Study on the non-periodicity of the generalized Thue-Morse sequences generated by cyclic permutations.

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Abstract

We generalize the Thue-Morse sequence (the generalized Thue-Morse sequences), and consider the necessary-sufficient condition that it is non-periodic. Moreover if the generalized Thue-Morse sequence is not periodic, then all equally spaced subsequences of the generalized Thue-Morse sequences are not periodic.

1 Introduction

According to the line in [3], let us introduce the Thue-Morse sequence.

Set \( f : \{1, 0\} \rightarrow \{0, 1\} \) by \( f(0) = 1 \), \( f(1) = 0 \). For a finite sequence \( A := (a_1, a_2, \ldots, a_n) \) where \( a_j \in \{1, 0\} \), we define by \( f(A) \) and \( Af(A) \)

\[
\begin{align*}
    f(A) &:= (f(a_1), f(a_2), \ldots, f(a_n)), \\
    Af(A) &:= (a_1, a_2, \ldots, a_n, f(a_1), f(a_2), \ldots, f(a_n)).
\end{align*}
\]

We recursively define a sequence \( A_{n+1} \) by

\[
A_0 = 0, \quad A_{n+1} = A_n f(A_n) \quad (n = 0, 1, \ldots).
\]

The sequence \( A_{n+1} \) consists of the number of \( 2^{n+1} \) of 1 and 0. We consider the limit of \( A_n \)

\[
A_\infty := \lim_{n \to \infty} A_n,
\]

which is referred to as the Thue-Morse sequence. It is known that this sequence \( A_\infty \) does not have a period [5].

In this article, we generalize the Thue-Morse sequence (Thue-Morse sequences of type \((L, p, \kappa)\)), where \( L \) and \( p \) denote integers greater than 1, and \( \kappa \) is a map

\[
\kappa : \{1, \ldots, p - 1\} \times \mathbb{N} \rightarrow \{0, 1, \ldots, L - 1\}.
\]
(\mathbb{N} denotes the set of non-negative integers (or, natural numbers)), and propose a new method for proving non-periodicity of the sequences. The original Thue-Morse sequence corresponds to that case that \((L, p) = (2, 2)\) and \(\kappa(n) = 1\) \((n \in \mathbb{N})\). This method is different from that used in [5]. We show that, through consideration on a generating function of the sequence of type \((L, p, \kappa)\), the \(p\)-adic expansion of natural numbers are deeply related to non-periodicity of the sequence, and prove a necessary-sufficient condition for the non-periodicity of that.

This paper is organized as follows: In section 2 we review the basic concepts about the periodicity of sequences, and give the definition a Thue-Morse sequence of type \((L, p, \kappa)\). For a sequence \((a(n))_{n=0}^{\infty}\), we set its generating function \(g(z) \in \mathbb{C}[z]\) by

\[
g(z) := \sum_{n=0}^{\infty} a(n) z^n.
\]

For a generalized Thue-Morse sequence, one can prove that the generating function is convergent, and that it has an infinite product expansion. In section 3 first we prove that lemma of the \(p\)-adic expansion of natural numbers. Then we will use this lemma and the infinite product expansion of the generating function of a generalized Thue-Morse sequence to prove a necessary-sufficient condition for the non-periodicity of a generalized Thue-Morse sequence.

2 Definition of the generalized Thue-Morse sequences and their generating functions

**Definition 2.1** Let \((a(n))_{n=0}^{\infty}\) be a sequence. We say that it is is periodic if there exist non negative integers \(N\) and \(l(0 < l)\) such that

\[
a(n) = a(n + l) \quad (\forall n \geq N).
\]

**Definition 2.2** Let \((a(n))_{n=0}^{\infty}\) be a sequence. An equally spaced subsequence of \((a(n))_{n=0}^{\infty}\) is defined to be a subsequence such as \((a(N+tl))_{t=0}^{\infty}\), where \(N \geq 0\) and \(l > 0\).

