More on general $p$-brane solutions

D. Gal’tsov$^a,d$, S. Klevtsov$^{a,b,c}$, D. Orlov$^a$ and G. Clément$^d$

$^a$Department of Theoretical Physics, Moscow State University, 119899, Moscow, Russia
$^b$ITEP, 25 B. Cheremushkinskaya, Moscow 117259, Russia
$^c$Department of Physics and Astronomy, Rutgers University, Piscataway, NJ 08854-8019, USA
$^d$Laboratoire de Physique Théorique LAPTH (CNRS), B.P.110, F-74941 Annecy-le-Vieux cedex, France

E-mail: galtsov@mail.phys.msu.ru, klevtsov@itep.ru, orlov_d@mail.ru, gclement@lapp.in2p3.fr

ABSTRACT: Recently it was found that the complete integration of the Einstein-dilaton-antisymmetric form equations depending on one variable and describing static singly charged $p$-branes leads to two and only two classes of solutions: the standard asymptotically flat black $p$-brane and the asymptotically non-flat $p$-brane approaching the linear dilaton background at spatial infinity. Here we analyze this issue in more details and generalize the corresponding uniqueness argument to the case of partially delocalized branes. We also consider the special case of codimension one and find, in addition to the standard domain wall, the black wall solution. Explicit relations between our solutions and some recently found $p$-brane solutions “with extra parameters” are presented.

KEYWORDS: String and Brane Theory, $p$-branes
1. Introduction

Classical supergravity solutions describing $p$-branes were extensively studied during the past decade [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13]. It is generally believed that a singly charged static $p$-brane solution depends on two parameters, the mass and the charge (densities) of the brane (up to the dilaton value at infinity) [1, 3], these solutions are asymptotically flat and possess a regular event horizon. Black $p$-branes have the $ISO(p) \times R$ symmetry of the world volume (with $R$ corresponding to the time direction), which is enhanced to the full Poincaré symmetry $ISO(p,1)$ in the extremal (BPS) case. Recently the uniqueness of these solutions was investigated in more detail [14, 15], and it was shown that, once the absence of naked singularities is required, the generic solution must be either asymptotically flat (standard) or to approach the linear dilaton asymptotic at spatial infinity [16, 17, 18]. In ten dimensions the latter class corresponds to the near-horizon limit of near-extremal $p$-branes and provides the holographic counterpart to the thermal phase of the dual conformal field theories with sixteen supercharges [19, 20, 21, 22].
Meanwhile some apparently different claims can be encountered in the recent literature. The complete integration of the Einstein-dilaton-antisymmetric form system for a single brane was performed \cite{23} and a family of ISO\((p) \times R\) solutions was presented, containing four free parameters. An interpretation of one of the extra parameters was attempted in \cite{24} (see also \cite{25, 26, 27}): the ISO\((p, 1)\) subfamily of the solutions of \cite{23} was treated as describing the brane-antibrane system in the sense of Sen \cite{28}, the corresponding extra degree of freedom being associated with the tachyon. The solutions of \cite{23} were also invoked in some recent attempts to find the supergravity description for stable non-BPS branes of string theory \cite{29, 30, 31, 32}. Other generalizations of the ISO\((p, 1)\) solution were given in \cite{33}. Besides containing additional parameters, the solutions presented in this paper also describe a more general structure of the transverse space, namely, \(SO(k) \times R^q, q = d - p - k - 2\) (cylinder), \(R^{q+k+1}\), as well as the case of the hyperbolic geometry \(SO(k-1, 1) \times R^q\).

However, the geometric structure of the \(p\)-brane type solutions with extra parameters, in particular the nature of singularities, was not sufficiently investigated so far.

The purpose of the present paper is to generalize the analysis of \cite{14, 15} to the case of a more general structure of the transverse space, extend the uniqueness proof for partially localized branes, consider the particular case of domain walls, and to clarify the relationship with other “general” \(p\)-brane solutions. We confirm our previous uniqueness argument saying that there are two and only two classes of solutions without naked singularities: the standard asymptotically flat branes (black or BPS), and black branes on the linear dilaton background.

The paper is organized as follows. In the next two sections we describe the system and its complete integration via reduction to the separate Liouville equations. In Sec. 4 the special points of the generic solutions are analyzed by computing the Ricci and Kretschmann scalars and studying the radial null geodesics. We identify the horizons, set up the horizon regularity conditions, and list all possibilities when singularities are hidden inside the horizon. We find three different classes of solution one of which is compact singular, and two other correspond to either asymptotically flat case, or asymptotically linear dilaton backgroundd case. The general regular asymptotically flat black solution is constructed in Sec. 5, it contains three free parameters (mass, charge and the asymptotic value of the dilaton) and no any ’extra parameters’. The following Sec. 6 is devoted to the construction of the black branes on the linear dilaton background including their thermodynamical properties. The Sec. 7 contains an analysis of the case of the codimension one. Here in addition to the standard supergravity domain wall we obtain the black solution which can be interpreted also as the black wall on the linear dilaton background. In Sec. 8 we give explicit relations between our generic solution and some other previously found general solutions and review them in the spirit of our reasoning.

2. Setup

Consider the standard action containing the metric, the \(q\)-form field strength, \(F_{[q]}\), and the
dilaton, $\phi$, coupled to the form field with the coupling constant $a$

$$S = \int d^d x \sqrt{-g} \left( R - \frac{1}{2} \partial_\mu \phi \partial^\nu \phi - \frac{1}{2q!} e^{a\phi} F_{[q]}^2 \right).$$  \hspace{1cm} (2.1)

The corresponding equations of motion

$$R_{\mu\nu} - \frac{1}{2} \partial_\mu \phi \partial_\nu \phi - \frac{e^{a\phi}}{2(q-1)!} \left[ F_{\mu\nu_2...\nu_q} F_{\nu_1...\nu_q} - \frac{q-1}{q(d-2)} F_{[q]} g_{\mu\nu} \right] = 0,$$  \hspace{1cm} (2.2)

$$\partial_\mu \left( \sqrt{-g} e^{a\phi} F_{\mu\nu} \right) = 0,$$  \hspace{1cm} (2.3)

$$\frac{1}{\sqrt{-g}} \partial_\mu \left( \sqrt{-g} \partial^\mu \phi \right) - \frac{a}{2q!} e^{a\phi} F_{[q]}^2 = 0,$$  \hspace{1cm} (2.4)

are invariant under the discrete S-duality:

$$g_{\mu\nu} \rightarrow g_{\mu\nu}, \quad F \rightarrow e^{-a\phi} * F, \quad \phi \rightarrow -\phi,$$  \hspace{1cm} (2.5)

where $*$ denotes a $d$-dimensional Hodge dual, so we will restrict ourselves to purely magnetic solutions.

We consider black $p$-brane solutions with a $p+1$ dimensional world volume and a transverse space being the $q$ dimensional space $\Sigma_{k,\sigma} \times \mathbb{R}^{q-k}$ $(p + q = d - 2)$:

$$ds^2 = -e^{2B} dt^2 + e^{2D}(dx_1^2 + \ldots + dx_p^2) + e^{2A} dr^2 + e^{2C} d\Sigma_{k,\sigma}^2 +$$

$$+ e^{2E}(dy_1^2 + \ldots + dy_{q-k}^2),$$  \hspace{1cm} (2.6)

parametrized by five functions of the radial variable $A(r)$, $B(r)$, $C(r)$, $D(r)$ and $E(r)$. The space $\Sigma_{k,\sigma}$ for $\sigma = 0, +1, -1$ is a $k$-dimensional constant curvature space — a flat space, a sphere and a hyperbolic space respectively:

$$d\Sigma_{k,\sigma}^2 = \bar{g}_{ab} dz^a dz^b = \begin{cases} 
    d\varphi^2 + \sin^2 \varphi d\Omega_{(k-1)}^2, & \sigma = -1, \\
    d\varphi^2 + \varphi^2 d\Omega_{(k-1)}^2, & \sigma = 0, \\
    d\varphi^2 + \sin^2 \varphi d\Omega_{(k-1)}^2, & \sigma = +1,
\end{cases}$$  \hspace{1cm} (2.7)

satisfying

$$\bar{R}_{ab} = \sigma(k-1)\bar{g}_{ab}.$$  \hspace{1cm} (2.8)

In the case of codimension one (a domain wall) $q = k = 0$, that is the $\Sigma$ and $y$-parts in (2.6) are absent and the coordinate $r$ varies on the full line $(-\infty, \infty)$.

The magnetically charged $p$-brane is supported by the form field

$$F_{[q]} = b \text{vol}(\Sigma_{k,\sigma}) \wedge dy_1 \wedge \ldots \wedge dy_{q-k},$$  \hspace{1cm} (2.9)

satisfying Eq. (2.3), where $b$ is the field strength parameter, with $\text{vol}(\Sigma_{k,\sigma})$ denoting the volume form of the space $\Sigma_{k,\sigma}$.
The Ricci tensor for the metric (2.6) has the non-vanishing components

\[
R_{tt} = e^{2(B-A)} \left[ B'' + B'(B' - A' + (q-k)E' + kC' + pD') \right], \tag{2.10}
\]

\[
R_{\alpha\beta} = -e^{2(D-A)} \left[ D'' + D'(B' - A' + (q-k)E' + kC' + pD') \right] \delta_{\alpha\beta}, \tag{2.11}
\]

\[
R_{rr} = -B'' - B'(B' - A') - k(C'' + C'^2 - A'C') - (q-k)(E'' + E'^2 - A'E') - p(D'' + D'^2 - A'D'), \tag{2.12}
\]

\[
R_{ab} = - \left\{ e^{2(C-A)} \left[ C'' + C'(B' - A' + (q-k)E' + kC' + pD') \right] - \sigma (k-1) \right\} \hat{g}_{ab}, \tag{2.13}
\]

\[
R_{ij} = -e^{2(E-A)} \left[ E'' + E'(B' - A' + (q-k)E' + kC' + pD') \right] \delta_{ij}, \tag{2.14}
\]

where primes denote derivatives with respect to \( r \).

