\textbf{L}^p\text{-Boundedness of Wave Operators for 2D Schrödinger Operators with Point Interactions}

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Dedicated to Professor Arne Jensen on the occasion of his 70th birthday.

\textbf{Abstract.} For two dimensional Schrödinger operator $H$ with point interactions, we prove that wave operators of scattering for the pair $(H, H_0)$, $H_0$ being the free Schrödinger operator, are bounded in the Lebesgue space $L^p(\mathbb{R}^2)$ for $1 < p < \infty$ if and only if there are no generalized eigenfunctions of $H u(x) = 0$ which satisfy $u(x) = C|x|^{-1} + o(|x|^{-1})$ as $|x| \to \infty$, $C \neq 0$. Otherwise they are bounded for $1 < p \leq 2$ and unbounded for $2 < p < \infty$.

1. Introduction

We consider Schrödinger operators in $\mathcal{H} = L^2(\mathbb{R}^2)$ with point interactions (SOPI in short) at $Y = \{y_1, \ldots, y_N\} \subset \mathbb{R}^2$ with strength $\alpha = (\alpha_1, \ldots, \alpha_N) \in \mathbb{R}^N$, $1 \leq N < \infty$ which are defined symbolically by

$$H_{\alpha,Y} = \left( -\Delta + \sum_{j=1}^{N} \alpha_j \delta(x - y_j) \right)$$

and which will shortly be defined rigorously. Solutions of $H_{\alpha,Y} u = 0$ which are bounded as $|x| \to \infty$ are called (threshold) resonances. We show that a resonance satisfies $u(x) = a + b \cdot x/|x|^2 + O(|x|^{-2})$ as $|x| \to \infty$ for a constant $a \in \mathbb{C}$ and a vector $b \in \mathbb{C}^2$; we call it $s$-wave resonance if $a \neq 0$, $p$-wave resonance if $a = 0$ and $b \neq 0$; it is a zero energy eigenfunction if both $a$ and $b$ vanish but $u \neq 0$. It will be shown which kind of resonances $H_{\alpha,Y}$ can possess is controled by the three $N \times N$ symmetric matrices defined below by (1.11) and (1.12) in terms of $\alpha$ and $Y$. We then prove that the wave operators

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of scattering for the pair \((H_{\alpha,Y}, H_0), H_0 = -\Delta\) being the free Schrödinger operator, are bounded in the Lebesgue space \(L^p(\mathbb{R}^2)\) for all \(1 < p < \infty\) if and only if \(p\)-wave resonances are absent from \(H_{\alpha,Y}\) and, otherwise they are bounded for \(1 < p \leq 2\) and unbounded for \(2 < p < \infty\).

For the roles played by SOPI in physics, in nuclear and solid state physics in particular, and for the history of its mathematical studies, we refer to the seminal monograph \([2]\), the introduction of \([4]\) and references therein and we start with reviewing the rigorous definition of \(H_{\alpha,Y}\) and some of its basic properties \((\[2\])\). The resolvent \(G(z) = (H_0 - z^2)^{-1}\) of the free Schrödinger operator with the momentum parameter \(z \in \mathbb{C}^+ = \{z \in \mathbb{C} : \Im z > 0\}\) is the convolution operator with

\[
G_z(x) \overset{\text{def}}{=} \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} \frac{e^{ix \cdot \xi} d\xi}{\xi^2 - z^2} = \frac{1}{4} H^{(1)}_0(z|x|),
\]

where \(H^{(1)}_0(z)\) is the Hankel function of the first kind:

\[
\frac{1}{4} H^{(1)}_0(z) = \left( -\frac{1}{2\pi} \log \left( \frac{z}{2i} \right) - \frac{\gamma}{2\pi} \right) \sum_{k=0}^{\infty} \frac{(-1)^k}{(k!)^2} \left( \frac{z^2}{4} \right)^k - \frac{1}{2\pi} \left( \frac{3z^2}{(1!)^2} - \left( 1 + \frac{1}{2} \right) \left( \frac{3z^2}{(2!)^2} \right) + \left( 1 + \frac{1}{2} + \frac{1}{3} \right) \left( \frac{3z^2}{(3!)^2} \right) - \cdots \right)
\]

\[
= \frac{e^{iz}}{2\pi} \int_0^\infty e^{-t} t^{-\frac{1}{2}} \left( \frac{t}{2} - iz \right)^{-\frac{1}{2}} dt, \quad z \in \mathbb{C}^+ \setminus \{0\},
\]

where \(\gamma\) is Euler’s constant \((\[26\])\). We denote the prefactor in \((1.3)\) by \(g(z)\):

\[
g(z) = -\frac{1}{2\pi} \log \left( \frac{z}{2} \right) + \frac{i}{4} - \frac{\gamma}{2\pi},
\]

where \(\log(z/2)\) is real for \(z > 0\). Notice that

\[
g(z|x-y|) = g(z) + N_0(x-y), \quad N_0(x) = -(2\pi)^{-1} \log |x|
\]

gives the leading two terms of the asymptotic expansion of \(G_z(x-y)\) as \(z \to 0\).

Define \(N \times N\) matrix \(\Gamma_{\alpha,Y}(z)\) for \(z \in \mathbb{C}^+ = \{z \in \mathbb{C} : \Im z \geq 0\}\) by

\[
\Gamma_{\alpha,Y}(z) = \left\{ (\alpha_j - g(z)) \delta_{jk} - G_z(y_j - y_k) \delta_{jk} \right\},
\]

where \(\delta_{jk}\) is the Kronecker delta and \(\delta_{jk} = 1 - \delta_{jk}\). It is shown \((\text{cf.} \[2\])\) that \(\Gamma_{\alpha,Y}(z)\), \(z \in \mathbb{C}^+\) is non-singular outside a finite subset \(E \subset i(0,\infty)\) and the operator valued function \(R(z^2)\) defined for \(z \in \mathbb{C}^+ \setminus E\) by

\[
R(z^2) = (H_0 - z^2)^{-1} + \sum_{j,k=1}^N \left[ \Gamma_{\alpha,Y}(z)^{-1} \right]_{jk} G_z(\cdot - y_j) \otimes G_z(\cdot - y_k)
\]

is the resolvent of the selfadjoint operator \(H_{\alpha,Y}\) in \(\mathcal{H}\), \(u \otimes v\) being the operator of rank one define by \((u \otimes v)(f) = u(f, v)_{L^2}: R(z^2) = (H_{\alpha,Y} - z^2)^{-1}; H_{\alpha,Y}\) is
the selfadjoint extension of $-\Delta|_{C^\infty_0(\mathbb{R}^2 \setminus \Omega)}$ formally defined by (1.1); it is a real local operator; domain $D(H_{\alpha,Y})$ is the set of $u$’s of the form

$$u(x) = v(x) + \sum_{j,k=1}^{N} [\Gamma_{\alpha,Y}(z)^{-1}]_{jk} v(y_k) G_z(x - y_j), \ v \in H^2(\mathbb{R}^2);$$

(1.9)

the function $u$ determines $v$ uniquely in (1.9) and $(H_{\alpha,Y} - z^2)u = (H_0 - z^2)v$, $H^2(\mathbb{R}^2)$ being the Sobolev space of second order.

The spectrum of $H_{\alpha,Y}$ consists of the absolutely continuous (AC for short) part $[0, \infty)$ and at most $N$ non-positive eigenvalues. The Definition (1.8) shows that the rank of $R(z^2) - (H_0 - z^2)^{-1}$ is $N$ and Kato–Rosenblum theorem ([19, 23]) implies that the wave operators defined by the strong limits in $L^2(\mathbb{R}^2)$:

$$W_{\alpha,Y}^\pm = \lim_{t \to \pm \infty} e^{itH_{\alpha,Y}} e^{-itH_0}$$

(1.10)

exist and are complete in the sense that Range $W_{\alpha,Y}^\pm = L^2_{ac}(H_{\alpha,Y})$, the AC subspace of $L^2(\mathbb{R}^2)$ for $H_{\alpha,Y}$. In this paper we study if the wave operators $W_{\alpha,Y}^\pm$ are bounded in $L^p(\mathbb{R}^2)$, $1 \leq p \leq \infty$.

In what follows we shall often write $\lambda$ for $z \in \mathbb{C}^+$ when we emphasize $z$ can also be real not only $z \in \mathbb{C}^+$.

$$a \leq |1| \ b \ for \ |a| \leq |b|.$$  

For a matrix-valued smooth function $F(\lambda) = (F_{mn}(\lambda))$ and a complex valued function $f(\lambda)$ defined on $\{ \lambda \in \mathbb{C}^+ : 0 < |\lambda| < \varepsilon \}$, $F(\lambda) = O(f(\lambda))$ means that $F^{(j)}_{mn}(\lambda) = (d/d\lambda)^j F_{mn}(\lambda)$ satisfy for some $\varepsilon_j > 0$ and $C_j > 0$ that

$$F^{(j)}_{mn}(\lambda) \leq |1| \ C_j f(\lambda) \lambda^{-j}, \ j = 0, 1, \ldots, \ 0 < |\lambda| < \varepsilon_j, \ \lambda \in \mathbb{C}^+.$$  

Note that $F(\lambda) = O(f(\lambda))$ and $N(\lambda) = O(h(\lambda))$ imply $F(\lambda)N(\lambda) = O(f(\lambda)h(\lambda))$ and, if $f(\lambda) \to 0$ as $\lambda \to 0$, then $(1 + F(\lambda))^{-1} = 1 + O(f(\lambda))$.

We introduce the three real symmetric matrices which will play important roles in the rest of the paper:

$$\hat{D} = \left( \frac{\delta_{jk}}{2\pi} \log |y_j - y_k| \right), \ \mathcal{G}_1(Y) = \frac{-\delta_{jk}}{4N} |y_j - y_k|^2,$$

(1.11)

$$\mathcal{G}_2(Y) = \frac{-\delta_{jk}}{8\pi N} |y_j - y_k|^2 \log \left( \frac{e}{|y_j - y_k|} \right).$$

(1.12)

They appear in the asymptotic expansion as $\lambda \to 0$ of $\Gamma(\lambda)$:

$$\Gamma(\lambda) = -Ng(\lambda) \left( P - \frac{g(\lambda)^{-1}\hat{D}}{N} + \lambda^2 \mathcal{G}_1(Y) + \lambda^2 g(\lambda)^{-1} \mathcal{G}_2(Y) + O(\lambda^4) \right),$$

(1.13)

where $P$ and $S$ are projections in $\mathbb{C}^N$:

$$e = \frac{1}{\sqrt{N}} \mathbf{1}, \ \mathbf{1} = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}, \ P = e \otimes e, \ S = 1 - P.$$  

(1.14)
It is known ([4]) that these matrices control also the asymptotic behavior as \( z \to 0 \) of \((H_{\alpha,Y} - z^2)^{-1}\) and threshold resonances which \( H_{\alpha,Y} \) can have. We shall make the latter point clear by defining the resonances as solutions of \( H_{\alpha,Y} \varphi = 0 \) in a weighted \( L^2 \) space.

The following is the main theorem of this paper. It will be stated by using the matrices defined above and it appears slightly differently from what is stated at the beginning of the paper, however, it will shortly become clear that they are actually equivalent. We shall assume that linear map \( S \bar{D} S \) is singular in \( SC^N \). Let \( T \) be the orthogonal projection in \( SC^N \) onto \( \text{Ker}_{SC^N} S \bar{D} S \). If \( T \bar{D}^2 T \) is non-singular in \( TC^N \), then \( W_{\alpha,Y}^{\pm} \) are bounded from \( L^p(\mathbb{R}^2) \) to itself for all \( 1 < p < \infty \).

**Remark 1.2.**

1. Suppose that linear map \( S \bar{D} S \) in \( SC^N \) is non-singular. Then \( W_{\alpha,Y}^{\pm} \) are bounded from \( L^p(\mathbb{R}^2) \) to itself for all \( 1 < p < \infty \).

2. Suppose that \( S \bar{D} S \) is singular in \( SC^N \). Let \( T \) be the orthogonal projection in \( SC^N \) onto \( \text{Ker}_{SC^N} S \bar{D} S \). If \( T \bar{D}^2 T \) is non-singular in \( TC^N \), then \( W_{\alpha,Y}^{\pm} \) are bounded from \( L^p(\mathbb{R}^2) \) to itself for all \( 1 < p < \infty \).

3. Suppose that \( S \bar{D} S \) is singular in \( SC^N \) and that \( T \bar{D}^2 T \) is also singular in \( TC^N \). Let \( T_p \) be the orthogonal projection in \( TC^N \) onto \( \text{Ker}_{TC^N} T \bar{D}^2 T \). If \( T_p G_1(Y)T_p \) is non-singular in \( T_p C^N \), then \( W_{\alpha,Y}^{\pm} \) are bounded from \( L^p(\mathbb{R}^2) \) to itself for \( 1 < p \leq 2 \) but are unbounded for \( 2 < p < \infty \).

4. Suppose that \( S \bar{D} S \) is singular in \( SC^N \), \( T \bar{D}^2 T \) in \( TC^N \) and \( T_p G_1(Y)T_p \) in \( T_p C^N \). Let \( T_e \) be the orthogonal projection in \( T_p C^N \) onto \( \text{Ker}_{T_p C^N} T_p G_1(Y)T_p \). If \( T_e \neq T_p \), viz. \( T_p G_1(Y)T_p \neq 0 \), then \( W_{\alpha,Y}^{\pm} \) are bounded from \( L^p(\mathbb{R}^2) \) to itself for \( 1 < p \leq 2 \) but are unbounded for \( 2 < p < \infty \).

5. Suppose that \( S \bar{D} S \) is singular in \( SC^N \), \( T \bar{D}^2 T \) in \( TC^N \) and \( T_p G_1(Y)T_p \) = 0. Then \( W_{\alpha,Y}^{\pm} \) are bounded from \( L^p(\mathbb{R}^2) \) to itself for all \( 1 < p < \infty \).

**Remark 1.2.**

1. The orthogonal projection \( T \) may of course be expressed in the form \( T = e_1 \otimes e_1 + \cdots + e_k \otimes e_k \), \( k = \text{rank} \ T \) in terms of an orthonormal basis \( \{ e_1, \ldots, e_k \} \) of \( TC^N \). However, it is also an orthonormal system in \( SC^N \) as well as in \( C^N \) and \( T \) is considered as an orthogonal projection in \( C^N \). It is then obvious \( TS = ST = T \). Likewise, we consider \( T_p, T_e \) also as orthogonal projections in \( C^N \) so that we have \( ST_p = T_p S = T_p \) and etc. This convention is employed in the theorem and will be so throughout the paper.

2. Statements (3) and (4) may of course be unified simply by assuming \( T_p G_1(Y)T_p \neq 0 \). We state Theorem 1.1 in this way for a later convenience.

3. It is known ([4]) that rank \( T \bar{D}^2 T = 1 \) under the condition of statement (2), that \( T_e G_2(Y)T_e \) is necessarily non-singular in \( T_e C^N \) in statements (4) and (5) and that the situation of (4) or (5) does not happen if \( N = 2 \).

For regular Schrödinger operators \( H = -\Delta + V(x) \) on \( \mathbb{R}^d \), \( L^p \)-boundedness of wave operators has long been studied and many results are known under various assumptions on \( V \). Results depend on the dimensions \( d \) and on the existence/absence of eigenvalue and/or resonances at \( z = 0 \). We list here some results. In the following it is assumed that \( |V(x)| \leq C \langle x \rangle^{-\sigma} \) for \( \sigma > 2 \) or for a larger \( \sigma \), \( \langle x \rangle = (1 + |x|^2)^{1/2} \).
(1) If \( d = 1 \), \( W^\pm \) are bounded in \( L^p(\mathbb{R}) \) for \( 1 < p < \infty \) but not for \( p = 1 \) and \( p = \infty \) ([6,13,27]).

(2) If \( H \) has no eigenvalue nor resonances at \( z = 0 \), \( W^\pm \) are bounded in \( L^p(\mathbb{R}^d) \) for \( 1 \leq p \leq \infty \) if \( d \geq 3 \) and for \( 1 < p < \infty \) if \( d = 2 \) ([18,28,29]).

(3) If \( H \) has an eigenvalue or resonances at \( z = 0 \), much is known if \( d \geq 5 \) or \( d = 3 \) and the results depend on \( d \) and the types of singularities of the resolvent at \( z = 0 \) ([10–12,14,30,31]).

(4) If \( d = 4 \) and \( H \) has an eigenvalue but no resonances at \( 0, W^\pm \) are bounded in \( L^p(\mathbb{R}^4) \) for \( 1 \leq p \leq 4 \) ([14,17]).

(5) If \( d = 2 \) and \( H \) has an s wave resonances or only eigenvalue at 0, \( W^\pm \) are bounded in \( L^p(\mathbb{R}^2) \) for \( 1 < p < \infty \) ([11]).

For SOPI, \( W_{\alpha,Y}^\pm \) are bounded in \( L^p(\mathbb{R}) \) for \( 1 < p < \infty \) for all \( \alpha \) and \( Y \) if \( d = 1 \) ([9]); if \( H_{\alpha,Y} \) has no eigenvalue nor resonances at zero, then \( W_{\alpha,Y}^\pm \) are bounded in \( L^p(\mathbb{R}^2) \) for \( 1 < p < \infty \) if \( d = 2 \) ([4]) and in \( L^p(\mathbb{R}^3) \) for \( 1 < p < 3 \) if \( d = 3 \) ([8]). Thus, Theorem 1.1 gives a complete result for SOPI in two dimensions, however, the problem for the end points \( p = 1 \) and \( p = \infty \) is still open. We mention that for Schrödinger operators with regular potentials in two dimensions, no results have been obtained when \( H \) has \( p \)-wave resonances which corresponds to the case of statements (3) and (4) of Theorem 1.1.

We introduce some notation. For \( \sigma \in \mathbb{R} \), \( L^2_{\sigma}(\mathbb{R}^2) \) and \( H^2_{\sigma}(\mathbb{R}^2) \) are weighted spaces:

\[
L^2_{\sigma}(\mathbb{R}^2) \overset{\text{def}}{=} \{ \langle x \rangle^{-\sigma} u(x) : u \in L^2(\mathbb{R}^2) \}, \quad ||u||_{L^2_{\sigma}} \overset{\text{def}}{=} ||\langle x \rangle^{\sigma} u||_{L^2},
\]

\[
H^2_{\sigma}(\mathbb{R}^2) \overset{\text{def}}{=} \{ \langle x \rangle^{-\sigma} u : u \in H^2(\mathbb{R}^2) \}, \quad ||u||_{H^2_{\sigma}} \overset{\text{def}}{=} ||\langle x \rangle^{\sigma} u||_{H^2}.
\]

For \( y \in \mathbb{R}^2 \), \( \tau_y u(x) = u(x - y) \) is the translation by \( y \) and we set

\[
\tau_Y v(x) = \begin{pmatrix} \tau_{y_1} v(x) \\ \vdots \\ \tau_{y_N} v(x) \end{pmatrix}, \quad G_{z,Y}(x) = \tau_Y G_z(x), \quad N_{0,Y}(x) = \tau_Y N_0(x)
\]

(1.15)

In terms of these vectors, the domain of \( H_{\alpha,Y} \) is given by

\[
D(H_{\alpha,Y}) = \{ u(x) = v(x) + \langle \Gamma_{\alpha,Y}(z)^{-1} v_Y, G_{z,Y}(x) \rangle : v \in H^2(\mathbb{R}^2) \}.
\]

(1.16)

Here and hereafter \( v_Y = \begin{pmatrix} v(y_1) \\ \vdots \\ v(y_N) \end{pmatrix} \) and \( \langle a, b \rangle = a_1 b_1 + \cdots + a_N b_N \) without complex conjugation.

In view of the proof of the corresponding statement for \( H_{\alpha,Y} \) in [2] the following lemma should be obvious and the proof will be omitted.

