A study about teaching quadratic functions using mathematical models and free software

T. V. Nepomucena, A. C. da Silva, D. F. Jardim, J. M. da Silva
Institute for Science, Engineering and Technology. Federal University of Jequitinhonha and Mucuri Valleys. Campus Mucuri. Rua do Cruzeiro, 01. Jardim São Paulo. Teófilo Otoni. Minas Gerais. Brazil
E-mail: thamaranepo@gmail.com

Abstract. In the face of the reality of teaching Mathematics in Basic Education in Brazil, specially relating teach functions focusing their relevance to the student’s academic development in Basic and Superior Education, this work proposes the use of educational software to help the teaching of functions in Basic Education since the computers and software show as an outstanding option to help the teaching and learning processes. On the other hand, the study also proposes the use of Didactic Transposition as a methodology investigation and research. Along with this survey, some teaching interventions were applied to detect the main difficulties in the teaching process of functions in the Basic Education, analyzing the results obtained along the interventions in a qualitative form. Considering the discussion of the results at the end of the didactic interventions, it was verified that the results obtained were satisfactory.

1. Introduction
The mathematical teaching in basic education in most of the Brazilian public schools do not reach their goal. Several researchers have shown that students finish their high school studies and start their studies in the university without any basic knowledge about functions. The difficulties that secondary students experience in applying mathematics to real-life or context-based tasks are a long-standing problem in educational research and that limitation to use mathematics, as Paulos [1] observed, limits an individuals career aspirations, social well-being, and financial security. In this sense, [2] shows in Freudenthal that understands mathematization as the local structuring of mathematical and nonmathematical fields employing mathematical tools for which the direction from reality to mathematics is highly relevant.

The lack of interest can be one of the main factors that cause a small achievement whether they are students children, young people or adults and this can be strongly related to the teaching strategy.

In general, the traditional teaching method does not stimulate the mathematical learning, especially because it deals with a new students generation that is used to working with technology in their everyday life. In this sense, computational software and mathematical models have become essential tools to help the students learning and also can facilitate not only the teaching of quadratic functions but also other mathematical basic contents.

GeoGebra, for example, as a dynamic mathematical software very known in mathematical teaching, is a great teaching tool for quadratic functions, especially because it contains interactive features, such as the slider, which enables dynamic and instantaneous parameter changes, as well
as the visualization of the graphical features, clarifying and refining students’ mental models of
the situation providing them work with the maximum and minimum points concepts of the
quadratic function more easily.

The purpose of this work is to understand and discuss the student’s difficulties related to the
quadratic functions. As a secondary purpose, it analyzes the student’s comprehension of the
fundamental concepts and to evaluate the use of mathematical and computational tools with
modeling approach as a teaching tool capable of providing an understanding of some problems
that are presented in secondary school textbooks. The work was developed with a group of
students enrolled in the second year of a public school in the state of Minas Gerais and a
mathematics teacher, the second author of this paper.

After analyzing the students difficulties, based on the didactic transposition process, the
models were used to stimulate learning in some intervention activities with the objective of
complementing the teaching. The discussion is focused in two activities related to physics, one
that investigates the launch of an object from the ground (Activity 1) and another that examines
the peace of metal in a dilatation phenomena (Activity 2).

2. The use of modeling approach to teaching quadratic functions

2.1. Modelling Approach associated with Didactic Transposition

Some mathematical concepts are commonly considered quite abstract and the use of
mathematical and computational models for teaching brings to the students some mathematical
problems applied to their reality. Kaiser and Sriraman pointed in [3] the importance to use
mathematics as science and humanistic ideals of education with focus on the ability of learners
to create relations between mathematics and reality

Traditional teaching, especially in Brazil, where classrooms are quite numerous and with a few
pedagogical resources, no longer attracts the attention of young students, especially because in
these days they are used to deal with technological resources. On the other hand, many teachers
who complete their undergraduate courses are still poorly trained and often fail to transform
the mathematical contents presented in the textbooks into knowledge to be understood by the
students. Thus, the choice of an adequate didactic approach for teaching mathematics can help
in the process of mediating the knowledge between teacher and students.

An interesting research proposal about a better adequacy for teaching, in the sense, so the
teacher language can be understood by the students, is the process called Didactic Transposition
(DT), proposed by Michel Verret in 1975 and thoroughly researched by Yves Chevallard in
1991. The computational resource, through the modeling of situations-problems described by
mathematics, according to the principles of DT, can give a more didactic dressing to the content
explored by the teacher in the classroom.

