Tailoring higher-order exceptional points toward enhanced sensitivity

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Non-Hermitian degeneracies, in the form of exceptional points (EPs), are unique characteristics of typical PT-symmetric systems. At EPs, two or more eigenvalues and their corresponding eigenvectors coalesce. While second-order EPs are mostly explored, recently, higher-order EPs are garnering extensive attention in various multidisciplinary fields, particularly owing to their possible applications for extremely high-sensitive devices. Here, we present a simple yet effective scheme to enhance the device sensitivity by slightly deviating the gain-neutral-loss linear configuration to a triangular one, resulting in an abrupt phase transition from third-order to second-order EPs. Our analysis demonstrates that the EPs can be tailored by judicious tuning of the coupling parameters of the system, resulting in enhanced sensitivity to a small perturbation. The tunable coupling also leads to a sharp change in the sensitivity slope, enabling the perturbation to be measured precisely as a function of coupling. This two-way detection of the perturbation opens up a rich landscape toward ultra-sensitive measurements, which could be applicable to a wide range of non-Hermitian ternary platforms.

I. INTRODUCTION

Non-Hermitian Hamiltonian with Parity-Time (PT) symmetry was initially studied as a mathematical curiosity, and it resulted in a very different perspective of quantum mechanics [1, 2]. And today, there is hardly any branch of physics which is not influenced by these concepts [3, 4]. The area which is most influenced both experimentally and theoretically is optics. In fact, optics turns out to be the test-bed for many key concepts related to parity-time symmetry. Till date, the experimental demonstration of non-reciprocity [5], loss-induced transparency [6], optical solitons [7], PT-symmetric microring lasers [8] and so on [9] has been carried out using various PT-symmetric optical platforms. Indeed, PT symmetry induced effects have been experimentally observed in other branches of physics as well, such as electronics [10], mechanics [11], atomic lattices [12], acoustics [13], Bose-Einstein condensates [14] and so on. In the context of non-Hermitian PT-symmetric systems, the so-called exceptional points (EPs) are of particular interest due to numerous possible applications that come through its exploitations [15]. It is worthwhile to note that a non-Hermitian Hamiltonian, $H$, is said to be PT-symmetric, provided, $[H, \mathcal{P}T] = 0$. Here, $\mathcal{P}$ refers to the parity operator that simply interchanges two of the constituent modes of the system, while $\mathcal{T}$ is the time-reversal operator that takes $i \rightarrow -i$. One prominent aspect of such Hamiltonians is the breaking of PT symmetry, in which the eigenspectra switches from being entirely real to completely imaginary. Such abrupt PT phase transition is marked by the presence of an EP [3, 5]. EPs of a PT-symmetric Hamiltonian are the ones at which two or more eigenvalues and their associated eigenvectors coalesce simultaneously and become degenerate [15]. EPs open doors for completely new functionalities and performance. Nonreciprocal light propagation [16], laser mode control [8, 17, 19], unidirectional invisibility [20, 23], optical sensing [24, 26], light stopping [27] and structuring [28], and delaying sudden death of entanglement [29] are some notable phenomena triggered by EPs.

Recently, it has been realized that the sensitivity of a non-Hermitian PT-symmetric system or device could be enhanced significantly by judicious engineering of higher-order EPs. In this regard, the observation of higher-order EPs in a coupled cavity arrangement, specifically, a ternary PT-symmetric photonic laser molecule with a carefully tailored gain–loss distribution, has been experimentally demonstrated by a group of researchers [25]. Their work reveals that the response of the ternary laser molecule exhibits cube-root behavior, which could be used to improve the sensing performance of microresonator arrangements. In our work we consider different three-channel PT-symmetric gain-neutral-loss configurations with linear and triangular arrangements of multi-core fibers owing to the ease of fabrication of such structures, apart from its immense utility in fiber-optic communication systems, which could be extended to other physical systems. We work out the constraints under which such a system could be PT-symmetric. We investigate how a small displacement of the neutral channel from the linear configuration changes the eigenvalues and the nature of the EPs. For that, we analyze the non-Hermitian system by introducing a triangular configuration. We find that the PT-symmetric three-core structure exhibits higher-order EPs of order three that transform to order two with only a very slight deviation from the linear geometry. This abrupt phase transition affects the eigenvalue bifurcation diagram and the sensitivity to small perturbations, which could lead to ultra-sensitive device applications. Furthermore, we investigate the device sensitivity to small perturbations in both linear and triangular configurations by tailoring the EPs, which leads to an efficient way of measuring small perturbations by simply changing the coupling.

