Operator Product Expansions in the Two-Dimensional O(N) Non-Linear Sigma Model

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The short-distance singularity of the product of a composite scalar field that deforms a field theory and an arbitrary composite field can be expressed geometrically by the beta functions, anomalous dimensions, and a connection on the theory space. Using this relation, we compute the connection perturbatively for the O(N) non-linear sigma model in two dimensions. We show that the connection becomes free of singularities at zero temperature only if we normalize the composite fields so that their correlation functions have well-defined limits at zero temperature.

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1. Introduction

In refs. [1], [2], and [3] it was found that the singularities of a conjugate field that deforms a field theory and an arbitrary composite field admit a geometrical expression. This paper is a continuation of the study of the geometrical structure. The simple example of the four-dimensional $\phi^4$ theory has been examined in ref. [2]. In the present paper we will study a more non-trivial example of the two-dimensional non-linear sigma model in some detail.

Let us briefly summarize the geometrical structure obtained in refs. [1], [2], and [3]. We consider a finite dimensional theory space with local coordinates $g^i (i = 1, ..., N)$, which are nothing but the parameters of a renormalized euclidean field theory in $D$-dimensions. Let the scalar field conjugate to $g^i$ be $O_i$. The conjugate fields form a basis of the tangent vector bundle of the theory space. The linearly independent composite fields $\{\Phi_a\}_g$ make a basis of an infinite dimensional vector bundle.

We denote the renormalization group (RG) equations of the parameters by

$$\frac{d}{dt} g^i = \beta^i (g). \quad (1.1)$$

The beta functions $\beta^i$ form a vector field on the theory space. Eq. (1.1) implies that the conjugate fields satisfy

$$\frac{d}{dt} O_i = \left( D\delta^j_i - \frac{\partial \beta^j}{\partial g^i} \right) O_j. \quad (1.2)$$

We denote the RG equations of the composite fields by

$$\frac{d}{dt} \Phi_a = \Gamma^b_a (g) \Phi_b. \quad (1.3)$$

Let us introduce the operator product expansion (OPE) of a conjugate field and a composite field:

$$O_i(r) \Phi_a(0) = \frac{1}{\text{vol}(S^{D-1})} (C_i)_a^b (r; g) \Phi_b(0) + o \left( \frac{1}{r^D} \right), \quad (1.4)$$

where we take the angular average over $r$, and we keep only the part which cannot be integrated over in space. Then, the matrix

$$H_i(g) \equiv C_i(r = 1; g) \quad (1.5)$$
is a tensor on the theory space. In ref. [3] the following geometrical expression for the tensor $H_i$ has been written:

$$H_i(g) = \partial_i \Psi(g) + [c_i(g), \Psi(g)] + \beta^j(g) \Omega_{ji}(g),$$  \hspace{1cm} (1.6)

where we define

$$\Psi(g) \equiv \Gamma(g) + \beta^i(g)c_i(g),$$  \hspace{1cm} (1.7)

and

$$\Omega_{ij}(g) \equiv \partial_i c_j - \partial_j c_i + [c_i, c_j].$$  \hspace{1cm} (1.8)

The matrix $c_i(g)$ is a connection on the theory space. It is a connection, since it transforms as

$$c_i(g) \rightarrow N(g)(\partial_i + c_i(g))(N(g))^{-1}$$  \hspace{1cm} (1.9)

under an arbitrary change of basis

$$\Phi_a \rightarrow (N(g))_a^b \Phi_b.$$  \hspace{1cm} (1.10)

It is easy to check that $\Psi(g)$ is a tensor. Eq. (1.8) gives a curvature of the connection $c_i$. The curvature can be obtained as the double integral over connected three-point functions [3]:

$$\Omega_{ji} \langle \Phi \rangle_g = \int_{1 \geq r} d^D r \ F.P. \int_{1 \geq r'} d^D r' \times \left( \left( O_i(r) \left( O_j(r') - \frac{1}{\text{vol}(S^{D-1})} C_{j}(r') \right) - (r \leftrightarrow r') \right) \Phi(0) \right)_g^c,$$  \hspace{1cm} (1.11)

where F.P. stands for taking the integrable part with respect to $r$.

The connection $c_i$ first appeared as finite counterterms in the so-called variational formula [4]. (In two dimensions a connection had been introduced by Kutasov [4].) The spatial integral over the conjugate field $O_i$ gives a field theoretic realization of the partial derivative with respect to the coordinate $g^i$, and finite counterterms are necessary to compensate the arbitrariness involved in the short-distance regularization of the spatial integral. The geometrical formula (1.6) implies that the connection $c_i$ actually controls the short-distance physics together with such obviously important quantities like the beta functions and anomalous dimensions.

The purpose of the paper is to study the connection $c_i$ in the two-dimensional $O(N)$ non-linear sigma model. The model is interesting in its similarity to the four-dimensional
non-abelian gauge theories: both are asymptotic free, and the relevant symmetries are realized non-linearly. We will find that the behavior of the connection \( c_i \) at low temperatures (equivalently, at short distances) contains information on the zero temperature limit of the theory. More specifically, we will find that the elements of the connection \( c_i \) are finite at zero temperature only if we normalize the composite fields such that their correlation functions have well-defined and non-trivial zero temperature limits.

The paper is organized as follows. In sect. 2 we describe the model. Since the model is well-known, we will be brief. In sect. 3 we give the results of perturbative calculations of the coefficients in the OPE’s of the conjugate fields and composite fields. The details are given in the appendices. In sect. 4 we determine the elements of the connection \( c_i \) perturbatively. We find that some elements diverge at zero temperature. In sect. 5 we study the low temperature behavior of the theory to understand the physical meaning of the divergences encountered in sect. 4. We conclude the paper in sect. 6.

2. The model

We define the theory in \( D = 2 + \epsilon \) dimensions using dimensional regularization and the minimal subtraction (MS) scheme. The lagrangian is

\[
\mathcal{L} = \frac{1}{g_0} \left( \frac{1}{2} \partial_\mu \Phi_0^I \partial_\mu \Phi_0^I - m_0^2 \Phi_0^N \right),
\]

where \( \Phi_0^I (I = 1, \ldots, N) \) is the bare field, normalized by

\[
\sum_{I=1}^{N} \Phi_0^I \Phi_0^I = 1,
\]

and \( g_0, m_0^2 \) are the bare temperature and magnetic field in the N-th direction. The renormalized parameters are

\[
g_0 = Z_g(\epsilon; g) g, \quad m_0^2 = Z_m(\epsilon; g) m^2.
\]

We will suppress the renormalization scale \( \mu \). In actual calculations we choose a particular value for \( \mu \) to simplify the results (see sect. 3). Let us denote the RG equations for the renormalized parameters by

\[
\frac{d}{dt} g = \beta_g(g), \quad \frac{d}{dt} m^2 = (2 + \beta_m(g)) m^2
\]
in the limit $\epsilon \to 0$.

The renormalized O(N) vector field $\Phi^I$ is defined by

$$\Phi^I_0 = \sqrt{Z_\Phi(\epsilon; g)} \Phi^I,$$  \hspace{1cm} \text{(2.5)}

where the renormalization constant $Z_\Phi$ satisfies the relation \[5\]

$$Z_m \sqrt{Z_\Phi} = Z_g.$$  \hspace{1cm} \text{(2.6)}

The fields conjugate to $g, m^2$, denoted by $O_g, O_m$, are defined by

$$O_g \equiv \frac{\partial L}{\partial g} \bigg|_{m^2, \Phi_0} = - \frac{1}{g^2} B_0 + \frac{m^2}{g^2} A_1,$$

$$O_m \equiv \frac{\partial L}{\partial m^2} \bigg|_{g, \Phi_0} = - \frac{1}{g} A_1,$$  \hspace{1cm} \text{(2.7)}

where we define the renormalized fields $B_0, A_1$ as

$$B_0 = \left[ \frac{1}{2} \partial_{\mu} \Phi^I \partial_{\nu} \Phi^I \right] \equiv \left( 1 + g \frac{\partial \ln Z_g}{\partial g} \right) \frac{Z_\Phi}{Z_g} \frac{1}{2} \partial_{\mu} \Phi^I \partial_{\nu} \Phi^I - m^2 g \frac{\partial \ln Z_\Phi}{\partial g} \frac{1}{2} \Phi^N,$$

$$A_1 \equiv \Phi^N.$$  \hspace{1cm} \text{(2.8)}

(The bracket indicates a renormalized field in the MS scheme.)

