Transport in strongly coupled quark matter

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Deconfined quark matter may be present in the cores of massive neutron stars, and is in addition expected to be produced in binary neutron star mergers. While the bulk thermodynamic properties of the medium may undergo only moderate changes at the transition density, transport properties are expected to be qualitatively different in the nuclear and quark matter phases, possibly leading to observable effects. Motivated by recent advances in neutron star observations, we perform the first-ever systematic study of the bulk and shear viscosities as well as the thermal and electrical conductivities in dense unpaired quark matter using holography. Working in the bottom-up V-QCD model and comparing our results to the predictions of a top-down D3-D7 system, we arrive at results that are in qualitative disagreement with the available leading-order predictions from perturbative QCD. Our findings highlight the differing transport properties of weakly and strongly interacting systems, and call for caution in the use of the perturbative results in neutron-star applications.

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INTRODUCTION

The first recorded observations of binary neutron star (NS) mergers, including both gravitational wave (GW) and electromagnetic (EM) signals, have opened intriguing new avenues for the study of strongly interacting matter at ultrahigh densities. While most of the attention in the field has so far been directed to the macroscopic properties of the stars and correspondingly to the bulk thermodynamics of NS matter (see e.g. and references therein), there exists ample motivation to inspect also the transport properties of dense QCD matter. This is in particular due to the fact that while the Equations of State (EoS) of nuclear and quark matter may largely resemble each other near the transition point, transport properties are expected to react much more dramatically to possible phase transitions in the system, potentially enabling a direct detection of quark matter (QM) either in quiescent NSs or their binary mergers. In addition, understanding the relative magnitudes of different transport coefficients may turn out useful for the relativistic hydrodynamic simulations of NS mergers.

In the QM phase, expected to be found in the cores of massive NSs and created in NS mergers, very few robust results exist for even the most central transport coefficients, including the bulk and shear viscosities and the electrical and heat conductivities. In fact, the only first-principles determination of these quantities dates back to the early 1990s, amounting to a leading-order perturbative QCD (pQCD) calculation in the unpaired phase of the theory. Owing to the strongly coupled nature of QM in the density regime relevant for NSs, these results are, however, expected to have limited predictive value. A nonperturbative analysis would therefore be required for robust predictions, but the Sign Problem of lattice QCD unfortunately prohibits the use of this standard tool at large densities.

Another complication in the determination of transport quantities in QM — related to the lack of nonperturbative field theory tools at finite density — has to do with the fact that the physical phase of QCD realized at moderate densities is at present unknown. While the general expectation is that some type of pairing is likely present all the way down to the deconfinement transition at low temperatures, it is currently unclear, which particular phases are realized in Nature. Assuming that the physical moderate-density phase contains at least some nonzero fraction of unpaired quarks, it is, however, often considered a reasonable approximation to first inspect transport coefficients in the somewhat
In the paper at hand, we approach the most important transport properties of dense unpaired QM with the only first-principles machinery capable of describing strongly coupled quantum field theories at high baryon density: the gauge/gravity duality or, in short, holography (see e.g. \cite{17 21 22} for reviews). In the context of heavy-ion physics, many qualitative and even quantitative insights have been gained through the study of questions that are difficult to address with traditional field theory machinery, such as the details of equilibration dynamics (see e.g. the recent review \cite{20}). In addition, the conjectured lower limit of the shear-viscosity-to-entropy ratio \cite{24}, \( \eta/s \geq 1/(4\pi) \), has hinted towards universality in strongly coupled systems, which has had a profound effect on many subfields of theoretical physics.

In the context of NS physics, promising progress has been achieved in applying holographic methods to the bulk thermodynamic quantities and phase structure of the theory \cite{25 29}, but no attempts have been made yet to analyze transport in strongly coupled dense QM. Here, we shall take the first steps in this direction by utilizing the most highly-developed bottom-up framework designed to mimic a gravity dual of QCD \cite{30}, the Veneziano limit of the Improved Holographic QCD (IHQCD) model, V-QCD \cite{31 33}. Interestingly, the results obtained for the most important transport coefficients of dense QM are in qualitative disagreement with known perturbative predictions, calling for caution in the application of the latter results to phenomenological studies of NSs.

