INITIAL-BOUNDARY VALUE AND INVERSE PROBLEMS FOR SUBDIFFUSION EQUATIONS IN $\mathbb{R}^N$

RAVSHAN ASHUROV AND RAXIM ZUNNUNOV

Abstract. An initial-boundary value problem for a subdiffusion equation with an elliptic operator $A(D)$ in $\mathbb{R}^N$ is considered. Uniqueness and existence theorems for a solution of this problem are proved by the Fourier method. Considering the order of the Caputo time-fractional derivative as an unknown parameter, the corresponding inverse problem of determining this order is studied. It is proved, that the Fourier transform of the solution $\hat{u}(\xi, t)$ at a fixed time instance recovers uniquely the unknown parameter. Further, a similar initial-boundary value problem is investigated in the case when operator $A(D)$ is replaced by its power $A^\sigma$. Finally, existence and uniqueness theorems for the solution of the inverse problem of determining both the orders of fractional derivatives with respect to time and the degree of $\sigma$ are proved.

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1. Introduction and main results

The theory of differential equations with fractional derivatives has gained significant popularity and importance in the last few decades, mainly due to its applications in many seemingly distant fields of science and technology (see, for example, [1] - [6]).

One of the most important time-fractional equations is the subdiffusion equation, which models anomalous or slow diffusion processes. This equation is a partial integro-differential equation obtained from the classical heat equation by replacing the first-order derivative with a time-fractional derivative of the order $\rho \in (0, 1)$.

When considering the subdiffusion equation as a model equation in the analysis of anomalous diffusion processes, the order of the fractional derivative is often unknown and difficult to measure directly. To determine this parameter, it is necessary to investigate the inverse problems of identifying these physical quantities based on some indirectly observable information about solutions (see a survey paper Li, Liu and Yamamoto [7]).

In this paper, we investigate the existence and uniqueness of solutions to initial-boundary value problems for subdiffusion equations with the Caputo derivative and the elliptic operator $A(D)$ in $\mathbb{R}^N$ with constant coefficients. The inverse problems of determining the order of the fractional derivative with respect to time and with respect to the spatial variable will also be investigated.

Let us proceed to a rigorous formulation of the main results of this article.

1. Let $A(D) = \sum_{|\alpha|=m} a_\alpha D^\alpha$ be a homogeneous symmetric elliptic differential expression of even order $m = 2l$, with constant coefficients, i.e. $A(\xi) > 0$, for all $\xi \neq 0$, where $\alpha = (\alpha_1, \alpha_2, ..., \alpha_N)$ - multi-index and $D = (D_1, D_2, ..., D_N)$, $D_j = \frac{\partial}{\partial x_j}$.

The fractional integration in the Riemann - Liouville sense of order $\rho < 0$ has the form

$$\partial^\rho_t h(t) = \frac{1}{\Gamma(-\rho)} \int_0^t \frac{h(\xi)}{(t-\xi)^{\rho+1}} d\xi, \quad t > 0,$$
provided the right-hand side exists. Here \( \Gamma(\rho) \) is Euler’s gamma function. Using this definition one can define the Caputo fractional derivative of order \( \rho \), \( 0 < \rho < 1 \), as

\[
D^\rho_t h(t) = \partial_t^{\rho-1} \frac{d}{dt} h(t).
\]

Let \( \rho \in (0, 1) \) be a given number. Consider the initial-boundary value problem

\[
\begin{align*}
D^\rho_t u(x, t) + A(D)u(x, t) &= 0, & x \in \mathbb{R}^N, & 0 < t \leq T, \\
\lim_{|x| \to \infty} D^\alpha u(x, t) &= 0, & |\alpha| \leq l - 1, & 0 < t \leq T, \\
u(x, 0) &= \varphi(x), & x \in \mathbb{R}^N,
\end{align*}
\]

where \( \varphi(x) \) is a given continuous function.

We call problem (1.1) - (1.3) the forward problem.

**Definition 1.1.** A function \( u(x, t) \) with the properties

\[
D^\rho_t u(x, t) \text{ and } A(D)u(x, t) \in C(\mathbb{R}^N \times (0, T])
\]

and satisfying conditions (1.1) - (1.3) is called the classical solution (or simply, solution) of the forward problem.

Denoting the Sobolev classes by \( L^2_2(\mathbb{R}^N) \) (see the definition in the next section), we can state an existence theorem for this problem.

**Theorem 1.2.** Let \( \tau > \frac{N}{2} \) and \( \varphi \in L^2_2(\mathbb{R}^N) \). Then the forward problem has a solution in the form

\[
u(x, t) = \int_{\mathbb{R}^N} E_\rho(-A(\xi)t^\rho) \hat{\varphi}(\xi)e^{ix\xi}d\xi.
\]

The integral uniformly converges with respect to \( x \in \mathbb{R}^N \) and for each \( t \in (0, T] \), where \( \hat{\varphi}(\xi) \) is the Fourier transform of \( \varphi \).

If the solution of the forward problem \( u(x, t) \in L^2(\mathbb{R}^N), t \in (0, T] \), then we may define the Fourier transform

\[
\hat{u}(\xi, t) = (2\pi)^{-N} \int_{\mathbb{R}^N} u(x, t)e^{-ix\xi}dx.
\]

The corresponding uniqueness theorem has the form.

**Theorem 1.3.** Let the following conditions be satisfied for all \( t \in (0, T] \)

1. \( \varphi \in C(\mathbb{R}^N) \),
2. \( \lim_{|x| \to \infty} D^\alpha u(x, t) = 0, \quad l \leq |\alpha| \leq m - 1, \)
3. \( D^\alpha u(x, t) \in L^2(\mathbb{R}^N), \quad |\alpha| \leq m, \)
4. \( \hat{u}(\xi, t) \in L^1(\mathbb{R}^N). \)

Then there can be only one solution to the forward problem.

