Abstract
The relativistic quantum mechanics of two interacting particles is considered. We first present a covariant formulation of kinematics and of reduced phase space, giving a short outline of the classical results. We then quantize the systems for the scalar-scalar, fermion-scalar and fermion-fermion cases. We study the spectrum and the spherical waves solutions of the free case. The interaction with central scalar and vector potentials is introduced and the explicit equations are deduced. The one particle and the non relativistic limits are recovered and the general lines for the solution of the boundary value problems are given. We make a numerical analysis of the first two cases with Coulomb interaction. For the two fermions we largely revisit the model we had previously derived in order to uniformize the description for all the three cases. In order to give a complete review we report in Appendix some of the most interesting results obtained for atomic and mesonic systems with Coulomb and Cornell potential interactions respectively.

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Keywords
Covariance; Two-body; Wave-equation; Scalar; Fermion.

1. Introduction
Just after the formulation of special relativity, the motion of a single material point, either free or in an external field, was given a variational formulation, both Lagrangian and Hamiltonian [1–3]. The difficulties of describing the dynamics of many bodies in the new theoretical framework were also very soon realized. Even the kinematics of two mass points posed the question of the status of the independent time coordinates. The problem was then dealt with by reducing the system to a non-relativistic approximation where the energy, given by the sum of two square roots, was expanded in powers of \( \frac{v}{c} \) and treated by the usual methods of particle dynamics [4]. The rise of Quantum Mechanics shifted the main interest towards the search of a relativistically correct quantization procedure. The Dirac equation was an essential achievement in this direction and, in addition to countless results in fundamental physics, it stimulated many attempts to find an extension for many fermion systems and their bound states [5]. Darwin gave a quantum version of his previous paper [6], but the most interesting results were probably produced by Breit [7]. Following suggestions independently formulated by Heisenberg and Pauli, Breit assumed as electron velocity the Dirac \( \alpha \)-matrices, substituted them in the Liénard-Wiechert potentials and gave a first order correction to the Coulomb potential of the Darwin Lagrangian. In this way he highlighted the strict relationship between the electron motion, the spin and the magnetic moment as a relativistic feature. Subsequently he understood that, while the Coulomb potential should be treated exactly, the magnetic term had to be taken only at a first perturbation order [8]. In a paper of the same year [9] Plesset proved a result contrary to a naïf intuition based on a non-relativistic picture: namely he showed that asymptotically unbounded positive potentials in vector coupling have no bound states. More detailed descriptions of the continuous spectrum for such and similar situations have been given in [10–12]. In fact the covariance properties were not the first concern of those papers and the equations assumed explicitly the reference frame with vanishing total momentum. Again in the same year, however, a joint paper of Dirac, Fock and Podolsky [13], proposed the introduction of a separate time for each particle to achieve an “obvious relativistic invariance”.

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A huge breakthrough into the subject was given by a great paper of Wigner [14], whose analysis of the relativistically admissible physical states concerned both Quantum Field Theory and Relativistic Quantum Mechanics. Some interesting papers were then devoted to the relativistic definition of the center of mass [15] and to the localization of elementary states [16]. The latter influential paper was published in the special issue for the Einstein 70th birthday: in the same volume Dirac made an analysis of the forms in which the interactions could be introduced in the relativistic classical Hamiltonian mechanics [17], while the Dirac constraints theory was published one year later [18]. A classic paper in this context was the one by Bakamjian and Thomas [19] on relativistic particle dynamics. The authors built a two interacting Dirac particles model, in which the two-body theory was for the first time coherently based on the algebra of the Poincaré generators and a canonical separation between global and relative operators was tried, still assuming for the time an *a priori* different status. Interesting results were obtained at the same time by Pirenne [20] and Berestetski and Landau [21], who used the semiclassical Breit approximation and the Schrödinger Coulomb wave functions in order to estimate the hyperfine shifts of the parapositronium and of the ground state of the ortopositronium, including the annihilation term. Few years later, exploiting the QFT advances [22, 23], these shifts were calculated up to the fifth order in the fine structure constant a [24, 25]. However, in those calculations and in many of the others that followed, the final results were obtained in the semiclassical approximation: a crucial role was played by the value at the origin of the non relativistic Coulomb wave function, raising a long lasting question on the opportunity of adopting some kind of smearing of the corresponding δ-function [26, 27].

In parallel with the investigations of the relativistic few body systems, the interest increased also for the analysis of their non-quantum counterpart, mainly based on the group symmetries and on the constraint theory [28–30]: the hope was to be able to catch the essence of the relationship between canonicity and relativistic covariance, in order to have sound guidelines to the quantization procedure. An interesting step forward was the so called “no interaction theorem” [31]. It established the impossibility for the relativistic dynamics of two or more massive particles of being given a Hamiltonian canonical formulation in explicit covariant form, unless the particles are free. The meaning of such a result was made obvious in a Lagrangian context [32, 33], proving the impossibility of arbitrary independent time reparametrizations of the world lines of two interacting relativistic particles. It was then clarified that the angular momentum was a major obstacle to a canonical and explicitly covariant formulation of the motion and it seemed that quantum field theories only could overcome this difficulty. However, obtaining bound states in QFT was - and, in some measure, keeps being - a difficult task, because of the perturbative nature of the theory. The great and mainly unsolved problems of the wishfully promising Bethe-Salpeter approach to bound states are an example of these obstacles [34]. In any case, the QED corrections of higher and higher order made on interacting particle models, even when constructed in a semi-relativistic framework, have been able to produce very accurate results for the levels and the states of simple atoms [35, 36].

The situation is even more difficult with the quarkonium models [37], used for a long time to investigate the spectrum of meson and baryon masses. The first attempt to get at least semi-quantitative results dates back from the seventies of last century. The Schrödinger equation with a potential was initially used in order to fit the charmonium states. It was soon realized that relativicity could not be neglected and the first pioneering work [38] was followed by many other papers where relativistic effects were added. Therefore, although generally starting from a Schrödinger equation, the models have been made more and more relativistic by adding perturbative corrections of higher order. [39–51]. The quantum chromodynamics (QCD) was also introduced starting from a non-relativistic treatment and expanding the interactions up to the second order in $v/c$ [52]. In such a way an effective theory, called non relativistic QCD or NRQCD, was built and used for the lattice and the continuum calculations [53, 54]. From the NRQCD another effective theory was then derived, the potential non relativistic QCD or pNRQCD [55], which is among the most popular methods for calculating meson spectra and decays [56, 57]. The lattice program has independently grown very consistently and is now able to deduce good estimates of hyperfine splittings by taking into account relativistic and QCD radiative effects [58, 61].

Papers which have proposed a covariant theory are present in literature. Many approaches originated from the Bethe-Salpeter equation and are connected with field theory: the spectra of the corresponding equations are not so easy to compute. Fewer models, often obtained from constrained dynamics, deal with a consistent relativistic quantum mechanical description. Extraneous arguments, however, are sometimes introduced, such as the confinement obtained by a cutoff or by a time-like potential, a different treatment of the Breit term depending upon the component masses, some *ad hoc* contact interaction [52, 57]. Those models generally encounter greater difficulties with the spectra of mesons formed by different mass quarks and, specially, when calculating the masses of the light mesons. The relativity is obviously essential for light quarks and must therefore be treated with the appropriate accuracy. It was also argued that the relativistic influence is stronger for systems whose components have different masses [68].
Our initial point of view originated from the massive entrance of differential geometry in classical mechanics. The main ideas came from group actions on symplectic manifolds, geometric quantization, reductions of Hamiltonian systems and more generally from the use of the induced representations on homogeneous spaces [69–78], also extended to more general and abstract contexts [79–82]. The analysis of the coadjoint orbits of the Poincaré and Weyl group [83] allowed to construct Hamiltonian systems corresponding to group representations, but the most relevant result was that the realization of the particle dynamics on the mass-shell is obtained by a symplectic reduction of the original phase space [71, 72]. A similar procedure is made possible for a system of two particles since the relative time variable becomes cyclic and disappears in the reduced space. Therefore one can describe a canonical consistent relativistic dynamics with a single time coordinate where the interactions are introduced on the reduced phase space by potentials depending upon the modulus of the relative position, which is Lorentz invariant in the reduced space. Moreover, the no-interaction theorem is respected, because the global position coordinate is a Newton-Wigner vector on which there is a well defined action of the kinematic group.

These nice properties suggested the possibility of adopting the two-body relativistic classical dynamics as a starting point to construct a relativistic wave equation for a two particle system. The case we started with concerned two spin one-half particles [84], since most of the elementary and fundamental two body bound systems are formed like that. The quantization was realized in analogy to the Dirac procedure for the single particle. In such a way we deduced a 16-dimensional vector equation reproducing at the quantum level the same nice properties of the classical systems. The wave equation was written in spherical coordinates for dealing with central interactions, determining the states in terms of energy, global angular momentum and parity. The rotation invariance reduces to eight for each parity the number of independent radial functions and since the latter satisfy four algebraic relations, the ultimate boundary value problem to be solved has order four. While the two free fermions can be easily discussed, the addition of central interactions originates a wave equation whose analytical discussion is too much demanding and a numerical treatment is almost compulsory. What can still analytically proved is the existence and the form of two different limits: one is the single particle Dirac limit, obtained when the mass of one of two components goes to infinity while the second remains fixed, giving the scale of mass; the other one is the Schrödinger limit, found by letting the ratio \( v/c \) be vanishing. The former of these limits constitutes a severe problem for the treatments originating from the Bethe-Salpeter equation, which is not able to reproduce it. In the papers [85–88] we tested the physical reliability of our wave equation by numerical calculations. In the area of atomic physics we determined the hyperfine shifts of the simple Hydrogen-like atoms with fermionic nucleus. Within the high energy framework we calculated the masses of the different families of mesons in the quarkonium model using the Cornell potential [89] and the Breit term as interactions. Electromagnetic transition rates were also calculated. The results are generally in excellent agreement with the experimental measures, proving that the correct inclusion of relativity is able to provide very accurate answers in contexts of different nature and for energies varying by several orders of magnitude.

