Free-energy distribution functions for the randomly forced directed polymer

V.S. Dotsenko\textsuperscript{a}, V.B. Geshkenbein\textsuperscript{b,c}, D.A. Gorokhov\textsuperscript{b}, and G. Blatter\textsuperscript{b}

\textsuperscript{a}LPTL, Université Paris VI, 75252 Paris, France
\textsuperscript{b}Theoretische Physik, ETH-Zurich, 8093 Zurich, Switzerland and
\textsuperscript{c}L.D. Landau Institute for Theoretical Physics, 119334 Moscow, Russia

(Dated: July 7, 2010)

We study the 1 + 1-dimensional random directed polymer problem, i.e., an elastic string $\phi(x)$ subject to a Gaussian random potential $V(\phi, x)$ and confined within a plane. We mainly concentrate on the short-scale and finite-temperature behavior of this problem described by a short- and finite-ranged disorder correlator $U(\phi)$ and introduce two types of approximations amenable to exact solutions. Expanding the disorder potential $V(\phi, x) \approx V_0(x) + f(x)\phi(x)$ at short distances, we study the random force (or Larkin) problem with $V_0(x) = 0$ as well as the shifted random force problem including the random offset $V_0(x)$; as such, these models remain well defined at all scales. Alternatively, we analyze the harmonic approximation to the correlator $U(\phi)$ in a consistent manner. Using direct averaging as well as the replica technique, we derive the distribution functions $P_{L,y}(F)$ and $P_{L}(F)$ of free energies $F$ of a polymer of length $L$ for both fixed ($\phi(L) = y$) and free boundary conditions on the displacement field $\phi(x)$ and determine the mean displacement correlators on the distance $L$. The inconsistencies encountered in the analysis of the harmonic approximation to the correlator are traced back to its non-spectral correlator; we discuss how to implement this approximation in a proper way and present a general criterion for physically admissible disorder correlators $U(\phi)$.

PACS numbers: 05.20.-y 75.10.Nr 74.25.Wx 61.41.+e

I. INTRODUCTION

Directed polymers subject to a quenched random potential have been the subject of intense investigations during the past two decades\textsuperscript{1}. Diverse physical systems such as domain walls in magnetic films\textsuperscript{2}, vortex in superconductors\textsuperscript{3}, wetting fronts on planar systems\textsuperscript{4}, or Burgers turbulence\textsuperscript{5} can be mapped to this model, which exhibits numerous non-trivial features deriving from the interplay between elasticity and disorder. The best understanding, so far, has been reached for the $(1 + 1)$-dimensional case, i.e., a string confined to a plane, and it is this geometry we study in the present paper. Specifically, we analyze the situation illustrated in Fig. 1 an elastic string (with elasticity $c$) of finite length $L$ within an interval $[0,L]$ directed along the $x$-axis. The disorder potential $V(\phi, x)$ drives a finite displacement field $\phi(x)$, which is counteracted by the elastic energy density $c(\partial_x \phi)^2/2$. The problem is conveniently defined through its Hamiltonian

$$H[\phi(x); V] = \int_0^L dx \left\{ \frac{c}{2} [\partial_x \phi(x)]^2 + V[\phi(x), x] \right\};$$

(1)

the disorder potential $V(\phi, x)$ is Gaussian distributed with a zero mean $\tilde{V}(\phi, x) = 0$ and a correlator

$$\tilde{V}(\phi, x)\tilde{V}(\phi', x') = \delta(x - x') U(\phi - \phi'),$$

(2)

with $U(\phi)$ the correlation function. In the present work, we are mainly interested in short-range correlated disorder potentials, which we characterize by their extension $\xi$ and the strength $U_0 = U(0)$; these parameters then combine in a curvature $u = -U''(0) \approx U(0)/\xi^2$.

FIG. 1: Thermally averaged trajectory $\langle \phi(x) \rangle_{\text{th}}$ of a random directed polymer in a fixed disorder potential $V(\phi, x)$ starting in $(x, \phi) = (0, 0)$ and ending in $(L, y)$. The free energy associated with such a configuration is denoted by $F$. The random choice of the underlying disorder potential $V(\phi, x)$ defines a random process; the free energy then turns into a random variable, whose distribution function $P_{L,y}(F)$ we seek to calculate.

Quantities of central interest are the scaling behavior of the mean-squared displacement $\langle \phi^2 \rangle(L)$ of the polymer with length $L$ and the distribution function $P(F)$ of the polymer’s free energy $F$. For free boundary conditions at $x = 0, L$, the former is defined through the expression $\langle \phi^2 \rangle(L) \equiv \langle (\phi(L) - \phi(0))^2 \rangle$, where $\langle \cdots \rangle$ and $\langle \cdots \rangle$ denote thermal (temperature $T$) and disorder (random potential $V$) averages, respectively. Its dependence on the distance $L$ exhibits a scaling form $\langle \phi^2 \rangle(L) \propto L^{2\zeta}$, with the so-called wandering exponent $\zeta$ to be determined\textsuperscript{6}.

The polymer’s free energy $F$ is defined via its partition function

$$Z[L, y; V] = \int_{\phi(0) = 0}^{\phi(L) = y} D[\phi(x)] \exp(-\beta H[\phi(x); V]),$$

(3)

where $\beta = 1/T$ denotes the inverse temperature (we set the Boltzmann constant to unity), from which the free
energy
\[ F[L, y; V] = -T \ln(Z[L, y; V]) \] (4)
follows immediately. The free energy \( F \) in Eq. (4) is defined for a specific realization of the random potential \( V \) and thus defines a random variable; given the above (Gaussian) distributed disorder potential, the task then is to determine the distribution function \( P_{L,y}(F) \). In Eq. (3), we have considered a string starting at \((x, \phi) = (0, 0)\) and ending in a fixed position \((x, \phi) = (L, y)\) a distance \( L \) away, but other cases, e.g., a free boundary condition at \( x = L \), see below, may be studied.

Two types of analytic solutions are known for the \((1 + 1)\)-dimensional random polymer problem with a short-range, in fact, \( \delta \)-correlated, potential \( U(\phi) = u\delta(\phi) \): mapping the replicated problem to interacting quantum bosons and using the Bethe-Ansatz technique, one can find the spectrum and eigenfunctions of the interacting quantum many-body problem, from which the distribution function \( P_L(F) \) for the free energy \( F \) of a polymer of length \( L \) and fixed endpoint at \( \phi(L) = 0 \) can be obtained; we call this the ‘longitudinal problem’. Restricting the solution to the ground state wave function only permits the determination of the far-left tail. First indications that the full distribution function should be of the Tracy-Widom form derived from the work of Prâhofer and Spohn on polynuclear growth, a model in the universality class of the Kardar-Parisi-Zhang (KPZ) model, to which the random directed polymer problem belongs as well. Recently, both tails of the free-energy distribution function \( P_{\phi}(F) \) have been found using instanton techniques, with results consistent with those in Ref. 9.

Finally, the full distribution function (in the Tracy-Widom form) has been obtained independently by several groups, by Dotsenko and Calabrese et al. using the replica technique including the full spectrum, by Sasamoto and Spohn deriving the exact KPZ solution from the corner growth model, and by Amir et al. An alternative (exact) result has been obtained using a mapping to the Burgers equation via the Cole-Hopf transformation, making use of an invariant distribution, the distribution function \( P_{\phi}(F') \) for the free-energy difference \( F' \) between two configurations with endpoints separated by \( 2y \) has been found (the so-called ‘transverse’ problem). Both approaches have been helpful in finding the wandering exponent \( \zeta = 2/3 \) of transverse fluctuations of the polymer; it is generally believed that this value is universal for short-range correlated disorder potentials, i.e., for rapidly decaying correlator functions \( U(\phi) \to \infty \to 0 \).

Recently, some of us have studied the joint distribution function \( P_{L,y}(\bar{F},F') \) for a polymer of length \( L \) which involves two configurations of the string ending in points separated by \( 2y \). An interesting result is found for the \( \delta \)-correlated potential, where \( P_{L,y}(\bar{F},F') = P_{L,y}(\bar{F}) P_{\phi}(F') \) separates for large \( L \) and large negative values of the mean free energy \( \bar{F} \), with the factor \( P_{L,y}(\bar{F}) \) reproducing Zhang’s tail, for \( P_L(F) \) and the transverse part \( P_{\phi}(F') \) coinciding with the stationary distribution function \( P_{\phi}(F') \) of the Burgers problem, thus placing previously known results into a common context.

The above solutions focus on the \( \delta \)-correlated disorder potential which exhibits a singular zero-temperature limit; the problem with a finite-width correlator \( U(\phi) \) of the random potential has remained unsolved so far. In the present paper, we study another problem, originally proposed by Larkin, that is amenable to a complete exact solution. This problem deals with the polymer’s behavior at short distances and also exhibits a regular low-temperature limit. The basic idea is to linearize the problem, either by an expansion of the original random potential \( V(\phi, x) \) for small values of \( \phi \),

\[ V(\phi, x) \approx V_0(x) + f(x)\phi(x), \] (5)
or by an expansion of the correlator \( U(\phi) \), see below. Studying the destruction of long-range order due to the presence of quenched disorder, i.e., the behavior of the displacement field \( \phi(x) \) at large distances \( x \), the random shift \( V_0(x) \) can be dropped and one arrives at the Larkin or random-force model described by the Hamiltonian

\[ \tilde{V}(\phi, x) = f(x)\phi(x), \] (6)

where \( f(x) \) denotes a (Gaussian) random force field with zero mean \( \bar{f}(x) = 0 \) and a correlator

\[ \bar{f}(x)f(x') = u \delta(x - x'). \] (7)

Its free-energy distribution function \( P_{\phi,L}(F) \) has been calculated by Gorokhov and Blatter for fixed boundary conditions; here, we extend this analysis to describe the case of free boundary conditions. Furthermore, in our analysis of the Larkin model as a short-scale approximation for the random potential problem, we are not allowed to ignore the random shift \( V_0(x) \); in our study, we assume the latter to be Gaussian correlated,

\[ V_0(x)V_0(x') = U_0 \delta(x - x'), \] (8)

and uncorrelated with the force, \( V_0(x)f(x') = 0 \); we call this approximation the shifted random force model. Both models not only serve as approximations to the random polymer problem on short scales but also describe physical problems where the underlying randomness is described by a (shifted) random force field on all length scales. Therefore, below we will quote results for these models for arbitrary lengths \( L \).

