QUANTUM SYMMETRIES ON NONCOMMUTATIVE COMPLEX SPHERES WITH PARTIAL COMMUTATION RELATIONS

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Abstract. We introduce the notion of noncommutative complex spheres with partial commutation relations for the coordinates. We compute the corresponding quantum symmetry groups of these spheres, and this yields new quantum unitary groups with partial commutation relations. We also discuss some geometric aspects of the quantum orthogonal groups associated with the mixture of classical and free independence discovered by Speicher and Weber. We show that these quantum groups are quantum symmetry groups on some quantum spaces of spherical vectors with partial commutation relations.

1. Introduction

This paper introduces a new class of noncommutative spheres and discusses the associated quantum symmetry groups. The quantization of classical spheres was initiated in the work of Podlés [Pod87, Pod95]. The theory of various noncommutative spheres and their quantum symmetries has been then remarkably developed in the past decades (see for example [CL01, CDV02, Gos09, BG10, Ban15, Ban16, Ban17] and references therein).

In a recent work [SW16], Speicher and Weber introduced a new class of noncommutative spheres with partial commutation relations, and computed the corresponding quantum symmetry group. This also leads to new versions of quantum orthogonal groups which do not interpolate between the classical and universal versions of orthogonal groups.

In this note we will continue the project proposed by [SW16]. We will discuss the complex versions of noncommutative spheres with partial commutation relations. We will compute the quantum symmetry groups of these objects. Compared to the real case studied in [SW16], the complex case involves more subtlety such as the mixture of normal and non-normal generators. Similarly as in the real case, we obtain new examples of quantum unitary groups with partial commutation relations. We refer to Section 2 for all details.

On the other hand, we also answer some unsolved problems in [SW16] regarding the real case. In [SW16], by virtue of the mixture of independences in quantum probability, some quantum orthogonal groups with partial commutation relations are introduced. However the geometric aspects of these quantum groups were not clear in their work. In this note we will construct some quantum tuples of noncommutative spheres so that the corresponding quantum symmetry groups are exactly those studied in [SW16]. The result will be given in Section 3.

2. The noncommutative complex spheres and quantum symmetries

In this section we let $\varepsilon = (\varepsilon_{ij})_{i,j\in\{1,...,n\}}$ and $\eta = (\eta_{kl})_{k,l\in\{1,...,n\}}$ be two symmetric matrices with $\varepsilon_{ij} \in \{0,1\}$, $\varepsilon_{ii} = 0$ and $\eta_{kl} \in \{0,1\}$.
2.1. Noncommutative complex \((\varepsilon, \eta)-spheres\). We consider the universal C*-algebra

\[ C^*(x_1, \ldots, x_n \mid \sum_{i=1}^{n} x_i^* x_i = 1, x_i x_j = x_j x_i \text{ if } \varepsilon_{ij} = 1, x_i^* x_j = x_j x_i^* \text{ if } \eta_{ij} = 1). \]

As an intuitive notation, we denote the above C*-algebra by \(C(S_{\mathbb{C}, \varepsilon, \eta}^{n-1})\) and we say that \(S_{\mathbb{C}, \varepsilon, \eta}^{n-1}\) is a noncommutative complex \((\varepsilon, \eta)-sphere\). Note that if all non-diagonal entries of \(\varepsilon\) and all entries of \(\eta\) are 1, then we obtain the algebra \(C(S_{\mathbb{C}}^{n-1})\) of continuous functions over the complex sphere \(S_{\mathbb{C}}^{n-1} \subset \mathbb{C}^n\). If all entries of \(\varepsilon\) and \(\eta\) are 0, we get the Banica’s free version of complex spheres in [Ban15].

Compared to the real spheres studied in [SW16], we consider two matrices \(\varepsilon\) and \(\eta\) rather than one in order to include the case where the generators \(x_i\) are non-normal. Note that the diagonal entries of \(\eta\) are related to the normality of the generators. We give the following remarks.

**Lemma 1.** (1) Let \(1 \leq i \leq n\). If \(x_i\) is normal (in other words if \(\eta_{ii} = 1\)), then for any \(j\),

\[ x_i x_j = x_j x_i \quad \text{iff} \quad x_i^* x_j = x_j x_i^*. \]

(2) Assume that \(\varepsilon_{ij} = \eta_{ij} = 1\) whenever \(\eta_{ii} = \eta_{jj} = 0\) and \(i \neq j\). Then \(x_i\) are normal for all \(1 \leq i \leq n\).

**Proof.** The assertion (1) follows immediately from the Fuglede theorem (see for example [Rud91] 12.16). Let us prove the assertion (2). Without loss of generality, let us assume that \(\eta_{ii} = 0\) for \(1 \leq i \leq k\), \(\eta_{ii} = 1\) for \(k + 1 \leq i \leq n\), and \(\varepsilon_{ij} = \eta_{ij} = 1\) for \(1 \leq i, j \leq k\) and \(i \neq j\), and show that \(x_1\) is normal.

