Multi–dimensional IWP Solutions for Heterotic String Theory

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Abstract

We present extremal stationary solutions that generalize the Israel–Wilson–Perjés class for the $d+3$–dimensional low–energy limit of heterotic string theory with $n \geq d+1$ $U(1)$ gauge fields compactified on a $d$–torus. A rotating axisymmetric dyonic solution is obtained using the matrix Ernst potential formulation and expressed in terms of a single $d+1 \times d+1$–matrix harmonic function. By studying the asymptotic behaviour of the field configurations we define the physical charges of the field system. The extremality condition makes the charges to saturate the Bogomol’nyi–Prasad–Sommerfield (BPS) bound. The gyromagnetic ratios of the corresponding field configurations appear to have arbitrary values. A subclass of rotating dyonic black hole–type solutions arises when the NUT charges are set to zero. In the particular case $d = 1, n = 6$, which correspond to $N = 4, D = 4$ supergravity, the found dyon reproduces the supersymmetric dyonic solution constructed by Bergshoeff et al.
1 Introduction

In effective low energy theories of gravity derived from superstring theory Einstein gravity is supplemented by additional fields such as the Kalb–Ramond, gauge fields, and the scalar dilaton which couples in a non–trivial way to other fields [1]. One of these theories is the bosonic sector of the heterotic string. This model, when compactified from $D = d + 3$ dimensions on a $d$–torus, can be parametrized by the $(d+1) \times (d+1)$ and $(d+1) \times n$ Matrix Ernst Potentials (MEP) $\mathcal{X}$ and $\mathcal{A}$ [2–3], where $n$ is the number of Abelian vector fields. In order to have a self–consistent quantum theory we must set $D = 10$ and $n = 16$, but in this letter we shall leave these parameters arbitrary for the sake of generality. However, there is one condition to be satisfied by them in order to have a solution (see [3] for details), namely, $n \geq d + 1$. The critical case is well described in the framework of this formalism as well as the $N = 4, D = 4$ supergravity case, when $d = 1$ and $n = 6$. Thus, the number of gauge fields is bounded below.

This letter is organized as follows. In Sec. 2 we present the effective action of the low energy limit of heterotic string in terms of the MEP. This fact allows one to map this action onto the stationary Einstein–Maxwell (EM) action and apply classical procedures, commonly used with the EM theory, to the heterotic string theory; in the framework of this approach we obtain in Sec. 3 a stationary class of rotating dyonic solutions that generalizes the Israel–Wilson–Perjés (IWP) class of the EM theory [5] by considering a linear dependence between the asymptotically flat potentials $\mathcal{X}$ and $\mathcal{A}$. Then we define the physical charges of the field system by studying the asymptotical behaviour of the 3–fields. Furthermore, we show that the physical charges of the obtained solutions saturate the BPS bound as a consequence of the extremality condition. Among them we identify axisymmetric rotating dyonic black hole–type solutions, endowed with rotating axion, dilaton and Kalb–Ramond fields, when the NUT charges vanish. In Sec. 4 we consider the particular case $N = 4, D = 4$ supergravity and show that our solutions reproduce the supersymmetric dyonic solutions of [4]. Sec. 5 contains some concluding remarks and a brief discussion.

2 Matrix Ernst Potentials

The effective action of low energy limit of heterotic string theory is

$$\begin{align*}
S^{(D)} &= \int d^{(D)}x \left| G^{(D)} \right|^{\frac{1}{2}} e^{-\phi^{(D)}} \left[ R^{(D)} + \phi_{,M}^{(D)}\phi^{(D);M} - \frac{1}{12} H^{(D)}_{MNP} H^{(D)MNP} - \frac{1}{4} F^{(D)I}_M F^{(D)IMN} \right],
\end{align*}$$

(1)

where

$$
F^{(D)I}_M = \partial_M A^{(D)I}_N - \partial_N A^{(D)I}_M, \quad H^{(D)}_{MNP} = \partial_M B^{(D)}_{NP} - \frac{1}{2} A^{(D)I}_M F^{(D)I}_N + \text{cycl. perms. of M,N,P.}
$$

