Irreducibility of G-varieties defined by quadrics.

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Abstract

Let \( g \) be a complex simple Lie algebra, \( G \) a simply connected Lie group with Lie algebra \( g \) and \( V \) a module. We will study the irreducibility of \( G \)-varieties defined by quadrics in \( \mathbb{P}V \).

Keywords: Irreducibility, Varieties defined by quadrics, Simple Lie algebra, Orbit closure

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Introduction.

We will study a question raised in the exercise [4, 15.44 Hard Exercise]. Let \( V = \mathbb{C}^n \) be the standard representation of \( \mathfrak{sl}_n(\mathbb{C}) \) and consider the following decomposition

\[ S^2(\wedge^k V) = \bigoplus_{i \geq 0} \Theta_{2i}, \]

where \( \Theta_{2i} \) is an irreducible representation of \( \mathfrak{sl}_n(\mathbb{C}) \). Let \( C^p(Gr^k(V)) \) be the \( p \)-restricted chordal variety of the Grassmannian of subspaces of dimension \( n - k \): that is, the union of chords \( LM \) joining pair of planes meeting in a subspace of dimension at least \( k - 2p + 1 \). In the exercise we must prove that the ideal in degree two of \( C^p(Gr^k(V)) \) is

\[ I(C^p(Gr^k(V)))_2 = \bigoplus_{i \geq p} \Theta_{2i}, \]

and the authors asked what is the actual zero locus of these quadrics? In the present paper we will generalize the situation to the following:

Let \( g \) be a simple Lie algebra and let \( G \) be the simply connected Lie group with Lie algebra \( g \), let \( V \) be a representation and \( G \cdot y \subseteq \mathbb{P}V \) be the closure of an orbit in the projective space \( \mathbb{P}V \).

Theorem. The zero locus of quadrics in \( I(G \cdot y)_2 \) is an irreducible variety.

As an application of this result, we will prove that there exist \( y \in C^p(Gr^k(V)) \) such that \( I(C^p(Gr^k(V)))_2 = I(G \cdot y)_2 \) and then, the zero locus of \( I(C^p(Gr^k(V)))_2 \) is an irreducible variety. This gives an answer to the question in [4, 15.44 Hard Exercise].

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After the previous motivation let’s present our notations. We will work with a simple Lie algebra $g$, a module $V$ and a simply connected Lie group $G$ with Lie algebra $g$. For each $y \in PV$ consider the orbit $G.y \subseteq PV$ and the zero locus of its ideal in degree two,

$$M_y = \{ x \in PV \mid q(x) = 0 \forall q \in I(G.y) \}.$$ 

We will prove that $M_y$ is irreducible. If the closure of the orbit $G.y$ is defined by quadrics, the variety $M_y$ is obviously irreducible. It is equal to $G.y$. Also, if the vector $y$ corresponds to a maximal weight vector of $V$, then the orbit is automatically closed, [4, p.388, claim 23.52], for example, the Veronese variety, the Grassmannian and partial frags varieties [3, §9.3]. In these cases, the variety $M_y$ is irreducible by trivial reasons (it is an orbit).

This article is divided in four sections. In section one we give some preliminaries. In section two we define the notion of a multi-matrix. The space of multi-matrixes arise naturally in the proof of the irreducibility of $M_y$. In section three we prove that the variety $M_y$ is irreducible (see 11). First we show in 6 that for every $y \in V$ there exist a multi-matrix $A$ such that

$$M_y \cong \{ ABA' \mid B \in \text{Cat}, \, \text{rk}(ABA') \leq 1 \},$$

where $\text{Cat}$ is the space of catalectic multi-matrixes (see definitions in 2). Second, in 9 we give a characterization of the space $\{ ABA' \mid B \in \text{Cat} \}$, and with this, we prove in 11 that $M_y$ is isomorphic to the irreducible variety $\{ P^2 \mid P \in \text{im}A' \}$ where $P^2$ is the square of the polynomial $P$. In section four we give some applications of the result.

1. Preliminaries.

We will work with the universal enveloping algebra of $g$,

$$Ug = \left( \bigoplus_{n \geq 0} g^\otimes n \right) / I, \quad I = \langle D \otimes E - E \otimes D - [D, E] \rangle.$$

Elements of $Ug$ are the classes of non-commutative polynomials in $g$. Our goal in this section is to prove the following

**Theorem.** Let $y \in V$, then there exists $r \in \mathbb{N}$, $(D_1, \ldots, D_r) \in g^r$, $N \in \mathbb{N}_0^r$ such that for every $Q \in Ug$, we have

$$Q(yy) = \sum_{i_1, j_1=0}^{N_1} \cdots \sum_{i_r, j_r=0}^{N_r} b_{i_1+j_1, \ldots, i_r+j_r} \frac{D_1^{i_1} \cdots D_r^{i_r} y \cdot D_1^{j_1} \cdots D_r^{j_r} y}{i_1! \cdots i_r! j_1! \cdots j_r!},$$

where the coefficients $b_{0,0}, \ldots, b_{2N_1, \ldots, 2N_r}$, depends on $Q$. In a more compact form, the formula is

$$Q(yy) = \sum_{i, j=0}^{N} b_{i+j} \frac{D_i^y D_j^y}{i! j!}.$$

This theorem will help us to study the irreducibility of $M_y$. But first let’s start with the following Lemmas.
We will consider the basis of

\[ \frac{D_1^n \ldots D_r^p(y y)}{n_1! \ldots n_r!} = \sum_{i+j=n_1, \ldots, i+r=n_r} \frac{D_1^i \ldots D_r^j y y}{i_1! \ldots i_r! j_1! \ldots j_r!}. \]

