Scattering of two-level atoms by delta lasers: exactly solvable models in atom optics

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Abstract
We study the scattering of two-level atoms at narrow laser fields, modelled by a \(\delta\)-shape intensity profile. The unique properties of these potentials allow us to give simple analytic solutions for one or two field zones. Several applications are studied: a single \(\delta\)-laser may serve as a model for atom detection and arrival-time measurements, either by means of fluorescence or variations in occupation probabilities. We show that, in principle, this ideal detector can measure the particle density, the quantum mechanical flux, arrival-time distributions or local kinetic energy densities. Moreover, two spatially separated \(\delta\)-lasers are used to investigate quantized-motion effects in Ramsey interferometry.

1. Introduction

In standard scattering theory, \(\delta\)-potentials are useful models to study the properties of scattering solutions, bound states \([1, 2]\) or inverse scattering \([3, 4]\), and to check the validity or illustrate different approximations, concepts or techniques \([5, 6]\). Because of the unique properties of these potentials, analytical solutions are easy to obtain without the use of special functions or extensive calculations. In many cases, \(\delta\)-potentials give a qualitative understanding to more complex scattering systems. For example, the Kronig–Penney model in solid state physics \([7]\), consisting of a lattice of \(\delta\)-functions, is very successful in describing energy gaps for the free electron gas in crystals. Further examples are finite lattices \([8]\), time-dependent \(\delta\)-interactions \([9]\), nonlinear delta interactions \([10]\) or exactly solvable models of few-body and many-body systems\(^3\) \([11, 13]\). The Tonks–Girardeau gas of one-dimensional repulsive bosons subjected to effective \(\delta\)-interactions \([14]\), in particular, has been recently realized experimentally with

\(^3\) For an extensive list of \(\delta\)-function potentials in many-body systems, see \([12]\).
ultracold atoms, which are quite suitable for delta-interaction models because of their large de Broglie wavelengths.

In all the above applications, the particles are formally structureless. Due to the rapid experimental progress in quantum optics and atom optics in the limit of ultracold conditions, there is a recent interest in solving multi-channel scattering problems at localized fields taking into account the internal level structure of atoms. For example, a series of papers has been devoted to the reflection and transmission of slow atoms from micromaser barriers [15]. However, exact solutions for a nonresonant barrier soon become unwieldy, although two-channel recurrence relations may be used to put multiple barrier problems down to the single barrier case [16]. For that reason, the study of \( \delta \)-laser models is extremely useful to understand atom–field interactions on a manageable level when taking into account the atomic quantized motion. A \( \delta \)-laser will not necessarily reproduce a real setup in its full glory, but it provides a valuable tool to check approximations, concepts and ideal limits of operational quantities. A previous analysis of the multi-channel \( \delta \)-potential problem can be found in [17].

In this paper we first derive the stationary solutions for the scattering of two-level atoms by one or two \( \delta \)-laser fields, including a treatment of spontaneous decay. The results are then applied to atom detection, to quantum time measurements and to matter-wave interferometry.

### 2. Interaction between a two-level atom and a \( \delta \)-laser

#### 2.1. Stationary solutions

Let us consider a moving two-level atom with internal states \(|1\rangle = (1 0)^T\) and \(|2\rangle = (0 1)^T\), interacting with a narrow, non-resonant and \( \delta \)-like laser field located at \( x = \xi \). The direction of the laser beam is assumed to be perpendicular to the motion of the atom. Note that in our one-dimensional treatment only the longitudinal motion is quantized but transverse momentum transfer is neglected. These recoil effects have been studied in detail in the context of Ramsey interferometry [18]. Their neglect is reasonable if the atoms are confined in a narrow waveguide in the Lamb–Dicke regime, moving freely only in one direction [16]. The Hamiltonian becomes time independent by using the standard field-adapted interaction picture. To incorporate decay of the upper level \(|2\rangle\) with a decay rate \( \gamma \), we use the quantum jump approach\(^4\) [19, 20]. Here, the evolution of the atom before the first emission of a photon and in the dipole and rotating-wave approximation is given by the non-Hermitian (‘conditional’) Hamiltonian

