FUSION PROCEDURE FOR COXETER GROUPS OF TYPE B
AND COMPLEX REFLECTION GROUPS $G(m, 1, n)$

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Abstract. A complete system of primitive pairwise orthogonal idempotents for the Coxeter groups of type $B$ and, more generally, for the complex reflection groups $G(m, 1, n)$ is constructed by a sequence of evaluations of a rational function in several variables with values in the group ring. The evaluations correspond to the eigenvalues of the two arrays of Jucys–Murphy elements.

1. Introduction

A. Jucys [13] gave a construction of a complete system of pairwise orthogonal primitive idempotents of the group ring of the symmetric group. The construction, now called the fusion procedure, involves a rational function in several variables, and the idempotents are obtained by taking certain limiting values of this function. We refer to [1, 3, 5, 10, 16, 17, 19] for different aspects and applications of the fusion procedure for the symmetric groups. There are analogues of the fusion procedure, in the spirit of [1], for the Coxeter group of type $B$ [2], the Hecke algebra of type $A$ [4, 18] and the spinor extension of the symmetric group [11, 15] (see [12] for its $q$-analogue).

A version of the fusion procedure for the symmetric group was given by A. Molev in [14]. Here the idempotents are obtained by consecutive evaluations of the rational function. An analogue of this fusion procedure was developed for the Hecke algebra [7], the Brauer algebra [6, 8] and the Birman–Murakami–Wenzl algebra [9].

The aim of this paper is to give a fusion procedure, in the spirit of [14], for the complex reflection groups $G(m, 1, n)$. As in [14], and later [6, 9], we use the Jucys–Murphy elements. They were introduced for $G(m, 1, n)$ independently in [22] and [24]. The Jucys–Murphy elements of $G(m, 1, n)$ form two arrays, $j_i$ and $\tilde{j}_i$, $i = 1, \ldots, n$, and their union is the maximal commutative set in $CG(m, 1, n)$; see [21, 22]. An irreducible representation of the group $G(m, 1, n)$ is coded by an $m$-tuple of partitions, and its basis vectors correspond to standard $m$-tuples of tableaux. The eigenvalues of $j_i$ carry information about the “position” - the place of a tableau in the $m$-tuple - while the eigenvalues of $\tilde{j}_i$ are related to the classical contents of nodes. In the work [21] both sets appeared as classical limits of simple expressions involving the single set of the Jucys–Murphy elements of the cyclotomic Hecke algebra, the flat deformation of $CG(m, 1, n)$. By the maximality,
all diagonal matrix units of $\mathbb{C}G(m, 1, n)$ can be expressed in terms of the Jucys–Murphy elements $j_i$ and $\tilde{j}_i$, $i = 1, \ldots, n$. Then we translate this expression as a fusion procedure: any diagonal matrix unit can be obtained by a sequence of evaluations of a certain rational function with values in $\mathbb{C}G(m, 1, n)$. The arrays $j_i$ and $\tilde{j}_i$ play different roles: the positions can be evaluated simultaneously, while the contents should then be evaluated consequently from 1 to $n$.

The group $G(1, 1, n)$ is isomorphic to the symmetric group $S_n$ and our fusion procedure for $m = 1$ reproduces the fusion procedure of [14].

The group $G(2, 1, n)$ is isomorphic to the hyperoctahedral group $B_n$, the Coxeter group of type $B$. Thus, in particular, we obtain a fusion procedure for the Coxeter group of type $B$ and a description of a complete set of pairwise orthogonal primitive idempotents of $B_n$ in terms of a single rational function with values in the group algebra $\mathbb{C}B_n$. A version of the fusion procedure for type $B$ was developed in [2]. Our fusion procedure is different from the one in [2] in that it uses another way to Baxterize the generators of the symmetric group.

For clarity of the exposition, we first describe the fusion procedure for the Coxeter group $B_n$. This is done not only for aesthetic reasons: the rational function with values in $\mathbb{C}B_n$ leading to the complete set of idempotents can be viewed as a word for the longest element of $B_n$ in which certain entries are “Baxterized”, similarly to the rational function for type $A$. Although $G(m, 1, n)$ is not a Coxeter group for $m > 2$ and the notion of length of an element is not defined, there is an analogue of the longest element: it is longest with respect to the normal form of [21] (the classical limit of the normal form for the cyclotomic Hecke algebra $H(m, 1, n)$ [20]); the rational function with values in $\mathbb{C}G(m, 1, n)$ leading to the complete set of idempotents can again be viewed as a word for the longest element with certain entries Baxterized.

We stress two features of our construction. First, the groups $G(m, 1, n)$ admit a fusion procedure for any positive integer $m$, and our construction is uniform for all $m$. Second, the new phenomenon - compared to the symmetric group - appears: for $m > 1$ only some well-chosen generators (appearing in the expression for the analogue of the longest element) are Baxterized.

The paper is organized as follows. Section 2 contains necessary definitions and notation about the groups $B_n$ and their representations. The Jucys–Murphy elements for $B_n$ were defined in [22–24]; we describe the diagonal matrix units in terms of them. In Section 3 we prove Theorem 3.2 which gives the fusion procedure for the groups $B_n$. In Section 4 in Theorem 4.2 we extend the results to the general case of the groups $G(m, 1, n)$. As the proofs work mainly along the same lines as for $B_n$, we only indicate the necessary modifications.

