FLAT BUNDLES WITH COMPLEX ANALYTIC
HOLONOMY

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Abstract. Let $G$ be a connected complex Lie group. We show that any flat principal $G$-bundle over any finite CW-complex pulls back to a trivial bundle over some finite covering space of the base space if and only if each real characteristic class of positive degree of $G$ vanishes. A third equivalent condition is that the derived group of the radical of $G$ is simply connected. As a corollary, the same conditions are equivalent if $G$ is a connected amenable Lie group. In particular, if $G$ is a connected compact Lie group then any flat principal $G$-bundle over any finite CW-complex pulls back to a trivial bundle over some finite covering space of the base space.

1. Introduction

Let $G$ be a Lie group. A principal $G$-bundle over a connected CW-complex $X$ is called flat, if there is a homomorphism

$$\rho : \pi_1(X) \to G,$$

the holonomy of the flat bundle, such that the given bundle is equivalent to the $G$-bundle $\tilde{X} \times_\rho G \to X$ canonically associated with the universal cover $\tilde{X}$ of $X$; the notation $\tilde{X} \times_\rho G$ refers to the orbit space of $\tilde{X} \times G$ under the $\pi_1(X)$-action given by

$$\gamma(x, g) = (\gamma x, \rho(\gamma)g).$$

Flat $G$-bundles are characterized by the fact that the classifying map $\theta : X \to BG$ factors as

$$X \to B\pi_1(X) \to BG,$$

where the first arrow classifies the universal cover of $X$ and the second one is $B\rho$. Equivalently, if $G^\delta$ denotes the group $G$ with the discrete topology and $\iota : G^\delta \to G$ denotes the identity map, a principal $G$-bundle over $X$ is flat, if and only if it is classified by a map $\theta : X \to BG$ which factors through

$$B\iota : BG^\delta \to BG.$$

We refer the reader to [15] for more details on the above facts.
A principal $G$-bundle over $X$ is called virtually trivial if its pull-back to some finite covering space of $X$ is trivial.

Under which conditions on a connected Lie group $G$ is any flat principal $G$-bundle, over any finite CW-complex, virtually trivial?

A necessary condition is that each real characteristic class of $G$ in $H^*(BG^δ, \mathbb{R})$, in the sense of [11, p. 23], of positive degree vanishes; that is the map

$$H^*(BG, \mathbb{R}) \to H^*(BG^δ, \mathbb{R})$$

induced by $B\iota$ is zero if $* > 0$. This necessary condition is fulfilled if $G$ is a complex reductive group; this follows from a result of Kamber and Tondeur [16, Theorem 3.5]. A well-known result of Deligne and Sullivan states that any flat principal $GL(n, \mathbb{C})$-bundle over any finite CW-complex is virtually trivial [5].

Before we state our main result, we recall that the radical $R$ of a connected Lie group $G$ is its maximal connected normal solvable subgroup. It is always a closed subgroup of $G$ but its commutator subgroup $[R, R]$ is in general not closed in $G$.

**Theorem 1.1.** Let $G$ be a connected complex Lie group. The following conditions are equivalent.

1. Any flat principal $G$-bundle over any finite CW-complex is virtually trivial.
2. The map $H^*(BG, \mathbb{R}) \to H^*(BG^δ, \mathbb{R})$ is zero in positive degree.
3. The map $H^2(BG, \mathbb{R}) \to H^2(BG^δ, \mathbb{R})$ is zero.
4. The derived subgroup $[R, R]$ of the radical $R$ of $G$ is simply connected.

According to Gotô [10, Theorem 25], a connected solvable Lie group $R$ is linear if and only if the closure of its derived subgroup is simply connected. As the map between fundamental groups

$$\pi_1([R, R]) \to \pi_1([R, R])$$

induced by the inclusion, is one-to-one, it follows that a connected complex Lie group $G$ whose radical is linear satisfies the equivalent conditions of the theorem. The chain of implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) holds for any connected Lie group. The proof of (3) $\Rightarrow$ (4) is given in [3, Proof of Theorem 2.2] and is based on a construction of Goldman [9]. Hence, in order to prove the theorem, it is enough to show that if $G$ is a connected complex Lie group and if the derived subgroup of the radical of $G$ is simply connected then any flat principal $G$-bundle over any finite CW-complex is virtually trivial.
The main steps in the proof are the following. All real characteristic classes of a connected Lie group $G$ are bounded, when viewed as classes in $H^*(BG^d, \mathbb{R})$, if and only if the derived subgroup of the radical of $G$ is simply connected [3]. Combining this fact with Gromov’s *Mapping Theorem* [11, Section 3.1], we reduce the problem to the case of semisimple groups. A connected complex semisimple Lie group $C$ has a unique complex algebraic structure and there exists a Chevalley integral group scheme $G_\mathbb{Z}$, whose set of $\mathbb{C}$-points $G_\mathbb{Z}(\mathbb{C})$ in its Lie group topology, $G_\mathbb{Z}(\mathbb{C})_{\text{Lie}}$, is as a complex Lie group isomorphic to $C$ (for the existence of $G_\mathbb{Z}$ see [6]; see also [8]). As explained in [7], this opens the way to the application of Sullivan’s completion techniques, in a similar way as in [5]: the Hasse principle applies and solves the problem.