**Remark 1.4.** We will prove that under the condition of Theorem 1.2 for initial function \( \varphi \), all four conditions of Theorem 1.3 are also satisfied. Thus, if we add these four conditions to Definition 1.1 then Theorem 1.3 guarantees both the existence and the uniqueness of such a solution.

In recent years, numerous works of specialists have appeared, where they study various initial-boundary value problems for various subdiffusion equations. Let us mention only some of these works. Basically, the case of one spatial variable \( x \in \mathbb{R} \) and subdiffusion equation with "the elliptical part" \( u_{xx} \) were considered (see, for example, handbook Machado, editor [1], book of A.A. Kilbas et al. [3] and monograph of A. V. Pskhu [8], and references in these works). The paper Gorenflo, Luchko and Yamamoto [9] is devoted to the study of subdiffusion equations in Sobolev spaces. In the paper by Kubica and Yamamoto [10], initial-boundary value problems for equations with time-dependent coefficients are considered. In multidimensional case \( x \in \mathbb{R}^N \) instead of the differential expression \( u_{xx} \) authors considered either the Laplace operator (3), (11) - (13), or pseudo-differential operators with constant coefficients in the whole space \( \mathbb{R}^N \).
(Umarov [14]). In the last paper the initial function \( \varphi \in L_p(\mathbb{R}^N) \) is such, that the Fourier transform \( \hat{\varphi} \) is compactly supported. The authors of the recent paper [15] considered initial-boundary value problems for subdiffusion equations with arbitrary elliptic differential operators in bounded domains.

2. Determining the correct order of an equation in applied fractional modeling plays an important role. The corresponding inverse problem for subdiffusion equations has been considered by a number of authors (see a survey paper Li, Liu and Yamamoto [7] and references therein, [16] - [21]). Note that in all known works the subdiffusion equation was considered in a bounded domain \( \Omega \subset \mathbb{R}^N \). In addition, it should be noted that in publications [16] - [19] the following relation was taken as an additional condition

\[
\begin{align*}
&u(x_0, t_0) = h(t), \quad 0 < t < T,
&\end{align*}
\]

at a monitoring point \( x_0 \in \Omega \). But this condition, as a rule (an exception is work [19] by J. Janno, where both the uniqueness and the existence are proved), can ensure only the uniqueness of the solution of the inverse problem [16] - [18]. Authors of paper Ashurov and Umarov [20] considered as an additional information the value of projection of the solution onto the first eigenfunction of the elliptic part of subdiffusion equation. Note, results of paper [20] are applicable only in case, when the first eigenvalue is equal to zero. The uniqueness and existence of an unknown order of the fractional derivative in the subdiffusion equation were proved in the recent work of Alimov and Ashurov [21]. In this case, the additional condition is \( |u(x, t_0)|^2 = d_0 \), and the boundary condition is not necessarily homogeneous.

Now let us consider the order of fractional derivative \( \rho \) in equation (1.1) as an unknown parameter. We formulate our inverse problem in the following way. Let us fix a vector \( \rho \in (0, 1) \).

**Theorem 1.8.** Let \( \varphi \in L_2(\mathbb{R}^N) \), \( \tau > \frac{N}{2} \), and \( t_0 \geq T_0 \). Then the inverse problem has a unique solution \( \{u(x, t), \rho\} \) if and only if

\[
\begin{align*}
&e^{\lambda_0} < \frac{d_0}{|\hat{\varphi}(t_0)|} \leq E_{\rho_0}(-\lambda_0 t_0^{\rho_0}).
&\end{align*}
\]

3. Finally, we will consider another inverse problem of determining both the orders of fractional derivatives with respect to time and the spatial derivatives in the subdiffusion equations.
For the best of our knowledge, only in the following two papers \[22\] and \[24\] such inverse problems were studied and only the uniqueness theorems were proved (note that uniqueness is a very important property of a solution from an application point of view). In paper \[22\] by Tatar and Ulusoy it is considered the initial-boundary value problem for differential equation

\[
\partial_t^\alpha u(t, x) = -(-\Delta)^\sigma u(t, x), \quad t > 0, \quad x \in (0, 1),
\]

where \(\Delta^\alpha\) is the one-dimensional fractional Laplace operator, \(\rho \in (0, 1)\) and \(\sigma \in (1/4, 1)\). The authors have proved that if the initial function \(\varphi(x)\) is sufficiently smooth and all its Fourier coefficients are positive, then the two-parameter inverse problem with additional information \[15\] has a unique solution. As for physical backgrounds for two-parameter differential equations, see, for example, \[24\].

In \[23\], M. Yamamoto proved the uniqueness theorem for the above two-parameter inverse problem in \(N\)-dimensional bounded domain \(\Omega\) with smooth boundary \(\partial\Omega\). The conditions on the initial function found in this work are less restrictive, for example, if \(\varphi\) is zero on \(\partial\Omega\), \(\varphi \in L_2^2(\Omega)\), \(\tau > N/2\), \(\varphi \geq 0\) in \(\Omega\) and \(\varphi(x_0) \neq 0\), then the uniqueness theorem is true.