**Definition 2.3** Let \((a(n))_{n=0}^{\infty}\) be a sequence with values in \(\mathbb{C}\). The sequence \((a(n))_{n=0}^{\infty}\) is called almost everywhere nonperiodic if all equally spaced subsequences of \((a(n))_{n=0}^{\infty}\) do not take on only one value.

Now we show some lemmas on the almost everywhere nonperiodic sequences.

**Lemma 2.1** Let \((a(n))_{n=0}^{\infty}\) be almost everywhere nonperiodic. Then \((a(n))_{n=0}^{\infty}\) is not periodic.

**Proof.** We show in contraposition. Assume \((a(n))_{n=0}^{\infty}\) is periodic. By definition, there exist non negative integers \(N\) and \(l(0 < l)\) such that

\[
a(n) = a(n + l), \quad (\forall n \geq N).
\]
Then consider \((a(N+tl))_{t=0}^{\infty}\). This equally spaced subsequence take on only one value. This completes the proof. \(\square\)

**Lemma 2.2** Let \((a(n))_{n=0}^{\infty}\) be almost everywhere nonperiodic. Then all equally spaced subsequence of \((a(n))_{n=0}^{\infty}\) are almost everywhere nonperiodic.

**Proof.** We show in contraposition. Assume \((a(N+tl))_{t=0}^{\infty}\) is not almost everywhere nonperiodic. Then there exist non negative integers \(k\) and and \(J(0 < J)\) such that \((a(N+kl+mJl))_{n=0}^{\infty}\) takes on only one value. \((a(N+kl+mJl))_{n=0}^{\infty}\) is equally spaced subsequence of \((a(n))_{n=0}^{\infty}\), too. Then \((a(n))_{n=0}^{\infty}\) is not almost everywhere nonperiodic. \(\square\)

**Corollary 2.1** \((a(n))_{n=0}^{\infty}\) is almost everywhere nonperiodic if and only if all equally spaced subsequences of \((a(n))_{n=0}^{\infty}\) are not periodic.

**Proof.** Assume \((a(n))_{n=0}^{\infty}\) is almost everywhere nonperiodic. By Lemma 2.1, 2.2, all equally spaced subsequences of \((a(n))_{n=0}^{\infty}\) are not periodic. The converse is proved similarly. \(\square\)

**Definition 2.4** Let \(L\) be be an integer greater than 1. Let \(a_1, a_2, \ldots, a_L\) be different complex numbers. Define map \(f : \{a_1, a_2 \ldots a_L\} \rightarrow \{a_1, a_2 \ldots a_L\}\) as follow,

\[
f(a_i) = a_{i+1},
\]

where the indices \(i\) is defined \(\mod L\). \(f^k\) stand for a \(k\) times composed mapping of \(f\). \(f^0\) means an identity mapping. \(A := (b_1, b_2, \ldots, b_n)\) where \(b_i \in \{a_1, a_2, \ldots, a_L\}\), we define by \(f^k(A)\) and \(Af^{\kappa_1}(A) \ldots f^{\kappa_n}(A)\) as follow,

\[
f^k(A) := (f^k(b_1), \ldots, f^k(b_n)), \quad \text{ (2.4)}
\]

\[
Af^{\kappa_1}(A) \ldots f^{\kappa_n}(A) := (b_1, \ldots, b_n, f^{\kappa_1}(b_1), \ldots, f^{\kappa_1}(b_n), \ldots, f^{\kappa_n}(b_1), \ldots, f^{\kappa_n}(b_n)). \quad \text{ (2.5)}
\]

\(A_0 = a_1\). Let \(p\) be an integer greater than 1, \(\mathbb{N}\) be the set of non negative integer. Let \(\kappa\) be map \(\kappa : \{1, \ldots, p-1\} \times \mathbb{N} \rightarrow \{0, \ldots, L-1\}\). We define \(W_m\), a space of words, by

\[
W_m := \{a_{i_1}a_{i_2} \ldots a_{i_m} | \text{ length of word generated by } a_j \text{ is } m\}. \quad \text{ (2.6)}
\]

