An Extension of Barta’s Theorem
and Geometric Applications

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Abstract

We prove an extension of a theorem of Barta then we make few geometric applications. We extend Cheng’s lower eigenvalue estimates of normal geodesic balls. We generalize Cheng-Li-Yau eigenvalue estimates of minimal submanifolds of the space forms. We prove an stability theorem for minimal hypersurfaces of the Euclidean space, giving a converse statement of a result of Schoen. Finally we prove a generalization of a result of Kazdan-Kramer about existence of solutions of certain quasi-linear elliptic equations.

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1 Introduction

The fundamental tone $\lambda^*(M)$ of a smooth Riemannian manifold $M$ is defined by

$$\lambda^*(M) = \inf \left\{ \frac{\int_M |\text{grad} f|^2}{\int_M f^2} ; f \in L^2_{1,0}(M) \setminus \{0\} \right\}$$

where $L^2_{1,0}(M)$ is the completion of $C^\infty_0(M)$ with respect to the norm $\|\varphi\|^2 = \int_M \varphi^2 + \int_M |\nabla \varphi|^2$. When $M$ is an open Riemannian manifold, the fundamental tone $\lambda^*(M)$ coincides with the greatest lower bound

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inf Σ of the spectrum Σ ⊂ [0, ∞) of the unique self-adjoint extension of the Laplacian ∆ acting on $C_0^\infty(M)$ also denoted by ∆. When $M$ is compact with piecewise smooth boundary $\partial M$ (possibly empty) then $\lambda^*(M)$ is the first eigenvalue $\lambda_1(M)$ of $M$ (Dirichlet boundary data if $\partial M \neq \emptyset$). A natural question is what bounds can one give for $\lambda^*(M)$ in terms of Riemannian invariants? Or if $M$ is an open manifold, for what geometries does $M$ have $\lambda^*(M) > 0$? See [2], [5], [10] or [18]. A simple method for giving bounds the first Dirichlet eigenvalue $\lambda_1(M)$ of a compact smooth Riemannian manifold $M$ with piecewise smooth boundary$^1 \partial M \neq \emptyset$ is Barta’s Theorem. All Riemannian manifolds in this paper are smooth and connected.

**Theorem 1.1 (Barta, [1])** Let $M$ be a bounded Riemannian manifold with piecewise smooth non-empty boundary $\partial M$ and $f \in C^2(M) \cup C^0(M)$ with $f|M > 0$ and $f|\partial M = 0$ and $\lambda_1(M)$ be the first Dirichlet eigenvalue of $M$. Then

$$\sup_M (-\triangle f/f) \geq \lambda_1(M) \geq \inf_M (-\triangle f/f).$$

With equality in (1) if and only if $f$ is a first eigenfunction of $M$.

**Remark 1.2** To obtain the lower bound for $\lambda_1(M)$ we may suppose that $f|\partial M \geq 0$.

Cheng applied Barta’s Theorem in a beautiful result known as Cheng’s eigenvalue comparison theorem.

**Theorem 1.3 (Cheng, [6])** Let $N$ be a Riemannian $n$-manifold and $B_N(p,r)$ be a geodesic ball centered at $p$ with radius $r < \text{inj}(p)$. Let $c$ be the least upper bound for all sectional curvatures at $B_N(p,r)$ and let $\mathbb{N}^n(c)$ be the simply connected $n$-space form of constant sectional curvature $c$. Then

$$\lambda_1(B_N(p,r)) \geq \lambda_1(\mathbb{N}^n(c))(r).$$

In particular, when $c = -1$ and $\text{inj}(p) = \infty$, Cheng’s inequality becomes $\lambda^*(N) \geq \lambda^*(\mathbb{H}^n(−1))$ which is McKean’s inequality, see [14].

Our main result gives lower bound for the fundamental tone $\lambda^*(M)$ of an arbitrary smooth Riemannian manifolds $M$ in terms of the divergence of certain vector fields regardless the smoothness degree of its boundary.

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$^1$Piecewise smooth boundary here means that there is a closed set $Q \subset \partial M$ of $(n−1)$-Hausdorff measure zero such that for each point $q \in \partial M \setminus Q$ there is a neighborhood of $q$ in $\partial M$ that is a graph of a smooth function over the tangent space $T_q\partial M$, see Whitney [20] pages 99-100.
Definition 1.4 Let $M$ be a Riemannian manifold and a vector field $X \in L^1_{loc}(M)$ (meaning that $|X| \in L^1_{loc}(M)$). A function $g \in L^1_{loc}(M)$ is a weak divergence of $X$ if

$$\int_M \phi g = -\int_M \langle \text{grad} \phi, X \rangle, \forall \phi \in C^\infty_0(M).$$

It is clear that there exists at most one weak divergence $g \in L^1_{loc}(M)$ for a given $X \in L^1_{loc}(M)$ and we may write $g = \text{Div} X$. For $C^1$ vector fields $X$ the classical divergence $\text{div} X$ and the weak divergence $\text{Div} X$ coincide.

Definition 1.5 Let $W^{1,1}(M)$ denote the Sobolev space of all vector fields $X \in L^1_{loc}(M)$ possessing weak divergence $\text{Div} X$.

Remark 1.6 If $X \in W^{1,1}(M)$ and $f \in C^1(M)$ then $fX \in W^{1,1}(M)$ with $\text{Div} (fX) = \langle \text{grad} f, X \rangle + f \text{Div} X$. In particular for $f \in C^\infty_0(M)$ we have that

$$\int_M \text{Div} (fX) = \int_M \langle \text{grad} f, X \rangle - \langle \text{grad} f, X \rangle = 0. \quad (3)$$

Conversely, if $fX \in W^{1,1}(M)$ for all $f \in C^\infty_0(M)$ then $X \in W^{1,1}(M)$.

Our main result is the following theorem.

Theorem 1.7 Let $M$ be a Riemannian manifold. Then

$$\lambda^*(M) \geq \sup_{W^{1,1}} \{\inf_M (\text{Div} X - |X|^2)\}. \quad (4)$$

If $M$ is a compact Riemannian manifold with smooth boundary then

$$\lambda_1(M) = \sup_{W^{1,1}} \{\inf_M (\text{Div} X - |X|^2)\}. \quad (5)$$

Our first geometric application of Theorem (1.7) is an extension of Cheng’s lower eigenvalue estimates. We show that inequality (2) is valid for arbitrary geodesic balls $B_N(p, r)$ provided the $(n-1)$-Hausdorff measure $\mathcal{H}^{n-1}(\text{Cut}(p) \cap B_N(p, r)) = 0$, where $\text{Cut}(p)$ is the cut locus of $p$. Moreover, we show that equality of the eigenvalues occurs if and only if $B_N(p, r)$ and $B_{\mathcal{H}^{n-1}(\cdot)}(r)$ are isometric.
Theorem 1.8 Let $N$ be a Riemannian $n$-manifold with radial sectional curvature $K(x)(\partial t, v) \leq c$, $x \in B_N(p, r) \setminus \text{Cut}(p)$, $v \perp \partial t$ and $|v| \leq 1$. Let $\mathbb{N}^n(c)$ be the simply connected $n$-space form of constant sectional curvature $c$ and suppose that $\mathcal{H}^{n-1}(\text{Cut}(p) \cap B_N(p, r)) = 0$. Then
\begin{equation}
\lambda^*(B_N(p, r)) \geq \lambda_1(B_{\mathbb{N}^n(c)}(r)).
\end{equation}
Equality in (6) holds iff $B_N(p, r)$ and $B_{\mathbb{N}^n(c)}(r)$ are isometric.

Our second geometric application (Theorem 1.10) is a generalization of the following Cheng-Li-Yau’s result proved in [8].

