Exponential Convergence Rates of Second Quantization Semigroups and Applications

Chang-Song Deng\textsuperscript{a)} and Feng-Yu Wang\textsuperscript{a),b)}\footnote{Supported in part by WIMCS and SRFDP.}†

\textsuperscript{a)}School of Mathematical Sciences, Beijing Normal University, Beijing 100875, China
\textsuperscript{b)}Department of Mathematics, Swansea University, Singleton Park, SA2 8PP, UK

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Abstract

Exponential convergence rates in the $L^2$-tail norm and entropy are characterized for the second quantization semigroups by using the corresponding base Dirichlet form. This supplements the well known result on the $L^2$-exponential convergence rate of second quantization semigroups. As applications, birth-death type processes on Poisson spaces and the path space of Lévy processes are investigated.

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1 Introduction

Let $E$ be a Polish space with Borel $\sigma$-field $\mathcal{F}$. Let $\mu$ be a non-trivial $\sigma$-finite measure on $(E, \mathcal{F})$. Let $(\mathcal{E}_0, \mathcal{D}(\mathcal{E}_0))$ be a symmetric Dirichlet form on $L^2(\mu)$. Consider the configuration space

$$\Gamma := \left\{ \gamma = \sum_i \delta_{x_i} \text{ (at most countable)} : x_i \in E \right\},$$
where \( \delta_x \) is the Dirac measure at \( x \) and \( \sum_\emptyset \) is regarded as the zero measure on \( E \). Let \( \mathcal{F}_\Gamma \) be the \( \sigma \)-field induced by \( \{ \gamma \mapsto \gamma(A) : A \in \mathcal{F} \} \). The Poisson measure with intensity \( \mu \), denoted by \( \pi_\mu \), is the unique probability measure on \( (\Gamma, \mathcal{F}_\Gamma) \) such that for any disjoint sets \( A_1, \cdots, A_n \in \mathcal{F} \) with \( \mu(A_i) < \infty \), \( 1 \leq i \leq n \),

\[
\pi_\mu \left( \{ \gamma \in \Gamma : \gamma(A_i) = k_i, 1 \leq i \leq n \} \right) = \prod_{i=1}^{n} \frac{\mu(A_i)^{k_i}}{k_i!}, \quad k_i \in \mathbb{Z}_+, 1 \leq i \leq n.
\]

This measure has the Laplace transform

\[
\pi_\mu(e^{t(f)}) = \exp \left[ \mu(e^t - 1) \right], \quad f \in L^1(\mu) \cap L^\infty(\mu),
\]

where \( \langle \gamma, f \rangle := \gamma(f) = \int_E f \, d\gamma \).

The second quantization of \((\mathcal{E}_0, \mathcal{D}(\mathcal{E}_0))\) is a symmetric conservative Dirichlet form on \( L^2(\pi_\mu) \) given by (see e.g. \cite[Lemma 6.3]{13})

\[
\mathcal{D}(\mathcal{E}) := \left\{ F \in L^2(\pi_\mu) : D.F(\gamma) := F(\gamma + \delta) - F(\gamma) \in \mathcal{D}(\mathcal{E}_0), \pi_\mu \text{-a.e. } \gamma, \quad \mathcal{E}_0(D.F, D.F) \in L^1(\pi_\mu) \right\},
\]

\[
\mathcal{E}(F, G) := \int_\Gamma \mathcal{E}_0(D.F(\gamma), D.G(\gamma))\pi_\mu(d\gamma), \quad F, G \in \mathcal{D}(\mathcal{E}),
\]

where \( \mathcal{D}_e(\mathcal{E}_0) \) is the extended domain of \( \mathcal{E}_0 \) (see \cite{11}).

Let \( P_t^0 \) and \( P_t \) be the semigroups associated to \((\mathcal{E}_0, \mathcal{D}(\mathcal{E}_0))\) on \( L^2(\mu) \) and \((\mathcal{E}, \mathcal{D}(\mathcal{E}))\) on \( L^2(\pi_\mu) \) respectively. We aim to investigate the convergence rate of \( P_t \) to \( \pi_\mu \) as \( t \to \infty \) by using properties of the base Dirichlet form.

We would like to consider the following three kinds of exponential convergence rates:

1. **Exponential convergence in the \( L^2 \)-norm:** let \( \lambda_L \) be the largest constant such that

\[
\| P_t - \pi_\mu \|_{L^2(\pi_\mu) \to L^2(\pi_\mu)} \leq e^{-\lambda_L t}, \quad t \geq 0,
\]

where \( \pi_\mu \) is regarded as a linear operator from \( L^2(\pi_\mu) \) to \( \mathbb{R} \) by letting \( \pi_\mu(F) = \int_F F \, d\pi_\mu \).

2. **Exponential convergence in the \( L^2 \)-tail norm:** let \( \lambda_T \) be the largest constant such that

\[
\| P_t \|_T := \lim_{n \to \infty} \sup_{\pi_\mu(F^2) \leq 1} \left\{ \frac{1}{1^{|P_t F| \geq n}} \| P_t F \|_{L^2(\pi_\mu)} \right\} e^{-\lambda_T t}, \quad t \geq 0.
\]

3. **Exponential convergence in entropy:** let \( \lambda_E \) be the largest constant such that

\[
\pi_\mu((P_t F) \log P_t F) \leq \pi_\mu(F \log F) e^{-\lambda_E t}, \quad t \geq 0, F \geq 0, \pi_\mu(F) = 1.
\]
The exponential convergence rate in the $L^2$-norm is already well described by the exponential decay rate of $P^0_t$, i.e. (see [5])

\[(1.2) \quad \lambda_L = \lambda_{L,0} := \inf \{ \mathcal{E}_0(f, f) : f \in \mathcal{D}(\mathcal{E}_0), \mu(f^2) = 1 \}. \]

It is well known that $\lambda_{L,0}$ is the largest number such that

\[
\|P_0^t f\|_{L^2(\mu)} \leq \|f\|_{L^2(\mu)} e^{-\lambda_{L,0} t}, \quad t \geq 0, f \in L^2(\mu)
\]

holds. See [7] and [13] for a criterion of the weak Poincaré inequality for second quantization Dirichlet forms.

Due to the above fact, in this paper we will only consider $\lambda_T$ and $\lambda_E$. To study these two quantities, we first describe them by using the Dirichlet form.

Since $\pi_\mu$ is a probability measure, by [10, Theorem 3.3] for $\phi \equiv 1$ we conclude that $\lambda_T$ is the largest number such that for any $C_1 > \lambda_T^{-1}$ the defective Poincaré inequality

\[
\pi_\mu(F^2) \leq C_1 \mathcal{E}(F, F) + C_2 \pi_\mu(|F|)^2, \quad F \in \mathcal{D}(\mathcal{E})
\]

holds for some constant $C_2 > 0$. Consequently,

\[(1.3) \quad \lambda_T = \lim_{n \to \infty} \inf \{ \mathcal{E}(F, F) + n \pi_\mu(|F|)^2 : F \in \mathcal{D}(\mathcal{E}), \pi_\mu(F^2) = 1 \}. \]

The quantity $\lambda_T$ is also related to the essential spectrum $\sigma_{\text{ess}}(\mathcal{L})$ of the generator $\mathcal{L}$ associated to $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$. Precisely, we have

\[
\lambda_T \geq \inf \sigma_{\text{ess}}(-\mathcal{L})
\]

and the equality holds provided for some $t > 0$ the operator $P_t$ has an asymptotic density w.r.t. $\pi_\mu$ (see [11, Theorem 3.2.2]).