Define \(A_n \in W_{p^n+1}\) recursively by

\[
A_{n+1} := A_n f^{\kappa(1,n)}(A_n) \ldots f^{\kappa(p-1,n)}(A_n), \quad \text{ (2.7)}
\]

and denote the limit of \(A_n\) by

\[
A_\infty := \lim_{n \rightarrow \infty} A_n. \quad \text{ (2.8)}
\]

We call \(A_\infty\) generalized Thue-Morse sequence of type \((L, p, \kappa)\). \((L, p, \kappa)\)-TM denote Thue-Morse sequence of type \((L, p, \kappa)\).
Example 2.1 Let $L = 2$, $a_1 = 0$, $a_2 = 1$, $\kappa(1,n) = 1$ for all $n$, then we have

$$A_0 = 0, A_1 = 01, A_2 = 0110, A_3 = 01101001,$$

$$A_\infty = 0110100110110011100110110110110101101010110\cdots.$$

This is the Thue-Morse sequence.

Definition 2.5 Let $(a(n))_{n=0}^\infty$ be a sequence with values in $\mathbb{C}$. The generating function of $(a(n))_{n=0}^\infty$ is a formal power series $g(z) \in \mathbb{C}[[z]]$ by $g(z) := \sum_{n=0}^\infty a(n)z^n$.

The following lemma will be used in the next section.

Lemma 2.3 Let $A_\infty$ be $(L,p,\kappa)-TM$. Substitute $\exp \frac{2\pi\sqrt{-1}}{L}$ for $a_j$. Let $g(z)$ be the generating function of $A_\infty$. Then $g(z)$ has infinite product on $|z| < 1$ as follow,

$$g(z) = \prod_{t=0}^\infty (1 + \sum_{s=1}^{p-1} \exp \frac{2\pi\sqrt{-1}\kappa(s,t)}{L}z^{sp'}).$$

(2.9)

Proof. Assume $a_j = \exp \frac{2\pi\sqrt{-1}}{L}$, then we have

$$f(a_j) = \exp \frac{2\pi\sqrt{-1}}{L}a_j.$$

(2.10)

$(L,p,\kappa)-TM$ take only finite values. By Cauchy-Hadamard theorem, $g(z)$ and $\prod_{t=0}^\infty (1 + \sum_{s=1}^{p-1} \exp \frac{2\pi\sqrt{-1}\kappa(s,t)}{L}z^{sp'})$ converge absolutely on the unit circle.

Let $g_{A_\infty}(z)$ be generating function of $A_n$. First, we show (2.11) by induction on $n$

$$g_{A_n}(z) = \prod_{t=0}^{n-1} (1 + \sum_{s=1}^{p-1} \exp \frac{2\pi\sqrt{-1}\kappa(s,t)}{L}z^{sp'}).$$

(2.11)

First we check the case $n=1$. By definition $A_1$, we have

$$g_{A_1}(z) = 1 + \sum_{s=1}^{p-1} \exp \frac{2\pi\sqrt{-1}\kappa(s,0)}{L}z^s.$$

(2.12)

Thus the case $n=1$ is true. By the induction hypothesis we may assume as follow,

$$g_{A_j}(z) = \prod_{t=0}^{j-1} (1 + \sum_{s=1}^{p-1} \exp \frac{2\pi\sqrt{-1}\kappa(s,t)}{L}z^{sp'}).$$

(2.13)
Then
\[ g_{A_{j+1}}(z) = \sum_{m=0}^{p-1} g_{f^{m}(A_{j})}(z)z^{mp^t}. \]  

(2.14)

