Application of Quasisubordination to Certain Classes of Meromorphic Functions

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Inequalities play a fundamental role in many branches of mathematics and particularly in real analysis. By using inequalities, we can find extrema, point of inflection, and monotonic behavior of real functions. Subordination and quasisubordination are important tools used in complex analysis as an alternate of inequalities. In this article, we introduce and systematically study certain new classes of meromorphic functions using quasisubordination and Bessel function. We explore various inequalities related with the famous Fekete-Szego inequality. We also point out a number of important corollaries.

1. Introduction

A complex valued function is said to be meromorphic if it has poles as its only singularities. Let \( \Sigma_1 \) denotes the class of all of meromorphic functions which has a simple pole at \( \omega = 0 \) and has Laurent series expansion of the form:

\[
\lambda(\omega) = \frac{1}{\omega} + \sum_{i=0}^{\infty} a_i \omega^i, \tag{1}
\]

which are analytic in the punctured open unit disc \( U^* = \{ \omega : \omega \in \mathbb{C} \text{ and } 0 < |\omega| < 1 \} = \overline{U} - \{0\} \), as open unit disc \( U = U^* \cup \{0\} \).

Here, we are listing some important subclasses of meromorphic functions which will be used in our subequal work. In 1936, Robertson [1] introduced the classes of meromorphic starlike and meromorphic convex functions of order \( \alpha \).

By \( \Sigma^{MS}_1(\alpha) \), we mean the subclass of \( \Sigma_1 \) consisting of all meromorphic starlike functions of order \( \alpha \). Analytically, \( \lambda(\omega) \in \Sigma^{MS}_1(\alpha) \) if

\[
\Re \left( \frac{\omega \lambda'(\omega)}{\lambda(\omega)} \right) < -\alpha, \quad (0 \leq \alpha < 1; \omega \in U^*). \tag{2}
\]

A closely related class of meromorphic convex functions of order \( \alpha \) is denoted by \( \Sigma^{MC}_1(\alpha) \) and defined as

\[
\lambda(\omega) \in \Sigma^{MC}_1(\alpha) \iff -\omega \lambda'(\omega) \in \Sigma^{MS}_1(\alpha). \tag{3}
\]

In 1952, W. Kaplan [2] introduced and studied an important class of analytic functions known as close-to-convex functions in the open unit disc \( U \). A function \( \lambda \) belongs to \( \Sigma_1 \), is in class \( \Sigma^{MC}_1(\alpha, \beta) \), of meromorphic close-to-convex functions of order \( \alpha \) and type \( \beta \) if there exist \( \delta(\omega) \in \Sigma^{MS}_1(\beta) \), and

\[
\Re \left( \frac{\omega \lambda'(\omega)}{\delta(\omega)} \right) < -\alpha. \tag{4}
\]
Let $\delta(\omega) \in \Sigma_1$ and having series representation of the form

$$\delta(\omega) = \frac{1}{\omega} + \sum_{t=0}^{\infty} b_t \omega^t. \quad (5)$$

Then, the convolution of $\lambda$ and $\delta$ as denoted by $\lambda * \delta$ is defined as

$$(\lambda * \delta)(\omega) = \frac{1}{\omega} + \sum_{t=0}^{\infty} a_t b_t \omega^t = (\delta * \lambda)(\omega), \quad (6)$$

where $\lambda$ is given by (1).

A function $\lambda$ is subordinate to $\delta$ in $U^*$ and written as $\lambda(\omega) < \delta(\omega)$, if there exists a Schwarz function $k(\omega)$, which is holomorphic in $U$ with $k(0) = 0$, such that $\lambda(\omega) = \delta(k(\omega))$. Let $\phi(\omega)$ be an analytic function with positive real part on $U$ satisfies $\phi(0) = 1$ and $\phi'(0) > 0$ which maps $U$ which is star shape with respect to $\omega = 1$, also symmetric with respect to the real axis. We denote $\overline{\Sigma}(\phi)$ be the class of function $\lambda \in \Sigma_1$ for which $-\omega \lambda'(\omega)/\lambda(\omega) < \phi(\omega)$, $\omega \in U^*$. The class $\overline{\Sigma}(\phi)$ was introduced and studied by Silverman et al. [3] (see also [4]). The class $\Sigma(\alpha)$ is a special case of the class $\Sigma(\phi)$ when $\phi(\omega) = 1 + (1 - 2\alpha)/\omega/1 - \omega (0 \leq \alpha < 1)$.