Theorem 1.9 (Cheng-Li-Yau, [8]) Let $M^m \subset \mathbb{N}^n(c)$ be an immersed $m$-dimensional minimal submanifold of the $n$-dimensional space form of constant sectional curvature $c \in \{-1, 0, 1\}$ and $D \subset M^m$ a $C^2$ compact domain. Let $a = \inf_{p \in D} \sup_{z \in D} \text{dist}_{\mathbb{N}^n(c)}(p, z) > 0$ be the outer radius of $D$. If $c = 1$ suppose that $a \leq \pi/2$. Then\begin{equation}
\lambda_1(D) \geq \lambda_1(B_{\mathbb{N}^n(c)}(a)).
\end{equation}
Equality in (7) holds iff $M$ is totally geodesic in $\mathbb{N}^n(c)$ and $D = B_{\mathbb{N}^n(c)}(a)$.

Theorem 1.10 Let $N$ be a Riemannian $n$-manifold with radial sectional curvature $K(x)(\partial t, v) \leq c$, for all $x \in B_N(p, r) \setminus \text{Cut}(p)$, and all $v \perp \partial t$ with $|v| \leq 1$. Let $M \subset N$ be an $m$-dimensional minimal submanifold and $\Omega \subset M \cap B_N(p, r)$ be a connected component. Suppose that the $(m-1)$-Hausdorff measure $\mathcal{H}^{m-1}(\Omega \cap \text{Cut}_N(p)) = 0$. If $c > 0$, suppose in addition that $r < \pi/2\sqrt{c}$. Then\begin{equation}
\lambda^*(\Omega) \geq \lambda_1(B_{\mathbb{N}^m(c)}(r)),
\end{equation}
where $B_{\mathbb{N}^m(c)}(r)$ is the geodesic ball with radius $r$ in the simply connected space form $\mathbb{N}^m(c)$ of constant sectional curvature $c$. If $\Omega$ is bounded then equality in (8) holds iff $\Omega = B_{\mathbb{N}^m(c)}(r)$ and $M = \mathbb{N}^m(c)$.

After Nadirashvili’s bounded minimal surfaces in $\mathbb{R}^3$, (see [16]), Yau in [19] asked if the spectrum of a Nadirashvili minimal surface was discrete. A more basic question is if the lower bound of the spectrum of a Nadirashvili minimal surface is positive. The following corollary shows that this is the case.
Corollary 1.11 Let $M \subset B_{\mathbb{R}^3}(r)$ be a complete bounded minimal surface in $\mathbb{R}^3$. Then

$$\lambda^*(M) \geq \lambda_1(D(r)) = c/r^2.$$ 

Where $c > 0$ is an absolute constant.

Let $M \subset \mathbb{R}^3$ be a minimal surface with second fundamental form $A$ and $B_M(p,r)$ be a stable geodesic ball with radius $r$. Schoen, [17] showed that $\|A\|^2(p) \leq c/r^2$ for an absolute constant $c > 0$. We have a converse of Schoen’s result for minimal hypersurfaces of the Euclidean space.

Corollary 1.12 Let $M \subset \mathbb{R}^{n+1}$ be a minimal hypersurface with second fundamental form $A$ and $B_M(r)$ be a geodesic ball with radius $r$. If

$$\sup_{B_M(r)} \|A\|^2 \leq \lambda_1(B_{\mathbb{R}^n}(0, r)) = c(n)/r^2,$$

then $B_M(r)$ is stable. Here $B_{\mathbb{R}^n}(0, r)$ is a geodesic ball of radius $r$ in the Euclidean space $\mathbb{R}^n$ and $c(n) > 0$ is a constant depending on $n$.

Finally we apply Barta’s Theorem and Theorem (1.7) to the theory of quasi-linear elliptic equations.

Theorem 1.13 Let $M$ be a bounded Riemannian manifold with smooth boundary and $F \in C^0(M)$. Consider this problem,

$$\begin{cases} 
\triangle u - |\text{grad} \, u|^2 = F \text{ in } M \\
u = +\infty \text{ on } \partial M.
\end{cases}$$

(9)

If (9) has a smooth solution then $\inf_M F \leq \lambda_1(M) \leq \sup_M F$. If either $\inf_M F = \lambda_1(M)$ or $\lambda_1(M) = \sup_M F$ then $F = \lambda_1(M)$. On the other hand if $F = \lambda$ is a constant then the problem (9) has solution if and only if $\lambda = \lambda_1(M)$.

Now if we allow continuous boundary data on problem (9) we have the following generalization of a result of Kazdan-Kramer [11].

Theorem 1.14 Let $M$ be a bounded Riemannian manifold with smooth boundary and $F \in C^0(M)$ and $\psi \in C^0(\partial M)$. Consider the problem

$$\begin{cases} 
\triangle u - |\text{grad} \, u|^2 = F \text{ in } M \\
u = \psi \text{ on } \partial M.
\end{cases}$$

(10)

then if $\sup_M F < \lambda_1(M)$ then (10) has solution. Moreover if (10) has solution then $\inf_M F < \lambda_1(M)$.
Remark 1.15

1 If we set $f = e^{-u}$ then (9) becomes

$$\begin{cases}
\triangle f + Ff &= 0 \text{ in } M \\
 f &= 0 \text{ on } \partial M.
\end{cases}$$

(11)

Kazdan-Warner in [12] studied this problem (11) and they showed that if $F \leq \lambda_1(M)$, then (11) has solution, with $\sup F = \lambda_1(M)$. Thus Theorem (1.13) is a complementary result to Kazdan-Warner result.

2 If we impose Dirichlet boundary data ($u = 0$ on $\partial M$) on problem (9) then there is a solution if $F \leq \lambda_1(M)$ with strict inequality in a positive measure subset of $M$. This was proved by Kazdan-Kramer in [11].

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2 An extension of Barta’s theorem

We can consider Theorem (1.7) as an extension of Barta’s theorem for if $M$ is a bounded Riemannian manifold with piecewise smooth boundary $\partial M \neq \emptyset$ and $f \in C^2(M) \cup C^0(\overline{M})$ is a positive function on $M$ then for the vector field $X = -\text{grad log } f$ we obtain that $\text{div } X - |X|^2 = -\triangle f/f$.

Theorem (1.7) Let $M$ be a Riemannian manifold. Then

$$\lambda^*(M) \geq \sup_{\mathcal{W}^{1,1}(M)} \{\inf_M \text{div } X - |X|^2\}.$$  

(12)

If $M$ is compact with smooth non-empty boundary then

$$\lambda_1(M) = \sup_{\mathcal{W}^{1,1}(M)} \{\inf_M \text{div } X - |X|^2\}.$$  

(13)

Proof: Let $X \in \mathcal{W}^{1,1}(M)$ and $f \in C^\infty_0(M)$. As observed in the Remark (1.6) we have that $\int_M \text{div } (f^2X) = 0$. On the other hand we have that
\[ 0 = \int_M \text{Div} (f^2X) = \int_M \langle \text{grad} f^2, X \rangle + \int_M f^2 \text{Div} (X) \]
\[ \geq - \int_M |\text{grad} f^2| \cdot |X| + \int_M f^2 \text{Div} X \]
\[ = - \int_M 2 \cdot |f| \cdot |X| \cdot |\text{grad} f| + \int_M f^2 \text{Div} X \]
\[ \geq - \int_M f^2 \cdot |X|^2 - \int_M |\text{grad} f|^2 + \int_M f^2 \text{Div} X \]
\[ = \int_M (\text{Div} X - |X|^2) \cdot f^2 - \int_M |\text{grad} f|^2 \]
\[ \geq \inf_M (\text{Div} X - |X|^2) \int_M f^2 - \int_M |\text{grad} f|^2. \]

Then
\[ \int_M |\text{grad} f|^2 \geq \inf_M (\text{Div} X - |X|^2) \int_M f^2, \]

and thus
\[ \int_M |\text{grad} f|^2 \geq \sup_{W^{1,1} M} \inf_M (\text{Div} X - |X|^2) \int_M f^2. \]

Therefore
\[ \lambda^*(M) \geq \sup_{W^{1,1} M} \inf_M (\text{Div} X - |X|^2). \]

This proves (12). Suppose that \( M \) is compact with smooth non-empty boundary and let \( v \) be its first eigenfunction. If we set \( X_0 = -\text{grad} (\log v) \) then we have that \( \text{div} X_0 - |X_0|^2 = -\Delta v/v = \lambda_1(M) \) and (13) is proven.

