Bounding the Size of a Network Defined By Visibility Property

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Abstract Phylogenetic networks are mathematical structures for modeling and visualization of reticulation processes in the study of evolution. Galled networks, reticulation visible networks, nearly-stable networks and stable-child networks are the four classes of phylogenetic networks that are recently introduced to study the topological and algorithmic aspects of phylogenetic networks. We prove the following results.

1. A binary galled network with \( n \) leaves has at most \( 2(n - 1) \) reticulation nodes.
2. A binary nearly-stable network with \( n \) leaves has at most \( 3(n - 1) \) reticulation nodes.
3. A binary stable-child network with \( n \) leaves has at most \( 7(n - 1) \) reticulation nodes.

Keywords phylogenetic network · galled network · reticulation visibility · nearly-stable property · stable-child property

1 Introduction

Reticulation processes refer to the transfer of genetic material between living organisms in a non-reproduction manner. Horizontal gene transfer is believed to be a highly significant reticulation process occurring between single-cell organisms (Doolittle and Bapteste 2007; Treangen and Rocha 2011). Other reticulation processes include introgression, recombination and hybridization (Fontaine et al. 2015; McBreen and Lockhart 2006; Marcussen et al. 2014). In the past two decades, phylogenetic networks have often been seen for the
modeling and visualization of reticulation processes (Gusfield 2014; Huson et al. 2011).

Galled trees, galled networks, reticulation visible networks are three of the popular classes of phylogenetic networks introduced to study the combinatorial and algorithmic perspectives of phylogenetics (Wang et al. 2001; Gusfield et al. 2004; Huson and Kloeper 2007; Huson et al. 2011). Reticulation visible networks include galled trees and galled networks. They are tree-based (Gambette et al. 2015). The tree-based networks are introduced by Francis and Steel (2015) recently.

It is well known that the number of internal nodes in a phylogenetic tree with \( n \) leaves is \( n - 1 \). In contrast, an arbitrary phylogenetic network with 2 leaves can have as many internal nodes as possible. Therefore, one interesting research problem is how large a phylogenetic network in a particular class can be. For example, it is well known that a tree-child network with \( n \) leaves has \( 3(n - 1) \) non-leaf nodes at most. A regular network with \( n \) leaves has \( 2^n \) nodes at most (Willson 2010). To investigate whether or not the tree containment problem is polynomial time solvable, surprisingly, Gambette et al. (2015) proved that a reticulation visible network with \( n \) leaves has at most \( 9(n - 1) \) non-leaf nodes. The class of nearly-stable networks was also introduced in their paper. They also proved the existence of a linear upper bound on the number of reticulation nodes in a nearly-stable network.

In the present paper, we establish the tight upper bound for the size of a network defined by a visibility property using a sub-tree technique that was introduced in Gambette et al. (2015). The rest of this paper is divided into six sections. Section 2 introduces concepts and notation that are necessary for our study. Recently, Bordewich and Semple (2015) proved that there are at most \( 3(n - 1) \) reticulation nodes in a reticulation visible network. In Section 3 we present a different proof of the \( 3(n - 1) \) tight bound for reticulation visible networks. Section 4 proves that there are at most \( 2(n - 1) \) reticulation nodes in a galled networks with \( n \) leaves. Section 5 and 6 establish the tight upper bounds for the sizes of nearly-stable and stable-child networks, respectively. In Section 7, we conclude the work with a few remarks.

2 Basic concept

2.1 Phylogenetic Networks

An acyclic digraph is a simple connected digraph with no directed cycles.

Let \( D = (V(D), E(D)) \) be an acyclic digraph and let \( u \) and \( v \) be two nodes in \( D \). If \( (u, v) \in E(D) \), it is called an outgoing edge of \( u \) and incoming edge of \( v \); \( u \) and \( v \) are said to be the tail and head of the edge. The numbers of incoming and outgoing edges of a node are called its indegree and outdegree, respectively. \( D \) is said to be rooted if there is a unique node \( \rho(D) \) with indegree 0; \( \rho(D) \) is called the root of \( D \). Note that in a rooted acyclic digraph there exists a directed path from the root to every other node.
For $E \subseteq \mathcal{E}(D)$, $D - E$ denotes the digraph with the same node set and the edge set $\mathcal{E}(D) - E$. For $V \subseteq \mathcal{V}(D)$, $D - V$ denotes the digraph with the node set $\mathcal{V}(D) - V$ and the edge set $\{(u, v) \in \mathcal{E}(D) | u \notin V \text{ and } v \notin V\}$. If $D'$ and $D''$ are subdigraphs of $D$, $D' + D''$ denotes the subdigraph with the node set $\mathcal{V}(D') \cup \mathcal{V}(D'')$ and the edge set $\mathcal{E}(D') \cup \mathcal{E}(D'')$.

A phylogenetic network on a finite set of taxa, $X$, is a rooted acyclic digraph in which each non-root node has either indegree 1 or outdegree 1 and there are exactly $|X|$ nodes of outdegree 0 and indegree 1, called leaves, that correspond one-to-one with the taxa in the network.

In a phylogenetic network, a node is called a tree node if it is either the root or a node having indegree one; it is called a reticulation node if its indegree is greater than one. Note that leaves are tree nodes and a tree node may have both indegree and outdegree one. A non-leaf node is said to be internal. A phylogenetic network without reticulation nodes is simply a phylogenetic tree.

For a phylogenetic network $N$, we use the following notation:

- $\rho(N)$: the root of $N$.
- $\mathcal{V}(N)$: the set of nodes.
- $T(N)$: the set of tree nodes.
- $R(N)$: the set of reticulation nodes.
- $\mathcal{E}(N)$: the set of edges.
- $L(N)$: the set of leaves.

For two nodes $u, v$ in $\mathcal{V}(N)$, if $(u, v) \in \mathcal{E}(N)$, $u$ is said to be a parent of $v$ and, equivalently, $v$ is a child of $u$. In general, if there is a directed path from $u$ to $v$, $u$ is an ancestor of $v$ and $v$ is a descendant of $u$. We sometimes say that $v$ is below $u$ when $u$ is an ancestor of $v$.

Let $P$ and $Q$ be two simple paths from $u$ to $v$ in $N$. We use $\mathcal{V}(P)$ and $\mathcal{V}(Q)$ to denote their node sets, respectively. They are internally disjoint if $\mathcal{V}(P) \cap \mathcal{V}(Q) = \{u, v\}$.

Finally, a phylogenetic network is binary, if its root has outdegree 2 and indegree 0, all internal nodes have degree 3, and all the leaves have indegree one. Here, we are interested in how large a binary phylogenetic network can be.

In the rest of the paper, a binary phylogenetic work is simply called a network and a phylogenetic tree a tree. For sake of convenience for discussion, we also add an open edge entering the root of a network.

### 2.2 Visibility Properties

A node $v$ in a network is visible (or stable) with respect to a leaf $\ell$ if $v$ is in every path from the network root to $\ell$. We say $v$ visible if it is visible with respect to some leaf in the network.

**Lemma 2.1** Let $N$ be a network and $N'$ a subnetwork of $N$ with the same root and leaves as $N$. Then, a node is visible in $N'$ if it is visible in $N$. Equivalently, a node is not visible in $N'$ if it is not visible in $N$.
Theorem 2.2 indicates that every reticulation visible network is tree-based. However, nearly-stable networks and stable-child networks are not necessarily tree-based.

Proof Let \( v \in V(N) \) be visible with respect to a leaf \( \ell \) in \( N \). For each path \( P \) from \( \rho(N') \) to \( \ell \), since it is also a path from \( \rho(N) \) to \( \ell \) in \( N \), it must pass through \( v \). Thus, \( v \) is also visible with respect to the same leaf in \( N' \). \( \square \)

Reticulation visible networks are networks in which reticulation nodes are all visible (Huson et al. 2011). They are also called stable networks by Gambette et al. (2015).

A network is galled if every reticulation node \( r \) has an ancestor \( a \) such that there are two disjoint tree paths from \( a \) to \( r \) (Huson and Kloepper 2007). Here, a path is tree path if its internal nodes are all tree nodes in the network. Galled networks are reticulation visible and are also known as level-1 networks.

 Nearly-stable networks are networks in which for every pair of nodes \( u \) and \( v \), either \( u \) or \( v \) is visible if \( (u,v) \) is an edge (Gambette et al. 2015).

 Stable-child networks are networks in which every node has a visible child.

Tree-based networks comprise another interesting class of networks that is introduced recently (Francis and Steel 2015). A network is tree-based if it can be obtained from a tree with the same leaves by the insertion of a set of edges between different edges in the tree.

Theorem 2.2 (Gambette et al. 2015) For every reticulation-visible network \( N \), there exists a subset of edges \( E \subseteq E(N) \) such that \( E \) contains exactly an incoming edge for each reticulation node and \( N - E \) is a subtree with the same leaves as \( N \).