Let us denote by \(A\) an operator in \(L_2(\mathbb{R}^N)\) with domain of definition \(D(A) = C_0^\infty(\mathbb{R}^N)\), acting as \(Af(x) = A(D)f(x)\). It is easy to verify that the closure \(\hat{A}\) of operator \(A\) is positive and selfadjoint. Therefore, by virtue of the von Neumann theorem, for any \(\sigma > 0\), we can introduce the degree of the operator \(\hat{A}\) as

\[
\hat{A}^\sigma f(x) = \int_0^\infty \lambda^\sigma dP_\lambda f(x) = \int_{\mathbb{R}^N} A^\sigma(\xi)\hat{f}(\xi)e^{ix\xi}d\xi,
\]

where projectors \(P_\lambda\) defined as

\[
P_\lambda f(x) = \int_{A(\xi)<\lambda} \hat{f}(\xi)e^{ix\xi}d\xi.
\]

The domain of definition of this operator is determined from the condition \(\hat{A}^\sigma f(x) \in L_2(\mathbb{R}^N)\) and has the form

\[
D(\hat{A}^\sigma) = \{f \in L_2(\mathbb{R}^N) : \int_{\mathbb{R}^N} A^{2\sigma}(\xi)|\hat{f}(\xi)|^2d\xi < \infty\}.
\]

Suppose first that \(\rho \in (\rho_0, 1)\) and \(\sigma \in (0, 1)\) are given numbers and consider the initial-boundary value (the second forward) problem

\[
D_t^\rho v(x, t) + A^\rho v(x, t) = 0, \quad x \in \mathbb{R}^N, \quad 0 < t \leq T,
\]

\[
\lim_{|x| \to \infty} v(x, t) = 0, \quad 0 < t \leq T,
\]

\[
v(x, 0) = \varphi(x), \quad x \in \mathbb{R}^N,
\]

where \(\varphi(x)\) is a given continuous function.

The solution to this problem is defined similarly to the solution to problem \[1.1\] - \[1.3\] (see Definition \[1.1\]). In exactly the same way as Theorem \[1.2\] it is proved that if a \(\varphi \in L_2^2(\mathbb{R}^N)\) and \(\tau > N/2\), then the solution of the second forward problem has the form

\[
v(x, t) = \int_{\mathbb{R}^N} E_\rho(-A^\rho(\xi)t^\sigma)\hat{\varphi}(\xi)e^{ix\xi}d\xi,
\]

where the integral uniformly converges in \(x \in \mathbb{R}^N\) and for each \(t \in (0, T]\).

Now suppose, that parameters \(\rho\) and \(\sigma\) are unknown. To find these numbers one obviously needs two extra conditions. It should be noted, that the proposed method, for simultaneously finding both the order of fractional differentiation \(\rho\) and the power \(\sigma\) is applicable if there exists such \(\xi_0 \in \mathbb{R}^N\), so that \(A(\xi_0) = 1\) and \(\hat{\varphi}(\xi_0) \neq 0\). Let \(\xi_0\) be one of such a vector. We consider the following information as additional conditions:

\[
V(\xi_0, t_0, \rho, \sigma) = |\hat{v}(\xi_0, t_0)| = d_0, \quad t_0 \geq T_0(1, \rho_0),
\]

\[
V(\xi_1, t_1, \rho, \sigma) = |\hat{v}(\xi_1, t_1)| = d_1, \quad A(\xi_1) = \lambda_1(\neq 1) \geq \Lambda_1(\rho, \sigma_0), \quad t_1 > 0,
\]

where \(\xi_1\) is such that \(\hat{\varphi}(\xi_1) \neq 0\) and \(\Lambda_1\) is defined in \[4.2\].
We call problem \(1.8 - 1.10\) together with extra conditions \(1.12\) and \(1.13\) the second inverse problem.

Note that \(V(\xi, t_0, \rho, \sigma)\) is actually independent of \(\sigma\). Therefore, to solve the second inverse problem, we first find the unique \(\rho^*\) that satisfies the relation \((1.12)\). Then, assuming that \(\rho^*\) is already known, using relation \((1.13)\), we find the second unknown parameter \(\sigma^*\).

**Theorem 1.9.** Let \(\varphi \in L^2_\alpha(\mathbb{R}^N)\) and \(\tau > \frac{N}{2}\). Then there exists unique \(\rho^*\), satisfying \((1.12)\), if and only if \(d_0\) satisfies inequalities \((1.7)\) with \(\lambda_0 = 1\). For \(\sigma^*\) to exist, it is necessary and sufficient that \(d_1\) satisfies the inequalities

\[
E_{\rho^*}(-\lambda_1 t_0^*) < \frac{d_1}{|\hat{\varphi}(\xi_1)|} < E_{\rho^*}(-\lambda_1^a t_0^*).
\]

It should also be noted that the theory and applications of various inverse problems, on determining the coefficients of the equation, the right-hand side, and also on determining the initial or boundary functions for differential equations of integer order are discussed in Kabanikhin [25] (see also references therein).

2. **Forward problems**

In the present section we prove Theorems 1.2 and 1.3 and equation \((1.11)\).

The class of functions \(L^2_\alpha(\mathbb{R}^N)\) which for a given fixed number \(a > 0\) make the norm

\[
||f||^2_{L^2_\alpha(\mathbb{R}^N)} = \int_{\mathbb{R}^N} (1 + |\xi|^2)^{\alpha}\hat{f}(\xi)e^{i\xi\xi}||^2d\xi = \int_{\mathbb{R}^N} (1 + |\xi|^2)^\alpha|\hat{f}(\xi)|^2d\xi
\]

finite is termed the Sobolev class \(L^2_\alpha(\mathbb{R}^N)\). Since for \(\tau > 0\) and some constants \(c_1\) and \(c_2\) one has the inequality

\[
c_1(1 + |\xi|^2)^{\tau} \leq 1 + A^\tau(\xi) \leq c_2(1 + |\xi|^2)^{\tau},
\]

then \(D(\hat{A}) = L^2_\alpha(\mathbb{R}^N)\).

Let \(I\) be the identity operator in \(L^2_\alpha(\mathbb{R}^N)\). Operator \((\hat{A} + I)^\tau\) is defined in the same way as operator \(\hat{A}^\tau\).

**Proof of Theorem 1.2** is based on the following lemma (see M.A. Krasnoselski et al. [26], p. 453), which is a simple consequence of the Sobolev embedding theorem.

