Multiplication of solutions for linear overdetermined systems of partial differential equations

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Abstract
A large family of linear, usually overdetermined, systems of partial differential equations that admit a multiplication of solutions, i.e., a bilinear and commutative mapping on the solution space, is studied. This family of PDE’s contains the Cauchy–Riemann equations and the cofactor pair systems, included as special cases. The multiplication provides a method for generating, in a pure algebraic way, large classes of non-trivial solutions that can be constructed by forming convergent power series of trivial solutions.

1 Introduction
In this paper we study a wide class of linear first order systems of partial differential equations, that allow a bi-linear multiplication in the space of solutions. The simplest example is the Cauchy–Riemann equations. We know that two holomorphic functions, \( f = V + i\tilde{V} \) and \( g = W + i\tilde{W} \), can be multiplied in order to produce a new holomorphic function \( fg = VW - \tilde{V}\tilde{W} + i(V\tilde{W} + \tilde{V}W) \). In terms of the Cauchy–Riemann equations

\[
\begin{align*}
\frac{\partial V}{\partial x} &= \frac{\partial \tilde{V}}{\partial y} \\
\frac{\partial V}{\partial y} &= -\frac{\partial \tilde{V}}{\partial x}
\end{align*}
\]

this multiplication can be expressed in the following way: two solutions \((V, \tilde{V})\) and \((W, \tilde{W})\) prescribe, in a bi-linear way, a new solution \((VW - \tilde{V}\tilde{W}, VW +\)
From the basic theory of holomorphic functions, we know that any solution of the Cauchy–Riemann equations can be expressed locally as a convergent power series of a simple solution with respect to the described multiplication.

The Cauchy–Riemann equations provide the simplest example of a system of PDE’s that has a multiplication on its solution set, but there are more sophisticated examples. One such example is the multiplication of cofactor pair systems, discovered by Lundmark in [5].

A cofactor pair system (or bi-cofactor system) is a dynamical system

\[ \dot{q}^h + \Gamma^h_{ij} \dot{q}^i \dot{q}^j = F^h, \quad h = 1, \ldots, n, \]

where \( F = F(q) \) has two different cofactor formulations

\[ F(q) = (\text{cof } J)^{-1} \nabla V = (\text{cof } \tilde{J})^{-1} \nabla \tilde{V}, \]

where \( J \) and \( \tilde{J} \) are independent special conformal Killing tensors of type \((1,1)\), \( V \) and \( \tilde{V} \) are smooth real-valued functions, \( \text{cof } J = (\text{det } J)^{-1} J \), and \( \nabla \) is the gradient \((\nabla V)^i = g^{ij} \partial_j V\). Cofactor pair systems have several desirable properties, in general they are completely integrable, they admit a bi-Hamiltonian formulation, and they are equivalent (or correspondent) to separable Lagrangian systems [1, 2, 6, 7, 8, 9].

A cofactor pair system is characterized by a pair of functions \( V \) and \( \tilde{V} \), and a pair of special conformal Killing tensors \( J \) and \( \tilde{J} \), that satisfy the relation

\[ (\text{cof } J)^{-1} \nabla V = (\text{cof } \tilde{J})^{-1} \nabla \tilde{V}. \]  

For fixed special conformal Killing tensors \( J \) and \( \tilde{J} \), the equation (1) constitutes a system of first order linear PDE’s for two functions \( V \) and \( \tilde{V} \). In [5], Lundmark found that the equation (1) allows a multiplication of solutions. When \( n = 2 \) the multiplication formula is given by

\[ (V, \tilde{V}) \ast (W, \tilde{W}) = \left( VW - \text{det } (\tilde{J}^{-1} J) \tilde{V} \tilde{W}, \quad V \tilde{W} + \tilde{V} W - \text{tr}(\tilde{J}^{-1} J) \tilde{V} \tilde{W} \right), \]

where \( (V, \tilde{V}) \) and \( (W, \tilde{W}) \) are solutions of (1). We see that when \( \text{det } (\tilde{J}^{-1} J) \) and \( \text{tr}(\tilde{J}^{-1} J) \) are not both constant, we can choose trivial (constant) solutions \( (V, \tilde{V}) \) and \( (W, \tilde{W}) \) of (1) and obtain non–trivial solutions through the multiplication. When \( n > 2 \) a multiplication also exists, but one has to consider the related parameter–dependent system

\[ (\text{cof}(J + \mu \tilde{J}))^{-1} \nabla V_\mu = (\text{cof } \tilde{J})^{-1} \nabla \tilde{V}, \]

which can also be written as

\[ (\tilde{J}^{-1} J + \mu I) \nabla V_\mu = \text{det } (\tilde{J}^{-1} J + \mu I) \nabla \tilde{V}, \]

where \( V_\mu \) is polynomial in the real parameter \( \mu \) (note that throughout this paper, we use the notation \( V_\mu \) rather than \( V(\mu) \) to indicate dependence on the parameter \( \mu \)).
The most interesting property of this multiplication is that it provides a tool for producing new cofactor pair systems from known ones. Especially, infinite families of separable potentials can be constructed. For example, the Jacobi, Neumann, and parabolic families of separable potentials are all constructed in [10] through a recursive process that is a special case of the multiplication of cofactor pair systems.

Remark 1. In [3], equations of the form (1), considered on a real or complex vector space where $J$ and $\tilde{J}$ are constant matrices, are studied, and the general analytic solution is described.

In order to gain better understanding of this multiplication, systems of the form

$$(X + \mu I)\nabla V_\mu = \det (X + \mu I)\nabla \tilde{V},$$

(2)

defined on a general (pseudo-) Riemannian manifold, were studied in [4], without referring to any underlying dynamical system. By analyzing the corresponding equations at each degree of $\mu$ in the equation (2), it becomes obvious that the equation is satisfied if and only if the degree of $V_\mu$ is $n$ and the left hand side $(X + \mu I)\nabla V_\mu$ can be written as a product of the scalar $\det (X + \mu I)$ and some 1–form which is constant in $\mu$. We can therefore rewrite the equation (2) as

$$(X + \mu I)\nabla V_\mu = 0 \pmod{\det (X + \mu I)}.$$  

(3)

It turned out that the system (2) allows for a multiplication of solutions, similar to the one existing for cofactor pair systems, if and only if the tensor $X$ satisfies the equation

$$(X + \mu I)\nabla \det (X + \mu I) = \det (X + \mu I)\nabla \text{tr}(X + \mu I).$$

(4)

Several classes of solutions of (4) were discovered, and it became apparent that systems of the form (1) and the Cauchy–Riemann equations only constitute special cases of a much larger family of systems of PDE’s that admit a multiplicative structure on the solution space.

It was also remarked in [4], that by considering more general systems than (2), one finds other new classes of systems that allow multiplication. In this paper, we will examine that subject. The linear systems of PDE’s that we consider are in general impossible to solve, but the multiplication provides a non-trivial superposition principle (on top of the ordinary linear superposition) that, given to solutions, prescribes a new solution in a bi-linear and pure algebraic way. With this superposition principle, large classes of new solutions can be generated from known solutions. In particular, we can construct non–trivial solutions by forming convergent power series of a simple solution. The question then arises for which systems of linear PDE’s these power series constitute all solutions, like in the case of the Cauchy–Riemann equations where all holomorphic functions admit a power series representation. Besides providing us with
more systems of PDE’s that admit a multiplicative structure on the solution set, the generalization helps us to better understand the multiplication for the systems already known (in particular the puzzling multiplication of cofactor pair systems).

This paper is organized as follows. In section 2 we formulate an abstract framework for characterizing the class of systems of PDE’s that admit multiplication. We define the \( \ast \)–operator and give a characterization of those systems that admit \( \ast \)–multiplication on the set of solutions. The multiplication provides a method for generating, in a pure algebraic and non-trivial way, new solutions from known solutions. A second formulation of the systems, using related matrices, is introduced. In this matrix notation the Euclidean algorithm for polynomial division, which is closely related to the \( \ast \)–multiplication, can be encoded in an explicit polynomial of matrices. Some algebraic properties of the multiplication are also mentioned in this section. In section 3 we investigate the explicit forms of systems that admit multiplication. The study splits into different cases depending on relations among certain discrete parameters that appear in the studied class of systems. Especially, some typical (generic) systems with multiplication are derived and examined. The most interesting property of the multiplication of solutions is that we can construct large classes of solutions by forming power series, with respect to the \( \ast \)–multiplication, of trivial solutions. Section 4 is devoted to study such power series solutions. The problem of constructing systems with \( \ast \)–multiplication is in general quite complicated. In section 5 several methods for constructing systems with \( \ast \)–multiplication are described. The last section 6 contains concluding remarks and natural questions raised by the study presented in this paper.

