Shape and soft functions of HQET and SCET in the ’t Hooft Model

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The main application of Heavy Quark Effective Theory (HQET) and of Soft Collinear Effective Theory (SCET) is in establishing factorization theorems for exclusive and semi-inclusive decays of heavy mesons. However, the calculation of the soft factors from the HQET or SCET factorization relations is, as usual, impeded by the non-perturbative aspect of the strong interactions. In the hope of gaining some insights into some of these quantities we compute them in the ’t Hooft model. We find that the $B$-meson shape function is exactly given by the square of the $B$-meson light-cone wave-function. The structure of the $B \to \pi$ structure function is more complicated: it is given by the product of wave-functions or by a resonant sum depending on the kinematics. The result simplifies dramatically in the chiral limit, where it can be compared with general arguments based on Heavy Meson-Chiral Perturbation theory. No attempt is made to use these results for applications to phenomenology.

I. INTRODUCTION

One of the most celebrated applications of Heavy Quark Effective Theory (HQET) is to the calculation of the rate for the semileptonic decay of a $B$-meson. It is found that the differential decay rate (with respect to a single kinematic variable) is systematically predicted as an expansion in inverse powers of the heavy quark mass, $1/m_b$, with the leading term completely fixed by symmetry, and hence calculable. In the case of charmless semileptonic $B$-meson decays the systematic expansion breaks down in the “end-point region,” namely, when the leptonic energy is as large as it can be. For $x \equiv 2E/m_B \approx 1 - (\Lambda/m_B)^2$ convergence is hopeless, while for $1 - x$ of order unity the expansion is very good. There is a region, $1 - x \approx \Lambda/m_B$, however, for which all orders in the HQET expansion are equally large. The most singular terms in the expansion can be formally re-summed into a so-called “shape-function” $n$.

Since the shape function involves the sum of an infinite number of terms, each characterized by an unknown constant (the matrix element of a local operator between $B$-meson states), it is largely unknown. Some basic properties can be readily derived (e.g., normalization follows from current conservation), but in practice this is far from sufficient to determine it with the precision required for applications. The favored solution to this impasse is to show that the shape-function is approximately universal and then (approximately) cancel it from measurements of, say, charmless semileptonic and radiative $B$-decays.

In this short note we propose to compute the shape function in a the large $N_c$-limit of QCD in $1 + 1$ dimensions, the ’t Hooft model. While the model can not be used to compute parameters of phenomenological interest, it has been extensively studied in the past and has given remarkable insights into the dynamics of the strong interactions. In the context of heavy mesons, calculations in the ’t Hooft model gave some of the earliest indications that the heavy quark expansion converges rather more quickly for some quantities than for others: relations among form factors for $B \to D\ell\nu$ and the predicted normalization of these form factors at zero recoil hold more accurately than relations among heavy meson decay constants. Some results are interesting but are unlikely to have a counterpart in four dimensions: in Ref. it was shown that in the chiral limit the form factors for the decay $B \to \pi\ell\nu$ are exactly given by a single pole formula (and the residue is fixed by symmetry considerations). And sometimes the results have served to raise warning flags about possibly unjustified assumptions in the four-dimensional analysis. For example, in contrast to what has been argued informally, a numerical solution to the model shows that the $1/m_b$ expansion of the lifetime of the $B$-meson contains corrections of first order in the expansion parameter, that is, $(1/m_b)^n$ with $n = 1$. To be sure, this result is controversial. It is trivial to show analytically that the $(1/m_b)^n$, $n = 1$ corrections are absent in the chiral limit, but attempts to extend this result to the non-chiral limit (where the result of Ref. applies) are far from rigorous. Moreover, the numerical result in is consistent with the theoretical observations in Ref. that the $1/m_b$ expansion for the smeared $B$-width has no $(1/m_b)^1$ term. In sum, the ’t Hooft model is a suitable tool for testing proposed methods and mechanisms in 4-dimensional QCD that should just as well apply in 2-dimensional QCD, but it is not likely a good model for phenomenological applications.

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There is another quantity of interest we will compute in this note. The shape-function can be described as the expectation value in a $B$-meson state of a non-local b-quark bilinear operator, with a Wilson line between quarks to ensure gauge independence. A similar quantity of interest is the matrix element between different states, say, a $B$-meson and a $\pi$-meson, of a non-local quark bilinear (with properly chosen flavor quantum numbers). This generalized parton distribution (GPD) plays a central role in the recently established soft pion theorems for certain soft factors in the Soft Collinear Effective Theory (SCET)\cite{21,22,23,24,25,26,27,28,29,30,31,32} description of exclusive $B$ decays to final states containing, possibly, several soft pions. This, in turn, is useful in extending the applicability of SCET to the calculation of non-resonant final states, as in, for example, $B \to K\pi\ell\nu$ or $B \to K\pi\gamma$\cite{33,34,35}.

The paper is organized as follows. We begin in section II with a review of the ‘t Hooft model. This is intended to establish notation and to review some techniques, but it is not recommended as a place to learn about the model, which can be best done by going to the original sources\cite{36,37,38,39}. Section III shows that the ‘t Hooft wave-functions for a meson are, up to normalization, the light-cone wave-functions for that meson. This is an old result, but we use it here as as simple example of the calculational techniques used in Secs. IV and V where the shape and soft functions are computed, respectively. In Sec. V we endeavor to determine the chiral limit of the shape functions obtained for general values of the quark masses. This requires some knowledge of Heavy-Light form factors, and of how the chiral limit is approached for these form factors, so we have included a review of those results as the first sub-section in Sec. V and we have separated the chiral limit of the soft functions into a separate sub-section as well.

Section VII is the most interesting: we discuss the interpretation of the results just obtained and compare them to the literature. We find that the method for computing soft functions proposed in Ref. \cite{19} is not quite correct. Moreover, we propose a simple (but incomplete) fix. And we find that the shape function is simply the square of the light-cone wave-function. These two results are tightly connected in the ‘t Hooft model and, as we remark in the concluding section VII it would be interesting to determine if this connection persists in 4-dimensional QCD. This would lead to interesting phenomenology. Finally, some technical issues, concerning scaling functions, have been left to an appendix.

## II. REVIEW OF THE MODEL

This model has been extensively studied and our work relies on technology pioneered by ‘t Hooft\cite{30}, Callan, Coote, and Gross\cite{37}, and Einhorn\cite{38}. In these papers the bound state equations were derived: and it was shown that the scattering amplitudes—and the form factor in particular—can be written entirely in terms of interactions among the meson bound states, with no quarks in the spectrum or in the singularity structure of the amplitudes.

We recall the features of the model which make it solvable, and refer the reader to the original papers for details. The dynamics are defined by the Lagrangian,

\[ \mathcal{L} = -\frac{1}{4} \text{Tr} F_{\mu\nu} F_{\mu\nu} + \sum_a \bar{\psi}_a (\gamma^\mu (i\partial_\mu - g_0 A_\mu) - m_a) \psi_a, \]

where $A_\mu$ is an $SU(N_c)$ gauge field, $F_{\mu\nu}$ is its field strength and $\psi_a$ is a Dirac fermion of mass $m_a$. In the large-$N_c$ limit, the gauge coupling is scaled with $N_c$: $g^2 = g_0^2 N_c$ is held fixed as $N_c \to \infty$. The label $a$ runs over flavors of quark, with bare masses $m_a$.

The theory is most conveniently quantized in light-cone gauge. Because there are no transverse dimensions, setting $A_- = 0$ eliminates the gluon self-coupling. It also serves to project gamma matrices onto a single component in any Feynman graph that has just gluon vertices and (-) component current insertions on fermion lines. In this gauge the gluon propagator is the inverse of $\partial^2$. The infrared divergence in the gluon propagator, $i/k^2$, is regulated by taking the principal value at the pole.

