IMMERSED DISKS, SLICING NUMBERS AND CONCORDANCE UNKNOTTING NUMBERS

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Abstract. We study three knot invariants related to smoothly immersed disks in the four-ball. These are the four-ball crossing number, which is the minimal number of normal double points of such a disk bounded by a given knot; the slicing number, which is the minimal number of crossing changes to a slice knot; and the concordance unknotting number, which is the minimal unknotting number in a smooth concordance class. Using Heegaard Floer homology we obtain bounds that can be used to determine two of these invariants for all prime knots with crossing number ten or less, and to determine the concordance unknotting number for all but thirteen of these knots. We also give a strengthened version of Ozsváth and Szabó’s obstruction to unknotting number one.

1. Introduction

The unknotting number $u(K)$ of a knot $K$ in $S^3$ is the minimal number of crossing changes required to convert it to an unknot. The trace of a regular homotopy realizing the crossing changes yields a normally immersed annulus $A$ in $S^3 \times [0,1]$ with a singularity for each crossing change. We say a surface is normally immersed if the immersion is proper and the only singularities are normal double points (also known as normal crossings), i.e. transverse double points in the interior of the surface. Since the other boundary of $A$ is an unknot we can complete $A$ to a normally immersed disk $\Delta$ in $B^4$ with boundary $K$ and $u(K)$ double points. Minimising the number of double points in any such disk with boundary $K$ gives a concordance invariant of $K$, the 4-ball crossing number $c^*(K)$ [22, 9, 5], also referred to as the 4-dimensional clasp number. Recall that $K$ and $K'$ are concordant if they cobound a properly embedded annulus in $S^3 \times I$. A knot $K$ is slice if it is concordant to the unknot or equivalently if it bounds a smoothly embedded disk in $B^4$.

Since by Proposition [2, 1] any normally immersed disk in $B^4$ can be factored into a concordance, followed by the trace of a regular homotopy, followed by a nullconcordance, it is natural to consider two intermediate invariants. The first is the slicing number $u_s(K)$ [22, 21, 11, 6, 13] which is the minimal number of crossing changes required to obtain a slice knot. This clearly gives rise to a normally immersed disk, obtained by capping off the trace of the crossing-change homotopy by a slice disk.

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The second intermediate invariant is the concordance unknotting number $u_c(K)$ which is the minimal unknotting number of any knot in the concordance class of $K$. Noting that a normal double point may be resolved at the cost of increasing the genus of the immersed surface by one, we have the following inequalities:

\[(1) \quad u(K) \geq u_c(K), u_s(K) \geq c^*(K) \geq g^*(K) \geq |\sigma(K)|/2,\]

where $g^*(K)$ denotes the smooth 4-ball genus of $K$, and the last inequality is due to Murasugi [10]. In this article we develop some tools to calculate these invariants, and we determine each of $u_s$ and $c^*$ for all prime knots with 10 or fewer crossings, and $u_c$ for all but three knots up to 9 crossings and all but ten 10-crossing knots. Our results make use of Montesinos’ trick [8] which states that the double branched cover of an unknotting number one knot is given by a half-integer surgery on a knot in $S^3$, and also theorems of Cochran and Lickorish [3], Ozsváth and Szabó [17], and Ni and Wu [11].

Using Theorem 2.2 we obtain the following result which shows that the slicing number obstruction given in [13, Theorem 2] in fact applies to the 4-ball crossing number.

**Theorem 1.** Suppose that $K$ bounds a normally immersed disk with $r_+$ positive double points and $r_- = \sigma(K)/2$ negative double points. Then the branched double cover $\Sigma(K)$ bounds a positive-definite smooth four-manifold $X$ with $b_2(X) = 2(r_+ + r_-)$ whose intersection form $Q_X$ is of half-integer surgery type, with exactly $r_+$ of the diagonal entries odd, and $\det Q_X$ divides $\det K$ with quotient a square.

Recall that a quadratic form over the integers is of half-integer surgery type if with respect to some basis for the lattice it is represented by a matrix of the form

\[
\begin{bmatrix}
A & I \\
I & 2I
\end{bmatrix},
\]

where $I$ denotes the identity matrix.

In case of $c^*(K) = 1$ we can express the above obstruction explicitly in terms of the correction terms of the double branched cover of $K$. Recall that the set of spin$^c$ structures on a 3-manifold $Y$ is an affine copy of $H_1(Y; \mathbb{Z})$, with additional structure given by conjugation of spin$^c$ structures, fixed points of which are spin structures on $Y$. When $Y$ is the double cover of $S^3$ branched along a knot there is a canonical identification of Spin$^c(Y)$ with the homology group of $Y$ by using the unique spin structure as the origin; we use this implicitly throughout. Recall also that for a spin$^c$ structure $\mathfrak{s}$ on a rational homology three-sphere $Y$, Ozsváth and Szabó [15] defined the correction term or $d$-invariant, $d(Y, \mathfrak{s})$, as the absolute grading of a particular subgroup of the Heegaard Floer homology group of $(Y, \mathfrak{s})$. The $d$-invariants are rational numbers which are computable in many cases.

**Corollary 2.** Let $K \subset S^3$ be a knot with four-ball crossing number one. Suppose $K$ has signature $\sigma(K) = 2$ and let $Y = \Sigma(K)$ be the double branched cover of $K$. Then
for some factorization $\det K = rs^2$ there exists an order $rs$ subgroup $H \leq \text{Spin}^c(Y)$ and an epimorphism $\phi: H \to \mathbb{Z}/r\mathbb{Z}$ such that the normalized $d$-invariants

$$\tilde{d}_i = -d(Y, t) + i^2/2r - \begin{cases} 
0 & i \equiv (r-1)/2 \pmod{2} \\
1/2 & i \equiv (r+1)/2 \pmod{2}
\end{cases}$$

for $i = 0, \ldots, (r-1)/2$ and for every $t \in \phi^{-1}(i)$ satisfy

(i) positivity: $\tilde{d}_i \geq 0$;
(ii) evenness: $\tilde{d}_i \in 2\mathbb{Z}$.

If there exists an epimorphism $\phi$ as above, we say that $Y$ admits a positive even subgroup matching.

If $K$ has signature zero, then at least one of $\pm \Sigma(K)$ admits a positive even subgroup matching.

Note that by conjugation invariance of $d$-invariants, the constraint given in Corollary 2 extends to $-(r-1)/2 \leq i \leq (r-1)/2$. This applies to Theorems 3 and 4 below also.

The following theorem concerning manifolds given as half-integer surgery on a knot in $S^3$ is an extension of results of Ozsváth and Szabó [17], where the first three conditions are established.

**Theorem 3.** Suppose that a rational homology sphere $Y$, with $|H_1(Y)| = r$ odd, is given by $r/2$ surgery on a knot in $S^3$. Then there exists a group isomorphism

$$\phi : \text{Spin}^c(Y) \to \mathbb{Z}/r\mathbb{Z}$$

such that the normalized $d$-invariants

$$\tilde{d}_i = -d(Y, \phi^{-1}(i)) + i^2/2r - \begin{cases} 
0 & i \equiv (r-1)/2 \pmod{2} \\
1/2 & i \equiv (r+1)/2 \pmod{2}
\end{cases}$$

for $i = 0, \ldots, (r-1)/2$ satisfy the following conditions:

(i) positivity: $\tilde{d}_i \geq 0$;
(ii) evenness: $\tilde{d}_i \in 2\mathbb{Z}$;
(iii) symmetry:

$$\tilde{d}_{2j} = \tilde{d}_{2j+1} \text{ for } 0 \leq j \leq (r-5)/4 \text{ if } r \equiv 1 \pmod{4},$$
$$\tilde{d}_{2j-1} = \tilde{d}_{2j} \text{ for } 1 \leq j \leq (r-3)/4 \text{ if } r \equiv -1 \pmod{4};$$
(iv) monotonicity: $\tilde{d}_i \leq \tilde{d}_{i+1}$ for $0 \leq i < (r-1)/2$;
(v) boundedness: $\tilde{d}_{i+1} \leq \tilde{d}_i + 2$ for $0 \leq i < (r-1)/2$.