In a more compact notation we have,

\[ \frac{D^r(y y)}{n!} = \sum_{i+j=n} \frac{D^i y D^j y}{i! j!}, \]

where \( D^r := D_1^r \ldots D_r^r \) and \( k! := k_1! k_2! \ldots k_i! \).

**Proof.** Given \( D_r \in \mathfrak{g} \), we have

\[ D_r^k(ab) = \sum_{l=0}^k \binom{k}{l} (D_r^l a)(D_r^{k-l} b) \]

The result follows by induction. \( \square \)

We are assuming that the Lie algebra \( \mathfrak{g} \) is simple. For each positive root \( \beta \), let \( X_\beta \in \mathfrak{g}^\beta, Y_\beta \in \mathfrak{g}^{-\beta} \) and \( H_\beta \in \mathfrak{h} \) such that \([X_\beta, Y_\beta] = H_\beta\). From a result in [7, p.57] we know that if \( W \) is irreducible and has a maximal weight vector \( w \), then \( Y_{\beta_1}^{m_1} \ldots Y_{\beta_k}^{m_k} w, m_i \in \mathbb{N}_0 \), generates \( W \) as a vector space. In the next Lemma we will prove that there is a similar result without the hypothesis on \( w \).

**Lemma 1.** Let \( W \) be a finite dimensional representation. Given \( w \in W \) there exists \( r \in \mathbb{N} \), \((D_1, \ldots, D_r) \in \mathfrak{g}'\) with

\[ D_i \in \{X_{\beta_1}, Y_{\beta_1}, \ldots, X_{\beta_k}, Y_{\beta_k}\}, \]

such that \( \{D_1^{m_1} \ldots D_r^{m_r} w\}_{m_1, \ldots, m_r \geq 0} \) generates \( U \mathfrak{g} w \) as a vector space. In a more compact form, the set may be written as \( \{D^r w\}_{m \geq 0} \).

**Proof.** Let \( p_1, \ldots, p_s \in U \mathfrak{g} w \) be the maximal weight vectors of the representation \( U \mathfrak{g} w \) and let \( P_1, \ldots, P_s \in U \mathfrak{g} \) be the non-commutative polynomials such that \( P_i w = p_i \). By the Poincar-Birkhoff-Witt Theorem, [4, p.486], if \( E_1, \ldots, E_s \) is a basis of \( \mathfrak{g} \) then every element in \( U \mathfrak{g} \) is a linear combination of the monomials

\[ \{E_1^{m_1} \ldots E_s^{m_s}\}, \quad m_1, \ldots, m_s \geq 0. \]

We will consider the basis of \( \mathfrak{g} \) obtained by the root decomposition,

\[ \{X_{\beta_1}, Y_{\beta_1}, H_{\beta_1}, \ldots, X_{\beta_k}, Y_{\beta_k}, H_{\beta_k}\}. \]

Given that \( H = [X, Y] = XY - YX \) in \( U \mathfrak{g} \) we may suppose that \( H \) does not appear in the monomials of \( P_i \). Let \( D_1 = Y_{\beta_1} \), \( \ldots, D_k = Y_{\beta_k} \) and let’s define \( D_{k+1}, \ldots, D_r \). Let \( D_{k+1} \) be the first variable of the first monomial of \( P_1 \), \( D_{k+2} \) the second variable of the first monomial of \( P_1 \), finally let \( D_r \) be the last variable of the last monomial of \( P_s \).

Note that with all the polynomials formed with the monomials of the form \( D_1^{m_1} \ldots D_r^{m_r} w \), we obtain, in particular, the polynomials

\[ Y_{\beta_1}^{m_1} \ldots Y_{\beta_k}^{m_k} P_i w = Y_{\beta_1}^{m_1} \ldots Y_{\beta_k}^{m_k} P_i \]

that generates, as a vector space, the whole representation \( U \mathfrak{g} p_i \). \( \square \)
**Lemma 3.** Let \( V \) be a finite dimensional representation, \( r \in \mathbb{N} \) and \( (D_1, \ldots, D_r) \in g' \).

\[
D_i \in \{X_{\beta_i}, Y_{\beta_i}, \ldots, X_{\beta_i}, Y_{\beta_i}\}.
\]

Given \( u \in V \) there exist \( N \in \mathbb{N}_0 \) such that

\[
D_1^{N_i+\ell_i} D_2^{N_i+\ell_2} \cdots D_r^{N_i+\ell_r} u = 0, \quad \forall k_1, \ldots, k_r \geq 0.
\]

In a more compact form, we may write

\[
D^{N+k} u = 0, \quad \forall k \geq 0.
\]

**Proof.** Assume first that \( u \) has a particular weight \( \mu \), that is, \( u \in V^\mu \). If \( D_r = X_\beta \) then \( D_r^\mu u \in V^{\mu+\beta} \), else if \( D_r = Y_\beta \) then \( D_r^\mu u \in V^{\mu-\beta} \). Given that \( V \) is finite dimensional it has finite weights, then there exist \( \ell \in \mathbb{N} \) such that \( D_\ell^\mu u = 0 \).