Denote \(y = \sum_{i=2}^{k} x_i\). Note that

\[ \sum_{i=1}^{k} x_i^* x_i = 1 - \sum_{i=k+1}^{n} x_i^* x_i = 1 - \sum_{i=k+1}^{n} x_i x_i^* = \sum_{i=1}^{k} x_i^* x_i. \]

So together with our assumption, we have

\[ (x_1 + y)^*(x_1 + y) = (x_1 + y)(x_1 + y)^*, \]

\[ (x_1 - y)^*(x_1 - y) = (x_1 - y)(x_1 - y)^*, \]

\[ (x_1 + y)(x_1 - y) = (x_1 - y)(x_1 + y). \]

By the Fuglede theorem we deduce that

\[ (x_1 + y)^*(x_1 - y) = (x_1 - y)(x_1 + y)^*. \]

Therefore

\[ x_1 x_1 = \frac{1}{4}[(x_1 + y) + (x_1 - y)]^*[x_1 + y] + (x_1 - y)]^* = x_1 x_1, \]

as desired. \(\square\)

**Remark 2.** By the above lemma, if all the generators \(x_i\) are normal (\(\eta_{ii} = 1\) for all \(1 \leq i \leq n\)), the C*-algebra \(C(S_{\mathbb{C}, \varepsilon, \eta}^{n-1})\) can be simply determined by the entries of \(\varepsilon\), that is,

\[ C(S_{\mathbb{C}, \varepsilon, \eta}^{n-1}) = C^*(x_1, \ldots, x_n \mid x_i^* x_i = x_i x_i^*, \sum_{i=1}^{n} x_i^* x_i = 1, x_i x_j = x_j x_i \text{ if } \varepsilon_{ij} = 1). \]
However, it is still worth considering two different matrices $\varepsilon, \eta$ rather than one since there exist non-trivial representations of $C(S^{n-1}_{C,\varepsilon,\eta})$ with $\varepsilon_{ij} \neq \eta_{ij}$. For instance, take

$$\varepsilon = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \eta = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},$$

then there exists a representation

$$\pi : C(S^{1}_{C,\varepsilon,\eta}) \to M_{4}(\mathbb{C}), \quad \pi(x_1) = a, \pi(x_2) = b$$

such that $a$ and $b$ are not normal and

$$ab = ba(\neq 0), \quad ab^* \neq b^*a.$$

Indeed, it suffices to take

$$a = \begin{bmatrix} 0 & 0 & 0 & \sqrt{2} \\ 0 & 0 & 0 & \sqrt{2} \\ 1 & 0 & 0 & 0 \\ \sqrt{2} & \sqrt{2} & \sqrt{2} & \sqrt{2} \end{bmatrix}, \quad b = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ \sqrt{2} & \sqrt{2} & \sqrt{2} & \sqrt{2} \end{bmatrix}.$$

### 2.2. Quantum symmetries.

Now we introduce the corresponding quantum groups. By virtue of Lemma 11, we make the convention that

(2.1) \[ \varepsilon_{ij} = \eta_{ij}, \quad \text{if } \eta_{ii} = 1 \text{ or } \eta_{jj} = 1, \]

and for all $i \neq j$ with $\eta_{ii} = \eta_{jj} = 0$, there exists $i_1, \ldots, i_r$ with $i_1 = i, i_r = j$ such that

(2.2) \[ \varepsilon_{i_s i_{s+1}} = 0 \text{ or } \eta_{i_s i_{s+1}} = 0, \quad 1 \leq s \leq r - 1. \]

For simplicity, we say that the pair $(\varepsilon, \eta)$ is regular if (2.1) and (2.2) hold. In spite of Lemma 11, the convention (2.2) still excludes a few examples of noncommutative complex $(\varepsilon, \eta)$-spheres. We keep this convention in order to avoid some technical complication. Indeed our arguments may work for more general cases, and we will explain it later in Remark 5.

We refer to [Wor87, Wor98, Tim08] for any unexplained notation and terminology on compact matrix quantum groups. Define the universal $C^*$-algebra

$$C(U^{\varepsilon,\eta}_{R}) = C^*(u_{ij}, i, j = 1, \ldots, n \mid u \text{ and } \bar{u} \text{ are unitary}, R^\varepsilon \text{ and } R^\eta \text{ hold}),$$

where $R^\varepsilon$ are the relations

$$u_{ik}u_{jl} = \begin{cases} u_{ij}u_{ik} & \text{if } \varepsilon_{ij} = 1 \text{ and } \varepsilon_{kl} = 1 \\ u_{jk}u_{il} & \text{if } \varepsilon_{ij} = 1 \text{ and } \varepsilon_{kl} = 0 \\ u_{il}u_{jk} & \text{if } \varepsilon_{ij} = 0 \text{ and } \varepsilon_{kl} = 1 \end{cases},$$

and $R^\eta$ are the relations

(2.3) \[ u_{ik}^*u_{jl} = u_{jl}^*u_{ik}^*, \quad \text{if } \eta_{ij} = \eta_{kl} = 1, \]

(2.4) \[ u_{ik}^*u_{jl} = 0, \quad u_{ik}u_{jl}^* = 0, \quad \text{if } \eta_{ij} = 1, \eta_{kl} = 0, k \neq l, \]

(2.5) \[ u_{ik}^*u_{jl} = 0, \quad u_{ik}u_{jl}^* = 0, \quad \text{if } \eta_{ij} = 0, \eta_{kl} = 1, i \neq j, \]

(2.6) \[ u_{ik}^*u_{jk} = u_{il}^*u_{ji} = u_{jk}u_{ik}^* = u_{lj}u_{il}^*, \quad \text{if } \eta_{ij} = 1, \eta_{kk} = \eta_{ll} = 0, \]

(2.7) \[ u_{ki}^*u_{kj} = u_{li}^*u_{ij} = u_{kj}u_{ki}^* = u_{jl}u_{li}^*, \quad \text{if } \eta_{ij} = 1, \eta_{kk} = \eta_{ll} = 0. \]
So for \( \eta_{kj} = 1 \) we may define
\[
X_{ij} = u_{ik}^* u_{jk} = u_{jk} u_{ik}^*, \quad Y_{ij} = u_{ki}^* u_{kj} = u_{kj} u_{ki}^*,
\]
where \( k \) is any index satisfying \( \eta_{kk} = 0 \). Note that this definition does not depend on the choice of \( k \) by virtue of (2.6) and (2.7).

