NULL CONTROLLABILITY OF ONE DIMENSIONAL DEGENERATE PARABOLIC EQUATIONS WITH FIRST ORDER TERMS

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Abstract. In this paper we present a null controllability result for a degenerate semilinear parabolic equation with first order terms. The main result is obtained after the proof of a new Carleman inequality for a degenerate linear parabolic equation with first order terms.

1. Introduction and main result. In this paper we are interested in null controllability properties of a degenerate semilinear parabolic equation. We consider \( a \in C[0,1], \ a > 0 \) in \((0,1), \ a(0) = 0 \). Let us fix \( T > 0 \) and a non-empty open subset \((\alpha, \gamma) = \omega \subset (0,1), \) with \( \alpha > 0 \). The degenerate parabolic equation we want to analyze is:

\[
\begin{aligned}
\begin{cases}
  y_t - (a(x)y_x)_x + f(x,t,y,y_x) = h1_\omega \text{ in } Q = (0,1) \times (0,T), \\
  y(1,t) = 0 \quad \text{and} \quad \begin{cases}
  y(0,t) = 0 \quad \text{for (WDP)}, \\
  (ay_x)(0,t) = 0 \quad \text{for (SDP)}, \quad t \in (0,T), \\
  y(x,0) = y_0(x), \quad x \in (0,1).
\end{cases}
\end{cases}
\end{aligned}
\]

(1)

Here, \( h \in L^2(Q) \) is the control function to be determined, \( 1_\omega \) the characteristic function of the set \( \omega \), \( y_0 \in L^2(0,1) \) and \( f \) is a globally Lipschitz function. The boundary conditions, weak degenerate problem (WDP), or strong degenerate problem (SDP), depend on the behavior of \( a \) close to \( x = 0 \).

On one hand, we consider that the problem is weakly degenerate (WDP) if

\[
\begin{aligned}
\begin{cases}
  (i) \quad a \in C[0,1] \cap C^1(0,1), \ a > 0 \text{ in } (0,1), \ a(0) = 0, \\
  (ii) \quad \exists K \in [0,1) \text{ such that } xa'(x) \leq Ka(x) \quad \forall x \in [0,1].
\end{cases}
\end{aligned}
\]

(2)

Here we consider Dirichlet boundary conditions \( y(0) = 0 \). Notice that, under these assumptions, \( x^K/a(x) \) is not decreasing and then, since \( 0 \leq K \leq 1 \), \( \frac{1}{\sqrt{a}}, \frac{1}{a} \in L^1(0,1) \).

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On the other hand, when the problem is strongly degenerate (SDP), we assume

(i) \( a \in C^4((0,1], a > 0 \text{ in } (0,1], a(0) = 0, \)
(ii) \( \exists K \in [1,2) \text{ such that } xa'(x) \leq Ka(x) \ \forall x \in [0,1], \)
(iii) \( \left\{ \begin{array}{ll}
        \exists \sigma \in (1,K] x \to a(x) is not decreasing close to 0, & \text{if } K > 1, \\
        \exists \sigma \in (0,1] x \to a(x) is not decreasing close to 0, & \text{if } K = 1,
    \end{array} \right. \) (3)

the natural boundary condition at \( x = 0 \) will be of Neumann type:

\[
(au_x)(0,t) = 0, \quad t \in (0,T).
\]

We observe that we cannot deduce that \( \frac{1}{x} \in L^1(0,1), \) however \( \frac{1}{\sqrt{x}} \in L^1(0,1), \) as a consequence of (3)(ii). For details, see [1].

Our aim is to give conditions on \( f \) in such a way that system (1) is null controllable. That is, give \( H \) a Hilbert space such that for any \( y^0 \in H \) it exists \( h \in L^2(\omega \times (0,T)) \) such that the corresponding solution to (1) satisfies

\[
y(T) = 0.
\]

In order to present our main result we need to introduce the Hilbert spaces \( H^1_a(0,1) \) where problem is well posed:

**WDP CASE**

\[
H^1_a(0,1) := \{ u \in L^2(0,1) \mid u \text{ absolutely continuous in } [0,1], \ 
\sqrt{a}u_x \in L^2(0,1) \text{ and } u(0) = u(1) = 0 \},
\]

and

\[
H^2_a(0,1) := \{ u \in H^1_a(0,1) \mid au_x \in H^1(0,1) \}.
\]

**SDP CASE**

\[
H^1_a(0,1) := \{ u \in L^2(0,1) \mid u \text{ locally absolutely continuous in } (0,1), \ 
\sqrt{a}u_x \in L^2(0,1) \text{ and } u(1) = 0 \},
\]

and

\[
H^2_a(0,1) := \{ u \in L^2(0,1) \mid u \text{ locally absolutely continuous in } (0,1), \ 
au \in H^1(0,1), au_x \in H^1(0,1) \text{ and } (au_x)(0) = 0 \},
\]

with norms

\[
\| u \|^2_{H^2_a} := \| u \|^2_{L^2(0,1)} + \| \sqrt{a}u_x \|^2_{L^2(0,1)}, \quad \text{and} \quad \| u \|^2_{H^2_a} := \| u \|^2_{H^2_a} + \| (au_x)_x \|^2_{L^2(0,1)}.
\]

In the last fifteen years the study of null controllability properties of degenerate parabolic equations has been intense. See e.g. [1], [8], [10], [2] and [14]. In these papers the authors studied null controllability properties of linear degenerate one dimensional equations without first order terms. That is, \( f \) is of the form \( f(x,t,y,y_x) = b(x,t)y \). In [6], the authors studied the null controllability properties when \( f(x,t,y,y_x) = b(x,t)y + c(x)y_x \), so a first order term is included but the coefficient does not depend on \( t \). This condition prevents of obtaining a result when the nonlinearity \( f \) depends on the first order term.

The results we are presenting here are the generalization of the ones presented in [11] in which \( a(x) = x^\alpha \) and \( f \) is linear. Also, we improve the results in [5] where a “regional persistent null controllability” is proved for system (1). That is, under appropriate assumptions, it is proved that for \( T' > T > 0 \) and any \( y^0 \in L^2(0,1), \ 0 < \alpha < \gamma < 1, \delta \in (0,\gamma - \alpha) \) it exists a control \( h \in L^2(\omega \times (0,T')) \) such that \( y(x,t) = 0, \forall (x,t) \in (\alpha + \delta,1) \times (T,T') \).
In order to present our result we need to introduce some hypothesis on \( f \). To this end fix a function \( \beta(x) \) such that

\[
\frac{\beta(x)}{x} \in L^\infty(0,1). \tag{5}
\]

**Assumptions A** Let \( f : [0,1] \times [0,T] \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R} \) be such that

a) \( f(x,t;0,0) = 0 \ \forall \ (x,t) \in Q. \)

b) \( f(\cdot; s,p) \in L^\infty(Q) \ \forall \ (s,p) \in \mathbb{R}^2. \)

c) \( f(\cdot; \cdot) \) is globally Lipschitz for every \((x,t)\in Q\) with Lipschitz constant independent of \((x,t)\). 

d) \( f(\cdot; s,p) = g(\cdot; s,p) + G(\cdot; s,p)p \) for every \((s,p)\in \mathbb{R}^2\), where \( g(\cdot; s,p) \in L^\infty(Q), \forall (s,p) \in \mathbb{R}^2 \) and

\[
\left| \frac{G(x,t; s,p)}{\beta(x)} \right| \leq C \ \text{almost everywhere} \ (x,t) \in Q, \forall (s,p) \in \mathbb{R}^2. 
\]

Our main result is:

**Theorem 1.1.** Fix \( T > 0 \) and \( y_0 \in L^2(0,1) \). Let us assume (2) (in the WDP) or (3) (in the SDP), (5), and assumptions A. Then, system (1) is null controllable, that is, it exists \( h \in L^2(\omega \times (0,T)) \) such that the solution \( y \) to (1) satisfies

\[
y(x,T) = 0 \ \text{for every} \ x \in [0,1]. \tag{6}
\]

Moreover, it exists a constant \( C > 0 \) only depending on \( T \), such that

\[
\int_0^T \int_\omega |h|^2 dx dt \leq C \int_0^1 y_0^2(x) dx.
\]

**Remark 1.** Observe that the assumptions on \( \beta(x) \), guarantee that

\[
\frac{\beta^2(x)}{a(x)} \leq C \frac{x^2}{a(x)} \leq \frac{C}{a(1)} \ \text{a.e.} \ x \in (0,1). \tag{7}
\]

In particular, we get \( |\beta(x)| \leq L \sqrt{a(x)} \), which is required to get an existence result for the linearized problem, see Theorem 2.1 below.

