric basis $T^{A_1(s_1), \ldots, A_q(s_q)}$ or in antisymmetric basis $T^{A_1(h_1) \ldots, A_q(h_{s_q})}$. The decomposition results from the following procedure. Every $o(d-1,2)$ index of $T$ has one component along $V^A$ and $d$ components orthogonal to $V^A$. In the former case we cancel a cell of the corresponding $o(d-1,2)$ Young tableau, while in the latter we keep it. A number of indices along $V^A$ cannot exceed $s_1$ because the symmetrization of more than $s_1$ indices in $T$ gives zero by the defining property of the Young tableau $Y_{d-1,2}(s_1, \ldots, s_q, 0, \ldots, 0)$. Moreover, no two cut cells can belong to the same column as is obvious from the realization of a Young tableau in the antisymmetric basis because the tensor $V^AV^B$ is symmetric. Otherwise, any set of indices 0 to $s_1$ can be aligned along $V^A$. Therefore any number of cells from 0 to $s_1$ can be cut under the condition that no two cells are cut from the same column. Of course only such cuts are allowed that give rise to a proper Young tableau. (Otherwise the resulting tensor is identically zero.)

The particular $o(d-1,1)$ tensors result from contractions of some of the indices of the original $o(d-1,2)$ tensor with the compensator $V^A$ followed by the proper
(anti)symmetrizations and projecting to the $V^A$-transversal components with respect to the rest of indices. Note that, in the symmetric basis, all contractions with the compensator are equivalent to some its contractions with indices of the first row because $V^AV^BV^C \ldots$ is a totally symmetric tensor.

The resulting list of the $o(d-1,1)$ components consists of the Young tableaux drawn in bold on the right hand side of the decomposition

\[
\begin{array}{c}
\text{o}(d-1,2) \text{ Young tableau} \\
\end{array} \quad \Rightarrow \quad \bigoplus_{\{r_I\}} \begin{array}{c}
\text{o}(d-1,1) \text{ Young tableaux} \\
\end{array}
\]

where $r_I$ denotes a number of cells cut from the last row of the $I$-th horizontal block of the original $o(d-1,2)$ Young tableau

\[
0 \leq r_I \leq (\tilde{s}_I - \tilde{s}_{I+1}), \quad 1 \leq I \leq k, \quad (3.17)
\]

with the convention that $\tilde{s}_{k+1} = 0$. In the $o(d-1,1)$ tableaux (3.16), the indices contracted with the compensator are denoted by $\circ$. Taking into account that the compensator is $o(d-1,1)$ invariant, they disappear from the $o(d-1,1)$ tableau according to the decomposition procedure described above.

It is sometimes convenient to use another parametrization by introducing the parameters

\[
t_I \equiv (\tilde{s}_I - \tilde{s}_{I+1}) - r_I : \quad 0 \leq t_I \leq (\tilde{s}_I - \tilde{s}_{I+1}). \quad (3.18)
\]

Note that the $o(d-1,1)$ tableau with the minimal number of cells corresponds to all $t_I = 0$. The parameters $t_I$ measure a deviation from the minimal number of cells.

As an illustration, let us consider the example of a $o(d-1,2)$ traceless tensor $T^{A(2),B}$ with the symmetry of the three-cell "hook" Young tableau $Y_{d-1,2}(2,1,0,\ldots,0)$, using for definiteness the symmetric basis. Its $o(d-1,1)$ decomposition gives four traceless tensors having the Young symmetries of $Y_{d-1,1}(2,1,0,\ldots,0)$, $Y_{d-1,1}(2,0,\ldots,0)$, $Y_{d-1,1}(1,1,0,\ldots,0)$ and $Y_{d-1,1}(1,0,\ldots,0)$

\[
T^{A(2),B} = A^{A(2),B} \oplus B^{A(2)} \oplus C^{A,B} \oplus D^A, \quad (3.19)
\]

that are $V^A$-transversal

\[
A^{A(2),B}V_A = 0, \quad A^{A(2),B}V_B = 0, \quad B^{A(2)}V_A = 0, \quad C^{A,B}V_B = 0, \quad D^AV_A = 0. \quad (3.20)
\]
Note that the second condition in (3.20) follows from the first one by virtue of the Young symmetry properties of $A^{(2),B}$. The explicit form of the projectors to these $o(d-1,1)$ components reads as

$$D^A = T^{B(2),A}V_{(BVB)} ,$$
$$B^{A(2)} = T^{A(2),B}V_B + D^{(AVA)} ,$$
$$C^{A,B} = \left(T^{AC,B} - T^{BC,A}\right)V_C + \frac{3}{2}D^{A}V_B - D^{B}V^{A} ,$$
$$A^{A(2),B} = T^{A(2),B} - \left(B^{A(2)}V_B - B^{B(AVA)}\right) + C^{B,(AVA)} + \left(D^{(AVA)}V_B - D^{BV(AVA)}\right) .$$

(3.21)

4 Higher-spin gauge fields in $AdS_d$

In this section we introduce following to [1] the frame-like fields and related metric-like fields for HS fields of general symmetry type.

4.1 Example of gravity

Many features of the frame-like formulation are illustrated by the example of Einstein gravity with the cosmological term in the formulation of MacDowell and Mansouri [39, 45, 47]. Here the gravitational field is described by the 1-form

$$\Omega^{AB}(x) = -\Omega^{BA}(x) = d\Omega^{AB}(x) ,$$

(4.1)

which is the gauge connection of the $AdS_d$ algebra $o(d-1,2)$. By introducing the $o(d-1,2)$ covariant derivative $D$

$$DT^A = dT^A + \Omega^A_{\quad B}T^B ,$$

(4.2)

one defines the gauge transformation law as

$$\delta \Omega^{AB} = D\xi^{AB} \equiv d\xi^{AB} + \Omega^A_{\quad C}\xi^{CB} + \Omega^B_{\quad C}\xi^{AC} ,$$

(4.3)

where $d = dx^m \partial_m$ is the exterior differential and $\xi^{AB}(x) = -\xi^{BA}(x)$ is a 0-form gauge parameter.

To establish a precise relationship between the metrics $g_{mn}$ and the connection $\Omega^{AB}$, the latter should be decomposed into Lorentz $o(d-1,1) \subset o(d-1,2)$ components which are the frame 1-form $e^a$ and the Lorentz connection $\omega^{ab}$. To make this decomposition $o(d-1,2)$ covariant it is convenient to use the compensator formalism described in section 3.3. Namely, $o(d-1,2)$ covariant versions of the frame field and Lorentz connection are defined as follows [16]

$$\lambda E^A = DV^A \equiv dV^A + \Omega^{AB}V_B , \quad \omega^{AB} = \Omega^{AB} - \lambda \left(E^A V_B - E^B V^A\right) ,$$

(4.4)
where the dimensionful parameter $\lambda$ is introduced to make the frame field dimensionless. Conventional Lorentz-covariant fields $e^a$ and $\omega^{ab}$ result from these formulae with the compensator in the standard gauge $V^A = \delta^A_d$. The metrics
\[ g_{mn} = \eta_{AB} E^A_m E^B_n , \] (4.5)
is the $o(d-1,2)$ invariant extension of the standard formula $g_{mn}(x) = \eta_{ab} e^a_m(x) e^b_n(x)$, which is (4.5) in the standard gauge. Note that the definitions (4.4) comply with Lorentz invariance of the compensator $D V^A \equiv d V^A + \omega^{AB} V^B = 0$.

The curvature associated with the gauge connection (4.1) is
\[ R^{AB} = d \Omega^{AB} + \Omega^A_C \wedge \Omega^{CB} . \] (4.6)
The remarkable property of the formulation of gravity in terms of $o(d-1,2)$ connection is that the anti-de Sitter geometry is described by a nondegenerate flat connection $\Omega^A_B = (h^A, \omega^{AB})$ that satisfies
\[ \text{rank}(h^A_m) = d , \] (4.7)
\[ R^{AB}(\Omega_0) = 0 . \] (4.8)
(The notation $h^A$ is used for the background AdS frame.) In terms of the covariant derivative $D_0$ with respect to the background connection $\Omega_0$, the zero-curvature condition (4.8) implies
\[ D^2_0 = 0 . \] (4.9)

Consider the perturbation expansion $\Omega^{AB} = \Omega^A_B + \Omega_1^{AB}$, where $\Omega_1$ describes dynamical fluctuations around the background connection $\Omega_0$. From (4.3) and (4.6) it follows that the linearized gauge transformation and curvature have the form
\[ \delta_0 \Omega^{AB}_1 = D_0 \delta^{AB} , \quad R^{AB}_1 = D_0 \Omega^{AB}_1 . \] (4.10)
From (4.9) it follows then that the linearized curvature $R_1$ is gauge invariant
\[ \delta_0 R^{AB}_1 = 0 . \] (4.11)

Assuming that the compensator is of order zero, the dynamical frame and Lorentz connection are
\[ \Omega^{AB}_1 = \omega^{AB}_1 + \lambda \left( E^A_1 V^B - E^B_1 V^A \right) , \] (4.12)
where $E^A_1 V_A = 0$ and $\omega^{AB}_1 V_B = 0$.

From the formula (4.5), one finds that the fluctuational part $g_{1mn}$ of the metrics is
\[ g_{1mn} = \eta_{AB} \left( E^A_1 h^B_n + E^A_1 h^B_m \right) = E^A_1 h^B_n + E^B_1 h^A_m , \] (4.13)
where $E^A_1 h^B_n = \eta_{AB} E^A_1 h^B_n$. The inverse frame $h^A_m$
\[ h^A_m h^B_m = \delta^A_m \] (4.14)
exists due to the nondegeneracy condition (1.7). Its form is fixed uniquely by requiring
\[ h^B_n h^B_n = \delta^B_A - V_A V^B. \] (4.15)

One can rewrite (4.13) as
\[ g_{1AB} = E_{1A;B} + E_{1B;A}, \] (4.16)
where \( g_{1A,B} = g_{1nm} h^n_A h^n_B \) and \( E_{1A;B} = E_{1nm} h^n_A \). The gauge transformation law for \( E_{1A;B} \), that follows from (4.10), has the form
\[ \delta E_{1A;B} = h^n_A D_n \varepsilon_B + \varepsilon_{AB}, \] (4.17)
where \( D \) is Lorentz covariant derivative evaluated with respect to \( \omega_{AB} \) while \( \varepsilon_A, \varepsilon_{AB} \) are Lorentz components of the gauge parameter \( \xi^{AB} \), i.e. \( \xi^{AB} = \varepsilon^{AB} + \lambda(\varepsilon^{AV} B - \varepsilon^{BV} A) \), \( \varepsilon^A V_A = 0 \), \( \varepsilon^{AB} V_B = 0 \).

That the metric fluctuation is described by the symmetric component of the frame field (i.e., \( E_{1A;B} + E_{1B;A} \)) is consistent with the fact that, as follows from (4.17), the antisymmetric component of the frame \( E_{1A;B} - E_{1B;A} \) is compensated by local Lorentz transformations with the gauge transformation parameter \( \varepsilon^{AB} \). The gauge transformation of the symmetric component \( E_{1A;B} + E_{1B;A} \) induced from (4.17) reproduces the linearized diffeomorphism with the gauge parameter \( \varepsilon^A \).

The formulae (4.10), (4.12), (4.16) for the gravitational field admit a straightforward generalization to higher spins. HS fields \( \Phi_{m,n,k,...}(x) \) which generalize the fluctuational part of the metrics in gravitation will be referred to as metric-like HS fields. Analogously, their \( p \)-form cousins \( \Omega_{(p)}^{A,B,C,...}(x) \) will be referred to as frame-like HS fields. As a preparation to the general case, let us consider bosonic symmetric HS fields.

### 4.2 Symmetric higher-spin gauge fields

The metric-like approach to totally symmetric bosonic massless fields of all spins was developed by Fronsdal both in flat [3] and AdS space [4]. Here an integer spin \( s \) massless field is described by a totally symmetric rank \( s \) \( o(d - 1, 1) \) tensor
\[ \Phi^{a_1...a_s}(x) \equiv \Phi^{a(s)}(x) \] (4.18)
subject to the Fronsdal double tracelessness condition [3] [4]
\[ \eta_{b_1 b_2} \eta_{b_3 b_4} \Phi^{b_1 b_2 b_3 b_4 (s-4)} = 0, \] (4.19)
which is nontrivial for \( s \geq 4 \). The HS gauge transformation is
\[ \delta \Phi^{a(s)}(x) = D^a \varepsilon^{a(s-1)}(x), \quad D^a = h^a_n D_n \] (4.20)
where the parameter \( \varepsilon^{a(s-1)} \) is a rank \( s - 1 \) symmetric traceless \( o(d - 1, 1) \) tensor and \( D_n \) is the background Lorentz derivative.
The frame-like formulation operates in terms of a 1-form frame-like HS gauge field \[ e^{a(s-1)} = d x^a e^{a(s-1)} \] (4.21) that is traceless in the tangent indices \[ e_b^{ba(s-3)} = 0. \] (4.22)

The HS gauge transformation law is
\[
\delta e^{a(s-1)} = D e^{a(s-1)} + h^a_{\ b} \varepsilon^{a(s-1), b}, \tag{4.23}
\]
where \( h^a \) is the background frame 1-form. The totally symmetric traceless 0-form gauge parameter \( \varepsilon^{a(s-1)}(x) \) is equivalent to that of the Fronsdal’s formulation. The 0-form gauge parameter \( \varepsilon^{a(s-1), b}(x) \) is also traceless having the symmetry type \( Y_{d-1,1}(s - 1, 1, 0, ..., 0) \) which means that \( \varepsilon^{a(s-1), a}(x) = 0 \). It is the HS generalization of the parameter \( \varepsilon^{a,b} \) of the local Lorentz transformations in gravity (\( s = 2 \)). The Lorentz type gauge ambiguity related to \( \varepsilon^{a(s-1), b} \) can be fixed by requiring the frame-like HS gauge field to be totally symmetric by setting
\[
h_{\ nu}^{\ a} e^{a(s-1)} = \Phi^{a(s-1)b}, \tag{4.24}
\]
where the symmetric tensor field
\[
\Phi^{a(s)} = h_{\ nu}^{\ a} e^{a(s-1)} \tag{4.25}
\]
identifies with the metric-like field of the Fronsdal formulation. Note that \( \Phi^{a(s)} \) is double traceless as a consequence of (4.22).

