 BERGMAN AND CARATHEODORY METRICS OF
THE KOHN-NIRENBERG DOMAINS

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ABSTRACT. The Kohn-Nirenberg domains are unbounded domains in $\mathbb{C}^2$ upon which many outstanding questions are yet to be explored (cf., e.g., [5]). The primary aim of this article is to demonstrate that the Bergman and Caratheodory metrics of any Kohn-Nirenberg domain are positive and complete.

1. INTRODUCTION

The following definition of the Kohn-Nirenberg domains is due to Fornæss [5].

Definition 1.1. A Kohn-Nirenberg domain (or, a KN-domain for the sake of brevity) is defined to be the set

$$\Omega = \{(z_1, z_2) \in \mathbb{C}^2 : \text{Re } z_2 + P_{2k}(z_1) < 0\},$$

where $P_{2k}$ is a real-valued polynomial in $z_1$ and $\bar{z}_1$ satisfying the following two conditions:

(i) There exists an integer $k > 1$ such that $P_{2k}(rz) = r^{2k}P_{2k}(z)$ for any $r \in \mathbb{R}$ and $z \in \mathbb{C}$.

(ii) $\frac{\partial^2 P_{2k}}{\partial z_1 \partial \bar{z}_1}|_{z_1 > 0} > 0$ for every $z_1 \neq 0$.

The primary result of this paper is

Theorem 1.2. The Bergman and Caratheodory metrics of any Kohn-Nirenberg domain are positive and complete.

Deferring further details to the next section, we clarify some terminology: (1) the positivity of a metric means that the length of any nonzero tangent vector is positive, and (2) the completeness of a metric means that the space equipped with its integrated distance is Cauchy-complete.

The study of KN-domains was initiated by the well-known work of Kohn and Nirenberg [14] which demonstrated the existence of pseudoconvex domains with a boundary point that does not admit, even locally, any holomorphic support functions and consequently, the boundary cannot be convexifiable there, via any change of holomorphic local coordinates. A representing example is the domain $\Omega = \{(z_1, z_2) \in \mathbb{C}^2 : \text{Re } z_2 + |z_1|^8 + \frac{15}{2}|z_1|^2 \text{ Re } z_1^6 < 0\}$ with the boundary point $(0,0) \in \partial \Omega$.

Thereafter, many more questions have been posed on these domains, some of which still remain open. One remarkable result is by Fornæss [5], which proves

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the sup-norm estimate for the $\bar{\partial}$-operator near the origin, a typical boundary point with no holomorphic support functions.

It is still an open question whether every Kohn-Nirenberg domain is biholomorphic to a bounded domain. Another question along the same line was whether their Bergman metric is positive (‘positive-definite’, in some literatures) and complete. Notice that Theorem 1.2 answers this question affirmatively.

Then, one recalls that [14] also introduces the domain

$$W_{KN} := \{(z_1, z_2) \in \mathbb{C}^2 : \text{Re} z_2 + |z_1 z_2|^2 + |z_1|^8 + \frac{15}{7}|z_1|^2 \text{Re} z_1^6 < 0\}.$$  

This domain is special because it still does not allow any holomorphic local support function at the origin, despite the fact that all of its boundary points except the origin are now strongly pseudoconvex. Notice however that this modified defining function is not weighted-homogeneous any more. Similar domains with a degree 6 polynomial defining function of this type were discovered by Fornæss [4]; the domain

$$W_{For} := \{(z_1, z_2) \in \mathbb{C}^2 : \text{Re} z_1 + |z_1 z_2|^2 + |z_1|^6 + t|z_1|^2 \text{Re} z_1^4 < 0\}$$

for any $t$ satisfying $1 < t < 9/5$ enjoys the same property. Consequently, one naturally asks whether the conclusion of the theorem above continues to hold for these domains.

It turns out that the weighted homogeneity of the defining polynomial is not essential in the proof of Theorem 1.2 and that a slight modification (cf. Proposition 3.4) yields

**Theorem 1.3.** The Bergman and Caratheodory metrics of the domains $W_{KN}$ and $W_{For}$ are positive and complete.

It seems to us that unbounded domains have emerged recently and formed a territory in Several Complex Variables of high research interests (cf., e.g., [11]). In particular, it is not at all clear whether the unbounded domains admit sufficiently many, independent, $L^2$ or $L^\infty$, holomorphic functions so that their Caratheodory and Bergman metrics can be seen to be positive, and furthermore, complete.

