Filtered Observer-Based IDA-PBC Control for Trajectory Tracking of a Quadrotor

MARÍA-EUSEBIA GUERRERO-SÁNCHEZ1, OMAR HERNández-GONZÁLEZ1, GUILLERMO VALENcIA-PALOMO2, FRANCISCO-ronay LÓPEZ-estRADA3, ABRAHAM-EfrAIM rODRíGUEZ-MATA4, and JAVIER gARRIDO5

1Tecnológico Nacional de México, Instituto Tecnológico Superior de Coatzacoalcos, Coatzacoalcos, Veracruz 96536, Mexico
2Tecnológico Nacional de México, IT Hermosillo, Hermosillo 83170, Mexico
3Tecnológico Nacional de México, IT Tuxtla Gutiérrez, TURIX Diagnosis and Control Group, Tuxtla Gutiérrez 29050, Mexico
4Tecnológico Nacional de México, IT de Chihuahua, Chihuahua 31310, Mexico
5Facultad de Ingeniería, Universidad Veracruzana, Coatzacoalcos, Veracruz 96535, Mexico
Corresponding author: Omar Hernández-González (ohernandezg@itesco.edu.mx)

ABSTRACT In this paper, a new filtered observer-based IDA-PBC (Interconnection and Damping Assignment-Passivity Based Control) strategy is developed for trajectory tracking of a quadrotor in the presence of disturbances and model uncertainties. The proposed algorithm allows the control of the quadrotor in all its states. It can deal with the noisy output measurements and uncertainties in the translational and rotational dynamics, as unmodeled dynamics inherent to real systems or unknown external signals (exogenous signals). The designed filtered observer estimates the state from noisy output measurements, and it depends only on two design parameters. Numerical simulation tests are carried out to highlight the overall controller approach in a realistic scenario.

INDEX TERMS Observer-based control IDA-PBC, quadrotor, filtered observer, model uncertainties.

I. INTRODUCTION
Quadrotors are widely used in everyday life, they have been used in many applications, e.g., surveillance, photography, transport of packages, agriculture, rescue, etc, [1], [2]. However, the performance of these vehicles can be degraded by the presence of disturbances, such as measurement noise and uncertainties. For the control of quadrotors, it is necessary to have specific characteristics of smoothness in both control signals and sensors, for this has been used innumerable techniques using filters [3], [4].

The use and research of observers for state estimation, perturbations, and the calculation of output controllers in drones is still very active, e.g., [5]–[8]. Given the complexity and its enormous applications, this research is still open [9]–[12]. To the best of our knowledge, there are few works oriented to the use of the novel filtered observers to applications in drones or quadrotors as [13].

Thus, the design problem of controllers and observers for quadrotors has been widely researched in the literature and is a novel subject in the current research. Within the framework of this theme, a continuous high-gain observer-based PD (Proportional Derivative) control for a quadrotor vehicle is synthesized in [14], where the high-gain observer estimates the state vector from the position coordinates and only one angle (yaw), the interesting of this paper is that the continuous-time estimation (the roll and pitch angles and all velocities) is achievable by using sampled measurements and a PD control. Other recent work [15] includes the problem of fault estimation for a quadrotor using a robust $H_{\infty}$ observer, where the supposed faults are external disturbances, parameter uncertainties, and nonlinear terms. In [16] a cascade active disturbance rejection control and a backstepping sliding-mode control are applied to achieve the desired trajectory tracking performance of a quadrotor in the presence of external disturbances and model uncertainties. [17] presents an integral sliding mode and backstepping sliding mode controllers in a double loop control structure, which ensure the trajectory tracking capability for the desired position of a quadrotor. Also, [18] developed an observer to estimate the external wrench forces applied on a UAV (Unmanned Aerial Vehicle). The lagrangian representation of the UAV is used, which permits to shape the interactive behavior of the quadrotor using an IDA-PBC (Interconnection and Damping Assignment Passivity Based Control).
The IDA-PBC strategy has been successfully applied to mechatronic systems, for example: [19] presents a dynamic feedback control by the IDA-PBC algorithm, to stabilize a class of weakly-coupled electromechanical systems as the electrostatic MEMS devices and the magnetic levitation. [20] designs two adaptive control techniques to handle uncertainties caused by parametric and modeling errors in the Acrobat and non-prehensile planar rolling robotic (disk-on-disk) systems. The techniques use the PCH (Port-controlled Hamiltonian) modelling framework and the IDA-PBC algorithm. [21] introduces an alternative construction that yields a continuous IDA-PBC law.

Other important works are: [22], [23]. [22] presents a linear matrix inequality-based second-order sliding set control for the uncertain systems with time-varying uncertainties, nonlinearities and external disturbances. [23] develops a passivity cascade technique-based control for a nonlinear inverted pendulum system with uncertainties and exogenous disturbances.

Other approaches have addressed the observer-based control design. For example, in [24] the design problem of the observer-based adaptive sliding mode control is studied and applied to a fixed-wing UAV. In [25] an observer-based backstepping control for a quadrotor in the presence of disturbances and parametric uncertainties is proposed. The disturbance observer is used to compensate the unknown disturbance, however, it only compensates small measurement noises. [26] develops an observed-based control strategy for position-yaw tracking of quadrotors. A disturbance observer-based nonlinear controller for the trajectory tracking of a quadrotor in the presence of noisy output measurements is introduced in [27], where the translational positions have a good performance, however, the rotational positions have highly significant variations.

It is noteworthy that several relatively recent works have been focused on the improvement of the high-gain observer performances, especially in the presence of measurement noises, by proposing versions with a time-varying design parameter or by slightly changing the structure of the gain in the observer [13], [28], [29]. This new work falls within the framework of this last approach.

A very common practical situation in the real-time implementation of any observer-based control is that output measurements can be strongly disturbed by noisy measurements, resulting in high estimation errors, or worse, unstable estimation behavior. In this situation, the main problem is to prove the exponential error convergence to a neighborhood of the origin for any initial conditions of a proposed observer that provides a free noise estimation of the state vector for a quadrotor. In this sense, the main focus of this work consists in the synthesis of a novel filtered observer-based IDA-PBC control for trajectory tracking of a quadrotor, which by the observer will be capable of dealing with the noisy output measurements and uncertainties in the translational and rotational dynamics, as unmodeled dynamics inherent to real systems or unknown external signals (exogenous signals).

There is currently little work on control algorithms applied to quadrotors that can handle and thus perform well in the presence of relatively large noise signals in the output measurements and uncertainties, in fact, most of the few current designs perform well only for small values of noise signals that are present in the output measurements, e.g. [14], [25]. Furthermore, current approaches do not consider all the system dynamics to propose estimates, e.g. [24]. Moreover, most of the works consider only the presence of noise in the rotation dynamics, e.g. [27].

Unlike many works published in the literature, the proposed observer system only needs two tuning parameters, one to adjust the state estimate and the other to filter the measurement noise. Traditional observer systems use several tuning parameters and most of them perform a linearization, e.g. [9], [10]. Also, some works consider more than 4 (out of 12) elements of the state vector as measured output, e.g. [26].

Thus, motivated by the aforementioned works, in this paper, we offer an alternative approach for trajectory tracking of a quadrotor in the presence of measurement noise and uncertainty in the states by using an observer-based IDA-PBC strategy. In order to design a filtered observer that estimates the state from noisy output measurements, one represents the quadrotor dynamic model as a non-uniformly observable nonlinear system with coupled blocks, based on the proposed observer from [14], for nonlinear systems with block-state-affine cascade structure, where a sufficient persistent excitation condition is provided. In the case when the noise signal is of high intensity or is strongly variable, the existing observer-based control design results fail to provide a good behavior. However, this paper introduces an algorithm that works for relatively large noise signals, which are present in the output measurements, and it also deals with uncertainties.