The aim of this paper is to prove Theorem 1.1. To this end we study the linearized degenerate parabolic equation and with a fixed point argument we obtain our main result. The rest of the paper is organized as follows. In the next chapter we present the linear problem. We prove a new Carleman inequality for the adjoint system associated with the linear one. In section 3, we prove our main result.

### 2. Linear problem

In this section we study the null controllability of the degenerate linear equation

\[
\begin{cases}
  y_t - (a(x)y_x)_x + b(x,t)y + \beta(x)c(x,t)y_x = h1_\omega & \text{in} \ Q, \\
  y(1,t) = 0 & \text{for} \ (WDP),
  y(0,t) = 0 & \text{for} \ (SDP), \\
  (ag_x)(0,t) = 0 & \text{for} \ (SDP), \\
  y(x,0) = y_0(x), & \text{in} \ (0,1).
\end{cases} \tag{8}
\]

The following existence and uniqueness results of solutions to (8) is well known, see e.g. [5].
Theorem 2.1. Suppose that \( b, c \in L^\infty(Q) \), \( a \) satisfies (2) (in the (SDP)) or (3) (in the (SDP)) and \( \beta \) satisfies (5), \( h \in L^2(\omega \times (0, T)) \), then for every \( y_0 \in L^2(0, 1) \), (8) has a unique solution
\[
y \in \mathcal{U} := C([0, T]; L^2(0, 1)) \cap L^2(0, T; H^1_a(0, 1)).
\]
Moreover, if \( y_0 \in H^1_a(0, 1) \), then
\[
y \in C([0, T]; H^1_a(0, 1)) \cap L^2(0, T; H^2_a(0, 1)) \cap H^1(0, T; L^2(0, 1)),
\]
and it exists a positive \( C_T \) such that
\[
\|v\|_{C([0, T]; H^1_a(0, 1))} + \|v\|_{L^2(0, T; H^2_a(0, 1))} + \|v\|_{H^1(0, T; L^2(0, 1))}^2
\leq C_T(\|y_0\|_{H^1_a(0, 1)}^2 + \|h\|_{L^2(\omega \times (0, T))}^2).
\]

It is by now well understood that the null controllability properties of system (8) can be characterized in terms of the adjoint system for every solution to (10). This is done in the next subsection.

\[
\begin{cases}
v_t + (a(x)v_x)_x - b(x, t)v + (\beta(x)c(x, t)v)_x = 0 & \text{in } Q, \\
v(1, t) = 0 & \text{for } (\text{WDP}), \\
v(0, t) = 0 & \text{for } (\text{SDP}), \\
v(x, T) = v_T(x), & \text{in } (0, 1).
\end{cases}
\]

So the aim of this section is to prove the following observability inequality:

Theorem 2.2. It exists a constant \( C > 0 \) only depending on \( T > 0 \) such that for every solution to (10) the following holds:
\[
\|v(0)\|_{L^2(0, 1)} \leq C \int_0^T \int_\omega |v|^2dxdt.
\]

In order to prove inequality (11) we will need to prove a new Carleman inequality for system (10). This is done in the next subsection.

2.1. Carleman estimates for degenerate parabolic equations. Our main result is a consequence of a Carleman inequality. We introduce some functions.

We define \( \phi(x) = c_1 \left( c_2 - \int_0^x \frac{y}{a(y)} dy \right), \forall x \in [0, 1] \).

If \( c_1 > 0 \) and \( c_2 > \frac{1}{a(1)(2-K)} \) (where \( K \) is the constant given in (2) and (3) (ii) according to the WDP or the SDP), then,
\[
\psi(x) > 0,
\]
for every \( x \in [0, 1] \). In fact, assumptions (2) and (3) (ii) implies that it exists \( K \in [0, 2) \) such that
\[
\frac{1}{a(\tau)} \leq \frac{1}{a(1)\tau^K},
\]
for every \( \tau \in (0, 1) \). Then,
\[
\int_0^x \frac{\tau}{a(\tau)} d\tau \leq \int_0^x \frac{x^{1-K}}{a(1)} d\tau = \frac{1}{a(1)(2-K)} < c_2,
\]
for every \( x \in [0, 1] \).

For \( \omega = (\alpha, \gamma) \) let us call \( \kappa^- = \frac{2\alpha + 3}{3}, \kappa^+ = \frac{\alpha + 2\gamma}{3} \), and let \( \xi \in C^2(\mathbb{R}) \) be such that \( 0 \leq \xi \leq 1 \) and
\[
\xi(x) = \begin{cases} 1 & \text{if } x \in (0, \kappa^-), \\
0 & \text{if } x \in (\kappa^+, 1). \end{cases}
\]
Now, for \( \omega' = (\alpha', \gamma') \subset (\kappa^-, \kappa^+) \subset \omega \) we will consider the function given in [12] for proving Carleman inequalities for a non degenerate parabolic equation. That is, we take \( \rho(x) \in C^2[0, 1] \) such that
\[
\rho(x) > 0, \quad x \in (0, 1); \quad \rho(0) = \rho(1) = 0 \text{ and } \rho_x \neq 0 \text{ in } (0, \alpha') \cup (\gamma', 1).
\]
We define now \( \psi(x) = e^{2\lambda \| \rho \|_\infty} - e^{\lambda \rho(x)} \) and
\[
\eta(x) = \phi(x) \xi(x) + (1 - \xi(x)) \psi(x).
\]
Observe that
\[
\eta'(x) = \phi'(x) \xi(x) + \phi(x) \xi'(x) - \xi'(x) \psi(x) + (1 - \xi(x)) \psi'(x).
\]
So for \( x \in (\kappa^+, \gamma) \), \( \eta'(x) \neq 0 \) We define
\[
\varphi(x, t) = \eta(x) \theta(t),
\]
with
\[
\theta(t) = \frac{1}{(t(T - t))^4} \forall t \in (0, T).
\]
Let us consider
\[
\begin{align*}
& v_t + (a(x)v_x) = F_0 + (\beta(x)F_1)_x \\
& v(1, t) = 0 \quad \text{and} \quad v(0, t) = 0 \quad \text{for the (WDP)} \\
& v(x, T) = v_T(x),
\end{align*}
\]
with \( F_0, F_1 \in L^2(Q) \) and \( v_T \in L^2(0, 1) \).

Our main result in this subsection is the following:

**Theorem 2.3.** Suppose that \( \omega = (\alpha, \gamma) \) with \( \alpha > 0 \). Assume that (2) or (3) and (5) are verified and let \( T > 0 \) be given. Then, there exists two positive constants \( C \) and \( s_0 \), such that every solution \( v \) to (12) satisfies
\[
\begin{align*}
& \int_0^T \int_0^1 \left( s\theta_0(x)v_x^2 + s^3\theta_0^3 \frac{x^2}{a(x)} v^2 \right) e^{-2s\varphi(x, t)} \, dx \, dt \\
& \leq C \left( \int_0^T \int_\omega e^{-2s\varphi(x, t)} v^2 \, dx \, dt + \int_0^T \int_0^1 \left( F_0^2 + s^2\theta_0^3 \frac{\beta(x)^2}{a(x)} F_1^2 \right) e^{-2s\varphi(x, t)} \, dx \, dt \right),
\end{align*}
\]
for every \( s \geq s_0 \).

The proof of Theorem 2.3 is given at the end of this section as a consequence of the following result for the degenerate parabolic system:
\[
\begin{align*}
& v_t + (a(x)v_x) = F \\
& v(1, t) = 0 \quad \text{and} \quad v(0, t) = 0 \quad \text{for (WDP)} \\
& v(x, T) = v_T(x),
\end{align*}
\]
with \( a \) satisfies (2) for the (WDP) and (3) for the (SDP), and \( F \in L^2(Q) \).