By transforming the radial coordinate, one can fix one of the metric functions in (2.6). This freedom can be encoded in the following gauge function \( \mathcal{F} \)

\[
\ln \mathcal{F} = -A + B + kC + pD + (q-k)E. \tag{2.15}
\]

Fixing \( \mathcal{F} \) we thereby choose some gauge. First we will find the general solutions without imposing any gauge, i.e. with arbitrary \( \mathcal{F} \). From the Einstein equations we find four equations for \( B, C, D \) and \( E \) with similar differential operators

\[
B'' + B' \frac{F'}{\mathcal{F}} = \frac{(q-1)b^2e^G}{2(d-2)\mathcal{F}^2}, \tag{2.16}
\]

\[
C'' + C' \frac{F'}{\mathcal{F}} = -\frac{(p+1)b^2e^G}{2(d-2)\mathcal{F}^2} + \sigma (k-1)e^{2(A-C)}, \tag{2.17}
\]

\[
D'' + D' \frac{F'}{\mathcal{F}} = \frac{(q-1)b^2e^G}{2(d-2)\mathcal{F}^2}, \tag{2.18}
\]

\[
E'' + E' \frac{F'}{\mathcal{F}} = -\frac{(p+1)b^2e^G}{2(d-2)\mathcal{F}^2}, \tag{2.19}
\]

where

\[
G = a\phi + 2B + 2pD, \tag{2.20}
\]

and the following equation

\[
(A + \ln \mathcal{F})'' - A'(A + \ln \mathcal{F})' + B'^2 + kC'^2 + pD'^2 + (q-k)E'^2 + \frac{1}{2} \phi'^2 = \frac{(q-1)b^2e^G}{2(d-2)\mathcal{F}^2}. \tag{2.21}
\]

The dilaton equation Eq.(2.4) takes a similar form

\[
\phi'' + \phi' \frac{F'}{\mathcal{F}} = \frac{ab^2e^G}{2\mathcal{F}^2}. \tag{2.22}
\]

To simplify the system we introduce a new function instead of \( A \):

\[
\mathcal{A} = A + \ln \mathcal{F} = B + kC + pD + (q-k)E, \tag{2.23}
\]


and a new variable \( \tau \), satisfying

\[
\frac{d\tau}{dr} = \frac{(k-1)}{\mathcal{F}}. \tag{2.24}
\]

Then, denoting the derivatives with respect to \( \tau \) by a dot, we obtain the following system:

\[
\ddot{B} = \frac{(q-1)b^2}{2(k-1)^2(d-2)} e^G, \tag{2.25}
\]

\[
\ddot{C} = -\frac{(p+1)b^2}{2(k-1)^2(d-2)} e^G + \frac{\sigma}{(k-1)} e^{2(A-C)}, \tag{2.26}
\]

\[
\ddot{D} = \frac{(q-1)b^2}{2(k-1)^2(d-2)} e^G, \tag{2.27}
\]

\[
\ddot{E} = -\frac{(p+1)b^2}{2(k-1)^2(d-2)} e^G, \tag{2.28}
\]

\[
\ddot{\phi} = \frac{ab^2}{2(k-1)^2} e^G, \tag{2.29}
\]

and

\[
\ddot{A} - \ddot{\mathcal{A}}^2 + \ddot{B}^2 + k\ddot{C}^2 + p\ddot{D}^2 + (q-k)\ddot{E}^2 + \frac{1}{2} \ddot{\phi}^2 = \frac{(q-1)b^2}{2(k-1)^2(d-2)} e^G. \tag{2.30}
\]

After solving the system with respect to \( \mathcal{A} \), this last equation becomes a constraint equations for the other functions involved.

3. General solution

The integration procedure is similar to that used in [15]. First we observe that the functions \( D, B \), and the quantities \(-(q-1)E/(p+1)\) and \((q-1)\phi/a(d-2)\) may differ only by a solution of the homogeneous equation, which is a linear function of \( \tau \). Therefore

\[
D = B + d_1 \tau + d_0, \tag{3.1}
\]

\[
E = -\frac{p+1}{q-1} B + e_1 \tau + e_0, \tag{3.2}
\]

\[
\phi = \frac{a(d-2)}{q-1} B + \phi_1 \tau + \phi_0, \tag{3.3}
\]

where \( d_0, d_1, e_0, e_1, \phi_0, \phi_1 \) are free constant parameters. Substituting this into (2.20) one finds :

\[
G = \frac{\Delta(d-2)}{(q-1)} B + g_1 \tau + g_0, \tag{3.4}
\]

where

\[
g_{0,1} = a\phi_{0,1} + 2pd_0, \tag{3.5}
\]

\[
\Delta = a^2 + \frac{2(p+1)(q-1)}{d-2}, \tag{3.6}
\]
so the Eq. (2.25) becomes a decoupled equation for $G$

$$
\ddot{G} = \frac{b^2 \Delta}{2(k-1)^2} e^G. \tag{3.7}
$$

Its general solution, depending on two integration constants $\alpha$ and $\tau_0$, reads

$$
G = \ln \left( \frac{\alpha^2 (k-1)^2}{\Delta b^2} \right) - \ln \left[ \sinh \left( \frac{\alpha}{2} (\tau - \tau_0) \right) \right], \tag{3.8}
$$

with $\alpha^2 > 0$ being equal to the first integral of Eq. (3.7)

$$
\dot{G}^2 - \frac{b^2 \Delta}{(k-1)^2} e^G = \alpha^2. \tag{3.9}
$$

Using this we integrate the remaining equations (2.26, 2.30) with account for the definition of the gauge function (2.15), which with the substitution (3.1, 3.2) reads

$$
A - kC = \frac{(p+1)(k-1)}{q-1} B + p(d_1 \tau + d_0) + (q-k)(e_1 \tau + e_0). \tag{3.10}
$$

Then the linear combination

$$
H = 2(A - C) \tag{3.11}
$$

will satisfy the second decoupled Liouville equation

$$
\dot{H} = 2\sigma e^H, \tag{3.12}
$$

whose first integral is

$$
\dot{H}^2 - 4\sigma e^H = \beta^2. \tag{3.13}
$$

Its general solution depending on two parameters $\beta$, $\tau_1$ reads, for $\beta^2 > 0$,

$$
H = \begin{cases} 
2 \ln \frac{\beta}{2} - \ln \left( \sinh \left[ \beta (\tau - \tau_1) / 2 \right] \right), & \sigma = 1, \\
\pm \beta (\tau - \tau_1), & \sigma = 0, \\
2 \ln \frac{\beta}{2} - \ln \left( \cosh \left[ \beta (\tau - \tau_1) / 2 \right] \right), & \sigma = -1,
\end{cases} \tag{3.14}
$$

Finally, expressing the metric functions $A, C$ from (3.10, 3.11), one can write the entire solution in terms of $G, H$ as follows:

$$
\begin{align*}
A &= \frac{k}{2(k-1)} H - \frac{p+1}{\Delta (d-2)} G - \ln \mathcal{F} + c_1 \tau + c_0, \tag{3.15} \\
B &= \frac{(q-1)}{\Delta (d-2)} (G - g_1 \tau - g_0), \tag{3.16} \\
C &= \frac{1}{2(k-1)} H - \frac{p+1}{\Delta (d-2)} G + c_1 \tau + c_0, \tag{3.17} \\
D &= \frac{(q-1)}{\Delta (d-2)} (G - g_1 \tau - g_0) + d_1 \tau + d_0, \tag{3.18} \\
E &= -\frac{p+1}{\Delta (d-2)} (G - g_1 \tau - g_0) + e_1 \tau + e_0, \tag{3.19} \\
\phi &= \frac{a}{\Delta} G + f_1 \tau + f_0, \tag{3.20}
\end{align*}
$$
where the following linear combinations of the previously introduced parameters are used
\[ g_{0,1} = a \phi_{0,1} + 2 p d_{0,1}, \]
\[ c_{0,1} = \frac{(p + 1) a \phi_{0,1}}{\Delta(d - 2)} - \frac{p(a^2(d - 2) + 2(p + 1)(q - k))}{\Delta(d - 2)(k - 1)} d_{0,1} - \frac{(q - k)}{k - 1} e_{0,1}, \]
\[ f_{0,1} = (1 - \frac{a^2}{\Delta}) \phi_{0,1} - \frac{2 p a}{\Delta} d_{0,1}. \]

Our solution depends on eleven parameters: \( b, d_0, d_1, e_0, e_1, \phi_0, \phi_1, \tau_0, \tau_1, \alpha, \beta \), which satisfy a constraint following from the Eq. (2.30)
\[ -k \beta^2 \frac{\alpha^2}{4(k - 1)} + \frac{\alpha^2}{2 \Delta} + (k - 1) c_1^2 + \left( \frac{(q - 1) g_1}{\Delta(d - 2)} \right)^2 + p \left( d_1 - \frac{g_1(q - 1)}{\Delta(d - 2)} \right)^2 + (q - k) \left( e_1 + \frac{(p + 1) g_1}{\Delta(d - 2)} \right)^2 + \frac{1}{2} f_1^2 = 0. \]

The coordinate \( \tau \) is defined by (2.24) only up to a translation, so without loss of generality we can set
\[ \tau_1 = 0. \]

Also, we can always rescale the coordinates \( x_\alpha \) and \( y_i \) of the flat spaces \( R^p \) and \( R^{q - k} \) so that
\[ d_0 = e_0 = 0. \]

We then remain with seven independent parameters. Note that the constraint involves only five of these. Considering the signs of the different terms in Eq. (3.24) we find three nontrivial cases: \( \alpha \) and \( \beta \) are both real, \( \alpha \) is pure imaginary and \( \beta \) real, and \( \alpha \) and \( \beta \) are both pure imaginary. For imaginary values of parameters \( \alpha \) and \( \beta \) one should replace \( \sinh, \cosh \rightarrow \sin, \cos \). In what follows we will deal mostly with the case of real \( \alpha \) and \( \beta \).