**Remark 2.1** This same proof above shows that
\[ \lambda^*(M) \geq \sup_{W^{1,1} M \setminus G} \inf (\text{Div} X - |X|^2), \]

where \( G \) has zero Lebesgue measure.
3 Geometric Applications

In the geometric applications of the Theorem (1.7) we need to know when a given vector field \(X\) is in the Sobolev space \(W^{1,1}(M)\). The following lemma give sufficient conditions.

**Lemma 3.1** Let \(\Omega \subseteq M\) be a bounded domain with piecewise smooth boundary \(\partial \Omega\) in a smooth Riemannian manifold \(M\). Let \(G \subseteq \Omega\) be a closed subset with \((n-1)\)-Hausdorff measure \(H^{n-1}(G) = 0\). Let \(X\) be a vector field of class \(C^1(\Omega \setminus G) \cap L^\infty(\Omega)\) such that \(\text{div}(X) \in L^1(\Omega)\).

Then \(X \in W^{1,1}(\Omega)\).

**Proof:** In fact we are going to show that

\[
\int_\Omega \text{div}(X) = \int_{\partial \Omega} \langle X, \nu \rangle, \tag{14}
\]

where \(\nu\) is the outward unit normal vector field on \(\partial \Omega \setminus Q\) and \(Q \subseteq \partial \Omega\) is a closed subset with \(H^{n-1}(Q) = 0\), see the footnote (1) on page 2. The equation (14) implies that \(X \in W^{1,1}(\Omega)\) since for all \(\phi \in C_0^\infty\) we have that

\[
\int_\Omega \text{div}(\phi X) = \int_\Omega \phi \text{div}(X) + \int_\Omega \langle \text{grad} \phi, X \rangle
\]

and \(\int_\Omega \text{div}(\phi X) = 0\) by (14). Suppose first that \(M = \mathbb{R}^n\). We may assume that \(G\) is connected otherwise we work with each connected component. By Whitney’s Theorem there exists a smooth function (we may suppose to be non-negative) \(f: \mathbb{R}^n \rightarrow \mathbb{R}\) such that \(G = f^{-1}(0)\). By Sard’s theorem we can pick a sequence of regular values \(\epsilon_i \rightarrow 0\) such that \(f^{-1}(\epsilon_i)\) is a smooth \((n-1)\)-dimensional submanifold \(\partial N_i\) bounding a connected set \(N_i = f^{-1}([0, \epsilon_i])\) containing \(G\), moreover, \(N_i \subseteq \Omega\) for \(\epsilon_i\) sufficiently small. Set \(\Omega_i = \Omega \setminus N_i\) and let \(\chi_i\) its characteristic function. Then \(\chi_i \cdot \text{div} X \rightarrow \text{div} X\) a.e. in \(\Omega\) and \(\chi_i \cdot \text{div} X \leq \text{div} X \in L^1(\Omega)\). By the Lebesgue Convergence Theorem \(\int_{\Omega_i} \text{div} X = \int_{\Omega} \chi_i \cdot \text{div} X \rightarrow \int_{\Omega} \text{div} X\). On the other hand applying the divergence theorem to \(X\) on \(\Omega_i\) we obtain

\[
\int_{\Omega_i} \text{div} X = \int_{\partial \Omega_i} \langle X, \nu \rangle
\]

\[
= \int_{\partial \Omega} \langle X, \nu \rangle - \int_{\partial N_i} \langle X, \nu_i \rangle,
\]

\(\nu_i\) is the outward (pointing toward \(G\)) unit vector field normal to \(\partial N_i\).

But \(\left|\int_{\partial N_i} \langle X, \nu_i \rangle\right| \leq Vol_{n-1}(\partial N_i) \|X\|_\infty = Vol_{n-1}(f^{-1}(\epsilon_i)) \|X\|_\infty\) and
$Vol_{n-1}(f^{-1}(\epsilon_i)) \to \mathcal{H}^{n-1}(G) = 0$ as we will show later. Passing to the limit we have
\[
\int_{\Omega} \text{div} X = \lim_{i \to \infty} \int_{\Omega_i} \text{div} X = \int_{\partial \Omega} \langle X, \nu \rangle = \int_{\partial \Omega_i} \langle X, \nu_i \rangle
\]
To show $Vol_{n-1}(f^{-1}(\epsilon_i)) \to \mathcal{H}^{n-1}(G) = 0$ recall the $(n-1)$-dimensional spherical measure of $A \subset \mathbb{R}^n$ is defined by $S^{n-1}(A) = \sup_{\delta > 0} S^0_{\delta}(A) = \lim_{\delta \to 0} S^0_{\delta}(A)$, where $S^0_{\delta}(A) = \inf \sum \text{diam}(B_j)^{n-1}$ the infimum taken over all coverings of $A$ by balls $B_j$ with diameter $\text{diam}(B_j) \leq \delta$. The $(n-1)$-spherical measure is related to $(n-1)$-Hausdorff measure by $\mathcal{H}^{n-1}(A) \leq S^{n-1}(A) \leq 2^{n-1}\mathcal{H}^{n-1}(A)$, $\forall A \subset \mathbb{R}^n$, see [15] for more details. Since $S^{n-1}(G) = 0$ we have that $S^0_{\delta}(G) = 0$ for all $\delta > 0$. Then for $\delta > 0$ and all $k > 0$ there exists a (finite) covering of $G$, $(G$ is compact), by closed balls $\{B_{kj}\}$ of diameter $\text{diam}(B_{kj}) \leq \delta$ such that $\sum_{j} \text{diam}(B_{kj})^{n-1} \leq 1/k$. For $\epsilon_i$ sufficiently small, say $\epsilon_i \leq \epsilon_{i_0}$, the balls $B_{kj}$ are also a covering for the submanifold $f^{-1}(\epsilon_i)$. This means that $S^0_{\delta}(f^{-1}(\epsilon_i)) \leq 1/k$. Taking a sequence $\delta_i \to 0$ we can find a sequence of pairs $(k_{l_i}, \epsilon_{l_i})$, $k_{l_i} \to \infty$, $\epsilon_{l_i} \to 0$ such that $S^0_{\delta_i}(f^{-1}(\epsilon_i)) \leq 1/k_{l_i}$ for all $\epsilon_i \leq \epsilon_{l_i}$. Thus we have that $\lim_{l_i \to \infty} S^0_{\delta_i}(f^{-1}(\epsilon_i)) = 0$ and since $S^0(f^{-1}(\epsilon_i)) \leq S^0_{\delta_i}(f^{-1}(\epsilon_i))$ we have that $\lim_{l_i \to \infty} S^0(f^{-1}(\epsilon_i)) = 0$. On the other $S^0(f^{-1}(\epsilon_i)) \geq \mathcal{H}^{n-1}(f^{-1}(\epsilon_i)) = c(n) \cdot \text{vol}_{n-1}(f^{-1}(\epsilon_i))$. This shows that $\text{vol}_{n-1}(f^{-1}(\epsilon_i)) \to 0$. This completes the proof of Lemma [3.1] for $M = \mathbb{R}^n$. The general case is done similarly using partition of unit.

### 3.1 Geodesic coordinates

Let $M$ be a Riemannian manifold and a point $p \in M$. For each vector $\xi \in T_pM$, let $\gamma_\xi$ the unique geodesic satisfying $\gamma_\xi(0) = p$, $\gamma_\xi'(0) = \xi$ and $d(\xi) = \sup\{t > 0 : \text{dist}_{M}(p, \gamma_\xi(t)) = t\}$. Consider the largest open subset $D_p = \{t \xi \in T_pM : 0 \leq t < d(\xi), |\xi| = 1\}$ of $T_pM$ such that for any $\xi \in D_p$ the geodesic $\gamma_\xi(t) = \exp_p(t \xi)$ minimizes the distance from $p$ to $\gamma_\xi(t)$ for all $t \in [0, d(\xi)]$. The cut locus of $p$ is given by $\text{Cut}(p) = \{\exp_p(d(\xi) \xi), \xi \in T_pM, |\xi| = 1\}$ and $M = \exp_p(D_p) \cup \text{Cut}(p)$. The exponential map $\exp_p : D_p \to \exp_p(D_p)$ is a diffeomorphism and is called the geodesic coordinates of $M \setminus \text{Cut}(p)$. 


