V.I. Arnold’s “Global” KAM Theorem and geometric measure estimates

L. Chierchia*, C. E. Koudjinan†

October 27, 2020

Abstract
This paper continues the discussion started in [CK19] concerning Arnold’s legacy on classical KAM theory and (some of) its modern developments. We prove a detailed and explicit ‘global’ Arnold’s KAM Theorem, which yields, in particular, the Whitney conjugacy of a non–degenerate, real–analytic, nearly–integrable Hamiltonian system to an integrable system on a closed, nowhere dense, positive measure subset of the phase space. Detailed measure estimates on the Kolmogorov’s set are provided in the case the phase space is: (A) a uniform neighbourhood of an arbitrary (bounded) set times the $d$–torus and (B) a domain with $C^2$ boundary times the $d$–torus. All constants are explicitly given.

MSC2010 numbers: 37J40, 37J05, 37J25, 70H08

Keywords: Nearly–integrable Hamiltonian systems; perturbation theory; KAM Theory; Arnold’s scheme; Kolmogorov’s set; primary invariant tori; Lagrangian tori; measure estimates; small divisors; integrability on nowhere dense sets; Diophantine frequencies.

1 Introduction

a. In [CK19], we revised Arnold’s original analytic ‘KAM scheme’ [Arn63] and showed, in particular, how to implement it so as to get the optimal relation between the size of the perturbation $\varepsilon$ and the Diophantine constant $\alpha$ associated to a persistent integrable torus (for generalities, we refer to the Introduction in [CK19]).

*Luigi Chierchia: Dipartimento di Matematica e Fisica, Università “Roma Tre”, Largo San Leonardo Murialdo 1, I-00146 Roma (Italy); luigi@mat.uniroma3.it
†Comlan Edmond Koudjinan: Institute of Science and Technology Austria (IST Austria), Am Campus 1, 3400 Klosterneuburg, Austria; edmond.koudjinan@ist.ac.at
In the present paper we show how Arnold’s ‘pointwise theorem’ (Theorem A in [CK19]) leads, naturally, to a ‘global theorem’, unifying and improving various previous versions of such a result: compare, in particular, with [Nei81], [Pös82], [CG82], [Pös01]. The term ‘global’ refers, here, to the simultaneous (and ‘smooth’) construction, in phase space, of all persistent KAM tori having a prefixed Diophantine constant. The main theorem (Theorem 1 below) is formulated in terms of a (Whitney) symplectic transformation conjugating a given (Kolmogorov non–degenerate) analytic, nearly–integrable Hamiltonian system to a Hamiltonian system integrable on a closed, nowhere dense set. All constants involved in Theorem 1 are explicitly computed, and, in particular, the optimal relation between $\varepsilon$ and $\alpha$ is retained.

b. An immediate corollary of ‘Arnold’s global theorem’ is that measure estimates of the (complement of the) Kolmogorov’s set (i.e., the set of all persistent integrable tori of a nearly–integrable Hamiltonian system) become essentially trivial (since symplectic transformations preserve Liouville measure on phase space). The problem of finding explicit measure estimates of the Kolmogorov’s set in terms of the structure of the phase space is, therefore, reduced to a purely geometrical problem. In particular, as in [BC18], we are interested in analyzing how such measure estimates depend upon general geometric properties of the action domain, an issue which is particularly relevant in developing KAM theory for secondary tori (i.e., those invariant Lagrangian tori which arise because of the perturbation and are not a continuation of integrable tori); compare [BC15], [BC17], [BC20].

In this paper, we shall discuss detailed measure estimates in two different cases, namely:

(A) (General case) The Hamiltonian is ‘uniformly real–analytic’ on $\mathcal{D} \times \mathbb{T}^d$, with action domain $\mathcal{D} \subseteq \mathbb{R}^d$ being a completely arbitrary bounded set and the unperturbed frequency map is a local diffeomorphism; ‘uniformly analytic’ means that the Hamiltonian is real–analytic on the union of complex balls with centers in $\mathcal{D}$ and fixed radius $R > 0$. In this case the phase space will be $\mathcal{D} \times \mathbb{T}^d$, where $\mathcal{D}$ is a suitable (‘minimal’) open cover of $\mathcal{D}$. This set–up is similar to that considered in [BC18].

(B) (Smooth case) The Hamiltonian is real–analytic on a phase space $\mathcal{D} \times \mathbb{T}^d$ with $\mathcal{D}$ being a bounded, connected, open set with $C^2$ boundary and the unperturbed frequency map is a global diffeomorphism on $\mathcal{D}$.

c. Let us briefly describe the type of measure estimates we get.

\footnote{Indeed, closed sets of uniform Diophantine numbers may have, in general, isolated points; compare [Arg20].}
Case (A): As usual in classical KAM theory, we consider real–analytic Hamiltonians
\[ H : (y, x) \in \mathcal{D} \times \mathbb{T}^d \mapsto H(y, x) := K(y) + \varepsilon P(y, x) \in \mathbb{R}, \quad (*) \]
where \((y, x) \in \mathbb{R}^d \times \mathbb{T}^d\) are standard action–angle variables (i.e., the phase space is endowed with the standard symplectic form \(dy \wedge dx\)), \(\varepsilon\) a small parameter, and \(H\) is real–analytic on the union of \(\mathbb{R}\)–balls with centers in some bounded set \(\mathcal{D} \subseteq \mathbb{R}^d\), while \(\mathcal{D}\) is suitable neighbourhood of \(\mathcal{D}\) (see below). The integrable Hamiltonian \(K\) is assumed to be Kolmogorov non–degenerate on \(\mathcal{D}\) (i.e., the frequency map \(y \mapsto \omega = \partial_y K(y)\) is a real–analytic local diffeomorphism). Let us denote by \(\mathcal{K}_{\mathcal{D}}(\alpha, \tau)\) the set of Lagrangian graphs over \(\mathbb{T}^d\) in \(\mathcal{D} \times \mathbb{T}^d\), which are invariant for the flow governed by \(H\) and on which the flow is analytically conjugated to the Kronecker flow \(x \mapsto x + \omega t\), with \(\omega \in \mathbb{R}^d\) \((\alpha, \tau)\)–Diophantine\(^2\), for some \(\tau > d - 1\). Then, there exist positive numbers \(C_*, \alpha_*, \varepsilon_*\) and \(r \leq R/9\), depending only on \(d, \tau, K\) and \(P\) (and explicitly given in Theorem 4 below), such that, if \(0 < \varepsilon < \varepsilon_*\), then
\[ \text{meas} \left( (\mathcal{D} \times \mathbb{T}^d) \setminus \mathcal{K}_{\mathcal{D}}(\alpha_* \sqrt{\varepsilon}, \tau) \right) \leq C_* N^\text{int}_r(\mathcal{D}) \sqrt{\varepsilon}, \]
where \(N^\text{int}_r(\mathcal{D})\) is the so–called \(r\)–internal covering number of \(\mathcal{D}\) and \(\mathcal{D}\) is a \(\mathbb{R}\)–neighbourhood of a minimal \(\mathbb{R}\)–internal cover of \(\mathcal{D}\) (compare § 3.1 for precise definitions).

Case (B): Here \(H\) is as above but \(\mathcal{D}\) is assumed to be an open, bounded, connected set with \(C^2\) boundary; \(H\) is \(\mathbb{R}\)–uniformly real–analytic on \(\mathcal{D}\) and the unperturbed frequency map is assumed to be a global diffeomorphism on \(\mathcal{D}\). Let
\[ r := \min \{ R, \ \min \text{foc} (\partial \mathcal{D}), 1/\kappa \} / \sqrt{d}, \]
where ‘\(\min \text{foc}\)’ denotes the so–called minimal focal distance, and \(\kappa\) is the maximum modulus of the principal curvatures of \(\partial \mathcal{D}\). Then, there exist positive numbers \(C_*, \alpha_*, \varepsilon_*\) depending only on \(d, \tau, K\) and \(P\) (and explicitly given in Theorem 5 below) such that, if \(0 < \varepsilon < \varepsilon_*\), then
\[ \text{meas} \left( (\mathcal{D} \times \mathbb{T}^d) \setminus \mathcal{K}_{\mathcal{D}}(\alpha_* \sqrt{\varepsilon}, \tau) \right) \leq C_* \max \left\{ \sec_{d-1}(\mathcal{D}), \mathcal{H}^{d-1}(\partial \mathcal{D}) \right\} \sqrt{\varepsilon}, \]
where \(\sec_{d-1}(\mathcal{D})\) is the measure of the maximal \((d - 1)\)–dimensional section of \(\mathcal{D}\) and \(\mathcal{H}^{d-1}\) denotes the \((d - 1)\)–dimensional Hausdorff measure (compare § 3.2 for precise definitions).

\(^{2}\)I.e., \(|\omega \cdot k| \geq \alpha/|k|^\tau\), for any \(k \in \mathbb{Z}^d \setminus \{0\}\).
d. Remarks

(i) For the optimality of the relation between $\varepsilon$ and $\alpha$ (and the reason for choosing $\alpha = \alpha_* \sqrt{\varepsilon}$ in the Kolmogorov’s set), see item d in the Introduction of [CK19].

(ii) Theorem 4 below extends and generalizes the main result (Theorem 1) in [BC18].

(iii) In Appendix A (see, in particular, Remark A.5), we correct a small flaw (concerning the choice of some constants) in [CK19].

(iv) In Remark A.4 (Appendix A) all constants appearing in the proof are explicitly given.

e. The paper is organized as follows.

In § 2.1 we introduce some of the notation used in the paper and in § 2.2 we state the ‘global Arnold’s theorem’ (Theorem 1). The statement of such a theorem is quite detailed; in particular, the introduction of apparently arbitrary sets or parameters (such as $D_0$ or $\rho$) allows to make applications in quite different circumstances (such as cases (A) and (B) mentioned above). On the other hand, the proof of this theorem does not really contain novel ideas and it is based on the schemes in [Arn63], [Kou19] and [CK19]. However, since we put some emphasis in making everything explicit, we felt necessary to outline the proof, detailing, in particular, the choice of the (many) parameters involved (this is done in Appendix A).

§ 3 is devoted to measure estimates and, in particular, to the statements and proofs of Theorem 4 and 5, which have been briefly explained in item c above.

Finally, Appendix B contains some of the technical tools used in the paper, namely:

B.1 Classical estimates (Cauchy, Fourier)
B.2 An Inverse Function Theorem
B.3 Internal coverings
B.4 Extensions of Lipschitz continuous functions
B.5 Lebesgue measure and Lipschitz continuous map
B.6 Lipeomorphisms “close” to identity
B.7 Whitney smoothness
B.8 Measure of tubular neighbourhoods of hypersurfaces
B.9 Kolmogorov non-degenerate normal forms
2 Arnold’s Global KAM Theorem

2.1 Notations

- \( \mathbb{N} := \{1, 2, 3, \cdots \} \) and \( \mathbb{N}_0 := \{0, 1, 2, 3, \cdots \} \).
- For \( d \in \mathbb{N} \) and \( x, y \in \mathbb{C}^d \), we let \( x \cdot y := x_1 \bar{y}_1 + \cdots + x_d \bar{y}_d \) be the standard inner product (the bar denotes complex conjugate). We denote, respectively, the sup–norm, the 1–norm and the Euclidean norm, by:
  \[
  |x| := \max_{1 \leq j \leq n} |x_j|, \quad |x|_1 := \sum_{j=1}^{d} |x_j|, \quad |x|_2 := \sqrt{\sum_{j=1}^{d} |x_j|^2}.
  \]
- \( T^d := \mathbb{R}^d / 2\pi \mathbb{Z}^d \) is the \( d \)–dimensional (flat) torus.
- Given \( \alpha > 0, \tau \geq d - 1 \geq 1 \), we denote by
  \[
  \text{Dioph}_{\alpha, \tau}^d := \{ \omega \in \mathbb{R}^d : |\omega \cdot k| \geq \frac{\alpha}{|k|_1}, \quad \forall \ 0 \neq k \in \mathbb{Z}^d \},
  \]
  the set of \( (\alpha, \tau) \)–Diophantine vectors in \( \mathbb{R}^d \).
- For \( r, s > 0, y_0 \in \mathbb{C}^d, \emptyset \neq D \subseteq \mathbb{C}^d \), we denote:
  \[
  \begin{align*}
  & \mathbb{B}_r(y_0) := \{ y \in \mathbb{R}^d : |y - y_0| < r \}, \quad (y_0 \in \mathbb{R}^d), \\
  & \mathbb{B}_r(D) := \bigcup_{y_0 \in D} \mathbb{B}_r(y_0), \quad (D \subseteq \mathbb{R}^d), \\
  & \mathbb{B}_r(y_0) := \{ y \in \mathbb{C}^d : |y - y_0| < r \}, \\
  & \mathbb{B}_r(D) := \bigcup_{y_0 \in D} \mathbb{B}_r(y_0), \\
  & T^d_s := \{ x \in \mathbb{C}^d : |\text{Im} \ x| < s \} / 2\pi \mathbb{Z}^d, \\
  & \mathbb{B}_{r,s}(y_0) := \mathbb{B}_r(y_0) \times T^d_s, \\
  & \mathbb{B}_{r,s}(D) := \mathbb{B}_r(D) \times T^d_s;
  \end{align*}
  \]
  we shall also denote, in bold face characters, Euclidean balls:
  \[
  \begin{align*}
  & \mathbb{B}_r(y_0) := \{ y \in \mathbb{R}^d : |y - y_0|_2 < r \}, \quad (y_0 \in \mathbb{R}^d), \\
  & \mathbb{B}_r(D) := \bigcup_{y_0 \in D} \mathbb{B}_r(y_0), \quad (D \subseteq \mathbb{R}^d).
  \end{align*}
  \]
If \( \mathbb{I}_d \) := \text{diag}(1) is the unit \((d \times d)\) matrix, we denote the standard symplectic matrix by
\[
\mathbb{J} := \begin{pmatrix} 0 & -\mathbb{I}_d \\ \mathbb{I}_d & 0 \end{pmatrix}.
\]

For \( D \subseteq \mathbb{R}^d \), \( r \geq 0 \) and \( s > 0 \), \( \mathcal{B}_{r,s}(D) \) denotes the Banach space of real–analytic functions
\[
f : \mathcal{B}_r(D) \times \mathbb{T}_s \rightarrow \mathbb{C}
\]
with bounded holomorphic extensions to \( \mathcal{B}_{r,s}(D) \), with uniform norm
\[
\|f\|_{r,s} := \|f\|_{r,s,D} := \sup_{\mathcal{B}_{r,s}(D)} |f| < \infty.
\]

Analogously, \( \mathcal{B}_r(D) \) denotes the Banach space of real–analytic functions
\[
f : \mathcal{B}_r(D) \rightarrow \mathbb{C}
\]
with bounded holomorphic extensions to \( \mathcal{B}_r(D) \), with
\[
\|f\|_r := \|f\|_{r,D} := \sup_{\mathcal{B}_r(D)} |f| < \infty.
\]

For a differentiable function \( f : A \subseteq \mathbb{C}^d \times \mathbb{C}^d \ni (y, x) \mapsto f(y, x) \in \mathbb{C} \), its gradient/Jacobian is denoted by \( \nabla f \) or by \( f' \).

