Method of construction of the Riemann function for a second-order hyperbolic equation

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Abstract. A linear hyperbolic equation of the second order in two independent variables is considered. The Riemann function of the adjoint equation is shown to be invariant with respect to the fundamental solutions transformation group. Symmetries and symmetries of fundamental solutions of the Euler–Poisson–Darboux equation are found. The Riemann function is constructed with the aid of fundamental solutions symmetries. Examples of the application of the algorithm for constructing Riemann function are given.

1. Introduction
In the study [1] B. Riemann suggested the Riemann’s principle of integrating that was applied to a hyperbolic second-order partial equation with two independent variables. In order to apply the method it is necessary to construct Riemann function that is a solution of Cauchy special characteristic problem. General method for Riemann function construction does not exist. In [2] an extensive analysis of six certain methods for creation Riemann function of particular types of equations. Ibragimov recommended to find Riemann function with the aid of equation symmetries [3] basing on Ovsyannikov study result [4] in group classification of homogeneous hyperbolic second-order equations. In the present study the method for Riemann function construction that is based on the usage of fundamental solutions symmetries is offered.

2. The Riemann’s method
Let us consider the general linear hyperbolic equation of the second order in two independent variables

\[ Lu = u_{xy} + a(x, y)u_x + b(x, y)u_y + c(x, y)u = f(x, y). \] (1)

Riemann’s method is based on the following identity

\[ 2(vLu - uL^*v) = (vu_y - uv_y + 2auv)_x + (vu_x - uv_x + 2buv)_y, \]

where \( L^*v = v_{xy} - (av)_x - (bv)_y + cv \) is the adjoint operator. Riemann’s method allows us to convert the integration problem of Eq. (1) to construction of the intermediary Riemann function \( v(x, y; x_0, y_0) \), that obeys the following homogeneous adjoint equation of variables \( x, y \)

\[ L^*R = 0 \]
and the following conditions at the characteristics:

\[(R_y - aR)|_{x=x_0} = 0, \quad (R_x - bR)|_{y=y_0} = 0, \quad R(x_0, y_0; x_0, y_0) = 1.\]

General solutions of Cauchy problem and Goursat problem are constructed with the aid of Riemann function for Eq. (1) [2].

Riemann function \( u = R^*(x, y; x_0, y_0) \) of the adjoint equation satisfies homogeneous equation (4)

\[LR^* = 0\]

and following conditions at characteristics

\[(R_y^* + aR^*)|_{x=x_0} = 0, \quad (R_x^* + bR^*)|_{y=y_0} = 0, \quad R^*(x_0, y_0; x_0, y_0) = 1.\] (2)

Riemann functions \( R \) and \( R^* \) have properties of reciprocity

\[R^*(x, y; x_0, y_0) = R(x_0, y_0; x, y).\] (3)

3. Symmetries of fundamental solutions

Fundamental solutions of linear partial differential equations are frequently invariant under transformations admitted by the original equation [4]. Below, a fundamental solution is constructed using the algorithm from [5] proposed for finding fundamental solutions of linear partial differential equations. The algorithm makes use of the symmetries admitted by a linear partial differential equation with a delta function on its right-hand side. Let us briefly describe the main result of this work.

Consider the \( p \)th-order linear partial differential equation

\[Au \equiv \sum_{\alpha=1}^{p} B_{\alpha}(x)D^{\alpha}u = 0, \quad x \in R^m.\] (4)

Here, the standard notation is used: \( \alpha = (\alpha_1, \ldots, \alpha_m) \) is a multi-index with nonnegative integer components, \( \alpha = \alpha_1 + \cdots + \alpha_m \), and

\[D^{\alpha} \equiv \left( \frac{\partial}{\partial x^1} \right)^{\alpha_1} \cdots \left( \frac{\partial}{\partial x^m} \right)^{\alpha_m}.\]

The fundamental solutions of Eq. (1) are solutions of the equation

\[Au = \delta(x - x_0).\] (5)

It was shown in [6] that Eq. (4) with \( p \geq 2 \) and \( m \geq 2 \) can admit only symmetry operators of the form

\[Y = \sum_{i=1}^{m} \xi^i(x) \frac{\partial}{\partial x^i} + \eta(x, u) \frac{\partial}{\partial u}, \quad \frac{\partial^2 \eta}{\partial u^2} = 0.\]