Spinors are conveniently split into L and R components,

\[ \psi_R \equiv P_R \psi = \frac{1}{2} \gamma^+ \gamma_- \psi, \quad \psi_L \equiv P_L \psi = \frac{1}{2} \gamma^- \gamma_+ \psi, \]

and in $A_- = 0$ gauge $\psi_L$ is not independent,

\[ \psi_L = -\frac{i m}{2} \gamma_- \psi_R. \]

The leading term of the $1/N_c$ expansion is the sum of planar graphs. The equations for the full propagator and self-energy can be solved exactly, with an extremely simple result: the net effect of all gluons starting and ending on the same fermion line is just to renormalize the quark mass appearing in the propagator,

\[ m_a^2 \to \tilde{m}_a^2 = m_a^2 - g^2/\pi, \]

and in $\partial^2$. The infrared divergence in the gluon propagator, $i/k^2$, is regulated by taking the principal value at the pole.

Spinors are conveniently split into L and R components,
so the full quark propagator is

$$S^{(a)}(k) = \frac{i k_-}{k^2 - \tilde{m}_a^2 + i \epsilon}.$$  (5)

After making this shift, all remaining gluon interactions enter as ladder-type exchanges. Crossings would require either gluon self-couplings, which are absent, or non-planar graphs, which are higher order in $1/N_c$. The Yang-Mills coupling constant $g$ has dimensions of mass, and we choose units such that $g^2/\pi = 1$, leaving $\tilde{m}_a^2$ as the dimensionless numbers parametrizing the theory.

The full meson-quark vertex, $\Gamma(p, k)$ (see Fig. 1), satisfies a Bethe-Salpeter-like equation: a gluon exchange between incoming and outgoing quarks leads to

$$\Gamma^{(12)}(p, k) - 1 = \frac{i}{\pi} \int \frac{d^2 q}{(q_+ - k_-)^2} S^{(1)}(q) \Gamma^{(12)}(p, q) S^{(2)}(q - p).$$  (6)

Here and below, the mark through the integral sign reminds us to take the principal value of the integrand. Eq. (6) shows that $\Gamma(p, k)$ only depends on $k$ through the variable $x = k_- / p_-$, so we define $\Gamma(p^2, x) = \Gamma(p, k)$. The equation for the vertex function is solved in terms of the solutions of the associated eigenvalue problem, which has the physical interpretation of the bound state equation:

$$\mu_n^2 \phi_n(x) = \left( \frac{\tilde{m}_1^2}{x} + \frac{\tilde{m}_2^2}{1 - x} \right) \phi_n(x) - \int_0^1 \frac{dy}{(y - x)^2} \phi_n(y).$$  (7)

The function $\phi_n(x)$ is commonly referred to as the 't Hooft wave-function of the $n^{th}$ eigenstate, with mass $\mu_n$, and $x = k_- / p_-$ is the fraction of light-cone momentum carried by the out-going quark. The out-going and in-going quarks may have different flavor, and hence their masses $m_1$ and $m_2$ are in general distinct. When needed we will explicitly denote the wave-function by the flavor of the quarks, as in $\phi_n^{(12)}$. It is conventional to choose a standard normalization, $\int_0^1 dx \phi_n(x) \phi_m(x) = \delta_{nm}$. The $\phi_n(x)$ vanish at the boundaries, and consistency of (7) requires that as $x \to 0$, $\phi_n(x) \sim x^{\beta_1}$, with

$$\pi \beta_1 \cot \pi \beta_1 = 1 - m_1^2,$$  (8)

and similarly with the replacements $m_1 \to m_2$ and $x \to 1 - x$ as $x \to 1$, as dictated by the boundary behavior of the Hilbert transform. The bound state equation does not have solutions in terms of known functions, but may be readily solved numerically.

The range of $x$ for the bound states is always in the interval $[0, 1]$, and $\phi_n = 0$ outside of this range; but determines as well the full meson-fermion-antifermion vertex,

$$\Phi_n(z) = \int_0^1 \frac{dy}{(y - z)^2} \phi_n(y),$$  (9)

for values of $z \notin [0, 1]$. This includes $z$ complex, corresponding to the general case where one or more of the lines of the meson-quark vertex is not on-shell in its physical region. $\Phi_n(z)$ is analytic in the complex plane, with a cut on the real axis from 0 to 1. When $x \in [0, 1]$, $\Phi_n(x)$ is defined by the principal value prescription, and

$$\Phi_n(x) = \left( -\mu_n^2 + \frac{\tilde{m}_1^2}{x} + \frac{\tilde{m}_2^2}{1 - x} \right) \phi_n(x),$$  (10)
in accordance with (7). Since $\phi_n(x)$ is finite, $\Phi_n(x)$ has zeros where the first factor on the right vanishes, and these are the values $x_{\pm}$ where the quarks would be on-shell. These zeros of the vertex function cancel quark poles in the propagators of loop amplitudes to ensure that no quark singularities appear in gauge-invariant Green functions.

All loop integrations are simplified by the fact that $\phi_n(x)$ is a function of $x = k_-/p_-$ only and is independent of $p_+$. When wave functions and propagators appear in loop integrals, only the latter depend on $p_+$, so the $\int dp_+$ is over rational functions and can be computed explicitly by contour integration, leaving a single integral over one real variable.

The full meson-quark vertex is now easily expressed in terms of the eigenfunctions,

$$\Gamma(p^2, x) = 1 - \sum_n \frac{c_n \phi_n(x)}{p^2 - \mu_n^2},$$

(11)

where

$$c_n = \int_0^1 dx \phi_n(x).$$

(12)

In the interval $0 \leq x \leq 1$ a handy alternative expression is available:

$$\Gamma(p^2, x) = \left(p^2 - \frac{\tilde{m}_1^2}{x} - \frac{\tilde{m}_2^2}{1-x}\right) \sum_n \frac{c_n \phi_n(x)}{p^2 - \mu_n^2}.$$  

(13)

The four quark connected scattering amplitude can be similarly obtained:

$$T = -\frac{i g^2}{2s^2} \left(s^2 - \frac{\tilde{m}_1^2}{x} - \frac{\tilde{m}_2^2}{1-x}\right) \sum_n \frac{\phi_n(x) \Phi_n^*(z)}{s^2 - \mu_n^2.}$$

(14)

Here the kinematics is as indicated in Fig. 2 and we have defined $s = p - k$, $x = r_-/s_-$ and $z = p_-/s_-.$

Below we will compare some of our results with those obtained from general considerations using heavy quark and chiral symmetries. It is therefore useful to present the ’t Hooft wave-functions in the limiting case of one heavy quark (and a light anti-quark). In heavy quark effective theory (HQET) Green functions are expanded about the large momentum of the heavy quark. If the momentum of the heavy meson of mass $\mu_n$ is $p$, then $v = p/\mu_n$ is its velocity. We take $m_1$ to be the heavy quark mass, $m_1 \gg 1$ while keeping $m_2$ fixed, and write $p = m_1 v + \hat{p}$ and $k = m_1 v + \hat{k}$; see Fig. 1. Then the argument of the ’t Hooft wave-function is $x = k_-/p_- = 1 + (\hat{k}_- - \hat{p}_-)/m_1 v_- + \cdots$ and the large mass limit is obtained by writing a function of $\hat{k}_- = (1 - x)m_1 v_-:

$$\psi_n(\hat{k}_-) \equiv \phi_n(1 - k_-/m_1 v_-)$$

(15)

(we have dropped the tilde since $\hat{k}_-$ is a dummy variable). Note that $\hat{k}_- - \hat{p}_- = k_- - p_-$ is the light quark momentum going into the graph. The restriction $x = k_-/p_- \leq 1$ is then the same as the statement that the wave-fucntion is non-vanishing only if the light quark momentum points out (when the heavy meson momentum points in).

