A geometric inequality for circle packings

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Abstract

A geometric inequality among three triangles, originating in circle packing problems, is introduced. In order to prove it, we reduce the original formulation to the nonnegativity of a particular polynomial in four real indeterminates. Techniques based on sum of squares decompositions, semidefinite programming, and symmetry reduction are then applied to provide an easily verifiable nonnegativity certificate.

1 Introduction

In this paper we prove the following geometric inequality: suppose that we have three triangles. One with sides of lengths $X, Y$ and $Z$, a second with sides of lengths $U, V$ and $W$ and a third triangle with sides of lengths $(X + U), (Y + V)$ and $(Z + W)$. Let us denote by $\alpha$ the angle in the first triangle between the sides of lengths $X$ and $Y$. Let $\beta$ be the corresponding angle in the second triangle between $U$ and $V$ and let $\gamma$ be the corresponding angle between $(X + U)$ and $(Y + V)$ in the third triangle. Then

$$\alpha \cdot (X + Y - Z) + \beta \cdot (U + V - W) \leq \gamma \cdot ((X + U) + (Y + V) - (Z + W)).$$

It turns out that proving this inequality is not at all simple. The need for this inequality originates in [10]. This last paper describes a new approach to circle packings [1, 7, 11]. The main features of this approach are the theory of Perron-Frobenius for non-negative matrices, [3, 6, 13] and fixed-point theory, [8, 2]. In particular our approach uses a converse of the contraction principle as appears in [2]. A central object that is introduced in [10] is the $\pi$-mapping, $f_\pi : \mathbb{R}^{+|V|} \to \mathbb{R}^{+|V|}$, of a graph embedding. This is a variant of Thurston’s relaxation mapping. A key property of the $\pi$-mapping, $f_\pi$, is its super-additivity,

$$\forall \bar{\tau}, \bar{s} \in \mathbb{R}^{+|V|}, \ f_\pi(\bar{\tau}) + f_\pi(\bar{s}) \leq f_\pi(\bar{\tau} + \bar{s}).$$

It turns out that this property is implied by the above geometric inequality. A very interesting feature of our proof of this inequality is the use of semidefinite programming

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based algorithms for producing representations of non-negative polynomials as sums of squares \([4, 9]\).

We make a reduction of the original inequality to the non-negativity of a certain real polynomial of degree 20 in 4 real indeterminates. Although instances of this size are near the limits of what can be achieved using generic methods, our particular polynomial enjoys certain convenient sparsity and symmetry properties. These allow the use of new algorithms, introduced in \([5]\), customized for polynomials with an invariant structure. Even though the computational procedures in its current form use floating point arithmetic to arrive at the result, the final solution can be easily verified in a completely independent fashion. The methods work quite nicely in the problem at hand, producing a concise representation of the polynomial as a sum of five squares, thereby concluding the proof.

2 The super-additivity of \(f_{\pi}\)

As proved in \([10]\), the super-additivity of \(f_{\pi}\) follows from

**Theorem 1** If \(a, b, c, d, R, S > 0\), then

\[
R \sin^{-1} \left\{ \frac{ab}{(R + a)(R + b)} \right\} + S \sin^{-1} \left\{ \frac{cd}{(S + c)(S + d)} \right\} \leq (R + S) \sin^{-1} \left\{ \frac{(a + c)(b + d)}{(R + S + a + c)(R + S + b + d)} \right\}.
\]

The theorem has two simple geometric interpretations:

(1) Three circles of radii \(R, a\) and \(b\) that are mutually tangent to one another from the outside form an Euclidean triangle. The vertices of the triangle are the centers of the circles. The sides of the triangle have the following lengths: \(R + a\), \(a + b\) and \(R + b\). Similarly three circles of radii \(S, c\) and \(d\) form a triangle of sides \(S + c\), \(c + d\) and \(S + d\). Finally, a third such triangle is formed by three circles of radii \(R + S + a + c\), \(a + c + b + d\) and \(R + S + b + d\). We note that the sides of the third triangle have lengths which are the sums of the corresponding sides of the first two triangles. On the other hand the three sets of triples of circles form also three circular triangles. The vertices of these triangles are the tangency points of pairs of circles in each triple. The lemma implies that the circular sides of the third (largest) triangle are greater than or equal to the sums of the corresponding circular sides of the first two circular triangles.

