Ultrametric dynamics for the closed fractal-cluster resource models

V.T. Volov

Physics Department, Samara State University of Railway Transport,
Samara, Russia, e-mail: vtvolov@mail.ru

A.P. Zubarev

Physics Department, Samara State University of Railway Transport,
Samara, Russia, e-mail: apzubarev@mail.ru

(Dated: May 11, 2014)

The evolutilonal scenario of the resource distribution in the fractal-cluster systems which is identified as an "organism" has been suggested. We propose a model in which the resource redistribution dynamics in the closed system is determined by the ultrametric structure of the system’s space. Moreover, each cluster has its own character time of a transfer to the equilibrium state which is determined by the ultrametric size of the cluster. The general equation which determines this dynamics has been written. For the determined type of the resource transitions among clusters, the solution to this equation has been numerically received. The problem of the parameter’s identification modeling for the real systems has been discussed.

I. INTRODUCTION

The progress of mathematics for modeling systems which have a visible or hidden hierarchical structure is important for the wide class of systems and processes in different spheres of physics, biology, economics and sociology: the spin glasses, biopolymers, the theory of optimization, the taxonomy, the evolutionary biology, the cluster and factor analysis and etc. [1, 2].

In fact any biological or socio-economic system has an evident hierarchical character of interaction among its subsystems and, this way, carries inside itself elements of the hierarchical structure. If the initial objects or states of the system have the hierarchical
structure then the structure is visible. However, all systems with a hidden hierarchical structure present more interest for the researchers. In these systems the hierarchical structure is not sought in the initial variables, but they have been observed after transition to some effective variables. As a rule, the number of these effective variables is essentially smaller than the number of the degrees of freedom for the whole system. Well-known members of the same systems are spin glasses [3], proteins [4–6]. There are reasons for the identification of hierarchical structures among the social-economical systems [7–9].

Adequate mathematical apparatus for the modelling systems which can be both visible and hidden hierarchical structures is the ultrametric analysis [2, 10–13].

One of the interesting facts which we have observed on the empirical researches, which relates to the reasonably wide class of complex natural and artificial genetic systems, is the certain system’s type being manifested. These systems have a strongly marked hierarchical structure for the recourse distribution on the functional indication [14]. It has been observed in a range of the systems (biological, technical, social) which have been evolutied for billions of years, but are now in a stable functioning state. These systems have five basic subsystems (cluster) which can be classified with their target functions: energy, transport, technological, ecological and informational clusters, where each of the ones have a certain share of "the resource". For such systems, the special term – "an organism" has been determined. For the social-economical systems which belong to the special class of "an organism" and it is these systems which have the most interest for the investigation in this article. The clusters will be identified on the target distribution for the extensive parameters as staff, industry funds, financial activities etc.

For the range of systems which relate to the class of "the organisms", each cluster can present itself so the functioning subsystem which is "an organism" too. Ergo we can divide the one to five subsystems (subclusters), which have the same target functions for each system inside itself of the one. For example, the resource in the energetic cluster can be to share for the own energetic, transport, ecology, technological and informational supporting. Such a decomposition can be extended to subclusters. Namely, each of the subclusters of given level can be considered as the union of the five high level subclusters. Such systems are called fractal-cluster systems (see [15–17]). Thus the space of the resource distribution for the same systems has an hierarchical structure which can be dis-
cribed by the hierarchical tree with a fixed number of branches $p=5$. It should be noted that identification of the subcluster in the particular fractal-cluster system depends on the type of "the resource", which has not always had a clear relationship to the observed characteristics of the system. Practically under the fractal-cluster modeling of the real social-economical, biological, technological and other systems of this type we have the possibility to describe only two or three hierarchical levels of the ones of the clusterization. Even such a classification of the resource distribution allows the production of the functioning estimation effectiveness for any real complex systems. Nevertheless, by using the mathematical apparatus, we can research the resource dynamics in the abstract systems with an infinite number of nested clusters (subclusters).

Statistical analysis of the empirical data on the resource distribution in the complex social-economical, technological, biological and other systems of the ones type allows to determine the ideal values for the resource distribution in the first level clusters \cite{15–17}. Under these values, the system’s development will be stable and more energy efficient. First of all, in the anthropological and technological systems, the statistical analysis allows to obtain the average values of the ideal distribution. These values for energetical, transport, ecological, technological and informational clusters are equal 0.38, 0.27, 0.16, 0.13, 0.06 respectively. For the system of another nature (biological, social and economical) these ideal volumes have approximately the same values.

