DECOMPOSITION MATRICES FOR THE GENERIC HECKE ALGEBRAS
ON 3 STRANDS IN CHARACTERISTIC 0

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Abstract. We classify all the decomposition matrices of the generic Hecke algebras on 3 strands in characteristic 0. These are the generic Hecke algebras associated to the exceptional complex reflection groups $G_4$, $G_8$ and $G_{16}$. We prove that for every choice of the parameters that define these algebras, all ordinary representations are obtained as modular reductions of irreducible representations.

1. Introduction

Between 1994 and 1998, Broué, Malle, and Rouquier generalized the definition of the Iwahori-Hecke algebra to the case of an arbitrary complex reflection group $W$ (see [4]). This generalized algebra, which we denote by $H^W$, is known as the generic Hecke algebra. It is defined over the Laurent polynomial ring $\mathbb{Z}[u_1^\pm, \ldots, u_m^\pm]$, where $\{u_i\}_{1 \leq i \leq m}$ is a set of parameters whose cardinality depends on $W$. In 1999, Malle proved that $H^W$ is split semisimple when defined over the field $\mathbb{C}(v_1, \ldots, v_m)$, where each parameter $v_i$ arises in a specific way from the parameter $u_i$ (see [10], theorem 5.2). As a result, Tits’ deformation theorem yields a bijection between the set of irreducible characters of $H^W$ and the set $\text{Irr}(W)$ of irreducible characters of $W$.

It is natural to wonder how the irreducible characters behave after specializing the parameters $v_i$ to arbitrary complex numbers. If the specialized Hecke algebra is semisimple, Tits’ deformation theorem still applies; the irreducible characters of the specialized Hecke algebra are parametrized again by $\text{Irr}(W)$. However, this is not always the case.

If the specialized algebra is not semisimple, one needs to take a different approach. The irreducible characters of the semisimple Hecke algebra may not remain irreducible after specialization, however they are a linear combination of irreducible characters of the specialized algebra.

One can define the decomposition matrix, which records the coefficients of this linear combination. The rows of the decomposition matrix are indexed by the ordinary characters and its columns by the modular irreducible ones. This matrix offers an optical depiction of the representation theory of the specialized algebra.

In 2011 M. Chlouveraki and H. Miyachi worked with the cyclotomic Hecke algebras for $d$-Harish-Chandra series of rank 2 (see [7]). In this case, the algebra depends only on one parameter. Giving specific values to this parameter, they managed to classify the decomposition matrices for these cases. At this point, a number of questions arise: why do these values provide different decomposition matrices? Is this a classification in the cyclotomic case outside the $d$-Harish-Chandra series or are there any other matrix models that are not described by M. Chlouveraki and H. Miyachi?

What happens in the generic case, where there are more than one parameters?

Let $n$ be the number of the irreducible characters of the semisimple Hecke algebra and $m$ the number of the irreducible characters of the specialized algebra. The main theorem of this paper is the following:

Theorem 1.1. Let $W$ be one of the exceptional groups $G_4$, $G_8$ and $G_{16}$. When the decomposition matrix is not the identity, we have $m < n$. Moreover, one can reorder the rows of the matrix, so that it takes the following form:
The appearance of the identity matrix proves that all ordinary representations are obtained as modular reductions of irreducible representations. This result is not always the case (see, for example, §18.6 in [15] or Exercise 8.9 (e) in [9]). Moreover, the fact that $m < n$ may have as explanation that the center of $H_W$ is reduction stable (see theorem 7.5.6 in [9]). However, the center of $H_W$ is not known yet, and such a condition couldn’t be checked.

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2. Preliminaries

2.1. Generic Hecke algebras on 3 strands. Let $W$ be a complex reflection group on a finite dimensional $\mathbb{C}$-vector space $V$ and let $B$ denote the complex braid group associated to $W$, in the sense of Broué-Malle-Rouquier (see [3]). A pseudo-reflection $s$ is called distinguished if its only nontrivial eigenvalue on $V$ equals $e^{-2\pi i k/e_s}$, where $i$ denotes a chosen imaginary unit and $e_s$ the order of $s$ in $W$. To every distinguish pseudo-reflection $s$, one can associate homotopy classes in $B$, that we call braided reflections (for more details one may refer to [4]).

Let $R$ denote the Laurent polynomial ring $\mathbb{Z}[u_{s,1}, u_{s,1}^{-1}]$, where $r$ runs over the set of the distinguished pseudo-reflections of $W$, $1 \leq i \leq e_s$, and $u_{s,i} = u_{t,i}$ if $s$ and $t$ are conjugate in $W$. The generic Hecke algebra $H$ associated to $W$ with parameters $u_{s,1}, \ldots, u_{s,e_s}$, is the quotient of the group algebra $RB$ of $B$ by the ideal generated by the elements of the form

$$(\sigma - u_{s,1})(\sigma - u_{s,2})\ldots(\sigma - u_{s,e_s}),$$

where $s$ runs over the conjugacy classes of the set of the distinguished pseudo-reflections of $W$ and $\sigma$ over the set of braided reflections associated to $s$. It is enough to choose one such relation per conjugacy class, since the corresponding braided reflections are conjugate in $B$.

Let $B_3$ be usual braid group on 3 strands, given by generators the braids $s_1$ and $s_2$ and the (single) relation $s_1s_2s_1 = s_2s_1s_2$. We denote by $W_k$ the quotients of $B_3$ by the additional relation $s_i^k = 1$, for $i = 1, 2$. Due to a Coxeter’s theorem (see §10 in [8]), these quotients are finite if and only if $k \in \{2, 3, 4, 5\}$. Apart from the case where $k = 2$, which leads to the symmetric group $S_3$, we encounter the exceptional complex reflection groups $G_4, G_8$ and $G_{16}$, for $k = 3, 4, 5$, respectively, as they are known in the Shephard-Todd classification (see [16]).

Definition 2.1. The generic Hecke algebra $H_k$ associated to $W_k$, $k = 3, 4, 5$, is called the generic Hecke algebra on 3 strands.

Remark 2.2. By the relation $s_1s_2s_1 = s_2s_1s_2$ we have that $s_1$ and $s_2$ are conjugate in $W_k$ and, hence, the algebra $H_k$ is defined over the ring $R_k := \mathbb{Z}[u_1^{\pm 1}, \ldots, u_k^{\pm 1}], k = 3, 4, 5$.

The following theorem is theorem 1.1 in [5].

Theorem 2.3. $H_k$ is a free $R_k$-module of rank $|W_k|$.

For the rest of this paper, we make the following assumption. His assumption has been verified for the case of $H_3$ (see Proposition 2.3 (ii) in [12]).

Assumption 2.4. $H_k$ has a unique symmetrizing trace $t_k : H_k \to R_k$ having the properties described in [3], theorem 2.1.
2.2. Schur elements. We denote by $K_k$, $k = 3, 4, 5$, the field of definition of $W_k$ (for more details see [1]). We denote by $\mu(K_k)$ the group of all roots of unity of $K_k$ and, for every integer $m > 1$, we set $\zeta_m := \exp(2\pi i / m)$, where $i$ denotes a square root of -1.

Let $v = (v_1, ..., v_k)$ be a set of $k$ indeterminates such that, for every $i \in \{1, ..., k\}$, we have $v_i^{\mu(K_k)} = \zeta_k^{-i}u_i$. By extension of scalars we obtain the algebra $\mathbb{C}(v)H_k := H_k \otimes_{R_k} \mathbb{C}(v)$, which is split semisimple (see [10], theorem 5.2). Hence, by Tits' deformation theorem (see theorem 7.4.6 in [9]), the specialization $v_i \mapsto 1$ induces a bijection $\text{Irr}(\mathbb{C}(v)H_k) \rightarrow \text{Irr}(W_k)$, $\chi_k \mapsto \chi$. By theorem 7.2.6 in [9] we have:

$$t_k = \sum_{\chi \in \text{Irr}(W_k)} \frac{1}{s_\chi} \chi_k,$$

where $s_\chi$ denotes the Schur element of $H_k$ associated to $\chi \in \text{Irr}(W_k)$, with respect to the symmetric form $t_k$.

M. Chlouveraki has shown that these elements are products of cyclotomic polynomials over $K_k$, evaluated on monomials of degree 0 (see theorem 4.2.5 in [6]). One can refer to J. Michel’s version of CHEVIE package of GAP3 (see [14]) for this factorization.

Example 2.5. We consider the case of $G_4$, which is we denote in this paper as $W_3$. In CHEVIE the parameters must be in $M\wp$ form (which stands for multivariate polynomials). We type:

```gap
gap> W_3:=ComplexReflectionGroup(4);;
gap> H_3:=Hecke(W_3,[Mwp(“u1"), Mwp(“u2"), Mwp(“u3")]);;
gap> CharNames(W_3);
[ "phi(1,0)", "phi(1,4)", "phi(1,8)", "phi(2,5)", "phi(2,3)", "phi(2,1)", "phi(3,2)" ]
```

We see that $W_3$ admits 7 irreducible representations, which are symbolized by $\phi_{i,j}$ in GAP notation, with $i$ denoting the degree and $j$ the fake degree of the representation.

We will find now the factorization of the Schur element of $H_3$ associated to the character $\phi_{1,4}$. This character is in the second position in the above list. We type:

```gap
gap> S:=FactorizedSchurElements(H_3);;
gap> S[2];
-u1^-4u2^-5u3^-1P1P6(u2u3^-1)P1P6(u1u2^-1)P1P6(u1u3^-1)
```

Hence, the Schur element $s_{\phi_{1,4}}$ is the following:

$$-u_3^5 u_2^2 \Phi_2(u_1 u_2^{-2} u_3) \Phi_1(u_1 u_2) \Phi_0(u_2 u_3^{-2} u_1) \Phi_1(u_2 u_3) \Phi_0(u_2 u_3),$$

where $\Phi_1$, $\Phi_2$ and $\Phi_0$ denote the cyclotomic polynomials $x-1$, $x+1$, and $x^2 - x + 1$, respectively. \hfill \square

2.3. Decomposition matrix. Let $\theta : \mathbb{C}[v, v^{-1}] \rightarrow \mathbb{C}$ be a ring homomorphism, such that $\theta(u_i) \in \mathbb{C}$. We call such morphism a specialization of $\mathbb{C}[v, v^{-1}]$. We set $\mathbb{C}H_k := H_k \otimes_{\mathbb{C}} \mathbb{C}$. This algebra is split, since it is a $\mathbb{C}$-algebra (and, hence, assumption 7.4.1 (a) in [9] is satisfied).

We suppose now that $\mathbb{C}H_k$ is also semisimple. Using Tits’ deformation theorem again, we obtain a canonical bijection between the set of irreducible characters of $\mathbb{C}H_k$ and the set of irreducible characters of $\mathbb{C}(v)H_k$, which are in bijection with the irreducible characters of $W_k$, as mentioned in section 2.2.

We will now examine the behavior of the irreducible representations of $\mathbb{C}H_k$ in the non-semisimple case. Let $R^+_0(\mathbb{C}(v)H_k)$ (respectively $R^+_0(\mathbb{C}H_k)$) denote the subset of the Grothendieck group of the category of finite dimensional $\mathbb{C}(v)H_k$ (respectively $\mathbb{C}H_k$)-modules. This set consists of elements $[V]$, where $V$ is a $\mathbb{C}(v)H_k$ (respectively $\mathbb{C}H_k$)-module, with relations $[V] = [V'] + [V'']$, for each exact sequence $0 \rightarrow V' \rightarrow V \rightarrow V'' \rightarrow 0$ (for more details, one may refer to 7.3 in [9]).

By theorem 7.4.3 in [9] we obtain a well-defined decomposition map

$$d_0 : R^+_0(\mathbb{C}(v)H_k) \rightarrow R^+_0(\mathbb{C}H_k).$$

Compared with the semisimple case, where every module can be written as a direct product of irreducible ones, in the non-semisimple case we have an analogous with the expressions $d_0([V])$. More precisely, let $V_{\chi}$ be an irreducible $\mathbb{C}(v)H_k$-module with character $\chi$. We set $[M] := d_0([V_{\chi}])$. We suppose now that $M$ is not irreducible. Then, it admits an irreducible submodule $S_M$, which gives rise to the exact sequence $0 \rightarrow S_M \rightarrow M \rightarrow M/S_M \rightarrow 0$. By definition, $[M] =$
Our goal is to write $[M]$ as a sum of irreducible classes of $\mathbb{C}H_k$-modules. If the module $M/S_M$ is irreducible, the result is obvious. If not, we have a new exact sequence for the module $M/S_M$ and, repeating the procedure, we finally obtain

$$d_\theta([V_\chi]) = \sum_{\phi} d^\theta_{\chi,\phi}[V_\phi].$$

where $d^\theta_{\chi,\phi}$ are non-negative integers and $V_\phi$ denotes the irreducible $\mathbb{C}H_k$-module that corresponds to character $\phi$. Notice that the above procedure terminates, since the dimension of the quotient module is strictly decreasing, and every 1-dimensional module is necessarily irreducible.

**Definition 2.6.** The $\text{Irr}(\mathbb{C}(v)H_k) \times \text{Irr}(\mathbb{C}H_k)$ matrix $(d^\theta_{\chi,\phi})$ is called the decomposition matrix associated to $d_\theta$.

We say that the $\mathbb{C}(v)H_k$-modules $V_\chi, V_\psi$ belong to the same block if there is a $\phi \in \text{Irr}(\mathbb{C}H_k)$ such that $d^\theta_{\chi,\phi} \neq 0 \neq d^\theta_{\psi,\phi}$. If an irreducible $\mathbb{C}(v)H_k$-module is alone in its block, then we call it a module of defect 0.

**2.4. Optimal basic sets.** We recall that in the semisimple case, the irreducible representations of $\mathbb{C}H_k$ are parametrized by the irreducible representations of $W_k$. The definition of optimal basic sets, in the sense of Chlouveraki-Miyachi (see [7]), give ways to parametrize irreducible representations of $\mathbb{C}H_k$ in the non-semisimple case. The following definition is definition 1 in [7].

**Definition 2.7.** An optimal basic set $B^{opt}$ for $\mathbb{C}H_k$ with respect to $\theta$ is a subset of $\text{Irr}(W_k)$ such that the following two conditions are satisfied:

1. For all $\phi \in \text{Irr}(\mathbb{C}H_k)$ there exists $\chi_\phi \in B^{opt}$ such that
   a) $d^\theta_{\chi_\phi,\phi} = 1$,
   b) If $d^\theta_{\chi_\phi,\phi} \neq 0$ for some $\psi \in \text{Irr}(W_k)$, then either $\psi = \chi_\phi$ or $\psi \not\in B^{opt}$.
2. The map $\text{Irr}(\mathbb{C}H_k) \rightarrow B^{opt}, \phi \mapsto \chi_\phi$ is a bijection.

Hence, an optimal basic set for $\mathbb{C}H_k$ is a special labeling of the columns of the decomposition matrix such that, after reordering its rows, its upper part is the identity matrix.

**Remark 2.8.** If the algebra $\mathbb{C}H_k$ is semisimple, $B^{opt} = \text{Irr}(W_k)$.

The rest of this paper is devoted to the proof of the existence of an optimal basic set for $\mathbb{C}H_k$, $k = 3, 4, 5$ in the non-semisimple case, with respect to any specialization $\theta$.

3. A classification of the decomposition matrices of $W_k$

3.1. **Notation.** Following the notation in GAP, we denote by $E(n)$, $n \in \mathbb{N}$, the primitive $n$-th root of unity $e^{2\pi i/n}$.