(2) Let us consider three Euclidean triangles. One with sides of lengths \(X, Y\) and \(Z\), and an angle \(\alpha\) between \(X\) and \(Y\). A second triangle with sides of lengths \(U, V\) and \(W\), and an angle \(\beta\) between \(U\) and \(V\). A third triangle with sides of lengths \((X + U), (Y + V)\) and \((Z + W)\), and an angle \(\gamma\) between \((X + U)\) and \((Y + V)\). Then

\[
\alpha \cdot (X + Y - Z) + \beta \cdot (U + V - W) \leq \gamma \cdot ((X + U) + (Y + V) - (Z + W)).
\]
3 Reducing to algebraic inequalities

We now make a reduction of the inequality of Theorem 1. There are two ideas involved in it. The first idea is summarized in the following,

Lemma 1 Suppose that there exists a twice differentiable, surjective and strictly increasing function \( f : I \rightarrow [0,1] \) which satisfies the following two conditions:

(1) \[ f''(1 - f^2) + f \cdot (f')^2 \leq 0 \text{ on } I. \]

(2) \[
\left( \frac{R}{R + S} \right) f^{-1} \left( \sqrt{\frac{ab}{(R + a)(R + b)}} + \frac{S}{(R + S)} \right)
\leq f^{-1} \left( \sqrt{\frac{(a+c)(b+d)}{(R + S + a + c)(R + S + b + d)}} \right),
\]

for all \( a, b, c, d, R, S > 0 \). Then, the inequality of Theorem 1 holds true.

Proof.
Consider the function \( y = \sin^{-1} f(x) \) for \( x \in I \). Then,

\[
\frac{dy}{dx} = \frac{f'}{\sqrt{1 - f^2}},
\]

\[
\frac{d^2y}{dx^2} = \frac{f''(1 - f^2) + f \cdot (f')^2}{(1 - f^2)^{3/2}}.
\]

By condition (1) we get \( \frac{d^2y}{dx^2} \leq 0 \) on \( I \) and hence \( y \) is concave in \( I \). So for any \( x, z \in I \) and for any \( 0 \leq t \leq 1 \), we have

\[
t \sin^{-1} f(x) + (1 - t) \sin^{-1} f(z) \leq \sin^{-1} f(tx + (1 - t)z). \tag{1}
\]

We make the following choice,

\[
x = f^{-1} \left( \sqrt{\frac{ab}{(R + a)(R + b)}} \right), \quad z = f^{-1} \left( \sqrt{\frac{cd}{(S + c)(S + d)}} \right), \quad t = \left( \frac{R}{R + S} \right).
\]

Then, by inequality (1) we get,

\[
\left( \frac{R}{R + S} \right) \sin^{-1} \left( \sqrt{\frac{ab}{(R + a)(R + b)}} \right) + \left( \frac{S}{R + S} \right) \sin^{-1} \left( \sqrt{\frac{cd}{(S + c)(S + d)}} \right) \leq \sin^{-1} f(tx + (1 - t)z).
\tag{2}
\]

By condition (2) we have,

\[
tx + (1 - t)z \leq f^{-1} \left( \sqrt{\frac{(a+c)(b+d)}{(R + S + a + c)(R + S + b + d)}} \right).
\]
and since $f$ is increasing and also $\sin^{-1}$ is increasing, we get,

$$
\sin^{-1} f(tx + (1-t)y) \leq \sin^{-1}\left(\sqrt{\frac{(a+c)(b+d)}{(R+S+a+c)(R+S+b+d)}}\right).
$$

(3)

Theorem 1 follows by inequalities (2) and (3).

