Higher security in one-sided device independent quantum key distribution from more uncertain systems

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We derive a fine-grained uncertainty relation for the measurement of two incompatible observables on a single quantum system of continuous variables, and show that the range of the sum of certainties is smaller than the corresponding range derived through fine-graining for discrete variable systems. The increment of uncertainty in continuous variable systems imposes a greater restriction on the information leakage to the eavesdropper compared to discrete variable systems, and is borne out by an enhanced lower bound on the key rate of a one-sided device independent quantum key distribution protocol. Such a lower bound on the key rate is obtained using a monogamy relation for a steering inequality derived using our uncertainty relation. The steering inequality presented here is able to capture the steerability of N00N states that have not been revealed hitherto using other criteria such as entropic steering relations.

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First, Heisenberg in his seminal paper [1] noted that two non-commuting observables could not be measured with arbitrary precision in context of a thought experiment. This phenomena is called the uncertainty principle which introduces a sharp distinction between quantum and classical physics. Subsequently, a number of studies [2–6] have provided improved uncertainty relations from different perspectives. The presence of uncertainty relations in quantum mechanics offers it significant advantages over classical mechanics for performing different information processing tasks. For example, various versions of uncertainty relations have been used to detect quantum entanglement [5, 7], to classify mixedness of quantum states [8], to categorize different physical theories according to their strength of nonlocality [9, 10], and to bound information leakage in quantum key distribution [11, 12].

Our focus here is on the application of uncertainty relations to quantum key distribution (QKD). The goal of any QKD protocol is to generate a key string between two distant parties (say, Alice and Bob) such that it remains secret from any eavesdropper (say, Charlie). In the first QKD protocol (BB84) proposed by Bennett and Brassard [15], security is based on the uncertainty of the outcome of incompatible spin measurements chosen randomly along x- and z-directions. The security of standard QKD protocols are based on certain idealistic assumptions [10]. These assumptions may be minimized in the so-called device independent QKD (DIQKD) [17] where it is no longer required to fully trust the devices used by Alice and Bob. However, practical and loophole free implementations of DIQKD protocols may be difficult since they require demonstration of nonlocality.

Recently, one-sided device independent quantum key distribution (1s-DIQKD) [12] protocols have been proposed in which, only one of the parties say, Bob, is trusted to make quantum measurements on his quantum system. These protocols are intermediate between standard QKD and DIQKD protocols. The security of 1s-DIQKD relies on demonstration of quantum steering [15] which, in the hierarchy of quantum correlations [19] lies between entanglement required for QKD and nonlocality for DIQKD. Steering is an off-shoot of the famous EPR paradox [20], that was first formulated for experimental realization [21] based on the Heisenberg uncertainty relation. Later, entropic steering criteria [22] have been formulated using entropic uncertainty relations [3, 4], which are able to reveal steering in states possessing higher order correlations [23]. Due to the recent demonstrations of loophole free steering [24], 1s-DIQKD protocols are regarded to be practically more viable.

The tightest steering inequality in discrete variable systems is obtained [14] through the application of the fine-grained uncertainty principle [6]. Fine-graining makes it possible to distinguish the uncertainty inherent in obtaining any particular combination of outcomes for different measurements. Application of fine-graining leads to discrimination between various physical theories on the basis of their degree of nonlocality [6, 9], and also provides an optimal lower bound of entropic uncertainty in presence of quantum memory [13]. In the present work we derive a fine-grained form of uncertainty relation for continuous variable systems (uncertainty is measured for particular outcome or outcomes restricted by certain rules), and apply it to the secret key rate of a 1s-DIQKD protocol.

Continuous variable QKD has attracted a lot of attention in recent times [25]. The development of effi-
cient error correction codes [26] has paved the way for secure data transfer over large distances and with high bit rates [27]. It has been claimed that continuous variable 1s-DIQKD protocols are reasonably robust to losses and practically more feasible compared to their discrete variable counterparts [28]. Very recently, Gaussian quantum steering and its application to the key rate in 1s-DIQKD has been studied [29]. In the present work using our derived uncertainty relation, we formulate a continuous variable fine-grained steering criterion, and show that this criterion is monogamous, i.e., Alice’s steerability of Bob to a particular state chosen randomly from the eigenstates of two incompatible observables restricts the control of Bob’s system by the eavesdropper. This leads to a bound on the secret key rate between Alice and Bob in a 1s-DIQKD protocol.

