Nelsonian Mechanics Revisited

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Abstract

In de Broglie and Bohm’s pilot-wave theory, as is well known, it is possible to consider alternative particle dynamics while still preserving the $|\psi|^2$ distribution. I present the analogous result for Nelson’s stochastic theory, thus characterising the most general diffusion processes that preserve the quantum equilibrium distribution, and discuss the analogy with the construction of the dynamics for Bell’s beable theories. I briefly comment on the problem of convergence to $|\psi|^2$ and on possible experimental constraints on the alternative dynamics.

1 Introduction

A well-known feature of de Broglie–Bohm pilot-wave theory (de Broglie, 1928; Bohm, 1952), first pointed out already by de Broglie (ibid.), is that the dynamics preserves the ‘quantum equilibrium’ distribution $|\psi|^2$. This property does not characterise the dynamics uniquely, however, and alternative dynamics were discussed by Bohm and Hiley (1993) in the context of the Pauli equation. Recently, Deotto and Ghirardi (1998) have given a general discussion of these alternative dynamics, showing how to construct additional velocity fields satisfying several important physical constraints.
In this note, I wish to point out that the analogous question can be asked also in Nelson’s stochastic mechanics (Nelson, 1966, 1985) and related theories, and that Deotto and Ghirardi’s results have a direct bearing also to this case. I further wish to point out that such a treatment is analogous to that of dynamics in BBB (de Broglie–Bohm–Bell) theories (or ‘beable’ theories), as discussed by Bell (1984), who constructed such dynamics in the first place from the requirement that the quantum distribution be preserved in time. More detailed discussion of such dynamics was given by Vink (1993) and by Bacciagaluppi and Dickson (1997).

I briefly review de Broglie–Bohm theory and the Nelson-like stochastic theories in Section 2, then summarise Deotto and Ghirardi’s results for de Broglie–Bohm theory, and present the analogous results for the stochastic theories in Section 3. In Section 4, following Carlen (1984), I show existence and uniqueness of solutions for the resulting generalised theories. Section 5 spells out the analogy with BBB theories. Finally, I conclude with some remarks on the distribution postulate and possible experimental constraints in Section 6.

2 Configuration-Space Theories

In de Broglie–Bohm theory, where for simplicity we consider a single particle, the state of the system at any time $t$ is given by the particle’s position $x$. The quantum state $\psi$, satisfying the Schrödinger equation

$$i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \nabla^2 \psi + V\psi,$$

has the role of a *pilot wave* or *guiding field* for the particle, and the fundamental equation of motion is de Broglie’s *guidance equation*

$$\dot{x} = \frac{\hbar}{m} \nabla S,$$

the Pauli equation (Bohm and Hiley, 1993, Sections 10.2 and 10.4). Bohm and Hiley’s treatment parallels the discussion by Gurtler and Hestenes (1975); see also Holland (1993, p. 394). Squires (1996) also points out the existence of alternative currents satisfying the continuity equation, with a brief discussion. Finally, in the stochastic case the possibility of additional velocity terms is mentioned by Nelson (1985, p. 55).
where $S$ is the phase of the wave function (not divided by $\hbar$). Straightforward inclusion of a vector potential in (1) and (4) (omitted here) makes the theory gauge invariant.

If a distribution $P(x, t)$ over the position of the particles is given, we obtain from (2) the following expression for a probability current:

$$j(x, t) = \frac{\hbar}{m} (\nabla S) P(x, t).$$

(3)

And since probability is always conserved, we obtain a continuity equation,

$$\frac{\partial P}{\partial t} + \nabla \cdot j = 0,$$

(4)

or

$$\frac{\partial P}{\partial t} + \nabla \cdot \left( \frac{\hbar}{m} (\nabla S) P \right) = 0.$$  

(5)

It is easy to show that the quantum equilibrium distribution $|\psi|^2$ is a solution of (5), using the Schrödinger equation (1) and writing $\psi = |\psi| \exp(iS)$.

Note that equation (3) becomes singular on the set $N$ of points where $\psi(x) = 0$ (the nodal set of $\psi$). Nonetheless, as has been shown by Berndl, Dürr, Goldstein, Peruzzi and Zanghì (1995; see also Berndl, 1996), the particle has probability zero of entering the nodal set from outside. The global existence and uniqueness of solutions is then guaranteed for all sufficiently regular initial $\psi$ and all but a set of $|\psi|^2$-measure zero initial $x$, namely the nodal set of $\psi$.

