EQUIDISTRIBUTION OF CLOSED GEODESICS ALONG RANDOM WALK TRAJECTORIES WITH RESPECT TO THE HARMONIC INVARIANT MEASURE

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Abstract. We prove that for suitable random walks on isometry groups of CAT(-1) spaces, typical sample paths eventually land on loxodromic elements which equidistribute with respect to a flow invariant measure on the unit tangent bundle of the quotient space.

1. Introduction and statement of results

Let \((X,d)\) be a CAT(-1) space and \(G < Isom(X)\) a nonelementary discrete subgroup. Let \(M = X/G\). Let 
\[T^1X = \partial^2 X \times \mathbb{R}\]
and
\[T^1M = T^1X/G\]
the unit tangent bundles of \(X\) and \(M\) respectively. Let \(\pi : T^1X \to X\) be the canonical projection. Let \(\tilde{g}_t\) be the geodesic flow on \(T^1X\) and \(g_t\) the geodesic flow on \(T^1M\). Let \(\tilde{d}_T\) be an \(Isom(X)\) invariant metric on \(T^1X\) satisfying
\[c^{-1}d(x,y) \leq \inf_{p \in \pi^{-1}x, q \in \pi^{-1}y} \tilde{d}_T(p,q) \leq cd(x,y).\]
For \(x \in X\) and \(\zeta \in X \cup \partial X\) let \(\gamma_{x,\zeta}\) be the unit speed geodesic ray from \(x\) in the direction of \(\zeta\), considered as a subset of \(T^1X\). The translation length of \(g \in G\) is defined as \(l(g) = \inf_{x \in X} d(x,gx)\). If \(l(g) > 0\) then the infimum is realized and \(g\) is called a loxodromic isometry of \(X\). The set of points in \(X\) realizing the infimum is a geodesic \(s_g\) in \(X\), and can be naturally considered a subset of \(T^1X\). Let \(\gamma = \gamma_g\) be the associated unit speed closed geodesic on \(M = X/G\) and \(D_\gamma = D_g\) the arclength Lebesgue measure on \(\gamma_g\), considered as a measure on \(T^1M\) normalized to unit mass. The Bowen-Margulis measure \(m\) on \(T^1M\) is the geodesic flow invariant measure corresponding to the Patterson-Sullivan geodesic current on \(\partial X \times \partial X\) (see [7] for details).

Roblin [7] proved the following equidistribution result for closed geodesics on \(M\) with respect to \(m\). More precisely, assume that \(m\) is finite, and normalized to have unit mass. Let \(L_R\) be the set of closed geodesics on \(M\) with length at most \(R\).
Then the measures on $T^1M$ defined by
$$L_R = \frac{1}{hR} \sum_{\gamma \in L_R} D_\gamma$$
weakly converge to $m$ as $R \to \infty$.

In this note, we prove a different sort of equidistribution result: along typical trajectories of random walks on $G$, closed geodesics associated to loxodromic elements equidistribute with respect to the "harmonic invariant measure."

Let $\mu$ be a probability measure on $G$ whose support generates $G$ as a semigroup. Let $\hat{\mu}(g) = \mu(g^{-1})$ be the reflected measure of $\mu$.

Let $\mu^N$ be the product measure on $G^N$. Let $T : G^N \to G^N$ be the transformation that takes the one-sided infinite sequence $(h_i)_{i \in \mathbb{N}}$ to the sequence $(\omega_i)_{i \in \mathbb{N}}$ with
$$\omega_n = h_1 \cdots h_n.$$

Let $P$ be the pushforward measure $T_\ast \mu^N$.

The measure $P$ describes the distribution of $\mu$ sample paths, i.e. of products of independent $\mu$-distributed increments.

The following results were proved by Maher-Tiozzo [6] in the more general setting of geodesic Gromov hyperbolic spaces.

**Proposition 1.1.** For any $x \in X$ and $P$ a.e. sample path $\omega = (\omega_n)_{n \in \mathbb{N}}$ of the random walk on $(G, \mu)$, the sequence $(\omega_n x_0)_{n \in \mathbb{N}}$ converges to a point $\omega_+ = bnd \omega \in \partial X$. If in addition $\mu$ has finite first moment with respect to the metric $(X, d)$, then there exists $L > 0$ such that for $P$-a.e. sample path $\omega$ and for every $x_0 \in X$ one has
$$\lim_{n \to \infty} \frac{d(X(x_0, \omega_n x_0))}{n} = L$$

The measure $\nu = bnd_\ast P$ is the unique $G$ stationary measure on $\partial X$.

