Braided Chains of q-Deformed Heisenberg Algebrae

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Abstract

Given $M$ copies of a $q$-deformed Weyl or Clifford algebra in the defining representation of a quantum group $G_q$, we determine a prescription to embed them into a unique, inclusive $G_q$-covariant algebra. The different copies are “coupled” to each other and are naturally ordered into a “chain”. In the case $G_q = SL_q(N)$ a modified prescription yields an inclusive algebra which is even explicitly $SL_q(M) \times SL_q(N)$-covariant, where $SL_q(M)$ is a symmetry relating the different copies. By the introduction of these inclusive algebrae we significantly enlarge the class of $G_q$-covariant deformed Weyl/Clifford algebrae available for physical applications.

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1 Introduction

Weyl and Clifford algebras (respectively denoted by $A_+, A_-$ in the sequel, and collectively as “Heisenberg algebras”) are at the hearth of quantum physics. The most useful Heisenberg algebras are the ones endowed with definite transformation properties under the action of some symmetry Lie group $G$ (or Lie algebra $g$).

The idea that quantum groups \cite{1} could generalize Lie groups in describing symmetries of quantum physical systems has attracted much interest in the past decade. Mathematically speaking, a quantum group can be described as a deformation $\text{Fun}(G_q)$ of the algebra $\text{Fun}(G)$ of regular functions on $G$ or, in the dual picture, as a deformation $U_q g$ of the universal enveloping algebra $U g$, within the category of (quasitriangular) Hopf algebras; here $q = e^{i\hbar}$, and $\hbar$ is the deformation parameter. These $q$-deformations induce matched $q$-deformations of all $\text{Fun}(G_q)$-comodule algebras \cite{i}, i.e. of the algebrae whose generators satisfy commutation relations that are preserved by the $\text{Fun}(G_q)$-coaction, in particular of $G$-covariant Heisenberg algebras. $q$-Deformed Heisenberg algebrae corresponding to a simple Lie algebra $g$ in the classical series $A_n$, $B_n$, $D_n$ were introduced in Ref. \cite{2, 3, 4, 5} in the restricted case that the generators $A_i^+$, $A_i^-$ belong respectively to the defining corepresentation $\phi_d$ of $\text{Fun}(G_q)$ and to its contragradient $\phi_d^\vee$.

In general, we shall denote by $A_{\pm,G,\phi}$ the Weyl/Clifford algebra with generators $a^i$, $a^+_i$ belonging respectively to some corepresentation $\phi$ of $G$ and to its contragradient $\phi^\vee$ and fulfilling the canonical (anti)commutation relations

\begin{align}
  a^+_i a^+_j \mp a^+_j a^+_i &= 0 \quad (1.1) \\
  a^i a^j \mp a^j a^i &= 0 \quad (1.2) \\
  a^i a^+_j - \delta^i_j \mathbf{1} \mp a^+_j a^i &= 0. \quad (1.3)
\end{align}

The purpose of this work is to find out if there exists some $G_q$-covariant deformation of $A_{\pm,G,\phi}$ (which we will denote by $A_{\pm,G,\phi}^q$) having the same Poincaré series as $A_{\pm,G,\phi}$. We shall denote the generators of $A_{\pm,G,\phi}^q$ by $A^i$, $A^+_i$.

As a preliminary result we show (Sect. \ref{sect:preliminary}) that, beside $A_{\pm,SL(N),\phi_d}^q$, $A_{\pm,SO(N),\phi_d}^q$, also $A_{\pm,Sp(n),\phi_d}^q$ can be defined. The first major result is however that one can embed $M$ identical copies of $A_{\pm,G,\phi_d}^q$ (resp. $A_{\pm,G,\phi_d}^q$) into a unique, well-defined algebra $A_{\pm,G,\phi_M}^q$ (resp. $A_{\pm,G,\phi_M}^q$), or more generally $M' < M$ copies of $A_{\pm,G,\phi_M}^q$ and $A_{\pm,G,\phi_M}^q$.
\((M-M')\) copies of \(\mathcal{A}^q_{-G,\phi_d}\) into a unique, well-defined deformed superalgebra \(\mathcal{A}^q_{G,\phi_M}\); \(\phi_M\) denotes here the direct sum of \(M\) copies of \(\phi_d\). Due to the rules of braiding \(\text{[3]}\), the different copies do not commute with each other; consistent commutation relations between the latter require the introduction of an ordering: we call the ordered sequence a “braided chain”.

The use of the symbols \(a^i, a^+_i, A^i, A^+_i\) etc. does not necessarily mean that we are dealing with creators & annihilators; the latter fact is rather determined by the choice of the \(*\)-structure, if any. In section \(\text{[4]}\) we consider the natural \(*\)-structures giving the generators the meaning of creation & annihilation operators, or \(\text{e.g.}\) of coordinates and derivatives.

The second major result (Sect. \(\text{[3]}\)) is that if \(G_q = SL_q(N)\) one can modify the \(A-A^+\) commutation relations of \(\mathcal{A}^q_{\pm SL(N),\phi_M}\) in such a way that the generators become explicitly \(GL_q(M) \times SL_q(N)\)-covariant\(\text{[3]}\). The additional symmetry \(GL_q(M)\) transforms the different copies into each other, as in the classical case.

