Szemerédi-type theorems for subsets of locally compact abelian groups of positive upper Banach density

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Abstract

By using ergodic theoretic techniques following Hillel Fürstenberg, we prove that measurable subsets of a locally compact abelian group of positive upper density contain Szemerédi-wise configurations defined by an arbitrary compact subset of the group.

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1 Introduction

E. Szemerédi’s theorem proved in 1975 says that if a set $E \subseteq \mathbb{Z}$ is of positive upper Banach density, i.e.,

$$
\text{BD}^*(E) := \limsup_{N-M \to \infty} \frac{|E \cap [M, N)|}{N-M} > 0,
$$

then $E$ contains arbitrarily long arithmetic progressions (cf. [8, 3]). By using their multiple recurrence theorem, H. Fürlsenberg and Y. Katznelson in 1978 proved a multidimensional version of Szemerédi’s theorem: If $E \subseteq \mathbb{Z}^n$ is of positive upper Banach density and $F$ is a finite subset of $\mathbb{Z}^n$, then for some vector $u \in \mathbb{Z}^n$ and integer $d \geq 1$, $u + dF \subset E$ (cf. [4] or [3, Theorem 7.16]).

Based on the above multidimensional version, moreover, Hillel Fürlsenberg proved that if $E \subseteq \mathbb{R}^n$ is of positive upper Banach density with respect to the Lebesgue measure on $\mathbb{R}^n$ and $F$ is a finite subset of $\mathbb{R}^n$, then for some vector $u \in \mathbb{R}^n$ and integer $d \geq 1$, $u + dF \subset E$ (cf. [3, Theorem 7.17]).

In the more general case that $\mathbb{R}^n$ is replaced by an abelian discrete additive group $(G, +)$, using the Stone-Čech compactification $\beta G$ of $G$ and Fürlsenberg’s multiple recurrence theorem, Hindman and Strauss in 2006 proved that if $E \subseteq G$ is of positive upper density relative to some Følner net in $G$, then for any $a \in G$ and $l \in \mathbb{N}$, \{ $u \in G \mid u + d\{a, 2a, \ldots, la\} \subseteq E$ \} has positive upper density relative to the same Følner net for some $d \in \mathbb{N}$ (cf. [5, Theorems 5.5 and 5.6]).

In this note, we will extend the usual notion of upper Banach density in §2 and then by using ergodic theoretic techniques following Hille Fürlsenberg, we shall derive more generalizations of Szemerédi’s theorem for any finite configurations not limited to the form $\{a, 2a, \ldots, la\}$ (cf. Theorems 3.1 and 3.3 and Corollary 3.5 in [3], for any locally compact Hausdorff, not necessarily discrete, abelian group $G$ with any fixed Haar measure.

2 Følner sequences and upper density

In this section we will introduce some basic notions and preliminary lemmas to state and prove our main theorems in the next section.

2.1 Basic notions

Let $(G, +)$ be a locally compact Hausdorff additive topological group. According to Haar’s theorem (cf. [6, Theorem 29C]), there exists a left-invariant
Haar measure on $(G, +)$, which we denote by $|\cdot|$ or $dg$. When $G$ is discrete, then $|\cdot|$ is just the usual counting measure on $G$.

A sequence of compact subsets $(F_n)_{n=1}^\infty$ in $G$ is called a classical Følner sequence in $(G, +, |\cdot|)$ if and only if

$$
\lim_{n \to \infty} \frac{|(g + F_n) \triangle F_n|}{|F_n|} = 0 \quad \forall g \in G.
$$

It is a well-known fact that if $G$ is a locally compact $\sigma$-compact Hausdorff abelian group like $(\mathbb{Z}^m, +)$ with the discrete topology and $(\mathbb{R}^m, +)$ with the Euclidean metric topology, then it has classical Følner sequences. Although a discrete uncountable abelian group does not have any classical Følner sequences, yet it always has Følner nets (cf. [1, 5]).

Since an uncountable abelian group has no classical Følner sequence under the discrete topology, we need to introduce a nonclassical Følner sequence in any locally compact Hausdorff group $(G, +, |\cdot|)$.

**Definition 2.1.** Given any subset $F \subseteq G$, a sequence of compact subsets $\mathcal{F} = (F_n)_{n=1}^\infty$ in $(G, +, |\cdot|)$ is called an $F$-Følner sequence in $(G, +, |\cdot|)$ if and only if

$$
\lim_{n \to \infty} \frac{|(g + F_n) \triangle F_n|}{|F_n|} = 0 \quad \forall g \in F.
$$

Notice here that (2.1) holds only for $g \in F$ but not for any $g \in G$.