If there exists an isomorphism $\phi$ satisfying the conditions of Theorem 3 we say that $Y$ admits a positive even symmetric monotone matching. In case the manifold $Y$ is an $L$-space the normalized $d$-invariants $\tilde{d}_i$ of Theorem 3 are given by the torsion
coefficients defined using the Alexander polynomial of the surgery knot, as in Theorem 1.2 of [19]. Conditions (iv) and (v) follow easily in that case. For many examples previously obstructed using the symmetry condition, the monotonicity condition may be substituted. In fact, we have not yet found an example which is obstructed by symmetry but not by monotonicity, or vice versa. More interestingly perhaps, the obstruction given by conditions (i), (ii) and (iv) of Theorem 3 extends over rational homology cobordisms as in the following statement.

**Theorem 4.** Suppose that a rational homology sphere $Y$, with $|H_1(Y)|$ odd, is rational homology cobordant to positive half-integer surgery on a knot in $S^3$. Then for some factorisation $|H_1(Y)| = rs^2$ there exists an order $rs$ subgroup $H \leq \text{Spin}^c(Y)$ and an epimorphism $\phi: H \to \mathbb{Z}/r\mathbb{Z}$ such that the $d$-invariants of $Y$ are constant on the fibers of $\phi$. Moreover, the normalized $d$-invariants

$$\tilde{d}_i = -d(Y, \phi^{-1}(i)) + i^2/2r - \begin{cases} 0 & i \equiv (r-1)/2 \pmod{2} \\ 1/2 & i \equiv (r+1)/2 \pmod{2} \end{cases}$$

for $i = 0, \ldots, (r-1)/2$ satisfy the following conditions:

(i) positivity: $\tilde{d}_i \geq 0$;

(ii) evenness: $\tilde{d}_i \in 2\mathbb{Z}$;

(iii) monotonicity: $\tilde{d}_i \leq \tilde{d}_{i+1}$ for $0 \leq i < (r-1)/2$.

Here we are using $d(Y, \phi^{-1}(i))$ to denote the constant value of the $d$-invariant on the fiber $\phi^{-1}(i)$ for $i \in \mathbb{Z}/r\mathbb{Z}$. If there exists an epimorphism $\phi$ satisfying the conditions of Theorem 4 we say that $Y$ admits a positive even monotone subgroup matching.

Applying these results to knots and keeping track of signs we obtain

**Corollary 5.** Let $K \subset S^3$ be a knot with unknotted number one. If $K$ has signature $\sigma(K) = 2$, then the double branched cover $\Sigma(K)$ admits a positive even symmetric monotone matching. If $K$ has signature zero, then at least one of $\pm \Sigma(K)$ admits a positive even symmetric monotone matching.

Suppose $K$ has concordance unknotted number one. If $K$ has signature two, then $\Sigma(K)$ admits a positive even monotone subgroup matching. If $K$ has signature zero, then at least one of $\pm \Sigma(K)$ admits a positive even monotone subgroup matching.

For each of the inequalities in (11) except for $u_s \geq c^*$ there exist examples for which the inequality is strict. One may ask whether in fact $u_s(K) = c^*(K)$ for all knots. We will see in Section 4 that this equality holds for all prime knots with 10 or fewer crossings. Another question which seems to be open is whether the slicing number $u_s$ is a concordance invariant. It turns out these questions are related to a generalisation of Fox’s slice-ribbon question, which asks whether every slice knot is in fact ribbon (i.e. admits a slice disk which is ribbon). An immersed or embedded disk in $B^4$ is called ribbon if the restriction of the radial distance function is Morse without local maxima.
Proposition 6. Let $c^*_r(K)$ denote the minimal number of double points in a normally immersed ribbon disk bounded by $K$ in $B^4$, and let SRC be the Slice-Ribbon Conjecture, which states that all slice knots are ribbon. Then:

$$c^* = c^*_r \iff \text{SRC and } u_s = c^* \iff \text{SRC and } u_s \text{ is a concordance invariant}.$$ 

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2. Geometric constructions

In this section we collect some results regarding properly normally immersed surfaces in $B^4$, and prove Theorem 1, Corollary 2 and Proposition 6.

Recall that a crossing change in a link $L$ may be recorded by placing a framed arc or equivalently a band connecting two arcs of the link. The crossing change operation consists of replacing the two arcs of the band on $L$ with a full positive twist as in Figure 1.

![Figure 1. Encoding a crossing change with a band.](image)

Proposition 2.1. Let $F$ be a connected surface normally properly immersed in $S^3 \times I$ with $L = F \cap S^3 \times \{0\} \neq \emptyset$. After an isotopy rel boundary we may assume that $F \cap S^3 \times [0, 1/3]$ is a concordance, $F \cap S^3 \times [1/3, 2/3]$ is the trace of a regular homotopy, and $F \cap S^3 \times [2/3, 1]$ is a smoothly embedded ribbon surface.

In particular, if $F$ is an immersed disk, then it can be factored into a concordance, followed by the trace of a regular homotopy, followed by a ribbon null concordance.

Proof. After a preliminary isotopy we may assume the projection to the interval is a Morse function $h$ on $F$. Then generic level sets $F \cap S^3 \times \{t\}$ are smooth links in the three-sphere, and a sequence of diagrams of such links gives a “movie presentation”
of the surface $F$. We may further assume that neighborhoods of double points of $F$ in the movie presentation of $F$ relative to $h$ correspond to crossing changes. Successive frames in the movie picture of $F$ are then obtained by crossing changes, Morse moves and isotopies. We may assume all the minima of $h$ occur before and all the maxima occur after all the other modifications which form a subsurface $F' \subset F$.

A complete description of the surface $F'$ can be encoded in a single diagram of the link $L'$ which consists of $L$ and the unlink of co-attaching circles for 0-handles, as follows. Following the movie we encode each crossing change encountered by adding a red band as in Figure 1 and each 1-handle addition (a band move) by adding a blue one. By shortening the arcs and pulling the rest of $L'$ along we can construct a diagram for $L'$ relative to which the added colored bands lie in a plane, are nonoverlapping, and intersect $L'$ only in attaching arcs. It is clear from this picture that we can now implement the crossing changes and band moves in any order we desire.

Since $F$ is connected we can first add 1-handles to $L'$ connecting components corresponding to the boundaries of 0-handles to components of $L$ thus forming a concordance, which is the first stage of factorisation. Next perform all the crossing changes encoded by the red bands, giving the second stage. The remaining 1-handles and 2-handles form a ribbon surface which is the last stage of factorisation. □

We note in passing that the above proposition tells us that the 4-ball crossing number of a knot is equal to the minimal slicing number of any knot in its concordance class.

Proof of Proposition 6. A knot $K$ is slice if and only if $c^*(K) = 0$, and is ribbon if and only if $c_r^*(K) = 0$; thus $c^* = c_r^*$ implies SRC. It was observed by Shibuya and Rudolph [22, 21] (and follows easily from Proposition 2.1) that $c_r^*(K)$ is equal to $u_r(K)$ which is the minimal number of crossing changes from $K$ to a ribbon knot. Note that SRC implies $u_s = u_r$. The equivalence

$$c^* = c_r^* \iff \text{SRC and } u_s = c^*$$

now follows by considering equalities among the quantities $c^*, c_r^*, u_r$ and $u_s$. We next show that

SRC and $u_s = c^* \iff \text{SRC and } u_s$ is a concordance invariant.