The computational modeling of some activities taken from textbooks of the basic education
allows us to simulate situations that go beyond what the exercise proposes, exploring different
concepts and transforming “taught knowledge”, which is limited to the bibliographic content
used by the teacher, in “used knowledge” whose object is the student [4] and[5]. For this purpose,
GeoGebra software has proved to be an excellent tool, easy to use by students, very intuitive
and allows the development of dynamic interactive models [8], [9] and [10].

2.2. Investigating the teaching of functions in secondary school

A careful research about the difficulties of a small group of students of the second year of the
middle level at the Geraldo de Souza Norte school pointed out a deficiency in their comprehension
about functions. Some students were able to associate the functions with situations of their
everyday life, but they were not able to manipulate these functions conveniently, to associate
the values of maximum and minimum with the particularities of each case, to understand the
implications of the differences between the concavities, among other things.
Quadratic functions are approached in different ways in the Brazilian school’s textbooks, but usually, the books have an exhaustive amount of activities only for fixation of the contents. Usually, these activities seek to bring real situations to be discussed in the classroom, but not always the teacher can transform the information into knowledge so the students can finally comprehend it. Therefore, working with these activities in a more attractive way, with the use of mathematical and computational modeling approach, should contemplate DT.

There are interesting applications of quadratic functions in physics, which could be worked with the help of the modeling process and the visual resource of GeoGebra software, allowing to explore the interdisciplinary character between mathematics and physics. For example, the phenomena of launching an object under the gravitational effect, use two quadratic functions. One of these two quadratic functions deals with the behavior of the object trajectory in the plane and the other quadratic function refers to the vertical position of the object with respect to the time.

In mathematics textbooks it is always possible to find such questions, as it can be found in [11], [12], [13] and [14]. But whatever exercise is taken from the textbook, to be modeled through software, it will be necessary to evaluate how the process of DT can be the bridge that leads to the students learning.

3. Activities with the Modelling Approach

3.1. Activity proposed to the students

The first stage of this research discusses the concepts of the quadratic function of an object launched from the ground, according to the following proposal:

**Activity 1** Consider a ball thrown vertically upwards from the ground. The ball position at each instant follows the function \( h(t) = 40 - 5t^2 \), where \( h \) is expressed in meters and \( t \) in seconds.

Determine: (a) The height at which the ball lies 1 second after the throw; (b) The instant at which it is 75m from the ground; (c) The maximum height reached by the ball and (d) The instant when the ball returns to the ground.

The students would have to answer the questions proposed in the exercise without the use of modeling process. The items (a) and (b) were easily answered by the students because it was necessary only a simple substitution of values and the determination of the other parameter. However, the students presented difficulties to solve especially the item (c), which needed the use of a concept of the maximum point of a quadratic function for the resolution. The item (d) was solved by some students, but most of them did not understand that they needed to determinate the roots of the function to solve the problem.

With the use of modeling process as a teaching tool, it is possible to deal with the issues that were misunderstood in the problem above and to explore it a little more, as shown in Figure 1 below.

Figure 1 shows that the computational model simulates the problem. Through the “Play” button, it shows the ball going up and down while the graph of the function \( h(t) \) is constructed. In this sense, it is possible to explore the concept of maximum point of the function and also show that it is associated with the maximum height that the ball reaches. Likewise, the roots of the function can be observed in the graph and also associated with the moment when the ball lies on the ground just before climbing and at the exact moment it returns to the ground.

Then, with the use of the modeling approach, it is possible to work on the domain of the quadratic function and the constraints from the real situations, showing the student the importance of understanding how to use Mathematics as a tool for solving real problems. For example, if you wish to analyze only the ball going up, or just descending, the visual feature of
Figure 1. The picture represents the simulation for the vertical upward launch of a ball. It was built with the ball movement. In (a) the ball in the upward movement; (b) the ball reaches the maximum high; (c) the ball is going down and (d) it returns to the ground. The pictures show and explore the mathematical concepts discussed in the classroom activity.

The model shows that the domain of the function is restricted to the ball behavior in the real world, even though mathematically the domain of the function is the real numbers set.