In order to highlight the possibility of greater security using continuous variables, we make a comparison of the uncertainty relation derived here with the fine-grained steering relation for discrete variables [14]. Such a comparison is facilitated by using the Wigner function for computing various probabilities associated with continuous variables. There exists an analogy between the measurement of spin-1/2 projectors and the parity operator, since the measurement outcomes for both are dichotomic. It is well known [30] that the Wigner function expressed as an expectation value of a product of displaced parity operators can be used to derive Bell-CHSH inequalities [31] for continuous variables. Here we use the Wigner function formalism [30] to derive a steering inequality for continuous variables. There exists an analogy between the measurement outcomes for both being dichotomic.

function formalism [30] to derive a steering inequality for continuous variables. We show that fine-graining leads to a novel manifestation of higher uncertainty in continuous variable systems compared to discrete variable systems, thereby enhancing the key rate of a continuous variable 1s-DIQKD protocol.

Let us begin with a brief description of EPR-steering by considering the following game [19]. Alice prepares two systems $A$ and $B$ in the state $\rho_{AB}$ and sends the system $B$ to Bob. Alice’s task is to convince Bob that the prepared state $\rho_{AB}$ is entangled. Bob does not trust Alice, but he trusts that he receives a quantum system $B$. Bob agrees with Alice only when the correlation between his outcome $b$ for the measurement chosen randomly from the set $\{\beta_1, \beta_2\}$ and Alice’s outcome $a$ for the measurement chosen randomly from the set $\{\alpha_1, \alpha_2\}$ can not be explained by a local hidden state (LHS) model, i.e.,

$$P(a_A, b_B) = \sum_{\lambda} P(\lambda)P(a_A|\lambda)P_Q(b_B|\lambda),$$

where $P(\lambda)$ is a positive valued distribution over a set of hidden variables $\lambda$, and $P_Q$ denotes quantum probability, viz. the corresponding outcome is obtained from a quantum measurement. The difference of the game considered here with the game considered in [14] is the assumption here throughout of Alice’s knowledge about Bob’s set of observables.

Now, from Eq. (1) it is easy to derive the relation

$$P(b_B|a_A) \leq \max_{\lambda} [P_Q(b_B|\lambda) - P_Q(b_B|\lambda_{\max})]$$

using $\sum_i x_i y_i \leq \max_i [x_i] \sum_i y_i \forall \{x_i, y_i\} \geq 0$. Similarly, with the help of $\sum_i x_i y_i \geq \min_i [x_i] \sum_i y_i \forall \{x_i, y_i\} \geq 0$, Eq. (1) becomes

$$P(b_B|a_A) \geq \min_{\lambda} [P_Q(b_B|\lambda) - P_Q(b_B|\lambda_{\min})].$$

Hence, combining the relations (2) and (3), the sum of conditional probability distributions according to the LHS model is bounded by

$$\min_{\beta_1, \beta_2} [P_Q(b_{\beta_1}|\lambda_{\min}) + P_Q(b_{\beta_2}|\lambda_{\min})] \leq P(b_{\beta_1}|a_{\alpha_1}) + P(b_{\beta_2}|a_{\alpha_2}) \leq \max_{\beta_1, \beta_2} [P_Q(b_{\beta_1}|\lambda_{\max}) + P_Q(b_{\beta_2}|\lambda_{\max})].$$

In order to obtain the lower and upper bounds of the algebraic inequality (4), we first formulate the fine-grained uncertainty principle in continuous variable systems and then evaluate the bounds using the derived uncertainty relation.

The well known uncertainty relations, e.g., Heisenberg uncertainty relation [11] and entropic uncertainty relations [3, 4] are coarse-grained. These give average uncertainties, where the average is taken over all possible measurement outcomes. For example, for the entropic uncertainty relation [4], the average uncertainty is measured by the Shannon entropy which is computed as $H(A) = \sum_x P(x) [-\log_2 P(x)]$, where $\{x\}$ is the set of possible measurement outcomes for the measurement of observable $A$, and “$-\log_2 P(x)$” is the measure of uncertainty of getting outcome $x$. The concept of fine-graining in uncertainty relations was first introduced by Oppenheim and Wehner in their work [5] to explain the failure of quantum theory to show the full non-local strength as allowed by no-signaling theory. They bound an event (which is defined by the outcomes chosen using imposed restrictions or conditions) by its minimum possible uncertainty, or maximum possible certainty, i.e., in quantum theory, an event never occurs with probability unity for two incompatible observables.

