The Amalgamated Product Structure of the Tame Automorphism Group in Dimension Three

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Abstract

It is shown the tame subgroup $TA_3(\mathbb{C})$ of the group $GA_3(\mathbb{C})$ of polynomials automorphisms of $\mathbb{C}^3$ can be realized as the product of three subgroups, amalgamated along pairwise intersections, in a manner that generalizes the well-known amalgamated free product structure of $TA_2(\mathbb{C})$ (which coincides with $GA_2(\mathbb{C})$ by Jung’s Theorem). The result follows from defining relations for $TA_3(\mathbb{C})$ given by U. U. Umirbaev.

1 Polynomial automorphism groups

For a commutative ring $R$, we write $R[\!\![X_1, \ldots, X_n]\!\!]$ for the polynomial ring in $n$ variables over $R$. We will have occasion to refer to the subalgebra $R[\!\![X_1, \ldots, X_{i-1}, X_{i+1}, \ldots, X_n]\!\!]$ for $i \in \{1, \ldots, n\}$, so we will use the shorter notation $R[\!\![X_i]\!\!]$ to denote it.

The symbol $GA_n(R)$ denotes the general automorphism group, by which we mean the automorphism group of Spec $R[\!\![n]\!\!]$ over Spec $R$. As such, it is anti-isomorphic to the group of $R$-algebra automorphisms of $R[\!\![n]\!\!]$. An element of $GA_n(R)$ is represented by a vector $\varphi = (F_1, \ldots, F_n) \in (R[\!\![n]\!\!])^n$; we will consistently use Greek letters to denote automorphisms.

The general linear group $GL_n(R)$ is contained in $GA_n(R)$ in an obvious way. Another familiar subgroup is $EA_n(R)$, the group generated by elementary automorphisms, i.e., those of the form $(X_1, \ldots, X_{i-1}, X_i + f, X_{i+1}, \ldots, X_n)$ for some $i \in \{1, \ldots, n\}$, $f \in R[X_i]$. The subgroup of tame automorphisms, denoted $TA_2(R)$, is the subgroup generated by $GL_n(R)$ and $EA_n(R)$. Other subgroups of interest are the affine group $Af_n(R)$, which is the group generated by $GL_n(R)$ together with the
translations, i.e., those automorphisms of the form \((X_1 + a_1, \ldots, X_n + a_n)\) with \(a_1, \ldots, a_n \in R\).

For \(K\) a field, \(\mathrm{GA}_n(K)\) is sometimes called the affine Cremona group. It sits naturally as a subgroup of the full Cremona group \(\mathrm{Cr}_n(K)\), which is the group of birational automorphisms of affine (or projective) \(n\)-space. The Jung-Van der Kulk Theorem ([5],[6]) states that \(\mathrm{TA}_2(K) = \mathrm{GA}_2(K)\). Shestakov and Umirbaev ([9]) showed that \(\mathrm{TA}_3(K) \neq \mathrm{GA}_3(K)\) when \(K\) has characteristic zero. This paper deals with the structure of \(\mathrm{TA}_3(K)\), when \(\text{char}(K) = 0\), based of work of Umirbaev in [11].

## 2 Amalgamated Products of Groups

We begin with the definition of an amalgamation of groups.

**Definition 2.1.** Suppose we are given groups \(A_i\) for each \(i \in \{1, \ldots, n\}\) and for each \(i, j \in \{1, \ldots, n\}\) with \(i \neq j\) we have groups \(B_{ij} = B_{ji}\) with injective homomorphisms \(\varphi_{ij} : B_{ij} \to A_i\) which are compatible, meaning if \(i, j, k\) are distinct then \(\varphi_{ij}^{-1}(\varphi_{ik}(B_{ik})) = \varphi_{ji}^{-1}(\varphi_{jk}(B_{jk}))\) and on this group \(\varphi_{ik}^{-1}\varphi_{ij} = \varphi_{jk}^{-1}\varphi_{ji}\). This gives set-theoretic gluing data by which we can compatibly glue \(A_i\) to \(A_j\) along \(B_{ij}\) via \(\varphi_{ij}^{-1}\varphi_{ji}\) forming an amalgamated union \(S\) of the sets \(A_1, \ldots, A_n\). We then form the free group \(F\) on \(S\), denoting the group operation on \(F\) by \(*\). For \(i \in \{1, \ldots, n\}\) and \(x, y \in A_i \subset S\), we let \(r_{x,y} = x*y*(xy)^{-1} \in F\)

(\text{where } xy \text{ is the product in } A_i).\) Finally we let \(G\) be the quotient of \(F\) by all the relations \(r_{x,y}\). The group \(G\) is called the amalgamated product of the groups \(A_i, i \in \{1, \ldots, n\}\) along the groups \(B_{ij}, i, j \in \{1, \ldots, n\}\). There are natural group homomorphisms \(i_i : A_i \to G\) with \(i_i \varphi_{ij} = i_j \varphi_{ji}\) on \(B_{ij}\).

The group \(G\) has the following universal property: Given a group \(H\) and maps \(\rho_i : A_i \to H\) for \(i \in \{1, \ldots, n\}\) such that \(\rho_i \varphi_{ij} = \rho_j \varphi_{ji}\) on \(B_{ij}\) for all \(i, j \in \{1, \ldots, n\}\), then there is a unique map \(\Phi : G \to H\) with \(\rho_i \iota_i\) for all \(i\).