3.2. **Methodology.** Motivated by the idea of M. Chlouveraki and H. Miyachi in [7] §3.1 we use the following criteria in order to calculate the decomposition matrix for $W_k$, $k = 3, 4, 5$. We have also used some of these criteria in [5], in order to classify the irreducible representations of $B_3$.

**Criterion 3.1.** Every 1-dimensional $\mathbb{C}H_k$-module $V_\phi$ is irreducible. Moreover, there is an 1-dimensional character $\chi \in \text{Irr}(\mathbb{C}(v)H_k)$, such that $d_\theta([V_\chi]) = [V_\phi]$.

**Proof.** By definition, the algebra $\mathbb{C}H_k$ is the quotient of the group algebra $\mathbb{C}B_3$ by the relations $(s_i - \theta(u_i)) \ldots (s_i - \theta(u_k)) = 0$, $i = 1, 2$. As a result, the 1-dimensional $\mathbb{C}H_k$-modules are of the form $s_1, s_2 \mapsto (\theta(u_j)), j = 1, \ldots, k$.

**Criterion 3.2.** 2-dimensional modules admit only 1-dimensional submodules.

We recall that $s_\chi$ denotes the Schur element associated to $\chi$. The next criterion summarized the results of G. Malle and R. Rouquier (see [11], Lemma 2.6) and M. Geck and G. Pfeiffer (see [9], theorem 7.5.11).
Criterion 3.3. \( \theta(s_i) \neq 0 \) if and only if \( V_\chi \) is a module of defect 0. Moreover, the decomposition matrix is of the form

\[
\begin{pmatrix}
0 \\
\vdots \\
\vdots \\
0 \\
x & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \\
\vdots \\
0 \\
\end{pmatrix}
\]

The next criterion follows directly from Lemma 7.5.10 in [9].

Criterion 3.4. If \( V_\chi, V_\psi \) are in the same block, then \( \theta(\omega_\chi(z_0)) = \theta(\omega_\psi(z_0)) \), where \( \omega_\chi, \omega_\psi \) are the corresponding central characters and \( z_0 \) is the central element \((s_1s_2)^3\).

Criterion 3.5. Modular restrictions of the irreducible characters of \( \mathbb{C}(v)H_k \) can be written uniquely as \( \mathbb{N} \)-linear combinations of the irreducible characters of \( \mathbb{C}H_k \).

Proof. Every \( \mathbb{C}(v)H_k \)-character can be written as \( \mathbb{N} \)-linear combination of the irreducible characters of \( \mathbb{C}H_k \). It remains to prove that the irreducible characters of \( \mathbb{C}H_k \) are linearly independent. Since the algebra \( \mathbb{C}H_k \) is split, the linear independence follows directly from Lemma 4.36 in [13].

Notice that the above criteria can be used for any finite dimensional, symmetric algebra defined over a field. The following propositions are applied only to the generic Hecke algebra on 3 strands, and give us a necessary and sufficient condition for a 2-dimensional, a 3-dimensional, and a 4-dimensional \( \mathbb{C}H_k \)-module to be irreducible.

Let \( a_i := \theta(v_i), i = 1, \ldots, k \), where \( \theta \) is the specialization defined in 2.3 and \( v_i \) the parameters defined in 3.2. We endow \( \{a_1, a_2, \ldots, a_k\} \) with the total order \( a_1 < a_2 < \cdots < a_k \). Let \( V^m, m \in \{2, 3, 4\} \), be an irreducible \( m \)-dimensional \( \mathbb{C}(v)H_k \)-module. We denote by \( U^m_{b_1, b_2, \ldots, b_m} \) the \( m \)-dimensional \( \mathbb{C}H_k \)-module, such that \( \theta(V^m) = [U^m_{b_1, b_2, \ldots, b_m}] \), where \( b_1, b_2, \ldots, b_m \in \{a_1, a_2, \ldots, a_k\} \) with \( b_1 < b_2 < \cdots < b_m \). Notice that the coefficient of the matrix models of these modules depend only on \( b_1, b_2, \ldots, b_m \). One can find these matrix models in [2] or with the use of GAP. For example, when \( k = 4 \), one can type:

```gap
gap> W:=ComplexReflectionGroup(3);
gap> a:=Mwp("a");
gap> b:=Mwp("b");
gap> c:=Mwp("c");
gap> d:=Mwp("d");
gap> H:=Hecke(W,[[a,b,c,d]]);
gap> R:=Representations(H,4);
```

and ask, for example, for the matrix \( R[5] \) that corresponds to the 2-dimensional module \( U_{a_1,a_2}^2 \).

**Proposition 3.6.** The \( \mathbb{C}H_k \)-module \( U_{b_1,b_2}^2 \) is irreducible if and only if \( b_1^2 - b_1b_2 + b_2^2 \neq 0 \).

Proof. We prove the case where \( k = 3 \). The other cases can be proven similarly. The matrix form of the \( \mathbb{C}H_k \)-module \( U_{b_1,b_2}^2 \) is the following:

\[
s_1 \mapsto A := \begin{pmatrix}
b_1 & 0 \\
-1 & b_2 \\
\end{pmatrix}, \quad s_2 \mapsto B := \begin{pmatrix}
b_2 & b_1 \\
0 & 0 \\
\end{pmatrix}.
\]

This module is irreducible if and only if it doesn’t admit 1-dimensional subrepresentations (criterion 3.2). This is equivalent to the fact that the matrices \( A \) and \( B \) don’t have a common eigenvector. The eigenvalues of \( A \) are \( b_1 \) and \( b_2 \) with corresponding eigenvectors \( v_{b_1} = (b_1^{-1}(b_2 - b_1) \ 1)^\top \) and \( v_{b_2} = (0 \ 1)^\top \). It is easy to check that \( v_{b_2} \) is not an eigenvector for \( B \). Moreover, we have \( Bv_{b_1} = (b_1^{-1}b_2^2 \ b_1) = b_1(b_1^{-2}b_2^2 \ 1)^\top \), which means that \( v_{b_1} \) is not an eigenvector for \( B \) if and only if \( b_1^{-2}b_2^2 \neq b_1^{-1}(b_2 - b_1) \), which concludes the proof. □
Corollary 3.7. Let \( \{a_1, a_2, \ldots, a_k\} \) has cardinality at least 2. Then, for every specialization \( \theta \), at least one \( U_{b_1, b_2}^2 \) is irreducible.

Proof. Suppose that there is a specialization \( \theta \) such that all the \( CH_k \)-modules \( U_{b_1, b_2} \) are non-irreducible. Due to proposition \( \text{[20]} \), we have that \( b_2^2 - b_1 b_2 + b_1^2 = 0 \) for every \( b_1, b_2 \in \{a_1, a_2, \ldots, a_k\} \) with \( b_1 < b_2 \). Since \( \{a_1, a_2, \ldots, a_k\} \) has cardinality at least 2, we can assume that \( a_2 \neq a_3 \). Since \( a_2^2 - a_1 a_2 + a_2^2 = a_3^2 - a_1 a_3 + a_2^2 = 0 \) and \( a_2 \neq a_3 \), we may assume that \( a_2 = -E(3)a_1 \) and \( a_3 = -E(3)a_1 \). Therefore, we have \( a_2^2 - a_2 a_3 + a_3^2 \neq 0 \), which contradicts the hypothesis. \( \square \)

Proposition 3.8. The \( CH_k \)-module \( U_{b_1, b_2, b_3}^2 \) is irreducible if and only if

\[
(b_1^2 - b_2 b_3)(b_2^2 - b_1 b_3)(b_3^2 - b_1 b_2) \neq 0.
\]

Proof. In general, a 3-dimensional module is irreducible if and only if it doesn’t admit 1-dimensional and 2-dimensional submodules. Let \( s_1 \mapsto A \) and \( s_2 \mapsto B \) the matrix form of the \( CH_k \)-module \( U_{b_1, b_2, b_3}^2 \). The existence of an 1-dimensional submodule translates into the existence of a common eigenvector for the matrices \( A \) and \( B \).

Let \( DU_{b_1, b_2, b_3}^2 := \text{Hom}_C(U_{b_1, b_2, b_3}^2, C) \) the \( CH_k^{op} \)-module, with action \((f \circ h)(u) = f(hu)\), for every \( f \in DU_{b_1, b_2, b_3}^2, h \in CH_k \) and \( u \in U_{b_1, b_2, b_3}^2 \). Since \( CH_k \) is a finite dimensional algebra defined over a field, the existence of a 2-dimensional \( CH_k \)-submodule (and, hence, the existence of a 1-dimensional \( CH_k \)-quotient) yields to the existence of an 1-dimensional \( CH_k^{op} \)-submodule. As a result, the transposed matrices \( A^T \) and \( B^T \) must have a common eigenvector.

Summing up, the \( CH_k \)-module \( U_{b_1, b_2, b_3}^2 \) is irreducible if and only if the matrices \( A \) and \( B \), on one hand, and the matrices \( A^T \) and \( B^T \) on the other, don’t have a common eigenvector. Using the method described in the proof of proposition \( \text{[20]} \), we conclude the proof. \( \square \)

Proposition 3.9. Let \( k \neq 3 \). The \( CH_k \)-module \( U_{b_1, b_2, b_3, b_4}^4 \) is irreducible if and only if

\[
(b_1^3 - b_2 b_3)(b_2^3 - b_1 b_3)(b_3^3 - b_1 b_2)(b_4^3 - b_1 b_2 b_3 + b_1 b_2 b_4 + b_1^2 b_3^2) \neq 0,
\]

for every \( m, r, l, s \in \{1, 2, 3, 4\} \), such that the set \( \{m, r, l, s\} \) is of cardinality 4.

Proof. In general, a 4-dimensional module is irreducible if and only if it doesn’t admit 1-dimensional, 2-dimensional and 3-dimensional submodules. Let \( s_1 \mapsto A \) and \( s_2 \mapsto B \) the matrix form of the \( CH_k \)-module \( U_{b_1, b_2, b_3, b_4}^4 \). The existence of an 1-dimensional submodule translates into the existence of a common eigenvector, for the matrices \( A \) and \( B \). As in proof of proposition \( \text{[20]} \), the existence of a 3-dimensional submodule means that the transposed matrices \( A^T \) and \( B^T \) must have a common eigenvector. Following the method we explained in proof of proposition \( \text{[20]} \), we conclude that there aren’t any 1-dimensional and 3-dimensional submodules if and only if \( b_1^3 - b_2 b_3 b_4 \neq 0 \), with \( m, r, l, s \) as in hypothesis.

It remains to examine the existence of a 2-dimensional submodule. Let \( W \) be 2-dimensional \( CH_k \)-submodule of \( U_{b_1, b_2, b_3, b_4}^4 \). As \( C \)-vector spaces, let \( U_{b_1, b_2, b_3, b_4}^4 = \langle u_1, u_2, u_3, u_4 \rangle \) and \( W = \langle w_1, w_2 \rangle \). We write \( w_1 \) and \( w_2 \) as \( C \)-linear combinations of \( u_1, \ldots, u_4 \) and we have \( w_1 = \sum_{i=1}^{4} x_i u_i \) and \( w_2 = \sum_{i=1}^{4} y_i u_i \). Since \( s_1 w_1 \in W \), there are \( \alpha, \beta \) such that \( \sum_{i=1}^{4} x_i (s_1 u_i) = \alpha w_1 + \beta w_2 \). Let \( A := (a_{ij}) \).

We have \( \sum_{i=1}^{4} x_i \left( \sum_{j=1}^{4} a_{ij} u_j \right) = \alpha \sum_{i=1}^{4} x_i u_i + \beta \sum_{i=1}^{4} y_i u_i \). The last equation gives \( \sum_{i=1}^{4} x_i a_{ij} = \alpha x_j + \beta y_j \), for every \( j \in \{1, 2, 3, 4\} \). Equivalently,

\[
(A^T - \alpha I_4) \begin{pmatrix} x_1 & x_2 & x_3 & x_4 \end{pmatrix} = \beta \begin{pmatrix} y_1 & y_2 & y_3 & y_4 \end{pmatrix}.
\]

Similarly, since \( s_1 w_2 \in W \), we have

\[
(A^T - \delta I_4) \begin{pmatrix} y_1 & y_2 & y_3 & y_4 \end{pmatrix} = \gamma \begin{pmatrix} x_1 & x_2 & x_3 & x_4 \end{pmatrix},
\]

for some \( \gamma, \delta \in C \). As a result, we have:

\[
(A^T - \alpha I_4)(A^T - \delta I_4) = \beta \gamma \begin{pmatrix} y_1 & y_2 & y_3 & y_4 \end{pmatrix},
\]

which contradicts the hypothesis. \( \square \)
meaning that \((y_1 \ y_2 \ y_3 \ y_4)^T\) is an eigenvector for the matrix \((A^T - \alpha I_4)(A^T - \delta I_4)\). Similarly one can prove that there are \(\alpha', \delta' \in \mathbb{C}\) such that \((y_1 \ y_2 \ y_3 \ y_4)^T\) is an eigenvector for the matrix \((B^T - \alpha' I_4)(B^T - \delta' I_4)\).

Summing up, the \(CH_k\)-module \(U_{b_1,b_2,b_3,b_4}^4\) admits a 2-dimensional submodule if and only if there are \(a, a', \beta, \beta' \in \mathbb{C}\) such as the matrices \((A^T - \alpha I_4)\) \((A^T - \delta I_4)\) and \((B^T - \alpha' I_4)(B^T - \delta' I_4)\) have a common eigenvector (not necessarily for the same eigenvalue). Using Maple we proved that these matrices don’t admit a common eigenvector if and only if \(b_m^2b_r^2 + b_1b_2b_3b_4 + b_l^2b_s^2 \neq 0\), with \(m, r, l, s\) as in hypothesis. 

### 3.3. The case \(k = 3\)

We recall that the complex reflection group \(W_3\) is the one denoted by \(G_4\) in the Shephard-Todd classification and admits the Coxeter-like presentation

\[
\left\langle s_1, s_2 \mid s_1^3 = s_2^3 = 1, s_1s_2s_1 = s_2s_1s_2 \right\rangle.
\]

The Hecke algebra \(H_3\) is defined over the Laurent polynomial ring \(R_3 := \mathbb{Z}[u_1^\pm, u_2^\pm, u_3^\pm]\). We identify \(s_i\) to their images in \(H_3\), and the latter admits the presentation

\[
\left\langle s_1, s_2 \mid s_1s_2s_1 = s_2s_1s_2, (s_i - u_1)(s_i - u_2)(s_i - u_3) = 0 \right\rangle, \text{ for } i = 1, 2.
\]

We fix a specialization \(\theta : \mathbb{C}[v_1^\pm, v_2^\pm, v_3^\pm] \rightarrow \mathbb{C}\) of \(\mathbb{C}[v_1^\pm, v_2^\pm, v_3^\pm]\) (the parameters \(v_i\) defined as in \[22\]) such that \(u_1 \mapsto a, u_2 \mapsto b,\) and \(u_3 \mapsto c\).

As we saw in example \[25\] \(W_3\) admits 7 irreducible characters, which are symbolized by \(\phi_{i,j}\), with \(i\) denoting the degree and \(j\) the fake degree of the representation. More precisely, we have three 1-dimensional characters (the characters \(\phi_{1,0}, \phi_{1,4}\) and \(\phi_{1,8}\)), three 2-dimensional characters (the characters \(\phi_{2,5}, \phi_{2,3}\) and \(\phi_{2,1}\)) and one 3-dimensional character (the character \(\phi_{3,2}\)).

We will now classify the decomposition matrices, by distinguishing the following cases. Notice that this classification is up to reordering of the characters.

