TRIPLE CHORDS AND STRONG (1, 2) HOMOTOPY

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Abstract. A triple chord $f$ is a sub-diagram of a chord diagram that consists of a circle and finitely many chords connecting the preimages for every double point on a spherical curve. This paper describes some relationships between the number of triple chords and an equivalence relation called strong (1, 2) homotopy, which consists of the first and one kind of the second Reidemeister moves involving inverse self-tangency if the curve is given any orientation. We show that a prime knot projection is trivialized by strong (1, 2) homotopy, if it is a simple closed curve or a prime knot projection without 1- and 2-gons whose chord diagram does not contain any triple chords. We also discuss the relation between Shimizu’s reductivity and triple chords.

1. Introduction

Sakamoto and Taniyama [6] characterized the sub-chord diagrams $\otimes$ (cross chord) and $\boxplus$ ($H$ chord), embedded in a chord diagram associated with a generic plane curve, where a chord diagram is a circle with the preimages of each double point of the curve connected by a chord. For example, a chord diagram of a plane curve contains $\boxplus$, if and only if the plane curve is not equivalent to any connected sum of plane curves, each of which is either the simple closed curve $\bigcirc$, the curve that appears similar to $\infty$, or a standard torus knot projection [6, Theorem 3.2].

This paper aims to obtain a similar characterization of the triple chord $f$, stated in Theorem 1. A knot projection is a generic spherical curve that is a regular projection image on $S^2$ of a knot. For a knot projection $P$, a chord diagram $CD_P$ is defined as a circle with the preimages of each double point of the knot projection connected by a chord. A knot projection is called prime, if it is not the connected sum of two knot projections. Let $P'$ be a unique knot projection with no 1- or 2-gons obtained by a finite sequence of the first and second Reidemeister moves always decreasing double points in an arbitrary manner for an arbitrary knot projection $P$ (for the uniqueness of $P'$, see [5, 3]).

Theorem 1. If the chord diagram $CD_P$ of a knot projection $P$ has no triple chord $\boxplus$, and $P'$ is a prime knot projection or a simple closed curve, then there exists a finite sequence consisting of local replacements $1a$ and $s2a$ shown in Fig. 1 from a simple closed curve $\bigcirc$ to $P$.

We define strong (1, 2) homotopy equivalence as follows: two knot projections $P$ and $P'$ are strong (1, 2) homotopy equivalent, if and only if $P$ is related to $P'$

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by a finite sequence consisting of local replacements 1 and s2, as shown in Fig. 2.

Corollary 1 from Theorem 1 helps in understanding the relation between the triple chords and strong (1, 2) homotopy.

**Corollary 1.** If the chord diagram of an arbitrary prime knot projection $P$ with no 1- or 2-gons has no triple chord $\otimes$ or $P$ is a simple closed curve, then $P$ is equivalent to a simple closed curve $\bigcirc$ under strong (1, 2) homotopy.

![Figure 1](image1.png)

**Figure 1.** Local replacements 1a (left) and s2a (right). Dotted arcs show the connections of non-dotted arcs.

The remainder of this paper contains the following sections. Sec. 2 states our conventions. Sec. 3 and Sec. 4 provide proofs of Theorems 1 and 2, respectively. Sec. 5 mentions a relation between Shimizu’s reductivity of knot projections and the triple chord.

2. Preliminary

*Reidemeister moves*, which are three local replacements on an arbitrary knot projection, are defined by Fig. 3. It is known that there exists a finite sequence of Reidemeister moves between any two knot projections. Shown left to right in

![Figure 3](image3.png)

**Figure 3.** First (left), second (center), and third (right) Reidemeister moves.

Fig. 3 are the first, second, and third Reidemeister moves. There are two types of the second Reidemeister moves, local replacement, s2, shown in Fig. 2 and $w2$, shown in Fig. 4. Now, we define the notion of reducible and reduced knot projection.

**Definition 1** (Reducible and reduced knot projection). A knot projection $P$ is *reducible*, if there is a double point $d$, called a reducible crossing, in $P$, as shown in Fig. 5. If a knot projection is not reducible, it is called a *reduced knot projection*. 
From this definition, we obtain Lemma 1 which is easy to prove and is used often throughout this paper.

