NUMBER OF BOUND STATES OF THE SCHRÖDINGER OPERATOR OF A SYSTEM OF THREE BOSONS IN AN OPTICAL LATTICE.

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ABSTRACT. We consider the Hamiltonian $\hat{H}_\mu$ of a system of three identical particles (bosons) on the $d-$dimensional lattice $\mathbb{Z}^d$, $d = 1, 2$ interacting via pairwise zero-range attractive potential $\mu < 0$. We describe precise location and structure of the essential spectrum of the Schrödinger operator $H_\mu(K), K \in \mathbb{T}^d$ associated to $\hat{H}_\mu$ and prove the finiteness of the number of bound states of $H_\mu(K), K \in \mathbb{T}^d$ lying below the bottom of the essential spectrum. Moreover, we show that bound states decay exponentially at infinity and eigenvalues and corresponding bound states of $H_\mu(K), K \in \mathbb{T}^d$ are regular as a function of center of mass quasi-momentum $K \in \mathbb{T}^d$.

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1. Introduction

Cold atoms loaded in an optical lattice provide a realization of a quantum lattice gas. The periodicity of the potential gives rise to a band structure for the dynamics of the atoms.

The dynamics of the ultracold atoms loaded in the lower band is well described by the Bose-Hubbard hamiltonian [20]; we give in section 2 the corresponding Schrödinger operator.

In the continuous case [6,7,14] the energy of the center-of-mass motion can by separated out from the total Hamiltonian, i.e., the energy operator can by split into a sum of a center-of-mass motion and a relative kinetic energy. So that the three-particle bound states are eigenvectors of the relative kinetic energy operator.

The kinematics of the quantum particles on the lattice is rather exotic. The discrete Laplacian is not translationally invariant and therefore one cannot separate the motion of the center of mass.

One can rather resort to a Bloch-Floquet decomposition. The three-particle Hilbert space $\mathcal{H} \equiv \ell^2(\mathbb{Z}^d)^3$ is decomposed as direct integral associated to the representation of the discrete group $\mathbb{Z}^d$ by shift operators.

$$\ell^2[(\mathbb{Z}^d)^3] = \int_{\mathbb{T}^d} \oplus \ell^2[(\mathbb{Z}^d)^2] \eta(dK),$$

where $\eta(dp)$ is the (normalized) Haar measure on the torus $\mathbb{T}^d$. Hence the total three-body Hamiltonian $H$ of a system of three particles on $d-$dimensional lattice $\mathbb{Z}^d$, $d \geq 1$ interacting via pairwise short range attractive potential $V$ appears to be decomposable

$$H = \int_{\mathbb{T}^d} \oplus H(K) \eta(dK).$$
The fiber hamiltonians $H(K)$ depends parametrically on the quasi momentum $K \in T^d \equiv \mathbb{R}^d/(2\pi \mathbb{Z}^d)$. It is the sum of a free part depending on $K$ continuously and an interaction term, both bounded.

Bound states $\psi_{E,K}$ are solution of the Schrödinger equation

$$H(K)\psi_{E,K} = E\psi_{E,K} \quad \psi_{E,K} \in \ell^2((\mathbb{Z}^d)^2).$$

The Efimov effect is one of the remarkable results in the spectral analysis of Hamiltonians associated to a system of three-particles moving on the three-dimensional Euclid space: if none of the three two-particle Schrödinger operators (associated to the two-particle subsystems of a three-particle system) has negative eigenvalues, but at least two of them have zero energy resonance, then the three-particle Schrödinger operator has an infinite number of discrete eigenvalues, accumulating at zero [13, 4, 5, 8, 16, 17, 19].

The finiteness of eigenvalues (absence Efimov’s effect) have been proved for the Hamiltonian of a system of three particles moving on $d = 1, 2$ dimensional Euclid space $\mathbb{R}^d$ in [18].

We consider the Hamiltonian $\hat{H}_\mu$ of a system of three identical particles (bosons) on $d$-dimensional lattice $\mathbb{Z}^d$, $d = 1, 2$, interacting via pairwise zero range attractive potential $\mu < 0$.

We prove the finiteness of the number of bound states (absence Efimov’s effect) of the Schrödinger operator $H_\mu(K)$, $K \in G \subseteq T^d$, $d = 1, 2$, where $G \subseteq T^d$ is a region, associated to the Hamiltonian $\hat{H}_\mu$.

We describe a precise location and structure of the essential spectrum of the Schrödinger operator $H_\mu(K)$, $K \in T^d$ and $H_\mu(K)$, $K \in T^d$ are regular as a function of center of mass quasi-momentum $K \in T^d$.

In [12] finiteness of the eigenvalues of the discrete Schrödinger operator associated to a system of three-bosons on one dimensional lattice $\mathbb{Z}^1$ has been shown.

Section 1 is an introduction. In Section 2 we introduce the Hamiltonians of systems of two and three-particles in coordinate and momentum representations as bounded self-adjoint operators in the corresponding Hilbert spaces.

In Section 3 we introduce the total quasi-momentum, decompose the energy operators into von Neumann direct integrals, introduce discrete Schrödinger operators $h_\mu(k)$, $k \in T^d$ and $H_\mu(K)$, $K \in T^d$, choosing relative coordinate system.

We state the main results in Section 4.

We introduce the channel operators and describe the essential spectrum of $H_\mu(K)$ by means of the discrete spectrum of the two particle Schrödinger operators $h_\mu(k)$, $k \in T^d$ (Theorem 4.2) in section 5.

In Section 6 we prove the finiteness of the number of eigenvalues of the three-particle Schrödinger operator $H_\mu(K)$, $K \in T^d$ (Theorem 4.3) and finiteness of isolated bands in a system of three particles in an optical lattice.

2. Hamiltonians of three identical particles on a lattices in the coordinate and momentum representations

Let $\mathbb{Z}^d$, $d = 1, 2$ be the $d$-dimensional lattice. Let $\ell^2((\mathbb{Z}^d)^m), d = 1, 2$ be Hilbert space of square-summable functions $\hat{\varphi}$ defined on the Cartesian power of $(\mathbb{Z}^d)^m, d = 1, 2$ and let $\ell^{2,s}((\mathbb{Z}^d)^m) \subset \ell^2((\mathbb{Z}^d)^m)$ be the subspace of symmetric functions.
Let $\Delta$ be the lattice Laplacian, i.e., the operator which describes the transport of a particle from one site to another site:

$$(\Delta \hat{\psi})(x) = \frac{1}{2} \sum_{|s|=1} [\hat{\psi}(x) - \hat{\psi}(x + s)], \quad \hat{\psi} \in \ell^2(\mathbb{Z}^d).$$

The free Hamiltonian $\hat{h}_0$ of a system of two identical quantum mechanical particles with mass $m = 1$ on the $d$-dimensional lattice $\mathbb{Z}^d$, $d = 1, 2$ in the coordinate representation is associated to the self-adjoint operator $\hat{h}_0$ in the Hilbert space $\ell^2, \tau(\mathbb{Z}^d)^2)$:

$$\hat{h}_0 = \Delta \otimes I + I \otimes \Delta,$$

where $\otimes$ denotes the tensor product and $I$ is the identity operator on $L^2(\mathbb{Z}^d)$. The Hamiltonian $\hat{h}_\mu$ of a system of two identical particles with the two-particle pair zero-range attractive interaction $\mu \hat{v}$ is a bounded perturbation of the free Hamiltonian $\hat{h}_0$ on the Hilbert space $\ell^2, \tau(\mathbb{Z}^d)^2)$

$$\hat{h}_\mu = \hat{h}_0 + \mu \hat{v}.$$ 

Here $\mu < 0$ is coupling constant and

$$\hat{v}(x_1, x_2) = \delta_{x_1, x_2} \hat{\psi}(x_1, x_2), \quad \hat{\psi} \in \ell^2, \tau(\mathbb{Z}^d)^2),$$

where $\delta_{x_1, x_2}$ is the Kronecker delta.