- **The set \(\{a, b, c\}\) has cardinality 1:** In this case, the 1-dimensional characters correspond to the same module, as well as the 2-dimensional ones. Due to criterion \[5.1\] and propositions \[3.6\] and \[3.8\] all the characters remain irreducible. As a result, the decomposition matrix is the following:

\[
\begin{pmatrix}
\phi_{1,0} & 1 & 1 & 1 & 1 \\
\phi_{2,5} & 1 & -1 & 1 & -1 \\
\phi_{3,2} & 1 & 1 & 1 & 1 \\
\phi_{1,4} & 1 & -1 & 1 & -1 \\
\phi_{1,8} & 1 & 1 & 1 & 1 \\
\phi_{2,3} & 1 & -1 & 1 & -1 \\
\phi_{2,1} & 1 & 1 & 1 & 1 \\
\end{pmatrix}
\]

- **The set \(\{a, b, c\}\) has cardinality 2:** Without loss of generality we assume that \(a = b \neq c\). As a result, the 1-dimensional characters \(\phi_{1,0}\) and \(\phi_{1,4}\) correspond to the same module, as well as the 2-dimensional characters \(\phi_{2,5}\) and \(\phi_{2,3}\). Moreover, due to proposition \[3.6\] the character \(\phi_{2,1}\) remains irreducible. We distinguish the following cases, based on whether or not the other two 2-dimensional characters, which correspond to the same module, remain irreducible:

\[C1. \ a^2 - ac + c^2 = 0 \] Due to proposition \[3.6\] the character \(\phi_{2,5}\) does not remain irreducible. We use criterion \[5.3\] to write the class of the corresponding module as a sum of two classes of 1-dimensional \(CH_3\)-modules.

\[
gap\text{T:=CharTable(H_3).irreducibles;} \\
gap\text{t:=List(T,i->List(1,j->Value(j,"b", a, "c", -E(3)*a))));} \\
gap\text{t[5]+t[1]+t[3];} \\
gap\text{true}
\]

Due to criterion \[5.3\] we have \(d_\theta([V_{\phi_{2,3}}]) = d_\theta([V_{\phi_{1,0}}]) + d_\theta([V_{\phi_{1,8}}])\). Moreover,

\[
gap\text{s:=SchurElements(H_3);} \\
gap\text{List(s,i->Value(i,"b","c",-E(3)*a));} \\
[0, 0, 0, 0, 0, E3, 3 ]
\]
Due to criterion 3.3, the characters $\phi_2,1$ and $\phi_3,2$ are of defect 0. As a result, the decomposition matrix is the following:

\[
\begin{pmatrix}
\phi_{1,0} & \phi_{1,8} \\
\phi_{2,1} & I_4 \\
\phi_{3,2} & \\
\phi_{4,1} & 1 & 1 & 1 \\
\phi_{4,2} & 1 & 1 & 1 \\
\phi_{4,3} & 1 & 1 & 1 \\
\phi_{5,1} & 1 & 1 & 1 \\
\end{pmatrix}
\]

C2. $a^2 - ac + c^2 \neq 0$: Due to proposition 3.6 the character $\phi_{2,5}$ is irreducible. It remains to investigate the behavior of the character $\phi_{3,2}$. Due to proposition 3.8 one has to examine the following cases:

$\triangleright a = -c$.

Using GAP and criterion 3.3 as before, we have $d_\theta([V_{\phi_{3,2}}]) = d_\theta([V_{\phi_{1,8}}]) + d_\theta([V_{\phi_{2,1}}])$. As a result, the decomposition matrix is the following:

\[
\begin{pmatrix}
\phi_{1,0} & \phi_{1,8} \\
\phi_{2,1} & I_4 \\
\phi_{3,2} & \\
\phi_{4,1} & 1 & 1 & 1 \\
\phi_{4,2} & 1 & 1 & 1 \\
\phi_{4,3} & 1 & 1 & 1 \\
\phi_{5,1} & 1 & 1 & 1 \\
\end{pmatrix}
\]

$\triangleright a^2 = -c^2$.

$\triangleright (a + c)(a^2 + c^2) \neq 0$. Due to proposition 3.8 the character $\phi_{3,2}$ remains irreducible and, as a result, the decomposition matrix is the following:

\[
\begin{pmatrix}
\phi_{1,0} & \phi_{1,8} \\
\phi_{2,5} & I_4 \\
\phi_{3,2} & \\
\phi_{4,1} & 1 & 1 & 1 \\
\phi_{4,2} & 1 & 1 & 1 \\
\phi_{4,3} & 1 & 1 & 1 \\
\phi_{5,1} & 1 & 1 & 1 \\
\end{pmatrix}
\]

- The set $\{a, b, c\}$ has cardinality 3: In this case all the characters correspond to distinct irreducible modules of $CH_3$. We distinguish the following cases, based on whether or not the 2-dimensional modules remain irreducible. Due to corollary 3.7 at least one of them must remain irreducible. Due to proposition 3.6 and without loss of generality, we have the following cases:

$\triangleright a^2 - ac + c^2 = b^2 - bc + c^2 = 0$: Since $a \neq b$ we may assume $c = -E(3)a$ and $b = -E(3)c$. 

\[
\begin{pmatrix}
\phi_{1,0} & \phi_{1,8} \\
\phi_{2,1} & I_5 \\
\phi_{3,2} & \\
\phi_{4,1} & 1 & 1 & 1 \\
\phi_{4,2} & 1 & 1 & 1 \\
\end{pmatrix}
\]
As a result, the decomposition matrix is the following:

\[
\begin{bmatrix}
\phi_{1,0} & \phi_{1,4} & I_4 \\
\phi_{2,1} & & \\
\phi_{2,5} & -1 & 1 & 1 \\
\phi_{2,3} & 1 & -1 & 1 \\
\phi_{3,2} & 1 & 1 & 1 \\
\end{bmatrix}
\]

\(b^2 - bc + c^2 = 0\) and \((a^2 - ac + c^2)(a^2 - ab + b^2) \neq 0\): Due to proposition 3.6 and 3.8, the characters \(\phi_{2,3}, \phi_{2,1}\) and \(\phi_{3,2}\) are irreducible. We type:

\[
\text{gap> T:=CharTable(H_3).irreducibles;};
\text{gap> t:=List(T,i->List(i,j->Value(j,["b", -E(3)c]))));}
\text{gap> t[4]=t[2]+t[3]; true}
\]

Therefore, the decomposition matrix is the following:

\[
\begin{bmatrix}
\phi_{1,0} & \phi_{1,4} & I_6 \\
\phi_{2,1} & & \\
\phi_{2,5} & -1 & 1 & 1 \\
\phi_{2,3} & 1 & -1 & 1 \\
\phi_{3,2} & 1 & 1 & 1 \\
\end{bmatrix}
\]

\((b^2 - bc + c^2)(a^2 - ac + c^2)(a^2 - ab + b^2) \neq 0\): Due to proposition 3.6, all the 2-dimensional characters are irreducible. If the character \(\phi_{3,2}\) is also irreducible, then we are in the semisimple case and the decomposition matrix is the identity matrix \(I_7\). We suppose now that \(\phi_{3,2}\) is not irreducible. Due to proposition 3.8, it will be sufficient to examine the case where \((a^2 + bc)(b^2 + ac)(c^2 + ab) = 0\). Without loss of generality, we assume that \(a^2 + bc = 0\). We type:

\[
\text{gap> T:=CharTable(H_3).irreducibles;};
\text{gap> t:=List(T,i->List(i,j->Value(j,["b", -E(3)c]))));}
\text{gap> t[7]=t[4]+t[1]; true}
\]

As a result, the decomposition matrix is of the form:

\[
\begin{bmatrix}
\phi_{1,0} & \phi_{1,4} & I_6 \\
\phi_{2,1} & & \\
\phi_{2,5} & -1 & 1 & 1 \\
\phi_{2,3} & 1 & -1 & 1 \\
\phi_{3,2} & 1 & 1 & 1 \\
\end{bmatrix}
\]

3.4. The case \(k = 4\). The complex reflection group \(W_4\) is the group \(G_8\) in the Shephard-Todd classification, with Coxeter-like presentation

\[
\langle s_1, s_2 \mid s_1^2 = s_2^4 = 1, s_1 s_2 s_1 = s_2 s_1 s_2 \rangle.
\]
The Hecke algebra $H_4$ is defined over the ring $R_4 := \mathbb{Z}[u_1^\pm, u_2^\pm, u_3^\pm, u_4^\pm]$. We identify again $s_i$ to their images in $H_4$, and we have that

$$H_4 = \langle s_1, s_2 \mid s_1 s_2 s_1 = s_2 s_1 s_2, (s_i - u_1)(s_i - u_2)(s_i - u_3)(s_i - u_4) = 0 \rangle,$$

for $i = 1, 2$.

We fix now a specialization $\theta : \mathbb{C}[v_1^\pm, v_2^\pm, v_3^\pm, v_4^\pm] \to \mathbb{C}[v_1^\pm, v_2^\pm, v_3^\pm, v_4^\pm]$ (the parameters $v_i$ defined as in [2.2]) such that $u_1 \mapsto a$, $u_2 \mapsto b$, $u_3 \mapsto c$, and $u_4 \mapsto d$.

In order to find the irreducible representations of $W_4$ we use GAP, as we explain in the example [2.5]. We have 16 irreducible characters, which we symbolize here as $\phi_1, \ldots, \phi_{16}$.

3.5. **The 1-dimensional characters.** Without loss of generality, we focus on character $\phi_1$. We distinguish the following cases:

- **C1.** $a, b, c, d$ are distinct: In this case, the characters $\phi_1, \ldots, \phi_4$ correspond to distinct 1-dimensional modules. Due to proposition 3.6, the decomposition matrix is of the form:

$$
\begin{pmatrix}
1 & 1 \\
1 & 1 \\
\vdots & \\
\end{pmatrix}
$$

- **C2.** At least two of the complex numbers $a, b, c, d$ are equal: Without loss of generality, we assume that $a = b$. Therefore, the characters $\phi_1$ and $\phi_2$ correspond to the same 1-dimensional module. We use proposition 3.4 again, and the decomposition matrix is of the form:

$$
\begin{pmatrix}
1 \\
1 \\
\vdots \\
\end{pmatrix}
$$

3.6. **The 2-dimensional characters.** Without loss of generality, we focus on the character $\phi_5$. We distinguish the following cases, based on proposition 3.6.

- **C1.** $a^2 - ab + b^2 \neq 0$: In this case, the character $\phi_5$ is irreducible and, hence, the decomposition matrix is of the form

$$
\begin{pmatrix}
1 \\
* \\
* \\
* \\
\vdots \\
\end{pmatrix}
$$

where $\ast$ denotes a placeholder for one of two values, 0 or 1. The sum of these values in each line must equal 1 (see C1 and C2 of section 3.5).

- **C2.** $a^2 - ab + b^2 = 0$: Due to proposition 3.6, the character $\phi_5$ is not irreducible. We recall that $E(n)$, $n \in \mathbb{N}$, denotes a $n$-th primitive root of unity. We have $b = -E(3)a$ and we use GAP, as we did with $W_3$. However, in the case of $W_4$ we have the appearance of square roots. CHEVIE tries to extract automatically these roots, which may unavoidably be inconsistent with our expectations sometimes. For this reason, we use variables representing roots of the parameters:

```gap
gap> W_4 := ComplexReflectionGroup(8);
gap> x := Mvp("x");
gap> y := Mvp("y");
```
At this point, we must choose the square root of each parameter $a$, $b$, $c$ and $d$. If someone chooses different square roots, what is changing is just the labeling of the characters $\phi_{15}$ and $\phi_{16}$ (which depends on a specialization of the roots of the parameters, not of the parameters themselves). We type:

\[
gap T:=\text{CharTable}(H_4).\text{irreducibles};;
\]

\[
t:=\text{List}(T,i->\text{List}(i,j->\text{Value}(j,\text{["y", E(4)*E(6)*x]})));;
\]

\[
t[5]=t[1]+t[2];
\]

true

Therefore, due to criterion 3.5 we have: $d_\theta([V_{\phi_1}]) = d_\theta([V_{\phi_5}]) + d_\theta([V_{\phi_2}])$. Taking also into account C1 of section 3.5, the decomposition matrix is of the form:

\[
\begin{pmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
\end{pmatrix}
\]

Due to proposition 3.10 we have a clear picture to the maximal number of 2-characters that are not irreducible. Since $b = -E(3)a$, we have that $2 \leq |\{a, b, c, d\}| \leq 4$. We distinguish the following cases:

1. The cardinality of the set $\{a, b, c, d\}$ is 2: Since $a^2 - ab + b^2 = 0$, we have $\{a, b, c, d\} = \{a, b\}$. It is enough to find the decomposition matrix in the following two cases. Any other option falls to these two cases, in the sense that the decomposition matrix is of the same form, if we reorder the characters.

2. $b = c = d$:
   - The 1-dimensional characters $\phi_2, \phi_3$ and $\phi_4$ correspond to the same module, which is distinct of the module to which the character $\phi_1$ corresponds.
   - The 2-dimensional characters $\phi_5, \phi_6, \phi_7$ correspond to the same module, as well as the 2-dimensional characters $\phi_8, \phi_9$ and $\phi_{10}$. These two modules are distinct, since $b \neq a$. Due to proposition 3.6, these modules are irreducible.
   - The 3-dimensional characters $\phi_{12}, \phi_{13}$ and $\phi_{14}$ correspond to the same module, which is distinct of the module to which the character $\phi_{11}$ corresponds, since $b \neq a$. Due to proposition 3.8, these modules are irreducible.
   - The 4-dimensional characters $\phi_{15}$ and $\phi_{16}$ correspond to distinct modules. We type:

\[
gap s:=\text{SchurElements}(H_4);;
\]

\[
\text{List}(s,i->\text{Value}(i, \text{["y",E(4)*E(8)*x,"z",E(4)*E(8)*x,"t",E(4)*E(8)*x]}));
\]

\[
[0,0,0,0,0,0,0,0,-16,0,0,0,0,-208-120ER(3),-208+120ER(3)]
\]

and, therefore, due to criterion 3.3 the characters $\phi_{11}, \phi_{15}$ and $\phi_{16}$ correspond to modules of defect 0.

Summing up, the decomposition matrix is the following:

\[
\begin{pmatrix}
I_7 \\
\phi_9 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
\phi_4 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
\phi_5 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
\phi_6 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
\phi_7 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
\phi_9 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
\phi_{10} & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
\phi_{11} & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
\phi_{12} & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
\phi_{13} & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
\phi_{14} & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
\end{pmatrix}
\]

where the upper part of the matrix is indexed by the characters $\phi_1, \phi_2, \phi_8, \phi_{11}, \phi_{12}, \phi_{15}$ and $\phi_{16}$. 
The 1-dimensional characters $\phi_1, \phi_4$ correspond to the same module, as well as the characters $\phi_2$ and $\phi_3$. These two modules are distinct.

