Random-matrix ensembles in $p$-wave vortices

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Abstract

In disordered vortices in $p$-wave superconductors the two new random-matrix ensembles may be realized: $B$ and $D_{III}$-odd (of $so(2N+1)$ and $so(4N+2)/u(2N+1)$ matrices respectively). We predict these ensembles from an explicit analysis of the symmetries of Bogoliubov-deGennes equations in three examples of vortices with different $p$-wave order parameters. A characteristic feature of the novel symmetry classes is a quasiparticle level at zero energy. Class $B$ is realized when the time-reversal symmetry is broken, and class $D_{III}$-odd when the time-reversal symmetry is preserved. We also suggest that the main contribution to disordering the vortex spectrum comes from the distortion of the order parameter around impurities.

Since Wigner's modeling Hamiltonians of complex nuclei by random matrices [1], the random-matrix theory (RMT) has played an important role in studying mesoscopic systems. In many cases, a chaotic (non-integrable) mesoscopic system may be accurately described by RMT. The supersymmetric technique by Efetov [2] provides a microscopic explanation of the RMT approximation for disordered systems.

According to the RMT approximation, the only characteristics of the system affecting the eigenvalue correlations at small energy scales are its symmetries. Therefore, a problem arises of classifying symmetries of random-matrix ensembles. It has been suggested by several authors [3, 4, 5] that random-matrix ensembles may be classified as corresponding symmetric spaces. The symmetric spaces may be divided into twelve infinite series reviewed in Table 1 (we split class $D_{III}$ in two subclasses: $D_{III}$-even and $D_{III}$-odd) [7]. Each of the symmetry classes may occur in one of the three forms: positive-curvature, negative-curvature and flat [7]. The corresponding Jacobians in the matrix space are expressed in terms of trigonometric, hyperbolic, and polynomial functions respectively. In the present paper we shall only discuss the RMT for Hamiltonians forming a linear space and therefore described by zero-curvature (flat) versions of RMT.

The simplest examples of the RMT symmetry classes are the three Wigner-Dyson classes: unitary, orthogonal and symplectic ($A$, $A_I$, and $A_{II}$, respectively, in Cartan’s notation) [8]. In these classes, the energy level correlations are invariant under translations in energies (at energy scales much smaller than the spectrum width), and the joint probability distribution of the energy levels $\omega_i$ is

$$
\frac{dP_\omega}{\prod_i d\omega_i} \propto \prod_{i<j} |\omega_i - \omega_j|^\beta.
$$

(1)

The parameter $\beta$ determines the strength of level repulsion and takes values 2, 1, and 4 for the unitary, orthogonal, and symplectic ensembles, respectively.
Table 1: Symmetric spaces and universality classes of random-matrix ensembles: (a) the three Wigner-Dyson classes $A$ (unitary), $A_I$ (orthogonal), $A_{II}$ (symplectic) [8]; (b) chiral classes $A_{III}$ (unitary), $BD_I$ (orthogonal), $C_{II}$ (symplectic), with dimensions $p \leq q$ have $q - p$ zero modes [9]; (c) superconducting classes $C$, $D$, $CI$, $DIII$-even [3]; (d) $p$-wave vortex classes $DIII$-odd, $B$ have one zero mode.

| Cartan class | Symmetric space | Dimension | Rank | $\beta$ | $\alpha$ |
|--------------|----------------|-----------|------|---------|---------|
| $A$ [GUE]    | SU($N$)        | $N^2-1$  | $N-1$ | 2       | –       |
| $AI$ [GOE]   | SU($N$)/SO($N$) | $(N-1)(N+2)/2$ | $N-1$ | 1       | –       |
| $AII$ [GSE]  | SU(2$N$)/Sp($N$) | $(N-1)(2N+1)$ | $N-1$ | 4       | –       |
| $A_{III}$ [chGUE] | SU($p$+$q$)/S(U($p$)$\times$U($q$)) | $2pq$ | $p$ | 2 | $1+2(q-p)$ |
| $BDI$ [chGOE] | SO($p$+$q$)/SO($p$)$\times$SO($q$) | $pq$ | $p$ | 1 | $q-p$ |
| $CII$ [chGSE] | Sp($p$+$q$)/Sp($p$)$\times$Sp($q$) | $4pq$ | $p$ | 4 | $3+4(q-p)$ |
| $C$          | Sp($N$)        | $N(2N+1)$ | $N$  | 2       | 2       |
| $D$          | SO($2N$)       | $N(2N-1)$ | $N$  | 2       | 0       |
| $CI$         | Sp($N$)/U($N$) | $N(N+1)$ | $N$  | 1       | 1       |
| $DIII$-even  | SO($4N$)/U($2N$) | $2N(2N-1)$ | $N$  | 4       | 1       |
| $DIII$-odd   | SO($4N$+$2$)/U($2N$+$1$) | $2N(2N+1)$ | $N$  | 4       | 5       |
| $B$          | SO($2N$+$1$)   | $N(2N+1)$ | $N$  | 2       | 2       |