Robertson [5] gave the idea of quasisubordination. For any two functions $\lambda(\omega)$ and $\delta(\omega)$, holomorphic in $U$, the function $\lambda(\omega)$ is said to be quasi-subordinate to the function $\delta(\omega)$ written as $\lambda(\omega) < q \delta(\omega)$, if there exists two holomorphic functions $\lambda(\omega)$ and $\phi(\omega)$, with $|\phi(\omega)| \leq 1$, $\lambda(\omega)/\phi(\omega)$ is holomorphic in $U^*$ and such that $\lambda(\omega) = \phi(\omega) \delta(k(\omega))$. In particular, if $\phi(\omega) = 1$, then quasisubordination reduces to subordination. Furthermore, if $k(\omega) = 1$, the quasisubordination becomes the majorization, (see [6]), which implies

$$\lambda(\omega) < q \delta(\omega) \Rightarrow \lambda(\omega) = \phi(\omega) \delta(\omega) \Rightarrow \lambda(\omega) < \phi(\omega(\omega \in U^*). \quad (7)$$

For recent work on meromorphic functions, we refer [7–18].

Motivated from the above cited work, we introduce the following subclasses of meromorphic functions. Throughout in this paper, we shall assume $0 \leq \gamma < 1$, $\gamma \neq 1/2$, $0 \leq \eta < 1$, $\omega \in U^*$, $\lambda, \delta \in \Sigma_1$, and $\phi(\omega)$ be an analytic function with positive real part on $U$ that satisfies $\phi(0) = 1$ and $\phi'(0) > 0$ which maps $U$ which is star shape with respect to $\omega = 1$, and also symmetric with respect to the real axis unless otherwise mentioned.

**Definition 1.** Let $\Sigma_q^{MC}(\phi, \gamma)$ be the class of functions $\lambda(\omega) \in \Sigma_1$ and satisfy

$$(1 - \gamma) \left( \frac{\omega \lambda'(\omega)}{\lambda(\omega)} \right) + \gamma \left( 1 + \frac{\omega \lambda''(\omega)}{\lambda(\omega)} \right) - 1 < \phi(\omega) - 1, \quad (\omega \in U^*). \quad (8)$$

The abovementioned class $\Sigma_q^{MC}(\phi, \gamma)$ is the meromorphic analogue of the class $S_q(\phi)$ introduced and studied by Mohd and Darus [19]. For $\gamma = 0$, the class $\Sigma_q^{MC}(\phi)$ was studied by Zayed et al. [20].

**Definition 2.** Let $\Sigma_q^{MK}(\phi, \eta)$ be the subclass of $\Sigma_1$ consisting of all functions $\lambda(\omega)$ for which their exist $\delta(\omega) \in \Sigma_q^{MS}(\phi)$ and satisfy

$$-\omega \lambda'(\omega) \left( \frac{1}{(1 - \eta) \delta(\omega) + \eta \omega \delta'(\omega)} \right) - 1 < \phi(\omega) - 1, \quad (\omega \in U^*). \quad (9)$$

For $\eta = 0$ and $\delta(\omega) = \lambda(\omega)$, the class $\Sigma_q^{MS}(\phi)$ was studied by Zayed et al. [20].

In this paper, we obtain the Fekete-Szegö inequality for meromorphic functions belonging to above defined classes. Let $k \in \Omega$ denote the class of functions of the form $k(\omega) = k_1 \omega + k_2 \omega^2 + k_3 \omega^3 + \cdots$, satisfying $|k(\omega)| < 1$, for $\omega \in U^*$. For more details, see [21–23]. To prove our main results, we need the following lemma.

**Lemma 3** [24]. If $k \in \Omega$, then for any complex number $u$, $|u^2 - k_2| \leq \max \|1 ; u\|$, the result is sharp for the functions given by $k(\omega) = \omega$ or $k(\omega) = \omega^2$.

