NON-NEGATIVE GLOBAL WEAK SOLUTIONS FOR A DEGENERATED PARABOLIC SYSTEM APPROXIMATING THE TWO-PHASE STOKES PROBLEM

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Abstract. We establish the existence of non-negative global weak solutions for a strongly couple degenerated parabolic system which was obtained as an approximation of the two-phase Stokes problem driven solely by capillary forces. Moreover, the system under consideration may be viewed as a two-phase generalization of the classical Thin Film equation.

1. Introduction and the main result

In this paper we study the following system of one-dimensional degenerated parabolic equations

\[
\begin{align*}
\frac{\partial f}{\partial t} &= -\partial_x \left( f^3 \partial_x^3 f + \frac{R}{2} \left( 2f^3 + 3f^2g \right) \partial_x^2 (f + g) \right), \\
\frac{\partial g}{\partial t} &= -\partial_x \left( \frac{3}{2} f^2 g \partial_x^3 f + \frac{R}{2} \left( 2\mu g^3 + 3f^2 g + 6fg^2 \right) \partial_x^2 (f + g) \right)
\end{align*}
\]

(1.1a)

for \((t,x) \in (0, \infty) \times I, \) where \(I : = (0, L)\) for some \(L > 0.\) This system models the motion of the interfaces of two immiscible fluid layers of width \(f\) and \(g,\) respectively. The layer of width \(f\) is located on an impermeable horizontal bottom, identified with the line \(y = 0,\) and the layer of width \(g\) is located on top of the first one. The system (1.1a) has been recently derived in [10] as a thin film approximation of the two-phase Stokes problem when capillary is the sole driving mechanism. The constants \(R\) and \(\mu,\) which are both assumed to be positive, are determined by material properties of the fluids and are given by

\[
\mu : = \frac{\mu_f}{\mu_g} \quad \text{and} \quad R : = \frac{\gamma_g}{\gamma_f},
\]

with \(\mu_f\) [resp. \(\mu_g\)] denoting the viscosity coefficient of the fluid layer of width \(f\) [resp. \(g\)]. Moreover, \(\gamma_f\) [resp. \(\gamma_g\)] is the surface tension coefficient at the interface \(y = f(t,x) + g(t,x).\) The system (1.1a) is supplemented by the initial conditions

\[
f(0) = f_0, \quad g(0) = g_0 \quad \text{in } I,
\]

(1.1b)

whereby \(f_0\) and \(g_0\) are assumed to be known non-negative functions, and by no-flux boundary conditions

\[
\partial_x f = \partial_x g = \partial_x^2 f = \partial_x^2 g = 0, \quad x = 0, L.
\]

(1.1c)
Let us first observe that if one of the fluid layer has constant zero width, then the system (1.1a) becomes, up to a scaling factor, the well-known Thin Film equation
\[ \partial_t h = \partial_x^2 (h^m \partial_x h), \quad m > 0, \tag{1.2} \]
with \( m = 3 \). The theory of existence of weak solutions for the Thin Film equation (1.2) is well-established nowadays, cf. [1, 2, 3, 4, 5, 20], to mention just some of the most important contributions. We emphasize that it has been rigorously proved in [14, 18] (see also [13]) that suitably rescaled solutions of the Stokes and of the Hele-Shaw problem converge towards corresponding solutions of the equation (1.2) with \( m = 3 \) and \( m = 1 \), respectively. Compared to the Thin Film equation (1.2), the system (1.1) is much more complex because it exhibits a strong coupling as both equations contain highest order derivatives of all the unknowns. There are also two sources of degeneracy because both interfaces may vanish on subsets of the interval \( I \).

It is worth mentioning that there exists also a two-phase generalization corresponding to the thin-film equation (1.2) with \( m = 1 \), which has been derived in [9] for flows with capillary and gravity effects. The resulting system, which has been investigated in [11, 16, 17] in the presence of capillary and in [8, 15] for flows driven only by gravity, appears as the thin layer approximation of the the two-phase Muskat problem. Compared to (1.1a), the parabolic system obtained in [9] has much more structure: there are two energy functionals available and, furthermore, the system can be interpreted as a gradient flow for the \( L_2 \)-Wasserstein distance in the space of probability measures with finite second moment. There are not many systems of equations which enjoy this nice geometric property. We mention that the parabolic-parabolic Keller-Segel system which has a mixed \( L_2 \)-Wasserstein gradient flow structure [6]. As far as we know, the two-phase generalization of the thin-film equation (1.2) with \( m \notin \{1,3\} \) has not been discovered yet.

When studying the problem (1.1), one has to rely only on the energy functional
\[ \mathcal{E}(f, g) : = \frac{1}{2} \int_I |\partial_x f|^2 + R |\partial_x(f + g)|^2 \, dx, \]
a fact which forces us to introduce here a weaker notion of solutions than that in [15, 17]. Indeed, it is not difficult to see that the functional \( \mathcal{E} \) decreases along smooth solutions of (1.1), as we have
\[ \frac{d\mathcal{E}(f, g)}{dt} = \int_I ((1 + R) \partial_x^2 f + R \partial_x^2 g) \partial_x^2 (\partial_t f) + R (\partial_x^2 (f + g)) \partial_x^2 (\partial_t g) \, dx \\
= - \int_I ((1 + R) \partial_x^2 f + R \partial_x^2 g) \partial_t f + R (\partial_x^2 (f + g)) \partial_t g \, dx \\
= - \int_I (1 + R) \partial_x^2 (f + g) \left( f^3 \partial_x^3 f + \frac{R}{2} (2f^3 + 3f^2 g) \partial_x (f + g) \right) \, dx \\
- \int_I R \partial_x^3 (f + g) \left( \frac{3}{2} f^2 g \partial_x^2 f + \frac{R}{2} (2f^3 + 3f^2 g + 6fg^2) \partial_x (f + g) \right) \, dx \\
= - \mu R^2 \int_I g^3 |\partial_x^3 (f + g)|^2 \, dx - \int_I f \partial_x^3 (f + g) \left( \frac{R}{2} (2f^3 + 3fg^2) \partial_x^2 (f + g) \right)^2 \, dx \\
- \frac{3R^2}{4} \int_I fg^2 [\partial_x^2 (f + g)]^2 \, dx. \tag{1.3} \]
Introducing a suitable regularized version of (1.1a), we construct first, for non-negative initial data, globally defined Galerkin approximations which are found to converge towards weak solutions of the approximating systems. On the other hand, the energy functional $E$ may be used to obtain estimates for the solutions of the approximating systems and we obtain sufficient information to prove that they converge towards weak solutions of the original problem (1.1). Though it is a priori not clear whether the weak solutions of the approximating systems are non-negative, we prove that the weak solutions of (1.1) have this property. This differs from the framework of thin fluid models with capillary effects and insoluble surfactant [7, 12] where the approximating regularized problems may be constructed such that starting from non-negative initial data the associated weak solutions are also non-negative.

Given $T \in (0, \infty]$, let $Q_T := (0, T) \times \mathcal{I}$. The main result of this paper is the following theorem, establishing the existence of global and non-negative weak solutions for the problem (1.1) that start from arbitrary non-negative initial data.