**special cases.**

(I) $f(x) = \sin x$, $I = [0, \pi/2]$. Then in this case we have,

$$
f''(1 - f^2) + f \cdot (f')^2 = -\sin x \cos^2 x + \sin x \cos^2 x = 0
$$

and condition (1) of the theorem is satisfied. Condition (2) is the inequality of Theorem 1 and so the theorem is correct trivially in this case.

(II) $f(x) = 1 - 1/x$, $I = [1, \infty]$. Then in this case we have,

$$
f''(1 - f^2) + f \cdot (f')^2 = -\frac{2}{x^3} \left(1 - \left(1 - \frac{1}{x}\right)^2\right) + \left(1 - \frac{1}{x}\right) \cdot \frac{1}{x^4} = \\
= \frac{1}{x^3} - \frac{3}{x^4} < 0,
$$

for $x \geq 1$. So condition (1) is satisfied. Condition (2) and the conclusion of the theorem prove the following,

**Lemma 2** If for every $a, b, c, d, R, S > 0$ the following inequality is true,

$$
\left(\frac{R}{R+S}\right) \left(\frac{1}{1 - \sqrt{(ab)/(R+a)(R+b)}}\right) + \\
+ \left(\frac{S}{R+S}\right) \left(\frac{1}{1 - \sqrt{(cd)/(S+c)(S+d)}}\right) \leq \\
\leq \frac{1}{1 - \sqrt{((a+c)(b+d))/((R+S+a+c)(R+S+b+d))}}
$$

then, the inequality of Theorem 1 is true.

The second idea in this approach (after that of Lemma 1) is an elementary trick to get rid of the square root functions in Lemma 2. Let us denote,

$$
\alpha = \sqrt{\frac{a}{R+a}}, \quad \beta = \sqrt{\frac{b}{R+b}}, \quad \gamma = \sqrt{\frac{c}{S+c}}, \quad \delta = \sqrt{\frac{d}{S+d}}.
$$

Then $0 \leq \alpha, \beta, \gamma, \delta \leq 1$. Also $\alpha, \beta$ are independent except for $\alpha = 1$ iff $\beta = 1$. That happens only if $R = 0$. $\gamma, \delta$ are independent except for $\gamma = 1$ iff $\delta = 1$. That happens only if $S = 0$. For the inverse transformations we have,

$$
a = \left(\frac{\alpha^2}{1 - \alpha^2}\right) R, \quad b = \left(\frac{\beta^2}{1 - \beta^2}\right) R, \quad c = \left(\frac{\gamma^2}{1 - \gamma^2}\right) S, \quad d = \left(\frac{\delta^2}{1 - \delta^2}\right) S.
$$
With these notations, the left hand side of the inequality in Lemma 2 is,
\[
\left( \frac{R}{R+S} \right) \left( \frac{1}{1-\alpha \beta} \right) + \left( \frac{S}{R+S} \right) \left( \frac{1}{1-\gamma \delta} \right) = \frac{R(1-\gamma \delta) + S(1-\alpha \beta)}{(R+S)(1-\alpha \beta)(1-\gamma \delta)}.
\]

As for the right hand side, we have,
\[
I_1 = \sqrt{\frac{a+c}{R+S+a+c}} = \sqrt{\frac{\alpha^2(1-\gamma \delta^2)R+\gamma^2(1-\alpha^2)S}{(1-\gamma \delta)R+(1-\alpha^2)S}},
\]
\[
I_2 = \sqrt{\frac{b+d}{R+S+b+d}} = \sqrt{\frac{\beta^2(1-\delta^2)R+\delta^2(1-\beta^2)S}{(1-\delta^2)R+(1-\beta^2)S}}.
\]

Plugging these into the inequality of Lemma 2 we get,
\[
\frac{R(1-\gamma \delta) + S(1-\alpha \beta)}{(R+S)(1-\alpha \beta)(1-\gamma \delta)} \leq \frac{1}{1-I_1I_2}.
\]

