ON THE FROBENIUS STABLE PART OF WITT VECTOR
COHOMOLOGY

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Abstract. For a proper (not necessarily smooth) variety over a finite field
with q elements, Berthelot-Bloch-Esnault proved a trace formula which com-
putes the number of rational points modulo q in terms of the Witt vector coho-
mology. We show the analogous formula for Witt vector cohomology of finite
length. In addition, we prove a vanishing result for the compactly supported
étale cohomology of a constant p-torsion sheaf on an affine Cohen-Macaulay
variety.

Introduction

Let p be a prime number and let k be a finite field with p^n elements. For a
proper (not necessarily smooth) scheme X over k we know (at least) two congruence
formulas for the number of k-rational points #X(k) modulo powers of p. The first
formula, by Katz [Kat73], states that
\[ \sum_{i \geq 0} (-1)^i \text{Tr}(F^a | H^i(X, \mathcal{O}_X)) \equiv #X(k) \mod p. \]

Here F denotes the absolute Frobenius with its Frobenius linear operation on coho-
mology, but F^a is linear. The second formula is due to Bloch-Illusie in the smooth
case and Berthelot-Bloch-Esnault [BBE07] in general. If WO_X = \varprojlim W_n\mathcal{O}_X
denotes the sheaf of Witt vectors on X with the Frobenius endomorphism F then
\[ \sum_{i \geq 0} (-1)^i \text{Tr}(F^a | H^i(X, WO_X) \otimes \mathbb{Q}) \equiv #X(k) \mod p^n. \]

In this paper we study this trace formula on a finite level, i.e. for W_n\mathcal{O}_X where
n \geq 1 is an integer. For a fixed n we obtain the following result.

Theorem 1 (cf. Corollary 3.7.2). If the cohomology groups H^i(X, W_n\mathcal{O}_X) are free
W_n(k)-modules for all i \geq 0 then
\[ \sum_{i \geq 0} (-1)^i \text{Tr}(F^a | H^i(X, W_n\mathcal{O}_X)) \equiv #X(k) \mod p^{\min\{a,n\}}. \]

However, the assumption of the theorem is non-trivial for n \geq 2. In particular,
it implies that the Frobenius is bijective on H^i(X, W_n\mathcal{O}_X) for all i.

More generally, the purpose of this paper is to study the maximal subpace
of H^*(X, W_n\mathcal{O}_X) (and H^*(X, WO_X)) on which the Frobenius is a bijection, we
call it the Frobenius stable Witt vector cohomology. By using compactifications
we extend the definition of Frobenius stable Witt vector cohomology to separated
schemes of finite type over a perfect field \( k \) (\( \text{char}(k) = p \)), we denote these groups by \( H^*_c(X, W_n \mathcal{O}_X) \) and \( H^*_c(X, W \mathcal{O}_X) \) (Definition 1.3.2). In contrast to usual Witt vector cohomology (for proper schemes), the groups \( H^*_c(X, W \mathcal{O}_X) \) are always finitely generated \( W(k) \)-modules. The first result is a weak Lefschetz-type statement.

**Theorem 2** (cf. Theorem 1.4.9). Let \( k \) be a perfect field of positive characteristic \( p \). Let \( X \) be an affine scheme of finite type over \( k \). Suppose \( X \) is equidimensional of dimension \( d \) and suppose that \( X \) is Cohen-Macaulay. Then

\[
H^i_c(X, \mathcal{O}_X) = 0 \quad \text{for all } i \neq d.
\]

The \( W_n(k) \)-modules \( H^*_c(X, W_n \mathcal{O}_X) \), together with the Frobenius endomorphism, are closely related to compactly supported étale cohomology \( H^*_c(X \times_k \mathbb{F}_p, \mathbb{Z}/p^n) \) equipped with the operation of the Galois group \( \text{Gal}(\bar{k}/k) \) (see Proposition 2.1.1 for a precise statement). Via this correspondence the Theorem 2 asserts that

\[
H^1_{c, \mathrm{rig}}(X, \mathbb{Z}/p) = 0 \quad \text{for all } i \neq d.
\]

We denote by \( K \) the quotient field of \( W(k) \). By using the comparison theorem of [BBE07] between Witt vector cohomology and compactly supported rigid cohomology we prove that \( H^*_c(X, W \mathcal{O}_X)_s \otimes_{W(k)} K \) is the slope zero part of \( H^*_c(X/K) \).

In order to prove Theorem 1 it is not sufficient to work with cohomology groups, but instead we have to work with perfect complexes. We observe that the Frobenius stable Witt vector cohomology of \( X \) has a natural interpretation as Witt vector cohomology. This paper is inspired by the paper [BBE07]. In particular, the definition of Frobenius stable Witt vector cohomology for non-proper varieties follows [BBE07].

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1. **The stable part of Witt vector cohomology**

1.1. Let \( k \) be a perfect field of positive characteristic \( p \). We denote by \( W(k) \) the ring of Witt vectors and by \( W_n(k) \) the ring of Witt vectors of length \( n \). The Frobenius automorphism on \( W(k) \) and \( W_n(k) \) is denoted by \( \sigma \). We denote by \( D \) the Dieudonné ring \( D = W(k)[F, V] \) with relations

\[
F a = \sigma(a) F, \quad a V = V \sigma(a), \quad F V = V F = p,
\]

for all \( a \in W(k) \). For an integer \( n \geq 1 \), we define \( D_n := D/p^n \).

By a \( D \)-module we will mean a left \( D \)-module. Given a \( D \)-module \( M \) we obtain a \( \sigma \)-linear (and a \( \sigma^{-1} \)-linear) map \( F : M \to M \) (and \( V : M \to M \), respectively).

For a \( W(k) \)-module \( M \) we define

\[
\sigma_* M := W(k) \otimes_{\sigma^{-1}, W(k)} M.
\]
In other words, the multiplication in $\sigma_*(M)$ is twisted by $\sigma$: $a \otimes m = 1 \otimes \sigma(a)m$. If $M$ is equipped with a $\sigma$-linear map $F$ then $\sigma \otimes F$ defines a $\sigma$-linear map $\sigma_* M \to \sigma_* M$. Similarly, if $M$ is a $D$-module then $\sigma_* M$ inherits a $D$-module structure.

**Definition 1.1.1.** Let $M$ be a $W(k)$-module together with a $\sigma$-linear map $F : M \to M$. We define

$$M_s := \bigcap_{a \geq 1} F^a(M).$$

We call $M_s$ the (Frobenius) stable part of $M$. We call $M$ stable if $M_s = M$.

Obviously, $F(M_s) \subset M_s$, and thus $M_s$ is equipped with the map $F$. If $M$ is a $D$-module then $M_s$ is a $D$-module.

**Proposition 1.1.2.** Let $M, M', M''$ be $W(k)$-modules equipped with $\sigma$-linear maps $F, F', F''$. Let $n \geq 1$ be an integer.

(i) The equality $(\sigma_*(M))_s = \sigma_*(M_s)$ holds.

(ii) If $M$ is a finite and free $W_n(k)$-module then $M_s$ is a finite and free $W_n(k)$-module.

(iii) If $M$ is a finite $W(k)$-module then $M_s \xrightarrow{\cong} \lim_{\leftarrow n} (M/p^n)_s$ is an isomorphism.

(iv) If $M$ is a finite $W(k)$-module then $F : M_s \to M_s$ is bijective.

(v) Suppose that

$$0 \to M' \to M \to M'' \to 0$$

is a short exact sequence (compatible with the $\sigma$-linear maps) and $M$ is a finite $W(k)$-module. Then

$$0 \to (M'_s) \to M_s \to (M''_s) \to 0$$

is exact.

**Proof.** The assertion (i) is obvious. The statement (ii) follows from the elementary fact stated in Lemma 1.1.3. By assumption $k$ is perfect and this implies that $M_s$ and $M_{\text{nil}}$ are sub-$W_n(k)$-modules. Therefore $M_s$ is a projective $W_n(k)$-modules and thus free.

For (iii): The $\sigma$-linear map on $M/p^n$ is induced by $M$. If $M$ is a finite $W(k)$-module then $M \to \lim_{\leftarrow n} M/p^n$ is an isomorphism. We get inclusions

$$M_s \subset \lim_{\leftarrow n} (M/p^n)_s \subset M.$$

By Lemma 1.1.3 $F$ is bijective on $\lim_{\leftarrow n} (M/p^n)_s$, thus $\lim_{\leftarrow n} (M/p^n)_s \subset M_s$.

For (iv): Follows from (iii) and the fact that $F$ is bijective on $(M''/p^n)_s$.