The conjugate fields satisfy the RG equations (in the limit $\epsilon \to 0$):

$$\frac{d}{dt} O_m = - \beta_m O_m,$$  \hspace{1cm} \text{(2.9)}

$$\frac{d}{dt} O_g = (2 - \beta'_g) O_g - m^2 \beta_m O_m.$$

We cannot possibly consider all the composite fields. As examples, we examine the four fields $A_1, A_2, \partial^2 A_1$, and $B_0$, where we define the (N,N)-component of a symmetric traceless O(N) tensor field $A_2$ by

$$A_2 \equiv Z_{A_2} \left( \Phi_0^N \Phi_0^N - \frac{1}{N} \right).$$  \hspace{1cm} \text{(2.10)}

The O(N) symmetry restricts the RG equations in the following form:

$$\frac{d}{dt} \Phi = \Gamma(g, m^2) \Phi,$$  \hspace{1cm} \text{(2.11)}
where

\[ \Phi \equiv (1, A_1, A_2, B_0, \partial^2 A_1)^T, \]
\[ \Gamma(g, m^2) \equiv \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & \gamma_{A_1} & 0 & 0 & 0 \\ 0 & 0 & \gamma_{A_2} & 0 & 0 \\ 0 & m^2 \gamma_{B_0} & 0 & 2 + \gamma_{B_0} & 0 \\ 0 & 0 & 0 & 0 & 2 + \gamma_{A_1} \end{pmatrix}. \]

(2.12)

Note that Eqs. (2.6), (2.7), and (2.9) imply

\[ \gamma_{A_1} = \frac{\beta_g}{g} - \beta_m \]
\[ \gamma_{B_0} = \frac{\beta_g}{g} \]
\[ \gamma_{B_0} = -g^2 \frac{\partial}{\partial g} \frac{\beta_g}{g^2}. \]

(2.14)

Similarly, the O(N) invariance implies

\[ (C_m)(r; g) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & (C_m)^{A_1}(r; g) & 0 & 0 \\ 0 & (C_m)^B_0(r; g) & 0 & 0 & 0 \\ (C_m)^{1\partial^2 A_1}(r; g) & 0 & (C_m)^{2\partial^2 A_1}(r; g) & 0 & 0 \end{pmatrix}, \]

(2.15)

and the nonvanishing elements of the matrix \( C_g(r; g, m^2) \) are

\[ (C_g)^{A_1}(r; g), \quad (C_g)^{A_2}(r; g), \]
\[ (C_g)^B_0(r; g), \quad (C_g)^{A_1}(r; g, m^2), \quad (C_g)^B_0(r; g), \]
\[ (C_g)^{1\partial^2 A_1}(r; g, m^2), \quad (C_g)^{A_1}(r; g), \quad (C_g)^{A_2}(r; g, m^2), \quad (C_g)^{2\partial^2 A_1}(r; g), \]

where we have taken the angular average over \( r \), and the terms less singular than \( 1/r^2 \) have been dropped. All the unwritten elements of \( C_g \) vanish. Following (1.5) we define

\[ H_m(g) \equiv C_m(r = 1; g), \quad H_g(g, m^2) \equiv C_g(r = 1; g, m^2). \]

(2.17)

We introduce the connection \( (c_m(g), c_g(g, m^2)) \), which has non-vanishing elements exactly where the OPE coefficients have non-vanishing elements.

In this particular model, Eq. (1.9) implies

\[ H_m(g) = \frac{\partial}{\partial m^2} \Psi + [c_m, \Psi] + \beta_g \Omega_{gm}, \]
\[ H_g(g, m^2) = \frac{\partial}{\partial g} \Psi + [c_g, \Psi] - (2 + \beta_m)m^2 \Omega_{gm}, \]

(2.18)
where
\[
\Psi(g, m^2) \equiv \Gamma + (2 + \beta_m)m^2c_m + \beta_g c_g ,
\]
\[
\Omega_{gm}(g) \equiv \frac{\partial}{\partial g} c_m - \frac{\partial}{\partial m^2} c_g + [c_g, c_m] .
\]

Eq. (1.11) gives
\[
\Omega_{gm}(g) \langle \Phi \rangle_{g,m^2} = \int_{1 \geq r} d^2 r \text{ F.P.} \int_{1 \geq r'} d^2 r' \times \left( \left( \mathcal{O}_m(r) \left( \mathcal{O}_g(r') - \frac{1}{2\pi} C_g(r') \right) - (r \leftrightarrow r') \right) \Phi(0) \right)^c .
\]

In the next two sections we will compute the OPE coefficients \( H_m, H_g \) and the curvature \( \Omega_{gm} \) perturbatively and determine the connection \( c_m, c_g \) by solving Eqs. (2.18) and (2.20).

3. Perturbative calculations

The technique of perturbative calculations is well-known. In order to obtain OPE coefficients in the coordinate space, we have used the formula
\[
\int \frac{d^2+\epsilon k}{(2\pi)^2+\epsilon} \mu^{-\epsilon} \frac{e^{ikr}}{(k^2)^n} \frac{1}{\pi r^2} \frac{(r/2)^{2n} \left( e^{2\epsilon} r \right)^{\epsilon} \Gamma(1+\epsilon/2-n) \Gamma(n)}{\Gamma(n)},
\]
where we have chosen the renormalization scale as
\[
\mu^2 = \frac{e^{-\gamma}}{\pi}
\]
so that
\[
\int \frac{d^2+\epsilon k}{(2\pi)^2+\epsilon} \mu^{-\epsilon} \frac{e^{ikr}}{k^2} \frac{1}{2\pi \epsilon} = -\frac{1}{2\pi} \ln r
\]
in the limit \( \epsilon \to 0 \).

We only give the final results, leaving the intermediate results to appendices A,B. First, we find the following beta functions and anomalous dimensions:
\[
\beta_m \simeq \frac{g}{\pi} N - \frac{3}{4} + \left( \frac{g}{\pi} \right)^2 N - \frac{2}{4},
\]
\[
\beta_g \simeq \frac{g^2}{\pi} N - \frac{2}{4} + \frac{g^3}{\pi^2} N - \frac{2}{4},
\]
\[
\gamma_{A_1} = \frac{g}{\pi} N - \frac{1}{4} + O(g^3), \quad \gamma_{A_2} = \frac{g}{\pi} N - \frac{1}{2} + O(g^3)
\]
\[
\gamma_{B_0} \simeq \frac{g^2}{\pi^2} N + 2 - \frac{N + 2}{4}, \quad \gamma_{B_1} = \frac{g}{\pi} N - \frac{1}{4} + O(g^3).
\]
Second, we find the following results for $H_m, H_g$:

$$(H_m)_{B^0}^{A_1}(g) = \frac{g}{\pi} \frac{N-1}{4} (1 + O(g^2))$$  \hspace{1cm} (3.5)

$$(H_m)_{\partial^2 A_1}^1 \simeq \frac{g}{\pi} \frac{-(N-1)}{2N\pi} \left( 1 + \frac{g}{\pi}(N-1)a \right)$$  \hspace{1cm} (3.6a)

$$(H_m)_{\partial^2 A_1}^{A_2} \simeq \frac{g}{\pi} \frac{-(N-1)}{2} \left( 1 - \frac{g}{\pi}a \right),$$  \hspace{1cm} (3.6b)

where $a$ is an unknown constant, and

$$(H_g)_{A_1}^{A_1}(g) = \frac{N-1}{4\pi} (1 + O(g^2))$$  \hspace{1cm} (3.7)

$$(H_g)_{A_2}^{A_2}(g) = \frac{N}{2\pi} (1 + O(g^2))$$  \hspace{1cm} (3.8)

