Research Article

Existence, Nonexistence, and Stability of Solutions for a Delayed Plate Equation with the Logarithmic Source

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1. Introduction

In this article, we consider a plate equation with frictional damping, delay, and logarithmic terms as follows:

\[ \begin{cases} 
\frac{\partial u}{\partial t} + \Delta^2 u + \alpha u(t) + \beta u(x, t - \tau) = u \ln |u|^k & \text{for } (x, t) \in \Omega \times (0, \infty), \\
\frac{\partial u}{\partial t} \bigg|_{\partial \Omega} = 0 & \text{for } (x, t) \in \partial \Omega \times (0, \infty), \\
u(x, 0) = u_0(x) & \text{for } x \in \Omega, \\
\frac{\partial u}{\partial t} \bigg|_{\partial \Omega} = j_0(x, t) & \text{for } (x, t) \in \Omega \times (-\tau, 0), 
\end{cases} \]

where \( \Omega \subset \mathbb{R}^N \), \( N \geq 1 \), is a bounded domain with smooth boundary \( \partial \Omega \). \( \tau > 0 \) denotes time delay, and \( \alpha, \beta, \text{ and } \gamma \) are real numbers that will be specified later. Generally, logarithmic nonlinearity seems to be in supersymmetric field theories and in cosmological inflation. From quantum field theory, that kind of \( (u|u|^n \ln |u|^k) \) logarithmic source term seems to be in nuclear physics, inflation cosmology, geophysics, and optics (see [1, 2]). Time delays often appear in various problems, such as thermal, economic, biological, chemical, and physical phenomena. Recently, partial differential equations have become an active area with time delay (see [3, 4]). In 1986, Datko et al. [5] indicated that, in boundary control, a small delay effect is a source of instability. Generally, a small delay can destabilize a system which is uniformly stable [6]. To stabilize hyperbolic systems with time delay, some control terms will be needed (see [7–9] and references therein).

For the literature review, firstly, we begin with the studies of Bialynicki-Birula and Mycielski [10, 11]. The authors investigated the equation with the logarithmic term as follows:

\[ u_{tt} - u_{xx} + u - \epsilon u \ln |u|^2 = 0, \]

where the authors proved that, in any number of dimensions, wave equations including the logarithmic term have localized, stable, soliton-like solutions.

In 1980, Cazenave and Haraux [12] studied the equation as follows:

\[ u_{tt} - \Delta u = u \ln |u|^k, \]
where the authors in [12] proved the existence and uniqueness of the solutions for equation (3). Gorka [2] obtained the global existence results of solutions for one-dimensional equation (3). Bartkowski and Górk [1] considered the weak solutions and obtained the existence results.

In [13], Hiramatsu et al. studied the equation as follows:

\[ u_{tt} - \Delta u + u + ut^2|u| = u \ln u. \]  

(4)

In [14], Han established the global existence of solutions for equation (4).

In [15], Al-Gharabli and Messaoudi were concerned with the plate equation with the logarithmic term as follows:

\[ u_{tt} + \Delta^2 u + h(u_t) = ku \ln |u|. \]  

(5)

They established the existence results by the Galerkin method and obtained the explicit and decay of solutions utilizing the multiplier method for equation (5).

In [16], Liu introduced the plate equation with the logarithmic term as follows:

\[ u_{tt} + \Delta^2 u + |u_t|^{m-2}u_t = |u|^{p-2}u \log |u|^k. \]  

(6)

The author proved the local existence by the contraction mapping principle. Also, he studied the global existence and decay results. Moreover, under suitable conditions, the author proved the blow-up results with the condition \( \gamma \). The constant \( \delta \) is a constant and

\[ \tau'(t) \leq d < 1, \quad \forall t > 0. \]  

(10)

In [26], Kafini and Messaoudi studied wave equations with delay and logarithmic terms as follows:

\[ u_{tt} - \Delta u + \mu_1u_t(x, t) + \mu_2u_t(x, t - \tau) = |u|^{p-2}u \log |u|^k. \]  

(11)

The authors proved the local existence and blow-up results for equation (11).

In [27], Park considered the equation with delay and logarithmic terms as follows:

\[ u_{tt} - \Delta u + a\beta u_t(t) + \beta u_t(x, t - \tau) = u \ln |u|^v. \]  

(12)

The author showed the local and global existence results for equation (12). Also, the author investigated the decay and nonexistence results for equation (12). In recent years, some other authors investigate hyperbolic-type equations with delay terms (see [28–33]).

In this work, we studied the local existence, global existence, nonexistence, and stability results of plate equation (1) with delay and logarithmic terms, motivated by the above works. There is no research to our best knowledge, related to plate equation (1) with the delay \( \beta u_t(x, t - \tau) \) term and logarithmic \( u \ln |u|^v \) source term; hence, our work is the generalization of the above studies.

This work consists of five sections in addition to the introduction. Firstly, in Section 2, we recall some assumptions and lemmas. Then, in Section 3, we obtain the local and global existence of solutions. Moreover, in Section 4, we establish the nonexistence results. Finally, in Section 5, we get the stability of solutions.