The forward implication is clear since $c^*$ is a concordance invariant. Now suppose we have a normally immersed disk $F$ in $B^4$ bounded by $K$ with $c^*(K)$ double points. By Proposition 2.1 this may be factored into a concordance from $K$ to a knot $K'$, followed by crossing changes and then a nullconcordance. It follows that

$$u_s(K') \leq c^*(K) \leq u_s(K),$$

and if $u_s$ is a concordance invariant then all of the above are in fact equal. □

The following theorem implies Theorem 1 and may be used to give a new proof of [13 Theorem 2], noting that the trace of a homotopy given by a crossing change
sequence from $K$ to a slice knot $J$ may be glued to a slice disk for $J$ to give an immersed disk $\Delta$ in $B^4$ bounded by $K$. There is a sign error in Remark 3.5 of that paper: changing a positive (respectively negative) crossing in $K$ results in a positive (respectively negative) double point of $\Delta$.

**Theorem 2.2.** Let $\Delta$ be a normally properly immersed disk in the 4-ball with $r$ double points and boundary $K$. Then the double branched cover $\Sigma(K)$ of $K$ bounds a manifold with $b_2 = 2r$ and signature $\sigma(K) + 2r_+$, where $\sigma(K)$ denotes the signature of $K$ and $r_+$ the number of positive double points in $\Delta$. Moreover, the intersection pairing of this manifold is of half-integer surgery type and the number of odd squares in such a representation of the pairing is equal to $r_+$.

*Proof.* We follow the construction in [3, Theorem 3.7] where all the properties except those in the last sentence of the theorem are established. We briefly recall the construction. Blow up $B^4$ at every double point of $\Delta$ to obtain $\tilde{W} = B^4 \# r\mathbb{CP}^2$ with an embedded disk $\tilde{\Delta}$ and let $\tilde{W}$ be the double branched cover of $W$ with branch set $\tilde{\Delta}$. Then $b_1(\tilde{W}) = 0$ and hence $b_2(\tilde{W}) = 2r$. We exhibit below a collection of $2r$ independent homology classes in $H_2(\tilde{W}; \mathbb{Z})$ with the claimed type of pairing. It follows then from [13, Proposition 2.4] that the pairing on $\tilde{W}$ is of the same type.

Let $p$ be a double point of $\Delta$ and let $B$ be a small ball around $p$; we may assume $\Delta$ intersects the boundary of $B$ in two great circles. Denote by $E$ the total space of the degree one $B^2$-bundle over $S^2$ that replaces $B$ after the blow-up at $p$. The proper transform of $\Delta$ thus contains two fibres of $E$ and in the double branched cover $\tilde{W}$ the zero section of $E$ lifts to a sphere $S$ of self-intersection two.

Choose a loop $C$ in $\Delta$ that is the image of an arc in $B^2$ connecting the two points mapping to $p$ under the immersion to $\Delta$. We may assume $C$ intersects the boundary of $B$ in two great circles. Denote by $E$ the total space of the degree one $B^2$-bundle over $S^2$ that replaces $B$ after the blow-up at $p$. The proper transform of $\Delta$ thus contains two fibres of $E$ and in the double branched cover $\tilde{W}$ the zero section of $E$ lifts to a sphere $S$ of self-intersection two.

Choose a loop $C$ in $\Delta$ that is the image of an arc in $B^2$ connecting the two points mapping to $p$ under the immersion to $\Delta$. We may assume $C$ does not contain any other double point of $\Delta$. Choose a (short) vector $v_p$ transverse to both sheets of $\Delta$ at $p$ and extend it to a (short enough) nonvanishing normal vector field $v$ to $\Delta$ along $C$. Then $C$ and its pushoff $C_v$ along $v$ cobound an embedded annulus $A$. Since the first homology of the complement of $\Delta$ is generated by the meridian $\mu$ of $\Delta$, it follows that $C_v$ is homologous to $k[\mu]$ for some $k \in \mathbb{Z}$. Choose $|k|$ disjoint curves $L \subset \Delta - B$, parallel to a component of $\Delta \cap \partial B$ and let $F_0$ be an embedded surface bounded by $C_v \cup L$ that intersects $\Delta$ only in $L$. Then $F_0 \cup A$ is a surface (with possible selfintersections) with boundary $C \cup L$. We may replace it by a surface $F$ with the same boundary and only (transverse) selfintersections along the boundary. We may further assume that $F \cap \partial B$ is a section $s_0$ of $E$ over an arc in the base sphere connecting the branch points. Extend $s_0$ to a section $s$ of $E$ that intersects the zero section transversely in one point. Then the preimage of $F$ along with a half of the preimage of $s$ (bounded by the preimage of $s_0$) determines a closed normally immersed surface $\tilde{F}$ in $\tilde{W}$. By construction the intersection number of $\tilde{F}$ with $S$ is $\pm 1$. 

To determine the parity of the self-intersection of \(\widetilde{F}\) construct a pushoff as follows. Choose a short nonvanishing normal vector field \(u\) along each component of \(\partial F\) in \(\Delta\) so that the pushoff \(\partial F_u \subset \Delta\) is an embedding of \(\partial F\) disjoint from \(\partial F\). Extend \(u\) to a normal vector field to \(F\) that is transverse to \(F\). Again we may assume that \(F_u \cap \partial B\) is a section \(s'_0\) of \(E\) over an arc in the base sphere connecting the branch points and extend \(s'_0\) to a section \(s'\) of \(E\) that intersects the zero section transversely in one point. The preimage of \(F_u\) along with a half of the preimage of \(s'\) then determines a pushoff \(\widetilde{F}'\) of \(\widetilde{F}\) in \(\widetilde{W}\). Note that any intersections between \(F\) and \(F_u\) contribute an even number to the intersection number \(I = \widetilde{F} \cdot \widetilde{F}'\) thus the parity of \(I\) depends on the intersections between the lifts of \(s\) and \(s'\). If \(p\) is a positive double point then \(s'_0\) can be thought of as \(s_0\) rotated by a small angle, i.e. multiplied by a unit complex number \(\xi\) close to 1. Hence we can choose \(s'\) to be \(\xi s\) and the contribution of the lifts of \(s\) and \(s'\) to \(I\) is \(\pm 1\). If on the other hand \(p\) is a negative double point, then the pushoff \(s'_0\) at the endpoints of the arc lies on the opposite sides of \(s_0\) and this additional linking between the lifts of \(s_0\) and \(s'_0\) contributes another \(\pm 1\) to \(I\) making it even in this case.

\[\Box\]

**Proof of Corollary** If \(\sigma(K) = 2\) and \(K\) bounds an immersed disk in the four-ball with one double point then it must be a negative double point. This follows from Proposition 2.1 and the fact that changing a positive crossing cannot decrease the signature (see [3, Proposition 2.1], also [23, Theorem 5.1]). According to Theorem 1 the double branched cover \(Y\) of \(K\) bounds a positive definite four-manifold \(X\) with intersection form \(Q = \begin{bmatrix} a & 1 \\ 1 & 2 \end{bmatrix}\) which presents a cyclic subgroup of \(H_1(Y; \mathbb{Z})\). By a standard algebraic topology argument (see e.g. [14]) the determinant \(r\) of \(Q\) divides \(\det K = |H_1(Y; \mathbb{Z})|\) with quotient a square, \(s^2\), and the image of the restriction homomorphism \(\text{Spin}^c(X) \to \text{Spin}^c(Y)\) has order \(rs\). For each \(\text{spin}^c\) structure \(t\) on \(Y\) that extends over \(X\) it follows from [13, Theorems 1.2 and 9.6] that

\[
\tilde{d}(Y, t) := -d(Y, t) + \min_s \frac{c_1(s)^2 - 2}{4}
\]

is a nonnegative even integer, where the minimum is taken over all \(s \in \text{Spin}^c(X)\) that restrict to \(t\). Changing the \(\text{spin}^c\) structure on \(X\) by a torsion element does not change the square of its Chern class. The formula follows by noting that the minimum computes the \(d\)-invariant of a \(\text{spin}^c\) structure on \(r/2\) surgery on the unknot \([16, \text{Corollary 1.5}]\). By the recursive formula for correction terms of lens spaces given in [15], with the labelling shifted by \((r + 1)/2\) so that the spin structure on the lens space is labelled by \(i = 0\), these are

\[
d(S^3_{r/2}(O), i) = i^2/2r - \begin{cases} 0 & i \equiv (r - 1)/2 \pmod{2} \\ 1/2 & i \equiv (r + 1)/2 \pmod{2} \end{cases}
\]

\[\text{(3)}\]
(Alternatively one can compute the minimum in (2) directly using a suitable set of short characteristic covectors for the form $Q$ as in [14] or [17].)