We equip \( \mathbb{C}^d \times \mathbb{C}^d \) (and its subsets) with the canonical symplectic form
\[
\omega := dy \wedge dx = dy_1 \wedge dx_1 + \cdots + dy_d \wedge dx_d,
\]
and denote by \( \phi^t_H \) the associated Hamiltonian flow governed by the Hamiltonian \( H(y, x) \), \( y, x \in \mathbb{C}^d \), i.e., \( z(t) := \phi^t_H(z) \) is the unique solution of
\[
\dot{z} = \mathbb{J} \nabla H, \quad z(0) = z.
\]

Given a linear operator \( L \) from the normed space \((V_a, \| \cdot \|_a)\) into the normed space \((V_b, \| \cdot \|_b)\), its “operator–norm” is given by
\[
\|L\| := \sup_{x \in V_a \setminus \{0\}} \frac{\|Lx\|_b}{\|x\|_a}, \quad \text{so that} \quad \|Lx\|_b \leq \|L\| \|x\|_a \quad \text{for any} \quad x \in V_a.
\]
Given \( \omega \in \mathbb{R}^d \), the directional derivative of a \( C^1 \) function \( f \) with respect to \( \omega \) is given by

\[
D_\omega f := \omega \cdot f_x = \sum_{j=1}^d \omega_j f_{x_j}.
\]

If \( f \) is a (smooth or analytic) function on \( \mathbb{T}^d \), its Fourier expansion is given by

\[
f = \sum_{k \in \mathbb{Z}^d} f_k e^{i k \cdot x}, \quad f_k := \frac{1}{(2\pi)^d} \int_{\mathbb{T}^d} f(x) e^{-i k \cdot x} \, dx,
\]

(where, as usual, \( e := \exp(1) \) denotes the Neper number and \( i \) the imaginary unit). We also set:

\[
\langle f \rangle := f_0 = \frac{1}{(2\pi)^d} \int_{\mathbb{T}^d} f(x) \, dx, \quad T_N f := \sum_{|k| \leq N} f_k e^{i k \cdot x}, \quad N > 0.
\]

For a function \( f : (\mathcal{M}, d_1) \to (\mathcal{M}, d_2) \), where \( (\mathcal{M}_j, d_j), \, j = 1, 2 \) are metric spaces, we denote

\[
\text{Lip}_{\mathcal{M}_1}(f) := \|f\|_{\text{Lip}_{\mathcal{M}_1}} := \sup_{x \neq x' \in \mathcal{M}_1} \frac{d_2(f(x), f(x'))}{d_1(x, x')} \leq \infty,
\]

and \( f \) is said Lipschitz continuous on \( \mathcal{M}_1 \) if \( \text{Lip}_{\mathcal{M}_1}(f) < \infty \). If \( \mathcal{M}_1 = \mathbb{R}^d \), we usually denote \( \text{Lip}_{\mathbb{R}^d}(f) = \text{Lip}(f) \).

\( C^k_W(D) \) denotes the set of functions which are \( C^k \) in the sense of Whitney on the set \( D \). A \( C^1_W \) map \( \phi : D \times \mathbb{T}^d \to \mathbb{R}^d \times \mathbb{T}^d \), is symplectic if the Whitney–gradient \( \nabla \phi = (\partial_y \phi, \partial_x \phi) \) satisfies \( (\nabla \phi)J(\nabla \phi)^T = J \) on \( D \times \mathbb{T}^d \). For more details, see Appendix B.7.

The \( s \)-dimensional Hausdorff measure on \( \mathbb{R}^d \) will be denoted by \( \mathcal{H}^s \); in particular \( \mathcal{H}^d \), which coincides with the \( d \)-dimensional outer Lebesgue measure, will be denoted by ‘meas’.

### 2.2 KAM Theorem

Given an open set \( \mathcal{D} \subseteq \mathbb{R}^d \) and a real–analytic Hamiltonian \( H : \mathcal{D} \times \mathbb{T}^d \to \mathbb{R} \), we say that \( \mathcal{T} \subseteq \mathcal{D} \times \mathbb{T}^d \) is a (primary\(^3\)) Kolmogorov (or ‘KAM’) torus for \( H \) if \( \mathcal{T} \) is a real–analytic Lagrangian embedded torus \( \mathcal{T} = \phi(\mathbb{T}^d) \), which is a graph over \( \mathbb{T}^d \), and such that

\[
\phi^t_H(\phi(\theta)) = \phi(\theta + \omega t), \quad \forall \theta \in \mathbb{T}^d, \, t \in \mathbb{R},
\]

\(^3\)As opposed to secondary tori (same definition but removing the graph assumption); for a KAM Theory for secondary tori, see [BC15]. In this paper, we shall only consider primary KAM tori.
for a given Diophantine ‘frequency vector’ \( \omega \in \text{Dioph}_n^\alpha \) (for some \( \alpha, \tau > 0 \)).

**Theorem 1** Let \( d \geq 2; \ R > 0; \ 0 < s \leq 1; \ \mathcal{D} \neq \mathcal{D} \subseteq \mathbb{R}^d; \ \varepsilon, \alpha > 0 \). Let the ‘integrable Hamiltonian’ \( K \in B_\mathcal{D}(\mathcal{D}) \) be a uniformly (Kolmogorov) non-degenerate (i.e. \( \det K_{yy} \neq 0 \)) and let the ‘perturbation’ \( P \) belong to \( B_{s_\mathcal{D}}(\mathcal{D}) \). Define

\[
M := \|K_{yy}\|_{\mathcal{R}, \mathcal{D}}, \quad L := \|K_{yy}^{-1}\|_{\mathcal{R}, \mathcal{D}}, \quad P := \|P\|_{\mathcal{R}, s, \mathcal{D}}, \quad \theta := ML, \quad \epsilon := \frac{\varepsilon MP}{\alpha^2}. \tag{2}
\]

Choose \( 0 < \rho < r \leq R, \ D_0 \subseteq \mathcal{D}, \ \tau \geq d - 1 \); define the following ‘action domains’:

\[
\mathcal{D} := B_r(D_0), \quad \widehat{\mathcal{D}} := B_{r-\rho}(D_0), \quad \mathcal{D}^* := \{ y \in \widehat{\mathcal{D}} : K_y(y) \in \text{Dioph}_{s\mathcal{D}}^\tau \}, \tag{3}
\]

and consider the ‘nearly–integrable’, non–degenerate Hamiltonian given by

\[
H : (y, x) \in \mathcal{D} \times \mathbb{T}^d \mapsto H(y, x) := K(y) + \varepsilon P(y, x) \in \mathbb{R};
\]

the ‘phase space’ \( \mathcal{D} \times \mathbb{T}^d \) being endowed with the standard symplectic form \( \omega \). Fix \( 0 < s_* < s \).

There exist constants \( c_*, c_0, c_1, c_2, c_3, c_4 > 1 \), depending only on \( d \) and \( \tau \), such that, if

\[
\alpha \leq \frac{c_0 \rho}{L}; \quad \epsilon \leq \epsilon_* := \frac{(s - s_*)^{\alpha}}{c_\theta}; \tag{4}
\]

with \( a := 7\nu + 4d + 2 \) and \( \nu := \tau + 1 \), then, the following statements holds.

There exists a nowhere dense set \( \mathcal{D}_* \subseteq B_{r-\frac{\epsilon}{2}}(D_0) \subseteq \mathcal{D} \), a lipeomorphism

\[
Y^* : \mathcal{D}^* \xrightarrow{\text{onto}} \mathcal{D}_*,
\]

a function \( K_* \in C_C^\infty(\mathcal{D}_*) \) and a \( C_C^\infty \)-symplectic transformation

\[
\phi_* := \text{id} + (v_*, u_*); \quad \mathcal{D}_* \times \mathbb{T}^d \rightarrow \mathcal{K} := \phi_*(\mathcal{D}_* \times \mathbb{T}^d) \subseteq \mathcal{D} \times \mathbb{T}^d; \tag{5}
\]

real–analytic in \( x \in \mathbb{T}^d \), such that\(^4\)

\[
\partial_y^* K_* \circ Y^* = \partial_y K, \quad \text{on } \mathcal{D}^*, \tag{6}
\]

\[
\partial_y^\beta(H \circ \phi_*)(y_*, x) = \partial_y^\beta K_*(y_*), \quad \forall (y_*, x) \in \mathcal{D}_* \times \mathbb{T}^d, \quad \forall \beta \in \mathbb{N}_0^d. \tag{7}
\]

\(^4\)\(y_*\)-derivatives are Whitney–derivatives.
Furthermore, the following estimates hold:

\[ \|Y^* - \text{id}\|_{\mathcal{D}^*} \leq c_1 (s - s_*)^\nu \frac{\varepsilon \log \epsilon}{\alpha}, \]
\[ \text{Lip}_{\mathcal{D}^*}(Y^* - \text{id}) \leq c_2 \theta^3 (s - s_*)^{-1} \frac{M \varepsilon P}{\alpha^2} \left( \log \frac{M \varepsilon P}{\alpha^2} \right)^\nu \leq \frac{1}{4d}, \]
\[ \max \left\{ \|u_*\|_*, 2d \sqrt{2} \frac{M \varepsilon P}{\alpha^2} \|v_*\|_* \right\} \leq c_3 \ell^\nu \theta^2 \frac{M \varepsilon P}{\alpha^2}, \]
\[ \|\partial_x u_*\|_* \leq c_4 \theta_0^2 \ell^\nu \frac{M \varepsilon P}{\alpha^2} \leq \frac{1}{4(18d^3 + 70d)} \]

where

\[ \| \cdot \|_* := \sup_{\mathcal{D}^* \times Td^d} | \cdot |, \quad \ell := 8(s - s_*)^{-1} \log \epsilon^{-1}. \]

The ‘Kolmogorov set’ $\mathcal{K}$ defined in (5) is foliated, as $y_* \in \mathcal{D}_*$, by Kolmogorov tori $T_* := \phi_*(\{y_*\} \times T^d)$, which are Kolmogorov non-degenerate\(^5\).

The proof of this theorem is based upon Arnold’s original KAM scheme, revised and improved in [CK19], where, in particular all constants are computed and optimal smallness conditions concerning the relation between small divisors and smallness of the perturbation are given. Since, essentially, no new ideas are needed, details are deferred to Appendix A.

However, let us make, here, a few observations.

**Remark 2**

(i) The hypotheses on $H$ can be rephrased by saying that $H$ is $\mathbb{R}$–uniformly real–analytic on $\mathcal{D}$. Notice that $\mathcal{D}$ can be a completely arbitrary subset of $\mathbb{R}^d$, but $\mathcal{D}$ and $\mathcal{G}$ are open sets.

(ii) The introduction of $\mathcal{D}_0$ and $\rho$ is made in order to be able to apply the theorem in quite different contexts; compare, e.g., next section on measure estimates.

(iii) Even if $\mathcal{D}_0$ is a single point, the theorem guarantees, in general, a set of positive measure of Kolmogorov tori for $H$, since the set $\mathcal{D}$ is a set of positive measure, provided $\tau > d - 1$ and $\alpha$ is small enough. Precise measure estimates are one of the objectives of this paper and will be given in next section.

(iv) The parameter $\theta$ defined in (2) measures the ‘torsion’ of the unperturbed system and it is always greater or equal than 1; indeed, for any $y_0 \in \mathcal{D}$, denoting $T(y) := K_{yy}(y)^{-1}$, one has:

\[ \theta := \text{LM} \geq \|T(y_0)\| \|K_{yy}(y_0)\| = \|T(y_0)\| \|T(y_0)^{-1}\| \geq 1. \]

(v) The constants $c_i$ appearing in the theorem are explicitly given in Appendix A; compare, in

\(^5\)For the precise definition, see Appendix A and B.9.
particular, Eq. (A.38).
(vi) $\mathcal{D}^*$ and $\mathcal{D}_*$ are closed, nowhere dense sets, but may have isolated points.$^6$

3 Measure estimates

The fact that the Kolmogorov set $\mathcal{K}$ in Theorem 1 is the image of a (Whitney) symplectic map, leads to straightforward measure estimates of its complement:

**Theorem 3** Under the same notations and assumptions of Theorem 1, let

$$
\beta := (1 + 2 \text{Lip}_{\mathcal{D}^*}(Y^* - \text{id}))^d (2\pi)^d,
\mathcal{T}_\rho := B_{r+\rho}(D_0) \setminus B_{r-\rho}(D_0),
\mathcal{R}_\alpha := \{ y \in \mathcal{D} : K_y(y) \notin \text{Dioph}_\alpha \}.
$$

Then, one has

$$
\text{meas}(\mathcal{D} \times \mathbb{T}^d \setminus \mathcal{K}) \leq \beta \text{meas}(B_\mathcal{D}(\mathcal{D}) \setminus \mathcal{D}^*) \leq \beta(\text{meas}(\mathcal{T}_\rho) + \text{meas}(\mathcal{R}_\alpha)).
$$

**Proof** By Theorem B.4, we can extend $Y^* - \text{id}$ component–wise to obtain a global Lipschitz continuous function $f : \mathbb{R}^d \supset$ satisfying $f|_{\mathcal{D}^*} = Y^* - \text{id}$ and

$$
\sup_{\mathbb{R}^d} |f| = \sup_{\mathcal{D}^*} |Y^* - \text{id}| \leq \frac{\rho}{2}, \quad \text{Lip}_{\mathbb{R}^d}(f) = \text{Lip}_{\mathcal{D}^*}(Y^* - \text{id}) < \frac{1}{4d}.
$$

Set $g := f + \text{id}$. Then, by Lemma B.6 and (15), one has$^7$

$$
\mathcal{D} \subseteq g(B_{\mathcal{D}}(\mathcal{D})).
$$

$^6$See [Arg20].
$^7$The bar on sets denotes closure.
Notice also that, by (15) and Lemma B.6, \( g \) is a lipeomorphism of \( \mathbb{R}^d \). Consequently,

\[
\text{meas}(\mathcal{D} \times \mathbb{T}^d \setminus \mathcal{K}) := \text{meas}(\mathcal{D} \times \mathbb{T}^d) - \text{meas}(\phi_*(\mathcal{D}_* \times \mathbb{T}^d)) \\
= \text{meas}(\mathcal{D} \times \mathbb{T}^d) - \text{meas}(\mathcal{D}_* \times \mathbb{T}^d) \\
= (2\pi)^d \left( \text{meas}(\mathcal{D}) - \text{meas}(\mathcal{D}_*) \right)
\]

\[
\leq (2\pi)^d \left( \text{meas}(g(B_{\frac{1}{2}}(\mathcal{D}))) - \text{meas}(\mathcal{D}_*) \right) \\
= (2\pi)^d \text{meas}(g(B_{\frac{1}{2}}(\mathcal{D})) \setminus g(\mathcal{D}_*)) \\
= (2\pi)^d \text{meas}(g(B_{\frac{1}{2}}(\mathcal{D})) \setminus \mathcal{D}_*) \quad \text{(because \( g \) is injective)}
\]

\[
\leq (2\pi)^d (\text{Lip} g)^d \text{meas}(B_{\frac{1}{2}}(\mathcal{D}) \setminus \mathcal{D}_*)
\]

\[
\leq (2\pi)^d (1 + 2 \text{Lip}(Y^* - \text{id}))^d \text{meas}(B_{\frac{1}{2}}(\mathcal{D}) \setminus \mathcal{D}_*).
\]

Finally, recalling that \( \mathcal{D} = B_{\rho}(\mathcal{D}_0) \) and (3), one sees that

\[
B_{\frac{1}{2}}(\mathcal{D}) \setminus \mathcal{D}_* = B_{\frac{1}{2}}(\mathcal{D}) \setminus \mathcal{D}_* \\
= B_{\rho + \frac{1}{2}}(\mathcal{D}_0) \setminus B_{\rho}(\mathcal{D}_0) \cup \{ y \in B_{\rho}(\mathcal{D}_0) : K_y(y) \notin \text{Dioph}_\alpha \} \\
\subseteq \mathcal{J}_\rho \cup \mathcal{R}_\alpha,
\]

from which, the second inequality in (14) follows at once. \( \blacksquare \)

Theorem 3 reduces the problem of estimating the measure of the complement of the Kolmogorov set \( \mathcal{K} \) to the estimate on the measure of the complement of Diophantine numbers in a given set and to the purely geometrical problem of estimating the measure of the tubular neighbourhood \( \mathcal{J}_\rho \) of the boundary of \( \mathcal{D} = B_{\rho}(D_0) \). Therefore, concrete measure estimates will depend upon the structure of the action domain \( \mathcal{D} \) and of the (unperturbed) frequency map

\[
y \in \mathcal{D} \mapsto \omega_0(y) := K_y(y) \in \mathbb{R}^d.
\]

We shall discuss, in detail, two different cases:

(A) (General case) \( \mathcal{D} \) is an arbitrary bounded set, \( H \) uniformly real–analytic on \( \mathcal{D} \times \mathbb{T}^d \) and \( \omega_0 \) is a local diffeomorphism on \( \mathcal{D} \) (which is always the case, if the unperturbed Hamiltonian is assumed to be Kolmogorov non–degenerate). In this case, as phase space we shall consider a ‘minimal’ (in a suitable sense) open cover of \( \mathcal{D} \) times \( \mathbb{T}^d \). This set–up is analogous to that considered in [BC18].