Fix a vector $\xi \in T_pM$, $|\xi| = 1$ and denote by $\xi^\perp$ the orthogonal complement of $\{R\xi\}$ in $T_pM$ and let $\tau_t : T_pM \to T_{\exp_p(t\xi)}M$ be the parallel translation along $\gamma_\xi$. Define the path of linear transformations $\mathcal{A}(t, \xi) : \xi^\perp \to \xi^\perp$

by

$$\mathcal{A}(t, \xi)\eta = (\tau_t)^{-1}Y(t)$$

where $Y(t)$ is the Jacobi field along $\gamma_\xi$ determined by the initial data $Y(0) = 0$, $(\nabla_\gamma^\prime \xi Y)(0) = \eta$. Define the map $R(t) : \xi^\perp \to \xi^\perp$

by

$$R(t)\eta = (\tau_t)^{-1}R(\gamma_\xi^\prime(t), \tau_t \eta)\gamma_\xi^\prime(t),$$

where $R$ is the Riemann curvature tensor of $M$. It turns out that the map $R(t)$ is a self adjoint map and the path of linear transformations $\mathcal{A}(t, \xi)$ satisfies the Jacobi equation $\mathcal{A}'' + RA = 0$ with initial conditions $\mathcal{A}(0, \xi) = 0$, $A'(0, \xi) = I$. On the set $\exp_p(D_p)$ the Riemannian metric of $M$ can be expressed by

$$ds^2(\exp_p(t\xi)) = dt^2 + |A(t, \xi)d\xi|^2. \quad (15)$$

Setting $\sqrt{g(t, \xi)} = \det A(t, \xi)$ we have by Rauch comparison theorem this following comparison theorem due to R. Bishop [4], see also [5].

**Theorem 3.2 (Bishop, [4])** If the radial sectional curvatures along $\gamma_\xi$ satisfies $\langle R(\gamma_\xi^\prime(t), v)\gamma_\xi^\prime(t), v \rangle \leq c|v|^2$, $\forall t \in (0, r)$ and if $S_c(t)$ does not vanishes on $(0, r)$ then

$$[\sqrt{g(t, \xi)/S_c^{n-1}(t)}]' \geq 0 \text{ on } (0, r)$$

$$\sqrt{g(t, \xi)} - S_c^{n-1}(t) \geq 0 \text{ on } (0, r) \quad (16)$$

Moreover equality occurs in one of these two inequalities at a point $t_0 \in (0, r)$ iff $R = c \cdot I$ and $A = S_c \cdot I$ on all $[0, t_0]$.

Where $C_c(t) = S_c'(t)$ and $S_c(t)$ is given by

$$S_c(t) = \begin{cases} 
\frac{1}{\sqrt{c}} \sin(\sqrt{c}t), & \text{if } c > 0 \\
\frac{1}{\sqrt{-c}} \sinh(\sqrt{-c}t), & \text{if } c < 0 \\
t, & \text{if } c = 0
\end{cases} \quad (17)$$
3.2 Proof of Theorem 1.8

**Theorem 1.8** Let \( N \) be a Riemannian \( n \)-manifold with radial sectional curvature \( K(x)(\partial t, v) \leq c, x \in B_N(p, r) \setminus \text{Cut}(p) \) and \( v \in T_x N \cap (\partial t)^\perp \) with \( |v| \leq 1 \). Let \( \mathbb{N}^n(c) \) be the simply connected \( n \)-space form of constant sectional curvature \( c \) and suppose that \( \mathcal{H}^{n-1}(\text{Cut}(p) \cap B_N(p, r)) = 0 \). Then

\[
\lambda^*(B_N(p, r)) \geq \lambda_1(B_{\mathbb{N}^n(c)}(r)). \tag{18}
\]

Equality in (18) holds iff \( B_N(p, r) \) and \( B_{\mathbb{N}^n(c)}(r) \) are isometric.

**Proof:** Observe that if \( r > \pi/\sqrt{c} \) there is nothing to prove. Because in this case \( \mathbb{N}^n(c) = \mathbb{S}^n(c) = B_{\mathbb{N}^n(c)}(r) \) and \( \lambda_1(B_{\mathbb{N}^n(c)}(r)) = 0 \). Hence, we may assume that \( r < \pi/\sqrt{c} \) if \( c > 0 \). Let \( v \) be a positive first eigenfunction of \( B_{\mathbb{N}^n(c)}(r) \). It is well known that \( v \) is a radial function satisfying the differential equation

\[
v''(t) + (n - 1) \frac{C_c(t)}{S_c(t)} v'(t) + \lambda_1(B_{\mathbb{N}^n(c)}(r)) v(t) = 0, \quad \forall t \in [0, r] \tag{19}\]

with \( v'(0) \leq 0 \) and with \( v(t) = 0 \) iff \( t = 0 \). Define \( u : B_N(p, r) \to [0, \infty) \) by

\[
u(x) = \begin{cases} v(t) & \text{if } x = \exp_p(t \xi), \ t \in [0, d(\xi)] \cap [0, r] \\ 0 & \text{if } x = \exp_p(d(\xi) \xi) \in \text{Cut}(p) \end{cases}
\]

Set \( X(x) = -\text{grad} \log u(x) \) if \( x \in B_N(p, r) \setminus \{p\} \cup \text{Cut}(p) \) and \( X(x) = 0 \) if \( x \in B_N(p, r) \cap \{p\} \cup \text{Cut}(p) \). The vector field \( X \) is expressed in geodesic coordinates by

\[
X(x) = \begin{cases} -\frac{v'(t)}{v(t)} \cdot \partial_t & \text{if } x = \exp_p(t \xi), \ t \in (0, d(\xi)) \cap (0, r) \\ 0 & \text{if } x = p, \text{ or } x = \exp_p(d(\xi) \xi) \end{cases}
\]

The vector field \( X \in \mathcal{W}^{1,1}(B_N(p, r)) \) as we will prove it later. Setting \( B \setminus G = B_N(p, r) \setminus \{p\} \cup \text{Cut}(p) \) for simplicity of notation we have by Theorem 1.7 and Remark 2.1 we have that

\[
\lambda^*(B_N(p, r)) \geq \inf_{B \setminus G} \{\text{Div} X - |X|^2 \} = \inf_{B \setminus G} \{\text{div} X - |X|^2 \} = \inf_{B \setminus G} -\frac{\Delta u}{u}
\]

By (16) and (19) we have that for \( 0 < t < d(\xi) \)

\[
-\frac{\Delta u}{u}(\exp_p(t \xi)) = -\frac{1}{v(t)} \left\{ v''(t) + \frac{v'(	frac{\xi}{\sqrt{g(t, \xi)}})}{\sqrt{g(t, \xi)}} v'(t) \right\}
\]
\[
\begin{align*}
&= \frac{1}{v(t)} \left\{ v''(t) + (n-1) \frac{S'_c(t)}{S_c(t)} v'(t) \right\} \\
&= \lambda_1(B_{N^v(c)}(r)) \tag{20}
\end{align*}
\]

Thus
\[
\lambda^*(B_N(p,r)) \geq \inf_{B \not\subset G} \left[ -\frac{\Delta u}{u} \right] \geq \lambda_1(B_{N^v(c)}(r))
\]