If $\sigma(K) = 0$ then at least one of $K$ or its mirror $\overline{K}$ bounds a disk with a positive double point (and no other double points) to which the above may be applied. \hfill \Box

3. Heegaard-Floer obstructions to (concordance) unknotting number one

In this section we prove Theorems 3 and 4 and Corollary 5.

Proof of Theorem 3. The first three conditions in the theorem are due to Ozsváth and Szabó [17]. For completeness we give a proof of all five based on the formula of Ni and Wu [11, Proposition 1.6] for $d$-invariants of a positive Dehn surgery on a knot $K$ in the 3-sphere: for $p, q > 0$

\begin{equation}
\begin{aligned}
d(S^3_{p/q}(K), i) &= d(S^3_{p/q}(O), i) - 2 \max\{V_{i/q}, H_{(i-p)/q}\}, & i = 0, \ldots, p-1, \\
\end{aligned}
\end{equation}

where $i$ enumerates relative spin$^c$ structures on the complement of $K$ and $O$ denotes the unknot. We define normalized $d$-invariants as

\begin{equation}
\begin{aligned}
\tilde{d}_i := -d(S^3_{p/q}(K), i) + d(S^3_{p/q}(O), i) = 2 \max\{V_{i/q}, H_{(i-p)/q}\}, \\
\end{aligned}
\end{equation}

for $i = 0, \ldots, p-1$.

The sequences $V_j$ and $H_j$ ($j \in \mathbb{Z}$) are determined by the knot Floer chain complex $C = \text{CFK}^\infty(K)$ of $K$ and we recall their construction. In [13] Ozsváth and Szabó define quotient complexes $A^+_j = C\{k \geq 0 \text{ or } l \geq j\}$ and $B^+_j = B^+ = C\{k \geq 0\}$ where $(k, l) \in \mathbb{Z} \oplus \mathbb{Z}$ denotes the bidegree on $C$, and $U$-equivariant maps $v^+_j : A^+_j \to B^+$ and $h^+_j : A^+_j \to B^+$. The $U$-action lowers the bidegree by $(1, 1)$ and $v^+_j$ is the obvious projection to the quotient complex; we will not need the description of $h^+_j$ but note that it is essentially the projection to $C\{l \geq j\}$. The homology of $B^+$ is isomorphic to $HF^+(S^3) \cong \mathcal{T}^+ = \mathbb{Z}[U, U^{-1}]/U\mathbb{Z}[U]$. Similarly there is a $\mathcal{T}^+$ summand, which we denote $\mathcal{T}^+_j$, in the homology of $A^+_j$ (noting that the homology of $C$ is isomorphic to $HF^\infty(S^3) \cong \mathcal{T} = \mathbb{Z}[U, U^{-1}]$, and the projection from $C$ to $A^+_j$ induces an isomorphism on homology in high degrees). Since both $v^+_j$ and $h^+_j$ induce isomorphisms on the chain level in sufficiently high degrees, their induced maps on homologies, restricted to $\mathcal{T}^+_j$, are given by multiplication by $U^{V_j}$ and $U^{H_j}$. Moreover, for $j$ greater than the genus of $K$, $A^+_j = B^+$ and hence $V_j = 0$ in this range. We restrict our analysis to $V_j$ for $j \geq 0$ as these determine $H_{-j}$ (see below) and all of these together are the only values of $V$'s and $H$'s relevant to (1).

We first show that $H_{-j} = V_j$ for $j \geq 0$. For $j = 0$ this was observed by Ni and Wu [11]. As noted in [20] Corollary 2.3 if $n$ is a large positive integer, then $V_i = 0$ for $i > n/2$ and $H_{i-n} = 0$ for $i \leq n/2$, and thus the normalized $d$-invariants $\tilde{d}_i$ of $S^3_n(K)$ are equal to either $2V_i$ for $i \leq n/2$ or $2H_{i-n}$ for $i > n/2$. Since in the case
of an integer surgery $i = 0$ corresponds to a spin structure, conjugation invariance $\tilde{d}_i = \tilde{d}_{a-i}$ of (normalized) $d$-invariants yields the stated equality.

Next we claim that $V_j - 1 \leq V_{j+1} \leq V_j$ for all $j$. Let $K_j$ denote the kernel of the projection $A_j^+ \to A_{j+1}^+$. Since the $U$-action on the left complex in the $U$-equivariant short exact sequence

$$0 \to K_j \to A_j^+ \to A_{j+1}^+ \to 0$$

is trivial, it follows that in homology the map from $\mathcal{T}_{j+1}^+$ is trivial and the map into $\mathcal{T}_j^+$ can contain in its image only the kernel of the $U$-action. Hence the map $\mathcal{T}_j^+ \to \mathcal{T}_{j+1}^+$ is either the identity or multiplication by $U$. This proves the claim.

We now specialize to $p/q = r/2$. At this point we know that $V_j (j \geq 0)$ is a nonincreasing and $H_j (j \leq 0)$ a nondecreasing sequence of nonnegative integers which yields the first two conditions. From the behaviour of $V_j$ and $H_j$ it follows that $d_{2i} = d_{2i+1}$ and $d_{-2i} = d_{-2i+1}$ for small $i$. Note that in this labelling of spin$^c$ structures the spin structure is labelled by $i = (r+1)/2$. We claim that the symmetry condition above extends to the spin structure from both ends. Denote by $\ell$ the largest value of $i$ for which $V_{[i/2]} \geq H_{[i-(i-1)/2]}$. If $\ell \geq (r+1)/2$ then the symmetry condition holds up to the spin structure whereas for $\ell < (r+1)/2$ it holds above it. In both cases using conjugation invariance of $d$-invariants it follows that the symmetry holds on both sides of the spin structure.

Next note that the monotonicity conditions on $V_j$ and $H_j$ imply that $\tilde{d}_i$ is nonincreasing up to the spin structure (the symmetry shows us that $\tilde{d}_{2i} = \tilde{d}_{2i+1}$, and the inequality $\tilde{d}_{2i+1} \leq \tilde{d}_{2i+2}$ in this range follows immediately from (5)) and nondecreasing after that. Moreover, since $V_j$ and $H_j$ can jump by at most 1, the last condition also follows.

Finally we substitute for $d(S^3_{r/2}(O), i)$ using (3). The isomorphism $\phi$ accounts for the fact that the given labelling of spin$^c$ structures on $Y$ may not agree with the one assumed above. □

**Proof of Theorem 4.** Let $Y' = S^3_{l/2}(C)$ with $l > 0$ and let $W$ denote the rational homology cobordism with $\partial W = Y \sqcup -Y'$. Let $X$ denote the surgery nullcobordism bounded by $Y'$; this is given by a Kirby diagram with framing $(l+1)/2$ on $C$ and framing 2 on a meridian of $C$. The restriction map from $\text{Spin}^c(X)$ to $\text{Spin}^c(Y')$ is surjective since $H^2(X; \mathbb{Z})$ is torsion-free. We need to understand the image of the restriction map $\text{Spin}^c(W) \to \text{Spin}^c(Y) \times \text{Spin}^c(Y')$, or equivalently from $\text{Spin}^c(X \cup W)$. This image is conjugation invariant and has an odd number of elements. Thus it contains a conjugation fixed element, which is the pair of spin structures on the two boundary components. It follows that it suffices to understand the restriction map on $H^2$.