Other symmetry classes appear when there exists an additional symmetry relating energies $E$ and $-E$. In the corresponding random-matrix ensembles, the levels $\omega_i$ repel not only each other, but also their mirror images $-\omega_i$, and the Jacobian has the form

$$dP[\omega_i] \propto \prod_i (\omega_i^2 - \omega_j^2) \prod_i \omega_i^{\beta}. \quad (2)$$

It is now characterized by the two parameters $\beta$ and $\alpha$ (the latter responsible for suppressing the density of states near $E = 0$).

The three chiral classes ($A_{III}$, $BDI$, and $CII$) appear in systems with a chiral symmetry anti-commuting with the Hamiltonian [5]. Four more classes ($C$, $D$, $CI$, and $DIII$) describe mesoscopic superconducting systems (with or without spin-rotational and time-reversal symmetries) [3]. The present paper is devoted to the last two lines in Table 1: classes $DIII$-odd and $B$. We shall demonstrate that these two classes correspond to the symmetries of a $p$-wave vortex with or without time-reversal symmetry, respectively (the symmetry class $B$ in $p$-wave vortices has also been predicted in [10]; a partial account of the present work appeared as a preprint [11]).

Below we consider three particular $p$-wave vortices and describe their symmetries: the single-quantum vortex in the $A$-phase with a fixed spin orientation of the order parameter, the half-quantum vortex in the $A$-phase with rotating orientation of the order parameter, and the spin (fluxless) vortex in the $B$-phase. We explicitly demonstrate that in the first two examples the
symmetries of the Hamiltonian are of type $B$, and in the last example – of type $D_{III}$-odd. Then we briefly discuss microscopic requirements for the RMT limit and show that the main contribution to the level mixing comes from inhomogeneous suppression of the order parameter by impurities. Our estimates suggest that the RMT limit may be achieved in a moderately clean superconductor.

1 Single-Quantum Vortex

In this section we consider a vortex in the order parameter similar to the $A$ phase of $^3$He. Namely, we assume that the condensate wave function has the form

$$\Psi_\pm = e^{i\varphi} \left[ d_x \left( |\uparrow\uparrow\rangle + |\downarrow\downarrow\rangle \right) + id_y \left( |\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle \right) \right] (k_x \pm i k_y) \right],$$

and that the vector $d$ defining the orientation of the triplet is fixed (without loss of generality, take it to be in the $z$ direction. We also assume a fixed chirality of the order parameter and take positive sign in place of $\pm$ in (3)). Then the only allowed vortices are those involving a rotation of the phase of the order parameter, and the magnetic flux in such vortices is quantized in conventional superconducting flux quanta $\Phi_0 = \hbar c/2e$.