The possibility of using the fractal-cluster models for the resource distribution analysis is based on the researches of the range investigations \cite{15–17}, in which researches the optimal resource distribution control methods had been presented. These methods are based on the formal analogies with the thermodynamic method in its informational interpretation. Such an approach has allowed us to receive the solutions close to the ideal resource distribution state. However, we have an open question – what is the way the same systems which are placed out the equilibrium (the ideal state) go to the ideal state soon. In this article we suggest the scenario of the closed fractal-cluster system’s evolution from the arbitrary state to the ideal state. We have postulated that dynamics of the resource redistribution in the closed systems under the condition of the external control factors absence is wholly determined by the ultrametric structure of the fractal-cluster space inside of the one this resource is allocated. It has been shown that for each cluster
we have own characteristic time transition to the ideal substate. The characteristic time transition is determined by the ultrametric size of the cluster (subcluster). From the beginning, the subclusters of the highest levels transfer to the ideal state, then the ones of more lower level make the same action and so on. This hierarchical structure of the characteristic transition time of all subclusters to the ideal substates has been determined for the whole system’s dynamics. The general equation which describes such dynamics has been written. For certain types of resource transitions among the clusters we have numerical researches for solving this equation. Also we have discussed the problem of the parameter model’s identification for the real systems.

II. MODEL OF RESOURCE DISTRIBUTION DYNAMICS FOR THE CLOSED FRACTAL-CLUSTER SYSTEMS

We are considering a graph which is the \( n \)-level hierarchical tree with the one root vertex, which is the center of the graph. Let \( p \) be the index of the tree branching, \( n \) is the hierarchical level number.

![Hierarchical tree](image)

Figure 1: Hierarchical tree which corresponds to the two level the fractal-cluster system. Here \((a_1, a_2), a_1, a_2 = 1, \ldots, p, p = 5\) is the boundary points (the second level clusters) tree parameterization.

The sample of this tree has presented on the fig. II (here \( p = 5, n = 2 \)). The ensemble of the boundary points tree has indentificated as \( U_n \). Let \( x \) is a point of the boundary. Then assignment \( x \) is equivalent to setting

\[ x = (a_1, a_2, \ldots, a_n), \]

where \( a_i = 1, \ldots, p, i = 1, \ldots, n \).
On the $U_n$ the ultrametric distance $d$ is putting. Namely, for any two points $x = (a_1, a_2, \ldots, a_n)$ and $y = (b_1, b_2, \ldots, b_n)$ one has:

$$d(x, y) = d(a_1, a_2, \ldots, a_n|b_1, b_2, \ldots, b_n) = p^{n-\gamma+1}.$$ 

Here $\gamma$ is determined by comparing the sets $(a_1, a_2, \ldots, a_n)$ and $(b_1, b_2, \ldots, b_n)$: if $a_1 = b_1$, $a_2 = a_2, \ldots, a_{j-1} = b_{j-1}, a_j \neq b_j$, then it’s took $\gamma = j$. For the similar elements this distance equals to zero.

As it is known [10, 11], any element $x$ of the $p$-adic numbers field $\mathbb{Q}_p$ can be presented in the following way $x = p^{-\gamma} (b_0 + b_1p + b_2p^2 + \cdots + b_np^n + \cdots)$, where $\gamma \in \mathbb{Z}$, $b_0 = 1, \ldots, p-1$, and for $i \neq 0$ $b_i = 0, \ldots, p-1$. $p$-adic norm of the element $x$ is given as $|x|_p = p^\gamma$. For two elements $x, y \in \mathbb{Q}_p$ distance between them is the ultrametric and is determined as $d(x, y) = |x - y|_p$. The subset $B_n = \{x \in \mathbb{Q}_p : |x|_p \leq p^n\}$ named the ball of radius $p^n$ in $p$-adic numbers $\mathbb{Q}_p$. The subset $B_n/B_0$ consists from the elements of view $p^{-n} (b_0 + b_1p + b_2p^2 + \cdots + b_{n-1}p^{n-1})$. It has one-to-one correspondence between the subset $B_n/B_0$ of the $p$-adic numbers field $\mathbb{Q}_p$ and the set of boundary points of the tree $U_n$:

$$U_n \ni (a_1, a_2, \ldots, a_n) \longleftrightarrow p^{-n} \left( b_0 + b_1p + b_2p^2 + \cdots + b_{n-1}p^{n-1} \right) \in B_n/B_0,$$

$$b_0 = a_1 - 1, b_1 = a_2 - 1, \cdots b_{n-1} = a_n.$$ 

The $i$-level cluster, $i = 1, \ldots, n$, will be to name the subsets of the set $U_i$, $i = 0, 1, \ldots, n-1$, such that $\forall x, y \in U_n d(x, y) \leq p^i$. Any point $x \equiv (a_1, a_2, \ldots, a_n) \in U_n$ is the cluster $U_0 = U_0(a_1, a_2, \ldots, a_n)$, such clusters (top points of the ultrametric tree) we will be to name by the highest level clusters or simply by points.

Let $F$ is some extensive parameter of the fractal-cluster system. The distribution function $f(x)$ of the on the fractal-cluster space $U_n$ will be to name the non-negative function $f(x) \equiv f(a_1, a_2, \ldots, a_n)$, which satisfies the following condition:

$$\sum_{x \in U_n} f(x) = 1,$$

therefore the value $F_i = \sum_{x \in U_i} f(x)$ is the value $F$, which relates to the cluster $U_i$.

Let one has non-negative numbers $q_1, q_2, \ldots, q_p$, which satisfies to the following condition:

$$\sum_{a=1}^p q_a = 1.$$
We will be to name the distribution \( f^{ss}(a_1, a_2, \ldots, a_n) \) the self-similar if
\[
f^{ss}(a_1, a_2, \ldots, a_n) = q_{a_1}q_{a_2} \cdots q_{a_n},
\]
Equation (1)

For the case \( p = 5 \) we will be to name a self-similar distribution \( f^{id}(a_1, a_2, \ldots, a_n) = q_{a_1}^{id}q_{a_2}^{id} \cdots q_{a_n}^{id} \) by the ideal one if
\[
q_{i}^{id} = 0.38, \quad q_{2}^{id} = 0.27, \quad q_{3}^{id} = 0.16, \quad q_{4}^{id} = 0.13, \quad q_{5}^{id} = 0.06.
\]

Next we will research the model of the fractal-cluster system’s evolution which is estimated by the distribution \( f(x, t) = f(a_1, a_2, \ldots, a_n, t) \), which depends from the time \( t \).

We will assume the following suggestions:

1. For any moment of the time \( t \) the whole system’s resource is constant:
\[
\sum_{x \in U_n} f(x, t) = 1.
\]
   Equation (2)

If the suggestion (2) is realized for the some time interval then this system is named as the closed system relatively of this resource for this time interval.

2. For any initial distribution the system transfers to the ideal state \( f^{id}(x) \):
\[
\lim_{t \to \infty} f(x, t) = f^{id}(x).
\]
   Equation (3)

3. The resource quantity which is transiting per the unit of time from the highest level cluster (point) \( y \) to any highest level cluster (point) \( x \) of the system decreases with increasing the ultrametric distance \( d(x, y) \) between these points:
\[
\frac{df(x, t)}{dt} \bigg|_{y \to x} \sim K(d(x, y)) f(y),
\]

where \( K(\lambda) \) is some positive decreasing function of the argument \( \lambda \).

These three suggestions allow to write of the dynamic equation for the \( f(x, t) \):
\[
\frac{\partial}{\partial t} f(x, t) = \sum_{y \in U_n, y \neq x} K(d(x, y)) \left( f^{id}(x)f(y, t) - f^{id}(y)f(x, t) \right).
\]
   Equation (4)

Summation of the right and left parts of the equation (4) on \( x \in U_n \) gives us the conservation law of the whole system’s resource:
\[
\frac{\partial}{\partial t} \sum_{x \in U_n} f(x, t) = 0.
\]
Obviously the function $f^{id}(x)$ is the stationary solution $[1]$. This ensures that the condition $[3]$ is hold for any solution $f(x, t)$.