- There are three distinct 2-dimensional characters; the characters $\phi_5$, $\phi_6$, $\phi_{10}$, which correspond to the same module, the character $\phi_7$ and the character $\phi_8$. The last two characters are irreducible, due to proposition 3.6.
- The 3-dimensional characters $\phi_{11}$ and $\phi_{14}$ correspond to the same module, as well as the characters $\phi_{12}$ and $\phi_{13}$. These two modules are distinct and irreducible, due to proposition 3.8.
- The 4-dimensional characters $\phi_{15}$ and $\phi_{16}$ correspond to distinct modules. Moreover, due to criterion 3.3 the character $\phi_{16}$ corresponds to a module of defect 0. The character $\phi_{15}$ is of dimension 4 and it is not of defect 0. Therefore, it is not irreducible. We type:

```gap
gap> s:=SchurElements(H_4);;
gap> List(s,i->Value(i,\{y,x,z,E(12)^11*x,t,E(12)^11*x\}));
[ 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, -324 ]
```

and, therefore, due to criterion 3.3 the character $\phi_{16}$ corresponds to a module of defect 0. The character $\phi_{15}$ is of dimension 4 and it is not of defect 0. Therefore, it is not irreducible. We type:

```gap
gap> T:=CharTable(H_4).irreducibles;;
gap> t:=List(T,i->List(i,j->Value(j,\{y,x,z,E(12)^11*x,t,E(12)^11*x\})));
gap> t[15]=t[7]+t[8];
true
```

Summing up, the decomposition matrix is the following:

$$
\begin{pmatrix}
I_7 \\
\phi_3 & 1 & . & . & . & . & . & . \\
\phi_4 & 1 & . & . & . & . & . & . \\
\phi_5 & 1 & 1 & . & . & . & . & . \\
\phi_6 & 1 & 1 & . & . & . & . & . \\
\phi_9 & 1 & 1 & . & . & . & . & . \\
\phi_{10} & 1 & 1 & 1 & . & . & . & . \\
\phi_{13} & 1 & 1 & 1 & 1 & . & . & . \\
\phi_{14} & 1 & 1 & 1 & 1 & 1 & . & . \\
\phi_{15} & 1 & 1 & 1 & 1 & 1 & 1 & . \\
\end{pmatrix}
$$

where the upper part of the matrix is indexed by the characters $\phi_1, \phi_2, \phi_7, \phi_8, \phi_{11}, \phi_{12}$ and $\phi_{16}$.

- The cardinality of the set \{a, b, c, d\} is 3: Since $a^2 - ab + b^2 = 0$ we assume that \{a, b, c, d\} = \{a, b, d\}. It is enough to find the decomposition matrix in the following cases:

$b = c$ and $d = -E(3)^2a$: Due to propositions 3.6 and 3.6 we can identify the non-irreducible 2 and 3-dimensional modules. Using criterions 3.3 and 3.5 as in the previous cases, we have that the decomposition matrix is the following:

$$
\begin{pmatrix}
I_9 \\
\phi_3 & 1 & . & . & . & . & . & . \\
\phi_4 & 1 & 1 & . & . & . & . & . \\
\phi_5 & 1 & 1 & . & . & . & . & . \\
\phi_6 & 1 & 1 & . & . & . & . & . \\
\phi_{10} & 1 & 1 & 1 & . & . & . & . \\
\phi_{12} & 1 & 1 & 1 & 1 & . & . & . \\
\phi_{13} & 1 & 1 & 1 & 1 & 1 & . & . \\
\end{pmatrix}
$$

The upper part of the matrix is indexed by the characters $\phi_1, \phi_2, \phi_4, \phi_8, \phi_9, \phi_{11}, \phi_{14}, \phi_{15}$, and $\phi_{16}$.
\[ b = c \text{ and } d \neq -E(3)^2a: \text{ Due to proposition 3.6 we have:} \]

\[
\begin{pmatrix}
1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 \\
\vdots & \vdots & \vdots & \vdots \\
\end{pmatrix}
\]

Later, we will study in detail characters of dimension more than 2 and we will see that the upper part of this matrix by adding some other characters, forms the identity matrix.

\[ c = d = -E(3)^2a: \text{ Due to proposition 3.6 the characters } \phi_8 = \phi_9 \text{ and } \phi_{10} \text{ are irreducible.} \]

We also have \( \theta(s_{\phi_i}) \neq 0 \), for \( i = 10, 11, 12, 15, 16 \). Due to criterion 3.3 these characters are of defect 0. Moreover, due to proposition 3.8 and the fact that \( c = d \), the characters \( \phi_{13} \) and \( \phi_{14} \) correspond to the same non-irreducible module. We type:

\[
\text{gap> } T:=\text{CharTable}(H_4).\text{irreducibles;;} \\
\text{gap> } t:=\text{List}(T,i->\text{List}(i,j->\text{Value}(j,\text{["y",E(4)*E(6)*x,"z",E(4)*E(6)^2*x,"t",E(4)*E(6)^2*x]})))); \\
\text{gap> } t[13]=t[1]+t[2]+t[3]; \\
\text{true}
\]

where the upper part of the matrix is indexed by the characters \( \phi_1, \phi_2, \phi_3, \phi_8, \phi_{10}, \phi_{11}, \phi_{12}, \phi_{15}, \) and \( \phi_{16} \).

\[ c = d = E(3)^2a: \text{ We obtain the same decomposition matrix as in case } c = d = -E(3)^2a, \]

by reordering the character \( \phi_1 \) with \( \phi_2 \) and the character \( \phi_6 \) with \( \phi_8 \).

\[ c = d \neq \pm E(3)^2a: \text{ The decomposition matrix is of the form:} \]

\[
\begin{pmatrix}
1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 \\
\vdots & \vdots & \vdots & \vdots \\
\end{pmatrix}
\]

\[ \textbf{The cardinality of the set } \{a, b, c, d\} \text{ is 4: Due to proposition 3.6 we have that at least 3 of the characters } \phi_6, \ldots, \phi_9 \text{ are irreducible. Without loss of generality, we} \]
suppose that \( \phi_6, \phi_7 \) and \( \phi_8 \) are irreducible. Hence, the decomposition matrix is of the following form:

\[
\begin{pmatrix}
\phi_1 & 1 \\
\phi_2 & \\
\phi_3 & \\
\phi_4 & \\
\phi_5 & \\
\phi_6 & 1 \\
\phi_7 & 1 \\
\phi_8 & 1 \\
\phi_9 & \ast \\
\phi_{10} & \ast \\
\phi_{11} & \\
\phi_{12} & \\
\phi_{13} & 1
\end{pmatrix}
\]

where \( \ast \) denotes again a placeholder for one of two values, 0 or 1, depending on whether or not the characters \( \phi_9 \) and \( \phi_{10} \) are irreducible, as we explained in C1 and in the beginning of C2 of this section.

The case where neither of these characters is irreducible is easy to be studied, due to proposition 3.6. Indeed, since \( a \) and \( d \) are distinct and since \( \phi_6 \) is irreducible (i.e. \( a^2 - ac + c^2 \neq 0 \)) the only possibility that both characters are not irreducible is when \( d = E(3)^2 a \) and \( c = -a \). We type:

```gap
T:=CharTable(H_4).irreducibles;;
t:=List(T,i->List(i,j->Value(j,
["y",E(4)*E(6)*x,"z",E(4)*x,"t",E(6)^2*x])));;
gap> t[11]=t[2]+t[3]+t[4];
true
gap> t[13]=t[1]+t[2]+t[4];
true
gap> t[15]=t[1]+t[2]+t[3]+t[4];
true
gap> t[16]=t[7]+t[8];
true
```

Due to criterion 3.5 the decomposition matrix is the following:

\[
\begin{pmatrix}
I_9 & \\
\phi_5 & 1 \\
\phi_6 & 1 \\
\phi_{10} & 1 \\
\phi_{11} & 1 \\
\phi_{13} & 1 \\
\phi_{15} & 1 \\
\phi_{16} & 1
\end{pmatrix}
\]

where the upper part of the matrix is indexed by the characters \( \phi_1, \phi_2, \phi_3, \phi_4, \phi_6, \phi_7, \phi_8, \phi_{12}, \phi_{14} \).

3.7. The 3-dimensional characters. Without loss of generality, we focus on the character \( \phi_{13} \). We distinguish the following cases, based on proposition 3.8.

C1. \( (a^2 + bd)(b^2 + ad)(d^2 + ab) \neq 0 \): In this case, the character \( \phi_{13} \) is irreducible and, hence, the decomposition matrix is of the form

\[
\begin{pmatrix}
\phi_{13} & \cdots & 0 & 1 & 0
\end{pmatrix}
\]

C2. \( (a^2 + bd)(b^2 + ad)(d^2 + ab) = 0 \): Let \( d^2 + ab = 0 \). We type:

```gap
T:=CharTable(H_4).irreducibles;;
t:=List(T,i->List(i,j->Value(j,
["y",E(4)*t^2*x^-1]));
gap> t[13]=t[4]+t[5];
true
```
At this point, we have the following cases, depending on whether or not the character \( \phi_5 \) is irreducible (see section 3.6):

\( a^2 - ab + b^2 \neq 0 \): The decomposition matrix is of the form

\[
\begin{pmatrix}
\phi_4 \\
\phi_5 \\
\phi_{13} \\
\vdots
\end{pmatrix}
\]

\( a^2 - ab + b^2 = 0 \): Since \( b = -E(3)a \) and \( d^2 + ab = 0 \) we have that \( d = \pm E(3)^2a \). The decomposition matrix has been described in 3.6. We give the general form here. Notice that since \( a, b \) and \( d \) are distinct, the characters \( \phi_1, \phi_2 \) and \( \phi_4 \) correspond to distinct modules. Up to reordering, it will be sufficient to examine the case, where \( d = -E(3)^2a \). Due to proposition 3.6 we have that the character \( \phi_7 \) is not irreducible. Moreover, due to the same proposition, the character \( \phi_9 \) is irreducible. Therefore, we have:

\[
\begin{pmatrix}
\phi_1 \\
\phi_2 \\
\phi_4 \\
\phi_5 \\
\phi_7 \\
\phi_9 \\
\phi_{13} \\
\vdots
\end{pmatrix}
\]

**Proposition 3.10.** Suppose that one 3-dimensional character breaks up into three 1-dimensional characters and a second 3-dimensional character is not irreducible. Then, the second character also breaks up into three 1-dimensional characters. The rest of the 3-dimensional characters are of defect 0.

**Proof.** Let \( \phi_{13} \) be the 1-dimensional character that breaks up into three 1-dimensional characters, as described in C2. We suppose now that \( \phi_{14} \) is not irreducible. As we explained in C2, we can assume that \( b = -E(3)a \) and \( d = -E(3)^2a \). As a result, it will be sufficient to check the following two cases (see proposition 3.8):

- \( c = d = -E(3)^2a \): The character \( \phi_6 \) is not irreducible (proposition 3.6). Moreover, using GAP, we can see that \( \theta(s_{\phi_i}) \neq 0 \), for \( i = 10, 11, 12, 15, 16 \). Due to criterion 3.3 these characters are of defect 0. As a result, the decomposition matrix is the following:

\[
\begin{pmatrix}
I_9 \\
\phi_4 \\
\phi_5 \\
\phi_6 \\
\phi_7 \\
\phi_8 \\
\phi_{13} \\
\phi_{14} \\
\vdots
\end{pmatrix}
\]

The upper part of the matrix is indexed by the characters \( \phi_1, \phi_2, \phi_3, \phi_9, \phi_{10}, \phi_{11}, \phi_{12}, \phi_{15}, \) and \( \phi_{16} \).

- \( c = E(3)^2a \): The character \( \phi_6 \) is irreducible and the character \( \phi_8 \) is not (see section 3.6). Moreover, \( \theta(s_{\phi_i}) \neq 0 \), for \( i = 10, 11, 12 \). Due to criterion 3.3 these characters are of defect 0. We also type:

```gap
t:=CharTable(H_4).irreducibles;;
t:=List(t,i->List(i,j->Value(j,\"y\", E(4)*E(6)*a, "z", E(6)^2*a, "t", E(4)*E(6)^2*a)));
t[15]=t[6]+t[9];
true
t[16]=t[1]+t[2]+t[3]+t[4];
true
```
As a result, the decomposition matrix is the following:

\[
\begin{pmatrix}
\phi_1 & 1 & 1 & \cdots & \cdots \\
\phi_2 & 1 & 1 & \cdots & \cdots \\
\phi_3 & 1 & 1 & 1 & \cdots \\
\phi_4 & 1 & 1 & 1 & 1 & \cdots \\
\phi_5 & 1 & 1 & 1 & 1 & 1 \\
\phi_6 & 1 & 1 & 1 & 1 & 1 \\
\phi_7 & 1 & 1 & 1 & 1 & 1 \\
\phi_8 & 1 & 1 & 1 & 1 & 1 \\
\phi_9 & 1 & 1 & 1 & 1 & 1 \\
\phi_{10} & 1 & 1 & 1 & 1 & 1 \\
\phi_{11} & 1 & 1 & 1 & 1 & 1 \\
\phi_{12} & 1 & 1 & 1 & 1 & 1 \\
\phi_{13} & 1 & 1 & 1 & 1 & 1 \\
\phi_{14} & 1 & 1 & 1 & 1 & 1 \\
\phi_{15} & 1 & 1 & 1 & 1 & 1 \\
\phi_{16} & 1 & 1 & 1 & 1 & 1 \\
\end{pmatrix}
\]

The upper part of the matrix is indexed by the characters \( \phi_1, \phi_2, \phi_3, \phi_4, \phi_5, \phi_{10}, \phi_{11}, \phi_{12} \).

3.8. **The 4-dimensional characters.** Due to proposition \textbf{3.9} the 4-dimensional characters cannot be in the same block. Without loss of generality, we examine the case of \( \phi_{15} \). Due to proposition \textbf{3.9} again, this character is irreducible if and only if \( \theta(s_{\phi_{15}}) \neq 0 \). As a result, we distinguish the following cases:

\begin{enumerate}
  \item[(C1)] \((a^3 - bcd)(b^3 - acd)(c^3 - abd)(d^3 - abc) = 0\): Let \( c^3 - abd = 0 \). We type:

\begin{verbatim}
gap> T:=CharTable(H_4).irreducibles;; gap> t:=List(T,i->List(i,j->Value(j,\["y", z^3*x^-1*t^-1\])));; gap> T:=CharTable(H_4).irreducibles;; gap> t:=List(T,i->List(i,j->Value(j,\["y", z^3*x^-1*t^-1\])));; gap> t[15]=t[3]+t[13]; true
\end{verbatim}

Notice here that if we chose \( y^3 = -xyt \) the character \( \phi_{15} \) will reorder with the character \( \phi_{16} \). We distinguish the following cases, depending on whether or not the character \( \phi_{13} \) is irreducible (see section \textbf{3.7}):

\begin{enumerate}
  \item[C1.1] \((a^2 + bd)(b^2 + ad)(d^2 + ab) \neq 0\): The decomposition matrix is of the form:

\[
\begin{pmatrix}
\phi_3 & 1 & 1 & \cdots \\
\phi_{13} & 1 & 1 & \cdots \\
\phi_{15} & 1 & 1 & \cdots \\
\end{pmatrix}
\]

\item[C1.2] \((a^2 + bd)(b^2 + ad)(d^2 + ab) = 0\): Following section \textbf{3.7} we assume that \( d^2 + ab = 0 \). Since \( c^3 - abd = 0 \) it follows that \( c^3 = -d^3 \). According to section \textbf{3.7} we distinguish the following cases:

\begin{itemize}
  \item \( a^2 - ab + b^2 \neq 0 \): Since \( d^2 + ab = c^3 + d^3 = 0 \) we have \( (a^2 \pm ac + c^2)(a^2 \pm ad + d^2) \neq 0 \) and, due to proposition \textbf{3.6} the characters \( \phi_5, \phi_6 \) and \( \phi_7 \) are irreducible. Their central characters are \( -a^3b, -a^3c, \) and \( -a^3d^3 \), respectively. Since \( c^3 \neq d^3 \), the characters \( \phi_6 \) and \( \phi_7 \) are not in the same block (criterion \textbf{3.3}).