Assume now that \( u \) is a general vector of \( V \), so we can decompose it as \( u = \sum u_i \) where each \( u_i \) has weight \( \mu_i \). From the previous paragraph we know that for each \( i \) there exist \( \ell_i \) such that \( D_\ell^\mu u_i = 0 \). So if we take the maximum of \(|\ell_i|\), there exist \( \ell \in \mathbb{N} \) such that \( D_\ell^\mu u = 0 \).

Finally, let’s see that for a given \( u \in V \), there exist \( (N_1, \ldots, N_r) \in \mathbb{N}_0^r \) such that

\[
D_1^{N_1+\ell_1} D_2^{N_2+\ell_2} \cdots D_r^{N_r+\ell_r} u = 0, \quad \forall k_1, \ldots, k_r \geq 0.
\]

Let \( N_r \) be such that \( D_r^{N_r} u = 0 \). Let \( N_{r-1} \) be the maximum of \(|\ell_i|\) where \( \ell_i \) is such that \( D_\ell^\mu u_i = 0 \) for \( 0 \leq i \leq N_r \). In general, let \( N_i \) be such that \( D_r^{N_r}(D_\ell^{N_1} \cdots D_r^\mu u) = 0 \) for all \( 0 \leq i_j \leq N_i \).

**Theorem 4.** Let \( y \in V \), then there exists \( r \in \mathbb{N} \) and \( (D_1, \ldots, D_r) \in g' \) such that for every \( Q \in U_g \), we have

\[
Q(yy) = \sum_{i,j=0}^{N} b_{ij} \frac{D_i^y D_j^y}{i! j!}.
\]

where the coefficients \( b_0, \ldots, b_{2N} \) depends on \( Q \).

**Proof.** From [1] there exists \((D_1, \ldots, D_r) \in g' \) such that \( \{D^\mu(yy)\}_{\mu=0} \) generates \( U_g(yy) \) as a vector space. From [2] there exist \( N \) big enough such that \( \{D^\mu(yy)\}_{\mu=0}^{2N} \) still generates \( U_g(yy) \) and also \( D^{N+k}y = 0 \) for \( k \geq 0 \). Finally,

\[
Q(yy) = \sum_{\mu=0}^{2N} b_{\mu} D^\mu(yy) = \sum_{\mu=0}^{2N} \sum_{i+j=\mu} b_{i} D_i^y D_j^y = \sum_{\mu=0}^{N} b_{\mu} \frac{D_i^y D_j^y}{i! j!}.
\]

The first equality follows because \( \{D^\mu(yy)\}_{\mu=0}^{2N} \) generates \( U_g(yy) \) as a vector space, the second equality follows from [1] and the last equality follows from the fact that \( D^{N+k}y = 0 \) for every \( k \geq 0 \).
2. Multi-matrixes.

The definitions of multi-matrix and multi-vector given here are a particular case of the definition of matrix in [1].

Let \( r \in \mathbb{N} \), for each \( N = (N_1, \ldots, N_r) \in \mathbb{N}_0^r \) let

\[ N := \{(i_1, \ldots, i_r) | 0 \leq i_k \leq N_k \}. \]

A multi-vector is a function \( v : N \rightarrow \mathbb{C} \), equivalently, an element of \( \mathbb{C}^N \). A multi-matrix is an element \( A \in \mathbb{C}^{N \times N} \).

For each \( i, j \in \mathbb{N}_0^r \), let \( A_{ij} := A(i, j) \) be the coordinates of the multi-matrix \( A \).

In the vector space of multi-matrixes we have operations of addition, product and transpose.

The addition is defined if the multi-matrixes are of the same size. The product \( AA' \) is defined if \( A \in \mathbb{C}^{N_1 \times N_2}, A' \in \mathbb{C}^{N_2 \times N_3} \).

\[
(A + A')_{ij} = A_{ij} + A'_{ij}, \quad (AA')_{ij} = \sum_{k=0}^{N} A_{ik}A'_{kj}, \quad (A')_{ij} = A_{ji}
\]

The notation \( \sum_{k=0}^{N} \) means \( \sum_{k \in \mathbb{N}} \).

A multi-matrix \( B \in \mathbb{C}^{N \times N} \) is catalectic if \( B_{ij} = b_{i+j} \) for some \( b \in \mathbb{C}^{2N} \). The projective space of catalectic multi-matrixes is

\[
\text{Cat} := \{ (B) \in \mathbb{C}^{N \times N} | B_{ij} = b_{i+j}, \ b \in \mathbb{C}^{2N} \}.
\]

Note that a catalectic multi-matrix \( B \) is symmetric, \( B' = B \).