Making this change of variables, writing $\mu_n = m_1 + \Lambda_n$ and taking the $m_1 \to \infty$ limit, the ’t Hooft wave-function takes the form

$$2\Lambda_n \psi(\hat{k}_-) = \left(\frac{k_-}{v_-} + \frac{\tilde{m}_2 v_-}{\hat{k}_-}\right) \psi_n(\hat{k}_-) - v_- \int_0^\infty \frac{dq_-}{(q_- - k_-)^2} \psi_n(q_-)$$

(16)

It is easy to check that this is the equation one obtains starting directly from the HQET Lagrangian [12].
III. LIGHT-CONE WAVE-FUNCTION

Define the light cone wave-functions for a meson $M_n$ by

$$\tilde{\phi}_\pm (\xi) = -\frac{i}{f_n} \int \frac{dx^-}{2\pi} e^{-i(1-\xi)x^- p^-} \langle 0|W_\pm (x^-)|M_n(p)\rangle$$

(17)

where the gauge invariant non-local operators $W_\pm$, defined by

$$W_\pm (x) \equiv \tilde{\psi}(x) P [e^i \int_x A^I] \gamma_\pm \gamma_5 \psi(0)$$

(18)

have been evaluated along the $x^+ = 0$ line. In the definition of $W_\pm$, $\int A = \int dy^\mu A_\mu (y)$, so in our computations, which are performed in light cone gauge, this is $\int dy^+ A^-$. The decay constant $f_n$ is defined through

$$\langle 0|W_\mu (0)|M_n(p)\rangle = \begin{cases} 
  ip_\mu f_n & \text{if } n \text{ is even (odd parity)}, \\
  i\epsilon_{\mu\nu} p^\nu f_n & \text{if } n \text{ is odd (even parity)}.
\end{cases}$$

(19)

The antisymmetric two index tensor $\epsilon_{\mu\nu}$ makes two dimensions special: it allows us to use an axial current as interpolating field for scalar mesons (and, similarly, to use a vector current for pseudo-scalar mesons). It is an easy exercise to compute the two-current correlator from which it follows that the decay constant is related to the integral of the wave-function in Eq. (18)

$$f_n = \sqrt{\frac{N_c}{4\pi}} c_n$$

(20)

The light-cone wave-function $\tilde{\phi}_-$ is given, up to normalization, by the 't Hooft wave-function. It is useful to review the derivation of this fact since the procedure is the same as in the more involved calculation of other soft functions below. Consider

$$\int d^2 y \ e^{-ip y} \langle 0|T[W(x^-)\tilde{\psi}(y)]|0\rangle.$$ 

(21)

In light-cone gauge this is really a three point function. To see this, take the path in the integral in Eq. (18) to be a straight line of $y^+$ constant, from $y^- = 0$ to $y^- = x^-$. After some spinor algebra, we have

$$W_-(x^-) = \psi_R^\dagger (x^-) \psi_R (0),$$

$$W_+(x^-) = \psi_L^\dagger (x^-) \psi_L (0).$$

(22)

Therefore, the correlator in Eq. (21) is related to the diagram in Fig. 1 by Fourier transformations. In order to perform the computation a simple trick proves useful. From the three point function

$$\Gamma (p, k) = \int d^2 x \ d^2 y \ e^{-ip x + i(k-p) y} \langle 0|T[\psi_R^\dagger (y) \psi_R (0) \psi_L^\dagger (x) \psi_L (0)]|0\rangle,$$

(23)

which is a non-gauge independent quantity given by Eq. (11), determine

$$\int d^2 y \ e^{i(k-p) y} \langle 0|T[\psi_R^\dagger (y) \psi_R (0)]|M_n(p)\rangle,$$

(24)

by extracting (and properly normalizing) the residue of the pole for the $n$-th meson state $|M_n(p)\rangle$. At this point one can Fourier transform back to $x$ space by integrating over $\int d^2 k \ e^{-ik x}$ and obtain $\langle 0|W(x^-)|M_n(p)\rangle$ by setting $x^+ = 0$. This is the gauge invariant integrand of the definition of the light-cone wave-function in Eq. (17). The light-cone wave-function is the the Fourier transform of this with respect to $x^-$ (which undoes one of the previous Fourier transforms).

Let’s see how this works, explicitly. Extracting the pole, expression (23) is

$$\int d^2 y \ e^{i(k-p) y} \langle 0|T[\psi_R^\dagger (y) \psi_R (0)]|M_n(p)\rangle = \sqrt{4\pi N_c} \left| f_n(\xi) \right| \frac{k_- - p_-}{k^2 - m_1^2 + i\epsilon (k - p)^2 - m_2^2 + i\epsilon}$$

(25)
where \( \xi = k_-/p_- \). Following the strategy outlined above, we invert the Fourier transform and set \( y^+ = 0 \),

\[
\langle 0|T[\psi_R^+(y^-)\psi_R(0)]|n\rangle = \int \frac{d^2k}{(2\pi)^2} e^{-i(k_- - p_-)y^-} \frac{4\pi N_c}{p_-} \Phi_n(\xi) \frac{k_-}{k^2 - \tilde{m}_+^2 + i\epsilon} \frac{k_- - p_-}{k^2 - \tilde{m}_+^2 + i\epsilon}
\]

(26)

The integral over \( k_+ \) can be performed explicitly, by appropriately choosing a semicircle at infinity in the complex \( k_+ \) plane to close the contour of integration,

\[
\langle 0|T[\psi_R^+(y^-)\psi_R(0)]|n\rangle = i\sqrt{\pi N_c} \int_0^{\mu_-} \frac{dk_-}{2\pi} e^{-i(k_- - p_-)y^-} \Phi_n(\xi) \frac{1}{\mu_-^2 - \tilde{m}_+^2 + i\xi - \mu_-^2 + i\xi}
\]

(27)

Since the argument of \( \Phi_n \) is in the unit interval we can use Eq. (10) to relate it to the 't Hooft wave-function,

\[
\langle 0|T[\psi_R^+(y^-)\psi_R(0)]|n\rangle = -ip_+ \sqrt{\frac{N_c}{4\pi}} \int_0^1 d\xi e^{i(1-\xi)p_-y^-} \phi_n(\xi)
\]

(28)

The coefficient in front is related to the decay constant by Eq. (20), and the integral over \( \xi \) is undone by the Fourier transform that defines the light-cone wave function, Eq. (17). The result is zero for \( \xi < 0 \) or \( \xi > 1 \) and, in the unit interval,

\[
\tilde{\phi}_n(\xi) = \phi_n(\xi)/c_n
\]

(29)

As a fairly trivial check of the calculation, note that \( \int_0^1 d\xi \tilde{\phi}_n(\xi) = 1 \). As advertised, up to the normalization factor, the light-cone wave-function is the 't Hooft wave-function. This, of course, is a well known fact, but the method exemplified here with this calculation is precisely what we will use to calculate the shape and soft functions.

The computation of the light-cone wave-function \( \tilde{\phi}_n \) proceeds in an entirely analogous manner. It follows from Eq. (3) that the result is obtained by including in the computation above an additional factor of \( m_1 m_2/4p_1 p_- \), where \( p_i \) are the quark momenta. Therefore,

\[
\tilde{\phi}_n(\xi) = -\frac{m_1 m_2}{2p_+ \xi(1-\xi)} \phi_n(\xi)/c_n
\]

(30)

Again we can check this by computing its integral. We need the useful relation

\[
\int_0^1 dx \phi_n(x) \frac{m_1 m_2}{x(1-x)} = (-1)^n \mu_n^2 c_n
\]

(31)

from which it follows that

\[
\int_0^1 d\xi \tilde{\phi}_n(\xi) = (-1)^{n+1} \frac{\mu_n^2}{2p_+^2} = (-1)^{n+1} \frac{p^+}{p_-}
\]

(32)

On the other hand, the integral of Eq. (17) gives

\[
\int_0^1 d\xi \tilde{\phi}_n(\xi) = -i \frac{1}{p_-} \int_0^1 d\xi \langle 0|W_+(0)|M_n(p)\rangle
\]

(33)

which agrees with Eq. (32) after use of Eq. (13).

