Entanglement system, Casimir energy and black hole

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Abstract

We investigate the connection between the entanglement system in Minkowski spacetime and the black hole using the scaling analysis. Here we show that the entanglement system satisfies the Bekenstein entropy bound. Even though the entropies of two systems are the same form, the entanglement energy is different from the black hole energy. Introducing the Casimir energy of the vacuum energy fluctuations rather than the entanglement energy, it shows a feature of the black hole energy. Hence the Casimir energy is more close to the black hole than the entanglement energy. Finally, we find that the entanglement system behaves like the black hole if the gravitational effects are included properly.

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1 Introduction

The fact that the black hole has entropy [1] and it can radiate [2] has led to many debates about its origin of quantum gravity for decades [3]. Although string theory is a strong candidate for the quantum gravity, it is so far incomplete to explain the quantum gravity well. Instead the holographic principle provides important progress in quantum gravity. If a system includes a gravity, guided by the black hole entropy, the system should obey a suitable entropy bound [4, 5]. The AdS/CFT correspondence is a realization of the holographic principle [6]. Also there were much progress in cosmic holography [8].

Usually the local field theory overcounts available degrees of freedom because it fails to account the effects of gravity appropriately [9]. If the gravity is included, not all degrees of freedom conjectured by the local field theory can be used for generating entropy or storing information. This is why we need to introduce the holographic principle.

On the other hand, the holography could be realized in Minkowski spacetime. These are the entanglement system and Casimir effect. The main features of entanglement system are summarized by the observer-dependent entropy, the role of quantum correlations across the boundary of the system, and the non-extensiveness [10, 11]. Actually, the entanglement entropy which is proportional to the area of the boundary is a measure of an observer’s lack of information regarding the quantum state of the other system in an inaccessible region. In general, the Casimir effect appears if the system described by the local (conformal) field theory has the boundary [12]. In this work we define the Casimir energy by the non-extensive scaling behavior of $E_C \rightarrow \lambda^{1-\frac{d}{2}} E_C$. The Casimir energy could be found from the vacuum energy fluctuations of the local field theory in the entanglement system [13]. Also this energy appears as the result of finite-size effects in the conformal field theory defined on the Einstein static universe [14]. Here the former means the Casimir energy in the bulk, while the latter is related to the CFT on the boundary.

In this work we study the connection between the entanglement system in Minkowski spacetime and the black hole using the scaling analysis. We clarify the distinction between the entanglement energy and the black hole energy. Introducing the Casimir energy instead of the entanglement energy, it shows a main feature of the black hole energy. Finally, we show that the entanglement system behaves like the black hole if one takes the gravitational effects into account properly.
2 Entanglement and black hole thermodynamics

For an entanglement system, we consider the three-dimensional spherical volume $V$ and its enclosed boundary $B$ in the flat spacetime. We assume that this system has the local (quantum) field theory with the IR cutoff $R$ and the UV cutoff $\Lambda = 1/a$. If the vacuum energy density $\rho_{\Lambda} = \Lambda^4$ of the system does not diverge, the vacuum energy and entropy take the forms of $E_{\Lambda} \simeq \Lambda^4 R^3$ and $S_{\Lambda} \simeq \Lambda^3 R^3$.

We start by noting the difference between the entanglement thermodynamics in the flat spacetime and the Schwarzschild black hole thermodynamics [15]. Here we introduce the first-law of thermodynamics for both sides:

$$dE_{\text{ENT}} = T_{\text{ENT}} dS_{\text{ENT}}, \quad dE_{\text{BH}} = T_{\text{BH}} dS_{\text{BH}}.$$  \hfill (1)

