Open-independent, open-locating-dominating sets

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\section*{Abstract}
A distinguishing set for a graph $G = (V, E)$ is a dominating set $D$, each vertex $v \in D$ being the location of some form of a locating device, from which one can detect and precisely identify any given ”intruder” vertex in $V(G)$. As with many applications of dominating sets, the set $D$ might be required to have a certain property for $\langle D \rangle$, the subgraph induced by $D$ (such as independence, paired, or connected). Recently the study of independent locating-dominating sets and independent identifying codes was initiated. Here we introduce the property of open-independence for open-locating-dominating sets.

\textbf{Keywords:} distinguishing sets, open-independent sets, open-locating-dominating sets, open-independent, open-locating-dominating sets

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\section{1. Introduction}
For a graph $G = (V, E)$ that represents a facility, an “intruder” in the system might be a thief, saboteur or fire. If $G$ represents a multiprocessor network with each vertex representing one processor, an “intruder” might be a malfunctioning processor. We assume that certain vertices will
be the locations of detectors, each detector having some capability to identify the location of an intruder vertex.

For $u, v \in D$, let $d(u, v)$ denote the distance in $G$ between $u$ and $v$. Some detectors, like sonar devices, can be assumed to determine the distance to the intruder vertex $x$ anywhere in the system. Much work has been done on locating sets as introduced in Slater [36] (and also called metric bases as independently introduced in Harary and Melter [11]). An (ordered) set $X = \{x_1, x_2, \ldots, x_k\} \subseteq V(G)$ is a locating set if for every $w \in V(G)$ the ordered $k$-tuple $(d(x_1, w), d(x_2, w), \ldots, d(x_k, w))$ uniquely determines $w$. We say that a vertex $x$ resolves vertices $u$ and $v$ if $d(x, u) \neq d(x, v)$. Then $X$ is locating if for every two vertices $u$ and $v$ at least one $x_i \in X$ resolves $u$ and $v$. For the recently introduced centroidal bases described in Foucaud, Klasing and Slater [9] the set of detectors in $X$ provide just an ordering of the relative distances to an intruder vertex, not the exact distances.

Some detectors (heat sensors, motion detectors, etc.) have a limited range. The open neighborhood of vertex $v$ is $N(v) = \{w \in V(G) : d(w, v) = 1\}$, and the closed neighborhood $N[v] = N(v) \cup \{v\} = \{w \in V(G) : d(w, v) \in \{0, 1\}\}$. Vertex set $D \subseteq V(G)$ is dominating if $\cup_{x \in D} N[x] = V(G)$. For $S \subseteq V(G)$ the distance $d(w, S) = \min \{d(x, w) : x \in S\}$, so $D$ is dominating if for every $w \in V(G)$ we have $d(w, D) \in \{0, 1\}$. Vertex set $D$ is an open dominating set (also called a total dominating set) if $\cup_{x \in D} N[x] = V(G)$, that is for every vertex $w$ (including $w \in D$) there is a vertex $x \in D$ with $d(w, x) = 1$.

For the case in which a detector at $v$ can determine if the intruder is at $v$ or if the intruder is in $N(v)$ (but which element in $N(v)$ cannot be determined), as introduced in Slater [37, 38, 39], a locating-dominating set $L \subseteq V(G)$ is a dominating set for which, given any two vertices $u$ and $v$ in $V(G) - L$, one has $N(u) \cap L \neq N(v) \cap L$, that is, for any two distinct vertices $u$ and $v$ (including ones in $L$) there is a vertex $x \in L$ with $d(x, u) \in \{0, 1\}$ and $d(x, u) \neq d(x, v)$ or $d(x, v) \in \{0, 1\}$ and $d(x, u) \neq d(x, v)$. Every graph $G$ has a locating-dominating set, namely $V(G)$, and the locating-dominating number $LD(G)$ is the minimum cardinality of such a set. See, for example, [3, 8, 17].

As introduced by Karpovsky, Charkrabarty and Levitin [22], an identifying code $C \subseteq V(G)$ is a dominating set for which given any two vertices $u$ and $v$ in $V(G)$ one has $N[u] \cap C \neq N[v] \cap C$, that is, there is a vertex $x \in C$ with $d(x, u) \leq 1$ and $d(x, v) \geq 2$ or $d(x, v) \leq 1$ and $d(x, u) \geq 2$. See, for example, [2, 4, 25]. Graph $G$ has an identifying code when for every pair of vertices $u$ and $v$ we have $N[u] \neq N[v]$, and the identifying code number $IC(G)$ is the minimum cardinality of such a set.