### 2. Main Results

In this section, we explore certain Fekete-Szegö-related inequalities for the class $\Sigma_q^{MC}(\phi, \gamma)$ and $\Sigma_q^{MK}(\phi, \eta)$.

**Theorem 4.** Let $\phi(\omega) = 1 + B_1 \omega + B_2 \omega^2 + \cdots, B_i > 0$, and $\phi(\omega) = c_0 + c_1 \omega + c_2 \omega^2 + \cdots$ if $\lambda(\omega)$ given by (1) be in the class $\Sigma_q^{MC}(\phi, \gamma)$, and $\mu$ is a complex number, then

$$|a_i - \mu a_i^2| \leq \frac{B_i}{2(1 - \gamma)} \left[ 1 + \max \left\{ 1, \frac{|B_i|}{|B_1|} + \frac{1}{|1 - \gamma| - 2\mu \left( \frac{1 - 2\gamma}{1 - \gamma} \right)^2} \right\} \right]. \quad (10)$$

The inequalities are sharp for $k(\omega) = \omega$ or $k(\omega) = \omega^2$.

**Proof.** Let $\lambda(\omega) \in \Sigma_q^{MC}(\phi, \gamma)$, then there exist analytic functions $\phi(\omega)$ and $k(\omega)$, with $|\phi(\omega)| < 1$, $k(0) = 0$, and $k(\omega) < 1$ such that

$$\left[ (1 - \gamma) \left( \frac{\omega \lambda'(\omega)}{\lambda(\omega)} \right) + \gamma \left( 1 + \frac{\omega \lambda''(\omega)}{\lambda(\omega)} \right) \right] - 1 \leq \phi(\omega) - 1, \quad (\omega \in U^*). \quad (11)$$

Taking first and second derivative of (1), and use in the left hand side of above equation, we obtain
then implies
\[
- \left[ (1 - \gamma) \left( \frac{\omega \lambda'(\omega)}{\lambda(\omega)} \right) + \gamma \left( 1 + \frac{\omega \lambda''(\omega)}{\lambda'(\omega)} \right) \right] = (1 - \gamma) [1 - a_0 \omega + (a_0^2 - 2a_1) \omega^2] + \gamma(1 + 2a_1 \omega^2) + \ldots,
\]

(12)

By using this fact and the well-known inequality \(|k_1| \leq 1\), we get
\[
|a_1 - \mu a_0^2| \leq \frac{B_i}{2(1 - 2\gamma)} \left[ 1 + \frac{B_2}{B_1} \left( \frac{1}{1 - \gamma} \right)^2 - 2\mu \left( \frac{1 - 2\gamma}{1 - \gamma} \right)^2 \right].
\]

(20)

Corollary 5. For \(\phi(\omega) = 1\) and \(\gamma = 0\) in Theorem 4, we obtain the result by Silverman et al. [3] (see Theorem 4).

For \(k(\omega) = \omega\) and repeating steps of Theorem 4, we obtain the following corollary.

Corollary 6. Let \(\lambda(\omega) \in \mathcal{S}_1\) satisfies
\[
- \left[ (1 - \gamma) \left( \frac{\omega \lambda'(\omega)}{\lambda(\omega)} \right) + \gamma \left( 1 + \frac{\omega \lambda''(\omega)}{\lambda'(\omega)} \right) \right] = 1 < \phi(\omega) - 1, \quad (\omega \in U^*),
\]

(21)

then for any complex number \(\mu\),
\[
|a_1 - \mu a_0^2| \leq \frac{B_i}{2(1 - 2\gamma)} \left[ 1 + \frac{B_2}{B_1} + \left( \frac{1}{1 - \gamma} - 2\mu \left( \frac{1 - 2\gamma}{1 - \gamma} \right)^2 \right) \right].
\]

(22)