The thermodynamic quantities are given by

$$S_{\text{ENT}} \sim A, \quad E_{\text{ENT}} \sim A, \quad T_{\text{ENT}} \sim A^0,$$  \hfill (2)

$$S_{\text{BH}} \sim A_{\text{EH}}, \quad E_{\text{BH}} \sim R_{\text{EH}}, \quad T_{\text{BH}} \sim \frac{1}{R_{\text{EH}}},$$  \hfill (3)

where $A = 4\pi R^2$ is the area of the boundary $B$ of the system while $A_{\text{EH}} = 4\pi R_{\text{EH}}^2$ is the area of the event horizon. Eq. (2) shows a universal behavior for all entanglement systems in Minkowski spacetime. Here we note that the zero-point (vacuum) energy of the system was subtracted in the calculation of the entanglement energy $E_{\text{ENT}}$, thus degrees of freedom on the boundary contributes to giving $E_{\text{ENT}} \sim A$. It seems that its areal behavior is compatible with the concept of the entanglement. On the black hole side, however, we have a linear behavior of the energy. The entanglement entropy behaves universally as $S_{\text{ENT}} \sim A$, which takes the same form as that of the black hole. The entanglement temperature is independent of the radius $R$ of system, whereas the temperature of black hole is given by $1/R_{\text{EH}}$.

The authors in [15] have explained this discrepancy by noting that the entanglement quantities are calculated in the flat spacetime and thus these do not include any gravitational effects. In order to compare these with those for the black hole, one has to make corrections to $E_{\text{ENT}}$ and $T_{\text{ENT}}$ by replacing these by $E_{\text{ENT}}^{\text{new}} = \sqrt{-g_{tt}}E_{\text{ENT}}$ and $T_{\text{ENT}}^{\text{new}} = \sqrt{-g_{tt}}T_{\text{ENT}}$ with $\sqrt{-g_{tt}} \simeq a/R$. It shows how an inclusion of gravity alters thermodynamics of the entanglement system. Considering the connection between $R \leftrightarrow R_{\text{EH}}$ and $a \leftrightarrow l_{\text{pl}}$, the new energy and temperature are the nearly same as those of the black

\footnote{In this work we use notations: $\sim$ for a comparison with the IR cutoffs ($R, R_{\text{EH}}$) only; $\simeq$ for a comparison with both the IR cutoffs ($R, R_{\text{EH}}$) and the UV cutoffs ($\Lambda, m_{\text{pl}}$).}
hole. Here one observes that the new entanglement temperature is red-shifted by inserting the factor of $\sqrt{-g_{tt}}$ at $r = R + a^2/R$. Simultaneously, the gravitational redshift effect modifies the area dependence of $E_{\text{ENT}} \sim A$ into the linear dependence of $E_{\text{ENT}}^{\text{new}} \sim R$, which is the case of the black hole. On later, they calculated the entanglement energy in the Schwarzschild background without introducing the red-shifted factor \cite{16}.

However, the matching procedure between two systems will not end at this stage. The above procedure is not enough to compare the entanglement system with the black hole. If one incorporates the gravity to any system, the resulting system necessarily follow the holographic principle. This is why we need to study entropy and energy bounds on matter. Also the AdS/CFT correspondence provides a guideline to study the entanglement system when including the gravity effects.

### 3 Entropy bounds

In this section we introduce a few of entropy bounds. First of all, for a weakly gravitating system in asymptotically flat space, Bekenstein has proposed that the generalized second law implies the bound \cite{17}

$$S \leq S_B = \frac{2k_B \pi E R}{\hbar c},$$

where Newton’s constant $G$ does not enter here. This bound is powerful for the system with relatively low density or small volume. For a black hole with $E_{\text{BH}} = M c^2 = c^4 R_{\text{BH}}/2G$, we find $S_{\text{BH}} = S_B$ which means that the Bekenstein bound is saturated when choosing the black hole as a matter.

On the other hand, for a strongly gravitating matter system in the curved spacetime, 't Hooft and Susskind has shown that the holographic principle implies the holographic entropy bound \cite{4, 5}

$$S \leq S_{\text{HOB}} = \frac{k_B c^3 A}{4G \hbar} = S_{\text{BH}},$$

where Newton’s constant $G$ is made explicitly.