$$(H_g)_{B^0}^{B_0}(g) \simeq -\frac{(N-2)}{2\pi}$$  \hspace{1cm} (3.9a)

$$(H_g)_{B^0}^{1}(g) \simeq -\frac{(N-1)}{2\pi}$$  \hspace{1cm} (3.9b)

$$(H_g)_{B^0}^{A_1}(g, m^2) \simeq m^2 \frac{g}{\pi} \frac{-(N-1)(N-2)}{8\pi}$$  \hspace{1cm} (3.9c)

$$(H_g)_{\partial^2 A_1}^{A_1}(g) \simeq \frac{N-1}{4\pi} \left( 1 - \frac{g}{\pi}(N-2) \right)$$  \hspace{1cm} (3.10a)

$$(H_g)_{\partial^2 A_1}^{A_2}(g) \simeq \frac{N-1}{\pi} \left( 1 - \frac{g}{\pi}(N-2) \right)$$  \hspace{1cm} (3.10b)

$$(H_g)_{\partial^2 A_1}^{1}(g, m^2) \simeq m^2 \frac{-(N-1)(N-3)}{4\pi N} \left( 1 + \frac{g}{\pi} b_1 \right)$$  \hspace{1cm} (3.10c)

$$(H_g)_{\partial^2 A_1}^{A_2} \simeq m^2 \frac{3(N-1)}{4\pi} \left( 1 + \frac{g}{\pi} b_2 \right),$$  \hspace{1cm} (3.10d)

where the constants $b_1, b_2$ are related by

$$-(N-3)b_1 + 3(N-1)b_2 = N(N-2).$$  \hspace{1cm} (3.11)

The unknown constants $a, b_1, b_2$ appear due to

$$A_2 - \frac{N-1}{N} 1 = O(g).$$  \hspace{1cm} (3.12)

In the next section we will be able to relate $b_1, b_2$ to the constant $a$. Third, from Eq. (2.21) we find the following curvature (see appendix C for more details):

$$(\Omega_{gm})_{B^0}^{A_1}(g) \simeq \frac{N-1}{8\pi}$$  \hspace{1cm} (3.13)

$$(\Omega_{gm})_{\partial^2 A_1}^{1}(g) + (\Omega_{gm})_{\partial^2 A_1}^{A_2}(g) \frac{N-1}{N} \simeq \frac{-(N-1)}{4\pi}.$$
4. Determination of the connection

The matrix elements of the connection $c_g$, $c_m$ depend on the convention adopted. First of all, under a change of basis

$$\Phi \rightarrow \Phi' \equiv N(g, m^2)\Phi,$$

the connection transforms as

$$c_g \rightarrow c'_g = N(g, m^2) \left( \partial_g N(g, m^2)^{-1} + c_g(g, m^2)N(g, m^2)^{-1} \right)$$

$$c_m \rightarrow c'_m = N(g, m^2) \left( \partial_m N(g, m^2)^{-1} + c_m(g)N(g, m^2)^{-1} \right).$$

Second, under a coordinate change

$$g \rightarrow \tilde{g} \equiv f(g), \quad m^2 \rightarrow \tilde{m}^2 \equiv m^2 h(g),$$

where

$$f'(0) = h(0) = 1,$$

the conjugate fields transform as

$$O_g \rightarrow \tilde{O}_{\tilde{g}} = \left( \frac{\partial \tilde{g}}{\partial g} \frac{\partial^2}{\partial \tilde{m}^2} \right)_{\tilde{m}^2} O_g + \left( \frac{\partial \tilde{m}^2}{\partial \tilde{g}} \frac{\partial \tilde{m}^2}{\partial \tilde{g}} \right)_{\tilde{m}^2} O_m = \frac{1}{f'(g)} \left( O_g - \frac{h'(g)}{h(g)} \frac{m^2}{2} O_m \right)$$

$$O_m \rightarrow \tilde{O}_{\tilde{m}} = \left( \frac{\partial \tilde{g}}{\partial \tilde{m}^2} \frac{\partial}{\partial \tilde{g}} \right)_{\tilde{m}^2} O_g + \left( \frac{\partial \tilde{m}^2}{\partial \tilde{g}} \frac{\partial \tilde{m}^2}{\partial \tilde{g}} \right)_{\tilde{m}^2} O_m = \frac{1}{h(g)} O_m.$$

If we do not change the basis $\Phi$, then the connection simply transforms as

$$c_{\tilde{g}} = \frac{1}{f'(g)} \left( c_g - \frac{h'(g)}{h(g)} m^2 c_m \right), \quad c_{\tilde{m}} = \frac{1}{h(g)} c_m.$$

Since the conjugate fields transform as (4.3), the connection for the conjugate fields transforms inhomogeneously as

$$\begin{align*}
(c_{\tilde{g}})_{\tilde{g}} &= \frac{1}{f'(g)} \left( (c_g)_g^g + \frac{f''(g)}{f'(g)} \right) \\
(c_{\tilde{g}})_{\tilde{m}} &= \frac{\tilde{m}^2}{(f'(g))^2} \left( \frac{h'(g)}{h(g)} ((c_g)_g^g - 2(c_g)_m^m) + \frac{1}{m^2} (c_g)_g^m \\
&\quad + \frac{h''(g)}{h(g)} - 2 \left( \frac{h'(g)}{h(g)} \right)^2 \right) \\
(c_{\tilde{g}})^1_{\tilde{g}} &= \frac{1}{f'(g)2} (c_g)^1_g \\
(c_{\tilde{g}})^{\tilde{m}} = (c_{\tilde{m}})^{\tilde{m}} &= \frac{1}{f'(g)} \left( (c_m)_g^m + \frac{h'(g)}{h(g)} \right).
\end{align*}$$
The results we present below correspond to a specific choice of the parameters \( g, m^2 \) and fields \( \Phi \) in the MS scheme with the renormalization scale \( \mu \) chosen by Eq. (3.2).

Using the OPE coefficients \( H_m, H_g \) and the curvature \( \Omega_{gm} \) that we have computed in the previous section, we can determine the connection \( c_m, c_g \) from Eqs. (2.18), (2.20).

We find the following matrix elements for \( c_m \):

\[
(c_m)_{B_0}^{A_1}(g) = O(g^2) \quad (4.8)
\]

\[
(c_m)_{\partial^2 A_1}(g) \simeq \frac{N-1}{N} \left( 1 - \frac{g}{\pi} \left( \frac{1}{2(N-3)} K \right) \right) \quad (4.9a)
\]

\[
(c_m)_{\partial^2 A_1}(g) \simeq -1 + \frac{g}{\pi} \left( \frac{1}{2(N-3)} K \right) , \quad (4.9b)
\]

where the constant \( K \) is related to the constant \( a \) in Eqs. (3.6a, b) by

\[
K \equiv N - 2 + 2a(N - 1) . \quad (4.10)
\]

We cannot determine the constant \( K \) to the order we have calculated.

We find the following matrix elements for \( c_g \):

\[
(c_g)^{A_1}(g, m^2) = O(g) , \quad (c_g)^{A_2}(g) = O(g) \quad (4.11)
\]

\[
(c_g)_{B_0}^{B_0}(g) \simeq -\frac{1}{g} + \frac{N-2}{4\pi} \quad (4.12a)
\]

\[
(c_g)_{B_0}^{1}(g) \simeq \frac{N-1}{4\pi} \left( 1 - \frac{g}{\pi} \left( \frac{N-2}{2} \right) \right) \quad (4.12b)
\]

\[
(c_g)_{B_0}^{A_1}(g, m^2) \simeq -m^2 \left( \frac{N-1}{8\pi} \right) \quad (4.12c)
\]

\[
(c_g)_{\partial^2 A_1}(g) \simeq -\frac{1}{g} \quad (4.13a)
\]

\[
(c_g)_{\partial^2 A_1}(g) \simeq -\frac{N-1}{2\pi} \left( 1 - \frac{g}{\pi} \left( \frac{N-2}{2} \right) \right) \quad (4.13b)
\]

\[
(c_g)_{\partial^2 A_1}(g, m^2) \simeq -m^2 \left( \frac{N-1}{g} \right) \left( 1 - \frac{g}{\pi} \left( \frac{N-1 - \frac{K}{2(N-3)}}{4} \right) \right) \quad (4.13c)
\]

\[
(c_g)_{\partial^2 A_1}(g, m^2) \simeq \frac{m^2}{g} \left( 1 + \frac{g}{\pi} \left( \frac{1 + \frac{K}{2(N-3)}}{4} \right) \right) , \quad (4.13d)
\]

where we could eliminate the constants \( b_1, b_2 \), which are now related to the sole constant \( K \) of Eq. (1.10) by

\[
b_1 = \frac{1}{N-3} \left( -N^2 + \frac{11}{2} N - \frac{13}{2} + \frac{2(N-2)}{2N-3} K \right) \quad (4.14)
\]

\[
b_2 = \frac{1}{3(N-1)} \left( \frac{7N-13}{2} + \frac{2(N-2)}{2N-3} K \right) .
\]
thanks to Eq. (2.20).