\[
H_c = \frac{\hat{p}^2}{2m} I_2 + \frac{\hbar u}{2} \delta(\hat{x} - \xi) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} - \frac{\hbar}{2} \begin{pmatrix} 0 & 0 \\ 0 & i \gamma + 2\Delta \end{pmatrix},
\]

where \( \Delta = \omega_L - \omega_{21} \) is the detuning between laser and atomic frequency and \( u \) controls the strength of the \( \delta \)-laser and has dimensions of velocity. An approximate physical realization corresponds to a square laser profile of width \( l \) and Rabi frequency \( \Omega \) such that \( \Omega l = u \), with the \( \delta \)-limit achieved as \( l \to 0 \). In terms of the physical process of atom-light interactions, a plane wave with velocity \( v \) undergoes less than one Rabi oscillation if \( u \ll v \) (semiclassical regime). However, the opposite case \( (u \gg v) \) is strongly quantum and there is no correspondence to the temporal Rabi oscillations for an atom at rest since the replacement \( t \mapsto l/v \) is not valid

\(^4\) The quantum jump approach is essentially equivalent to the Monte Carlo wavefunction approach of Dalibard \textit{et al} [21] and to the quantum trajectories of Carmichael [22].
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Figure 1. Scattering probabilities of a two-level atom (\(^{87}\)Rb, \(m = 1.4 \times 10^{-25}\) kg) at a \(\delta\)-laser, \(|r_{11}'|^2\) (solid line), \(|r_{11}|^2\) (dashed line), \(\frac{1}{2}|r_{12}|^2 = \frac{1}{2}|r_{11}|^2\) (dashed-dotted line). Parameters are \(u = 0.1\) MHz \(\times 10\) \(\mu\)m = 1 m s\(^{-1}\), \(\gamma = 0\), \(\Delta = 100\) Hz.

anymore. The eigenfunctions of \(H_c\) with energy \(E = \hbar^2 k^2/2m = mv^2/2\) for a ground state plane wave incoming from the left are given by

\[
\Phi_k(x) = \begin{cases} 
(e^{ikx} + r_{11}e^{-ikx}) & x \leq \xi \\
(r_{12}'e^{ikx}) & x \geq \xi,
\end{cases}
\]

where \(q^2 = k^2 + m(i\gamma + 2\Delta)/\hbar\) and the scattering amplitudes \(r_{ij}', t_{ij}'\) from channel \(i\) to \(j\) for left incidence are determined by the matching conditions at \(x = \xi\). They read

\[
\begin{align*}
    r_{11}' &= -m^2 u^2 \exp(2ik\xi)/d, \\
    t_{11}' &= 4\hbar^2 kq/d, \\
    r_{12}' &= -2i\hbar mu \exp[i(k + q)\xi]/d, \\
    t_{12}' &= -2i\hbar mu \exp[i(k - q)\xi]/d,
\end{align*}
\]

where \(d = 4\hbar^2 kq + m^2 u^2\). For a later purpose, we note that the corresponding scattering coefficients for an incoming excited wave (first index equals 2) can be obtained simply by exchanging \(k\) and \(q\) in the above expressions and the scattering coefficients for right incidence, \(r_{ij}', t_{ij}'\), are obtained by replacing \(\xi\) by \(-\xi\). According to (3c) and (3d), excitation by scattering at the \(\delta\)-laser is suppressed for slow atoms \((v \ll u/2)\) and for fast atoms \((v \gg u/2)\) (see figure 1), whereas it is easy to show that maximal excitation is achieved if \(|r_{11}'| = |t_{11}'|\).