For example, in qubit systems the following game [6] is considered. A binary question $q \in \{0, 1\}$ is given randomly to a player, say, Bob who measures $\sigma_z$ when $q = 0$ ($1$) is received. Here, the average uncertainty of getting spin up measurement outcome (labeled by “$b = 0$”) irrespective of the given question (or indeed, the average certainty) where the average is taken over all possible choice of questions, i.e., measurements, is bounded by

$$\frac{1}{2} - \frac{1}{2\sqrt{2}} \leq \frac{1}{2} [P(b_{\sigma_z} = 0) + P(b_{\sigma_z} = 0)] \leq \frac{1}{2} + \frac{1}{2\sqrt{2}},$$

where the equalities occur for maximally certain states. Here, they are the eigenstates of the observables ($\sigma_z + \sigma_z)/\sqrt{2} \land (\sigma_z - \sigma_z)/\sqrt{2}$ for the upper and lower
bounds, respectively. For another possible condition in qubit systems, i.e., for the spin down outcome \( b = 1 \), the certainty is again bounded by \( \left( \frac{1}{2} - \frac{1}{2\sqrt{2}} \right) \). Hence, in discrete variable systems, the shared state is steerable if the value of \( \frac{1}{2} [P(b_\alpha = 0) + P(b_\beta = 0)] \) lies outside the range \( \left[ \frac{1}{2} - \frac{1}{2\sqrt{2}}, \frac{1}{2} + \frac{1}{2\sqrt{2}} \right] \), where Alice has prior knowledge of Bob’s measurement settings.

In continuous variable systems, Bell’s inequality is shown to be violated using the Wigner function formalism \([30]\). Fine-graining connects uncertainty with nonlocality, and hence for a given Bell-CHSH inequality one can formulate a FUR for a bipartite system \([30]\). Similar considerations hold true for single particle quantum systems, thus making it possible to construct uncertainty relations for single systems using the Wigner distribution corresponding to a state, representing the average of displaced parity measurement over that state. Let us here label the outcome of even parity measurement by “0”.

The corresponding projection operator is given by

\[
\Pi^+ (\beta) = D(\beta) \left( \sum_{n=0}^{\infty} |2n\rangle \langle 2n| \right) D^\dagger (\beta),
\]

where \( D(\beta) (= \exp(\beta b^\dagger - \beta^* b)) \), is the displacement operator with coherent displacement \( \beta \), and \( b^\dagger \) and \( b \) the annihilation and creation operators, respectively. Similarly, the projection operator corresponding to the odd parity measurement outcome labeled by “1” is given by

\[
\Pi^- (\beta) = D(\beta) \left( \sum_{n=0}^{\infty} |2n+1\rangle \langle 2n+1| \right) D^\dagger (\beta).
\]

The observable associated with the Wigner function is given by

\[
\hat{W}(\beta) = \Pi^+ (\beta) - \Pi^- (\beta),
\]

which can be realized using detectors with the capability of distinguishing the number of absorbed photons \([30]\). We will take \( \beta \)'s to be real displacements in the rest of this work.

Here, we are interested in a fine-grained form of uncertainty relation similar in form to the fine-grained uncertainty relation for discrete variables where the range of certainty is given by the inequality \([5]\). The average certainty of the parity measurement outcome \( b \) over displacements \( \alpha \) and \( \beta \) is \( \frac{1}{2} [P(b_\alpha) + P(b_\beta)] \). As we saw in the case of discrete variables, the average certainty is bounded by the minimum uncertainty states, or, the maximally certain states, viz., the eigenstates of \( (\sigma_+ + \sigma_-)/\sqrt{2} \) there \([6]\). In continuous variable systems it is well known that the coherent states

\[
|\gamma\rangle = \exp\left(-\frac{|\gamma|^2}{2}\right) \sum_{m=0}^{\infty} \frac{\gamma^m}{\sqrt{m!}} |m\rangle
\]

correspond to the minimum uncertainty states in phase-space. Therefore, we obtain the fine-grained bounds on \( \frac{1}{2} [P(b_\alpha) + P(b_\beta)] \) using the coherent states.