Next, it is easy to check that $\lambda_E$ is the largest number such that the $L^1$ log-Sobolev inequality

\[
\text{Ent}_{\pi_\mu}(F) := \pi_\mu(F \log F) - \pi_\mu(F) \log \pi_\mu(F)
\]

\[
\leq \frac{1}{\lambda_E} \mathcal{E}(F, \log F), \quad F \in \mathcal{D}(\mathcal{E}), \inf F > 0
\]

holds. That is (see [14, Theorem 1.1]),

\[(1.5) \quad \lambda_E = \inf \left\{ \frac{\mathcal{E}(F, \log F)}{\text{Ent}_{\pi_\mu}(F)} : \inf F > 0, F \in \mathcal{D}(\mathcal{E}), \text{Ent}_{\pi_\mu}(F) > 0 \right\}. \]

We remark that for $F \in \mathcal{D}(\mathcal{E})$ with $\inf F > 0$, one has $\log F \in \mathcal{D}(\mathcal{E})$ so that $\mathcal{E}(F, \log F)$ exists.
Finally, we would like to mention that the log-Sobolev inequality introduced in [2]

\[ \text{Ent}_{\pi_n}(F^2) \leq C\mathcal{E}(F, F), \quad F \in \mathcal{D}(\mathcal{E}) \]  

(1.6)

for some constant \( C > 0 \) implies that \( \lambda_E \geq 4/C \) (see e.g. [14, Theorem 1.2]). But it is easy to see that the second quantization Dirichlet form does not satisfy the log-Sobolev inequality (see [9] and the first page of [12]). Indeed, given n onnegative function \( f \in L^\infty(\mu) \cap L^1(\mu) \cap \mathcal{D}(\mathcal{E}_0) \), applying (1.6) to \( F(\gamma) := e^{\gamma f} \) and using (1.1) we obtain

\[
\int_E (2f e^{2f} - e^{2f} + 1) \, d\mu \leq C \mathcal{E}_0(e^f - 1, e^f - 1).
\]

Replacing \( f \) by \( \log(nf + 1) \) which is once again in \( L^\infty(\mu) \cap L^1(\mu) \cap \mathcal{D}(\mathcal{E}_0) \), we obtain

\[
\frac{1}{n^2 \log n} \int_E \{2(nf + 1)^2 \log(nf + 1) - (nf + 1)^2 + 1\} \, d\mu \leq \frac{C}{\log n} \mathcal{E}_0(f, f).
\]

Letting \( n \to \infty \) we arrive at \( \mu(f^2) \leq 0 \) which is impossible if \( f \) is non-trivial.

It is now the place to state our main result of the paper where \( \lambda_E \) and \( \lambda_T \) are described by using the base Dirichlet form \((\mathcal{E}_0, \mathcal{D}(\mathcal{E}_0))\).

**Theorem 1.1.** We have

\[ \lambda_E = \inf \left\{ \frac{\mathcal{E}_0(f^2, f)}{\mu(f^2)} : f \in \mathcal{D}(\mathcal{E}_0) \cap L^\infty(\mu), \mu(f^2) > 0 \right\} \]  

(1.7)

and

\[ \lambda_{L,0} \leq \lambda_T \leq \lambda_{T,0} := \lim_{n \to \infty} \inf \left\{ \mathcal{E}_0(f, f) + n \mu(|f|)^2 : f \in \mathcal{D}(\mathcal{E}_0), \mu(f^2) = 1 \right\}. \]  

(1.8)

To derive the exact value of these two quantities, let us decompose the Dirichlet form \( \mathcal{E}_0 \) into three parts: the diffusion part, the jump part and the killing part. We will see in the next result that in many cases \( \lambda_E \) is determined merely by the killing term.

Let \( W \) be a nonnegative measurable function on \( E, \mathcal{A} \subset L^1(W\mu) \cap L^\infty(\mu) \) be a linear subspace, \( q \geq 0 \) be a symmetric measurable function on \( E \times E \), and \( \Gamma_1 : \mathcal{A} \times \mathcal{A} \to L^1(\mu) \) be a nonnegative definite bilinear map such that

(i) \( \mathcal{A} \) is dense in \( L^2((1 + W)\mu); \)

(ii) If \( f \in \mathcal{A} \) and \( \phi : \mathbb{R} \to \mathbb{R} \) is Lipschitz continuous with \( \phi(0) = 0 \), then \( \phi(f) \in \mathcal{A}; \)

(iii) For any \( f \in \mathcal{A}, \int_{E \times E} |f(x) - f(y)|^2 q(x, y) \mu(dx) \mu(dy) < \infty; \)
(iv) $\Gamma_1(f, \phi(g)) = \phi'(g)\Gamma_1(f, g)$ holds for any $\phi \in C^1(\mathbb{R})$ with $\phi(0) = 0$ and any $f, g \in \mathcal{A}$.

Consider the following diffusion-jump type quadric form with potential:

$$
\mathcal{E}_0(f, g) := \mu \left( \Gamma_1(f, g) + Wfg \right) + \frac{1}{2} \int_{E \times E} (f(x) - f(y))(g(x) - g(y))q(x, y)\mu(dx)\mu(dy), \quad f, g \in \mathcal{A}.
$$

(1.9)

Assume that $(\mathcal{E}_0, \mathcal{A})$ is closable such that its closure $(\mathcal{E}_0, \mathcal{D}(\mathcal{E}_0))$ is a Dirichlet form on $L^2(\mu)$. When $\Gamma_1 = 0$, $q = 0$ and $W \equiv 1$, the framework goes back to [12] where the Poincaré inequality and the $L^1$ log-Sobolev inequality with constant 1 are proved. The contribution of our next result is to confirm that these inequalities are sharp under a more general framework.

**Corollary 1.2.** Let $(\mathcal{E}_0, \mathcal{D}(\mathcal{E}_0))$ be given in (1.9) such that (i)–(iv) hold.

1. If there exists a sequence of nonnegative functions $\{f_n\}_{n \geq 1} \subset \mathcal{A}$ such that $\{f_n > 0\} \uparrow E$ as $n \uparrow \infty$, then $\lambda_E = \text{ess}_\mu \inf W$.  

2. Let $\Gamma_1 = 0$ and $q = 0$, and let $\mu$ be finite on bounded sets. If $\text{supp} \mu \cap \{W < \varepsilon\}$ is uncountable whenever $\mu(W < \varepsilon) > 0$ (it is the case if $\mu$ does not have atom), then $\lambda_L = \lambda_T = \text{ess}_\mu \inf W$.

To conclude this section, we present below an example to illustrate Corollary 1.2(1).

**Example 1.1.** Let $E$ be a connected (not necessarily complete) Riemannian manifold and $V$ a locally bounded measurable function. Let $\mu(dx) = e^{V(x)}dx$ with $dx$ the volume measure. Then we take $\mathcal{A}$ to be the set of all Lipschitz continuous functions on $E$ with compact supports. It is trivial that conditions (i) and (ii) hold and $\mathcal{A} \subset L^1(W\mu) \cap L^\infty(\mu)$ provided $W$ is locally bounded. Define

$$
\Gamma_1(f, g) = \langle \nabla f, \nabla g \rangle, \quad f, g \in \mathcal{A}.
$$

Then condition (iv) holds. Finally, let $\rho(x, y)$ be the Riemannian distance between $x$ and $y$. If $q(x, y)$ satisfies

$$
(1.10) \quad \int_{K \times E} (\rho(x, y)^2 \wedge 1)q(x, y)\mu(dx)\mu(dy) < \infty
$$

for any compact subset $K$ of $E$, then (iii) is satisfied. Thus, by Corollary 1.2(1) where the required sequence $\{f_n\}_{n \geq 1}$ automatically exists according to the definition of $\mathcal{A}$, we have

$$
\lambda_E = \text{ess}_\mu \inf W.$$

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In particular, let $\mu$ be the Lebesgue measure and $E$ a bounded open domain in $\mathbb{R}^d$ (it is complete under a compatible metric), a typical choice of $q(x, y)$ such that (1.10) holds is
\[
\frac{1}{|x - y|^{d + \alpha}}
\] for $\alpha \in [0, 2)$. Moreover, if $E = \mathbb{R}^d$ and $\mu(dx) = dx$, then (1.10) holds for this $q(x, y)$ with $\alpha \in (1, 2)$.