A classical Følner sequence in $(G, +, |\cdot|)$ is just a $G$-Følner sequence. Let us consider an example in order to show an $F$-Følner sequence in $(G, +, |\cdot|)$ is not necessarily a classical Følner sequence for $F \neq G$. Let $(G, +) = (\mathbb{R}^2, +)$ with $|\cdot| = dx dy$ under the standard Euclidean topology and $\varepsilon > 0$. Define a sequence of thin rectangles $F_n = \{(x, y) : |x - n| \leq \varepsilon, |y| \leq n\}$. Clearly, $|F_n| = 4n\varepsilon \to \infty$ as $n \to \infty$. It is easy to see that for $F = \{0\} \times \mathbb{R}$, $(F_n)_{n=1}^\infty$ is an $F$-Følner sequence but not a classical Følner sequence in $(G, +, |\cdot|)$.

**Definition 2.2.** Given any $F \subseteq G$, for any $F$-Følner sequence $\mathcal{F} = (F_n)_{n=1}^\infty$ in $(G, +, |\cdot|)$ and any $|\cdot|$-measurable subset $E \subseteq G$, we set

$$
D_\mathcal{F}^*(E) = \limsup_{n \to \infty} \frac{|E \cap F_n|}{|F_n|},
$$

which is called the upper density of $E$ corresponding to $\mathcal{F}$ in $(G, +, |\cdot|)$.

We shall say that $E$ is of positive upper Banach density in $(G, +, |\cdot|)$, write $\text{BD}^*(E) > 0$, if $D_\mathcal{F}^*(E) > 0$ corresponding to some classical Følner
sequence $\mathcal{F} = (F_n)_{n=1}^\infty$ in $(G, +, |\cdot|)$. We shall be concerned with measurable sets $E \subseteq G$ having $\text{BD}^*(E) > 0$.

The locally compact Hausdorff additive topological group $(G, +)$ is said to be *amenable* if the so-called Følner condition holds; that is, for any compact set $K \subseteq G$ and any $\varepsilon > 0$, there exists some compact set $F \subseteq G$ such that

$$\frac{|(K + F) \triangle F|}{|F|} < \varepsilon.$$

See, e.g., [7]. Each locally compact Hausdorff abelian topological group is amenable. Of course, any amenable group does not need to have a classical Følner sequence in it if no the $\sigma$-compact condition [7]. However, there always exists a $K$-Følner sequence in the sense of Definition 2.1 for any compact subset $K$ of an amenable group $G$.

### 2.2 Preliminary lemmas

We can then obtain the following simple result from the definition of amenable group:

**Lemma 2.3.** If $(G, +)$ is an amenable and locally compact Hausdorff topological group, then for any compact set $K \subseteq G$ there exists a $K$-Følner sequence $(F_n)_{n=1}^\infty$ in $(G, +, |\cdot|)$.

**Proof.** Since $G$ is amenable and locally compact, then by [7, Theorem 4.10] it follows that for any compact set $K \subseteq G$ and $\varepsilon_n > 0$, there exists some compact set $F_n \subseteq G$ such that

$$\frac{|(g + F_n) \triangle F_n|}{|F_n|} < \varepsilon_n \quad \forall g \in K.$$

Letting $\varepsilon_n \to 0$ implies the desired result. \qed

This lemma enables us to choose an $F$-Følner sequence $\mathcal{F} = (F_n)_{n=1}^\infty$ in $(G, +, |\cdot|)$ for any compact set $F \subseteq G$ in Theorems 3.1 and 3.3 below.