The Lorentz-like HS symmetry with the parameter \( \varepsilon^{a(s-1), b} \) assumes a HS Lorentz connection-like 1-form \( \omega^{a(s-1), b} \). The analysis of its transformation law shows [9, 10] that, for \( s > 2 \), some additional gauge connections and symmetry parameters have to be introduced. As a result, the full set of HS frame-like fields associated with a spin \( s \) massless field consists of the 1-forms
\[
\Upsilon^{a(s-1), b(t)}(x) = d x^a \Upsilon^{a(s-1), b(t)}(x) , \tag{4.26}
\]
that take values in all traceless tensor representations of the Lorentz algebra \( o(d - 1, 1) \) described by the Young tableaux \( Y_{d-1,1}(s - 1, t, 0, \ldots, 0) \) with at most two rows, such that the upper row has length \( s - 1 \)
\[
o(d - 1, 1) : \begin{array}{c|c|c|c|c|c|c|c|c|c} \hline & & & & & & & & & \\ \hline & & & & & & & & & \\ \hline & & & & & & & & & \\ \hline & & & & & & & & & \\ \hline & & & & & & & & & \\ \hline & & & & & & & & & \\ \hline & & & & & & & & & \\ \hline & & & & & & & & & \\ \hline \end{array} \tag{4.27}
\]

The field \( \Upsilon^{a(s-1)} \) with \( t = 0 \) identifies with the physical spin \( s \) frame-like field \( e^{a(s-1)} \). The case of \( t = 1 \) corresponds to the auxiliary Lorentz-like field \( \omega^{a(s-1), b} \). The remaining fields (4.26) with \( t \geq 2 \) are called extra fields [9, 10].
The frame-like formalism works both in the $AdS_d$ and in the flat space. In the $AdS_d$ case it is convenient to use the observation of [47] that the set of the HS 1-forms $\Upsilon^{a(s-1),b(t)}$ with all $0 \leq t \leq s - 1$ results from the 1-form

$$\Omega_{(1)}^{A(s-1),B(s-1)} = dx^a \Omega_{a(s-1),B(s-1)}$$

that carries the traceless tensor representation of $o(d-1,2)$ described by the length $s - 1$ two-row rectangular Young tableau

$$o(d-1,2) : \begin{array}{c} s-1 \\ \end{array}$$

i.e. symmetrization of any $s$ indices of $\Omega_{(1)}^{A(s-1),B(s-1)}$ gives zero.

The Lorentz-irreducible HS fields $\Upsilon^{a(s-1),b(t)}$ result from the field $\Omega_{(1)}^{A(s-1),B(s-1)}$ by means of the reduction procedure described in section 3.3. In particular, the component of $\Omega_{(1)}^{A(s-1),B(s-1)}$, that is most parallel to the compensator $V^A$, is the physical frame-like field

$$\lambda^{s-1} e^{A(s-1)} = \Omega_{(1)}^{A(s-1),B(s-1)} V_B \cdots V_B .$$

(4.30)

(Note that a contraction of $s$ or more indices of $\Omega_{(1)}^{A(s-1),B(s-1)}$ with the compensator gives zero by the Young properties.) The less $V^A$-longitudinal components identify with the other fields in the set (4.26).

The linearized curvature is defined as the $o(d-1,2)$ covariant derivative of the HS connection 1-form

$$R_{(2)}^{A(s-1),B(s-1)} = D_0 \Omega_{(1)}^{A(s-1),B(s-1)} .$$

(4.31)

Due to the zero-curvature condition (4.9), the curvature (4.31) is invariant under the linearized HS gauge transformations

$$\delta \Omega_{(1)}^{A(s-1),B(s-1)} = D_0 \xi^{(0)}_{A(s-1),B(s-1)}$$

(4.32)

with the traceless 0-form gauge parameter of the Young symmetry (4.29).

Being decomposed into Lorentz components, the gauge transformation law (4.32) reproduces gauge transformations for the fields $\Upsilon$. In particular, the maximally $V$-tangential part of (4.32) reproduces the gauge transformation (4.23) for the frame-like field $e^{A(s-1)}$.

The formulation in terms of the Lorentz 1-forms $\Upsilon$ is equivalent to that in terms of the 1-form field $\Omega_{(1)}$. The advantage of the latter formulation is that it has simple algebraic meaning, operating with the HS connection that takes values in a single irreducible representation of the $AdS$ algebra $o(d-1,2)$. 
4.3 Mixed-symmetry higher-spin gauge fields in AdS basis

The described approach admits a straightforward generalization to massless HS fields of any symmetry type. Consider a \( AdS_d \) spin \( \mathbf{s} = (s, \ldots, s, s_{p+1}, \ldots, s_q, 0, \ldots, 0) \) massless field characterized by the following \( o(d-1) \) Young tableau of the lowest energy (vacuum) state

\[
\begin{array}{cccccccccc}
& & & & & & & & & \\
& & & & & & & & & \\
& & & & & & & & & \\
& & & & & & & & & \\
& & & & & & & & & \\
\end{array}
\]

\[
(4.33)
\]

that has the upper block of length \( s \) and height \( p \). As shown in [1], in the frame-like formulation, its field dynamics can be described by a \( p \)-form gauge field

\[
\Omega_{(p)} A_0(s-1), \ldots, A_p(s-1), A_{p+1}(s_{p+1}), \ldots, A_q(s_q)(x),
\]

\[
(4.34)
\]

that takes values in the traceless \( o(d-1,2) \) tensor module corresponding to the Young tableau \( Y(s-1, \ldots, s-1, s_{p+1}, \ldots, s_q, 0, \ldots, 0) \)

\[
\begin{array}{cccccccccc}
& & & & & & & & & \\
& & & & & & & & & \\
& & & & & & & & & \\
& & & & & & & & & \\
& & & & & & & & & \\
\end{array}
\]

\[
(4.35)
\]

A simple mnemonic rule is that, to obtain the \( AdS_d \) tensor representation carried by a HS gauge connection \( (4.34) \), one adds the longest row to the \( o(d-1) \) Young tableau of the vacuum energy representation under consideration \( (4.33) \) and then cuts the shortest column. The resulting gauge field is a \( p \)-form where \( p \) is the height of the rightmost column of the original vacuum representation.

The gauge transformation is

\[
\delta \Omega_{(p)} A_0(s-1), \ldots, A_q(s_q) = D_0 \xi_{(p-1)} A_0(s-1), \ldots, A_q(s_q),
\]

\[
(4.36)
\]

where a traceless tensor \( \xi_{(p-1)} A_0(s-1), \ldots, A_q(s_q) \) is a \( (p-1) \)-form gauge parameter that takes values in the same representation of \( o(d-1,2) \). There is a set of level-l \( (1 \leq l \leq p-1) \) gauge parameters and gauge transformations

\[
\delta \xi_{(p-l)} A_0(s-1), \ldots, A_q(s_q) = D_0 \xi_{(p-l-1)} A_0(s-1), \ldots, A_q(s_q), \quad l = 1, \ldots, p-1.
\]

\[
(4.37)
\]
From the zero-curvature condition (4.9) it follows that the \((p+1)\)-form curvature associated with the \(p\)-form gauge field

\[ R_{(p+1)} A_0(s-1), ..., A_q(s_q) = D_0 \Omega_{(p)} A_0(s-1), ..., A_q(s_q) \]  

(4.38)
is invariant under the gauge transformations (4.36)

\[ \delta R_{(p+1)} = 0 \]  

(4.39)
and satisfies the Bianchi identities

\[ D_0 R_{(p+1)} A_0(s-1), ..., A_q(s_q) = 0 . \]  

(4.40)

Also, we shall use the \(p\)-form gauge field (4.34) rewritten in the antisymmetric basis

\[ \Omega_{(p)} A_{[\tilde{h}_1]}, ..., A_{s-1}[\tilde{h}_{s-1}] . \]  

(4.41)
Here \(\tilde{h}_1 \geq \cdots \geq \tilde{h}_{s-1} \geq p + 1\) are the heights of the columns of the \(o(d-1, 2)\) Young tableau (4.35). In the sequel we shall switch freely between the symmetric and antisymmetric descriptions of the frame-like HS fields.

### 4.4 Mixed-symmetry higher-spin gauge fields in Lorentz basis

The dynamical content of the frame-like formulation of massless HS fields is most conveniently analyzed in terms of Lorentz-tensor components of a HS field. To this end, the \(o(d-1, 2)\)-module carried by the \(p\)-form field \(\Omega_{(p)}\) should be decomposed into \(o(d-1, 1)\)-modules. According to section 3.3, the result of the decomposition of the \(p\)-form field \(\Omega_{(p)}\) with tangent indices associated with traceless AdS Young tableau (4.35) into a set of Lorentz-covariant \(p\)-form fields is

\[ \Omega_{(p)} = \bigoplus_{(t_1, \ldots, t_k)} \lambda^{(s-1-\sum_I t_I)} \Upsilon_{(p)}^{(t_1, \ldots, t_k)} , \]  

(4.42)
where fields \(\Upsilon_{(p)}^{(t_1, \ldots, t_k)}\), parameterized by integers \(t_I\) (3.18), have tangent indices corresponding to various traceless \(o(d-1, 1)\) Young tableaux of the form (3.16). The \(\lambda\)-dependent factors are introduced to highlight that different Lorentz-covariant fields will have different mass dimensions compatible with the flat limit \(\lambda \to 0\).

The following classification of Lorentz-covariant \(p\)-form fields is motivated by their different dynamical roles [1, 2]. The field of the set (4.42) with the minimal number of cells is called physical. Auxiliary fields are those with the Lorentz-covariant components \(\{ \Upsilon_{(p)}^{(0, \ldots, 0, 1, 0, \ldots, 0)} \}\). In other words, auxiliary fields have one more Lorentz index compared to the physical field. We distinguish between relevant auxiliary field, that has an additional cell in the first column, and irrelevant auxiliary fields that have additional cells in any other columns. Extra fields are those that have two or more Lorentz indices compared to the physical field. More precisely,
• Physical field $\Upsilon^{(0,\ldots,0)}_{(p)} \equiv e_{(p)}$ has tangent Lorentz indices described by the $o(d-1,1)$ Young tableau of the form

![Young tableau](image)

$\lambda^{s-1} e_{(p)} A_1(s-1), \ldots, A_p(s-1), A_{p+1}(s+1), \ldots, A_q(s_q) = V_{A_0} \cdots V_{A_{s-1}} \Omega(e_{(p)} A_0(s-1), A_1(s-1), \ldots, A_q(s_q)).$ \hfill (4.44)

Recall that contraction of any $s$ indices of $\Omega(e_{(p)} A_0(s-1), A_1(s-1), \ldots, A_q(s_q))$ with $V^A$ gives zero because of the Young properties of $\Omega(e_{(p)} A_0(s-1), A_1(s-1), \ldots, A_q(s_q))$. This $V^A$ transversality effectively means that all indices of the physical field $e_{(p)}$ (4.44) are Lorentz indices.

In the antisymmetric basis (4.41), the analog of (4.44) is

$\lambda^{s-2} \omega^{(p)} A_1[\tilde{h}_1-1], \ldots, A_{s-1}[\tilde{h}_{s-1}-1] = V_{A_1} \cdots V_{A_{s-1}} \Omega^{(p)} A_1[\tilde{h}_1], A_2[\tilde{h}_2], \ldots, A_{s-1}[\tilde{h}_{s-1}].$ \hfill (4.45)

• Relevant auxiliary field $\Upsilon^{(0,\ldots,0,1)}_{(p)} \equiv \omega_{(p)}$ is described by the $o(d-1,1)$ Young tableau of the form

![Young tableau](image)

$\lambda^{s-2} \omega^{(p)} A_1[\tilde{h}_1], A_2[\tilde{h}_2-1], \ldots, A_{s-1}[\tilde{h}_{s-1}-1] = V_{A_2} \cdots V_{A_{s-1}} \Omega^{(p)} A_1[\tilde{h}_1], A_2[\tilde{h}_2], \ldots, A_{s-1}[\tilde{h}_{s-1}], -\tilde{h}_1 V^A A_1 V_{A_2} \cdots V_{A_{s-1}} V_B \Omega^{(p)} B A_1[\tilde{h}_1-1], \ldots, A_{s-1}[\tilde{h}_{s-1}].$ \hfill (4.47)
It is easy to see that the tensor on the right hand side of (4.47) has the correct Young symmetry and is $V^A$-transversal. The $o(d - 1, 2)$ covariant expression for the relevant auxiliary field in the symmetric basis is not given here because it is more involved, requiring explicit Young symmetry projectors. Note that in the case of gravity, the formula (4.47) gives, as expected, $\omega^{AB}_1 = \Omega^{AB}_1 - \lambda (E_1^A V^B - E_1^B V^A)$ (4.12).

- Irrelevant auxiliary fields $\{\Upsilon_p(0,\ldots,0,1,0,\ldots,0)\} \equiv \{\omega'_p\}$ have an additional cell compared to the Young tableau of the physical field, that is situated in any column except for the first one. Generally, if a field under consideration is described by an $AdS_d$ tangent Young tableau which consists of $N$ blocks, it gives rise to $N - 1$ irrelevant auxiliary fields. In particular, for the case of totally symmetric fields $N = 1$ and, in agreement with [10], there is only one auxiliary field, namely, the relevant one, while the irrelevant auxiliary fields do not appear. The same is true for any HS field described in the frame-like formalism by a $p$-form that takes values in some rectangular Young tableau.

- Extra fields $\{\Upsilon_p(t_1,\ldots,t_k), \sum_t t_I \geq 2\} \equiv w(p)$ have two or more additional cells compared to Young tableau of the physical field.

HS curvatures and gauge parameters admit the decompositions analogous to (4.42)

$$R_{(p+1)} = \bigoplus_{(t_1,\ldots,t_k)} \lambda^{(s-1-\sum_t t_I)} R_{(p+1)}^{(t_1,\ldots,t_k)}, \quad (4.48)$$

$$\xi_{(p-1)} = \bigoplus_{(t_1,\ldots,t_k)} \lambda^{(s-1-\sum_t t_I)} \varepsilon_{(p-1)}^{(t_1,\ldots,t_k)}. \quad (4.49)$$

From these decompositions and (4.38) it follows that Lorentz-covariant components of the HS curvature $R_{(p+1)}^{(t_1,\ldots,t_k)}$ have the form

$$R_{(p+1)}^{(t_1,\ldots,t_k)} = D(\Upsilon_p)^{(t_1,\ldots,t_k)} + \sum_{I=1}^k \left( \sigma_{-}^{(I)}(\Upsilon_p)^{(t_1,\ldots,t_k)} + \lambda^2 \sigma_{+}^{(I)}(\Upsilon_p)^{(t_1,\ldots,t_k)} \right), \quad (4.50)$$

where the sigma-operators have the following structure

$$\sigma_{-}^{(I)}(\Upsilon_p)^{(t_1,\ldots,t_I,\ldots,t_k)} \equiv \alpha_I(t) P_{-}^{(I)} \left( h \wedge \Upsilon_p^{(t_1,\ldots,t_{I+1},\ldots,t_k)} \right), \quad (4.51)$$

$$\sigma_{+}^{(I)}(\Upsilon_p)^{(t_1,\ldots,t_I,\ldots,t_k)} \equiv \beta_I(t) P_{+}^{(I)} \left( h \wedge \Upsilon_p^{(t_1,\ldots,t_{I-1},\ldots,t_k)} \right). \quad (4.52)$$

Here the background frame 1-form $h$ contracts an extra index of $\Upsilon_p^{(t_1,\ldots,t_{I+1},\ldots,t_k)}$ in (4.51) and adds a missed index of $\Upsilon_p^{(t_1,\ldots,t_{I-1},\ldots,t_k)}$ in (4.52), $P_{\pm}^{(I)}$ are the projectors to the irreducible $o(d - 1, 1)$ module carried by $\Upsilon_p^{(t_1,\ldots,t_{I},\ldots,t_k)}$ and $\alpha_I(t)$ and $\beta_I(t)$ are some coefficients.
The gauge transformations have analogous form

$$\delta \Upsilon_{(p)}^{(t_1, \ldots, t_k)} = \mathcal{D}(\varepsilon_{(p-1)})^{(t_1, \ldots, t_k)} + \sum_{I=1}^{k} \left( \sigma_{\pm}^{(I)}(\varepsilon_{(p-1)})^{(t_1, \ldots, t_k)} + \lambda^2 \sigma_{\pm}^{(I)}(\varepsilon_{(p-1)})^{(t_1, \ldots, t_k)} \right)$$

(4.53)

As a consequence of the flatness of AdS covariant derivative, $D_0^2 = 0$, the sigma-operators satisfy the relations

$$\mathcal{D}^2 + \lambda^2 \sum_{I=0}^{k} \{\sigma_{\pm}^{(I)}, \sigma_{\pm}^{(I)}\} = 0, \quad \{\sigma_{\pm}^{(I)}, \sigma_{\pm}^{(J)}\} = 0, \quad \{\mathcal{D}, \sigma_{\pm}^{(I)}\} = 0.$$  

(4.54)

Other way around, these conditions determine the coefficients $\alpha_i(t)$ and $\beta_i(t)$ of sigma-operators (4.51) and (4.52) up to free parameters which manifest the rescaling ambiguity of $\Upsilon_{(p)}^{(t_1, \ldots, t_1, \ldots, t_k)}$.