2. Idempotents and Jucys–Murphy elements of the group $B_n$

2.1. Definitions. The Coxeter group $S_{n+1}$ of type $A$ (the symmetric group on $n + 1$ letters) is generated by the elements $s_1, \ldots, s_n$ with the defining relations:

\begin{align}
  s_i^2 &= 1, \\
  s_is_{i+1}s_i &= s_{i+1}s_is_{i+1} \quad \text{for } i = 1, \ldots, n, \\
  s_is_j &= s_js_i \quad \text{for } i, j = 1, \ldots, n \text{ such that } |i - j| > 1.
\end{align}

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The Coxeter group $B_{n+1}$ of type $B$ (also called the hyperoctahedral group) is generated by the elements $s_1, \ldots, s_n$ and $t$ with the defining relations (2.1):

\begin{align}
(t s_1 t s_1 = s_1 t s_1 t, \\
  s_i t = t s_i \quad \text{for} \ i = 2, \ldots, n;
\end{align}

and

\begin{equation}
t^2 = 1.
\end{equation}

For any $i = 1, \ldots, n$, set

\begin{equation}
s_i(p, p', a, a') := s_i + \frac{\delta_{p,p'}}{a - a'},
\end{equation}

where $\delta_{p,p'}$ is the Kronecker delta. For $p = p'$ the elements (2.4) are called Baxter-ized elements; the parameters $a$ and $a'$ are referred to as spectral parameters. We also define

\begin{equation}
t(p) := \frac{1}{2}(1 + pt).
\end{equation}

The following relation is satisfied and will be used later:

\begin{equation}
s_i(p, p', a, a')s_i(p', p, a', a) = \frac{(a - a')^2 - \delta_{p,p'}}{(a - a')^2} \quad \text{for} \ i = 1, \ldots, n.
\end{equation}

Define the elements $j_i, i = 1, \ldots, n + 1$, and $\tilde{j}_i, i = 1, \ldots, n + 1$, of the group algebra $\mathbb{C}B_{n+1}$ by the following initial conditions and recursions:

\begin{equation}
j_1 = t, \quad j_{i+1} = s_i j_i s_i \quad \text{and} \quad \tilde{j}_1 = 0, \quad \tilde{j}_{i+1} = s_i \tilde{j}_i s_i + \frac{1}{2}(s_i + j_i s_i j_i).
\end{equation}

The elements $j_i$ and $\tilde{j}_i$ are analogues, for the group $B_{n+1}$, of the Jucys–Murphy elements. The elements $j_i, i = 1, \ldots, n + 1$, and $\tilde{j}_i, i = 1, \ldots, n + 1$, form a maximal commutative set in $\mathbb{C}B_{n+1}$ (see [21,23]); in addition, $j_i$ and $\tilde{j}_i$ commute with all generators $s_k$, except $s_i$ and $s_{i-1}$:

\begin{equation}
j_i s_k = s_k j_i \quad \text{and} \quad j_i s_k = s_k j_i, \quad \text{if} \ k \neq i - 1, i.
\end{equation}

Let $\lambda \vdash n + 1$ be a partition of $n + 1$, that is, $\lambda = (\lambda_1, \ldots, \lambda_k)$, where $\lambda_j, j = 1, \ldots, k$, are positive integers, $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k$ and $n + 1 = \lambda_1 + \cdots + \lambda_k$. We identify partitions with their Young diagrams: the Young diagram of $\lambda$ is a left-justified array of rows of nodes containing $\lambda_j$ nodes in the $j$-th row, $j = 1, \ldots, k$; the rows are numbered from top to bottom.

A 2-partition, or a Young 2-diagram, of $n + 1$ is a pair of partitions such that the total number of boxes in the associated Young diagrams is equal to $n + 1$; e.g. the Young 2-diagram ($\square, \square$) represents the 2-partition $((2), (1))$ of $3$.

A 2-node $\alpha^{(2)}$ is a pair $(\alpha, k)$ consisting of a usual node $\alpha$ and an integer $k = 1, 2$. The integer $k$ will be called the position of the 2-node. A set of 2-nodes can be equivalently described by an ordered pair of sets of nodes (the integer $k$ of a 2-node $(\alpha, k)$ indicates to which set the node $\alpha$ belongs). A 2-partition $\lambda^{(2)}$ is a set of 2-nodes such that the subset consisting of the 2-nodes having position $p$ is a usual partition, $p = 1, 2$.

For a 2-node $\alpha^{(2)} = (\alpha, k)$ lying in the line $x$ and the column $y$ of the $k$-th diagram, we denote by $c(\alpha^{(2)})$ the classical content of the node $\alpha$, $c(\alpha^{(2)}) := c(\alpha) = y - x$. Let $\{\xi_1, \xi_2\}$ be the set of distinct square roots of unity, ordered arbitrarily; we also define $p(\alpha^{(2)}) := \xi_k$. 

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For a 2-partition \( \lambda^{(2)} \), a 2-node \( \alpha^{(2)} \) of \( \lambda^{(2)} \) is called *removable* if the set of 2-nodes obtained from \( \lambda^{(2)} \) by removing \( \alpha^{(2)} \) is still a 2-partition. A 2-node \( \beta^{(2)} \) not in \( \lambda^{(2)} \) is called *addable* if the set of 2-nodes obtained from \( \lambda^{(2)} \) by adding \( \beta^{(2)} \) is still a 2-partition. For a 2-partition \( \lambda^{(2)} \), we denote by \( \mathcal{E}_{-}(\lambda^{(2)}) \) the set of removable 2-nodes of \( \lambda^{(2)} \) and by \( \mathcal{E}_{+}(\lambda^{(2)}) \) the set of addable 2-nodes of \( \lambda^{(2)} \). For example, the removable/addable 2-nodes (marked with \(-/+\)) for the 2-partition \((\square, \square)\) are 

\[
\begin{pmatrix}
\square & - & + \\
+ & - & +
\end{pmatrix}.
\]

Let \( \lambda^{(2)} \) be a 2-partition of \( n+1 \). A standard 2-tableau of shape \( \lambda^{(2)} \) is obtained by placing the numbers \( 1, \ldots, n+1 \) in the 2-nodes of the diagrams of \( \lambda^{(2)} \) in such a way that the numbers in the nodes ascend along rows and columns in every diagram. For a standard 2-tableau \( \mathcal{T} \) of shape \( \lambda^{(2)} \) let \( \alpha_{i}^{(2)} \) be the 2-node of \( \mathcal{T} \) with number \( i \), \( i = 1, \ldots, n+1 \); we set \( c(\mathcal{T}|i):=c(\alpha^{(2)}_{i}) \) and \( p(\mathcal{T}|i):=p(\alpha^{(2)}_{i}) \). For example, take the standard 2-tableau \( \mathcal{T}=\left(\begin{array}{ll}
1 & 3 \\
2 & 3
\end{array}\right)\); then we have 

\[
c(\mathcal{T}|1)=0, \quad c(\mathcal{T}|2)=0, \quad c(\mathcal{T}|3)=1 \quad \text{and} \quad p(\mathcal{T}|1)=\xi_{1}, \quad p(\mathcal{T}|2)=\xi_{2}, \quad p(\mathcal{T}|3)=\xi_{1}.
\]