**Lemma 2.4.** Let us assume that (2) (WDP) or (3) (SDP) hold true and let \( T > 0 \) be given. Then, there exists two positive constants \( C \) and \( s_0 \), such that every solution \( v \) to (14), satisfies
\[
\begin{align*}
& \int_0^T \int_0^1 \left( s\theta_0(x)v_x^2 + s^3\theta_0^3 \frac{x^2}{a(x)} v^2 \right) e^{-2s\varphi(x, t)} \, dx \, dt \\
& \leq C \left( \int_0^T \int_\omega e^{-2s\varphi(x, t)} F^2 \, dx \, dt + \int_0^T \int_0^1 e^{-2s\varphi(x, t)} v^2 \, dx \, dt \right)
\end{align*}
\]
for every $s \geq s_0$.

Proof. We define $w(t,x) = e^{-s\varphi(t,x)}v(t,x)$. Then, $w$ solves

$$(e^{s\varphi}w)_t + (a(x)(e^{s\varphi}w)_x)_x = F.$$ 

Clearly we get

$$s\varphi_t w + w_t + s(a(x)\varphi_x)_x w + s^2 a(x)\varphi_x^2 w + 2sa(x)\varphi_x w_x + (a(x)w_x)_x = e^{-2s\varphi}F.$$ 

Following the ideas in [8] we define

$$P^+_s(w) = s\varphi_t w + s^2 a(x)\varphi_x^2 w + (a(x)w_x)_x$$

$$P^-_s(w) = w_t + s(a(x)\varphi_x)_x w + 2sa(x)\varphi_x w_x.$$ 

We want to estimate the $L^2$-scalar product $(P^+_s, P^-_s)$. We define

$$Q_1 = \int_0^T \int_0^1 P^+_s(w)w_t,$$

$$Q_2 = 2\int_0^T \int_0^1 P^+_s(w)s(a(x)\varphi_x)_x,$$

and

$$Q_3 = \int_0^T \int_0^1 P^+_s(w)s(a(x)\varphi_x)_x.$$ 

Integrating by parts, we get

$$Q_1 = -\frac{1}{2} \int_0^T \int_0^1 w^2(s\varphi_t\eta + 2s^2 a(x)\theta_t\eta_x^2) + \int_0^T aw_x w_t \bigg|_0^1.$$ 

Observe that $w_t(1) = 0$. To compute the boundary condition at $x = 0$ we see that in the WDP case $w_t(0) = 0$ and in the SDP situation we have $aw_x(0) = -s\theta a\varphi_x w(0)$. By definition of $\varphi$, $\varphi(0) = 0$. We proceed as in [1] p. 184. Observe that $a\varphi_x = -c_1 x$, $x \in (0, \tau^-)$. When $x \to 0$, $a\varphi_x \to 0$. So we don’t have a contribution of the boundary term.

After integration by parts, we get,

$$Q_2 = \int_0^T s^2 a\varphi_x \varphi_t w^2 + s^2 a\varphi_x^2 w^2 + s\varphi_x (aw_x)^2 \bigg|_0^1$$

$$- s^2 \int_0^T \int_0^1 (a\varphi_x \varphi_t + \varphi_x \varphi_{xt}) w^2 - s^3 \int_0^T \int_0^1 [(a\varphi_x)_x a\varphi_x + (a\varphi_x^2)(a\varphi_x)_x] w^2$$

$$- s \int_0^T \int_0^1 \varphi_{xx} (aw_x)^2.$$ 

(16)

We proceed again as in [1] to estimate the boundary terms at $x = 0$, taking in mind the definition of $\varphi$ close to $x = 1$. It is clear that at $x = 1$, $w(1) = 0$. In the (WDP) we get

$$b.t. = \int_0^T s\varphi_x(1)(a(1)w_x(1))^2 - \int_0^T s\varphi_x(0)(a(0)w_x(0))^2$$

$$= -s \int_0^T \theta \psi_x(1)(a(1)w_x(1))^2 - s \int_0^T \theta \psi_x(0)(a(0)w_x(0))^2$$

$$= B_1 + B_2.$$ 

By construction, $B_1 \geq 0$ and since close to $x = 0$, $a(x)\varphi_x(x) = -c_1 x$, $B_2 = 0$.
In the case (SDP) we get \( aw_x(0) = -sa\phi_x\theta w(0) \) so
\[
\int_0^T \theta \phi_x(0)(a(0)w_x(0))^2 = 0
\]
and the term \( B_1 \geq 0. \)

Similarly, we get
\[
Q_3 = -\int_0^T \int_0^1 (\varphi_x(a\varphi_x)_x + s^3 a(x)\varphi_x^2(a\varphi_x)_x)w^2 - s\int_0^T \int_0^1 [(a\varphi_x)_x a\varphi_x + (a\varphi_x^2)(a\varphi_x)_x]w^2 + \int_0^T (a\varphi_x)_x aw_x \bigg|_0^1.
\]

Proceeding as before it is easy to see that the boundary terms are zero. (See also p.184 [1])

Altogether we get that
\[
(P_{s+}^+, P_{s-}^-) \geq -\frac{s}{2} \int_0^T \int_0^1 \theta_{tt} \eta w^2 + 2s^2 \int_0^T \int_0^1 a \theta_t \eta \eta_x w^2 - s^2 \int_0^T \int_0^1 a \eta_{xx} \eta \theta_t w^2 - 2s \int_0^T \int_0^1 \theta_{xx} (aw_x)^2 - s^3 \int_0^T \int_0^1 [(a\varphi_x)_x a\varphi_x + (a\varphi_x^2)(a\varphi_x)_x]w^2 - s \int_0^T \int_0^1 \theta_t (a\eta_x)_{xx} w - a_x \eta_x w_x aw_x - s^3 \int_0^T \int_0^1 \theta^3 a^2 \eta_x^2 \eta_{xx} w^2 + \int_0^T \int_0^1 \theta_t \theta \eta (a\eta_x)_{xx}w^2.
\]

We can decompose the right hand side of this inequality as the sum of integrals over \([0, \kappa^-], [\kappa^-, \kappa^+] \) and \([\kappa^+, 1] \). Observe that our choice of weight functions allows us to reason on \([0, \kappa^-] \) similarly to the derivation of (3.10) in [1]. On the other hand, on \((\kappa^+, 1) \) we obtain classical Carleman estimates with terms on the gradient and the function over the set \( \bar{\omega} \). Proceeding as in [1] to bound the terms over \((0, \kappa^-) \) and as traditional Carleman estimates (see e.g. [12]) for the terms over \((\kappa^+, 1) \) and having in mind that \( a(x) \) is bounded below by a positive constant in \((\kappa^+, 1) \) we obtain, with appropriate constants:
\[
\int_Q \left( s \theta a(x)v_x^2 + s^3 \theta^3 \frac{x^2}{a(x)} v^2 \right) e^{-2s\varphi} \, dx \, dt \leq C \left( \int_Q e^{-2s\varphi}F^2 \, dx \, dt + \int_0^T \int_\omega e^{-2s\varphi}v_x^2 \, dx \, dt + \int_0^T \int_\omega e^{-2s\varphi}v^2 \, dx \, dt \right).
\]

To get (15) we eliminate the term \( \int_0^T \int_\omega e^{-2s\varphi}v_x^2 \, dx \, dt \) performing local energy estimates and “growing” \( \bar{\omega} \) to \( \omega \). That is, we use Cacciopoli’s inequality, which is:
\[
\int_0^T \int_\omega e^{-2s\varphi} v_x^2 \, dx \, dt \leq C \left( \int_0^T \int_\omega e^{-2s\varphi} v_x^2 \, dx \, dt + \int_0^T \int_\Omega e^{-2s\varphi} F^2 \, dx \, dt \right).
\]

This completes the proof.

2.2. **Proof of Theorem 2.3.** Here we use the technique introduced in [13]; that is, we use Carleman inequality (15) to obtain two auxiliary results. That is, we prove the null controllability of degenerate parabolic equations with particular source terms and use these results in the proof of the new Carleman inequality (13). In
fact, we will use source terms with coefficients that appear in the left hand side of
the Carleman inequality (2.4), that is, we consider the following systems:
\[
\begin{cases}
z_t - (a(x)z_x)_x = s^3\theta^3 \frac{x^2}{a(x)} e^{-2s\varphi(x,t)} f + u_1, \\
z(0, t) = 0 & \text{for the (WDP),} \\
z(1, t) = 0 & \text{for the (SDP),} \\
z(x, 0) = 0,
\end{cases}
\quad \text{in } Q,
\quad \text{in } (0, T),
\quad \text{in } (0, 1),
\]
and
\[
\begin{cases}
z_t - (a(x)z_x)_x = s\theta e^{-2s\varphi(x,t)} a\varphi_x + u_1, \\
z(0, t) = 0 & \text{for the (WDP),} \\
z(1, t) = 0 & \text{for the (SDP),} \\
z(x, 0) = 0,
\end{cases}
\quad \text{in } Q,
\quad \text{in } (0, T),
\quad \text{in } (0, 1),
\]
with \( f \in L^2(Q) \).