**Lemma 1.3.** Let \( 1 < \sigma < 2 \) and \( z \in \mathbb{C}^+ \setminus \mathcal{E} \). The operator \( R(z^2) \) defined by (1.8) can be extended to a bounded operator in \( L^2_{-\sigma}(\mathbb{R}^2) \) by continuity, which we denote by \( R_{-\sigma}(z^2) \). Then, \( R_{-\sigma}(z^2) \) is the resolvent of a closed operator.
$H_{\alpha,Y}^\sigma$ in $L^2_\sigma(\mathbb{R}^2)$. The domain of $H_{\alpha,Y}^\sigma$ is given independently of $z \in \mathbb{C}^+ \setminus \mathcal{E}$ by

$$\text{Image } R_{z^2} = \{ u = v + \langle \Gamma_{\alpha,Y}(z)^{-1} v_Y, \mathcal{G}_{z,Y} \rangle : v \in H^2_{z^2}(\mathbb{R}^2) \}, \quad (1.17)$$

where $v \in H^2_{z^2}(\mathbb{R}^2)$ is uniquely determined by $u$ and

$$(H_{\alpha,Y}^\sigma - z^2)u = (H_0 - z^2)v. \quad (1.18)$$

**Lemma 1.4.** The null space of $H_{\alpha,Y}^\sigma$ is given independently of $1 < \sigma < 2$ by

$$\text{Ker } H_{\alpha,Y}^\sigma = \left\{ \varphi(x) = \frac{\langle \hat{D}a, 1 \rangle}{N} - \frac{1}{2\pi} \sum_{j=1}^{N} a_j \log |x - y_j| : a \in \text{Ker } S\hat{D} \right\}, \quad (1.19)$$

where $a_1, \ldots, a_N$ are components of $a \in \mathbb{C}^N$.

We prove Lemma 1.4 and the following two theorems in the next section. We denote by $C_b(\mathbb{R}^2 \setminus Y)$ the set of continuous functions in $\mathbb{R}^2 \setminus Y$ which are bounded outside a bounded open set containing $Y$ and define

$$\mathcal{R}_{\alpha,Y} = \text{Ker } H_{\alpha,Y}^\sigma \cap C_b(\mathbb{R}^2 \setminus Y),$$

which is independent of $1 < \sigma < 2$ by virtue of Lemma 1.4. We define $\hat{x} = x/|x|$ for $x \neq 0$.

**Theorem 1.5.** The space $\mathcal{R}_{\alpha,Y}$ is equal to

$$\left\{ \varphi(x) = \frac{\langle \hat{D}a, 1 \rangle}{N} - \frac{1}{2\pi} \sum_{j=1}^{N} a_j \log |x - y_j| : a \in \text{Ker } S\hat{D}S \cap \mathbb{C}^N \right\}. \quad (1.20)$$

The function $\varphi(x)$ of (1.20) satisfies

$$\varphi(x) = \frac{\langle \hat{D}a, 1 \rangle}{N} + \frac{1}{2\pi} \sum_{j=1}^{N} \frac{\langle \hat{x}, a_j y_j \rangle}{|x|} + O(|x|^{-2}) \quad (|x| \to \infty). \quad (1.21)$$

In particular, $\mathcal{R}_{\alpha,Y} = \{0\}$ if only if $S\hat{D}S$ is non-singular in $\mathbb{C}^N$.

**Definition 1.6.** A function $\varphi \in \mathcal{R}_{\alpha,Y}$ is called (threshold) resonance of $H_{\alpha,Y}$. (1) $H_{\alpha,Y}$ is said be regular at zero if $\mathcal{R}_{\alpha,Y} = \{0\}$, otherwise singular at zero. (2) $\varphi \in \mathcal{R}_{\alpha,Y}$ of (1.20) is an $s$-wave resonance if $\langle \hat{D}a, 1 \rangle \neq 0$ and $p$-wave resonance if $\langle \hat{D}a, 1 \rangle = 0$ but $\sum_{j=1}^{N} a_j y_j \neq 0$. (3) $\varphi \in \mathcal{R}_{\alpha,Y} \setminus \{0\}$ of in (1.21) is an eigenfunction of $H_{\alpha,Y}$ with eigenvalue 0 if $\langle \hat{D}a, 1 \rangle = 0$ and $\sum_{j=1}^{N} a_j y_j = 0$. In the following theorem we use the notation of Theorem 1.1.

**Theorem 1.7.** Suppose that $S\hat{D}S$ is singular in $\mathbb{C}^N$. Then:

(1) $s$-wave resonances exist if and only if $\hat{P}\hat{D}T \neq 0$.

(2) All $\varphi \in \mathcal{R}_{\alpha,Y}$ are $s$-wave resonances if and only if $T\hat{D}^2T$ is non-singular in $T\mathbb{C}^N$. 
(3) Suppose that $T\hat{D}^2T$ is singular in $T\mathbb{C}^N$. Then, $\varphi \in \mathcal{R}_{\alpha,Y}$ of (1.20) is
(a) an s-wave resonance if $a \in T\mathbb{C}^N \setminus T_p\mathbb{C}^N$,
(b) a p-wave resonance if $a \in T_p\mathbb{C}^N \setminus T_e\mathbb{C}^N$,
(c) an eigenfunction with eigenvalue 0 if $a \in T_e\mathbb{C}^N$.

(4) The eigenspace of $H_{\alpha,Y}$ associated with eigenvalue zero is the set of all $\varphi(x)$ in (1.20) with $a \in T_e\mathbb{C}^N$.

In virtue of Theorems 1.1, 1.5 and 1.7, wave operators are bounded in $L^p(\mathbb{R}^2)$ for all $1 < p < \infty$ if $H_{\alpha,Y}$ has no p-wave resonances otherwise they are bounded only for $1 < p \leq 2$.

We briefly record here the result of [4] on the threshold behavior of $(H_{\alpha,Y} - 2^2)^{-1}$ to show its relation to Theorem 1.1. We refer to [4] for more precise result. For Schrödinger operators with regular potentials, the relation between resonances, the threshold behavior of the resolvent and the large time behavior of solutions of time dependent Schrödinger equations is extensively studied (see e.g. [10,11,15,16,21,24]).

Let $\sigma > 1$ and let $B_\sigma$ be the Banach space of bounded operators from $L^2(\mathbb{R}^2)$ to $L^2_\sigma(\mathbb{R}^2)$. Then, the well-known limiting absorption principle for $(H_0 - 2^2)^{-1}$ ([1,20]) and the behavior of the Hankel function ([26]) imply that $(H_{\alpha,Y} - 2^2)^{-1}$ regarded as a $B_\sigma$-valued function of $z \in \mathbb{C}^+ \setminus E$ can be continuously extended to $(\mathbb{C}^+ \setminus (E \cup \{0\})$.

**Theorem 1.8** ([4]). (1) Suppose that $H_{\alpha,Y}$ is regular at zero, then $(H_{\alpha,Y} - 2^2)^{-1}$ can be extended continuously to 0.

(2) Suppose that the condition of Theorem 1.1 (2) is satisfied. Then, $T = f \otimes f$ for a normalized $f \in T\mathbb{R}^N$ and

$$(H_{\alpha,Y} - \lambda^2)^{-1} = a^{-2}g(\lambda)\varphi \otimes \varphi + O(1) \quad (\lambda \to 0),$$

where $\varphi(x)$ is an s-wave resonance defined in (1.20) with $f$ in place of $a$.

(3) Suppose that the condition of Theorem 1.1 (3) is satisfied. Then

$$(H_{\alpha,Y} - \lambda^2)^{-1} = -(Ng(\lambda)\lambda^2)^{-1} \sum_{j=1}^n a_j \varphi_j(x)\varphi_j(y) + O((\lambda g(\lambda))^{-2}) \quad (\lambda \to 0),$$

where $n = \text{rank } T_p$ and $\varphi_j$, $j = 1, \ldots, n$ are p-wave resonances.

(4) Suppose that the condition of Theorem 1.1 (4) is satisfied. Then,

$$(H_{\alpha,Y} - \lambda^2)^{-1}(x,y) = -(N\lambda^2)^{-1}\langle T_e N_{0,Y}(x), [T_e \mathcal{G}_2(Y)T_e]^{-1}T_e N_{0,Y}(y) \rangle + O(\lambda^{-2}g(\lambda)^{-1}).$$

The rest of the paper is devoted to the proof of the lemmas and theorems (but not of Theorem 1.8). In Sect. 2, we prove results on the resonances, Lemmas 1.3, 1.4 and Theorem 1.5. In Sect. 3, we collect results necessary for proving Theorem 1.1. We first recall from [4] the stationary and the product decomposition formulas for the wave operators and the result that the high energy part $W_{\alpha,Y}^\pm (|D|)$ is bounded in $L^p(\mathbb{R}^2)$ for all $1 < p < \infty$. We then examine the result in [4] on the behavior of $\Gamma(\lambda)^{-1}$ as $\lambda \to 0$ and give an estimate on the Fourier transform of a logarithmic function. We prove in Sect. 4
the statements of Theorem 1.1 separately. In virtue of the high energy results mentioned above we prove them for the low energy part $W_{a,Y}^{\pm}(\|D\|)$ only. Statement (1) is a direct result of the product formula and Mikhlin’s theorem on Fourier multiplier. Proof of statements (2) to (5) uses the cancellation property produced by linear operators $S, T, T_p$ and $T_c$.

2. Proof of Results on Resonances

In this section we prove Lemma 1.4, Theorems 1.5 and 1.7.

Proof of Lemma 1.4. Define for $a \in \mathbb{C}^N$ and $z \in \mathbb{C}^+$,

$$h_z(x) \overset{\text{def}}{=} \langle a, N_0,Y(x) - G_{z,Y}(x) \rangle. \quad (2.1)$$

We have $h_z(x) \in H^2_{-\sigma}(\mathbb{R}^2)$ for $1 < \sigma < 2$ because (1.3) implies the log-singularities at $x = y_j$ cancel and it grows only logarithmically as $|x| \to \infty$. Moreover, $N_0(x - y_j) - G_z(x - y_j) = -g(z) + O(|x - y_j|^2)$ as $x \to y_j$ and, the definitions of $\Gamma_{a,Y}(z)$ and $D$ imply

$$h_z(y_j) = \sum_{k=1}^{N} a_k \left( -\frac{1}{2\pi} \log|y_k - y_j| - G_z(y_j - y_k) \right) \delta_{jk} - g(z) \delta_{jk}$$

$$= [\Gamma_{a,Y}(z) - D]a_j, \quad j = 1, \ldots, N. \quad (2.2)$$

(a) We first show that $\varphi(x)$ defined by (1.19) with $a \in \text{Ker} S\hat{D}$ is a solution of $H_{a,Y}^{-\sigma}\varphi = 0$. Let $C = N^{-1} \langle Da, 1 \rangle$. By virtue of (2.1)

$$\varphi(x) = v(x) + \langle a, G_{z,Y}(x) \rangle \quad \text{with} \quad v(x) \overset{\text{def}}{=} h_z(x) + C \in H^2_{-\sigma}(\mathbb{R}^2) \quad (2.4)$$

and, we have $\Gamma_{a,Y}(z)a = v_Y$ because (2.3) and $S\hat{D}a = 0$ or $\hat{D}a \in PC^N$ imply $v_Y - \Gamma_{a,Y}(z)a = -\hat{D}a + C 1 = (-N^{-1} \langle Da, 1 \rangle + C)1 = 0$. It follows $\varphi \in D(H_{a,Y}^{-\sigma})$. Moreover, (1.18) and (2.4) imply $H_{a,Y}^{-\sigma}\varphi = 0$ because

$$(H_{a,Y}^{-\sigma} - z^2)\varphi(x) = (-\Delta - z^2)v(x) = -z^2(C + \langle a, N_0,Y(x) \rangle) = -z^2\varphi(x).$$

(b) Assume conversely that $\varphi(x)$ satisfies $H_{a,Y}^{-\sigma}\varphi = 0$. We show that $\varphi(x)$ is necessarily of the form (1.19) with $a$ such that $S\hat{D}a = 0$. Since $\varphi \in D(H_{a,Y}^{-\sigma})$, for $z \in \mathbb{C}^+ \setminus \mathcal{E}$ there must exist $v \in H^2_{-\sigma}(\mathbb{R}^2)$ such that

$$\varphi(x) = v(x) + \langle a_z, G_{z,Y}(x) \rangle \quad \text{where} \quad a_z = \Gamma_{a,Y}(z)^{-1} v_Y \quad (2.5)$$

and $(H_{a,Y}^{-\sigma} - z^2)\varphi = (-\Delta - z^2)v + z^2 \varphi(x) = (-\Delta - z^2)v(x)$. Hence,

$$-\Delta v(x) = z^2(v(x) - \varphi(x)) = -z^2\langle a_z, G_{z,Y}(x) \rangle. \quad (2.6)$$

We observe that $(a_z)_j = -\lim_{x \to y_j}(2\pi)(\log|x - y_j|)^{-1}\varphi(x)$ because $v \in H^2_{-\sigma}(\mathbb{R}^2)$ is continuous and $a_z$ is independent of $z$. Thus, we write $a$ for $a_z$ and define $h_z(x)$ by (2.1) with this $a$. We show $v(x) = h_z(x) + C$ for a constant $C$ and, hence $\varphi(x) = \langle a, N_0,Y(z) \rangle + C$, or it must be of the form in (1.19). Indeed, it follows from (2.6) that

$$-\Delta (v + \langle a, G_{z,Y} \rangle) = \langle a, (-\Delta - z^2)G_{z,Y} \rangle = \langle a, \delta_Y \rangle = \langle a, -\Delta N_0,Y \rangle$$
and \(0 = -\Delta(v(x) + \langle a, \mathcal{G}_{z,Y}(x) - N_{0,Y}(x)\rangle) = -\Delta(v(x) - h_z(x))\). Thus, \(v(x) - h_z(x)\) must be a harmonic polynomial which belongs to \(H^2_\sigma(\mathbb{R}^2)\) and, hence \(v(x) - h_z(x) = C\). For determining \(C\), we recall (2.3), which implies
\[
C1 = v_Y - (h_z)_Y = \Gamma_{\alpha,Y}(z) a - (\Gamma_{\alpha,Y}(z) - \mathcal{D}) a = \mathcal{D}a.
\]
It follows that \(a\) must be such that \(\mathcal{D}a \in PC^N\) and \(C = N^{-1} \langle \mathcal{D}a, 1 \rangle\). \(\Box\)

**Proof of Theorem 1.5.** The function of (1.19), \(\varphi(x) = C + \langle a, N_{0,Y}(x)\rangle\) with \(S\mathcal{D}a = 0\), is evidently continuous in \(\mathbb{R}^2\setminus Y\). It is also evident that \(\varphi(x)\) is bounded near infinity if and only if \(\langle 1, a \rangle = 0\) or \(a \in SC^N\). This implies that \(\mathcal{R}_{\alpha,Y}\) is given by (1.20). The relation (1.21) is evident since \(a \in SC^N\) implies \(\langle 1, a \rangle = a_1 + \cdots + a_N = 0\). \(\Box\)

**Proof of Theorem 1.7.** (1) If \(H_{\alpha,Y}\) has an s-wave resonance, then there must exist nonzero \(a \in TC^N = SC^N \cap \text{Ker} S\mathcal{D}\) such that \(P\mathcal{D}a \neq 0\), hence \(P\mathcal{D}T \neq 0\). Conversely, if \(P\mathcal{D}T \neq 0\), then there exists \(a \in TC^N\) such that \(P\mathcal{D}a \neq 0\) and \(\varphi(x)\) defined by (1.20) with this \(a\) is an s-wave resonance.

(2) If \(TD^2T\) is non-singular in \(TC^N\), \(\text{Ker} |_{TC^N}TD^2T = \text{Ker} |_{TC^N}\mathcal{D}T = \{0\}\) since \(\mathcal{D}\) is real symmetric and, for \(a \in TC^N \setminus \{0\}\) we have \(P\mathcal{D}a = \mathcal{D}a \neq 0\). Thus all resonances are s-wave resonances. If \(TD^2T\) is singular in \(TC^N\) then, \(\mathcal{D}a = 0\) for an \(a \in TC^N\) which trivially satisfies \(P\mathcal{D}a = 0\) and, (1.20) with this \(a\) produces either a p-wave resonance or an eigenfunction.

(3a) If \(a \in TC^N\) is such that \(TD^2T a \neq 0\), then \(\mathcal{D}a \neq 0\) and \(P\mathcal{D}a \neq 0\) as \(S\mathcal{D}a = 0\). Thus, \(\varphi(x)\) produced by \(a\) by (1.20) is an s-wave resonance.

(3b,c) For \(a \in TPc^N \setminus \{0\}\), we have \(\mathcal{D}a = \mathcal{D}T a = 0\) and trivially \(P\mathcal{D}a = 0\). If \(T_p\mathcal{G}_1(Y)T_p a \neq 0\), \(\langle \mathcal{G}_1(Y)a, a \rangle_{C^N} = (2N)^{-1} |\sum_{j=1}^{N} a_j y_j|^2 \neq 0\) and \(\varphi(x)\) produced by \(a\) by (1.20) is a p-wave resonance; if \(T_p\mathcal{G}_1(Y)T_p a = 0\), then \(\sum_{j=1}^{N} a_j y_j = 0\), \(\varphi(x) = O(|x|^{-2})\) as \(|x| \to \infty\) and it is an eigenfunction.

(4) If \(a \in T_pC^N \subset TPc^N\), then we have seen in (3b) that \(\langle \mathcal{D}a, 1 \rangle = 0\) and \(\sum_{j=1}^{N} a_j y_j = 0\). Hence \(\varphi(x)\) (an eigenfunction. Conversely, eigenfunction is also of the form (1.20) with \(a \in TC^N\) by virtue of Theorem 1.5 and \(a\) must further satisfy \(\langle \mathcal{D}a, 1 \rangle = 0\) and \(\sum_{j=1}^{N} a_j y_j = 0\). Since \(S\mathcal{D}a = 0\), the former implies \(\mathcal{D}a = 0\), hence \(TD^2T a = 0\) and \(a \in TPc^N\). The latter implies \(\langle a, T_p\mathcal{G}_1(Y)T_p a \rangle = 0\) and \(T_p\mathcal{G}_1(Y)T_p a = 0\) since \(T_p\mathcal{G}_1(Y)T_p\) is non-negative. Thus, we have \(a \in T_pC^N\). \(\Box\)

### 3. Preliminaries for the Proof of Theorem 1.1

In this section, we collect several lemmas which we use for proving Theorem 1.1, some of which are well known and are recorded for the reader’s convenience. The strength \(\alpha\) and points of interaction \(Y\) will be fixed hereafter and will be often omitted from various formulas unless no confusion is feared. We prove Theorem 1.1 for \(W^-\). We have \(W^+ = CW^-C^{-1}\) by the complex conjugation \(Cu(x) = \overline{u(x)}\) and the result for \(W^+\) immediately follows from that for \(W^-\).
Hereafter we shall often write $x\xi$ for $x \cdot \xi$ for simplicity. We set

$$D_* = \{ u \in S(\mathbb{R}^2) : \mathcal{F}u \in C^\infty_0 (\mathbb{R}^2 \setminus \{0\}) \}. \quad (3.1)$$

For Borel functions $f$, $f(D)$ is the Fourier multiplier by $f(\xi): f(D)u(x) = \mathcal{F}^{-1}(f(\xi)\hat{u}(\xi))(x)$. The space $D_*$ is dense in $L^p(\mathbb{R}^2)$ for any $1 < p < \infty$. For any $\lambda \in \mathbb{R}$, there exists a constant $\Gamma(\lambda) = \Gamma(\lambda)^{-1}$ such that

$$\Gamma(\lambda) = \Gamma(\lambda)^{-1}$$

and $\Gamma(|D|) = \Gamma_jk(|D|)$ is the operator matrix of Fourier multipliers $\Gamma_jk(|D|)$.