In this paper we present a general theory for obtaining the relativistic wave equations of two particles with spin 0 and 1/2. Besides the two fermion case, therefore, we determine the equations for two scalars and for a scalar and a fermion. We use the geometrical and kinematical properties so far developed in order to couple the single particle relativistic wave equation corresponding to each component. The particular features are different, but the general method is the same for all the possible cases. We give for the first time the treatment of the two scalars and scalar-fermion systems, adding a fresh and largely revisited exposition of the two interacting Dirac fermions, uniformized to the general scheme we are developing. As we stated above, at a fundamental level the most diffuse two body systems are formed by two fermions. Those composed of two scalar particles are rare and the corresponding measurements are generally lacking. In the realm of the recently developing physics of pionic atoms one could think of the ion formed by an \( \alpha \)-particle and a \( \pi^- \) meson. Slightly more frequently elementary scalar-fermion systems are met: for instance the \( {\mathrm{^4He}}^- \) ion or the (proton-\( \pi^- \)) hydrogenic atom, which has recently raised a considerable interest [90]. One could mention also cases, like the deuterium atom, where one of the partners is a vector particle for which the Proca equation should be used. This last argument and the phenomenological analysis of the two former ones are excluded from this paper and deferred to future research. However, for the sake of completeness and for showing the effectiveness of the approach we have presented, we report in Appendix a choice of the results obtained in [84–88], concerning the hyperfine structure of the Hydrogen like atoms and the mass spectrum of mesons in the quarkonium model. We only give some bare-bone indications of the phenomenological meaning of the data, referring for details to our previous papers. Some final observations conclude the paper.
2. Kinematics and classical systems

We introduce the basic kinematic definitions that allow to develop a covariant description of two relativistic particles interacting through potentials uniquely dependent upon the difference of their coordinates. This means that no external field is present and that the total momentum is conserved. We call \( m_1 \) and \( m_2 \) the particle masses and we assume, without loss in generality, that \( m_1 \geq m_2 \). The corresponding coordinates and the momenta will be \( \vec{x}_i, p_i \), with \( i = 1, 2 \), \( \mu = 0, 1, 2, 3 \). The metric tensor \( \eta_{\mu \nu} \) has the usual signature \((+,-,-,-)\), we denote in boldface the 3-vectors and we use units with \( c = 1 \). We next introduce the conjugate pairs of the so called ‘global’ and ‘relative’ coordinates

\[
\begin{align*}
P^\mu &= p_1^\mu + p_2^\mu, & X^\mu &= (1/2) (\vec{x}_1 + \vec{x}_2), \\
\tilde{q}^\mu &= (1/2) (p_1^\mu - p_2^\mu), & \tilde{r}^\mu &= \tilde{x}_1^\mu - \tilde{x}_2^\mu.
\end{align*}
\]

As we stated in the introduction, the relative time and the corresponding relative energy pose a problem of physical interpretation, connected with the existence of the ‘no interaction theorem’ \([31, 32]\). The Lagrangian formulation \([32]\) relates the absence of interaction to the existence of independent time coordinates of the particles: it is natural – and it has been proved – that interaction determines correlations among the world lines, incompatible with arbitrary choices of the parametrizations in time.

For the Hamiltonian reduction of the phase space, needed in the present framework, is not necessary such a complex formalism as in the general case of a Lie group action on the phase space \([77]\). Indeed, since the main problem is given by the presence of relative time and relative energy, it is sufficient to give the explicit canonical transformation (whichever the way it has been obtained) turning the relative time into a cyclic variable and then apply the elementary theory of the canonical reduction. The relative time has thus no consequences on the development of the dynamics and plays the role of a gauge function to be chosen \textit{a posteriori} in order to recover a complete Minkowski description of the world lines of the two particles. Starting from the two mass shell conditions written in terms of the coordinates \([24]\) and taking their sum and difference, we have

\[
(P \cdot \tilde{q}) = (1/2) (m_1^2 - m_2^2), \quad (1/2) P^2 + 2\tilde{q}^2 = m_1^2 + m_2^2.
\]

We find it natural to choose as canonical coordinate the variable \((P^2)^{-1/2} (P \cdot \tilde{q})\), which is just the relative energy in the \( P = 0 \) frame. To complete the set of the new relative canonical momenta it is therefore coherent to add the relative 3-momentum \( \tilde{q} \) boosted to the \( P = 0 \) frame. It is obvious that this is not a choice of a particular reference frame but only a definition of a new set of canonical coordinates to which we add the total 4-momentum \( P \) in order to exhaust the momentum variables. We formally complete the canonical transformation by determining the corresponding position variables by the standard method of the generating function \([92]\). Geometrically the matrix of the Lorentz transformation \( L^{-1}(P) \), which appears in the canonical variables, plays the role of a set of vierbein

\[
\epsilon_A^\mu(P) = \frac{P^\mu}{\sqrt{P^2}} \quad \eta_{\mu \alpha} \epsilon_A^\alpha(P) = \eta_{\alpha \beta} \quad \text{and} \quad \eta_{\alpha \beta} \epsilon_B^\alpha(P) \epsilon_B^\beta(P) = \eta_{\mu \nu}.
\]

The conjugate pairs of canonical variables are:

\[
\begin{align*}
q_0 &= \epsilon_0^\mu(P) \tilde{q}_\mu, & r_0 &= \epsilon_0^\mu(P) \tilde{r}_\mu, \\
r_A &= \epsilon_A^\mu(P) \tilde{q}_\mu, & \tilde{r}_A &= \epsilon_A^\mu(P) \tilde{r}_\mu, \\
P^\mu &= P^\mu, & Z^\mu &= X^\mu + \left( P^2 \right)^{-1} \left( (P \cdot \tilde{q}) \rho_\mu - (P \cdot \tilde{r}) \tilde{q}_\mu \right) + \left( P^2 + (P^2)^{1/2} \right)^{-1} W^{0\mu}.
\end{align*}
\]

where the Pauli-Lubin tensor is defined as \( W_{\mu \nu} = (P^2)^{-1} \epsilon_{\mu \nu \rho \sigma} P^\rho W^{\sigma} \), with \( W_{\mu} = \epsilon_{\mu \nu \rho \sigma} P^\rho \tilde{q}^\sigma \). The action of the Poincaré group is properly defined: this is due to the geometrical nature originating the canonical transformation and it is well established by the general theory \([77]\). Moreover \( \epsilon_A^\mu(P) W_{\mu} = \epsilon_{ABC} r_B q_C \equiv L_A \) is the orbital angular momentum appearing in our previous papers. Indeed \( Z_0 \) is a Newton-Wigner position vector for a particle of angular momentum \( L_A \) and \( Z^\mu \) has the covariance of the time component of a Lorentz 4-vector. This is necessary if \( Z^0 \) has to be assumed as evolution parameter of the system. Moreover \( q \) and \( r \) are Wigner vectors of spin one with Lorentz invariant modulus \( q = (q_A q_A)^{1/2} \), \( r = (r_A r_A)^{1/2} \). \( q_0 \) and \( r_0 \) are Lorentz invariant also. The canonical coordinate \( Z^\mu \) in \([2,4]\) differs by a term \((P^2)^{-1/2} P^\mu q_0 r_0\) from the expression written in our previous papers. While the present choice is
functions and reproduces, for of vector or scalar nature, respectively coupled to the energy or to the mass. In this section we begin by studying the quantization of a system of two free scalar particles with masses concerned with particles having a definite internal tensorial structure, namely scalars and fermions, and with potentials been studied in many papers using different techniques. For a potential \( V(r) \) in the vector coupling, the energy (2.6) is simply modified by the addition of a non-trivial interaction built in terms of coupling, the energy (2.6) is simply modified by the addition of a non-trivial interaction built in terms of.

So far we have examined a system of two free particles. On the reduced phase space it is straightforward to introduce a non trivial interaction built in terms of \( r \), and possibly \( q \), which produces a Lorentz covariant dynamics. A simplest choice is to use a potential function in vector, scalar or even tensor coupling. In the following sections we shall study the first two choices, dealing with atoms and mesons in a quantum mechanical context. For a potential \( V(r) \) in the vector coupling, the energy (2.6) is simply modified by the addition of \( V(r) \). Using the spherical coordinates for the radial part, we easily find the radial momentum \( q_r \), given by

\[
q_r = \left( \frac{(\lambda + V(r))^2 - (m_1^2 + m_2^2)}{2(\lambda + V(r))} \right)^{1/2} - \left( \frac{m_1^2 m_2^2}{(\lambda + V(r))^2} + \frac{L^2}{r^2} \right)^{1/2}
\]

and introducing \( L \) the absolute value of the conserved angular momentum.

Choosing the Coulomb potential \( V(r) = -\alpha/r \) and introducing \( u(r) = \lambda + V(r) \), we obtain the equation for the trajectory

\[
d\theta/dr = (2L/\alpha) u \left( u^4 - (2L/\alpha)^2 u^2 (\lambda - u)^2 - 2(m_1^2 + m_2^2)u^2 + (m_1^2 - m_2^2)^2 \right)^{-1/2}
\]

It is integrated in terms of elliptic functions. For equal masses \( m_1 = m_2 = m \) the solution is expressed by elementary functions and reproduces, for \( L \geq \alpha/2 \), the elliptic, parabolic and hyperbolic solutions respectively.