We define \( P_W = \{ p \in C^2(Q) \mid p(0) = 0, p(1) = 0 \} \) for the (WDP) case and
\( P_S = \{ p \in C^2(Q) \mid (ap_x)(0) = 0, p(1) = 0 \} \) for the (SDP) case. Let \( \mathcal{L} \) and \( \mathcal{L}^* \) be two linear operators defined as:
\[
\mathcal{L} p = p_t - (a(x)p_x)_x
\]
and
\[
\mathcal{L}^* p = p_t + (a(x)p_x)_x,
\]
for every \( p \in P_W \) for the (WDP) case and \( p \in P_S \) for the (SDP) case.

We define
\[
\lambda(p, p') = \int_0^T \int_0^1 e^{-2s\varphi(x,t)} \mathcal{L} p \mathcal{L}^* p' \, dxdt + \int_0^T \int_0^1 e^{-2s\varphi(x,t)} pp' \, dxdt,
\]
for every \( p, p' \) in \( P_W \) for the (SDP) case and for every \( p, p' \) in \( P_S \) for the (SDP) case. It is not difficult to check that \( \lambda(\cdot, \cdot) \) is a bilinear, positive and symmetric form. Then, \( \lambda(\cdot, \cdot) \) defines an internal product in \( P_W \) and in \( P_S \). We define \( P_W \) as the closure of \( P_W \) with the norm \( \| p \|_{P_W} = (\lambda(p, p))^{1/2} \) in the case (WDP) and \( P_S \) as the closure of \( P_S \) with the norm \( \| p \|_{P_S} = (\lambda(p, p))^{1/2} \) in the case (SDP).

As a consequence of Carleman inequality (15), we get the following null controllability result.

**Theorem 2.5.** Let \( T > 0 \) and \( f \in L^2(Q) \) be given. Then, we have that:

1. For system (19), it exists a control \( u \) and a state \( z \), such that \( z(x, T) = 0 \) in \( (0, 1) \) and
\[
\int_0^T \int_0^1 e^{2s\varphi(x,t)} z^2 \, dxdt + \int_0^T \int_\omega e^{2s\varphi(x,t)} u^2 \, dxdt 
\leq C \int_0^T \int_0^1 s^3\theta^3 \frac{x^2}{a(x)} e^{-2s\varphi(x,t)} f^2 \, dxdt \tag{21}
\]
is satisfied.

2. For system (20), it exists a control \( u \) and a state \( z \), such that \( z(x, T) = 0 \) in \( (0, 1) \) and the following is satisfied
\[
\int_0^T \int_0^1 e^{2s\varphi(x,t)} z^2 \, dxdt + \int_0^T \int_\omega e^{2s\varphi(x,t)} u^2 \, dxdt
\leq C \int_0^T \int_0^1 s\theta e^{-2s\varphi(x,t)} f^2 \, dxdt. \tag{22}
\]
Proof. Given $f \in L^2(Q)$ and $T > 0$ we consider the following problem:

\[
\begin{aligned}
\ell&_e(e^{-2s\varphi(x,t)}L^*p) - s^3\varphi^3 \frac{x^2}{a(x)} e^{-2s\varphi(x,t)} f = e^{-2s\varphi(x,t)}p\chi_\omega \quad \text{in } Q \\
e^{-2s\varphi(x,t)}L^*p(0) = 0 & \quad \text{for the WDP,} \\
e^{-2s\varphi(x,t)}L^*a p_x(0) = 0 & \quad \text{for the SDP,} \\
e^{-2s\varphi(x,t)}L^*p(x,0) = e^{-2s\varphi(x,t)}L^*p(x,T) = 0 & \quad \text{in } (0,1).
\end{aligned}
\]

We will show that (23) has a solution in the WDP case (the argument in the SDP case is analogous).

Carleman inequality (15) implies

\[
\int_0^T \int_0^1 \left( s\theta(x)p_x^2 + s^3\theta^3 \frac{x^2}{a(x)} p^2 \right) e^{-2s\varphi(x,t)} dx dt \\
\leq C \left( \int_0^T \int_0^1 e^{-2s\varphi(x,t)} |L^*p|^2 dx dt + \int_0^T \int_0^1 e^{-2s\varphi(x,t)} p^2 dx dt \right),
\]

for every $p \in \mathcal{P}_W$. In consequence, for every $p \in \mathcal{P}_W$

\[
\int_0^T \int_0^1 \left( s\theta(x)p_x^2 + s^3\theta^3 \frac{x^2}{a(x)} p^2 \right) e^{-2s\varphi(x,t)} dx dt < \infty.
\]

On the other hand,

\[
\ell(p) = -\int_0^T \int_0^1 s^3\theta^3 \frac{x^2}{a(x)} e^{-2s\varphi(x,t)} f p dx dt \\
\leq C \left( \int_0^T \int_0^1 s^3\theta^3 \frac{x^2}{a(x)} e^{-2s\varphi(x,t)} f^2 \right)^{1/2} ||p||_{\mathcal{P}_W}
\]

and then $\ell$ is linear and continuous on $\mathcal{P}_W$. From Lax-Milgram Theorem it exists a unique $\bar{p} \in \mathcal{P}$ solution to the problem

\[
\lambda(\bar{p},p') = \ell(p') \quad \forall p' \in \ell(p').
\]

In consequence, $\bar{p}$ is a weak solution of (23). It can be shown that if $p$ is a classical solution, then it satisfies $e^{-2s\varphi(x,t)}L^*p(0) = 0$, $e^{-2s\varphi(x,t)}L^*p(1) = 0$. Observe that $e^{-2s\varphi(x,t)}L^*p(x,0) = 0$, $e^{-2s\varphi(x,t)}L^*p(x,T) = 0$. In fact, multiplying (23) by $p' \in \mathcal{P}_W$ and integrating by parts we get $\forall p' \in \mathcal{P}_W$

\[
\int_0^1 e^{-2s\varphi(x,t)}L^*pp' \bigg|_0^T dx + \int_0^T a(x)e^{-2s\varphi(x,t)}L^*p(x,t)p'_x \bigg|_0^1 dt + \ell(p') = \lambda(p,p').
\]

Taking an appropriate $p' \in \mathcal{P}_W$ we conclude that $e^{-2s\varphi(x,t)}L^*p(x,t) = 0$ in $\partial Q$.

We define now

\[
\bar{z} = e^{-2s\varphi(x,t)}L^*\bar{p}; \quad \bar{u} = e^{-2s\varphi(x,t)}\bar{p}\chi_\omega.
\]

Then, $\bar{z}$ solves

\[
\begin{aligned}
\bar{z}_t - (a(x)\bar{z}_x) &= s^3\theta^3 \frac{x^2}{a(x)} e^{-2s\varphi(x,t)} f + \bar{u}\chi_\omega \quad \text{in } Q, \\
\bar{z}(1) &= 0 \quad \text{and } \bar{z}(0) = 0 \quad \text{WDP,} \\
\bar{z}(x,0) &= \bar{z}(x,T) = 0.
\end{aligned}
\]

From (24),

\[
\bar{p}_t + (a(x)\bar{p}_x)_x = e^{2s\varphi(x,t)}\bar{z}.
\]