Proof Let \( \ell \in L(N) \). Assume \( \ell \) is below some \( r_1 \in R \). Then, there is a path \( P(r_1, \ell) \) from \( r_1 \) to \( \ell \) that avoids \( u \). Since \( R \) is finite and \( N \) is acyclic, there exists a series of reticulation nodes, \( r_1, r_2, \ldots, r_k \) such that:

(i) each \( r_j \) has a parent \( p_j \) below \( r_{j+1} \) for \( j = 1, 2, \ldots, k - 1 \), and

(ii) the node \( r_k \) has a parent \( p_k \) such that there is a path \( P(\rho(N), p_k) \) from \( \rho(N) \) to \( p_k \) that avoids \( u \).

Since \( p_j \) is below \( r_{j+1} \), there exists a path \( P(r_{j+1}, p_j) \) from \( r_{j+1} \) to \( p_j \) for each \( j < k \). Since \( N \) is acyclic and \( r_{j+1} \) is below \( u \), the path \( P(r_{j+1}, p_j) \) avoids \( u \). Concatenating these paths, we obtain the following path

\[
P(\rho(N), p_k) + (p_k, r_k) + P(r_k, p_{k-1}) + (p_{k-1}, r_{k-1}) + \cdots + (p_1, r_1) + P(r_1, \ell)
\]

from \( \rho \) to \( \ell \) that avoids \( u \). \( \square \)
3 Reticulation visible networks

Gambette et al. (2015) proved that there are at most $4(n - 1)$ reticulation nodes in a reticulation visible network with $n$ labeled leaves. On the other hand, there are as many as $3(n - 1)$ reticulations in the reticulation visible network in Figure 1. So, what is the tight upper bound on the number of reticulation nodes? Interestingly, $3(n - 1)$ is the tight upper bound, which was independently proved by Bordewich and Semple (2015) using the induction approach. Here, we present an alternative proof to illustrate our approach.

Given a reticulation visible network $N$ with $n$ leaves, we let $E$ be a set of edges such that $N - E$ is a subtree with the same root and leaves as $N$ (Theorem 2.2). Since $N - E$ has $n$ leaves, there are exactly $n - 1$ nodes of degree 3. Thus, there are $2n - 2$ paths whose internal nodes are of degree 2, starting at a degree-3 node and terminating at either another node of degree 3 or a leaf. Let these $2n - 2$ paths be $P_1, P_2, \cdots, P_{2n-2}$.

The edges of $N - E$ not in $\bigcup_{1 \leq i \leq 2n-2} E(P_i)$ make up a path $P_0$ that contains the root $\rho(N)$ (Figure 1). If $\rho(N)$ is of degree 2, $P_0$ passes through $\rho(N)$ and terminates at a degree-3 node. If $\rho(N)$ is of degree 3, $P_0$ is simply the open edge entering $\rho(N)$. Altogether, these $2n - 1$ paths are called the trivial paths of $N - E$. Note that $E(N - E) = \bigcup_{0 \leq i \leq 2n-2} E(P_i)$.

It is not hard to see that, for each edge in $E$, its head and tail are both found in these trivial paths. An edge $(u, v) \in E$ is called a cross edge if $u \in V(P_i)$ and $v \in V(P_j)$ for $i \neq j$; it is called non-cross edge otherwise. The facts in the following proposition appear in the proof of Theorem 1 in Gambette et al (2015).

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![Figure 1](image_url)  
**Fig. 1** (a) A reticulation visible network with 4 leaves that has as many reticulation nodes as possible. (b) A subtree of the network that has the same leaves, in which only $u_1, u_2, u_3$ are of degree 3.
Proposition 3.1

(1) No two cross edges $e \in E$ have their heads in the same trivial path in $N - E$.

(2) For each non-cross edge $(u, v)$ such that $u, v \in P_i$ for some $i \geq 0$, there are at least one cross edge $(w, x) \in E$ such that $w$ is between $u$ and $v$ in $P_i$.

For a cross edge $(u, v)$ such that $u \in V(P_i)$ and $v \in V(P_j) (i \neq j)$, we say $(u, v)$ leaves $P_i$ and enters $P_j$. For a non-cross path $(u, v)$ and a cross edge $(w, x)$, if $u$ and $v$ are in $P_i$ and $w$ is a node between $u$ and $v$ in $P_i$, we say $(u, v)$ jumps over $(w, x)$.

It is trivial to see that no cross edge enters the trivial path $P_0$. Proposition 3.1 suggests that $E$ contains at most $2n - 2$ cross edges and thus at most $2n - 2$ non-cross edges. By Theorem 2.2, $|\mathcal{R}(N)| = |E| \leq 4n - 4$.

To obtain the tight upper bound $3n - 3$ for $|\mathcal{R}(N)|$, we define the cost $c(e)$ of a cross edge $e \in E$ as:

$$c(e) = \begin{cases} 2 & \text{if there is a non-cross edge jumping over the tail of } e, \\ 1 & \text{otherwise}. \end{cases}$$

We will charge the cost of a cross edge to the trivial path it enters and call it the weight of the trivial path. If no cross edge enters a trivial path, the weight of this trivial path is set to be 0. By Proposition 3.1, the weight of a trivial path is at most 2. We use $w(P_i)$ to denote the weight of a trivial path $P_i$, $0 \leq i \leq 2n - 2$.

For an internal node $t$ of degree 3 in $N - E$, we use $P_{13}$ to denote the trivial path entering $t$ and $P_{11}, P_{12}$ to denote the two trivial paths leaving $t$.

Proposition 3.2

Let $t$ be a degree-3 node in $N - E$.

Fig. 2 Three cases that are considered in the proof of Proposition 3.2. $P_{11}, P_{12}, P_{13}$ are the three trivial paths incident to a degree-3 node $t$; $(w_i, x_i)$ is the cross edge ending at $v_i$ in $P_i$, and the non-cross edge $(u_i, v_i)$ jumps over $(w_i, x_i)$ for $i = 1, 2$. Here, $(u_2, v_2)$ is not drawn. (a) $w_1$ and $v_1$ are both between $t$ and $x_1$ in $P_{11}$. (b) $w_1$ is between $t$ and $x_2$ in $P_{12}$, but $v_1$ is below $x_2$ in $P_{12}$. (c) The node $w_1$ is below $x_2$ and $w_2$ is below $x_1$. This case is impossible to occur, as there is a directed cycle.
(i) If \( P_{t3} \neq P_0 \) and \( w(P_{t1}) = w(P_{t2}) = 2 \), then \( w(P_{t3}) = 0 \).
(ii) If \( P_{t3} = P_0 \), then \( P_{t3} = 0 \) and \( w(P_{t1}) + w(P_{t2}) \leq 3 \).

Proof For sake of simplicity, we let \( T = N - E \) and use \( P_T(z', z'') \) to denote the unique path from \( z' \) to \( z'' \) for a node \( z' \) and a descendant \( z'' \) of \( z' \) in \( T \). \( N \) and \( T \) have the same root and leaves. The common root of \( N \) and \( T \) is written \( \rho \).

(i) Assume that \( P_{t3} \neq P_0 \) and \( w(P_{t1}) = w(P_{t2}) = 2 \). Then, there exists a cross edge \((w_j, x_j)\) entering \( P_t \) and a non-cross edge \((u_j, v_j)\) jumping over \((w_j, x_j)\) for each \( j = 1, 2 \). We shall prove that \( w(P_{t3}) = 0 \) by showing that no cross edge enters \( P_{t3} \).

Assume \( w_1 \) is between \( t \) and \( x_2 \) in \( P_{t2} \). When \( v_1 \) is below \( w_1 \) in \( P_T(t, x_2) \) (Figure 2a), there are two cases. If \( w_2 \) is in the path \( P_t \) or below it, then, \( P_T(\rho, w_2) \) does not pass \( v_1 \). If \( w_2 \) is not below \( v_1 \), \( P_T(\rho, w_2) \) does not pass \( t \) and \( v_1 \). Therefore, by Lemma 2.3 there is path from \( \rho \) to every leaf below \( x_2 \) that does not pass \( v_1 \). For any leaf \( \ell \) not below \( x_2 \) in \( T \), \( P_T(\rho, \ell) \) avoids \( v_1 \). Hence, \( v_1 \) is a reticulation node in \( N \), but not visible. This is a contradiction.

When \( v_1 \) is below \( x_2 \) in \( T \) (Figure 2b), \( v_1 \) is below \( x_2 \) as a reticulation node. Since \( P_T(\rho, v_1) \) does not pass \( x_2 \), by Lemma 2.3 there is a path from \( \rho \) to a leaf below \( x_2 \) that does not pass \( x_2 \). For any leaf \( \ell \) not below \( x_2 \) in \( T \), \( P_T(\rho, \ell) \) avoids \( x_2 \). Hence, \( x_2 \) is not visible, a contradiction.

We have proved that \( w_1 \) is not between \( t \) and \( x_2 \) in the tree path \( P_{t2} \). By symmetry, \( w_2 \) is not between \( t \) and \( x_1 \) in \( P_{t1} \).