In the following sections we will determine the wave equations for two interacting relativistic particles in the three cases of scalar-scalar, scalar-fermion and fermion-fermion systems. A unit system with \( h = c = 1 \) is used. We will denote by \( \sigma_i, i = x, y, z \), and \( \sigma_i = \sigma_i \pm i\sigma_i \) the Pauli matrices. For later convenience we also introduce the \( n \times n \) identity matrix \( I_n \) and the matrices \( I_{n,n} = \text{diag}(I_n, -I_n) \).

3. The wave equation for two scalars

The quantization of Hamiltonians including the sum of two square roots with the possible addition of a potential has been studied in many papers using different mathematical techniques (see, e.g., [93]). Here and in the following we are concerned with particles having a definite internal tensorial structure, namely scalars and fermions, and with potentials of vector or scalar nature, respectively coupled to the energy or to the mass. In this section we begin by studying the quantization of a system of two free scalar particles with masses \( m_1 \) and \( m_2 \), \( m_1 \geq m_2 \). A vector interaction – in particular Coulomb – will then be added. This case is the easiest from the point of view of the wave function structure. As we shall see, however, it contains a peculiar feature due to the presence of the Laplace operator. It has been known since a long time that the Hamiltonian form of the wave equation for scalar particles is better obtained by using a 2-dim formulation for the Klein-Gordon equation

\[
i \partial \Phi/\partial t - H \Phi = 0, \quad \Phi = \tau(\phi_1, \phi_2)
\]

where \( \tau A \) denotes transposition of a vector or matrix \( A \). In the Feshbach-Villars representation the Hamiltonian \( H \) of (3.1) for a mass \( m \) scalar particle is

\[
H_{FV} = \tau p^2/2m + \sigma_i m, \quad p_k = \partial/\partial x_k, \quad \tau = \sigma_x + i\sigma_y
\]
For future convenience we prefer to use a different basis. The matrix $\tau$ is similar to the Pauli $\sigma_-$ by the transformation generated by $T_\tau = \sigma_\tau + \sigma_\tau$. The Hamiltonian becomes then
\[ H = -\sigma_- \left( \nabla^2 / 2m \right) + (\sigma_\tau + 2\sigma_\tau) m = \begin{pmatrix} m & 2m \\ -\nabla^2 / 2m & -m \end{pmatrix}. \] 
(3.3)

Assuming a vector coupling with a central potential $V(r)$, in particular a Coulomb potential $-q/r$, we define
\[ \lambda(r) = \lambda - V(r), \quad \text{in particular} \quad \lambda(r) = \lambda + \alpha/r. \] 
(3.4)

The actual form of the system (3.1) reads
\[ \left( I_2 \lambda(r) - H \right) \Phi(r) = \begin{pmatrix} \lambda(r) - m & -2m \\ -\nabla^2 / 2m & \lambda(r) + m \end{pmatrix} \begin{pmatrix} \phi_1(r) \\ \phi_2(r) \end{pmatrix} = 0 \] 
(3.5)

The Klein-Gordon equation is recovered by solving in $\phi_2(r)$ the first equation and substituting in the second one.

3.1. The states and the wave equation.

The eigenvalue problem for two scalar particles interacting through a Coulomb-like potential is written in the form
\[ \left( I_1 \lambda(r) - \left( H_1 \otimes I_2 + I_2 \otimes H_2 \right) \right) \Phi(r) = 0 \] 
(3.6)

where $H_i, i = 1, 2$, is the Hamiltonian (3.4) for the particle with mass $m_i$ and $\Phi = \Phi_1 \otimes \Phi_2 = \{ \phi_1, \phi_2, \phi_3, \phi_4 \}$ is the state of the system.

We use the canonical variables (3.4). As we said in Section 3.2 the action of the Lorentz group on the system is well defined: for simplicity and without loss of generality, we can choose the reference system in which the global spatial momentum is vanishing. The eigenvalue system of the two scalar particles specifies then to
\[ \begin{pmatrix} \lambda(r) - m_1 - m_2 & -2m_2 \\ -\nabla^2 / 2m_2 & \lambda(r) - m_1 + m_2 \\ -\nabla^2 / 2m_1 & 0 \\ 0 & -\nabla^2 / 2m_1 \end{pmatrix} \begin{pmatrix} \phi_1(r) \\ \phi_2(r) \\ \phi_3(r) \\ \phi_4(r) \end{pmatrix} = 0 \] 
(3.7)

$-\nabla^2$ being the Laplacian $\nabla^2$ with respect to the relative coordinate. The matrix operator in (3.7) is not Hermitian, although all the matrix elements are such. This is due to the representation (3.3) of the Klein-Gordon equation, sharing the same features.

Let us look for a reduction of the system (3.7). The first line yields directly an algebraic relation
\[ \left( \lambda(r) - m_1 - m_2 \right) \phi_1(r) = 2m_2 \phi_2(r) - 2m_1 \phi_3(r) \] 
(3.8)

From the second and third lines we derive another algebraic relation
\[ m_2 \left( \lambda(r) - m_1 + m_2 \right) \phi_2(r) - m_1 \left( \lambda(r) + m_1 - m_2 \right) \phi_3(r) = 0 \] 
(3.9)

By (3.8) and (3.9) we express $\phi_2(r)$ and $\phi_3(r)$ in terms of $\phi_1(r)$:
\[ \phi_2(r) = \left( 4m_2 \lambda(r) \right)^{-1} \left( \left( \lambda(r) - m_2 \right)^2 - m_1^2 \right) \phi_1(r), \quad \phi_3(r) = \left( 4m_1 \lambda(r) \right)^{-1} \left( \left( \lambda(r) - m_1 \right)^2 - m_2^2 \right) \phi_1(r). \] 
(3.10)

In order to proceed with the reduction process we need a prolongation of the system (3.5); this is due to the presence of the Laplacian in (3.7) and it is not needed in the cases of scalar-fermion and fermion-fermion systems we will treat later on. The prolongation is obtained by multiplying the matrix (3.7) to the left by the diagonal operator $\text{diag}(q^2, 1, 1, 1)$. Requiring the maintenance of the Hermiticity of the matrix elements, in the $(1,1)$ place we take the product in the form $-q \left( \lambda(r) - m_1 - m_2 \right) q$. For a Coulomb potential the same result is obtained by using the symmetrized form.
A simple calculation leads to

\[
(\lambda(r) - m_1 - m_2) \nabla^2 \phi_1(r) + \nabla \lambda(r) \cdot \nabla \phi(r) + 4m_1m_2(\lambda(r) + m_1 + m_2) \phi_2(r) = 0
\]

(3.11)

By isolating \(\phi_2(r)\) from the second line of (3.7) and substituting \(\phi_2(r)\) from (3.10) we find the final wave equation for the two scalar particles. Letting \(\phi_1(r) \equiv \phi(r)\) the interacting wave-equation for the two scalars is:

\[
\nabla^2 \phi(r) + (2 \lambda(r) - 1) \nabla \lambda(r) \cdot \nabla \phi(r) + \eta^2(r) \phi(r) = 0
\]

(3.12)

By specifying (3.12) to the Coulomb potential, so that \(\lambda(r) = \lambda + \alpha/r\), writing the Laplace operator in spherical coordinates and letting \(\phi(r) = u(r) Y_{\ell,m}(\theta, \phi)\), we find the radial equation

\[
\left( \frac{d^2}{dr^2} + \frac{2}{r} \frac{d}{dr} \right) u(r) + \frac{1}{4} \left( \lambda + \frac{\alpha}{r} \right)^2 + \frac{\left( m_1^2 - m_2^2 \right)^2}{4 \left( \lambda + \alpha/r \right)^2} - \frac{m_1^2 + m_2^2}{2} - \frac{\ell (\ell + 1)}{r^2} u(r) = 0
\]

(3.13)

The eigenvalues \(\lambda\) give the total energies of the two scalars. When \(\alpha = 0\) into (3.13) we have the free equation whose eigenvalues are

\[
\lambda = \pm \left( q_A q_A + m_1^2 \right)^{1/2} \pm \left( q_A q_A + m_2^2 \right)^{1/2} \quad \text{(all possible sign combinations)}
\]

(3.14)

in agreement with (2.6).

3.2. Limits

It is straightforward to take the limit of (3.12) for \(m_1 \to \infty\). Indeed, recalling that

\[
\lambda = m_1 + m_2 + E,
\]

(3.15)

when \(m_1\) becomes infinite, the limiting equation reduces to

\[
\nabla^2 \phi(r) + \left( (\lambda' + \alpha/r)^2 - m_2^2 \right) \phi(r)
\]

(3.16)

with \(\lambda' = \lambda - m_1 = m_2 + E\). Equation (3.16) is thus the Klein Gordon equation for a scalar with mass \(m_2\).