2. Preliminaries

In this part, we show the norm of \( X \) by \( \| \cdot \|_X \) for a Banach space \( X \). We give the scalar product in \( L^2(\Omega) \) by \( \langle \cdot, \cdot \rangle \). We show \( \| \cdot \|_2 \) by \( \| \cdot \| \), for brevity. Let \( B_1 \) be the constant of the embedding inequality

\[ \| u \|^2 \leq B_1 \| \Delta u \|^2 \quad \text{for} \ u \in H^1_0(\Omega). \]  

(13)

We have the following assumptions related to problem (1):

(H1). The weights of delay and dissipation satisfy

\[ 0 < |\beta| < \alpha. \]  

(14)

(H2). The constant \( \gamma \) in (1) satisfies

\[ 0 < \gamma < \pi e^{2(N+1)/N}. \]  

(15)

To get the main result, we have the lemmas as follows.
Lemma 1 (see [34, 35]) (Logarithmic Sobolev inequality). For any \( u \in H^2_0(\Omega) \),
\[
\int_{\Omega} u^2 \ln|u| \, dx \leq \frac{1}{2} \|u\|^2 \ln \|u\|^2 + \frac{k^2}{2n} \|\nabla u\|^2 - \frac{N}{2} (1 + \ln k) \|u\|^2,
\]
where \( k \) is a positive real number.

Remark 3. Assume that inequality (17) holds for all \( k > 0 \), and we choose the constant \( k \) that satisfies
\[
0 < k < 1.
\]

Lemma 4 (see [12]) (Logarithmic Gronwall inequality). Suppose that \( c > 0 \) and \( i \in L^1(0, T; R^+) \). If a function \( f : [0, T] \rightarrow [1, \infty) \) satisfies
\[
f(t) \leq c \left( 1 + \frac{\int_0^t i(s) \ln f(s) \, ds}{f(s)} \right), \quad 0 \leq t \leq T,
\]
then
\[
f(t) \leq ce^{\int_0^t \frac{i(s) \, ds}{f(s)}}, \quad 0 \leq t \leq T.
\]

We define
\[
J(\nu) = \frac{1}{2} \|\Delta \nu\|^2 - \frac{1}{2} \int_{\Omega} \nu^2 \ln |\nu|^2 \, dx + \frac{\gamma \nu}{4} \|\nu\|^2,
\]
\[
I(\nu) = \frac{1}{2} \|\Delta \nu\|^2 - \int_{\Omega} \nu^2 \ln |\nu|^2 \, dx,
\]
for \( \nu \in H^2_0(\Omega) \); then,
\[
J(\nu) = \frac{1}{2} I(\nu) + \frac{\gamma}{4} \|\nu\|^2.
\]
Suppose that
\[
d = \inf_{\nu \in H^2_0(\Omega) \setminus \{0\}} \sup_{\lambda > 0} J(\nu).
\]

Then, it satisfies (see, e.g., [36–38])
\[
0 < d = \inf_{\nu \in \mathcal{N}} J(\nu),
\]
where \( \mathcal{N} \) is the well-known Nehari manifold, denoted by
\[
\mathcal{N} = \{ \nu \in H^2_0(\Omega) \setminus \{0\} \mid I(\nu) = 0 \}.
\]

Lemma 5. \( I \) and \( J \) are the functions that satisfy
\[
I(\lambda \nu) = \frac{\lambda}{\lambda \nu} \|\nu\|^2 - \frac{\gamma \nu}{4} \|\nu\|^2,
\]
for any \( \nu \in H^2_0(\Omega) \) with \( \|\nu\| \neq 0 \), where
\[
\lambda^* = \exp \left( \frac{\|\nu\|^2 - \int_{\Omega} \nu^2 \ln |\nu|^2 \, dx}{\gamma^2 \|\nu\|^2} \right).
\]

Proof. We obtain, for \( \lambda \geq 0 \),
\[
\lambda \frac{d}{d\lambda} J(\nu) = \lambda \left( \lambda \|\Delta \nu\|^2 - \lambda \int_{\Omega} \nu^2 \ln |\nu|^2 \, dx + \gamma \nu \|\nu\|^2\right) - \gamma \nu \int_{\Omega} \nu^2 \ln |\nu|^2 \, dx
\]
\[
= \lambda^2 \left( \|\Delta \nu\|^2 - \int_{\Omega} \nu^2 \ln |\nu|^2 \, dx - \gamma \nu \ln |\nu| \int_{\Omega} \nu^2 \, dx \right)
\]
\[
= I(\nu),
\]
and therefore, we obtain the desired result.

Remark 6. \( J(\lambda \nu) \) has the absolute maximum value at \( \lambda^* \), such that
\[
\sup_{\lambda > 0} J(\lambda \nu) = J(\lambda^* \nu) = \exp \left( \frac{2\|\Delta \nu\|^2 - 2\int_{\Omega} \nu^2 \ln |\nu|^2 \, dx}{\gamma^2 \|\nu\|^2} \right) \frac{\gamma^2}{4} \|\nu\|^2,
\]
for \( \nu \in H^2_0(\Omega) \).