Similarly, the free Hamiltonian $\hat{H}_0$ of a system of three identical particles on the $d$-dimensional lattice $\mathbb{Z}^d$ with mass $m = 1$ is defined on the Hilbert space $\ell^2, \tau(\mathbb{Z}^d)^3)$:

$$\hat{H}_0 = \Delta \otimes I + I \otimes \Delta \otimes I + I \otimes I \otimes \Delta.$$

The Hamiltonian $\hat{H}_\mu$ of a system of three identical particles with the two-particle pair zero-range interactions $\hat{v} = \hat{v}_\alpha \tau_{\beta, \gamma, \alpha, \beta, \gamma = 1, 2, 3}$ is a bounded perturbation of the free Hamiltonian $\hat{H}_0$

$$\hat{H}_\mu = \hat{H}_0 + \mu \hat{V},$$

where $\hat{V} = \sum_{\alpha=1}^{3} \hat{V}_\alpha$, $V_\alpha = \hat{V}, \alpha = 1, 2, 3$ is the multiplication operator on $\ell^2, \tau(\mathbb{Z}^d)^3)$ defined by

$$(\hat{V}_\alpha \hat{\psi})(x_1, x_2, x_3) = \delta_{x_1, x_2, x_3} \hat{\psi}(x_1, x_2, x_3),$$

$$\alpha < \beta < \gamma < \alpha, \alpha, \beta, \gamma = 1, 2, 3, \hat{\psi} \in \ell^2, \tau(\mathbb{Z}^d)^3).$$

### 2.1. The momentum representation.

Let $\mathbb{T}^d = (-\pi, \pi]^d$ be the $d$-dimensional torus and $L^2, \tau(\mathbb{T}^d)^m \subset L^2(\mathbb{T}^d)^m$ be the subspace of symmetric functions defined on the Cartesian power $(\mathbb{T}^d)^m, m \in \mathbb{N}$.

Let $\mathcal{F} : L^2(\mathbb{T}^d) \to \ell^2(\mathbb{Z}^d)$ be the standard Fourier transform

$$\mathcal{F} : \ell^2(\mathbb{Z}^d) \to L^2(\mathbb{T}^d), \quad [\mathcal{F}(f)](p) := \sum_{x \in \mathbb{Z}^d} e^{-i(p,x)} f(x)$$

with the inverse

$$\mathcal{F}^* : L^2(\mathbb{T}^d) \to \ell^2(\mathbb{Z}^d), \quad [\mathcal{F}^*(\psi)](x) := \int_{\mathbb{T}^d} e^{i(p,x)} \psi(p) \eta(dp),$$

and $\eta(dp) = \frac{dp}{(2\pi)^d}$ is the (normalized) Haar measure on the torus.

It easily can be checked that Fourier transform

$$\hat{\Delta} = \mathcal{F} \Delta \mathcal{F}^*$$
of the Laplacian $\Delta$ is the multiplication operator by the function $\varepsilon(\cdot)$:

$$(\hat{\Delta} f)(p) = \varepsilon(p) f(p), \quad f \in L^2(\mathbb{T}^d),$$

where

$$\varepsilon(p) = \sum_{i=1}^{d} (1 - \cos p(i)), \quad p = (p^{(1)}, \ldots, p^{(d)}) \in \mathbb{T}^d.$$  

The two-particle total Hamiltonian $h_{\mu}$ is given by

$$h_{\mu} = h_0 + \mu \nu,$$

where $\hat{I}$ is the identity operator on $L^2(\mathbb{T}^d)$. It is easy to see that the operator $h_0$ is the multiplication operator by the function $\varepsilon(k_1) + \varepsilon(k_2)$:

$$(h_0 f)(k_1, k_2) = [\varepsilon(k_1) + \varepsilon(k_2)] f(k_1, k_2), \quad f \in L^{2,\infty}([\mathbb{T}^d]^2)$$

and $\nu$ is the convolution type integral operator

$$(\nu f)(k_1, k_2) = \int_{(\mathbb{T}^d)^2} \delta(k_1 + k_2 - k'_1 - k'_2) f(k'_1, k'_2) \eta(dk'_1) \eta(dk'_2)$$

$$= \int_{\mathbb{T}^d} f(k'_1, k_1 + k_2 - k'_1) \eta(dk'_1), \quad f \in L^{2,\infty}([\mathbb{T}^d]^2),$$

where $\delta(\cdot)$ is the $d-$dimensional Dirac delta function.

The three-particle Hamiltonian in the momentum representation is given as bounded self-adjoint operator on the Hilbert space $L^{2,\infty}([\mathbb{T}^d]^3)$

$$H_{\mu} = H_0 + \mu (V_1 + V_2 + V_3),$$

where $H_0$ is of the form

$$H_0 = \hat{\Delta} \otimes \hat{I} \otimes \hat{I} + \hat{I} \otimes \Delta \otimes \hat{I} + \hat{I} \otimes \hat{I} \otimes \hat{\Delta},$$

i.e., the free Hamiltonian $H_0$ is the multiplication operator by the function $\sum_{\alpha=1}^{d} \varepsilon(k_{\alpha})$:

$$(H_0 f)(k_1, k_2, k_3) = \left[ \sum_{\alpha=1}^{3} \varepsilon(k_{\alpha}) \right] f(k_1, k_2, k_3),$$

and $V_\alpha = V, \alpha = 1, 2$ are convolution type integral operators

$$(V_\alpha f)(k_\alpha, k_\beta, k_\gamma) = \int_{(\mathbb{T}^d)^3} \delta(k_\alpha - k'_\alpha) \delta(k_\beta + k_\gamma - k'_\beta) f(k'_\alpha, k'_\beta, k'_\gamma) \eta(dk'_\alpha) \eta(dk'_\beta) \eta(dk'_\gamma)$$

$$= \int_{\mathbb{T}^d} f(k_\alpha, k'_\beta, k_\beta + k_\gamma - k'_\beta) \eta(dk'_\beta), \quad f \in L^{2,\infty}([\mathbb{T}^d]^3).$$
3. Decomposition of the energy operators into von Neumann direct integrals. Quasi-momentum and coordinate systems

Let \( k = k_1 + k_2 \in \mathbb{T}^d \) resp. \( K = k_1 + k_2 + k_3 \in \mathbb{T}^d \) be the two- resp. three-particle quasi-momentum and the set \( Q_k \) resp. \( Q_K \) is defined as follows

\[
Q_k = \{(k_1, k - k_1) \in (\mathbb{T}^d)^2 : k_1 \in \mathbb{T}^d, k - k_1 \in \mathbb{T}^d\}
\]
resp.

\[
Q_K = \{(k_1, k_2, K - k_1 - k_2) \in (\mathbb{T}^d)^3 : k_1, k_2 \in \mathbb{T}^d, K - k_1 - k_2 \in \mathbb{T}^d\}.
\]

We introduce the mapping

\[
\pi_1 : (\mathbb{T}^d)^2 \to \mathbb{T}^d, \quad \pi_1(k_1, k_2) = k_1
\]
resp.

\[
\pi_2 : (\mathbb{T}^d)^3 \to (\mathbb{T}^d)^2, \quad \pi_2(k_1, k_2, k_3) = (k_1, k_2).
\]

Denote by \( \pi_k, k \in \mathbb{T}^d \) resp. \( \pi_K, K \in \mathbb{T}^d \) the restriction of \( \pi_1 \) resp. \( \pi_2 \) onto \( Q_k \subset (\mathbb{T}^d)^2 \), resp. \( Q_K \subset (\mathbb{T}^d)^3 \), that is,

\[
\pi_k = \pi_1|_{Q_k} \quad \text{and} \quad \pi_K = \pi_2|_{Q_K}.
\]

It is useful to remark that \( Q_k, k \in \mathbb{T}^d \) resp. \( Q_K, K \in \mathbb{T}^d \) are the \( d \) resp. \( 2d \) dimensional manifold isomorphic to \( \mathbb{T}^d \) resp. \( (\mathbb{T}^d)^2 \).