Another approximation discussed in the literature deals with the expansion of the correlator \( U(\phi - \phi') \) rather than the potential,

\[ U(\phi - \phi') \approx U_0(\phi - \phi') = U_0 - \frac{1}{2} u(\phi - \phi')^2; \] (9)

also, models have been studied with ad hoc power-law expressions for the correlator. Here we point out, that
expanding the correlator $U(\phi)$ of the random potential or choosing arbitrary forms for the correlator is a problematic step, as this action may generate a non-spectral correlator introducing an ill-defined Gaussian measure and thus lead to an unphysical model at the very start, cf. Sec. [IV] below for a detailed discussion. As an immediate consequence, the mapping to the quantum boson problem requiring integration over the random potential $V(\phi, x)$ fails. On the other hand, performing the integration over the random potential $V(\phi, x)$ with a well defined correlator $U(\phi)$ first and expanding the resulting interacting $-\beta^2 U(\phi)$ both on bosons only thereafter is a perfectly admissible program producing identical results; in this case, however, we know that the (perfectly well defined) quantum polymer problem does not describe the random polymer problem on scales where the approximate quadratic correlator deviates strongly from the original correlator.

In the following, we will discuss the harmonic-correlator approximation of the original random polymer problem with the understanding, that the harmonic approximation is done after the mapping to bosons. This approximation then relates to the above shifted random force model with a force correlator $\partial_\phi V(\phi, x) \partial_\phi V(\phi', x')|_{\phi, \phi' = 0} = [-U''(0)] \delta(x - x')$ and the shift described by $V(\phi, x)V(\phi', x')|_{\phi, \phi' = 0} = U_0 \delta(x - x')$ with the additional advantage to preserve the translation invariance of the problem (note, that the shifted random force model involves only the constant and mixed $\langle u\phi \phi' \rangle$ terms in the correlator, with the quadratic terms $\propto \langle \phi^2 + \phi'^2 \rangle$ absent). Furthermore, in Sec. [IV] we will present a criterion assuring the consistent definition of a correlator: such correlators have to be spectral. Also, we emphasize the difference in terminology introduced above: while the random force and shifted random force approximations to the random potential $V(\phi, x)$ define proper models of disordered elastic systems, this is not the case when expanding the potential correlator $U(\phi)$: this is why we refrain from considering a model with a harmonically correlated random potential but prefer to talk about a harmonic approximation to the correlator.

The (shifted) random force models and the harmonic-correlator approximation produce similar results for various quantities defined at short scales, such as the mean free energy and the displacement correlator. While all displacement correlators are identical for the two models and the approximation, the random force model supplies us with the distribution function for the relaxational free energy (i.e., the free energy of the distorted string reduced by the energy of the straight string), whereas the shifted random force model and the harmonic-correlator approximation provide the distribution function for the total free energy. Comparing the latter two, we note that the results for the harmonic-correlator approximation provide a more consistent description of the original problem Eq. (2) at short distances, as the correlator remains translation invariant, while some terms in the expansion of the random potential are dropped for the shifted random force problem. On the other hand, the applicability of the harmonic-correlator approximation is limited to short scales, as the free-energy distribution function $P_L(F)$ suffers from a negative second moment whenever large displacements show up, e.g., at large distances $L$ or at high temperatures $T$; this is to be expected as the harmonic correlator deviates strongly from the original correlator $U(\phi)$ (and eventually turns negative) at large arguments.

For a system with quenched disorder it is usually extremely difficult to find averaged physical quantities, e.g., the mean free energy $F = -T \ln Z$. Replica theory, requiring calculation of the disorder-averaged $n$-th power of the partition function $Z^n$ then comes in as a helpful technique. Usually, it is the limit $\lim_{n \to 0} (Z^n - 1)/n = \ln Z$, to be calculated after analytic continuation of $n$, which is of fundamental interest. It turns out (see below) that the same quantity $Z^n$ and its analytic continuation is relevant in the calculation of the free-energy distribution function $F(F)$, since the latter is nothing but the inverse Laplace transform of the former, hence replica theory seems the technique of choice for the solution of the present problem as well. However, the shifted random force model defines a quadratic problem that can be analyzed in a straightforward manner, i.e., the partition function $Z(f(x))$ (involving an integration over the field) can be found for any configuration $f(x)$ of the random force and the disorder average of its $n$-th power can be done in the end. This is opposite to the replica approach where the integrations are interchanged, with the first integration over the disorder of the replicated system generating an interacting imaginary-time quantum boson problem, which then is solved in a second step (corresponding to the integration over the field). Below, we will discuss both procedures for the Larkin model and find that they provide similar challenges.

The disorder $(u$ and $U_0)$ and elastic $(c)$ parameters of the above random polymer problems define convenient and physically relevant length and energy scales: The ratio of $U_0$ and $u$ defines the transverse length scale $\xi$ where the shifted random force model approximates well the random polymer problem,

$$\xi = \left(\frac{U_0}{u}\right)^{1/2}. \quad (10)$$

Comparing the elastic energy $E_c = c\xi^2/L = cU_0/ul$ with the disorder energy $E_I = \sqrt{U_0L}$ accumulated over a distance $L$, one obtains the corresponding longitudinal scale $L_c$,

$$L_c = \left(\frac{c^2U_0}{u^2}\right)^{1/3} = \left(\frac{c^2\xi^2}{u}\right)^{1/3}. \quad (11)$$

Finally, the energy scale associated with these length scales is

$$U_c = \left(\frac{cU_0^2}{u}\right)^{1/3} = \frac{c\xi^2}{L_c}. \quad (12)$$
Note that the longitudinal \((L_c)\) and transverse \((\xi)\) scales define the limits of validity where our expansions describe the original random polymer problem. The parameters are not fully appropriate to describe the results of the Larkin model, as the latter is characterized by one disorder parameter \((u)\) only—to allow for proper comparison, below, we will nevertheless express all physical results through \(\xi, L_c,\) and \(U_c\). For the Larkin model, these parameters will combine to expressions containing only \(u\) and \(c\).

Besides providing new results for the Larkin model and a discussion of its use as an approximation to the random polymer problem at short scales, the present study also has its merits from a methodological point of view, since it is the only case where the entire analysis (direct and via replica) could be carried through in a complete and consistent manner. Below, we introduce the formalism (Sec. \[\text{II}\]) and then apply it to the (shifted) random force models (Sec. \[\text{III}\]). We then analyze the harmonic-correlator approximation (section \[\text{IV}\]), analyze its failure via replica (section \[\text{V}\]), and via replica) could be carried through in a complete setting. Collapsing the end-points \(\{\xi(r)\}\) below, we will nevertheless express all physical results in \(\mathbb{Z}(\eta; L, y)\) to its right. Furthermore, taking the \(k\)-fold derivatives of \(\mathbb{Z}(\eta; L, y)\) with respect to \(\eta\) provides us with all the moments

\[
\mathbb{Z}^{(k)}(\eta; L, y) = (-1)^k \frac{\partial^k \mathbb{Z}(\eta; L, y)}{\partial \eta^k} \bigg|_{\eta=0}. \tag{16}
\]

The calculation of the moments \(\mathbb{Z}^{(n)}(L, y)\) involves integrations over the displacement field \(\phi(x)\) and over the distribution function \(P[V(\phi, x)]\) of the disorder potential,

\[
\mathbb{Z}(L, y) = \int \mathcal{D}[V(\phi, x)] P[V(\phi, x)] \prod_{a=1}^n \int \mathcal{D}[\phi_a(x)] \exp\left(-\beta \sum_{a=1}^n H[\phi_a(x)]\right). \tag{17}
\]

For the random force or Larkin model, these integrations can be done straightforwardly in the sequence above using the distribution function for the random force

\[
P[f(x)] \propto \exp\left(- \int dx f^2(x)/2a\right); \tag{18}
\]

this program will be carried through in section \[\text{III A}\] below. Fixed and free boundary conditions are conveniently imposed by the requirements \(\phi(0) = 0, \phi(L) = y\) and \(\phi(0) = 0, \partial_x \phi(L) = 0\).

On the other hand, for the general situation with a random potential \(V(\phi, x)\), the integration in Eq. \[17\] over the displacement fields \(\phi_a(x)\) cannot be done. Interchanging the integrations over \(V(\phi, x)\) and \(\phi_a(x)\) takes us directly to the replica technique: performing first the integration over the disorder potential \(V(\phi, x)\), the remaining integrations over the fields \(\phi_a(x)\) have to be done with the replica Hamiltonian \(H_n[\{\phi_a\}]\),

\[
\Psi(\{y_a\}; L) = \left[\prod_{a=1}^n \int_{\phi_a(0) = 0} \mathcal{D}[\phi_a(x)] \right] \times \exp\left(-\beta H_n[\{\phi_a(x)\}]\right), \tag{19}
\]

\[
H_n[\{\phi_a(x)\}] = \int dx \left\{ \frac{c}{2} \sum_{a=1}^n [\partial_x \phi_a(x)]^2 - \frac{\beta}{2} \sum_{a,b=1}^n U[\phi_a(x) - \phi_b(x)] \right\}. \tag{20}
\]

Here, we have allowed the individual replicas of the elastic string to end in different locations \(y_a\). The expression Eq. \[19\] is identical with the imaginary time \((x)\) propagator \(\Psi(\{y_a\}; x)\) of a many body problem in a path integral setting. Collapsing the end-points \(y_a = y\), this propagator coincides with the \(n\)-th moment \((\text{17})\) of the partition function,

\[
\Psi(\{y_a = y\}; x = L) = Z_r(n; L, y) = Z^{(n)}(L, y), \tag{21}
\]

where the last equation holds, provided that the integration over the disorder potential \(V\) can be exchanged with the integration over the field \(\phi_a\). Note that it is this
mapping from the polynomial statistical mechanics problem to the quantum boson problem which fails when the correlator is non-spectral, e.g., for the (naive version of) harmonic-correlator approximation.

