Abstract

We deduce a kernel that allows the Moyal quantization of the cylinder (as phase space) by means of the Stratonovich–Weyl correspondence.

1 Introduction

Moyal’s work \cite{Moyal1949} showed how the Weyl correspondence \cite{Weyl1927} may develop Quantum Mechanics (QM) as a theory of functions on phase space. The algebra of these functions is endowed with a non-commutative product (star or Moyal product), while the physical states are represented by the Wigner functions \cite{Wigner1932}.

This formulation of QM has not had a success like those of Heisenberg, Schrödinger or Feynman. Difficulties like the extension of the theory to particles with spin or to relativistic particles were some of the reasons for this lack of success. In the last years, these difficulties have been solved \cite{Schwinger1951}, \cite{Wightman1965} by using a new approximation to the problem: the Stratonovich–Weyl (SW) correspondence.

The seminal ideas of this procedure appeared many years ago in Ref. \cite{Marmo1978}. The SW correspondence is applied to physical systems with a (connected) Lie group of symmetries, \( G \). The basic point is the SW kernel, that maps the points of a coadjoint orbit (phase space) of \( G \) into the set of operators acting on the carrier Hilbert space of the projective unitary representation associated to this coadjoint orbit.

However, all the cases studied in the above mentioned papers correspond to free particles. The method has been applied in more general situations using central extensions of the symmetry group, since in some cases, these extensions can be interpreted as constant forces. Thus, some particular interactions have been incorporated to the theory\cite{Simoni1995}, \cite{Simoni1996}. On the other hand, in recent works \cite{Olmo1996}, \cite{Martín1996}, the problem of contracting representations of groups has been studied for some kinematical groups and the results have been applied to the contraction of SW kernels. In physical terms, we have studied the contraction of classical systems together with the contraction of their quantized counterparts.

Nevertheless, not all the results have been satisfactory. In Ref. \cite{Martín1996} one of the groups under consideration was the euclidean group of the plane, and the construction of the SW kernel for one of the coadjoint orbits of this group (a cylinder) failed. Till now all

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subsequent attempts have been useless. In this work we obtain a SW kernel for that coadjoint orbit of the euclidean group by means of a constructive method.

The paper is organized as follows. Section 2 is devoted to present two short reviews about Moyal quantization and the SW correspondence. In Section 3 we introduce the SW kernel for the cylinder and give the basic hints for its construction. The paper ends with some remarks and conclusions.

2 Moyal quantization

From the beginning of QM it has had a great interest on reinterpreting the standard formulation in terms of Hilbert spaces and operators using the “classical language” of phase spaces and functions defined on them, trying to reduce it to a classical statistical theory. Moyal in 1949 [1] took into account this philosophy and showed how to construct such a theory for the case of euclidean phase space, i.e., $\mathbb{R}^{2n}$, making use of the Weyl correspondence [2] and the Wigner functions [3] (see [13] for a review).

2.1 Moyal formulation of Quantum Mechanics

Moyal’s formulation considers both observables and states as (generalized) functions on a given phase space $M$, in such a way that we are able to compute expected values by integrating over the phase space as in classical statistical mechanics

$$\langle f \rangle_\rho = \frac{\int_M f(u) \rho(u) du}{\int_M \rho(u) du} \quad (2.1)$$

The process of quantization is introduced considering an associative but non commutative “twisted product”

$$(f \times h)(u) = \int_M \int_M L(u, v, w) f(v) h(w) dv dw, \quad (2.2)$$

where $L(u, v, w)$ is a non-local integral kernel to be defined.

The problem of quantization is much more clarified when the system under study has a Lie group $G$ as set of symmetries. This group acts on the phase space by symplectomorphisms, and induces a compatibility condition on the twisted product $(f^g \times h^g = (f \times h)^g$, with $f^g(u) = f(g^{-1}u)$) that is reflected on the trikernel $L$ in the following covariance condition

$$L(g \cdot u, g \cdot v, g \cdot w) = L(u, v, w), \quad \forall g \in G. \quad (2.3)$$

In this case the (elementary) classical system can be identified with a coadjoint orbit of $G$ (or of a central extension of it) which will be the phase space, and the quantum analog (in the standard sense) with a projective unitary representation of $G$. The passage from the classical to the quantum system is easily achieved with the aid of the Kirillov theory [14] for the case of nilpotent groups, in other cases the problem is more difficult.