For comparison and completeness, we shall contrast our V-QCD results not only to pQCD but also to the most widely studied top-down holographic model describing deconfined fermionic matter, i.e. the D3-D7 system in the quenched approximation \cite{34 35}. The latter provides reliable results for quantities that can be derived from the free energy \cite{36}, which in the present case translates to the shear viscosity, obtainable via the relation \( \eta/s = 1/(4\pi) \). Other physical quantities are on the other hand expected to receive corrections from the backreaction of fermionic matter — an effect we shall indeed witness in our results.

**SETUP**

As explained above, we approach the description of dense strongly coupled matter via the string-theory inspired V-QCD model \cite{31 33}. The way we have set up our analysis is, however, more general, and can accommodate other holographic models as well, such as the probe-brane limit of the D3-D7 system \cite{35 37}. Both setups have been thoroughly applied to the study of the bulk properties of NS matter, and we refer the interested reader to Refs. \cite{13 14 23 25 33 35}.

A significant difference between two holographic models lies in how the effects of the flavor sector are treated. V-QCD has these effects systematically built in, whereas the probe D3-D7 system works only at leading order in the ratio between the numbers of flavors and colors, \( N_f/N_c \) (note, however, that for massless quarks backreaction has been considered in \cite{39 43}). Despite this difference, we can define the two models via the same bulk action, consisting of two terms \( S_{\text{total}} = S_g + S_t \), where

\[
S_g = N_f^2 M_{\text{Pl}}^3 \int d^5 x \sqrt{-g} \left( R - \frac{1}{2} \partial_{\mu} \phi \partial^{\mu} \phi - V(\phi) \right) \tag{1}
\]

\[
S_t = -N_f N_c M_{\text{Pl}}^3 \int d^5 x Z(\phi, \chi) \sqrt{-\det (\Gamma_{\mu\nu})} \ . \tag{2}
\]

Here, \( M_{\text{Pl}} \) denotes the rescaled five-dimensional Planck mass, \( R \) is the Ricci scalar, and we have defined

\[
\Gamma_{\mu\nu} = g_{\mu\nu} + \kappa(\phi, \chi) \partial_{\mu} \chi \partial_{\nu} \chi + W(\phi, \chi) F_{\mu\nu} \ . \tag{3}
\]

Of the two parts of the action, \( S_g \) is related to the glue sector of a gauge theory with rank \( N_c \). The scalar field \( \phi \) is identified with the dilaton, which according to the holographic dictionary is dual to the Yang–Mills Lagrangian and its coupling constant. In the glue sector of V-QCD \cite{43 44}, the potential is chosen to reproduce the known physics of Yang-Mills theory upon comparison with perturbative results and lattice data \cite{46 47}. In the D3-D7 model, \( S_g \) on the other hand descends from the closed string sector of type IIB supergravity by reducing on a five-dimensional compact manifold with the corresponding potential \( V(\phi) \) originating from higher-dimensional fluxes upon dimensional reduction. To fix units, we demand that the asymptotically \( AdS \) space has unit radius.

At the same time, the physics of \( N_f \) flavors of fundamental quarks is captured by the DBI action \( S_t \), where the tachyon field \( \chi \) is dual to the chiral condensate \( \bar{q} q \), whose boundary value is related to the masses of the quarks \cite{45}. We set the quark masses to zero in V-QCD, but keep them non-vanishing in the D3-D7 model to achieve the breaking of conformal symmetry. In the D3-D7 model, a quark mass corresponds to the energy necessary to introduce an additional quark over the ground state, implying that it corresponds to a constituent quark mass, with a value of the order of \( 1/N_c \) times the baryon mass \cite{25}. The field strength \( F_{\mu\nu} = \partial_{\mu} A_{\nu} - \partial_{\nu} A_{\mu} \) on the other hand provides the dynamics for the U(1)_{B} gauge field corresponding to the conserved baryonic charge of the dual field theory.