The equivalence to a quantum many body problem becomes more obvious when going from the path-integral Eq. (19) to an operator formalism; the evaluation of the path-integral Eq. (19) then is equivalent to the solution of the imaginary-time Schrödinger equation

\[-\partial_t \Psi(y_a; t) = \hat{H} \Psi(y_a; t)\]  

with the Hamiltonian

\[\hat{H} = -\frac{1}{2}\beta c \sum_{a=1}^{n} \frac{\partial^2_{y_a}}{2} - \frac{\beta^2}{2} \sum_{a,b=1}^{n} U(y_a - y_b).\]  

The Hamiltonian (22) describes \(n\) particles of mass \(\beta c\) interacting via the attractive two-body potential \(-\beta^2 U(y)\); the propagation in (22) starts in the origin at time \(x = 0\),

\[\Psi(y_a; 0) = \Pi_{a=1}^{n} \delta(y_a),\]  

and ends at different coordinates \(\{y_a\}\) after propagation during the time \(x\). To keep up the formal distinction between the two quantities, we denote the direct physical trajectories \(\phi(x)\) involving either direct integration over the field \(\phi\), raising the result to the power \(n\)-fold replication followed by averaging over disordered \(V\) and denote the replica expression (\(n\)-fold replication followed by averaging over disorder \(V\) and integration over the fields \(\phi_a\) done in the end) by \(Z_{\phi}(n; L, y)\).

Finally, we note that in the replica technique, free boundary conditions at the endpoint are more conveniently implemented through an integration over \(y\); the partition function for the polymer with free boundary conditions assumes the form

\[Z[L; V] = N \int_{-\infty}^{+\infty} dy \, Z[L, y; V]\]  

with \(N\) a suitable normalization constant. Taking the \(n\)-th power and averaging over the disorder potential \(V\) provides us with the moments \(Z^n[L]\). Following the replica procedure, after replication and integration over \(V\), one arrives at the free-boundary replicated partition function through integration over the set \(\{y_a\}\) of \(n\) different endpoints,

\[Z_{\phi}(n; L) = \left[ \prod_{a=1}^{n} N \int_{-\infty}^{+\infty} dy_a \right] \Psi(\{y_a\}; x = L).\]  

In the next sections, we are going to apply the above general schemes for the calculation of the free-energy distribution functions \(P_{L, y}(F)\) and \(P_{L}(F)\) for fixed and free boundary conditions, respectively, of the (shifted) random force model Eqs. (3) and of the random directed polymer model Eq. (1) with the parabolic approximation for the correlation function, Eq. (3), done after averaging over the disorder. Before doing so, we briefly discuss the results for the free string which determines our normalization \(N\).

### A. Free string

The path integrals Eqs. (3), (17), and (19) over trajectories \(\phi(x)\) involve an arbitrary measure of integration. Here, we choose a particular normalization such that the partition function \(Z_0(L, y)\) (or wave function \(\Psi(y, x = L)\)) of the free polymer problem with fixed boundary conditions \(\phi(0) = 0\) and \(\phi(L) = y\) assumes the form

\[Z_0(L, y) = \Psi(y; L)\]  

\[= \int_0^y D[\phi(x)] \exp \left[ -\frac{\beta c}{2} \int_0^L dx \left[ \partial_x \phi(x) \right]^2 \right] \equiv \exp \left( -\frac{\beta c}{2L} y^2 \right);\]  

the corresponding free energy then is given by

\[F_0(L, y) = \frac{c}{2L} y^2.\]  

For the partition function of the free polymer with free boundary conditions we choose the normalization \(N = (2\pi L / \beta c)^{1/2}\) and obtain

\[Z_0(L) = \sqrt{\frac{2\pi L}{\beta c}} \int_{-\infty}^{+\infty} dy \, Z_0(L, y) = 1\]  

and the free energy \(F_0(L) = 0\). These results will be helpful in the interpretation of the free-energy distribution functions for the random force model calculated below. With this normalization, all our free energies \(F\) are measured with respect to the free thermal energy \(F_0^{\text{rel}} = T \ln \sqrt{2\pi LT / \beta c}\) of the free string due to its entropy.

### III. RANDOM FORCE MODEL

We select the simplest case, the Larkin model, for the discussion of the two methodological approaches involving either direct integration over the field \(\phi\) and subsequent disorder average over \(V\) or the route following the replica approach. While the first route is preferably done in Fourier space, the replica calculation will be formulated in real space. Also, note that the analysis for the Larkin- or random force model provides the distribution function for the relaxation free-energy free energy \(F - E_0\) rather than the (total) free energy \(F\) of the polymer,

\[Z[L, y; V] = \exp^{-\beta F[L, y; V]}\]  

\[= \int_{\phi(0) = 0}^{\phi(L) = y} D[\phi(x)] \exp(-\beta H[\phi; V_0 + f \phi])\]  

\[= e^{-\beta E_0} \int_{\phi(0) = 0}^{\phi(L) = y} D[\phi(x)] \exp(-\beta H[\phi; f \phi]),\]  

with \(E_0 = \int dx V_0(x)\) the disorder energy of a straight string. This latter remark is relevant in the comparison of the random force and the harmonic models.
A further speciality of the Larkin model is the separation between the thermal and the quenched disorder. Indeed, splitting the displacement field \( \phi(x) \) into the Hamiltonian’s minimizer \( \phi_0(x) \),

\[
eq c\partial_x^2 \phi_0(x) = f(x),
\]
and fluctuations \( \delta \phi(x) \) around it, \( \phi(x) = \phi_0(x) + \delta \phi(x) \), we can decompose the Hamiltonian into the free part \( H_0[\delta \phi(x)] \) and the energy of the minimizer \( H[\phi_0(x)] \),

\[
H_0[\delta \phi(x)] = \int_0^L dx \left\{ \frac{c}{2} [\partial_x \delta \phi(x)]^2 \right\},
\]
and the partition function reads

\[
Z(L, y; f) = \exp\left(-\frac{\beta c y^2}{2L}\right) \prod_{m=1}^{\infty} \exp\left(\frac{\beta f_m^2}{4cLk_m^2} + \frac{\beta y}{Lk_m} (-1)^m f_m\right).
\]

The disorder average in the partition function Eqs. (34) or (35) has to be taken over the distribution function for the random force \( f \), cf. Eq. (18), or in Fourier space,

\[
P(f_m) = \frac{1}{\sqrt{4\pi uL}} \exp(-f_m^2/4uL).
\]

Taking the result (38) to the \( n \)-th power and integrating over the force distribution Eq. (39), we obtain the intermediate result

\[
Z^n(L, y) = \exp\left(-\frac{\beta n c y^2}{2L}\right) \times \prod_{m=1}^{\infty} \left[1 - \frac{s}{\pi^2 m^2}\right]^{-1/2} \exp\left(\sum_{m=1}^{\infty} \frac{\beta n c}{2L} \frac{y^2 s^2}{\pi^2 m^2 - s}\right)
\]
with \( s = \frac{\beta n uL^2}{c} \).

Using the product and partial fraction expansion of circular functions,

\[
-\sin \sqrt{s} = \prod_{m=1}^{\infty} \left[1 - \frac{s}{\pi^2 m^2}\right],
\]

\[
\tan \sqrt{s} = 1 + \sum_{m=1}^{\infty} \frac{2s}{s - m^2 \pi^2},
\]
we obtain the final result

\[
Z^n(L, y) \equiv Z(s; \epsilon) = \left(\frac{\sqrt{s}}{\sin \sqrt{s}}\right)^{1/2} \exp\left[-\epsilon \frac{s \sqrt{s}}{\tan \sqrt{s}}\right],
\]
with the dimensionless displacement parameter

\[
\epsilon = \frac{c^2 y^2}{2uL^3}.
\]

With our normalization, the partition sum does not depend on temperature any more (note that in the calculation of the free-energy distribution function, the variable \( s \) will be integrated over, cf. Eq. (60)). The result Eq. (44) is well defined provided that \( 0 < s < \pi^2 \); the singularity at \( s = \pi^2 \) will determine the shape of the left tail in the free-energy distribution function, see (67) below.

As a simple application, we can use the partition function Eq. (38) to find the free energy \( \langle F \rangle = -T \ln Z(L, 0; f_m) \) of the string starting and ending in \( \phi = 0 \). Taking the disorder average over the term \( \sum_m \beta f_m^2/4cLk_m^2 \) in the logarithm of the partition function Eq. (38), we obtain the result

\[
\langle F \rangle = -\frac{uL^2}{2c\pi^2} \sum_{m=1}^{\infty} \frac{1}{m^2} = -\frac{U_c}{12} \left(\frac{L}{L_c}\right)^2,
\]
where we have used the Riemann Zeta function \(\zeta(2) = \pi^2/6\) and the definitions Eqs. \((11)\) and \((12)\). Alternatively, we can use the Eqs. \((10)\) and \((11)\) and calculate \(\langle F \rangle(L, 0) = -(uL^2/c)\partial_x Z(s; 0)|_{s=0}\). With \(Z(s; 0) \approx 1 + s/12\) we then easily recover the above result. Note that the result Eq. \((10)\) measures the free energy \(\mathcal{F}_1 = T \ln \sqrt{2\pi L/T/c}\) of the free string.

The result for the free boundary condition \(\partial_x \phi|_{x=L} = 0\) is obtained by using an alternative expansion: first, we symmetrically extend the system from the interval \([0, L]\) to the interval \([0, 2L]\) with the definitions \(\phi(L + x) \equiv \phi(L - x)\), \(f(L + x) \equiv f(L - x)\). Second, we expand the solution to the interval \([-2L, 2L]\) using the same anti-symmetric extension as above. As a result, we can expand the displacement and force fields into modes \(\sin(q_m L)\) with \(q_m = (2m - 1)\pi/2L\), \(m = 1, \ldots, \infty\) and hence zero slope at \(x = L\). Following the same steps as above, we arrive at Eq. \((10)\) with \(y = 0\) and the product corresponding to the expansion of the cosine

\[
\cos \sqrt{s} = \prod_{n=1}^{\infty} \left[ 1 - \frac{4s}{\pi^2(2n - 1)^2} \right].
\]

The final result for the partition function with free boundary conditions then reads

\[
Z(s) = \frac{1}{\sqrt{\cos \sqrt{s}}},
\]

where the regime of applicability is restricted to the domain \(0 < s < \pi^2/4\); again, the singularity at \(s = \pi^2/4\) determines the shape of the left tail in the free-energy distribution function, cf. \((71)\). The alternative procedure of realizing the free boundary condition via integration over the end-point coordinate \(y\), cf. Eq. \((25)\), provides the identical result, although via a much more tedious calculation of determinants.