When a group \(G\) is the amalgamation of two subgroups \(A_1\) and \(A_2\) along a common subgroup \(B\), the two groups inject into the amalgamated product and a very strong factorization theorem holds. Moreover the Bass-Serre tree theory of groups acting on trees (see [8]) provides a tree on which \(G\) acts without inversion, having a fundamental domain consisting of a single edge with its end vertices, the stabilizers of the vertices being \(A_1\) and \(A_2\) and the stabilizer of the edge the common subgroup \(B\).

Such theorems do not hold in general for amalgamations of three or more groups along pairwise intersections. The groups \(A_i\) may not map injectively into \(G\), and in fact \(G\) may be the trivial group when none of the groups \(A_i\) are trivial, as the following example from [10] shows.
Example 2.2. For $\{i, j, k\} = \{1, 2, 3\}$ let $B_{ij}$ be the infinite cyclic group generated by $b_k$. Let

$$
A_1 = \langle b_2, b_3 \mid b_2 b_3 b_2^{-1} = b_3^2 \rangle \\
A_2 = \langle b_3, b_1 \mid b_3 b_1 b_3^{-1} = b_1^2 \rangle \\
A_3 = \langle b_1, b_2 \mid b_1 b_2 b_1^{-1} = b_2^2 \rangle
$$

Then $B_{ij}$ is a common subgroup of $A_i$ and $A_j$ and we can form the amalgamation $G$ of the groups $A_i$ along the groups $B_{ij}$. It can be shown that in this case $G$ is the trivial group.

Whether such amalgamation data gives rise to the group acting on a simplicial complex is not easy to detect (see, for example, [10], [4], and [1]). It occurs precisely when each of the groups $A_{ij}$ maps injectively to $G$, and in this situation, the amalgamated union $S$ maps injectively to $G$ as well. The $n$-simplex of groups arising from this data is called developable by Haefliger ([1]) in case of this occurrence.

However, if the groups $A_i$ are subgroups of a given group $G$ and if we take $B_{ij}$ to be $A_i \cap A_j$ and $\varphi_{ij}$ the inclusion map within $G$, then clearly there exists a homomorphism $\Phi : G \to G$ restricting to the identity on each $A_i$, which shows that in this case the amalgamated union $S$ maps injectively to $G$. The map $\Phi$ will be surjective precisely when $G$ is generated by the subgroups $A_1, \ldots, A_n$. If $\Phi$ is an isomorphism, then the structure of $G$ arises from the action of $G$ on an $n$-dimensional simply connected simplicial complex, with a single simplex serving as a fundamental domain.

Automorphism groups of various kinds can be realized as amalgamations of groups. Some examples are given below.

Example 2.3. $\text{SL}_2(\mathbb{Z}) = (\mathbb{Z}/4\mathbb{Z}) \ast_{\mathbb{Z}/2\mathbb{Z}} (\mathbb{Z}/6\mathbb{Z})$ acts on the upper half plane. The generator of $\mathbb{Z}/4\mathbb{Z}$ and $\mathbb{Z}/6\mathbb{Z}$ can be taken to be $\left( \begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array} \right)$ and $\left( \begin{array}{cc} 1 & -1 \\ 1 & 0 \end{array} \right)$, respectively. Here the translates of the circular arc $z = e^{i\theta}$ with $\pi/3 \leq \theta \leq \pi/2$ form a tree with this arc as a fundamental domain, and this is the tree given by the Bass-Serre theory.

Example 2.4 (Nagao’s Theorem [7]). For $K$ a field we have $\text{GL}_2(K[X]) = \text{GL}_2(K) \ast_{B_2(K)} B_2(K[X])$, with $B_2$ denoting the lower triangular group. This structure can be realized via the Bass-Serre theory by the action of $\text{GL}_2(K[X])$ on a tree whose vertices are $\mathcal{O}$-lattices in the rank two vector space over $K(X)$. Here the fundamental domain is not just a single edge, but an edge connected to a “directed geodesic.”
Example 2.5 (Jung-Van der Kulk Theorem [5, 8]). For $K$ a field, the group $GA_2(K)$ of polynomial automorphisms of the affine plane has the structure $GA_2(K) = Af_2(K) * Bf_2(K) BA_2(K)$. Here $BA_2$ is the group of automorphisms of the form $(X_1 + \alpha, X_2 + f(X_1))$, $f(X_1) \in K[X_1]$ and $Bf_2(K) = Af_2(K) \cap BA_2(K)$. Again, this structure arises from the action of $GA_2(K)$ on a tree whose vertices are certain complete algebraic surfaces realized as collections of local rings (“models”) inside the function field $K(X_1, X_2)$ (see [12]).

Example 2.6. The full Cremona group $Cr_2(K)$ over an algebraically closed field $K$ is the amalgamation of three groups: the automorphism group of $\mathbb{P}_K^2$, which is $PGL_2(K)$, the automorphism group of $\mathbb{P}_K^1 \times \mathbb{P}_K^1$, and thirdly the $K$-automorphism group of $\mathbb{P}_L^1$ where $L = K(t)$, with $t$ transcendental over $K$. There is a naturally realizable simplicial complex of triangles $\mathcal{C}$ on which $Cr_2(K)$ acts which yields this structure and also contains the tree of Example 2.5 with the action of $GA_2(K)$ being the restriction of the action of $Cr_2(K)$ on $\mathcal{C}$. See [12] for details.

3 Polynomial Automorphisms in Dimension Three

A major breakthrough came in 2004 when Shestakov and Umirbaev showed that the automorphism group $GA_3(K)$ properly contains the tame subgroup $TA_3(K)$ when $K$ is a field of characteristic zero ([9]). Specifically they showed that the automorphism

$$(X + Z(Y Z + X^2), Y - 2X(Y Z + X^2) - X(Y Z + X^2)^2, Z)$$

is not tame, resolving a conjecture of Nagata from 1972. The group $GA_3(K)$ remains a mystery, as no describable set of generators has been given.