In the V-QCD model, the functions \( Z, \kappa, \) and \( W \) are fixed to reproduce the desired features of QCD both in the UV \cite{31 44} and IR \cite{23 33 45 49 51}, and to match with lattice data for thermodynamic quantities at small chemical potentials \cite{38}. In this way, the model in effect extrapolates the lattice results to the regime relevant for neutron star cores. We also determine \( M_{\text{Pl}} \) by using lattice data and set \( N_f/N_c = 1 \), and furthermore employ the
couplings of the fits 5b, 7a, and 8b given in Appendix A of [33].

For the D3-D7 model, supergravity on the other hand implies the simple relations

\[ M_p^2 = \frac{1}{8\pi^2}, \quad Z = \frac{\lambda_{YM}}{2\pi^2} \cos^3 \chi, \quad W = \frac{2\pi}{\sqrt{\lambda_{YM}}}, \quad k = 1, \tag{4} \]

where \( \lambda_{YM} = 4\pi\alpha_s N_c \) is the 't Hooft coupling of the dual field theory. Following [25], we fix \( \lambda_{YM} \approx 10.74 \) so that the pressure matches the Stefan-Boltzmann value at asymptotically large chemical potentials.

For the metric, the Ansatz corresponding to a homogeneous, rotationally-invariant background reads

\[ ds^2 = g_{tt}(r)dt^2 + g_{xx}(r)dx^2 + g_{rr}(r)dr^2. \tag{5} \]

In addition, we use the fact that the gauge potential and scalar fields are only functions of the holographic radial coordinate, \( A_i = A_i(r) \), \( \phi = \phi(r) \), \( \chi = \chi(r) \). Owing to the fact that we are interested in the deconfined phase of the field theory, there is a black hole of radius \( r_H \) in the interior, with \( r_D \) determined by the condition \( g_{tt}(r_H) = 0 \). This will be utilized in the following, when we write down closed formulas for the transport coefficients in terms of the potentials and fields evaluated at \( r = r_H \), with subscript \( H \) referring to this evaluation.

For future use, we also recall generic expressions for the temperature \( T \), entropy density \( s \), and charge density \( \rho \):

\[ 4\pi T = \left[ \frac{d}{dr} \left( -\frac{g_{tt}(r)}{g_{rr}(r)} \right) \right]_{r=r_H}, \quad s = 4\pi N_c^2 M_p^2 (g_{xx}(r_H))_1^{3/2}, \]

\[ 4\pi \rho = -\frac{N_f}{N_c} Z H W^2 F^H \sqrt{1 - \frac{F^H}{H^2 (F^H)^2}}. \tag{6} \]

In contrast, the quark chemical potential \( \mu \equiv \mu_B/N_f \) is determined by the boundary value of the gauge potential, \( \mu = A_i(\infty) \), given the regularity condition \( A_i(r_H) = 0 \).

The thermodynamic energy density \( \varepsilon \) and the pressure \( p \) can finally be derived from the thermodynamic relations \( \varepsilon + p = Ts + \mu \rho \) and \( \partial_T p = s \). Alternatively, \( p \) may also be computed by evaluating the action of the holographic model in the on-shell limit. To solve the metric and thermodynamics in the V-QCD setup, we used the Mathematica package available at [52].