The physical relevance of the case that \(\phi\) is a direct sum of many copies of \(\phi_d\)’s is easily understood once one notes that the different copies could correspond to different particles, crystal sites or space(time)-points, respectively in quantum mechanics, condensed matter physics or quantum field theory. The coupling (\(\text{i.e.}\) non-commutativity) between the different copies can be interpreted as a naturally built-in form of interaction between them. In the particular case that \(\mathcal{A}^q_{\pm G,\phi}\) (with \(q \in \mathbb{R}\)) is a \(q\)-deformation of the \(*\)-algebra \(\mathcal{A}_{\pm G,\phi}\) with \((a^i)^\dagger = a^+_i\), then the physical interpretation of \(A^i, A^+_j\) as annihilators and creators does not necessarily requires the introduction of particles with \text{exotic statistics}. Indeed, it is possible to adopt ordinary boson/fermion statistics \(\text{[8, 9]}\), whereby \(A^+_i, A_i\) are to be interpreted as “composite operators” creating and destroying some sort of “dressed states” of bosons/fermions.

\section{2 Preliminaries}

For a simple Lie group \(G\) the algebra \(\text{Fun}(G_q)\) \(\text{[10]}\) is generated by \(N^2\) objects \(T^i_j\), \(i, j = 1, \ldots, N\), fulfilling the commutation relations

\begin{equation}
\hat{R}^{ij}_{hk} T^h_l T^k_m = T^i_l T^j_k \hat{R}^{hk}_{lm}. \tag{2.1}
\end{equation}

\(2\) The result regarding \(\mathcal{A}^q_{\pm SL(N),\phi_M}\) was essentially already found in \(\text{[3]}\), whose author we thank for drawing our attention to this point.
\(N\) is the dimension of the defining representation of \(G\), \(\hat{R}\) the corresponding ‘braid matrix’ [10], i.e. a numerical matrix fulfilling the ‘braid equation’

\[
\hat{R}_{12} \hat{R}_{23} \hat{R}_{12} = \hat{R}_{23} \hat{R}_{12} \hat{R}_{23}.
\]

(2.2)

Here we have used the conventional tensor notation \((M_{12})^{ij}_{lmn} = M^{ij}_{lm} \delta^k_n\), etc. Because of eq.’s (2.2), (2.1) \(\text{Fun}(G_q)\) is also a bialgebra with coproduct and counit respectively given by \(\Delta(T^i_j) = T^i_h \otimes T^h_j\) and \(\varepsilon(T^i_j) = \delta^i_j\).

A (right) comodule algebra of \(\text{Fun}(G_q)\) is an algebra \(\mathcal{M}\) equipped with a ‘corepresentation’ \(\phi\), i.e. with an algebra homomorphism \(\phi : \mathcal{M} \to \mathcal{M} \otimes \text{Fun}(G_q)\) such that \((\text{id} \otimes \Delta) \circ \phi = (\phi \otimes \text{id}) \circ \phi\). For any polynomial function \(f(t)\) in one variable, the algebra \(\mathcal{M}\) generated by \(N\) objects \(A^+_i\) fulfilling the quadratic relations

\[
[f(\hat{R})]^{ij}_{hk} A^+_i A^+_j = 0
\]

and equipped with the algebra homomorphism \(\phi_d(A^+_i) := A^+_j \otimes T^i_j\) is a comodule algebra [10].

By adding to the quadratic conditions (2.1) some suitable inhomogeneous condition [10], \(\text{Fun}(G_q)\) can be endowed also with an antipode \(S\) and therefore becomes a Hopf algebra [10]. Then the algebra \(\mathcal{M}'\) generated by \(N\) objects \(A^i\) fulfilling the quadratic relations

\[
[f(\hat{R})]^{ij}_{hk} A^i A^j = 0
\]

(2.3)

and equipped with the algebra homomorphism \(\phi^\vee_d(A^i) := A^j \otimes ST^i_j\) is a comodule algebra with inverse transformation properties of \(\mathcal{M}\); therefore the corepresentation \(\phi^\vee_d\) can be called the contragradient of \(\phi_d\).

To go on, we need to recall some specific information regarding each quantum group \(G_q\). The braid matrix \(\hat{R}\) of the quantum group \(\text{Fun}(G_q)\) is a \(N^2 \times N^2\) matrix that admits the following projector decomposition [10]

\[
\hat{R} = q\mathcal{P} - q^{-1}\mathcal{P}^A \quad \text{if } G = SL(N)
\]

\[
\hat{R} = q\mathcal{P} - q^{-1}\mathcal{P}^A + q^{1-N}\mathcal{P}^t \quad \text{if } G = SO(N)
\]

\[
\hat{R} = q\mathcal{P}' - q^{-1}\mathcal{P}' - q^{1-N}\mathcal{P}' \quad \text{if } G = Sp(n), \quad N = 2n,
\]

(2.5)

\(^3\) In the case \(G_q = SL_q(N)\) this condition reads \(\det_q T = 1\), where \(\det_q T\) is the \(q\)-deformed determinant of \(T\). One can also define a Hopf algebra \(\text{GL}_q(N)\) by using the same \(\hat{R}\)-matrix, introducing a new generator \(t\) that is central and group-like, together with its inverse \(t^{-1}\), and then imposing the weaker condition \(\det_q T = t\).
with
\[ P^\mu P^\nu = \delta^{\mu\nu}, \quad \sum_\mu P^\mu = 1. \]  
(2.6)