The hook of a node \( \alpha \) of a partition \( \nu \) is the set of nodes of \( \nu \) consisting of the node \( \alpha \) and the nodes which lie either under \( \alpha \) in the same column or to the right of \( \alpha \) in the same row; the hook length \( h_{\nu}(\alpha) \) of \( \alpha \) is the cardinality of the hook of \( \alpha \). We extend this definition to 2-nodes. For a 2-node \( \alpha^{(2)}=(\alpha,k) \) of a 2-partition \( \nu^{(2)} \), the hook length of \( \alpha^{(2)} \) in \( \nu^{(2)} \), which we denote by \( h_{\nu^{(2)}}(\alpha^{(2)}) \), is the hook length of the node \( \alpha \) in the \( k \)-th partition of \( \nu^{(2)} \). Let 

\[
f_{\nu^{(2)}}:=\left( \prod_{\alpha^{(2)}\in\nu^{(2)}} h_{\nu^{(2)}}(\alpha^{(2)}) \right)^{-1}.
\]

2.2. Idempotents of the group \( B_{n+1} \). The representation theory of the Coxeter groups of type \( B \) was developed by A. Young [25]; see also [21][23]. The irreducible representations of \( B_{n+1} \) are in bijection with 2-partitions of \( n+1 \). The basis vectors of the irreducible representation of \( B_{n+1} \) corresponding to the 2-partition \( \lambda^{(2)} \) are parameterized by standard 2-tableaux of shape \( \lambda^{(2)} \). The Jucys–Murphy elements are diagonal in this basis. For a standard 2-tableau \( \mathcal{T} \), we denote by \( E_{\mathcal{T}} \) the primitive idempotent of \( B_{n+1} \) corresponding to \( \mathcal{T} \). We have, for any \( i=1,\ldots,n+1 \), 

\[
\begin{align*}
\tilde{j}_{i}E_{\mathcal{T}} & =E_{\mathcal{T}}j_{i}=p_{i}E_{\mathcal{T}} \quad \text{and} \quad \tilde{j}_{i}E_{\mathcal{T}} =E_{\mathcal{T}}\tilde{j}_{i}=c_{i}E_{\mathcal{T}}.
\end{align*}
\]

Here we set \( p_{i}:=p(\mathcal{T}|i) \) and \( c_{i}:=c(\mathcal{T}|i) \) for all \( i=1,\ldots,n+1 \) for brevity. Due to the maximality of the commutative set \( \{j_{i},\tilde{j}_{i}\}_{i=1,\ldots,n+1} \) of Jucys–Murphy elements, the idempotent \( E_{\mathcal{T}} \) can be expressed in terms of \( j_{i},\tilde{j}_{i}, i=1,\ldots,n+1 \). Let \( \alpha^{(2)} \) be the 2-node of \( \mathcal{T} \) with the number \( n+1 \). As the tableau \( \mathcal{T} \) is standard, the 2-node \( \alpha^{(2)} \) of \( \lambda^{(2)} \) is removable. Let \( \mathcal{U} \) be the standard 2-tableau obtained from \( \mathcal{T} \) by removing the 2-node \( \alpha^{(2)} \) and let \( \mu^{(2)} \) be the shape of \( \mathcal{U} \). The inductive formula for \( E_{\mathcal{T}} \) in terms of the Jucys–Murphy elements reads: 

\[
E_{\mathcal{T}} = E_{\mathcal{U}} \prod_{\beta^{(2)} \in \mathcal{E}_{+}(\mu^{(2)})} \frac{\tilde{j}_{n+1} - c(\beta^{(2)})}{c(\beta^{(2)}) - c(\alpha^{(2)})} \prod_{\beta^{(2)} \in \mathcal{E}_{+}(\mu^{(2)})} \frac{j_{n+1} - p(\beta^{(2)})}{p(\alpha^{(2)}) - p(\beta^{(2)})}.
\]
Note that the second product in the right-hand side of (2.11) contains only one term (this will not be so for the cyclotomic groups \(G(m, 1, n)\) with \(m > 2\)). We have \(B_0 \cong \mathbb{C}\) and \(E_{U_0} = 1\) for the (unique) 2-tableau \(U_0\) of 0.

Let \(\{T_1, \ldots, T_k\}\) be the set of pairwise different standard 2-tableaux that can be obtained from \(U\) by adding a 2-node with the number \(n + 1\). The formula

\[
E_U = \sum_{i=1}^{k} E_{T_i},
\]

together with (2.10), implies that the rational function

\[
E_U \frac{u - c_{n+1}}{u - j_{n+1}} \frac{v - p_{n+1}}{v - j_{n+1}}
\]

is non-singular at \(u = c_{n+1}\) and \(v = p_{n+1}\), and, moreover,

\[
E_U \frac{u - c_{n+1}}{u - j_{n+1}} \frac{v - p_{n+1}}{v - j_{n+1}} \bigg|_{u = c_{n+1}} = E_T.
\]

Since \(j_{n+1}\) takes values \(\pm 1\), the rational function \(\frac{v - p_{n+1}}{v - j_{n+1}}\) is non-singular at \(v = p_{n+1}\) and

\[
\frac{v - p_{n+1}}{v - j_{n+1}} \bigg|_{v = p_{n+1}} = \frac{1}{2} (1 + p_{n+1} j_{n+1}).
\]

For the clarity of the calculations in the sequel, we define, generalizing (2.5),

\[
\bar{j}_i(p) := \frac{1}{2} (1 + p j_i) \quad \text{for} \quad i = 1, \ldots, n + 1.
\]

Combining (2.12) and (2.13), we obtain the following formula for the idempotent \(E_T\):

\[
E_T = E_U \bar{j}_{n+1} (p_{n+1}) \frac{u - c_{n+1}}{u - j_{n+1}} \bigg|_{u = c_{n+1}}.
\]

3. Fusion formula for idempotents of \(B_{n+1}\)

We start with the following lemma which will be useful in the sequel.