The primary aim of unmanned aerial vehicles is to follow a trajectory within the desired operating and safety limits. However, in many cases this is not possible due to the presence of measurement noise and uncertainty. It is evident that the quadrotor needs a robust controller to compensate for disturbances. Faced with this problem, the novelty of this work is to design a robust observer-based IDA-PBC control with respect to uncertainty in states and external signals that disturb the system, such as measurement noise, for trajectory tracking of a quadrotor. The contribution is then, the observer design that considers the nonlinearities of the model and that is able to estimate with a good performance, despite the presence of relatively large measurement noise and uncertainty in the state, this causes the controller performance to be adequate, i.e. even if the controller is not robust to measurement noise and uncertainty in the states, the observer will feedback to the controller signals free of measurement noise and uncertainty. In the case of many physical systems, the above is very important, for example, in the quadrotor the noise in the tracker positions has a significant impact on the accuracy of the position measurement.
The mathematical model of the quadcopter for the translational and rotational motions is represented as follows [14]:

\[
\begin{bmatrix}
v
\dot{v}
\end{bmatrix} = \begin{bmatrix}
0 & 0 & g
\end{bmatrix} + \begin{bmatrix}
s_\psi s_\phi + c_\psi c_\phi s_\theta & -c_\phi s_\theta & s_\psi c_\phi s_\theta \\
-c_\psi s_\phi + s_\psi c_\phi s_\theta & c_\phi c_\theta & -c_\psi s_\phi + s_\psi c_\phi s_\theta \\
c_\phi c_\theta & c_\phi s_\theta & c_\phi c_\theta 
\end{bmatrix} \begin{bmatrix}
f
m
\end{bmatrix},
\]

(1)

where the mass of the quadcopter is introduced by \( m \), the gravity acceleration is represented by \( g \), the constants \( a_1, a_2, a_3, b_1, b_2 \) and \( b_3 \) are obtained as:

\[
\begin{align*}
a_1 &= \frac{I_y - I_z}{I_z}, & a_2 &= \frac{I_x - I_z}{I_y}, & a_3 &= \frac{I_x - I_y}{I_z}, \\
b_1 &= \frac{1}{I_x}, & b_2 &= \frac{1}{I_y}, & b_3 &= \frac{1}{I_z},
\end{align*}
\]

where \( I_x, I_y \) and \( I_z \) represent the moments of inertia of the vehicle with respect to frame \( B \).

According to \( q = [v \ a]^T \), the mathematical model (1) can be rewritten as follows:

\[
\begin{bmatrix}
\dot{x}
\dot{y}
\dot{z}
\end{bmatrix} = \frac{f}{m} \left( s_\psi s_\phi + c_\psi c_\phi s_\theta \right) + \begin{bmatrix}
a_1 \dot{\theta} & + b_1 \tau_0
\end{bmatrix} \\
\begin{bmatrix}
\dot{\psi}
\dot{\theta}
\end{bmatrix} = \frac{a_2 \dot{\phi} \psi + b_2 \tau_0}{m} \\
\dot{\phi} = a_3 \dot{\theta} \phi + b_3 \tau_3
\]

(2)

III. IDA-PBC STRATEGY FOR A QUADROTOR

We can represent (2) as:

\[
M \ddot{q} + C(q)\dot{q} + G = B(q)u
\]

where the inertia matrix \( M \) is constant and independent of \( q \), \( C(q) \) represents the Coriolis matrix, \( G \) is the gravitational vector and \( B(q) \) takes the form:

\[
B(q) = \begin{bmatrix}
s_\psi s_\phi + c_\psi c_\phi s_\theta & 0 & 0 & 0 \\
-c_\psi s_\phi + s_\psi c_\phi s_\theta & c_\phi c_\theta & -c_\psi s_\phi + s_\psi c_\phi s_\theta & 0 \\
c_\phi c_\theta & c_\phi s_\theta & c_\phi c_\theta & 0
\end{bmatrix}
\]

The total energy is

\[
H(q, p) = \frac{1}{2} p^T M^{-1} p + V(q)
\]

(3)

where the total energy \( H(q, p) \) is a Hamiltonian function, \( q \in \mathbb{R}^6 \) represents the generalized position and \( p \in \mathbb{R}^6 \) introduces the generalized momentum. Then, the dynamic system is defined as

\[
\begin{bmatrix}
\dot{q}
\dot{\theta}
\end{bmatrix} = \begin{bmatrix}
0
-I_n
\end{bmatrix} \begin{bmatrix}
\nabla_q H
\nabla_\theta H
\end{bmatrix} + \begin{bmatrix}
0
\end{bmatrix} B(q) u
\]

(4)

where \( I_n \in \mathbb{R}^{6 \times 6} \) is the identity matrix, \( \nabla_q H = \partial H / \partial q \) and \( \nabla_\theta H = \partial H / \partial \theta \).
The desired Hamiltonian function can be written as:

\[ H_d(q, p) = \frac{1}{2} p^T M_d^{-1} p + V_d(q) \quad (5) \]

where \( M_d = M_d^T > 0 \) introduces the closed loop inertia matrix and \( V_d \) expresses the desired potential energy. It is necessary that the energy \( V_d \) have an isolated minimum at \( q_\star \), i.e., \( q_\star = \arg \min V_d(q) \).

The PBC control signal is the sum of energy shaping and injects the damping:

\[ u = u_{es}(q, p) + u_{dl}(q, p) \quad (6) \]

The desired port controlled Hamiltonian dynamics is taken as [30]:

\[ \begin{bmatrix} \dot{q} \\ \dot{p} \end{bmatrix} = [J_d(q, p) - R_d(q, p)] \begin{bmatrix} \nabla_q H_d \\ \nabla_p H_d \end{bmatrix} \quad (7) \]

where

\[ J_d = -J_d^T = \begin{bmatrix} 0 & -M_d M^{-1} J_2(q, p) \\ -M_d M^{-1} J_2(q, p)^T & 0 \end{bmatrix} \]

\[ R_d = R_d^T = \begin{bmatrix} 0 & B(q) K_v B(q)^T \\ 0 & 0 \end{bmatrix} \geq 0 \]

\( J_2 \) is a skew-symmetric matrix and \( K_v = K_v^T > 0 \), both containing design parameters.

The damping injection term is given by

\[ u_{dl}(q, p) = -K_v B(q)^T \nabla_p H_d \quad (8) \]

To compute \( u_{es} \) we substitute (6) and (8) in (4) and make it equal to (7), then

\[ \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix} \begin{bmatrix} \nabla_q H \\ \nabla_p H \end{bmatrix} + \begin{bmatrix} 0 & B(q) \end{bmatrix} u_{es} = \begin{bmatrix} 0 & M^{-1} M_d \\ -M^{-1} M_d & J_2(q, p) \end{bmatrix} \begin{bmatrix} \nabla_q H_d \\ \nabla_p H_d \end{bmatrix} \quad (9) \]

One can observe from (9) that the first row is clearly satisfied. In the second set of equations the PDEs (Partial Differential Equations) yield to the energy shaping expression given by

\[ u_{es} = (B(q)^T B(q))^{-1} B(q)^T (\nabla_q H - M_d M^{-1} \nabla_q H_d + J_2 M^{-1} p) \quad (10) \]

The PDEs (9) can be separated into the equations

\[ B(q)^\perp \left\{ \nabla_q (p^T M_d^{-1} p) - M_d M^{-1} \nabla_q (p^T M_d^{-1} p) \right\} + 2J_2 M_d^{-1} p = 0 \]

\[ B(q)^\perp \left\{ \nabla_q V - M_d M^{-1} \nabla_q V_d \right\} = 0 \quad (11) \]

where \( B(q)^\perp \) is the full rank left annihilator of \( B(q) \).

To obtain the energy-shaping, we consider \( J_2 = 0 \) and \( M_d \) a constant matrix of the form:

\[ M_d = \begin{bmatrix} a_1 & 0 & 0 & a_7 & 0 & 0 \\ 0 & a_2 & 0 & 0 & a_8 & 0 \\ 0 & 0 & a_3 & 0 & 0 & 0 \\ a_7 & 0 & 0 & a_4 & 0 & 0 \\ 0 & a_8 & 0 & 0 & a_5 & 0 \\ 0 & 0 & 0 & 0 & 0 & a_6 \end{bmatrix} \quad (12) \]

With \( B(q)^\perp = \begin{bmatrix} c_\psi & c_\phi \end{bmatrix} \), the potential energy PDE (11) takes the form:

\[ c_\phi \nabla_\phi \left( \frac{a_2}{m} \frac{\partial V_d}{\partial y} + \frac{a_8}{I_y} \frac{\partial V_d}{\partial \phi} \right) + c_\phi c_\psi \left( \frac{a_1}{m} \frac{\partial V_d}{\partial \psi} + \frac{a_2}{I_x} \frac{\partial V_d}{\partial \theta} \right) \]

\[ + M_g s_d - \frac{a_3}{m} \frac{\partial V_d}{\partial z} = 0 \quad (13) \]

which is solved and thus the desired potential energy is computed by

\[ V_d = \frac{m g I_z}{a_8 c_\psi} \ln c_\phi + \Phi (\bullet) \quad (14) \]

where \( \Phi(q) \) is an arbitrary differentiable function. For this, the necessary condition \( \nabla_q V_d(q_\star) = 0 \) and the sufficient condition \( \nabla_q^2 V_d(q_\star) > 0 \) will hold if the Hessian of \( \Phi(q) \) at \( q_\star \) is positive. We choose \( \Phi(q) \) to be a quadratic function which leads to

\[ V_d = \frac{m g I_z}{a_8 c_\psi} \ln c_\phi + \frac{k_{px}}{2} \left( \theta - a_7 m \right)^2 + \frac{k_{py}}{2} \left( \phi - a_8 m \right)^2 \]

\[ + k_{pz} \left( z - \frac{m I_y}{a_3 a_8 c_\psi} \ln c_\phi \right)^2 \quad (15) \]

where \((x_d, y_d, z_d)\) denotes the equilibrium configuration and the \( k_{px}, k_{py}, k_{pz} \) terms are proportional gains and are used as tuning parameters. In order to determine the final control law, we first compute the energy-shaping expression \( u_{es} \) from (10) and second, we obtain the damping injection expression \( u_{dl} \) from (8).