For (v): We only need to prove that $M_s \to (M''_s)$ is surjective. Consider the exact sequence

$$0 \to K_n \to M/p^n \to M''/p^n \to 0.$$

Lemma 1.1.3 implies that

$$0 \to (K_n)_s \to (M/p^n)_s \to (M''/p^n)_s \to 0$$

is again exact. The projective system $(K_n)$ satisfies the Mittag-Leffler condition, and taking the limit $\lim_{\leftarrow n}$ implies the claim. $\square$
**Lemma 1.1.3.** Let $M$ be an abelian group. Let $F : M \to M$ be an endomorphism. Suppose that there are integers $n, m \geq 1$ such that $\ker(F^n) = \ker(F^{n+1})$ and $F^m(M) = F^{m+1}(M)$. Then there exists a unique decomposition

$$M = M_s \oplus M_{\text{nil}}$$

with $F(M_s) \subset M_s$, $F(M_{\text{nil}}) \subset M_{\text{nil}}$, such that the restriction of $F$ to $M_s$ is bijective, and the restriction of $F$ to $M_{\text{nil}}$ is nilpotent. Moreover,

$$M_s = F^{n+m}(M), \quad M_{\text{nil}} = \ker(F^{n+m}).$$

**Proof.** We leave the proof to the reader. \qed

1.2. Let $X$ be a separated scheme of finite type over $k$, we denote by $W_n(\mathcal{O}_X)$ the Witt sheaf of rings of $X$ of length $n$. We have the Frobenius endomorphism

$$F : W_n(\mathcal{O}_X) \to W_n(\mathcal{O}_X), \quad (a_1, \ldots, a_n) \mapsto (a_1^p, \ldots, a_n^p).$$

For $X = \text{Spec}(k)$ we keep the notation $\sigma$ for $F$. Of course, $W_n(\mathcal{O}_X)$ is an $W_n(k)$ module. We have the Verschiebung

$$V : W_n(\mathcal{O}_X) \to W_n(\mathcal{O}_X), \quad (a_1, \ldots, a_n) \mapsto (0, a_1, \ldots, a_{n-1}),$$

and the relation $V \circ F = F \circ V = p$.

If $X$ is proper then the cohomology groups $H^i(X, W_n(\mathcal{O}_X))$ are finite $W_n(k)$ modules. Moreover, $F$ (and $V$) induces a $\sigma$-linear map (a $\sigma^{-1}$-linear map, respectively,

$$H^i(X, W_n(\mathcal{O}_X)) \to H^i(X, W_n(\mathcal{O}_X)), \quad \text{for all } i.$$ 

For simplicity we write $F = H^i(F)$ and $V = H^i(V)$.

1.3. Let $X$ be a separated (not necessarily proper) scheme of finite type over $k$. We may choose a compactification $Y$ of $X$. Let $Z = Y \setminus X$ and choose a ideal sheaf $\mathcal{I}$ for $Z$. Denoting

$$W_n(\mathcal{I}) := \{(a_1, \ldots, a_n) \in W_n(\mathcal{O}_X); a_i \in \mathcal{I}\}$$

we get by restriction a Frobenius and a Verschiebung endomorphism.

The groups $H^i(Y, W_n(\mathcal{I}))$ are finite $W_n(k)$-modules and equipped with the $\sigma$-linear map $F$ and the $\sigma^{-1}$-linear map $V$ satisfying $FV = VF = p$. Thus they are $D_n$-modules.

In general, $H^*(Y, W_n(\mathcal{I}))$ depends on the choice of the ideal and the compactification. However, we will see that the stable part does not.

**Lemma 1.3.1.** Let $n \geq 1$ be an integer.

(i) If $\mathcal{I}'$ is another ideal for $Z$ such that $\mathcal{I}' \subset \mathcal{I}$ then

$$H^i(Y, W_n(\mathcal{I}'))_s \cong H^i(Y, W_n(\mathcal{I}))_s$$

is an isomorphism of $D_n$-modules for all $i$.

(ii) Suppose $Y'$ is another compactification of $X$ with a morphism $g : Y' \to Y$ which induces the identity on $X$. Then

$$g^*: H^i(Y, W_n(\mathcal{I}))_s \to H^i(Y', W_n(\mathcal{O}_{Y'}))_s$$

is an isomorphism of $D_n$-modules for all $i$. 

Proof. There is a short exact sequence

\begin{equation}
0 \to W_{n-1}(\mathcal{I}) \xrightarrow{V} W_n(\mathcal{I}) \xrightarrow{R^{n-1}} \mathcal{I} \to 0,
\end{equation}

where \( V \) is the Verschiebung \((a_1, \ldots, a_{n-1}) \mapsto (0, a_1, \ldots, a_{n-1})\) and \( R^{n-1} \) is the projection \((a_1, \ldots, a_n) \mapsto a_1\). This gives a long exact sequence of \( D_n \)-modules:

\[
\ldots \to H^{i-1}(Y, \mathcal{I}) \to \sigma_* H^i(W_{n-1}(\mathcal{I})) \xrightarrow{V} H^i(W_n(\mathcal{I})) \xrightarrow{R^{n-1}} H^i(\mathcal{I}) \to \ldots
\]

The maps in (i) and (ii) are compatible with this long exact sequence, thus it is sufficient to show the assertion in the case \( n = 1 \).

For (i). For \( a \geq 1 \) the morphism \( \text{Frob}^a : \mathcal{I} \to \text{Frob}_\mathcal{I}^a \mathcal{I}, r \mapsto r^p \), factors through

\begin{equation}
\mathcal{I} \xrightarrow{\phi^a} \text{Frob}_\mathcal{I}^a (\mathcal{I}^p) \xrightarrow{\sigma_*} \text{Frob}_\mathcal{I}^a \mathcal{I},
\end{equation}

which induces a morphism of \( D_1 \)-modules (\( V \) acts as zero)

\[
\phi^a : H^1(Y, \mathcal{I}) \to \sigma_* H^1(Y, \mathcal{I}^p).
\]

Since \( F \) is bijective on the stable part,

\[
H^1(Y, \mathcal{I}) \xrightarrow{\phi^a} \sigma_* H^1(Y, \mathcal{I}^p) \xrightarrow{\sigma_* (\text{inclusion})} \sigma_* H^1(Y, \mathcal{I})
\]

is surjective. Since \( \mathcal{I}^p \subset \mathcal{I}' \) for some \( a \) this implies the surjectivity of

\begin{equation}
H^1(Y, \mathcal{I}') \to H^1(Y, \mathcal{I})_s.
\end{equation}

We also obtain that \( H^1(Y, \mathcal{I}^p)_s \to H^1(Y, \mathcal{I}')_s \) is surjective when \( \mathcal{I}^p \subset \mathcal{I}' \).

In order to prove the injectivity of \( \text{Frob}^a \) it is sufficient to prove that \( \phi^a \) is surjective. This follows from the commutative diagram

\[
\begin{array}{ccc}
H^1(Y, \mathcal{I})_s & \xrightarrow{\phi^a} & \sigma_* H^1(Y, \mathcal{I}^p)_s \\
\downarrow \text{inclusion} & & \downarrow \sigma_* \phi^a \\
H^1(Y, \mathcal{I}^p)_s & \xrightarrow{F^a} & \sigma_* H^1(Y, \mathcal{I}^p)_s
\end{array}
\]

and the surjectivity of \( F^a \).

For (ii). We prove the claim in two steps. In the first step we show that

\begin{equation}
H^1(Y, \mathcal{I})_s \xrightarrow{\sim} H^1(Y, g_* \mathcal{I} \mathcal{O}_{Y'})_s,
\end{equation}

and in the second step we prove

\begin{equation}
H^1(Y, g_* \mathcal{I} \mathcal{O}_{Y'})_s \xrightarrow{\sim} H^1(Y, \mathcal{I} \mathcal{O}_{Y'})_s.
\end{equation}

Defining

\[
K_n = \ker(\mathcal{I}^n \to g_* \mathcal{I}^n \mathcal{O}_{Y'}), \quad C_n = \text{coker}(\mathcal{I}^n \to g_* \mathcal{I}^n \mathcal{O}_{Y'}),
\]

the \( \oplus_{n \geq 0} \mathcal{I}^n \) modules \( \oplus_{n \geq 0} K_n \) and \( \oplus_{n \geq 0} C_n \) are finitely generated [Gro61, 3.3.1]. Thus there exists \( d \) such that for all \( n \geq d \) the following maps are surjective

\[
\mathcal{I} \otimes \mathcal{O}_Y, K_n \to K_{n+1}, \quad \mathcal{I} \otimes \mathcal{O}_Y, C_n \to C_{n+1}.
\]

Since \( K_n, C_n \) are supported in \( Y \setminus X \) (for \( n \leq d \)), we can choose an integer \( e \geq 1 \) such that \( \mathcal{I}^e K_n = 0 = \mathcal{I}^e C_n \) for all \( n \). It follows that for \( m = e + d \) the maps

\begin{equation}
K_{n+m} \to K_n, \quad C_{n+m} \to C_n
\end{equation}

vanish for all \( n \).
From the commutative diagram

\[
\begin{array}{ccc}
\text{Frob}^a \mathcal{I} & \rightarrow & \text{Frob}^a g_* \mathcal{O}_{Y'} \\
\uparrow & & \uparrow \\
\text{Frob}^a \mathcal{T}^a & \rightarrow & \text{Frob}^a g_* \mathcal{T}^a \mathcal{O}_{Y'} \\
\phi^a & \rightarrow & g_* \phi^a \\
\downarrow & & \downarrow \\
\mathcal{I} & \rightarrow & g_* \mathcal{I} \mathcal{O}_{Y'} \\
\end{array}
\]

we obtain induced morphisms \( \text{Frob}^a : K_1 \rightarrow \text{Frob}^a K_1 \) and \( \text{Frob}^a : C_1 \rightarrow \text{Frob}^a C_1 \) which factor through \( \text{Frob}^a(K_{p^n}) \subset \text{Frob}^a(K_1) \) and \( \text{Frob}^a(C_{p^n}) \rightarrow \text{Frob}^a(C_1) \), respectively. Thus Frob is nilpotent on \( K_1 \) and \( C_1 \) by 1.3.1.6 This proves 1.3.1.4.