Here $G^{(D)}_{MN}$ is the multidimensional metric, $B^{(D)}_{MN}$ is the anti–symmetric Kalb-Ramond field, $\phi^{(D)}$ is the dilaton and $A^{(D)I}_M$ denotes a set of $U(1)$ gauge fields ($I = 1, 2, ..., n$).
After the Kaluza-Klein compactification on a \(d\)-torus, one obtains the following set of three-dimensional fields [1]-[7]:

a) scalar fields

\[
G = (G_{pq} \equiv G^{(D)}_{p+3,q+3}), \quad B = (B_{pq} \equiv B^{(D)}_{p+3,q+3}), \quad A = (A^I_p \equiv A^{(D)}_{p+3}), \quad \phi = \phi^{(D)} - \frac{1}{2} \ln |\det G|, \quad (2)
\]

where the subscripts \(p, q = 1, 2, \ldots, d\).

b) tensor fields

\[
g_{\mu\nu} = e^{-2\phi} \left( G^{(D)}_{\mu\nu} - G^{(D)}_{p+3,\nu} G^{(D)}_{q+3,\mu} G^{pq} \right), \quad B_{\mu\nu} = B^{(D)}_{\mu\nu} - 4 B_{pq} A^p_\mu A^q_\nu - 2 \left( A^p_\mu A^{p+d}_\nu - A^p_\nu A^{p+d}_\mu \right),
\]

(we shall consider the ansatz when \(B_{\mu\nu} = 0\) in view of its non-dynamical properties).

c) vector fields

\[
(A_1)^{\mu} = \frac{1}{2} G^{pq} G^{(D)}_{q+3,\mu}, \quad (A_3)^{I+2d} = -\frac{1}{2} A^{d(I)}_I + A^I_\mu A_\mu^{q+3}, \quad (A_2)^{p+d} = \frac{1}{2} B^{(D)}_{p+3,\mu} - B_{pq} A_\mu^{q+3} + \frac{1}{2} A^I_\mu A^{I+2d}_\mu,
\]

which can be dualized on-shell as follows

\[
\begin{align*}
\nabla \times \vec{A}_1 & = \frac{1}{2} e^{2\phi} G^{-1} \left( \nabla u + (B + \frac{1}{2} A A^T) \nabla v + A \nabla s \right), \\
\nabla \times \vec{A}_3 & = \frac{1}{2} e^{2\phi} (\nabla s + A^T \nabla v) + A^T \nabla \times \vec{A}_1, \\
\nabla \times \vec{A}_2 & = \frac{1}{2} e^{2\phi} G \nabla v - (B + \frac{1}{2} A A^T) \nabla \times \vec{A}_1 + A \nabla \times \vec{A}_3.
\end{align*}
\]

(3)

The columns \(u\) and \(v\) have dimension \(d\) with components \(u_1, u_r\) and \(v_1, v_r\) \((r = 2, 3, \ldots, d)\), respectively, whereas the dimension of the column \(s\) is \(n\). Thus, the final system is defined by the field variables \(G, B, A, \phi, u, v\) and \(s\).

At this stage it is convenient to introduce the matrix Ernst potentials

\[
\mathcal{X} = \left( \begin{array}{c} -e^{-2\phi} + v^T X v + v^T A s + \frac{1}{2} s^T s \\ X v + u + A s \\ v^T X - u^T X \end{array} \right), \quad \mathcal{A} = \left( \begin{array}{c} s^T + v^T A \\ A \end{array} \right),
\]

(4)

where the \(d \times d\) matrix potential \(X = G + B + \frac{1}{2} A A^T\). This pair of potentials allows us to express the 3-dimensional action in a quasi-EM form [3]:

\[
S^{(3)} = \int d^3x \ g \left\{ -\mathcal{R} + \text{Tr} \left[ \frac{1}{4} (\nabla \mathcal{X} - \nabla \mathcal{A} A^T) \mathcal{G}^{-1} (\nabla \mathcal{X}^T - \mathcal{A} \nabla \mathcal{A}^T) \mathcal{G}^{-1} + \frac{1}{2} \nabla \mathcal{A}^T \mathcal{G}^{-1} \nabla \mathcal{A} \right] \right\},
\]

(5)

where \(\mathcal{G} = \frac{1}{2} \left( \mathcal{X} + \mathcal{X}^T - \mathcal{A} \mathcal{A}^T \right)\). The equations of motion of the matter part of this action have the Ernst form [3]:

\[
\begin{align*}
\nabla^2 \mathcal{X} - 2 (\nabla \mathcal{X} - \nabla \mathcal{A} A^T) (\mathcal{X} + \mathcal{X}^T - \mathcal{A} \mathcal{A}^T)^{-1} \nabla \mathcal{X} & = 0, \\
\nabla^2 \mathcal{A} - 2 (\nabla \mathcal{X} - \nabla \mathcal{A} A^T) (\mathcal{X} + \mathcal{X}^T - \mathcal{A} \mathcal{A}^T)^{-1} \nabla \mathcal{A} & = 0.
\end{align*}
\]