Finally, using Eqs. (2.7) and the transformation properties (4.2), we obtain

\[
\begin{align*}
(c_g)_m^m (g) &= (c_m)_g^m (g) = \frac{1}{g} + (c_g)^{A_1} (g) = \frac{1}{g} + O(g) \\
(c_g)_g^g (g) &= \frac{2}{g} + (c_g)^{B_0} (g) \approx \frac{1}{g} + \frac{N - 2}{4\pi} \\
(c_g)_g^A_1 (g) &= -\frac{1}{g^2} (c_g)^{B_0} (g) \approx -\frac{1}{g^2} \frac{N - 1}{4\pi} + \frac{1}{g} \frac{(N - 1)(N - 2)}{8\pi^2} \\
(c_g)_g^m (g, m^2) &= \frac{1}{g} \left( (c_g)^{A_1}_{B_0} (g, m^2) + m^2 \left( (c_g)^{B_0} (g) - (c_g)^{A_1} (g) \right) \right) \\
&\approx m^2 \left( -\frac{1}{g^2} + \frac{1}{g} \frac{N - 3}{8\pi} \right). \quad (4.15d)
\end{align*}
\]

5. Zero temperature limit

By examining the connection \(c_m, c_g\) obtained in the previous section, we notice that many matrix elements diverge at zero temperature \(g = 0\). This is disturbing; the connection controls the short-distance properties of the theory, and we expect their matrix elements to behave in the same way as the beta functions and anomalous dimensions at low temperatures, i.e., we expect them to be finite polynomials of the mass parameter \(m^2\) whose coefficients are smooth functions of \(g\) and admit perturbative expansions around \(g = 0\). But most of the divergences at \(g = 0\) are simply due to the wrong normalization of the composite fields.

We can see this as follows. Consider composite fields \(\Phi_a\). Suppose the diagonal elements of the connection \(c_g\) diverge as

\[
(c_g)_{\phi^b_a} (g) = \frac{n}{g} \delta^b_a + O \left( g^0 \right). \quad (5.1)
\]

Then Eq. (4.2) implies that for the renormalized fields \(g^n \Phi^a\) the diagonal elements become finite:

\[
(c_g)_{g^n \phi^b_a} (g) = -\frac{n}{g} \delta^b_a + (c_g)_{\phi^b_a} (g) = O \left( g^0 \right). \quad (5.2)
\]

We can identify another source of divergences in the connection \(c_g, c_m\) as the wrong normalization of the parameter \(m^2\). The field \(O_m\), which is conjugate to \(m^2\), is given by the second of Eqs. (2.7). The finiteness of \((c_g)^{A_1} (g)\) at \(g = 0\) implies that it is the field
$A_1$, but not $O_m$, which has proper normalization. Hence, we should redefine the mass parameter (or external magnetic field) by

$$\tilde{m}^2 \equiv \frac{m^2}{g}.$$  

(5.3)

This is nothing but a reduced external magnetic field. Then, Eq. (4.5d) gives the new conjugate field as

$$O_{\tilde{m}} = g O_m = -A_1,$$  

(5.4)

where we have kept the temperature intact:

$$\tilde{g} = g.$$  

(5.5)

Eq. (4.5a) gives

$$O_{\tilde{g}} = O_g + \frac{m^2}{g} O_m = -\frac{1}{g^2} B_0.$$  

(5.6)

We now redefine the fields $\Phi^T = (1, A_1, A_2, B_0, \partial^2 A_1)$ by

$$\Phi'^T = \left(1, A_1, A_2, B'_0 \equiv \frac{1}{g} B_0, \frac{1}{g} \partial^2 A_1 \right),$$  

(5.7)

and use $\tilde{g} \equiv g$ and $\tilde{m}^2$ as the parameters. Then we find the following matrix elements for the connection $c_{\tilde{m}}, c_{\tilde{g}}$:

$$(c_{\tilde{m}}) B_0^A(g) = O(g^2)$$  

(5.8)

$$(c_{\tilde{m}}) \frac{1}{g} \partial^2 A_1 (g) \sim \frac{N - 1}{N}$$  

(5.9a)

$$(c_{\tilde{m}}) \frac{A_2}{g} \partial^2 A_1 (g) \sim -1$$  

(5.9b)

$$(c_{\tilde{g}}) A_1 A_1 (g) = O(g)$$  

(5.10)

$$(c_{\tilde{g}}) A_2 A_2 (g) = O(g)$$  

(5.11)

$$(c_{\tilde{g}}) B'_0 A_2 (g) \sim \frac{N - 2}{4\pi}$$  

(5.12a)

$$(c_{\tilde{g}}) B'_0 (g) \approx \frac{1}{g} \left( \frac{N - 1}{4\pi} \left( 1 - \frac{g}{\pi} \frac{N - 2}{2} \right) \right)$$  

(5.12b)

$$(c_{\tilde{g}}) \frac{1}{g} \partial^2 A_1 (g, \tilde{m}^2) \sim -\frac{N - 1}{8\pi} \tilde{m}^2$$  

(5.12c)

$$(c_{\tilde{g}}) \frac{1}{g} \partial^2 A_1 (g) = O(1)$$  

(5.13a)

$$(c_{\tilde{g}}) \frac{A_1}{g} \partial^2 A_1 (g) \sim \frac{1}{g} \frac{-(N - 1)}{2\pi}$$  

(5.13b)

$$(c_{\tilde{g}}) \frac{A_1}{g} \partial^2 A_1 (g, \tilde{m}^2) = \tilde{m}^2 O(g^0)$$  

(5.13c)

$$(c_{\tilde{g}}) \frac{A_2}{g} \partial^2 A_1 (g, \tilde{m}^2) = \tilde{m}^2 O(g^0).$$  

(5.13d)
We find that all the matrix elements, except the \((B_0', 1)\) and \((\frac{1}{g} \partial^2 A_1, A_1)\) elements of \(c_\tilde{g}\), become finite at \(g = 0\) as functions of \(g, \tilde{m}^2\). All the maximal elements, i.e., those relating two fields which can mix under the RG, are finite at \(g = 0\). In fact the finiteness of the maximal elements at \(g = 0\) specifies the new basis \(\Phi'\) uniquely up to a linear transformation (4.1), where \(N\), as a function of \(g, \tilde{m}^2\), must be invertible even at the origin \(g = 0\). The proper normalization of the basis is necessary in order to assure that the vector bundle of the composite fields is well-defined in a neighborhood of the origin \(g = 0\). We can also explain the singularities of the non-maximal elements (5.12b) and (5.13b), but we leave it to appendix D since the explanation requires the familiarity with the variational formula [8], which we have not introduced in this paper.