Below we analyze the low-lying energy levels obtained as solutions to Bogoliubov–deGennes equations for the mean-field Hamiltonian

$$H = \sum_\alpha \Psi_\alpha^\dagger \left[ \frac{(p - eA)^2}{2m} + V(r) - \varepsilon \right] \Psi_\alpha + \Psi_\uparrow^\dagger \left( \Delta_x \star \frac{p_x}{k_F} + \Delta_y \star \frac{p_y}{k_F} \right) \Psi_\downarrow + h.c.,$$

where $\Psi_\alpha$ are the electron operators ($\alpha$ is the spin index), $V(r)$ is the external potential of impurities, $A(r)$ is the electromagnetic vector potential, $\Delta_x(r)$ and $\Delta_y(r)$ are the coordinate-dependent components of the superconducting gap. In the bulk, the preferred superconducting order is one of the two chiral components $\eta_\pm = \Delta_x \pm i \Delta_y$, but in inhomogeneous systems, such as a vortex core, an admixture of the opposite component may be self-consistently generated. Star ($\star$) denotes the symmetrized ordering of the gradients $p_\mu$ and the order parameters $\Delta_\mu$ [definition: $A \star B \equiv (AB + BA)/2$]. At infinity, the order parameter imposes the vortex boundary conditions:

$$\Delta_x(r \to \infty, \phi) = \Delta_0 e^{\pm i \phi}, \quad \Delta_y(r \to \infty, \phi) = i \Delta_0 e^{\pm i \phi},$$

where $r$ and $\phi$ are polar coordinates. Plus or minus signs in the exponent correspond to a positive or a negative single-quantum vortex. For an axially-symmetric vortex with the chirality of the order parameter non-self-consistently fixed ($\Delta_y \equiv i \Delta_x$), without the vector-potential $A(r)$ and without disorder $V(r)$, the low-lying eigenstates of the Hamiltonian have been found by Kopnin and Salomaa [13]. The spectrum is

$$E_n = n \omega_0, \quad (p - \text{wave}), \quad n = 0, \pm 1, \pm 2, \ldots$$

with $\omega_0 \sim \Delta^2/\varepsilon_F$. This result should be compared with the spectrum of the vortex core in a $s$-wave superconductor [14]:

$$E_n = \left( n + \frac{1}{2} \right) \omega_0, \quad (s - \text{wave}), \quad n = 0, \pm 1, \pm 2, \ldots$$
The common feature of the spectra in the s-wave and p-wave cases is the symmetry with respect to zero energy. If we interpret holes in the negative-energy levels as excitations with positive energies (and with opposite spin), then this symmetry implies that the excitations are doubly degenerate in spin: to each spin-up excitation there corresponds a spin-down excitation at the same energy. For the s-wave vortex, this degeneracy is due to the full spin-rotation $SU(2)$ symmetry. The p-wave Hamiltonian $H$ has a reduced spin symmetry. Namely, it has the symmetry group $O(2)$ generated by rotations about the $z$-axis ($\Psi_+ \mapsto e^{i\alpha}\Psi_+$, $\Psi_- \mapsto e^{-i\alpha}\Psi_-$) and by the spin flip $\Psi_+ \mapsto \Psi_-$, $\Psi_- \mapsto \Psi_+$. This non-abelian group causes the two-fold degeneracy of all levels, except for the zero-energy level(s) where the symmetry $O(2)$ may mix the creation and annihilation operators for the same state. This symmetry is crucial for our discussion. Note that we have not included in the Hamiltonian neither the spin-orbit term ($U_{SO} \cdot [\sigma \times \mathbf{p}]$), nor the Zeeman splitting $H(\mathbf{r}) \cdot \sigma$. Either of these terms would break the spin symmetry $O(2)$, which would eventually result in a different universality class of the disordered system (type $D$ with non-degenerate levels), if these terms are sufficiently strong.

The difference between the s- and p-wave vortices is the zero-energy level in the p-wave case. It has been shown by Volovik that this level has a topological nature [15]. Indeed, suppose we gradually increase disorder in the Hamiltonian (4). The levels shift and mix, but the degeneracy of the levels remains the same as long as the symmetry $O(2)$ is preserved. The total number of levels remains odd, and therefore the zero-energy level cannot shift if the final Hamiltonian is a continuous deformation of the original one (without disorder), i.e. if the topological class of the boundary conditions (5) remains the same.