Moreover, Tiozzo proved that sample paths sublinearly track geodesics.

**Proposition 1.2.** Assume $\mu$ has finite first moment with respect to $(X, d)$. For $P$-a.e. sample path $\omega$ and every $x \in X$ one has
$$\frac{d(\omega_n x, \gamma x, \omega_+)}{n} \to 0.$$  

Moreover, Dahmani-Horbez [2] proved (again in the more general setting of Gromov hyperbolic spaces) that a typical sample path eventually lands on loxodromic elements which pass close to the basepoint.

**Proposition 1.3.** Let $G \acts X$ be a nonelementary discrete action on a CAT($-1$) space and $\mu$ a measure with finite second moment on $G$ whose support generates $G$ as a semigroup. Then:

a) For $P$-a.e. $\omega \in G^N$

$$L = \lim_{n \to \infty} \frac{l(\omega_n)}{n}.$$
In particular $\omega_n$ is loxodromic for large enough $n$.

b) For any $c > 0$, for all $\varepsilon > 0$ and for $P$-a.e. sample path $\omega$ there is an $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$, $\gamma(\varepsilon Ln, (1 - \varepsilon)Ln)$ is contained in the $c$ neighborhood in $T^1X$ of the axis of $\omega_n$ where $\gamma$ is the unit speed geodesic from a basepoint $o \in X$ to $\omega_+ \in \partial X$.

Proof. a) This is a special case of [2, Theorem 0.2].

b) The corresponding statement for some $c > 0$ is a special case of [2, Proposition 1.9]. We then can obtain it for arbitrarily small $c > 0$ by using the following fact, which follows from comparison geometry: for any $c_1 > c_2 > 0$ there is a $t_0 > 0$ such that whenever $\gamma_1$ and $\gamma_2$ are parametrized unit speed geodesics in a $CAT(-1)$ space such that $d(\gamma_1(t), \gamma_2(t)) \leq c_1$ for all $|t| < N$ we have $d(\gamma_1(t), \gamma_2(t)) \leq c_2$ for all $|t| < N - t_0$. □

In this note, we are interested in the asymptotic behavior, as $n \to \infty$ of the closed geodesic corresponding to the loxodromic element $\omega_n$ along typical sample paths of the random walk. In particular, we will show they equidistribute with respect to a certain measure on the unit tangent bundle of $M$.

Let $\nu$ and $\hat{\nu}$ be respectively the $\mu$ and $\hat{\mu}$ stationary probability measures on $\partial X$ respectively.

The action $G \acts X \times X$ preserves the measure class of $\nu \times \nu$ and is ergodic with respect to it [5].

We will make the following assumption about $G$ and $\mu$.

Axiom 1.4. There exists a $G$ invariant and geodesic flow invariant Radon measure $\tilde{m}$ on the unit tangent bundle

$$T^1X = \partial^2X \times \mathbb{R},$$

in the measure class of $\hat{\nu} \times \nu \times \text{Leb}$ (where $\text{Leb}$ is the Lebesgue measure on $\mathbb{R}$) which projects to a probability measure $m$ on the unit tangent bundle $T^1M = T^1X/G$ of $M$. We call $m$ the harmonic invariant measure of $(G, \mu)$.

Axiom 1.4 is known to hold in the following cases.

- When $\mu$ is the discretization of Brownian motion on the universal cover of a compact manifold of negative curvature (Anderson-Schoen [1]).
- When $G < Isom(X)$ is convex cocompact and $\mu$ has finite support. ([Kaimanovich [4, Theorem 3.1]).
- When $G < Isom(X)$ is geometrically finite and $\mu$ has finite support (Gekhtman-Gerasimov-Potyagailo-Yang [3, Theorem 1.8]). In this setting, when $G$ contains parabolic elements [3, Theorem 1.5] implies that the harmonic invariant measure is singular to the Bowen-Margulis measure.

Recall a family $\sigma_n$ of Borel probability measures on a metrizable space $Z$ weakly converges to a measure $\sigma$ on $Z$, if for every bounded continuous
function \( g \) on \( Z \),
\[
\int g d\sigma_n \to \int g d\sigma.
\]
By the Portmanteau theorem, this is equivalent to: for any Borel \( A \subset Z \),
\[
\limsup \sigma_n(A) \leq \sigma(A)
\]
and to: for any Borel \( A \subset Z \),
\[
\liminf \sigma_n(A) \geq \sigma(\text{int}A)
\]
where \( \text{int}A \) is the interior of \( A \).