It is well known that any discrete countable abelian group $G$ has $G$-Følner sequences. Although this is not the case for any discrete abelian group, yet we can obtain the following

**Lemma 2.4.** Let $(G, +)$ be a discrete abelian group. Then for any finite subset $F \subseteq G$ and any Følner net $(F_\theta)_{\theta \in \Theta}$ in $(G, +, |\cdot|)$, there exists an $F$-Følner sequence $(F_n)_{n=1}^\infty$ in $(G, +, |\cdot|)$ such that $(F_n)_{n=1}^\infty \subseteq (F_\theta)_{\theta \in \Theta}$.
Proof. Let \((F_\theta)_{\theta \in \Theta}\) be a Følner net in \((G, +, |.|)\); that is, \((\Theta, \geq)\) is a directed index set and \(F_\theta\) is a finite subset of \(G\) such that
\[
\lim_{\theta \in \Theta} \frac{|(g + F_\theta) \Delta F_\theta|}{|F_\theta|} = 0 \quad \forall g \in G.
\]
Then for any \(\epsilon > 0\), there is some \(\theta_\epsilon \in \Theta\) such that
\[
\frac{|(g + F_\theta) \Delta F_\theta|}{|F_\theta|} < \epsilon \quad \forall g \in F \text{ and } \theta \geq \theta_\epsilon.
\]
Letting \(\epsilon \to 0\) we can find a desired \(F\)-Følner sequence \((F_n)_{n=1}^\infty\) in \((G, +, |.|)\). This thus proves Lemma 2.4.

Given any \(K \subseteq G\), by \(\langle K \rangle\) we denote the subgroup of \((G, +)\) spanned by \(K\); that is,
\[
\langle K \rangle = \{k_1 + \cdots + k_n \mid n \geq 1, k_i \in K \cup (-K)\},
\]
where \(-K = \{-k \mid k \in K\}\). The following is a simple observation.

**Lemma 2.5.** Let \((G, +)\) be an amenable locally compact Hausdorff topological group and \(K \subseteq G\) any compact set. If \((F_n)_{n=1}^\infty\) is a \(K\)-Følner sequence in \((G, +, |.|)\), then it is also a \(\langle K \rangle\)-Følner sequence in \((G, +, |.|)\).

**Proof.** Given any \(g, g_1, g_2 \in K\), by (2.1) with \(K\) in place of \(F\), we have
\[
\lim_{n \to \infty} \frac{|(g_2 + g_1 + F_n) \Delta F_n|}{|F_n|} \leq \lim_{n \to \infty} \frac{|(g_2 + g_1 + F_n) \setminus F_n + |F_n \setminus (g_2 + g_1 + F_n)|}{|F_n|} = \lim_{n \to \infty} \frac{|(g_2 + g_1 + F_n) \setminus (g_2 + F_n) + |(g_2 + F_n) \setminus (g_2 + g_1 + F_n)|}{|F_n|} = 0
\]
and
\[
\lim_{n \to \infty} \frac{|(-g + F_n) \Delta F_n|}{|F_n|} = \lim_{n \to \infty} \frac{|F_n \Delta (g + F_n)|}{|F_n|} = 0.
\]
Thus \((F_n)_{n=1}^\infty\) is a \(\langle K \rangle\)-Følner sequence in \((G, +, |.|)\).

We notice here that, as a \(\sigma\)-compact topological group, \((\langle K \rangle, +)\) itself does not need to be amenable, since the subgroup \(\langle K \rangle\) is not necessarily a closed subset of \(G\). In addition, \((F_n \cap \langle K \rangle)_{n=1}^\infty\) does not need to be a Følner sequence in \(\langle K \rangle\). In fact, \(F_n \cap \langle K \rangle = \emptyset\) in the example constructed before. Moreover, it is possible that \(E \cap \langle K \rangle = \emptyset\) in Definition 2.2.
3 Fürstenberg correspondence principle

Let \((G, +)\) be a locally compact Hausdorff additive topological group with the zero element \(o\). Given any compact Hausdorff space \(X\), we shall say that

\[ T: G \times X \to X \quad \text{or simply write} \quad G \curvearrowright_T X \]

is a Borel \(G\)-action on \(X\) or \(X\) is called a Borel \(G\)-space if it holds that

- \(T_g: x \mapsto T(g, x)\) is a continuous selfmap of \(X\), for each \(g \in G\);
- \(T: (g, x) \mapsto T(g, x)\) is jointly Borel measurable;
- \(T_0x = x\) for each \(x \in X\) and \(T_{g+h} = T_g \circ T_h\) for all \(g, h \in G\).

In this section, we will consider Fürstenberg correspondence principles between configurations in subsets of \(G\) and Borel \(G\)-space \(X\) associated to \(G\).