(B) (Smooth case) \( \mathcal{D} \) is a bounded, connected, open set with \( C^2 \) boundary and \( \omega_0 \) is a global diffeomorphism on \( \mathcal{D} \). In this case the phase space is just \( \mathcal{D} \times \mathbb{T}^d := \mathcal{D} \times \mathbb{T}^d \).
3.1 General case

In order to state the result for case (A), let us give two definitions.

- Given a bounded non-empty set $\mathcal{D} \subseteq \mathbb{R}^d$, and given $r > 0$, an \textit{r–internal covering} of $\mathcal{D}$ is a subset $\mathcal{D}_0$ of $\mathcal{D}$ such that

\[
\mathcal{D} \subseteq B_r(\mathcal{D}_0) = \bigcup_{y \in \mathcal{D}_0} B_r(y);
\]

the \textit{r–internal covering number} of $\mathcal{D}$, is defined as

\[
N_r^{\text{int}}(\mathcal{D}) := \min \{ n \in \mathbb{N} : \{y_1, \ldots, y_n\} \text{ is an } r–\text{internal covering of } \mathcal{D}\};
\]

an $r$–internal cover $\mathcal{D}_0$ of $\mathcal{D}$ with cardinality equal to the $r$–internal covering number will be called \textbf{a minimal} $r$–internal covering of $\mathcal{D}$.

- Given a real–analytic Hamiltonian $H : \mathcal{D} \times \mathbb{T}^d \to \mathbb{R}$, we denote the set of KAM tori for $H$ with frequency vector in Dioph $\tau$ by

\[
\mathcal{K}_\mathcal{D}(\alpha, \tau) := \{ T \subseteq \mathcal{D} \times \mathbb{T}^d : T \text{ is a KAM torus for } H \text{ with frequency } \omega \in \text{Dioph}_\alpha \}. \tag{20}
\]

\textbf{Theorem 4} Let $\mathcal{D}$ be an arbitrary bounded non–empty set in $\mathbb{R}^d$, $\tau > d - 1 \geq 1$, $R, s > 0$. Let $K \in \mathcal{B}_R(\mathcal{D})$ be a uniformly (Kolmogorov) non–degenerate, $P \in \mathcal{B}_{R,s}(\mathcal{D})$ and let $M$, $L$, $\rho$ as in (2). Let $c_0$ and $c_*$ be as in Theorem 1. Fix $0 < s_* < s$, let $\epsilon_*$ be as in (4) and define:

\[
r := \frac{R}{1 + 2d^2 \theta}, \quad \alpha_* := \sqrt{\frac{MP}{\epsilon_*}}, \quad \epsilon_* := \left(\frac{c_0 r}{L \alpha_*}\right)^2, \quad \delta_0 := \inf_{B_r(\mathcal{D})} \left| \det K_{yy} \right|, \quad \theta_0 := \max \left\{ \frac{M}{\delta_0}, \theta \right\}, \quad \rho := \frac{\alpha L}{c_0} \sqrt{\epsilon}.
\]

Let $\mathcal{D}_0 \subseteq \mathcal{D}$ be a minimal $r$–internal covering of $\mathcal{D}$, $\mathcal{D} := B_r(\mathcal{D}_0)$ and let $\mathcal{K}_\mathcal{D}(\alpha_* \sqrt{\epsilon}, \tau)$ be as in (20) with $H = K + \epsilon P$. Then, if $0 < \epsilon < \epsilon_*$, one has

\[
\text{meas} \left( (\mathcal{D} \times \mathbb{T}^d) \setminus \mathcal{K}_\mathcal{D}(\alpha_* \sqrt{\epsilon}, \tau) \right) \leq \bar{c}_* \theta_0 N_r^{\text{int}}(\mathcal{D}) \, M^{-1} \, r^{d-1} \, \alpha_* \sqrt{\epsilon}, \tag{22}
\]

with

\[
\bar{c}_* := \frac{5}{4} \left(2\pi d\right)^d \left(\frac{d^2 2d^3}{c_0} + 2d \frac{d-1}{2} \sum_{k \in \mathbb{Z}^d \setminus \{0\}} \frac{1}{|k|_1 |k|_2}\right). \tag{23}
\]

\[N_r^{\text{int}}(\mathcal{D}) \] is finite if and only if $\mathcal{D}$ is bounded. A simple upper bound on $N_r^{\text{int}}(\mathcal{D})$ for bounded domains $\mathcal{D}$ is: $N_r^{\text{int}}(\mathcal{D}) \leq (\lceil \text{diam}(\mathcal{D})/r \rceil + 1)^d$; compare [BC18] or Appendix B, § B.3.
Proof\ Let $\alpha := \alpha_* \sqrt{\varepsilon}$. Then, $\rho = \mathbf{L} \alpha/c_0$, so that the first inequality in (4) is satisfied (with the equal sign). Furthermore, with the above positions, $\epsilon$ in (2) is given by

$$
\epsilon = \frac{\mathbf{MP}}{\alpha_*^2 \tau},
$$

so that also the second inequality in (4) is satisfied (with the equal sign). Finally, the relation $\rho < r$ is equivalent to $\varepsilon < \varepsilon_*$, which is satisfied by hypothesis. Hence, all the assumptions of Theorem 1 are satisfied and therefore the measure estimate (14) holds with $\mathcal{K}$ as in (5).

We proceed to estimate the two terms in the right hand side of (14) separately. Let us first discuss the measure of $\mathcal{R}_\alpha$. We claim that the map $y \in B_r(y_0) \mapsto \omega_0(y)$ is a diffeomorphism for every $y_0 \in \mathcal{D}$. To see this, we shall apply the quantitative Inverse Function Theorem B.2 to $f(y) = K_y(y)$. In such a case, we can take $T = K_{yy}(y_0)^{-1}$ (using Cauchy estimates, see Lemma B.1),

$$
\|1_d - TK_{yy}(y)\| \leq \|T\|\|K_{yy}(y_0) - K_{yy}(y)\| \leq d^2 \mathbf{L} \|\partial_y K_{yy}\| r \leq d^2 \mathbf{L} \frac{\|K_{yy}\|_\mathcal{K}}{R - r} \leq d^2 \mathbf{L} \mathbf{M} \frac{r}{R - r} = \frac{1}{2},
$$

where $\|\partial_y K_{yy}\|_r := \sup_{B_r(y_0)} \max\{|\partial^3_{y_i y_j y_k} K| : i, j, k = 1, \ldots, d\}$. Hence, by Theorem B.2, $\omega_0$ is invertible on any ball $B_r(y_0)$ with $y_0 \in \mathcal{D}$, as claimed.

Now, let $\mathcal{D}_0 = \{y_1, \ldots, y_{n_0}\}$ with $n_0 := N^\text{int}_r(\mathcal{D})$. Then:

$$
\text{meas}(\mathcal{R}_\alpha) \leq \sum_{j=1}^{n_0} \text{meas}\left(\{y \in B_r(y_j) : |\omega_0(y) \cdot e_k| \leq \frac{\alpha}{|k|_1 |k|_2}\}\right),
$$

where $e_k := \frac{k}{|k|_2}$. Since on $B_r(y_j)$, $y \mapsto \omega_0(y)$ is a diffeomorphism, by the change of variables $y = \omega_0^{-1}(\omega)$, we find

$$
\text{meas}\left(\{y \in B_r(y_j) : |\omega_0(y) \cdot e_k| \leq \frac{\alpha}{|k|_1 |k|_2}\}\right) \leq \delta_0^{-1} \text{meas}\left(\{\omega \in \omega_0(B_r(y_j)) : |\omega \cdot e_k| \leq \frac{\alpha}{|k|_1 |k|_2}\}\right) \leq \delta_0^{-1} \left(\text{diam } \omega_0(B_r(y_j))\right)^{-d-1} \frac{2\alpha}{|k|_1^d |k|_2} \leq \delta_0^{-1} (\mathbf{M}^2 \sqrt{d} \tau)^{-d-1} \frac{2\alpha}{|k|_1^d |k|_2}.
$$
Summing up over \(j\) and \(k\), one gets

\[
\text{meas}(\mathcal{R}_\alpha) \leq \left(2^d \frac{d^{d-1}}{\pi} \sum_{\kappa \in \mathbb{Z}^d \setminus \{0\}} \frac{1}{|k|_1^2 |k|_2^2} \right) n_0 \delta_0^{-1} M^{d-1} r^{d-1} \alpha. \tag{24}
\]

Let us turn to the estimate of \(\text{meas}(\mathcal{T}_\rho)\). Observing that

\[
\mathcal{T}_\rho := B_{r+\rho}(D_0) \setminus B_{r-\rho}(D_0) \subseteq \bigcup_{j=1}^{n_0} B_{r+\rho}(y_j) \setminus B_{r-\rho}(y_j),
\]

one finds

\[
\text{meas}(\mathcal{T}_\rho) \leq \sum_{j=1}^{n_0} \text{meas} \left( B_{r+\rho}(y_j) \setminus B_{r-\rho}(y_j) \right) = n_0 2^d ((r+\rho)^d - (r-\rho)^d) \leq d 2^d n_0 \rho r^{d-1} = \frac{d 2^d n_0}{c_0} \frac{\theta}{M} r^{d-1} \alpha. \tag{25}
\]

Observing that \(\mathcal{K} \subseteq \mathcal{K}_\rho(\alpha_{\sqrt{\varepsilon}} \tau)\) and that, by (9), \(\beta\) in (13) satisfies \(\beta < \frac{\bar{c}}{4}(2\pi)^d\), one sees that (24) and (25) imply (22) with \(\bar{c}\) as in (23).

### 3.2 Smooth case

In order to state the result for case (B), we need the following definitions.

- Let \(S\) be a compact and connected \(C^2\)–hypersurface of \(\mathbb{R}^d\). The **minimal focal distance of** \(S\) is defined as:

  \[
  \text{minfoc}(S) := \min \left\{ \inf \{e_c(u, \nu^+(u)) : u \in S\} , \inf \{e_c(u, \nu^-(u)) : u \in S\} \right\},
  \]
  
  where \(\nu^\pm(u)\) denotes the outwards/inwards normal to \(S\) at \(u\) and

  \[
  e_c(u,v) := \sup \{t > 0 : \text{dist}_2(u + tv, S) = t\},
  \]
  
  \(\text{dist}_2\) being euclidean distance.

- Given any bounded set \(D\) in \(\mathbb{R}^d\), we define the (measure of the) **maximal** \((d-1)\)–dimensional section of \(D\) as

  \[
  \text{sec}_{d-1}(D) := \sup_{\lambda \in \Lambda^{d-1}} \mathcal{H}^{d-1}(\lambda \cap D)
  \]
  
  where \(\Lambda^{d-1}\) denotes the set of all hyperplanes in \(\mathbb{R}^d\) and \(\mathcal{H}^{d-1}\) the \((d-1)\)–dimensional Hausdorff measure.
Given a set $D \subseteq \mathbb{R}^d$ and $\rho > 0$, we define $\rho$–inner domains of $D$ (which depend upon the choice of the metric) as:

$$D'_\rho := \{ y \in D : B_\rho(y) \subseteq D \},$$  
$$D''_\rho := \{ y \in D : B_\rho(y) \subseteq D \},$$

(26)

**Theorem 5** Let $\mathcal{D} \subseteq \mathbb{R}^d$ be a open and bounded set with $C^2$, compact and connected boundary. Let $\tau > d - 1 \geq 1$, $s > 0$. Let $K \in \mathcal{B}_s(\mathcal{D})$ be uniformly (Kolmogorov) non-degenerate and so that the unperturbed frequency map $y \in \mathcal{D} \mapsto \omega_0(y) := K_0(y) \in \mathbb{C}^d$ is a global diffeomorphism. Let $P \in \mathcal{B}_s(\mathcal{D})$ and let $M, L, P, \theta$ as in (2) and define

$$r := \min \{ \rho, \ \min \{ \tau \mathcal{D} \}, \ \min \{ \frac{1}{\kappa} \} / \sqrt{d} \},$$

(27)

where $\kappa := \sup_{\partial \mathcal{D}} \max_{1 \leq j \leq d-1} |\kappa_j|$, $\kappa_j$’s being the principal curvatures of $\partial \mathcal{D}$. Let $c_0$ and $c_\star$ be as in Theorem 1; fix $0 < s_\star < s$, let $\epsilon_\star$ be as in (4); let $\alpha_\star$, $\epsilon_\star$, $\delta_0$, $\theta_0$ and $\rho$ be as in (21). Let $\mathcal{K}_\mathcal{D}(\alpha_\star \sqrt{\epsilon}, \tau)$ be as in (20) with $H = K + \epsilon P$. Then, if $0 < \epsilon < \epsilon_\star$, one has

$$\text{meas} \left( (\mathcal{D} \times \mathbb{T}^d) \backslash \mathcal{K}_\mathcal{D}(\alpha_\star \sqrt{\epsilon}, \tau) \right) \leq \hat{c}_\star \theta_0 \ M^{-1} \ \max \{ \text{sec}_{d-1}(\mathcal{D}) , \mathcal{H}^{d-1}(\partial \mathcal{D}) \} \ \alpha_\star \sqrt{\epsilon},$$

(28)

with

$$\hat{c}_\star := \frac{5}{2} \ (2\pi)^d \ \left( \frac{2d}{\sqrt{d} \ c_0} + \sum_{k \in \mathbb{Z}^d \setminus \{0\}} \frac{1}{|k|^2} \right).$$

(29)

**Proof** The idea is again to apply Theorem 1 and Theorem 3. Let $\mathcal{D}_0 := \mathcal{D}''_{\sqrt{dr}}$. Since $\sqrt{dr} \leq \min \{ \tau \mathcal{D} \},$ by Lemma B.10,

$$B_r(\mathcal{D}_0) \subseteq B_{\sqrt{dr}}(\mathcal{D}''_{\sqrt{dr}}) = \mathcal{D}, \ \ \text{and} \ \ \hat{\mathcal{D}} = B_{r-\rho}(\mathcal{D}_0) \supseteq B_{\sqrt{dr}}(\mathcal{D}_0) = \mathcal{D}''_{\sqrt{r-1}r+\rho}.$$

(30)

As in the proof of Theorem 4, we let $\alpha := \alpha_\star \sqrt{\epsilon}$, so that $\rho = \frac{L \alpha}{\alpha_0}$ and $\epsilon = \frac{MP}{\alpha^2_\star}$ (cfr. (2)). Then, the inequalities in (4) hold with the equal sign. The relation $\rho < r$ is equivalent to $\epsilon < \epsilon_\star$, which is satisfied by hypothesis. Hence, all the assumptions of Theorem 1 are satisfied and the measure estimate (14) holds with $\mathcal{K}$ as in (5).

By hypothesis the frequency map $y \mapsto \omega_0(y)$ is a diffeomorphism on $\mathcal{D}$, so we can repeat the estimate on the measure of $\mathcal{R}_\alpha$ done in the proof of Theorem 4 without the need of

---

Recall that $B_\rho$ denotes a ball with respect to the sup norm $|\cdot| = |\cdot|_\infty$, while $B_\rho$ denotes a ball with respect to the Euclidean norm $|\cdot|_2$. 