(26)
In order to estimate the norms of $\bar{z}$ and of the control $\bar{u}$ we multiply (26) by $\bar{z}$. Then,
\[
\int_0^T \int_0^1 \left( \bar{p}_t + (a(x)\bar{p}_x)_x \right) \bar{z} \, dx \, dt = \int_0^T \int_0^1 e^{2s\varphi(x,t)} \bar{z}^2 \, dx \, dt.
\]
Integrating by parts in space and time and using (25) we see that
\[
- \int_0^T \int_0^1 s^3 \theta^3 \frac{x^2}{a(x)} e^{-2s\varphi(x,t)} f \bar{p} \, dx \, dt - \int_0^T \int \omega \bar{u} \bar{p} \, dx \, dt = \int_0^T \int_0^1 e^{2s\varphi(x,t)} \bar{z}^2 \, dx \, dt,
\]
\[
I_1 + I_2 = \int_0^T \int_0^1 e^{2s\varphi(x,t)} \bar{z}^2 \, dx \, dt.
\]
We know that $\bar{u} = e^{-2s\varphi(x,t)} \bar{p}_\omega$, then
\[
I_2 = - \int_0^T \int \omega e^{-2s\varphi(x,t)} \bar{p}^2 \, dx \, dt.
\]
On the other hand,
\[
I_1 \leq \frac{1}{2} \delta \int_0^T \int_0^1 s^3 \theta^3 \frac{x^2}{a(x)} e^{-2s\varphi(x,t)} f^2 \, dx \, dt + \frac{\delta}{2} \int_0^T \int_0^1 s^3 \theta^3 \frac{x^2}{a(x)} e^{-2s\varphi(x,t)} \bar{p}^2 \, dx \, dt.
\]
Since $\bar{p}$ is solution to (26) we can apply Carleman inequality (15) with right hand side $e^{2s\varphi(x,t)} \bar{z}$. We have that
\[
\frac{\delta}{2} \int_0^T \int_0^1 s^3 \theta^3 \frac{x^2}{a(x)} e^{-2s\varphi(x,t)} \bar{p}^2 \, dx \, dt \leq \frac{C\delta}{2} \left( \int_0^T \int_0^1 e^{-2s\varphi(x,t)} e^{4s\varphi(x,t)} \bar{z}^2 \, dx \, dt + \int_0^T \int_\omega e^{-2s\varphi(x,t)} \bar{p}^2 \, dx \, dt \right).
\]
From (24) we get
\[
\int_0^T \int_\omega e^{-2s\varphi(x,t)} \bar{p}^2 \, dx \, dt = \int_0^T \int_\omega e^{2s\varphi(x,t)} \bar{u}^2 \, dx \, dt.
\]
In conclusion, we got
\[
\int_0^T \int_\omega e^{2s\varphi(x,t)} \bar{u}^2 \, dx \, dt + \int_0^T \int_0^1 e^{2s\varphi(x,t)} \bar{z}^2 \, dx \, dt \leq C \int_0^T \int_0^1 s^3 \theta^3 \frac{x^2}{a(x)} e^{-2s\varphi(x,t)} f^2 \, dx \, dt.
\]
With this we conclude the proof of (21). The proof of (22) is similar, we only need to consider the functional
\[
\ell(p) = \int_0^T \int_0^1 s\theta e^{-2s\varphi(x,t)} a(x) f p \, dx \, dt.
\]
\[\square\]

**Proof of Theorem 2.3.** The proof is given in two steps:

**Step 1. Two auxiliary null controllability problems**

We apply the previous result to $v \in L^2(Q)$ solution to (10). We deduce the existence of a control $\hat{v}$ and a state $\hat{z}$ such that
\[
\begin{cases}
\hat{z}_t - (a(x)\hat{z}_x)_x = s^3 \theta^3 \frac{x^2}{a(x)} e^{-2s\varphi(x,t)} v + \hat{v}1_\omega & \text{in } Q, \\
\hat{z}(1, t) = 0 & \text{WDP, } t \in (0, T), \\
\hat{z}(0, t) = 0 & \text{SDP, } \hat{z}(0, t) = 0 & \text{in } (0, 1),
\end{cases}
\]

(27)

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and
\[ \int_0^T \int_0^1 e^{2s\varphi(x,t)} \dot{x}^2 dx dt + \int_0^T \int_0^1 e^{2s\varphi(x,t)} \dot{z}^2 dx dt \leq C \int_0^T \int_0^1 s^3 \theta^3 \frac{x^2}{a(x)} e^{-2s\varphi(x,t)} v^2 dx dt. \] (28)

If we multiply by \( s^{-2}\theta^{-3} e^{2s\varphi(x,t)} \dot{z} \) equation (27) and we integrate by parts, we conclude that
\[ \int_0^T \int_0^1 s^{-2}\theta^{-3} a(x) e^{2s\varphi(x,t)} \dot{x}^2 dx dt = - \int_0^T \int_0^1 s^{-2}\theta^{-3} e^{2s\varphi(x,t)} \dot{z} \dot{z}_t dx dt \]
\[ - \int_0^T \int_0^1 s^{-2}\theta^{-3} a(x) e^{2s\varphi(x,t)} \dot{z}_x \dot{z} dx dt + \int_0^T \int_0^1 s^{-2} \frac{x^2}{a(x)} v \dot{z}_x dx dt \]
\[ + \int_0^T \int_0^1 s^{-2}\theta^{-3} e^{2s\varphi(x,t)} \dot{v} \dot{z} dx dt = H_1 + H_2 + H_3 + H_4. \]

We observe that for every \( (x, t) \in Q_1 \),
\[ |(\theta^{-3} e^{2s\varphi(x,t)})_t| \leq C s e^{2s\varphi(x,t)} \]
and
\[ |(e^{2s\varphi(x,t)})_x| \leq C \theta e^{2s\varphi(x,t)} \]
and for every \( x \in (0, 1) \), \( x^2/a(x) \leq C \). Then, applying (28), we get
\[ H_1 = - \frac{1}{2} \int_0^T \int_0^1 s^{-2}\theta^{-3} a(x) e^{2s\varphi(x,t)} \frac{d}{dt} [\dot{z}^2] dx dt \leq \frac{1}{2} \int_0^T \int_0^1 s^{-2} |\theta^{-3} e^{2s\varphi(x,t)}|_t \dot{z}^2 dx dt \]
\[ \leq C \int_0^T \int_0^1 s^{-1} e^{2s\varphi(x,t)} \dot{z}^2 dx dt \leq C \int_0^T \int_0^1 s^3 \theta^3 \frac{x^2}{a(x)} e^{-2s\varphi(x,t)} v^2 dx dt \] (29)

and
\[ H_2 \leq C \int_0^T \int_0^1 s^{-1} \theta^{-2} a(x) e^{2s\varphi(x,t)} |\dot{z}_x| |\dot{z}| dx dt \]
\[ = C \int_0^T \int_0^1 s^{-1} \theta^{-3/2} a(x) e^{2s\varphi(x,t)} \theta^{-1/2} |\dot{z}_x| |\dot{z}| dx dt \] (30)
\[ \leq \frac{1}{2} \int_0^T \int_0^1 s^{-2}\theta^{-3} a(x) e^{2s\varphi(x,t)} \dot{z}_x^2 dx dt + C \int_0^T \int_0^1 e^{2s\varphi(x,t)} \dot{z}^2 dx dt \]
\[ \leq \frac{1}{2} \int_0^T \int_0^1 s^{-2}\theta^{-3} a(x) e^{2s\varphi(x,t)} \dot{z}_x^2 dx dt + C \int_0^T \int_0^1 s^3 \theta^3 \frac{x^2}{a(x)} e^{-2s\varphi(x,t)} v^2 dx dt. \]

Assuming \( s_0 \geq 1 \), we have that
\[ H_3 \leq \int_0^T \int_0^1 s^{3/2} \frac{x^2}{a(x)} v \dot{z} dx dt \]
\[ \leq C \left( \int_0^T \int_0^1 s^3 \theta^3 \frac{x^2}{a(x)} e^{-2s \varphi(x,t)} v^2 dx dt \right)^{1/2} \left( \int_0^T \int_0^1 e^{2s \varphi(x,t)} \dot{z}^2 dx dt \right)^{1/2} \]
\[ \leq C \int_0^T \int_0^1 s^3 \theta^3 \frac{x^2}{a(x)} e^{-2s \varphi(x,t)} v^2 dx dt \] (31)
and

$$H_4 \leq \int_0^T \int_0^\omega e^{2s\varphi(x,t)}|\hat{\omega}|dxdt$$

\[
\leq C \left( \int_0^T \int_0^1 e^{2s\varphi(x,t)}\hat{\omega}^2 dt \right)^{1/2} \left( \int_0^T \int_0^1 e^{2s\varphi(x,t)}\hat{\omega}_t^2 dt \right)^{1/2}
\]

\[
\leq C \int_0^T \int_0^1 s^3 \theta a(x) e^{-2s\varphi(x,t)} v^2 dxdt.
\]

Applying (29), (30), (31) and (32), we conclude

$$\int_0^T \int_0^1 s^{-2} \theta^{-3} a(x) e^{2s\varphi(x,t)} \hat{\omega}_x^2 dxdt \leq C \int_0^T \int_0^1 s^3 \theta^3 \frac{x^2}{a(x)} e^{-2s\varphi(x,t)} v^2 dxdt,$$

which combined with (28) gives

\[
\int_0^T \int_0^1 e^{2s\varphi(x,t)} \hat{\omega}_x^2 dxdt + \int_0^T \int_\omega e^{2s\varphi(x,t)} \hat{\hat{\omega}}^2 dxdt
\]

\[
+ \int_0^T \int_0^1 s^{-2} \theta^{-3} a(x) e^{2s\varphi(x,t)} \hat{\omega}_x^2 dxdt \leq C \int_0^T \int_0^1 s^3 \theta^3 \frac{x^2}{a(x)} e^{-2s\varphi(x,t)} v^2 dxdt
\]

for every \( s \geq s_0 \).