IV. SHAPE FUNCTIONS

The shape functions are defined by

\[
f^n_\nu(\xi) = \frac{1}{4\pi} \int dx^- e^{-i\xi x^- p_-} \langle B_n(p) | \bar{b}(x) P[\epsilon^+ \bar{c} d] \Gamma b(0) | B_n(p) \rangle \bigg|_{x^+ = 0}
\]

(34)

The notation suggests the meson, \( B \), has a heavy \( b \)-quark and a light \( u \)-quark as constituents. In fact, our computation is valid for arbitrary masses of the quarks, but the notation was adopted since it is in the context of heavy mesons that shape functions often arise in practice.
It is convenient to specialize to the case $\Gamma = \gamma_-$. Then, using $\mu = -$ vector-currents to interpolate for the mesons, we are led to consider a correlator of right-handed quark fields,

$$
\int d^2x d^2y d^2z \ e^{ip(x-z)+i ky} \langle 0| T[b_R^\dagger(y)b_R(0)b_R^\dagger(x)u_R(x)u_R^\dagger(z)]b_R(z)|0\rangle.
$$

(35)

As in the previous section, this quantity can be readily computed but is not gauge-invariant. However, undoing the Fourier transform over $y$ and specializing to the line $y^+ = 0$ does give a gauge independent quantity.

Two distinct contributions to (35) are shown in Fig. 3. It is easy to check that the second graph, involving the full scattering kernel, gives a vanishing contribution to the shape function. The first graph gives

$$
N_n \Gamma^{(12)}(p, k)\Gamma^{(21)}(-p, k-p)S^{(1)}(k)S^{(2)}(k-p).
$$

(36)

We have indicated with superscript the quark flavors in these functions, with $i = 1, 2$ denoting $q = b, u$, respectively. For the vertex function the superscript (12) indicates an outgoing $b$-quark and an incoming $u$-quark, and we adopt the same notation for other related quantities, e.g., the ’t Hooft wave-functions. Applying LSZ reduction and integrating over $k_+$ this gives

$$
\phi_n^{(12)}(\xi)\phi_n^{(21)}(1-\xi)\theta(\xi(1-\xi)),
$$

(37)

where $\xi = k_-/p_-$. We have kept the subscript $n$ arbitrary, indicating that the result is more general than we set out to obtain, namely, it is valid for the shape function of any of the states in the tower of which the $B$ meson is the ground state (corresponding to $n = 0$). Using $\phi_n^{(21)}(1-\xi) = \phi_n^{(12)}(\xi)$ we finally have

$$
f^{(n)}_-(\xi) = (\phi_n^{(12)}(\xi))^2
$$

(38)

where it is understood the support is for $0 \leq \xi \leq 1$ and the subscript in $f$ reminds us of the choice $\Gamma = \gamma_-$. From the normalization of the ’t Hooft wave-functions it follows that

$$
\int_0^1 d\xi f^{(n)}_-(\xi) = 1
$$

(39)

upon integration of Eq. (38). This is a minimal test on our calculations since the normalization of the shape function follows from charge (b-number) conservation.

Finally we express this result in terms of functions in the heavy quark limit. It is customary to write the shape function in terms of the residual momentum $\hat{k}_-$ of the $b$-quark. Our expression for the heavy quark ’t Hooft wave-function, Eq. (15), has the light quark momentum as argument. To write a heavy quark limit formula for the shape function as a function of the $b$-quark residual momentum $\hat{k}_-$ we use $\Lambda_n = \mu_n - m_1$, so that $\xi = k_-/p_- = (m_1 v_- + \hat{k}_-)/(m_1 v_- + \Lambda_n v_-) = 1 + (k_- - \Lambda_n v_-)/m_1 v_- + \cdots$. We obtain

$$
f^{(n)}_-(\hat{k}_-) = [\psi_n(\Lambda_n - \hat{k}_-)]^2
$$

(40)

Note that this has support only for $\hat{k}_- < \Lambda_n$. 
where the label \( n \) is interpreted as triple meson couplings, are given by triple overlaps of wave-functions. In terms of \( \psi_R \) while for odd \( n \) is a \( b \)-quark, the incoming external line is a \( u \)-quark and the internal line is a \( d \)-quark.

V. “HEAVY-LIGHT” SOFT FUNCTION

A. “Heavy-light” Form Factors

In preparation for our study of the soft function we review some basic results for the transition current form factors. We consider mesons with different flavor, a \( b\bar{u} \)-meson that we refer to, in analogy with it’s four dimensional counterpart, as the “\( B \)-meson,” and a \( u\bar{d} \) meson, the “pion.” Flavors \( b, d, u \) will be denoted by a flavor index \( a = 1, 2, 3 \), respectively. While we refer to \( b \) as the “heavy” quark and to \( u \) and \( d \) as the “light” quarks, we do not make assumptions on the relative sizes of the masses, except when we consider the chiral limit for which we take \( m_2 = m_3 \to 0 \) \( (m_u = m_d \to 0) \) holding \( m_1 \) fixed. The form factors \( f_\pm \) are defined by \[ f_+(q^2) = \sum_{\text{even } n} \frac{c_n g_{\pi B_n}(q^2)}{q^2 - \mu_n^2} \]
\[ f_-(q^2) = \sum_{\text{odd } n} \frac{c_n g_{\pi B_n}(q^2)}{q^2 - \mu_n^2} - \sum_{\text{even } n} \frac{c_n g_{\pi B_n}(q^2)(\mu_B^2 - \mu_n^2)}{q^2(q^2 - \mu_n^2)}, \]

where the label \( n \) refers to the tower of states with quantum numbers \( 12 = b\bar{d} \). The quantities \( g_{\pi B_n} \), which are interpreted as triple meson couplings, are given by triple overlaps of wave-functions. In terms of \( \tilde{\omega} = p_-/q_- \), which is a function of \( q^2 \) only, it was found that, for even \( n \)
\[ g_{\pi B_n}(q^2) = \frac{-2\mu_n^2}{q^2\tilde{\omega} - \mu_B^2/\omega} \tilde{g}_n \]
while for odd \( n \)
\[ g_{\pi B_n}(q^2) = -2\mu_n^2 \tilde{g}_n \]

where
\[ \tilde{g}_n = \frac{1}{1 - \tilde{\omega}} \int_0^{\tilde{\omega}} dz \phi^{(13)}(\frac{z}{\omega}) \Phi^{(32)}(\frac{z - \tilde{\omega}}{1 - \tilde{\omega}}) \phi_n(z) - \frac{1}{\tilde{\omega}} \int_0^1 dz \Phi^{(13)}(\frac{z}{\omega}) \phi^{(32)}(\frac{z - \tilde{\omega}}{1 - \tilde{\omega}}) \phi_n(z). \]

As shown in Ref. \[ 38 \] the form factors are super-convergent: at large \( q^2 \)
\[ f_\pm \sim \frac{1}{|q^2|^{1+\beta_2}}. \]
By considering the integral \( \oint_C dz f_k(z)/(z - q^2) \) over a contour \( C \) consisting of a circle at infinity deformed on the real line to avoid all the poles, one can show that the \( q^2 \) dependent couplings may be replaced by their value on-shell, e.g.,

\[
f_+(q^2) = \sum_{n} \frac{c_n g_{\tau B_n}(q^2)}{q^2 - \mu_n^2} = \sum_{n} \frac{c_n g_{\tau B_n}(\mu_n^2)}{q^2 - \mu_n^2}
\]

(49)

This is useful because one can show \[13\] that in the chiral limit, \( m_2 = m_3 \to 0^+ \), one has \( g_{\tau B_n}(\mu_n^2) \to 0 \) for \( n \neq 0 \), and \( g_{\tau B B}(\mu_B^2) \to -\mu_B^2/\pi \) where \( c_\pi = \frac{\gamma_0}{\gamma} \) is the normalized “pion” decay constant. Therefore, in the chiral limit

\[
f_+(q^2) = -f_-(q^2) = \frac{f_\pi/f_\pi}{1 - q^2/\mu_B^2}
\]

(50)

**B. Soft Functions**

Next we turn our attention to the non-diagonal analog of the shape function, the “soft function” defined, in analogy with the shape function, by

\[
F_\mu \equiv \int \frac{dx^-}{4\pi} e^{-ixp\cdot x^-} \langle \pi(p')|\bar{u}(x^-)P[\gamma^\mu f_\mu^{(A)}|\gamma_\mu \bar{b}(0)]B(p) \rangle .
\]

(51)

The soft function arises naturally in computations of \( B \)-meson decay amplitudes in SCET in which a pion is produced and is soft in the rest frame of the decaying \( B \)-meson \[11\ [33] [34].