---
localizing the actions. Letting, as above, \( e_k := \frac{k}{|k|_2} \), we find
\[
\text{meas}(\mathcal{R}_\alpha) = \text{meas}\left( \{ y \in \mathcal{D} : \omega_0(y) \notin \text{Dioph}_\alpha^+ \} \right)
\leq \sum_{k \in \mathbb{Z}^d \setminus \{0\}} \text{meas}\left( \{ y \in \mathcal{D} : |\omega_0(y) \cdot e_k| \leq \frac{\alpha}{|k|_1 |k|_2} \} \right).
\]
\[
\leq \sum_{k \in \mathbb{Z}^d \setminus \{0\}} \delta_0^{-1} \text{meas}\left( \{ \omega \in \omega_0(\mathcal{D}) : |\omega \cdot e_k| \leq \frac{\alpha}{|k|_1 |k|_2} \} \right).
\]
\[
\leq \delta_0^{-1} M^{d-1} \sec_{d-1}(\mathcal{D}) \sum_{k \in \mathbb{Z}^d \setminus \{0\}} \frac{2\alpha}{|k|_1 |k|_2}.
\]
\[
\leq \theta_0 M^{-1} \sec_{d-1}(\mathcal{D}) \sum_{k \in \mathbb{Z}^d \setminus \{0\}} \frac{2\alpha}{|k|_1 |k|_2}.
\]

The estimate on the measure of \( \mathcal{T}_\rho \) follows from Lemma B.11. Indeed, if we denote \( \mathfrak{X}_\rho(S) := \{ u \in \mathbb{R}^d : \text{dist}_2(u, S) < \rho \} \), we have (compare (B.23))
\[
\mathcal{T}_\rho = B_{r+\rho}(D_0) \setminus B_{r-\rho}(D_0) \subseteq B_{\sqrt{\alpha} \rho}(\mathcal{D}) \setminus \mathcal{D}_{(\sqrt{d} \alpha)^{1/2} r + \rho} \subseteq \mathfrak{X}_{\sqrt{\alpha} \rho}(\partial \mathcal{D}),
\]
Since \( r \leq \min\{ \min\text{foc}(\partial \mathcal{D})/\sqrt{d}, 1/(\sqrt{d} \kappa) \} \), by (B.24), we get
\[
\text{meas}(\mathcal{T}_\rho) \leq \text{meas}(\mathfrak{X}_{\sqrt{\alpha} \rho}(\partial \mathcal{D}))
\leq \frac{2}{d} \frac{(1 + \sqrt{d} \kappa)^d - 1}{\kappa} \mathcal{H}^{d-1}(\partial \mathcal{D})
\leq \frac{2^{d+1} \rho}{\sqrt{d}} \mathcal{H}^{d-1}(\partial \mathcal{D})
= \frac{2^{d+1}}{\sqrt{d}} \theta M^{-1} \mathcal{H}^{d-1}(\partial \mathcal{D}) \alpha.
\]
Since \( \alpha = \alpha_0 \sqrt{\varepsilon} \), (28) follows, with \( \hat{c}_* \) as in (29).

Appendix

A Proof of Theorem 1

In this appendix we provide the details needed to prove Arnold’s Global KAM Theorem (Theorem 1). The main point is the choice of the various parameters and sequences
involved in the Newton–like procedure based on the iteration of a ‘KAM step’ (in turn, based upon the original scheme by Arnold; compare [Arn63] and its revisions in [Kou19] and [CK19]). Although the main ideas are well known, some details are needed, especially in order to compute explicitly constants and to keep the optimal relation between $\varepsilon$ and $\alpha$. Furthermore, the construction of the ‘integrating map’ also require a discussion. All this is done in the present appendix.

By following [Kou19, Chap. 6], one gets the following:

**General step of the KAM scheme**

**Lemma A.1 (KAM step)** Let $r > 0$, $0 < 2\sigma < s \leq 1$, $\mathcal{D}_r \subseteq \mathbb{R}^d$ be a non-empty, bounded domain. Consider the Hamiltonian parametrized by $\varepsilon \in \mathbb{R}$

$$H(y, x; \varepsilon) := K(y) + \varepsilon P(y, x),$$

where $K, P \in B_{r,s}(\mathcal{D}_r)$. Assume that\(^{10}\)

$$\begin{align*}
\det K_{yy}(y) &\neq 0, \\
\|K_{yy}\|_{\mathcal{D}_r} &\leq M, \\
\|P\|_{r,s,\mathcal{D}_r} &\leq P, \\
T(y) &:= K_{yy}(y)^{-1}, \quad \forall \ y \in \mathcal{D}_r, \\
\|T\|_{\mathcal{D}_r} &\leq L, \\
K_y(\mathcal{D}_r) &\subseteq \Delta_{\alpha}^\mathcal{D}.
\end{align*}$$

(A.1)

Fix $\varepsilon \neq 0$ and assume that

$$\lambda \geq \log \left( \frac{\sigma^{2\nu + d} \alpha^2}{\varepsilon PM} \right) \geq 1.$$  

(A.2)

Let

$$\ell := 4\sigma^{-1} \lambda, \quad \tilde{r} \leq \frac{r}{32dLM}, \quad \tilde{r} \leq \min \left\{ \frac{\alpha}{2dM\ell}, \tilde{r} \right\},$$

$$\tilde{r} := \frac{\tilde{r}\sigma}{16dLM}, \quad \bar{s} := s - \frac{2}{3}\sigma, \quad s' := s - \sigma,$$

and\(^{11}\)

$$\mathbf{p} := P \max \left\{ \frac{16L}{r\tilde{r}} \sigma^{-(\nu + d)}, \frac{C_4}{\alpha\tilde{r}} \sigma^{-2(\nu + d)} \right\}. $$

Assume:

$$\varepsilon \mathbf{p} \leq \frac{\sigma}{3}. $$

(A.4)

\(^{10}\)In the sequel, $K$ and $P$ stand for generic real analytic Hamiltonians which, later on, will respectively play the roles of $K_j$ and $P_j$, and $y_0, r$, the roles of $y_j, r_j$ in the iterative step.

\(^{11}\)Notice that $\mathbf{p} \geq \sigma^{-d} \mathbf{p} \geq \mathbf{p}$ since $\sigma \leq 1$. Notice also that $LM \geq 1$, so that $\frac{16L}{r\tilde{r}} \sigma^{-(\nu + d)} > \frac{16L}{r\tilde{r}} \geq \frac{4}{LM\ell}$. 

17
Then, there exists a diffeomorphism $G : \mathbb{B}_r(\mathcal{D}_t) \to G(\mathbb{B}_r(\mathcal{D}_t))$, a symplectic change of coordinates
\[ \phi' = \text{id} + \varepsilon \tilde{\phi} : \mathbb{B}_{r/2,s'}(\mathcal{D}_t') \to \mathbb{B}_{2r/3,s}(\mathcal{D}_t), \]
(A.5)
such that
\[ \begin{cases} H \circ \phi' =: H' =: K' + \varepsilon^2 P' , \\ \partial_y K' \circ G = \partial_y K , \quad \det \partial_y K' \circ G \neq 0 \quad \text{on } \mathcal{D}_t, \end{cases} \]
(A.6)
with $K'(y') := K(y') + \varepsilon \tilde{K}(y') := K(y') + \varepsilon \langle P(y', \cdot) \rangle$. Moreover, letting $(\partial_y^2 K'(y'))^{-1} =: T(y') + \varepsilon \tilde{T}(y')$, $y' \in G(\mathcal{D}_t)$, the following estimates hold.

\[ \begin{cases} \| \partial_y^2 \tilde{K} \|_{r/2, \mathcal{D}_t} \leq M_p , \\ \| G - \text{id} \|_{r, \mathcal{D}_t} \leq \sigma^{\nu+d} r \varepsilon p , \\ \| \tilde{T} \|_{\mathcal{D}_t} \leq L_p , \\ \max \left\{ \frac{C_2}{C_4} \| \nabla \tilde{\phi} \nabla^{-1} \|_{r/2,s', \mathcal{D}_t'}, \| \nabla \tilde{\phi} \|_{r/2,s', \mathcal{D}_t'} \right\} \leq \sigma^{d} p , \\ \| P' \|_{r/2,s', \mathcal{D}_t'} \leq p P , \end{cases} \]
(A.7)
where
\[ \mathcal{D}_t' := G(\mathcal{D}_t), \quad (\partial_y^2 K'(y'))^{-1} =: T \circ G^{-1}(y') + \varepsilon \tilde{T}(y'), \quad \forall y' \in \mathcal{D}_t', \]
\[ W := \text{diag} (\tilde{r}^{-1} 1_d, 1_d) , \quad \overline{W} := \text{diag} (\sigma^{-\tau} \tilde{r}^{-1} 1_d, 1_d) . \]

**Implementation**

As in [CK19], we shall separate the first step from the others. Let $H$, $K$, $P$, $\rho$, $s$, $s_*$, $W$, $P$, $M$, $L$, $\theta$, $\epsilon$ be as in §2. Set
\[ \sigma_0 := (s - s_*)/2 , \quad \epsilon_0 := \epsilon , \quad \theta_0 := \theta , \quad r_0 := \rho , \quad L_0 := L , \quad M_0 := M , \quad P_0 := P , \quad W_1 := W , \]
\[ \lambda_0 := \log \epsilon_0^{-1} , \quad \lambda_* := C_\gamma \sigma_0^{-(4\nu+2d+1)} \theta_0^2 \lambda_0^{2\nu} , \quad \theta_* := 2^{2\nu+2d+1} C_5 \theta_0^2 , \quad \ell_0 := 4 \sigma_0^{-1} \lambda_0 , \]
\[ K_0 := K , \quad P_0 := P , \quad H_0 := H , \quad \mathcal{D}_0 := \mathcal{D}_* . \]

**First Step**

Let
\[ \begin{align*}
 s_1 & := s_0 - \sigma_0 , \quad \tilde{r}_1 := \frac{r_0}{64d \theta_0} , \quad \tilde{r}_1 := \frac{r_1 \sigma_0}{32d \theta_0} , \quad r_1 := \frac{1}{2} \min \left\{ \frac{\alpha}{2d \sqrt{2M_0 \ell_0^2} r_1} , \tilde{r}_1 \right\} , \\
 M_1 & := \left( 1 + \frac{\sigma_0}{3} \right) M_0 , \quad L_1 := \left( 1 + \frac{\sigma_0}{3} \right) L_0 , \quad \tilde{\epsilon}_0 := C_8 \sigma_0^{-(3\nu+2d+1)} \epsilon_0^{1/2} , \quad P_1 := \frac{\tilde{\epsilon}_0 P_0}{\epsilon} , \\
p_0 & := p_0 \max \left\{ \frac{8L_0}{r_0^2 \sigma_0^{-(\nu+d)}}, \frac{C_4}{2 \alpha r_1 \sigma_0^{-2(\nu+d)}} \right\} .
\end{align*} \]

18
Lemma A.2 Under the above assumptions and notations, if
\[
\alpha \leq \frac{C_4 r_0}{16 L_0} \quad \text{and} \quad \max \{ e, \epsilon_0 \} \leq 1 ,
\]
then, there exist \( \mathcal{D}_1 \subseteq \mathcal{D} \), a real-analytic diffeomorphism
\[
G_1 : \mathbb{B}_{\tilde{r}_1} (\mathcal{D}^*) \to G_1 (\mathbb{B}_{\tilde{r}_1} (\mathcal{D}^*))
\]
and a real-analytic symplectomorphism
\[
\phi_1 : \mathbb{B}_{r_1,s_1} (\mathcal{D}_1) \to \mathbb{B}_{r_0,s_0} (\mathcal{D}_0)
\]
such that
\[
G_1 (\mathcal{D}^*) = \mathcal{D}_1 \quad \text{(A.10)}
\]
\[
\partial_{y_1} K_1 \circ G_1 = \partial_y K_0 \quad \text{(A.11)}
\]
\[
H_1 := H_0 \circ \phi_1 =: K_1 + \varepsilon^2 P_1 \quad \text{on } \mathbb{B}_{r_1,s_1} (\mathcal{D}_1)
\]
(A.12)
and\(^{12}\)
\[
\mathcal{D}_1 \subseteq \mathcal{D}_{r_1} \quad \text{(A.13)}
\]
\[
\| \partial_{y_1}^2 K_1 \|_{r_0/4, \mathcal{D}_1} \leq M_0 \quad \| T_{1} \|_{\mathcal{D}_1} \leq L_1 \quad \text{on } \mathbb{B}_{\tilde{r}_1} (\mathcal{D}^*) \quad \text{(A.14)}
\]
\[
\| P_t \|_{r_1,s_1, \mathcal{D}_1} \leq P_1 \quad \text{(A.15)}
\]
\[
\| G_1 - \text{id} \|_{r_1, \mathcal{D}_1} \leq 2 \sigma_0 \quad \text{(A.16)}
\]
\[
\| \partial_G G_1 - 1_d \|_{r_1/2, \mathcal{D}_1} \leq 2 \delta^4 \tilde{\theta} \sigma_0 \quad \text{(A.17)}
\]
\[
\max \{ C_{12} C_4^{-1} \| W_1 \|_{r_1,s_1, \mathcal{D}_1} \| W_1^{-1} \|_{r_1,s_1, \mathcal{D}_1} \} \leq \sigma_0 \quad \text{(A.18)}
\]
\(^{12}(A.17)\) follows trivially \((A.16)\) using Cauchy’s estimate.
Second step, iteration and convergence

For a given \( j \geq 1 \), define\(^{13}\)
\[
\sigma_j := \frac{\sigma_0}{2^j}, \quad s_{j+1} := s_j - \sigma_j = s_* + \frac{\sigma_0}{2^j}, \quad \bar{s}_j := s_j - \frac{2\sigma_j}{3}, \quad \ell_j := 4^j \ell_0,
\]
\[
M_{j+1} := M_0 \prod_{k=0}^{j} \left( 1 + \frac{\sigma_k}{3} \right) < M_0 \sqrt{2}, \quad L_{j+1} := L_0 \prod_{k=0}^{j} \left( 1 + \frac{\sigma_k}{3} \right) < L_0 \sqrt{2},
\]
\[
\epsilon_j := \frac{M_0 \varepsilon^2 j^2 P_j}{\alpha^2}, \quad \tilde{r}_{j+1} := \frac{r_j}{64d\theta_0}, \quad \tilde{r}_{j+1} := \tilde{r}_{j+1} \sigma_j, \quad r_{j+1} := \frac{1}{2} \min \left\{ \frac{\alpha}{2d \sqrt{2M_0 \ell_n}}, \frac{r_j}{64d\theta_0} \right\},
\]
\[
P_{j+1} := \lambda_* \theta_*^{-1} \frac{M_0 P_j}{\alpha^2}, \quad \hat{\epsilon}_j := \lambda_* \theta_*^j \epsilon_j, \quad \mathcal{W}_{j+1} := \text{diag} \left( \left( 2r_{j+1} \right)^{-1} 1_d, 1_d \right),
\]
\[
\mathcal{W}_{j+1} := \text{diag} \left( \sigma_j^{3/2} \left( 2r_{j+1} \right)^{-1} 1_d, 1_d \right), \quad p_j := P_j \max \left\{ \frac{8L_0 \sqrt{2} \sigma_j^{-(\nu+d)}}{r_j r_{j+1}}, \frac{C_4}{2\alpha r_{j+1}} \sigma_j^{-2(\nu+d)} \right\}.
\]

Observe that, for any \( j \geq 1 \),
\[
\hat{\epsilon}_{j+1} = \lambda_* \theta_*^{j+1} \hat{\epsilon}_{j+1} = \lambda_* \theta_*^{j+1} \frac{M_0 \varepsilon^{2j+1} P_{j+1}}{\alpha^2} = \lambda_* \theta_*^{j+1} \frac{M_0 \varepsilon^{2j+1} P_{j+1}}{\alpha^2} \lambda_* \theta_*^{-1} \frac{M_0 P_j}{\alpha^2} \left( \lambda_* \theta_*^j \epsilon_j \right)^2 = \hat{\epsilon}_{j+1}^2,
\]
\[\text{i.e.} \quad \hat{\epsilon}_j = \hat{\epsilon}_{j-1}^2.\]