2 Multiplication of solutions for linear systems of PDE’s

Let \( Q \) be a \( n \)–dimensional differentiable real manifold. Consider equations of the form

\[
A_\mu dV_\mu \equiv 0 \pmod{Z_\mu},
\]

where \( A_\mu \) is a \((1,1)\)–tensor depending polynomially on the real parameter \( \mu \), and \( Z_\mu \), \( V_\mu \) are real-valued \( C^1 \) functions on \( Q \) that also depend polynomially on \( \mu \). The expression \( A_\mu dV_\mu \) is a \( 1 \)–form which components are polynomial in \( \mu \), and the unknown function \( V_\mu \) is a solution of the equation (5) if these components are all divisible (when considered as polynomials in \( \mu \)) by the fixed function \( Z_\mu \). Let \( Z_\mu = Z_0 + \mu Z_1 + \cdots + \mu^{m-1} Z_{m-1} + \mu^m \), be a polynomial of degree \( m \), then there is no restriction to assume that \( A_\mu = A_0 + \mu A_1 + \cdots + \mu^k A_k \) has degree at most \( m-1 \) (otherwise we can reduce it modulo \( Z_\mu \)). In order to simplify the description of systems admitting a multiplication, we also assume that \( V_\mu = V_0 + \mu V_1 + \cdots + \mu^{m-1} V_{m-1} \) has degree \( m-1 \).
Remark 2. The system (5) should be compared with system (3). We see that (3) generalizes (5) in several ways. First of all, no metric is specified on the manifold $Q$ corresponding to the system (5), and we consider a system of equations for 1-forms using the exterior differential operator $d$, rather than a system of equations for vector fields expressed with the gradient operator $\nabla$. Moreover, in (5) the polynomial $X + \mu I$ is of degree one and contains only one arbitrary tensor $X$, while in (3) we consider a more general polynomial $A_\mu$, and instead of $\det (X + \mu I)$, we consider an arbitrary polynomial $Z_\mu$ that does not have to be related to $A_\mu$.

Since the highest order coefficient of $Z_\mu$ is a unit, for each polynomial $P_\mu$ with coefficients in the commutative ring of real-valued functions on $Q$, there exists unique polynomials $Q_\mu$ and $R_\mu$ such that $P_\mu = Q_\mu Z_\mu + R_\mu$, where $\deg R_\mu < m$. Thus, for each function $V_\mu$, there exists unique 1-forms $B_0, \ldots, B_{m-1}$ such that

$$A_\mu dV_\mu \equiv B_0 + \mu B_1 + \cdots + \mu^{m-1} B_{m-1} \pmod{Z_\mu},$$

and the equation (5) can be written as $B_0 = B_1 = \cdots = B_{m-1} = 0$. Thus, in local coordinates $q^1, \ldots, q^n$, the system (5) constitutes a, usually overdetermined, system of $nm$ first order linear partial differential equations for $m$ dependent variables. We will see that there exist non-trivial systems (5) admitting a multiplicative structure on its solution set.

Define a bilinear operation $\ast$, on the set of all real-valued functions on $Q$ that are polynomial in $\mu$, by letting $V_\mu \ast W_\mu$ be the residue of the ordinary product $V_\mu W_\mu$ modulo $Z_\mu$. In other words, $V_\mu \ast W_\mu$ is the unique polynomial of degree less than $m$ that can be written as $V_\mu W_\mu - Q_\mu Z_\mu$, for some polynomial $Q_\mu$. For certain choices of $A_\mu$ and $Z_\mu$, the $\ast$-multiplication maps solutions of (5) to new solutions:

**Theorem 1** ($\ast$-Multiplication). Let $S$ denote the solution set of (5). Then $\ast$ is a bilinear operation on $S$ if and only if

$$A_\mu dZ_\mu \equiv 0 \pmod{Z_\mu},$$

i.e., if and only if $Z_\mu - \mu^n \in S$.

**Proof.** Given two solutions $V_\mu, W_\mu \in S$, let $Q_\mu$ be the polynomial such that the product $V_\mu W_\mu$ can be written as $V_\mu W_\mu = Q_\mu Z_\mu + V_\mu \ast W_\mu$. Then, we have

$$A_\mu d(V_\mu \ast W_\mu) = W_\mu A_\mu dV_\mu + V_\mu A_\mu dW_\mu - Z_\mu A_\mu dQ_\mu - Q_\mu A_\mu dZ_\mu \equiv -Q_\mu A_\mu dZ_\mu \pmod{Z_\mu}.$$ 

Thus, we see that $A_\mu dZ_\mu \equiv 0$ is a sufficient condition for the existence of the bi-linear operation $\ast$ on $S$. To see that it is also a necessary condition, consider the trivial solutions $V_\mu = \mu$ and $W_\mu = \mu^{m-1}$. For this choice of solutions, the polynomial $Q_\mu$ becomes a non-zero constant, which forces the relation (7) to be satisfied in order for $V_\mu \ast W_\mu$ to be a solution. \qed
The following algebraic properties of $*$ are immediate consequences of the corresponding properties of multiplication in general quotient rings of polynomials:

**Corollary 2.** The solution set $S$ together with the scalar multiplication, addition (defined in the obvious way) and multiplication $*$ is an algebra over $\mathbb{R}$, where the $*$-multiplication is associative and commutative.

To calculate the $*$-product $V_\mu * W_\mu$, we form the ordinary product

$$V_\mu W_\mu = V_0 W_0 + (V_0 W_1 + V_1 W_0) \mu + \ldots + (V_{m-2} W_{m-1} + V_{m-1} W_{m-2}) \mu^{2m-3} + V_{m-1} W_{m-1} \mu^{2m-2},$$

and replace $\mu^m, \ldots, \mu^{2m-2}$ with their residues modulo $Z_\mu$:

$$\mu^m \equiv -Z_0 - Z_1 \mu - \ldots - Z_{m-1} \mu^{m-1},$$

$$\mu^{m+1} \equiv Z_0 Z_{m-1} + (Z_1 Z_{m-1} - Z_0) \mu + \ldots + (Z_{m-1} Z_{m-1} - Z_{m-2}) \mu^{m-1},$$

$$\vdots$$

In general it is hard to find non-trivial solutions of the system (5), but having the $*$-operator we can generate (in a pure algebraic way) an infinite family of non-trivial solutions by starting with trivial solutions. For example, we can construct non-trivial solutions by forming polynomials, or convergent power series, of the trivial solution $\mu \in S$:

$$V_\mu = \sum_r a_r \mu^*_r, \quad \text{where} \quad \mu^*_r := \underbrace{\mu * \cdots * \mu}_{r \text{ factors}}, \quad \text{with} \quad a_r \text{ being real constants.} \quad (8)$$

We note that, since all trivial solutions can be expressed as polynomials in $\mu$, every solution that is a sum of products of trivial solutions has the form (8) again. As long as we consider domains in the manifold $Q$ where $Z_0 \neq 0$, we can also allow negative powers $\mu^{-a}$ in (5), for every natural number $a$, by defining $\mu^{-a} := (\mu^{-1})^a$ and

$$\mu^{-1} := -\frac{1}{Z_0} \left(1 + \cdots + Z_{m-1} \mu^{m-2} + \mu^{m-1}\right),$$

so that $\mu * \mu^{-1} = 1$.

The following two examples illustrate how the Cauchy–Riemann equations and the cofactor pair systems can be considered as special cases of systems of the form (5) that admit a $*$-multiplication of the kind described in theorem 1:

**Example 1** (Cauchy–Riemann equations). Let $Q$ be the 2-dimensional Euclidean space with Cartesian coordinates $(x, y)$, $Z_\mu = 1 + \mu^2$ and

$$A_\mu = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} + \mu I,$$
where \( I \) is the identity matrix. Then the system (5) reduces to the Cauchy–Riemann equations, and the ordinary multiplication of holomorphic functions follows from the \( *- \) multiplication:

\[
(V, \tilde{V}) * (W, \tilde{W}) = \left( VW - \tilde{V}\tilde{W}, V\tilde{W} + \tilde{V}W \right).
\]

Every solution \((V, \tilde{V})\) that satisfies the Cauchy–Riemann equations in the origin, can be expressed in a neighborhood of the origin as a power series of the simple solution \((x, y)\):

\[
V_\mu = V + \mu \tilde{V} = \sum_{r=0}^{\infty} a_r (x + y\mu)^r_*,
\]

where \((x + y\mu)^r_*\) again denotes the \( r \)’th power with respect to the \( *- \) multiplication.

Example 2 (Multiplication of cofactor pair systems). Let \( m = n \) and suppose that the tensor \( A_\mu \) is linear in \( \mu \) and has the identity mapping as the highest order coefficient, i.e. \( A_\mu = X + \mu I \). The system (5) can for this special case be written as

\[
X dV_i + dV_{i-1} = Z_i dV_{n-1}, \quad i = 0, \ldots, n-1,
\]

where \( V_{-1} := 0 \). If we also let \( Z_\mu = \det (X + \mu I) \), the system (5) reduces to the system (2) when we specify a metric on \( Q \) and consider the equivalent “vector version” \( A_\mu \nabla V_\mu \equiv 0 \) of (5). Restricting the attention to the case \( Z_\mu = \det (X + \mu I) \) is quite natural since, if we also assume that the coefficients \( Z_0, \ldots, Z_{n-1} \) of \( Z_\mu \) are functionally independent, it is a necessary condition for the equation (7) to be satisfied. To see this, choose coordinates \( q^1, \ldots, q^n \) as \( q^i = Z_{i-1} \). Then the equation (7), or equivalently \( X dZ_i + dZ_{i-1} = Z_i dZ_{n-1}, \) reduces to \( X_j^i = q^j \delta^n_i - \delta^i_j \), where \( \delta \) is the Kronecker delta symbol. In other words, \(-X\) must in these coordinates be the companion matrix (see (10)) of the polynomial \( Z_\mu \), and therefore it follows that \( Z_\mu = \det (X + \mu I) \).

The \( *- \) multiplication reduces to Lundmarks multiplication of cofactor pair systems if we let \( X = \bar{J}^{-1} J \), where \( J \) and \( \bar{J} \) are special conformal Killing tensors. In (2), several other families of tensors \( X \), that satisfy the equation (4) have been found.