We also find, from Eqs. (4.7), the following matrix elements of the connection for the redefined conjugate fields:

\[
(c_\tilde{g})_{\tilde{g}}^\tilde{g}(g) = (c_g)_{g}^g(g) \simeq \frac{1}{g} + \frac{N - 2}{4\pi} \quad (5.14a)
\]

\[
(c_\tilde{g})_{\tilde{m}}(g, \tilde{m}^2) = \tilde{m}^2 \left(-\frac{1}{g} ((c_g)^g - 2(c_g)^m) + \frac{1}{m^2} (c_g)^m\right) \\
\simeq \frac{\tilde{m}^2 - N + 1}{g} \quad (5.14b)
\]

\[
(c_\tilde{g})_1^1(g) = (c_g)_1^g(g) \simeq -\frac{1}{g^2} \frac{N - 1}{4\pi} \left(1 - \frac{g}{\pi} \frac{N - 2}{2}\right) \quad (5.14c)
\]

\[
(c_\tilde{g})_{\tilde{m}}(g) = (c_\tilde{m})_{\tilde{g}}^{\tilde{m}}(g) = (c_m)_{g}^m - \frac{1}{g} = O(g) \quad (5.14d)
\]

We have two maximal elements to pay attention to: \((c_\tilde{g})_{\tilde{g}}^\tilde{g}\) and \((c_\tilde{g})_{\tilde{m}}\). Only the second is finite at zero temperature. The divergence of the element \((c_\tilde{g})_{\tilde{g}}^\tilde{g}\) comes from the singular relation of the conjugate field \(\mathcal{O}_\tilde{g}\) to the well-normalized field \(B_0'\):

\[
\mathcal{O}_\tilde{g} = -\frac{1}{g} B_0' \quad (5.15)
\]

The redefinition of \(\tilde{g} = g\) by

\[
g' = \ln g \quad (5.16)
\]

makes the diagonal element \((c_g')_{g'}^g'\) regular at \(g = 0\), but the non-analytic transformation (5.16) is illegal as a coordinate transformation near the origin \(g = 0\). At the end of appendix D we give a technical reason for the necessity of \(\frac{1}{g}\) in Eq. (5.15): without the factor the conjugate field is a total derivative at \(g = 0\), and it does not deform the theory.
We reconcile ourselves with the zero-temperature singularity of \((c_{\tilde{g}})_{\tilde{g}}\) by regarding it as a unique feature of a theory with its symmetry non-linearly realized: we can trace the source of the singularity to the bare temperature \(g_0\) in the denominator of the lagrangian (2.1).

Finally, we must understand the physical meaning of the normalization of the redefined composite fields \(\Phi'\) (Eq. (5.7)). We will show that the fields \(\Phi'\) have well-defined and non-trivial correlation functions at low temperatures in the absence of an external magnetic field, i.e., \(m^2 = 0\). We do not have a general proof, and we will be content with verifying this claim only for the two-point functions. Perturbative calculations give the following results:

\[
\langle \Phi^I(r)\Phi^J(0) \rangle_g = \frac{\delta^{IJ}}{N} \left( 1 + \frac{g}{\pi} \frac{1 - N}{2} \ln r + O(g^2) \right) \quad (5.17)
\]

\[
\langle B'_0(r)B'_0(0) \rangle_g = \frac{N - 1}{4\pi^2r^4} + O(g) \quad (5.18)
\]

\[
\langle \frac{1}{g} \partial^2 \Phi^I(r) \frac{1}{g} \partial^2 \Phi^J(0) \rangle_g = \delta^{IJ} \frac{1}{\pi^2 r^4} \left( \frac{N - 1}{N} \left( 1 + \frac{g}{\pi} (N - 3) \left( \frac{1}{2} \ln r - 1 \right) + O(g^2) \right) \right). \quad (5.19)
\]

(We did not try to compute the two-point function of \(A_2\).) We recall that the correlation functions must be \(O(N)\) invariant at \(m^2 = 0\).

Hence, we find that the connection \(c_{\tilde{g}}, c_{\tilde{m}}\) has finite matrix elements (excluding the non-maximal elements) at \(g = 0\) if we normalize the fields so that their correlation functions have well-defined zero-temperature limits.

In order to make the above finding more firm, let us consider one more field that we have not considered before, i.e., the spatial derivative of the \(O(N)\) vector, \(\partial_\mu \Phi^I\). By taking the second-order derivative of (5.17), we obtain

\[
\left\langle \frac{1}{\sqrt{g}} \partial_\mu \Phi^I(r) \frac{1}{\sqrt{g}} \partial_\nu \Phi^J(0) \right\rangle_g \simeq \delta^{IJ} \frac{1}{\pi^2 r^4} \left( \frac{N - 1}{2N} \left( \delta_{\mu\nu} - 2 \frac{r_\mu r_\nu}{r^2} \right) \right). \quad (5.20)
\]

Therefore, the field \(\frac{1}{\sqrt{g}} \partial_\mu \Phi^I\) is properly normalized. To show that the connection \(c_{\tilde{g}}\) for \(\frac{1}{\sqrt{g}} \partial_\mu \Phi\) is finite at \(g = 0\), we must compute the OPE coefficient in

\[
O_{\tilde{g}}(r) \partial_\mu \Phi^I(0) \simeq \frac{1}{2\pi} \left( C_{\tilde{g}} \right)_{\partial_\mu \Phi^I} (r; g) \partial_\nu \Phi^J(0). \quad (5.21)
\]

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This is done in appendix E. We find
\[
(H_{\tilde{g}}) \frac{1}{\sqrt{g}} \partial_{\nu} \Phi^J (g) = (H_g) \partial_{\nu} \Phi^J (g)
\]
\[= \delta^I_J \delta_\mu^\nu \frac{1}{2} \left( (H_g) A_1^A (g) + (H_g) \partial^2 A_1^A (g) \right) \simeq \delta^I_J \delta_\mu^\nu \frac{1}{8\pi}.
\]
Since
\[
\frac{d}{dt} \frac{1}{\sqrt{g}} \partial_{\mu} \Phi^I = \delta^I_J \delta_\mu^\nu \left( 1 + \gamma A_1 - \frac{\beta g}{2g} \right) \frac{1}{\sqrt{g}} \partial_{\nu} \Phi^J,
\]
Eqs. (2.18) imply
\[
(H_{\tilde{g}}) \frac{1}{\sqrt{g}} \partial_{\nu} \Phi^J (g) = \frac{d}{dg} \left( \delta^I_J \delta_\mu^\nu \left( \gamma A_1 - \frac{\beta g}{2g} \right) + \beta g \left( c_{\tilde{g}} \right) \frac{1}{\sqrt{g}} \partial_{\nu} \Phi^J \right).
\]
Hence, we find, from Eqs. (3.4) and (5.22),
\[
(c_{\tilde{g}}) \frac{1}{\sqrt{g}} \partial_{\nu} \Phi^J = \delta^I_J \delta_\mu^\nu \text{O}(g^0).
\]
Thus, the connection is finite at \(g = 0\) for the field \(\frac{1}{\sqrt{g}} \partial_{\mu} \Phi^I\), which has a well-defined two-point function in the limit \(g = 0\).

6. Conclusion

In this paper we have calculated the connection \(c_g, c_m\) for the composite fields \(\Phi\) (Eq. (2.12)). From the behavior of the matrix elements near zero temperature \(g = 0\), we have concluded that the reduced magnetic field \(\tilde{m}^2 \equiv \frac{m^2}{g}\) is more natural at low temperatures, and that the matrix elements are finite at \(g = 0\) only if we normalize the composite fields so that their correlation functions have well-defined and non-trivial zero temperature limits. Since we take these limits in the absence of an external magnetic field, the correlation functions are O(N) invariant even at zero temperature.

We find especially the second conclusion pleasing, since the connection controls the short-distance physics (equivalently low temperature physics) together with the beta functions and anomalous dimensions. Anything to do with short-distance physics should be given as finite polynomials of the reduced magnetic field \(\tilde{m}^2\) whose coefficients can be formally expanded in powers of \(g\).

We believe it would be interesting to examine non-abelian gauge theories which also realize the relevant symmetry non-linearly [7]. We expect to find that the behavior of the connection at short distances dictates how to take the short-distance limit of the BRST invariant theories.