Lemma A.3 Assume (A.12) \((A.15)\) with some \( \varepsilon \neq 0 \) and
\[
\max \left\{ \epsilon, \epsilon_0, 2^{11} d^2 \theta_0 \sigma_0^{\nu+d} \ell/3, 2C_0 \theta_0 \hat{\epsilon}_1 \right\} \leq 1.
\]
Then, one can construct a sequence of real-analytic diffeomorphisms
\[
G_j : \mathcal{B}_{r_j}(\mathcal{D}_{j-1}) \rightarrow G_j(\mathcal{B}_{r_j}(\mathcal{D}_{j-1})) , \quad j \geq 2
\]
and of real-analytic symplectic transformations
\[
\phi_j : \mathcal{B}_{r_j, s_j}(\mathcal{D}_j) \rightarrow \mathcal{B}_{r_{j-1}, s_{j-1}}(\mathcal{D}_{j-1}) , \quad (A.20)
\]
such that
\[
G_j(\mathcal{D}_{j-1}) = \mathcal{D}_j \subseteq \mathcal{D}_{r_j},
\]
\[
\partial_y K_{j+1} \circ G_{j+1} = \partial_y K_j ,
\]
\[
H_j := H_{j-1} \circ \phi_j =: K_j + \varepsilon^{2j} P_j \quad \text{on} \quad \mathcal{B}_{r_j, s_j}(\mathcal{D}_j),
\]
converge uniformly. More precisely, we have the following:

\(^{13}\)Notice that \( s_j \downarrow s_* \) and \( r_j \downarrow 0 \).
We can now complete the proof of Theorem 1. First of all, observe that
\[(\log t)^a \leq \left(\frac{2a}{e}\right)^a \sqrt{t}, \quad \forall t \geq e, \quad \forall a > \frac{1}{2}, \tag{A.26}\]
and from the proof, we have
\[
\epsilon p_0 (3\sigma_{i-1}^{-1}) \stackrel{(A.8)}{=} 6 d C_4 \sqrt{2\sigma_0^{-2(\nu + d) - 1}} \frac{K_0 p_0 \ell_0}{\alpha^2} \tag{A.27}
\]
\[
\leq \epsilon_0 \leq 1, \tag{A.28}
\]

\[\text{Observe that (A.22) follows (A.21) using Cauchy's estimate.}\]
and, for $j \geq 1$, 
\[ \varepsilon^{2^j} p_j(3\sigma_j^{-1}) \leq \varepsilon_1^{2^j-1}/\theta_* . \]  
(A.29)

Let $\phi_* := \phi_1 \circ \phi^*$. Thus, uniformly on $\mathbb{D}_* \times \mathbb{T}_{\mathcal{T}_{s^*}}$, \(^{15}\)

\[ |W_1(\phi_* - \text{id})| \leq |W_1(\phi_1 \circ \phi^* - \phi^*)| + |W_1(\phi^* - \text{id})| \]
\[ \leq \|W_1(\phi_1 - \text{id})\|_{\mathcal{T}_{s^*}, \mathbb{D}_*} + \|W_1W_2^{-1}\| |W_2(\phi^* - \text{id})| \]
\[ \leq \sigma_0 \varepsilon p_0 + \frac{2\sigma_0^{d+1} \varepsilon_1}{3 \theta_*} \]
\[ (A.26)+(A.27) \]
\[ \leq 6dC_4 \sqrt{2} \sigma_0^{-(2\nu+2d+1)} \epsilon_0 \ell_0^\nu + \left( \frac{\nu}{2e} \right) \nu \sigma_0^{-(\nu+4d+2)} \theta_0^2 \ell_0^\nu \epsilon_0 \]
\[ \leq C_9 \theta_0^2 \ell_0 \epsilon_0 , \]

\textit{i.e.} (10). Moreover, setting $G_0 := \text{id}$, we have for any $i \geq 3$, 
\[ \|G^i - \text{id}\|_{\mathcal{D}_*} \leq \sum_{j=0}^{i-1} \|G^{j+1} - G^j\|_{\mathcal{D}_*} = \sum_{j=0}^{i-1} \|G_j - \text{id}\|_{\mathcal{D}_*, \mathcal{D}_*} \leq 2 \sum_{j=0}^{i-1} r_j \sigma_j \varepsilon^{2^j} p_j \]
\[ (A.28)+(A.29) \]
\[ \leq 2 r_1 \sigma_0^\nu \sum_{j=0}^{\infty} \sigma_j \varepsilon^{2^j} p_j \]
\[ (A.28)+(A.29) \]
\[ \leq 2 r_1 \sigma_0^\nu \cdot C_9 \theta_0^2 \ell_0 \epsilon_0 , \]

and then passing to the limit, we get 
\[ \|Y^* - \text{id}\|_{\mathcal{D}_*} \leq 2^{2r+1/2} d^{-1} C_9 \sigma_0^\nu \theta_0^2 \varepsilon \frac{P_0}{\alpha} , \]

\textit{i.e.} (8). Now, observing that, for any $j \geq 1$, $\nabla \phi^{j+1} = \nabla \phi^j \nabla \phi_{j+1}$, $\|W_j W_{j+1}^{-1}\| = 1$ and $\|W_{j+1} W_j^{-1}\| \leq C_5 \theta_0$, we obtain 
\[ \|W_1(\nabla \phi^{j+1} - \mathbb{1}_{2d}) W_{j+1}^{-1}\| \leq \left( \|W_1(\nabla \phi^j - \mathbb{1}_{2d}) W_j^{-1}\| + 1 \right) \left( \|W_{j+1}(\nabla \phi_{j+1} - \mathbb{1}_{2d}) W_{j+1}^{-1}\| + 1 \right) \]
\[ - 1 \]
\[ \leq \left( \|W_1(\nabla \phi^j - \mathbb{1}_{2d}) W_j^{-1}\| + 1 \right) \left( \frac{C_4}{C_{12}} \sigma_j \varepsilon^{2^j} p_j + 1 \right) - 1 , \]

\(^{15}\)Observe that $\lambda
^{\nu \ell_0} \leq (4\nu)^{2\nu} \sqrt{\ell_0} \leq (4\nu)^{2\nu} (2^{11} d^2 C_5)^{-1/2} \theta_0^{-1} \sigma_0^{(\nu+4d+2)/2}$. 

22
which iterated yields\textsuperscript{16}

\[
\|\mathcal{W}_1(\nabla \phi^{j+1} - 1)\mathcal{W}^{-1}_{j+1}\|_* \leq \prod_{j=1}^{\infty} \left( \frac{C_4}{C_{12}} \sigma_{j-1}^{d_j} e^{2^{j-1}} p_{j-1} + 1 \right) - 1 \\
\leq \exp \left( \sum_{j=1}^{\infty} \frac{C_4}{C_{12}} \sigma_{j-1}^{d_j} e^{2^{j-1}} p_{j-1} \right) - 1 \\
\overset{\text{(A.28)+(A.29)}}{\leq} \exp(C_4 C_9 C_{12}^{-1} \theta_0^2 \epsilon_0^\mu \epsilon_0) - 1 \\
\leq \exp((4d)^{-1}) C_4 C_9 C_{12}^{-1} \theta_0^2 \epsilon_0^\mu \epsilon_0 \\ \\
\overset{\text{(4)+(A.26)}}{\leq} \frac{1}{4(18d^3 + 70)\theta},
\]

and letting \( j \to \infty \), we obtain

\[
\|\partial_x u_*\|_* \leq \exp((4d)^{-1}) C_4 C_9 C_{12}^{-1} \theta_0^2 \epsilon_0^\mu \epsilon_0 \leq \frac{1}{4(18d^3 + 70)\theta}.
\]

\textit{i.e.} (11).

Next, we show that \( \operatorname{Lip}(\mathcal{D}^*) \) \( Y^* - \text{id} \) \( < 1 \), which will imply that\textsuperscript{17} \( Y^* : \mathcal{D}^* \stackrel{\text{onto}}{\longrightarrow} \mathcal{D}_* \) is a lipeomorphism. For, observe first that, for any \( j \geq 1, 0 < r < \tilde{r}_j/2, y_{j-1} \in \mathcal{D}_{j-1} \) and any \( y \in \mathbb{B}_r(y_{j-1}) \), we have

\[
|G_j(y) - G_j(y_{j-1})| \leq |(G_j(y) - y) - (G_j(y_{j-1}) - y_{j-1})| + |y - y_{j-1}| \overset{\text{(A.22)+(A.19)}}{\leq} \frac{1}{2} |y - y_{j-1}| + r < 2r,
\]

so that

\[
G_j(\mathbb{B}_r(\mathcal{D}_{j-1})) \subseteq \mathbb{B}_{2r}(G_j(\mathcal{D}_{j-1})) = \mathbb{B}_{2r}(\mathcal{D}_j).
\]

(A.31)

Thus, as the sequence \( \tilde{r}_j \) is strictly decreasing, for any \( j \geq k \geq 1, G^k \) is well-defined on \( \mathbb{B}_{2^{-j-1} \tilde{r}_{j+1}}(\mathcal{D}_0) \) and we have

\[
G^k(\mathbb{B}_{2^{-j-1} \tilde{r}_{j+1}}(\mathcal{D}_0)) \subseteq G_j \circ \cdots \circ G_2(\mathbb{B}_{2^{-j-1} \tilde{r}_{j+1}}(\mathcal{D}_1)) \subseteq \cdots \subseteq \mathbb{B}_{2^{-j-1} \tilde{r}_{k+1}}(\mathcal{D}_k) \subseteq \mathbb{B}_{2^{-k-1} \tilde{r}_{k+1}}(\mathcal{D}_k).
\]

(A.32)

\textsuperscript{16}Use: \( e^t - 1 \leq t e^t \), for any \( t \geq 0 \).

\textsuperscript{17}See Proposition II.2 in [Zeh10].
Therefore, for any $j \geq 2$,
\[
\|G^j - \text{id}\|_{L, \mathbb{B}_{2^{-j-1}2^{j+1}}(\varphi^*)} + 1 \leq \|(G_j - \text{id}) \circ G^{j-1} + (G^{j-1} - \text{id})\|_{L, \mathbb{B}_{2^{-j-1}2^{j+1}}(\varphi)} + 1
\leq \|(G_j - \text{id})\|_{L, \mathbb{B}_{2^{-j-1}2^{j+1}}(\varphi)}(\|G^{j-1} - \text{id}\|_{L, \mathbb{B}_{2^{-j-1}2^{j+1}}(\varphi)}) + 1
\leq \|(G_j - \text{id})\|_{L, \mathbb{B}_{2^{-j}2^{j}}(\varphi)}(\|G^{j-1} - \text{id}\|_{L, \mathbb{B}_{2^{-j}2^{j}}(\varphi)}) + 1
\leq (\|G_j - \text{id}\|_{L, \mathbb{B}_{2^{-j}}(\varphi)} + 1)(\|G^{j-1} - \text{id}\|_{L, \mathbb{B}_{2^{-j}}(\varphi)} + 1)
\]
which iterated leads to\(^{18}\)
\[
\|G^j - \text{id}\|_{L, \varphi^*} \leq -1 + (1 + (32d)^{-1}) \prod_{i=2}^{\infty} (2^5 d \theta_0 \sigma_{\theta}^{d+1} e^{2^i-1} L_{i-1} + 1)
\leq -1 + \exp((32d)^{-1} + 2^5 d \theta_0 \sum_{i=1}^{\infty} \sigma_{\theta_0}^{d+1} e^{2^i} L_i)
\leq -1 + \exp((32d)^{-1} + 2^5 d \theta_0 \sum_{i=1}^{\infty} \sigma_{\theta_0}^{d} e^{2^i} L_i)
\leq -1 + \exp\left((32d)^{-1} + 2^5 d \theta_0 \frac{2^{\sigma_{\theta_0}^{d+1} e^{2^i}}}{3 \theta_*^i}\right)
\leq -1 + \exp\left((32d)^{-1} + (32d)^{-1}\right) \leq e^{1/(16d)/(16d)} < \frac{1}{4d}. \quad (A.33)
\]
Hence, letting $j \to \infty$, we get that $Y^*$ is Lipschitz continuous, with Lip_{\varphi^*}(Y^* - \text{id}) satisfying (9) as
\[
2^5 d C_1 \sqrt{2} \theta_0 \sigma_0^{d+1} e^{-\nu} e L_0 + \sum_{j \geq 2} 2^5 d \theta_0 \sigma_{\theta_0}^{d+1} e^{2^j-1} L_{j-1} \leq c_2 \theta^3 (s-s_*)^{-1} \frac{M_{\varphi}}{\alpha^2} \left(\log \frac{\alpha^2}{M_{\varphi}}\right)^{\nu}. \quad (A.19)
\]
\(^{18}\)Use, again, $e^t - 1 \leq t e^t, \forall t \geq 0$, and $2^5 d C_2 \sqrt{2} \theta_0 \sigma_0^{d+1} e^{-\nu} e L_0 \leq \sqrt{2} \theta d^2 C_2 \theta \sigma_0^{-\nu} e L_0 < (A.27)$. 
\[2^7 d^2 C_2 \theta \sigma_0^{-\nu} e L_0 < (32d)^{-1}. \quad (A.19)
\]
Next, we show that $\phi_* \in C_8^\infty(\mathcal{D}_* \times \mathbb{T}^d)$. For any $n, j \geq 1$, we have

$$\|G^{n+j} - G^j\|_{\phi_*} \leq \sum_{k=j}^{n+j-1} \|G^{k+1} - G^k\|_{\phi_*} \tag{A.21}$$

$$\leq 2r_{j+1} \sigma_j \sum_{k \geq 1} \sigma_k \varepsilon^{2k} L_k$$

$$\leq 2r_{j+1} \sigma_j \frac{2 \sigma_0 \varepsilon^{d+1}}{3 \theta_*} \tag{A.19}$$

$$< \sigma_j \tilde{r}_{j+1}.$$

Now, letting $n \to \infty$, we get

$$\|Y^* - G^j\|_{\phi_*} < \sigma_j \tilde{r}_{j+1} < \frac{\tilde{r}_{j+1}}{4}. \tag{A.34}$$

Hence\(^\text{19}\), for any $j \geq 1$,

$$\mathbb{B}_{\tilde{r}_{j+1}}(G^j(\mathcal{D}^*)) \subseteq \mathbb{B}_{\tilde{r}_{j+1}}(\mathcal{D}_*) \subseteq \mathbb{B}_{r_{j+1}}(\mathcal{D}_j) \subseteq \mathbb{B}_{r_j}(\mathcal{D}_j). \tag{A.35}$$

Therefore, for any $n \geq 1$, we have

$$\sum_{j \geq 3} \|W_2(\phi^j - \phi^{j-1})\|_{\tilde{r}_{j+1/2, s_j, \mathcal{D}_*}} \left(\frac{\tilde{r}_{j+1}}{2}\right)^{-n} \tag{A.35} \leq (2^{12} d^2 \theta^2_0)^n \sum_{j \geq 3} \|W_2(\phi^j - \phi^{j-1})\|_{r_{j, s_j, \mathcal{D}_j}} (r_j \sigma_j)^{-n} \tag{A.24}$$

$$\leq C_6 a_2 \sum_{j \geq 3} \left(\frac{1}{\sqrt{2}} \sigma_0 \varepsilon^{d+1} \right)^{2j-2} (2a_1)^{n(j-1)} < +\infty,$$

since, for $j$ sufficiently large,

$$\left(\frac{1}{\sqrt{2}} \sigma_0 \varepsilon^{d+1} \right)^{2j-1} (2a_1)^{nj} < \left(\frac{1}{\sqrt{2}} \sigma_0 \varepsilon^{d+1} \right)^{2j-1} \tag{A.19} \leq (1/\sqrt{2})^{2j-1}.$$

\(^\text{19}\)Recall that, by definition, $G^i(\mathcal{D}^*) = \mathcal{D}_j$ and $Y^*(\mathcal{D}^*) = \mathcal{D}_*$. 

25
Thus, letting $\Phi_j := \phi_1 \circ \phi^j$ and using the Mean Value theorem, we have
\[
\sum_{j \geq 2} \|W_2(\Phi_j - \Phi_{j-1})\|_{\tilde{r}_{j+1/2}, s_j, \mathcal{D}*} \left(\frac{\tilde{r}_{j+1}}{2}\right)^{-n} \leq \|W_2 \nabla \phi_1 W_2^{-1}\|_{r_{1,s_1,\mathcal{D}*}} \times \\
\times \sum_{j \geq 3} \|W_2(\phi^j - \phi^{j-1})\|_{\tilde{r}_{j+1/2}, s_j, \mathcal{D}*} \left(\frac{\tilde{r}_{j+1}}{2}\right)^{-n} < \infty .
\]
Consequently, writing
\[
\Phi_j = (\Phi_j - \Phi_{j-1}) + \cdots + (\Phi_3 - \Phi_2) , \quad j \geq 2 ,
\]
and invoking Lemma B.8 (see Appendix B.7), we conclude that $\phi_* = \lim \Phi_j \in C_\mathcal{W}^C(\mathcal{D}* \times \mathbb{T}^d)$.