3.2. Activity produced by the students

The following model, presented in Figure 2, was constructed by one of the students along the classroom activities, using the following proposed activity:

**Activity 2** The dilation and contraction behavior of metal, measured in a laboratory with a test specimen. It was exposed to a sudden temperature variation along 10 minutes, is in accordance with the function $T(t) = t^2 - 12t + 32$, where $T$ is the temperature measured in Celsius and $t$ measured in minutes. What is the lowest temperature recorded in this experiment and at what time this temperature occur?

The idea of putting students in the condition of pro activity, building their model, allowed us to verify how this activity could stimulate their thinking and increase their interest in study mathematics, especially when is related to the content of the functions.

As expected, with the computational resource, the students presented no difficulty in determining that the lowest temperature was -4 Celsius and that this occurred when the time was equal to 6 min. Thus, it was possible to approach the concept of maximum and minimum of a quadratic function and the students realized that the minimum value that the function could
Figure 2. The model constructed by the students. It represents the behavior of the temperature in the metal under dilation and contraction phenomena observed along the time.

assume was the value corresponding to the image of the parabola that describes the function behavior.

Besides, the visual resource allowed the students to determine the time interval in which the temperature presented negative values. With the free parameter “t” for the time representation, it was possible to determine the exact values of this interval, that is, 4 and 8 minutes after the start of the experiment. It was also possible to analyze that if the temperature becomes negative only after four minutes, the only possibility is to have a parabola with concavity upwards, having the function a minimum and not a maximum point.

Then, the use of sliders, whose free parameters in the modeling are “a”, “b” and “c”, allow us to evaluate, for example, if the problem would accept a negative value for the coefficient of the quadratic term. Then the parabola would have concavity down, and what that would represent. It is also possible to open a discussion with the students to evaluate the changes that would occur in the proposed activity, dynamically changing these parameters through sliders and verifying their implications.

The discussion for this case would go a long way towards showing that even though it is possible mathematically, physical explanation does not always allow for any situation, often without even a real interpretation, requiring the student to reflect on each subject, as in the case of obtaining a negative value for the time.

4. Results and Discussions
The difficulty presented by the students about the understanding of quadratic functions motivated the study, and the results are presented in this work. The investigation about the students’ knowledge related to the basic concepts of quadratic functions, as the minimum or maximum points, the function roots and the implications of these concepts analyzed in real problems, showed that these fixation activities found in the books and handled by the teacher in the classroom were not enough efficient to promote the students comprehension of these concepts.

On the other hand, the search of an updated teaching methodology, using the technological resource, led to different activities with the GeoGebra software producing interactive models based on the exercises of the books. That was a good result because the teacher could finally work with the quadratic functions using real-world applications.
The activity performed for the students, described in section 3.1, generated some expectations regarding the use of modeling approach with the teaching of functions, both by the teacher and by the students. The visual resource using technology as an educational tool is fundamental when you want to explore something that using only pencil and paper is not possible. The model of the ball that was going up and down and the graph of the function produced simultaneously, provides a dynamical effect with a positive impact on the students, growing their curiosity and rousing their attention.

The teacher, using the modeling approach as a teaching tool, stimulates the students about the learning of mathematics, promotes interdisciplinary application and makes the process of teaching, in the DT sense, closer to the student’s reality.

In general, with the modeling activities described in this work, it was observed that the students understood the implication of the quadratic function signal. Indeed, since they managed to identify, even intuitively, the interval at which the function, particularly in the exercise described in Section 3.2, assumes negative or positive values, what means, the range which the temperature was present.

Therefore, the modeling process allowed students to quickly identify the minimum or maximum points of a quadratic function and their respective values, as well as to better understand concepts such as the function roots, the signs and also the concavities of the quadratic function, even if this does not have used technical terms. So, as Kaiser says in [3], the central character of the realistic or applied perspective formulated by [6], can be stated that modeling is understood as an activity to solve authentic problems and not as a development of mathematical theory.

5. Final Considerations
The search for different methods to approach the basic contents of mathematics for secondary education has led the scientific community to propose new didactic and teaching practices. In particular, DT is understood as a process that acts to provide the reflection about the best teaching practice in the classroom (taught knowledge) that leads to understanding by the student (used knowledge)[4]. In this case, modeling, as long as associated with well-developed pedagogical activities, plays the role of restructuring the “hard” context of the textbooks into contents that can be learned by the student, helping him to achieve a necessary level of knowledge to use mathematics effectively in real-life situations. This modeling association with pedagogical activities is a pedagogical and psychological goal mostly discussed in [7] although the author had identified and described the nuances differently.