\( P^A, P^S \) are \( SL_q(N) \)-covariant \( q \)-deformations of the antisymmetric and symmetric projectors respectively; they have dimensions \( \frac{N(N-1)}{2} \) and \( \frac{N(N+1)}{2} \) respectively. \( P^a, P^t, P^s \) are \( SO_q(N) \)-covariant \( q \)-deformations of the antisymmetric, trace, and symmetric trace-free projectors respectively; they have dimensions \( \frac{N(N-1)}{2}, 1 \) and \( \frac{N(N+1)}{2} - 1 \) respectively. \( P^s', P^t', P^a' \) are \( Sp_q(n) \)-covariant \( q \)-deformations respectively of the symmetric, symplectic, antisymmetric symplectic-free projectors; they have dimensions \( \frac{N(N+1)}{2}, 1 \) and \( \frac{N(N-1)}{2} - 1 \) respectively. Setting

\[ P^+ = P^S \quad \text{if} \quad G = SL(N) \]
\[ P^+ = P^s + P^t \quad \text{if} \quad G = SO(N) \]
\[ P^+ = P^s' \quad \text{if} \quad G = Sp(n) \]
\[ P^- = P^A \quad \text{if} \quad G = SL(N) \]
\[ P^- = P^a \quad \text{if} \quad G = SO(N) \]
\[ P^- = P^a' + P^t' \quad \text{if} \quad G = Sp(n) \]

we obtain \( \text{Fun}(G_q) \)-covariant deformations \( P^+, P^- \) of the \( \frac{N}{2}(N+1) \)-dim symmetric and \( \frac{N}{2}(N-1) \)-dim antisymmetric projector respectively.

In the sequel we shall need also the explicit expression for the \( \hat{R} \) matrix of \( SL_q(N) \) and for its inverse:

\[ \hat{R} = q \sum_{i=1}^{N} e_i^i \otimes e_i^i + \sum_{i \neq j} e_i^i \otimes e_j^j + (q - q^{-1}) \sum_{i<j} e_i^i \otimes e_j^j. \]
\[ \hat{R}^{-1} = q^{-1} \sum_{i=1}^{N} e_i^i \otimes e_i^i + \sum_{i \neq j} e_i^j \otimes e_j^i + (q^{-1} - q) \sum_{i>j} e_i^i \otimes e_j^j. \]

Here we have used the conventional tensor notation and denoted by \( e_i^j \) the \( N \times N \) matrix with \( (e_i^j)^h_k = \delta^{ih}\delta_{jk} \).

By repeated application of the equations (2.2), (2.1) we find

\[ f(\hat{R})_{hk}^{ij} T^l_k T^m_l = T^l_k f(\hat{R})_{hk}^{ij}, \]
\[ f(\hat{R}_{12}) \hat{R}_{23} \hat{R}_{12} = \hat{R}_{23} \hat{R}_{12} f(\hat{R}_{23}) \]

for any polynomial function \( f(t) \) in one variable, in particular for those \( f \)'s yielding \( f(\hat{R}) = P^\mu \) or \( f(\hat{R}) = \hat{R}^{-1} \). The Equations (2.2), (2.1) and (2.10) are evidently satisfied also after the replacement \( \hat{R} \to \hat{R}^{-1} \).
If in relations (2.3), (2.4) one chooses \( f(\hat{R}) = P^\mp \), then these equations become
the Fun(\( G_q \))-covariant deformations of the (anti)commutation relations (1.1), (1.2):
\[
\begin{align*}
P^\mp_{\pm \frac{1}{2}} A_i^\pm A_j^\pm &= 0, \quad (2.11) \\
P^\mp_{\pm \frac{1}{2}} A_i^\pm A_k^h &= 0. \quad (2.12)
\end{align*}
\]
Relations (1.1), (1.2), (2.11), (2.12) amount each to \( N(N-1)/2 \) or to \( N(N+1)/2 \) independent relations, respectively if the upper or the lower sign is considered. The algebrae \( \mathcal{M}, \mathcal{M}' \) defined resp. by (2.11), (2.12) have [10, 11] the same Poincaré series as the algebrae defined by resp. by (1.1), (1.2).

To obtain Fun(\( G_q \))-covariant deformations \( \mathcal{A}_q^{\pm, G, \phi_d} \) of the classical Heisenberg algebrae described in Section 1 one still has to deform relations (1.3). For \( G_q = SL_q(N), SO_q(N) \) this was done in Ref. [2, 3, 4, 5]. The natural ansatz is to look for quadratic cross commutation relations, in the form
\[
A_i^\pm A_j^\pm = \delta_{ij} \mathbf{1} \pm S^{ih}_{jk} A_i^+ A_k^h. \quad (2.13)
\]
The inhomogeneous term is fixed by the requirement that \( \{A^i\} \) is the basis dual to \( \{A_i^\pm\} \). The numerical matrix \( S \) has to be determined imposing Fun(\( G_q \))-covariance and that that \( \mathcal{A}_q^{\pm, G, \phi_d} \) itself has the same Poincaré series as its classical counterpart \( \mathcal{A}_{\pm, G, \phi_d} \). It will be convenient to use the following general

