INTEGRAL IDENTITIES FOR THE BOUNDARY OF A CONVEX BODY

TATIANA MOSEEVA

Abstract. We present the multidimensional versions of the Pleijel and Ambartzumian–Pleijel identities. We also obtain the generalization of both the Blaschke–Petkantschin and Zähle formulae considering the case when some points are chosen inside the convex body and some on the boundary. Moreover, a version of the Zähle formula for the polytopes is derived.

1. Introduction and main results

For a convex body $K$ in $\mathbb{R}^d$ denote by $|K|$ its volume (the $d$-dimensional Lebesgue measure). Denote by $\partial K$ the boundary of $K$ equipped with the $(d-1)$-dimensional Hausdorff measure $\sigma_{\partial K}$. When it causes no confusion, for abbreviation we use just $\sigma$ instead of $\sigma_{\partial K}$. Also for simplicity we write $|\partial K|$ instead of $\sigma(\partial K)$.

For $l \in \{1, \ldots, d\}$ denote by $A_{d,l}$ the set of all affine $l$-planes in $\mathbb{R}^d$ equipped with the unique Haar measure $\mu_{d,l}$ invariant with respect to the rigid motions and normalized by

$$\mu_{d,l}(\{E \in A_{d,l} : E \cap \mathbb{B}^d \neq \emptyset\}) = \kappa_{d-l},$$

where $\mathbb{B}^k$ is the $k$-dimensional unit ball and $\kappa_k := |\mathbb{B}^k|$. Also let $\omega_k := |S^{k-1}| = k\kappa_k$, where $S^{k-1} = \partial \mathbb{B}^k$ is the $(k-1)$-dimensional unit sphere.

Given a finite number of points $x_1, \ldots, x_n \in \mathbb{R}^d$ denote by $[x_1, \ldots, x_n]$ their convex hull. In particular, $[x_1, x_2]$ is the segment with endpoints $x_1$ and $x_2$.

We start with the case $d = 2$. Let $K$ be a planar convex body with $C^1$ boundary. Consider a line $G$ intersecting $K$ chosen with respect to the measure $\mu_{2,1}$. In [3] Pleijel discovered an identity expressing the integral functional of the length of the chord $G \cap K$ in terms of the double integration over the boundary of $K$:

$$\int_{G \cap K \neq \emptyset} h(|G \cap K|) \mu_{2,1}(dG) = \frac{1}{2} \int_{(\partial K)^2} h'(|x_1 - x_2|) \cos \alpha_1 \cos \alpha_2 \sigma(dx_1)\sigma(dx_2),$$

where $h : \mathbb{R}_+ \to \mathbb{R}$ is any function with continuous derivative such that $h(0) = 0$ and $\alpha_1, \alpha_2$ are angles between tangents at points $x_1, x_2$ and the chord $[x_1, x_2]$. Here we consider the tangents lying
on the same side of the chord. Moreover, Pleijel showed that

\[
\int_{G \cap K \neq \emptyset} h(|G \cap K|) \mu_{2,1}(dG) = \int_{G \cap K \neq \emptyset} h'(|G \cap K|) |G \cap K| \cot \alpha \cot \mu_{2,1}(dG).
\]

From (1.2) it is possible to derive (see, e.g., [3, Section 6.9]) an explicit form of the defect in the isoperimetric inequality:

\[
|\partial K|^2 - 4\pi |K| = 2 \int_{(\partial K)^2} \sin^2 \frac{\alpha_1 - \alpha_2}{2} \sigma(dx_1) \sigma(dx_2).
\]

In [2], Ambartzumian gave a combinatorial proof of the Pleijel identity and presented its version for the convex planar polygons which is now known as the Ambartzumian–Pleijel identity: if \( P \) is a convex polygon with side lengths \( a_1, \ldots, a_n \), then

\[
\int_{G \cap P \neq \emptyset} h(|G \cap P|) \mu_{2,1}(dG) = \int_{G \cap P \neq \emptyset} h'(|G \cap P|) |G \cap P| \cot \alpha \cot \mu_{2,1}(dG) + \sum_{i=1}^{n} a_i \int_{0}^{h(t)} dt.
\]

There exists an analogue of the Pleijel identity for the convex bodies with smooth boundary in \( \mathbb{R}^3 \), see [1]:

\[
\int_{G \cap K \neq \emptyset} h(|G \cap K|) \mu_{3,1}(dG) = 4 \int_{(\partial K)^2} \frac{h'(|x_1 - x_2|)}{|x_1 - x_2|} \cos \alpha \cos \phi \sigma(dx_1) \sigma(dx_2),
\]

where \( \alpha, \alpha \) are the angles between the line containing the chord \( [x_1, x_2] \) and its projections onto the tangent planes, and \( \phi \) is the angle between projections of normal vectors at points \( x_1 \) and \( x_2 \) onto the orthogonal complement of the line passing through the points \( x_1, x_2 \).

Our first result is the following generalization of the Pleijel identity to arbitrary dimension.

