ON POSSIBLE VALUES OF THE INTERIOR ANGLE BETWEEN INTERMEDIATE SUBALGEBRAS

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Abstract. We show that all values in the interval $[0, \frac{\pi}{2}]$ can be attained as the interior angle between intermediate subalgebras (as introduced in [3]) of a certain inclusion of simple unital $C^*$-algebras. We also calculate the interior angle between intermediate crossed product subalgebras of any inclusion of crossed product algebras corresponding to any action of a countable discrete group and its subgroups on a unital $C^*$-algebra.

1. Introduction

In any category, in order to classify its objects, the analysis of the relative positions of the subobjects of an object has proved to be a very rewarding approach. In the same vein, in the category of operator algebras, a great deal of work has been done by some eminent mathematicians - see, for instance, [1, 3, 6, 7, 8, 11] and the references therein. The theory of subfactors and, more generally, the theory of inclusions of (simple) $C^*$-algebras are two limelights of this aspect.

In this article, our focus lies only on unital $C^*$-algebras and their subalgebras. Over the years, various significant tools and theories have been developed to understand the relative positions of subalgebras of a given unital $C^*$-algebra. Among them, Watatani’s notions of finite-index conditional expectations and $C^*$-basic construction with respect to a finite-index conditional expectation ([11]) have proved to be fundamental in the development of the theory of inclusions of $C^*$-algebras - see [11, 9, 7, 6, 3]. Based on these two notions, and motivated by [1], very recently, Bakshi and the first named author, in [3], introduced the notions of interior and exterior angles between intermediate $C^*$-subalgebras of a given inclusion $B \subset A$ of unital $C^*$-algebras with a finite-index conditional expectation. As an application of the notion of interior angle, the authors in [3] were able to improve Longo’s upper bound for the cardinality of the lattice of intermediate $C^*$-subalgebras of any irreducible inclusion of simple unital $C^*$-algebras.

Apart from the above mentioned quantitative application of the notion of interior angle, we expect some significant qualitative consequences too to be visible soon. In this direction, it is then quite natural to first ask whether one can make some concrete calculations of these angles and the possible values that they can attain. This article essentially answers these questions to a certain level of satisfaction. Being precise, through some elementary calculations, we are able to show that all values in the interval $[0, \frac{\pi}{2}]$ are attained as the interior angles between intermediate subalgebras of a certain inclusion of simple unital $C^*$-algebras. Further, motivated by [2], we also calculate the interior angle between intermediate crossed product subalgebras of any inclusion of crossed product algebras corresponding to any action of a countable discrete group and its subgroups on a unital $C^*$-algebra.

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The article is organized as follows:

After the introduction, we have a relatively longer section on preliminaries wherein we recall and derive some basic nuances related to finite-index conditional expectations and Watatani’s \( C^* \)-basic construction related to inclusions of unital \( C^* \)-algebras. This discussion is fundamental to the formalism of the interior and exterior angles, which we briefly recall in Section 3; and, in the same section, we also derive some useful expressions related to them. Then, in Section 4, we prove that for any \( t \in [0, \frac{\pi}{2}] \), there exists a \( 2 \times 2 \) unitary matrix \( u \) such that the interior angle \( \alpha(\Delta, u\Delta u^*) = t \) with respect to the canonical conditional expectation from \( M_2(\mathbb{C}) \) onto \( \mathbb{C} \), where \( \Delta \) denotes the diagonal subalgebra of \( M_2(\mathbb{C}) \); thereby, establishing that all values in the interval \( [0, \frac{\pi}{2}] \) are attained as the interior angles between intermediate subalgebras. Finally, in Section 5, as an application of some expressions derived in Section 4, we prove that for any \( x \in A - \text{see [11, Proposition 2.10.2]} \). Such a set \( \{\lambda_1, \ldots, \lambda_n\} \) is called a quasi-basis for \( E \) and the Watatani index of \( E \) is defined as

\[
\text{Ind}(E) = \sum_{i=1}^{n} \lambda_i \lambda_i^*.
\]

It is known that \( \text{Ind}(E) \) is a positive invertible element of \( \mathcal{Z}(A) \) and is independent of the quasi-basis \( \{\lambda_i\} \) - see [11 §2]. Also, \( E \) is faithful, \( E(1_A) = 1_B \) and \( \text{Ind}(E) \geq 1 \).

**Remark 2.1.**

(1) If \( B \subset C \subset A \) are inclusions of unital \( C^* \)-algebras with \( 1_A \in B \) and \( E : A \to B, F : A \to C \) and \( G : C \to B \) are faithful conditional expectations satisfying \( E = G \circ F \), then \( E \) has finite index if and only if \( F \) and \( G \) have finite index - see [9 Proposition 3.5].

(2) For an inclusion \( B \subset A \), in general, if \( E, E' : A \to B \) are two conditional expectations, one may be of finite index and the other may fail - see [11 Example 2.10.1].

Interestingly, if there exists a finite index conditional expectation from \( A \) onto \( B \), then all faithful conditional expectations from \( A \) onto \( B \) are of finite index if the centralizer of \( B \) in \( A \), i.e., \( \mathcal{C}_A(B) := \{ x \in A : xb = bx \text{ for all } b \in B \} \), is finite dimensional - see [11 Propositions 2.10.2].

Thus, when \( \mathcal{C}_A(B) \) is finite dimensional, one can roughly say that the property of ‘finite index’ is an intrinsic property of the inclusion \( B \subset A \) and not of a conditional expectation from \( A \) onto \( B \).
(3) There exist finite-index conditional expectations even when the corresponding centralizers are not finite dimensional. For instance, see [11, Example 2.6.7].

Let $A = C(X)$ and $B := A^\circ$, where $X$ is an infinite compact Hausdorff space and $\alpha$ is a free action of a finite group $G$ on $A$. Define $E : A \to B$ by

$$E(f) = \sum_{g \in G} \alpha_g(f), \quad f \in A.$$  

Then, $E$ has finite-index and $\text{Ind}(E) = |G|$ - see [11, Proposition 2.8.1] - whereas $C_A(B)$ is infinite dimensional as $A = C(X)$ is a commutative $C^*$-algebra.

2.1.2. Watatani’s $C^*$-basic construction. Let $B \subseteq A$ be an inclusion of unital $C^*$-algebras with common unit and suppose $E : A \to B$ is a faithful conditional expectation. Let $A_1$ denote the Watatani’s $C^*$-basic construction of the inclusion $B \subseteq A$ with respect to the conditional expectation $E$, i.e., in short, one essentially shows the following:

(1) $A$ is a pre-Hilbert $B$-module with respect to the $B$-valued inner product given by

$$\langle a, a' \rangle_B := E(a^*a') \quad \text{for } a, a' \in A;$$

and, if $\mathfrak{A}$ denotes the Hilbert $B$-module completion of $A$, then

(2) the space of adjointable maps on $\mathfrak{A}$, denoted by $\mathcal{L}_B(\mathfrak{A})$, is a unital $C^*$-algebra (with the usual operator norm) and $A$ embeds in it as a unital $C^*$-subalgebra (and, by a slight abuse of notation, we identify $A$ with its image in $\mathcal{L}_B(\mathfrak{A})$);

(3) there exists a projection $e_B \in \mathcal{L}_B(\mathfrak{A})$ (called the Jones projection associated to $E$) such that $e_B a e_B = E(a) e_B$ for all $a \in A$ (it is standard to denote $e_B$ by $e_1$ as well); and

(4) one considers $A_1 := \overline{\text{span}\{xe_B y : x, y \in A\}} \subseteq \mathcal{L}_B(\mathfrak{A})$, which turns out to be a $C^*$-algebra (not always unital) and is called the $C^*$-basic construction of the inclusion $B \subseteq A$.

The system $(A, B, E, e_B, A_1)$ has the following natural universal property.

**Theorem 2.2.** [11, Proposition 2.2.11] Let $B \subseteq A$ be an inclusion of unital $C^*$-algebras with a faithful conditional expectation $E : A \to B$. Suppose that $A$ acts faithfully on some Hilbert space $H$ and $e$ is a projection on $H$ satisfying $e a e = E(a) e$ for all $a \in A$. If the linear map $B \ni b \mapsto eb \in B(H)$ is injective, then there is a $*$-isomorphism $\theta : A_1 \to A e A \subseteq B(H)$ such that $\theta(x e_B y) = x y e$ for all $x, y \in A$.