Now, we prove $Y* \in C_\mathcal{W}^C(\mathcal{D}*)$ analogously. For any $j \geq 2$ and $n \geq 1$, we have
\[
G^j = (G^j - G^{j-1}) + \cdots + (G^2 - G^1) ,
\]
and, thanks to (A.31), $G^{j+1} - G^j$ is well–defined on $\mathcal{B}_{2^{-j-2}}(\mathcal{D}*)$, for any $j \geq 1$, so that
\[
\sum_{j \geq 1} \|G^{j+1} - G^j\|_{\tilde{r}_{j+2}, \mathcal{D}*} \left(\frac{\tilde{r}_{j+2}}{2^{j+2}}\right)^{-n} = \sum_{j \geq 1} (2^{j+2} \tilde{r}_{j+2}^{-1})^n \|G_{j+1} - \text{id}\circ G^j\|_{\tilde{r}_{j+2}, \mathcal{D}*} \\
\leq \sum_{j \geq 2} (2^{j+2} \tilde{r}_{j+2}^{-1})^n \|G_{j+1} - \text{id}\|_{\tilde{r}_{j+1}, \mathcal{D}*} \\
\leq 2 \sum_{j \geq 1} (2^{j+2} \tilde{r}_{j+2}^{-1})^n r_{j+1} \sigma_j^\nu d^2 \varepsilon^2 |L_j| \\
\leq (A.19) < \infty ,
\]
which proves that $Y* \in C_\mathcal{W}^C(\mathcal{D}*)$.

Finally, we prove Kolmogorov’s non–degeneracy$^{20}$ of the Kolmogorov tori $\phi_*(\mathcal{D}* \times \mathbb{T}^d)$. Fix $y_* \in \mathcal{D}$. Let $y_0 := (Y*)^{-1}(y_*)$ and
\[
\hat{\varepsilon} := \frac{1}{4(18d^3 + 70)\theta} .
\]
$^{20}$See Appendix B.9.
Since \( \| \partial_x u_* \|_* \leq \epsilon \) (11) \( \leq 1/2 \), then the map \( x \mapsto x + u_*(y_*, x) \) is a diffeomorphism of \( \mathbb{T}^d \).

Letting 
\[
(\partial_x (\text{id} + u_*)(y_*, x))^{-1} =: 1_d + A(y_*, x),
\]
we have
\[
\| A \|_* \leq 2 \| \partial_x u_* \|_* \leq 2 \epsilon < 1; \quad \| v_* \|_* \leq \frac{C_9 \sqrt{2}}{4d} \frac{\theta^2 \epsilon P}{\alpha} \leq \frac{C_4 C_9 \sqrt{2}}{2^5 d C_*} \rho < \frac{\rho}{8}. \tag{A.36}
\]
Moreover, write \( K_{yy}(y_*) = K_{yy}(y_0)(1_d + K_{yy}(y_0)^{-1}(K_{yy}(y_*) - K_{yy}(y_0))) \) and observe
\[
dist(y_0, \partial \mathcal{D}) \geq \rho \quad \text{and} \quad \| y_* - y_0 \| \leq \frac{2 \rho - 5 C_4 C_9 \sqrt{2}}{d C_*} \rho < \frac{\rho}{64d},
\]
so that
\[
dist(y_*, \partial \mathcal{D}) \geq \frac{\rho}{2}. \tag{A.37}
\]
Thus, by the Mean Value Theorem, we have
\[
\| K_{yy}(y_0)^{-1}(K_{yy}(y_*) - K_{yy}(y_0)) \| \leq \frac{d^2 K}{\rho^2} |y_* - y_0| \tag{A.37}
\]
\[
\leq \frac{2^{r+1/2} d^2 C_9 \theta^3}{\alpha \rho} \leq \frac{2^{r+15/2} d^2 C_4 C_9}{C_*} \leq \frac{1}{2}.
\]
Hence, \( K_{yy}(y_*) \) is invertible and \( \| K_{yy}(y_*)^{-1} \| \leq 2 \| K_{yy}(y_0)^{-1} \| \leq 2T \).
In [Sal04] it is proven that the map
\[
\phi^{y*}(y, x) := (y_* + v_*(y_*, x) + y + A^T y, x + u_*(y_*, x)).
\]
is symplectic. Then,
\[
H \circ \phi^{y*}(y, x) = E^{y*} + \omega^{y*} \cdot y + Q^{y*}(y, x)
\]
with:
\[
E^{y*} = K_{yy}(y_*) \quad \omega^{y*} := K_{y}(y_0), \quad \langle Q^{y*}_{yy}(0, \cdot) \rangle = K_{yy}(y_*) + \langle \mathcal{M} \rangle,
\]
\[
\mathcal{M} := \partial^2_y \left( K_{yy}(y_* + v_*) + y + A^T y \right) - \frac{1}{2} y^T K_{yy}(y_*) y \bigg|_{y=0} + \partial^2_y (\epsilon P \circ \phi) \bigg|_{y=0},
\]
\[
\| K_{yy}(y_*)^{-1} \mathcal{M} \|_* \leq 2T \mathcal{M} \tag{A.36}
\]
\[
\leq 2(18d^3 + 70) \epsilon \theta = 1/2,
\]
which show that \( \langle Q^{y*}_{yy}(0, \cdot) \rangle \) is invertible.
Remark A.4 Here we list all the constants, which appear in the above proof and give the explicit expression for the constants $c_k$’s appearing in the statement of Theorem 1. Recall that $\tau > d - 1 \geq 1$ and notice that all the $C_i$’s are greater than 1 and depend only upon $d$ and $\tau$.

$$\nu := \tau + 1, \quad C_0 := 4 \left( \frac{3}{2} \right)^{2\nu + d} \int_{\mathbb{R}^d} \left( |y|^{\nu} + d |y|^{2\nu} \right) e^{-|y|^2} dy, \quad C_1 := 2 \left( \frac{3}{2} \right)^{\nu + d} \int_{\mathbb{R}^d} |y|^{\nu} e^{-|y|^2} dy,$$

$$C_2 := 2^{3d} d, \quad C_3 := d^2 C_1^2 + 6dC_1 + C_2, \quad C_4 := \max \left\{ (1 + d^2)C_0, C_3 \right\}, \quad C_5 := \max \left\{ 2^{2\nu}, 2^7 d \right\},$$

$$C_6 := \left( 2^{-d} C_5 \right)^{\frac{1}{4}}, \quad C_7 := 3 \cdot 2^{4\nu + 2d + 3d} \max \left\{ 2^{2\nu + 6d}, C_4/2 \right\} C_5, \quad C_8 := 3 \cdot 2^{3\nu + 1} \nu e^{-\nu} dC_4 \sqrt{2},$$

$$C_9 := 3 \cdot 2^{-4(4\nu + 2d)} dC_4 \sqrt{2} + 2^{-\nu} \nu e^{-\nu} C_7 C_8, \quad C_{10} := 2 \left( \frac{3}{2} \right)^{\nu + d + 1} \int_{\mathbb{R}^d} |y|^\nu e^{-|y|^2} dy,$$

$$C_{11} := 8 \left( \frac{3}{2} \right)^{3\tau + d + 2} \int_{\mathbb{R}^d} (2|y|^\tau + 3|y|^{2\tau + 1} + |y|^{3\tau + 2}) e^{-|y|^2} dy, \quad C_{12} := \max \left\{ 2C_{10}, 2C_{11}, 12C_0 \right\},$$

$$C_* := \max \left\{ 2^{11\nu + 6d + 4} \nu e^{-\nu} C_5^2 C_6 C_7 C_8, \left( 2^{\nu/2 - 2d + 2}(18d^3 + 70) \nu e^{-\nu} C_4 C_9 C_{12} \right)^2, 2^{7 + 8d^2} C_4 C_9 \sqrt{2} \right\}.$$

Then,

$$c_* := C_*, \quad c_0 := 2^{-4} C_4, \quad c_1 := 2^{7 - 1/2 - d - 1} C_9,$$

$$c_2 := 2^{2\nu + 6} dC_9, \quad c_3 := C_9, \quad c_4 := e^{(4d)^{-1}} C_4 C_9 C_{12}^{-1}. \quad (A.38)$$

Remark A.5 There is a small flaw in [CK19]: The parameter L chosen in [CK19, Lemma 1] is not big enough to ensure that the new perturbation $P'$ and the symplectic change of coordinates $\phi$ are well-defined on $D_{\bar{r}^k \times \mathbb{R}^d}$ for $\bar{r}_k$. The right choice is the following

$$L := P \max \left\{ \frac{40 dT^2 \sigma^{\nu + d}}{r \sigma^{(2\nu + d)}}, \frac{2C_4}{\alpha \sigma^{2(\nu + d)}} \right\}, \quad W := \text{diag} \left( r^{-1} \mathbb{1}_d, \mathbb{1}_d \right), \quad \hat{\epsilon}_0 := C_9 \sigma^{(2 - d - 1)} e_0 \theta_0^2 \lambda_\nu,$$

$$L_j := \max \left\{ \frac{80 d^2 T_0 \theta_0}{r_j \sigma^{\nu + d}}, \frac{C_4}{\alpha \sigma^{2(\nu + d)}} \right\}, \quad W_j := \text{diag} \left( 2r_j^{-1} \mathbb{1}_d, \mathbb{1}_d \right), \quad \hat{\epsilon}_{j+1} := \frac{K_0 e^{2 + 1} P_{j+1}}{\alpha^2},$$

$$P_{j+2} := \lambda_* \theta_j^2 \frac{K_0 P_{j+1}^2}{\alpha^2}, \quad \hat{\epsilon}_{j+1} := \lambda_* \theta_j^2 \hat{\epsilon}_{j+1}.$$

Of course, one needs, then, to change accordingly (and in a straightforward way) the constants involved, as follows:

\[\text{[In the present remark, we will adopt the notations of [CK19].}\]
Lemma B.1 (B Tools)

(ii) For any $a \in \mathbb{R}^d$ with $|a| < 1$, we have

$$C_0 := 4\sqrt{2} \left( \frac{3}{2} \right)^{2\nu + d} \int_{\mathbb{R}^d} (|y|^\nu + |y|^{2\nu}) e^{-|y|_1} dy,$$

$$C_1 := 2 \left( \frac{3}{2} \right)^{\nu + d} \int_{\mathbb{R}^d} |y|^\nu e^{-|y|_1} dy,$$

$$C_2 := 2^{3d} d, \quad C_3 := (d^2 C_4^2 + 6d C_4 + C_4) \sqrt{2}, \quad C_4 := \max \{6d^2 C_0, C_3\},$$

$$C_5 := \frac{3 \cdot 2^5 d}{5}, \quad C_6 := \max \{2^{2\nu}, C_5\}, \quad C_7 := 3d \cdot 2^{4\nu + 2} \sqrt{2} \max \{640d^2, C_4\},$$

$$C_8 := \left( 2^{-d} C_6 \right)^{1/8}, \quad C_9 := 3d \cdot 2^{\nu + 2} \sqrt{2} \max \{80d \sqrt{2}, C_4\},$$

$$C_{10} := (4\nu e^{-1})^{2\nu} \left( 1 + 2^{\nu + 2d + 2}(\nu e^{-1})^{2\nu} C_4^2 C_7 \right) C_9 / (3d^2),$$

$$C_{11} := (5d \cdot 2^{3(\nu + 1)})^{-1} C_{10},$$

$$C_{12} := 2^{2(5\nu + 4d + 2)} C_4^2 C_7 C_8 C_9, \quad C_{13} := C_{10} + C_{11}, \quad C_{14} := C_{12},$$

$$C_{15} := 18d^3 + 70, \quad C_{16} := (6\nu e^{-1})^{4\nu}, \quad C := \max \{3C_{10}, C_{13}\},$$

$$C_* := \max \left\{ C_{16} C_{14}^{2/3}, 6C_{15} C_6 C_2, 2^{2(4\nu + 2d + 1)} C_{16} C_6^2, C_{10}^2 \right\}.$$

The smallness condition (14) and the estimate (16) become, respectively,

$$\alpha \leq \frac{r}{\nu} \quad \text{and} \quad \epsilon \leq \epsilon_* := \frac{(s - s_*)^a}{C_6 \theta^6},$$

and

$$\max \left\{ \|u_\ast\|_{s_\ast}, \|\partial_x u_\ast\|_{s_\ast}, \frac{K}{\alpha} (\log \epsilon)^{1/2} \|v_\ast\|_{s_\ast} \right\} \leq \frac{C \theta^3}{(s - s_*)^{a/2}} \epsilon (\log \epsilon)^{1/2} \leq \frac{1}{4} \epsilon,$$

where $a := 6\nu + 3d + 2$.

B Tools

B.1 Classical estimates (Cauchy, Fourier)

Lemma B.1 ([CC95]) Let $p \in \mathbb{N}$, $r, s > 0$, $y_0 \in \mathbb{C}^d$ and $f$ a real-analytic function $\mathbb{B}_{r,s}(y_0)$ with $\|f\|_{r,s} := \sup_{\mathbb{B}_{r,s}(y_0)} |f| < \infty$. Then,

(i) For any multi-index $(l, k) \in \mathbb{N}^d \times \mathbb{N}^d$ with $|l|_1 + |k|_1 \leq p$ and for any $0 < r' < r, 0 < s' < s$,

$$\|\partial_y^l \partial z^k f\|_{r', s'} \leq p! \|f\|_{r,s} (r - r')! |l|_1 (s - s')! |k|_1.$$

(ii) For any $k \in \mathbb{Z}^d$ and any $y \in \mathbb{B}_r(y_0)$

$$|f_k(y)| \leq e^{-|k|_1 s} \|f\|_{r,s}.$$

---

As usual, $\partial_y^l := \frac{\partial^{l_1}}{\partial y_1^{l_1} \ldots \partial y_d^{l_d}}$, $\forall y \in \mathbb{R}^d, l \in \mathbb{Z}^d.$
B.2 An Inverse Function Theorem

**Theorem B.2** Let $D$ be a convex subset of $\mathbb{C}^d$, $y_0 \in D$ and let $f \in C^1(D, \mathbb{C}^d)$ such that\(^{23}\) $\det f'(y_0) \neq 0$. Assume
\[
\varrho := \sup_{y \in D} \|1 - T f'(y)\| < 1, \quad T := (f'(y_0))^{-1}. \tag{B.1}
\]
Then, $\det f'(y) \neq 0$, for each $y \in D$ and
\[
\|f'(y)\| \leq \lambda := \frac{\|T\|}{1 - \varrho}. \tag{B.2}
\]
Moreover, $f$ in injective on $D$ and its inverse function $g : f(D) \to D$ satisfies
\[
\text{Lip}_{f(D)}(g) \leq \lambda. \tag{B.3}
\]
Furthermore, if $D := B_r(y_0)$, $\rho := r/\lambda$ and $z_0 := f(y_0)$, then
\[
B_\rho(z_0) \subseteq f(D). \tag{B.4}
\]

**Proof** For every $y \in D$, we have $f'(y) = f'(y_0)(1 - A)$, where $A := 1 - T f'(y)$ with $\|A\| \leq \varrho < 1$. Thus, $f'(y)$ is invertible and
\[
\|f'(y)\|^{-1} = \|\sum_{n=0} A^nT\| \leq \frac{\|T\|}{1 - \varrho},
\]
proving (B.2). Now, consider the auxiliary map $F : D \ni y \mapsto y - T f(y)$. We have $F \in C^1(D, \mathbb{C}^d)$ and $\sup_D \|F'\| \leq \varrho$. Thus, for every $y, \bar{y} \in D$ with $y \neq \bar{y}$, we have
\[
\|T\| f(y) - f(\bar{y})\| \overset{(B.1)}{\geq} \|T(f(y) - f(\bar{y}))\|
= \|(y - \bar{y}) + (F(\bar{y}) - F(y))\|
\overset{}{\geq} \|y - \bar{y}\| - \|y - \bar{y}\| \sup_{D} \|F'\|
\overset{}{\geq} \|y - \bar{y}\|(1 - \varrho) \overset{(B.1)}{\geq} 0, \tag{B.5}
\]
which shows that $f$ is injective on $D$ and, hence, that (B.3) holds.
To show (B.4) in the case $D := B_r(y_0)$ and $\rho := r/\lambda$, fix $\eta \in \mathbb{C}^d$ with $\|\eta - z_0\| < \rho$. We have to show that there exists $\bar{y} \in D$ such that $f(\bar{y}) = \eta$. Define the map
\[
\Phi : y \in D \mapsto \Phi(y) := y - T(f(y) - \eta) \in Y. \tag{B.6}
\]
\(^{23}\) $f'$ being the Jacobian matrix of $f$. 