For \( c = -d \) we have that the character \( \phi_{10} \) is irreducible (proposition \textbf{3.6}). Since \( c \neq d \) the characters \( \phi_3 \) and \( \phi_4 \) are distinct. Moreover, since \( d^2 + ab = 0 \), we also have \( c^2 + ab = 0 \). Due to proposition \textbf{3.12} we have that the character \( \phi_{14} \) is not irreducible (see section \textbf{3.7}). As a result, the decomposition matrix is of the form:

\[
\begin{pmatrix}
\phi_3 & 1 & 1 & \cdots \\
\phi_4 & 1 & 1 & \cdots \\
\phi_5 & 1 & 1 & \cdots \\
\phi_6 & 1 & 1 & \cdots \\
\phi_7 & 1 & 1 & \cdots \\
\phi_{10} & 1 & 1 & \cdots \\
\phi_{13} & 1 & 1 & \cdots \\
\phi_{14} & 1 & 1 & \cdots \\
\phi_{15} & 1 & 1 & \cdots \\
\end{pmatrix}
\]

where * denotes a placeholder for one of two values, 0 or 1. The sum of these values in each line equals 1. Moreover, the characters \( \phi_5, \phi_6 \) and \( \phi_7 \) cannot be altogether in the same block.

For \( c^2 - cd + d^2 = 0 \) the character \( \phi_{10} \) is not irreducible (see proposition \textbf{3.6}).
\end{itemize}
\end{enumerate}
Moreover, we have that $a \neq c$ and $a \neq d$. If, for example, $a = c$ we have that $a^2 - ad + d^2 = 0$, which contradicts the fact that $a^2 \pm ad + d^2 \neq 0$. Summing up, the decomposition matrix is of the form:

$$
\begin{pmatrix}
\phi_5 & \phi_1 & \phi_4 & \phi_5 & \phi_8 & \phi_{11} & \phi_{14} & \phi_{16} \\
1 & 1 & 1 & \ast & \ast & \ast & \ast & \\
\phi_5 & \phi_1 & \phi_4 & \phi_5 & \phi_8 & \phi_{11} & \phi_{14} & \phi_{16} \\
1 & 1 & 1 & \ast & \ast & \ast & \ast & \\
\phi_{10} & \phi_{13} & \phi_{15} \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & \\
\end{pmatrix}
$$

where $\ast$ denotes a placeholder for one of two values, 0 or 1. The sum of these values in each line equals 1. Again, the characters $\phi_5$, $\phi_6$ and $\phi_7$ cannot be altogether in the same block.

$\triangleright$ $a^2 - ab + b^2 = 0$: There are two cases (up to reordering) to be examined.

**Case 1:** $b = -E(3)^2a$, $c = -E(3)a$ and $d = E(3)a$. This case has been examined (up to reordering) in [3.6](#) (case where the cardinality of $\{a,b,c,d\}$ is 4).

**Case 2:** $b = -E(3)^2a$, $c = -E(3)^2a$ and $d = E(3)a$. We first notice that the characters $\phi_7$, $\phi_8$, $\phi_{11}$, $\phi_{14}$ and $\phi_{16}$ are of defect 0, due to criterion [3.3](#). According to sections [3.6](#) and [3.7](#) the decomposition matrix is the following:

$$
\begin{pmatrix}
I_9 \\
\phi_5 & \phi_1 & \phi_4 & \phi_5 & \phi_8 & \phi_{11} & \phi_{14} & \phi_{16} \\
1 & 1 & 1 & \ast & \ast & \ast & \ast & \\
\phi_5 & \phi_1 & \phi_4 & \phi_5 & \phi_8 & \phi_{11} & \phi_{14} & \phi_{16} \\
1 & 1 & 1 & \ast & \ast & \ast & \ast & \\
\phi_{10} & \phi_{13} & \phi_{15} \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & \\
\end{pmatrix}
$$

where the upper part of the matrix is indexed by the characters $\phi_1$, $\phi_2$, $\phi_4$, $\phi_7$, $\phi_8$, $\phi_{11}$, $\phi_{14}$, and $\phi_{16}$.

C2. $(a^2b^2 - abcd + c^2d^2)(a^2c^2 - abcd + b^2d^2)(a^2d^2 - abcd + b^2c^2) = 0$: Let $ab = E(3)cd$. We type:

```gap
gap> T:=CharTable(H_4).irreducibles;;
gap> t:=List(T,i->List(i,j->Value(j,["t", E(6)*x*y*z^-1])));;
gap> t[15]=t[5]+t[10];
true
```

Since $ab \neq cd$ the characters $\phi_5$ and $\phi_{10}$ correspond to distinct modules. We distinguish the following cases, depending on whether or not these characters are irreducible (see section [3.6](#)).

$\triangleright$ $(a^2 - ab + b^2)(c^2 - cd + d^2) \neq 0$: Due to proposition [3.9](#) the characters $\phi_5$ and $\phi_{10}$ are irreducible. As a result, the decomposition matrix is of the form:

$$
\begin{pmatrix}
\phi_5 & \phi_{10} & \phi_{15} \\
1 & 1 & 1 \\
\phi_{10} & \phi_{15} \\
1 & 1 & 1 \\
\phi_{15} \\
1 & 1 \\
\end{pmatrix}
$$

$\triangleright$ $(a^2 - ab + b^2)(c^2 - cd + d^2) = 0$: This case is (up to reordering) case C1.2.

C3. $(a^3 - bcd)(b^3 - acd)(c^3 - abd)(d^3 - abc)(a^2b^2 - abcd + c^2d^2)(a^2c^2 - abcd + b^2d^2)(a^2d^2 - abcd + b^2c^2) \neq 0$:

Due to criterion [3.3](#) the characters $\phi_{15}$ and $\phi_{16}$ are of defect zero. As a result, the decomposition matrix is of the form:

$$
\begin{pmatrix}
\phi_{15} & \phi_{16} \\
1 & 1 \\
\end{pmatrix}
$$
3.9. **The case** $k = 5$. The complex reflection group $W_5$ is the group $G_{16}$ in the Shephard-Todd classification, with Coxeter-like presentation

$$\langle s_1, s_2 \mid s_1^5 = s_2^5 = 1, s_1 s_2 s_1 = s_2 s_1 s_2 \rangle.$$ 

The Hecke algebra $H_5$ is defined over the ring $R_5 := \mathbb{Z}[u_1^\pm, u_2^\pm, u_3^\pm, u_4^\pm, u_5^\pm]$. We identify again $s_i$ to their images in $H_5$, and we have that

$$H_5 = \langle s_1, s_2 \mid s_1 s_2 s_1 = s_2 s_1 s_2, (s_i - u_1)(s_i - u_2)(s_i - u_3)(s_i - u_4)(s_i - u_5) = 0 \rangle,$$

for $i = 1, 2$.

We fix now a specialization $\theta : \mathbb{C}[v_1^\pm, v_2^\pm, v_3^\pm, v_4^\pm, v_5^\pm] \to \mathbb{C}$ of $\mathbb{C}[v_1^\pm, v_2^\pm, v_3^\pm, v_4^\pm, v_5^\pm]$ (the parameters $v_i$ defined as in 2.2) such that $u_1 \mapsto a, u_2 \mapsto b, u_3 \mapsto c, u_4 \mapsto d, \text{ and } u_5 \mapsto 5$.

We have 45 irreducible characters, which we symbolize here as $\phi_i, i = 1, \ldots, 45$. We have 5 characters of dimension 1 (the characters $\phi_1, \ldots, \phi_5$), 10 of dimension 2 (the characters $\phi_6, \ldots, \phi_{15}$), 10 of dimension 3 (the characters $\phi_{16}, \ldots, \phi_{27}$), 10 of dimension 4 (the characters $\phi_{28}, \ldots, \phi_{38}$), 5 of dimension 5 (the characters $\phi_{39}, \ldots, \phi_{44}$), and 5 of dimension 6 (the characters $\phi_{45}$).

3.10. **The 1-dimensional characters**. Without loss of generality, we focus on character $\phi_1$. As in section 3.5 we distinguish the following cases:

- **C1.** At least two of the complex numbers $a, b, c, d, e$ are equal: We obtain the same matrix model as in C2 of section 3.5.

- **C2.** $a, b, c, d, e$ are distinct: In this case, the characters $\phi_1, \ldots, \phi_5$ correspond to distinct 1-dimensional modules. Due to criterion 3.1 the decomposition matrix is of the form:

$$
\begin{pmatrix}
\phi_1 & 1 & 1 & 1 & 1 \\
\phi_2 & & & & \\
\phi_3 & & & & \\
\phi_4 & & & & \\
\phi_5 & & & & \\
\vdots & & & & \\
\end{pmatrix}
$$

3.11. **The 2-dimensional characters**. Without loss of generality, we focus on the character $\phi_6$. As in section 3.6 we distinguish the following cases:

- **C1.** $a^2 - ab + b^2 \neq 0$: Due to 3.6 the character $\phi_6$ is irreducible and, hence, the decomposition matrix is of the form:

$$
\begin{pmatrix}
\phi_6 & 0 & 0 & 0 & 0 & 0 \\
\vdots & & & & & \\
\end{pmatrix}
$$

- **C2.** $a^2 - ab + b^2 = 0$: Due to proposition 3.6 the character $\phi_6$ is not irreducible. We recall that $E(n), n \in \mathbb{N}$, denotes a $n$-th primitive root of unity. We have $b = -E(3)a$ and we use this time variables representing 10th roots of the parameters:

```gap
gap> W_5:=ComplexReflectionGroup(16);;
gap> x:=Mvp("x");;
gap> y:=Mvp("y");;
gap> z:=Mvp("z");;
gap> t:=Mvp("t");;
gap> w:=Mvp("w");;
gap> H_5:=Hecke(W_5,[[x^10,y^10,z^10,t^10,w^10]]);
gap> T:=CharTable(H_5).irreducibles;;
gap> t:=List(T, i->List(i,j->Value(j, ["y", E(12)*7*x]))));
gap> t[6]=t[1]+t[2];
gap> true
```

Due to criterion 3.6 and the fact that $a \neq b$, the decomposition matrix is of the form

$$
\begin{pmatrix}
\phi_1 & 1 & 1 \\
\phi_2 & 1 & 1 \\
\phi_6 & 1 & 1 \\
\vdots & & \\
\end{pmatrix}
$$
3.12. The 3-dimensional characters. Without loss of generality, we focus on the character $\phi_{16}$. We distinguish the following cases, based on proposition 3.8:

C1. $(a^2 + bc)(b^2 + ac)(c^2 + ab) \neq 0$: In this case, the character $\phi_{16}$ is irreducible and, hence, the decomposition matrix is of the form

\[
\begin{pmatrix}
\vdots & \vdots & \vdots & \vdots & \vdots \\
\phi_{16} & 0 & 0 & 1 & 0 & 0 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\end{pmatrix}
\]

C2. $(a^2 + bc)(b^2 + ac)(c^2 + ab) = 0$: Let $c^2 + ab = 0$. We type:

```gap
gap> T:=CharTable(H_5).irreducibles;;
gap> t:=List(T,i->List(i,j->Value(j,\"y", E(20)*z^2*x^-1))));;
gap> t[16]=t[3]+t[6];
true
```

At this point, we have the following cases, depending on whether or not the character $\phi_6$ is irreducible:

- $a^2 - ab + b^2 = 0$: Since $b = -E(3)a$ and $c^2 + ab = 0$ we have that $c = \pm E(3)^2a$. Moreover, $a$, $b$ and $c$ are distinct and, hence, the characters $\phi_1$, $\phi_2$ and $\phi_3$ correspond to distinct modules. Up to reordering, it will be sufficient to examine the case $c = -E(3)^2a$. Due to proposition 3.6, the character $\phi_7$ is not irreducible whereas the character $\phi_{10}$ is. Hence:

\[
\begin{pmatrix}
\phi_1 & \phi_2 & \phi_3 & \phi_6 & \phi_7 & \phi_{10} & \phi_{16} \\
1 & 1 & 1 & 1 & 1 & 1 & \vdots \\
\end{pmatrix}
\]

- $a^2 - ab + b^2 \neq 0$: Due to proposition 3.6 the character $\phi_6$ is irreducible. Moreover, since $c^2 + ab = 0$, we have that $(c^2 + ac + c^2)(c^2 - ac + a^2) \neq 0$. Since $c^2 - ac + a^2 \neq 0$ proposition 3.4 is applicable and we have that the character $\phi_7$ is irreducible. Furthermore, $(c^2 - cb + b^2)(c^2 + cb + b^2) \neq 0$. Indeed, if $c^2 \pm cb + b^2 = 0$ we have that $c = \pm E(3)b$ and, hence, due to the fact that $c^2 = -ab$, we obtain $a = -E(3)^2b$, which contradicts the hypothesis. Therefore, due to proposition 3.6 the character $\phi_{10}$ is irreducible.

We now distinguish the following cases, based on whether or not the characters $\phi_6$ and $\phi_{10}$ are in the same block. We have $\theta(\omega_{\phi_6}) = -a^3b^3$, $\theta(\omega_{\phi_{10}}) = -a^3c^3$ and $\theta(\omega_{\phi_{16}}) = -b^3c^3$.

Notice that criterion 3.3 provides a necessary but not sufficient condition for two characters being in the same block. As a result, one should also use criterion 3.5 in order to check if the aforementioned characters are in the same block.

- $a = c = -b$:

\[
\begin{pmatrix}
\phi_1 & \phi_2 & \phi_3 & \phi_6 & \phi_7 & \phi_{10} & \phi_{16} \\
1 & 1 & 1 & 1 & 1 & 1 & \vdots \\
\end{pmatrix}
\]

We obtain the same form of the matrix for $b = c = -a$.

- $a = b = \pm E(4)c$:

\[
\begin{pmatrix}
\phi_1 & \phi_2 & \phi_3 & \phi_6 & \phi_7 & \phi_{10} & \phi_{16} \\
1 & 1 & 1 & 1 & 1 & 1 & \vdots \\
\end{pmatrix}
\]
• \((c-a)(c-b)(a-b) \neq 0:\)

\[
\begin{array}{cccc}
\phi_1 & \phi_2 & \phi_3 & \phi_6 \\
\phi_7 & \phi_{10} & \phi_{16} & \\
\ldots & & & \\
\end{array}
\]

\[
\begin{pmatrix}
1 & 1 & 1 & \\
1 & 1 & 1 & \\
1 & 1 & 1 & \\
\end{pmatrix}
\]

3.13. The 4-dimensional characters. Without loss of generality, we study the character \(\phi_{35}\). We distinguished the following cases, based on proposition \textit{3.9}

C1. \((a^3 - bcd)(b^3 - acd)(c^3 - abd)(d^3 - abc) = 0:\) Let \(d^3 - abc = 0\). We type:

\[
\text{gap> } T:=\text{CharTable}(H_5).\text{irreducibles};;
\text{gap> } t:=\text{List}(T,i->\text{List}(i,j->\text{Value}(j,\text{"y", t^3*x^-1*z^-1}))));;
\text{gap> } t[35]=t[4]+t[16];
\]

true

We distinguish the following cases, depending on whether or not the character \(\phi_{16}\) is irreducible:

C1.1 \((a^2 + bc)(b^2 + ac)(c^2 + ab) \neq 0:\) Since \(d^3 - abc = 0\), we have that \((a^3 + d^3)(b^3 + d^3)(c^3 + d^3) \neq 0\). Due to propositions \textit{3.6} and \textit{3.8} the characters \(\phi_8, \phi_{11}, \phi_{13}\) and \(\phi_{16}\) are irreducible. As a result, the decomposition map is of the form:

\[
\begin{array}{cccc}
\phi_4 & \phi_8 & \phi_{11} & \phi_{13} \\
\phi_{16} & \phi_{35} & & \\
\ldots & & & \\
\end{array}
\]

\[
\begin{pmatrix}
1 & 1 & 1 & \\
* & * & 1 & \\
1 & 1 & 1 & \\
\end{pmatrix}
\]

where * denotes a placeholder for one of two values, 0 or 1. The sum of these values in each line equals 1.