**Lemma 3.1.** The mapping \( \pi_k, k \in \mathbb{T}^d \) resp. \( \pi_K, K \in \mathbb{T}^d \) is bijective from \( Q_k \subset (\mathbb{T}^d)^2 \) resp. \( Q_K \subset (\mathbb{T}^d)^3 \) onto \( \mathbb{T}^d \) resp. \( (\mathbb{T}^d)^2 \) with the inverse mapping given by

\[
(\pi_k)^{-1}(k_1) = (k_1, k - k_1)
\]
resp.

\[
(\pi_K)^{-1}(k_1, k_2) = (k_1, k_2, K - k_1 - k_2).
\]

Let \( L^{2,\sigma}(\mathbb{T}^d) \subset L^2(\mathbb{T}^d) \) be the subspace of even functions. Decomposing the Hilbert space \( L^{2,\sigma}((\mathbb{T}^d)^2) \) resp. \( L^{2,\sigma}((\mathbb{T}^d)^3) \) into the direct integral

\[
L^{2,\sigma}((\mathbb{T}^d)^2) = \int_{k \in \mathbb{T}^d} \oplus L^{2,\sigma}(\mathbb{T}^d)\eta(dk)
\]
resp.

\[
L^{2,\sigma}((\mathbb{T}^d)^3) = \int_{K \in \mathbb{T}^d} \oplus L^{2,\sigma}((\mathbb{T}^d)^2)\eta(dK)
\]
yields the decomposition of the Hamiltonian \( h_\mu \) resp. \( H_\mu \) into the direct integral

\[
h_\mu = \int_{k \in \mathbb{T}^d} \oplus \tilde{h}_\mu(k)\eta(dk)
\]
resp.

\[
H_\mu = \int_{K \in \mathbb{T}^d} \oplus \tilde{H}_\mu(K)\eta(dK).
\]
3.1. The discrete Schrödinger operators. The fiber operator $\tilde{h}_\mu(k), k \in \mathbb{T}^d$ is unitarily equivalent to the operators $h_\mu(k), k \in \mathbb{T}^d$ acting in $L^{2,\epsilon}(\mathbb{T}^d) \subset L_2(\mathbb{T}^d)$:

$$h_\mu(k) = h_0(k) + \mu \epsilon.$$ 

The operator $h_0(k)$ is the multiplication operator by the function $E_k(p)$:

$$(h_0(k)f)(p) = E_k(p)f(p), \quad f \in L^{2,\epsilon}(\mathbb{T}^d),$$

where

$$E_k(p) = \varepsilon\left(\frac{k}{2} - p\right) + \varepsilon\left(\frac{k}{2} + p\right)$$

and

$$(vf)(p) = \int_{\mathbb{T}^d} f(q)d\eta(q), \quad f \in L^{2,\epsilon}(\mathbb{T}^d).$$

The fiber operator $\tilde{H}_\mu(K), K \in \mathbb{T}^d$ is unitarily equivalent to the operator $H_\mu(K), K \in \mathbb{T}^d$ given by

$$H_\mu(K) = H_0(K) + \mu(V_1 + V_2 + V_3), V_\alpha = V, \quad \alpha = 1, 2, 3.$$ 

The operator $H_0(K), K \in \mathbb{T}^d$ acts in the Hilbert space $L^{2,\sigma}(\mathbb{T}^d)2$ and is of the form

$$(H_0(K)f)(p, q) = E(K; p, q)f(p, q), \quad f \in L^{2,\sigma}(\mathbb{T}^d),$$

where

$$E(K; p, q) = \epsilon(K - p - q) + \epsilon(p) + \epsilon(q).$$

The operator $V = V_1 + V_2 + V_3$ acts in $L^{2,\sigma}(\mathbb{T}^d)2$ and in the coordinates $(p, q) \in (\mathbb{T}^d)^2$ can be written as follows

$$(Vf)(p, q) = \int_{\mathbb{T}^d} f(p, t)\eta(dt) + \int_{\mathbb{T}^d} f(t, q)\eta(dt) + \int_{\mathbb{T}^d} f(t, K - p - q)\eta(dt), f \in L^{2,\sigma}(\mathbb{T}^d).$$

4. Statement of the main results

According to Weyl’s theorem [15] the essential spectrum $\sigma_{\text{ess spec}}(h_\mu(k))$ of the operator $h_\mu(k), k \in \mathbb{T}^d$ coincides with the spectrum $\sigma(h_0(k))$ of $h_0(k)$. More precisely,

$$\sigma_{\text{ess spec}}(h_\mu(k)) = [\mathcal{E}_{\text{min}}(k), \mathcal{E}_{\text{max}}(k)],$$

where

$$\mathcal{E}_{\text{min}}(k) \equiv \min_{p \in \mathbb{T}^d} \mathcal{E}_k(p) = 2 \sum_{i=1}^{d} \left[1 - \cos\left(\frac{k^{(i)}}{2}\right)\right]$$

$$\mathcal{E}_{\text{max}}(k) \equiv \max_{p \in \mathbb{T}^d} \mathcal{E}_k(p) = 2 \sum_{i=1}^{d} \left[1 + \cos\left(\frac{k^{(i)}}{2}\right)\right].$$

The following Theorem states the existence of a unique eigenvalue of the operator $h_\mu(k)$.

**Theorem 4.1.** For any $\mu < 0$ and $k \in \mathbb{T}^d$, $d = 1, 2$ the operator $h_\mu(k)$ has a unique eigenvalue $e_\mu(k)$, which is even on $\mathbb{T}^d$ and satisfies the relations $e_\mu(k) < \mathcal{E}_{\text{min}}(k), k \in \mathbb{T}^d$ and $e_\mu(0) < e_\mu(k), k \neq 0$. Moreover, for any $\mu < 0$ the eigenvalue $e_\mu(k)$ is regular function in $k \in \mathbb{T}^d$. 
The associated eigenfunction \( f_{\mu, e_{\mu}}(k)(\cdot) \), \( k \in \mathbb{T}^d \) is a regular function on \( \mathbb{T}^d \) and has the form

\[
    f_{\mu, e_{\mu}}(k)(\cdot) = \frac{\mu c(k)}{E_k(\cdot) - e_{\mu}(k)},
\]

where \( c(k) \neq 0, k \in \mathbb{T}^d \) is a normalizing constant. Moreover, the vector valued mapping

\[
    f_{\mu} : \mathbb{T}^d \to L^2[\mathbb{T}^d, \eta(dk); L^{2,v}(\mathbb{T}^d)], \quad k \to f_{\mu, e_{\mu}}(k)
\]

is regular on \( \mathbb{T}^d \).

Theorem 4.2 can be proven in the same way as in [9].

