Super–Toda Models Associated to Any (super–)Lie Algebra

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Abstract

It is shown how to obtain superconformal Toda models as reductions of WZNW theories based on any Lie or super–Lie algebra.

Introduction

Two-dimensional integrable theories are now widely studied in the high energy physicists’ community due to their applications to string theory.

Two interrelated kinds of such theories are basically considered, namely the non-relativistic integrable equations in 1 + 1 dimensions (KdV-type systems) and the 2-dimensional relativistic equations of Toda type, whose simplest example is the Liouville theory, and which can be regarded as Hamiltonian-constrained WZNW models [1].

For what concerns the supersymmetric extension of such models it was known that supersymmetric integrable systems could be obtained, in both cases, from super–Lie algebras of a particular kind, i.e. the class of super–Lie algebras admitting a Dynkin-diagram presentation which involves fermionic simple roots only [2], [3].

Later it was understood [4], [5] that in the case of supersymmetric integrable equations of KdV (better to say of NLS)-type such requirement was too stringent. A general prescription to realize non-relativistic supersymmetric integrable systems from any bosonic or super Lie algebra was given in [6].

In collaboration with D. Sorokin we have further investigated the situation for Toda models [7] and [8] and realized that there exists a nice algebraic setting which
allows to construct superconformal Toda theories as WZNW reduction from any
super and bosonic Lie algebra. Here I will present this algebraic characterization,
together with some formulas not published elsewhere.

**The algebraic setting.** Let us recall some basic fact concerning (super-)Lie
algebras.

Let \( \mathcal{G} \) be a given (super-)Lie algebra having rank \( r \). In a convenient basis it can
be reconstructed through the following set of relations (together with the Serre’s
relations which for our purposes we do not need to specify):

\[
\begin{align*}
[h_i, h_j] &= 0 \\
[h_i, e_j] &= a_{ij}e_j \\
[h_i, f_j] &= -a_{ij}f_j \\
[e_i, f_j] &= \delta_{ij}h_j
\end{align*}
\]

(1)

Here \( i, j = 1, 2, ..., r \).
The \( h_i \)'s are the Cartan generators, while we denote with \( e_i \)'s, \( f_i \)'s respectively the
positive and negative simple roots. The matrix \( a_{ij} \) is the so-called Cartan matrix of
\( \mathcal{G} \).

Some rank \( r = 2 \) (super-)algebras are given by \( A_2 \equiv sl(3) \) bosonic Lie algebra,
admitting 8 generators and 2 invariants having degree 2, 3, the \( B_2 \equiv sp(4) \) bosonic
Lie algebra, with 10 generators and 2 invariants having degree 2, 4, the super-Lie al-
gebra \( B(0, 2) \equiv osp(1|4) \), having 10 bosonic plus 4 fermionic generators (one bosonic
and one fermionic simple root). For completeness let us present the fundamental
representations of the above (super-)algebras.

Let us first denote with \( e_{ij} \) the matrices having entries \( c_{ij} = \delta_{ij} \). It turns out that:
the \( sl(3) \) algebra is explicitly given by the following 3 × 3 matrices:

\[
\begin{align*}
h_1 &= e_{11} - e_{22}; \\
e_1 &= e_{12}; \\
f_1 &= e_{21};
\end{align*}
\]

while the remaining generators associated to the maximal positive and negative
roots are respectively

\[
\begin{align*}
e_3 &= e_{13}; \\
f_3 &= e_{31}
\end{align*}
\]

Similarly the fundamental representation of the \( osp(1|4) \) superalgebra is realized by
the following set of \( (4 + 1) \times (4 + 1) \) supermatrices (4 bosonic and 1 fermionic index).
We have for Cartan generators and simple roots:

\[
\begin{align*}
   h_1 &= e_{11} - e_{22} - e_{33} + e_{44}; \quad h_2 = e_{33} - e_{44} \\
   e_1 &= e_{13} + e_{42}; \quad f_1 = e_{24} + e_{31} \\
   e_2 &= e_{35} + e_{54}; \quad f_2 = e_{53} - e_{45}
\end{align*}
\]

\((e_1, f_1)\) are here bosonic while \(e_2, f_2\) are fermionic.