Furthermore, Brunstein and Veneziano has proposed the causal entropy bound \cite{18}

$$S_{\text{matter}} \leq S_{\text{CEB}} = \frac{c}{2\hbar} \sqrt{\frac{k_B E V}{G}},$$

which is given by the geometric mean of $S_B$ and $S_{\text{HOB}}$. They showed that for a weakly gravitating system\textsuperscript{2}, there exists an important sequence

$$S_B \leq S_{\text{CEB}} \leq S_{\text{HOB}}.$$  

\textsuperscript{2}For the static case, we split all systems which are asymptotically flat into the weakly self-gravitating
This means that for isolated systems of weakly gravity, the strongest bound is the Bekenstein bound while the weakest one is the holographic entropy bound.

In order to carry out the scaling analysis, we use the Planck units as

\[ \hbar = G = c = k_B = 1 \]  

in the \((n + 1)\)-dimensional spacetimes with \(n = 3\). Here we need to check whether or not Newton’s constant \(G\) exists for any relations. This procedure is important because the presence of \(G\) represents an inclusion of gravity. For this purpose, we study the scaling behavior of the thermodynamic quantities \(S\) and \(E\) upon choosing Eq. (8) \[14\]. For example, we have the extensive scaling behavior for the vacuum energy and entropy:

\[ E_\Lambda \rightarrow \lambda E_\Lambda, \quad S_\Lambda \rightarrow \lambda S_\Lambda \quad \text{under} \quad V \rightarrow \lambda V. \]

The Bekenstein entropy is appropriate for describing a weakly self-gravitating system because \(S_B\) is super-extensive: it scales \(S_B \rightarrow \lambda^{1+\frac{1}{n}}\) under \(V \rightarrow \lambda V, E \rightarrow \lambda E\). On the other hand, the holographic entropy bound is suitable for a strongly self-gravitating system because \(S_{HOB}\) is sub-extensive: it scales \(S_{HOB} \rightarrow \lambda^{1-\frac{1}{n}}S_{HOB}\) under \(V \rightarrow \lambda V\). The covariant entropy bound is unclear since \(S_{CEB}\) is extensive: it scales \(S_{CEB} \rightarrow \lambda S_{CEB}\) under \(V \rightarrow \lambda V, E \rightarrow \lambda E\). However, \(S_{CEB}\) includes “\(G\)”, which implies that it carries with the effect of gravity.

We often say that the sub-extensive quantity includes the effects of gravity, while the super-extensive one does not. Explicitly, a scaling representation of Eq. (9) for a weakly self-gravitating system is given by

\[ \lambda^{1+\frac{1}{n}} \leq \lambda \leq \lambda^{1-\frac{1}{n}} \]  

while for a strongly self-gravitating system\(^3\), it takes the form

\[ \lambda^{1-\frac{1}{n}} \leq \lambda \leq \lambda^{1+\frac{1}{n}}. \]  

The above shows clearly that the sub-extensive quantities represent the strong gravity, while the super-extensive quantities denote the weak gravity or “no gravity”. Here we system with \(R \geq R_{EH}\) and the strongly self-gravitating system with \(R \leq R_{EH}\) \[3\]. For the dynamic case (e.g., cosmology) based on the closed FRW space of \(ds^2_{FRW} = -dt^2 + R(t)^2d\Omega^2_n\), we split all systems into the weakly self-gravitating system with \(HR \leq 1\) and the strongly self-gravitating system with \(HR \geq 1\) \[14\]. Here \(H(R)\) are the Hubble parameter (scale parameter).

\(^3\)For example, in cosmology, they found a sequence \(S_{HEB} \leq S_{BOB}, S_{CEB} \leq S_B\), where \(S_{HEB} = (n - 1)HV/4\hbar\) is the Hubble entropy to define the Hubble entropy bound of \(S \leq S_{HEB}\) \[13\] and \(S_{BCB} = A(B)/4\hbar\) is the entropy to define the Bousso’s covariant entropy bound of \(S[L(B)] \leq S_{BCB}\) on the light sheet \(L\) of a surface \(B\) \[13\]. Here the Hubble entropy bound is the strongest one, while the weakest one is the Bekenstein bound.
include the case of “no gravity” because Bousso derived the Bekenstein bound from the generalized covariant entropy bound (GCEB) \[20\]. In his derivation, gravity \((G)\) plays a crucial role. Combining the GCEB involving \(1/G\hbar\) \[21\] with the Einstein equation involving \(G\) leads to the Bekenstein bound in Eq. (4) without \(G\). Hence this bound can be tested entirely within the local field theory without any use of the laws of gravity.