We note that the spectrum \( \sigma_{\text{spec}}(H_0(K)) \) of the operator \( H_0(K) \), \( K \in \mathbb{T}^d \) is the segment \( [E_{\min}(K), E_{\max}(K)] \), where

\[
    E_{\min}(K) \equiv \min_{p,q \in \mathbb{T}^d} E(K; p, q), \quad E_{\max}(K) \equiv \max_{p,q \in \mathbb{T}^d} E(K; p, q).
\]

We describe the essential spectrum of the three-particle operator \( H_{\mu}(K) \), \( K \in \mathbb{T}^d \) by the spectrum of the non perturbed operator \( H_{0}(K) \), \( K \in \mathbb{T}^d \) and the discrete spectrum of the two-particle operator \( h_{\mu}(k) \), \( k \in \mathbb{T}^d \) in the following theorem.

**Theorem 4.3.** For any \( \mu < 0 \) the essential spectrum \( \sigma_{\text{essspec}}(H_{\mu}(K)) \) of \( H_{\mu}(K) \), \( K \in \mathbb{T}^d \) is described as follows

\[
    \sigma_{\text{essspec}}(H_{\mu}(K)) = \cup_{k \in \mathbb{T}^d} \{ e_{\mu}(k) + \varepsilon(K - k) \} \cup [E_{\min}(K), E_{\max}(K)],
\]

where \( e_{\mu}(k) \) is a unique eigenvalue of the operator \( h_{\mu}(k) \), \( k \in \mathbb{T}^d \).

The next theorem states the finiteness of the number of eigenvalues for the Schrödinger operator \( H_{\mu}(K) \) and the analyticity of the eigenvalues and associated eigenfunctions.

Let \( U_{\delta}(K)[p_{\mu}(K)] = \{ K \in \mathbb{T}^d : |K - p_{\mu}(K)| < \delta \} \) be \( \delta = \delta(K) - \) neighborhood of the point \( p_{\mu}(K) \in \mathbb{T}^d \).

**Theorem 4.3.** Let \( d = 1, 2 \) and \( \mu < 0 \). Then

(i) There exists \( \delta > 0 \) such that for each \( K \in U_{\delta}[0] \) the operator \( H_{\mu}(K) \) has finite number of eigenvalues \( E_{1,\mu}(K), \ldots, E_{n_{\mu}}(K) \) lying below the bottom of the essential spectrum \( \sigma_{\text{essspec}}(H_{\mu}(K)) \) with the associated bound states

\[
    \psi_{\mu, E_{1,\mu}}(K)(\cdot), \ldots, \psi_{\mu, E_{n_{\mu}}}(K)(\cdot) \in L^{2,v}(\mathbb{T}^d)^2, \quad K \in U_{\delta}[0].
\]

(ii) The eigenfunction \( f_{\mu, E_{\mu}}(K)(\cdot, \cdot') \in L^{2,v}(\mathbb{T}^d)^2 \) of \( H_{\mu}(K) \) associated to the eigenvalue \( E_{\mu}(K) \), \( K \in U_{\delta}[0] \) is regular in \( (p, q) \in (\mathbb{T}^d)^2 \). Moreover, each eigenvalue \( E_{\mu}(K) \), \( K \in U_{\delta}[0] \) of \( H_{\mu}(K) \) is a regular function in \( K \in U_{\delta}[0] \) and the vector valued mapping

\[
    f_{\mu} : U_{\delta}[0] \to L^2[U_{\delta}[0], \eta(dK); L^{2,v}(\mathbb{T}^d)^2], \quad K \to f_{\mu, E_{\mu}}(K)
\]

is also regular on \( \mathbb{T}^d \).

**Corollary 4.4.** The two-particle Hamiltonian \( h_{\mu} \) has a unique isolated band spectrum and the three-particle Hamiltonian \( H_{\mu} \) have a finite number band spectrum.

Denote by \( \tau_{\text{spec}}(H_{\mu}(K)) \) resp. \( \tau_{\text{essspec}}(H_{\mu}(K)) \) the bottom of the spectrum resp. essential spectrum of the three-particle Schrödinger operator \( H_{\mu}(K) \), \( K \in \mathbb{T}^d \), i.e.,

\[
    \tau_{\text{spec}}(H_{\mu}(K)) = \inf_{\|f\| = 1} (H_{\mu}(K)f, f).
\]
\begin{equation}
\tau_{\text{essspec}}(H_{\mu}(K)) = \inf \sigma_{\text{essspec}}(H_{\mu}(K)).
\end{equation}

Let
\[\sigma_{\text{estwo}}(H_{\mu}(K)) = \cup_{k \in \mathbb{T}^d} \{ \epsilon_{\mu}(k) + \varepsilon(K - k) \},\]

resp.
\[\sigma_{\text{esthree}}(H_{\mu}(K)) = [E_{\text{min}}(K), E_{\text{max}}(K)]\]

be the two-particle resp. three-particle essential spectrum of \(H_{\mu}(K), K \in \mathbb{T}^d\) and
\[\tau_{\text{estwo}}(H_{\mu}(K)) = \inf \sigma_{\text{estwo}}(H_{\mu}(K))\]

resp.
\[\tau_{\text{esthree}}(H_{\mu}(K)) = E_{\text{min}}(K) = \inf_{||f||=1} \langle (H_0(K)f, f) \rangle\]

be the bottom of the two-particle resp. three-particle essential spectrum.

**Remark 4.5.** For the operator \(H_{\mu}(K)\) associated to a system of three bosons on the lattice \(\mathbb{Z}^d, d = 1, 2\) Theorems 4.1 and 4.2 give
\[\sigma_{\text{estwo}}(H_{\mu}(K)) \neq \emptyset\]
and
\[\tau_{\text{estwo}}(H_{\mu}(K)) < \tau_{\text{esthree}}(H_{\mu}(K))\]

and hence
\[\sigma_{\text{esthree}}(H_{\mu}(K)) \subset \sigma_{\text{essspec}}(H_{\mu}(K)).\]

Consequently, the inequality
\[\tau_{\text{essspec}}(H_{\mu}(K)) < \tau_{\text{esthree}}(H_{\mu}(K))\]

holds, which allows to prove the finiteness of the number of bound states of three interacting bosons on the lattice \(\mathbb{Z}^d, d = 1, 2\).

**Remark 4.6.** We remark that for the three-particle Schrödinger operator \(H_{\mu}(K)\), associated to a system of three bosons in the three-dimensional lattice \(\mathbb{Z}^3\), there exists \(\mu_0 > 0\) such that
\[\sigma_{\text{essspec}}(H_{\mu_0}(0)) = \sigma_{\text{esthree}}(H_{\mu_0}(0))\]

and hence
\[\tau_{\text{essspec}}(H_{\mu_0}(0)) = \tau_{\text{esthree}}(H_{\mu_0}(0)).\]

At the same time for any nonzero \(K \in \mathbb{T}^3\) the following relation
\[\sigma_{\text{estwo}}(H_{\mu_0}(K)) \neq \emptyset\]
holds and hence
\[\tau_{\text{estwo}}(H_{\mu_0}(K)) < \tau_{\text{esthree}}(H_{\mu_0}(K))\]

and
\[\sigma_{\text{esthree}}(H_{\mu_0}(K)) \subset \sigma_{\text{essspec}}(H_{\mu_0}(K)).\]

Thus only the operator \(H_{\mu_0}(0)\) may have an infinite number of eigenvalues below the bottom of the three-particle continuum (Efimov’s effect) \[3, 8\], which yields the existence of an infinite number of bound states.
5. The Essential Spectrum of the Operator $H_\mu(K)$.