For general complex manifolds, there are well-known theorems based upon negativity of curvatures, e.g., [7]. But for domains in $\mathbb{C}^n$, it is natural to ask whether their Bergman metric, for instance, can be seen to be positive and complete, directly from their defining functions.

One notable recent result in this direction is the following

**Theorem 1.4** (Chen-Kamimoto-Ohsawa [3]). If $\rho: \mathbb{C}^n \to \mathbb{R}$ is a nonnegative continuous plurisubharmonic function satisfying $\lim_{\|z\| \to \infty} \rho(z) = +\infty$, then the Bergman metric of the domain $\Omega := \{(z, w) \in \mathbb{C}^n \times \mathbb{C} : \text{Re} w + \rho(z) < 0\}$ in $\mathbb{C}^{n+1}$ is positive-definite and complete.

Notice however that many Kohn-Nirenberg domains are not covered by this theorem. Furthermore, the arguments and methods of this paper are different from those of [3].

2. Basic observations for KN-domains

2.1. The Bergman kernel. The Bergman kernel of a domain $W$ in $\mathbb{C}^n$, denoted by $K_W(z, w)$, for $z, w \in W$, is constructed upon the Hilbert space $A^2(W)$ of $L^2$
holomorphic functions of $W$. (This space may happen to be trivial, however, since $W$ could be unbounded. In case $W$ is a complex manifold of dimension $n$, $A^2(W)$ shall mean the Hilbert space of holomorphic $(n,0)$-forms, say $\alpha$, satisfying $\int_W \alpha \wedge \bar{\alpha} < +\infty$.) Then with the complete orthonormal system \{\varphi_j: j = 1, 2, \ldots \} one has

$$K_W(z,w) = \sum_{j=1}^{\infty} \varphi_j(z) \bar{\varphi}_j(w).$$

(For the manifold case, $\sum \varphi_j(z) \wedge \bar{\varphi}_j(w)$.) Then the Bergman metric is defined to be, if $K_W(z,z) > 0$,

$$\beta^W_z := \sum_{j,k=1}^{n} \frac{\partial^2 \log K_W(z,z)}{\partial z_j \partial \bar{z}_k} \bigg|_z d\bar{z}_j \otimes d\bar{z}_k.$$

While this $(1,1)$-tensor is known to define a positive-definite metric (a positive metric, in our terminology) if $W$ is a bounded domain in $\mathbb{C}^n$ by a theorem of Bergman himself, its positivity is not at all clear even when $K_W(z,z) > 0$ for every $z \in W$. Nevertheless, we shall call this tensor the Bergman metric following the usual convention.

In case the Bergman metric is positive, its real part defines a Riemannian metric of $W$. Thus it generates the length of piecewise $C^1$ curves and consequently the distance function, according to the usual routine of Riemannian geometry. We say therefore that the Bergman metric is complete if $W$, equipped with this distance, is Cauchy-complete as a metric space.

Now let $\Omega := \{(z_1, z_2) \in \mathbb{C}^2: \text{Re } z_2 + P_{2k}(z_1) < 0\}$ be a Kohn-Nirenberg domain in $\mathbb{C}^2$ as in Definition 1 above. Denote by $K_\Omega$ the Bergman kernel function of $\Omega$. Then

**Proposition 2.1.** If $\Omega$ is a Kohn-Nirenberg domain, then it admits a nowhere vanishing square-integrable holomorphic function. In particular, $K_\Omega(p,p) > 0$ for every $p \in \Omega$.

**Proof.** For this domain, there is a holomorphic peak function constructed by Bedford and Fornæss [1] as follows: for sufficiently small a positive constant $\eta$, let

$$\Omega_\eta := \{(z_1, z_2) \in \mathbb{C}^2: \text{Re } z_2 + P_{2k}(z_1) < \eta(|z_2| + |z_1|^{2k})\}.$$  

Then, by Main Theorem of Bedford and Fornæss [1] p. 559, there exists a holomorphic function $\varphi: \Omega_\eta \to \mathbb{C}$ satisfying:

1. There exists a constant $C > 1$ such that

$$\frac{1}{C}(|z_2| + |z_1|^{2k}) \leq |\varphi(z)| \leq C(|z_2| + |z_1|^{2k}), \; \forall z = (z_1, z_2) \in \overline{\Omega_\eta}.$$

2. For a sufficiently large integer $N > 1$ there exists a branch of $\sqrt[N]{\varphi}$ such that $\arg \sqrt[N]{\varphi} \in [-\pi/4, \pi/4]$.