Now let us discuss the non-gravitational systems. The entanglement entropy \(S_{\text{ENT}}\) and the Casimir entropy \(S_C\) show the areal behavior

\[
S_{\text{ENT}} \sim A \sim \lambda^{1-\frac{1}{n}}, \quad S_C \sim A \sim \lambda^{1-\frac{1}{n}} \quad (11)
\]

which are sub-extensive even though they have nothing to do with the gravity. Here we assume that \(S_C \sim E_C R\) for the bulk Casimir entropy.

In this sense these scalings are related to the holography induced by non-gravitational mechanisms. In case of the entanglement system, requiring the tensor product structure of the Hilbert space which is caused by the boundary between two regions (like the event horizon in the black hole) leads to the entanglement entropy \[22\]. That is, the entangled nature of quantum state of the system inside and outside the boundary leads to an areal scaling. On the other hand, the Casimir energy of \(E_C \sim R\) comes from the finite-size effects in the quantum fluctuations of the local field theory, and thus disappears unless the system is finite \[23, 24\].

We classify the scaling behaviors of various entropies into three cases:

- super – extensive \(\implies\) no gravity or weak gravity \((S_B)\)
- extensive \(\implies\) no gravity \((S_B)\) or gravity \((S_{\text{CEB}})\)
- sub – extensive \(\implies\) strong gravity \((S_{\text{HOB}})\) or holography \((S_{\text{ENT}}, S_C)\).

Although the entanglement entropy \(S_{\text{ENT}}\) and the black hole entropy \(S_{\text{BH}}\) have the same scaling dimension, they are quite different because \(S_{\text{ENT}}\) does not include “\(G\)”. Now we check which entropy bound is suitable for describing the entanglement system. We note again that the entanglement system is in the flat spacetime. Hence we could use the Bekenstein bound for it. Considering Eq. (1), one finds a relation

\[
S_{\text{ENT}} \sim A \leq 2\pi E_{\text{ENT}} R \sim V, \quad (12)
\]

which implies that the entanglement system satisfies the Bekenstein bound very well. Therefore, the entanglement system is one of holographic systems in Minkowski spacetime. Also the thermal radiation in a cavity in the flat spacetime respects this Bekenstein bound \[25\]. However, the Casimir entropy satisfies the holographic entropy bound of \(S_C \leq S_{\text{HOB}}\).
4 Energy bounds

We are in a position to discuss the energy bounds. In this section we restore “$G = 1/m^2_{pl}$”.

First of all, let us check the scaling dimension of various energies: the black hole energy of $E_{BH} \simeq m^2_{pl} R$; the Casimir energy of $E_C \sim R$; the entanglement energy of $E_{ENT} \simeq \Lambda^3 R^2$; the vacuum energy of $E_\Lambda \simeq \Lambda^4 R^3$. Under $V \rightarrow \lambda V$, we have the following behavior:

$$E_{BH} \sim \lambda^{1-\frac{2}{n}}, \quad E_C \sim \lambda^{1-\frac{2}{n}}, \quad E_{ENT} \sim \lambda^{1-\frac{1}{n}}, \quad E_\Lambda \sim \lambda,$$  \hspace{1cm} (13)

where all except the vacuum energy are sub-extensive. The black hole includes obviously the gravitational self-energy. The Casimir energy\(^4\) and entanglement energy have no gravitational effects. They become sub-extensive due to their own properties.