The use of technology in the classroom, no matter if is only as a demonstrative tool, where only the teacher manipulates the model and explores it, or if is an activity in which the student constructs his model, is fundamental in the present times. where the students learn quickly to deal with the computer. In this case, the modeling gains more dynamism with the resources of the GeoGebra software, being possible to work concepts that are hardly fully exploited only by the traditional method, with the use of pencil and paper. But it is important to say that, as [16] pointed, the modeling activity is this changed by the presence of the software.

Therefore, the modeling process, particularly if is used to teach quadratic functions and problems involving this content, complements an outstanding teaching pedagogical practice in this modern world in which technology is an indisputable reality.

Acknowledgments
Thanks to Capes for the fellowship to one of the authors that belong to the PROFMAT post graduating program at UFVJM. Thanks to FAPEMIG for the financial support.
References

[1] Paulos J A (2000). Innumeracy: mathematical illiteracy and its consequences. London: Penguin.

[2] Freudenthal H (1973). Mathematics as an Educational Task. Dordrecht: Reidel.

[3] Kaiser G and Sriraman B. A global survey of international perspectives on modelling in mathematics education. ZDM - The International Journal on Mathematics Education. 2006. 38(3), 302-310.

[4] Chevallard Y. On Didactic Transposition Theory: Some Introductory Notes. Communication à l’International Symposium on Selected Domains of Research and Development in Mathematics Education (Bratislava, 3-7 aot 1988). Paru dans les Proceedings de ce symposium (Bratislava, 1989), pp. 51-62.

[5] Chevallard Y. La Transposition Didactique: Du Savoir Savant au Savoir Ensigné. Grenoble, La pensée Sauvage, 1991.

[6] Haines C P and Crouch R. (2005). Getting to grips with real world contexts: Developing research in mathematical modelling. (To appear in) Proceedings of the 4th European Congress of Mathematics Education, in St. Feliu de Guixols, Spain, Feb 16-22, 2005.

[7] Blum W. (1996). Anwendungsbezüge im Mathematikunterricht Trends und Perspektiven. In G. Kadunz, H. Kautschitsch, G. Ossimitz and E. Schneider (Ed.), Trends und Perspektiven (pp. 15-38). Wien: Hölder-Pichler-Tempsky.

[8] Silva J M, Jardim D F, Carius A C (2016). O ensino e a aprendizagem de conceitos de Cálculo usando modelos matemáticos e ferramentas tecnológicas. Revista de Ensino de Engenharia, 35(02):70-80.

[9] Silva J M, Jardim D F, Carius A C, Silva A C. (2017). Teaching derivatives concepts with computational techniques. In Conference Proceedings. New Perspectives in Science Education, page 237. libreriauniversitaria. it Edizioni.

[10] Jardim D F, Silva J M, Pereira M M, Junior E A S, Nepomucena T V, Pinheiro, T R. (2015). Estudando Limites com o GeoGebra. Vozes dos Vales, 08(08):119.

[11] Dante L R. (2014). Matemática Contexto e aplicações, volume 01. Ática, São Paulo, 2a edition.

[12] Iezzi G, Dolce O, Degenszajn D, Périgo R, Almeida N. (2013). Matemática Ciência e Aplicações, volume 01. Saraiva, São Paulo, 7 edition.

[13] Souza J. (2013). Coleção: Novo Olhar-Matemática., volume 1. FTD, São Paulo.

[14] Diniz M and Stocco K. (2004). Matemática (Ensino M´dio), volume 1. Saraiva, São Paulo, 4a ed. reformulada edition.

[15] Araujo J L, Campos I S. (2015). Negotiating the Use of Mathematics in a Mathematical Modeling Project. In: Stillman, G. and Blum, W. and Biembengut, M. S. and (Eds). Mathematical Modelling in Education Research and Practice. New York: Springer. Cap.23, p. 283-292.

[16] Stillman G A, Blum W., Biembengut M S. (2015). Cultural, Social, Cognitive and Research Influences on Mathematical Modelling Education. In: Stillman, G. and Blum, W. and Biembengut, M. S. and (Eds). Mathematical Modelling in Education Research and Practice. New York: Springer. Cap.1, p. 1-32.