From the equation $[4]$ it follows that the value

$$
\tau_i = \frac{1}{K(p^i)}
$$

is the characteristic transition time to the ideal state of the $i$-level cluster i.e. the function

$$
f(a_1, \ldots, a_i, t) = \sum_{a_{i+1}, \ldots, a_n} f(a_1, \ldots, a_i, a_{i+1}, \ldots, a_n, t)
$$

Let’s mark that under $f^{id}(x) = \text{const}$ the equation $[4]$ coincides in form with the equation of the random walk on the ultrametric tree which is the Kolmogorov & Feller’s equation $[18]$ for the homogenous Markov’s processes distribution function. The last equation in the limit $n \to \infty$ becomes the ultrametric equation of the random walk on the $p$-adic numbers field (Vladimirov’s equation) $[5, 11]$. We have to mark that in our interpretation the function $f(x)$ is not density of the distribution probability. It determines a point of the system configurational space which is the space of functions $f(x)$ satisfying the condition $\sum_{x \in \Omega_n} f(x) = 1$. In this case $f(x, t)$ has determined the path in the configurational space and therefore the equation $[4]$ describes the deterministic dynamics of the fractal-cluster system.

Let discuss the equation $[4]$ solution for the 3-levels system ($n = 3$) with self-similar $[11]$ and homogenous ($q_a = \frac{1}{5^a}$, $a = 1, \ldots, 5$) initial distribution on the clusters. We choose the function $K(\lambda)$, which determines the resource transfers from the highest level cluster $y$ to the any highest level cluster $x$ per the time unite in the following form:

$$
K(\lambda) = \frac{1}{T \lambda^b},
$$

where $b$ is the model parameter, which has characterized "an activity" of the resource redistribution, $T$ is a parameter which determines the time scale. Let the initial distribution is uniform:

$$
f(a_1, a_2, \ldots, a_n, 0) = \frac{1}{5^a}.
$$

We will be interested by the resource distributional dynamics in the first level clusters i.e. the functions

$$
f(a, t) = \sum_{a_2, a_3, \ldots, a_n} f(a, a_2, a_3, \ldots, a_n, t).
$$
The solution of the equation (4) with the initial condition (7) found numerically. On fig. II the resource value from the time trend \( f(1,1,1,t) \) in the subcluster of highest level \((1,1,1)\) for the 3-level system under certain fixed values of the model’s parameters has been presented. It is clear that transfer of this cluster to the ideal state is realized on the character times by order \( \tau_3 \sim 10^3 \). So far, on the character times by order \( \tau_2 \sim 10^6 \) are realized transfers to the ideal state of the second level subclusters and at the end on the character transfer times by order \( \tau_1 \sim 10^9 \) are realized by transfers to the ideal state of the first level subclusters, i.e. transfer of the whole system.

![Figure 2: Dependence \( f(1,1,1,t) \) for \( p = 5, n = 3, b = 4, T = 1 \). Here are clearly visible characteristic transition times in clusters: \( \tau_3 \sim 10^3, \tau_2 \sim 10^6, \tau_1 \sim 10^9 \).](image)

The practical application of this model demands of its parameter’s identification. In this case, the function \( K(\lambda) \) choice depends from the system type and have to be determined by an empirical way. For example, besides the power form, it is possible to derive a choice of the function (6) in exponential \( \frac{1}{T \exp (b\lambda)} \) or in logarithmic \( \frac{1}{T \log (1 + b\lambda)} \) forms. For the adequate choice of this function it is necessary for each researched system to have empirically determined values of the characteristic transition times \( \tau_i \ i = 1, \ldots, n \) for the clusters of all level to the equilibrium state. For example for the 3–level system
under three known conditions, the empirical values \( \tau_1, \tau_2 \) and \( \tau_3 \) with the help of \( \tau \) it is possible to realize the optimal choice of the function \( K(\lambda) \) type and identification of its two parameters – \( T \) and \( b \).