Now we can identify the symmetry class of the Bogoliubov–deGennes Hamiltonian. The only symmetry of the mean-field Hamiltonian (4) is the spin symmetry $O(2)$. The time-reversal symmetry is already broken by the vortex and by the pairing, and therefore neither the vector potential $\mathbf{A}(x)$ nor local deformations of $\Delta_0$ can reduce the symmetry of the Hamiltonian. When projected onto spin-up excitations $\gamma_+^\dagger = \int [u(\mathbf{r})\Psi_+^\dagger(\mathbf{r}) + v(\mathbf{r})\Psi_-^\dagger(\mathbf{r})]d^2\mathbf{r}$, the Bogoliubov–deGennes Hamiltonian for the two-component vector $(u, v)$ takes the form:

$$H_{\text{BdG}} = \left( \begin{array}{cc} \frac{-i\nabla - e\mathbf{A}}{2m}^2 + V(\mathbf{r}) - \varepsilon_F & \frac{\Delta_x}{k_F} \star (-i\nabla_x) + \frac{\Delta_y}{k_F} \star (-i\nabla_y) \\ \frac{\Delta_x}{k_F} \star (-i\nabla_x) + \frac{\Delta_y}{k_F} \star (-i\nabla_y) & \frac{-i\nabla + e\mathbf{A}}{2m}^2 + V(\mathbf{r}) - \varepsilon_F \end{array} \right).$$

In an arbitrary orthonormal basis of electronic states, this Hamiltonian may be written as a matrix

$$H_{\text{BdG}} = \left( \begin{array}{cc} h & \Delta \\ \Delta^\dagger & -h^* \end{array} \right).$$

From the hermiticity of the Hamiltonian, it follows that $h^\dagger = h$. From the explicit form of the p-wave pairing, $\Delta = -\Delta^T$ (it is here that the p-wave structure of the pairing is important; for s-wave pairing we would have $\Delta = \Delta^T$ instead). These are the only restrictions on the Hamiltonian (4). If we define

$$U_0 = \frac{1}{\sqrt{2}} \left( \begin{array}{cc} 1 & 1 \\ i & -i \end{array} \right),$$

the restrictions on the Hamiltonian (4) are equivalent to the condition that the rotated matrix $iU_0H_{\text{BdG}}U_0^{-1}$ is real antisymmetric, i.e. it belongs to the Lie algebra $so(M)$, where $M$ is the dimension of the Hilbert space (the same rotation of the Hamiltonian was used in [3] to identify the $D$ universality class).
The last step in our argument is to note that, under the vortex boundary conditions, the dimension of the Hamiltonian (9) is odd, not even (this may be difficult to visualize from the particle-hole representation (9), but easier from the rotated Hamiltonian $U_0 H_{\text{BdG}} U_0^{-1}$). Thus for a single-quantum vortex, we identify the space of the Hamiltonians as $so(2N+1)$ (class $B$ in Cartan’s notation, see Table 1).

2 Half-Quantum Vortex

Now let us consider a slightly different order parameter: suppose that the condensate wave function is described by the same expression (3), but now the vector $d$ is able to rotate (either in plane or in all three dimensions). The order parameter does not change under simultaneous change of sign of the vector $d$ and phase shift by $\pi$ of the phase $\phi$: $(\phi, d) \sim (\phi + \pi, -d)$. This makes possible a half-quantum vortex with magnetic flux $\Phi_0/2$ [16]. On going around such a vortex, both the vector $d$ and the phase $\phi$ rotate by $\pi$ (Fig. 1a) (see also discussion of such vortices in [17, 18]).

Consider first the case when the vector $d$ rotates in a plane (without loss of generality, let $d$ rotate in the $x$–$y$ plane). Then the condensate wave function (3) is the sum of two decoupled components containing $|\uparrow\uparrow\rangle$ and $|\downarrow\downarrow\rangle$ pairings:

$$
\Psi(r, \theta) = \left[ \Delta_+^{(1)}(r, \theta) |\uparrow\uparrow\rangle + \Delta_+^{(2)}(r, \theta) |\downarrow\downarrow\rangle \right] (k_x + ik_y)
$$