When a detection device at vertex $v$ can determine if an intruder is in $N(v)$ but will not/cannot report if the intruder is at $v$ itself, then we are interested in open-locating-dominating sets as introduced for the $k$-cubes $Q_k$ by Honkala, Laihonen and Ranto [21] and for all graphs by Seo and Slater [26, 27]. An open dominating set $S \subseteq V(G)$ is an open-locating-dominating set if for all $u$ and $v$ in $V(G)$ one has $N(u) \cap S \neq N(v) \cap S$, that is, there is a vertex $x \in S$ with $d(x, u) = 1 \neq d(x, v)$ or $d(x, v) = 1 \neq d(x, u)$. A graph $G$ has an open-locating-dominating set when no two vertices have the same open neighborhood, and $OLD(G)$ is the minimum cardinality of such a set. See, for example, [5, 16, 21, 28, 29, 30, 31, 32, 33]. Lobstein [24] maintains a bibliography, currently with more than 300 entries, for work on these topics.

Dominating sets $D$ have many applications (see Haynes, Hedetniemi and Slater [12, 13]), and in many cases the subgraph generated by $D$, denoted $\langle D \rangle$, is required to have an additional property
such as independence, paired, or connected. Recently, independent locating-dominating sets and independent identifying codes have been introduced in Slater [42]. Not all graphs have independent locating-dominating sets (respectively, independent identifying codes), and there is no forbidden subgraph characterization of such graphs. In fact, we have the following.

**Theorem A** (Slater [42]) Simply deciding, for a given input graph $G$, if $G$ has an independent locating-dominating set is NP-complete.

**Theorem B** (Slater [42]) Simply deciding, for a given input graph $G$, if $G$ has an independent identifying code is NP-complete.

Note that, by definition, an open dominating set $S$ can not be independent, each $v \in S$ must be open dominated by some $x \in N(v)$. In this paper we consider “open-independence” and introduce open-independent, open-locating-dominating sets.

2. **Open-independent sets; open-independent-dominating sets; open-independent, open dominating sets**

Assuming every vertex is the possible location of an intruder and that a detector at vertex $v$ can not detect an intruder at $w \in V(G)$ if $d(v, w) \geq 2$, in order for every intruder to be detectable we require a dominating set for the detectors. Vertex set $D \subseteq V(G)$ is dominating if every vertex $w$ not in $D$ is adjacent to a vertex $v \in D$, equivalently, (a) $\bigcup_{x \in D} N[x] = V(G)$ or (b) $V(G) - D$ is enclaveless (Note that a set $E \subseteq V(G)$ is defined to be enclaveless if every vertex in $E$ is adjacent to at least one vertex $V(G) - E$). Also, $S \subseteq V(G)$ is independent if no two vertices in $S$ are adjacent. Now, $R \subseteq V(G)$ is dominating when condition (1) below holds, and $R$ is independent when (2) below holds.

1. for every $v \in V(G)$, $|N[v] \cap R| \geq 1$.
2. for every $v \in R$, $|N[v] \cap R| \leq 1$.

Obviously every $v \in R$ satisfies $|N[v] \cap R| \geq 1$, so condition (2) could be replaced with $v \in R$ implies $|N[v] \cap R| = 1$. We use $\leq$ for what follows in (4).

For open domination, one assumes that a vertex $v$ does not dominate itself. An intruder (thief, saboteur, fire) at $v$ might prevent its own detection; a malfunctioning processor might not detect its own miscalculations. Vertex set $R \subseteq V(G)$ is open-dominating if $\cup_{v \in R} N(v) = V(G)$ or, equivalently, if condition (3) holds.

3. for every $v \in V(G)$, $|N(v) \cap R| \geq 1$.

Now we define $R \subseteq V(G)$ to be open-independent if (4) holds. That is, $R$ is independent if each vertex $v \in R$ is dominated by $R$ at most (equivalently, exactly) once, and $R$ is open-independent if each vertex $v \in R$ is open-dominated by $R$ at most once.