The remainder of the paper is organized as follows. In Section 2 complete proofs of Theorem 1.1 and Corollary 1.2 are presented; In Section 3 the exponential convergence rates are considered for birth-death type Dirichlet forms on $L^2(\pi \mu)$ with a weighted function on $\Gamma \times E$; and in Section 4 results derived in Section 3 are applied to the path space of Lévy processes by following the line of [12].

2 Proofs of Theorem 1.1 and Corollary 1.2

Proof of (1.7). We first remark that for any $f \in \mathcal{D}(\mathcal{E}_0) \cap L^\infty(\mu)$ one has $e^f - 1 \in \mathcal{D}(\mathcal{E}_0)$, since the function $\phi(r) := e^r - 1$ is locally Lipschitz continuous and $\phi(0) = 0$. Therefore, it suffices to show that for any $\lambda > 0$, the $L^1$ log-Sobolev inequality
\[
\text{Ent}_{\pi \mu}(F) \leq \frac{1}{\lambda} \mathcal{E}(F, \log F), \quad F \in \mathcal{D}(\mathcal{E}), \inf F > 0
\]
is equivalent to
\[
\mu(f e^f - e^f + 1) \leq \frac{1}{\lambda} \mathcal{E}_0(e^f - 1, e^f), \quad f \in \mathcal{D}(\mathcal{E}_0) \cap L^\infty(\mu).
\]

(a) (2.2) implies (2.1). It suffices to prove (2.1) for $F \in \mathcal{D}(\mathcal{E}) \cap L^\infty(\pi \mu)$ with $\inf F > 0$. In this case we have $g_\gamma := F(\gamma + \delta z) - 1 \in \mathcal{D}(\mathcal{E}_0)$ for $\pi_\mu$-a.e. $\gamma \in \Gamma$. Since $\sup F \geq g_\gamma + 1 > 0$, it follows that $f_\gamma := \log(g_\gamma + 1) \in \mathcal{D}(\mathcal{E}_0) \cap L^\infty(\mu)$ for $\pi_\mu$-a.e. $\gamma \in \Gamma$. By (2.2) which holds also for $f \in \mathcal{D}(\mathcal{E}_0) \cap L^\infty(\mu)$, we have
\[
\lambda \int_E (f_\gamma e^{f_\gamma} - e^{f_\gamma} + 1) \, d\mu \leq \mathcal{E}_0(e^{f_\gamma} - 1, e^{f_\gamma}) = \mathcal{E}_0(g_\gamma, \log(g_\gamma + 1)).
\]
On the other hand, by the modified log-Sobolev inequality presented in [12, Theorem 1.1] (note that $\Phi(r) = r \log r$ therein), it holds that
\[
\text{Ent}_{\pi \mu}(F) \leq \int_{\Gamma} \pi_\mu(d\gamma) \int_{E} \{D_z(F \log F)(\gamma) - (1 + \log F(\gamma)) D_z F(\gamma)\} \mu(dz).
\]
Since
\[
D_z F(\gamma) = F(\gamma)(e^{f_\gamma}(z) - 1), \quad \log \frac{F(\gamma + \delta z)}{F(\gamma)} = f_\gamma(z),
\]
Thus, by (2.6) and the dominated convergence theorem we arrive at
\[
D_z(F \log F)(\gamma) - (1 + \log F(\gamma))D_zF(\gamma) = F(\gamma + \delta_z) \log \frac{F(\gamma + \delta_z)}{F(\gamma)} - D_zF(\gamma)
\]
\[
= (D_zF(\gamma)) \left( \log \frac{F(\gamma + \delta_z)}{F(\gamma)} - 1 \right) + F(\gamma) \log \frac{F(\gamma + \delta_z)}{F(\gamma)}
\]
\[
= F(\gamma) \{(e^{f_\gamma} - 1)(f_\gamma - 1) + f_\gamma\} = F(\gamma)(f_\gamma e^{f_\gamma} - e^{f_\gamma} + 1)(z).
\]
Combining this with (2.3) and (2.4), we obtain
\[
\lambda \text{Ent}_{\pi_\mu}(F) \leq \lambda \int_\Gamma F(\gamma) \pi_\mu(d\gamma) \int_E (f_\gamma e^{f_\gamma} - e^{f_\gamma} + 1)d\mu
\]
\[
\leq \int_\Gamma F(\gamma) \mathcal{E}_0(g_\gamma, \log(g_\gamma + 1)) \pi_\mu(d\gamma)
\]
\[
= \int_\Gamma \mathcal{E}_0(D.F, D.\log F)d\pi_\mu = \mathcal{E}(F, \log F).
\]

(b) (2.1) implies (2.2). We first consider \( f \in \mathcal{D}(\mathcal{E}_0) \cap L^\infty(\mu) \cap L^1(\mu) \). Let \( F(\gamma) = e^{\gamma(f)} \).
By (1.1) we have \( F \in L^2(\pi_\mu) \) and
\[
(2.5) \quad \text{Ent}_{\pi_\mu}(F) = \pi_\mu(F) \int_E (fe^f - e^f + 1) d\mu.
\]
Moreover, for any \( \varepsilon > 0 \) one has \( F + \varepsilon \in \mathcal{D}(\mathcal{E}) \), \( \inf(F + \varepsilon) > 0 \) and
\[
\mathcal{E}(F + \varepsilon, \log(F + \varepsilon))
\]
(2.6)
\[
= \int_\Gamma F(\gamma) \left\{ \mathcal{E}_0(e^f - 1, f) + \mathcal{E}_0(e^f - 1, \log \frac{e^{\gamma(f)} + \varepsilon e^{-f}}{e^\gamma + \varepsilon} ) \right\} \pi_\mu(d\gamma).
\]
Since \( \phi(s) := \log \frac{e^{\gamma(f)} + \varepsilon e^{-f}}{e^{\gamma(f)} + \varepsilon} \) satisfies \( \phi(0) = 0 \) and \( |\phi'(s)| \leq 1 \), we get
\[
\left| \mathcal{E}_0(e^f - 1, \log \frac{e^{\gamma(f)} + \varepsilon e^{-f}}{e^{\gamma(f)} + \varepsilon} ) \right| \leq \sqrt{\mathcal{E}_0(e^f - 1, e^f - 1) \mathcal{E}_0(\phi(f), \phi(f))}
\]
\[
\leq \sqrt{\mathcal{E}_0(e^f - 1, e^f - 1) \mathcal{E}_0(f, f)} < \infty.
\]
Thus, by (2.6) and the dominated convergence theorem we arrive at
\[
\lim_{\varepsilon \downarrow 0} \mathcal{E}(F + \varepsilon, \log(F + \varepsilon))
\]
(2.7)
\[
= \int_\Gamma F(\gamma) \mathcal{E}_0(e^f - 1, f) \pi_\mu(d\gamma) = \int_\Gamma F(\gamma) \lim_{\varepsilon \downarrow 0} \mathcal{E}_0(e^f - 1, \log \frac{e^{\gamma(f)} + \varepsilon e^{-f}}{e^\gamma + \varepsilon} ) \pi_\mu(d\gamma)
\]
\[
= \pi_\mu(F) \mathcal{E}_0(e^f - 1, f).
\]
Therefore, first applying \((2.1)\) to \(F + \varepsilon\) then letting \(\varepsilon \downarrow 0\), we obtain \((2.2)\) from \((2.5)\) and \((2.7)\).

In general, for any \(f \in \mathcal{D}(\mathcal{E}_0) \cap L^\infty(\mu)\), let

\[ f_n = \left( f - \frac{1}{n} \right)^+ - \left( f + \frac{1}{n} \right)^- \quad n \geq 1. \]

Then it is easy to see that \(f_n \in \mathcal{D}(\mathcal{E}_0) \cap L^\infty(\mu) \cap L^1(\mu)\) and \(f_n \to f\) in \(\mathcal{D}(\mathcal{E}_0) \cap L^\infty(\mu)\). Therefore, \((2.2)\) holds.