Assume there is a cross edge enters \( P_{t3} \). Let \( r \) be the lowest reticulation node in \( P_{t3} \). Then, \( w_1 \) and \( w_2 \) are both not in \( P_T(r, t) \). Otherwise, either \( v_1 \) or \( v_2 \) is between \( r \) and \( t \), contradicting that \( r \) is the lowest reticulation node in \( P_{t3} \). Combining this fact with that \( w_j \) is not between \( t \) and \( x_{3-j} \) in \( T \) for \( j = 1, 2 \), we conclude that either \( w_j \) is below \( x_{3-j} \) or there is a path from \( \rho \) to \( w_j \) not passing \( r \) for each \( j = 1, 2 \). Hence, by Lemma 2.3 \( r \) is not visible with respect to any leaf below \( r \). For any leaf \( \ell \) not below \( r \) in \( T \), the tree path \( P_T(\rho, \ell) \) avoids \( r \). Hence, \( r \) is not visible, a contradiction.

We have proved that \( w(P_{t3}) = 0 \).

(ii) If \( P_{t3} = P_0 \), then \( t \) is an ancestor of any other degree-3 node in \( N - E \). Since \( N \) is acyclic, there does not exist \((u, v) \in E \) such that \( u \in P_i \) for some \( i > 0 \) and \( v \in P_0 \). Hence, \( w(P_0) = 0 \).

Assume on the contrary the weights of \( P_{t1} \) and \( P_{t2} \) are both \( 2 \). Then, \( w_j \) is not between \( t \) and \( x_{3-j} \) for \( j = 1, 2 \), proved above. If \( w_1 \) or \( w_2 \) is in \( P_0 \), the lowest reticulation in \( P_0 \) is not visible, a contradiction. Otherwise, \( w_1 \) is below \( x_2 \) and \( w_2 \) is below \( x_1 \), implying a cycle in \( N \) (Figure 2). This is a contradiction. Hence, either \( P_{t1} \) or \( P_{t2} \) has weight less than \( 2 \). □

Theorem 3.3 Let \( N \) be a reticulation visible network with \( n \) leaves. Then,

\[ |\mathcal{R}(N)| \leq 3(n - 1). \]

Proof Let \( V \) denote the set of \((n - 1)\) internal nodes of degree 3 in \( N - E \). Note that any trivial path other than \( P_0 \) starts with a node in \( V \). Define:

\[ V_k = \{ v \in V \mid w(P_{v1}) + w(P_{v2}) = k \} \]
for $0 \leq k \leq 4$. Clearly, $V_k$’s are pairwise disjoint and hence

$$|V_0| + |V_1| + |V_3| + |V_4| = |V| = n - 1.$$ 

When $v \in V_4$, $w(P_{v1}) = w(P_{v2}) = 2$. By Proposition 3.2, $P_{v3} \neq P_0$. Let $p(v)$ be the start node of $P_{v3}$ for each $v \in V_4$. Again, by Proposition 3.2, $p(v) \in V_0 \cup V_1 \cup V_2$. It is clear that under the map $p(\cdot)$, at most two nodes in $V_4$ are mapped to the same node in $V_0$, and different nodes in $V_4$ are mapped to different nodes in $V_1 \cup V_2$. Thus, $|V_4| \leq 2|V_0| + |V_1| + |V_2|$. Since $w(P_0) = 0$, the inequality implies that

$$|R(N)| = \sum_{v \in V}[w(P_{v1}) + w(P_{v2})] = |V_4| + 2|V_2| + 3|V_3| + (3|V_4| + |V_4|) \leq 2(|V_0| + |V_1|) + 3(|V_2| + |V_3| + |V_4|) \leq 3(|V_0| + |V_1| + |V_2| + |V_3| + |V_4|) = 3(n - 1),$$

where the first inequality is derived from the substitution of $2|V_0| + |V_1| + |V_2|$ for $|V_4|$.

\[\square\]

4 Galled networks

Galled networks form a subclass of reticulation visible networks (Huson et al. 2011). In this section, we shall show that there are at most $2(n - 1)$ reticulations in a galled network with $n$ leaves. Given that the galled network shown in Figure 3a has exactly $2(n - 1)$ reticulations, $2(n - 1)$ is the tight bound on the number of reticulation nodes in a galled network with $n$ leaves.

**Theorem 4.1** For a galled network $N$ with $n$ leaves, $|R(N)| \leq 2(n - 1)$.

![Fig. 3](image)

Fig. 3 (a) A galled network with 4 leaves that has as many reticulation nodes as possible. (b) and (c) are two cases considered in the proof of Theorem 4.1: there is a non-cross edge $(u, v)$ in $E$ such that $u$ and $v$ are in $P_0$, and there is a cross-edge edge $(u_1, v_1)$ and a non-cross edge $(u_2, v_2)$ both ending at a node in a trivial path other than $P_0$, where $u_1$ is not drawn. In (b) and (c), solid straight and curve arrows represent edges and paths in $N - E$, respectively; round dot arrows represent edges in $E$, respectively.
Proof Let $N$ be a galled network with $n$ leaves and let $\rho = \rho(N)$. Since $N$ is reticulation visible, by Theorem 2.2 there is a set of edges $E$ such that (a) $E$ contains exactly one incoming edge for each reticulation node and (b) $N - E$ is a subtree with the same leaves as $N$.

We use the same notation as in Section 3. $P_0$ denotes the trivial path whose first edge is the open edge entering $\rho$; $P_1, \ldots, P_{2n-2}$ denote the other $2n - 2$ trivial paths in $N - E$. We prove the result by showing that $E$ does not contain any non-cross edges and only one cross-edge can end at a node in each $P_i$ for $i > 0$.

If $P_0$ contains only the open edge entering $\rho$, there is no edge in $E$ that enters $P_0$. We first prove that this fact is also true even if $P_0$ contains other edges below $\rho$.

Since $N$ is acyclic and there is a directed path from the end of $P_0$ to a node in $P_i$ for any $i > 0$, there is no cross edge $(u,v) \in E$ such that $u$ is in $P_i$ and $v$ is in $P_0$.

If there is a non-cross edge $(u,v)$ such that $u,v$ are in $P_0$ (Figure 3b), we let $w$ be the other child of $u$ in $P_0$. Then, $w$ must be a tree node such that $(w,x) \in E$, where $x$ is a reticulation node in some trivial path $P_i$, $i > 0$. (If $w$ is a reticulation, it is not visible, a contradiction.) Since $N$ is galled and $x$ is a reticulation node, there exist two paths $P'$ and $P''$ from a common tree node to $x$ in $N$ such that (i) they are internally disjoint and (2) $x$ is the only reticulation node in them. Note that no edges in $E$ other than $(w,x)$ can appear in $P'$ and $P''$. Otherwise, either $P'$ and $P''$ contain another reticulation node. Thus, $P' + P'' - (w,x)$ is a subtree of $N - E$. This implies that one of $P'$ and $P''$ is the single edge $(w,x)$ and the other is $P_{N-E}(w,x)$, the unique path from $w$ to $x$ in the tree $N - E$. This is impossible, as the reticulation node $v$ is in $P_{N-E}(w,x)$.

We have shown that there is no edge in $E$ that enters $P_0$. Next, we show that there is at most one edge in $E$ that enters $P_i$ for each $1 \leq i \leq 2n - 2$.

Assume that $(u_1,v_1)$ and $(u_2,v_2)$ are two edges in $E$ such that $v_2$ is below $v_1$ in some $P_i$ ($i > 0$) (Figure 3c). Then, $(u_2,v_2)$ must be a non-cross edge and $u_2$ is also below $v_1$. (Otherwise, $v_1$ is not visible.) Again, by Fact (2) in Proposition 3.1, there is a cross edge $(w,x)$ such that $w$ is between $u_2$ and $v_2$ in $P_i$ and $x$ is in $P_j$, $j \neq i$. Since $x$ is a reticulation node and $N$ is galled, there are two internally disjoint paths $P'$ and $P''$ from a common tree node to $x$ in which any nodes other than $x$ are a tree node. If $P' + P''$ contains an edge in $E$ other than $(w,x)$, the head of the edge is a reticulation node and appears in either $P'$ or $P''$, a contradiction. Hence, $P' + P'' - (w,x)$ is a subtree of $T$. Without loss of generality, we may assume $P'$ contains $(w,x)$. That is, $(w,x)$ is the last edge of $P'$. Note that $v_1, u_2, w, v_2$ are all nodes in $P_i$, ordered from top to bottom. If $P'$ contains more than one edge in $T$, it must pass through $v_1$, a contradiction. If $P'$ is equal to $(w,x)\), then $P''$ must pass through $v_2$, a contradiction. Therefore, there is at most one edge in $E$ whose head is in each trivial path $P_i$, $i > 0$.

In summary, there are $2n - 2$ trivial paths other than $P_0$ and there is at most one edge in $E$ entering each of them. Hence, $|\mathcal{R}(N)| = |E| \leq 2(n - 1)$. \qed
5 Nearly-stable network

In this section we will give a tight bound for the number of reticulations in a nearly-stable network. The class of nearly-stable networks is different from the class of reticulation visible networks, but surprisingly the tight upper bound is also $3(n - 1)$. The network shown in Figure 4a is an example for a nearly-stable network with $3(n - 1)$ reticulations. We need the following fact, proved by Gambette et al. (2015).

