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Nonlocal fractional system involving the fractional $p, q$-Laplacians and singular potentials

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Abstract In this paper, we will focus on following nonlocal quasilinear elliptic system with singular nonlinearities:

\[
\begin{align*}
(-\Delta)^{s_1}_p u &= \frac{1}{u^{\alpha_1}} + v^{\beta_1} \quad \text{in } \Omega, \\
(-\Delta)^{s_2}_q v &= \frac{1}{v^{\alpha_2}} + u^{\beta_2} \quad \text{in } \Omega, \\
u, v &= 0 \quad \text{in } (\mathbb{R}^N \setminus \Omega), \\
u, v &> 0 \quad \text{in } \Omega,
\end{align*}
\]

where $\Omega \subset \mathbb{R}^N$ be a smooth bounded domain, $s_1, s_2 \in (0, 1)$, $\alpha_1, \alpha_2, \beta_1, \beta_2$ are suitable positive constants, $(-\Delta)^{s_1}_p$ and $(-\Delta)^{s_2}_q$ are the fractional $p$ - Laplacian and $q$ - Laplacian operators. Using approximating arguments, Rabinowitz bifurcation Theorem, and fractional Hardy inequality, we are able to show the existence of positive solution to the above system.

Mathematics Subject Classification 35B51 · 74G10 · 55Q25 · 47G20

1 Introduction

In this work, we consider the existence of positive solution of the following nonlocal quasilinear system:

\[
\begin{align*}
(-\Delta)^{s_1}_p u &= \frac{1}{u^{\alpha_1}} + v^{\beta_1} \quad \text{in } \Omega, \\
(-\Delta)^{s_2}_q v &= \frac{1}{v^{\alpha_2}} + u^{\beta_2} \quad \text{in } \Omega, \\
u, v &= 0 \quad \text{in } (\mathbb{R}^N \setminus \Omega), \\
u, v &> 0 \quad \text{in } \Omega,
\end{align*}
\]

where $\Omega$ be a smooth bounded domain of $\mathbb{R}^N$, $s_1, s_2 \in (0, 1)$ with $s_1 \neq 1, \alpha_1, \alpha_2, \beta_1$ and $\beta_2$ are positive constants. Here, $(-\Delta)^{s_1}_p$ ( resp. $(-\Delta)^{s_2}_q$ ) is the fractional $p$ - Laplacian (resp. $q$ - Laplacian), defined by

\[
(-\Delta)^{s_i}_{\text{\textit{g_i}}} u(x) := \text{P.V. } \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^{s_i - 2}(u(x) - u(y))}{|x - y|^{N+s_i \cdot \text{\textit{g_i}}}} \, dy, \quad i = 1, 2,
\]

where $(t_1, g_1) = (s_1, p)$, $(t_2, g_2) = (s_2, q)$ with $p, q > 1$ and P.V is the principal value.

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In the local case, $t_i = 1$ for $i = 1, 2$, the operator defined in (1.2) is reduced to $\Delta_{\alpha} u = div(\nu |\nabla u|^{\alpha-2} \nabla u)$ that the well-known $\Delta_{\alpha}$ Laplacian operator with $\alpha_i > 1$ and $\alpha_i \in (p, q)$.

Before giving our main results, let us briefly recall literature.

- **Equation**: Notice that System (1.1) can be seen as a version of the singular scalar equations

\[
\begin{align*}
(-\Delta)^s_p u &= \frac{\lambda}{u^\alpha} + u^\beta \quad \text{in } \Omega, \\
u &= 0 \quad \text{in } (\mathbb{R}^N \setminus \Omega), \\
u &> 0 \quad \text{in } \Omega^2 ,
\end{align*}
\]  

where $s \in (0, 1)$, $\alpha, \beta > 0$, $p > 1$ and $\lambda$ is a real positive parameter. Several works are devoted to classes of problems (1.3).

For $s = 1$, $1 < p < N$ and $\lambda = 0$, the existence of weak solution and regularity of solutions have been widely studied in [6,7,9,11,21,29] and the references therein. In the case $s = 1$, $p = 2$, and $\lambda \neq 0$, problem (1.3) has been treated in [18], where the authors have used the variational method to show that for $0 < \lambda < \Lambda < \infty$, problem (1.3) has two solutions. This paper was generalized for $p -$Laplacian in [16] where the authors have showed the existence of two solutions using the variational method for $0 < \alpha < 1$ and $p - 1 < \beta < \frac{pN}{N-p} - 1$ (see also [4]). Other related works can be found [3, 14, 15, 32] and their corresponding references.

Recently, the study of fractional elliptic equations with singular nonlinearity attracted lot of interests by researchers in nonlinear analysis. In [5], for $p = 2$ and $0 < s < 1$, the authors studied the existence of distributional solutions of problem (1.3) using the uniform estimates of $\{u_n\}$ which are solutions of the regularized problems with singular term $u^{-\alpha}$ replaced by $(u_n + \frac{1}{N})^{-\alpha}$ (see also [12, 27, 30]) for more general context. The cases, when $0 < s < 1$ and $p \neq 2$, have been considered in [26] where the authors have showed the existence of multiple solutions to (1.3) using variational methods. Readers may refer to the work in [13, 31] and the references therein.