**Proof of \((1.8)\).** Since it is well known that

\[ \lambda_L = \inf \{ \mathcal{E}(F, F) : F \in \mathcal{D}(\mathcal{E}), \pi_\mu(F^2) - \pi_\mu(F)^2 = 1 \}, \]

\((1.2)\) and \((1.3)\) imply \(\lambda_T \geq \lambda_{L,0}\). So, it remains to prove \(\lambda_T \leq \lambda_{T,0}\). If \(0 < \lambda < \lambda_T\), then there exists \(C > 0\) such that

\[ (2.8) \quad \pi_\mu(F^2) \leq \frac{1}{\lambda} \mathcal{E}(F, F) + C\pi_\mu(F)^2, \quad F \in \mathcal{D}(\mathcal{E}), F \geq 0. \]

For any \(f \in \mathcal{D}(\mathcal{E}_0)\), letting \(F(\gamma) = \gamma(|f|)\) we have \(\mathcal{E}(F, F) = \mathcal{E}_0(|f|, |f|) \leq \mathcal{E}_0(f, f)\) and (see e.g. \cite{7} Proof of Lemma 7.2)

\[ \pi_\mu(F^2) = \mu(f^2) + \mu(|f|^2), \quad \pi_\mu(F) = \mu(|f|). \]

Therefore, it follows from \((2.8)\) that

\[ \mu(f^2) \leq \frac{1}{\lambda} \mathcal{E}_0(f, f) + (C - 1)\mu(|f|^2), \quad f \in \mathcal{D}(\mathcal{E}_0). \]

This implies that \(\lambda_{T,0} \geq \lambda\) holds for any \(\lambda < \lambda_T\). Hence, \(\lambda_T \leq \lambda_{T,0}\). \(\square\)

To prove Corollary \((1.2)\), we need the following fundamental lemma. We include a simple proof for completeness.

**Lemma 2.1.** Let \(\nu\) be a measure on \(E\) such that \(\nu\) is finite on bounded sets. If there exists a constant \(c > 0\) such that \(\nu(f^2) \leq c\nu(|f|^2)\) holds for all \(f \in L^2(\nu)\), then \(\text{supp}\nu\) is at most countable. If moreover \(\nu(E) < \infty\) then \(\text{supp}\nu\) is finite.

**Proof.** Since \(\nu\) is finite on bounded sets and \(E\) is separable, there exists a sequence of open sets \(\{G_n\}_{n \geq 1}\) such that \(\bigcup_{n \geq 1} G_n = E\) and \(\nu(G_n) < \infty\) for \(n \geq 1\). Now we fix \(n \geq 1\). Suppose there are \(m\) many different points \(\{x_i\}_{i=1}^m\) in \(\text{supp}\nu \cap G_n\), where \(m \geq 1\). For each \(i\) there exists \(r_i > 0\) such that \(B_i := \{x : d(x, x_i) < r_i\} \subseteq G_n\) and \(\{B_i\}_{i=1}^m\) are disjoint.
Since \( x_i \) is in the support of \( \nu \), we have \( \nu(B_i) > 0 \) for each \( i \in \{1, \cdots, m\} \). Moreover, since
\[
\sum_{i=1}^{m} \nu(B_i) = \nu \left( \bigcup_{i=1}^{m} B_i \right) \leq \nu(G_n) < \infty,
\]
there exists \( i_0 \in \{1, \cdots, m\} \) such that
\[
0 < \nu(B_{i_0}) \leq \frac{1}{m} \nu(G_n).
\]
But applying \( \nu(f^2) \leq cv(|f|^2) \) to \( f = 1_{B_{i_0}} \) we obtain \( \nu(B_{i_0}) \geq \frac{1}{c} \) \( \nu(G_n) \).

Therefore, it follows from (1.7) that \( \lambda_E \geq \text{ess}_\mu \inf W \).

On the other hand, let \( g \in \mathcal{A} \) be a fixed nonnegative function. For any \( n \geq 1 \), applying (1.7) to \( f := 2 \log(n g + 1) \in \mathcal{A} \subset \mathcal{D}(\mathcal{E}_0) \cap L^\infty(\mu) \) and noting that by (iv)
\[
\Gamma_1((ng + 1)^2 - 1, 2 \log(n g + 1)) = 4n^2 \Gamma_1(g, g),
\]
we obtain
\[
\begin{align*}
\lambda_E & \int_E \{(ng + 1)^2 \log [(ng + 1)^2] - (ng + 1)^2 + 1 \} \, d\mu \\
& \leq \mathcal{E}_0((ng + 1)^2 - 1, 2 \log(n g + 1)) \\
& = \int_E \{4n^2 \Gamma_1(g, g) + W(n^2 g^2 + 2ng) \log [(ng + 1)^2] \} \, d\mu \\
& \quad + \int_{E \times E} \{(ng(x) + 1)^2 - (ng(y) + 1)^2\} \left( \log \frac{ng(x) + 1}{ng(y) + 1} \right) q(x, y) \mu(dx) \mu(dy).
\end{align*}
\]
Multiplying both sides by \(\frac{1}{n^2 \log n}\) and letting \(n \to \infty\), by the dominated convergence theorem we arrive at

\[
2\mu(g^2(\lambda_E - W)) \leq \limsup_{n \to \infty} \int_{E \times E} G_n(x, y)q(x, y)\mu(dx)\mu(dy),
\]

where

\[
0 \leq G_n(x, y) := \frac{(n^2g(x) + 1)^2 - (n^2g(y) + 1)^2}{n^2 \log n} \log \frac{n^2g(x) + 1}{n^2g(y) + 1} \leq \frac{(ng(x) + ng(y) + 2\log(n[g(x) \vee g(y)]) + 1)}{n \log n} |g(x) - g(y)|
\]

\[
\leq c|g(x) - g(y)|
\]

for \(\mu\)-a.e. \(x, y \in E\) and some constant \(c > 0\) since \(g \in L^\infty(\mu)\). Thus, by (iii) and the dominated convergence theorem it follows that

\[
\lim_{n \to \infty} \int_{E \times E} G_n(x, y)q(x, y)\mu(dx)\mu(dy)
\]

\[=
\int_{E \times E} \lim_{n \to \infty} G_n(x, y)q(x, y)\mu(dx)\mu(dy)
\]

\[=
\int_{E \times E} (g(x)^2 - g(y)^2)(1_{\{g > 0\}}(x) - 1_{\{g > 0\}}(y))q(x, y)\mu(dx)\mu(dy).
\]

Combining this with (2.9) and using the symmetry of \(q(x, y)\) we get

\[
\mu(g^2(\lambda_E - W)) \leq \frac{1}{2} \int_{E \times E} (g(x)^2 - g(y)^2)(1_{\{g > 0\}}(x) - 1_{\{g > 0\}}(y))q(x, y)\mu(dx)\mu(dy)
\]

\[=
\int_{\{g > 0\} \times \{g \geq 0\}} g(x)^2q(x, y)\mu(dx)\mu(dy), \quad g \in \mathcal{A}, \ g \geq 0.
\]

Next, let \(E_n = \{f_n > 0\}\). For any \(n, m \geq 1\), applying (2.10) to \(g_{nm} := g + f_n/m\) we have

\[
\mu(g_{nm}^2(\lambda_E - W)) \leq \int_{(E_n \cup \{g > 0\}) \times (E_n^c \cap \{g \geq 0\})} g_{nm}(x)^2q(x, y)\mu(dx)\mu(dy)
\]

\[\leq \left(\|g\|_\infty + \frac{\|f_n\|_\infty}{m}\right) \int_{\{g > 0\} \times (E_n^c \cap \{g \geq 0\})} \left\{|g(x) - g(y)| + \frac{1}{m}|f_n(x) - f_n(y)|\right\}q(x, y)\mu(dx)\mu(dy)
\]

\[\leq \frac{1}{m}\|f_n\|_\infty \int_{(E_n \setminus \{g > 0\}) \times (E_n^c \cap \{g \geq 0\})} |f_n(x) - f_n(y)|q(x, y)\mu(dx)\mu(dy).
\]
Consider the quadric form \( \lambda_E \) for any \( E \). According to Propositions 3.3 and 3.4 below, \( \psi \) is Lipschitz continuous with \( \psi(0) = 0 \), it holds that \( \mu(\lambda_E - W) \leq 0 \) for any \( g \in \mathcal{A} \). Noting that \( \mathcal{A} \) is dense in \( L^2((1 + W)\mu) \), then it is trivial to see that \( \lambda_E \leq \text{ess}_\mu \inf W \). This completes the proof.