(6)
3 Dyonic Rotating Black Hole–type Solutions

In this Sec. we obtain a class of extremal solutions for the equations of motion (6) which generalize the IWP class of the EM theory following the procedure indicated in [3]. We consider a linear dependence between the potentials \( A \) and \( X \), and require them to be asymptotically flat, i.e., \( X_\infty \to \Sigma \) and \( A_\infty \to 0 \), where \( \Sigma = \text{diag}(-1, -1, 1, 1, \ldots, 1) \). Thus, the matrix Ernst potentials are related by

\[
A = (\Sigma - X)b, \quad (7)
\]

where \( b \) is an arbitrary constant \( d + 1 \times n \)-matrix. By substituting (7) into the action (5) and setting the Lagrangian of the system to zero (it implies that \( R_{ij} = 0 \)), we get the following condition to be satisfied

\[
bb^T = -\Sigma/2. \quad (8)
\]

Indeed, both equations of motion (6) reduce to the Laplace equation in Euclidean 3–space

\[
\nabla^2[(\Sigma + X)^{-1}] = 0. \quad (9)
\]

The solutions for this equation are well known and in the simplest case one can consider the harmonic function

\[
\frac{2}{\Sigma + X} = \Sigma + \Re \frac{M}{R}, \quad \text{where} \quad R^2 = x^2 + y^2 + (z + i\alpha)^2, \quad (10)
\]

\( M \) is a complex \( d + 1 \)-dimensional constant matrix with arbitrary components \( m_{\tilde{p},\tilde{q}} = \tilde{m}_{\tilde{p},\tilde{q}} + i\tilde{n}_{\tilde{p},\tilde{q}} \), and \( \alpha \) is a real constant. We choose \( M \) and \( R \) in this way in order to deal with rotating black hole–type solutions (in this case we have a ring singularity) when the NUT charges of the field system are set to zero. This is in contrast with the results obtained in [3] and [10], where the four– and five–dimensional stationary classes of solutions become static when the NUT charges vanish. In a forthcoming paper we will investigate solutions where both \( M \) and \( b \) have a more general form. Such an ansatz leads to a richer class of solutions for the theory under consideration.

In order to obtain a real value of the potential \( A \) (see Eqs. (7) and (8)) we can require just the first two rows of \( b \) to be real (leaving the remaining rows imaginary), then we perform the matrix product (7) and set the factors that multiply the imaginary components of \( b \) to zero. It turns out that this condition imposes the following restriction on the matrix \( M \)

\[
M = \begin{pmatrix}
m_{11} & m_{12} & 0 & \cdots & 0 \\
m_{21} & m_{22} & 0 & \cdots & 0 \\
m_{r+1,1} & m_{r+1,2} & 0_{d-1}
\end{pmatrix}, \quad (11)
\]

where \( 0_{d-1} \) denotes a \((d - 1)\)-dimensional square array of zeroes. It is not difficult to check that this procedure leads to real solutions for the potential \( A \).
At this stage one is able to calculate the 3–fields \(G, B, A, \phi, u, v\) and \(s\). By studying their asymptotic behaviour one can establish the following relation between the integration constants and the physical parameters of the theory

\[
G \sim \left( -\left( 1 + \frac{2\tilde{m}_{t+1,2}}{R_{as}} \right) \frac{\tilde{m}_{r+1,2}}{R_{as}} \right) = \left( -\left( 1 - \frac{2m}{\tilde{C}_r} \right) \frac{C_r}{R_{as}} \right),
\]

\[
B \sim \left( \begin{array}{cc}
0 & -\frac{\tilde{m}_{r+1,2}}{R_{as}} \\
\frac{\tilde{m}_{r+1,2}}{R_{as}} & 0_{d-1}
\end{array} \right) = \left( \begin{array}{cc}
0 & -\frac{C_r}{R_{as}} \\
\frac{C_r}{R_{as}} & 0_{d-1}
\end{array} \right), \quad \phi \sim -\tilde{m}_{11} = \frac{D}{R_{as}},
\]

\[
A = \left( \begin{array}{c}
A^I_r \\
A^I_t
\end{array} \right) \sim \left( \begin{array}{c}
2(\tilde{m}_{21}b_{1I} + \tilde{m}_{22}b_{2I})/R_{as} \\
-2(\tilde{m}_{r+1,1}b_{1I} + \tilde{m}_{r+1,2}b_{2I})/R_{as}
\end{array} \right) = \left( \begin{array}{c}
Q^I_r/R_{as} \\
Q^I_t/R_{as}
\end{array} \right),
\]

