Homotopy perturbation method for the Wu-Zhang equation in fluid dynamics

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Abstract: He's homotopy perturbation method, which does not need a small parameter in an equation, is implemented to finding the soliton solutions of the Wu-Zhang equation arising in fluid dynamics. In this method, a homotopy is first constructed for the equation. An initial approximation can be chosen with possible unknown constants that can be determined by imposing the boundary and initial conditions. The results reveal that the method is very effective, convenient and quite accurate to systems of nonlinear equations.

1. Introduction
It is well known that nonlinear phenomena are very important in a variety of scientific fields, especially in fluid mechanics, solid state physics, plasma physics, plasma waves and chemical physics. In many different fields of science and engineering, it is very important to obtain exact or numerical solutions of nonlinear partial differential equations. Searching for exact and numerical solutions, especially, for traveling wave solutions, of nonlinear equations in mathematical physics plays an important role in soliton theory [1,2]. Recently, many new approaches to nonlinear equations were proposed, such as Bäcklund transformation [3], Hirota's bilinear method [3], the homogeneous balance method [4], the Riccati expansion method [5] and the variational iteration method [6,7]. Among all of the analytical methods in open literature, the homotopy perturbation method (HPM) [8], proposed first by He in 1998 and was further developed and improved by He [9-11]. The method yields a very rapid convergence of the solution series in the most cases. The main application of the HPM shows miraculous exactness and convenience compared to other methods [12-15].

In this paper, we apply He's homotopy perturbation method to find the approximation solution and numerical solutions for the following Wu-Zhang (WZ) equation [16]:

\[ u_t + uu_x + vu_y + w_z = 0, \]
\[ v_t + uv_x + vv_y + w_y = 0, \]
\[ w_t + (uw)_x + (vw)_y + \frac{1}{3}(u_{xxx} + u_{xxy} + v_{xyy} + v_{yyy}) = 0, \]  

(1)
which is also called as the (2+1)-dimensional dispersive long wave equation (here \( u \equiv u(x, y, t), v \equiv v(x, y, t) \) and \( w \equiv w(x, y, t) \)). The WZ equation describes nonlinear and dispersive long gravity waves traveling in two horizontal directions on shallow waters of uniform depth. In Eq. (1), \( w - 1 \) is the elevation of the water wave, \( u \) is the surface velocity of water along the \( x \) direction, and \( v \) is the surface velocity of water along the \( y \) direction.

The explicit solutions of Eq. (1) are very helpful for coastal and civil engineers to apply the nonlinear water wave model in harbor and coastal design. Therefore, finding explicit solutions and numerical results of Eq. (1) are of fundamental interest in fluid dynamics. In [17], Chen, Tang and Lou have proven that Eq. (1) has no Painlevé property, and presented two types of periodic wave solutions as well as the corresponding solitary wave solutions by using extended Painlevé truncated expansion method.

2. Basic idea of He's homotopy perturbation method

To illustrate the basic ideas of this method, we consider the following nonlinear differential equation [10]:

\[
A(u) - f(r) = 0, \quad r \in \Omega,
\]

with the boundary conditions

\[
B(u, \partial u / \partial n) = 0, \quad r \in \Gamma,
\]

where \( A \) is a general differential operator, \( B \) is a boundary operator, \( f(r) \) is a known analytical function, \( \Gamma \) is the boundary of the domain \( \Omega \) and \( \partial / \partial n \) denoted differential along the normal drawn outwards from \( \Omega \).

Generally speaking, the operator \( A \) can be divided into two parts \( L \) and \( N \), where \( L \) is linear, but \( N \) is nonlinear. Equation (2) can therefore be rewritten as follows:

\[
L(u) + N(u) - f(r) = 0.
\]

By the homotopy technique, we construct a homotopy \( H(V, p) : \Omega \times [0, 1] \rightarrow \mathbb{R} \) which satisfies

\[
H(V, p) = (1 - p)[L(V) - L(u_0)] + p[A(V) - f(r)] = 0, \quad r \in \Omega,
\]

or

\[
H(V, p) = L(V) - L(u_0) + pL(u_0) + p(N(V) - f(r)) = 0,
\]

where \( p \in [0, 1] \) is an embedding parameter, \( u_0 \) is an initial approximation of Eq. (2), which satisfies the boundary conditions. Hence, it is obvious that

\[
H(V, 0) = L(V) - L(u_0) = 0,
\]

\[
H(V, 1) = A(V) - f(r) = 0,
\]

and the changing process of \( p \) from zero to unity is just that of \( H(V, p) \) from \( L(V) - L(u_0) \) to \( A(V) - f(r) \). In topology, this is called deformation, and \( L(V) - L(u_0) \) and \( A(V) - f(r) \) are called homotopy.