[here $r$ and $\theta$ are the polar coordinates]. In a half-quantum vortex, one of the pairing components $\Delta_+^{(1)}$ and $\Delta_+^{(2)}$ has an odd winding number of the phase around the vortex, while the other has an even winding number of the phase. In the simplest vortex of this type, the even winding number is zero, the corresponding component of the order parameter is nearly uniform and does not produce states near Fermi energy. The component with an odd winding number (without loss of generality, let it be $\Delta_+^{(1)}$, and the winding number be one) has vortex boundary conditions. The mean-field Hamiltonian decouples and the vortex part of the Hamiltonian (spin-up sector) takes the form [we
have also assumed an axially symmetric order parameter:

\[ H = \int d^2 r \left[ \Psi_\uparrow^\dagger \left( \frac{P_\uparrow^2}{2m} - \varepsilon_F \right) \Psi_\uparrow + e^{i\theta} \Delta(r) \Psi_\uparrow^\dagger \left( \nabla_x + i \nabla_y \right) \Psi_\uparrow + \text{h.c.} \right]. \]  

(12)

In this form the half-quantum vortex is equivalent to a single-quantum vortex in a superconductor of spinless fermions (such a superconductor must have odd pairing). The quasiparticles do not have a definite spin projection, but mix particles and holes in the same spin-up sector: particles in negative-energy levels are identical to those for a single-quantum vortex, and the solution of Kopnin and Salomaa [13] is equally applicable to this half-quantum vortex resulting in the same spectrum (6). However, because of the relation \( \gamma'(E) = \gamma(-E) \), the number of subgap states in the half-quantum vortex is one half of those in a single-quantum vortex: particles in negative-energy levels are identical to holes in positive-energy levels. The zero-energy state is a Majorana fermion [13, 14].

Now add a disorder term to the Hamiltonian of a clean vortex (12). Consider the following perturbations:

- potential scattering:  \( H_V = \Psi_\alpha^\dagger V(r) \Psi_\alpha \),
- Zeeman field:  \( H_h = \Psi_\alpha^\dagger (h(r) \sigma_{\alpha\beta}) \Psi_\beta \),
- vector potential:  \( H_a = \Psi_\alpha^\dagger (a(r) \cdot p) \Psi_\alpha \),
- spin-orbit scattering:  \( H_U = \Psi_\alpha^\dagger (U_{\text{SO}}(r) \ast [p \times \sigma_{\alpha\beta}]) \Psi_\beta \),
- deformation of order parameter:

\[ H_\Delta = \Psi_\alpha^\dagger \left( \Delta_\alpha^{(x)}(r) \ast p_x + \Delta_\alpha^{(y)}(r) \ast p_y \right) \Psi_\beta + \text{h.c.} \]

We assume that the perturbation may be switched on adiabatically in such a way that subgap levels stay localized. One possible way to fulfill this condition is to require that the perturbation is introduced only in a finite region around the vortex core.

The symmetry classification proceeds slightly differently depending on whether the perturbation preserves the decoupling of the Hamiltonian into spin-up and spin-down sectors.

The Hamiltonian remains decoupled under the perturbations \( H_V, H_a, H_h \) with \( \mathbf{h} \parallel \hat{z} \), \( H_U \) with \( U_{\text{SO}} \perp \hat{z} \), and \( H_\Delta \) with diagonal \( \Delta_{\alpha\beta} \). With these perturbations, the “spin-up” sector of the Hamiltonian in the basis \( (u, v) \) preserves its form (13) with the same symmetries as for a single-flux vortex. The symmetry class of the Hamiltonian is therefore identified as \( \text{so}(2N + 1) \) (class \( B \)).

If other perturbations are present, so that the Hamiltonian no longer splits into “spin-up” and “spin-down” parts, the full matrix of the Hamiltonian needs to be considered. Without any spin structure, the Hamiltonian breaking time-reversal and spin symmetries belongs to the \( \text{so}(M) \) symmetry class, as shown by Altland and Zirnbauer [13]. Their argument also works in our case, with the reservation that the number of levels \( M \) is odd (with each level counted twice: once as a particle and the other time as a hole). It follows from the usual continuity consideration (the Majorana fermion at zero energy survives any local perturbation). Therefore the symmetry class is again \( \text{so}(2N + 1) \) (class \( B \)). In the examples considered by Altland and Zirnbauer, the number of levels was even, which lead to the symmetry class \( D \) or \( \text{so}(2N) \).