2.2. Particle detection by spontaneous emission

The quantum description of detection and, in particular, of arrival times is still subject to discussion, see [23] for reviews. In a series of recent papers, the detection process has been modelled by means of the fluorescence from a two-level atom interacting with a spatially confined laser field [20, 24, 25]. This model has led to new insight into the connection between operational quantities (defined for the system in interaction with a measuring apparatus) and
ideal arrival-time distributions (defined for the particle in isolation). It is shown in the following that δ-lasers can be used to model point detectors and that the outcome of such a detector can be related to various ideal quantities by taking appropriate limits. Other point-like detector models with δ-potentials have been studied in [26–28].

We consider a two-level atom in the ground state, incoming from the far left and impinging on a δ-laser localized at $x = \xi$. Before the first spontaneous emission, the atom is described by a conditionally evolved wave packet

$$\langle x | \Psi(t) \rangle = \Psi(x, t) = \int_0^\infty dk \tilde{\psi}(k) e^{-i\hbar k^2 t / 2m} \Phi_k(x).$$

(4)

Here, $\langle k | \psi \rangle = \tilde{\psi}(k)$ denotes the momentum amplitude that the initial ground state wave packet would have at $t = 0$ in the absence of the laser. Note that for $\gamma \neq 0$ the norm of $|\Psi(t)\rangle$ decreases in time and it gives the probability of observing no photon until time $t$. The interaction with the laser field will probably excite the atom and it may emit a photon subsequently. The probability density of the first photon emitted at time $t$ is given by [20]

$$\Pi(t) = -\frac{d}{dt} \|\Psi(t)\|^2 = \gamma \int_{-\infty}^\infty dx |\langle 2 | \Psi(x, t) \rangle|^2.$$  

(5)

To relate this operational quantity to ideal distributions, we consider limiting cases with respect to the parameters $\gamma$, $u$ and $\Delta$. Let us assume zero detuning, $\Delta = 0$, and strong coupling ($u \gg v$) between the atom and the detector to assure immediate detection without delay. This strong coupling regime, however, leads to a decrease in the detection amplitude due to an enhanced fraction of atoms which is reflected off the field in the ground state (see figure 1) and will never emit a photon. To correct this decrease, it is natural and simple to define a normalized detection rate by

$$\Pi_N(t) = \Pi(t) \left( \int_{-\infty}^\infty dt \Pi(t) \right)^{-1}.$$  

(6)

More sophisticated normalization procedures which compensate for the detection losses in a more careful way and preserve the bilinear form properties of the distribution are discussed below.

With respect to $\Pi_N$ we obtain two main results. First, for strong decay, $\hbar \gamma \gg mu^2 \gg mv^2$, the normalized detection rate in leading order becomes

$$\Pi_N(t) \simeq \langle v^{-1} \rangle^{-1} |\psi_{\text{free}}(\xi, t)\rangle^2,$$

(7)

where $\psi_{\text{free}}(x, t) = \int_0^\infty dk \tilde{\psi}(k) \exp(ikx - i\hbar k^2 t / 2m)$ is the freely evolving wave packet and the normalization constant is the mean inverse velocity of this packet. Thus, by means of the fluorescence measurement, one obtains the density of the unperturbed wave packet at the position where the δ-laser is located. With respect to arrival-time measurements, we note that (7) has been shown in [30] to be the zeroth order of an expansion of Kijowski’s arrival-time distribution [31] for nearly monochromatic wave packets, and it has been recently used to study arrival times in the presence of interactions [26].

The second result is obtained in parameter regimes for which $mu^2 \gg \hbar \gamma \gg mv^2$. In that case, the leading order of equation (6) gives

$$\Pi_N(t) \simeq \frac{2}{p_0} \langle \tilde{K}(\xi) \rangle,$$

(8)

where $\tilde{K}(\xi) = \tilde{p}\delta(\tilde{x} - \xi) \hat{p}/2m$ is the local kinetic energy operator at $x = \xi$, the expectation value is taken over the freely evolving packet $\psi_{\text{free}}(t)$ and $p_0$ is its mean momentum. Thus,

5 For a treatment of detection delay, see [20].
a strongly coupled $\delta$-laser provides a way to measure the local kinetic energy density of the particle [30, 32].