Hence,
\[
I_1I_2 \geq \frac{R\alpha \beta (1-\gamma \delta) + S\gamma \delta (1-\alpha \beta)}{R(1-\gamma \delta) + S(1-\alpha \beta)}.
\]

On squaring both sides we conclude that in order to prove Theorem 1, it suffices to prove the following,

**Lemma 3** If \( R, S > 0 \) and \( 0 < \alpha, \beta, \gamma, \delta < 1 \), then,
\[
\left( \frac{\alpha^2(1-\gamma \delta^2)R+\gamma^2(1-\alpha^2)S}{(1-\gamma \delta)R+(1-\alpha^2)S} \right) \left( \frac{\beta^2(1-\delta^2)R+\delta^2(1-\beta^2)S}{(1-\delta^2)R+(1-\beta^2)S} \right) \geq \left( \frac{R\alpha \beta (1-\gamma \delta) + S\gamma \delta (1-\alpha \beta)}{R(1-\gamma \delta) + S(1-\alpha \beta)} \right)^2.
\]

**Proof:** Let us define,
\[
E = \left( \frac{\alpha^2(1-\gamma \delta^2)R+\gamma^2(1-\alpha^2)S}{(1-\gamma \delta)R+(1-\alpha^2)S} \right) \left( \frac{\beta^2(1-\delta^2)R+\delta^2(1-\beta^2)S}{(1-\delta^2)R+(1-\beta^2)S} \right) - \\
\quad - \left( \frac{R\alpha \beta (1-\gamma \delta) + S\gamma \delta (1-\alpha \beta)}{R(1-\gamma \delta) + S(1-\alpha \beta)} \right)^2.
\]

Then,
\[
E = \frac{RS(R+S)[(1-\gamma \delta)L \cdot R + (1-\alpha \beta)M \cdot S]}{[(1-\gamma \delta)R+(1-\alpha \beta)S][(1-\delta^2)R+(1-\beta^2)S][(1-\gamma \delta)R+(1-\alpha \beta)S]^2},
\]

where we have,
\[
L = \alpha^2 \beta^2 (\alpha - \beta)^2 + (\alpha - \beta)^2 \gamma^3 \delta^3 + \\
\quad + ((\alpha \delta)^2 (1-\alpha \beta)(1-\alpha \beta^2) + (\alpha \gamma)(2-4\alpha \beta + \beta \alpha^3 + \alpha \beta^3) + \\
\quad + (\beta \gamma)^2 (1-\alpha \beta)(1+\alpha \beta - 2\alpha^2)) + \\
\quad + \gamma^2 \beta (1-\alpha \beta)(2\alpha - \beta - \alpha \beta^2) - \gamma \delta (\alpha^2 + \beta^2 + 2\alpha^3 \beta^3 - 4\alpha^2 \beta^2),
\]
\[\delta^2 \alpha (1-\alpha \beta)(2\beta - \alpha - \alpha^2 \beta) \gamma \delta,\]
or as a polynomial in \(\gamma\) and \(\delta\),
\[L = \alpha^2 \beta^2 (\alpha - \beta)^2 + \beta^2 (1 - \alpha \beta)(1 + \alpha \beta - 2\alpha^2) \gamma^2 +\]
\[\alpha^2 (1 - \alpha \beta)(1 + \alpha \beta - 2\beta^2) \delta^2 - \alpha \beta (2 + \alpha \beta^2 - 4\alpha \beta + \alpha^3 \beta) \gamma \delta +\]
\[\beta (1 - \alpha \beta)(2\alpha - \beta - \alpha \beta^2) \gamma \delta - (\alpha^2 + \beta^2 + 2\alpha^3 \beta^3 - 4\alpha^2 \beta^2) \gamma^2 \delta^2 +\]
\[\alpha (1 - \alpha \beta)(2\beta - \alpha - \alpha^2 \beta) \gamma \delta + (\alpha - \beta)^2 \gamma^2 \delta^2,\]
and where \(M = M(\alpha, \beta, \gamma, \delta) = L(\gamma, \delta, \alpha, \beta).\) Thus it suffices to prove that for any \(0 < \alpha, \beta, \gamma, \delta < 1\) we have \(L(\alpha, \beta, \gamma, \delta) \geq 0.\) For this will also imply that \(M(\alpha, \beta, \gamma, \delta) \geq 0\) for any such a choice. This, in turn, will show that \(E \geq 0\) for every choice of \(R, S > 0\) and \(0 < \alpha, \beta, \gamma, \delta < 1\) and hence will prove Lemma 3. To check the non-negativity of \(L(\alpha, \beta, \gamma, \delta)\) we make the substitutions
\[(\alpha, \beta, \gamma, \delta) = (\frac{x^2}{1+x^2}, \frac{y^2}{1+y^2}, \frac{z^2}{1+z^2}, \frac{w^2}{1+w^2}) \quad (4)\]
and clear the denominators. This will give us a polynomial in \(\mathbb{R}[x, y, z, w].\) In fact, this polynomial is
\[P(x, y, z, w) =\]
\[= L \left( \frac{x^2}{1+x^2}, \frac{y^2}{1+y^2}, \frac{z^2}{1+z^2}, \frac{w^2}{1+w^2} \right)(1 + x^2)^4(1 + y^2)^4(1 + z^2)^3(1 + w^2)^3. \quad (5)\]
As a consequence, it suffices to check that \(P(x, y, z, w)\) is non-negative for all real values of its indeterminates. We conclude the proof of Lemma 3 in the following section, after a brief detour explaining the sum of squares based methods we have used.

