On the formula of regularized traces

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January 18, 2013

We consider an operator \( L \) generated by a 2\( m \)-order differential expression

\[
ly \equiv (-1)^m D^{2m}y + \sum_{k=0}^{2m-2} p_k(x) D^k y
\]

and by \( m \) boundary conditions

\[
P_j(D)y(0) = 0, \quad j = 1, \ldots, m.
\]

Here \( p_k \in L_{1,\text{loc}}(\mathbb{R}_+) \) are real functions while \( P_j \) is a polynomial of degree \( k_j \); moreover, the system of boundary conditions is assumed normalised, i.e. \( 0 \leq k_1 < k_2 < \ldots < k_m \leq 2m - 1 \).

Suppose this operator is self-adjoint in the space \( L_2(\mathbb{R}_+) \), semibounded from below and has purely discrete spectrum \( \{\lambda_n\}_{n=1}^{+\infty} \) enumerated in ascending order according to the multiplicity.

Let \( Q \) be an operator of multiplication by a real function \( q \in L_\infty(\mathbb{R}_+) \). Then an operator \( L + Q \) also has a purely discrete spectrum \( \{\mu_n\}_{n=1}^{+\infty} \).

A number of papers beginning from the pioneering work [1] are devoted to the calculation of spectral functions and regularized traces of differential operators. The aim of our paper is to prove the following statement.

**Theorem 1.** Let \( q \) have a bounded support\(^1\), and the function \( \psi(x) = \frac{1}{x} \int_0^x q(t)dt \) has a bounded variation at zero. Then the following relation holds:

\[
S_1 \equiv \sum_{n=1}^{+\infty} \left[ \mu_n - \lambda_n - \frac{c_n}{\pi} \int_0^\infty q(t)dt \right] = -\psi(0+) \cdot \left( \frac{m}{2} - \frac{1}{4} - \frac{\kappa}{2m} \right),
\]

where

\[
c_1 = \frac{1}{\lambda_1^{2m}}; \quad c_n = \frac{1}{\lambda_n^{2m}} - \frac{1}{\lambda_{n-1}^{2m}}, \quad n > 1; \quad \kappa = \sum_{j=1}^m k_j.
\]

\(^*\)A.N. was partially supported by grant NSh4210.2010.1. D.S. and P.Z. were supported by the Chebyshev Laboratory (Dept. of Mathematics and Mechanics, St.Petersburg State University) under the grant 11.G34.31.0026 of the Government of the Russian Federation.

\(^1\)This condition is used only to conclude the relation (4) from Proposition 1. It seems that this condition may be weakened. To do this one needs to obtain a global estimate of the difference of spectral functions to operators \( L \) and \( \tilde{L} \). However, up to the moment we do not know such an estimate.
Remark. Formula (3) was conjectured by A.I. Nazarov at the conference in Moscow, 2007, during the A.S. Pechentsov lecture. For one-term boundary conditions $P_j(D) = D^{k_j}$ this formula was proved in the preprint [2].

We introduce two auxiliary operators:

\begin{itemize}
  \item $\tilde{L}$ is a $2m$-order operator with boundary conditions (2), self-adjoint in $L_2(R_+)$; its lower-order coefficients are compactly supported and coincide with lower-order coefficients in (1) on the segment $[0, R] \supset \text{supp}(q)$;
  \item $L_0$ is the operator $(-1)^m D^{2m}$ with boundary conditions (2).
\end{itemize}

For $\lambda \in \mathbb{R}$ we denote by $\theta(x, y, \lambda)$ the spectral function of the operator $L$, i.e. the kernel of its spectral projector $E_\lambda$, see [6]. In a similar way, $\tilde{\theta}(x, y, \lambda)$ is the spectral function of the operator $\tilde{L}$. Also we denote by $H_0(x, y, \tau)$ the Green function of the operator $L_0 - \tau$ (for $\tau \not\in \mathbb{R}_+$).