\[
u_1 \sim \frac{\tilde{m}_{12} - \tilde{m}_{21}}{R_{as}} = \frac{N}{R_{as}}, \quad 
u_1 \sim \frac{\tilde{m}_{12} + \tilde{m}_{21}}{R_{as}} = \frac{C_1}{R_{as}},
\]

\[
u_r = \nu_r \sim \frac{\tilde{m}_{r+1,1}}{R_{as}} = \frac{N_r}{R_{as}}, \quad s^I \sim 2\tilde{m}_{11}b_{1I} + \tilde{m}_{12}b_{2I} = \frac{Q^I_m}{R_{as}},
\]

where \(b_{ij} = \delta_{ij}/2, i, j = 1, 2\); \(m\) is the ADM mass, \(D\) is the dilaton, \(N\) and \(N_r\) are \(d\) NUT charges, \(C_1\) is a scalar (axion) charge, \(C_r\) are \(d-1\) Kaluza–Klein charges, \(Q^I_r\) and \(Q^I_m\) are two sets of \(n\) electric and magnetic charges, and \(Q^I_r\) are \(d-1\) sets of \(n\) charges that come from the extra dimensions of the electromagnetic sector; and \(R_{as} = \sqrt{x^2 + y^2 + z^2}\). The extremality character of the found solutions makes these charges to saturate the BPS bound

\[
4(D^2 + m^2) + 2(N^2 + C_1^2) + \sum_{r=2}^d (Q^I_r)^2 = (Q^I_m)^2 + (Q^I_m)^2 + 4\sum_{r=2}^d (N_r^2 + C_r^2),
\]

where a summation under \(I\) is understood. This means that the attractive forces are precisely balanced by the repulsive forces in the field configuration.

In order to write down the explicit form of a single point–like solution in terms of the multidimensional variables we must calculate all vector 3–fields using the dualization formulae (3). After some algebraic manipulations we obtain

\[
2\nabla \times \vec{A}_1 = Re\nabla \left[ \left( \frac{m_{21} - m_{12}}{m_{r+1,1}} \right) \frac{1}{R} \right] + \left( \frac{\sigma}{R} \right) Im \left( \frac{1}{R} \nabla \frac{1}{R} \right),
\]

\[
2\nabla \times \vec{A}_2 = Re\nabla \left[ \left( \frac{-(m_{12} + m_{21})}{m_{r+1,1}} \right) \frac{1}{R} \right] + \left( \frac{\sigma}{R} \right) Im \left( \frac{1}{R} \nabla \frac{1}{R} \right),
\]

\]

\[
5
\]
Then, the 3–interval reads

\[ \nabla \times \mathbf{A}_3^I = \left\{ Re \left( \nabla \frac{m_{11}}{R} \right) b_{1I} + \left[ Re \left( \nabla \frac{m_{12}}{R} \right) - \sigma Im \left( \frac{1}{R} \nabla \frac{1}{R} \right) \right] b_{2I} \right\}, \]

where \( \sigma = \bar{m}_{11} \bar{n}_{12} + \bar{m}_{21} \bar{n}_{22} - \bar{m}_{12} \bar{n}_{11} - \bar{m}_{22} \bar{n}_{21} + \sum_r (\bar{m}_{r+1,2} \bar{n}_{r+1,1} - \bar{m}_{r+1,1} \bar{n}_{r+1,2}); \) \( b_{1I} \) and \( b_{2I} \) being the first two rows of the matrix \( b \).

It is natural to write down the solutions in terms of oblate spheroidal coordinates defined by

\[ x = \sqrt{\rho^2 + \alpha^2 \sin \theta \cos \varphi}, \quad y = \sqrt{\rho^2 + \alpha^2 \sin \theta \sin \varphi}, \quad z = \rho \cos \theta. \]