**Proof of Corollary 1.2 (2).** Let \( \Gamma_1 = 0 \) and \( q = 0 \). Then \( \mathcal{E}_0(f, g) = \mu(Wfg) \). In this case, we have

\[
\lambda_{L,0} = \inf_{f \in L^2(\mu), \mu(f^2) > 0} \frac{\mu(Wf^2)}{\mu(f^2)} = \text{ess}_\mu \inf \mu W.
\]

So, by Theorem 1.1, it suffices to show that \( \lambda_{T,0} \leq \text{ess}_\mu \inf W \). If \( \lambda_{T,0} > \text{ess}_\mu \inf W \) then there exist \( 0 < r < \{\text{ess}_\mu \inf W\}^{-1} \) and \( c > 0 \) such that

\[
(2.11) \quad \mu(f^2) \leq r\mathcal{E}_0(f, f) + c\mu(|f|^2) = r\mu(Wf^2) + c\mu(|f|^2), \quad f \in L^2(\mu)
\]

holds. Take \( \varepsilon \in (0, r^{-1}) \) such that \( \mu(W < \varepsilon) > 0 \). Let \( \mu_\varepsilon = 1_{\{W < \varepsilon\}}\mu \). Using \( f1_{\{W < \varepsilon\}} \) to replace \( f \), we obtain from (2.11) that

\[
\mu_\varepsilon(f^2) \leq c \frac{1}{1 - r\varepsilon}\mu_\varepsilon(|f|^2), \quad f \in L^2(\mu_\varepsilon).
\]

Thus, according to Lemma 2.1, \( \text{supp}\mu_\varepsilon \) is at most countable. This is contradictory to the assumption that \( \text{supp}\mu \cap \{W < \varepsilon\} \) is uncountable.

**3 Birth-death type Dirichlet forms on \( L^2(\pi_\mu) \)**

Let \( \psi \) be a nonnegative measurable function on \( \Gamma \times E \) such that

\[
\psi_\mu(z) := \int_\Gamma \psi(\gamma, z)\pi_\mu(d\gamma) < \infty, \quad \mu\text{-a.e. } z \in E.
\]

Consider the quadric form

\[
\mathcal{E}_\psi(F, G) := \int_{\Gamma \times E} \left( F(\gamma + \delta_z) - F(\gamma) \right) \left( G(\gamma + \delta_z) - G(\gamma) \right) \psi(\gamma, z)\pi_\mu(d\gamma)\mu(dz),
\]

\[
\mathcal{D}(\mathcal{E}_\psi) := \{F \in L^2(\pi_\mu) : \mathcal{E}_\psi(F, F) < \infty\}.
\]

According to Propositions 3.3 and 3.4 below, \( (\mathcal{E}_\psi, \mathcal{D}(\mathcal{E}_\psi)) \) is a conservative symmetric Dirichlet form on \( L^2(\pi_\mu) \), which is regular provided \( \mu(\psi_\mu) < \infty \). Obviously, if \( \psi(\gamma, z) \) does not depend on \( \gamma \), then \( \mathcal{E}_\psi \) goes back to the second quantization Dirichlet form for \( \mathcal{E}_0(f, g) := \mu(\psi fg) \) with \( \mathcal{D}(\mathcal{E}_0) = L^2((1 + \psi)\mu) \).
**Theorem 3.1.** Let $\lambda_L(\psi), \lambda_T(\psi)$ and $\lambda_E(\psi)$ be, respectively, the exponential convergence rates in the $L^2$-norm, the $L^2$-tail norm and entropy for the semigroup associated to $(S^\psi, \mathcal{D}(S^\psi))$.

(1) In general, we have $\text{ess}_{\pi_\mu} \inf \psi \leq \lambda_L(\psi), \lambda_E(\psi) \leq \text{ess}_\mu \inf \psi$. If $\psi(\gamma, z)$ is independent of $\gamma$, then $\lambda_L(\psi) = \lambda_E(\psi) = \text{ess}_\mu \inf \psi$.

(2) Let $\mu$ do not have atom and be finite on bounded sets. Then $\text{ess}_{\pi_\mu} \inf \psi \leq \lambda_T(\psi) \leq \text{ess}_\mu \inf \psi$. If moreover $\psi(\gamma, z)$ does not depend on $\gamma$, then $\lambda_T(\psi) = \text{ess}_\mu \inf \psi$.

**Proof.** (1) Let $\mathcal{E}$ be the second quantization Dirichlet form for $\mathcal{E}_0(f, g) := (\text{ess}_{\pi_\mu} \inf \psi) \mu(f g)$. Obviously, we have $\mathcal{E}_\psi \geq \mathcal{E}$. Combining this with Corollary [1.2] and (1.2) we conclude that $\lambda_L(\psi) \wedge \lambda_E(\psi) \geq \text{ess}_{\pi_\mu} \inf \psi$.

Consequently, it suffices to prove the desired upper bound estimate.

Taking $F(\gamma) = \gamma(f)$ for nonnegative $f \in L^1(\mu) \cap L^\infty(\mu)$, we see that the defective Poincaré inequality

\[
\pi_\mu(F^2) \leq C_1 \mathcal{E}_\psi(F, F) + C_2 \pi_\mu(F)^2
\]

implies that

\[
\mu(f^2) \leq C_1 \mu(\psi_f f^2) + (C_2 - 1) \mu(f)^2.
\]

Thus, (3.1) for $C_2 = 1$ (i.e. the Poincaré inequality) implies that $C_1 \geq (\text{ess}_\mu \inf \psi)^{-1}$. That is, $\lambda_L(\psi) \leq \text{ess}_\mu \inf \psi$.

On the other hand, according to (b) in the proof of (1.4), the $L^1$ log-Sobolev inequality

\[
\pi_\mu(F \log F) \leq \lambda \mathcal{E}_\psi(F, \log F) + \pi_\mu(F) \log \pi_\mu(F)
\]

for $F(\gamma) := e^{\gamma(f)}$ implies that

\[
\mu(f e^f - e^f + 1) \leq \lambda \mu(\psi_{\mu}(e^f - 1) f), \quad f \in L^\infty(\mu) \cap L^1(\mu).
\]

Hence, by the proof of Corollary [1.2] for $W = \psi_{\mu}$, $\Gamma_1 = 0$ and $q = 0$, we conclude that (3.3) implies $\lambda \geq (\text{ess}_\mu \inf \psi)^{-1}$. This means that $\lambda_E(\psi) \leq \text{ess}_\mu \inf \psi$.

(2) Assume that $\mu$ does not have atom and is finite on bounded sets. According to Theorem [1.1] we obtain

\[
\lambda_T \geq \lambda_{L,0} = \text{ess}_{\pi_\mu} \inf \psi.
\]

Finally, by Lemma [2.1] (3.2) for any $C_2 > 0$ implies that $C_1 \geq (\text{ess}_\mu \inf \psi)^{-1}$. Now we conclude that $\lambda_T(\psi) \leq \text{ess}_\mu \inf \psi$ and the proof is completed. □
The remainder of this section devotes to characterizing the form \((\mathcal{E}^\psi, \mathcal{D}(\mathcal{E}^\psi))\). To see that it is a Dirichlet form on \(L^2(\pi_\mu)\), we need the following quasi-invariant property of the map \(\gamma \mapsto \gamma + \delta_z\).

**Lemma 3.2.** If \(A \in \mathcal{F}_\Gamma\) is a \(\pi_\mu\)-null set, then

\[
\tilde{A} := \{ (\gamma, z) \in \Gamma \times E : \gamma + \delta_z \in A \}
\]

is a \((\pi_\mu \times \mu)\)-null set.