The open-independence number for a graph $G$ denoted by $OIND(G)$ is the maximum cardinality of an open-independent set for $G$. Note that $OIND(G) \geq \beta(G)$, where $\beta(G)$ denotes the maximum cardinality of an independent set for $G$.

4. for every $v \in R$, $|N(v) \cap R| \leq 1$.  

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The domination number \( \gamma(G) \) is the minimum cardinality of a dominating set, a dominating set of cardinality \( \gamma(G) \) being called a \( \gamma \)-set whereas any dominating set is called a \( \gamma \)-set. Similar terminology is used for other parameters. The independent domination number (which could be denoted \( \gamma_{IND}(G) \)) is traditionally denoted by \( i(G) \) and is the minimum cardinality of a dominating set \( D \) for which every component of \( \langle D \rangle \) is a singleton. We let \( \gamma_{OIND}(G) \) denote the minimum cardinality of an open-independent dominating set \( D \), a dominating set \( D \) for which each component of \( \langle D \rangle \) has cardinality at most two, \( \langle D \rangle = jK_1 \cup kK_2 \). Clearly \( \gamma(G) \leq \gamma_{OIND}(G) \leq i(G) \).

The open (or total) domination number, the minimum cardinality of an open dominating set is denoted \( \gamma_t \) or \( \gamma_{OP} \). We let \( \gamma_{OIND}^{OP} \) denote the open-independent, open domination number, the minimum cardinality of an open dominating set \( D \) for which every component of \( \langle D \rangle \) is a \( K_2 \), when such a set exists. If so, then \( \gamma_{OP}(G) \leq \gamma_{OIND}^{OP}(G) \). Note, for example, that the 5-cycle \( C_5 \) does not have an open-independent, open dominating set.

For the graph \( G_{11} \) in Figure 1(a) the set \( \{u, v, w\} \) is the minimum dominating set which is open-independent and \( \gamma(G_{11}) = \gamma_{OIND}(G_{11}) = 3; i(G) = 4 = |\{u, w, x, y\}|; \) and \( \gamma_{OP}(G_{11}) = 4 = |\{u, v, w, z\}| = \gamma_{OIND}^{OP}(G_{11}) \). In Figure 1(b) the graph \( G_{12} \) has the minimum dominating set \( \{f, g, h\} \) and a minimum open independent dominating set \( \{g, h, i, j, k\} \), so \( \gamma(G_{12}) = 3 < 5 = \gamma_{OIND}(G_{12}) \), and the graph \( G_7 \) has the minimum open dominating set \( \{a, b, c\} \) and the minimum open independent, open dominating set \( \{a, b, d, e\} \), with \( \gamma_{op}(G_7) = 3 < 4 = \gamma_{OIND}^{OP}(G_7) \).

Open-independent, open dominating sets have been considered in another context by Studer, Haynes, and Lawson [43]. As introduced in Haynes and Slater [14, 15], a paired dominating set \( D \) is a dominating set for which \( \langle D \rangle \) has a perfect matching. Studer, et al. [43] define an open-independent, open dominating set as an induced-paired dominating set.

As noted, in this paper we are interested in distinguishing sets and will consider open-independent, open-locating-dominating sets.

3. Open-independent, open-locating-dominating sets

For an open-locating-dominating set \( S \) each \( v \in V(G) \) has a distinct set of detectors, \( N(v) \cap S \). A graph \( G \) has an open-locating-dominating set (OLD-set) if and only if no two vertices \( u \) and \( v \)
have the same open neighborhood, that is $N(u) \neq N(v)$. Clearly, $OLD(G) \leq OLD_{OIND}(G)$ in this case. For an open-independent, OLD-set $S$, the subgraph $<S>$ must have each component of order two. We let $OLD_{OIND}(G)$ be the minimum cardinality of an open-independent $OLD(G)$-set when such a set exists. For the tree $T_8$ in Figure 2, $OLD(T_8) = 5$ and $OLD_{OIND}(T_8) = 6$.

![Figure 2](image.png)

Figure 2. $\gamma^{OP}(T_8) = OLD(T_8) = 5$ and $OLD_{OIND}(T_8) = 6$.

For the tree $T_9$ in Figure 3, there is an open-independent, open-dominating set of size four, but there does not exist an OLD-set (and, hence, no $OLD_{OIND}$-set). Note that the 5-cycle does not have an open-independent, open-dominating set (and, hence, no $OLD_{OIND}$-set).