**Lemma 3.1.** For any integer \(l, 1 \leq l \leq n\), we have

(i) \(\bar{j}_{n+1} = s_n s_{n-1} \ldots s_l \bar{j}_l s_l \ldots s_{n-1} s_n + \frac{1}{2} \sum_{i=l}^{n} s_n \ldots s_{i+1} s_i s_{i+1} \ldots s_n (1 + j_{n+1} j_i)\),

(ii) \(\bar{j}_i (p) s_l \ldots s_{n-1} s_n \bar{j}_{n+1}

\quad = \bar{j}_i (p) j_l s_l \ldots s_{n-1} s_n + \frac{1}{2} \sum_{i=l}^{n} s_l s_{i+1} \ldots s_{i-1} s_{i+1} \ldots s_n s_n \bar{j}_i (p) (1 + j_{n+1} j_i)\).

The product \(s_l s_{i+1} \ldots s_{i-1}\) in the right-hand side of (ii) is understood to be equal to 1 if \(i = l\).

**Proof.** We prove (i) by induction on \(n - l\). The basis of the induction is, for \(l = n\), the formula \(\bar{j}_{n+1} = s_n \bar{j}_n s_n + \frac{1}{2} (s_n + s_n j_{n+1} j_n)\), which follows from the definition (2.7) of the Jucys–Murphy elements, namely from \(\bar{j}_{n+1} = s_n \bar{j}_n s_n + \frac{1}{2} (s_n + j_n s_n j_n)\) and \(j_n s_n = s_n j_{n+1}\). Now assume that

\[
\bar{j}_{n+1} = s_n s_{n-1} \ldots s_{l+1} \bar{j}_{l+1} s_{l+1} \ldots s_{n-1} s_n + \frac{1}{2} \sum_{i=l+1}^{n} s_n \ldots s_{i+1} s_i s_{i+1} \ldots s_n (1 + j_{n+1} j_i),
\]
and replace \( \tilde{j}_{l+1} \) by \( s_l \tilde{j}_l + \frac{1}{2} (s_l + j_l s_l \tilde{j}_l) \). One obtains the assertion (i) using that 
\[
\tilde{j}_l s_{l+1} \ldots s_{n-1} s_n = s_{l+1} \ldots s_{n-1} s_n \tilde{j}_l \quad \text{and} \quad \tilde{j}_l s_{l+1} \ldots s_{n-1} s_n = s_l s_{l+1} \ldots s_{n-1} s_n \tilde{j}_{l+1}.
\]
Using (i), we find
\[
\tilde{j}_l (p) s_l \ldots s_{n-1} s_n \tilde{j}_{l+1} = \tilde{j}_l (p) \left( \tilde{j}_l s_l \ldots s_{n-1} s_n + \frac{1}{2} \sum_{i=1}^{n} s_l s_{l+1} \ldots s_{i-1} s_{i+1} \ldots s_{n-1} s_n (1 + \tilde{j}_{n+1} s_i) \right),
\]
and (ii) follows since \( j_l s_{l+1} \ldots s_{i-1} = s_l s_{l+1} \ldots s_{i-1} \tilde{j}_l \) and \( \tilde{j}_l \) commutes with \( s_{i+1} \ldots s_n s_n \).

Let \( \phi_1 (v, u) := t(v); \) for \( k = 1, \ldots, n \) define
\[
\phi_{k+1} (v_1, \ldots, v_k, v, u_1, \ldots, u_k, u) := s_k (v, v_k, u_k) \phi_k (v_1, \ldots, v_{k-1}, v, u_1, \ldots, u_{k-1}, u) s_k
\]
(3.1) \( = s_k (v, v_k, u, u_k) s_{k-1} (v, v_{k-1}, u, u_{k-1}) \ldots s_1 (v, v_1, u, u_1) t(v) s_1 \ldots s_{k-1} s_k. \)

Define the following rational function with values in the group ring of \( B_{n+1} \):
\[
\Phi (v_1, \ldots, v_{n+1}, u_1, \ldots, u_{n+1}) := \prod_{k=0, \ldots, n}^{\pi} \phi_{k+1} (v_1, \ldots, v_k, v_{k+1}, u_1, \ldots, u_k, u_{k+1});
\]
the arrow over \( \prod \) indicates that the (non-commuting) factors are taken in descending order.

Let \( \lambda^{(2)} \) be a 2-partition of \( n + 1 \) and \( \mathcal{T} \) a standard 2-tableau of shape \( \lambda^{(2)} \). For \( i = 1, \ldots, n + 1 \), set \( p_i := p(\mathcal{T} | i) \) and \( c_i := c(\mathcal{T} | i) \).

**Theorem 3.2.** The idempotent \( E_{\mathcal{T}} \) corresponding to the standard 2-tableau \( \mathcal{T} \) of shape \( \lambda^{(2)} \) can be obtained by the following consecutive evaluations:
\[
E_{\mathcal{T}} = f_{\lambda^{(2)}} (\Phi (v_1, \ldots, v_{n+1}, u_1, \ldots, u_{n+1}) | v_i = p_i, \quad i = 1, \ldots, n+1 | u_1 = c_1 \ldots | u_n = c_n | u_{n+1} = c_{n+1} \).
\]

*Proof.* Define
\[
F_{\mathcal{T}} (u) := \frac{u - c_{n+1}}{u} \prod_{i=1}^{n} \frac{(u - c_i)^2}{(u - c_i)^2 - \delta_{p_i, p_{n+1}}}.
\]
Let \( \mathcal{U} \) be the standard 2-tableau obtained from \( \mathcal{T} \) by removing the 2-node with the number \( n + 1 \) and let \( \mu^{(2)} \) be the shape of \( \mathcal{U} \).

**Proposition 3.3.** We have
\[
F_{\mathcal{T}} (u) \phi_{n+1} (p_1, \ldots, p_n, p_{n+1}, c_1, \ldots, c_n, u) E_{\mathcal{U}} = \frac{u - c_{n+1}}{u - \tilde{j}_{n+1}} j_{n+1} (p_{n+1}) E_{\mathcal{U}}.
\]

*Proof.* We prove (3.5) by induction on \( n \). As \( c_1 = 0 \) and \( \tilde{j}_1 = 0 \), the basis of induction for \( n = 0 \) is the formula \( t(p_1) = j_1 (p_1) \) which is satisfied by definition; see (2.5) and (2.15).