Theorem 7. Let \((\omega) = 1 + B_1 \omega + B_2 \omega^2 + \ldots, B_i > 0\), and \(\phi(\omega) = c_0 + c_1 \omega + c_2 \omega^2 + \ldots\), if \(\lambda(\omega), \delta(\omega)\), given by (1) and (5) be in the class \(\sum_{MK}(\phi, \eta)\), and \(\mu\) is a complex number, then
\[
\left\{ b_i + \frac{a_i}{2} - a_1, \eta - \mu b_0^2 \right\} \leq \frac{B_i}{2} \left[ 1 + \max \left\{ 1, \frac{B_2}{B_1} + \left( \frac{1}{1 - \eta} \right)^2 - 2\mu \left( \frac{1 - 2\gamma}{1 - \gamma} \right)^2 \right\} \right].
\]

(23)

The inequalities is sharp for \(k(\omega) = \omega\) or \(k(\omega) = \omega^2\).

Proof. Let \(\lambda(\omega), \delta(\omega) \in \sum_{MK}(\phi, \eta)\), then there exist analytic functions \(\phi(\omega)\) and \(k(\omega)\), with \(|\phi(\omega)| < 1\), \(k(0) = 0\), and \(k(\omega) < 1\) such that
\[
-\omega \lambda'(\omega) - 1 = \phi(\omega), \quad (\omega \in U^*).
\]

(24)

Taking first derivative of (1) and (5), and use in the left hand side of above equation, we obtain
\[-\omega \lambda'(\omega)\]
\[\frac{(1-\eta)\delta(\omega) + \eta \omega \delta'(\omega)}{(1-\eta)\delta(\omega) + \eta \omega \delta'(\omega)} = 1 + 2\eta(1+2\eta) + b_0(1-\eta)(-1-4\eta)\omega + [b_0^2(1-\eta)^2 - b_1 + 2a_i\eta - a_i^2] \omega^2 + \ldots,\]

(25)

then implies

\[-\omega \lambda'(\omega)\]
\[\frac{(1-\eta)\delta(\omega) + \eta \omega \delta'(\omega)}{(1-\eta)\delta(\omega) + \eta \omega \delta'(\omega)} = \frac{1}{2} + \frac{a_i}{2} - \frac{a_i \eta - \mu b_0^2}{(1-\eta)^2(-1-4\eta)^2}.\]

(26)

\[\phi(k(\omega)) = 1 + k_1B_11 + (k_2B_2 + k_3B_1)\omega^2 + (\omega B_1 + 2k_1B_1 + 2k_2B_1)\omega^3 + \ldots,\]

(27)

which implies

\[\phi(\omega)[\phi(k(\omega)) - 1] = c_0k_1B_1\omega + (c_0k_2B_2 + c_0k_2B_1 + c_1k_1B_1)\omega^2 + \ldots.\]

(28)

Comparing (26) and (28), we get

\[b_0 = \frac{c_0k_1B_1}{(1-\eta)(1-4\eta)}, \quad b_1 + a_i(1-\eta) = \frac{B_1 c_0}{2} \left[ k_2 + k_1c_0 \right] + \frac{k_2B_1}{(1-4\eta)^2},\]

(29)

Thus,

\[\frac{b_1 + a_i}{2} - \frac{a_i \eta - \mu b_0^2}{2} = \frac{B_1 c_0}{2} \left[ k_2 + k_1c_0 \right] + \frac{k_2B_1}{(1-4\eta)^2},\]

(30)

Since \(\phi(\omega)\) is analytic and bounded in \(U^*\) (see [25]), we have

\[|c_n| \leq 1 - |c_0|^2 \leq 1, \quad (n > 0).\]

(31)

By using this fact and the well-known inequality \(|k_1| \leq 1,\) we get

\[\frac{b_1 + a_i}{2} - \frac{a_i \eta - \mu b_0^2}{2} \leq \frac{B_1 c_0}{2} \left[ k_2 + k_1c_0 \right] + \frac{k_2B_1}{(1-4\eta)^2} + \frac{2\mu}{(1-\eta)^2(1-4\eta)^2} c_0 k_1 B_1.\]

(32)

**Corollary 8.** For \(\phi(\omega) = 1, \delta(\omega) = \lambda(\omega),\) and \(\eta = 0\) in Theorem 7, we obtain the result by Silverman et al. [3] (see Theorem 7). For \(k(\omega) = \omega\) and repeating steps of Theorem 7, we obtain the following corollary.