The following is a slight modification of the result of Lemmas 4.3 and 4.4 of [4]. Define the operator $K$ by

$$K u(x) = \frac{1}{\tau^2} \int_0^\infty \mathcal{F}u(\lambda \omega) d\omega \quad (3.4)$$

**Lemma 3.1.** The wave operator $W^-$ may be represented in the form

$$W^- u = u + \frac{1}{\pi i} \int_0^\infty \int_{\mathbb{R}^2} \lambda \tilde{\Gamma}(\lambda) G_{\lambda,Y}(x), G_{\lambda,Y}(y) - G_{-\lambda,Y}(y)) u(y) dy \, d\lambda \quad (3.2)$$

for $u \in D_*$. Equivalently $W^- u = u + \sum_{j,k=1}^N \tau_{y_j} \Omega_{jk} \tau_{y_k}^*$ where

$$\Omega_{jk} u(x) = \frac{1}{\pi i} \int_0^\infty \lambda \tilde{\Omega}_{jk}(\lambda) G_{\lambda}(x) \left( \int_{\mathbb{R}^2} (G_{\lambda}(y) - G_{-\lambda}(y)) u(y) dy \right) d\lambda. \quad (3.3)$$

**Decomposition formula.** The following is a slight modification of the result of Lemmas 4.3 and 4.4 of [4]. Define the operator $K$ by

$$K u(x) = \frac{1}{\pi^2} \int_0^\infty \mathcal{F}u(\lambda \omega) d\omega \quad (3.4)$$

**Lemma 3.2.** (1) For $j, k = 1, \ldots, N$, $\Omega_{jk}$ is the product of $\tilde{\Gamma}_j k(|D|)$ and $K$:

$$(\Omega_{jk} u)(x) = (K \circ \tilde{\Gamma}_j k(|D|)) u(x), \quad u \in D_* \quad (3.5)$$

(2) $K$ is a singular integral operator:

$$K u(x) = \lim_{\varepsilon \downarrow 0} \frac{2}{\pi^2} \int_{\mathbb{R}^2} \frac{u(y) dy}{x^2 - y^2 + i\varepsilon}. \quad (3.6)$$

(3) For any $1 < p < \infty$, there exists a constant $C_p > 0$ such that

$$\|K u\|_{L^p(\mathbb{R}^2)} \leq C_p \|u\|_{L^p(\mathbb{R}^2)}, \quad u \in D_* \quad (3.7)$$
We recall the well-known result (e.g. [25]) on the Fourier multiplier which will be very often used in what follows.

**Lemma 3.3** (Mikhlin). Let \( m \in C^2(\mathbb{R}^2 \setminus \{0\}) \) satisfy \( |\partial_\xi^\alpha m(\xi)| \leq C|\xi|^{-|\alpha|} \) for \( |\alpha| \leq 2 \). Then, \( m(D) \in \mathcal{B}(L^p(\mathbb{R}^2)) \) for \( 1 < p < \infty \).

In what follows \( \chi \) will stand for the real function \( \chi \in C_0^\infty(\mathbb{R}) \) which satisfies

\[
\chi(\lambda) = 1 \text{ for } |\lambda| < 1 \quad \text{and} \quad \chi(\lambda) = 0 \text{ for } |\lambda| > 2
\]

(3.8)

and, for \( \varepsilon > 0 \), we define

\[
\chi_{\leq \varepsilon}(\lambda) = \chi(\lambda/\varepsilon), \quad \chi_{\geq \varepsilon}(\lambda) = 1 - \chi_{\leq \varepsilon}(\lambda).
\]

(3.9)

When \( \varepsilon > 0 \) is fixed, then we often write \( \chi_{\leq \varepsilon}(\lambda) = \chi_{\leq \varepsilon}(\lambda) \) and \( \chi_{\geq \varepsilon}(\lambda) = \chi_{\geq \varepsilon}(\lambda) \) omitting \( \varepsilon > 0 \). We shall often say that a matrix function \( M(\lambda) \) is a *good multiplier near zero* when \( M_{nm}(|\xi|) \chi_{\leq \varepsilon}(|\xi|) \) satisfies the condition of Lemma 3.3 for some \( \varepsilon > 0 \), \( M_{nm}(|\xi|) \) being the entries of \( M(|\xi|) \). If \( M(\lambda) = O(1) \), then it is a good multiplier near zero.

### 3.2. High Energy Part \( W^- \chi_{\geq \varepsilon}(|D|) \)

The \( L^p \) property of the high energy part of \( W^- \) does not depend on the small \( \lambda \) behavior of \( (H_{\alpha,Y} - \lambda^2)^{-1} \) and the following is proved in [4].

**Theorem 3.4.** For any \( \varepsilon > 0 \), \( W^-_{\alpha,Y}\chi_{\geq \varepsilon}(|D|) \) is bounded from \( L^p(\mathbb{R}^2) \) to itself for any \( 1 < p < \infty \).

We say for simplicity that an operator is a *good operator* if it is bounded from \( L^p(\mathbb{R}^2) \) to itself for any \( 1 < p < \infty \). By virtue of Theorem 3.4, we need consider only the low energy part \( W_{\text{low}} = W^-\chi_{\leq \varepsilon}(|D|) \).

### 3.3. Expansion of \( \Gamma(\lambda)^{-1} \)

For proving Theorem 1.1 we need precise information on the behavior of \( \tilde{\Gamma}(\lambda) = \Gamma(\lambda)^{-1} \) as \( \lambda \downarrow 0 \). We have already obtained some results in [4, 5], however, because we shall need some more precise results and the ones that are buried in proofs there, we have decided to completely redo it. The low energy behavior of \( \Gamma(\lambda)^{-1} \) is different depending on the conditions of statements of Theorem 1.1 and we split the subsection into five paragraphs accordingly. We remark that the conditions in each steps are mutually exclusive. Hereafter we shall indiscriminately denote by \( M(\lambda) \) a good multiplier near zero which may differ at each appearance.

We shall repeatedly use the following lemma due to Jensen and Nenciu [16]. The following trivial identities for matrices will be frequently used:

\[
(1 + X)^{-1} = 1 - X(1 + X)^{-1} = 1 - (1 + X)^{-1}X,
\]

(3.10)

\[
(1 + X)^{-1} = 1 - X + X(1 + X)^{-1}X.
\]

(3.11)
Lemma 3.5. Let $A$ be a closed operator in a Hilbert space $\mathcal{H}$ and $S$ a projection. Suppose $A + S$ has a bounded inverse. Then, $A$ has a bounded inverse if and only if

$$B = S - S(A + S)^{-1}S$$

has a bounded inverse in $S\mathcal{H}$ and, in this case,

$$A^{-1} = (A + S)^{-1} + (A + S)^{-1}SB^{-1}S(A + S)^{-1}. \quad (3.12)$$

We also use the well-known Feshbach formula but only at the last step. In what follows we use the notation of Theorem 1.1. Identity matrices of various subspaces are indiscriminately denoted by 1 and as remarked previously orthogonal projections $P$ in subspaces $V$ will be often regarded as the projection $P \oplus 0$ in the full space $\mathbb{C}^N = V \oplus V^\perp$. We shall often say $\lambda \in \mathbb{C}^+$ is small when $0 < |\lambda|$ is small.

Step 1. Define $A(\lambda)$ by

$$\Gamma(\lambda) = -Ng(\lambda)A(\lambda)$$

and

$$F = -N^{-1}\tilde{D}, \quad R(\lambda) = \mathcal{G}_1(Y) + g(\lambda)^{-1}\mathcal{G}_2(Y). \quad (3.13)$$

We study the behavior of $A(\lambda)^{-1}$ as $\lambda \to 0$ via Lemma 3.5. (1.13) implies

$$A(\lambda) + S = 1 + g(\lambda)^{-1}F + \mathcal{M}_0(\lambda), \quad \mathcal{M}_0(\lambda) \stackrel{\text{def}}{=} \lambda^2R(\lambda) + O(\lambda^4). \quad (3.14)$$

Thus, $A(\lambda) + S = (1 + \mathcal{M}_0(\lambda)(1 + g(\lambda)^{-1}F)^{-1})(1 + g(\lambda)^{-1}F)$ is non-singular for small $\lambda \in \mathbb{C}^+$ and by virtue of (3.11),

$$(A(\lambda) + S)^{-1} = (1 + g(\lambda)^{-1}F)^{-1}$$

$$- (1 + g(\lambda)^{-1}F)^{-1}\mathcal{M}_0(\lambda)(1 + O(\lambda^2))(1 + g(\lambda)^{-1}F)^{-1}. \quad (3.15)$$

In particular $(A(\lambda) + S)^{-1}$ is a good multiplier near zero and

$$(A(\lambda) + S)^{-1}S = S + O(g(\lambda)^{-1})S, \quad S(A(\lambda) + S)^{-1} = S + SO(g(\lambda)^{-1}). \quad (3.16)$$

In view of Lemma 3.5, we define the operator $B(\lambda)$ in $S\mathbb{C}^N$ by

$$B(\lambda) = S - S(A(\lambda) + S)^{-1}S. \quad (3.17)$$

Substituting (3.15) for $(A(\lambda) + S)^{-1}$ in (3.17), we find the leading term as $\lambda \to 0$ of $B(\lambda)$ is given by

$$S - S(1 + g(\lambda)^{-1}F)^{-1}S = -N^{-1}g(\lambda)^{-1}S\tilde{D}(1 + g(\lambda)^{-1}F)^{-1}S.$$

We therefore consider $A_1(\lambda) = -Ng(\lambda)B(\lambda)$ instead of $B(\lambda)$. If we define

$$L(\lambda) = S\tilde{D}(1 + g(\lambda)^{-1}F)^{-1}S$$

and

$$\lambda^2R_1(\lambda) \stackrel{\text{def}}{=} NS(1 + g(\lambda)^{-1}F)^{-1}\mathcal{M}_0(\lambda)(1 + O(\lambda^2))(1 + g(\lambda)^{-1}F)^{-1}S, \quad (3.18)$$

then we have $R_1(\lambda) = O(1)$ and
\[ A_1(\lambda) = L(\lambda) - g(\lambda)\lambda^2 R_1(\lambda) \]  
\[ = S\tilde{D}S + g(\lambda)^{-1}N^{-1}S\tilde{D}^2(1 + g(\lambda)^{-1}F)^{-1}S - g(\lambda)\lambda^2 R_1(\lambda). \]

(3.19)  
(3.20)

We have the following result.

**Lemma 3.6.** If $S\tilde{D}S$ is non-singular in $\mathbb{S}\mathbb{C}^N$, then $\Gamma(\lambda)^{-1}$ is a good multiplier near zero.

**Proof.** (3.20) implies $A_1(\lambda) = S\tilde{D}S + O(g(\lambda)^{-1})$. Hence, $A_1(\lambda)^{-1}$ exists in $\mathbb{S}\mathbb{C}^N$ for small $\lambda$ and it is a good multiplier near zero. It follows by virtue of (3.12) of Lemma 3.5 that $\Gamma(\lambda)^{-1}$ is given by

\[ -N^{-1}g(\lambda)^{-1}(A(\lambda) + S)^{-1} + (A(\lambda) + S)^{-1}SA_1(\lambda)^{-1}S(A(\lambda) + S)^{-1}. \]

(3.21)

Since $(A(\lambda) + S)^{-1}$ is a good multiplier near zero, (3.21) implies the lemma. \qed

**Step 2.** Suppose next $S\tilde{D}S$ is singular in $\mathbb{S}\mathbb{C}^N$ and let $T$ be the orthogonal projection in $\mathbb{S}\mathbb{C}^N$ onto $\text{Ker}_{\mathbb{S}\mathbb{C}^N} S\tilde{D}S$. From the definition of $L(\lambda)$ (see (3.20)) we have

\[ L(\lambda) + T = (S\tilde{D}S + T) + N^{-1}g(\lambda)^{-1}S\tilde{D}(1 + g(\lambda)^{-1}F)^{-1}\tilde{D}S \]

and $S\tilde{D}S + T$ is clearly invertible in $\mathbb{S}\mathbb{C}^N$. Hence

\[ L(\lambda) + T = (1 + g(\lambda)^{-1}\tilde{F}(\lambda))(S\tilde{D}S + T). \]

(3.22)

where $\tilde{F}(\lambda) = O(1)$ is defined by

\[ \tilde{F}(\lambda) \overset{\text{def}}{=} N^{-1}S\tilde{D}(1 + g(\lambda)^{-1}F)^{-1}\tilde{D}S(S\tilde{D}S + T)^{-1}. \]

(3.23)

It follows that $L(\lambda) + T$ is also invertible in $\mathbb{S}\mathbb{C}^N$ for small $\lambda \in \mathbb{C}^+$ and

\[ (L(\lambda) + T)^{-1} = (S\tilde{D}S + T)^{-1} - g(\lambda)^{-1}(S\tilde{D}S + T)^{-1}\tilde{F}(\lambda) \]

\[ + g(\lambda)^{-2}(S\tilde{D}S + T)^{-1}\tilde{F}(\lambda)^2(1 + g(\lambda)^{-1}\tilde{F}(\lambda))^{-1} \]

(3.24)

by virtue of (3.11). Then, (3.19) implies that $A_1(\lambda) + T$ is also invertible in $\mathbb{S}\mathbb{C}^N$ and

\[ (A_1(\lambda) + T)^{-1} = (L(\lambda) + T)^{-1}(1 - g(\lambda)^2R_1(\lambda)(L(\lambda) + T)^{-1})^{-1} \]

\[ = (L(\lambda) + T)^{-1} + g(\lambda)\lambda^2(L(\lambda) + T)^{-1}(R_1(\lambda) + O(\lambda^2 g(\lambda)))(L(\lambda) + T)^{-1}. \]

(3.25)  
(3.26)

Equations (3.24) and (3.26) imply that $(A_1(\lambda) + T)^{-1}$ is a good multiplier near zero and that

\[ (A_1(\lambda) + T)^{-1}T = T + O(g(\lambda)^{-1}), \quad T(A_1(\lambda) + T)^{-1} = T + O(g(\lambda)^{-1}). \]

(3.27)

For studying $A_1(\lambda)^{-1}$ via Lemma 3.5, define $B_1(\lambda)$ in $T\mathbb{C}^N$ by

\[ B_1(\lambda) = T - T(A_1(\lambda) + T)^{-1}T. \]

(3.28)

Inserting (3.26) for $(A_1(\lambda) + T)^{-1}$ in (3.28), we have

\[ B_1(\lambda) = T - T(L(\lambda) + T)^{-1}T - g(\lambda)\lambda^2 R_2(\lambda), \]

(3.29)
where we defined $\mathcal{R}_2(\lambda)$ by
\begin{equation}
\mathcal{R}_2(\lambda) \overset{\text{def}}{=} T(L(\lambda) + T)^{-1}(\mathcal{R}_1(\lambda) + O(g(\lambda)\lambda^2))(L(\lambda) + T)^{-1}T. \tag{3.30}
\end{equation}
Since $T(S\tilde{D}S + T)^{-1} = T$ and $TS = ST = T$, (3.24) and (3.23) imply that
\begin{equation}
T - T(L(\lambda) + T)^{-1}T \\
= g(\lambda)^{-1}T\tilde{F}(\lambda)T - g(\lambda)^{-2}T\tilde{F}(\lambda)^2(1 + g(\lambda)^{-1}\tilde{F}(\lambda))^{-1}T
\end{equation}
which we write
\begin{equation}
N^{-1}g(\lambda)^{-1}(T\tilde{D}^2T + g(\lambda)^{-1}F_2(\lambda)),
\end{equation}
by defining $F_2(\lambda)$ by
\begin{equation}
F_2(\lambda) = N^{-1}T\tilde{D}^2(1 + g(\lambda)^{-1}F)^{-1}\tilde{D}T - NT\tilde{F}(\lambda)^2(1 + g(\lambda)^{-1}\tilde{F}(\lambda))^{-1}T.
\end{equation}
With these definition $B_1(\lambda) = N^{-1}g(\lambda)^{-1}A_2(\lambda)$, where
\begin{equation}
A_2(\lambda) = T\tilde{D}^2T + g(\lambda)^{-1}F_2(\lambda) - g(\lambda)^2N\lambda^2\mathcal{R}_2(\lambda). \tag{3.31}
\end{equation}

**Lemma 3.7.** Suppose $T\tilde{D}^2T$ is non-singular in $T\mathbb{C}^N$. Then $\dim T\mathbb{C}^N = 1$, rank $T\tilde{D}^2T = 1$ and $H_{\alpha,Y}$ has an s-wave resonance only. Moreover,
\begin{equation}
\Gamma(\lambda)^{-1} = Ng(\lambda)T(T\tilde{D}^2T)^{-1}T + M(\lambda). \tag{3.32}
\end{equation}

**Remark 3.8.** In (3.32) and in the expressions of $\Gamma(\lambda)^{-1}$ which will appear in what follows it is important that the singular terms, $Ng(\lambda)T(T\tilde{D}^2T)^{-1}T$ here, are always accompanied by projections $S, T, T_p$ or $T_e$. These projections will produce cancellations of the singularities when it is substituted for $\Gamma(\lambda)^{-1}$ in the stationary formula (3.2) of the wave operator.

**Proof.** The first part of the lemma is proved in [5]. We prove (3.32). If $T\tilde{D}^2T$ is non-singular in $T\mathbb{C}^N$, then (3.31) implies $A_2(\lambda)$ and, hence $B_1(\lambda)$ are non-singular in $T\mathbb{C}^N$, $A_2(\lambda)^{-1} = (T\tilde{D}^2T)^{-1} + O(g(\lambda)^{-1})$,
\begin{equation}
B_1(\lambda)^{-1} = Ng(\lambda)((T\tilde{D}^2T)^{-1} + O(g(\lambda)^{-1})) \tag{3.33}
\end{equation}
and, by virtue of Lemma 3.5
\begin{equation}
A_1(\lambda)^{-1} = (A_1(\lambda) + T)^{-1} + (A_1(\lambda) + T)^{-1}TB_1(\lambda)^{-1}T(A_1(\lambda) + T)^{-1}. \tag{3.34}
\end{equation}
Combining (3.21) and (3.34), we see that $\Gamma(\lambda)^{-1}$ is equal to
\begin{equation}
-N^{-1}g(\lambda)^{-1}(A(\lambda) + S)^{-1} + (A(\lambda) + S)^{-1}S(A_1(\lambda) + T)^{-1}S(A(\lambda) + S)^{-1} \tag{3.35}
\end{equation}
\begin{equation}
+ Ng(\lambda)(A(\lambda) + S)^{-1}S(A_1(\lambda) + T)^{-1}TA_2(\lambda)^{-1}T(A_1(\lambda) + T)^{-1}S(A(\lambda) + S)^{-1}. \tag{3.36}
\end{equation}
Since $(A(\lambda) + S)^{-1}$ and $(A_1(\lambda) + T)^{-1}$ are good multipliers near zero as has been proved previously, the two terms in (3.35) enjoy the same property. Equations (3.33), (3.27) and (3.16) imply that (3.36) is equal to the sum of $Ng(\lambda)ST(T\tilde{D}^2T)^{-1}TS$ and a good multiplier near zero. This proves (3.32).
Step 3. We next assume that $S\hat{D}S|_{SCN}$ and $T\hat{D}^2T|_{TCN}$ are both singular. Let $T_p$ be the orthogonal projection onto $\text{Ker} T\hat{D}^2T|_{TCN}$. Recall that we irrespectively write $M(\lambda)$ for a good multiplier near zero. Recalling (3.31), we define

$$\tilde{A}_2(\lambda) \overset{\text{def}}{=} T\hat{D}^2T + g(\lambda)^{-1}F_2(\lambda)$$

so that $A_2(\lambda) = \tilde{A}_2(\lambda) - g(\lambda)^2\lambda^2N\mathcal{R}_2(\lambda)$. It is evident that $T\hat{D}^2T + T_p$ is non-singular in $TC^N$. Hence both of $\tilde{A}_2(\lambda) + T_p$ and $A_2(\lambda) + T_p$ are invertible in $TC^N$ for small $\lambda \in \mathbb{C}^+$ and

$$(A_2(\lambda) + T_p)^{-1} = (\tilde{A}_2(\lambda) + T_p)^{-1}
+ g(\lambda)^2\lambda^2N(\tilde{A}_2(\lambda) + T_p)^{-1}\mathcal{R}_2(\lambda)(1 + O(\lambda^2g(\lambda)^2))(\tilde{A}_2(\lambda) + T_p)^{-1}$$

$$(T\hat{D}^2T + T_p)^{-1} + O(g(\lambda)^{-1})$$

(3.38)

In particular $(A_2(\lambda) + T_p)^{-1}$ is a good multiplier near zero.