C1.2 \((a^2 + bc)(b^2 + ac)(c^2 + ab) = 0:\) Let \(c^2 + ab = 0\). Since \(d^3 = abc\) we have that \(c^3 + d^3 = 0\). As we explained in section \textit{3.12}, \(\phi_{16} = \phi_3 + \phi_6\). We distinguish the following cases (up to reordering), based on whether or not the character \(\phi_6\) is irreducible.

\(\triangleright b = -E(3)a, c = -E(3)^2a, d = E(3)a:\) Using propositions \textit{3.6} and \textit{3.8} as well as criterion \textit{3.9} we have:

\[
\begin{array}{cccc}
\phi_1 & \phi_2 & \phi_3 & \phi_4 \\
\phi_{10} & \phi_6 & \phi_{17} & \phi_{22} \\
\phi_{11} & \phi_{12} & \phi_{13} & \phi_{14} \\
\phi_{15} & \phi_{16} & \phi_{19} & \phi_{20} \\
\phi_{21} & \phi_{23} & \phi_{24} & \phi_{25} \\
\ldots & & & \\
\end{array}
\]

\[
\begin{pmatrix}
1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 \\
\end{pmatrix}
\]

\(\triangleright b = -E(3)a, c = -E(3)^2a, d = a:\) Using again propositions \textit{3.6} and \textit{3.8} and criterion \textit{3.9} the decomposition matrix is of the following form:
Moreover, we have $\phi_{18} = \phi_{24}$, $\phi_{20} = \phi_{25}$, $\phi_{26} = \phi_{34}$, $\phi_{20} = \phi_{31}$, and $\phi_{41} = \phi_{44}$. We give in the next section the final form of this matrix, by examining the behavior of the 5-dimensional characters.

\[ a^2 - ab + b^2 \neq 0: \] Since $c^3 + d^3 = 0$ and $c^2 + ab = 0$, we have that $(c^3 \pm a^3)(c^3 \pm a^3) \neq 0$. As a result, we have that the characters $\phi_6$, $\phi_7$, $\phi_8$, $\phi_{10}$ and $\phi_{11}$ are irreducible. Moreover, we have that $\theta(\omega_{66}(z_0)) = -a^3b^3$, $\theta(\omega_{77}(z_0)) = -a^3e^3$, $\theta(\omega_{88}(z_0)) = -a^3d^3$, $\theta(\omega_{1010}(z_0)) = -c^3b^3$, and $\theta(\omega_{1111}(z_0)) = -d^3b^3$. As a result, the characters $\phi_6$, $\phi_7$ and $\phi_8$ are distinct, as well as the characters $\phi_{10}$ and $\phi_{11}$ (criterion [3.1]). As a result, the decomposition matrix is of the following form:

\[
\begin{pmatrix}
\phi_1 & 1 & 1 & 1 & 1 \\
\phi_2 & 1 & 1 & 1 & 1 \\
\phi_3 & 1 & 1 & 1 & 1 \\
\phi_4 & 1 & 1 & 1 & 1 \\
\phi_5 & 1 & 1 & 1 & 1 \\
\phi_6 & 1 & 1 & 1 & 1 \\
\phi_7 & 1 & 1 & 1 & 1 \\
\phi_8 & 1 & 1 & 1 & 1 \\
\phi_{10} & 1 & 1 & 1 & 1 \\
\phi_{11} & 1 & 1 & 1 & 1 \\
\phi_{12} & 1 & 1 & 1 & 1 \\
\phi_{14} & 1 & 1 & 1 & 1 \\
\phi_{15} & 1 & 1 & 1 & 1 \\
\phi_{35} & 1 & 1 & 1 & 1 \\
\end{pmatrix}
\]

where $*$ denotes a placeholder for one of two values, 0 or 1. The sum of these values in each line and in each column equals 1.

C2. $(a^2b^2 - abcd + c^2d^2)(a^2c^2 - abcd + b^2d^2)(a^2d^2 - abcd + b^2c^2) = 0$: Let $ab = E(3)cd$. We type:

```
gap T:=CharTable(H_4).irreducibles;;
gap t:=List(T,i->List(i,j->Value(j,["t", E(6)*x*y*z^-1])));
gap t[36]=t[6]+t[13];
gap true
```

Since $ab \neq cd$ the characters $\phi_6$ and $\phi_{13}$ correspond to distinct modules. We distinguish the following cases, depending on whether or not these characters are irreducible (see section [3.0]):

\[ (a^2 - ab + b^2)(c^2 - cd + d^2) \neq 0: \] Due to proposition [3.6] the characters $\phi_6$ and $\phi_{13}$ are irreducible. As a result, the decomposition matrix is of the form:

\[
\begin{pmatrix}
\phi_6 & 1 & 1 \\
\phi_{13} & 1 & 1 \\
\phi_{35} & 1 & 1 \\
\end{pmatrix}
\]

\[ (a^2 - ab + b^2)(c^2 - cd + d^2) = 0: \] This case is (up to reordering) case C1.2.
C3. \((a^3-bcd)(b^3-acd)(c^3-abd)(d^3-abc)(a^2b^2-abcd+a^2c^2-a^2d^2)ab^2+bd^2-c^2d^2-abcd+b^2c^2)\neq 0:

Due to criterion 3.3, the characters \(\phi_{30}\) and \(\phi_{35}\) are irreducible. As we explained in the beginning of this section, these characters are not in the same block. As a result, the decomposition matrix is of the form:

\[
\begin{pmatrix}
\phi_{30} \\
\phi_{35}
\end{pmatrix} = \begin{pmatrix}
1 \\
1
\end{pmatrix}
\]

3.14. The 5-dimensional characters. We recall that there are five 5-dimensional characters, the characters \(\phi_{36} \ldots \phi_{40}\). For \(i, j \in \{36, 37, \ldots, 40\}\) we have:

\[\theta(\omega_{\phi_i}(z_0)) = \begin{cases}
\theta(\omega_{\phi_j}(z_0)), & \text{for } i = j \\
-\theta(\omega_{\phi_j}(z_0)), & \text{for } i \neq j
\end{cases}\]

Hence, due to criterion 3.4, the 5-dimensional characters are not in the same block. Moreover, the matrix models of the corresponding modules depend on \(a, b, c, d, e\) and on the choice of a 5th root of \(abcde\), which we denote by \(r\). Let \(U^r\) the corresponding \(CH_5\)-module. We have the following proposition:

**Proposition 3.11.** The \(CH_5\)-module \(U^r\) is irreducible if and only if \((r^2 + \alpha r + \alpha^2)(r^2 + \alpha \beta)\neq 0\), for every \(\alpha, \beta \in \{a, b, c, d, e\}\) with \(\alpha \neq \beta\).

**Proof.** In general, a 5-dimensional module is irreducible if and only if it doesn’t admit 1-dimensional, 2-dimensional, 3-dimensional and 4-dimensional submodules. Let \(s_1 \mapsto A\) and \(s_2 \mapsto B\) the matrix form of the \(CH_5\)-module \(U^r\). As we explained in the proof of proposition 3.9, the existence of a 1-dimensional and 4-dimensional submodule translates into the existence of a common eigenvector, for the matrices \(A\) and \(B\) and a common eigenvector for the matrices \(A^r\) and \(B^r\). Following the method we explained in proof of proposition 3.10, we conclude that there aren’t any 1-dimensional and 4-dimensional submodules if and only if \(r^2 + \alpha r + \alpha^2 \neq 0\), for every \(\alpha, \beta \in \{a, b, c, d, e\}\).

It remains to exclude the existence of 2-dimensional and 3-dimensional submodules. Similarly as the proof of proposition 3.9, the \(CH_5\)-module \(U^r\) admits a 2-dimensional submodule if and only if there are \(\lambda_1, \lambda_2, \lambda_3, \lambda_4 \in \mathbb{C}\) and matrices \((A^T - \lambda_1 I_4)(A^T - \lambda_2 I_4)\) and \((B^T - \lambda_1 I_4)(B^T - \lambda_2 I_4)\) have a common eigenvector (not necessarily for the same eigenvalue). The existence of a 3-dimensional submodule and, hence, a 3-dimensional quotient, translates into the existence of a 2-dimensional \(CH_5^2\)-submodule (see proof of proposition 3.8). As a result, there are \(\mu_1, \mu_1', \mu_2, \mu_2' \in \mathbb{C}\) such as the matrices \((A - \mu_1 I_4)(A - \mu_2 I_4)\) and \((B - \mu_1' I_4)(B - \mu_2' I_4)\) have a common eigenvector (not necessarily for the same eigenvalue). Using Maple we proved that there aren’t any common eigenvectors for the aforementioned matrices if and only if \(r^2 + \alpha \beta \neq 0\), for every \(\alpha, \beta \in \{a, b, c, d\}\) with \(\alpha \neq \beta\).

Without loss of generality, we study now the behavior of the character \(\phi_{40}\). We distinguish the following cases, based on proposition 3.11

**C1.** \(r^2 + \alpha r + \alpha^2 = 0\), for some \(\alpha \in \{a, b, c, d, e\}\): Let \(r^2 + er + e^2 = 0\). Hence, \(r = E(3)e \Rightarrow abdec = E(3)^2e^5 \Rightarrow e^4 = E(3)abcd\). We type:

```
gap> T:=CharTable(H_5).irreducibles;;
gap> t:=List(T,i->List(i,j->Value(j,\"t\",E(3)\"2*x^1*y^1*z^1*1*w^4\\))));;
gap> t[40]=t[5]+t[35];
true
```

We now distinguish the following cases (up to reordering), based on section 3.13. Notice that, in order to have \(\phi_{40} = \phi_5 + \phi_{35}\) and not \(\phi_{40} = \phi_5 + \phi_{30}\), it is important to choose \(t = E(3)x^{-1}y^{-1}z^{-1}w^4\) instead of \(t = -E(3)x^{-1}y^{-1}z^{-1}w^4\). Since we work with 10th roots of the parameters and not with the parameters themselves, one may notice that there are less choices for the parameter \(e\).

**C1.1** \(e^4 = E(3)^2d^3\) and \((a^2 + bc)(b^2 + ac)(c^2 + ab)\neq 0\): Since \(e \neq d\), the characters \(\phi_4\) and \(\phi_5\) are distinct. Following C1.1 of section 3.13, the decomposition matrix is of the form:
The upper part of the matrix is labeled by the characters \( \phi_1, \phi_2, \phi_3, \phi_4, \phi_5, \phi_8, \phi_{10}, \phi_{15}, \phi_{17}, \phi_{18}, \phi_{22} \), and \( \phi_{41} \) together with the rest of the characters \( \phi_9, \phi_{11}, \phi_{20}, \phi_{21}, \phi_{27}, \phi_{28}, \phi_{29}, \phi_{32}, \phi_{33}, \phi_{34}, \phi_{36}, \phi_{37}, \phi_{38}, \phi_{39}, \) and \( \phi_{44} \), which are of defect 0.

### C1.2 \( b = -E(3)a, c = -E(3)^2a, d = E(3)a, e = a \):
The choice of the 10th roots of the parameters are: \( y = E(12)^7x \), \( z = E(60)^{19}x \), \( t = -E(15)^8x \), \( w = E(5)^2x \).

\[
\begin{pmatrix}
\phi_5 & 1 \\
\phi_6 & 1 & 1 \\
\phi_7 & 1 & 1 \\
\phi_{12} & 1 & 1 \\
\phi_{13} & 1 & 1 \\
\phi_{14} & 1 & 1 \\
\phi_{16} & 1 & 1 \\
\phi_{19} & 1 & 1 \\
\phi_{23} & 1 & 1 \\
\phi_{24} & 1 & 1 \\
\phi_{25} & 1 & 1 \\
\phi_{26} & 1 & 1 \\
\phi_{29} & 1 & 1 \\
\phi_{30} & 1 & 1 \\
\phi_{31} & 1 & 1 \\
\phi_{35} & 1 & 1 \\
\phi_{40} & 1 & 1 \\
\phi_{42} & 2 & 1 & 1 \\
\phi_{43} & 2 & 1 & 1 \\
\phi_{45} & \ldots & \ldots & 1 \\
\end{pmatrix}
\]

The upper part of the matrix is labeled by the characters \( \phi_1, \phi_2, \phi_3, \phi_4, \phi_5, \phi_8, \phi_{10}, \phi_{11}, \phi_{14}, \phi_{17}, \phi_{20}, \phi_{21}, \) and \( \phi_{22} \), and the rest of the characters \( \phi_9, \phi_{15}, \phi_{18}, \phi_{27}, \phi_{28}, \phi_{29}, \phi_{32}, \phi_{33}, \phi_{34}, \phi_{36}, \phi_{37}, \phi_{38}, \phi_{39}, \phi_{42}, \) and \( \phi_{44} \), which are of defect 0.

### C1.3 \( b = -E(3)a, c = -E(3)^2a, d = E(3)a, e = -a \):
The choice of the 10th roots of the parameters are: \( y = E(12)^7x \), \( z = E(60)^{19}x \), \( t = -E(15)^8x \), \( w = E(20)^3x \).

\[
\begin{pmatrix}
\phi_6 & 1 & 1 \\
\phi_7 & 1 & 1 \\
\phi_{13} & 1 & 1 \\
\phi_{15} & 1 & 1 \\
\phi_{16} & 1 & 1 \\
\phi_{19} & 1 & 1 \\
\phi_{23} & 1 & 1 \\
\phi_{25} & 1 & 1 \\
\phi_{27} & 1 & 1 \\
\phi_{28} & 1 & 1 \\
\phi_{30} & 1 & 1 \\
\phi_{32} & 1 & 1 \\
\phi_{35} & 1 & 1 \\
\phi_{40} & 1 & 1 \\
\phi_{43} & 1 & 1 \\
\phi_{44} & \ldots & \ldots & 1 \\
\end{pmatrix}
\]

The upper part of the matrix is labeled by the characters \( \phi_1, \phi_2, \phi_3, \phi_4, \phi_5, \phi_8, \phi_{10}, \phi_{11}, \phi_{14}, \phi_{17}, \phi_{20}, \phi_{21}, \phi_{27}, \phi_{28}, \phi_{29}, \phi_{32}, \phi_{33}, \phi_{34}, \phi_{36}, \phi_{37}, \phi_{38}, \phi_{39}, \phi_{42}, \\phi_{44} \), and \( \phi_{45} \), which are of defect 0.

### C1.4 \( b = -E(3)a, c = -E(3)^2a, d = a, e = E(3)a \):
The choice of the 10th roots of the parameters are: \( y = E(12)^7x \), \( z = E(60)^{19}x \), \( t = -E(5)^5x \), \( w = E(60)^{14}x \).
The upper part of the matrix is labeled by the characters $\phi_1$, $\phi_2$, $\phi_3$, $\phi_5$, $\phi_9$, $\phi_{10}$, $\phi_{17}$, $\phi_{18}$, $\phi_{23}$, and $\phi_{41}$ together with the characters $\phi_8$, $\phi_{12}$, $\phi_{19}$, $\phi_{21}$, $\phi_{27}$, $\phi_{28}$, $\phi_{30}$, $\phi_{32}$, $\phi_{33}$, $\phi_{36}$, $\phi_{37}$, $\phi_{38}$, $\phi_{39}$, and $\phi_{45}$, which are of defect 0.