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FIG. 1. (Color online) Schematic diagram of three-channel $\mathcal{PT}$-symmetric couplers with linear and triangular configurations. Here, $\kappa$ is the coupling between the gain (G) and the loss (L) channels and $\alpha_1$ ($\alpha_2$) is the coupling between the neutral (N) channel and the gain (loss) channel. Different component of $\mathcal{PT}$-symmetric devices (electronic trimer circuit, multi-core fibers or PCFs, and three closely spaced single-core fibers) where coupling coefficients can be controlled from the linear to triangular geometry for practical applications.

II. THE MODEL

To describe the problem, we consider two possible types of three-channel $\mathcal{PT}$-symmetric structures with gain-neutral-loss configurations, as shown in Fig. 1. Here, the coupling coefficients can be tailored by changing the relative distance between the ports, and we get the two possible configurations, namely, the triangular and linear configurations. Here, in general, there could be different coupling coefficients between two neighboring channels. However, in order for the system to be operated in the $\mathcal{PT}$ regime, these coupling coefficients must follow some relations, which we discuss in the next section. From the fabrication point of view, there are different fiber structures with three core $\mathcal{PT}$-symmetric configurations that are easy to fabricate (either three cores surrounded by a common cladding or three solid cores in a photonic-crystal fiber (PCF) placed linearly/triangularly or a setup with three single-core fibers that are closely spaced and mutually coupled) than other configurations where loss/gain quadrupole have been realized [25]. On top of that, single-core fibers have been widely used over the years for various sensing applications that rely on mode switching, refractive index change, and other fiber parameters that interact with the environment [31]. A $\mathcal{PT}$-symmetric electronic circuit [10, 31, 33] with three components can also be used to describe such ternary systems, where mutual inductance $M$ and a controllably coupled capacitor $C_c$ act as respective coupling. As an alternative, a variable coupled inductive circuit between gain and loss components can be used as the coupling $\kappa$. For the generalized triangular configuration, shown in Fig. 1, one can construct the dimensionless coupled-mode equations that describe the evolution of field envelopes inside the first gain core ($a_g$), second neutral core ($b_n$), and third loss core ($c_l$) in the linear domain as:

$$i \frac{\partial a_g}{\partial \xi} = i\Gamma a_g + \alpha_1 b_n + \kappa c_l, \quad (1a)$$

$$i \frac{\partial b_n}{\partial \xi} = \alpha_1 a_g + \alpha_2 c_l, \quad (1b)$$

$$i \frac{\partial c_l}{\partial \xi} = -i\Gamma c_l + \alpha_2 b_n + \kappa a_g. \quad (1c)$$

Here, $\xi$ is the dimensionless propagation distance, which could be time or any other parameter of the system under consideration. $\alpha_1$, $\alpha_2$, and $\kappa$ are the rescaled (by the characteristic length of the system) linear coupling coefficients between the neutral—gain ports, neutral—loss ports, and gain—loss ports, respectively. Also, $\Gamma$ is the rescaled balanced linear gain/loss coefficient. It is to be noted that when the coupling between loss and gain is zero, the triangular configuration becomes the well-known linear configuration [25], and the corresponding pulse propagation equations are obtained by setting $\kappa = 0$ in Eqs. (1a) and (1c).