Let $\zeta$ be the value of $\tau$ such that $\arg(\zeta) \in [0, \pi m]$, while $z = \exp(\frac{i\pi}{m})$. We introduce the matrix $B(\zeta) = [P_j(iz^{j-1} \zeta)]_{\ell,j=1}^m$ and define $\Delta(\zeta) = \det(B(\zeta))$.

The following statements were proved in [7], see also [8]. We state them with some redefinitions.

**Proposition 1.** A ([7, Ch. 1, Sec. 2]). Let the following condition hold:

A. The matrix $B(\zeta)$ is non-degenerate, and the elements of the inverse matrix satisfy the estimate $[B^{-1}(\zeta)]_{\ell j} = O(|\zeta|^{-k_j})$ as $|\zeta| \to \infty$.

Then the relation

\[ \tilde{\theta}(x, y, \lambda) = -\frac{1}{2\pi i} \int_{|\tau| = \lambda} H_0(x, y, \tau) d\tau + O(\lambda^{-\frac{1}{m}}), \quad \text{as } \lambda \to +\infty, \]

holds uniformly on $\mathbb{R}_+^2$.

B ([7, Ch. 1, Sec. 4]). Let condition A be satisfied. Then the relation

\[ \theta(x, y, \lambda) = \tilde{\theta}(x, y, \lambda) + o(1), \quad \text{as } \lambda \to +\infty, \]

holds uniformly on $[0, R]^2$.

First, we note that he assumptions of Proposition 1 may be weakened.

**Lemma 1.** For $|\zeta|$ large enough, condition A is always satisfied. Thus, this condition in Proposition 1 can be omitted.

Proof. We claim that $\Delta(\zeta)$ is a $\kappa$-degree polynomial of $\zeta$. Indeed, the columns of $B(\zeta)$ are $k_j$-degree polynomials, while the highest degrees cannot reduce since the corresponding coefficient is the Vandermonde determinant $\det(W(z^{k_1}, \ldots, z^{k_m})) \neq 0$. Thus, $\Delta(\zeta) \neq 0$ for sufficiently large $|\zeta|$.

Further, the cofactor of $b_{\ell j}$ is evidently a polynomial of $\zeta$, and its degree does not exceed $\kappa - k_j$. The statement now follows from the Cramer formula. $\Box$

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1. Three particular cases: 1) $k_j = 2j - 2$; 2) $k_j = 2j - 1$; 3) $k_j = j - 1$ were considered earlier in papers [3], [4], [5].

2. Note that this operator does not need to be self-adjoint!
Proof of Theorem \[7\]. Note that the left-hand side of (3) can be rewritten as follows, see \[5\], proof of Theorem 1:

\[
S_1 = \lim_{\lambda \to +\infty} \int_0^{+\infty} q(x) \cdot \left( \theta(x, x, \lambda) - \frac{\lambda^{\frac{1}{2m}}}{\pi} \right) dx.
\]

By Lemma \[1\] this formula can be transformed to

\[
S_1 = \lim_{\lambda \to +\infty} \int_0^{+\infty} q(x) \cdot \left( \frac{1}{2\pi i} \int_{|\tau|=\lambda} H_0(x, x, \tau) d\tau - \frac{\lambda^{\frac{1}{2m}}}{\pi} \right) dx. \tag{4}
\]

Now we use the explicit formula for \(H_0\) obtained in \[3\], Lemma 1. We state them with some redefinitions.

\[
H_0(x, x, \tau) = \frac{i}{2m\zeta^{2m-1}} \cdot \sum_{\alpha=1}^{m} z^{\alpha-1} \cdot \left( 1 - \frac{1}{\Delta(\zeta)} \right) \cdot \sum_{\beta=1}^{m} \exp(i(z^{\alpha-1} + z^{\beta-1}) x\zeta) \cdot \Delta_{\alpha\beta}(\zeta),
\]

where the determinant \(\Delta_{\alpha\beta}(\zeta)\) is obtained from \(\Delta(\zeta)\) if we substitute \(P_j(-iz^{\alpha-1}\zeta)\) for \(P_j(iz^{\beta-1}\zeta)\) in \(\beta\)-th line. Surely, for this formula we need \(\Delta(\zeta) \neq 0\) but this is the case for \(|\tau|\) large enough.