Since we are considering the system of identical particles, there is only one channel operator $H_{\mu,\text{ch}}(K), K \in \mathbb{T}^d, \ d = 1, 2$ defined in the Hilbert space $L^{2,\sigma}[(\mathbb{T}^d)^2] = L^2(\mathbb{T}^d) \otimes L^{2,\sigma}(\mathbb{T}^d)$ as

$$H_{\mu,\text{ch}}(K) = H_0(K) + \mu V.$$ 

The operators $H_0(K)$ and $V = V_\alpha$ act as follows

$$(H_0(K)f)(p, q) = \mathcal{E}(K; p, q)f(p, q), \ f \in L^{2,\sigma}[(\mathbb{T}^d)^2],$$

where

$$\mathcal{E}(K; p, q) = \varepsilon(K - p) + \varepsilon(p - q) + \varepsilon(q + p)$$

and

$$(Vf)(p, q) = \int_{\mathbb{T}^d} f(p, t)\eta(dt), \ f \in L^{2,\sigma}[(\mathbb{T}^d)^2].$$

The decomposition of the space $L^{2,\sigma}[(\mathbb{T}^d)^2]$ into the direct integral

$$L^{2,\sigma}[(\mathbb{T}^d)^2] = \int_{k \in \mathbb{T}^d} \oplus L^{2,\sigma}(\mathbb{T}^d)\eta(dk)$$

yields for the operator $H_{\mu,\text{ch}}(K)$ the decomposition

$$H_{\mu,\text{ch}}(K) = \int_{k \in \mathbb{T}^d} \oplus h_\mu(K, k)\eta(dk).$$

The fiber operator $h_\mu(K, k)$ acts in the Hilbert space $L^{2,\sigma}(\mathbb{T}^d)$ and has the form

$$h_\mu(K, k) = h_\mu(k) + \varepsilon(K - k)I,$$

where $I = I_{L^{2,\sigma}(\mathbb{T}^d)}$ is the identity operator and $h_\mu(k)$ is the two-particle operator defined by (3.1). The representation (5.1) of the operator $h_\mu(K, k)$ and Theorem 4.1 yield for the spectrum of operator $h_\mu(K, k)$ the equality

$$\sigma(h_\mu(K, k)) = Z_\mu(K, k) \cup [\varepsilon_{\min}(k), \varepsilon_{\max}(k)],$$

where

$$Z_\mu(K, k) = e_\mu(k) + \varepsilon(K - k)$$

and $e_\mu(k)$ is the unique eigenvalue of the operator $h_\mu(k)$.

The spectrum of the channel operator $H_{\mu,\text{ch}}(K), K \in \mathbb{T}^d$ is described in the following

**Lemma 5.1.** The equality holds

$$\sigma(H_{\mu,\text{ch}}(K)) = \cup_{k \in \mathbb{T}^d} \{Z_\mu(K, k)\} \cup [\varepsilon_{\min}(K), \varepsilon_{\max}(K)].$$

**Proof.** The theorem (see, e.g., [13]) on the spectrum of decomposable operator and the structure (5.2) of the spectrum of $h_\mu(K, k)$ give the proof. \hfill \Box

The essential spectrum of $H_\mu(K), K \in \mathbb{T}^d$ is described in the following

**Theorem 5.2.** The equality

$$\sigma_{\text{ess}}(H_\mu(K)) = \sigma(H_{\mu,\text{ch}}(K))$$

holds.

**Proof.** Theorem 5.2 can be proven by the same way as Theorem 3.2 in [13]. \hfill \Box
Theorems 4.1 and 5.2 yield that the bottom $\tau_{\text{essspec}}(H_\mu(K))$ of the essential spectrum of the operator $H_\mu(K)$ is less than the bottom $\tau_{\text{spec}}(H_0(K)) = \tau_{\text{essthree}}(H_\mu(K))$ of the spectrum of the non-perturbed operator $H_0(K)$, which is attributed to the three-particle Schrödinger operators on the lattice $\mathbb{Z}^d$ and Euclid space $\mathbb{R}^d$ in dimensions $d = 1, 2$.

**Lemma 5.3.** For any $\mu < 0$ and $K \in \mathbb{T}^d$, $d = 1, 2$ the bottom of the essential spectrum of $H_\mu(K)$ satisfies the relations

$$
\tau_{\text{essspec}}(H_\mu(K)) < \tau_{\text{essthree}}(H_\mu(K)) = \tau_{\text{spec}}(H_0(K)) = E_{\text{min}}(K)
$$

holds, where $\tau_{\text{essspec}}(H_\mu(K))$ is defined in [4.1].

**Proof.** Theorem 4.1 yields that for any $k \in \mathbb{T}^d$ the operator $h_\mu(k)$ has a unique eigenvalue $e_\mu(k) < 2\varepsilon(\frac{k}{2}) = E_{\text{min}}(k)$.

Hence

$$
Z_\mu(K, k)\bigg|_{k = \frac{2K}{3}} = e_\mu\left(\frac{2K}{3}\right) + \varepsilon\left(\frac{K}{3}\right) < 2\varepsilon\left(\frac{K}{3}\right) + \varepsilon\left(\frac{K}{3}\right) = 3\varepsilon\left(\frac{K}{3}\right) = E_{\text{min}}(K).
$$

The definition of $\tau_{\text{essspec}}(H_\mu(K))$ gives

$$
\tau_{\text{essspec}}(H_\mu(K)) = \tau_{\text{spec}}(H_\mu^{ch}(K))
$$

$$
= \inf_{k \in \mathbb{T}^d} \sigma(H_\mu^{ch}(K)) = \inf_{k \in \mathbb{T}^d} Z_\mu(K, k) \leq e_\mu\left(\frac{2K}{3}\right) + \varepsilon\left(\frac{K}{3}\right) < 3\varepsilon\left(\frac{K}{3}\right),
$$

which proves Lemma 5.3. \hfill \square

6. PROOF OF THE MAIN RESULTS

For any $K, k \in \mathbb{T}^d$, $d = 1, 2$ the essential spectrum $\sigma_{\text{essspec}}(h_\mu(K, k))$ of the operator $h_\mu(K, k)$, $K, k \in \mathbb{T}^d$ coincides with the spectrum $\sigma(h_0(K, k))$ of $h_0(K, k)$. More precisely,

$$
\sigma_{\text{essspec}}(h_\mu(K, k)) = [E_{\text{min}}(K, k), E_{\text{max}}(K, k)].
$$

$$
E_{\text{min}}(K, k) = \min_q E(K, k; q) = \min_q \varepsilon_k(q) + \varepsilon(K - k) = 2\varepsilon\left(\frac{k}{2}\right) + \varepsilon(K - k)
$$

$$
E_{\text{max}}(K, k) = \max_q E(K, k; q) = \max_q \varepsilon_k(q) + \varepsilon(K - k) = [2d - \varepsilon\left(\frac{k}{2}\right)] + \varepsilon(K - k).
$$

The determinant $\Delta_\mu(K, k; z)$, $K, k \in \mathbb{T}^d$, $d = 1, 2$ associated to the operator $h_\mu(K, k)$ can be defined as a regular function in $C \setminus \{E_{\text{min}}(K, k), E_{\text{max}}(K, k)\}$ as

$$
\Delta_\mu(K, k; z) = 1 + \mu \int_{\mathbb{T}^d} \frac{\eta(dk)}{E(K, k; q) - z}.
$$

Let $L_\mu(K, z)$, $K \in \mathbb{T}^d$, $z < \tau_{\text{essspec}}(H_\mu(K))$ be a self-adjoint operator defined in $L^2(\mathbb{T}^d)$ as

$$
[L_\mu(K, z)w](p) = -\mu \int_{\mathbb{T}^d} \frac{\Delta_\mu^{\frac{1}{z}}(K, p; z)\Delta_\mu^{\frac{1}{z}}(K, q; z)}{E(K; p, q) - z} w(q)\eta(dq), w \in L_2(\mathbb{T}^d).
$$