\section{IV. DESIGN OF THE FILTERED OBSERVER}

The excessive cost of the motion capture systems have motivated researchers to propose estimation and control approaches, which are able to handle this condition. Thus, a simple and cost-effective way to obtain the position \( \nu \) of the quadrotor is to use a Global Positioning System (GPS), in addition, the yaw angle \( \psi \) can be obtained from a digital compass. Another obstacle of the robust control design for the quadrotor is the presence of the adverse effects as disturbance, uncertainty and noisy measurements. The output measurements are usually transmitted with measurement noise, there is therefore a peaking phenomenon of the state estimates affecting the control performance.

Then, in order to propose an observer with filtering capabilities for the system (2), firstly, one defines the continuous-time measured output of the quadrotor system as: \( Y = [x \ y \ z \ \psi] \).
With the aim of providing a filtered observer based IDA-PBC control for the system (2) and (3), one should consider a change of coordinates \( \Phi: \mathbb{R}^{12} \to \mathbb{M} \subset \mathbb{R}^{12} \), these set are compact and where \( Z = \Phi(q) \), in such a way that the quadrator system (2) in their coordinates can be rendered a canonical form. Thus, the new vector of coordinates is given: 
\[
z_i = \begin{bmatrix} z_1^i \ z_2^i \ \cdots \ z_{12}^i \end{bmatrix}^T, i = 1, 2, \text{ and the other two is } z_i = \begin{bmatrix} z_1^i \ z_2^i \ \cdots \ z_{12}^i \end{bmatrix}^T, i = 3, 4.
\]

Now, the new system of coordinates is defined as follows:
\[
\begin{align*}
z_1^1 &= x & z_2^1 &= y & z_3^1 &= z \\
z_1^2 &= \dot{x} & z_2^2 &= \dot{y} & z_3^2 &= \dot{z} \\
z_1^3 &= s_\theta & z_2^3 &= y s_\theta & z_3^3 &= -c_\theta s_\phi \\
z_1^4 &= \dot{\theta} s_\theta & z_2^4 &= \dot{\theta} c_\theta s_\phi & z_3^4 &= \dot{\phi} c_\theta c_\phi \\
z_1^5 &= \dot{\phi} c_\theta c_\phi
\end{align*}
\]

Consider the quadrator positions \( y_1 = x \) and \( y_2 = y \) be the first two measured outputs, therefore, \( z_i \) for \( i = 1, 2 \) and it yields:
\[
\begin{bmatrix}
\dot{z}_1^1 = z_1^1 \\
\dot{z}_2^1 = z_2^1 \\
\dot{z}_3^1 = z_3^1 \\
\dot{z}_4^1 = \alpha_2 \dot{\phi} \psi + b_2 \tau_y & s_\theta c_\phi & \psi \\
\dot{z}_5^1 = \alpha_3 \dot{\phi} \dot{\theta} & c_\phi & \dot{\phi}
\end{bmatrix}
\]

where \( x = (x_1, \ldots, x_k)^T \in \mathbb{R}^n \) represents the state vector, with \( x^k = (x_{1,k}, \ldots, x_{9,k})^T \in \mathbb{R}^n \), where \( x_1^k = (x_{1,1}, \ldots, x_{9,1})^T \in \mathbb{R}^n \) with \( x_{ij}^k \in \mathbb{R} \) for \( k = 1, \ldots, q \). 
\[
\begin{bmatrix}
\dot{y}_1 = y_1 \\
\dot{y}_2 = y_2 \\
\dot{y}_3 = y_3 \\
\dot{y}_4 = y_4 \\
\dot{y}_5 = y_5
\end{bmatrix}
\]

\[
A(u) = \text{diag} \begin{bmatrix} A_1 & \cdots & A_q \end{bmatrix}, \quad C = \text{diag} \begin{bmatrix} C_1 & \cdots & C_q \end{bmatrix}
\]

The unknown function is \( \psi = (\psi^1, \ldots, \psi^q) \in \mathbb{R}^q \) and the noise signal is \( w = (w^1, \ldots, w^q) \in \mathbb{R}^q \).

The matrix B is defined as follows:
\[
B = [B_1 \cdots B_q]^T, \quad B_k = [0 \cdots 0 I_{p_k \times 1}]^T
\]

The nonlinear function \( \varphi(u, x) \) is given by the expression:
\[
\begin{bmatrix}
\varphi^1(u, x) \\
\varphi^2(u, x) \\
\varphi^3(u, x) \\
\varphi^4(u, x)
\end{bmatrix} \in \mathbb{R}^n,
\]

where the function \( \varphi^k(u, x) \in \mathbb{R}^{p_k} \) is differentiable w.r.t. \( x \) and assumes a structural dependence on the state variables.

Additionally, in order to provide a filtered observer, the following classical assumptions are considered (see [13]):

\( A1 \) The state vector and the control input are bounded, i.e., \( x \in X \) and \( u \in U \).

\( A2 \) The function \( \varphi(x, u) \) is Lipschitz w.r.t. \( x \) uniformly w.r.t. \( u \), with a Lipschitz constant \( L_\varphi \).

\( A3 \) The upper bound of the function \( A_k(u) \) is denoted by \( A_M = \sup_{x \in X} \| A_k(u) \| \).\]

\( A4 \) The unknown function \( \varepsilon \) and the noise signal \( w \) are essentially bounded, i.e., \( \exists \varepsilon > 0 \),...
where $\delta_k$ denotes the input. The matrix $K_k(\Theta)$ is defined as:

$$K_k(S_k) = \text{diag} \left( S_k^{-1} C_k^{-T} \right)$$

The observer gain is

$$K_k(S_k) = \text{diag} \left( S_k^{-1} C_k^{-T} \right)$$

Following the above, the next main theorem can be established:

**Theorem 1:** Consider the uncertain nonlinear system with noisy measurement outputs (18), satisfying assumptions A1-A5. If there exists two tuning parameters $\Theta^*$ and $\mu^*$ for sufficiently values of $\Theta$ and $\mu$, such that $\Theta > \Theta^*$ and $\mu > \mu^*$, system (25)-(27) provides a free noise estimation of the state vector for system (18) with an exponential error convergence to a neighborhood of the origin for any initial conditions $(\hat{x}(0), \eta(0)) \in X$ and $\hat{\eta}(0) = 0$, thus, the observation error $\hat{x} - x$ closes to zero, depending on $\hat{e}$ that is the ultimate limit of uncertainties and the noise signal $\hat{w}$.

Proof of Theorem 1: Now, we prove the convergence to a neighborhood of the origin of the i-th component of the observation error $\hat{e}^i \in \mathbb{R}^{n_k}$ where $\hat{e}^i = \hat{x}^i - x^i$. Let $\hat{e}^i$ be the k-th subcomponent of $\hat{e}$, which is the estimation error $\hat{e} = \hat{x} - x$. Now we obtain:

$$\hat{e}^i = A_k(u)\hat{e}^i + \hat{v}^i(u, \hat{x}, x) - \Theta^k \Delta_k^{-1}(\Theta)K_k(S_k)\eta_k - B_k \hat{e}_k$$

where $\hat{v}^i(u, \hat{x}, x) = \psi^i(u, \hat{x}) - \psi^i(u, x)$.
For $k = 1, \ldots, q$ set:

\[
\dot{\mathbf{e}}^k = \Lambda_k(\Theta)\mathbf{e}^k \quad \text{and} \quad \dot{\mathbf{y}}^k = D_k(\Theta)\mathbf{y}^k
\]  

(30)

where $\Lambda_k(\Theta)$ is given by (19) and $D_k(\Theta)$ is given:

\[
D_k(\Theta) = \text{diag} \left[ I_{p_1} \quad I_{p_1}/\Theta^{\delta_k} \ldots I_{p_1}/(\Theta^{\delta_k}(\lambda - 1)) \right]
\]  

(31)

It is therefore easy to deduce that $D_k(\Theta)C_n^T C_k = C_k C_n^T \Lambda_k(\Theta)$.