**Theorem 1.1.** Let $f_0, g_0 \in H^1(\mathcal{I})$ be two non-negative functions. Then, there exists at least a weak global solutions $(f, g)$ of problem (1.1) with the following properties:

(a) $f, g \in L^\infty(0, T; H^1(\mathcal{I})) \cap (\cap_{\alpha \in [0, 1/2]} C([0, T], C^\alpha(\mathcal{I})))$ for all $T > 0$;

(b) $(f, g)(0) = (f_0, g_0)$ and $f \geq 0$, $g \geq 0$ in $(0, \infty) \times \mathcal{I}$;

(c) the mass of the fluids is conserved, that is for every $t > 0$
$$\|f(t)\|_{L^1} = \|f_0\|_{L^1} \quad \text{and} \quad \|g(t)\|_{L^1} = \|g_0\|_{L^1};$$

(d) defining for every $T > 0$ the sets
$$\mathcal{P}_f := \{(t, x) \in Q_T : f(t, x) > 0\}, \quad \mathcal{P}_g := \{(t, x) \in Q_T : g(t, x) > 0\},$$
we have $\partial^3_x f, \partial^3_x g \in L^2_{loc}(\mathcal{P}_f \cap \mathcal{P}_g)$ and there exists functions $j_f, j_g, j_f, g \in L^2(Q_T)$ with
$$j_f = f^{1/2} (f \partial^3_x f + \frac{4R}{\mu} (2f + 3fg) \partial^3_x (f + g)),$$
$$j_g = g^{3/2} \partial^3_x (f + g), \quad j_f, g = f^{1/2} g \partial^3_x (f + g)$$
a.e. in $\mathcal{P}_f \cap \mathcal{P}_g$,

and such that
$$H_f := f^{3/2} j_f, \quad H_g := \mu R g^{3/2} j_g + \frac{3R}{4} f^{1/2} g j_f, g + \frac{3}{2} f^{1/2} gg j_f$$
belong to $L^2(Q_T)$, and

$$\int_{Q_T} f \partial_t \xi \, dx \, dt + \int_{Q_T} H_f \partial_x \xi \, dx \, dt = \int_{\mathcal{I}} f(T, x) \xi(T, x) \, dx - \int_{\mathcal{I}} f_0 \xi(0, x) \, dx,$$

$$\int_{Q_T} g \partial_t \xi \, dx \, dt + \int_{Q_T} H_g \partial_x \xi \, dx \, dt = \int_{\mathcal{I}} g(T, x) \xi(T, x) \, dx - \int_{\mathcal{I}} g_0 \xi(0, x) \, dx$$
for all $\xi \in C^\infty(\overline{Q_T})$;

(e) the energy inequality
$$E(f(T), g(T)) + \mu R \int_{\mathcal{P}_f \cap \mathcal{P}_g} |j_f|^2 + \frac{3R^2}{4} |j_f, g|^2 + |j_f|^2 \, dx \, dt \leq E(f_0, g_0)$$
is satisfied for almost all $T > 0$. 


We emphasize that due to the lack of regularity of the weak solutions \((f, g)\) found in Theorem 2.1, which is mainly due to the strong coupling of the system (1.1a), we can identify the function \(H_f\) only in \(P_g\) and \(H_g\) only in \(P_f\):

\[
H_f = \left( f^3 \partial_x^3 f + \frac{R}{2} \left( 2f^3 + 3f^2 g \right) \partial_x^3 (f + g) \right) 1_{(0, \infty)}(f) \quad \text{a.e. in } P_g,
\]

\[
H_g = \left( \frac{3}{2} f^2 g \partial_x^3 f + \frac{R}{2} \left( 2\mu g^3 + 3f^2 g + 6fg^2 \right) \partial_x^3 (f + g) \right) 1_{(0, \infty)}(g) \quad \text{a.e. in } P_f.
\]

Particularly, if the test function \(\xi\) in (1.4) satisfies additionally \(\text{supp} \partial_x \xi \subset P_g\), then (1.4) is exactly the equation one obtains when multiplying the first equation by (1.1a) by \(\xi\) and integrating by parts (similarly for (1.5)).

The outline of the paper is as follows: in Section 2 we regularize the problem and construct global weak solutions for the approximating regularized systems. Furthermore, we prove that any accumulation point of the set of approximating weak solutions has to be non-negative. Based upon the estimates deduced for this family of weak solutions, we prove in Section 3 that certain sequences of approximating weak solutions converge, when letting the regularization parameter go to zero, towards non-negative weak solutions of the original problem (1.1).

2. The regularized approximating problems

In this section we construct a family of regularized systems approximating in the limit the original system (1.1a). This is done in such a manner that the energy functional \(E\) still decreases along solutions of the regularized system. The advantage of such a construction is twofold: first it enables us to find globally defined Galerkin approximations which are shown to converge towards weak solutions of the approximating system, and secondly it provides us with sufficient information in order to find accumulation points of this family of weak solutions which solve the problem (1.1) in the weak sense defined in Theorem 1.1.

To proceed we define for every \(\varepsilon \in (0, 1]\), the Lipschitz continuous function \(a_\varepsilon : \mathbb{R} \rightarrow \mathbb{R}\) by the relation

\[
a_\varepsilon(s) := \varepsilon + \max\{0, s\} \quad \text{for } s \in \mathbb{R}.
\]

With this notation, we introduce the following modified version of the original problem (1.1a)

\[
\begin{align*}
\partial_t f_\varepsilon &= -\partial_x \left( a_\varepsilon^3(f_\varepsilon) \partial_x^3 f_\varepsilon + \frac{R}{2} \left( 2a_\varepsilon^3(f_\varepsilon) + 3a_\varepsilon^2(f_\varepsilon) a_\varepsilon(g_\varepsilon) \right) \partial_x^3 (f_\varepsilon + g_\varepsilon) \right), \\
\partial_t g_\varepsilon &= -\partial_x \left( \frac{3}{2} a_\varepsilon^2(f_\varepsilon) a_\varepsilon(g_\varepsilon) \partial_x^3 f_\varepsilon \right. \\
&\quad + \frac{R}{2} \left( 2\mu a_\varepsilon^3(g_\varepsilon) + 3a_\varepsilon^2(f_\varepsilon) a_\varepsilon(g_\varepsilon) + 6a_\varepsilon(f_\varepsilon) a_\varepsilon^2(g_\varepsilon) \right) \partial_x^3 (f_\varepsilon + g_\varepsilon) \right),
\end{align*}
\]

(2.1)

which is more regular than (1.1a) in the sense that the coefficients of the fourth-order derivatives are bounded from below by a positive constant depending only on \(\varepsilon\). The system (2.1) is supplemented by the initial and boundary conditions (1.1b) and (1.1c).

The main result of this section is the following theorem, ensuring the solvability of the regularized approximating problem (2.1), (1.1b), and (1.1c) for any \(\varepsilon \in (0, 1]\).
Theorem 2.1. Let \( f_0, g_0 \in H^1(\mathcal{I}) \) be two non-negative functions and \( \varepsilon \in (0, 1] \) be fixed. Then, there exists at least a couple of functions \((f_\varepsilon, g_\varepsilon)\) having the following regularity

- \( f_\varepsilon, g_\varepsilon \in L_\infty(0, T; H^1(\mathcal{I})) \cap L_2(0, T; H^3(\mathcal{I})) \cap (\cap_{\alpha \in [0, 1/2]}C([0, T], C^\alpha(\mathcal{I}))) \),
- \( \partial_t f_\varepsilon, \partial_t g_\varepsilon \in L_2(0, T; (H^1(\mathcal{I}))') \),

and satisfying

\[
\int_0^T \langle \partial_t f_\varepsilon(t)|\xi(t) \rangle \, dt = \int_{Q_T} \left( a_\varepsilon^2(f_\varepsilon) \partial_x^2 f_\varepsilon + \frac{R}{2} \left( 2a_\varepsilon^2(f_\varepsilon) + 3a_\varepsilon^2(f_\varepsilon) a_\varepsilon(g_\varepsilon) \right) \partial_x^2(f_\varepsilon + g_\varepsilon) \right) \partial_x \xi \, dxdt,
\]

\[
\int_0^T \langle \partial_t g_\varepsilon(t)|\xi(t) \rangle \, dt = \int_{Q_T} \left( \frac{R}{2} \left( 2\mu a_\varepsilon^2(g_\varepsilon) + 3a_\varepsilon^2(f_\varepsilon) a_\varepsilon(g_\varepsilon) + 6a_\varepsilon(f_\varepsilon) a_\varepsilon^2(g_\varepsilon) \right) \partial_x^3(f_\varepsilon + g_\varepsilon) + \frac{3}{2} a_\varepsilon^2(f_\varepsilon) a_\varepsilon(g_\varepsilon) \partial_x^2 f_\varepsilon \right) \partial_x \xi \, dxdt
\]