If \( p_{n+1} \neq p_i, \) \( i = 2, \ldots, n \), then fix \( l = 1 \). Otherwise fix \( l \) such that \( p_{n+1} = p_l \) and \( p_{n+1} \neq p_i, \) \( i = l + 1, \ldots, n \).

Define \( \mathcal{V} \) to be the standard 2-tableau obtained from \( \mathcal{U} \) by removing the 2-nodes containing the numbers \( l + 1, \ldots, n \) and \( \mathcal{W} \) to be the standard 2-tableau obtained...

\[
E_{\mathcal{T}} = f_{\lambda^{(2)}} (\Phi (v_1, \ldots, v_{n+1}, u_1, \ldots, u_{n+1}) | v_i = p_i, \quad i = 1, \ldots, n+1 | u_1 = c_1 \ldots | u_n = c_n | u_{n+1} = c_{n+1} \).
\]
from $\mathcal{V}$ by removing the 2-node with the number $l$. We will use that $E_{\mathcal{V}}E_{\mathcal{U}} = E_{\mathcal{U}}$ and that $E_{\mathcal{V}}$ commutes with $s_i$, for $i = l, l+1, \ldots, n$. We rewrite the left-hand side of (3.5) as 

$$F_\mathcal{T}(u)s_n \cdots s_{l+1}s_l(p_{n+1}, p_l, u, c_l)\phi_l(p_1, \ldots, p_{l-1}, p_{n+1}, c_1, \ldots, c_{l-1}, u)E_{\mathcal{V}} \cdot s_l \cdots s_n E_{\mathcal{U}}.$$ 

If $l > 1$, then we use the induction hypothesis, namely 

$$\phi_l(p_1, \ldots, p_{l-1}, p_l, c_1, \ldots, c_{l-1}, u)E_{\mathcal{V}} = (F_\mathcal{V}(u))^{-1} \frac{u - c_l}{u - j_l} j_l(p_l)E_{\mathcal{V}},$$

and we notice that $p_{n+1} = p_l$.

If $l = 1$, then $E_{\mathcal{V}}E_{\mathcal{U}} = 1$, $F_\mathcal{V}(u) = 1$ and, by definition, $\phi_1(p_{n+1}, u)$ is equal to $j_1(p_{n+1})$. Thus, in both situations (for $l = 1$ and for $l > 1$), we obtain for the left-hand side of (3.5):

$$F_\mathcal{T}(u)(F_\mathcal{V}(u))^{-1}s_n \cdots s_{l+1}s_l(p_{n+1}, p_l, u, c_l)\frac{u - c_l}{u - j_l} j_l(p_{n+1})s_l s_{l+1} \cdots s_n E_{\mathcal{U}}.$$

Therefore, the equality (3.5) is equivalent to (3.6)

$$F_\mathcal{T}(u)(F_\mathcal{V}(u))^{-1}(u - c_l) j_l(p_{n+1}) s_l s_{l+1} \cdots s_n (u - j_{n+1}) E_{\mathcal{U}} = \frac{(u - c_{n+1})(u - c_l)^2}{(u - c_l)^2 - \delta_{p_l,p_{n+1}}} (u - j_{n+1}) s_l(p_l, p_{n+1}, c_l, u) s_l \cdots s_n j_{n+1}(p_{n+1}) E_{\mathcal{U}},$$

where, in moving $s_n \cdots s_{l+1}s_l(p_{n+1}, p_l, u, c_l)\frac{u - c_l}{u - j_l}$ to the right-hand side and $\frac{1}{u - j_{n+1}}$ to the left-hand side, we have used that $\tilde{j}_{n+1}$ commutes with $j_{n+1}$ and $E_{\mathcal{U}}$, and also the formula (2.6) to take the inverse of $s_l(p_{n+1}, p_l, u, c_l)$.

To prove the equality (3.6), first notice that we have

$$F_\mathcal{T}(u)(F_\mathcal{V}(u))^{-1}(u - c_l) = (u - c_{n+1}) \prod_{i=1}^{n} \frac{(u - c_i)^2}{(u - c_i)^2 - \delta_{p_i,p_{n+1}}} \prod_{i=1}^{l-1} \frac{(u - c_i)^2 - \delta_{p_i,p_{n+1}}}{(u - c_i)^2},$$

which gives, since $p_i \neq p_{n+1}$ if $i > l$ and $p_l = p_{n+1}$ if $l > 1,$

$$F_\mathcal{T}(u)(F_\mathcal{V}(u))^{-1}(u - c_l) = \frac{(u - c_{n+1})(u - c_l)^2}{(u - c_l)^2 - \delta_{p_l,p_{n+1}}}.$$

So it remains to prove that (3.8)

$$j_l(p_{n+1}) s_l s_{l+1} \cdots s_n (u - j_{n+1}) E_{\mathcal{U}} = (u - \tilde{j}_l) s_l(p_l, p_{n+1}, c_l, u) s_l \cdots s_n j_{n+1}(p_{n+1}) E_{\mathcal{U}}.$$ 

Expand $s_l(p_l, p_{n+1}, c_l, u)$ in the right-hand side of (3.8):

$$((u - \tilde{j}_l)s_l - \delta_{p_l,p_{n+1}} \frac{\tilde{j}_l - u}{c_l - u}) s_{l+1} \cdots s_n j_{n+1}(p_{n+1}) E_{\mathcal{U}}.$$ 

As $\tilde{j}_l$ commutes with $s_{l+1} \cdots s_n$ and $j_{n+1}$ and $\tilde{j}_l E_{\mathcal{U}} = c_l E_{\mathcal{U}}$, we find that the expression (3.9) equals

$$\left( (u - \tilde{j}_l)s_l s_{l+1} \cdots s_n j_{n+1}(p_{n+1}) - \delta_{p_l,p_{n+1}} s_{l+1} \cdots s_n j_{n+1}(p_{n+1}) \right) E_{\mathcal{U}}.$$ 

