Design of control laws for rotary inverted pendulum based on LQR and Lyapunov function

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Abstract. In this paper, a new technique for synthesizing control laws of stabilizing rotary inverted pendulum is proposed. The key of the proposed method is combination of linear quadratic regulator (LQR) method and Lyapunov function such that the control system can achieve higher performance. The Lyapunov function-based methods guarantee the globally approximate stability; however, they face to the difficulty of constructing an effective Lyapunov function. On the other hand, the LQR-based methods have been shown the ability to achieve high performance only if the control system has a weak nonlinearity and/or operate around equilibrium point. Combining them may partially deal with their limitations. The effectiveness of the proposed method is examined by experiments on the rotary inverted pendulum and compared with the LQR method. The results show that the proposed method has achieved a better performance.

1. Introduction

The Rotary Inverted Pendulum, which was first proposed by Furuka et al. [1], is a well-known test platform to verify the control theories due to its static instability. Designing controllers for many real-life applications such as aerospace vehicles and robotics [2-3] has shown that studying on the rotary inverted pendulum plays a key role to go further.

There are several methods, which have been proposed for the swing-up and stabilizing control of the rotary inverted pendulum in the literature. Akhtaruzzaman and Shafie [4] proposed a method that used proportional - integral – derivative (PID), fuzzy control and LQR techniques to balance the pendulum in its upright position. Chiem at al. [5] presented a method that uses quasi-time optimal control approach. Wen at al. [6] used Lyapunov control method to design control laws for stabilizing the rotary inverted pendulum. Rojas-Moreno at al. [7] designed a fractional order-based LQR controller for the system. However, many limitations of those methods such as complexity and strictly operated conditions are still presented and need to be solved.

Among many approaches, the Lyapunov method is an effective method, which can be applied to both linear and nonlinear systems. Especially, performance and characteristics of a complex system can be analysed by using analytic expression of designed control laws. In addition, the LQR method is an optimal control method, which was originally designed for linear systems and applied in many applications. To guarantee both stabilization and achieving high operating performance of the rotary inverted pendulum as illustrated in Figure 1 around equilibrium point, a method that combines the LQR and Lyapunov methods is proposed in this work.
The remaining of this paper is organized as follow. Section 2 introduces the mathematical model of the rotary inverted pendulum, in which the linear model is obtained. The control laws, which are designed by combining the LQR and Lyapunov methods, are presented in Section 3. Section 4 illustrates some simulation results as well as discussions. Finally, Section 5 concludes the paper.

**Figure 1.** Schematic presentation of the rotary inverted pendulum

### 2. Mathematical model of rotary inverted pendulum

The rotary inverted pendulum illustrated in Figure 1, includes a rotating arm, which is driven by a motor, and a pendulum mounted on arm’s rim. The pendulum moves as an inverted pendulum in a plane perpendicular to the rotating arm. $\alpha$ (angular displacement of the arm) and $\beta$ (angular displacement of the pendulum) are used as the generalized coordinates to present the inverted pendulum system. Parameter settings for the system are set according to Table 1.

| Parameters | Values | Functions (unit) |
|------------|--------|------------------|
| L          | 0.2    | Length of half pendulum (m) |
| M          | 0.1    | Mass of pendulum (kg) |
| $J_0$      | 0.001  | Moment of inertia at the load (kg.m$^2$) |

Assuming that friction at all connected points is zero, applying Lagrange method, the mathematical model of the rotary inverted pendulum can be described as follows [8]:

\[
\begin{align*}
(J_0 + m r^2)\ddot{\alpha} - mLr \cos(\beta) \dot{\beta} + mLr \sin(\beta) \beta^2 &= \tau \\
-mLr \cos(\beta) \ddot{\alpha} + \frac{4}{3} mL^2 \ddot{\beta} - mgL \sin(\beta) &= 0
\end{align*}
\]

where, $r$ is rotating arm length; $g$ is gravitational constant; $u$ is moment generated by motor.

Converting Eq. (1) into the state space from, we obtained:

\[
x = f(x) + B(x)\tau
\]

where
\[ f(x) = \begin{bmatrix} x_2 \\ mr \sin(x_2) [-4Lx_4^2 + 3g \cos(x_3)] \\ A(x_3) \\ x_4 \\ 3 \sin(x_2) [Jg + mgr^2 - Lmr^2 x_4^2 \cos(x_3)] \\ B(x_3) \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 4 \\ A(x_3) \\ 0 \\ r \cos(x_3) \\ B(x_3) \end{bmatrix} \]

with

\[ x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} \alpha \\ \dot{\alpha} \\ \beta \\ \dot{\beta} \end{bmatrix} \]

\[ B(x_3) = 4LJ + 4Lmr^2 - 3Lmr^2 \cos^2(x_3) \]

\[ A(x_3) = 4J + 4mr^2 - 3mr^2 \cos^2(x_3) \]

\[ J = m(2L)^2 / 3 \]

Linearizing Eq. (2) at the equilibrium point, \( x = [0, 0, 0]^T \), it becomes

\[ \dot{x} = A_0 x + B_0 u = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{3mgr}{4J + 4mr^2 - 3mr^2} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & \frac{3(Jg + mgr^2)}{4LJ + 4Lmr^2 - 3Lmr^2} & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{4}{4J + 4mr^2 - 3mr^2} \\ 0 \\ \frac{3r}{4LJ + 4Lmr^2 - 3Lmr^2} \end{bmatrix} \]