The positive (negative) non-simple roots are given by \(p_{kl} (n_{kl})\), where \(k,l\) label the decomposition in terms of the simple roots, bosonic and fermionic respectively:

\[
\begin{align*}
   p_{02} &= 2e_{34}; \quad n_{02} = -2e_{43} \\
   p_{11} &= e_{15} - e_{52}; \quad n_{11} = -(e_{31} + e_{21}) \\
   p_{12} &= 2(e_{14} - e_{32}); \quad n_{12} = 2(e_{41} - e_{23}) \\
   p_{22} &= -4e_{12}; \quad n_{22} = 4e_{21}
\end{align*}
\]

The fundamental representation for the \(sp(4)\) subalgebra of \(osp(1|4)\) is recovered from the above formulas discarding the fermionic generators \(e_2, f_2, p_{11}, n_{11}\). It is therefore realized in terms of \(4 \times 4\) bosonic matrices. It should be noticed that it can be taken \(e_1\) and \(p_{02}\) as the two simple positive roots of \(sp(4)\).

Let us introduce now the differential operator \(d\), nilpotent and fermionic, mapping functions into 1-forms; it turns out

\[
   d^2 = 0 \quad (4)
\]

In the case of one variable \(z\), \(d = dz \frac{\partial}{\partial z}\). Let us denote with \(G(z)\) the functions valued in the (super-) group \(G\), admitting \(\mathcal{G}\) as Lie (super-)algebra.

The \(\mathcal{G}\)-Lie algebra valued Cartan form is introduced through the position

\[
   \Omega =_{\text{def}} dG \cdot G^{-1} \quad (5)
\]

As a consequence of the above definition and the (\[\]) property of \(d\) we get the Maurer-Cartan equation

\[
   d\Omega - \Omega \cdot \Omega = 0 \quad (6)
\]

which can also be written, exploiting the Lie-algebraic properties, as

\[
   d\Omega - \frac{1}{2}[\Omega, \Omega]_+ = 0 \quad (7)
\]

Where the anticommutator is understood in the Lie-algebraic context (remember that \(\Omega\) is a Grassmann 1-form).

The above formula can be trivially extended to the multivariables case, as well as to the superspace formulation.
Let us introduce the $N = 1$ superspace with bosonic coordinate $z$ and fermionic $\theta$. The fermionic $D$ derivative is given by

$$D = \frac{\partial}{\partial \theta} + i \theta \partial_z \quad (8)$$

Therefore it follows that we can define a nilpotent Grassmann differential $d$

$$d = \text{def} \ (dz - i \theta d\theta) \partial_z + d\theta D \quad (9)$$

it is easily checked that $d^2 = 0$ (recall that $dz$ is Grassmann but now $d\theta$ is bosonic).

We can introduce as before a Cartan superform which still satisfies the Maurer-Cartan equation. Let us denote as $\tau^\alpha$ the (super-) generators of the (super-) Lie algebra $G$. We can therefore set

$$\Omega = (dz - i \theta d\theta) J + i d\theta \Psi \quad (10)$$

where $J, \Psi$ are $G$-valued:

$$J = J_\alpha \tau^\alpha = \text{def} \ \partial G \cdot G^{-1}$$

$$\Psi = \Psi_\alpha \tau^\alpha = \text{def} \ -i DG \cdot G^{-1} \quad (11)$$

As a consequence of the Maurer-Cartan equation satisfied by $\Omega$ the $J_\alpha$ superfields are not independent, but are constructed from the $\Psi_\alpha$ superfields:

$$J = D\Psi - i \frac{1}{2} [\Psi, \Psi]_+ \quad (12)$$

where the anticommutator is in the Lie algebra.