Cohen \textit{et al.} have shown that at the saturation of the holographic entropy bound in Eq. (5) \cite{26}, it includes many states with $R < R_{EH}$ which corresponds to the strongly self-gravity condition. Therefore, the energy of most states will be so large that they will collapse to a black hole which is larger than the system. This is hard to be accepted. Explicitly, the local field theory with $E_\Lambda \simeq \Lambda^4 R^3$ and $S_\Lambda \simeq \Lambda^3 R^3$ is able to describe a thermodynamic system at temperature $T$ provided that $T \leq \Lambda$. If $T \gg 1/R$, the energy and entropy will be those for thermal radiation: $E_{RAD} \simeq T^4 R^3$ and $S_{RAD} \simeq T^3 R^3$. At the saturation of $S_\Lambda = S_{HOB}$ with $T_{SAT} = \Lambda$, one has $T_{SAT}^3 \simeq m^2_{pl} / R^{4/3}$. Considering $R_{EH} \simeq T_{SAT}^3 R^2$, one obtains a strongly self-gravity relation of $R_{EH} \simeq m^2_{pl} R^{5/3} \gg R$ which is not the case. In order to eliminate these higher states, they proposed a stronger energy bound

$$E_\Lambda \leq E_{BH},$$  \hspace{1cm} (14)

where the energy bound is saturated if the system is replaced by the system-size black hole. At the saturation of $E_\Lambda = E_{BH}$, one finds a relation of $R_{EH} \simeq m_{pl}^{1/2} R^{3/2} > R$ with $T_{SAT}^3 \simeq (m_{pl}/R)^{1/2}$. In addition, Eq. (14) corresponds to a more restrictive entropy bound of $S_\Lambda \leq A^{3/4}$. Thus this energy bound is stronger than the holographic entropy bound. However, it will not eliminate all states with $R < R_{EH}$ lying within $R_{EH}$ completely.

One may accept the weakly self-gravity relation of $R_{EH} \leq R$ as a energy bound. If one uses the Planck units, this reduces to $E_{BH} = M \leq R$. This is called the Schwarzschild limit \cite{4} and the gravitational stability condition \cite{9, 19, 20}. In addition, this corresponds to the small self-gravitation \cite{3}, the limited-gravity \cite{18, 23}, and the no-collapse criterion \cite{27, 28}. We note that this relation includes the gravity effect of $G$. Hence this

\(^4\)More precisely, if one introduces the vacuum energy fluctuations in Ref.\cite{13} instead of $E_C$, one finds a compact relation of Eq. (13).
relation has nothing to do with physics in the flat spacetime. One may propose a relation of $E \leq R$ in Minkowski spacetime. However, one should be careful in using this relation because there does not exist such a linear behavior of the energy except the Casimir energy and a special case in [5]. For example, if one uses the saturation ($E = R$, average total energy of the system) of this bound $E \leq R$ in calculating the entanglement entropy, then one finds the maximal entropy $S_{MAX} \sim A^{3/4}$ but not the entanglement entropy $S_{ENT} \sim A$ [28].

It would be better to express the weakly and strongly gravitating systems in terms of the system energy $E$ and the black hole energy $E_{BH}$ [14, 29]

- weak gravity condition : $E \leq E_{BH}$
- strong gravity condition : $E \geq E_{BH}$.

We note that $E_{ENT} \sim A > R$. It seems that the entanglement energy satisfies the strong gravity condition. However, this view is incorrect because the entanglement system has nothing to do with the gravity. The holographic nature of its energy comes from the concept of entanglement in the flat spacetime. The scaling relations between the energy and entropy are given by

$$E_{BH}R_{EH} \simeq S_{BH}, \quad E_{C}R \sim S_C, \quad E_{ENT}R \simeq S^{1+\frac{1}{n}}_{ENT}, \quad E_{\Lambda}R \simeq S^{1+\frac{1}{n}}_{\Lambda}. \quad (15)$$

Here it seems that the Casimir system is closer to the black hole than the entanglement system without gravity effect. In order to get the closest case to the black hole, we introduce the vacuum energy fluctuations of a free massless scalar field in Minkowski space. The origin of such energy fluctuations is similar to the entanglement entropy and energy but the following is different [30]: the trace over degrees of freedom is not performed on them and quantum expectations values in a pure state are used for calculation rather than using statistical averages over a mixed state [15].