3. The irreducibility of \( M_y \).

Let \( G \) be a simple Lie group with Lie algebra \( g \), let \( V \) be a finite dimensional representation and let \( y \in V \) be a non-zero vector. Recall the definition of \( M_y \),

\[ M_y = \{ x \in \mathbb{P}V \mid q(x) = 0 \ \forall q \in I(G.y) \}. \]

Lemma 5. The variety \( M_y \) may be defined as

\[ M_y = \{ (x) \in \mathbb{P}V \mid xx \in Ug(yy) \}. \]

where \( Ug(yy) \) is the smallest \( g \)-module that contains \( yy \in S^2(V) \).

Proof. Consider the vector space generated by the elements of the form \( g.yy \in S^2(V) \),

\[ S = \langle g.yy \mid g \in G \rangle \subseteq S^2(V). \]

The vector space \( S \) is the smallest \( G \)-module that contains \( yy \). Using the \( G \)-isomorphism \( \phi : S^2(V^\vee) \rightarrow S^2(V)^{\vee} \) we can identify a quadratic polynomial \( q \in I(G.y) \) with a linear functional \( \phi_q \) such that \( \phi_q(xx) = 2q(x) \). In fact we have the following,

\[ S^\vee := \{ \phi \in S^2(V)^{\vee} \mid \phi(s) = 0 \ \forall s \in S \} = \]
\{\phi \in S^2(V) \mid \phi(gy, gy) = 0 \forall g \in G\} \cong \{q \in S^2(V^\vee) \mid q(\gamma y) = 0 \forall \gamma \in G\} = I(G, y)_2.

Given that \(S\) is the smallest \(G\)-module that contains \(yy\), it is equal to the \(g\)-module \(Ug(yy)\), then
\[
M_y = \{x \in PV \mid \phi(x) = 0 \forall \phi \in I(G, y)_2\} = \{x \in PV \mid \phi(xx) = 0 \forall \phi \in I(G, y)_2\} = \{x \in PV \mid xx \in S\} = \{x \in PV \mid xx \in Ug(yy)\}.
\]

\[\]

**Theorem 6.** Let \(y \in V\) and \(\ell = \dim V\), then there exists a multi-matrix \(A \in \mathbb{C}^{\ell \times N}\) depending on \(y\) such that
\[
M_y \cong \phi_A(Cat) \cap V
\]

where \(\phi_A(B) = ABA'\) and \(V\) is the Veronese variety. \(V = \{(xx') \mid x \in \mathbb{C}^{\ell \times 1}\}\). Note that the space \(\phi_A(Cat) \cap V\) consist of symmetric \(\ell \times \ell\)-matrixes, i.e. \(\phi_A(Cat) \cap V \subseteq PS^2(\mathbb{C}^\ell)\).

**Proof.** Consider the Veronese map \(v_2\),
\[
v_2 : PV \longrightarrow PS^2(V), \quad (x) \longmapsto (xx).
\]
Its image is the Veronese variety \(V\). From [5, exercise 2.8] we know that \(M_y \cong v_2(M_y)\),
\[
v_2(M_y) = \{(xx) \mid (x) \in M_y\} = \{(xx) \mid xx \in Ug(yy)\} = \{(xx) \mid xx = Q(yy)\}, \quad Q \in Ug.
\]

Fix a basis of \(V\), \([v_1, \ldots, v_N]_I\), then we can write the elements \(D^I y/j!\),
\[
\frac{D^I y}{j!} = \sum_{k=1}^\ell b_{ik} v_k,
\]
then, by \([\ref{bcv}]\) we have
\[
Q(yy) = \sum_{i,j=0}^N b_{ij} D^I y/j! = \ell \sum_{k,l=1}^\ell \left( \sum_{i,j=0}^N b_{ij} a_k a_l \right) v_k v_l.
\]

The element \(Q(yy)\) is of the form \(xx\), where \(x = \sum_{k=1}^\ell \lambda_k v_k\) if and only if,
\[
\left( \sum_{k=1}^\ell \lambda_k v_k \right) = Q(yy) \quad \Longleftrightarrow \sum_{i,j=0}^N b_{ij} a_k a_l = \lambda_k \lambda_l, \quad \forall k, l \Longleftrightarrow \langle ABA' \rangle \in \mathcal{V}.
\]

where \(B \in \mathbb{C}^{N \times N}\) and \(A \in \mathbb{C}^{\ell \times N}\) are such that \(R_{ij} = b_{ij}, A_{ki} = a_{ik}\).

With this theorem at hand, we can now prove the irreducibility of \(M_y\). First we will characterize \(\phi_A(Cat)\) and then its intersection with \(V\). Let’s introduce some notations,
Notation 7. Let $V_1, V_2 \subseteq V$ be two linear subspaces, then we will denote

$$V_1 \oplus V_2 := V_1 \otimes V_2 \bigoplus V_2 \otimes V_1 \subseteq S^2(V).$$

Another notation that we are going to use is the map $\mu : S^2(\mathbb{C}^N) \rightarrow \mathbb{C}^{2N}$. Given two multi-vectors $f, g \in \mathbb{C}^N$, let $\mu(f, g)$ be the multi-vector in $\mathbb{C}^{2N}$ defined by

$$\mu(f, g)_{\alpha} = \sum_{\alpha_1 \alpha_2 = \alpha} f_{\alpha_1} g_{\alpha_2}.$$ 

We will call $\mu$ the polynomial multiplication or the convolution product. This is because an element $f \in \mathbb{C}^N$ may be considered as a polynomial in $r$ variables of degree $N$; the coefficient of the monomial $\alpha = (i_1, \ldots, i_r) \in N$ is $f(\alpha)$, then for $f, g \in \mathbb{C}^N$, we may consider the product $\mu(f, g) \in \mathbb{C}^{2N}$.