for all \( T > 0 \) and all \( \xi \in L_2(0, T; H^1(\mathcal{I})) \), whereby \( \langle \cdot | \cdot \rangle \) is the pairing between \( H^1(\mathcal{I}) \) and \((H^1(\mathcal{I}))'\). Moreover, \((f_\varepsilon, g_\varepsilon)(0) = (f_0, g_0)\),

\[
\int_I f_\varepsilon(T) \, dx = ||f_0||_{L_1} \quad \text{and} \quad \int_I g_\varepsilon(T) \, dx = ||g_0||_{L_1} \quad \text{for all} \ T \geq 0,
\]

\[
\partial_x f_\varepsilon(T) = \partial_x g_\varepsilon(T) = 0 \quad \text{at} \ x = 0, L \ \text{for almost all} \ T > 0,
\]

and the energy inequality

\[
\mathcal{E}(f_\varepsilon(T), g_\varepsilon(T)) + \int_{Q_T} a_{\varepsilon}(f_\varepsilon) a_{\varepsilon}(f_\varepsilon) \partial_x^2 f_\varepsilon + \frac{R}{2} \left( 2a_{\varepsilon}(f_\varepsilon) + 3a_{\varepsilon}(g_\varepsilon) \right) \partial_x^2(f_\varepsilon + g_\varepsilon) \right)^2 \, dxdt
\]

\[
+ \mu R^2 \int_{Q_T} a_{\varepsilon}^2(g_\varepsilon) \partial_x^3(f_\varepsilon + g_\varepsilon) \right)^2 \, dxdt + \frac{3R^2}{4} \int_{Q_T} a_{\varepsilon}(f_\varepsilon) a_{\varepsilon}^2(g_\varepsilon) \partial_x^3(f_\varepsilon + g_\varepsilon) \right)^2 \, dxdt
\]

\[
\leq \mathcal{E}(f_0, g_0)
\]

is satisfied for almost all \( T > 0 \).

2.1. Approximations of the weak solutions of (2.1) by Fourier series expansions.

In the remaining of this section \( \varepsilon \in (0, 1] \) is arbitrary but fixed. In a first step we construct Galerkin approximations for the weak solution of the problem determined by (2.1), (1.1b) and (1.1c). Letting

\[
\phi_0 := \sqrt{1/L} \quad \text{and} \quad \phi_k := \sqrt{2/L \cos(k \pi x/L)}, \ k \geq 1,
\]

denote the normalized eigenvectors of the operator \(-\partial_x^2 : H^2(\mathcal{I}) \rightarrow L_2(\mathcal{I})\) which satisfy zero Neumann boundary conditions, it is well-known that any function which belongs to \( H^1(\mathcal{I}) \) can be represented in \( H^1(\mathcal{I}) \) by its trigonometric series. Let thus \( f_0, g_0 \) be two non-negative functions from \( H^1(\mathcal{I}) \). For each fixed \( n \in \mathbb{N} \), we consider the partial sums

\[
f^n_0 := \sum_{k=0}^n f_{0k} \phi_k, \quad g^n_0 := \sum_{k=0}^n g_{0k} \phi_k
\]
of the series expansions for the initial conditions \( (f_0, g_0) \), and we seek for continuously differentiable functions
\[
f_\varepsilon^n(t, x) := \sum_{k=0}^{n} F_\varepsilon^k(t)\phi_k(x), \quad g_\varepsilon^n(t) := \sum_{k=0}^{n} G_\varepsilon^k(t)\phi_k(t, x)
\]
for \( t \geq 0 \) and \( x \in \mathbb{T} \), which solve (2.1) when testing with functions from the linear subspace \( \langle \phi_0, \ldots, \phi_n \rangle \), and which satisfy initially
\[
f_\varepsilon^n(0) = f_0^n, \quad g_\varepsilon^n(0) = g_0^n.
\]
By construction the functions \( (f_\varepsilon^n, g_\varepsilon^n) \) satisfy the boundary conditions (1.1c) and, if we test both equations of (2.1) with \( \phi_0 \), it follows at once that \( F_\varepsilon^0 \) and \( G_\varepsilon^0 \) are constant in time, that is
\[
F_\varepsilon^0(t) = f_0^0, \quad G_\varepsilon^0(t) = g_0^0, \quad t \geq 0.
\]
Additionally, testing the system (2.1) successively with \( \phi_1, \ldots, \phi_n \), it follows that the \( 2n \)-tuple \( (\mathbb{F}, \mathbb{G}) := (F_\varepsilon^1, \ldots, F_\varepsilon^n, G_\varepsilon^1, \ldots, G_\varepsilon^n) \) is the solution of the initial value problem
\[
(\mathbb{F}, \mathbb{G})'(t) = \Psi(\mathbb{F}, \mathbb{G}), \quad (\mathbb{F}, \mathbb{G})(0) = (f_{01}, \ldots, f_{0n}, g_{01}, \ldots, g_{0n}), \quad (2.7)
\]
whereby the function \( \Psi := (\Psi_1, \Psi_2) : \mathbb{R}^{2n} \rightarrow \mathbb{R}^{2n} \) is given by
\[
\Psi_{1,j}(p, q) = \sum_{k=1}^{n} p_k \int_{0}^{L} a_\varepsilon^3(\Xi_f(p)) \partial_x^3 \phi_k \partial_x \phi_j \, dx
\]
\[
+ R \sum_{k=1}^{n} (p_k + q_k) \int_{0}^{L} \left( a_\varepsilon^3(\Xi_f(p)) + \frac{3}{2} a_\varepsilon^2(\Xi_f(p)) a_\varepsilon(\Xi_g(q)) \right) \partial_x^2 \phi_k \partial_x \phi_j \, dx,
\]
and
\[
\Psi_{2,j}(p, q) = \frac{3}{2} \sum_{k=1}^{n} p_k \int_{0}^{L} a_\varepsilon^2(\Xi_f(p)) a_\varepsilon(\Xi_g(q)) \partial_x^2 \phi_k \partial_x \phi_j \, dx
\]
\[
+ R \sum_{k=1}^{n} (p_k + q_k) \int_{0}^{L} \left( \mu a_\varepsilon^3(\Xi_g(q)) + \frac{3}{2} a_\varepsilon^2(\Xi_f(p)) a_\varepsilon(\Xi_g(q)) \right) \partial_x^2 \phi_k \partial_x \phi_j \, dx
\]
for \( j \in \{1, \ldots, n\} \) and \( (p, q) \in \mathbb{R}^{2n} \). We used here the shorthand
\[
\Xi_f(p) = f_{00}\phi_0 + \sum_{l=1}^{n} p_l \phi_l, \quad \Xi_g(q) = g_{00}\phi_0 + \sum_{l=1}^{n} q_l \phi_l
\]
for all \( p, q \in \mathbb{R}^n \). Recalling that \( a_\varepsilon \) is a Lipschitz continuous function, we deduce that \( \Psi \) is locally Lipschitz continuous in \( \mathbb{R}^{2n} \), and therefore the initial value problem (2.7) possesses a unique solution \( (\mathbb{F}, \mathbb{G}) \) defined on a maximal time interval \([0, T_\varepsilon^n] \). In order to prove that the solution is global, that is \( T_\varepsilon^n = \infty \) for all \( n \in \mathbb{N} \), we show that the energy functional \( \mathcal{E} \) decreases
along \((f^n_\epsilon, g^n_\epsilon)\). Indeed, since \(\partial^2_x f^n_\epsilon(t), \partial^2_x g^n_\epsilon(t) \in \langle \phi_0, \ldots, \phi_n \rangle\) for all \(t \in [0, T^n_\epsilon]\), we may use them as test functions for (2.1). Integrating by parts, we then find
\[
\frac{d}{dt} (\mathcal{E}(f^n_\epsilon, g^n_\epsilon)) = \int_I (1 + R) \partial_x f^n_\epsilon R \partial_x g^n_\epsilon \partial_x (\partial_t f^n_\epsilon) + R(\partial_x f^n_\epsilon + \partial_x g^n_\epsilon) \partial_x (\partial_t g^n_\epsilon) dx
\]
\[
= - \int_I (1 + R) \partial^2_x f^n_\epsilon R \partial^2_x g^n_\epsilon \partial_t f^n_\epsilon + R(\partial^2_x f^n_\epsilon + g^n_\epsilon) \partial_t g^n_\epsilon dx
\]
\[
= - \mu R^2 \int_I \partial^3_x (g^n_\epsilon) \partial^3_x (f^n_\epsilon + g^n_\epsilon) dx - \frac{3R^2}{4} \int_I a_\epsilon(f^n_\epsilon) a_\epsilon(g^n_\epsilon) \partial^2_x (f^n_\epsilon + g^n_\epsilon) dx
\]
\[
- \int_I a_\epsilon(f^n_\epsilon) \partial^3_x (f^n_\epsilon) a_\epsilon(g^n_\epsilon) + \frac{R}{2}(2a_\epsilon(f^n_\epsilon) + 3a_\epsilon(g^n_\epsilon)) \partial^3_x (f^n_\epsilon + g^n_\epsilon) dx
\]
for all \(t \in [0, T^n_\epsilon]\), the last equality following similarly (1.3). Particularly, relation (2.8) ensures the boundedness of the function \((\mathcal{F}, \mathcal{G})\) on \([0, T^n_\epsilon]\). This means that the Galerkin approximations \((f^n_\epsilon, g^n_\epsilon)\) exist globally in time for all \(n \in \mathbb{N}\).