Needless to say, the references mentioned above do not exhaust the rich literature on the subject.

- **System**: The case of systems with $p, q -$Laplacians and $s_1 = s_2 = 1$, System (1.1) with singular nonlinearities was treated in [1], the authors have showed using Rabinowitz bifurcation theorem and a Hardy–Sobolev inequality the existence of the weak solution, for every $(\alpha_i, \beta_i) \in (0, \theta_i)$ with $i = 1, 2$ and

\[
\theta_1 = \min \left\{ \frac{p^*}{q}, p - 1, 1 \right\} \quad \text{and} \quad \theta_2 = \min \left\{ \frac{q^*}{p'}, q - 1, 1 \right\} .
\]

We refer the readers, [2, 17, 22, 25] for more general context and the references therein.

Recently, System (1.1) has been treated by another type of operator, notably an anisotropic operator; see [8].

Our main interest in this work is to analyze System (1.1). We will consider principally nonlinearities with concave–convex structure. It is clear that one of the main difficulties to show some control of the singular term near the boundary of the domain. The existence of solutions will be proved using approximation technics, the classical Rabinowitz bifurcation Theorem, and Hopf’s lemma. Our main existence result is stated in the following theorem.

**Theorem 1.1** Let $\Omega$ be a bounded regular domain in $\mathbb{R}^N$, $s_1, s_2 \in (0, 1), p \in [1, \frac{N}{s_1}), q \in [1, \frac{N}{s_2})$, $p' \in [1, p^*_{s_1}]$, $q' \in [1, q^*_{s_2}]$ where $p'$ and $q'$ are conjugate exponents of $p$ and $q$, respectively. Assume that $\alpha_i, \beta_i \in (0, \gamma_i)$ for $i = 1, 2$, such that

\[
\gamma_1 = \min \left\{ \frac{p'}{q'}, p - 1, \frac{s_1}{s_1} \right\} \quad \text{and} \quad \gamma_2 = \min \left\{ \frac{q'}{p'}, q - 1, \frac{s_2}{s_2} \right\} .
\]

Then, System (1.1) possesses a nontrivial solution in $W^{s_1,p}_0(\Omega) \times W^{s_2,q}_0(\Omega)$.

The paper is organized as follows. In the next section, we recall some basic notions and properties like fractional Sobolev spaces, notion of solution, and beside that some inequalities and useful lemmas are included, as well as strong maximum principle and Rabinowitz bifurcation Theorem that will be used along in this paper. In the last section, we prove the main existence results of this work.
2 The functional setting and tools

In this section, we collect some well-known results on Sobolev spaces and give some tools as they are needed to prove our main results.

Let \( \Omega \subset \mathbb{R}^N \) be an arbitrary open-bounded set. For \( p > 1 \) and \( s \in (0, 1) \), we denoted by

\[
W^{s, p}(\Omega) := \left\{ u \in L^p(\Omega) : \iint_{\Omega \times \Omega} \frac{|u(x) - u(y)|^p}{|x - y|^{N + sp}} \, dx \, dy < \infty \right\},
\]

the fractional order Sobolev space endowed with the norm

\[
\|u\|_{W^{s, p}(\Omega)} := \left( \int_{\Omega} \int_{\Omega} \frac{|u(x) - u(y)|^p}{|x - y|^{N + sp}} \, dx \, dy \right)^{\frac{1}{p}}.
\]

We set

\[
W_0^{s, p}(\Omega) := \left\{ u \in W^{s, p}(\mathbb{R}^N) : u = 0 \text{ in } \mathbb{R}^N \setminus \Omega \right\}.
\]

Then, \( W_0^{s, p}(\Omega) \) endowed with the norm

\[
\|u\|_{W_0^{s, p}(\Omega)} := \left( \int_{D_\Omega} \int_{D_\Omega} \frac{|u(x) - u(y)|^p}{|x - y|^{N + sp}} \, dx \, dy \right)^{\frac{1}{p}},
\]

where

\[
D_\Omega := (\mathbb{R}^N \times \mathbb{R}^N) \setminus (\Omega \times \mathbb{R}^N) = (\Omega \times \mathbb{R}^N) \cup (\mathbb{R}^N \times \Omega),
\]
is Banach space; we refer to [10, 23] for more details and properties of the fractional Sobolev spaces.

Theorem 2.1 (Fractional Sobolev inequality [10]) Assume that \( 0 < s < 1, \ p > 1 \) satisfy \( ps < N \). Then, there exists a positive constant \( S = S(N, s, p) \), such that for all \( v \in C_0^\infty(\mathbb{R}^N) \)

\[
\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|v(x) - v(y)|^p}{|x - y|^{N + ps}} \, dx \, dy \geq S \left( \int_{\mathbb{R}^N} |v(x)|^p \, dx \right)^{\frac{p}{p_s}},
\]

where \( p_s^* = \frac{pN}{N - ps} \) is critical Sobolev exponent.