### 4.5 Frame versus metric

To explain how a metric-like HS field of a given spin is encoded in the corresponding physical frame-like HS field let us analyze its gauge transformation law. Introducing schematic notations for gauge parameter associated with the physical field, $\varepsilon_{(p-1)}$, and with the auxiliary fields, $\varepsilon_{(p-1)}^{(I)}$, from (4.53) we obtain

$$\delta \varepsilon_{(p)} = \mathcal{D}\varepsilon_{(p-1)} + \sum_{I=1}^{k} \sigma_{-}^{(I)} \varepsilon_{(p-1)^{I}}.$$  

(4.55)

The part of the gauge transformation with the parameters $\varepsilon_{(p-1)}^{(I)}$ is Stueckelberg, allowing to gauge away some of components of the physical field. The metric-like field is the part of the physical field invariant under the Stueckelberg symmetries, pretty much like the usual metric tensor can be understood as a part of the frame invariant under the Lorentz transformations.

To find out which components of the physical field are invariant under the Stueckelberg shift transform it is convenient to convert all world indices of the physical field and gauge parameters in (4.55) into the tangent ones. For the physical field the result is

$$e_{[n_1 \ldots n_p] ; a_1(s-1), \ldots, a_p(s-1), a_{p+1}(s_{p+1}), \ldots, a_q(s_q)} =$$

$$= h_{m_1}^{n_1} \ldots h_{m_p}^{n_p} e_{[m_1 \ldots m_p] ; a_1(s-1), \ldots, a_p(s-1), a_{p+1}(s_{p+1}), \ldots, a_q(s_q)}.$$  

(4.56)

Conversions of the gauge parameters is analogous.

The metric-like HS field $\Phi^{a(s), \ldots, a_q(s_q)}(x)$ is the following component of the physical field

$$\Phi^{a_1(s-1), \ldots, a_p(s-1), a_{p+1}(s_{p+1}), \ldots, a_q(s_q)} = e_{[a_1 \ldots a_p] ; a_1(s-1), \ldots, a_p(s-1), a_{p+1}(s_{p+1}), \ldots, a_q(s_q)},$$  

(4.57)
i.e. it results from symmetrization of the form indices with tangent indices of the first $p$ rows of $e_{(p)}$.

To check the invariance of the metric-like field defined according to the formula (4.57) under Stueckelberg shift symmetry (4.55) with the gauge parameter $\varepsilon^{(I)}_{(p-1)}$

$$\delta_{(I)} \Phi^{a_{1}(s),...,a_{p}(s),a_{p+1}(s_{p+1})...,a_{q}(s_{q})} \equiv$$

$$\equiv \delta_{(I)} e^{[a_{1}...a_{p}]; a_{1}(s-1),...,a_{p}(s-1),a_{p+1}(s_{p+1})...,a_{q}(s_{q})} = 0$$

one rewrites (4.55) as

$$\delta_{(I)} e^{n[p]; a_{1}(s-1),...,a_{p}(s-1),a_{p+1}(s_{p+1})...,a_{q}(s_{q})} =$$

$$= \mathcal{P}_{(I)} \left( \varepsilon^{(I)} n[p-1]; a_{1}(s-1),...,a_{p}(s-1),a_{p+1}(s_{p+1})...,a_{q}(s_{q}) \right),$$

where $\mathcal{P}_{(I)}$ projects r.h.s. of (4.59) on the Young tableau associated with the indices of the field $e_{(p)}$. Substituting the expression (4.59) into the variation of the metric-like field (4.58) one obtains that the index $n$ from the $I$-th row of the Stueckelberg gauge parameter in the formula (4.59) is symmetrized with all indices from a row of its uppermost rectangular block what gives zero by virtue of Young symmetry properties.

Also the invariance (4.58) can be proved by checking that all remaining components of the physical field $e_{(p)}$ are Stueckelberg with respect to the gauge parameters $\varepsilon^{(I)}_{(p-1)}$. This can be achieved by comparing the contents of the tensor product of the antisymmetric world Lorentz modules associated with the differential forms with the tangent Lorentz modules carried, respectively by the physical $p$-form on the one side and the auxiliary $p-1$ form gauge parameters on the other side (modulo the level 2 gauge symmetries).

As a consequence of the formula (4.57), the metric-like field $\Phi$ satisfies some tracelessness conditions because the physical frame-like field is traceless in the tangent Lorentz indices. Namely, it follows that the tensor $\Phi^{a_{1}(s),...,a_{q}(s_{q})}$ satisfies the tracelessness conditions (3.12) and (3.13) [1], i.e. the metric-like field $\Phi$ belongs to the tensor space $B^{d-1,1}_{p}(s,...,s_{q},0,...,0)$ of section 3.2. The Fronsdal double tracelessness condition for totally symmetric fields $\Phi^{a(s)} \in B^{d-1,1}_{1}(s,0,...,0)$, is the particular case of the tracelessness conditions for a general mixed-symmetry field.

As an illustration, let us consider a spin $(2,1,0,...,0)$ mixed-symmetry field described by the three-cell ”hook” tableau (for more examples, see [1, 36]). Its frame-like description gives rise to the physical 1-form field $e_{m}^{a,b}$ which is an antisymmetric Lorentz tensor. There is just one auxiliary field carrying three antisymmetrized indices. The corresponding gauge transformation (4.55) is

$$\delta e_{m}^{a,b} = D_{m}^{a} \varepsilon_{a,b} + h_{m,c} \varepsilon_{a,b,c},$$

(4.60)

or, converting world indices into tangent ones,

$$\delta e^{m:a,b} = D^{m} \varepsilon^{a,b} + \varepsilon^{m,b,c}.$$
A set of traceless $o(d - 1, 1)$ tensor components contained in the physical field is given by the following tensor product

$$
\begin{array}{ccc}
\otimes & \oplus & \oplus \\
\end{array}
$$

(4.62)

By virtue of the Stueckelberg symmetry part of (4.61) with the antisymmetric parameter $\varepsilon^{a,b,c}$ the second component in (4.62) can be gauged away. The remaining first and third components form a tracefull tensor $\Phi^{ab,c}$ given by (4.57),

$$
\Phi^{ab,c} = \frac{1}{2}(e^{a;b,c} + e^{b;a,c}).
$$

(4.63)

The gauge transformation law for the field $\Phi^{ab,c}$ that results from (4.61) is

$$
\delta \Phi^{ab,c} = \mathcal{D}^{a}_{\varepsilon^{b},c} + \mathcal{D}^{b}_{\varepsilon^{a},c}.
$$

(4.64)

Other way around, the metric-like formulation can be taken as a starting point of the derivation of the frame-like formulation of HS fields. Namely, given a metric-like HS gauge field, one adds Stueckelberg components so that together with the original metric-like field they form a $p$-form gauge field which carries tangent Lorentz indices associated with some irreducible $o(d - 1, 1)$-module. This $p$-form field is the physical field and its transformation law is postulated to be (4.55). The frame-like machinery evolves further by introducing the new gauge fields associated with Stueckelberg shift parameters. These will be auxiliary fields with the gauge transformation law that contains the Lorentz derivative acting on the Stueckelberg shift parameters. In addition, there will be some new shift parameters in the gauge transformation of the auxiliary fields. In their turn, these new shift parameters require new gauge fields which are extra fields. This procedure continues further to obtain a full set of physical, auxiliary and extra fields necessary to construct curvature $(p + 1)$-forms manifestly invariant under the full set of gauge symmetries. As explained in section 4.4 the resulting set forms a $p$-form gauge field taking values in the irreducible $o(d - 1, 2)$-module described by the Young tableau (4.35).

5 General properties of a higher-spin action

5.1 Background

As shown in [24, 25] generic massless fields in $AdS_d$ are different from the massless fields in Minkowski space in the sense that an irreducible gauge field in $AdS_d$ reduces in the flat limit to a number of massless fields of different types in Minkowski space. This effect has clear interpretation in terms of representations of the $AdS_d$ algebra $o(d - 1, 2)$.

As discussed in section 4.4 the space of single-particle states of a given relativistic field with the energy bounded from below forms a lowest weight $o(d - 1, 2)$-module,
$D(E_0, s)$. Its lowest weight is defined in terms of the lowest energy $E_0$ and spin $s$ associated with the weights of the maximal compact subalgebra $o(2) \oplus o(d - 1) \subset o(d - 1, 2)$. For quantum-mechanically consistent fields, the modules $D(E_0, s)$ correspond to unitary representations of $o(d - 1, 2)$. Unitarity requires $E_0 \geq E_0(s)$ where $E_0(s)$ is some function of the spin $s$ found for the case of $d = 4$ in [23] and for the general case in [24]. At $E_0 = E_0(s)$ null states appear that form a submodule to be factored out. These signal a gauge symmetry of the system. The corresponding fields are the usual massless fields.

Modules with lower energies $E < E_0(s)$ correspond to nonunitary (ghost) massive fields. At certain singular values $E_i(s)$ of $E$ the corresponding nonunitary module may contain a submodule that again signals a gauge symmetry in the field-theoretical description. Such modules correspond either to the partially massless fields [26, 27, 28] or to “non-unitary massless fields”. More specifically, one can see that a $o(d - 1)$ tensor module carried by the vacuum space of a submodule (called singular space) corresponds to the $o(d - 1, 1)$ tensor module carried by the associated gauge symmetry parameter while a level, at which the singular space appears, equals to a highest order of derivatives that act on the gauge parameter in the gauge transformation law. The parameters that enter the gauge transformation law with one derivative correspond to different massless fields in $AdS_d$. Those that enter the gauge transformation law with two or more derivatives correspond to partially massless fields.

Since singular energies $E_i(s)$ are scaled in units of $AdS_d$ curvature $\lambda$, i.e.

$$E_i(s) = \lambda e_i(s),$$

where $e_i$ is $\lambda$-independent, all special energies tend to zero in the flat limit $\lambda \to 0$ so that all gauge symmetries are inherited by one massless theory in Minkowski space. More precisely, different gauge symmetries of unitary and non-unitary massless theories in $AdS_d$ become different gauge symmetries of the same massless theory in Minkowski space while the flat limit of partially massless gauge symmetries seem to correspond to particular flat massless gauge symmetries with the gauge parameters expressed via derivatives of some other tensors identified with the gauge parameters of partially massless models in $AdS_d$.

The gauge parameter associated with the unitary massless field is most antisymmetric, resulting from cutting a cell from the shortest column of the $o(d - 1)$ Young tableau of the vacuum space of $D(E_0, s)$. All other gauge symmetries resulting from different cell cuts correspond to “non-unitary massless fields”. These are absent in a consistent $AdS_d$ massless theory but may re-appear in its flat limit.

### 5.2 Degrees of freedom

An action functional that describes one or another dynamical system should exhibit appropriate global symmetries and describe a correct number of degrees of freedom. An irreducible dynamical system carries a minimal possible number of degrees of
freedom. The reduction of a number of degrees of freedom is achieved either via
gauge symmetries or via constraints.

Generally, gauge symmetries kill more degrees of freedom than constraints because they require $q + 1$ gauge conditions for a gauge transformation $\delta \phi = \partial^i \epsilon$ that contains $q$ time derivatives of the gauge parameter. This fact is most obviously seen in terms of the Dirac constraint dynamics \cite{50, 51}, where gauge symmetries correspond to first-class constraints while constraint field equations correspond to second-class constraints. The general situation can be illustrated by the example of massive and massless spin one field in flat space.

The Proca equation for a massive spin one field is

$$\Box A_\mu(x) - \partial_\mu \partial^\nu A_\nu(x) + m^2 A_\mu(x) = 0 , \quad \nu = 0, ..., d - 1. \quad (5.2)$$

For $m \neq 0$, the model has no gauge symmetry and describes $d - 1$ physical degrees of freedom. Indeed, taking divergency of the left hand side of this equation with $m \neq 0$, one obtains the Lorentz condition

$$m^2 \partial_\nu A_\nu(x) = 0. \quad (5.3)$$

As a result, the Lagrangian equation (5.2) is equivalent to

$$\Box A_\mu(x) + m^2 A_\mu(x) = 0 , \quad (5.4)$$

and (5.3), thus describing $d - 1$ degrees of freedom in $d$ dimensions.

Maxwell equations have the form (5.2) with $m = 0$

$$\Box A_\mu(x) - \partial_\mu \partial^\nu A_\nu(x) = 0 . \quad (5.5)$$

In this case, differentiating the left hand side by $\partial^\mu$ one obtains the Bianchi identity which manifests the spin one gauge symmetry

$$\delta A_\mu = \partial_\mu \epsilon . \quad (5.6)$$

This gauge symmetry kills two degrees of freedom so that a massless spin one particle carries $d - 2$ degrees of freedom. Indeed, let us impose the Coulomb gauge condition

$$\partial^i A_i = 0 \quad i = 1, 2 \ldots, d - 1 . \quad (5.7)$$

Then the equation (5.5) with $\mu = 0$ gives the constraint

$$\triangle A_0 = 0 , \quad (5.8)$$

where $\triangle = \partial_0 \partial^i$ is the $(d - 1)$-dimensional Laplace operator, which does not contain time derivatives. For $A_0$ vanishing at space infinity this implies

$$A_0 = 0 . \quad (5.9)$$
Together with \((5.7)\), \((5.8)\) reduces a number of degrees of freedom of the Maxwell theory to \(d - 2\). (Note that, in presence of currents, the charge density appears on the right hand side of the equation \((5.8)\) which reconstructs the electric potential in terms of electric charge.)

We see that in the both cases it is crucial that the highest derivative part of the field equations has the correct form. If a relative coefficient between the two second-order derivative terms in \((5.2)\) was changed, both the constraint in the massive case and the gauge symmetry in the massless case would be lost and the system would describe some more degrees of freedom (usually ghost-like).

More generally, the condition that a system describes a minimal possible number of degrees of freedom is that the term with highest derivatives should satisfy the Bianchi identities that, for a Lagrangian system, is equivalent to the condition that it is gauge invariant. This can be equivalently formulated in the form that a limit of a theory with all dimensionful parameters tending to zero (defined so that only the highest derivative terms survive) should give a gauge theory with a maximal possible number of gauge symmetries. For a theory in a curved space with a dimensionful parameter \(\lambda\) which characterizes its curvature, the analogous condition should be imposed on the part of the action that survives in the flat limit \(\lambda \to 0\). So, the condition that a theory in curved space should exhibit enhancement of gauge symmetries in a flat limit

\[
\delta_\varepsilon \Phi(x) = \sum_I \partial \varepsilon_I(x), \tag{5.10}
\]

is analogous to the gauge symmetry enhancement of the Proca equation in the massless limit. The gauge parameters \(\varepsilon_I(x)\) in \((5.10)\) are described by the Young tableaux resulting from that of the field \(\Phi(x)\) by cutting off a cell from the last row of any \(I\)-th horizontal block.