Lemma 7. The potential depth \( d \) in (25) satisfies
\[
d \geq \frac{\gamma}{4} \exp \left( \frac{\pi^2}{\gamma} \right)^{N/2} = E_1.
\]
Proof. By Corollary 2, (13), and (18), we have

$$I(v) \geq \left(1 - \frac{k^2 y}{2 \pi}\right) ||\Delta v||^2 + \frac{N y}{2} (1 + \ln k) ||v||^2 - \frac{y}{2} ||v||^2 \ln ||v||^2$$

$$> \frac{N y}{2} (1 + \ln k) ||v||^2 - \frac{y}{2} ||v||^2 \ln ||v||^2.$$  

(33)

Taking the limit $k \rightarrow \sqrt{\pi y}$, we obtain

$$I(v) \geq \left\{ \frac{N y}{2} \left(1 + \ln \sqrt{\pi y}\right) - \frac{y}{2} \ln ||v||^2 \right\} ||v||^2.$$  

(34)

Taking into consideration this and (28), we get

$$0 = I(\lambda^* v) \geq \left\{ \frac{N y}{2} \left(1 + \ln \sqrt{\pi y}\right) - \frac{y}{2} \ln ||\lambda^* v||^2 \right\} ||\lambda^* v||^2,$$

(35)

and therefore,

$$||\lambda^* v||^2 \geq \varepsilon^N \left(\frac{\pi}{y}\right)^{N/2}. \tag{36}$$

Hence, we have by (24) and (31)

$$\sup_{\lambda \in \mathcal{L}} I(\lambda v) = I(\lambda^* v) = \frac{1}{2} I(\lambda^* v) + \frac{y}{4} ||\lambda^* v||^2 = \frac{y}{4} ||\lambda^* v||^2 \geq \frac{y}{4} \varepsilon^N \left(\frac{\pi}{y}\right)^{N/2}. \tag{37}$$

From the definition of $d$ given in (25), we obtain the result. \hfill \Box

3. Existence

In this part, we have studied the local existence, global existence, nonexistence, and stability results of plate equation (1) with delay and logarithmic terms, motivated by the above works. There is no research, to our best knowledge, related to plate equation (1) with delay and logarithmic terms, motivated by the above studies. Firstly, we introduce the new function

$$y(x, \eta, t) = u_i(x, t - \eta \tau) \quad \text{for} \quad (x, \eta, t) \in \Omega \times [0, 1] \times (0, \infty).$$  

(38)

Hence, problem (1) takes the form

$$\begin{aligned}
&u_{tt} + \Delta^2 u + au_i(x, t) + b y(x, 1, t) = u \ln |u|^\gamma \quad \text{for} \quad (x, t) \in \Omega \times (0, \infty), \\
y(x, \eta, t) + y(x, \eta, 0) = 0 \quad \text{for} \quad (x, \eta, t) \in \partial \Omega \times (0, \infty), \\
u(x, t) = \frac{\partial u_i(x, t)}{\partial v} = 0 \quad \text{for} \quad (x, t) \in \partial \Omega \times (0, \infty), \\
u(x, 0) = u_i(x, 0) = u_i(x) \quad \text{for} \quad x \in \Omega, \\
y(x, \eta, 0) = y_i(x, \eta, 0) = y_i(x, \eta) \quad \text{for} \quad (x, \eta) \in \partial \Omega \times (0, 1),
\end{aligned} \tag{39}$$

Definition 8. Assume that $T > 0$, $(u, y)$ is a local solution of problem (39) if it satisfies

$$u \in C([0, T]; H^2_0(\Omega)) \cap C^1([0, T]; L^2(\Omega)) \cap C^2([0, T]; H^{-2}(\Omega)),$$

$$(u_{tt}, v) + (\Delta u_t, \Delta v) + \alpha(u_i(t), v) + \beta(y(1, t), v) = (u \ln |u|^\gamma, v) \quad \text{for} \quad v \in H^2_0(\Omega),$$

$$\int_0^1 (y_i(\eta, t), \varphi(\eta)) d\eta + \int_0^1 (v_i(\eta, t), \varphi(\eta)) d\eta = 0 \quad \text{for} \quad \forall \varphi \in L^2(\Omega \times (0, 1)),$$

$$u(0) = u_0 \quad \text{in} \quad H^2_0(\Omega),$$

$$u_i(0) = u_i \quad \text{in} \quad L^2(\Omega),$$

$$y(0) = y_i \quad \text{in} \quad L^2(\Omega \times (0, 1)). \tag{40}$$

3.1. Local Existence. In this part, we establish the local existence results similar to [8, 39].

Theorem 9. Suppose that (H1) and (H2) are satisfied. Then, for the initial data $u_0 \in H^2_0(\Omega)$, $u_i \in L^2(\Omega)$, and $y_0 \in L^2(\Omega \times (0, 1))$, there exists a local solution $(u, y)$ for problem (39).

Proof. Let $\{v_i\}_{i \in \mathbb{N}}$ be the orthogonal basis of $H^2_0(\Omega)$ that is orthonormal in $L^2(\Omega)$. Define $\varphi_i(x, 0) = v_i(x)$, and we extend $\varphi_i(x, 0)$ by $\varphi_i(x, \eta)$ over $L^2(\Omega \times (0, 1))$. We denote $V_n = \text{span}\{v_1, v_2, \ldots, v_n\}$ and $W_n = \text{span}\{\varphi_1, \varphi_2, \ldots, \varphi_n\}$ for $n \geq 1$. We consider the Faedo-Galerkin approximation solution $(u^n, y^n) \in V_n \times W_n$ of the form

$$u^n = \sum_{i=1}^n h^n_i(t) v_i(x), \quad \tag{41}$$

$$y^n(x, \eta, t) = \sum_{i=1}^n g^n_i(t) \varphi_i(x, \eta),$$

$n = 1, 2, \ldots$, solving the approximate system

$$\begin{aligned}
&\int_0^1 u^n_t \ln |u^n|^\gamma v_i dx \quad \text{for} \quad v \in V_n,
\end{aligned} \tag{42}$$