**Lemma 1** Let \( \hat{R} = \sum_{\mu} c_\mu P^\mu \) be the projector decomposition of the braid matrix \( \hat{R} \), and let \( \mathcal{P}^+ := \sum_{\mu: c_\mu > 0} P^\mu \) and \( \mathcal{P}^- := \sum_{\mu: c_\mu < 0} P^\mu \) be the corresponding deformed symmetric and antisymmetric projectors respectively. Assume that relations (2.11), (2.12) define algebrae \( \mathcal{M}, \mathcal{M}' \) with the same Poincaré series as their classical counterparts. In order that relations (2.11-2.13) define a deformed Weyl algebra \( \mathcal{A}_q^+ \) (resp. Clifford algebra \( \mathcal{A}_q^- \)) with the same Poincaré series as its classical counterpart \( \mathcal{A}_+ \) (resp. \( \mathcal{A}_- \)) there must exist exactly one negative (resp. positive) \( c_\mu \), say \( c_- \) (resp. \( c_+ \)), and the commutation relations (2.13) have to take one of the two following forms
\[
\begin{align*}
A_i^\pm A_j^\pm &= \delta_{ij} \mathbf{1} - (c_+)^{-1} \hat{R}_{jk} A_k^+ A_i^+, \quad (2.14) \\
A_i^\pm A_j^\pm &= \delta_{ij} \mathbf{1} - c_- \hat{R}_{jk} A_k^+ A_i^+. \quad (2.15)
\end{align*}
\]
Proof. Let us multiply eq. (2.11) by \( A^i \) from the left. We easily find

\[
0 = A^i \mathcal{P}_L^{ij} \hat{h}_k A^k_i A^+_j
\]

(2.13)

\[
[\mathcal{P}_L (1 + S)]^{ij} \hat{h}_k A^k_i \rightarrow \text{relations}
\]

In order that the second term vanishes without introducing new, third degree relations (which would yield a different Poincaré series) it must be either \( S \propto \hat{R} \) or \( S \propto \hat{R}^{-1} \), so that

\[
(S_{12} S_{23} \mathcal{P}_L^{ij} A^k_i A^+_j A^m)L = 0
\]

These correspond to the two possible braiding \( \hat{R} \). If \( S = b \hat{R} \), then the first term vanishes iff

\[
0 = \mathcal{P}^+ (1 + S) = \sum_{\mu: \pm c_\mu > 0} \mathcal{P}^{\mu} (1 + c_\mu b) \quad \Leftrightarrow \quad 1 + c_\mu b \quad \forall \mu: \pm c_\mu > 0.
\]

Thus there must exist exactly one positive (resp. negative) \( c_\mu \) and relation (2.14) must hold. Similarly one proves relation (2.13) if \( S = b \hat{R}^{-1} \). \( \square \)

As immediate consequences of this lemma and of the decompositions (2.7) we find:

- there exist no satisfactory definitions of the \( q \)-deformed algebras \( A^q_{+,SO(N),\phi_d} \), \( A^q_{+,sp(n),\phi_d} \), since these correspond respectively to the projectors (2.7)2, (2.7)6;
- there exist satisfactory definitions of the \( q \)-deformed algebra \( A^q_{+,SL(N),\phi_d} \), \( A^q_{-,SL(N),\phi_d} \), \( A^q_{+,SO(N),\phi_d} \), \( A^q_{-,sp(n),\phi_d} \), since these are the algebras corresponding respectively to the projector (2.7)2, (2.7)1, (2.7)5, (2.7)3 (up to our knowledge, the latter has never been considered before in the literature).

3 Main embedding prescription

We would like to generalize the construction of the preceding section to the case in which \( A^+_i \), \( A^i \) belong respectively to corepresentations \( \phi_M, \phi_M^\dagger \) that are direct sums of \( M \geq 1 \) copies of \( \phi_d, \phi_d^\dagger \). Let \( A^q_{\pm,G,\phi_d} (\alpha = 1, 2, ..., M) \) be \( G \)-covariant \( q \)-deformed Heisenberg algebra with generators \( 1, A^{\alpha,i}, A^{\alpha,i}_+ \), \( i = 1, ..., N \), and relations

\[
\mathcal{P}^{(\alpha)h_k}_{ij} A^{\alpha}_+ A^{\alpha}_i = 0 \quad \text{(3.16)}
\]

\[
\mathcal{P}^{(\alpha)ij}_{hk} A^{\alpha} A^{\alpha} = 0 \quad \text{(3.17)}
\]

\[
A^{\alpha,i}_+ A^{\alpha,i}_+ - \delta^i_j \frac{1}{1} - (-1)^{\alpha} [(q^{-1} - 2q) \hat{R}]^{\alpha}_i \mathcal{P}^{(\alpha)ij}_{hk} A^{\alpha} A^{\alpha} = 0. \quad \text{(3.18)}
\]
According to the last remark in the previous section, let \( \epsilon_{\alpha} \) take the values \( \epsilon_{\alpha} \equiv 0 \) if \( G = SO(N) \), \( \epsilon_{\alpha} \equiv 1 \) if \( G = Sp(n) \), and \( \epsilon_{\alpha} = 0, 1 \) if \( G = SL(N) \); \( \epsilon_{\alpha} = 0, 1 \) correspond to Weyl, Clifford respectively. Moreover, let

\[
\mathcal{P}^{(\alpha)} = \begin{cases} 
\mathcal{P}^+ & \text{if } \epsilon_{\alpha} = 0 \\
\mathcal{P}^- & \text{if } \epsilon_{\alpha} = 1;
\end{cases}
\]

(3.19)