We consider the comultiplication defined by
\[
\Delta : C(U_n^{P,\eta}) \rightarrow C(U_n^{P,\eta}) \otimes C(U_n^{P,\eta}) \quad u_{ij} \mapsto \sum_{k=1}^n u_{ik} \otimes u_{kj}.
\]
Note that if all non-diagonal entries of \( \varepsilon \) and all entries of \( \eta \) are 1, then we obtain the usual unitary group \( U_n \) of degree \( n \). If all entries of \( \varepsilon \) and \( \eta \) are 0, we get the free unitary group \((C(U_n^+), \Delta)\) introduced by Shuzhou Wang \[VDW96\].

**Proposition 3.** \( U_n^{P,\eta} \) is a compact matrix quantum group.

**Proof.** It suffices to prove that \( \Delta \) defines a *-homomorphism on \( C(U_n^{P,\eta}) \). By the universality of \( C(U_n^{P,\eta}) \), it remains to verify that the elements \( u_{ij}' := \Delta(u_{ij}) \), \( 1 \leq i, j \leq n \) satisfy the relations in the definition of \( C(U_n^{P,\eta}) \). It is routine to see that the matrix \( u' = [u_{ij}'] \) and its conjugate are unitary. The verification for \( R^P \) follows the same pattern as in \[SW16\], and we omit the details. We are left with verifying the relations \( R_n^P \).

In order to prove (2.3) for \( u_{ij}' \), we assume that \( \eta_{ij} = \eta_{kl} = 1 \). We have
\[
u_{ik}^* u_{jl}^* = \sum_{1 \leq r, p \leq n} u_{ir}^* u_{jp} \otimes u_{rk}^* u_{pl}.
\]
By (2.4) and (2.5), we see that \( u_{ir}^* u_{jp} = u_{rk}^* u_{pl} = 0 \) if \( p \neq r \) and \( \eta_{rp} = 0 \). Hence the above equality can be rewritten as
\[
u_{ik}^* u_{jl}^* = \sum_{r, p : \eta_{rp} = 1} u_{ir}^* u_{jp} \otimes u_{rk}^* u_{pl} + \sum_{r, p : \eta_{rr} = 0} u_{ir}^* u_{jp} \otimes u_{rk}^* u_{rl}.
\]
where the last equality follows from (2.3). Similarly we have
\[
u_{ij}^* u_{ik}^* = \sum_{r, p : \eta_{rp} = 1} u_{jp} u_{ir}^* \otimes u_{pl} u_{rk}^* + \{r : \eta_{rr} = 0\} X_{ij} \otimes Y_{kl}.
\]
Thus we obtain \( u_{ik}^* u_{jl}^* = u_{ij}' u_{ik}^* \).

For (2.4), assume that \( \eta_{kj} = 1, \eta_{kl} = 0 \) with \( k \neq l \). Then for any pair \( (r, p) \) with \( r \neq p \), either \( u_{ir}^* u_{jp} = 0 \) or \( u_{rk}^* u_{pl} = 0 \) according to (2.3). Hence we have
\[
u_{ik}^* u_{jl}^* = \sum_{1 \leq r, p \leq n} u_{ir}^* u_{jp} \otimes u_{rk}^* u_{pl} = \sum_{r, p : \eta_{rr} = 0} u_{ir}^* u_{jp} \otimes u_{rk}^* u_{rl}.
\]
where the last equality follows from the fact that \( u_{rk}^* u_{rl} = 0 \) for \( \eta_{rr} = 1 \) according to (2.4). In the same way we see that \( u_{ik}^* u_{jl}^* = 0 \). The case \( \eta_{ij} = 0, \eta_{kl} = 1, i \neq j \) is similar.
It remains to deal with the relations (2.6) and (2.7). Assume that \( \eta_{ij} = 1 \) and \( \eta_{kk} = 0 \). We have

\[
\begin{align*}
\sum_{r:p;\eta_{pp}=0} u_{rk}^* u_{jk}^* &= \sum_{r:p;\eta_{pp}=1} u_{rk}^* u_{jp} \otimes u_{rpk} + \sum_{r:p;\eta_{pp}=0} u_{ir}^* u_{jp} \otimes u_{r_ku_{pk}} \\
&= \sum_{r:p;\eta_{pp}=1} u_{ir}^* u_{jp} \otimes X_{rp} + \sum_{r:p;\eta_{pp}=0} u_{ir}^* u_{jr} \otimes u_{rk} \otimes u_{r_k}
\end{align*}
\]

which yields that \( u_{rk}^* u_{jk}^* = u_{jk}^* u_{rk}^* \). Moreover, we note that the right hand side of the above formula does not depend on \( k \). Therefore we see that for \( \eta_{ll} = 0 \), we have

\[
\begin{align*}
\sum_{r:p;\eta_{pp}=1} u_{ir}^* u_{jr} \otimes X_{rp} + X_{ij} \otimes (1 - \sum_{r:p;\eta_{pp}=1} u_{rk}^* u_{rk})
\end{align*}
\]

Similarly we obtain

\[
\begin{align*}
\sum_{r:p;\eta_{pp}=1} u_{jp}^* u_{ir} \otimes X_{rp} + X_{ij} \otimes (1 - \sum_{r:p;\eta_{pp}=1} X_{rr}),
\end{align*}
\]

as desired. The case for (2.7) is similar. \( \square \)

Now we will prove that \( U_{n}^{\varepsilon;\eta} \) is the quantum symmetry group of \( S_{n}^{n-1} \). We refer to [SW16, Remark 4.10] for more explanation on the notion of quantum symmetries in our setting.