### B. Replica approach

The replica Hamiltonian Eq. \((20)\) for the random force problem Eq. \((3)\) reads

\[
H_n[\{\phi_a\}] = \frac{1}{2} \int_0^L dx \left\{ c \sum_{a=1}^n [\partial_x \phi_a(x)]^2 - \beta u \sum_{a,b=1}^n \phi_a(x) \phi_b(x) \right\}
\]

\[
= -\frac{1}{2} \int_0^L dx \sum_{a,b=1}^n \phi_a(x) U_{ab} \phi_b(x)
\]

with the matrix

\[
U_{ab} = c \delta_{ab} \partial_x^2 + \beta u.
\]

Accounting for the random shift \(V_0(x)\), cf. Eq. \((8)\), adds an additional term \(-n^2 \beta U_0 L/2\) to the Hamiltonian \((19)\).

The matrix \(U_{ab}\) can be easily diagonalized and we find one \((n - 1)\)-fold degenerate eigenvalue \(\lambda_1 = c \partial_x^2\) pertinent to the free string with the \((n - 1)\) orthonormal eigenvectors \(\xi_a\) obeying the constraint \(\sum_{a=1}^n \xi_a^2 = 0\), \(i = 1, \ldots, n - 1\). The \(n\)-th eigenvalue \(\lambda_2 = c \partial_x^2 + \beta n u\) is non-degenerate and appertains to an inverted harmonic potential problem; the associated eigenvector is \(\xi_n = 1/\sqrt{n}\), \(a = 1, \ldots, n\). The coefficients \(\xi_i\) of the \((n \times n)\) transformation matrix \(\xi_i\) satisfy the conditions \(\sum_{a=1}^n \xi_a^2 = \delta_{ij}\) (completeness) and \(\sum_{i=1}^n \xi_i^2 \xi_i^b = \delta_{ab}\) (orthonormality). In terms of the new fields \(\varphi_i(x)\) and boundary conditions \(q_i = \varphi_i(L)\),

\[
\varphi_i(x) = \sum_{a=1}^n \xi_a^i \phi_a(x), \quad q_i = \sum_{a=1}^n \xi_a^i y_a, \quad (52)
\]

the wave function (or propagator) Eq. \((19)\) takes the form

\[
\Psi(\{q_i\}; L) = \prod_{i=1}^{n-1} \int_0^{q_i} D[\varphi_i(x)] \int_0^{q_i} D[\varphi_n(x)] \exp \left[ -\frac{\beta c}{2} \int_0^L dx \sum_{i=1}^{n-1} [\partial_x \varphi_i(x)]^2 \right] \exp \left[ -\frac{\beta c}{2} \int_0^L dx \left[ [\partial_x \varphi_n(x)]^2 - \varphi_n^2(x)/\lambda_2 \right] \right],
\]

where we have introduced the length parameter \(\lambda\), cf. Eq. \((11)\).

\[
\lambda^2 = c \frac{L^2}{\beta n u} = \frac{s}{\beta n u}.
\]

The \((n - 1)\) free propagators are given by \(\Psi_0(q_i; L), \quad i = 1, \ldots, n - 1\), cf. Eq. \((25)\). The propagator \(\Psi_{1n}(q_n; L)\) for the inverted harmonic potential problem is obtained by solving the imaginary-time Schrödinger equation

\[
\partial_x \Psi_{1n}(q; x) = \frac{1}{2} \left[ \frac{1}{\beta c} \partial_x^2 + \frac{\beta c}{\lambda_2} q^2 \right] \Psi_{1n}(q; x)
\]

with the initial condition \(\Psi_{1n}(q; x = 0) = \delta(q)\). With the Gaussian Ansatz \(\Psi_{1n}(q; x) = \chi(x) \exp[-a(x) q^2/2]\) and proper accounting of the initial condition, we find the solution (cf. Ref. \(24)\)

\[
\Psi_{1n}(q_n; L) = \left( \frac{\sqrt{s}}{\sin \sqrt{s}} \right)^{1/2} \exp \left( -\frac{\sqrt{s}}{2L \tan \sqrt{s}} q_n^2 \right).
\]

Inserting the free (Eq. \((27)\)) and harmonic (Eq. \((30)\)) factors into the full propagator Eq. \((53)\) and transforming back to original variables, \(\sum_{i=1}^{n-1} q_i^2 = \sum_{i=1}^{n-1} q_i^2 - q_n^2 = \sum_{i=1}^{n-1} q_i^2 - (1/n)(\sum_{i=1}^{n-1} y_i)^2\), we obtain the result

\[
\Psi(\{y_a\}; L) = \prod_{a=1}^n \Psi_0(y_a; L) \frac{\Psi_{1n}(\sum_{i=1}^{n-1} y_i, \sqrt{n} L)}{\Psi_0(\sum_{a=1}^n y_a, \sqrt{n} L)}.
\]


Choosing the appropriate boundary conditions $y_a = y, a = 1, \ldots, n$, we obtain the replica partition function Eqs. (13) and (21) identical to the previous result Eq. (11), $Z_r(s; \epsilon) = Z(s; \epsilon), 0 < s < \pi^2$.

The result for the (replica) partition function has been derived for positive integer $n$. Since $Z(s; \epsilon)$ depends on $n$ only via the parameter $s$, the expression Eq. (11) can be analytically continued to the complex half-plane restricted by the condition Re$[s] < \pi^2$.

For the free boundary condition, we obtain the replica partition function via integration of Eq. (57) over all end-points $\{y_a\}$, cf. Eq. (29) to obtain the result identical to Eq. (48), $P_r(s) = 1/\sqrt{\cos s}, 0 < s < \pi^2/4$. Furthermore, the analytic continuation to real negative values of the parameter $n$ provides the expression

$$Z(s) = \frac{1}{\sqrt{\cosh s}} \quad (s < 0); \quad \text{(58)}$$

Alternatively, this result is obtained via the solution of the Schrödinger equation Eq. (60) for negative $n$ involving a summation over the discrete spectrum of the parabolic potential, see Appendix A.

C. Distribution function: fixed boundary condition

We now turn to the calculation of the free-energy distribution function $P_{L,y}(F)$ from the partition function $Z^n(L, y)$. Following the procedure described in Sec. II specifically Eqs. (14) and (15), the Laplace transform and its inverse assume the form

$$Z(s; \epsilon) = \int_{-\infty}^{+\infty} df \, p_c(f) \exp(-sf), \quad \text{(59)}$$

$$p_c(f) = \frac{1}{2\pi i} \int_{-i\infty}^{+i\infty} ds \, Z(s; \epsilon) \exp(sf), \quad \text{(60)}$$

where

$$f(F, L) = \frac{F}{F_f(L)}, \quad F_f(L) = \frac{u}{c} L^2 = U_c \left( \frac{L}{L_c} \right)^2, \quad \text{(61)}$$

$$\epsilon(y, L) = \frac{c^2 y^2}{2a L^2}, \quad \text{(62)}$$

are the rescaled free energy of the system and the rescaled displacement parameter; the original free-energy distribution function $P_{L,y}(F)$ then derives from the rescaled expression $p_c(f)$ through the relation

$$P_{L,y}(F) = \frac{1}{F_f(L)} p_c(y, L)(f(F, L)). \quad \text{(63)}$$

Note, that the parameter $\xi$ drops out in the combination $U_c/L_c^2$, as has to be the case for the random force model where the disorder is characterized by only one parameter, its strength $u$. Or in other words, using the results below for a random force (rather than a random potential) problem, these are valid for all length scales. For the relaxational free-energy distribution function of the system with fixed boundary condition, we obtain the expression

$$p_{\epsilon}(f) = \frac{1}{2\pi i} \int_{-i\infty}^{+i\infty} ds \left( \frac{\sqrt{s}}{\sin \sqrt{s}} \right)^{1/2} \exp(-\epsilon \frac{s\sqrt{s}}{\tan \sqrt{s}} + fs), \quad \text{(64)}$$

which simplifies drastically for the special case of fixed boundary conditions with $\phi(0) = \phi(L) = 0$,

$$P_{\epsilon}(f) = \frac{1}{2\pi i} \int_{-i\infty}^{+i\infty} ds \left( \frac{\sqrt{s}}{\sin \sqrt{s}} \right)^{1/2} \exp(fs). \quad \text{(65)}$$

The above result already expresses an important property of the distribution function $P_{\epsilon}(f)$: for $f > 0$ the expression under the integral is analytic and quickly goes to zero at $s \to -\infty$, hence the contour of integration in the complex plane can be safely shifted to $-\infty$. This implies that the function $P_{\epsilon}(f)$ must be equal to zero for $f > 0$ and the relaxational free energy of the directed polymer with zero boundary conditions is bounded from above, $F < 0$. This constraint then is easily understood, as the presence of a random force can only reduce the relaxational free energy of the directed polymer.

![FIG. 2: Relaxational free-energy distribution function $p_{\epsilon}(f)$ for the randomly forced directed polymer for several values of the dimensionless displacement parameter $\epsilon = (y/\xi)^2(L_c/L)^2/2$: $\epsilon = 0, 0.001, 0.005, 0.01$.](image-url)
distribution function $p_c(f)$ in the limits $f \to \pm \infty$ and $f \to -0$ are derived via a saddle-point integration and read,

\begin{align}
    p_c(f \to -\infty) &\sim \exp\left(-\pi^2 |f| \right), \\
    p_c(f \to +\infty) &\sim \exp\left(-\frac{4}{27\epsilon^2} |f|^3 \right), \quad (67) \\
    p_{c=0}(f \to -0) &\sim \exp\left(-\frac{1}{32 |f|} \right); \quad (68)
\end{align}

note that the shape of the left tail is determined by the singularity of $Z(s; \epsilon)$ at $s = \pi^2$, cf. Eq. (13). The above results agree with those obtained before in Ref. [20].

D. Distribution function: free boundary conditions

The result [18] provides us with all the moments of the relaxational free-energy distribution function, of which the first one, the average free energy, is given by

$$
\langle F \rangle = -U_c(L/L_c)^2 [\delta s Z(s)]_{s=0} = -\frac{U_c}{4} \left( \frac{L}{L_c} \right)^2. \quad (69)
$$

In order to obtain the full distribution function, we perform the inverse Laplace transform $\langle P_L(F) = p(f = F/F_f(L))/F_f(L) \rangle$

$$
p(f) = \frac{1}{2\pi i} \int_{-i\infty}^{+i\infty} ds \frac{1}{\sqrt{\cos \sqrt{s}}} \exp(f s). \quad (70)
$$

Given the scaling form in $f = F/F_f(L)$, the result is valid at all scales. Again, for $f > 0$ the integrand is analytic and rapidly approaches zero as $s \to -\infty$ and hence the function $p(f)$ must vanish identically for $f > 0$. The functional form for $f < 0$ is found as before, see Appendix [13]. The relaxational free-energy distribution function $\mathcal{P}(f)$ assumes a universal form with no parameters; it vanishes identically for $f > 0$ and its overall form is shown in Fig. 3.