3. $Q := \exp(-\sqrt[N]{\varphi})$ is a holomorphic peak function at $(0,0)$, i.e., $Q$ is holomorphic on $\Omega_\eta$, continuous on the closure $\overline{\Omega_\eta}$, $Q(0,0) = 1$ and $|Q(z_1, z_2)| < 1$ for every $(z_1, z_2) \in \overline{\Omega_\eta} \setminus \{(0,0)\}$.

Since this $Q(z)$ decays exponentially as $|z| \to \infty$ in $\Omega_\eta$, it is square-integrable on $\Omega_\eta$. Since $\Omega \subset \Omega_\eta$, $Q$ is also square-integrable on $\Omega$. Since $Q$ does not vanish anywhere on $\Omega$, the proof follows immediately. \qed
Notice that the same conclusion holds for the Bergman kernel functions $K_{W_{KN}}$ and $K_{W_{Tor}}$ of the domains $W_{KN}$ and $W_{Tor}$, respectively, since these domains are subdomains of KN-domains.

2.2. The Caratheodory metric. Recall the following classical concepts: for a complex manifold $M$, denote by $\mathcal{H}(M, \Delta)$ the set of holomorphic functions from $M$ into the unit open disc $\Delta$ in $\mathbb{C}$. Let $p \in M$ and $v \in T_p M$. Then the Caratheodory pseudo-metric (metric, if positive) of $M$ is defined by
\[
F_M^C(p, v) = \sup\{|df_p(v)|: f \in \mathcal{H}(M, \Delta), f(p) = 0\}.
\]
This induces the Caratheodory pseudo-distance (distance, if positive)
\[
\rho_M^C(p, q) = \inf \int_0^1 F_M^C(\gamma(t), \gamma'(t))dt,
\]
where the infimum is taken over all the piecewise $C^1$ curves $\gamma: [0, 1] \to M$ with $\gamma(0) = p$, $\gamma(1) = q$. Of course, if we denote by $d^P_M$ the Poincaré distance of the unit disc $\Delta = \{z \in \mathbb{C}: |z| < 1\}$, then it is well-known that $\rho_M^C = \beta_M^P$.

Remark 2.2. Throughout this paper, we consider only the Caratheodory pseudo-distance introduced just now. But there is another equally well-known Caratheodory distance:
\[
d_M^C(p, q) = \sup\{d_M^P(f(p), f(q)) : f \in \mathcal{H}(M, \Delta)\}.
\]
We shall, however, deal with this pseudo-distance only almost at the end of this article, in Remark 4.3. We point out on the other hand, in most other literatures, our Caratheodory pseudo-distance is usually called the integrated Caratheodory pseudo-distance.

Now we present

**Proposition 2.3.** If $\Omega$ is a Kohn-Nirenberg domain, then $F_M^C(p, v) > 0$ for every $(p, v) \in \Omega \times (\mathbb{C}^2 \setminus \{(0, 0)\})$.

**Proof.** For $p = (p_1, p_2)$ and $v = (v_1, v_2)$ take $g(z_1, z_2) := Q(z_1, z_2) \cdot (\bar{v}_1(z_1 - p_1) + \bar{v}_2(z_2 - p_2))$. Then $g$ is a bounded holomorphic function of $\Omega$, since $Q$ decays exponentially at infinity in $\Omega$ and is continuous in the closure. Moreover $g(p) = 0$ and $|dg_p(v)| = Q(p)||v||^2 > 0$. Hence the proof is complete. $\square$

2.3. The Hahn-Lu comparison theorem. The following theorem is very mild a modification of the comparison theorem by Hahn [8, 9, 10], and Lu [15] which compares the Caratheodory metric and the Bergman metric:

**Theorem 2.4** (Hahn-Lu comparison theorem). If $M$ is a complex manifold such that

(i) its Caratheodory metric $F_M^C$ is positive, and
(ii) its Bergman kernel $K_M$ satisfies $K_M(p, p) \neq 0$ for every $p \in M$,

then its Bergman metric $\beta_M^B(v, w)$ satisfies the inequality
\[
(F_M^C(p, v))^2 \leq \beta_M^B(p, v),
\]
for any $p \in M$ and $v \in T_p M$. In particular, this implies that the Bergman metric is positive.
Proof. We shall only prove it for the case when $M = \Omega$ is a domain in $\mathbb{C}^n$, fitting to the purpose of this article; the manifold case uses essentially the same arguments except some simplistic adjustments.