### 2.2. Convergence of the Galerkin approximations.

We next identify an accumulation point of the family \(\((f^n_\epsilon, g^n_\epsilon)\)_n\), which is shown subsequently to be a weak solution of the regularized system in the sense of Theorem 2.1. To this end, let \(T \in (0, \infty)\) be an arbitrary constant. Invoking (2.8), we deduce the uniform boundedness\(^1\) of
\[
\partial_x f^n_\epsilon, \partial_x g^n_\epsilon \quad \text{in} \quad L_\infty(0, T; L_2(I)),
\]
\[
a^{3/2}_\epsilon(g^n_\epsilon) \partial^3_x (f^n_\epsilon + g^n_\epsilon), a^{1/2}_\epsilon(f^n_\epsilon) a_\epsilon(g^n_\epsilon) \partial^3_x (f^n_\epsilon + g^n_\epsilon) \quad \text{in} \quad L_2(Q_T),
\]
\[
a^{1/2}_\epsilon(f^n_\epsilon) \left(a_\epsilon(f^n_\epsilon) \partial^3_x f^n_\epsilon + \frac{R}{2}(2a_\epsilon(f^n_\epsilon) + 3a_\epsilon(g^n_\epsilon)) \partial^3_x (f^n_\epsilon + g^n_\epsilon)\right) \quad \text{in} \quad L_2(Q_T),
\]
while from (2.6) we obtain that
\[
\int_I f^n_\epsilon(t) dx = \|f_0\|_{L_1} \quad \text{and} \quad \int_I g^n_\epsilon(t) dx = \|g_0\|_{L_1} \quad \text{for all} \quad t \geq 0.
\]

First, we observe that Poincaré’s inequality combined with (2.9) and (2.12) imply that
\[
f^n_\epsilon, g^n_\epsilon \quad \text{are bounded in} \quad L_\infty(0, T; H^1(I)).
\]
On the other hand, by construction we know that \(a_\epsilon \geq \epsilon\), and we infer from the relations (2.10)-(2.12), by using Poincaré’s inequality again and the uniform boundedness of \((f^n_\epsilon)_n\) and \((g^n_\epsilon)_n\) in \(C(Q_T)\), cf. (2.13), that
\[
f^n_\epsilon, g^n_\epsilon \quad \text{are bounded in} \quad L_2(0, T; H^3(I)).
\]

In the next step, we derive uniform bounds for the time derivatives of the Galerkin approximations. In order to do so, we observe that the relation of (2.1) may be written in a more

\(^1\)All the bounds in this section are uniform in \(n \in \mathbb{N}\).
We claim that a similar estimate is valid also for
integration with respect to time, yields
Consequently, for every \( t \)
we finally conclude that

The relations (2.11) and (2.13) yield now that the sequence \((H^\varepsilon_{f,n})_n\) is bounded in \(L_2(Q_T)\).
Given \( \zeta \in H^1(\mathcal{I}) \), we define for each \( n \in \mathbb{N} \) the truncation

Integration by parts then implies that

Consequently, for every \( t \in [0, T] \), the function \( \partial_t f^n_\varepsilon(t) \) belongs to the dual \((H^1(\mathcal{I}))'\) of \(H^1(\mathcal{I})\)
and, integration with respect to time, yields

We claim that a similar estimate is valid also for \( \partial_t g^n_\varepsilon \). Indeed, the second relation of (2.1) may be recast as the equation \( \partial_t g^n_\varepsilon = -\partial_x H^\varepsilon_{g,n} \), whereby

Gathering (2.10), (2.11), and (2.13), we see that \((H^\varepsilon_{g,n})_n\) is a bounded sequence in \(L_2(Q_T)\), and we finally conclude that

Using an argument based on the Aubin-Lions Lemma, cf. Corollary 4 in [19], together with the continuity of the embeddings

we conclude from (2.14) and (2.16) that the sequences \((f^n_\varepsilon)_n\) and \((g^n_\varepsilon)_n\) are both relatively compact in \( C([0, T], C^\alpha(\mathcal{I})) \cap L_2(0, T; C^{2+\alpha}(\mathcal{I})) \) for all \( \alpha \in [0, 1/2) \). Hence, using a diagonal procedure, we find functions \( f_\varepsilon \) and \( g_\varepsilon \) and subsequences of \((f^n_\varepsilon)_n\) and \((g^n_\varepsilon)_n\) (not relabeled) such that

\[ f^n_\varepsilon \to f_\varepsilon \quad \text{and} \quad g^n_\varepsilon \to g_\varepsilon \quad \text{in} \quad C([0, T], C^\alpha(\mathcal{I})) \cap L_2(0, T; C^{2+\alpha}(\mathcal{I})) \]  

\[ (2.17) \]
for all \( \alpha \in [0,1/2] \). Furthermore, let us observe that the relations (2.13), (2.14), and (2.17) ensure that the limit functions belong also to \( f_\varepsilon, g_\varepsilon \in L_\infty(0,T; H^1(I)) \cap L_2(0,T; H^3(I)) \) and that
\[
\partial_x^k f^n_\varepsilon \to \partial_x^k f_\varepsilon \quad \text{and} \quad \partial_x^k g^n_\varepsilon \to \partial_x^k g_\varepsilon \quad \text{in} \quad L_2(Q_T) \quad \text{for} \quad k = 1, 2, 3. \tag{2.18}
\]

Finally, from (2.16) we obtain that \( \partial_t f^n_\varepsilon, \partial_t g^n_\varepsilon \in L_2(0,T; (H^1(I))' \), and
\[
\partial_t f^n_\varepsilon \to \partial_t f_\varepsilon, \quad \partial_t g^n_\varepsilon \to \partial_t g_\varepsilon \quad \text{in} \quad L_2(0,T; (H^1(I))'). \tag{2.19}
\]
The fact that \( (f_\varepsilon, g_\varepsilon) \) can be defined globally follows by using a standard Cantor diagonal argument (choosing a sequence \( T_n \nearrow \infty \)).

2.3. Construction of the weak solutions for the regularized system.

In this last part of Section 2 we prove that the functions \( (f_\varepsilon, g_\varepsilon) \) constructed in (2.17) are weak solutions of (2.1), (1.1b), and (1.1c), and enjoy all the properties stated in Theorem 2.1. First, let us observe that the functions \( f_\varepsilon \) and \( g_\varepsilon \) possess the regularity and integrability properties required in Theorem 2.1. Moreover, because \( f_0 \in H^1(I) \) and due to (2.17), we have that \( f_\varepsilon(0) = f_0 \) and \( g_\varepsilon(0) = g_0 \). Furthermore, combining (2.12) and (2.17), it follows that the identities (2.3) are satisfied.