III. HAMILTONIAN ANALYSIS, EXCEPTIONAL POINTS, AND STABILITY

From the above-mentioned coupled three-channel $\mathcal{PT}$-symmetric configurations (schematically depicted in Fig. 1), we can construct a generalized Hamiltonian for both linear and triangular geometry using the set of equations [Eq. (1)]. The Hamiltonian can be written in the following form:

$$\hat{H} = \begin{pmatrix} i\Gamma & \alpha_1 & \kappa \\ \alpha_1 & 0 & \alpha_2 \\ \kappa & \alpha_2 & -i\Gamma \end{pmatrix}. \quad (2)$$

The Hamiltonian $\hat{H}$ describing the above-mentioned systems will be $\mathcal{PT}$-symmetric if $\hat{H}$ satisfies the commutation relation: $[\hat{H}, \mathcal{PT}] = 0$. It can be directly calculated from the commutation relation that, when $\alpha_1 = \alpha_2 = \alpha$, i.e., the couplings between the neutral port and the other two ports are equal, then the three-channel coupled configurations (shown in Fig. 1) become $\mathcal{PT}$-symmetric. The complex eigenvalues corresponding to the $\mathcal{PT}$-symmetric Hamiltonian $H^{\mathcal{PT}} = \begin{pmatrix} i\Gamma & \alpha & \kappa \\ \alpha & 0 & \alpha \\ \kappa & \alpha & -i\Gamma \end{pmatrix}$ can be obtained by the direct method of diagonalization of the Hamiltonian as

$$\lambda_1 = \frac{3^{1/3} f_1 + f_2^{2/3}}{3^{2/3} f_2^{1/3}},$$

$$\lambda_2 = (\lambda_3) = \frac{-3^{1/3}(1 + i\sqrt{3}) f_1 - (1 - i\sqrt{3}) f_2^{2/3}}{2 \cdot 3^{2/3} f_2^{1/3}}, \quad (3)$$
where \( f_1 = 2\alpha^2 + \kappa^2 - \Gamma^2 \) and \( f_2 = 9\alpha^2\kappa + \sqrt{729\alpha^4\kappa^2 - 27f_3^3}/3 \). Using this set of eigenvalues [Eq. (3)], we analyze the bifurcations and resonance EPs in detail. We also investigate the stability and response of the system against a slight perturbation around the EPs using the system Hamiltonian \((H^{PT})\). This analysis along with the transition from the linear to the triangular configurations give critical information with broad applications in non-Hermitian measurement devices with enhanced sensitivity.

### A. Tailoring Exceptional points

To demonstrate the behavior of the three-channel \(PT\)-symmetric system, we plot the eigenvalues as a function of \(\Gamma\) in Figs. 2(a)–(d) for different sets of coupling coefficients. These eigenvalues could be energy or frequency spectra or any other physical parameters of the system. From Figs. 2(a,b,d), it is evident that the eigenvalues are completely real below a specific threshold value of \(\Gamma\), the EPs, denoted by arrows. Here we observe that for triangular configurations \((\kappa \neq 0)\), two of the three eigenvalues collapse to a single \(\Gamma\) point (EP) from where the complex eigenvalues emerge. However, the third eigenvalue remains completely non-zero real. With increasing \(\Gamma\), the real part of the degenerate eigenvalues and the non-degenerate real eigenvalues approach to zero. In the case of triangular configuration, this EP is a second-order EP (EP2), which, unlike the pure dipole \(PT\) case, does not coalesce to a point on the zero eigenvalue axis. It instead falls on a line perpendicular to the \(\Gamma\)-axis. This is analogous to a two atomic system where a third atom acts as an impurity that changes the properties of the system. Now, in the case of a gain-neutral-loss linear quadrupole system \((\kappa = 0)\) shown in Fig. 2(d), this EP is a higher-order EP in nature, also known as the three-fold EP (EP3). Unlike triangular configurations, here, only two real eigenvalues are non-zero that collapse at the EP3. From Eq. (3), the generalized analytical expression of the EPs on the \(\Gamma\)-axis can be calculated as

\[
\Gamma_{EP} = \alpha \sqrt{2 + \left(\frac{\kappa}{\alpha}\right)^2 - 3 \left(\frac{\kappa}{\alpha}\right)^{2/3}}
\]  