Start with the following quantities developed by Bergman [2]:

$$
\mathcal{B}_0(p) = \sup \left\{ |\psi(p)|^2 : \psi \text{ holomorphic, } \int_{\Omega} |\psi|^2 \leq 1 \right\}
$$

$$
\mathcal{B}_1(p, v) = \sup \left\{ |\partial_v \varphi|_p^2 : \varphi \text{ holomorphic, } \varphi(p) = 0, \int_{\Omega} |\varphi|^2 \leq 1 \right\},
$$

where $\partial_v \varphi |_p = \sum_{j=1}^n v_j \frac{\partial \varphi}{\partial z_j} |_p$. These concepts are significant because $\mathcal{B}_1(p, v) = \mathcal{B}_0(p) \cdot \beta_p^\varOmega(v, v)$, when $\mathcal{B}_0(p) > 0$.

Following [10], consider an $L^2$-holomorphic function $\hat{\psi}$ on $\Omega$ with $\|\hat{\psi}\|_{L^2(\Omega)} \leq 1$ satisfying $|\hat{\psi}(p)|^2 = \mathcal{B}_0(p)$. Then Montel’s theorem on normal families implies the existence of $\eta \in \mathcal{H}(\Omega, \Delta)$ on $\Omega$ with $\eta(p) = 0$ and $|\partial_v \eta|_p = |\partial_v \eta |_p = F_{\varOmega}^C(p, v)$, the Caratheodory length of $v$ at $p$. Since $|\eta \hat{\psi}| \leq |\hat{\psi}|$, one obtains $|\partial_v (\eta \hat{\psi}) |_p^2 \leq \mathcal{B}_1(p, v)$. Since the function $\eta \hat{\psi}$ is holomorphic on $\Omega$ with $\|\eta \hat{\psi}\|_{L^2(\Omega)} \leq \|\hat{\psi}\|_{L^2(\Omega)} \leq 1$ and vanishes at $p$, one immediately obtains that $|\partial_v (\eta \hat{\psi}) |_p = |\partial_v \eta |_p |\hat{\psi}(p)|$ and more importantly, that $\mathcal{B}_1(p, v) \geq F_{\varOmega}^C(p, v)^2 \mathcal{B}_0(p)$. Hence the comparison

$$
\beta_p^\varOmega(v, v) = \frac{\mathcal{B}_1(p, v)}{\mathcal{B}_0(p)} \geq \frac{F_{\varOmega}^C(p, v)^2}{\mathcal{B}_0(p)}
$$

follows as desired. \qed

The original statement required positivity of both metrics. But the proof above (with no changes from the arguments by Hahn [10]) clearly shows that not all those assumptions are necessary. In fact these thoughts yield:

Proposition 2.5. If $W$ is a Kohn-Nirenberg domain in the sense of Definition [1] or one of the domains $W_{\text{KN}}$ and $W_{\text{For}}$, then their Bergman and Caratheodory metrics are positive. Moreover, if we denote by $\beta_p^W(v, w)$ the Bergman metric at $p$, then

$$
\beta_p^W(v, v) \geq F_{\varOmega}^C(p, v)^2
$$

for every $(p, v) \in W \times \mathbb{C}^2$.

3. Construction of peak functions and Completeness

We establish, in this section, the completeness of the Bergman and Caratheodory metrics of the Kohn-Nirenberg domains. Thanks to the comparison theorem of Hahn-Lu, we are only to show that any KN-domain (as well as $W_{\text{KN}}$ and $W_{\text{For}}$) equipped with its Caratheodory distance is Cauchy-complete.

Towards this goal, the following statement plays a crucial role.

Theorem 3.1. Every boundary point of the Kohn-Nirenberg domain $\Omega$ admits a holomorphic peak function. More precisely, every boundary point $p \in \partial \Omega$ admits a holomorphic function $f_p : \Omega \rightarrow \Delta$ satisfying:

1. $\lim_{z \rightarrow p} f_p(z) = 1$.
2. For any $r > 0$ there exists $s > 0$ such that $|f_p(w)| < 1 - s$ whenever $w \in \Omega \setminus B(p, r)$.
Proof. As before, our Kohn-Nirenberg domain $\Omega$ is defined by the inequality $\Re z_2 + P_{2k}(z_1) < 0$. We construct a holomorphic peak function, say $f_p \in \mathcal{H}(\Omega, \Delta)$, at every boundary point $p \in \partial \Omega$.