By (2.10) and the fact \( g_{f^{m}(A_{j})}(z) = \exp \frac{2\pi \sqrt{-1} \kappa(m,j)}{L} A_{j}(z) \), we have
\[ g_{A_{j+1}}(z) = g_{A_{j}}(z)(1 + \sum_{m=1}^{p-1} \exp \frac{2\pi \sqrt{-1} \kappa(m,t)}{L} z^{mp^t}) = \prod_{t=0}^{j} (1 + \sum_{s=1}^{p-1} \exp \frac{2\pi \sqrt{-1} \kappa(s,t)}{L} z^{sp^t}). \]  

(2.15)

(2.11) is shown. Then we compare the coefficients of both sides \( z^j \) of (2.9). The coefficient of right side \( z^j \) of (2.9) is determined by \( g_{A_{N}}(z) \) which \( N \) is large enough. By definition of \( A_{\infty} \), the first \( p^N \) words of \( A_{\infty} \) is \( A_{N} \). By (2.11), the coefficients of both sides \( z^j \) of (2.9) coincide. This completes the proof. □

Definition 2.6 Let \( (a(n))_{n=0}^{\infty} \) be a sequence with values in \( \mathbb{C} \). Let \( g(z) \) be the generating function of \( (a(n))_{n=0}^{\infty} \). We say \( (a(n))_{n=0}^{\infty} \) is \( p \)-adic expansion sequence if the \( g(z) \) has the following infinite product expansion for an integer \( p \) greater than 1 and all \( t, s, j \neq 0 \),
\[ g(z) = \prod_{j=0}^{\infty} (1 + \sum_{s=1}^{p-1} t_{s,j}z^{sp^t}). \]  

(2.16)

3 The necessary-sufficient condition for the non-periodicity of a generalized Thue-Morse sequence

This section we consider the multiple of a period of \( (L, p, \kappa) \)-TM which is periodic. First, we show the following lemma.

Lemma 3.1 Let \( p \) be an integer greater than 1 and \( x, m \) be an integer greater than zero. Then \( xm \) represents as follow,
\[ xm = \sum_{q=1}^{finite} s_{x}p^{w_{x}(q)}, \]  

(3.1)
where \( 1 \leq s_{x} \leq p-1 \) and \( w_{x}(q+1) > w_{x}(q) \geq 0 \). Let \( t \) be an non negative integer. Then there exists an integer \( x \) such that,
\[ xm = \sum_{q=1}^{finite} s_{x}p^{w_{x}(q)}, \]  

(3.2)
where \( s_{x_1} = 1, w_x(2) - w_x(1) > t, w_x(q + 1) > w_x(q) \geq 0 \). Moreover Let \( t' \) be other non negative integer. Then there exists an integer \( X \) such that,
\[
Xm = \sum_{q=1}^{finite} s_{X_q}p^{w_X(q)},
\]
where \( s_{X_1} = 1, w_X(2) - w_X(1) > t', w_X(q + 1) > w_X(q) \geq 0, w_X(1) = w_X(1) \).

Proof. Denote factorization into prime factors of \( p \) and \( m \) as follow,
\[
p = \prod_{t=1}^{k} p_t^{y_t}, \quad m = \prod_{h=1}^{l} m_h^{z_h}.
\]
\( m = G\prod_{u=1}^{n} p_{u}^{-x_u} \) where \( G \) and \( p \) are coprime, \( p_{u} \in \{p_t|1 \leq t \leq k\} \). By the fact \( G \) and \( p \) are coprime, then there exist integers \( D \) and \( E \) such that \( DG = 1 - p^{t+1} \). Let
\[
F := \max\{A|x_u = y_uA + H, 0 \leq H < y_u, 1 \leq u \leq n\}.
\]
By definition of \( F \), \( p^{F+1}\prod_{u=1}^{n} p_{u}^{-x_u} \) is natural number. Then we have
\[
mD^2Gp^{F+1}\prod_{u=1}^{n} p_{u}^{-x_u} = p^{F+1}D^2G^2
\]
By \( D^2G^2 = 1 + p^{t+1}E(p^{t+1}E - 2) \), \( E(p^{t+1}E - 2) \) is a natural number. If \( E(p^{t+1}E - 2) > 0 \), \( p^{F+1}D^2G^2 \) satisfies lemma. If \( E(p^{t+1}E - 2) = 0 \), then \( G = 1 \). \( p^{F+1}(1 + p^{t+1}) \) satisfies lemma. Since \( F + 1 \) independent of \( t \), then the second claim is trivial. \( \square \)