The computation of the soft functions proceeds in complete analogy to that of the previous section. The first contribution to \( g_\tau \), shown in Fig. 4, is similar to what we found for the shape function above in the region \( 0 < k_- < p_- \). But now there is an important difference, the first graph in Fig. 4 also gives a non-vanishing contribution from the region \( p'_- - p_- < k_- < 0 \). This new term cancels against a similar term from the second graph in Fig. 4. The result is

\[
F_- = \phi^{(13)}(1 + \xi - \omega)\phi^{(32)}(1 - \xi/\omega)\theta(\omega - \xi)\theta(\xi) + \sum_n \phi_n^{(12)}(1 + \xi/\omega)g_n(q^2)\theta(1 + \xi - \omega)\theta(-\xi).
\]

(52)

We have introduced \( \omega = p'_- / p_- \) and assumed \( \omega < 1 \). The meaning of \( \xi \) is clear, \( \xi = k_- / p_- \). The functions \( \phi^{(13)} \) and \( \phi^{(32)} \) correspond to the incoming and outgoing \( B \) and \( \pi \) mesons, respectively. The masses \( \mu_n \) are for the tower of states composed of \( b \) and \( \bar{u} \) quarks, and \( g_n(q^2) \) are functions of \( q^2 = (p - p')^2 \) given by

\[
g_n(q^2) = \frac{1}{\omega} \int_0^1 dv \phi^{(13)}(1 - v)\Phi^{(32)}(v/\omega)\phi_n^{(12)} \left( \frac{1 - v}{1 - \omega} \right) - \frac{1}{1 - \omega} \int_0^\omega dv \phi^{(13)}(1 - v)\phi^{(32)}(v/\omega)\Phi_n^{(12)} \left( \frac{1 - v}{1 - \omega} \right).
\]

(53)

It is straightforward to check that this result gives correctly the current form factor. Upon integration over \( \xi \):

\[
\langle \pi(p')|\bar{u}(x^-)\gamma_-\psi^{(1)}|B(p) \rangle = p_-\kappa(q^2) + (p - p')_- \sum_n \frac{c_n g_n(q^2)}{q^2 - \mu_n^2}
\]

(54)

where \( g_n \) was defined in \[33] and \( \kappa \) is given by

\[
\kappa(q^2) = \int_0^\omega d\xi \phi^{(13)}(1 + \xi - \omega)\phi^{(32)}(1 - \xi/\omega),
\]

(55)

in agreement with the classical result of \[38\].

It will be useful to consider the crossed channel soft function. This is interesting in its own right, but more importantly, it will be needed to investigate the chiral limit of the soft functions. The graphs are again given by Fig. 4 except with the direction of the pion momentum reversed. By direct computation we find

\[
F_- = -\sum_n \phi_n(x)\bar{g}_n(q^2)\theta(x)\theta(1 - x),
\]

(56)
where \( x \) is the momentum fraction of the outgoing light quark, \( x = 1 + \hat{\omega} \xi = -k_-/ (p_- + p'_-) \), and \( \hat{g}_n(q^2) \) is given by Eq. (17). It is straightforward to check that this is in fact the analytic continuation of the last term on the right hand side of Eq. (16), that is, the soft function in the region \(-k_- < 0\).

We pause to make an interesting, though peripheral observation. The soft function in Eq. (16) vanishes for \( k_- > 0 \), while the function in Eq. (15) is explicitly non-vanishing in that region. Clearly one is not the analytic continuation (over \( q^2 \) for fixed \( k_- \)) of the other. This violation of crossing symmetry is not particularly worrisome. Crossing symmetry is not a fundamental property of quantum field theories (and is not a "symmetry"). It is established on a case by case basis (see, e.g., [40, 41]). It is surprising to see it fail because one has learned from experience that without exception one can show it. Except in the case at present. We suspect this case does not satisfy crossing symmetry because we are not considering an S-matrix element or a matrix element of a local operator but rather the matrix element of a non-local operator partially Fourier transformed. As soon as one integrates over \( k_- \) crossing is recovered: one obtains the form factors (44) and (54) that can be easily checked to be the analytic continuation of the same as in (56) but with an additional factor of \( m \). Hence we obtain, in the "crossed" channel,

\[
F_+ = \frac{m_1 m_2}{2 x (1-x)} \sum_n \frac{\phi_n(x) \hat{g}_n(q^2)}{q^2 - \mu_n^2} \theta(x) \theta(1-x),
\]

where \( x \) is the momentum fraction of the outgoing light quark, \( x = 1 + \hat{\omega} \xi = -k_-/ (p_- + p'_-) \). It is now easy to check that this gives the current form factor \( f_+ \) upon integration over \( x \). One only needs to use Eq. (51) and the result, from Ref. [12], that

\[
\sum_n \frac{e_n \pi B_n}{\mu_n^2} = 0.
\]

C. Chiral Limit

Since the form factors simplify tremendously in the chiral limit, it is natural to ask whether this is also the case of the soft functions. Moreover, the chiral limit is well understood in four dimensions, and the methods used in understanding the chiral limit in four dimensions should apply as well in two dimensions.

Consider the function

\[
G(x, q^2) = \sum_n \frac{\phi_n(x) \hat{g}_n(q^2)}{q^2 - \mu_n^2},
\]

which appears in (56). We show in the Appendix that for fixed \( x \) this sum is super-convergent, that is

\[
G(x, q^2) \sim \frac{1}{(q^2)^{(1+\beta_2)}}, \quad \text{as } q^2 \to \infty.
\]

Hence one may integrate this in the complex \( q^2 \) plane over a closed contour that avoids the positive real axis and closes on a circle at infinity to show, just as for the form factor, that the numerators can be replaced by residues:

\[
G(x, q^2) = \sum_n \frac{\phi_n(x) \hat{g}_n(\mu_n^2)}{q^2 - \mu_n^2},
\]

We can now use the result that in the chiral limit the couplings to higher resonances vanish to simplify the soft functions of Eqs. (56) and (57), as follows:

\[
F_- \to -\frac{\phi_B(x) \hat{g}_0(\mu_B^2)}{q^2 - \mu_B^2} \theta(x) \theta(1-x),
\]

\[
F_+ \to \frac{m_1 m_2}{2 x (1-x)} \frac{\phi_B(x) \hat{g}_0(\mu_B^2)}{q^2 - \mu_B^2} \theta(x) \theta(1-x).
\]
We have retained the vanishingly small mass $m_2$ in the expression for $F_+$ above. The chiral limit should be understood as having arbitrarily small mass, but care must be exercised in taking the limit $m_2 = 0$. For example, $m_2$ plays a role regulating the behavior of $F_+$ as $x \to 0$. One can integrate over $x$ to obtain a form factor, and then take the limit $m_2 \to 0$ smoothly.