For 1.3.1.5 it is sufficient to show that Frob acts as a nilpotent endomorphism on \( R^i g_* (\mathcal{I} \mathcal{O}_{Y'}) \) for \( i > 0 \). Then the assertion follows from the Leray spectral sequence. In view of [D66, Appendix-Proposition 5] the map \( R^i g_* (\mathcal{I} \mathcal{O}_{Y'}) \rightarrow R^i g_* (\mathcal{I} \mathcal{O}_{Y'}) \), induced by \( \mathcal{T}^a \mathcal{O}_{Y'} \subset \mathcal{I} \mathcal{O}_{Y'} \), vanish if \( a \) is sufficiently large. Thus the factorization 1.3.1.2 implies the claim.

For two ideals of \( Z = Y' \setminus X \) we can take the intersection, and two compactifications can be dominated by a third one, thus Lemma 1.3.1 shows that \( H^i(Y, W_n(\mathcal{I}))_s \) is independent of the choice of \( \mathcal{I} \) and \( Y \).

**Definition 1.3.2.** Let \( Y \) be a compactification of \( X \) and \( \mathcal{I} \) an ideal for \( Y' \setminus X \). For all \( i \) we denote by \( H^i_c(X, W_n \mathcal{O}_X)_s \) the \( D_n \)-module \( H^i(Y, W_n(\mathcal{I}))_s \). If \( X \) is compact we will omit the index \( c \).

1.4. Let \( X \) be of finite type and separated over \( k \). Choose a compactification \( Y \) and an ideal \( \mathcal{I} \) for \( Y' \setminus X \). The restriction of the Frobenius to the nilradical \( \mathcal{N} \) of \( \mathcal{O}_Y \) is nilpotent and from the short exact sequence

\[
0 \rightarrow \mathcal{I} \cap \mathcal{N} \rightarrow \mathcal{I} \rightarrow \mathcal{I} \mathcal{O}_{\text{red}} \rightarrow 0
\]

we conclude

\[
H^i_c(X, W_n \mathcal{O}_X)_s \xrightarrow{\cong} H^i_c(X_{\text{red}}, W_n(\mathcal{O}_{X_{\text{red}}}))_s \quad \text{for all } i, n.
\]

Let \( U \subset X \) be an open subset and choose an ideal \( \mathcal{J} \) for \( Y \setminus U \). In view of the short exact sequence

\[
0 \rightarrow \mathcal{J} \rightarrow \mathcal{I} \rightarrow \mathcal{I}/\mathcal{J} \rightarrow 0
\]

we get a long exact sequence

\[
(1.4.0.1) \quad \cdots \rightarrow H^i_c(U, W_n \mathcal{O}_U)_s \rightarrow H^i_c(X, W_n \mathcal{O}_X)_s \rightarrow H^i_c(X \setminus U, W_n \mathcal{O}_{X \setminus U})_s \rightarrow H^{i+1}_c(U, W_n \mathcal{O}_U)_s \rightarrow \cdots
\]

Let \( U_1, U_2 \) be open sets of \( X \) such that \( X = U_1 \cup U_2 \). Choose ideals \( \mathcal{I}_1 \) and \( \mathcal{I}_2 \) for \( Y \setminus U_1 \) and \( Y \setminus U_2 \), respectively. Then \( \mathcal{I}_1 + \mathcal{I}_2 \) is an ideal for \( Y \setminus X \) and \( \mathcal{I}_1 \cap \mathcal{I}_2 \) is an ideal for \( Y \setminus (U_1 \cap U_2) \). From the short exact sequence

\[
0 \rightarrow W_n(\mathcal{I}_1 \cap \mathcal{I}_2) \rightarrow W_n(\mathcal{I}_1) \oplus W_n(\mathcal{I}_2) \rightarrow W_n(\mathcal{I}_1 + \mathcal{I}_2) \rightarrow 0
\]
we get a long exact sequence

\[(1.4.0.2)\]
\[
\ldots \to H_i^s(U_1 \cap U_2, W_n \mathcal{O}_{U_1 \cap U_2})_s \to H_i^s(U_1, W_n \mathcal{O}_{U_1})_s \oplus H_i^s(U_2, W_n \mathcal{O}_{U_2})_s \to H_i^s(X, W_n \mathcal{O}_X)_s \to H_i^{s+1}(U_1 \cap U_2, W_n \mathcal{O}_{U_1 \cap U_2})_s \to \ldots
\]

**Definition 1.4.1.** Let \(X\) be a separated scheme of finite type over a perfect field \(k\) of positive characteristic \(p\). By taking the inverse limit we define

\[H_i^s(X, W \mathcal{O}_X)_s = \lim_{n} H_i^s(X, W_n \mathcal{O}_X)_s \quad \text{for all} \quad i.\]

If \(X\) is compact we will omit the index \(c\).

We have a natural inclusion

\[(1.4.1.1)\]
\[H_i^s(X, W \mathcal{O}_X)_s = \lim_{n} H^i(Y, W_n \mathcal{I})_s \subset \lim_{n} \mathcal{H}^i(Y, W_n \mathcal{I}) = \mathcal{H}^i(Y, W \mathcal{I}),\]

and after inverting \(p\) we obtain in the notation of [BBE07, 2.13]:

\[H_i^s(X, W \mathcal{O}_X)_s \otimes W(k) K \subset \mathcal{H}^i(Y, W \mathcal{I}) \otimes W(k) K =: H_i^s(X, W \mathcal{O}_X)_s.\]

In general, the \(D\)-module \(\mathcal{H}^i(Y, W \mathcal{I})\) is not a finite \(W(k)\)-module. However, the \(K\)-vector space \(\mathcal{H}^i(Y, W \mathcal{I}) \otimes W(k) K\) is finite dimensional and independent of the choice of the compactification \(Y\) and the ideal \(\mathcal{I}\) (BBE07 §2).

**Proposition 1.4.2.** Via the inclusion \((1.4.1.1)\) we get \(H_i^s(X, W \mathcal{O}_X)_s = \mathcal{H}^i(Y, W \mathcal{I})_s\) for all \(i\).

**Proof.** The inclusion \(\subset\) follows immediately from the definitions, and \(\supset\) follows since \(F\) is bijective on \(H_i^s(X, W \mathcal{O}_X)_s\).

**Proposition 1.4.3.** For all \(n \geq 1\) there is a long exact sequence of \(D\)-modules

\[
\ldots \to H_i^s(X, W \mathcal{O}_X)_s \xrightarrow{F^n} H_i^{s+1}(X, W \mathcal{O}_X)_s \to H_i^{s+1}(X, W \mathcal{O}_X)_s \to \ldots
\]

**Proof.** For all \(m > n\) we have a short exact sequence (notation as in Definition 1.3.2)

\[(1.4.3.1)\]
\[0 \to W_{m-n}(\mathcal{I}) \xrightarrow{V^n} W_m(\mathcal{I}) \xrightarrow{R^{m-n}} W_n(\mathcal{I}) \to 0,
\]

with

\[V^n(a_0, \ldots, a_{m-n-1}) = (0, \ldots, 0, a_1, \ldots, a_{m-n-1}),\]

\[R^{m-n}(a_0, \ldots, a_{m-1}) = (a_0, \ldots, a_{m-1}).\]

Now, \((1.4.3.1)\) induces a long exact sequence of \(D_m\)-modules

\[
\ldots \to \sigma^n \mathcal{H}^i(Y, W_{m-n}(\mathcal{I})) \xrightarrow{V^n} \mathcal{H}^i(Y, W_{m}(\mathcal{I})) \to \mathcal{H}^i(Y, W_n(\mathcal{I})) \to \ldots
\]

Taking stable part and using the isomorphism

\[F^n : \mathcal{H}^i(Y, W_{m-n}(\mathcal{I}))_s \xrightarrow{\cong} \sigma^n \mathcal{H}^i(Y, W_{m-n}(\mathcal{I}))_s,
\]

we obtain a long exact sequence

\[(1.4.3.2)\]
\[
\ldots \to H_i^s(Y, W_{m-n}(\mathcal{I}))_s \xrightarrow{V^n \circ F^n} H_i^s(Y, W_m(\mathcal{I}))_s \to H_i^s(Y, W_n(\mathcal{I}))_s \to \ldots
\]
Since all groups are finite $W_m(k)$-modules the projective limit $\varprojlim_{n} H^i_c(X, W_n\mathcal{O}_X)$ is exact, and $V \cdot F = p$ implies the claim. \hfill\qed

**Corollary 1.4.4.** Let $X$ be a separated scheme of finite type over a perfect field $k$ of positive characteristic $p$. The $W(k)$-module $H^i_c(X, W\mathcal{O}_X)_s$ is finitely generated for all $i$.