Proposition 3.1 Let \((a(n))_{n=0}^{\infty}\) be \( p \)-adic expansion sequence and \( g(z) \) be the generating function of \((a(n))_{n=0}^{\infty}\). If there exists an equally spaced subsequence of \((a(n))_{n=0}^{\infty}\) which is periodic, then there exist a non negative integer \( A \) and constant \( h \) which satisfy
\[
g(z) = (\sum_{n=0}^{p^A-1} a(n)z^n)\prod_{y=0}^{\infty}(1 + \sum_{l=1}^{p-1} h^{lp^y}z^{p^{A+y}}).
\]

Proof. Assume there exists equally spaced subsequence of \((a(n))_{n=0}^{\infty}\) which is periodic. By Corollary 2.1, \((a(n))_{n=0}^{\infty}\) is not almost everywhere nonperiodic. Then there exist non negative integers \( N \) and \( m(0 < m) \) such that
\[
a(N) = a(N + tm) \quad (\forall t \in \mathbb{N}).
\]
Let \( p \)-adic expansion of \( N \) be as follow,
\[
N = \sum_{q=1}^{N(p)} s_{N_q}p^{w_N(q)} \quad 1 \leq s_{N_q} \leq p - 1, 0 \leq w_N(q) < w_N(q + 1).
\]
By the fact uniqueness of $p$-adic expansion, we have
\[ a(N) = a(N + p^r tm) = a(N)a(p^r tm) \quad (\forall r > w_N(N(p))). \tag{3.9} \]

By (3.9) and the fact $a(N) \neq 0$,
\[ a(p^r tm) = 1. \tag{3.10} \]

@ By lemma 3.1, there exists an integer greater than zero $x$ such that
\[ xm = \sum_{q=1}^{x_m(p)} s_{x_q}p^{w_x(q)}. \tag{3.11} \]

$0 \leq w_x(q) < w_x(q+1)$, $1 \leq s_{x_q} \leq p-1$, $s_{x_1} = 1$ and $w_x(2) - w_x(1) > 1$.
Moreover by lemma 3.1, there exists an integer greater than zero $X$ such that
\[ Xm = \sum_{q=1}^{X_m(p)} s_{X_q}p^{w_X(q)}. \tag{3.12} \]

$0 \leq w_X(q) < w_X(q+1)$, $1 \leq s_{X_q} \leq p-1$, $s_{X_1} = 1$, $w_X(2) - w_X(1) > 0$.

Let an integer $r = \frac{w_N(N(p))}{w_x(1)} + 1$. Let $l(l \in \{1, \ldots, p-1\})$ be any integer.

By (3.10) and the fact uniqueness of $p$-adic expansion, we have
\[ 1 = a(p^r xm). \tag{3.13} \]
\[ 1 = a(p^r lXm). \tag{3.14} \]
\[ 1 = a(p^r xm + p^r lXm). \tag{3.15} \]

By (3.13), (3.14), (3.15), definition of $xm$, definition of $Xm$ and the fact uniqueness of $p$-adic expansion, we have
\[ a(p^r)a(p^r xm - p^r) = 1. \tag{3.16} \]
\[ a(lp^r)a(lXmp^r - lp^r) = 1. \tag{3.17} \]
\[ 1 = a(p^r(l + 1))a(xmp^r - p^r)a(lXmp^r - lp^r). \tag{3.18} \]

By (3.16), (3.17) and (3.18), we have
\[ a(p^r(l + 1)) = a(p^r)a(p^r l). \tag{3.19} \]