The non relativistic limit is found by explicitly reintroducing the speed of light \(c\) and rescaling the variables as follows:

\[
M \to M c^2, \quad \mu \to \mu c^2, \quad \alpha \to \alpha/c, \quad r \to r/c
\]

(3.17)

Taking the limit \(c \to \infty\) of (3.13), we recover the radial Schrödinger equation with the reduced mass \(m_R = m_1m_2/(m_1+m_2)\)

\[
\left( \frac{d^2}{dr^2} + \frac{2}{r} \frac{d}{dr} + 2m_R \left( E + \frac{\alpha}{r} - \frac{\ell (\ell + 1)}{r^2} \right) u(r) = 0
\]

(3.18)

3.3. Solutions

The solutions of the free equation are given in terms of spherical Bessel functions \(j_{\ell}(z) = (\pi/2z)^{1/2} J_{\ell+1/2}(z)\), and read

\[
u(r) = A j_{\ell}(kr), \quad k = (2\lambda)^{-1} \left( \lambda^2 - \lambda^2 \right)^{1/4} \left( \lambda^2 - \lambda^2 \right)^{1/4}
\]

(3.19)

There exists an analytical solution for the interacting equation (3.13) also. It is expressed as a combination of confluent Heun functions \(H_c(\eta, \pm \beta, \gamma, \delta, \zeta, -\lambda r/\alpha)\)

(3.20)

where \(A\) is an integration constant and the parameters of \(H_c\) are defined as

\[
\eta = (\alpha/\lambda^2) \left[ \left( \lambda^2 - \lambda^2 \right) \left( \lambda^2 - \lambda^2 \right) \right]^{1/2}
\]

\[
\beta = (1/4 + 4j(j+1) - \alpha^2)^{1/2}
\]

\[
\gamma = (2\lambda^2)^{-1} \left( \lambda^4 - 4\alpha^2(\lambda^2 - \lambda^2) \right)^{1/2}
\]

\[
\delta = -(2\lambda^2)^{-1} \alpha^2 \left( \lambda^4 - \lambda^2 \right)^{1/2}
\]

\[
\zeta = 1/8 + \alpha^2/2
\]

(3.21)

It is clear from (3.15) that for bound states, \(E < 0\), the parameter \(\eta\) is real, such as all the other ones.

In Table I here below we present the results for the lowest levels of two scalar systems with different mass components, comparing the Schrödinger, the Klein-Gordon and the two-body results.
For the free Dirac equation, letting

\[ p = \sqrt{\frac{2m}{\hbar}} \]

equation, again with the reduced mass, according to the well known expression

\[ m = \frac{m_1 m_2}{m_1 + m_2} \]

\[ \Omega \]

We next study a free system formed by a scalar particle and a fermion with respective masses \( m_1 \) and \( m_2 \). This latter mass gives the scale of the energies of the columns 3-5, expressed in units of \( m_2 c^2 \). In the third column we give the Schrödinger levels \( E_n = -m_2 c^2 \ell^2/2\hbar^2 \), \( m_2 \) being the classical reduced mass. In the fourth exact result of spectrum of the Klein-Gordon equation, again with the reduced mass, according to the well known expression

\[ E_{KG}(n, \ell) = (m_2 c^2) \left[ -1 + \left( 1 + \alpha^2 \left( n - \ell - \frac{1}{2} + \left( \ell + \frac{1}{2} \right)^2 - \alpha^2 \right)^{1/2} \right) \right]^{-1/2}. \]

(3.22)

In the last column we give the levels of equation \( (3.13) \). The results have been calculated numerically. Indeed, the spectral theory of the Heun’s equation and its polynomial solutions imply the discussion of a three term recursion relations. The truncation of the series solution requires to satisfy certain conditions which are not met in \( (3.13) \). Cases in which such a discussion has been made possible by the special values assumed by the parameters of the equations are found in \( [98] \).

| \( n, \ell \) | \( m_1/m_2 \) | \( E_{Sch} \) | \( E_{KG} \) | \( E_{num} \) |
|----------------|----------------|-------------|-------------|-------------|
| (1,0)          | 1              | -1.331283 10^{-3} | -1.331372 10^{-3} | -1.331323 10^{-3} |
|                | 100            | -2.636205 10^{-5} | 2.636381 10^{-5} | 2.636361 10^{-5} |
| (2,0)          | 1              | -3.328210 10^{-4} | -3.328354 10^{-4} | -3.328268 10^{-4} |
|                | 100            | -6.590514 10^{-6} | -6.590799 10^{-6} | -6.590793 10^{-6} |
| (2,1)          | 1              | -3.328210 10^{-4} | -3.328235 10^{-4} | -3.328216 10^{-4} |
|                | 100            | -6.590514 10^{-6} | -6.590565 10^{-6} | -6.590563 10^{-6} |

Table I

In the first column of Table I we indicate the state defined by the principal quantum number \( n \) and by the total angular momentum \( \ell \). In the second one the ratio of the heavier mass \( m_1 \) to the lighter \( m_2 \). This latter mass gives the scale of the energies of the columns 3-5, expressed in units of \( m_2 c^2 \). In the third column we give the Schrödinger levels \( E_n = -m_2 c^2 \ell^2/2\hbar^2 \), \( m_2 \) being the classical reduced mass. In the fourth exact result of spectrum of the Klein-Gordon equation, again with the reduced mass, according to the well known expression

\[ E_{KG}(n, \ell) = (m_2 c^2) \left[ -1 + \left( 1 + \alpha^2 \left( n - \ell - \frac{1}{2} + \left( \ell + \frac{1}{2} \right)^2 - \alpha^2 \right)^{1/2} \right) \right]^{-1/2}. \]

(3.22)

In the last column we give the levels of equation \( (3.13) \). The results have been calculated numerically. Indeed, the spectral theory of the Heun’s equation and its polynomial solutions imply the discussion of a three term recursion relations. The truncation of the series solution requires to satisfy certain conditions which are not met in \( (3.13) \). Cases in which such a discussion has been made possible by the special values assumed by the parameters of the equations are found in \( [98] \).

4. The wave equation for a scalar and a fermion

We next study a free system formed by a scalar particle and a fermion with respective masses \( m_S \) and \( m_F \). The internal geometrical structure of the fermion adds some slight complications to the simpler treatment of Section 3. The first question to unravel concerns the form of the states for which angular momentum and parity are diagonal. In fact the task is not so complicated, due to the scalar nature of one of the components.

Recall first the basic method leading to the solution of the Coulomb problem for the Dirac equation \( [99] \). Using the standard representation \( \psi = \left( \phi, \chi \right) \) for the Dirac spinor, the angular dependence of the components \( \phi \) and \( \chi \) are given by:

\[ \phi(r) = r^\ell \hat{r} Y_{\ell m}(\theta, \phi), \quad \chi(r) = -ir^\ell \hat{r} Y_{\ell m}(\theta, \phi) \]

(4.1)

By space inversion \( \phi(r) \rightarrow i\phi(-r) \) and \( \chi(r) \rightarrow -i\chi(-r) \). The parity of spherical spinors is opposite when \( |\ell - \ell'| = 1 \). Therefore, if the spinor \( \psi \) has definite parity and \( \phi \) depends upon \( \Omega_{\ell m} \) with \( \ell = j \pm 1/2 \), then \( \chi \) depends necessarily on \( \Omega_{\ell' m}' \), with \( \ell' = j \mp 1/2 \). The general form of the Dirac spinor is then as follows:

\[ \psi = \left( \phi, \chi \right) = \left( a(r) \Omega_{\ell m}(\theta, \phi), \quad b(r) \Omega_{\ell' m}'(\theta, \phi) \right) \]

(4.2)

For the free Dirac equation, letting \( p = |p| \), the radial functions are

\[ a(r) = A R_{p}(r), \quad b(r) = B R_{p}(r), \quad R_{p}(r) = 2 p^{1/2} j_{\ell}(pr), \]

(4.3)

where \( A \) and \( B \) are integration constants.
4.1. The states and the wave equation

Let us switch to the scalar-fermion system describing the scalar, as in Section 3, by the two component formalism and the fermion by the Dirac equation. According to the two component representation \( \Phi_a \) of the Klein-Gordon equation, the states of definite angular momentum and parity will now be the eight-component vectors. For fixed \( j, \ell, m \) the states \( \Phi_1 \) and \( \Phi_2 \) of opposite parity have the form

\[
\Phi_1(r) = T \left( a_1(r) \Omega_{\ell m}(\theta, \phi), \quad a_2(r) \Omega_{\ell' m'}(\theta, \phi), \quad a_3(r) \Omega_{\ell m}(\theta, \phi), \quad a_4(r) \Omega_{\ell' m'}(\theta, \phi) \right)
\]

\[
\Phi_2(r) = T \left( b_1(r) \Omega_{\ell m}(\theta, \phi), \quad b_2(r) \Omega_{\ell' m'}(\theta, \phi), \quad b_3(r) \Omega_{\ell m}(\theta, \phi), \quad b_4(r) \Omega_{\ell' m'}(\theta, \phi) \right)
\]

(4.4)

The explicit form of the Klein-Gordon and Dirac Hamiltonians \( H_S \) and \( H_F \) of the two particles, after the factorization of the global part and with \( q_A = -i \partial/\partial r_A \), \( A = x, y, z \), is

\[
H_S = \begin{pmatrix} m_S + m_F & 0 & q_0 & \sqrt{2} q_x - 2m_S & 0 & 0 & 0 & 0 \\ 0 & m_S - m_F & -\sqrt{2} q_x & 0 & 2m_S & 0 & 0 & 0 \\ q_0 & \sqrt{2} q_x & m_S - m_F & 0 & 0 & 2m_S & 0 & 0 \\ -\sqrt{2} q_x & 0 & -m_S + m_F & 0 & 0 & 0 & 2m_S & 0 \\ 0 & 0 & 0 & -m_S + m_F & 0 & q_0 & \sqrt{2} q_x & -q_0 \\ q_0^2/(2m_S) & 0 & 0 & 0 & -m_S + m_F & -\sqrt{2} q_x & -q_0 & 0 \\ 0 & q_0^2/(2m_S) & 0 & 0 & 0 & -m_S - m_F & 0 & 0 \\ 0 & 0 & 0 & q_0^2/(2m_S) & -\sqrt{2} q_x & -q_0 & 0 & -m_S - m_F \end{pmatrix}
\]