**Proof.** Proposition 1.4.3 implies that
\[ H^i_c(X, W\mathcal{O}_X)_s/p^nH^i_c(X, W\mathcal{O}_X)_s \subset H^i_c(X, W_n\mathcal{O}_X)_s \]
for all $i, n$. Therefore
\[ H^i_c(X, W\mathcal{O}_X)_s = \lim_{n} \left( H^i_c(X, W\mathcal{O}_X)_s/p^nH^i_c(X, W\mathcal{O}_X)_s \right), \]
and the assertion follows since $H^i_c(X, \mathcal{O}_X)_s$ is a finite dimensional $k$-vector space. \hfill\qed

**Remark 1.4.5.** Corollary 1.4.4 follows immediately from [Ser58 §5, Proposition 3]. We include a proof for the convenience of the reader.

**Proposition 1.4.6.** Suppose $X$ is proper over $k$. Suppose that $H^i(X, \mathcal{O}_X)_s = H^i(X, \mathcal{O}_X)$ for all $i$. Then $H^i(X, W_n\mathcal{O}_X)_s = H^i(X, W_n\mathcal{O}_X)$ for all $i$ and $n \geq 1$. In particular, $H^i(X, W\mathcal{O}_X)_s = H^i(X, W\mathcal{O}_X)$ for all $i$.

**Proof.** The equality $H^i(X, W_n\mathcal{O}_X)_s = H^i(X, W_n\mathcal{O}_X)$ for all $i$, follows by induction on $n$ from the short exact sequence
\[ 0 \to W_{n-1}(\mathcal{O}_X) \xrightarrow{V} W_n(\mathcal{O}_X) \xrightarrow{R^{n-1}} \mathcal{O}_X \to 0. \]
\hfill\qed

**Proposition 1.4.7.** Let $X$ be a separated scheme of finite type over a perfect field $k$ of positive characteristic $p$. Let $n \geq 1$. The following statements are equivalent:

(i) For all $i$, $H^i_c(X, W_n\mathcal{O}_X)_s$ is a free $W_n(k)$-module.

(ii) For all $i$, $H^i_c(X, W_n\mathcal{O}_X)_s$ is a free $W_n(k)$-module of rank $\dim_k H^i_c(X, \mathcal{O}_X)_s$.

(iii) For all $i$, the map $H^i_c(X, W_n\mathcal{O}_X)_s \to H^i_c(X, \mathcal{O}_X)_s$ is surjective.

**Proof.** Obviously (ii) implies (i). Now, suppose (i) holds. It is clear from the long exact sequence 1.4.3.2 that
\[ \text{length}(H^i_c(X, W_m\mathcal{O}_X)_s) \leq m \cdot \dim_k H^i_c(X, \mathcal{O}_X)_s \]
for all $m \leq n$. If the canonical map $H^i_c(X, W_n\mathcal{O}_X)_s/p \to H^i_c(X, \mathcal{O}_X)_s$ is surjective (which holds for $i \geq \dim X$) then equality holds in (1.4.7.1) for $m = n$, and thus
\[ H^i_c(X, W_{n-1}\mathcal{O}_X)_s \xrightarrow{V \cdot F} H^i_c(X, W_n\mathcal{O}_X)_s \]
is injective. It follows that $H^i_c(X, W_n\mathcal{O}_X)_s/p \to H^i_c(X, \mathcal{O}_X)_s$ is surjective. By descending induction on $i$ (starting from $i = \dim X$) we see that (ii) holds.

Now, suppose that (iii) holds. By induction on $n$ we may suppose that for all $i$, $H^i_c(X, W_{n-1}\mathcal{O}_X)_s$ is free of rank $\dim_k H^i_c(X, \mathcal{O}_X)_s$. Since $H^i_c(X, W_{n-1}\mathcal{O}_X)_s/p \to H^i_c(X, \mathcal{O}_X)_s$ is surjective (by assumption (iii)) it is an isomorphism. It follows that
\[ H^i_c(X, W_n\mathcal{O}_X)_s \to H^i_c(X, W_{n-1}\mathcal{O}_X)_s \]
is surjective. From the long exact sequence 1.4.3.2 we get short exact sequences
\[ 0 \to H^i_c(X, W_{n-1}\mathcal{O}_X)_s \xrightarrow{V \cdot F} H^i_c(X, W_n\mathcal{O}_X)_s \to H^i_c(X, \mathcal{O}_X)_s \to 0. \]
The surjectivity of $H_i(X, W_n \mathcal{O}_X)_s$ and $V \circ F = p$ on $H_i^s(X, W_n \mathcal{O}_X)_s$ implies
\[
H_i^s(X, W_{n-1} \mathcal{O}_X)_s = p \cdot H_i^s(X, W_n \mathcal{O}_X)_s
\]
which yields (together with $H_i^s(X, W_n \mathcal{O}_X)_s / p \cong H_i^c(X, \mathcal{O}_X)_s$) that $H_i^c(X, W_n \mathcal{O}_X)_s$ is free of rank $\dim_k H_i^c(X, \mathcal{O}_X)_s$. □

**Proposition 1.4.8.** (cf. [Ser58, §5, Corollaire 2]) Suppose $X$ is proper and $n \geq 2$. The following statements are equivalent:

(i) For all $i$, $H^i(X, W_n \mathcal{O}_X)$ is a free $W_n(k)$-module.

(ii) For all $i$, $H^i(X, \mathcal{O}_X)_s = H^i(X, \mathcal{O}_X)$ and $H^i(X, W_n \mathcal{O}_X) \rightarrow H^i(X, \mathcal{O}_X)$ is surjective.

(iii) For all $i$, $H^i(X, W_n \mathcal{O}_X)_s = H^i(X, W_n \mathcal{O}_X)$ and $H^i(X, W_n \mathcal{O}_X)_s$ is a free $W_n(k)$-module.

**Proof.** Proposition 1.4.6 and 1.4.7 imply (ii) ⇒ (iii). Obviously, (iii) ⇒ (i).

Now, suppose that (i) holds. From the long exact sequence associated to 1.4.3.1 we obtain
\[
\text{length}(H^i(X, W_m \mathcal{O}_X)) \leq m \cdot \dim_k H^i(X, \mathcal{O}_X)
\]
for all $m$. If $H^i(X, W_n \mathcal{O}_X)/p \rightarrow H^i(X, \mathcal{O}_X)$ is surjective then in 1.4.8.1 equality holds; this implies that
\[
V : \sigma_* H^i(X, W_{n-1} \mathcal{O}_X) \rightarrow H^i(X, W_n \mathcal{O}_X)
\]
is injective. Thus $V(\sigma_* H^i(X, W_{n-1})) = p H^i(X, W_n \mathcal{O}_X)$ and in particular, we see that $H^i(X, W_{n-1} \mathcal{O}_X)$ is free. We also obtain $H^{i-1}(X, W_n \mathcal{O}_X)/p \xrightarrow{\cong} H^{i-1}(X, \mathcal{O}_X)$. By induction on $i$ we get $H^i(X, W_n \mathcal{O}_X)/p \xrightarrow{\cong} H^i(X, \mathcal{O}_X)$ for all $i$. We conclude that $H^i(X, W_m \mathcal{O}_X)$ is free of rank $\dim_k H^i(X, \mathcal{O}_X)$ for all $i, m \leq n$. Consider the case $m = 2$: we have a short exact sequence
\[
0 \rightarrow \sigma_* H^i(X, \mathcal{O}_X) \xrightarrow{\nu} H^i(X, W_2 \mathcal{O}_X) \xrightarrow{\beta} H^i(X, \mathcal{O}_X) \rightarrow 0
\]
for all $i$. The composition $H^i(X, \mathcal{O}_X) \xrightarrow{\nu} \sigma_* H^i(X, \mathcal{O}_X) \xrightarrow{\nu} H^i(X, W_2 \mathcal{O}_X)$ equals $p \cdot R^{-1}$. Because $H^i(X, W_2 \mathcal{O}_X)$ is free, $V$ and thus $F$ is injective, which proves $H^i(X, \mathcal{O}_X)_s = H^i(X, \mathcal{O}_X)$. □

**Theorem 1.4.9.** Let $k$ be a perfect field of positive characteristic $p$. Let $X$ be an affine scheme of finite type over $k$. Suppose $X$ is equidimensional of dimension $d$ and suppose that $X$ is Cohen-Macaulay. Then
\[
H_i^c(X, \mathcal{O}_X)_s = 0 \quad \text{for all} \quad i \neq d.
\]

**Proof.** Let $Y$ be a compactification of $X$. By blowing up the complement $Y \setminus X$ we may suppose that we can find an ideal $\mathcal{I}$ for $Y \setminus X$ which is a Cartier divisor. It is sufficient to prove that the projective system $(H^i(Y, \mathcal{I}^n))_n$ is essentially zero if $i < d$. Let $\omega_Y$ be the dualizing complex of $Y$. We obtain
\[
H^i(Y, \mathcal{I}^n)^Y = H^{-i}(Y, \omega_Y \otimes \mathcal{I}^{-n}),
\]
thus we need to prove that the inductive system $(H^{-i}(Y, \omega_Y \otimes \mathcal{I}^{-n}))_n$ is essentially zero. Since
\[
\lim_n \omega_Y \otimes \mathcal{I}^{-n} = j_*(\omega_Y|_X),
\]
with \( j : X \to Y \) the open immersion, we get
\[
\lim_{n \to \infty} H^{-i}(Y, \omega_Y \otimes \mathcal{I}^{-n}) = H^{-i}(Y, j_*(\omega_Y|_X)) = H^{-i}(Y, Rj_*(\omega_Y|_X)) = H^{-i}(X, \omega_Y|_X).
\]
By assumption \( X \) is Cohen-Macaulay, and therefore \( \omega_Y|_X \cong \omega_X \) is concentrated in degree \(-d\).