**Lemma 2.1.** Let \(\nu > 1 + \frac{N}{2m}\). Then for any \(|\alpha| \leq m\) operator \(D^\alpha(\hat{A} + I)^{-\nu}\) continuously maps from \(L^2_\alpha(\mathbb{R}^N)\) into \(C(\mathbb{R}^N)\) and moreover the following estimate holds true

\[
||D^\alpha(\hat{A} + I)^{-\nu}f||_{C(\mathbb{R}^N)} \leq C||f||_{L^2_\alpha(\mathbb{R}^N)}.
\]

**Proof.** For any \(a > N/2\) one has the Sobolev embedding theorem: \(L^2_\alpha(\mathbb{R}^N) \rightarrow C(\mathbb{R}^N)\), that is

\[
||D^\alpha(\hat{A} + I)^{-\nu}f||_{C(\mathbb{R}^N)} \leq C||D^\alpha(\hat{A} + I)^{-\nu}f||_{L^2_\alpha(\mathbb{R}^N)}.
\]

Therefore, it is sufficient to prove the inequality

\[
||D^\alpha(\hat{A} + I)^{-\nu}f||_{L^2_\alpha(\mathbb{R}^N)} \leq C||f||_{L^2_\alpha(\mathbb{R}^N)}.
\]

But this is a consequence of the estimate

\[
\int_{\mathbb{R}^N} |\hat{f}(\xi)|^2|\xi|^{2a}(1 + A(\xi))^{-2\nu}(1 + |\xi|^2)^\nu d\xi \leq C \int_{\mathbb{R}^N} |\hat{f}(\xi)|^2d\xi,
\]

that is valid for \(\frac{N}{2} < a \leq \nu m - |\alpha|\).

To prove the existence of the forward problem’s solution we remind the following estimate of the Mittag-Leffler function with a negative argument (see, for example, [8], p.29)

\[
|E_{\rho}(t)| \leq \frac{C}{1+t}, \quad t > 0.
\]

In accordance with Definition 1.4, we will first show that for function \(1.24\) one has \(A(D)u(x, t) \in C(\mathbb{R}^N \times (0, T])\) (that is one can validly apply the operators \(D^\alpha\), \(|\alpha| \leq m\), to the series in \((1.4)\) term-by-term).
Consider the truncated integral

\[ S_{\mu}(x, t) = \int_{A(\xi) \leq \mu} E_{\mu}(-A(\xi)t^\mu) dP_{\lambda} \varphi(x) = \int_{A(\xi) \leq \mu} E_{\mu}(-A(\xi)t^\mu) \hat{\varphi}(\xi)e^{ix\xi} d\xi. \]

Let \( \tau > \frac{N}{2} \) and \( \nu = 1 + \frac{m}{m} > 1 + \frac{N}{2m} \). Then

\[ S_{\mu}(x, t) = (\hat{A} + I)^{-\tau/m-1} \int_{0}^{\mu} (\lambda + 1)^{\tau/m+1} E_{\mu}(-\lambda t^\nu) dP_{\lambda} \varphi(x). \]

Therefore by virtue of Lemma 2.1 one has

\[ \left| D^\alpha S_{\mu}(x, t) \right|_{C(\mathbb{R}^N)}^2 = \left| D^\alpha (\hat{A} + I)^{-\tau/m-1} \int_{0}^{\mu} (\lambda + 1)^{\tau/m+1} E_{\mu}(-\lambda t^\nu) dP_{\lambda} \varphi(x) \right|_{C(\mathbb{R}^N)}^2 \leq C\int_{A(\xi) \leq \mu} \left| (A(\xi) + 1)^{\tau/m+1} E_{\mu}(-A(\xi)t^\nu) \hat{\varphi}(\xi) \right|^2 d\xi. \]

Applying the inequality (2.3) gives \( \left| (A(\xi) + 1)E_{\mu}(-A(\xi)t^\nu) \right| \leq Ct^{-\nu} \). Therefore,

\[ \left| D^\alpha S_{\mu}(x, t) \right|_{C(\mathbb{R}^N)}^2 \leq C\int_{A(\xi) \leq \mu} \left| (A(\xi) + 1)^{\tau/m+1} \hat{\varphi}(\xi) \right|^2 d\xi \leq Ct^{-2\nu} \int_{A(\xi) \leq \mu} \left| (A(\xi) + 1)^{\tau/m+1} \hat{\varphi}(\xi) \right|^2 d\xi \leq C_{\tau} \left| \hat{\varphi} \right|_{L_2^2(\mathbb{R}^N)}^2. \]

This implies the uniform in \( x \in \mathbb{R}^N \) convergence of the differentiated sum (2.4) in the variables \( x_j \) for each \( t \in (0, T] \).

Further, from equation (1.1) one has \( D^\alpha_x S_{\mu}(x, t) = -A(D)S_{\mu}(x, t) \). Therefore, proceeding as above, we arrive at \( D^\alpha u(x, t) \in C(\mathbb{R}^N \times (0, T]) \).

It is not difficult to verify that equation (1.1) and the initial condition (1.3) are satisfied (see, for example, [6], page 173 and (22)).

Let us show that the inclusion \( \varphi \in L_2^2(\mathbb{R}^N), \tau > N/2, \) implies \( \hat{\varphi} \in L_1(\mathbb{R}^N) \). Indeed,

\[ \int_{\mathbb{R}^N} |\varphi(\xi)|^2 d\xi = \int_{\mathbb{R}^N} |\varphi(\xi)|(1 + |\xi|^2)^r/2(1 + |\xi|^2)^{-r/2} d\xi \leq C_r \int_{\mathbb{R}^N} |\varphi(\xi)|^2 d\xi \]

Therefore, by virtue of inequality (2.3), one has \( E_{\mu}(-A(\xi)t^\nu) \hat{\varphi}(\xi) \in L_1(\mathbb{R}^N) \). Similarly, inequalities (2.4) and (2.5) imply \( \xi^\alpha E_{\mu}(-A(\xi)t^\nu) \hat{\varphi}(\xi) \leq C |A(\xi)|E_{\mu}(-A(\xi)t^\nu) \hat{\varphi}(\xi) \) for all \( |\alpha| \leq m \). Hence, \( D^\alpha u(x, t) \), as a function of \( x \), is the Fourier transform of a \( L_1^\tau \) function. Obviously, this implies both (1.2) and condition (2) of Theorem 1.3.