Recalling that the comodules of \( \text{Fun}(G_q) \) belong to a braided monoidal category, we know that consistent commutation relations between the generators of \( ^{\alpha}A_{\pm,G,\phi_d}^q \), \( ^{\beta}A_{\pm,G,\phi_d}^q \), \( \alpha \neq \beta \), are given by the two possible braidings (the latter correspond to the quasitriangular structures \( \mathcal{R}, \mathcal{R}^{-1}_{21} [6] \)). Accordingly, the commutation relations between \( A_{\alpha,i}^+, A_{\beta,j}^+ \) for instance may become

\[
\begin{align*}
\text{either} & \quad A_{\alpha,i}^+, A_{\beta,j}^+ \propto \hat{R}_{ij}^{hk} A_{\beta,h}^+ A_{\alpha,k}^+ \\
\text{or} & \quad A_{\alpha,i}^+, A_{\beta,j}^+ \propto \hat{R}_{ij}^{-1hk} A_{\beta,h}^+ A_{\alpha,k}^+.
\end{align*}
\]

There are \( M(M-1)/2 \) different pairs \( (\alpha, \beta) \); if we could choose for each pair the upper or lower solution independently we would have \( 2^{M(M-1)/2} \) different versions of the deformed commutation relations. We claim that, in fact, only \( M! \) are allowed, in other words that, up to a reordering (\( i.e. \) a permutation of the \( \alpha \)'s), the only consistent way is:

**Proposition 1** Without loss of generality we can assume

\[
A_{\alpha,i}^+, A_{\beta,j}^+ = (-1)^{\epsilon_{\alpha} \epsilon_{\beta}} c_{\alpha \beta} \hat{R}_{ij}^{hk} A_{\beta,h}^+ A_{\alpha,k}^+, \quad \text{if } \alpha < \beta,
\]

with \( c_{\alpha \beta} \xrightarrow{q \rightarrow 1} 1 \).

(We have factorized the overall sign necessary to get the correct commutation relations between fermionic or bosonic variables in the classical limit).

**Proof.** The claim can be proved inductively. It is obvious if \( M = 2 \). Assume now that the claim is true when \( M = P \), and call \( A_{\alpha,i}^+ \) the generators of the \( (P+1) \)-th additional subalgebra. We need to prove that

\[
\begin{align*}
A_{\beta,i}^+, A_{\alpha,j}^+ \propto \hat{R}_{ij}^{hk} A_{\alpha,h}^+ A_{\beta,k}^+ & \quad \Rightarrow \quad A_{\alpha,i}^+, A_{\beta,j}^+ \propto \hat{R}_{ij}^{hk} A_{\alpha,h}^+ A_{\beta,k}^+ \quad \forall \alpha < \beta \\
A_{\gamma,i}^+, A_{\alpha,j}^+ \propto \hat{R}_{ij}^{-1hk} A_{\alpha,h}^+ A_{\gamma,k}^+ & \quad \Rightarrow \quad A_{\beta,i}^+, A_{\gamma,j}^+ \propto \hat{R}_{ij}^{-1hk} A_{\alpha,h}^+ A_{\gamma,k}^+ \quad \forall \delta > \gamma.
\end{align*}
\]

Let \( A_{\beta,i}^+, A_{\alpha,j}^+ = V_{ij}^{hk} A_{\alpha,h}^+ A_{\beta,k}^+ \); we can invert the order of the factors in the product \( A_{\alpha,h}^+, A_{\beta,i}^+, A_{\gamma,j}^+ \) either by permuting the first two factors, then the last two, finally the
first two again, or by permuting the last two factors, then the first two, finally the
last two again; to get the same result we need that \( \hat{R}_{12} V_{23} \hat{R}_{12} = \hat{R}_{23} V_{12} \hat{R}_{23} \). This
equation is satisfied iff \( V \propto \hat{R} \). Thus we have proved the first implication. Similarly
one proves the second. \( \square \)

Eq. (3.20) and the condition that \( A^{\alpha,i} \) are the dual generators of \( A^+_{\alpha,i} \) implies
(for \( \alpha < \beta \))

\[
A^{\alpha,j} A^{\beta,i} = (-1)^{\epsilon_{\alpha} \epsilon_{\beta}} c_{\alpha \beta} \hat{R}^{ij}_{hk} A^{\beta,k} A^{\alpha,h}
\]  
(3.21)

As for the remaining relations, we shall look for them in the form \( A^{\beta,j} A^+_{\alpha,i} = M^i_{jk} A^+_{\alpha,h} A^{\beta,k} \). It is easy to check that from either of the previous relation and
the commutation relations of \( \alpha A^q \pm, G, \phi_d \) it follows (for \( \alpha < \beta \)):

\[
A^{\beta,j} A^+_{\alpha,i} = (-1)^{\epsilon_{\alpha} \epsilon_{\beta}} c_{\alpha \beta} \hat{R}^{ij}_{hk} A^+_{\alpha,h} A^{\beta,k}
\]  
(3.22)

\[
A^{\alpha,i} A^+_{\beta,j} = (-1)^{\epsilon_{\alpha} \epsilon_{\beta}} c_{\alpha \beta} \hat{R}^{-1}_{jk} A^+_{\beta,h} A^{\alpha,k}
\]  
(3.23)

For instance, relation (3.22) is derived by consistency when requiring that one gets
the same result from \( A^{\alpha,i} A^+_{\alpha,j} A^+_{\beta,k} \) either by permuting the first two factors, then
the last two, finally the first two again, or by permuting the last two factors, then
the first two, finally the last two again.