If $p = (0,0)$, this is already done in [1]; the function $Q$ in the proof of Proposition 2.1 suffices. Then the translation along the $\Im z_2$ direction yields holomorphic peak functions at the points $p = (0, ib) \in \partial \Omega$ for every $b \in \mathbb{R}$.

If $p \in \partial \Omega \setminus \{(0, ib): b \in \mathbb{R}\}$, $\partial \Omega$ is strongly pseudoconvex at $p$. Then there is of course a holomorphic local-peak function, i.e., a holomorphic function $g_{p,r} : B(p, r) \to \mathbb{C}$ for some $r > 0$ such that $g_{p}(p) = 1$ and $|g_{p}(z)| < 1$ for every $z \in B(p, r) \cap \overline{\Omega} \setminus \{p\}$. However, since the domain is unbounded, care has to be taken in extending $g_p$ to a global peak function.

Take constants $0 < r_1 < r_2 < r$ and then choose $\varepsilon > 0$ so that the domain $\Omega^\varepsilon = \{z \in \mathbb{C}^2: \Re z_2 + P_{2k}(z_1) < \varepsilon\}$ is strongly pseudoconvex at every $q \in \partial \Omega^\varepsilon \cap B(p, r_2)$ and that

$$\{z \in B(p, r): g_p(z) = 1\} \cap \{\overline{B}(p, r_2) \setminus B(p, r_1)\} \subset \mathbb{C}^2 \setminus \Omega^\varepsilon.$$

Then take the $C^\infty$ cut-off function $\chi : \mathbb{C}^2 \to [0, 1]$ satisfying $\chi \equiv 1$ on $B(p, r_1)$ and $\text{supp} \chi \subset B(p, r_2)$.

Now we exploit the $\bar{\partial}$-problem on $\Omega^\varepsilon$ with an $L^2$-estimate with an arbitrary plurisubharmonic weight ([12], Theorem 4.4.2). First let

$$\alpha_p(z) = \begin{cases} \bar{\partial} \left( \frac{\chi(z)}{(1 - g_{p}(z))^{\Omega(z)}} \right) & \text{if } z \in \Omega^\varepsilon \cap \{\overline{B}(p, r_2) \setminus B(p, r_1)\} \\ 0 & \text{if } z \in (\Omega^\varepsilon \setminus B(p, r_2)) \cup B(p, r_1). \end{cases}$$

(We acknowledge at this point that the set-up of this differential form has been influenced by the methods of Fornaess and McNeal [3].)

Then consider the function $u_p : \Omega^\varepsilon \to \mathbb{C}$ that solves the equation $\bar{\partial}u_p = \alpha_p$ with the estimate (with the zero plurisubharmonic weight)

$$\int_{\Omega^\varepsilon} \frac{|u_p(z)|^2}{(1 + |z|^2)^2} \, d\mu(z) \leq \int_{\Omega^\varepsilon} |\alpha_p(z)|^2 \, d\mu(z).$$

Notice that this $\alpha_p$ is a bounded-valued $\bar{\partial}$-closed smooth $(0, 1)$-form with bounded support inside $\Omega^\varepsilon$. Consequently the right-hand side of the preceding inequality is bounded, say, by a positive constant $A$.

We would like to obtain a pointwise estimate for $|u_p(z)|$ at every $z \in \Omega$ with $||z|| \gg 1$. Take a positive constant $R_p$ sufficiently large that $B(p, r_2) \subset B(0, R_p - 1)$. Since the defining function is a polynomial, one sees that, for any $\xi \in \Omega \setminus B(0, R_p)$, there exists a uniform $c_\varepsilon > 0$ independent of $\xi$ such that $B(\xi, c_\varepsilon ||\xi||^{-2k}) \subset \Omega^\varepsilon$ and that the support of the function $\chi$ has no intersection with $B(\xi, c_\varepsilon ||\xi||^{-2k})$. In particular, $u_p$ is holomorphic on $B(\xi, c_\varepsilon ||\xi||^{-2k})$. 
Thus, for any \( \xi = (\xi_1, \xi_2) \in \Omega \setminus B(0, R_p) \),
\[
A \geq \int_{\Omega^r} |\alpha_p(z)|^2 d\mu(z) \\
\geq \int_{\Omega^r} \frac{|u_p(z)|^2}{(1 + \|z\|^2)^2} d\mu(z) \\
\geq \int_{B(\xi, c \|\xi\|^{-2k})} \frac{|u_p(z)|^2}{(1 + \|z\|^2)^2} d\mu(z) \\
\geq \frac{1}{4\|\xi\|^4} \int_{B(\xi, c \|\xi\|^{-2k})} |u_p(z)|^2 d\mu(z).
\]
Since \( u_p \) is holomorphic on \( B(\xi, c \|\xi\|^{-2k}) \), the sub mean-value inequality implies
that \( |u_p(\xi)|^2 \leq A_p \|\xi\|^{8k+4} \), for some positive constant \( A_p \) independent of \( \xi \). In
particular, \( |u_p| \) grows at most polynomially (of degree \( 8k + 4 \)) at infinity.

