ON THE SYMMETRY INTEGRAL

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to my Angel

ABSTRACT. We give a level one result for the “symmetry integral”, say \( I_f(N, h) \), of essentially bounded \( f : \mathbb{N} \to \mathbb{R} \); i.e., we get a kind of “square-root cancellation” bound for the mean-square (in \( N < x \leq 2N \)) of the “symmetry sum” of, say, the arithmetic function \( f := g * 1 \), where \( g : \mathbb{N} \to \mathbb{R} \) is such that \( \forall \varepsilon > 0 \) we have \( g(n) \ll_{\varepsilon} n^{\varepsilon} \), and supported in \([1, Q]\), with \( Q \ll N \) (so, the exponent of \( Q \) relative to \( N \), say the level \( \lambda := (\log Q)/(\log N) \) is \( \lambda < 1 \)), where the symmetry sum weights the \( f \)-values in (almost all, i.e. all but \( o(N) \) possible exceptions) the short intervals \([x - h, x + h] \) (with positive/negative sign at the right/left of \( x \)), with mild restrictions on \( h \) (say, \( h \to \infty \) and \( h = o(\sqrt{N}) \), as \( N \to \infty \)).

1. Introduction and statement of the results.

We give upper bounds for the symmetry integral (compare [C-S], [C], [C1], [C2], [C3], [C5], [C6], esp.)

\[
I_f(N, h) \overset{\text{def}}{=} \sum_{x,N} \left| \sum'_{|n-x| \leq h} \text{sgn}(n-x)f(n) \right|^2,
\]

with \( x \sim N \) for \( N < x \leq 2N \), \( \text{sgn}(r) := \frac{r}{|r|} \forall r \in \mathbb{R}, r \neq 0 \), \( \text{sgn}(0) := 0 \) and the dash means: the terms \( n = x \pm h \) have to be halved (say, \( \sum_{|n-x| \leq h} \text{sgn}(n-x)f(n) = \sum_{|n-x| \leq h} \text{sgn}(n-x)f(n) - \frac{f(x+h)-f(x-h)}{2} \)); this is one of the possible definitions: comparing the one(s) in the papers quoted above, the difference involved will be soon shown negligible, at least for the (large) class of essentially bounded arithmetic functions \( f : \mathbb{N} \to \mathbb{R} \).

Here we’ll use the term “essentially” to leave (negligible) multiplicative factors bounded by arbitrary small powers of \( N \). For example,

\( f \) is essentially bounded (abbrev. \( f \ll_{\varepsilon} 1 \)) \( \iff \forall \varepsilon > 0 \ f(n) \ll_{\varepsilon} N^{\varepsilon} \)

(here, \( \forall n \leq 2N + h \), since we “don’t see” \( f \) any further), with the usual Vinogradov notation (i.e., \( \ll_{\varepsilon} \) means “bounded in absolute value with a constant factor depending on \( \varepsilon \))’; and we’ll abbreviate

\[
F(N, h) \ll G(N, h) \overset{\text{def}}{=} \forall \varepsilon > 0 \ |F(N, h)| \ll_{\varepsilon} N^{\varepsilon} G(N, h).
\]

For example, to see that our new definition is “close” to the old one (an integral !), whenever \( f \ll 1 \),

\[
\int_N^{2N} \left| \sum_{|n-x| \leq h} \text{sgn}(n-x)f(n) \right|^2 \, dx \ll \sum_{N \leq x < 2N} \left| \sum'_{|n-x| \leq h} \text{sgn}(n-[x])f(n) \right|^2 \, dx + N \ll I_f(N, h) + N + h^2
\]

where \([x] \overset{\text{def}}{=} \text{integer part of } x \in \mathbb{R} \) and both the remainders \( \ll N \) and \( \ll h^2 \) are clearly negligible: below the “diagonal” remainders, i.e. \( \ll Nh \) (when \( h \to \infty \) and \( h = o(N) \), for \( N \to \infty \), as we’ll assume henceforth).

The symmetry integral (now on, \( I_f(N, h) \) defined above) measures, for \( f \), the almost-all (i.e., leave eventual \( o(N) \) exceptions) symmetry (around \( x \)) in the short (since \( h = o(x) \)) intervals \([x - h, x + h] \).

For a motivation to study \( I_f \), see esp. [C]. Our present arguments closely resemble [C7] ones.

Our main result is the following.