**Theorem 1.1.** Let \( K \) be a convex body in \( \mathbb{R}^d \) with \( C^1 \) boundary. Then for every function \( h : \mathbb{R} \to \mathbb{R} \) with continuous derivative such that \( h(0) = 0 \) we have

\[
\int_{G \cap K \neq \emptyset} h(|G \cap K|) \mu_{d,1}(dG) = \frac{1}{(d-1)\omega_d} \int_{(\partial K)^2} \frac{h'(|x_1 - x_2|)}{|x_1 - x_2|^{d-2}} \cos \alpha \cos \phi \sigma(dx_1) \sigma(dx_2),
\]

where \( \alpha, \alpha \) are the angles between the line containing the chord \( [x_1, x_2] \) and its projections onto the tangent planes, and \( \phi \) is the angle between projections of normal vectors at points \( x_1 \) and \( x_2 \) onto the orthogonal complement of the line passing through the points \( x_1, x_2 \).
To formulate our further results, we need to extend some of our notation to subsets of lower dimension. Given some $E \in A_{d,l}$ and $k \in \{1, \ldots, l\}$ denote by $A_{E,k}$ the set of all affine $k$-planes in $E \cong \mathbb{R}^l$ equipped with the unique Haar measure $\mu_{E,k}$ invariant with respect to the rigid motions in $E$ and normalized the same way as in (1.1). Now for an arbitrary subset $M \subset \mathbb{R}^d$ we write

$$A_{M,k} = A_{\text{aff } M,k}, \quad \mu_{M,k} = \mu_{\text{aff } M,k},$$

where $\text{aff } M$ denotes the affine span of $M$, i.e. the intersection of all affine planes containing $M$.

We also denote by $\lambda_{E}$ the $d$-dimensional Lebesgue measure in $E$ and we denote $\lambda_E(|K \cap E|)$ briefly by $|K \cap E|$. Analogously, given points $x_0, \ldots, x_l \in \mathbb{R}^d$, we write $|[x_0, \ldots, x_l]|$ for the $d$-dimensional Lebesgue measure of their convex hull.

Our next result generalizes the Ambartzumian–Pleijel identity (see (1.4)) to higher dimensions. Given a convex polytope $P$ denote by $\mathcal{F}(P)$ the set of its facets.

**Theorem 1.2.** Let $P$ be a convex polytope in $\mathbb{R}^d$. Then for every function $h : \mathbb{R} \to \mathbb{R}$ with continuous derivative such that $h(0) = 0$ we have

$$\int_{G \cap P \neq \emptyset} h([G \cap P])\mu_{d,1}(dG) = \frac{1}{(d-1)!} \int_{G \cap P \neq \emptyset} h'([G \cap P])|G \cap P| \cot \alpha_1 \cot \alpha_2 \cos \phi_0 \mu_{d,1}(dG) + \sum_{F \in \mathcal{F}(P)} H([G \cap F])\mu_{F,1}(dG),$$

where $\alpha_1, \alpha_2$ are the angles between the line $G$ and its projections onto the tangent planes, $\phi_0$ is the angle between projections of normal vectors at endpoints of $G \cap P$ onto the orthogonal complement of line $G$, and $H$ is the antiderivative of $h$.

The following Blaschke–Petkantschin formula (see, e.g., [7], Theorem 7.2.7) is widely used when dealing with the convex hulls of random points in a given convex body: if $h : (\mathbb{R}^d)^{l+1} \to \mathbb{R}$ is a non-negative measurable function, $l \in \{1, \ldots, d\}$, then

$$\int_{(\mathbb{R}^d)^{l+1}} h(x_0, \ldots, x_l)dx_0 \ldots dx_l = (d-l)! b_{d,l} \int_{A_{d,l}} h(x_0, \ldots, x_l)[x_0, \ldots, x_l]^{d-l} \lambda_E(dx_0) \ldots \lambda_E(dx_l) \mu_{d,l}(dE),$$

where $b_{d,l} = \frac{\omega_{d-k+1} \ldots \omega_d}{\omega_1 \ldots \omega_d}$.

To prove Theorem 1.2 we will need the following Zähle formula which is a counterpart of the Blaschke–Petkantschn formula for points distributed over the boundary of the convex body ([3], [6]): if $K$ is a convex body in $\mathbb{R}^d$ and $h : (\mathbb{R}^d)^{l+1} \to \mathbb{R}$ is a measurable function, $l \leq d$, then

$$\int_{A_{d,l}(E \cap \partial K)^{l+1}} h(x_0, \ldots, x_l) \mathbb{1}_{\{x_0, \ldots, x_l\ \text{in general position}\} \sigma_{E \cap \partial K}(dx_0) \ldots \sigma_{E \cap \partial K}(dx_l) \mu_{d,l}(dE)$$

$$= \frac{1}{(d-l)! b_{d,l}} \int_{(\partial K)^{l+1}} h(x_0, \ldots, x_l) \mathbb{1}_{\{x_0, \ldots, x_l\ \text{in general position}\} \frac{1}{|[x_0, \ldots, x_l]|^{d-l}}$$

$$\times \prod_{j=0}^l \|P_{\text{aff } (x_0, \ldots, x_j)}(n_K(x_j))\| \sigma(dx_0) \ldots \sigma(dx_l),$$

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where $n_K(x)$ denotes the outer unit normal vector to $\partial K$ at $x$ and $P_E$ is the orthogonal projection onto the plane $E$.