To make this result completely explicit it is necessary to understand the chiral limit of $\hat{g}_0(\mu_B^2)$. There is a subtlety here that must be addressed with care. For small but nonzero “pion” mass, $\mu_\pi = \mu_0^{(23)}$, the pole at $q^2 = \mu_B^2$ falls below threshold. Therefore the value of the variable $\tilde{\omega}$ at $q^2 = \mu_B^2$ is complex, and satisfies

$$(1 - \tilde{\omega})^2 = -\frac{\mu_\pi^2}{\mu_B^2} + \mathcal{O}\left(\frac{\mu_\pi}{\mu_B}\right)^3$$

(64)

Comparing the exact expression for the form factor $f_+$, Eq. (14), with the result in the chiral limit, Eq. (50), and using the explicit form of the triple boson coupling given in Eqs. (51) – (54), we obtain the chiral limit for $\hat{g}_0$:

$$\hat{g}_0(\mu_B^2) \rightarrow -\frac{\mu_\pi^2}{c_\pi(1 - \tilde{\omega})}.$$  

(65)

One can then check consistency by examining the chiral limit of the second form factor, $f_-$. A similar computation yields

$$\hat{g}_0(\mu_B^2) \rightarrow -\frac{\mu_\pi^2}{c_\pi(1 - \tilde{\omega})},$$

(66)

which is seen to coincide with Eq. (65) when use is made of the resonance value of $\tilde{\omega}$ as given in Eq. (64).

Collecting results and expressing them in terms of the light-cone wave-functions, we see that, in the crossed channel,

$$F_- \rightarrow 2\mu_B^2 \frac{f_B}{f_\pi} \frac{p'_-}{p_- + p'_-} \tilde{\phi}_{B-}(x),$$

(67)

$$F_+ \rightarrow -2\mu_B^2 \frac{f_B}{f_\pi} \frac{p'_+}{p_+ + q^2 - \mu_B^2} \tilde{\phi}_{B+}(x),$$

(68)

Finally, we can use these results to find the chiral limit of the soft functions in the “normal” channel. Since $F_-$ in the crossed channel as given by Eq. (56) is the analytic continuation of the second term in Eq. (52), we may express the soft function in the “normal” channel simply as

$$-2\mu_B^2 \frac{f_B}{f_\pi} \frac{p'_-}{p_- + p'_-} \frac{\tilde{\phi}_{B-}(y)}{(p_- + p'_-)^2 - \mu_B^2},$$

(69)

for $-(p_- - p'_-) < k_- < 0$. Here $y$ is the argument of $\phi_n^{(12)}$ in (52), namely, $y = 1 + \xi/(1 - \omega) = (p_- - p'_- + k_-)/(p_- - p'_-)$, which is interpreted as the momentum fraction carried by the $b$ quark. Moreover, for $0 < k_- < p'_-$ we had the result

$$\phi^{(13)}(1 + \xi - \omega)\phi^{(32)}(1 - \xi/\omega)$$

(70)

which can be slightly simplified in the chiral limit, since the pion wave-function simplifies in that limit, $\phi_\pi \to 1$. Combining partial results we have as our final result the chiral limit of the soft function in the normal channel:

$$F_- \rightarrow \frac{f_B}{f_\pi} \left[ \phi_{B-}(-(1 - \omega)y) \theta(\xi)\theta(\omega - \xi) - 2\mu_B^2 \frac{p'_-}{p_- + p'_-} \frac{\tilde{\phi}_{B-}(y)}{(p_- + p'_-)^2 - \mu_B^2} \theta(-\xi)\theta(1 + \xi - \omega) \right]$$

(71)

Here, we remind the reader, $\omega = p'_-/p_-$, $\xi = k_-/p_-$, $y = 1 + \xi/(1 - \omega) = (p_- - p'_- + k_-)/(p_- - p'_-)$ are the relevant momentum fractions and we assumed $\omega < 1$.

VI. DISCUSSION

A. Shape Functions

The result for the shape function, Eq. (38), is rather surprising. The shape function “factorizes” into the product of light-cone wave functions for the incoming and outgoing mesons, but there is no reason to suspect this factorization
a priori. Certainly naïve factorization would not have given this result. Naïve factorization prescribes that one is to insert the vacuum in all possible ways into the operator that is being sandwiched with meson states, and extract the color singlet component only. But in this case the operator is a non-local quark bilinear. If one were to insert the vacuum between quark fields the resulting operator would have to be set to zero by this prescription, since it has no color singlet. Worse, vanishing of this quantity is protected also by conservation of quark number.

There is little guidance for phenomenological models of the shape function. If one could only proof that the factorization observed in this paper is a more general consequence of the large $N_c$ limit, beyond the simple 1+1 dimensional case, it would be interesting to use this as a first approximation to the shape function, with corrections that vanish in the large $N_c$ limit. Alternatively, since the shape function is directly measurable (up to power corrections) one could use this relation to infer the light-cone wave-function, which appears in many calculations but cannot be measured directly. Brave souls may pursue this approximation as a conjecture, awaiting a proof in 4 dimensions.

Let us review the results from that paper.

This relation may give interesting constraints on both functions since both are separately constrained both by theory (e.g., through moments) and phenomenologically. For example, consider popular models for the light-cone wave-function and shape function of Refs. [42] and [43], respectively:

$$f_{B^-}(k_-) = \kappa \left[ \tilde{\psi}_{B^-}(\Lambda - k_-) \right]^2 \theta(\Lambda - k_-)$$  \hspace{1cm} (72)

where $f_B$ and $\tilde{\psi}_B$ stand for the shape and light-cone wave-functions, $\tilde{\psi} \equiv \Lambda_0 = \mu_B - m_1$, and $\kappa^{-1} = \int_{-\Lambda}^{\infty} \tilde{\psi}_{B^-}^2(k_-)$.

B. Soft function

It was shown in [19] that the soft functions can be expressed in terms of the light-cone wave-functions in the chiral limit. Let us review the results from that paper.

We begin by reviewing the effective Lagrangian for heavy mesons interacting with low energy pseudo-Goldstone bosons, the so called Heavy Meson Chiral Perturbation Theory (HMCHPT) [44, 45, 46]. The effective Lagrangian, adapted to the two dimensional theory, is written in terms of a heavy meson super-field,

$$H_a = - \left( \frac{1 + f}{2} \right) \gamma_5 B_a,$$

where $B_a = (b_u, b_d, B_s)$ are the positive energy fields for the $B$ mesons, and a matrix of pseudo-Goldstone bosons,

$$M = \left( \begin{array}{ccc} \frac{1}{\sqrt{2}} n_0 + \frac{1}{\sqrt{6}} \eta & \pi^+ & K^+ \\ \pi^- & -\frac{1}{\sqrt{2}} n_0 + \frac{1}{\sqrt{6}} \eta & K^0 \\ K^- & \bar{K}^0 & -2 \sqrt{2} \eta \end{array} \right).$$

We are considering the case of three light flavors, since this is what is phenomenologically relevant. Our computations in previous sections apply since the third light flavor does not contribute in any of the processes we have considered (to leading order in $1/N_c$). The matrix of pseudo-Goldstone bosons appears in the Lagrangian through $\xi = e^{iM/f}$ (and $\Sigma = \xi \bar{\xi}$). Under $SU(3)_L \times SU(3)_R$ these transform as,

$$\Sigma \rightarrow L \Sigma R^\dagger, \quad \xi \rightarrow L \xi U^\dagger = U \xi R^\dagger \quad \text{and} \quad H_a \rightarrow H_b U^\dagger_{ba},$$

where the transformation $U$, defined through Eq. [74], depends non-linearly on the pseudo-Goldstone bosons. In terms of these fields, the effective Lagrangian is, to lowest order in a momentum expansion and $1/m_b$,

$$\mathcal{L} = -2i \text{Tr} [\tilde{H}^{(Q)a} v_\mu \partial^\mu H^{(Q)}_a] + \frac{f^2}{8} \text{Tr} \left( \partial^\nu \Sigma \partial_\nu \Sigma^\dagger \right) + \lambda_0 \text{Tr} \left[ m_\phi \Sigma + m_\phi \Sigma^\dagger \right] + \frac{g}{2} \text{Tr} [\tilde{H}^{(Q)a} H^{(Q)}_a \xi^\dagger (\xi^\dagger \partial_\mu \xi - \xi \partial_\mu \xi^\dagger)] + \cdots$$  \hspace{1cm} (78)
The coupling constant $g$ determines the coupling of $B$ mesons to pseudo-Goldstone bosons. The $\pi - B - B$ interaction term in the Lagrangian is

$$-i g \frac{\epsilon_{\mu\nu}^\pi}{f} B_\mu \partial^\nu M_{ba} B_a^\dagger$$

(79)