Using the identities (22) and (30), into equation (29), it yields:

\[
\dot{\mathbf{e}}^k = \Theta^{\delta_k} A_k(u)\mathbf{e}^k - \Theta^{\delta_k} K_k(S_k)\mathbf{y}^k_k \\
+ \Lambda_k(\Theta)\phi^k(u, \hat{x}^k, x^k) - \Lambda_k(\Theta)B_k\mathbf{e}^k_k \\
= \Theta^{\delta_k} A_k(u)\mathbf{e}^k - \Theta^{\delta_k} K_k(S_k)D_k(\Theta)\mathbf{y}^k \\
+ \Lambda_k(\Theta)\phi^k(u, \hat{x}^k, x^k) - \Lambda_k(\Theta)B_k\mathbf{e}^k_k \\
= \Theta^{\delta_k} A_k(u)\mathbf{e}^k - \Theta^{\delta_k} K_k(S_k)\mathbf{y}^k \\
+ \Lambda_k(\Theta)\phi^k(u, \hat{x}^k, x^k) - \Lambda_k(\Theta)B_k\mathbf{e}^k_k \\
= \Theta^{\delta_k} A_k(u)\mathbf{e}^k - \Theta^{\delta_k} K_k(S_k)\mathbf{y}^k \\
+ \Lambda_k(\Theta)\phi^k(u, \hat{x}^k, x^k) - \Lambda_k(\Theta)B_k\mathbf{e}^k_k \\
\]  

(32)

Now, from the equations (18) and (27), and also using the identities (29), one gets:

\[
\hat{\mathbf{n}}_k = -\Theta^{\delta_k} \mu^{\delta_k} \left( (I_k - A^T_k)\mathbf{y}^k - D_k(\Theta)C_n^T C_k(\mathbf{e}^k - \mathbf{w}_k) \right) \\
= -\Theta^{\delta_k} \mu^{\delta_k} \left( (I_k - A^T_k)\mathbf{y}^k + C_n^T C_k \Lambda_k(\Theta)\mathbf{y}^k - \mathbf{w}_k \right) \\
= -\Theta^{\delta_k} \mu^{\delta_k} \left( (I_k - A^T_k)\mathbf{y}^k - C_n^T C_k \mathbf{e}^k + C_n^T C_k \mathbf{w}_k \right) \\
\]  

(33)

Adding and subtracting the term $\Theta^{\delta_k} S_k^{-1} C_n^T C_k \mathbf{e}^k$, one obtains the following error dynamics:

\[
\dot{\mathbf{e}}^k = \Theta^{\delta_k} \left( A_k(u) - S_k^{-1} C_n^T C_k \right)\mathbf{e}^k - \Theta^{\delta_k} K_k(S_k)\mathbf{y}^k \\
+ \Lambda_k(\Theta)\phi^k(u, \hat{x}^k, x^k) - \Lambda_k(\Theta)B_k\mathbf{e}^k_k \\
+ \Theta^{\delta_k} S_k^{-1} C_n^T C_k \mathbf{e}^k \\
\]  

(34)

From the observation gain $K_k(S_k)$ in (28), one has:

\[
K_k(S_k)U_k = S_k^{-1} C_n^T C_k \\
\]  

(35)

which permits to rewrite the error dynamics:

\[
\dot{\mathbf{e}}^k = \Theta^{\delta_k} \left( A_k(u) - S_k^{-1} C_n^T C_k \right)\mathbf{e}^k - \Theta^{\delta_k} K_k(S_k)\mathbf{z}^k \\
+ \Lambda_k(\Theta)\phi^k(u, \hat{x}^k, x^k) - \Lambda_k(\Theta)B_k\mathbf{e}^k_k \\
\]  

(36)

where $\mathbf{z}^k = \mathbf{y}^k - U_k\mathbf{e}^k$.

From the equations (33) and (36), one obtains the time derivative of $\mathbf{z}^k$, as follows:

\[
\dot{\mathbf{z}}^k = \mathbf{n}^k - U_k\mathbf{e}^k \\
\]  

(37)

From (35), one verifies that $(I_k - A^T_k)U_k = C_n^T C_k$, therefore one gets the following dynamic equation:

\[
\dot{\mathbf{z}}^k = \Theta^{\delta_k} \mu^{\delta_k} \left( -(I_k - A^T_k) + \frac{1}{\mu^{\delta_k}} U_k K_k(S_k) \right)\mathbf{e}^k \\
+ \Theta^{\delta_k} \left( (I_k - A^T_k) - U_k K_k(S_k) \right)\mathbf{w}_k \\
\]  

(38)

Now, consider the following Lyapunov candidate function:

\[
V(\mathbf{e}, \mathbf{z}, t) = V_1(\mathbf{x}) + V_2(\mathbf{z}) \\
= \mathbf{e}^T \mathbf{S}^e \mathbf{e}^k + \mathbf{z}^T \mathbf{S}^z \mathbf{z}^k \\
= \sum_{k=1}^{q} V_{1,k}(\mathbf{e}^k) + \sum_{k=1}^{q} V_{2,k}(\mathbf{z}^k) \\
\]  

(39)

where $V_{1,k}(\mathbf{e}^k) = \mathbf{e}^T \mathbf{S}^e \mathbf{e}^k$ and $V_{2,k}(\mathbf{z}^k) = \mathbf{z}^T \mathbf{S}^z \mathbf{z}^k$.

Firstly, one focuses on the term $V_{1,k}(\mathbf{e}^k)$, differentiating $V_{1,k}(\mathbf{e}^k)$ along the trajectories of the system:

\[
\dot{V}_{1,k}(\mathbf{e}^k) = 2\mathbf{e}^T \mathbf{S}^e \mathbf{e}^k + \mathbf{e}^T \mathbf{S}^e \mathbf{e}^k \\
\]  

(40)

where $S_k$ is given by (26).

Introducing the ODE Lyapunov equation (36) into (40), it yields:

\[
\dot{V}_{1,k}(\mathbf{e}^k) = -\Theta^{\delta_k} \mathbf{e}^T \mathbf{S}^e \mathbf{e}^k - \Theta^{\delta_k} \mathbf{e}^T \mathbf{S}^e \mathbf{e}^k \\
+ 2\mathbf{e}^T \mathbf{S}^e \Lambda_k(\Theta)\phi^k(u, \hat{x}, x) \\
- 2\mathbf{e}^T \mathbf{S}^e \Lambda_k(\Theta)B_k\mathbf{e}^k - \Theta^{\delta_k} \mathbf{e}^T \mathbf{S}^e \mathbf{e}^k \\
\leq -\Theta^{\delta_k} \mathbf{e}^T \mathbf{S}^e \mathbf{e}^k \\
+ 2\sqrt{\lambda_M(S_k)V_{1,k}(\mathbf{e}^k)} \sum_{i=1}^{\lambda_k} \frac{1}{\eta_i} \left\| \phi_i^k(u, \hat{x}, x) \right\| \\
+ 2\sqrt{\lambda_M(S_k)V_{1,k}(\mathbf{e}^k)} \Theta^T \mathbf{e}^k + (\lambda - 1)\mathbf{e}^k \\
+ 2\sqrt{\lambda_M(S_k)V_{1,k}(\mathbf{e}^k)} \left\| K_k(S_k) \right\| \left\| \mathbf{e}^k \right\| \left\| \mathbf{e}^k \right\| \\
\]  

(41)

where $\sigma^k = \sigma^k + (i - 1)\delta_k$.

In addition, considering Assumption A1-A2, i.e. the boundedness of $\frac{\partial \psi}{\partial x}$, one has:

\[
\dot{V}_{1,k}(\mathbf{e}^k) \leq -\Theta^{\delta_k} V_{1,k}(\mathbf{e}^k) \\
+ 2L_\phi \sqrt{\lambda_M(S_k)} \sqrt{V_{1,k}(\mathbf{e}^k)} \\
\times \sum_{i=1}^{\lambda_k} \sum_{j=1}^{\lambda_i} \int_{0}^{\infty} \sigma^j \eta_i \left\| \mathbf{e}^k \right\| \left\| \mathbf{e}^k \right\| \\
\]  

(42)
+ 2\sqrt{\lambda_M(S_k)} \sqrt{V_{1,k}(\tilde{e}^k)} \frac{\tilde{\epsilon}}{\sigma^2_0 + (\lambda_k - 1)\delta_k} \\
+ 2\sqrt{\lambda_M(S_k)} \sqrt{V_{1,k}(\tilde{e}^k)} ||K_k(S_k)|| \|\tilde{e}^k\| 

(42)

where \(L_{\tilde{q}k}\) comes from Assumption \textbf{A2} and the \(\chi^k_{j,l}\)'s are defined as in Lemma 1 in [31].