We have obtained the general solution for the three cases \( \sigma = 1, 0 \) and \( -1 \) (spherical, toroidal or hyperbolic sector \( \Sigma \)). One can show that there are no asymptotically flat stationary solutions for \( \sigma = 0 \) and \( -1 \). In the remainder of the paper we restrict our analysis to the case of \( SO(k) \) symmetry, i.e. \( \sigma = 1 \).

4. Special points

The solution constructed above have several special points: \( \tau = 0, \tau = \tau_0, \tau = \pm \infty \). To reveal their physical meaning consider the Ricci scalar which can be expressed using the field equations as follows
\[ R = \frac{(k - 1)^2}{2} e^{-2A} \left( \dot{\phi}^2 + \frac{b^2(d - 2q)}{(d - 2)(k - 1)^2} e^G \right). \]

Note that for \( d = 2q \) and a constant dilaton (which is consistent with the field equations if \( a = 0 \)), the scalar curvature is identically zero, as a consequence of the conformal invariance of the corresponding form field.
Substituting here the solution (3.15-3.20) one obtains
\[
R = \frac{(k-1)^2}{2}e^{-2c_0-2c_1\tau} \left[ \frac{\alpha(k-1)}{\sqrt{\Delta b}} \right]^2 \left[ \frac{2\sinh(\frac{\alpha}{2}\tau)}{\beta} \right] \left[ \sinh \left( \frac{\alpha}{2}(\tau-\tau_0) \right) \right]^{-4\frac{\alpha+1}{\Delta(d-2)}} \times
\]
\times \left\{ \left( f_1 \sinh \left( \frac{\alpha}{2}(\tau-\tau_0) \right) - \alpha \frac{a}{\Delta} \cosh \left( \frac{\alpha}{2}(\tau-\tau_0) \right) \right)^2 + \alpha^2 \frac{d-2q}{\Delta(d-2)} \right\}.
\]
(4.2)

An additional information about singularities can be found from the square of the Riemann tensor (the Kretschmann scalar) \( K = R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta} \). To present it in a compact notation, we introduce the four-component quantities \( Y_i = \{B, D, C, E\}, l_i = \{1, p, k, q - k\}, i = 1, 2, 3, 4 \). Then the Kretschmann scalar takes the following form:
\[
K = 4(k-1)^4 e^{-4A} \left( \sum_i l_i(-\ddot{Y}_i - \dot{Y}_i^2 + \ddot{A})^2 + \frac{1}{2} \sum_{i \neq j} l_i l_j \dot{Y}_i \dot{Y}_j + \frac{1}{2} \sum_i l_i (l_i - 1) \dot{Y}_i^2 - \frac{k}{k-1} C^2 e^{2(A-C)} + \frac{1}{2} \frac{k}{(k-1)^3} e^{4(A-C)} \right).
\]
(4.3)

From these formulas one can see that \( \tau = \tau_0 \) generically is the singular point of the geometry unless \( \tau_0 = 0 \), in which case the singularity can be avoided for some choice of parameters. One of the points \( \tau = \pm \infty \) generically is also singular. Namely, if the parameters in the exponential are chosen so that one of them is regular, another will be singular and vice versa. Another special feature associated with these points is that the metric coefficient
\[
g_{tt} = e^{2B} = \left[ \frac{\alpha^2(k-1)^2 e^{-(g_0+g_1\tau)}}{\Delta b^2 \sinh^2(\alpha(\tau-\tau_0)/2)} \right]^{\frac{2(\alpha-1)}{\Delta(d-2)}}
\]
may vanish there, corresponding to horizons of the geometry. This may happen for
\[
\tau = +\infty, \text{ if } |\alpha| > -g_1
\]
and
\[
\tau = -\infty, \text{ if } |\alpha| > g_1.
\]
(4.5, 4.6)

In what follows we will choose the second option \( \tau = -\infty \) for the location of the regular event horizon, in which case the point \( \tau = +\infty \) generically will be singular (unless some special values of the parameters are chosen).

Further information about the geometrical structure of the solution can be extracted from the analysis of radial geodesics \( x^\mu(\lambda) = (t(\lambda), r(\lambda)) \)
\[
t'' + 2\frac{dB}{dr}r' = 0,
\]
\[
r'' + \frac{dA}{dr}r^2 + e^{2(B-A)} \frac{dB}{dr}t^2 = 0,
\]
(4.7)

where primes denote the derivatives with respect to the affine parameter \( \lambda \). The first integral of this system reads
\[
t' = e^{-2B+C_1}, r' = \pm e^{C_1} \sqrt{e^{-2(A+B)} + C_2 e^{2(A+C)}},
\]
(4.8)
where $C_{1,2}$ are integration constants. Null radial geodesics correspond to $C_2 = 0$, in which case the second equation reduces (after a rescaling of the affine parameter) to

$$r' = \frac{1}{k - 1} e^{-(A + B)}, \quad (4.9)$$

Passing to our new variable $\tau$, related to $r$ via $d\tau/dr = (k - 1)/F$, we can rewrite the null radial geodesic equation as

$$\tau' = e^{-(A + B)}, \quad (4.10)$$

so that

$$\lambda \sim \int e^{A+B} d\tau. \quad (4.11)$$

We wish to identify the values $\tau = \tau_\infty$ which may correspond to points at an infinite distance from the horizon. For this we evaluate the integrals in the vicinity of $\tau_\infty$ obtaining the leading term

$$\lambda \sim \int (\tau - \tau_\infty)^{-1} d\tau \sim (\tau - \tau_\infty)^\gamma. \quad (4.12)$$

If $\gamma < 0$, the affine distance to this point will be infinite. We have three particular cases:

1) $\tau_0 \neq 0$, $\tau_\infty = 0$, $\gamma = -\frac{1}{k - 1} < 0$, \quad (4.13)

2) $\tau_0 \neq 0$, $\tau_\infty = \tau_0$, $\gamma = \frac{a^2(d - 2) + 2(pq + 1)}{\Delta(d - 2)} > 0$, \quad (4.14)

3) $\tau_0 = 0$, $\tau_\infty = 0$, $\gamma = \frac{-a^2(d - 2) + 2(q - 1)(k - 1) + 2(p + 1)(q - k)}{\Delta(d - 2)(k - 1)} < 0$. \quad (4.15)

Thus, $\tau = 0$ is an infinitely distant point in both cases $\tau_0 = 0$ and $\tau_0 \neq 0$. One can further show that in the case $\tau_0 \neq 0$ both the Ricci and Kretschmann scalars vanish at $\tau = 0$; we will see that in this case the metric is asymptotically flat. In the second case (4.14), the non-zero $\tau_0$, which is a singular point of the solution, is located at a finite affine distance, thus representing a true singularity.

Now let us investigate the regularity of the event horizon $\tau = -\infty$. Near a regular horizon the metric coefficient $e^{2B}$ must behave \cite{13} as

$$e^{2B} \sim \lambda^n, \quad (4.16)$$

where $\lambda = 0$ corresponds to the position of the horizon, and the integer $n$ is equal to unity for a non-degenerate horizon, and $n \geq 2$ for a degenerate horizon. Differentiating (4.16) we obtain the following condition

$$e^{-(A + 2n - 2B)} \frac{d}{d\tau} e^{2B} \sim \text{const} \quad (4.17)$$
at the horizon. For the solution (3.15-3.20) we get
\[ e^A \sim e^{\frac{2n}{n-1} B}. \] (4.18)

Taking into account this equation we can rewrite the equation (2.30) near the horizon as
\[ \frac{4(n-1)}{n^2} \dot{B}^2 + k \dot{C}^2 + p \dot{D}^2 + (q-k) \dot{E}^2 + \frac{1}{2} \dot{\phi}^2 \sim 0. \] (4.19)

The left-hand side being a sum of squares, all of them must vanish, so that we obtain the following conditions on the behavior of the metric exponents at the horizon for \( n = 1 \):
\[ \dot{C} = \dot{D} = \dot{E} = \dot{\phi} = 0, \quad \text{at} \quad \tau = -\infty, \] (4.20)
which are equivalent for the solution (3.15-3.20) to the following conditions on the free parameters:
\[ |\alpha| = |\beta| = -2d_1 = \frac{2(q-1)}{p+1} e_1 = -\frac{2(q-1)}{a(d-2)} \phi_1. \] (4.21)

In what follows without loss of generality we will choose \( \alpha \) to be non-negative. The calculation shows that under these conditions the constraint (3.24) is satisfied automatically.