**Proof.** We shall make use of the Mecke identity [5] (see also [6]), i.e.

\[
\int_{\Gamma \times E} H(\gamma + \delta_z, z)\pi(\gamma)(dz)\mu(d\gamma) = \int_{\Gamma \times E} H(\gamma, z)\pi(\gamma)(dz)\mu(d\gamma)
\]

holds for any measurable function \(H\) on \(\Gamma \times E\) such that one of the above integrals exists. Applying (3.4) to \(H(\gamma, z) = 1\) \(A(\gamma)\) and noting that \(\pi(\mu)(A) = 0\), we obtain

\[
(\pi_\mu \times \mu)(\tilde{A}) = \int_{\Gamma \times E} 1_A(\gamma + \delta_z)\pi_\mu(d\gamma)(dz)
\]

\[
= \int_{\Gamma \times E} 1_A(\gamma)\gamma(dz)\pi_\mu(d\gamma)
\]

\[
= \int_A \gamma(E)\pi_\mu(d\gamma) = 0.
\]

\[
\square
\]

**Proposition 3.3.** \((\mathcal{E}^\psi, \mathcal{D}(\mathcal{E}^\psi))\) is a conservative symmetric Dirichlet form on \(L^2(\pi_\mu)\) with \(\mathcal{D}(\mathcal{E}^\psi)\) including the family of cylindrical functions

\[
\mathcal{F}_\mu^C := \left\{ \gamma \mapsto f(\gamma(h_1), \cdots, \gamma(h_m)) : m \geq 1, f \in C^1_b(\mathbb{R}^m), h_i \in L^1(\mu) \cap L^\infty(\mu), ||\psi_\mu 1_{h_i \neq 0}||_\infty < \infty \right\},
\]

where \(\| \cdot \|_\infty\) is the \(L^\infty(\mu)\)-norm.

**Proof.** According to Lemma 3.2, for \(F, G \in \mathcal{D}(\mathcal{E}^\psi), \mathcal{E}^\psi(F, G)\) is finite and does not depend on \(\pi_\mu\)-versions of \(F\) and \(G\). Thus, \((\mathcal{E}^\psi, \mathcal{D}(\mathcal{E}^\psi))\) is a well defined positive bilinear form on \(L^2(\pi_\mu)\). Since \(\mathcal{F}_\mu^C\) is dense in \(L^2(\pi_\mu)\) and the normal contractivity property is trivial by the definition of \(\mathcal{E}^\psi\), it remains to show \(\mathcal{D}(\mathcal{E}^\psi) \supset \mathcal{F}_\mu^C\) and the closed property of the form. We prove these two points separately.

(a) Let \(F \in \mathcal{F}_\mu^C\) with

\[
F(\gamma) = f(\gamma(h_1), \cdots, \gamma(h_m)), \ \gamma \in \Gamma,
\]
which is well defined in $L^2(\pi_\mu)$ since $\gamma(K) < \infty$ for $\pi_\mu$-a.e. $\gamma \in \Gamma$ and any compact subset $K$ of $E$. We intend to show that $\mathcal{E}^\psi(F, F) < \infty$. Since $f \in C_0^1(\mathbb{R}^m)$, $h_i \in L^1(\mu) \cap L^\infty(\mu)$, and there exists $n \geq 1$ such that

$$\mu(h_i \neq 0, \psi_\mu > n) = 0, \quad i = 1, \ldots, m,$$

we obtain

$$\mathcal{E}^\psi(F, F) = \int_{\Gamma \times (\bigcup_{i=1}^m \{h_i \neq 0\})} \left[ f(\gamma(h_1) + h_1(z), \ldots, \gamma(h_m) + h_m(z)) - f(\gamma(h_1), \ldots, \gamma(h_m)) \right]^2 \psi(\gamma, z) \pi_\mu(d\gamma) \mu(dz)
\leq ||\nabla f||^2 \int_{\Gamma \times \{\psi \leq n\}} \sum_{i=1}^m h_i(z)^2 \psi_\mu(\gamma, z) \pi_\mu(d\gamma) \mu(dz)
= ||\nabla f||^2 \sum_{i=1}^m \int_{\{\psi \leq n\}} h_i(z)^2 \psi_\mu(z) \mu(dz)
\leq n ||\nabla f||^2 \sum_{i=1}^m \mu(h_i^2) \leq n ||\nabla f||^2 \sum_{i=1}^m ||h_i|| \mu(|h_i|) < \infty.

(b) Let $\{F_n\}_{n \geq 1}$ be an $\mathcal{E}^\psi$-Cauchy sequence. We shall find $F \in \mathcal{D}(\mathcal{E}^\psi)$ such that $\mathcal{E}^\psi_1(F_n - F, F_n - F) := \mathcal{E}^\psi(F_n - F, F_n - F) + \pi_\mu(|F_n - F|^2) \to 0$ as $n \to \infty$. Since $\{F_n\}_{n \geq 1}$ is a Cauchy sequence in $L^2(\pi_\mu)$ (which is complete), there exists $F \in L^2(\pi_\mu)$ such that $F_n \to F$ in $L^2(\pi_\mu)$. Now we can choose a subsequence $\{F_{n_k}\}_{k \geq 1}$ such that $F_{n_k} \to F$ $\pi_\mu$-a.e. By Lemma 3.2 we have $F_{n_k}(\gamma + \delta_z) \to F(\gamma + \delta_z)$ for $(\pi_\mu \times \mu)$-a.e. $(\gamma, z) \in \Gamma \times E$. Therefore, it follows from the Fatou lemma that

$$\mathcal{E}^\psi(F_n - F, F_n - F)
= \int_{\Gamma \times E \times \{n_k \to \infty\}} \lim_{n \to \infty} \inf \left[ (F_n - F_{n_k})(\gamma + \delta_z) - (F_n - F_{n_k})(\gamma) \right]^2 \psi(\gamma, z) \pi_\mu(d\gamma) \mu(dz)
\leq \lim_{n \to \infty} \inf \mathcal{E}^\psi(F_n - F_{n_k}, F_n - F_{n_k}).$$

Since $\{F_n\}_{n \geq 1}$ is an $\mathcal{E}^\psi$-Cauchy sequence and $F_n \to F$ in $L^2(\pi_\mu)$, this implies that

$$\lim_{n \to \infty} \mathcal{E}^\psi_1(F_n - F, F_n - F) = 0.$$

Combining this with the fact that

$$\mathcal{E}^\psi(F, F) \leq 2 \mathcal{E}^\psi(F_n - F, F_n - F) + 2 \mathcal{E}^\psi(F_n, F_n), \quad n \geq 1,$$

we conclude that $F \in \mathcal{D}(\mathcal{E}^\psi)$ and $F_n \to F$ in $\mathcal{D}(\mathcal{E}^\psi)$ as $n \to \infty$. \qed
The next result provides a criterion for the regularity of the Dirichlet form, which ensures the existence of the associated Markov process according to the Dirichlet form theory (see [1, 4]). To this end, we first reduce \( \Gamma \) to a locally compact subspace \( \Gamma_\mu \).

Since \( \Gamma \) is a Polish space such that the set \( \{ \pi_\mu \} \) of single probability measure is tight, we can choose an increasing sequence \( \{ K_n \}_{n \geq 1} \) consisting of compact subsets of \( \Gamma \) such that \( \pi_\mu(K_n) \leq 1/n \) for any \( n \geq 1 \). Then \( \pi_\mu \) has full measure on \( \Gamma_\mu := \bigcup_{n=1}^\infty K_n \), which is a locally compact separable metric space.

**Proposition 3.4.** If \( \psi \in L^1(\pi_\mu \times \mu) \), then \( (\mathcal{E}^\psi, \mathcal{D}(\mathcal{E}^\psi)) \) is a regular Dirichlet form on \( L^2(\Gamma_\mu; \pi_\mu) \).

