Jordan-Wigner Dualities for Translation-Invariant Hamiltonians in Any Dimension: Emergent Fractons that are Fermions

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Inspired by recent developments generalizing Jordan-Wigner dualities to higher dimensions\textsuperscript{1–3}, we develop a framework for such dualities for translation-invariant Hamiltonians using the algebraic formalism proposed by Haah\textsuperscript{4}. We prove that given a translation-invariant fermionic system with general $q$-body interactions, where $q$ is even, a local mapping preserving global fermion parity to a dual Pauli spin model exists and is unique up to a choice of basis. Furthermore, the dual spin model is constructive, and we present various examples of these dualities. As an application, we bosonize fermionic systems where fermion parity is also conserved on submanifolds such as higher-form, line, planar or fractal symmetry. For some cases in 3+1D, bosonizing such system can give rise to fracton models where the emergent particles are immobile, but yet can behave in certain ways like fermions. These models may be examples of new non-relativistic ‘t Hooft anomalies. Furthermore, fermionic subsystem symmetries are also present in various Majorana stabilizer codes, such as the color code or the checkerboard model, and we give examples where their duals are cluster states or new fracton models distinct from their doubled CSS codes.

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I. INTRODUCTION

The Kramers-Wannier (KW) and Jordan-Wigner (JW) transformations are important dualities in one-dimensional systems; A one dimensional transverse-field Ising model with global $\mathbb{Z}_2$ symmetry is KW dual to itself with an inverse Ising coupling. At the same time, it is also JW dual to a spinless-fermionic system with global fermion parity.

There are generalizations of both KW and JW dualities, though those of the former are much better understood. KW dualities have been extended to transverse field Ising models in arbitrary dimensions\textsuperscript{5,6}, which gives rise to dual gauge theories whose ground states can exhibit topological order\textsuperscript{7,8}. At the same time, generalized Ising models with arbitrary Ising-interactions can exhibit extra symmetries in addition to the usual global symmetry, such as higher-form symmetries\textsuperscript{9–12} or subsystem symmetries\textsuperscript{13–17}. KW dualities have been generalized to theories with such symmetries\textsuperscript{15,18–23}, and the dual theories can exhibit new types of ordered states, such as fracton topological order\textsuperscript{15,24,25}.

JW dualities can also be generalized to higher dimensions, though the intricacies have only been fully understood recently. The parallel to KW dualities has been established as an exact bosonization introduced in Refs. 1–3 (see also, Ref. 26) in the case of a global fermion parity symmetry. However, the construction is more subtle than its bosonic counterpart. The duality is only well-defined if the fermion theory is put on a spin manifold and explicitly depends on a choice of spin structure. Furthermore, for spatial dimension $d > 1$, the resulting gauge theory is different from the usual gauge theory obtained by a KW duality. The gauge constraints can be understood as an anomalous higher-form symmetry that is required so that a spin system can support emergent excitations with fermionic statistics\textsuperscript{27}.

It is natural to ask whether there is a similar analogue of the KW dualities between generalized Ising models with many-body Ising interactions. That is, whether one can generalize JW dualities to fermionic systems with arbitrary $q$-body interactions where $q$ is even. In this paper, we answer this question in the positive, and explicitly construct an exact bosonization for any such in-
FIG. 1. An example of an exactly solvable model with fracton excitations that are fermions, which is a twisted version of the X-Cube stabilizer code. The blue and red lines respectively denote Pauli $Z$ and $X$’s living on the edges of a cubic lattice. The model can be thought of as the result of gauging a fermion system which conserves fermion parity in the (001), (010) and (001) planes. We discuss how to obtain this model in Section IVF.

TABLE I. Summary of JW Dualities considered, and the corresponding ground states. ?? means the dual bosonic ground state has not been identified. The subscript $F$ denotes a twisted version where emergent excitations are fermions. TC, TO, and SSB stand for Toric Code, Topological Order, and Spontaneous Symmetry Breaking, respectively.

| Section | $d$ | $\mathbb{Z}_2^d$ | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ | Anomalous | Fermionic Ground State | Bosonic Ground state |
|---------|-----|-----------------|----------------|----------------|-----------|------------------------|---------------------|
| IV A    | 2   | Global          | 1-form         | Yes            | Product   | Majorana Color Code$^{28}$ | 2D TC$^*$           |
|         |     |                 | $\mathbb{Z}_2$ |                |           |                        | $\mathbb{Z}_2^d$ TO |
| IV B    | 3   | Global          | 2-form         | Yes            | Product   | Majorana Checkerboard$^{29}$ | 3D TC$^{2}$         |
|         |     |                 | Majorana codes 2-5$^{29}$ |          |           | Semiionic X-cube$^{30}$ $\otimes$ 3D TC$^F$ |
| IV C    | 3   | 1-form          | 1-form         | Yes?           | Product   | 3D TC                  |
| IV D    | 2   | Line($\times2$) | Line ($\times2$) | No             | Product   | SSB$^{33}$             |
|         |     |                 | $\mathbb{Z}_2$ |                |           |                        | Wen plaquette$^{32}$ |
| IV E    | 2   | Line ($\times2$) | Line($\times2$) | No             | Product   | SSBB$^{a}$             |
|         |     |                 | $\mathbb{Z}_2$ |                |           |                        | Product             |
|         |     |                 | Majorana color code |            |           |                        | SSPT$^{16,31}$      |
| IV F    | 3   | Planar ($\times3$) | Rank-2         | Yes?           | Product   | X-cube$^{(F^\perp)}$ |
|         |     |                 | Majorana code 3 |                |           |                        | ??                  |
| IV G    | 3   | Planar ($\times6$) | Planar ($\times6$) | No             | Product   | SSB                    |
|         |     |                 | Majorana Checkerboard |            |           |                        | 3D Cluster state    |
| IV H    | 2   | 2D Fractal      | 2D Fractal     | Yes            | Product   | SSB                    |
| IV I    | 3   | 2D Fractal stacks | Higher-rank fractal | Yes          | Product   | Yoshida’s fractal code$^F$ |
|         |     |                 | Majorana Checkerboard |            |           |                        | ??                  |
| IV J    | 3   | 3D Fractal      | Higher-rank fractal | Yes?          | Product   | Haah’s code$^{(F^\perp)}$ |
|         |     |                 | Majorana code 5 |                |           |                        | ??                  |
| V       | 3   | Planar ($\times3$) | Rank-2          | Yes?           | Product   | Checkerboard$^{(F^\perp)}$ |
|         |     |                 | Majorana Checkerboard |            |           |                        | ??                  |

interaction assuming translation invariance. The key step in establishing such a duality takes advantage of an algebraic representation of translation invariant Pauli and Majorana Hamiltonians introduced in Refs. 4 and 29, respectively. Furthermore, the constructed model can be shown to be unique up to a choice of basis. More specifically, all possible dual models only differ by a finite-depth translation-invariant Clifford circuit.

Interestingly, for certain types of fermion parity symmetries in 3D, our generalized JW duality allows us to construct gauge theories whose excitations are immobile in the deconfined phase, that is, the ground state exhibits fracton order. However, they are also fermions, in the sense that they cannot be condensed due to the non-commutativity of the operators that proliferate them. Moreover, it is understood that the fermions can only be condensed if paired up with another excitation (either physical or emergent) exhibiting the same anomaly$^{27,33}$. An example of a model which exhibits such property is a twisted X-cube model shown in Figure 1. This opens a question of whether there are meaningful statistical processes that one can perform to detect whether such immobile excitations are fermionic. A closely related question is whether the gauge constraints of these fracton models, when considered as a higher-rank symmetry$^{34–39}$, has an associated ’t Hooft anomaly that generalize those discussed in Ref. 27.

An important application of these dualities (and its predecessors) is that they are local maps, and are hence useful for simulating arbitrary interacting fermionic systems in any dimension with qubits. For example, any translation-invariant Majorana code$^{28,29,40}$ can be locally
mapped to a translation-invariant Pauli stabilizer code, and any operations used to perform the computation also map accordingly\textsuperscript{41}.

This paper is structured as follows: in Section II, we review the algebraic formalism of translation-invariant Pauli Hamiltonians and use it to construct a KW duality for a generalized transverse-field Ising model in any dimension. In close parallel, Section III constructs the generalized JW duality for a fermion model with arbitrary q-body interactions, where we prove its existence and uniqueness up to a choice of basis. Section IV discuss various examples, from reviewing the dualities with global fermion parity in 2D and 3D, to new dualities where the fermionic system has additional higher-form or subsystem fermion parity symmetry. A summary of the dualities and models considered in this section are summarized in Table I. Section V outlines a general procedure to construct twisted versions of translation-invariant CSS codes. Readers interested in fracton models can directly go to sections, IVF, IVI, IVJ, and V. In Section VI, we conjecture and give supporting arguments that for certain JW dualities, the gauge constraints of the dual spin models exhibit an anomaly associated to the fact that the emergent particles are fermions. We also give a concrete example where an anomalous fractal symmetry can be realized as an effective symmetry action on the boundary of a bulk Symmetry-Protected Topological (SPT) phase. We conclude in Section VII with various open questions.

II. KRAMERS-WANNIER DUALITY

Before stating our construction of the JW duality, we find it insightful to first introduce the KW duality. Although there are many formulations of such duality in the literature, we will focus on presenting this duality from the algebraic formalism, which we will review in detail in section II A. In section II B, we construct a generalized Ising model in this formalism, and show that it can be dualized into another generalized Ising model in section II C. Along the way, we demonstrate the KW duality for the 1D transverse field Ising model and work through a similar 2D example in section II D.

A. Algebraic Formalism

We begin by reviewing the algebraic formalism of translation invariant Pauli Hamiltonians introduced in Ref. 4. Our Hilbert space is a $d$-dimensional cubic lattice with qubits placed on each vertex. Fixing a point of origin, the position of each qubit can be labeled by its coordinates $i = (i_1, ..., i_d) \in \mathbb{Z}^d$, which we will represent using a monomial $x_1^{i_1} \cdots x_d^{i_d}$. A generic Pauli operator in this Hilbert space can then be denoted by the presence of Pauli matrices at each coordinate. We designate these positions by a polynomial with coefficients only taking values 0 or 1 mod 2, obtained by summing over all monomials at which the Pauli matrices are present.

It is sufficient to use two polynomials to present a general Pauli operator in the Hilbert space up to a sign, since Pauli Y’s can be obtained as products of Pauli Z’s and Pauli X’s. For example, the Pauli $X_0, Y_0,$ and $Z_0$ located at the origin can respectively be represented as two-component vectors

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

and a nearest neighbor Ising coupling in the $x$ direction $Z_0 Z_x$ can be represented as

$$\begin{pmatrix} 1 + x \\ 0 \end{pmatrix}.$$  

In general, a Pauli operator can be uniquely decomposed up to a sign as

$$\mathcal{O} \propto \bigotimes_i Z_i^{a_i} X_i^{b_i},$$

where $a_i, b_i \in \{0, 1\}$. Therefore, we can represent this operator as the vector

$$\left( \sum_i a_i x_1^{i_1} \cdots x_d^{i_d} \right) \left( \sum_i b_i x_1^{i_1} \cdots x_d^{i_d} \right).$$

The action of translation acts naturally in this formalism as a multiplication by a monomial. For example, translating the operator by $n$ sites in the $i^{th}$ spatial direction can be obtained by multiplying the vector with $x_i^n$.

Let us call the space of the polynomial representation of all Pauli operators $P$. Mathematically, the polynomials live in a Laurent polynomial ring $R = \mathbb{F}_2[x_1^{±1}, ..., x_d^{±1}]$, and the action of this ring as translation on $P = R^2$ gives it the structure of a module, called the Pauli module. In this paper, objects in $P$ will always be denoted in bold, and represented by a vector with a horizontal line to visually separate Pauli $Z$ and Pauli $X$ operators. Maps into $P$ will also feature such line for clarity. In general, maps between modules will also be bolded, but will be represented as matrices without such lines.

In general, any lattice can be recast into a cubic lattice with $N$ spins per unit cell, for some $N$. In such case, we can represent any operator as a length $2N$ vector, with the first $N$ entries denoting the positions of the Pauli $Z$’s, and the next $N$ corresponding to Pauli $X$’s.

Commutation relations of operators $A$ and $B$ – algebraically represented as $A$ and $B$ – at different positions are efficiently calculated via a symplectic inner product

$$\langle A, B \rangle = A^\dagger \lambda_N B$$

called the commutation value. Here, the $\dagger$ superscript denotes the matrix transpose, followed by the antipode
map \( x_i \rightarrow \bar{x}_i \equiv x_i^{-1} \) (i.e., an inversion with respect to the origin), and \( \lambda_N \) is the symplectic form\(^{12} \)

\[
\lambda_N = \begin{pmatrix}
0_{N \times N} & 1_{N \times N} \\
1_{N \times N} & 0_{N \times N}
\end{pmatrix}.
\] (6)

The role of this symplectic form is to count the number of sites where a Pauli \( Z \) from \( A \) overlaps with a Pauli \( X \) from \( B \) and vice versa. In addition, the commutation value \( \langle A, B \rangle \) tells us the positions of the operator \( A \) that anticommute with operator \( B \) placed at the origin.

Let us demonstrate with an example of operators in a 1D transverse field Ising model. The Ising coupling \( Z_0 Z_1 \) and the transverse field \( X_0 \) are respectively represented by vectors

\[
Z = \begin{pmatrix} 1 + x \\ 0 \end{pmatrix}, \quad X = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.
\] (7)

The commutation value of \( Z \) and \( X \) is

\[
\langle Z, X \rangle = 1 + \bar{x},
\] (8)

which means that the Ising coupling placed at the origin \( (Z_0 Z_1) \), and the one shifted one site to the left \( (Z_1 Z_0) \) anticommutes with the transverse field term sitting at the origin \( (X_0) \).

A translation-invariant Hamiltonian with fully-commuting terms can be represented by a matrix \( \sigma \), where each column of \( \sigma \) denotes each term in the Hamiltonian. \( \sigma \) is called the stabilizer map which maps from the generator-label module \( G \), whose rows labels the stabilizers, to the Pauli module \( P \). Because all columns of \( \sigma \) represent pairwise-commuting Pauli operators, it follows that \( \langle \sigma, \sigma \rangle = 0 \).