Lemma 3.9. (1) The projection $T_p$ annihilates $\hat{D}, \tilde{F}(\lambda), L(\lambda), F_2(\lambda)$ and $\tilde{A}_2$:

$$\hat{D}T_p = T_p\hat{D} = 0, \quad T_p\tilde{F}(\lambda) = \tilde{F}(\lambda)T_p = 0, \quad T_pL(\lambda) = L(\lambda)T_p = 0, \quad T_pF_2(\lambda) = F_2(\lambda)T_p = 0, \quad \tilde{A}_2(\lambda)T_p = T_p\tilde{A}_2(\lambda) = 0.$$  

(3.39)

(2) We have the following identities:

$$(A(\lambda) + S)^{-1}S(A_1(\lambda) + T)^{-1}T(A_2(\lambda) + T_p)^{-1}T_p$$

$$= T_p + \lambda^2g(\lambda)^2SM(\lambda)T_p + \lambda^2g(\lambda)M(\lambda)T_p.$$  

(3.40)

$$T_p(A_2(\lambda) + T_p)^{-1}T(A_1(\lambda) + T)^{-1}S(A(\lambda) + S)^{-1}$$

$$= T_p + \lambda^2g(\lambda)^2T_pM(\lambda)S + \lambda^2g(\lambda)T_pM(\lambda).$$  

(3.41)

**Proof.** (1) Since $\hat{D}$ is real symmetric, we have $\text{Ker} T\hat{D}^2T = \text{Ker} \hat{D}T$ and $\hat{D}T_p = T_p\hat{D} = 0$. Other identities of (1) follows from this immediately. It follows from (3.38) that $T_p(A_2(\lambda) + T_p) = T_p + T_pO(\lambda^2g(\lambda)^2)$, $(A_2(\lambda) + T_p)T_p = T_p + O(\lambda^2g(\lambda)^2)T_p$ and hence

$$T_p(A_2(\lambda) + T_p)^{-1} = T_p + T_pO(\lambda^2g(\lambda)^2),$$

$$(A_2(\lambda) + T_p)^{-1}T_p = T_p + O(\lambda^2g(\lambda)^2)T_p.$$  

(3.43)

Likewise we have, respectively, from (3.20) and (3.14) that

$$\begin{cases}
(A_1(\lambda) + T)^{-1}T_p = T_p + O(\lambda^2g(\lambda)), \\
T_p(A_1(\lambda) + T)^{-1} = T_p + T_pO(\lambda^2g(\lambda)).
\end{cases}$$  

(3.45)

Then, we apply (3.43), (3.44) and (3.45) consecutively in this order and then (3.27) and (3.16) to see that the left of (3.40) is equal to

$$(A(\lambda) + S)^{-1}S(A_1(\lambda) + T)^{-1}T_p + (A(\lambda) + S)^{-1}S(A_1(\lambda) + T)^{-1}TO(\lambda^2g(\lambda)^2)T_p$$

$$= (A(\lambda) + S)^{-1}S(T_p + O(\lambda^2g(\lambda))T_p)$$
We substitute (3.51) for Lemma 3.10. It follows by applying (3.47) and (3.48) to (3.46) that

\[ T_p(A_2(\lambda) + T_p)^{-1} = T_p + \mathcal{O}(\lambda^2 g(\lambda)^2)T_p. \]

This yields (3.40). (3.42) is the conjugate of (3.40). □

We study \( A_2(\lambda)^{-1} \) via Lemma 3.5 and define

\[ B_2(\lambda) = T_p - T_p(A_2(\lambda) + T_p)^{-1}T_p. \]  (3.46)

By virtue of (3.39) we have \( T_p(\tilde{A}_2(\lambda) + T_p)^{-1} = (\tilde{A}_2(\lambda) + T_p)^{-1}T_p = T_p \) and, the first line of (3.38) implies

\[ T_p(A_2(\lambda) + T_p)^{-1}T_p = T_p + \mathcal{O}(\lambda^2 g(\lambda)^2)T_p. \]  (3.47)

Equations in (3.39) imply

\[ T_p(L(\lambda) + T)^{-1} = (L(\lambda) + T)^{-1}T_p = T_p, \]
\[ T_p(1 + g(\lambda)^{-1}F)^{-1} = (1 + g(\lambda)^{-1}F)^{-1}T_p = T_p \]

and, recalling the Definitions (3.30), (3.18) and (3.13), we see that

\[ T_p\mathcal{R}_2(\lambda)T_p = N\mathcal{T}_p(\mathcal{R}(\lambda) + \mathcal{O}(g(\lambda)\lambda^2))T_p \]
\[ = \mathcal{N}\mathcal{T}_p(\mathcal{G}_1(Y) + g(\lambda)^{-1}\mathcal{G}_2(Y) + \mathcal{O}(g(\lambda)\lambda^2))T_p. \]  (3.48)

It follows by applying (3.47) and (3.48) to (3.46) that

\[
\begin{align*}
B_2(\lambda) &= -N^2 g(\lambda)^2 \lambda^2 A_3(\lambda), \\
A_3(\lambda) &= T_p\mathcal{G}_1(Y)T_p + g(\lambda)^{-1}T_p\mathcal{G}_2(Y)T_p + T_p\mathcal{O}(\lambda^2 g(\lambda)^2)T_p.
\end{align*}
\]  (3.49)

**Lemma 3.10.** Suppose \( T\bar{D}^2T \) is singular in \( T\mathbb{C}^N \) and let \( T_p \) be the projection onto \( \text{Ker} T\bar{D}^2T \) in \( T\mathbb{C}^N \). Suppose \( T_p\mathcal{G}_1(Y)T_p \) is non-singular in \( T_p\mathbb{C}^N \). Then,

\[ \Gamma(\lambda)^{-1} = -N^{-1} g(\lambda)^{-1}\lambda^{-2}T_p(\mathcal{G}_1(Y)T_p + g(\lambda)^{-1}T_p\mathcal{G}_2(Y)T_p)^{-1}T_p \]
\[ + g(\lambda)SM(\lambda)S + M(\lambda). \]  (3.50)

**Proof.** If \( T_p\mathcal{G}_1(Y)T_p \) is non-singular in \( T_p\mathbb{C}^N \), then \( B_2(\lambda) \) and \( A_3(\lambda) \) are invertible in \( T_p\mathbb{C}^N \) by virtue of (3.49) and Lemma 3.5 implies that \( A_2(\lambda)^{-1} \) is equal to

\[ (A_2(\lambda) + T_p)^{-1} - N^{-2}g(\lambda)^{-2}\lambda^{-2}(A_2(\lambda) + T_p)^{-1}T_pA_3(\lambda)^{-1}T_p(A_2(\lambda) + T_p)^{-1}. \]  (3.51)

We substitute (3.51) for \( A_2(\lambda)^{-1} \) of (3.36). Then, modulo \( M(\lambda) \) which is produced by (3.35), \( \Gamma(\lambda)^{-1} \) is equal to

\[ \begin{align*}
N g(\lambda)(A(\lambda) + S)^{-1}S(A_1(\lambda) + T)^{-1}T(A_2(\lambda) + T_p)^{-1} \\
\times T(A_1(\lambda) + T)^{-1}S(A(\lambda) + S)^{-1} \\
- N^{-1} g(\lambda)^{-1}\lambda^{-2}(A(\lambda) + S)^{-1}S(A_1(\lambda) + T)^{-1}T(A_2(\lambda) + T_p)^{-1} \\
\times T_pA_3(\lambda)^{-1}T_p(A_2(\lambda) + T_p)^{-1}T(A_1(\lambda) + T)^{-1}S(A(\lambda) + S)^{-1}.
\end{align*} \]  (3.52)

Substituting (3.38) that \( (A_2(\lambda) + T_p)^{-1} = (T\bar{D}^2T + T_p)^{-1} + \mathcal{O}(g(\lambda)^{-1}) \) for (3.52) and applying the relations (3.16) and (3.27), we obtain

\[ (3.52) = Ng(\lambda)S(T\bar{D}^2T + T_p)^{-1}TS + M(\lambda). \]  (3.54)
To study (3.53) we introduce a short hand notation
\[ G_1 \overset{\text{def}}{=} T_p \mathcal{G}_1(Y)T_p, \quad G_2 \overset{\text{def}}{=} T_p \mathcal{G}_2(Y)T_p, \quad \mathcal{R}(\lambda) \overset{\text{def}}{=} T_p \mathcal{R}(\lambda)T_p = G_1 + g(\lambda)^{-1} G_2. \]

The assumption of the lemma implies \( \mathcal{R}(\lambda) \) is invertible in \( T_p \mathbb{C}^N \) and so is \( A_3(\lambda) \) and, \( A_3(\lambda)^{-1} = \mathcal{R}(\lambda)^{-1} + O(\lambda^2 g(\lambda)^2) \), which we substitute for \( A_3(\lambda)^{-1} \) in (3.53). Then (3.53) becomes
\[
-N^{-1} g(\lambda)^{-1} \lambda^{-2} (A(\lambda) + S)^{-1} S(A_1(\lambda) + T)^{-1} T(A_2(\lambda) + T_p)^{-1}
\]
\[
\times T_p (\mathcal{R}(\lambda)^{-1} + O(\lambda^2 g(\lambda)^2)) T_p (A_2(\lambda) + T_p)^{-1} T(A_1(\lambda) + T)^{-1} S(A(\lambda) + S)^{-1}.
\]

Then, we replace functions on each side of \( (\mathcal{R}(\lambda)^{-1} + O(\lambda^2 g(\lambda)^2)) \) by the corresponding functions of Lemma 3.9 (2). The term \( O(\lambda^2 g(\lambda)^2) \) produces \( T_p O(g(\lambda)) T_p + O(\lambda^2 g(\lambda)^3) = g(\lambda) S M(\lambda) S + M(\lambda) \) and \( \mathcal{R}(\lambda)^{-1} \) does
\[
-N g(\lambda)^{-1} \lambda^{-2} T_p \mathcal{R}(\lambda)^{-1} T_p + g(\lambda) S M(\lambda) \mathcal{R}(\lambda)^{-1} T_p + T_p \mathcal{R}(\lambda)^{-1} M(\lambda) S + M(\lambda)
\]
Combining these results with (3.54), we obtain (3.50).

**Step 4.** Next we assume in addition to those of Step 3 that \( G_1 \) is singular in \( T_p \mathbb{C}^N \) but \( G_1 \neq 0 \). We let \( T_e \) be the projection in \( T_p \mathbb{C}^N \) onto \( \text{Ker} T_p \mathbb{C}^N G_1 \). We recall from [4] that, then \( T_e G_2 T_e \) is necessarily non-singular in \( T_e \mathbb{C}^N \). In the following lemma we write \( T_e^\perp \) for \( T_p \ominus T_e \) and we define the operator \( \mathcal{B} \) on \( T_p \mathbb{C}^N \) and \( D_0(\lambda) \) on \( (T_p \ominus T_e) \mathbb{C}^N \), respectively, by
\[
\mathcal{B} = T_e^\perp G_2 T_e (T_e G_2 T_e)^{-1} T_e,
\]
\[
D_0(\lambda) = (T_e^\perp G_1 T_e^\perp + g(\lambda)^{-1} (T_e^\perp G_2 T_e^\perp - T_e^\perp G_2 T_e (T_e G_2 T_e)^{-1} T_e G_2 T_e^\perp))^{-1}.
\]

By the remark above, \( \mathcal{B} \) is well defined and, since \( T_e^\perp G_1 T_e^\perp \) is clearly invertible in \( (T_p \ominus T_e) \mathbb{C}^N \), \( D_0(\lambda) \) is also well defined for small \( \lambda \in \mathbb{C}^+ \).

**Lemma 3.11.** Let \( G_1 \) be singular in \( T_p \mathbb{C}^N \) but \( G_1 \neq 0 \) and, \( T_e \) be the projection in \( T_p \mathbb{C}^N \) onto \( \text{Ker} T_p \mathbb{C}^N G_1 \). Then, \( \Gamma(\lambda)^{-1} \) has the following expression as \( \lambda \to 0 \):
\[
-N^{-1} \lambda^{-2} T_e (T_e G_2 T_e)^{-1} T_e - N^{-1} g(\lambda)^{-1} \lambda^{-2} T_p (T_e^\perp - B^*) T_e^\perp D_0(\lambda) T_e^\perp (T_e^\perp - B) T_p
\]
\[
+ g(\lambda)^3 T_p M(\lambda) T_p + g(\lambda)^2 S M(\lambda) S + g(\lambda) S M(\lambda) + g(\lambda) M(\lambda) S + M(\lambda).
\]

(3.55)

We use the following well-known formula of linear algebra.

**Lemma 3.12.** Let \( A \) be a block decomposed matrix:
\[
A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \quad a_{jk}: \mathcal{H}_j \to \mathcal{H}_k
\]

Suppose that \( a_{22}^{-1} \) exists. Then \( A^{-1} \) exists if and only if \( d = (a_{11} - a_{12} a_{22}^{-1} a_{21})^{-1} \) exists. In this case we have
\[
A^{-1} = \begin{pmatrix} d & -da_{12} a_{22}^{-1} \\ -a_{22}^{-1} a_{21} d & a_{22}^{-1} a_{21} da_{12} a_{22}^{-1} + a_{22}^{-1} \end{pmatrix}.
\]

(3.56)
Proof. We may repeat the argument of Step 3 up to (3.49) which implies that, as long as \( A_3(\lambda)^{-1} \) exists in \( T_p \mathbb{C}^N \), \( \Gamma(\lambda)^{-1} = (3.52) + (3.53) \) and that (3.52) = \( g(\lambda)SM(\lambda)S + M(\lambda) \) (see (3.54)), which we put into \( g(\lambda)SM(\lambda) + M(\lambda) \) of (3.55). Thus, we have only to study (3.53). We study \( A_3(\lambda)^{-1} \) in \( T_p \mathbb{C}^N \) by using Lemma 3.12. We write \( A_3(\lambda) \) in the block matrix in the direct decomposition \( T_p \mathbb{C}^N = (T_p \otimes T_e) \mathbb{C}^N \oplus T_e \mathbb{C}^N \):

\[
A_3(\lambda) = \begin{pmatrix} T_e G_1 T_e & g(\lambda)^{-1} T_e G_2 T_e \\ g(\lambda)^{-1} T_e G_2 T_e & g(\lambda)^{-1} T_e G_2 T_e \end{pmatrix} + O(g(\lambda)^2 \lambda^2)
\]

(3.57)

Then, \( a_{22} = g(\lambda)^{-1} T_e G_2 T_e \) is invertible in \( T_e \mathbb{C}^N \) as mentioned above;

\[
a_{11} - a_{12}a_{22}^{-1} a_{21} = T_e G_1 T_e + g(\lambda)^{-1} (T_e G_2 T_e - T_e G_2 T_e (T_e G_2 T_e)^{-1} T_e G_2 T_e) \]

is also invertible for small \( \lambda \in \mathbb{C}^+ \) in \( T_e \mathbb{C}^N \) because \( \text{Ker} G_1 \cap T_e \mathbb{C}^N = \{0\} \) by the definition of \( T_e \). Then, Lemma 3.12 implies that the matrix in the right of (3.57) is invertible in \( T_p \mathbb{C}^N \) and the inverse is equal to

\[
\begin{pmatrix} D_0(\lambda) & -D_0(\lambda) B \\ -B^* D_0(\lambda) & B^* D_0(\lambda) B + g(\lambda)(T_e G_2 T_e) \end{pmatrix} = O(g(\lambda)) \cdot
\]

(3.58)

Then, the standard perturbation theory implies

\[
A_3(\lambda)^{-1} = \begin{pmatrix} D_0(\lambda) & -D_0(\lambda) B \\ -B^* D_0(\lambda) B^* D_0(\lambda) B + g(\lambda)(T_e G_2 T_e) \end{pmatrix} + T_p O(g(\lambda)^4 \lambda^2) T_p
\]

\[
= g(\lambda)(T_e G_2 T_e)^{-1} + (T_e - B^*) D_0(\lambda)(T_e - B) + T_p O(g(\lambda)^4 \lambda^2) T_p.
\]

(3.59)

We substitute (3.59) for \( A_3(\lambda)^{-1} \) in (3.53) and denote by \( \Gamma_1(\lambda) \) and \( \Gamma_2(\lambda) \) the functions produced, respectively, by \( g(\lambda)(T_e G_2 T_e)^{-1} \) and by the other two terms. Then, by virtue of Lemma 3.9 (2), \( \Gamma_1(\lambda) \) is equal to

\[
- N^{-1} \lambda^{-2} (1 + \lambda^2 g(\lambda)^2 SM(\lambda) + \lambda^2 g(\lambda) M(\lambda))
\times T_e (T_e G_2 T_e)^{-1} T_e (1 + \lambda^2 g(\lambda)^2 M(\lambda) S + \lambda^2 g(\lambda) M(\lambda))
\]

\[
= - N^{-1} \lambda^{-2} T_e (T_e G_2 T_e)^{-1} T_e
\]

\[
+ g(\lambda)^2 (SM(\lambda) T_e + T_e M(\lambda) S) + g(\lambda) (M(\lambda) T_e + T_e M(\lambda)) + M(\lambda)
\]

(3.60)

Likewise, denoting \( C = (T_e - B^*) D_0(\lambda)(T_e - B) \), \( \Gamma_2(\lambda) \) is equal to

\[
- N^{-1} \lambda^{-2} g(\lambda)^{-1} (1 + \lambda^2 g(\lambda)^2 SM(\lambda) + \lambda^2 g(\lambda) M(\lambda)) T_p (C + O(\lambda^2 g(\lambda)^4))
\times T_p (1 + \lambda^2 g(\lambda)^2 M(\lambda) S + \lambda^2 g(\lambda) M(\lambda))
\]

\[
= - N^{-1} \lambda^{-2} g(\lambda)^{-1} T_p C T_p + g(\lambda)^3 T_p M(\lambda) T_p
\]

\[
+ g(\lambda) (SM(\lambda) T_p + T_p M(\lambda) S) + M(\lambda).
\]

(3.62)

Combining (3.60), (3.62) with the remark stated at the beginning, we obtain (3.55) and conclude the proof.
Step 5. We finally assume in addition to conditions of Step 3 and 4 that $G_1 = 0$. Then, $T_p = T_e$ and $H_{\alpha, \gamma}$ has no $p$-wave resonances.