According to assumption \textbf{A5}, one clearly has:

\[
\lambda_m(S_k) \geq \frac{e^{-1}\delta_0}{\alpha(\Theta)} 
\]

(43)

Also, \(\sigma_j^l\) is defined as in (21):

\[
\sigma_j^l = \sigma_1^l + (j + 1)\delta_l 
\]

(44)

Finally, one obtains:

\[
\dot{V}_{1,k}(\tilde{e}^k) \leq -\theta^{q_k} V_{1,k}(\tilde{e}^k) + 2L_{\tilde{q}k}\mu_s \\
+ \sqrt{V_{1,k}(\tilde{e}^k)} \sum_{l=1}^{q_k} \sqrt{V_{1,k}(\tilde{e}^k)} \frac{\tilde{\epsilon}}{\sigma^2_0 + (\lambda_k - 1)\delta_k} \\
+ 2\sqrt{\lambda_M(S_k)} \sqrt{V_{1,k}(\tilde{e}^k)} ||K_k(S_k)|| ||\tilde{e}^k|| 

(45)

where:

\[
\mu_s = \frac{\lambda_M(S_k)}{\lambda_m(S_k)} 
\]

(46)

Moreover, according to Lemma 1 in [31], one has:

\[
\sigma_j^l - \sigma_1^l - \frac{\delta_l}{2} - \frac{\delta_k}{2} \leq -\frac{n}{2q} \\
\sigma_1^l + (\lambda_k - 1)\delta_k \geq (\lambda_1 - 1)\delta_1 
\]

(47)

Considering (45) and (47), one gets:

\[
\dot{V}_{1,k}(\tilde{e}^k) \leq -\theta^{q_k} V_{1,k}(\tilde{e}^k) \\
+ 2L_{\tilde{q}k}\mu_s \theta^{-\frac{n}{2}} \sqrt{\theta^{q_k} V_{1,k}(\tilde{e}^k)} \sum_{l=1}^{q_k} \sum_{l=1}^{q_k} \sqrt{\theta^{q_k} V_{1,k}(\tilde{e}^k)} \\
+ 2\sqrt{\lambda_M(S_k)} \sqrt{V_{1,k}(\tilde{e}^k)} \frac{\tilde{\epsilon}}{\sigma^2_0 + (\lambda_k - 1)\delta_k} \\
+ 2\sqrt{\lambda_M(S_k)} \sqrt{V_{1,k}(\tilde{e}^k)} ||K_k(S_k)|| ||\tilde{e}^k|| 

(48)

Moreover, let:

\[
V_{1,k}(\tilde{e}^k) = \theta^{q_k} V_{1,k}(\tilde{e}^k) \\
V_{1}(\tilde{e}) = \sum_{k=1}^{q} V_{1,k}(\tilde{e}^k) 
\]

(49)

Since \(\delta_q = 1\), one has:

\[
V_{1}(\tilde{e}) \geq \theta^{q_k} V_{1}(\tilde{e}) = \Theta V_{1}(\tilde{e}) 
\]

(50)

And inequality (48) becomes:

\[
\dot{V}_{1,k} \leq -V_{1,k}^* \\
+ 2L_{\tilde{q}k}\mu_s \theta^{-\frac{n}{2}} \sqrt{V_{1,k}(\tilde{e}^k)} \sum_{l=1}^{q_k} \sum_{l=1}^{q_k} \sqrt{V_{1,k}(\tilde{e}^k)} \\
+ 2\sqrt{\lambda_M(S_k)} \sqrt{V_{1,k}(\tilde{e}^k)} \frac{\tilde{\epsilon}}{\sigma^2_0 + (\lambda_k - 1)\delta_k} \\
+ 2\sqrt{\lambda_M(S_k)} \sqrt{V_{1,k}(\tilde{e}^k)} ||K_k(S_k)|| ||\tilde{e}^k|| 

(51)

Then,

\[
\dot{V}_{1,k} \leq -V_{1,k}^* + 2L_{\tilde{q}k}\mu_s \lambda_k \theta^{-\frac{n}{2}} \sqrt{V_{1,k}(\tilde{e}^k)} \sum_{l=1}^{q_k} \sum_{l=1}^{q_k} \sqrt{V_{1,k}(\tilde{e}^k)} \\
+ 2\sqrt{\lambda_M(S_k)} \sqrt{V_{1,k}(\tilde{e}^k)} \frac{\tilde{\epsilon}}{\sigma^2_0 + (\lambda_k - 1)\delta_k} \\
+ 2\sqrt{\lambda_M(S_k)} \sqrt{V_{1,k}(\tilde{e}^k)} ||K_k(S_k)|| ||\tilde{e}^k|| 

\]

\[
\leq -V_{1,k}^* + 2nL_{\tilde{q}k}\mu_s \lambda_k \theta^{-\frac{n}{2}} \sqrt{V_{1,k}(\tilde{e}^k)} \sqrt{V_{1}(\tilde{e})} \\
+ 2\sqrt{\lambda_M(S_k)} \sqrt{V_{1}(\tilde{e})} \frac{\tilde{\epsilon}}{\sigma^2_0 + (\lambda_k - 1)\delta_k} \\
+ 2\sqrt{\lambda_M(S_k)} \sqrt{V_{1}(\tilde{e})} ||K_k(S_k)|| ||\tilde{e}^k|| 

(52)

Hence,

\[
\dot{V}_{1}(\tilde{e}) \leq -V_{1}(\tilde{e}) + 2nL_{\tilde{q}}\mu_s \lambda_k \theta^{-\frac{n}{2}} \sqrt{V_{1}(\tilde{e})} \\
+ 2\sqrt{\lambda_M(S_k)} \sqrt{V_{1}(\tilde{e})} \frac{\tilde{\epsilon}}{\sigma^2_0 + (\lambda_k - 1)\delta_k} \\
+ 2\sqrt{\lambda_M(S_k)} \sqrt{V_{1}(\tilde{e})} ||K_k(S_k)|| ||\tilde{e}^k|| 

\]

\[
\leq -V_{1}(\tilde{e}) + 2nL_{\tilde{q}}\mu_s \lambda_k \theta^{-\frac{n}{2}} \sqrt{V_{1}(\tilde{e})} \\
+ 2\sqrt{\lambda_M(S_k)} \sqrt{V_{1}(\tilde{e})} \frac{\tilde{\epsilon}}{\sigma^2_0 + (\lambda_k - 1)\delta_k} \\
+ 2\sqrt{\lambda_M(S_k)} \sqrt{V_{1}(\tilde{e})} ||K_k(S_k)|| ||\tilde{e}^k|| 

(53)

where \(L_{\tilde{q}} = \max\{L_{\tilde{q}k}; 1 \leq k \leq q\}\).

Furthermore, using (50) for \(\theta\) high enough such that:

\[
1 - n^2 \mu_s \theta^{-\frac{n}{2}} L_{\tilde{q}} > 0 
\]

(54)
According to assumption A3 and (54), one gets:
\[
\dot{V}_1(\bar{e}) \leq -\Theta \left( 1 - \sqrt{\frac{\alpha(\Theta)}{\Theta, \delta_0}} \sqrt{\frac{n^4 e\lambda(M(S))}{\delta_0^2}} L_{\phi} \right) V_1(\bar{e}) \\
+ 2q\sqrt{\lambda(M(S))} \sqrt{V_1(\bar{e})} \frac{\bar{e}}{\Theta(\delta_0^2 - 1)\delta_0^2} \\
+ 2\sqrt{\lambda(M(S))} \sqrt{V_1(\bar{e})} \|K_k(S_k)\| \|\bar{z}\| (55)
\]
Now, we focus on the term \(V_{2,k}(\bar{z})^2 \leq \bar{z}^2 \leq \|\bar{z}\|^2\), one gets the following differential equation:
\[
\dot{V}_{2,k}(\bar{z}) = \Theta_k \mu_k \bar{z}^T \left( -2I_k + (A_k^T + A_k) \right) \bar{z} \\
+ 2\Theta_k \mu_k \bar{z}^T U_k K_k(S_k) \bar{z} \\
+ 2\Theta_k \mu_k \bar{z}^T (U_k A_k(u) + U_k K_k(S_k) U_k) \bar{z} \\
- 2\Theta_k \mu_k \bar{z}^T U_k \Lambda_k(\Theta) \varphi(u, \bar{z}, x) \\
- 2\Theta_k \mu_k \bar{z}^T \mu_k \bar{z} \bar{C}_k \bar{w}_k \\
\leq -\Theta_k \mu_k \bar{z}^T U_k K_k(S_k) \bar{z} \\
+ 2\Theta_k \mu_k \bar{z}^T U_k A_k(u) + U_k K_k(S_k) U_k \|\bar{z}\|^2 \\
- 2\Theta_k \mu_k \bar{z}^T U_k \Lambda_k(\Theta) \varphi(u, \bar{z}, x) \\
- 2\Theta_k \mu_k \bar{z}^T \mu_k \bar{z} \bar{C}_k \bar{w}_k (56)
\]
where \(\rho_k\) denotes the smallest eigenvalue of the SPD matrix \(2I_k - (A_k^T + A_k)\).