The condition that the particles be distributed according to $|\psi|^2$ can be taken to imply that the theory is empirically equivalent to standard quantum mechanics, namely if one argues that all measurement results in standard quantum mechanics can be described using positions of particles — which are all there is in pilot-wave theory. (The general theory of measurement in pilot-wave theory was achieved by Bohm (1952, Part II).) A question arises thus as to whether there are any dynamical equations other than (2) that preserve $|\psi|^2$.

An example of such a theory is obtained if one modifies de Broglie–Bohm theory by assuming that the particle is undergoing not a deterministic evolution but a diffusion process, so that the dynamics is given by a stochastic
guidance equation,
\[ dx = b dt + \sqrt{\alpha} d\omega \]  \hspace{1cm} (6)
(in Itô form), with \( b = \frac{\hbar}{m} \nabla S + \alpha \frac{\hbar}{2m} |\psi|^2 \), and where \( d\omega \) is a Wiener process with
\[ \langle d\omega \rangle = 0, \quad \langle (d\omega)^2 \rangle = \frac{\hbar}{m}. \]  \hspace{1cm} (7)
Here, \( b \) is the drift velocity of the process and \( \nu := \alpha \frac{\hbar}{2m} \) is the diffusion coefficient, treated as a free parameter. Such theories have been discussed by Bohm and Hiley (1989) under the heading of the stochastic interpretation of quantum mechanics, and by Peruzzi and Rimini (1996), who talk about hidden configurations theories. Nelson’s mechanics is recovered if we set \( \alpha = 1 \), and de Broglie–Bohm theory for \( \alpha = 0 \).

The conservation equation for such a process can be written as a Fokker–Planck equation:
\[ \frac{\partial P}{\partial t} + \nabla \cdot (b P - \nu \nabla P). \]  \hspace{1cm} (8)
But now, by substituting \( P = |\psi|^2 \) one easily sees that (8) reduces to
\[ \frac{\partial |\psi|^2}{\partial t} + \nabla \cdot \left( \frac{\hbar}{m} (\nabla S) |\psi|^2 \right) = 0, \]  \hspace{1cm} (9)
which, as mentioned above, is always satisfied. Thus, \( \rho := |\psi|^2 \) is a solution to the Fokker–Planck equation, and, indeed, it can be understood as an equilibrium solution, for which the so-called osmotic current \( u P \), where \( u \) is

\[ \text{This approach differs from that of Nelson and its generalisation for arbitrary } \alpha \text{ by Davidson (1979) in that it postulates a wave function } \psi \text{ obeying the Schrödinger equation. Nelson’s (1966) original paper proposes to derive the Schrödinger equation from the stochastic particle dynamics. And, as essentially shown by Davidson (1979) — but one should use the drift velocity } b \text{ instead of his equation (18) — one can generalise Nelson’s derivation of the Schrödinger equation to the case of arbitrary non-zero } \alpha. \text{ However, these derivations of the Schrödinger equation — as all derivations based on the supposed equivalence with the pair of equations consisting of the continuity equation and a Hamilton–Jacobi-type equation — need to be qualified, since they rely on the tacit assumption that the resulting wave function be single-valued. As pointed out by Wallström (1994), this is a non-trivial requirement, which is in fact equivalent to the Bohr–Sommerfeld stability condition, i.e. to a standard quantisation condition.} \]
the osmotic velocity
\[ u := \alpha \frac{\hbar}{2m} \nabla |\psi|^2 \] (10)

and the diffusion current \( \nu \nabla P \) balance each other exactly. The resulting average velocity
\[ v := \frac{\hbar}{m} \nabla S \] (11)
is called the current velocity.

Again, when \( \psi = 0 \) the guidance equation becomes singular. On the other hand, also in the case of Nelson’s mechanics it has been shown that under certain regularity conditions on the initial wave function and the potential \( V \) in the Schrödinger equation (1), the particle will have probability zero of entering the nodal set \( \mathcal{N} \), and global existence and uniqueness hold for all initial \( x \not\in \mathcal{N} \) (Carlen, 1984; Nelson, 1985, Sections 11 and 15). As remarked by Peruzzi and Rimini (1996), these results are valid for arbitrary \( \alpha \) (cf. below, Section 4).