Concerning (2.4), we note that (2.17) guarantees that \( f^n_\varepsilon(T) \to f_\varepsilon(T) \) in \( C^2(I) \) for almost all \( T \geq 0 \). Because \( \partial_x f^n_\varepsilon(T) = 0 \) at \( x = 0,L \), the desired claim (2.4) for \( f_\varepsilon \) (and similarly for \( g_\varepsilon \)) is immediate.

We next prove that the energy estimate (2.5) is satisfied by the functions \( (f_\varepsilon, g_\varepsilon) \). To this end, we infer from the relations (2.10), (2.11), (2.17), and (2.18), by using also the Lipschitz continuity of the map \( a_\varepsilon \) and after extracting further subsequences of \( (f^n_\varepsilon) \) and \( (g^n_\varepsilon) \) (not relabeled) that we have the following weak convergences in \( L_2(Q_T) \):
\[
\begin{align*}
a^{3/2}(g^n_\varepsilon) & \partial^3_x (f^n_\varepsilon + g^n_\varepsilon) \to a^{3/2}(g_\varepsilon) \partial^3_x (f_\varepsilon + g_\varepsilon), \\
a^{1/2}(f^n_\varepsilon) a_\varepsilon (g^n_\varepsilon) & \partial^2_x (f^n_\varepsilon + g^n_\varepsilon) \to a^{1/2}(f_\varepsilon) a_\varepsilon (g_\varepsilon) \partial^2_x (f_\varepsilon + g_\varepsilon), \\
a^{1/2}(f^n_\varepsilon) & \left( a_\varepsilon (f^n_\varepsilon) \partial^2_x f^n_\varepsilon + \frac{R}{2} (2a_\varepsilon (f^n_\varepsilon) + 3a_\varepsilon (g^n_\varepsilon)) \partial^2_x (f^n_\varepsilon + g^n_\varepsilon) \right) \\
& \to a^{1/2}(f_\varepsilon) \left( a_\varepsilon (f_\varepsilon) \partial^2_x f_\varepsilon + \frac{R}{2} (2a_\varepsilon (f_\varepsilon) + 3a_\varepsilon (g_\varepsilon)) \partial^2_x (f_\varepsilon + g_\varepsilon) \right).
\end{align*}
\]
Recalling that \( f^n_\varepsilon(t) \to f_\varepsilon(t), \ g^n_\varepsilon(t) \to g_\varepsilon(t) \) in \( C^2(I) \) for almost all \( t \geq 0 \), and that the initial data \( f_0, g_0 \) belong to \( H^1(I) \), we conclude after integrating (2.8) with respect to time and passing to \( \liminf_{n \to \infty} \) that the desired energy inequality (2.5) is satisfied.

To finish the proof of Theorem 2.1, we are only left to prove relations (2.2). Let therefore \( \xi \in L_2(0,T; H^1(I)) \) be given, and define for each \( n \in \mathbb{N} \) the truncation
\[
\xi^n(t,\cdot) := \sum_{k=0}^{n} (\xi(t,\cdot) | \phi_k)_{L_2} \phi_k, \quad t \in (0,T).
\]
Using integrating by parts, we find in a similar way as before that
\[
\int_I \partial_t f^n_\varepsilon(t) \xi^n(t) \, dx = - \int_I \xi^n(t) \partial_x H^n_\varepsilon(t) \, dx = \int_I H^n_\varepsilon(t) \partial_t \xi^n(t) \, dx,
\]
whence, we have
\[
\int_0^T \langle \partial_t f_\varepsilon^n(t) | \xi^n(t) \rangle \, dt = \int_0^T \langle \partial_t f_\varepsilon^n(t) | \xi^n(t) \rangle_{L^2} \, dt = \int_{Q_T} H^\varepsilon_n(t) \partial_x \xi^n(t) \, dx \tag{2.20}
\]
for all \( n \geq 0 \). Since by Lebesgue’s dominated convergence \( \xi^n \to \xi \) in \( L^2(0,T;H^1(I)) \) we find, together with (2.19), that
\[
\int_0^T \langle \partial_t f_\varepsilon^n(t) | \xi^n(t) \rangle \, dt \to \int_0^T \langle \partial_t f_\varepsilon(t) | \xi(t) \rangle \, dt. \tag{2.21}
\]
Furthermore, (2.10), (2.13), (2.17), and (2.18) ensure that, after extracting further subsequences, we have \( H^\varepsilon_{f,n} \to H^\varepsilon_f \) in \( L^2(Q_T) \), whereby we set
\[
H^\varepsilon_f := a^3_\varepsilon(f_\varepsilon) \partial_x^3 f_\varepsilon + \frac{R}{2} (2a^2_\varepsilon(f_\varepsilon) + 3a^2_\varepsilon(f_\varepsilon) a_\varepsilon(g_\varepsilon)) \partial_x^3 (f_\varepsilon + g_\varepsilon). \tag{2.22}
\]
Letting now \( n \to \infty \) in (2.20), we obtain from (2.21) and (2.22) the first relation of (2.2). On the other hand, combining the relations (2.10), (2.11), (2.13), (2.17), and (2.18) we may assume that \( H^\varepsilon_{g,n} \to H^\varepsilon_g \) in \( L^2(Q_T) \), with \( H^\varepsilon_g \) given by
\[
H^\varepsilon_g := 3 a^2_\varepsilon(f_\varepsilon) a_\varepsilon(g_\varepsilon) \partial_x^3 f_\varepsilon + \frac{R}{2} (2\mu a^2_\varepsilon(g_\varepsilon) + 3a^2_\varepsilon(f_\varepsilon) a_\varepsilon(g_\varepsilon) + 6a^2_\varepsilon(f_\varepsilon) a_\varepsilon^2(g_\varepsilon)) \partial_x^3 (f_\varepsilon + g_\varepsilon).
\]
Repeating the arguments presented above we conclude that the second identity of (2.2) is also satisfied, and the proof of Theorem 2.1 is complete.

Let us remark that we do not know whether the weak solutions \((f_\varepsilon,g_\varepsilon)\) found in Theorem 2.1 are non-negative. The next lemma though, together with the convergence results that we will provide in the next section yields the non-negativity of the weak solutions of problem (1.1), which are found as being accumulation points of the family \(((f_\varepsilon,g_\varepsilon))_{\varepsilon \in (0,1]}\), cf. Lemma 2.2 and Corollary 2.3 below. To this end, we introduce the following notation. Pick a function \( \varphi \in C^\infty(\mathbb{R}) \), which is non-negative, has support contained in \([-1,0]\), and satisfies
\[
\int_{\mathbb{R}} \varphi(x) \, dx = 1.
\]
Moreover, let the function \( \chi_1 : \mathbb{R} \to \mathbb{R} \) be defined by the relation
\[
\chi_1(x) := - \int_0^x \int_s^\infty \varphi(\tau) \, d\tau ds \quad \text{for } x \in \mathbb{R},
\]
and \((\chi_\delta)_{\delta > 0}\) be the associated mollifier, that is \( \chi_\delta(x) := \delta \chi_1(x/\delta) \) for \( x \in \mathbb{R} \) and \( \delta > 0 \). The following properties of \((\chi_\delta)_{\delta > 0}\) play an important role in the proof of Lemma 2.2:
\[
\| \chi_\delta - \max\{-1,0\} \|_{L^\infty(\mathbb{R})} \leq \delta, \tag{2.23}
\]
\[
\| \chi_\delta \|_{L^\infty(\mathbb{R})} \leq 1, \quad \| \chi_\delta^\prime \|_{L^\infty(\mathbb{R})} \leq \delta^{-1} \| \varphi \|_{L^\infty(\mathbb{R})}, \quad \text{and} \quad \| \chi_\delta^\prime \|_{L^\infty(\mathbb{R})} \leq \delta^{-2} \| \varphi \|_{L^\infty(\mathbb{R})}, \tag{2.24}
\]
for all \( \delta > 0 \).