![Figure 3](image.png)

Figure 3. $\gamma^{OP}_{OIND}(T_9) = 4$ and $OLD(T_9)$ is not defined.

**Proposition 3.1.** If $S$ is any $OLD_{OIND}$-set for a graph $G$ and $v$ is an endpoint, $deg_Gv = 1$, with $N(v) = \{w\}$, then $\{v, w\} \subseteq S$. In particular, $\{v, w\}$ is contained in any $OLD_{OIND}(G)$-set.

**Proof.** Because $N(v) = w$, any open dominating set $S$ must contain $w$. Because $S$ is open-independent, $N(w)$ contains exactly one element of $S$, and because $S$ is open-locating if $N(w) \cap$
\[
S = \{x\} \text{ with } x \neq v \text{ we have the contradiction that } N(v) \cap S = \{w\} = N(x) \cap S. \text{ Hence, } v \in S.
\]

**Figure 4.** \(H, G_1[H]\) and \(G_2[H]\).

For any connected graph \(H\) of order \(n \geq 2\), let \(G_1[H]\) be obtained by adding for each \(v \in V(H)\) two vertices \(v'\) and \(v''\) and edges \(vv'\) and \(v'v''\), and let \(G_2[H]\) be obtained from \(H\) by further adding vertices \(v'''\) and \(v''''\) and edges \(vv'''\) and \(v''''v'''''\). Then every \(G_1[H]\) and \(G_2[H]\) have OLD-sets, and \(G_2[H]\) has an \(OLD_{OIND}\)-set while \(G_1[H]\) does not.

Hence, we have the following.

**Theorem 3.1.** For every graph \(H\) there are graphs \(G_1\) and \(G_2\) with \(H\) as an induced subgraph where \(G_1\) does not have an \(OLD_{OIND}\)-set but \(G_2\) does have an \(OLD_{OIND}\)-set.

There is no forbidden subgraph characterization of the set of graphs which have \(OLD_{OIND}\)-sets, nor of the set of graphs which do not have \(OLD_{OIND}\)-sets. In fact, simply deciding for a given graph \(G\) if \(G\) has an \(OLD_{OIND}\)-set is an NP-complete problem. As noted in Garey and Johnson [10], Problem 3-SAT is NP-complete.

**3-SAT**

**INSTANCE.** Sets \(U = \{u_1, u_2, \ldots, u_n\}\) and \(\overline{U} = \{\overline{u}_1, \overline{u}_2, \ldots, \overline{u}_n\}\) and collection \(C = \{c_1, c_2, \ldots, c_m\}\) of 3-element subsets of \(U \cup \overline{U}\).

**QUESTION.** Does there exist a satisfying truth assignments for \(C\), that is, a subset \(S\) of \(U \cup \overline{U}\) of order \(n\) with \(|S \cap \{u_i, \overline{u}_i\}| = 1\) for \(1 \leq i \leq n\) with \(S \cap c_j \neq \emptyset\) for \(1 \leq j \leq m\)?

**XOIOILD** (existence of an open-independent, open-locating-dominating set)

**INSTANCE.** A graph \(G\).

**QUESTION.** Does \(G\) have an \(OLD_{OIND}\)-set?
Theorem 3.2. Simply deciding, for a given graph $G$, if $G$ has an open-independent, OLD-set is an NP-complete decision problem. That is, $XOIOOLD$ is NP-complete.

Proof. One can easily verify in polynomial time if a given set $S \subseteq V(G)$ is an $OLD_{OIND}$-set, so $XOIOOLD \in NP$.

We can reduce the known NP-complete 3-SAT problem to $XOIOOLD$ in polynomial time as follows. For each $u_i \in U$ let $G_i$ be the 6-vertex graph illustrated in Figure 5 with $V(G_i) = \{u_i, \overline{u}_i, v_i, w_i, x_i, y_i\}$ and $E(G_i) = \{u_i\overline{u}_i, u_iv_i, \overline{u}_iv_i, v_iw_i, w_ix_i, x_iy_i\}$. For each clause $c_j \in C$ let $H_j$ be the 3-vertex graph with $V(H_j) = \{a_j, b_j, d_j\}$ and $E(H_j) = \{a_jb_j, b_jd_j\}$. Interconnect the clause components and literal components by adding edges $d_jc_{j,1}, d_jc_{j,2}$ and $d_jc_{j,3}$ for $1 \leq j \leq m$ where $c_j = \{c_{j,1}, c_{j,2}, c_{j,3}\} \in C$, as illustrated in Figure 5. Let $G$ be the resulting graph of order $6n + 3m$.