Thus Theorem 1.2 and condition (2) of Theorem 1.3 are proved.

Consider the other three conditions of Theorem 1.3. The inclusion \( D^\alpha u(x, t) \in L_2(\mathbb{R}^N), |\alpha| \leq m \), for all \( t \in (0, T] \), is a consequence of condition \( \varphi \in L_2(\mathbb{R}^N) \). Indeed, using inequalities 2.1 and (2.2) we arrive at

\[ \left| D^\alpha u(x, t) S_{\mu}(x, t) \right|_{L_2(\mathbb{R}^N)} = \int_{A(\xi) \leq \mu} \left| \xi^\alpha E_{\mu}(-A(\xi)t^\nu) \hat{\varphi}(\xi) \right|^2 d\xi \leq C \int_{A(\xi) \leq \mu} \left| A(\xi) E_{\mu}(-A(\xi)t^\nu) \hat{\varphi}(\xi) \right|^2 d\xi \leq C t^{-2\nu} \int_{\mathbb{R}^N} \left| D_{\xi}^\alpha \varphi \right|^2 d\xi. \]

The property of function \( \varphi : \hat{\varphi} \in L_1(\mathbb{R}^N) \), established above, implies condition (4):

\[ |\hat{\varphi}(\xi, t)| = |(2\pi)^{-N} E_{\mu}(-A(\xi)t^\nu) \hat{\varphi}(\xi)| \leq C |\hat{\varphi}(\xi)| \in L_1(\mathbb{R}^N). \]

As regards condition (1) of Theorem 1.3, it is a direct consequence of the Sobolev embedding theorem and the condition \( \varphi \in L_2^2(\mathbb{R}^N), \tau > \frac{N}{2} \), of Theorem 1.2.
Proof of Theorem 1.3. Let conditions (1)-(4) of Theorem 1.3 are satisfied. Observe, as it was shown above, Theorem 1.2 guarantee the fulfillment of these conditions. Suppose that problem (1.1)-(1.3) has two solutions $u_1(x,t)$ and $u_2(x,t)$. Our aim is to prove that $u(x,t) = u_1(x,t) - u_2(x,t) \equiv 0$. Since the problem is linear, then we have the following homogenous problem for $u(x,t)$:

\begin{equation}
D_t^\alpha u(x,t) + A(D)u(x,t) = 0, \quad x \in \mathbb{R}^N, \quad 0 < t \leq T;
\end{equation}

\begin{equation}
\lim_{|x| \to \infty} D^\alpha u(x,t) = 0, \quad |\alpha| \leq l - 1, \quad 0 < t \leq T;
\end{equation}

\begin{equation}
u(x,0) = 0, \quad x \in \mathbb{R}^N.
\end{equation}

Let $u(x,t)$ be a solution of problem (2.8). Since $u(x,t) \in L_2(\mathbb{R}^N)$, $t \in (0,T]$ (see condition (3) of Theorem 1.3), we may define the Fourier transform $\hat{u}(\xi,t)$ and according to condition (4) one has $\hat{u}(\xi,t) \in L_1(\mathbb{R}^N)$. Therefore, by virtue of Fubini’s theorem, the following function of $t$ exists for almost all $\lambda$:

\begin{equation}
w_\lambda(t) = \int_{\mathbb{R}^N} \hat{u}(\xi,t)d\sigma_\lambda(\xi),
\end{equation}

where $d\sigma_\lambda(\xi)$ is the corresponding surface element.

Since $u(x,t)$ is a solution of equation (2.8), then (note, $A(D)u(x,t) \in L_2(\mathbb{R}^N)$)

\begin{equation}D_t^\alpha w_\lambda(t) = -(2\pi)^{-N} \int_{\mathbb{R}^N} A(D)u(x,t)e^{-ix\xi}dx \sigma_\lambda(\xi).
\end{equation}

The inner integral exists as the Fourier transform of $L_2$-function. In this integral, we integrate by parts. We will take into account the following: $A(D)$ is a homogeneous symmetric and even order differential expression; conditions (2) of Theorem 1.3 and (2.6). Then

\begin{equation}D_t^\alpha w_\lambda(t) = -(2\pi)^{-N} \int_{\mathbb{R}^N} A(-i\xi)u(x,t)e^{-ix\xi}dx \sigma_\lambda(\xi) = -\lambda w_\lambda(t).
\end{equation}

Therefore, we have the following Cauchy problem for $w_\lambda(t)$:

\begin{equation}D_t^\alpha w_\lambda(t) + \lambda w_\lambda(t) = 0, \quad t > 0; \quad w_\lambda(0) = 0.
\end{equation}

This problem has the unique solution; hence, the function defined by (2.8), is identically zero (see, for example, [6], p. 173 and [27]): $w_\lambda(t) \equiv 0$ for almost all $\lambda > 0$. Integrating the equation (2.8) with respect to $\lambda$ over the domain $(0, +\infty)$ we obtain

\begin{equation}\int_{\mathbb{R}^N} \hat{u}(\xi,t)d\xi = 0, \quad t > 0.
\end{equation}

Therefore, $\hat{u}(\xi,t) = 0$ for almost all $\xi$, or $u(x,t) = 0$ for almost all $x$ and since $u(x,t)$ continuous on $x$, then $u(x,t) = 0$ for all $x$ and $t$. Thus Theorem 1.3 is proved.

Formula (1.11) for the solution of the second forward problem is established in exactly the same way with formula (1.7).

3. First inverse problem

Lemma 3.1. Given $\rho_0$ from the interval $0 < \rho_0 < 1$, there exists a number $T_0 = T_0(\lambda_0, \rho_0)$, such that for all $t_0 \geq T_0$ and $\lambda \geq \lambda_0$ function $e_\lambda(\rho) = E^{\rho}(\rho^\nu_0)$ is positive and monotonically decreasing with respect to $\rho \in [\rho_0,1)$ and $e_\lambda(1) < e_\lambda(\rho) \leq e_\lambda(\rho_0)$.