**Theorem 4.** Assume that \((\varepsilon, \eta)\) is regular. Then \( U_{n}^{\varepsilon;\eta} \) is the quantum symmetry group of \( S_{n}^{n-1} \) in the sense that \( U_{n}^{\varepsilon;\eta} \) acts on \( S_{n}^{n-1} \) by homomorphisms

\[
\alpha, \beta : C(S_{n}^{n-1}) \rightarrow C(U_{n}^{\varepsilon;\eta}) \otimes C(S_{n}^{n-1})
\]

\[
\alpha(x_i) = \sum_{j} u_{ij} \otimes x_j, \quad \beta(x_i) = \sum_{k} u_{ki} \otimes x_k,
\]

and for any compact matrix quantum group \( \mathbb{G} \) acting on \( S_{n}^{n-1} \) in the above way, \( \mathbb{G} \) is a compact matrix quantum subgroup of \( U_{n}^{\varepsilon;\eta} \).

**Proof.** Following the same pattern as in the proof of Proposition 3, it is easy to check that the actions \( \alpha \) and \( \beta \) for \( U_{n}^{\varepsilon;\eta} \) exist. In the following we only prove the maximality. In other words, let \( \mathbb{G} \) be another \( n \times n \) compact matrix quantum group with matrix coefficients \( \{u_{ij} : 1 \leq i, j \leq n\} \), acting on \( S_{n}^{n-1} \) via the actions

\[
\alpha', \beta' : C(S_{n}^{n-1}) \rightarrow C(\mathbb{G}) \otimes C(S_{n}^{n-1})
\]

\[
\alpha'(x_i) = \sum_{j} u_{ij} \otimes x_j, \quad \beta'(x_i) = \sum_{k} u_{ki} \otimes x_k.
\]

We need to show that the unitary conditions and the relations \( R^\varepsilon \) and \( R^\eta \) hold for the generators \( u_{ij} \). Let us verify the relations \( R^\eta \). To this end, for any \( k, l \) with \( k \neq l \), we introduce the homomorphism

\[
\pi_{kl} : C(S_{n}^{n-1}) \rightarrow C(S_{n}^{n-1}), \quad \pi_{kl}(x_k) = x_1, \pi_{kl}(x_l) = x_2, \pi_{kl}(x_i) = 0, \quad i \neq k, l
\]

where \( \tilde{\varepsilon}_{12} = \varepsilon_{kl}, \tilde{\eta}_{12} = \eta_{kl}, \tilde{\eta}_{11} = \eta_{kk}, \tilde{\eta}_{22} = \eta_{ll} \).
Take \(i, j\) with \(\eta_{ij} = 1\). In particular, for any \(k, l\) with \(k \neq l\), we have
\[
(id \otimes \pi_{kl}) \circ \alpha'(x_i^* x_j) = (id \otimes \pi_{kl}) \circ \alpha'(x_j x_i^*),
\]
which means
\[
\begin{align*}
& u_{ik}^* u_{jl} \otimes x_i^* x_2 + u_{il}^* u_{jk} \otimes x_2^* x_1 + u_{ik}^* u_{jk} \otimes x_1^* x_2 + u_{il}^* u_{jl} \otimes x_2^* x_2 \\
& = u_{ij}^* u_{ik} \otimes x_2^* x_1 + u_{jk}^* u_{il} \otimes x_1^* x_2 + u_{ik}^* u_{jl} \otimes x_1^* x_1 + u_{il}^* u_{jl} \otimes x_2^* x_2.
\end{align*}
\]
By definition, \(x_1 \mapsto -x_1\) gives a homomorphism of \(C(S^n_{\xi, \eta})\), so the above equality still hold when replacing \(x_1\) by \(-x_1\). Combining these two equalities we obtain
\[
(2.9) \quad u_{ik}^* u_{jl} \otimes x_i^* x_2 = u_{il}^* u_{jk} \otimes x_2^* x_1 + u_{ij}^* u_{ik} \otimes x_2^* x_2 + u_{il}^* u_{jl} \otimes x_2^* x_2.
\]
Recall that \(x_1^* x_1 + x_2^* x_2 = x_1^* x_1^* + x_2^* x_2^* = 1\). The second equality above can be written as
\[
(2.10) \quad u_{ik}^* u_{jk} - u_{jk}^* u_{ik} + (u_{il}^* u_{jl} - u_{il}^* u_{jk}) \otimes x_2^* x_2 + (u_{jk}^* u_{ik} - u_{jl}^* u_{il}) \otimes x_2^* x_2 = 0.
\]
It is obvious to see that the unit element 1 is linearly independent from \(\{x_2^* x_2, x_2^* x_2\}\). Therefore we have
\[
(2.11) \quad u_{ik}^* u_{jk} = u_{jk}^* u_{ik}.
\]
Now we prove (2.3). Assume \(k \neq l\) and \(\eta_{kl} = 1\). In this case we consider the torus \(T^2 = \{(z_1, z_2) \in \mathbb{C}^2 : |z_1| = |z_2| = 1\}\), and we have a homomorphism
\[
\pi_{T^2} : C(S^1_{\xi, \eta}) \rightarrow C(T^2), \quad \pi_{T^2}(x_1) = f_1, \pi_{T^2}(x_2) = f_2,
\]
de where \(f_i(z_1, z_2) = \frac{\sqrt{2}}{2} \tilde{f}_i (i = 1, 2)\) are the coordinate functions. Applying \(id \otimes \pi_{T^2}\) to (2.9), we have
\[
\begin{align*}
\tilde{f}_i u_{ik}^* u_{jl} - u_{ij}^* u_{ik}^* & \otimes \tilde{f}_1 f_2 - (u_{ij}^* u_{il}^* - u_{il}^* u_{jk}^*) \otimes \tilde{f}_1 f_2 = 0.
\end{align*}
\]
Note that \(\tilde{f}_1 f_2\) and \(f_1 \tilde{f}_2\) are linearly independent. Therefore we obtain
\[
(2.12) \quad u_{ik}^* u_{jl} = u_{il}^* u_{ik}, \quad u_{jk}^* u_{il} = u_{il}^* u_{jk}.
\]
Together with (2.11) we obtain (2.3).