However, the tame subgroup $TA_3(K)$ is (by definition) generated by the elementary and linear automorphisms, which are familiar. Moreover a set of generating relations has been given by Umirbaev in [11]. This paper will show that an amalgamated product structure for $TA_3(K)$ results from Umirbaev’s relations. We begin by presenting those results.

For $\varphi = (F_1, \ldots, F_n) \in GA_n(K)$ and $f \in K^n$ we write $f(\varphi)$ for $f(F_1, \ldots, F_n)$. This defines an action (on the right) of $GA_n(K)$ on $K^n$.

For $i \in \{1, \ldots, n\}$, $\alpha \in K$, and $f \in K[\hat{i}, \hat{f}]$, consider the automorphism

$$\sigma_{i, \alpha, f} = (X_1, \ldots, X_{n-1}, \alpha X_i + f, X_{i+1}, \ldots, X_n),$$

(1)
which is easily seen to be tame. Given \( k, \ell \in \{1, \ldots, n\} \), \( k \neq \ell \), we define a tame automorphism \( \tau_{k, \ell} \) by

\[
\tau_{k, \ell} = \sigma_{\ell,1}X_k\sigma_{k,1}X_\ell\sigma_{\ell,-1}X_k
\]

(2)

A simple calculation shows that \( \tau_{k, \ell} \) is the transposition switching the \( X_k \) and \( X_\ell \) coordinates.

One can check directly that

\[
\sigma_{i,\alpha,f}\sigma_{i,\beta,g} = \sigma_{i,\alpha\beta,f} + \alpha g.
\]

(3)

Also, if \( i, j \in \{1, \ldots, n\} \), \( i \neq j \), and if \( f \in K[X, \hat{i}] \cap K[X, \hat{j}] \), \( g \in K[X, \hat{j}] \), then

\[
\sigma_{i,\alpha,f}^{-1}\sigma_{j,\beta,g}\sigma_{i,\alpha,f} = \sigma_{j\beta,g}(\sigma_{i,\alpha,f})
\]

(4)

It follows that if \( g \in K[X, \hat{i}] \cap K[X, \hat{j}] \) then \( \sigma_{i,\alpha,f} \) and \( \sigma_{i,\beta,g} \) commute.

Let \( k, \ell \in \{1, \ldots, n\} \), \( k \neq \ell \). For \( i \in \{1, \ldots, n\} \) let \( j \) be the image of \( i \) under the permutation which switches \( k \) and \( \ell \), in other words, the element of \( \{1, \ldots, n\} \) for which \( X_j = X_i(\tau_{k, \ell}) \). Then we have

\[
\tau_{k, \ell}\sigma_{i,\alpha,f}\tau_{k, \ell} = \sigma_{j,\alpha,f}(\tau_{k, \ell})
\]

(5)

Theorem 4.1 of [11] asserts the following.

**Theorem 3.1** (Umirbaev). *Let \( K \) be a field of characteristic zero. The relations (3), (4), and (5) are defining relations for \( TA_3(K) \) with respect to the generators \( \sigma_{i,\alpha,f} \) defined in (1). Here \( \tau_{k, \ell} \) in (5) is defined formally in terms of these generators by (2).*

This will be the key tool in the proof of Theorem 4.1 which is the main result of this paper.

### 4 Subgroups of Interest

For \( i \in \{1, \ldots, n\} \), let \( V_i \) be the sub-vector space of \( K[\hat{i}] \) generated by \( K \) and the variables \( X_1, \ldots, X_i \), i.e.,

\[
V_i = K \oplus KX_1 \oplus \cdots \oplus KX_i.
\]

(6)

Then \( H_i \) is defined to be the stabilizer of \( V_i \) in \( GA_n(K) \) via the action defined in (3), i.e.,

\[
H_i = \{ \varphi \in GA_n(K) \mid f(V_i) = V_i \}.
\]

(7)
Note that $H_n$ is the affine group $A_f_n(K)$. More generally, the subgroup of $H_i$ that fixes each of the variables $X_{i+1}, \ldots, X_n$ can be identified with $A_f_i(K)$. In fact, $H_i$ retracts onto $A_f_i(K)$ via the map $\varphi = (F_1, \ldots, F_n) \mapsto (F_1, \ldots, F_i)$, and the kernel of this retraction is the subgroup of $H_i$ consisting of the elements that fix each of the variables $X_1, \ldots, X_i$, which is $GA_{n-1}(K[X_1, \ldots, X_i])$. Thus $H_i$ has the semidirect product structure

$$H_i = A_f_i(K) \rtimes GA_{n-i}(K[X_1, \ldots, X_i])$$

(8)

(where, for $i = n$, we read this as $H_n = A_f_n(K)$). These subgroups are defined in [2], p. 23, where it is conjectured that together they generate $GA_n(K)$ (Conjecture 14.1) and that (whether or not that conjecture is true) the subgroup generated by $H_1, \ldots, H_n$ is the amalgamated product of these groups along pairwise intersections (Conjecture 14.2). It should be noted that Freudenburg produced an example (see [3], p. 121) of an automorphism in $GA_3(K)$ which has not been shown to lie in this subgroup.

Furthermore the groups $\widetilde{H}_i$ are defined by

$$\widetilde{H}_i = H_i \cap TA_n(K),$$

which are easily seen to generate $TA_n(K)$. We can surmise from (8) that

$$\widetilde{H}_i \supset A_f_i(K) \rtimes TA_{n-i}(K[X_1, \ldots, X_i]).$$

(9)

For $i = n$ equality holds trivially and we have $\widetilde{H}_n = H_n$, both being equal to $A_f_n(K)$. For $i = n-1$ it is also easily seen that equality holds in (9) and moreover we have $\widetilde{H}_{n-1} = H_{n-1}$ since $TA_1$ and $GA_1$ coincide over an integral domain (even a reduced ring).