30
Then, $\Phi$ is a contraction on $D$. Indeed, $\Phi$ is $C^1$, $\Phi'(y) = 1 - Tf'(y)$ and
\[
\text{Lip}_D \Phi = \sup_D \|\Phi'\| = \rho < 1. \tag{B.7}
\]

Furthermore, $\Phi : D \to D$, since, if $y \in D$, then
\[
\|\Phi(y) - y_0\| \leq \|\Phi(y) - \Phi(y_0)\| + \|\Phi(y_0) - y_0\| \leq \rho r + \|T\|\|\eta - z_0\| < \rho r + \|T\| \rho = r.
\]

Hence, by the contraction Lemma, $\Phi$ has a (unique) fixed point $\bar{y} \in D$, but $\Phi(\bar{y}) = \bar{y}$ means $f(\bar{y}) = \eta$. □

### B.3 Internal coverings

Given any non–empty subset $D$ of $\mathbb{R}^d$, and given $r > 0$, a $r$–internal covering of $D$ is a subset $P$ of $D$ such that $D \subseteq \bigcup_{y \in P} B_r(y)$; the $r$–internal covering number of $D$, denoted $N^\text{int}_r(D)$, is the minimal cardinality of any $r$–internal cover.

In [BC18] the following simple upper bound (having fixed the sup norm in $\mathbb{R}^d$) on $N^\text{int}_r(D)$ for bounded sets $D$ is given:

**Lemma B.3** Let $D \subseteq \mathbb{R}^d$ be a non–empty bounded set. Then, for any $r > 0$, one has
\[
N^\text{int}_r(D) \leq \left(\left\lceil\frac{\text{diam} D}{r}\right\rceil + 1\right)^d. \tag{B.8}
\]

For convenience of the reader, we reproduce here the elementary proof of the lemma.

**Proof** It is enough to produce a $r$–internal cover of $D$ with cardinality $N$ bounded by the right hand side of (B.8).

If $D$ is a singleton, the claim is obvious with $N = 1$. Assume, now, $\delta := \text{diam} D > 0$, and let $M := [\delta/r] + 1$ and $z_i = \inf\{x_i| x \in D\}$. Then, $D \subseteq K := z + [0, \delta]^d$ and one can find $0 < r' < r$ close enough to $r$ so that $[\delta/r'] \leq [\delta/r] + 1 = M$. Then, one can cover $K$ with $M^d$ closed, contiguous cubes $K_j$, $1 \leq j \leq M^d$, with edge of length $r'$. Let $j_i$ be the indices such that $K_{j_i} \cap D \neq \emptyset$ and pick a $y_i \in K_{j_i} \cap E$; let $1 \leq N \leq M^d$ be the number of such cubes. Observe that, since we have chosen the sup–norm in $\mathbb{R}^d$, one has $K_{j_i} \subseteq B_r(y_i)$ and (B.8) follows. □

---

$24[x]$ denotes the integer–part (or “floor”) function $\max\{n \in \mathbb{Z}| n \leq x\}$, while $[x]$ denote the “ceiling function” $\min\{n \in \mathbb{Z}| n \geq x\}$; observe that $[x] \leq [x] + 1$. 

31
B.4 Extensions of Lipschitz continuous functions

Here we recall a Theorem due to Minty according to which a Lipschitz continuous function can be extended keeping unchanged both the sup–norm and the Lipschitz constant.

**Theorem B.4 (G. J. Minty[Min70])** Let \((V, \langle \cdot, \cdot \rangle)\) be a separable inner product space, \(\emptyset \neq A \subseteq V\), \(L > 0\), \(0 < \alpha \leq 1\) and \(g: A \to \mathbb{R}^d\) a \((L, \alpha)\)–Lipschitz–Hölder continuous function on \(A\), namely, \(g\) satisfies

\[
|g(x_1) - g(x_2)|_2 \leq L \|x_1 - x_2\|^\alpha, \quad \forall x_1, x_2 \in A,
\]

where \(\| \cdot \|\) denotes the norm on \(V\) induced by the inner product. Then, there exists a global \((L, \alpha)\)–Lipschitz–Hölder continuous function \(G: V \to \mathbb{R}^d\) such that \(G|_A = g\). Furthermore, \(G\) can be chosen in such way that \(G(V)\) is contained in the closed convex hull of \(g(A)\). Hence, in particular,

\[
\sup_{x \in V} |G(x)|_2 = \sup_{x \in A} |g(x)|_2 \quad \text{and} \quad \sup_{x_1 \neq x_2 \in V} \frac{|G(x_1) - G(x_2)|_2}{\|x_1 - x_2\|^\alpha} = \sup_{x_1 \neq x_2 \in A} \frac{|g(x_1) - g(x_2)|_2}{\|x_1 - x_2\|^\alpha}.
\]

(B.10)

B.5 Lebesgue measure and Lipschitz continuous map

**Lemma B.5** Let \(\emptyset \neq A \subseteq \mathbb{R}^d\) be a Lebesgue–measurable set and \(f: A \to \mathbb{R}^d\) be Lipschitz continuous. Then,

\[
\text{meas } (f(A)) \leq \text{Lip}_A(f)^d \text{meas}(A)
\]

(B.11)

and\(^{25}\)

\[
|\text{meas}(f(A)) - \text{meas}(A)| \leq ((1 + \delta)^d - 1) \text{meas}(A).
\]

(B.12)

where

\[
\delta := \text{Lip}_A(f - \text{id})
\]

(B.13)

**Proof** Eq. (B.11) is standard: see, e.g., Theorem 2, Sec 2.2 and Theorem 1, Sec 2.4 in [EG15].

Let us prove (B.13). By Theorem B.4, \(f - \text{id}\) can be extended to a Lipschitz continuous \(g: \mathbb{R}^d \to \mathbb{R}^d\) with

\[
\text{Lip}(g) = \text{Lip}_A(f - \text{id}) = \delta.
\]

\(^{25}\)I.e., satisfying (B.9) on \(V\).

\(^{26}\)Inequality (B.12) is sharp as shown by the example \(f = (1 + \delta) \text{id}\).
By Rademacher’s Theorem, there exists a set \( N \subseteq \mathbb{R}^d \) with \( \text{meas}(N) = 0 \) such that \( g \) is differentiable on \( \mathbb{R}^d \setminus N \) and

\[
\| g_y \|_{\mathbb{R}^d \setminus N} \leq \text{Lip}_{\mathbb{R}^d \setminus N}(g) \leq \text{Lip}(g) = \delta .
\]

Now pick \( y \in \mathbb{R}^d \setminus N \). Then,

\[
| \det(\mathbb{1}_d + g_y(y)) - 1 | = \left| \int_0^1 \frac{d}{dt} \det(\mathbb{1}_d + t g_y) dt \right| = \left| \int_0^1 \text{tr} ( \text{Adj} (\mathbb{1}_d + t g_y) g_y ) dt \right|
\leq \int_0^1 d \| \mathbb{1}_d + t g_y \|^{d-1} \| g_y \| dt \leq \int_0^1 d (1 + \delta t)^{d-1} \delta dt = (1 + \delta)^d - 1 .
\]

Thus, by the change of variable (or area) formula\(^{27}\), we have

\[
| \text{meas}(f(A)) - \text{meas}(A) | = \left| \int_{(\mathbb{1}_d + g)(A)} dy - \int_A dy \right|
= \left| \int_{(\mathbb{1}_d + g)(A \setminus N)} dy - \int_{A \setminus N} dy \right| = \left| \int_{A \setminus N} | \det(\mathbb{1}_d + g_y) | dy - \int_{A \setminus N} dy \right|
\leq \int_{A \setminus N} | \det(\mathbb{1}_d + g_y) - 1 | dy \leq ((1 + \delta)^d - 1) \text{meas}(A) .
\]

### B.6 Lipeomorphisms “close” to identity

**Lemma B.6** Let \( g : \mathbb{C}^d \to \mathbb{C}^d \) be a Lipschitz continuous function such that

\[
\delta := \sup_{\mathbb{R}^d} | g - \mathbb{1} | < \infty , \quad (B.14)
\]

\[
\theta := \text{Lip}_{\mathbb{R}^d}(g - \mathbb{1}) < 1 . \quad (B.15)
\]

Then, \( g \) has a Lipschitz global inverse \( G \) satisfying

\[
\sup_{\mathbb{R}^d} | G - \mathbb{1} | \leq \delta , \quad (B.16)
\]

\[
\text{Lip}_{\mathbb{R}^d}(G - \mathbb{1}) < \frac{1}{1 - \theta} . \quad (B.17)
\]

Furthermore, for any \( \emptyset \neq A \subseteq \mathbb{C}^d \),

\[
A \subseteq g \left( \overline{B}_\delta(A) \right) . \quad (B.18)
\]

\(^{27}\text{See [EG15], §3.3.}\)
**Proof** Let $f := g - \text{id}$, then, for any $x_1 \in \mathbb{R}^d$, one has
\[
|g(x_1) - g(x_2)| = |x_1 - x_2 + (f(x_1) - f(x_2))| \overset{(B.15)}{=} |x_1 - x_2| - \theta |x_1 - x_2| \leq |x_1 - x_2| - \theta |x_1 - x_2|,
\]
which proves injectivity of $g$ and that
\[
\inf_{x_1 \neq x_2} \frac{|g(x_1) - g(x_2)|}{|x_1 - x_2|} \geq 1 - \theta > 0. \quad \text{(B.19)}
\]
Let us now prove (B.18). Let $\bar{y} \in A$. It is enough to show that there exists $|y| \leq \delta$ such that $\bar{y} = g(y + \delta)$ i.e. $y = f(y + \delta)$ i.e. $y$ is a fixed point of the map
\[
h: \overline{B_\delta}(0) \ni y \mapsto -f(y + \delta).
\]
But, for any $y \in \overline{B_\delta}(0)$,
\[
|h(y)| = |f(y + \delta)| \leq \|f\|_{\mathbb{R}^d} \overset{(B.14)}{\leq} \delta,
\]
i.e. $h: \overline{B_\delta}(0) \rightarrow \overline{B_\delta}(0)$. Moreover, $h$ is a contraction since $\text{Lip}_{\overline{B_\delta}(0)}(h) \leq \text{Lip}_{\mathbb{R}^d}(f) \overset{(B.15)}{<} 1$.
Thus, by Banach’s fixed point Theorem, we see that (B.18) holds.
From (B.18) it follows at once that $g$ is onto $\mathbb{R}^d$.
Now, (B.16) and (B.17) follows easily from, respectively (B.14) and (B.19). 

**B.7 Whitney smoothness**

**Definition B.7** Let $A \subseteq \mathbb{R}^d$ be non–empty and $n \in \mathbb{N}_0$, $m \in \mathbb{N}$. A function $f: A \rightarrow \mathbb{R}^m$ is said $C^n$ on $A$ in the Whitney sense, with Whitney derivatives $\left( f_\nu \right)_{\nu \in \mathbb{N}_0^d, |\nu|_1 \leq n}$, $f_0 = f$, and we write $f \in C^m_W(A, \mathbb{R}^m)$, if for any $\varepsilon > 0$ and $y_0 \in A$, there exists $\delta > 0$ such that, for any $y, y' \in A \cap B_\delta(y_0)$ and $\nu \in \mathbb{N}_0^d$, with $|\nu|_1 \leq n$,
\[
\left| f_\nu(y') - \sum_{\mu \in \mathbb{N}_0^d, |\mu|_1 \leq n - |\nu|_1} \frac{1}{\mu!} f_{\nu+\mu}(y)(y')^\mu \right| \leq \varepsilon |y' - y|^{n - |\nu|_1}. \quad \text{(B.20)}
\]

**Lemma B.8** ([Chi86, Kou19]) Let $A \subseteq \mathbb{R}^d$ be non–empty and $n \in \mathbb{N}_0$. For $m \in \mathbb{N}$, let $f_m$ be a real–analytic function with holomorphic extension to $D_{r_m}(A)$, with $r_m \downarrow 0$ as $m \rightarrow \infty$. Assume that
\[
a := \sum_{m=1}^{\infty} \| f_m \|_{r_m A} r_m^n < \infty, \quad \| f_m \|_{r_m A} := \sup_{B_{r_m}(A)} |f_m| . \quad \text{(B.21)}
\]
Then \( f := \sum_{m=1}^{\infty} f_m \in C^\infty_W(A, \mathbb{R}) \) with Whitney derivatives \( f_\nu := \sum_{m=1}^{\infty} \partial_{y_\nu} f_m \).

For completeness, we recall the beautiful Whitney extension theorem.

**Theorem B.9** ([Whi34]) Let \( A \subseteq \mathbb{R}^d \) be a closed set and \( f \in C^n_W(A, \mathbb{R}) \), \( n \in \mathbb{N}_0 \). Then there exists \( \tilde{f} \in C^n(\mathbb{R}^d, \mathbb{R}) \), real–analytic on \( \mathbb{R}^d \setminus A \) and such that \( D^\nu \tilde{f} = f_\nu \) on \( A \), for any \( \nu \in \mathbb{N}_0^d \), with \( |\nu|_1 \leq n \).

### B.8 Measure of tubular neighbourhoods of hypersurfaces

Recall the definitions of minimal focal distance and of inner domains given in § 3.2. The first elementary remark is that, for smooth domains, taking \( \rho \)-inner domains is the inverse operation of taking \( \rho \)-neighborhood:

**Lemma B.10** Let \( \mathcal{D} \subseteq \mathbb{R}^d \) be an open and bounded set with \( C^2 \) boundary \( \partial \mathcal{D} = S \) compact and connected. Then, for any \( 0 < \rho' < \rho \leq \text{minfoc}(S) \), one has

\[
B_\rho(\mathcal{D}'_\rho) = \mathcal{D}, \quad \text{and} \quad B_{\rho-\rho'}(\mathcal{D}'_\rho) = \mathcal{D}'_{\rho'}.
\]

**Proof** We start proving the first part of (B.22). By definition, \( B_\rho(\mathcal{D}'_\rho) \subseteq \mathcal{D} \). Thus, it remains only to show that \( \mathcal{D}'\setminus\mathcal{D}'_{\rho'} \subseteq B_\rho(\mathcal{D}'_{\rho'}) \).

Let then \( y_0 \in \mathcal{D}'\setminus\mathcal{D}'_{\rho'} \). As \( S \) is compact and \( \text{dist}_2 \) is continuous, there exists \( \bar{y}_0 \in S \) such that \( \text{dist}_2(y_0, \mathcal{D}' \setminus \mathcal{D}) = \text{dist}_2(\bar{y}_0, S) = |y_0 - \bar{y}_0|_2 \). The vector \( \nu := (y_0 - \bar{y}_0)/|y_0 - \bar{y}_0|_2 \) is the inward unit normal to \( \partial \mathcal{D} = S \) at \( \bar{y}_0 \). Indeed, for any smooth curve \( \gamma: [0,1] \to S \) with \( \gamma(0) = \bar{y}_0 \), 0 is a minimum of the smooth map \( f(t) := |\gamma(t) - y_0|_2^2 \). Thus,

\[
0 = f'(0) = 2\dot{\gamma}(0) \cdot (y_0 - \bar{y}_0).
\]

which, by the arbitrariness of \( \gamma \), implies that the line \( (\bar{y}_0y_0) \) is perpendicular to the tangent space to \( S \) at \( \bar{y}_0 \) and, therefore \( \nu \) is the inward unit normal to \( \partial \mathcal{D} \) at \( \bar{y}_0 \). Let \( y_1 := \bar{y}_0 + \rho \nu \). By assumption, we have \( \text{dist}_2(y_1, S) = \rho \), and, therefore, \( y_1 \in \mathcal{D} \). In addition, \( y_1 \in \mathcal{D}'_{\rho'} \).