H. S. thanks Terry Tomboulis for discussions.
Appendix A. Renormalized composite fields

For perturbative calculations, we introduce $N - 1$ independent scalar fields $\phi^i (i = 1, ..., N - 1)$ such that

$$\Phi^i = \sqrt{g} \phi^i \quad (i = 1, ..., N - 1), \quad \Phi^N = \sqrt{1 - Z_\Phi g \phi^2 / Z_\Phi}. \quad (A.1)$$

The following composite fields are renormalized:

$$\left[ \frac{\phi^2}{2} \right]_1 \equiv \left( 1 - \frac{g}{2\pi\epsilon} \right) \left( \frac{\phi^2}{2} + \frac{N - 1}{4\pi\epsilon} \right) \quad (A.2)$$

$$\left[ \frac{(\phi^2)^2}{8} \right]_0 \equiv \left( \frac{(\phi^2)^2}{8} \right) + \frac{N + 1}{4\pi\epsilon} \phi^2 + \frac{(N - 1)(N + 1)}{32\pi^2\epsilon^2} \quad (A.3)$$

$$\left[ \frac{1}{2} \partial \phi \cdot \partial \phi \right]_1 = \frac{1}{2} \partial \phi \cdot \partial \phi - \frac{N - 1}{4\pi\epsilon} m^2$$

$$+ g \left( \frac{N - 2}{2\pi\epsilon} \frac{1}{2} \partial \phi \cdot \partial \phi + m^2 \frac{N + 1}{4\pi\epsilon} \phi^2 - m^2 \frac{(N - 1)(N - 5)}{16\pi^2\epsilon^2} \right) \quad (A.4)$$

$$\left[ \frac{1}{2} \phi \cdot \partial \mu \phi \right]_0 \equiv \frac{1}{2} \phi \cdot \partial \mu \phi + \frac{1}{2\pi\epsilon} \frac{1}{2} \partial \phi \cdot \partial \phi$$

$$- \frac{m^2}{2\pi\epsilon} \phi^2 - \frac{(N - 1)m^2}{8\pi^2\epsilon^2}, \quad (A.5)$$

where $[(\text{field})]_n$ is renormalized to order $g^n$. Since

$$\partial^2 \frac{\phi^2}{2} = 2 \left( \frac{1}{2} \partial \phi \cdot \partial \phi + m^2 \frac{\phi^2}{2} \right) + O(g), \quad (A.6)$$

we find

$$\partial^2 \left[ \frac{(\phi^2)^2}{8} \right]_0 \equiv \partial^2 \left( \frac{(\phi^2)^2}{8} \right) + \frac{N + 1}{2\pi\epsilon} \left( \frac{1}{2} \partial \phi \cdot \partial \phi + m^2 \frac{\phi^2}{2} \right). \quad (A.7)$$

Using Eqs. (3.1), (3.2), and (3.3), we can compute the OPE of the above renormalized composite fields as follows:

$$\left[ \frac{1}{2} \partial \phi \cdot \partial \phi \right]_1 (r) \left[ \frac{\phi^2}{2} \right]_1 (0)$$

$$\simeq \frac{N - 1}{8\pi^2 r^2} \left( 1 + \frac{g}{\pi} (N - 2) \ln r \right) \left( 1 - \frac{N + 1}{4\pi^2 r^2} \right) \left[ \frac{\phi^2}{2} \right]_0 (0) \quad (A.8a)$$
$$\left[ \frac{1}{2} \partial \tilde{\phi} \cdot \partial \tilde{\phi} \right]_0 \left( r \right) \left[ \frac{\tilde{\phi}^2}{8} \right]_0 \approx \frac{N + 1}{8 \pi^2 r^2} \left[ \frac{\tilde{\phi}^2}{2} \right]_0 \quad (A.8b)$$

$$\left[ \frac{1}{2} (\phi \cdot \partial \mu \tilde{\phi})^2 \right]_0 \left( r \right) \left[ \frac{\phi^2}{2} \right]_0 \approx \frac{1}{4 \pi^2 r^2} \left[ \frac{\phi^2}{2} \right]_0 \quad (A.8c)$$

$$\left[ \frac{1}{2} \partial \tilde{\phi} \cdot \partial \phi \right]_1 \left( r \right) \left[ \frac{1}{2} \partial \tilde{\phi} \cdot \partial \phi \right]_1 \left( 0 \right) \approx \left( N - 1 \right) \left[ \left( \frac{1}{4 \pi^2 r^4} - \frac{m^2}{8 \pi^2 r^2} \right) - \frac{g}{4 \pi^3} \left( \frac{N - 2}{r^4} \right) \right] \left( 1 \right) \quad (A.8d)$$

$$\left[ \frac{1}{2} (\phi \cdot \partial \mu \phi)^2 \right]_0 \left( r \right) \left[ \frac{1}{2} (\phi \cdot \partial \mu \phi)^2 \right]_0 \left( 0 \right) \approx \left( \frac{1}{2 \pi^2 r^4} - \frac{m^2}{4 \pi^2 r^2} \right) \left[ \frac{\phi^2}{2} \right]_0 \quad (A.8e)$$

$$\left[ \frac{1}{2} (\phi \cdot \partial \mu \phi)^2 \right]_0 \left( r \right) \left[ \frac{1}{2} (\phi \cdot \partial \mu \phi)^2 \right]_0 \left( 0 \right) \approx \left( \frac{1}{2 \pi^2 r^4} - \frac{m^2}{4 \pi^2 r^2} \right) \left[ \frac{\phi^2}{2} \right]_0 \quad (A.8f)$$

$$\partial^2 \left[ \frac{\tilde{\phi}^2}{8} \right]_0 \left( r \right) \left[ \frac{\phi^2}{2} \right]_0 \approx \frac{N + 1}{4 \pi^2 r^2} \left[ \frac{\tilde{\phi}^2}{2} \right]_0 \quad (A.8g)$$

$$\partial^2 \left[ \frac{\tilde{\phi}^2}{8} \right]_0 \left( r \right) \partial^2 \left[ \frac{\tilde{\phi}^2}{8} \right]_0 \left( 0 \right) \approx N + \frac{1}{\pi^2} \left[ \left( \frac{1}{2 r^4} + \frac{m^2}{4 r^2} \right) \left[ \frac{\phi^2}{2} \right]_0 \left( 0 \right) + \frac{1}{4 r^2} \left[ \frac{1}{2} \partial \tilde{\phi} \cdot \partial \phi \right]_0 \left( 0 \right) \right] \quad (A.8h)$$

$$\partial^2 \left[ \frac{\tilde{\phi}^2}{8} \right]_0 \left( r \right) \left[ \frac{1}{2} \partial \tilde{\phi} \cdot \partial \phi \right]_0 \left( 0 \right) \approx \frac{N + 1}{\pi^2} \left[ \frac{\tilde{\phi}^2}{2} \right]_0 \left( 0 \right), \quad (A.8i)$$

where we have taken the angular average over $r$.