The following Proposition gives an isomorphism between $\phi_A(Cat)$ and $\mathbb{P}(\mu(\text{im} A', \text{im} A'))^\vee$. This isomorphism is the restriction of the following one:

**Lemma 8.** There exist an isomorphism between the projective space of catalectic multi-matrixes, Cat, and the dual of the image of $\mu : S^2(\mathbb{C}^N) \rightarrow \mathbb{C}^{2N}$

$$\phi_A(Cat) \rightarrow \mathbb{P}(\mu(S^2(\mathbb{C}^N)))^\vee, \quad B \rightarrow \tilde{B}.$$ 

where $b \in \mathbb{C}^{2N}$ is a multi-vector that defines $B$ and the associated linear functional $\tilde{B} : \mathbb{C}^{2N} \rightarrow \mathbb{C}$ is $\tilde{B}(x) = b'x$.

Even more, $B : S^2(\mathbb{C}^N) \rightarrow \mathbb{C}$ as a symmetric form, $x'By$, is equal to the symmetric form $\tilde{B} \circ \mu : S^2(\mathbb{C}^N) \rightarrow \mathbb{C}$ given by $\tilde{B}(\mu(x, y))$.

**Proof.** First of all, let’s see that the map $\mu$ is surjective. Let $x_i$ be the multi-vector that has a 1 in the $i$-th place and 0 in the rest. The multi-vectors $[x_i]_{i \in \mathbb{C}^N}$ generate $\mathbb{C}^{2N}$ and also, $x_i \in \text{im}(\mu)$. Then $\mu$ is surjective.

The definition of a catalectic multi-matrix $B$ implies that there exist a multi-vector $b \in \mathbb{C}^{2N}$ such that $B_{ij} = b_{i+j}$. It is easy to see that this multi-vector $b$ is unique, suppose $b, b'$ defines the same catalectic multi-matrix, then

$$b_i = B_{ij} = b_i', \quad b_{N+j} = B_{N+j} = b_{N+j}'.$$

Note that the multi-vector $b$ is the concatenation of the 0-row and $N$-column of $B$.

Finally, we have defined a linear isomorphism $B \rightarrow \tilde{B}$ and also,

$$x'_i B x_j = B_{ij} = b_{i+j} = \tilde{B}(\mu(x_i, x_j)).$$

\[\square\]

**Proposition 9.** Let $\mu : S^2(\mathbb{C}^N) \rightarrow \mathbb{C}^{2N}$ be the polynomial multiplication. Let $A \in \mathbb{C}^{N \times N}$, then

$$\phi_A(Cat) \rightarrow \mathbb{P}(\mu(\text{im} A', \text{im} A'))^\vee, \quad ABA' \rightarrow \tilde{B}|_{\mu(\text{im} A', \text{im} A')}.$$ 

is a linear isomorphism. We will identify multi-matrixes $ABA'$ with functionals on $\mu(\text{im} A', \text{im} A')$. 

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Proof. Let $B$ be a catalectic multi-matrix, $b \in \mathbb{C}^\infty_N$ its associated multi-vector and let $A \in \mathbb{C}^{\times N}$. Let $x_i$ be the multi-vector that has a 1 in the $i$-th place and a 0 in the rest, then

$$(ABA')_{ij} = x_i^t ABA' x_j = (x_i A')^t B (A' x_j) = \overline{b}(\mu(A' x_i, A' x_j)).$$

Let’s see that the following linear map is well defined and has an inverse,

$$\phi_{A} (\text{Cat}) \to \mathbb{P}(\mu(\text{im}A', \text{im}A)^\vee),$$
$$ABA' \to \overline{b}_{[\mu(\text{im}A', \text{im}A)^\vee]}.$$

Suppose $AB_1A' = AB_2A'$ then

$$\overline{b}_1(\mu(A' x_i, A' x_j)) = (AB_1A')_{ij} = (AB_2A')_{ij} = \overline{b}_2(\mu(A' x_i, A' x_j)).$$

Suppose $\overline{b}_1_{[\mu(\text{im}A', \text{im}A)^\vee]} = \overline{b}_2_{[\mu(\text{im}A', \text{im}A)^\vee]}$ then

$$(AB_1A')_{ij} = \overline{b}_1(\mu(A' x_i, A' x_j)) = \overline{b}_2(\mu(A' x_i, A' x_j)) = (AB_2A')_{ij}.$$ 

\[\square\]

By now we have the isomorphism $\phi_{A} (\text{Cat}) \cong \mathbb{P}(\mu(\text{im}A', \text{im}A)^\vee)$, but we need to characterize those linear functionals that corresponds to multi-matrices $(ABA') \in \mathcal{V}$. In the next Theorem, we will prove that the following map parameterize them.