There is one other case where the containment of (9) is known to be an equality. Namely, for $n = 3$ and $K$ of characteristic zero we have $\widetilde{H}_1 = A_f_1(K) \rtimes TA_2(K[X_1])$. This follows from Corollary 10 of [9], a very deep result asserting that in $GA_3(K)$ we have

$$GA_2(K[X_1]) \cap TA_3(K) = TA_2(K[X_1]).$$

This together with the known proper containment $TA_2(K[X_1]) \subsetneq GA_2(K[X_1])$ tells us that $\widetilde{H}_1 \subsetneq H_1$ for $n = 3$. It is not known whether $\widetilde{H}_1 \subsetneq H_1$ when $n > 3$.

It is conjectured that $TA_n(K)$ is the amalgamated product of the subgroups $\widetilde{H}_1, \ldots, \widetilde{H}_n$ along pairwise intersections ([2], Conjecture 14.3). The main result of this paper is that this conjecture is true for $n = 3$ and $K$ a field of characteristic zero. In light of the above observations, for $n = 3$ we have $\widetilde{H}_2 = H_2$ and $\widetilde{H}_3 = H_3$ (but not $\widetilde{H}_1 = H_1$), so this can be stated as:

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1This example is also of interest because it has not been shown to be stably tame.
Theorem 4.1. For $K$ a field of characteristic zero, $TA_3(K)$ is the amalgamated product of the three groups $\tilde{H}_1, H_2, H_3$ along their pairwise intersections.

This will be proved in the next section.

5 Proof of Theorem 4.1

The main tool in the proof is Theorem 3.1, which asserts that $TA_2(K)$ is generated by the elements $\sigma_{i,\alpha,f}$ as defined in (1) subject to the relations (3), (4), and (5).

Let $F$ be the free group generated by the formal symbols $[\sigma_{i,\alpha,f}]$, with $i \in \{1, \ldots, n\}, f \in K[X, \hat{i}]$. Accordingly, we rewrite the relations (3), (4), and (5) replacing each $\sigma$ by its corresponding formal symbol $[\sigma]$:

\[ [\sigma_{i,\alpha,f}] [\sigma_{i,\beta,g}] = [\sigma_{i,\alpha\beta,f+\alpha g}] \]  
\[ [\sigma_{i,\alpha,f}]^{-1} [\sigma_{j,\beta,g}] [\sigma_{i,\alpha,f}] = [\sigma_{j,\beta,g}(\sigma_{i,\alpha,f})] \]  
\[ [\tau_{k,l}] [\sigma_{i,\alpha,f}] [\tau_{k,l}] = [\sigma_{j,\alpha,f}(\tau_{k,l})] \]

where, in (R2), $i \neq j$, $f \in K[X, \hat{i}] \cap K[X, \hat{j}]$, $g \in K[X, \hat{j}]$, and, in (R3), $k \neq \ell$, $j$ is the image of $i$ under the permutation which switches $k$ and $\ell$, and

\[ [\tau_{k,l}] = [\sigma_{\ell,1,X_k}] [\sigma_{k,1,-X_k}] [\sigma_{\ell,1,-X_k}] \]

(after (2)). Let $N$ be the normal subgroup of $F$ generated by (R1), (R2), and (R3).

Theorem 3.1 says that the homomorphism from $F$ to $TA_3(K)$ sending $[\sigma_{i,\alpha,f}]$ to $\sigma_{i,\alpha,f}$ induces an isomorphism

\[ F/N \cong TA_3(K). \]  

Let $G$ be the amalgamated product of $\tilde{H}_1, H_2, H_3$ along their pairwise intersections. The inclusions of $\tilde{H}_1, H_2, H_3$ in $TA_3(K)$ induce a group homomorphism $\Phi : G \to TA_3(K)$ which is surjective since the three subgroups generate $TA_3(K)$ (in fact any two of them generate). We will define a group homomorphism from $TA_3(K)$ to $G$ using the isomorphism (11) and show that it is inverse to $\Phi$, thus proving the theorem.

We first define a homomorphism $\tilde{\Psi} : F \to G$, which is accomplished by specifying the images of the free generators $[\sigma_{i,\alpha,f}]$. According to the discussion in Section 2, $G$ contains the amalgamated union of $\tilde{H}_1, H_2, H_3$ as does $TA_3(K)$, with $\Phi$ restricting to the identity map on this set. Let us denote by $\tilde{\mathcal{H}}_1, \tilde{\mathcal{H}}_2, \tilde{\mathcal{H}}_3$ the isomorphic copies of $\tilde{H}_1, H_2, H_3$, respectively, that lie inside $G$. 


It is important to keep in mind that \( \tilde{H}_1 \cup \tilde{H}_2 \cup \tilde{H}_3 \) maps bijectively to \( \tilde{H}_1 \cup H_2 \cup H_3 \) via \( \Phi \).