**Remark 2.3.** If $E : A \to B$ has finite index with a quasi-basis $\{\lambda_i\}$, then

(1) the two norms $\|\cdot\|_A$ and $\|\cdot\|$ are equivalent on $A$ (where $\|x\|_A := \|E_B(x^* x)\|^{1/2}$) - see [11] or the proof of [3, Lemma 2.11]; in particular, $A$ itself is a Hilbert $B$-module;

(2) $A_1$ is unital and is equal to $C^*(A, e_B)$ - see [11, Proposition 1.5];

(3) there exists a finite-index conditional expectation $E_1 : A_1 \to A$ (called the dual conditional expectation) with a quasi-basis $\{\lambda_i e_B(\text{Ind}(E))^{1/2}\}$ which satisfies the equation

$$E_1(x e_B y) = \text{Ind}(E)^{-1} x y$$

for all $x, y \in A$ and $\text{Ind}(E_1) = \sum_i \lambda_i \text{Ind}(E) e_B \lambda_i^*; \quad$ moreover, if $\text{Ind}(E) \in B$, then $\text{Ind}(E_1) = \text{Ind}(E)$ - see [11, Propositions 2.3.2 & 2.3.4]; and,

(4) if $F : A \to B$ is another finite-index conditional expectation and $C^*(A, f_B)$ denotes the corresponding $C^*$-basic construction, then there exists a $*$-isomorphism $\theta : A_1 \to C^*(A, f_B)$ such that $\theta(e_B) = f_B$ and $\theta(a) = a$ for all $a \in A$ - [11, Proposition 2.10.11]; and,

(5) $A_1 = \overline{\text{span}\{xe_B y : x, y \in A\}} =: A e B A$ - see [11, Lemma 2.2.2].
2.2. **Intermediate $C^*$-subalgebras.** Throughout this subsection, we let $B \subset A$ be an inclusion of unital $C^*$-algebras, $E : A \to B$ be a finite-index conditional expectation with a quasi-basis $\{\lambda_i : 1 \leq i \leq n\}$, $A_1 := \text{Ae}_B A = (C^*(A, e_B))$ denote the $C^*$-basic construction of $B \subset A$ with respect to $E$ and $E_1 : A_1 \to A$ denote the dual conditional expectation.

As in [7], let $\text{IMS}(B, A, E)$ denote the set of intermediate $C^*$-subalgebras $C$ between $B$ and $A$ with a conditional expectation $F : A \to C$ satisfying the compatibility condition $E = E_{1|C} \circ F$.

**Remark 2.4.**

1. If $C \in \text{IMS}(B, A, E)$ with respect to two compatible conditional expectations $F, F' : A \to C$, then $F = F'$ - see [7] Page 3.

2. If $C \in \text{IMS}(B, A, E)$ with respect to the compatible conditional expectation $F : A \to C$, then $F$ is faithful (since $E$ is so) and, therefore, by Remark 2.3(1), $F$ has finite index.

3. It must be mentioned here that it was presumed (without mention) in [3] that the compatible conditional expectation has finite index and was implicitly used while defining the notions of interior and exterior angles between intermediate subalgebras of an inclusion of unital $C^*$-algebras.

4. For $C \in \text{IMS}(B, A, E)$ with respect to the compatible conditional expectation $F : A \to C$, we observe that $A$ is a Hilbert $C^*$-module (Remark 2.3(2)); we let $e_C$ denote the corresponding Jones projection in $\mathcal{L}_C(A)$ and $C_1$ denote the Watatani basic construction of the inclusion $C \subset A$; thus, $C_1 = C^*(A, e_C) \subseteq \mathcal{L}_C(A)$.

**Remark 2.5.** In general, if $Q \subset P$ is an inclusion of unital $C^*$-algebras with a finite-index conditional expectation $G : P \to Q$, then not every intermediate $C^*$-subalgebra $R$ of $Q \subset P$ belongs to $\text{IMS}(Q, P, G)$ - see [7] Example 2.5. In fact, the example given in [7] illustrates that there need not exist even a single conditional expectation from $P$ onto $R$.

Izumi observed that the intermediate subalgebras of an inclusion of simple $C^*$-algebras have certain specific structures.

**Proposition 2.6.** [8] Let $B \subset A$ be an inclusion of unital $C^*$-algebras with a finite-index conditional expectation $E : A \to B$. If either $A$ or $B$ is simple then, every $C$ in $\text{IMS}(B, A, E)$ is a finite direct sum of simple closed two-sided ideals.

**Proof.** Let $C \in \text{IMS}(B, A, E)$ with respect to the compatible conditional expectation $F : A \to C$. Then, by Remark 2.3(2) and [11] Proposition 2.1.5, $F$ and $E_{1|C}$ satisfy the Pimsner-Popa inequality. Further, since $A$ or $B$ is simple and unital, it then follows from [6] Theorem 3.3 that $C$ is a finite direct sum of simple closed two-sided ideals. $\Box$

The following useful observations will be needed ahead when we recall and derive some generalities related to the notions of interior and exterior angles.

**Proposition 2.7.** Let $B \subset A$ be an inclusion of unital $C^*$-algebras, $E : A \to B$ be a finite-index conditional expectation with a quasi-basis $\{\lambda_i : 1 \leq i \leq n\}$, $A_1$ denote the $C^*$-basic construction of $B \subset A$ with respect to $E$, $E_1 : A_1 \to A$ denote the dual conditional expectation and $C \in \text{IMS}(B, A, E)$ with respect to the compatible finite-index conditional expectation $F : A \to C$. Then,

1. $\mathcal{L}_C(A) \subset \mathcal{L}_B(A)$;
2. $C_1 \subset A_1$, so that $e_C \in A_1$;
3. $e_C e_B = e_B = e_B e_C$;
4. $E_{1|C}$ has finite index with a quasi-basis $\{F(\lambda_i)\}$ and $e_C = \sum \mu_j e_B \mu_j^*$ for any quasi-basis $\{\mu_j\}$ of the conditional expectation $E_{1|C}$;
5. $E_1(e_B) = \text{Ind}(E)^{-1} \in \mathcal{Z}(A)$;
(6) $E_1(e_C) = \text{Ind}(E)^{-1}\text{Ind}(E_{1_C}) \in Z(C)$; and,

(7) in addition, if $\text{Ind}(E_{1_C}) \in Z(A)$, then

(a) $\text{Ind}(E) = \text{Ind}(F)\text{Ind}(E_{1_C})$;

(b) $E_{1C1} = F_1$, where $F_1$ denotes the dual conditional expectation of $F$; and

(c) $C_1 \in \text{IMS}(A, A_1, E_1)$ with respect to the conditional expectation $G : A_1 \to C_1$ satisfying $G(xe_{B_y}) = \text{Ind}(E_{1C})^{-1}xe_{C_y}$ for all $x, y \in A$ and has a quasi-basis

$\{\lambda_i e_B \text{Ind}(E_{1C})^{1/2} : 1 \leq i \leq n\}$.

In particular, we then have $E_1(e_C) = \text{Ind}(F)^{-1}$.

Proof. (1): Let $T \in \mathcal{L}_C(A)$ and $T^*$ denote its adjoint in $\mathcal{L}_C(A)$. Then, we see that

$$c(T(x), y)_B = E(T(x)^*y) = (E_{1c} \circ F)(T(x)^*y) = E_{1c}(T(x), y)_C$$

$$= E_{1c}(\langle x, T^*(y) \rangle_C) = (E_{1c} \circ F)(xT^*(y)) = \langle x, T^*(y) \rangle_B$$

for all $x, y \in A$. Hence, $T \in \mathcal{L}_B(A)$.

Because of (1), (2) now follows on the lines due to Lemma 4.2.

(3): Clearly, $e_{C}e_B = e_B$ (as $B \subset C$). Next, we observe that

$$e_{B}e_{C}(a) = e_B(F(a)) = E(F(a)) = (E_{1c} \circ F)(a) = E(a) = e_B(a)$$

for all $a \in A$. Thus, $e_{B}e_{C} = e_B$.

(4): That $E_{1c}$ has finite-index with quasi-basis $\{F(\lambda_i)\}$ follows from Proposition 1.7.2. Further, for any quasi-basis $\{\mu_j\}$ for $E_{1c}$, we have

$$\langle \sum \mu_j e_B \mu_j^* \rangle(a) = \sum \mu_j e_B \mu_j^*(a)$$

$$= \sum \mu_j E(\mu_j^*(a))$$

$$= \sum \mu_j (E_{1c} \circ F)(\mu_j^*(a))$$

$$= \sum \mu_j E_{1c}(\mu_j^*F(a))$$

$$= F(a)$$

$$= e_C(a)$$

for all $a \in A$. Hence, $e_C = \sum \mu_j e_B \mu_j^*$.

(5): See Proposition 2.3.2.