Let $\Lambda_X = (H_2(X; \mathbb{Z}), Q_X)$ denote the intersection lattice of $X$, with dual lattice $\Lambda^*$ which is $H^2(X; \mathbb{Z})$ with the induced pairing. Let $\Lambda = (H_2(X \cup W; \mathbb{Z})/\text{Tors}, Q_{X \cup W})$ denote the intersection lattice of the manifold $X \cup W$ bounded by $Y$, and let $\Lambda^*$ be
its dual lattice (which is $H^2(X \cup W; \mathbb{Z})/\text{Tors}$ with the induced pairing). We have natural inclusions

$$\Lambda_X \subset \Lambda \subset \Lambda^* \subset \Lambda^*_X,$$

the last of which comes from the restriction map $H^2(X \cup W; \mathbb{Z}) \to H^2(X; \mathbb{Z})$. Let $r$ be the determinant of $\Lambda$ (that is, the order of $\Lambda^*/\Lambda$) and let $t$ be the index of $\Lambda_X$ in $\Lambda$. Using the long exact sequence of the pair $(X, Y')$ we see that $\Lambda^*_X/\Lambda_X \cong H^2(Y'; \mathbb{Z})$ is cyclic of order $l = rt^2$. The image of $H^2(X \cup W; \mathbb{Z})$ in $H^2(Y'; \mathbb{Z})$ is the index $t$ subgroup $H' \cong \Lambda^*/\Lambda_X \subset \Lambda^*_X/\Lambda_X$.

Using the long exact sequence of the pair $(X \cup W, Y)$ as in e.g. [14], the order of $H_1(Y)$ is $rs^2$ for some $s$. Moreover the image $H$ of $H^2(X \cup W; \mathbb{Z})$ in $H^2(Y; \mathbb{Z})$ has order $rs$, the image $T$ of the torsion subgroup of $H^2(X \cup W; \mathbb{Z})$ in $H$ has order $s$, and $H/T \cong \Lambda^*/\Lambda \cong \mathbb{Z}/rt\mathbb{Z}$. The composite surjection $H^2(X \cup W; \mathbb{Z}) \to \Lambda^* \to \Lambda^*/\Lambda \cong \mathbb{Z}/rt\mathbb{Z}$ thus factors through each of $H$ and $H'$.

Given a spin$^c$ structure on $X \cup W$, its restriction to $Y$ may be changed by any given element of $T$ using the action of the torsion subgroup of $H^2(X \cup W; \mathbb{Z})$, leaving the restriction to $Y'$ unchanged. Since $d$-invariants are spin$^c$ rational homology cobordism invariants [15], this proves the first statement of the theorem. Choose a labelling $\text{Spin}^c(Y') \cong \mathbb{Z}/rt\mathbb{Z}$ as in Theorem 3. For each $i \in \mathbb{Z}/rt\mathbb{Z}$ choose a spin$^c$ structure $s_i \in \text{Spin}^c(X \cup W)$ whose restriction to $Y'$ is labelled by $ti$, and let $\phi(s_i|_Y) = i$. From the discussion above, this is the quotient homomorphism $H \to H/T$ composed with an automorphism of $\mathbb{Z}/rt\mathbb{Z}$. Then

$$d(Y', ti) = d(Y, s_i|_Y)$$

and the second conclusion of the theorem now follows from the conclusion of Theorem 3 applied to $Y'$.

Note that since in the above proof we are using the conclusions of Theorem 3 on a subgroup of spin$^c$ structures on the surgery manifold, we lose the symmetry and boundedness conditions from Theorem 3.

In the proof of Corollary 5 we need to know whether the double branched cover of an unknotting number one knot is a positive or a negative surgery on some knot. The exact information is given by the following signed refinement of Montesinos’ trick, which is proved in [17, Theorem 8.1]. However, we only need to pin down the sign in case the signature is nonzero and we give an alternative argument for that.

**Lemma 3.1.** Let $K$ be a knot in $S^3$ with determinant $l$ and unknotting number one. Suppose that either

(i) $K$ has signature 2, or
(ii) $K$ has signature 0 and may be unknotted by changing a positive crossing to a negative crossing.

Then $\Sigma(K) = S^3_{l/2}(C)$. 

□
Proof of (i). If the signature of \( K \) is two then \( Y = \Sigma(K) \) is the boundary of a spin 4-manifold \( W \) with signature 2. Indeed, \( W \) can either be taken to be the double branched cover of a Seifert surface for \( K \) pushed into the four-ball or the manifold from Theorem 1. In the latter case note that since by [3, Theorem 3.7] \( H_1(W; \mathbb{Z}/2\mathbb{Z}) = 0 \) and the intersection form is even (only changing a negative crossing can unknot \( K \)), the manifold is spin. Moreover, since \( \sigma(K) = 2 \), \( \det(K) \equiv -1 \) (mod 4) by [10, Theorem 5.6], the surgery manifold \( X \), corresponding to the positive surgery, that \( \pm Y \) bounds, is also spin with signature 2. If \( -Y \) were the positive surgery, then \( W \cup X \) would be a spin manifold of signature 4, a contradiction. \( \square \)

Proof of Corollary 3. Suppose \( K \) is a knot with unknotting number one and let \( Y = \Sigma(K) \). If \( K \) has signature zero, then either \( Y \) or \( \Sigma(K) = -Y \) is a positive surgery on a knot; if necessary we replace \( K \) by its mirror so that \( Y \) is a positive surgery. If \( K \) has signature two then it follows from Lemma 3.1(i) that \( Y \) is a positive surgery. In each case we then apply Theorem 3.

Now suppose \( K \) is concordant to a knot \( K' \) with unknotting number one. The argument in the preceding paragraph may be applied to \( K' \), which has the same signature as \( K \), and we then apply Theorem 4 using the fact that the double branched cover \( W \) of a concordance between \( K \) and \( K' \) gives a rational homology cobordism between \( \Sigma(K) \) and \( \Sigma(K') \). \( \square \)

4. Examples

Table 2 lists the determinant, unknotting number, concordance unknotting number, slicing number, 4-ball crossing number, 4-ball genus and half of the absolute value of the signature for knots with up to 9 crossings, with incomplete information for three knots. In this section we describe how the previously unknown values in the table have been obtained.

The slicing number of \( 7_4 \) was determined by Livingston [6] and the 4-ball crossing number of \( 8_{16} \) was determined by Murakami and Yasuhara [9]. Apart from the values of \( u_c, u_s \) and \( c^* \), the data in the table is taken from [2].

For any knot for which the unknotting number is equal to the 4-ball genus, it follows from (1) that \( u = u_c = u_s = c^* = g^* \). Also any slice knot has \( u_c = u_s = c^* = g^* = 0 \). We have highlighted the knots in Table 2 for which neither of these situations apply, and we give some details about each in what follows.

At the end of the section we give a brief account of the computation of the invariants in Table 3 of knots with 10 crossings.