**Lemma 3.13.** Suppose $G_1 = 0$. Then, $T_e = T_p$ and $\Gamma(\lambda)^{-1}$ is equal to

\[
-N^{-1}\lambda^{-2} T_p G_2^{-1} T_e + g(\lambda)^3 T_e M(\lambda) T_e + g(\lambda)^2 (T_e M(\lambda) S + S M(\lambda) T_e) + g(\lambda) (T_e M(\lambda) + M(\lambda) T_e) + S M(\lambda) S + M(\lambda).
\]

(3.63)

**Proof.** Since $G_1$ is trivially singular, $T_e G_2 T_e$ is non-singular as mentioned above. It follows from (3.49) that $A_3(\lambda)^{-1}$ exists in $T_p C^N$ and

\[
A_3(\lambda)^{-1} = (g(\lambda)^{-1} G_2 + T_p O(\lambda^2 g(\lambda)^2) T_p)^{-1} = g(\lambda) G_2^{-1} + O(\lambda^2 g(\lambda)^4).
\]

(3.64)

Thus, we still have the expression $\Gamma(\lambda)^{-1} = (3.52) + (3.53)$ and (3.52) satisfies the estimate (3.54) which we put into the last two terms of (3.63). We substitute (3.64) in (3.53) and apply Lemma 3.9 (2). Then (3.53) becomes

\[
-N^{-1} g(\lambda)^{-1} \lambda^{-2} (T_p + \lambda^2 g(\lambda)^2 S M(\lambda) + \lambda^2 g(\lambda) M(\lambda)) T_p \\
\times (g(\lambda) G_2^{-1} + O(\lambda^2 g(\lambda)^4)) T_p (T_p + \lambda^2 g(\lambda)^2 M(\lambda) S + \lambda^2 g(\lambda) M(\lambda))
\]

\[
= -N^{-1} \lambda^{-2} (T_p + \lambda^2 g(\lambda)^2 S M(\lambda) + \lambda^2 g(\lambda) M(\lambda)) G_2^{-1} \\
\times (T_p + \lambda^2 g(\lambda)^2 M(\lambda) S + \lambda^2 g(\lambda) M(\lambda)) + g(\lambda)^3 T_p M(\lambda) T_p + M(\lambda)
\]

\[
= -N^{-1} \lambda^{-2} T_p G_2^{-1} T_p - N^{-1} T_p G_2^{-1} (g(\lambda)^2 M(\lambda) S + g(\lambda) M(\lambda))
\]

\[
- N^{-1} (g(\lambda)^2 S M(\lambda) + g(\lambda) M(\lambda)) G_2 T_p + g(\lambda)^3 T_p M(\lambda) T_p + M(\lambda).
\]

As $T_p = T_e$, this implies (3.63). □

### 3.4. Fourier Transform of a Logarithmic Function

Let $g(z)$ be the function of (1.5). In the following section we need a pointwise estimate on the Fourier transform

\[
F(x) = \frac{1}{2\pi} \int_{\mathbb{R}^2} e^{ipx} \chi_{\leq \varepsilon}(|p|) dp.
\]

(3.65)

We remark that Hausdorff–Young’s inequality implies $F \in L^q(\mathbb{R}^2)$ for $2 \leq q \leq \infty$. The following estimate must be well known and we give a proof for reader’s convenience.

**Lemma 3.14.** Let $\chi_{\leq \varepsilon}$ be defined by (3.9). Then for $0 < \varepsilon \leq 1$ there exists a constant $C_\varepsilon > 0$ such that

\[
|F(x)| \leq \frac{C_\varepsilon}{\langle x \rangle \log(2 + |x|)}
\]

(3.66)

**Proof.** Since $F(x)$ is evidently a smooth function, it suffices to show (3.66) when $|x|$ is sufficiently large and we assume $|x| > (100e^{10\gamma} + 4\pi/\varepsilon^2)$, $\gamma = 0.577\ldots$ being Euler’s constant so that $2\pi/|x| < \varepsilon/2$ and $\sqrt{|x|} < \varepsilon|x|$. Since $|g(r)| \geq (\log |r|)/200\pi + 1/8$ for $0 < r < 2\pi/|x|$, we have

\[
\frac{1}{2\pi} \int_{\mathbb{R}^2} e^{ix\xi} \chi_{\leq 2\pi/|x|}(|\xi|) \chi_{\leq \varepsilon}(|\xi|) d\xi \leq \int_{0}^{2\pi/|x|} \frac{dr}{|g(r)|} \leq \frac{C}{\langle x \rangle \log(2 + |x|)}
\]
for a constant $C > 0$. Thus, it suffices to prove the lemma after inserting $\chi_{\geq 2\pi/|x|}(|\xi|)$ in the integrand of (3.66). We denote the function thus obtained by $\tilde{F}(x)$. The Bessel function satisfies (see e.g. [25], page 338):

$$J_0(r) = \frac{1}{\pi} \int_0^\pi e^{ir\theta} \cos \theta \, d\theta = (2/\pi)^{1/2} r^{-1/2} \cos(r - \pi/4) + O(r^{-3/2}), \quad r \to \infty.$$ 

Let $\chi_{\leq \varepsilon_1}(r) = \chi_{\geq \varepsilon_1}(r) \chi_{\leq \varepsilon_2}(r)$. Then, after a change variable

$$\tilde{F}(x) = \frac{1}{\sqrt{2\pi}} \int_0^\infty \chi_{[2\pi/|x|, \varepsilon]}(r) \left( \frac{\cos(|x|r - \pi/4)}{|x|^r} + O(|x|r^{-3/2}) \right) dr = \frac{1}{\sqrt{2\pi|x|}} \int_0^\infty \chi_{[2\pi/|x|, \varepsilon]}(r) \left( \frac{\cos(r - \pi/4)}{\sqrt{r}} + O(r^{-3/2}) \right) dr. \quad (3.67)$$

Splitting the integral, we write $\tilde{F}(x) = F_1(x) + F_2(x)$ where

$$F_1(x) = \frac{1}{\sqrt{2\pi |x|}} \int_0^\infty \chi_{[2\pi/|x|, \varepsilon]}(r) \frac{\cos(r - \pi/4)}{\sqrt{r}} dr,$$

$$F_2(x) = \frac{1}{\sqrt{2\pi |x|}} \int_0^\infty \chi_{[2\pi/|x|, \varepsilon]}(r) O(r^{-3/2}) dr.$$

Write $\tilde{\chi}$ for $\chi_{[2\pi/|x|, \varepsilon]}$. We estimate $F_2(x)$ first. Since $|g(r)| \geq 1/4$ for any $r > 0$ and $|g(r/|x|)| \geq C \log |x|$ for $2\pi \leq r < \sqrt{x}$ as $|x| > 100 e^{10\gamma}$, we have

$$|F_1(x)| \leq \frac{C}{|x|} \left( \int_0^{\sqrt{x}} + \int_{\sqrt{x}}^{|x|} \right) \frac{\tilde{\chi}(r)}{g(r/|x|)} (r)^{-3/2} dr$$

$$\leq \frac{2}{|x|} \int_0^{\sqrt{x}} \frac{1}{g(r/|x|)} (r)^{-3/2} dr + \frac{C}{|x|} \int_{\sqrt{x}}^{e|x|} \frac{1}{g(r/|x|)} (r)^{-3/2} dr \leq \frac{C}{(x) \log(2 + |x|)}.$$

Since $\cos(s + \pi) = - \cos(s)$, $F_1(x)$ is equal to $(2\pi)^{-1/2}$ times

$$\frac{1}{|x|} \int_0^\infty \frac{\tilde{\chi}(r)}{g(r/|x|)} \cos(r - \pi/4) \frac{\cos(r - \pi/4)}{r^{1/2}} dr = - \frac{1}{|x|} \int_0^\infty \frac{\tilde{\chi}(r + \pi)}{g((r + \pi)/|x|)} \cos(r - \pi/4) \frac{\cos(r - \pi/4)}{(r + \pi)^{1/2}} dr$$

$$= \frac{1}{2|x|} \int_0^\infty \cos(r - \pi/4) \left( \frac{\tilde{\chi}(r)}{g(r/|x|)} - \frac{\tilde{\chi}(r + \pi)}{g((r + \pi)/|x|)} \right) (r)^{-1/2} dr$$

The function inside $(\cdots)$ is equal to $K_1 + K_2 + K_3$ where

$$K_1(r, x) = \frac{\tilde{\chi}(r) - \tilde{\chi}(r + \pi)}{g(r/|x|)} r^{-1/2},$$

$$K_2(r, x) = \tilde{\chi}(r + \pi) \left( \frac{1}{g(r/|x|)} - \frac{1}{g((r + \pi)/|x|)} \right) r^{-1/2},$$

$$K_3(r, x) = \frac{\tilde{\chi}(r + \pi)}{g((r + \pi)/|x|)} (r^{-1/2} - (r + \pi)^{-1/2}).$$

Since $\tilde{\chi}(r) - \tilde{\chi}(r + \pi) \neq 0$ only on $[\pi, 2\pi]$ and on $[\varepsilon|x| - \pi, \varepsilon|x|]$ and, $|g(r/|x|)| \geq c \log |x|$ on $[\pi, 2\pi]$ and $|g(r/|x|)| \geq C$ on $[\varepsilon|x| - \pi, \varepsilon|x|]$, it follows that

$$\frac{1}{2|x|} \int_0^\infty \cos(r - \pi/4) K_1(r, x) dr \leq \frac{1}{|x|} \left( \int_0^{2\pi} \frac{dr}{\log |x|} + \int_{\varepsilon|x| - \pi}^{\varepsilon|x|} \frac{dr}{r^{1/2}} \right).$$
and this is bounded in modulus by $C|x|^{-1}(2 + \log |x|)^{-1}$ as desired. Note that $K_2(r, x) \neq 0$ for $\pi < r < \varepsilon |x| - \pi$ and we estimate
\[
\frac{1}{g(r/|x|)} - \frac{1}{g((r + \pi)/|x|)} \leq |\cdot| \cdot \frac{\log(r + \pi) - \log r}{2\pi g(r/|x|) \cdot g((r + \pi)/|x|)}
\]
\[
= \frac{\log(1 + \frac{\pi}{r})}{2\pi g(r/|x|) \cdot g((r + \pi)/|x|)} \leq \begin{cases} 
\frac{C_\varepsilon}{(\log |x|)^2 r}, & 2\pi < r < \sqrt{|x|}, \\
\frac{C_\varepsilon}{r}, & \sqrt{|x|} \leq r \leq \varepsilon |x|.
\end{cases}
\]
It follows for large $|x|$ that
\[
\frac{1}{2|x|} \int_0^\infty \cos(r - \frac{\pi}{4})K_2(r, x)dr \leq |\cdot| \frac{C}{|x|} \int_0^\sqrt{x} \frac{1}{(\log |x|)^2 r^{3/2}}dr + \frac{C}{|x|} \int_{\sqrt{x}}^{\varepsilon |x|} \frac{1}{r^{3/2}}dr \leq \frac{C_1}{|x|(|\log |x||)^2}.
\]
We likewise estimate for $\pi \leq r \leq \varepsilon |x| - \pi$
\[
K_3(r, x) \leq |\cdot| \frac{\pi \chi(r + \pi)}{r^{1/2}(r + \pi)|g((r + \pi)/|x|)|} \leq \begin{cases} 
\frac{C}{r^{3/2} \log |x|}, & \pi \leq r \leq \sqrt{|x|}, \\
\frac{C}{r^{3/2}}, & \sqrt{|x|} < r \leq \varepsilon |x| - \pi
\end{cases}
\]
and
\[
\frac{1}{2|x|} \int_0^\infty \cos(r - \frac{\pi}{4})K_3(r, x)dr \leq |\cdot| \frac{C}{|x|} \left( \int_\pi^{\sqrt{|x|}} \frac{dr}{\log |x| r^{3/2}} + \int_\pi^{\varepsilon |x| - \pi} \frac{dr}{r^{3/2}} \right) \leq \frac{C}{\langle x \rangle \log(2 + |x|)}.
\]
Adding these up, we complete the proof. \hfill \square

4. Proof of Theorem 1.1

In this section we prove Theorem 1.1. By virtue of Theorem 3.4 it suffices to prove the statements for the lower energy part $W^-_{\text{low}} = W_\text{low}^\pi = W_\pi^\pi - \chi_{\leq \varepsilon}(|D|)$. By virtue of (3.2), $W^-_{\text{low}} = \chi_{\leq \varepsilon}(|D|)u + \Omega_{\text{low}}u$, where $\Omega_{\text{low}}u(x)$ is given by
\[
\frac{1}{\pi i} \int_0^\infty \chi_{\leq \varepsilon}(\lambda)\lambda(\tilde{\Gamma}(\lambda)G_{\lambda,Y}(x), \int_{\mathbb{R}^2} (G_{\lambda,Y}(y) - G_{-\lambda,Y}(y))u(y)dy)_{C_N} \lambda d\lambda
\]
and we have only to study $\Omega_{\text{low}}$. Recall $\tilde{\Gamma}(\lambda) = \Gamma(\lambda)^{-1}$. $\|u\|_p$ is the norm of $L^p(\mathbb{R}^2)$ and $\|u\|_2 = \|u\|_2$. $(u, v)$ is the coupling without complex conjugation. The inner product of $L^2$ will be denoted by $(u, v)$. Recall that the space $D_*$ defined by (3.1) is dense in $L^p(\mathbb{R}^2)$ for any $1 < p < \infty$. In what follows we always assume that $u \in D_*$. 

4.1. Proof of Statement (1)

If $S^*DS$ is non-singular in $S\mathbb{C}^N$, then Lemma 3.6 implies that $\tilde{\Gamma}(\lambda)\chi_{\leq \varepsilon}(\lambda)$ for a small $\varepsilon > 0$ is a good multiplier. The product formula (3.5) implies

$$(\Omega_{jk}\chi_{\leq \varepsilon}(|D|)u)(x) = \pi^2(K \circ \tilde{\Gamma}_{jk}(|D|)\chi_{\leq \varepsilon}(|D|))u(x)$$

and Lemmas 3.2 and 3.3 imply statement (1). This has been proved already in [4].

4.2. Proof of Statement (2)

Under the condition of statement (2), Lemma 3.7 is satisfied and we have (3.32) for $\lambda$ in the support of the function $\chi_{\leq \varepsilon}$. We write $B$ for $(T\hat{D}^2T)^{-1}$ and substitute (3.32) for $\tilde{\Gamma}(\lambda)$ in (4.1). The term $M(\lambda)$ produces a good operator as in the proof of statement (1) and we are left with

$$\frac{N}{i\pi} \int_0^\infty \chi_{\leq \varepsilon}(\lambda)\lambda g\langle TBTG_{\lambda,Y}(x), \int_{\mathbb{R}^2}(G_{\lambda,Y}(y) - G_{-\lambda,Y}(y))u(y)dy\rangle d\lambda. \quad (4.2)$$

For (4.2), we may still apply the product decomposition of Lemma 3.2, however, the multiplier $\chi_{\leq \varepsilon}(\lambda)g(\lambda)$ is not bounded near $\lambda = 0$ and Mikhlin’s theorem does not apply. To get around this we use the cancellation produced by the projector $T$ among the components of $G_{\lambda,Y}(y) - G_{-\lambda,Y}(y)$. Define the vector function $I_Y(\lambda)$ by

$$I_Y(\lambda) = \frac{1}{i\pi} \int_{\mathbb{R}^2} (G_{\lambda,Y}(y) - G_{-\lambda,Y}(y))u(y)dy = \int_{S^1} \left( \begin{array}{c} e^{iy_1\lambda} \\ \vdots \\ e^{iy_N\lambda} \end{array} \right) \hat{u}(\lambda\omega)d\omega. \quad (4.3)$$

Since $T = ST$, we may insert the orthogonal projection $S$ in front of $I_Y(\lambda)$ without changing (4.2). Since $NS = N - 1 \otimes 1$, we then have

$$NSI_Y u(\lambda) = \int_{S^1} \sum_{k=1}^N \left( e^{iy_1\omega\lambda} - e^{iy_k\omega\lambda} \right) \hat{u}(\lambda\omega)d\omega. \quad (4.3)$$

Recall that $R = \left( \begin{array}{c} R_1 \\ R_2 \end{array} \right)$ and $R_l = R_l(D) = F^{-1}(\xi_l/|\xi|)F$, $l = 1, 2$ are Riesz transforms and $\langle a, Ru \rangle = a_1R_1u + a_2R_2u$ for a vector $a \in \mathbb{R}^2$. Then, Taylor’s formula implies that the $j$-th component of (4.3) is equal to

$$i \sum_{k=1}^N \int_{S^1} \left( \int_0^1 e^{i(\theta y_j + (1-\theta)y_k)\lambda}(y_j - y_k, \omega)\lambda \hat{u}(\lambda\omega)d\omega \right)d\omega$$

$$= i\lambda \sum_{k=1}^N \int_{S^1} F \left( \langle y_j - y_k, R_{\tau_yu}(y_k, \lambda\omega) \rangle(\lambda\omega)d\omega \right)d\theta, \quad (4.4)$$

where $\tau_y u(x) = u(x - y)$ is the translation by $y$. Define

$$m(\lambda) = \lambda g(\lambda)\chi_{\leq \varepsilon}(\lambda). \quad (4.5)$$
Then, \(m(|\xi|)\) is a good multiplier, \(m(|D|)\) commutes with translations and the Riesz transform \(R\) and, we have
\[
m(\lambda)\mathcal{F}(\langle a, R\tau_y u \rangle)(\lambda \omega) = \mathcal{F}(\langle a, R\tau_y m(|D|)u \rangle)(\lambda \omega). \tag{4.6}
\]
Thus, if we define \((V_{jk} u)(x) = \int_0^1 \langle y_j - y_k, R\tau_{\theta y_j + (1-\theta)y_k} m(|D|)u \rangle d\theta\) for \(j, k = 1, \ldots, N\), then, \(V_{jk}\) are evidently good operators and (3.4) implies
\[
\int_0^\infty \chi_{\leq \varepsilon}(\lambda) \lambda g(\lambda) \langle TBTg_{\lambda, Y}(x), NSI_{Y} u(\lambda) \rangle d\lambda. m. \tag{4.2}
\]
This is a good operator and statement (2) is proved. \(\square\)

4.3. Proof of Statement (3)
We next assume that \(T_p \varphi_1(Y)T_p = G_1\) is non-singular in \(T_p \mathbb{C}^N\) and \(\Gamma(\lambda)^{-1}\) satisfies Lemma 3.10. We substitute \(3.50\) for \(\hat{\Gamma}(\lambda)\) in (4.1), which produces three operators. The proof of statements (1) and (2) implies that \(g(\lambda)SM(\lambda)S\) and \(M(\lambda)\) produce good operators. Define
\[
B_*(\lambda) = (G_1 + g(\lambda)^{-1}G_2)^{-1}.
\]
and write the integral produced by \(-N^{-1}g(\lambda)^{-1}\lambda^{-2}T_pB_*(\lambda)T_p\) in the form
\[
\hat{\Omega}_{\text{low}} u(x) = \int_0^\infty \chi_{\leq \varepsilon}(\lambda) \lambda^{-1} g(\lambda)^{-1} \langle T_pB_*(\lambda)T_pg_{\lambda, Y}(x), NSI_{Y} u(\lambda) \rangle_{\mathbb{C}^N} d\lambda, \tag{4.8}
\]
where unimportant constants are ignored. Recall that \(ST_p = T_pS = T_p\) and remark that we have inserted \(S\) in front of \(I_{Y} u(\lambda)\) which is allowed by the presence of \(T_p\). Notice the presence of strong singularities \(\lambda^{-1}g(\lambda)^{-1}\).

4.3.1. Decomposition into Good Part and Bad Parts. By further expanding the exponential functions, we decompose \(NSI_{Y} u(\lambda)\) into “good” and “bad” parts \(\hat{g}(\lambda)\) and \(\hat{b}(\lambda)\) as follows (sorry for using \(g\) to denote the good part and we hope that this causes no confusion):
\[
(NSI_{Y} u)(\lambda) = \hat{g}(\lambda) + \hat{b}(\lambda) = \begin{pmatrix} g_1(\lambda) \\ \vdots \\ g_N(\lambda) \end{pmatrix} + \begin{pmatrix} b_1(\lambda) \\ \vdots \\ b_N(\lambda) \end{pmatrix}, \tag{4.9}
\]
\[
g_j(\lambda) = \sum_{k=1}^N g_{jk}(\lambda), \quad b_j(\lambda) = \sum_{k=1}^N b_{jk}(\lambda) \quad j = 1, \ldots, N \tag{4.10}
\]
\[
g_{jk}(\lambda) \overset{\text{def}}{=} i \int_{\mathbb{S}^1} \left( \int_0^1 (e^{i(\theta y_j + (1-\theta)y_k)\omega \lambda} - 1)(y_j - y_k)\omega \lambda \hat{u}(\lambda \omega) d\theta \right) d\omega, \tag{4.11}
\]
\[ b_{jk}(\lambda) \overset{\text{def}}{=} i \int_{S^1} (y_j - y_k) \omega \lambda \hat{u}(\lambda \omega) d\omega \]  

(4.12)

Then, substituting (4.9) for \((NSI_Y u)(\lambda)\) in (4.8), we obtain

\[ \tilde{\Omega}_{\text{low}} u = \tilde{\Omega}_{\text{low},g} u + \tilde{\Omega}_{\text{low},b} u \]  

(4.13)

where definition of \(\tilde{\Omega}_{\text{low},g} u\) and \(\tilde{\Omega}_{\text{low},b} u\) should be obvious.