In the degenerate case \( n \geq 2 \), we obtain from (4.19) the additional condition
\[ \dot{B} = 0, \quad \text{at} \quad \tau = -\infty. \] (4.22)

Combined with the conditions (4.21), this leads to
\[ \alpha = \beta = d_1 = \phi_1 = e_1 = 0. \] (4.23)

Note that in the case \( \alpha = \beta = 0 \), (3.8) and (3.14) should be replaced by
\[ G = -2 \ln |\tau - \tau_0| + \ln(4(k-1)^2/b^2 \Delta), \quad H = -2 \ln |\tau|, \] (4.24)
leading to the behavior near the horizon \( \tau \to -\infty \)
\[ A \sim \mu \ln |\tau|, \quad B \sim \nu \ln |\tau|, \] (4.25)
with
\[ \mu = \frac{2(p+1)}{\Delta(d-2)} - \frac{k}{k-1}, \quad \nu = -\frac{2(q-1)}{\Delta(d-2)}. \] (4.26)

From (4.11), \( d\lambda \sim \tau^{\mu+\nu} d\tau \), leading to
\[ e^{2B} \sim \tau^{2\nu} \sim \lambda^{2\nu/(\mu+\nu+1)}. \] (4.27)

This is consistent with \( e^{2B} \sim \lambda^2 \) only if \( \mu + 1 = 0 \), leading to the conditions for the existence of regular degenerate horizons
\[ a^2(d-2) + 2(p+1)(q-k) = 0 \quad \Leftrightarrow \quad (a^2 = 0, \quad q = k). \] (4.28)
Coming back to the non-degenerate case, we note that once the position of the regular event horizon is chosen as $\tau = -\infty$, the Ricci scalar will be finite there, while at $\tau = +\infty$ (an inner horizon) it will diverge as

$$ R \sim (ae^{\alpha \tau} + \text{const}) e^{\left[ \frac{2(a^2(p+q)+2(p+1)(q-k))}{(q-1)\Delta(p+q)} \right] \alpha \tau}, $$

(4.29)

since the coefficient in the exponential is non-negative (for our choice $\alpha \geq 0$). In the special case where the coefficient vanishes, i.e. $a = 0$, $q = k$, the inner horizon is apparently regular (obviously, the degenerate horizon discussed above will arise when this inner horizon coincides with the outer horizon). However, a closer look reveals that the Kretschmann scalar (4.3), which for $a = 0$, $q = k$ diverges at $\tau = \infty$ as

$$ K \sim \alpha^4(k - 1)^4 \frac{p^2(p - 1)}{(p + 1)^3} e^{2\alpha \tau} $$

(4.30)

can be finite only for $p = 0$ or $p = 1$ (see next section).

Let us finally comment on the possibility of imaginary values of the parameters $\alpha$ and $\beta$. As previously mentioned, for imaginary values of $\alpha$ one should replace in the solution the hyperbolic functions sinh, cosh by the trigonometric functions sin, cos. In this case we will have an infinite sequence of singularities, located at the points $\tau = \tau_0 + 2\pi n/|\alpha|$ (n integer), together with an infinite sequence of points at infinity $\tau = 2\pi m/|\alpha|$ (m integer) if $\beta$ is also imaginary. In any sector between two consecutive singularities (or between a singularity and the next point at infinity), the metric function is bounded, so that no horizon can occur. In what follows we restrict ourselves to real values of these parameters.

5. The black solutions in Schwarzschild-like coordinates

As discussed in the previous section, the nondegenerate black solution (3.15-3.20) with $\tau_1 = 0$ and conditions (4.21) has a horizon at $\rho \to -\infty$, a point at infinity at $\tau = 0$, and a singularity at $\tau = \tau_0$ if $\tau_0 \neq 0$. According to the sign of the integration constant $\tau_0$, there are two black solution branches:

a) $\tau_0 > 0$ At the end-point $\tau = 0$, the solution is asymptotically flat. We will check in the following that this can be gauge transformed to the standard black brane solution.

b) $\tau_0 < 0$ In this case the spacetime ends at the point singularity $\tau = \tau_0$, a “bag-of-gold”-like black solution.

At the boundary between these two cases, $\tau_0 = 0$, lies a critical solution which is geodesically complete outside the horizon, but is not asymptotically flat, generalizing the black brane solution with linear dilaton asymptotics of [14].

With a view to transform these solutions to a Schwarzschild-like gauge, it is convenient to map the coordinate $\tau < 0$ to a new radial coordinate $\xi$, such that the horizon $\tau \to -\infty$ maps to a finite value $\xi = \xi_+$, and the asymptotic infinity $\tau = 0$ maps to $\xi = +\infty$. This is achieved by the map

$$ e^{\alpha \tau} = \frac{\xi - \xi_+}{\xi - \xi_-}, $$

(5.1)
with $\xi > \xi_+ > \xi_-$. To extend this to the full line of $\xi$ one has to consider complex $\tau$ as shown on Fig. 1. This map defines $\xi$ (and its special values $\xi_\pm$) only up to linear transformations. We first fix the common scale of $\xi$, $\xi_+$ and $\xi_-$ by choosing

$$\xi_+ - \xi_- = \alpha.$$  \hspace{1cm} (5.2)

The image of $\tau_0$ by the map (5.1) is $\xi_0$:

$$e^{\alpha\tau_0} = \frac{\xi_0 - \xi_+}{\xi_0 - \xi_-}. \hspace{1cm} (5.3)$$

Introducing the notation

$$f_\pm(\xi) = 1 - \frac{\xi_\pm}{\xi}, \quad f_0(\xi) = 1 - \frac{\xi_0}{\xi}, \hspace{1cm} (5.4)$$

we can express the functions $G$ and $H$ as

$$H = \ln(\xi_2^2 f_+ f_-), \hspace{1cm} (5.5)$$

$$G = \ln \left( \frac{f_+ f_-}{f_0} \right) + G_0, \hspace{1cm} (5.6)$$

with

$$e^{G_0} = \frac{\alpha^2 (k-1)^2}{\Delta b^2} \sinh^2 \left( \frac{\alpha\tau_0}{2} \right) = \frac{4(k-1)^2(\xi_+ - \xi_0)(\xi_- - \xi_0)}{\Delta b^2}, \hspace{1cm} (5.7)$$

and present the generic black solution as follows

$$e^{2A} = |\xi|^{\frac{2k}{k-1}} f_+ f_-^{\frac{2k}{k-1}} \frac{4(p+1)}{\Delta (d-2)} \frac{2(p+1)}{\Delta (d-2)} \frac{1}{\lambda_0},$$

$$e^{2B} = f_+ f_-^{1+\frac{4(q-1)}{\Delta(d-2)}} \frac{2(q-1)}{\Delta(d-2)} \frac{1}{\lambda_0},$$

$$e^{2C} = |\xi|^{\frac{2k}{k-1}} f_-^{\frac{2k}{k-1}} \frac{4(p+1)}{\Delta (d-2)} \frac{2(p+1)}{\Delta (d-2)} \frac{1}{\lambda_0},$$

$$e^{2D} = f_-^{\frac{4(q-1)}{\Delta(d-2)}} \frac{2(q-1)}{\Delta(d-2)} \frac{1}{\lambda_0},$$

$$e^{2E} = f_-^{\frac{4(p+1)}{\Delta(d-2)}} \frac{2(p+1)}{\Delta(d-2)} \frac{1}{\lambda_0},$$

$$e^{\alpha \phi} = e^{\alpha \phi_0} f_+^{\frac{2a^2}{\Delta}} f_-^{\frac{2}{\Delta}},$$

$$F_{[q]} = 2(k-1) \sqrt{\frac{(\xi_+ - \xi_0)(\xi_- - \xi_0)}{\Delta}} e^{-G_0/2} \frac{\text{vol}(\Sigma_{k,1}) \wedge dy_1 \wedge ... \wedge dy_{q-k}}{\Delta b^2}, \hspace{1cm} (5.10)$$

with

$$\chi_0(\xi) = e^{\alpha \phi_0 - G_0} f_0^2(\xi) \hspace{1cm} (5.11)$$

(recall that we have chosen $d_0 = e_0 = 0$, so that $g_0 = a \phi_0$). The curvature scalar reads, in these coordinates,

$$R = \frac{2(k-1)^2}{\Delta} |\xi|^{-\frac{2k}{k-1}} \chi_0^{-\frac{2(p+1)}{\Delta(d-2)}} f_-^{2 - \frac{2}{\Delta}} \frac{4(p+1)}{\Delta(d-2)} f_0^{-2} \times$$

$$\times \left[ \frac{d - 2q}{d - 2} (\xi_+ - \xi_0)(\xi_- - \xi_0) + \frac{a^2}{\Delta}(\xi_- - \xi_0)^2 f_- \right]. \hspace{1cm} (5.12)$$
Consider first the asymptotically flat case $\tau_0 > 0$. In this case, one can choose Minkowskian coordinates at $\tau = 0$ by imposing the conditions

$$A, B, D, E \to 0, \quad C \to \ln r,$$

(5.13)

for $r \to \infty$, where $r$ is our original radial coordinate, related to $\tau$ by (2.24). We find that the functions $B, D$ and $E$ all vanish at infinity provided

$$a\phi_0 = G_0.$$  

(5.14)

From the behavior of the functions $A$ and $C$ one can recover the asymptotic behavior of the gauge function $F$ corresponding to this solution,

$$\ln F = k \ln r, \quad \tau = -r^{-(k-1)}.$$  

(5.15)

This asymptotic behavior is consistent with our coordinate transformation (5.1) provided

$$\xi = r^{k-1},$$  

(5.16)

which corresponds to fixing the following gauge function

$$F = r\xi f_+ f_-.$$  

(5.17)

Now we consider the generic case $\tau_0 \neq 0$, with both signs of $\tau_0$ allowed. We recall that the map (5.1) together with the relation (5.2) define $\xi$, $\xi_+$ and $\xi_-$ only up to a common additive constant. Without loss of generality, we can choose this constant so that the image of $\tau_0$ is

$$\xi_0 = 0,$$  

(5.18)

leading to $f_0 = 1$. With the choice (5.14), the generic $\tau_0 \neq 0$ solution is then obtained by setting $\chi_0 = 1$ in (5.8):

$$ds^2 = -f_+ f_-^{-1} + \frac{4(q-1)}{A(d-2)} dt^2 + f_+^{-\frac{4(q-1)}{A(d-2)}} (dx_1^2 + \ldots + dx_p^2) +$$

$$+ f_+^{-1} f_-^{-\frac{2}{k-1}} \frac{4(p+1)}{A(d-2)} dr^2 + r^2 f_+^{-\frac{2}{k-1}} \frac{4(p+1)}{A(d-2)} d\Omega_k^2 +$$

$$+ f_-^{-\frac{2}{A(d-2)}} (dy_1^2 + \ldots + dy_{q-k}^2),$$  

(5.19)

$$e^{a\phi} = e^{a\phi_0} f^\frac{2k}{A},$$  

(5.20)

$$F_{[q]} = 2(k-1)^{(k-1)} \frac{(r+r_-)^{\frac{k-1}{2}}}{\sqrt{\Delta}} e^{-\frac{2}{A} \phi_0} \text{vol}(\Sigma_{k,1}) \wedge dy_1 \wedge \ldots \wedge dy_{q-k}.$$  