In Lemma 5.1 below, we show that if $G$ is a connected amenable Lie group, then its universal complexification $\gamma_G : G \to G^+$, (see Section 5 below) is one-to-one and is a homotopy equivalence (these two properties on the universal complexification characterize connected amenable Lie groups among connected Lie groups, but we won’t need this fact). As a consequence, Theorem 1.1 has the following corollary. (Notice that none of the conditions in Theorem 1.1 refers to a complex structure on the Lie group.)

**Corollary 1.2.** For a connected amenable Lie group, the four conditions in Theorem 1.1 are equivalent.

The proof of Corollary 1.2 is explained in Section 5 below. In [9], Goldman proved $(1) \iff (4)$ for connected solvable Lie groups. The finiteness assumption on the $CW$-complex which is the base of the bundle is not stated explicitly in [9] but is necessary, as the following example shows: the flat principal $S^1$-bundle over $K(\mathbb{Q}/\mathbb{Z}, 1)$ with classifying map induced by the inclusion $\mathbb{Q}/\mathbb{Z} \subset S^1$ is not virtually trivial because its classifying map $K(\mathbb{Q}/\mathbb{Z}, 1) \to BS^1 = K(\mathbb{Z}, 2)$ corresponds to an element of infinite order in $H^2(K(\mathbb{Q}/\mathbb{Z}, 1), \mathbb{Z}) \cong \lim_{\leftarrow} \mathbb{Z}/n\mathbb{Z}$.

That $(2) \iff (4)$ for connected solvable Lie groups follows from [3, Theorem 1.1] because a solvable group with the discrete topology is amenable and the real cohomology of amenable groups vanishes [17].

A compact Lie group is amenable as a Lie group, and the radical of a compact Lie group is abelian, hence the above corollary implies the following.
Corollary 1.3. Let $G$ be a connected compact Lie group. Then any flat principal $G$-bundle over any finite CW-complex is virtually trivial.

A flat bundle whose holonomy is non-amenable and has no complex structure may fail to be virtually trivial even if all its real characteristic classes vanish in positive degree. The cohomology ring

$$H^*(BSO(2n+1), \mathbb{R})$$

is generated by Pontrjagin classes [19, Theorem 15.9] and Pontrjagin classes of flat $GL(n, R)$-bundles vanish [19, Appendix C, Corollary 2] hence

$$H^*(BSL(2n+1), \mathbb{R}) \to H^*(BSL^\delta(2n+1), \mathbb{R})$$

vanishes in positive degree. But Millson [18] and Deligne [4] have constructed, for each $n \geq 3$, flat principal $SL(n, R)$-bundles over finite CW-complexes which are not virtually trivial.

The paper is organized as follows. In Section 2 we prove Theorem 1.1 for the case of a complex semisimple Lie group. In Section 3 we prove a lemma closely related to Goldman’s result [9], the main difference being that it also applies to bundles which are not necessarily flat. In Section 4 we explain how the results of the previous sections imply Theorem 1.1 in full generality. In Section 5 we prove Corollary 1.2.

2. The complex semisimple case

First, we fix some notation and recall some facts concerning Sullivan’s completion functor [20]. Let $p$ be a prime. We can think of Sullivan’s $p$-adic completion as a functor $X \mapsto \hat{X}_p$ on the homotopy category of connected CW-complexes, together with a natural transformation $X \to \hat{X}_p$ which for $X$ a simply connected CW-complex of finite type induces isomorphisms

$$\pi_i(X) \otimes \hat{\mathbb{Z}}_p \to \pi_i(\hat{X}_p), \ i \geq 2,$$

with $\hat{\mathbb{Z}}_p$ denoting the ring of $p$-adic integers. We will need the following basic fact.

Lemma 2.1. [20, Thm. 3.2]. Let $X$ be a finite CW-complex and $Y$ a simply connected CW-complex of finite type. A map

$$f : X \to Y$$

is homotopic to a constant map if and only if for every prime $p$ the map

$$\hat{f}_p : X \to Y \to \hat{Y}_p$$

is homotopic to a constant map.
The point here is that the space \( X \) in the lemma does not need to be simply connected (or nilpotent).