Note that if \( i = 2 \) or \( i = 3 \) then \( \sigma_{i,\alpha,f} \) lies in \( \tilde{H}_1 \) and if \( \deg f \leq 1 \) then \( \sigma_{i,\alpha,f} \) lies in \( H_3 \), so in each of these cases \( \sigma_{i,\alpha,f} \) can be viewed as an element of the union \( \tilde{H}_1 \cup \tilde{H}_2 \cup \tilde{H}_3 \subset \mathfrak{S} \). To avoid confusion, we will denote these elements of \( \mathfrak{S} \) by \( s_{i,\alpha,f} \). Thus it makes sense to make the assignments

\[
\hat{\Psi}([\sigma_{i,\alpha,f}]) = s_{i,\alpha,f} \in \tilde{H}_1 \quad \text{for } i = 2, 3. \tag{12}
\]

\[
\hat{\Psi}([\sigma_{1,\alpha,f}]) = s_{1,\alpha,f} \in H_3 \quad \text{for } \deg f \leq 1. \tag{13}
\]

Since the factors of (10) involve only polynomials of degree \( \leq 1 \), \( \hat{\Psi}([\tau_{k,\ell}]) \) is defined by applying \( \hat{\Psi} \) to those factors using (12) and (13) above. We will denote the resulting element of \( \mathfrak{S} \) by \( t_{k,\ell} \). Thus:

\[
\hat{\Psi}([\tau_{k,\ell}]) = t_{k,\ell} = s_{\ell,1,X_k} s_{k,1,-X_\ell} s_{\ell,-1,X_k}. \tag{14}
\]

and this is the just the permutation in \( \mathfrak{S}_3 \cong \text{Alt}_3(K) \) that switches \( k \) and \( \ell \).

It remains to define \( \hat{\Psi}([\sigma_{1,\alpha,f}]) \) for arbitrary \( f \in K[X_2, X_3] \). This we do as follows:

\[
\hat{\Psi}([\sigma_{1,\alpha,f}]) = t_{1,3} s_{3,\alpha,f(X_2,X_1)} t_{1,3}. \tag{15}
\]

The reader will easily verify that this assignment coincides with (13) in the case \( \deg f \leq 1 \), since both occur in \( \mathfrak{S}_3 \).

Thus we have defined \( \hat{\Psi} : \mathcal{F} \rightarrow \mathfrak{S} \), and we must now show that the subgroup \( \mathcal{N} \) lies in the kernel of \( \hat{\Psi} \), i.e., that equations (R1), (R2), and (R3) hold replacing \( \sigma \) by \( s \) and \( \tau \) by \( t \). This gets a bit tedious because of the asymmetry in the definitions of \( \hat{\Psi}([\sigma_{i,\alpha,f}]) \) depending on \( i \).

We begin with (R1). Note that if \( i = 2 \) or \( i = 3 \), then according to (12), this amounts to showing that

\[
s_{i,\alpha,f} s_{i,\beta,g} = s_{i,\alpha\beta,f+ag} \quad \text{for } i = 2, 3. \tag{16}
\]

But this is a relation that takes place in \( \tilde{H}_1 \), so it holds in \( \mathfrak{S} \). For \( i = 1 \) we must use (15). For \( f, g \in K[X_2, X_3] \) we have

\[
\hat{\Psi}([\sigma_{1,\alpha,f}]) \hat{\Psi}([\sigma_{1,\beta,g}]) = (t_{1,3} s_{3,\alpha,f(X_2,X_1)} t_{1,3}) (t_{1,3} s_{3,\beta,g(X_2,X_1)} t_{1,3})
\]

\[
= t_{1,3} s_{3,\alpha,f(X_2,X_1)} s_{3,\beta,g(X_2,X_1)} t_{1,3}
\]

\[
(\text{since } t_{1,3}^2 = 1 \text{ in } \mathfrak{S}_3)
\]

\[
= t_{1,3} s_{3,\alpha\beta,f+ag} t_{1,3} \quad \text{by (16)}
\]

\[
= \hat{\Psi}([\sigma_{1,\alpha\beta,f+ag}]), \quad \text{by (15)}
\]
Applying (18) to $\Psi$ that has been substituted. the next line; the overbrace in the next line marks the equivalent expression permutation relation, which holds in the symmetric group $S$ holds in $t$

Using the relation $\tilde{\Psi}$, but, again, this is a relation that holds in $t$

Using the relation $t$

completing the proof that the relation (11) is respected by $\tilde{\Psi}$.

We now address (R2). If $\{i, j\} = \{2, 3\}$ we must show, again appealing to (12), that

$$g_{i,\alpha,f}^{-1} g_{j,\beta,g} g_{i,\alpha,f} = g_{j,\beta,g(\sigma_{i,\alpha,f})} \quad \text{for } i = 2, 3.$$  \hspace{1cm} (17)

But, again, this is a relation that holds in $\mathcal{H}_1$, hence in $\mathcal{G}$.

We now consider the case $i = 1, j = 3$. We will use the following basic permutation relation, which holds in the symmetric group $\mathcal{G}_3 \subset \mathcal{H}_3$ (hence it holds in $\mathcal{G}$) for $\{k, \ell, m\} = \{1, 2, 3\}$:

$$t_{k,\ell} = t_{k,m} t_{m,\ell} t_{k,m}. \hspace{1cm} (18)$$

In the equations below the underbrace indicates what will be replaced in the next line; the overbrace in the next line marks the equivalent expression that has been substituted.