Assume there is a satisfying truth assignment $S \subseteq U \cup \overline{U}$. Form $W \subseteq V(G)$ by letting $\{y_i, x_i, v_i, a_j, b_j\} \subseteq W$ for $1 \leq i \leq n, 1 \leq j \leq m$. For $1 \leq i \leq n$, add $u_i$ to $W$ if the literal $u_i \in S$, otherwise the literal $\overline{u}_i \in S$ and one adds $\overline{u}_i$ to $S$. Then $<W>$ consists of $4n + 2m$ vertices inducing $2n + m$ independent edges. Note that $N(a_j) = \{b_j\} \subseteq W, b_j \in N(d_j) \cap W$ but $|N(d_j) \cap W| \geq 2$ because $S$ is satisfying. It is easily seen that $G$ has $W$ as an open-independent, OLD-set.

Assume $G$ has an $OLD_{OIND}$-set $W$. By Proposition 3.1 we have $\{y_i, x_i\} \subseteq W$ with $w_i \notin W$ for $1 \leq i \leq n$, and $\{a_j, b_j\} \subseteq W$ with $d_j \notin W$ for $1 \leq j \leq m$. Because $N(y_i) \cap W = \{x_i\}$ and $N(w_i) \cap W \neq N(y_i) \cap W$, each $v_i \in W$. Now $v_i \in W$ implies the open-independent, dominating set $W$ has $|N(v_i) \cap W| = 1$, so $W$ contains exactly one of $u_i$ and $\overline{u}_i$. Let $S = W \cap (U \cup \overline{U})$. Because $W$ is an OLD-set $N(a_j) \cap W = \{b_j\} \subseteq N(d_j) \cap W$, and we have $N(d_j) \cap (U \cup \overline{U}) \neq \emptyset$. That is, $S$ must be a satisfying truth assignment.

Figure 5. $c_1 = \{u_1, \overline{u}_2, u_3\}, c_2 = \{u_1, u_2, \overline{u}_3\}$, etc.
Theorem 3.3. If the girth of $G$ satisfies $g(G) \geq 5$ and $W \subseteq V(G)$, then $W$ is an OLD$_{OIND}$-set if and only if (1) each $v \in W$ is open-dominated exactly once, and (2) each $v \notin W$ is open-dominated at least twice.

Proof. Assume $W$ is an OLD$_{OIND}$-set for $G$. Because $W$ is open-independent and open-dominating, each $v \in W$ has $|N(v) \cap W| = 1$. If $v \notin W$, then $W$ open-dominates $v$ implies there is a vertex $w \in N(v) \cap W$. As noted $|N(w) \cap W| = 1$, say $N(w) \cap W = \{x\}$. Then $N(x) \cap W = \{w\} \neq N(v) \cap W$ implies that $|N(v) \cap W| \geq 2$.

Assume conditions (1) and (2) hold for $W \subseteq V(G)$. Then $W$ is open-dominating. Assume $v \in W$ has $|N(v) \cap W| = 1$. For $N(v) \cap W = \{w\}$, we have $N(w) \cap W = \{v\}$ and $x \in N(w) - \{v\}$ implies that $x \notin W$, so $x$ is open-dominated at least twice. Thus $v$ is the only vertex with $N(v) \cap W = \{w\}$. Assume $v \notin W$, let $\{x, y\} \subseteq N(v) \cap W$. No other vertex $u$ has $\{x, y\} \subseteq N(u) \cap W$ or else $u, x, v, y$ is a 4-cycle and $g(G) \leq 4$. Thus $N(v) \cap W$ uniquely distinguishes $v$. \hfill \Box

Assume $W$ is an OLD$_{OIND}$-set for path $P_n : v_1, v_2, ..., v_n$. By Proposition 3.1 we have $\{v_1, v_2\} \subseteq W, v_3 \notin W, \{v_4, v_5\} \subseteq W, v_6 \notin W, ... \{v_{n-1}, v_n\} \subseteq W$.