**Lemma 1.** An arbitrary prime knot projection with no 1-gons is a reduced knot projection.

**Proof.** To establish the claim, it is sufficient to show that (*) if an arbitrary knot projection with no 1-gons is reducible, then knot projection is non-prime. We will now show (*). Let $P$ be an arbitrary knot projection with no 1-gons. Assume that $P$ is reducible. Then, $P$ can be presented in Fig. 5. If the two faces having the point $d$ of $P$, as in Fig. 5, are not 1-gons, then $T$ and $T'$ are not simple arcs. Thus, $P$ is non-prime. This completes the proof. □

### 3. Proof of Theorem 1

To establish Theorem 1, we prove Theorem 2. If a knot projection is not a simple closed curve $\models$, we call it a non-trivial knot projection.

**Theorem 2.** A prime non-trivial knot projection with no 1- or 2-gons contains at least one triple chord.

Now, we deduce Theorem 1 from Theorem 2.

**Proof.** Based on our assumption in Theorem 1, a knot projection $P$ has no triple chords in $CD_P$. For $P$, we can consider $P'$, the unique knot projection with no 1- or 2-gons by a finite sequence consisting of the first and second Reidemeister moves decreasing the number of double points [5, Theorem 2.2] or [3, Theorem 2.2]. By the assumption of Theorem 1, $P'$ is a prime knot projection with no 1- and 2-gons or a simple closed curve $\models$. Thus, by Theorem 2, $P' = \models$.

Recover $P$ from $P'$ using the sequence consisting of $1a$ and $2a$, where $1a$ (resp. $2a$) is the first (resp. second) Reidemeister move always increasing a double point (resp. double points). If at least one $2a$ in the sequence is $w2$, then there exists at least one triple chord $\bullet$ in $CD_P$. This is because a 2-gon raised by $w2$ can be represented as shown in Fig. 6 and the corresponding chord diagram is shown at the left of Fig. 6. We can see that the point contained in both dotted arcs exists. Thus, we can find $\bullet$ in $CD_P$.

However, the existence of $\bullet$ contradicts the assumption that $P$ has no triple chords. Thus, the sequence consisting of $1a$ and $2a$ must consist of $1a$ and $s2a$. We conclude that Theorem 2 implies Theorem 1. □
In the next section, we present the proof of Theorem 2.

4. Proof of Theorem 2

To prove Theorem 2, we first recall Fact 1. Fact 1 and its proof were obtained by A. Shimizu [7, Proof of Prop. 3.1].

Fact 1 (7). A reduced knot projection $P$ contains at least one element of the following set:

\[
\{ \begin{array}{c}
\includegraphics[width=1cm]{3gon1}, \includegraphics[width=1cm]{3gon2}, \includegraphics[width=1cm]{3gon3}, \includegraphics[width=1cm]{3gon4}
\end{array}\}
\]

We must also check the following Lemma 2. Recall that a knot projection that is not a simple closed curve $\bigcirc$ is called a non-trivial knot projection.

Lemma 2. (a): A non-trivial knot projection with no 1- or 2-gons has at least eight 3-gons. (Cf. [1, Theorem 2.2].)

(b): If a non-trivial prime knot projection $P$ with no 1- or 2-gons has at least one 3-gon in $\{A, B, C\}$ in the following, then $P$ has a triple chord in $CD_P$.

\[
\text{Figure 7. All types of 3-gons. Dotted arcs show the connections of arcs.}
\]

Proof. (a): Let $V$ be the number of double points (i.e., vertices), $E$ the number of edges, and $F$ the number of faces. Let $p_k$ be the number of $k$-gons. For a non-trivial knot projection $P$ with no 1- or 2-gons,

\[
\begin{align*}
\sum_{k \geq 3} kp_k &= 2E, \\
\sum_{k \geq 3} p_k &= F.
\end{align*}
\]
Now, we consider knot projections that are graphs on $S^2$ such that every vertex has four edges. Thus,

\[
\begin{align*}
4V &= 2E, \\
V - E + F &= 2.
\end{align*}
\]

Formula (2) implies $4F - 2E = 8$. Substituting formula (1) into $2E$ and $F$ of $4F - 2E = 8$, we have

\[p_3 + \sum_{k \geq 4} (4-k)p_k = 8.\]

Then, we have $p_3 \geq 8$. This completes the proof.