### 4 Sums of squares

An obvious sufficient condition for non-negativity is to represent \(P(x, y, z, w)\) as a sum of squares of real polynomials. The connections between sums of squares and non-negativity have been extensively studied since the end of the 19th century, when Hilbert showed that in the general case the two conditions are not equivalent. We refer the reader to the wonderful survey \([12]\) by Reznick on the available results and history of Hilbert’s 17th problem. In the work of Choi, Lam, and Reznick \([4]\) the algebraic structure of sums of squares decompositions is fully analyzed, and the important “Gram matrix” method is introduced. On the computational side, convex optimization approaches to this problem originate in the early work of Shor \([14]\). Recently, efficient techniques using semidefinite programming and exploiting problem structure have been developed in \([1, 5]\). A brief description of the methods follows, referring the reader to the cited works, and the references therein, for the full algorithmic details.

We explain next the general idea of the Gram matrix method. Given a multivariate polynomial \(F(x)\) for which we want to decide whether a sum of squares decomposition exists, we attempt to express it as a quadratic form in a new set of variables \(u.\) A judicious choice of these new variables will depend on both the sparsity structure and symmetry properties of \(F [4, 5].\) For instance, for the simplest case of a generic dense
polynomial of total degree $2d$, the variables $u$ will be all the monomials (in the variables $x$) of degree less than or equal to $d$. Consequently, we try to represent $F(x)$ as:

$$F(x) = u^T Q u$$  \hspace{1cm} (6)$$

where $Q$ is a constant matrix. Since in general the variables $u$ will not be algebraically independent, the matrix $Q$ in the representation (6) is not unique. In fact, there is an affine subspace of matrices $Q$ that satisfy the equality, as can be easily seen by expanding the right-hand side and equating term by term. If in the representation above the matrix $Q$ can be chosen to be positive semidefinite, then a factorization of the matrix $Q$ directly provides a sum of squares decomposition of $F(x)$. Conversely, if $F$ is a sum of squares, then such a $Q$ can always be constructed by expanding the terms in monomials. Therefore, the problem of checking if a polynomial can be decomposed as a sum of squares is equivalent to verifying whether a certain affine matrix subspace intersects the cone of positive definite matrices. This latter class of convex optimization problems is known as semidefinite programs (SDP) [16], and can be efficiently solved using a variety of numerical algorithms, mainly based on interior point methods.