For $f \in K[X_2]$ and $g \in K[X_1, X_2]$,

$$\tilde{\Psi} (\sigma_{1,\alpha,f(X_2)})^{-1} [\sigma_{3,\beta,g(X_1, X_2)} [\sigma_{1,\alpha,f(X_2)}])$$

$$= \tilde{\Psi} (\sigma_{1,\alpha,f(X_2)})^{-1} \tilde{\Psi} (\sigma_{3,\beta,g(X_1, X_2)}) \tilde{\Psi} (\sigma_{1,\alpha,f(X_2)})$$

$$= \left( t_{1,3} g_{3,\alpha,f(X_2)}^{-1} t_{1,3} \right) \left( g_{3,\beta,g(X_1, X_2)} \right) \left( t_{1,3} g_{3,\alpha,f(X_2)} t_{1,3} \right) \quad \text{by (12) and (15)}$$

Applying (18) to $t_{1,3}$:

$$= t_{1,2} t_{2,3} t_{1,2} g_{3,\alpha,f(X_2)}^{-1} t_{1,2} t_{2,3} t_{1,2} g_{3,\beta,g(X_1, X_2)} t_{1,2} t_{2,3} t_{1,2} g_{3,\alpha,f(X_2)} t_{1,2} t_{2,3} t_{1,2} t_{2,3} t_{1,2}$$

Using the relation $t_{1,2} g_{3,\alpha,f(X_2)} t_{1,2} = g_{3,\alpha,f(X_1)}$ from $\mathcal{H}_2$:

$$= t_{1,2} t_{2,3} g_{3,\alpha,f(X_1)}^{-1} t_{1,2} t_{2,3} g_{3,\beta,g(X_1, X_2)} t_{1,2} t_{2,3} g_{3,\alpha,f(X_1)} t_{2,3} t_{1,2}$$

Using the relation $t_{1,2} g_{3,\beta,g(X_1, X_2)} t_{1,2} = g_{3,\beta,g(X_2, X_1)}$ from $\mathcal{H}_2$:

$$= t_{1,2} t_{2,3} g_{3,\alpha,f(X_1)}^{-1} t_{2,3} g_{3,\beta,g(X_2, X_1)} t_{2,3} g_{3,\alpha,f(X_1)} t_{2,3} t_{1,2}$$

Using the relation $t_{2,3} g_{3,\beta,g(X_2, X_1)} t_{2,3} = g_{2,\beta,g(X_3, X_1)}$ from $\mathcal{H}_1$:

$$= t_{1,2} t_{2,3} g_{3,\alpha,f(X_1)}^{-1} g_{2,\beta,g(X_3, X_1)} g_{3,\alpha,f(X_1)} t_{2,3} t_{1,2}$$
Applying (17):

\[ t_{1,2} t_{2,3} s_{2,3,\beta,g(\alpha X_3+f(X_1),X_1)} t_{2,3} t_{1,2} \]

Using the relation \( t_{2,3} s_{2,3,\beta,g(\alpha X_3+f(X_1),X_1)} t_{2,3} = s_{3,\beta,g(\alpha X_2+f(X_1),X_1)} \) from \( \tilde{\mathcal{F}}_1 \):

\[ t_{1,2} s_{3,\beta,g(\alpha X_2+f(X_1),X_1)} t_{1,2} \]

\[ = s_{3,\beta,g(\alpha X_1+f(X_2),X_2)} \] from \( \tilde{\mathcal{F}}_2 \)

\[ = \hat{\Psi} \left( [\sigma_{3,\beta,g(\alpha X_1+f(X_2),X_2)]} \right) \]

which accomplishes our goal.

Now let \( i = 3, j = 1 \). For \( f \in K[X_2] \) and \( g \in K[X_2, X_3] \),

\[ \hat{\Psi} \left( [\sigma_{3,\alpha,f(X_2)}]^{-1} [\sigma_{1,\beta,g(X_2,X_3)}] [\sigma_{3,\alpha,f(X_2)}] \right) \]

\[ = t_{1,3} t_{1,3} s_{3,\alpha,f(X_2)}^{-1} t_{1,3} s_{3,\alpha,f(X_2)} \] by (12) and (15)

\[ = t_{1,3} t_{1,3} s_{3,\alpha,f(X_2)}^{-1} t_{1,3} s_{3,\alpha,f(X_2)} \]

Applying (18):

\[ t_{1,2} t_{2,3} t_{1,2} s_{3,\alpha,f(X_2)}^{-1} t_{2,3} t_{1,2} s_{3,\beta,g(X_2,X_1)} \]

Using the relation \( t_{1,2} s_{3,\alpha,f(X_2)} t_{1,2} = s_{3,\alpha,f(X_1)} \) in \( \tilde{\mathcal{F}}_2 \):

\[ t_{1,3} t_{1,2} s_{3,\alpha,f(X_2)}^{-1} t_{1,2} s_{3,\beta,g(X_2,X_1)} t_{1,2} t_{2,3} s_{3,\alpha,f(X_1)} t_{2,3} t_{1,2} t_{1,3} \]

Using the relation \( t_{2,3} s_{3,\alpha,f(X_1)} t_{2,3} = s_{2,\alpha,f(X_1)} \) in \( \tilde{\mathcal{F}}_1 \):

\[ t_{1,3} t_{1,2} s_{2,\alpha,f(X_1)}^{-1} t_{1,2} s_{3,\beta,g(X_2,X_1)} t_{1,2} s_{2,\alpha,f(X_1)} t_{1,2} t_{1,3} \]
Using the relation $t_{1,2} s_{3,\beta,g}(X_2, X_1) t_{1,2} = s_{3,\beta,g}(X_1, X_2)$ in $\mathfrak{H}_2$:

$$= t_{1,3} t_{1,2} s_{2,\alpha,f}(X_1)^{-1} s_{3,\beta,g}(X_1, X_2) s_{2,\alpha,f}(X_1) t_{1,2} t_{1,3}$$

Applying (17):

$$= t_{1,3} t_{1,2} s_{3,\beta,g}(X_1, \alpha X_2 + f(X_1)) t_{1,2} t_{1,3}$$

Using the relation $t_{1,2} s_{3,\beta,g}(X_1, \alpha X_2 + f(X_1)) t_{1,2} = s_{3,\beta,g}(X_2, \alpha X_1 + f(X_2))$ in $\mathfrak{H}_2$:

$$= t_{1,3} s_{3,\beta,g}(X_2, \alpha X_1 + f(X_2)) t_{1,3}$$

$$= \tilde{\Psi} \left( \left[ \sigma_{1,\beta,g}(X_2, \alpha X_1 + f(X_2)) \right] \right)$$

$$= \tilde{\Psi} \left( \left[ \sigma_{1,\beta,g}(\sigma_{3,\alpha,f}(X_2)) \right] \right) \quad \text{by (15)},$$

as desired.