**Proposition 5.1** Let $N$ be a nearly-stable network with $n$ leaves. There exists a set $E$ of edges such that (a) $N - E$ is a reticulation visible subnetwork over the same leaves as $N$, and (b) $E$ contains exactly one incoming edge for each reticulation node that is not visible in $N$.

Let $E$ be the set of edges satisfying the two properties in Proposition 5.1 and let $N' = N - E$ (Figure 4b). The edges in $E$ are said to be NS-edges. We remark that $N'$ is a subdivision of a binary reticulation visible network. That is, the reticulation visible network can be obtained from $N'$ by replacing some paths whose internal nodes are of degree 2 with directed edges with the same orientation. Hence, $N'$ contains degree-2 nodes if $E$ is not empty.

For a path $P$, we use $IV(P)$ to denote the set of its internal nodes. Since $N'$ is a subdivision of a binary reticulation visible network with the same leaves as $N$, by Theorem 2.2 there is a set $P$ of paths in $N'$ such that (i) each path $P \in P$ is from a degree-3 tree node to a visible reticulation node in $N'$ and its internal nodes are all of degree-2 in $N'$, and (ii) $N' - \bigcup_{P \in P} IV(P) - \bigcup_{P \in P} E(P)$ is a subtree with the same leaves as $N$.

Let $T = N' - \bigcup_{P \in P} IV(P) - \bigcup_{P \in P} E(P)$. $T$ is obtained from the removal of the internal nodes and edges of the paths in $P$. We can classify the paths in $P$ as **cross paths** and **non-cross paths** accordingly as in Section 3 (Figure 4b).

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**Fig. 4** (a) A nearly-stable network $N$ with four leaves. It has nine reticulations (shaded circles), five of which are not visible. The round dot edges are those removed to obtain the reticulation visible network $N'$ in (b). The dashed paths in $N'$ are the cross and non-cross paths removed to obtain a subtree with the same leaves as $N$. (c) and (d) are two cases considered in the proof of the part (c) in Lemma 5.2: a non-cross path from $u$ and $v$ contains a tree node $z$ of $N$, and it contains a reticulation node $z$ of $N$. 

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Lemma 5.2 Let $N$ be a nearly-stable network and let $E$, $N'$, $T$ and $P$ be defined above.

(a) Every internal node in a path in $P$ is not visible in $N$.

(b) Each cross path in $P$ consists of either a single edge or two edges in $N$.

(c) Each non-cross path in $P$ is simply an edge in $N$.

(d) If $P$ is a cross path in $P$ from $w$ to $x$ and $P''$ is a non-cross path in $P$ from $u$ to $v$ such that $w$ is between $u$ and $v$ (Figure 4), then $P$ and $P''$ are both a single edge in $N$.

(e) Every two distinct paths in $P$ are node disjoint.

Proof We remark that $P_T(x,y)$ denotes the unique path from $x$ to $y$ for any two nodes $x$ and $y$ in $T$.

(a) Let $P$ be a path in $P$ and let $y$ be an internal node of it. For any leaf $\ell \in L(T)$, the unique path $P_T(\rho, \ell)$ does not pass $y$ in $T$. Hence, $y$ is not visible in $N$.

(b) If there are two or more internal nodes in a path in $P$, by (a), they are consecutive and not visible in $N$, contradicting that $N$ is nearly-stable.

(c) We use $\rho$ to denote the root of $N$, which is also the root of $N'$ and $T$.

Let $P$ be a non-cross path between $u$ and $v$, where $u$ and $v$ are in some path $P_0$ in $T$. Note that $P_T(u,v)$ is a sub-path of $P_0$ and is internally disjoint from $P$. By Fact (2) in Proposition 3.1, there is an internal node $w$ in $P_T(u,v)$ that is the start node of a cross path $P(w,x)$ in $P$.

First, any node $y$ between $u$ and $v$ in $P_T(u,v)$ is not visible. This is because for any network leaf $\ell$ not below $v$ in $T$, $P_T(\rho, \ell)$ does not pass through $y$, and for any network leaf $\ell$ below $v$ in $T$, $P_T(\rho, \ell) = P_T(u,v) + P$ is a path not passing through $y$. Therefore, $y$ must be the unique internal node of $P_T(u,v)$.

That is, $w$ is the child of $u$ and the parent of $v$ in $P_T(u,v)$.

Assume that $P$ is not an edge in $N$. By (a), there is a unique degree-2 node $z$ between $u$ and $v$ in $P$. We consider the following two cases.

If $z \in T(N)$ (Figure 4), then the other outgoing edge $(z, z')$ had been removed to obtain $N'$. That is, $(z, z') \in E$. By the definition of $E$, $z'$ is a reticulation node and not visible in $N$. That $z$ and $z'$ are both not visible contradicts that $N$ is nearly-stable.

If $z \in R(N)$ (Figure 4), then the other incoming edge $(z'', z)$ had been removed to obtain $N'$. Note that $z'' \neq u$ and $z'' \neq w$, as $w$ has degree 3 in $N'$. In addition, $z''$ is not an internal node of a path in $P$. (Otherwise, by (a), $z$ and $z''$ are both not visible). So $z''$ is a node in $T$. Clearly $z''$ is not below $v$ and hence not below $w$ in $T$. (Otherwise $N$ has a cycle.) Hence, $P_T(\rho, z'')$ does not pass through $u$.

Consider a network leaf $\ell \in L(N)$. If it is not below $v$, then $P_T(\rho, \ell)$ does not pass through $u$. If $\ell$ is below $v$, then $P_T(\rho, z'') + (z'', z) + (z, v) + P_T(v, \ell)$ is a path not passing through $u$ in $N$. Therefore, $u$ is not visible. That $u$ and $w$ are both not visible in $N$ contradicts that $N$ is nearly-stable.
Fig. 5 The two cases considered in the proof of Lemma 5.3. Solid arrows and curves represent the edges and paths in $T$, while square dot arrows and curves represent the removed edges and paths. $f_1$ is the reticulation child of $t$ in a trivial path $P_{t1}$ leaving $t$. $f_2$ is the child of $t$ in the trivial path $P_{t2}$. The path from $w_2$ to $x_2$ is a cross path entering $x_2$. (a) A cross path from $w_1$ to $x_1$ and $f$ is a node between $t$ and $x_1$, where $x_1$ is in $P_{t1}$. (b) The unique tree node $f_2$ between $t$ and $x_2$ is also a parent of $f_1$ in $N$.

(d) By the proof of (c), $P'$ is a single edge in $N$ and $w$ is the only node in $P$ and not visible. Thus $P$ must be an edge in $N$. (Otherwise by (a) $w$ and its child in $P$ are not visible, contradicting that $N$ is nearly-stable.)

(e) It can be easily derived from the definition of the cross path. □

Let $C \in \mathcal{P}$ be a cross path from $w$ to $x$. Then, $x$ is a visible reticulation node in $N$. It may have as many as two reticulation parents that are not visible. Let $U_x = \text{parent}(x) \cap \mathcal{UR}(N)$, where $\text{parent}(x)$ is the set of all parents of $x$ and $\mathcal{UR}(N)$ is the set of all reticulation nodes that are not visible in $N$. $|U_x| = 0, 1, \text{ or } 2$. Define the cost of $C$ as:

$$c(C) = \begin{cases} \ 2 + |U_x| & \text{if there is a non-cross edge jumping over } w, \\ \ 1 + |U_x| & \text{otherwise,} \end{cases}$$

(1)

where 2 is used to count $x$ and the other child of $w$ which is a visible reticulation node if there is a non-cross edge jumping over $w$.

As in Section 3, we let $P_0$ denote the trivial path whose first edge is the incoming edge to $\rho$ and let $P_1, \ldots, P_{2n-2}$ denote the other $2n - 2$ trivial paths in $T$. We charge the cost of a cross path to the trivial path $P_i$ in $T$ in which the cross path enters and call it the weight of $P_i$. The weight of $P_i$ is denoted by $w(P_i)$. If a trivial path does not contain any end node of the cross paths in $\mathcal{P}$, its weight is set to be 0.

Each visible reticulation node contributes to at least one unit of weight. By the definition of nearly-stable networks, any reticulation node that is not visible must have a visible reticulation node as its child, and by the proof of Lemma 5.2(c), any reticulation node that is not visible in $N$ must be in some $U_x$, $x$ being the end node of a cross path, so it also contributes to at least one unit weight. Therefore, $|\mathcal{R}(N)| \leq \sum_{i=0}^{2n-2} w(P_i)$. To bound this, we first establish a useful lemma.

As in Section 3 we use $P_{t3}$ to denote the trivial path entering $t$ and $P_{t1}$, $P_{t2}$ to denote the trivial paths leaving $t$ for a node $t$ of degree 3 in $T$.