It is insightful to define the excitation map \( \epsilon = \sigma^\dagger \lambda_N \), from \( P \) to an excitation module \( E \). Because \( \epsilon \) is just the adjoint under the symplectic inner product \( \langle \cdot, \cdot \rangle \), it therefore labels which stabilizers are excited when a certain Pauli operator in \( P \) is applied. The commutativity condition of the Hamiltonian (i.e. being a stabilizer code) is equivalent to the statement that \( \epsilon \circ \sigma = 0 \), i.e.,

\[
G \xrightarrow{\sigma} P \xrightarrow{\epsilon} E
\] (9)

is a complex.

Lastly, a notion that will become useful later on is a basis transformation. Operators can be represented differently by changing the basis of the wavefunction. This is implemented in the Heisenberg picture as a unitary operation which maps between Pauli operators. Let \( I \) be the algebraic representation of such unitary, which is an isomorphism of \( P \). Since it must preserve the commutation relations of all Pauli operators represented in \( P \), this requires \( \langle IA, IB \rangle = \langle A, B \rangle \), which implies that \( I^\dagger \lambda_N I = \lambda_N \). Hence, \( I \) is a symplectic transformation. When there is no translation involved, the unitaries that realize such transformation between stabilizer Hamiltonians are Clifford unitaries.

### B. Generalized Transverse Field Ising Model

Let us now define a generalized transverse field Ising model within the algebraic formalism. Consider two sets of operators, which we will call transverse fields and Ising interactions, denoted algebraically by \( X \) and \( Z \), respectively. We denote \( K \) the number of sites per unit cell (and hence the number of independent transverse fields) and \( N \) the number of types of Ising interactions. Without loss of generality, we choose a basis in which the transverse fields point in the \( x \) direction. Hence, we can algebraically represent the transverse fields as single Pauli \( X \)’s

\[
X = \begin{pmatrix} 0_{K \times K} \\ 1_{K \times K} \end{pmatrix}.
\] (10)

On the other hand, the \( N \) Ising terms can be arbitrary interactions that all mutually commute, but do not commute with the transverse fields. Thus, \( Z \) is a \( 2K \times N \) matrix. Together with \( X \), they satisfy

\[
\langle Z, Z \rangle = \langle X, X \rangle = 0, \quad \langle Z, X \rangle \neq 0.
\] (11)

Because a general Hamiltonian with both transverse field and Ising terms is not simultaneously diagonalizable, a generic excitation cannot be labeled by eigenstates of these operators. However, it is still useful to define the map \( \sigma = (Z \ X) \), the augmented matrix of \( Z \) and \( X \) and its corresponding excitation map \( \epsilon = \sigma^\dagger \lambda_N \). We will call \( \sigma \) the Ising map instead of the stabilizer map to emphasize that \( \sigma \) does not realize a stabilizer code. Evidently, this is because

\[
\langle \sigma, \sigma \rangle = \begin{pmatrix} 0_{N \times N} & \langle Z, X \rangle \\ \langle X, Z \rangle & 0_{K \times K} \end{pmatrix}
\] (12)

is non-zero. The above matrix will be called the commutation matrix of the Ising map \( \sigma \). Because this matrix is not identically zero, it follows that the sequence (9) in the case of the Ising map is not a complex. In general, any local operator constructed from terms in \( \sigma \) is an allowed term in the Hamiltonian. Such terms are precisely the image of \( \sigma \), which forms a submodule of \( P \).

An important information about such map is the kernel of \( \epsilon \). It is a submodule of \( P \) which represents the set of operators that commute with all operators in the Hamiltonian (i.e., \( \im \sigma \)). Therefore, \( \ker \epsilon \) generates the symmetries of the Hamiltonian. Let us illustrate with the transverse field Ising model in 1D. In this case, \( K = N = 1 \). Using Eq. (7) we can write the Ising map

\[
\sigma = \begin{pmatrix} 1 + x \\ 0 \end{pmatrix}.
\] (13)

The corresponding excitation map is

\[
\epsilon = \sigma^\dagger \lambda_1 = \begin{pmatrix} 0 & 1 + \bar{x} \\ 1 & 0 \end{pmatrix}.
\] (14)
Here, we will slightly abuse notation and represent the kernel of a map via its generators in the algebraic formalism (that is, the kernel is written as the columns which span it). In this case, there is only one generator:

$$\ker \epsilon = \left( \frac{0}{\sum_i x^i} \right).$$

As Pauli matrices, this is $\prod_i X_i$, the $\mathbb{Z}_2$ global spin-flip symmetry of the Ising model.

C. Kramers-Wannier duality

Given an Ising model represented algebraically by the Ising map $\sigma$, one could ask whether there exists another set of operators, which have the same commutation relations as those of the given Ising model. That is, whether there is a dual Ising map $\tilde{\sigma} : G \rightarrow \tilde{P}$, such that $(\sigma, \sigma) = (\tilde{\sigma}, \tilde{\sigma})$. Defining $\tilde{\epsilon} = \tilde{\sigma}^\dagger \lambda_N$, this is equivalent to finding a $\tilde{\sigma}$ such that the diagram

$$\begin{align*}
G & \xrightarrow{\sigma} P \xrightarrow{\epsilon} E \\
\tilde{G} & \xrightarrow{\tilde{\sigma}} \tilde{P} \xrightarrow{\tilde{\epsilon}} \tilde{E}
\end{align*}$$

commutes.

The KW dual is a certain specific choice of the dual Ising map $\tilde{\sigma}$ given by $\tilde{\sigma} = (\tilde{X} \tilde{Z})$, where

$$\tilde{X} = \left( \begin{array}{c} 0_{N \times N} \\ 1_{N \times N} \end{array} \right), \quad \tilde{Z} = \left( \begin{array}{c} \langle Z, X \rangle \\ 0_{K \times N} \end{array} \right).$$

Important, the role of the transverse fields and Ising terms in the dual model are swapped: there are now $N$ sites per unit cell (hence $N$ transverse field terms) and $K$ Ising terms. Furthermore, it follows that $P$ and $\tilde{P}$ are different modules (meaning the two Ising models can live in different Hilbert spaces) when $N \neq K$.

Let us verify that the choice above is a valid dual Ising model.

**Proposition 1.** The map $\tilde{\sigma} = (\tilde{X} \tilde{Z})$ is a valid KW dual.

*Proof.* Since $\langle \tilde{Z}, \tilde{Z} \rangle = \langle \tilde{X}, \tilde{X} \rangle = 0$, and $\langle Z, X \rangle = \langle Z, X \rangle^\dagger |0_{K \times N} \rangle \langle 0_{N \times N} | X_{N \times N} = \langle X, Z \rangle$. Therefore,

$$\langle \tilde{\sigma}, \tilde{\sigma} \rangle = \left( \frac{\langle \tilde{X}, \tilde{X} \rangle \langle \tilde{Z}, \tilde{Z} \rangle}{\langle Z, X \rangle \langle Z, Z \rangle} \right) = \left( \frac{0_{N \times N} \langle Z, X \rangle}{\langle X, Z \rangle 0_{K \times N}} \right) = (\sigma, \sigma)$$

as desired.

To illustrate, we continue with the 1D example. Inserting Eq.(7) into Eq. (17), the dual operators are

$$\tilde{X} = \left( \begin{array}{c} 0 \\ 1 \end{array} \right), \quad \tilde{Z} = \left( \begin{array}{c} 1 + \bar{x} \\ 0 \end{array} \right).$$

which are the same set of operators up to a shift of $\tilde{Z}$ by $x$. Therefore, the Ising model is self-dual under the KW duality. Similarly, the dual model also has a $\mathbb{Z}_2$ symmetry given by

$$\ker \tilde{\epsilon} = \left( \frac{0}{\sum_i x^i} \right)$$

We summarize the relevant operators under this duality in Table II.

To conclude this section, we note that the duality is much more constrained at the level of states because of the symmetry constraints. Only $\mathbb{Z}_2$ even states on both sides are allowed to map to each other in this duality. With this in mind, we mention an important related fact about KW dualities. Consider the set of operators in the Ising model that product to the identity. Because it trivially commutes with all other operators, its dual operator is a non-identity operator that commutes with all other operators in the dual Ising model, and is therefore a symmetry. The KW duality provides a stronger statement. Operators that product to the identity exactly generate the symmetry constraints of the dual theory and vice versa. Mathematically, this amounts to the following statements:

**Proposition 2.** $\ker \epsilon = \sigma \ker \tilde{\sigma}$

**Proposition 3.** $\ker \tilde{\epsilon} = \tilde{\sigma} \ker \sigma$

We provide proofs of the above in Appendix A.

To demonstrate this for the 1D KW duality, the product of all Ising couplings along the 1D chain product to identity, meaning

$$\sigma \left( \sum_i x^i \right) = Z \sum_i x^i = 0.$$

In fact, this is the only relation and therefore we have that

$$\ker \sigma = \left( \frac{\sum_i x^i}{0} \right)$$

From Prop. 3, we can calculate the dual symmetry as

$$\ker \tilde{\epsilon} = \tilde{\sigma} \ker \sigma = \left( \frac{0 1 + \bar{x}}{1 0} \right) \left( \frac{\sum_i x^i}{0} \right) = \left( \frac{0}{\sum_i x^i} \right)$$

in agreement with Eq. 20.

D. Example: Transverse field Ising model on a square lattice

To demonstrate this machinery and later parallel the analogy to JW transformations, we explicitly write down the duality of a transverse field Ising model in 2D.
TABLE II. Summary of operators under the (generalized) KW duality in the algebraic notation. Objects on the left map to the objects in the corresponding row on the right (except the symmetry constraints in the last row).

| Operators                     | Transverse field | Dual Operators |
|-------------------------------|------------------|----------------|
| Ising coupling $Z = \begin{pmatrix} \bar{Z}_X \\ \bar{Z}_Z \end{pmatrix}$ | $\bar{X} = \begin{pmatrix} 0_{N \times N} \\ \mathbb{1}_{N \times N} \end{pmatrix}$ | $\tilde{Z} = \begin{pmatrix} \bar{Z}_X \\ \bar{Z}_Z \end{pmatrix}$ |
| Transverse field $X = \begin{pmatrix} 0_{K \times K} \\ \mathbb{1}_{K \times K} \end{pmatrix}$ | Ising coupling $\tilde{\sigma} = \begin{pmatrix} \bar{Z}_X \\ \bar{Z}_Z \end{pmatrix}$ | $\tilde{\sigma} = \begin{pmatrix} \bar{Z}_X \\ \bar{Z}_Z \end{pmatrix}$ |
| Ising map $\sigma = (Z \ X)$ | Ising map $\tilde{\sigma} = (X \ Z)$ | $\tilde{\sigma} = (X \ Z)$ |
| Excitation map $\epsilon = \sigma^\dagger \lambda_K = \begin{pmatrix} \bar{Z}_X \\ \bar{Z}_Z \end{pmatrix}$ | Excitation map $\tilde{\epsilon} = \sigma^\dagger \lambda_N = \begin{pmatrix} 0_{N \times N} \\ 0_{K \times N} \end{pmatrix}$ | $\tilde{\epsilon} = \sigma^\dagger \lambda_N = \begin{pmatrix} 0_{N \times N} \\ 0_{K \times N} \end{pmatrix}$ |
| Symmetry $\ker \epsilon = \sigma \ker \sigma$ | Symmetry $\ker \tilde{\epsilon} = \sigma \ker \sigma$ |

Condition: $\langle Z, Z \rangle = \bar{Z}_X Z_X + \bar{Z}_Z Z_Z = 0$

Qubits are placed on vertices of a square lattice, with usual translations vectors generated by $x$ and $y$. Since there is one site per unit cell, the transverse field is given by

$$X = \begin{pmatrix} 0 \\ 1 \\ x \\ 1+y \\ 0 \\ 1 \end{pmatrix}. \tag{24}$$

On the other hand, there are two types of nearest neighbors Ising couplings: one for each link orientation

$$Z = \begin{pmatrix} 1+x \\ 1+y \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}. \tag{25}$$

Therefore, the Ising and excitation maps are

$$\sigma = \begin{pmatrix} 1+x \\ 1+y \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad \epsilon = \begin{pmatrix} 0 \\ 1+x \\ 0 \\ 1 \\ y \end{pmatrix} \tag{26}$$

and

$$\ker \epsilon = \begin{pmatrix} 0 \\ \sum_{ij} x_i y_j \end{pmatrix}, \tag{27}$$

corresponding to a global $\mathbb{Z}_2$ symmetry.

Using the KW duality, the dual operators are then

$$\bar{X} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}, \quad \bar{Z} = \begin{pmatrix} 0 \\ 1+x \\ 1+y \\ 0 \\ 0 \end{pmatrix}. \tag{28}$$

The dual Ising and excitation maps are

$$\tilde{\sigma} = \begin{pmatrix} 0 \\ 0 \\ 1+x \\ 0 \\ 0 \\ 1+y \end{pmatrix}, \quad \tilde{\epsilon} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1+x \end{pmatrix}. \tag{29}$$

Using Prop. 3, we can read out the dual symmetry by dualizing the set of operators that product to the identity. On a torus, there are three generators:

$$\ker \tilde{\epsilon} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1+y \sum_i x_i \end{pmatrix} \begin{pmatrix} \sum_j y_j \end{pmatrix}. \tag{30}$$

The first column is a local symmetry constraint, which corresponds to enforcing the Gauss law in a (strict) $\mathbb{Z}_2$ gauge theory. Together with two codimension one constraints which wrap around the torus, the dual model has a 1-form symmetry. When only $\bar{X}$ is present, the model is confined, as the ground state is polarized with a magnetic field in the $x$ direction. On the other hand, when only $\bar{Z}$ is present, the model is deconfined and the ground state exhibits a $\mathbb{Z}_2$ topological order. We note that the deconfined phase can also be considered a phase where the 1-form symmetry is spontaneously broken.