3. Design of control laws based on LQR method and Lyapunov function

3.1 Design of control laws using the LQR method

To regulate the system about the upright equilibrium point, control laws for the rotary inverted pendulum are designed by using the LQR method. Particularly, the control signal: \( u(t) = -Kx(t) \) is designed such that objective function JLQR is minimum.

\[ J_{LQR} = \int_0^{\infty} (x^T Q + u^T R u) dt \]

where \( Q \) and \( R \) are positive matrices. \( K = R^{-1}B_0^T P \) is defined by solving algebraic Ricatti equation, in which \( P \) matrix has to be satisfied the following condition:

\[ PA_0 + A_0^T P + Q - PB_0 R^{-1} B_0^T P = 0 \]

By substituting the system parameters as given in Table 1, and \( A_0, B_0 \) in Eq. (4), we get \( P \) as follows:


\[ P = \begin{bmatrix}
1.3528 & 0.4151 & -3.4828 & -0.5545 \\
0.4151 & 0.4897 & -4.1572 & -0.6544 \\
-3.4828 & -4.1572 & 38.3713 & 5.5650 \\
-0.5545 & -0.6544 & 5.5650 & 0.8764
\end{bmatrix} \]  \hspace{1cm} (7)

with \( Q \) and \( R \) are defined as follows:

\[ Q = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 2 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}; \hspace{0.5cm} R = 1; \]

Hence, the control signal is presented by:

\[ u_{LQR}(t) = -0.4142x_1(t) - 0.5607x_2(t) + 8.3021x_3(t) + 1.4376x_4(t) \]  \hspace{1cm} (8)

### 3.2 The proposed method for designing control laws

Based on works from Le Tr Th. [9-10] for designing control laws using Lyapunov function, Eq. (2) for affine nonlinear systems can be rewritten as follows:

\[ x = f(x) + B(x)u_f \]  \hspace{1cm} (9)

where \( f(x) \) is a differentiable and nonlinear function with \( f(0) = 0 \).

In this work, we propose to define \( f(x) \), \( B(x) \) and \( u_f \) with following forms:

\[ f(x) = A_0x + f_0(x) \]
\[ B(x) = B_0 + \overline{B}(x) \]  \hspace{1cm} (10)

\[ u_f = u_{LQR} + u \]

By substituting Eq. (10) to Eq. (9), we obtained:

\[ x = (A_0 - B_0K^T)x + \left[ f_0(x) - \overline{B}(x)K^Tx \right] + B(x)u = \left[ A_1x + f_1(x) \right] + B(x)u \]  \hspace{1cm} (11)

The control laws derived from the Eq. (11) ensure the rotary inverted pendulum stable around the equilibrium point. In order to obtain high accuracy and fast convergence, Lyapunov function is selected by using quadratic function \( V(x) \) as follows:

\[ V(x) = x^TPx \]  \hspace{1cm} (12)

where \( P \) is a matrix obtained by solving Ricatti equation as in Eq. (6). Calculating the first derivative of \( V(x) \), we get:

\[ \dot{V}(x) = x^T(A_0^TP + PA_0)x + 2(x^TPf(x) + x^TPB(x)u) \]  \hspace{1cm} (13)

Since \( A_0^TP + PA_0 = -Q + PB_0B_0^TP \) is negative matrix, according to Lyapunov stability theorem [11], the system is stable if and only if the selected \( u(x) \) make \( \dot{V}(x) \leq 0 \) in all system trajectories. Hence, the control law, \( u_f \), is designed as the following form:
where $\epsilon$ is a small constant that making $u$ does not go to infinity; $\alpha$ is a constant in $[0, 0.5)$.

Note that, the detail of $u$ is not presented here for the purpose of simplifying and compact.

4. Simulation results
In this section, the simulation results for two cases: the initial point of the pendulum is close to and far from the equilibrium point, will be represented. In each case, the system response will be estimated for both the LQR method and the proposed control law.

In the first case, the initial position of the pendulum is set with $x_0 = [0, 0, \pi/18, 0]$. Figure 2 and Figure 3 indicate that both methods can stabilize the rotary inverted pendulum with similar system responses. This is consistant with the theorical analysis in Section 3: around the equilibrium point, $u_{LQR}$ will dominate in the control signal.

In the second case, the initial position of the pendulum is set with $x_0 = [0, 0, \pi/5, 0]$. Figure 4 and Figure 5 show the arm and pendulum responses for the considering methods. From the results, we can observe that the proposed method outperforms the LQR method in all performance criteria such as overshoot, settling time. However, the figures also reveal that the switching step of the proposed control law has relay form. As a result, oscillation will be occured in the process that the system converges to the equilibrium point. This is an undesired phenomenon of mechanical systems.

![Image](image_url)

**Figure 2.** $\beta$ and $\dot{\beta}$ responses for the LQR and proposed methods of the first case
5. Conclusion
In this paper, a new method for designing control laws of stabilizing rotary inverted pendulum is represented. In essence, the proposed method is the combination of LQR method and Lyapunov function-based method such that the advantages of each method can be enhanced, simultaneously, their disadvantages may be eliminated. While the Lyapunov function-based method can guide the system to the equilibrium point effectively, the LQR method can help the system operating with high performance around the equilibrium point. The proposed control law has been theoretically analysed to prove the capability of globally approximate stabilization. The simulation results also reveal that the proposed method outperforms its counterpart. However, it still exist a limitation which is introducing oscillation when switching control signals. As a future work, we will consider solutions to get rid of the limitation. In addition, stability and robustness of the system will be examined in the presence of noise.
Figure 5. $\alpha$ and $\dot{\alpha}$ responses for the LQR and proposed methods of the second case

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