This is to be compared with the coupling $g_{\pi BB}$ which corresponds to a Lagrangian interaction term between three pseudo-scalar mesons

$$\hat{g}_{ijk} \epsilon_{\mu\nu}^\pi \partial^\mu \varphi_i \partial^\nu \varphi_j \varphi_k .$$

(80)

Here we have used the notation of Ref. [13]. In particular, we have a dimensionful normalization constant relating the coupling with a carat and the couplings in Eqs. (45) and (46), $g_{\pi BB} = m_B^2 \hat{g}_{\pi BB}$. Setting $\varphi_i(x) = \pi(x)$, $\varphi_j(x) = e^{-im_{\pi}x} B(x)/\sqrt{m_B}$ and $\varphi_k(x) = e^{im_{\pi}x} B^\dagger(x)/\sqrt{m_B}$ we see this is exactly the same interaction as in the chiral Lagrangian with the identification

$$g_{\pi BB} = m_B^2 \hat{g}_{\pi BB} = m_B^2 g^B_\pi .$$

(81)

Below we use this relation together with the 't Hooft model result $c_\pi \hat{g}_{\pi BB} = -1$ to eliminate $g$ from the final expression for the soft function; see Eq. (80).

Operators in the theory are also constrained by the symmetries. For example, the left handed current $L'_\mu = \bar{q}_a \gamma_\mu P_L b$ can be expressed in the low energy effective theory as [44, 45, 46]

$$L'_\mu = -i \alpha \text{Tr} [\gamma^\nu P_L H_b \xi_{ba}^\dagger] .$$

(82)

The proportionality constant $\alpha$ can be fixed by requiring the correct matrix element between a $B$-meson state and the vacuum, $\alpha = \sqrt{m_B} f_B$ (we have adopted here non-relativistic normalization of the $B$ states, as is standard practice in HQET).

Following Ref. [19] we apply this technique to the non-local operators in the definition of the soft functions. To follow the notation in [19] more closely, we introduce the null vectors $n^\mu = (1/\sqrt{2}, 1/\sqrt{2})$ and $\bar{n}^\mu = (1/\sqrt{2}, -1/\sqrt{2})$. These have $n^+ = \bar{n}^- = 1$ and $n^- = \bar{n}^+ = 0$. The non-local operators are

$$O^{\mu}_{La}(k_-) = \int \frac{dx^-}{2\pi} e^{-ix^- k_- T} \left[ \bar{q}_a(x^-) e^{i \int_0^{x^-} A_\mu} P_R \gamma^\mu b(0) \right],$$

(83)

$$O^{\mu}_{Ra}(k_-) = \int \frac{dx^-}{2\pi} e^{-ix^- k_- T} \left[ \bar{q}_a(x^-) e^{i \int_0^{x^-} A_\mu} P_L \gamma^\mu b(0) \right],$$

which transform under the chiral group as $(3_L, 1)$ and $(1, \bar{3}_R)$, respectively. The corresponding operators in the effective theory then are of the form

$$O^{\mu}_{La}(k_-) = \text{Tr} [\alpha_L(k_-) P_R \gamma^\mu H_b \xi_{ba}^\dagger]$$

$$O^{\mu}_{Ra}(k_-) = \text{Tr} [\alpha_R(k_-) P_L \gamma^\mu H_b \xi_{ba}]$$

(84)

Using the fact that $\hat{\delta} H = H$, the most general form of $\alpha_{L,R}$ is a linear combination of $\hat{\Psi}$ and $\hat{\Psi}$. Matching to the matrix element between the vacuum and a $B$ meson we obtain

$$\alpha_L(k_-) = \alpha_R(k_-) = -i f_B / \sqrt{m_B} [\hat{\Psi}_{B^+} + \hat{\Psi}_{B^-}] .$$

(85)
It is now a straightforward exercise to compute the soft functions in the effective theory. To compare with our earlier results for $P_-$ we consider the sum of operators $O_L + O_R$. There are two contributions, as shown in Fig. 5, the first is a contact term obtained by retaining a pion field in the expansion of $\xi_{ba}$ and the second is a pole term from the no pion term in $\xi_{ba}$. We obtain

$$i\sqrt{m_B} \langle \pi(p')|(O_L + O_R)-|B(p)\rangle = \frac{f_B}{f_\pi} \tilde{\psi}_B(-k_-) \left[ 1 + \frac{2}{v} e_{\mu\nu} v^\mu p'^\nu \right].$$  \hspace{1cm} (86)

This is to be compared with our result, Eq. (74). For the comparison we substitute $m_B v + p$ for $p$, so the meaning of $p$ becomes the residual momentum, and retain only the leading term in an expansion in inverse powers of the heavy mass:

$$F_- = \frac{f_B}{f_\pi} \left[ \tilde{\psi}_B((p'_- - k_-)\theta(k_-)\theta(p'_- - k_-) - \frac{p'_-}{v}\tilde{\psi}_B(-k_-)\theta(-k_-) \right].$$  \hspace{1cm} (87)

Using $v_- = v_+ = 1/\sqrt{2}$ and neglecting $p'_- = 2\mu^2/\mu_-$ we see that except for the argument of the light-cone wave-function and the presence of $\theta$-functions the two terms in $\text{86}$ agree with the corresponding terms in $\text{87}$.

The result is somewhat puzzling, since the form of Eq. (86) was obtained through general arguments in Ref. [19]. However, the arguments presented there are not quite rigorous. We can partially resolve the discrepancy as follows.

Consider gauging the vector flavor symmetries of the QCD Lagrangian. We gauge both the full, non-abelian, flavor symmetry of the light quarks and the flavor symmetry of the heavy quark, “$b$-number.” The gauge fields are taken as background fields and are introduced solely for the purpose of maintaining local invariance, but will be set to zero at the end of the calculation. Now, instead of the integrated operators in $\text{83}$, we consider the un-integrated, non-local operators

$$O^\mu_{La}(x^-,0) = T(\bar{q}_a(x^-)e^{-i\int_0^\xi A^\mu} P_R^\gamma b(0),$$

$$O^\mu_{Ra}(x^-,0) = T(\bar{q}_a(x^-)e^{-i\int_0^\xi A^\mu} P_L^\gamma b(0).$$  \hspace{1cm} (88)

Under a local vector flavor symmetry transformation these transform as

$$O^\mu_{L,R} \rightarrow B(0)O^\mu_{L,R}V(x^-)^\dagger,$$  \hspace{1cm} (89)

where $B$ and $V$ stand for the unitary transformations of $b$-number and light flavor, respectively. The corresponding operators representing these in the chiral Lagrangian should transform the same way. Hence we write

$$O^\mu_{La}(x^-,0) = \text{Tr}[\alpha_L(0,x^-) P_R^\gamma c_{ab}(0,x^-)\xi_{ba}(x^-)],$$

$$O^\mu_{Ra}(x^-,0) = \text{Tr}[\alpha_R(0,x^-) P_L^\gamma c_{ab}(0,x^-)\xi_{ba}(x^-)].$$  \hspace{1cm} (90)

Here $\alpha_{L,R}$ are coefficient functions, to be determined, and $\beta(0,x^-) = T \exp(-i \int_0^{x^-} A^V)$, where $A^V_\mu$ is the background vector gauge field associated with the light quark flavor vector symmetry, transforms as

$$\beta(0,x^-) \rightarrow V(0)\beta(0,x^-)V^\dagger(x^-).$$  \hspace{1cm} (91)

Turning off the background fields, Fourier transforming these operators and repeating the argument above we now obtain

$$i\sqrt{m_B} \langle \pi(p')|(O_L + O_R)-|B(p)\rangle = \frac{f_B}{f_\pi} \left[ \tilde{\psi}_B((p'_- - k_-)\theta(k_-)\theta(p'_- - k_-) + 2\tilde{\psi}_B(-k_-)\theta(-k_-) e_{\mu\nu} v^\mu p'^\nu \right].$$  \hspace{1cm} (92)

This only fails to reproduce Eq. (87) in that the first term (the “contact” interaction) is missing the $\theta$-function enforcing $k_- > 0$. In the ’t Hooft model calculation the origin of this restriction is clear: the contact terms involves the product of the wave-functions of the $B$- and $\pi$-mesons, and the latter requires $k_- > 0$. However, the chiral Lagrangian operator we have constructed has lost this information. It would be interesting to see how this problem could be fixed. It seems likely that some understanding of the representation of non-local purely light-quark operators in the chiral/SCET Lagrangian is necessary, since this should carry information about the pion wave-function. However, since the physical basis for the restriction that $k_- > 0$ is pretty clear, it does make sense to adopt this result in the full 4-dimensional analysis, now modified by replacing the non-local operators $\text{90}$ for $\text{81}$.
VII. CONCLUSIONS

We have computed the analogues of the shape function and soft functions of 4-dimensional HQET/SCET in QCD in 1+1 dimensions (the ‘’t Hooft Model’’). These are matrix elements between mesons of non-local operators.