Although the dynamics is stochastic, Nelson is careful not to pick out a preferred direction in time (in this, he is not followed by Bohm and Hiley). The time-reversed process is in fact also required to be a diffusion with the same osmotic velocity and diffusion coefficient, or equivalently to satisfy also the so-called backward Fokker–Planck equation,
\[ \frac{\partial P}{\partial t} + \nabla \cdot \left( \mathbf{b}^* P + \nu^* \nabla P \right), \] (12)

where
\[ \mathbf{b}^* = \frac{\hbar}{m} \nabla S - \nu \frac{\nabla \rho}{\rho} \quad \text{and} \quad \nu^* = \nu. \] (13)

Thus,
\[ \frac{\partial P}{\partial (-t)} + \nabla \cdot \left[ (-v + u)P - \nu \nabla P \right], \] (14)
so, indeed, in the time-reversed process only the current velocity changes sign, the osmotic velocity and diffusion coefficient remaining the same.
3 Generalisation of the Dynamics

Deotto and Ghirardi’s (1998) question is whether there are any other physically reasonable velocity fields $v$ apart from (2) that yield

$$\frac{\partial |\psi|^2}{\partial t} + \nabla \cdot (v |\psi|^2) = 0. \tag{15}$$

They point out that, quite obviously, any velocity field of the form $v = \frac{\hbar}{m} \nabla S + v_{DG}$ (my notation) with

$$v_{DG} = \frac{j_{DG}}{|\psi|^2}, \tag{16}$$

where

$$\nabla \cdot j_{DG} = 0, \tag{17}$$

also satisfies (15). They then proceed to constrain the extra velocity fields by requiring Galilei covariance, the possibility of defining an effective wave function and other physically motivated conditions. They also note that $j_{DG}$ needs to be chosen carefully, so as to ensure in addition that global uniqueness and existence are still satisfied (1998, Section 12). They show that all these constraints can be simultaneously satisfied and thus show that there are in fact infinitely many deterministic pilot-wave-like theories compatible with the condition that the particle distributions be given by $|\psi|^2$ at all times.

Deotto and Ghirardi’s question can now be modified as follows: what is the most general diffusion process (of the form (3)) that preserves the distribution $|\psi|^2$?

Take again the Fokker–Planck equation (8),

$$\frac{\partial P}{\partial t} + \nabla \cdot (bP - \nu \nabla P). \tag{18}$$

As in Carlen’s (1984) proof of global existence and uniqueness, we shall be concerned with the dynamics on the complement of the nodal set of $\psi$ (and show below that Carlen’s results generalise to our case), so that we can define

$$w := b - \nu \frac{\nabla |\psi|^2}{|\psi|^2}. \tag{19}$$
We now solve (18) for $w$ with $|\psi|^2$ substituted for $P$:

\[
\begin{align*}
\frac{\partial |\psi|^2}{\partial t} + \nabla \cdot \left( w |\psi|^2 + \nu \frac{\nabla |\psi|^2}{|\psi|^2} |\psi|^2 - \nu \nabla |\psi|^2 \right) &= 0 \\
= \frac{\partial |\psi|^2}{\partial t} + \nabla \cdot \left( w |\psi|^2 \right) &= 0,
\end{align*}
\]

(20)

so that

\[
\begin{align*}
w &= \frac{\hbar}{m} \nabla S + v_{DG},
\end{align*}
\]

(21)

as above. Thus,

\[
\begin{align*}
b &= \frac{\hbar}{m} \nabla S + v_{DG} + \nu \frac{\nabla |\psi|^2}{|\psi|^2},
\end{align*}
\]

(22)

and our desired diffusion process has Fokker–Planck equation

\[
\begin{align*}
\frac{\partial P}{\partial t} + \nabla \cdot \left( \frac{\hbar}{m} \nabla S + v_{DG} + \alpha \frac{\hbar}{2m} \nabla |\psi|^2 \right) P - \alpha \frac{\hbar}{2m} \nabla P = 0
\end{align*}
\]