It deserves being noticed that $\Psi_\alpha$ have opposite statistics, while $J_\alpha$ have the same statistics of their corresponding $\tau^\alpha$ generators in $G$.

To be definite let us take for instance the $A_1 \equiv \text{sl}(2)$ algebra, admitting as generators $H, E_+, E_-$ ($\equiv \tau^0, \tau^+, \tau^-$ respectively) and structure constants given by

$$[H, E_\pm] = \pm 2 E_\pm$$

$$[E_+, E_-] = H \quad (13)$$

We obtain

$$J_+ = D\Psi_+ - 2i \Psi_0 \Psi_+$$

$$J_- = D\Psi_- + 2i \Psi_0 \Psi_-$$

$$J_0 = D\Psi_0 - i \Psi_+ \Psi_- \quad (14)$$
As discussed in our previous paper we can impose on the $N = 1$ affine $sl(2)$ algebra a superconformal constraint given by
\[ J_- = 1 \] (15)
which allows us imposing a further gauge-fixing
\[ J_0|_{\theta=0} = 0 \] (16)
Despite the fact that the above gauge-fixing is not manifestly supersymmetric it turns out to be indeed superconformal, for details see [7].

The above constraint and gauge-fixing can be explicitly solved in terms of the component fields entering the $\Psi_i$ superfields: Let
\[ \Psi_i = \xi_i(z) + \theta j_i(z) \] (17)
(here $i = 0, \pm$). In the $sl(2)$ case we are left with 3 fundamental unconstrained fields, two fermionic and one bosonic, given by $\xi_-, \xi_+$ and $j_+$, with spin dimension respectively $-(\frac{1}{2}), \frac{3}{2}$ and 2.

The remaining fields are expressed through these ones according to:
\[ j_- = 1 - i\partial \xi_- \cdot \xi_- \]
\[ \xi_0 = \frac{1}{2} \partial \xi_- \]
\[ j_0 = i \xi_+ \xi_- \] (18)

Let us now discuss the general procedure to impose superconformal constraints on the $N = 1$ affinization of a generic (super)-Lie algebra $G$ of rank $r$ with $n_b$ bosonic and $n_f$ fermionic simple roots (therefore $n_b + n_f = r$). In the following we will take letters from the middle of the alphabet, either Latin ($m, n$) or Greek ($\mu, \nu$), to denote respectively bosonic and fermionic simple roots and their corresponding generators obtained as commutators in the Cartan subalgebra (therefore $m, n = 1, 2, ..., n_b, \mu, \nu = 1, 2, ..., n_f$).

The set of constraints is imposed through the following positions:
\[ i) \text{ set } \Psi_a = 0 \text{ all superfield associated to every negative non-simple root (deg}(\tau^a) < -1). \]
\[ ii) a) \text{ set } \Psi_{-\mu} = 1 \text{ all superfields associated to a fermionic negative simple root } \tau^{-\mu}. \]
\[ ii) b) \text{ set } J_{-m} = 1 \text{ all composite superfields (14) associated to a bosonic negative simple root.} \]

The above constraints allows to impose further gauge-fixing restrictions which as before turn out to be superconformal. For the Cartan sector we get
\[ \Psi_{0,\mu} = 0 \text{ for any } \mu = 1, ..., n_f \]
\[ J_{0,m}|_{\theta=0} = 0 \text{ for any } m = 1, ..., n_b \] (19)
Further gauge-fixing conditions can be imposed on the positive-root sector. We will not specify them in the general case, we will present them just for the $osp(1|4)$ superalgebra.