More precisely, a HS action in \(AdS_d\) has the form

\[
S_{2_{AdS}} \sim \int \mathcal{D}\Phi \mathcal{D}\Phi + \lambda^2 \Phi^2. \tag{5.11}
\]

The mass-like terms \(\lambda^2 \Phi^2\) break down all the gauge symmetries \((5.10)\) of the action \(S_{2_{flat}}\) except for that associated with the \(AdS_d\) gauge parameter. However, in the flat limit \(\lambda \to 0\), the action \(S_{2_{flat}} \sim \int \partial \Phi \partial \Phi\) exhibits enhancement of gauge symmetries.

Let us stress that the precise form of the kinetic term of the action \((5.11)\) is not uniquely fixed by the gauge invariance condition with respect to the true \(AdS_d\) gauge symmetry. So the requirement of the flat limit gauge symmetry enhancement is an important additional condition that determines the structure of the action. Of course, when there are several different terms of subleading orders of derivatives, these should also be adjusted in a way implying the maximal reduction of degrees of freedom, i.e. preserving a maximal possible number of gauge symmetries and constraints.

Once the correct model of an irreducible massless HS theory of a given spin \(s\) is formulated in terms of a field \(\Phi(x)\), it should be possible to impose various covariant
irreducibility tracelessness and Lorentz-type conditions

\[ \text{tr} \Phi(x) = 0, \quad \mathcal{D}\Phi(x) = 0, \quad (5.12) \]

which are either gauge conditions for the gauge symmetry transformations of the model

\[ \delta_\epsilon \Phi(x) = \mathcal{D}\epsilon(x) \quad (5.13) \]

or constraints which follow from the field equations as the spin one Lorentz condition follows from the Proca equation. As a result, the remaining field equations get the Klein-Gordon form

\[ (\mathcal{D}^2 + \lambda^2 M(d, s)) \Phi(x) = 0, \quad (5.14) \]

where the explicit form of \( M(d, s) \)

\[ M(d, s) = \sum_{i=1}^{q} s_i - (s - p - 1)(s - p - 2 + d) \quad (5.15) \]

was found by Metsaev in [24]. This formula provides one more check whether or not a theory under consideration describes properly a massless field of a given type.

In fact, \( (5.14) \) fixes the quadratic Casimir operator of the \( AdS_d \) algebra realized on Lorentz-covariant tensor fields satisfying the constraints \( (5.12) \). Note that the equation \( (5.14) \) possesses a leftover gauge symmetry with the parameter \( \epsilon(x) \) satisfying the irreducibility conditions

\[ \text{tr} \epsilon(x) = 0, \quad \mathcal{D}\epsilon(x) = 0 \quad (5.16) \]

along with certain differential conditions of the type \( (5.14) \). The gauge parameters of this symmetry form a \( o(d - 1, 2) \)-module which is the singular submodule to be factored out (i.e., gauged away) to obtain the irreducible massless module.

### 5.3 Higher-spin action in the frame-like formalism

The form of a HS action is to large extent fixed by the gauge symmetry principle. The frame-like formalism is convenient in first place because it allows us to have true HS symmetries manifest.

It is natural to search for a HS action for a given gauge HS field in the form \([9, 10, 47]\)

\[ S_2 = \lambda^{-2(s-1)} \int_{M^d} \sum H^{\cdots} E^{\cdots}_{0} \wedge \cdots \wedge E^{\cdots}_{0} \wedge R_{(p+1)} \wedge R_{(p+1)} \wedge \cdots \wedge R_{(p+1)} \wedge \cdots . \quad (5.17) \]

Here factor \( \lambda^{-2(s-1)} \) is introduced to provide correct flat limit of \( (5.17) \) (see section \[8\]) and \( H^{\cdots} \) are some coefficients built of the Levi-Civita tensor and the compensator \( V^A(x) \), which parameterize various possible contractions of tangent indices. Since
the curvatures $R_{(p+1)}$ \[4.38\] are by construction invariant under the gauge transformations \[4.36\] of the $p$-form gauge fields $\Omega_{(p)}$, the action $S_2$ is gauge invariant for any choice of the coefficients $H$.

Any action of the form \[5.17\] is manifestly invariant both under diffeomorphisms (because of using the exterior algebra formalism) and under the local $o(d-1,2)$ symmetry that acts on the tangent indices $A, B, \ldots = 0, 1, \ldots$. Since the the compensator $V^A$ and background gravitation fields, which enter the covariant derivative $D_0$, are supposed to take some fixed values, these symmetries are broken. (Note that, in a full interacting theory, this breakdown will be spontaneous, induced by the vacuum expectation values of the gravitational and compensator fields.) However, the action \[5.17\] turns out to be invariant under the global $o(d-1,2)$ symmetry which is a combination of diffeomorphisms and local $o(d-1,2)$ transformations that leaves the background gravitation fields and the compensator invariant. (For more detail we refer the reader to [43].)

Let us now discuss which additional conditions should be imposed on the coefficients $H$ to guarantee that the action \[5.17\] indeed describes a given massless HS field. The correct action should

(i) be expressible in terms of the physical frame-like field and its first derivatives,

(ii) exhibit gauge symmetry enhancement in the flat limit.

Note that, taking into account the HS gauge invariance of the action and that the metric-like field identifies with the frame-like physical field modulo Stueckelberg gauge symmetries, the first of these conditions means that the action is expressible in terms of the metric-like field and its first derivatives.

The condition (i) is not automatically satisfied because the HS curvatures depend on physical, auxiliary and extra fields in the terminology of section \[4.4\]. It will be fulfilled, however, if the coefficients $H$ are such that the action is independent of the extra fields

$$\frac{\delta S_2}{\delta w_{(p)}} \equiv 0,$$

\[5.18\]

and the auxiliary fields satisfy algebraic field equations that express them in terms of first derivatives of the physical field. As we shall see, the latter condition turns out to be automatically true once \[5.18\] is satisfied. The extra fields should not appear at the free field level because they bring in extra degrees of freedom both if treated as independent fields and if expressed via derivatives of the physical fields by some constraints. In the latter case, carrying two more Lorentz indices compared to the physical field, extra fields can only be expressed via second derivatives of the physical field thus bringing higher derivatives into the action if \[5.18\] is not true.

To summarize, the extra field decoupling condition \[5.18\] guarantees that extra fields enter the action through the total derivative terms and do not contribute to nontrivial equations of motion.

It turns out however that the extra field decoupling condition \[5.18\] alone does not fix the coefficients uniquely, admitting a $N$-parametric family of $AdS_d$ gauge-
invariant actions, where $N$ is a number of different auxiliary 1-form connections. This means that there exist $N$ different actions expressible in terms of the metric-like field and its first derivatives which are all gauge invariant under the necessary $AdS_d$ gauge symmetry. This is the manifestation of the fact that the $AdS_d$ gauge symmetry alone does not fix uniquely a form of the kinetic term for a generic mixed-symmetry field. To determine which particular combination of these $N$ actions is the correct one, the condition (ii) should be imposed.

It turns out \cite{2} that the condition (ii) is satisfied if all irrelevant auxiliary fields also do not contribute to the action, i.e.

$$\frac{\delta S_2}{\delta \omega^i_{(p)}} \equiv 0. \quad (5.19)$$

The proof of this fact is given in section 8.

The extra field decoupling condition (5.18) together with the irrelevant auxiliary field decoupling condition (5.19) fix uniquely the coefficients $H^{\cdots}$ in the action (5.17), which has the following structure

$$S_2 \sim \int_{\mathcal{M}^d} \left( \omega_{(p)} (De_{(p)} - \omega_{(p)}) + \lambda^2 e_{(p)} e_{(p)} \right), \quad (5.20)$$

where $e_{(p)}$ is the physical field and $\omega_{(p)}$ is the relevant auxiliary field. From this form of the action (5.17) it follows that the relevant auxiliary field $\omega_{(p)}$ is expressed by virtue of its equation of motion through the first derivatives of the physical field (modulo pure gauge parts). Plugging the resulting expression back into the action gives rise to the second-order action for the metric-like HS field that has necessary gauge symmetry and the correct kinetic term.

### 6 Action and $Q$-complex

#### 6.1 Fock space notations

To simplify the analysis of the HS action it is convenient to reformulate the problem in terms of a certain Fock space \cite{2}. This approach generalizes that applied to totally symmetric fields in \cite{10,11}.

The analysis of the HS dynamics is somewhat simpler in the antisymmetric basis for Young tableaux where the expression (4.47) for the relevant auxiliary field is particularly simple. To have antisymmetries manifest, let us introduce the set of fermionic oscillators

$$\psi^A_\alpha = (\psi_i^A, \bar{\psi}_j^A) \quad \text{and} \quad \bar{\psi}_\alpha^A = (\bar{\psi}_i^A, \bar{\psi}_j^A),$$

where $i, j = 1 \div (s - 1), \ \alpha = 1 \div 2(s - 1)$. These oscillators satisfy the anticommutation relations

$$\{\psi_i^A, \bar{\psi}_j^B\} = \delta^i_j \eta^{AB}, \quad \{\psi^i_A, \bar{\psi}_j^B\} = \delta^i_j \eta^{AB} \quad (6.1)$$
with all other anticommutators equal to zero.

Also we introduce fermionic oscillators $\theta^A$ and $\bar{\theta}^B$ that satisfy anticommutation relations

$$\{\theta^A, \bar{\theta}^B\} = \eta^{AB}, \quad \{\theta^A, \theta^B\} = 0, \quad \{\bar{\theta}^A, \bar{\theta}^B\} = 0$$

(6.2)

and anticommutate with $\psi_\alpha^A$ and $\bar{\psi}_\alpha^A$.

Let us define the left and right Fock vacua by

$$\langle 0 | \psi_\alpha^A = 0, \quad \langle 0 | \bar{\psi}_\alpha^A = 0,$$

(6.3)

$$\bar{\psi}_\alpha^A | 0 \rangle = 0, \quad \bar{\theta}^A | 0 \rangle = 0$$

(6.4)

along with

$$\langle 0 | \theta^{A_1} \cdots \theta^{A_{d+1}} | 0 \rangle = \epsilon^{A_1 \cdots A_{d+1}}, \quad \langle 0 | \theta^{A_1} \cdots \theta^{A_k} | 0 \rangle = 0 \quad \text{for} \quad k \neq d + 1.$$ (6.5)

The oscillators $\theta$ provide a convenient way to introduce the $o(d - 1, 2)$ Levi-Civita tensor via formula (6.3).

In our construction, a $p$-form $o(d - 1, 2)$ gauge field will be described as Fock vectors of two types $|\Omega(p)\rangle = \Omega(p)|0\rangle$ or $|\bar{\Omega}(p)\rangle = \bar{\Omega}(p)|0\rangle$, where

$$\Omega(p) = \Omega(p) A_1[h_1] \cdots A_{s-1}[h_{s-1}] (\psi_{A_1}^1) h_1 \cdots (\psi_{A_{s-1}}^{s-1}) h_{s-1}^1,$$

(6.6)

$$\bar{\Omega}(p) = \Omega(p) A_1[h_1] \cdots A_{s-1}[h_{s-1}] (\bar{\psi}_1^A) h_1 \cdots (\bar{\psi}_{s-1}^A) h_{s-1}^1.$$ More generally, operators $\hat{A}_{(m)}$ and $\hat{\bar{A}}_{(m)}$ will be assumed to be analogously constructed from a $m$-form $A_{(m)}$ instead of $\Omega(p)$.

Note that the proposed approach is different from that used for the totally symmetric HS fields \cite{10}. Indeed, in Ref. \cite{10}, HS fields were considered as elements of left and right Fock modules, i.e. $|\Omega\rangle$ and $|\bar{\Omega}\rangle$. In our approach HS fields are elements of the tensor product $|\Omega \otimes \bar{\Omega}\rangle$.

The Young symmetry and tracelessness conditions on the $p$-form gauge fields imply

$$l^i_j |\Omega(p)\rangle = 0, \quad \bar{s}_{ij} |\Omega(p)\rangle = 0,$$

(6.7)

$$l^i_i |\Omega(p)\rangle = 0, \quad \bar{s}^i_j |\Omega(p)\rangle = 0,$$

(6.8)

$$l^i_1 |\Omega(p)\rangle = \hat{h}_1 |\Omega(p)\rangle, \quad l^1_i |\Omega(p)\rangle = \bar{h}_i |\Omega(p)\rangle,$$

(6.9)

where

$$l_{\alpha\beta} = \eta_{AB} \psi_\alpha^A \bar{\psi}_\beta^B, \quad \bar{s}_{\alpha\beta} = \eta_{AB} \bar{\psi}_\alpha^A \psi_\beta^B.$$ (6.10)

The linearized curvatures \cite{138} in the antisymmetric basis are

$$|\mathcal{R}_{(p+1)}\rangle = \hat{\mathcal{R}}_{(p+1)} |0\rangle = D_0 |\Omega(p)\rangle, \quad |\mathcal{R}_{(p+1)}\rangle = \bar{\mathcal{R}}_{(p+1)} |0\rangle = D_0 |\bar{\Omega}(p)\rangle.$$ (6.11)

Here the $o(d - 1, 2)$ covariant background derivative is

$$D_0 = d + \Omega_0^A B \psi^i_A \bar{\psi}_i^B + \Omega_0^A B \psi^i_A \bar{\psi}_i^B + \Omega_0^A B \theta_A \bar{\theta}^B, \quad D_0^2 = 0,$$ (6.12)
where $\Omega^A_{0B}$ is the background $AdS_d$ gauge field satisfying the zero-curvature condition (4.19). The gauge transformations (4.36) and the Bianchi identities (4.40) are

$$
\delta|\hat{\Omega}_{(p)}\rangle = D_0|\hat{\xi}_{(p-1)}\rangle, \quad \delta|\hat{\Omega}_{(p)}\rangle = D_0|\hat{\xi}_{(p-1)}\rangle, \quad (6.13)
$$

$$
D_0|\hat{R}_{(p+1)}\rangle = 0, \quad D_0|\hat{R}_{(p+1)}\rangle = 0. \quad (6.14)
$$

In the sequel we make use of the following operators

$$
\bar{\eta}_a = \bar{\psi}^A_A \theta_A, \quad \bar{v}_a = \bar{\psi}^A_A V_A, \quad \chi = \theta^A V_A, \quad E_0 = E^A_A \theta_A. \quad (6.15)
$$

where $V^A$ and $E^A_0$ are the compensator and the background frame field, respectively.