**Lemma 2.2.** Let \( C \) be a connected complex semisimple Lie group and \( X \) a connected finite CW-complex. Let \( P : E \to X \) be a flat principal \( C \)-bundle. Then \( P \) is virtually trivial.

**Proof.** We can assume that \( C \) is isomorphic to \( G_{\mathbb{Z}}(\mathbb{C})_{\text{Lie}} \) for some Chevalley group scheme \( G_{\mathbb{Z}} \). Let \( \psi : \overline{X} \to X \to BC \) be the classifying map for \( P \) pulled back to a finite covering space \( \overline{X} \) of \( X \). Because \( BC \) is simply connected and of finite type and \( \overline{X} \) is a finite complex, we can prove that \( \psi \) is homotopic to a constant map for a particular \( \overline{X} \) by showing that for every prime \( p \), the map \( \hat{\psi}_p : \overline{X} \to \overline{BC}_p \) into the \( p \)-adic completion of \( BC \) is homotopic to a constant map (Lemma 2.1). Let \( \pi \) be the fundamental group of \( X \) and \( \rho : \pi \to C = G_{\mathbb{Z}}(\mathbb{C})_{\text{Lie}} \) the holonomy of the bundle \( P \). Since \( \pi \) is finitely generated, there exist a subring \( \Lambda \subset \mathbb{C} \) of finite type over \( \mathbb{Z} \) such that the image of \( \rho \) is contained in \( G_{\mathbb{Z}}(\Lambda) \). Choose a maximal ideal \( m \subset \Lambda \) such that the finite field \( \mathbb{F} = \Lambda/m \) has characteristic \( q \) different from any torsion prime occurring in the finite torsion subgroup of \( \bigoplus \mathbb{Z} H^i(X, \pi_{i-1}(C)) \). Let \( \overline{\mathbb{F}} \) be an algebraic closure of \( \mathbb{F} \) and \( H \subset \mathbb{C} \) a strict Henselization of \( \Lambda \) in \( \mathbb{C} \), with residue field \( \mathbb{F} \). We then obtain a diagram of group homomorphisms

\[
\begin{array}{cccccc}
\pi & \longrightarrow & G_{\mathbb{Z}}(\Lambda) & \longrightarrow & G_{\mathbb{Z}}(H) & \longrightarrow & G_{\mathbb{Z}}(\mathbb{C}) & \longrightarrow & C \\
\phi & & & & \downarrow & & \simeq & \downarrow & \left( (BG_{\mathbb{F}})_{\text{et}} \right)_{\ell} \\
& & \simeq & & & & & \\
& & G_{\mathbb{Z}}(\overline{\mathbb{F}}) & & & & & \\
\end{array}
\]

such that the image of the composite map \( \phi : \pi \to G_{\mathbb{Z}}(\overline{\mathbb{F}}) \) is finite, because \( \pi \) is finitely generated and \( G_{\mathbb{Z}}(\overline{\mathbb{F}}) \) is a locally finite group. Let \( \overline{X} \) be the finite covering space of \( X \) corresponding to the kernel of \( \phi \). We will show that the bundle \( P \) pulled back to \( \overline{X} \) is trivial. Let \( \psi : \overline{X} \to BC \) be the classifying map for that bundle. For every prime \( \ell \) different from the characteristic \( q \) of \( \mathbb{F} \) the map \( \hat{\psi}_\ell : \overline{X} \to \overline{BC}_\ell \) is homotopically trivial, because up to homotopy it can be factored through the homotopically trivial map \( \overline{X} \to BG_{\mathbb{Z}}(H) \to BG_{\mathbb{Z}}(\overline{\mathbb{F}}) \), using natural maps

\[
\begin{array}{cccccc}
\overline{X} & \longrightarrow & BG_{\mathbb{Z}}(\Lambda) & \longrightarrow & BG_{\mathbb{Z}}(H) & \longrightarrow & BG_{\mathbb{Z}}(\mathbb{C}) & \longrightarrow & \overline{BC}_\ell \\
\downarrow & & & \downarrow & & \simeq & \downarrow & & \\
BG_{\mathbb{Z}}(\mathbb{F}) & \longrightarrow & ((BG_{\mathbb{F}})_{\text{et}})_{\ell} & & & & & & \\
\end{array}
\]
For the maps and notation see page 432 of [7]. It remains to deal with the prime $\ell = q$: we need to show that $\hat{\psi}_q : \overline{X} \to \overline{BC}_q$ is homotopically trivial too. Because the bundle $P$ is flat with connected complex reductive structure group, the rational characteristic classes of $P$ are all zero (the map $H^*(BC, \mathbb{Q}) \to H^*(X, \mathbb{Q})$ is zero in positive degrees cf. Theorem 3.5 of Kamber-Tondeur [16]). As a result, the obstructions to trivializing the $\hat{C}_q$-fibration classified by $X \to \overline{BC}_q$, are all $q$-torsion. These obstructions lie in the group $\bigoplus_i H^i(X, \pi_{i-1}(C) \otimes \mathbb{Z})$, but this group is torsion-free by the choice of $q$. We conclude that the map $\hat{\psi}_q : \overline{X} \to \overline{BC}_q$ is homotopically trivial and the Hasse principle (Lemma 2.1) applied to the map $\psi : \overline{X} \to BC$ implies therefore that $\psi$ must be homotopically trivial too.

