A quantum-geometrical description of the statistical laws of nature

Wellington da Cruz  
Departamento de Física,  
Universidade Estadual de Londrina, Caixa Postal 6001,  
Cep 86051-970 Londrina, PR, Brazil  
E-mail address: wdacruz@exatas.uel.br  
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Abstract

We consider the fractal characteristic of the quantum mechanical paths and we obtain for any universal class of fractons labeled by the Hausdorff dimension defined within the interval $1 < h < 2$, a fractal distribution function associated with a fractal von Neumann entropy. Fractons are charge-flux systems defined in two-dimensional multiply connected space and they carry rational or irrational values of spin.

This formulation can be considered in the context of the fractional quantum Hall effect-FQHE, where we discovered that the quantization of the Hall resistance occurs in pairs of dual topological quantum numbers, the filling factors. In this way, these quantum numbers get their topological character from the Hausdorff dimension associated with the fractal quantum path of such particles termed fractons. On the other hand, the universality classes of the quantum Hall transitions can be classified in terms of $h$. Another consequence of our approach, which is supported by symmetry principles, is the prediction of the FQHE. The connection between Physics and Number Theory appears naturally in this context.

keywords: Fractal distribution function; fractal von Neumann entropy; Fractons; fractional quantum Hall effect; number theory.
We make out a review of some concepts introduced by us in the literature, such as [1–8]: fractons, universal classes $h$ of particles, fractal spectrum, duality symmetry between classes $h$ of particles, fractal supersymmetry, fractal distribution function, fractal von Neumann entropy, fractal index etc. We apply these ideas in the context of the FQHE and number theory.

Fractons are charge-flux systems which carry rational or irrational values of spin. These objects are defined in two-dimensional multiply connected space and are classified in universal classes $h$ of particles or quasiparticles, with the fractal parameter or Hausdorff dimension $h$, defined in the interval $1 < h < 2$. It is related to the quantum paths and can be extracted from the propagators of the particles in the momentum space [2,9]. The particles are collected in each class take into account the fractal spectrum

$$h - 1 = 1 - \nu, \quad 0 < \nu < 1; \quad h - 1 = \nu - 1, \quad 1 < \nu < 2;$$

$$h - 1 = 3 - \nu, \quad 2 < \nu < 3; \quad h - 1 = \nu - 3, \quad 3 < \nu < 4; \text{etc.}$$

(1)

and the spin-statistics relation $\nu = 2s$, valid for such fractons. The fractal spectrum establishes a connection between $h$ and the spin $s$ of the particles: $h = 2 - 2s, \ 0 \leq s \leq \frac{1}{2}$. Thus, there exists a mirror symmetry behind this notion of fractal spectrum. Given the statistical weight for these classes of fractons

$$W[h, n] = \frac{[G + (nG - 1)(h - 1)]!}{[nG]![G + (nG - 1)(h - 1) - nG]!}$$

(2)

and from the condition of the entropy be a maximum, we obtain the fractal distribution function [2]

$$n[h] = \frac{1}{Y[\xi] - h}$$

(3)

The function $Y[\xi]$ satisfies the equation

$$\xi = \left\{Y[\xi] - 1\right\}^{h-1} \left\{Y[\xi] - 2\right\}^{2-h},$$

(4)

with $\xi = \exp\{(\epsilon - \mu)/KT\}$. We understand the fractal distribution function as a quantum-geometrical description of the statistical laws of nature, since the quantum path is a fractal curve and this reflects the Heisenberg uncertainty principle.