(23)

(where we have written again $\nu = \alpha \frac{\hbar}{2m}$), and corresponding Itô equation

\[
\begin{align*}
dx = \left[ \frac{\hbar}{m} \nabla S + v_{DG} + \alpha \frac{\hbar}{2m} \nabla |\psi|^2 \right] dt + \sqrt{\alpha} d\omega
\end{align*}
\]

(24)

We see that for $v_{DG} = 0$ we recover Peruzzi and Rimini’s hidden configuration theories, in particular Nelson’s stochastic mechanics for $\alpha = 1$, while for $v_{DG} \neq 0$ and $\alpha = 0$ the theory collapses to Deotto and Ghirardi’s deterministic theories.

Nothing prevents us, however, from interpreting $v_{DG}$ as part of the osmotic velocity rather than the current velocity, or to split $w$ into a $v_{DG}$ and a $u_{DG}$ that separately (for reasons to become apparent) satisfy

\[
\begin{align*}
\nabla \cdot \left( v_{DG} |\psi|^2 \right) &= \nabla \cdot \left( u_{DG} |\psi|^2 \right) = 0.
\end{align*}
\]

(25)

We also see that if (24) allows for unique global solutions under the same conditions as (13) — in particular, for all $x$ not in the nodal set $N$ — then our derivation (in particular (19)) is self-consistent, in the sense that the values
of \( \mathbf{w} \) and \( \mathbf{b} \) on the nodal set are irrelevant. We shall now show that under the regularity assumptions needed for the standard proof of global existence and uniqueness in Nelson’s mechanics (Carlen, 1984), the following further condition is sufficient:

\[
\int_s^t \int |\mathbf{v}_{DG}|^2 |\psi|^2 d\mathbf{x} dt' < \infty \quad \text{and} \quad \int_s^t \int |\mathbf{u}_{DG}|^2 |\psi|^2 d\mathbf{x} dt' < \infty \tag{26}
\]

for any finite time interval \([s, t]\) (cf. Carlen, 1984, p. 298; Nelson, 1985, p. 57). Deotto and Ghirardi’s example of a \( \mathbf{v}_{DG} \) for which in the deterministic case global existence and uniqueness hold (1998, Section 12) satisfies (26), so that we can use it to construct an explicit example also for the stochastic case.

### 4 Existence and Uniqueness of Solutions

Carlen (1984) shows that the Fokker–Planck equations (8) and (12), with \( \nu = \frac{\hbar}{2m} \), allow for unique global solutions if the following two conditions hold:

\[
\int_s^t \int (|\mathbf{u}|^2 + |\mathbf{v}|^2) \rho d\mathbf{x} dt' < \infty, \tag{27}
\]

and, for any bounded \( f \) with bounded continuous first derivatives, \( \int f(\mathbf{x}) \rho d\mathbf{x} \) is differentiable for almost all \( t \) and

\[
\frac{\partial}{\partial t} \int f(\mathbf{x}) \rho d\mathbf{x} = \int (\nabla f) \cdot \mathbf{v} \rho d\mathbf{x} \tag{28}
\]

for almost all \( t \). Here, \( \rho = |\psi|^2 \), \( \mathbf{b} = \mathbf{v} + \mathbf{u} \), \( \mathbf{b}_* = \mathbf{v} - \mathbf{u} \). (28) yields an interpretation of \( \mathbf{v} \) as the current velocity, because it is a weakened form of the continuity equation

\[
\frac{\partial \rho}{\partial t} + \nabla \cdot (\mathbf{v} \rho) = 0, \tag{29}
\]

obtained using \( f \) as a test function and integrating by parts. With \( \mathbf{u} = \frac{\hbar}{2m} \nabla \rho \) and \( \mathbf{v} = \frac{\hbar}{m} \nabla S \) and under the appropriate regularity assumptions for \( \psi \), (24) and (28) are shown to hold. (And, as remarked by Peruzzi and Rimini (1996), the generalisation to arbitrary \( \nu \) is straightforward.)
We thus need to show that (27) and (28) are satisfied also when we take
\( u = \nu \frac{\nabla \rho}{\rho} + u_{DG} \) and \( v = \frac{\hbar}{m} \nabla S + v_{DG} \).