4.3.2. Good Part Produces a Good Operator.

**Lemma 4.1.** For \(1 < p < \infty\), there exists a constant \(C_p > 0\) such that

\[ \|\tilde{\Omega}_{\text{low},g} u\|_p \leq C_p \|u\|_p, \quad u \in D_*. \]  

(4.14)

**Proof.** Taylor’s formula implies

\[ e^{i(\theta y_j + (1 - \theta) y_k) \omega \lambda} - 1 = i \lambda \langle \theta y_j + (1 - \theta) y_k, \omega \rangle \cdot \int_0^1 e^{i \mu (\theta y_j + (1 - \theta) y_k) \omega \lambda} d\mu, \]

which produces the additional \(\lambda\) factor in (4.11) and \(g_{jk}(\lambda)\) becomes the integral \(\int_0^1 \int_{[0,1]^2} \tau_{\mu(\theta y_j + (1 - \theta) y_k)}(\langle y_k, R \rangle (y_j - y_k, R) + \theta (y_j - y_k, R)^2) u(x) d\theta d\mu.\)

Since \(\{\tau_y : y \in \mathbb{R}^2\}\) is uniformly bounded in \(B(L^p)\), the operator

\[ W_{jk} u(x) \overset{\text{def}}{=} \int_{[0,1]^2} \tau_{\mu(\theta y_j + (1 - \theta) y_k)}(\langle y_k, R \rangle (y_j - y_k, R) + \theta (y_j - y_k, R)^2) u(x) d\theta d\mu. \]

is a good operator. Define \(m(\lambda) = g(\lambda)^{-1} \chi_{\leq \epsilon}(\lambda)\). \(m(\lambda)\) is a good multiplier and we can express \(\tilde{\Omega}_{\text{low},g} u(x)\) is the form

\[ \sum_{j,k,l=1}^N \int_0^\infty \lambda(T_p B_*(\lambda) T_p)_{jk} \mathcal{G}_\lambda(x - y_k) \left( \int_{S^1} \mathcal{F}(W_{jl} m(|D|) u)(\lambda \omega) d\omega \right) d\lambda. \]

It follows by virtue of the Definition (3.4) of \(K\) that

\[ \tilde{\Omega}_{\text{low},g} u(x) = \pi^2 \sum_{j,k,l=1}^N \tau_{y_k}(K \circ W_{jl} m(|D|)(T_p B_*(|D|) T_p)_{jk}) u(x). \]  

(4.15)

Since \(B_*(\lambda)\) is a good multiplier under the assumption, Lemma 3.2 implies the lemma. \(\square\)
4.3.3. Decomposition of the Bad Part. We decompose \( \tilde{\Omega}_{\text{low},b}u(x) \) into the low and high energy parts:

\[
\tilde{\Omega}_{\text{low},b}u(x) = \int_{0}^{\infty} \lambda^{-1} \chi_{\leq \varepsilon}(\lambda) g(\lambda) \lambda^{-1} \langle T_p B_{\ast}(\lambda) T_p G_{\lambda,Y}(x), \hat{b}(\lambda) \rangle d\lambda \tag{4.16}
\]

\[
= \chi_{\geq 2\varepsilon}(|D|) \tilde{\Omega}_{\text{low},b}u(x) + \chi_{\leq 2\varepsilon}(|D|) \tilde{\Omega}_{\text{low},b}u(x). \tag{4.17}
\]

Note that supports of \( \chi \) and \( \chi_{\leq \varepsilon} \) do not intersect.

Lemma 4.2. For any \( \varepsilon > 0 \), \( \chi_{\geq 2\varepsilon}(|D|) \tilde{\Omega}_{\text{low},b} \) is bounded from \( L^p(\mathbb{R}^2) \) to itself for \( 1 < p \leq 2 \).

Proof. Denote \( \tilde{B}_{jk}(\lambda) = \langle T_p B_{\ast}(\lambda) T_p \rangle_{jk} \) and express \( \chi_{\geq 2\varepsilon}(|D|) \tilde{\Omega}_{\text{low},b}u(x) \) as the sum over \( j, k = 1, \ldots, N \) of

\[
X_{jk}u(x) \overset{\text{def}}{=} \int_{0}^{\infty} \lambda^{-1} \chi_{\leq \varepsilon}(\lambda) g(\lambda) \lambda^{-1} \tilde{B}_{jk}(\lambda) \tau_{y_j} \chi_{\geq 2\varepsilon}(|D|) G_{\lambda}(x) b_k(\lambda) d\lambda. \tag{4.18}
\]

Define \( \mu(\xi) = \chi_{\geq 2\varepsilon}(|\xi|)|\xi|^{-2} \). \( \mu \) is a good multiplier. By applying the inverse Fourier transform to \( \chi_{\geq 2\varepsilon}(\xi)(\xi^2 - \lambda^2)^{-1} = \mu(\xi) + \lambda^2 \mu(\xi)(\xi^2 - \lambda^2)^{-1} \) we have

\[
\chi_{\geq 2\varepsilon}(|D|) G_{\lambda}(x) = (2\pi)^{-1} \hat{\mu}(x) + \mu(|D|) \lambda^2 G_{\lambda}(x). \tag{4.19}
\]

We substitute (4.19) for the \( \chi_{\geq 2\varepsilon}(|D|) G_{\lambda}(x) \) in (4.18). The second summand \( \mu(|D|) \lambda^2 G_{\lambda}(x) \) cancels the singularity \( \lambda^{-1} \) and produces

\[
\sum_{l=1}^{N} \mu(D) \int_{0}^{\infty} \lambda G_{\lambda}(x - y_j) \left( \int_{\mathbb{S}^1} i \langle y_k - y_l, \omega \rangle \rho_{jk}(\lambda) \hat{u}(\lambda \omega) d\omega \right) d\lambda \tag{4.20}
\]

where \( \rho_{jk}(\lambda) = \lambda \chi_{\leq \varepsilon}(\lambda) g(\lambda) \lambda^{-1} \tilde{B}_{jk}(\lambda) \) is a good multiplier. By using \( K \) of (3.4), we may express (4.20) in the form

\[
i\pi^2 \sum_{l=1}^{N} \mu(D) \tau_{y_j} K \circ (\langle y_k - y_l, R \rangle \rho_{jk}(|D|)) u
\]

and Lemma 3.2 implies that this is a good operator.

The first summand \( (2\pi)^{-1} \hat{\mu}(x) \) of (4.19) produces for (4.18)

\[
\sum_{l=1}^{N} \int_{0}^{\infty} \chi_{\leq \varepsilon}(\lambda) g(\lambda) \lambda^{-1} \hat{\mu}(x - y_j) \tilde{B}_{jk}(\lambda) \left( \int_{\mathbb{S}^1} i \langle y_k - y_l, \omega \rangle \hat{u}(\lambda \omega) d\omega \right) \frac{d\lambda}{(2\pi)^2}. \tag{4.21}
\]

Note that \( \hat{\mu}(x) \) is \( \lambda \)-independent and that, if \( \xi = \lambda \omega \), \( \lambda > 0 \) and \( \omega \in \mathbb{S}^1 \) in the polar coordinates, then \( d\xi = \lambda d\lambda d\omega \). Thus, by using the Parseval formula, we may express

\[
(4.21) = \frac{1}{(2\pi)^2} \sum_{l=1}^{N} i \langle y_k - y_l, R \rangle u \langle \mathcal{F} \tilde{\rho}_{jk}, \langle y_k - y_l, R \rangle u \rangle_{L^2}, \tag{4.22}
\]

where \( \tilde{\rho}_{jk}(\xi) = |\xi|^{-1} \chi_{\leq \varepsilon}(|\xi|) g(|\xi|) \lambda^{-1} \tilde{B}_{jk}(|\xi|) \). It is evident that \( \tilde{\rho}_{jk} \in L^p(\mathbb{R}^2) \) for \( 1 \leq p \leq 2 \) and, we have \( \hat{\mu} \in L^p(\mathbb{R}^2) \) for \( 1 \leq p < \infty \) because \( \hat{\mu}(x) = \)
Because of the presence of $\sum_{j,k}$ and (4.21) is bounded in $N^2$, $K. Yajima Ann. Henri Poincaré$

Thus Hölder’s and Hausdorff-Young’s inequalities and $L^p$-boundedness of Riesz transforms imply for $1 < p \leq 2$ and $q = (p - 1)/p$ that

$$
\| \hat{\mu}(x - y_j) (F \hat{p}_{jk}, \langle y_k - y_l, R \rangle u) \|_p \\
\leq \| \hat{\mu} \|_p \| F \hat{p}_{jk} \|_q \| \langle y_k - y_l, R \rangle u \|_p \leq C \| \hat{\mu} \|_p \| F \hat{p}_{jk} \|_p \| u \|_p
$$

and (4.21) is bounded in $L^p(\mathbb{R}^2)$ for $1 < p \leq 2$. \hfill \Box

We next show that $\chi_{\geq 2\varepsilon}(|D|)\tilde{\Omega}_{\text{low,b}}$ is unbounded in $L^p(\mathbb{R}^2)$ for $2 < p < \infty$. In the proof of following lemma we shall ignore unimportant constants.

**Lemma 4.3.** Suppose $\varepsilon > 0$ is sufficiently small. Then, $\chi_{\geq 2\varepsilon}(|D|)\tilde{\Omega}_{\text{low,b}}$ is unbounded from $L^p(\mathbb{R}^2)$ to itself for any $2 < p < \infty$. Since $\chi_{\geq 2\varepsilon}(|D|)$ is bounded in $L^p(\mathbb{R}^2)$, it follows that $\tilde{\Omega}_{\text{low,b}}$ must be unbounded in $L^p(\mathbb{R}^2)$ for any $2 < p < \infty$.

**Proof.** By virtue of the proof of previous Lemma 4.2, it suffices to show that the sum over $j, k = 1, \ldots, N$ of (4.21) or (4.22) is unbounded in $L^p$ for $p > 2$. Introduce the notation: $\rho(\lambda) = \chi_{\leq \varepsilon}(\lambda)g(\lambda)^{-1}|\lambda|^{-1}$ and

$$
\hat{\mu}_Y(x) = \begin{pmatrix}
\hat{\mu}(x - y_1) \\
\vdots \\
\hat{\mu}(x - y_N)
\end{pmatrix}, \quad R_Y(\omega) = i \begin{pmatrix}
\langle y_1, \omega \rangle \\
\vdots \\
\langle y_N, \omega \rangle
\end{pmatrix}, \quad (4.24)
$$

Because of the presence of $T_p$ in $\tilde{B} = T_pB_*(\lambda)T_p$ which annihilates $P$, we have again with the vector notation and $\omega = \xi/|\xi|$ that

$$
\sum_{jkl} \langle 4.21 \rangle = \int_0^\infty \rho(\lambda) \langle (T_pB_*(\lambda)T_p)\hat{\mu}_Y(x), \hat{b}(\lambda) \rangle d\lambda
$$

$$
= \langle T_p\hat{\mu}_Y(x), \int_0^\infty \rho(\lambda)\lambda \left( \int_{S^1} B_*(\lambda)T_pR_Y(\omega)\hat{u}(\lambda \omega)d\omega \right) d\lambda \rangle
$$

$$
= \langle T_p\hat{\mu}_Y(x), \int_{\mathbb{R}^2} T_pB_*(|\xi|)T_pR_Y(\omega)\rho(|\xi|)\hat{u}(\xi) d\xi \rangle. \quad (4.25)
$$

Define the vector of linear functionals $\ell(u)$ by

$$
\ell(u) = \int_{\mathbb{R}^2} T_pB_*(|\xi|)T_pR_Y(\omega)\rho(|\xi|)\hat{u}(\xi) d\xi
$$

$$
= \int_{\mathbb{R}^2} F(T_pB_*(|\xi|)T_pR_Y(\omega)\rho(|\xi|))(y)u(y)dy \in \mathbb{C}^N \quad (4.26)
$$

where the second identity follows via the Parseval identity. Take an orthonormal basis $\{e_1, \ldots, e_n\}$ of $T_p\mathbb{C}^N$ such that $T_pG_1(Y)T_pe_j = \kappa_j e_j$, $j = 1, \ldots, n$, $n = \text{rank } T_p$ (we shall see $\kappa_j > 0$ shortly) so that

$$
\langle 4.25 \rangle = \sum_{j=1}^n \langle \hat{\mu}_Y(x), e_j \rangle \langle e_j, \ell(u) \rangle. \quad (4.27)
$$
As we shall see below, \( \langle e_1, \hat{\mu}_Y(x) \rangle, \ldots, \langle e_n, \hat{\mu}_Y(x) \rangle \) are linearly independent in \( L^p(\mathbb{R}^2) \). It follows by virtue of the Hahn-Banach theorem that, if \( (4.27) \) defines a bounded operator in \( L^p(\mathbb{R}^2) \), then it must be that for \( j = 1, \ldots, n \)

\[
\langle e_j, \ell(u) \rangle = \int_{\mathbb{R}^2} \mathcal{F}(\langle e_j, T_p B_*(|\xi|) T_p R_Y(\omega) \rangle \rho(|\xi|))(y) u(y) dy
\]

are bounded functionals on \( L^p(\mathbb{R}^2) \), hence via the Riesz representation theorem that \( \mathcal{F}(\langle e_j, T_p B_*(|\xi|) T_p R_Y(\omega) \rangle \rho(|\xi|))(y) \in L^q(\mathbb{R}^2) \). Then, Hausdorff-Young’s inequality implies \( \langle e_j, T_p B_*(|\xi|) T_p R_Y(\omega) \rangle \rho(|\xi|) \in L^p(\mathbb{R}^2) \) for \( j = 1, \ldots, n \) and

\[
T_p B_*(|\xi|) T_p R_Y(\omega) \rho(|\xi|) \in L^p(\mathbb{R}^2). \quad (4.28)
\]

However, \( T_p B_*(|\xi|) T_p \) is invertible in \( T_p \mathbb{C}^N \) for small \(|\xi| > 0\) and \( (T_p B_*(|\xi|) T_p)^{-1} \) is a good multiplier near zero. It follows from \( (4.28) \) that \( T_p R_Y(\omega) \rho(|\xi|) \in L^p(\mathbb{R}^2) \) and, hence

\[
\langle e_j, T_p R_Y(\omega) \rho(|\xi|) \rangle = i \sum_{l=1}^{N} \langle e_l^{(j)} y_l, \omega \rangle \rho(|\xi|) \in L^p(\mathbb{R}^2), \quad j = 1, \ldots, n, \quad (4.29)
\]

where \( e_l^{(j)} \) is the \( l \)-th component of \( e_j \). However, this is impossible because

\[
\kappa_j = \langle e_j, T_p G_1(Y) T_p e_j \rangle = -\frac{1}{4N} \sum_{k,l=1}^{N} |y_k - y_l|^2 e_k^{(j)} e_l^{(j)} = \frac{1}{2N} \left( \sum_{l=1}^{N} e_l^{(j)} y_l \right)^2 > 0,
\]

\[
(4.30)
\]

which implies \( \sum_{l=1}^{N} e_l^{(j)} y_l \neq 0 \) and because for any \( a \in \mathbb{R}^{2} \setminus \{0\} \) and \( \varepsilon > 0 \)

\[
\int_{|\xi| < \varepsilon} \frac{|\langle a, \omega \rangle|^p}{|\xi|^p |g(|\xi|)|^p} d\xi = |a| \left( \int_{0}^{\pi} |\cos \theta|^p d\theta \right) \left( \int_{0}^{\varepsilon} \frac{dr}{r^{p-1} |g(r)|^p} \right) = \infty.
\]

To see \( \langle e_1, \hat{\mu}_Y(x) \rangle, \ldots, \langle e_n, \hat{\mu}_Y(x) \rangle \) are linearly independent, we first note that, if \( y_j \neq y_k \) for \( j \neq k \), any non-trivial linear combination of \( \hat{\mu}(x - y_1), \ldots, \hat{\mu}(x - y_N) \) does not vanish because

\[
\mathcal{F}(c_1 \hat{\mu}(x - y_1) + \cdots + c_N \hat{\mu}(x - y_N))(\xi) = |\xi|^{-2} \chi_{\geq 2\varepsilon}(\xi) \sum_{j=1}^{N} c_j e^{iy_j \xi} = 0
\]

will imply \( \sum_{j=1}^{N} c_j e^{iy_j \xi} = 0 \) for \( |\xi| \geq 2\varepsilon \) and hence \( c_1 = \cdots = c_N = 0 \). Thus, if \( \sum_{j=1}^{n} c_j \langle e_j, \hat{\mu}_Y(x) \rangle = 0 \), then \( \sum_{j=1}^{n} c_j e_j = 0 \) and \( c_1 = \cdots = c_n = 0 \). This completes the proof. \( \Box \)

The proof of statement (3) will be finished if we have proven the following lemma.

**Lemma 4.4.** Operator \( \chi_{\leq 2\varepsilon}(|D|) \tilde{\Omega}_{\text{low},b} \) is bounded in \( L^p(\mathbb{R}^2) \) if \( 1 < p < 2 \).

**Proof.** The proof uses the fact that \( \tilde{B}(\lambda) = T_p B_*(\lambda) T_p \) has the factor \( T_p \) also on the right. \( \chi_{\leq 2\varepsilon}(|D|) \tilde{\Omega}_{\text{low},b}(x) \) may be expressed in the form

\[
\int_{0}^{\infty} \int_{S^1} \chi_{\leq \varepsilon}(\lambda) g(\lambda)^{-1} \langle \tilde{B}(\lambda) \chi_{\leq 2\varepsilon}(|D|) G_{\lambda,Y}(x), R_Y(\omega) \rangle \hat{u}(\lambda \omega) d\lambda d\omega \quad (4.31)
\]
Recall that we are assuming $u \in D_*$ and $(\xi^2 - \eta^2 - i0)^{-1}$ in (4.32) is well defined as the limit $\kappa \downarrow 0$ of $(\xi^2 - \eta^2 - i\kappa)^{-1}$. Hereafter in the proof we write $\rho(jk) = \chi_{\leq \varepsilon}B(jk)\rho(jk)g(\lambda)^{-1}$. $\rho(jk)$ is a good multiplier.