(6): For any quasi-basis $\{\mu_j\}$ for the conditional expectation $E_{1c}$, we have $e_C = \sum \mu_j e_B \mu_j^*$. Hence,

$$E_1(e_C) = E_1(\sum \mu_j e_B \mu_j^*)$$

$$= \sum E_1(\mu_j e_B \mu_j^*)$$

$$= \text{Ind}(E)^{-1} \sum \mu_j \mu_j^*$$

$$= \text{Ind}(E)^{-1}\text{Ind}(E_{1C}).$$

(7a): Let $\{\mu_1, \mu_2, ..., \mu_n\}$ be a quasi-basis for $E_{1c}$ and $\{\gamma_1, \gamma_2, ..., \gamma_m\}$ be a quasi-basis for $F$. Then, it is (known and can be) easily seen that $\{\gamma_i \mu_j : 1 \leq i \leq m, 1 \leq j \leq n\}$ is a quasi-basis.
for $E$ - see also [11, Proposition 1.7.1]. Thus,

$$
\text{Ind}(E) = \sum_{i,j} (\gamma_i \mu_j)(\gamma_i \mu_j)^*
= \sum_{i} \gamma_i (\sum_{j} \mu_j \mu_j^*) \gamma_i^*
= \text{Ind}(E|_{C}) \text{Ind}(F).
$$

(7b): We have $C_1 = \text{span}\{ xe_{CY} : x, y \in A \}$. Fix a quasi-basis $\{ \mu_j \}$ for $E_{|C}$. Then, for every pair $x, y \in C$, we observe that

$$
E_1(xe_{CY}) = E_1(x \sum_j \mu_j e_B \mu_j^* y)
= \text{Ind}(E)^{-1} \sum_j x \mu_j \mu_j^* y
= \text{Ind}(E)^{-1} x \text{Ind}(E|_{C}) y = \text{Ind}(F)^{-1} xy
= F_1(xe_{CY}),
$$

where the second last equality follows from (7a). Hence, $(E_1)|_{C_1} = F_1$.

(7c): We have $A_1 = \text{span}\{ xe_{BY} : x, y \in A \}$. Consider the linear map $G : A_1 \to C_1$ given by

$$
G(\sum_i x_i e_{BY_i}) = \text{Ind}(E|_{C})^{-1} \sum_i x_i e_{CY_i}
$$

for $x_i, y_i \in A, i = 1, \ldots, n$.

We first assert that $G$ is a conditional expectation of finite-index.

Fix a quasi-basis $\{ \mu_j \}$ for $E_{|C}$. Then, for any $x, y \in A$, by Item (4), we have

$$
G(xe_{CY}) = G(x \sum_j \mu_j e_B \mu_j^* y)
= \text{Ind}(E_{|C})^{-1} \sum_j x \mu_j e_B \mu_j^* y
= \text{Ind}(E_{|C})^{-1} \sum_j x \mu_j \mu_j^* e_{CY} \quad \text{(since $e_C \in C' \cap C_1$)}
= \text{Ind}(E_{|C})^{-1} x \text{Ind}(E_{|C}) e_{CY}
= xe_{CY}. \quad \text{(since $\text{Ind}(E_{|C}) \in \mathcal{Z}(A)$)}
$$

This implies that $G^2 = G$. Further, for any $\sum_i x_i e_{BY_i} \in A_1$, we have

$$
G\left( \sum_i x_i e_{BY_i} \right) \left( \sum_i x_i e_{BY_i} \right) = G\left( \sum_{i,j} y_i^* E(x_i^* x_j) e_{BY_j} \right)
= \text{Ind}(E_{|C})^{-1} \sum_{i,j} y_i^* e_C E(x_i^* x_j) e_{CY_j}.
$$

Then, taking $a_{i,j} := e_C E(x_i^* x_j) e_C \in C_1, i, j = 1, \ldots, n$, we have

$$
[a_{i,j}] = \text{diag}(e_C, \ldots, e_C)[E(x_i^* x_j)]\text{diag}(e_C, \ldots, e_C).
$$

By [11] Lemma 3.1, $[x_i^* x_j]$ is positive in $M_n(A)$ and since $E : A \to B$ is completely positive, it follows that $[E(x_i^* x_j)]$ is positive in $M_n(B)$. Hence, $[a_{i,j}]$ is positive in $M_n(C_1)$.

Thus, by [11] Lemma 3.2, it follows that $\sum_{i,j} y_i^* e_C E(x_i^* x_j) e_{CY_j} \geq 0$ in $C_1$. Further, since
\[ \text{Ind}(E_{\mathcal{I}|_{C}})^{-1} \in \mathcal{Z}(A) \cap \mathcal{Z}(C) \text{ and } e_{C} \in C' \cap C_{1}, \text{ it follows that } \text{Ind}(E_{\mathcal{I}|_{C}})^{-1} \text{ commutes with } \sum_{i,j} y_{i} e_{C} E(x_{i} x_{j}) e_{C} y_{j} \text{ and hence } \]
\[ G((\sum_{i} x_{i} e_{B} y_{i})^* (\sum_{i} x_{i} e_{B} y_{i})) \geq 0. \]

Thus, \( G : A_{1} \to C_{1} \) is positive and, therefore, it is a conditional expectation.

Further, \( G : A_{1} \to C_{1} \) has finite index with quasi basis \( \{ \lambda_{i} e_{B} \text{Ind}(E_{\mathcal{I}|_{C}})^{1/2} : 1 \leq i \leq n \} \) because, for any \( x, y \in A \), we have
\[
\sum_{i} \lambda_{i} e_{B} \text{Ind}(E_{\mathcal{I}|_{C}})^{1/2} G((\text{Ind}(E_{\mathcal{I}|_{C}})^{1/2} e_{B} \lambda_{x} x e_{B} y))
\]
\[
= \sum_{i} \lambda_{i} e_{B} \text{Ind}(E_{\mathcal{I}|_{C}})^{1/2} G((\text{Ind}(E_{\mathcal{I}|_{C}})^{1/2}) E(\lambda_{x} x) e_{B} y)
\]
\[
= \sum_{i} \lambda_{i} e_{B} \text{Ind}(E_{\mathcal{I}|_{C}})^{1/2} \text{Ind}(E_{\mathcal{I}|_{C}})^{-1} \text{Ind}(E_{\mathcal{I}|_{C}})^{1/2} E(\lambda_{x} x) e_{C} y
\]
\[
= x e_{B} y. \quad \text{(since } e_{C} \in C_{1} \cap B' \text{, } e_{B} e_{C} = e_{B} \text{ and } e_{B} \in A_{1} \cap B')
\]

Finally, since \( E_{1|_{C_{1}}} = F_{1} \), we observe that
\[
(E_{1|_{C_{1}}} \circ G)(x e_{B} y) = E_{1}(\text{Ind}(E_{\mathcal{I}|_{C}})^{-1} x e_{C} y)
\]
\[
= F_{1}(\text{Ind}(E_{\mathcal{I}|_{C}})^{-1} x e_{C} y)
\]
\[
= \text{Ind}(E_{\mathcal{I}|_{C}})^{-1} \text{Ind}(F)^{-1} x y
\]
\[
= \text{Ind}(E)^{-1} x y
\]
\[
= E_{1}(x e_{B} y)
\]

for all \( x, y \in A \), where the second last equality follows from Item (7a). Hence, \( (E_{1|_{C_{1}}} \circ G) = E_{1} \).

These show that \( C_{1} \in \text{IMS}(A, A_{1}, E_{1}) \) with respect to the finite-index conditional expectation \( G : A_{1} \to C_{1} \).

Recall that for an inclusion \( B \subset A \) of unital \( C^* \)-algebras, the normalizer of \( B \) in \( A \) is defined as
\[
N_{A}(B) := \{ u \in \mathcal{U}(A) : u B u^* = B \},
\]
where \( \mathcal{U}(A) \) denotes the group of unitaries in \( A \); and, (as already mentioned above) the centralizer of \( B \) in \( A \) is defined as
\[
C_{A}(B) := \{ a \in A : a b = b a \text{ for all } b \in B \}.
\]
Clearly, \( \mathcal{U}(B) \) is a normal subgroup of \( N_{A}(B) \) and \( C_{A}(B) \) is a unital \( C^* \)-subalgebra of \( A \), which is also denoted by \( B' \cap A \) and is called the relative commutant of \( B \) in \( A \).

The following observation provides us with some easy examples of elements in \( \text{IMS}(B, A, E) \).

**Lemma 2.8.** Let \( B \subset A \) be an inclusion of unital \( C^* \)-algebras with a finite-index conditional expectation \( E : A \to B \). Let \( C \in \text{IMS}(B, A, E) \) with respect to the compatible finite-index conditional expectation \( F : A \to C \) and \( u \in \mathcal{U}(A) \). Then,

1. \( F_{u} : A \to u C u^* \) given by \( F_{u} = \text{Ad}_{u} \circ F \circ \text{Ad}_{u}^* \), i.e., \( F_{u}(a) = u F(u^* a u) u^* \) for \( a \in A \), is a finite-index conditional expectation with a quasi-basis \( \{ \eta_{i} : 1 \leq i \leq n \} \), where \( \{ \eta_{i} : 1 \leq i \leq n \} \) is a quasi basis for \( F \); \( \text{Ind}(F_{u}) = \text{Ind}(F) \); and,
2. in addition, if \( u \in N_{A}(B) \) and \( E \) satisfies the tracial property, i.e., \( E(x y) = E(y x) \) for all \( x, y \in A \), then \( u C u^* \in \text{IMS}(B, A, E) \) with respect to \( F_{u} \).
Proof. (1) is a straight forward verification.