$$\begin{aligned}
&u_{tt}^n + \Delta^2 u^n + au_i(x, 1, t) + b y^n(x, 1, t) = u \ln |u|^\gamma \quad \text{for} \quad (x, t) \in \Omega \times (0, \infty), \\
y^n(x, \eta, 1) = y_i(x, \eta, 0) = y_i(x, \eta) \quad \text{for} \quad (x, \eta) \in \partial \Omega \times (0, 1),
\end{aligned} \tag{43}$$

$$\begin{aligned}
&u^n(x, 0) = u_i(x, 0) = u_i(x) \quad \text{for} \quad x \in \Omega,
\end{aligned} \tag{44}$$

$$\begin{aligned}
&y^n(x, \eta, 0) = y_i(x, \eta, 0) = y_i(x, \eta) \quad \text{for} \quad (x, \eta) \in \partial \Omega \times (0, 1),
\end{aligned} \tag{45}$$

$$\begin{aligned}
&u^n_t(\eta, t) + \Delta u^n_t(\eta, t) + \alpha(u^n_i(\eta, t), v_i) + \beta(y^n(\eta, 1, t), v_i) = (u \ln |u|^\gamma, v_i) \quad \text{for} \quad v \in V_n,
\end{aligned} \tag{46}$$

$$\begin{aligned}
&u^n(0) = u_0 \quad \text{in} \quad H^2_0(\Omega),
\end{aligned} \tag{47}$$

$$\begin{aligned}
&u^n_i(0) = u_i \quad \text{in} \quad L^2(\Omega),
\end{aligned} \tag{48}$$

$$\begin{aligned}
&y^n(0) = y_i \quad \text{in} \quad L^2(\Omega \times (0, 1)).
\end{aligned} \tag{49}$$

These approximate systems are uniformly bounded and converges to a local solution $(u, y)$ of problem (39).
where
\[ u_0^n \to u_0 \text{ in } H^1_0(\Omega), \]
\[ u_1^n \to u_1 \text{ in } L^2(\Omega), \]
\[ y_0^n \to y_0 \text{ in } L^2(\Omega \times (0, 1)). \]

Since problem (42)–(44) is a normal system of ordinary differential equations, there exists a solution \((u^n, y^n)\) on the interval \([0, t_n]\), \(t_n \in (0, T]\). The extension of that solution to the \([0, T]\) is a consequence of the estimate below.

By replacing \(v\) by \(u_1^n(t)\) in (42) and by using the relation
\[
\int_{\Omega} u^n |u^n|^r u^n dx = \frac{d}{dt} \left\{ \frac{1}{2} \int_{\Omega} (u^n)^2 \ln |u^n|^r dx - \frac{r}{4} |u^n|^2 \right\},
\]
we have
\[
\frac{d}{dt} \left\{ \frac{1}{2} \|u^n\|^2 + \frac{1}{2} \|\Delta u^n\|^2 + \frac{r}{4} |u^n|^2 - \frac{1}{2} \int_{\Omega} (u^n)^2 \ln |u^n|^r dx \right\} = -\alpha \|u^n(t)\|^2 - \beta (y^n(1, t), u_1^n(t)).
\]

By replacing \(\varphi\) by \(\omega y^n(\eta, t)\) in (43), we see that
\[
\frac{\omega r}{2} \frac{d}{dt} \int_0^1 \left( y^n(\eta, x, t) \right)^2 d\eta dx = -\frac{\omega}{2} \|y^n(1, t)\|^2 + \frac{\omega}{2} \|y^n(0, t)\|^2.
\]

Summing (47) and (48), we obtain
\[
\frac{d}{dt} E^n(t) = -\alpha \|u^n\|^2 - \beta (y^n(1, t), u_1^n(t)) - \frac{\omega}{2} \|y^n(1, t)\|^2 + \frac{\omega}{2} \|y^n(0, t)\|^2,
\]
where
\[
E^n(t) = \frac{1}{2} \|u^n\|^2 + \frac{1}{2} \|\Delta u^n\|^2 + \frac{r}{4} |u^n|^2 - \frac{1}{2} \int_{\Omega} (u^n)^2 \ln |u^n|^r dx + \frac{\omega r}{2} \|y^n\|^2_{L^2(\Omega \times (0, 1))},
\]
\[
\|u^n\|^2 + \|\Delta u^n\|^2 + \|u^n\|^2 + \int_0^t \|u_1^n(s)\|^2 ds + \int_0^t \|y^n(1, s)\|^2 ds \leq 2E^n(0) + \frac{Y}{2} \|u^n\|^2 \ln \|u^n\|^2.
\]

By using (18), we obtain
\[
1 - \frac{Y^2}{2\pi} > 0,
\]
\[
\frac{Y}{2} (1 + N(1 + \ln k)) > 0,
\]
and therefore,
\[
\|u^n\|^2 + \|\Delta u^n\|^2 + \|u^n\|^2 + \int_0^t \|u_1^n(s)\|^2 ds + \int_0^t \|y^n(1, s)\|^2 ds \leq c_1 \left( 1 + \|u^n\|^2 \ln \|u^n\|^2 \right),
\]
where the sequel \(c_j, j = 1, 2, \cdots\), shows a positive constant. Also, we know that
\[
u^n(x, t) = u^n(0, t) + \int_0^t u_1^n(x, s) ds.
\]