On the other hand, applying part 2 of Theorem 2.5 for \( f = v_x \in L^2(Q) \), where \( v \) is the solution to (10), we can deduce the existence of a control \( \hat{v} \) and a state \( \hat{z} \) such that

\[
\begin{cases}
\hat{z}_t - (a(x) \hat{z}_x)_x = s\theta(e^{-2s\varphi(x,t)} a(x) v_x)_x + \hat{v}1_\omega & \text{in } Q, \\
\hat{z}(1, t) = 0 & \text{WDP}, \\
\hat{z}(0, t) = 0 & \text{SDP}, \\
\hat{z}(x, 0) = \hat{z}(x, T) = 0, & \text{in } (0, 1),
\end{cases}
\]

and

\[
\int_0^T \int_0^1 e^{2s\varphi(x,t)} \hat{\omega}_x^2 dxdt + \int_\omega e^{2s\varphi(x,t)} \hat{\hat{\omega}}^2 dxdt
\]

\[
\leq C \int_0^T \int_0^1 s\theta a(x) e^{-2s\varphi(x,t)} v_x^2 dxdt
\]

for every \( s \geq s_0 \).

It is not difficult to see that

\[
\int_0^T \int_0^1 s^{-2} \theta^{-2} a(x) e^{2s\varphi(x,t)} \hat{\omega}_x^2 dxdt = - \int_0^T \int_0^1 s^{-2} \theta^{-2} e^{2s\varphi(x,t)} \hat{\omega}_x dxdt
\]

\[
+ \int_0^T \int_0^1 e^{2s\varphi(x,t)} v_x \hat{z} dxdt + \int_\omega e^{2s\varphi(x,t)} \hat{\hat{\omega}} \hat{z} dxdt
\]

\[
= L_1 + L_2 + L_3 + L_4 + L_5.
\]

Proceeding as before, we conclude that

\[
\int_0^T \int_0^1 s^{-2} \theta^{-2} a(x) e^{2s\varphi(x,t)} \hat{\omega}_x^2 dxdt \leq C \int_0^T \int_0^1 s\theta a(x) e^{-2s\varphi(x,t)} v_x^2 dxdt,
\]
combined with inequality (35) gives
\[
\int_0^T \int_0^1 e^{2s\varphi(x,t)} z^2 dx dt + \int_0^T \int_0^1 e^{2s\varphi(x,t)} \bar{v}^2 dx dt \\
+ \int_0^T \int_0^1 s^{-2}a(x)e^{2s\varphi(x,t)} z_s^2 dx dt \leq C \int_0^T \int_0^1 s\theta a(x)e^{-2s\varphi(x,t)} v_s^2 dx dt
\]  
for every \( s \geq s_0 \).

**Step 2. Proof of Carleman inequality (13)**  
We multiply (27) by \( v \) solution to (10). Then, integrating by parts and using Hölder’s inequality, we obtain
\[
\int_0^T \int_0^1 s^3\beta^3 \frac{x^2}{a(x)} e^{-2s\varphi(x,t)} v^2 dx dt = -\int_0^T \int_0^1 \hat{z}v dx dt - \int_0^T \int_0^1 \hat{z}(a(x)v_x)_x dx dt \\
- \int_0^T \int_0^1 \hat{v} v dx dt - \int_0^T \int_0^1 \beta(x) F_1 \hat{z} dx dt - \int_0^T \int_0^1 \hat{v} v dx dt \leq \\
\left( \int_0^T \int_0^1 e^{-2s\varphi(x,t)} F_0^2 dx dt + \int_0^T \int_0^1 \frac{s^2\beta^3}{a(x)} e^{-2s\varphi(x,t)} F_0^2 dx dt \\
+ \int_0^T \int_0^1 \frac{s^2\beta^3}{a(x)} e^{-2s\varphi(x,t)} F_1^2 dx dt + \int_0^T \int_0^1 \frac{s^2\beta^3}{a(x)} e^{-2s\varphi(x,t)} v^2 dx dt \right)^{1/2}
\]

From (33), we deduce
\[
\int_0^T \int_0^1 \frac{s^3\beta^3}{a(x)} x^2 e^{-2s\varphi(x,t)} v^2 dx dt \leq C \left( \int_0^T \int_0^1 e^{-2s\varphi(x,t)} F_0^2 dx dt \\
+ \int_0^T \int_0^1 \frac{s^2\beta^3}{a(x)} e^{-2s\varphi(x,t)} F_1^2 dx dt + \int_0^T \int_0^1 \frac{s^2\beta^3}{a(x)} e^{-2s\varphi(x,t)} v^2 dx dt \right).
\]  

Analogously, if we multiply by \( v \) system (34) and integrate by parts, we conclude
\[
\int_0^T \int_0^1 s\theta a(x)e^{-2s\varphi(x,t)} v_s^2 dx dt = -\int_0^T \int_0^1 \hat{z}v dx dt - \int_0^T \int_0^1 \hat{z}(a(x)v_x)_x dx dt \\
- \int_0^T \int_0^1 \hat{v} v dx dt - \int_0^T \int_0^1 \beta(x) F_1 \hat{z} dx dt - \int_0^T \int_0^1 \hat{v} v dx dt \leq \\
\left( \int_0^T \int_0^1 e^{-2s\varphi(x,t)} F_0^2 dx dt + \int_0^T \int_0^1 \frac{s^2\beta^3}{a(x)} e^{-2s\varphi(x,t)} F_0^2 dx dt \\
+ \int_0^T \int_0^1 \frac{s^2\beta^3}{a(x)} e^{-2s\varphi(x,t)} F_1^2 dx dt + \int_0^T \int_0^1 \frac{s^2\beta^3}{a(x)} e^{-2s\varphi(x,t)} v^2 dx dt \right)^{1/2}.
\]

Considering (36), we obtain
\[
\int_0^T \int_0^1 s\theta a(x)e^{-2s\varphi(x,t)} v_s^2 dx dt \leq C \left( \int_0^T \int_0^1 e^{-2s\varphi(x,t)} F_0^2 dx dt \\
+ \int_0^T \int_0^1 \frac{s^2\beta^3}{a(x)} e^{-2s\varphi(x,t)} F_1^2 dx dt + \int_0^T \int_0^1 \frac{s^2\beta^3}{a(x)} e^{-2s\varphi(x,t)} F_1^2 dx dt \right),
\]
combined with (37), this implies (13), completing the proof. □
3. Null controllability of semilinear degenerate parabolic equations. In this section we prove our main result Theorem 1.1. First, using our Carleman inequality (13) we give a sketch of the proof Theorem 2.2. As a consequence of this result, we state the null controllability of linear degenerate parabolic equations but we do not give its proof since nowadays it is classical. See e.g. [16], [12], [13] for classical results. Finally we conclude the section with the non linear result.