Notice that \( u_p \) is smooth in \( \Omega^r \) and hence smooth on the closure of \( \Omega \). Since the
Bedford-Fornæss peak function \( Q \) enjoys the exponential decay estimate at infinity,
we may take sufficiently small a constant \( c_p > 0 \) so that
\[
c_p |Q(z)u_p(z)| < \frac{1}{2}, \quad \forall z \in \Omega.
\]
In particular, we have
\[
(3.1) \quad \text{Re} \left( c_p Q(z)u_p(z) - 1 \right) < -\frac{1}{2}, \quad \forall z \in \Omega.
\]
Since \( u_p \) satisfies the equation \( \partial u_p = \alpha_p \), it follows that the function
\[
\frac{\chi(z)}{1 - \mathfrak{g}_p(z)} - Q(z)u_p(z)
\]
is holomorphic on \( \Omega \).

Altogether, if we define \( f_p : \Omega \to \mathbb{C} \) with the positive constant \( c_p \) by
\[
f_p(z) = \exp \left( \frac{\mathfrak{g}_p(z) - 1}{c_p \chi(z) - (1 - \mathfrak{g}_p(z))[c_p Q(z)u_p(z) - 1]} \right),
\]
then \( f_p \in \mathcal{A}(\Omega) \). Moreover, for the positive constant \( c_p \), one easily obtains by (3.1)
that
\[
\text{Re} \left( \frac{\mathfrak{g}_p(z) - 1}{c_p \chi(z) - (1 - \mathfrak{g}_p(z))[c_p Q(z)u_p(z) - 1]} \right) < 0
\]
for every \( z \in \Omega \).

Finally, \( \lim_{0 \geq z \rightarrow p} f_p(z) = e^0 = 1 \). Notice that \( p \) is the only boundary point
that has this property for \( f_p \). The other condition is also easily checked since this last
estimate receives a definite negative upper bound if \( z \) is at a positive distance away
from \( p \). Therefore, \( f_p \) is the desired global peak function at \( p \) for \( \Omega \). \( \square \)

**Remark 3.2.** Notice that the homogeneity of the polynomial \( P_{2k} \) is not essential
in extending the holomorphic local-peak function \( \mathfrak{g}_p \) to the global peak function
\( f_p \) when \( p \in \partial \Omega \) is a strongly pseudoconvex point. The same extension procedure
works for any domain \( W_S : = \{(z_1, z_2) \in \mathbb{C}^2 : \text{Re} z_1 + S(z_1, z_2) + P_{2k}(z_1) < 0 \} \) defined
by the polynomial defining function, whenever \( P_{2k} \) is as in Definition 1.1 as long
as \( S \) satisfies the following technical conditions:
(i) $S$ is a nonnegative real-valued polynomial in $z_1, \bar{z}_1, z_2,$ and $\bar{z}_2$ of degree strictly less than $2k$ with $S(0) = 0$; and

(ii) $W_S$ is strongly pseudoconvex at every boundary point possibly except the origin.

Notice that the domains $W_{\text{KN}}$ and $W_{\text{For}}$ belong to such a collection of domains. Thus the conclusion of the theorem above holds in particular for the domains $W_{\text{KN}}$ and $W_{\text{For}}$.

Now we prove

**Theorem 3.3.** Every Kohn-Nirenberg domain defined in Definition 1.1, equipped with its Carathéodory distance, is a complete metric space.

**Proof.** Let $(q_n)_n$ be a Cauchy sequence in the Kohn-Nirenberg domain $\Omega$ with respect to the Carathéodory distance $\rho^C_{\Omega}$. We now pose:

**Claim.** There exists a compact subset $X$ of $\mathbb{C}^2$ such that $q_n \in X \subset \Omega$ for every $n = 1, 2, \ldots$

Note that this implies that $(q_n)_n$ has a subsequence $(q_{n_k})_k$ convergent to $\hat{q} \in X$ with respect to the Euclidean distance. However, since the Carathéodory distance $\rho^C_{\Omega}$ is continuous, $(q_{n_k})_k$ converges to $\hat{q}$ with respect to $\rho^C_{\Omega}$, and hence $(q_n)_n$ converges to $\hat{q}$ with respect to $\rho^C_{\Omega}$ as well. Therefore, it suffices to establish this claim.