To recapitulate, the duality maps the 2D Ising model

$$H = -\sum_i \begin{pmatrix} \bar{X}_i X_{i+1} + g_x Z_{i} \bar{Z}_{i+1} + g_y \bar{Z}_i Z_{i+1} \end{pmatrix} \tag{31}$$

with global (0-form) $\mathbb{Z}_2$ symmetry constraint $\prod_{ij} X_{ij} = 1$ to the dual Ising model

$$\tilde{H} = -\sum_i \begin{pmatrix} \bar{Z}_i Z_{i+1} + g_x X_{i} \bar{X}_{i+1} + g_y \bar{X}_i X_{i+1} \end{pmatrix}, \tag{32}$$

with 1-form symmetry constraints

$$\bar{X}_i X_{i+1} = \cdots X_{i-2} X_{i-1} X_i \cdots = 1. \tag{33}$$

It is important to emphasize that although the local and non-local symmetry generators above correspond to the plaquette terms and Wilson loops in the toric code, the dual model is not the toric code Hamiltonian since it contains no plaquette terms. Rather, the plaquette terms are
the symmetries of the Hamiltonian, and only states which not charged under this 1-form symmetry are allowed to map under the KW duality. To properly obtain a duality to the toric code Hamiltonian, which is a dynamical $\mathbb{Z}_2$ gauge theory. One must properly couple the transverse-field Ising model to a dynamical gauge field living on links. We refer to Ref. 45 for a thorough treatment of such a duality.

### III. JORDAN-WIGNER DUALITY

We will now generalize the formalism described in order to bosonize a fermionic model into a spin model. We review a similar algebraic formalism for fermions in Section IIIA. With this notation, we write down a fermionic review a similar algebraic formalism for fermions in Section IIIB. We then introduce the JW duality in Section IIIC and discuss existence and uniqueness up to a phase

### A. Algebraic Formalism for Fermions

Hamiltonians for translation-invariant fermions can also be efficiently represented in the algebraic representation. A complex fermion operator $c_i$ at each site can be decomposed into two real (Majorana) fermions

$$\gamma_i = c_i + c_i^\dagger, \quad \gamma'_i = (c_i - c_i^\dagger)/i,$$

so that the local fermion parity is

$$P_i = 1 - 2c_i^\dagger c_i = -i\gamma_i \gamma'_i.$$  \hspace{1cm} (35)

Similarly, in the algebraic formalism, Majorana operators can be represented up to a sign by respectively representing $\gamma$ and $\gamma'$ at the origin as $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$, in a Majorana module $M$. The local fermion parity at each site can be represented as

$$P = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$ \hspace{1cm} (36)

up to a phase $\pm i$. In general, when there are $K$ sites per unit cell,

$$P = \begin{pmatrix} 1_K \times K \\ 1_K \times K \end{pmatrix}.$$ \hspace{1cm} (37)

Because the algebra of fermionic operators is $\mathbb{Z}_2$ graded, while bosonic operators are not, we can only find a mapping between parity even operators to bosonic operators. Therefore, we will from now on assume that all fermionic operators are even.

The commutation value of two $\mathbb{Z}_2$ even operators represented in the algebraic representation as $A$ and $B$ can be calculated via the inner product

$$\langle A, B \rangle_F = A^\dagger B.$$ \hspace{1cm} (38)

Here, the subscript $F$ stands for “Fermion” and denotes the orthogonal (as opposed to symplectic) inner product. Descriptively, this inner product counts the number of overlapping $\gamma$’s and $\gamma'$’s between $A$ and $B$. This is the number of commutations, which gets subtracted from the total number of anticommutations when $A$ is commuted through $B$. A rigorous proof can be found in the supplementary material of Ref. 29.

A commuting Majorana model can be described by a stabilizer map $\sigma_F : G \rightarrow M$, which satisfies $\langle \sigma_F, \sigma_F \rangle_F = 0$. The excitation map is given by the adjoint of $\sigma_F$ under the orthogonal inner product $\epsilon_F = \sigma_F^\dagger$, and the commuting condition can be recast as the complex

$$G \xrightarrow{\sigma_F} M \xrightarrow{\epsilon_F} E.$$ \hspace{1cm} (39)

In short, the only main changes from bosons to fermions are a redefinition of symbols and the type of inner product.

### B. Fermion model with $q$-body interactions

In the same way that the local fermion parity acts as the transverse field in the Ising model, a fermion model can have interaction terms which play a similar role to the Ising interactions. For example, the Hamiltonian of the 1D toy model of a p-wave superconductor:

$$H = \sum_i (-i\gamma'_i \gamma_i + \Delta c_i c_{i+1} + h.c.) - \mu \left( c_i^\dagger c_i - \frac{1}{2} \right)$$ \hspace{1cm} (40)

in terms of Majorana fermions for $t = \Delta$ reads

$$H = \frac{i}{2} \sum_i 2t\gamma'_i \gamma_i - \mu \gamma_i \gamma'_i.$$ \hspace{1cm} (41)

The interaction term in this case is the Majorana bilinear $\gamma'_i \gamma_i$ acting on nearest neighbor sites. We represent this algebraically with an interaction term $S$.

$$S = \begin{pmatrix} 1 \\ i \end{pmatrix}.$$ \hspace{1cm} (42)

In general, we define an interaction term to be any even $q$-body interaction in the Hamiltonian which does not commute with local fermion parity operators (the chemical potential terms). A fermion system with $q = 2$ is a free-fermion system, but in general we allow $q$ to be any even number (which we will from now on assume). Therefore, each interaction term can be represented as a column of a matrix $S$ which satisfies $\langle P, S \rangle_F \neq 0$. The number $q$ is equal to the number of terms appearing
TABLE III. Summary of operators under the (generalized) JW duality in the algebraic notation. Objects on the left map to the objects in the corresponding row on the right (except the symmetry constraints in the last row).

| Interaction term | Operators | Dual Operators |
|------------------|-----------|----------------|
| $S = \begin{pmatrix} S_Z \\ S_X \end{pmatrix}$ | $\tilde{X} = \begin{pmatrix} T_{N \times N} \\ \mathbb{1}_{N \times N} \end{pmatrix}$ |
| Local Parity | $P = \begin{pmatrix} \mathbb{1}_{K \times K} \\ \mathbb{1}_{K \times K} \end{pmatrix}$ | $\tilde{Z} = \begin{pmatrix} (S, P)_F \mathbb{1}_{N \times N} \\ 0_{K \times N} \end{pmatrix}$ |
| Fermion map $\sigma_F = (S \ P)$ | Ising map $\hat{\sigma} = (\tilde{X} \ \tilde{Z})$ |
| Excitation map $\epsilon_F = \sigma_F^\dagger = \begin{pmatrix} S_Z^\dagger \\ \mathbb{1}_{K \times K} \end{pmatrix}$ | Excitation map $\tilde{\epsilon} = \tilde{\sigma}_N^\dagger = \begin{pmatrix} \mathbb{1}_{N \times N} \\ T_{N \times N} \end{pmatrix}$ |
| Symmetry $\ker \epsilon_F = \sigma_F \ker \sigma$ | Symmetry $\ker \tilde{\epsilon} = \sigma_F \ker \sigma$ |

Condition: $(S, S)_F = S_Z^2 + S_X^2 = T + T^\dagger$

in each column of $S$, and in general can vary between columns.

Given a set of interacting terms represented by $S$, we will assume that $P$ and $S$ generate all possible allowed operators in this system. Hence, defining the augmented matrix $\sigma_F = (S \ P)$, the allowed terms in the fermionic Hamiltonian are images of $\sigma_F$, which form a submodule of the Majorana module $M$. We will call $\sigma_F$ the fermion map.

Again, we similarly define the commutation matrix of the fermion map as $\langle \sigma_F, \sigma_F \rangle_F \neq 0$, and $\ker \epsilon_F$ represent the symmetries of the fermionic Hamiltonian. Because all the operators are even, $\ker \epsilon_F$ will always include the total fermion parity $\sum_{i_1, \ldots, i_d} x_{i_1}^+ \cdots x_{i_d}^+ P$. Nevertheless, as we will see, the fermionic Hamiltonian might have further symmetries depending on the type of interaction terms, such as conservation for individual lines, planes, or even fractal submanifolds.

C. Jordan-Wigner Duality

Similarly to the KW duality, the JW duality maps fermionic operators to dual Pauli operators which have the same commutations relations. In the algebraic language, given a fermion map $\sigma_F$, we would like to find a dual map $\tilde{\sigma}$ such that $\langle \sigma_F, \sigma_F \rangle_F = \langle \tilde{\sigma}, \tilde{\sigma} \rangle$. That is, the following diagram commutes

$$\begin{array}{ccc}
G & \xrightarrow{\sigma_F} & M \\
\xrightarrow{\tilde{\sigma}} & E \\
\xrightarrow{\tilde{\epsilon}_F} & \tilde{P}
\end{array}$$

However, interaction terms, unlike the Ising interactions in the KW duality, do not generally need to commute\textsuperscript{1–3}. That is, one might have $\langle S, S \rangle \neq 0$. Nevertheless, we will demonstrate that such a local mapping from even fermionic operators to bosonic operators always exists. The expression of the bosonic operators is almost identical to the KW duality. We can write $\tilde{\sigma} = (\tilde{X} \ \tilde{Z})$ where

$$\tilde{X} = \begin{pmatrix} T_{N \times N} \\ \mathbb{1}_{N \times N} \end{pmatrix}, \quad \tilde{Z} = \begin{pmatrix} (S, P)_F \mathbb{1}_{N \times N} \\ 0_{K \times N} \end{pmatrix},$$

for some matrix $T$ which satisfies $T + T^\dagger = (S, S)_F$. We will call $T$ the transmutation matrix. As the name suggests, the importance of $T$ is to modify the statistics of the excitations of the dual model from bosons to fermions when $(S, S)_F \neq 0$. Let us prove that the given map works.

**Proposition 4.** $\tilde{\sigma} = (\tilde{X} \ \tilde{Z})$ as given by Eq. (44) where $T$ satisfies $T + T^\dagger = (S, S)_F$ is a valid JW dual.

**Proof.** We compute $\langle \tilde{Z}, \tilde{Z} \rangle = 0$, $\langle \tilde{X}, \tilde{X} \rangle = T + T^\dagger$, and $\langle \tilde{Z}, \tilde{X} \rangle = (S, P)_F \mathbb{1}_{N \times N} = (P, S)_F$.

We will set $T = 0$, and the dual operators are

$$\tilde{X} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \tilde{Z} = \begin{pmatrix} 1 + \tilde{\epsilon} \\ 0 \end{pmatrix},$$

with a global $\mathbb{Z}_2$ symmetry. This is precisely the same as the KW dual of the 1D Ising model (19). In this duality, the trivial “product state” of fermions maps to the symmetry-broken phase, while the “Majorana” phase maps to the paramagnetic phase.

We remark that this mapping is the opposite of the duality usually discussed in 1D, since the Majorana edge
mode is often associated to the ground state degeneracy due to spontaneous symmetry breaking. We present two comments regarding this issue.

First, the duality presented has symmetry constraints at the level of states, meaning that we only allowed to map parity even fermionic states to $\mathbb{Z}_2$ symmetric states and vice versa. Therefore, only the symmetric combinations map to each other and there is no degeneracy in this restricted Hilbert space.

Second, the two different JW dualities differ precisely by an additional KW duality. However, as we will see in higher dimensions, this extra step is not always possible if $T \neq 0$ since $X$ is no longer a transverse field. (This can also be seen as an indication of an ’t Hooft anomaly). Therefore, the JW duality presented here is the natural one to generalize.

Similarly to the KW duality, we also have that the symmetry constraints are exactly induced by the set of dual operators that product to the identity. That is,

**Proposition 5.** $\ker \epsilon_F = \sigma_F \ker \sigma$

**Proposition 6.** $\ker \tilde{\epsilon} = \tilde{\sigma} \ker \sigma_F$

Again, we defer the proofs to Appendix A. A summary of the relevant operators under the duality is listed in Table III.

**D. Choices of the Interacting Terms and Transmutation Matrix**

We will now turn to discuss the nuances of the JW duality. The first main difference from the KW duality is the necessity of the transmutation matrix $T$. First, we must show that for any given choice of interaction terms $S$, we can always construct such a $T$. The proof is actually constructive. Denote $S_i$ as the $i^{th}$ column of $S$. Since $\langle S, S \rangle_F$ is Hermitian, we can explicitly construct $T$ as an upper triangular matrix, keeping only entries $\langle S_j, S_k \rangle_F$ for $j < k$. The diagonal elements $T_{jk}$ can be constructed by picking “half” the entries in $\langle S_j, S_j \rangle_F$.

The argument above is formally shown below.

**Lemma 7.** Given $S$, there always exists a matrix $T$ such that $T + T^\dagger = \langle S, S \rangle_F$

**Proof.** Since the diagonal elements $\langle S_j, S_j \rangle_F$ has a unique decomposition

$$\langle S_j, S_j \rangle_F = \sum_i c_i^{(j)} x_1^{i_1} \cdots x_d^{i_d}$$

for coefficients $c_i^{(j)} \in \mathbb{F}_2$, and $\langle S_j, S_j \rangle_F = \langle S_j, S_j \rangle_F$, it follows that $c_i^{(j)} = c_i^{(j)}$, where $-i = (-i_1, \cdots, -i_d)$. In particular, $c_i^{(j)} = 0$. Therefore, define

$$b_i^{(j)} = \begin{cases} c_i^{(j)}, & \text{if the smallest } l \text{ where } i_l \neq 0 \text{ satisfies } i_l > 0 \\ 0, & \text{else} \end{cases}$$

Then $T$ defined as

$$T_{jk} = \begin{cases} \langle S_j, S_k \rangle_F, & j < k \\ \sum_i b_i^{(j)} x_1^{i_1} \cdots x_d^{i_d}, & j = k \\ 0, & j > k \end{cases}$$

has the desired property.

The choice of $T$ constructed above is not unique, however, we show in the following that such an ambiguity is not important.

**Proposition 8.** 1. All choices of $T$ differ by a Hermitian matrix $A = A^\dagger$.
2. Different choices of $T$ give rise to dual symmetries that are related by a basis transformation.

**Proof.** 1. Let $T$ and $T'$ be two valid choices, and $A = T' + T$. Then $A + A^\dagger = (T + T^\dagger) + (T' + T'^\dagger) = 0$.
2. Since $\tilde{\epsilon}$ has the form

$$\tilde{\epsilon} = \tilde{\sigma}^\dagger \lambda_N = \begin{pmatrix} \mathbb{1} & T^\dagger \\ 0 & (P, S_j)_F \end{pmatrix},$$

one can verify that

$$\tilde{\epsilon}' = \tilde{\sigma} \begin{pmatrix} \mathbb{1} & A \\ 0 & \mathbb{1} \end{pmatrix}.$$

Since $\begin{pmatrix} \mathbb{1} & A \\ 0 & \mathbb{1} \end{pmatrix}$ is its own inverse, one concludes that

$$\ker \tilde{\epsilon}' = \begin{pmatrix} \mathbb{1} & A \\ 0 & \mathbb{1} \end{pmatrix} \ker \tilde{\epsilon}.$$

One can verify that above matrix is a symplectic transformation when $A$ is Hermitian, and therefore relates the two symmetries by a basis transformation.