Then, the 3–interval reads

\[ ds_3^2 = (\rho^2 + \alpha^2 \cos^2 \theta)(\rho^2 + \alpha^2)^{-1} d\rho^2 + (\rho^2 + \alpha^2 \cos^2 \theta) d\theta^2 + (\rho^2 + \alpha^2) \sin^2 \theta d\varphi^2 \]

and only the \( A_{\varphi}^{(a)} \) components do not vanish \(^3\):

\[
2A_{1\varphi} = (\rho^2 + \alpha^2 \cos^2 \theta)^{-1} \left\{ \left( \frac{\bar{m}_{21} - \bar{m}_{12}}{\bar{m}_{r+1,1}} \right) (\rho^2 + \alpha^2) \cos \theta + \left[ \left( \frac{\bar{n}_{12} - \bar{n}_{21}}{\bar{n}_{r+1,1}} \right) + \frac{1}{2} \left( \sigma \rho \right) \right] \alpha \sin^2 \theta \right\},
\]

\[
2A_{2\varphi} = (\rho^2 + \alpha^2 \cos^2 \theta)^{-1} \left\{ \left( -\frac{\bar{m}_{12} + \bar{m}_{21}}{\bar{m}_{r+1,1}} \right) (\rho^2 + \alpha^2) \cos \theta + \left[ \left( \frac{\bar{n}_{12} + \bar{n}_{21}}{\bar{n}_{r+1,1}} \right) + \frac{1}{2} \left( \sigma \rho \right) \right] \alpha \sin^2 \theta \right\},
\]

\[
A_3^I \varphi = (\rho^2 + \alpha^2 \cos^2 \theta)^{-1} \left\{ \left[ \bar{n}_{11} (\rho^2 + \alpha^2) \cos \theta - \bar{n}_{11} \rho \alpha \sin^2 \theta \right] b_{1I} + \left[ \bar{m}_{12} (\rho^2 + \alpha^2) \cos \theta - (\bar{n}_{12} \rho + \sigma/2) \alpha \sin^2 \theta \right] b_{2I} \right\}.
\]

From here we see that there exist \( 2d + n \) angular momenta defined by

\[ A_{\varphi}^{(a)} \sim - \frac{\bar{n}_{11} \alpha \sin^2 \theta}{\rho} = \frac{2J^{(a)} \sin^2 \theta}{\rho}, \]

where the parameters \( \bar{n}_{11}^{(a)} \) are arbitrary in the general case and hence, the gyromagnetic ratios of the corresponding field configurations turn out to be arbitrary as well in the context of this approach. In the general case the expressions for the multidimensional fields depend on the arbitrary parameters \( \bar{n}_{pq} \), as it takes place for the \( A_{\varphi}^{(a)} \) components. However, in order to obtain the four–dimensional solutions constructed in \(^3\) (and their direct generalization to the multidimensional case) we set

\[ 2\bar{n}_{11} = (C_1 - N), \quad \bar{n}_{12} = -m, \quad \bar{n}_{21} = D, \quad 2\bar{n}_{22} = -(C_1 + N), \quad \bar{n}_{r+1,1} = C_r, \quad \bar{n}_{r+1,2} = -N_r \]

and quote the full solution corresponding to this special case:

\[ ds^2 = G_{MN} dx^M dx^N = G_{pq} \left( dx^{p+3} + \omega^{(p)} d\varphi \right) \left( dx^{q+3} + \omega^{(q)} d\varphi \right) + e^{2\psi} g_{\mu\nu} dx^\mu dx^\nu, \]

\(^3\)In fact we have imposed the axial symmetry with respect to \( z \) and have chosen \( R = \rho + i \alpha \cos \theta \).
where the symmetric matrix $G_{pq}$ has the components

\[ G_{11} = -\Delta^{-2} \left[ (\rho^2 + \alpha^2 \cos^2 \theta) + 2D \rho + (N - C_1) \alpha \cos \theta + D^2 + (C_1 - N)^2 / 4 \right] (\rho^2 + \alpha^2 \cos^2 \theta), \]

\[ G_{1r} = \Delta^{-2} \left\{ (N_r \rho + C_r \alpha \cos \theta) \left[ (C_1 \rho + (D - m) \alpha \cos \theta) + D(C_1 + N)/2 + m(C_1 - N)/2 \right] + (C_r \rho - N_r \alpha \cos \theta) \left[ (\rho^2 + \alpha^2 \cos^2 \theta) + 2D \rho + (N - C_1) \alpha \cos \theta + D^2 + (C_1 - N)^2 / 4 \right] \right\}, \]