**Remark 1.4.10.** In general the statement of the Theorem 1.4.9 fails if \( X \) is not Cohen-Macaulay. For example consider two affine planes glued at a point 0, \( X = \mathbb{A}^2 \cup_0 \mathbb{A}^2 \). Then \( H^1_\varepsilon(X, \mathcal{O}_X)_s \cong k \).

**Corollary 1.4.11.** With the assumptions of Theorem 1.4.9. The group \( H^i_\varepsilon(X, W\mathcal{O}_X)_s \) vanishes if \( i \neq d \), and is a finite free \( W(k) \)-module for \( i = d \).

**Proof.** Follows from Proposition 1.4.3 and 1.4.7.

1.5. Next, we want to show that
\[
H^i_\varepsilon(X, W\mathcal{O}_X)_s \otimes_{W(k)} K \cong H^i_{\text{rig}, c}(X/K)_{[0]}
\]
where the right hand side is the slope = 0 part of compactly supported rigid cohomology. We will need the following Lemma.

**Lemma 1.5.1.** Suppose \( k \subset L \) is an extension of perfect fields. Let \( (M_n) \) be a projective system of \( W(k) \)-modules such that \( p^n M_n = 0 \) and \( M_n \) is a finite \( W_n(k) \) module. We set \( M = \varprojlim_n M_n \) and consider the natural map
\[
(1.5.1.1) \quad M \otimes_{W(k)} W(L) \to \varprojlim_n (M_n \otimes_{W_n(k)} W_n(L)).
\]

(i) If \( M \) is a finite \( W(k) \)-module then the map \( (1.5.1.1) \) is an isomorphism.

(ii) The map \( (1.5.1.1) \) is injective.

**Sketch of proof.** Since \( k \) and \( L \) are perfect we conclude that \( W_n(L) \) is flat over \( W_n(k) \). Assertion (i) follows by reduction to the case \( M_n = M/p^n \). Statement (ii) follows from (i). We leave the details to the reader.

**Proposition 1.5.2.** Let \( k \) be a perfect field of positive characteristic \( p \), with ring of Witt vectors \( W = W(k) \) and \( K = \text{Frac}(W(k)) \). Let \( X \) be a separated scheme of finite type over \( k \). For all \( i \), there is an isomorphism
\[
H^i_\varepsilon(X, W\mathcal{O}_X)_s \otimes_{W(k)} K \to H^i_{\text{rig}, c}(X/K)_{[0]}
\]
which is compatible with the \( F \)-operation.

**Proof.** Let \( Y \) be a compactification of \( X \) and \( \mathcal{I} \) an ideal for \( Y \setminus X \). By the work of Berthelot, Bloch and Esnault [BBE07], there is an isomorphism of \( F \)-isocrystals
\[
H^i(Y, W(\mathcal{I})) \otimes_{W(k)} K \to H^i_{\text{rig}, c}(X/K)_{<1},
\]
where the right hand side is the slope \(< 1\) part of (compactly supported) rigid cohomology. By definition we have
\[
H^i_\varepsilon(X, W(\mathcal{O}_X))_s \otimes_{W(k)} K \subset H^i(Y, W(\mathcal{I})) \otimes_{W(k)} K,
\]
and since \( F \) is bijective on the finite \( W(k) \)-module \( H^i_\varepsilon(X, W(\mathcal{O}_X))_s \) we obtain
\[
(1.5.2.1) \quad H^i_\varepsilon(X, W(\mathcal{O}_X))_s \otimes_{W(k)} K \subset H^i_{\text{rig}, c}(X/K)_{[0]}.
\]
We set $\bar{K} = \text{Frac}(W(\bar{k}))$. In order to prove the surjectivity of $1.5.2.1$, we need to show that every $v \in H^i_{\text{rig},c}(X/K)[0] \otimes_K \bar{K}$ with $(F \otimes \sigma)(v) = v$ is contained in $H^i_c(X, W(O_X))_s \otimes_W \bar{K}$. Multiplying $v$ by a suitable power of $p$ we may assume that $v$ lies in the image of $H^i(Y, W(\mathcal{I})) \otimes_W W(\bar{k})$. Choose a preimage $v'$ of $v$; $v'$ is well-defined up to $p$-power torsion. Again, by multiplying $v'$ with a power of $p$ we may assume that $(F \otimes \sigma)(v') = v'$. Denote by $v'_n$ the image of $v'$ in $H^i(Y, W_n(\mathcal{I})) \otimes_W W_n(\bar{k})$, it is obviously contained in the Frobenius stable part (for $F \otimes \sigma$). It is easy to see that

$$(H^i(Y, W_n(\mathcal{I})) \otimes_W W_n(\bar{k}))_s = H^i(Y, W_n(\mathcal{I}))_s \otimes_W W_n(\bar{k}),$$

and $v' \in H^i_c(X, W(O_X))_s \otimes_W W(\bar{k})$ follows from Lemma $1.5.1$, (ii). \hfill $\square$

**Corollary 1.5.3.** Under the assumption of Proposition $1.5.2$. If $H^i_c(X, O_X)_s = 0$ for all $i$ then $H^i_{\text{rig},c}(X/K)[0]$ for all $i$.

**Proof.** The vanishing of $H^i_c(X, O_X)_s$ implies the vanishing of $H^i_c(X, W O_X)_s$ by Proposition $1.4.3$ and Corollary $1.4.4$. \hfill $\square$

## 2. Comparison with étale $p$-adic cohomology

2.1. Let $k$ be a perfect field of positive characteristic $p$. We fix an algebraic closure $\bar{k}$. Let $G$ be the Galois group of $\bar{k}$ over $k$. We denote by $\mathcal{D}(k)$ and $\mathcal{D}(\bar{k})$ the Dieudonné ring of $k$ and $\bar{k}$, respectively (see Section I.1). We have a functor

$$\mathcal{G}: (\text{Frobenius stable } \mathcal{D}(k)\text{-modules } M \text{ s.t. } M \text{ is a finite } W(k)\text{-module}) \rightarrow (\text{finite } \mathbb{Z}_p\text{-modules with } G\text{-action})$$

which is defined as follows. For $M$ we denote by

$$\tilde{M} := \mathcal{D}(\bar{k}) \otimes_{\mathcal{D}(k)} M = W(\bar{k}) \otimes_{W(k)} M$$

the induced module over $\mathcal{D}(\bar{k})$. It is equipped with the obvious $G$-action, and the $G$-action commutes with $F, V$. Moreover, we have $\tilde{M}^G = M$. The functor

$$N \mapsto N^{1-F} := \ker(1 - F : N \rightarrow N)$$

is exact when restricted to the category of $\mathcal{D}(\bar{k})\text{-modules } N$ such that $N$ is a finite $W(\bar{k})\text{-module}$ (see [Ill79] II, Lemme 5.3]). Moreover, if $N$ is stable then

$$(2.1.0.1) \quad W(\bar{k}) \otimes_{\mathbb{Z}_p} N^{1-F} \xrightarrow{\cong} N,$$

thus $N^{1-F}$ is a finite $\mathbb{Z}_p\text{-module } (W(\bar{k})$ is faithfully flat over $\mathbb{Z}_p$). We set

$$\mathcal{G}(M) = \tilde{M}^{1-F}.$$

The functor $\mathcal{G}$ is exact and fully faithful.

Let $M$ be again a stable $\mathcal{D}(k)$-module such that $M$ is a finite $W(k)$-module. Suppose that $M = \lim_n M_n$ for stable $\mathcal{D}_n(k)$-modules $M_n$ such that $M_n$ is a finite $W_n(k)$-module. Lemma $1.5.1$, (i) implies that $M = \lim_n \tilde{M}_n$ and thus

$$(2.1.0.2) \quad \mathcal{G}(M) = \lim_n \mathcal{G}(M_n).$$

**Proposition 2.1.1.** Let $k$ be a perfect field of positive characteristic $p$ with ring of Witt vectors $W = W(k)$ and $K = \text{Frac}(W(k))$. Let $X$ be a separated scheme of finite type over $k$. 