It is important however, that to reach the RMT limit in the latter case, such a system must have sufficiently strong disorder to bring the quasiparticles from the “spin-down” sector (gapped in the clean vortex) to the Fermi energy and to mix them strongly with the quasiparticles in the “spin-up” sector. In the present paper, we do not discuss conditions for reaching this limit.
3 Spin Vortex

In this section we consider a hypothetical phase of a triplet superconductor analogous to the phase \( B \) of \(^3\)He. Namely, the condensate wave function is assumed to be

\[
\Psi = \Delta e^{i\phi} \left[ d_x(k) \left( |\uparrow\uparrow\rangle + |\downarrow\downarrow\rangle \right) + i d_y(k) \left( |\uparrow\uparrow\rangle - |\downarrow\downarrow\rangle \right) \right], \quad d(k) = R_\theta(k),
\]

and \( R_\theta \) is a rotation in the plane by angle \( \theta \) (like in the previous sections, we consider a two-dimensional problem, and \( k \) is a two-dimensional vector). In this phase, the vector \( d \) rotates a full turn as the vector \( k \) goes around the circular Fermi surface. The parameter \( \theta \) denotes the angle between vectors \( k \) and \( d \). We neglect spin-orbit interactions and assume that all values of \( \theta \) are allowed.

It is important that this phase preserves the time-reversal symmetry. Moreover, a vortex structure is possible without time-reversal symmetry breaking. It is realized by rotating the angle \( \theta \) a full turn of \( 2\pi \) when going around the vortex center and keeping the phase \( \phi \) constant (Fig. 1b). This vortex preserves the time-reversal symmetry and is a topological defect in the spin structure of the triplet pairing, thus we shall call it “spin vortex”.

The Hamiltonian of the clean axially symmetric spin vortex splits into “spin-up” and “spin-down” sectors, similarly to the case of the half-quantum vortex considered in the previous section. In the spin vortex, both the “spin-up” and the “spin-down” components have vortex structure. The “spin-up” Hamiltonian is identical to the Hamiltonian \( H \) of (12), while the “spin-down” component is its complex conjugate \( H^\ast \) (with spin reversed) — or vice versa.

In a clean spin vortex, the spectrum of each of the two vortices (in the “spin-up” and the “spin down” sectors) are identical to that in the half-quantum vortex considered in the previous section, and therefore the total spectrum coincides with that of the single-flux vortex of Section I (with the two Majorana fermions combining into a single zero-energy level). In contrast to the single-quantum vortex, it is now not the spin symmetry which is responsible for the double degeneracy of the levels with non-zero energy, but the time-reversal symmetry (the degeneracy is due to Kramers’ theorem).

Now suppose that the Hamiltonian is perturbed by a disorder term preserving the time-reversal symmetry. The allowed perturbations include potential scattering \( H_V \), spin-orbit scattering \( H_U \), as well as deformation of the order parameter \( H_\Delta \) in a time-reversal invariant way. Then the levels with non-zero energy stay doubly degenerate, and the zero-energy level cannot shift (because it is non-degenerate). The symmetry class of the perturbed Hamiltonian may be identified from the argument of Altland and Zirnbauer as \( D_{\text{III}} \) (time-reversal symmetry present, spin symmetry broken) [3]. In the \( p \)-wave vortex, the number of subgap levels is odd, and therefore the symmetry class is \( D_{\text{III-odd}} \), in contrast to class \( D_{\text{III-even}} \) studied by Altland and Zirnbauer [similarly to distinguishing between classes \( B \) and \( D \), we find it useful to distinguish between classes \( D_{\text{III-even}} \) and \( D_{\text{III-odd}} \): they have different parameters \( \alpha \), as a consequence of the zero-energy level in odd dimensions].

4 Level Mixing by Disorder

In this section we discuss whether the RMT limit may be realized in disordered vortices. We suppose that the superconductor contains a finite concentration \( n_{\text{imp}} \) of spinless impurities with strong electron scattering (Born parameter is of order one). Then the effect of impurities is
corresponding matrix elements. The deformation of the order parameter also suppress the
superconducting order parameter thus inducing an inhomogeneous pairing perturbation \( H_{\Delta} \).
We shall see that the latter effect has a much stronger influence on shifting the energy levels of the
subgap states.