We can obtain for any class its distribution function considering Eq.(3) and Eq.(4). For example, the universal class $h = \frac{3}{2}$ with distinct values of spin $\left\{\frac{1}{4}, \frac{3}{4}, \frac{5}{4}, \ldots\right\}_{h=\frac{3}{2}}$, has a specific fractal distribution

$$n\left[\frac{3}{2}\right] = \frac{1}{\sqrt{\frac{1}{4} + \xi^2}}$$

(5)

We also have

$$\xi^{-1} = \left\{\Theta[Y]\right\}^{h-2} - \left\{\Theta[Y]\right\}^{h-1}$$

(6)
where

$$\Theta[\mathcal{Y}] = \frac{\mathcal{Y}[\xi] - 2}{\mathcal{Y}[\xi] - 1}$$  \hspace{1cm} (7)$$

is the single-particle partition function. We verify that the classes \( h \) satisfy a duality symmetry defined by \( h = 3 - k \). So, fermions and bosons come as dual particles. As a consequence, we extract a fractal supersymmetry which defines pairs of particles \( \left( s, s + \frac{1}{2} \right) \). In this way, the fractal distribution function appears as a natural generalization of the fermionic and bosonic distributions for particles with braiding properties. Therefore, our approach is a unified formulation in terms of the statistics which each universal class of particles satisfies: from a unique expression we can take out any distribution function. In some sense, we can say that fermions are fractons of the class \( h = 1 \) and bosons are fractons of the class \( h = 2 \).

The free energy for particles in a given quantum state is expressed as

$$\mathcal{F}[h] = K T \ln \Theta[\mathcal{Y}].$$ \hspace{1cm} (8)

Hence, we find the average occupation number

$$n[h] = \xi \frac{\partial}{\partial \xi} \ln \Theta[\mathcal{Y}].$$ \hspace{1cm} (9)

The fractal von Neumann entropy per state in terms of the average occupation number is given as [1,2]

$$S_G[\mathcal{Y}, n] = K \left[ [1 + (h - 1)n] \ln \left( \frac{1 + (h - 1)n}{n} \right) - [1 + (h - 2)n] \ln \left( \frac{1 + (h - 2)n}{n} \right) \right]$$ \hspace{1cm} (10)

and it is associated with the fractal distribution function Eq.3.

The entropies for fermions \( \left\{ \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \cdots \right\}_{h=1} \) and bosons \( \left\{ 0, 1, 2, \cdots \right\}_{h=2} \), can be recovered promptly

$$S_G[1] = -K \left\{ n \ln n + (1 - n) \ln(1 - n) \right\}$$ \hspace{1cm} (11)

and

$$S_G[2] = K \left\{ (1 + n) \ln(1 + n) - n \ln n \right\}.$$ \hspace{1cm} (12)

Now, as we can check, each universal class \( h \) of particles, within the interval of definition has its entropy defined by the Eq.(10). Thus, for fractons of the self-dual class \( \left\{ \frac{1}{4}, \frac{3}{4}, \frac{5}{4}, \cdots \right\}_{h=\frac{3}{2}} \), we have

$$S_G \left[ \frac{3}{2} \right] = K \left\{ (2 + n) \ln \sqrt{\frac{2 + n}{2n}} - (2 - n) \ln \sqrt{\frac{2 - n}{2n}} \right\}.$$ \hspace{1cm} (13)

We have also introduced the topological concept of fractal index, which is associated with each class. As we saw, \( h \) is a geometrical parameter related to the quantum paths of the particles and so, we define [3]
\[ i_f[h] = \frac{6}{\pi^2} \int_{\infty(T=0)}^{1(T=\infty)} \frac{d\xi}{\xi} \ln \{ \Theta[Y(\xi)] \}. \]  
\hspace{1cm} (14)

We obtain for the bosonic class \( i_f[2] = 1 \), for the fermionic class \( i_f[1] = 0.5 \) and for some classes of fractons, we have \( i_f[3/2] = 0.6, i_f[5/3] = 0.56, i_f[5/3] = 0.656 \). For the interval of the definition \( 1 \leq h \leq 2 \), there exists the correspondence \( 0.5 \leq i_f[h] \leq 1 \), which signalizes the connection between fractons and quasiparticles of the conformal field theories, in accordance with the unitary \( c < 1 \) representations of the central charge. For \( \nu \) even it is defined by

\[ c[\nu] = i_f[h, \nu] - i_f \left[ h, \frac{1}{\nu} \right] \]  
\hspace{1cm} (15)

and for \( \nu \) odd it is defined by

\[ c[\nu] = 2 \times i_f[h, \nu] - i_f \left[ h, \frac{1}{\nu} \right], \]  
\hspace{1cm} (16)

where \( i_f[h, \nu] \) means the fractal index of the universal class \( h \) which contains the particles with distinct values of spin, which obey specific fractal distribution function. For example, we obtain the results

\[ c[0] = i_f[2, 0] - i_f[h, \infty] = 1; \]
\[ c[1] = 2 \times i_f[1, 1] - i_f[1, 1] = 0.5; etc. \]  
\hspace{1cm} (17)

