Non-Gaussian fluctuations near the QCD critical point

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We study the effect of the QCD critical point on non-Gaussian moments (cumulants) of fluctuations of experimental observables in heavy-ion collisions. We find that these moments are very sensitive to the proximity of the critical point, as measured by the magnitude of the correlation length $\xi$. For example, the cubic central moment of multiplicity $\langle (\delta N)^3 \rangle_c \sim \xi^{4.5}$ and the quartic cumulant $\langle (\delta N)^4 \rangle_c \sim \xi^7$. We estimate the magnitude of critical point contributions to non-Gaussian fluctuations of pion and proton multiplicities.

Introduction — Mapping the QCD phase diagram is one of the fundamental goals of heavy-ion collision experiments. QCD critical point is a distinct singular feature of the phase diagram, the existence of which is a ubiquitous property of QCD models [1] based on chiral dynamics. Locating the point using first-principle lattice calculations is a formidable challenge and, while recent progress and results are encouraging, much work needs to be done to understand and constrain systematic errors [2]. If the critical point is located in the region accessible to heavy-ion collision experiments it can be discovered experimentally. The search for the critical point is planned at the Relativistic Heavy Ion Collider (RHIC) at BNL, the Super Proton Synchrotron (SPS) at CERN and the future Facility for Antiproton and Ion Research (FAIR) at GSI.

This Letter focuses on the experimental observables needed to locate the critical point in heavy-ion collisions. Locating the point requires a scan of the phase diagram, by varying the initial collision energy $\sqrt{s}$. The characteristic signature is the non-monotonic behavior, as a function of $\sqrt{s}$, of the experimental observables sensitive to the proximity of the critical point to the point where freeze-out occurs for a given $\sqrt{s}$. [3, 4]

The most characteristic feature of a critical point is increase and divergence of fluctuations. Most fluctuation measures discussed to-date can be related to quadratic fluctuations of pion and proton multiplicities.

Near the critical point $m_{\sigma} \ll T$, so the mode $\sigma$ can be treated as a classical field.

Calculating 2-point correlator $\langle \sigma(x)\sigma(0) \rangle$ we find that the correlation length $\xi = m_{\sigma}^{-1}$. For correlation functions of the zero momentum mode $\sigma_0 \equiv \int d^4x \sigma(x)/V$ we find

$$\kappa_2 = \langle \sigma_0^2 \rangle = \frac{T}{V} \xi^2; \quad \kappa_3 = \langle \sigma_0^3 \rangle = \frac{2\lambda_3 T}{V} \xi^6; \quad \kappa_4 = \langle \sigma_0^4 \rangle - \langle \sigma_0^2 \rangle^2 = \frac{6T}{V}[2(\lambda_3\xi)^2 - \lambda_4]\xi^8. \quad (3)$$

The critical point is characterized by $\xi \to \infty$. The central observation in this Letter is that the higher moments of the fluctuation measures depend on $\xi$ too weakly.

In this Letter we point out that higher, non-Gaussian, moments of the fluctuations are significantly more sensitive to the proximity of the critical point than the commonly employed measures based on quadratic moments.
Critical contribution to experimental observables — We shall now estimate the effect of the critical point fluctuations on the observables such as the pion multiplicity fluctuations. Using similar approach, it should be straightforward to construct corresponding estimates for such observables as charge, proton number, transverse momentum fluctuations, etc., as well as to take into account acceptance cuts.

We shall focus on the most singular contribution, proportional to a power of the correlation length $\xi$. This contribution can be found using an intuitive picture described in Ref. [4]. In this picture one considers a joint probability distribution for the occupation numbers $n_p$ of observed particles (such as pions) together with the value of the the critical mode field $\sigma_0$, the latter treated as classical. Due to coupling of the critical mode of the type $\sigma\pi\pi$, the fluctuations of the occupation numbers receive additional contribution, proportional to the corresponding correlation functions (moments) of the fluctuations of $\sigma_0$ given by Eq. (3). In this Letter, however, it will be more convenient to use instead more formal diagrammatic method developed in Ref. [7].