According to the HPM, we can first use the embedding parameter \( p \) as a “small parameter”, and assume that the solution of Eqs. (5) or (6) can be written as a power series in \( p \):
Setting \( p \to 1 \) results in the approximate solution of Eq. (2):

\[
u = \lim_{p \to 1} V = V_0 + V_1 + V_2 + \cdots.
\]

The combination of the perturbation method and the homotopy method is called the homotopy perturbation method (HPM), which has eliminated the limitations of the traditional perturbation methods. On the other hand, this technique can have full advantage of the traditional perturbation techniques.

3. Analysis of the method

To investigate the solitary wave solution of Eq. (1) through the HPM, we first construct a homotopy as follows:

\[
(1 - p)(v_{1,x} - u_{0,x}) + p(v_{1,x} + v_{1,y,x} + v_{2,y} + v_{3,y}) = 0,
\]

\[
(1 - p)(v_{2,x} - u_{0,x}) + p(v_{2,x} + v_{2,y,x} + v_{2,y} + v_{3,y}) = 0,
\]

\[
(1 - p)(v_{3,x} - w_{0,x}) + p(v_{3,x} + (v_{1}v_{3})_{x} + (v_{2}v_{3})_{y} + \frac{1}{3}(v_{1,xx} + v_{1,xy} + v_{2,xy} + v_{2,yy}))) = 0,
\]

and the initial approximations are as follows:

\[
v_{10}(x, y, t) = u_{0}(x, y, t) = u(x, y, 0),
\]

\[
v_{20}(x, y, t) = v_{0}(x, y, t) = v(x, y, 0),
\]

\[
v_{30}(x, y, t) = w_{0}(x, y, t) = w(x, y, 0),
\]

and

\[
v_{1} = v_{10} + pv_{11} + p^{2}v_{12} + p^{3}v_{13} + p^{4}v_{14} + \cdots,
\]

\[
v_{2} = v_{20} + pv_{21} + p^{2}v_{22} + p^{3}v_{23} + p^{4}v_{24} + \cdots,
\]

\[
v_{3} = v_{30} + pv_{31} + p^{2}v_{32} + p^{3}v_{33} + p^{4}v_{34} + \cdots,
\]

where \( v_{ij} \) \((i = 1, 2, 3, j = 1, 2, 3, 4, \cdots)\) are functions of the independent variables \( \{x, y, t\} \) yet to be determined. Substituting Eqs. (12) and (13) into Eq. (11) and arranging the coefficients of \( p \) powers, we have:

\[
(\nu_{11,x} + v_{10}v_{10,x} + v_{20}v_{10,y} + v_{10}v_{10,x} + v_{30,y})p
\]

\[
+(v_{11}v_{10,x} + v_{20}v_{11,y} + v_{10}v_{11,x} + v_{12,x} + v_{21}v_{10,y} + v_{31,x})p^{2}
\]

\[
+(v_{11}v_{10,x} + v_{21}v_{11,y} + v_{13,x} + v_{10}v_{12,x} + v_{12,x} + v_{20}v_{12,y} + v_{31,x} + v_{22}v_{10,y})p^{3} = 0,
\]

\[

\]
In order to obtain the unknowns of \((u_i, v_i, t_i)\) \((i, j = 1, 2, 3)\), we must construct and solve the following system which includes nine equations with nine unknowns, considering the initial conditions of \(u_{i0}(x, y, 0)\) \((i = 1, 2, 3)\), and having the initial approximations of Eq. (12):

\[
\begin{align*}
(v_{21r} + v_{10r}v_{20r} + v_{20r}v_{20r} + v_{20r}v_{30r})p + (v_{11r}v_{20r} + v_{20r}v_{21r} + v_{10r}v_{21r} + v_{22r} + v_{21r} + v_{31r} + v_{30r} + v_{32r} + v_{20r}v_{22r} + v_{21r}v_{22r} + v_{21r}v_{22r} + v_{11r}v_{21r} + v_{22r}v_{22r})p^2 + \cdots & = 0, \\
(v_{10r}v_{30r} + v_{20r}v_{30r} + v_{20r}v_{30r} + v_{30r} + v_{30r} + v_{30r} + v_{10r}v_{30r})p + (v_{11r}v_{30r} + v_{31r} + v_{30r} + v_{31r} + v_{32r} + v_{11r}v_{30r} + v_{21r}v_{30r} + v_{22r} + v_{22r} + v_{32r})p^2 + \cdots & = 0.
\end{align*}
\]