We change the variable \(\tau = \zeta^{2m}\) in the inner integral in (4). It is easy to see that

\[
-\frac{1}{2\pi i} \int_{|\tau|=\lambda} \frac{i}{2m\zeta^{2m-1}} \sum_{\alpha=1}^{m} z^{\alpha-1} = -\int_{\Gamma_\lambda} \frac{d\zeta}{\pi(1 - z)} = \frac{\lambda^{\frac{1}{2m}}}{\pi}
\]

(here \(\Gamma_\lambda\) is the arc of the circle \(\{\zeta = \lambda^{\frac{1}{2m}} e^{i\phi} : \phi \in [0, \frac{\pi}{m}]\}\). Therefore (4) is rewritten as follows:

\[
S_1 = \frac{1}{2\pi} \lim_{\lambda \to +\infty} \int_0^{+\infty} q(x) \cdot \int_{\Gamma_\lambda} \sum_{\alpha,\beta=1}^{m} z^{\alpha-1} \exp(i(z^{\alpha-1} + z^{\beta-1}) x\zeta) \cdot \frac{\Delta_{\alpha\beta}(\zeta)}{\Delta(\zeta)} d\zeta dx. \tag{5}
\]

We denote \(B_{\alpha\beta} = \lim_{\zeta \to \infty} \frac{\Delta_{\alpha\beta}(\zeta)}{\Delta(\zeta)}\) and claim that

\[
S_1 = \frac{1}{2\pi} \lim_{\lambda \to +\infty} \int_0^{+\infty} q(x) \cdot \int_{\Gamma_\lambda} \sum_{\alpha,\beta=1}^{m} z^{\alpha-1} \exp(i(z^{\alpha-1} + z^{\beta-1}) x\zeta) \cdot B_{\alpha\beta} d\zeta dx. \tag{6}
\]

Indeed, since \(\Delta_{\alpha\beta}\) is a polynomial of \(\zeta\) and its degree does not exceed \(\varkappa\), the relation

\[
\frac{\Delta_{\alpha\beta}(\zeta)}{\Delta(\zeta)} = B_{\alpha\beta} + O(\zeta^{-1}) \tag{7}
\]

\[4\] In \[4\] Theorem 2] a formula similar to (5) was obtained under additional assumption: \(\Delta(\zeta) \neq 0\) for \(\tau \in \mathbb{R} \setminus \{0\}\). This formula contains the real part of the integral over the segment \([0, \lambda^{\frac{1}{2m}}]\) instead of integral over \(\Gamma_\lambda\). In fact, this formula is correct only under a stronger assumption: \(\Delta(\zeta) \neq 0\) for \(\tau \neq 0\). In this case \[4\] formula (0.5) is reduced to (5) by the Cauchy theorem.
holds as $|\zeta| \to \infty$. Further, for any $1 \leq \alpha, \beta \leq m$ the inequality $0 \leq \arg(z^{\alpha-1} + z^{\beta-1}) \leq \frac{m-1}{m} \pi$ holds true. Therefore, the integrals

$$I_{\alpha\beta}(x) = \int_{\Gamma_\lambda} z^{\alpha-1} \exp(i(z^{\alpha-1} + z^{\beta-1}) x \zeta) \cdot \left( \frac{\Delta_{\alpha\beta}(\zeta)}{\Delta(\zeta)} - B_{\alpha\beta} \right) d\zeta$$

are bounded by (7) uniformly with respect to $x \geq 0$ and tend to zero as $x \to +\infty$, $\lambda \to +\infty$, by the Jordan Lemma. By the Lebesgue Dominated Convergence Theorem, $\lim_{\lambda \to +\infty} \int_0^\infty q(x) \cdot I_{\alpha\beta}(x) \, dx = 0$, and (6) follows.