Utilizing Cauchy-Schwarz’s inequality and (57), we obtain
\[
\|u^n\|^2 < \omega < 2\alpha - |\beta|.
\]
\[
\|u^n(t)\|^2 = 2\|u^n(0)\|^2 + 2T \int_0^t \|u^n(s)\|^2 \, ds \\
\leq 2\|u^n(0)\|^2 + 2T \int_0^t c_1 \left(1 + \|u^n(s)\|^2 \ln \|u^n(s)\|^2\right) \, ds \\
\leq c_2 \left(1 + \int_0^t \|u^n(s)\|^2 \ln \|u^n(s)\|^2 \, ds\right).
\]

(59)

From Lemma 4, we arrive at
\[
\|u^n(t)\|^2 \leq c_3 e^{c_1 t}.
\]

(60)

If \(f(s) = s \ln s\) is the function which is continuous on \((0, \infty)\), \(\lim_{s \to 0} f(s) = 0\), \(\lim_{s \to \infty} f(s) = +\infty\), and \(f\) decreases on \((0, e^{-1})\) and increases on \((e^{-1}, +\infty)\); hence, we get by (57) and (60)
\[
\|u^n\|^2 + \|\Delta u^n\|^2 + \|u^n\|^2 + \int_0^t \|u^n(s)\|^2 \, ds + \int_0^t \|y^n(s)\|^2 \, ds \\
+ \|y^n\|^2_{L^2(\Omega \times (0,1))} \leq c_4.
\]

(61)

Hence, there exists a subsequence of \((u^n, y^n)\), which we still denote \((u^n, y^n)\), such that
\[
\begin{align*}
  u^n &\rightharpoonup u \text{ weakly star in } L^\infty(0, T; H^2_0(\Omega)), \\
  u^n_t &\rightharpoonup u_t \text{ weakly star in } L^\infty(0, T; L^2(\Omega)), \\
  y^n &\rightharpoonup y \text{ weakly star in } L^\infty(0, T; L^2(\Omega \times (0, 1))), \\
  y^n(1) &\rightharpoonup y(1) \text{ weakly in } L^2(0, T; L^2(\Omega)).
\end{align*}
\]

(62)

Utilizing the Aubin-Lions compactness theorem, we conclude that
\[
\begin{align*}
  u^n &\to u \text{ strongly in } L^2(0, T; L^2(\Omega)), \\
  u^n_t &\to u_t \quad \text{a.e. in } \Omega \times (0, T).
\end{align*}
\]

(63)

The function \(s \to s \ln |s|^y\) is continuous on \(R\); hence,
\[
\begin{align*}
  u^n \ln |u^n|^y &\to u \ln |u|^y \text{ a.e. in } \Omega \times (0, T).
\end{align*}
\]

(64)

Let
\[
\begin{align*}
  \Omega_1 &\equiv \{x \in \Omega \mid |u^n| < 1\}, \\
  \Omega_2 &\equiv \{x \in \Omega \mid |u^n| \geq 1\}.
\end{align*}
\]

(65)

Thus, we obtain
\[
\int_\Omega (u^n \ln |u^n|^y)^2 \, dx = \int_\Omega (u^n \ln |u^n|)^2 \, dx + \int_\Omega (u^n \ln |u^n|^y)^2 \, dx \\
\leq \gamma^2 \left(\int_\Omega (u^n \ln |u^n|)^2 \, dx + \int_\Omega (u^n \ln |u^n|^y)^2 \, dx\right) \\
\leq \gamma^2 \left(e^{-2} |\Omega|_1 + e^{-2} \left(\frac{2}{q - 2}\right)^2 \int_\Omega (u^n)^2 \, dx\right)
\]

for any \(q > 2\),

(66)

where we used
\[
\begin{align*}
  |s \ln s| &\leq \frac{1}{c_1} \quad \text{for } 0 < s < 1, \\
  s^{-\kappa} &\leq \frac{1}{c_1} \quad \text{for } s \geq 1 \text{ and } \kappa > 0.
\end{align*}
\]

(67)

By (57) and (66), we conclude that
\[
\int_\Omega (u^n \ln |u^n|^y)^2 \, dx \leq \gamma^2 \left(e^{-2} |\Omega|_1 + e^{-2} \left(\frac{2}{q - 2}\right)^2 B^2_{q_1} |\Delta u^n|^2\right) \leq c_6,
\]

(68)

where \(B_{q_1}\) is the Sobolev imbedding constant of
\[
H^2_0(\Omega) \subset L^q(\Omega) \quad \text{for } q > 2, \text{ if } N = 1, 2, 3, 4; 2 < q < \frac{2N}{N - 4}, \text{ if } N \geq 5.
\]

(69)

Therefore, we get from (68)
\[
u^n \ln |u^n|^y \quad \text{which is uniformly bounded in } L^\infty(0, T; L^2(\Omega)).
\]

(70)

From the Lebesgue bounded convergence theorem, (64), and (70), we arrive at
\[
u^n \ln |u^n|^y \rightharpoonup u \ln |u|^y \quad \text{strongly in } L^2(0, T; L^2(\Omega)).
\]

(71)

We pass the limit \(m \to \infty\) in (42) and (43). The remainder of the proof is standard and similar to [39, 40]. \(\Box\)

3.2. Global Existence. In this part, we obtain the global existence results for problem (39). For this goal, we define the energy functional of problem (39):
\[
E(t) = \frac{1}{2} \|u_t\|^2 + \frac{1}{2} \|\Delta u\|^2 + \frac{y}{4} \|u\|^y - \frac{1}{2} \int_\Omega u^2 \ln |u|^y \, dx \\
+ \frac{\omega_T}{2} \|y\|^2_{L^2(\Omega \times (0,1))},
\]