**Proof.** Since \( \psi \in L^1(\pi_\mu \times \mu) \), we have \( \mathcal{B}_b(\Gamma_\mu) \subset \mathcal{D}(\mathcal{E}^\psi) \), where \( \mathcal{B}_b(\Gamma_\mu) \) is the set of all bounded measurable functions on \( \Gamma_\mu \). In particular, \( C_0(\Gamma_\mu) \subset \mathcal{D}(\mathcal{E}^\psi) \). Thus, it suffices to prove that \( C_0(\Gamma_\mu) \) is dense in \( \mathcal{D}(\mathcal{E}^\psi) \) w.r.t. the \( \mathcal{E}^\psi \)-norm, i.e. for any \( F \in \mathcal{D}(\mathcal{E}^\psi) \), one may find a sequence \( \{ F_n \}_{n \geq 1} \subset C_0(\Gamma_\mu) \) such that \( \mathcal{E}^\psi_1(F_n - F, F_n - F) \to 0 \) as \( n \to \infty \).

Since \( \mathcal{B}_b(\Gamma_\mu) \cap \mathcal{D}(\mathcal{E}^\psi) \) is dense in \( \mathcal{D}(\mathcal{E}^\psi) \) (see e.g. [4, Proposition I.4.17]), we may assume that \( F \in \mathcal{B}_b(\Gamma_\mu) \). Moreover, since \( C_0(\Gamma_\mu) \) is dense in \( L^2(\Gamma_\mu; \pi_\mu) \), we may find a sequence \( \{ F_n \}_{n \geq 1} \subset C_0(\Gamma_\mu) \) such that \( \sup_{n \in \mathbb{N}} \| F_n \|_\infty \leq \| F \|_\infty \) and \( \pi_\mu(\| F_n - F \|^2) \to 0 \) as \( n \to \infty \). Without loss of generality, we assume furthermore that \( F_n \to F \) \( \pi_\mu \)-a.e. By Lemma 3.2 \( F_n(\gamma + \delta_z) \to F(\gamma + \delta_z) \) and \( (F_n - F)^2(\gamma + \delta_z) \leq \| F_n \|_\infty + \| F \|_\infty \|^2 \leq 4\| F \|_\infty^2 \) for \( \pi_\mu \)-a.e. \( (\gamma, z) \in \Gamma \times E \).

Note that (we do not have to distinguish integrals on \( \Gamma_\mu \) and \( \Gamma \) since \( \pi_\mu(\Gamma_\mu) = 0 \))

\[
\mathcal{E}^\psi(F_n - F, F_n - F) \\
\leq 2 \int_{\Gamma \times E} (F_n - F)^2(\gamma + \delta_z) \psi(\gamma, z) \pi_\mu(d\gamma) \mu(dz) \\
+ 2 \int_{\Gamma \times E} (F_n - F)^2(\gamma) \psi(\gamma, z) \pi_\mu(d\gamma) \mu(dz).
\]

Since \( \psi \in L^1(\pi_\mu \times \mu) \), by the dominated convergence theorem we obtain

\[
\lim_{n \to \infty} \mathcal{E}^\psi(F_n - F, F_n - F) = 0.
\]

Combining this with \( \pi_\mu(\| F_n - F \|^2) \to 0 \), we conclude that

\[
\lim_{n \to \infty} \mathcal{E}^\psi_1(F_n - F, F_n - F) = 0,
\]

which completes the proof. \( \square \)

Finally, we consider the generator \( (\mathcal{L}^\psi, \mathcal{D}(\mathcal{L}^\psi)) \) of the Dirichlet form \( (\mathcal{E}^\psi, \mathcal{D}(\mathcal{E}^\psi)) \). For a measurable function \( F \) on \( \Gamma \), let

\[
\mathcal{L}^\psi_b F(\gamma) = \int_E (F(\gamma + \delta_z) - F(\gamma)) \psi(\gamma, z) \mu(dz),
\]

\[
\mathcal{L}^\psi_d F(\gamma) = \int_E 1_{\{\gamma \geq \delta_z\}} (F(\gamma - \delta_z) - F(\gamma)) \psi(\gamma - \delta_z, z) \gamma(dz), \quad \gamma \in \Gamma
\]
provided the integrals above exist.

**Proposition 3.5.** Suppose \( F \in \mathcal{D}(\mathcal{E}^\psi) \) such that \( \mathcal{L}_b^\psi F, \mathcal{L}_d^\psi F \in L^2(\pi_\mu) \). Then \( F \in \mathcal{D}(\mathcal{L}^\psi) \) and \( \mathcal{L}^\psi F = \mathcal{L}_d^\psi F + \mathcal{L}_b^\psi F \). In particular, if \( \mu \) is locally finite and

(3.5) \[ \int_{\Gamma} \psi(\gamma, \cdot)^2 \pi_\mu(d\gamma) \in L^1_{\text{loc}}(\mu), \]

then

\[ \mathcal{D}(\mathcal{L}^\psi) \supset \left\{ \gamma \mapsto f(\gamma(h_1), \cdots, \gamma(h_m)) : m \geq 1, f \in C^1_b(\mathbb{R}^m), h_i \in C_0(E) \right\}. \]

**Proof.** (1) For any \( F \in \mathcal{D}(\mathcal{E}^\psi) \) such that \( \mathcal{L}_b^\psi F, \mathcal{L}_d^\psi F \in L^2(\pi_\mu) \), by the Mecke identity (3.3) for

\[ H(\gamma, z) = F(\gamma)1_{\{\gamma \geq \delta_z\}}(F(\gamma - \delta_z) - F(\gamma))\psi(\gamma - \delta_z, z), \]

we obtain

\[ -\mathcal{E}^\psi(F, F) \]

\[ = \int_{\Gamma \times E} F(\gamma) (F(\gamma + \delta_z) - F(\gamma))\psi(\gamma, z)\pi_\mu(d\gamma)\mu(dz) \]

\[ + \int_{\Gamma \times E} F(\gamma + \delta_z) (F(\gamma) - F(\gamma + \delta_z))\psi(\gamma, z)\pi_\mu(d\gamma)\mu(dz) \]

\[ = \int_{\Gamma \times E} F(\gamma) (F(\gamma + \delta_z) - F(\gamma))\psi(\gamma, z)\pi_\mu(d\gamma)\mu(dz) \]

\[ + \int_{\Gamma \times E} F(\gamma)1_{\{\gamma \geq \delta_z\}}(F(\gamma - \delta_z) - F(\gamma))\psi(\gamma - \delta_z, \gamma(z)\pi_\mu(d\gamma) \]

\[ = \int_{\Gamma} F(\gamma)(\mathcal{L}_b^\psi F + \mathcal{L}_d^\psi F)(\gamma)\pi_\mu(d\gamma). \]

Hence, the first assertion follows.

(2) Let

\[ F(\gamma) = f(\gamma(h_1), \cdots, \gamma(h_m)), \quad \gamma \in \Gamma, \]

where \( f \in C^1_b(\mathbb{R}^m), h_i \in C_0(E) \) and \( m \geq 1 \). By the Schwartz inequality we have

\[ \int_{\Gamma \times E} (F(\gamma + \delta_z) - F(\gamma))^2 \psi(\gamma, z)^2 \pi_\mu(d\gamma)\mu(dz) \]

\[ = \int_{\Gamma \times (\bigcup_{i=1}^m \text{supp} h_i)} \left[ f(\gamma(h_1) + h_1(z), \cdots, \gamma(h_m) + h_m(z)) \right. \]

\[ - f(\gamma(h_1), \cdots, \gamma(h_m))) \left. \right]^2 \psi(\gamma, z)^2 \pi_\mu(d\gamma)\mu(dz) \]
\[\leq \|\nabla f\|^2 \sum_{i=1}^{m} \int_{\Gamma \times (\bigcup_{i=1}^{m} \text{supp} h_i)} h_i(z)^2 \psi(\gamma, z)^2 \pi_{\mu}(d\gamma) \mu(dz)\]
\[\leq \|\nabla f\|^2 \left(\sum_{i=1}^{m} \|h_i\|^2_{\infty}\right) \int_{\Gamma \times (\bigcup_{i=1}^{m} \text{supp} h_i)} \psi(\gamma, z)^2 \pi_{\mu}(d\gamma) \mu(dz)\]
\[< \infty,\]

where the last step is due to (3.5). Then \(L^\psi_b F \in L^2(\pi_{\mu})\) since

\[\|L^\psi_b F\|^2_{L^2(\pi_{\mu})} \leq \int_{\Gamma \times E} (F(\gamma + \delta z) - F(\gamma))^2 \psi(\gamma, \delta z)^2 \pi_{\mu}(d\gamma) \mu(dz) < \infty.\]