4.1. Knots with \( u = c^* \). If a knot has \( u = c^* \) then these also are equal to \( u_c \) and \( u_s \) by (1). Using Theorem 1 in place of [13, Theorem 2] we may adapt Corollaries 2 and 3 of [13] to conclude that \( 7_4, 8_{16}, 9_5, 9_{15}, 9_{17} \) and \( 9_{31} \) all have \( u = c^* = 2 \) and \( 9_{10}, 9_{13}, \) and \( 9_{38} \) have \( u = c^* = 3 \).
For the knot $K = 8_{18}$, the homology group of $Y = \Sigma(K)$ is isomorphic to $\mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z}/15\mathbb{Z}$. The signature of this knot is zero. The only factorisation $45 = rs^2$ for which there is an epimorphism from an order $rs$ subgroup of $\mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z}/15\mathbb{Z}$ onto $\mathbb{Z}/r\mathbb{Z}$ is $r = 5$ and $s = 3$. The $d$-invariants of the unique order 5 subgroup of $\text{Spin}^c(Y)$ may be calculated as in [17, Proposition 3.2] to be

$$[0, 4/5, -4/5, -4/5, 4/5].$$

This subgroup gives a full set of representatives for the order 5 quotient of any of the possible order 15 subgroups of $\text{Spin}^c(Y)$. Thus by Corollary 2, if $c^*(K) = 1$ there would exist a positive even matching between these numbers (up to overall sign change) and the $d$-invariants of $5/2$ surgery on the unknot, which are

$$[0, -2/5, 2/5, 2/5, -2/5].$$

No such matching exists; in fact no integral matching exists, so that $Y$ is obstructed by the linking form from bounding a topological manifold with intersection form

$$Q = \begin{pmatrix} 3 & 1 \\ 1 & 2 \end{pmatrix}.$$ 

We conclude that $u = c^* = 2$ for this knot.

Similar reasoning applies to the knot $9_{40}$ for which the homology group is $\mathbb{Z}/5\mathbb{Z} \oplus \mathbb{Z}/15\mathbb{Z}$ and the $d$-invariants on the unique $\mathbb{Z}/3\mathbb{Z}$ subgroup are $[-1/2, 5/6, 5/6]$, while those of $3/2$ surgery on the unknot are $[-1/2, 1/6, 1/6]$.

The knot $K = 9_{49}$ has signature 4 and determinant 25. If $c^*(K) = 2$ then as in the proof of Corollary 2, $K$ must bound an immersed disk with 2 negative (and no positive) double points. Now by Theorem 1, $\Sigma(K)$ bounds a positive-definite even form of half-integer surgery type of rank 4 and determinant 1 or 25. However no such forms exist (this may be checked as in [12, §6]) and we conclude that $u = c^* = 3$.

**4.2. Knots with $u_c = u_s = c^* = 1$.** The knots $8_{10}$ and $9_{37}$ each have $u_c = u_s = 1$ as may be seen from Figure 2; the former is concordant to the trefoil and is one crossing change from $3_1 \# 3\bar{1}$, the latter is concordant to $4_1$ and gives $6_1$ after changing the indicated crossing. Comparing with the proof of Proposition 2.1 one may observe that in each case the crossing change and the concordance combine to give an immersed disk with a single double point.

**4.3. Knots with $u_c = 2$, $u_s = c^* = 1$.** The knots listed in Table 1 each have $u_s = 1$ as may be seen by changing the crossing circled in Figure 2 in each case the slice knot thus obtained is given in the table.

We consider first the knot $K = 8_3$, which is the two-bridge knot $S(17, 4)$ with determinant 17 and signature 0. The $d$-invariants may be computed using either the recursive formula from [15] or the Goeritz matrix from an alternating diagram as in [17]. In cyclic group order starting at the spin structure these are

$$[0, 4/17, 16/17, 2/17, -4/17, -2/17, 8/17, -8/17, -16/17, -8/17, 8/17, -2/17, -4/17, 2/17, 16/17, 4/17].$$
Table 1. Data for knots with \( u_c = 2 \) and \( u_s = 1 \), and with cyclic \( H_1(\Sigma(K)) \).

| Knot | \( \vec{d} \) | Slice knot |
|------|-------------|------------|
| 8_3  | \( 0^6, 2, 0, 2 \) | \( 6_1 \) |
| 8_4  | \( 0^5, 2, 0, 2^3 \) | \( 6_1 \) |
| 8_6  | \( 0^7, 2, 0, 2^3 \) | \( 6_1 \) |
| 8_{12}| \( 0^8, 2, 0, 2^4, 4 \) | \( 6_1 \) |
| 9_8  | \( 0^7, 2, 0, 2^4, 4, 2, 4 \) | \( 6_1 \) |
| 9_{25}| \( 0^{11}, 2, 0, 2^4, 4, 2, 4^4, 6 \) | \( 6_1 \) |
| 9_{29}| \( 0^9, 2, 0, 2^6, 4, 2, 4^4, 6^3 \) | \( 3_1 \# 3_1 \) |
| 9_{32}| \( 0^{11}, 2, 0, 2^6, 4, 2, 4^4, 6^4, 8 \) | \( 6_1 \) |

Since the determinant is square-free we are looking for a positive even matching on the whole group; we must compare these \( d \)-invariants with those of \( 17/2 \) surgery on the unknot which are

\[
[0, -\frac{8}{17}, \frac{2}{17}, -\frac{4}{17}, \frac{8}{17}, \frac{4}{17}, \frac{18}{17}, \frac{16}{17}, \frac{18}{17}, \frac{17}{17}, \frac{16}{17}, \frac{18}{17}, \frac{8}{17}, -\frac{4}{17}, \frac{2}{17}, -\frac{8}{17}] .
\]

Since the signature is zero we are free to switch the sign of the first list above. We find there are two positive even matchings \( \phi : \mathbb{Z}/17\mathbb{Z} \to \mathbb{Z}/17\mathbb{Z} \). The first is multiplication by 5 and applies to the list above, and the second is multiplication by 3 and applies to the list with the opposite sign. Both result in the same ordered list of \( \vec{d} \) invariants as in Theorem 4 which are

\[0, 0, 0, 0, 0, 0, 2, 0, 2.\]

This fails monotonicity and so \( u_c(K) = u(K) = 2 \).

Similar analysis applies to each of the knots listed in Table 1. In the second column of the table we have used an abbreviated notation in which for example \( 0^6 \) stands for \( 0, 0, 0, 0, 0, 0 \).

4.4. Knots with unknown values. The knots \( 9_{47} \) and \( 9_{48} \) have \( u_s = 1 \) as may be seen by changing a crossing as in Figure 2 resulting in \( 6_1 \) and \( 8_{20} \) respectively.

We next consider \( K = 9_{35} \), which is the pretzel knot \( P(3, 3, 3) \). This has signature 2 and \( H_1(\Sigma(K)) \cong \mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z}/9\mathbb{Z} \). We give two proofs that \( u_c(K) > 1 \). The \( d \)-invariants (multiplied by 18) are

\[
\begin{bmatrix}
-9 & 19 & -5 & 27 & 7 & 7 & 27 & -5 & 19 \\
3 & -5 & 7 & 3 & 19 & 19 & 3 & 7 & -5 \\
3 & -5 & 7 & 3 & 19 & 19 & 3 & 7 & -5
\end{bmatrix} .
\]
Here the rectangular array shows the group structure with the invariant coming from
the spin structure in the top left position. There are four subgroups of order 9; none
of these admits an epimorphism onto \( \mathbb{Z}/3\mathbb{Z} \) with constant \( d \)-invariant on fibres. This
is easy to see since the \( d \)-invariant of the spin structure is not repeated in any other
spin\( ^c \) structure. Thus there does not exist a positive even subgroup matching, so by
Corollary 5 we see that \( u_c(K) > 1 \).