The operator $L_\mu(K, z)$ is a lattice analogue of the Birman-Schwinger operator that has been introduced in [8] to investigate Efimov’s effect for the three-particle lattice Schrödinger operator $H_\mu(K)$. 
For a bounded self–adjoint operator \( A \) in a Hilbert space \( \mathcal{H} \) and for each \( \gamma \in \mathbb{R} \) we define the number \( n_+[\gamma, A] \) resp. \( n_-[\gamma, A] \) as

\[
n_+[\gamma, A] : = \max \{ \dim \mathcal{H}_A^+(\gamma) : \mathcal{H}_A^+(\gamma) \subset \mathcal{H} \text{ subspace with } \langle A\varphi, \varphi \rangle > \gamma, \varphi \in \mathcal{H}_A^+(\gamma), ||\varphi|| = 1 \} \]

resp.

\[
n_-[\gamma, A] : = \max \{ \dim \mathcal{H}_A^-(\gamma) : \mathcal{H}_A^-(\gamma) \subset \mathcal{H} \text{ subspace with } \langle A\varphi, \varphi \rangle < \gamma, \varphi \in \mathcal{H}_A^-(\gamma), ||\varphi|| = 1 \} .
\]

If some point of the essential spectrum of \( A \) is greater resp. smaller than \( \gamma \), then \( n_+[\gamma, A] \) resp. \( n_-[\gamma, A] \) is equal to infinity. If \( n_+[\gamma, A] \) resp. \( n_-[\gamma, A] \) is finite, then it is equal to the number of the eigenvalues (counting multiplicities) of \( A \), which are greater resp. smaller than \( \gamma \) (see, for instance, Glazman lemma [13]).

**Remark 6.1.** Theorem 6.2 yields that for any \( K \in \mathbb{T}^d \) the operator \( H_\mu(K) \) has no essential spectrum below \( \tau_{\text{essspec}}(H_\mu(K)) \).

**Lemma 6.2.** (The Birman-Schwinger principle). For each \( \mu < 0 \), \( K \in \mathbb{T}^d \) and \( z < \tau_{\text{essspec}}(H_\mu(K)) \) the operator \( L_\mu(K, z) \) is compact and the equality

\[
n_-[z, H_\mu(K)] = n_+(1, L_\mu(K, z)].
\]

holds. Moreover for any \( \mu < 0 \), \( K \in \mathbb{T}^d \) the operator \( L_\mu(K, z) \) is continuous in \( z \in (-\infty, \tau_{\text{essspec}}(H_\mu(K)) \).

**Proof.** We first verify the equality

(6.1)

\[
n_-[z, H_\mu(K)] = n_+(1, -3\mu R_0^{\frac{3}{2}}(K, z)VR_0^{\frac{1}{2}}(K, z)].
\]

Assume that \( u \in \mathcal{H}_{H_\mu(K)}(z) \subset L^2([\mathbb{T}^d])^2 \), that is, \( \langle (H_0(K) - z)u, u \rangle < -3\mu(Vu, u) \). Then

\[
(y, y) < (-3\mu R_0^{\frac{3}{2}}(K, z)VR_0^{\frac{1}{2}}(K, z)y, y), \quad y = R_0^{\frac{1}{2}}(K, z)u,
\]

where \( R_0(K, z) \) is the resolvent of the \( H_0(K). \) Hence

\[
n_-[z, H_\mu(K)] \leq n_+(1, -3\mu R_0^{\frac{3}{2}}(K, z)VR_0^{\frac{1}{2}}(K, z)].
\]

Reversing the argument we get the opposite inequality, which proves (6.1).

Note that any nonzero eigenvalue of \( R_0^{\frac{3}{2}}(K, z)VR_0^{\frac{1}{2}} \) is an eigenvalue for \( V^{\frac{1}{2}}R_0^{\frac{3}{2}}(K, z) \) as well, of the same algebraic and geometric multiplicities. Therefore we get

\[
n_+\left[1, -3\mu R_0^{\frac{3}{2}}(K, z)VR_0^{\frac{1}{2}}(K, z)\right] = n_+\left[1, -3\mu V^{\frac{1}{2}}R_0^{\frac{3}{2}}(K, z)V^{\frac{1}{2}}\right].
\]

Let us check the equality

\[
n_+\left[1, -3\mu V^{\frac{1}{2}}R_0^{\frac{1}{2}}(K, z)V^{\frac{1}{2}}\right] = n_+(1, L_\mu(K, z)].
\]

We show that for any

\[
u \in \mathcal{H}_{-3\mu V^{\frac{1}{2}}R_0^{\frac{1}{2}}(K, z)V^{\frac{1}{2}}}(1)
\]

there exists \( y \in \mathcal{H}_{L_\mu(K, z)}(1) \) such that \( (y, y) < (L_\mu(K, z)y, y) \).

Let \( u \in \mathcal{H}_{-3\mu V^{\frac{1}{2}}R_0^{\frac{1}{2}}(K, z)V^{\frac{1}{2}}}(1) \). Then

\[
(u, u) < -3\mu(V^{\frac{1}{2}}R_0^{\frac{1}{2}}(K, z)V^{\frac{1}{2}}u, u)
\]
and
\[(6.2) \quad ([I + \mu V^\frac{1}{2} R_0(K, z)V^\frac{1}{2}])u, u] < -2\mu(V^\frac{1}{2} R_0(K, z)V^\frac{1}{2}) u, u).\]

Since \(z < \tau_{\text{essspec}}(H_\mu(K))\) the operator \(I + \mu V^\frac{1}{2} R_0(K, z)V^\frac{1}{2}\) is invertible and positive the operator \(W_\mu^\frac{1}{2}(K, z) = (I + \mu V^\frac{1}{2} R_0(K, z)V^\frac{1}{2})^{-\frac{1}{2}}\) exists. Setting
\[y = (I + \mu V^\frac{1}{2} R_0(K, z)V^\frac{1}{2})^{\frac{1}{2}} u\]
gives us
\[(y, y) < -2\mu(W_\mu^\frac{1}{2}(K, z)V^\frac{1}{2} R_0(K, z)V^\frac{1}{2} W_\mu^\frac{1}{2}(K, z)y, y).\]
Since \(W_\mu^\frac{1}{2}(K, z)\) is the multiplication operator by the function \(\Delta_\mu^{-\frac{1}{2}}(K, p; z)\) the inequalities
\[(y, y) \leq (I_\mu(K, z)y, y)\]
and
\[n_+[1, -3\mu R_0^\frac{1}{2}(K, z)VR_0^\frac{1}{2}(K, z)] \leq n_+[1, I_\mu(K, z))\]
hold. By the same way one checks that
\[n_+(1, I_\mu(K, z)) \leq n_+(1, -3\mu R_0^\frac{1}{2}(K, z)VR_0^\frac{1}{2}(K, z)).\]

The following lemma gives the well known relation between the eigenvalues of \(h_\mu(K, k)\) and zeros of the determinant \(\Delta_\mu(K, k; z)\) \([3]\).

**Lemma 6.3.** For all \(K, k \in \mathbb{T}^d\) the number \(z \in C[\min E(K, k), \max E(K, k)]\) is an eigenvalue of the operator \(h_\mu(K, k)\) if and only if
\[\Delta_\mu(K, k; z) = 0.\]

The proof of Lemma 6.3 is usual and can be found \([9]\).

**Lemma 6.4.** The following assertions (i)–(iv) hold true.

(i) If \(f \in L^2,\sigma([\mathbb{T}^d]^2)\) solves \(H_\mu(K)f = zf, z < \tau_{\text{essspec}}(H_\mu(K))\) then
\[\psi(p) = \Delta_\mu^{-\frac{1}{2}}(K, p; z)\varphi(p), \text{ where } \varphi(p) = \int_{\mathbb{T}^d} f(p, t)\eta(dt) \in L^2(\mathbb{T}^d)\]
solves \(\psi = I_\mu(K, z)\psi\).

