THE 2-DIMENSION SERIES OF THE JUST-NONSOLVABLE BSV GROUP

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To Said N. Sidki, born January 23, 1941 in Al Quds (Palestine), for his 60th birthday

Abstract. I compute the structure of the restricted 2-algebra associated to a group first described by Andrew Brunner, Said Sidki and Ana Cristina Vieira, acting on the binary rooted tree [BSV99]. I show that its width is unbounded, growing logarithmically, and obeys a simple rule. As a consequence, the dimension of $\omega^n/\omega^{n+1}$ (where $\omega < F_2\Gamma$ is the augmentation ideal), is $p(0)+\cdots+p(n)$, the total number of partitions of numbers up to $n$.

1. Introduction

The purpose of this paper is to construct the restricted Lie algebra associated to the Brunner-Sidki-Vieira group $\Gamma$. I decide to state the auxiliary results in as little generality as possible, aiming to arrive as quickly as possible at an explicit description.

1.1. The group. Let $\Sigma = \{0,1\}$ be a two-letter alphabet. The rooted binary tree is the free monoid $\Sigma^*$. The group $\Gamma$ acts on $\Sigma^*$, and is generated by two elements $\tau, \mu$ defined as follows:

- $\tau$ is called the “dyadic adding machine”, or “odometer”.

$\Gamma$ is universal in the following sense: any finite-state state-closed infinite cyclic group of the automorphism group of the binary tree is generated by an $n$-th root of $\tau$ or $\mu$ for odd $n$. This singles out $\tau$ and $\mu$ as “special” generators. $\tau$ is called the “dyadic adding machine”, or “odometer”.

The standard Lie algebra is much more complicated; considering this restricted Lie algebra suppresses much of the 2-torsion.
It is known that this group is infinite, just-nonsolvable, and torsion-free \([\text{BSV99}]\).

1.2. The dimension series. The 2-dimension series of a group \(\Gamma\) is defined as follows: \(\Gamma_1 = \Gamma\), and \(\Gamma_n = [\Gamma, \Gamma_{n-1}]\Gamma_{n/2}^2\), where \(H^2\) denotes the subgroup of \(H\) generated by its squares. It can alternately be described, by a result of Lazard \([\text{Laz53}]\), as

\[
\Gamma_n = \prod_{2^i \geq n} \gamma_i(\Gamma)^{2^i},
\]

\(\gamma_i(\Gamma)\) being the lower central series, or as

\[
\Gamma_n = \{ g \in \Gamma | g - 1 \in \omega^n \},
\]

where \(\omega\) is the augmentation ideal of the group algebra \(\mathbb{F}_2 \Gamma\).

1.3. The Lie algebra. The 2-Lie algebra of \(\Gamma\) is the restricted graded Lie algebra

\[
\mathcal{L}(\Gamma) = \bigoplus_{n \geq 1} \Gamma_n / \Gamma_{n+1}.
\]

Its Lie bracket is induced by commutation in \(\Gamma\), and its Frobenius endomorphism is induced by squaring in \(\Gamma\).

The degree of \(g \in \Gamma\) is the maximal \(n \in \mathbb{N} \cup \{\infty\}\) such that \(g\) belongs to \(\Gamma_n\). A basis of the rank-\(n\) module of \(\mathcal{L}(\Gamma)\) can be found among elements of degree \(n\); and since \(\Gamma\) is residually-2, the only element of infinite degree is 1.

Following \([\text{KLP97}]\), say \(\Gamma\) has finite width if there is a constant \(W\) such that \(\ell_n := \dim \Gamma_n / \Gamma_{n+1} \leq W\) holds for all \(n\).

However, the width could also be defined with respect to the lower central series; I don’t know whether these definitions are equivalent.

The growth of \(\Gamma\) is unknown, and that motivated my computations — see equation (2). It is known that \(\Gamma\) doesn’t contain non-abelian free subgroups \([\text{Sil01}]\), but it is unknown whether it contains non-abelian free monoids.

\[\phi^\tau = \phi; \quad (0 \sigma_2 \ldots \sigma_n)^\tau = 1 \sigma_2 \ldots \sigma_n; \quad (1 \sigma_2 \ldots \sigma_n)^\tau = 0 (\sigma_2 \ldots \sigma_n)^\tau; \quad (0 \sigma_2 \ldots \sigma_n)^\mu = 0 (\sigma_2 \ldots \sigma_n)^{\mu-1}.\]
Consider the graded algebra

\[ F_2 \Gamma = \bigoplus_{n=0}^{\infty} \varpi^n/\varpi^{n+1}. \]

A fundamental result of Daniel Quillen \cite{Qui68} implies that \( F_2 \Gamma \) is the enveloping 2-algebra of \( L(\Gamma) \).