Let us introduce a notion of weak equality. Two polynomials $A(\bar{s}, \bar{\eta}, \bar{v})$ and $B(\bar{s}, \bar{\eta}, \bar{v})$ are weakly equivalent, $A \sim B$, if

$$
\langle 0 | (\wedge E_0)^{d-m-n} \chi (A - B) \wedge \hat{A}_{(m)} \wedge \hat{B}_{(n)} | 0 \rangle = 0 \quad (6.16)
$$

for any fields $A_{(m)}$ and $B_{(n)}$ that satisfy the Young symmetry and tracelessness conditions (6.7)-(6.9). In other words, the weak equivalence of two functions means that they differ by terms proportional to Young symmetrizers and trace operators which are zero by (6.7)-(6.9). A generic weakly zero function $W = W(\bar{s}, \bar{\eta}, \bar{v})$ can be cast into the following form

$$
W = \sum_{i,j=1}^{s-1} \mathcal{W}^{ij} \bar{s}^{ij} + \sum_{i,j=1}^{s-1} \mathcal{W}_{ij} \bar{s}_{ij} + \sum_{i,j=1, i \neq j}^{s-1} [\mathcal{W}^{ij}, l^j_i]
$$

$$
+ \sum_{i,j=1, i \neq j}^{s-1} [\mathcal{W}^j_i, l^i_j] + \sum_{i=1}^{s-1} \left( [\mathcal{W}_i, l^i_i] - \tilde{\eta}_i \mathcal{W}_i \right) + \sum_{i=1}^{s-1} \left( [\mathcal{W}_i, l^i_i] - \bar{\eta}_i \mathcal{W}_i \right), \quad (6.17)
$$

where $\mathcal{W}_{ij}, \mathcal{W}^{ij}, \mathcal{W}_j^i, \mathcal{W}^j_i, \mathcal{W}_i$ and $\mathcal{W}_i$ are arbitrary functions of $\bar{s}, \bar{\eta}$ and $\bar{v}$. Here the first two terms are weakly zero due to the tracelessness condition and the other terms are weakly zero due to the Young conditions (6.7), (6.8), (6.9). Note that any operator $\mathcal{Y}$ acting on polynomials of $\bar{s}, \bar{\eta}$ and $\bar{v}$, that commutes with $\bar{s}_{ij}, \bar{s}^{ij}, l^j_i$ and $l^i_j$

$$
[\bar{s}_{ij}, \mathcal{Y}] = [\bar{s}^{ij}, \mathcal{Y}] = [l^j_i, \mathcal{Y}] = [l^i_j, \mathcal{Y}] = 0, \quad (6.18)
$$

preserves the form (6.17) and thus maps weakly zero polynomials to weakly zero polynomials, i.e.

$$
\mathcal{Y} W \sim 0 \quad \forall \quad W \sim 0. \quad (6.19)
$$

### 6.2 General ansatz for a higher-spin action

The Fock space form of the frame-like action (5.17) is

$$
S_2 = \lambda^{-2(s-1)} \int_{M^d} \langle 0 | (\wedge E_0)^{d-2p-2} \chi H(\bar{s}, \bar{\eta}, \bar{v}) \wedge \hat{R}_{(p+1)} \wedge \hat{R}_{(p+1)} | 0 \rangle, \quad (6.20)
$$
where $H(\bar{s}, \bar{\eta}, \bar{v})$ is some polynomial of the commuting variables $\bar{s}_{\alpha\beta}$, $\bar{\eta}_\alpha$ and anti-commuting variables $\bar{v}_\alpha$. Oscillators $\theta$ contained in $H$ add up to $d-2p-1$ oscillators $\theta$ in $(E_0)^{d-2p-2}\chi$ to generate the Levi-Civita tensor via (6.5). From (6.5) it follows that the number of $\theta$'s in $H$ is $2p+2$ (otherwise the action (6.20) is zero) since these oscillators enter $H$ through the operators $\bar{\eta}_\alpha$, it follows that

$$\bar{\eta}_\alpha \frac{\partial}{\partial \bar{\eta}_\alpha} H = (2p+2)H$$

(6.21)

i.e., $H$ is a degree $2p+2$ homogeneous polynomial of $\bar{\eta}_\alpha$.

The operators $\bar{s}_{\alpha\beta}, \bar{\eta}_\alpha, \bar{v}_\alpha, \chi,$ and $E_0$ are responsible for the following contractions of indices in the action

- the operator $\bar{s}_{\alpha\beta}$ contracts two indices of $\hat{A}_{(m)}$ and/or $\tilde{A}_{(n)}$ placed in $\alpha$-th and $\beta$-th columns;
- the operator $\bar{\eta}_\alpha$ contracts an index of the Levi-Civita tensor with an index in $\alpha$-th column of $\hat{A}_{(m)}$ or $\tilde{A}_{(n)}$;
- the operator $\bar{v}_\alpha$ contracts the compensator $V^A$ with an index in $\alpha$-th column of $\hat{A}_{(m)}$ or $\tilde{A}_{(n)}$;
- the operator $\chi$ contracts the compensator $V^A$ with the Levi-Civita tensor;
- the operator $E_0$ contracts the frame $E^A_0$ with the Levi-Civita tensor.

This list exhausts all possible independent contractions between the constituents of the action. One can see that all other contractions either are zero (like $E^A_0 V_A = 0$) or can be reduced to the contractions listed above. For example, the term $\eta_{AB} E^A_0 \tilde{\psi}_\alpha$ responsible for contracting the index of frame with an index of $\hat{A}_{(m)}$ or $\tilde{A}_{(n)}$ reduces to the $E_0$-type terms by virtue of the following identity

$$E^A_0 \wedge (\wedge E_0)^m|0\rangle = \frac{1}{m+1} \delta^A (\wedge E_0)^{m+1}|0\rangle$$

(6.22)

along with the property that $\delta^A$ annihilates both left and right Fock vacua (6.3), (6.4).

Using the symmetry of (6.20) with respect to the exchange of the $(p+1)$-form HS curvatures, we require $H(s, \eta, v) = H(s_{\alpha\beta}, \eta_\alpha, v^i)$ to satisfy the symmetry property

$$H(s_{\alpha\beta}, \eta_\alpha, v^i, \tilde{v}^i) = (-1)^{P+P+1} H(-s_{\alpha\beta}, \eta_\alpha, \tilde{v}^i, \bar{v}_i),$$

(6.23)

where $P$ is defined by $\tilde{\Omega}_p(\psi) = (-1)^P \tilde{\Omega}_p(-\psi)$. 

36
6.3 \(Q\)-complex

Taking into account the definitions of the curvature (6.11) and the compensator (4.3), applying the Bianchi identities (6.14) and making use of the identity (6.22), the variation of the action can be reduced to the following form

\[
\delta S_2 = 2\lambda^{2(s-1)}(-1)^{d+p+1} \int_{M^d} \langle 0 | D_0 \left( (\wedge E_0)^{d-2p-2} \chi H \right) \wedge \hat{R}_{(p+1)} \wedge \delta \tilde{\Omega}_{(p)} | 0 \rangle = 
\]

\[
= 2(-1)^p \frac{\lambda^{2s-1}}{d-2p-1} \int_{M^d} \langle 0 | (\wedge E_0)^{d-2p-1} \chi Q H \wedge \hat{R}_{(p+1)} \wedge \delta \tilde{\Omega}_{(p)} | 0 \rangle, 
\]

where

\[
Q = \left( d - 1 + \bar{v}^\alpha \frac{\partial}{\partial \bar{v}^\alpha} - \bar{\eta}^\alpha \frac{\partial}{\partial \bar{\eta}^\alpha} \right) \bar{v}^\beta \frac{\partial}{\partial \bar{v}^\beta} + \bar{s}^{\alpha \beta} \frac{\partial^2}{\partial \bar{v}^\alpha \partial \bar{v}^\beta}. 
\]

Note that \(Q\) satisfies (6.18). As a result, \(QW \sim 0\) for any weakly zero function \(W \sim 0\).

The important fact is that

\[
Q^2 = 0 
\]

as one can check directly. (Essentially, \((6.26)\) is a consequence of \(D_0^2 = 0\). A natural guess is that, in an appropriate representation, \(Q\) can be rewritten as a de Rham operator. Indeed, one can see that

\[
\delta = \bar{v}^\alpha \frac{\partial}{\partial \bar{v}^\alpha} = A^{-1} Q A, 
\]

where

\[
A = \left( d - 1 + \bar{v}^\alpha \frac{\partial}{\partial \bar{v}^\alpha} - \bar{\eta}^\alpha \frac{\partial}{\partial \bar{\eta}^\alpha} \right) !\! \exp \left( \frac{1}{2} \bar{s}^{\alpha \beta} \frac{\partial^2}{\partial \bar{v}^\alpha \partial \bar{v}^\beta} \right). 
\]

The properties of the operator \(Q\) allow us to analyse the action and the de-coupling conditions in terms of the \(Q\)-complex. From the form of variation (6.24) it follows in particular that total derivative terms in the action are described by various \(Q\)-closed functions \(H\) (modulo weakly zero terms).

Suppose now that the variation of the action has the form

\[
\delta S_2 = \lambda^{2s-1} \langle 0 | (\wedge E_0)^{d-2p-1} \chi \mathcal{E}(\bar{s}, \bar{\eta}, \bar{v}) \wedge \hat{R}_{(p+1)} \wedge \delta \tilde{\Omega}_{(p)} | 0 \rangle 
\]

where \(\mathcal{E}(\bar{s}, \bar{\eta}, \bar{v})\) is some polynomial function which is \(Q\)-closed modulo weakly zero terms

\[
Q \mathcal{E} \sim 0. 
\]

The question is whether it is possible to reconstruct an action that leads to the field equations (6.29), i.e. to represent \(\mathcal{E}\) in the form

\[
\mathcal{E} \sim QH 
\]

with some \(H(\bar{s}, \bar{\eta}, \bar{v})\). The answer is yes and an explicit formula for \(H(\bar{s}, \bar{\eta}, \bar{v})\) in terms of \(\mathcal{E}\) is given in subsection 6.4. This result will allow us in section 7 to analyze
the decoupling conditions at the level of field equations, that is relatively simple, reconstructing the action afterwards.

Analogously, one can show that the total derivative terms in the Lagrangian are described by $Q$-exact coefficient function

$$H(s, \bar{\eta}, \bar{v}) = QT(s, \bar{\eta}, \bar{v})$$

for some $T(s, \bar{\eta}, \bar{v})$.

### 6.4 Action from field equations

To reconstruct the action function $H$ through the function $E$ in the variation of $H$ we have to solve the equation (6.31) for the known function $E$ satisfying the consistency condition (6.30). This can be easily done in the basis of $Q$ has a form of the standard de Rham operator

$$\delta = \bar{v}^\alpha \frac{\partial}{\partial \bar{v}^\alpha}, \quad \delta^2 = 0.$$ (6.33)

Introducing $H' = A^{-1}H$, $E' = A^{-1}E$ and taking into account that the operators $A$, $A^{-1}$ map weakly zero polynomials to weakly zero polynomials (see the arguments in the end of subsection 6.1) one rewrites equation (6.31) as

$$\delta H' \sim E'. $$ (6.34)

Let us solve first the strong equation

$$\delta F = G, $$ (6.35)

where $F = F(s, \bar{\eta}, \bar{v})$, $G = G(s, \bar{\eta}, \bar{v})$ are some polynomials and $G$ satisfies the compatibility condition

$$\delta G = 0.$$ (6.36)

Consider the operator $\delta^*$

$$\delta^* = \bar{\eta}^\alpha \frac{\partial}{\partial \bar{\eta}^\alpha}, \quad \delta^{*2} = 0.$$ (6.37)

Acting by $\delta^*$ on the both sides of (6.35) one obtains

$$\Delta F = \delta^* G + \delta \delta^* F,$$ (6.38)

where the operator

$$\Delta \equiv \{\delta, \delta^*\} = \bar{\eta}^\alpha \frac{\partial}{\partial \bar{\eta}^\alpha} + \bar{v}^\alpha \frac{\partial}{\partial \bar{v}^\alpha}$$ (6.39)

commutes with $\delta$ and $\delta^*$. As a result, neglecting a $\delta$-exact term, a partial solution of (6.35) is

$$F = \Delta^{-1} \delta^* G.$$ (6.40)
The operator $\Delta^{-1}$ admits the following integral realization
\[
\Delta^{-1}A(s, \bar{\eta}, \bar{v}) = \int_0^1 \frac{dt}{t} A(s, t\bar{\eta}, t\bar{v}) \tag{6.41}
\]
for a function $A(s, \bar{\eta}, \bar{v})$ such that $t^{-1}A(s, t\bar{\eta}, t\bar{v})$ is polynomial in $t$ (in the cases of interest this is always true because of $6.21$). Substituting (6.41) into (6.40) one obtains for the general solution of (6.35)
\[
\mathcal{F}(s, \bar{\eta}, \bar{v}) = \int_0^1 \frac{dt}{t} \bar{\eta}^\alpha \frac{\partial}{\partial \bar{v}^\alpha} G(s, t\bar{\eta}, t\bar{v}) + \bar{v}^\alpha \frac{\partial}{\partial \bar{\eta}^\alpha} T(s, \bar{\eta}, \bar{v}) \tag{6.42}
\]
with an arbitrary polynomial $T$. It is worth to note that since the operator $\Delta^{-1}\delta^*$ satisfies (6.18) the equation (6.35) with a weakly zero polynomial $G \sim 0$ always admits a weakly zero solution.

Now we are in a position to solve the weak equation (6.34) which has the form
\[
\delta H' = \mathcal{E}' + \mathcal{K}, \tag{6.43}
\]
where $\mathcal{K} \sim 0$. From (6.42) one obtains
\[
H'(s, \bar{\eta}, \bar{v}) = \int_0^1 \frac{dt}{t} \bar{\eta}^\alpha \frac{\partial}{\partial \bar{v}^\alpha} \left( \mathcal{E}'(s, t\bar{\eta}, t\bar{v}) + \mathcal{K}(s, t\bar{\eta}, t\bar{v}) \right) + \bar{v}^\alpha \frac{\partial}{\partial \bar{\eta}^\alpha} T(s, \bar{\eta}, \bar{v}) \tag{6.44}
\]
Acting on the both hand sides of (6.44) by the operator $A$ and neglecting the weakly zero term resulting from $\mathcal{K}$ and $\mathcal{Q}$-exact term resulting from $T$, we obtain
\[
H(s, \bar{\eta}, \bar{v}) \sim A\left[ \int_0^1 \frac{dt}{t} \bar{\eta}^\alpha \frac{\partial}{\partial \bar{v}^\alpha} \left( A^{-1} \mathcal{E} \right)(s, t\bar{\eta}, t\bar{v}) \right]. \tag{6.45}
\]
Substituting (6.45) into the action (6.20), we finally obtain the following expression for the action functional that gives rise to the variation (6.29)
\[
S_2 = \lambda^{-2(s-1)} \int_{\mathcal{M}^d} \langle \mathcal{E}(s, \bar{\eta}, \bar{v}) \rangle \wedge \mathcal{R}_{(p+1)} \wedge \mathcal{R}_{(p+1)} |0\rangle. \tag{6.46}
\]

It remains to find the function $\mathcal{E}(s, \bar{\eta}, \bar{v})$ that gives rise to the correct field equations of a HS gauge field to reconstruct its action by this formula.

## 7 Higher-spin equations of motion

In this section we find the function $\mathcal{E}$ that satisfies equation (6.30) and the decoupling conditions (5.18) and (5.19) which require that the variation (6.29) with respect to extra fields $w_{(\rho)}$ and irrelevant auxiliary fields $\omega_{(\rho)}$ should be identically zero. This condition fixes the dependence of $\mathcal{E}(s, \bar{\eta}, \bar{v})$ on $\bar{v}$ as follows
\[
\mathcal{E}(s, \bar{\eta}, \bar{v}) = \left( \bar{\eta}_1 \frac{\partial}{\partial \bar{v}_1} - \bar{\eta}_1 \frac{\partial}{\partial \bar{v}_1} \right) \mathcal{E}(s, \bar{\eta}) \bar{v}^{2(s-1)}, \tag{7.1}
\]
where

\[ \bar{v}^{2(s-1)} = \bar{v}_1 \cdots \bar{v}_{s-1} \bar{v}^1 \cdots \bar{v}^{s-1}. \]  

Indeed, substituting (7.1) into (6.29) one finds that the first term of (7.1) contains \((s-1)\) (maximal possible number) compensators contracted with \(\delta \Omega_{(p)}\) and therefore corresponds to the variation with respect to the physical field (cf. (4.45)). Analogously, the second term contains \((s-2)\) compensators contracted with all columns of \(\delta \Omega_{(p)}\) except for the first one and therefore corresponds to the variation with respect to the relevant auxiliary field (cf. (4.47)). (It is this place where the antisymmetric basis turns out to be most convenient). In the both cases, the remaining index in the first column of either \(R_{(p+1)}\) or \(\delta \Omega_{(p)}\) is contracted with the Levi-Civita tensor via \(\bar{\eta}_n\) or \(\bar{\eta}^1\). The relative coefficient in (7.1) is fixed by the symmetry property of \(H_{(6.23)}\).