Let us consider now the following quasilinear problem:

\[
\begin{cases}
(-\Delta)_p^s u = f & \text{in } \Omega, \\
u = 0 & \text{in } \mathbb{R}^N \setminus \Omega,
\end{cases}
\tag{2.1}
\]

where \( \Omega \subset \mathbb{R}^N \) be open-bounded domain, \( 1 < p < \infty, \ 0 < s < 1, \ f \in W^{-s, p'}(\Omega) \) (we shall denoted \( W^{-s, p'}(\Omega) \) is the dual of the reflexive Banach space of \( W_0^{s, p}(\Omega) \) and \( p' = \frac{p}{p-1} \).

Definition 2.2 For \( f \in W^{-s, p'}(\Omega) \), we say that \( u \in W_0^{s, p}(\Omega) \) is a weak solution to (2.1) if

\[
\frac{C_{N, p, s}}{2} \iint_{D_\Omega} \frac{|u(x) - u(y)|^{p-2}(u(x) - u(y))(\phi(x) - \phi(y))}{|x - y|^{N + sp}} \, dx \, dy = \int_{\Omega} f \, v \, dx,
\]

for all \( v \in W_0^{s, p}(\Omega) \).

Proposition 2.3 [33]. Let \( s \in (0, 1) \) and \( 1 < p < \infty \). Then, for every \( f \in W^{-s, p'}(\Omega) \), the Dirichlet problem (2.1) has a unique weak solution \( u \in W_0^{s, p}(\Omega) \). Moreover

\[
\|u\|_{W_0^{s, p}(\Omega)} \leq \|f\|_{W^{-s, p'}(\Omega)}^{\frac{1}{p-1}}.
\tag{2.2}
\]
Proposition 2.4 [33]. Let $s \in (0, 1)$ and $1 < p < \infty$. Then, we have that

1. $(-\Delta)_p^s : W_0^{s,p}(\Omega) \to W^{-s,p'}(\Omega)$ is strictly monotone, continuous, coercive, and bounded.
2. $((\Delta)_p^s)^{-1} : W^{-s,p'}(\Omega) \to W_0^{s,p}(\Omega)$ is locally Lipschitz continuous if $p \in (1, 2)$ and is Lipschitz continuous if $p \geq 2$.
3. The composed operator $W^{-s,p'}(\Omega) \hookrightarrow W_0^{s,p}(\Omega) \hookrightarrow L^q(\Omega)$ is compact if $1 \leq q < \frac{pN}{N-ps}$.

Lemma 2.5 (Strong maximum principle [24]). Let $u \in W_0^{s,p}(\Omega)$ satisfy

$$
\begin{cases}
(-\Delta)_p^s u \geq 0 & \text{in } \Omega, \\
u \geq 0 & \text{in } \mathbb{R}^N \setminus \Omega.
\end{cases}
$$

Then, $u$ has lower semi-continuous representative in $\Omega$, which is either identically 0 or positive.

Theorem 2.6 (Hardy inequality [19]). Let $0 < s < 1$ and $1 < p < \infty$ be such that $sp < N$. Assume that $\Omega \subset \mathbb{R}^N$ is a (bounded) uniform domain with a (locally) $(s, p)$-uniformly fat boundary. Then, $\Omega$ admits an $(s, p)$-Hardy inequality, that is, there is a constant $C > 0$, such that

$$
C \int_{\Omega} \frac{|u(x)|^p}{d^p(x)} \, dx \leq \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^p}{|x - y|^{N+ps}} \, dy \quad \text{for every } u \in W_0^{s,p}(\Omega).
$$

Finally, we recall the classical Rabinowitz result, see [28], that will be used systematically in this paper.

Theorem 2.7 Let $E$ be a Banach space and $T : \mathbb{R}^+ \times E \to E$ a continuous and compact operator, such that $T(0, u) = 0$ for all $u \in E$. Then, the equation

$$u = T(\lambda, u),$$

possesses an unbounded continuum $F \subset \mathbb{R}^+ \times E$ of solutions with $(0, 0) \in F$.

3 Proof of the main result

In this section, we focus to prove the existence of nontrivial solution to (1.1) under some hypothesis on $\alpha_1, \alpha_2, \beta_1, \beta_2, s_1,$ and $s_2$.

Before proving Theorem 1.1, we begin with the following auxiliary system:

$$
\begin{align*}
(-\Delta)_p^{s_1} u &= \lambda \left[ \frac{1}{\sigma + u^{\alpha_1}} + u^{\beta_1} \right] & \text{in } \Omega, \\
(-\Delta)_p^{s_2} v &= \lambda \left[ \frac{1}{\delta + u^{\alpha_2}} + u^{\beta_2} \right] & \text{in } \Omega, \\
u, v &= 0 & \text{in } (\mathbb{R}^N \setminus \Omega), \\
u, v &= > 0 & \text{in } \Omega.
\end{align*}
$$

(3.1)

First, we begin by the following Lemma.