2.1 Matrix notation

For the purpose of further study of systems of PDE’s of the form (5), we shall introduce a new kind of matrix formulation for these systems and for the corresponding \( *- \) multiplication. The matrix formulation makes it possible to give an explicit formula for calculating powers of solutions, with respect to the \( *- \) multiplication.
The idea is to consider the column matrix 

\[ V = [V_0, V_1, \ldots, V_{m-1}]^T \]

instead of the polynomial 

\[ V_\mu = V_0 + \mu V_1 + \cdots + \mu^{m-1} V_{m-1}, \]

and to observe that 

\[ V = V_C e_1 = (V_0 C_0 + V_1 C_1 + \cdots + V_{m-1} C^{m-1}) e_1 \]

with \( V_C = V_\mu = C \) where we have formally substituted the parameter \( \mu \) with the companion matrix 

\[ C := C[Z_\mu] = \begin{bmatrix}
0 & 0 & \cdots & 0 & -Z_0 \\
1 & 0 & \cdots & 0 & -Z_1 \\
0 & 1 & \ddots & \vdots \\
\vdots & \ddots & 0 & -Z_{n-2} \\
0 & 0 & \cdots & 1 & -Z_{n-1}
\end{bmatrix} \]

of \( Z_\mu \), and where \( e_1 = [1, 0, \ldots, 0]^T \). The most important advantage of the matrix notation is that we can express the Euclidean algorithm of polynomial division in a more explicit way. For any polynomial 

\[ P_\mu = P_0 + \cdots + P_\mu^t, \]

the residue modulo \( Z_\mu \) can be written as 

\[ R_\mu = R_0 + \cdots + R_{m-1} \mu^{m-1}, \]

where 

\[ [R_0, R_1, \ldots, R_{m-1}]^T = P_C e_1. \]

Thus, in the matrix notation, the \(*\)-multiplication of two solutions \( V \) and \( W \) of (5) can be written as 

\[ V * W = V C W C e_1. \]

In particular, the matrix version of \( \mu \)-\( a \) is 

\[ C a^e_1. \]

The system (5) can also be expressed in terms of matrices as 

\[ 0 = \sum_{i=0}^{k} \sum_{j=0}^{m-1} C^{i+j} e_1 [\partial_1 V_j, \ldots, \partial_n V_j] A_i \]

\[ = \sum_{i=0}^{k} C^i \left( \sum_{j=0}^{m-1} C^j e_1 [\partial_1 V_j, \ldots, \partial_n V_j] \right) A_i \]

\[ = \sum_{i=0}^{k} C^i V^' A_i, \]

where we consider \( A_i \) as the matrix with elements \((A_i)^a_b\) and \( V^' \) is the functional matrix 

\[ V^' = \begin{bmatrix}
\partial_1 V_0 & \cdots & \partial_n V_0 \\
\vdots & \ddots & \vdots \\
\partial_1 V_{m-1} & \cdots & \partial_n V_{m-1}
\end{bmatrix}. \]

The equation (11) is indeed independent of coordinates. The expression on the right-hand side of equation (11) is a \( m \times 1 \)-matrix consisting of the 1-forms 

\[ \sum_{i,j} (C^i)_a A_i dV_{j-1}, \]

where \((C^i)_a\) is the element in the row \( a \) and the column \( j \) of the matrix \( C^i \). The \(*\)-multiplication theorem can then be expressed as:
Proposition 3 (Explicit criterion for existence of \( \ast \)-multiplication). Let \( V \) and \( W \) be solutions of \( 0 = \sum_{i=0}^{k} C^i V^i A_i \), then \( V \ast W := V C W C e_1 \) is also a solution if and only if
\[
0 = \sum_{i=0}^{k} C^i Z^i A_i. \tag{12}
\]

3 Explicit form of linear PDE’s admitting \( \ast \)-multiplication

The system (5) contains three parameters: \( n \) – the dimension of the manifold \( Q \); \( m \) – the polynomial degree of the function \( Z_\mu \); \( k \) – the polynomial degree of the tensor \( A_\mu \). In this section we will discuss how the form of the system (5), or (11) in matrix notation, and of the related \( \ast \)-multiplication depends on these numbers. We will also, for different choices of \( n, m, k \), specify the structure of typical (generic) systems that allow \( \ast \)-multiplication. This is done by choosing the functions \( Z_i \) as coordinates.

When \( k = 0 \), the system (5) can be written as \( A_0 dV_i = 0, i = 0, \ldots, m - 1 \) and the multiplication becomes trivial. When \( n = 1 \), the system (5) reduces to a quite simple system of ordinary differential equations. Therefore, we consider only cases where \( k > 0 \) and \( n > 1 \).

The \( \ast \)-product \( V \ast W \) of two solutions is a collection of \( m \) functions, each being a sum of functions of the form \( P_{ij} V_i W_j \), where \( P_{ij} \) is a polynomial expression of the variables \( Z_k \). The structure of the \( \ast \)-multiplication formula depends only on the parameter \( m \), not on \( n \) or \( k \). Since the degree of the polynomials \( P_{ij} \) will not exceed \( m - 1 \), the multiplication will be more complex for higher values of the parameter \( m \). For the simplest case \( m = 2 \), we have
\[
V \ast W = V C W C e_1 = (V_0 W_0 C^0 + (V_0 W_1 + V_1 W_0) C + V_1 W_1 C^2) e_1 = \begin{bmatrix} V_0 W_0 - Z_0 V_1 W_1 \\ V_0 W_1 + V_1 W_0 - Z_1 V_1 W_1 \end{bmatrix},
\]
and for \( m = 3 \) the \( \ast \)-product \( V \ast W \) is given by
\[
\begin{bmatrix}
V_0 W_0 - Z_0 V_1 W_2 - Z_0 V_2 W_1 + Z_1 Z_2 V_1 W_2 \\
V_0 W_1 + V_1 W_0 - Z_1 V_1 W_2 - Z_1 Z_2 + (Z_0 + Z_1 Z_2) V_2 W_2 \\
V_0 W_2 + V_1 W_1 + V_2 W_0 - Z_2 V_1 W_2 - Z_2 V_2 W_1 + (Z_1 + Z_2^2) V_2 W_2
\end{bmatrix} \tag{13}
\]

3.1 Generic cases for different choices of \( (n, m, k) \)

Our approach to find explicit forms of equations admitting \( \ast \)-multiplication, is to choose some generic coordinates in which the system (5), equipped with
*—multiplication, takes a simple form. Since the functions $Z_i$ play a fundamental role for the multiplication, we will assume that as many of these functions as possible are functionally independent and take them as local coordinates on $Q$. Since there are at most $n$ different functionally independent functions, the relation between $m$ (the number of functions $Z_i$) and $n$ will be crucial for specifying each generic case.

When $m = n$, we can choose as generic coordinates $q^1 = Z_0, \ldots, q^n = Z_{m-1}$ if the functions $Z_i$ are functionally independent. We consider the case when $m = n$ as our main case since the system (5) takes a simpler form than in the other cases.

When $m < n$, the functions $Z_i$ are too few to form a complete set of coordinates. Instead we choose generic coordinates $q^1, \ldots, q^n$ such that $q^1 = Z_0, \ldots, q^m = Z_{m-1}$, without specifying the last $n - m$ coordinates $q^{m+1}, \ldots, q^n$.

When $m > n$, the functions $Z_i$ must be functionally dependent. For the generic case we assume that $Z_0, \ldots, Z_{n-1}$ are functionally independent, and choose them as generic coordinates.

We shall present below an explicit form of the system (5) in generic coordinates for the cases $m = n$, $m < n$, and $m > n$. For each case we will also consider simpler sub-cases according to the following schematic diagram (14) for the triples $(n,m,k)$

\[
\begin{array}{ccc}
(n,m<n,k) & (n,m=n,k) & (n,m>n,k) \\
\downarrow & \downarrow & \downarrow \\
(n,2,1), (n,3,2) & (n,m=n,1) & (2,3,1) \\
\downarrow & \downarrow & \\
(3,2,1) & (2,2,1) & \\
\end{array}
\]

(14)

For the sake of simplicity, we will consider the case when $A_{m-1}$ is non-singular. It is then no restriction to assume that $A_{m-1}$ is the identity mapping.

3.2 $m = n$

A generic case with the simplest structure is obtained when $m = n$. According to the discussion above, $(n,m,k) = (2,2,1)$ is the lowest value of the parameters for which the multiplication is non-trivial. It is also the best case to study in order to get a good understanding of the mechanism of the multiplication. We will investigate this case in detail, and after that some of the ideas will be generalized to the cases $(n = m, m, k)$ and $(n = m, m, k = 1)$. 