The next step is to find such a function \(\tilde{E}(\bar{s}, \bar{n})\) that \(E(\bar{s}, \bar{n}, \bar{v})\) of (7.1) satisfies the weak closedness condition (6.30).

Using the antisymmetric basis, let us arrange tangent indices of the \(p\)-form field \(\Omega_{(p)}\) into vertical blocks \((m_I, \tilde{h}_I), I = 1, \ldots, N\) as explained in section 3.1.2. Let \(\mu_I\) be a number of the first column of the \(I\)-th vertical block. Since the length of the uppermost row of \(\Omega_{(p)}\) is \(s-1\), we have

\[ \sum_{I=1}^{N} m_I = s-1, \quad \mu_{I+1} - \mu_I = m_I, \quad I = 1, \ldots, N - 1. \]

Now we use the Young symmetries of the HS connections to choose a particular basis in the space of different terms that contribute to the action. Algebraically, this is equivalent to choosing a representative of \(\tilde{E}(\bar{s}, \bar{n})\) modulo weakly zero-terms. The important fact is that, by adding weakly zero terms in the variation (6.29), the function \(\tilde{E}\) in (7.1) can always be chosen in the form

\[ \tilde{E}(\bar{s}, \bar{n}) = \tilde{E}(\bar{u}, \bar{n}), \]  

where the operators

\[ \bar{u}_i = \bar{s}_i^i, \quad \bar{n}_I = \bar{\eta}_{\mu_I} \bar{\eta}^{\mu_I}, \]  

(\(no\ sums\ over\ repeated\ indices\) realize column-to-column contractions between \(R_{(p+1)}\) and \(\delta \Omega_{(p)}\) and contractions of the Levi-Civita tensor with the first columns of \(I\)-th vertical blocks of \(R_{(p+1)}\) and \(\delta \Omega_{(p)}\).)

It follows that the variables (7.4) enter the function (7.3) through the combinations of the form

\[ \left( \prod_{i=\mu_I}^{\mu_I+m_I-1} \bar{u}_i \right)^{\tilde{h}_I-1} \left( \frac{\bar{n}_I}{\bar{u}_{\mu_I}} \right)^k \]  

for some \(k \leq p\). In terms of tensors this means that the indices of, say, \(\delta \Omega_{(p)}\) in \(I\)-th vertical block of length \(m_I\) and height \(\tilde{h}_I\) are contracted as follows. One index from every column is contracted with the compensator (or with the Levi-Civita tensor for the second term in (7.1) and \(i = 1\)). The contractions of the remaining \((\tilde{h}_I - 1) \times m_I\) indices depend on whether they belong to the first column of \(I\)-th block or not. The
remaining indices of columns $\mu_I + 1, \ldots, \mu_I + m_I - 1$ are column-to-column contracted with the corresponding indices of $R_{(p)}$. The remaining indices of the first column of the $I$-th block $\mu_I$ are either contracted with $k_I$ indices of the Levi-Civita tensor or contracted with $\tilde{h}_I - 1 - k_I$ indices of the $\mu_I$-th column of $R_{(p+1)}$. Note that, due to (6.29), a function $E$ in the variation (6.29) should have $2p$ operators $\bar{\eta}$ to contract the remaining $2p$ indices of the Levi-Civita tensor. These indices are contracted with $k_I$ indices of the first columns of the $i$-th vertical block of both $R_{(p+1)}$ and $\delta \Omega_{(p)}$ so that $\sum_{I=1}^N k_I = p$.

As a result, the function $\tilde{E}$ has the form

$$
\tilde{E}(\bar{u}, \bar{n}) = \left( \prod_{i=1}^{s-1} (\bar{u}_i)^{\bar{h}_i-1} \right) \tilde{e}(t),
$$

$$
\tilde{e}(t) = \sum_{k_I \geq 0, I=1+ N \atop k_1 + \cdots + k_N = p} \rho(k_1, \ldots, k_N) t_1^{k_1} \cdots t_N^{k_N},
$$

where

$$
t_I = \frac{\bar{n}_I}{\bar{u}_{\mu_I}}, \quad I = 1, \ldots, N.
$$

The coefficients $\rho(k_1, \ldots, k_N)$ parameterize various types of contractions between $2p$ indices of the Levi-Civita tensor and those of $R_{(p+1)}$ and $\delta \Omega_{(p)}$.

For example, for the gauge field $\Omega_{(p)}$ with tangent indices described by a rectangular $o(d-1, 2)$ Young tableau, the function $\tilde{E}$ is

$$
\tilde{E} = \rho \prod_{i=1}^{s-1} (\bar{u}_i)^{\bar{h}_i-1} t_1^p
$$

with an arbitrary constant $\rho$. As can be easily checked, the corresponding function $E$ (7.1) satisfies equation (6.30).

In the rest of this section we show that the coefficient function $\rho(k_1, \ldots, k_N)$, is uniquely fixed by the compatibility condition (6.30) on the function $E$ and has the form

$$
\rho(k_1, \ldots, k_N) =
\frac{\rho \delta(p - \sum_{I=1}^N k_I)}{(k_1!)^2 (k_1 + 1) (h_1 - k_1 - 1)! \prod_{I=2}^N (k_I!)^2 (h_I - k_I - 1)!} \frac{\left( \vartheta(I) - \sum_{j=I}^N k_J \right)!}{\left( \vartheta(I) + m_I - \sum_{j=I+1}^N k_J \right)!},
$$

where $\rho$ is an arbitrary constant and

$$
\vartheta(I) = s - \mu_I - m_I + \tilde{h}_I - 1, \quad \vartheta(I) \geq p.
$$

The formula (7.10) is obtained as follows. Using the explicit form (6.25) of the
operator $\mathcal{Q}$, equation (6.30) gives for the function $\mathcal{E}$ of the form (7.1)

$$
\left( \bar{s}_i^j \left( \bar{\eta}_i \frac{\partial^2}{\partial \bar{v}_i \partial \bar{v}_1} - \bar{\eta}_1 \frac{\partial^2}{\partial \bar{v}_1 \partial \bar{v}_1} \right) \bar{v}^{2(s-1)} \frac{\partial}{\partial \bar{\eta}_j} - \bar{s}_i^j \left( \bar{\eta}_1 \frac{\partial^2}{\partial \bar{v}_i \partial \bar{v}_1} - \bar{\eta}_1 \frac{\partial^2}{\partial \bar{v}_1 \partial \bar{v}_1} \right) \bar{v}^{2(s-1)} \frac{\partial}{\partial \bar{\eta}_j} \right)
$$

$$
\bar{v}^2(s-1) \left( \bar{\Gamma}_i \left( \bar{\eta}_1 \frac{\partial^2}{\partial \bar{v}_i \partial \bar{v}_1} - \bar{\eta}_1 \frac{\partial^2}{\partial \bar{v}_1 \partial \bar{v}_1} \right) \bar{v}^{2(s-1)} \frac{\partial}{\partial \bar{\eta}_j} - \bar{s}_i^j \left( \bar{\eta}_1 \frac{\partial^2}{\partial \bar{v}_i \partial \bar{v}_1} - \bar{\eta}_1 \frac{\partial^2}{\partial \bar{v}_1 \partial \bar{v}_1} \right) \bar{v}^{2(s-1)} \frac{\partial}{\partial \bar{\eta}_j} \right) \bar{\mathcal{E}}(\bar{\eta}, \bar{n}) \sim 0.
$$

(7.12)

Naively, the operators $\bar{s}_i^j$ at $i \neq j$ spoil the column-to-column character of contractions encoded in the form of function (7.3). However, by using the Young symmetry properties (i.e., by adding proper weakly zero terms) it is possible to rewrite (7.12) in terms of the variables $\bar{u}$ and $\bar{n}$. Namely, after some algebra based on identities (A.1)-(A.4) given in Appendix, equation (7.12) takes the form

$$
\left( (\bar{N}_I + 2)(\bar{N}_I + m_I) \frac{\bar{u}_{\mu_I}}{\bar{U}_I} \frac{\partial}{\partial \bar{n}_I} + \sum_{J=2}^{I-1} (\bar{N}_J + 1)(\bar{N}_I + m_I) \frac{\bar{u}_{\mu_J}}{\bar{U}_J} \frac{\partial}{\partial \bar{n}_J} + \left( \sum_{J=I+1}^{n} \bar{N}_J \frac{\bar{U}_I}{U_I} - \frac{s - \mu_I - m_I}{U_I} - 1 \right) \bar{u}_{\mu_I} \frac{\partial}{\partial \bar{n}_I} \right) \tilde{\mathcal{E}}(\bar{\eta}, \bar{n}) = 0, \quad 2 \leq I \leq n,
$$

(7.13)

where

$$
\bar{N}_I = \bar{n}_I \frac{\partial}{\partial \bar{n}_I}, \quad \bar{U}_I = \bar{u}_{\mu_I} \frac{\partial}{\partial \bar{u}_{\mu_I}}, \quad \text{no summation}.
$$

(7.14)

Plugging the ansatz (7.6) into (7.13) one obtains the equation

$$
\sum_{J=2}^{I-1} A_J z_J + B_I z_I = -z_1, \quad I = 2, \ldots, N,
$$

(7.15)

where we use notations

$$
\bar{T}_I = \bar{t}_I \frac{\partial}{\partial \bar{t}_I},
$$

(7.16)

$$
- \frac{(\bar{T}_1 + 2)}{(h_1 - \bar{T}_1 - 1)} \frac{\partial}{\partial \bar{t}_1} \tilde{e}(t) = z_1, \quad \frac{\partial}{\partial \bar{t}_I} \tilde{e}(t) = z_I, \quad I = 2, \ldots, N,
$$

(7.17)

$$
A_I = \frac{(\bar{T}_I + 1)}{(h_I - \bar{T}_I - 1)}, \quad B_I = A_I \left( \sum_{J=1}^{N} \frac{T_J - \partial(I)}{\bar{T}_I + m_I} \right).
$$

(7.18)

Equation (7.15), which has the triangle form, implies

$$
z_I = (-)^{I+1} \frac{\prod_{J=2}^{I-1} (A_J - B_J)}{\prod_{J=2}^{I} B_J} z_1.
$$

(7.19)
This gives the following equations on the function $\tilde{e}(t)$:

$$\frac{\partial}{\partial t} \tilde{e}(t) = \frac{(\tilde{T}_I - \tilde{h}_I + 1)(\tilde{T}_I + m_I)(\tilde{T}_I + 2)}{(h_I - \tilde{T}_1 - 1)(\tilde{T}_I + 1)} \frac{\prod_{I=2}^{I-1} \left( \sum_{K=J+1}^{N} \tilde{T}_K - \vartheta(J) - m_J \right)}{\prod_{J=2}^{I} \left( \sum_{K=J}^{N} \tilde{T}_K - \vartheta(J) \right)} \frac{\partial}{\partial t} \tilde{e}(t),$$

(7.20)

for $I = 2, \ldots, N$.

The substitution of $\tilde{e}(t)$ (7.7) into (7.20) gives the following equations on the coefficient function $\rho(k_1, \ldots, k_N)$

$$\rho(k_1 - 1, \ldots, k_I + 1, \ldots, k_N) = G_I(k_1, ..., k_I, ..., k_N) \rho(k_1, \ldots, k_I, \ldots, k_N), \quad I = 2 \div N,$$

(7.21)

$$\rho(p, 0, \ldots, 0) = \rho,$$

(7.22)

where $\rho$ is an arbitrary constant and

$$G_I(k_1, ..., k_n) = \frac{(k_I - \tilde{h}_I + 1)(k_I + m_I) k_I(k_I + 1)}{(k_I + 1)^2} \frac{\prod_{J=2}^{I-1} \left( \sum_{K=J+1}^{N} k_K - \vartheta(J) - m_J \right)}{\prod_{J=2}^{I} \left( \sum_{K=J}^{N} k_K - \vartheta(J) \right)},$$

(7.23)

$$k_1 \geq 1, \quad k_I \geq 0, \quad I = 2 \div N, \quad k_1 + \ldots + k_N = p,$$

(7.24)

The equations (7.21)-(7.24) give

$$\rho(k_1, ..., k_N) = \rho \prod_{I=2}^{N} \rho_I(p - \sum_{J=I+1}^{N} k_J, k_2, ..., k_I, 0, ..., 0),$$

(7.25)

where $\rho_I(p - \sum_{J=I+1}^{N} k_J, k_2, ..., k_I, 0, ..., 0)$ is a solution of (7.21) with fixed $I$ and $k_J = 0$ at $J > I$. This gives the final result (7.10). It is elementary to check that (7.10) does solve the system (7.21)-(7.24).

To summarize, the expressions (7.1), (7.6), (7.7), (7.10) determine the function $\mathcal{E}$ that satisfies the weak $Q$-closedness equation (6.30) and gives rise to the action (6.46) satisfying the decoupling conditions (5.18), (5.19). The constructed function $\mathcal{E}$ is (weakly) unique up to a normalization factor $\rho$.

8 Gauge symmetry enhancement at $\lambda = 0$

As explained in section 5.2 to describe correctly dynamics of a mixed-symmetry gauge field, the $AdS_d$ HS action should admit additional gauge symmetries (5.10) in the flat limit $\lambda = 0$. In this section we show that the $AdS_d$ action (6.46) constructed with the function $\mathcal{E}$ (7.1), (7.6), (7.7), (7.10) indeed exhibits the flat space gauge symmetry enhancement with traceless gauge parameters $S_I, I = 1, \ldots, N$. 43
To analyze the flat limit we choose the compensator in the standard form $V^A = \delta^A_d$ so that $A = (a, d + 1)$ where $a$ is a $O(d - 1, 1)$ vector index while value $d + 1$ corresponds to the last $(d + 1)$-th direction. Choosing Cartesian coordinate system with background frame field $E^a_0 = \delta^a_0$ and Lorentz spin connection $\omega^a_{\mu} = 0$, we replace the Lorentz covariant derivative $D$ by the flat derivative $\partial$. In the subsequent analysis we identify the world and tangent Lorentz indices with the help of the background frame field $\delta^a_0$.