(5.21)

The coordinate $r$ in (5.8) and (5.19) is related to $\xi$ by

$$r = |\xi|^{1/(k-1)},$$  

(5.22)

so as to accommodate both signs of $\xi$. For $\tau_0 > 0$,

$$0 < \xi_- < \xi_+$$  

(5.23)
\(\xi > 0\), this is the standard black brane solution \[3\], with a spacelike central singularity at \(\xi = \xi_-\), except in the cases \((a = 0, q = k, p = 0 \text{ or } 1)\) where the central singularity is located at \(\xi = 0\) and is timelike. In both cases the singularities are hidden behind the event horizon \(\xi = \xi_+\). For \(\tau_0 < 0\),

\[
\xi_- < \xi_+ < 0
\]

\(\xi < 0\), there are generically two central singularities, a timelike singularity at \(\xi = 0\) separated from the (generic) spacelike singularity at \(\xi = \xi_-\) by the horizon at \(\xi = \xi_+\).

We discuss briefly the cases \(a = 0, q = k\), and \(p = 0 \text{ or } 1\) where the inner horizon \(\xi = \xi_-\) becomes regular. For \(p = 0\), the solution \((5.19)\) reduces to the Reissner-Nordstrom-like form

\[
ds^2 = - f_+ f_- dt^2 + f_+^{-1} f_-^{-1} dr^2 + r^2 d\Omega^2_k.
\]

Clearly, this is invariant under the involution (preserving the order \(\xi_- < \xi_+)\)

\[
\xi \to -\xi, \quad \xi_\pm \to -\xi_\mp
\]

which transforms the \(\tau_0 < 0\) solution into the asymptotically flat \(\tau > 0\) solution. For \(p = 1\), the solution \((5.19)\) reduces to

\[
ds^2 = - f_- dt^2 + f_+ dx^2 + f_+^{-1} f_-^{-1} dr^2 + r^2 d\Omega^2_k.
\]

For \(\tau_0 > 0\) \((\xi_+ > \xi_- > 0)\), this is stationary in both sectors \(\xi > \xi_+\) and \(0 < \xi, \xi_-\), however in the inner stationary sector the role of time is taken up by the coordinate \(x\). The solution for \(\tau_0 < 0\) \((\xi_+ > \xi_- > 0)\) is transformed by the involution \((5.26)\) into

\[
ds^2 = - f_- dt^2 + f_+ dx^2 + f_+^{-1} f_-^{-1} dr^2 + r^2 d\Omega^2_k,
\]

which is just the asymptotically flat solution \((5.27)\) with \(\xi_+\) and \(\xi_-\) exchanged. When \(\xi\) decreases from \(+\infty\), first the outer horizon \(\xi = \xi_+\) is crossed, with \(g_{tt}\) keeping its sign, but \(g_{xx}\) and \(g_{rr}\) both flipping signs, so that in the intermediate sector between the two horizons there are three time coordinates*. Then the inner horizon \(\xi = \xi_-\) is crossed, leading to the inner sector with only one time coordinate \(x\), and the central singularity \(\xi = 0\).

Returning to the general case, we recall that the generic solution \((3.15-3.20)\) had seven independent parameters. We imposed three relations \((4.21)\) from the condition of regularity of the horizon (the fourth relation \((4.21)\) follows from the constraint \((3.24)\)), and the relations \((5.14)\) to ensure the asymptotic Minkowskian behavior. The coordinate transformation \((5.1)\) introduced two more parameters \(\xi_\pm\), corresponding to the location of the horizons, which were then constrained by the relations \((5.2)\) and \((5.18)\). Therefore the solution \((5.19)\) depends on \(7 - 4 = 3\) independent parameters: \(\xi_+, \xi_-\) and \(\phi_0\). This last parameter corresponds to the value of the dilaton at infinity. The other two parameters

*In the event that the range of the coordinate \(x\) is compact, the region \(\xi < \xi_+\) will contain closed timelike curves.
enter the expressions for the mass and charge of the solution. Indeed, following [34, 35] we can define the ADM mass per unit volume of the brane by

\[
M_{\text{vol(p-brane)}} = 2\Omega_k L_{q-k} r^k \times \left[ \frac{k}{r} (e^{A} - e^{C - \ln r} - pD' - (q - k) E' - k(C - \ln r)^{\prime}) \right] \bigg|_{r \to \infty},
\]

where the prime denotes \( r \)-derivative, \( \Omega_k \) is the volume of the unit \( k \)-sphere, \( L_{q-k} \) is the volume of the flat part of the transverse space of dimension \( q - k \) and we have set the Newton constant \( G_d = 1/16\pi \).

The ADM charge (normalized in a similar way) is defined up to the normalization factor by (2.9). For the solution (5.19) we get

\[
M_{\text{vol(p-brane)}} = \Omega_k L_{q-k} \left[ k(\xi_+ - \xi_-) + \frac{4(k - 1)}{\Delta} \xi_- \right],
\]

\[
P_{\text{vol(p-brane)}} = 2(k - 1)\Omega_k L_{q-k} \sqrt{\frac{\xi_+ \xi_-}{\Delta}} e^{-\frac{\phi_0}{2}}.
\]

The calculation of the entropy per unit volume of the brane and temperature of the solution along the lines described in [35] gives

\[
T = \frac{k - 1}{4\pi} |\xi_+|^{\frac{2}{\Delta}} (\xi_+ - \xi_-)^{-\frac{1}{k-1} \frac{2}{\sqrt{\Delta}}} \frac{1}{2},
\]

\[
S_{\text{vol(p-brane)}} = 4\pi \Omega_k L_{q-k} |\xi_+|^{\frac{2}{\Delta}} (\xi_+ - \xi_-)^{\frac{1}{k-1} - \frac{2}{\sqrt{\Delta}}}.
\]

The magnetic potential \( W \) is obtained by computing the dual electric form

\[
\tilde{F}_{[p+2]} = e^{\alpha \phi} \star F_{[q]} = dW_{[p+1]},
\]

with

\[
W_{[p+1]} = W dt \wedge dx_1 \wedge ... \wedge dx_p, \quad W = 2 \left( \frac{\xi_+ \xi_-}{\Delta} \right)^{\frac{1}{2}} |\xi|^{-\frac{1}{k-1}} e^{\frac{\phi_0}{2}}.
\]

In accordance with the first law of thermodynamics for black branes, the following identity holds

\[
dM = TdS + W_{|\xi = \xi_+} dP.
\]

6. The critical solution \( \tau_0 = 0 \): black branes with linear dilaton asymptotics

From (5.3), we see that the value \( \tau_0 = 0 \) corresponds to the limit \( \xi_0 \to \pm \infty \) (see Fig. 2). In this limit, the function \( \chi_0 \) defined by (5.11) becomes

\[
\chi_0(\xi) = \frac{\Delta b^2 e^{\alpha \phi_0}}{4(k - 1)^2} \xi^{-2}.
\]
We note that this function enters the generic solution (5.8) through the product

\[ f_2^2 - \Delta b^2 e^{\alpha \phi_0} (\xi - \xi_-)^2. \]  

(6.2)

Choosing now (without loss of generality)

\[ \xi_- = 0, \]  

(6.3)

fixing again the gauge function (5.17), which gives \( \xi = r^{k-1} \), and putting

\[ \frac{\Delta b^2 e^{\alpha \phi_0}}{4(k-1)^2} = r_0^{2(k-1)}, \quad \xi_+ = c, \]  

(6.4)

we find that the solution reduces to the black p-brane in the linear dilaton background [14] delocalized in part of the transverse space of dimension \( q - k \)

\[ ds^2 = \left( \frac{r}{r_0} \right)^{\frac{4(q-1)(k-1)}{\Delta(d-2)}} \left( -\left(1 - \frac{c}{r^{k-1}}\right) dt^2 + dx_1^2 + \ldots + dx_p^2 \right) + \right. 
+ \left( \frac{r_0}{r} \right)^{\frac{4(p+1)(k-1)}{\Delta(d-2)}} \left( 1 - \frac{c}{r^{k-1}} \right)^{-1} dr^2 + r^2 d\Omega_k^2 + dy_1^2 + \ldots + dy_{q-k}^2, \]  

(6.5)

\[ e^{\alpha \phi} = e^{\alpha \phi_0} \left( \frac{r}{r_0} \right)^{\frac{2(k-1)a^2}{\Delta}}, \]  

(6.6)

\[ F[q] = \frac{2(k-1)e^{-\alpha \phi_0/2}r_0^{k-1}}{\sqrt{\Delta}} \text{vol}(\Sigma_{k,1}) \wedge dy_1 \wedge \ldots \wedge dy_{q-k}, \]  

(6.7)

where the radial coordinate \( r \) is again related to \( \xi \) by (5.16).