A physical picture of the above limits can be given with the help of figure 1. In the regime $u \gg v$ (to the very left of the crossing point), ground state reflection dominates, but the normalization ‘selects’ the small fraction of particles which are reflected or transmitted in the excited state and will emit a photon subsequently. For $mu^2 \ll h\gamma$, this part will be larger than in the case $mu^2 \gg h\gamma$ for a fixed value of $\gamma$ such that the decay of the spatial excited state probability as a function of $x$ is slower.

2.3. Operator normalization

The normalization procedure given in (6) gives more weight to the momenta at which no detection losses (by reflection or transmission) occur. Moreover, it does not provide a distribution which is bilinear in the state. This is in contrast to ideal bilinear quantities such as the quantum mechanical flux or Kijowski’s distribution [31]. To preserve these bilinear form properties, Brunetti and Fredenhagen proposed a normalization formalism ‘on the level of operators’ [33]. Their idea has been recently applied to quantum arrival times [29]. In this section, we will briefly review the basic idea and will present results for the first-photon distribution at $\delta$-lasers.

To begin, we define an operator $\tilde{B} = \tilde{1} - \tilde{\delta} \tilde{S}$, where $\tilde{S}$ is the usual collision operator, relating the incoming asymptotes to the outgoing asymptotes. Since in the presence of decay the potential is purely absorptive, $\tilde{S}$ will not conserve the norm and will not be unitary. In this case, $\tilde{B}$ becomes the positive absorption operator [34] and the expectation value of $\tilde{B}$ with respect to the incoming state provides the total detection probability $1 - \int_{-\infty}^{\infty} dt \Pi_1(t)$.

Now, instead of the incoming ground state $|\psi\rangle|1\rangle$, we consider the new incoming state $|\psi_1\rangle|1\rangle = \tilde{B}^{-1/2}|\psi\rangle|1\rangle$. One can show that the inverse square root of $\tilde{B}$ exists since $|r_{11}(k)|^2 + |r_{12}(k)|^2 < 1$. The action of $\tilde{B}^{-1/2}$ on the incident state gives $\tilde{\psi}_1(k) = (k|\tilde{B}^{-1/2}\psi) = (1 - |r_{11}(k)|^2 - |r_{12}(k)|^2)^{-1/2}\tilde{\psi}(k)$. This transformation (‘filtering’) compensates for losses in the detection probability by amplifying (relative to others) the very slow and very fast momentum components which are preferentially reflected or transmitted in the ground state without emitting a photon [29]. Therefore, the filtered probability distribution $\Pi_1(t)$, obtained in analogy to (4) and (5), is normalized to one and becomes operationally meaningful. This allows us to consider again the strong coupling regime $v \ll u, E \ll h\gamma$ as in the derivation of the density. Interestingly, one obtains in leading order that $\Pi_1(t)$ agrees with the ideal arrival-time distribution of Kijowski [31], $\Pi_K(t)$, for the free incoming state $\tilde{\psi}_{\text{free}}$:

$$\Pi_1(t) \simeq \Pi_K(t) = \frac{1}{m} \langle \hat{p}^{1/2} \delta (x - \xi) \hat{p}^{1/2} \rangle_t, \quad v \ll u, E \ll h\gamma. \tag{9}$$

The form of Kijowski’s arrival-time distribution can be understood as a positive symmetrization of the classical flux. Originally, it was derived for the free motion case only and based on axiomatic arguments. We therefore emphasize the fact that our operational approach based on the normalization procedure above can also be used to generalize $\Pi_K(t)$ for particles moving under the influence of an additional external potential [27].