The basic Lie algebra of symmetry operators of Eq. (4) regarded as a vector space is the direct sum of two subalgebras: one consisting of operators of the form

\[X = \sum_{i=1}^{m} \xi^i(x) \frac{\partial}{\partial x^i} + \zeta(x) u \frac{\partial}{\partial u}.\] (6)
and the infinite-dimensional subalgebra generated by the operators

\[ X_\infty = \varphi(x) \frac{\partial}{\partial u}, \]  

(7)

where \( \varphi(x) \) is an arbitrary solution of Eq. (4). Note that operators (7) are symmetry operators of Eq. (5). In what follows, we consider only symmetry operators of form (6). Let denote \( X_p \) an extension of order \( p \) of symmetry operator (6).

**Proposition 1.** The infinitesimal operator given by (6) is a symmetry operator of Eq. (4) if and only if there exists a function \( \lambda(x) \) satisfying the identity

\[ X_p(Au) \equiv \lambda(x)Au \]  

(8)

for any function \( u = u(x) \) from the domain of Eq. (4).

**Theorem 1.** The Lie algebra of symmetry operators of Eq. (5) is a subalgebra of the Lie algebra of symmetry operators of Eq. (4) and is defined by the relations

\[ \xi^i(x_0) = 0, \quad i = 1, \ldots, m, \]

\[ \lambda(x_0) + \sum_{i=1}^{m} \frac{\partial \xi^i(x_0)}{\partial x^i} = 0. \]  

(9)

Let us describe an algorithm for finding fundamental solutions by applying symmetries [5]:

1. Find a general symmetry operator of Eq. (4) and the corresponding function \( \lambda(x) \) satisfying identity (9).
2. Use this operator and relations (9) to obtain the basis for the Lie algebra of symmetry operators of Eq. (5).
3. Construct invariant fundamental solutions with the help of the symmetries of Eq. (5).
4. Obtain new fundamental solutions from the known ones with the help of the symmetries of Eq. (5) (production of solutions).

**Remark 1.** To find generalized invariant fundamental solutions, we need to search for invariants in the class of generalized functions.

4. The main result

Symmetry operator of homogeneous equation (1) has a form

\[ X = \xi^1(x) \frac{\partial}{\partial x} + \xi^2(y) \frac{\partial}{\partial y} + \zeta(x, y) u \frac{\partial}{\partial u}, \]  

(10)

and as this takes place the following relations must be hold

\[ \frac{\partial \zeta}{\partial x} + \frac{\partial (b \xi^1)}{\partial x} + \xi^2 \frac{\partial b}{\partial y} = 0, \]

\[ \frac{\partial \zeta}{\partial y} + \frac{\partial (a \xi^2)}{\partial y} + \xi^1 \frac{\partial a}{\partial x} = 0, \]  

(11)

\[ \frac{\partial^2 \zeta}{\partial x \partial y} + a \frac{\partial \zeta}{\partial x} + b \frac{\partial \zeta}{\partial y} + \frac{\partial (c \xi^1)}{\partial x} + \frac{\partial (c \xi^2)}{\partial y} = 0. \]

Function \( \lambda = \lambda(x, y) \) that satisfies the identity law \( X_2(Lu) = \lambda Lu \) has the form

\[ \lambda = \zeta - \frac{d \xi^1}{dx} - \frac{d \xi^2}{dy}. \]  

(12)
Let us consider the equation

\[ Lu = \delta(x - x') \delta(y - y') , \]

that describes fundamental solutions of homogeneous Eq. (1). So the symmetry operators of fundamental solutions (or the symmetries of Eq. (13)) satisfy following additional conditions as Theorem 1 takes place

\[ \xi^1(x') = 0, \quad \xi^2(y') = 0, \]

\[ \lambda(x', y') + \frac{d\xi^1(x')}{dx'} + \frac{d\xi^2(y')}{dy'} = 0 . \]

Show that conditions at characteristics (2) are invariant under symmetry operator (10) at (11), (12) and (14). Note that characteristics \( x = x', y = y' \) are invariant under operators of the symmetry of the fundamental solutions. \( \zeta(x', y') = 0 \) results from relations (12) and the second one of (14). This implies that the latest one of relations (2) is invariant.