A link between this geometric quantization and Moyal’s one is provided by the SW correspondence which is explained in the next subsection.
2.2 The Stratonovich–Weyl correspondence

The SW correspondence contains, as a particular case, the Weyl correspondence rule assigning linear operators on the standard Hilbert space of QM to functions defined on a flat phase space. In this sense, we can consider the SW correspondence as a generalization of Weyl’s one, that allows us to extend Moyal QM to other situations (spin, relativistic systems, systems under interaction, etc.). It is defined by means of the SW kernel, \( \Omega \), through the integral formula

\[
A = \int_O f(u)\Omega(u)d\mu(u).
\]

The construction of \( \Omega \) exploits the fact that the physical system under study has a (connected) Lie group \( G \) as symmetry group. The idea is to consider a projective unitary irreducible representation (PUIR), \( U \), of \( G \) in a separable Hilbert space, \( H \), and a coadjoint orbit, \( O \), of \( G \) (taken as phase space) associated with it. The SW kernel is a mapping transforming each point \( u \) of \( O \) into an operator, \( \Omega(u) \), acting on \( H \) and verifying the following properties:

1. \( u \mapsto \Omega(u) \) is one to one.
2. \( \Omega(u) \) is selfadjoint, \( \forall u \in O \).
3. \( \text{tr}[\Omega(u)] = 1, \forall u \in O \). This trace is usually defined in a generalized sense.
4. Traciality:

\[
\int_O \text{tr}[\Omega(u)\Omega(v)]\Omega(v)d\mu(v) = \Omega(u),
\]

where \( \mu \) is the \( G \)-invariant measure on \( O \). This property means that \( \text{tr}[\Omega(u)\Omega(v)] \) behaves like a Dirac delta \( \delta(u - v) \) with respect to the measure \( \mu(v) \).

5. Covariance:

\[
U(g)\Omega(u)U(g^{-1}) = \Omega(gu), \forall g \in G, \forall u \in O,
\]

with \( gu \) the transformed point of \( u \) by the coadjoint action of \( g \).

The construction of \( \Omega \) presents some problems. The first one is related with the use of PUIR’s: usually, it is easier to handle with linear representations than with projective ones. This difficulty is avoided by considering another group, \( \overline{G} \), that linearizes the problem. The representation or splitting group \( \overline{G} \) of \( G \) is defined as the minimal connected and simply connected central extension of \( G \) such that any PUIR of \( G \) can be lifted to a linear unitary irreducible representation (LUIR) of \( \overline{G} \) and, reciprocally, every LUIR of \( \overline{G} \) provides a PUIR of \( G \).

The second problem appears as a consequence of the necessary relationship between coadjoint orbits and representations. They can be considered as the classical and quantum version, respectively, of the “same” physical elementary system, and therefore we should assign to each LUIR of \( \overline{G} \) a coadjoint orbit of \( \overline{G} \). This association is carried out by the method of Kirillov [14] for constructing induced representations in the case of nilpotent
groups. For non nilpotent groups the association is not immediate, but can be achieved in many cases.

The property of traciality permits to obtain an inversion formula for the correspondence

\[ W_A(u) \equiv \text{tr}[A\Omega(u)] = \int_O f(v)\text{tr}[\Omega(u)\Omega(v)]d\mu(v) = f(u). \quad (2.7) \]

The function \( W_A(u) \) is usually called the Wigner function of \( A \), with reference to the functions obtained by inverting the Weyl mapping, as Moyal pointed out. Traciality also yields the following expression

\[ \text{tr}[AB] = \int_O W_A(u)W_B(u)d\mu(u), \quad (2.8) \]

allowing to obtain quantum averages as in classical statistical mechanics, the centerpiece of Moyal’s formulation.

Physical calculations based on the SW correspondence are made by means of a non-commutative product, the so-called star or twisted product, for generalized functions on phase space. This is equivalent to the product of operators on its corresponding Hilbert space. We can define the twisted product of two functions \( f(u) \) and \( g(u) \) on \( O \) as

\[ (f \ast g)(u) = \int_O \int_O \text{tr}[\Omega(u)\Omega(v)\Omega(w)] f(v)g(w)d\mu(v)d\mu(w). \quad (2.9) \]

It is easy to verify

\[ (W_A \ast W_B)(u) = W_{AB}(u), \quad (2.10) \]

and

\[ \int_O (f \ast g)(u)d\mu(u) = \int_O f(u)g(u)d\mu(u). \quad (2.11) \]

The term \( \text{tr}[\Omega(u)\Omega(v)\Omega(w)] \) is called the tri-kernel of the SW correspondence and is the main ingredient for quantization, together with the coadjoint orbit (phase space).

