Analysis of Cascaded Brillouin Scattering in Optical Fibres

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Abstract: Cascaded Brillouin scattering is increasingly becoming of interest in many areas of photonics (e.g. see [1] and [2]) and has been studied experimentally by many groups. In gain assisted experiments for example up to nearly 800 distinct lines have been observed[3] while in passive cavities, mode locking of a few Brillouin lines has recently been demonstrated[4]. We show that although each pair of lines interact through the creation of an acoustic phonon the resulting acoustic field nevertheless can be written as the sum of only two waves (a forward and backward propagating wave). This new analysis results in a set of strongly coupled amplitude equations for the fields that are derived here for the first time in the context of optical fibres. Moreover we show that previous authors have misunderstood the nature of the acoustic field resulting in incorrect sets of coupled equations.

Index Terms: Brillouin Scattering, Fiber Optics

Stimulated Brillouin scattering (SBS) in transparent materials has long been studied due to its ubiquity and low threshold. In optical fibres for example the threshold can be as low as a few mW making it the dominant nonlinear process when dealing with the propagation of narrow linewidth sources[5]. In Brillouin scattering an optical field generates a co-propagating acoustic field that in turn, changes the refractive index leading to Bragg scattering and thus a strongly reflected Stokes field. Conservation of energy requires that the Stokes field is downshifted by the acoustic frequency which in fibres is typically around 11 GHz. This reflected field can in turn be strong enough to create a second Stokes field leading to a cascade of multiple Stokes waves each an identical frequency separation. Cascaded Brillouin scattering is especially strong in fibre resonators where anywhere up to nearly 800 distinct Brillouin lines can be seen [3] and more recently some groups have demonstrated that these lines can be phase-locked resulting in a 11GHz pulse train (see [4] and [2]). Given the importance of and growing interest in Brillouin resonators it is perhaps surprising that a detailed theoretical analysis of cascaded Brillouin scattering has not been performed. We present here a new analysis of the acoustic fields present in the fibre and show that this results in a new set of coupled equations for the amplitudes of the optical fields. Moreover this analysis shows that previous researchers have incorrectly generalised the equations for Brillouin scattering when trying to analyse cascaded Brillouin generation leading to flawed results. We start by summarising the well known physics of simulated Brillouin scattering before extending it to cascaded Brillouin scattering and finally discuss the differences between our work and previous analyses.

1. Standard Analysis of Brillouin Scattering

SBS is a three wave process in which a strong pump field \( P \) creates a co-propagating phonon field \( Q \) and a backwards propagating Stokes field \( B_0 \) that is down shifted by the acoustic resonance frequency \( \Omega \) of \( \sim 11 \text{GHz} \). For the case of a single Stokes wave the density fluctuations \( \rho \) in the
fibre can be written as [5]

\[ \rho(r, t) = F_A(x, y)Q(z, t)e^{-i(\Omega t - k_A z)} + c.c. \]  

(1)

where \( Q \) is the slowly varying acoustic field at frequency \( \Omega \) propagating in the \(+z\) direction with wave vector \( k_A \). The acoustic mode profile is given by \( F_A \) and is assumed constant down the fibre.

The phonon field \( Q \) acts to couple the pump and Stoke’s wave leading to the standard coupled amplitude equations:

\[ \frac{\partial P}{\partial z} + \frac{1}{v_g} \frac{\partial P}{\partial t} = -\alpha P + i\kappa B_0 Q \]  

(2)

\[ -\frac{\partial B_0}{\partial z} + \frac{1}{v_g} \frac{\partial B_0}{\partial t} = -\alpha B_0 + i\kappa P^* \]  

(3)

\[ \frac{\partial Q}{\partial t} + v_a \frac{\partial Q}{\partial z} = -\Gamma_A Q + \frac{i\delta}{A_{eff}} PB_0^* \]  

(4)

where \( v_g \) is the group velocity of the electric field at frequency \( \omega_0 \), \( \alpha \) is the loss and \( \kappa \) represents the coupling strength. The acoustic wave propagates at a velocity of \( v_a \) and is strongly damped by \( \Gamma_A \) and is driven by the product of the electric fields through the electrostriction term represented by \( \delta \). \( A_{eff} \) represents the effective area of the fibre.

In analysing Brillouin generation it is usual to make the assumption that the fields are time-independent and furthermore the spatial derivative of \( Q \) can be dropped due to the slow speed of the phonon field in which case Eq. (4) can be solved for \( Q \) and the usual equations for the power of the pump and stokes waves can be derived [5]

\[ \frac{dI_p}{dz} = -g_B I_p I_s - 2\alpha I_p \]  

(5)

\[ -\frac{dI_s}{dz} = g_B I_p I_s - 2\alpha I_s \]  

(6)

where \( g_B \) is the standard Brillouin gain and \( I_p = P P^* \) and \( I_s = B_0 B_0^* \).