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3.2.1 \((n, m, k) = (2, 2, 1)\)

For this choice of parameters \(n, m\) and \(k\), we have \(Z_\mu = Z_0 + \mu Z_1 + \mu^2\), \(A_\mu = A_0 + \mu A_1\), \(V_\mu = V_0 + \mu V_1\) and (5) can be written as

\[
\begin{align*}
A_0 dV_0 &= Z_0 A_1 dV_1 \\
A_1 dV_0 &= (Z_1 A_1 - A_0) dV_1,
\end{align*}
\]

(15)

or, as we have seen, in matrix notation as \(0 = V' A_0 + CV' A_1\). Thus, in local coordinates, (15) constitutes a system of four partial differential equations for two unknown functions \(V_0, V_1\) of two independent variables \(x, y\). Since the number of equations exceeds the number of dependent variables, this system will in general be overdetermined. If we assume that \(A_1\) is non-singular, we can instead of (15) consider the equivalent system

\[
\begin{align*}
0 &= (A^2 - Z_1 A + Z_0 I) dV_1 \\
dV_0 &= (Z_1 I - A) dV_1,
\end{align*}
\]

(16)

where \(A = A_0 A_1^{-1}\), or in matrix notation \(0 = V' A + CV'\). We assume now that the functions \(x = Z_0, y = Z_1\) are functionally independent and consider the system (16) in the generic coordinates \(x, y\). In these coordinates the relation (12), that guarantees a \(*\)–multiplication of the corresponding system (16), reduces to \(A = -C\). Thus, in generic coordinates, if we require existence of \(*\)–multiplication, the first equation of system (16) is a consequence of the Cayley–Hamilton theorem so the system reduces to the system \(dV_0 = (y I + C) dV_1\), that has components:

\[
\begin{align*}
\frac{\partial V_0}{\partial x} &= y \frac{\partial V_1}{\partial x} + \frac{\partial V_1}{\partial y} \\
\frac{\partial V_0}{\partial y} &= -x \frac{\partial V_1}{\partial x}.
\end{align*}
\]

Thus, the generic \((2, 2, 1)\)–case constitutes in fact a determined system of two partial differential equations for two unknown functions of two independent variables. As we have seen, the \(*\)–product of two solutions can in this case be written as

\[
V * W = V C W C e_1 = \begin{bmatrix} V_0 W_0 - x V_1 W_1 \\
V_0 W_1 + V_1 W_0 - y V_1 W_1 \end{bmatrix}.
\]
The simplest non-trivial solutions obtained by taking powers of the trivial solution \((0, 1)\) are

\[
(0, 1)^{-2} = \left( -\frac{y^2 - x}{x^2}, -\frac{y}{x^2} \right)
\]

\[
(0, 1)^{-1} = \left( -\frac{y}{x}, -\frac{1}{x} \right)
\]

\[
(0, 1)^0 = (1, 0)
\]

\[
(0, 1)^1 = (0, 1)
\]

\[
(0, 1)^2 = (-x, -y)
\]

\[
(0, 1)^3 = (xy, -x + y^2)
\]

If the roots of the polynomial \(Z_\mu\) (or the eigenvalues of the companion matrix \(C\)) are functionally independent, we can instead define local coordinates through \(Z_0 = xy, Z_1 = x + y\). The condition (7) can then be expressed as \(A_0 = DA_1\), where \(D = \text{diag}(x, y)\). If we moreover assume that \(A_1\) is non-singular, the system (5) reduces to \(DdV_0 = xydV_1\), or in components:

\[
\begin{align*}
\frac{\partial V_0}{\partial x} &= y \frac{\partial V_1}{\partial x} \\
\frac{\partial V_0}{\partial y} &= x \frac{\partial V_1}{\partial y}
\end{align*}
\]

This system has the general solution

\[
V_0 = \frac{x\phi(y) - y\psi(x)}{x - y}, \quad V_1 = \frac{\phi(y) - \psi(x)}{x - y},
\]

where \(\phi\) and \(\psi\) are arbitrary functions of one variable.

### 3.2.2 \((n, m, k) = (m, m, k)\)

We will now study the generic case of the more general situation when the only restriction for the parameters \(n, m, k\) is that \(n = m\). We assume now that \(q^1 = Z_0, \ldots, q^n = Z_{m-1}\) are functionally independent and constitute a complete set of coordinates. The condition (12), which guarantees existence of multiplication for the system [3], now attains the simple form \(AC = 0\) since \(Z_i\) in these coordinates becomes the identity matrix. Hence, the system (5) admits \(*\)–multiplication if and only if it, in the matrix notation with the coordinates \(q^i = Z_{i-1}\), can be written as

\[
0 = \sum_{i=1}^{k} \left( C^i V' - V'C^i \right) A_i
\]

where the tensors \(A_1, \ldots, A_k\) are arbitrary.
3.2.3 \((n, m, k) = (m, m, 1)\)

When \(k = 1\), the equation (19) becomes a remarkably simple equation \(0 = (CV' - V'C)A_1\) in terms of the generic coordinates. Thus, if we also assume that \(A_1\) is non-singular, we obtain in the generic \((m, m, 1)\)-case, the equation

\[CV' = V'C.\] (20)

By calculating the residue of \(A_{\mu}dV_{\mu}\) modulo \(Z_{\mu}\), we see that the equation (20) can also be written as

\[-CdV_i + dV_{i-1} = q_i^{i+1}dV_{n-1}, \quad i = 0, 1, \ldots, n - 1, \quad V_{-1} := 0\] (21)

However, the equation for which \(i = 0\) in (21) can be discarded since it is a consequence of the other equations and of the Cayley–Hamilton theorem for the companion matrix \(C\). This is realized by adding to the equation in (21) for which \(i = 0\) the equation for which \(i = 1\) multiplied with \(C\), then adding the equation for which \(i = 2\) multiplied with \(C^2\), and so on. In components, the equations (21) can then be written as

\[
\begin{align*}
-\frac{\partial V_i}{\partial q^2} + \frac{\partial V_{i-1}}{\partial q^1} &= q_i^{i+1}\frac{\partial V_{n-1}}{\partial q^1}, \\
-\frac{\partial V_i}{\partial q^3} + \frac{\partial V_{i-1}}{\partial q^2} &= q_i^{i+1}\frac{\partial V_{n-1}}{\partial q^2}, \\
&\vdots \\
q_i^{i+1}\frac{\partial V_i}{\partial q^1} + \cdots + q_i^n\frac{\partial V_i}{\partial q^n} &= q_i^{i+1}\frac{\partial V_{n-1}}{\partial q^n},
\end{align*}
\]

where \(i = 1, \ldots, n - 1\). Thus, we see that the generic \((m, m, 1)\)-case consists of \(m(m - 1)\) PDE’s for \(m\) dependent variables, and is therefore an overdetermined system when \(m > 1\), that nevertheless has non-trivial solutions, e.g., \(V_i = Z_i = q_i^{i+1}\). If we let \(U = V \ast W\), each entry \(U_a\) of the 1-column matrix \(U\) is a sum of terms \(P_{ij}V_iW_j\), where \(P_{ij}\) is a polynomial of degree at most \(m - 1\) in the coordinates \(q^1, \ldots, q^n\). We note also that for every solution \(V\) of equation (20), each term of the sum in equation (19) vanishes. Thus, every solution \(V\) in the generic \((m, m, 1)\)-case also solves the generic \((m, m, k)\)-case for arbitrary \(k\). The lowest order \(*\)-powers of the trivial solution \(\mu\) (or \([0, 1, 0, \ldots, 0]^T\) in matrix
notation) are given by:

\[ \mu_1^* = [0, 1, 0, \ldots, 0]^T \]
\[ \mu_2^* = [0, 0, 1, \ldots, 0]^T \]
\[ \vdots \]
\[ \mu_{m-1}^* = [0, 0, \ldots, 1]^T \]
\[ \mu_m^* = -[q^1, q^2, \ldots, q^m]^T \]
\[ \mu_{m+1}^* = [q^1 q^m, -q^1 + q^2 q^m, \ldots, -q^{m-1} + (q^m)^2]^T \]
\[ \mu_{m+2}^* = [q^1 (q^{m-1} - (q^m)^2), q^1 q^m + q^2 (q^{m-1} - (q^m)^2), \ldots, \]
\[ -q^{m-2} + q^m (2q^{m-1} - (q^m)^2)]^T \]

### 3.3 \( m < n \)

Suppose that \( Z_0, \ldots, Z_{m-1} \) are functionally independent and consider local coordinates \( q^1, \ldots, q^n \) such that \( q^1 = Z_0, \ldots, q^m = Z_{m-1} \), without specifying the other coordinates \( q^{m+1}, \ldots, q^n \). Now the relation (12) is equivalent to

\[ 0 = \sum_{i=0}^{k} \begin{bmatrix} C_i | 0 \end{bmatrix} A_i, \]

where \( \begin{bmatrix} C_i | 0 \end{bmatrix} \) denotes a \( m \times n \)-matrix constructed by writing \( n - m \) zero-columns right to the matrix \( C_i \). This condition determines the first \( m \) rows of \( A_0 \) uniquely, while the last rows as well as \( A_1, A_2, \ldots, A_k \) are arbitrary.

One should note that when \( m < n \), since \( Z_i \) are the only functions except \( V_i \) and \( W_i \) that appear in \( V^* W \), this product will not depend on certain coordinates unless \( V \) or \( W \) are themselves functions depending on these coordinates. Thus, a product of two trivial solutions will will never depend on these missing coordinates.