**Corollary 9.** Let \(\lambda(\omega)\) and \(\delta(\omega) \in \Sigma_1\) satisfy

\[-\omega \lambda'(\omega)\]
\[\frac{(1-\eta)\delta(\omega) + \eta \omega \delta'(\omega)}{(1-\eta)\delta(\omega) + \eta \omega \delta'(\omega)} \leq 1 - \gamma \phi(\omega) - 1, \quad (\omega \in U^*),\]

(33)

then for any complex number \(\mu,\)

\[\frac{b_1 + a_i}{2} - \frac{a_i \eta - \mu b_0^2}{2} \leq \frac{B_1}{2} \left[ 1 + \frac{B_2}{B_1} + \frac{1}{(1-4\eta)^2} \right] + \frac{2\mu}{(1-\eta)^2(1-4\eta)^2}.\]

(34)

3. Meromorphic Functions Related with the Bessel Function

Let us consider the second order linear homogenous differential equation (see, Baricz [26])

\[\omega^2 k''(\omega) + a_1 k'(\omega) + [b_0 \omega^2 - \eta^2 + (1-a)] k(\omega) = 0.\]

(35)

The function

\[k_{\nu, \omega, \beta}(\omega) = \sum_{n=0}^{\infty} \frac{(-\beta)^n}{\omega^{(n+1)}} \Gamma\left(\nu + 1 + (\alpha + 1/2)\right) \frac{\omega^{n+\nu}}{n!},\]

(36)

is known as generalized Bessel's function of first kind and is the solution of differential equation given in (35). If we denote

\[\xi_{\nu, \omega, \beta}(\omega) = \frac{\omega^2}{\omega^{(n+1)}} \Gamma\left(\nu + 1 + (\alpha + 1/2)\right) \frac{\omega^{n+\nu}}{n!},\]

(37)

where \(\nu, \alpha,\) and \(\beta\) are positive real numbers. The operator \(\xi_{\nu, \omega, \beta}\) is a meromorphic analogue introduced by Deniz [27] (see also Baricz et al. [28]) for analytic functions. In terms of convolution, \(\xi_{\nu, \omega, \beta}\) is given by

\[(\xi_{\nu, \omega, \beta}\lambda)(\omega) = k_{\nu, \omega, \beta}(\omega) * \lambda(\omega) = \frac{1}{\omega} + \sum_{n=0}^{\infty} \frac{(-\beta)^n}{\omega^{(n+1)}} \Gamma\left(\nu + 1 + (\alpha + 1/2)\right) \frac{\omega^{n+\nu}}{n!} \delta_n(\omega).\]

(38)

The operator \(\xi_{\nu, \omega, \beta}\) was introduced and studied by Mostafa et al. [29]. For more details, see [30, 31] and references cited therein. Motivated from the above cited work, we introduce the following classes of meromorphic functions.

**Definition 10.** A function \(\lambda(\omega) \in \Sigma_1\) given by (1) is said to belong to the class \(\Sigma_{\nu, \omega, \beta}(\phi, \gamma)^\beta\) if

\[-(1-\gamma)\left(\frac{\omega (\xi_{\omega, \omega, \beta}(\omega))}{(\xi_{\nu, \omega, \beta}(\omega))} \right) + \gamma \left(1 + \frac{\omega (\xi_{\omega, \omega, \beta}(\omega))}{(\xi_{\nu, \omega, \beta}(\omega))}\right) - 1 < \phi(\omega) - 1.\]

(39)
For $\gamma = 0$, the class $\sum_{V, a, b}^{MS(q)}(\phi)$ was studied by Zayed et al. [20].

**Definition 11.** Let $\sum_{V, a, b}^{MK(q)}(\phi, \eta)$ be the subclasses of $\sum_{V, a, b}^{MS(q)}(\phi)$ consisting of all functions $\lambda(\omega) \in \sum_{V, a, b}^{MS(q)}(\phi)$ for which there exist $\delta(\omega) \in \sum_{V, a, b}^{MS(q)}(\phi)$ and

$$
-\omega(\zeta_{V, a, b}^{\lambda})'(\omega) \\
(1-\eta)(\zeta_{V, a, b}^{\phi})(\omega) + \eta \omega(\zeta_{V, a, b}^{\lambda})'(\omega) - 1, \quad (\omega \in U^*).
$$

(40)

For $\eta = 0$ and $\delta(\omega) = \lambda(\omega)$, the class $\sum_{V, a, b}^{MS(q)}(\phi)$ was studied by Zayed et al. [20].