For even parity measurement \( b = 0 \) at the displacement chosen from \{\( \alpha, \beta \)\}, the average certainty becomes

\[
\frac{1}{2} [P(b_\alpha = 0) + P(b_\beta = 0)] = \langle \gamma | \frac{\Pi^+ (\alpha) + \Pi^+ (\beta)}{2} | \gamma \rangle = \frac{1}{2} (\exp[-|\gamma - \alpha|^2] \cosh[|\gamma - \beta|^2] + \exp[-|\gamma - \beta|^2] \cosh[|\gamma - \beta|^2]),
\]

where, similar to the displacements \( \beta \) and \( \alpha \), we choose \( \gamma \) also to be real. Note that in the above equation the situation when \( \alpha = \beta = \gamma \) needs to be avoided in order to ensure that the average certainty of getting even parity for the zero photon state does not always stay close to 1, (this is similar to getting, say, spin up outcome in discrete variables for spin measurements along directions \( i \) and \( j \) when \( i \rightarrow j \)). For calculational convenience we henceforth set \( \alpha = -\beta \), and compute the probability distribution \([P(b_\gamma = 0) + P(b_{-\gamma} = 0)]\). The validity of the FUR \([4]\) is ensured by avoiding the region where both \( \beta \rightarrow 0 \) and \( \gamma \rightarrow 0 \), as shown in the Fig. \([1]\). The above probability distribution \([P(b_\beta = 0) + P(b_{-\beta} = 0)]\) is bounded by \( \frac{1}{4} \), where the maximum occurs for \( \beta = \gamma \).

Similarly, coherent states bound the average certainty of odd parity measurements for the displacement \( \beta \) and \( -\beta \), i.e.,

\[
\frac{1}{2} [P(b_\beta = 1) + P(b_{-\beta} = 1)] = \langle \gamma | \frac{\Pi^- (\beta) + \Pi^- (-\beta)}{2} | \gamma \rangle = \frac{1}{2} (\exp[-(\gamma - \beta)^2] \sinh[(\gamma - \beta)^2] + \exp[-(\gamma + \beta)^2] \sinh[(\gamma + \beta)^2]).
\]

In this case the bounds are \( \frac{1}{4} \leq \frac{1}{2} [P(b_\beta = 1) + P(b_{-\beta} = 1)] \leq \frac{1}{2} \), except for the condition \( \gamma \rightarrow 0 \) \& \( \beta \rightarrow 0 \) at which the probability of getting odd counts for the zero photon state approaches zero, as shown in the Fig. \([1]\).

Combining the cases of odd and even parities, one therefore finds that the fine-grained uncertainty relation (FUR) bounds the certainty for the measurement of two incompatible observables on single quantum systems in continuous variable by

\[
\frac{1}{4} \leq \frac{1}{2} [P(b_\beta) + P(b_{-\beta})] \leq \frac{3}{4}.
\]

One may note that our restriction on having \( \gamma \rightarrow 0 \) \& \( \beta \rightarrow 0 \) simultaneously limits the value of \( \gamma \) to \( |\gamma|^2 \geq 0 \) when \( \beta \rightarrow 0 \). In the Fig. \([1]\), we plot the infimum value of \( \frac{1}{2} [P(b_\beta) + P(b_{-\beta})] \) with \( \gamma \). One sees that when \( \beta \rightarrow 0 \) the FUR \([12]\) may be regarded to valid for a source with at least a single average photon number, i.e., \( |\gamma|^2 \geq 1 \). Note also, that the range of certainty in discrete variable systems given by Eq. \([5]\) \([14]\) is \( \left[ \frac{1}{2} - \frac{1}{2\sqrt{2}}, \frac{1}{2} + \frac{1}{2\sqrt{2}} \right] \) which is higher than that in continuous variable systems.
This feature reflecting more uncertainty in continuous variable systems helps to improve the secret key rate, since higher uncertainty enables less information to the eavesdropper.

![Plot](image)

**FIG. 1:** Plot of $\frac{1}{4} \inf_{\beta} [P(b_{\beta} = 0) + P(b_{\beta} = 0)]$ with $\gamma$. The solid curve is for $\frac{1}{4} \max_{\beta} [P(b_{\beta} = 0) + P(b_{\beta} = 0)]$, and the dashed curve is for $\frac{1}{4} \min_{\beta} [P(b_{\beta} = 1) + P(b_{\beta} = 1)]$. The solid and dashed lines correspond, respectively, to the upper and lower bounds of certainty in regions of validity of the FUR [12].

Now, with the help of the FUR given by [12], the steering inequality [4] becomes

$$\frac{1}{4} \leq \frac{1}{2} [P(b_{\beta} | a_{\alpha_1}) + P(b_{\beta} | a_{\alpha_2})] \leq \frac{3}{4}. \quad (13)$$

The violation of the above inequality [13] indicates that the measurement correlations are unable to be explained with the help of an LHS model, i.e., the state is said to be steerable if and only if the probability distribution $\frac{1}{2} [P(b_{\beta} | a_{\alpha_1}) + P(b_{\beta} | a_{\alpha_2})]$ lies outside the region $[\frac{1}{4}, \frac{3}{4}]$.