(1) For all $i$ we have

$$G(H^i_c(X, W(\mathcal{O}_X)))_s \cong H^i_{\acute{e}t,c}(X \times_k \bar{k}, \mathbb{Z}_p).$$

(2) For all $i$ and $n \geq 1$ we have

$$G(H^i_c(X, W_n(\mathcal{O}_X)))_s \cong H^i_{\acute{e}t,c}(X \times_k \bar{k}, \mathbb{Z}/p^n).$$

Proof. Again, let $Y$ be a compactification of $X$ and $\mathcal{I}$ an ideal for $Z = Y \setminus X$. We denote by $\bar{X} := X \times_k \bar{k}$ the base change, and set $\bar{\mathcal{I}} := \mathcal{I} \otimes_k \bar{k}$.

Since $W_n(\bar{k})$ is flat over $W_n(k)$ the natural map

$$(2.1.1.1) \quad H^i(Y, W_n(\mathcal{I}))_s \otimes_{W_n(k)} W_n(\bar{k}) \xrightarrow{\cong} H^i(\bar{Y}, W_n(\bar{\mathcal{I}}))_s$$

is an isomorphism. In view of Lemma 1.5.1(i) we obtain

$$H^i_c(X, \mathcal{O}_X)_s \otimes_{W(k)} W(\bar{k}) \xrightarrow{\cong} H^i_c(\bar{X}, \mathcal{O}_{\bar{X}})_s.$$

Let $j : \bar{X} \to \bar{Y}$ be the open immersion. We have an exact sequence of sheaves on the étale site $\bar{Y}_{\acute{e}t}$,

$$(2.1.1.2) \quad 0 \to j^! \mathbb{Z}/p^n \to W_n(\bar{\mathcal{I}}) \xrightarrow{1-F} W_n(\bar{\mathcal{I}}) \to 0.$$

Indeed, 2.1.1.2 is exact because there is a commutative diagram

\[
\begin{array}{ccc}
0 & \xrightarrow{i_*} & \mathbb{Z}/p^n & \xrightarrow{1-F} & W_n(\mathcal{O}_\bar{Z}) & \xrightarrow{1-F} & W_n(\mathcal{O}_Z) & \to 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & \\
0 & \xrightarrow{j^!} & \mathbb{Z}/p^n & \xrightarrow{1-F} & W_n(\mathcal{O}_\bar{Y}) & \xrightarrow{1-F} & W_n(\mathcal{O}_Y) & \to 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & \\
0 & \xrightarrow{j^!} & \mathbb{Z}/p^n & \xrightarrow{1-F} & W_n(\bar{\mathcal{I}}) & \xrightarrow{1-F} & W_n(\bar{\mathcal{I}}) & \\
\end{array}
\]

with exact first two lines and $i : \bar{Z} = \bar{Y} \setminus \bar{X} \to \bar{Y}$ being the closed immersion.

If $M$ is a finite $W_n(k)$-module equipped with a $\sigma$-linear endomorphism $F$ then the map $1 - F : M \to M$ is surjective [Ill79 II, Lemme 5.3]. Thus 2.1.1.2 yields (2). The compatibility with the Galois action is readily verified. The statement (1) follows from the compatibility of $G$ with projective limits (see 2.1.0.2). \qed

Remark 2.1.2. Proposition 2.1.1(1) and (2) is well-known for proper schemes. The case $n = 1$ is Proposition 2.2.5 in [Kat73].

Corollary 2.1.3. Let $k$ be a perfect field of positive characteristic $p$. Let $X$ be an affine scheme of finite type over $k$. Suppose $X$ is equidimensional of dimension $d$ and suppose that $X$ is Cohen-Macaulay. Then

$$H^i_{\acute{e}t,c}(X \times_k \bar{k}, \mathbb{Z}/p) = 0 \quad \text{for all } i \neq d.$$

Proof. This follows from Theorem 1.4.9 and Proposition 2.1.1 \qed
3. The formal Euler characteristic of the slope zero part of rigid cohomology modulo powers of \( p \).

3.1. Let \( k \) be a perfect field of positive characteristic \( p \). We denote by \( W(k) \) and \( K \) the ring of Witt vectors and its quotient field, respectively.

**Notation 3.1.1.** In order to simplify the notation we will write \( \Lambda \) for \( W(k) \) or \( W_n(k) \) or \( K \). We denote by \( D_\Lambda \) the ring \( D \) for \( \Lambda = W(k) \), the ring \( D_n \) for \( \Lambda = W_n(k) \), and the ring \( D \otimes_{W(k)} K \) for \( \Lambda = K \).

**Definition 3.1.2.** We denote by \( K_\Lambda \) the quotient of the free abelian group generated by Frobenius stable \( D_\Lambda \)-modules \( M \) (see Definition 1.1.1) which are finite and free as \( \Lambda \)-modules, modulo the relations \( [M] - [M'] - [M''] \) for every exact sequence \( 0 \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0 \) as \( D_\Lambda \)-modules. Note that since \( M, M', M'' \) are free the sequence is split as sequence of \( \Lambda \)-modules.

In order to get a comparison theorem we need to find natural perfect complexes \( R\Gamma(X, W\mathcal{O}_X)_s \) and \( R\Gamma(X, W_n\mathcal{O}_X)_s \) with cohomology groups \( H^*_c(X, W\mathcal{O}_X)_s \) and \( H^*_c(X, W_n\mathcal{O}_X)_s \), respectively.

3.2. Recall that we have a functor

\[(\text{Schemes}/\mathbb{F}_p) \rightarrow (\text{Schemes}/\mathbb{F}_p), \quad X \mapsto X^{\text{perf}},\]

defined by \( \text{Spec}(A) \mapsto \text{Spec}(A^{\text{perf}}) \) on affine schemes, and

\[A^{\text{perf}} = \lim_{\text{Frob}} A := \lim_{\text{Frob}} (A \xrightarrow{\text{Frob}} A \xrightarrow{\text{Frob}} \ldots).\]

In general \( X^{\text{perf}} \) is not noetherian, but the underlying topological spaces of \( X \) and \( X^{\text{perf}} \) are identified via the natural map \( X^{\text{perf}} \rightarrow X \).

Let \( X \) be separated and of finite type over a perfect field \( k \). Choose a compactification \( Y \) of \( X \) and an ideal \( \mathcal{I} \) for the complement \( Y \setminus X \). We define

\[H^i_c(X, W_n\mathcal{O}_{X^{\text{perf}}}) := H^i(Y, W_n(\lim_{\text{Frob}} \mathcal{I})).\]

The next proposition implies the independence of \( Y, \mathcal{I} \).

**Proposition 3.2.1.** Let \( k \) be a perfect field of positive characteristic \( p \). Let \( X \) be separated and of finite type over \( k \). The natural map

\[H^i_c(X, W_n\mathcal{O}_X)_s \rightarrow H^i_c(X, W_n\mathcal{O}_{X^{\text{perf}}})\]

is an isomorphism for all \( i \) and all \( n \geq 1 \).

**Proof.** Since the topological space of \( Y^{\text{perf}} \) is noetherian we obtain

\[H^i(Y, W_n(\lim_{\text{Frob}} \mathcal{I})) = H^i(Y, \lim_{\text{Frob}} W_n(\mathcal{I})) = \lim_{\text{Frob}} H^i(Y, W_n(\mathcal{I})).\]

In view of Lemma 1.1.3 we see that

\[H^i(Y, W_n(\mathcal{I})) \rightarrow \lim_{\text{Frob}} H^i(Y, W_n(\mathcal{I}))\]

induces an isomorphism with the stable part \( H^i(Y, W_n(\mathcal{I}))_s \). \( \Box \)
3.3. An $\mathbb{F}_p$-algebra $R$ is called perfect if Frob induces an automorphism of $R$, equivalently $R = R^{\text{perf}}$; an ideal $I \subset R$ is called perfect if Frob induces an automorphism on $I$. For any $\mathbb{F}_p$-algebra $R$ and any ideal $I \subset R$ the ideal $\lim_{\text{Frob}} I \subset R^{\text{perf}}$ is perfect.

Lemma 3.3.1. Let $R$ be a perfect $k$-algebra. Let $I \subset R$ be a perfect ideal. The following holds:

(i) For all $m \geq 1$: $W(I) \otimes_{W(k)} W_m(k) = W_m(I)$.
(ii) For all $n \geq m \geq 1$: $W_n(I) \otimes_{W_n(k)} W_m(k) = W_m(I)$.