\[\square\]

3. Bundles with solvable holonomy

The following is a characterization of virtually trivial principal bundles over finite connected CW-complexes, in case the structural group is a connected solvable Lie group. It can be viewed as a variation of a theorem due to Goldman [9], but without assuming that the bundle in question is flat.

**Lemma 3.1.** Let $R$ be a solvable connected Lie group and $P : E \to X$ a principal $R$-bundle over the connected CW-complex $X$, with $\psi : X \to BR$ the classifying map. Assume that $H_1(X, \mathbb{Z})$ is finitely generated. Then the bundle $P$ is virtually trivial if and only if $\psi^* : H^2(BR, \mathbb{R}) \to H^2(X, \mathbb{R})$ is the 0-map.

**Proof.** If $\psi^* \neq 0$ then for any finite covering space $\pi : \overline{X} \to X$ the composition

$$H^2(BR, \mathbb{R}) \xrightarrow{\psi^*} H^2(X, \mathbb{R}) \xrightarrow{\pi^*} H^2(\overline{X}, \mathbb{R})$$

is non-zero too, because $\pi^* : H^*(X, \mathbb{R}) \to H^*(\overline{X}, \mathbb{R})$ is injective. It follows that $P$ cannot pull back to a trivial bundle on some finite covering space of $X$. Conversely, assume that $\psi^* = 0$. Because $R$ is homotopy equivalent to a maximal compact subgroup $T \subset R$, $T$ a torus, $BR$ is homotopy equivalent to $K(\mathbb{Z}^n, 2)$ where $\pi_1(T) \cong \mathbb{Z}$. It follows that there is a single obstruction $\omega \in H^2(X, \pi_1(R))$ to the existence of a section for $P$. Because of our assumption on $X$, the kernel of the natural map $H^2(X, \mathbb{Z}) \to H^2(X, \mathbb{R})$ is finite (isomorphic to the torsion subgroup of $H_1(X, \mathbb{Z})$) and thus the hypothesis that $\psi^* = 0$ implies that $\omega$ must be a torsion class. From the universal coefficient theorem...
we see that therefore
\[ \omega \in \text{Ext}(H_1(X,\mathbb{Z}),\pi_1(R)) \rightarrow H^2(X,\pi_1(R)) \rightarrow \text{Hom}(H_2(X,\mathbb{Z}),\pi_1(R)). \]
Let \( \text{Tor} \subset H_1(X,\mathbb{Z}) \) be the finite torsion subgroup and choose a surjection \( \theta : \pi_1(X) \rightarrow \text{Tor} \). Let \( f : \overline{X} \rightarrow X \) denote the covering space corresponding to the kernel of \( \theta \). It follows that
\[ f^*(\omega) = 0 \in \text{Ext}(H_1(\overline{X},\mathbb{Z}),\pi_1(R)). \]
But \( f^*(\omega) \) is the only obstruction to the existence of a section for the principal \( R \)-bundle \( f^*P : f^*E \rightarrow \overline{X} \), showing that \( f^*P \) is trivial and thus completing the proof of the lemma. \( \square \)

4. Proof of Theorem 1.1

We will need the following two auxiliary results.

**Lemma 4.1.** Let \( R \) be a solvable connected Lie group and let \( P : E \rightarrow Z \) be a principal \( R \)-bundle over the finite connected complex \( Z \), classified by \( \kappa : Z \rightarrow B R \). Let \( G \) be a connected Lie group containing \( R \) as a normal, closed subgroup and denote by \( \iota : R \rightarrow G \) the inclusion. Assume that the principal \( G \)-bundle over \( Z \) classified by \( (B\iota) \circ \kappa : Z \rightarrow BG \) satisfies \( \kappa^* \circ (B\iota)^* = 0 : H^2(BG,\mathbb{R}) \rightarrow H^2(Z,\mathbb{R}) \). Then the principal \( R \)-bundle \( P \) is virtually trivial.