Lemma 3.1 Under the hypothesis of Theorem 1.1. Then, for $\delta, \sigma > 0$ fixed and for all $\lambda > 0$, System (3.1) possesses a nontrivial solution in $W_0^{s_1,p}(\Omega) \times W_0^{s_2,q}(\Omega)$.

Proof We consider now the following approximating system:

$$
\begin{align*}
(-\Delta)_p^{s_1} u &= \lambda \left[ \frac{1}{|\psi|^{\alpha_1} + \sigma} + |\psi|^{\beta_1} \right] & \text{in } \Omega, \\
(-\Delta)_p^{s_2} v &= \lambda \left[ \frac{1}{|\phi|^{\alpha_2} + \delta} + |\phi|^{\beta_2} \right] & \text{in } \Omega, \\
u, \psi &= 0 & \text{in } (\mathbb{R}^N \setminus \Omega), \\
u, \psi &= > 0 & \text{in } \Omega.
\end{align*}
$$

(3.2)

where $(\phi, \psi) \in L^{p'}(\Omega) \times L^{q'}(\Omega)$ are be fixed.

First of all, we observe that:
For $\psi \in L^q'(\Omega)$ and $\sigma > 0$, we have that $\frac{1}{|\psi|^{q_2+\sigma}} \in L^{q'}(\Omega)$.

On the other hand, by hypothesis, $p' \beta_1 < q'$, then we get,

$$L^q'(\Omega) \hookrightarrow L^{q'\beta_1}(\Omega).$$

Hence, $|\psi|^{\beta_1} \in L^{q'}(\Omega)$.

By same way as before, we obtain that $\frac{1}{|\psi|^{q_2+\sigma}} \in L^{q'}(\Omega)$ and $|\phi|^{\beta_2} \in L^q(\Omega)$ for $\beta_2 q' < p'$.

Now, using Proposition 2.3 for each $(\lambda, \phi, \psi) \in IR^+ \times L^{p'}(\Omega) \times L^q(\Omega)$, system (3.2) possesses a unique weak solution $(u, v)$ in $W_0^{s_1,p}(\Omega) \times W_0^{s_2,q}(\Omega)$, that is

$$C_{N,p,s}\int_{\Omega} |u(x)-u(y)|^{p-2}\frac{(u(x)-u(y)(\xi(x)-\xi(y)))}{|x-y|^{N+sp}} \frac{1}{|\psi|^{q_1+\sigma}} + |\psi|^{\beta_1} d\xi d\lambda,$$

for all $\xi \in W_0^{s_1,p}(\Omega)$ and

$$C_{N,q,s}\int_{\Omega} |u(x)-u(y)|^{q-2}\frac{(u(x)-u(y)(\xi(x)-\xi(y)))}{|x-y|^{N+sq}} \frac{1}{|\phi|^{q_2+\delta}} + |\phi|^{\beta_2} d\xi d\lambda,$$

for all $\xi \in W_0^{s_2,q}(\Omega)$.

Hence, the following operator:

$$S : IR^+ \times L^{p'}(\Omega) \times L^q(\Omega) \rightarrow W_0^{s_1,p}(\Omega) \times W_0^{s_2,q}(\Omega)$$

$$(\lambda, \phi, \psi) \mapsto (T(\lambda, \phi, \psi)) = (u, v).$$

is well defined. Let us show that

$$S : IR^+ \times L^{p'}(\Omega) \times L^q(\Omega) \rightarrow L^{p'}(\Omega) \times L^q(\Omega)$$

$$(\lambda, \phi, \psi) \mapsto (\lambda, \phi, \psi) = (u, v)$$

is compact.

In fact, let $\{(\lambda_n, \phi_n, \psi_n)\}$ be a bounded sequence in $IR^+ \times L^{p'}(\Omega) \times L^q(\Omega)$, such that

$$\begin{align*}
(-\Delta)^{s_1}_{p} u_n &= \lambda_n \left[ \frac{1}{|\psi_n|^{q_1+\sigma}} + |\psi_n|^{\beta_1} \right] \\
(-\Delta)^{s_2}_{q} v_n &= \lambda_n \left[ \frac{1}{|\phi_n|^{q_2+\delta}} + |\phi_n|^{\beta_2} \right]
\end{align*}$$

(3.3)

Therefore, it follows that $\{u_n\}$ and $\{v_n\}$ are bounded, respectively, in $W_0^{s_1,p}(\Omega)$ and $W_0^{s_2,q}(\Omega)$.