There is no canonical way of constructing SW kernels and no theoretical result is known assuring the existence or unicity of these kernels. However, for many physical systems the problem has been solved (Ref. [6], [8]) following three simple steps: 1) Choose an arbitrary point \( u_0 \) of \( O \) as origin. 2) Make an Ansatz for a selfadjoint operator, \( \Omega(u_0) \), of trace one (with respect to a suitable trace). 3) Finally, define the kernel on the whole \( O \) by

\[ \Omega(u) = \Omega(gu_0) = U(g)\Omega(u_0)U(g^{-1}), \quad (2.12) \]

where \( g \) is an element of \( \mathcal{C} \) such that \( gu_0 = u \). Note that this kernel is well defined if and only if

\[ \Omega(u_0) = U(\gamma)\Omega(u_0)U(\gamma^{-1}), \quad \forall \gamma \in \Gamma_{u_0}, \quad (2.13) \]

where \( \Gamma_{u_0} \) is the isotopy group of \( u_0 \), i.e., \( \Gamma_{u_0} = \{ \gamma \in \mathcal{C} \mid \gamma u_0 = u_0 \} \). This property, proved in Ref. [9], implies that \( \Omega(u) \), defined as above, is covariant. Reciprocally, if \( \Omega(u) \) is covariant, the latter property holds. Remark that the covariance property guarantees that the SW kernel is well defined on the coadjoint orbit \( O \), in other words, it is independent of the choice of a section from \( O \) on \( \mathcal{C} \).

**Example 2.1.-** We can give an interesting and simple example to improve the understanding of the previous construction for the standard quantum theory.
The basic tool in the Moyal formulation of QM [15, 16] is the twisted product (also Moyal product) for functions on phase space. This product can be defined by using the Weyl mapping, i.e., a linear isomorphism between the space of the above mentioned functions and the space of the operators on a standard Hilbert space. The Weyl mapping can be introduced through the Grossmann–Royer operators [17, 18], which are defined as follows

\[
K(q, p)\varphi(x) = 2^n e^{2i p \cdot (x - q)} \varphi(2q - x),
\]

where the standard IR^{2n} phase space with canonical coordinates (q, p) is assumed. These operators act as integral kernels in such a way that to a function \( f \) corresponds the operator

\[
W(f) = \frac{1}{2\pi} \int_{IR^{2n}} f(q, p)K(q, p)dq dp.
\]

The mapping is invertible, so the Moyal product can be defined by

\[
f \ast g = W^{-1}(W(f)W(g)),
\]

whose explicit expression is

\[
(f \ast g)(u) = \frac{1}{\pi} \int_{IR^{4n}} f(v)g(w) \exp[i(u J v + v J w + w J u)]dv dw,
\]

where \( J \) is the matrix \( \left( \begin{array}{cc} 0 & I_n \\ -I_n & 0 \end{array} \right) \), \( I_n \) is the \( n \times n \) identity, and \( u, v \) and \( w \) stand for (q, p), (q', p') and (q'', p''), respectively.

Now, we can construct the SW correspondence for the Heisenberg group \( H^{2n+1} \), i.e., the set \( IR^{2n+1} \) endowed with the following composition law

\[
(a, b, c)(a', b', c') = (a + a', b + b', c + c' + \frac{1}{2}(a \cdot b' - a' \cdot b)),
\]

with \( a, a', b, b' \in IR^n \) and \( c, c' \in IR \).

The corresponding Lie algebra, \( \mathcal{H}^{2n+1} \), is a (2n + 1)–dimensional one, generated by \( I, Q \) and \( P \). In the standard formulation of QM these generators are represented as the identity, position and momentum operators on \( L^2(IR^n) \), respectively, with nonvanishing commutation relations \([Q_i, P_j] = iI\delta_{ij}\).

The coadjoint action is given by

\[
x' = x + zb, \quad y' = y - za, \quad z' = z,
\]

where (x, y, z) are the coordinates of a point of \( (\mathcal{H}^{2n+1})^* \) in a basis dual of the basis \{Q_i, P_i, I\}.