(2): Let \( D := uCu^* \). Since \( u \in \mathcal{N}_A(B) \), \( B = uBu^* \subset uCu^* \) and we have

\[
E_{1D} \circ F_u(a) = E_{1D}(uF(u^*au)u^*) = E(u^*uF(u^*au)) = E_C \circ F(u^*au) = E(u^*au) = E(uu^*a) \quad \text{(by tracial property again)} = E(a)
\]

for all \( a \in A \). Hence, \( uCu^* \in \text{IMS}(B,A,E) \) with respect to \( F_u \). \( \square \)

Remark 2.9. Note that \( ue_Cu^* \) is a projection in \( C_1 \) (as \( u \in A \subset C_1 \)) and, for each \( x \in A \), we have

\[
(ue_Cu^*)x(ue_Cu^*) = uF(u^*xu)e_Cu^* = F_u(x)ue_Cu^*.
\]

So, it is quite tempting to think that maybe the basic construction of \( B \subset uCu^* \) is given by \( (uCu^*)_1 = C_1 \) (as the \( C^* \)-algebra) with Jones projection \( e_{uCu^*} = ue_Cu^* \). However, this is not the case.

For instance, if we let \( A, B, C, E : A \rightarrow B \) and \( F : A \rightarrow C \) be the same as in Section 4, then taking \( u = \begin{bmatrix} 1 & \sqrt{2} & i/2 \\ i/2 & \sqrt{2} & 0 \\ \sqrt{2} & i/2 & 0 \end{bmatrix} \), we observe that

\[
e_{uCu^*} = \begin{bmatrix} 1/2 & 0 & 0 & 1/2 \\ 0 & 1/2 & 1/2 & 0 \\ 0 & 1/2 & 0 & 1/2 \\ 1/2 & 0 & 0 & 1/2 \end{bmatrix}
\]

(\text{using values of } e_C \text{ from Lemma 4.2 and } e_{uCu^*} \text{ from Lemma 4.3}).

Remark 2.10. (1) In general, the dual conditional expectation of a tracial conditional expectation need not be tracial.

For instance, consider the inclusion \( B = \mathbb{C} \ni \lambda \mapsto (\lambda, \lambda) \in A = \mathbb{C} \oplus \mathbb{C} \) with respect to the conditional expectation \( E : A \rightarrow B \) given by \( E((\lambda, \mu)) = \frac{(\lambda+\mu)}{2} \). Clearly, \( E \) is a finite-index tracial conditional expectation and we see that one can identify \( A_1 \) with \( M_2(\mathbb{C}) \) and then the dual conditional expectation \( E_1 : A_1 \rightarrow A \) is given by \( E_1([a_{ij}]) = (a_{11}, a_{22}) \). Clearly, \( E_1(AB) \neq E_1(BA) \) for \( A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \) and \( B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \).

(2) It is natural to wonder whether traciality of \( E \) can be dropped or not while showing that \( uCu^* \) belongs \( \text{IMS}(B,A,E) \) with respect to \( F_u \). And, it turns out that it can't be dropped always.

For instance, consider \( A = M_2(\mathbb{C}) \) and \( B = \mathbb{C}I_2 \) with the conditional expectation \( E : A \rightarrow B \) given by \( E([a_{ij}]) = a_{11}t + a_{22}(1-t) \) where \( t \neq 1/2 \) is fixed. Let \( C = \{ \text{diag}(\lambda, \mu) : \lambda, \mu \in \mathbb{C} \} \) and \( F : A \rightarrow C \) be the conditional expectation defined by \( F([a_{ij}]) = \text{diag}(a_{11}, a_{22}) \). Clearly, \( E \) and \( F \) are finite-index conditional expectations with quasi-bases \( \{ \sqrt{t}e_{11}, \sqrt{1-t}e_{12}, \sqrt{t}e_{21}, \sqrt{1-t}e_{22} \} \) and \( \{ e_{ij} : 1 \leq i, j \leq 2 \} \).
respectively, and \( E_{1C} \circ F = E \). If \( u = \begin{bmatrix} 1 & 1 \\ \sqrt{2} & \sqrt{2} \end{bmatrix} \), then \( u \in U(2) \) and

\[
F_u([a_{ij}]) = \begin{bmatrix} (a_{11} + a_{22})/2 & (a_{12} - a_{21})/2 \\ (a_{21} - a_{12})/2 & (a_{11} + a_{22})/2 \end{bmatrix}
\]

for all \([a_{ij}] \in A\). Thus, \( E_{i_{axC}} \circ F_u([a_{ij}]) = \frac{a_{11} + a_{22}}{2} \), which is not equal to \( E([a_{ij}]) \) (as \( t \neq 1/2 \)).

### 3. Interior and Exterior Angles

Let \( B \subset A \) be an inclusion of unital \( \mathcal{C}^* \)-algebras with a conditional expectation \( E : A \to B \). Then, for the \( B \)-valued inner product \( \langle \cdot , \cdot \rangle_B\) on \( A \) given by \( \langle x, y \rangle_B = E(x^*y) \), one has the following well known analogue of the Cauchy-Schwarz inequality

\[
\| \langle x, y \rangle_B \| \leq \| x \|_A \| y \|_A \quad \text{for all } x, y \in A,
\]

where \( \| x \|_A := \| E_B(x^*x) \|^{1/2} \). And, unlike for usual inner products, equality in \( \text{3.1} \) does not imply that \( \{ x, y \} \) is linearly dependent. For instance, consider \( B = \{ \text{diag}(\lambda, \mu) : \lambda, \mu \in \mathbb{C} \} \) in \( A = M_2(\mathbb{C}) \) with the natural finite-index conditional expectation \( E : A \to B \) given by \( E([x_{ij}]) = \text{diag}(x_{11}, x_{22}) \). Then, for \( x = \text{diag}(1, 1) \) and \( y = \text{diag}(i, 1) \) in \( A \), one easily verifies that \( \| \langle x, y \rangle_B \| = \| x \|_A \| y \|_A \) whereas \( \{ x, y \} \) is linearly independent.

Employing Inequality \( \text{3.1} \), motivated by \( \text{1} \), Bakshi and the first named author introduced the following definitions of the interior and exterior angles between intermediate \( \mathcal{C}^* \)-subalgebras.

**Definition 3.1.** \( \text{[3]} \) Let \( B \subset A \) be an inclusion of unital \( \mathcal{C}^* \)-algebras with a finite-index conditional expectation \( E : A \to B \). Then, for \( C, D \in \text{IMS}(B, A, E) \setminus \{ B \} \), the interior angle between \( C \) and \( D \) (with respect to \( E \)), denoted as \( \alpha(C, D) \), is given by the expression

\[
\cos(\alpha(C, D)) = \frac{\| (e_C - e_D, e_D - e_B)_A \|}{\| e_C - e_B \|_A \| e_D - e_B \|_A};
\]

and, for \( C, D \in \text{IMS}(B, A, E) \setminus \{ A \} \) with \( C_1, D_1 \in \text{IMS}(A, A_1, E_1) \), the exterior angle between \( C \) and \( D \) is defined as

\[
\beta(C, D) = \alpha(C_1, D_1),
\]

where \( \alpha(C_1, D_1) \) is defined with respect to the dual conditional expectation \( E_1 : A_1 \to A \).

By definition, both angles are allowed to take values only in the interval \( [0, \frac{\pi}{2}] \).

**Remark 3.2.**

1. Note that if \( \text{Ind}(E_{1D}) , \text{Ind}(E_{1C}) \in \mathcal{Z}(A) \), then by Proposition \( \text{2.7(7)} \), \( C_1, D_1 \in \text{IMS}(A, A_1, E_1) \). Thus, \( \beta(C, D) \) is defined for such intermediate subalgebras.

2. If \( B \subset C, D \subset A \) is a quadruple of simple unital \( \mathcal{C}^* \)-algebras, then \( \beta(C, D) \) is always defined.

We now derive some useful formulae for the interior and exterior angles in terms of certain related quasi-bases.