Indeed, (28) is obviously satisfied, since by partial integration,
\[
\int (\nabla f) \cdot v \rho \, dx = - \int f \nabla \cdot (v \rho) \, dx = - \int f \left[ \nabla \cdot \left( \frac{\hbar}{m} \nabla S \rho \right) + \nabla \cdot (v_{DG} \rho) \right] \, dx,
\]
and the second term in the integrand vanishes because of (25). Further, (28) is satisfied, because
\[
\int_s^t \int (|u|^2 + |v|^2) \rho \, dx \, dt' = \int_s^t \int \left[ \left( \nu \frac{\nabla \rho}{\rho} \right)^2 + \left( \frac{\hbar}{m} \nabla S \right)^2 \right] \rho \, dx \, dt' +
\]
\[
+ 2 \int_s^t \int \frac{\nabla \rho}{\rho} \cdot u_{DG} \rho \, dx \, dt' +
\]
\[
+ 2 \int_s^t \int \frac{\hbar}{m} \nabla S \cdot v_{DG} \rho \, dx \, dt' +
\]
\[
+ \int_s^t \int |u_{DG}|^2 \rho \, dx \, dt' +
\]
\[
+ \int_s^t \int |v_{DG}|^2 \rho \, dx \, dt', \quad (31)
\]
where the first integral is finite if Carlen’s regularity conditions are satisfied, the second and third are shown to vanish by partial integration using (25) (notice that \( \frac{\nabla \rho}{\rho} = \nabla \log \rho \)), and the last two are finite by assumption (26). This establishes the desired result.

5 BBB Theories

BBB theories are the discrete and stochastic analogue of de Broglie–Bohm theory. That is, they are theories in which the state of the system is given by some eigenprojection \( P_i \) of some observable \( R \), representing possession of the \( i \)th eigenvalue. Thus, \( R \) is a beable rather than an observable, in
Bell’s (1987, *passim*) terminology. Beable theories have been championed by Sudbery (1986, 1987) and have been proposed as a general framework for interpreting quantum mechanics in Bub (1997), which is the first systematic exposition of beable theories in book form.

When Bell (1984) first proposed such a theory, he also set out to construct a *dynamics* — which due to the discreteness of the beable had to be stochastic — that would preserve for all times the ‘quantum equilibrium’ distribution for the values of $R$:

$$p_i(t) = \langle \psi(t)|P_i|\psi(t) \rangle.$$  

(Bell’s problem was thus analogous to the question posed by Deotto and Ghirardi (1998), and which we have taken up in the previous section, of constructing a (most general) dynamics of a certain form that respects quantum equilibrium. Bell’s (1984) treatment of dynamics was further elaborated by Vink (1993) and generalised to the case of time-dependent beables $R(t)$ by Bacciagaluppi and Dickson (1997) and by Sudbery (s.d.).)

I here follow Bacciagaluppi and Dickson (1997).

If one considers a closed system, the evolution may be supposed to be Markovian (a condition which will then generally be violated by the induced evolution of subsystems), and the corresponding process can be canonically reconstructed from its finite-time *transition probabilities*. Under the appropriate conditions these can be recovered in turn from the *infinitesimal* transition probabilities (or *infinitesimal parameters*) of the process.

If $p_{ji}(t, s)$ is the transition probability from state $i$ at time $s$ to state $j$ at $t$

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3Bell’s (1984) original proposal envisaged a field quantity as beable (see also Sudbery, 1987). A dissenting opinion — one which I do not endorse — as to the adequacy of such a choice *vis-à-vis* the measurement problem has been recently put forward by Saunders (s.d.).

4This generalisation makes it applicable to the so-called modal interpretation of quantum mechanics. Sudbery (s.d.) now prefers this interpretation to an interpretation with a time-independent beable. See Dieks and Vermaas (1998) for a state-of-the-art collection of papers on the modal interpretation, including discussions of dynamics, of the problems of Lorentz invariance and, most importantly, of empirical adequacy; for another recent overview see Bacciagaluppi (s.d.).
time \( t \) (where \( t > s \)), we have:

\[
p_j(t) - p_j(s) = \sum_i p_{ji}(t, s)p_i(s) - p_{ij}(t, s)p_j(s),
\]

(33)

and under certain assumptions, say, that the transition probabilities be partially differentiable with respect to \( t \), we also have the following master equation:

\[
\dot{p}_j(t) = \sum_i t_{ji}(t)p_i(t) - t_{ij}(t)p_j(t),
\]

(34)

where the infinitesimal parameters \( t_{ji}(t) \) are in fact the partial derivatives \( \frac{\partial}{\partial t_1} p_{ji}(t_1, t_2)|_{t_1, t_2=t} \).