Cubic cumulant — The 3-particle correlator receives the following most singular contribution from the $\sigma$ fluctuations, given by the diagram in Fig. 1

$$\langle \delta n_p \delta n_p \delta n_p \rangle_\sigma = \frac{2\lambda_3}{\sqrt{2}T} \left( \frac{G}{m_\sigma^2} \right)^3 \frac{v^2_{p1} v^2_{p2} v^2_{p3}}{\omega_{p1} \omega_{p2} \omega_{p3}}$$

where, subscript $\sigma$ indicates that only contribution of the critical mode is considered and, as in Refs. [4, 7], we denoted $\sigma\pi\pi$ coupling by $G$ and introduced a shorthand notation for the variance of the occupation number distribution: $\delta n_p = \bar{n}_p(1 \pm \bar{n}_p)$, where the “+” is for the Bose particles.

Since the total multiplicity is just the sum of all occupation numbers and thus

$$\delta N = \sum_p \delta n_p$$

the cubic moment of the pion multiplicity distribution is given by

$$\langle (\delta N)^3 \rangle = V^3 \int_{p_1} \int_{p_2} \int_{p_3} \langle \delta n_{p1} \delta n_{p2} \delta n_{p3} \rangle$$

and find

$$\omega_3(N) = \frac{\langle (\delta N)^3 \rangle}{N}$$

Quartic cumulant — The leading contribution to the connected 4-particle correlator is given by the sum of two types of diagrams in Fig. 2

$$\langle \delta n_{p1} \delta n_{p2} \delta n_{p3} \delta n_{p4} \rangle_{c,\sigma} = \frac{6}{V^3 T} \left\{ \left( \frac{\lambda_3}{m_\sigma} \right)^2 - \lambda_4 \right\} \left( \frac{G}{m_\sigma^2} \right)^4 \frac{v^2_{p1} v^2_{p2} v^2_{p3} v^2_{p4}}{\omega_{p1} \omega_{p2} \omega_{p3} \omega_{p4}}$$

The quartic cumulant of multiplicity fluctuations is given by

$$\langle (\delta N)^4 \rangle_c = V^4 \int_{p_1 p_2 p_3 p_4} \langle \delta n_{p1} \delta n_{p2} \delta n_{p3} \delta n_{p4} \rangle_c$$

This cumulant also scales as $V^1$ in thermodynamic limit. As in Eq. (7) we define a ratio whose $V \to \infty$ limit is finite: $\omega_4(N) \equiv \langle (\delta N)^4 \rangle_c/N$, and find

$$\omega_4(N)_\sigma = \frac{6}{V T} \left\{ \left( \frac{\lambda_3}{m_\sigma^2} \right)^2 - \lambda_4 \right\} \left( \frac{G}{m_\sigma^2} \right)^4 \left( \int_p \bar{n}_p \right)^{-1}$$

Estimate of the effect — In order to estimate the magnitude of the effect, we need to estimate the values of the coupling constants $\lambda_3$ and $\lambda_4$. The main uncertainty in the estimate will come, however, from the uncertainty of the value of $G$, which enters in a large power. This constant is known only roughly — the estimate was made in Ref. [4]. Therefore, a crude estimate for $\lambda$’s suffices.
Near the critical point both $\lambda_3$ and $\lambda_4$ vanish with a power of $\xi$ given by (neglecting $\eta \ll 1$):

$$\lambda_3 = \tilde{\lambda}_3 T \cdot (T \xi)^{-3/2}, \quad \text{and} \quad \lambda_4 = \tilde{\lambda}_4 \cdot (T \xi)^{-1},$$  \hspace{1cm} (12)

where dimensionless couplings $\tilde{\lambda}_3$ and $\tilde{\lambda}_4$ are universal, and for the Ising universality class they have been measured (see, e.g., Ref. [8] for a review). $\tilde{\lambda}_3$ varies from 0 to about 8 depending on the direction of approach to the critical point (crossover or first-order transition side). The coupling $\tilde{\lambda}_4$ varies from about 4 to about 20. Since the freeze-out occurs somewhere between these two extremes (as illustrated in Fig. 3) we shall pick some mid-range values for our estimates. The main point is the strong dependence of the effect on $\xi$.