**Proposition 3.3.** Let \( B \subset A \) be an inclusion of unital \( \mathcal{C}^* \)-algebras with a finite index conditional expectation \( E : A \to B \) with quasi-basis \( \{ \lambda_i : 1 \leq i \leq p \} \). Let \( C, D \in \text{IMS}(B, A, E) \setminus \{ B \} \) with respect to the conditional expectations \( F : A \to C \) and \( F' : A \to D \), respectively. Let \( \{ \mu_j : 1 \leq \mu_j \leq m \} \) and \( \{ \delta_k : 1 \leq \delta_k \leq n \} \) be quasi-bases for \( E_{1C} \) and \( E_{1D} \), respectively. Then, we have the following:
(1) The interior angle between $C$ and $D$ is given by
\[
\cos(\alpha(C, D)) = \frac{\|(\text{Ind}(E))^{-1}\left(\sum_{j,k} \mu_j E(\mu_j^* \delta_k) \delta_k^* \right) - 1\|}{\|\text{Ind}(E)\|^{-1}\|\text{Ind}(E_{|C}) - 1\|^{1/2}\|\text{Ind}(E_{|D}) - 1\|^{1/2}}.
\]

In particular, if $\text{Ind}(E)$ is a scalar, then
\[
\cos(\alpha(C, D)) = \frac{\|\sum_{j,k} \mu_j E(\mu_j^* \delta_k) \delta_k^* - 1\|}{\|\text{Ind}(E_{|C}) - 1\|^{1/2}\|\text{Ind}(E_{|D}) - 1\|^{1/2}}.
\]

(2) Whenever $\text{Ind}(E_{|C})$ and $\text{Ind}(E_{|D})$ belong to $Z(A)$, the exterior angle between $C$ and $D$ can be derived from (3.2) using the following expressions:
\[
\langle e_C - e_2, e_{D_1} - e_2 \rangle_{A_1} = (\text{Ind}(E_1))^{-1}\left[(\text{Ind}(E_{|C}))^{-2}(\text{Ind}(E_{|D}))^{-1}\sum_{i,i'} \lambda_i e_C \text{Ind}(F') \right.
\]
\[
\times \sum_{j,k} \mu_j E(\mu_j^* \lambda_i^* \lambda_i \delta_k) \delta_k^*) e_D \lambda_i^* - 1\left],
\]
\[
\|e_C - e_2\|_{A_2} = \left\|\left(\text{Ind}(E_1)\right)^{-1}\left[(\text{Ind}(E_{|C}))^{-1}\left(\sum_i \lambda_i F(\text{Ind}(F)) e_C \lambda_i^*\right) - 1\right]\right\|^{1/2}
\]
\[
\text{and a similar expression for } \|e_{D_1} - e_2\|_{A_2}.
\]

Proof. (1) follows immediately by substituting the expressions for $e_C, e_{D_1}$, as obtained in Proposition 2.7, (4),(5),(6), in the definition of interior angle (3.2).

(2) Note that the dual conditional expectation $E_1 : A_1 \to A$ is of finite index with a quasi-basis $\{\lambda_i e_B(\text{Ind}(E))^{1/2}\} - \text{see Remark 2.3(3)}$. Further, from Proposition 2.7(4), a quasi-basis for $E_{1|C_1}$ is given by $\{w_i := G(\lambda_i e_B(\text{Ind}(E))^{1/2}) : 1 \leq i \leq n\}$, where $G : A_1 \to C_1$ is the conditional expectation as in the proof of Proposition 2.7(7). Thus, by Proposition 2.7(4), we see that
\[
e_{C_1} = \sum_i w_i e_2 w_i^*
\]
\[
= \sum_i \lambda_i e_C(\text{Ind}(E_{|C}))^{-1}(\text{Ind}(E))^{1/2} e_2(\text{Ind}(E))^{1/2}(\text{Ind}(E_{|C}))^{-1} e_C \lambda_i^*,
\]
since $\text{Ind}(E_{|C}) \in Z(A) \cap Z(C)$ and $e_C \in C' \cap C_1$. Thus,
\[
E_2(e_{C_1}) - E_2(e_2) = (\text{Ind}(E_1))^{-1}\left[(\sum_i \lambda_i e_C(\text{Ind}(E_{|C}))^{-1}(\text{Ind}(E))^{1/2} e_2(\text{Ind}(E))^{1/2}(\text{Ind}(E_{|C}))^{-1} e_C \lambda_i^*) - 1\right]
\]
\[
= (\text{Ind}(E_1))^{-1}\left[(\sum_i \lambda_i e_C(\text{Ind}(E_{|C}))^{-2}(\text{Ind}(E)) e_C \lambda_i^*) - 1\right] - 1\right]
\]
\[
= (\text{Ind}(E_1))^{-1}\left[(\sum_i \lambda_i e_C(\text{Ind}(E_{|C}))^{-1}(\text{Ind}(F)) e_C \lambda_i^*) - 1\right] (\text{by Prop. 2.7(7a)})
\]
\[
= (\text{Ind}(E_1))^{-1}\left[(\text{Ind}(E_{|C}))^{-1}\sum_i \lambda_i F(\text{Ind}(F)) e_C \lambda_i^*) - 1\right],
\]
which shows that
\[ \|e_{C_1} - e_2\|_{A_2} = \|(\text{Ind}(E_1))^{-1} \left[ (\text{Ind}(E_{1|C}))^{-1} \left( \sum_i \lambda_i F(\text{Ind}(F)) e_{C i} \right) - 1 \right] \|^{1/2}. \]

Further, as above, we have
\[ e_{D_1} = \sum_i \lambda_i e_D(\text{Ind}(E_{1|D}))^{-1}(\text{Ind}(E))^{1/2} e_2(\text{Ind}(E))^{1/2}(\text{Ind}(E_{1|D}))^{-1} e_D \lambda^*_i; \]
so that
\[ e_{C_1} e_{D_1} = \sum_{i,i'} \lambda_i e_C(\text{Ind}(E_{1|C}))^{-1}(\text{Ind}(E))^{1/2} E_1 \left[ (\text{Ind}(E))^{1/2}(\text{Ind}(E_{1|C}))^{-1} e_C \lambda^*_i \lambda^*_i e_D \right. \]
\[ \left. \times (\text{Ind}(E_{1|D}))^{-1}(\text{Ind}(E))^{1/2} e_2(\text{Ind}(E))^{1/2}(\text{Ind}(E_{1|D}))^{-1} e_D \lambda^*_i \right] = \sum_{i,i'} \lambda_i e_C(\text{Ind}(E_{1|C}))^{-1} \text{Ind}(F) E_1(\text{Ind}(E_{1|D}))^{-1} \text{Ind}(F') e_2 e_D \lambda^*_i, \]
where the last equality holds because of Proposition 2.7 (7)(a). Then, we obtain
\[ E_2(e_{C_1} e_{D_1}) = E_2(e_2) \]
\[ = (\text{Ind}(E_1))^{-1} \sum_{i,i'} \lambda_i e_C(\text{Ind}(E_{1|C}))^{-1}(\text{Ind}(F)) E_1(\text{Ind}(E_{1|D}))^{-1} \]
\[ \times \text{Ind}(F') e_D \lambda^*_i - (\text{Ind}(E_1))^{-1} \]
\[ = (\text{Ind}(E_1))^{-1} \left[ (\text{Ind}(E_{1|C}))^{-2}(\text{Ind}(E_{1|D}))^{-1} \sum_i \lambda_i e_C \text{Ind}(F') \right. \]
\[ \left. \times \sum_{i,i'} \mu_{ij} e(\mu^*_i \lambda^*_i \lambda^*_i \delta_k \delta_{ijk}) e_D \lambda^*_i - 1 \right]. \]
Since \( (e_{C_1} - e_2, e_{D_1} - e_2) A_1 = E_2(e_{C_1} e_{D_1}) - E_2(e_2), \) we are done.

\[ \square \]

**Remark 3.4.** A priori, for \( C, D \in \text{IMS}(B, A, E) \setminus \{ B \}, \) it is not clear whether \( \alpha(C, D) = 0 \)
implies \( C = D \) or not. Though, when \( B \subset A \) is an irreducible inclusion of simple unital \( \text{C}^* \)-algebras, then it is known to be true - see [3, Proposition 5.10]. Also, this phenomenon holds for a certain collection of intermediate subalgebras even in some non-irreducible set up as we shall see ahead in Corollary 4.9.

4. Possible values of the interior angle

Throughout this section, we let \( A = M_2(\mathbb{C}), B = CI_2, \Delta = \{ \text{diag}(\lambda, \mu) : \lambda, \mu \in \mathbb{C} \}, \)
\( E : A \to B \) denote the canonical (tracial) conditional expectation given by
\[ E([a_{ij}]) = \frac{(a_{11} + a_{22})}{2} I_2 \text{ for } [a_{ij}] \in A \]
and \( F : A \to \Delta \) denote the conditional expectation given by \( F([a_{ij}]) = \text{diag}(a_{11}, a_{22}). \)

The following useful observations are standard - see [11, Example 2.4.5].

**Lemma 4.1.** With running notations, the following hold:

1. \( E \) is a finite-index conditional expectation with a quasi-basis
\[ \{ \sqrt{2} e_{ij} : 1 \leq i, j \leq 2 \}, \]
where \( \{ e_{ij} : 1 \leq i, j \leq 2 \} \) denotes the set of standard matrix units of \( M_2(\mathbb{C}). \)
Lemma 4.3. For every unitary \( u \) that
\[
\text{Ind}(E) = 4 \quad \text{and} \quad E \text{ is the (unique) minimal conditional expectation from } A \text{ onto } B.
\]
(3) The \( C^* \)-basic construction \( A_1 \) for \( B \subset A \), with respect to the conditional expectation \( E \), can be identified with \( M_4(\mathbb{C}) \) and the Jones projection \( e_1 \) corresponding to the conditional expectation \( E \) is given by
\[
e_1 = \frac{1}{2} \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}.
\]
(4) Identifying \( M_4(\mathbb{C}) \) with \( M_2(\mathbb{C}) \otimes M_2(\mathbb{C}) \), the dual conditional expectation \( E_1 : A_1 \to A \) is given by \( E_1 = \text{id}_{M_4} \otimes E \); thus,
\[
E_1(X) = \begin{bmatrix} E(X_{(1,1)}) & E(X_{(1,2)}) \\ E(X_{(2,1)}) & E(X_{(2,2)}) \end{bmatrix}, X \in M_4(\mathbb{C}),
\]
where \( X_{(i,j)} \) denotes the \((i,j)\)th \(2 \times 2\) block of a matrix \( X \in M_4(\mathbb{C})\).