From Eq. (13), if the first three approximations are sufficient, we will obtain:

\[
u(x, y, t) = \lim_{n \to \infty} v_i(x, y, t) \approx \sum_{k=0}^{3} v_{ik}(x, y, t),
\]
\[ v(x, y, t) = \lim_{p \to 1} v_2(x, y, t) \approx \sum_{k=0}^{3} v_{2k}(x, y, t), \]  
\[ w(x, y, t) = \lim_{p \to 1} v_3(x, y, t) \approx \sum_{k=0}^{3} v_{3k}(x, y, t). \]  

4. Application

Firstly, we consider the solutions of Eq. (1) with the initial conditions:

\[ u(x, y, 0) = -\frac{k_1 + k_2 b_0}{k_1} + \frac{2\sqrt{3}}{3} k_1 \tanh(k_1 x + k_2 y), \]
\[ v(x, y, 0) = b_0 + \frac{2\sqrt{3}}{3} k_2 \tanh(k_1 x + k_2 y), \]
\[ w(x, y, 0) = \frac{2}{3} (k_1^2 + k_2^2) \text{sech}^2(k_1 x + k_2 y), \]  
where \( b_0, k_1, k_2 \) and \( k_3 \) are arbitrary constants. To calculate the terms of the homotopy series (16) for \( u(x, y, t), v(x, y, t) \) and \( w(x, y, t) \), we substitute the initial conditions (17) into the system (15) and using Maple, the solutions of the equations can be obtained as follows:

\[ v_{10}(x, y, t) = u(x, y, 0) = -\frac{k_1 + k_2 b_0}{k_1} + \frac{2\sqrt{3}}{3} k_1 \tanh(k_1 x + k_2 y), \]
\[ v_{11}(x, y, t) = \frac{2\sqrt{3}}{3} k_1 k_3 \text{sech}^2(k_1 x + k_2 y)t, \]
\[ v_{12}(x, y, t) = -\frac{2\sqrt{3}}{3} k_1 k_3^2 \tanh(k_1 x + k_2 y)\text{sech}^2(k_1 x + k_2 y)t^2, \]
\[ v_{13}(x, y, t) = \frac{2\sqrt{3}}{9} k_1 k_3^3 (3 \tanh^2(k_1 x + k_2 y) - 1) \text{sech}^2(k_1 x + k_2 y)t^3, \]
\[ v_{20}(x, y, t) = v(x, y, 0) = b_0 + \frac{2\sqrt{3}}{3} k_2 \tanh(k_1 x + k_2 y), \]
\[ v_{21}(x, y, t) = \frac{2\sqrt{3}}{3} k_2 k_3 \text{sech}^2(k_1 x + k_2 y)t, \]
\[ v_{22}(x, y, t) = \frac{2\sqrt{3}}{3} k_2 k_3^2 \tanh(k_1 x + k_2 y)\text{sech}^2(k_1 x + k_2 y)t^2, \]
\[ v_{23}(x, y, t) = \frac{2\sqrt{3}}{9} k_2 k_3^3 (3 \tanh^2(k_1 x + k_2 y) - 1) \text{sech}^2(k_1 x + k_2 y)t^3, \]
\[ v_{30}(x, y, t) = w(x, y, 0) = \frac{2}{3}(k_1^2 + k_2^2) \text{sech}^2(k_1x + k_2y), \]
\[ v_{31}(x, y, t) = -\frac{4}{3}(k_1^2 + k_2^2)k_1 \tanh(k_1x + k_2y) \text{sech}^2(k_1x + k_2y)t, \]
\[ v_{32}(x, y, t) = \frac{2}{3}(k_1^2 + k_2^2)k_3^2(3 \tanh^2(k_1x + k_2y) - 1) \text{sech}^2(k_1x + k_2y)t^2, \]
\[ v_{33}(x, y, t) = -\frac{8}{9}(k_1^2 + k_2^2)k_3^3 \tanh(k_1x + k_2y)(3 \tanh^2(k_1x + k_2y) - 2) \text{sech}^2(k_1x + k_2y)t^3. \]

In this manner, the other components can be obtained. Substituting the above components into Eq. (16) and using the Taylor series, we obtain the closed form solutions as follows:
\[ u(x, y, t) = -\frac{k_1 + k_2b_0}{k_1} + \frac{2\sqrt{3}}{3}k_1 \tanh(k_1x + k_2y + k_3t), \]
\[ v(x, y, t) = b_0 + \frac{2\sqrt{3}}{3}k_2 \tanh(k_1x + k_2y + k_3t), \]
\[ w(x, y, t) = \frac{2}{3}(k_1^2 + k_2^2) \text{sech}^2(k_1x + k_2y + k_3t), \]

where \( b_0, k_1, k_2 \) and \( k_3 \) are arbitrary constants. With the initial conditions (17), the solitary wave solutions of Eq. (1) of kink-type for \( u(x, y, t) \) and \( v(x, y, t) \) and bell-type for \( w(x, y, t) \).