The coadjoint orbits of dimension greater than zero are all of them isomorphic to \( IR^{2n} \), and yield the same kind of induced representations. A LUIR of \( H^{2n+1} \) associated to the coadjoint orbit specified by \( z = 1 \) is

\[
[U(a, b, c)\varphi](\xi) = \exp[-i(c + b \cdot \xi + \frac{1}{2}a \cdot b)] \varphi(a + \xi),
\]
where \( \varphi \in L^2(\mathbb{R}^n) \).

Taking \((q = x/z, \ p = y)\) as canonical coordinates on the orbit and choosing the point \((0, 0)\) as origin \((u_0)\) we can make the Ansatz

\[
[\Omega(u_0)\varphi](\xi) = 2^n \varphi(-\xi),
\]

and obtain from it the SW kernel and the trikernel, which coincide with (2.14) and the integral kernel in (2.17), respectively.

3  Stratonovich–Weyl kernel for the cylinder

In this section we will finally accomplish the quantization of the cylinder (considered as a coadjoint orbit of the euclidean group) by means of the construction of an appropriate SW kernel. We start presenting some basic notions about the euclidean group that will be used later. A brief review about the preceding attempt [8] for quantizing the cylinder is also included for the sake of completeness.

3.1  The Euclidean group of the plane

The euclidean group \(E(2)\) is the group of motions of the plane, i.e., the set of transformations leaving invariant the euclidean length in \(\mathbb{R}^2\), roughly speaking, translations and rotations.

The corresponding Lie algebra, \(e(2)\), has dimension three and is spanned by the infinitesimal generators of translations, \(P_1\) and \(P_2\), and rotations around an axis perpendicular to the plane, \(J\). Their commutation relations are

\[
[J, P_i] = \epsilon_{ij} P_j, \quad [P_1, P_2] = 0. \quad i, j = 1, 2,
\]

where \(\epsilon_{ij}\) is the totally skewsymmetric tensor. The algebra \(e(2)\) admits a maximal non-trivial central extension by \(\mathbb{R}\), associated to a new non-zero Lie bracket: \([P_1, P_2] = I\), where \(I\) is the new central generator.

Integrating the preceding commutators we get the group law for the extended euclidean group \(\widetilde{E}(2)\),

\[
g'g = (\eta', a', \phi')(\eta, a, \phi) = (\eta' + \eta + \frac{1}{2}a^\phi \times a, a' + a^\phi, \phi' + \phi),
\]

where \(g \equiv (\eta, a, \phi) = e^{\eta I} e^{a P} e^{\phi J}\) \((\eta \in \mathbb{R}, \ a \in \mathbb{R}^2, \ \phi \in [0, 2\pi])\), and \(a^\phi\) stands for the transformed of the vector \(a\) by a rotation of angle \(\phi\).

The coadjoint action of \(\widetilde{E}(2)\) on \(\widetilde{e}(2)^*\), the dual of \(\widetilde{e}(2)\), can be readily computed. Its expression, using coordinates \((\beta, p, j)\) in the dual basis to \((I, P, J)\) is

\[
\beta' = \beta, \quad p' = p^\phi + \beta a^\phi, \quad j' = j + a \times p^\phi + \frac{1}{2} \beta a^2.
\]

This action splits \(\widetilde{e}(2)^*\) into several orbits, which can be classified according to the invariants \(\beta\) and \(p^2 - 2\beta j\) as follows:
(i) $\beta \neq 0$, this kind of orbits are 2D paraboloids.

(ii) $\beta = 0$ and $p^2 = r^2 \neq 0$, these are 2D cylinders parallel to the $j$-axis, which will be denoted by $O_r$.

(iii) $\beta = 0$ and $p^2 = 0$, each point $(0, 0, j)$ is a zero dimensional orbit.

Orbits of type (i) were studied in [8] where their complete Moyal quantization program was accomplished. However, for the second kind of them the results were not completely satisfactory because the proposed kernel did not verify traciality. In subsect. 3.3 we will present a suitable SW kernel for these orbits. The third kind of orbits are meaningless from the physical point of view because they represent systems without dynamics.