(4.5)

The global Hamiltonian \( H = H_S \otimes I_4 + I_2 \otimes H_F \) thus reads

\[
H = \begin{cases}
\begin{array}{cccccccc}
m_S + m_F & 0 & q_0 & \sqrt{2} q_x & 2m_S & 0 & 0 & 0 \\
0 & m_S - m_F & -\sqrt{2} q_x & 0 & 2m_S & 0 & 0 & 0 \\
q_0 & \sqrt{2} q_x & m_S - m_F & 0 & 0 & 2m_S & 0 & 0 \\
-\sqrt{2} q_x & 0 & -m_S + m_F & 0 & 0 & 0 & 2m_S & 0 \\
0 & 0 & 0 & -m_S + m_F & 0 & q_0 & \sqrt{2} q_x & -q_0 \\
q_0^2/(2m_S) & 0 & 0 & 0 & -m_S + m_F & -\sqrt{2} q_x & -q_0 & 0 \\
0 & q_0^2/(2m_S) & 0 & 0 & 0 & -m_S - m_F & 0 & 0 \\
0 & 0 & 0 & q_0^2/(2m_S) & -\sqrt{2} q_x & -q_0 & 0 & -m_S - m_F
\end{array}
\end{cases}
\]

(4.6)

In (4.6) \( q_x, q_0 \) are the spherical differential operators

\[
q_x = -\left( \pm q_x + iq_y \right)/\sqrt{2}, \quad q_0 = q_z, \quad q_A = -i \partial/\partial r_A
\]

(4.7)

In order to determine the radial functions, we work out explicitly the eigenvalue problem relative to the Hamiltonian (4.6) for the states (4.4). We obtain a system of equations by imposing the vanishing of the coefficients of the different spherical harmonics for all the components of the resulting vector. From the relations thus found, only four linearly independent equations can be extracted for each parity. We make the change of variables \( a_2(r) \to -ia_2(r) \) and \( a_4(r) \to -ia_4(r) \). For the first state (4.4), \( \Phi_1(r) \), we have

\[
\begin{align*}
\left( d/dr - r^{-1} (j - 1/2) \right) a_1(r) + (\ell r - m_S + m_F) a_2(r) - 2m_S a_4(r) &= 0 \\
\left( d/dr + r^{-1} (j + 3/2) \right) a_2(r) - (\ell r - m_S - m_F) a_4(r) + 2m_S a_3(r) &= 0 \\
\left( d/dr + r^{-1} (j + 3/2) \right) a_4(r) - (\ell r + m_S - m_F) a_3(r) - (2m_S)^{-1} \nabla^2 a_1(r) &= 0 \\
\left( d/dr - r^{-1} (j - 1/2) \right) a_3(r) + (\ell r + m_S + m_F) a_4(r) + (2m_S)^{-1} \nabla^2 a_2(r) &= 0
\end{align*}
\]

(4.8)

By a direct computation it can be seen that the equations for the second state (4.4), \( \Phi_2(r) \), are obtained from (4.8) by simply changing \( m_F \) into \( -m_F \).

As it occurs in the case of two scalars, here also the system can be reduced because of the presence of some algebraic relations. The procedure is a bit lengthy but straightforward. Isolate \( a_4(r) \) from the first and \( a_2(r) \) from the second of equations (4.8) and substitute in the third and fourth. When the explicit expression of the Laplacian operator is taken into
account the second derivatives cancel and the angular parts contribute with terms \( J^2/r^2 \). The value of the constant \( J^2 \) is different for the two equations. In the third equation, when calculating \( \nabla^2 a_1(r) \), we have \( J^2 = (j - 1/2) + (j + 1/2) \); in the fourth one, when calculating \( \nabla^2 a_2(r) \), \( J^2 = (j + 1/2) + (j + 3/2) \). Letting \( a_1(r) = f(r) \) and \( a_2(r) = g(r) \), for the state \( \Phi(r) \) we get the reduced system

\[
\begin{align*}
\frac{d f(r)}{dr} & - \left( \frac{j - 1/2}{r} - \frac{\lambda(r)/dr}{2\lambda(r)} \right) f(r) - \left( \frac{\lambda}{2} + m_f + \frac{m_f^2 - m_s^2}{2\lambda} \right) g(r) = 0 \\
\frac{d g(r)}{dr} & + \left( \frac{j + 3/2}{r} + \frac{\lambda(r)/dr}{2\lambda(r)} \right) g(r) + \left( \frac{\lambda}{2} - m_f + \frac{m_f^2 - m_s^2}{2\lambda} \right) f(r) = 0
\end{align*}
\]

The reduced system for \( \Phi_\ell(r) \) is again found from \( 4.9 \) by changing \( m_F \) into \(-m_F\). In particular, for a Coulomb potential, the system \( 4.9 \) becomes

\[
\begin{align*}
\frac{d f(r)}{dr} & - \frac{1}{r} \left( j - \frac{1}{2} + \frac{\alpha}{2(\lambda + \alpha)} \right) f(r) - \left( \frac{\lambda}{2} + m_F + \frac{\alpha}{2r} + \frac{m_F^2 - m_s^2}{2(\lambda + \alpha)} \right) g(r) = 0 \\
\frac{d g(r)}{dr} & + \frac{1}{r} \left( j + \frac{3}{2} - \frac{\alpha}{2(\lambda + \alpha)} \right) g(r) + \left( \frac{\lambda}{2} - m_F - \frac{\alpha}{2r} + \frac{m_F^2 - m_s^2}{2(\lambda + \alpha)} \right) f(r) = 0
\end{align*}
\]

### 4.2. Limits

Let us look at the two separate limits of \( 4.8 \) for the scalar or the fermion mass tending to infinity. In both cases the first step is to substitute \( \lambda(r) = m_s + m_F + E - V(r) \) into \( 4.8 \).

Consider \( m_F \to \infty \). The first two equations give immediately \( a_3(r) = a_4(r) = 0 \) and the last two ones become identities. Defining \( \lambda_F(r) = m_F + E - V(r) \), we are therefore left with

\[
\begin{align*}
(\frac{d}{dr} - r^{-1} (j - 1/2)) a_1(r) + (\lambda_F(r) + m_F) a_2(r) & = 0 \\
(\frac{d}{dr} + r^{-1} (j + 3/2)) a_2(r) - (\lambda_F(r) - m_F) a_1(r) & = 0
\end{align*}
\]

which is exactly the Dirac equation for the spinor \( 4.2 \) with \( j = \ell + 1/2 \). The same procedure for the state with opposite parity leads to the Dirac equation with \( j = \ell - 1/2 \).

Take now \( m_s \to \infty \) again in \( 4.8 \). Substitute the first order expansions \( a_0(r) = m_F a_0(1) + a_{0,0}(r) \) for the radial functions and consider the different orders in \( m_F \). From the first and the last equation of \( 4.8 \) we have \( a_{2,1} = a_{4,1} = 0 \). At order one in \( m_F \):

\[
\begin{align*}
a_{2,0}(r) & = -(1/2) \left( \frac{d}{dr} - r^{-1} (j - 1/2) \right) a_{1,1}(r) \quad a_{0,0}(r) = -(1/2) \left( \frac{d}{dr} - r^{-1} (j - 1/2) \right) a_{3,1}(r) \\
a_{3,1}(r) & = (2m_S)^{-1} \left( E - V(r) \right) a_{1,1}(r) \quad (\nabla^2 / 2m_S) a_{1,1}(r) + \left( 2m_S + E - V(r) \right) a_{3,1}(r) = 0
\end{align*}
\]

We therefore see that \( a_{2,0}(r), a_{3,1}(r), a_{4,0}(r) \) are all determined by \( a_{1,1}(r) \) which satisfies the Klein-Gordon equation

\[
\nabla^2 a_{1,1}(r) + (\lambda_S(r)^2 - m_S^2) a_{1,1}(r) = 0
\]

where \( \lambda_S(r) = m_S + E - V(r) \). Analogous results are obtained for the state \( \Phi_\ell(r) \).

The non relativistic limit is easily seen to reproduce the Schrödinger equation \( 3.18 \).

### 4.3. Solutions

The free wave equation, obtained from \( 4.9 \) with \( \lambda(r) = \lambda = constant \), has the solution

\[
f(r) = A \ j_j(kr) \quad g(r) = 2A \left( \lambda + m_F \right)^{-1} k \ j_{j+1}(kr)
\]

where \( k \) is as in \( 3.19 \) with \( m_1 = m_F, m_2 = m_S \).

The interacting problem does not admit solutions in terms of special functions but for equal masses, in which case the solution is again a combination of Heun’s confluent functions. In Table II here below we give the numerical results for the spectrum, calculated by deriving a second order equation from \( 4.10 \). The actual values of the levels are obtained multiplying the data by reduced mass \( m_F m_S / (m_F + m_S) \).
Some comments are in order. Concerning the numerical precision, the data have been calculated in such a way that the figures appearing in the table are all meaningful. The different properties of level structure with varying mass ratio appear clearly from the data. Indeed, coherently with what we have shown for infinite mass limits, for increasing \( m_F / m_F \) we pass from a Klein-Gordon (second column) to a Dirac (last column) behavior, switching from an approximate degeneracy in \( \ell \) to an approximate degeneracy in \( j \) (columns 3-5). When the level energy increases these degeneracies enhance, so that, at the accuracy presented in the table, the levels appear to be coincident. Instead, for equal fermion and scalar masses, the degeneracy of the levels with opposite parity is exact. This is due to the fact that the parity is changed by letting \( m_F \rightarrow -m_F \) and for \( m_F = m_S \) the resulting second order equations are identical. Finally we see that a term crossings are present exactly at \( m_S / m_F = 1 \). This occurs also for the two fermion systems [34]. In that case, however, the hyperfine interaction described by the Breit term removes the crossings. In the present situation the hyperfine interactions are absent and higher order corrections should be considered to see whether crossings survive.