Putting together estimates of $\lambda$, $G$ (from [4]) and $\xi$ (from [2]) we find for pions at $T \approx 120$ MeV (in full acceptance, for a single pion species, see also Eq. (20)):

$$\omega_3(N_\pi) \approx 1. \left( \frac{\tilde{\lambda}_3}{4.7} \right) \left( \frac{G}{300 \text{ MeV}} \right)^3 \left( \frac{\xi}{3 \text{ fm}} \right)^{9/2}$$ \hspace{1cm} (13)

$$\omega_4(N_\pi) \approx 12. \left( \frac{2\tilde{\lambda}_3^2 - \tilde{\lambda}_4}{50.} \right) \left( \frac{G}{300 \text{ MeV}} \right)^4 \left( \frac{\xi}{3 \text{ fm}} \right)^7$$ \hspace{1cm} (14)

Because of large powers of the coupling $G$, which is known only poorly, the uncertainty in this result is significantly larger than that of similar estimates of the quadratic moments of fluctuations [4].

The most significant feature of this result is the strong dependence on $\xi$ which makes the cubic and quartic cumulants very sensitive signatures of the critical point.

Another example: proton multiplicity fluctuations — The above analysis carries over to the proton multiplicity fluctuations. The fluctuations of the net proton number is a good proxy to the baryon number fluctuations, whose magnitude, proportional to the baryon number susceptibility, must diverge at the critical point Ref. [3]. This susceptibility, as well as “kurtosis”, $\kappa_4/\kappa_2$, have been studied on the lattice using Taylor expansion around $\mu_B = 0$ and in QCD models [2, 10]. Here, for simplicity, we shall present the results for the proton only multiplicity, which is easier to measure in experiments.

To adapt equation such as Eq. (4) to protons, one needs to substitute $G$ with the coupling $g$ of the critical mode $\sigma$ to protons (times the mass $m_p$ of the proton): $G \rightarrow gm_p$. The estimate for this coupling can be taken from the sigma-model to be roughly $g \approx m_p/f_\pi \approx 10$. The variance of the occupation number distribution is $v^2_p = \bar{n}_p(1 - \bar{n}_p)$ and $\bar{n}_p = (e^{w_p - \mu_B}/T + 1)^{-1}$, where $\mu_B$ is baryochemical potential. There is also a factor of $2k^{−1}$ for $\omega_k$ (as in Eq. (20)) because of the proton spin degeneracy. Putting this together one finds, e.g., for protons at SPS freeze-out conditions $(T, \mu_B) \approx (168, 266)$ MeV [11]

$$\omega_3(N_p) \approx 6. \left( \frac{\tilde{\lambda}_3}{4.7} \right) \left( \frac{g}{10} \right)^3 \left( \frac{\xi}{1 \text{ fm}} \right)^{9/2}$$ \hspace{1cm} (15)

$$\omega_4(N_p) \approx 46. \left( \frac{2\tilde{\lambda}_3^2 - \tilde{\lambda}_4}{50.} \right) \left( \frac{g}{10} \right)^4 \left( \frac{\xi}{1 \text{ fm}} \right)^7$$ \hspace{1cm} (16)

Note that the effect is much larger on the proton multiplicity fluctuations, compared to the pion multiplicity.

Similar to quadratic fluctuations [9], the exponents in Eqs. (15, 16) agree (up to $\eta \ll 1$) with the critical exponents of the baryon number cumulants dictated by scaling and universality:

$$\langle (\delta N_B)^k \rangle_c \approx VT^{k-1} \frac{\partial^k F(T, \mu_B)}{\partial \mu_B^k} \sim \xi^k \cdot \eta^{k(5-\eta)/2-3}.$$ \hspace{1cm} (17)