Finally, to obtain numerical results, we use the values \( \Lambda_{UV} = 226.24, 210.76, \) and \( 156.68 \) MeV for the fits 5b, 7a, and 8b in the V-QCD model, following the choices made in [13, 33]. The quark mass in the D3-D7 model is on the other hand given two values — \( M_q = 210.76 \) MeV for direct comparison with the V-QCD potential 7a [33], and \( M_q = 308.55 \) MeV following the logic of ref. [24] — which are used to (partially) probe the systematic uncertainties of this model. In the D3-D7 system, the quark mass is determined from the asymptotic expansion of the scalar [54]:

\[ M_q/T = \lambda_{YM}^{1/2}/2 \lim_{r \to \infty} r \chi(r)/r_H. \]

An important point to note is that in both models, we consistently work with three mass-degenerate quark flavors assuming beta equilibrium, whereby all quark flavors share the same chemical potential \( \mu = \mu_B/3 \). It is important to note that this implies the presence of no electrons in the system, which would only change upon taking flavor-dependent masses into account.

**VISCOSITIES**

The shear and bulk viscosities of dense QCD matter describe the resistance of the system to deformations, and become relevant in settings where NSs are either strongly deformed or their interiors taken out of thermal equilibrium, both of which occur in different stages of binary NS mergers [10, 55]. In addition, viscosities play a role in determining the damping of unstable r-modes in rapidly rotating stars [50, 61].

Viscosities appear in contributions to stress forces due to an inhomogeneous motion of the fluid. Letting \( v_i \), \( i = 1, 2, 3 \), be the components of the velocity of a fluid moving at low velocities, the resulting stress becomes

\[ T_{ij} = -2\eta \partial_i v_j - \left( \frac{\zeta - 2\eta}{3} \right) \delta_{ij} \partial k v^k, \tag{7} \]

where \( \eta \) and \( \zeta \) are the shear and bulk viscosities, respectively.

In many holographic models, the shear viscosity saturates the KSS bound [24], \( \eta/s = 1/(4\pi) \). This is in particular the case for both of our holographic models V-QCD and D3-D7, which allows us to evaluate the quantity with ease. The resulting expressions are plotted in Fig. 1 (left) together with the pQCD result for unpaired quark matter [21, 15],

\[ \eta \approx 4.4 \times 10^{-3} \frac{\mu_B m_D^{2/3}}{\alpha_{QCD}^2 T^{5/3}}. \tag{8} \]

In evaluating this expression, we have used one-loop results for both the Debye mass \( m_D^2 = 2\alpha_s(N_f \mu^2 + 2(N_c + N_f)\pi^2 T^2/3))/\pi \) (see e.g. [62]) and the strong coupling \( \alpha_s^{-1} = 22N_c - 4N_f/12\pi \log \left( -\frac{(3\pi T)^2 + (2\mu)^2}{\Lambda_{QCD}} \right) \). Here, we set \( N_c = N_f = 3, \Lambda_{QCD} = 176 \) MeV, and vary \( x \) — a number parametrizing the renormalization scale dependence of the pQCD result — between 1/2 and 2 [63].

We observe that at high temperatures the shear viscosity of the strongly coupled fluid is qualitatively larger than the perturbative result, while at low temperatures it approaches a constant with the crossing of the holographic and pQCD results taking place around \( T \sim 25 - 50 \) MeV. We also note that in agreement with our naive expectation, both holographic models give comparable results, as the D3-D7 calculation is not hampered by problems related to backreaction in this case.
In addition, we have determined the QCD contribution to the bulk viscosity $\zeta$, which however represents only a subdominant contribution to r-mode damping \cite{64}. Here, we use the Eling-Oz formula \cite{65},

$$\frac{\zeta}{\eta} = \left( s \frac{\partial H}{\partial s} + \rho \frac{\partial H}{\partial \rho} \right)^2 + \frac{N_f}{N_c} c_H \left( s \frac{\partial \chi_H}{\partial s} + \rho \frac{\partial \chi_H}{\partial \rho} \right)^2 ,$$

(9)

where $c_H = \kappa_H Z_H / \sqrt{1 - W^2(F_H^r)^2}$. The corresponding results are shown in Fig. 1 (right), where we observe that in V-QCD $\zeta$ is highly suppressed in comparison with $\eta$ at low temperatures, but reaches a value of around 10% of the shear viscosity at high temperatures. The D3-D7 model on the other hand leads to a very flat curve in the range of temperatures studied, but the validity of this model is clearly questionable due to the lack of fermionic backreaction. In leading-order pQCD, the bulk viscosity finally vanishes when the quark masses are negligible compared to the chemical potentials, and remains small even at high temperatures \cite{66} (note however that non-perturbative effects are expected to increase the value somewhat \cite{67, 68}), which explains the lack of a pQCD curve in this figure.