We close this section by noting that a non-positive version of the operator normalization based on the Rivier symmetrization rule leads to the quantum mechanical flux [29], but an operational understanding of this derivation is difficult. The flux is the classical result for an arrival-time distribution, but its quantum version is not a positive distribution even for particles with only positive momenta because of the backflow effect [35]. A measurement model of the quantum mechanical flux by means of a fluorescence experiment has been given in [20, 25].
2.4. Particle detection by variations in occupation probabilities

In this section we present results for another operational quantity that may be used for particle detection, namely variations in the occupation probabilities. For a two-level atom, let us consider

$$\Pi(t) = \frac{dP_2}{dt} = \frac{d}{dt} \|2|\Psi(t)\|_2^2,$$  \hspace{1cm} (10)

which is measurable by probing the excited state at different times. For vanishing decay, $\gamma = 0$, the time evolution is unitary and with the Hamiltonian (1), $\Pi(t)$ becomes

$$\Pi(t) = u \ Im((1|\Psi(\xi, t)\rangle \langle \Psi(\xi, t)|2)).$$  \hspace{1cm} (11)

Interestingly, the normalized version of this quantity leads to the same ideal distributions as in the fluorescence measurement, but now for different parameter regimes. For $\mu^2 < m^2 \ll \bar{\hbar}/\Delta_1$, we find the density

$$\Pi_N(t) \simeq \langle v^2 - 1 \rangle^{-1} |\psi_{\text{free}}(\xi, t)|^2.$$  \hspace{1cm} (12)

In spite of the weak potential which leads to the dominance of ground state transmission, we emphasize the fact that the motion in this regime cannot be considered semiclassical since the detuning term in the Hamiltonian dominates the kinetic energy.

On the other hand, for $v \ll u$, $E \ll \bar{\hbar}/\Delta_1$ and $\mu u^2 / (\bar{\hbar}/\Delta_1) \sim 1$, we end up with the local kinetic energy density

$$\Pi_N(t) \simeq \frac{2}{p_0} \langle \hat{K}(\xi) \rangle.$$  \hspace{1cm} (13)

As above, both results are with respect to the freely evolving wave packet.

3. Interaction between a two-level atom and two $\delta$-lasers

3.1. Stationary solutions

Let us now consider a two-level atom, interacting with two separated laser fields with $\delta$-shape. This setup provides a toy model to study Ramsey’s method of matter-wave interferometry with separated fields including effects of quantized motion. The conditional Hamiltonian for the atom is given in analogy to (1) by

$$H_c = \frac{\hbar^2}{2m} \mathbf{1}_2 + \frac{\hbar u}{2} [\delta(\hat{x}) + \delta(\hat{x} - L)] \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} - \frac{\hbar}{2} \begin{pmatrix} 0 & 0 \\ 0 & i \gamma + 2\Delta \end{pmatrix},$$  \hspace{1cm} (14)

where $\xi = 0$ has been set for convenience and $L$ is the distance between both lasers. The asymptotic eigenfunctions of $H_c$ for a plane wave incoming from the left and in the ground state are given by

$$\Phi_{\pm}(x) = \begin{cases} e^{ikx} + R_{11} e^{-ikx}, & x \leq 0 \\ R_{12} e^{-iqx}, & x \geq L, \end{cases}$$  \hspace{1cm} (15)

where now the amplitudes have to be determined by the matching conditions at $x = 0$ and $x = L$, and they read

$$R_{11} = -m^2 u^2 [\hbar^2 k q (1 + e^{2ikL} + 2e^{2ikqL}) + m^2 u^2 (e^{2ikL} - 1)(e^{2iqL} - 1)]/D,$$  \hspace{1cm} (16a)

$$R_{12} = -2i \hbar k mu [\hbar^2 k q (1 + e^{2ikL}) + m^2 u^2 (e^{2ikL} - 1)(e^{2iqL} - 1)]/D.$$  \hspace{1cm} (16b)
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\[ T_{11} = 4\hbar^2 k q e^{-ikL}[4\hbar^2 k q e^{ikL} + 2im^2u^2 e^{iqL} \sin(kL)]/D, \quad (16c) \]

\[ T_{12} = -8i\hbar^3 k^2 qmu e^{-iqL}(e^{ikL} + e^{iqL})/D \quad (16d) \]

with the common denominator

\[ D = 16\hbar^4 k^2 q^2 + 8\hbar^2 kqm^2u^2(1 + e^{i(k+q)L}) + m^4u^4(e^{2ikL} - 1)(e^{2iqL} - 1). \quad (17) \]