This proves (18). To handle the equality case in (18), we observe that the set of non-smooth points of the boundary \( \partial B_N(p,r) \) is exactly the intersection \( \text{Cut}(p) \cap \partial B_N(p,r) \) but since \( \mathcal{H}^{n-1}(\text{Cut}(p) \cap \partial B_N(p,r)) = 0 \) the Hausdorff measure \( \mathcal{H}^{n-1}(\text{Cut}(p) \cap \partial B_N(p,r)) = 0 \). Thus \( B_N(p,r) \) has piecewise smooth boundary (see [20] pages 99-100). Hence there exists a positive Dirichlet eigenfunction for \( B_N(p,r) \) with eigenvalue \( \lambda_1(B_N(p,r)) = \lambda^*(B_N(p,r)) \). The equality \( \lambda^*(B_N(p,r)) = \lambda_1(B_{N^v(c)}(r)) \) implies that \( u \) is a first eigenfunction of \( B_N(p,r) \), see Barta’s Theorem [11]. Looking at (20) we see that the equality implies that
\[
\sqrt{g(t,\xi)}' \sqrt{g(t,\xi)} = (n-1) \frac{S'_c(t)}{S_c(t)}
\]
on all \([0, r] \). Thus by Bishop’s Theorem (3.2) we have that \( R = c \cdot I \) and \( A = S_c \cdot I \). This is saying that \( B_N(p,r) \) is isometric to \( B_{N^v(c)}(r) \).

To finish the proof of Theorem (1.8) we need to show that the vector field \( X \in \mathcal{W}^{1,1}(B_N(p,r)) \). As observed in Remark (1.6) it suffices to show that \( f^2X \in \mathcal{W}^{1,1}(B_N(p,r)) \) for all \( f \in C^{\infty}_0(B_N(p,r)) \). Since the \((n-1)\)-Hausdorff measure \( \mathcal{H}^{n-1}(\{p\} \cup \text{Cut}(p)) = 0 \) by the Lemma (3.1) it is sufficient to show that
\[
(i) \ \text{div} (f^2X) \in L^1(B_N(p,r)) \\
(ii) \ f^2X \in C^1(B_N(p,r) \setminus \{p\} \cup \text{Cut}(p)) \cap L^\infty(B_N(p,r)).
\]
The vector field \( X \) is clearly smooth on \( B_N(p,r) \setminus \{p\} \cup \text{Cut}(p) \) thus on this set \( \text{div} X = \text{Div} X \) and \( \text{div} (f^2X) = \langle \text{grad} f^2, X \rangle + f^2 \text{div} X \). Integrating over \( B_N(p,r) \cap \text{Supp}(f) \) we have
\[
\int_{B_N(p,r) \cap \text{Supp}(f)} |\text{div} (f^2X)| \leq \int_{B_N(p,r) \cap \text{Supp}(f)} |\text{grad} f^2||X| \\
+ \int_{B_N(p,r) \cap \text{Supp}(f)} |f^2||\text{div} X| \tag{21}
\]
The first term of (21) is finite
\[
\int_{B_N(p,r) \cap \text{Supp}(f)} |\text{grad } f^2| |X| \leq \sup_{\text{Supp}(f)} \{|\text{grad } f^2| \cdot |X|\} \cdot \text{vol}(B_N(p,r)) < \infty
\]
since \( |X(x)| \leq |(v'/v)(t)| < \infty \) for \( x = \exp_p(t\xi) \in B_N(p,r) \cap \text{Supp}(f) \).
We have that
\[
(f^2 \text{div } X)(x) = f^2(x)[-\frac{v''}{v}(t) + \frac{v'^2}{v^2}(t) - \frac{v'}{v}(t) \sqrt{g(t,\xi)' \frac{1}{g(t,\xi)}}].
\]
Integrating over \( B_N(p,r) \cap \text{Supp}(f) \) we have
\[
\int_{B_N(p,r) \cap \text{Supp}(f)} |f^2||\text{div } X| \leq \int_{t(\xi)} \int_0^{t(\xi)} \left(|\frac{v''}{v}| + |\frac{v'^2}{v^2}| + \frac{v'}{v}(t) \sqrt{g(t,\xi)' \frac{1}{g(t,\xi)}}\right) dt d\xi < \infty
\]
that the second term on the right hand side of (21) is also finite. Where \( t(\xi) \) is the largest \( t < \min\{d(\xi), r\} \) such that \( \exp_p(t\xi) \in \text{Supp}(f) \). This shows that \( \text{div } (f^2 X) \in L^1(B_N(p,r)) \). Showing the item (ii) is a trivial task.

### 3.3 Proof of Theorem 1.10

**Theorem 1.10** Let \( N \) be a Riemannian \( n \)-manifold with radial sectional curvature \( K(x)(\partial_t, v) \leq c \), for all \( x \in B_N(p,r) \setminus \text{Cut}(p) \), and all \( v \perp \partial_t \) with \( |v| \leq 1 \). Let \( M \subset N \) be an \( m \)-dimensional minimal submanifold and \( \Omega \subset M \cap B_N(p,r) \) be a connected component. Suppose that the \((m-1)\)-Hausdorff measure \( \mathcal{H}^{m-1}(\Omega \cap \text{Cut}_N(p)) = 0 \). If \( c > 0 \), suppose in addition that \( r < \pi/2\sqrt{c} \). Then
\[
\lambda^*(\Omega) \geq \lambda_1(B_{N^m(c)}(r)), \tag{22}
\]
where \( B_{N^m(c)}(r) \) is the geodesic ball with radius \( r \) in the simply connected space form \( N^m(c) \) of constant sectional curvature \( c \). If \( \Omega \) is bounded then equality in (22) holds iff \( \Omega = B_{N^m(c)}(r) \) and \( M = N^m(c) \).

**Proof:** Let \( v : B_{N^m(c)}(r) \to \mathbb{R} \) be a positive first Dirichlet eigenfunction of \( B_{N^m(c)}(r) \). It is known that \( v \) is radial with \( v'(t) \leq 0 \) and \( v'(t) = 0 \) iff
where $\phi$ such that we have that $|X \mathsf{div} \ e$ and it can be shown that the vector field $X$ by $\triangle \ t \ \xi$. Recall that for each $G$ is not smooth at $\phi$ $u$ $\gamma$ the distance from $\varphi$ and $\psi$ $\in \gamma$. $|X \mathsf{div} \ e$ is given by the following formula, (see [3], [7], [13]), and satisfies (14) and by Theorem (1.7) we have that

$$\lambda^*(\Omega) \geq \inf_{\Omega \backslash G} \left[ \text{Div} X - |X|^2 \right] = \inf_{\Omega \backslash G} \left[ \text{div} X - |X|^2 \right] = \inf_{\Omega \backslash G} \left[ -\triangle \psi / \psi \right].$$

Where $\triangle \psi$ is given by the following formula, (see [3], [7], [13]),

$$\triangle \psi(x) = \sum_{i=1}^{m} \text{Hess} \ u(\varphi(x))(e_i, e_i) + \langle \text{grad} \ u, \vec{H} \rangle$$

where $\varphi(x) = \exp_p(t \xi)$, $\vec{H} = 0$ is the mean curvature vector of $\Omega$ at $\varphi(x)$ and $\{e_1, \ldots, e_m\}$ is an orthonormal basis for $T_{\varphi(x)} \Omega$. Choose this basis such that $e_2, \ldots e_m$ are tangent to the distance sphere $\partial B_N(p, t) \subset N$ and $e_1 = \cos(\beta(x)) \partial / \partial t + \sin(\beta(x)) \partial / \partial \theta$, where $\partial / \partial \theta \in [[e_2, \ldots e_m]]$, $|\partial / \partial \theta| = 1$. From (24) we have for $\varphi(x) \in \Omega \backslash G$ that

$$\triangle \psi(x) = \sum_{i=1}^{m} \text{Hess} \ u(\varphi(x))(e_i, e_i)$$

$$= v''(t)(1 - \sin^2 \beta(x))$$

$t = 0$. We can normalize $v$ such that $v(0) = 1$. The differential equation

$$\Delta_{\mathcal{N}(x)} v(t) + \lambda_1(B_{\mathcal{N}(x)}(r)) v(t) = 0 \ is \ expressed \ in \ geodesic \ coordinates \ by \ on \ \mathfrak{M}(x) \ v''(t) + \lambda_1(B_{\mathcal{N}(x)}(r)) v(t) = 0, \ \forall \ t \in [0, r]. \ (23)$$