(Eq. 4)

From Eq. (4), one can get at \(\kappa = \alpha\) there would not be any nonzero real eigenvalues, suggesting that the unbroken regime is forbidden when the system is an equilateral triangle in terms of coupling coefficients. It is to be noted from Eq. (4) that, in the absence of neutral channel in between the gain and the loss channels, i.e., \(\alpha = 0\), the system becomes a standard two-core \(PT\)-symmetric coupler for which the value of gain/loss strength at the EP2 is \(\Gamma_{EP2} = \kappa\). Thus, a specific choice of the coupling coefficients \(\alpha\) and \(\kappa\) directly controls the EPs of non-Hermitian systems. Alternative representations of the eigenvalues and EPs are shown in Fig. 2(e). Here all the three real versus imaginary parts of the \(\lambda_n\) (thin-light-dashed lines) meet at the zero-crossing of the two axes (EP3) for the ternary linear arrangement \((\kappa = 0)\). However, for triangular configuration with increasing \(\kappa/\alpha\), the system abruptly changes its phase, and the three curves (broad-solid curves) move away from the center. In this case, the EP2 (indicated by an arrow) falls on the \(\text{Im}[\lambda_n] = 0\) line, and can be tuned further by changing the coupling contrast \(\kappa/\alpha\). Next, in Fig. 2(f) we plot the rescaled contrast of \(\Gamma_{EP}\) as a function of \(\kappa/\alpha\), implying that the system experiences a wide range of tunable EPs obtained by changing the geometry of the ternary alignment while maintaining isosceles triangular \(PT\) structures. Note that, for the linear configuration we have only one controlling parameter \(\alpha\) apart from \(\Gamma\), which results in an isolated fixed EP3, \(\Gamma_{EP3}/\alpha = \sqrt{2}\) [solid red-circle in Fig. 2(f)]. In our case, however, with the triangular configuration, we have an additional controlling

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**FIG. 2.** (Color online) The real (solid curves) and imaginary parts (dashed curves) of the eigenvalues \((\lambda_n)\) as a function of the gain/loss parameter \(\Gamma\) for \(PT\)-symmetric \((a,b,c)\) triangular and \((d)\) linear configurations. The EPs [Eq. (4)] are denoted by arrows. (e) An alternative phase-space representation of the eigenvalues and EPs. (f) Manipulation of EPs by controlling the coupling contrast \(\kappa/\alpha\). Here the red-circle represents the isolated fixed EP3.
parameter $\kappa$ that tailors the EP2. Also, from Fig. 2(f) we can conclude that in the limit $\kappa/\alpha \ll 1$ the influence of the neutral core can be ignored, and the system behaves as a pure gain-loss binary $\mathcal{PT}$-symmetric system ($\Gamma = \Gamma_{\text{EP2}}$). The dynamics around EP2 and EP3 can also be efficiently tailored by deviating from the linear geometry, and the sensitivity of the system to perturbations can be significantly enhanced through the sudden phase transition; additionally, the strength of the perturbation can be precisely measured as a function of coupling coefficient contrast $\kappa/\alpha$, which we discuss below.