(b): • A-type 3-gon. Observe the figure of the spherical curve that contains dotted arcs and an A-type 3-gon shown in Fig. 7. From the assumption,

\[\text{Figure 8. A-type 3-gon having the dotted arc labeled } \alpha \text{ (left), the corresponding chord diagram (center), and chord diagram with a triple chord (right).}\]

a knot projection $P$ containing an A-type 3-gon is prime. Then, the $\alpha$-part of Fig. 8 must intersect at least one of the other dotted arcs. Similar to Fig. 8, there exists $\bowtie$ in CD$_P$.

• B-type 3-gon. Note the spherical curve $P$ that contains dotted arcs and a B-type 3-gon shown in Fig. 7. The corresponding chord diagram CD$_P$ is shown at the right of Fig. 9. In CD$_P$, we can find $\bowtie$, since

\[\text{Figure 9. B-type 3-gon, which must have a red double point, and its chord diagram.}\]

there are two dotted arcs in a B-type 3-gon that must intersect (Fig. 9 left).

• C-type 3-gon. If a knot projection $P$ contains a C-type 3-gon, then CD$_P$ immediately has a triple chord. The consideration of the three cases completes the proof.

Now, we prove Theorem 2.

Proof. By Fact 1, a knot projection $P$, which we have considered, contains at least one of the elements mentioned in Fact 1. Thus, we consider the possibilities that $P$ contains the first, second, third, or fourth of those elements. In what follows,
Figure 10. Third element (left) of the set of Fact 1 and the case having a D-type 3-gon (right). Dotted arcs show the connections of non-dotted arcs.

checking the possibility of the first (resp. second, third, or fourth) element is called the first (resp. second, third, or fourth) element case.

The first element case.

By assumption, \( P \) has no 2-gon. Thus, there is no possibility of the existence of the first element of the set of Fact 1.

The second element case.

If \( P \) has the second element (i.e., two neighboring 3-gons) from the left-hand side of the set shown in Fact 1, assume that one of the two neighboring 3-gons is D type. In this case, another 3-gon in the two neighborhood 3-gons is type B. This implies that \( P \) has at least a type A, B, or C 3-gon, from which we conclude that \( P \) has triple chords in \( CD_P \) by Lemma 2. Thus, it is sufficient to consider the two cases of the third or the fourth figure from the left-hand side in the set of Fact 1. Below, we consider these figures.

The third element case.

By Lemma 2, if a knot projection contains the part shown in Fig. 10, we can assume that the 3-gon is type D from that figure. Since the 3-gon is type D, dotted arcs arise as shown at the right-hand side. Thus, we distinguish the following cases in which a dotted arc contains the arc DG shown in the figure:

- Arc number 1 contains DG (Case A, B),
- Arc number 2 (or 3) contains DG (Case C, D).

In the remainder of the proof, the symbol \((X, Y)\) (resp. \((x \sim y)\)) means we connect a point \(X\) with a point \(Y\) (resp. a vertex \(x\) with a vertex \(y\)) via a route outside the fixed part of a knot projection, e.g., as seen below, Case A and Fig. 11.

- Case A is defined by \((A, G), (B, D), (C, E), \) and \((F, H)\). See Fig. 11.
- Case B is defined by \((A, D), (B, G), (C, E), \) and \((F, H)\). See Fig. 12.
- Case C is defined by \((A, B), (C, D), (E, G), \) and \((F, H)\). See Fig. 13.
Case D is defined by (A, B), (C, G), (D, E), and (F, H) as shown in Fig. 14. This knot projection $P$ is a prime knot projection with no 1- or 2-gons; hence, $P$ is reduced (Lemma 1). Thus, $(a\sim a)$ intersects another dotted arc $(*)$. If $(*)$ is $(b\sim e)$ or $(d\sim e)$, $P$ has a triple chord in $CD_P$. If $(*)$ is neither $(b\sim e)$ nor $(d\sim e)$, but is $(c\sim c)$, the knot projection $P$ and its $CD_P$ appears as Fig. 15 and thus, there exists a triple chord in $CD_P$.

In summary, if a knot projection $P$ has the third element of the set of Fig. 1, then $P$ has a triple chord in $CD_P$. Then, we are left with only the case of the fourth element of the set of Fact 1.
The fourth element case. Start by setting the symbols for points to be connected and vertices as in Fig. 16. By Lemma 1, we can fix the 3-gon in Fig. 16 as type $D$. Then, we can draw dotted arcs as in the figure. Next, we consider the dotted arcs that contain the non-dotted arcs $DG$ and $FI$. Based on this consideration, we prove the claim case by case. Since there is the symmetry between arc numbers 2 and 3, it is sufficient to consider the following four groups, each of which contains eight cases (in total, 32 cases). The points of grouping are as follows.