**Example 1** Consider the quartic form in two variables described below, and define $u = [x^2, y^2, xy]^T$.

$$F(x, y) = 2x^4 + 2x^3y - x^2y^2 + 5y^4$$

$$= \begin{bmatrix} x^2 \\ y^2 \\ xy \end{bmatrix}^T \begin{bmatrix} q_{11} & q_{12} & q_{13} \\ q_{12} & q_{22} & q_{23} \\ q_{13} & q_{23} & q_{33} \end{bmatrix} \begin{bmatrix} x^2 \\ y^2 \\ xy \end{bmatrix}$$

$$= q_{11}x^4 + q_{22}y^4 + (q_{33} + 2q_{12})x^2y^2 + 2q_{13}x^3y + 2q_{23}xy^3$$

Therefore, in order to have an identity, the following linear equalities should hold:

$$q_{11} = 2, \quad q_{22} = 5, \quad q_{33} + 2q_{12} = -1, \quad 2q_{13} = 2, \quad 2q_{23} = 0. \quad (7)$$

A positive semidefinite $Q$ that satisfies the linear equalities can then be found using semidefinite programming. A particular solution is given by:

$$Q = \begin{bmatrix} 2 & -3 & 1 \\ -3 & 5 & 0 \\ 1 & 0 & 5 \end{bmatrix} = L^T L, \quad L = \frac{1}{\sqrt{2}} \begin{bmatrix} 2 & -3 & 1 \\ 0 & 1 & 3 \end{bmatrix},$$

and therefore we have the sum of squares decomposition:

$$F(x, y) = \frac{1}{2}(2x^2 - 3y^2 + xy)^2 + \frac{1}{2}(y^2 + 3xy)^2.$$
some as a result of the substitution (4). Concretely, it is easy to see that $P$ is invariant under the transformations:

$$
(x, y, z, w) \rightarrow (y, x, w, z) \quad (8)
$$

$$
\rightarrow (\pm x, \pm y, \pm z, \pm w) \quad (9)
$$

The first property is a clear consequence of the symmetry of our original geometric inequality with respect to interchange of the two triangles. The second one is a side effect of our choice for modeling the nonnegativity constraints. The transformations given above generate a symmetry group $G$ with 32 elements and 14 irreducible representations: eight one-dimensional and six two-dimensional ($8 \cdot 1^2 + 6 \cdot 2^2 = 32$). As explained extensively by Gatermann and Parrilo in [5], these symmetries can be exploited very successfully in reducing the computational requirements.

To do this, the approach in [5] relies on a crucial property of convex optimization problems invariant under a group action, namely the fact that the optimal solution can always be restricted to the fixed-point subspace. Using Schur’s lemma of representation theory, it is shown there that by using an appropriate symmetry-adapted coordinate transformation, the original semidefinite program can be decomposed into a collection of smaller coupled problems, of cardinality equal to the number of irreducible representations of the group. This reduces both the size and the number of variables in the problem, and as a consequence notably enhances both the accuracy and conditioning of the solution.

Attempting to directly establish the nonnegativity of $P$ without taking into account both the sparsity and symmetries can be a difficult (or even impossible) task for current SDP solvers, both in terms of memory requirements and accuracy. A naive approach, using only degree information but no structure whatsoever, would require solving a semidefinite program of dimension 1001 $\times$ 1001 and 10626 constraints. By exploiting only the sparsity of $P$, but not its symmetry, the problem is reduced to dimension 137 $\times$ 137 and 1328 constraints. Adding the simplifications resulting from the symmetries, the problem is further simplified to a much more manageable one with 14 coupled SDP (one for each irreducible representation), of dimensions ranging between 2 $\times$ 2 and 11 $\times$ 11 (see Table [1]). For instance, for the trivial irreducible representation (# 1 in the table), the corresponding new variables $u$ are invariant under the group action, and given by:

$$
\begin{align*}
&y^2 z^2 + x^2 w^2 \\
x^2 z^2 w^2 + y^2 z^2 w^2, & y^4 z^2 + x^4 w^2, & x^2 y^2 z^2 + x^2 y^2 w^2, & x^4 y^2 + x^2 y^4 \\
x^2 y^2 z^2 w^2, & x^4 z^2 w^2 + y^4 z^2 w^2, & x^2 y^4 z^2 + x^4 y^2 w^2, & x^4 y^2 z^2 + x^2 y^4 w^2.
\end{align*}
$$