The basis transformation as a Clifford unitary can be read off from the symplectic matrix. It is most intuitively depicted by edges of a translation-invariant graph that connect vertices at the origin to the locations determined by $A$. The unitary is a diagonal operator, consisting of $S$ gates where edges connect the same vertex, and Controlled-Z gates on all other edges.

Lastly, we discuss the effects of modifying the interaction terms $S$. For example, in the 1D chain, we defined the interaction term as $i_\gamma^{(i)}(\gamma_{i+1})$. However, one could claim that $i_\gamma^{(i)}(\gamma_{i+1})$ is also equally valid as a interaction term, which will change $S$ and its dual $X$. Although that is indeed the case, redefining the interaction terms by attaching local fermion parities will not change the dual symmetry. Below, we implicitly sum over the index $k$, which runs over the labels of sites in the unit cell.

**Proposition 9.** The dual symmetry is invariant (up to a basis transformation) under $S_i \rightarrow S_i' = S_i + P_k f_k i$, for any $f_k \in \mathbb{F}_2[x_1^{i_1}, \cdots, x_d^{i_d}]$. 
The fermion map and excitation maps are
\[ \sigma_F = \begin{pmatrix} 1 & 1 & 1 \\ x & y & 1 \end{pmatrix}, \quad \epsilon_F = \begin{pmatrix} 1 & \bar{x} \\ \bar{y} & 1 \end{pmatrix}. \] (58)

Evidently, the only symmetry is the total fermion parity
\[ \ker \epsilon_F = \left( \frac{\sum_{ij} x^i y^j}{\sum_{ij} x^j y^i} \right). \] (59)

To perform the JW duality, we first compute the commutation matrix
\[ \langle S, S \rangle_F = \begin{pmatrix} 0 & 1 + \bar{x} y \\ 1 + x \bar{y} & 0 \end{pmatrix}, \] (60)
and choose a corresponding transmutation matrix as
\[ T = \begin{pmatrix} 0 & \bar{x} y \\ 1 & 0 \end{pmatrix}. \] (61)

Therefore, the dual operators are
\[ \tilde{X} = \begin{pmatrix} 0 & \bar{x} y \\ 1 & 0 \end{pmatrix}, \quad \tilde{Z} = \begin{pmatrix} 1 + \bar{x} \\ 0 \end{pmatrix}. \] (62)

From Prop. 6, the dual symmetries can be calculated from dualizing the fermionic operators that product to identity. Similarly to the KW duality, there is one local and two non-local relations.
\[ \ker \sigma_F = \begin{pmatrix} 1 + y \sum_i x^i \\ 1 + x \sum_i y^i \\ 0 \end{pmatrix}, \] (63)

Therefore, we have
\[ \ker \tilde{\epsilon} = \tilde{\sigma} \ker \sigma_F = \begin{pmatrix} 1 + x \sum_i y^i \\ 1 + y \sum_i x^i \\ 0 \end{pmatrix}. \] (64)

To summarize visually, the duality maps the fermionic Hamiltonian
\[ H = -\sum_i \begin{bmatrix} \ldots & p_{\ldots} & g_x \gamma \ldots \gamma' & g_y \gamma' \gamma \ldots \end{bmatrix} \] (65)
with global \( \mathbb{Z}_2^F \) symmetry constraint \( \prod_i P_i = 1 \) to
\[ \tilde{H} = -\sum_i \begin{bmatrix} \ldots & Z & \ldots & g_x Z \ldots & g_y Z \ldots \end{bmatrix} \] (66)
with 1-form symmetries

\[
\frac{\alpha X \gamma}{\gamma \gamma' X} = \cdots \frac{\gamma' X \gamma}{\gamma \gamma' X} = \frac{\gamma' X \gamma}{\gamma \gamma' X} = 1.
\]

As an application, we can also dualize the Majorana color code\cite{15, 28}. With the choice of unit cell as given in Figure 2, the stabilizer can be written as

\[
H = \sigma_\mathbf{F} \left( \begin{array}{ccc}
1 + \bar{x} & 1 + \bar{y} & 1 + \bar{z} \\
1 + x & 1 + y & 1 + z \\
1 + \bar{x} & 1 + \bar{y} & 1 + \bar{z}
\end{array} \right).
\]

First, we remark that the model alone has a larger symmetry, which preserves fermion parity in individual diagonal lines. However, we explicitly break those by allowing the interaction terms given by \( S \). The duality in the case of subsystem symmetry can be found in Section IV.E.

Dualizing this stabilizer gives

\[
\tilde{H} = \sigma_\mathbf{F} \left( \begin{array}{ccc}
1 + \bar{x} & 1 + \bar{y} & 1 + \bar{z} \\
1 + x & 1 + y & 1 + z \\
1 + \bar{x} & 1 + \bar{y} & 1 + \bar{z}
\end{array} \right).
\]

The ground state of this model under the 1-form symmetry exhibits a \( \mathbb{Z}_3^2 \) topological order. The corresponding Wilson loops are the non-local line operators in Eq. (67), and the dual of the line operators of the Majorana color code. Their \( \xi \) Hooft loops are obtained in a similar fashion.

It is interesting in its own right to analyze the stabilizer code

\[
H = - \sum_i \frac{X Y Z}{Y Z X} + \frac{Z X Y}{X Y Z}.
\]

This model can be thought of as the result of “gauging” the global fermion parity symmetry of the Majorana color code. Since this stabilizer has the same topological order as the (Pauli) color code\cite{48}, it would be interesting to compare their performances.

**B. JW for Global Fermion Parity in 3D**

The same exercise can be done for a 3D cubic lattice. There are now three interaction terms

\[
S = \left( \begin{array}{ccc}
\frac{x+y}{z} & \frac{y+z}{x} & \frac{z+x}{y} \\
\frac{x+y}{z} & \frac{y+z}{x} & \frac{z+x}{y} \\
\frac{x+y}{z} & \frac{y+z}{x} & \frac{z+x}{y}
\end{array} \right),
\]

and the corresponding commutation matrix is

\[
\langle S, S \rangle_\mathbf{F} = \left( \begin{array}{ccc}
0 & 1 + \bar{x} y & 1 + \bar{x} z \\
1 + x \bar{y} & 0 & 1 + \bar{y} z \\
1 + x \bar{z} & 1 + y \bar{z} & 0
\end{array} \right).
\]

We choose\cite{49}

\[
T = \left( \begin{array}{ccc}
\frac{x+\bar{y}}{z} & \frac{\bar{y}+z}{x} & \frac{z+x}{\bar{y}} \\
\frac{1}{x} & \frac{\bar{y}}{z} & \frac{z+x}{\bar{y}} \\
\frac{1}{x} & \frac{\bar{y}}{z} & \frac{z+x}{\bar{y}}
\end{array} \right),
\]

so that the result is invariant under a \( C_3 \) rotation around the \((1, 1, 1)\) axis, and shifting \( x \rightarrow y \rightarrow z \rightarrow x \). The dual operators are

\[
\tilde{X} = \left( \begin{array}{ccc}
0 & \bar{x} y & 1 \\
1 & \bar{y} z & 0 \\
\bar{x} x & 0 & 1
\end{array} \right), \quad \tilde{Z} = \left( \begin{array}{ccc}
1 + \bar{x} & 1 + \bar{y} & 1 + \bar{z} \\
1 & 0 & 0 \\
0 & 1 & 0
\end{array} \right).
\]

There are three local and three non-local symmetry constraints. A similar calculation as the 2D case gives
The symmetries listed here are a 2-form symmetry. When $Z$ is the stabilizer, it is deconfined and the ground state has the same topological order as a “twisted” 3D toric code where the emergent point particle has fermionic statistics$^{10,50}$.

As an application, we dualize the six Majorana Hamiltonians with extensive ground state degeneracy proposed in Ref. 15, the first of which is the Majorana checkerboard model. Like the 2D case, these models alone actually have a larger symmetry, but we explicitly break them by adding the interaction terms as shown above.

The results are summarized in Table VII of Appendix B.

### C. 1-form fermion parity in 3D

Lattice models with higher-form symmetry can be constructed for spin models$^{10-1251}$. For example, the KW dual of the 2D and 3D transverse field Ising models have 1-form and 2-form symmetries respectively.

Here, we consider a 3D fermionic model with 1-form fermion parity symmetry. We place fermions on the edges of a cubic lattice and consider interaction terms to be generated by Majorana operators of four links surrounding a plaquette as shown in Table IV. The local 1-form symmetries are generated by a product of six fermion parity operators on links surrounding a vertex, and the non-local symmetries form non-trivial 2-cycles around the torus in the dual lattice.

In the algebraic notation, there are three sites per unit cell: one per each link. We can write

$$S = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & z & y \\ z & 0 & x \\ y & x & 0 \end{pmatrix} \quad \text{and} \quad P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}. \quad (76)$$

The 1-form symmetries are generated by

$$\ker \epsilon_F = \begin{pmatrix} 1 + z + \bar{x}z(1 + y) & z(1 + \bar{x}) & 1 + x & 1 + x & \bar{y} & \sum_i z^i \\ 1 + y & 1 + x + \bar{y}x(1 + z) & x(1 + \bar{y}) & 0 & \sum_i x^i z^i & 0 \\ y(1 + \bar{z}) & 1 + z & 1 + y + \bar{y}y(1 + x) & 0 & \sum_i x^i y^i & \sum_i z^i \\ \bar{x} & 0 & 0 & 0 & 1 & \bar{x} \\ \bar{z} & 0 & 0 & 0 & 0 & \bar{z} \end{pmatrix}. \quad (77)$$

We choose the following transmutation matrix

$$T = \begin{pmatrix} 0 & x\bar{y} & 1 \\ 1 & 0 & y\bar{z} \\ z\bar{x} & 1 & 0 \end{pmatrix}. \quad (78)$$

Therefore, the dual operators are

$$\tilde{X} = \begin{pmatrix} 0 & x\bar{y} & 1 \\ 1 & 0 & y\bar{z} \\ z\bar{x} & 1 & 0 \end{pmatrix}, \quad \tilde{Z} = \begin{pmatrix} 0 & 1 + \bar{z} & 1 + \bar{y} \\ 1 + z & 0 & 1 + \bar{x} \\ 1 + \bar{y} & 1 + \bar{x} & 0 \end{pmatrix}. \quad (79)$$

The duality is depicted visually in Table IV. The local dual symmetry constraints are generated from

$$\tilde{G} = \tilde{\sigma} = \begin{pmatrix} 1 + x \\ 1 + y \\ 1 + z \\ y + z \\ x + z \\ x + y \end{pmatrix} \begin{pmatrix} 1 + x + y + x\bar{z} \\ 1 + y + z + y\bar{x} \\ 1 + z + x + z\bar{y} \\ 1 + x \\ 1 + y \\ 1 + z \end{pmatrix} \quad (80)$$

The ground state of this dual spin model when $Z$ dominates is actually just the 3D toric code. This is because the local symmetry constraint can be written as the vertex terms attached with one plaquette term per orientation as shown above.

It is not clear whether this 1-form symmetry is anomalous. If it is so, we conjecture from the bulk-boundary correspondence that the anomaly should be matched by a bulk SPT with $\mathbb{Z}_2$ 1-form symmetry. Such SPT in 4+1D is classified by $\mathbb{Z}_2^{52,53}$, with response to a background $\mathbb{Z}_2$ 2-form $B$ given by $B \cup S^1 B$, where $S^1$ is the 1st Steenrod square.
TABLE IV. Duality of operators (up to a sign) for a fermion system with 1-form symmetry in 3D to a spin system with $Z_2$ 1-form symmetry. The interaction terms living on plaquettes are sent to the red edges in the dual lattice. Here, the blue and red lines represent Pauli $Z$’s and $X$’s, respectively.

| Fermion | Spin | Fermion | Spin |
|---------|------|---------|------|
| $\gamma'$ | $\gamma$ | $P$ | $P$ |
| $\gamma'$ | $\gamma$ | $P$ | $P$ |
| $\gamma'$ | $\gamma$ | $P$ | $P$ |

D. Subsystem Fermion parity in 2D

The prototypical example of a 2D system with line subsystem symmetry is the Plaquette Ising a.k.a. the Xu-Moore model\textsuperscript{14}. Such models with subsystem symmetry can host subsymmetry-broken phases, subsystem symmetry-protected phases\textsuperscript{16,31}, or topological order\textsuperscript{32,54}. Here, we consider an analogous fermionic system, which will turn out to be JW dual to such a spin system with line symmetry.

Consider a square lattice with interaction term

$$S = \gamma' \gamma P,$$

or in the algebraic notation,

$$S = \begin{pmatrix} x(1+y) \\ 1+y \end{pmatrix}.$$

The Hamiltonian has subsystem fermion parity symmetry, defined as the product of the local fermion parity operators on each individual vertical and horizontal lines. They correspond to

$$\ker \mathbf{\epsilon}_F = \begin{pmatrix} \sum_i x^i y^i \\ \sum_i x^i \end{pmatrix}.$$

To perform the JW duality, we compute

$$\langle S, S \rangle_F = \langle 0 \rangle = 0,$$

that is, all the interaction terms commute, and so we can trivially choose $T = 0$. The dual operators are therefore

$$\hat{X} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \hat{Z} = \begin{pmatrix} (1 + \bar{x})(1 + \bar{y}) \\ 0 \end{pmatrix}.$$

and the dual symmetry is

$$\ker \hat{\mathbf{\epsilon}} = \ker \begin{pmatrix} 1 \\ 0 \\ (1 + x)(1 + y) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \sum_i x^i \sum_i y^i \end{pmatrix}.$$

Thus, we see that fermion parity and the interaction term respectively map to the Ising term and the transverse field in the Xu-Moore model. Pictorially,

$$i\gamma' \gamma \rightarrow Z \ Z, \quad \gamma' \gamma \rightarrow X,$$

Since the interaction term fully commutes, the Hamiltonian consisting only of this term is exactly solvable. However, it is symmetry breaking. On a torus of size $L_x \times L_y$, the product of $S$ along any column vanishes. More precisely,

$$\sum_i y^i S = 0.$$

Hence, the stabilizer has an extensive ground state degeneracy of $L_x$, which can be labeled by the eigenvalues of the $L_x$ vertical line symmetries. The degeneracy can be broken by explicitly breaking the vertical symmetries with $\gamma' \gamma$, which commutes with $S$. The model is still exactly solvable, with remaining horizontal symmetries, and the ground states are decoupled horizontal Majorana chains.