We calculate the inner integral in (6) and rewrite the formula for $S_1$ as follows:

$$S_1 = \frac{1}{2\pi i} \cdot \lim_{s \to \infty} \int_0^\infty q(x) g(sx) s \, dx,$$

where

$$g(y) = \sum_{\alpha, \beta=1}^m \mathbb{P}_{\beta \alpha} B_{\alpha\beta} \frac{\exp(i(z^\alpha + z^\beta) y) - \exp(i(z^{\alpha-1} + z^{\beta-1}) y)}{y},$$

while

$$\mathbb{P}_{\beta \alpha} = \frac{z^{\alpha-1}}{z^{\alpha-1} + z^{\beta-1}} = \frac{1}{1 + z^{\beta-\alpha}}.$$  

To pass to the limit in (8) in terms with $\alpha = \beta = 1$ and $\alpha = \beta = m$, one uses the Riemann localization principle, see, e.g., [10, Ch. I, Sec. 33], and the Vallée-Poussin test of convergence, see [10, Ch. III, Sec. 3]. Other terms, after integration by parts, are covered by the Lebesgue Theorem, since the exponents have negative real parts. Thus we arrive at

$$S_1 = \frac{1}{2\pi i} \cdot \psi(0+) \int_0^\infty g(y) \, dy = -\frac{\psi(0+)}{2m} \cdot \text{Sp}(PB)$$

(the last equality follows from [11, 3.434.2]).

Note that the entries of matrix $B$ are quotients of determinants composed of the leading coefficients of polynomials that are entries of determinants $\Delta_{\alpha\beta}(\zeta)$ and $\Delta(\zeta)$. Direct calculation via the Cramer formula gives

$$B = W \cdot \text{diag}(-1)^{k_1}, \ldots, (-1)^{k_m}] \cdot W^{-1}$$

(we recall that $W = (W_{ij})$ is the Vandermond matrix generated by the numbers $w_j = z^{kj}$, $j = 1, \ldots, m$).

Formula (11) shows that if $e^{(j)} = [1, w_j, \ldots, w_j^{m-1}]^T$ is $j$-th column of matrix $W$ then $Be^{(j)} = (-1)^{k_j}e^{(j)}$. Thus, vectors $e^{(j)}$ form the eigen basis of $B$.

Next, by the geometrical progression sum formula, we rewrite the matrix $\mathbb{P}$ as follows:

$$\mathbb{P} = \lim_{\rho \to 1-} \sum_{n=0}^{\infty} (-1)^n \rho^n \varphi_n \varpi_n^T, \quad (11)$$

where $\varphi_n = (1, z^n, \ldots, z^{(m-1)n})^T$.

Denote by $\mathbb{K}$ the set $\{k_j + 2mp | 1 \leq j \leq m, \ p \in \mathbb{Z}_+ \}$. 

\footnote{Note that passage from (8) to (9) follows also from [5, Lemma 2].}
Lemma 2. If $n \in \mathbb{K}$ then $\text{Sp}(\varphi_n \varphi_n^T \mathbb{B}) = (-1)^n m$. If $n \notin \mathbb{K}$ then $\text{Sp}(\varphi_n \varphi_n^T \mathbb{B}) = (-1)^{n+1} m$.

To prove this lemma we need an obvious proposition.

**Proposition 2.** Let $u, v$ be elements of Hilbert space and let $a, b \in \mathbb{C}$. Then $(au + bv, u) = 0$ implies $(au + bv, -au + bv) = |au + bv|^2$.