**CONDUCTIVITIES**

A newly formed NS undergoes a cooling process mainly through the emission of neutrinos from its interior. The neutrinos transport heat to the surface, where the energy is emitted as radiation. A closer inspection of this process shows that the thermal evolution of the star depends on several quantities, including the heat conductivity that determines the heat flux to the surface and the electrical conductivity that determines the magnitude of Joule heating through the decay of magnetic fields \cite{69}–\cite{71}. In the postmerger phase of a NS binary merger, the electrical and thermal conductivities are furthermore relevant for equilibration and the evolution of magnetic fields \cite{10}–\cite{72}. Finally, these quantities may in principle prove useful in distinguishing between different phases of QCD through the observation of thermal radiation.

In the strongly coupled theories that we study in the present work, matter resides in a state that can be described as a relativistic fluid. This implies that the electrical and thermal conductivities are not independent, but are determined by a single coefficient $\sigma$, defined by the constitutive relation of the current arising from a gradient of the chemical potential,

$$J_x = -\sigma \partial_x \left( \frac{\mu}{T} \right) .$$

(10)

The electrical conductivity, defined as the ratio between the current and the electric field $E_x$, can be shown to take the form (see Appendix A)

$$\sigma^{xx} = \frac{J^x}{E_x} = \frac{\varepsilon + p}{T s} \sigma ,$$

(11)

while the thermal conductivity, defined as the ratio between the heat current $Q^x$ and the temperature gradient, becomes

$$\kappa^{xx} = \frac{Q^x}{-\partial_x T} = \frac{\mu \varepsilon + p}{T \rho} \sigma = \frac{\mu s}{\rho} \sigma^{xx} .$$

(12)

It should be stressed that these expressions hold only for a steady state, where the gradients of the temperature and chemical potential balance each other, so that the transport does not occur via convection.

With the above results established, we see that it suffices to compute the electrical conductivity in the holographic models, which can be done by extending the methods of \cite{73}–\cite{74} to a generic DBI action, such as
Our main results are depicted in Figs. 1 and 2. Inspecting these plots, an issue that stands out immediately is the stark contrast between the V-QCD and pQCD predictions for all quantities, for which both results are available: in V-QCD, the transport coefficients typically increase with temperature, being strongly suppressed in the $T = 0$ limit, while a qualitatively different behavior is predicted by pQCD. In addition, while we observe a good agreement between the V-QCD and D3-D7 predictions for the shear viscosity, the same is not true for the other three quantities studied. This discrepancy can be interpreted to reflect the importance fermionic backreaction, missing from the D3-D7 calculation.

Our results for the two conductivities are displayed in Fig. 2 together with the perturbative results for unpaired quark matter \cite{7} \cite{15}.

\begin{align}
\sigma^x x &= N_f N_c M_1^3 \frac{s T}{\varepsilon + p} \sqrt{1 - \frac{W^2_H}{F^2_{rt}}} .
\end{align}

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\begin{align}
\sigma^x x &\approx 0.01 \frac{\mu^2 m_D^2}{\alpha_s T^{5/3}} , \quad \kappa^x x \approx 0.5 \frac{m_D^2}{\alpha_s^2} .
\end{align}

Here, we continue to use the same values for $m_D$ and $\alpha_s$ as listed in the previous section, and the electrical conductivity is given in units of $e^2/(\hbar c)$.