Proposition 3.2. Path $P_n$ has an OLD$_{OIND}$-set $W$ if and only if $n \equiv 2(\text{mod } 3)$ and OLD$_{OIND}(P_{3k+2}) = 2k + 2$.

![Figure 6. Some trees with OLD$_{OIND}$-sets.](image)

Theorem 3.4. (Seo and Slater [26]) A tree $T$ has an OLD-set if and only if no two endpoints of $T$ have the same neighbor.
Similar to the characterization given in Studer, et al. [43] for open-independent dominating sets in trees, we can recursively define the collection of pairs \((T, W)\) where \(T\) is a tree and \(W\) is the unique \(OLD_{OIND}(T)\)-set. First note that the tree \(A_t\) of order \(2t + 1\) in Figure 6 has \(OLD_{OIND}(A_t) = 2t\) and \(V(A_t) - x\) is the unique \(OLD_{OIND}(A_t)\)-set for \(t \geq 2\).

**Theorem 3.5.** If \(T_n\) is a tree of order \(n\) with an \(OLD_{OIND}\)-set, then the \(OLD_{OIND}(T)\)-set \(W\) is unique and \(T_n\) can be obtained recursively from \(P_2\) by a sequence of operations \(OP1\) and \(OP2\) defined as follows.

\((OP1)\) Let \(T^*\) be a tree with \(OLD_{OIND}(T)\)-set \(W\) and let \(z \in W\). The tree \(T\) is obtained from \(T^*\) by adding a \(P_3: x, w, v\) and adding the edge \(zx\).

\((OP2)\) Let \(T^*\) be a tree with \(OLD_{OIND}(T)\)-set \(W\) and let \(z\) be any vertex in \(T^*\). The tree \(T\) is obtained from \(T^*\) by adding an \(A_t\) with \(t \geq 2\) and adding the edge \(zx\).

**Proof.** We first observe that if \(T\) is obtained from \(T^*\) by \((OP1)\), then \(W \cup \{w, v\}\) is an \(OLD_{OIND}\)-set for \(T\), and if \(T\) is obtained from \(T^*\) by \((OP2)\), then \(W \cup \{w_1, v_1, ..., w_t, v_t\}\) is an \(OLD_{OIND}\)-set for \(T\).

Assume tree \(T\) has \(OLD_{OIND}\)-set \(W\). If \(T\) is a path, then Proposition 3.2 shows that \(T\) can be obtained from \(P_2\) by a sequence of \((OP)\)-operations and there is a unique \(OLD_{OIND}\)-set for \(T\). If \(T\) is not a path, select a vertex \(y\) with \(\text{deg} y \geq 3\) where all or all but one of the branches at \(y\) are paths. Suppose \(y, u_1, u_2, ..., u_j\) is a branch path with \(j \geq 3\). By Proposition 1 we must have \(\{u_{j-1}, u_j\} \subseteq W\) and \(u_{j-2} \notin W\). Also \(u_{j-3}\) (possibly \(u_{j-3} = y\)) must be in \(W\) or else \(N(u_j) \cap W = N(u_{j-2}) \cap W = \{u_{j-1}\}\). Let \(T^* = T - \{u_j, u_{j-1}, u_{j-2}\}\). Since \(W\) is an \(OLD_{OIND}\)-set of \(T\) and \(N(u_j) \cap V(T^*) = \emptyset\) and \(N(u_{j-1}) \cap V(T^*) = \emptyset\), \(W - \{u_j, u_{j-1}\}\) is an \(OLD_{OIND}\)-set of \(T^*\). So \(T\) is obtainable from \(T^*\) by \((OP1)\) where \(z = u_{j-3}\). Because \(W\) is an \(OLD_{OIND}\)-set, \(y\) can not be the support vertex of two or more endpoints. If \(y\) is adjacent to an endpoint \(x\) and \(y, u_1, u_2\) is a branch path, Proposition 1 would imply that \(\{u_1, u_2\} \subseteq W\) and \(\{x, y\} \subseteq W\), so \(W\) would not be open-independent. Now \(y\) can be assumed to have \(\text{deg} y - 1 = b\) branch paths of length two. We have a subgraph \(A_b\) with vertices \(\{y, w_1, v_1, ..., w_b, v_b\}\) with \(b \geq 2\). Let \(N(y) = \{w_1, w_2, ..., w_b, z\}\), and \(T\) can be obtained from \(T^* = T - \{y, w_1, v_1, ..., w_b, v_b\}\) by \((OP2)\). \(\square\)

4. **OLD_{OIND} % for infinite grids**

Much work has been done on distinguishing sets (LD-sets, IC-sets and OLD-sets) in infinite grids (hexagonal, square, triangular, tumbling block, etc). See, for example, [1, 6, 7, 18, 19, 20, 22, 23, 26, 28, 29, 30, 40, 41].