From (16a)–(17), it can be seen that \( T_{12} \) and \( R_{12} \) tend to zero for \( k \to 0 \) and \( k \to \infty \), respectively. Thus, also for a double-laser setup the excitation probability vanishes for very slow and for very fast particles.

3.2. Ramsey interferometry

Atom interferometry based on Ramsey’s method with separated fields \([36]\) is an important tool of modern precision measurements and the basis of atomic clocks. An essential feature of the observed Ramsey interference fringe is that its width is simply the inverse of the time taken by the atoms to cross the intermediate region. This motivates the use of slow (ultracold) atoms \([37]\). But, if the kinetic energy becomes comparable with the atom–field interaction energy, one has to take into account the quantized centre-of-mass motion of the atom and the well-known semiclassical results have to be corrected \([16]\). A useful toy model to analytically study these effects is provided by the double \( \delta \)-laser setup.

The measured quantity in a Ramsey interferometry experiment is the transmission probability of excited atoms, \( P_{12}(\Delta) \), as a function of the detuning \( \Delta \). This function is easily obtained in the semiclassical regime where \( E \gg mu^2, E \gg \hbar \Delta \), and the centre-of-mass motion can be treated independently of the internal dynamics. In this regime, \( P_{12}(\Delta) \) has been derived in \([36]\) for rectangular field shapes. Applying the \( \delta \)-limit to this expression yields

\[ P_{12}^{\text{scl}}(\Delta) = 4\sin^2 \left( \frac{u}{2v} \right) \cos^2 \left( \frac{u}{2v} \right) \cos^2 \left( \frac{\Delta L}{2v} \right). \quad (18) \]

From (18) we find the Ramsey fringes to be of \( \cos^2 \) shape and their width to be \( 2\pi/T \), where \( T = L/v \) is the semiclassical crossing time of the intermediate region. Note that in contrast to rectangular field shapes, \( P_{12}^{\text{scl}} \) does not drop to zero for large detuning.

However, if the kinetic energy of the atom is comparable with the interaction energy, the semiclassical approach is not valid anymore and a full quantum mechanical solution is required. In that case the excitation probability is given in terms of the scattering amplitudes derived above, where \( \gamma = 0 \) is assumed in the following. Since the outgoing excited state component becomes evanescent for \( \Delta \) smaller than the critical value \( \Delta_{\text{cr}} = -\hbar k^2/2m \) due to the wavenumber \( q \), we finally obtain the quantum result to be

\[ P_{12}(\Delta) = \frac{q}{k} |T_{12}|^2 \quad \text{for} \quad \Delta > \Delta_{\text{cr}} \quad (19) \]

and zero elsewhere \([16]\). From this one can study the transition from the semiclassical to the quantum regime.

For \( v \to \infty \), (19) reduces to \( P_{12} = u^2v^{-2}\cos^2[\Delta L/(2v)] \) which agrees in leading order of \( u/v \) with the semiclassical expression (18), as expected. A numerical example is shown in figure 2(a). For slower and slower atoms, \( P_{12} \) differs from \( P_{12}^{\text{scl}} \) and changes its behaviour from interference fringes to scattering resonances, see figures 2(b)–(d). The reason for this is the increasing dominance of quantum reflections at the field that have been neglected in the semiclassical description. For kinetic energies of the order of the interaction energies, the maxima of the quantum fringes are slightly displaced with respect to the semiclassical result.
which may become important as a systematic frequency uncertainty in future time standards using ultracold atoms (figure 2(b)).