The following Szemerédi-type theorem (Theorem 3.1) is one of our main results, in the proof of which there are two ingredients in our Fürstenberg correspondence principle: (1) the compact Hausdorff \(G\)-space \(X\) is not necessarily metrizable; and (2) although the topology of \((G, +, |\cdot|)\) does not need to be discrete, yet to define the associated \(G\)-action we will employ the discrete topology that is not necessarily compatible with the fixed Haar measure \(|\cdot|\) on \(G\).

**Theorem 3.1.** Let \((G, +)\) be a locally compact Hausdorff abelian topological group and \(F \subseteq G\) a compact subset. If a measurable \(E \subseteq G\) is of positive upper density corresponding to an \(F\)-Følner sequence, \(\mathcal{F} = (F_n)_{n=1}^{\infty}\), in \((G, +, |\cdot|)\), then for any \(g_1, \ldots, g_l \in \langle F \rangle\),

\[ \text{BD}_\ast \left( \{d \in \mathbb{Z} : D^F_{\mathbb{Z}}(\{u \in E : u + d\{g_1, \ldots, g_l\} \subseteq E\}) > 0\} \right) > 0. \]

Here \(\text{BD}_\ast\) denotes the lower Banach density of sets in \((\mathbb{Z}, +, |\cdot|)\) and \(\langle F \rangle\) stands for the subgroup of \(G\) generated by \(F\).

**Proof.** By refining the \(F\)-Følner sequence \(\mathcal{F} = (F_n)_{n=1}^{\infty}\) if necessary, we may assume

\[ D^F_{\mathbb{Z}}(E) = \lim_{n \to \infty} \frac{|E \cap F_n|}{|F_n|}. \]

Let \(X = \prod_{g \in G} \{0, 1\}\) be the Cartesian product endowed with the standard pointwise convergence topology. Then \(X\) is a compact Hausdorff space. For
any \( x \in X \), we may identify it with the function \( x(\, \cdot \,): g \mapsto x(g) \) from \( G \) into the discrete space \( \{0, 1\} \). Note that \( x(\, \cdot \,) \in X \) is not necessarily a measurable function from \( G \) to \( \{0, 1\} \) under the locally compact Hausdorff topology of \( G \) and that the compact Hausdorff topology of \( X \) is independent of the topology of \( G \).

Let \( \chi \in X \) be given by \( \chi(g) = 1 \) if and only if \( g \in E \). Since by hypothesis \( E \) is \( |\, \cdot \,|\)-measurable, hence \( \chi(\, \cdot \,): G \to \{0, 1\} \) is \( |\, \cdot \,|\)-measurable under the locally compact Hausdorff topology of \( G \).

Define the clopen cylinder set of \( X \), \([1]\) = \( \{ x \in X \mid x(o) = 1 \} \), which is a compact \( G_\delta \)-set and so is a Baire set of \( X \). Then the characteristic function \( 1_{[1]}(x) \) of \([1]\) is a continuous function on \( X \), i.e., \( 1_{[1]}(x) \in C(X) \).

As in the usual Fürstenberg correspondence principle, under the discrete topology of \( G \) we may now define a \( G \)-action on \( X \) as follows:

\[
T: G \times X \to X \quad \text{or write} \quad G \curvearrowright_T X; \quad (g, x) \mapsto T_g x = x(\, \cdot \, + g)
\]

where \( x(\, \cdot \, + g) \in X \) is defined by

\[
x(\, \cdot \, + g): t \mapsto x(t + g).
\]

It should be noted that the continuity of \( T_g: X \to X \) is obvious under the product topology of \( X \). Thus, under the discrete topology of \( G \), \( G \curvearrowright_T X \) is a canonical \( G \)-action; in other words, \( X \) is a Borel \( G \)-space.

By the Riesz representation theorem we can identify Baire measures on \( X \) with positive functionals on \( C(X) \). Using the refined \( F \)-Følner sequence \( \mathcal{F} = (F_n)_{n=1}^\infty \) in \((G, +, |\, \cdot \,|)\) we shall define a Baire probability on the product space \( X \). Namely

\[
\mu_n(\varphi) = \frac{1}{|F_n|} \int_{F_n} \varphi(T_g \chi) dg \quad \forall \varphi \in C(X),
\]

noting that \( \varphi(T, \chi): g \mapsto \varphi(T_g \chi) \) is \( |\, \cdot \,|\)-measurable and \( dg \)-integrable from \( F_n \) to \( \{0, 1\} \) under the topology of \((G, +, |\, \cdot \,|)\) because of the measurability of \( E \) and that the Haar measure \( |\, \cdot \,| \) is not defined by the newly introduced discrete topology of \( G \).