Theorem 3.4 implies that this sequence has no subsequence that approaches a boundary point of $\Omega$ arbitrarily closely in the Euclidean distance, due to the distance-decreasing property of the Carathéodory distance. Hence it remains to show that no subsequence of $(q_n)_n$ can diverge indefinitely far away from the origin with respect to the Euclidean distance.

Assume the contrary. Then choosing a subsequence we may assume without loss of generality that $\lim_{n \to \infty} \|q_n\| = \infty$. Let $q_n := (a_n, b_n)$ for every $n$. Notice that, for every $n = 1, 2, \ldots$, there exists a unique positive number $t_n$ satifying the equation

$$t_n^2|a_n|^2 + t_n^{4k}|b_n|^2 = 1.$$

Define the map $\varphi_n : \mathbb{C}^2 \to \mathbb{C}^2$ by $\varphi_n(z_1, z_2) = (t_n z_1, t_n^{2k} z_2)$ for each $n$. Then $\varphi_n \in \text{Aut}(\Omega)$.

Observe that $\lim_{n \to \infty} \varphi_n(q_1) = (0, 0)$. If we denote by $S = \{(z_1, z_2) \in \mathbb{C}^2 : |z_1|^2 + |z_2|^2 = 1\}$, then $\varphi_n(q_n) \in S \cap \Omega$, for every $n$. Therefore,

$$\rho^C_{\Omega}(q_1, q_n) = \rho^C_{\Omega}(\varphi_n(q_1), \varphi_n(q_n)) \geq \rho^C_{\Delta}(Q(\varphi_n(q_1)), Q(\varphi_n(q_n))),$$

and $\|\varphi_n(q_1) - \varphi_n(q_n)\| > \frac{1}{2}$ for $n \gg 1$. Then Proposition 3.3 yields a constant $s$ with $0 < s < 1$ independent of $n$ with $|Q(\varphi_n(q_n))| < 1 - s$. Therefore one sees that

$$\rho^C_{\Delta}(Q(\varphi_n(q_1)), Q(\varphi_n(q_n))) \geq \inf_{t \in \mathbb{R}} \rho^C_{\Delta}(Q(\varphi_n(q_1)), (1 - s)e^{it}).$$

But, since $\lim_{n \to \infty} \varphi_n(q_1) = (0, 0)$, the preceding inequality yields that

$$\lim_{n \to \infty} \rho^C_{\Omega}(q_1, q_n) = +\infty.$$

So $(q_n)_n$ fails to be a bounded sequence with respect to the Carathéodory distance, and consequently cannot be a Cauchy sequence with respect to the Carathéodory distance $\rho^C_{\Omega}$. This contradicts the original hypothesis that $(q_n)_n$ was $\rho^C_{\Omega}$-Cauchy. Thus the proof is complete. \hfill $\Box$

Furthermore, we obtain
Proposition 3.4. Let $\Omega := \{(z_1, z_2) \in \mathbb{C}^2: \operatorname{Re} z_2 + P_{2k}(z_1) < 0\}$ be a Kohn-Nirenberg domain. If $S(z_1, z_2)$ is a nonnegative polynomial of degree strictly less than $2k$, with $S(0) = 0$, such that the domain

$$W_S := \{(z_1, z_2) \in \mathbb{C}^2: \operatorname{Re} z_2 + S(z_1, z_2) + P_{2k}(z_1) < 0\}$$

is strictly pseudoconvex everywhere except at the origin $(0, 0)$, then the domain $W_S$ equipped with its Carathéodory distance is Cauchy-complete.

Proof. Since $W_S$ is contained in the Kohn-Nirenberg domain $\Omega$, $\rho_{W_S}^\mathcal{C} \geq \rho_{\Omega}^\mathcal{C}$ and consequently $\rho_{W_S}^\mathcal{C}$ is positive. By Remark 3.2, $W_S$ admits a holomorphic peak function at every strongly pseudoconvex point. Since the origin is a common boundary point to $W_S$ and $\Omega$, the peak function for $\Omega$ at the origin is also a peak function of $W_S$ at the origin. Thus no Cauchy sequence of $W_S$ with respect to its Carathéodory distance $\rho_{W_S}^\mathcal{C}$ can accumulate at a boundary point of $W_S$. Finally, since the Carathéodory distance $\rho_{W_S}^\mathcal{C}$ is larger than or equal to $\rho_{\Omega}^\mathcal{C}$, no Cauchy sequence with respect to $\rho_{W_S}^\mathcal{C}$ can diverge indefinitely far away from the origin. Altogether, it follows that every Cauchy sequence of $W_S$ is bounded, and bounded away from the boundary. Now the proof follows by the continuity of the Carathéodory distance. \(\Box\)

Altogether the proofs of Theorems 1.2 and 1.3 now follow by the Hahn-Lu comparison theorem (Theorem 2.4). Of course the domains described in this proposition also have their Bergman metrics positive and complete.