Mean transverse momentum — From the expression for the correlators Eq. (11) or [10] one can similarly estimate the effect of the critical point on other observables, for example, higher moments of the fluctuation of mean transverse momentum $p_T$. For example, the cubic moment $\kappa_3$ of the mean $p_T$ distribution around the all-event mean $\bar{p}_T$ can be expressed as

$$\kappa_3(\delta p_T) \equiv \langle (p_T - \bar{p}_T)^3 \rangle = \sum_{p_1, p_2, p_3} \langle |p_1|\cdot|p_2|\cdot|p_3|\rangle \langle \delta n_{p_1, \delta n_{p_2, \delta n_{p_3}} \rangle \times \langle |p_2|\cdot|p_3|\rangle \langle |p_1|\cdot|p_2|\rangle \rangle$$ \hspace{1cm} (18)

and estimated using Eq. (11). Normalizing this variable similarly to the variable $F$ proposed in Ref. [4] removes $N$ scaling and makes it less sensitive to the effect of the flow (“blue shift” of momenta):

$$F_3 \equiv \frac{\kappa_3(\delta p_T)}{N_{\text{inc}}^{3/2}(\delta p_T)}$$ \hspace{1cm} (19)

where $v_{\text{inc}}(\delta p_T}$ is the variance of the inclusive (single-particle) $p_T$ distribution. We leave this to future work.
Comments and discussion — It is worth noting that, even though the $\xi$ dependence of $\omega_4$ is stronger, its measurement involves subtraction of two contributions, $(\delta N)^4 - 3(\delta N)^2$, each of which is order $N$ times larger than their difference, which might dilute the signal-to-noise ratio in experimental measurement.

Since the freeze-out occurs, generically, somewhat off the crossover line, as illustrated in Fig. 3, one should expect the critical point contribution to fluctuations to be skewed. In this case, the deviations from the Gaussian shape are dominated by the cubic moment, or $\omega_3$.

What is the sign of $\omega_3(N)_\sigma$? One can anticipate it by using the following, admittedly crude, argument. The skewness of the distribution of the order parameter near the critical point is a “shadow” of a second peak. This peak corresponds to the phase on the other side of the first-order transition line (quark-gluon plasma phase at higher $T$ and $\mu_B$ on Fig. 3). This phase has higher entropy and baryon number, thus fluctuations of these quantities must be skewed toward higher values: $\omega_3 > 0$.

Since pion and proton numbers are rough proxies to entropy and baryon number respectively their skewness should be also positive.

What is the “natural”, background value one should expect for, e.g., $\omega_3$? For a gas of classical free particles (Poisson distribution) $\omega_3(N) = 1$. Bose statistics increases this by a factor $\omega_3(N)_{\text{BE}} = (1 + n_p)(1 + 2n_p)$, e.g., approximately 1.3 for pions at $T = 120$ MeV.

More importantly, quantum statistics only correlates fluctuations of particles of the same species, thus $\omega_3(N)_{\text{BE}}$ is the same for all charge (i.e., $N = N_{\pi^+} + N_{\pi^-}$) and single charge multiplicity fluctuations. In contrast, the critical point contribution correlates also $\pi^+$ with $\pi^-$, thus making $\omega_3$ 4 times larger for all charge vs single charge (for $N_{\pi^+} = N_{\pi^-}$):

$$\omega_k(N_{\pi^+} + N_{\pi^-})_\sigma = 2^{k-1} \omega_k(N_{\pi^+})_\sigma. \quad (20)$$

Eq. (20) can help separate critical point contribution from contributions due to quantum statistics.

It is important to note that other sources may and do contribute to the skewness and kurtosis: remnants of initial fluctuations, flow, jets – to name just a few obvious contributors. Identification and evaluation of these contributions is a task well beyond this paper. This serves to emphasize that the energy scan of the QCD phase diagram is needed to separate such background contributions from the genuine critical point effect, the latter being non-monotonous function of the initial collision energy $\sqrt{s}$ as the critical point is approached and then passed. The fact that non-Gaussian moments have stronger dependence on $\xi$ than, e.g., quadratic moments, makes those higher moments more sensitive signatures of the critical point.

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[12] Strictly speaking, the correlation functions scale slightly differently than Eqs. 3 might suggest, e.g., $\langle \sigma_2^2 \rangle \sim \xi^{2-\eta}$.

Since the anomalous dimension $\eta \approx 0.04$ is very small, the difference between the actual asymptotic scaling and the scaling in Eq. 3 becomes discernible only for very large values of $\xi$, irrelevant in the context of this study. More importantly, the parameters $\lambda_3$ and $\lambda_4$ also scale with $\xi$ (see Eq. (12)).