Lemma 5.3 Let $P_{tj}$ be a trivial path defined above and let $C_j$ be a cross path from $w_j$ to $x_j$, where $x_j$ is in $P_{tj}$ and $j \in \{1, 2\}$. Define $j' = 3 - j$. 

Let $C_j$ be a cross path from $w_j$ to $x_j$, where $x_j$ is in $P_{tj}$ and $j \in \{1, 2\}$. Define $j' = 3 - j$. 

Let $C_j$ be a cross path from $w_j$ to $x_j$, where $x_j$ is in $P_{tj}$ and $j \in \{1, 2\}$. Define $j' = 3 - j$.
Proof Note that \( \{j, 3 - j\} = \{1, 2\} \) for \( j = 1, 2 \). Without loss of generality, we may assume that \( P_{1j} = P_{11} \) and \( P_{2j} = P_{12} \), that is \( j = 1 \) and \( 3 - j = 2 \).

(a) Let \( f \) be a node between \( t \) and \( x_1 \) in \( P_{11} \) (Figure 5a) and let \( \ell \) be a leaf in \( N \). If \( \ell \) is not below \( x_1 \) in \( T \), the path \( P_T(\rho, \ell) \) does not pass through \( f \).

Let \( \ell \) be a leaf below \( x_1 \) in \( T \). Since \( w_1 \) is not in \( P_{11} \) in \( T \), the tree path \( P_T(\rho, w_1) \) does not pass \( f \). By Lemma 2.3 there is a path from \( \rho \) to \( \ell \) that avoids \( f \). Therefore, \( f \) is not visible.

Since \( N \) is nearly-stable, there is at most one node in \( P_T(t, x_1) \), as each internal node is not visible.

(b) Suppose on the contrary, there is a cross path \( C_2 \) from \( w_2 \) to \( x_2 \) entering \( P_{12} \), where \( x_2 \) is in \( P_{12} \). By (a), \( x_2 \) is a child of \( t \) or there is a unique node \( f_2 \) between \( t \) and \( x_2 \) in \( P_{12} \). We first show that \( t \) is not visible in \( N \).

If \( x_2 \) is a child of \( t \) or there is a node \( f_2 \) in \( P_T(t, x_2) \) such that \( f_2 \) is a reticulation node in \( N \), \( t \) has two reticulation children in \( N \). By Lemma 2.3 \( t \) is not visible.

If \( P_{12} \) contains a node \( f_2 \) between \( t \) and \( x_2 \) in \( N \), \( (f_2, f_1) \) must not be an edge in \( N \). Otherwise, as shown in Figure 5b, \( f_1 \) and \( f_2 \) are then not visible, contradicting that \( N \) is nearly-stable.

Let \( (y_1, f_1) \) be the edge removed from \( f_1 \) in the process of transforming \( N \) to \( N' \). Since \( y_1 \neq f_2 \), either \( y_1 \) is below \( x_2 \) or there is a path from \( \rho \) to \( y_1 \) that avoids \( t \).

Since \( w_2 \) is in another trivial path and there is no node between \( t \) and \( f_1 \) in \( P_{11} \), \( w \) is either below \( f_1 \) or the path \( P_T(\rho, w_2) \) does not pass \( t \).

Since the reticulation nodes \( f_1, x_2 \) are below \( t \) and satisfy the condition in Lemma 2.3 there is a path from \( \rho \) to \( \ell \) that avoids \( t \) for any leaf \( \ell \) below \( f_1 \) or \( x_2 \). For any leaf \( \ell \) below neither \( f_1 \) nor \( x_2 \), it is not below \( t \) and the path \( P_T(\rho, \ell) \) does not pass through \( t \). Therefore, \( t \) is also not visible.

The fact that \( t \) and \( f_1 \) are both not visible contradicts that \( N \) is nearly-stable. This implies that there is no cross path entering \( P_{12} \).

(c) If \( x_2 \) is the child of \( t \) in \( P_{12} \), the case is trivial.

Assume that there is an internal node \( f_2 \) between \( t \) and \( x_2 \) in \( P_{12} \). By the fact (a), \( f_2 \) is not visible. If \( w_1 = f_2 \), then \( w_1 \) and its child in \( P_1 \) are both not visible, contradicting \( N \) is nearly-stable network. \( \square \)

Proposition 5.4 For an internal node \( t \) of degree 3 in \( T \),

(a) \( w(P_{1j}) \leq 3 \) and \( w(P_{2j}) \leq 3 \).
(b) if \( w(P_{1j}) = 3 \), then \( w(P_{1(3-j)}) = 0 \), where \( j \in \{1, 2\} \).
(c) if \( P_{t3} \neq P_t \) and \( w(P_{t1}) = w(P_{t2}) = 2 \), then \( w(P_{t3}) = 0 \). Moreover, assume \( p(t) \) is the degree-3 ancestor of \( t \) such that \( P_{t3} = P_{p(t)1} \). Then \( w(P_{p(t)2}) \leq 2 \).

(d) if \( P_{t3} = P_0 \), then \( w(P_{t3}) = 0 \) and \( w(P_{t1}) + w(P_{t2}) \leq 3 \).

Proof (a) We only prove that \( w(P_{t1}) \leq 3 \). If there is no non-cross edge jumping over the start node of the cross path entering \( P_{t1} \), by Eqn. (1), the weight of \( P_{t1} \) is at most 3.

If there is a non-cross edge jumping over the start node \( w_1 \) of the cross-path \( C \) ending at a node \( x_1 \) in \( P_{t2} \), then \( P_{t2} \) is equal to the single edge \( (w_1, x_1) \). Therefore, \( x_1 \) has at most one reticulation parent, which is in \( P_{t1} \) if exists. By Eqn. (1), \( w(P_{t1}) \leq 3 \).

(b) Assume \( w(P_{t1}) = 3 \). Then, there is a cross path \( C \) from \( w_1 \) to \( x_1 \) where \( x_1 \) is in \( P_{t1} \). If there is no non-cross edge jumping over \( w_1 \), by Eqn. (1), \( x_1 \) has two reticulation parents (Figure 6).

If there is a non-cross edge jumping over \( w_1 \), by the fact (d) of Lemma 5.2 \( C \) is equal to a single edge \( (w_1, x_1) \), and by Eqn. (1), \( x_1 \) has one reticulation parent \( f_1 \) in \( P_{t1} \) (Figure 6). By the fact (b) of Lemma 5.3, there is no cross path that enters \( P_{t2} \), implying \( w(P_{t2}) = 0 \).

(c) Assume \( P_{t3} \neq P_0 \) and \( w(P_{t1}) = w(P_{t2}) = 2 \). Let \( C_j \) be the cross path from \( w_j \) to \( x_j \), with \( x_j \) in \( P_{tj} \), \( j = 1, 2 \). Since \( w(P_{t1}) = w(P_{t2}) = 2 \), by the fact (b) of Lemma 5.3, there is no reticulation node between \( t \) and \( x_j \) for each \( j \). Hence, for each \( j \), either the parent of \( x_j \) in \( C_j \) is a reticulation node and not visible (Figure 6c), or there is a non-cross edge \( (u_i, v_i) \) jumping over \( w_i \) (Figure 6).

By the facts (a) and (b) of Lemma 5.3 either \( x_j \) is the child of \( t \) in \( P_{tj} \) or there is a tree node \( f_j \) between \( t \) and \( x_j \) in \( P_{tj} \) for \( j \in \{1, 2\} \).

Assume that there is a tree node \( f_j \) between \( t \) and \( x_j \) in \( P_{tj} \), \( j \in \{1, 2\} \). Let \( j' = 3 - j \). If \( C_{j'} \) has an internal node that is a reticulation, by the fact (c) of Lemma 5.3 \( w_{j'} \neq f_j \).

If there is a non-cross edge jumping over \( w_{j'} \), by the fact (d) of Lemma 5.2 that \( w_{j'} \neq f_j \) implies that the endpoints of the non-cross edge are also between \( t \) and \( x_j \). This is impossible, as there is only \( f_j \) between \( t \) and \( x_j \). Therefore, \( w_{j'} \neq f_j \). Similarly, \( w_j \neq f_j \).

Fig. 6 (a)-(b) Two types of trivial paths of weight 3. (c)-(e) Three types of trivial paths of weight 2. Solid arrows and curves represent the edges and paths in \( T \), while square dot arrows and curves represent the removed edges and paths. The path from \( w_1 \) to \( x_1 \) is the cross path ending at a node in a trivial path leaving \( t \).
We have proved that for \( j = 1, 2, w_j \) is not between \( t \) and \( x_j \). Thus, \( w_j \) is either below \( x_j \) or there is a path from \( \rho \) to \( w_j \) that does not pass \( t \). Therefore, by Lemma 2.3, there is a path from \( \rho \) to \( \ell \) not passing through \( t \) for any leaf below either \( x_1 \) or \( x_2 \). For any leaf \( \ell \) below neither \( x_1 \) nor \( x_2 \), since it is not below \( t \) in \( T \), \( P_T(\rho, \ell) \) does not contain \( t \). Therefore, \( t \) is not visible. This also implies that \( x_1 \) and \( x_2 \) are children of \( t \).