Proof. We leave the proof to the reader. \qed

3.4. Let $X$ be separated and of finite type over a perfect field $k$. Choose a compactification $Y$ of $X$ and an ideal $\mathcal{I}$ for the complement $Y \setminus X$. Attached to a finite affine covering $\{U_i\}_i$ of $Y$ we get the Cech-complex

$$(3.4.0.1) \quad R\Gamma([U_i], W(\lim_{\text{Frob}} \mathcal{I})) \in D^b(W(k)-\text{modules}).$$

We have

$$R\Gamma([U_i], W(\lim_{\text{Frob}} \mathcal{I})) = \lim_{\text{Frob}} R\Gamma([U_i], W_n(\lim_{\text{Frob}} \mathcal{I})) = R\lim_{\text{Frob}} R\Gamma([U_i], W_n(\lim_{\text{Frob}} \mathcal{I})).$$

Indeed, the first equality is obvious and the second follows because

$$R\Gamma([U_i], W_n(\lim_{\text{Frob}} \mathcal{I})) \to R\Gamma([U_i], W_m(\lim_{\text{Frob}} \mathcal{I}))$$

is surjective on the components. Proposition 3.2.1 implies

$$H^i(R\Gamma([U_i], W(\lim_{\text{Frob}} \mathcal{I}))) \cong H^i \lim_{\text{Frob}} R\Gamma([U_i], W_n(\lim_{\text{Frob}} \mathcal{I})).$$

for all $i$. For $(*)$ we used

$$H^i R\Gamma([U_i], W_n(\lim_{\text{Frob}} \mathcal{I})) = H^i R\Gamma([U_i], W_n(\lim_{\text{Frob}} \mathcal{I}))) \cong H^i(X, W_n \mathcal{O}_X)_s = H^i_c(X, W \mathcal{O}_X)_s$$

and the fact that the projective system

$$n \mapsto H^i(Y, W_n(\lim_{\text{Frob}} \mathcal{I})) \cong H^i_c(X, W_n \mathcal{O}_X)_s$$

satisfies the Mittag-Leffler condition.

Thus the complex $[3.4.0.1]$ is in $D^b(W(k)-\text{modules})$ up to canonical isomorphisms independent of the choice of $Y, \mathcal{I}$, and the covering $\{U_i\}$.

Definition 3.4.1. We define

$$R\Gamma_c(X, W \mathcal{O}_X)_s := R\Gamma([U_i], W(\lim_{\text{Frob}} \mathcal{I})),$$

and similarly for all $n \geq 1$,

$$R\Gamma_c(X, W_n \mathcal{O}_X)_s := R\Gamma([U_i], W_n(\lim_{\text{Frob}} \mathcal{I})).$$

If $X$ is proper we drop the index $c$. 
Since the components of $3.4.0.1$ are flat by Lemma $3.3.1$, $R\Gamma_c(X, W\mathcal{O}_X)$ is a perfect complex (see [171, I] for the definition of a perfect complex which should not be confused with the notion of a perfect ring) and

$$(3.4.1.1) \quad R\Gamma_c(X, W\mathcal{O}_X)_s \otimes^L_{W(k)} W_n(k) = R\Gamma_c(X, W_n\mathcal{O}_X)_s.$$ 

The Frobenius morphism

$$F : R\Gamma((U_i, W(\lim T)) \mapsto \sigma_* R\Gamma((U_i, W(\lim T))$$

induces an isomorphism in $D^b(W(k))$-modules:

$$F : R\Gamma_c(X, W\mathcal{O}_X)_s \mapsto \sigma_* R\Gamma_c(X, W\mathcal{O}_X)_s,$$

which agrees with our $F$-operation on the cohomology.

3.5. As a unifying notation we write $\Lambda$ for $W(k)$ and $W_n(k)$ in the following. Suppose $N \in D^b(\Lambda)$-modules is a perfect complex together with a quasi-isomorphism $F : N \mapsto \sigma_* N$. By definition we can find

$$M \in K^b(\text{free and finite } \Lambda)-\text{modules} =: K^b(\text{ff-}\Lambda)$$

together with $F_M : M \mapsto \sigma_* M$ and a quasi-isomorphism $\psi_M : M \mapsto N$ such that $\sigma_* (\psi_M) \circ F_M = F \circ \psi_M$ in $D^b(\Lambda)$. Induced by $F_M$ we get a $\sigma$-linear map $F_M^i$ on the components $M^i$ of $M$. Taking the Frobenius-stable part is an exact functor when restricted to finite $\Lambda$-modules and preserves free modules (see Proposition 3.1.2), therefore $M_s \mapsto M$ is an isomorphism in $K^b(\text{ff-}\Lambda)$. For all $i$ we get a stable $D_\Lambda$-module $M^i_s$ with $F$-operation induced by $F_M^i$, and $V = pF^{-1}$. We define

$$(3.5.0.2) \quad e(M, F_M) = \sum_i (-1)^i [M^i_s] \in K_\Lambda.$$

**Proposition 3.5.1.** The class $e(M, F_M, \psi_M)$ depends only on $N$ and the morphism $F : N \mapsto \sigma_* N$.

**Proof.** First, we observe that if $M \cong 0$ in $K^b(\text{ff-}\Lambda)$ then $e(M, F_M) = 0$ for any $F_M$.

Let $M_1, M_2 \in K^b(\text{ff-}\Lambda)$ and $F_M : M_i \mapsto \sigma_* M_i, i = 1, 2$, with a morphism of complexes $\psi : M_1 \mapsto \sigma_* M_2$ such that $\sigma_* (\psi_M) \circ F_M = F \circ \psi_M$ in $D^b(\Lambda)$. Induced by $F_M$ we get a $\sigma$-linear map $F_M^i$. In particular, there is a homotopy $K : M_1 \mapsto \sigma_* M_2[-1]$ such that

$$F_M^2 \circ \psi - \sigma_* \psi \circ F_M^1 = K \circ d_{M_1} + d_{M_2} \circ K.$$ 

We need to show that $e(M_1, F_{M_1}) = e(M_2, F_{M_2})$.

We define a morphism of complexes

$$F_3 : \text{cone}(\psi) \mapsto \text{cone}(\sigma_* \psi) = \sigma_* \text{cone}(\psi)$$

where $F_3^i : M^i_2 \oplus M^{i+1}_1 \mapsto \sigma_* M^i_2 \oplus \sigma_* M^{i+1}_1$ is of the form

$$F_3^i = \begin{pmatrix} F^i_2 & K^{i+1} \\ 0 & F^{i+1}_1 \end{pmatrix}.$$ 

Therefore we get short exact sequences which are compatible with the $F$-operations

$$0 \rightarrow (M^2_2, F^2_2) \rightarrow (M^2_2 \oplus M^{i+1}_1, F_3) \rightarrow (M^{i+1}_1, F^{i+1}_1) \rightarrow 0$$

$$0 \rightarrow (M^2_2, F^2_2)_s \rightarrow (M^2_2 \oplus M^{i+1}_1, F_3)_s \rightarrow (M^{i+1}_1, F^{i+1}_1)_s \rightarrow 0.$$ 

Since cone($\psi$) $\cong 0$ this implies the claim. □

In view of this proposition we write $e(N, F)$ for the class constructed in $3.5.0.2$. 

3.6. Recall that we have obvious maps:

\[
\begin{array}{c}
\mathcal{K}_{W(k)} \otimes_{W(k)} K \\
\downarrow \\
\mathcal{K}_{W_n(k)}
\end{array}
\]

**Proposition 3.6.1.** Let \( X \) be separated and of finite type over \( k \).

(i) For all \( n \geq 1 \):

\[
e(R\Gamma_c(X, W_0X)_s, F) \otimes_{W(k)} W_n(k) = e(R\Gamma_c(X, W_nO_X)_s, F).
\]

(ii) The following equality holds

\[
e(R\Gamma_c(X, W_0X)_s, F) \otimes_{W(k)} K = \sum_i (-1)^i [H^i_c(X, W_0X)_s \otimes_{W(k)} K] \in \mathcal{K}_K.
\]

(iii) Let \( n \geq 1 \). If the cohomology groups \( H^i_c(X, W_0X)_s \) are free \( W_n(k) \)-modules for all \( i \) then

\[
e(R\Gamma_c(X, W_nO_X)_s, F) = \sum_i (-1)^i [H^i_c(X, W_nO_X)_s] \in \mathcal{K}_{W_n(k)}.
\]

**Proof.** Statement (i) follows from [3.4.11]. Statements (ii) and (iii) are obvious. \( \square \)

3.7. Let \( k \) be a finite field with \( q = p^a \) elements. Let \( r \) be an integer \( r \geq 1 \). We define a homomorphism

\[
\text{Tr}_A^r : \Lambda \to \Lambda, \quad [M] \mapsto \text{Tr}(F^{ar} | M).
\]

Obviously, for an element \( M \in \mathcal{K}_W(k) \) we get

\[
\text{Tr}_A^r(M \otimes_{W(k)} K) = \text{Tr}^r_{W(k)}(M), \quad \text{Tr}^r_{W_n(k)}(M \otimes_{W(k)} W_n(k)) = \text{Tr}^r_{W_n(k)}(M) \mod p^a.
\]

**Corollary 3.7.1.** Let \( k \) be a finite field with \( q = p^a \) elements. Let \( X \) be a proper scheme over \( k \). We denote by \( H^\ast_{rig}(X/K)_{[0]} \) the slope zero part of rigid cohomology. For an integer \( r \geq 1 \) we denote by \( k_r \) the field extension of \( k \) of degree \( r \). Let \( n \geq 1 \) be an integer.