### 5. The wave equation for two fermions

We finally quantize the system of two relativistic fermions of spin 1/2 in terms of two coupled Dirac equations. As stated in the Introduction, this is the only case we had considered [34, 38] and we present it here because of the ample revisitation of its construction, so to give a coherent and uniform treatment of all the simplest two body relativistic wave equations. Being a two body problem, the Hilbert space of the states is given by the tensor product of the spaces of the two Dirac spinors. We consider the two independent Dirac operators in the global and relative coordinates (2.1):

\[
D_1 = \left( P_\mu / 2 + \tilde{q}_\mu \right) \tilde{\gamma}^{(1)}_{\lambda}\gamma_\mu - m_1, \quad D_2 = \left( P_\mu / 2 - \tilde{q}_\mu \right) \tilde{\gamma}^{(2)}_{\lambda}\gamma_\mu - m_2
\]  

(5.1)

The matrices \( \tilde{\gamma}^{(1)}_{\lambda} = \gamma_\mu \otimes I_4 \) and \( \tilde{\gamma}^{(2)}_{\lambda} = I_4 \otimes \gamma_\mu \) operate on each fermion space. Using the canonical coordinates (2.4) and coherently defining \( \gamma_{(i)0}(P) = \epsilon^{(i)}_{\lambda \mu}(P) \tilde{\gamma}_{i\lambda\mu} \), \( \gamma_{(i)A}(P) = \epsilon^{(i)}_{\lambda \mu}(P) \tilde{\gamma}_{i\lambda\mu} \), we have the operator relations

\[
(1/2) \lambda \gamma_{(1)0} + q_0 \gamma_{(1)\lambda} - q_{A\lambda} \gamma_{(1)A} = m_1, \quad (1/2) \lambda \gamma_{(2)0} + q_0 \gamma_{(2)\lambda} - q_{A\lambda} \gamma_{(2)A} = m_2
\]

(5.2)

As long as \( P \) is conserved the gamma matrices can be given the usual representation. We solve \( 5.2 \) in \( \lambda \) and \( q_0 \):

\[
\lambda = \left( \gamma_{(1)0} \gamma_{(1)A} - \gamma_{(2)0} \gamma_{(2)A} \right) q_A + \gamma_{(1)0} m_1 + \gamma_{(2)0} m_2, \quad q_0 = \left( 2 \lambda \right)^{-1} \left( m_1^2 - m_2^2 \right)
\]  

(5.3)

The spectrum of the free system is obtained from the first of equations (5.3). The free relative momentum components \( q_A \) being conserved, the spectrum is formed by the four eigenvalues \( 5.3 \) each one of them with multiplicity four. Defining \( M = m_1 + m_2 \) and \( \mu = m_1 - m_2 \), we see that \( \pm M, \mu, \pm \mu \) are the eigenvalues of the two fermions at relative rest. The components of the spinor tensor product will be reordered so that in the eigenstate of the system at rest the eigenvalues appear in the order \( M, -M, \mu, -\mu \). The global angular momentum and parity are conserved. A parity transformation is the composition of a spatial inversion \( r \rightarrow -r \) and an internal parity transformation represented by the matrix \( \gamma_0 \otimes \gamma_0 = I_{8,8} \). We denote by I or II the two opposite parities.
The state of opposite parity is obtained as

\[ \Psi_{(\ii, \ii)} = T \left( \psi^{(M)}_{(\ii, \ii)}, \psi^{(-M)}_{(\ii, \ii)}, \psi^{(\mu)}_{(\ii, \ii)} \right), \quad \Psi^{(\ii, \ii)} = T \left( \psi^{(\ii, \ii)}_{0}, \psi^{(\ii, \ii, 1)}, \psi^{(\ii, \ii, 1)}_{1} \right) = \pm M, \pm \mu. \]  

(5.4)

The eigenstates of angular momentum are obtained by introducing the ‘spherical singlets and triplets’, which play the same role as the spherical spinors \( \{ \Omega_{1}, \Omega_{2}, \Omega_{3} \} \). The angular part of the singlets is either simply given by the spherical harmonic \( Y_{x}^{m}(\theta, \phi) \) or vanishing. In terms of the usual \( \langle j_{1}, m_{1}, j_{2}, m_{2} | J, M \rangle \) Clebsch-Gordan coefficients the triplets are

\[
\Omega^{(b)}_{m}(\theta, \phi) = \mathcal{Hm} \left( \begin{array}{c} \sqrt{(j_{1} - j_{2} + l + 1)(j_{1} + j_{2} + l + 1)} Y_{j_{1}+1}^{m}_{j_{1}+1}(\theta, \phi), \sqrt{(j_{1} + j_{2} + l + 1)(j_{1} - j_{2} + l + 1)} Y_{j_{1}+1}^{m}_{j_{1}+1}(\theta, \phi) \end{array} \right)
\]

(5.5)

Their explicit expressions therefore read

\[
\Omega^{(b)}_{m}(\theta, \phi) = \mathcal{Hm} \left( \begin{array}{c} \sqrt{(j_{1} - j_{2} + l + 1)(j_{1} + j_{2} + l + 1)} Y_{j_{1}+1}^{m}_{j_{1}+1}(\theta, \phi), \sqrt{(j_{1} + j_{2} + l + 1)(j_{1} - j_{2} + l + 1)} Y_{j_{1}+1}^{m}_{j_{1}+1}(\theta, \phi) \end{array} \right)
\]

(5.5)

Using the previous spherical singlets and triplets, the components \( \Psi^{(\ii)}_{1} \) for the state with the first parity are

\[
\Psi^{(M)}_{1} = T \left( a_{0}(r) Y_{m}(\theta, \phi), b_{0}(r) \Omega^{(b)}_{m}(\theta, \phi) \right), \quad \Psi^{(-M)}_{1} = T \left( a_{1}(r) Y_{m}(\theta, \phi), b_{1}(r) \Omega^{(b)}_{m}(\theta, \phi) \right)
\]

(5.7)

The state of opposite parity is obtained as \( \Psi_{\ii} = (\sigma \otimes 1_{S}) \Psi_{1} \). In (5.7) the eight unknown radial functions \( a_{i}(r), b_{i}(r), c_{i}(r), d_{i}(r), i = 1, 2 \), replace the functions \( a(r) \) and \( b(r) \) of (4.2). Introducing the interaction requires a precise knowledge of its tensorial nature. For Hydrogen-like atoms the interaction is represented by a Coulomb potential in vector coupling; for mesons in the Quarkonium model it is described by the Cornell potential, which has a Coulomb-like vector component and a scalar confining linear term. The general two body interacting wave equation we consider is therefore the following:

\[
\left[ \left( \gamma_{1}(1) \gamma_{1}(1) a - \gamma_{2}(2) \gamma_{2}(2) a \right) + \left( 1/2 \right) \left( \gamma_{1}(1) + \gamma_{2}(2) \right) \left( M + \sigma \sigma \right) + \left( 1/2 \right) \left( \gamma_{1}(1) - \gamma_{2}(2) \right) \mu + \hat{\alpha} \right] \Psi(r) = 0.
\]

(5.8)

In the case of Hydrogen-like atoms \( \sigma = 0 \) and \( \alpha = Z \alpha_{n} \), \( Z \) being the atomic number of the nucleus and \( \alpha_{n} \) the fine structure constant. For the Quarkonium model \( \sigma \) is known as the ‘strong tension’ and \( \alpha = (4/3) \alpha_{S} \), the latter being the strong running coupling constant \( \tilde{\alpha} \). In both cases we can add the Breit term \( V_{B}(r) \), responsible for the spin-spin interaction, which has to be treated at the first perturbation order:

\[
V_{B}(r) = (g/2r) \left( \gamma_{1}(1) \gamma_{1}(1) a \gamma_{2}(2) \gamma_{2}(2) a + \gamma_{1}(1) \gamma_{2}(2) \gamma_{2}(2) \gamma_{1}(1) a \right) = \frac{g}{2r} \left( \gamma_{1}(1) \gamma_{1}(1) a \gamma_{2}(2) \gamma_{2}(2) a + \gamma_{1}(1) \gamma_{2}(2) \gamma_{2}(2) \gamma_{1}(1) a \right).
\]

(5.9)
The analogous relations for $W$ are chosen as independent unknown functions \((H \Psi = 0)\) for all the components of the resulting vector. Eight linearly independent equations can be extracted from all the relations through this. Introducing the sum and difference notation $\xi_{+}(r) = \xi_{+}(r) = \xi_{-}(r)$, \((\xi = a, b, c, d)\) and the operators

\[
D_{[j_{1}, j_{2}]} = (2 j + 1)^{-1/2} \frac{d}{dr} \left( d/dr + j_{2}/r \right)
\]

the radial boundary eigenvalue problem for $\Psi_{1}$ reduces to the system

\[
\begin{align*}
D_{[j_{1}, j_{1}]} a_{s}(r) &= D_{[j_{1}, j_{1}]} b_{s}(r), & c_{s}(r) &= c_{s}(r) \\
D_{[j_{1}, j_{1}]} a_{c}(r) &= D_{[j_{1}, j_{1}]} b_{c}(r), & c_{c}(r) &= c_{c}(r) \\
D_{[j_{1}, -j_{1}]} a_{s}(r) &= D_{[j_{1}, -j_{1}]} b_{s}(r), & c_{s}(r) &= c_{s}(r) \\
D_{[j_{1}, -j_{1}]} a_{c}(r) &= D_{[j_{1}, -j_{1}]} b_{c}(r), & c_{c}(r) &= c_{c}(r) \end{align*}
\]

\[
D_{[j_{1}, j_{2}]} c_{s}(r) = D_{[j_{1}, j_{2}]} d_{s}(r), & D_{[j_{1}, j_{2}]} c_{c}(r) = D_{[j_{1}, j_{2}]} d_{c}(r),
\]

The system for $\Psi_{1}$ is obtained by means of the parity transformation. It can be seen that this amounts to substituting $M + \sigma r \rightarrow -\mu$ and $\mu \rightarrow -(M + \sigma r)$.