This solution is asymptotically nonsingular, as can be seen from the following expression for the curvature scalar

\[ R = \frac{4(p+1)(k-1)^2}{\Delta^2(d-2)r^2} \left( \frac{r}{r_0} \right)^{\frac{4(p+1)(k-1)}{\Delta(d-2)}} \left( \Delta - (q-1) - \frac{(d-2)a^2c}{2(p+1)r^{k-1}} \right), \]  

(6.8)

since

\[ 2(p+1)(k-1) - \Delta(d-2) = -(a^2(d-2) + 2(p+1)(q-k)) < 0. \]  

(6.9)

Following Ref. [14, 36, 37, 38, 39] one can get the following expression for the mass per volume of the p-brane

\[ \frac{M}{\text{vol(p-brane)}} = 2 \int_{\partial S(r=\infty)} e^B (K - K_0) \sqrt{\sigma d^{d-2}x}, \]  

(6.10)

where \( K = -\sigma^{\mu\nu}D_\nu n_\mu \) is the trace of the extrinsic curvature of the boundary, \( K_0 = K|_{c=0} \) is its background value, \( D_\nu \) is the covariant derivative with respect to the metric \( h_{\mu\nu} = g_{\mu\nu} + u_\mu u_\nu \) induced on the space-like hypersurface \( S \) of dimension \( d-1 \), \( u^\mu \) is the normal vector to \( S \), \( \sigma_{\mu\nu} = h_{\mu\nu} - n_\mu n_\nu \) is the metric induced on the boundary of \( S \) with the normal vector \( n^\mu \).
The mass corresponding to the solution obtained reads:

\[ \frac{M}{\text{vol}(\text{p-brane})} = \Omega_k L_{q-k} \left( k - \frac{2(k-1)}{\Delta} \right) c. \] (6.11)

The entropy and the temperature of the brane can be calculated as usual (see e.g. the Ref. [35]) giving

\[ T = \frac{(k-1)c^{\frac{2}{k}} - \frac{1}{k-1} r_0^{-\frac{2(k-1)}{\Delta}}}{4\pi}, \] (6.12)

\[ \frac{S}{\text{vol}(\text{p-brane})} = 4\pi c^{\frac{-2}{k}} + \frac{k}{k-1} r_0^{-\frac{2(k-1)}{\Delta}} \Omega_k L_{q-k}. \] (6.13)

According to the first law of thermodynamics for black branes the following identity should hold

\[ dM = T dS. \] (6.14)

Applying this to the p-brane with the linear dilaton asymptotic we get

\[ \frac{dM}{\text{vol}(\text{p-brane})} = \Omega_k L_{q-k} \left( -\frac{2(k-1)}{\Delta} + k \right) dc, \] (6.15)

\[ \frac{T dS}{\text{vol}(\text{p-brane})} = \Omega_k L_{q-k} \left\{ \left( -\frac{2(k-1)}{\Delta} + k \right) dc + \frac{2(k-1)^2 \Delta}{c} \frac{dr_0}{r_0} \right\}. \] (6.16)

Therefore the first law of thermodynamics is true only if we do not vary the parameter \( r_0 \), proportional from (6.4) to the charge \( b \). This is due to the fact that the charge is not associated with the black brane, but rather with the linear dilaton background [14].

In the case of zero dilaton coupling \( a = 0 \) the metric takes the form

\[ ds^2 = \left( \frac{r}{r_0} \right)^{\frac{2(k-1)}{(p+1)}} \left( -\left( 1 - \frac{c}{r^{k-1}} \right) dt^2 + dx_1^2 + \ldots + dx_p^2 \right) + \right. \]
\[ + \left. \left( \frac{r_0}{r} \right)^{\frac{2(k-1)}{(q-1)}} \left( \frac{1}{1 - \frac{c}{r^{k-1}}} \right)^{-1} dr^2 + r^2 d\Omega_k^2 + dy_1^2 + \ldots + dy_{q-k}^2 \right), \] (6.17)

while the curvature scalar is

\[ R = \frac{(k-1)^2(p-q+2)}{(q-1)(p+1)r_0^{\frac{2(k-1)}{q-1}}} r^{-\frac{2(q-k)}{q-1}}. \] (6.18)

For \( q \neq k \) we have a singularity at \( r = 0 \) and vanishing of \( R \) at infinity. Asymptotically, as \( r \to \infty \), the metric becomes:

\[ ds^2 = \left( \frac{\tilde{r}}{r_0} \right)^{\frac{2(k-1)(q-1)}{(p+1)(q-k)}} \left( -dt^2 + dx_1^2 + \ldots + dx_p^2 \right) + \right. \]
\[ + \left. \left( \frac{r_0}{\tilde{r}} \right)^{\frac{2(k-1)}{q-k}} (dy_1^2 + \ldots + dy_{q-k}^2) \right], \] (6.19)

in terms of the new radial coordinate

\[ \tilde{r} = \left( \frac{r}{r_0} \right)^{\frac{q-k}{q-1}} r_0. \] (6.20)
7. Domain Walls

Our general analysis is not valid in the special case of codimension one, i.e., for domain walls. Consider now the domain wall for \( p = d - 2, \ k = q = 0 \). The metric has the form

\[
 ds^2 = -e^{2B}dt^2 + e^{2D} \left( dx_1^2 + \ldots + dx_p^2 \right) + e^{2A}dy^2, \tag{7.1}
\]

where the coordinate \( y \) now varies on the full axis. In the magnetic sector, we have the trivial solution for the zero-form:

\[
 F[0] = b = \text{const.} \tag{7.2}
\]

Note that the term \( F^2 \) enters the action as a cosmological constant \( \Lambda = -b^2/2 \), and it is common to consider both signs for it. Accordingly, in the following we will consider \( b^2 \) real.

Also, the parameter

\[
 \Delta = a^2 - 2p + 1 \tag{7.3}
\]

now can be positive, negative or zero. The equations of motion now read

\[
 \ddot{B} = -\frac{b^2}{2p} e^G,
\]

\[
 \ddot{D} = -\frac{b^2}{2p} e^G,
\]

\[
 \ddot{\phi} = \frac{a b^2}{2} e^G, \tag{7.4}
\]

where

\[
 G = 2B + 2pD + a\phi, \tag{7.5}
\]

and

\[
 \ddot{A} - \dot{A}^2 + \dot{B}^2 + p\dot{D}^2 + \frac{1}{2} \dot{\phi}^2 = -\frac{b^2}{2p} e^G. \tag{7.6}
\]

Summing up the equations with the corresponding coefficients, we get the following equation for \( G \)

\[
 \ddot{G} = \frac{b^2 \Delta}{2} e^G. \tag{7.7}
\]

For \( \Delta b^2 > 0 \), this has the solution parametrized by \( \alpha \geq 0 \):

\[
 G = \ln \left[ \frac{\alpha^2}{\Delta b^2} \right] - \ln \left[ \sinh^2 \left( \frac{\alpha}{2} (\tau - \tau_0) \right) \right]. \tag{7.8}
\]

In the case \( \Delta = 0 \ (a^2 = 2(p + 1)/p) \), the solution of (7.7) is a linear function:

\[
 G = \alpha(\tau - \tau_0). \tag{7.9}
\]
In what follows we set $\tau_0 = 0$ without loss of generality, and assume

$$\Delta b^2 \geq 0$$

(for $\Delta b^2 < 0$, there are no physically interesting solutions).

In terms of $G$ the solution for $\Delta \neq 0$ reads

$$B = -\frac{1}{\Delta p} G + b_1 \tau + b_0,$$
$$D = B + d_1 \tau + d_0,$$
$$\phi = \frac{a}{\Delta} G + f_1 \tau + f_0,$$
$$A = (p + 1) B + p(d_1 \tau + d_0),$$

(7.10)

where $A = A + \ln F$, with the relation between the integration constants

$$b_{0,1} = -\frac{1}{2(p + 1)} [2pd_{0,1} + af_{0,1}].$$

(7.11)

Half of the constants are subject to the constraint following from the Eq. (2.30):

$$\frac{\alpha^2}{2\Delta} + \frac{p}{p + 1} (d_1^2 - \Delta f_1^2/4) = 0.$$  

(7.12)

If $\Delta = 0$, the solution is:

$$B = -\frac{b^2}{2\alpha^2 p} e^{\alpha \tau} + b_1 \tau + b_0,$$
$$D = -\frac{b^2}{2\alpha^2 p} e^{\alpha \tau} + d_1 \tau + d_0,$$
$$\phi = \frac{ab^2}{2\alpha^2} e^{\alpha \tau} + f_1 \tau + f_0,$$
$$A = -\frac{b^2(p + 1)}{2\alpha^2 p} e^{\alpha \tau} + (b_1 + pd_1) \tau + (b_0 + pd_0),$$

(7.13)

(7.14)

(7.15)

(7.16)

with the parameters $b_{0,1}, d_{0,1}, \phi_{0,1}$ satisfying the conditions

$$2b_0 + 2pd_0 + af_0 = 0, \quad 2b_1 + 2pd_1 + af_1 = \alpha$$

(7.17)

and the constraint

$$2pad_1 = (apd_1 + f_1)^2.$$  

(7.18)

### 7.1 Standard solution

Consider the case $\Delta \neq 0$. Without loss of generality we can set

$$d_0 = f_0 = 0.$$  

(7.19)

Assuming now

$$\alpha = d_1 = f_1 = 0$$  

(7.20)
we find that Eq. (7.8) reduces to
\[ G = -2 \ln |q\tau|, \quad (7.21) \]
with
\[ q^2 = \Delta b^2 / 4, \quad (7.22) \]
and the solution reads
\[ e^{2A} = F^{-2} |q\tau|^{\frac{4(p+1)}{2p}}, \]
\[ e^{2B} = e^{2D} = |q\tau|^{\frac{4}{2p}}, \]
\[ e^{a\phi} = |q\tau|^{-\frac{2a^2}{2p}}. \quad (7.23) \]

It has three special points: \( \tau = 0, \pm \infty \). Consider the behavior of radial null geodesics and the scalar curvature in their vicinity. The affine parameter in terms of \( \tau \) will read
\[ \lambda \sim |\tau|^{\frac{a^2 + 2/p}{\Delta}}, \quad (7.24) \]
while the curvature scalar is
\[ R \sim |\tau|^{-\frac{2a^2}{2p}}. \quad (7.25) \]

One can see that for \( \Delta > 0 \) the infinities \( \tau = \pm \infty \) are at an infinite affine distance and the curvature scalar is zero there. At \( \tau = 0 \), which is at a finite affine distance, \( R \) diverges (for \( a \neq 0 \)). To exclude the singularity we then have to cut the solution at some finite \( \tau \) introducing a material brane.