Then using that $s_l s_{l+1} \cdots s_n j_{n+1} = j_l s_l s_{l+1} \cdots s_n$, we obtain for the right-hand side of (3.8):

$$((u - \tilde{j}_l) j_l(p_{n+1}) s_l s_{l+1} \cdots s_n - \delta_{p_l,p_{n+1}} s_{l+1} \cdots s_n j_{n+1}(p_{n+1})) E_{\mathcal{U}}.$$
Using Lemma 3.1(ii), we write the left-hand side of (3.8) in the form
\[
\left( j_1(p_{n+1}) - \tilde{j}_1 \right) s_1 s_{l+1} \cdots s_n - \frac{1}{2} \sum_{i=l}^{n} s_i s_{l+1} \cdots s_{i-1} \cdot s_{i+1} \cdots s_{n-1} s_n h_1 \left( 1 + j_{n+1}^1 \right) E_{\mathcal{U}}.
\]
(3.11)

As \( j_i E_{\mathcal{U}} = \alpha_i E_{\mathcal{U}}, i = 1, \ldots, n \), the expression (3.11) is equal to
\[
\left( j_1(p_{n+1}) - \tilde{j}_1 \right) s_1 s_{l+1} \cdots s_n - \frac{1}{2} \sum_{i=l}^{n} s_i s_{l+1} \cdots s_{i-1} \cdot s_{i+1} \cdots s_{n-1} s_n \frac{1}{2} (1 + p_{i} p_{n+1}) (1 + p_{i} j_{n+1}) E_{\mathcal{U}}.
\]
(3.12)

Since \( p_k p_{n+1} = -1, k = l+1, \ldots, n \), we finally obtain for the left-hand side of (3.8):
\[
\left( j_1(p_{n+1}) - \tilde{j}_1 \right) s_1 s_{l+1} \cdots s_n - \delta_{p_{l}, p_{n+1}} s_{l+1} \cdots s_{n-1} s_n h_{n+1}^{1} (p_1) E_{\mathcal{U}}.
\]
(3.13)

The comparison of (3.13) and (3.11) proves the equality (3.8). \( \square \)

**Proposition 3.4.** The rational function \( F_\mathcal{T}(u) \), defined by (3.2), is regular at \( u = c_{n+1} \) and, moreover,
\[
F_\mathcal{T}(c_{n+1}) = f_{\lambda(2)}(f_{\mu(2)})^{-1}.
\]
(3.14)

**Proof.** The proposition will directly follow from the result, used in [14], concerning usual tableaux. Let \( \lambda \) be a partition of \( n+1 \), with \( n \leq n \). Let \( S \) be a subset of \( \{1, \ldots, n+1\} \) such that \( S \) contains the number \( n+1 \) and has a cardinal equal to \( n+1 \). Let \( \mathcal{T} \) be a tableau of shape \( \lambda \) filled with numbers belonging to \( S \) such that the numbers in the nodes are in strictly ascending order along rows and columns in the right and down directions. Let \( \gamma \) be the node of \( \mathcal{T} \) with the number \( n+1 \) and \( \mu \) be the shape of the tableau obtained from \( \mathcal{T} \) by removing the node \( \gamma \). Define the following rational function:
\[
\tilde{F}_\mathcal{T}(u) = \frac{u - c(\gamma)}{u} \prod_{\alpha \in \mu} \frac{(u - c(\alpha))^2}{(u - c(\alpha) - 1)(u - c(\alpha) + 1)}.
\]
(3.15)

The product in the right-hand side of (3.15) depends only on the shape \( \mu \), and one has
\[
\prod_{\alpha \in \mu} \frac{(u - c(\alpha))^2}{(u - c(\alpha) - 1)(u - c(\alpha) + 1)} = u \prod_{\beta \in \mathcal{E}_-(\mu)} (u - c(\beta)) \prod_{\alpha \in \mathcal{E}_+(\mu)} (u - c(\alpha))^{-1},
\]
where \( \mathcal{E}_-(\mu) \) (respectively, \( \mathcal{E}_+(\mu) \)) is the set of removable (respectively, addable) nodes of \( \mu \). Therefore the rational function \( \tilde{F}_\mathcal{T}(u) \) is regular at \( u = c(\gamma) \) and, moreover,
\[
\tilde{F}_\mathcal{T}(c(\gamma)) = \prod_{\beta \in \mathcal{E}_-(\mu)} (c(\gamma) - c(\beta)) \prod_{\alpha \in \mathcal{E}_+(\mu) \setminus \{\gamma\}} (c(\gamma) - c(\alpha))^{-1}.
\]
(3.16)

It is known that the right-hand side of (3.16) is equal to
\[
\prod_{\alpha \in \lambda} (h_\lambda(\alpha))^{-1} \prod_{\alpha \in \mu} h_\mu(\alpha).
\]
(3.17)

Define \( \mathcal{T} \) to be the tableau of the standard 2-tableau \( \mathcal{T} \) which contains the node with number \( n+1 \). The assertion of Proposition 3.4 follows; the only observation one has to make is that the 2-nodes \( (\alpha, \gamma) \) with \( p_k \neq p_{n+1} \) do not contribute to (3.14). \( \square \)
4. Fusion procedure for the complex reflection group $G(m, 1, n + 1)$

We extend the results of the previous section to the complex reflection groups $G(m, 1, n + 1)$ for all positive integers $m$. We skip the proofs when they are completely similar to the proofs in the preceding section; we only indicate modifications.

4.1. Definitions. The complex reflection group $G(m, 1, n + 1)$ is generated by the elements $s_1, \ldots, s_n$ and $t$ with the defining relations (2.1), (2.2) and

\[ t^m = 1. \]

In particular, the group $G(1, 1, n + 1)$ is isomorphic to the symmetric group $S_{n+1}$ and the group $G(2, 1, n + 1)$ to $B_{n+1}$.