Indeed, for any \( y \in B_\rho(y_1) \), \( \text{dist}_2(y, \mathcal{D}' \setminus \mathcal{D}) \geq \text{dist}_2(y_1, \mathcal{D}' \setminus \mathcal{D}) - |y_1 - y|_2 = \text{dist}_2(y_1, S) - |y_1 - y|_2 = \rho - |y_1 - y|_2 > 0 \). Thus, as \( \mathcal{D}' \setminus \mathcal{D} \) is a closed set, \( y \notin \mathcal{D}' \setminus \mathcal{D} \) i.e. \( y \in \mathcal{D} \). Hence, \( B_\rho(y_1) \subseteq \mathcal{D} \) i.e. \( y_1 \in \mathcal{D}'_{\rho'} \). In particular, the argument above shows that:

\[\text{dist}_2(y, \mathcal{D}' \setminus \mathcal{D}) \geq \rho \] implies that \( y \in \mathcal{D}'_{\rho'} \). Thus, as \( y_0 \in \mathcal{D}'\setminus\mathcal{D}'_{\rho'} \), we have \( \text{dist}_2(y_0, \mathcal{D}' \setminus \mathcal{D}) < \rho \).

---

\[\text{Actually, one checks easily that } \partial \mathcal{D}'_{\rho'} = \{ y \in \mathbb{R}^d : \text{dist}_2(y, \mathcal{D}' \setminus \mathcal{D}) = \rho \} \text{ and } \text{int}(\mathcal{D}'_{\rho'}) = \{ y \in \mathbb{R}^d : \text{dist}_2(y, \mathcal{D}' \setminus \mathcal{D}) > \rho \}, \text{ int}(\mathcal{D}'_{\rho'}) \text{ being the interior of } \mathcal{D}'_{\rho'}.\]
which means \( y_0 \) is in the open segment \((\bar{y}_0, y_1)\). Therefore, \(|y_0 - y_1|_2 < |\bar{y}_0 - y_1|_2 = \rho \) i.e. \( y_0 \in B_\rho(y_1) \subseteq B_\rho(D^n) \).

We now prove the second part of (B.22). We have \( B_{\rho - \rho'}(D^n_\rho) \subseteq D^n_\rho \). Indeed, for any \( y_0 \in D^n_\rho, y_1 \in B_{\rho - \rho'}(y_0) \) and \( y \in B_\rho(y_1) \),

\[
|y - y_0| \leq |y - y_1| + |y_1 - y_0| < \rho' + (\rho - \rho') = \rho \quad \text{i.e.} \quad y \in B_\rho(y_0),
\]

which implies \( B_{\rho - \rho'}(D^n_\rho) \subseteq D^n_\rho \). It remains to show that \( D^n_\rho \setminus D^n_\rho \subseteq B_{\rho - \rho'}(D^n_\rho) \). The proof follows in analogous to the previous. Let \( y_0 \in D^n_\rho \setminus D^n_\rho \) and \( \bar{y}_0 \in S \) such that \( \text{dist}_2(y_0, \mathbb{R}^d \setminus D) = \text{dist}_2(y_0, S) = |y_0 - \bar{y}_0|_2 \). Then, \( \rho' \leq |y_0 - \bar{y}_0|_2 < \rho \), and the vector \( \nu := (y_0 - \bar{y}_0)/|y_0 - \bar{y}_0|_2 \) is the inward unit normal to \( \partial D = S \) at \( \bar{y}_0 \). Set \( y'_1 := \bar{y}_0 + \rho' \nu \). Thus, \(|y'_1 - \bar{y}_0|_2 = \rho' \leq |y_0 - \bar{y}_0|_2 \) and, hence, \( y'_1 \in D^n_\rho \) and \( y'_1 \) is in the semi open segment \((\bar{y}_0, y_0]\). Therefore, \(|y'_1 - \bar{y}_0|_2 = |y_0 - \bar{y}_0|_2 - |y'_1 - \bar{y}_0|_2 < \rho - \rho' \). Hence, \( y_0 \in B_{\rho - \rho'}(y'_1) \subseteq B_{\rho - \rho'}(D^n_\rho) \) i.e. \( D^n_\rho \setminus D^n_\rho \subseteq B_{\rho - \rho'}(D^n_\rho) \). \( \blacklozenge \)

Next result, gives a precise evaluation of tubular domains in the case the metric is the euclidean one. Define

\[
\Sigma_\rho(S) := \{ u \in \mathbb{R}^d : \text{dist}_2(u, S) < \rho \} .
\]  

(B.23)

**Lemma B.11** Let \( D \subseteq \mathbb{R}^d \) be a bounded set with \( C^2 \) boundary \( \partial D = S \) compact and connected. Then, for any \( 0 < \rho \leq \min \text{foc}(S) \), then,

\[
\text{meas}(\Sigma_\rho(S)) \leq \frac{2}{d} \frac{(1 + \rho \kappa)^d - 1}{\kappa} \mathcal{H}^{d-1}(S) ,
\]  

(B.24)

where \( \kappa := \sup_{x} \max_{1 \leq j \leq d-1} |\kappa_j| \) with \( \kappa_j \) the principal curvatures of \( S \), while \( \mathcal{H}^{d-1} \) denotes the \((d-1)\)-dimensional Hausdorff measure (‘surface area’).

**Proof**\( ^{29} \) We will estimate the ‘inner tubular neighbourhoods’

\[
\Sigma'_\rho(S) := \{ y \in D : \text{dist}_2(y, S) < \rho \} ,
\]

as the argument for ‘outer tubular neighbourhood’ \( \{ y \notin D : \text{dist}_2(y, S) < \rho \} \), is completely analogous.

Since \( S \) is compact and connected, we may assume that \( S = f^{-1}(\{0\}) \) with \( f \in C^2(\mathbb{R}^d, \mathbb{R}) \) and 0 a regular value for \( f \). Set

\[
\nu(x) = \frac{\nabla f}{|\nabla f|_2} , \quad | \cdot |_2 := \text{dist}_2(\cdot, 0) ,
\]  

\( ^{29} \)Compare [Ste12], Ch. 1.
and replacing eventually \( f \) by \(- f\), we can assume that \( \nu \) is the inwards unit normal vector fields of \( S \). Let \( \{ \phi_j : U_j \to \mathbb{R}^m \}_{j=1}^p \) be an atlas of \( S \),
\[
\Psi_j(u, t) := \phi_j(u) + t\nu(\phi_j(u)), \quad O_j := \Psi_j(U_j \times [0, \rho]),
\]
and observe that\(^{30}\)
\[
\mathcal{T}^\prime_\rho(S) = \bigcup_{j=1}^p O_j.
\]
Let \( \{ \psi_j \}_{j=1}^p \) be a partition of unity subordinated to the open covering of \( \{ O_j \}_{j=1}^p \) of \( \mathcal{T}^\prime_\rho(S) \) i.e.

(i) \( \psi_j \in C^\infty_c(\mathcal{T}^\prime_\rho(S)) \);

(ii) \( 0 \leq \psi_j \leq 1 \);

(iii) \( \text{supp} \psi_j \subseteq O_j \);

(iv) \( \sum_{j=1}^p \psi_j \equiv 1 \) on \( \mathcal{T}^\prime_\rho(S) \).

Given \( 1 \leq j \leq p \), define \( n_j : U_j \longrightarrow S^d = \{ x \in \mathbb{R}^d : |x|_2 = x_1^2 + \cdots + x_d^2 = 1 \} \subseteq \mathbb{R}^d \) as
\[
n_j := \nu \circ \phi_j,
\]
and \( K_j : U_j \longrightarrow T^*S \) such that\(^{31}\),
\[
K_j(u) := -\nu'(\phi_j(u)).
\]
Then, \( K_j \) is symmetric\(^{32}\) and therefore diagonalizable, with eigenvalues \( \kappa_i \circ \phi_j^{-1} \), \( 1 \leq i \leq d-1 \) and satisfies
\[
\frac{\partial n_j}{\partial u} = -K_j \frac{\partial \phi_j}{\partial u}. \tag{B.25}
\]

\(^{30}\)As \( S = \bigcup_{j=1}^p \phi_j(U_j) \), we have \( \mathcal{T}^\prime_\rho(S) = \bigcup_{j=1}^p O_j \), for any \( 0 < \rho \leq \min foc(S) \).

\(^{31}\)\( T^*S \) being the cotangent bundle of \( S \).

\(^{32}\)\( K_j \) is actually the Weingarten map \( W_x = -\nu'(x) \) “written in the local chart” \( (U_j, \phi_j) \).
Thus, recalling that $0 = \partial_x \nu^2 = 2\nu' \cdot \nu$, we have

$$\text{meas}(\mathcal{C}_p^\prime(S)) = \sum_{j=1}^{p} \int_{O_j} \psi_j \, dudt$$

$$= \sum_{j=1}^{p} \int_{\psi_j(U_j \times [0, \rho])} \psi_j \, dudt$$

$$= \sum_{j=1}^{p} \int_{U_j \times [0, \rho]} \Psi_j^p(\psi_j dudt)$$

$$= \sum_{j=1}^{p} \int_{U_j \times [0, \rho]} \psi_j \circ \Psi_j \left| \det \left( \frac{\partial \psi_j}{\partial (u, t)} \right) \right| \, dudt$$

$$\leq \int_0^\rho \sum_{j=1}^{p} \int_{U_j} \psi_j \left( \phi_j(u) + t\nu(\phi_j(u)) \right) \left| \det \left( \frac{\partial \phi_j}{\partial u} \right) \nu(\phi_j(u)) \right| \, du \, (1 + \kappa \delta)^{-1} dt$$

$$= \int_0^\rho \sum_{j=1}^{p} \int_{U_j} \psi_j \left( x + t\nu(x) \right) d\mathcal{H}^{d-1}(x) \, (1 + \kappa \delta)^{-1} dt \quad \text{(see [EG15, Theorem 2, pg. 99])}$$

$$\leq \int_0^\rho \sum_{j=1}^{p} \int_{\cup_{i=1}^{p} \phi_i(U_i)} \psi_j \left( x + t\nu(x) \right) d\mathcal{H}^{d-1}(x) \, (1 + \kappa \delta)^{-1} dt$$

$$= \int_0^\rho \sum_{j=1}^{p} \psi_j \left( x + t\nu(x) \right) d\mathcal{H}^{d-1}(x) \, (1 + \kappa \delta)^{-1} dt$$

$$= \int_0^\rho \sum_{j=1}^{p} d\mathcal{H}^{d-1}(x) \, (1 + \kappa \delta)^{-1} dt$$

$$= \frac{(1 + \rho \kappa)^d - 1}{d \kappa} \mathcal{H}^{d-1}(S).$$
B.9 Kolmogorov non–degenerate normal forms

Let $H: \mathcal{M} := \mathbb{R}^d \times \mathbb{T}^d \to \mathbb{R}$ be a $C^2$–Hamiltonian. An embedded torus $\mathcal{T}$ in $\mathcal{M}$ is said $H$–Kolmogorov non–degenerate if there exists a neighborhood $\mathcal{M}_0$ of $\{0\} \times \mathbb{T}^d$ in $\mathcal{M}$, a symplectic change of coordinates $\phi: \mathcal{M}_0 \to \mathcal{M}$ with $\phi(\{0\} \times \mathbb{T}^d) = \mathcal{T}$, a constant $E \in \mathbb{R}$, a vector $\omega \in \mathbb{R}^d$ and a function $Q: \mathcal{M}_0 \to \mathbb{R}$ of class $C^2$ such that

$$H \circ \phi(y, x) = E + \omega \cdot y + Q(y, x) \quad \text{and} \quad \partial_y^\mu Q(0, \cdot) \equiv 0, \ \forall \mu \in \mathbb{N}_0^d, \ |\mu|_1 \leq 1, \ (B.26)$$

and

$$\det \left( \partial_{yy}^\mu Q(0, \cdot) \right) \neq 0. \quad (B.27)$$

A Hamiltonian $H$ in the form (B.26) is said in Kolmogorov normal form. The Kolmogorov normal form is said non–degenerate if, in addition, the quadratic (in $y$) part $Q$ satisfies (B.27).

References

[Arg20] F. Argentieri. Isolated points of Diophantine sets, Preprint, 2020.

[Arn63] V.I. Arnold. Proof of A.N. Kolmogorov’s theorem on the conservation of conditionally periodic motions with a small variation in the Hamiltonian. Russian Math. Surv, 18(9), 1963.

[BC15] Luca Biasco and Luigi Chierchia. On the measure of Lagrangian invariant tori in nearly-integrable mechanical systems. Atti Accad. Naz. Lincei Rend. Lincei Mat. Appl., 26(4):423–432, 2015.

[BC17] Luca Biasco and Luigi Chierchia. Kam 2017 theory for secondary tori. arXiv (arXiv:1702.06480v1), 2017.

[BC18] L. Biasco and L. Chierchia. Explicit estimates on the measure of primary KAM tori. Annali di Matematica Pura ed Applicata (1923-), 197(1):261–281, 2018.

[BC20] L. Biasco and L. Chierchia. On the topology of nearly-integrable Hamiltonians at simple resonances. Nonlinearity, 33(7):3526–3567, 2020.

[CC95] A. Celletti and L. Chierchia. A constructive theory of Lagrangian tori and computer-assisted applications. In Dynamics reported, pages 60–129. Springer, 1995.

[CG82] L. Chierchia and G. Gallavotti. Smooth prime integrals for quasi-integrable Hamiltonian systems. Nuovo Cimento B (11), 67(2):277–295, 1982.
[Chi86] L. Chierchia. Quasi-periodic Schroedinger operators in one dimension, absolutely continuous spectra, bloch waves, and integrable hamiltonian systems. Technical report, New York Univ., NY (USA), 1986.

[CK19] L. Chierchia and C.E. Koudjinan. V.I. Arnold’s “pointwise” KAM Theorem. *Regular and Chaotic Dynamics*, 24(6):583–606, 2019.

[EG15] L. C. Evans and R. F. Gariepy. *Measure Theory and fine properties of functions*. CRC press, 2015.

[Kou19] C.E. Koudjinan. *Quantitative KAM normal forms and sharp measure estimates*. PhD thesis, Università degli Studi Roma Tre, March 2019. arxiv.org/abs/1904.13062.

[Min70] G. J. Minty. On the extension of Lipschitz, Lipschitz–Hölder continuous, and monotone functions. *Bulletin of the American Mathematical Society*, 76(2):334–339, 1970.

[Nei81] A.I. Neishtadt. Estimates in the Kolmogorov theorem on conservation of conditionally periodic motions. *Journal of Applied Mathematics and Mechanics*, 45(6):766–772, 1981.

[Pös82] J Pöschel. Integrability of hamiltonian systems on cantor sets. *Communications on Pure and Applied Mathematics*, 35(5):653–696, 1982.

[Pös01] J. Pöschel. A Lecture on the Classical KAM Theorem. In *Proc. Symp. Pure Math*, volume 69, pages 707–732, 2001.

[Sal04] D. Salamon. The Kolmogorov–Arnold–Moser Theorem. *Math. Phys. Electron. J*, 10(3):1–37, 2004.

[Ste12] S. Sternberg. *Curvature in Mathematics and Physics*. Courier Corporation, 2012.

[Whi34] H. Whitney. Analytic extensions of differentiable functions defined in closed sets. *Transactions of the American Mathematical Society*, 36(1):63–89, 1934.

[Zeh10] E. Zehnder. *Lectures on dynamical systems: Hamiltonian vector fields and symplectic capacities*, volume 11. European Mathematical Society, 2010.