Assume \( p(t) \) is the start node of \( P_{t3} \) and \( P_{t3} = P_{p(t)1} \). We further prove that \( P_{t3} \) consists of only an edge \((p(t), t)\) in \( N \).

Assume on the contrary there are nodes between \( p(t) \) and \( t \) in \( P_{p(t)1} \). We consider the parent \( y (\neq p(t)) \) of \( t \) in the trivial path \( P_{p(t)1} \). If \( y \) is a reticulation node, that is not visible implies that \( y \) is also not visible, a contradiction. Hence \( y \) must be a tree node in \( N \). We consider the following two cases.

**Case 1.** \( y \) is equal to \( w_j \) or equal to the other parent of the internal node of \( C_j \) for some \( j \in \{1, 2\} \).

Without loss of generality, we may assume \( j = 1 \) (Figure 7a and b). This implies that there is no non-cross edge jumping over the cross path \( C_1 \) and there is a reticulation node \( z_1 \) in \( C_1 \).

When \( y = w_1 \), let \((z_1', z_1)\) be the edge removed from \( z_1 \) in the first stage. Since \( z_1' \) is a parent of \( z_1 \), if \( z_1' \) is below \( y \), it must be below \( x_2 \). When \( y \neq w_1 \), \( w_1 \) is below \( x_2 \) if it is below \( y \).

Similarly, \( w_2 \) is below \( x_1 \) and thus below \( z_1 \) if it is below \( y \).

The set of reticulation nodes \( \{z_1, x_2\} \) and \( y \) and satisfy the condition in Lemma 2.3, so there is a path from \( \rho \) to \( \ell \) that avoids \( y \) for any leaf \( \ell \) below \( z_1 \) and \( x_2 \). If \( \ell \) is below neither \( z_1 \) nor \( x_2 \), it is not below \( y \), and \( P_T(\rho, \ell) \) does not pass through \( y \). Hence, \( y \) is not visible.

**Case 2.** \( y \) is neither \( w_j \) nor the other parent of the internal node of \( C_j \) for each \( j = 1, 2 \) (Figure 7c).

In this case, for each \( j = 1, 2, x_j \) is either below \( x_j' \) or there is a path from \( \rho \) to \( w_j \) that avoids \( y \). Applying Lemma 2.3 on the set of reticulations \( \{x_1, x_2\} \) and \( y \), we conclude that there is a path from \( \rho \) to \( \ell \) that avoids \( y \) for any leaf \( \ell \) below \( x_1 \) or \( x_2 \). Clearly, for any leaf \( \ell \) not below \( x_1 \) or \( x_2 \), \( P_T(\rho, \ell) \) avoids \( y \). Therefore, \( y \) must be not visible. That \( y \) and \( t \) are two consecutive nodes and not visible contradicts that \( N \) is nearly-stable.

After proving that the path \( P_{p(t)1} \) is actually an edge \((p(t), t)\), we now prove that \( w(P_{p(t)1}) \leq 2 \). Assume on the contrary \( w(P_{p(t)1}) \geq 3 \). Then, the child \( f_3 \) of \( p(t) \) in \( P_{p(t)2} \) must be a reticulation node (Figure 7d). Then, the set of reticulations \( \{f_3, x_1, x_2\} \) and \( p(t) \) satisfy the conditions in Lemma 2.3, so there exists a path from \( \rho \) to \( \ell \) that does not pass through \( p(t) \) for any leaf \( \ell \) below \( p(t) \) in \( T \). For any leaf \( \ell \) not below \( p(t) \), the tree path \( P_C(\rho, \ell) \) does not pass through \( p(t) \). Hence, \( p(t) \) is not visible. That \( p(t) \) and \( t \) are not visible contradicts that \( N \) is nearly-stable network.

(d) If \( P_{t3} = P_0 \), then \( t \) is an ancestor of any degree-3 node in \( T \). Since \( N \) is acyclic, there does not exist any cross path \( C \in P \) from \( w \) to \( x \), such that \( x \in P_0 \) while \( w \in P_i \) for \( i > 0 \). Hence \( w(P_0) = 0 \).

If the weight of \( P_{t1} \) and \( P_{t2} \) are both \( 2 \), and if \( w_i \) is the start node of the cross path \( C_i \) that enters \( P_{t_i} \) for \( i = 1, 2 \), either \( w_1 \) or \( w_2 \) is a node in \( P_0 \).
Following the proof of fact (c), we conclude that $t$ and its parent in $P_0$ are both not visible, a contradiction.

\[\square\]

**Theorem 5.5** Let $N$ be a nearly-stable network with $n$ leaves. Then, $|\mathcal{R}(N)| \leq 3(n - 1)$.

**Proof** Let $V$ denote the set of internal nodes of degree 3 in $T$, and let

$$V_i = \{v \in V \mid w(P_{v1}) + w(P_{v2}) = i\}.$$

For any $v \in V$, we define $p(v)$ to be the start node of the trivial path $P_{v3}$ that enters $v$. By Proposition 5.4 (c) and (d), that $v \in V_4$ implies $p(v) \in V_0 \cup V_1 \cup V_2$. Additionally, there are at most two different nodes $v'$ and $v''$ in $V_4$ such that $p(v') = p(v'') \in V_0$, as there are only two trivial paths leaving a degree-3 tree node in $T$; for different $v'$ and $v''$, if $p(v')$ and $p(v'')$ are in $V_1 \cup V_2$, then $p(v') \neq p(v'')$. Taken together, the two facts imply that $|V_4| \leq 2|V_0| + |V_1| + |V_2|$. Since $w(P_{v3}) = 0$, we have

$$|\mathcal{R}(N)| = \sum_{v \in V} [w(P_{v1}) + w(P_{v2})] = |V_1| + 2|V_2| + 3|V_3| + 4|V_4| 
\leq 2|V_0| + 2|V_1| + 3|V_2| + 3|V_3| + 3|V_4| 
\leq 3(|V_0| + |V_1| + |V_2| + |V_3| + |V_4|) 
= 3(n - 1).$$

\[\square\]

### 6 Stable-child network

The stable-child network shown in Figure 8 has as many as $7n - 1$ reticulation nodes. In this section, we shall prove that a stable-child network can have $7n - 1$ reticulation nodes at most.
We first transform a stable-child network to a reticulation visible network and then to a binary tree with the same leaves by removing some edges into reticulations nodes.

**Proposition 6.1** Let $N$ be a stable-child network. There is a set of edges $E$ such that (1) $N - E$ is a subdivision of a reticulation visible network over the same leaves as $N$, and (2) $E$ contains exactly an incoming edge for a reticulation node if it is not visible in $N$.

**Proof** For a reticulation node $r$ that is not visible in $N$, its unique child must be a visible reticulation node. Furthermore, since each node has a visible child, its parents both have a visible child other than $r$ and are both a tree node. To transform $N$ into a reticulation visible network, we will delete one or two edges around a reticulation if it is not visible.

For each reticulation node $r$ that is not visible, we consider the following three cases. If $r$ and its unique child $c$ have a common parent $p$ (Figure 8b), then $(p, r)$ is removed.

If $r$ and its child $c$ do not have a common parent, but $r$ has a reticulation sibling $r'$ such that the parent $w$ of the common parent $p$ of $r$ and $r'$ is the other parent of $r'$ (Figure 8c), $(p, r)$ and $(w, r')$ are then deleted at the same time.

When neither occurs (Figure 8d), we arbitrarily select an incoming edge of $r$ to remove.

The edges removed in the above process is called **SC-edges**. Each SC-edge is from a tree node to a reticulation node. A SC-edge is **concealed** if the head is a visible reticulation node; it is **revealed** otherwise. Note that a concealed SC-edge is deleted only when the case shown in Figure 8c is satisfied. Therefore, a concealed SC-edge jumps over the associated revealed SC-edge that is removed at the same time. It is not hard to see that the SC-edges that are removed when different reticulation nodes are considered are different. Let $E$ be the set of SC-edges.
First, we only deleted an incoming edge for each reticulation node and did not delete the incoming edge for each tree node, so the resulting network $N - E$ is connected. Second, $N - E$ has the same leaves as $N$. The reasons for this include that (i) we do not remove any outgoing edge of a reticulation node, and (ii) for any tree node $t$, if an outgoing edge of it is removed, the other outgoing edge enters another tree node and thus has never not been removed.

Now, we show that $N - E$ is a subdivision of a binary reticulation visible network. Since we had deleted an incoming edge for a reticulation node if it is not visible, all the remaining reticulation nodes are visible. $N - E$ is reticulation visible. We can also show that there are no two internally disjoint paths from a common tree node to a common reticulation node in which each internal node is of degree 2, implying that $N - E$ is a subdivision of a binary reticulation visible network.