On the other hand, using the Mecke identity (3.4) for

\[H(\gamma, z) = 1_{\{\gamma \geq \delta z\}} (F(\gamma - \delta z) - F(\gamma))^2 \psi(\gamma - \delta z, z)^2,\]

we arrive at

\[\|L^\psi_d F\|^2_{L^2(\pi_{\mu})} \leq \int_{\Gamma \times E} 1_{\{\gamma \geq \delta z\}} (F(\gamma - \delta z) - F(\gamma))^2 \psi(\gamma - \delta z, z)^2 \pi_{\mu}(d\gamma) \mu(dz)\]
\[= \int_{\Gamma \times E} (F(\gamma + \delta z) - F(\gamma))^2 \psi(\gamma, z)^2 \pi_{\mu}(d\gamma) \mu(dz) < \infty.\]

Consequently, \(L^\psi_d F \in L^2(\pi_{\mu})\) and the proof is now completed according to the first assertion.

\[\square\]

4 The path space of Lévy processes

Let \(X = \{X_t : t \geq 0\}\) be the Lévy process on \(\mathbb{R}^d\) starting from 0 with a constant drift \(b \in \mathbb{R}^d\) and the Lévy measure \(\nu\), which satisfies \(\nu(\{0\}) = 0\) and

\[\int_{\mathbb{R}^d} (|z|^2 \wedge 1) \nu(dz) < \infty.\]

So, \(X_t\) is generated by

\[\mathcal{L} f = \langle b, \nabla f \rangle + \int_{\mathbb{R}^d} \{f(z + \cdot) - f - \langle \nabla f, z \rangle 1_{\{|z| \leq 1\}}\} \nu(dz),\]

which is well defined for \(f \in C^2_b(\mathbb{R}^d)\).

Let \(\Lambda\) be the distribution of \(X\), which is a probability measure on the path space

\[W := \{w : [0, \infty) \to \mathbb{R}^d \mid w \text{ is right continuous having left limits}\}.\]
It is well known that \( W \) is a Polish space under the Skorokhod metric
\[
\text{dist}(v, w) := \inf \left\{ \delta > 0 : \text{there exist } n \geq 1, 0 = s_0 < s_1 < \cdots < s_n, \text{ and } 0 = t_0 < t_1 < \cdots < t_n \text{ such that } \left| t_i - s_i \right| \leq \delta \text{ and } \sup_{s \in [s_{i-1}, s_i), t \in [t_{i-1}, t_i)} 1 \wedge |v_s - w_t| \leq \delta \text{ hold for all } 1 \leq i \leq n \right\}.
\]

Let \( \tilde{\psi} \in L^1(\Lambda \times \nu \times dt) \) be a nonnegative measurable function on \( W \times (\mathbb{R}^d \setminus \{0\}) \times [0, \infty) \) such that
\[
\tilde{\psi}_{\nu \times dt}(x, t) := \int_W \tilde{\psi}(w, x, t) \Lambda(\text{d}w) < \infty, \quad (\nu \times dt)\text{-a.e. } (x, t) \in (\mathbb{R}^d \setminus \{0\}) \times [0, \infty).
\]
Consider
\[
\tilde{\mathcal{E}}(\tilde{\psi})(F, G) := \int_{W \times \mathbb{R}^d \times [0, \infty)} \left( F(w + x1_{[t, \infty)}) - F(w) \right) \left( G(w + x1_{[t, \infty)}) - G(w) \right) \times \tilde{\psi}(w, x, t) \Lambda(\text{d}w) \nu(\text{d}x) \text{d}t
\]
for
\[
F, G \in \mathcal{D}(\tilde{\mathcal{E}}(\tilde{\psi})) := \{ F \in L^2(\Lambda) : \tilde{\mathcal{E}}(\tilde{\psi})(F, F) < \infty \}.
\]

To apply the known Poincaré inequality on Poisson space, we follow the line of \([12]\) by constructing the Lévy process using Poisson point processes. Let \( E = (\mathbb{R}^d \setminus \{0\}) \times [0, \infty) \), which is a Polish space by taking the following complete metric on \( \mathbb{R}^d \setminus \{0\} \):
\[
\rho(x, y) := \sup \left\{ |f(x) - f(y)| : |\nabla f(z)| \leq \frac{1}{|z|} \vee 1, \quad z \in \mathbb{R}^d \setminus \{0\}, f \in C^1(\mathbb{R}^d \setminus \{0\}) \right\}.
\]

Next, let \( \mu = \nu \times dt \), which is finite on bounded subsets of \( E \) and does not have atom. Let \( \pi_\mu \) be the Poisson measure with intensity \( \mu \), which is a probability measure on the configuration space
\[
\Gamma := \left\{ \sum_{i=1}^n \delta_{(x_i, t_i)} : x_i \in \mathbb{R}^d \setminus \{0\}, t_i \in [0, \infty), 1 \leq i \leq n, n \in \mathbb{Z}_+ \cup \{\infty\} \right\}.
\]
Then on the probability space \((\Gamma, \mathcal{F}_\Gamma, \pi_\mu)\), the Lévy process \( X_t \) can be formulated as (see \([2]\))
\[
X_t(\gamma) = bt + \int_{\{|z|>1\} \times [0, t]} z \gamma(\text{d}z, \text{d}s) + \int_{\{|z|\leq1\} \times [0, t]} z(\gamma - \mu)(\text{d}z, \text{d}s), \quad t \geq 0,
\]
where the second term in the right hand side above is the Stieltjes integral, and the last term is the Itô integral. Therefore,

\[ (4.1) \quad \Lambda = \pi_\mu \circ X^{-1}. \]

Combining this with the Mecke identity (3.4), we obtain

\[ (4.2) \quad \int_W \sum_{\triangle w_t \neq 0} h(w, \triangle w_t, t) \Lambda(dw) \]

\[ = \int_{W \times (\mathbb{R}^d \setminus \{0\}) \times [0, \infty)} h(w + x1_{[t,T]}, x, t) \Lambda(dw) \nu(dx) dt \]

for any non-negative measurable function \( h \) on \( W \times \mathbb{R}^d \times [0, \infty) \). Due to (4.1) and (4.2), arguments used in Section 3 also work for \( (\hat{\delta}^\psi, \mathcal{D}(\hat{\delta}^\psi)), \Lambda \) and \( \hat{\psi} \) in place of \( (\delta^\psi, \mathcal{D}(\delta^\psi)), \pi_\mu \) and \( \psi \) respectively. In particular, letting \( \hat{\lambda}_L(\hat{\psi}), \hat{\lambda}_T(\hat{\psi}) \) and \( \hat{\lambda}_E(\hat{\psi}) \) be, respectively, the exponential convergence rates in the \( L^2 \)-norm, the \( L^2 \)-tail norm and entropy for the semigroup associated to \( (\hat{\delta}^\psi, \mathcal{D}(\hat{\delta}^\psi)) \), we obtain the following result.

**Theorem 4.1.** We have

\[ \text{ess}_{\Lambda_x \mu} \inf \hat{\psi} \leq \hat{\lambda}_L(\hat{\psi}), \hat{\lambda}_T(\hat{\psi}), \hat{\lambda}_E(\hat{\psi}) \leq \text{ess}_\mu \inf \hat{\psi}_\mu, \]

and the equalities hold provided \( \hat{\psi}(w, x, t) \) does not depend on \( w \).

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