\[ G_{rr'} = \delta_{r+1,r'+1} - \Delta^{-2} (\rho^2 + \alpha^2 \cos^2 \theta)^{-2} \left\{ \left[ (N_r \rho + C_r \alpha \cos \theta)(\rho^2 + \alpha^2 \cos^2 \theta) + \Delta_{2r} \right] \times \left[ (N_r \rho + C_r \alpha \cos \theta)(\rho^2 + \alpha^2 \cos^2 \theta) + \Delta_{2r'} \right] + \left[ (C_r \rho - N_r \alpha \cos \theta)(\rho^2 + \alpha^2 \cos^2 \theta) + \Delta_{1r} \right] \times \left[ (C_r \rho - N_r \alpha \cos \theta)(\rho^2 + \alpha^2 \cos^2 \theta) + \Delta_{1r'} \right] \right\}, \]

the conformal multiplier has the form

\[ e^{2\phi} = 1 + \frac{2D \rho + (N - C_1) \alpha \cos \theta}{\rho^2 + \alpha^2 \cos^2 \theta} + \frac{\delta_0}{(\rho^2 + \alpha^2 \cos^2 \theta)^2} \]

(21)

and the components of the rotational vector are defined by

\[ \omega^{(l)} = -(\rho^2 + \alpha^2 \cos^2 \theta)^{-1} \left\{ N(\rho^2 + \alpha^2) \cos \theta + [(m + D) \rho - \sigma/2] \alpha \sin^2 \theta \right\}, \]

\[ \omega^{(r)} = (\rho^2 + \alpha^2 \cos^2 \theta)^{-1} \left[ N_r(\rho^2 + \alpha^2) \cos \theta - C_r \alpha \rho \sin^2 \theta \right], \]

where we have introduced the notations \( \Delta = \rho^2 + \alpha^2 \cos^2 \theta + (D + m) \rho + N \alpha \cos \theta + D m - (C_1^2 - N^2)/4 \),

\( 2\Delta_{1r} = (C_r \rho - N_r \alpha \cos \theta)(2D \rho + (N - C_1) \alpha \cos \theta) + (N_r \rho + C_r \alpha \cos \theta) [(C_1 + N) \rho - 2m \alpha \cos \theta], \)

\( 2\Delta_{2r} = (N_r \rho + C_r \alpha \cos \theta)(2m \rho + (C_1 + N) \alpha \cos \theta) + (C_r \rho - N_r \alpha \cos \theta) [(C_1 - N) \rho + 2D \alpha \cos \theta], \)

\( \delta_0 = [D^2 + (C_1 - N)^2/4(\rho^2 + \alpha^2 \cos^2 \theta) - (N_r \rho + C_r \alpha \cos \theta)^2 \] and \( \sigma = 2mD + C_r^2 + N_r^2 + (N^2 - C_1^2)/2. \)

From Eq. (20) we see that the interval adopts the form of an axisymmetric rotating black hole solution when the NUT charges vanish.

The only non–vanishing components of the multidimensional matter fields are

\[ B_{1r} = \Delta^{-1} (\rho^2 + \alpha^2 \cos^2 \theta)^{-1} \left[ (C_r \rho - N_r \alpha \cos \theta)(\rho^2 + \alpha^2 \cos^2 \theta) + \Delta_{1r} \right], \]