\textbf{Lemma 2.2.} The functions \((f_\varepsilon,g_\varepsilon)\) found in Theorem 2.1 satisfy
\[
\left| \int_I \chi_\varepsilon \varphi(f_\varepsilon(T)) \, dx \right| \leq C \sqrt{T} \varepsilon \quad \text{and} \quad \left| \int_I \chi_\varepsilon \varphi(g_\varepsilon(T)) \, dx \right| \leq C \sqrt{T} \varepsilon \tag{2.25}
\]
for all \( \varepsilon \in (0,1] \) and all \( T \geq 0 \).
Before proving Lemma 2.2 let us draw the conclusion that all accumulation points of the family \((f_\varepsilon,g_\varepsilon)\) in \(C(Q_T,\mathbb{R}^2)\), with \(T > 0\), are non-negative functions.

**Corollary 2.3.** Assume that there exists a sequence \((\varepsilon_k)_k \subset (0,1)\) with \(\varepsilon_k \searrow 0\) and a pair \((f,g)\) \(\in C(Q_T,\mathbb{R}^2)\) such that
\[
(f_{\varepsilon_k},g_{\varepsilon_k}) \to (f,g) \quad \text{in } C(Q_T,\mathbb{R}^2).
\] (2.26)

Then, \(f\) and \(g\) are both non-negative functions in \(Q_T\).

**Proof.** In virtue of (2.24), we have
\[
\|\chi_{\sqrt{\varepsilon_k}}(f_{\varepsilon_k}) - \max\{-f,0\}\|_{L_\infty(Q_T)}
\leq \|\chi_{\sqrt{\varepsilon_k}}(f_{\varepsilon_k}) - \chi_{\sqrt{\varepsilon_k}}(f)\|_{L_\infty(Q_T)} + \|\chi_{\sqrt{\varepsilon_k}}(f) - \max\{-f,0\}\|_{L_\infty(Q_T)}.
\]
Whence, our assumption (2.26) guarantees the convergence \(\chi_{\sqrt{\varepsilon_k}}(f_{\varepsilon_k}) \to \max\{-f,0\}\) in \(C(Q_T)\). Letting now \(k \to \infty\) in the first inequality of (2.25) yields
\[
\int_I \max\{-f(t),0\} \, dx = 0
\]
for all \(t \in [0,T]\). This is the desired assertion for \(f\). The proof of the non-negativity of \(g\) follows similarly. \(\square\)

**Proof of Lemma 2.2.** Let \(\delta > 0\) be given. Since \(\chi'_\delta(f^n_\varepsilon(t)) \in H^1(I)\), we compute that
\[
\frac{d}{dt} \int_I \chi_\delta(f^n_\varepsilon(t)) \, dx = \int_I \chi'_\delta(f^n_\varepsilon(t)) \partial_t f^n_\varepsilon(t) \, dx = \int_I \partial_t f^n_\varepsilon(t) \sum_{k=0}^n (\chi'_\delta(f^n_\varepsilon(t)) | \phi_k)_{L_2} \phi_k \, dx
\]
\[
= \int_I H^\varepsilon_n(f_\varepsilon) \sum_{k=0}^n \partial_x ((\chi'_\delta(f^n_\varepsilon(t)) | \phi_k)_{L_2} \phi_k) \, dx,
\]
relation which is satisfied for all \(t \geq 0\). The assertions (2.25) are obviously true when \(T = 0\), so let us assume that \(T > 0\). Integration the previous identities with respect to time on \([0,T]\) shows that
\[
\int_I \chi_\delta(f^n_\varepsilon(T)) \, dx = \int_I \chi_\delta(f^n_\varepsilon(0)) \, dx + \int_{Q_T} H^\varepsilon_n \sum_{k=0}^n (\chi'_\delta(f^n_\varepsilon) | \phi_k)_{L_2} \partial_x \phi_k \, dx dt.
\] (2.27)

In order to let \(n \to \infty\) in (2.27), we first observe
\[
\sum_{k=0}^n (\chi'_\delta(f^n_\varepsilon) | \phi_k)_{L_2} \partial_x \phi_k \to \chi''_\delta(f_\varepsilon) \partial_x f_\varepsilon \quad \text{in } L_2(Q_T).
\] (2.28)

Indeed, we have
\[
\chi''_\delta(f_\varepsilon) \partial_x f_\varepsilon - \sum_{k=0}^n (\chi'_\delta(f^n_\varepsilon) | \phi_k)_{L_2} \partial_x \phi_k = \left(\chi''_\delta(f_\varepsilon) \partial_x f_\varepsilon - \sum_{k=0}^n (\chi'_\delta(f_\varepsilon) | \phi_k)_{L_2} \partial_x \phi_k\right)
\]
\[
+ \sum_{k=0}^n (\chi'_\delta(f_\varepsilon) - \chi'_\delta(f^n_\varepsilon) | \phi_k)_{L_2} \partial_x \phi_k,
\]
and the convergence of the first term to zero follows by using Lebesgue’s dominated convergence theorem together with the fact that $\chi''(f_\varepsilon(t)) \in H^1(I)$ for all $t \geq 0$. On the other hand, the
reminding sum is the truncation of the Fourier series of $\chi''(f_\varepsilon) \partial_x f_\varepsilon^n - \chi''(f_\varepsilon) \partial_x f_\varepsilon$ and, using (2.24), may be estimated as follows

$$\left\| \sum_{k=0}^{n} (\chi'_\delta(f_\varepsilon^n) - \chi'_\delta(f_\varepsilon) \phi_k) \partial_x \phi_k \right\|_{L^2(Q_T)}^2 \leq \left\| \chi''(f_\varepsilon) \partial_x f_\varepsilon^n - \chi''(f_\varepsilon) \partial_x f_\varepsilon \right\|_{L^2(Q_T)}^2$$

$$\leq 2 \left\| \chi''(f_\varepsilon) \right\|_{L^\infty(Q_T)}^2 \left\| \partial_x f_\varepsilon^n \right\|_{L^2(Q_T)}^2 + 2 \left\| \chi''(f_\varepsilon) \right\|_{L^\infty(Q_T)}^2 \left\| \partial_x f_\varepsilon - \partial_x f_\varepsilon \right\|_{L^2(Q_T)}^2$$

$$\leq 2 \delta^{-4} \left\| \varphi' \right\|_{L^\infty(R)}^2 \left\| f_\varepsilon^n - f_\varepsilon \right\|_{L^2(Q_T)}^2 \left\| \partial_x f_\varepsilon \right\|_{L^2(Q_T)}^2 + 2 \delta^{-2} \left\| \varphi' \right\|_{L^\infty(R)}^2 \left\| \partial_x f_\varepsilon - \partial_x f_\varepsilon \right\|_{L^2(Q_T)}^2,$$

the desired estimate (2.28) being now a consequence of (2.13) and (2.17).