For (2.4), we assume \(k \neq l\) and \(\eta_{kl} = 0\). If \(\xi_{kl} = 0\), we consider the full group C*-algebra \(C^*(\mathbb{F}_2)\) of the free group with two generators. Denote by \(u_1, u_2\) the corresponding free unitary generators. Note that \(u_1\) and \(u_2\) are normal. We have a homomorphism
\[
(2.13) \quad \pi_{\mathbb{F}_2} : C(S^1_{\xi, \eta}) \rightarrow C^*(\mathbb{F}_2), \quad \pi_{\mathbb{F}_2}(x_1) = \frac{\sqrt{2}}{2} u_1, \pi_{\mathbb{F}_2}(x_2) = \frac{\sqrt{2}}{2} u_2.
\]
Note that the elements \(u_1^* u_2, u_1 u_2^*, u_2 u_1^*, u_2 u_1\) are linearly independent. Therefore, applying \(id \otimes \pi_{\mathbb{F}_2}\) to (2.9), we have
\[
\begin{align*}
& u_{ik}^* u_{jl} = u_{il}^* u_{jk} = u_{ij}^* u_{ik} = u_{jk}^* u_{il} = 0.
\end{align*}
\]
If \(\xi_{kl} = 1\), then by our convention (2.11) we have \(\eta_{kk} = \eta_{ll} = 0\). Then we have a homomorphism given in Remark (2.1)
\[
(2.14) \quad \pi : C(S^1_{\xi, \eta}) \rightarrow M_4(\mathbb{C}), \quad \pi(x_1) = a, \pi(x_2) = b.
\]
Here we see from Remark (2) that the elements \(a^* b, ab^*, b^* a, ba^*\) are linearly independent. So applying \(id \otimes \pi\) to (2.9), we have
\[
\begin{align*}
& u_{ik}^* u_{jl} = u_{il}^* u_{jk} = u_{ij}^* u_{ik} = u_{jk}^* u_{il} = 0.
\end{align*}
\]
Thus we proved (2.1) as desired. By performing similar computations for \( \beta' \), we also obtain the relation (2.5).

For (2.6), assume \( \eta_{kk} = \eta_{ll} = 0 \). By virtue of the convention (2.2), we assume without loss of generality that \( \eta_{kl} = 0 \) or \( \varepsilon_{kl} = 0 \). If \( \eta_{kl} = 0 \), again we apply the above homomorphism \( \text{id} \otimes \pi \) to (2.10). Note that the elements \( a^*a \) and \( aa^* \) are linearly independent in \( \mathbb{M}_4(\mathbb{C}) \). Hence we obtain

\[
\sum_{i=1}^{n} u_{il}^* u_{jl} - u_{ik}^* u_{jk} = u_{jk} u_{ik}^* - u_{jl} u_{il}^* = 0.
\]

Together with (2.11) we obtain the desired relation (2.6). If \( \varepsilon_{kl} = 0 \), it suffices to consider the noncommutative sphere \( S^{1}_{\mathbb{C}, \tilde{\eta}, \tilde{x}} \) instead of \( S^{1}_{\mathbb{C}, \tilde{\eta}, \tilde{x}} \) and replace \( x_1 \) by its adjoint \( x_1^* \) in (2.11).

Then the same arguments as before yield the relation (2.6). The relation (2.7) follows similarly by computations for \( \beta' \).

The relations \( R^e \) can be proved by similar arguments as in the proof of [SW16, Theorem 4.7]. The only non-obvious ingredient is that \( x_k x_1 \) and \( x_1 x_k \) are linearly independent for any choice of \( \eta \) whenever \( \varepsilon_{kl} = 0 \). However this follows similarly as what we did for \( \{ x_k^*, x_l \} \) in the case \( \eta_{kl} = 0 \). Indeed, as is pointed out before, it suffices to consider the noncommutative sphere \( S^{1}_{\mathbb{C}, \tilde{\eta}, \tilde{x}} \) instead of \( S^{1}_{\mathbb{C}, \tilde{\eta}, \tilde{x}} \), and replace \( x_1 \) by its adjoint \( x_1^* \) in (2.13) and (2.14). We leave the details to the reader.