\[ A_{1r} = \Delta^{-1} \left\{ [(C_1 - N) \rho + 2D \alpha \cos \theta] b_{11} - \left[ 2m \rho + (C_1 + N) \alpha \cos \theta + 2mD + (N^2 - C_1^2)/2 \right] b_{21} \right\}, \]
\[ A'_I = -2{\Delta}^{-1}(\rho^2 + \alpha^2 \cos^2 \theta)^{-1} \left\{ \left[ N_r \rho + C_r \alpha \cos \theta \right] (\rho^2 + \alpha^2 \cos^2 \theta) + \Delta_{2r} \right\} b_{1I} + \\
\left( C_r \rho - N_r \alpha \cos \theta \right) (\rho^2 + \alpha^2 \cos^2 \theta) + \Delta_{1r} \right\} b_{2I} \right\}, \]
\[ \phi^{(D)} = \ln\left\{ \Delta^{-1}(\rho^2 + \alpha^2 \cos^2 \theta)^{-1} \left[ (\rho^2 + \alpha^2 \cos^2 \theta)^2 + (2D \rho + (N - C_1) \rho \cos \theta) (\rho^2 + \alpha^2 \cos^2 \theta) + \delta_0 \right] \right\}, \]
\[ B_{\iota \varphi}^{(D)} = (\rho^2 + \alpha^2 \cos^2 \theta)^{-1} \left\{ -A'_I \left[ Q_m^I (\rho^2 + \alpha^2) \cos \theta - \left( Q_c^{IT} \rho + \sigma b_{2I}^T / 2 \right) \alpha \sin^2 \theta \right] / 2 + \\
B_{r 1}^T \left[ N_r (\rho^2 + \alpha^2) \cos \theta - C_r \alpha \rho \sin^2 \theta \right] - C_1 (\rho^2 + \alpha^2) \cos \theta + ((m - D) \rho - \sigma / 2) \alpha \sin^2 \theta \right\}, \]
\[ B_{r \varphi}^{(D)} = (\rho^2 + \alpha^2 \cos^2 \theta)^{-1} \left\{ -A'_I \left[ Q_m^I (\rho^2 + \alpha^2) \cos \theta - \left( Q_c^{IT} \rho + \sigma b_{2I}^T / 2 \right) \alpha \sin^2 \theta \right] / 2 - \\
B_{r 1} \left[ N_r (\rho^2 + \alpha^2) \cos \theta - ((m + D) \rho + \sigma / 2) \alpha \sin^2 \theta \right] + \left[ N_r (\rho^2 + \alpha^2) \cos \theta - C_r \alpha \rho \sin^2 \theta \right] \right\}, \]
\[ A_{\varphi}^{(D)I} = (\rho^2 + \alpha^2 \cos^2 \theta)^{-1} \left\{ - \left[ Q_m^I (\rho^2 + \alpha^2) \cos \theta - \left( Q_c^{IT} \rho + \sigma b_{2I}^T / 2 \right) \alpha \sin^2 \theta \right] + A_c^{IT} \times \\
\left[ N_r (\rho^2 + \alpha^2) \cos \theta - C_r \alpha \rho \sin^2 \theta \right] - A'_I \left[ N_r (\rho^2 + \alpha^2) \cos \theta - ((m + D) \rho + \sigma / 2) \alpha \sin^2 \theta \right] \right\} \right\}, \]
where \( b_{1I} = -\frac{1}{4 \Delta_{12}} [(C_1 + N) Q_e^I + 2m Q_m^I] \), \( b_{2I} = -\frac{1}{4 \Delta_{12}} [2D Q_e^I + (C_1 - N) Q_m^I] \), and \( \Delta_{12} = mD - (C_1^2 - N^2) / 4 \). One can see that the multidimensional components of the fields \( B_{\iota \varphi}^{(D)} \), \( B_{r \varphi}^{(D)} \) and \( A_{\varphi}^{(D)I} \) are non–trivial at spatial infinity as it takes place for the magnetically charged configurations of the ordinary EM theory.

By counting the number of independent parameters which parametrize the physical charges of the solution we see that matrix \( M \) contributes with \( 2(d + 1) \) items in the framework of our ansatz. On the other hand, matrix \( b \) provides \( 2n - 3 \) independent parameters since only its first two rows affect the solution (these rows are normalized and orthogonal to each other in view of Eq. (8)). We have the rotational parameter \( \alpha \) as well. Thus we have in total \( 2(d + n) \) independent integration constants which define charges of the field system.

Thus, our solution can be interpreted as an asymptotically Taub–NUT rotating field configuration with axial symmetry formed by: the Einstein mass \( m_E = D + m \), the Kaluza–Klein charges \( C_r \), their corresponding NUT charges \( N \) and \( N_r \), the multi–dimensional dilaton with charge \( D^{(D)} = D - m \), the axion charge \( C_1 \), the electromagnetic charges \( Q_c^I \), \( Q_m^I \) and \( Q_e^I \) (the fields with magnetic charges are non–trivial at spatial infinity as in ordinary EM theory) and the antisymmetric Kalb–Ramond charges \( C_r \) which turn out to be equal to the Kaluza–Klein charges in our ansatz. The total number of independent charges is equal to \( 2(d + n) \) in the framework of the ansatz under consideration.
4 \ N=4, \ D=4 \ Supergravity

In the particular case \( d = 1, \ n = 6 \) the considered action corresponds to the bosonic sector of \( N = 4, \ D = 4 \) supergravity. A supersymmetric generalization of the IWP solutions for such a theory was constructed by Bergshoeff et al. \[4\] choosing as ansatz two arbitrary complex harmonic functions. In this Sec. we show that in the case of a single point–like source, our solutions reproduce these solutions. In order to do so, it is convenient to switch to the Einstein frame:

\[
ds_E^2 = e^{-\phi^{(4)}} \, ds_{str}^2 = -\frac{(\rho^2 + \alpha^2 \cos^2 \theta)}{\Delta} (dt - \omega_\varphi d\varphi)^2 + \frac{\Delta}{(\rho^2 + \alpha^2 \cos^2 \theta)} g_{\mu\nu} dx^\mu dx^\nu, \tag{23}
\]