Write the invariance condition at the characteristic \( x = x' \)

\[ X(u_y + au) \bigg|_{x = x'} = 0 \]

\[ \{ \left( \zeta - \frac{d\xi^2}{dy} \right) (u_y + au) + u \left[ \frac{\partial\zeta}{\partial y} + \frac{\partial(a\xi^2)}{\partial y} + \xi^1 \frac{\partial a}{\partial x} \right] \} \mid_{x = x'} = 0 \]

Invariance condition (15) is realized owing to (11) and (2) at the characteristic \( x = x_0 \). Invariance of the condition at the characteristic \( y = y_0 \) is convinced analogous.

Hence the theorem that presents the main result of the section is proved.

**Theorem 2.** Symmetries of the fundamental solutions of the second-order linear hyperbolic equation in two independent variables retain Riemann function of adjoint equation invariant.

Theorem 2 implies that algorithm of the adjoint equation is invariant under the symmetries of fundamental solutions and the solution of the original equation. Hence, Riemann function of the original equation is obtained from the reciprocity relation (3).

Formulate the algorithm of Riemann function construction based on the usage of the fundamental solutions symmetries:

1. Solving for symmetries of linear homogeneous equation (1).
2. Computation of the fundamental solutions symmetries.
3. Construction of invariant solutions with the aid of fundamental solutions symmetries.
4. Extraction of Riemann function from the computed invariant solutions invoking continuity condition of Riemann function and its first derivatives at the point \((x_0, y_0)\) and the condition \(R(x_0, y_0; x_0, y_0) = 1\).

**Remark 2.** The algorithm allows to find Riemann function of hyperbolic equation not going to the characteristic variables. This lays emphasis on the invariant nature of the given method for the construction of Riemann function.

Let us consider the equation

\[ Lu = \delta(x - x_0) \delta(y - y_0) , \]

**Remark 3** As Adamar noticed in [7] the fundamental solution of Eq. (4) can be written as

\[ u = R^\star \theta(x - x_0) \theta(y - y_0) . \]
Next, as Theorem 2 takes place, its right part is invariant of fundamental solutions symmetries.

Consider examples of the method application.

**Example 1.** Consider an equation

\[ u_{xy} + u = 0. \]  

(16)

Symmetry operators basis of the finite-dimensional part of Lie algebra of Eq. (16) has the form

\[ X_1 = \frac{\partial}{\partial x}, \quad X_1 = \frac{\partial}{\partial y}, \quad X_1 = x \frac{\partial}{\partial x} - y \frac{\partial}{\partial y}, \quad X_1 = u \frac{\partial}{\partial u}. \]

Therefore, function \( \lambda = a_4 \) (see (8)) and equation

\[ u_{xy} + u = \delta(x - x_0)\delta(y - y_0) \]

admits the symmetry operator

\[ Y = (x - x_0)\frac{\partial}{\partial x} - (y - y_0)\frac{\partial}{\partial y}. \]

(17)

The solution that is invariant under symmetry operator (17) is found as

\[ u = f(z), \quad z = (x - x_0)(y - y_0). \]

Here function \( f = f(z) \) is a solution of the ordinary differential equation

\[ zf'' + f' + f = 0. \]

(18)

The general solution of Eq. (18) has the form

\[ f = C_1J_0(2\sqrt{z}) + C_2Y_0(2\sqrt{z}), \]

where \( C_1, C_2 \) are arbitrary constants; \( J_0(z), \ Y_0(z) \) are Bessel functions [8]. The condition \( f(0) = 1 \) follows

\[ R^*(x, y; x_0, y_0) = R(x, y; x_0, y_0) = J_0(2\sqrt{(x - x_0)(y - y_0)}). \]

**Example 2.** Consider an equation

\[ u_{xy} + \frac{1}{4(x + y)^2} u = 0. \]  

(19)

Symmetry operators basis of the finite-dimensional part of Lie algebra of (19) has the form

\[ X_1 = \frac{\partial}{\partial x} - \frac{\partial}{\partial y}, \quad X_2 = x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y}, \quad X_3 = x^2 \frac{\partial}{\partial x} - y^2 \frac{\partial}{\partial y}, \quad X_4 = u \frac{\partial}{\partial u}. \]