From Assumption A2, one obtains the following expressions:
\[
2\bar{z}^T U_k A_k(\Theta) \varphi(u, \bar{z}, x) \\
\leq 2n\Theta \mu_k \sqrt{V_{2,k}(\bar{z})} \sqrt{\frac{V_1(\bar{e})}{\lambda(M(S_k))}} (57)
\]
\[
2\Theta_k \mu_k \|U_k A_k(u) + U_k K_k(S_k) U_k \| \|\bar{z}\|^2 \\
\leq 2\Theta_k \sqrt{n} \sqrt{\lambda(M(A_M + K_M))} \sqrt{\frac{V_{2,k}(\bar{z})}{\lambda(M(S_k))}} (58)
\]
where \(A_M = \sup_{u \in U(A(u)}(u)\) and \(K_M = \sup_{s \in [0,1]}(K_k(s_k))\).

Then, substituting the above expression into (56), it yields:
\[
\dot{V}_{2,k}(\bar{z}) \leq -\Theta_k \mu_k \rho_k V_{2,k}(\bar{z}) + 2\Theta_k \sqrt{n} K_M \sqrt{V_{2,k}(\bar{z})} \\
+ 2\Theta_k \sqrt{n} \sqrt{\lambda(M(A_M + K_M))} \sqrt{\frac{V_{2,k}(\bar{z})}{\lambda(M(S_k))}} \\
+ 2n\Theta \mu_k \sqrt{V_{2,k}(\bar{z})} \sqrt{\frac{V_1(\bar{e})}{\lambda(M(S_k))}} \\
+ 2\Theta_k \sqrt{n} \sqrt{V_{2,k}(\bar{z})} \sqrt{\frac{V_1(\bar{e})}{\lambda(M(S_k))}} \bar{w} \\
\leq -\Theta_k \mu_k \rho_k \left( 1 - 2 \frac{K_M \sqrt{n}}{\mu_k \rho_k} \right) V_{2,k}(\bar{z}) \\
+ 2\sqrt{n} \left( \sqrt{\frac{n^4 e\lambda(M(S))}{\delta_0^2}} \frac{V_1(\bar{e})}{\lambda(M(S_k))} \right) \\
+ 2\bar{w} \sqrt{\lambda(M(S_k))} \sqrt{V_1(\bar{e})} \sqrt{V_{2,k}(\bar{z})} \sqrt{\frac{V_1(\bar{e})}{\lambda(M(S_k))}} (59)
\]

The value of \(\mu\) can be chosen as \(\left( 1 - 2 \frac{K_M \sqrt{n}}{\mu_k \rho_k} \right) \geq \frac{1}{2}\), thus, \(\mu \geq 2\sqrt{n} \frac{K_M}{\rho_k}\) and \(\Theta_k \leq 1\). Therefore, the above inequality can be rewritten as:
\[
\dot{V}_{2,k}(\bar{z}) \leq -\Theta_k \mu_k \rho_k V_{2,k}(\bar{z}) + 2\Theta \mu \sqrt{V_{2,k}(\bar{z})} \bar{w} \\
+ 2\sqrt{n} \sqrt{\frac{n^4 e\lambda(M(S))}{\delta_0^2}} \frac{V_1(\bar{e})}{\lambda(M(S_k))} \sqrt{V_{2,k}(\bar{z})} \sqrt{\frac{V_1(\bar{e})}{\lambda(M(S_k))}} \\
\times \Theta \sqrt{\lambda(M(S_k))} \sqrt{V_{2,k}(\bar{z})} \sqrt{\frac{V_1(\bar{e})}{\lambda(M(S_k))}} \\
\leq -\Theta_k \mu_k \rho_k V_{2,k}(\bar{z}) + 2\Theta \mu \sqrt{V_{2,k}(\bar{z})} \bar{w} \\
+ 2\sqrt{n} \sqrt{\frac{n^4 e\lambda(M(S))}{\delta_0^2}} \frac{V_1(\bar{e})}{\lambda(M(S_k))} \sqrt{V_{2,k}(\bar{z})} \sqrt{\frac{V_1(\bar{e})}{\lambda(M(S_k))}} \\
\times \Theta \sqrt{\lambda(M(S_k))} \sqrt{V_{2,k}(\bar{z})} \sqrt{\frac{V_1(\bar{e})}{\lambda(M(S_k))}} (60)
\]
Similarly as in the previous term \(V_{k,n}(\bar{z})\), one considers that \(V_{k,n}^s(\bar{z}) = \Theta_k \mu_k V_{k,n}(\bar{z})\), therefore, \(V_{2,k}(\bar{z}) = \sum_{k=1}^n V_{k,n}(\bar{z})\), and knowing that \(\delta_k = 1\), one deduces that
\[
V_{2,k}(\bar{z}) \geq \Theta_k \mu_k V_{2,k}(\bar{z}) = \Theta V_{2,k}(\bar{z}) (61)
\]
Hence,
\[
\dot{V}_2(\bar{e}) \leq -\Theta \mu \rho \frac{V_2(\bar{e}) + 2\Theta \mu \sqrt{V_2(\bar{e})} \bar{w}}{2} \\
+ 2\sqrt{n} \sqrt{\frac{n^4 e\lambda(M(S))}{\delta_0^2}} \frac{V_1(\bar{e})}{\lambda(M(S_k))} \sqrt{V_2(\bar{e})} \sqrt{\frac{V_1(\bar{e})}{\lambda(M(S_k))}} \\
\times \Theta \sqrt{\lambda(M(S_k))} \sqrt{V_2(\bar{e})} \sqrt{\frac{V_1(\bar{e})}{\lambda(M(S_k))}} (62)
\]
where \(c_0 = 2 \sqrt{n} \sqrt{\frac{n^4 e\lambda(M(S_k))}{\delta_0^2}} \frac{V_1(\bar{e})}{\lambda(M(S_k))}\).

Now, using inequality (55), it leads to:
\[
\dot{V}_1(\bar{e}) \leq -\Theta \left( 1 - \sqrt{\frac{\alpha(\Theta)}{\Theta, \delta_0}} \sqrt{\frac{n^4 e\lambda(M(S))}{\delta_0^2}} L_{\phi} \right) V_1(\bar{e}) \\
+ 2q\sqrt{\lambda(M(S))} \sqrt{V_1(\bar{e})} \frac{\bar{e}}{\Theta(\delta_0^2 - 1)\delta_0^2} \\
+ 2\sqrt{\lambda(M(S))} \sqrt{V_1(\bar{e})} \|K_k(S_k)\| \|\bar{z}\| \bar{w} (63)
\]
where \(c_1 = 2 \sqrt{\lambda(M(S_k))}\) and \(\omega(\Theta) = \sqrt{\frac{\alpha(\Theta)}{\Theta, \delta_0}} \sqrt{\frac{n^4 e\lambda(M(S))}{\delta_0^2}} L_{\phi}\).

\[\text{VOLUME 9, 2021} \quad \text{114829} \]

M. E. Guerrero-Sánchez et al.: Filtered Observer-Based IDA-PBC Control for Trajectory Tracking
Accordingly, it is possible to set:

$$\dot{V}(\bar{e}, \bar{z}, t) = \ddot{V}_1(\bar{e}) + \ddot{V}_2(\bar{z})$$  \hspace{1cm} (65)$$

where $\ddot{V}_1 = \Theta \omega(\Theta) V_1(\bar{e})$ and $\ddot{V}_2 = \Theta \mu \Omega \mu(\bar{z})$. Therefore, one states $\dot{V}_1(\bar{e}) \leq \dot{V}(\bar{e}, \bar{z}, t)$ and $\dot{V}_2(\bar{z}) \leq \dot{V}(\bar{e}, \bar{z}, t)$.