**Theorem 12.** Let $(\omega) = 1 + B_1 \omega + B_2 \omega^2 + \cdots, B_i > 0$, and $\varphi(\omega) = c_0 + c_1 \omega + c_2 \omega^2 + \cdots, \text{ if } \lambda(\omega) \text{ given by (1) be in the class } \sum_{V, a, b}^{MC(q)}(\phi, \gamma)$, and $\mu$ be a complex number, then

$$
|a_1 - \mu a_0^2| \leq \frac{4(h + (a + 1/2))(v + (a + 1/2)B_i)}{\beta_1(1 - 2y)} \\
\times \left[ 1 + \max \left\{ 1, \frac{B_i}{B_1} \right\} - \mu \left( \frac{2v + (a + 1/2)(1 - 2y)}{(v + (a + 1/2))(1 - y)} \right)^{-1/2} \right].
$$

(41)

**Proof.** Let $\lambda(\omega) \in \sum_{V, a, b}^{MC(q)}(\phi, \gamma)$, then there exist analytic functions $\varphi(\omega)$ and $k(\omega)$, with $|\varphi(\omega)| < 1, k(0) = 0$, and $k(\omega) < 1$ such that

$$
-\left[ (1-y) \left( \frac{\omega(\zeta_{V, a, b}^{\lambda})'(\omega)}{(\zeta_{V, a, b}^{\lambda})(\omega)} \right) + y \left( 1 + \frac{\omega(\zeta_{V, a, b}^{\lambda})'(\omega)}{(\zeta_{V, a, b}^{\lambda})(\omega)} \right) \right] - 1
= \varphi(\omega)|\phi(k(\omega)) - 1|. \quad (42)

Taking first and second derivative of (38), in use of the left side of the above equation, we obtain

$$
-\left[ (1-y) \left( \frac{\omega(\zeta_{V, a, b}^{\lambda})'(\omega)}{(\zeta_{V, a, b}^{\lambda})(\omega)} \right) + y \left( 1 + \frac{\omega(\zeta_{V, a, b}^{\lambda})'(\omega)}{(\zeta_{V, a, b}^{\lambda})(\omega)} \right) \right] \\
= (1-y) \left[ 1 + \frac{\beta_0 \omega}{4(v + (a + 1/2))} \right. \\
\left. - \frac{2\beta^2_1 a_i \omega^2}{4^2 \times 2(v + 1 + (a + 1/2))(v + (a + 1/2))} \right. \\
\left. + \frac{\beta^2_2 a_0 \omega^3}{4^2(v + (a + 1/2))^2} \cdots \right] \\
+ y \left[ 1 + \frac{2\beta^2_1 a_i \omega^2}{4^2 \times 2(v + 1 + (a + 1/2))(v + (a + 1/2))} \cdots \right],
$$

(43)

thus implies

$$
\left[ (1-y) \left( \frac{\omega(\zeta_{V, a, b}^{\lambda})'(\omega)}{(\zeta_{V, a, b}^{\lambda})(\omega)} \right) + y \left( 1 + \frac{\omega(\zeta_{V, a, b}^{\lambda})'(\omega)}{(\zeta_{V, a, b}^{\lambda})(\omega)} \right) \right] - 1 \\
= \frac{\beta_0 a_i}{4(v + (a + 1/2))(1 - y)\omega} \\
+ \frac{\beta_1 a_0}{4^2(v + (a + 1/2))(1 - y)}(1 - y) \\
- \frac{2\beta_2 a_i \omega^2}{4^2 \times 2(v + 1 + (a + 1/2))(v + (a + 1/2))(1 - 2y)} \omega^2 \cdots,
$$