Note that the free energy of the ‘trivial’ configuration $\phi(x) \equiv 0$ is equal to zero and any deviation due to the action of the random force can only reduce the energy, providing a simple explanation for the cutoff at positive energies. The asymptotic behavior in the limits $f \to -\infty$ and $f \to -0$ can be easily estimated by a saddle-point calculation,

\begin{align}
    p(f \to -\infty) &\sim \exp\left(-\frac{\pi^2}{4} |f| \right), \\
    p(f \to -0) &\sim \exp\left(-\frac{1}{32 |f|} \right). \quad (72)
\end{align}

E. Shifted random force model

In order to find the distribution function for the total free energy (rather than its relaxational part), we have to account for the random shift $V_0(x)$, cf. Eq. (58). Here, we concentrate on the situation with free boundary conditions. The multiplication of Eq. (18) with the Gaussian $\exp(\beta^2 n^2 U_0 L/2) = \exp[s^2 (L_c/L)^3/2]$ and subsequent Laplace transform of

$$
Z_r(s) = \frac{1}{\sqrt{\cos \sqrt{s}}} \sqrt{\frac{2}{\beta^2 \epsilon^2}} \quad (73)
$$
generates the (rescaled) free-energy distribution function $\mathcal{P}^r(f)$ shown in Fig. 4.

FIG. 3: Relaxational free-energy distribution function $p(f)$ of the randomly forced directed polymer with free boundary conditions.

FIG. 4: Free-energy distribution function $\mathcal{P}^r(f)$ of the randomly forced directed polymer with free boundary conditions including the random shift $V_0(x)$. For $L \gg L_c$, the relaxational part of the free energy dominates the distribution; in the regime $L < L_c$, where the shifted random force model provides an approximation to the random polymer problem, the free-energy distribution function is dominated by the Gaussian part originating from the shift $V_0(x)$.

The (total) free-energy distribution function $\mathcal{P}^t$ derives from a convolution of the distribution function $\mathcal{P}^r(F)$ of the relaxational free energy, cf. Eq. (70) and Fig. 3 and
the factor $P^V$ originating from the random shift $V_0(x)$,

$$P^V(F) = \int_{-\infty}^{\infty} dF' P^I(F') P^V(F - F'); \quad (74)$$

the contribution $P^V$ from the random shift assumes the simple Gaussian form

$$P^V(F) = \frac{1}{\sqrt{2\pi} F_V(L)} \exp[-(F/F_V(L))^2/2] \quad (75)$$

with the scaling parameter

$$F_V(L) = U_c \sqrt{L/L_c}. \quad (76)$$

For $L \gg L_c$, the result coincides with that for the relaxational energy, cf. Eq. (70) and Fig. 3 and scales as $F/F_V(L)$. In the limit of short lengths $L \lesssim L_c$, where the model can be used as an approximation of the random potential problem, the distribution function is dominated by the Gaussian due to the random shift $V_0(x)$. More specifically, we find the width of the distribution’s body through the calculation of the second cumulant: expanding $Z_r(s)$ for small values of $s$ and using Eq. (16), we obtain

$$Z_r(s) \approx 1 + \frac{s}{4} + \frac{s^2}{2} \left[ \frac{7}{48} + \left(\frac{L}{L_c}\right)^3 \right], \quad (77)$$

$$\langle F \rangle = -\frac{U_c}{4} \left(\frac{L}{L_c}\right)^2, \quad (78)$$

$$\overline{F^2 - F^2} = U_c^2 \frac{L}{L_c} \left[ 1 + \frac{1}{12} \left(\frac{L}{L_c}\right)^3 \right]. \quad (79)$$

The leading term $\propto U_c/L^2 = u/c$ in first moment Eq. (78) derives from the random force part of the disorder. This is different from the second cumulant in Eq. (79), where the first term $\propto U_c^2/L_c = U_0$ derives from the random shift $V_0(x)$ and dominates over the contribution from the random force (second term $\propto (u/c)^2$) at short lengths $L < L_c$, hence the width of $P^I$ is given by $F_V$ (see below for a discussion of the corrections $\propto (L/L_c)^3$).

Besides the first two moments/cumulants, we can easily determine the scaling of higher moments. Starting from the convolution Eq. (74), we note the different scaling of the arguments in the two distribution functions, $\propto F/L$ for the relaxational part $P^I$ and $\propto F/\sqrt{L}$ for the random shift part $P^V(F)$. Hence, for small distances $L$, the function $P^I(F')$ peaks narrowly near zero, while $P^V(F - F')$ retains a broader shape; expanding $P^V(F - F')$ around $F$ and integrating over $F'$, we obtain the following expansion for the total distribution function,

$$P^I(F) \approx P^V(F) - P^{V'}(F) \overline{F^2} + \frac{1}{2} P^{V''}(F) \overline{F^2} + \ldots \quad (80)$$

where $P^{V'}(F)$ is the derivative of $P^V$ with respect to the argument $F$ and $(\ldots)'$ denotes averaging over the random force part $P^I$. Using the scaling $\overline{F^2} \propto u L^2$ and $\overline{F^2} \propto u^2 L^4$ for the moments of the relaxational free-energy, we can calculate the dependence of the moments $\overline{F^k}$ on the length $L$. Thereby, we exploit the fact that the leading term $P^V(F)$ in the expansion (70) is symmetric in $F$, cf. Eq. (74), and hence determines the even moments, while the next term is anti-symmetric and generates the odd moments; finally, the third term provides the corrections to the even moments. The combination of the scaling of $P^V(F)$ (deriving from the random shift $V_0(x)$) and of the first two moments $\overline{F^I}$ and $\overline{F^2}$ (deriving from the random force $f(x)$) then generates the following non-trivial scaling of the moments with different powers in $L$ for the even and odd moments,

$$\overline{F^2k} \propto L^k + O(L^{k+3}), \quad (81)$$

$$\overline{F^{2k+1}} \propto L^{k+2}, \quad (82)$$

where $O(L^{k+3})$ denotes a correction term $\propto L^{k+3}$. In particular, $\overline{F} \propto u L^2$, $\overline{F^2} \propto u_0 L + O(u^2 L^3)$, $\overline{F^4} \propto u_0 L^3$, $\overline{F^6} \propto u_0^2 L^4$ and $\overline{F^8} \propto O(u^2 L^4)$.

Finally, we can estimate the tails of $P^I$ from the convolution Eq. (74) using the asymptotic behavior of $P^I$ and $P^V$ and find a left tail $P^I(F < -F_{\text{tail}}) \propto \exp(F/F_V)$ and a Gaussian tail on the right, $P^I(F > F_{\text{tail}}) \propto \exp[-(F/F_V)^2/2]$, where $F_{\text{tail}} = F_V[1 + (L/L_c)^{3/2}]$. On short scales $L < L_c$, we have $F_{\text{tail}} \approx F_V(L/L_c)^{3/2} > F_V$ and the random force behavior appears only quite beyond the body.

### F. Joint distribution function

We add a note on the joint free-energy distribution function $P_{L,y}(F, F')$, where $F = (F_+ + F_-)/2$ and $F' = (F_+ - F_-)/2$ denote the mean free energy and the free-energy difference for two polymer trajectories starting at the origin $\phi = 0$ at $x = 0$ and ending in the symmetric points $\phi = \pm y$ at $x = L$, $F_{\pm} = F(L, \pm y; V)$. Opposite to the $\delta$-correlated potential, cf. Ref. [16] the present case of the random force model is less revealing and we keep the discussion short.

Starting with the original (random) Hamiltonian $H[\phi(x); V]$ with the random force potential Eq. (65), we account for the boundary condition $\phi(L) = y$ through the shift $\phi(x) \rightarrow (y/L)x + \phi(x)$ (the $T = 0$ solution for the string ending in $\phi(L) = y$ derives from the solution ending in $\phi(L) = 0$ by adding the shift $(y/L)x$) and obtain the Hamiltonian described by Eqs. (53) and (57). The relaxational free energy of the system with the boundary condition $\phi(L) = y$ separates into the terms

$$F[L, y; f] = \frac{c y^2}{2L} + \frac{y}{L} \int_0^L dx x f(x) + F[L, 0; f]. \quad (83)$$

The first term is the trivial part of the elastic energy, the second is a random constant, and finally, the third is the
(random) relaxational free energy of the polymer with zero boundary conditions; its randomness is correlated with the randomness in the second term. Then, for the free energies $\bar{F}$ and $F'$ introduced above, we find that

$$F'[L, y; f] = \frac{y}{L} \int_0^L dx \, x f(x), \quad (84)$$

$$\bar{F}[L, y; f] = \frac{cy^2}{2L} + F[L, 0; f], \quad (85)$$

and hence $F'$ and $\bar{F}$ carry the information on the second and third terms in Eq. (63), respectively. Although the joint distribution function for the random and correlated quantities $\bar{F}$ and $F'$ must be non-trivial, we can conclude that the separate statistics of $\bar{F}$ and $F'$ must be simple: according to Eq. (64), the distribution for $F'$ is Gaussian with zero mean and width $(T')^2 = y^2uL/3$, while the distribution for $\bar{F}$ must coincide with that for the free energy with zero boundary conditions $P_{L, y=0}(F)$, cf. Eq. (65), shifted by the trivial elastic term $cy^2/2L$. Also note, that a change in the final coordinate $y$ modifies the polymer's trajectory over the entire length $L$ and hence the joint distribution function is not expected to factorize, in contrast to the results found for the short-range correlated random polymer problem.33 The detailed replica calculation, which represents a straightforward extension of the above analysis, produces results in full agreement with these simple arguments.