The differential order for both parity systems is actually four due to four algebraic relations among the unknown functions. These are more simply written by introducing the following linear combinations:

\[
\begin{align*}
\xi_{+}(r) &= \frac{1}{2} (\xi^{\dagger}(r) + \xi(r)), \xi_{-}(r) = \frac{1}{2} (\xi^{\dagger}(r) - \xi(r))
\end{align*}
\]

where we have defined $\xi_{+} = j_{1/2}(2 j + 1)^{-1/2}$ and $\xi_{-} = (j + 1/2)(2 j + 1)^{-1/2}$. The algebraic relations for the state $\Psi_{1}$ are

\[
a_{s}(r) = \lambda(r) \xi_{+}(r), \quad \xi_{+}(r) = -\xi_{-}(r), \quad \xi_{-}(r) = -\xi_{+}(r)
\]

The analogous relations for $\Psi_{2}$ are again obtained by substituting $M + \sigma r \rightarrow -\mu$ and $\mu \rightarrow -(M + \sigma r)$ in (5.13). We choose the independent unknown functions \((a_{s}(r), b_{s}(r), a_{c}(r), b_{c}(r))\) which we respectively arrange in a vector $Y(r) \equiv T(y_{1}(r), y_{2}(r), y_{3}(r), y_{4}(r))$. When $Y(r)$ is known, the state $\Psi_{1}$ is reconstructed by the inverse transformation. The analogous procedure applies to the odd parity. The eigenfunctions for both fermions are thus obtained by solving

\[
\begin{pmatrix}
0 & E(r, \lambda) & -F(r, \lambda) & 0 \\
E(r, \lambda) & 1 / r & 0 & F(r, \lambda) \\
G(r, \lambda) & 2 / r & E(r, \lambda) & 0 \\
0 & -G(r, \lambda) & E(r, \lambda) & 1 / r
\end{pmatrix}
\begin{pmatrix}
y_{1}(r) \\
y_{2}(r) \\
y_{3}(r) \\
y_{4}(r)
\end{pmatrix}
= 0
\]

(5.15)
With the usual notation (3.4) for the Coulomb potential, the matrix elements for the two parities are
\[
E_1(r, \lambda) = \frac{(j + 1)!}{2 \lambda} \mu, \quad F_1(r, \lambda) = \frac{\lambda}{2} - \frac{\mu^2}{2 \lambda}, \quad G_1(r, \lambda) = \frac{\lambda}{2} - \frac{2j(j + 1)}{r^2 \lambda} - \frac{(M + \sigma r)^2}{2 \lambda}
\]
\[
E_\Pi(r, \lambda) = -\frac{(j + 1)!}{2 \lambda} (M + \sigma r), \quad F_\Pi(r, \lambda) = \frac{\lambda}{2} - \frac{(M + \sigma r)^2}{2 \lambda}, \quad G_\Pi(r, \lambda) = \frac{\lambda}{2} - \frac{2j(j + 1)}{r^2 \lambda} - \frac{\mu^2}{2 \lambda}
\]  
\tag{5.16}

5.2. Limits

Let us look at two limiting cases of (5.15). We first consider the limit for \(m_1 \to \infty\), or ‘Dirac limit.’ In this case and with \(\lambda_2(r) = m_2 + E + \alpha/r\), the coefficient matrices (5.13) for the first parity become
\[
E_1^D(r, \lambda) = \frac{(j + 1)!}{r}, \quad \lambda_{12}^D(r) = \lambda_{21}^D(r) = \lambda_2(r) + m_2, \quad G_2^D(r, \lambda) = \lambda_2(r) - (m_2 + \sigma r)
\]

The limiting equations obtained by using (5.17) in (5.15) are equivalent to pairs of Dirac equations. A mixing is necessary in order to decouple the fourth order system. In the two parity cases, with \(s_0, s_1\) as above, the mixing transformations are respectively generated by the 2 × 2 block matrices
\[
T_1 = \begin{pmatrix} 0 & s_0a_{21} + s_1a_{22} \\ s_1 & 0 \end{pmatrix}, \quad T_2 = \begin{pmatrix} 0 & s_0b_{21} + s_1b_{22} \\ s_1 & 0 \end{pmatrix}
\]  
\tag{5.18}

In both cases \((y_1(r), y_2(r))\) decouple from \((y_2(r), y_3(r))\) and give Dirac equations
\[
\frac{d}{dr} \begin{pmatrix} r f(r) \\ r g(r) \end{pmatrix} + \begin{pmatrix} \kappa/r & -\lambda_2(r) - m_2 - \sigma r \\ \lambda_2(r) - m_2 - \sigma r & -\kappa/r \end{pmatrix} \begin{pmatrix} r f(r) \\ r g(r) \end{pmatrix} = 0
\]  
\tag{5.19}

For the first parity, taking the unknown functions \((f(r), g(r))\) equal to \((r y_2(r), r y_1(r))\) or to \((r y_3(r), -r y_2(r))\) we get (5.19) with \(\kappa = -(j + 1)\) or \(\kappa = j\), the two values corresponding to orbital angular momentum \(\ell = j\). In the odd case for \((f(r), g(r))\) equal to \((r y_1(r), -r y_2(r))\) or \((r y_2(r), r y_3(r))\), we get \(\kappa = -j\) or \(\kappa = j + 1\), the two values corresponding to \(\ell = j + 1\) and \(\ell = j - 1\) respectively.

In order to recover the Schrödinger equation we reintroduce explicitly the factor \(c\). The limit is obtained from (5.15)- (5.17) using the rescaling (3.17) together with \(\sigma \to \sigma c\). Eliminating \(y_3(r)\) and \(y_4(r)\) we get a system of two second order differential equations for \(y_1(r)\) and \(y_2(r)\). When \(c \to \infty\) both \(y_1(z)\) and \(y_2(z)\) satisfy the Schrödinger equation
\[
\left( \frac{d^2}{dr^2} + \frac{2}{r} \frac{d}{dr} + m_r (E - \sigma r + \frac{\alpha}{r} - \frac{j(j + 1)}{r^2}) \right) u(r) = 0
\]  
\tag{5.20}

For the first parity we have two decoupled equations with \(\ell = j\) in both cases. For the second parity, a transformation generated by the matrix \(T = (j + 1)! I_2 + i j^{\frac{1}{2}} \gamma_5\) is needed and produces two equations (5.20) with \(\ell = j \pm 1\).

5.3. Solutions

Analytical solutions are available for the free system (5.15)- (5.16) with \(\alpha = \sigma = 0\). They are expressed in terms of spherical Bessel functions. With integration constants \(A, B, k\) as in (3.19), for parity I we have:
\[
y_1(r) = A j_j(kr) \quad y_2(r) = B j_j(kr)
\]
\[
y_3(r) = A j_j(kr - \mu) j_j(kr - \mu) \quad y_4(r) = -A j_j(kr - \mu) j_j(kr - \mu)
\]  
\tag{5.21}

The solutions for \(\Psi_\Pi\) are obtained by substituting \(M \to -\mu\) and \(\mu \to -M\) in (5.21).

The interacting models have been discussed numerically. The analysis is conducted within the spectral theory in Hilbert spaces by the ‘double shooting method’ with a very high accuracy obtained by Padé approximants. The matching of the four components of the system at a crossing point \(0 < r_c < \infty\) determines a linear system of four algebraic equations whose determinant constitutes the spectral condition.