Proof. Let us denote by $\delta(1;\beta)$ a contour oriented by non-decreasing $\arg \zeta$ consisting of the following parts: the ray $\arg \zeta = -\beta$ with $|\zeta| \geq 1$, the arc $-\beta \leq \arg \zeta \leq \beta, |\zeta| = 1$, and the ray $\arg \zeta = \beta, |\zeta| \geq 1$. If $0 < \beta < \pi$, then the contour $\delta(1;\beta)$ divides the complex $\zeta$-plane into two unbounded parts, namely $G^{(1)}(1;\beta)$ to the left of $\delta(1;\beta)$ by orientation, and $G^{(2)}(1;\beta)$ to the right of it. The contour $\delta(1;\beta)$ is called the Hankel path.
Let \( \beta = \frac{2\pi}{\lambda \rho}, \rho \in [\rho_0, 1) \). Then by the definition of this contour \( \delta(1; \beta) \), we arrive at (note, \( -\Lambda_0^\rho \in C^{(1)}(1; \beta) \), see [5], p. 27)

\[
E_\rho(-\lambda t_0^\rho) = \lambda t_0^\rho \Gamma(1-\rho) - \int_{\delta(1, \beta)} \frac{e^{\zeta^{1/\rho}}}{\zeta + \lambda t_0^\rho} d\zeta = f_1(\rho) + f_2(\rho).
\]

Let \( \Psi(\rho) \) be the logarithmic derivative of the gamma function \( \Gamma(\rho) \) (for the definition and properties of \( \Psi \) see [28]). Then \( \Gamma'(\rho) = \Gamma(\rho)\Psi(\rho) \), and therefore,

\[
f_1'(\rho) = -\frac{\ln t_0 - \Psi(1-\rho)}{\lambda t_0^\rho \Gamma(1-\rho)}.
\]

Since

\[
\frac{1}{\Gamma(1-\rho)} = \frac{1}{\Gamma(2-\rho)} \cdot \Psi(1-\rho) = \Psi(2-\rho) - \frac{1}{1-\rho},
\]

the function \( f_1'(\rho) \) can be represented as follows

\[
f_1'(\rho) = -\frac{1}{\lambda t_0^\rho} \frac{(1-\rho)\ln t_0 - \Psi(2-\rho) + 1}{\Gamma(2-\rho)}.
\]

If \( \gamma \approx 0, 57722 \) is the Euler-Mascheroni constant, then \( -\gamma \leq \Psi(2-\rho) < 1 - \gamma \). By virtue of this estimate we may write

\[
- f_1'(\rho) \geq \frac{(1-\rho)\ln t_0 - (1-\gamma)}{\Gamma(2-\rho)\lambda t_0^\rho} \geq \frac{1}{\lambda t_0^\rho},
\]

provided \( \ln t_0 > 1 - \gamma \) or \( t_0 \geq 2 \).

On the other hand one has

\[
f_2'(\rho) = \frac{1}{2\pi i \lambda t_0^\rho} \int_{\delta(1, \beta)} \frac{e^{\zeta^{1/\rho}}}{\zeta + \lambda t_0^\rho} \left[ -\frac{1}{\rho^\rho} |\zeta|^{1/\rho} \ln |\zeta| + i \beta \right] - \ln t_0 - \frac{\lambda t_0^\rho}{\rho^\rho} \ln t_0 d\zeta.
\]

Note, since \( \beta = \frac{2\pi}{\lambda \rho} \) and \( \rho_0 < \rho < 1 \), then for a negative number \( z < 0 \) the following inequality holds

\[
\min_{\zeta \in \delta(1, \beta)} |\zeta - z| \geq |z|.
\]

Therefore,

\[
|f_2'(\rho)| \leq \frac{1}{\rho^\rho} \int_{\delta(1, \beta)} |e^{\zeta^{1/\rho}}||\zeta| \left[ \frac{1}{\rho^\rho} |\zeta|^{1/\rho} \ln |\zeta| + \frac{\beta}{I \cdot \rho^\rho} \right] |\zeta| d\zeta = \frac{1}{I \cdot \rho^\rho} \int_{\delta(1, \beta)} |e^{\zeta^{1/\rho}}||\zeta|^{1/\rho+1} \ln |\zeta| |d\zeta| + \frac{\beta}{I \cdot \rho^\rho} \int_{\delta(1, \beta)} |e^{\zeta^{1/\rho}}||\zeta|^{1/\rho+1} |d\zeta| + \frac{2\ln t_0}{I} \int_{\delta(1, \beta)} |e^{\zeta^{1/\rho}}||\zeta| |d\zeta|,
\]

where \( I = 2\pi \rho (\lambda t_0^\rho)^2 \). Let us denote the last three integrals by \( J_j, j = 1, 2, 3 \), correspondingly.

**Lemma 3.2.** Let \( 0 < \rho \leq 1 \) and \( m \in \mathbb{N} \). Then

\[
I(\rho) = \frac{1}{\rho} \int_1^\infty e^{-\frac{2\pi}{\lambda \rho} s^{m+1}} ds \leq C_m.
\]

**Proof.** Set \( r = s^{\frac{1}{\rho}} \). Then

\[
s = r^\rho, \quad ds = \rho r^{\rho-1} dr.
\]

Therefore,

\[
I(\rho) = \int_1^\infty e^{-\frac{2\pi}{\lambda \rho} r^{m+1+2\rho}} dr \leq \int_1^\infty e^{-\frac{2\pi}{\lambda \rho} r^{m+1}} dr = C_m.
\]