### Appendix B. OPE coefficients $C_m, C_g$

The $O(N)$ covariant composite fields are related to the composite fields of appendix
A in the following way:

\[ A_1 \simeq 1 - g \left[ \frac{\vec{\phi}^2}{2} \right]_1 - g^2 \left[ \frac{(\vec{\phi}^2)^2}{8} \right]_0 \]  \hspace{1cm} (B.1)

\[ A_2 \simeq \frac{N - 1}{N} - 2g \left[ \frac{\vec{\phi}^2}{2} \right]_1 \]  \hspace{1cm} (B.2)

\[ B_0 \simeq g \left( \frac{1}{2} \partial \vec{\phi} \cdot \partial \vec{\phi} \right)_1 + g \left[ \frac{1}{2}(\vec{\phi} \cdot \partial_{\mu} \vec{\phi})^2 \right]_0 \]  \hspace{1cm} (B.3)

\[ \partial^2 A_1 \simeq g \left( -2 \left[ \frac{1}{2} \partial \vec{\phi} \cdot \partial \vec{\phi} \right]_1 - 2m^2 \left[ \frac{\phi^2}{2} \right]_1 \right) \]
\[ + g^2 \left( \partial^2 \left[ \frac{(\vec{\phi}^2)^2}{8} \right]_0 - 4 \left[ \frac{1}{2}(\vec{\phi} \cdot \partial_{\mu} \vec{\phi})^2 \right]_0 - 4m^2 \left[ \frac{(\vec{\phi}^2)^2}{8} \right]_0 \right) \]  \hspace{1cm} (B.4)

\[ O_m \simeq -\frac{1}{g} + \left[ \frac{\phi^2}{2} \right]_1 + g \left[ \frac{(\vec{\phi}^2)^2}{8} \right]_0 \]  \hspace{1cm} (B.5)

\[ O_g \simeq -\frac{1}{g} \left[ \frac{1}{2} \partial \vec{\phi} \cdot \partial \vec{\phi} \right]_1 - \left[ \frac{1}{2}(\vec{\phi} \cdot \partial_{\mu} \vec{\phi})^2 \right]_0 \]
\[ + \frac{m^2}{g} - \frac{m^2}{g} \left[ \frac{\phi^2}{2} \right]_1 - m^2 \left[ \frac{(\vec{\phi}^2)^2}{8} \right]_0 \]  \hspace{1cm} (B.6)

Now it is straightforward to compute the coefficient functions in the products \( O_m(r)\Phi(0) \) and \( O_g(r)\Phi(0) \) using the above relations and Eqs. (A.8). The OPE coefficients \( C_m \) are obtained as

\[ (C_m)_{A_1}^{A_1}(r; g) \simeq \frac{1}{r^2} \frac{g(N - 1)}{4\pi} \left( 1 + \frac{g}{\pi} (N - 2) \ln r \right) \]  \hspace{1cm} (B.7)

\[ (C_m)_{\partial^2 A_1}(r; g) \simeq \frac{g(N - 1)}{2\pi r^2} \]  \hspace{1cm} (B.8a)
\[ \times \left( 1 + \frac{g}{\pi} \left( (N - 1)a + \frac{N - 3}{2} \ln r \right) \right) \]

\[ (C_m)_{\partial^2 A_1}(r; g) \simeq \frac{g(N - 1)}{2\pi r^2} \]  \hspace{1cm} (B.8b)
\[ \left( 1 + \frac{g}{\pi} \left( -a + \frac{2N - 3}{2} \ln r \right) \right) \]

where the constant \( a \) is undetermined, since

\[ A_2 - \frac{N - 1}{N} - 1 = O(g). \]  \hspace{1cm} (B.9)

The OPE coefficients \( C_g \) are obtained as

\[ (C_g)_{A_1}^{A_1}(r; g) \simeq \frac{N - 1}{4\pi r^2} \left( 1 + \frac{g}{\pi} (N - 2) \ln r \right) \]  \hspace{1cm} (B.10)
\begin{equation}
(C_g)_{B_0}^{A_1}(r;g,m^2) \simeq \frac{m^2}{2 \pi r^2} \left(\frac{(N-1)(N-3)}{4N} \left(1 + \frac{g}{\pi} (b_1 + (N-2) \ln r)\right) \right)
\end{equation}

where the constants \(b_1, b_2\) are related only by
\[-(N-3)b_1 + 3(N-1)b_2 = N(N-2).\]

An ambiguity is left due to Eq. (B.9).

Appendix C. Curvature

At tree level we find

\begin{equation}
\langle (\mathcal{O}_m(r)\mathcal{O}_g(r') - (r \leftrightarrow r')) B_0(0) \rangle_{g,m^2}^{c}
\end{equation}

\begin{align*}
&\simeq \left\langle \left(\frac{1}{2} \partial \vec{\phi} \cdot \partial \vec{\phi}(r) + \overline{\partial \vec{\phi}^2} \right)(r') - (r \leftrightarrow r') \right\rangle \frac{1}{2} \partial \vec{\phi} \cdot \partial \vec{\phi}(0) \right\rangle^{c}_{0,0} \\
&= - \frac{N-1}{(2\pi)^3} \frac{(r-r')_{\mu}}{(r-r')^2} \left( \frac{r_{\mu} r_{\nu}}{r^2 r'^2} - 2 \frac{r_{\mu} r_{\nu}' r_{\nu}'}{r^2 r'^4} + (r \leftrightarrow r') \right).
\end{align*}

This has no unintegrable singularity at either \(r = 0\) or \(r' = 0\). Hence, Eq. (2.21) gives

\begin{equation}
\langle \mathcal{O}_{gm}^{A_1}(g,m^2) \mathcal{O}_{g}^{A_1}(g,m^2) \rangle_{g,m^2}
\end{equation}

\begin{align*}
&\simeq - \frac{N-1}{(2\pi)^3} \int_{1 \geq r} d^2 r \text{ F.P.} \int_{1 \geq r'} d^2 r' \\
&\times \frac{(r-r')_{\mu}}{(r-r')^2} \left( \frac{r_{\mu} r_{\nu}}{r^2 r'^2} - 2 \frac{r_{\mu} r_{\nu}' r_{\nu}'}{r^2 r'^4} + \frac{r_{\mu} r_{\nu}}{r^2 r'^2} - 2 \frac{r_{\mu} r_{\nu}' r_{\nu}'}{r^2 r'^4} \right)
\end{align*}

\begin{equation}
= - \frac{N-1}{(2\pi)^3} \int_{1 \geq r} d^2 r \left(-\pi\right) = \frac{N-1}{8\pi}.
\end{equation}
The curvature for $\partial^2 A_1$ can be reduced to exactly the same integral up to a factor $-2$.

**Appendix D. The singularities of the connection at $g = 0$**

We owe the singularities of the non-maximal elements of the connection $c_{\tilde{g}}$ at $g = 0$ to the singular normalization of the conjugate field $\mathcal{O}_{\tilde{g}}$ (see Eq. (5.15)). To see this, we must go back to the variational formula in which the connection plays the role of finite counterterms [3].

Let us consider the derivative of the expectation value of $B_0'$ in the absence of an external magnetic field. The variational formula gives

\[
-\partial_g \langle B'_0 \rangle_g = \int_{r \geq \epsilon} d^2 r \left\langle \mathcal{O}_{\tilde{g}}(r) B'_0(0) \right\rangle^c_g \\
+ \left( (c_{\tilde{g}}) B'_0(g) - \int_{1 \geq r \epsilon} \frac{d^2 r}{2\pi} (C_{\tilde{g}}) B'_0(r; g) \right) \langle B'_0 \rangle_g \\
+ \left( (c_{\tilde{g}}) \frac{1}{B'_0}(g) - \int_{1 \geq r \epsilon} \frac{d^2 r}{2\pi} (C_{\tilde{g}}) \frac{1}{B'_0}(r; g) \right) \cdot 1 .
\] (D.1)

For this to be finite at $g = 0$ (in fact it vanishes), the terms proportional to $\frac{1}{g}$ on the right-hand side must vanish. To order $\frac{1}{g}$ we find

\[
\left\langle \mathcal{O}_{\tilde{g}}(r) B'_0(0) \right\rangle^c_g \simeq -\frac{1}{g} \left\langle \left[ \frac{1}{2} \partial \phi \cdot \partial \bar{\phi} \right] (r) \left[ \frac{1}{2} \partial \phi \cdot \partial \bar{\phi} \right] (0) \right\rangle_0 = -\frac{1}{g} \frac{N - 1}{4\pi^2 r^4} 
\] (D.2)

from Eq. (B.3). Hence, the UV subtracted integral becomes

\[
\int_{r \geq \epsilon} d^2 r \left\langle \mathcal{O}_{\tilde{g}}(r) B'_0(0) \right\rangle^c_g - \int_{1 \geq r \geq \epsilon} \frac{d^2 r}{2\pi} (C_{\tilde{g}}) \frac{1}{B'_0}(r; g) \\
\simeq \int_{r \geq 1} d^2 r \frac{-1}{g} \frac{N - 1}{4\pi^2 r^4} = -\frac{1}{g} \frac{N - 1}{4\pi} .
\] (D.3)