Lemma 4.2.
(1) \( F \) has finite index with a quasi-basis \( \{e_{ij} : 1 \leq i, j \leq 2\} \) and scalar index equal to 2.
(2) \( \Delta \in \text{IMS}(B, A, E) \) with respect to the conditional expectation \( F \) and the corresponding Jones projection in \( D_1 \) \((\subset A_1 = M_4(\mathbb{C}))\) is given by
\[
e_\Delta = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.
\]

Lemma 4.3. For every unitary \( u \) in \( A \),
(1) the map \( F_u : A \to u\Delta u^* \) given by \( F_u = \text{Ad}_u \circ F \circ \text{Ad}_u^* \) is a finite-index conditional expectation with a quasi-basis \( \{e_{ij} : 1 \leq i, j \leq 2\} \) and \( \text{Ind}(F_u) = 2 \);
(2) \( D := u\Delta u^* \in \text{IMS}(B, A, E) \) with respect to the conditional expectation \( F_u \); and,
(3) if \( u = [\lambda_{ij}] \), then the corresponding Jones projection in \( D_1 \) \((\subset A_1 = M_4(\mathbb{C}))\) is given by \( e_D = [x_{ij}] \), where
\[
x_{11} = |\lambda_{11}|^4 + |\lambda_{12}|^4, x_{12} = \lambda_{21}\lambda_{11}(|\lambda_{11}|^2 - |\lambda_{12}|^2), x_{14} = 2|\lambda_{11}|^2|\lambda_{12}|^2,
x_{22} = 2|\lambda_{11}|^2|\lambda_{21}|^2, x_{23} = 2\lambda_{21}^2\lambda_{11}^2
\]
and the remaining entries are given by \( x_{12} = x_{13} = x_{21} = x_{31} = -x_{24} = -x_{42}, x_{41} = x_{14}, x_{33} = x_{22}, x_{32} = x_{23} \) and \( x_{44} = x_{11} \).

Proof. (1): Clearly, the map \( F_u : A \to D \) is a conditional expectation and we can easily verify that
\[
x = \sum_{i,j} e_{ij} u F_u(u^* e_{ij}^* x) = \sum_{i,j} F_u(x e_{ij} u) u^* e_{ij}^* x
\]
for all \( x \in A \). Thus, \( \{e_{ij} u : 1 \leq i, j \leq 2\} \) is a quasi-basis for \( F_u \) and \( \text{Ind}(F_u) = 2 = \text{Ind}(F) \).
Since \( E \) satisfies the tracial property and \( \mathcal{N}_A(B) = A \), (2) follows from Lemma 4.2.
(3): After some routine calculation, for any \( \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in A \), we obtain
\[
F_u \left( \begin{bmatrix} a & b \\ c & d \end{bmatrix} \right) = \begin{bmatrix} x|\lambda_{11}|^2 + y|\lambda_{12}|^2 & x\lambda_{21}\lambda_{11} + y\lambda_{22}\lambda_{12} \\ x\lambda_{21}\lambda_{11} + y\lambda_{22}\lambda_{12} & x|\lambda_{12}|^2 + y|\lambda_{11}|^2 \end{bmatrix},
\]
where \( x = a|\lambda_{11}|^2 + d|\lambda_{12}|^2 + b\lambda_{22}\bar{\lambda}_{11} + c\bar{\lambda}_{21}\lambda_{11} \) and \( y = a|\lambda_{12}|^2 + d|\lambda_{11}|^2 + b\lambda_{22}\bar{\lambda}_{12} + c\bar{\lambda}_{21}\lambda_{12} \). Since \( u \) is a unitary, we have \( \lambda_{12}\lambda_{22} = -(\lambda_{11}\lambda_{21}), \lambda_{12}\lambda_{22} = -(\lambda_{11}\lambda_{21}), |\lambda_{11}|^2 = |\lambda_{22}|^2 \) and \( |\lambda_{12}|^2 = |\lambda_{21}|^2 \); thus, we further deduce that
\[
x|\lambda_{11}|^2 + y|\lambda_{12}|^2 = a(|\lambda_{11}|^4 + |\lambda_{12}|^4) + 2d|\lambda_{11}|^2|\lambda_{12}|^2 + c\lambda_{21}\lambda_{11}(|\lambda_{11}|^2 - |\lambda_{12}|^2) + b\lambda_{21}\lambda_{11}(|\lambda_{11}|^2 - |\lambda_{12}|^2); \quad \text{and}
\]
\[
x|\lambda_{12}|^2 + y|\lambda_{11}|^2 = 2a|\lambda_{11}|^2|\lambda_{12}|^2 + d(|\lambda_{11}|^4 + |\lambda_{12}|^4) + b\lambda_{21}\lambda_{11}(|\lambda_{12}|^2 - |\lambda_{11}|^2) + c\lambda_{21}\lambda_{11}(|\lambda_{12}|^2 - |\lambda_{11}|^2).
\]

Then, using the above expression for \( F_u([a_{ij}]) \) for \([a_{ij}] \in A \) and the matrix \( eD \) as given in the statement, we can easily verify that
(i) \( e_D xe_D = F_u(x)e_D \) and
(ii) \( e_D(x) = F_u(x) \)
for all \( x \in A \). This completes the proof. \( \square \)

We are now all set to derive a concrete expression for the interior angle between \( \Delta \) and its conjugate \( u\Delta u^* \), in terms of the entries of \( u \).

**Theorem 4.4.** If \( u = [\lambda_{ij}] \in U(2), \) then
\[
\cos(a(\Delta, u\Delta u^*)) = \sqrt{1 - (2|\lambda_{11}||\lambda_{12}|)^4}.
\]

**Proof.** Let \( D = u\Delta u^* \). From the matrix expressions of \( e_1, e_\Delta \) and \( e_D \) obtained above, we easily see that \( E_1(e_1) = \frac{1}{2}I_2, E_1(e_\Delta) = \frac{1}{2}I_2, E_1(e_D) = \frac{1}{2}I_2 \) and
\[
E_1(e_\Delta e_D) = \begin{bmatrix}
\frac{(|\lambda_{11}|^4 + |\lambda_{12}|^4)/2}{\lambda_{21}\lambda_{11}(|\lambda_{12}|^2 - |\lambda_{11}|^2)/2}, & \frac{\lambda_{21}\lambda_{11}(|\lambda_{11}|^2 - |\lambda_{12}|^2)/2}{(|\lambda_{11}|^4 + |\lambda_{12}|^4)/2}
\end{bmatrix}.
\]

Thus,
\[
||e_\Delta - e_1||_A_1 = ||E(e_\Delta - e_1)|| = \frac{1}{2} = ||E(e_D - e_1)|| = ||e_D - e_1||_A_1.
\]

Next, we calculate \( ||E_1(e_\Delta e_D - e_1)|| \). Let
\[
T = E_1(e_\Delta e_D - e_1) = \begin{bmatrix}
\frac{(|\lambda_{11}|^4 + |\lambda_{12}|^4)/2 - 1/4}{\lambda_{21}\lambda_{11}(|\lambda_{12}|^2 - |\lambda_{11}|^2)/2}, & \frac{\lambda_{21}\lambda_{11}(|\lambda_{11}|^2 - |\lambda_{12}|^2)/2}{(|\lambda_{11}|^4 + |\lambda_{12}|^4)/2 - 1/4}
\end{bmatrix}.
\]