\( \square \)
It is not hard to verify, that
\[ C_2 = \frac{2^4 \cdot 16}{e} \approx 94.2, \quad C_1 = \frac{2^3 \cdot 5}{e} \approx 14.72, \quad C_0 = \frac{2^2 \cdot 2}{e} \approx 3. 
\]

Consider the integral \( J_1 \). Due to the presence of \( \ln |\zeta| \), the integrand \( J_1 \) is equal to 0 for \( |\zeta| = 1 \). Moreover, on the rays \( \arg \zeta = \pm \beta, \beta = \frac{\ln (\lambda t)}{\rho} \), one has
\[ |e^{\frac{\beta}{\rho}}| = \exp \left( \cos \frac{\beta}{\rho} |\zeta| \right) = e^{-\frac{1}{2}(\ln |\zeta|)^2}. \]

Hence (note \( \ln |\zeta|^\frac{1}{\rho} < |\zeta|^\frac{1}{\rho} \)) by virtue of Lemma 3.2
\[ J_1 = \frac{1}{I \cdot \rho} \int_{\delta(1,\beta)} \left| e^{\frac{1}{\rho}} \right| |\zeta|^{1/\rho + 1} \ln |\zeta|^\frac{1}{\rho} \, d\zeta = \frac{2}{I \cdot \rho} \int_1^\infty \left[ e^{-\frac{1}{2} \beta \rho} s^{1/\rho + 1} + \beta \right] \ln |\zeta|^\frac{1}{\rho} \, d\zeta \leq \frac{2}{I} \int_1^\infty \left[ e^{-\frac{1}{2} \beta \rho} s^{1/\rho + 1} + \beta \right] \, ds \leq \frac{C_2}{\pi \rho (\lambda t_0)^2}.
\]

The integrands in \( J_2 \) and \( J_3 \) do not vanish on the sphere \( \{|\zeta| = 1\} \) and the measure of the corresponding arc \( -\beta \leq \arg \zeta \leq \beta, |\zeta| = 1 \), is equal to \( 2\beta \). Therefore, using the same technique as above, we obtain
\[ J_2 = \frac{\beta}{I \cdot \rho^2} \int_{\delta(1,\beta)} \left| e^{\frac{1}{\rho}} \right| |\zeta|^{1/\rho + 1} \, d\zeta = \frac{2 \beta}{I \cdot \rho^2} \int_1^\infty \left[ e^{-\frac{1}{2} \beta \rho} s^{1/\rho + 1} + \beta \right] \, ds \leq \frac{2 \beta}{2 \pi \rho^3 (\lambda t_0)^2} \cdot \frac{3(1 + \frac{\beta}{\rho})}{4 \rho (\lambda t_0)^2}.
\]

Similarly,
\[ J_3 = \frac{2 \ln t_0}{I} \int_{\delta(1,\beta)} \left| e^{\frac{1}{\rho}} \right| |\zeta| \, d\zeta = \frac{4 \ln t_0}{I} \int_1^\infty \left[ e^{-\frac{1}{2} \beta \rho} s + \beta \right] \, ds \leq \frac{2 \ln t_0 (C_0 + \frac{\beta}{\rho})}{\pi (\lambda t_0)^2}.
\]

Thus we have
\[ |f_2'(\rho)| \leq \frac{B_1/\rho + B_2 \ln t_0}{(\lambda t_0)^2}, \]
where \( B_1 = 43 \) and \( B_2 = 4.6 \).

Therefore, taking into account estimate (3.2), we have
\[ \frac{d}{d\rho} e^{\lambda}(\rho) < -\frac{1}{\lambda t_0} + \frac{B_1/\rho + B_2 \ln t_0}{(\lambda t_0)^2}. \]

In other words, this derivative is negative if
\[ \frac{d}{d\rho} e^{\lambda}(\rho) < \frac{B_1/\rho + B_2 \ln t_0}{\lambda t_0}, \]
for all \( \rho \in [\rho_0, 1] \) or, which is the same,
\[ \rho_0 > \frac{B_1/\rho_0 + B_2 \ln t_0}{\lambda}. \]

Next, consider two cases: \( B_1/\rho_0 > B_2 \ln t_0 \) and \( B_1/\rho_0 \leq B_2 \ln t_0 \). Recall that to satisfy inequality (3.2) we assumed that \( t_0 \geq 2 \).

**Case 1.** Let \( B_1/\rho_0 > B_2 \ln t_0 \). Then
\[ \ln t_0 > \frac{B_1}{B_2}, \quad \text{or} \quad t_0 > \frac{B_1}{B_2} e^{\frac{B_1}{B_2}} \]
(the latter inequality is satisfied, say if \( t_0 \leq e^{B_1/B_2} \)). Therefore, inequality (3.4) is satisfied if
\[ t_0 > \frac{2B_1}{\rho_0 \lambda}. \]

Thus, from the last two inequalities it follows that if
\[ \rho_0 \cdot \lambda_0 > 2B_1 \cdot e^{\frac{B_1}{B_2}} \quad \text{and} \quad 2 \leq t_0 \leq e^{B_1/B_2} \]
(3.5)
then derivative \((3.3)\) is negative for all \(\lambda \geq \lambda_0\) and \(\rho \in [\rho_0, 1]\). Note \(2B_1 \cdot e^{-\frac{\rho_1}{\rho^2}} < 0.0075\) (see Remark 1.7).

If \(\rho_0\) and \(\lambda_1\) are such small numbers, that the first inequality of \((3.3)\) does not hold true, then consider Case 2. In this case, \(t_0\) should be chosen large enough.