### 3.2 Previous results about cylinder quantization

The $\tilde{E}(2)$-homogeneous space can be endowed naturally with a symplectic structure. A set of canonical coordinates for this structure is $\alpha = \text{Arg}(p)$ and $j$, i.e., $\{\alpha, j\} = 1$ as is easy to check.

The representation $U_r$ associated to $O_r$ is obtained by Kirillov’s theory [14]

$$[U_r(\eta, a, \phi, \psi]) = e^{i\alpha t} \psi(\theta - \phi), \quad (3.4)$$

where $t = r(\cos \theta, \sin \theta)$ and $\psi$ belongs to $L^2(S^1)$.

In order to construct the kernel $\Omega$ we take into account that in other physical situations [8] parity-like operators (2.21) have played a remarkable role in the definition of the SW kernel at the origin point of the orbit. Hence, it is reasonable to consider in this case the operator

$$[\Omega(0, 0)\psi](\theta) = N\psi(-\theta), \quad (3.5)$$

where $(0, 0)$ are the canonical coordinates of the chosen origin point, and $N$ is an arbitrary normalization constant. This operator verifies property (2.6) and, henceforth, it induces a well defined object on the whole orbit:

$$[\Omega(\alpha, j)\psi](\theta) = Ne^{2ij\sin(\alpha - \theta)}\psi(2\alpha - \theta). \quad (3.6)$$

Operators just calculated are self-adjoint and of unit trace when $N = 2$. In addition, they satisfy the covariance property but, unfortunately, they do not satisfy traciality, i.e., we have

$$\text{tr}[\Omega(\alpha', j')\Omega(\alpha, j)] = 4\pi \delta(\alpha - \alpha')J_0(2(j - j')), \quad (3.7)$$

with $J_0$ the first-kind Bessel function.

Other parity-like operators as

$$[\Omega(0, 0)\psi](\theta) = N\psi(\theta + \pi), \quad \text{or} \quad [\Omega(0, 0)\psi](\theta) = N\psi(-\theta + \pi) \quad (3.8)$$

lead to lose even the covariance.
3.3 SW kernel for the cylinder

A new approach to this subject consists in taking an arbitrary operator at the origin point of the orbit and then to impose the conditions 1)–5) to it. The most restrictive conditions are covariance and traciality. Next proposition is the result of requiring covariance.

**Proposition 3.1.-** The following statements are equivalents:

(i). $\Omega$ is covariant.

(ii). $U(\gamma)\Omega(u_0)U(\gamma^{-1}) = \Omega(u_0)$, $\forall \gamma \in \Gamma_{\mu_0}$.

(iii). $[U(X), \Omega(u_0)] = 0$, $\forall X \in \text{Lie}(\Gamma_{\mu_0})$.

**Proof.** The equivalence of (i) and (ii) can be find in Ref. [8]. We leave the rest of the demonstration to the reader.

For the case we are interested in, the isotopy group $\Gamma_{\mu_0}$ is generated by $P_1$. The space $L^2(S^1)$ admits a discrete basis given by the functions $\{f_n(\theta) = \frac{1}{\sqrt{2\pi}} e^{in\theta}; \ n \in \mathbb{Z}\}$ which, hereafter, will be written in the bracket notation $\{|n\rangle; \ n \in \mathbb{Z}\}$.

Substituting $\Omega(u_0)$ by a generic operator $A$ in statement (iii) of Prop. 4.1, and computing $\langle r| [P_1, A]|s\rangle$ we obtain

$$\langle r| [P_1, A]|s\rangle = A_{s,r+1} + A_{s,r-1} - A_{s+1,r} - A_{s-1,r} = 0. \quad (3.9)$$

Using techniques from the theory of finite difference equations we find the general solution of the preceding system

$$A_{r,s} = a_{r+s} + b_{r-s}, \quad (3.10)$$

where $a$ and $b$ are arbitrary functions on $\mathbb{Z}$, that can also be considered as Fourier coefficients of two functions $a$ and $b$ on $S^1$.

The explicit action of $A$ on a generic function $f \in L^2(S^1)$ is

$$[Af](\theta) = a(\theta)f(\theta) + b(\theta)f(\theta). \quad (3.11)$$

Thus, we obtain the more general covariant operator-valued function on the cylinder

$$[\Omega(\alpha, j)f](\theta) = e^{2ij\sin(\theta-\alpha)}a(\theta-\alpha)f(2\alpha-\theta) + b(\theta-\alpha)f(\theta). \quad (3.12)$$

Imposing the hermiticity property to this operator we get that the function $b$ must be zero and that $a$ verifies the relation $a(-\theta) = \overline{a(\theta)}$, where the bar means complex conjugation.