4. Higher Dimensions

The method of this paper up to this point is not restricted to dimension two, except at the places where the Bedford-Fornæss peak functions were exploited. But that is also not a strong restriction. In fact the following existence theorem of peak functions at the origin of certain domains in higher dimensional cases were established by Noell [16]:

Let $U$ be a domain in $\mathbb{C}^n$. Denote by $z = (z', z_n)$ the complex variable(s) for $\mathbb{C}^n$ with $z' = (z_1, \ldots, z_{n-1})$, and by $\mathcal{A}(U)$ the set of continuous functions on $\overline{U}$, holomorphic on the domain $U$.

Theorem 4.1 (Noell). Suppose that $P_{2k}$ is a homogeneous plurisubharmonic polynomial of degree $2k$ on $\mathbb{C}^{n-1}$, and assume that $P_{2k}$ is not harmonic along any complex line through the origin of $\mathbb{C}^{n-1}$. For a sufficiently small positive constant $\eta$, let

$$\Omega_\eta := \{(z', z_n) \in \mathbb{C}^n: \operatorname{Re} z_n + P_{2k}(z') < \eta |z_n| + \eta \|z'\|^{2k}\}.$$

Then, there exists a function $f \in \mathcal{A}(\Omega_\eta)$ with the following properties:

1. For some constant $C > 1$,

$$\frac{1}{C} (|z_n| + \|z'\|^{2k}) \leq |f(z)| \leq C (|z_n| + \|z'\|^{2k}),$$

for all $z = (z', z_n) \in \Omega_\eta$.

2. For a sufficiently large integer $N > 1$ there exists a branch of $\sqrt[n]{f}$ such that $\arg \sqrt[n]{f} \in [-\pi/4, \pi/4]$. 


(3) For all $i_1, \ldots, i_n \geq 0$ there exist constants $C_{i_1, \ldots, i_n}$ and $N_{i_1, \ldots, i_n}$ such that
\[
\left| \frac{\partial^{i_1 + \cdots + i_n}}{\partial z_{i_1}^{e_{i_1}} \cdots \partial z_{i_n}^{e_{i_n}}} f \right| \leq C_{i_1, \ldots, i_n} \|z\|^{-N_{i_1, \ldots, i_n}} \quad \text{when } \|z\| \leq 1.
\]
(4) $\exp(-\sqrt{f})$ is a peak function at 0 for $\mathcal{A}(\Omega_\eta)$.
(5) $\exp(-1/\sqrt{f}) \in C^\infty(\Omega_\eta)$ and $f$ is a separating function at the origin for $\mathcal{A}(\Omega_\eta)$.

Thus we immediately obtain

**Theorem 4.2.** The Bergman metric and the Carathéodory metric of any domain $\Omega$ in the statement of the preceding theorem are positive and complete.

Of course, all other theorems in this article also generalize to these domains.

**Remark 4.3.** Another well-known Carathéodory pseudo-distance is defined by
\[
d_{CM}(p, q) = \sup \left\{ d_P(\Delta)(f(p), f(q)) \mid f : M \to \Delta \text{ holomorphic} \right\},
\]
where $d_P(\Delta)$ is the Poincaré distance of the unit disc $\Delta$. This Carathéodory distance $d_{CM}$ is in general smaller than the (integrated) Carathéodory distance $\rho_{CM}$. Hence one might like to ask whether KN-domains as well as the higher dimensional domains just mentioned here, when equipped with the Carathéodory distance $d_C$, are also complete. They indeed are.

Using peak functions one can show that the Cauchy sequences cannot approach the boundary. The sequence cannot diverge infinitely far from the origin by the proof-arguments of Theorem 3.3. The only remaining point to show now should be the positivity, i.e., $d^{CM}_{\Omega}(p, q) > 0$ if $p \neq q$. But this was established earlier by Yu [17].

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