In the end we show that \( u \) and \( \bar{u} \) are unitary. We have

\[
1 \otimes 1 = \alpha' \left( \sum_{i=1}^{n} u_{i}^* x_{i} \right) = \sum_{k,l=1}^{n} \sum_{i=1}^{n} u_{ik}^* u_{il} \otimes x_{k}^* x_{l}.
\]

Take an arbitrary \( 1 \leq k \leq n \). Consider the circle \( \mathbb{T} = \{ z \in \mathbb{C} : |z| = 1 \} \) and the homomorphism \( \pi_k : C(S^{n-1}_{\mathbb{C}, \tilde{\eta}, \tilde{x}}) \rightarrow C(\mathbb{T}) \), \( \pi_k(x_k) = f \), \( \pi_k(x_i) = 0 \), \( i \neq k \), where \( f(z) = z, z \in \mathbb{T} \). Applying \( \text{id} \otimes \pi_k \) to the equality (2.15) we obtain

\[
\sum_{i=1}^{n} u_{ik}^* u_{ik} = 1.
\]

Together with (2.15) this also implies that

\[
\sum_{1 \leq k,l \leq n, k \neq l} \sum_{i=1}^{n} u_{ik}^* u_{il} \otimes x_{k}^* x_{l} = 0.
\]

Take arbitrary \( 1 \leq k, l \leq n \) with \( k \neq l \) and consider the homomorphism \( \pi_{kl} \) introduced before. Applying \( \text{id} \otimes \pi_{kl} \) to the above equality, we get

\[
\sum_{i=1}^{n} u_{ik}^* u_{il} \otimes x_{1}^* x_{2} + \sum_{i=1}^{n} u_{il}^* u_{ik} \otimes x_{2}^* x_{1} = 0.
\]

We have already seen that \( x_{1}^* x_{2} \) and \( x_{2}^* x_{1} \) are linearly independent in terms of the homomorphism \( \pi_{T^2} \). Hence we deduce that

\[
\sum_{i=1}^{n} u_{ik}^* u_{il} = 0.
\]

Together with (2.16) we see that \( u^* u = 1 \). Considering the action on \( x_i x_i^* \), we see also that \( uu^* = 1 \). Hence \( u \) is unitary. Similar arguments for \( \beta \) yields that \( \bar{u} \) is unitary. Therefore the proof is finished.

\[ \square \]
Remark 5. The theorem indeed holds for more general cases beyond the regular assumption (2.2). For example, by the similar argument we may show that the result still holds if the elements \( x_1^* x_1, x_2^* x_1, \ldots, x_{i-1}^* x_1, x_{i-1}^* x_{i-1} \) are linearly independent, where \( \{x_i, \ldots, x_i\} \) are the family of all non-normal generators. However, it seems that the detailed verification of the linear independence is quite technical depending on the choice of the parameters \((\varepsilon, \eta)\), so we would prefer not to go further.

Remark 6. It is easy to see that there is a homomorphism
\[
\phi : C(S^{n-1}_{\mathbb{C}, \varepsilon, \eta}) \to C(U^{\varepsilon, \eta}_n), \quad x_i \mapsto u_{i1}.
\]
Intuitively speaking, the sphere \( S^{n-1}_{\mathbb{C}, \varepsilon, \eta} \) can be viewed as a quantum space determined by the relations of the first column of the quantum symmetry group which acts on it. A complete theory towards this direction, in the setting of easy quantum groups, has been recently developed by [JW18]. In general, it is unclear whether the natural homomorphism in the form of \( \phi \) is an isomorphism (see the comments in [JW18, Section 2]). Here we may provide a non-isomorphic example in our setting of mixed relations. More precisely, let \( n = 2 \) and
\[
\varepsilon = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad \eta = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},
\]
Then the natural homomorphism
\[
\phi : C(S^1_{\mathbb{C}, \varepsilon, \eta}) \to C(U^{\varepsilon, \eta}_2), \quad x_i \mapsto u_{i1}, \quad i = 1, 2
\]
is non-injective. Indeed, since \( \eta_{11} = \eta_{22} = 0 \), by the unitary condition of \( u \) and \( \bar{u} \) we have
\[
0 = u_{11} u_{21}^* + u_{12} u_{22}^* = 2X_{12}.
\]
In particular
\[
\phi(x_1 x_2^*) = u_{11} u_{21}^* = X_{12} = 0.
\]
However, we have \( x_1 x_2^* \neq 0 \). Indeed, consider the matrices \( a, b \) given in Remark 2. Then instead of \( \pi \), there is a representation
\[
\pi' : C(S^1_{\mathbb{C}, \varepsilon, \eta}) \to \mathbb{M}_4(\mathbb{C}), \quad \pi(x_1) = a, \pi(x_2) = b^*.
\]
and
\[
\pi(x_1 x_2^*) = ab = \begin{bmatrix} 0 & 0 \\ 0 & \frac{1}{2} \end{bmatrix} \neq 0.
\]
Therefore \( \phi \) is not injective.