We observe that both in the perturbative and D3-D7 calculations the conductivities either decrease with temperature or are largely independent of it, while in the V-QCD model they are increasing functions of $T$. This qualitative discrepancy can be easily understood: both in the pQCD and D3-D7 calculations, the momentum dissipation of charged particles is suppressed in an unphysical way, either through the assumption of a negligibly small coupling or by only retaining the leading order effect in an $N_f/N_c$ expansion \cite{70}. There are strong reasons to expect that properly including backreaction in this holographic model would bring the D3-D7 result into at least qualitative agreement with the V-QCD one (see e.g. the discussion in \cite{77}).

**DISCUSSION**

A first-principles microscopic determination of the fundamental properties of dense QCD matter is a notoriously difficult problem, not least due to the strongly coupled nature of the system at phenomenologically relevant energies. In recent years, the gauge/gravity duality has shown considerable promise as a potential tool, as it allows approaching the problem from an angle complementary to those of traditional field theory methods, typically applicable only at low or very high densities. Indeed, promising results have already been obtained for many bulk thermodynamic quantities in particular in the deconfined phase of the theory, leading to predictions for observables such as the neutron star mass-radius relation and the phase diagram of the theory \cite{13} \cite{25} \cite{33}.

In the paper at hand, we have used the holographic machinery to tackle a more challenging class of physical quantities characterizing the response of the QCD medium to external perturbations. In particular, we have studied the behavior of the transport coefficients most relevant for the physics of NSs and their mergers, i.e. the shear and bulk viscosities and the thermal and electrical conductivities. All these quantities have been evaluated in the most highly developed bottom-up framework mimicking a gravity dual for QCD, V-QCD, and the corresponding results subsequently compared to those from the D3-D7 probe brane setup and perturbative QCD \cite{15}.

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temperatures reflects the decreasing drag force, which is however expected to be cut off by radiation effects not captured by our analysis \cite{70,80,81}. Furthermore, at very low temperatures, the quenched approximation is expected to break down altogether \cite{40,41}. The V-QCD model on the other hand does not suffer from these issues, as flavors are unquenched, and consequently the trends exhibited by both conductivities can be expected to reflect the true behavior of these quantities in unpaired quark matter at sizable coupling strength.

For the viscosities, the situation is somewhat different. In both holographic setups, the shear viscosity is essentially a bulk thermodynamic quantity, with a result directly proportional to the entropy density, which leads to a fair agreement between the two predictions. On the other hand, in the pQCD calculation flavor contributions give rise to a strong increase of the quantity in the $T \to 0$ limit, leading to a stark disagreement with the holographic results. In contrast, for the bulk viscosity, for which no pQCD result is available, we witness a marked sensitivity of the result to the pattern in which conformal invariance is broken in our holographic models, which essentially invalidates the prediction of the D3-D7 model for this quantity.

In summary, we have seen both the perturbative and quenched approximations modify the qualitative behavior of transport coefficients at low temperatures, leading to results dramatically different from the true behavior of a strongly coupled unquenched theory, represented by the V-QCD model in our calculation. This observation clearly calls for significant caution in the application of the perturbative transport results in any phenomenological study within neutron star physics, and highlights the necessity of further developing the holographic approach to the problem.

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Appendix A. Conductivities in a relativistic fluid

In the background metric $G_{\mu\nu}$ and in the presence of a background gauge field $A_\mu$, the constitutive relations for the energy-momentum tensor and charge current of a relativistic fluid read

$$T^{\mu\nu} = (\varepsilon + p)u^\mu u^\nu + p G^{\mu\nu} + \tau^{\mu\nu}, \quad J^\mu = \rho u^\mu + \nu^\mu, \tag{15}$$

where $u^\mu$ is the fluid velocity ($u^\mu u_\mu = -1$), $p$ the pressure, $\varepsilon$ the energy density, and $\rho$ the charge density. The thermodynamic potentials depend on the temperature $T$ and chemical potential $\mu$ that will be the dynamical variables together with the velocity.