Note that, \( T^*T \) turns out to be a scalar matrix with eigenvalue \( \lambda \), where
\[
\lambda = \left( \frac{(|\lambda_{11}|^4 + |\lambda_{12}|^4) - 1}{4} \right)^2 + \frac{|\lambda_{11}|^2|\lambda_{21}|^2(|\lambda_{12}|^2 - |\lambda_{11}|^2)^2}{4}
\]
\[
= \frac{1}{16} \left( 2(|\lambda_{11}|^4 + |\lambda_{12}|^4) - 1 \right)^2 + \frac{|\lambda_{11}|^2|\lambda_{21}|^2(|\lambda_{12}|^2 - |\lambda_{11}|^2)^2}{4}
\]
\[
= \frac{1}{16} \left( 2(|\lambda_{11}|^4 + |\lambda_{12}|^4) - (|\lambda_{11}|^2 + |\lambda_{12}|^2)^2 \right)^2 + \frac{|\lambda_{11}|^2|\lambda_{12}|^2(|\lambda_{12}|^2 - |\lambda_{11}|^2)^2}{4}
\]
\[\text{(since } |\lambda_{11}|^2 + |\lambda_{12}|^2 = 1 \text{ and } |\lambda_{21}| = |\lambda_{12}|)\]
\[
= \left( \frac{|\lambda_{11}|^2 - |\lambda_{12}|^2}{4} \right)^2 + \frac{|\lambda_{11}|^2|\lambda_{12}|^2(|\lambda_{12}|^2 - |\lambda_{11}|^2)^2}{4}
\]
\[
= \left( \frac{(|\lambda_{11}|^2 - |\lambda_{12}|^2)^2}{4} \right)^2 \left( 1 + 4|\lambda_{11}|^2|\lambda_{12}|^2 \right)
\]
\[
= \frac{1}{16} \left( |\lambda_{11}|^2 + |\lambda_{12}|^2 \right)^2 - 4|\lambda_{11}|^2|\lambda_{12}|^2 \left( 1 + 4|\lambda_{11}|^2|\lambda_{12}|^2 \right)
\]
\[
= \frac{1}{16} \left( 1 - 2|\lambda_{11}||\lambda_{12}| \right)^4.
\]
Thus,
\[ \|e_\Delta - e_1 e_D - e_1\|_A = \|E_1(e_\Delta e_D - e_1)\| = \|T\| = \left(\sqrt{1 - (2|\lambda_{11}|\|\lambda_{12}\|)^2}\right)/4.\]

Finally, substituting the above values of \(\|e_\Delta - e_1\|_A, \|e_D - e_1\|_A,\) and \(\|\langle e_\Delta - e_1 e_D - e_1\rangle\|_A\) in (3.2), we obtain
\[ \cos(\alpha(\Delta, u\Delta u^*)) = \sqrt{1 - (2|\lambda_{11}|\|\lambda_{12}\|)^4}.\]

Recall that a unitary matrix whose all entries have the same modulus is called a Hadamard matrix. Also, if \((B, C, D, A)\) is a quadruple of finite von Neumann algebras (i.e., \(B \subset C, D \subset A\)) with a faithful normal tracial state \(\tau : A \to \mathbb{C}\), then \((B, C, D, A)\) is said to be a commuting square if \(E_C^A E_D^A \subset E_B^A E_D^A\), where \(E_X^A : A \to X\) denotes the unique \(\tau\)-preserving conditional expectation from \(A\) onto any von Neumann subalgebra \(X\) of \(A\).

**Corollary 4.5.** Let \(u = \in U(2)\). Then,
1. \(\alpha(\Delta, u\Delta u^*) = \frac{\pi}{2}\) if and only if \(u\) is a Hadamard matrix if and only if \((B, \Delta, u^*\Delta u, A)\) is a commuting square; and,
2. if \(u = [\lambda_{ij}]\), then \(\alpha(\Delta, u\Delta u^*) = 0\) if and only if either \(u \in \Delta\) or \(\lambda_{11} = 0 = \lambda_{22}\).

In particular, \(\alpha(\Delta, u\Delta u^*) = 0\) if and only if \(\Delta = u\Delta u^*\).

**Proof.** (1): From Theorem 4.4, we observe that
\[ \cos(\alpha(\Delta, u\Delta u^*)) = 0 \iff \sqrt{1 - (2|\lambda_{11}|\|\lambda_{12}\|)^4} = 0 \]
\[ \iff |\lambda_{11}|\|\lambda_{12}\| = \frac{1}{2} \]
\[ \iff |\lambda_{11}| = |\lambda_{12}| \quad (\text{since } |\lambda_{11}|^2 + |\lambda_{12}|^2 = 1) \]
\[ \iff |\lambda_{11}| = |\lambda_{12}| = |\lambda_{21}| = |\lambda_{22}| \]
\[ \iff u \text{ is Hadamard Matrix.} \]

Note that \(F : M_2 \to \Delta\) and \(F_u : M_2 \to u\Delta u^*\) are the unique trace preserving conditional expectations. And, it is a well known fact - see, for instance, [5.52.2] - that \((B, \Delta, u^*\Delta u, A)\) is a commuting square if and only if \(u\) is a Hadamard matrix.

(2): Again, from Theorem 4.4
\[ \cos(\alpha(\Delta, u\Delta u^*)) = 1 \iff |\lambda_{11}|\|\lambda_{12}\| = 0 \]
\[ \iff |\lambda_{11}| = 0 \quad \text{or} \quad |\lambda_{12}| = 0 \]
\[ \iff |\lambda_{11}| = 0 = |\lambda_{22}| \quad \text{or} \quad u \text{ is diagonal. (as } u \text{ is unitary)} \]

We can now deduce our assertion that the interior angle attains all values in \([0, \pi/2]\).

**Corollary 4.6.**
\[ \{\alpha(\Delta, u\Delta u^*) : u \in U(2)\} = \left[0, \frac{\pi}{2}\right]. \]

**Proof.** Note that, for each \(u = [\lambda_{ij}] \in U(2), 0 \leq (2|\lambda_{11}|\|\lambda_{12}\|)^4 \leq 1.\) Thus, we can define a map \(\varphi : U(2) \to [0, 1]\) given by
\[ \varphi([\lambda_{ij}]) = \sqrt{1 - (2|\lambda_{11}|\|\lambda_{12}\|)^4}. \]

Clearly, \(\varphi\) is a continuous function. Since \(U(2)\) is connected, it follows that \(\varphi(U(2))\) is also connected. Note that, from Corollary 4.5 we have \(\varphi(u) = 0\) for any complex Hadamard matrix.
u ∈ U(2) and ϕ(I_2) = 1. Hence, ϕ(U(2)) = [0, 1]. In particular, in view of Theorem 4.4 we obtain
\[ \{ α(Δ, uΔu^*) : u ∈ U(2) \} = [0, \frac{π}{2}], \]
as was desired. \hfill \Box

**Corollary 4.7.** There exist C, D ∈ IMS(B, A, E) such that e_C e_D ≠ e_D e_C.

**Proof.** Fix a u = [λ_{ij}] ∈ U(2) and let C = Δ and D = uΔu*. Then, both C, D ∈ IMS(A, B, E) and using the values of e_C and e_D from Lemmas 4.2 and 4.3 we obtain
\[ e_C e_D = \begin{bmatrix} |λ_{11}|^4 + |λ_{12}|^2 & 0 & 2|λ_{11}|^2|λ_{12}|^2 \\ 0 & 0 & 0 \\ 2|λ_{11}|^2|λ_{12}|^2 & 0 & |λ_{11}|^4 + |λ_{12}|^4 \end{bmatrix} \]
and
\[ e_D e_C = \begin{bmatrix} |λ_{11}|^4 + |λ_{12}|^2 & 0 & 0 \\ 0 & 0 & 2|λ_{11}|^2|λ_{12}|^2 \\ 2|λ_{11}|^2|λ_{12}|^2 & 0 & |λ_{11}|^4 + |λ_{12}|^4 \end{bmatrix}. \]
Thus, if u is neither a diagonal matrix nor a Hadamard matrix nor λ_{11} = 0 = λ_{22}, then we see that C ≠ D and e_C e_D ≠ e_D e_C. \hfill \Box

5. Angles between intermediate crossed product subalgebras of crossed product inclusions

Recall that if a countable discrete group G acts on a unital C*-algebra P via a map α : G → Aut(P), then the space C_c(G, P) consisting of compactly supported P-valued functions on G can be identified with the space \{ ∑_{finite} a_g g : a_g ∈ P, g ∈ G \} of formal finite sums and is a unital *-algebra with respect to (the so called twisted) multiplication given by the convolution operation
\[ \left( ∑_{s ∈ I} a_s s \right) \left( ∑_{t ∈ J} b_t t \right) = ∑_{s ∈ I, t ∈ J} a_s α_s(b_t)st \]
and involution given by
\[ \left( ∑_{s ∈ I} a_s s \right)^* = ∑_{s ∈ I} a_{s^{-1}}(a_s^*)s^{-1} \]
for any two finite sets I and J in G. Further, the reduced crossed product P ×_{α,r} G and the universal crossed product P ×_{α} G are defined, respectively, as the completions of C_c(G, P) with respect to the reduced norm
\[ \| ∑_{finite} a_g g \|_r := ∑_{finite} π(a_g)(1 ⊗ λ_g) \|_{B(H ⊗ ℓ^2(G))}, \]
where P ⊂ B(H) is a (equivalently, any) fixed faithful representation of A, λ : G → B(ℓ^2(G)) is the left regular representation and π : A → B(H ⊗ ℓ^2(G)) is the representation satisfying π(a)(ξ ⊗ δ_g) = α_g^{-1}(a)ξ ⊗ δ_g for all ξ ∈ H and g ∈ G; and the universal norm
\[ \| x \|_u := sup \| π(x) \| \text{ for } x ∈ C_c(G, P), \]
where the supremum runs over all (cyclic) *-homomorphisms π : C_c(G, P) → B(H), respectively. We suggest the reader to see [4, 12] for more on crossed products.
When $G$ is a finite group, then it is well known that the reduced and universal norms coincide on $C_c(G, P)$ and $C_c(G, P)$ is complete with the common norm; thus, $P \rtimes_{\alpha, r} G = C(G, P) = P \rtimes_{\alpha} G$ (as $*$-algebras).