3. Remarks on the orthogonal cases

In this section we would like to discuss some related questions appeared in [SW16]. In [SW16] another version of commutation relations for quantum orthogonal groups is proposed. More precisely, we consider the corresponding quantum group given by the universal C*-algebra
\[
C(\hat{O}_n^\varepsilon) = C^*(u_{ij}, i, j = 1, \ldots, n \mid u_{ij} = u_{ij}^*, u \text{ is orthogonal, } \hat{R}^\varepsilon \text{ holds}),
\]
where $\hat{R}^\varepsilon$ denotes the relations

$$u_{ik}u_{jl} = \begin{cases} u_{jl}u_{ik} & \text{if } \varepsilon_{ij} = 1 \text{ and } \varepsilon_{kl} = 1 \\ 0 & \text{if } \varepsilon_{ij} = 1 \text{ and } \varepsilon_{kl} = 0 \\ 0 & \text{if } \varepsilon_{ij} = 0 \text{ and } \varepsilon_{kl} = 1 \end{cases}.$$ 

The quantum space on which $\hat{O}_n^\varepsilon$ acts maximally was left unsolved in [SW16]. In the following we will briefly answer this question in terms of quantum tuples of noncommutative spheres, inspired by [JW18]. Consider the universal C*-algebra $C(\mathbb{X}_n^\varepsilon)$ generated by $x_{ij}, i, j = 1, \ldots, n$ with relations

$$x_{ij} = x_{ij}^*, \quad \sum_i x_{ik}x_{il} = \delta_{kl},$$

and

$$x_{ik}x_{jl} = \begin{cases} x_{ji}x_{ik} & \text{if } \varepsilon_{ij} = 1 \text{ and } \varepsilon_{kl} = 1 \\ 0 & \text{if } \varepsilon_{ij} = 1 \text{ and } \varepsilon_{kl} = 0 \\ 0 & \text{if } \varepsilon_{ij} = 0 \text{ and } \varepsilon_{kl} = 1 \end{cases}.$$ 

We remark that in the case where $\varepsilon_{ij} = 1$ for all $i \neq j$, we see that $\hat{O}_n^\varepsilon$ equals the classical hyperoctahedral group $H_n$, and $\mathbb{X}_n^\varepsilon$ is simply the space of $n \times n$ orthogonal matrices with cubic columns. Note that the set of cubic vectors $I_n \subset \mathbb{R}^n$ consists of the points $(0, \ldots, 0, \pm 1, 0, \ldots, 0)$ on each axis of $\mathbb{R}^n$. It is well-known that $H_n$ is the symmetry group of $I_n$. If $\varepsilon_{ij} = 0$ for all $i, j$, then $\hat{O}_n^\varepsilon$ is the free quantum orthogonal group $O_n^+$ introduced in [VDW96] (see also [BS09]), and $\mathbb{X}_n^\varepsilon$ is the partition quantum space $X_{n,n}(\Pi)$ introduced in [JW18], where in our setting $\Pi$ is the set of non-crossing pair partitions.

**Theorem 7.** $\hat{O}_n^\varepsilon$ is the quantum symmetry group of $\mathbb{X}_n^\varepsilon$, in the sense that $\hat{O}_n^\varepsilon$ acts on $\mathbb{X}_n^\varepsilon$ by homomorphisms

$$\alpha, \beta : C(\mathbb{X}_n^\varepsilon) \rightarrow C(\hat{O}_n^\varepsilon) \otimes C(\mathbb{X}_n^\varepsilon),$$

$$\alpha(x_{ik}) = \sum_j u_{ij} \otimes x_{jk}, \quad \beta(x_{ik}) = \sum_j u_{ji} \otimes x_{jk},$$

and for any compact matrix quantum group $\mathbb{G}$ acting on $\mathbb{X}_n^\varepsilon$ in the above way, $\mathbb{G}$ is a compact matrix quantum subgroup of $\hat{O}_n^\varepsilon$.

**Proof.** We first check that the actions $\alpha$ and $\beta$ are well-defined. It is a standard argument to see that $\alpha(x_{ij}) = \alpha(x_{ij})^*$ and $\sum_i \alpha(x_{ik})\alpha(x_{il}) = \delta_{kl}$ using the orthogonal relations of $\hat{O}_n^\varepsilon$.

Also, according to the relations $\hat{R}^\varepsilon$, for $\varepsilon_{ij} = 1, \varepsilon_{kl} = 1$,

$$\alpha(x_{ik})\alpha(x_{jl}) = \sum_{p,q; \varepsilon_{pq}=1} u_{ip}u_{jq} \otimes x_{pk}x_{ql} = \sum_{p,q; \varepsilon_{pq}=1} u_{jq}u_{ip} \otimes x_{ql}x_{pk} = \alpha(x_{jl})\alpha(x_{ik}),$$

and for $\varepsilon_{ij} = 0, \varepsilon_{kl} = 1$,

$$\alpha(x_{ik})\alpha(x_{jl}) = \sum_{p,q; \varepsilon_{pq}=0} u_{ip}u_{jq} \otimes x_{pk}x_{ql} = 0,$$

and for $\varepsilon_{ij} = 1, \varepsilon_{kl} = 0$,

$$\alpha(x_{ik})\alpha(x_{jl}) = \sum_{p,q; \varepsilon_{pq}=1} u_{ip}u_{jq} \otimes x_{pk}x_{ql} = 0.$$ 