(44)

$$
\varphi(\omega)(k(\omega)) = 1 + k_1 B_1 \omega + (k_1^2 B_2 + k_1 B_1) \omega^2 \\
+ (k_1 B_1 + 2k_2 k_2 B_2 + k_1^3 B_3) \omega^3 \cdots,
$$

(45)

which implies

$$
\varphi(\omega)(k(\omega)) - 1 = c_0 k_2 B_1 \omega + c_0^2 k_2^2 B_2 + c_0 k_1 B_1 \omega^2 \cdots. \quad (46)
$$

Comparing (44) and (46), we get

$$
da_0 = \frac{4(v + (a + 1/2))c_0 k_1 B_1}{\beta_1(1 - y)},
$$

(47)

$$
da_1 = -\frac{4^2(v + 1 + (a + 1/2))(v + (a + 1/2)B_i)}{\beta_1^2(1 - 2y)} \cdot \left[ c_0 k_2 + c_1 k_1 + k_1^2 \left( \frac{c_0 B_2}{B_1} - \frac{c_0 B_1}{1 - y} \right) \right].
$$

Thus,

$$
da_1 - \mu a_0^2 \leq \frac{4^2(v + 1 + (a + 1/2))(v + (a + 1/2)B_i)}{\beta_1^2(1 - 2y)} \times \left[ c_0 k_2 + c_1 k_1 + k_1^2 \left( \frac{c_0 B_2}{B_1} - \frac{c_0 B_1}{1 - y} \right) \right].
$$

(48)

Since $\varphi(\omega)$ is analytic and bounded in $U^*$ (see [25]), we have $|c_0| \leq 1$, $|c_0^2| \leq 1$, $(n > 0)$.

By using this fact and the well-known inequality $|k_i| \leq 1$, we get

$$
da_1 - \mu a_0^2 \leq \frac{4^2(v + (a + 1/2))(v + (a + 1/2)B_i)}{\beta_1^2(1 - 2y)} \times \left[ 1 + \max \left\{ 1, \frac{B_i}{B_1} \right\} - \mu \left( \frac{v + (a + 1/2)(1 - 2y)}{v + (a + 1/2)(1 - y)} \right)^{-1/2} \right].
$$

(49)

We have thus completed the proof of Theorem 12.
For $k(\omega) = \omega$ and repeating steps of Theorem 12, we obtain the following corollary.

**Corollary 13.** Let $\lambda(\omega) \in \sum_1$ satisfies

$$- \left[ (1 - \gamma) \left( \frac{\omega(\zeta_{\nu, \varphi})}{(\zeta_{\nu, \varphi})} \right) + \gamma \left( I + \frac{\omega(\zeta_{\nu, \varphi})}{(\zeta_{\nu, \varphi})} \right) \right] - I \ll \phi \cdot (\omega) - I (\omega \in U^*).$$

Then, for any complex number $\mu$,

$$|a_1 - \mu a_0^2| \leq \frac{4^2(v + (\alpha + 1/2))(v + 1 + (\alpha + 1/2))B_1}{B^2(1 - 2\gamma)} \cdot \left[ 1 + \frac{B_2}{B_1} + B_1 \left( \frac{(v + (\alpha + 1/2))(1 - 2\gamma)}{(v + 1 + (\alpha + 1/2))(1 - \eta)^2} \right) \right].$$

**Theorem 14.** If $\lambda(\omega)$ and $\phi(\omega)$ be in the class $\sum_{\nu, \alpha, \beta} (\phi, \eta)$ and $\mu$ is a complex number, then

$$|a_1 - \mu a_0^2| \leq \frac{4^2(v + (\alpha + 1/2))(v + 1 + (\alpha + 1/2))B_1}{B^2(1 - 2\gamma)} \cdot \left[ 1 + \frac{B_2}{B_1} + B_1 \left( \frac{(v + (\alpha + 1/2))(1 - 2\gamma)}{(v + 1 + (\alpha + 1/2))(1 - \eta)^2} \right) \right].$$