IV. HARMONIC-CORRELATOR APPROXIMATION

We consider the random directed polymer described by the Hamiltonian Eq. (1) and approximate the interaction $-\beta^2 U$ in the replica Hamiltonian Eq. (20) by the harmonic expression Eq. (9) to arrive at,

$$H_n[\{\phi_a\}] = \int_0^L dx \left\{ \frac{c}{2} \sum_{a=1}^n [\partial_x \phi_a(x)]^2 + \frac{\beta u}{4} \sum_{a,b=1}^n [\phi_a(x) - \phi_b(x)]^2 \right\} - \frac{n^2}{2} \beta U_0 L$$

$$= -\frac{1}{2} \int_0^L dx \sum_{a,b=1}^n \phi_a(x) \partial_x \phi_b(x) - \frac{n^2}{2} \beta U_0 L$$

with the matrix $\partial_x \phi_b(x)$ of (c$\partial_x^2 - \beta nu$) $\delta_{ab}$ + $\beta u$ (note that the parabolic approximation of the correlator should be implemented after the integration over the disorder potential). Diagonalization produces the $(n - 1)$-fold degenerate eigenvalue $\lambda_1 = c \partial_x^2 - \beta nu$ of the harmonic oscillator problem with the $(n - 1)$-th orthonormal eigenvectors $\xi_i^{(\alpha)}$ constrained by the condition $\sum_{\alpha=1}^n \xi_i^{(\alpha)} = 0$, $i = 1, \ldots, n - 1$, and one non-degenerate eigenvalue $\lambda_2 = c \partial_x^2$ of the free problem with the eigenvector $\xi_n^{(\alpha)} = 1/\sqrt{n}$, $a = 1, \ldots, n$.

The propagator for the harmonic-correlator approximation then assumes the form (cf. (57))

$$\Psi(\{y_a\}; L) = \prod_{a=1}^n \Psi_h(y_a; L) \Psi_0(\sum_b y_b/\sqrt{n}; L)$$

where $\Psi_h$ derives from $\Psi_{ih}$ by the substitution $\lambda \rightarrow i\lambda$. For simplicity, we only consider the model with free boundary conditions and find the shifted random force result Eq. (73) replaced by the expression $(s = L^2/\lambda^2 = n(U_c/T)^2(L/Lc)^2)$; see also Appendix A.

$$\bar{Z}_r(s) = \left[ \frac{1}{\sqrt{\cosh \sqrt{s}}} \right]^{(n-1)} \exp \left[ \frac{s^2(L_c/L)^3}{2} \right]$$

Although the inverse Laplace transform can be performed, the resulting (total) free-energy probability distribution $\bar{p}(f)$ develops a negative right tail at zero and low temperatures, see Fig. 5 at large temperatures $T \gg U_c$, the right tail exhibits pronounced oscillations. These unphysical results are due to the departure of the approximate harmonic interaction $U_p(\phi)$ from the true interaction $U(\phi)$, becoming relevant at large scales $L > L_c$, $\phi > \xi$, $F > U_c$, and the large fluctuations of the string at high temperatures $T \gg U_c$. Note that the inverse Laplace transform cannot be performed at all in case the random shift $V_0(x)$ is ignored.

![Figure 5: Free-energy distribution function $\bar{p}(f)$ for the directed polymer with free boundary conditions using the harmonic-correrlator approximation. Solid curves refer to $T = 0$, while the dashed curve attains to $T = 2U_c$.](image)
and using Eq. (10), we find the average free energy \( \overline{F} = -U_c(L/L_c)^3/4 \) (cf. Eq. (78)) and the second cumulant reads

\[
\overline{F^2} - \overline{F^2}^2 = U_c^2 \frac{L}{L_c} \left[ 1 - \frac{T}{2U_c} \frac{L}{L_c} - \frac{1}{12} \left( \frac{L}{L_c} \right)^3 \right]. \tag{90}
\]

Comparing with the result Eq. (79) for the shifted random force, we note the additional dependence on temperature and the sign-change in the correction term \( (L/L_c)^3 \). This decrease in width is in accord with the behavior of the free energy fluctuations in the random directed polymer problem (2), which scales as \( \delta F \propto L^{K-1} \) at large distances; with a wandering exponent \( \zeta = 2/3 \), we have \( \delta F \propto L^{1/3} \). The negative correction \( (L/L_c)^3 \) to the linear growth in \( \delta F^2 \) observed in Eq. (90) then is consistent with the sublinear growth \( \delta F^2 \propto L^{2/3} \) of the exact solution.

As before, we can analyze the higher moments of the distribution function and compare the results for the harmonic-correlator approximation at \( T = 0 \) with those obtained for the shifted random force model. Making use of the product form of \( \tilde{Z} \), and expanding each factor in \( s \), we find identical leading terms for all even and odd moments (corresponding to equal expressions for the first two terms in Eq. (S8)), while the corrections to the even moments (described by the third term in Eq. (S9)) are different. Furthermore, we note that the even moments \( \overline{F^{2k}} \propto U_c^2(\langle L/L_c \rangle)^k \left[ 1 - \mathcal{O}(L^3/L_c^3) \right] \propto U_c^2 L^k \) are large with a small negative correction, while the odd moments \( \overline{F^{2k+1}} \propto \delta F^{(2k+1)} \propto U_c^2(\langle L/L_c \rangle)^{k+2} \left[ 1 + \mathcal{O}(L^3/L_c^3) \right] \propto uU_c^2 L^{k+2} \) are small, their ratios being \( \left( \overline{F^{2k+1}} \right)^2 / \overline{F^{2k+1}} \propto (L/L_c)^3 \). To leading order, the free-energy distribution function for the random potential model at small scales then is a trivial Gaussian generated by \( V_0(x) \), with a small negative shift and a small reduction in width due to the random force term in the potential.

At \( T = 0 \), the second cumulant turns negative for \( L > 2\sqrt{3/2} L_c \) and the result Eq. (90) makes no longer any sense, hence the harmonic approximation to a random potential problem cannot be used on scales larger then \( L \) (along the transverse direction) or \( L_c \) (along the longitudinal direction); at finite temperatures the regime of validity is further reduced.

Although the results for the shifted random force remain valid at any length \( L \), we emphasize that the harmonic correlator provides a better approximation for the behavior of the short-range correlated random polymer: Both results agree in lowest order, providing the same first moment \( \tilde{F} \) due to the random force \( f(x) \) and the same leading term in the second cumulant \( \overline{F^2} - \overline{F}^2 \) generated by the random shift \( V_0(x) \). The correction \( \propto (L/L_c)^3 \) in the second cumulant is due to the random force \( f(x) \) and contributes with the opposite sign in the shifted random force model as compared to the harmonic correlator approximation. While the shifted random force result Eq. (79) is correct (at all scales) when dealing with a true random force model, the correction \( \propto (L/L_c)^3 \) carries the wrong sign when used as an approximation to the random potential model and it is the result of the harmonic correlator approximation Eq. (90) which should be trusted. Indeed, the harmonic correlator preserves the translation invariance of the problem, whereas some quadratic terms are dropped from the shifted random force model. Expanding the potential to second order,

\[
V(\phi, x) = V_0(x) + f(x)\phi(x) - \frac{1}{2}g(x)\phi^2(x), \tag{91}
\]

we identify the terms in \( \overline{V(\phi, x)V(\phi', x')} \) with the harmonic expansion Eq. (9) to obtain the correlators Eqs. (7) and (8), \( V_0(x)g(x') = u\delta(x-x') \), and vanishing mixed terms \( V_0(x)f(x') = f(x)g(x') = 0 \). A scaling estimate of the second moment \( \overline{F^2} \) using Eq. (91) then provides a leading term \( \propto L \) from \( V_0 \) and subleading terms \( \propto L^4 \) from \( f \) and \( g \). The contribution from the random force provides the positive contribution \( U_c(\langle L/L_c \rangle)^4/12 \) in the cumulant Eq. (79), while the mixed terms \( V_0(x)g(x') \) contribute with a negative weight, generating the negative correction \( \propto L^4 \) in Eq. (91). Note that higher order terms do not change this result but contribute to the next order term \( \propto L^7 \).

Given that the harmonic correlator provides the better approximation to the random polymer problem at short scales, one may wonder why we end up with unphysical results (negative distribution function, negative second moment) at larger scales. Also, different types of correlators, e.g., power-law type, have been studied in the past, cf. Refs. 1 and 21, and one would like to know, what properties of a disorder correlator guarantee consistent results; this question is addressed in the following section.

### A. Correlators with non-positive spectrum

It is important to identify problematic correlators right from the beginning; indeed, the proper definition of the disorder potential is subject to important constraints\(^{25}\) regarding its shape \( U(\phi) \) and failure to respect these constraints may lead to unphysical results. Consider a random potential \( V(\phi) \) and its Fourier representation

\[
V(q) = \int dq V(q) \exp(-iq\phi); \tag{92}
\]

then the Gaussian distribution function of the random function \( V(q) \) has the form

\[
P[V(q)] = P_0 \exp \left( - \int \frac{dq}{2\pi} \frac{|V(q)|^2}{2G(q)} \right); \tag{93}
\]

the width \( G(q) \) has to be positive and relates to the correlation function \( U(\phi) \) via

\[
U(\phi) = \int \frac{dq}{2\pi} G(q) \exp(iq\phi). \tag{93}
\]
Expanding both sides in powers of $\phi$,

$$U(0) + \sum_{k=1}^{\infty} \frac{U^{(2k)}(0)}{(2k)!} \phi^{2k}$$  \hspace{1cm} (94)

and comparing coefficients, we find that the $2k$-th derivative of $U(\phi)$ in the origin relates to the integral

$$\int dq \, G(q) \phi^{2k} = 0 \quad \text{for} \quad k \geq k^*,$$  \hspace{1cm} (95)

which cannot be satisfied for a positively defined $G(q)$.

Obviously then, choosing a parabolic correlator $U_p(\phi)$ as in Eq. (9) is in severe conflict with the constraint Eq. (95). The averaging over the disorder potential $V(\phi, x)$ is undefined for those modes (in Fourier space) where $G(q)$ is negative. Hence, going over from the disordered directed polymer (a statistical mechanics problem) to the quantum boson problem is an ill-defined step and the results cannot be trusted any longer. On the other hand, performing the integration over the random potential $V(\phi, x)$ with a well defined, i.e., spectral, correlator $U(\phi)$ and expanding the resulting interaction $-\beta^2 U(\phi)$ between bosons is perfectly admissible and leads to an identical result; in this case, however, we know that the quantum boson problem does not describe the random polymer problem on scales where the approximate quadratic correlator deviates strongly from the original correlator. Nevertheless, in the end we have to appreciate, that the harmonic-correlator approximation (9), although breaking down at lengths beyond $L_c$, does produce more accurate approximate results for the short-range correlated random potential problem (2) than the shifted random force model (3), although the latter remains formally valid at all length scales $L$. The (shifted) random force model then should be used whenever the disorder landscape is given by a force field as defined by Eqs. (3), (7), and (8), but not as an approximation to a random potential problem.