**Case 2.** Let \(B_1/\rho_0 \leq B_2 \ln t_0\) or, which is the same, \(t_0^{\rho_0} \geq e^{B_2/\rho_0}\). From \((3.4)\) one has
\[
t_0^{\rho_0} \geq \frac{2B_2 \ln t_0}{\lambda}.
\]
Thus, in Case 2 in order for the derivative \((3.3)\) to be negative for all \(\lambda \geq \lambda_0\) and \(\rho \in [\rho_0, 1]\), it is sufficient that the following inequality takes place \(t_0 \geq T_0\), where (see Remark 1.7)
\[
T_0 = e^k, \quad k \geq \frac{1}{\rho_0} \max \left\{ \frac{B_1}{B_2} \ln \frac{2B_2k}{\lambda_0} \right\}.
\]
Finally, by virtue of inequality \(e_\lambda(1) = e^{-\lambda_x} > 0\), one has \(e_\lambda(\rho) > 0\).

Since
\[
U(t, \rho) = |\hat{u}(\xi_0, t)| = E_\rho(-A(\xi_0)t^\rho)|\hat{\phi}(\xi_0)| = E_\rho(-\lambda_0 t^\rho)|\hat{\phi}(\xi_0)|,
\]
Lemma 1.8 follows immediately from Lemma 3.1. Theorem 1.8 is an easy consequence of these two lemmas.

In conclusion, we make the following remark. If the elliptic polynomial \(A(\xi)\) is nonhomogeneous, that is \(A(\xi) = \sum a_\lambda \xi^\lambda\) and moreover, \(A(\xi) \geq \lambda_0 > 0\), then from Lemma 3.1 it follows:

If \(t_0 \geq T_0\) and \(T_0\) is as above, then \(E_\rho(-A(\xi)t^\rho)\), as a function of \(\rho\), is positive and decreases monotonically in \(\rho \in [\rho_0, 1]\) for any \(\xi \in \mathbb{R}^N\).

Therefore, in this case you can also consider various options for the function \(U(t, \rho)\). Examples
\[
U(t, \rho) = ||Au(x, t)||^2, \quad U(t, \rho) = ||u(x, t)||^2, \quad U(t, \rho) = (u, \varphi).
\]

4. Second inverse problem

To prove Theorem 1.9 we first find the unknown parameter \(\rho^*\). Suppose, as required by Theorem 1.3 that \(d_0\) satisfies condition (1.7) with \(\lambda_0 = A(\xi_0) = 1\). Then, as it follows from Lemma 3.1 for all \(t_0 \geq T_0(1, \rho_0)\) the equation
\[
V(\xi_0, t_0, \rho, \sigma) = |\hat{u}(\xi_0, t_0)| = E_\rho(-t^\rho)|\hat{\phi}(\xi_0)| = d_0
\]
has the unique solution \(\rho^* \in (\rho_0, 1)\).

Now let us define \(\sigma^* \in [\sigma_0, 1]\), which corresponds to the already found \(\rho^*\) and satisfies condition \((1.3)\). Let \(\beta = \frac{\lambda_0}{\ln \rho^*}\). Then formula \((5.1)\) will have the form
\[
E_{\rho^*}(\lambda^\rho t_0^\rho) = \frac{1}{\lambda^\rho t_0^\rho \Gamma(1 - \rho^*)} - \frac{1}{2\pi i \rho^* \lambda^\rho t_0^\rho} \int_{\delta(1, \beta)} e^{\zeta^\rho \rho^*} \zeta d\zeta = g_2(\sigma) + g_2(\sigma).
\]
One has
\[
g_2'\left(\sigma\right) = -\frac{\ln \lambda}{\lambda^\rho t_0^\rho \Gamma(1 - \rho^*)}
\]
and
\[
g_2'\left(\sigma\right) = \frac{(1 + t_0^\rho)\ln \lambda}{2\pi i \rho^* \lambda^\rho t_0^\rho} \int_{\delta(1, \beta)} e^{\zeta^\rho \rho^*} \zeta d\zeta.
\]
It is not hard to verify, that \(g_2'(\lambda)\) has the estimate (is proved in a completely similar way to estimate of \(J_3\))
\[
|g_2'(\sigma)| \leq \frac{(1 + t_0^\rho)\ln \lambda}{\pi(\lambda^\rho t_0^\rho)^2} \left( C_0 + \frac{4}{3}\pi \right) < \frac{2.3(1 + t_0^\rho)\ln \lambda}{(\lambda^\rho t_0^\rho)^2}.
\]
Therefore, for all \(t_0 > 1\) we have
\[
\frac{d}{d\sigma} e^{\lambda^\rho \rho^*} < -\frac{\ln \lambda}{\lambda^\rho t_0^\rho \Gamma(1 - \rho^*)} + \frac{5\ln \lambda}{\lambda^2 \rho^* t_0^\rho}.
\]
Hence this derivative is negative if
\[ \lambda'' \geq \lambda'' \sigma \geq \lambda'' \sigma_0 \geq 5 \Gamma(1 - \rho^*). \]

Thus, if \( \lambda_1 \geq \Lambda_1 \), where (see Remark 1.7)
\[ (4.2) \Lambda_1 = e^{n}, \quad n \geq \frac{\ln(5 \Gamma(1 - \rho^*))}{\sigma_0}, \]
then \( e_{\lambda_1}(\rho^*) \), as a function of \( \sigma \in [\sigma_0, 1) \), strictly decreases for all \( t_0 > 1 \).

Moreover, for all \( \lambda_1 \geq \Lambda_1 \) and \( \sigma \in [\sigma_0, 1) \) the following estimate is fulfilled
\[ e_{\lambda_1}(\rho^*) < e_{\lambda_1}(\rho^*) < e_1(\rho^*). \]

The last estimate shows that if \( d_1 \) satisfies condition (1.14), then, assuming \( \rho^* \) has already been found, we can easily determine the parameter \( \sigma^* \) from equality (1.13), that is, from
\[ e_{\lambda_1}(\rho^*)|\tilde{\varphi}(\xi_1)| = d_1. \]

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INSTITUTE OF MATHEMATICS, UZBEKISTAN ACADEMY OF SCIENCE
Current address: Institute of Mathematics, Uzbekistan Academy of Science, Tashkent, 81 Mirzo Ulugbek str. 100170
E-mail address: ashurovr@gmail.com