**Proof of Lemma 2** First,

$$\text{Sp}(\varphi_n \varphi_n^T \mathbb{B}) = \text{Sp}(\varphi_n^T \mathbb{B} \varphi_n) = (\mathbb{B} \varphi_n, \varphi_n).$$

We consider two subspaces

$$U = \text{Lin}\{e^{(j)} \mid k_j \equiv 1 \pmod{2}\}; \quad V = \text{Lin}\{e^{(j)} \mid k_j \equiv 0 \pmod{2}\}.$$  

Note that $U + V = \mathbb{C}^m$, since the vectors $e^{(j)}$ form a basis. Thus, there exists a decomposition $\varphi_n = u + v$ with $u \in U$, $v \in V$. Further, we have shown that $\mathbb{B}|_U$ is an identity operator while $\mathbb{B}|_V$ multiplies by $-1$. This implies $\mathbb{B}\varphi_n = -u + v$.

1. Let $n \in \mathbb{K}$. Then $\varphi$ is an eigenvector of $\mathbb{B}$ corresponding to the eigenvalue $(-1)^n$. Therefore $(\mathbb{B}\varphi_n, \varphi_n) = (-1)^n(\varphi_n, \varphi_n) = (-1)^n m$.

2. Let $n \notin \mathbb{K}$ and $n \equiv 1 \pmod{2}$. Then direct calculation shows that $\varphi \perp U$. By Proposition 2 $(\mathbb{B}\varphi_n, \varphi_n) = |\varphi_n|^2 = m$.

3. In a similar way, if $n \notin \mathbb{K}$ and $n \equiv 0 \pmod{2}$ then $\varphi \perp V$. By Proposition 2 $-(\mathbb{B}\varphi_n, \varphi_n) = |\varphi_n|^2 = m$. \hfill \Box

To complete the proof of Theorem we define the set $K = \{k_1, \ldots, k_m\}$. Then formula (11) and Lemma 2 imply

$$\text{Sp}(\mathbb{B}) = \lim_{\rho \to 1-} \sum_{n=0}^{\infty} (-1)^n \rho^n \text{Sp}(\varphi_n \varphi_n^T \mathbb{B}) =$$

$$= m \lim_{\rho \to 1-} \left( \sum_{n \in \mathbb{K}} (-1)^n \rho^n (-1)^n + \sum_{0 \leq n \notin \mathbb{K}} (-1)^n \rho^n (-1)^{n+1} \right)$$

$$= m \lim_{\rho \to 1-} \left( \sum_{j \in K} \sum_{p=0}^{\infty} \rho^{j+2mp} - \sum_{0 \leq j < 2m, j \notin K} \sum_{p=0}^{\infty} \rho^{j+2mp} \right)$$

$$= m \lim_{\rho \to 1-} \left( \sum_{j \in K} \frac{\rho^j}{1 - \rho^{2m}} - \sum_{0 \leq j < 2m, j \notin K} \frac{\rho^j}{1 - \rho^{2m}} \right).$$

Since the set $K$ contains exactly $m$ elements, we get

$$\text{Sp}(\mathbb{B}) = m \lim_{\rho \to 1-} \left( \sum_{j \in K} \frac{-j}{2m} - \sum_{0 \leq j < 2m, j \notin K} \frac{-j}{2m} \right) =$$

$$= \frac{1}{2} \sum_{0 \leq j < 2m} j - \sum_{j \in K} j =$$

$$= \frac{m(2m - 1)}{2} - \sum_{j=1}^{m} k_j. \quad (12)$$

Theorem follows immediately from (9) and (12). \hfill \Box
We are grateful to I.A. Sheipak and A.A. Shkalikov who gave us the opportunity to learn the text of A.G. Kostyuchenko’s dissertation; to V.A. Kozlov and A.N. Podkorytov for useful discussions; to A.S. Pechentsov and A.I. Kozko who provided us with the texts of papers [5] and [9].

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