### B. Displacement correlator

Another quantity of interest in the random polymer problem is the displacement correlator $\langle \phi(L) \rangle_L(\phi(0))$, with $\langle \ldots \rangle$ and $\langle \ldots \rangle$ denoting thermal and disorder averages, respectively. Choosing free boundary conditions with $\phi(0) = 0$ and an arbitrary position $\phi(L) = y$ for the end-point, the averages $\langle \bar{y}^2 \rangle$ and $\langle \bar{y} \rangle^2$ are easily calculated within replica theory \[24\]. Defining

$$\langle \bar{y}_a \bar{y}_b \rangle = \left[ \prod_{c=1}^{n} \int dy_c \right] \bar{y}_a \bar{y}_b \Psi(\{y_c\}; L),$$  \hspace{1cm} (96)

we obtain the two types of averages

$$\langle \bar{y}^2 \rangle = \lim_{\tilde{n} \to 0} \langle \bar{y}_a \bar{y}_a \rangle |_{a=b},$$  \hspace{1cm} (97)

$$\langle \bar{y} \rangle^2 = \lim_{\tilde{n} \to 0} \langle \bar{y}_a \bar{y}_b \rangle |_{a \neq b}.$$

The Hamiltonians for the shifted random force model and the harmonic-correlator approximation differ only by the term $(\beta n/2) \sum_{a=1}^{\tilde{n}} \delta^2$, which vanishes in the limit $n \to 0$, hence both schemes produce identical results for the displacement correlators in Eq. (97). We then concentrate on the random force case and calculate the expression

$$\langle \bar{y}_a \bar{y}_b \rangle = C \left[ \prod_{c=1}^{n} \int dy_c \right] \bar{y}_a \bar{y}_b \exp \left[ -\frac{1}{2} \sum_{c,d} K_{cd} y_c y_d \right]$$  \hspace{1cm} (98)

with $K_{cd} = A \delta_{cd} + B$ and

$$A = \frac{\beta c}{L}, \hspace{1cm} B = \frac{\beta c}{n L} \left[ \frac{\sqrt{s}}{\tan \sqrt{s}} - 1 \right],$$

$$C = \left( \frac{\beta c}{2 \pi L} \right)^{n/2} \left( \frac{\sqrt{s}}{\sin \sqrt{s}} \right)^{1/2}.$$

In the calculation of $\langle \bar{y}_a \bar{y}_b \rangle |_{a \neq b}$, we combine all diagonal terms into a sum $(D/2) \sum c y_c^2$ with $D = A + B$, leaving the non-diagonal in the form $(B/2) \sum_{c \neq d} y_c y_d$: the non-diagonal average then follows from the derivative

$$\langle \bar{y}_a \bar{y}_b \rangle |_{a \neq b} = -\frac{2}{n(n-1)} \frac{\partial}{\partial B} \left[ \prod_{c=1}^{n} \int dy_c \right] \Psi(\{y_c\}; L) \Big|_{D},$$

while the diagonal average is given by the derivative

$$\langle \bar{y}^2 \rangle = \frac{2}{n} \frac{\partial}{\partial A} \left[ \prod_{c=1}^{n} \int dy_c \right] \Psi(\{y_c\}; L) \Big|_{B}.$$  \hspace{1cm} (99)

The final results assume the form

$$\langle \bar{y}^2 \rangle = \lim_{\tilde{n} \to 0} \left( \frac{1}{A} - \frac{B}{A^2} \right), \hspace{1cm} \langle \bar{y} \rangle^2 = \frac{\xi^2 T}{U_c L_c} + \frac{\xi^2}{3} \left( \frac{L}{L_c} \right)^3,$$

$$\langle \bar{y}^2 \rangle = \lim_{\tilde{n} \to 0} \frac{B}{(D - B)^2} = -\lim_{\tilde{n} \to 0} \frac{B}{A^2} = \frac{\xi^2}{3} \left( \frac{L}{L_c} \right)^3;$$

the relation $\langle \bar{y}^2 \rangle - \langle \bar{y} \rangle^2 = \langle \bar{y}^2 \rangle |_{V=0} = TL/c$ (here, $\langle \bar{y}^2 \rangle |_{V=0}$ denotes the thermal average in the absence of any disorder, $V = 0$), is a constraint holding true for any disorder potential uncorrelated in $x$, cf. Refs. 20 and 27.
V. SUMMARY AND CONCLUSIONS

The (shifted) randomly forced polymer model and the random disordered polymer described through a harmonic-correlator approximation define quadratic problems and hence admit exact solutions. For the random force models, different approaches can be taken, either a direct integration of the path-integrals within a Fourier representation or using the (real space) replica technique; in retrospect, the preferred method is a matter of taste. We have determined the free-energy distribution functions $P_{L,y}(F)$ and $P_L(F)$ for fixed and free boundary conditions. This calculation necessitates the determination of all powers $Z_r(n; L, y) = Z^n[L, y; V]$ (rather than the usual $n \to 0$ limit) and subsequent inverse Laplace transformation of the analytically continued replica partition function $Z_r(n \in C; L, y)$. The displacement correlators $(\langle y^2 \rangle)$ and $(\langle y \rangle^2)$ have been found as well. The simplicity of the quadratic models allows to carry through the entire program and thus serves to study not only the physical properties of the problem but its methodological aspects as well.

Regarding the shape of the distribution functions for the random force model, a number of interesting features has been obtained: for the free boundary, the probability to find a positive free energy $F$ vanishes exactly, with an essential singularity appearing in $P_L(F) \propto \exp(-uL^2/32c|F|)$ as $F$ approaches zero from the left, cf. Eq. (72). For fixed boundary conditions, a similar result has been found for $P_{L,y=0}(F)$, see also Ref. [20]. Furthermore, the left and right tails provide a consistent scaling $F \propto L^2$ and $y \propto L^{3/2}$, $P_{L,y}(F \to -\infty) \propto \exp(-\pi^2c|F|/uL^2)$ and $P_{L,y}(F \to \infty) \propto \exp(-16/27|F|^3/ucy^4)$, cf. Eqs. (66) and (67).

When interested in the short distance behavior of the random directed polymer Eq. (2), two types of approximations offer a drastic simplification of the problem: these are the expansion of the random potential $V(\phi, x)$ according to Eq. (5) (generating the shifted random force problem) or the expansion of the correlator Eq. (9) (leading to the harmonic-correlator approximation). While both approximations generate the same results for the even and odd moments to leading order, the next to leading order terms turn out different. In this situation, the results of the harmonic-correlator approximation have to be trusted, as it consistently accounts for the relevant terms preserving the translation invariance of the problem. Collecting all results, we find that the free-energy distribution function for the random potential model at $T = 0$ and) small scales is a trivial Gaussian of width $U_c\sqrt{L/L_c}$ generated by $V_0(x)$, with a small negative shift $-U_c(L/2L_c)^2$ and a small reduction $-(U_c/24)(L/L_c)^7/2$ in width due to the random force term in the potential.

Finally, we mention a few useful insights regarding the replica technique which derive from our analysis above. The replica technique provides a link between two seemingly unrelated problems, the classical statistical theory of disordered polymers and the quantum many-body theory of attractive bosons. Several stumbling blocks can be eliminated by properly appreciating the subtleties in this mapping. As is well known, after the mapping from polymers to bosons the disorder correlator assumes the role of the interaction potential. While many shapes for the interaction potential may produce meaningful results for the quantum boson problem, only a restricted set of them (those describing a correlator with positive spectrum) relate to a meaningful random polymer problem. Hence, the original choice of physical correlators and any modification thereof during the calculation should be done with great care; in particular, a simple power-law form might not work. E.g., there is nothing wrong in studying quantum bosons with a simple harmonic interaction $U(\phi) = -U_0 + u\phi^2/2$ and the results obtained for the quantum propagator are perfectly acceptable for any constant shift $U_0$. However, interpreting the result for the propagator in terms of a replica partition function and transforming back (via the inverse Laplace transformation) to random polymers, the resulting distribution function becomes unphysical when setting $U_0 = 0$; dropping a shift $U_0$ in the potential for the bosons is a trivial shift in energy, while ignoring the same shift in the correlator produces unphysical results for the polymer problem after Laplace transformation.

Acknowledgments

We thank Sergey Korshunov for numerous enlightening discussions and acknowledge financial support from the Pauli Center at the ETH Zurich and the Swiss National Foundation.

Appendix A: Negative replica number

We determine the replica partition function $Z_r(s)$, Eq. (53), for the polymer with free boundary conditions via direct solution of the Schrödinger equation Eq. (54) for negative $n$. We confirm, that the result analytically continued from positive $n$ agrees with the one obtained for negative $n$. The wave function $\Psi(q, x)$ satisfies the Schrödinger equation (cf. Eq. (53))

$$\partial_x \Psi(q, x) = \frac{1}{2} \left[ \frac{1}{\beta c} \frac{\partial^2}{q^2} - \frac{\beta c}{\lambda^2} q^2 \right] \Psi(q, x), \quad (A1)$$

with $\lambda^2 = c/\beta|n|u$. We are seeking the solution

$$\Psi(q, x) = \sum_{k=0}^{\infty} A_k \exp(-E_k x) \Psi_k(q), \quad (A2)$$

satisfying the initial condition $\Psi(q; x = 0) = \delta(q)$; the energies and corresponding orthonormal eigenfunctions
$E_k$ and $\Psi_k(q)$ satisfy the stationary equation,

$$E_k\Psi_k(q) = -\frac{1}{2} \left[ \frac{1}{\beta c} q^2 + \frac{\beta c}{\lambda^2} q^2 \right] \Psi_k(q). \quad (A3)$$