Table 1. Each of four groups having eight cases. Dotted arcs show the connections of arcs.

- Dotted arc number 1 contains both $DG$ and $FI$ (Cases 1–8). This condition fixes $(C, E)$ and $(H, J)$.
- Dotted arc number 2 contains both $DG$ and $FI$ (Cases 9–16). This condition fixes $(A, B)$ and $(H, J)$.
  (Replacing 2 with 3 replicates the discussion as a result of their symmetry; hence, we omit the respective cases.)
- Arc number 1 contains exactly one non-dotted arc (i.e., DG or FI), and arc number 2 contains exactly one non-dotted arc (Cases 17–24). This condition fixes $(H, J)$.
  (Replaying 2 with 3 replicates the discussion as a result of their symmetry, hence we omit the respective cases.)
- Arc number 2 contains exactly one non-dotted arc (i.e., DG or FI) and arc number 3 contains exactly one non-dotted arc (Cases 25–32). This condition fixes $(A, B)$.
- Cases 1–8. If arc number 1 contains both two arcs DG and FI, then we can automatically fix $(C, E)$ and $(J, H)$ (Table 1, Cases 1–8). Table 2 shows how arcs

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure16.png}
\caption{Fourth element (left) of the set of Fact 1 and the case having a D-type 3-gon (right). Dotted arcs show the connections of non-dotted arcs.}
\end{figure}
connect, considering all possibilities. Recall that the symbol \( (X, Y) \) means that we connect \( X \) and \( Y \). For every case 1–8, a knot projection \( P \) has at least one triple chord in \( CD_P \). See Table 3.  

- **Cases 9–18.** If arc number 2 contains both arcs \( DG \) and \( FI \), we can automatically fix connections \( (A, B) \) and \( (H, J) \) (Table 1, Cases 9–16). Table 4 shows how arcs connect, considering all possibilities. Except for Cases 10 and 16, the existence of a triple chord is directly proved by Table 5.  

- **Case 10** (not easily proved). Observe the figure in Case 10 on the bottom line of Table 4. First, this knot projection \( P \) is a prime knot projection with no 1- or 2-gons. Thus, \( P \) is reduced (Lemma 1). Therefore, the dotted arc \( (a \sim a) \) must intersect at least one of the other dotted arcs. If \( (a \sim a) \) intersects \( (b \sim c) \), \( (d \sim e) \), or

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**Table 2.** Method to split into Cases 1–8.

| (B, D) | (G, F)(I, A) (Case 1) | (B, F) | (I, D)(G, A) (Case 3) |
|--------|----------------------|--------|----------------------|
|        | (G, I)(F, A) (Case 2) |        | (I, G)(D, A) (Case 4) |
| (B, G) | (D, F)(I, A) (Case 5) | (B, I) | (F, D)(G, A) (Case 7) |
|        | (D, I)(F, A) (Case 6) |        | (F, G)(D, A) (Case 8) |

**Table 3.** Easy cases to prove. Cases 1–8.

| Case 1 | Case 2 | Case 3 |
|--------|--------|--------|
| ![Diagram](image1) | ![Diagram](image2) | ![Diagram](image3) |
| Case 4 | Case 5 | Case 6 |
| ![Diagram](image4) | ![Diagram](image5) | ![Diagram](image6) |
| Case 7 | Case 8 |
| ![Diagram](image7) | ![Diagram](image8) |

**Table 4.** Method to split into Cases 9–16.

| (C, D) | (G, F)(I, E) (Case 9) | (C, F) | (I, D)(G, E) (Case 11) |
|--------|----------------------|--------|----------------------|
|        | (G, I)(F, E) (Case 10) |        | (I, G)(D, E) (Case 12) |
| (C, G) | (D, F)(I, E) (Case 13) | (C, I) | (F, D)(G, E) (Case 15) |
|        | (D, I)(F, E) (Case 14) |        | (F, G)(D, E) (Case 16) |

• **Cases 9–18.** If arc number 2 contains both arcs \( DG \) and \( FI \), we can automatically fix connections \( (A, B) \) and \( (H, J) \) (Table 1, Cases 9–16). Table 4 shows how arcs connect, considering all possibilities. Except for Cases 10 and 16, the existence of a triple chord is directly proved by Table 5.