We may also use a result from [13] based on Donaldson’s diagonalisation theorem
\[ \text{[4]} \] to show that in fact \( c^*(K) > 1 \). Combining the proof of [13, Corollary 5] with
Theorem 1 we find that any normally immersed disk in \( B^4 \) bounded by the knot \( 7_4 \)
has at least two negative double points. Since \( 7_4 \) is the \( P(1, 3, 3) \) pretzel, we obtain
\( K = 9_{35} \) from it by changing a positive crossing. Thus an immersed disk bounded by
\( K \) gives rise to one for \( 7_4 \) with one additional positive double point. We conclude that
\( c^*(K) > 1 \); moreover if \( c^*(K) = 2 \) then any immersed disk realising this bound has
two negative double points. An argument of Traczyk [25] using the Jones polynomial
shows that \( 9_{35} \) cannot be unknotted by changing two negative crossings. However
as we see from Figure 3 it is possible to go from \( 9_{35} \) to the slice knot \( 8_{20} \), which
incidentally is the pretzel \( P(3, 2, -3) \), by two crossing changes. Thus \( u_s = c^* = 2 \) for
this knot.

4.5. Knots with 10 crossings. Table 3 lists invariants for 10 crossing knots; we
have omitted slice knots or knots for which \( u_c, u_s \) and \( c^* \) are computable from (1). Here we briefly indicate how the data in the table was compiled.

The knots 10\_19, 10\_20, 10\_24, 10\_36, 10\_68, 10\_69, 10\_86, 10\_97, 10\_105, 10\_109, 10\_116, 10\_121,
10\_122, 10\_144, 10\_163, 10\_165 are obstructed from having \( c^* = 1 \); 10\_53, 10\_101 and 10\_120 are similarly obstructed from having \( c^* = 2 \). This follows as in Corollaries 2 and 3 of [13],
using Theorem 1 in place of [13, Theorem 2].

The knots 10\_40, 10\_65, 10\_67, 10\_74, 10\_77, 10\_103 and 10\_106 are concordant to unknotting
number one knots \[7\] and hence have \( u_c = 1 \).

All remaining knots in Table 3 with slice genus one are obstructed from having concordance unknotting number one by Theorem 1 with the exception of 10\_158, which
is not obstructed from being concordant to a knot with determinant 5s^2 and unknotting number one. The correction terms of the double branched cover of 10\_158 were
computed using the method from [17].

For all knots in Table 3 with \( u > c^* \) (or \( u \) unknown), we have found an appropriate set of crossing changes to convert to a slice knot. For the most part these are exhibited
in the minimal diagram listed in [2]; the exceptions are shown in Figure 3.

---

\[^1\]There is an oversight in the proof of [13, Corollary 5]: using the notation therein, there is more
than one embedding of the lattice \( L'_n \) in \( \mathbb{Z}^m \), however the conclusion that the orthogonal complement
does not contain a finite index sublattice of half-integer type is correct for any such embedding.
Figure 2. Crossing changes giving slice knots, and band moves giving concordances to unknotted number one knots.
Figure 3. Crossing changes in nonminimal diagrams giving slice knots.
| Knot | det | $u$ | $u_c$ | $u_s$ | $c^*$ | $g^*$ | $|\sigma|/2$ |
|------|-----|-----|-------|-------|------|------|-------------|
| $3_1$ | 3   | 1   | 1     | 1     | 1     | 1    | 1           |
| $4_1$ | 5   | 2   | 2     | 2     | 2     | 2    | 2           |
| $5_1$ | 7   | 1   | 1     | 1     | 1     | 1    | 1           |
| $6_1$ | 9   | 1   | 0     | 0     | 0     | 0    | 0           |
| $7_1$ | 13  | 1   | 1     | 1     | 1     | 1    | 1           |
| $8_1$ | 15  | 2   | 2     | 2     | 2     | 2    | 2           |
| $9_1$ | 17  | 2   | 2     | 2     | 2     | 2    | 2           |
| $10_1$ | 19  | 1   | 1     | 1     | 1     | 1    | 1           |
| $11_1$ | 21  | 1   | 1     | 1     | 1     | 1    | 1           |
| $12_1$ | 23  | 2   | 2     | 2     | 2     | 2    | 2           |
| $13_1$ | 25  | 2   | 2     | 2     | 2     | 2    | 2           |
| $14_1$ | 27  | 2   | 2     | 2     | 2     | 2    | 2           |
| $15_1$ | 29  | 2   | 2     | 2     | 2     | 2    | 2           |
| $16_1$ | 31  | 2   | 2     | 2     | 2     | 2    | 2           |
| $17_1$ | 33  | 2   | 2     | 2     | 2     | 2    | 2           |
| $18_1$ | 35  | 2   | 2     | 2     | 2     | 2    | 2           |
| $19_1$ | 37  | 2   | 2     | 2     | 2     | 2    | 2           |
| $20_1$ | 39  | 2   | 2     | 2     | 2     | 2    | 2           |
| $21_1$ | 41  | 2   | 2     | 2     | 2     | 2    | 2           |
| $22_1$ | 43  | 2   | 2     | 2     | 2     | 2    | 2           |
| $23_1$ | 45  | 2   | 2     | 2     | 2     | 2    | 2           |
| $24_1$ | 47  | 2   | 2     | 2     | 2     | 2    | 2           |
| $25_1$ | 49  | 2   | 2     | 2     | 2     | 2    | 2           |
| $26_1$ | 51  | 2   | 2     | 2     | 2     | 2    | 2           |
| $27_1$ | 53  | 2   | 2     | 2     | 2     | 2    | 2           |
| $28_1$ | 55  | 2   | 2     | 2     | 2     | 2    | 2           |
| $29_1$ | 57  | 2   | 2     | 2     | 2     | 2    | 2           |
| $30_1$ | 59  | 2   | 2     | 2     | 2     | 2    | 2           |
| $31_1$ | 61  | 2   | 2     | 2     | 2     | 2    | 2           |
| $32_1$ | 63  | 2   | 2     | 2     | 2     | 2    | 2           |
| $33_1$ | 65  | 2   | 2     | 2     | 2     | 2    | 2           |
| $34_1$ | 67  | 2   | 2     | 2     | 2     | 2    | 2           |
| $35_1$ | 69  | 2   | 2     | 2     | 2     | 2    | 2           |
| $36_1$ | 71  | 2   | 2     | 2     | 2     | 2    | 2           |
| $37_1$ | 73  | 2   | 2     | 2     | 2     | 2    | 2           |
| $38_1$ | 75  | 2   | 2     | 2     | 2     | 2    | 2           |
| $39_1$ | 77  | 2   | 2     | 2     | 2     | 2    | 2           |
| $40_1$ | 79  | 2   | 2     | 2     | 2     | 2    | 2           |
| $41_1$ | 81  | 2   | 2     | 2     | 2     | 2    | 2           |
| $42_1$ | 83  | 2   | 2     | 2     | 2     | 2    | 2           |
| $43_1$ | 85  | 2   | 2     | 2     | 2     | 2    | 2           |
| $44_1$ | 87  | 2   | 2     | 2     | 2     | 2    | 2           |
| $45_1$ | 89  | 2   | 2     | 2     | 2     | 2    | 2           |
| $46_1$ | 91  | 2   | 2     | 2     | 2     | 2    | 2           |
| $47_1$ | 93  | 2   | 2     | 2     | 2     | 2    | 2           |
| $48_1$ | 95  | 2   | 2     | 2     | 2     | 2    | 2           |
| $49_1$ | 97  | 2   | 2     | 2     | 2     | 2    | 2           |
| $50_1$ | 99  | 2   | 2     | 2     | 2     | 2    | 2           |
| $51_1$ | 101 | 2   | 2     | 2     | 2     | 2    | 2           |
| $52_1$ | 103 | 2   | 2     | 2     | 2     | 2    | 2           |
| $53_1$ | 105 | 2   | 2     | 2     | 2     | 2    | 2           |
| $54_1$ | 107 | 2   | 2     | 2     | 2     | 2    | 2           |
| $55_1$ | 109 | 2   | 2     | 2     | 2     | 2    | 2           |
| $56_1$ | 111 | 2   | 2     | 2     | 2     | 2    | 2           |
| $57_1$ | 113 | 2   | 2     | 2     | 2     | 2    | 2           |
| $58_1$ | 115 | 2   | 2     | 2     | 2     | 2    | 2           |
| $59_1$ | 117 | 2   | 2     | 2     | 2     | 2    | 2           |
| $60_1$ | 119 | 2   | 2     | 2     | 2     | 2    | 2           |
| $61_1$ | 121 | 2   | 2     | 2     | 2     | 2    | 2           |
| $62_1$ | 123 | 2   | 2     | 2     | 2     | 2    | 2           |
| $63_1$ | 125 | 2   | 2     | 2     | 2     | 2    | 2           |
| $64_1$ | 127 | 2   | 2     | 2     | 2     | 2    | 2           |
| $65_1$ | 129 | 2   | 2     | 2     | 2     | 2    | 2           |
| $66_1$ | 131 | 2   | 2     | 2     | 2     | 2    | 2           |
| $67_1$ | 133 | 2   | 2     | 2     | 2     | 2    | 2           |
| $68_1$ | 135 | 2   | 2     | 2     | 2     | 2    | 2           |
| $69_1$ | 137 | 2   | 2     | 2     | 2     | 2    | 2           |
| $70_1$ | 139 | 2   | 2     | 2     | 2     | 2    | 2           |