Assume on the contrary there are two internally disjoint path $P_1$ and $P_2$ between $u$ and $v$ such that their internal nodes are all of degree 2. If neither $P_1$ nor $P_2$ is a single edge, then the two children of $u$ in $P_1$ and $P_2$ are both not visible, contradicting that $N$ is a stable-child network. Therefore, either $P_1$ or $P_2$ is a single edge from $u$ and $v$.

Without loss of generality, we may assume that $P_2$ is equal to the edge $(u, v)$. According to the three rules which we used to remove the edges in $E$, if an incoming edge of a node is removed, its child in $N - E$ is visible in $N$. This implies that $N - E$ does not contain a path consisting of two or more degree-2 nodes that are not visible in $N$. Therefore, $P_1$ has exactly one internal node $x$. If $x$ is a tree node in $N$, then, we removed an outgoing edge of $x$ according to either the second or third case. In the former case, we remove $(u, v)$ at the same time. In the later case, $(u, v)$ does not exist in $N$. This is impossible. When $x$ is a reticulation node, we removed an incoming edge of it. Again, the edge $(u, v)$ does not exist in $N$ in each possible case, a contradiction.

We have proved that $N - E$ is a subdivision of a binary network. \[\square\]

Let $N' = N - E$ be the subnetwork obtained after the removal of the edges in $E$. $N'$ is a subdivision of a reticulation visible network. By Theorem 2.2 there exist a set of paths $\mathcal{P}$ such that (i) $T := N - E - \cup_{P \in \mathcal{P}} IV(P) - \cup_{P \in \mathcal{P}} E(P)$ is a subtree of $N$ with the same leaves and (ii) all the internal nodes in each path in $\mathcal{P}$ are of degree 2.

Again, we use $P_0, P_1, ..., P_{2n-2}$ to denote the trivial paths in $T$, where $P_0$ denotes the trivial path starting with $\rho(N)$. As in the last section, a path in $\mathcal{P}$ is called a non-cross path if its start and end nodes are both in $P_j$ for some $j$; it is called a cross path otherwise.

**Lemma 6.2** Let $P$ be a path in $\mathcal{P}$.

(a) Every internal node in $P$ is not visible in $N$.

(b) If $P$ is a non-cross path, it is simply an edge.

(c) If $P$ is a cross path and ends at a node $x$ in the trivial path $P_j$, every node between the start node of $P_j$ and $x$ in $T$ is not visible in $N$. 
(d) If \( P \) is a cross path and there is a non-cross path \( P' \) jumping over it, then either \( P \) is an edge or the start node of \( P \) is the parent of the end node of \( P' \) in \( T \).

**Proof** (a) and (b) are essentially the restatement of the fact (a) and (c) in Lemma 5.2.

(c) Let \( y \) be a node between the start node of \( P_1 \) and \( x \) in \( T \). For any leaf \( \ell \) that is not below \( x \) in \( T \), \( P_T(\rho, \ell) \) is a path that does not pass \( x \) and hence \( y \). For any leaf \( \ell \) below \( x \) in \( T \), the path \( P_T(\rho, w) + P + P_T(x, \ell) \) avoids \( y \), as \( w \) is the start node of \( P \) in a trivial path different from \( P_1 \). Hence, \( y \) is not visible in \( N \).

(d) By (b), \( P' \) is simply an edge \( (u, v) \) in \( N' \). Let \( P \) start at a node \( w \). If neither \( P_T(w, v) \) nor \( P \) is a single edge, the two children of \( w \) in \( P \) and \( P_T(w, v) \) are both not visible, contradicting that \( N \) is stable-child. \( \Box \)

Let \( r \) be a reticulation node and not visible in \( N \). Then, a revealed SC-edge \( e_r \) was removed from \( r \) to obtain \( N' \) from \( N \). We define the cost \( c(r) \) of \( r \) to be:

\[
e(r) = \begin{cases} 2 & \text{if a concealed SC-edge is associated with } e_r, \\ 1 & \text{otherwise.} \end{cases}
\]

Recall that \( U_s = \text{parent}(s) \cap \mathcal{UR}(N) \). We can define the cost of a cross path \( C \in P \) from \( w \) to \( x \) as follows:

\[
e(C) = \begin{cases} 2 + \sum_{r \in U_s \cup U_v} c(r) & \text{if a non-cross edge } (u, v) \text{ jumps over } w, \\ 1 + \sum_{r \in U_s} c(r) & \text{otherwise.} \end{cases}
\]

We further charge the cost of \( C \) to the trivial path \( P_t \) to which \( x \) belongs and call it the weight of \( P_t \), written \( w(P_t) \). If there is no cross entering \( P_t \), the weight of \( P_t \) is set to be 0.

As for nearly-stable networks, we have that \( |\mathcal{R}(N)| \leq \sum_{i=0}^{2n-2} w(P_i) \).

For an internal node \( t \) with degree-3 in \( T \), we still use \( P_{11} \) and \( P_{12} \) to denote the trivial paths leaving \( t \) and \( P_{13} \) to denote the trivial path entering \( t \).

**Proposition 6.3** For each internal node \( t \) of degree 3 in \( T \),

(a) \( w(P_{11}) \leq 6, j = 1, 2 \).

(b) \( w(P_{11}) + w(P_{12}) \leq 10 \).

(c) If \( P_{13} \neq P_0 \) and \( w(P_{11}) + w(P_{12}) \geq 8 \), then \( w(P_{13}) = 0 \). Moreover, assume \( p(t) \) is the start node of \( P_{13} \) and \( P_{13} = P_{p(t)1} \). Then \( w(P_{p(t)1}) \leq 4 \).

(d) If \( P_{13} = P_0 \), then \( w(P_{13}) = 0 \) and \( w(P_{11}) + w(P_{12}) \leq 7 \).

**Proof** (a) We will only prove that \( P_{11} \leq 6 \). Let \( C_1 \) denote the cross path sending at a node \( x_1 \) in \( P_{11} \). Let \( w_1 \) be the start node of \( C_1 \) in a trivial path different from \( P_{11} \). Note that \( U_s = \mathcal{UR}(N) \cap \text{parent}(s) \).

If there is no non-cross edge jumping over \( w_1 \), then there are at most 2 elements in \( U_s \), and each element can have two unit cost at most. Thus, by Eqn. 2, \( w(P_{11}) \leq 5 \).
If there is a non-cross edge \((u_1, v_1)\) jumping over \(w_1\). By the fact (b) in Lemma 6.2, \(U_{v_1}\) is empty or a singleton. Moreover, by the fact (d) in Lemma 6.2, \(C_1\) is an edge, or \((u_1, v_1)\) is an edge in \(T\). If \(C_1\) is an edge, then \(|U_{x_1}| \leq 1\). If \((u_1, v_1)\) is an edge, \(|U_{v_1}| = 0\). Both implies that \(|U_{x_1}| + |U_{v_1}| \leq 2\). Therefore, by Eqn. 2, \(w(P_{11}) \leq 2 + 2(|U_{x_1}| + |U_{v_1}|) \leq 6\).

We remark that, if the parent of \(x_1\) in \(T\) is not a reticulation node in \(N\), then \(|U_{x_1}| + |U_{v_1}| \leq 1\), and therefore \(w(P_{11}) \leq 4\). Equality holds only if there is a non-cross edge jumping over \(x\).

(b) If \(w(P_{11}) \neq 0\) and \(w(P_{12}) \neq 0\), we assume that the cross path \(C_i\) ending at a node \(x_i\) in \(P_i\), starts at \(w_i\) for \(i = 1, 2\). By the fact (c) of Lemma 6.2, every internal node in \(P_T(t, x_1)\) and \(P_T(t, x_2)\) is not visible. If there is a node between \(t\) and \(x_i\) for each \(i\), then the two children of \(t\) are not visible in \(N\), contradicting that \(N\) is stable-child. So \(t\) is the parent of either \(x_1\) or \(x_2\). Without loss of generality, we may assume that \(t\) is the parent of \(x_1\) in \(T\). By the remark in the end of the proof of (a), \(w(P_{11}) \leq 4\) and hence \(w(P_{11}) + w(P_{12}) \leq 4 + 6 = 10\).

(c) Assume that \(P_3 = \rho_{p(t)}\). If \(w(P_{11}) + w(P_{12}) \geq 8\), by (a), the weights of \(P_{11}\) and \(P_{12}\) are both not zero. Hence, there is a cross path \(C_i\) ending at a node \(x_i\) in \(P_i\) and starting at a node \(w_i\) in a trivial path different from \(P_i\) for each \(i = 1, 2\).

We first show that \(w_i\) either (i) below \(x_{3-i}\), or (ii) neither in \(P_3\) nor below \(t\) for \(i = 1\) and 2. Without loss of generality, we assume \(i = 1\).

**Case 1.** \(w_1\) is in \(P_T(t, x_2)\) (Figure 5b).

If there is a non-cross edge \((u_1, v_1)\) jumping over \(w_1\), \(v_1\) is either in \(P_T(u_1, x_2)\) or \(v_1\) is below \(x_2\) in \(P_2\). The former implies that \(v_1\) is not visible, whereas the latter implies that \(x_2\) is not visible. This contradicts that both \(v_1\) and \(x_2\) are visible in \(N\).