(72)

where \(\omega\) is the positive constant given in (51). We see that
\[
E(t) = \frac{1}{2} \|u_t\|^2 + J(u(t)) + \omega_T \|y\|^2_{L^2(\Omega \times (0,1))} = \frac{1}{2} \|u_t\|^2 \\
+ \frac{1}{2} J(u(t)) + \frac{y}{4} \|u\|^y + \frac{\omega_T}{2} \|y\|^2_{L^2(\Omega \times (0,1))}.
\]

(73)
By the same arguments similar to (52), we infer that
\[
\frac{d}{dt} E(t) \leq -C_1 \|u_t\|^2 - C_2 \|y(1,t)\|^2 \leq 0,
\]
where \(C_1\) and \(C_2\), given in (54), are positive constants.

Lemma 10. Suppose that (H1) and (H2) are satisfied. If \(E(0) < d\) and \(I(u_0) > 0\), then the solution \(u\) of problem (1) satisfies
\[
I(u(t)) > 0 \quad \text{for} \ t \in [0, T),
\]
where \(T\) is the maximal existence time of the solutions.

Proof. We know that \(I(u_0) > 0\) and \(u\) is continuous on \([0, T)\); hence, we have
\[
I(u(t)) > 0 \quad \text{for some interval} \ [0, t_1) \subset [0, T).
\]
Let \(t_0\) be the maximum of \(t_1\) satisfying (76). Assume that \(t_0 < T\); then, \(I(u(t_0)) = 0\), that is,
\[
u(t_0) \in \mathcal{N}.
\]
Therefore, we obtain by (26)
\[
J(u(t_0)) \geq \inf_{v \in \mathcal{N}} J(v) = d.
\]
We see that this is in contradiction to the relation as follows:
\[
J(u(t_0)) \leq E(t_0) \leq E(0) < d.
\]
By (74) and Lemma 10, we see that \(E(t)\) is a nondecreasing function. \(\square\)

Theorem 11. The solution \(u\) is global, under the conditions of Lemma 10.

Proof. It suffices to show that \(\|u_t\|^2 + \|\Delta u\|^2\) is bounded independent of \(t\). By Lemma 10, (73), and (74), we get
\[
\|u_t\|^2 \leq \|u\|^2 + I(u(t)) \leq 2E(t) \leq 2E(0) < 2d.
\]
In a similar way, we get
\[
\|u\|^2 < \|u\|^2 + \frac{2}{\gamma} I(u(t)) = \frac{4}{\gamma} I(u(t)) \leq \frac{4}{\gamma} E(t) \leq \frac{4}{\gamma} E(0) < \frac{4d}{\gamma}.
\]
By Corollary 2 and (23), we conclude that
\[
\|\Delta u\|^2 = I(u(t)) + \int_\Omega u^2 \ln |u| dx \leq 2E(t) + \frac{\gamma}{2} \|u\|^2 \ln \|u\|^2
\]
\[
+ \frac{k_2^2 \gamma}{2\pi} \|\Delta u\|^2 - \frac{\gamma}{2} \left(1 + \ln k\right) \|u\|^2.
\]
By taking the limit \(k \to 0^+\) in this inequality and from (81), we obtain
\[
\left(1 - \frac{\gamma^2 \rho^2}{2\pi}\right) \|\Delta u\|^2 \leq 2E(t) + \frac{\gamma}{2} \left(\|u\|^2 - N(1 + \ln \rho)\|u\|^2\right)
\]
\[
< 2d + \frac{\gamma}{2} \left(\ln \left(\frac{4d}{\gamma} e^{-N\rho}\right)\right) - N(1 + \ln \rho)\|u\|^2
\]
\[
= 2d + \frac{\gamma}{2} \left(\ln \left(\frac{4d}{\gamma} e^{-N\rho}\right)\right)\|u\|^2.
\]
By Lemma 7 and (18), we get
\[
\ln \left(\frac{4d}{\gamma} e^{-N\rho}\right) \geq \ln \left(\frac{\pi}{\gamma}\right)^{N/2} (1 - \rho^2)\|u\|^2
\]
\[
= \ln \left(\frac{\sqrt{\pi}}{\sqrt{\gamma}}\right)^N \ln 1 = 0.
\]
Therefore, we see by (81) and (83) that
\[
\left(1 - \frac{\gamma^2 \rho^2}{2\pi}\right) \|\Delta u\|^2 < 2d + 2d \ln \left(\frac{4d}{\gamma} e^{-N\rho}\right).
\]
Hence, we conclude that
\[
\|\Delta u\|^2 < 2d \left(1 - \frac{\gamma^2 \rho^2}{2\pi}\right)^{-1} \left(1 + \ln \left(\frac{4d}{\gamma} e^{-N\rho}\right)\right).
\]
Therefore, we complete the proof by (80) and (86). \(\square\)

4. Nonexistence

In this part, similar to [41–43], we get the nonexistence results for problem (1). Firstly, we need the lemma as follows.

Lemma 12. Assume that (H1) and (H2) are satisfied. If \(E(0) < E_i\) and \(I(u_0) < 0\), then the solution \(u\) of problem (1) satisfies
\[
I(u(t)) < 0 \quad \text{for} \ t \in [0, T), \quad (87)
\]
\[
\|u(t)\|^2 > \frac{4E_i}{\gamma} \quad \text{for} \ t \in [0, T), \quad (88)
\]
where \(T\) is the maximal existence time of the solutions.