Due to the constraints and gauge-fixings we have that

$$J_{-m} = D\Psi_{-m} + i \sum_{n=1,\ldots,n_b} a_{nm} \Psi_{0,n} \Psi_{-m}$$

(with $a_{nm}$ elements of the Cartan matrix), while

$$J_{0,m} = D\Psi_{0,m} - i\Psi_{+m} \Psi_{-m}$$

In the particular $osp(1|4)$ case, as already mentioned, we deal with one positive bosonic simple root ($e_1$) and one fermionic ($e_2$). It turns out that a consistent gauge-fixing condition on the positive sector is given by the condition that all superfields $\Psi_{p,kl}$ are set equal to zero apart $\Psi_{p,02}$, $\Psi_{p,22}$ associated to a $deg = +2, +4$ root respectively, as well as $\Psi_{p,12}$ which however is subjected to the gauge-condition $J_{p,12} = 0$ (the corresponding root has degree $deg = +3$).

The above system of constraints and gauge-fixings can be explicitly solved in terms of a set of fundamental unconstrained component fields. It turns out that we are left with 5 surviving fields (3 fermionic and 2 bosonic) with spin dimension given by

$$(-\frac{1}{2}, \frac{3}{2}, 2, \frac{5}{2}, 4)$$

They are respectively denoted as

$$(\xi_{f1}, \xi_{p,02}, \xi_{p,22})$$

The remaining field components entering the superfields $\Psi_i$ are expressed through the above ones via the relations:

$$\begin{align*}
\dot{j}_{f1} &= 1 - i\partial \xi_{f1} \cdot \xi_{f1} \\
\xi_{h1} &= \left(\frac{1}{2}\right)\partial \xi_{f1} \\
\dot{j}_{p,12} &= i\xi_{f1} \partial \xi_{p,12} \\
\xi_{p,22} &= \xi_{f1} \dot{j}_{p,22} + \left(\frac{1}{2}\right)(1 + i\partial \xi_{f1} \cdot \xi_{f1})\dot{\partial} \xi_{p,12}
\end{align*}$$

(22)

All the other component fields are vanishing apart from

$$\xi_{f2} = 1$$
In the above formulas the meaning of the indices is obvious.

As already mentioned such set of constraints and gauge-fixings is superconformal. This statement could be proven explicitly working at the purely Lie-algebraic level we have discussed so far. However the computations, even if straightforward, are rather cumbersome and it is for that reason more convenient to discuss such topics in the context of the relation between our Lie-algebraic framework and the Lie-group properties (this connection is what we need indeed to define super-Toda models).

**Super–Toda models.**

Let us come back to the formula which relates the superfields $\Psi_i$ to the functions on the group:

$$\Psi = \Psi_\alpha \tau^\alpha = \text{def} -iDG \cdot G^{-1}$$

According to such a formula the superfield currents in $\Psi$ can be expressed in terms of what we can call the Gauß superfields entering the Gauß decomposition of the group-valued function $G$:

$$G = M \cdot \Phi \cdot N$$

where $\Phi$ is given by the Cartan (0-grading) sector of $G$

$$\Phi = e^{\sum_{i=1,...,r} \Phi_i h_i}$$

and $M$ ($N$) are strictly upper (lower) triangular matrices (we can assume working in a given, let’s say the fundamental, representation for the group). Therefore

$$M = 1 + M_+ + ...$$

and similarly

$$N = 1 + N_- + ...$$

with $M_+$ ($N_-$) representing the contribution from the $\text{deg} = +1$ ($\text{deg} = -1$) elements of the triangular matrices, while the dots represent the contributions from the higher (lower) grading elements.

The WZNW theory can be regarded as a two-dimensional field theory which, in the supersymmetric case, is determined by the group-valued fields $G(Z, \overline{Z})$ which, besides the chiral $N=1$ superspace coordinate $Z \equiv z, \theta$, depends also on the antichiral superspace coordinate $\overline{Z} \equiv \overline{z}, \overline{\theta}$. An antichiral fermionic derivative $\overline{D}$,

$$\overline{D} = \frac{\partial}{\partial \overline{\theta}} + i\overline{\theta} \partial_{\overline{z}}$$

(24)
should be introduced.