The goal of this note is to prove the following.

**Theorem 1.5.** Suppose \( X \) is a proper \( \text{CAT}(-1) \) space, \( G < \text{Isom}(X) \) is a nonelementary discrete subgroup, and \( \mu \) a measure on \( G \) with finite second moment with respect to \( (X,d) \) and satisfying Axiom [\ref{axiom}] Then for \( P \)-almost every sample path \( \omega \in G^\mathbb{N} \) the measures \( D_{\omega,n} \) weakly converge to the harmonic invariant measure \( m \) as \( n \to \infty \).

2. Proof of Theorem [\ref{axiom}]

For a Borel subset \( A \subset T^1X/G \) let \( \tilde{A} \) be the \( G \) invariant preimage of \( A \) in \( T^1X \). Let \( N_rA \) and \( I_rA \) the \( r \) neighborhood and \( r \) interior of \( A \) respectively, with respect to the metric \( d_T \). The measure \( \tilde{\nu} \times \nu \) is known to be ergodic with respect to the \( G \) action, and hence \( m \) is ergodic with respect to the geodesic flow (see [\ref{4}]).

Thus the Birkhoff ergodic theorem implies
\[
\frac{1}{T}|\{t \in [0,T] : g_tq \in A\}| \to m(A)
\]
for \( m \)-a.e. \( q \in T^1M \).

Consequently,
\[
\frac{1}{T}|\{t \in [0,T] : \tilde{g}_tq \in \tilde{A}\}| \to m(A)
\]
for \( \tilde{m} \)-a.e. \( q \in T^1X \). Consequently, since geodesics converging to the same boundary point are asymptotic, we have that for any \( r > 0 \), for \( P \) a.e. \( \omega \in G^\mathbb{N} \), any limit point as \( T \to \infty \) of
\[
\frac{1}{T}|\{t \in [0,T] : g_{\omega,t}q \in \tilde{A}\}|
\]
is bounded between \( m(I_rA) \) and \( m(N_rA) \). In particular for large \( n \) we have
\[
m(I_{2r}A) \leq \frac{1}{Ln}|\{t \in [0,Ln] : g_{\omega,t} \in \tilde{A}\}| \leq m(N_{2r}A)
\]
and thus
\[
m(I_{2r}A) - 2r \leq \frac{1}{Ln}|\{t \in [rLn,(1-r)Ln] : g_{\omega,t} \in \tilde{A}\}| \leq m(N_{2r}A)
\]
By Proposition \[\ref{prop1.2}\] \(d(\omega_n^0, \gamma_0, \omega_n^+ (Ln))/n \to 0\), so since \(X\) is CAT\((-1)\) we have for all large enough \(n\):

\[d(\gamma_0, \omega_n^+ (t), \gamma_0, \omega_n(t)) \leq r\]

for all \(0 \leq t \leq (1 - r)Ln\). Thus,

\[m(I_{3r}A) - 2r \leq \frac{1}{Ln} \{ t \in [rLn, (1 - r)Ln] : \gamma_0, \omega_n^+ (t) \in \tilde{A} \} \leq m(N_{3r}A)\]

Thus, for a suitable unit speed parametrization of the axis \(s = s_{\omega_n}\) of \(\omega_n\), Proposition \[\ref{prop1.5}\] implies for large enough \(n\):

\[m(I_{3r}A) - 3r \leq \frac{1}{Ln} \{ t \in [rLn + r, (1 - r)Ln - r] : s(t) \in \tilde{A} \} \leq m(N_{3r}A)\]

and thus

\[m(I_{3r}A) - 3r \leq \frac{1}{Ln} \{ t \in [0, Ln] : s(t) \in \tilde{A} \} \leq m(N_{3r}A) + 3r\].

Since \(l(\omega_n)/n \to L\) this implies for large enough \(n\):

\[m(I_{3r}A) - 3r \leq \frac{1}{l(\omega_n)} \gamma_{\omega_n} \cap A \leq m(N_{3r}A) + 3r\].

Letting \(r \to 0\) gives

\[m(int(A)) \leq \lim\inf_{n \to \infty} \frac{1}{l(\omega_n)} |\gamma_{\omega_n} \cap A|\]

and

\[\lim\sup_{n \to \infty} \frac{1}{l(\omega_n)} |\gamma_{\omega_n} \cap A| \leq m(\overline{A})\]

completing the proof.

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### References

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