Thus, letting $n \to \infty$ in (2.27) and taking into account that $f_\varepsilon(0) = f_0 \geq 0$, we obtain the following identity for the weak solution of (2.1) found in Theorem 2.1

$$\int_I \chi_\delta(f_\varepsilon(T)) \, dx = \int_{Q_T} H_f^\varepsilon \chi''(f_\varepsilon) \partial_x f_\varepsilon \, dx dt.$$

Since $\chi''_\delta = 0$ on $R \setminus (-\delta, 0)$, H"older’s inequality leads us to

$$\left( \int_I \chi_\delta(f_\varepsilon(T)) \, dx \right)^2 \leq \left( \int_{[-\delta \leq f_\varepsilon \leq 0]} \left| H_f^\varepsilon \chi''(f_\varepsilon) \partial_x f_\varepsilon \right| \, dx dt \right)^2$$

$$\leq \int_{[-\delta \leq f_\varepsilon \leq 0]} a_\varepsilon(f_\varepsilon) \left| a_\varepsilon(f_\varepsilon) \partial_x^3 f_\varepsilon + \frac{R}{2} \left( 2a_\varepsilon(f_\varepsilon) + 3a_\varepsilon(g_\varepsilon) \right) \partial_x^3 f_\varepsilon + g_\varepsilon \right|^2 \, dx dt$$

$$\times \int_{[-\delta \leq f_\varepsilon \leq 0]} a_\varepsilon^2(f_\varepsilon) \chi''_\delta(f_\varepsilon)^2 \left| \partial_x f_\varepsilon \right|^2 \, dx dt.$$

We choose now $\delta := \sqrt{\varepsilon}$. Recalling that $a_\varepsilon \equiv \varepsilon$ on $(-\infty, 0]$, the energy inequality (2.5) together with (2.24) imply that

$$\left| \int_I \chi_{\sqrt{\varepsilon}}(f_\varepsilon(T)) \, dx \right| \leq C \left( \int_{[-\sqrt{\varepsilon} \leq f_\varepsilon \leq 0]} \varepsilon^3 \chi''_{\sqrt{\varepsilon}}(f_\varepsilon^2) \left| \partial_x f_\varepsilon \right|^2 \, dx dt \right)^{1/2}$$

$$\leq C \varepsilon \left\| \varphi \right\|_{L^\infty(R)} \left( \int_{Q_T} \left| \partial_x f_\varepsilon \right|^2 \, dx dt \right)^{1/2} \leq C \sqrt{T} \varepsilon,$$

which is the desired estimate (2.25) for $f_\varepsilon$. Concerning the second estimate of (2.25), similar arguments to those presented above yield that

$$\int_I \chi_\delta(g_\varepsilon(T)) \, dx = \int_{Q_T} H_g^\varepsilon \chi''_\delta(g_\varepsilon) \partial_x g_\varepsilon \, dx dt.$$
for all $T > 0$ and $\delta > 0$. Writing $H^g_T$ as the sum of three terms, cf. (2.15), we obtain from Hölder’s inequality and the estimate (2.5) the following inequalities

$$\left| \int_I \chi_{\delta}(g_\varepsilon(T)) \, dx \right|$$

$$\leq \mu R \int_{[-\delta \leq g_\varepsilon \leq 0]} a_\varepsilon^{3/2}(g_\varepsilon) \chi''_\delta(g_\varepsilon) |\partial_x g_\varepsilon| \left| a_\varepsilon^{3/2}(g_\varepsilon) \partial_x^2 (f_\varepsilon + g_\varepsilon) \right| \, dx \, dt$$

$$+ \frac{3R}{4} \int_{[-\delta \leq g_\varepsilon \leq 0]} a_\varepsilon^{1/2}(f_\varepsilon^n) a_\varepsilon(g_\varepsilon^n) \chi''_\delta(g_\varepsilon) |\partial_x g_\varepsilon| \left| a_\varepsilon^{1/2}(f_\varepsilon^n) a_\varepsilon(g_\varepsilon^n) \partial_x^2 (f_\varepsilon^n + g_\varepsilon^n) \right| \, dx \, dt$$

$$+ \frac{3}{2} \int_{[-\delta \leq g_\varepsilon \leq 0]} a_\varepsilon^{1/2}(f_\varepsilon^n) a_\varepsilon(g_\varepsilon^n) \chi''_\delta(g_\varepsilon) |\partial_x g_\varepsilon|$$

$$\times \left| a_\varepsilon^{1/2}(f_\varepsilon^n) \left( a_\varepsilon(f_\varepsilon^n) \partial_x^2 f_\varepsilon^n + \frac{R}{2} (2 a_\varepsilon(f_\varepsilon^n) + 3 a_\varepsilon(g_\varepsilon^n)) \partial_x^2 (f_\varepsilon^n + g_\varepsilon^n) \right) \right| \, dx \, dt$$

$$\leq C \left( \int_{[-\delta \leq g_\varepsilon \leq 0]} a_\varepsilon^2(g_\varepsilon)(a_\varepsilon(f_\varepsilon) + a_\varepsilon(g_\varepsilon)) \chi''_\delta(g_\varepsilon) |\partial_x g_\varepsilon|^2 \, dx \, dt \right)^{1/2},$$

and, when $\delta = \sqrt[\varepsilon]{\varepsilon}$, we arrive at the following estimate

$$\left| \int_I \chi_{\sqrt[\varepsilon]{\varepsilon}}(g_\varepsilon(T)) \, dx \right| \leq C \sqrt[\varepsilon]{\| \varphi \|_{L_\infty(\mathbb{R})}} \left( \int_{Q_T} |\partial_x g_\varepsilon|^2 \, dx \, dt \right)^{1/2} \leq C \sqrt{T \varepsilon}.$$

This proves the lemma. □

3. Existence of weak solutions for the original problem

This last section is devoted to the proof of our main result Theorem 1.1. Therefore, we collect first some estimates for the family of weak solutions $((f_\varepsilon, g_\varepsilon))_{\varepsilon \in (0, 1]}$ of the approximating problems (2.1), (1.1b), and (1.1c). Considering now $\varepsilon \in (0, 1]$ as a parameter, we deduce from (2.2), (2.3), and (2.5) the uniform boundedness of

$$\partial_x f_\varepsilon, \quad \partial_x g_\varepsilon \quad \text{in} \quad L_\infty(0, T; L_2(I)), \quad \text{for all} \quad T > 0. \quad (3.1)$$

$$\partial_t f_\varepsilon, \quad \partial_t g_\varepsilon \quad \text{in} \quad L_2(0, T; (H^1(I))^\prime), \quad \text{for all} \quad T > 0. \quad (3.2)$$

$$a_\varepsilon^{3/2}(g_\varepsilon) \partial_x^2 (f_\varepsilon + g_\varepsilon), \quad a_\varepsilon^{1/2}(f_\varepsilon) a_\varepsilon(g_\varepsilon) \partial_x^2 (f_\varepsilon + g_\varepsilon) \quad \text{in} \quad L_2(Q_T), \quad \text{for all} \quad T > 0. \quad (3.3)$$

$$a_\varepsilon^{1/2}(f_\varepsilon) \left( a_\varepsilon(f_\varepsilon) \partial_x^2 f_\varepsilon + \frac{R}{2} (2 a_\varepsilon(f_\varepsilon) + 3 a_\varepsilon(g_\varepsilon)) \partial_x^2 (f_\varepsilon + g_\varepsilon) \right) \quad \text{in} \quad L_2(Q_T) \quad (3.4)$$

for all $T > 0$. Recalling also (2.3), the arguments used in the previous section ensure the existence of a sequence $(\varepsilon_k)_k \subset (0, 1]$ with $\varepsilon_k \searrow 0$ and functions $f, g \in L_2(0, T; H^1(I)) \cap (\cap_{\alpha \in (0, 1/2]} C([0, T], C^\alpha(I)))$, having the property that

$$f_{\varepsilon_k} \to f, \quad g_{\varepsilon_k} \to g \quad \text{in} \quad C([0, T], C^\alpha(I)) \quad \text{for all} \quad \alpha \in [0, 1/2); \quad (3.5)$$

$$f_{\varepsilon_k} \to f, \quad g_{\varepsilon_k} \to g \quad \text{in} \quad L_2(0, T; H^1(I)). \quad (3.6)$$

$$f_{\varepsilon_k} \to f, \quad g_{\varepsilon_k} \to g \quad \text{in} \quad L_2(0, T; H^1(I)). \quad (3.7)$$
Particularly, (3.5) implies that for almost every $t \in [0, T]$ we have
\[
\partial_x f_{\xi_k}(t) \rightharpoonup \partial_x f(t), \quad \partial_x g_{\xi_k}(t) \rightharpoonup \partial_x g(t) \quad \text{in } L_2(\mathcal{I}),
\]}
and it follows now directly from (2.5) that $f, g \in L_\infty(0, T; H^1(\mathcal{I}))$. Moreover, the convergence (3.5) together with the Corollary 2.3 yield the non-negativity of the limits $f$ and $g$. The latter property combined with the relation (2.3) ensure the desired mass conservation property claimed by Theorem 1.1 (c). Let us also observe that the claim (b) of Theorem 1.1 is a simple consequence of the convergence (3.5) and of the relations $f_{\varepsilon}(0) = f_0$ and $g_{\varepsilon}(0) = 0$ for all $\varepsilon \in (0, 1]$.