(ii) If \(\psi \in L^2(\mathbb{T}^d)\) solves \(\psi = I_\mu(K, z)\psi\), then
\[f(p, q) = \mu[\varphi(p) + \varphi(q) + \varphi(K - p - q)] \in L^2,\sigma([\mathbb{T}^d]^2),\]
where \(\varphi(p) = \Delta_\mu^{-\frac{1}{2}}(K, p; z)\varphi(p), \text{ solves the equation } H_\mu(K)f = zf\).

(iii) For any \(\mu < 0\) the eigenvalue \(E_\mu(K) < \tau_{\text{essspec}}(H_\mu(K))\) of the operator \(H_\mu(K)\) and the associated eigenfunction \(f \in L^2,\sigma([\mathbb{T}^d]^2)\) are regular in \(K \in \mathbb{T}^d\).

**Proof.**

(i) Let for some \(K \in \mathbb{T}^d\) and \(z < \tau_{\text{essspec}}(H_\mu(K))\) the equation
\[(6.3) \quad (H_\mu(K)\psi)(p, q) = z\psi(p, q),\]
i.e., the equation
\[
(6.4) \quad [E(K; p, q) - z] f(p, q) = -\mu \int_\mathbb{T}^d f(p, t)\eta(dt) + \int_\mathbb{T}^d f(t, q)\eta(dt) + \int_\mathbb{T}^d f(K - p - q, t)\eta(dt)]
\]
has a solution \( f \in L^2,\ast(\mathbb{T}^d)^2 \).

Denoting by
\[
\varphi(p) = \int_\mathbb{T}^d f(p, t)\eta(dt)
\]
we rewrite the equation (6.4) as follows
\[
(6.5) \quad f(p, q) = -\mu \frac{\varphi(p) + \varphi(q) + \varphi(K - p - q)}{E(K; p, q) - z} \in L^2,\ast(\mathbb{T}^d)^2,
\]
which gives for \( \varphi \in L^2(\mathbb{T}^d) \) the equation
\[
(6.6) \quad \varphi(p) = -\mu \int_\mathbb{T}^d \frac{\varphi(p) + \varphi(t) + \varphi(K - p - t)}{E(K; p, t) - z} \eta(dt).
\]

Since the function \( E(K; p, t) \) is invariant under \( K - p - t \to t \), we have
\[
\varphi(p) \left[ 1 + \mu \int_\mathbb{T}^d \frac{dq}{E(K; p, q) - z} \right] = 2\mu \int_\mathbb{T}^d \frac{\varphi(q)}{E(K; p, q) - z} \eta(dq)
\]
Denoting by \( \Delta^{\frac{1}{2}}_\mu(K, p; z) \varphi(p) = \psi(p) \) and taking into account the inequality \( \Delta_\mu(K, p; z) \neq 0, z < \tau_{\text{essspec}}(H_\mu(K)) \) we get the equation
\[
(6.7) \quad \psi(p) = -2\mu \int_\mathbb{T}^d \frac{\Delta^{\frac{1}{2}}_\mu(K, p; z) \Delta^{\frac{1}{2}}_\mu(K, q; z) \psi(q)}{E(K; p, q) - z} \eta(dq).
\]

(ii) Assume that for some \( z < \tau_{\text{essspec}}(H_\mu(K)) \) the function \( \psi \in L^2(\mathbb{T}^d) \) is a solution of the equation (6.7). Then \( \varphi(p) = \Delta^{-\frac{1}{2}}_\mu(K, p; z) \psi(p) \in L^2(\mathbb{T}^d) \) is a solution of the equation (6.6). Hence the function defined by (6.5) belongs \( L^2,\ast(\mathbb{T}^d)^2 \) and is a solution of the Schrödinger equation \( H_\mu(K)f = zf \), i.e., \( f \) is an eigenfunction of the operator \( H_\mu(K) \) associated to the eigenvalue \( z \prec \tau_{\text{essspec}}(H_\mu(K)) \).

(iii) For all \( \mu < 0, K \in \mathbb{T}^d \) and \( z \prec \tau_{\text{essspec}}(H_\mu(K)) \) the kernel function
\[
L_\mu(K, z; p, q) = -2\mu \frac{\Delta^{\frac{1}{2}}_\mu(K, p; z) \Delta^{\frac{1}{2}}_\mu(K, q; z)}{E(K; p, q) - z}
\]
of the operator \( L_\mu(K, z) \) is continuous in \( p, q \in \mathbb{T}^d \). Hence, for any \( \mu < 0 \) and \( K \in \mathbb{T}^d \) the Fredholm determinant \( D_\mu(K, z) = \det[I - L_\mu(K, z)] \) associated to \( L_\mu(K, z; p, q) \) is real and regular function in \( z \in (-\infty, \tau_{\text{essspec}}(H_\mu(K))) \).

Lemma 6.4 and the Fredholm theorem yield that each eigenvalue \( E_\mu(K) \in (-\infty, \tau_{\text{essspec}}(H_\mu(K))) \) of the operator \( H_\mu(K) \) is a zero of the determinant \( D_\mu(K, z) \) and vice versa. Consequently, the compactness of the torus \( \mathbb{T}^d \) and implicit function theorem yield that for each \( \mu < 0 \) the eigenvalue \( E_\mu(K) \) of \( H_\mu(K) \) is a regular function in \( K \in \mathbb{T}^d, d = 1, 2 \).

Since the functions \( \Delta_\mu(K, p; E_\mu(K)) \) and \( E(K; p, q) - E_\mu(K) \) are regular in \( K \in \mathbb{T}^d \) the solution \( \psi \in L^2(\mathbb{T}^d) \) of the equation (6.7) and hence the function
Therefore for any function \( \varphi \) are regular in \( K \in T^d \). Hence, the eigenfunction (6.5) of the operator \( H_\mu(K) \) associated to eigenvalue \( E_\mu(K) < \tau_{\text{essspec}}(H_\mu(K)) \) is also regular in \( K \in T^d \).

Consequently, the vector valued mapping

\[
f_\mu : T^d \rightarrow L^2[T^d, \eta(dK); L^{2,s}(T^d, \eta(dK))], K \rightarrow f_{\mu,K}(\cdot, \cdot)
\]

is regular in \( T^d \).

Now we are going to proof the finiteness of the number \( N(K, \tau_{\text{essspec}}(H_\mu(K))) \) of eigenvalues of the three-particle Schrödinger operator \( H_\mu(K) \), \( K \in U_\delta(0) \).

We postpone the proof of the main theorem after the following two lemmas.

**Lemma 6.5.** Let \( d = 1, 2 \). For any \( K \in U_\delta(0) \) there are positive nonzero constants \( C_1 \) and \( C_2 \) depending on \( K \) and a neighborhood \( U_\delta(K)[p_\mu(K)] \) of the point \( p_\mu(K) \in T^d \) such that for all \( p \in U_\delta(K)[p_\mu(K)] \) the following inequalities

\[
C_1 |p - p_\mu(K)|^2 \leq \Delta_\mu(K, p, \tau_{\text{essspec}}(H_\mu(K))) \leq C_2 |p - p_\mu(K)|^2
\]

hold.