**Proof.** Let \( Q = G/R \). Since for any connected Lie group the second homotopy group vanishes and the fundamental group is abelian, we have a short exact sequence of abelian groups
\[ 0 \rightarrow \pi_1(R) \rightarrow \pi_1(G) \rightarrow \pi_1(Q) \rightarrow 0, \]
inducing a split short exact sequence of \( \mathbb{R} \)-vector spaces
\[ 0 \rightarrow \text{Hom}(\pi_1(Q),\mathbb{R}) \rightarrow \text{Hom}(\pi_1(G),\mathbb{R}) \xrightarrow{\Phi} \text{Hom}(\pi_1(R),\mathbb{R}) \rightarrow 0. \]
For any connected Lie group \( L \), the group \( H_2(BL,\mathbb{R}) \) is naturally isomorphic to \( H_1(L,\mathbb{R}) \cong \pi_1(L) \otimes \mathbb{R} \). It follows that the natural map \( (B\iota)^* : H^2(BG,\mathbb{R}) \rightarrow H^2(BR,\mathbb{R}) \) corresponds to the surjective map \( \Phi \). Therefore, the vanishing of
\[ \kappa^* \circ (B\iota)^* : H^2(BG,\mathbb{R}) \rightarrow H^2(Z,\mathbb{R}) \]
implies the vanishing of
\[ \kappa^* : H^2(BR,\mathbb{R}) \rightarrow H^2(Z,\mathbb{R}). \]
Using Lemma 3.1 we conclude that the principal \( R \)-bundle \( P \) is virtually trivial. \( \square \)
Lemma 4.2. Let $G$ be a connected Lie group and let $R$ be its radical. Suppose that its derived group $[R, R]$ is simply connected in its Lie group topology and that $G/R$ has a finite fundamental group. Let $G^\delta$ denote the group $G$ with the discrete topology. Then the identity map on the underlying sets $\iota_G : G \to G$ induces the zero map $\iota_\ast_G : H^2(BG, \mathbb{R}) \to H^2(BG^\delta, \mathbb{R})$.

Proof. There is a short exact sequence of Lie groups

\[ 0 \to R \xrightarrow{\iota} G \xrightarrow{\pi} Q \to 0 \]

with $R$ the radical of $G$ and $Q$ semisimple.

Split case. We first assume that the short exact sequence (1) is split, with $\sigma : Q \to G$ a splitting. For a discrete group $D$ we write $H^*_{\text{b}}(D, \mathbb{R})$ for its bounded real cohomology and we denote by $\theta_D : H^*_{\text{b}}(D, \mathbb{R}) \to H^*(D, \mathbb{R})$ the forgetful map. Because $R^\delta$ is an amenable discrete group, the inflation map $\pi_\ast : H^*_{\text{b}}(Q^\delta, \mathbb{R}) \to H^*_{\text{b}}(G^\delta, \mathbb{R})$ is an isomorphism (cf. Ivanov [14, Theorem 3.8.4], see also Gromov’s Mapping Theorem [11, Section 3.1]).

Therefore, the induced maps

\[ \pi_\ast : H^*_{\text{b}}(Q^\delta, \mathbb{R}) \to H^*_{\text{b}}(G^\delta, \mathbb{R}), \quad \sigma_\ast : H^*_{\text{b}}(G^\delta, \mathbb{R}) \to H^*_{\text{b}}(Q^\delta, \mathbb{R}) \]

are inverse isomorphisms. We write

\[ \pi_\ast : H^*(Q^\delta, \mathbb{R}) \to H^*(G^\delta, \mathbb{R}), \quad \sigma_\ast : H^*(G^\delta, \mathbb{R}) \to H^*(Q^\delta, \mathbb{R}) \]

and

\[ \pi_\ast : H^*(BQ, \mathbb{R}) \to H^*(BG, \mathbb{R}), \quad \sigma_\ast : H^*(BG, \mathbb{R}) \to H^*(BQ, \mathbb{R}) \]

for the maps induced by $\sigma$ respectively $\pi$ in these cohomology groups.

We then have a commutative diagram

\[
\begin{array}{ccc}
H^2(BG, \mathbb{R}) & \xrightarrow{\iota_{\ast_B}} & H^2(BQ, \mathbb{R}) \\
\downarrow_{\theta_G} & & \downarrow_{\theta_Q} \\
H^2(BG^\delta, \mathbb{R}) & \xleftarrow{\pi_\ast} & H^2(BQ^\delta, \mathbb{R}) \\
\end{array}
\]

Let $x \in H^2(BG, \mathbb{R})$. We need to show that $\iota_{\ast_B}^*(x) = 0$. Since $[R, R]$ is simply connected, $\iota_{\ast_B}^*(x)$ is bounded, meaning that it lies in the image
of $\theta_G$ (see Theorem 1.1 of [3]). By assumption, $\pi_1(Q)$ is finite. Thus $H^2(BQ, \mathbb{R}) \cong \text{Hom}(\pi_1(Q), \mathbb{R}) = 0$ which implies that $\iota^*_Q = 0$ in the diagram above. Choose $y$ such that $\theta_G(y) = \iota^*_G(x)$. Because $y = \pi^*_1\sigma^*_y(y)$, we have

$$\iota^*_G x = \theta_G y = \theta_G(\pi^*_1\sigma^*_y y) = \pi^*_1\theta_Q\sigma_y = \pi^*_1\sigma_y^*(\theta_G y) =$$

$$= \pi^*_1(\sigma^*_y\iota^*_G x) = \pi^*_1(\iota^*_Q\sigma_{\text{top}}^*x) = 0,$$

because $\iota^*_Q = 0$.