For \( \Delta < 0 \) the scalar curvature is zero at \( \tau = 0 \) (if \( a \neq 0 \)), and diverges at \( \tau = \pm \infty \). The surface \( \tau = 0 \) is at an infinite affine distance, and one can cut the solution to avoid singularities at \( \tau = \pm \infty \).

Now, if the singularities are at a finite affine distance, consider the following coordinate transformation
\[ -\tau = q^{-1} H^\epsilon, \quad H = c + m|y|, \quad m = q^\epsilon, \quad (7.26) \]
where \( \epsilon = \pm 1 \). Then for \( c > 0 \) the singularity will be cut out. The free parameter \( \epsilon \) can be used as follows:

a) If \( \tau = \pm \infty \) is at infinite affine distance (\( \Delta > 0 \)), we choose \( \epsilon = +1 \) so that the region \( \tau \in (-\infty, -c/m] \) maps to the semi-axes \( y \neq 0 \). The domain wall locates at \( \tau = -c/m \).

b) If \( \tau = 0 \) is at infinity (\( \Delta < 0 \)), we choose in (7.26) \( \epsilon = -1 \), in which case the region \( \tau \in [-m/c, 0) \) maps to the semi-axes \( y \neq 0 \), the domain wall will then be at \( \tau = -m/c \).

In terms of these coordinates we arrive at the standard domain wall solution of Ref. [21]:
\[ ds^2 = H^{\frac{2q^\epsilon}{2p}} (-dt^2 + dx_1^2 + \ldots + dx_p^2) + \frac{m^2}{q^2} H^{-\frac{4(p+1)\epsilon}{2p}} + 2(\epsilon-1) dy^2, \quad (7.27) \]
\[ e^{a\phi} = H^{-\frac{2a^2}{2p}}, \quad (7.28) \]
\[ R = \frac{b^2}{2} \left( a^2 + \frac{p + 2}{p} \right) H^{-\frac{2a^2}{2p}}. \quad (7.29) \]
7.2 Black solution

Now let us seek for a more general solution without naked singularities, but possibly endowed with a horizon. Repeating the previous analysis, we see that in the case $\Delta \neq 0$ the horizons correspond to $\tau = \pm \infty$. Let $\tau = -\infty$ be the event horizon, then from regularity we get the conditions:

\[ d_1 = -\frac{\alpha}{2}, \quad f_1 = -\frac{a\alpha}{\Delta}, \] (7.30)

and the constraint (7.12) holds automatically. Taking into account (7.30), setting $d_0 = 0$ by a rescaling of $\{x^i\}$, and trading the two integration constants $f_0$ and $\alpha$ for

\[ y_0 = q^{-1}e^{-a\phi_0/2} \left( \phi_0 = -\frac{\Delta p}{2(p+1)} f_0 \right), \] (7.31)

\[ \mu = \alpha y_0^2, \] (7.32)

we arrive at the solution:

\[ e^{2B} = \left( \frac{4y_0^2}{\mu^2} e^{a\tau} \sinh^2(\alpha\tau/2) \right)^{\frac{2}{3\mu}} e^{a\tau}, \] (7.33)

\[ e^{2D} = \left( \frac{4y_0^2}{\mu^2} e^{a\tau} \sinh^2(\alpha\tau/2) \right)^{\frac{2}{3\mu}} e^{a\tau}, \] (7.34)

\[ e^{2A} = \left( \frac{4y_0^2}{\mu^2} e^{a\tau} \sinh^2(\alpha\tau/2) \right)^{\frac{2(\Delta + 1)}{3\mu}} e^{a\tau}, \] (7.35)

\[ e^{a\phi} = \left( \frac{4y_0^2}{\mu^2} e^{a\tau} \sinh^2(\alpha\tau/2) \right)^{-\frac{a^2}{\alpha}} e^{a\phi_0}. \] (7.36)

Passing to $y$ via the map

\[ \tau = \frac{1}{\alpha} \ln \left( 1 - \frac{\mu}{y} \right) \] (7.37)

we obtain

\[ ds^2 = \left( \frac{y}{y_0} \right)^{-\frac{4}{3\mu}} \left[ -\left( 1 - \frac{\mu}{y} \right) dt^2 + dx^2 \right] + \left( \frac{y}{y_0} \right)^{-2\left(1 + \frac{a^2}{\alpha} \right)} \left( 1 - \frac{\mu}{y} \right)^{-1} dy^2, \] (7.38)

\[ e^{a(\phi - \phi_0)} = \left( \frac{y}{y_0} \right)^{\frac{2a^2}{\alpha}}. \] (7.39)

This solution can be interpreted as a black domain wall. For $\mu = 0$ it reduces to the standard solution (7.27)-(7.28) with $c = 0$, which is identical with the linear dilaton background locally. So we can regard the solution (7.39) in the same way as our previous solutions: a black domain wall on the linear dilaton background. For $\Delta < 0$ ($b^2 < 0$), the linear dilaton asymptotic region $y \to \infty$ is at infinite affine distance, with the other asymptotic region $y = 0$ corresponding to a singularity hidden behind the horizon. On the contrary,
for $\Delta > 0$ ($b^2 > 0$), the spacetime ends at the point singularity $y \to +\infty$ (bag-of-gold black domain wall), with the other asymptotic region $y = 0$ at infinite affine distance.

In the case $\Delta = 0$ one finds the possible horizons at $\tau = \pm \infty$ as well. Choosing as the horizon $\tau = -\infty$, we find the regularity conditions on the parameters:

$$b_1 = \frac{\alpha}{2}, \quad d_1 = f_1 = 0.$$  \hfill (7.40)

Choosing $\alpha = 1$, and transforming the radial coordinate to $y = e^\tau$, the corresponding black domain wall solution is (up to rescalings)

$$ds^2 = e^{-\frac{b^2}{p}y} \left(-ydt^2 + dx^2\right) + e^{-\frac{b^2(p+1)}{p}y} \left(dy^2 + \frac{d\tau^2}{y}\right),$$  \hfill (7.41)

$$e^{a\phi} = e^{-\frac{b^2(p+1)}{p}y}.$$  \hfill (7.42)

In this case the affine parameter for null radial geodesics reads, in the asymptotic region $y \to +\infty$,

$$\lambda \propto e^{-\frac{b^2(p+2)}{2p}y},$$  \hfill (7.43)

so that, again, for $b^2 < 0$ the manifold extends to infinity, with a singularity at $y \to -\infty$, while for $b^2 > 0$ it ends at the point singularity $y \to +\infty$, with $y \to -\infty$ at infinite affine distance.

8. Relation to $p$-brane solutions with extra parameters

The full $p$-brane solution was first derived in Ref. [23], using a rather intricate integration method and a gauge different from the present one. The solution contained four independent parameters, but it remained unclear whether it was free of naked singularities. Later this solution was reproduced and analyzed in [24] where it was suggested that extra parameters may occur, then it was rederived with different variations and analyzed in a number of papers (see for example [40, 41]). It was stated that the full (asymptotically flat) solution has four independent parameters. However, all the previous solutions have defects, such as naked singularities. In this section we will show that to avoid naked singularities, one should fix additional parameters, and that the actual number of parameters in good solutions (i.e. asymptotically flat, satisfying cosmic censorship and having regular horizon) reduces to two.

Let us show this for the solution of Ref. [23] (all the other solutions are related to this solution by simple change of notations). To match our notations with those of Ref. [23] one should set $E = 0, d_0 = 0, q = k$ and set the dilaton to zero at infinity, or equivalently

$$\phi_0 = 0, \quad \frac{\alpha^2(k-1)^2}{\Delta b^2} = \sinh^2\left(\frac{1}{2} \alpha \tau_0 \right).$$  \hfill (8.1)

Making the following change of variables in (3.15-3.20)

$$\tau = -\frac{1}{2} r_0^{k-1} \ln \left(1 - \left(\frac{r}{r_0}\right)^{k-1}\right), \quad \text{and} \quad \mathcal{F} = -r^k \left(1 - \frac{r_0^{2(k-1)}}{r^{2(k-1)}}\right).$$  \hfill (8.2)
and replacing our parameters by those of Ref. \cite{23}

\begin{align}
\beta &= 4 r_0^{k-1}, \quad \coth \left( \frac{\alpha r_0}{2} \right) = c_3, \quad d_1 = -c_2 r_0^{k-1}, \\
\phi_1 &= 2 c_1 r_0^{k-1} - \frac{a(d-3)}{(k-1)} c_2 r_0^{k-1}, \quad \frac{\alpha}{\beta} = \tilde{k},
\end{align}