Under the usual weak-* topology of Baire probability measures on \( X \), we can find a net \( (\mu_\theta)_{\theta \in \Theta} \), which is a subnet of the Baire probability measure sequence \((\mu_n)_{n=1}^\infty\), such that

\[
\text{weakly-*} \lim_{\theta \in \Theta} \mu_\theta = \mu
\]

for some Baire probability measure \( \mu \) on \( X \). By Lemma [2.5], it is routine to check that \( \mu \) is \( T_g \)-invariant for each \( g \in \langle F \rangle \) (not for any \( g \in G \)).
Since \( D^*_F(E) > 0 \), hence \( \mu([1]_o) > 0 \). Indeed, by \( 1_{[1]_o}(\cdot) \in C(X) \) and 
\( 1_{[1]_o}(T_g \chi) = 1_E(g) \),

\[
\mu([1]_o) = \lim_{\theta \in \Theta} \frac{1}{|F_\theta|} \int_{F_\theta} 1_{[1]_o}(T_g \chi) dg \\
= \lim_{\theta \in \Theta} \frac{|E \cap F_\theta|}{|F_\theta|} \\
= D^*_F(E).
\]

Let \( g_1, \ldots, g_l \in \langle F \rangle \) be arbitrarily given. Then by Fürstenberg’s multiple
recurrence theorem (cf. [4, 3]), it follows that

\[
D = \left\{ d \in \mathbb{Z} \mid \mu \left( [1]_o \cap T_{g_1}^{-d}[1]_o \cap \cdots \cap T_{g_l}^{-d}[1]_o \right) > 0 \right\}
\]
is of positive lower Banach density in \((\mathbb{Z}, +, | \cdot |_\mathbb{Z})\). Set \( K = \{ g_1, \ldots, g_l \} \).
Next, given any \( d \in D \) we set

\[
A = \{ u \in E \mid u + dK \subseteq E \}
\]
and

\[
U = [1]_o \cap T_{g_1}^{-d}[1]_o \cap \cdots \cap T_{g_l}^{-d}[1]_o.
\]

Then \( U \) is clopen and so is a Baire set in \( X \) for \( T_g \) is continuous of \( X \) to itself and hence

\[
\mu(U) = \lim_{\theta \in \Theta} \frac{1}{|F_\theta|} \int_{F_\theta} 1_U(T_g \chi) dg \\
\leq \lim_{\theta \in \Theta} \frac{|A \cap F_\theta|}{|F_\theta|} \\
\leq \limsup_{n \to \infty} \frac{|A \cap F_n|}{|F_n|} \\
= D^*_F(A).
\]

This proves Theorem [3.1] \( \square \)

Let \( \mathcal{B}a(X) \) be the \( \sigma \)-algebra of all Baire subsets of \( X \). It should be noted
that for \( G \curvearrowright_T (X, \mathcal{B}a(X), \mu) \) associated to \( E \) in the proof of Theorem [3.1] \( X \)
is never metrizable if \( G \) is uncountable. Further since in our context there is
no the ergodic decomposition theorem for \((X, \mathcal{B}a(X), \mu)\) is not (isomorphic to)
a Polish probability space and so no quasi-generic point, hence the proof
of \( D^*_F(\{ u \in E \mid u + d\{g_1, \ldots, g_l\} \subseteq E \}) > 0 \) is of interest.

An interesting consequence of the above theorem is the following, which
is a generalization of the classical Szemerédi theorem for \( G = \mathbb{Z} \) due to
E. Szemerédi [8], for $G = \mathbb{Z}^m$ due to Fürstenberg and Katzenelson [4], and for $G = \mathbb{R}^m$ with the Euclidean metric topology due to H. Fürstenberg [3, Theorem 7.17].

**Corollary 3.2.** Let $(G, +)$ be a second countable locally compact Hausdorff abelian topological group. If a measurable set $E \subseteq G$ is of positive upper Banach density, i.e., $BD^*(E) > 0$, then for any $g_1, \ldots, g_l \in G$

$$BD_*\left(\{d \in \mathbb{Z} \mid BD^*(\{u \in E : u + d\{g_1, \ldots, g_l\} \subseteq E\}) > 0\}\right) > 0.$$  

**Proof.** Since every second countable locally compact Hausdorff group is $\sigma$-compact, hence $G$ has a classical Følner sequence. Then $BD^*(E)$ makes sense and the statement follows at once from Theorem 3.1 with $F = G$. □

If we now utilize Bergelson-Leibman polynomial multiple recurrence theorem (cf. [2]) instead of Fürstenberg’s multiple recurrence theorem in the proof of Theorem 3.1, then we can easily obtain the following.