Proof. We know that \(I(u_0) < 0\) and \(u\) is continuous on \([0, T)\); hence, we have
\[
I(u(t)) < 0 \quad \text{for some interval} \ [0, t_1) \subset [0, T). \quad (89)
\]
Let \(t_0\) be the maximal time satisfying (89) and assume that \(t_0 < T\); then, \(I(u_0) = 0\), such that
\[
u(t_0) \in \mathcal{N}. \quad (90)
\]
Therefore, we obtain
\[ d \leq J(u(t_0)) = \frac{1}{2} I(u(t_0)) + \frac{\gamma}{4} \| u(t_0) \|^2 \leq E(u(t_0)) \leq E(0) < E_1. \]  
\[ (91) \]

This is in contradiction to Lemma 7. Thus, (87) is proved. By Lemma 7, (31), and (87), we conclude that
\[ E_1 \leq d \leq J(\lambda^* u(t)) = \exp \left( 2 \| \Delta u \|^2 - 2 \int_{\Omega} \int u^2 \ln \| u' \| dx \right) \frac{\gamma}{4} \| u \|^2 \]
\[ < \frac{\gamma}{4} \| u \|^2. \]
\[ (92) \]

Therefore, the proof is completed.

\[ \text{Theorem 13. Suppose that (H1) and (H2) are satisfied. Let } E(0) < \zeta E_1, \text{ where } 0 < \zeta < 1, \text{ and } I(u_0) < 0. \text{ Then, the solution of problem (1) blows up at infinity.} \]

Proof. Firstly, we set
\[ F(t) = \xi E_1 - E(t). \]
\[ (93) \]

By (74), we obtain
\[ F'(t) = -E'(t) \geq C_1 \| u_t \|^2 + C_2 \| y(1, t) \|^2 \geq 0. \]
\[ (94) \]

Utilizing (72), (88), and (94), we see that
\[ 0 \leq F(0) \leq \xi E_1 + \frac{1}{2} \int_{\Omega} u^2 \ln \| u' \|^2 dx < \frac{\gamma}{4} \| u \|^2 \]
\[ + \frac{1}{2} \int_{\Omega} u^2 \ln \| u' \|^2 dx. \]
\[ (95) \]

We define
\[ G(t) = F(t) + \epsilon (u, u_t) + \frac{\epsilon \alpha}{2} \| u \|^2. \]
\[ (96) \]

By (39) and (72), we get
\[ G'(t) = F'(t) + \epsilon \| u_t \|^2 - \epsilon \| \Delta u \|^2 - \epsilon \beta(u, y(1, t)) + \epsilon \int_{\Omega} u^2 \ln \| u' \|^2 dx = F'(t) + 2\epsilon \| u_t \|^2 - \epsilon \beta(u, y(1, t)) \]
\[ - 2\epsilon E(t) + \frac{\epsilon \gamma}{2} \| u \|^2 + \omega \epsilon \| y \|^2 \gamma_{\gamma(0,1)}. \]
\[ (97) \]

Utilizing Young’s inequality and (94), we obtain
\[ \beta(u, y(1, t)) \leq \beta \left( \delta \| u \|^2 + \frac{1}{40} \| y(1, t) \|^2 \right) \leq \delta \| u \|^2 + \frac{1}{40} \| y(1, t) \|^2 \beta F'(t). \]
\[ (98) \]

By adapting this to (97) and from (88) and (93), we have
\[ G'(t) \geq \left( 1 - \frac{\epsilon \beta}{40C_1} \right) F'(t) + 2\epsilon \| u_t \|^2 + \left( \frac{\epsilon \gamma}{2} - \epsilon \beta \right) \| u \|^2 \]
\[ + 2\epsilon E(t) - 2\epsilon \xi E_1 + \omega \epsilon \| y \|^2 \gamma_{\gamma(0,1)} \]
\[ \geq \left( 1 - \frac{\epsilon \beta}{40C_1} \right) F'(t) + 2\epsilon \| u_t \|^2 + \epsilon \left( 1 - \zeta \right) \| u \|^2 \]
\[ + 2\epsilon E(t) + \omega \epsilon \| y \|^2 \gamma_{\gamma(0,1)}. \]
\[ (99) \]

Firstly, fix $\delta > 0$ such that $(1 - \zeta)/(\gamma/2) - \beta \delta > 0$ and then choose $\epsilon > 0$ small enough so that $1 - (\epsilon \beta/40C_1) > 0$. Then, by (94), we get
\[ G'(t) \geq c_0 (F(t) + \| u_t \|^2 + \| u \|^2) \geq 0. \]
\[ (100) \]

Also, we conclude that
\[ G(0) = F(0) + \epsilon (u_0, u_1) + \frac{\epsilon \alpha}{2} \| u_0 \|^2 > 0. \]
\[ (101) \]

Taking $\epsilon > 0$ small enough again, we obtain
\[ G(t) \geq c_0 G(0) > 0. \]
\[ (102) \]

By (100) and (102), we get
\[ G(t) \geq G(0) > 0. \]
\[ (103) \]

Utilizing (100) and (101), we see that
\[ G'(t) \geq c_{10} G(t), \]
\[ (104) \]

and therefore,
\[ G(t) \geq \epsilon \omega^2 G(0) > 0. \]
\[ (105) \]

Therefore, $G(t)$ blows up at infinity. Consequently, the proof is completed.