### 8.1 Extended Fock space notations

Let us introduce new fermionic oscillators $\kappa^a_\alpha$, $\tilde{\kappa}^a_\alpha$ and $\kappa^a_\alpha$, $\bar{\kappa}^a_\alpha$, which anticommute with the previously introduced oscillators $\psi$, $\bar{\psi}$ (6.1) and $\theta$, $\bar{\theta}$ (6.2), and satisfy the anticommutation relations

$$\{\bar{\kappa}^a_\alpha, \kappa^b_\beta\} = \eta^{ab}_{\alpha\beta}, \quad \{\bar{\kappa}^a_\alpha, \kappa^b_\beta\} = \eta^{ab}_{\alpha\beta},$$

with other anticommutators being zero. The left and right Fock vacua are defined by (6.3)-(6.5) and

$$\langle 0 | \kappa^a_\alpha = 0, \quad \langle 0 | \kappa^a_\alpha = 0,$$

$$\bar{\kappa}^a_\alpha | 0 \rangle = 0, \quad \bar{\kappa}^a_\alpha | 0 \rangle = 0.$$

We contract new oscillators either with the world indices of forms or with the $s$-th column of the metric-like field $\Phi$, flat gauge parameter $S_I$, etc... For example, the operators $\mathcal{A}(p)$, $\mathcal{B}(p)$ associated with the $p$-form $A(p)$ now read as

$$\mathcal{A}(p) = A^a_{1} h_{1} \ldots a_{s-1} h_{s-1}; m[p] \left(\psi^{a}_{1} \bar{\psi}^{1} \ldots \left(\psi^{a}_{s-1} \bar{\psi}^{s-1}\right) h_{s-1}(\kappa^{m}_{\cdot})^{p}\right),$$

$$\mathcal{B}(p) = A^a_{1} h_{1} \ldots a_{s-1} h_{s-1}; m[p] \left(\bar{\psi}^{a}_{1} \psi^{1} \ldots \left(\bar{\psi}^{a}_{s-1} \psi^{s-1}\right) h_{s-1}(\kappa^{m}_{\cdot})^{p}\right).$$

For the metric-like fields the associated operators $\tilde{\Phi}$, $\Phi$ read as

$$\tilde{\Phi} = \Phi^a_{1} h_{1} \ldots a_{s-1} h_{s-1}; a_{s} p \left(\bar{\psi}^{a}_{1} \psi^{1} \ldots \left(\bar{\psi}^{a}_{s-1} \psi^{s-1}\right) h_{s-1}(\kappa^{a}_{\cdot})^{p}\right),$$

$$\Phi = \Phi^a_{1} h_{1} \ldots a_{s-1} h_{s-1}; a_{s} p \left(\psi^{a}_{1} \bar{\psi}^{1} \ldots \left(\psi^{a}_{s-1} \bar{\psi}^{s-1}\right) h_{s-1}(\kappa^{a}_{\cdot})^{p}\right).$$

We extend the sets of operators $s_{\alpha \beta}$ and $l_{\alpha \beta}$ (6.10) by the operators

$$s^i_{\beta} = \bar{\psi}^i_{\beta} \bar{\kappa}^a_{\beta} \eta^{ab}, \quad \bar{s}^i_{\alpha} = \bar{\bar{\psi}}^i_{\alpha} \bar{\kappa}^a_{\alpha} \eta^{ab},$$

$$\bar{s}^i_{\alpha} = \bar{\bar{\psi}}^i_{\alpha} \bar{\kappa}^a_{\alpha}, \quad \bar{\bar{s}}^i_{\alpha} = \bar{\psi}^i_{\alpha} \kappa^a_{\alpha}.$$

$$l^i_{\beta} = \psi^i_{\beta} \bar{\kappa}^a_{\beta}, \quad \bar{l}^i_{\alpha} = \bar{\psi}^i_{\alpha} \bar{\kappa}^a_{\alpha},$$

$$l^i_{\alpha} = \kappa^a_{\alpha} \psi^i_{\alpha}, \quad \bar{l}^i_{\alpha} = \kappa^a_{\alpha} \bar{\psi}^i_{\alpha}.$$
and introduce derivative operators

\[
D_i = \psi^i_a \partial_a, \quad D^i = \psi^i_a \partial^a, \quad \bar{D}_i = \bar{\psi}^i_a \partial_a, \quad \bar{D}^i = \bar{\psi}^i_a \partial^a, \quad D_\bullet = \kappa_a \partial_a, \quad D^\bullet = \kappa^a \partial^a, \quad (8.10)
\]

Finally let \( \nu_I \) denote the number of the last column of the \( I \)-th vertical block. Recall that by \( \mu_I \) we denote the number of the first column in the \( I \)-th vertical block. Thus

\[
\nu_I = \mu_{I+1} - 1, \quad I = 1, \ldots, N - 1. \quad (8.12)
\]

### 8.2 Variation of the flat higher-spin action

Taking into account the form of the function \( \mathcal{E}(\tilde{\mathcal{E}}) \), the variation of \( AdS_d \) HS action constructed by the formula (6.46) is

\[
\delta \mathcal{S}_2 \propto \int_{\mathcal{M}^d} \left( \langle 0 | (\wedge E_0)^{d-2p-1} \chi \tilde{\mathcal{E}}(\bar{u}, \bar{\eta}) \bar{\eta}_1 \right. \left. \wedge \hat{\mathcal{R}}_{(p+1)} \wedge \delta \hat{\mathcal{E}}_{(p)} | 0 \rangle + \right.
\]

\[
+ (-)^s \langle 0 | (\wedge E_0)^{d-2p-1} \chi \tilde{\mathcal{E}}(\bar{u}, \bar{\eta}) \bar{\eta}_1 \wedge \hat{\mathcal{R}}_{(p+1)} \wedge \delta \hat{\mathcal{E}}_{(p)} | 0 \rangle \right). \quad (8.13)
\]

Here Lorentz-covariant \( p \)-forms

\[
\lambda^{s-1} \hat{\mathcal{E}}_{(p)} | 0 \rangle = \bar{v}^1 \cdots \bar{v}^{s-1} \bar{\Omega}_s | 0 \rangle,
\]

\[
\lambda^{s-2} \hat{\mathcal{E}}_{(p)} | 0 \rangle = (1 - v_1 \bar{v}^1) \bar{v}^2 \cdots \bar{v}^{s-1} \bar{\Omega}_s | 0 \rangle \quad (8.14)
\]

are the physical and relevant auxiliary fields (cf. (4.43), (4.47)) and Lorentz-covariant \( (p+1) \)-forms

\[
\lambda^{s-1} \hat{\mathcal{R}}_{(p+1)} | 0 \rangle = \bar{v}^1 \cdots \bar{v}_{s-1} \hat{\mathcal{R}}_{(p+1)} | 0 \rangle,
\]

\[
\lambda^{s-2} \hat{\mathcal{R}}_{(p+1)} | 0 \rangle = (1 - v_1 \bar{v}_1) \bar{v}_2 \cdots \bar{v}_{s-1} \hat{\mathcal{R}}_{(p+1)} | 0 \rangle \quad (8.15)
\]

are the Lorentz components of the curvature \( \hat{R}_{(p+1)} \) associated with the physical and relevant auxiliary fields, respectively. Note that the factor of \( \lambda^{2s-1} \) in the variation (6.29) is cancelled by those of \( \hat{R}_{(p+1)} \wedge \hat{\omega}_{(p)} \) and \( \hat{\mathcal{R}}_{(p+1)} \wedge \hat{\mathcal{E}}_{(p)} \), making the flat limit of the variation well defined.

The variation over the relevant auxiliary field \( \omega_{(p)} \) gives rise to the equation of motion, that can be written in the form

\[
p_{a_1 [\bar{h}_{1-1}, \ldots, a_{s-1} [\bar{h}_{s-1-1}, m[p+1]]} = C_{a_1 [\bar{h}_{1-1}, \ldots, a_{s-1} [\bar{h}_{s-1-1}, m[p+1], (8.16)
\]

where the tensor \( C_{a_1 [\bar{h}_{1-1}, \ldots, a_{s-1} [\bar{h}_{s-1-1}, m[p+1]](x) \) is either zero if \( \bar{h}_{s-1} = p+1 \) or equals to the primary Weyl tensor if \( \bar{h}_{s-1} > p+1 \) \( \text{[I]} \) of Young symmetry type

\[
Y_{(d-1,1)}(s, \ldots, s, s_{p+2}, \ldots, s_\nu). \quad (8.17)
\]
The equation (8.16) expresses the relevant auxiliary field \( \omega(p) \) in terms of first derivative of the physical field \( e(p) \) up to pure gauge degrees of freedom. Using the extended Fock notations, equation (8.16) can be written as

\[
\hat{r}_{(p+1)}|0\rangle = \hat{C}|0\rangle, \tag{8.18}
\]

where the operator \( \hat{C} \) is related to the tensor \( C \) by (8.1). It is convenient to use the 1.5-formalism with the auxiliary field expressed implicitly by virtue of its equation of motion (8.16) through the physical field.

To check the invariance under the flat gauge symmetries one needs the explicit form of the physical field transformations induced by the corresponding transformations (5.10) of the metric-like field \( \Phi \). In this paper we assume that the flat gauge parameters \( S_I, I = 1, \ldots, N - 1 \) are traceless. Note that the gauge parameter \( S_N \) associated with the vertical block of the minimal height \( p \) corresponds to the physical AdS\(_d\) gauge parameter \( \varepsilon(p-1) \) (8.55). It follows that flat gauge transformations (5.10) of the metric-like field \( \Phi \) result from the following transformations of the physical field \( e(p) \)

\[
\delta_I \hat{e}(p)|0\rangle = \mathcal{P}_e \left( D^{\nu_I} \hat{S}_I \right)|0\rangle, \tag{8.19}
\]

where the operator \( \mathcal{P}_e = \mathcal{P}_e(\psi, \bar{\psi}) \) imposes the necessary trace and Young symmetry conditions, projecting to the traceless Young tableau associated with tangent indices of the field \( e(p) \). The explicit form of \( \mathcal{P}_e \) is complicated but, fortunately, it is not needed for our analysis.

Substituting (8.19) into (8.13) and neglecting terms with the variation of the relevant auxiliary field by using the 1.5-order formalism, one obtains for the gauge variation with the parameters \( S_I \)

\[
\delta_I \hat{S}^{\text{flat}}_2 = \int_{\mathcal{M}^d} \left( \alpha^1_I \Delta^1_I + \alpha^2_I \Delta^2_I + \alpha^3_I \Delta^3_I \right) \frac{1}{p} \langle 0|(E_0)^{d}\mathcal{F}_I(\bar{u}) \left( \bar{D}_I \bar{D}_{\nu_I} \hat{u}_\bullet \right) \hat{\omega}(p) \hat{S}_I |0\rangle, \tag{8.20}
\]

where \( \alpha^1,^2,^3_I \) are some coefficients determined by the form of the projector \( \mathcal{P}_e \) and function \( \mathcal{E} \), and

\[
\Delta^1_I = \frac{1}{p} \langle 0|(E_0)^{d}\mathcal{F}_I(\bar{u}) \left( \bar{D}_I \bar{D}_{\nu_I} \right) \hat{\omega}(p) \hat{S}_I |0\rangle, \tag{8.21}
\]

\[
\Delta^2_I = \langle 0|(E_0)^{d}\mathcal{F}_I(\bar{u}) \left( \bar{D}_I \bar{D}_{\nu_I} \bar{S}_\bullet \right) \hat{\omega}(p) \hat{S}_I |0\rangle, \tag{8.21}
\]

\[
\Delta^3_I = \langle 0|(E_0)^{d}\mathcal{F}_I(\bar{u}) \left( \bar{D}_{\nu_I} \bar{D}_I \bar{S}_\bullet \right) \hat{\omega}(p) \hat{S}_I |0\rangle \tag{8.22}
\]

with

\[
\mathcal{F}_I(\bar{u}) = \left( \prod_{k=1, k \neq \nu_I}^{s-1} \left( \bar{u}_{k} \right)^{h_{k-1}} \left( \bar{u}_{\nu_I} \right)^{h_{\nu_I} - 2} \left( \bar{u}_\bullet \right)^{p-1} \right). \tag{8.22}
\]
The operators $\Delta_1^{1,2,3}$ represent various types of contractions between the relevant auxiliary field $\omega(p)$, traceless flat gauge parameter $S_I$, and two flat space-time derivatives $\partial$. The important fact is that there are only three independent contractions of the type $\partial \partial \omega(p) S_I$.

Indeed relevant auxiliary field $\omega(p)$ has one additional index in the first column and one another index in the $\nu_I$-th column compared with those of the gauge parameter $S_I$. The number of world indices of $\omega(p)$ equals the number of indices in $s$-th column of $S_I$, $I \neq N$. Due to the Young symmetry and tracelessness conditions of $\omega(p)$ and $S_I$ all possible contractions of the type $\partial \partial \omega(p) S_I$ reduce to

$$\partial^{f_1} \partial^{f_2} \omega(p)_{g_1}^{a_1 \ldots a_{\nu_I} \ldots \ldots f_3 m \ldots} S_I^{a_1 \ldots a_{\nu_I} \ldots \ldots g_3 m \ldots},$$

where $\ldots$ denotes column-to-column contractions and the indices $f$ and $g$ should be contracted in all possible ways. Thus, there are three types of contractions that are represented in Fock space notations by $\Delta_1^{1,2,3}$. Using the ambiguity in adding total derivatives without loss of generality we can assume that flat derivatives act on $\omega(p)$.

### 8.3 Proof of invariance

The straightforward check of the invariance of the flat action, i.e. that $\delta I S_{\text{flat}}^2 = 0$, is complicated requiring explicit expressions for the relevant auxiliary field $\omega(p) = \omega(p) (\partial e(p))$ and the coefficients $\alpha_1^{1,2,3}$. Fortunately, there is a simpler proof using some relations among the operators $\Delta_1^{1,2,3}$ and the coefficients $\alpha_1^{1,2,3}$ which result from Bianchi identities (6.14) and the manifest HS gauge symmetries (6.13).

Consider the flat limit of the Bianchi identity for the physical curvature $r(p+1)$

$$\text{dr}_{(p+1)} + \frac{\sigma_\perp^{(1)}}{\partial} \mathcal{R}_{(p+1)} + \ldots = 0 \quad (8.24)$$

with

$$r_{(p+1)} = \text{de}_{(p)} + \frac{\sigma_\perp^{(1)}}{\partial} \omega_{(p)} + \cdot \cdot \cdot, \quad \mathcal{R}_{(p+1)} = \text{d}\omega_{(p)} + \cdot \cdot \cdot. \quad (8.25)$$

Here $\sigma_\perp^{(1)}$ is the operator that decreases a number of Lorentz indices of the first vertical block (4.51) by one. Dots denote the contributions of the irrelevant auxiliary fields and extra fields that can be discarded in the variation of the action by virtue of the decoupling conditions. In the extended Fock space notations the equations (8.24) and (8.25) read as

$$D^* \hat{r}_{(p+1)}|0\rangle + \left( \mathcal{P} e l^* \mathcal{R}_{(p+1)} + \cdot \cdot \cdot \right) |0\rangle = 0, \quad (8.26)$$

$$\hat{r}_{(p+1)}|0\rangle = \left( D^* \hat{e}_{(p)} + \mathcal{P} e l^* \hat{\omega}_{(p)} + \cdot \cdot \cdot \right) |0\rangle, \quad \mathcal{R}_{(p)}|0\rangle = \left( D^* \hat{\omega}_{(p)} + \cdot \cdot \cdot \right) |0\rangle. \quad (8.27)$$

Taking into account the equation of motion (8.18) which is a constraint on the relevant auxiliary field, i.e. applying the 1.5-order formalism, we obtain for (8.26)

$$D^* \hat{C}|0\rangle + \mathcal{P} e l^* \mathcal{R}_{(p+1)}|0\rangle = 0. \quad (8.28)$$
The application of the operator \( l^i \mathcal{P}_e l^{i*} \), \( i, j = 1, \ldots, s - 1 \) to the left-hand-side of (8.28) annihilates the term with \( \hat{C} \).