Using the Poincaré-Bikhoff-Witt isomorphism, Stephen Jennings \cite{Jen41} then showed that

\[ \sum_{n \geq 0} \dim(\varpi^n/\varpi^{n+1})h^n = \prod_{n \geq 1} (1 + h^n)\ell_n. \]

Of course, Jennings did that before Quillen, and quite unaware of the fact that his construction corresponded to PBW!

This shows that a residually-2 group either has polynomial growth (if \( \ell_n = 0 \) for some \( n \)) or growth at least \( e^{\sqrt{n}} \) (if \( \ell_n \geq 1 \) for all \( n \)).

It is also known \cite[Lemma 2.5]{BG00a} that if \( f_n \) denotes the number of group elements of \( \Gamma \) of length at most \( n \) in the generators, then

\[ f_n \geq \dim F_2 \Gamma/\varpi^{n+1} = \sum_{i=0}^{n} \dim(\varpi^n/\varpi^{n+1}). \]

1.4. The main result. Along with a description of \( L(\Gamma) \), I shall show:

**Theorem 1.1.** The ranks of successive quotients, \( \ell_n = \dim F_2 \Gamma_n/\Gamma_{n+1} \), satisfy

\[ \ell_n = \begin{cases} i + 2 & \text{if } n = 2^i \text{ for some } i; \\ \max \{i + 1 \mid 2^i \text{ divides } n\} & \text{otherwise.} \end{cases} \]

As a consequence, \( \dim \varpi^n/\varpi^{n+1} = \sum_{i=0}^{n} p(n) \), the number of partitions of numbers up to \( n \), and the word growth of \( \Gamma \) is at least \( e^{\sqrt{n}} \).

2. Lie graphs and Branch Portraits

The structure of \( L(\Gamma) \) can best be described using Lie graphs, introduced in \cite{Bar00}. The Lie graph of \( L(\Gamma) \) has as vertex set a basis \( V \) of \( L(\Gamma) \); in the favourable case that \([v, \tau] , [v, \mu] \) and \( v^2 \) are basis elements for all \( v \in V \), the Lie graph has for all \( v \in V \) an arrow labelled \( \tau \) from \( v \) to \([v, \tau] \), one labelled \( \mu \) leading to \([v, \mu] \) and one labelled 2 leading to \( v^2 \).

This is in fact just a simple representation of the structure constants of \( L(\Gamma) \).

At present, only the first four successive quotients of the lower central series are known: \( \gamma_1/\gamma_2 = \mathbb{Z}^2 \), \( \gamma_2/\gamma_3 = \mathbb{Z} \) and \( \gamma_3/\gamma_4 = \gamma_4/\gamma_5 = \mathbb{Z}/8\mathbb{Z} \).

This is the first group for which this miracle happens!
2.1. **Branch portraits.** The important “branching property” of \( \Gamma \) is that \( \Gamma' \) contains \( \Gamma' \times \Gamma' \), where in this last group the left and right factors act on \( 0\Sigma^* \) and \( 1\Sigma^* \) respectively; and furthermore \( \Gamma' = \langle c \rangle (\Gamma' \times \Gamma') \), with \( c = [\tau, \mu] \).
Therefore every $g \in \Gamma$ can be represented as $g = \tau^i\mu^jg_0$ for some $i, j \in \mathbb{Z}$ and $g_0 \in \Gamma'$; then inductively each $g_\sigma \in \Gamma'$ can be represented as $g_\sigma = c^{k_\sigma}(g_{0\sigma}, g_{1\sigma})$, and so every $g \in \Gamma$ gives rise to a labelling of the tree $\Sigma^*$, with label $(i, j, k_0) \in \mathbb{Z}^3$ at the root vertex $\emptyset$ and label $k_\sigma \in \mathbb{Z}$ at the vertex $\sigma$.