Non-split case. Suppose that the exact sequence (1) is non-split. Let $\tilde{Q} \to Q$ be the universal cover. The pull-back of $G \to Q$ over $\tilde{Q}$ yields a short exact sequence of Lie groups

$$R \to \tilde{G} \to \tilde{Q}$$

which is split because $\tilde{Q}$ is simply connected (see Lemma 14 of [2]). The natural map $\tilde{p} : \tilde{G} \to G$ is a surjective homomorphism of connected Lie groups with finite kernel $K$ isomorphic to $\pi_1(Q)$. Since $BK$ is $\mathbb{R}$-acyclic, the induced maps

$$p^*_\text{top} : H^*(BG, \mathbb{R}) \xrightarrow{\cong} H^*(B\tilde{G}, \mathbb{R}) \quad \text{and} \quad p^*_\delta : H^*(BG^\delta, \mathbb{R}) \xrightarrow{\cong} H^*(B\tilde{G}^\delta, \mathbb{R})$$

are isomorphisms. From the split case we infer that $\iota^*_G : H^2(B\tilde{G}, \mathbb{R}) \to H^2(B\tilde{G}^\delta, \mathbb{R})$ is the zero map, and thus the corresponding map $\iota^*_G$ is zero too.

Proof of Theorem 1.1.

Let $G$ be a connected complex Lie group. Its radical $R$ is a complex Lie subgroup and $G/R$ is complex semisimple and has therefore a finite fundamental group [12, Chapter XVII, Theorem 2.1]. As explained in the introduction, in order to prove the theorem, it is enough to assume that $[R, R]$ is simply connected (in its Lie group topology) and to show that if $P : E \to X$ is a flat principal $G$-bundle over a connected finite complex $X$, with classifying map $\alpha : X \to BG$, then there is a finite connected covering space $\beta : Y \to X$, such that the bundle $P$ pulled back to $Y$ is trivial, i.e. such that the map $\alpha \circ \beta : Y \to BG$ is homotopic to a constant map. Let $p : G \to Q$ be the projection and put $\gamma = Bp : BG \to BQ$. Then the map $\gamma \circ \alpha : X \to BQ$ classifies a principal $Q$-bundle over $X$ which is flat because $P$ is flat and the diagram

$$\begin{array}{ccc}
BG & \longrightarrow & BQ \\
\downarrow & & \downarrow \\
BG & \longrightarrow & BQ
\end{array}$$
commutes. By Lemma 2.2 we can find a finite connected covering space \( \delta : Z \to X \) such that the bundle classified by \( \gamma \circ \alpha \circ \delta : Z \to BQ \) is trivial. The lifting property of the fibration \( BR \to BG \to BQ \) implies that \( \alpha \circ \delta : Z \to BG \) factors through \( \epsilon = Bi : BR \to BG \), where \( i : R \to G \) stands for the inclusion. In other words, there is a map \( \kappa : Z \to BR \), with \( \epsilon \circ \kappa : Z \to BG \) homotopic to \( \alpha \circ \delta : Z \to BG \). We claim that the (not necessarily flat) principal \( R \)-bundle classified by \( \kappa : Z \to BR \) is virtually trivial. By Lemma 4.1 it suffices to show that 

\[
(\epsilon \circ \kappa)^* = (\alpha \circ \delta)^* = 0 : H^2(BG, \mathbb{R}) \to H^2(Z, \mathbb{R}) .
\]

As \( P \) is flat, \( \alpha^* : H^2(BG, \mathbb{R}) \to H^2(X, \mathbb{R}) \) factors through \( H^2(BG^\delta, \mathbb{R}) \) and since by assumption \([R, R]\) is simply connected and \( Q \) is complex semisimple, Lemma 4.2 applies and implies that

\[
H^2(BG, \mathbb{R}) \to H^2(BG^\delta, \mathbb{R})
\]

is the zero map. Thus \( (\alpha \circ \delta)^* = 0 \) and therefore, by Lemma 4.1, the bundle classified by \( \kappa : Z \to BR \) is virtually trivial. We now choose a finite connected covering space \( \mu : Y \to Z \) on which the \( R \)-bundle pulls back to a trivial bundle, i.e. \( \kappa \circ \mu \simeq \ast \). It then follows that the original \( G \)-bundle over \( X \) pulls back to the trivial bundle over the finite covering space \( \beta = \delta \circ \mu : Y \to X \).