The two cases $\{i, j\} = \{1, 3\}$ will employ the equality

$$t_{1,3} s_{2,\beta,g}(X_1, X_3) t_{1,3} = s_{2,\beta,g}(X_3, X_1), \quad (19)$$

which arises by conjugating the $\mathfrak{H}_2$ identity $t_{1,2} s_{3,\beta,g}(X_1, X_2) t_{1,2} = s_{3,\beta,g}(X_2, X_1)$ by $t_{2,3}$, evoking the $\mathfrak{H}_1$ identity $t_{2,3} s_{3,\beta,g}(X_1, X_2) t_{2,3} = s_{2,\beta,g}(X_1, X_3)$ and the $\mathfrak{H}_3$ identity $t_{2,3} t_{1,2} t_{2,3} = t_{1,3}$.

For $i = 1, j = 2$ we have

$$\tilde{\Psi} \left( \left[ \sigma_{1,\alpha,f}(X_3) \right]^{-1} \left[ \sigma_{2,\beta,g}(X_1, X_3) \right] \left[ \sigma_{1,\alpha,f}(X_3) \right] \right)$$

$$= \tilde{\Psi} \left( \left[ \sigma_{1,\alpha,f}(X_3) \right]^{-1} \tilde{\Psi} \left( \left[ \sigma_{2,\beta,g}(X_1, X_3) \right] \right) \tilde{\Psi} \left( \left[ \sigma_{1,\alpha,f}(X_3) \right] \right) \right)$$

$$= t_{1,3} s_{3,\alpha,f}(X_1) t_{1,3} s_{2,\beta,g}(X_1, X_3) t_{1,3} s_{3,\alpha,f}(X_1) t_{1,3} \quad \text{by (15)}$$

$$= t_{1,3} s_{3,\alpha,f}(X_1) t_{1,3} s_{2,\beta,g}(X_3, X_1) s_{3,\alpha,f}(X_1) t_{1,3} \quad \text{by (15)}$$

$$= t_{1,3} s_{2,\beta,g}(\alpha X_3 + f(X_1), X_1) t_{1,3} \quad \text{by (17)}$$

$$= s_{2,\beta,g}(\alpha X_1 + f(X_3), X_3) \quad \text{by (19)}$$

$$= s_{2,\beta,g} \left( \sigma_{1,\alpha,f}(X_3) \right)$$

$$= \tilde{\Psi} \left( \left[ \sigma_{2,\beta,g}(\sigma_{1,\alpha,f}(X_3)) \right] \right).$$

The case $i = 2, j = 1$ follows similarly:

$$\tilde{\Psi} \left( \left[ \sigma_{2,\alpha,f}(X_3) \right]^{-1} \left[ \sigma_{1,\beta,g}(X_2, X_3) \right] \left[ \sigma_{2,\alpha,f}(X_3) \right] \right)$$
= \tilde{\Psi} \left( [\sigma_{2,\alpha,f}(X_1)] \right)^{-1} \tilde{\Psi} \left( [\sigma_{1,\beta,g}(X_2,X_3)] \right) \tilde{\Psi} \left( [\sigma_{2,\alpha,f}(X_3)] \right) \\
= s_{2,\alpha,f}(X_3) \times t_{1,3} s_{3,\beta,g}(X_2,X_1) \times t_{1,3} s_{2,\alpha,f}(X_3) \quad \text{by (15)} \\
= t_{1,3} t_{1,3} s_{2,\alpha,f}(X_3) \times t_{1,3} s_{3,\beta,g}(X_2,X_1) \times t_{1,3} s_{2,\alpha,f}(X_3) t_{1,3} t_{1,3} \quad \text{since } t_{1,3}^2 = 1 \\
= t_{1,3} s_{2,\alpha,f}(X_3) \times s_{2,\alpha,f}(X_1) s_{3,\beta,g}(X_2,X_1) s_{2,\alpha,f}(X_3) t_{1,3} \quad \text{by (19)} \\
= t_{1,3} s_{3,\beta,g}(\alpha X_2+f(X_1),X_3) t_{1,3} \quad \text{by (17)} \\
= \tilde{\Psi} \left( [\sigma_{1,\beta,g}(\alpha X_2+f(X_3),X_3)] \right) \quad \text{by (15)},

completing the proof that the relation (R2) is respected by \( \tilde{\Psi} \).