Since $\tilde{B} = T_pB_*T_p$ and $T_p$ annihilates $1$ we may replace $G_{\lambda,Y}$ in (4.31) by $G_{\lambda,Y}(x) - N^{-1}(\sum_{l=1}^{N} G_{\lambda}(x - y_l))1$ without changing the result. However, this changes $e^{-iy_j\xi}$ in (4.32) to

$$\frac{1}{N}\sum_{l=1}^{N}(e^{-iy_j\xi} - e^{-iy_l\xi}) = \frac{-i}{N}\sum_{l=1}^{N}\int_{0}^{1} e^{-i\xi(\theta y_j + (1-\theta) y_l)}d\theta \cdot (y_j - y_l)\xi. \quad \text{(4.33)}$$

Then, (4.32) becomes the sum over $j, k, l = 1, \ldots, N$ of

$$U_{jl}\int_{\mathbb{R}^{6}}\chi_{\leq 2\varepsilon}(|\xi|)\rho(jk)(|\eta|)\frac{(y_j - y_l)\cdot \xi}{(\xi^2 - \eta^2 - i0)|\eta|} e^{ix\xi}(y_k, \hat{\eta})\hat{u}(\eta)d\eta d\xi, \quad \text{(4.34)}$$

where $U_{jl} = N^{-1}\int_{0}^{1} \tau_{\theta y_j + (1-\theta) y_l}d\theta$ is a good operator. Since $(y_j - y_l)\xi = (y_j - y_l)\hat{\xi}|\eta| + (|\xi| - |\eta|)$, $\hat{\xi} = \xi/|\xi|$, we have

$$\frac{(y_j - y_l)\xi}{\xi^2 - \eta^2 - i0} = (y_j - y_l)\hat{\xi} \cdot \frac{|\eta|}{\xi^2 - \eta^2 - i0} + (y_j - y_l)\hat{\xi} \cdot \frac{1}{|\xi| + |\eta|}, \quad \text{(4.35)}$$

which we use in the integral of (4.34). Then the first term yields

$$\langle y_j - y_l, R_x \rangle \int_{\mathbb{R}^{4}} e^{ix\xi}(\chi_{\leq 2\varepsilon}(|\xi|))\rho(jk)(|\eta|)\langle y_k, \hat{\eta} \rangle\hat{u}(\eta)d\eta d\xi \quad \text{(4.36)}$$

If we integrating with respect $\xi$ first and use the polar coordinate $\eta = \lambda \omega$, (4.36) becomes

$$\langle y_j - y_l, R_x \rangle \chi_{\leq 2\varepsilon}(|D|) \int_{0}^{\infty} \lambda G_{\lambda}(x) \left(\int_{\mathbb{S}^{1}} \mathcal{F}(\rho(jk)(|D|)\langle y_k, R \rangle u)(\lambda \omega)d\omega \right) d\lambda = \langle y_j - y_l, R_x \rangle \chi_{\leq 2\varepsilon}(|D|)K \circ (\rho(jk)(|D|)\langle y_k, R \rangle u)(x). \quad \text{(4.37)}$$

This is a good operator since $\rho(jk)(|\xi|)$ is a good multiplier. The operator produced by the second term may be expressed as follows:

$$\langle y_j - y_l, R_x \rangle \int_{\mathbb{R}^{2}} \left(\int_{\mathbb{R}^{4}} e^{ix\xi - i\eta y} \chi_{\leq 2\varepsilon}(|\xi|)\chi_{\leq 2\varepsilon}(|\eta|) d\eta d\xi \right) (\tilde{\rho}_{jk}(D)u)(y) dy$$

where $\tilde{\rho}_{jk}(D) = \tilde{B}(jk)(|D|)\chi_{\leq \varepsilon}(|D|)\langle y_k, R \rangle$ is a good operator. Recall that $\chi_{\leq \varepsilon}(\lambda) = \chi_{\leq \varepsilon}(\lambda)\chi_{\leq 2\varepsilon}(\lambda)$. Thus, the proof of Lemma 4.4 will be completed if we have proven the following lemma. \hfill \Box

**Lemma 4.5.** Let $L(x, y)$ be defined by

$$L(x, y) = \int_{\mathbb{R}^{4}} e^{ix\xi - i\eta y} \chi_{\leq 2\varepsilon}(|\xi|)\chi_{\leq 2\varepsilon}(|\eta|) d\eta d\xi. \quad \text{(4.38)}$$
Then, the integral operator
\[ Lu(x) = \int_{\mathbb{R}^2} L(x, y)u(y)\,dy \]
(4.39)
is bounded in \( L^p(\mathbb{R}^2) \) for \( 1 < p < 2 \).

Proof. The Fourier transform of \( \chi_{\leq 2\varepsilon}(|\xi|)\chi_{\leq 2\varepsilon}(|\eta|)(|\xi| + |\eta|)^{-1} \) is smooth and, in virtue of Lemma B of Appendix it is bounded by \( C\langle x \rangle^{-1}\langle y \rangle^{-1}\langle x + y \rangle^{-1} \).

It follows by virtue of Lemma 3.14 that
\[ L(x, y) \leq |x|C \int_{\mathbb{R}^2} \frac{C}{\langle x - y \rangle \langle x \rangle \langle y \rangle \log(1 + \langle y \rangle)}\,dy' \]
Then, by using Minkowski’s inequality twice we have for \( 1 < p < 2 \) that
\[
\begin{align*}
\left\| \int_{\mathbb{R}^2} L(x, y)|f(y)|\,dy \right\|_p &\leq \int_{\mathbb{R}^2} \left( \int_{\mathbb{R}^2} |L(x, y)|^pdx \right)^{1/p} |f(y)|\,dy \\
&\leq C \int_{\mathbb{R}^2} \left( \int_{\mathbb{R}^2} \frac{dx}{\langle x \rangle^p \langle x + y - y' \rangle^p} \right)^{1/p} |f(y)|\,dy' \\
&\leq C \int_{\mathbb{R}^2} \left( \int_{\mathbb{R}^2} \frac{dy'}{(y - y')^{3-\frac{2}{p}} \langle y' \rangle \log(1 + \langle y' \rangle)} \right) |f(y)|\,dy
\end{align*}
(4.40)

We show for the dual exponent \( q = p/(p-1) \) of \( 1 < p < 2 \),
\[ Q(y) \overset{\text{def}}{=} \int_{\mathbb{R}^2} \frac{dy'}{(y - y')^{3-\frac{2}{p}} \langle y' \rangle \log(1 + \langle y' \rangle)} \in L^q(\mathbb{R}^2), \]
(4.41)
which will prove that the right of (4.40) is bounded by \( \|Q\|_q\|f\|_p \) by Hölder’s inequality and which will complete the proof. Since \( p > 1, 3 - \frac{2}{p} > 1 \), Schwarz’s inequality implies
\[ |Q(y)| \leq \|\langle y \rangle^{-3+\frac{2}{p}}\|_2\|\langle y' \rangle \log(1 + \langle y' \rangle)^{-1}\|_2 < \infty \]
and \( Q(y) \in L^\infty(\mathbb{R}^2) \). Thus, we need show \( Q \in L^q(|y| > R) \) for a large \( R = 100 \) and we assume \( |y| > 100 \) in what follows in the proof. We split \( \mathbb{R}^2 \) into three regions \( D_1 = \{ y' : |y' - y| \leq |y|/2 \} \), \( D_2 = \{ y' : |y' - y| > |y|/2 \text{ and } |y'| \leq 2|y| \} \) and \( D_3 = \{ y' : |y'| > 2|y| \} \) so that
\[ Q(y) = \left( \int_{D_1} + \int_{D_2} + \int_{D_3} \right) \frac{(y - y')^{\frac{2}{p} - 3}dy'}{\langle y' \rangle \log(1 + \langle y' \rangle)} \overset{\text{def}}{=} A(y) + B(y) + C(y). \]

On \( D_1 \) we have \( |y|/2 < |y'| < 2|y| \) and \(-1 < \frac{2}{p} - 2 < 0 \) if \( 1 < p < 2 \). Hence
\[ A(y) \leq \frac{C}{\langle y \rangle \log(1 + \langle y \rangle)} \int_{0}^{\frac{|y|}{2}} \langle r \rangle^{\frac{2}{p} - 3}rdr \leq \frac{C\langle y \rangle^{\frac{2}{p} - 2}}{\log(1 + \langle y \rangle)} \in L^q(|y| > 100) \]
since \( q \left( \frac{2}{p} - 2 \right) = -2 \) and \( q > 2 \). For \( B(y) \) we have
\[ B(y) \leq \frac{C}{\langle y \rangle^{3-\frac{2}{p}}} \int_{0}^{2|y|} \frac{rdr}{\langle r \rangle \log(2 + r)} \leq \frac{C}{\langle y \rangle^{3-\frac{2}{p}}} \left( C + \int_{e^2}^{2|y|} \frac{dr}{\log r} \right) \]
Here, integration by parts shows
\[ \int_{e^2}^{2|y|} \frac{dr}{\log r} = \frac{r}{\log r} \bigg|_{e^2}^{2|y|} + \int_{e^2}^{2|y|} \frac{dr}{(\log r)^2} \leq \frac{2|y|}{\log 2|y|} + \frac{1}{2} \int_{e^2}^{2|y|} \frac{dr}{\log r} \]
and the integral is bounded by \(4|y|/(\log 2|y|)^{-1}\). It follows once more that
\[ B(y) \leq \frac{C}{\langle y \rangle^{2-\frac{2}{p}}} \in L^q(\{|y| > 100\}) \]
Finally, as \(|y'| > 2|y|\) implies \(|y - y'| > |y'|/2\) and \(3 - \frac{2}{p} > 1\)
\[ C(y) \leq \int_{2|y|}^{\infty} \frac{rdr}{\langle r \rangle^{4-\frac{2}{p}}} \log(1 + \langle r \rangle) \leq \frac{C}{\langle y \rangle^{2-\frac{2}{p}}} \in L^q(\{|y| > 100\}). \]
Thus, \(Q \in L^q(\mathbb{R}^2)\) as desired and the lemma is proved. \(\square\)

4.4. Proof of Statement (4)

We use the notation of Lemma 3.11. By virtue of Lemma 3.11 under the assumption of statement (4) \(\Gamma(\lambda)^{-1}\) is equal to
\[
-N^{-1}\lambda^{-2}T_e(T_eG_2T_e)^{-1}T_e - N^{-1}g(\lambda)^{-1}\lambda^{-2}T_p(T_e^\perp - B^*)T_e^\perp D_0(\lambda)T_e^\perp(T_e^\perp - B)T_p + g(\lambda)^3T_pMT_p + g(\lambda)^2SMS + g(\lambda)SM + g(\lambda)MS + M. \tag{4.42}
\]
We substitute (4.42) for \(\bar{\Gamma}(\lambda)\) in (4.1), which produces seven operators. The one produced by \(M(\lambda)\) is a good operator by the proof of statement (1); those produced by \(g(\lambda)SM\), \(g(\lambda)^2SMS\) and \(g(\lambda)^3T_pMT_p\) which have the factor \(S\) on the left are also good operators. This can be seen by repeating the proof statement (2) by observing that (i) out of two \(T\)'s in \(TBT\) of (4.2) the one on the left is used for introducing \(S\) in front of \(I_Y(\lambda)\), which produces (4.4) with the helpful factor \(\lambda\) and that (ii) \(\lambda g(\lambda)^j\chi_{\leq \varepsilon}(\lambda)M(\lambda)\) is a good multiplier for any \(j \in \mathbb{N}\) and it can play the role of \(m(\lambda)\) of (4.5).

For the operator produced by \(g(\lambda)M(\lambda)S\) which contains the singularity of \(g(\lambda)\) and the factor \(S\) on the right we have the following lemma.

Lemma 4.6. The operator \(\Omega_{rs}\) defined by (4.1) with \(g(\lambda)M(\lambda)S\) in place of \(\bar{\Gamma}(\lambda)\) is a good operator.

Proof. \(\Omega_{rs}u(x)\) is given by
\[
\frac{1}{\pi i} \int_0^\infty \chi_{\leq \varepsilon}(\lambda)g(\lambda)\lambda\langle M(\lambda)S\rangle_{\lambda,Y}(x), \int_{\mathbb{R}^2} (\mathcal{G}_{\lambda,Y}(y) - \mathcal{G}_{-\lambda,Y}(y))u(y)dy \rangle_{C^N}d\lambda.
\]
We split \(\Omega_{rs}u\) into the high and the low energy part as follows:
\[
\Omega_{rs}u = \chi_{\geq 2\varepsilon}(|D|)\Omega_{rs}u + \chi_{\leq 2\varepsilon}(|D|)\Omega_{rs}u.
\]
(1) We first prove that \(\chi_{\geq 2\varepsilon}(|D|)\Omega_{rs}u\) is a good operator. Since the components of \(M(\lambda)S\) are good multipliers for small \(\lambda > 0\) and translations are good operators, it suffices to show that the operator \(Zu(x)\) defined by
\[
Zu(x) = \int_0^\infty \chi_{\leq \varepsilon}(\lambda)\lambda g(\lambda)\chi_{\geq 2\varepsilon}(|D|)\mathcal{G}_{\lambda}(x) \left(\int_{\mathbb{S}} \hat{u}(\lambda \omega) \omega \right) d\lambda \tag{4.43}
\]

\[ (\mathcal{G}_{\lambda,Y}(y) - \mathcal{G}_{-\lambda,Y}(y))u(y)dy \]
is a good operator. We repeat the argument of the proof of Lemma 4.2: Substitute (4.19) for \( \chi_{\geq 2\varepsilon}(|D|)G\lambda(x) \) in (4.43), which produces two integrals. The one produced by \( \hat{\mu}(x) \) is equal to
\[
\hat{\mu}(x) \int_{0}^{\infty} \chi_{\leq \varepsilon}(\lambda)\lambda g(\lambda) \left( \int_{\mathbb{R}} (\mathcal{F}u)(\lambda\omega)d\omega \right) d\lambda = \hat{\mu}(x)\langle \mathcal{F}(\chi_{\leq \varepsilon}g(\lambda)), u \rangle.
\]
By virtue of (4.23) we have \( \hat{\mu} \in L^p(\mathbb{R}^2) \) for all \( 1 \leq p < \infty \) and
\[
|\mathcal{F}(\chi_{\leq \varepsilon}(\lambda)g(|\xi|))(x)| = |(2\pi)^{-1}(\hat{g} \ast \hat{\chi}_\varepsilon)(x)| \leq C\langle x \rangle^{-2}.
\]
Hence, (4.44) is a good operator. Define \( \tilde{\mu} \) as a good multiplier and the operator produced by \( \mu(|D|)\lambda^2G\lambda(x) \) of (4.19) is equal to
\[
\mu(|D|) \int_{0}^{\infty} \lambda G\lambda(x) \left( \int_{\mathbb{R}} \mathcal{F}(\mu(|D|)u)(\lambda\omega)d\omega \right) d\lambda = \mu(|D|)K\hat{\mu}(|D|)u
\]
(modulo a constant factor), which is a good operator. Thus, \( \chi_{\geq 2\varepsilon}(D_x)\Omega_{rs} \) is a good operator.

(2) We next show \( \chi_{\leq 2\varepsilon}(D_x)\Omega_{rs} \) is also a good operator. The proof below resembles the one of Lemma 4.4. Recall \( \tau_Yu(x) \) of (1.15). We have
\[
\chi_{\leq 2\varepsilon}(D_x)\Omega_{rs}u(x) = \int_{0}^{\infty} \chi_{\leq \varepsilon}(\lambda)\lambda g(\lambda)\left( M(\lambda)S\chi_{\leq 2\varepsilon}(|D|)G\lambda, Y(x), \mathcal{F}(\tau_Yu)(\lambda\omega)d\omega \right)_{CN} d\lambda.
\]
Here, \( S\chi_{\leq 2\varepsilon}(|D|)G\lambda, Y(x) \) is a vector whose \( j \)-th component is given by
\[
K_j(x, \lambda) = \frac{1}{N} \sum_{k=1}^{N} \frac{1}{2\pi} \int_{\mathbb{R}^2} \chi_{\leq 2\varepsilon}(|\xi|) e^{ix\xi}(e^{-iy_j\xi} - e^{-iy_k\xi}) \frac{\xi^2 - \lambda^2 - i\theta}{\xi^2} d\xi.
\]
Using Taylor’s formula, write \( e^{-iy_j\xi} - e^{-iy_k\xi} \) in the form
\[
-i(y_j - y_k)\xi + |\xi|^2 \int_{[0,1]^2} e^{-i\mu(\theta y_j + (1-\theta)y_k)}\xi(y_j - y_k, \hat{\xi})(\theta y_j + (1-\theta)y_k)\xi d\theta d\mu
\]
and define two functions \( L^{(1)}_\lambda(x) \) and \( L^{(2)}_\lambda(x) \) by
\[
L^{(1)}_\lambda(x) = \frac{1}{2\pi} \int_{\mathbb{R}^2} \chi_{\leq 2\varepsilon}(|\xi|) e^{ix\xi}|\xi|^2 d\xi, \quad L^{(2)}_\lambda(x) = \frac{1}{2\pi} \int_{\mathbb{R}^2} \chi_{\leq 2\varepsilon}(|\xi|) e^{ix\xi}|\xi|^2 d\xi.
\]
Then, \( K_j(x, \lambda) \) may be expressed as a sum
\[
K_j(x, \lambda) = \frac{1}{N} \sum_{k=1}^{N} (-i(y_j - y_k, R_x)L^{(1)}_\lambda(x) + U_{jk}L^{(2)}_\lambda(x)).
\]
Here, \( R_x = (R_{x_1}, R_{x_2}) \) is the Riesz transform and for \( j, k = 1, \ldots, N \)
\[
U_{kj} = \int_{0}^{1} \int_{0}^{1} \tau_{\mu(\theta y_k + (1-\theta)y_j)}(y_k - y_j, R_x)(\theta y_k + (1-\theta)y_j, R_x)d\mu d\theta.
\]
Let $M(\lambda) = (m_{jk}(\lambda))$. Then $\chi_{\leq 2\varepsilon(|D|)} \Omega_{rs} u(x)$ is equal to

$$
\sum_{j,k=1}^{N} \int_{0}^{\infty} \chi_{\leq \varepsilon(\lambda)} \lambda g(\lambda) K_{j}(x,\lambda) \left( \int_{\mathbb{R}} \mathcal{F}(m_{jk}(|D|) \tau_{y_{k}} u)(\lambda \omega) d\omega \right) d\lambda
$$

and $\langle y_{j} - y_{k}, R_{x} \rangle, U_{k_{j}}, m_{jk}(|D|)$ and $\tau_{y_{j}}$ are all good operators. Thus, we need only prove that $Q_{1}$ and $Q_{2}$ defined by

$$
Q_{j} u(x) = \int_{0}^{\infty} \chi_{\leq \varepsilon(\lambda)} \lambda g(\lambda) L_{\lambda}^{(j)}(x) \left( \int_{\mathbb{R}} \check{u}(\lambda \omega) d\omega \right) d\lambda, \quad j = 1, 2
$$

are good operators. As in (4.35) we have the identity:

$$
L_{\lambda}^{(1)}(x) = \lambda \chi_{\leq 2\varepsilon(|D|)} G_{\lambda}(x) + \frac{1}{2\pi} \int_{\mathbb{R}^{2}} \chi_{\leq 2\varepsilon(|\xi|)} e^{ix\xi} d\xi.
$$

Define $\nu(\lambda) = \lambda \chi_{\leq \varepsilon(\lambda)} g(\lambda)$. Then $\nu(\lambda)$ is a good multiplier and we have

$$
Q_{1} u(x) = \chi_{\leq 2\varepsilon(|D|)} (K \circ \nu(|D|)) u(x) + \check{L} \chi_{\leq \varepsilon(|D|)} u(x),
$$

where $\check{L}$ is the integral operator with the integral kernel

$$
\check{L}(x,y) = \int_{\mathbb{R}^{4}} e^{i\xi x - i\eta y} \frac{\chi_{\leq 2\varepsilon(|\eta|)} \chi_{\leq 2\varepsilon(|\eta|)} g(|\eta|)}{4\pi^{2}(|\xi| + |\eta|)} d\eta d\xi.
$$

It is evident that $\chi_{\leq 2\varepsilon(|D|)} K \circ (\nu(|D|)) u(x)$ is a good operator and the proof of Lemma 4.5 implies $\check{L}$ is a also good operator. Indeed, by using (4.45) and Lemma B in Appendix we obtain

$$
|\check{L}(x,y)| \leq C \int_{\mathbb{R}^{2}} \frac{dy'}{\langle x \rangle \langle y - y' \rangle \langle x \rangle + \langle y - y' \rangle \langle y' \rangle^{2}}.
$$

Then, the argument which led to (4.40) implies

$$
||\check{L}f||_{p} \leq C \int_{\mathbb{R}^{2}} \left( \int_{\mathbb{R}^{2}} \frac{dy'}{(y - y')^{2}} \right) |f(y)| dy.
$$

Then, Young’s inequality implies that the function in the parentheses is in $L^{q}(\mathbb{R}^{2})$ for $q = p/(p - 1)$. Thus, Hölder’s inequality implies that $\check{L}$ and, hence $Q_{1}$ is a good operator.