Thus, we get the existence a subsequence $(u_n, v_n)_{n}$ and $u, v \in W_0^{s_1,p}(\Omega) \times W_0^{s_2,q}(\Omega)$, such that

1. $u_n \rightharpoonup u$ weakly in $W_0^{s_1,p}(\Omega)$,
2. $u_n \rightharpoonup u$ in $L^{r'}$ strongly for every $r \in [1, p^*_s)$,
3. $v_n \rightharpoonup v$ weakly in $W_0^{s_2,q}(\Omega)$,
4. $v_n \rightharpoonup v$ in $L^q(\Omega)$ strongly for every $\theta \in [1, q^*_s)$.

Now, using $\xi = u_n - u$ as test function in first equation of system (3.3), we get

$$\left(\int_{\Omega} (u_n-u)((-\Delta)^{s_1}_{p} u_n - (-\Delta)^{s_1}_{p} u) dx \right) = \lambda_n \left[ \int_{\Omega} \frac{u_n - u}{|\psi_n|^{q_1+\sigma}} + \int_{\Omega} |\psi_n|^{\beta_1} (u_n - u) dx \right] + o(1)$$

$$\leq C|u_n - u|_{L_p(\Omega)} + C|u_n - u|_{L_p(\Omega)} \| \psi_n \|_{L^{q'}(\Omega)} + o(1),$$

$$\leq C|u_n - u|_{L_p(\Omega)} + C|u_n - u|_{L_p(\Omega)} \| \psi_n \|_{L^{q'}(\Omega)} + o(1),$$

$$\leq C|u_n - u|_{L_p(\Omega)} + o(1).$$
On the other hand, if $p \geq 2$, we obtain that

$$C||u_n - u||_{W_0^{s_1,p}(\Omega)}^p \leq \int_\Omega (u_n - u)((-\Delta)^{s_1}_p u_n - (-\Delta)^{s_1}_p u)dx.$$  

Thus

$$||u_n - u||_{W_0^{s_1,p}(\Omega)} \leq C||u_n - u||_{L^p(\Omega)}.$$  

Since $W_0^{s_1,p}(\Omega) \hookrightarrow L^p(\Omega)$ is compact, hence, it follows that $u_n \to u$ strongly in $W_0^{s_1,p}(\Omega)$. As before, by similar reasoning, we get $v_n \to v$ strongly in $W_0^{s_2,q}(\Omega)$. The case $1 < q < 2$ and $1 < p < 2$ is made using similar arguments, and we will omit its proof.

Therefore

$$||v_n||_{W_0^{s_2,q}(\Omega)} \to 0, \text{ as } n \to +\infty, \text{ in } W_0^{s_1,p}(\Omega) \times W_0^{s_2,q}(\Omega).$$

Consequently, mapping $S$ is compact, and claim follows.

Hence, using the same computation, we get easily that $S$ is continuous.

On the other hand, we observe that $(0, 0, 0) \in F$ and $S(0, u, v) = (0, 0)$, then we are in the conditions of Theorem 2.7. Hence, we get an component $S$ is continuous. and claim follows.

Using $u$ as test function of the first equation of (3.1), we obtain that

$$||u||_{W_0^{s_1,p}(\Omega)} = \lambda \left[ \int_\Omega \frac{u}{v^{s_1} + \sigma} dx + \int_\Omega v^{\beta_1} u dx \right]$$

$$\leq C||u||_{L^p(\Omega)}^\lambda + C||u||_{L^p(\Omega)} ||v||^\beta_1_{L^q(\Omega)},$$

$$\leq C||u||_{W_0^{s_1,p}(\Omega)}^\lambda + C||u||_{W_0^{s_1,p}(\Omega)} ||v||^\beta_1_{W_0^{s_2,q}(\Omega)},$$

where in the last inequalities, we have used the fact

$$W_0^{s_1,p}(\Omega) \hookrightarrow L^p(\Omega) \text{ and } W_0^{s_2,q}(\Omega) \hookrightarrow L^q(\Omega).$$

Similarly, using $v$ as test function in the second equation of (3.1) and by taking into consideration the following immersions:

$$W_0^{s_2,q}(\Omega) \hookrightarrow L^q(\Omega) \text{ and } W_0^{s_1,p}(\Omega) \hookrightarrow L^{p_1}(\Omega),$$

it follows that:

$$||v||_{W_0^{s_2,q}(\Omega)} \leq C||v||_{W_0^{s_2,q}(\Omega)}^\lambda + C||v||_{W_0^{s_2,q}(\Omega)} ||u||^\beta_2_{W_0^{s_1,p}(\Omega)}.$$  

Hence, combining the above estimate, we obtain that

$$||u||_{W_0^{s_1,p}(\Omega)}^p + ||v||_{W_0^{s_2,q}(\Omega)}^q \leq C||u||_{W_0^{s_1,p}(\Omega)}^\lambda + C||v||_{W_0^{s_2,q}(\Omega)}^\lambda$$

$$\leq \lambda ||u||_{W_0^{s_1,p}(\Omega)} + \beta_1 \lambda ||v||_{W_0^{s_2,q}(\Omega)}.$$

Since $p, q > 1, \beta_1 < p - 1$ and $\beta_2 < q - 1$, then from the last inequality, we get $||u||_{W_0^{s_1,p}(\Omega)}$ and $||v||_{W_0^{s_2,q}(\Omega)}$ are bounded and this provides the contradiction, and consequently, $F$ must be unbounded with respect to $\lambda$ and in particular for $\lambda = 1$, on have $(1, u, v) \in F$ which gives a solution to (3.1).  \[ \square \]
Now, we are able to prove our main result.