**Lemma 10.** Let $A \in \mathbb{C}^{\times N}$ then the following map is well defined

$$\Psi : \text{Gr}^1(\text{im}A') \to \mathbb{P}(\mu(\text{im}A', \text{im}A)^\vee), \quad W \to \mu(\text{W.im}A', \text{im}A)^\vee = \langle \phi \rangle,$$

where $\text{Gr}^1(\text{im}A')$ is the variety of hyperplanes in $\text{im}A'$ and the symbol $\circ$ is the annihilator of vector spaces $\mu(\text{W.im}A') \subseteq \mu(\text{im}A').$, that is, $\ker \phi = \mu(\text{W.im}A')$.

**Proof.** Let’s see first that the map

$$\text{Gr}^1(\text{im}A') \to \mathbb{P}(\mu(\text{im}A', \text{im}A)^\vee), \quad W \to \mu(\text{W.im}A', \text{im}A)^\vee = \langle \phi \rangle$$

is well defined. Let $W \in \text{Gr}^1(\text{im}A')$ and consider the following short exact sequence

$$0 \to \ker \mu \cap (\text{W.im}A') \to \text{W.im}A' \to \mu(\text{W.im}A') \to 0.$$ 

Let $K := \ker \mu \cap (\text{im}A')$ and given that $W \subseteq \text{im}A'$ we have

$$\ker \mu \cap (\text{W.im}A') = \ker \mu \cap (\text{im}A') \cap (\text{im}A'. \text{im}A') = K \cap (\text{W.im}A').$$

Let’s see that in fact $K \subseteq W\text{im}A'$. Let $v \in \text{im}A'$ be such that $(v) \circ W = \text{im}A'$, then $(v,v) \circ W\text{im}A' = \text{im}A'. \text{im}A'$. Given that $v \neq 0$ we have

$$\mu(\overline{v}v) = 0 \implies v,v \notin K \implies (v,v) \cap K = 0 \implies K \subseteq W\text{im}A'.$$

In other words, for any $W \in \text{Gr}^1(\text{im}A')$ we have $K \subseteq W\text{im}A'$ then the following two short exact sequences have the same kernel $K$,

$$0 \to K \to \text{W.im}A' \xrightarrow{\mu} \mu(\text{W.im}A') \to 0.$$
then \( \mu(W,\text{im}A') \) is a hyperplane of \( \mu(\text{im}A',\text{im}A') \), so the following morphism is well defined,

\[
Gr^1(\text{im}A') \to Gr^1(\mu(\text{im}A',\text{im}A')), \quad W \to \mu(W,\text{im}A').
\]

Identifying \( Gr^1(\mu(\text{im}A',\text{im}A')) \) with \( \mathbb{P}(\mu(\text{im}A',\text{im}A'))^\vee \), for every hyperplane \( W \subseteq \text{im}A' \), there exist a functional \( \phi: \mu(\text{im}A',\text{im}A') \to \mathbb{C} \) such that \( \mu(W,\text{im}A') = \ker \phi \), specifically,

\[
Gr^1(\text{im}A') \to \mathbb{P}(\mu(\text{im}A',\text{im}A'))^\vee, \quad W \to \mu(W,\text{im}A')^\vee \equiv \langle \phi \rangle.
\]

We are now in the position to prove the irreducibility of \( M_y \),

**Theorem 11.** Let \( A \in \mathbb{C}^{t\times N} \) then \( \phi_A(Cat) \cap \mathcal{V} \) is irreducible. As a corollary, given \( y \in \mathcal{V} \) there exist a multi-matrix \( A \) such that \( M_y \equiv \phi_A(Cat) \cap \mathcal{V} \) (see [5]), then the variety \( M_y \) is irreducible.

**Proof.** Let’s see first that the image of \( \Psi \) corresponds to multi-matrixes \( ABA' \in \mathcal{V} \) (for the definition of \( \Psi \) see [10]).

From [5] we know that the multi-matrix \( ABA' \) has associated a functional \( \hat{b}^\dagger|_{(\mu(\text{im}A',\text{im}A'))} \). Even more \( \text{rk}(ABA') \leq 1 \) if and only if there exist a codimension one hyperplane \( W \subseteq \text{im}A' \) such that \( x^tABA'y = 0 \) for all \( A'y \in W \) and for all \( A'x \in \text{im}A' \). This is equivalent to \( \hat{b}^\dagger(\mu(A'x,A'y)) = 0 \) for all \( A'y \in W \) and for all \( A'x \in \text{im}A' \), i.e.

\[
\text{rk}(ABA') \leq 1 \iff \exists W \in Gr^1(\text{im}A') \mid \mu(W,\text{im}A') \subseteq \ker(\hat{b}^\dagger|_{(\mu(\text{im}A',\text{im}A'))}),
\]

where \( Gr^1(\text{im}A') \) is the variety of hyperplanes in \( \text{im}A' \).