The SW correspondence associated to (3.12), once made $b = 0$, maps the set of operators on the Hilbert space $L^2(S^1)$ into the set of functions on the cylinder. These operators can be written in terms of the transition operators $P_{n,m} = |m\rangle\langle n|$. The associated symbol to $P_{n,m}$ is a function that can be factorized in the following way

$$W_{n,m}(\alpha, j) = \Theta_{m-n}(\alpha)L_{m+n}(j), \quad (3.13)$$

with

$$\Theta_n(\alpha) = \frac{1}{\sqrt{2\pi}} e^{in\alpha}, \quad L_n(j) = \int_{S^1} \frac{d\theta}{2\pi} e^{2ij\sin \theta} a(\theta)e^{-in\theta}. \quad (3.14)$$

Note that if we choose $a(\theta) = 1$, then we reproduce the case given by (3.3) and the term $(3.14b)$ reduces essentially to a Bessel function, i.e., $L_n(j) = J_n(2j)$. 

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Traciality property (2.5) can be rewritten, more conveniently for our purposes, in terms of the functions $W_{n,m}$ as
\[ \int d\mu(v) K(u,v) W_{n,m}(v) = W_{n,m}(u), \] (3.15)
where $K(u,v) = \text{tr}[\Omega(u)\Omega(v)]$. This last expression allows to interpret $K$ as a reproducing kernel on the space spanned by the functions $W_{n,m}$. Note that $K(u,v) = \sum_{l,k} W_{l,k}(u) W_{k,l}(v)$. Using covariance and the factorization (3.13), condition (3.15) can be reduced to
\[ \sum_s L_{2s+r}(u_0) \langle L_{2s+r} | L_r \rangle = L_r(u_0). \] (3.16)
Finally, a sufficient condition in order to get traciality is to impose the following condition on $a$:
\[ |a(\theta)|^2 + |a(\theta + \pi)|^2 = 4 |\cos\theta|. \] (3.17)

Summarizing, we can say that the operators defined by
\[ [\Omega(\alpha,j)\psi](\theta) = e^{2ij\sin(\theta - \alpha)} a(\theta - \alpha) \psi(2\alpha - \theta), \] (3.18)
with $a$ satisfying $a(-\theta) = \overline{a(\theta)}$ and (3.17), are covariant, self-adjoint and tracial.

The following two remarks concern the requirements about trace, 3), and injectivity, 1), that any SW kernel ought to verify:

i) The trace of the operator $\Omega(\alpha,j)$ is equal to $\frac{1}{2} a(0)$, so we must take $a(0) \neq 0$ in order to get a finite trace.

ii) Function $a$ can be written as
\[ a(\theta) = 2\sqrt{|\cos\theta|} \cos(\frac{\pi}{4} + h(\theta)) e^{i\varphi(\theta)}, \] (3.19)
where the functions $h : S^1 \rightarrow [-\frac{\pi}{4}, \frac{\pi}{4}]$ and $\varphi : S^1 \rightarrow (-\pi, \pi]$ satisfy the conditions $h(\pi + \theta) = -h(\theta)$ and $\varphi(-\theta) = -\varphi(\theta)$. When $h \neq 0$ the SW kernel is injective. In the case $h = 0$ we still have two possibilities depending on the value of $\varphi(\theta + \pi) - \varphi(\theta)$: it can be different from or equal to $2\sin\theta$. The kernel $\Omega$ is injective in the first situation, but is not in the second one.

4 Conclusions

The problem of the cylinder quantization has been solved by the SW procedure once the SW kernel was known. The constructive nature of the method, used here for obtaining the SW kernel, allows to get the general solution imposing successively the appropriate conditions to have suitable kernels, without making any Ansatz about the value of the kernel at a point. Our deduction of the SW kernel is strongly based on Prop. 3.1 by taking advantage of the infinitesimal version for covariance property established in it.

On the other hand, it is worthy to note that we have found not a single kernel but a whole family of SW kernels depending on a function. An open problem to investigate is whether different kernels give different quantizations or not. Another researching direction is to impose new conditions in order to guarantee the unicity of the kernel. Work along both lines is in progress.
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