The following diagram, with commuting squares up to homotopy, depicts, for the convenience of the reader, the maps described above:

\[
\begin{array}{cccc}
Y & \xrightarrow{\mu} & Z & \xrightarrow{\kappa} & BR \\
\downarrow & & \downarrow \delta & & \downarrow \epsilon \\
X & \xrightarrow{\alpha} & BG & \xrightarrow{\gamma} & BQ \\
\end{array}
\]

\( \square \)

5. Proof of Corollary 1.2

We first recall some facts on the complexification of a connected Lie group \( G \). We follow the notation used in Hochschild [13]; (see also Bourbaki [1, Chapter III, §6, Prop. 20]). To any Lie group corresponds a complex Lie group \( G^+ \) and a homomorphism of Lie groups

\[
\gamma_G : G \to G^+, 
\]

called the universal complexification of \( G \), with the property that, for every continuous homomorphism \( \eta \) of \( G \) into a complex Lie group \( H \),
there is one and only one complex analytic homomorphism $\eta^+ : G^+ \to H$ such that $\eta = \eta^+ \gamma_G$. In general $\gamma_G$ is not injective. Its kernel is a central (not necessarily discrete) subgroup of $G$. Let $R < G$ denote the radical of the connected Lie group $G$ and $L < G$ a Levi subgroup (a maximal connected semisimple subgroup). Then $G = RL$ and in case $L < G$ is closed, the kernel of $\gamma_G$ coincides with the kernel of $\gamma_L$ and is discrete in $G$ (see [13, Theorem 4]). Also, if $G$ is linear, $\gamma_G$ is injective and for $G$ compact, $\gamma_G$ maps $G$ isomorphically onto a maximal compact subgroup of $G$. Therefore, for compact $G$, the map $\gamma_G$ is a homotopy equivalence. As explained in [13], in the case $R$ is a connected solvable Lie group (not necessarily linear), $\gamma_R$ is injective and induces an isomorphism between fundamental groups $\pi_1(R) \to \pi_1(R^+)$. The universal covers of the solvable Lie groups $R$ and $R^+$ being contractible, it follows that $\gamma_R : R \to R^+$ is a homotopy equivalence.

**Lemma 5.1.** Let $G$ be a connected amenable Lie group. Then the complexification map $\gamma_G : G \to G^+$ is one-to-one and a homotopy equivalence.

**Proof.** A connected Lie group $G$ is amenable if and only if it fits in a short exact sequence

$$\{1\} \to R \to G \to Q \to \{1\},$$

where $R$ denotes the radical of $G$ and the quotient $Q$ is compact semisimple [21, Corollary 4.1.9]. Let $L < G$ be a Levi subgroup. Since $G/R = Q$ is compact and semisimple, its fundamental group is finite. Thus $L \to Q$, induced by the projection $G \to Q$, is a finite covering space and it follows that $L$ is compact, thus closed in $G$. Moreover, $L$ is linear and we conclude that $\gamma_L : L \to L^+$ is one-to-one. According to [13, Theorem 4, (2) ⇔ (5)], we conclude that $\gamma_G$ is injective too. Consider the commutative diagram

$$\begin{array}{ccc}
R & \xrightarrow{\iota} & G \\
\gamma_R \downarrow & & \downarrow \gamma_G \\
R^+ & \xrightarrow{\iota^+} & G^+ \xrightarrow{\pi^+} Q^+ \xrightarrow{\gamma_Q} Q.
\end{array}$$

As remarked above, $\gamma_R$ and $\gamma_Q$ are injective maps and homotopy equivalences. By [13, Theorem 4], $\iota^+$ maps $R^+$ isomorphically onto the radical of $G^+$. We claim that $G^+/R^+$ it is isomorphic to $Q^+$. To see this, we need to verify that this quotient has the universal property of $Q^+$. Let $\nu : G \to G^+ \to G^+/R^+$ be the natural map. Since $R \subset \ker \nu$, we obtain an natural map $\overline{\nu} : Q \to G^+/R^+$. Let $f : Q \to C$ be an analytic homomorphism into a complex Lie group $C$. Then $f \circ \pi : G \to Q \to C$
is trivial on $R$ and extends therefore uniquely to a complex analytic homomorphism $G^+ \to C$ which vanishes on $R^+$. It follows that the original map $f$ factors uniquely through $\overline{\nabla} : Q \to G^+/R^+$, showing that $G^+/R^+ \cong Q^+$. Note that both horizontal lines in the diagram above are fibration sequences. We conclude that $\gamma_G$ must be a homotopy equivalence too. \[\square\]

The following lemma is a general fact about universal complexifications of connected solvable Lie groups. It will be useful in the proof of Corollary 1.2.