B. Stability against small perturbations

To demonstrate the sensitivity of the system against perturbations, we choose the system to be operated near the EP3 by setting the controlling parameters, which leads to enhanced sensitivity. We introduce a small perturbation $\epsilon$ imposed on the gain channel. In doing so, the non-Hermitian Hamiltonian of the system is modified as

$$H' = \begin{pmatrix} i\Gamma + \epsilon & \alpha & \kappa \\ \alpha & 0 & \alpha \\ \kappa & \alpha & -i\Gamma \end{pmatrix}. \quad (5)$$

This perturbation could, in principle, be introduced in any other port, but in this process the sensitivity would be reduced [20]. In order to understand how the perturbation changes the dynamics of the system near the EPs, we first consider the linear configuration. In Fig. 3(a,b) the bifurcation diagram of the complex eigenvalues around a EP3 (indicated by solid red circle) are plotted from Hamiltonian $H'$ [Eq. (5)] with rescaled gain/loss contrast and detuning $\epsilon/\alpha$ for $\kappa = 0$.

Next, we consider the generalized case with $\kappa \neq 0$. We plot the bifurcation diagram of the eigenvalues around the EP2 (solid red circles) in the presence of perturbations in Fig. 3(c-f). As we mentioned earlier, for the triangular case, the EP2 falls on a line perpendicular to the $\Gamma/\alpha$ axis. The real parts of the two eigenvalues fall on the EP2 line, while imaginary parts of them coincide at $\lambda_n = 0$ on the line. Figure 3(c,d) also show that, unlike linear configuration, here, the imaginary planes are more wrapped in the presence of perturbations, and even at some point, they intersect each other. From these bifurcation diagrams, the variation of the eigenvalues at the EP3 and EP2 with respect to the perturbations are plotted in Fig. 3(a,b,c). The $\text{Re}[\lambda_n]$ values in Fig. 3(c) (solid curves) show a trend more like the linear configuration as one moves away from the EP2. However, when the system is observed close to the EP2, it behaves differently, and one of the real eigenvalues suddenly changes its curvature while remaining non-degenerate. To check how the system reacts to the perturbation around the EP2, we plot the sensitivity in Fig. 4(d,e), i.e., the difference between two eigenvalues $\text{Re}(\lambda_2 - \lambda_1)$ in linear and logarithmic scales. The finding is dramatic. Firstly, by changing the coupling contrast from zero to non-zero, we can change the bifurcation dynamics of the system that leads to the transition from EP3 to EP2. In doing so, the sudden change in the dynamics of the system can be detected through its eigenvalues, which is further controlled in the case of EP2 by changing the coupling contrast. Secondly, the transition from EP3 to EP2 also changes the sensitivity from $\sim \epsilon^{1/3}$ to $\sim \epsilon^{1/2}$ and vice versa. Furthermore, when the coupling contrast is very small, i.e., $\kappa/\alpha \ll 1$, there is a sharp change in slope in the logarithmic plot [Fig. 4(e)] from $1/2$ to $1/3$, with a case $\kappa/\alpha = 0.01$ represented by a solid red circle. This slope changing point is further plotted as a function of coupling contrast in Fig. 4(f) with solid red circles, which follows a linear relation from the fitting: $\epsilon/\alpha = 4 \kappa/\alpha$ (solid blue curve). The sensitivity or splitting can also be enhanced by tuning the $\kappa/\alpha$, as shown in Fig. 4(d). These findings are important because a slight variation of coupling contrast, $\kappa/\alpha$, on the order of $10^{-4}$ or less could make the transition from EP3 to EP2 of the non-Hermitian system and also change the sensitivity slope from $1/3$ to $1/2$ even for very small perturbation of the same order of magnitude as $\kappa/\alpha$. This way, a very small perturbation can be precisely measured by tuning the coupling contrast and looking at where the transition.

FIG. 3. (Color online) Bifurcation of complex eigenvalues around EPs in the presence of perturbation: (a,b) for linear configuration ($\kappa/\alpha = 0$); (c,d) and (e,f) for triangular configurations with $\kappa/\alpha = 0.5$ and $\kappa/\alpha = 0.01$, respectively. Here solid red dots represent the EPs.
from one slope to the other takes place. This controllable $\mathcal{PT}$-symmetric system with EP-manipulation could be applied in high-sensitivity measurement devices to detect a very tiny perturbation $\epsilon$ by slightly deviating the linear configuration through the controlling parameter $\kappa/\alpha$. This ternary system could also be useful in measuring forces through deflection of the neutral element, as EP and the deflection of the neutral component are directly related.