Now we can write (34) in analogy to (4) as a continuity equation for \( p_j(t) \):

\[
\dot{p}_j = \sum_i j_{ji},
\]

(35)

where we have defined

\[
j_{ji} := t_{ji}p_i - t_{ij}p_j.
\]

(36)

Thus, given that \( \dot{p}_j \) is known (by the Schrödinger equation and (32)), we can solve the linear system of equations (35) for \( j_{ji} \) and then (36) for \( t_{ji} \), and from the \( t_{ji} \) construct the stochastic process.

One has that

\[
\dot{p}_j(t) = 2\text{Im}\left[\langle \psi(t)|P_jHP_j|\psi(t)\rangle\right],
\]

(37)

and the general solution of (35) is

\[
j_{ji}(t) = 2\text{Im}\left[\langle \psi(t)|P_jHP_i|\psi(t)\rangle\right] +
\]

\[
+\langle \psi(t)|\left[\dot{P}_j(t)P_i(t) - \dot{P}_i(t)P_j(t)\right]|\psi(t)\rangle +
\]

\[
+J_{ji}^{\text{DG}}(t),
\]

(38)

where the first term is Bell’s choice of current, the second term is needed for time-dependent \( R(t) \), and the third term is a ‘Deotto–Ghirardi’ current satisfying

\[
\sum_i j_{ji}^{\text{DG}}(t) = 0.
\]

(39)
Once one chooses a current, one has a further freedom in choosing the
infinitesimal parameters $t_{ji}(t)$ in the solution of (36). Bell’s (1984) own choice
was

$$t_{ji} := \begin{cases} 
\frac{\dot{j}_{ji}}{p_i} & \text{for } j_{ji} > 0, \\
0 & \text{for } j_{ji} \leq 0.
\end{cases} \quad (40)$$

Any other solution takes the form:

$$t_{ji} \geq \frac{\dot{j}_{ji}}{p_i} \quad \text{for } j_{ji} > 0, \quad (41)$$

$$t_{ji} = \frac{t_{ij}p_j - \dot{j}_{ij}}{p_i} \quad \text{for } j_{ji} < 0, \quad (42)$$

and

$$t_{ji} = 0 \quad \text{for } j_{ji} = 0. \quad (43)$$

As usual, the dynamics becomes singular whenever $p_i(t) = 0$ and the
standard existence and uniqueness theorem (Feller, 1940) does not apply.
But again one can show that under the appropriate conditions the states
with zero quantum probability cannot be reached. A full proof will be given
elsewhere, but see Bacciagaluppi (s.d., Chapter 4) for a sketch.

Both Sudbery (1987) and Vink (1993) have addressed the question of the
continuous limit of BBB theories. Using appropriate limiting procedures one
indeed recovers both de Broglie–Bohm theory and, as Vink shows, Nelson’s
mechanics (Sudbery remarks that there are seemingly sensible limiting pro-
cedures for which this is not true).

I wish to emphasise that such a dynamics has applications beyond the
interpretational framework of beable theories. In particular, as mentioned
by Bacciagaluppi and Dickson (1997), it can be applied to the evolution of the
decohering variables in any discrete model of decoherence or whenever
else one considers effective or strict discrete superselection rules.
6 Convergence to Equilibrium and Experimental Constraints