Besides $\Psi$ an antichiral $G$-valued superfield $\overline{\Psi}$ can be defined

$$\overline{\Psi} = \overline{\Psi}_\alpha \tau^\alpha = \text{def} \ iG^{-1}\overline{DG}$$

(25)

The equations of motion for the unconstrained WZNW model on $G$ are simply

$$\overline{D} \Psi = D \overline{\Psi} = 0$$

(26)

In terms of the component fields the above equation implies that the fields entering $\Psi$ are chiral, those entering $\overline{\Psi}$ antichiral.

From the above positions and equations of motion we can extract the constrained WZNZ theory, obtained with the imposition of the above-discussed superconformal constraints and gauge-fixings on the chiral supercurrents $\Psi$ (and a similar set on the antichiral supercurrents $\overline{\Psi}$).

As a result the constrained WZNW theory turns out to be equivalent to a set of free-field equations expressing chirality (antichirality) conditions on the set of fields solving constraints and gauge-fixings, so for instance we get in the $N = 1$ constrained $sl(2)$ case the free equations of motion

$$\frac{\partial}{\partial z} \xi_-(z) = \frac{\partial}{\partial z} \xi_+(z) = \frac{\partial}{\partial z} j_+(z) = 0$$

(27)

and analogous equations for the antichiral fields.

These free equations are translated into non-trivial non-linear equations for the Gauß fields entering $G$. Let us present here for completeness the relations between superfields in the unconstrained WZNW model based on $sl(2)$.

In this case we have

$$G = e^{ME^+}e^{PH}e^{NE^-}$$

(28)

and we get

$$
\begin{align*}
\Psi_+ & = -i(DM - 2MD\Phi - M^2DNe^{-2\Phi}) \\
\Psi_0 & = -i(D\Phi + MDNe^{-2\Phi}) \\
\Psi_- & = -i(DNe^{-2\Phi})
\end{align*}$$

(29)

(there are similar equations for the antichiral superfields).

It should be noticed that when discussing the dynamics of the constrained supersymmetric WZNW we have a certain freedom in deciding which fields should be assumed to be the fundamental ones. We can for instance express all the dynamics
in terms of the Gauß fields, or to assume the free currents as the fundamental fields (in this case the dynamics of the theory gets trivial), but we have also the freedom to perform a mixed choice, assuming some of the Gauß fields and some of the currents as fundamental fields. It is indeed convenient to make such a choice as it will be clear later.

Let us discuss now the equations of motion for the constrained WZNW models. Let

\[ \Psi = \Psi_\prec + \Psi_\prec + \Psi_\succ \]

where \( \Psi_\prec, \Psi_\prec, \Psi_\succ \) denote the projections of \( \Psi \) onto the negative roots, the Cartan subalgebra and the positive roots respectively. Since

\[ \Psi = -i(DM \cdot M^{-1} - MD\Phi \cdot \Phi^{-1}M^{-1} + M\Phi DN \cdot N^{-1} \Phi^{-1}M^{-1}) \]

we get that negative-root terms are obtained only from the third term. Defined \( K \) as

\[ K =_{def} -i\Phi DN \cdot N^{-1} \Phi^{-1} \]

and taking into account the constraint on negative non-simple roots we get

\[ \Psi_\prec = K \]

For a generic (super-)Lie algebra we have, due to the constraint

\[ K = \sum_{\mu=1,\ldots,n_f} \tau^{-\mu} + \sum_{m=1,\ldots,n_b} \Psi_{-m} \tau^{-m} \]

The superfields \( \Psi_{-m} \) are chiral, but constrained. In the \( osp(1|4) \) case we get for instance

\[ \Psi_\prec = \chi f_2 + f_1 \]

(for simplicity we have set \( \chi =_{def} \Psi_{f_2} \)).