In this section, analogous to [2 Proposition 2.7], we derived a concrete value for the interior angle between intermediate crossed product subalgebras of an inclusion of crossed product algebras.

The following important observations are well known.

**Proposition 5.1.** [5 9] Let $G$ be a countable discrete group and $H$ be its subgroup. Let $P$ be a unital $C^*$-algebra such that $G$ acts on $P$ via a map $\alpha : G \to \text{Aut}(P)$. Let $A := P \rtimes_{\alpha, r} G$ (resp., $A := P \rtimes G$) and $B := P \rtimes_{\alpha, r} H$ (resp., $B := P \rtimes H$). Then,

1. the canonical injective $*$-homomorphism

$$C_c(H, P) \ni \sum_{\text{finite}} a_h h \mapsto \sum_{\text{finite}} a_h h \in C_c(G, P)$$

extends to an injective $*$-homomorphism from $B$ into $A$; and

2. the natural map

$$C_c(G, P) \ni \sum_{\text{finite}} a_g g \mapsto \sum_{\text{finite}} a_h h \in C_c(H, P)$$

extends to a conditional expectation $E : A \to B$.

Moreover, $E$ has finite index if and only if $[G : H] < \infty$ and in that case a quasi-basis for $E$ is given by $\{g_i : 1 \leq i \leq [G : H]\}$ for any set $\{g_i\}$ of left coset representatives of $H$ in $G$ and $E$ has scalar index equal to $[G : H]$.

**Proof.** (1) follows from [5] (also see [9 Remark 3.2]) and the first part of the proof of [9 Proposition 3.1], respectively.

(2): Consider the canonical $C_c(H, P)$-bilinear projection $E_0 : C_c(G, P) \to C_c(H, P)$ given by

$$E_0 \left( \sum_{\text{finite}} a_g g \right) = \sum_{\text{finite}} a_h h.$$

Then, from [5] (also see [9 Remark 3.2]) and [9 Proposition 3.1, Remark 3.2], respectively, it follows that $E_0$ extends to a conditional expectation from $A$ onto $B$. Also, from [9 Theorem 3.4], it follows that $E$ has finite index (with a quasi-basis as in the statement) if and only $[G : H] < \infty$. \hfill $\square$

**Proposition 5.2.** Let $G, H, P, \alpha, A, B$ and $E$ be as in Proposition 5.1 with $[G : H] < \infty$ and $\{g_i : 1 \leq i \leq [G : H]\}$ be a set of left coset representatives of $H$ in $G$. Let $K$ and $L$ be proper intermediate subgroups of $H \subset G$ and let $C := P \rtimes_{\alpha, r} K$ (resp., $P \rtimes_{\alpha} K$) and $D := P \rtimes_{\alpha, r} L$ (resp., $P \rtimes_{\alpha} L$). Then, $C, D \in \text{IMS}(B, A, E) \setminus \{A, B\}$ and the interior angle between them is given by

$$\cos(\alpha(C, D)) = \frac{[K \cap L : H] - 1}{\sqrt{[K : H] - 1} \sqrt{[L : H] - 1}}.$$  

**Proof.** Note that, $B \subset C, D \subset A$, by Proposition 5.1. Also, $[G : K]$ and $[G : L]$ are both finite as $[G : H]$ is finite. So, $C, D \in \text{IMS}(B, A, E)$ with respect to the natural finite-index conditional expectations guaranteed by Proposition 5.1.

Fix left coset representatives $\{k_r : 1 \leq r \leq [K : H]\}$ and $\{l_s : 1 \leq s \leq [L : H]\}$ of $H$ in $K$ and $L$, respectively. Then, it is readily seen that $E_{1C} : C \to B$ and $E_{1D} : D \to B$ have
quasi-bases \( \{k_r : 1 \leq r \leq [K : H]\} \) and \( \{l_s : 1 \leq s \leq [L : H]\} \), respectively. Then, from (5.3), we obtain

\[
\cos(\alpha(C, D)) = \frac{\| \sum_{r,s} k_r^*E(k_r^*l_s^*) - 1 \|}{\| \sqrt{[K : H]} - 1 \| \| \sqrt{[L : H]} - 1 \|}
\]

\[
= \frac{\| \sum_{(r,s)} k_r^*k_r^*l_s^* - 1 \|}{\sqrt{[K : H]} - 1 \sqrt{[L : H]} - 1}
\]

where the last equality holds because the map

\[
\{ (r,s) : k_rH \cap l_sH \neq \emptyset \} \ni (r,s) \mapsto k_rH = l_sH \in (K \cap L)/H
\]
is a bijection. \(\square\)

**Corollary 5.3.** Let the notations be as in Proposition 5.2. Then,

1. \(\alpha(C, D) = \frac{2}{\pi}\) if and only if \(K \cap L = H\); and,
2. \(\alpha(C, D) = 0\) if and only if \(K = L\).

In particular, if \(C_g := P \times_{\alpha} (g^{-1}Kg)\) (resp., \(P \times_{\alpha, r} (g^{-1}Kg)\)) then \(\alpha(C, C_g) = 0\) for all \(g \in G\) if and only if \(K\) is normal in \(G\).

**Proof.** (1) is straight forward and, for (2), we just need to show the necessity.

Note that \(\alpha(C, D) = 0\) implies that \(\frac{\| (K \cap L) : H \| - 1}{\sqrt{[K : H] - 1 \sqrt{[L : H] - 1}} = 1\), which then implies that

\[
\left( \sqrt{\frac{[K \cap L] : H}{[K : H] - 1}} \right) \left( \sqrt{\frac{[K \cap L] : H}{[L : H] - 1}} \right) = 1
\]

and that \([K \cap L : H] \neq 1\). Note that

\[
0 < \frac{[K \cap L] : H}{[K : H] - 1}, \frac{[K \cap L] : H}{[L : H] - 1} \leq 1;
\]

so, it follows that

\[
\frac{[K \cap L] : H}{[K : H] - 1} = 1 = \frac{[K \cap L] : H}{[L : H] - 1}.
\]

Hence, \(K = K \cap L = L\). \(\square\)

Recall that for a subgroup \(H\) of a group \(G\), its normalizer is given by

\[
\mathcal{N}_G(H) = \{ g \in G : g^{-1}Hg = H \}.
\]

**Corollary 5.4.** Let \(G, H\) and \(K\) be as in Proposition 5.2. If \(g \in \mathcal{N}_G(H)\), then \(\alpha(C, C_g) = 0\) if and only if \(g \in \mathcal{N}_G(K)\), where \(C_g\) is same as in Corollary 5.3.

**Proof.** Let \(L := g^{-1}Kg\). Since \([K : H] = [L : H]\), from Expression (5.1), we obtain

\[
\cos(\alpha(C, C_g)) = \frac{[K \cap L] : H}{[K : H] - 1}.
\]

Thus, \(\alpha(C, C_g) = 0\) if and only if \(K \cap (g^{-1}Kg) = K\) if and only if \(g \in \mathcal{N}_G(K)\). \(\square\)

Note that, if \(P = C\) and \(\alpha : G \to \text{Aut}(C)\) is the trivial representation, then we know that \(C_r^* = C \times_{\alpha, r} G\) and \(C^* = C \times_{\alpha} G\). Thus, we readily deduced the following:
Corollary 5.5. Let $G$ be a countable discrete group with subgroups $H, K$ and $L$ such that $H \subseteq K \cap L$, $H \neq K, L$ and $|G : H| < \infty$. Let $A := C^*_r(G)$ (resp., $C^*(G)$), $B := C^*_r(H)$ (resp., $C^*(H)$), $C := C^*_r(K)$ (resp., $C^*(K)$) and $D := C^*_r(L)$ (resp., $C^*(L)$). Then, $C, D \in \text{IMS}(B, A, E)$ and
\begin{equation}
\cos(\alpha(C, D)) = \frac{|K \cap L : H| - 1}{\sqrt{|K : H| - 1} \sqrt{|L : H| - 1}},
\end{equation}
where $E : A \to B$ is the conditional expectation as in Proposition 5.1 with $P = C$.

Example 5.6. Let $G = \mathbb{Z}_3 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_5 \oplus \mathbb{Z}_5$. Consider its subgroups $K = \mathbb{Z}_3 \oplus \mathbb{Z}_3 \oplus (0)$, $L = (0) \oplus \mathbb{Z}_3 \oplus (0)$ and $H = (0) \oplus (0) \oplus \mathbb{Z}_5 \oplus (0)$. Then,
$$\cos(\alpha(C[K], C[L])) = \frac{1}{2}.$$ 

Thus, $\alpha(C[K], C[L]) = \pi/3$.

In particular, this illustrates that if $B \neq C \subseteq D \subseteq A$, then $\alpha(C, D)$ need not be 0.

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