Recall that for each $\xi \in T_p N$, $|\xi| = 1$, $d(\xi) > 0$ is the largest real number (possibly $\infty$) such that geodesic $\gamma_\xi(t) = \exp_p(t \xi)$ minimizes the distance from $\gamma_\xi(0) = p$ to $\gamma_\xi(t)$ for all $t \in [0, d(\xi)]$. We have that $B_N(p, r) \cap \text{Cut}(p) = \exp_p(\{t \xi \in T_p N : 0 \leq t < \min\{r, d(\xi)\}, |\xi| = 1\})$. Define $u : B_N(p, r) \to \mathbb{R}$ by $u(\exp_p(t \xi)) = v(t)$ if $t < \min\{r, d(\xi)\}$ and $u(r \xi) = u(d(\xi) \xi) = 0$. Let $\Omega \subset M \cap B_N(p, r)$ be a connected component and $\psi : \Omega \to \mathbb{R}$ defined by $\psi = u \circ \varphi$, where $\varphi$ is the minimal immersion $\varphi : M \subset N$. The vector field $X = -\text{grad} \ log \ \psi$ identified with $d\varphi(X)$ is not smooth at $G = \Omega \cap (\{p\} \cup \text{Cut}(p))$. By hypothesis $\mathcal{H}^{m-1}(G) = 0$ and it can be shown that the vector field $X \in C^1(\Omega \backslash G) \cap L^\infty(\Omega)$ and $\text{div} X \in L^1(\Omega)$ thus $X \in \mathcal{W}^{1,1}$ and satisfies (14) and by Theorem (1.7) we have that

$$\lambda^*(\Omega) \geq \inf_{\Omega \backslash G} \left[ \text{Div} X - |X|^2 \right] = \inf_{\Omega \backslash G} \left[ \text{div} X - |X|^2 \right] = \inf_{\Omega \backslash G} \left[ -\triangle \psi / \psi \right].$$
\( v'(t) \sin^2 \beta(x) \text{Hess}(t)(\partial/\partial \theta, \partial/\partial \theta) \)

\( + v'(t) \sum_{i=2}^{m} \text{Hess}(t)(e_i, e_i) \)

where \( t = \text{dist}_{N}(p, x) \). Adding and subtracting \((C_c/S_c)(t) v'(t) \sin^2 \beta(x)\) and \((m-1)(C_c/S_c)(t) v'(t)\) in (25) we have

\[
\triangle \psi(x) = v''(t) + (m-1) \frac{C_c}{S_c}(t) v'(t)
\]

\[+ \left( \text{Hess}(t)(\partial/\partial \theta, \partial/\partial \theta) - \frac{C_c(t)}{S_c(t)} \right) v'(t) \sin^2 \beta(x) \]

\[+ \sum_{i=2}^{m} \left[ \text{Hess}(t)(e_i, e_i) - \frac{C_c}{S_c}(t) \right] v'(t) \]

\[+ \left( \frac{C_c(t)}{S_c(t)} v'(t) - v''(t) \right) \sin^2 \beta(x) \]

From (23) and (26) we have that

\[
- \frac{\Delta \psi}{\psi}(x) = \lambda_1(B_{\text{N}(c)}(r))
\]

\[- \left( \text{Hess}(t)(\partial/\partial \theta, \partial/\partial \theta) - \frac{C_c(t)}{S_c(t)} \right) \frac{v'(t)}{v(t)} \sin^2 \beta(x) \]

\[- \sum_{i=2}^{m} \left[ \text{Hess}(t)(e_i, e_i) - \frac{C_c}{S_c}(t) \right] \frac{v'(t)}{v(t)} \]

\[- \frac{1}{v(t)} \left( \frac{C_c(t)}{S_c(t)} v'(t) - v''(t) \right) \sin^2 \beta(x) \]

Since the radial curvature \( K(x)(\partial t, v) \leq c \) for all \( x \in B_{N}(p, r) \setminus \text{Cut}(p) \) and all \( v \perp \partial t \) with \(|v| \leq 1\) then by the Hessian Comparison Theorem (see [18]) we have that \( \text{Hess}(t(x))(v, v) \geq (C_c/S_c)(t) \) for all \( v \perp \partial t, \ t(x) = t, \ x = \exp_p(t \xi) \). But \( v'(t) \leq 0 \) then we have that the second and third terms of (27) are non-negative. If the fourth term of (27) is non-negative then we would have that

\[
- \frac{\Delta \psi}{\psi}(x) \geq \lambda_1(B_{\text{N}(c)}(r))
\]
By Theorem (1.7) we have that
\[ \lambda^*(\Omega) \geq \inf(-\frac{\Delta \psi}{\psi}) \geq \lambda_1(B_{\mathbb{W}^m(c)}(r)). \]  
(28)

This proves (22). We can see that
\[ -\left( \frac{C_c(t)}{S_c(t)} \frac{v'(t)}{v(t)} - \frac{v''(t)}{v(t)} \right) \sin^2 \beta(x) \geq 0 \]
is equivalent to
\[ m \frac{C_c(t)}{S_c(t)} v'(t) + \lambda_1(B_{\mathbb{W}^m(c)}(r)) v(t) < 0, \quad t \in (0, r). \]  
(29)

To prove (29) we will assume without loss of generality that \( c = -1, 0, 1 \). Let us consider first the case \( c = 0 \) that presents the idea of the proof. The other two remaining cases \( c = -1 \) and \( c = 1 \) we are going to treat (quickly) with the same idea. When \( c = 0 \) the inequality (29) becomes
\[ \frac{mv'(t)}{t} + \lambda_1 v(t) < 0, \quad t \in (0, r), \]  
(30)

where \( \lambda_1 := \lambda_1(B_{\mathbb{W}^m(c)}(r)). \) Let \( \mu(t) := \exp\{-\frac{\lambda_1 t^2}{2m}\}. \) The functions \( v \) and \( \mu \) satisfy the following identities,
\[ (t^{m-1}v'(t))' + \lambda_1 t^{m-1}v(t) = 0 \]
\[ (t^{m-1}\mu'(t))' + \lambda_1 t^{m-1}(1 - \frac{\lambda_1 t^2}{m^2})\mu(t) = 0 \]  
(31)

In (31) we multiply the first identity by \( \mu \) and the second by \(-v\) adding them and integrating from 0 to \( t \) the resulting identity we obtain,
\[ t^{m-1}v'(t)\mu(t) - t^{m-1}v(t)\mu'(t) = -\frac{\lambda_1^2}{m^2} \int_0^t \mu(t)v(t) \, dt < 0, \quad \forall t \in (0, r). \]

Then \( \mu(t)v'(t) < \mu'(t)v(t) \) and this proves (30).
Assume that now that \( c = -1 \). The inequality (29) becomes
\[ m \frac{C_{-1}(t)}{S_{-1}(t)} v'(t) + \lambda_1 v(t) < 0 \]  
(32)

Set \( \mu(t) := C_{-1}(t)^{-\lambda_1/m}. \) The functions \( v \) and \( \mu \) satisfy the following identities
\[ (S_{-1}^{m-1}v')' + \lambda_1 S_{-1}^{m-1}v = 0 \]
\[ (S_{-1}^{m-1}\mu')' + \lambda_1 S_{-1}^{m-1} \left( \frac{m-1}{m} - \frac{1}{mC_{-1}^2} - \frac{\lambda_1 S_{-1}^2}{m^2 C_{-1}^2} \right) \mu = 0 \]  
(33)
In (33) we multiply the first identity by $\mu$ and the second by $-v$ adding them and integrating from 0 to $t$ the resulting identity we obtain

$$S_{m-1}^{-1} (v'\mu - \mu'v) (t) + \int_0^t \lambda_1 S_{m-1}^{-1} \left[ \frac{1}{m} - \frac{1}{mC_{m-1}^2} + \frac{\lambda_1 S_{m}^2}{m^2 C_{m-1}^2} \right] \mu v = 0$$

The term $S_{m-1}^{-1} \left[ \frac{1}{m} - \frac{1}{mC_{m-1}^2} + \frac{\lambda_1 S_{m}^2}{m^2 C_{m-1}^2} \right] \mu v$ is positive (one can easily check) therefore we have that $(v'\mu - \mu'v)(t) < 0$ for all $t \in (0, r)$. This proves (32).