It therefore turns out that

\[ \Psi_\prec = -\Phi DN \cdot N^{-1} + [M_+, K] \]

Applying to the above relation the (chiral) equations of motion for \( \Psi \) we get that the Cartan superfields satisfy

\[ -i\overline{D}D( \sum_{i=1,\ldots,r} \Phi_i h_i) + [\overline{D}M_+, K]^+_+ = 0 \]
In order to get the equations of motion for the Cartan superfields in closed form we need to specify $\mathcal{D}M_+$; this can be done by looking at the antichiral supercurrents, repeating the same steps as before. At the end we get, for the $osp(1|4)$ case

$$\mathcal{D}M_+ \propto e^{\Phi_{h_2} - \Phi_{h_1}}e_2 + \overline{\chi}e^{2\Phi_{h_1} - \Phi_{h_2}}e_1$$

(38)

and are led to the following set of equations of motion:

$$D\chi + 2D\Phi_1\chi = 1$$
$$\mathcal{D}\chi + 2\mathcal{D}\Phi_1\chi = 1$$
$$\mathcal{D}D\Phi_1 = e^{2\Phi_1 - \Phi_2}\overline{\chi}\chi$$
$$\mathcal{D}D\Phi_2 = ie^{\Phi_2 - \Phi_1}$$

(39)

(for simplicity we have set $\Phi_{h_{1,2}} \equiv \Phi_{1,2}$). A normalization choice, which implies in particular the appearance of the $i$ in the r.h.s. has been taken in order to guarantee the reality of the superfields $\Phi_1, \Phi_2$.

It should be noticed that the above set of equations can be considered as a subset of the equations of motion for the constrained WZNW model. Indeed this is a closed set of equations involving the Cartan (i.e. Gauß superfields) $\Phi_{1,2}$ together with the supercurrents $\chi, \overline{\chi}$. However, as it will be clear when performing the analysis in terms of field components, some extra fields solving the WZNW constraints, do not enter the above superfields (these are the spin $\frac{3}{2}$ fields satisfying the free (chiral and antichiral) equations. These considerations are particularly important when performing the hamiltonian derivation of the constrained WZNW system. Indeed the above equations alone can not be derived from a hamiltonian.

The reason why some extra fields are present is simply due to the fact that in the above derivation we have not used all properties and equations in our theory (only the zero-grading component equations for the Cartan superfields have been taken into account).

Before going ahead with the presentation of the above system in terms of component fields, let us present here their superconformal properties.

For a single chirality let us consider the infinitesimal local transformations

$$\theta \mapsto \hat{\theta} = \theta + \epsilon_f + \theta \partial_z \epsilon_b$$
$$z \mapsto \hat{z} = z + \epsilon_b + i\theta \epsilon_f$$

(40)

parametrized by the infinitesimal bosonic and fermion functions $\epsilon_b, \epsilon_f$ which can be accomodated in the infinitesimal superfield $\Lambda = i\theta \partial_f z + \partial \epsilon_b$. It can be explicitly checked that the equations (38) are invariant under the following transformations

$$\Phi_1(Z) \mapsto \hat{\Phi}_1(\hat{Z}) = \Phi_1(Z) - 3\Lambda - \frac{5}{2}D\Lambda\chi$$
\[
\Phi_2(Z) \mapsto \hat{\Phi}_2(\hat{Z}) = \Phi_2 - 4\Lambda - \frac{5}{2} D\Lambda \chi
\]
\[
\chi(Z) \mapsto \hat{\chi}(\hat{Z}) = \chi(Z) + \Lambda \chi
\]
\[
D_Z \mapsto \hat{D}_Z = D_Z - \Lambda D Z
\]

(41)

In component fields we have

\[
\chi(Z) = \xi(z) + \theta j(z)
\]
\[
\overline{\chi}(Z) = \overline{\xi}(\overline{z}) + \overline{\theta} j(\overline{z})
\]
\[
\Phi_k(Z, \overline{Z}) = \phi_k + \theta \psi_k + \overline{\theta} \overline{\psi}_k + \theta \overline{\theta} B_k \text{ for } k = 1, 2
\]

(42)

where \(\xi, \overline{\xi}, \psi_k, \overline{\psi}_k\) are fermionic, \(j, \overline{j}, \phi_k, B_k\) bosonic; moreover all of them are real.