Let $a(p^r) = h$ and using notation of definition 2.6. Then by uniqueness of $p$-adic expansion,
\[ a(lp^j) = t_{l,j}. \tag{3.20} \]
\[ t_{l,r} = h^l. \tag{3.21} \]
\[ t_{1,r+1} = h^p. \]  
(3.22)

By (3.10) and inductively,
\[ t_{l,r+y} = h^{lp^y} \quad (\forall y \in \mathbb{N}). \]  
(3.23)

This completes the proof. □

**Theorem 3.1** \((L, p, \kappa)\)-TM is periodic if and only if there exists an integer \(A\) and all natural number \(y\), all \(l(1 \leq l \leq p - 1)\) which satisfy
\[ \kappa(l, A + y) \equiv \kappa(1, A)lp^y \pmod{L}. \]  
(3.24)

Moreover if \((L, p, \kappa)\)-TM is not periodic, then all equally spaced subsequences of \((L, p, \kappa)\)-TM are not periodic.

**Proof.** By lemma 2.3, \((L, p, \kappa)\)-TM is \(p\)-adic expansion sequence. Then (3.24) is necessary condition. We will show converse. Assume \((L, p, \kappa)\)-TM satisfies (3.24), then there exists a non negative integer \(A\) such that
\[ t_{l,A+y} = h^{lp^y} \quad (\forall y \in \mathbb{N}). \]  
(3.25)

Then \(g(z)\) has infinite product expansion as follow,
\[ g(z) = \left( \sum_{n=0}^{p^A-1} a(n)z^n \right) \prod_{y=0}^{\infty} (1 + \sum_{l=1}^{p-1} (hz^{p^A})^{lp^y}). \]  
(3.26)

Let \(Z = hz^{p^A}\). By lemma 3.1 (especially \(\kappa\) is zero map) and the fact \(h\) is \(L\)-th root of \(1\), we have
\[ \prod_{y=0}^{\infty} (1 + \sum_{l=1}^{p-1} Z^{lp^y}) = \sum_{n=0}^{\infty} Z^n \quad \text{on } |Z| < 1. \]  
(3.27)

Let \(G(z) = \sum_{n=0}^{p^A-1} a(n)z^n\). Then \(g(z)\) satisfies as follow,
\[ g(z) = G(z)(1 + \sum_{n=1}^{\infty} (hz^{p^A})^n) = G(z) + \sum_{n=1}^{\infty} G(z)(hz^{p^A})^n. \]  
(3.28)

By the fact \(h\) is \(L\)-th root of \(1\), we have
\[ (G(z) \sum_{n=0}^{L-1} (hz^{p^A})^n)(1 + \sum_{s=1}^{\infty} z^{sLp^A}). \]  
(3.29)

The degree of \(G(z)\) is \(p^A - 1\), then the sequence which satisfies (3.24) has period \(Lp^A\). By the fact mentioned above and Propotition 3.1, if \((L, p, \kappa)\)-TM is not periodic then all equally spaced sequences of \((L, p, \kappa)\)-TM are not periodic. This completes the proof. □
If \((L, p, \kappa)\)-TM independent of \(n\), then \((\kappa(1), \kappa(2), \cdots, \kappa(p-1))\)-L denote \((L, p, \kappa)\)-TM.

**Corollary 3.1** \((\kappa(1), \kappa(2), \cdots, \kappa(p-1))\)-L is periodic if and only if all \(\kappa(l)(0 < l < p)\) which satisfy

\[
l\kappa(1) \equiv \kappa(l), \kappa(p-1) \equiv 0 \pmod{L}.
\]

Moreover if \((\kappa(1), \kappa(2), \cdots, \kappa(p-1))\)-L is not periodic, then all equally spaced subsequences of \((\kappa(1), \kappa(2), \cdots, \kappa(p-1))\)-L are not periodic.

**Proof.** By theorem 3.1, we have

\[
\kappa(1, A+1) \equiv \kappa(1, A)p \pmod{L}, \kappa(p-1) \equiv (p-1)\kappa(1) \equiv 0 \pmod{L}.
\]

This completes the proof. \(\square\)

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