• **Case 10** (not easily proved). Observe the figure in Case 10 on the bottom line of Table 4. First, this knot projection \( P \) is a prime knot projection with no 1- or 2-gons. Thus, \( P \) is reduced (Lemma 1). Therefore, the dotted arc \( (a \sim a) \) must intersect at least one of the other dotted arcs. If \( (a \sim a) \) intersects \( (b \sim c) \), \( (d \sim e) \), or
Table 5. Cases easily proved: Case 9, Cases 11–15. Non-easy cases: Case 10 and its additional figure Case 10a, Case 16.

| Case 9 | Case 11 | Case 12 |
|--------|---------|---------|
| ![Diagram](image1) | ![Diagram](image2) | ![Diagram](image3) |

| Case 13 | Case 14 | Case 15 |
|---------|---------|---------|
| ![Diagram](image4) | ![Diagram](image5) | ![Diagram](image6) |

| Case 10 | Case 10a | Case 16 |
|---------|---------|---------|
| ![Diagram](image7) | ![Diagram](image8) | ![Diagram](image9) |

(e∼f), then P has a triple chord in CD_P. Therefore, we can assume that (a∼a) intersects (c∼f). In this case, observe the figure in Case 10a on the bottom line of Table 5.

Next, since the knot projection we considered is a prime knot projection with no 1- or 2-gons, the dotted arc (b∼c) intersects at least one of the other dotted arcs.

- If (b∼c) intersects (a∼a) or (c∼f), then P has triple chords in CD_P.
- If (b∼c) intersects (e∼f), but not (a∼a) or (c∼f), then (e∼f) intersects (a∼a) or (c∼f). However, in each of these two cases, P has a triple chord in CD_P.
- If (b∼c) intersects (d∼e), but not (a∼a) or (c∼f), then (d∼e) intersects (a∼a) or (c∼f). However, in each of these two cases, P has a triple chord in CD_P.

Therefore, when (b∼c) intersects another dotted arc, a knot projection P that we considered has a triple chord in CD_P.

- **Case 16** (not easily proved). Observe the right-bottom figure of Table 5. The existence of a triple chord this case is proved in the same way as Case D, by omitting dotted 1-gons (f∼f). Compare Fig. 14 with Case 16 in Table 5.
- **Cases 17–24.** Dotted arc numbers 1 and 2 each contains an instance of DG and FI. This case implies fixing (H, J), i.e., H must connect with J (Table 1, Cases 17–24). In this case, Table 6 shows how the case is split into eight cases, and it is easy to show that each knot projection P of those cases has a triple chord in CD_P. See Table 7.
- **Cases 25–32.** Dotted arc numbers 2 and 3 each contains exactly one of the non-dotted arcs. Hence, we can automatically fix the connection (A, B) (Table 1, Cases 25–32). Table 8 shows how arcs connect considering all possibilities.
Table 6. Method to split into Cases 17–24.

| (B, D)(G, A) | (C, F)(I, E) | (Case 17) |
|-------------|-------------|-----------|
|             | (C, I)(F, E) | (Case 18) |
| (B, F)(I, A) | (C, D)(G, E) | (Case 21) |
|             | (C, G)(D, E) | (Case 22) |
| (B, G)(D, A) | (C, F)(I, E) | (Case 19) |
|             | (C, I)(F, E) | (Case 20) |

Table 7. Cases easily proved.

Table 8. Method to split into Cases 25–32.

- **Case 25** (not easily proved). Observe the figure for Case 25 in the lower part of Table 9. First, since the considered knot projection $P$ is reduced (cf. Lemma 1), the dotted arc $(a \sim a)$ intersects one of the other dotted arcs. If $(a \sim a)$ intersects $(b \sim c)$ or $(d \sim e)$, then $P$ has a triple chord in $CD_P$. Therefore, we can assume that $(a \sim a)$ intersects $(c \sim f)$ (the figure Case 25a of Table 9). Here, note that the assumption that $(a \sim a)$ intersects $(e \sim f)$ is equivalent to the assumption that $(a \sim a)$ intersects $(c \sim f)$ by symmetry, hence we omit the case $(a \sim a)$ intersects $(e \sim f)$.

Next, consider Case 25a in Table 9. Since $P$ is a prime knot projection with no 1- or 2-gons, $(b \sim c)$ must intersect one of the other dotted arcs.