Due to the equations of motion it turns out that \(j, \overline{j}, \phi_k, B_k\) are algebraically determined from the remaining fields and their derivatives:

\[
j = 1 - i \partial \xi \cdot \xi
\]
\[
\overline{j} = 1 - i \overline{\partial} \overline{\xi} \cdot \overline{\xi}
\]
\[
\psi_1 = \left(\frac{1}{2}\right) \partial \xi + \partial \phi_1 \cdot \xi
\]
\[
\overline{\psi}_1 = \left(\frac{1}{2}\right) \overline{\partial} \overline{\xi} + \overline{\partial} \overline{\phi}_1 \overline{\xi}
\]
\[
B_1 = -ie^{\phi_1 - \phi_2} \xi
\]
\[
B_2 = e^{\phi_2 - \phi_1}
\]

(43)

The closed system of equations is given by the following set:

\[
\overline{\partial} \xi = 0
\]
\[
\partial \overline{\xi} = 0
\]
\[
\overline{\partial} \psi_2 = e^{\phi_2 - \phi_1} (\psi_2 - \left(\frac{1}{2}\right) \overline{\partial} \overline{\xi} - \overline{\partial} \phi_1 \cdot \overline{\xi})
\]
\[
\partial \overline{\psi}_2 = -e^{\phi_2 - \phi_1} (\overline{\psi}_2 - \left(\frac{1}{2}\right) \partial \xi - \partial \phi_1 \cdot \xi)
\]
\[
\Box \phi_1 = -ie^{\phi_1 \overline{\xi}} e^{-\phi_1 - \phi_2} (1 - i \psi_2 \xi) (1 - i \overline{\psi}_2 \overline{\xi})
\]
\[
\Box \phi_2 = -e^{2(\phi_2 - \phi_1)} + ie^{\phi_1 \overline{\xi}} + ie^{\phi_2 - \phi_1} (\psi_2 - \left(\frac{1}{2}\right) \partial \xi - \partial \phi_1 \cdot \xi)(\overline{\psi}_2 - \left(\frac{1}{2}\right) \overline{\partial} \overline{\xi} - \overline{\partial} \phi_1 \cdot \overline{\xi})
\]

(44)

Concerning the above equations some remarks are in order: the system as already discussed is superconformal. Due to the non-linear transformation properties the supersymmetry is spontaneously broken. Setting \(\xi = \overline{\xi} = 0\) implies reducing the supersymmetric system to a system based on standard (not superfields) fields valued on \(osp(1|4)\) which does not admit supersymmetry. Introducing the \(\xi, \overline{\xi}\) fields allows constructed an enlarged supersymmetric system.
When setting all fermionic fields $\psi_2, \bar{\psi}_2, \xi, \bar{\xi}$ equal to zero we are led with a coupled system of bosonic equations for $\phi_1, \phi_2$ which is nothing else than the Toda model associated to the $sp(4)$ bosonic subalgebra of $osp(1|4)$.

Moreover since $osp(1|4)$ bosonic subalgebra $osp(1|2)$ as subalgebra(associated to the standard superLiouville theory) we can ask under which limit the superLiouville theory can be recovered from our system. This is just obtained as follows: at first one has to set $\xi, \bar{\xi} = 0$, and $\phi = \text{def} \phi_2 - \phi_1$. Realizing that the limit $\phi_1 \to -\infty$ is a solution of the above equations (with the conditions $\xi = \bar{\xi} = 0$), then the closed system involving $\phi, \psi_2, \bar{\psi}_2$ is nothing else than the superLiouville theory (it should be noticed that $\phi_1$ should decouple in this case because it is associated to the bosonic simple root which does not belong to the $osp(1|2)$ subalgebra of $osp(1|4)$).

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