where \( e^{\phi^{(4)}} = \Delta^{-1}[\rho^2 + \alpha^2 \cos^2 \theta + 2D\rho + (N - C_1)\alpha \cos \theta + D_2 + (C_1 - N)^2/4] \) is the conformal factor and \( \omega_\varphi = (\rho^2 + \alpha^2 \cos^2 \theta)^{-1}\{N(\rho^2 + \alpha^2) \cos \theta + [m_E\rho - mD + (C_1^2 - N^2)/4] \alpha \sin^2 \theta\} \) is the angular velocity of our rotating object; from here we see that \( m_E/2, N/2 \) and \( (D^{(4)} + iC_1)/2 \) are the mass, the NUT charge and complex axion–dilaton charges of \[4\], respectively. The non–trivial components of the matter fields are given by the second, fourth, fifth and seventh relations of Eq. (22) when the extra–dimensional parameters vanish.

From Eq. (18) it is clear that the rotating axion charge generates the dipole momentum

\[
J_a = D^{(4)} \alpha/4, \tag{24}
\]

whereas the rotating \( n \) electric charges induce a magnetic field and originate the momenta

\[
J^l = -Q^l_m \alpha/4. \tag{25}
\]

Thus, this particular solution corresponds to an asymptotically Taub–NUT rotating field configuration with axial symmetry where the Einstein mass is endowed with the NUT charge, the axion and dilaton charges and two sets of \( n \) electric and \( n \) magnetic charges which rotate together with it. The fields generated by the charges \( N, C_1 \) and \( Q^l_m \) do not vanish at spatial infinity having a Dirac string peculiarity; they are the NUT charge, the charge of the background axion field and \( n \) magnetic charges, respectively. When the NUT parameter is set to zero, a rotating dyonic black hole solution endowed with a rotating axion–dilaton field arises.

5 \ Conclusions and Discussion

In this letter we have obtained a class of stationary extremal solutions that generalize the IWP class of EM theory for the \( d + 3 \)--dimensional heterotic string compactified on a \( d \)--torus using the MEP formalism. The physical charges of the field system saturate the BPS bound as a consequence of the extremal character of the found solutions. These solutions
are expressed in terms of $2(d + n)$ ($n \geq d + 1$ being the number of Abelian vector fields) independent real parameters related to physical charges of the field system.

In a special case the found solutions correspond to a dyonic asymptotically Taub–NUT rotating field configuration with axial symmetry. This object is formed by the following rotating fields: the gravitational and Kalb–Ramond fields, the axion/dilaton, as well as by $(d + 1) \times n$ electromagnetic charges.

Among these solutions we identify (by requiring the asymptotic flatness condition to be satisfied) a class of rotating dyonic black hole–type configurations. All the four–dimensional rotating black hole solutions of this type develop naked singularities before the BPS bound is reached. So there is no horizon hiding the singularity. This fact makes impossible the study of thermal properties of extreme rotating objects since their entropy is proportional to the horizon area. If indeed, the rotational parameter vanishes, the solutions become static. The thermodynamical properties of these objects are well–known (see [11], for instance).

In principle, the MEP formalism allows one to extract a richer class of solutions by considering a more complete ansatz (with more general complex $M$ and $b$). In this case, the Kaluza–Klein and antisymmetric fields will have different charges, for example. Moreover, the number of independent parameters will increase since we will have more components of $M$ and more independent electromagnetic charges. On the other hand, one can consider multi–center solutions or linear combinations of Legendre polynomials as solutions of the Laplace equation.

At the end of the letter we would like to stress one special property of our solutions. One can see that the electric potentials $A_I^d$ from Eqs. (22) have no dipole momentum in contrast with the components of the vector potentials $A_{(D)I}^{\varphi}$. This is not an intrinsic property of the equations of motion. Actually, it is well known that the equations of motion posses the $SL(2, \mathbb{R})$ symmetry that interchanges the electric and magnetic sectors. In this work we show how in the framework of the constructed solutions one can interprete the magnetic dipole momenta as a result of the rotation of the electric charges (a similar situation takes place for the axion field, where its dipole momentum is defined by the rotating dilaton charge). To achieve this symmetry in the solution one must choose a more general solution of the Laplace equation (10). It seems that taking into account the next dipole term in the solution (10) it is possible to recover the lost symmetry. However this requires further investigation.

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