We have

$$
L_{\lambda}^{(2)}(x) = \chi_{\leq 2\varepsilon} \lambda^{2} x \leq 2\varepsilon(|D|) G_{\lambda}(x).
$$

Define $\mu(\xi) = \chi_{\leq \varepsilon(|\xi|)} g(|\xi|)$ and $\check{\mu}(\xi) = |\xi|^{2} \mu(\xi)$. Then $Q_{2} u(x)$ is expressed in the form

$$
Q_{2} u(x) = \chi_{\leq 2\varepsilon} \left( \int_{\mathbb{R}^{2}} \check{\mu}(y) u(y) dy \right) + \chi_{\leq 2\varepsilon(|D|)} K \circ \check{\mu}(|D|) u.
$$

Here, $\check{\mu} \in L^{p}(\mathbb{R}^{2})$ for any $1 < p < \infty$ and $\check{\mu}$ is a good multiplier. Thus, $Q_{2}$ is also a good operator and the proof of Lemma 4.6 is completed.

**Lemma 4.7.** Let $\Omega_{e}$ be defined by (4.1) with $-N^{-1} \lambda^{-2} T_{e} (T_{e} G_{2} T_{e})^{-1} T_{e}$ in place of $\check{\Gamma}(\lambda)$. Then, $\Omega_{e}$ is a good operator.
Proof. Write $H$ for $(T_e G_2 T_e)^{-1}$ to shorten formulas. Let $\{e_1, \ldots, e_n\}$ be an orthonormal basis of $T_e \mathbb{C}^N$, $n = \text{rank } T_e$ and $H_{lm} = \langle e_l, H e_m \rangle$, $l, m = 1, \ldots, n$. Then, $\Omega_e u(x)$ is equal to

$$\frac{-1}{N} \int_0^\infty \chi_{\leq \varepsilon}(\lambda) \lambda^{-1} \langle HT_e G_{\lambda, y}(x), \int_{S^1} T_e \tau_y \hat{u}(\lambda \omega) d\omega \rangle_{\mathbb{C}^N} d\lambda$$

$$= \sum_{lm} \frac{-1}{N} \int_0^\infty \chi_{\leq \varepsilon} \lambda^{-1} H_{lm} \langle e_m, G_{\lambda, y}(x) \rangle \left( \int_{S^1} \langle e_l, \tau_y \hat{u}(\lambda \omega) \rangle_{\mathbb{C}^N} d\omega \right) d\lambda$$

(4.51)

Since $S e_l = e_l$, we may replace in (4.51) $\tau_y \hat{u}(\lambda \omega)$ by $S \tau_Y \hat{u}(\lambda \omega)$ whose $j$-th component is given by $N^{-1} \sum_{k=1}^N (e^{iy_j \lambda \omega} - e^{iy_k \lambda \omega}) \hat{u}(\lambda \omega)$. As previously we have

$$e^{iy_j \lambda \omega} - e^{iy_k \lambda \omega} = -i \langle y_j - y_k, \lambda \omega \rangle + \lambda^2 v_{jk}(\lambda \omega),$$

$$v_{jk}(\lambda \omega) = \int_{[0,1]^2} e^{-i\mu(y_j+(1-\theta)y_k)\lambda \omega} \langle y_j - y_k, \omega \rangle \langle \theta y_j + (1-\theta)y_k, \omega \rangle d\theta d\mu.$$

Since $e_l = (e^{(l)}_1, \ldots, e^{(l)}_N) \in T_e \mathbb{C}^N$, we have $\sum_{j=1}^N e^{(l)}_j = 0$ and

$$0 = \langle G_1(Y) e_l, e_l \rangle = -\frac{1}{4N} \sum_{k,l=1}^N \|y_j - y_k\|^2 e^{(l)}_j e^{(l)}_k = \frac{1}{2N} \left( \sum_{j=1}^N e^{(l)}_j y_j \right)^2 = 0.$$

It follows that $\sum_{j,k=1}^N e^{(l)}_j (y_j - y_k) = 0$ and

$$\langle e_l, \tau_Y \hat{u}(\lambda \omega) \rangle = \frac{\lambda^2}{N} \sum_{j,k=1}^N e^{(l)}_j v_{jk}(\lambda \omega) \hat{u}(\lambda \omega) = \frac{\lambda^2}{N} \mathcal{F}(\mathcal{V}_l u)(\lambda \omega)$$

(4.52)

where $\mathcal{V}_l$ is a good operator defined by

$$V_l u(x) = \sum_{j,k=1}^N e^{(l)}_j \int_{[0,1]^2} \tau_{\mu(\theta y_j + (1-\theta)y_k)}(y_j - y_k, R) \langle \theta y_j + (1-\theta)y_k, R \rangle u(x).$$

Thus, $\Omega_e u(x)$ is equal to

$$\frac{-1}{N^2} \sum_{lmn} \int_0^\infty \chi_{\leq \varepsilon}(\lambda) \lambda H_{lm} e^{(m)}_n G_{\lambda}(x - y_n) \left( \int_{S^1} \mathcal{F}(\mathcal{V}_l u)(\lambda \omega) d\omega \right) d\lambda$$

$$= \frac{-1}{N^2} \sum_{lmn} H_{lm} e^{(m)}_n \tau_{y_n} \mathcal{C} (\chi_{\leq \varepsilon}(|D|) \mathcal{V}_l u)(x)$$

and, is a good operator. □

The next lemma completes the proof of statement (4) of Theorem 1.1. To shorten the formulas we write

$$B_{ss}(\lambda) = -\frac{1}{iN\pi} (T_e \perp B^*) T_e \perp D_0(\lambda) T_e \perp (T_e \perp B).$$

Recall that $B = T_e \perp G_2 T_e (T_e G_2 T_e)^{-1} T_e$. 

Lemma 4.8. Let $\Omega_{md}$ be defined by (4.1) with $g(\lambda)^{-1}\lambda^{-2}T_pB_{**}(\lambda)T_p$ in place of $\Omega(\lambda)$. Then, $\Omega_{md}$ is bounded in $L^p(\mathbb{R}^2)$ for $1 < p \leq 2$ but is unbounded for $2 < p < \infty$.

Proof. By the definition $\Omega_{md}u(x)$ is equal to

$$\int_0^\infty \chi_{\leq \varepsilon}(\lambda)g(\lambda)^{-1}\lambda^{-1}\langle T_pB_{**}(\lambda)T_pG_{\lambda,Y}(x), i\pi SI_Y u(\lambda)\rangle_{CN} d\lambda \quad (4.53)$$

Then, we repeat the argument of Sect. 4.3 for $\tilde{\Omega}_{low}$ of (4.8) replacing $B_*$ by $-i\pi NB_{**}$. Replacing $SI_Y u(\lambda)$ by $N^{-1} \times (4.9)$, we decompose $\Omega_{md} = \Omega_{md,g} + \Omega_{md,b}$ as in (4.13).

(i) Proofs of Lemmas 4.1 and 4.2, respectively, imply without any more change that $\Omega_{md,g}$ is a good operator and that $\chi_{\geq 2\varepsilon(|D|)}\Omega_{md,b}$ is bounded in $L^p(\mathbb{R}^2)$ for $1 < p < 2$.

(ii) The proof of Lemma 4.4 implies that $\chi_{\leq 2\varepsilon(|D|)}\Omega_{md,b}$ is bounded in $L^p(\mathbb{R}^2)$ for $1 < p < 2$.

(i) and (ii) should be obvious because the only property of $B_*$ used in the proof of these lemmas is that $T_pB_{**}(\lambda)T_p$ is a good multiplier which is shared by $T_pB_{**}(\lambda)T_p$.

(iii) We prove that $\chi_{\geq 2\varepsilon(|D|)}\Omega_{md,b}$ is unbounded $L^p(\mathbb{R}^2)$ for $2 < p < \infty$ by slightly modifying the argument of the proof of Lemma 4.3 as follows:

Let $l = \text{rank} T^\perp_e G_1 T^\perp_e$ and take an orthonormal basis $\{e_1, \ldots, e_n\}$ of $T_pC^N$ such that $e_1, \ldots, e_l$ are eigenvectors of $T^\perp_e G_1 T^\perp_e$ with positive eigenvalues and $e_{l+1}, \ldots, e_n \in T_eC^N$. Then, the argument of the proof of Lemma 4.3 implies that, if $\chi_{\leq 2\varepsilon(|D|)}\Omega_{md,b}$ is bounded in $L^p(\mathbb{R}^2)$ for $2 < p < \infty$, then it must be that $B_{**}(|\xi|)^* R_Y(\omega)\rho(|\xi|) \in L^p(\mathbb{R}^2)$. However, (4.30) implies $\sum_{j=1}^N e_j^{(k)} y_j = 0$ for $e_k \in T_eC^N$, $k = l+1, \ldots, n$ and hence $T_eR_Y(\omega) = 0$. It follows

$$B_{**}(|\xi|)^* R_Y(\omega)\rho(|\xi|) = (T^\perp_e - B^*)T^\perp_e D_0(\lambda)T^\perp_e R_Y(\omega)\rho(|\xi|) \quad (4.54)$$

and images of $B^* = T_e(T_2T_2T_e)^{-1}T_2T_2T_e$ and $T^\perp_e$ are orthogonal to each other. Thus, we must have $T^\perp_e D_0(\lambda)T^\perp_e R_Y(\omega)\rho(|\xi|) \in L^p(\mathbb{R}^2)$ and, since $T_eD_0(\lambda)T^\perp_e$ has an inverse in $T^\perp_C C^N$ which is bounded for $\rho(\lambda) \neq 0$, it must be that $T^\perp_e R_Y(\omega)\rho(|\xi|) \in L^p(\mathbb{R}^2, T_pC^N)$ and, hence $\langle e_j, R_Y(\omega)\rho(|\xi|) \rangle \in L^p(\mathbb{R}^2)$, $j = 1, \ldots, l$. However, we have shown in the last part of the proof of Lemma 4.3 that this is impossible. Thus, $\chi_{\geq 2\varepsilon(|D|)}\Omega_{md,b}$ must be unbounded in $L^p(\mathbb{R}^2)$ for $2 < p < \infty$. \hfill $\square$

4.5. Proof of Statement (5) of Theorem 1.1

Under the condition of statement (5), $T_p = T_e$ and $\Gamma(\lambda)^{-1}$ satisfies (3.63), which we substitute in (4.1). Since $T_pS = ST_p = T_p$, the argument of Sect. 4.4 implies that the operator produced by $g(\lambda)^2(T_pM(\lambda)S + SM(\lambda)T_p)^* \langle T_pM(\lambda) + g(\lambda)M(\lambda)T_p, M(\lambda) \rangle$ is a good operator. An easy modification of the argument in Sect. 4.2 implies that $g(\lambda)^3 T_pM(\lambda)T_p$ produces a good operator. The argument of the proof of Lemma 4.7 applies to show
that $-N^{-1}\lambda^{-2}T_eG^{-1}_eT_e$ also produces a good operator. We skip the repetitive details. This proves statement (5) and completes the proof of Theorem 1.1.

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Appendix

In this appendix we show the following lemma:

Lemma A. For any $\epsilon > 0$ there exists a constant $C_\epsilon > 0$ such that

$$\int_{\mathbb{R}^4} e^{ix\xi - ipy} \chi_{\xi \leq \epsilon(|\xi|) \chi_{\xi < \epsilon(|p|)}} \frac{C_\epsilon}{|\xi| + |p|} d\xi dp \leq |\cdot| \frac{C_\epsilon}{(\langle x \rangle + \langle y \rangle)^3} \log \left( \frac{(\langle x \rangle + \langle y \rangle)^2}{\langle x \rangle \langle y \rangle} \right).$$

(4.55)

Proof. If we use the identity

$$\frac{1}{|\xi| + |p|} = \int_0^\infty e^{-t(|\xi| + |p|)} dt$$

and Fubini’s theorem, then the left side of (4.55) becomes

$$\int_0^\infty \left( \frac{1}{2\pi} \int_{\mathbb{R}^2} e^{ix\xi - t|\xi|} \chi_{\xi \leq \epsilon(|\xi|)} d\xi \right) \left( \frac{1}{2\pi} \int_{\mathbb{R}^2} e^{-ipy - t|p|} \chi_{\xi \leq \epsilon(|p|)} dp \right) dt.$$  

(4.56)

The functions inside parentheses are convolutions of the Poisson kernel with bump functions $F\chi_{\xi \leq \epsilon}(x)$ and $F\chi_{\xi \leq \epsilon}(y)$, respectively, (see [25], p. 61). They are bounded by $C_1 t((\langle x \rangle^2 + t^2)^{3/2}$ and $C_2 t((\langle y \rangle^2 + t^2)^{3/2}$, respectively. It follows by changing variable $t$ to $\langle x \rangle^{1/2} \langle y \rangle^{1/2} t$ that

$$(4.56) \leq |\cdot| C \int_0^\infty \frac{t^2 dt}{(\langle x \rangle^2 + t^2)^{3/2} (\langle y \rangle^2 + t^2)^{3/2}} = \frac{C}{\langle x \rangle^{3/2} \langle y \rangle^{3/2}} \int_0^\infty \frac{t^2 dt}{(t^4 + s^2 t^2 + 1)^{3/2}}, \quad s = \frac{\langle x \rangle^2 + \langle y \rangle^2}{\langle x \rangle \langle y \rangle}.$$  

(4.57)

We estimate the integral in the right hand side of (4.57) by slitting (0, $\infty$) into intervals into (0, 1/s), (1/s, s) and (s, $\infty$) where the denominator is bounded from below by 1, $s^3 t^3$ and $t^6$, respectively. Then

$$\int_0^\infty \frac{t^2 dt}{(t^4 + s^2 t^2 + 1)^{3/2}} \leq \int_0^{1/s} t^2 dt + \int_{1/s}^s \frac{dt}{s^3 t} + \int_s^{\infty} \frac{dt}{t^4} = \frac{2}{3} (1 + 3 \log s).$$

Since $s \geq \sqrt{3}$, the right side may be further estimated by $Cs^{-3} \log s^2$ and $s^2 \geq (\langle x \rangle^2 + \langle y \rangle^2)^2 / 2 \langle x \rangle \langle y \rangle$. Combining this with (4.57), we obtain the lemma. \hfill \Box

For applications in the text we need only the following weaker version which trivially follows from Lemma A.

Lemma B. For any $\epsilon > 0$ there exits a constant $C_\epsilon > 0$ such that

$$\frac{1}{(2\pi)^2} \int_{\mathbb{R}^4} e^{ix\xi - ipy} \chi_{\xi \leq \epsilon(|\xi|) \chi_{\xi < \epsilon(|p|)}} \frac{C_\epsilon}{|\xi| + |p|} d\xi dp \leq |\cdot| \frac{C_\epsilon}{\langle x \rangle \langle y \rangle (\langle x \rangle + \langle y \rangle)}.$$  

(4.58)
References

[1] Agmon, S.: Spectral properties of Schrödinger operators and scattering theory. Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) 2, 151–218 (1975)
[2] Albeverio, S., Gesztesy, F., Høegh-Krohn, R., Holden, H.: Solvable Models in Quantum Mechanics, 2nd edn. AMS Chelsea Publishing, Providence (2005)
[3] Beceanu, M., Schlag, W.: Structure formulas for wave operators under a small scaling invariant condition. J. Spectr. Theory 9(3), 967–990 (2019)
[4] Cornean, H.D., Michelangeli, A., Yajima, K.: Two dimensional Schrödinger operators with point interactions, threshold expansions and $L^p$-boundedness of wave operators. Rev. Math. Phys. 31(4), 32 (2019)
[5] Cornean, H.D., Michelangeli, A., Yajima, K.: Errata: Two dimensional Schrödinger operators with point interactions, threshold expansions and $L^p$-boundedness of wave operators. Rev. Math. Phys. 32(4), 5 (2020)
[6] Dancona, P., Fanelli, L.: $L^p$-boundedness of the wave operator for the one dimensional Schrödinger operator. Commun. Math. Phys. 268(2), 415–438 (2006)
[7] Digital Library of Mathematical Functions. https://dlmf.nist.gov/
[8] Dell’Antonio, G., Michelangeli, A., Scandone, R., Yajima, K.: The $L^p$-boundedness of wave operators for the three-dimensional multi-centre point interaction. Ann. Inst. H. Poincaré 19, 283–322 (2018)
[9] Duchêne, V., Marzuola, J.L., Weinstein, M.I.: Wave operator bounds for one-dimensional Schrödinger operators with singular potentials and applications. J. Math. Phys. 52, 013505, 17 (2011)
[10] Erdoğan, M.B., Green, W.R.: Dispersive estimates for Schrödinger operators in dimension two with obstructions at zero energy. Trans. Am. Math. Soc. 365, 6403–6440 (2013)
[11] Erdoğan, M.B., Goldberg, M., Green, W.R.: On the $L^p$ boundedness of wave operators for two-dimensional Schrödinger operators with threshold obstructions. J. Funct. Anal. 274, 2139–2161 (2018)
[12] Finco, D., Yajima, K.: The $L^p$ boundedness of wave operators for Schrödinger operators with threshold singularities. II. Even dimensional case. J. Math. Sci. Univ. Tokyo 13(3), 277–346 (2006)
[13] Galtbayar, A., Yajima, K.: The $L^p$-continuity of wave operators for one dimensional Schrödinger operators. J. Math. Sci. Univ. Tokyo 7(2), 221–240 (2000)
[14] Goldberg, M., Green, W.R.: The $L^p$ boundedness of wave operators for Schrödinger operators with threshold singularities. Adv. Math. 303, 360–389 (2016)
[15] Jensen, A., Kato, T.: Spectral properties of Schrödinger operators and time-decay of the wave functions. Duke Math. J. 46(3), 583–611 (1979)
[16] Jensen, A., Nenciu, G.: A unified approach to resolvent expansions at thresholds. Rev. Math. Phys. 13(6), 717–754 (2001)
[17] Jensen, A., Yajima, K.: On $L^p$ boundedness of wave operators for 4-dimensional Schrödinger operators with threshold singularities. Proc. Lond. Math. Soc. (3) 96(1), 136–162 (2008)
[18] Jensen, A., Yajima, K.: A remal on the $L^p$-boundedness of wave operators for two dimensional Schrödinger operators. Commun. Math. Phys. 225(3), 633–637 (2002)
[19] Kato, T.: Perturbation of Linear Operators. Springer, Heidelberg (1966)
[20] Kuroda, S.T.: Introduction to Scattering Theory, Lecture Notes, Matematisk
Institut, Aarhus University (1978)
[21] Murata, M.: Asymptotic expansions in time for solutions of Schrödinger-type
equations. J. Funct. Anal. 49(1), 10–56 (1982)
[22] Peral, J.C.: $L^p$ estimate for the wave equation. J. Funct. Anal. 36, 114–145
(1980)
[23] Reed, M., Simon, B.: Methods of Modern Mathematical Physics II, Fourier Analysis,
Selfadjointness. Academic Press, New York (1975)
[24] Schlag, W.: Dispersive estimates for Schrödinger operators in dimension two.
Commun. Math. Phys. 257, 87–117 (2005)
[25] Stein, E.M.: Harmonic Analysis: Real-Variable Methods, Orthogonality, and
Oscillatory Integrals. Princeton University Press, Princeton (1993)
[26] Watson, G.N.: Theory of Bessel Functions. Cambridge Univ. Press, London
(1922)
[27] Weder, R.: The $W^{k,p}$-continuity of the Schrödinger wave operators on the line.
Commun. Math. Phys. 208(2), 507–520 (1999)
[28] Yajima, K.: The $W^{k,p}$-continuity of wave operators for Schrödinger operators.
J. Math. Soc. Jpn. 47(3), 551–581 (1995)
[29] Yajima, K.: $L^p$ boundedness of wave operators for two dimensional Schrödinger
operators. Commun. Math. Phys. 208(1), 125–152 (1999)
[30] Yajima, K.: $L^1$ and $L^\infty$-boundedness of wave operators for three dimensional
Schrödinger operators with threshold singularities. Tokyo J. Math. 41(2), 385–406 (2018)
[31] Yajima, K.: Remarks on $L^p$-boundedness of wave operators for Schrödinger operators
with threshold singularities. Doc. Math. 21, 391–443 (2016)

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