We will call \( A^q_{G,\phi,M} \) the unital algebra generated by \( 1, A^{\alpha,i}, A^+_{\alpha,i}, \alpha = 1, 2, ..., M, \)
\( i = 1, ..., N \) and commutation relations (3.16-3.18), (3.20-3.23). We have thus
proved

\[ \text{Proposition 2} \] \( A^q_{G,\phi,M} \) has the same Poincaré series as its classical counterpart
\( A_{G,\phi,M} \).

4 \( \ast \)-Structures

Let \( \text{Fun}(G_q) \) be a Hopf \( \ast \)-algebra and assume that \( A^q_{\pm, G, \phi_d} \) are \( \text{Fun}(G_q) \)-comodule
\( \ast \)-algebras:

\[
\phi_d(b^{\ast}) = \left[ \phi_d(b) \right]^{*_{\alpha} \ast_{\ast}}, \quad b \in A^q_{\pm, G, \phi_d},
\]  
(4.1)

(here \( \ast_{\alpha} \) denotes the \( \ast \) of \( A^q_{\pm, G, \phi_d} \)). Can we use the \( \ast_{\alpha} \)’s to build a \( \ast \)-structure \( \ast \)
of the whole \( A^q_{\pm, G, \phi} \)?

In the case that \( \ast \) realizes the compact real section of \( \text{Fun}(G_q) \) (what requires
\( q \in \mathbb{R}^+ \)), then the simplest \( \ast \)-structure in \( A^q_{\pm, G, \phi_d} \) is

\[
(A^i)^{\ast} = A^+_i.
\]  
(4.2)
It is immediate to check that the Ansatz \((A^i,\alpha)^* = A^+_i,\alpha\) would be compatible with relations (3.16-3.18), but inconsistent with relations (3.20-3.23). Therefore let us choose the Ansatz
\[(A^i,\alpha)^* = A^+_i,\pi(\alpha),\quad (4.3)\]
where \(\pi\) is some permutation of \((1,\ldots,M)\). It is easy to check that consistency with relations (3.20-3.23) requires
\[\pi(\alpha) = M - \alpha + 1, \quad (4.4)\]
\[\eta_{\pi(\alpha)} = \eta_\alpha, \quad c_{\pi(\alpha)\pi(\beta)} = c_{\beta\alpha}, \quad \epsilon_{\pi(\alpha)} = \epsilon_\alpha. \quad (4.5)\]
Eq. (4.4) shows that \(\pi\) must be the inverse-ordering permutation; Eq. (4.5) amounts to say that \(\star\) must preserve the bosonic or fermionic character of the generators.
\[A^q_{+,SO(N),\phi_d}\] admits also an alternative \(\ast\)-structure compatible with \(\phi_d\), namely
\[(A^+_i)^* = A^*_j C_{ji}, \quad (4.6)\]
together with a nonlinear transformation for \((A^i)^*\). Here \(C_{ij}\) is the \(q\)-deformed metric matrix, which is related to the projector \(P^t\) appearing in (2.3) through the formula \(P^t_{ij} = C_{ij} C_{jk} C_{kl} C_{lm}\). It is easy to check that the Ansatz
\[(A^+_{i,\alpha})^* = A^+_{j,\pi(\alpha)} C_{ji}^\alpha, \quad (4.7)\]
together the corresponding nonlinear one for \((A^{i,\alpha})^*\), defines a consistent \(\ast\)-structure of \(A^q_{+,SO(N),\phi_d}\) provided relations (4.4), (4.5) hold (with \(\epsilon_\alpha \equiv 0 \ \forall \alpha\)).

5 Modified prescription: \(GL_q(M) \times G_q\)-covariant algebras

If all the generators of \(A_{\pm,G,\phi}\) have the same Grassman parity, they belong to a corepresentation of \(GL(M) \times G\). The coaction of the group \(GL(M)\) amounts to a linear invertible transformation \(T\) of the \(a^{\alpha,i}\) and of the \(a^{+\alpha,i}\):
\[a^{\alpha,i} \rightarrow a^{\beta,i} T^\alpha_{\beta}, \quad a^+_{\alpha,i} \rightarrow a^+_{\beta,i} T^{-1\beta\alpha}, \quad (5.1)\]
which leaves the commutation relations (1.1-1.3) invariant. [If in addition we require some \(\ast\)-structure to be preserved, then \(T\) has to belong to some suitable subgroup]
of $GL(M)$; e.g. $T \in U(M)$ if $(a^\dagger)^t = a^\dagger$. We try to construct now a variant of the algebra of section 3 having explicitly $GL_q(M) \times G_q$-covariant generators$^4$.