Table 1: The HPM results for \( u(x, y, t), v(x, y, t) \) and \( w(x, y, t) \) for the first thirty approximation in comparison with the analytical solutions when \( b_0 = 0.1, k_1 = 0.05, k_2 = 0.1, k_3 = 0.3 \) and \( t = 2.5 \) for the solitary wave solutions with the initial conditions (17) of Eq. (1), respectively.

| \((x, y)\)          | \(u_{\text{exact}} - u_{\text{homotopy}}\) | \(v_{\text{exact}} - v_{\text{homotopy}}\) | \(w_{\text{exact}} - w_{\text{homotopy}}\) |
|---------------------|-----------------------------------------------|-----------------------------------------------|-----------------------------------------------|
| (0.001,-40)         | 0.340E-7                                      | 0.800E-10                                    | 0.200E-12                                    |
| (0.002,-30)         | 0.400E-8                                      | 0.400E-8                                     | 0.600E-12                                    |
| (0.003,-20)         | 0.750E-7                                      | 0.500E-11                                    | 0.270E-10                                    |
| (0.004,-10)         | 0.270E-7                                      | 0.330E-9                                     | 0.360E-10                                    |
| (0.005,0)           | 0.100E-8                                      | 0.100E-9                                     | 0.670E-10                                    |
| (0.006,10)          | 0.100E-8                                      | 0.100E-9                                     | 0.241E-10                                    |
| (0.007,20)          | 0.710E-8                                      | 0.200E-9                                     | 0.280E-11                                    |
| (0.008,30)          | 0.180E-7                                      | 0.700E-9                                     | 0.110E-12                                    |

5. Numerical results of the HPM
To demonstrate the convergence of the HPM, the results of the numerical example are presented and only few terms are required to obtain accurate solutions. The accuracy of the HPM for the Wu-Zhang equation is controllable, and absolute errors are very small with the present choice of \( x, y \) and \( t \). The results are listed in Table 1. Moreover, as the decomposition method does not require discretization of the variables time and space, it is not affected by computation round off errors and one is not faced with the necessity of large computer memory and time. For an initial condition, we achieve a very good approximation to the partial exact solution by using only 30 terms of the decomposition series,
which shows that the speed of convergence of this method is very fast. It is evident that the overall errors can be made smaller by adding new terms of the decomposition series.

6. Summary and discussion
In this paper, we considered a numerical treatment for the solutions of the WZ equation using the HPM. To the best of our knowledge, this is the first result on the application of the approach to this equation. This method transforms Eq. (1) into a recursive relation.

The obtained numerical results compared with the analytical solution show that the method provides remarkable accuracy. Generally speaking, the HPM provides analytic, verifiable, rapidly convergent approximation that yields insight into the character and the behavior of the solution just as in the closed form solution. It solves nonlinear problems without requiring linearization, or unjustified assumption that may change the problem being solved. The method can also easily be extended to other similar physical equations, with the aid of Maple (or Matlab, Mathematica, etc.), the course of solving nonlinear evaluation equations can be carried out in a computer.

References
[1] Debarnath L 1997 Nonlinear partial differential equations for scientists and engineers. Birkhäuser, Boston
[2] Wazwaz A M 2002 Partial differential equations: Methods and applications. Balkema, Rotterdam
[3] Hirota R 1971 *Phys. Rev. Lett.* 27 1192
[4] Wang M L 1996 *Phys. Lett. A* 213 279
[5] Yan Z Y and Zhang H Q 2001 *Phys. Lett. A* 285 355
[6] He J H 1997 *Commun Nonlinear. Sci.* 2 235
[7] He J H 1998 *Comput. Method. Appl. M.* 167 69
[8] He J H 1999 *Comput. Method. Appl. M.* 178 257
[9] He J H 1998 *Comput. Method. Appl. M.* 167 57
[10] He J H 2000 *Int. J. Nonlinear. Mech.* 35 37
[11] He J H 2004 *Appl. Math. Comput.* 156 527
[12] He J H 2005 *Chaos. Soliton. Fract.* 26 827
[13] He J H 2004 *Appl. Math. Comput.* 156 591
[14] He J H 2006 *Phys. Lett. A* 350 87
[15] Abbasbandy S 2006 *Chaos. Soliton. Fract.* 30 1206
[16] Wu T Y and Zhang J E 1996 On modeling nonlinear long wave. PA: SIAM, Philadelphia
[17] Chen C L, Tang X Y and Lou S Y 2002 *Phys. Rev. E* 66 036605