#### 3.3.1 \( (n, m, k) = (n, 2, 1) \)

We will now choose the lowest possible values for \( m \) and \( k \) and let \( n \) be arbitrary. We note that the \((3, 2, 1)\)-case, which is the simplest possible case for which \( m < n \), is included. In generic coordinates, the relation (12) is equivalent to the following relation between the components of the tensors \( A_0 \) and \( A_1 \):

\[
\begin{bmatrix} (A_0)^1_1 & \cdots & (A_0)^1_n \\ (A_0)^2_1 & \cdots & (A_0)^2_n \end{bmatrix} = C \begin{bmatrix} (A_1)^1_1 & \cdots & (A_1)^1_n \\ (A_1)^2_1 & \cdots & (A_1)^2_n \end{bmatrix},
\]

where \( C \) denotes the companion matrix of \( Z_\mu \). We see that, even though we restrict our attention to the case \( A_1 = I, n(n-2) \) components of \( A_0 \) can still be chosen arbitrary. However, for some choices of these components, the system (5) will not depend on some of the coordinates and can therefore be reduced to a lower dimensional problem with a smaller number of independent variables. We illustrate this phenomenon in the case \((3, 2, 1)\).
3.3.2 \((n, m, k) = (3, 2, 1)\)

According to the discussion above, we can assume that

\[
A_0 = \begin{bmatrix}
0 & x & 0 \\
-1 & y & 0 \\
0 & a & b & c
\end{bmatrix},
\]

in generic coordinates \((x, y, z)\) where \(a, b, c\) are arbitrary functions. By analyzing the corresponding system \([5]\), one can see that either all solutions \(V_0, V_1\) are constant with respect to the variable \(z\) and the system then reduces to the generic \((2, 2, 1)\)-case, or otherwise we must have \(y^2 > 4x\), \(b = a(c - y)\), and \(c = y \pm (1/2)\sqrt{y^2 - 4x}\), and the system can then be written as \(A_0 dV_0 = x dV_1\), or in components

\[
\begin{align*}
-x \frac{\partial V_0}{\partial y} + a(x, y, z) \frac{\partial V_0}{\partial z} &= x \frac{\partial V_1}{\partial x} \\
x \frac{\partial V_0}{\partial x} + y \frac{\partial V_0}{\partial y} + a(x, y, z) \left( -y \pm \frac{\sqrt{y^2 - 4x}}{2} \right) \frac{\partial V_0}{\partial z} &= x \frac{\partial V_1}{\partial y} \\
\left( \frac{y \pm \sqrt{y^2 - 4x}}{2} \right) \frac{\partial V_0}{\partial z} &= x \frac{\partial V_1}{\partial z},
\end{align*}
\]

where \(a = a(x, y, z)\) is an arbitrary function. Thus, we see that the generic \((3, 2, 1)\)-case involves an arbitrary function, which was not possible for the \((m, m, 1)\)-case. We note also that all solutions \(V\) of the generic \((2, 2, 1)\)-case also solve the generic \((3, 2, 1)\)-case. Since the multiplication coincides with the multiplication in the \((2, 2, 1)\)-case, we see that unless \(V\) or \(W\) depend on \(z\), the product \(V \ast W\) will not depend on \(z\). Especially, the solutions obtained by taking powers of the trivial solution \(\mu\) are again given by \([17]\).

3.3.3 \((n, m, k) = (n, 3, 2)\)

When we consider higher values of the number \(k\), the corresponding systems become harder to analyze. One reason is that when we increase \(k\) by one, we add a new tensor \(A_k\) which means that we add \(n^2\) new components. Another reason is that with higher values of \(k\), we get higher order polynomials in \(Z_i\). Already for \(k = 2\), such systems become quite hard to handle. In the general \((n, 3, 2)\)-case for example, the system \([6]\) can be written as

\[
\begin{align*}
0 &= A_0 dV_0 - Z_0 A_1 dV_2 - Z_0 A_2 dV_1 + Z_0 Z_2 A_2 dV_2 \\
0 &= A_0 dV_1 + A_1 dV_0 - Z_1 A_1 dV_2 - Z_1 A_2 dV_1 + (Z_1 Z_2 - Z_0) A_2 dV_2 \\
0 &= A_0 dV_2 + A_1 dV_1 + A_2 dV_0 - Z_2 A_1 dV_2 - Z_2 A_2 dV_1 + (Z_2^2 - Z_1) A_2 dV_2.
\end{align*}
\]

Even if we assume that \(A_2 = I\) and that \(q^{i+1} = Z_i\) are functionally independent, we still have \(2n^2\) arbitrary functions in the picture (the components of \(A_0\) and \(A_1\)). When the condition \([12]\), which in generic coordinates is equivalent to \(0 = \sum_{i=1}^{k} [C^i \mid 0] A_i\), is satisfied we still have \(n(2n - m)\) arbitrary functions.
3.4 $m > n$

When $m > n$, the generic case becomes more complicated than for $m \leq n$. Consider the generic case when $q^1 = Z_0, \ldots, q^n = Z_{n-1}$ are functionally independent and take $q^1, \ldots, q^n$ as local coordinates. In these coordinates we have

\[ Z' = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \partial_1 Z_n & \partial_2 Z_n & \cdots & \partial_n Z_n \\ \vdots & \vdots & \ddots & \vdots \\ \partial_1 Z_{m-1} & \partial_2 Z_{m-1} & \cdots & \partial_n Z_{m-1} \end{bmatrix}. \]

Thus, the relation (12), which in the generic cases for $m \leq n$ became a set of algebraic equations for the components of $A_i$, becomes now a complicated differential relation between the components of $A_i$ and the functions $Z_n, \ldots, Z_{m-1}$.

3.4.1 $(n, m, k) = (2, 3, 1)$

We consider the simplest case for which $m > n$, i.e., when $(n, m, k) = (2, 3, 1)$. The system (5) can then be written as

\[
\begin{align*}
0 &= A_0 dV_0 - Z_0 A_1 dV_2 \\
0 &= A_1 dV_0 + A_0 dV_1 - Z_1 A_1 dV_2 \\
0 &= A_1 dV_1 + (A_0 - Z_2 A_1) dV_2.
\end{align*}
\]

The condition $m > n$, implies that the functions $Z_0, Z_1, Z_2$ are functionally dependent. For the sake of convenience, we assume in the generic case that the functions $Z_0, Z_1$ are functionally independent and that $Z_2 = \phi(Z_0, Z_1)$ for some function $\phi$. We also assume that $A_1 = I$. Thus, in generic coordinates $x = Z_0$, $y = Z_1$, the condition (12) is satisfied if and only if $A := A_0$ is given by

\[ A = \begin{bmatrix} x \partial_x \phi & x \partial_y \phi \\ y \partial_x \phi - 1 & y \partial_y \phi \end{bmatrix}, \]

where $\phi$ satisfies the non-linear partial differential equations

\[
\begin{align*}
0 &= x \left( \frac{\partial \phi}{\partial x} \right)^2 - \phi \frac{\partial \phi}{\partial x} + y \frac{\partial \phi}{\partial x} \frac{\partial \phi}{\partial y} - \frac{\partial \phi}{\partial y} \\
0 &= 1 + x \frac{\partial \phi}{\partial x} \frac{\partial \phi}{\partial y} + y \left( \frac{\partial \phi}{\partial y} \right)^2 - \phi \frac{\partial \phi}{\partial y}.
\end{align*}
\]

Solutions of the equations (23) exist, for instance $\phi = ay - a^2 x + a^{-1}$ solves (23) for any non-zero real constant $a$. In the generic case, the system (22) is
equivalent to (note that the tensor $A$ is non-singular when $x \neq 0$)

\[
\begin{align*}
0 &= AdV_0 - xdV_2 \\
0 &= (xI - yA + \phi A^2 - A^3) dV_2 \\
0 &= dV_1 + (A - \phi I) dV_2.
\end{align*}
\]

The Cayley–Hamilton theorem together with the assumption that $\phi$ satisfies the equations (23) implies that the expression $xI - yA + \phi A^2 - A^3$ vanishes. Hence, the system (22) reduces to

\[
\begin{align*}
AdV_0 &= xdV_2 \\
dV_1 &= (\phi I - A) dV_2.
\end{align*}
\]

In the case when $\phi = ay - a^2x + a^{-1}$, the system (22) can explicitly be written as the overdetermined system

\[
\begin{align*}
0 &= a^2 x \frac{\partial V_0}{\partial x} + (ay^2 + 1) \frac{\partial V_0}{\partial y} + x \frac{\partial V_2}{\partial x} \\
0 &= -ax \frac{\partial V_0}{\partial x} - ay \frac{\partial V_0}{\partial y} + x \frac{\partial V_2}{\partial y} \\
0 &= -\frac{\partial V_1}{\partial x} + (ay + a^{-1}) \frac{\partial V_2}{\partial x} + (a^2 y + 1) \frac{\partial V_2}{\partial y} \\
0 &= -\frac{\partial V_1}{\partial y} - ax \frac{\partial V_2}{\partial x} + (a^{-1} - a^2x) \frac{\partial V_2}{\partial y}.
\end{align*}
\]

For this system, the $\ast$–product of trivial solutions gives in general non-trivial solutions. For example, when $V = [0, 1, 0]^T$ and $W = [0, 0, 1]^T$ (or simply $V_\mu = \mu$ and $W_\mu = \mu^2$ using the $\mu$–notation), their $\ast$–product becomes

\[
V \ast W = \begin{bmatrix}
-x \\
y \\
a^2x - ay - a^{-1}
\end{bmatrix}.
\]

### 3.5 Summary

The results about the possible structures of the system of linear PDE’s (5), and of the corresponding relation (7) (that characterizes the existence of $\ast$–multiplication), can for non-singular $A_k$ be summarized as follows:

1. $m = n$. Suppose that $Z_0, \ldots, Z_{m-1}$ are functionally independent. In the generic coordinates $q^i = Z_{i-1}$, the relation (7) is satisfied if and only if $A_C = A_0C^0 + \cdots + A_kC^k = 0$, and the system (11) can be written as

\[
0 = \sum_{i=1}^k (C^iV' - V'C^i) A_i,
\]
where the tensors $A_1, \ldots, A_k$ are arbitrary. In the generic case when $k = 1$, the system (11) takes the remarkably simple form $CV' = V'C$, for which every solution $V$ is also a solution in each generic $(n, m = n, k)$-case with arbitrary $k > 0$.