**Proof.** We prove Lemma 6.5 for the case \( d = 2 \). The point \( p = 0 \) is the non degenerate minimum of the function \( \varepsilon(p) \), i.e.,

\[
\varepsilon(p) = \frac{1}{2} p^2 + O(|p|^3) \text{ as } p \to 0.
\]

Since the eigenvalue \( e_\mu(p) \) lying below the essential spectrum is a unique zero of the determinant \( \Delta_\mu(p, z) \) associated to operator \( h_\mu(p) \), simple computations gives

\[
\left( \frac{\partial^2 e_\mu(p)}{\partial p^{(i)} \partial p^{(j)}} \right)_{i,j=1}^2 = C \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad C > 0.
\]

Analogously, the eigenvalue \( Z_\mu(0, p) \) of the operator \( h_\mu(0, p) \), lying below the essential spectrum, is unique zero of the determinant \( \Delta_\mu(0, p, z) \) and hence the point \( p = p_\mu(0) = 0 \in T^2 \) is non-degenerate minimum of

\[
Z_\mu(0, p) := e_\mu(p) + \varepsilon(p).
\]

Therefore for any \( K \in U_\delta(0) \) the point \( p_\mu(K) \in U_\delta(0) \) is non degenerate minimum of the function \( Z_\mu(K, p) \) and the matrix

\[
B(K) = \left( \frac{\partial^2 Z_\mu(K, p_\mu(K))}{\partial p^{(i)} \partial p^{(j)}} \right)_{i,j=1}^2
\]

is positive definite. Hence, the eigenvalue \( Z_\mu(K, p) \) has following asymptotics

\[
(6.8) \quad Z_\mu(K, p) = \tau_{\text{essspec}}(H_\mu(K)) + (B(K)(p-p_\mu(K)), p-p_\mu(K)) + o(|p-p_\mu(K)|^2),
\]

as \( |p-p_\mu(K)| \to 0 \), where \( \tau_{\text{essspec}}(H_\mu(K)) = Z_\mu(K, p_\mu(K)) \).

For any \( K, p \in T^d \) there exists a \( \gamma = \gamma(K, p) > 0 \) neighborhood \( W_\gamma(Z_\mu(K, p)) \) of the point \( Z_\mu(K, p) \in C \) such that for all \( z \in W_\gamma(Z_\mu(K, p)) \) the following equality holds

\[
\Delta_\mu(K, p, z) = \sum_{n=1}^{\infty} C_n(\mu, K, p)(z - Z_\mu(K, p))^n,
\]

where
Lemma 6.6. Let $K \in U_\delta(0)$. The operator $L_\mu(K, z)$ can be represented as sum of the two operators

$$ L_\mu(K, z) = L_\mu^{(1)}(K, z) + L_\mu^{(2)}(K, z), $$

where the operator $L_\mu^{(1)}(K, z)$, $z < \tau_{\text{essspec}}(H_\mu(K))$ has finite rank and $L_\mu^{(2)}(K, z)$, $z \leq \tau_{\text{essspec}}(H_\mu(K))$ belongs to the Hilbert-Schmidt class.

Proof. We represent the operator $L_\mu(K, z)$ as sum of two operators

$$ L_\mu(K, z) = L_\mu^{(1)}(K, z) + L_\mu^{(2)}(K, z), $$

where

$$ [L_\mu^{(1)}(K, z)w](p) = 2\mu \int_{\mathbb{T}^d} L_\mu^{(1)}(K, z; p, q)w(q)\eta(dq), $$

is the finite rank operator, where

$$ L_\mu^{(1)}(K, z; p, q) = \frac{1}{E(K; p, p_\mu(K)) - z} + \frac{1}{E(K; p_\mu(K), q) - z} - \frac{1}{E(K; p_\mu(K), p_\mu(K)) - z} $$

and

$$ [L_\mu^{(2)}(K, z)w](p) = 2\mu \int_{\mathbb{T}^d} L_\mu^{(2)}(K, z; p, q)w(q)\eta(dq), $$

where

$$ L_\mu^{(2)}(K, z; p, q) = \frac{1}{E(K; p, q) - z} - L_\mu^{(1)}(K, z; p, q). $$

For any $z < \tau_{\text{essspec}}(H_\mu(K))$ the kernel $L_\mu^{(2)}(K, z; p, q)$ of the operator $L_\mu^{(2)}(K, z)$ is a regular function at the point $(p_\mu(K), p_\mu(K))$ and $L_\mu^{(2)}(K, z; p_\mu(K), p_\mu(K)) = 0$. Lemma 6.5 yields the inequality

$$ E(K; p, q) - \tau_{\text{essspec}}(H_\mu(K)) \geq E_{\text{min}}(K) - \tau_{\text{essspec}}(H_\mu(K)) > 0. $$

Hence the functions

$$ [E(K; p, q) - \tau_{\text{essspec}}(H_\mu(K))]^{-1} - L_\mu^{(1)}(K, \tau_{\text{essspec}}(H_\mu(K)); p, q) $$

is regular in $(p, q) \in \mathbb{T}^2$.

So, the operator $L_\mu^{(2)}(K, \tau_{\text{essspec}}(H_\mu(K)))$, $K \in U_\delta(0)$ belongs to the Hilbert-Schmidt class. \qed
Proof of Theorem 4.3 For any compact operators $A_1$, $A_2$ and positive numbers $\lambda_1$, $\lambda_2$ Weyl’s inequality
\[ n_+[\lambda_1 + \lambda_2, A_1 + A_2] \leq n_+ [\lambda_1, A_1] + n_+ [\lambda_2, A_2] \]
gives that for all $z < \tau_{\text{essspec}}(H_\mu(K))$, $K \in U_\delta(0)$ the relations
\[ n_+ [1, L_\mu^2(K, z)] \leq n_+ \left[ \frac{1}{3}, L_\mu^1(K, z) \right] + n_+ \left[ \frac{2}{3}, L_\mu^2(K, z) \right] \leq \]
\[ n_+ \left[ \frac{1}{3}, L_\mu^1(K, z) \right] + n_+ \left[ \frac{1}{3}, L_\mu^2(K, z) \right] + n_+ \left[ \frac{1}{3}, L_\mu^2(K, z) - L_\mu^2(K, \tau_{\text{essspec}}(H_\mu(K))) \right] \]
hold. The compactness of $L_\mu^2(K, \tau_{\text{essspec}}(H_\mu(K))), K \in U_\delta(0)$ yields
\[ n_+ \left[ \frac{1}{3}, L_\mu^2(K, \tau_{\text{essspec}}(H_\mu(K))) \right] < \infty. \]
The operator $L_\mu^2(K, z)$ is continuous in $z \leq \tau_{\text{essspec}}(H_\mu(K))$ and converges to $L_\mu^2(K, \tau_{\text{essspec}}(H_\mu(K)))$ in uniform operator topology. Therefore for all sufficiently small $|\tau_{\text{essspec}}(H_\mu(K)) - z|$ we have
\[ n_+ \left[ \frac{1}{3}, L_\mu^2(K, z) - L_\mu^2(K, \tau_{\text{essspec}}(H_\mu(K))) \right] = 0. \]
The dimension of rank of $L_\mu^1(K, z), z < \tau_{\text{essspec}}(H_\mu(K))$ does not depend on $z < \tau_{\text{essspec}}(H_\mu(K))$ and
\[ n_+ \left[ \frac{1}{3}, L_\mu^1(K, z) \right] < \infty. \]
Lemma 6.2 yields that the operator $L_\mu(K, \tau_{\text{essspec}}(H_\mu(K))), K \in U_\delta(0)$ has a finite number eigenvalues greater than 1 and consequently by Lemma 6.2 the operator $H_\mu(K), K \in U_\delta(0)$ has a finite number of eigenvalues in the interval $(-\infty, \tau_{\text{essspec}}(H_\mu(K)))$.

From Lemma 6.6 one concludes that if $E_\mu(K) < \tau_{\text{essspec}}(H_\mu(K))$ is an eigenvalue of the operator $H_\mu(K)$ then the associated eigenfunction is of the form $(6.5)$.

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