Next, we provide an application of our derived steering inequality by discussing the steerability of the NOON state. We show the monogamy of our steering inequality and derive the lower bound of the secret key rate under individual attack.

NOON states [32] are regarded to be of high utility in quantum metrology for making precise interferometric measurements. Such a state is a maximally path-entangled two-mode number state of continuous variables is given by [33]

$$|N\text{N}N\rangle = \frac{1}{\sqrt{2}} (|N, 0\rangle - |0, N\rangle), \quad (14)$$

where $N$ is the number of photons that can be found either in the first or the second mode. These states have been experimentally realized up to $N = 5$ [34]. The entanglement of NOON states given in terms of their logarithmic negativity is independent of the value of $N$, and Bell’s inequality is maximally violated for all $N \geq 3$ by NOON states. However, they do not violate the entropic steering inequality for $N \geq 2$ [35]. We will show now that the above state is steerable for $N \geq 2$ using our derived steering inequality.

In order show the violation of our steering inequality [13] by NOON states, the two cases - I. $N$ even, and II. $N$ odd, may be considered separately. When $N$ is even, the maximum violation of the inequality [13] on the upper side occurs when $\frac{1}{4} [P(b_{\beta} | a_{\alpha_1}) + P(b_{\beta} | a_{\alpha_2})] = 1$ for the choices of the parameters given by $\{b = 0, a = 0\}$, $\{b = 0, a = 1\}$. Similarly, the maximum violation on the lower side occurs when $\frac{1}{4} [P(b_{\beta} | a_{\alpha_1}) + P(b_{\beta} | a_{\alpha_2})] = 0$ for the choices of the parameters give by $\{b = 1, a = 0\}$, $\{b = 1, a = 1\}$. For the odd case, the maximum violations on the upper side are 1 for the choices $\{b = 0, a = 1\}$, $\{b = 1, a = 0\}$, and on the lower side are 0 for the choices $\{b = 0, a = 0\}$, $\{b = 1, a = 1\}$. In the Fig. [2], we provide a 3-dimensional plot of the quantity $\frac{1}{4} [P(b_{\beta} = 0 | a_{\alpha_1} = 1) + P(b_{\beta} = 0 | a_{\alpha_2} = 1)]$ with respect to $\beta$ and $\alpha$ for $N = 2, 4, 6$. One sees that the violation of the steering inequality occurs maximally for $N \geq 2$ in the region $|\beta| \to 0$. As stated earlier, the condition for validity of our FUR, viz., $\gamma \gg 0$, when $\beta \to 0$, is ensured since the average photon number $(N/2)$ here is greater than 1.

![Plot](image)

**FIG. 2:** Coloronline. The variation of $\frac{1}{4} [P(b_{\beta} = 0 | a_{\alpha_1} = 1) + P(b_{\beta} = 0 | a_{\alpha_2} = 1)]$ with respect to $\beta$ and $\alpha$ for three different values of $N$. (i) The red colored curve corresponds to the value $N = 2$; (ii) the green colored curve is for $N = 4$; and the blue colored curve is for $N = 6$.

Monogamy of non-local correlations certifies the security of QKD [36]. To develop a monogamy relation associated with the upper bound of our steering inequality [13], let us consider that three parties Alice, Bob and Charlie share a tripartite state $\rho_{ABC}$ for which the inequality

$$\frac{1}{2} (\Sigma_{BA} + \Sigma_{BC}) \leq \frac{3}{2} \quad (15)$$

is satisfied, where $\Sigma_{BA} = P(b_{\beta_1} | a_{\alpha_1}) + P(b_{\beta_2} | a_{\alpha_2})$ and $\Sigma_{BC} = P(b_{\beta_1} | c_{\gamma_1}) + P(b_{\beta_2} | c_{\gamma_2})$. $c$ is Charlie’s outcome for the measurement chosen from the set $\{|\gamma_1\rangle, |\gamma_2\rangle\}$. The above inequality [15] is a monogamy relation of steerability. The proof comes from contradiction. Let $\frac{1}{2} (\Sigma_{BA} + \Sigma_{BC}) > \frac{3}{2}$. The above sum can be written as the sum of $[P(b_{\beta_1} | a_{\alpha_1}) + P(b_{\beta_2} | c_{\gamma_2})]$ and $[P(b_{\beta_1} | c_{\gamma_1}) + P(b_{\beta_2} | a_{\alpha_2})]$. As the average of the above two terms is greater than 3/2, one of the terms, say, the first is greater than 3/2. Then, it is possible to find a conditional Bob’s...
state for which $|P(b_{21}) + P(b_{23})| > 3/2$, which contradicts the FUR given by inequality (12). Similarly, using the lower bound of our steering inequality (13), one can obtain $\frac{1}{2} [\Sigma_{BA} + \Sigma_{BC}] \geq \frac{1}{2}$, which together with the relation (15) gives