Thus,

$$\dot{V}_1(\bar{e}) \leq -\ddot{V}_1(\bar{e}) + c_1 \Theta \sqrt{\dot{V}_1(\bar{e}) - \ddot{V}_1(\bar{z})} \sqrt{\ddot{V}_2(\bar{z})} + 2q \sqrt{\lambda M(\bar{e}) \dot{V}_1(\bar{e})} \sqrt{\ddot{V}_2(\bar{z})}$$

$$\leq -\ddot{V}_1(\bar{e}) + c_1 \sqrt{\mu} \sqrt{\frac{2}{\omega(\Theta) \rho}} \ddot{V}(\bar{e}, \bar{z}, t) + 2q \sqrt{\lambda M(\bar{e}) \dot{V}_1(\bar{e})} \frac{\ddot{V}(\bar{e}, \bar{z}, t)}{\Theta(\lambda_1 - 1) \lambda_1}$$

(66)

Since $\dot{V}_1(\bar{e}) \leq \ddot{V}(\bar{e}, \bar{z}, t)$, it yields:

$$\dot{V}_1(\bar{e}) = -\ddot{V}_1(\bar{e}) + c_1 \sqrt{\mu} \sqrt{\frac{2}{\omega(\Theta) \rho}} \ddot{V}(\bar{e}, \bar{z}, t) + 2q \sqrt{\lambda M(\bar{e}) \dot{V}_1(\bar{e})} \frac{\ddot{V}(\bar{e}, \bar{z}, t)}{\Theta(\lambda_1 - 1) \lambda_1}$$

(67)

Similarly, one obtains

$$\dot{V}_2(\bar{z}) \leq -\ddot{V}_2(\bar{z}) + 2\Theta \mu \sqrt{\dot{V}_2(\bar{z})} \sqrt{\ddot{V}_1(\bar{e})}$$

(68)

$$\leq -\ddot{V}_2(\bar{z}) + 2\Theta \mu \sqrt{\ddot{V}_1(\bar{e})} \sqrt{\frac{2}{\omega(\Theta) \rho}} \ddot{V}(\bar{e}, \bar{z}, t)$$

Set $\dot{V}(\bar{e}, \bar{z}, t) = V_1(\bar{e}) + V_2(\bar{z})$ as the candidate Lyapunov equation. From equations (67) and (68), and considering $\sqrt{\dot{V}_1(\bar{e})} \leq \dot{V}$ and $\sqrt{\dot{V}_2(\bar{z})} \leq \sqrt{\dot{V}^2}$, one gets:

$$\dot{V}(\bar{e}, \bar{z}, t) = -\dot{V} + \frac{2}{\mu} \sqrt{\omega(\Theta) \rho} \left( c_1 + c_0 \sqrt{\omega(\Theta)} \right) \ddot{V}$$

(69)

$$+ 2 \left( q \sqrt{\lambda M(\bar{e}) \Theta(\lambda_1 - 1) \lambda_1} + \Theta \mu \dot{W} \right) \sqrt{\dot{V}}$$

For $\Theta$ and $\mu$ sufficiently high, one has:

$$\lim_{\Theta \to \infty} \frac{\alpha(\Theta)}{\Theta} = \lim_{\mu \to \infty} \frac{1}{\mu} = 0.$$  \hspace{1cm} (70)

That allows us to consider that $0 < \xi < 1$, thus:

$$1 - \sqrt{\frac{2}{\omega(\Theta) \rho} \left( c_1 + c_0 \sqrt{\omega(\Theta)} \right) > \xi}$$  \hspace{1cm} (71)

Likewise, one has $\Theta \omega(\Theta) \leq \Theta \mu \rho \bar{z}$ which implies that

$$\Theta \omega(\Theta) \Theta \mu \rho \bar{z} \leq \ddot{V} \leq \frac{\Theta \mu \rho \bar{z}}{2} \dot{V}$$  \hspace{1cm} (72)

Hence, from (69), (71) and (72), one has:

$$\dot{V}(\bar{e}, \bar{z}, t) = -\xi \Theta \omega(\Theta) \dot{V}$$

$$+ 2 \left( q \sqrt{\lambda M(\bar{e}) \Theta(\lambda_1 - 1) \lambda_1} + \Theta \mu \dot{W} \right) \sqrt{\dot{V}}$$

(73)

Therefore,

$$\frac{d}{dt} \sqrt{\dot{V}(\bar{e}, \bar{z}, t)} = -\xi \Theta \omega(\Theta) \dot{V}$$

$$+ 2 \left( q \sqrt{\lambda M(\bar{e}) \Theta(\lambda_1 - 1) \lambda_1} + \Theta \mu \dot{W} \right)$$

(74)

According to (64), (70) and (71), allowing us to show that for large sufficiently values of $\Theta$ and $\mu$, $\dot{V}(\bar{e}, \bar{z}, t)$ exponentially converges to a neighbourhood around zero, as it is shown in (74), the rate of convergence depends on the upper bounds of the uncertainties and the noise signals $\ddot{V}$ and $\dot{W}$. This concludes the proof of Theorem 1.

**Remark 1:** The complexity of the proposed algorithm depends on the size of the uncertainty, which can be reduced to a certain value through the parameter $\Theta$. Nevertheless, there is a trade-off between the sensitivity of the measurement noise and the convergence error. The sensitivity can be compensated by the observer immersed in the controller by means
of the adjustment parameter $\mu$, however, the uncertainty is limited to a certain value.

V. SIMULATIONS
To validate the performance of the novel observer-based IDA-PBC, some simulation tests have been performed. The observer given by (25)-(27) is applied to the quadrotor dynamical model (18). An IDA-PBC controller with gravity compensation is used to stabilize the vehicle’s position and attitude dynamics.

In simulation tests, the model parameters have been taken close to real aerial platforms, such parameters are given by:

$$m = 0.56\text{Kg}$$
$$I_x = I_y = 14.2e^{-3}\text{Kgm}^2$$
$$d = 0.21m$$
$$g = 9.81m/s^2$$
$$I_z = 2I_x.$$

From the block structure (18), it has $p = 4$, since the available outputs are $[x \ y \ z \ \psi]$, one has four subsystems (16) and (17), thus $\lambda_1 = \lambda_2 = 4$ and $\lambda_3 = \lambda_4 = 2$, therefore, according (20) one obtains $\delta_1 = 5$, $\delta_2 = \delta_3 = \delta_4 = 1$. The tuning parameters are taken as $\Theta = 1.2$ and $\mu = 7$. The output vector is $y(t) = [x, y, z, \psi]^T + w(t)$ and represents the noisy output measurements.

A. PERFORMANCE IN PRESENCE OF MEASUREMENT NOISE
In order to highlight the performance of the proposed control law in the presence of the different noise signals, two experimental results are provided. In the first test, the noise signal $w(t)$ is a Gaussian noise with a variance equal to 0.0005, this results in the noisy output measurements plotted in Figure 2. In the second test, a noise signal $w(t)$ is given by $0.155 \sin(20t)$, the perturbed trajectories are plotted in Figure 3. In both simulations, the uncertainty is $\epsilon(t) = \epsilon_1 = \epsilon_2 = \epsilon_3 = \epsilon_4 = 0.1 \cos(20t)$, this uncertainties can be attached to unmodeled dynamics, such as wind frictions and structural vibrations, as [25].

The performance of the novel observer-based IDA-PBC control in the presence of the Gaussian noise is presented in Figure 4, in addition, the behavior of the design in the presence of the sinusoidal noise signal is illustrated in Figure 5. Note that in both experiments the proposed
observer-based control provides a good trajectory tracking in presence of uncertainties and disturbances as the measurement noise (Gaussian noise and Sinusoidal noise). Considering all these non-negligible effects, the novel approach will make easier the practical control implementation. It is also important to highlight that the performance of the proposed design has a slight performance degradation in the presence of the Gaussian noise, as shown in Figure 6, this degradation relates in particular to high-frequency measurement noise, in order to reduce this effect, some works add a low-pass filter of high order [29]. However, this may be impractical to compute for high-order systems as the quadrotor. Furthermore, these designs address only the SISO uniformly observable systems, this is not the case for quadrotor dynamics (MIMO non-uniformly observable system).

**B. COMPARISON TO A STANDARD DESIGN**

In order to compare the behavior of the standard and proposed control design in the presence of the high amplitude noise signals, a comparison test is performed. For this test, the noise signal \( w(t) \) is a sum of a Gaussian noise with a variance equal to 0.001 and a signal given by \( 0.25 \sin(20t) \), this results in the noisy output measurements plotted in Figure 7 (proposed design) and Figure 8 (standard design). Moreover, this test considers the same uncertainty of the previous tests. Figure 9 shows that the proposed controller can reach the reference in a very short time in the presence of the high-amplitude noise.