For the sake of completeness we reorganize and report in Appendix some results obtained in (84-88) for atom and meson systems.
6. Conclusions

Relativistic quantum mechanics has generally been considered a preliminary step, naturally and unavoidably leading to the quantization of fields. The history of physics demonstrates that this idea has produced the fantastic achievements obtained during the whole last century and, undoubtedly, keeps being very fruitful. There are however some subjects that are better dealt with directly in a quantum mechanical framework rather than in a field theoretical one. This happens in the study of the bound states of composite relativistic systems when a high degree of accuracy is required. It is not only the case of high energy objects, as the hadrons, but also that of atoms, due to the higher and higher precision recently reached by the measurements. The development of relativistic quantum mechanics for an arbitrary number of interacting bodies, however, has met some problems for which a complete and convincing solution, provided it exists, has not yet been given. In particular, difficulties have always been found in reconciling the covariance with the evolution in time if an interaction is present. This is well explained by Wigner (1969), when he says: ‘It appears reasonable to state that the problem of the motion of several interacting relativistic particles in classical and quantum mechanics has not been solved. In constructing such a theory one encounters relative time coordinates whose meaning is often obscure’ (quoted by Cook [100]). We have given our point of view on the restricted case of systems composed of two particles only, by presenting a procedure that allows to reduce the phase space so to make the relative time a cyclic variable. We must admit that for two bodies the reduction procedure works fine, but unfortunately we do not have the general solution when the number of the particles is larger. In the restricted case we have given a covariant treatment of the problem both at a classical and at a quantum level and we have tested it, with positive confirmation, against the experimental data taken on physical objects of different nature and ranging on different scales of energy. The quantum system of two interacting particles is defined by the total angular momentum, the parity and the invariant mass, which is the eigenvalue of the Hamiltonian, while the masses of the free fermions enter as independent parameters. In this way we have built a composite object such that each of its eigenstates corresponds to a representation of the Poincaré group. The comparison with the experimental data has implied the recourse to a numerical analysis of our model. We believe that a more general and deeper mathematical investigation, in analogy to what has been done for the Dirac equation [101] should be worthwhile. The main interest could come from the fact that we are studying a fourth order problem: this, indeed, presents aspects that are not always intuitive, due to the commonly rooted habit of reasoning in terms of second order – and mainly elliptic – equations. From a more physical point of view, we believe that the method could be generalized to decays [87, 88] and to include the higher order corrections of the fields. The two body equation would then provide a better starting point for obtaining a higher accuracy in the phenomenological analysis of physical systems and in the comparison with experimental data.

7. Appendix

7.1. Atoms

The current theoretical calculations found in literature reproduce rather well the measured quantities. The agreement with data improves by adding effects of different nature by perturbation expansions, generally starting from a non-relativistic description of the physical system. Here we report our data exclusively obtained from equation (5.15) and the Breit term (5.9) treated at the first perturbation order. The constants are fixed at their measured physical values: $m_1$ and $m_2$ the masses of the two particles, $b = a = Ze^2/\hbar c$, $e$ being the electron charge, $g = \kappa_1 \kappa_2 a$ where $\kappa_1$ and $\kappa_2$ are the factors accounting for the anomalous magnetic moments of the two fermions. The latter are assumed to be point-like, neglecting finite radius corrections even for massive nuclei as $^3\text{He}^+$. The model is therefore very sharp, without any free parameter and with a completely clear physical content. The results show that the complete inclusion of covariance properties and spin-spin interactions, even without further QED corrections, produce by themselves very high accuracy. The Lamb shift can be introduced in an effective way (we did it in [87]), but it is not considered here, since it gives very very small corrections to the data we present in Table III, concerning the hyperfine splittings (HFS) of the levels.
We have considered the meson mass spectrum in the Quarkonium model using the Cornell potential, formed by a
7.2. Mesons
The values of the proton, electron and muon mass, $m_p$, $m_e$ and $m_μ$ are 1.6726218986 MeV, 0.5109989474 eV and
-23.678
24.
The first column of Table III indicates the simple atom which we refer to. The following columns respectively give
Table III
Theoretical results obtained by different methods for the HFS of the $s$ and $p$ levels of simple
Hydrogen-like atoms are found in the papers [102, 107]. More details on our results are found in [67].
7.2. Mesons
We have considered the meson mass spectrum in the Quarkonium model using the Cornell potential, formed by a
Coulomb-like term $α/r$ and a confining scalar term $σr$. $σ$ is called the ‘string tension’, while $α = (4/3)α_s$ where
$α_s$ is the strong running coupling constant. Both these parameters are fixed at a constant value for each meson family.
The spin dependent interactions are modeled by a Breit perturbation term.
Table IV
In Table IV we report in MeV the masses of the heavy mesons formed by Bottom, Charm and Strange quarks, $bb$, $cc$ and
$s\bar{s}$. The first column contains term symbol and $I^G(JPC)$ numbers. In the last three columns we give the particle name,
are not available since the beginning. The masses of the component quarks, in MeV, are: \( m_b = 4725.5, m_c = 1394.5.5, m_s = 134.27 \). For \( b\bar{b}, c\bar{c} \) the string tension is \( \sigma = 1.111 \) GeV/fm; for \( s\bar{s} \), \( \sigma = 1.34 \) GeV/fm. Moreover \( \alpha_S = 0.3272, 0.435, 0.6075 \) respectively. [108].

### Table V

| State          | Exp | Num | State          | Exp | Num |
|----------------|-----|-----|----------------|-----|-----|
| \((1^1s_0)\) 0\((0^{--})\) \(\pi^0\) | 139.57±0.0035 | 616.45 | \((1^1s_0)\) 0\((0^+)\) \(B^0_r\) | 5366.77±0.24 | 5387.41 |
| \((1^3s_1)\) 1\((1--')\) \(\rho(770)\) | 775.49±0.39 | 826.14 | \((1^3s_1)\) 0\((1--)\) \(B^+_i\) | 5415.40±2.1 | 5434.34 |
| \((1^1p_0)\) 1\((0^{++})\) \(a_0(980)\) | 980±20 | 970.34 | \((1^3p_1)\) 0\((1^+)\) \(B_i(5830)^0\) | 5829.40±0.70 | 5817.80 |
| \((1^3p_1)\) 1\((1++)\) \(a_1(1260)\) | 1230±40 | 1204.66 | \((1^3p_2)\) 0\((2^+)\) \(B_{2s}(5840)^0\) | 5839.70±0.60 | 5829.33 |
| \((1^3p_1)\) 1\((1++)\) \(b_1(1235)\) | 1229.5±3.2 | 1274.76 | \((1^1s_0)\) 0\((0^-)\) \(D^+_s\) | 1968.49±3.2 | 1961.24 |
| \((1^3p_2)\) 1\((2++)\) \(a_2(1320)\) | 1318.3±0.60 | 1325.40 | \((1^3s_1)\) 0\((1^-)\) \(D_i^{(*)}\) | 2112.30±0.50 | 2101.78 |
| \((2^1s_0)\) 1\((0^{--})\) \(\pi(1300)\) | 1300±100 | 1337.36 | \((1^3p_0)\) 0\((0^+)\) \(D_{2s}(2317)^\pm\) | 2317.80±0.6 | 2339.94 |
| \((2^3s_1)\) 1\((1--)\) \(\rho(1450)\) | 1465±25 | 1497.63 | \((1^3p_1)\) 0\((1^+)\) \(D_{1i}(2460)^\pm\) | 2459.60±0.6 | 2466.15 |
| \((1^3d_1)\) 1\((1--)\) \(\rho(1570)\) | 1570\(^\pm\) | 1565.42 | \((1^3p_1)\) 0\((1^+)\) \(D_{1i}(2536)^\pm\) | 2535.12±0.13 | 2535.82 |
| \((3^2s_0)\) 1\((0^{++})\) \(\pi(1800)\) | 1812±12 | 1882.30 | \((1^1p_2)\) 0\((2^+)\) \(D_{2s}(2573)\) | 2571.90±0.8 | 2574.92 |

In Table V we give the levels of \( Bc, Bs, Ds \) and of the light \( u\bar{d} \) mesons in MeV. For the first three families we have \( \sigma = 1.111, 1.111, 1.227 \) GeV/fm and \( \alpha = 0.3591, 0.3975, 0.5344 \) respectively. For \( u\bar{d} \) family \( m_u = 2.94 \) and \( m_d = 6.1 \) MeV, \( \sigma = 1.34 \) GeV/fm and \( \alpha = 0.656 \). The \( \alpha_S \) curve has a steep increase for low masses [108]; fixing it at a constant value for the whole family induces a large error in the pion mass. The pion experimental mass is reproduced by taking \( \alpha = 0.99 \). At a smaller extent the same can be said for \( \rho(770) \). See [109,112] for results using different methods.

Although the similarities between the atom and the meson spectroscopy are evident, there are deep differences for e two cases. In atomic systems all the relevant physical parameters (masses, coupling constant, anomalous magnetic moments) are fixed at the measured values and the problem is completely determined up to the precision, actually very high, at which the radiative effects of the electromagnetic field, the finite dimensions of the particles, the electroweak unification and so on, give non negligible corrections. For Quarkonium models, on the contrary, the fundamental physical parameters are not available since the beginning. The masses of the component quarks, \( \sigma \) and \( \alpha_S \) must be determined by means of a self-consistent fit on the spectral data, calculated with the inclusion of the essential contribution of the Breit term. Limitations to potential models are due to the asymptotic freedom at short distances and to the creation of light quarks for increasing energy of the interaction: attempts have been made to describe these effects by softening the Coulomb-like potential at the origin and screening the scalar term at infinity. Each correction, however, introduces new parameters which are fitted so to enhance the agreement with the experimental data. We could argue that the effectiveness of a model can roughly be appraised by looking at the accuracy and the quantity of data it is able to reproduce with the least number of fitted parameters. In this sense our covariant wave equation is very effective. We have indeed used the least number of fitted parameters: the same constant values of \( \sigma \) and \( \alpha_S \) within each meson family and the quark masses. The ‘flavor independence’ due QCD, implies, moreover, that the string tension should be expected to be constant, at least for heavy quarks. The separate fits for the families \( b\bar{b}, c\bar{c} \) and \( b\bar{s} \), yield values of \( \sigma \) that turn out to be the same within the limits of the computational error. Finally we have been able to reproduce also the masses of the light \( u\bar{d} \) mesons, for which potential models generally fail: in this case, in addition to relativity, the contribution of the Breit term is determinant. Moreover the masses for the \( u \) and \( d \) quarks produced by the fit are small and very close to the current algebra masses, in contrast with the much higher values of the constituent masses used for potential models. For greater detail from a technical and phenomenological point of view we refer to [86,88].
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