In fibres and in particular resonant Brillouin lasers it is common that the Stokes wave becomes sufficiently intense that it creates a second phonon wave and thus a second Stokes wave that is co-propagating with the pump and offset by twice the acoustic resonance frequency (2\( \Omega \)). Denoting the second acoustic wave by \( R \) and the second Stokes wave as \( F_0 \) we get the coupled equations:

\[ \frac{\partial P}{\partial z} + \frac{1}{v_g} \frac{\partial P}{\partial t} = -\alpha P + i\kappa B_0 Q \]  

(7)

\[ -\frac{\partial B_0}{\partial z} + \frac{1}{v_g} \frac{\partial B_0}{\partial t} = -\alpha B_0 + i\kappa (P Q^* + F_0 R) \]  

(8)

\[ \frac{\partial F_0}{\partial z} + \frac{1}{v_g} \frac{\partial F_0}{\partial t} = -\alpha F_0 + i\kappa (B_0 R^*) \]  

(9)

\[ \frac{\partial Q}{\partial t} + v_a \frac{\partial Q}{\partial z} = -\Gamma_A Q + \frac{i\delta}{A_{eff}} PB_0^* \]  

(10)

\[ \frac{\partial R}{\partial t} - v_a \frac{\partial R}{\partial z} = -\Gamma_A R + \frac{i\delta}{A_{eff}} B_0 F_0^* \]  

(11)

Again in the time-independent case and when the spatial derivatives of the phonon waves can be neglected we can solve for \( Q \) and \( R \) as

\[ Q = i\frac{\delta}{\Gamma_A A_{eff}} PB_0^* \]  

(12)

\[ R = i\frac{\delta}{\Gamma_A A_{eff}} B_0 F_0^* \]  

(13)
Substituting the expressions for $Q$ and $R$ into the Eq. (7)–(9) and assuming continuous wave interactions leads to the coupled power equations

$$\frac{dI_p}{dz} = -g_B I_p I_{s0} - 2\alpha I_p$$  \hspace{1cm} (14)

$$\frac{-dI_{s0}}{dz} = g_B (I_p I_{s0} - I_{s1} I_{s0}) - 2\alpha I_{s0}$$  \hspace{1cm} (15)

$$\frac{-dI_{s1}}{dz} = g_B (I_{s0} I_{s1}) - 2\alpha I_{s1}$$  \hspace{1cm} (16)

where $I_p$ is the power in the pump beam, $I_{s0}$ is the power in the backwards propagating first Stokes wave and $I_{s1}$ is the power in the forward propagating 2nd Stokes beam. We emphasise here that the correct equations are Eqs. (7)–(11) rather than the coupled power equations which do not describe the full dynamics of the system (see Ref. [6] for example). Furthermore once a third stokes wave is introduced, then as we will show below, it is no longer possible to write down a set of coupled power equations and thus any analysis that begins with such a set of equations is wrong.

2. Generation of Multiple Brillouin Lines

Continuing the analysis from the previous section suppose that there are now $2N - 1$ Brillouin lines generated in a fibre ($N$ can be over 390 see for example Ref. [3]). The fundamental question is how many acoustic waves are present in the fibre? Since each pair of lines both generate and are coupled by an acoustic wave it might initially appear that there are $2N - 1$ acoustic waves present of which $N$ are propagating in the forward direction and $N - 1$ are propagating in the backward direction. Conservation of energy and momentum implies that the frequency $\Omega_j$ of the $j^{th}$ acoustic wave will be given by Stokes shift between the $j^{th}$ and $(j+1)^{th}$ electromagnetic wave. Thus we can write the total acoustic perturbation in the fibre as

$$\rho(z,t) = FA(x,y) \left( \sum_j Q_j(z,t)e^{-i(\Omega_j t - k_A(\Omega_j)z)} + \sum_m R_m(z,t)e^{-i(\Omega_m t + k_A(\Omega_m)z)} \right) + c.c.$$  \hspace{1cm} (17)

where $Q_j$ and $R_m$ represent the forward and backward propagating acoustic waves respectively.

In a fibre or waveguide the Brillouin gain is relatively narrow (~ 17 MHz in silica for example) compared to the wavelength shift (11 GHz) and so the standard assumption is that all of the Brillouin lines are equally spaced in frequency implying that

$$\Omega_l = \Omega, \quad l = 1, 2, ..., 2N - 1$$  \hspace{1cm} (18)

and thus the sums in Eq. (17) each collapse to a single term leading to the simplified expression

$$\rho(z,t) = FA(x,y) \left( Q(z,t)e^{-i(\Omega t - k_A(\Omega)z)} + R(z,t)e^{-i(\Omega + k_A(\Omega)z)} \right) + c.c.$$  \hspace{1cm} (19)

for the acoustic field. We note that measurements of the Brillouin shift in fibres for multiple cascaded Brillouin lines has found that Eq. (19) is correct[1].