Actually, for the case of $R\Lambda > \pi$ [13], the energy dispersion of vacuum energy fluctuations is given by [31, 32]

$$\Delta E_{\Lambda} \simeq \Lambda^2 R, \quad (16)$$

where $\Delta E_{\Lambda}$ is defined by

$$\Delta E_{\Lambda} \equiv \sqrt{\langle (H_V - \langle H_V \rangle)^2 \rangle}. \quad (17)$$

Here we choose $E_{C} \simeq \Delta E_{\Lambda}$ as the Casimir energy for the system in Minkowski space. The weak gravity condition holds for the Casimir energy too

$$E_{C} \leq E_{BH}, \quad (18)$$
which means that the Casimir energy by itself cannot be sufficient to form a system-size black hole. From Eq. (18), one finds the upper bound on the UV cutoff

$$\Lambda \leq m_{pl}$$

(19)

which may be related to the gravitational stability condition if one includes a large number of massless fields [13]. In addition, we obtain the bound on the Casimir energy density of

$$\rho_C \equiv E_C/V$$

(20)

$$\rho_C \simeq \frac{\Lambda^2}{R^2} \leq \rho_{BH} \simeq \frac{m_{pl}^2}{R^2}.$$  

On the other hand, there exists the vacuum energy bound from Eq. (14) as

$$\rho_\Lambda \equiv \Lambda^4 \leq \rho_{BH}.$$  

(21)

This implies that the vacuum energy by itself cannot be sufficient to form a system-size black hole. At the saturation of these bounds, we obtain the important holographic energy density $\rho_{HOG} = m_{pl}^2/R^2$, which was used widely to explain the dark energy in cosmology [33, 34, 29]. A relation of $\rho_{HOL} = \rho_{BH}$ implies that the whole universe is dominated by black hole states. Thus it is called the maximal darkness conjecture [35].

5 Where is the discrepancy?

Up to now we do not introduce any effect of gravity in the entanglement system. It was shown that the entanglement system behaves like the black hole if the gravitational effects are included properly [15]. We wish to reconcile our picture with the AdS/CFT correspondence, where the gravitational blue/redshift play the important role in establishing the holographic principle. For this purpose, we introduce the $(n+2)$-dimensional AdS-black hole with the metric element [17]

$$ds^2_{AdS} = -\left[1 + \frac{r^2}{l^2} - \frac{\omega_n M}{r^{n-1}}\right]dt^2 + \frac{dr^2}{1 + \frac{r^2}{l^2} - \frac{\omega_n M}{r^{n-1}}} + r^2 d\Omega_n$$

(22)

where $l$ is the AdS-curvature radius related to the cosmological constant $\Lambda_{n+2} = -(n+1)/2l^2$ and $\omega_n = 16\pi G_{n+2}/nV_n$ with $G_{n+2}$ the $(n+2)$-dimensional Newton constant and $V_n$ the volume of unit $S^n$. Here the bulk thermodynamic quantities of black hole energy, Hawking temperature and Bekenstein-Hawking entropy are given by

$$E = M = \frac{r_+^{n-1}}{\omega_n}\left[\frac{r_+^2}{l^2} + 1\right], \quad T_H = \frac{1}{4\pi} \left[\frac{(n+1)r_+}{l^2} + \frac{(n-1)}{r_+}\right], \quad S_{BH} = \frac{V_n}{4G_{n+2}}r_+^n.$$  

(23)
According to Ref. [14], the Casimir energy is defined by the violation of the Euler identity as

\[ E_c^b = n(E - T_H S_{BH} + pV). \]  

(24)

Noting that the CFT is a radiation-like matter at high temperature, one has the equation of state for \( pV = E/n \). Then the Casimir energy in the bulk is given by

\[ E_c^b = (n + 1)E - nT_H S_{BH} = \frac{nV_n r_+^{n-1}}{8 \pi G_{n+2}}. \]  

(25)

On the other hand, the \((n + 1)\)-dimensional CFT near infinity is defined by the Einstein static universe [14]:

\[ ds_{ESU}^2 = -d\tau^2 + \rho^2 d\Omega_n^2. \]  