Notice in Table [1] that the combined total, taking into account multiplicities, is equal to 137, the dimension of the sparse version of the problem.

The resulting system of matrix inequalities can be solved with standard SDP solvers, such as SeDuMi [15]. The output provides a decomposition of $P$ as a sum of squares of polynomials, with coefficients given by floating point numbers. In this particular case, the computed values immediately suggest the existence of a solution, presented below, with polynomials having integer coefficients. The solution can be verified in a completely independent fashion, providing a mathematically correct certificate of the nonnegativity of the polynomial $P$. 

8
| Irr. Rep. # | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 |
|-------------|---|---|---|---|---|---|---|---|---|----|----|----|----|----|
| Multiplicity| 1 | 1 | 1 | 1 | 1 | 1 | 1 | 2 | 2 | 2  | 2  | 2  | 2  | 2  |
| Dim. SDP    | 9 | 6 | 6 | 4 | 8 | 5 | 3 | 2 | 11| 7  | 8  | 7  | 8  | 6  |

Table 1: Irreducible representations of $G$ and the corresponding SDP dimensions.

**Proof of Lemma 3: (continued)** Consider the following three polynomials:

\[ A(x, y, z, w) = -y^2z^2 - y^4z^2 + x^2w^2 + 2x^2y^2w^2 - 2x^2y^2z^2 - x^2y^4 - 2x^2y^4z^2 + x^4w^2 + 2x^4y^2w^2, \]

\[ B(x, y, z, w) = (1 + x^2 + y^2)(-x^2w^2 - x^2z^2w^2 - x^2y^2w^2 + x^2y^2z^2 + y^2z^2 + y^2z^2w^2), \]

and

\[ C(x, y, z, w) = (x - y)(x + y)(-x^2z^2w^2 + x^2y^2 + x^2y^2w^2 + x^2y^2z^2 - z^2w^2 - y^2z^2w^2), \]

Then we have the following identity

\[ P(x, y, z, w) = A(x, y, z, w)^2(z^2 + w^2 + 2z^2w^2) + B(x, y, z, w)^2 + C(x, y, z, w)^2. \]

Thus $P(x, y, z, w)$ is a sum of five squares of real polynomials and the proof of Lemma 3 is completed. ◇

Rewriting the obtained sum of squares decomposition in terms of the original variables, the following representation of $L$ can be obtained:

\[
L = L_1 + L_2 + L_3
\]

\[
L_1 = (\gamma + \delta)(-\alpha^2\beta + \alpha\beta^2 - \alpha\delta + \beta\gamma - \beta\gamma\delta + \alpha\delta\gamma - \alpha\beta^2\gamma + \alpha^2\beta\delta)^2
\]

\[
L_2 = (1 - \gamma)(1 - \delta)(\alpha\beta - 1)^2(\alpha\delta - \beta\gamma)^2
\]

\[
L_3 = (1 - \gamma)(1 - \delta)(\alpha - \beta)^2(\alpha\gamma - \beta\delta)^2.
\]

From this, stronger conclusions on the sign of $L$ can be immediately derived: not only it is nonnegative on the open unit hypercube $(0, 1)^4$ as needed for Lemma 3, but the same property holds on the much larger region $\mathbb{R} \times \mathbb{R} \times \{\gamma + \delta \geq 0, (1 - \gamma)(1 - \delta) \geq 0\}$.

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