**Theorem 3.3.** Let $(G, +)$ be a locally compact Hausdorff abelian group and $F \subseteq G$ a compact subset. If a measurable set $E \subseteq G$ is of positive upper density corresponding to an $F$-Følner sequence $F = (F_n)_{n=1}^\infty$ in $(G, +, |\cdot|)$. Then for any $g_1, \ldots, g_l \in \langle F \rangle$

$$BD_*\left(\{d \in \mathbb{Z} \mid D^*_F(\{u \in E : u + \{p_1(d)g_1, \ldots, p_l(d)g_l\} \subseteq E\}) > 0\}\right) > 0$$

for any $l$ polynomials $p_1(t), \ldots, p_l(t) \in \mathbb{Z}[t]$ with $p_i(0) = 0$ for $i = 1, \ldots, l$.

Recall that for any discrete abelian group $(G, +)$, $E \subseteq G$ is of positive upper Banach density if and only if there exists a Følner net $(F_\theta)_{\theta \in \Theta}$ in $(G, +, |\cdot|)$ such that

$$BD^*(E) = \lim_{\theta \in \Theta} \frac{|E \cap F_\theta|}{|F_\theta|} > 0.$$  

Then by Theorem 3.3 together with Lemma 2.4, we can obtain the following

**Corollary 3.4.** Let $(G, +)$ be a discrete abelian additive group and let $p_1(t), \ldots, p_l(t) \in \mathbb{Z}[t]$ with $p_i(0) = 0$ for $1 \leq i \leq l$. If $E \subseteq G$ is such that $BD^*(E) > 0$ with Følner nets, then for any $\{g_1, \ldots, g_l\} \subseteq G$ we have

$$BD_*\left(\{d \in \mathbb{Z} \mid BD^*(\{u \in E : u + p_i(d)g_i \in E \text{ for } i = 1, \ldots, l\}) > 0\}\right) > 0.$$  

**Proof.** For any $\{g_1, \ldots, g_l\} \subseteq G$, let $F = \{g_1, \ldots, g_l\}$. By Lemma 2.4, we can find some $F$-Følner sequence, say $F = (F_n)_{n=1}^\infty$, in $(G, +, |\cdot|)$ with $D^*_F(E) = BD^*(E)$. Then the statement follows from Theorem 3.3. □
We notice here that $G$ does not have any classical Følner sequence with the discrete topology when $G$ is uncountable and moreover the finite set \( \{g_1 = a, g_2 = 2a, \ldots, g_l = la\} \) in [5] is a special configuration in Corollary 3.2. A special case of Theorem 3.3 is the following:

**Corollary 3.5.** Let $(G, +)$ be an amenable group, $F \subset G$ a compact set and $E \subseteq G$ with $D^*_F(E) > 0$ respecting to some $F$-Følner sequence $F = (F_n)_{n=1}^\infty$ in $G$. Then for any $g \in \langle G \rangle$, $l \in \mathbb{N}$ and $p_1(t), \ldots, p_l(t) \in \mathbb{Z}[t]$ with $p_i(0) = 0$ for $i = 1, \ldots, l$, there is some $d \in \mathbb{N}$ such that

\[
D^*_F(\{u \in E : u + \{p_1(d)g, 2p_2(d)g, \ldots, lp_l(d)g\} \subseteq E\}) > 0.
\]

**Proof.** This follows obviously from applying Theorem 3.3 with the special case $g_1 = g, g_2 = 2g, \ldots, g_l = lg$ and noting $T_{g_1}, \ldots, T_{g_l}$ are commuting. \(\square\)

Finally we note that in Theorems 3.1 and 3.3 we cannot consider $(\langle F \rangle, +)$ as an independent abelian group, because the $F$-Følner sequence $(F_n)_{n=1}^\infty$ does only belong to $G$, not to $\langle F \rangle$, and moreover, $E$ we consider here may have a void intersection with $\langle F \rangle$ and $D^*_F(E)$ is associated to $(F_n)_{n=1}^\infty$ in $G$.

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