Thus $\alpha$ is a well-defined homomorphism. Similarly we see that the action $\beta$ exists as well.
Now assume that $\mathbb{G}$ is an arbitrary compact matrix quantum group acting on $\mathbb{X}^\varepsilon_n$ by homomorphisms
\[ \alpha', \beta' : C(\mathbb{X}_n^\varepsilon) \to C(O^\varepsilon_n) \otimes C(\mathbb{X}_n^\varepsilon), \]
\[ \alpha'(x_{ik}) = \sum_j u_{ij} \otimes x_{jk}, \quad \beta'(x_{ik}) = \sum_j u_{ji} \otimes x_{jk}. \]

Note that the diagonal $C^*$-subalgebra generated by $\{ x_{ii} : 1 \leq i \leq n \}$ in $C(\mathbb{X}_n^\varepsilon)$ satisfies the relations $x_{ii}x_{jj} = x_{jj}x_{ii}$ for $\varepsilon_{ij} = 1$. Restricting the homomorphisms $\alpha'$ and $\beta'$ to this subalgebra, the similar arguments as in [SW16 Theorem 4.7] yield that the generators $u_{ij}$ are self-adjoint, and the relation $u_{ik}u_{jl} = u_{jl}u_{ik}$ for $\varepsilon_{ij} = \varepsilon_{kl} = 1$ still holds, for which we omit the details. Now consider the case $\varepsilon_{ij} = 1, \varepsilon_{kl} = 0$. We have a priori

\[ 0 = \alpha'(x_{ik}x_{jk}) = \sum_{p,q=1}^n u_{ip}u_{jq} \otimes x_{pk}x_{qk} = \sum_{p,q: \varepsilon_{pq}=0} u_{ip}u_{jq} \otimes x_{pk}x_{qk}, \]

where we applied the relations $x_{pk}x_{qk} = 0$ for $\varepsilon_{pq} = 1$ since $\varepsilon_{kk} = 0$. If $k = l$, it is easy to see that there exists a homomorphism $\pi_1 : C(\mathbb{X}_n^\varepsilon) \to \mathbb{C}$ such that $\pi_1(x_{kk}) = 1$ and $\pi_1(x_{k'k}) = 0$ for $k' \neq k$. Applying the homomorphism $\text{id} \otimes \pi_1$, the equality (3.1) yields

\[ (3.1) \quad 0 = \alpha'(x_{ik}x_{jk}) = \sum_{p,q=1}^n u_{ip}u_{jq} \otimes x_{pk}x_{qk}, \]

where $u_{ij}$ is the usual defining matrix of $C(O^\varepsilon_n)$. Applying the homomorphism $\text{id} \otimes \pi_1$, the equality (3.1) yields

\[ (3.2) \quad u_{ik}u_{jk} = 0. \]

If $k \neq l$, we consider the surjective homomorphism $\pi_{O^\varepsilon_n^+} : C(\mathbb{X}_n^\varepsilon) \to \mathbb{C}$ such that

\[ \text{id} \otimes \pi_{O^\varepsilon_n^+} \left( \begin{bmatrix} x_{kk} & x_{kl} \\ x_{lk} & x_{ll} \end{bmatrix} \right) = \begin{bmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{bmatrix}, \]

where $v$ is the usual defining matrix of $C(O^\varepsilon_n)$. Applying the homomorphism $\text{id} \otimes \pi_1$, the equality (3.1) yields

\[ u_{ik}u_{jl} \otimes v_{11}v_{21} + u_{jl}u_{ik} \otimes v_{11}v_{21} = u_{ik}u_{jk} \otimes v_{11}^2 + u_{jk}u_{ik} \otimes v_{21}^2 = 0. \]

Moreover, together with (3.2), we see that

\[ u_{ik}u_{jl} \otimes v_{11}v_{21} + u_{jl}u_{ik} \otimes v_{21}v_{11} = 0. \]

It is well-known that $v_{11}v_{2j}$ and $v_{21}v_{11}$ are linearly independent (see for example a simple matrix model in [Ban17 Theorem 3.9]). Hence we have

\[ u_{ik}u_{jl} = u_{jl}u_{ik} = 0. \]

Continuing the similar arguments for the action $\beta'$, we obtain completely the relations $\tilde{R}^\varepsilon$.

Now the orthogonal relations for $\mathbb{G}$ follows easily. Note that we have $\sum x_{ik}^2 = 1$ for all $k$. Therefore

\[ 1 \otimes 1 = \alpha'(\sum_i x_{ii}^2) = \sum_{i,p} u_{ip}u_{ii} \otimes x_{pk}x_{qk} + \sum_p \sum_i u_{ip}^2 \otimes x_{pk}^2 \]

\[ = \sum_{p,q: \varepsilon_{pq}=0} \sum_i u_{ip}u_{iq} \otimes x_{pk}x_{qk} + \sum_p \sum_i u_{ip}^2 \otimes x_{pk}^2. \]

Using the homomorphism $\pi_1$ as above, we deduce that $\sum_i u_{ip}^2 = 1$, and hence

\[ \sum_{p,q: \varepsilon_{pq}=0} \sum_i u_{ip}u_{iq} \otimes x_{pk}x_{qk} = 0. \]
For \( k \neq l \) with \( \varepsilon_{kl} = 0 \), we use the homomorphism \( \pi_{O^+_2} \) as above and we obtain \( \sum_i u_{ik}u_{il} = 0 \).

And for \( \varepsilon_{kl} = 1 \), we see from the relation \( \tilde{R}^\varepsilon \) that \( \sum_i u_{ik}u_{il} = \sum_i 0 = 0 \). Repeating the similar arguments with the action \( \beta' \), we prove that \( u \) is orthogonal. The proof is complete.

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