Lastly we come to (R3). If \( \{k, \ell, i\} = \{2, 3\} \) then we also have \( j \in \{2, 3\} \) and we must show that \( t_{k,\ell} s_{i,\alpha,f} t_{k,\ell} = s_{j,\alpha,f}(\tau_k,\ell) \). But this relation holds in \( \tilde{\mathcal{S}}_1 \). Also, if \( i = 3 \) and \( \{k, \ell\} = \{1, 2\} \), then \( j = 3 \) and the relation holds in \( \mathcal{S}_2 \). Thus for \( i = 3 \) the only remaining case is \( \{k, \ell\} = \{1, 3\} \), which follows quickly from (15). To wit:

\[
\tilde{\Psi}(\{\tau, \omega, f(X_1,X_2)\}[\tau,\omega]) = \tilde{\Psi}(\{\tau, \omega, f(X_1,X_2)\}) \tilde{\Psi}(\{\tau, \omega\}) \\
= t_{1,3} s_{2,\alpha,f}(X_1,X_2) t_{1,3} \quad \text{by (12) and (13)} \\
= \tilde{\Psi}(\{\alpha, f(X_3)\}) \quad \text{by (15)} \\
= \tilde{\Psi}(\{\alpha, f(\tau, \omega)\}) 
\]

For \( i = 2 \) the remaining cases are \( \{k, \ell\} = \{1, 2\} \) and \( \{k, \ell\} = \{1, 3\} \). For the first:

\[
\tilde{\Psi}(\{\tau, \omega, f(X_1,X_2)\}[\tau,\omega]) = \tilde{\Psi}(\{\tau, \omega, f(X_1,X_2)\}) \tilde{\Psi}(\{\tau, \omega\}) \\
= t_{1,2} s_{2,\alpha,f}(X_1,X_3) t_{1,2} \quad \text{by (12) and (13)} 
\]

Using the relation \( s_{2,\alpha,f}(X_1,X_3) = t_{2,3} s_{3,\alpha,f}(X_1,X_2) t_{2,3} \) in \( \tilde{\mathcal{S}}_1 \):

\[
= t_{1,2} t_{2,3} s_{3,\alpha,f}(X_1,X_2) t_{2,3} t_{1,2} \\
= t_{1,3} t_{1,2} s_{3,\alpha,f}(X_1,X_2) t_{1,2} t_{1,3} \quad \text{using (18)}
\]

Using the relation \( t_{1,2} s_{3,\alpha,f}(X_1,X_2) t_{1,2} = s_{3,\alpha,f}(X_2,X_1) \) in \( \mathcal{S}_2 \):

\[
= t_{1,3} s_{3,\alpha,f}(X_2,X_1) t_{1,3}
\]
Thus we have verified all the cases when \( i = 2 \) or \( i = 3 \).

Finally we consider \( i = 1 \). If \( \{ k, \ell \} = \{ 2, 3 \} \), (13) is a consequence of (15):

\[
\hat{\Psi}([\tau_{1,3}][\sigma_{2,\alpha,f}(X_{1},X_{2})][\tau_{1,3}]) = \hat{\Psi}([\tau_{2,3}]) \hat{\Psi}([\tau_{1,3}][\sigma_{2,\alpha,f}(X_{1},X_{3})])[\tau_{1,3}]
\]

By (12) and (14)

\[
= t_{1,3} \hat{\Psi}_{2,\alpha,f}(X_{1},X_{3}) t_{1,3}
\]

Using (18)

\[
= \hat{\Psi}([\tau_{1,3}])
\]

If \( \{ k, \ell \} = \{ 1, 3 \} \) we have

\[
\hat{\Psi}([\tau_{1,3}][\sigma_{1,\alpha,f}(X_{2},X_{3})][\tau_{1,3}]) = \hat{\Psi}([\tau_{1,3}][\sigma_{1,\alpha,f}(X_{2},X_{3})])
\]

By (14) and (15)

\[
= t_{1,3} t_{1,3} \hat{\Psi}_{3,\alpha,f}(X_{2},X_{1}) t_{1,3} t_{1,3}
\]

Since \( t_{1,3}^2 = 1 \)

\[
= \hat{\Psi}([\tau_{1,3}])
\]

If \( \{ k, \ell \} = \{ 1, 2 \} \) we have

\[
\hat{\Psi}([\tau_{1,2}][\sigma_{1,\alpha,f}(X_{2},X_{3})][\tau_{1,2}]) = \hat{\Psi}([\tau_{1,2}][\sigma_{1,\alpha,f}(X_{2},X_{3})])
\]

By (14) and (15)
= \underbrace{t_1,3 \, s_{3,\alpha,f}(X_2, X_1)}_{t_2,3} \, t_1,3 \quad \text{using (18)}

Using the relation $t_{2,3} \, s_{3,\alpha,f}(X_2, X_1) \, t_{2,3} = s_{2,\alpha,f}(X_3, X_1)$ in $\tilde{\mathcal{H}}_1$:

\[
\begin{align*}
&= t_{1,3} \, s_{2,\alpha,f}(X_3, X_1) \, t_{1,3} \\
&= s_{2,\alpha,f}(X_1, X_3) \quad \text{by (19)} \\
&= \hat{\Psi} \left( [\sigma_{2,\alpha,f}(X_1, X_3)] \right) \\
&= \hat{\Psi} \left( [\sigma_{2,\alpha,f}(\tau_1, \tau_2)] \right).
\end{align*}
\]

The proof of Theorem 4.1 is now complete.

6 Concluding Remarks and Questions

The combinatoric upshot of Theorem 4.1 is that $TA_3(K)$ is the colimit of a “triangle of groups” in Stallings’ sense (see [10]), comprising $\tilde{H}_1$, $H_2$, and $H_3$, their pairwise intersections, and the intersection of all three. These groups form the stabilizers of the three vertices, three edges and face, respectively, of a simplex $f$ in a simply connected complex of triangles $\mathcal{D}$ on which $TA_3(K)$ acts, and for which $f$ serves as a fundamental domain. There are unanswered questions about $\mathcal{D}$. For example, is it 2-connected (i.e., does every continuous image of the 2-sphere in $\mathcal{D}$ contract in $\mathcal{D}$), and does it have infinite diameter (i.e., is the number of faces need to connect two arbitrary points unbounded)?

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