Now, consider the following operator, which is a 90° rotated interaction term

$$S' = \begin{pmatrix} \gamma' \gamma \\ \gamma' \gamma' \end{pmatrix}.$$

A Hamiltonian consisting of only this term spontaneously breaks the horizontal line symmetries. Interestingly, the result of bosonizing this operator using the map (88) is a 2D cluster state, which is the stabilizer for the $Z_2$ SSPT phase\textsuperscript{16,31} given by

$$\begin{pmatrix} Z & Z \\ Z & X \ Z \\ Z & Z \end{pmatrix}.$$

Furthermore, one can also consider the KW dual of the above stabilizer, which is the Wen-Plaquette model\textsuperscript{32}

$$\begin{pmatrix} Y & Z \\ Z & Y \end{pmatrix}.$$

The ground state of this Hamiltonian spontaneously breaks the $Z_2$ line symmetry, and is distinct from the
TABLE V. Duality of operators (up to a sign) for a fermion system with line symmetry to a spin system with $\mathbb{Z}_2$ line symmetry.

| Fermion | Comment       | Spin   | Comment           |
|---------|---------------|--------|-------------------|
| $i\gamma\gamma'$ | Trivial        | $\mathbb{Z} \times \mathbb{Z}$ | Symmetry breaking |
| $\gamma' \gamma$ | SS in $y$ direction | $X$ | Trivial |
| $\gamma' \gamma$ | SS in $x$ direction | $\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$ | SSPT (2D cluster state) |
| $\gamma' \gamma$ | $\mathbb{Z}_2$ Topological order | $Y \times \mathbb{Z}$ | $\mathbb{Z}_2$ Topological order (Wen plaquette) |

TABLE VI. Duality of operators (up to a sign) for a fermion system with (diagonal) line symmetry to a spin system with $\mathbb{Z}_2$ (diagonal) line symmetry.

| Fermion | Comment       | Spin   | Comment           |
|---------|---------------|--------|-------------------|
| $i\gamma\gamma'$ | Trivial        | $\mathbb{Z}$ | Symmetry breaking |
| $\gamma' \gamma$ | Symmetry Breaking | $X$ | Trivial |
| $\gamma' \gamma$ | Symmetry Breaking | $\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$ | $\mathbb{Z}_2$ SSPT |
| $\gamma' \gamma$ | $\mathbb{Z}_2$ Topological order (Majorana color code) | $\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$ | $\mathbb{Z}_2$ SSPT |

The fact that the interaction terms all commute allows us to further perform a KW duality on the spin system. In Appendix C, we show that the combined duality is actually a “naive” JW duality that one would do in a 2D system.

The symmetries in this system are diagonal line symmetries, with normals pointing in the $(1,1)$ and $(1,-1)$ directions. The stabilizers can be tripartited into three mutually commuting sets given by three alternating rows. The product of the stabilizers in every three rows is the total fermion parity. An identical calculation to that of the Majorana color code shows that the degeneracy of such a model on a torus is 4. Therefore, the JW dual of the Wen-plaquette model also realizes a $\mathbb{Z}_2$ topological order.

A summary of the fermionic operators and their dual is given in Table V.

E. $\mathbb{Z}_F^2$ line symmetry in diagonal directions in 2D

Consider a square lattice with interaction term

$$\gamma' \gamma' \gamma' \gamma'$$

or

$$S = \left(\frac{x + y}{x + y}\right)$$

in the algebraic notation.

The symmetries in this system are diagonal line symmetries, with normals pointing in the $(1,1)$ and $(1,-1)$ directions

$$\ker \epsilon_F = \left(\sum_{i}(xy)^{i\epsilon} \sum_{j}(xy)^{j\epsilon}\right).$$

Alternatively, by enlarging the unit cell to two sites, it can be viewed as $\mathbb{Z}_2^F \times \mathbb{Z}_2^F$ line symmetries in the vertical and horizontal directions as in the previous subsection. Again, because all interaction terms commute, we can dualize to spin operators

$$\tilde{X} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \tilde{Z} = \begin{pmatrix} x + y + \bar{x} + \bar{y} \\ 0 \end{pmatrix},$$

$$\gamma' \gamma' \gamma' \gamma'$$

(93)

(94)

(95)

(96)

(97)
There are also non-local gauge-constraints which can be generated by mapping a “belt” of interaction terms that wrap diagonal line symmetries. It is a 2D cluster state, given by the stabilizer
\[ \hat{H} = \left( \frac{x + y + \bar{x} + \bar{y}}{1} \right). \] (98)

The JW dual of this stabilizer is
\[ H_F = \left( \frac{1 + x + y}{1 + \bar{x} + \bar{y}} \right), \] (99)

which is exactly the Majorana color code \(^{28,29}\), (though inverted compared to Eq. (68).

One can also bosonize the 90° rotated interaction term. This turns out to be a different SSPT. The mapping of operators in this duality is summarized in Table VI.

\[
\{S, S\}_F = \begin{pmatrix}
(y + \bar{y})(z + \bar{z}) & (1 + x)(1 + \bar{y})(z + \bar{z}) & (1 + x)(y + \bar{y})(1 + \bar{z}) \\
(1 + \bar{x})(1 + y)(z + \bar{z}) & (x + \bar{x})(z + \bar{z}) & (x + \bar{x})(1 + y)(1 + \bar{z}) \\
(1 + \bar{x})(y + \bar{y})(1 + z) & (x + \bar{x})(1 + \bar{y})(1 + z) & (x + \bar{x})(y + \bar{y})
\end{pmatrix}.
\] (101)

We choose the following transmutation matrix
\[ T = \begin{pmatrix}
\bar{y}(z + \bar{z}) & (1 + x)(1 + \bar{y})\bar{z} & (1 + x)\bar{y}(1 + \bar{z}) \\
(1 + \bar{x})(1 + y)\bar{z} & (x + \bar{x})\bar{z} & \bar{x}(1 + y)(1 + \bar{z}) \\
(1 + \bar{x)}\bar{y}(1 + z) & \bar{x}(1 + \bar{y})(1 + z) & \bar{x}(y + \bar{y})
\end{pmatrix}, \] (102)

so that it is invariant under a \( C_3 \) rotation around the \((1,1,1)\) axis on the cubic lattice. Namely, a cyclic permutation of the rows and columns and the coordinates \(x \rightarrow y \rightarrow z \rightarrow x\) leaves \( T \) invariant.

To summarize, the JW dual has operators
\[
\hat{X} = \begin{pmatrix}
\bar{y}(z + \bar{z}) & (1 + x)(1 + \bar{y})\bar{z} & (1 + x)\bar{y}(1 + \bar{z}) \\
(1 + \bar{x})(1 + y)\bar{z} & (x + \bar{x})\bar{z} & \bar{x}(1 + y)(1 + \bar{z}) \\
(1 + \bar{x)}\bar{y}(1 + z) & \bar{x}(1 + \bar{y})(1 + z) & \bar{x}(y + \bar{y})
\end{pmatrix}, \quad \hat{Z} = \begin{pmatrix}
(1 + \bar{y})(1 + \bar{z}) \\
(1 + \bar{x})(1 + \bar{z}) \\
(1 + \bar{x})(1 + \bar{y})
\end{pmatrix}. \] (103)

The symmetries of the dual model includes both local and non-local constraints. The local gauge constraints \( \hat{G} \) are generated by
\[
\hat{G} = \begin{pmatrix}
(1 + x)(y\bar{z} + \bar{y}z) & (1 + x)(y\bar{z} + \bar{y}z) & (1 + x)z\bar{x} \\
0 & (1 + y)z\bar{x} & 0 \\
1 + x & 0 & 1 + y \\
1 + y & 1 + y & 0 \\
0 & 0 & 1 + z
\end{pmatrix} \] (104)

There are also non-local gauge constraints which can be generated by mapping a “belt” of interaction terms that wrap around the torus. Like the X-cube case, there are \( 6L - 3 \) such independent non-local constraints.

Pictorially, the duality maps the fermionic Hamiltonian
\[
\hat{H} = -\sum \left[ -i\gamma\gamma' + g_x \sqrt{g_y + g_x} + g_z \sqrt{g_y + g_z} \right], \] (105)

F. (100) Fermion Planar symmetry in a cubic lattice

The plaquette Ising model for a cubic lattice in 3D has a KW dual which is a gauge theory. In the deconfined phase, the ground state is the same as that of the X-cube model\(^{15,21}\). The gauge constraints are the “cross” terms in the X-cube model which, when enforced energetically, forbids lineon excitations.

We will now consider the fermion analog of this model and perform a JW duality. The interaction terms are four Majorana operators at the corners of each face.

\[
S = \begin{pmatrix}
(1 + y)(1 + z) & (1 + x)(1 + z) & (1 + x)(1 + y)
\end{pmatrix} \] (100)

The symmetries of this model are fermion parity conservation in each individual \( xy, yz, \) and \( xz \) plane.

To perform the duality, we first calculate the commutation matrix
with conserved fermion parity on individual planes to the following spin model on the dual lattice

\[
\hat{H} = -\sum \left[ \begin{array}{c}
\begin{array}{c}
\text{plane 1} \\
\text{plane 2} \\
\text{plane 3} \\
\text{plane 4} \\
\text{plane 5} \\
\text{plane 6}
\end{array}
\end{array}
+ g_x + g_y + g_z
\right],
\]

(106)

with local gauge constraints

\[
\begin{align*}
\begin{array}{c}
\text{plane 1} \\
\text{plane 2} \\
\text{plane 3} \\
\text{plane 4} \\
\text{plane 5} \\
\text{plane 6}
\end{array}
= \begin{array}{c}
1 \\
1 \\
1 \\
1 \\
1 \\
1
\end{array}
\end{align*}
\]

(107)

Here, the red and blue lines denote Pauli X and Z operators, respectively.

When \( g_x = g_y = g_z = 0 \), the model is deconfined exactly solvable. The emergent excitations from violating the cube terms are fractons. Furthermore, because the gauge constraints are modified “cross” terms of the X-cube model, the mobility of these fractons are exactly identical: four fractons can be created using the dual of the interaction terms \( \tilde{X} \), and pairs of fractons can move in a plane.

Under this duality, we can also dualize the Majorana model\(^{15} \) given by \( H_F = \left( f_3 \right) \), where \( f_3 = 1 + x + y + yz + xz \), since it can be written as

\[
H_F = \sigma_F \left( \begin{array}{c}
1 + yz \\
1 + xz \\
0
\end{array} \right).
\]

(108)

The resulting dual Hamiltonian is

\[
\hat{H} = \left( \begin{array}{c}
\begin{array}{c}
1 + x + \bar{x}y + x\bar{y} + y\bar{z} + x\bar{y}z + yz + x\bar{z} + \bar{x}z + \bar{y}z \\
1 + \bar{x} + \bar{y}z + y\bar{z} + x\bar{y}z + x\bar{z} + \bar{x}z + \bar{y}z \\
1 + x + \bar{x}y + x\bar{y} + y\bar{z} + x\bar{y}z + yz + x\bar{z} + \bar{x}z + \bar{y}z
\end{array}
\end{array} \right).
\]

(109)

As fermion models, \( H_F \) and \( P \) clearly realize different fracton phases since \( H_F \) has an extensive degeneracy, while \( P \) has a unique ground state. Therefore, the stabilizer code obtained by replacing \( \tilde{Z} \) with \( \hat{H} \) should also realize different fracton phases. It would be interesting to look into the properties of this model.

G. (110) Fermion planar symmetry in 3D

Consider the interaction term

\[
S = \left( \begin{array}{c}
x + y + z \\
\bar{x} + \bar{y} + \bar{z}
\end{array} \right).
\]

(110)

The symmetries of this system are six planar symmetries, given by the (110), (101), (101), (011), and (011) planes.

Since, \( \langle S, S \rangle_F = 0 \), the dual operators are

\[
\tilde{X} = \left( \begin{array}{c}
0 \\
1 \\
0 \\
0 \\
1 \\
0
\end{array} \right), \quad \tilde{Z} = \left( \begin{array}{c}
x + y + z + \bar{x} + \bar{y} + \bar{z} \\
0
\end{array} \right),
\]

(111)

and the dual symmetries are Pauli X operators acting on the six same planes. In this duality, the Majorana checkerboard model\(^{29} \) whose stabilizer is

\[
H_F = \left( \begin{array}{c}
1 + x + y + z \\
1 + \bar{x} + \bar{y} + \bar{z}
\end{array} \right)
\]

(112)

dualizes to the 3D cluster state on a cubic lattice

\[
\hat{H} = \left( \begin{array}{c}
x + y + z + \bar{x} + \bar{y} + \bar{z} \\
0
\end{array} \right).
\]

(113)

H. Fibonacci Fractal symmetry in 2D

We consider a fermion model with the following interaction term

\[
S = \gamma \gamma \gamma \gamma',
\]

(114)

which is represented by

\[
S = \left( \begin{array}{c}
y(1 + x + \bar{x}) \\
1
\end{array} \right)
\]

(115)
in the algebraic notation. The symmetry that protects this phase is generated by fermion parities placed in the fractal shape of the Fibonacci Cellular Automaton (CA)

$$\ker \epsilon_F = \left( \sum_i \bar{y}^i(1 + x + \bar{x})^i \right),$$

which is depicted visually in Figure 3. Since this CA is invertible, the symmetries are well defined on a torus.