Since \(w_1\) is an internal node between \(t\) and \(x_2\) in \(T\), then \(t\) is the parent of \(x_1\) in \(T\). By the fact (a) and (c) of Lemma 6.2, each internal node in \(C_1\) or \(P_T(t, x_1)\) is not visible. Thus, \((w_1, x_1)\) or \((w_1, x_2)\) is an edge. Otherwise, the two children of \(w_1\) are not visible, contradicting that \(N\) is stable-child network.

If \((w_1, x_1)\) is an edge in \(N\), then \(w(P_{11}) = 1\) and hence \(w(P_{11}) + w(P_{12}) \leq 7\), a contradiction. If \((w_1, x_2)\) is an edge, then \(w(P_{11}) \leq 3\). Since \(w_1\) is not a reticulation, by the remark at the end of the proof of (a), \(w(P_{12}) \leq 4\). Therefore, \(w(P_{11}) + w(P_{12}) \leq 7\). This is impossible.

**Case 2.** \(w_1\) is a node in the path \(P_3\).

Without loss of generality, we can assume \(w_1\) is lower than \(w_2\) if \(w_2\) is also in \(P_3\). We claim that \(t\) and all the internal nodes in \(P_T(u_1, t)\) are not visible. Let \(u\) be either \(t\) or an internal node in \(P_T(u_1, t)\). If \(u\) is in \(P_T(t, x_1)\), the case is symmetric to Case 1. So there are two cases to consider: either \(w_2\) is below \(x_1\) (Figure 5c), or \(w_2\) is not below \(w_1\) (Figure 5d). In both cases, the reticulation node set \(\{x_1, x_2\}\) are below the node \(u\), and satisfies the condition in Lemma 2.3, so \(u\) is not visible with respect to each leaf \(\ell\) below either \(x_1\) or \(x_2\). For any leaf \(\ell\) not below \(x_1\) or \(x_2\), the path \(P_T(\rho, \ell)\) avoids \(u\). Hence \(u\) is not visible in \(N\).

There are some observations from this result. First, there is no non-cross edge \((u_1, v_1)\) jumping over \(w_1\), otherwise \(v_1\) is not visible. Second, a child of
Fig. 9 Four cases that are considered in the proof of Proposition 6.3. (a) \( w_1 \) is between \( t \) and \( x_2 \), (b) \( w_1 \) is the trivial path \( P_{13} \) entering \( t \), and \( w_2 \) is below \( x_1 \). (c) \( w_2 \) is in \( P_{13} \), and \( w_2 \) is not below \( x_1 \). (d) \( w_1 \) is below \( w_2 \) and \( w_2 \) is also below \( x_1 \). This is impossible in a network

\( w_1 \) in \( P_{13} \) is not visible, and so the cross path \( C_1 \) is simply an edge. Otherwise by the fact (a) of Lemma 6.2, the two children of \( w_1 \) are both not visible in \( N \).

By (b) of Lemma 6.2, either \((t, x_1)\) or \((t, x_2)\) is an edge in \( T \). If \( t \) is the parent of \( x_1 \) in \( T \), then \( w(P_{11}) = 1 \) according to Eqn. 2. By (a), \( w(P_{11}) + w(P_{12}) \leq 7 \). If \( t \) is the parent of \( x_2 \), \( w(P_{11}) \leq 3 \) and \( w(P_{12}) \leq 4 \) according to the remark at the end of the proof of (a). Taken together, both facts imply that \( w(P_{11}) + w(P_{12}) \leq 7 \), which contradicts the assumption that \( w(P_{11}) + w(P_{12}) \geq 8 \).

**Case 3.** \( w_1 \) is below \( x_2 \) and \( w_2 \) is below \( x_1 \) (Figure 9d).

This case is impossible since there is a cycle in \( N \), contradicting \( N \) is acyclic.

To sum up, \( w(P_{11}) + w(P_{12}) \geq 8 \) implies that either (i) or (ii) is true. But in both cases, if we let \( u \) be either \( t \) or any internal node in \( P_{13} \), the set of reticulations \( \{x_1, x_2\} \) is below \( u \) and satisfies the condition in Lemma 6.2. Therefore, \( t \) and any internal node of \( P_{13} \) are not visible.

There are two observations from this result. First, \( w(P_{13}) = 0 \) because there is no cross path that ends at \( P_{13} \). (Otherwise the cross path enters \( P_{13} \) at a reticulation that is not visible in \( N \).) Second, the child of \( p(t) \) in \( P_{13} = P_{p(t)1} \) is not visible.

Clearly, \( w(P_{p(t)2}) = 0 \) if there is no cross path that ends at \( P_{p(t)2} \). Assume there is a cross path \( C_3 \) from \( w_3 \) to \( x_3 \) with \( x_3 \) in \( P_{p(t)2} \). By (c) of Lemma 6.2, each internal node in \( P_T(p(t), x_3) \) is not visible. But the child of \( p(t) \) in \( P_{p(t)1} \) is not visible. Hence, \( p(t), x_3 \) is an edge in \( T \). Then, by the remark in the end of proof of (a), \( w(P_{p(t)2}) \leq 4 \).

**d** If \( P_3 = P_0 \), then \( t \) is an ancestor of any degree-3 node in \( T \). Since \( N \) is acyclic, there does not exist any cross path \( P(u, v) \in P \) such that \( u \in P_1 \) for \( i > 0 \) and \( v \in P_0 \). Hence, \( w(P_0) = 0 \).

If \( w(P_{11}) + w(P_{12}) \geq 8 \), then, every node in \( P_0 \) is not visible in \( N \), shown in (c). This contradicts that \( P_0 \) contains the network root \( \rho(N) \) and \( \rho(N) \) is visible with respect to each leaf in \( N \).

**Theorem 6.4** Let \( N \) be a stable-child network with \( n \) leaves. Then, \( |R(N)| \leq 7(n - 1) \).
Proof Let $V$ denote the set of the $(n - 1)$ internal nodes of degree 3 in $T$. Define

$$V_i = \{v \in V | w(P_{v1}) + w(P_{v2}) = i\}.$$ 

By Proposition 6.3 (b), $w(P_{v1}) + w(P_{v2}) \leq 10$. Hence, $V = \bigcup_{i=0}^{10} V_i$, and thus $\sum_{i=0}^{10} |V_i| = n - 1$.

Let $p(v)$ be the start node of the trivial path entering $v$ in $T$. By Proposition 6.3 (c) and (d), if $v \in V_j$, then $p(v) \in V_0 \cup V_1 \cup V_2 \cup V_3 \cup V_4$ for each $j \in \{8, 9, 10\}$. By Proposition 6.3 (c), under the mapping $p(\cdot)$, at most two nodes in $V_8 \cup V_9 \cup V_{10}$ are mapped to the same node in $V_0$, and only one node can be mapped to $V_1 \cup V_2 \cup V_3 \cup V_4$. Thus,

$$|V_8| + 2|V_9| + 3|V_{10}| \leq 3(|V_8| + |V_9| + |V_{10}|) \leq 6|V_0| + 3(\sum_{i=1}^{4} |V_i|).$$

Since $w(P_0) = 0$, the above inequality implies that

$$|R(N)| = \sum_{v \in V} [w(P_{v1}) + w(P_{v2})]$$

$$= \sum_{i=0}^{10} i|V_i|$$

$$\leq |V_1| + 2|V_2| + 3|V_3| + 4|V_4| + 7 \sum_{i=5}^{10} |V_i| + 3|V_{10}|$$

$$\leq |V_1| + 2|V_2| + 3|V_3| + 4|V_4| + 7 \sum_{i=5}^{10} |V_i| + 6|V_0| + 3 \sum_{i=1}^{4} |V_i|$$

$$= 6|V_0| + 4|V_1| + 5|V_2| + 6|V_3| + 7|V_4| + 7 \sum_{i=5}^{10} |V_i|$$

$$\leq 7 \sum_{i=0}^{10} |V_i|$$

$$= 7(n - 1).$$

\( \square \)

7 Conclusion

We have established the tight upper bounds for the sizes of galled, nearly-stable, and stable-child networks. Since the number of internal tree nodes is equal to the number of leaves plus the number of reticulation nodes in a binary network, we can summarize our results in Table 1. Without question, these tight bounds provide insight to the study of combinatorial and algorithmic aspects of the network classes defined by visibility property.
Bounding the Size of a Network Defined By Visibility Property

Table 1 The tight upper bounds on the sizes of binary networks with n leaves defined by visibility property. The bound for reticulation visible network is found in Bordewich and Semple(2015).

| Network Type                  | No. of reticulation nodes | No. of internal tree nodes |
|-------------------------------|---------------------------|----------------------------|
| Galled Network                | $2(n - 1)$                | $3(n - 1)$                 |
| Reticulation visible network  | $3(n - 1)$                | $4(n - 1)$                 |
| Nearly-stable network         | $3(n - 1)$                | $4(n - 1)$                 |
| Stable-child network          | $7(n - 1)$                | $8(n - 1)$                 |

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