Let $T_\alpha^\beta$, $t = \det q || T_\alpha^\beta ||$ be the generators of the quasitriangular Hopf algebra $Fun[GL_q(M)]$, and $T_\alpha^a_b$ the generators of $Fun(G_q)$ $^1$. Let us introduce collective indices $A, B, ...$, denoting the pairs $(\alpha, a)$, $(\beta, b)$, .... The Hopf algebra $Fun(GL_q(M) \times G_q)$ can be defined as the algebra generated by objects $T_B^A$ satisfying commutation relations which can be obtained from (2.1) by the replacement

$$T_B^A \rightarrow T_\beta^\alpha T_\beta^a$$

by assuming that $[T_\alpha^\beta, T_\alpha^a_b] = 0$:

$$\hat{R}^{AB}_{\ CD} T^C_E T^D_F = T^A_C T^B_D \hat{R}^{CD}_{\ EF}.$$  

(5.3)

Here $\hat{R}$ is one of the matrices

$$\hat{R}^{AB}_{\ CD} := \hat{R}^{\pm \alpha \beta}_{\ M \gamma \delta} \hat{R}^{ab}_{\ cd} \equiv (\hat{R}^I_M \otimes \hat{R}^{AB})_{\ CD},$$

(5.4)

and $\hat{R}_M$ is the braid matrix $^2$ of $SL_q(M)$. $\hat{R}_\pm$ satisfies the braid equation, since $\hat{R}, \hat{R}_M$ do. The coproduct, counit, antipode and quasitriangular structure are introduced as in Sect. 2 by $\Delta(T_B^A) = T_C^A \otimes T_B^C$, $\varepsilon(T_B^A) = \delta_B^A$, $ST_B^A = T^{-1}_B$.

A (right) comodule algebra of $Fun(GL_q(M) \times G_q)$ can be associated to the defining corepresentation of the latter, $\phi_\gamma(A_\alpha^A) = A_B^+ \otimes T_B^A$, where $A_C^+$ denote the generators. The dual comodule algebra, with generators $A_C^-$, will be associated to the contragradient corepresentation $\phi_\gamma'(A^A) = A^B \otimes ST_B^A$. To find compatible quadratic commutation relations among the $A_C^+$'s (resp. $A^B$'s) we need the projector decomposition of $\hat{R}_\pm$, as in Sect. 4. For this scope we just need to write down the projector decompositions of both $\hat{R}^{\pm 1}_M$ and $\hat{R}$ and note that $\mathcal{P} := \mathcal{P}_M \otimes \mathcal{P}'$ is a projector $\mathcal{P}$ whenever $\mathcal{P}, \mathcal{P}'$ are.

We start with the case $G_q = SL_q(N)$. We find

$$\hat{R}_+ = (q \mathcal{P}_M^S - q^{-1} \mathcal{P}_M^A) \otimes (q \mathcal{P}_M^S - q^{-1} \mathcal{P}^A)$$

$$= -(\mathcal{P}_M^S \otimes \mathcal{P}^A + \mathcal{P}_M^A \otimes \mathcal{P}_M^S) + q^2 \mathcal{P}_M^S \otimes \mathcal{P}_M^S + q^{-2} \mathcal{P}_M^A \otimes \mathcal{P}_M^A$$

$$=: -\mathcal{P}^+ + q^2 \mathcal{P}_{S,1}^+ + q^{-2} \mathcal{P}_{S,2}^+$$

(5.5)

and

$$\hat{R}_- = (q^{-1} \mathcal{P}_M^S - q \mathcal{P}_M^A) \otimes (q \mathcal{P}_M^S - q^{-1} \mathcal{P}^A)$$

$^4$Or equivalently $SL_q(M) \times G_q$-covariance, if we impose also the unit condition on the $q$-

determinant of $GL_q(M)$.
\[
(P^A \otimes P^A + P^S \otimes P^S) - q^2 P^A_M \otimes P^S - q^{-2} P^S_M \otimes P^A
\]
\[=: P^+ - q^2 P^{A,1} - q^{-2} P^{A,2}. \tag{5.6}\]

We are in the condition to apply Lemma I. As a consequence, there exists a \(GL_q(M) \times SL_q(N)\)-covariant Weyl algebra \(A^q_{+,GL_q(M) \times SL_q(N),\phi_D}\), defined by the following commutation relations:

\[
P^{-CD}_{AB} A^+_C A^+_D = 0 \tag{5.7}
\]

\[
P^{-AB}_{CD} A^+_D A^+_C = 0 \tag{5.8}
\]

\[
A^A A^+_B - \delta^A_B 1 - \hat{R}^{AC}_{BD} A^+_C A^+_D = 0. \tag{5.9}
\]

Moreover, there exists a \(q\)-deformed \(SL_q(M) \times SL_q(N)\)-covariant Clifford algebra \(A^q_{-,SL_q(M) \times SL_q(N),\phi_D}\), defined by the following commutation relations:

\[
P^{+CD}_{AB} A^+_C A^+_D = 0 \tag{5.10}
\]

\[
P^{+AB}_{CD} A^+_D A^+_C = 0 \tag{5.11}
\]

\[
A^A A^+_B - \delta^A_B 1 + \hat{R}^{AC}_{BD} A^+_C A^+_D = 0. \tag{5.12}
\]

According to lemma II, one could give also alternative definitions with \(\hat{R}^{-1}\) instead of \(\hat{R}\) in relations (5.3), (5.12).

Let us verify that relations (5.7), (5.8), (5.10), (5.11) are of the kind considered in section I.