(26)

Now we can determine the boundary metric of Eq.(26) by using the bulk metric in Eq.(22) near infinity as

\[ ds_b^2 = \lim_{r \to \infty} \frac{\rho^2}{r^2} ds_{AdS}^2 = -d\tau^2 + \rho^2 d\Omega_n^2 \to ds_{ESU}^2, \quad \tau = \frac{\rho}{l} t. \]  

(27)

Using the Euclidean formalism, we find bulk-boundary relations [36]:

\[ T_{CFT} = \frac{l}{\rho} T_H = \frac{1}{4 \pi \rho} \left[ (n + 1) \hat{r} + \frac{n-1}{\hat{r}} \right], \quad E_{CFT} = \frac{l}{\rho} E = \frac{nV_n \kappa \hat{r}^{n-1}}{\rho} \left[ \hat{r}^2 + 1 \right] \]  

(28)

and

\[ E_{CFT}^c = \frac{l}{\rho} E_b^c = \frac{2nV_n \kappa \hat{r}^{n-1}}{\rho}, \quad S_{CFT} = S_{BH} = 4 \pi \kappa V_n \hat{r}^n \]  

(29)

with \( \hat{r} = r_+/l \) and \( \kappa = l^n/16 \pi G_{n+2} \). Here we can choose the radius \( \rho \) of \( S^n \) as we wish. Thus, considering the duality of \( r_+ \leftrightarrow \hat{r} \), we find the same forms for the CFT quantities as those in the bulk-AdS space. We note the different functional forms for a large black hole with \( r_+ \gg l \): \( E \sim r_+^{n+1} \), \( S_{BH} \sim r_+^n \), \( E_b^c \sim r_+^{n-1} \). Also we have the same relations for the dual CFT. This means that in the AdS spacetime and its dual CFT, the Casimir energy has the lowest power in compared with the energy and entropy. As an example, we have \( E_b^c \sim r_+ \) for \((1+3)\)-dimensional AdS spacetime and \( E_{CFT}^c \sim \hat{r} \) for \((1+2)\)-dimensional CFT.

We call these as either the UV/IR scaling transformation in the AdS/CFT correspondence or the Tolman redshift transformation on the gravity side [37]. The scaling factor of \( \sqrt{-g_{\infty}^{tt}} = l/\rho \) comes from the fact the Killing vector \( \partial/\partial t \) is normalized so that near infinity

\[ g\left( \frac{\partial}{\partial t}, \frac{\partial}{\partial t} \right) \to -\frac{\rho^2}{l^2}. \]  

(30)
This fixes the red-shifted CFT of $E$, $E^c$, and $T$, but $S$ is not scaled under the UV/IR transformation. Although there is no room to accommodate the AdS/CFT correspondence in the entanglement system, we use it to point out the discrepancy between the entanglement and black hole systems.

Now let us introduce the Tolman redshift transformation on the black hole system. In general, the local temperature observed by an observer at $r > R_{EH}$ in the Schwarzschild black hole background is defined by

$$T(r) = \frac{T_\infty}{\sqrt{-g_{tt}}} = \frac{1}{8\pi M} \frac{1}{\sqrt{1 - \frac{2M}{r}}} \quad (31)$$

where $T_\infty = 1/8\pi M$ is the Hawking temperature measured at infinity and the denominator of $\sqrt{-g_{tt}}$ is the red-shifted factor. Near the horizon (at $r = R_{EH} + l_p^2/R_{EH}$), this local temperature is given by $T = 1/8\pi l_p^2$ which is independent of the black hole mass $M$.

Similarly, the local energy is given by $E(r) = E_\infty \sqrt{-g_{tt}}$ with $E_\infty = M = R_{EH}/2$. Near the horizon, we have $E \propto A_{EH}$. However, there is no difference between the local entropy $S_{BH}$ near the horizon and the entropy $S_\infty$ at infinity.