We calculate the commutation matrix to be $\langle S, S \rangle_F = x^2 + \bar{x}^2$ and so we choose $T = x^2$. The dual operators are

$$\tilde{X} = \begin{pmatrix} x^2 \\ 1 \end{pmatrix}, \quad \tilde{Z} = \begin{pmatrix} 1 + \bar{y}(1 + x + \bar{x}) \\ 0 \end{pmatrix}. \quad (117)$$

The dual symmetries are also fractal.

$$\ker \tilde{\epsilon} = \left( x^2 \sum_i \bar{y}^i(1 + x + \bar{x})^i \right) \quad (118)$$

and can be thought of as Pauli-$X$ operators applied in the upside-down version of the Fibonacci fractal pattern above, followed by Pauli-$Z$ operators in the same pattern, but displaced two sites in the $-x$ direction, as shown in Figure 3. Visually, it is clear that since the positions of the Pauli-$Z$’s are translated to the left, and hence there is no translation invariant circuit that can remove the Pauli-$Z$’s. When $Z$ is the stabilizer, it is in the spontaneously broken phase.

I. Fibonacci Fractal symmetry in 3D

We now consider a similar model in 3D with an extra hopping (i.e., $q = 2$ interaction) in the $z$-direction. That is,

$$S = \begin{pmatrix} y(1 + x + \bar{x}) & 1 \\ 1 & z \end{pmatrix}. \quad (119)$$

The symmetries are now stacks of Fibonacci CA in the $z$-direction. The commutation matrix is

$$\langle S, S \rangle_F = \begin{pmatrix} x^2 + \bar{x}^2 & \bar{y}(1 + x + \bar{x}) + z \\ y(1 + x + \bar{x}) + \bar{z} & 0 \end{pmatrix}. \quad (120)$$

and so we choose the transmutation matrix

$$T = \begin{pmatrix} x^2 & \bar{y}(1 + x + \bar{x}) + z \\ 0 & 0 \end{pmatrix}. \quad (121)$$

The dual operators are

$$\tilde{X} = \begin{pmatrix} x^2 & \bar{y}(1 + x + \bar{x}) + z \\ 0 & 0 \end{pmatrix}, \quad \tilde{Z} = \begin{pmatrix} 1 + \bar{y}(1 + x + \bar{x}) \\ 1 + \bar{z} \\ 0 \end{pmatrix}. \quad (122)$$

In this model, there is a local dual symmetry

$$\tilde{G} = \begin{pmatrix} \bar{x}^2(1 + z) \\ (1 + y(1 + x + \bar{x}))(1 + \bar{z}) \\ 1 + z \\ 1 + \bar{y}(1 + x + \bar{x}) \end{pmatrix} \in \ker \tilde{\epsilon} \quad (123)$$

denoted in the first column. This term, along with $Z$, forms a stabilizer code

$$\begin{pmatrix} \bar{x}^2(1 + z) & 1 + \bar{y}(1 + x + \bar{x}) \\ (1 + y(1 + x + \bar{x}))(1 + \bar{z}) & 1 + \bar{z} \\ 1 + z & 0 \\ 1 + \bar{y}(1 + x + \bar{x}) & 0 \end{pmatrix},$$

which is a twisted version of Yoshida’s fractal spin model.

J. Haah’s Fractal symmetry

In analogy to the fractal Ising model, consider a fermionic system with interaction terms

$$S = \begin{pmatrix} f_1 & 0 \\ 0 & f_2 \end{pmatrix}. \quad (124)$$

where $f_1 = 1 + x + y + z$ and $f_2 = 1 + xy + yz + zx$. The symmetries of this model are local fermion parities acting on sites in the same fractal pattern as the fractal Ising model. Since $\langle S, S \rangle_F = \begin{pmatrix} f_1 f_1 & 0 \\ 0 & f_2 f_2 \end{pmatrix}$, we construct

$$T = \begin{pmatrix} t_1 & 0 \\ 0 & t_2 \end{pmatrix}. \quad (125)$$
where \( t_1 = x + y + z + x\bar{y} + y\bar{z} + z\bar{x} \) and \( t_2 = xy + yz + zx + x\bar{y} + y\bar{z} + z\bar{x} \). One can verify that \( t_1 + t_1 = f_1 \bar{f}_1 \) and \( t_2 + t_2 = f_2 \bar{f}_2 \). The dual variables are

\[
\tilde{X} = \begin{pmatrix} t_1 & 0 \\ 0 & t_2 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \tilde{Z} = \begin{pmatrix} \bar{f}_1 \\ \bar{f}_2 \\ f_2 \\ f_1 \end{pmatrix}. \tag{126}
\]

The local gauge constraints are generated from

\[
\tilde{G} = \begin{pmatrix} (t_1 + f_1 \bar{f}_1) f_2 \\ (t_2 + f_2 \bar{f}_2) f_1 \\ f_2 \\ f_1 \end{pmatrix}. \tag{127}
\]

Together, \( \tilde{Z} \) and \( \tilde{G} \) are stabilizers that realize a twisted Haah’s code. In general, these formulas hold for any model with two types of interaction terms, by replacing \( f_1, f_2 \) and solving for \( t_1, t_2 \).

The Majorana Hamiltonian given by \( H_F = (f_5 / f_5) \), where \( f_5 = 1 + x + y + z + xy + yz + zx \) can be generated from fermion parity and the interaction terms. In particular,

\[
H_F = \sigma_F \begin{pmatrix} 1 + \bar{x}\bar{y}\bar{z} \\ 1 + \bar{x}\bar{y}\bar{z} \\ xy + yz + zx + \bar{x}\bar{y}\bar{z} + \bar{y}\bar{z} + \bar{x}\bar{z} \end{pmatrix}. \tag{128}
\]

The resulting dual Hamiltonian is

\[
\hat{H} = \begin{pmatrix} x + y + z + xy + yz + zx + x\bar{y} + y\bar{z} + z\bar{x} + \bar{x}\bar{y}^2 + \bar{y}\bar{z}^2 + \bar{z}\bar{x}^2 + x\bar{y}\bar{z} + y\bar{z}x + z\bar{x}y + \bar{x}\bar{y}\bar{z}^2 + \bar{y}\bar{z}^2 + \bar{z}\bar{x}^2 + \bar{x}\bar{y}\bar{z} \\ 1 + xy + yz + zx + \bar{x}\bar{y} + \bar{y}\bar{z} + \bar{z}\bar{x} + \bar{x}\bar{y}^2 + \bar{y}\bar{z}^2 + \bar{z}\bar{x}^2 + \bar{x}\bar{y}^2 + \bar{y}\bar{z}^2 + \bar{z}\bar{x}^2 + \bar{x}\bar{y}\bar{z} \\ 1 + \bar{x}\bar{y}\bar{z} \end{pmatrix}. \tag{129}
\]

This stabilizer should result in a different phase than when \( \tilde{Z} \) is the stabilizer under this gauge constraint.

2. Perform a KW duality to obtain an Ising model. This is equivalent to the “ungauging” procedure of Ref. 20.

3. If the Ising terms contain an even number of Pauli-Z’s, construct a fermion model by replacing transverse fields \( X \) with fermion parity \( P \), and obtain interaction terms by replacing each Pauli-Z with either \( \gamma \) or \( \gamma’ \) (up to a sign \( \pm i \)).

4. Perform the JW duality on this new fermion model, and construct a new stabilizer model using the Ising terms (which will be identical to the Z stabilizers in the original CSS code), and local terms of the dual symmetry. If there is no choice such that \( \langle S, S \rangle_F = 0 \) up to attaching local fermion parity operators, then then new stabilizer code cannot be related to the original CSS code via a translation-invariant Clifford unitary.

In fact, many of the examples we have discussed in the previous section are obtained from this procedure. For example, starting from the 3D Toric Code, one can either “ungauge” the 2-form symmetry or 1-form symmetry to get a 0-form or 1-form Ising model, respectively. Replacing this bosonic system with a fermionic system with the same type of symmetries then gives the initial fermion systems for Sections IV B and IV C, respectively. We have also used this to construct the twisted X-cube, Yoshida’s fractal code, and Haah’s code.

We will further demonstrate this procedure with a class of examples. We will choose the CSS code to be a self-dual “doubled” Majorana code. As a byproduct, this procedure gives us a natural JW duality to bosonize the

FIG. 4. Constructing “twisted” stabilizer codes from known CSS codes

V. TWISTED CODES FROM CSS CODES

As an application of the KW and JW dualities, we present a general method of constructing twisted codes from CSS codes, which is illustrated in Figure 4.

1. Start from a CSS code and treat the stabilizers containing X terms as the dual symmetry. Add an X transverse field into the Hamiltonian to make it a dual transverse Ising model.

2. Perform a KW duality to obtain an Ising model. This is equivalent to the “ungauging” procedure of Ref. 20.

3. If the Ising terms contain an even number of Pauli-Z’s, construct a fermion model by replacing transverse fields \( X \) with fermion parity \( P \), and obtain interaction terms by replacing each Pauli-Z with either \( \gamma \) or \( \gamma’ \) (up to a sign \( \pm i \)).

4. Perform the JW duality on this new fermion model, and construct a new stabilizer model using the Ising terms (which will be identical to the Z stabilizers in the original CSS code), and local terms of the dual symmetry. If there is no choice such that \( \langle S, S \rangle_F = 0 \) up to attaching local fermion parity operators, then then new stabilizer code cannot be related to the original CSS code via a translation-invariant Clifford unitary.

In fact, many of the examples we have discussed in the previous section are obtained from this procedure. For example, starting from the 3D Toric Code, one can either “ungauge” the 2-form symmetry or 1-form symmetry to get a 0-form or 1-form Ising model, respectively. Replacing this bosonic system with a fermionic system with the same type of symmetries then gives the initial fermion systems for Sections IV B and IV C, respectively. We have also used this to construct the twisted X-cube, Yoshida’s fractal code, and Haah’s code.

We will further demonstrate this procedure with a class of examples. We will choose the CSS code to be a self-dual “doubled” Majorana code. As a byproduct, this procedure gives us a natural JW duality to bosonize the
Majorana code, and produces a new code which is distinct from both the original CSS code and the twisted code.

The Majorana codes introduced in Ref. 29 are given by a Majorana stabilizer of the form \( H = \left( \begin{array}{c} f \\ f \\ 0 \end{array} \right) \) for some polynomial \( f \). It satisfies \( \langle H, H \rangle_F = 0 \). In addition, we will require \( f \) to contain an even number of terms.

The doubled CSS code of such a Majorana code is given by the Pauli stabilizer

\[
\begin{pmatrix}
\tilde{f} & 0 \\
f & 0 \\
0 & \tilde{f}
\end{pmatrix}.
\] (130)

To perform a KW duality, we will treat the first column as the Ising term \( Z \) and the second column as the symmetry \( G \in \ker \tilde{\epsilon} \). The KW dual then gives

\[
X = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad Z = \begin{pmatrix} f & \tilde{f} \\ 0 & 0 \end{pmatrix}.
\] (131)

The next step is to replace Pauli operators with appropriate Majorana operators.

\[
P = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad S = \begin{pmatrix} f & 0 \\ 0 & \tilde{f} \end{pmatrix},
\] (132)

where we have replaced the first and second Ising terms with \( \gamma \) and \( \gamma' \), respectively. The reason of such choice is so that the two type of interaction terms automatically commute. The commutation matrix is

\[
\langle S, S \rangle_F = \begin{pmatrix} ff & 0 \\ 0 & ff \end{pmatrix}.
\] (133)

From Lemma 7, a transmutation matrix \( T \) exists. This implies that there is a polynomial \( t \) such that \( t + \tilde{t} = ff \). Therefore, we can choose

\[
T = \begin{pmatrix} t & 0 \\ 0 & \tilde{t} \end{pmatrix},
\] (134)

and the dual operators are given by

\[
\tilde{X}' = \begin{pmatrix} t & 0 \\ 0 & \tilde{t} \\ 0 & 1 \end{pmatrix}, \quad \tilde{Z} = \begin{pmatrix} \tilde{f} \\ f \end{pmatrix}.
\] (135)

Note that here, \( \tilde{Z} \) remains the same. The local gauge constraints are obtained by dualizing the relation \( S_1f + S_2\tilde{f} + Pf\tilde{f} = 0 \), which gives

\[
\tilde{G}' = \begin{pmatrix} (t + \tilde{f}) \tilde{f} \\ (\tilde{t} + ff) \tilde{f} \\ f \end{pmatrix}.
\] (136)

To conclude, the twisted stabilizer code is given by

\[
H = \begin{pmatrix} f \\ f \end{pmatrix},
\] (137)

In addition, the JW duality can also bosonize the original Majorana code. This is because \( H = S_1 + S_2 \). Hence, its dual is given by

\[
H = \begin{pmatrix} 1 \\ f \end{pmatrix},
\] (138)

and the new stabilizer code from gauging the original Majorana model is

\[
H = \begin{pmatrix} t (t + \tilde{f}) \tilde{f} \\ \tilde{t} (\tilde{t} + ff) \tilde{f} \\ f \end{pmatrix}.
\] (139)

An example of this calculation is to use \( f_1 = 1 + x + y + z \). The duality can be obtained by choosing \( t = x + y + z + xy + yz + zx + (x + \tilde{x})y\tilde{z} \). The symmetries of the fermion model are planar symmetries on an FCC lattice. The stabilizer given by Eq. (139) is the result of gauging the Majorana checkerboard model with such planar symmetries, and is a different model from the doubled (Pauli checkerboard) model.

One can also do the same calculation for the Majorana model given by \( f_4 = 1 + y + z + xy + yz + xz \). The JW transformation developed allows us to construct spin models which seem to have emergent fermions. However, defining what it means for such a particle to be a "fermion" seems to be a subtle issue. A fermion is usually defined via its exchange statistics, which can be computed via carefully designed braiding processes in the lattice model. However, in many of the cases we have considered (specifically where the emergent particles are also fractons), it is not clear how to exchange such particles if they are also immobile.

An alternative way that has been used to imply the existence of fermions is the existence of an anomaly. One method is to argue that the symmetry cannot be consistently coupled to a dynamical gauge field and so has an 't Hooft anomaly. This physically corresponds to the inability to condense the fermion. Alternatively, one can argue at the lattice model level that the symmetry cannot be realized in an onsite manner, and correspondences between the two have been established.
However, a field theory description for models with such exotic symmetries is still in development and it is not obvious how to properly define support for such symmetries. For example for fractal symmetries, sites outside the support of the original fractal could alternatively be considered inside the support of a different fractal. Unfortunately, rigorously arguing whether a subsystem or higher-rank symmetries has an anomaly by the above methods is beyond the scope of this work.