**Sketch of the proof of Theorem 2.2.** To prove Theorem 2.2 we proceed as in [1], pp.192-195. That is, we multiply \( v_t + (a(x)v_x)_x - b(x,t)v + (\beta(x)c(x,t)v)_x = 0 \) by \( v \) and we integrate over \((0,1)\). We get that,

\[
0 = \int_0^1 [v_t + (a(x)v_x)_x - b(x,t)v + (\beta(x)c(x,t)v)_x]vdx
\]

\[
= \frac{d}{dt} \int_0^1 v^2 dx - \int_0^1 a(x)v_x^2 dx - \int_0^1 b(x,t)v^2 dx - \int_0^1 \beta(x)c(x,t)vv_x dx.
\]

Observe that, for \( x \in [0,1], \ a(x) \geq a(1)C_\beta \beta^2(x) \) for some constant \( C_\beta > 0 \) as a consequence of (7). Then,

\[
\int_0^1 a(x)v_x^2 dx = \frac{d}{dt} \int_0^1 v^2 dx - \int_0^1 b(x,t)v^2 dx
\]

\[
- \int_0^1 (2a(1)C_\beta \beta^2(x)) \frac{1}{2} c(x,t) \frac{\sqrt{\beta}}{\sqrt{2a(1)C_\beta}} vv_x dx
\]

\[
\leq \frac{d}{dt} \int_0^1 v^2 dx + ||b||_\infty \int_0^1 v^2 dx + \frac{||c||_\infty^2}{2a(1)C_\beta} \int_0^1 v^2 dx + \int_0^1 a(1)C_\beta \beta^2(x)v_x^2 dx.
\]

This implies that,

\[
0 \leq 2 \int_0^1 (a(x) - a(1)C_\beta \beta^2(x)) v_x^2 dx
\]

\[
\leq \frac{d}{dt} \int_0^1 v^2 dx + \left( \frac{2||b||_\infty}{2a(1)C_\beta} + \frac{||c||_\infty^2}{2a(1)C_\beta} \right) \int_0^1 v_x^2 dx.
\]

Multiplying by \( e^{(2||b||_\infty + ||c||_\infty^2 / 2a(1)C_\beta)t} \), we get

\[
0 \leq e^{(2||b||_\infty + ||c||_\infty^2 / 2a(1)C_\beta)t} \left[ \frac{d}{dt} \int_0^1 v^2 dx + \left( \frac{2||b||_\infty}{2a(1)C_\beta} + \frac{||c||_\infty^2}{2a(1)C_\beta} \right) \int_0^1 v_x^2 dx \right].
\]

Therefore, for every \( t \in [0,T] \),

\[
0 \leq \frac{d}{dt} \int_0^1 v^2(x,t) dx - \frac{1}{T} \int_0^1 v^2(x,0) dx
\]

That means, that for a constant \( C > 0 \),

\[
\int_0^1 v^2(x,0) dx \leq C \int_0^1 v^2(x,t) dx, \quad \forall \ t \in [0,T].
\]

As in [1] it is needed to take two cases, \( K \neq 1 \) and \( K = 1 \), to get

\[
\int_0^1 v^2(x,t) dx \leq C \int_0^1 a(x)v_x^2(x,t) dx, \quad \forall \ t \in [0,T].
\]

Then, for every \( t \in [0,T] \)

\[
\int_0^1 v^2(x,0) dx \leq C \int_0^1 a(x)v_x^2(x,t) dx.
\]
As a consequence, integrating over \([T/4, 3T/4]\) and using Theorem 2.3, we have
\[
\int_0^1 v^2(x,0)dx \leq C \int_{T/4}^{3T/4} \int_0^1 a(x)v_x^2(x,t)dxdt
\]
\[
\leq C \int_{T/4}^{3T/4} \int_0^1 s\theta e^{2s\theta(x,t)}a(x)v_x^2(x,t)dxdt
\]
\[
\leq C \int_0^T \int_\omega v^2 dxdt.
\]
This concludes the proof of Theorem 2.2.}

Observe that the observability inequality implies the null controllability of system:
\[
\begin{cases}
y_t - (a(x)y_x)_x + b(x,t)y + \beta(x)c(x,t)y_x = h1_\omega & \text{in } Q = (0,1) \times (0,T),
y(0,t) = y(1,t) = 0 & t \in (0,T),
y(x,0) = y_0(x), & \text{in } (0,1).
\end{cases}
\] (38)

In fact, by a minimization argument, it can be proven that

**Theorem 3.1.** Given \(T > 0\) and \(y_0 \in L^2(0,1)\), it exists \(h \in L^2(\omega \times (0,T))\) such that the solution \(y\) corresponding to (38) satisfies
\[
y(x,T) = 0 \quad \text{for almost every } x \in [0,1].
\]

Moreover, it exists a positive constant \(C\), only depending on \(T\), such that
\[
\int_0^T \int_\omega |h|^2 dxdt \leq C \int_0^1 y_0^2(x)dx.
\]

**Proof of Theorem 1.1.** Let \(y_0 \in H^1_\omega(0,1)\). Let us assume that
\[
g(x, t; \cdot), G(x, t; \cdot) \in C^0(\mathbb{R}^2) \quad \forall(x, t) \in Q.
\]

We define \(Z = L^2(0,T; H^1_\omega(0,1))\). For every \(z \in Z\), we consider the linear system
\[
\begin{cases}
y_t - (a(x)y_x)_x + g(x, t; z, z_x)y + \beta(x)c(x,t)y_x = h1_\omega & \text{in } Q,
y_0 = 0 & \text{in } (0,1),
y(0,t) = y(1,t) = 0 & t \in (0,T),
y(x,0) = y_0(x), & (a_y)_t(0,t) = 0.
\end{cases}
\] (39)

We associate to \(z\) a family of controls \(U(z) \subset L^2\) that drives the corresponding solution to (39) to zero. Observe that (39) is of the form (8) with
\[
\begin{cases}
b = b_z = g(x,t; z, z_x) \in L^\infty(Q) \\
c = c_z = G(x,t; z, z_x) / \beta(x) \in L^\infty(Q).
\end{cases}
\]

From Theorem 3.1, we deduce the existence of a control \(\hat{h}_z \in L^2(\omega \times (0,T))\) such that the solution to (39) with \(h = \hat{h}_z\) satisfies
\[
\hat{y}_z(x,T) = 0 \quad \text{in } (0,1)
\]
and, furthermore
\[
\|\hat{h}_z\|_{L^2(\omega \times (0,T))} \leq C\|y_0\|_{L^2(0,1)}.
\] (40)

On the other hand, from Theorem 2.1, we obtain that
\[
\hat{y}_z \in L^2(0,T; H^1_\omega(0,1))
\]
\[\|\hat{y}_z\|_{L^2(0,T;H^1_0(0,1))} \leq C(\|y_0\|_{H^1_0(0,1)} + \|\hat{y}_z\|_{L^2(\omega \times (0,T))}).\]  

Estimates (40) and (41) can be written as

\[\|\hat{y}_z\|_{L^2(\omega \times (0,T))} \leq C\|y_0\|_{L^2(0,1)} \quad (42)\]

and

\[\|\hat{y}_z\|_Z \leq C\|y_0\|_{H^1_0(0,1)} \quad (43)\]

Given \(h \in L^2(\omega \times (0,T))\), let \(y_h \in Z\) be the solution to (39) in \(Q\) with right-hand side \(h\) (to simplify notation we omit the dependence on \(z\)). For every \(z \in Z\) we define

\[U(z) = \{h \in L^2(\omega \times (0,T)) : y_h(T) = 0, \quad \|h\|_{L^2(\omega \times (0,T))} \leq C\|y_0\|_{L^2(0,1)}\}\]

and

\[\Lambda(z) = \{y_h : h \in U(z), \quad \|y_h\|_Z \leq C\|y_0\|_{H^1_0(0,1)}\}.\]

In this way we introduced a multivalued application

\[z \mapsto \Lambda(z).\]

We will show that this application has a fixed point \(y\). Of course, this will imply the existence of \(h \in L^2(\omega \times (0,T))\) such that (1) has a solution that satisfies (6).