The Tolman redshift transformation is also possible between $E_{ENT}$, $T_{ENT}$, $S_{ENT}$ near the boundary $B$ and $E_{\infty}^{\infty}$, $T_{\infty}^{\infty}$, $S_{\infty}^{\infty}$ at infinity if the gravity effect is included in the entanglement system. Choosing the local temperature as $T_{ENT} \simeq 1/a$ at $r = R + a^2/R$, then the corresponding temperature at infinity is given by $T_{\infty}^{\infty} = \sqrt{-g_{tt}} T_{ENT} \simeq 1/R$ because of $\sqrt{-g_{tt}} \simeq a/R$ near the horizon. Furthermore, if the local energy is given by $E_{\infty}^{\infty} \simeq A/a^3$ at $r = R + a^2/R$, then the corresponding energy at infinity is $E_{\infty}^{\infty} \simeq \sqrt{-g_{tt}} E_{\infty} \simeq R/a^2$. The entanglement entropy remains unchanged under the transformation. This picture is consistent with Ref.[15]. That is, the entanglement system including the gravity effects shows the feature of the black hole.

However, there exists a discrepancy in the Casimir energy. Considering $E_C \sim R/a^2$ as the local Casimir energy, the Casimir energy at infinity is given by $E_C^\infty = \sqrt{-g_{tt}} E_C \simeq 1/a$. Now our question is whether or not Eq.(25) can apply to the asymptotically flat black hole in the $(2 + n)$-dimensional spacetime: $E_S^\infty = n V_n r_+^{n-1}/16\pi G_{n+2}$, $S = V_n r_+^n/4G_{n+2} \sim A$, $T_S^\infty = (n - 1)/4\pi r_+$. These are obtained from Eq.(23) together with $r_+ \ll l$. Let us assume that like the AdS-black hole, this black hole may be described by a radiation-like CFT in $(1 + n)$ dimensions [39]. Then, using Eq.(25), we have

$$\tilde{E}_C^\infty = \frac{n V_n}{8\pi G_{n+2}} r_+^{n-1} = 2E_S^\infty, \quad (32)$$

for the Schwarzschild black hole. This means that the Casimir energy is nearly the same form as in the black hole energy. Here we conjecture that the local Casimir energy is...
given by $\tilde{E}_C \sim A$. Then we have $\tilde{E}_C^\infty = \sqrt{-g_{tt}}\tilde{E}_C \sim R$ like the black hole energy in the (1 + 3)-dimensional spacetime. The discrepancy between $E_C^\infty$ and $\tilde{E}_C^\infty$ mainly arises from the handicap of the Schwarzschild black hole which is defined in asymptotically flat space. Actually Eq. (25) holds only if there exists its dual CFT. However, we do not know what is its dual CFT. As a result, we cannot establish the correct connection between the local Casimir energy and Casimir energy at infinity at this stage. This is mainly because the Schwarzschild black hole is too simple to split the energy into the black hole energy and Casimir energy, in contrast with the AdS-black hole.

6 Discussions

We have the two kinds of the holography. The first one is called the g-holography induced by the gravity which appeared in the black hole and cosmology. The second holography is shown by non-gravitational mechanisms in the flat spacetime. The sub-extensive entropy including the gravity effect “G” belongs to the g-holography. For example, these are $S_{BH} = S_{HOB}$ for the black hole and $S_{CEB}, S_{HEB}$ and $S_{BOB}$ for cosmology. On the other hand, the sub-extensive entropy defined without $G$ can be used to describe the holography realized in the flat spacetime. There are $S_{ENT}$ and $S_C$.

Now we discuss the connection between the entanglement system and black hole. There exists difference between two systems. The entanglement energy shows an areal behavior in contrast to the linear behavior of the black hole. The entanglement system satisfies the Bekenstein entropy bound which is suitable for either the no-gravity system or the weakly gravitational system. Also we find that the Casimir energy (vacuum energy fluctuations) is more close to the black hole energy than the entanglement energy.

However, two systems behave like the same if the gravity effects are included in the entanglement system. This is checked with the UV/IR scaling transformation in the AdS/CFT correspondence and the Tolman redshift transformation on the gravity system.

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