We extend the definition (2.4) to the generators $s_1, \ldots, s_n$ of $G(m, 1, n + 1)$:

\[ s_i(p, p', a, a') := s_i + \frac{\delta_{p, p'}}{a - a'}, \quad i = 1, \ldots, n. \]

The Jucys–Murphy elements for the group $G(m, 1, n + 1)$ are the elements $j_i$, $i = 1, \ldots, n + 1$, and $\tilde{j}_i$, $i = 1, \ldots, n + 1$, of the group ring defined inductively by the following initial conditions and recursions:

\[ j_1 = t, \quad j_{i+1} = s_i j_i s_i \quad \text{and} \quad \tilde{j}_1 = 0, \quad \tilde{j}_{i+1} = s_i \tilde{j}_i s_i + \frac{1}{m} \sum_{k=0}^{m-1} j_i^k s_i j_i^{m-k}. \]

For $m = 1$, that is, for $S_{n+1}$, $j_k = 1$, $k = 1, \ldots, n + 1$; the recursion formula for $\tilde{j}_{i+1}$ reduces to $\tilde{j}_{i+1} = s_i \tilde{j}_i s_i + s_i$.

As for $m = 2$, the elements $j_i$, $i = 1, \ldots, n + 1$, and $\tilde{j}_i$, $i = 1, \ldots, n + 1$, form a maximal commutative set in $CG(m, 1, n + 1)$; see [21, 22]. In addition, $j_i$ and $\tilde{j}_i$ commute with all $s_k$, except $s_i$ and $s_{i+1}$.

The definitions of a 2-partition, 2-node, standard 2-tableaux and hook length generalize naturally to any $m$; for example, an $m$-partition is an $m$-tuple of partitions and an $m$-node $\alpha^{(m)}$ is a pair $(\alpha, k)$, $k = 1, \ldots, m$. For an $m$-node $\alpha^{(m)} = (\alpha, k)$ of an $m$-partition $\lambda^{(m)}$ such that the node $\alpha$ lies in the line $x$ and the column $y$ of the $k$-th diagram, we define $c(\alpha^{(m)}) := c(\alpha) = y - x$. Let $\{\xi_1, \ldots, \xi_m\}$ be the set of distinct $m$-th roots of unity, ordered arbitrarily; we define $p(\alpha^{(m)}) := \xi_k$.

As for $m = 2$, we define

\[ f_{\nu^{(m)}} := \left( \prod_{\alpha^{(m)} \in \nu^{(m)}} h_{\nu^{(m)}}(\alpha^{(m)}) \right)^{-1}, \]

where $h_{\nu^{(m)}}(\alpha^{(m)})$ is the hook length of $\alpha^{(m)}$ calculated in $\nu^{(m)}$.

The irreducible representations of the group $G(m, 1, n + 1)$ are parameterized by $m$-partitions of $n + 1$. For a given $m$-partition $\lambda^{(m)}$, the basis vectors of the corresponding representation are indexed by standard $m$-tableaux of shape $\lambda^{(m)}$. We denote by $E_T$ the idempotent of the group ring corresponding to a standard $m$-tableau $T$.
4.2. **Fusion formula for idempotents of** $G(m, 1, n + 1)$. Let $\lambda^{(m)}$ be an $m$-partition of $n + 1$; fix a standard $m$-tableau $T$ of shape $\lambda^{(m)}$. Let $U$ be the standard $m$-tableau obtained by removing from $T$ the $m$-node containing $n + 1$ and let $\mu^{(m)}$ be the shape of $U$. Set, for $i = 1, \ldots, n + 1$, $p_i := p(T|i)$ and $c_i := c(T|i)$.

The same reasoning as in Subsection 2.2 leads to the formula for $E_T$ (cf. (2.12)):

(4.5) \[ E_U \left| \begin{array}{c} u = c_{n+1} \\ v = p_{n+1} \\ \mu^{(m)} \end{array} \right| \frac{v - p_{n+1}}{v - j_{n+1}} = E_T. \]

Since $j_{n+1}$ takes values in $\{\xi_1, \ldots, \xi_m\}$, the rational function $\frac{v - p_{n+1}}{v - j_{n+1}}$ is non-singular for $v = p_{n+1}$ and

(4.6) \[ \frac{v - p_{n+1}}{v - j_{n+1}} \bigg|_{v=p_{n+1}} = \frac{1}{m} \sum_{k=0}^{m-1} p_{n+1}^{m-k} j_{n+1}^k. \]

An analogue of (2.14) is

(4.7) \[ j_i(p) := \frac{1}{m} \sum_{k=0}^{m-1} p_{n+1}^{m-k} j_i^k \quad \text{for } i = 1, \ldots, n + 1. \]

For $i = 1$ we shall write $t(p) := \frac{1}{m} \sum_{k=0}^{m-1} p^{m-k} j_i^k$ instead of $j_i(p)$. As $p$ is always evaluated at an $m$-th root of unity, definition (4.7) agrees, for $m = 2$, with (2.14) and (2.15).

Combining (4.5) and (4.6), we obtain for the idempotent $E_T$ (cf. (2.15))

(4.8) \[ E_T = E_U \left| \begin{array}{c} u = c_{n+1} \\ v = p_{n+1} \\ \mu^{(m)} \end{array} \right| \frac{u - c_{n+1}}{u - j_{n+1}} \bigg|_{u=c_{n+1}}. \]

We generalize Lemma 3.1 to an arbitrary positive integer $m$.

**Lemma 4.1.** For any integer $l$, $1 \leq l \leq n$, we have

(i) \[ \tilde{j}_{n+1} = s_n s_{n-1} \cdots s_l j_{l-1} s_{l-1} \cdots s_1 s_0 \sum_{i=l}^n s_i s_{i+1} s_{i+1} \cdots s_n j_{n+1}^{m-k} j_i^k. \]

(ii) \[ j_i(p) s_l \cdots s_{n-1} s_n \tilde{j}_{n+1} = j_i(p) s_l \cdots s_{n-1} s_n + \sum_{i=l}^n s_i s_{i+1} \cdots s_{i-1} s_{i+1} \cdots s_n j_i(p) \sum_{k=0}^{m-1} j_{n+1}^{m-k} j_i^k. \]

The product $s_l s_{l+1} \cdots s_{i-1}$ in the right-hand side of (ii) is understood to be equal to 1 if $i = l$.