For a given vertex \(x\) in a dominating set \(D\) in a graph \(G\), the share \(sh(x; D)\) is defined in Slater [41] as a measure of how much domination the individual vertex \(x\) does. For example, in graph \(H1\) of Figure 7 we have \(N[3] = \{2, 3, 4, 5, 6, 9\}\) and \(sh(3; \{3, 4, 7\}) = 1/2 + 1/2 + 1/2 + 1/2 + 1/2 + 1/2 = 8/3\). Also, \(sh(4; \{3, 4, 7\}) = 1 + 1/2 + 1/2 + 1/2 + 1/2 + 1/2 + 1/2 = 11/3\), and \(sh(7; \{3, 4, 7\}) = 1/3 + 1/2 + 1 + 1/2 + 1/2 + 1/3 = 8/3\). Note that \(\sum_{v \in D} sh(v; D) = |V(G)| = n\) for any dominating set \(D\) and that \(|D| \geq |V(G)|/\text{MAX}_v sh(v; D)\).

Similarly, the open share \(sh^{op}(x; D)\) is defined in Seo and Slater [26] for open dominating set \(D\). Specifically, if \(D\) is open dominating and \(x \in D\) then, for each \(y \in N(x)\), let
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Figure 7. Graphs $H_1$ and $H_2$.

$sh^{op}(x; D(y)) = 1/|N(y) \cap D|$ and let $sh^{op}(x; D) = \Sigma_{y \in N(x)} sh^{op}(x; D(y))$. For example, in Figure 7 $D = \{3, 4\}$ is an open dominating set for the graph $H_2$. We have $N(3) = \{2, 4, 5\}$ and $sh^{op}(3; D) = sh^{op}(3; D(2)) + sh^{op}(3; D(4)) + sh^{op}(3; D(5)) = 1/2 + 1 + 1/2 = 2$. Also, $N(4) = \{1, 2, 3, 5\}$ and $sh^{op}(4; D) = sh^{op}(4; D(1)) + sh^{op}(4; D(2)) + sh^{op}(4; D(3)) + sh^{op}(4; D(5)) = 1 + 1/2 + 1 + 1/2 = 3$. Note that $\Sigma_{x \in D} sh^{op}(x; D) = |V(G)|$, and if $sh^{op}(w; D) \geq sh^{op}(x; D)$ for all $x \in D$, then $|D| \geq |V(G)|/sh^{op}(w; D)$.

In this paper, we will focus on open-locating-dominating sets along with open-shares of vertices.

Percentage parameters for measuring density for locally-finite, countably infinite graphs were defined in Slater [41]. For example, for the $\gamma(G)$ parameter we have $\gamma(G)$ defined as follows as the minimum possible percentage of vertices in a dominating set of $G$. The closed $k$-neighborhood of vertex $v$ is the set of vertices at distance at most $k$ from $v$, $N^k[v] = \{w \in V(G) : d(v, w) \leq k\}$. For $S \subseteq V(G)$, the density of $S$ is $dens(S) = \max_{v \in V(G)} \limsup_{k \to \infty} (|S \cap N^k[v]|/|N^k[v]|)$. Then, for example, the domination percentage of $G$ is $\gamma(G) = \min\{dens(S) : S \subseteq V(G) is dominating\}$. Let HEX, SQ, and TRI denote the infinite hexagonal, square and triangular grid graphs, respectively.

**Theorem 4.1.** (Seo and Slater [26]) $OLD\%(HEX) = 1/2$.

The darkened vertices of HEX in Figure 8 form an OLD\%(HEX)-set $D$ achieving the value $1/2$, and $D$ is an open-independent set. Hence we have the following.