In another way, the central charge \( c[\nu] \) can be obtained using the Rogers dilogarithm function, i.e.

\[ c[\nu] = \frac{L[x^\nu]}{L[1]}, \]  
\hspace{1cm} (18)

with \( x^\nu = 1 - x, \ \nu = 0, 1, 2, 3, etc. \) and

\[ L[x] = -\frac{1}{2} \int_0^x \left\{ \ln \left( \frac{1 - y}{y} \right) + \ln \frac{y}{1 - y} \right\} dy, \ 0 < x < 1. \]  
\hspace{1cm} (19)

Thus, we have established a connection between fractal geometry and number theory, given that the dilogarithm function appears in this context, besides another branches of mathematics [10].

Such ideas can be applied in the context of the FQHE. This phenomenon is characterized by the filling factor parameter \( f \), and for each value of \( f \) we have the quantization of Hall resistance and a superconducting state along the longitudinal direction of a planar system of electrons, which are manifested by semiconductor doped materials, i.e. heterojunctions, under intense perpendicular magnetic fields and lower temperatures [11].

The parameter \( f \) is defined by \( f = N \frac{\phi}{\phi_0} \), where \( N \) is the electron number, \( \phi_0 \) is the quantum unit of flux and \( \phi \) is the flux of the external magnetic field throughout the sample. The spin-statistics relation is given by \( \nu = 2s = 2 \frac{\phi}{\phi_0} \), where \( \phi \) is the flux associated with the charge-flux system which defines the fracton \((h, \nu)\). According to our approach there is a correspondence between \( f \) and \( \nu \), numerically \( f = \nu \). This way, we verify that the filling factors observed experimentally appear into the classes \( h \) and from the definition of
duality between the equivalence classes, we note that the FQHE occurs in pairs of these dual topological quantum numbers

\[(f, \tilde{f}) = \left(\frac{1}{5}, \frac{2}{5}\right), \left(\frac{5}{3}, \frac{4}{3}\right), \left(\frac{1}{3}, \frac{4}{3}\right), \left(\frac{2}{7}, \frac{5}{7}\right), \left(\frac{5}{9}, \frac{4}{9}\right), \left(\frac{1}{5}, \frac{4}{5}\right), \left(\frac{2}{5}, \frac{4}{5}\right), \left(\frac{5}{9}, \frac{8}{9}\right), \left(\frac{2}{5}, \frac{4}{5}\right), \left(\frac{5}{9}, \frac{8}{9}\right)\ etc.\]

We verify that all the experimental results satisfy this symmetry principle.

Finally, we observe again that our formulation to the universal class \( h \) of particles with any values of spin \( s \) establishes a connection between Hausdorff dimension \( h \) and the central charge \( c[\nu] \). Besides this, we have obtained a relation between the fractal parameter and the Rogers dilogarithm function, through the concept of fractal index, which is defined in terms of the partition function associated with each universal class of particles. Also we have established a connection between the fractal parameter \( h \) and the Farey sequences of rational numbers. Farey series \( F_n \) of order \( n \) is the increasing sequence of irreducible fractions in the range \( 0 - 1 \) whose denominators do not exceed \( n \). We have the following

**Theorem** [6]: The elements of the Farey series \( F_n \) of the order \( n \), belong to the fractal sets, whose Hausdorff dimensions are the second fractions of the fractal sets. The Hausdorff dimension has values within the interval \( 1 < h < 2 \), which are associated with fractal curves.

For a extended review about our work see Ref. [8].
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