In the above energy-momentum tensor, the terms $\tau^{\mu\nu}$ and $\nu^\mu$ contain derivatives of the fields. We work in the Landau frame, where

$$u_\mu \tau^{\mu\nu} = 0, \quad u_\mu \nu^\mu = 0. \tag{16}$$

We note that in the absence of parity breaking, the most general derivative terms in the current compatible with the second law of thermodynamics read, to first order in derivatives,

$$\nu^\mu = \sigma \left( E^\mu - T P^{\mu\nu} \nabla_\nu \left( \frac{\mu}{T} \right) \right). \tag{17}$$

Here, $P^{\mu\nu}$ is the projector transverse to the velocity, and $E^\mu = F^{\mu\nu} u_\nu$ is the electric field. The derivative terms of the energy-momentum tensor will not be relevant in the following.

The dynamics of the fluid is determined by the conservation equations

$$\nabla_\mu T^{\mu\nu} = F^{\mu\lambda} J_\lambda, \quad \nabla_\mu J^\mu = 0. \tag{18}$$

In the absence of sources $G_{\mu\nu} = \eta_{\mu\nu}, A_\mu = 0$, the energy and charge densities and the pressure are constant, and the fluid is at rest. We now turn on small homogeneous time-dependent perturbations $h_{\mu\nu}, a_\mu$:

$$G_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}(t), \quad A_\mu = a_\mu(t), \tag{19}$$

which will induce a small change in the hydrodynamic variables $T, \mu$, and $u^\mu$ that can be found by solving the hydrodynamic equations to linear order in the sources. The value of the energy-momentum tensor and the current in the presence of the external sources is then obtained by inserting the solutions for the hydrodynamic variables back in the constitutive relations and expanding to linear order. For the calculation of the conductivities, we can set $h_{00} = h_{ij} = a_0 = 0$.

The explicit dependence of the currents on the sources turns out to read

$$J_i = -\frac{p^2}{\varepsilon + p} a_i - \rho h_{0i} - \sigma \partial_\mu a_\mu,$$

$$T^0_i = -\rho a_i - (\varepsilon + p) h_{0i}. \tag{20}$$
while a constant electric field and temperature gradient \( \zeta_i = -\partial_i T/T \) correspond to sources linear in time, i.e.

\[
a_i = -t(E_i - \mu \zeta_i) , \quad h_{0i} = -t \zeta_i .
\]

Introducing these expressions in Eq. (20), we readily obtain

\[
J_i = \frac{\rho}{\varepsilon + \rho} t \left((\varepsilon + p - \mu \rho) \zeta_i + \rho E_i\right) + \sigma (E_i - \mu \zeta_i) ,
\]

\[
T_i^0 = t \left[\rho E_i + (\varepsilon + p - \mu \rho) \zeta_i\right] .
\]

One can impose the condition that the fluid remains at rest, \( T_i^0 = 0 \) (no convection), by imposing the following relation between the electric field and the gradient of temperature

\[
\rho E_i + T s \zeta_i = 0 ,
\]

where we have used the thermodynamic relation \( \varepsilon + p - \mu \rho = Ts \). Physically, the forces induced by the electric and temperature gradients compensate each other, so all transport will occur through diffusion.

The charge and heat currents finally become

\[
J_i = \sigma E_i - \mu \sigma \zeta_i
\]

\[
Q_i = T_i^0 - \mu J_i = -\mu \sigma E_i + \mu^2 \sigma \zeta_i .
\]

Using Eq. (23) to solve for \( \zeta_i \) in terms of \( E_i \), or vice versa, we then obtain for the currents

\[
J_i = \sigma \frac{\varepsilon + p}{T s} E_i
\]

\[
Q_i = T_i^0 - \mu J_i = \mu \sigma \frac{\varepsilon + p}{\rho} \zeta_i ,
\]

so that the electrical and thermal conductivities read

\[
\sigma^{ij} = \frac{\varepsilon + p}{T s} \delta^{ij} , \quad \kappa^{ij} = \frac{\mu \sigma}{T} \frac{\varepsilon + p}{\rho} \delta^{ij} .
\]
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