$$\frac{1}{2} \leq \frac{1}{2} [\Sigma_{BA} + \Sigma_{BC}] \leq \frac{3}{2}. \quad (16)$$

The monogamy relation given by inequality (15) is useful to find out the secret key rate in a 1s-DIQKD protocol. In this scenario Alice’s system is not trusted, whereas Bob’s system is trusted as quantum system. The lower bound of the secret key rate under individual attack is given by [37]

$$r \geq I(B : A) - I(B : C), \quad (17)$$

where $I$ is the mutual information. Let us consider that the upper bound of the steering inequality (13) is violated by an amount $\delta$, i.e., $\frac{1}{2} [P(b_{21} | a_{11}) + P(b_{23} | a_{23})] = \frac{3}{2} + \delta$, where $0 < \delta \leq \frac{1}{4}$. Then, the monogamy relation (15) implies $\frac{1}{2} [P(b_{21} | c_{11}) + P(b_{23} | c_{23})] \leq 0.75 - \delta$. Hence the lower bound of the key rate becomes

$$r \geq \log_2 \left[ \frac{0.75 + \delta}{0.75 - \delta} \right], \quad (18)$$

where the logarithm of base 2 is taken as the secret key rate is expressed in the units of bits per shared state. For the maximally entangled state, viz., the NOON state for which $\delta = 1/4$, the steering inequality (13) is maximally violated. Therefore, the lower bound of the secret key rate in 1s-DIQKD is unity. One may note here that in comparison the lower bound of the secret key rate in discrete variables is 1/2 [13]. Hence, the use of continuous variable systems in QKD offers more security in principle than the discrete variable systems.

In conclusion, in the present work we first derive a fine-grained uncertainty relation for continuous variable systems with the help of an operational interpretation of the Wigner function [30]. Our derived uncertainty relation provides a manifestation of higher uncertainty in continuous variable systems compared to discrete variable systems, since in the former the certainty is bounded by $[1/4, 3/4]$, whereas in the latter the certainty is bounded by $[1/2 - 1/(2\sqrt{2}), 1/2 + 1/(2\sqrt{2})]$. The increment of uncertainty in continuous variable systems restricts the amount of information leakage to the eavesdropper, making continuous variable systems more secure in principle, than discrete variable systems. Our derived steering inequality is hence stronger than that of discrete variable systems, as in the latter, Bob is convinced of the prepared state being entangled only when the average of the conditional probabilities is larger than $1/2 + 1/(2\sqrt{2})$ [13], whereas, in continuous variable systems it is 3/4. With the help of a derived monogamy relation corresponding to our steering inequality, we bound the key rate in the 1s-DIQKD protocol which is secured under individual attacks. Further, we show that our steering inequality improves a drawback of the entropic steering inequality which is not violated by the NOON state for $N \geq 2$ [23], in spite of its maximal Bell violation for all $N$ [33]. The fine-grained steering inequality derived here is capable of detecting maximal steerability of NOON states for $N \geq 2$.

The relation of the Wigner function with displaced parity operators [30] further facilitates comparison of the key rates in continuous and discrete variables. The lower bound of the secret key rate is unity for the shared maximally entangled state of continuous variables, which is at least double that for discrete variables [14] even when Alice knows Bob’s set of observables before preparation of the state. A significant corollary of our analysis is that knowing Bob’s set of observables in continuous variables does not help Alice to cheat, whereas it is indeed helpful in discrete variable systems [13]. The quasi-probability distributions used in the derivation of our steering inequality through the Wigner function formalism [30] may be reconstructed experimentally by homodyne detection techniques that are currently realizable with high efficiency [38]. Recent experiments [39] have indeed confirmed Bell’s nonlocality in continuous variable systems using similar techniques. It should thus be feasible to experimentally verify our steering inequality for continuous variables. To end, the experimental realization of quasi-probability functions may provide a boost for future communication technology based on continuous variable systems.

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