To this aim we use a version of Kakutani’s fixed point due to Fan and Glicksberg (see [17]) that can be applied to \(\Lambda\). Firstly, from (42) and (43) we deduce that \(\Lambda(Z)\) is, for every \(z \in Z\), a non empty set. Secondly, it is easy to check that \(\Lambda(z)\) is a uniformly bounded set, closed and convex in \(Z\). The regularity assumption on \(y_0\) and Theorem 2.1, implies

\[y \in C^0([0,T]; H^1_0(0,1)) \cap L^2(0,T; H^2_0(0,1)) \cap H^1(0,T; L^2(0,1)),\]

and it exists a constant \(C_T\) such that

\[\|y\|_{L^2(0,T;H^2_0(0,1))} + \|y_t\|_{L^1(Q)} \leq C_T\|y_0\|_{H^1_0(0,1)} \quad (44)\]

(where \(C_T\) is independent of \(z\)) for every \(y \in \Lambda(z)\). Furthermore, \(H^2_0(0,1)\) is compactly imbeded in \(H^1_0(0,1)\) (see, e.g. [1]). We can conclude that it exists a compact set \(K \subset Z\) such that

\[\Lambda(z) \subset K \quad \forall z \in Z \quad (\text{see} \ [15]). \]

Now we prove that \(z \mapsto \Lambda(z)\) is upper hemi-continuous, i.e. the real function

\[z \in Z \mapsto \sup_{y \in \Lambda(z)} \langle \mu, y \rangle\]

is upper semi-continuous for every \(\mu \in Z'\). In other words, we will check that

\[B_{\alpha,\mu} = \{z \in Z : \sup_{y \in \Lambda(z)} \langle \mu, y \rangle \geq \alpha\}\]

is a closed set of \(Z\) for every \(\alpha \in \mathbb{R}\). To this end, take \(\{z_n\}\) a sequence in \(B_{\alpha,\mu}\) such that

\[z_n \to z \quad \text{in} \quad Z.\]

We want to show that \(z \in B_{\alpha,\mu}\). We know, that it exists a subsequence \(\{z_{n_k}\}\) such that

\[z_{n_k}(x,t) \to z(x,t), \quad \text{almost everywhere in} \quad Q,\]

and

\[\sqrt{a(x)}z_{n_k}(x,t) \to \sqrt{a(x)}z(x,t), \quad \text{almost everywhere in} \quad Q.\]
Then, from the continuity assumptions on \( g \) and \( G \), we get
\[
g(x, t; z_{n_k}, z_{n_k,x}) \to g(x, t; z, z_x) \text{ in } L^\infty(Q),
\]
and
\[
\frac{G(x, t; z_{n_k}, z_{n_k,x})}{\beta(x)} \to \frac{G(x, t; z, z_x)}{\beta(x)} \text{ in } L^\infty(Q).
\]

On the other hand, since all the sets \( \Lambda(z_n) \) are compact and satisfy (45), we deduce that
\[
\alpha \leq \sup_{y \in \Lambda(z_n)} \langle \mu, y \rangle = \langle \mu, y_{n_k} \rangle \tag{46}
\]
for some \( y_{n_k} \in \Lambda(z_{n_k}) \). From the definition of \( \Lambda(z_{n_k}) \) and \( U(z_{n_k}) \), it exists \( h_{n_k} \in L^2(\omega \times (0, T)) \) such that the following equation is verified in \( Q \)
\[
y_{t, n_k} - (a(x)y_{x,n_k})_x + g(x, t; z_{n_k}, z_{x,n_k})y_{n_k} + \beta(x) \left[ \frac{G(x, t; z_{n_k}, z_{x,n_k})}{\beta(x)} \right] y_{x,n_k} = h_{n_k}1_\omega.
\]
with \( y_{n_k}(T) = 0 \) and
\[
\|h_{n_k}\|_{L^2(\omega \times (0, T))} \leq C\|y_0\|_{L^2(0, 1)},
\]
\[
\|y_{n_k}\|_Z \leq C\|y_0\|_{H^1_0(0, 1)}
\]
where \( y_{n_k} \) (resp. \( h_{n_k} \)) is uniformly bounded in \( Z \) (resp. \( L^2(\omega \times (0, T)) \)). Then,
\[
y_{n_k} \to \widehat{y} \text{ strongly in } Z
\]
and
\[
h_{n_k} \rightharpoonup \hat{h} \text{ weakly in } L^2(\omega \times (0, T)).
\]
It is not difficult to show that
\[
\left\{ \begin{array}{ll}
\widehat{y}_t - (a(x)\widehat{y}_x)_x + g(x, t; z, z_x)\widehat{y} + \beta(x) \left[ \frac{G(x, t; z, z_x)}{\beta(x)} \right] \widehat{y}_x = \hat{h}1_\omega & \text{in } Q, \\
\widehat{y}(1, t) = 0 \text{ and } (a\widehat{y}_x)(0, t) = 0 & \text{for (WDP), } t \in (0, T), \\
\widehat{y}(x, 0) = y_0(x), \quad \widehat{y}(x, T) = 0 & \text{in } (0, 1),
\end{array} \right.
\]
in the sense of distributions; that is \( \widehat{v} \in U(\omega) \) and \( \widehat{y} \in \Lambda(z) \). In consequence, we can take the limit in (46) to deduce that
\[
\alpha \leq \langle \mu, \widehat{y} \rangle \leq \sup_{y \in \Lambda(\omega)} \langle \mu, y \rangle
\]
that means that, \( z \in B_{\omega, \mu} \). We obtain that \( z \mapsto \Lambda(z) \) is upper hemi-continuous.

We assume now that \( g(x, t; \cdot) \) and \( G(x, t; \cdot) \) belong to \( L^\infty(\mathbb{R}^2) \). We introduce the function \( \rho(x, t; \cdot) \in C_\infty_c(\mathbb{R}^2) \) such that \( \rho(x, t; \cdot) \geq 0 \text{ in } \mathbb{R}^2 \), \( \text{supp } \rho(x, t; \cdot) \subset \overline{B}(0, 1) \)
and
\[
\int \int_{\mathbb{R}^2} \rho(x, t; s, p)dsdp = 1
\]
for every \((x, t) \in Q \). We consider the functions \( \rho_n, g_n \) and \( G_n \) \((n \geq 1)\), with
\[
\rho_n(x, t; s, p) = \frac{1}{n^2} \rho(x, t; ns, np) \quad \forall (s, p) \in \mathbb{R}^2,
\]
and
\[
g_n = \rho_n * g, \quad G_n = \rho_n * G.
\]
Then, it is not difficult to see that \( g_n \) and \( G_n \) satisfy:
\begin{enumerate}
\item \( g_n(x, t; \cdot), G_n(x, t; \cdot) \in L^\infty(\mathbb{R}^2) \) \((n \geq 1)\).
\item \( g_n(x, t; \cdot) \to g(x, t; \cdot) \) and \( G_n(x, t; \cdot) \to G(x, t; \cdot) \) uniformly in \( \mathbb{R}^2 \) for \((x, t) \in Q \).
\end{enumerate}
For every $n$, we obtain a control $h_n \in L^2(\omega \times (0, T))$ such that
\[
\begin{align*}
&y_{n}(t, x) - (a(x)y_{x,n})_x + f(x, t, y_n, y_{x,n}) = h_n 1_\omega & \text{in } Q, \\
&y_n(1, t) = 0 & \text{and } \begin{cases} y_n(0, t) = 0 & \text{for (WDP)}, \\ (a y_{x,n})(0, t) = 0 & \text{for (SDP)}, \end{cases} & t \in (0, T), \\
y_n(x, 0) = y_0(x), & \text{in } (0, 1),
\end{align*}
\]
has a least one solution $y_n \in Z$ that satisfies
\[
y_n(x, T) = 0 \quad \text{in } (0, 1),
\]
\[
\|h_n\|_{L^2(\omega \times (0, T))} \leq C \quad \text{and } \|y_n\|_Z \leq C \quad \forall \ n \geq 1.
\]
We have that $y_n \in K$ for every $n \geq 1$, with $K$ a compact subset of $Z$. Therefore, we can assume that at least for a subsequence
\[
y_n \to y \quad \text{strongly in } Z,
\]
\[
h_n \rightharpoonup h \quad \text{weakly in } L^2(\omega \times (0, T)).
\]
Passing to the limit in (47), we obtain a control $h \in L^2(\omega \times (0, T))$ such that it exists a control $\bar{h} \in L^2(Q)$ such that the corresponding solution $y$ to (1) satisfies (6). The regularizing effects of the degenerate parabolic equation allows to prove the result when $y_0 \in L^2(0, 1)$. In fact in this situation take $h = 0$ in a time interval $(0, t_0)$ with $t_0 < T$ and then control the equation in $(t_0, T)$, with initial datum $y(t_0)$. \hfill $\square$

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