(8.3)

we obtain the general asymptotically flat solution in the form of Ref. \cite{23}

\begin{align}
ds^2 &= -f(r) e^{2A(r)} dt^2 + e^{2A(r)} (dx_1^2 + \ldots + dx_p^2) + e^{2B(r)} (dr^2 + r^2 d\Sigma_\tilde{k}^2), \quad (8.4)
\end{align}

where

\begin{align}
f(r) &= \left[ 1 - \left( \frac{r_0}{r} \right)^{k-1} \right]^{-c_2} \left[ 1 + \left( \frac{r_0}{r} \right)^{k-1} \right], \quad (8.5) \\
A(r) &= \frac{(k-1)(ac_1 + (1 + \frac{a^2}{2(k-1)}) c_2)}{\Delta(d-2)} h(r) - \frac{2(k-1)}{\Delta(d-2)} \ln[\cosh(\tilde{k}h(r)) + c_3 \sinh(\tilde{k}h(r))], \quad (8.6) \\
B(r) &= \frac{1}{k-1} \ln \left[ 1 - \left( \frac{r_0}{r} \right)^{2(k-1)} \right] - \frac{2(a(k-1)c_1 - ac_2)}{2\Delta(d-2)} h(r) + \frac{2(p+1)}{\Delta(d-2)} \ln[\cosh(\tilde{k}h(r)) + c_3 \sinh(\tilde{k}h(r))], \quad (8.7) \\
\phi(r) &= -\frac{(k-1)(2(p+1)c_1 - ac_2)}{\Delta(d-2)} h(r) - \frac{2a}{\Delta} \ln[\cosh(\tilde{k}h(r)) + c_3 \sinh(\tilde{k}h(r))], \quad (8.8)
\end{align}

and

\begin{align}
h(r) &= \ln \left( \frac{1 - \left( \frac{r_0}{r} \right)^{k-1}}{1 + \left( \frac{r_0}{r} \right)^{k-1}} \right). \quad (8.9)
\end{align}

The constraint (3.24) transforms into the condition

\begin{align}
\frac{4}{\Delta} \tilde{k}^2 + c_1^2 - \frac{1}{\Delta} \left( ac_1 + \frac{(k-1)c_2}{d-2} \right)^2 - \frac{2k}{k-1} + \frac{(d-3)}{2(d-2)} c_2^2 = 0. \quad (8.10)
\end{align}

So the complete asymptotically flat solution has four independent parameters \( r_\pm, c_2 \) and \( \tilde{k} \), as was stated in \cite{23}.

To see the full structure of the solution (8.4) let us now make the transformation from the isotropic coordinate \( r \) to a Schwarzschild-type coordinate \( \tilde{r} \)

\begin{align}
r = \tilde{r} \left( \frac{\sqrt{f_+} + \sqrt{f_-}}{2} \right)^{\frac{2}{k-1}}, \quad (8.11)
\end{align}

where

\begin{align}
f_\pm &= 1 - \left( \frac{r_\pm}{r} \right)^{k-1}, \quad r_0^{k-1} = \frac{1}{4}(r_+^{k-1} - r_-^{k-1}), \quad c_3 = \frac{r_+^{k-1} + r_-^{k-1}}{r_+^{k-1} - r_-^{k-1}}. \quad (8.12)
\end{align}
The curvature has the following form in terms of the coordinate $\tilde{r}$

$$
R \sim \frac{(k-1)^2}{2k}(f_+ f_-)^k \left( \frac{f_+}{f_-} \right) \frac{\alpha^{(p+1)}}{\Delta(d-2)} \frac{1}{\Delta(d-2)} [r_+^{-1} f_+^{-1} - r_-^{-1} f_-^{-1}]^{-\frac{2(p+1)}{\Delta(d-2)}} - 1 \times
$$

$$
\times \left[ \text{const}_1 f_+^{-\tilde{k}} + \text{const}_2 f_-^{-\tilde{k}} + \frac{b^2(d-2k)}{(k-1)^2} \right].
$$

(8.13)

From this expression we see that the solution has an initial singularity at the point $\tilde{r} = 0$ and, for generic values of parameters, may be singular at $\tilde{r} = r_-$ and $\tilde{r} = r_+$. Therefore to get a physically interesting solution (i.e. satisfying the principle of cosmic censorship) we have to demand the point $r_+$ to be a regular horizon. The condition of its non-degeneracy ($n = 1$ in terms of formula (4.16)) gives us the following constraint on parameters

$$
2a(d-2)c_1 - a^2(d-2)c_2 - 2p(k-1)c_2 + 4(d-2)\tilde{k} - \frac{2\Delta k(d-2)}{k-1} = 0,
$$

(8.14)

using which we can rewrite (8.10) as

$$
\frac{pk}{2(d-2)}(c_2 + 2)^2 + \frac{1}{a^2(d-2)}(2k(p+1) + p(k-1)c_2 - 2(d-2)\tilde{k})^2 = 0.
$$

(8.15)

This condition fixes two of the four parameters

$$
c_2 = -2, \quad \tilde{k} = 1,
$$

(8.16)

and two independent parameters remain, in accordance with the result in section 6.

Let us make some remarks on the solution obtained in Ref. [41]. It can be derived from our generic solution (3.15–3.20) using the same gauge function as in (3.17) ($q = k$)

$$
\tau = \frac{2}{\beta} \ln \left( \frac{1 - (\frac{r}{\tilde{r}})^{q-1}}{1 + (\frac{r}{\tilde{r}})^{q-1}} \right) \equiv \frac{2}{\beta} \ln \frac{\tilde{H}_1}{H_1}, \quad \text{and} \quad \mathcal{F} = r^q \left( 1 - \frac{w^2(k-1)}{r^2(k-1)} \right) \equiv r^q \tilde{H}_1 \tilde{H}_1.
$$

(8.17)

where

$$
\beta = 4aq^{q-1}.
$$

(8.18)

Fixing the following set of parameters as

$$
d_{0,1} = e_{0,1} = 0, \quad g_1 = a\phi_1, \quad \frac{\alpha^2(q-1)^2}{\Delta b^2} = \sinh^2(\alpha \tau_0 / 2),
$$

(8.19)

identifying the remaining parameters with those of Ref. [11]

$$
\chi = \frac{\Delta(d-2)}{q-1}, \quad \cosh^2 \theta = \frac{1}{1 - e^{2\phi_1}}, \quad \delta = -\frac{2\phi_1}{\beta}
$$

(8.20)

and replacing the combinations $-(\alpha + a\phi_1)/\beta$ and $-(\alpha - a\phi_1)/\beta$ by $\alpha$ and $\beta$ respectively we obtain the solution in the form of [11]

$$
d s^2 = (H_1 \tilde{H}_1)^{\frac{4}{\chi(q-1)}} F_1^{\frac{4(q+1)}{\chi(q-1)}} (dr^2 + r^2 d\Sigma_0^2) + \frac{\tilde{H}_1}{\tilde{H}_1} \left(-dt^2 + dx_1^2 + \ldots + dx_p^2\right),
$$

(8.21)

$$
e^{\alpha \phi} = \left( \frac{H_1}{\tilde{H}_1} \right)^{\frac{a\delta}{2a^2(d-2)}} \frac{\tilde{H}_1^{\frac{4}{\chi(q-1)}}}{F_1^{\frac{4(q+1)}{\chi(q-1)}}},
$$

(8.22)
where

\[ F_1 = \cosh^2 \theta \left( \frac{H_1}{\tilde{H}_1} \right)^\alpha - \sinh^2 \theta \left( \frac{\tilde{H}_1}{H_1} \right)^\beta. \]  

(8.23)

This solution has three independent parameters: \( \delta, w \) and \( \theta \). However, as one can easily check (by the transformation to the Schwarzschild-type coordinate (8.11)), this solution has a horizon at \( r_+ \), which generically is not regular. Moreover, the parameters of the solution \( \text{[41]} \) cannot be adjusted to give a regular horizon, because, as was pointed out in \( \text{[41]} \), this solution corresponds to \( c_2 = 0 \) of Ref.\( \text{[23]} \), but we have found earlier in this section that the regularity of the horizon of Ref.\( \text{[23]} \) implies

\[ c_2 = -2. \]  

(8.24)

Indeed, one can check that, after transforming to Schwarzschild type coordinates and imposing the regularity conditions (4.21) one obtains a complex value for the parameter \( \delta \), which is not allowed by the construction of \( \text{[41]} \). Therefore within the parameter space of the solution of Ref. \( \text{[41]} \) we do not find solutions satisfying the conditions(4.21) necessary for regularity. The above reasoning applies to the first branch of the general solution of Ref. \( \text{[41]} \) (the one with the real functions \( H_1 \) and \( \tilde{H}_1 \)). We suppose that their second branch with imaginary \( H_1 \) and \( \tilde{H}_1 \) corresponds to imaginary values of our parameters \( \alpha \) and/or \( \beta \). In this case we did not find solutions with a regular horizon either.

9. Conclusions

The purpose of this paper was to clarify the meaning of special points in the general supergravity \( p \)-brane solution with a single charge and to find all possibilities for the values of parameters when the solution is free from naked singularities, though not necessarily asymptotically flat. We have found that there are three options depending on the value of the parameter \( \tau_0 \) marking the position of the (generic) singularity. For \( \tau_0 > 0 \), the singularity is hidden inside the event horizon, and the asymptotically flat solution coincides with the standard black \( p \)-brane of \( \text{[8]} \). For \( \tau_0 < 0 \), the solution has a regular horizon, but is singular at spatial infinity. For \( \tau_0 = 0 \), the regular black solution is not asymptotically flat, and constitutes a delocalized generalization of the black \( p \)-brane on the linear dilaton background discussed in \( \text{[14]} \). We also presented an explicit identification of the solution by Zhou and Zhu in terms of our solution, thus clarifying the nature of “the extra parameters”. In addition we compared our solution with the recent results by Lu and Roy. Our general conclusion is that no \( p \)-brane solutions possessing a regular event horizon and not plagued with naked singularities exist, other than the standard asymptotically flat black branes and the black branes on the linear dilaton background.

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Figure 1: Complex plane of $\tau$ for an asymptotically flat solution. In this case $\tau_0$ is strictly positive and corresponds to the central singularity. The real axis corresponds to regions outside the event horizon and inside the inner horizon. Between the horizons $\tau$ has an imaginary part (upper line). The dependence $\tau(r)$ corresponds to the map (5.1,5.22).

Figure 2: Complex plane of $\tau$ for the solution with the linear dilaton asymptotic. In this case $\tau_0$ is non-positive and to get a solution without naked singularities one has to take the limit $\tau_0 \to -0$. 