In the extreme case, \( u \gg v \), the double \( \delta \)-laser potential acts as a Fabry–Perot matter-wave interferometer (figure 2(d)). Taking into account the low-velocity behaviour of the \( \delta \)-laser scattering coefficients, \( t_{ii} \sim v^2/u^2, t_{ij} \sim v/u, r_{ij} \sim -1, r_{ij} \sim u/v \), the two-channel recurrence relations and the general expression for \( T_{12} \) derived in [16] may be used to show that in leading order \( (v^3/u^3) \), one has

\[
T_{12} \simeq \frac{t_{12}^i t_{22}^i}{1 - r_{22}^i t_{22}^i} + \frac{t_{11}^i t_{12}^i}{1 - r_{11}^i t_{11}^i} \left( 1 - r_{22}^i t_{22}^i \right) + \frac{t_{12}^i t_{12}^i}{1 - r_{11}^i t_{11}^i} + \frac{t_{12}^i t_{22}^i}{1 - r_{22}^i t_{22}^i},
\]

where the tilde denotes the scattering coefficients for the second \( \delta \)-potential at \( x = L \). Note the difference to the corresponding expression of a double-barrier potential given in [16], due to the non-generic properties of the \( \delta \)-potential scattering amplitudes.

A clear physical interpretation of equation (20) can be given since it describes the possible multiple-scattering paths through two \( \delta \)-lasers for \( u \gg v \). The four families of paths corresponding to the four terms in equation (20) are schematically shown in figure 3. Only the denominator \( 1 - r_{22}^i t_{22}^i \) leads to resonances with respect to \( \Delta \). Thus, as in [16] one may estimate the positions \( \Delta_n \) and the corresponding widths \( w_n \) of the resonances to obtain

\[
\Delta_n = \Delta_{cr} + \frac{\hbar}{2m} \left( \frac{n\pi}{L} \right)^2,
\]
4. Discussion

We have analytically investigated the one-dimensional scattering of two-level atoms at one and two δ-laser fields. The relatively simple expressions for the scattering amplitudes make this a valuable toy model to study applications of atom optics in the ultracold regime where the kinetic energy is comparable with the atom–field interaction energy.

The main objective of our analysis is to provide more insight into concepts and ideal limits of models in atom optics. A physical realization of the δ-laser, of course, might be difficult since this requires the de Broglie wavelength of the atom to be larger than the laser beam width and $\hbar/\Omega_1$ to be larger than the kinetic energy. However, with subrecoil cooling techniques and translational temperatures of the order of nK the corresponding de Broglie wavelength is of the order of $\mu$m which is not far away from possible laser beam waists.

We have shown that a single δ-laser may serve as a particle detector either through fluorescence measurements or through variations in the atomic occupation probabilities. The outcome of such a detector has been related to several ideal quantities, such as the probability density, the flux, Kijowski’s arrival-time distribution or local kinetic energy densities, by considering different parameter regimes and normalization procedures. These relations are of fundamental importance, since the antithetic questions ‘How to measure ideal quantities?’ and ‘What quantities an experiment really measures?’ are only insufficiently answered for quantum detection processes and quantum time measurements so far.

Moreover, by means of the double δ-laser setup we have studied Ramsey interferometry including quantum reflections at the fields. We obtained an exact quantum mechanical solution for the interference fringes that agrees in the weak coupling regime with the well-known expression of Ramsey, but shows deviations for stronger coupling or slower atoms. A peak shift of the central fringe at $\Delta = 0$ occurs which should be considered as a systematic frequency uncertainty in future atomic clocks with ultraslow atoms. For the extreme case that the atomic kinetic energy is much smaller than the atom–field interaction energy, the standard fringes are completely suppressed and the excitation probability $P_{12}(\Delta)$ exhibits a Fabry–Perot resonance structure as a function of $\Delta$. The characteristic features of these resonances have been quantified.