- If $(b \sim c)$ intersects $(a \sim a)$ or $(c \sim f)$, then $P$ has a triple chord in $CD_P$.
- If $(b \sim c)$ intersects $(e \sim f)$, then $P$ has a triple chord in $CD_P$. 
Table 9. Cases easily proved: Cases 26, 27 and Cases 29–32. Non-easy cases: Case 25 and its additional figure Case 25a, Case 28.

| Case 26 | Case 27 | Case 29 |
|---------|---------|---------|
| ![Diagram](image1) | ![Diagram](image2) | ![Diagram](image3) |

| Case 30 | Case 31 | Case 32 |
|---------|---------|---------|
| ![Diagram](image4) | ![Diagram](image5) | ![Diagram](image6) |

| Case 25 | Case 25a |
|---------|---------|
| ![Diagram](image7) | ![Diagram](image8) |

| Case 28 | Case 28a |
|---------|---------|
| ![Diagram](image9) | ![Diagram](image10) |

- If (b~c) intersects (d~e), but not (a~a) or (c~f), then (d~e) must intersect (a~a) or (c~f). However, whether (d~e) intersects (a~a) or (c~f), $P$ has a triple chord in $CD_P$.

Thus, if (b~c) intersects one of the other dotted arcs, a considered knot projection $P$ has a triple chord in $CD_P$.

- **Case 28.** See the bottom line of Fig. 9. By the assumption, the considered knot projection $P$ is a prime knot projection with no 1- or 2-gons. Thus, $P$ is reduced (Lemma 1), and the dotted arc (a~a) intersects the other dotted arcs.

  - If (a~a) intersects (b~f), then $P$ has a triple chord in $CD_P$.
  - If (a~a) intersects (d~f), then $P$ has a triple chord in $CD_P$.
  - If (a~a) intersects (c~c), but not (b~f) or (d~f), then $P$ has a triple chord in $CD_P$, as shown in the bottom line of Fig. 9.
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If \((a \sim a)\) intersects \((e \sim e)\), but not \((b \sim f)\) or \((d \sim f)\), then \(P\) has a triple chord in \(CD_P\) by the same reasoning as that of \((c \sim c)\) via their symmetry between \((c \sim c)\) and \((e \sim e)\).

These 32 cases complete the proof of Theorem 2.

5. Reductivity and Triple Chords

This section mentions a relation between the reductivity of a knot projection and triple chords. The reductivity is introduced by A. Shimizu \([7, \text{Sec. 1}]\) using local replacement \(A^{-1}\) (also called \(A^{-1}\) move in this paper) that appears in \([2]\).

**Definition 2.** The local replacement \(A^{-1}\) move at a double point is defined by Fig. 17.

The reductivity \(r(P)\) of a knot projection is the minimal number of \(A^{-1}\) moves to produce a reducible knot projection.

**Remark 1.** It is worthwhile mentioning the following fact here. Any reduced knot projections are related by a finite sequence consisting of \(A^{-1}\) moves and inverse moves, where every knot projection appearing in each step in the sequence is reduced \([2, \text{Corollary 1.2}]\). Therefore, it is natural to consider the notion of reductivity \([7]\).

We characterize knot projections with \(r(P) = 1\).

**Theorem 3.** For a reduced knot projection \(P\), there exists a circle with two double points as shown in Fig. 18 if and only if \(r(P) = 1\).

**Proof.** (If part) Assume that \(r(P) = 1\). Let \(P'\) be a reduced knot projection obtained from \(P\) by applying an \(A^{-1}\) move at a double point, say \(a\), of \(P\), and \(b\) a reducible crossing of \(P'\). Then it follows by definition that there exists a simple circle which intersects \(P\) with \(a\) and \(b\) only. There are four cases with respect to the connectivity among the paths at \(a\) and \(b\) as shown in Fig. 19.

Since \(P\) is an immersion of a single circle, we have the second and third cases.
• (Only if part) Assume that a reduced knot projection $P$ has two double points as shown in Fig. 18. By applying an $A^{-1}$ move at one of the double points, we have a reducible knot projection.

Corollary 2. A knot projection $P$ with $r(P) = 1$ has at least one triple chord in $CDP$.

Proof. If $P$ satisfies $r(P) = 1$, then it has two double points, say $a$ and $b$, as shown in Fig. 18. Let $x$ be a double point of $P$ in the region surrounded by the red circle. Then the double points $a$, $b$, and $x$ gives a triple chord in $CDP$.

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