We take first relations (5.7) into account. We find

\[
(q + q^{-1})^2 P^{-}_{ik} = (q + q^{-1})^2 [P^S \otimes P^A + P^A \otimes P^S] (q 1 - \hat{R}_M) \otimes (q^{-1} 1 + \hat{R}) + (q^{-1} 1 + \hat{R} M) \otimes (q 1 - \hat{R}) = 2 (1 \otimes 1 - \hat{R} M \otimes \hat{R}) + (q - q^{-1})(1 \otimes \hat{R} + \hat{R} M \otimes 1).
\]

Using relation (2.8) we can write \(\hat{R}_M\) explicitly and check that relations (5.7) amount to relations

\[
P^{-}_{ij} A^+_{a,h} A^+_{a,k} = 0, \tag{5.13}
\]

\[
A^+_{a,i} A^+_{a,j} - \hat{R}^{hk}_{ij} A^+_{a,h} A^+_{a,k} = 0, \quad \text{if } \alpha < \beta. \tag{5.14}
\]

Similarly one verifies that: 1) relations (5.8) amount to relations

\[
P^{-}_{ij} A^\alpha_{h,k} A^\alpha_{a,h} = 0 \tag{5.15}
\]

\[
A^\beta_{i,j} A^\alpha_{a,h} - \hat{R}^{ij}_{hk} A^\beta_{a,k} A^\alpha_{a,h} = 0 \quad \text{if } \alpha < \beta; \tag{5.16}
\]

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2) that relations (5.10) amount to relations

\[ P^{+hk}_{ij}A^+_{a,h}A^+_{a,k} = 0 \]  \hspace{1cm} (5.17)

\[ A_{a,i}^+, A_{\beta,j}^+ + \hat{R}^{-1hk}_{ij}A^+_{\beta,h}A^+_{a,k} = 0, \quad \text{if } \alpha < \beta; \]  \hspace{1cm} (5.18)

3) that relations (5.11) amount to relations

\[ P^{+ij}_{hk}A^\alpha,k A^\alpha,h = 0 \]  \hspace{1cm} (5.19)

\[ A_{\alpha,j}^+, A_{\beta,i}^+ + \hat{R}^{-1ij}_{hk}A^\beta,k A^\alpha,h = 0 \quad \text{if } \alpha < \beta. \]  \hspace{1cm} (5.20)

On the other hand, relations (5.9), (5.12) for \( \alpha \neq \beta \) are not of the type (3.22), (3.23) found in section 3; in fact, in a similar way one can show that relation (5.9) takes the form

\[ A^\alpha,a A^\alpha,b + \hat{R}^{ac}_{bd}A^\beta,c A^\alpha,d = 0 \quad \alpha \neq \beta, \]  \hspace{1cm} (5.21)

whereas relation (5.12) amounts to

\[ A^\alpha,a A^\alpha,b + \delta^\alpha_b \mathbf{1} - q\hat{R}^{ac}_{bd}A^\beta,c A^\alpha,d - (q - q^{-1}) \sum_{\beta > \alpha} \hat{R}^{ac}_{bd}A^\beta,c A^\beta,d = 0 \]  \hspace{1cm} (5.22)

Relations (5.21), (5.23) specialized to the case \( \alpha > \beta \) coincide with relations (3.22); specialized to the case \( \alpha < \beta \), they differ from relations (3.23). Relations (5.22), (5.24) differ from relations (3.18) by the additional terms with coefficient \((q - q^{-1})\).

The subalgebra \( \mathcal{M} \) (resp. \( \mathcal{M}' \)) generated by \( A^+_\alpha \)'s (resp. \( A^\alpha \)'s) has the same Poincaré series of the subalgebra generated by classical \( a^+_\alpha \)'s (resp. \( a^\alpha \)'s), because of relations (5.13), (5.14) [resp. (5.13), (5.16)] in the Weyl case and because of relations (5.17), (5.18) [resp. (5.19), (5.20)] in the Clifford case. Since relations (5.9), (5.12) allow to change the order of \( A^+_\alpha \)'s and \( A^\beta \)'s in any product, we conclude that

**Proposition 3** The algebras \( A^q_{\pm, GL_q(\mathcal{M}) \times SL_q(N), \phi_D} \) have the same Poincaré series as their classical counterparts.

\(^5\)These are of the type considered in Sect. 2, provided we invert the order of greek indices.
Finally, let us ask about \(\star\)-structures. When \(q \in \mathbb{R}^+\) the Hopf algebra \(GL_q(M) \times SL_q(N)\) admits the compact section \(U_q(M) \times SU_q(N)\) \([10]\). The deformed Heisenberg algebras defined by relations (5.7-5.12) admit a natural \(U_q(M) \times SU_q(N)\)-covariant \(\star\)-structure given by

\[(A^A)^\star = A^+_A; \tag{5.25}\]

this can be easily checked by applying this \(\star\) to relations (5.7-5.12) and by noting that \(\hat{R}^T = \hat{R}\) and therefore \(\hat{R}^{-1} \hat{R} = \hat{R}, P^T = P\).

Let us take now in consideration the cases that \(G_q = SO_q(N), Sp_q(n)\). The projector decomposition of \(\hat{R}_M \otimes \hat{R} = \sum_{\mu} \lambda_{\mu} P_{\mu}\) gives \(\lambda_{\mu} = q^2, q^{-2}, -1, \pm q^{2-N}, \mp q^{-N}\), where the upper and lower sign refer to \(G_q = SO_q(N)\) and \(Sp_q(n)\) respectively. The projector decomposition of \(\hat{R}_M^{-1} \otimes \hat{R} = \sum_{\mu} \lambda_{\mu} P_{\mu}\) gives \(\lambda_{\mu} = -q^2, -q^{-2}, 1, \mp q^{2-N}, \pm q^{-N}\). In both cases we always have more than one positive and more than one negative \(\lambda_{\mu}\). By Lemma \([1]\) no \(GLq(M) \times G_q\) covariant \(q\)-deformed Weyl/Clifford algebra can be built by this procedure.

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