We next establish the identities (d) of Theorem 1.1. For this let $\xi \in C^\infty(\overline{Q_T})$ be given and, for every $\varepsilon \in (0, 1]$, let $((f^n_{\varepsilon}, g^n_{\varepsilon}))_n$ be the sequence found in Section 2 to converge towards the weak solution $(f_{\varepsilon}, g_{\varepsilon})$ of problem (2.1). Integrating by parts, we then find that
\[
\int_0^T \int_{\mathcal{I}} \partial_t f^n_{\varepsilon} \xi \, dx \, dt = \int_0^T \int_{\mathcal{I}} f^n_{\varepsilon}(T, x) \xi(T, x) \, dx - \int_0^T \int_{\mathcal{I}} f^n_{\varepsilon}(0) \xi(0, x) \, dx - \int_0^T \int_{\mathcal{I}} f^n_{\varepsilon} \partial_\xi \xi \, dx \, dt
\]}
Hence, letting $n \to \infty$ in (3.9), we deduce in virtue of (2.17), (2.19), and of the first identity in (2.2) that
\[
\int_{Q_T} H^f_{\varepsilon} \partial_x \xi \, dx \, dt = \int_{\mathcal{I}} f_{\varepsilon}(T, x) \xi(T, x) \, dx - \int_{\mathcal{I}} f_{\varepsilon}(0) \xi(0, x) \, dx - \int_{Q_T} f_{\varepsilon} \partial_\xi \xi \, dx \, dt
\]}
Particularly, recalling (3.5), it is easy to see that along the subsequence $((f_{\varepsilon_k}, g_{\varepsilon_k}))_k$, the right hand side of the equality (3.10) converges towards the corresponding quantities appearing in the equation (1.4).

In order to study the behaviour of the left hand side of the relation (3.10), we define for every $m \in \mathbb{N}$, $m \geq 1$, the open subsets
\[
\mathcal{P}_f^m := \{(t, x) \in (0, T) \times \mathcal{I} : f(t, x) > 1/m\},
\]
\[
\mathcal{P}_g^m := \{(t, x) \in (0, T) \times \mathcal{I} : g(t, x) > 1/m\},
\]
of $Q_T$, and observe that $\mathcal{P}_f = \cup_{m=1}^\infty \mathcal{P}_f^m$ and $\mathcal{P}_g = \cup_{m=1}^\infty \mathcal{P}_g^m$. Let now $m \geq 1$ be fixed. Due to (3.5), we may find a positive integer $k_0$ with the property that $f_{\varepsilon_k}(t, x) > (2m)^{-1}$ and $g_{\varepsilon_k}(t, x) > (2m)^{-1}$ for all $(t, x) \in \mathcal{P}_f^m \cap \mathcal{P}_g^m$ and all $k \geq k_0$. Thanks to (3.3) and (3.4), the sequences $(\partial^3_x f_{\varepsilon_k})_k$ and $(\partial^3_x g_{\varepsilon_k})_k$ are both bounded in $L_2(\mathcal{P}_f^m \cap \mathcal{P}_g^m)$, and, up to the extraction of a diagonal subsequence, we may assume that
\[
\partial^3_x f_{\varepsilon_k} \rightharpoonup \partial^3_x f, \quad \partial^3_x g_{\varepsilon_k} \rightharpoonup \partial^3_x g \quad \text{in } L_2(\mathcal{P}_f^m \cap \mathcal{P}_g^m)
\]}
for all $m \geq 1$. Because $(H^f_{\varepsilon_k})_k$ and $(a_{\varepsilon_k}^{-3/2}(f_{\varepsilon_k}) H^f_{\varepsilon_k})_k$ are bounded in $L_2(Q_T)$, cf. (2.3), (3.1), and (3.4), we can also presuppose that there exist functions $H_f, j_f \in L_2(Q_T)$ such that
\[
H^f_{\varepsilon_k} \rightharpoonup H_f, \quad (a_{\varepsilon_k}(f_{\varepsilon_k}))^{-3/2} H^f_{\varepsilon_k} \rightharpoonup j_f \quad \text{in } L_2(Q_T).
\]}
Using the convergences (3.11) and (3.5), we may identify the weak limits in (3.12) in the set where $f$ and $g$ are both positive
\[
H_f = f^3 \partial^3_x f + \frac{R}{2} (2f^3 + 3f^2g) \partial^3_x (f + g)
\]
\[
j_f = f^{1/2} \left( f \partial^3_x f + \frac{R}{2} (2f + 3fg) \partial^3_x (f + g) \right)
\]
in $L_2(\mathcal{P}_f \cap \mathcal{P}_g)$. 

Moreover, because of $|a_{\varepsilon_k}(f_{\varepsilon_k}) - f| \leq \varepsilon_k + |f_{\varepsilon_k} - f|$ for all $k \geq 0$, we may identify $H_f$ in the large set $P_g$. Indeed, by the dominated convergence theorem $a_{\varepsilon_k}^{3/2}(f_{\varepsilon_k}) \to f^{3/2}$ in $L_2(Q_T)$, which shows, together with (3.12), that $H_f = f^{3/2} j_f$ in $L_2(Q_T)$. Summarizing, we have shown that

\[
H_f = \left( f^3 \partial_x^3 f + \frac{R}{2} (2f^3 + 3f^2 g) \partial_x^3 (f + g) \right) 1_{(0,\infty)}(f) \quad \text{in} \quad L_2(P_g), \tag{3.13}
\]

and the desired assertion (1.4) follows now at once. The identity (1.5) is obtained by using similar arguments. Indeed, in this case it is possible to identify first the weak limit $H_g$ of (a subsequence of) $(H_g^{\varepsilon_k})_k$ in $L_2(Q_T)$. On the other hand, because of (3.3), there exist functions $j_g, j_f, g \in L_2(Q_T)$ such that

\[
a_{\varepsilon_k}^{3/2}(g_{\varepsilon_k}) \partial_x^3 (f_{\varepsilon_k} + g_{\varepsilon_k}) \rightharpoonup j_g, \quad a_{\varepsilon_k}^{1/2}(f_{\varepsilon_k}) a_{\varepsilon_k}(g_{\varepsilon_k}) \partial_x^3 (f_{\varepsilon_k} + g_{\varepsilon_k}) \rightharpoonup j_f, g \quad \text{in} \quad L_2(Q_T). \tag{3.14}
\]

Again, due to (3.5) and (3.11), we identify $j_g = g^{3/2} \partial_x^3 (f + g)$ and $j_f, g = f^{1/2} g \partial_x^3 (f + g)$ in $L_2(P_f \cap P_g)$. Writing $H_g^{\varepsilon_k}$ in a similar manner as in (2.15), the dominated convergence theorem shows then

\[
H_g = \mu R g^{3/2} j_g + \frac{3R}{4} f^{1/2} g j_f g + \frac{3}{2} f^{1/2} g j_f \quad \text{in} \quad L_2(Q_T),
\]

and therefore

\[
H_g = \left( \frac{3}{2} f^2 g \partial_x^3 f + \frac{R}{2} (2\mu g^3 + 3f^2 g + 6fg^2) \partial_x^3 (f + g) \right) 1_{(0,\infty)}(g) \quad \text{in} \quad L_2(P_f).
\]

The assertion (1.5) is now immediate.

Finally, we collect (3.8), (3.12), and (3.14), and pass to $\liminf_{k \to \infty}$ in the energy inequality (2.5) to obtain the desired claim (e) of Theorem 1.1.

Because $T$ was chosen arbitrary, we may again pick a sequence $T_n \to \infty$ and, extracting a diagonal sequence of $((f_{\varepsilon_k}, g_{\varepsilon_k}))_k$ we may assume that $f$ and $g$ are globally defined and the claims of Theorem 1.1 are true for all $T > 0$.

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