**Lemma 5.2.** Let $R$ be a connected solvable Lie group. Then

$$[R, R]^+ = [R^+, R^+]$$

That is, the universal complexification of the derived subgroup of $R$ is isomorphic to the derived subgroup of the universal complexification of $R$.

**Proof.** As $R$ is solvable, $\gamma_R$ is one-to-one hence so is its restriction $\eta : [R, R] \to [R^+, R^+]$ to $[R, R]$. The universal property of

$$\gamma_{[R,R]} : [R, R] \to [R, R]^+,$$

implies the existence of a complex analytic homomorphism

$$\eta^+ : [R, R]^+ \to [R^+, R^+]$$

such that $\eta^+ \gamma_{[R,R]} = \eta$. Taking derivatives at the identities and using the fact that for any real Lie algebra $\mathfrak{r}$ we have

$$[\mathfrak{r} \otimes \mathbb{C}, \mathfrak{r} \otimes \mathbb{C}] = [\mathfrak{r}, \mathfrak{r}] \otimes \mathbb{C},$$

we deduce that $\eta^+$ is a local isomorphism, hence a covering homomorphism. This proves the lemma in the case $R$ is simply connected. Indeed, the inclusion $\gamma_R : R \to R^+$ is a homotopy equivalence, hence $R^+$ is also simply connected, and according to [2, Lemma 6] we have

$$\pi_1([R^+, R^+]) = \pi_1(R^+) \cap [R^+, R^+]$$

This shows that $[R^+, R^+]$ is also simply connected, hence $\eta^+$ is a global isomorphism. To handle the general case, let us show that the discrete kernel of $\eta^+$ is trivial. To that end, we show that the natural embeddings of fundamental groups in the centers of universal covers coincide. Let $\tilde{R}$ be the universal cover of $R$. It is obvious from the construction of the universal complexification that the universal cover $\tilde{R}^+$ of $R^+$
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coincides with \((\tilde{R})^+)\). We have:

\[
\pi_1([R, R]^+) = \pi_1([R, R]) = \pi_1(R) \cap [\tilde{R}, \tilde{R}] \\
= \pi_1(R) \cap [\tilde{R}, \tilde{R}]^+ = \pi_1(R^+) \cap [\tilde{R}, \tilde{R}]^+ \\
= \pi_1(R^+) \cap [(\tilde{R})^+, (\tilde{R})^+] = \pi_1(R^+) \cap [\tilde{R}, \tilde{R}]^+ \\
= \pi_1([R^+, R^+]).
\]

The first equality (as well as the fourth one) is true because the embedding of a connected solvable Lie group in its universal complexification is a homotopy equivalence, the second equality (as well as the last one) is a general fact (see [2, Lemma 6]) about closed normal subgroups in Lie groups, the third equality follows from the fact that 

\([\tilde{R}, \tilde{R}]^+ \cap \tilde{R} \subset [\tilde{R}, \tilde{R}]\)

which is deduced from the corresponding inclusion between Lie algebras. The fifth equality is true because we have already proved the lemma for simply connected solvable Lie groups hence 

\([\tilde{R}, \tilde{R}]^+ = [(\tilde{R})^+, (\tilde{R})^+]\). □

Proof of Corollary 1.2. As explained in the introduction, in order to prove that the four conditions are equivalent, it is enough to show that if \(G\) is a connected amenable Lie group with radical \(R\) such that \([R, R]\) is simply connected, then any flat principal \(G\)-bundle over a finite \(CW\)-complex is virtually trivial. As we have observed earlier, the complexification map \(R \to R^+\) is injective and, applying again [13, Theorem 4, (2) \(\iff\) (5)], we see that as \(G/R\) is compact semisimple, \(R^+\) maps isomorphically onto the radical of \(G^+\). As we have seen in the course of the proof of Lemma 5.1, the map \(\gamma_G\) restricted to \(R\) agrees with \(\gamma_R\). Thus, as \([R, R]\) is simply connected by hypothesis, and as \([R^+, R^+]\)/\(\gamma_G[R, R]\) is simply connected according to [13, Theorem 3] and Lemma 5.2, we deduce that \([R^+, R^+]\) is simply connected too. Let \(f : X \to BG\) classify a flat principal \(G\)-bundle over the finite connected \(CW\)-complex \(X\). Since \([R^+, R^+]\) is the commutator subgroup of the radical of \(G^+\) and \([R^+, R^+]\) is simply connected, we conclude by Theorem 1.1 that the flat bundle classified by \(B\gamma_G \circ f : X \to BG^+\) is virtually trivial. Because \(B\gamma_G\) is a homotopy equivalence (Lemma 5.1), the bundle classified by \(f\) is virtually trivial too. □

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