III. CONCLUSION

In this article the scenario of the resource distributional dynamics in those closed systems which relate to "an organism" class has been suggested. On the basis of the natural proposals the evolution equation for the distribution function in such a system has been written. Solutions of this equation for large time pass into the stationary solution corresponding to the ideal distribution of resources. The main principle of dynamics in the closed fractal-cluster system construction is the fact that the highest levels space of the clusters has the indexed hierarchical structure which generates the ultrametric structure. The general suggestion consists of the fact that dynamics of the resource distribution in the closed systems is determined by the whole ultrametric structure of the highest levels of the cluster’s space. Each cluster has its own characteristic time of transition to the ideal state. This time is determined by the ultrametric size of the cluster (it is a maximum distance \( d(x, y) \) between the higher level clusters \( x \) and \( y \), which include to this cluster through this function \( K(d(x, y)) \), and which determines the value of the resource, which transfers the resource from the higher level cluster \( y \) in any highest level cluster \( x \). We have a wide range of choice of the function \( K(d(x, y)) \). Under numerical analysis we use the power form of this function but we can choose the logarithmic and other forms of the one. In our opinion this choice has to be special for each particular system. Our suggestion is determined by the mechanics of the resource transfer dynamics in the closed system: from the beginning, the highest level clusters transfer to the ideal state then the clusters of lower do it and so on. Therefore one always has the hierarchy of characteristic transition times to the ideal state for all clusters. This hierarchy is determined by the whole system dynamics. This fact is demonstrated by the numeric solutions the equation for the resource distribution function. The particular form of the function \( K(d(x, y)) \) depends from the type of the modeling system and it must be determined on the best approximation of the characteristic transition time to the equilibrium state for the clusters.
of the all levels.

This work was partially supported by Russian Foundation for Basic Research grant (project 13-01-00790-a).

[1] Rammal R., Toulouse G., Virasoro M.A. Ultrametricity for physicists. Rev. Mod. Phys. 589 (1986) 765.
[2] Dragovich B., Khrennikov A.Yu., Kozyrev S.V., Volovich I.V. On p-adic mathematical physics. p-Adic Numbers, Ultrametric Analysis, and Applications Vol. 1, No 1, (2009) 1.
[3] Dotsenko V. An Introduction to the Spin Glasses and Neural Networks. (Singapure: World Scientific, 1994) 156 P.
[4] Ogielski A.T., Stein D.L. Dynamics on Ultrametric Spaces. Phys. Rev. Lett. 55 (1985) 1654.
[5] Avetisov V.A., Bikulov A.Kh., Kozyrev S.V., Osipov V.A. p-Adic Models of Ultrametric Diffusion Constrained by Hierarchical Energy Landscapes. J.Phys. A: Math. Gen. 35 (2002) 177.
[6] Avetisov V.A., Bikulov A.Kh., Osipov V.A. p-Adic description of characteristic relaxation in complex systems. J.Phys. A: Math. Gen. 36 (2003) 4239.
[7] Sornette D., Johansen A. A hierarchical model of financial crashes. Physica A 261 (1998) 581.
[8] Mantenga R.N., Stanley H.E. An Introduction to Econophysics. Correlations and Complexity in Finance. (Cambridge: Cambridge Univ. Press, 2000) x + 148 P.
[9] Bikulov A.H., Zubarev A.P., Kaidalova L.V. The hierarchical dynamics model of the financial market which the one is concerned to the defolt point state and the p-adiatic mathematical analysis. Vestnik Samarskogo gosydarstvenogo texnicheskogo universitets. Seria: Physiko-Mathematiceskie nauki, 42 (2006)135.
[10] Schikhof W.H. Ultrametric Calculus. An Introduction to p-adic Analysis. (Cambridge: Cambridge Univ. Press, 1984) viii+306 P.
[11] Vladimirov V.S., Volovich I.V., Zelenov E.I., p-Adic Analysis and Mathematical Physics. (Singapure: World Scientific Publishing, 1994), 340 P.
[12] Shelkovich V.H., Hrennicov A.V. The modern p-adiatic analysis and mathematical physics. Theory and applications (M.: Physmatlit, 2012) 452 P.

[13] Dolgopolov M.V., Zubarev A.P. Some Aspects of $\mathfrak{m}$-Adic Analysis and Its Applications to $\mathfrak{m}$-Adic Stochastic Processes. $p$-Adic Numbers, Ultrametric Analysis, and Applications, Vol. 3, No 1 (2011) p. 39.

[14] Burdakov U.P. Effictionly of the life. (M.: Energoizdat, 1997) 304 P.

[15] Volov V.T. Economics, fluctuations and thermodynamics. (Samara: SNC RAN, 2001) 222 p.

[16] Volov V.T. Fractal-cluster theory of the educational structure controlling. (Kazan: Kaz. Gos. Universitets, 2000) 387 p.

[17] Volov V.T. Fractal-cluster theory and thermodynamic principles of the control and analysis for the self-organizing systems. arXiv:1309.1415

[18] Gardiner C.W. Handbook of Stochastic Methods: For Physics, Chemistry and the Natural Sciences. (Springer, 1996) 442 P.