We may think of a few possible experimental scenarios in which our proposed idea of tailoring EPs may be applicable. In $\mathcal{PT}$-symmetric electrical circuits and circuit-QED systems, the three component coupled circuit (shown in Fig. 1) can be utilized, whose coupling contrast can be tuned by controlling the coupled capacitor $C_s$. Our idea of the triangular coupling scheme could be utilized in the divergent EP-system, where the addition of a third coupling would further improve the overall sensitivity. A system consisting of three closely aligned single-core mutually coupled fibers (schematically depicted in Fig. 1), waveguides, and resonators with the gain-neutral-loss configuration can be utilized for deflection or stress measurement. Additionally, gain-neutral-loss cores with linear and triangular alignment with different sets of $\kappa/\alpha$ can be fabricated in a single PCF or as multi-core fibers for various sensing applications. Other systems include optomechanics, atomic systems, acoustic vibration of membranes attached to mutually coupled inductances of a $\mathcal{PT}$-symmetric electrical circuit, and so on.

**IV. CONCLUSIONS**

To conclude, in this work we have studied how one can obtain excellent sensitivity against perturbations in a $\mathcal{PT}$-symmetric three-channel system through manipulation of higher-order EPs. We demonstrate that a tiny deviation from the linear to triangular configuration causes the phase transition of the system, resulting in a transition from EP3 to EP2. Besides that, the sensitivity of the system to a small perturbation can be enhanced by tuning the coupling contrast. Furthermore, the coupling contrast causes a sharp change of the slope of the sensitivity curve from $1/2$ to $1/3$. This two-way change of the system dynamics can be utilized to enhance the device sensitivity and efficiently measure the strength of perturbation. These findings could pave the way for manipulating the system dynamics and improving sensitivity in various non-Hermitian device applications, potentially opening up new opportunities in other interdisciplinary fields.

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[1] C. M. Bender and S. Boettcher, “Real Spectra in Non-Hermitian Hamiltonians Having $\mathcal{PT}$ Symmetry,” *Phys. Rev. Lett.* **80**, 5243 (1998).

[2] C. M. Bender, D. C. Brody, and H. F. Jones, “Complex Extension of Quantum Mechanics,” *Phys. Rev. Lett.* **89**, 270401 (2002).
[3] R. El-Ganainy, K. G. Makris, M. Khajavikhan, Z. H. Musslimani, S. Rotter, and D. N. Christodoulides, “Non-Hermitian physics and PT symmetry,” Nature Physics 14, 11 (2018).

[4] D. N. Christodoulides and J. Yang, Parity-time symmetry and its applications, Springer Nature, Singapore (2018).

[5] C. E. Rüter, K. G. Makris, R. El-Ganainy, D. N. Christodoulides, M. Segev, and D. Kip, “Observation of parity-time symmetry in optics,” Nat. Phys. 6, 192 (2010).

[6] A. Guo, G. J. Salamo, D. Duchesne, R. Morandotti, M. Volatier-Ravat, V. Aimez, G. A. Siviloglou, and D. N. Christodoulides, “Observation of PT-symmetry breaking in complex optical potentials,” Phys. Rev. Lett. 103, 093902 (2009).

[7] M. Wimmer, A. Regensburger, M. A. Miri, C. Bersch, D. N. Christodoulides, and U. Peschel, “Observation of optical solitons in PT-symmetric lattices,” Nature Comm. 6, 7782 (2015).

[8] H. Hodaei, M. A. Miri, M. Heinrich, D. N. Christodoulides, and M. Khajavikhan, “Parity-time-symmetric microring lasers,” Science 346, 975 (2014).