Theorem. Let \( N, h, Q \in \mathbb{N} \), with \( Q \ll N \) and \( h \to \infty \), \( h = o(\sqrt{N}) \) as \( N \to \infty \). Assume \( g : \mathbb{N} \to \mathbb{R} \) is independent of \( x, N, h \), essentially bounded and supported in \([1, Q]\); set \( f := g * 1 \). Then

\[
I_f(N, h) \ll Nh.
\]

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We then get square-root cancellation for the symmetry integral of $d_k$ (generating Dirichlet series $\zeta^k$):

**Corollary.** Fix $k \geq 1$ integer. Let $N, h \in \mathbb{N}$ with $h = o(\sqrt{N})$. Then (the $\ll -\text{const. depends on } k$)

$$I_{d_k}(N, h) \ll_k Nh.$$  

Also, for the von Mangoldt function $\Lambda(n) := \log p$ for $n = p^r$, $r \in \mathbb{N}$, $p$ prime, := 0 otherwise,

$$I_\Lambda(N, h) \ll Nh.$$  

We’ll not prove the Corollary. (A kind of “DIRICHLET HYPERBOLA TRICK” for $I_{d_k}$ gives $Q \ll N^{1-1/k}$, [C6].)

In passing, we note that an estimate of the kind in the Corollary, but for the Selberg integral of $d_k$ would give, applying [C4] bounds, the proof of the Lindelöf Hypothesis! (Also, we prove, with the Corollary, the almost-all symmetry of primes: for the corresponding Selberg integral, i.e. the classical Selberg integral of primes, this would give the Density Hypothesis!).

The paper is organized as follows:

- △ in section §2 we start proving the Theorem, then
- △ in next section, §3, we state and prove a small Lemma, in order to complete Theorem’s proof;
- △ we conclude with some comments, in the final section.

The author trusts the possibility to treat the classic Selberg integral through the approach outlined in [C].

**2. A weak majorant principle.**

We start proving the Theorem, giving at next section the necessary Lemma. Here, we closely follow [C7].

**Proof.** Assume now $Q \ll N$, $g : \mathbb{N} \to \mathbb{R}$, with supp $(g) \subset [1, Q]$, $g \ast 1 := f \ll 1$, $h \to \infty$ and $h = o(N)$:

$$I_f(N, h) = \sum_{x \sim N} \left| \sum_{q, q \leq x + h} g(q) \chi_q(x) \right|^2,$$

where we define, this time (compare [C-S], esp.), $\forall q \in \mathbb{N}$,

$$\chi_q(x) \overset{\text{def}}{=} -\sum_{\substack{|n-x| \leq h \\ n \equiv 0 (\text{mod } q)}}\text{sgn}(n-x) = \sum_{\ell \geq 1} \frac{\ell}{q} \sum_{j \leq \frac{h}{q}} c_{j, \ell}^\pm \sin \frac{2 \pi x j}{\ell},$$

and the **Fourier coefficients are positive**

$$c_{j, q}^\pm := \frac{4}{q} \cot \frac{\pi j}{q} \sin \frac{2 \pi j h}{q} := \frac{1}{q} F_{h, \frac{j}{q}} \geq 0 \quad \forall j \leq \frac{q}{2}$$

(better, the finite Fourier expansion has non-negative coefficients); here (see [C-S], apply Parseval identity)

$$\sum_{j \leq q} |c_{j, q}^\pm|^2 \ll \sum_{0 < j \leq q} |c_{j, q}^\pm|^2 \ll \left\| \frac{h}{q} \right\| \ll \min \left(1, \frac{h}{q} \right).$$

As usual, $\|\alpha\| := \min_{n \in \mathbb{Z}} |\alpha - n|$ is the **distance to the next integer**, $\forall \alpha \in \mathbb{R}$.

The **orthogonality of the additive characters** is the starting point (compare [C7]).

Here we’ll not define explicitly the **Ramanujan coefficients**, see [C7].

This time, not like in [C7], we use as a “majorant”, for our $f(n)$, the **divisor function**, $d(n)$. In fact, this one has, at least for $h = o(\sqrt{N})$, see [C-S] (compare [C2]), square-root cancellation on average over $x \sim N$. 