In this brief note, I have reviewed the derivations of dynamics compatible with the assumption of the quantum equilibrium distribution, in the contexts of deterministic and stochastic configuration-space theories and of beable theories. In de Broglie-Bohm theory this assumption is famously known as the distribution postulate, and a vigorous debate has raged over the need and means of its justification (Berndl, Daumer, Dürr, Goldstein, and Zanghì, 1995; Dürr, Goldstein, and Zanghì, 1992; Valentini, 1991, 1996; see also Barrett, 1995). One particular strategy has been that of modifying the de Broglie-Bohm theory to include, effectively or fundamentally, a stochastic element à la Nelson (Bohm, 1953; Bohm and Vigier, 1954; Bohm and Hiley, 1989). In fact, in a theory with stochastic dynamics, it makes perfect sense to say that the epistemic distribution over the states of the system changes in time, and in fact may approach asymptotically a distribution independent of any initial distribution (such distributions are known as ergodic distributions in the theory of Markov chains, especially homogeneous ones; see e.g. Fisz (1963, p. 256)). Whether or not this asymptotic behaviour is achieved depends crucially on the ‘mixing’ properties of the dynamics, as is very clear in Bohm and Vigier (1954, p. 211), who base their (at best unrigorous) derivation precisely on this assumption. Such asymptotic behaviour has been rigorously shown to hold in Nelson’s stochastic mechanics in the case of the one-dimensional harmonic oscillator (Cufaro Petroni and Guerra, 1995), and more generally for any one-dimensional system in a bound state of a Hamiltonian of the form

\[ H = -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} + V(x), \]  

(44)

with time-independent \( V(x) \) (Cufaro Petroni, De Martino and De Siena, s.d.). (There are systems for which it provably does not hold, for instance the free particle (Cufaro Petroni and Guerra, 1995).) A detailed discussion of the distribution postulate in the BBB theories will be given in Bacciagaluppi and Barrett (s.d.). Here I wish to remark the following.

In Nelson’s stochastic mechanics, as we have seen, the drift velocity \( b \) is
made up of a current velocity, which is equal to the de Broglie–Bohm velocity, and of an osmotic velocity of the form

$$u = \frac{\hbar}{2m} \nabla |\psi|^2.$$. \( (45) \)

The form of \( u \) makes it intuitively plausible that particles will be, indeed, driven away from regions where \(|\psi|^2\) is small, in a way as to achieve (asymptotic) convergence to the quantum equilibrium distribution (cf. Bohm and Hiley, 1989, p. 103).

While this is obviously a desirable property of the dynamics, one might be suspicious of the fact that postulating an osmotic velocity of the form \( (45) \) is in fact an ad hoc manoeuvre designed to favour precisely the convergence to \(|\psi|^2\). This worry now seems unjustified. In fact, we have shown that up to an arbitrary Deotto–Ghirardi term the form of \( u \) follows already from the weaker requirement that the diffusion process preserve the quantum equilibrium distribution \(|\psi|^2\), and any additional velocity fields would presumably further enhance any mixing properties of the dynamics.

From this point of view, the difference between the quantum and classical level lies in the different ability of the dynamics to mix at different scales (see the remarks in Bacciagaluppi, s.d., Chapters 4 and 5). While on the subquantum level one requires from the dynamics that it satisfy assumptions of some ergodic theorem, mixing behaviour must be confined to within the support of the effective wave function, so as to allow for experimental records to be permanent — or for Schrödinger’s cat to rest in peace. This is very welcome, since it means that there are situations in which one need not worry about proving the ‘metric indecomposability’ of the configuration space under the (time-dependent) quantum equilibrium measure: on the contrary, decomposability is necessary in order to recover empirical predictions.

This brings me to my final point, namely to the idea that there are experimental constraints on the choice of Deotto–Ghirardi currents, whether in the deterministic or stochastic configuration-space setting or in the beable setting. As in the case of the observed stability of experimental records, any such constraints must come from the consideration of individual systems (since any dynamics of the kind considered is by construction compatible with
the observable statistics of ensembles), the prime example of such systems being ions in traps.

In particular, Sudbery (s.d.) has shown that Bell’s choice of current and infinitesimal parameters correctly describes spontaneous decay and correctly prevents spontaneous excitation, and — more strikingly — correctly predicts the statistics of bright and dark periods of the ‘quantum telegraph’, or intermittent fluorescence phenomenon (Dehmelt, 1975; see Plenio and Knight (1998) for a recent review). Experiments such as the quantum telegraph clearly put experimental constraints on the (directly exhibited) dynamics of the ‘hidden’ quantities — in fact, going beyond the quantum mechanical predictions for ensembles, they represent novel test cases for any approach to the foundations of quantum mechanics.

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