In this paper, the best we are able argue that might be indicative of a anomaly of a $\mathbb{Z}_2$ symmetry is to argue that a non-anomalous symmetry can be written in a form that consists of only a single type of Pauli matrix (in this case, Pauli-$X$). This makes the symmetry factorizable into a tensor-product structure, and allows a KW duality, which suggests that it can be coupled to a dynamical gauge field. On the other hand, an obstruction to having such form is indicative of some anticommutations that results from having additional Pauli-$Z$’s in the symmetries that cannot removed. Furthermore, such a symmetry does not admit a KW dual in our formalism, and so in some sense can be related to the inability to condense the excitations.

If we assume this criteria as a partial indication of the anomaly, we are able to show the following

**Proposition 10.** The spin system JW dual to the fermion system as defined cannot have a corresponding KW dual unless $(S, S)_F = 0$.

**Proof.** The JW dual to the fermion system has dual symmetry

$$\ker \tilde{e} = \ker \left( \begin{pmatrix} \frac{1}{0} \\ 0 \end{pmatrix} T \left( P, S \right)_F \right).$$

(140)

Let us assume such a KW dual exists, i.e., there exists $\sigma = (X \ Z)$ such that the KW diagram (16) commutes. Then, up to a basis transformation we must also have

$$\ker \tilde{e} = \ker \left( \begin{pmatrix} \frac{1}{0} \\ 0 \end{pmatrix} (X, Z)_F \right),$$

(141)

which contains symmetries that have only Pauli-$X$’s. For this to be satisfied, there must exist a symplectic matrix $I = \begin{pmatrix} I_1 & I_2 \\ I_3 & I_4 \end{pmatrix}$ such that

$$\begin{pmatrix} \frac{1}{0} \\ 0 \end{pmatrix} (X, Z)_F I = \begin{pmatrix} \frac{1}{0} \\ 0 \end{pmatrix} T \left( P, S \right)_F .$$

(142)

Solving this gives $I_1 = 1$, $I_3 = 0$, $\langle X, Z \rangle I_4 = \left( P, S \right)_F$. Imposing $I$ is symplectic i.e. $I^T I = \lambda$, gives $I_4 = \lambda$ and $T + T^\dagger = (S, S)_F = 0$.

Therefore, we must show that it is impossible to choose the interaction terms in the fermion system such that they all commute. In Section IV, we have explicitly written down various models where $(S, S) \neq 0$. Hence, we need to justify that one cannot redefine commuting interaction terms simply by attaching local fermion parities.

From Prop 9, this implies that there are no polynomials $f_{ki} \in \mathbb{F}_2[x_1^{\pm 1}, \ldots, x_d^{\pm 1}]$ such that

$$\langle S_i, S_j \rangle_F + \langle S_i, P_k \rangle_F f_{kj} + \langle P_k, S_j \rangle_F \tilde{f}_{ki} = 0. \quad (143)$$

For the examples in this paper, we are able to prove this for the 2D and 3D twisted toric codes, and the 2D and 3D models with Fibonacci fractal symmetry. A proof (assuming translation invariance) can be found in Appendix D. This implies that the dual symmetries of the Fibonacci fractal models in Sections IVH and IVI are indeed anomalous. We relegate the task of determining whether the other models we have constructed are anomalous or not to future work.

### A. Anomaly cancellation example

Due to the bulk-boundary correspondence, theories with anomalies can be canceled by an SPT in one higher dimension. For example, the 1-form anomaly in 2D can be canceled by a 3D 1-form SPT\textsuperscript{10,12}. Furthermore, the SPT itself is dual to the ground state of a 3D twisted toric code, and so at low energies, it has an emergent anomalous 2-form symmetry, which can be canceled by a 4+1D bulk of a 2-form SPT. The hierarchy of anomalies and SPTs continues in this fashion ad infinitum\textsuperscript{57}.

Here, we will demonstrate an example with a similar hierarchy. The JW dual of the 2D Fibonacci fermion model in Section IVH lives naturally on the boundary of a 3D SPT with fractal symmetry. This SPT can then be KW dual to a model whose ground state realizes the twisted fractal spin model in Section IVI.

First, let us consider the 3D model with the Fibonacci Ising term in the $xy$ plane and a standard Ising term in the $z$ direction, reminiscent of the fermion model in Section IVI:

$$Z = \begin{pmatrix} 1 + y(1 + x + \bar{x}) & 1 + z \\ 1 + z & 0 \end{pmatrix}, \quad X = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \quad (144)$$

The symmetries are stacks of the Fibonacci CA in the $z$ direction.

$$\ker \mathbf{e} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \sum_{ij} y^i(1 + x + \bar{x}) y^j z^j . \quad (145)$$

The KW dual of this model is given by

$$\tilde{X} = \begin{pmatrix} 0 & 0 \\ 0 & \bar{x} \end{pmatrix}, \quad \tilde{Z} = \begin{pmatrix} 1 + \bar{y}(1 + x + \bar{x}) \\ 1 + \bar{z} \end{pmatrix}. \quad (146)$$

with symmetry constraints

$$\ker \tilde{e} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \bar{z} \\ 1 + z & \sum_i y^i(1 + x + \bar{x}) y^j \sum_i z^j \end{pmatrix}. \quad (147)$$
The first column of \( \ker \hat{\epsilon} \) and \( \hat{Z} \) are together the stabilizer code
\[
\begin{pmatrix}
0 \\
0 \\
\frac{1 + \tilde{g}(1 + z)}{1} \\
\frac{1 + \tilde{g}(1 + x + \tilde{x})}{1 + y(1 + x + \tilde{x})}
\end{pmatrix}
\]  
(148)
for the untwisted fractal spin model. Now, consider the following stabilizer:
\[
H = \sigma \begin{pmatrix}
1 + z + (1 + \tilde{z})\tilde{g}(1 + x + \tilde{x}) \\
1 \\
1 + \tilde{g}(1 + x + \tilde{x}) \\
x^2(1 + \tilde{z})
\end{pmatrix}
\]  
(149)
where \( f_{\text{SPT}} = x^2 + \tilde{x}^2 + (y + \tilde{y})(1 + x + \tilde{x}) + z(1 + x^2 + y(1 + x + \tilde{x}) + \tilde{z}(1 + \tilde{x}^2 + \tilde{g}(1 + x + \tilde{x})) \). Since it is made out of the Ising and transverse field terms, it respects the symmetry, and can be dualized to
\[
\hat{H} = \tilde{\sigma} \begin{pmatrix}
1 + z + (1 + \tilde{z})\tilde{g}(1 + x + \tilde{x}) \\
1 \\
1 + \tilde{g}(1 + x + \tilde{x}) \\
x^2(1 + \tilde{z})
\end{pmatrix}
\]
(150)
The first column of \( \ker \hat{\epsilon} \) and \( \hat{H} \) are together the stabilizer code
\[
\begin{pmatrix}
0 \\
0 \\
\frac{1 + \tilde{g}(1 + x + \tilde{x})}{1} \\
\frac{(1 + \tilde{g}(1 + x + \tilde{x}))(1 + z)}{x^2(1 + \tilde{z})}
\end{pmatrix}
\]  
(151)
which is exactly the twisted fractal spin model (123) up to inversion and an appropriate swap of rows and columns. When \( H \) is enforced energetically, the low-energy Hilbert space has an emergent symmetry (given by the stabilizers of \( H \)) which is anomalous.

Since the twisted and untwisted fractal spin models cannot be connected via a translation-invariant Clifford circuit, \( H \) in Eq. (149) is potentially an SPT protected by the fractal symmetry. In particular, since it is a cluster state, when the symmetry is explicitly broken, the model can be disentangled with Controlled-Z gates acting in a translation-invariant way according to the position of Pauli-Z’s which are given by the first row of \( H \).

Let us now look at the symmetry action on the boundary of \( H \). We consider a semi-infinite 3D system from \( z = -\infty \) which terminates at \( z = 0 \). The symmetry is given by
\[
\sum_i \sum_{j=-\infty}^0 \tilde{g}^i(1 + x + \tilde{x})^j z^j.
\]  
(152)
In the ground state, we can use \( H \) to substitute a Pauli \( X \) with Pauli Z’ s at the positions given by the first row of \( H \). However, since \( H \) spans three layers in the \( z \) direction, this substitution is only valid from the second layer downwards. A calculation in Appendix E shows that the effective symmetry action at the layer \( \tilde{z}^0 \) matches the anomalous symmetry action of the JW dual with Fibonacci symmetry (118) up to inversion. This confirms that \( H \) is indeed an SPT.

VII. DISCUSSION

Assuming translation invariance, we have constructed a generalization of the JW duality that performs an exact bosonization of a fermion system with arbitrary \( q \)-body interactions. Under this framework, we have proven the existence and uniqueness (up to a choice of basis) of the dual spin theory.

In the case of multibody interaction terms, the fermionic Hamiltonian has an additional higher-form or subsystem fermion parity symmetry and the dual spin theories can in some cases exhibit fracton topological order. Furthermore, starting from a CSS code, the dualities allow us to construct a new “twisted” stabilizer code with possible fermionic excitations, and at the same time bosonize Majorana codes in various ways.

We conclude by listing many open questions

1. Fermionic nature of fractons: From the duality, exactly solvable models (stabilizer codes) can be constructed by properly “gauging” the \( \mathbb{Z}_2 \) global/higher form/subsystem symmetries. In the case of planar or fractal symmetries in 3D, the resulting models are twisted models of the X-cube, Checkerboard, Haah’s code, and Yoshida’s fractal code. The fracton excitations are fermionic in the sense that there is an obstruction to condensing them. It would be interesting to see if there is a meaningful braiding procedure to detect the “fermionic statistics” of these fractons. We intend to address this in future work. Furthermore, it would be interesting to show whether these twisted models are in different phases from their usual counterparts, either in the usual sense, possibly including translation symmetries or in terms of foliated fracton order.

2. ’t Hooft anomalies: if the fractons truly have a fermionic nature, then there should be an associated ’t Hooft anomaly which generalize the Steenrod square topological action. As field theoretical methods are being developed to describe fracton phases, do our proposed lattice models, when transcribed into the field theory language, have the correct anomaly? A closely related question is due to the bulk-boundary correspondence: what is the corresponding SPT in one higher dimension that has the corresponding anomaly on its boundary?

3. Given a certain spin model, is there a way to check if it is dual to some fermionic system? For example, for models whose excitations are known to have fermion statistics, such as the Levin-Wen fermion model or spin models which admit Parton constructions in terms
of Majorana fermions\textsuperscript{63–66}, how does one determine the associated symmetries and JW dual of these models?

4. General lattices: Although translation invariance was a key assumption in establishing the dualities in this paper, the existence of dual Pauli operators with the same commutation relations in arbitrary lattices can be similarly argued to exist by replacing the commutation matrix \((S, S)_F\) with a large “adjacency matrix” between all interaction terms that anticommute in an arbitrary lattice, and constructing the analogue of the transmutation matrix \(T\) as an upper triangular matrix. Nevertheless, a necessary condition for a self-consistent duality in the global symmetry case is the vanishing of the 2nd Stiefel-Whitney of the manifold, and the duality depends on a choice of spin-structure\textsuperscript{27,67–70}. For general interaction terms, are there similar obstructions that generalize the notion of Stiefel-Whitney classes spin-structures? For example, the duality between the fermion model with planar symmetry in Section IV F and the twisted X-cube model – if well defined on a 3-torus – would actually depend on a choice of \(2^{6L-3}\) such “spin structures”. An interesting extension would be to determine the such obstructions in dualities for fracton models with arbitrary foliations\textsuperscript{71,72} or even perhaps arbitrary cellulations\textsuperscript{23}.

5. Fermionic higher-form/subsystem SPTs: So far, the Hamiltonians on fermionic side have been either symmetry-breaking or topological ordered. Are there examples of SPTs protected by higher form or subsystem symmetries i.e. those that are non-trivial solely by sub-dimenional fermion parity without any further symmetries? Can higher-form fermionic phases be classified by a variant of spin cobordism\textsuperscript{73,74}?

6. Parafermions: Generalizations to dualities between parafermions and \(Z_p\) clock models\textsuperscript{45,75} for prime \(p\) are straightforward in this formalism by instead working with the polynomial ring over \(\mathbb{F}_p\). However, in 3D, mobile particles cannot have emergent parafermionic statistics. If fermionic statistics can be properly defined for fractons, can they still be extended to parafermions?

7. It seems that the twisted X-cube model constructed can be obtained from a recent defect network construction in Ref. 76 by replacing the 3D toric code with the twisted 3D toric code. It would be interesting to see whether such defect construction can account in general for the twisted models we have presented here in a similar fashion.

Note Added: During the preparation of this manuscript, I became aware of related work by Wilbur Shirley\textsuperscript{77}, which constructs similar fracton models that have immobile fermion excitations. Our results were obtained independently.

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Although, they have been recently called faithful/non-faithful, respectively. In this paper, we define an inverse of a Majorana code into a self-dual CSS code, since the doubled code has only bosonic excitations.

More correctly, antihermitian if generalizing to relativistic n-symmetry as opposed to unfaithful/relativistic n-form symmetry to distinguish it from those considered in high energy.

In the KW duality, \( \ker \epsilon \supseteq \sigma \ker \tilde{\sigma} \).

**Proof.** Let \( a \in \ker \tilde{\sigma} \). It suffices to show that \( \epsilon(\sigma a) = 0 \).
Indeed, since the diagram commutes, $(\epsilon \circ \sigma)a = (\tilde{\epsilon} \circ \tilde{\sigma})a = 0$

Now we will prove the inclusion to the right, which assumes the form of the operators summarized in Tables II and III.