C1.5 $b = -E(3)a$, $c = -E(3)^2a$, $d = a$, $e = -E(3)a$: The choice of the 10th roots of the parameters are: $y = E(12)^7x$, $z = E(60)^{19}x$, $t = -E(5)x$, $w = -E(60)^{59}x$. 

The upper part of the matrix is labeled by the characters $\phi_1$, $\phi_2$, $\phi_3$, $\phi_8$, $\phi_{10}$, $\phi_{17}$, $\phi_{18}$, $\phi_{19}$, $\phi_{26}$, $\phi_{29}$, $\phi_{30}$, and $\phi_{42}$ together with the characters $\phi_{23}$, $\phi_{28}$, $\phi_{36}$, $\phi_{37}$, $\phi_{38}$, and $\phi_{39}$, which are of defect 0.
C1.6 $e^4 = E(3)d^4$, $c^2 + ab = 0$, and $a^2 - ab + b^2 \neq 0$: For $c \neq e$ the decomposition matrix is of the following form:

$$
\begin{pmatrix}
1 & 1 & 1 \\
1 & * & * \\
1 & * & * \\
1 & * & * \\
1 & 1 & 1 \\
\vdots & & \\
\end{pmatrix}
$$

where $*$ denotes a placeholder for one of two values, 0 or 1. The sum of these values in each line and in each column equals 1.

For $e = c$ we have a different approach. Notice that $c^3 = -d^3$. Since $e = c$ and $e^4 = E(3)d^4$ we obtain $c^4 = E(3)d^3d$, and, hence, $d = -E(3)^2c$. Due to proposition 3.10 the character $\phi_{13} = \phi_{15}$ is not irreducible. Due to the same proposition, the character $\phi_{14}$ is irreducible. As a result, the decomposition matrix is of the form:

$$
\begin{pmatrix}
1 & 1 & 1 \\
* & * & * \\
* & * & * \\
1 & 1 & 1 \\
1 & 1 & 1 \\
\vdots & & \\
\end{pmatrix}
$$

where $*$ denotes again a placeholder for one of two values, 0 or 1. The sum of these values in each line equals 1. We also have $\phi_{12} = \phi_{10}$, $\phi_{19} = \phi_{21}$, $\phi_{22} = \phi_{24}$, and $\phi_{28} = \phi_{30}$. The characters $\phi_{10}$ and $\phi_{11}$ are not in the same block.

C1.7 $ab = E(3)cd$ and $(a^2 - ab + b^2)(c^2 - cd + d^2) \neq 0$:

$$
\begin{pmatrix}
1 & 1 & 1 \\
1 & * & * \\
1 & * & * \\
1 & * & * \\
1 & 1 & 1 \\
\vdots & & \\
\end{pmatrix}
$$

C1.8 $ab = E(3)cd$ and $(a^2 - ab + b^2)(c^2 - cd + d^2) = 0$: This case falls (up to reordering) to cases C1.2–C1.5.

C1.9 $(a^3 - bcd)(b^3 - acd)(c^3 - abd)(d^3 - abc)(a^2b^2 - abcd + c^2d^2)(a^2c^2 - abcd + b^2d^2)(a^2d^2 - abcd + b^2c^2) \neq 0$:

$$
\begin{pmatrix}
1 & 1 & 1 \\
1 & 1 & 1 \\
1 & 1 & 1 \\
\vdots & & \\
\end{pmatrix}
$$

C2. $r^2 + \alpha \beta = 0$, for some $\alpha, \beta \in \{a, b, c, d, e\}$ distinct: Let $r^2 = -ab$. We type:

```gap
gap> T:=CharTable(H.5).irreducibles;;
gap> t:=List(T,i->List([j],j->Value(t,"w\w*\E(8)\x\y\z\t\d"))));;
gap> t[40]=t[6]+t[25];
gap> true
```

We now distinguish the following cases (up to reordering), based on sections 3.11 and 3.12.
C2.1 \((a^2 - ab + b^2)(c^2 + de)(d^2 + ce)(e^2 + dc) \neq 0\): Due to propositions 3.6 and 3.8 the characters \(\phi_6\) and \(\phi_{25}\) are irreducible. As a result, the decomposition matrix is of the form:

\[
\begin{array}{c}
\phi_6 \\
\phi_{25} \\
\phi_{40} \\
\vdots
\end{array}
\begin{pmatrix}
1 & 1 \\
1 & 1
\end{pmatrix}
\]

C2.2 \((a^2 - ab + b^2) = 0\) and \((c^2 + de)(d^2 + ce)(e^2 + dc) \neq 0\): Due to propositions 3.6 and 3.8 the decomposition matrix is of the form:

\[
\begin{array}{c}
\phi_1 \\
\phi_6 \\
\phi_{25} \\
\phi_{40} \\
\vdots
\end{array}
\begin{pmatrix}
1 & 1 \\
1 & 1 \\
1 & 1 \\
1 & 1 \\
\vdots
\end{pmatrix}
\]

C2.3 \((a^2 - ab + b^2) \neq 0\) and \((c^2 + de)(d^2 + ce)(e^2 + dc) = 0\): Without loss of generality, we assume that \(c^2 + de = 0\). Following section 3.12 we have that \(\phi_{25} = \phi_3 + \phi_{15}\). We distinguish the following cases, depending on the behavior of the 2-dimensional character \(\phi_{15}\).

- \(d^2 - de + e^2 \neq 0\): Due to proposition 3.6 the character \(\phi_{15}\) is irreducible.

The characters \(\phi_6\) and \(\phi_{15}\) correspond to the same module, if and only if the matrix models are the same. One can find these matrix models in GAP and notice that they are the same if and only if \(ab = de\) and \(a + b = d + e\). We multiply the second equation with \(a\) and we have \(a^2 + ab = ad + ae \Rightarrow a^2 + de = ad + ae \Rightarrow a \in \{d, e\}\).

If \(a = d\), since \(a + b = d + e\), we have \(b = e\). Similarly, if \(a = e\), we have \(b = d\). As a result, we can distinguish the following cases:

- If \(a \notin \{d, e\}\) or if \(b \notin \{d, e\}\) the characters \(\phi_6\) and \(\phi_{15}\) are not in the same block. As a result, the decomposition matrix is of the following form:

\[
\begin{array}{c}
\phi_3 \\
\phi_6 \\
\phi_{15} \\
\phi_{25} \\
\phi_{40} \\
\vdots
\end{array}
\begin{pmatrix}
1 & 1 \\
1 & 1 \\
1 & 1 \\
1 & 1 \\
1 & 1 \\
\vdots
\end{pmatrix}
\]

- We suppose now that \(a, b \in \{d, e\}\). The decomposition matrix is of the following form:

\[
\begin{array}{c}
\phi_3 \\
\phi_6 \\
\phi_{15} \\
\phi_{25} \\
\phi_{40} \\
\vdots
\end{array}
\begin{pmatrix}
1 & 1 \\
1 & 1 \\
1 & 1 \\
1 & 1 \\
1 & 2 \\
\vdots
\end{pmatrix}
\]

An interesting case that one can examine here, is \(a = b = d = e\). Since \(c^2 + ab = 0\) we have that \(c^2 + a^2 = 0\). Let \(x = x, y = x, z = E(8)x, t = x, w = x\) a choice of 10th roots of the parameters. The decomposition matrix is of the following form:
The decomposition matrix is of the following form:

\[
\begin{pmatrix}
\phi_2 & 1 & \cdots & \cdots & \cdots \\
\phi_4 & 1 & \cdots & \cdots & \cdots \\
\phi_5 & 1 & \cdots & \cdots & \cdots \\
\phi_8 & 1 & \cdots & \cdots & \cdots \\
\phi_9 & 1 & \cdots & \cdots & \cdots \\
\phi_{10} & \cdots & 1 & \cdots & \cdots \\
\phi_{11} & \cdots & 1 & \cdots & \cdots \\
\phi_{12} & \cdots & 1 & \cdots & \cdots \\
\phi_{13} & \cdots & 1 & \cdots & \cdots \\
\phi_{14} & \cdots & 1 & \cdots & \cdots \\
\phi_{15} & \cdots & \cdots & \cdots & \cdots \\
\phi_{16} & 1 & 1 & \cdots & \cdots \\
\phi_{17} & \cdots & 1 & \cdots & \cdots \\
\phi_{18} & \cdots & 1 & \cdots & \cdots \\
\phi_{19} & \cdots & 1 & \cdots & \cdots \\
\phi_{20} & \cdots & 1 & \cdots & \cdots \\
\phi_{21} & \cdots & \cdots & \cdots & \cdots \\
\phi_{22} & \cdots & 1 & \cdots & \cdots \\
\phi_{23} & \cdots & 1 & \cdots & \cdots \\
\phi_{24} & \cdots & 1 & \cdots & \cdots \\
\phi_{25} & \cdots & 1 & \cdots & \cdots \\
\phi_{26} & \cdots & \cdots & \cdots & \cdots \\
\phi_{27} & \cdots & \cdots & \cdots & \cdots \\
\phi_{29} & \cdots & \cdots & \cdots & \cdots \\
\phi_{30} & \cdots & \cdots & \cdots & \cdots \\
\phi_{31} & \cdots & \cdots & \cdots & \cdots \\
\phi_{32} & \cdots & \cdots & \cdots & \cdots \\
\phi_{33} & 1 & 1 & \cdots & \cdots \\
\phi_{34} & \cdots & 1 & \cdots & \cdots \\
\phi_{35} & \cdots & \cdots & \cdots & \cdots \\
\phi_{36} & 1 & 2 & \cdots & \cdots \\
\phi_{37} & \cdots & 1 & \cdots & \cdots \\
\phi_{38} & \cdots & \cdots & \cdots & \cdots \\
\phi_{39} & \cdots & \cdots & \cdots & \cdots \\
\phi_{40} & \cdots & \cdots & \cdots & 1 \\
\phi_{41} & \cdots & \cdots & \cdots & \cdots \\
\phi_{42} & \cdots & \cdots & \cdots & \cdots \\
\phi_{43} & \cdots & \cdots & \cdots & \cdots \\
\phi_{44} & \cdots & \cdots & \cdots & \cdots \\
\phi_{45} & \cdots & \cdots & \cdots & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
\end{pmatrix}
\]

where the upper part of the matrix is indexed by the characters \( \phi_1, \phi_3, \phi_6, \phi_7, \phi_{17}, \phi_{26}, \phi_{31}, \phi_{41} \), together with the characters \( \phi_{28}, \phi_{36}, \phi_{37}, \phi_{38}, \phi_{39}, \phi_{43} \), which are of defect 0.

\[ d^2 - dc + e^2 = 0 \]: The decomposition matrix is of the following form:

\[
\begin{pmatrix}
1 & 1 \\
& 1 \\
& & 1 \\
& & & 1 \\
& & & & 1 \\
& & & & & 1 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\end{pmatrix}
\]

C2.4 \((a^2 - ab + b^2) \neq 0\) and \((c^2 + dc)(a^2 + ce)(e^2 + dc) = 0\): The decomposition matrix is of the form described in C1.2–C1.6.

C3. \((r^2 + \alpha r + \alpha^2)(r^2 + \alpha \beta) \neq 0\), for every \( \alpha, \beta \in \{a, b, c, d\} \) with \( \alpha \neq \beta \): Due to proposition 3.11, the character \( \phi_{40} \) is irreducible and the decomposition matrix is of the form:

\[
\begin{pmatrix}
\vdots \\
\phi_{40} & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \\
\vdots \\
\end{pmatrix}
\]

3.15. The 6-dimensional characters. We recall that there are five 6-dimensional characters, the characters \( \phi_{41}, \ldots, \phi_{45} \). Each character depends on a value \( \alpha \in \{a, b, c, d, e\} \). Without loss of generality, we focus on character \( \phi_{43} \), which depends on \( c \). Since \( \phi_{43} \) is of maximal dimension, one should focus on the values of \( a, b, c, d, e \) that annihilate the corresponding Schur element \( s_{\phi_{43}} \).

One can see 4 types of monomials in \( s_{\phi_{43}} \). The first one is the \((c - a)(c - b)(c - d)(c - e)\), which is annihilated when \( c = \beta \), for \( \beta \in \{a, b, d, e\} \). This means that the character \( \phi_{43} \) coincides with the 6-dimensional character depending on \( \beta \).

We examine now the other 3 types of monomials by distinguishing the following cases:
C1. \((ab + de)(ad + bc)(ae + bd) = 0\): Without loss of generality, we suppose that \(ab = -de\).

We type:

```gap
> T:=CharTable(H_5).irreducibles;;
> t:=List(T,i->List(i,j->Value(j,\['t', E(3)^2*x^-1*y^-1*z^-1*w^4\])));
> t[43]=t[16]+t[25];
true
```

We also notice that, since \(ab \neq de\) the characters \(\phi_{16}\) and \(\phi_{25}\) don’t correspond to the same module (the matrix models are not the same). We follow now section 3.12 and we distinguish the following cases:

C1.1 The characters \(\phi_{16}\) and \(\phi_{25}\) are irreducible: The decomposition matrix is of the form:

\[
\begin{pmatrix}
\phi_{16} & 1 \\
\phi_{25} & 1 \\
\phi_{43} & 1 \\
\end{pmatrix}
\]

C1.2 Only one of the characters \(\phi_{16}\) and \(\phi_{25}\) is irreducible: Without loss of generality and following section 3.12, we suppose that \(c^2 = -ab\).

If \(a^2 - ab + b^2 \neq 0\) the decomposition matrix is of the form:

\[
\begin{pmatrix}
\phi_3 & 1 & 1 \\
\phi_6 & 1 & 1 \\
\phi_{16} & 1 & 1 \\
\phi_{43} & 1 & 1 \\
\end{pmatrix}
\]

We suppose now that \(a^2 - ab + b^2 = 0\). Following again 3.12, the decomposition matrix is of the form:

\[
\begin{pmatrix}
\phi_1 & 1 & 1 & 1 \\
\phi_2 & 1 & 1 \\
\phi_{25} & 1 & 1 \\
\phi_6 & 1 & 1 \\
\phi_{16} & 1 & 1 \\
\phi_{43} & 1 & 1 \\
\end{pmatrix}
\]

C1.3 Neither \(\phi_{16}\) nor \(\phi_{25}\) is irreducible: Since \(ab \neq de\), it remains to examine (up to re-ordering) the following cases:

\(\triangleright c^2 + ab = d^2 + ce = 0\): Since \(ab = -de\) we have that \(c^2 = de\). We also have \(d^2 = -ce\) and, hence, \(d^3 = -c^3\). As a result, \(d \neq c\), meaning that the characters \(\phi_3\) and \(\phi_4\) correspond to distinct modules. Let \(c = -E(3)^kd\), with \(k \in \{0, 1, 2\}\). Since \(d^2 = -ce\) we obtain \(e = E(3)^{-k}d\) and \(ab = -E(3)^{-k}d^2\).