[9] S. V. Suchkov, A. A. Sukhorukov, J. Huang, S. V. Dmitriev, C. Lee, and Yu. S. Kivshar, “Nonlinear switching and solitons in PT-symmetric photonic systems,” Laser Photonics Rev. 10, 177 (2016).

[10] J. Schindler, A. Li, M. C. Zheng, F. M. Ellis, and T. Kottos, “Experimental study of active LRC circuits with PT symmetries,” Phys. Rev. A 84, 040101 (2011).

[11] C. M. Bender, B. Berntson, D. Parker, and E. Samuel, “Observation of PT Phase Transition in a Simple Mechanical System,” Am. J. Phys. 81, 173 (2013).

[12] Z. Zhang, Y. Zhang, J. Sheng, L. Yang, M. A. Miri, D. N. Christodoulides, B. He, Y. Zhang, and M. Xiao, “Observation of parity-time symmetry in optically induced atomic lattices,” Phys. Rev. Lett. 117, 123601 (2016).

[13] C. Shi, M. Dubois, Y. Chen, L. Cheng, H. Ramezani, Y. Wang, and X. Zhang, “Accessing the exceptional points of parity-time symmetric acoustics,” Nat. Commun. 7, 11110 (2016).

[14] M. Kreibich, J. Main, H. Cartarius, and G. Wunner, “Realizing PT-symmetric non-Hermiticity with ultracold atoms and Hermitian multiwell potentials,” Phys. Rev. A 90, 033630 (2014).

[15] M. A. Miri and A. Alu, “Exceptional points in optics and photonics,” Science 363, eaar7709 (2019).

[16] B. Peng, S. K. Özdemir, F. Lei, F. Monifi, M. Gianfreda, G. L. Long, S. Fan, F. Nori, C. M. Bender, and L. Yang, “Parity-time-symmetric whispering-gallery microcavities,” Nat. Phys. 10, 394 (2014).

[17] L. Feng, Z. J. Wong, R.-M. Ma, Y. Wang, and X. Zhang, “Single-mode laser by parity-time symmetry breaking,” Science 346 972 (2014).

[18] H. Hodaei, A. U. Hassan, W. E. Hayenga, M. A. Miri, D. N. Christodoulides, and M. Khajavikhan, “Dark-state lasers: mode management using exceptional points,” Opt. Lett. 41, 3049 (2016).

[19] B. Peng, S. K. Özdemir, M. Liertzer, W. Chen, J. Kramer, H. Yilmaz, J. Wiersig, S. Rotter, and L. Yang, “Chiral modes and directional lasing at exceptional points,” Proc. Natl. Acad. Sci. 113, 6845 (2016).

[20] Z. Lin, H. Ramezani, T. Eichelkraut, T. Kottos, H. Cao, and D. N. Christodoulides, “Unidirectional Invisibility Induced by PT-Symmetric Periodic Structures,” Phys. Rev. Lett. 106, 213901 (2011).

[21] S. Longhi, “Invisibility in PT-symmetric complex crystals,” J. Phys. A: Math. Theor. 44, 485302 (2011).

[22] A. Regensburger, C. Bersch, M. A. Miri, G. Onishchukov, D. N. Christodoulides, and U. Peschel, “Parity–time-synthetic photonic lattices,” Nature 488, 167 (2012).

[23] L. Feng, Y.-L. Xu, W. S. Fegadolli, M.-H. Lu, J. E. B. Oliveira, V. R. Almeida, Y.-F. Chen, and A. Scherer, “Experimental demonstration of a unidirectional reflectionless parity-time metamaterial at optical frequencies,” Nat. Mater. 12, 108 (2013).

[24] J. Wiersig, “Enhancing the Sensitivity of Frequency and Energy Splitting Detection by Using Exceptional Points: Application to Microcavity Sensors for Single-Particle Detection,” Phys. Rev. Lett. 112, 203901 (2014).

