Low rank surrogates for fuzzy-stochastic partial differential equations

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We consider a particular fuzzy-stochastic PDE depending on the interaction of probabilistic and non-probabilistic (via fuzzy arithmetic in terms of possibility theory) influences. Such a combination is beneficial in an engineering context, where aleatoric and epistemic uncertainties appear simultaneously. The fuzzy-stochastic dependence is described in a high-dimensional parameter space, thus easily leading to an exponential complexity in practical computations. To alleviate this severe obstacle, a compressed low-rank approximation in form of Hierarchical Tucker representation of the desired parametric quantity of interest is derived. The performance of the proposed model order reduction approach is demonstrated.

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1 Introduction

In engineering problems the lack of knowledge, and simple randomness of physical processes are the two main sources for uncertainty. For both kinds of uncertainty it is important to choose appropriate models, for the first a possibilistic model is adequate and a probabilistic for the latter. See [5] for a clarification of these concepts.

The combination of both models leads to a nesting of computational methods. E.g. in a purely fuzzy setting an optimization algorithm is used to determine the membership function of a quantity. If probabilistic influences are added to the quantity, the membership function itself becomes a random variable. To compute the mean of this probabilistic membership function sufficiently many sample membership functions are needed. For each sample the optimization algorithm is employed. This introduces the curse of dimensionality, since the computational cost rise exponentially with each uncertainty. If different uncertainties interact this challenge is further aggravated, leading to the necessity to evaluate a huge number of solutions of the underlying mathematical model. As an alternative, given sufficient properties of the parameter to solution map, surrogate models can be employed. Here we will exploit two properties, first the regularity will be exploited by a tensorized Chebyshev arithmetic in terms of possibility theory) influences. Such a combination is beneficial in an engineering context, where

2 Parametric model equation

The model equation of interest reads for \( D = [0, 1]^2 \)

\[
- \operatorname{div} \kappa(x, y, z) \nabla u(x, y, z) = f(x) \quad \text{in } D,
\]

\[
\gamma_0 u(x, y, z) = g \quad \text{on } \Gamma_0 := [0, 1] \times \{0\},
\]

\[
\gamma_1^q [z] u(x, y, z) = g \quad \text{on } \Gamma_s := \partial D \setminus \Gamma_0,
\]

with a pathwise in \( z \) parametrized random field \( \kappa(\cdot, \cdot, z) \) modelled by a KLE, using \( iid \) uniform random variables \( \sim \mathcal{U}(-1, 1) \) and a fuzzy Gaussian kernel \( \epsilon(r, \bar{z}) := \exp(-r/\bar{z}) \) for the physical components. Here \( \bar{z} \) is a symmetric triangular fuzzy number with \( Z := \text{supp} \bar{z} = [0.5, 1] \), s.t. \( z \in Z \), denotes a realization, cf. [6]. We set \( g(x) = \sin(5\pi x_1) \) on \( \{1\} \times [0, 1] \), \( 0 \) else and \( f(x) = 10 \exp(-20(x_1 - 0.5)^2 + (x_2 - 0.5)^2) \).

Note that the parameter range for the fuzzy input has an influence on the truncation error of the KLE due to differences in the absolute decay of the eigenvalues. In the experiments we uniformly truncate after \( M = 10 \) terms, s.t. \( \lambda_M \in \mathcal{O}(1/(e^{5M/2}/2\Gamma(1/2M^{1/2})) \) for all \( z \in Z \), e.g. [4]. With this truncation we define \( \Gamma := [-1, 1]^M \), thus \( y \in \Gamma \). For the quantity of interest \( q: \Gamma \times Z \rightarrow \mathbb{R} : (y, z) \mapsto \int_{\Gamma_y} u_{y, z}^2 dS \), where \( u_{y, z} \) denotes the solution to Equation (1) for the chosen parameters \( y, z \).

Under the assumption of regularity, we construct a surrogate model from the space of tensorized Chebyshev polynomials. We restrict this space further by assuming a low-rank structure of the coefficients \( H \in \mathcal{H}(d, n) \). Here, \( \mathcal{H}(d, n) \) defines the Hierarchical Tucker Tensor space with dimension \( d \) and size \( n \) in each direction, cf. [2], so that the surrogate model \( \hat{q} \) may be written as

\[
\hat{q}(p_1, \ldots, p_d) = \sum_{(i_1, i_2, \ldots, i_d) \in \mathcal{I}} H_{i_1, i_2, \ldots, i_d} T_{i_1}(p_1) \cdots T_{i_d}(p_d), \quad \text{with } H \in \mathcal{H}(d, n) \text{ and multi-indexset } \mathcal{I}.
\]

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Finding the low-rank coefficients $H$ is achieved by a generalized cross approximation, cf. [3], on the tensorized Chebyshev grid.

The computational cost of one evaluation of the surrogate model are reduced drastically by several magnitudes. This enables the computation of polymorphic entities like the fuzzy cumulative distribution

$$P\left(q(\tilde{z}, \cdot) \leq t\right) = f(\tilde{z}, t),$$

which is a fuzzy function. Thus, for each value $t$ a membership function $\mu_f$ is determined by employing an $\alpha$-cut method, cf. [6], where for a finite set of membership levels $\alpha$ the optimization problem

$$\min_{z \in C_\alpha} f(z, t) \quad \text{and} \quad \max_{z \in C_\alpha} f(z, t), \text{ with } C_\alpha = \{ z \in Z | \mu_{\tilde{z}} \geq \alpha \}$$

is solved. For each approximation of $f(z, t)$ sufficient enough samples are needed to be drawn according to the probability distribution of the parameters $p_1, \ldots, p_d$.

### 3 Numerical example

The main part of the construction cost stem from the 20,100 evaluations of the original model needed to apply the generalized cross approximation. For the fuzzy parameter 15 Chebyshev points are used, for the first KLE term 6 points and for the 10-th expansion term only 2 points. The maximal rank on each transfer node was set to 15, but only an average rank of 7 was found by the generalized cross approximation. In the Figure on the left, 1000 random parameters were used to estimate the relative error. The difference in error between on and off the Chebyshev grid is marginal, which is a fact in favour for the chosen polynomial basis. For the computation of the fuzzy cumulative distribution function 415,029 evaluations of the surrogate model where used. This roughly reduced the computational cost by a factor of 20. We demonstrated that, first the observation of polymorphic uncertainties is susceptible to the curse of dimensionality and second that by the construction of suitable low-rank surrogate models alleviates the curse. Additional to the standard KLE example a fuzzy-stochastic elasticity problem was investigated in [1]. In summary, the research of suitable surrogate models appears to be highly relevant for the field of polymorphic uncertainty quantification.

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