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References

[1] Morse P M and Feshbach H 1953 Methods of Theoretical Physics (New York: McGraw-Hill) p 1644
[2] Gottfried K and Tung-Mow Y 2003 Quantum Mechanics: Fundamentals (New York: Springer) p 202
[3] Chand K and Sabatier P 1989 Inverse Problems in Quantum Scattering Theory (New York: Springer)
[4] Lamb G 1980 Elements of Soliton Theory (New York: Wiley)
[5] Muga J G, Delgado V and Snider R F 1995 Phys. Rev. B 52 16381
[6] Muga J G, Brouard S and Snider R F 1992 Phys. Rev. A 46 6075
[7] Berezin A A 1989 J. Phys. A: Math. Gen. 22 67
[8] Sprung D W L, Wu H and Martorell J 1993 Phys. Rev. A 46 6075
[9] Berezin A A 1986 Phys. Rev. B 33 2122
[10] Molina M I and Bustamante C A 2002 Am. J. Phys. 70 67
[11] Takahashi M 1999 Thermodynamics of One-Dimensional Solvable Models (Cambridge: Cambridge University Press) p 10
[12] Lapidus I R 1983 Am. J. Phys. 51 1036
[13] Muga J G and Snider R F 1998 Phys. Rev. A 57 3317
[14] Girardeau M 1960 J. Math. Phys. 1 516
[15] Englert B G, Schwinger J, Barut A O and Scully M O 1991 Europhys. Lett. 14 25
[16] Muga J G and Snider R F 1998 Phys. Rev. A 57 3317
[17] Hegerfeldt G C and Wilser T S 1991 Classical and quantum systems Proc. 2nd Int. Wigner Symp. ed H D Doebner, W Scherer and F Schroeck (Singapore: World Scientific) p 104
[18] Muga J G, Sala R and Egusquiza I L (ed) 2002 Time in Quantum Mechanics (Berlin: Springer) p 233
[19] Muga J G, Sala R and Egusquiza I L (ed) 2002 Time in Quantum Mechanics (Berlin: Springer) p 233
[20] Bordé C J 2001 C. R. Acad. Sci., Paris IV 2 509
[21] Hegerfeldt G C and Wässer T S 1991 Classical and quantum systems Proc. 2nd Int. Wigner Symp. ed H D Doebner, W Scherer and F Schroeck (Singapore: World Scientific) p 104
[22] Hegerfeldt G C 1993 Phys. Rev. A 47 449
[23] Hegerfeldt G C 2003 Irreversible Quantum Dynamics Springer LNP 622 ed F Benatti and R Floreanini (Berlin: Springer) p 233
[24] Aoki K, Horikoshi A and Nakamura E 2000 Phys. Rev. A 62 052104
[25] Dalibard J, Castin Y and Molmer K 1992 Phys. Rev. Lett. 68 580
[26] Artemov E, Boshier J D and Hegerfeldt G C 2000 Phys. Rev. A 62 052104
[27] Bondurant R S 2004 Phys. Rev. A 68 062104
[28] Muga J G, Seidel D and Hegerfeldt G C 2003 Phys. Rev. A 68 022111
[29] Muga J G, Seidel D and Hegerfeldt G C 2005 J. Phys. B: At. Mol. Opt. Phys. 38 535
[30] Kijowski J 1974 Rep. Math. Phys. 6 361
[31] Ayers P W, Parr R G and Nagy A 2002 Int. J. Quantum Chem. 90 309
[32] Brunetti R and Fredenhagen K 2002 Phys. Rev. A 66 044101
[33] Muga J G, Palao J P, Navarro B and Egusquiza I L 2004 Phys. Rep. 395 357
[34] Bracken A J and Melloy G F 1994 J. Phys. A: Math. Gen. 27 2197
[35] Ramsey N F 1950 Phys. Rev. 78 695
[36] Salomon C et al 2001 C. R. Acad. Sci., Paris IV 2 1313