2
Then (see [C7], proof of the Proposition: by the way, misprints occur there, missing \( \frac{1}{A} \) and \( \frac{1}{A} \))

\[
I_f(N, h) = D_f^\pm (N, h) + 2 \sum_{1<\ell,t \leq Q} \sum_{d \leq \frac{\ell}{A}} \sum_{q \leq \frac{\ell}{A}} g(\ell d) g(tq) \frac{\ell d t q}{\ell d t q} \sum_{j \leq \frac{\ell}{4}} \sum_{r \leq \frac{\ell}{4}} F_h^\pm \left( \frac{j}{\ell} \right) F_h^\pm \left( \frac{r}{t} \right) \sum_{x \sim N} \sin \frac{2\pi x j}{\ell} \sin \frac{2\pi x r}{t} =
\]

\[
= D_f^\pm (N, h) + \sum_{1<\ell,t \leq Q} \sum_{d \leq \frac{\ell}{A}} \sum_{q \leq \frac{\ell}{A}} g(\ell d) g(tq) \frac{\ell d t q}{\ell d t q} \sum_{j \leq \frac{\ell}{4}} \sum_{r \leq \frac{\ell}{4}} F_h^\pm \left( \frac{j}{\ell} \right) F_h^\pm \left( \frac{r}{t} \right) \left( \sum_{x \sim N} \cos 2\pi \delta x - \sum_{x \sim N} \cos 2\pi \sigma x \right),
\]

where we abbreviate

\[
\sum_{x \sim N} = \sum_{x \geq \ell d - h} \sum_{x \leq t q - h},
\]

indicating the present diagonal as

\[
D_f^\pm (N, h) \overset{def}{=} \sum_{1<\ell,t \leq Q} \sum_{d \leq \frac{\ell}{A}} \sum_{q \leq \frac{\ell}{A}} g(\ell d) g(tq) \frac{\ell d t q}{\ell d t q} \sum_{j \leq \frac{\ell}{4}} \sum_{r \leq \frac{\ell}{4}} F_h^\pm \left( \frac{j}{\ell} \right) F_h^\pm \left( \frac{r}{t} \right) \sum_{x \sim N} \sin^2 \frac{2\pi x j}{\ell},
\]

with \( \delta := j/\ell - r/t > 0 \) (see above) and

\[
\sigma := \left\| \frac{j}{\ell} + \frac{r}{t} \right\| \in \left[ 0, \frac{1}{2} \right].
\]

This time we appeal to the elementary Lemma of [C7] to treat the terms with well-spaced fractions (i.e., \( \delta > 1/A \) or \( \sigma > 1/A \), with \( 1/A = 1/NL, L := \log N, \) say), but we have a minus sign on the terms with \( \sigma \leq 1/A \). These are treated exactly, when \( 1/A = o(1/N) \), in the next small Lemma.

### 3. Statement and proof of the lemma to complete the Theorem Proof.

We can state and show our

**Lemma.** Let \( A, N \in \mathbb{N} \) with \( \frac{1}{A} = o(1/N) \) when \( N \to \infty \). Let \( Q \ll N \) and \( R_\ell, R_t, F_h^\pm \) be as above. Then

\[
\delta := \frac{j}{\ell} - \frac{r}{t}, \quad \sigma := \left\| \frac{j}{\ell} + \frac{r}{t} \right\| \Rightarrow \sum_{1<\ell,t \leq Q} R_\ell(f) R_t(f) \sum_{j \leq \frac{\ell}{4}} \sum_{r \leq \frac{\ell}{4}} F_h^\pm \left( \frac{j}{\ell} \right) F_h^\pm \left( \frac{r}{t} \right) \sum_{x \sim N} \cos 2\pi \sigma x = 0.
\]

**Proof.** Abbreviating \( \sigma \) as above we have

\[
\sigma \leq \frac{1}{A} \Rightarrow 0 < \frac{j}{\ell} + \frac{r}{t} \leq \frac{1}{A} \quad \text{or} \quad 0 \leq 1 - \frac{j}{\ell} - \frac{r}{t} \leq \frac{1}{A};
\]

first case gives an absurd: in particular \( 0 < j = o(\frac{\ell}{A}) = o\left( \frac{Q}{A} \right) = o(1), 0 < r = o\left( \frac{\ell}{A} \right) = o\left( \frac{Q}{A} \right) = o(1). \)