Proof of Theorem 1.1. Using Lemma 3.1, we deduce that the system

\[
\begin{align*}
(-\Delta)^{s_1} p u_n &= \frac{1}{v_n^{p_1} + \frac{1}{n}} + v_n^{\beta_1} \quad \text{in } \Omega, \\
(-\Delta)^{s_2} q v_n &= \frac{1}{u_n^{q_2} + \frac{1}{n}} + u_n^{\beta_2} \quad \text{in } \Omega, \\
\end{align*}
\]  

(3.4)

has a solution \((u_n, v_n) \in W_0^{s_1, p}(\Omega) \times W_0^{s_2, q}(\Omega)\).

Now, we claim

\[ u_n > 0, \quad v_n > 0 \quad \text{for all } n \in \mathbb{N}. \]

In fact, let \((w_1, w_2) \in W_0^{s_1, p}(\Omega) \times W_0^{s_2, q}(\Omega)\) are the nontrivial solutions to

\[
\begin{align*}
(-\Delta)^{s_1} p w_1 &= m_1, \quad \text{in } \Omega, \\
w_1 &= 0 \quad \text{in } (\mathbb{R}^N \setminus \Omega), \\
w_1 &> 0 \quad \text{in } \Omega. \\
\end{align*}
\]  

(3.5)

and

\[
\begin{align*}
(-\Delta)^{s_2} q w_2 &= m_2, \quad \text{in } \Omega, \\
w_2 &= 0 \quad \text{in } (\mathbb{R}^N \setminus \Omega), \\
w_2 &> 0 \quad \text{in } \Omega, \\
\end{align*}
\]  

(3.6)

where \(0 < m_1 = \min_{t \geq 0} \left\{ \frac{1}{\frac{1}{p_1 + 1} + t^{\beta_1}} \right\} \) and \(0 < m_2 = \min_{t \geq 0} \left\{ \frac{1}{\frac{1}{q_2 + 1} + t^{\beta_2}} \right\} \).

Since \(p, q > 1\), then using the comparison principle (see [20]) and Lemma 2.5, we get

\[ u_n \geq w_1 > 0 \quad \text{and } v_n \geq w_2 > 0 \quad \text{for every } n \in \mathbb{N} \]

as desired. Let us show that the sequences \(\{u_n\}_n\) and \(\{v_n\}_n\) are bounded in \(W_0^{s_1, p}(\Omega)\) and \(W_0^{s_2, q}(\Omega)\), respectively.

First, we take \(u_n\) as test function in first equation of (3.4), and we get

\[
\|u_n\|_{W_0^{s_1, p}(\Omega)}^p \leq \int \frac{u_n}{v_n^{\beta_1}} \, dx + \int u_n v_n^{\beta_1} \, dx. \quad (3.7)
\]

Since \(\beta_1 p' < q' \) and \(q' [1, q_{s_2}], \) thus, using Hölder and Sobolev inequalities, it follows that:

\[
\int \frac{u_n}{v_n^{\beta_1}} \, dx \leq C \|u_n\|_{L^p(\Omega)} \|v_n\|_{L^{q'}}(\Omega) \leq C \|u_n\|_{W_0^{s_1, p}(\Omega)} \|v_n\|_{W_0^{s_2, q}(\Omega)}. \]

Now, we will estimate the first integral in the right-hand side of inequality (3.7). By Hopf’s lemma (see [24]), we get that, \(w_2(x) \geq C d^{s_2}(x)\); therefore, \(v_n \geq w_2(x) \geq C d^{s_2}(x)\) in \(\Omega\).

Therefore, we get

\[
\int \frac{u_n}{v_n^{\beta_1}} \, dx \leq \int \frac{u_n}{w_2} \, dx \leq C \int \frac{u_n}{d^{s_2}(x)} \, dx.
\]

Since \(s_1 > \alpha_1 s_2\), then, using Hölder and Hardy inequalities, we reach that

\[
\int \frac{u_n}{v_n^{\beta_1}} \, dx \leq C \int \frac{u_n}{d^{s_1}(x)} \, dx \leq C \left( \int \frac{u_n}{d^{s_1}(x)} \, dx \right)^{\frac{1}{p}} \leq C \|u_n\|_{W_0^{s_1, p}(\Omega)}.
\]
Therefore, we conclude that
\[ ||u_n||_{W^{s_i,p}_0(\Omega)}^p \leq C ||u_n||_{W^{s_i,p}_0(\Omega)} + C ||u_n||_{W^{s_i,p}_0(\Omega)} ||v_n||_{W^{q_i,q}_{0^*}(\Omega)}^{\beta_i} \]  
(3.8)