Consider the group, written $\overline{\Gamma}$, of all infinite expressions of the form

$$\tau^*\mu^*c^*(c^*, c^*)(c^*, c^*, c^*)\cdots$$

for any choices of $* \in \mathbb{Z}$; this is an uncountable group containing $\Gamma$. The dimension series of $\overline{\Gamma}$ coincides with that of $\Gamma$, so we may perform the computations in $\overline{\Gamma}$, written again $\Gamma$ from now on.

Elements of $\Gamma$ may also be decomposed according to their action on $\Sigma^*$; to wit, given $g \in \Gamma$, choose $\epsilon^i \in \text{Sym}(2) = \langle \epsilon \rangle$ such that $g/\epsilon^i$ fixes the first letter of all words in $\Sigma^*$; then $g/\epsilon^i$ acts both on $0\Sigma^*$ and $1\Sigma^*$ as elements of $\Gamma$, written $g_0$ and $g_1$. In this way every $g \in \Gamma$ has an expression of the form

$$\epsilon^*(\epsilon^*, \epsilon^*)(\epsilon^*, \epsilon^*, \epsilon^*)\cdots;$$

I shall implicitly write elements in either form $g$ or $(g_0, g_1)\epsilon^i$. Using that notation,

$$\tau = (1, \tau)\epsilon, \quad \mu = (1, \mu^{-1})\epsilon, \quad c = \epsilon(1, \tau^{-1})\epsilon(1, \mu)(1, \tau)\epsilon(1, \mu^{-1})\epsilon = ((\mu\tau)^{-1}, \mu\tau).$$

2.2. The basis. Recalling that $\Gamma'$ contains $\Gamma' \times \Gamma'$, define the following two endomorphisms of $\Gamma'$:

$$0(g) = (g, 1), \quad 1(g) = (g, g).$$

These maps will be used to create arbitrary branch portraits; it’s much more clever to use these elements $(g, 1)$ and $(g, g)$ rather than the “obvious” $(g, 1)$ and $(1, g)$.

**Theorem 2.1.** The following elements form a basis of $\mathcal{L}(\Gamma)$:

- $\tau^{2^n}$ for all $n \in \mathbb{N}$, of degree $2^n$;
- $\mu^{2^n}$ for all $n \in \mathbb{N}$, of degree $2^n$;
- $W(c)^{2^n}$ for all $W \in \{0, 1\}^m$ and $n \in \mathbb{N}$, of degree

$$2^n\left(\sum_{i=1}^m W_i2^{i-1} + 2^m + 1\right).$$

That’s really reading ‘$1W0\ldots01$’ in base 2.
Figure 1. The beginning of the Lie graph of $L(\Gamma)$.

The arrows in the Lie graph are

\[ \tau^{2n} \xrightarrow{\mu} 0^n(c) \quad \mu^{2n} \xrightarrow{\tau} 0^n(c) \]

\[ 1^n(c) \xrightarrow{\tau,\mu} 0^{n+1}(c) \]

\[ v^{2n} \xrightarrow{2} v^{2n+1} \] for all basis vectors $v$

\[ 1^n0W(c) \xrightarrow{\tau,\mu} 0^n1W(c) \] for all $W \in \{0, 1\}^*$
Proof. The strategy of the proof strategy is simple, and follows ideas from [Bar00].

First the degrees of the basis vectors are as claimed: for this it suffices to check that the commutation relations are as described. Compute:

\[ [\mu^{2n}, \tau] = (\mu^{2n-1}, \mu^{2n-1}) \epsilon (1, \tau^{-1}) (\mu^{-2n-1}, \mu^{-2n-1}) (1, \tau) \epsilon \]
\[ = ([\mu^{-2n-1}, \tau], 1) = 0 [\mu^{-2n-1}, \tau] = \cdots = 0^n(c)^{(1)^n} \equiv 0^n(c); \]
\[ [1^n \delta W(c), \tau] = (1^{n-1} \delta W(c)^{-1}, 1^{n-1} \delta W(c)^{-1}) \epsilon (1, \tau^{-1}) (1^{n-1} \delta W(c), 1^{n-1} \delta W(c)^{-1}) \epsilon \]
\[ = ([1^{n-1} \delta W(c), \tau], 1) = \cdots = 0^n(\delta W(c), \tau) \]
\[ = 0^n(W(c)^{-1}, 1) \epsilon (1, \tau^{-1}) (W(c), 1)(1, \tau) \epsilon \]
\[ = 0^nW(c)/0^{n+1}W(c)^{2} \equiv 0^nW(c); \]
\[ [1^n(c), \tau] = (1^{n-1}(c)^{-1}, 1^{n-1}(c)^{-1}) \epsilon (1, \tau^{-1}) (1^{n-1}(c), 1^{n-1}(c)) (1, \tau) \epsilon \]
\[ = ([1^{n-1}(c), \tau], 1) = \cdots = 0^n[c, \tau] \]
\[ = 0^n(\mu \tau, (\mu \tau)^{-1}) \epsilon (1, \tau^{-1}) ((\mu \tau)^{-1}, \mu \tau)(1, \tau) \epsilon \]
\[ = 0^{n+1}(c^{-\tau^{-1}} \mu^{-2})/0^n(c)^2 \equiv 0^{n+1}(c); \]

and similar results hold for commutation with \( \mu \). Here the sign ‘\( \equiv \)’ means congruence modulo terms of greater degree, i.e. commutators or squares of the elements involved.

It remains to check that the vectors of degree \( n \) are independent modulo \( \Gamma_{n+1} \); this is done by induction on \( n \), the first two values following from [BSV99]. Consider first odd integers \( n > 1 \), when there is a single element \( W(c) \) to consider. If \( W(c) \in \Gamma_{n+1} \), then since \( W(c) = [V(c), \tau] = [V(c), \mu] \) for some \( V(c) \), this would imply \( V(c) \in \Gamma_n \) contrary to induction.

Consider now even integers \( n = 2m \), and the elements \( W(c), A^2 \) of our putative basis of degree \( n \), for some \( A \in \Gamma_m \). Among the linear combinations in \( \Gamma_{n+1} \), if \( W(c) \in \Gamma_{n+1} \), then again this would imply \( V(c) \in \Gamma_n \) for some \( V(c) \) of lesser weight; if \( A^2 \in \Gamma_{n+1} \), then \( A \in \Gamma_{m+1} \); if \( W(c)A^2 \in \Gamma_{n+1} \), then \( V(c) \in \Gamma_n \) and \( A \in \Gamma_{m+1} \). All these conclusions are contrary to induction, and are the only possible linear combinations among basis elements of degree \( n \).

These were guessed using Gap [S93], by computing the 2-series of the quotient of \( \Gamma \) acting on the eighth level \( \Sigma^8 \) of the tree. The use of 0 and 1 is motivated by the fact that \( [0v, \tau] \equiv 1v \).

It is there that “dropping” squares makes the computations so much simpler for the dimension series than for the lower central series.

Imagine the Lie graph as hanging from its top generators \( \tau, \mu \), with length-1 and length-doubling edges attaching the basis elements. The depth of an element is the maximal number of consecutive edges connecting it to the top. In this image all edges are under tension, uniquely defining the Lie graph structure. The Lie graph is determined by having degree-compatible edges, and being connected.
The proof of Theorem 1.1 now follows; indeed the basis vectors of degree $2^n$ are $\tau_{2^n}$, $\mu_{2^n}$ and the $1^i(c)^{n-i}$ for all $i \in \{0, \ldots, n-1\}$, making $n+2$ elements in all, and the basis vectors of degree $2^n m$, for odd $m$, are the $1^iW(c)^{n-i}$ for all $i \in \{0, \ldots, n\}$, where $W$ is the unique word in $\{0,1\}^*$ such that $W(c)$ is of degree $m$. Then by (1)

\[
\sum_{n \geq 0} \dim(\pi^n/\pi^{n+1}) h^n = \prod_{n \geq 0} (1 + h^{2^n}) \prod_{n \geq 0} \prod_{m \geq 1} (1 + h^{2^n m})
\]

\[= \frac{1}{1 - h} \prod_{m \geq 1} \frac{1}{1 - h^m} = \sum_{0 \leq i \leq n} p(i) h^n.\]

The asymptotics $p(n) \approx e^{\pi \sqrt{2n/3}}$ are well known to number theorists [HR18, Usp20] and imply by (2) the lower bound on the growth of $\Gamma$.

In fact, this lower bound is already known, because $\Gamma$ is not virtually nilpotent.

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