We know from [10] that \( \mu(W,\text{im}A') \) is an hyperplane of \( \mu(\text{im}A',\text{im}A') \), and then the kernel of \( \hat{b}^\dagger|_{(\mu(\text{im}A',\text{im}A'))} \) must be equal to \( \mu(W,\text{im}A') \). In other words, the image of \( \Psi \) corresponds to those functional whose kernel are equal to \( \mu(W,\text{im}A') \) for some \( W \in Gr^1(\text{im}A') \), equivalently, the image of \( \Psi \) corresponds to multi-matrixes \( ABA' \in \mathcal{V} \).

Summing up, we have the following isomorphisms that implies the irreducibility

\[
\phi_A(Cat) \cap \mathcal{V} = \{ (ABA') \mid \text{rk}(ABA') \leq 1 \} = \{ (ABA') \mid \exists W \in Gr^1(\text{im}A'), \mu(W,\text{im}A') \subseteq \ker(\hat{b}^\dagger) \equiv \{ (ABA') \mid \exists W \in Gr^1(\text{im}A'), \mu(W,\text{im}A') = \ker(\hat{b}^\dagger|_{(\mu(\text{im}A',\text{im}A'))}) \} = \{ \mu(W,\text{im}A')^\vee \in \mathbb{P}(\mu(\text{im}A',\text{im}A'))^\vee \mid W \in Gr^1(\text{im}A') \}.
\]

Let’s characterize the last variety. Consider an inner product in \( \text{im}A' \) and for every \( W \in Gr^1(\text{im}A') \) let \( (v) = W^\perp \). From the proof of [10] it is easy to see that \( \mu(v,v) \cap \mu(W,\text{im}A') = \mu(\text{im}A',\text{im}A') \), then

\[
\{ \mu(W,\text{im}A')^\vee \mid W \in Gr^1(\text{im}A') \} \equiv \{ (\mu(v,v)) \mid v \in \text{im}A' \} \subseteq \mathbb{P}(\mu(\text{im}A',\text{im}A')).
\]
4. Applications.

**Corollary 12.** Let $V$ be a representation of a simple Lie algebra $\mathfrak{g}$. Let $G$ be a simply connected Lie group with Lie algebra $\mathfrak{g}$ and let $X \subseteq \mathbb{P}V$ be a variety stable under $G$ with a dense orbit $G.y$. Then $M_y$ is the intersection of quadrics that contains $X$. $M_y$ is an irreducible variety and $I(X)_2 = I(M_y)_2$.

Proof. Follows from the fact that the smallest $\mathfrak{g}$-module that contains $yy \in S^2(V)$ is the same as the smallest $G$-module that contains $yy \in S^2(V)$, that is, $U\mathfrak{g}(yy) = \langle G.y \rangle$. Then

$I(X)_2 = \{q \mid q(x) = 0, x \in X\} = \{q \mid q(g.y) = 0, g \in G\} =$

$\{q \mid q((G.yy)) = 0\} = \{q \mid q(U\mathfrak{g}(yy)) = 0\} = U\mathfrak{g}(yy)^\circ = I(M_y)_2$.

Recall that $M_y$ is generated in degree two. □

**Remark 13.** In [2, 1.3.29] there is a sufficient condition for a variety to have a dense orbit. It says that when the action of $G$ in $V$ has a finite number of orbits, any irreducible $G$-stable variety $X \subseteq \mathbb{P}V$ is the closure of an orbit $G.y$.

In the next Theorem we will give another result that guarantees that the base-locus of quadrics containing a variety is irreducible. The hypothesis is over the module $V$ independently of the varieties. But first we will need a Lemma:

**Lemma 14.** Let $W$ be a $\mathfrak{g}$-module and let $w = w_1 + \ldots + w_k \in W$, with $w_i \in W_i$, $w_i \neq 0$ and $W_i$ a simple submodule of $W$ ($1 \leq i \leq k$). Suppose that $W_i \nleq W_j$ for $i \neq j$ then

$U\mathfrak{g}w = W_1 \oplus \ldots \oplus W_k$

Proof. Let $p_i$ be a maximal weight vector of $W_i$ of weight $\omega_i$ ($1 \leq i \leq k$). Given that $W_i \nleq W_j$ the weights $\omega_i \in \mathfrak{h}^\circ$ are all different ([7, p.58]).

Case one: Assume that $w = p_1 + \ldots + p_k$ is a sum of maximal weight vectors. Given that they are all different, there exist $P \in U\mathfrak{g}$ such that $P w = P p_i \neq 0$ for some $1 \leq i \leq k$. On the other hand, given that $P p_i \neq 0$, it generates the whole submodule $W_i$ and then there exist $Q \in U\mathfrak{g}$ such that $Q P w = p_i$. Finally we proceed by induction for $w - p_i$.

Case two: If $w = w_1 + \ldots + w_k$ then there exist $P \in U\mathfrak{g}$ such that $P w$ is a sum of maximal weight vectors. Then apply case one. □

**Theorem 15.** Let $V$ be a $G$-module such that $S^2(V) = W_1 \oplus \ldots \oplus W_k$, $W_i \nleq W_j$. Let $X \subseteq \mathbb{P}V$ be an irreducible $G$-stable variety. Then there exists a generic $y \in X$ such that

$M_y = \langle \langle x \rangle \in \mathbb{P}V \mid q(x) = 0 \forall q \in I(X)_2 \rangle$.