By the same computation as in above, if \( s_2 > \alpha_2 s_1, \beta_2 q' < p' \) and \( p' \in [1, p_i^s] \) are satisfies, we can show that
\[ ||v_n||_{W^{q_i,q}_{0^*}(\Omega)}^q \leq C ||v_n||_{W^{q_i,q}_{0^*}(\Omega)} + C ||v_n||_{W^{q_i,q}_{0^*}(\Omega)} ||u_n||_{W^{s_i,p}_0(\Omega)}^{\beta_i} \]  
(3.9)

Since \( p, q > 1 \) and \( \beta_1, \beta_2 \leq \min\{p - 1, q - 1\} \) and from (3.8) and (3.9), we reach that \( \{u_n\}_n \) and \( \{v_n\}_n \) are bounded in \( W^{s_1,p}_0(\Omega) \) and in \( W^{s_2,q}_{0^*}(\Omega) \), respectively. Therefore, there exist two measurable functions \( u \in W^{s_1,p}_0(\Omega) \) and \( v \in W^{s_2,q}_{0^*}(\Omega) \), such that

1. \( u_n \rightharpoonup u \) weakly in \( W^{s_1,p}_0(\Omega) \),
2. \( u_n \rightarrow u \) in \( L^\tau \) strongly for every \( \tau \in [1, p_i^s] \),
3. \( v_n \rightharpoonup v \) weakly in \( W^{s_2,q}_{0^*}(\Omega) \),
4. \( v_n \rightarrow v \) in \( L^\theta \) strongly for every \( \theta \in [1, q_i^{s*}] \),
5. \( u_n(x) \rightharpoondown u(x) \) a.e in \( \Omega \),
6. \( v_n(x) \rightharpoonup v(x) \) a.e in \( \Omega \).

Hence, using classical arguments, we get the desired result.

A direct consequence of our result in the case where \( s_1 = s_2 = s \) is the following.

**Corollary 3.2** Let \( \Omega \) be a bounded regular domain of \( \mathbb{R}^N \), \( s \in (0, 1) \), \( p \in [1, \frac{N}{s}] \), \( q \in [1, \frac{N}{s}] \), \( p' \in [1, p_i^s] \), \( q' \in [1, q_i^{s*}] \) where \( p' \) and \( q' \) are conjugate exponents of \( p \) and \( q \), respectively. Assume that \( \alpha_i, \beta_i \in (0, \gamma_i) \) for \( i = 1, 2 \), such that
\[ \gamma_1 = \min \left\{ \frac{p'}{q'}, p - 1, 1 \right\} \quad \text{and} \quad \gamma_2 = \min \left\{ \frac{q'}{p'}, q - 1, 1 \right\} . \]

Then, System (1.1) possesses a nontrivial solution in \( W^{s_1,p}_0(\Omega) \times W^{s_2,q}_{0^*}(\Omega) \).