Indeed, at least one of the operators \( l \) necessarily acts on \( \hat{C} \) in \( l^i \mathcal{P}_e l^{i*} \mathcal{D}^\bullet \hat{C} |0\rangle \) antisymmetrizing one index of the \( s \)-th column with all indices of the \( i \)-th or \( j \)-th column that is zero by the Young symmetry properties of \( C \) (8.17). It is important that the trace properties of the primary Weyl tensor are irrelevant in this analysis.

Taking into account (8.27) one obtains the relation

\[
l^i l^j \mathcal{P}_e l^{i*} \mathcal{D}^\bullet \hat{\omega}(p) |0\rangle = 0. \tag{8.29}
\]

The derivative \( \mathcal{D}^\bullet \) and the projector \( \mathcal{P}_e \) commute to each other since they are built of oscillators of different types, namely \( \kappa^\bullet \) and \( \psi, \bar{\psi} \), respectively. Multiplying (8.29) by the operator \( \hat{S}_I \) associated with the gauge parameter \( S_I \), which is built of the oscillators that (anti)commute to those in (8.29), one finally obtains that

\[
l^i l^j \mathcal{D}^\bullet \mathcal{P}_e l^{i*} \hat{\omega}(p) \hat{S}_I |0\rangle = 0. \tag{8.30}
\]

The identity (8.29) expresses specificities of the form of the expression of the relevant auxiliary field in terms of the physical field \( \omega(p) (e(p)) \). From (8.30) we now derive some useful relations on the operators \( \Delta_I^{1,2,3} \).

Let us first consider the case where either \( I \neq 1 \) or \( I = 1, m_1 > 1 \) (i.e. the first vertical block consists of more than one column). Acting on (8.30) with \( i = j = 1 \) by the operators

\[
\mathcal{F}(\bar{u}) \mathcal{D}^\bullet \bar{s}_1, \mathcal{F}(\bar{u}) \mathcal{D}^\bullet \bar{s}_1, \tag{8.31}
\]

and

\[
\mathcal{F}(\bar{u}) \bar{u} \mathcal{D}_1 \bar{s}_1, \tag{8.32}
\]

one finds that

\[
\left( \pi_1^{11} \Delta_I^1 + \pi_1^{12} \Delta_I^2 + \pi_1^{13} \Delta_I^3 \right) = 0, \tag{8.33}
\]

\[
\left( \pi_1^{21} \Delta_I^1 + \pi_1^{22} \Delta_I^2 + \pi_1^{23} \Delta_I^3 \right) = 0 \tag{8.34}
\]

with some sets of coefficients \( \pi_1^1 = (\pi_1^{11}, \pi_1^{12}, \pi_1^{13}) \) and \( \pi_1^2 = (\pi_1^{21}, \pi_1^{22}, \pi_1^{23}) \).

Let us show that vectors \( \pi_1^1 \) and \( \pi_1^2 \) are linearly independent, namely, that \( \pi_1^{11} = 0 \) while \( \pi_1^{21} \neq 0 \). Indeed, the expression resulting from the combination of (8.31) and (8.30) has the form

\[
\langle 0 | (E_0)^d \chi \frac{\mathcal{F}(\bar{u})}{\bar{u}_1} \mathcal{D}^\bullet \bar{s}_1 \left( \bar{s}_1 \mathcal{D}_{\nu_1} + \mathcal{D}_1 \bar{s}_{\nu_1} \right) \mathcal{P}_e l^{i*} \hat{\omega}(p) \hat{S}_I |0\rangle = 0, \tag{8.35}
\]

One observes that it contains the derivative \( \mathcal{D}^\bullet \) while the operator \( \Delta_I^1 \) does not. Recall that contrary to the operator \( \Delta_I^1 \) the operators \( \Delta_I^2 \) and \( \Delta_I^3 \) contain the derivative \( \mathcal{D}^\bullet \) (8.21). Therefore, the operator \( \Delta_I^1 \) cannot contribute to (8.34), i.e. \( \pi_1^{11} = 0 \). One can show that (8.34) \( \pi_1^{21} \neq 0 \) and the coefficients \( \pi_1^{12} \) and \( \pi_1^{13} \) are not both zero.

As a result, it follows that operators \( \Delta_I^{1,2,3} \) are proportional to each other

\[
\Delta_I^1 \propto \Delta_I^2 \propto \Delta_I^3 \tag{8.36}
\]
so that the variation (8.20) has the form
\[ \delta_1 \mathcal{S}_2^{\text{flat}} = \gamma \int_{\mathcal{M}^d} \Delta_1^1 \] (8.37)
with some coefficient \( \gamma \).

Now let us take into account that the variation (8.37) is invariant under the gauge transformation
\[ \delta \hat{\omega}_{(p)} |0\rangle = D^\bullet \xi_{(p-1)} |0\rangle \] (8.38)
with the \((p-1)\)-form gauge parameter \( \xi_{(p-1)} \) that corresponds to the relevant auxiliary field. Substituting (8.38) into (8.37) one finds that the coefficient \( \gamma \) in (8.37) is zero thus completing the proof of the fact that the flat limit of the constructed action is invariant under the additional flat gauge symmetries in the case \( I \neq 1 \) or \( I = 1, m_1 \neq 1 \).

Let us note that from the invariance of the Bianchi identity (6.14) under the gauge transformation (8.38) it follows that
\[ \Delta_1^1 = \Delta_1^2 = \Delta_1^3. \] (8.39)

Actually, taking into account obvious (anti)commutation relations of the operators \( \bar{s}, l, \bar{D} \) and \( D \) one finds that an expression of the form
\[ \beta_1^1 \Delta_1^1 + \beta_2^2 \Delta_1^2 + \beta_3^3 \Delta_1^3 \] (8.40)
is invariant under (8.38) provided that
\[ \beta_1^1 + \beta_2^2 + \beta_3^3 = 0. \] (8.41)
Along with (8.33), (8.34) this implies (8.39).

The case of \( I = 1, \ m_1 = 1 \) is special since the operators (8.31) and (8.32) vanish in this case. From the definition of \( \Delta_1^{1,2,3} \) one finds that in this case
\[ \Delta_1^1 = 0, \quad \Delta_1^2 = -\Delta_1^3. \] (8.42)

Acting on (8.30) with \( i = j = 2 \) by the operator
\[ \mathcal{F}(\bar{u}) \bar{D}^\bullet \bar{s}_1 \bar{l}_2 \bar{s}_{12} \] (8.43)
one finds that
\[ \frac{\mathcal{F}(\bar{u})}{\bar{u}_1} \bar{D}^\bullet \bar{D}_1 \bar{s}_1 \bar{s}_1 \bar{l}_1 \bar{P}_{e(p)} l^\bullet \hat{\omega}_{(p)} \bar{S}_1 |0\rangle = 0, \] (8.44)
from where it follows that
\[ \Delta_1^1 = \Delta_1^2 = \Delta_1^3 = 0. \] (8.45)
This completes the proof of the invariance of the flat HS action under enhanced flat gauge symmetries.
9 Conclusions and outlook

The description of massless fields in $(A)dS$ spacetime of any dimension in terms of gauge connections referred to as frame-like approach is extended to generic bosonic massless fields. Let us summarize some of the features of the frame-like formulation of the bosonic massless HS field dynamics considered in this paper.

- A given HS massless field of any symmetry type propagating on the $AdS_d$ background of any dimension $d$ is described as $p$-form gauge field

$$\Omega^I_{(p)}(x)$$

that takes values in an appropriate finite-dimensional $o(d-1,2)$-module $I$. The module $I$ is some traceless $o(d-1,2)$ tensor representation described by a Young tableau of the form uniquely defined by a spin of a HS field.

- With the $p$-form gauge field $\Omega^I_{(p)}(x)$ one associates gauge-invariant field strength (curvature) in a standard fashion as $R^I_{(p+1)}(x) = D_0 \Omega^I_{(p)}(x)$, where $D_0$ is the $o(d-1,2)$ covariant derivative that describes $AdS_d$ background via the flatness condition $D_0^2 = 0$. The HS gauge symmetries are defined as $\delta \Omega^I_{(p)}(x) = D_0 \xi^I_{(p-1)}(x)$.

- Being decomposed into $o(d-1,1) \subset o(d-1,2)$ components of the representation $I$, the $p$-form gauge field reduces to the set of various symmetry type Lorentz-covariant fields that have different dynamical roles. In particular, the Lorentz-covariant field with a minimal number of Lorentz tangent indices is the physical field generalizing the frame field in gravity.

- The manifestly gauge invariant free HS action functional is constructed in the form of specific bilinear combinations of the curvatures

$$S_2 = \int R_{(p+1)} R_{(p+1)} .$$

The coefficients in the action are fixed by the decoupling conditions guaranteeing that the action is free of higher derivatives and describes the correct number of degrees of freedom associated with the physical HS field.

- The flat limit of the $AdS_d$ theory exhibits the required flat gauge symmetry enhancement thus providing the consistency of the HS field dynamics both in $(A)dS$ and in Minkowski space.

- The reformulation of the action in terms of an auxiliary fermionic Fock space allows us to reduce the problem of reconstruction of a free field action to the analysis of an appropriate differential complex, with the derivation $Q$ associated with the variation of the action.
Among possible directions for the future research let us mention the following:

- the frame-like Lagrangian formulation for fermionic HS massless fields;
- the frame-like Lagrangian formulation for the partially massless fields of general symmetry type along the lines of [28];
- the frame-like Lagrangian formulation for \((A)dS_{d+1}\) (anti)selfdual bosonic and fermionic HS massless and partially massless fields;
- the unfolded formulation of HS (partially) massless fields of mixed-symmetry type by working out a structure of the infinite-dimensional modules associated with the generalized Weyl tensors.

Accomplishment of this programme will provide the full identification of connections, that take values in different finite-dimensional representations of the \((A)dS\) algebra \((o(d-1,2))o(d,1)\), with the different types of relativistic fields. Because the frame-like geometric approach makes global and local HS symmetries manifest, it plays a fundamental role for understanding a structure of consistent global HS symmetries and, at the later stage, of nonlinear HS gauge theories. Finally, a very interesting direction that may be important for understanding fundamental symmetries of string theory would be to extend the proposed formulation to massive HS fields.

**Appendix**

Consider an expression of the form

\[
\langle 0 | (\wedge E_0)^{d-m-n} \chi M \wedge \hat{A}_{(m)} \wedge \hat{B}_{(n)} | 0 \rangle
\]

defined with respect to any function \(M = M(\bar{u}, \bar{\eta}, \bar{v})\) and any \(\hat{A}_{(m)}, \hat{B}_{(n)}\) (for definiteness, let these forms have tangent indices described by Young tableau \((4.35)\)). Then, up to weakly zero terms, the following identity holds

\[
\bar{s}_i^j M(\bar{u}, \bar{\eta}, \bar{v}) \sim \delta_{ij} \bar{u}_i M(\bar{u}, \bar{\eta}, \bar{v})
- \theta(j - i - 1)(\bar{u}_i \frac{\partial}{\partial \bar{u}_i})^{-1}(\bar{\eta}^j \frac{\partial}{\partial \bar{\eta}^j} + \bar{v}^j \frac{\partial}{\partial \bar{v}^j}) \bar{u}_i M(\bar{u}, \bar{\eta}, \bar{v})
+ \theta(i - j - 1)(\bar{u}_j \frac{\partial}{\partial \bar{u}_j})^{-1}(\bar{\eta}^i \frac{\partial}{\partial \bar{\eta}^i} + \bar{v}^i \frac{\partial}{\partial \bar{v}^i}) \bar{u}_j M(\bar{u}, \bar{\eta}, \bar{v})
\]

(A.1)

where

\[
\theta(n) = \begin{cases} 
1, & n \geq 0, \\
0, & n < 0.
\end{cases}
\]
Let the function $M$ have the special form

$$M(s, \bar{\eta}, \bar{v}) = M(\bar{u}, n) \ \bar{\eta}_k \ \bar{\eta}_k \ \partial_{\bar{u}_k} (\bar{v}_2 \ldots \bar{v}_{s-1} \ \bar{v}^1 \ldots \bar{v}^{s-1}) ,$$

where $M(\bar{u}, n)$ is arbitrary. Then, up to weakly zero terms, the following identities hold within the interval $\mu_I < k < \mu_{I+1}$ (i.e., for all $k$ enumerating columns of the $I$-th rectangular block with exception of the first column):

\begin{align}
M(\bar{u}, n) \ \bar{\eta}_k \ \partial_{\bar{v}_k} (\bar{v}_2 \ldots \bar{v}_{s-1} \ \bar{v}^1 \ldots \bar{v}^{s-1}) & \sim \left( \frac{1}{N_I + 1} M(\bar{u}, n) \right) \ \bar{\eta}_k \ \bar{\eta}_{\mu_I} \ \partial_{\bar{v}_{\mu_I}} (\bar{v}_2 \ldots \bar{v}_{s-1} \ \bar{v}^1 \ldots \bar{v}^{s-1}) , \\
M(\bar{u}, n) \ \bar{\eta}_k \ \partial_{\bar{v}_k} (\bar{v}_1 \ldots \bar{v}_{s-1} \ \bar{v}^2 \ldots \bar{v}^{s-1}) & \sim \left( \frac{1}{N_I + 1} M(\bar{u}, n) \right) \ \bar{\eta}_k \ \bar{\eta}_{\mu_I} \ \partial_{\bar{v}_{\mu_I}} (\bar{v}_1 \ldots \bar{v}_{s-1} \ \bar{v}^2 \ldots \bar{v}^{s-1}) , \\
M(\bar{u}, n) \ \bar{\eta}_k \ \partial_{\bar{v}_k} (\bar{v}_2 \ldots \bar{v}_{s-1} \ \bar{v}^1 \ldots \bar{v}^{s-1}) & \sim \left( \frac{1}{N_I + 1} M(\bar{u}, n) \right) \ \bar{\eta}_k \ \bar{\eta}_{\mu_I} \ \partial_{\bar{v}_{\mu_I}} (\bar{v}_2 \ldots \bar{v}_{s-1} \ \bar{v}^1 \ldots \bar{v}^{s-1}) .
\end{align}

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Erratum: Frame-like formulation for free mixed-symmetry bosonic massless higher-spin fields in $AdS_d$

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In this paper and in hep-th/0501108 the Lagrangian formulation of mixed-symmetry field dynamics in $AdS_d$ space-time was proposed. Our construction contained the conjecture expressed by the formula (8.16) of Section 8 that torsion-like components of the linearized curvature are zero. However, recently, it was shown that for fields of general type this conjecture is not true [1]. Hence, we cannot claim that the action proposed in this paper works properly for general massless mixed symmetry fields. At this stage we can only claim that it does work for the particular class of mixed symmetry fields described by rectangular Young diagrams of arbitrary length $s$ and height $h_1 \leq [(d - 1)/2]$ and fields with spins described by Young diagrams composed of two horizontal rectangular blocks, the upper block is of arbitrary length $s$ and height $h_1$, the second block is a column of height $h_2$ provided $h_1 + h_2 \leq [(d - 1)/2]$. In the first case equation (8.16) is true because torsion-like components of the linearized curvature are absent. In the second case these components are non-zero but nonetheless they are consistently eliminated by virtue of Bianchi identities. The analysis of the general case is more complicated and requires further investigation.

Let us stress that despite the aforementioned problems, the set of gauge fields, their gauge symmetries and linearized curvatures suggested in our paper provide correct setting for the analysis of on-shell higher spin dynamics, as was checked in particular in [2, 3].

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