In other words, the intersection of the quadrics that contains $X$ is an irreducible variety.

Proof. Let $C \subseteq V$ be the irreducible cone associated to $X \subseteq \mathbb{P}V$. Let $S_X$ be the smallest submodule of $S^2(V)$ that contains $\{cc \mid c \in C\}$. Given $W_i \subseteq S_X$, let $\pi_i : S^2(V) \to W_i$ be the projection to $W_i$ and

$H_i := \{\pi_i = 0\} = \ker \pi_i$. 

10
Note that \( S_X \not\subsetneq H_i \) and given that \( H_i \) is a module, we have \( \{cc \mid c \in C\} \not\subsetneq H_i \).

Let \( H := \bigcup_i H_i \), then \( \{cc \mid c \in C\} \setminus H \) is a Zariski dense subset of \( \{cc \mid c \in C\} \). Then there exist a generic \( yy \not\in H \) such that \( y \in C \).

\[
yy = \sum a_iw_i, \quad a_i = \pi_i(yy) \neq 0 \implies U(y) = S_X.
\]

The last implication follows from [4]. Finally \( I(X)_2 = S_X^c = U(y) = I(M_{j_2}) \).

**Remark 16.** With this Theorem we can answer the question of the exercise [4, 15.44 Hard Exercise] (see the Introduction of this paper). Using the fact that \( S^2(\wedge^4 V) \) has a decomposition into non-isomorphic simple submodules,

\[
S^2(\wedge^4 V) = \bigoplus_{i \geq 0} \Theta_{2i},
\]

and that the \( p \)-restricted chordal variety \( C^p(Gr^4(V)) \) is irreducible we can say that the intersection of all the quadrics that contains \( C^p(Gr^4(V)) \) is an irreducible variety.

**Corollary 17.** Let \( V \) be a \( G \)-module such that \( S^2(V) = W_1 \oplus \ldots \oplus W_k, W_i \not\cong W_j \). Let \( X \subseteq PV \) be a \( G \)-stable variety defined by quadrics. Then there exists \( x_1, \ldots, x_r \in X \) such that the irreducible components of \( X \) are of the form \( X = M_{j_1} \cup \ldots \cup M_{j_r} \).  

**Proof.** Let \( X = X_1 \cup \ldots \cup X_s \) be the decomposition of \( X \) into irreducible components. Let \( x_1 \in X_1 \) be a generic element and consider the irreducible variety \( M_{j_1} \) defined by \( I(X)_1 \), then

\[
I(M_{j_1})_2 = I(X_1)_2 \supseteq I(X)_2.
\]

Given that \( M_{j_1} \) and \( X \) are defined by quadrics, \( M_{j_1} \subseteq X \), also, \( X_1 \subseteq M_{j_1} \). Being \( M_{j_1} \) irreducible, we have \( M_{j_1} = X_1 \). Repeat this for the remaining components \( X_i, 2 \leq i \leq s \).

**Corollary 18.** Let \( V \) be a \( G \)-module such that \( S^2(V^c) = W_1 \oplus \ldots \oplus W_k, W_i \not\cong W_j \). Let \( X \subseteq PV \) be a \( G \)-stable variety defined by

\[
I(X)_2 = W_2 \oplus \ldots \oplus W_k
\]

then \( X \) is irreducible. Even more, if the ideal in degree two is

\[
I(X)_2 = W_{s+1} \oplus \ldots \oplus W_k
\]

then \( X \) has at most \( \varphi(s) \) irreducible components (it could be irreducible like in [4], or even empty).

The set function \( \varphi(s) \) count the maximum number of subsets \( \{S_1, \ldots, S_{\varphi(s)}\} \) of a set of \( s \) elements such that \( S_i \not\subsetneq S_j \). We have \( \varphi(1) = 1, \varphi(2) = 2, \varphi(3) = 3, \varphi(4) = 6 \).

**Proof.** First note that \( S^2(V^c) \) has all the simple submodules non-isomorphic if and only if \( S^2(V) \) has all the simple submodules non-isomorphic. Assume now that \( I(X)_2 = W_2 \oplus \ldots \oplus W_k \) then the ideal in degree two of an irreducible component \( M_{j_1} \) contains \( I(X)_2 \).

\[
I(X)_2 \subseteq I(M_{j_1})_2.
\]

then the simple module \( W_1 \) is in \( I(M_{j_1})_2 \) or not. In both cases \( X \) is irreducible.

Assume now that \( I(X)_2 = W_{s+1} \oplus \ldots \oplus W_k \). Let \( X = M_{j_1} \cup \ldots \cup M_{j_s} \) be the irreducible decomposition of \( X \). The simple submodules of \( I(M_{j_1})_2 \) that are not contained in \( I(X)_2 \) determine a subset \( S_i \subseteq \{1, \ldots, s\} \). Note that \( M_{j_i} \not\subseteq M_{j_j} \) if and only if \( S_i \not\subseteq S_j \).
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