(i) The trace

\[
N := \sum_{i \geq 0} (-1)^i \text{Tr}(F^{ar} | H^i_{rig}(X/K)_{[0]}))
\]

lies in \( W(k) \), i.e. \( N \in W(k) \). For all \( r, n \) the following congruence holds:

\[
N \equiv \text{Tr}^r_{W_n(k)}(e(R\Gamma(X, W_nO_X)_s, F)) \mod p^n.
\]

(ii) There is an integer \( r_0 \) which depends on \( n \) and \( X \) such that for all \( r \geq r_0 \) the following congruence holds:

\[
\text{Tr}^r_{W_n(k)}(e(R\Gamma(X, W_nO_X)_s, F)) \equiv \#X(k_r) \mod p^n.
\]

(iii) Suppose that \( H^i(X, O_X) = H^i(X, O_X) \) for all \( i \). Then

\[
\text{Tr}^r_{W_n(k)}(e(R\Gamma(X, W_nO_X)_s, F)) \equiv \#X(k_r) \mod p^\min\{ra, n\}.
\]
Proof. For (i): Proposition \[3.6.1\] implies
\[
\begin{align*}
\text{Tr}_{W(k)}^r(e(RG(X, W\mathcal{O}_X)_s, F)) &= \sum_{i \geq 0}(-1)^i \text{Tr}(F^{\text{ar}} | H^i(X, W\mathcal{O}_X)_s \otimes W(k) K), \\
\text{Tr}_{W(k)}^r(e(RG(X, W\mathcal{O}_X)_s, F)) \equiv \text{Tr}_{W_n(k)}^r(e(RG(X, W_n\mathcal{O}_X)_s, F)) \mod p^n.
\end{align*}
\]
Thus the claim follows from Proposition \[1.5.2\].
For (ii): We have the trace formula for rigid cohomology:
\[
\sum_{i \geq 0}(-1)^i \text{Tr}(F^{\text{ar}} | H^i_{\text{rig}}(X/K)) = X(k_r).
\]
The isocrystal $H^*_{\text{rig}}(X/K)$ has a slope decomposition with slopes $\geq 0$. On the slope $\lambda$-part the eigenvalues of $F^a$ have $p$-adic valuation $a \cdot \lambda$. Let $\lambda > 0$ be the smallest slope $\neq 0$. Choose $r_0$ such that $a \cdot r_0 \cdot \lambda > n$. For every $r \geq r_0$ we get
\[
N := \sum_{i \geq 0}(-1)^i \text{Tr}(F^{\text{ar}} | H^i_{\text{rig}}(X/K)[0]) = X(k_r) \mod p^n,
\]
where $N \in W(k)$ by part (i). In view of (i), this proves the claim.
For (iii): Proposition \[1.4.8\] implies that
\[
H^i(X, W\mathcal{O}_X)_s = H^i(X, W\mathcal{O}_X) \quad \text{for all } i.
\]
In view of Proposition \[3.6.1\] (ii) we see that
\[
N := \text{Tr}_{W(k)}^r(e(RG(X, W\mathcal{O}_X)_s, F)) = \sum_{i \geq 0}(-1)^i \text{Tr}(F^{\text{ar}} | H^i(X, W\mathcal{O}_X) \otimes W(k) K)
\]
lies in $W(k)$. It is proved in \[BBE07\] Corollary 1.3] that $N \equiv \#X(k_r) \mod p^{r^2}$. Now, Proposition \[3.6.1\] (i) implies
\[
N \equiv \text{Tr}_{W_n(k)}^r(e(RG(X, W_n\mathcal{O}_X)_s, F)) \mod p^n,
\]
which proves the statement. \hfill \Box

Corollary 3.7.2. Let $k$ be a finite field with $q = p^a$ elements. Let $X$ be a proper scheme over $k$. Let $n \geq 1$ be an integer and suppose that $H^i(X, W_n\mathcal{O}_X)$ is a free $W_n(k)$-module for all $i$. Then
\[
\sum_{i \geq 0}(-1)^i \text{Tr}(F^a | H^i(X, W_n\mathcal{O}_X)) \equiv \#X(k) \mod p^{\min(a, n)}.
\]
Proof. The case $n = 1$ is Katz’s formula \[Kat73\]. In the case $n \geq 2$, Proposition \[1.4.8\] implies $H^i(X, \mathcal{O}_X)_s = H^i(X, \mathcal{O}_X)$ for all $i$. Proposition \[3.6.1\] (iii) and Corollary \[3.7.1\] (iii) imply the claim. \hfill \Box

Example 3.7.3. Let $k$ be finite field with $p^a$ elements. Let $G$ be a finite (abstract) group. For every integer $d \geq 1$, Serre shows the existence of a regular complete intersection $Y$ in projective space such that $\dim(Y) = d$ and $G$ acts freely on $Y$ \[Ser68\] §20.
Since $Y$ is a complete intersection, we get for all $n \geq 1$:
\[
H^i(Y, W_n\mathcal{O}_Y) = 0 \quad \text{for all } i \notin \{0, d\}, \quad H^d(Y, W_n\mathcal{O}_Y) = W_n(k).
\]
Because $W_n\mathcal{O}$ is an étale sheaf and étale cohomology for $W_n\mathcal{O}$ agrees with Zariski cohomology \[Hill79\] Proposition 0.1.5.8], we obtain a spectral sequence
\[
E_2^{p,q} = H^p(G, H^q(Y, W_n\mathcal{O}_Y)) \Rightarrow H^{p+q}(X, W_n\mathcal{O}_X),
\]
where $X := Y/G$. From the spectral sequence we obtain

\[(3.7.3.2) \quad H^i(X, W_n\mathcal{O}_X) = H^i(G, W_n(k)) \quad \text{for all } i < d.\]

In particular, $H^i(X, W_n\mathcal{O}_X) = H^i(X, W_n\mathcal{O}_X)_s$ for all $i < d$.

For $G = \mathbb{Z}/p^n$ we have

\[(3.7.3.3) \quad H^i(G, W_n(k)) = \begin{cases} W_n(k) & \text{for } i = 0, \\
\ker(p^n : W_n(k) \to W_n(k)) & \text{for } i \text{ odd}, \\
\coker(p^n : W_n(k) \to W_n(k)) & \text{for } i \text{ even}. \end{cases}\]

Suppose $d \geq 2$. We conclude that the cohomology groups $H^*(X, W_n\mathcal{O}_X)_s$ are free $W_n(k)$-modules if and only if $m \geq n$ (for $H^d$ we use Proposition 1.4.7). For the projective limit we have the following picture:

\[
H^i(X, W\mathcal{O}_X)_s = \begin{cases} 0 & \text{if } i \text{ is odd and } i < d, \\
W_m(k) & \text{if } i \text{ is even and } 0 < i < d. \end{cases}
\]

For $i = d$ we have to distinguish two cases:

1. If $d$ is odd then $H^d(X, W\mathcal{O}_X)_s$ is free of rank $= \dim H^d(X, \mathcal{O}_X)_s$.
2. If $d$ is even then $H^d(X, W\mathcal{O}_X)_s$ has $W_m(k)$ as torsion subgroup. The dimension of $H^d(X, W\mathcal{O}_X)_s \otimes \mathbb{Q}$ is $\dim H^d(X, \mathcal{O}_X)_s - 1$.

This follows from the long exact sequence in Proposition 1.4.3. Suppose $d$ is even. We claim that for $r$ sufficiently large and $m \geq n$:

\[
\text{Tr}(F^r|H^d(X, W\mathcal{O}_X) \otimes \mathbb{Q}) = \text{Tr}(F^r|H^d(X, W\mathcal{O}_X)_s \otimes \mathbb{Q}) \equiv \text{Tr}(F^r|H^d(X, W_n\mathcal{O}_X)_s) - 1 \mod p^n.
\]

The first equality follows (for $r \gg 0$) because $H^d(X, W\mathcal{O}_X)_s \otimes \mathbb{Q}$ is the slope zero part of $H^d(X, W\mathcal{O}_X) \otimes \mathbb{Q}$ (see the proof of Corollary 3.7.1). The second equality is a consequence of Proposition 3.6.1

\[
1 + \text{Tr}(F^r|H^d(X, W\mathcal{O}_X)_s \otimes \mathbb{Q}) = \sum_{i \geq 0} (-1)^i \text{Tr}(F^r|H^i(X, W\mathcal{O}_X)_s \otimes \mathbb{Q}) \equiv \sum_{i \geq 0} (-1)^i \text{Tr}(F^r|H^i(X, W_n\mathcal{O}_X)_s) = \text{Tr}(F^r|H^d(X, W_n\mathcal{O}_X)_s) \mod p^n.
\]

For the last equality we used 3.7.2, 3.7.3, and the assumption $m \geq n$.

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