2. $m < n$. Suppose that $Z_0, \ldots, Z_{m-1}$ are functionally independent. In the generic coordinates $q^1 = Z_0, \ldots, q^m = Z_{m-1}, q^{m+1}, \ldots, q^n$ (not specified), the relation (7) is satisfied if and only if $0 = \sum_{i=0}^{k} [C^i | 0] A_i$. It is characteristic for the case $m < n$ that $*$-products of trivial solutions will remain constant in some variables.

3. $m > n$. This is the hardest case to analyze since the relation (7) in the generic coordinates $q^1 = Z_0, \ldots, q^n = Z_{n-1}$ becomes a differential relation for the functions $Z_{n+1}, \ldots, Z_{m-1}$, while it leads to algebraic equations in the previous cases.

4 Power series

As mentioned above, we can take power series of the trivial solution $\mu$ to build up more complicated solutions of (5). In the matrix notation, such power series have the form

$$P = \sum_{r=0}^{\infty} a_r C^r e_1,$$

where $a_r$ are real constants. We will now investigate when these power series define new solutions of the system (5), i.e. when they are convergent and the summation and derivation commutes so that they define genuine solutions of the first order systems of PDE’s (5).

The companion matrix $C$ can be factorized as $C = T J T^{-1}$, where the matrix $J$ has the Jordan canonical form, i.e. $J = \text{diag}(J_1, \ldots, J_s)$, where

$$J_s = \begin{bmatrix}
\lambda_s & 1 \\
 & \ddots & 1 \\
 & \ddots & \ddots & 1 \\
 & \ddots & \ddots & \ddots & 1 \\
\end{bmatrix},$$

and $\lambda_s$ is an eigenvalue of $C$ (note that the eigenvalues of $C$ coincide with the roots of the polynomial $Z_\mu$). Thus, the partial sums of the power series can be written as

$$P_N = \sum_{r=0}^{N} a_r C^r e_1 = T \left( \sum_{r=0}^{N} a_r J^r \right) T^{-1} e_1.$$
Lemma 4. Let $J_s$ be the Jordan block defined by (24). The entries of powers of $J_s$ can be given by

$$(J_s^r)_{ij} = \binom{r}{j-i} \lambda_s^{r+i-j}.$$ 

Proof. The proof follows immediately by induction over $r$. \hfill $\square$

From the lemma, it is clear that all non-zero elements of the matrix $\sum_{r=0}^N a_r J^r$ have the form $\sum_{r=0}^N a_r \binom{r}{i} \lambda_s^{r-i}$, where $\lambda_s$ is an eigenvalue of $C$. All power series $\sum_{r=0}^\infty a_r \binom{r}{i} \lambda_s^{r-i}$ have the same radius of convergence as $\sum_{r=0}^\infty a_r t^r$. Hence, we conclude that at every point of $Q$ where all eigenvalues of $C$ belongs to the open set $]-R,R[ \ (\text{where } 1/R = \limsup |a_n|^{1/n})$, the power series $\sum_{r=0}^\infty a_r J^r$, and therefore also $\sum_{r=0}^\infty a_r C^r e_1$, will be convergent. The question that now remains, is what is required for a convergent power series $\sum_{r=0}^\infty a_r C^r e_1$ to be a solution to (5). The following theorem gives a sufficient condition:

Theorem 5. Consider a power series $P = \sum_{r=0}^\infty a_r C^r e_1$. Let $D \subset Q$ be a domain in which each eigenvalue $\lambda_s$ of the matrix $C$ is real and $\lambda_s \in [-R+\epsilon,R-\epsilon]$ for some $\epsilon > 0$, where $1/R = \limsup |a_n|^{1/r}$. Assume also that the geometrical multiplicity of each eigenvalue is constant in $D$. Then the power series $P$ defines a solution of (5) in $D$.

Remark 3. The assumption about the geometrical multiplicity guarantees that the structure of the Jordan canonical form of $C$ is preserved in $D$ (note that the case with simple eigenvalues is included). There is no indication that this assumption, or the assumption about eigenvalues being real, are necessary for the conclusion in theorem 5 but without them the proof would be technically more complicated.

Proof. $P$ is a solution of (5) if and only if

$$0 = \sum_{i=0}^k C^i P' A_i = \sum_{i=0}^k C^i \left( \sum_{r=0}^\infty a_r C^r e_1 \right)' A_i.$$ 

Thus, since $C^r e_1$ is a solution for any $r$, we see that it is enough to prove that $P' = \lim_N P'_N$, or in components, $\partial_i P_j = \lim_N \partial_i P'_N$, $i,j = 1,\ldots,n$, where $P'_j$ and $P'_N$ denote the $j$'th element of $P$ and $P_N$, respectively. Since $P_N$ converges to $P$ in every point, we only have to prove that $\partial_i P'_N$ converges uniformly in $D$.

According to (25) and lemma 4, each $P'_N$ consists of a finite sum of terms of the form $f(q) \sum_r b_r \lambda_s^r$, where $\lambda_s$ is an eigenvalue of $C$, $f$ is a real-valued function, and the power series $\sum_r b_r \lambda_s^r$ has $R$ as its radius of convergence. Thus, $\partial_i P'_N$ consists of a finite sum of terms of the form $(\partial_i f(q)) \sum_r b_r \lambda_s^r + f(q) (\partial_i \lambda_s) \sum_r r b_r \lambda_s^{r-1}$, which converge uniformly in $D$. \hfill $\square$
According to theorem 5, if the companion matrix $C$ has simple eigenvalues and $\sum a_r t^r$ is a power series with infinite radius of convergence, then $\sum a_r C^r e_1$ is a globally defined power series solution of (5). Thus, for example, we can construct the power series solutions

$$\exp C := \left( \sum_{r=0}^{\infty} \frac{1}{r!} C^r \right) e_1$$

$$\sin C := \left( \sum_{r=1}^{\infty} (-1)^{r-1} \frac{1}{(2r-1)!} C^{2r-1} \right) e_1$$

$$\cos C := \left( \sum_{r=0}^{\infty} (-1)^r \frac{1}{(2r)!} C^{2r} \right) e_1.$$

We end the discussion about power series with some examples.

**Example 3.** To illustrate the mechanism of generating power series solutions, with respect to the $*-$multiplication, we return to the system (15) in the $(2, 2, 1)$-case. We assume that $A_\mu$ and $Z_\mu$ satisfy the relation (7) so that the system has $*-$multiplication, and we also assume that the roots $\lambda_1, \lambda_2$, of $Z_\mu$ are simple. The companion matrix can then be diagonalized as

$$C = TDT^{-1} = \begin{bmatrix} \lambda_2 & \lambda_1 & 0 & 0 \\ -1 & -1 & \lambda_2 & \lambda_1 \end{bmatrix} \begin{bmatrix} \lambda_2 & \lambda_1 \\ -1 & -1 \end{bmatrix}^{-1}.$$

A power series $\sum a_r C^r e_1$ defines a solution in any domain $D \subset \mathbb{Q}$ in which $|\lambda_1|, |\lambda_2| < 1/ \limsup |a_r|^{1/r} - \epsilon$, and can be written as

$$\sum a_r C^r e_1 = T \left( \sum a_r D^r \right) T^{-1} e_1$$

$$= \frac{1}{\lambda_1 - \lambda_2} \begin{bmatrix} \lambda_2 & \lambda_1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} \sum a_r \lambda_1^r \\ 0 \end{bmatrix} - \lambda_2 \begin{bmatrix} \sum a_r \lambda_2^r \\ 0 \end{bmatrix} \begin{bmatrix} -1 & -\lambda_1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$= \frac{1}{\lambda_1 - \lambda_2} \begin{bmatrix} \lambda_1 \sum a_r \lambda_1^r - \lambda_2 \sum a_r \lambda_2^r \\ \sum a_r \lambda_1^r - \sum a_r \lambda_2^r \end{bmatrix}.$$

Thus, for example we have

$$\exp C = \frac{1}{\lambda_1 - \lambda_2} \begin{bmatrix} \lambda_1 \exp (\lambda_2) - \lambda_2 \exp (\lambda_1) \\ \exp (\lambda_1) - \exp (\lambda_2) \end{bmatrix}.$$

In the case when the eigenvalues of $C$ are constant (which is the case for the Cauchy–Riemann equations), these power series will only provide trivial solutions. Hence, in order to obtain interesting solutions for this case, one has to build power series solutions from a non-trivial solution (for instance $(x, y)$ in the Cauchy–Riemann case). The other extreme case is the generic situation when the eigenvalues are functionally independent and $A_1$ is non-singular. Then, as
remark earlier, the general solution of 18 can be written as

\[ V = \frac{1}{\lambda_1 - \lambda_2} \left[ \begin{array}{c} \lambda_1 \phi(\lambda_2) - \lambda_2 \psi(\lambda_1) \\ \psi(\lambda_1) - \phi(\lambda_2) \end{array} \right]. \] (26)

It is a remarkable property that every analytic solution in the generic case can be expressed by power series of the trivial solution \((0, 1)\). Namely, if \(V\) is a solution of the form 26 where \(\phi(t) = \sum a_r t^r\) and \(\psi(t) = \sum b_r t^r\) are analytic, then \(V\) can be expressed in terms of the two simple solutions \((\lambda_1 - \lambda_2)^{-1}(\lambda_1, -1)\) and \((\lambda_1 - \lambda_2)^{-1}(-\lambda_2, 1)\), and of power series of the trivial solution \((0, 1)\):

\[ V = \frac{1}{\lambda_1 - \lambda_2} \left[ \begin{array}{c} \lambda_1 - 1 \\ -1 \end{array} \right] \ast \left( \sum a_r C^r e_1 \right) + \frac{1}{\lambda_1 - \lambda_2} \left[ \begin{array}{c} -\lambda_2 \\ 1 \end{array} \right] \ast \left( \sum b_r C^r e_1 \right) \]

Hence, a significant part of the solution set of 18 can be expressed by power series in trivial solutions.