5. Stability

In this part, we obtain the stability of global solutions. Firstly, we define the perturbed energy by
\[ \Psi(t) = E(t) + \epsilon \Phi(t) + \epsilon \Xi(t), \]
\[ (106) \]

where $\phi > 0$, $\Phi(t) = (u_0, u_1)$, and $\Xi(t) = \int_{\Omega} \int_{\gamma(0,1)} e^{-\eta} y^2(x, \eta, t) d\eta dx.$

\[ \text{Lemma 14. Under the conditions of Lemma 10, for } C_3, C_4 > 0, \text{ we obtain} \]
\[ C_3 E(t) \leq \Psi(t) \leq C_4 E(t). \]
\[ (107) \]
Proof. Utilizing Lemma 10 and Young’s inequality, we have

\[
\|\Phi(t) + \Xi(t)\| \leq \frac{1}{2} \|u_t\|^2 + \frac{1}{2} \|u\|^2 + \|y\|^2_{L^2(\mathbb{R}^+ \times (0,1))} \\
\leq \frac{1}{2} \|u_t\|^2 + \frac{1}{2} \left( \frac{y}{4} \|u\|^2 + \frac{1}{2} I(u(t)) \right) \\
+ \|y\|^2_{L^2(\mathbb{R}^+ \times (0,1))} = \frac{1}{2} \|u_t\|^2 + \frac{1}{2} I(u(t)) \\
+ \|y\|^2_{L^2(\mathbb{R}^+ \times (0,1))} \leq C_2E(t).
\]

Adding and subtracting $\xi E(t)$ with $0 < \xi < 2\epsilon$, we get

\[
\Psi'(t) \leq -\xi E(t) - \left( C_1 - \epsilon - \epsilon \alpha^2 B_1 - \frac{\epsilon}{\tau} \right) \|u_t\|^2 \\
- \left( \frac{\epsilon}{2} - \frac{\xi}{2} \right) \|\Delta u\|^2 - (C_2 - \epsilon \beta^2 B_1) \|y(1,t)\|^2 \\
+ \left( \frac{\epsilon}{2} - \frac{\xi}{2} \right) \int_0^1 u^2 \ln |u| \, dx - \left( \epsilon \gamma - \frac{\xi}{2} \right) \|y\|^2_{L^2(\mathbb{R}^+ \times (0,1))}.
\]

Utilizing the logarithmic Sobolev inequality, we have

\[
\Psi'(t) \leq -\xi E(t) - \left( C_1 - \epsilon - \epsilon \alpha^2 B_1 - \frac{\epsilon}{\tau} \right) \|u_t\|^2 \\
- \left( \frac{\epsilon}{2} - \frac{\xi}{2} \right) \|\Delta u\|^2 - (C_2 - \epsilon \beta^2 B_1) \|y(1,t)\|^2 \\
+ \frac{Y}{2} \left( -\xi \right) \{\ln \|u\|^2 - N(1 + \ln k)\} \|u\|^2 \\
- (C_2 - \epsilon \beta^2 B_1) \|y(1,t)\|^2 - \left( \epsilon \gamma - \frac{\xi}{2} \right) \|y\|^2_{L^2(\mathbb{R}^+ \times (0,1))}.
\]

Now, choose $\epsilon > 0$ small enough, such that

\[
C_1 - \epsilon - \epsilon \alpha^2 B_1 - \frac{\epsilon}{\tau} > 0, \\
C_2 - \epsilon \beta^2 B_1 > 0.
\]

By taking $\xi > 0$ sufficiently small and noting that $(1/2) - (\gamma k^2/2\pi) > 0$ (see (18)), we infer that

\[
\Psi'(t) \leq -\xi E(t) + \frac{1}{2} \{\ln \|u\|^2 - N(1 + \ln k)\} \|u\|^2,
\]

where $0 < E(0) < E_1$; therefore, there exists $0 < \mu < 1$, that is, $E(0) = \mu E_1$. Therefore, we obtain by (81)

\[
\ln \|u\|^2 < \ln \left( \frac{4}{Y} E(t) \right) \leq \ln \left( \frac{4}{Y} E(0) \right) = \ln \left( \frac{4\mu E_k}{Y} \right) = \ln \left( \mu^{\gamma} \left( \frac{\pi}{Y} \right)^{N/2} \right).
\]

Hence, by (18), we arrive at

\[
\ln \|u\|^2 - N(1 + \ln k) \leq \ln \left( \mu^{\gamma} \left( \frac{\pi}{Y} \right)^{N/2} \right) - N(1 + \ln k)
\]

\[
= N \ln \left( \mu^{\gamma} \left( \frac{\pi}{Y} \right)^{N/2} \right) < N \ln 1 = 0.
\]
Substituting this into (116), we arrive at

\[ \Psi'(t) \leq -\xi E(t). \quad (119) \]

As a result, from Lemma 14, we completed the proof. □

6. Conclusions

Recently, there have been many published works related to wave equations with time delay. There were no local existence, global existence, nonexistence, and stability results of the plate equation with delay and logarithmic source terms, to the best of our knowledge. Firstly, we have obtained the local and global existence results. Then, we have obtained the nonexistence of solutions. Finally, we have proved stability results under sufficient conditions.

Data Availability

No data were used to support the study.

Conflicts of Interest

The authors declare that they have no competing interests.

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