**Theorem 4.2.** $OLD_{OIND}\%(HEX) = OLD\%(HEX) = 1/2$.

Figure 9(a) illustrates that $OLD\%(SQ) = 2/5$, but $OLD_{OIND}\%(SQ) > OLD\%(SQ)$.

**Theorem 4.3.** (Seo and Slater [26]) $OLD\%(SQ) = 2/5$. 

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Figure 8. $OLD(HEX) = 1/2 = OLD_{IND}(HEX)$. 

Figure 9. $OLD(SQ) = 2/5$ and $OLD_{IND}(SQ) = 3/7$. 

Figure 10. $OLD_{IND}(TRI) \leq 8/25$. 

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**Theorem 4.4.** \( \text{OLD}_{\text{OIND}}(SQ) = 3/7. \)

**Proof.** The set of darkened vertices in Figure 9(b) shows that \( \text{OLD}_{\text{OIND}}(SQ) \geq 3/7 \). To see that \( \text{OLD}_{\text{OIND}}(SQ) \geq 3/7 \) we use a straightforward share argument. Let \( D \) be an \( \text{OLD}_{\text{OIND}}(SQ) \)-set. We have \( V(SQ) = \mathbb{Z} \times \mathbb{Z} \) and \( N((i,j)) = \{(i-1,j), (i,j+1), (i+1,j), (i,j-1)\} \). We will show that \( sh_{\text{op}}^D(x;D) \leq 7/3 \) for every \( x \in D \), and hence \( \text{OLD}_{\text{OIND}}(SQ) \geq 7/3 \). Without loss of generality, assume \( x = (0,0) \in D \). Exactly one neighbor of \( x \) is also in \( D \), and we can assume that \( (1,0) \in D \). In particular, \( sh_{\text{op}}^D(x;D((1,0))) = 1 \). Any other vertex \( y \in N(x) \) is open dominated at least twice by \( D \), so \( sh_{\text{op}}^D(x;D(y)) \leq 1/2 \) for each \( y \in \{(−1,0),(0,1),(0,−1)\} \). Suppose all three of these vertices give \((0,0)\) a share value of \(1/2\).

Case 1. \( N((−1,0)) \cap D = \{(0,0), (−2,0)\} \). Then \((−1,1) \notin D \) with \( D \cap \{(−1,0),(0,1)\} = \emptyset \), and so \( N((−1,1)) \cap D = \{(−2,1), (−1,2)\} \). Similarly, \( N((0,1)) \cap D = \{(−2,−1), (−1,−2)\} \). But then \( \{(−2,1), (−2,0), (−2,−1)\} \subseteq D \), contradicting the open independence of \( D \).

Case 2. \( N((−1,0)) \cap D \neq \{(0,0), (−2,0)\} \). Then one of \((−1,1)\) and \((−1,−1)\) is in \( D \), say \((-1,1)\). Because \( sh_{\text{op}}^D((0,0);D((0,1))) = 1/2 \) we have \( N((0,1)) \cap D = \{(0,0), (−1,1)\} \). Also, \( sh_{\text{op}}^D((−1,0);D((−1,0))) = 1/2 \) implies \( N((−1,0)) \cap D = \{(0,0), (−1,1)\} = N(0,1) \cap D \), contradicting the fact that \( D \) must distinguish \((0,1)\) and \((−1,0)\).

Because \( sh_{\text{op}}^D(x;D) < 1 + 1/2 + 1/2 + 1/2 \), we have \( sh_{\text{op}}^D(x;D) \leq 1 + 1/2 + 1/2 + 1/3 = 7/3 \) for every \( x \in D \). As noted, this implies \( \text{OLD}_{\text{OIND}}(SQ) \geq 7/3. \)

\[ \square \]

**Theorem 4.5.** (Kincaid, Oldham, and Yu [23]) \( \text{OLD}(TRI) = 4/13. \)

Figure 10 shows that \( \text{OLD}_{\text{OIND}}(TRI) \leq 8/25 \). To date, the best we have is that: \( \text{OLD}_{\text{OIND}}(TRI) \in [4/13, 8/25] \).

5. Open independent sets

In this paper we focused on open-independence for OLD-sets. Of interest is the parameter \( OIND \) itself, as well as the lower open independence parameter \( oind \) where \( oind(G) \) is the minimum cardinality of a maximally open-independent set.

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