Our main results are as follows. First, the shape function, \( f_B(k_-) \), is, up to a multiplicative constant fixed by normalization conditions, the square of the the light-cone wave function, \( \tilde{\psi}_B(-k_-) \),

\[
 f_B(k_-) = \kappa \left[ \tilde{\psi}_B(-k_-) \right]^2 \theta(-k_-).
\]

Second, in the chiral and heavy meson combined limits the soft function for \( B \to \pi \) transitions (defined in (51)) is completely determined by the \( B \) meson light-cone wave-function, as argued on general grounds in Ref. [19]. However, we find that the representation of the non-local operator in the effective theory presented in Ref. [19] is not correct. A better representation of this operator is given by a non-local operator in the effective theory, e.g.,

\[
 O^\mu_L(x^-, 0) = \text{Tr} \left[ \alpha_L(0, x^-) P_R^\mu H_c(0) \beta_{cb}(0, x^-) \xi^\dagger_{ba}(x^-) \right].
\]

This operator almost correctly reproduces the result from direct computation in the ‘’t Hooft Model. It only fails to reproduce a \( \theta \)-function restricting the momentum of the light quark (see Sect. VI B for details), which however is easily understood on physical grounds and can be adopted (albeit in a somewhat ad-hoc fashion) in computations in four dimensional QCD.

We cannot argue that the first of these results is applicable in four dimensional QCD. However, the two results are tightly connected in the ‘’t Hooft model. This suggests that there may in fact exist a way to justify the first result in four dimensions. It is interesting to note in this regard that the same argument that led us to consider non-local operators in the effective theory for the soft-functions, gives that the effective operator in the effective theory for the shape function is also non-local, roughly

\[
 O^\Gamma_T(x^-, 0) = \text{Tr} \left[ \alpha(x^-, 0) H_a(x^-) \beta(x^-) \eta_\eta H_b(0) \right].
\]

Hence, if one could argue that \( \alpha(x^-, 0) \) factorizes, then the result \( f \propto \tilde{\psi}^2 \) would follow automatically. Were this established, interesting phenomenological constraints relating exclusive to inclusive \( B \)-decays would follow. Clearly this is a topic that deserves much further investigation.

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APPENDIX: SCALING THEOREMS

We follow the work of Einhorn closely[38]. Our scaling functions are similar but not the same as his. Let

\[
 G^{(12)}(x, y; q^2) = \sum_n \phi_{n}^{(12)}(x) \phi_{n}^{(12)}(y) \frac{q^2 - \mu_n^2}{q^2}.
\]  

(A.1)

One then shows this has a limit, which defines the scaling functions

\[
 h_{\pm}^{(1)}(\xi; y) = \lim_{q^2 \to \infty} q^2 G^{(12)}(\pm \frac{\xi}{q^2}, y; q^2).
\]  

(A.2)

Note that integrating \( h(\xi; y) \) over \( y \) gives Einhorn’s scaling function \( h(\xi) \). The scaling function satisfies

\[
 \left( 1 - \frac{\tilde{m}_1^2}{\xi} \right) h_{\pm}^{(1)}(\xi; y) + \int_0^1 d\eta \frac{h_{\pm}^{(1)}(\eta; y)}{(\eta - \xi)^2} = \delta(y),
\]  

(A.3)

from which it immediately follows that

\[
 h_{\pm}^{(1)}(\xi; y) \sim \xi^{\tilde{m}_1} \quad \text{as} \; \xi \to 0, \text{ fixed} \; y.
\]  

(A.4)
Knowing that $G$ has a finite limit one can determine the large $q^2$ scaling of $G(x, q^2)$ in Eq. (60). To this end write explicitly the first term in $g_n(q^2)$ from Eq. (47) in $G(x, q^2)$ as

$$
\frac{1}{1 - \bar{\omega}} \int_0^\infty dz \phi^{(13)}(z) \Phi^{(32)} \frac{z - \bar{\omega}}{1 - \bar{\omega}} \sum_n \frac{\phi_n^{(12)}(x) \phi_n^{(12)}(z)}{q^2 - \mu_n^2} = \frac{\bar{\omega}}{1 - \bar{\omega}} \int_0^1 dv \int_0^1 dt \frac{\phi^{(13)}(v) \phi^{(32)}(t) G^{(12)}(\bar{\omega} v, x; q^2)}{t + \frac{\bar{\omega}}{1 - \bar{\omega}} (1 - v)}
$$

In the second line we have changed integration variables ($v = z/\bar{\omega}$), used the integral representation of $\Phi$, and expressed the sum in terms of the function $G$ of (A.1). The third line uses a further change of variables $\tilde{\omega} = t(1 - \bar{\omega}) + \bar{\omega}$ so the denominator factors,

$$
(1 - \omega)t + \bar{\omega}(1 - v) = \bar{\omega}(u - v)^2,
$$

which leaves the integrand in a form suitable for exploration of the small $\bar{\omega}$ behavior. This corresponds to large $q^2, q^2 \sim m_B^2/\bar{\omega}$.

Using $\phi^{(32)}(x) \approx cx^{\beta_3}$ as $x \to 0$, the behavior of (A.5) is obtained by writing it in terms of the scaling function:

$$
\sim \frac{c (m_B^2)^{\beta_3}}{(q^2)^{1+\beta_3}} \int_0^1 dv \int_1^\infty du \frac{\phi^{(13)}(v)(u - 1)^{\beta_3} h_i^{(12)}(m_B^2 v; x)}{(u - v)^2}.
$$

Analogous steps are taken for the second term in $G(x; q^2)$:

$$
- \frac{1}{\bar{\omega}} \int_0^\infty dz \phi^{(13)}(z) \Phi^{(32)} \frac{z - \bar{\omega}}{1 - \bar{\omega}} \sum_n \frac{\phi_n^{(12)}(x) \phi_n^{(12)}(z)}{q^2 - \mu_n^2}
$$

$$
= \frac{1 - \bar{\omega}}{\bar{\omega}} \int_0^1 dv \int_0^1 dt \frac{\phi^{(13)}(t) \phi^{(32)}(v) G^{(12)}(\bar{\omega} + (1 - \bar{\omega}) v, x; q^2)}{(t - 1 - \bar{\omega} v)^2}
$$

$$
= - \int_0^1 dt \int_1^{1/\bar{\omega}} du \frac{\phi^{(13)}(t) \phi^{(32)}(\frac{u - 1}{1 - \bar{\omega}}) G^{(12)}(\bar{\omega} u, x; q^2)}{(u - t)^2}
$$

$$
\sim - \frac{c (m_B^2)^{\beta_3}}{(q^2)^{1+\beta_3}} \int_0^1 dv \int_1^\infty du \frac{\phi^{(13)}(t)(u - 1)^{\beta_3} h_i^{(12)}(m_B^2 u; x)}{(u - t)^2}.
$$

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