Consider first the potential term \( H_V \) with the impurity potential \( V(r) = \sum_i V_i \delta(r - r_i) \). The
matrix element of the impurity potential between the two localized states is equal to

\[
(H_V)_{mn} = \sum_i V_i \rho_{mn}(r_i) , \quad \rho_{mn}(r) = (u_m^*(r)u_n(r) - v_m^*(r)v_n(r)) .
\] (14)

From explicitly solving Bogoliubov–deGennes equations for a clean \( p \)-wave vortex [13], one finds
that \( u_n \) and \( v_n \) are proportional to each other, to the leading order in \( E/\Delta \) (where \( E = n\omega_0 \) is
the energy of the \( n \)-th level). Therefore the matrix element \( \rho_{mn} \) between any two low-lying states
nearly vanishes (this was noticed by Volovik in [13]). A more accurate calculation shows that matrix
elements of the charge density between states at energy \( E \) is of the order of \( \xi^{-2}(E/\Delta) \), which is
by the factor \( (E/\Delta) \) smaller than that in the \( s \)-wave vortex. As a consequence, even for strong
impurities (with \( V_i \sim \varepsilon_F k_F^{-2} \)), an extremely high impurity concentration \( (n_{\text{imp}} \sim k_F^2) \) is required
to mix low-energy levels — a much higher disorder than that destroying \( p \)-wave superconductivity
\( (n_{\text{imp}} \sim k_F^{-1}) \) [20, 21].

The suppression of the order parameter \( H_{\Delta} \) turns out to be a more important effect. In
unconventional superconductors, spinless impurities suppress pairing in the region of size \( \xi \) which
is of the same order as the size of the vortex core. For a homogeneous order parameter (without vortex),
the suppression of the order parameter \( \delta \Delta(r) \) decreases as \( R^{-1} \) (in our two-dimensional problem)
at distances up to \( \xi \), and is negligible at distances much larger than \( \xi \) [22, 23], and the integral suppression may be estimated as
\( \int \delta \Delta \ ds r \sim \Delta k_F^{-1} \). Also, in a chiral superconductor, the component of the order parameter of opposite chirality is admixed in the vicinity of impurity [24].

The two effects gives comparable contributions to the matrix elements between subgap states. An
accurate calculation of the impurity influence on the states in the vortex core requires a full self-
consistent solution for the order parameter taking into account both the vortex and the impurities.
This goes beyond the scope of this paper, and we only estimate the order of magnitude of the
corresponding matrix elements. The deformation of the order parameter \( H_{\Delta} \) has the matrix
elements between subgap states \( m \) and \( n \):

\[
(H_{\Delta})_{mn} = \int d^2r \left( u_m^*(r) \left[ \delta \Delta^{(\alpha)}(r) \ast (i\nabla) \right] u_n(r) + v_m^*(r) \left[ \delta \Delta^{(\alpha)*}(r) \ast (i\nabla) \right] v_n(r) \right) .
\] (15)

Unlike the case of the matrix elements of \( H_V \), there is no cancellation in \( (H_{\Delta})_{mn} \), and we may estimate, for one impurity in the vortex core

\[
(H_{\Delta})_{mn} \sim \frac{1}{\xi^2} \int \delta \Delta(r) d^2r \sim \Delta(k_F) \sim \omega_0 .
\] (16)

Thus, a single impurity introduces matrix elements of the order of interlevel spacing. These
estimates suggest that the RMT regime may be achieved when the number of impurities in the
vortex core is much greater than one (which corresponds to the moderately clean regime \( \Delta^{-1} \ll \tau \ll \omega_0^{-1} \)).
5 Summary and Discussion

We have shown that symmetries of vortex Hamiltonians in $p$-wave superconductors correspond to one of the two random-matrix ensembles: $B$ or DIII-odd. These two RMT ensembles are distinguished from the rest of ensembles by the presence of zero-energy modes (zero-energy modes are also present in chiral ensembles at $p \neq q$, see Table 1). Zero-energy levels appear in $p$-wave vortices as a consequence of odd pairing symmetry combined with the vortex (topologically nontrivial) boundary conditions. We have checked the symmetries of the Hamiltonian explicitly taking three vortices as examples: a single-quantum vortex with spin symmetry, a half-quantum vortex, and a flux-less (spin) vortex without time-reversal symmetry breaking. Based on these examples, we may conjecture that classes $B$ and DIII-odd appear in any vortex-like structure with odd pairing, whenever a zero-energy mode is present. In cases with time-reversal symmetry, the vortex belongs to the class DIII-odd; when the time-reversal symmetry is broken — to the class $B$. Of course, this can also be checked explicitly in any particular case.