Proof. Let $\lambda(\omega)$ and $\phi(\omega) \in \sum_{\nu, \alpha, \beta} (\phi, \eta)$ then there exist analytic functions $\varphi(\omega)$ and $k(\omega)$, with $|\varphi(\omega)| < 1$, $k(0) = 0$, and $k(\omega) < 1$ such that

$$|a_1 - \mu a_0^2| \leq \frac{4^2(v + (\alpha + 1/2))(v + 1 + (\alpha + 1/2))B_1}{B^2(1 - 2\gamma)} \cdot \left[ 1 + \frac{B_2}{B_1} + B_1 \left( \frac{(v + (\alpha + 1/2))(1 - 2\gamma)}{(v + 1 + (\alpha + 1/2))(1 - \eta)^2} \right) \right].$$

Taking first derivative of (5) and (38) in use of the left side of above equation, we have

$$-\omega(\zeta_{\nu, \varphi}) \left( \frac{\omega(\zeta_{\nu, \varphi})}{(\zeta_{\nu, \varphi})} \right) \left( \frac{(1 - \eta)(\zeta_{\nu, \varphi})}{(\zeta_{\nu, \varphi})} + \eta \omega(\zeta_{\nu, \varphi}) \right) (\omega) - 1 = \phi(\omega)[\phi(k(\omega)) - 1].$$

and this implies

$$-\omega(\zeta_{\nu, \varphi}) \left( \frac{\omega(\zeta_{\nu, \varphi})}{(\zeta_{\nu, \varphi})} \right) \left( \frac{(1 - \eta)(\zeta_{\nu, \varphi})}{(\zeta_{\nu, \varphi})} + \eta \omega(\zeta_{\nu, \varphi}) \right) (\omega) - 1 = \frac{\beta^2 b_0(1 - \eta)}{4(v + (\alpha + 1/2))} \cdot \frac{\omega}{\omega + 1} \frac{\beta^2 b_1(1 - 2\eta)}{4^2(v + (\alpha + 1/2)^2) - 4^2 	imes 2(v + 1 + (\alpha + 1/2))(v + (\alpha + 1/2))} - \frac{\beta^2 a_1}{4^2 	imes 2(v + 1 + (\alpha + 1/2))(v + (\alpha + 1/2))} \omega^2 + \cdots.$$
\[ \frac{b_1 + a_1}{2} - b_1 \eta \leq \frac{4^2(v + (a + 1/2))(v + 1 + (a + 1/2))B_1}{\beta^2} \]

\[ \times \left[ 1 + \max \left\{ 1, \left| \frac{B_2}{B_1} \right| + B_1 \right\}, 1 - \mu \left( \frac{(v + (a + 1/2))}{(v + 1 + (a + 1/2))(1 - \eta)} \right) \right] \].

(61)

We have thus completed the proof of Theorem 14. For \( k(\omega) = \omega \) and repeating steps of Theorem 14, we obtain the following corollary.

**Corollary 15.** Let \( \lambda(\omega) \) and \( \delta(\omega) \) satisfy

\[ \eta(\zeta_v,\alpha,\beta,\lambda)'(\omega) \]

\[ (1 - \eta)(\zeta_v,\alpha,\beta)'(\omega) + \eta(\zeta_v,\alpha,\beta)''(\omega) = 1 < \phi(\omega) - 1, \quad (\omega \in U^*), \]

then for any complex number \( \mu \),

\[ \frac{b_1 + a_1}{2} - b_1 \eta \leq \frac{4^2(v + (a + 1/2))(v + 1 + (a + 1/2))B_1}{\beta^2} \]

\[ \times \left[ 1 + \left| \frac{B_2}{B_1} \right| + B_1 \right], 1 - \mu \left( \frac{(v + (a + 1/2))}{(v + 1 + (a + 1/2))(1 - \eta)} \right) \].

(62)

\[ \text{4. Conclusion} \]

In our present investigation, we have defined and systematically studied the famous Fekete-szego inequality for some subclass of meromorphic functions by using quasi-subordination. It is important to mention that certain results in the literature, for example [3, 19, 20], are special cases of the results obtained by us.

**Data Availability**

No data is used.

**Conflicts of Interest**

The authors declare that they have no conflict of interest.

**Authors’ Contributions**

SH came with the main thoughts and helped to draft the manuscript. SGAS and AR proved the main theorems. ZS and MD revised the paper. All authors read and approved the final manuscript.

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