[25] H. Hodaei, A. U. Hassan, S. Wittek, H. García-Gracia, R. El-Ganainy, D. N. Christodoulides, and M. Khajavikhan, “Enhanced sensitivity at higher-order exceptional points,” Nature (London) 548, 187 (2017).

[26] W. Chen, S. K. Özdemir, G. Zhao, J. Wiersig, and L. Yang, “Exceptional points enhance sensing in an optical microcavity,” Nature (London) 548, 192 (2017).

[27] T. Goldzak, A. A. Mailybaev, and N. Moiseyev, “Light Stops at Exceptional Points,” Phys. Rev. Lett. 120, 013901 (2018).

[28] P. Miao, Z. Zhang, J. Sun, W. Walasik, S. Longhi, N. M. Litchinitser, and L. Feng, “Orbital angular momentum microlaser,” Science 353, 464 (2016).

[29] S. Chakraborty, and A. K. Sarma, “Delayed sudden death of entanglement at exceptional points,” Phys. Rev. A 100, 063846 (2019).

[30] H.-E. Joe, H. Yun, S.-H. Jo, M. B. G. Jun, and B.-K. Min, “A review on optical fiber sensors for environmental monitoring,” Int. J. of Precis. Eng. and Manuf.-Green Tech. 5, 173 (2018).

[31] J. Schindler, Z. Lin, J. M. Lee, H. Ramezani, F. M. Ellis, and T. Kottos, “PT-symmetric electronics,” J. Phys. A: Math. Theor. 45, 444029 (2012).

[32] M. Sakhdari, M. Hajizadegan, Q. Zhong, D. N. Christodoulides, R. El-Ganainy, and P.-Y. Chen, “Experimental Observation of PT Symmetry Breaking near Divergent Exceptional Points,” Phys. Rev. Lett. 123, 193901 (2019).

[33] A. Stegmaier, S. Imhof, T. Helbig, T. Hofmann, C. H. Lee, M. Kremer, A. Fritzsche, T. Feichtner, S. Klembt, S. Höfling, I. Boettcher, I. C. Fulga, L. Ma, O. G. Schmidt, M. Greiter, T. Kiiessler, A. Szameit, and R. Thomale, “Topological Defect Engineering and PT Symmetry in Non-Hermitian Electrical Circuits,” Phys. Rev. Lett. 126, 215302 (2021).

[34] A. Baust, E. Hoffmann, M. Haeblerlein, M. J. Schwarz, P. Eder, J. Goetz, F. Wulschner, E. Xie, L. Zhong, F. Quijandría, B. Peropadre, D. Zueco, J.-J. García Ripoll, E. Solano, K. Fedorov, E. P. Menzel, F. Deppe, A. Marx, and R. Gross, “Tunable and switchable coupling between two superconducting resonators,” Phys. Rev. B 91, 041405 (2015).

[35] F. Quijandría, U. Naether, S. K. Özdemir, F. Nori, and D. Zueco, “PT-symmetric circuit QED,” Phys. Rev. A 97, 053846 (2018).
[36] S. Longhi, “PT phase control in circular multi-core fibers,” Opt. Lett. 41, 1897 (2016).

[37] H. Jing, Ş. K. Özdemir, H. Lü, and F. Nori, “High-order exceptional points in optomechanics,” Sci. Rep. 7, 3386 (2017).

[38] W. Xiong, Z. Li, Y. Song, J. Chen, G.-Q. Zhang, and M. Wang, “Higher-order exceptional point in a pseudo-Hermitian cavity optomechanical system,” Phys. Rev. A 104, 063508 (2021).

[39] J. Sheng, M.-A. Miri, D. N. Christodoulides, and M. Xiao, “PT-symmetric optical potentials in a coherent atomic medium,” Phys. Rev. A 88, 041803(R) (2013).