Hence

\[
0 \leq \left( \frac{1}{2} - \frac{j}{\ell} \right) + \left( \frac{1}{2} - \frac{r}{t} \right) \leq \frac{1}{A} \Rightarrow 0 \leq \frac{1}{2} - \frac{j}{\ell} \leq \frac{1}{A}, 0 \leq \frac{1}{2} - \frac{r}{t} \leq \frac{1}{A};
\]

whence (use \( 1/A = o(1/N) \), here)

\[
\frac{\ell}{2} - \frac{\ell}{A} \leq j \leq \frac{\ell}{2}, \quad \frac{t}{2} - \frac{t}{A} \leq r \leq \frac{t}{2}, \quad \Rightarrow \quad j = \left\lfloor \frac{\ell}{2} \right\rfloor, r = \left\lfloor \frac{t}{2} \right\rfloor.
\]

3
which, together with previous ranges for \(1 - j/\ell - r/t\), give, again from \(\frac{1}{A} = o(\frac{1}{N})\),

\[
0 \leq \frac{1}{\ell} \left\lceil \frac{\ell}{2} \right\rceil + \frac{1}{t} \left\lceil \frac{t}{2} \right\rceil \leq \frac{1}{A} \Rightarrow 2|\ell, 2|t.
\]

Thus,

\[
\frac{j}{\ell} - \frac{r}{t} = \frac{1}{2}
\]

and the thesis, using \(F_h^{+} \left( \frac{1}{2} \right) = 0\). □

We may complete Theorem’s proof. In fact, applying [C7] well-spaced Lemma (the conditions in . . . give a small disturbance, since the same considerations of [C7] Proposition proof apply verbatim) with the same choice \(A = NL = N \log N\), say, from present Lemma (which, together with the [C7] one, says terms with the \(x\)-sum of \(\cos 2\pi\sigma x\) are negligible !)

\[
I_f(N, h) \ll Nh + \sum_{1 < t \leq Q} D_d^+(N, h) + \sum_{d \leq 2N+h} \sum_{q \leq \sqrt{2N+h}} \left( \sum_{j/t} \sum_{r/t} F_h^+ \left( \frac{j}{\ell} \right) F_h^+ \left( \frac{r}{t} \right) \right)
\]

\[
\ll I_d(N, h) + Nh,
\]

whence the thesis, since the symmetry integral of the function \(d(n)\), from [C-S] (requiring \(h = o(\sqrt{N})\), here), is

\[
\ll Nh. \kappa
\]

4. SOME COMMENTS.

The method outlined in [C7] and here treats more in general second moments over “long intervals” of “short interval” sums (or averages). Both these approaches use the non-negativity of the Fourier coefficients (say, \(\bar{F}_h\) there and \(F_h^{+}\) here) arising from the short interval inner average. The Selberg integral is another planet. I mean, no more free positivity there ! (No free meal...) However, there’s still some hope ! First of all, the present approach seems weaker than [C7] one, whose proofs are easier; but not a problem the estimate in our present Lemma. The real complication comes when facing changing signs of Fourier coefficients for the Selberg integral. They require another approach and, also, will set some limits on the ranges of the divisors, say \(Q\). Here we don’t have any troubles, since we have a REAL (though approximate !) majorant principle, i.e. we use brute force and “bound any essentially bounded function with the divisor function”. This is possible, since we KNOW the symmetry integral of \(d(n)\), not only, but it’s also fantastically small (it’s a square-root saving ! However, see my paper on Parma University Journal about square-free numbers: they’ve even better cancellation on their symmetry integral !).

The problem we have to face for both the original and the modified Selberg integral is the (already mentioned, see [C7] final comments) problem of the Wintner majorants that even here is represented (but immediately solved).

The majorating procedure is optimal (as we see now) in the case of the symmetry integral. Then, if you have a way to get non-trivial bounds from this (as Kaczorowski and Perelli do, as stated in [C]) for the Selberg integral, better to avoid a direct approach to \(J_f(N, h)\).

However, the modified Selberg integral has its own interest; and the further difficulty to build good Wintner majorants to get non-trivial bounds is happily faced, in view of the non-trivial results obtained !

The same is of course true whenever a kind of majorant principle will be available for the Selberg integral, but (due to Fourier coefficients non-constant signs) with limitations, due to the technical analytic treatment.

I expect (and have already some scratch calculations) the analytic part to be fair (not impossible, not easy); then, the “heavy” part is building (non-trivial!) majorants. And next ... estimate Selberg’s integral !
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