The coefficients $A_k$ in Eq. (A2) derive from the initial condition

$$A_k = \int_{-\infty}^{+\infty} dq \, \Psi_k(q) \Psi(q; x = 0) = \Psi_k^*(0).$$

The spectrum of the harmonic problem is given by $E_k = (k + 1/2)/\lambda$ and the corresponding eigenfunctions are (see, e.g., Ref. 28)

$$\Psi_k(q) = \left( \frac{\beta c/\lambda}{\sqrt{\pi 2^k} k!} \right)^{1/2} \exp\left[ -\left( \beta c/2\lambda \right) q^2 \right] H_k[\sqrt{\beta c/\lambda q}],$$

where $H_k(x)$ are the Hermite polynomials $H_k(x) = (-1)^k \exp(x^2) \frac{d^k}{dx^k} \exp(-x^2)$. Substituting $A_k$ and $\Psi_k$ into Eq. (A2) and taking into account that $H_{2k+1}(0) = 0$, we obtain the wave function

$$\Psi(q; x) = \sqrt{\frac{\beta c/\lambda}{\sqrt{\pi}} \sum_{l=0}^{\infty} \frac{(-1)^l}{2^{2l} (2l)!} \exp(-E_{2l} x)} \times \exp\left[ -\left( \beta c/\lambda \right) q^2 \right] H_{2l}[\sqrt{\beta c/\lambda q}] H_{2l}(0).$$

With the spectrum $E_k(\lambda)$ and the normalization $H_{2l}(0) = (-1)^l (2l)!/2^{2l}!$, we obtain the replica partition function for free boundary conditions (cf. Eq. 13)

$$Z_r(n; L) = Z(s) = \int_{-\infty}^{+\infty} dq \, \Psi(q; x = L) = \frac{\exp(-\sqrt{s}/2)}{\sqrt{\pi}} \sum_{l=0}^{\infty} \frac{(-1)^l}{2^{2l} (2l)!} \exp(-2\sqrt{s} l) C_{2l}$$

where $s = L^2/\lambda^2$ and

$$C_k = \int_{-\infty}^{+\infty} dx \exp(-x^2/2) H_k(x). \quad (A6)$$

Using the recurrence relation $H_{k+1}(x) = 2x H_k(x) - 2k H_{k-1}(x)$, we find that $C_{k+2} = 2(k + 1) C_k$ and with $C_0 = \sqrt{2 \pi}$, we obtain the coefficients $C_k = \sqrt{2 \pi (2l)!}/l!$. Substitution into Eq. (A5) provides the replica partition function in the form

$$Z(s) = \sqrt{s} \exp(-\sqrt{s}/2) R[\eta(s)]$$

with the function $R(\eta)$ defined by the series

$$R(\eta) = \sum_{l=0}^{\infty} \frac{(2l)!}{(l!)^2} \eta^l \quad (A8)$$

and we have introduced the shorthand $\eta(s) = -\exp(-2\sqrt{s})/4$.

In order to find the explicit form of the function $R(\eta)$, we implement the shift $l \to l + 1$ in the sum (A3) and obtain,

$$R(\eta) = 1 + \sum_{l=0}^{\infty} \frac{(2l)!}{(l!)^2} \eta^l = 1 + \sum_{l=0}^{\infty} \frac{(2l + 2)!}{(l!)^2} \eta^{l+1}$$

$$= 1 + 4\eta \sum_{l=0}^{\infty} \frac{(2l)!}{(l!)^2} \eta^l - 2 \sum_{l=0}^{\infty} \frac{(2l)!}{(l + 1)(l!)^2} \eta^{l+1}$$

$$= 1 + 4\eta R(\eta) - 2 S(\eta), \quad (A9)$$

$$S(\eta) = \sum_{l=0}^{\infty} \frac{(2l)!}{(l + 1)(l!)^2} \eta^{l+1}. \quad (A10)$$

With $R$ the derivative of $S$. $R(\eta) = \partial_\eta S(\eta)$, we obtain the differential equation $\partial_\eta S(\eta) = 1 + 4\eta \partial_\eta S(\eta) - 2 S(\eta)$ and the initial condition $S(0) = 0$ determines the solution $S(\eta) = (1 - \sqrt{1 - 4\eta})/2$, from which $R(\eta) = 1/\sqrt{1 - 4\eta}$ follows via integration. Substitution into Eq. (A7) produces the final result $Z(s) = 1/\cosh \sqrt{s}$, in agreement with Eq. (58).

**Appendix B: Inverse Laplace transformations**

The inverse Laplace transforms Eqs. (64) and (70) are reduced to the following expressions: Using the transformation $s = \rho \exp(\pm i/2)$ in Eq. (64), we analytically continue the expression for the distribution function $p_c(f)$ to the imaginary axis,

$$p_c(f) = \frac{1}{\pi} \text{Re} \int_0^{\infty} d\rho \left( \frac{\sqrt{\rho} \exp(i\pi/4)}{\sin[(1+i)\sqrt{\rho}/2]} \right)^{1/2}$$

$$\times \exp\left[ -\frac{\left( 1 - i \right) \rho \sqrt{\rho^2}}{\tan[(1+i)\sqrt{\rho^2}] + i f \rho} \right].$$

A change of the integration variable $\rho = 2t^2$ provides, after some algebra, the final expression

$$p_c(f) = \frac{\gamma^{5/2}}{\pi} \int_0^{\infty} dt \, t^{3/2} \exp[-\omega(t)]$$

$$\times \frac{\cos[\gamma(t)/2 + 2t^2 f + \pi/8 - \omega_+(t)]}{\sqrt{\Phi(t)}}. \quad (B2)$$

The functions $\Phi(t)$, $\omega_\pm(t)$, and $\gamma(t)$ are defined as,

$$\Phi(t) = \sqrt{\sin(t) \cosh(t)}^2 + [\cos(t) \sinh(t)]^2,$n

$$\omega_\pm(t) = t^2 \sinh(2t) \pm \sin(2t) \Phi(t),$$

$$\sin(\gamma(t)) = - \cos(t) \sinh(t)/\Phi(t),$$

$$\cos(\gamma(t)) = \sin(t) \cosh(t)/\Phi(t).$$

Similarly, substituting $s = 2t^2 \exp(\pm i/2)$ in Eq. (70), one obtains

$$p(f) = \frac{4}{\pi} \int_0^{\infty} dt \, \cos[\zeta(t)/2 + 2ft^2] \frac{\Phi(t)}{\sqrt{\Phi(t)}}. \quad (B3)$$
with the functions $\Psi(t)$ and $\zeta(t)$ defined by

$$\Psi(t) = \sqrt{[\cos(t) \cosh(t)]^2 + [\sin(t) \sinh(t)]^2},$$

$$\sin(\zeta(t)) = \sin(t) \sinh(t)/\Psi(t),$$

$$\cos(\zeta(t)) = \cos(t) \cosh(t)/\Psi(t).$$

The remaining integrals in Eqs. [152] and [153] have to be done numerically.

---

1. T. Halpin-Healy and Y-C. Zhang, Phys. Rep. 254, 215 (1995).
2. S. Lemerle, J. Ferré, C. Chappert, V. Mathet, T. Giamarchi, and P. Le Doussal, Phys. Rev. Lett. 80, 849 (1998).
3. G. Blatter, M.V. Feigel’man, V.B. Geshkenbein, A.I. Larkin, and V.M. Vinokur, Rev. Mod. Phys. 66, 1125 (1994).
4. D. Wilkinson and J.F. Willemsen, J. Phys. A 16, 3365 (1983).
5. J.M. Burgers, The Nonlinear Diffusion Equation (Reidel, Dordrecht, 1974).
6. D.A. Huse and C.L. Henley, Phys. Rev. Lett. 54, 2708 (1985); M. Kardar and Y-C. Zhang, Phys. Rev. Lett. 58, 2087 (1987).
7. M. Kardar, Nucl. Phys. B 290, 582 (1987).
8. Y-C. Zhang, Europhys. Lett. 9, 113 (1989).
9. M. Práchofer and H. Spohn, Phys. Rev. Lett. 84, 4882 (2000).
10. I.V. Kolokolov and S.E. Korshunov, Phys. Rev. B 75, 140201(R) (2007); I.V. Kolokolov and S.E. Korshunov, Phys. Rev. B 78, 024206 (2008); see also I.V. Kolokolov and S.E. Korshunov, Phys. Rev. E 80, 031107 (2009), where the tails in the free-energy distribution are calculated for the random polymer problem with a finite-width correlator.
11. V. Dotsenko, ‘Bethe Ansatz derivation of the Tracy-Widom distribution for one-dimensional directed polymers’, [arXiv:1003.4890] and [arXiv:1004.4459] (2010).
12. P. Calabrese, P. Le Doussal, A. Rosso, ‘Free-energy distribution of the directed polymer at high temperature’, [arXiv:1002.4560] (2010).
13. T. Sasamoto, H. Spohn, ‘Universality of the one-dimensional KPZ equation’, [arXiv:1002.1883] (2010) and ‘Exact height distributions for the KPZ equation with narrow wedge initial condition’, [arXiv:1002.1879] (2010).
14. G. Amir, I. Corwin, J. Quastel, ‘Probability Distribution of the Free Energy of the Continuum Directed Random Polymer in 1 + 1 dimensions’, [arXiv:1003.0443] (2010).
15. D.A. Huse, C.L. Henley, and D.S. Fisher, Phys. Rev. Lett. 55, 2924 (1985).
16. G. Parisi, J. Phys. France 51, 1595 (1990).
17. M. Práhofer and H. Spohn, J. Stat. Phys. 108, 1071 (2002).
18. V.S. Dotsenko, L.B. Ioffe, V.B. Geshkenbein, S.E. Korshunov and G. Blatter, Phys. Rev. Lett. 100, 050601 (2008).
19. A.I. Larkin, Zh. Eksp. Teor. Fiz. 58, 1466, 1970. [Sov. Phys. JETP 31, 784, 1970].
20. D.A. Gorokhov and G. Blatter, Phys. Rev. Lett. 82, 2705 (1999).
21. G. Parisi, Rend. Fis. Acc. Lincei IX-1, 277 (1990).
22. D.A. Gorokhov, Dynamics of Disordered Systems, PhD Thesis, ETH Zürich, Diss. ETH No. 13070.
23. Handbook of Mathematical Functions, edited by M. Abramowitz and I.A. Stegun (Dover, New York, 1965).
24. J. Zinn-Justin, Quantum Field Theory and Critical Phenomena (Clarendon, Oxford, 1993).
25. D.A. Gorokhov and G. Blatter, Phys. Rev. B 59, 32 (1999).
26. U. Schütz, J. Villain, E. Brézin, and H. Orland, J. Stat. Phys. 51, 1 (1988).
27. S.E. Korshunov, Phys. Rev. B 63, 174514 (2001).
28. L.D. Landau and E.M. Lifshitz, in Quantum Mechanics, Vol. 3 of Course in Theoretical Physics (Pergamon Press, London/Paris, 1958).