**Remark 4.** The solutions \((\lambda_1 - \lambda_2)^{-1}(\lambda_1, -1)\) and \((\lambda_1 - \lambda_2)^{-1}(-\lambda_2, 1)\) have some remarkable properties. They are idempotent and their sum is the identity, i.e.,

\[
\left( \frac{\lambda_1}{\lambda_1 - \lambda_2}, \frac{-1}{\lambda_1 - \lambda_2} \right)^2 = \left( \frac{\lambda_1}{\lambda_1 - \lambda_2}, \frac{-1}{\lambda_1 - \lambda_2} \right), \\
\left( \frac{-\lambda_2}{\lambda_1 - \lambda_2}, \frac{1}{\lambda_1 - \lambda_2} \right)^2 = \left( \frac{-\lambda_2}{\lambda_1 - \lambda_2}, \frac{1}{\lambda_1 - \lambda_2} \right), \\
\left( \frac{\lambda_1}{\lambda_1 - \lambda_2}, \frac{-1}{\lambda_1 - \lambda_2} \right) + \left( \frac{-\lambda_2}{\lambda_1 - \lambda_2}, \frac{1}{\lambda_1 - \lambda_2} \right) = (1, 0).
\]

**Example 4.** Let us instead consider the case where \(m = 2\) and the eigenvalues of \(C\) coincide everywhere, \(\lambda_1 = \lambda_2 =: \lambda\), i.e., the functions \(Z_0\) and \(Z_1\) are functionally dependent and related as \(Z_1^2 = 4Z_0\). Then we can factorize \(C\) as 

\[ C = T J T^{-1} = \left[ \begin{array}{cc} 1 & -\lambda \\ 0 & 1 \end{array} \right] \left[ \begin{array}{cc} \lambda & 0 \\ 1 & \lambda \end{array} \right] \left[ \begin{array}{cc} 1 & \lambda \\ 0 & 1 \end{array} \right], \]

and a general power series solution can be written as

\[ \sum_r a_r C^r e_1 = T \left( \sum_r a_r J^r \right) T^{-1} e_1 = \left[ \frac{\sum_r (1 - r) a_r \lambda^r}{\sum_r r a_r \lambda^{r-1}} \right]. \]

For this example, the exponential power series produces the solution \(\exp(\lambda)(1 - \lambda, 1)\).

**Example 5.** In 3, a matrix equation of the form

\[ \nabla f = M \nabla g, \] (27)

where \(M\) is a constant matrix, is studied in an open convex domain of a real or complex vector space. Here \(f(x)\) and \(g(x)\) are scalar-valued functions defined
on the vector space, and \( \nabla f = [\partial_{x_1} f, \ldots, \partial_{x_n} f]^T \), where \( x = [x_1, \ldots, x_n]^T \) is a coordinate vector with respect some basis. We give below an example of an equation of the form \( (27) \) that admits a \(*-\)multiplication, and we show that solutions can be represented by power series of simple solutions.

Let \( V \) be a real vector space of dimension four, and assume that the matrix \( M \) in \( (27) \) has eigenvalues \( \lambda, \lambda, \alpha \pm i\beta \), where \( \lambda, \alpha, \) and \( \beta \) are real constants. By performing a linear change of variables \( x \to Ax \), where \( A \) is a constant matrix, the equation \( (27) \) transforms to \( \nabla f = A^{-T} MA^T \nabla g \). Thus, we can assume that \( M \) is in canonical real normal form. We assume that the eigenvalue \( \lambda \) has geometrical multiplicity one, so that \( M \) is given by

\[
M = \begin{bmatrix}
\lambda & 0 & 0 & 0 \\
0 & \lambda & 0 & 0 \\
0 & 0 & \alpha & \beta \\
0 & 0 & -\beta & \alpha
\end{bmatrix}.
\]

According to \( [3] \), the general analytic solution of \( (27) \) can be decomposed as

\( f = f_1 + f_2, \ g = g_1 + g_2 \), where \( (f_1, g_1) \) and \( (f_2, g_2) \) are solutions of

\[
\nabla f_1 = \begin{bmatrix}
\lambda & 1 \\
0 & \lambda
\end{bmatrix} \nabla g_1, \quad \text{and} \quad \nabla f_2 = \begin{bmatrix}
\alpha & \beta \\
-\beta & \alpha
\end{bmatrix} \nabla g_2,
\]

respectively. By changing to new dependent variables

\[
\hat{f}_1 = f_1 - \lambda g_1, \quad \hat{g}_1 = g_1, \quad \hat{f}_2 = f_2 - \alpha g_2, \quad \hat{g}_2 = \beta g_2,
\]

it becomes obvious that there is no restriction to assume that \( \lambda = \alpha = 0 \) and \( \beta = 1 \). We note that the equation for \( f_2 \) and \( g_2 \) then reduces to the Cauchy–Riemann equations. For this choice of \( M \), it is trivial to obtain the general analytic solution of \( (27) \):

\[
\begin{cases}
f_1 = x_3 \phi'(x_1) + \psi(x_1) \\
g_1 = \psi(x_1) + c
\end{cases}
\quad \begin{cases}
f_2 = \text{Re} \left( F(x_3 + i x_4) \right) \\
g_2 = \text{Im} \left( F(x_3 + i x_4) \right)
\end{cases},
\]

where \( \psi \) and \( \phi \) are arbitrary analytic functions of one real variable, and \( F \) is an arbitrary holomorphic function of one complex variable. We have already seen that the general solution of the Cauchy–Riemann equations can be represented as a power series of the simple solution \( x_3 + i x_4 \) with respect to a \(*-\)multiplication (coinciding with multiplication of holomorphic functions). Also the first system in \( (29) \) admits a \(*-\)multiplication since it can be written as a system of the form \( (5) \) where

\[
A_\mu = \begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix} + \mu I, \quad V_\mu = f_1 + \mu g_1, \quad Z_\mu = \mu^2,
\]

and the relation \( (7) \) is trivially satisfied since \( Z_\mu \) is constant. It is remarkable that also for this system, the general analytic solution can written as a power
series of simple solutions. Namely, let the functions $\phi$ and $\psi$, from the general solution, be analytic with power series representations $\psi(s) = \sum_r a_r s^r$ and $\phi(s) = \sum_r b_r s^r$, respectively. Then, the solution $(f_1, g_1)$ is given by

$$f_1 + \mu g_1 = \sum_r (a_r + \mu b_r) \ast (x_1 + \mu x_2)^r.$$

Thus, when $M$ is given by (28), we have shown that the general analytic solution of (27) can be obtained by taking power series, with respect to $\ast$—multiplication, of simple solutions of the subsystems (29). Also for general systems of type (27), solutions can be built up by taking $\ast$—power series of simple non-trivial solutions.

5 How to find systems with multiplication

As we have seen, the problem of finding systems of the form (5) that allow a $\ast$—multiplication is equivalent to finding a $(1, 1)$—tensor $A_\mu$ and a function $Z_\mu$ such that $A_\mu dZ_\mu \equiv 0$. In this general form, the problem is hard to handle, since in coordinates we may need to solve a system of complicated non-linear PDE’s.

One way to construct systems of PDE’s allowing multiplication, is to choose the function $Z_\mu$ first and treat it as fixed. The equation (7), or equivalently (12), becomes then a system of linear algebraic equations for the components of $A_1$, which is easier to solve. It constitutes a system of $mn$ equations for $n^2(k+1)$ unknown functions. The number $n(k+1) - m$ decides about how large family of solutions that can be found for each $Z_\mu$. When $n(k+1) - m < 0$, the number of unknown functions is less than the number of equations and we may not expect to find solutions for every choice of $Z$. On the other hand, for large values of $n(k+1) - m$ we get many families of solutions whenever the system is consistent. We illustrate this process of finding systems with a $\ast$—multiplication by an example.