**Remark 1** Notice that, if we take, \( s_1 = s_2 = 1 \) in Theorem 1.1, we get the result obtained in [1].

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**Declarations**

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**References**

1. Alves, C.O.: Correa On the existence of positive solution for a class of singular systems involving quasilinear operator. Appl. Math. Comput. 185(1), 727–736 (2007)
2. Alves, C.O.; Corrêa, F.J.S.A.; Gonçalves, J.V.A.: Existence of solutions for some classes of singular Hamiltonian systems. Adv. Nonlinear Stud. 5, 265–278 (2005)
3. Arcoya, D.; Moreno-Mérida, L.: Multiplicity of solutions for a Dirichlet problem with a strongly singular nonlinearity. Nonlinear Anal. 95, 281–291 (2014)
4. Bal, K.; Garain, P.: Multiplicity of solution for a quasilinear equation with singular nonlinearity. Mediterr. J. Math. 17, 91 (2020). https://doi.org/10.1007/s00009-020-01515-5
5. Barrios, B.; DeBonis, I.; Medina, M.; Peral, I.: Semilinear problems for the fractional Laplacian with a singular nonlinearity. J. Open. Math. 13, 91–107 (2015)
6. Berdan, N.E.; Diáz, J.I.; Rakotoson, J.M.: The uniform Hopf inequality for discontinuous coefficients and optimal regularity in BMO for singular problems. J. Math. Anal. Appl. 437, 350–379 (2016)
7. Boccardo, L.; Orsina, L.: Semilinear elliptic equations with singular nonlinearities. Calculus Var. Partial Differ. Equ. 37(3/4), 363–380 (2010)
8. Boukabara, Y.O.: Anisotropic system with singular and regular nonlinearities. Complex Variables Elliptic Equ (2019). https://doi.org/10.1080/17476933.2019.1606802
9. Crandall, M.G.; Rabinowitz, P.H.; Tartar, L.: On a Dirichlet problem with a singular nonlinearity. Commun. Partial Differ. Equ. 2(2), 193–222 (1977)
10. Di Nezza, E.; Palatucci, G.; Valdinoci, E.: Hitchhiker’s guide to the fractional Sobolev spaces. Bull. Sci. Math. 136(2), 521–573 (2012)
11. Diaz, J.I.; Morel, J.M.; Oswald, L.: An elliptic equation with singular nonlinearity. Commun. Partial Differ. Equ. 12(12), 533–544 (1987)
12. Molica-Bisci, G.; Saoudi, K.: The Nehari manifold for a singular elliptic equation involving the fractional Laplace operator. J. Fract. Differ. Calc. 6(2), 201–217 (2016)
13. Ghanmi, A.; Saoudi, K.: A multiplicity results for a singular problem involving the fractional p-Laplacian operator. J. Complex Variables Elliptic Equ. 61(9), 1199–1216 (2016)
14. Giacomoni, J.; Saoudi, K.: Multiplicity of positive solutions for a singular and critical problem. Nonlinear Anal. 71(9), 4060–4077 (2010)
15. Giacomoni, J.; Sreenadh, K.: Multiplicity results for a singular and quasilinear equation. Annali della Scuola Normale Superiore di Pisa Tome 71(2), 193–222 (1971)
16. Ghanmi, A.; Saoudi, K.: The Nehari manifold for a singular elliptic equation involving the fractional Laplace operator. J. Fract. Differ. Calc. 6(2), 201–217 (2016)
17. Ghanmi, A.; Saoudi, K.: A multiplicity results for a singular problem involving the fractional p-Laplacian operator. J. Complex Variables Elliptic Equ. 61(9), 1199–1216 (2016)
18. Giacomoni, J.; Hernandez, J.; Sauvy, P.: Quasilinear and singular elliptic systems. Adv. Nonlinear Anal. 2(3), 487–512 (2013)
19. Haitao, Y.: Multiplicity and asymptotic behavior of positive solutions for a singular semilinear elliptic problem. J. Differ. Equ. 176(2), 511–531 (2001)
20. Jarohs, S.: Strong comparison principle for the fractional p-Laplacian and applications to starshaped rings. Adv. Nonlinear Stud. 18, 691–704 (2018)
21. Lazer, A.C.; McKenna, P.J.: On a singular nonlinear elliptic boundary-value problem. Proc. Am. Math. Soc. 111(3), 721–730 (1991)
22. Manouni, E.; Perera, K.; Shivaji, R.: On singular quasimonotone (p, q)-Laplacian systems. Proc. R. Soc. Edinb. Sect. A 142, 585–594 (2012)
23. Molica-Bisci, G.; Radulescu, V.D.; Servadei, R.: Variational Methods for Nonlocal Fractional Problems. Cambridge University Press, Cambridge (2016)
24. Mosconi, S.; Squassina, M.: Nonlocal problems at nearly critical growth. Nonlinear Anal. 136, 84–101 (2016)
25. Motreanu, D.; Moussaoui, A.: A quasilinear singular elliptic system without cooperative. Acta Math. Sci. 34(3), 905–916 (2014)
26. Mukherjee, T.; Sreenadh, K.: On Dirichlet problem for fractional p – Laplacian with singular nonlinearity. Adv. Nonlinear Anal. (2016). https://doi.org/10.1515/anona-2016-0113
27. Panda, A.; Choudhuri, D.; Kumar, R.: Existence of positive solutions for a singular elliptic problem with critical exponent and measure data. Rocky Mt. J. Math. 51(3), 973–988 (2021). https://doi.org/10.1216/rmj.2021.51.973
28. Rabinowitz, P.H.: Some global results for nonlinear eigenvalue problems. J. Funct. Anal. 7(3), 487–513 (1971)
29. Rosen, G.: Minimum value for c in the sobolev inequality |φ|^2 ≤ c|ψ|^2. SIAM J. Appl. Math. 21(1), 30–32 (1971)
30. Saoudi, K.: A critical fractional elliptic equation with singular nonlinearities. J. Fract. Differ. Appl. Anal. 20(6), 1507–1530 (2017)
31. Saoudi, K.; Ghosh, S.; Choudhuri, D.; Honder and Hölder regularity of solutions for a nonlocal elliptic PDE involving singularity. J. Math. Phys. 60, 101509 (2019). https://doi.org/10.1063/1.5107517
32. Sun, Y.; Shao, P.; Long, Y.: Combined effects of singular and superlinear nonlinearities in some singular boundary value problems. J. Differ. Equ. 176(2), 511–531 (2001)
33. Warma, M.: Local Lipschitz continuity of the inverse of the fractional p – Laplacian, Hölder type continuity and continuous dependence of solutions to associated parabolic equations on bounded domains. Nonlinear Anal. Theory Methods Appl. 135, 129–157 (2016)