Lemma 12. In the KW duality, $\ker \epsilon \subseteq \sigma \ker \sigma$

Proof. Let $b = \begin{pmatrix} b_Z \\ b_X \end{pmatrix} \in \ker \epsilon = \ker \left( \begin{array}{cc} Z_X^b \\ 0 \end{array} \right)$, then we must have

$$b_Z = Z_X^b b_X = 0.$$  \hspace{1cm} (A1)

We want to show that there exists $a \in \ker \tilde{\sigma}$ such that $\sigma a = b$. Indeed, let $a = \begin{pmatrix} 0 \\ b_X \end{pmatrix}$. Then

$$\tilde{\sigma} a = \begin{pmatrix} 0 \\ Z_X^b \end{pmatrix} \begin{pmatrix} 0 \\ b_Z \end{pmatrix} = 0, \quad \sigma a = \begin{pmatrix} 0 \\ Z_X^b \end{pmatrix} \begin{pmatrix} 0 \\ b_Z \end{pmatrix} = b,$$

as desired. \hfill \square

Lemma 13. In the KW duality, $\ker \tilde{\epsilon} \subseteq \tilde{\sigma} \ker \tilde{\sigma}$

Proof. Let $b = \begin{pmatrix} b_Z \\ b_X \end{pmatrix} \in \ker \tilde{\epsilon} = \ker \left( \begin{array}{cc} 1 \\ 0 \\ Z_Z \end{array} \right)$, then we must have

$$b_Z = Z_Z b_X = 0.$$  \hspace{1cm} (A4)

Let $a = \begin{pmatrix} b_X \\ Z_X b_X \end{pmatrix}$, then $a \in \ker \sigma$ since

$$\sigma a = \begin{pmatrix} Z_Z \\ Z_X \end{pmatrix} \begin{pmatrix} b_X \\ Z_X b_X \end{pmatrix} = 0 \quad \text{(A5)}$$

Furthermore, using $\langle Z, Z \rangle = Z_Z^b Z_X + Z_X^b Z_Z = 0$,

$$\tilde{\sigma} a = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \begin{pmatrix} b_X \\ Z_X b_X \end{pmatrix} = \begin{pmatrix} Z_X^b Z_Z b_X \\ b_X \end{pmatrix} = \begin{pmatrix} 0 \\ b_X \end{pmatrix} = b$$ \hspace{1cm} (A6)

The proofs for the JW dualities are nearly identical, with minor modifications.

Lemma 14. In the JW duality, $\ker \epsilon_F \subseteq \sigma_F \ker \tilde{\sigma}$

Proof. Let $b = \begin{pmatrix} b_Z \\ b_X \end{pmatrix} \in \ker \epsilon_F = \ker \left( \begin{array}{cc} S_Z^b \\ S_X^b \end{array} \right)$, then we must have

$$b_Z + b_X = (S_Z + S_X^b)b_Z = 0.$$  \hspace{1cm} (A7)

Let $a = \begin{pmatrix} 0 \\ b_Z \end{pmatrix}$, then

$$\tilde{\sigma} a = \begin{pmatrix} T S_Z^b + S_X^b \\ \frac{1}{0} \end{pmatrix} \begin{pmatrix} 0 \\ b_Z \end{pmatrix} = 0, \quad \sigma_F a = \begin{pmatrix} S_Z^b \frac{1}{S_X^b} \frac{1}{0} \end{pmatrix} \begin{pmatrix} 0 \\ b_Z \end{pmatrix} = b.$$ \hspace{1cm} (A8)

Lemma 15. In the JW duality, $\ker \tilde{\epsilon} \subseteq \sigma_F \ker \tilde{\sigma}$

Proof. Let $b = \begin{pmatrix} b_Z \\ b_X \end{pmatrix} \in \ker \tilde{\epsilon} = \ker \left( \begin{array}{cc} 1 \\ 0 \\ T \end{array} \right)$, then we must have

$$b_Z + T b_X = (S_Z + S_X)b_X = 0.$$ \hspace{1cm} (A10)

Let $a = \begin{pmatrix} b_X \\ S_X b_X \end{pmatrix}$, then $a \in \ker \sigma_F$ since

$$\sigma_F a = \begin{pmatrix} S_Z \\ S_X \end{pmatrix} \begin{pmatrix} b_X \\ S_X b_X \end{pmatrix} = 0.$$ \hspace{1cm} (A11)

Furthermore, using $T + T^\dagger = (S, S)_F = S_Z^b S_Z + S_X^b S_X$,

$$\tilde{\sigma} a = \begin{pmatrix} T S_Z^b + S_X^b \\ \frac{1}{0} \end{pmatrix} \begin{pmatrix} b_X \\ S_X b_X \end{pmatrix} = \begin{pmatrix} (T + S_X^b S_Z^b) b_X \\ b_X \end{pmatrix} = \begin{pmatrix} (T^\dagger + S_Z^b (S_Z + S_X^b)) b_X \\ b_X \end{pmatrix} = b.$$ \hspace{1cm} (A12)

Appendix B: Dual of Majorana codes in 3D with global symmetry

The models in Ref. 29 are defined as

$$H_F^{(i)} = \frac{f_i}{f_j}$$ \hspace{1cm} (B1)

For some polynomial $f_i \in F[z^\pm 1, y^\pm 1, z^\pm 1]$. To bosonize, we need to write the Hamiltonian in terms of $P$ and $S$ given by Eq. (71). That is, we need to find a vector $v \in G$ such that $\sigma_F v = H_F^{(i)}$. Doing so, we can then obtain the bosonized operator as $\tilde{H}^{(i)} = \tilde{\sigma} v$. The results are summarized in Table VII. Note that the dual operators are only unique up to multiplying by the local symmetry constraints, which are set to one by the duality.

We remark that since the Majorana checkerboard model $H_F^{(1)}$ is unitary equivalent to a product of the Semionic X-cube model with trivial fermions, the model given by $H^{(1)}$ and local constraints from Eq. (75) should be unitary equivalent to a product of the semionic X-cube model and the twisted 3D toric code.
This mapping can be obtained by a “naive” JW transformation in 2D as follows: we define a total ordering of sites row by row in a 2D square lattice as

\[
\cdots < (i-1, j-1) < (i, j-1) < (i+1, j-1) < \cdots
\]

\[
\cdots < (i-1, j) < (i, j) < (i+1, j) < \cdots
\]

\[
\cdots < (i-1, j+1) < (i, j+1) < (i+1, j+1) < \cdots
\]

Then the Majorana operators can be mapped as

\[
\gamma_{(i,j)} \rightarrow X_{(i,j)} \prod_{(k,l)<(i,j)} X_{(k,l)},
\]

\[
\gamma'_{(i,j)} \rightarrow Y_{(i,j)} \prod_{(k,l)<(i,j)} X_{(k,l)},
\]

\[
P_{(i,j)} = -i\gamma_{(i,j)}\gamma'_{(i,j)} \rightarrow X_{(i,j)}.
\]

which reproduces the duality above.

The obvious problem of this map for dualizing a fermionic system with global symmetry is that bilinears
of Majoranas between different rows get mapped to non-local spin operators. However, with fermion parity conservation on every horizontal line, such an operators are forbidden, and so all symmetric operators under subsystem fermion parity get mapped to local spin operators.

Appendix D: Non-Existence of Commuting interaction terms for Twisted Dual models

This Appendix is devoted to proving that for certain fermionic systems, one cannot redefine interaction terms $S$ by attaching local fermion parities so that $\langle S, S \rangle_F = 0$. The models where we are able to prove so are those with global symmetry in 2D and 3D, and those with Fibonacci symmetry in 2D and 3D.

1. 2D $\mathbb{Z}_2^F$ global symmetry

Let us demonstrate with the simplest example where it is known that the dual theory is necessarily anomalous. From the model in Section IV A, we have

$$\langle P, S \rangle = (1 + x + 1 + y),$$

$$\langle S, S \rangle_F = \begin{pmatrix}
0 & 1 + xy \\
1 + x y & 0
\end{pmatrix}.$$  \hfill (D1)

For $i = j = 1$, Eq. (143) reads

$$(1 + \bar{x})f_1 + (1 + x)f_1 = 0.$$  \hfill (D3)

where we have dropped the index $k$, since there is only one site per unit cell.

We wish to solve for solutions of such linear equation in $R = Frac[y, y, y, 1]$. A method of solving such equations is by first going to the fraction field Frac($R$), consisting of formal fractions of elements in an integral domain $R$. Formally,

$$\text{Frac}(R) = \{ f/g \mid f, g \in R, g \neq 0 \text{ and } f \sim g \text{ if } fg' = f'g \in R \}.$$ \hfill (D4)

The denominator denotes an equivalence class, where two fractions are equal if their cross-multiplications in $R$ are equal.

Since Frac($R$) is equipped with an involution, we can separate out its “symmetric” part. Let us define the symmetric subfield

$$S = \{ f \in \text{Frac}(R) \mid f = \bar{f} \}.$$ \hfill (D5)

It turns out that a single extra “antisymmetric” generator is sufficient to extend $S$ back to Frac($R$). Let us choose this generator to be $x$. To see why, we note that

$$\bar{x} = (x + \bar{x}) + (1)x,$$

$$x^2 = (1 + (x + \bar{x})x,$$

$$y = \left( \frac{xy + \bar{y}x}{x + \bar{x}} \right) + \left( \frac{y + \bar{y}}{x + \bar{x}} \right)x,$$ \hfill (D8)

where the quantities in brackets are in $S$. All other monomials can be constructed recursively from these relations.

We now solve Eq. (D3). First, we decompose $f_1$ into its symmetric and antisymmetric parts

$$f_1 = f_1^s + f_1^a x$$ \hfill (D9)

where $f_1^s, f_1^a \in S$. Inserting this, we obtain

$$(f_1^s + f_1^a)(x + \bar{x}) = 0.$$ \hfill (D10)

Thus, we need $f_1^a = f_1^a$, meaning

$$f_1 = (1 + x)f_1^s ; f_1^s \in S.$$ \hfill (D11)

Projecting solutions back to $R$, we require $f_1^a$ to be an element of $R$ that is invariant under the involution. An identical exercise shows that

$$f_2 = (1 + y)f_2^s ; f_2^s \in R.$$ \hfill (D12)

Inserting the solutions, we find

$$(1 + \bar{x})f_1 + (1 + y)f_2 = 1 + \bar{x}y.$$ \hfill (D13)

Since the left hand side is invariant under the involution, while the right hand side is not, there is no such solution.

The proof is identical for the 3D case with global symmetry.

2. Fibonacci fractal symmetry

We show that the fermion model protected by the Fibonacci fractal symmetry in Sections IV H and IV I cannot have commuting interaction terms. The interaction term is given by

$$S = \left( \frac{x(1 + y + \bar{y})}{1} \right).$$ \hfill (D15)

So that

$$\langle P, S \rangle = x(1 + y + \bar{y}), \quad \langle S, S \rangle_F = y^2 + \bar{y}^2$$ \hfill (D16)

and Eq. (143) reads

$$(1 + \bar{x}(1 + y + \bar{y}))f + (1 + x(1 + y + \bar{y}))\bar{f} = y^2 + \bar{y}^2.$$ \hfill (D17)

Inserting $f = f^s + x f^a$, we find

$$(f^s(1 + y + \bar{y}) + f^a(x + \bar{x}) = y^2 + \bar{y}^2.$$ \hfill (D18)

Although there exists solutions in Frac($R$), there are no valid solutions in $R$. This is because $(x + \bar{x})$ does not have an inverse in $R$.  

The proof above also shows that the stabilizers of the twisted fractal code cannot be connected to the usual fractal code in the presence of translation symmetry. It would be interesting to see if their ground states are actually distinct or not.
Appendix E: Symmetry Action on the boundary of a fractal SPT

In this Appendix, we calculate the symmetry action on the boundary of the 3D fractal SPT. The symmetry acts on the lower half region

\[ \left( \sum_i \sum_{j=-\infty}^0 \bar{y}^i (1 + x + \bar{x})^j z^j \right), \]  \hspace{1cm} (E1)\

and the stabilizer is given by

\[ H = \left( x^2 + \bar{x}^2 + (y + \bar{y})(1 + x + \bar{x}) + z(1 + x^2 + y(1 + x + \bar{x})) + \bar{z}(1 + \bar{x}^2 + \bar{y}(1 + x + \bar{x})) \right). \]  \hspace{1cm} (E2)

Since the stabilizer acts on three consecutive planes, we can use the stabilizer to substitute

\[ \begin{pmatrix} 0 \\ 1 \end{pmatrix} \rightarrow \begin{pmatrix} x^2 + \bar{x}^2 + (y + \bar{y})(1 + x + \bar{x}) + z(1 + x^2 + y(1 + x + \bar{x})) + \bar{z}(1 + \bar{x}^2 + \bar{y}(1 + x + \bar{x})) \\ 0 \end{pmatrix}. \]  \hspace{1cm} (E3)

from the second layer downwards (i.e. for any \( z^j \) where \( j < 0 \)). The symmetry is then

\[ \left( \sum_i \sum_{j=-\infty}^{-1} \bar{y}^i (1 + x + \bar{x})^j z^j \left[ x^2 + \bar{x}^2 + (y + \bar{y})(1 + x + \bar{x}) + z(1 + x^2 + y(1 + x + \bar{x})) + \bar{z}(1 + \bar{x}^2 + \bar{y}(1 + x + \bar{x})) \right] \right). \]  \hspace{1cm} (E4)

The sum for the positions of the Pauli-Z above separated by layer (i.e. by the degree of \( z \)) is

\[ z^0 \left[ \sum_i \bar{y}^i (1 + x + \bar{x})^i \left( 1 + \bar{x}^2 + y(1 + x + \bar{x}) \right) \right] + \bar{z} \left[ \sum_i \bar{y}^i (1 + x + \bar{x})^i \left( x^2 + \bar{x}^2 + (y + \bar{y})(1 + x + \bar{x}) + (1 + \bar{x}^2 + y(1 + x + \bar{x})) \right) \right] + \sum_{j=2}^{\infty} \bar{z}^j \left[ \sum_i \bar{y}^i (1 + x + \bar{x})^i \left( x^2 + \bar{x}^2 + (y + \bar{y})(1 + x + \bar{x}) + (1 + x^2 + y(1 + x + \bar{x})) + (1 + \bar{x}^2 + \bar{y}(1 + x + \bar{x})) \right) \right] \]  \hspace{1cm} (E5)

The last line cancels completely, while for the other two lines we can shift \( i \) to simplify the expression. The results in

\[ \frac{x^2 + \bar{x}^2 \sum_i \bar{y}^i (1 + x + \bar{x})^i}{\sum_i \bar{y}^i (1 + x + \bar{x})^i}. \]  \hspace{1cm} (E6)

There are remaining Pauli Z’s on the layer \( \bar{z} \), but they do not anticommute with anything else in that layer. Therefore, the algebra of the symmetry operators matches that of

\[ \frac{x^2 \sum_i \bar{y}^i (1 + x + \bar{x})^i}{\sum_i \bar{y}^i (1 + x + \bar{x})^i}, \]  \hspace{1cm} (E7)

which upon inversion is the anomalous symmetry of the 2D system given in Eq. (118) or Figure 3.