**Table 2.** Invariants of knots with at most 9 crossings. Knots for which calculation of $u_s$, $u_c$ or $c^*$ is nontrivial are highlighted.
Table 3. **Invariants of knots with 10 crossings.** Only knots for which calculation of $u_s$, $u_c$ or $c^*$ is nontrivial are listed.

| Knot | det | $u$ | $u_c$ | $u_s$ | $c^*$ | $g^*$ | $|\sigma|/2$ |
|------|-----|-----|-------|-------|------|-------|-----------|
| 104  | 27  | 2   | 1     | 1     | 1    | 1     | 1         |
| 106  | 37  | 3   | [2,3] | 2     | 2    | 2     | 2         |
| 108  | 43  | 2   | [2,3] | 1     | 1    | 1     | 1         |
| 109  | 47  | 2   | 1     | 1     | 1    | 1     | 1         |
| 112  | 55  | 2   | 2     | 2     | 1    | 1     | 1         |
| 113  | 55  | 2   | 2     | 2     | 1    | 1     | 1         |
| 114  | 43  | 2   | 1     | 1     | 1    | 1     | 1         |
| 115  | 47  | 2   | 1     | 1     | 1    | 1     | 1         |
| 116  | 51  | 2   | 2     | 2     | 1    | 1     | 1         |
| 118  | 35  | 2   | 2     | 2     | 1    | 1     | 1         |
| 120  | 55  | 2   | 2     | 2     | 1    | 1     | 1         |
| 121  | 57  | 2   | 1     | 1     | 1    | 1     | 1         |
| 122  | 43  | 2   | [2,3] | 1     | 1    | 1     | 1         |
| 123  | 47  | 2   | [2,3] | 1     | 1    | 1     | 1         |
| 124  | 43  | 2   | 1     | 1     | 1    | 1     | 1         |
| 125  | 47  | 2   | 1     | 1     | 1    | 1     | 1         |
| 126  | 43  | 2   | [2,3] | 1     | 1    | 1     | 1         |
| 127  | 47  | 2   | [2,3] | 1     | 1    | 1     | 1         |
| 128  | 43  | 2   | 1     | 1     | 1    | 1     | 1         |
| 129  | 47  | 2   | 1     | 1     | 1    | 1     | 1         |
| 130  | 43  | 2   | [2,3] | 1     | 1    | 1     | 1         |
| 131  | 47  | 2   | [2,3] | 1     | 1    | 1     | 1         |
| 132  | 43  | 2   | 1     | 1     | 1    | 1     | 1         |
| 133  | 47  | 2   | 1     | 1     | 1    | 1     | 1         |
| 134  | 43  | 2   | [2,3] | 1     | 1    | 1     | 1         |
| 135  | 47  | 2   | [2,3] | 1     | 1    | 1     | 1         |

*Knots for which calculation of $u_s$, $u_c$ or $c^*$ is nontrivial are listed.*
References

1. N. Askitas, Multi-# unknotting operations: a new family of local moves on a knot diagram and related invariants of knots, J. Knot Theory Ramifications 7 (1998), no. 7, 857–871.
2. J. C. Cha and C. Livingston, Table of knot invariants, http://www.indiana.edu/~knotinfo.
3. T. D. Cochran and W. B. R. Lickorish, Unknotting information from 4-manifolds, Trans. Amer. Math. Soc. 297 (1986), no. 1, 125–142.
4. S. K. Donaldson, The orientation of Yang-Mills moduli spaces and 4-manifold topology, J. Differential Geom. 26 (1987), no. 3, 397–428.
5. T. Kawamura, On unknotting numbers and four-dimensional clasp numbers of links, Proc. Amer. Math. Soc. 130 (2002), no. 1, 243–252 (electronic).
6. C. Livingston, The slicing number of a knot, Algebr. Geom. Topol. 2 (2002), 1051–1060 (electronic).
7. _____, The concordance genus of knots, Algebr. Geom. Topol. 4 (2004), 1–22.
8. J. M. Montesinos, Variedades de seifert que son cubiertas ciclicas ramificados de dos hojas, Bol. Soc. Mat. Mexicana (2) 18 (1973), 1–32.
9. H. Murakami and A. Yasuhara, Four-genus and four-dimensional clasp number of a knot, Proc. Amer. Math. Soc. 128 (2000), no. 12, 3693–3699.
10. K. Murasugi, On a certain numerical invariant of link types, Trans. Amer. Math. Soc. 117 (1965), 387–422.
11. Y. Ni and Z. Wu, Cosmetic surgeries on knots in $S^3$, arXiv:1009.4720; accepted for publication in J. Reine Angew. Math.
12. B. Owens, Unknotting information from Heegaard Floer homology, Adv. Math. 217 (2008), no. 5, 2353–2376.
13. _____, On slicing invariants of knots, Trans. Amer. Math. Soc. 362 (2010), no. 6, 3095–3106.
14. B. Owens and S. Strle, Rational homology spheres and the four-ball genus of knots, Adv. Math. 200 (2006), no. 1, 196–216.
15. P. Ozsváth and Z. Szabó, Absolutely graded Floer homologies and intersection forms for four-manifolds with boundary, Adv. Math. 173 (2003), no. 2, 179–261.
16. _____, On the Floer homology of plumbed three-manifolds, Geom. Topol. 7 (2003), 185–224 (electronic).
17. _____, Knots with unknotting number one and Heegaard Floer homology, Topology 44 (2005), no. 4, 705–745.
18. _____, Knot Floer homology and integer surgeries, Algebr. Geom. Topol. 8 (2008), no. 1, 101–153.
19. _____, Knot Floer homology and rational surgeries, Algebr. Geom. Topol. 11 (2011), no. 1, 1–68.
20. D. Ruberman and S. Strle, Concordance properties of parallel links, arXiv:1108.4476.
21. L. Rudolph, Braided surfaces and Seifert ribbons for closed braids, Comment. Math. Helv. 58 (1983), no. 1, 1–37.
22. T. Shibuya, Some relations among various numerical invariants for links, Osaka J. Math. 11 (1974), 313–322.
23. A. Stoimenow, Polynomial values, the linking form and unknotting numbers, Math. Res. Lett. 11 (2004), no. 5-6, 755–769.
24. F. Swenton, Kirby calculator, http://kirbycalculator.net.
25. P. Traczyk, A criterion for signed unknotting number, Low-dimensional topology (Funchal, 1998), Contemp. Math., vol. 233, Amer. Math. Soc., Providence, RI, 1999, pp. 215–220.
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