For $c = 1$ the inequality (29) becomes the following inequality

$$m C_1 v'(t) + \lambda_1 v(t) < 0, \quad 0 < t < \pi/2$$

(34)

Set $\mu(t) := C_1(t)^{-\lambda_{1/m}}$, $0 < t < \pi/2$. The functions $v$ and $\mu$ satisfy the following identities

$$(S_{m-1}^{-1} v')' + \lambda_1 S_{m-1}^{-1} v = 0$$

$$(S_{m-1}^{-1} \mu')' - \lambda_1 S_{m-1}^{-1} \left( \frac{m-1}{m} + \frac{1}{mC_1^2} + \frac{\lambda_1 S_{m}^2}{m^2 C_1^2} \right) \mu = 0$$

(35)

In (35) we multiply the first identity by $\mu$ and the second by $-v$ adding them and integrating from 0 to $t$ the resulting identity we obtain

$$S_{m-1}^{-1} (v'\mu - \mu'v) (t) + \int_0^t \lambda_1 S_{m-1}^{-1} \left[ 2 - \frac{1}{m} + \frac{1}{mC_1^2} + \frac{\lambda_1 S_{m}^2}{m^2 C_1^2} \right] \mu v = 0$$

The term $S_{m-1}^{-1} \left[ 2 - \frac{1}{m} + \frac{1}{mC_1^2} + \frac{\lambda_1 S_{m}^2}{m^2 C_1^2} \right] \mu v$ is positive therefore we have that $(v'\mu - \mu'v)(t) < 0$ for all $t \in (0, r)$. This proves (34) and thus the fourth term in (27) is non-negative.

To finishes the proof of the Theorem (1.10) we need to consider the equality case in (22) when $\Omega$ is bounded. From the spectral theory it is known that for a given bounded domain in a Riemannian manifold there is $u \in C^\infty(\Omega) \cap H_0^1(\Omega)$, positive in $\Omega$ satisfying $\Delta u + \lambda_1(\Omega) u = 0$, where $\lambda_1(\Omega) = \lambda^*(\Omega)$. This function is also called an eigenfunction. Observe that $u|_{\partial\Omega} = 0$ only a.e. thus $u$ is not considered a solution for the Dirichlet eigenvalue problem. The proof of existence of $u$ is the same proof of existence of Dirichlet eigenvalues for a smooth domain, since it (the proof) does need smoothness of the boundary but the boundedness of the domain. With an approximation argument Barta’s Theorem can be extend to arbitrary bounded open sets.
Proposition 3.3 Let $\Omega$ be a bounded domain in a smooth Riemannian manifold. Let $v \in C^2(\Omega) \cap C^0(\overline{\Omega})$, $v > 0$ in $\Omega$ and $v|\partial\Omega = 0$. Then

$$\lambda^*(\Omega) \geq \inf_{\Omega} (-\frac{\triangle v}{v}).$$

(36)

Moreover, $\lambda^*(\Omega) = \inf_{\Omega} (-\frac{\triangle v}{v})$ if and only if $v = u$, where $u$ is a positive eigenfunction of $\Omega$, i.e. $\triangle u + \lambda^*(\Omega) u = 0$.

Proof: Let $\epsilon_i \to 0$ be a sequence of positive regular values of $v$ and let $\Omega^v_{\epsilon_i} = \{ x \in \Omega; v(x) > \epsilon_i \}$. Applying Barta’s Theorem we have that

$$\lambda^*(\Omega^v_{\epsilon_i}) = \lambda_1(\Omega^v_{\epsilon_i}) \geq \inf_{\Omega^v_{\epsilon_i}} (-\frac{\triangle v}{v}) \geq \inf_{\Omega} (-\frac{\triangle v}{v})$$

(37)

But $\lim_{\epsilon_i \to 0} \lambda^*(\Omega^v_{\epsilon_i}) = \lambda^*(\Omega)$, see in [3], page 23. Let $u \in C^\infty \cap H^1_0(\Omega)$ be a positive eigenfunction of $\Omega$. Then

$$\lambda^*(\Omega) = -\frac{\triangle u}{u} = \frac{\triangle v}{v} + \frac{u \triangle v - v \triangle u}{uv}$$

(38)

Claim: $\int_{\Omega} (u \triangle v - v \triangle u) = 0$.

Proof: Let $\epsilon_i \to 0$ be a sequence of positive regular values of $u$ and let $\Omega^u_{\epsilon_i} = \{ x \in \Omega; u(x) > \epsilon_i \}$. Set

$$u_{\epsilon_i} = \begin{cases} u - \epsilon_i & \text{on } \Omega^u_{\epsilon_i} \\ 0 & \text{on } \Omega \setminus \Omega^u_{\epsilon_i} \end{cases}$$

(39)

One can show that $u_{\epsilon_i} \to u$ in $H^1(\Omega)$ using the Lebesgue Convergence Theorem. Therefore

$$\int_{\Omega^v_{\epsilon_i}} (u_{\epsilon_i} \triangle v - v \triangle u_{\epsilon_i}) = -\int_{\Omega^v_{\epsilon_i}} \langle \text{grad } u_{\epsilon_i}, \text{grad } v \rangle + \lambda^*(\Omega) \int_{\partial\Omega^v_{\epsilon_i}} uv$$

$$\to -\int_{\Omega} \langle \text{grad } u, \text{grad } v \rangle + \lambda^*(\Omega) \int_{\partial\Omega} uv = 0$$

(40)

Since $v \in H^1_0(\Omega)$ and $u$ is a weak solution of $\triangle u + \lambda^*(\Omega) u = 0$. On the other hand $\int_{\Omega^v_{\epsilon_i}} (u_{\epsilon_i} \triangle v - v \triangle u_{\epsilon_i}) \to \int_{\Omega} (u \triangle v - v \triangle u)$.
If $Ω$ is bounded we have that $\partial ϕ(Ω) \subset \partial B_N(p, r)$. This implies that the function $ψ = u \circ ϕ \in C^2(Ω) \cap C^0(Ω)$ is such that $ψ|\partial Ω = 0$. Suppose that $λ^*(Ω) = λ_1(B_{N^m(c)}(r))$. Then by Proposition 3.3, $ψ : Ω → \mathbb{R}$ is an eigenfunction of $Ω$ and we have that $λ^*(Ω) = −\triangle ψ/ψ$. From (27) we have that

$$\left( \text{Hess}(t)(∂/∂θ, ∂/∂θ) − C_c(t) \frac{v'(t)}{v(t)} \right) \sin^2 β(x) = 0$$

$$m \sum_{i=2}^{m} \left[ \text{Hess}(t)(e_i, e_i) − C_c(t) \frac{v'(t)}{v(t)} \right] = 0$$

$$\frac{1}{v(t)} \left( C_c(t) \frac{v'(t)}{S_c(t)} v''(t) \right) \sin^2 β(x) = 0,$$

for all $t$ such that $ϕ(x) = \exp_p(tξ) \in Ω$. This implies $\sin^2 β(x) = 0$ for all $x \in Ω$ and we have that $e_1(ϕ(x)) = ∂/∂t$. Integrating the vector field $∂/∂t$ we have a minimal geodesic (in $N \cap ϕ(Ω)$) joining $ϕ(x)$ to the center $p$. This imply that $Ω$ is the geodesic ball in $M$ centered at $ϕ^{−1}(p)$ with radius $r$ i.e. $Ω = B_M(ϕ^{−1}(p), r)$. Since $ψ$ is an eigenfunction with the same eigenvalue $λ_1(B_{N^m(c)}(r))$ we have that

$$\triangle_M v(t) = \triangle_{N^m(c)} v(t), \quad t = \text{dist}_N(p, ϕ(q)), \forall q \in Ω. \quad (42)$$

Rewriting this identity (42) in geodesic coordinates we have that

$$\frac{\sqrt{g(t, ξ)}}{\sqrt{g(t, ξ)}}(t, θ)v'(t) + v''(t) = (m − 1) \frac{C_c(t)}{S_c(t)} v'(t) + v''(t)$$

This imply that by Bishop Theorem $Ω = B_M(ϕ^{−1}(p)$ and $B_{N^m(c)}(r)$ are isometric. By analytic continuation $M = N^m(c)$.