This singularity is canceled by the counterterm $(c_{\tilde{g}}) \frac{1}{B'_0}$ as we can see from Eq. (5.12b). The $\frac{1}{g}$ singularity of the connection is needed to cancel the singularity in the definition of the conjugate field (5.13).
Similarly we can analyze the derivative of a correlation function involving the other field $\frac{1}{g} \partial^2 \Phi^I$:

$$
- \partial_g \left\langle \frac{1}{g} \partial^2 \Phi^I(0) \Phi^J(R) \right\rangle_g = \int_{|r-R| \geq \epsilon} d^2 r \left\langle (O_{\bar{g}}(r) - \langle O_{\bar{g}} \rangle_g) \frac{1}{g} \partial^2 \Phi^I(0) \Phi^J(R) \right\rangle_g \\
+ \left( \left( c_{\bar{g}} \right) \frac{1}{g} \partial^2 \Phi^K(g) - \int_{r \geq \epsilon} (C_{\bar{g}}) \frac{1}{g} \partial^2 \Phi^K(r; g) \right) \left\langle \frac{1}{g} \partial^2 \Phi^K(0) \Phi^J(R) \right\rangle_g \\
+ \left( \left( c_{\bar{g}} \right) \frac{1}{g} \partial^2 \Phi^K(g) - \int_{r \geq \epsilon} (C_{\bar{g}}) \frac{1}{g} \partial^2 \Phi^K(r; g) \right) \left\langle \Phi^K(0) \Phi^J(R) \right\rangle_g \\
+ \left( \left( c_{\bar{g}} \right) \frac{1}{g} \partial^2 \Phi^K(g) - \int_{r \geq \epsilon} (C_{\bar{g}}) \frac{1}{g} \partial^2 \Phi^K(r; g) \right) \left\langle \frac{1}{g} \partial^2 \Phi^I(0) \Phi^K(R) \right\rangle_g .
$$

At order $\frac{1}{g}$, the integrand is given by

$$
\left\langle O_{\bar{g}}(r) \frac{1}{g} \partial^2 \Phi^I(0) \Phi^J(R) \right\rangle_g \simeq - \frac{1}{g} \frac{\delta^{IJ}}{N} \left\langle \left[ \frac{1}{2} \partial \phi \cdot \partial \phi \right]_0 (r)(-2) \left[ \frac{1}{2} \partial \phi \cdot \partial \phi \right]_0 (0) \mathbf{1}(R) \right\rangle_0 
= \frac{\delta^{IJ}}{g} \frac{N-1}{N} \frac{1}{2\pi^2 r^4} .
$$

Hence, to order $\frac{1}{g}$, we obtain the following UV subtracted integral:

$$
\int_{|r-R| \geq \epsilon} d^2 r \left\langle (O_{\bar{g}}(r) - \langle O_{\bar{g}} \rangle_g) \frac{1}{g} \partial^2 \Phi^I(0) \Phi^J(R) \right\rangle_g \\
- \int_{r \geq \epsilon} (C_{\bar{g}}) \frac{1}{g} \partial^2 \Phi^K(r; g) \left\langle \Phi^K(0) \Phi^J(R) \right\rangle_g \\
\simeq \frac{1}{g} \int_{r \geq 1} d^2 r \frac{N-1}{N} \frac{\delta^{IJ}}{2\pi^2 r^4} = \frac{1}{g} \frac{N-1}{N} \frac{\delta^{IJ}}{2\pi} .
$$

This is again canceled by the counterterm

$$
\left( c_{\bar{g}} \right) \frac{1}{g} \partial^2 \Phi^K(g) \left\langle \Phi^K(0) \Phi^J(0) \right\rangle_g \simeq \frac{1}{g} \frac{-(N-1)}{2\pi} \frac{\delta^{IJ}}{N} ,
$$

where we have used Eq. (5.13b).

The above cancelations which assure the validity of the variational formula at $g = 0$ can be understood as follows. From Eqs. (B.3) and (A.6), we observe that the field $B'_0$ is a total derivative at lowest order in $g$:

$$
B'_0 \simeq \left[ \frac{1}{2} \partial \phi \cdot \partial \phi \right]_0 = \frac{1}{2} \partial^2 \left[ \frac{\phi^2}{2} \right]_0 .
$$

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Therefore, the UV subtracted integral in the variational formula reduces to surface terms at radius $|r| = 1$; we have seen two examples in the above. To cancel these local terms we must introduce finite counterterms of order $\frac{1}{g}$. Hence, we need the $\frac{1}{g}$ singularities of the connection. The only exception occurs when the UV divergence of the cutoff integral is logarithmic. Since $\frac{1}{r}$ cannot be given as a total derivative, no logarithmic divergence appears in the integral in the variational formula, and we do not need any subtraction or finite counterterm. Therefore, the maximal elements of the connection do not need $\frac{1}{g}$ contributions.

Eq. (D.8) also explains the presence of the extra singular factor $\frac{1}{g}$ in the definition of the conjugate field, Eq. (5.15). If $B'_0$ were the conjugate field to $g$, it would not deform the theory. But, with $\frac{1}{g}$, the conjugate field is not a total derivative at order $g^0$, and it deforms the theory non-trivially.

**Appendix E. OPE for the derivative field $\partial_\mu \Phi^N$**

Let us introduce the OPE

$$2\pi \mathcal{O}_g(r) \Phi^N(r') = (C_g)_{A_1}^A (r - r'; g) \Phi^N(r') + (C_g)_{A_1}^{\partial A_1} (r - r'; g) \partial_\mu \Phi^N(r')$$

$$+ (C_g)_{A_1}^{\partial^2 A_1} (r - r'; g) \partial^2 \Phi^N(r')$$

$$+ (C_g)_{A_1}^A (r - r'; g, m^2) 1 + (C_g)_{A_1}^{A_2} (r - r'; g, m^2) A_2(r') + o(|r - r'|^0), \quad (E.1)$$

where we have ignored the term proportional to $(2 \partial_\mu \partial_\nu - \delta_{\mu\nu}) A_1$, which turns out to be irrelevant. By taking the derivative with respect to $r'_\mu$ once and setting $r'_\mu = 0$ and taking the angular average over $r$, we obtain

$$2\pi \mathcal{O}_g(r) \partial_\mu A_1(0) = \delta^{\nu}_\mu (C_g)_{A_1}^{\partial A_1} (r; g) \partial_\nu A_1(0) + o \left( \frac{1}{r^2} \right), \quad (E.2)$$

where we define the angular average

$$\delta^{\nu}_\mu (C_g)_{A_1}^{\partial A_1} (r; g) = \delta^{\nu}_\mu (C_g)_{A_1}^{A_1} (r; g) - \int_{|r| = 1} \frac{d\theta}{2\pi} \partial_\mu (C_g)_{A_1}^{\partial A_1} (r; g). \quad (E.3)$$

By differentiating Eq. (E.1) twice with respect to $r'$ and setting $r' = 0$ and taking the angular average over $r$, we obtain

$$2\pi \mathcal{O}_g(r) \partial^2 A_1(0) = \partial^2 (C_g)_{A_1}^{A_1} (r; g) A_1(0) + \partial^2 (C_g)_{A_1}^{A_2} (r; g, m^2) 1$$

$$+ \partial^2 (C_g)_{A_1}^{A_2} (r; g, m^2) A_2(0) + (C_g)_{\partial^2 A_1}^{\partial^2 A_1} (r; g) \partial^2 A_1(0) + o \left( \frac{1}{r^2} \right), \quad (E.4)$$

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where

\[
(C_g) \frac{\partial^2 A_1}{\partial^2 A_1}(r; g) = 2(C_g) \frac{\partial A_1}{\partial A_1}(r; g) - (C_g) A_1 A_1(r; g) .
\] (E.5)

Therefore, we obtain

\[
(C_g) \frac{\partial A_1}{\partial A_1}(r; g) = \frac{1}{2} \left((C_g) A_1 A_1(r; g) + (C_g) \frac{\partial^2 A_1}{\partial^2 A_1}(r; g)\right) .
\] (E.6)
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