**Example 6.** Let $Q$ be a manifold of dimension two with local coordinates $x, y$ and suppose that $Z_\mu = x + y\mu + xy\mu^2 + \mu^3$ and $k = 1$. When $Z_\mu$ is given, we can find all tensors $A_\mu = A_0 + \mu A_1$ such that the corresponding system (5) is equipped with a $\ast$—multiplication, i.e., such that (7) is satisfied.

$$0 = Z' A_0 + CZ' A_1$$

$$= \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & y \\ y & x & \end{bmatrix} \begin{bmatrix} (A_0)^1_1 & (A_0)^2_1 \\ (A_0)^1_2 & (A_0)^2_2 \\ \end{bmatrix} +$$

$$\begin{bmatrix} 0 & 0 & -x \\ 1 & 0 & -y \\ 0 & 1 & -xy \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ y & x \end{bmatrix} \begin{bmatrix} (A_1)^1_1 & (A_1)^2_1 \\ (A_1)^1_2 & (A_1)^2_2 \\ \end{bmatrix}.$$
The solution of this system of linear equations is given by

\[
A_0 = \begin{bmatrix}
  xy & x^2 \\
y^2 - 1 & xy
\end{bmatrix}, \quad A_1 = \begin{bmatrix}
  (x^2y + 1)f & (x^2y + 1)g \\
x(1-y^2)f & x(1-y^2)g
\end{bmatrix}
\]

where \(f = f(x, y)\) and \(g = g(x, y)\) are two arbitrary functions. The system (30) takes a form that is independent of the choice of \(f\) and \(g\):

\[
\begin{align*}
0 &= x(y + x^2)\frac{\partial V_0}{\partial x} + (y^2 - 1)\frac{\partial V_0}{\partial y} - x(1 + x^2 y)\frac{\partial V_2}{\partial x} + x^2(y^2 - 1)\frac{\partial V_2}{\partial y} \\
0 &= (1 + x^2 y)\frac{\partial V_0}{\partial x} + x(1 - y^2)\frac{\partial V_0}{\partial y} + x(y + x^2)\frac{\partial V_1}{\partial x} + (y^2 - 1)\frac{\partial V_1}{\partial y} - \\
y(1 + x^2 y)\frac{\partial V_2}{\partial x} + xy^2(y^2 - 1)\frac{\partial V_2}{\partial y} \\
0 &= (1 + x^2 y)\frac{\partial V_1}{\partial x} + x(1 - y^2)\frac{\partial V_1}{\partial y} + x^3(1 - y^2)\frac{\partial V_2}{\partial x} + \\
(y^2 - 1)(x^2 y + 1)\frac{\partial V_2}{\partial y}
\end{align*}
\]

Although there are two arbitrary functions \(f\) and \(g\), every choice give rise to the same system. Thus, for this particular choice of \(Z_\mu\) and parameters \((n, m, k)\), (30) is the only system with \(*\)-multiplication. Since \(m = 3\), the corresponding multiplication formula is given by (42).

In [4], the problem of finding systems with multiplication on Riemannian manifolds was studied for the special case when \(A_\mu = X + \mu I\) and \(Z_\mu = \det A_\mu\). With those restrictions, the problem of finding a system equipped with a multiplication reduces to finding a tensor \(X\) that satisfies the equation (41). The following families of solutions were found in [4]:

1. \(X = \tilde{J}^{-1}J\) where \(J\) and \(\tilde{J}\) are arbitrary special conformal Killing tensors.

   In this case, the multiplication of cofactor pair systems [5] is reconstructed.

2. Every \(X\) with a vanishing Nijenhuis torsion \(N_X = 0\). This follows from the remarkable relation

   \[
   2(Xd(\det X) - \det Xd(\tr X))_i = (N_X)_i^k C^j_k,
   \]

   where \(C = \text{cof} X\), and that \(X\) and \(X + \mu I\) share the same torsion. This result holds also when no metric is specified on the manifold \(Q\).

3. If \(X\) is a non-singular solution, then \(X^{-1}\) is a solution as well.

4. In [4], a method for constructing solutions \(X\) consisting of smaller blocks that satisfy (41) is presented. A similar result is valid for the more general equation (7):

   **Theorem 6.** Suppose that \(A_\mu\) and \(Z_\mu\) satisfies the relation (42) and let \(\tilde{A}_\mu\) be a \((1, 1)\)-tensor, and \(\tilde{Z}_\mu\) a function on a different manifold \(\tilde{Q}\), satisfying the relation \(\tilde{A}_\mu d\tilde{Z}_\mu \equiv 0 \pmod{\tilde{Z}_\mu}\). Then the tensor \(A_\mu \oplus \tilde{A}_\mu\) is a \((1, 1)\)-tensor on the manifold \(Q \times \tilde{Q}\) and it satisfies the relation

   \[
   (A_\mu \oplus \tilde{A}_\mu) d(Z_\mu \tilde{Z}_\mu) \equiv 0 \pmod{Z_\mu \tilde{Z}_\mu}.
   \]
Proof.

\[(A_\mu \oplus \tilde{A}_\mu)d(Z_\mu \tilde{Z}_\mu) = (d_Q(Z_\mu \tilde{Z}_\mu) + d_{\tilde{Q}}(Z_\mu \tilde{Z}_\mu))\]

\[= A_\mu d_Q(Z_\mu \tilde{Z}_\mu) + \tilde{A}_\mu d_{\tilde{Q}}(Z_\mu \tilde{Z}_\mu) = \tilde{Z}_\mu A_\mu d_Q Z_\mu + Z_\mu \tilde{A}_\mu d_{\tilde{Q}} \tilde{Z}_\mu\]

\[= Z_\mu \tilde{Z}_\mu \alpha \equiv 0 \pmod{Z_\mu \tilde{Z}_\mu},\]

where \(\alpha\) is a certain 1-form, and \(d_Q\) and \(d_{\tilde{Q}}\) denote the exterior differential operator on \(Q\) and \(\tilde{Q}\), respectively.

5. If the non-singular tensor \(X\) is diagonal in the coordinates \(q^1, \ldots, q^n\), then \(X\) satisfies the equation (4) if and only if \(X = \text{diag}(X_1, \ldots, X_s)\), where \(X_a = \text{diag}(\phi_a, \ldots, \phi_a)\) is a square diagonal matrix of size \(n_a\) and \(\phi_a = \phi_a(q^{n_1+\cdots+n_a-1+1}, \ldots, q^{n_1+\cdots+n_a})\) is an arbitrary (sufficiently regular) function, depending only on the specified coordinates.

As we have seen, there are several ways to construct system of partial differential equations with multiplication of solutions. Nevertheless, for any given system of PDE’s, it is in general hard to settle whether it admits a \(*-\)multiplication.

6 Conclusions

The \(*-\)multiplication constitutes a powerful method for generating, in a pure algebraic way, new solutions from known solutions of certain linear systems of PDE’s. Especially by taking trivial solutions, we can construct large families of non-trivial solutions of systems for which non-trivial solutions are hard to obtain by other methods. By generalizing the ideas from \(\mathbb{H}\), we have significantly extended the class of systems of PDE’s admitting \(*-\)multiplication. By identifying which elements of the construction of systems of PDE’s with multiplication that are relevant, we have obtained much better understanding of the nature of the \(*-\)multiplication. Our insight into the mechanism of the multiplication has been obtained due to the matrix formulation of the problem and of encoding the Euclidean algorithm as a matrix polynomial (section 2.1), and due to the effective construction of power series solutions (section 4).

There are still many questions regarding the \(*-\)multiplication which it may be worth to study. Some examples of such questions are:

1. Which solutions can be represented as power series of trivial solutions, and when do they constitute all solutions? What can we say about power series of non-trivial but simple solutions (compare with the Cauchy–Riemann equations where all holomorphic functions are represented by power series of a linear polynomial solution)?
2. In section 5 the problem of finding systems equipped with $\ast-$multiplication was treated. But the opposite problem is also interesting. Given a linear system of PDE’s, determine if it has a multiplicative structure on the space of solutions.

3. Theorem 1 characterizes all systems that admit $\ast-$multiplication. We have also presented some typical (generic) systems with multiplication. However, in order to gain a better understanding of the systems that admit multiplication, it would be desirable to have some natural principle of classifying these systems.

4. In example 5 we saw that a particular matrix equation of the form $\nabla f = M \nabla g$ admits a $\ast-$multiplication and that the general analytic solution can be expressed by power series, with respect to the $\ast-$multiplication, of simple solutions. This is a much more general property, that can be generalized to every equation $\nabla f = M \nabla g$ where $M$ is a constant matrix with either real or complex entries. The work on this problem is already in progress, and the results are being prepared for publication.

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References

[1] S. Benenti. Special symmetric two-tensors, equivalent dynamical systems, cofactor and bi-cofactor systems. *Acta Appl. Math.*, 87(1-3):33–91, 2005.

[2] M. Crampin and W. Sarlet. A class of nonconservative Lagrangian systems on Riemannian manifolds. *J. Math. Phys.*, 42(9):4313–4326, 2001.

[3] M. Jodeit, Jr. and P. J. Olver. On the equation $\nabla f = M \nabla g$. *Proc. Roy. Soc. Edinburgh Sect. A*, 116(3-4):341–358, 1990.

[4] J. Jonasson. The equation $X \nabla \det X = \det X \nabla \text{tr} X$, multiplication of cofactor pair systems, and the Levi-Civita equivalence problem. *J. Geom. Phys.*, 57(1):251–267, 2006.

[5] H. Lundmark. *Newton systems of cofactor type in Euclidean and Riemannian spaces*. PhD thesis, Matematiska institutionen, Linköpings universitet, 2001. Linköping Studies in Science and Technology. Dissertations. No. 719.

[6] H. Lundmark. Higher-dimensional integrable Newton systems with quadratic integrals of motion. *Stud. Appl. Math.*, 110(3):257–296, 2003.
[7] H. Lundmark and S. Rauch-Wojciechowski. Driven Newton equations and separable time-dependent potentials. *J. Math. Phys.*, 43(12):6166–6194, 2002.

[8] S. Rauch-Wojciechowski, K. Marciniak, and H. Lundmark. Quasi-Lagrangian systems of Newton equations. *J. Math. Phys.*, 40(12):6366–6398, 1999.

[9] S. Rauch-Wojciechowski and C. Waksjö. Stäckel separability for Newton systems of cofactor type, 2003. arXiv:nlin.SI/0309048.

[10] S. Wojciechowski. Review of the recent results on integrability of natural Hamiltonian systems. In *Systèmes dynamiques non linéaires: intégrabilité et comportement qualitatif*, volume 102 of *Sém. Math. Sup.*, pages 294–327. Presses Univ. Montréal, Montreal, QC, 1986.