We first assume that \(k = 0\). As a result, \((a^2 - ab + b^2)(c^2 - ce + e^2) \neq 0\). If \(a \notin \{c,e\}\) or \(b \notin \{c,e\}\) the characters \(\phi_6\) and \(\phi_{14}\) correspond to distinct modules. As a result, the decomposition matrix is of the form:

\[
\begin{pmatrix}
\phi_3 & 1 & 1 \\
\phi_4 & 1 & 1 \\
\phi_6 & 1 & 1 \\
\phi_{14} & 1 & 1 \\
\phi_{16} & 1 & 1 \\
\phi_{25} & 1 & 1 \\
\phi_{43} & 1 & 1 \\
\end{pmatrix}
\]

Let now \(a, b \in \{c,e\}\). We notice that \(a \neq b\). Indeed, let \(a = b = c\). Since \(c = -d\) and \(e = d\) we have \(ab = c^2 = de\), which contradicts the fact that \(ab = -de\). Similarly, for \(a = b = e\). As a result, it remains to examine the case \(a = c = -d\)
and $b = e = d$:

$$
\begin{pmatrix}
\phi_3 & 1 & & & & & & \\
\phi_4 & 1 & & & & & & \\
\phi_5 & 1 & & & & & & \\
\phi_6 & 1 & & & & & & \\
\phi_7 & 1 & & & & & & \\
\phi_8 & 1 & & & & & & \\
\phi_9 & 1 & & & & & & \\
\phi_{10} & 1 & & & & & & \\
\phi_{11} & 1 & & & & & & \\
\phi_{12} & 1 & & & & & & \\
\phi_{13} & 1 & & & & & & \\
\phi_{14} & 1 & & & & & & \\
\phi_{15} & 1 & & & & & & \\
\phi_{16} & 1 & & & & & & \\
\phi_{17} & 1 & 1 & & & & & \\
\phi_{18} & 1 & 1 & & & & & \\
\phi_{19} & 1 & 1 & & & & & \\
\phi_{20} & 1 & 1 & & & & & \\
\phi_{21} & 1 & 1 & & & & & \\
\phi_{22} & 1 & 1 & & & & & \\
\phi_{23} & 1 & 1 & & & & & \\
\phi_{24} & 1 & 1 & & & & & \\
\phi_{25} & 1 & 1 & & & & & \\
\phi_{26} & 1 & 1 & & & & & \\
\phi_{27} & 1 & 1 & & & & & \\
\phi_{28} & & & & & & & \\
\phi_{29} & & & & & & & \\
\phi_{30} & 1 & 1 & & & & & \\
\phi_{31} & 1 & 1 & & & & & \\
\phi_{32} & 1 & 1 & & & & & \\
\phi_{33} & & & & & & & \\
\phi_{34} & & & & & & & \\
\phi_{35} & & & & & & & \\
\phi_{36} & & & & & & & \\
\phi_{37} & & & & & & & \\
\phi_{38} & & & & & & & \\
\phi_{39} & & & & & & & \\
\phi_{40} & & & & & & & \\
\phi_{41} & & & & & & & \\
\phi_{42} & & & & & & & \\
\phi_{43} & & & & & & & \\
\phi_{44} & & & & & & & \\
\phi_{45} & & & & & & & \\
\end{pmatrix}
$$

where the upper part of the matrix is indexed by the characters $\phi_1$, $\phi_2$, $\phi_6$, $\phi_{11}$, $\phi_{26}$, $\phi_{31}$, $\phi_{32}$, and $\phi_{42}$, together with the characters $\phi_7$, $\phi_{24}$, $\phi_{36}$, $\phi_{37}$, $\phi_{38}$, $\phi_{39}$, which are of defect 0.

We assume now that $k \neq 0$. Without loss of generality, let $k = 1$. Hence, $c^2 - ce + e^2 = 0$ and, due to proposition 3.6, the character $\phi_{14}$ is irreducible.

For $a^2 - ab + b^2 \neq 0$ the decomposition matrix is of the form:

$$
\begin{pmatrix}
\phi_3 & 1 & 1 & & & & & \\
\phi_4 & 1 & 1 & & & & & \\
\phi_5 & 1 & 1 & & & & & \\
\phi_6 & 1 & 1 & & & & & \\
\phi_7 & 1 & 1 & & & & & \\
\phi_8 & 1 & 1 & & & & & \\
\phi_9 & 1 & 1 & & & & & \\
\phi_{10} & 1 & 1 & & & & & \\
\phi_{11} & 1 & 1 & & & & & \\
\phi_{12} & 1 & 1 & & & & & \\
\phi_{13} & 1 & 1 & & & & & \\
\phi_{14} & 1 & 1 & & & & & \\
\phi_{15} & 1 & 1 & & & & & \\
\phi_{16} & 1 & 1 & & & & & \\
\phi_{17} & 1 & 1 & & & & & \\
\phi_{18} & 1 & 1 & & & & & \\
\phi_{19} & 1 & 1 & & & & & \\
\phi_{20} & 1 & 1 & & & & & \\
\phi_{21} & 1 & 1 & & & & & \\
\phi_{22} & 1 & 1 & & & & & \\
\phi_{23} & 1 & 1 & & & & & \\
\phi_{24} & 1 & 1 & & & & & \\
\phi_{25} & 1 & 1 & & & & & \\
\phi_{26} & 1 & 1 & & & & & \\
\phi_{27} & 1 & 1 & & & & & \\
\phi_{28} & & & & & & & \\
\phi_{29} & & & & & & & \\
\phi_{30} & 1 & 1 & & & & & \\
\phi_{31} & 1 & 1 & & & & & \\
\phi_{32} & 1 & 1 & & & & & \\
\phi_{33} & & & & & & & \\
\phi_{34} & & & & & & & \\
\phi_{35} & & & & & & & \\
\phi_{36} & & & & & & & \\
\phi_{37} & & & & & & & \\
\phi_{38} & & & & & & & \\
\phi_{39} & & & & & & & \\
\phi_{40} & & & & & & & \\
\phi_{41} & & & & & & & \\
\phi_{42} & & & & & & & \\
\phi_{43} & & & & & & & \\
\phi_{44} & & & & & & & \\
\phi_{45} & & & & & & & \\
\end{pmatrix}
$$

For $a^2 - ab + b^2 = 0$ the decomposition matrix is of the form described in C1.2 and C1.4 of section 3.14.

$a^2 + bc = d^2 + ce = 0$: We have $a^2 d^{-2} = be^{-1}$. Since $ab \neq de$ we have $a \neq d$ and, hence, the characters $\phi_1$ and $\phi_4$ correspond to distinct modules. Let $(a^2 - ab + b^2)(c^2 - ce + e^2) \neq 0$. For $b \neq e$ the characters $\phi_{10}$ and $\phi_{14}$ correspond to distinct modules. As a result, the decomposition matrix is of the form:

$$
\begin{pmatrix}
\phi_1 & 1 & 1 & & & & & \\
\phi_4 & 1 & 1 & & & & & \\
\phi_{10} & 1 & 1 & & & & & \\
\phi_{14} & 1 & 1 & & & & & \\
\phi_{16} & 1 & 1 & & & & & \\
\phi_{25} & 1 & 1 & & & & & \\
\phi_{43} & 1 & 1 & & & & & \\
\end{pmatrix}
$$
For $b = e$ we have:

$$
\begin{pmatrix}
\phi_1 & 1 & 1 \\
\phi_4 & 1 & 1 \\
\phi_{10} & 1 & 1 \\
\phi_{14} & 1 & 1 \\
\phi_{16} & 1 & 2 \\
\phi_{25} & 1 & 1 \\
\phi_{43} & 1 & 1 \\
\end{pmatrix}
$$

We assume now that $b^2 - bc + c^2 = 0$. Let $b = -E(3)c$. Since $a^2 = -bc$ we have that $a = \pm E(3)^2c$. As a result, the characters $\phi_1$, $\phi_2$ and $\phi_3$ are distinct. For $c^2 - ce + e^2 \neq 0$ and $d \notin \{a, b, c\}$ the decomposition matrix is the following:

$$
\begin{pmatrix}
\phi_1 & 1 & 1 & 1 \\
\phi_2 & 1 & 1 & 1 \\
\phi_3 & 1 & 1 & 1 \\
\phi_{10} & 1 & 1 & 1 \\
\phi_{14} & 1 & 1 & 1 \\
\phi_{25} & 1 & 1 & 1 \\
\phi_{43} & 1 & 1 & 1 \\
\end{pmatrix}
$$

Without loss of generality, we assume now that $d = a$. Since $ab = -de$ we obtain $e = -b$. As a result, we have $d = a = \pm E(3)^2c$, $b = -E(3)c$ and $e = E(3)c$. However, $d^2 = E(3)c^2 = ce$, which contradicts the fact that $d^2 + ce = 0$.

It remains to examine the case, where $b^2 - bc + c^2 = c^2 - ce + e^2 = 0$. Up to reordering, we have $b = -E(3)c$, $a = E(3)^2c$, $e = -E(3)c$ and $d = -E(3)^2c$. The decomposition matrix is of the form:

$$
I_{24}
$$

The upper part of the matrix is labeled by the characters $\phi_1$, $\phi_2$, $\phi_3$, $\phi_4$, $\phi_7$, $\phi_{11}$, $\phi_{17}$, $\phi_{18}$, $\phi_{19}$, $\phi_{26}$, $\phi_{29}$, and $\phi_{42}$ together with the characters $\phi_8$, $\phi_{12}$, $\phi_{23}$, $\phi_{24}$, $\phi_{26}$, $\phi_{28}$, $\phi_{29}$, $\phi_{31}$, $\phi_{33}$, $\phi_{36}$, $\phi_{37}$, $\phi_{38}$, $\phi_{39}$, and $\phi_{44}$, which are of defect 0.

C2. $(a^2c - bde)(b^2c - ade)(d^2c - ade)(e^2c - abd) = 0$: Without loss of generality, we suppose that $e^2c = abd$. We type:

```gap
t := CharTable(H_5).irreducibles;
next
s := List(T,i->List(j,Value(j,["t", -2*y^2+1*z^2-1]));;
next
s[43] = t[44] + t[35];
next
true
```

We distinguish the following cases, based on sections 3.11 and 3.13.
C2.1 The characters $\phi_{14}$ and $\phi_{35}$ are irreducible: As a result, the decomposition matrix is of the form:

$$
\begin{pmatrix}
\phi_{14} & 1 & 1 & 1 \\
\phi_{30} & 1 & 1 & 1 \\
\phi_{35} & 1 & 1 \\
\phi_{43} & 1 & 1 \\
\vdots & & & 
\end{pmatrix}
$$

C2.2 The character $\phi_{14}$ is irreducible and the character $\phi_{35}$ is not: The only case that hasn’t been studied in C1 is the case where the character $\phi_{35}$ breaks up into two 2-dimensional irreducible characters:

Due to proposition 3.6 and C2 of section 3.13 we assume that $ab = E(3)cd$ and $(a^2 - ab + b^2)(e^2 - cd + d^2) \neq 0$. We have that $\phi_{35} = \phi_6 + \phi_{13}$, where $\phi_6$ and $\phi_{13}$ are distinct 2-dimensional irreducible characters. We also notice that $ab \neq ce$. Indeed, let $ab = ce$. Since $e^2c = abd$ we obtain $e = d$. However, since $ab = E(3)cd$ we have $ce = E(3)ce$, which is a contradiction. As a result, the characters $\phi_6$ and $\phi_{14}$ are distinct.

We also notice that the characters $\phi_{13}$ and $\phi_{14}$ are distinct. Indeed, these characters correspond to the same module, if $e = d$. Since $e^2c = abd$ we have $dc = ab$. We use now the relation $ab = E(3)cd$ and we obtain $cd = E(3)cd$, which is a contradiction. As a result, the decomposition matrix is of the following form:

$$
\begin{pmatrix}
\phi_6 & 1 & 1 \\
\phi_{13} & 1 & 1 \\
\phi_{14} & 1 & 1 \\
\phi_{43} & 1 & 1 \\
\vdots & & & 
\end{pmatrix}
$$

C2.3 Neither the character $\phi_{35}$ nor the character $\phi_{14}$ is irreducible: This case coincide with C1.2 and C1.3.

C3. $c^4 + abdc = 0$: We type:

```
gap> T:=CharTable(H_5).irreducibles;;
gap> t:=List(T,i->List(i,j->Value(j,"w", E(4)*z^4*y^-1*t^-1*x^-1))));
gap> t[43]=t[3]+t[40];
true
```

Following section 3.13 we notice that the only case we haven’t seen in C1 and C2 is the case, where $\phi_{40}$ is irreducible and, hence, the decomposition matrix is of the form:

$$
\begin{pmatrix}
\phi_3 & 1 & 1 \\
\phi_{40} & 1 & 1 \\
\phi_{43} & 1 & 1 \\
\vdots & & & 
\end{pmatrix}
$$

Remark 3.12. In [5] §5 we have classified the irreducible representations of the braid group $B_3 = \langle s_1, s_2 \mid s_1s_2s_1 = s_2s_1s_2 \rangle$ for dimension $k = 2, 3, 4, 5$. Let $s_1 \mapsto A$ and $s_2 \mapsto B$ such a representation. In order to deal with the case $k = 5$, we made an assumption for the determinant of the matrix $A$ (which is the same as the determinant of the matrix $B$); we assumed that $\det A \neq$
\(-\lambda_i^6 \lambda_j^{-1}\), where \(\lambda_i, \lambda_j\) denote any eigenvalues of \(A\), not necessarily distinct. In this section we also considered the case where \(\det A = -\lambda_i^6 \lambda_j^{-1}\) and one can easily notice that the results presented in [5] are valid for this case, as well.

References

[1] David Bessis. Sur le corps de définition d’un groupe de réflexions complexe. Communications in Algebra, 25(8):2703–2716, 1997.
[2] Michel Broué and Gunter Malle. Zyklotomische Heckealgebren in Repräsentations unipotenten gendräiches et blocs des groupes réductifs finis. Astérisque, 212:119–189, 1993.
[3] Michel Broué, Gunter Malle, and Jean Michel. Towards Spetses I. Transformation groups, 4(2-3):157–218, 1999.
[4] Michel Broué, Gunter Malle, and Raphaël Rouquier. Complex reflection groups, braid groups, Hecke algebras. Journal für die Reine und Angewandte Mathematik, 500:127–190, 1998.
[5] Eirini Chavli. Universal deformations of the finite quotients of the braid group on 3 strands. Journal of Algebra, 459:238–271, 2016.
[6] Maria Chlouveraki. Blocks and families for cyclotomic Hecke algebras. Springer, 2009.
[7] Maria Chlouveraki and Hyohe Miyachi. Decomposition matrices for d-Harish-Chandra series: the exceptional rank two cases. LMS Journal of Computation and Mathematics, 14:271–290, 2011.
[8] HSM Coxeter. Factor groups of the braid group. In Proc. Fourth Canadian Math. Congress, Banff, pages 95–122, 1957.
[9] Meinolf Geck and Götz Pfeiffer. Characters of finite Coxeter groups and Iwahori-Hecke algebras. Number 21. Oxford University Press, 2000.
[10] Gunter Malle. On the Rationality and Fake Degrees of Characters of Cyclotomic Algebras. Journal of Mathematical Sciences-University of Tokyo, 6(4):647–678, 1999.
[11] Gunter Malle and Raphaël Rouquier. Familles de caractères de groupes de réflexions complexes. Representation Theory of the American Mathematical Society, 7(23):610–640, 2003.
[12] Ivan Marin and Emmanuel Wagner. Markov traces on the Birman-Wenzl-Murakami algebras. arXiv preprint arXiv:1409.4021, 2014.
[13] Pierre-Loic Meillot. Representation Theory of Symmetric Groups. CRC Press, 2017.
[14] Jean Michel. The development version of the CHEVIE package of GAP3. Journal of Algebra, 435:308–336, 2015.
[15] Jean-Pierre Serre. Linear representations of finite groups, volume 42. Springer Science & Business Media, 2012.
[16] Geoffrey C Shephard and John A Todd. Finite unitary reflection groups. Canad. J. Math, 6(2):274–301, 1954.