Therefore, function \( \lambda = -2a_2 - 2a_3(x - y) + a_4 \) and equation

\[ u_{xy} + \frac{1}{4(x + y)^2} u = \delta(x - x_0)\delta(y - y_0) \]

admits the symmetry operator

\[ Y = (x - x_0)(x + y_0)\frac{\partial}{\partial x} - (y - y_0)(y + x_0)\frac{\partial}{\partial y}. \]

(20)
The solution that is invariant under symmetry operator (20) is found as
\[ u = f(z), \quad z = \frac{(x - x_0)(y - y_0)}{(x_0 + y_0)(x + y)}. \]

Here function \( f = f(z) \) is a solution of the ordinary differential equation
\[ (z^2 + z)f'' + (2z + 1)f' + \frac{1}{4}f = 0. \]  
(21)

The general solution of Eq. (21) has the form
\[ f = C_1 \text{EllipticK}(i\sqrt{z}) + C_2 \text{EllipticCK}(i\sqrt{z}), \]
where \( C_1, C_2 \) are arbitrary constants; \( \text{EllipticK}(z) \) is complete elliptic integral of the first kind and \( \text{EllipticCK}(z) \) is complementary complete elliptic integral of the first kind [8]. From the condition \( f(0) = 1 \) follows
\[ R^* (x, y; x_0, y_0) = R (x, y; x_0, y_0) = 2\pi \text{EllipticK} \left( i \sqrt{\frac{(x - x_0)(y - y_0)}{(x_0 + y_0)(x + y)}} \right). \]

5. Riemann’s function for Euler–Poisson–Darboux equation

Let us consider Euler–Poisson–Darboux equation
\[ Lu = \frac{\partial^2 u}{\partial r^2} + \frac{2\alpha}{r} \frac{\partial u}{\partial r} - \frac{\partial^2 u}{\partial z^2} = 0. \]  
(22)

The symmetries of Eq. (22) can be found using the symmetry-finding algorithm from [4].

**Proposition 2.** In case of \( \alpha \neq 0 \) Eq. (22) admits the following basis of the finite part of Lie algebra symmetry operators
\[ Y_1 = \frac{\partial}{\partial z}, \quad Y_2 = r \frac{\partial}{\partial r} + z \frac{\partial}{\partial z}, \quad Y_3 = u \frac{\partial}{\partial u}, \]
\[ Y_4 = 2rz \frac{\partial}{\partial r} + (r^2 + z^2) \frac{\partial}{\partial z} - 2\alpha z u \frac{\partial}{\partial u}. \]

Let us consider the equation
\[ Lu = \delta (r - r_0) \delta (z - z_0). \]  
(23)

We use the results of [5] in order to find the symmetries of Eq. (23).

**Proposition 3.** Eq. (23) admits the symmetry operator
\[ Y = 2r(z - z_0) \frac{\partial}{\partial r} + [r^2 + (z - z_0)^2 - r_0^2] \frac{\partial}{\partial z} - 2\alpha (z - z_0) u \frac{\partial}{\partial u}. \]  
(24)

The symmetry operator (24) has two functionally independent invariants
\[ \xi = \frac{r^2 - (z - z_0)^2 + r_0^2}{2rr_0}, \quad \tau = r^\alpha u. \]

Invariant solutions are sought in the form
\[ \tau = f(\xi), \]
or

\[ u = r^{-\alpha} f(\xi). \]

**Proposition 4.** Euler–Poisson–Darboux solutions (22), that are invariant under symmetry operator (24), have the following form

\[ u = r^{-\alpha} \left[ C_1 P_{-\alpha}(\xi) + C_2 Q_{-\alpha}(\xi) \right], \]

where \( C_1, C_2 \) are arbitrary constants; \( P_{-\alpha}(\xi), \ Q_{-\alpha}(\xi) \) are Legendre functions of the first and second kind [8]. The condition \( f(0) = 1 \) gives us Riemann function \( R^* \).

**Proposition 5.** Riemann function of adjoint Eq. (22) appears as follows

\[ R^*(r, z; r_0, z_0) = \left( \frac{r}{r_0} \right)^{-\alpha} P_{-\alpha}(\xi). \] (25)

Riemann function (25) describes the solution of characteristic problem of the two centered depression waves interpenetration up to a constant multiplier \( t_0 \) [9].

6. Conclusion
The main result of the study is the suggested method for Riemann function construction based on the usage of the symmetries of fundamental solutions. Its effectiveness is performed by examples.

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