The two classes $B$ and DIII-odd considered in this paper are odd-dimensional counterparts of the classes $D$ and DIII-even, respectively (see Table 1), with the even-dimensional classes realized in conventional (singlet-pairing) normal-superconducting structures. Far from zero energy, the statistics of energy levels is not affected by the zero mode. In the immediate vicinity of zero energy, repulsion from the zero mode in classes $B$ and DIII-odd increases the exponent $\alpha$ (see Table 1), suppressing the average density of states $\langle \rho(\omega) \rangle$ as $\langle \rho(\omega) \rangle \propto \omega^2$ for class $B$, and $\langle \rho(\omega) \rangle \propto |\omega|^5$ for class DIII-odd.

Because of the zero modes, the classes $B$ and DIII-odd stand somewhat separately from the rest of RMT ensembles. In particular, they do not appear in the table of correspondence between symmetries of the Hamiltonian and those of the transfer matrix (for RMT ensembles without zero-energy levels, this table establishes a one-to-one correspondence). Neither these two ensembles are known to be derived from a supersymmetric sigma-model as other classes are. Understanding the latter fact remains an interesting problem, as well as a possible microscopic derivation of the random-matrix theory for $p$-wave vortices with the supersymmetric method, analogously to that for the $s$-wave vortex.

Several more comments can be made on comparing our results to the level statistics in $s$-wave vortices. The latter is known to belong to the symmetry class $C$ having spin-rotational symmetry and broken time-reversal symmetry. In contrast with the $SU(2)$ spin symmetry in conventional superconductors, the single-quantum vortex considered in our paper has the $O(2)$ spin symmetry instead, which leads to a somewhat different symmetry classification. The resulting symmetry is $so(N)$, which leads to classes $D$ (in even dimensions, no vortex) or $B$ (odd dimensions, vortex with zero mode).

The joint probability distributions of the energy levels for classes $B$ ($p$-wave vortex) and $C$ ($s$-wave vortex) coincide (with $\alpha = \beta = 2$), and only the zero-energy level distinguishes the two ensembles. In particular, the average density of states in the class $B$ random-matrix ensemble is

$$\langle \rho(\omega) \rangle = \frac{1}{\omega_0} \frac{\sin(2\pi \omega/\omega_0)}{2\pi \omega} + \delta(\omega).$$

(for class $C$ it is the same but without the $\delta(\omega)$ term).

In $s$-wave vortices, Koulakov and Larkin found that in a wide range of impurity concentrations, the level mixing does not lead to a class $C$, but has more symmetries. These symmetries arise from the fact that the impurities are local scatterers in a chiral quasi-one-dimensional two-channel system, resulting in an ensemble of $2 \times 2$ random matrices, instead of $N \times N$ matrices with large
Such effect seems improbable in $p$-wave vortices where the main contribution to level mixing comes from the distortion of the order parameter around impurities which is extended in space to distances of the order of the core size. In this paper we performed only an order-of-magnitude estimate suggesting that the RMT limit may be reached in moderately clean vortices (with the number of impurities per vortex core much greater than one). A more accurate self-consistent treatment of impurities in the vortex core is required for a quantitative study of the effects of disorder.

The author wishes to thank M. V. Feigel'man for suggesting this problem, discussions, and comments on the manuscript. At different stages of work, the author benefited from discussions with G. Blatter, V. Geshkenbein, D. Gorokhov, R. Heeb, L. Ioffe, C. Mudry, M. Skvortsov, G. E. Volovik, and M. Zhitomirsky. The author thanks Swiss National Foundation for financial support.

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