### 3.4 Proof of Corollary 1.11

Let $M \subset B_{\mathbb{R}^3}(r)$ be a complete bounded minimal surface in $\mathbb{R}^3$. Then

$$λ^*(M) ≥ λ_1(∅(r)) = c/r^2.$$

Where $c > 0$ is an absolute constant. The proof of this result follows directly from Theorem 1.10. The theorem says that the fundamental tone $λ^*(Ω) ≥ λ_1(B_{\mathbb{R}^3}(r))$ for any connected component of $M \cap B_{\mathbb{R}^3}(r)$. In particular, for $Ω = M$ we have that $λ^*(M) ≥ λ_1(B_{\mathbb{R}^3}(r)) = c(2)/r^2$, where $c(2)$ is the first zero of the Bessel function $J_0$, see [5], page 46.
3.5 Proof of Corollary 1.12

Let $\varphi : M \hookrightarrow N^{n+1}$ be a complete orientated minimal hypersurface and $A(X) = -\nabla_X \eta$ its second fundamental form, where $\eta$ is globally defined unit vector field normal to $\varphi(M)$. A normal domain $D \subset M$ is said to be stable if the first Dirichlet eigenvalue $\lambda_1^D = \inf\{-\int_D u \, Lu/\int_D u^2, u \in C^\infty_0(D)\}$ of the operator $L = \Delta + \text{Ric}(\eta) + \|A\|^2$ is positive. On the other hand we have that

$$-\int_D u \, Lu = \int_D \left[|\text{grad} \, u|^2 - (\text{Ric}(\eta) + \|A\|^2) \, u^2\right] \geq \int_D \lambda_1^\Delta(D) - (\text{Ric}(\eta) + \|A\|^2) \, u^2.$$

Therefore if $\lambda_1^\Delta(D) \geq \sup_{x \in D}\{\text{Ric}(\eta) + \|A\|^2(x)\}$ then $D$ is stable. In [3] we give estimates for $\lambda_1^\Delta(D)$ in submanifolds with locally bounded mean curvature, in particular minimal hypersurfaces. With those estimates we have obvious statements for stability theorems. If $D = B_{M}(p,r)$ is a ball in $M^n \hookrightarrow \mathbb{R}^{n+1}$ obviously that $\varphi(D) \subset B_{n+1}(\varphi(p),r)$. And we have that $\lambda_1(D) \geq \lambda^*(\varphi^{-1}(B_{n+1}(\varphi(p),r))) \geq \lambda_1(B_n(0,r))$. This proves the Corollary 1.12.

3.6 Quasilinear elliptic equations

In this section we want to apply Barta’s Theorem to study the existence of solutions to certain quasi-linear elliptic equations. Let $M$ be a bounded Riemannian manifold with smooth non-empty boundary $\partial M$ and $f \in C^2(M)$, $f > 0$. If we set $u = -\log f^2$, $f > 0$ then the problem $\Delta f + F f = 0$ becomes $\Delta u - |\text{grad} \, u|^2 = F$. Hence Barta’s Theorem 1.1 can be translated as

**Theorem 1.13** Let $M$ be a bounded Riemannian manifold with smooth boundary and $F \in C^0(\overline{M})$. Consider this problem,

$$\begin{cases}
\Delta u - |\text{grad} \, u|^2 = F & \text{in } M \\
u = +\infty & \text{on } \partial M.
\end{cases} \tag{43}$$

If (43) has a smooth solution then $\inf_M F \leq \lambda_1(M) \leq \sup_M F$. If either $\inf_M F = \lambda_1(M)$ or $\lambda_1(M) = \sup_M F$ then $F = \lambda_1(M)$. On the other hand if $F = \lambda$ is a constant the problem (43) has solution if and only if $\lambda = \lambda_1(M)$.

\(^2\)This transformation $u = -\log f$ we learned from Kazdan & Kramer [11] but it also appears in [9].
Likewise Theorem (1.7) can be translated into language of quasi-linear elliptic equations.

**Theorem 1.14** Let $M$ be a bounded Riemannian manifold with smooth boundary and $F \in C^0(M)$ and $\psi \in C^0(\partial M)$. Consider the problem

$$
\begin{cases}
\Delta u - |\text{grad } u|^2 = F & \text{in } M \\
u = \psi & \text{on } \partial M.
\end{cases}
$$

(44)

then

a) If $\sup_M F < \lambda_1(M)$ then (44) has solution.

b) If (44) has solution then $\inf_M F < \lambda_1(M)$.

**Proof:** The operator $L = -\Delta - F$ is compact thus its spectrum is a sequence of eigenvalues $\lambda_1^L < \lambda_2^L \leq \lambda_3^L \leq \cdots \rightarrow \infty$. Suppose that $\sup_M F < \lambda_1(M)$ then we have that

$$
\lambda_1^L = \inf \left\{ \int_M |\text{grad } u|^2 - F u^2 ; \ u \in H^1_0(M), \int_M u^2 = 1 \right\}
\geq \lambda_1(M) - \sup_M F > 0.
$$

Therefore $L$ is invertible. On the other hand, $u = -\log f$ is a solution of (44) if and only if $f$ is solution of

$$
\begin{cases}
L f = 0 & \text{in } M \\
f = e^{-\psi} \text{ on } \partial M
\end{cases}
$$

(45)

Consider the harmonic extension $v$ of $e^{-\psi}$ on $\partial M$, ($\Delta v = 0$ in $M$ and $v = e^{-\psi}$ on $\partial M$), and $h = f - v$. We have that $L h = F v$ in $M$ and $h = 0$ on $\partial M$. Then $h = L^{-1}(F v)$ and $f = -v + L^{-1}(F v)$ is solution of (45).

Item b. Suppose that the equation (44) has a smooth solution $u$. Let $\varphi$ be the first eigenfunction of the operator $-\Delta$, $(\Delta \varphi + \lambda_1(M) \varphi = 0$ in $M$ and $\varphi|\partial M = 0)$, and $f$ a solution of (44). By Green we have that

$$
\int_M f \Delta \varphi - \varphi \Delta f = \int_{\partial M} e^{-\psi} \frac{\partial \varphi}{\partial \eta} - \varphi \frac{\partial f}{\partial \eta} = \int_{\partial M} e^{-\psi} \frac{\partial \varphi}{\partial \eta}.
$$

Thus

$$
\int_{\partial M} e^{-\psi} \frac{\partial \varphi}{\partial \eta} = \int_M (F - \lambda_1) \varphi f
$$

21
if $v$ is the harmonic extension $v$ of $e^{-\psi}$ then we have

$$
\int_M -\lambda_1 f \varphi = \int_M v \Delta \varphi - \varphi \Delta v = \int_{\partial M} v \frac{\partial \varphi}{\partial \eta} - \varphi \frac{\partial v}{\partial \eta} = \int_{\partial M} e^{-\psi} \frac{\partial \varphi}{\partial \eta} < 0
$$

since $\inf_M v = \inf_{\partial M} v = \inf_{\partial M} e^{-\psi} > 0$. Therefore $\int_M (F - \lambda_1)|\varphi| f < 0$ and $\inf F < \lambda_1(M)$.

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