Eq. (18) represents the crux of our argument since if Eq. (18) is correct then there are only 2 acoustic waves in the fibre: one propagating in the forward direction and one in the backwards direction. Looking in the literature we have found only one paper by C. Montes [7] published in 1985 that discusses this situation (in the context of plasma physics) which he calls the strongly coupled case (see Eqs. 47–50 of Ref [7]). More recently Ogusu [8] for example keeps multiple acoustic waves despite the fact that he assumes that each Brillouin shift is identical. This approach is followed by Buttner et al. [2] who in their supplementary information (Eq. S1) state that the each Brillouin wave is shifted by an identical amount. A third recent example is Yuan et al. [9] who implicitly adopt this model in their numerical analysis of their results.
We argue here that if there is only a single acoustic wave propagating in each direction in the fibre then it must be driven by multiple sets of interacting fields. This has important implications for the derivation of the correct set of equations to describe cascaded Brillouin scattering. In particular it implies that unlike the case for 2 or 3 interacting waves coupled power equations cannot be used and instead coupled equations for the fields are required and the correct equations which are a generalisation of the work of Montes [7] are presented below.

In line with the notation introduced above we denote the pump beam by $P$, the backwards propagating Stokes waves by $B_i$ and the forward propagating Stokes waves by $F_i$ where $i$ runs from 0 to $N$. Using this notation the correct equations are:

\[
\frac{\partial P}{\partial z} + \frac{1}{v_g} \frac{\partial P}{\partial t} = -\alpha P + i\kappa B_0 Q \tag{20}
\]

\[
-\frac{\partial B_0}{\partial z} - \frac{1}{v_g} \frac{\partial B_0}{\partial t} = -\alpha B_0 + i\kappa (PQ^* + F_0 R) \tag{21}
\]

\[
-\frac{\partial B_i}{\partial z} + \frac{1}{v_g} \frac{\partial B_i}{\partial t} = -\alpha B_i + i\kappa (F_{i-1}Q^* + F_i R) \tag{22}
\]

\[
\frac{\partial F_i}{\partial z} + \frac{1}{v_g} \frac{\partial F_i}{\partial t} = -\alpha F_i + i\kappa (B_i R^* + B_{i+1} Q) \tag{23}
\]

\[
\frac{\partial F_N}{\partial z} + \frac{1}{v_g} \frac{\partial F_N}{\partial t} = -\alpha F_N + i\kappa (B_n R^*) \tag{24}
\]

for the optical fields (assuming an even number of fields). While the acoustic waves satisfy

\[
\frac{\partial Q}{\partial t} + v_a \frac{\partial Q}{\partial z} = -\Gamma_A Q + i\frac{\delta}{A_{eff}} \left( PB_0^* + \sum_{j=1}^{N-1} F_j B_j^{*+1} \right) \tag{25}
\]

\[
\frac{\partial R}{\partial t} - v_a \frac{\partial R}{\partial z} = -\Gamma_A R + i\frac{\delta}{A_{eff}} \left( \sum_{j=0}^{N} B_j F_j^* \right) \tag{26}
\]

Making the standard approximations for the acoustic waves leads to the equations for $Q$ and $R$:

\[
Q = i\frac{\delta}{\Gamma_A A_{eff}} \left( PB_0^* + \sum_{j=0}^{N-1} F_j B_j^{*+1} \right) \tag{27}
\]

\[
R = i\frac{\delta}{\Gamma_A A_{eff}} \left( \sum_{j=0}^{N} B_j F_j^* \right) \tag{28}
\]

which can then be substituted back into the equations for the fields [Eq. (20)–Eq. (24)] if desired. In that case it can be easily shown that the coupled equations conserve the total photon flux through the system.

### 3. Discussion and Conclusions

The new equations Eq. (20)–(26) are the correct and obvious generalisation of the coupled equations for the case of two Stokes waves when there are only two acoustic waves propagating in the fibre. However in the literature most authors have chosen, incorrectly, to generalise the coupled power equations [Eq. (14) –Eq (16)] or equivalently to include a separate acoustic wave for each new pair of Stokes waves. Not surprisingly, since this fundamental physical picture of
the interaction is wrong, it leads to an incorrect set of equations and thus the resulting analysis cannot be trusted. We emphasise here the point made by Montes in his original article that in case of more than 3 interacting optical fields it is not possible to write down coupled equations for the evolution of the power in the fields and rather the coupled amplitude equations must be used.

A direct consequence of the presence of only two acoustic fields is that all optical fields are coupled together via the two phonon fields. An effect of this is to alter the threshold required to see multiple lines since each field has a growth term proportional to the pump $P$ (forward propagating fields) or its complex conjugate (backward propagating fields). Thus one would expect that there would be more Brillouin lines in a resonator than would be expected using the analysis of Ogusu for example. We are currently looking at numerically solving the full coupled equations and will present a more detailed numerical analysis of the difference between the two sets of equations in a later paper.

We also note that it is relatively trivial to extend the above equations to include Kerr nonlinear terms and also to describe ring resonators or Fabry-Perot cavities such as those treated by both Buttner et al. and by Ogusu. Treating a Fabry-Perot cavity requires that the number of optical fields is doubled since each Stoke wave is reflected by the cavity mirrors giving rise to a forward and backward propagating wave at the same frequency. However even in this case the number of acoustic waves remains fixed at 2 and so we do not treat this case any further.

In conclusion we have highlighted a common error in the analysis of cascaded Brillouin generation in optical fibres and presented the correct analysis based on correctly identifying the acoustic fields in the fibre (following earlier analysis by Montes). We hope that this work will be of use to other researchers working in the field.

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