**FIGURE 7.** Trajectories of the observer based on IDA-PBC control with high amplitude noise.
VI. CONCLUSION

In order to avoid the heavy design effort and ensure the robust control strategy in the presence of adverse effects as disturbance, uncertainty, and noisy measurements, it has been presented a new observer-based IDA-PBC control for a quadrotor. The original idea of the proposed strategy of the observer-based control is that it is able to guarantee a near-zero steady-state error, in the presence of relatively large uncertainties and disturbances, as measurement noise and unmodeled dynamics of the quadrotor (wind frictions and structural vibrations). The features of the proposed strategy were further validated with simulation tests for a quadrotor, which show a remarkable improvement of the sensitivity of the estimator subject to measurement noise.
and thereby achieving a reduction of the steady-state error. The behavior of the proposed observer-based IDA-PBC control for small uncertainties and disturbances tends to be similar to the behavior of the typical approach, i.e., high-gain observer-based PD control without uncertainties and disturbances. Future work includes extending the scheme for the case of sampled output measurements and delayed output measurements, as well as the combination of both. Moreover, it will consider an IDA-PBC design with total energy-shaping.

REFERENCES

[1] M. F. F. Rahman, S. Fan, Y. Zhang, and L. Chen, “A comparative study on application of unmanned aerial vehicle systems in agriculture,” Agricult.
[2] P. Manju, D. Pooja, and V. Dutt, “Drones in smart cities,” in AI and IoT-Based Intelligent Automation in Robotics. Hoboken, NJ, USA: Wiley, 2021, ch. 12, pp. 205–228. [Online]. Available: https://onlinelibrary.wiley.com/doi/abs/10.1002/9781119711230.ch12. doi: 10.1002/9781119711230.ch12.
[3] K. Liu and R. Wang, “Antisaturation command filtered backstepping control based disturbance rejection for a quadrotor UAV,” IEEE Trans. Circuits Syst. I, Exp. Briefs, early access, Mar. 31, 2021, doi: 10.1109/TCSI.2021.3069967.
[4] J. Faraji and J. Keighobadi, “Design and Simulation of the integral backstepping sliding mode control and extended Kalman-Bucy filter for quadrotor,” J. Mech. Eng., vol. 50, no. 4, pp. 131–140, 2021.
[5] J. A. Guzmán-Rabasa, F. R. López-Estrada, B. M. González-Contreras, G. Valencia-Palomo, M. Chádli, and M. Pérez-Patricio, “Actuator fault detection and isolation on a quadrotor unmanned aerial vehicle modeled as a linear parameter-varying system,” Meas. Control, vol. 52, nos. 9–10, pp. 1228–1239, Nov. 2019.
[6] M.-E. Guerrero-Sánchez, O. Hernández-González, R. Lozano, C.-D. García-Beltrán, G. Valencia-Palomo, and F.-R. López-Estrada, “Energy-based control and LMI-based control for a quadrotor transporting a payload,” Mathematics, vol. 7, no. 1, p. 1090, Nov. 2019.
[7] S. Gómez-Periáte, F.-R. López-Estrada, G. Valencia-Palomo, D. Rotondo, and M.-E. Guerrero-Sánchez, “Actuator and sensor fault estimation based on a proportional multiple-integral sliding mode observer for linear parameter-varying systems with inexact scheduling parameters,” Int. J. Robust Nonlinear Control, to be published, doi: 10.1002/rnc.5371.
[8] M. E. Guerrero-Sánchez, R. Lozano, P. Castillo, O. Hernández-González, C. D. García-Beltrán, and G. Valencia-Palomo, “Nonlinear control strategies for a UAV carrying a load with swing attenuation,” Appl. Math. Model., vol. 91, pp. 709–722, Mar. 2021.
[9] W. You, F. Li, L. Liao, and M. Huang, “Data fusion of UWB and IMU based on unscented Kalman filter for indoor localization of quadrotor UAV,” IEEE Access, vol. 8, pp. 64971–64981, 2020.
[10] A. Kaba and E. Kiyak, “Optimizing a Kalman filter with an evolutionary algorithm for nonlinear quadrotor attitude dynamics,” J. Comput. Sci., vol. 39, Jan. 2020, Art. no. 101051.
[11] E. Lefeber, M. Greiff, and A. Robertsson, “Filtered output feedback tracking control of a quadrotor UAV,” IFAC-PapersOnLine, vol. 53, no. 2, pp. 5764–5770, 2020.
[12] G. Unal, “Integrated design of fault-tolerant control for flight control systems using observer and fuzzy logic,” Airc. Eng. Aerosp. Technol., vol. 93, no. 4, pp. 723–732, Jun. 2021.
[13] J. L. Robles-Magdaleno, A. E. Rodríguez-Mata, M. Farza, and M. M’saad, “A filtered high gain observer for a class of non uniformly observable systems—Application to a phytoplanktonic growth model,” J. Process Control, vol. 87, pp. 68–78, Mar. 2020.
[14] O. Hernández-González, M.-E. Guerrero-Sánchez, M. Farza, T. Ménard, M. M’saad, and R. Lozano, “High gain observer for a class of nonlinear systems with coupled structure and sampled output measurements: Application to a quadrotor,” Int. J. Control, vol. 50, no. 5, pp. 1089–1105, Apr. 2019.
[15] X.-L. Ren, “Observer design for actuator failure of a quadrotor,” IEEE Access, vol. 8, pp. 152742–152750, 2020.
[16] L.-X. Xu, H.-J. Ma, D. Guo, A.-H. Xie, and D.-L. Song, “Backstepping sliding-mode and cascade active disturbance rejection control for a quadrotor UAV,” IEEE/ASME Trans. Mechatronics, vol. 25, no. 6, pp. 2743–2753, Dec. 2020.
OMAR HERNÁNDEZ-GONZÁLEZ was born in Coatzacoalcos, Veracruz, Mexico, in 1983. He received the Engineering degree in electronics from the Instituto Tecnológico de Minatitlán, Mexico, in 2005, the M.Sc. degree in automatic control from the National Center of Research and Technological Development (CENIDET), Mexico, in 2008, and the Ph.D. degree in automatic from The University of Caen Normandie, France, in 2017. His research interests include nonlinear observers, delayed systems, and UAVs.

GUILLERMO VALENCIA-PALOMO received the Engineering degree in electronics from the Tecnológico Nacional de México-IT Mérida, Mexico, in 2003, the M.Sc. degree in automatic control from Cenidet, in 2006, and the Ph.D. degree in automatic control from The University of Sheffield, U.K., in 2010. Since 2010, he has been with Tecnológico Nacional de México, IT Hermosillo. He has led a number of funded research projects, and the funding for these projects comprises a mixture of sole investigator funding, collaborative grants, and funding from industry. He is the author/coauthor of more than 40 articles published in ISI-journals, several international conferences, and one patent in commercial exploitation. His research interests include predictive control, LPV systems, fault detection, fault-tolerant control systems, and their applications. He is an Associate Editor of IEEE ACCESS, International Journal of Aerospace Engineering, and IEEE LATIN AMERICA TRANSACTIONS. He is a member of the Editorial Board of Mathematical and Computational Applications.

FRANCISCO-RONAY LÓPEZ-ESTRADA received the Ph.D. degree in automatic control from the University of Lorraine, France, in 2014. Since 2008, he has been a Lecturer with the National Tecnológico Nacional de México, IT Tuxtla Gutiérrez. He has led several funded research projects, and the funding for these projects comprises a mixture of sole investigator funding and collaborative grants. He is the author/coauthor of more than 40 research articles published in ISI-journals and several international conferences. Furthermore, he collaborates with European universities as the University of Lorraine, University of Paris 2, University of Stavanger, Polytechnique University of Catalonia, and among others. His research interests include fault diagnosis and fault-tolerant control based on convex LPV and Takagi-Sugeno models and their applications. He is a part of the Editorial Board of the Mathematical and Computational Applications and International Journal of Applied Mathematics and Computer Science journals.

JAVIER GARRIDO received the B.Eng. degree in electronic engineering from the ITM, Mexico, in 2001, and the M.S. and Ph.D. degrees in automatic control from CINVESTAV, México, in 2003 and 2015, respectively. He is currently a Professor with Universidad Veracruzana, Mexico. His research interests include robotics, renewable energy, and identifying mechanical faults with motor current signature analysis.

VOLUME 9, 2021

ABRAHAM-EFRAÍM RODRÍGUEZ-MATA was born in Mexico City, in January 1986. He received the degree in chemical engineering, in 2009, and the master’s and Ph.D. degrees in automatic control from the CINVESTAV-IPN, Mexico, in 2012 and 2016, respectively. He is currently a Research Professor with the Tecnológico Nacional de México, IT Chihuahua. His research interests include robust control, active disturbance rejection control, and the design of nonlinear observers.