**Proof.** The proof is completely similar to the proof of Lemma 3.1

Let $\phi_1(v, u) := t(v)$; for $k = 1, \ldots, n$ define

\[
\phi_k+1(v_1, \ldots, v_k, v, u_1, \ldots, u_k, u) := s_k(v, v_k, u, u_k) \phi_k(v_1, \ldots, v_k+1, v, u_1, \ldots, u_k-1, u) s_k
\]

(4.9) \[ = s_k(v, v_k, u, u_k) s_{k-1}(v, v_{k-1}, u, u_{k-1}) \cdots s_1(v, v_1, u, u_1) t(v) s_1 \cdots s_{k-1} s_k. \]

Formula (4.9) reads the same way for any $m$; only the definition of $t(v)$ depends on $m$. 


Define the following rational function with values in the group ring of $G(m, 1, n)$:

$$
\Phi(v_1, \ldots, v_{n+1}, u_1, \ldots, u_{n+1}) := \prod_{k=0}^{n} \phi_{k+1}(v_1, \ldots, v_k, v_{k+1}, u_1, \ldots, u_k, u_{k+1});
$$

the arrow over $\prod$ indicates that the (non-commuting) factors are taken in descending order.

**Theorem 4.2.** The idempotent $E_T$ corresponding to the standard $m$-tableau $T$ of shape $\lambda(m)$ can be obtained by the following consecutive evaluations:

$$E_T = f_{\lambda(m)} \Phi(v_1, \ldots, v_{n+1}, u_1, \ldots, u_{n+1}) |_{v_i = p_i, \ i=1, \ldots, n+1} |_{u_1 = c_1, \ldots, u_n = c_n} |_{u_{n+1} = c_{n+1}}. $$

**Proof.** Define

$$F_T(u) := \frac{u - c_{n+1}}{u} \prod_{i=1}^{n} \frac{(u - c_i)^2}{(u - c_i)^2 - \delta_{p_i, p_{n+1}}}. $$

**Proposition 4.3.** We have

$$F_T(u) \phi_{n+1}(p_1, \ldots, p_n, p_{n+1}, c_1, \ldots, c_n, u) E_U = \frac{u - c_{n+1}}{u - j_{n+1}} j_{n+1}(p_{n+1}) E_U. $$

**Proof.** The proof follows the same lines as the proof of Proposition 3.3. Actually it is exactly the same until the calculation of $j_i(p_{n+1}) s_i s_{l+1} \ldots s_n (u - \tilde{j}_{n+1}) E_U$ just after formula (3.10). We give the modified end of the proof.

Here, we rewrite $j_i(p_{n+1}) s_i s_{l+1} \ldots s_n (u - \tilde{j}_{n+1}) E_U$ using Lemma 4.1(ii):

$$\left( j_i(p_{n+1})(u - \tilde{j}_{l}) s_i s_{l+1} \ldots s_n - \frac{1}{m} \sum_{i=l}^{n} s_l s_{l+1} \ldots s_{i-1} \cdot s_{i+1} \ldots s_n s_{n-1} s_n j_i(p_{n+1}) \sum_{k=0}^{m-1} j_{n+1} j_i^{m-k} \right) E_U. $$

As $j_i E_U = p_i E_U$ for $i = 1, \ldots, n$, the expression (4.14) is equal to

$$\left( j_i(p_{n+1})(u - \tilde{j}_{l}) s_i s_{l+1} \ldots s_n - \frac{1}{m} \sum_{i=l}^{n} s_l s_{l+1} \ldots s_{i-1} \cdot s_{i+1} \ldots s_n s_{n-1} s_n \sum_{k=0}^{m-1} p_{n+1} p_i^{m-k} \delta_{p_i, p_{n+1}} \right) E_U. $$

Since $\frac{1}{m} \sum_{k=0}^{m-1} p_{n+1} p_i^{m-k} = \delta_{p_i, p_{n+1}}$ we obtain that

$$j_i(p_{n+1}) s_i s_{l+1} \ldots s_n (u - \tilde{j}_{n+1}) E_U$$

equals

$$\left( j_i(p_{n+1})(u - \tilde{j}_{l}) s_i s_{l+1} \ldots s_n - \delta_{p_i, p_{n+1}} s_i s_{l+1} \ldots s_n s_{n-1} s_n j_{n+1}(p_{n+1}) \right) E_U. $$

This concludes the proof.

The analogue of Proposition 3.4 holds as well.

**Proposition 4.4.** The rational function $F_T(u)$, defined by (4.12), is regular at $u = c_{n+1}$ and, moreover,

$$F_T(c_{n+1}) = f_{\lambda(m)}(f_{\mu(m)})^{-1}. $$

**Proof.** The proof is completely similar to the proof of Proposition 3.4.

\[\square\]
Similarly to Theorem 3.2, Theorem 4.2 follows, using induction on \( n \), from formula (1.8), Proposition 4.3 and Proposition 4.4.

For calculations, it is sometimes useful to write the function \( \Phi \) in a slightly different form. Namely, let \( \tilde{\phi}_k(v, u) := 1 \) and define
\[
\tilde{\phi}_{k+1}(v_1, \ldots, v_k, v, u_1, \ldots, u_k) \quad (4.18)
\]
\[
= s_k(v, v_k, u, u_k) \tilde{\phi}_{k}(v_1, \ldots, v_{k-1}, v, u_1, \ldots, u_{k-1}, u) s_k
\]
for \( k = 1, \ldots, n \). The elements \( \tilde{\phi}_{k+1}(v_1, \ldots, v_k, v, u_1, \ldots, u_k, u) \) do not involve the generator \( t \), and \( \Phi(v_1, \ldots, v_{n+1}, u_1, \ldots, u_{n+1}) \), defined in (4.10), equals
\[
\prod_{k=0, \ldots, n} \tilde{\phi}_{k+1}(v_1, \ldots, v_k, v, u_1, \ldots, u_k) \cdot j_1(v_1)j_2(v_2) \cdots j_{n+1}(v_{n+1}).
\]

For example, let \( m = 2 \); choose the order \( \{1, -1\} \) on the set of square roots of 1. The primitive idempotent, corresponding to the standard 2-tableau \((1\ 3\ 2)\), reads \( s_2(1 + s_1)s_2j_1(1)j_2(-1)j_3(1)/2 \).

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