Let us present the following generalisation of both the Blaschke–Petkantschin and Zähle formulae for convex bodies with smooth boundary.

**Theorem 1.3.** Let $K$ be a convex body with smooth boundary and let $h(x_0,\ldots,x_l)$ be a continuous function, $l \leq d$. Then for all $k \in \{0,\ldots,l+1\}$ we have

\[
\int \int_{(\partial K)^k \cap (K)^{l-k+1}} h(x_0,\ldots,x_l)dx_0 \ldots dx_{l-k}\sigma(dx_{l-k+1}) \ldots \sigma(dx_l) \\
= (l!)^{d-l}b_{d,l} \int \int \int_{A_{d,1}(E \cap \partial K)^k \cap (E \cap K)^{l-k+1}} h(x_0,\ldots,x_l)[x_0,\ldots,x_l]^{d-l} \\
\times \prod_{j=l-k+1}^{l} \|P_E(n_K(x_j))\|^{-1} \lambda_E(dx_0) \ldots \lambda_E(dx_{l-k})\sigma_{E \cap \partial K}(dx_{l-k+1}) \ldots \sigma_{E \cap \partial K}(dx_l) \mu_{d,1}(dE),
\]

where $n_K(x)$ denotes the outer unit normal vector to $\partial K$ at $x$ and $P_E$ is the orthogonal projection onto the hyperplane $E$.

Consider the case $k = l = 1$ when we have one point on the boundary and one point inside $K$. Applying Theorem 1.3 to the function $h(x_0, x_1) = |x_0 - x_1|^n$ for some integer $n$ implies the following formula.

**Corollary 1.1.** For a convex body $K$ with smooth boundary we have

\[
\int \int_{\partial K} |x_0 - x_1|^n dx_0 \sigma(dx_1) = \frac{\omega_d}{4(n + d)} \int_{A_{d,1}} |G \cap K|^{n+d} \left( \frac{1}{\sin \alpha_1} + \frac{1}{\sin \alpha_2} \right) \mu_{d,1}(dG),
\]

where $\alpha_i$ is the angle between line $G$ and tangent hyperplane at point $x_i$.

This is a counterpart of Kingman’s formula [4], which states that

\[
\int_{K^2} |x_0 - x_1|^n dx_0 dx_1 = \frac{\omega_d}{(n + d)(n + d + 1)} \int_{A_{d,1}} |G \cap K|^{n+d+1} \mu_{d,1}(dG).
\]

**Proof of Corollary 1.1.** Applying Theorem 1.3 with $h(x_0, x_1) = |x_0 - x_1|^n$ gives

\[
\int \int_{\partial K} |x_0 - x_1|^n dx_0 \sigma(dx_1) \\
= \frac{\omega_d}{4} \int_{A_{d,1}} \left( \frac{1}{\sin \alpha_1} \int_{G \cap K} |x_0 - x_1|^{n+d-1} \lambda_G(dx_0) + \frac{1}{\sin \alpha_2} \int_{G \cap K} |x_0 - x_2|^{n+d-1} \lambda_G(dx_0) \right) \mu_{d,1}(dG).
\]

Noting that

\[
\int_{G \cap K} |x_0 - x_1|^{n+d-1} \lambda_G(dx_0) = \int_0^{G \cap K} x^{n+d-1} dx = \frac{|G \cap K|}{n + d}
\]

finishes the proof.
Applying (1.6) to the polytopes gives the following result.

**Theorem 1.4.** Let \( P \) be a convex polytope in \( \mathbb{R}^d \) and let \( h(x_0, \ldots, x_l) \) be a measurable function, \( l \leq d - 1 \). Then

\[
\int_{(\partial P)^{l+1}} h(x_0, \ldots, x_l)\sigma(dx_0)\cdots\sigma(dx_l)
\]

\[
= (l!)^{d-l} b_{d,l} \sum_{(F_0, \ldots, F_l)\in F^{l+1}(P)} \int_{E\cap F_0} \int_{E\cap F_1} \cdots \int_{E\cap F_l} h(x_0, \ldots, x_l)
\]

\[
\times \ |[x_0, \ldots, x_l]|^{d-l} \prod_{j=0}^l \|P_E(nP(x_j))\|^{-1} \lambda_{E\cap F_0}(dx_0) \cdots \lambda_{E\cap F_l}(dx_l) \mu_{d,l}(dE)
\]

\[
+ (l!)^{d-l-1} b_{d-1,l} \cdot \sum_{F\in F(P)} \int_{A_{d-1,F}} \int_{A_{d,F}} h(x_0, \ldots, x_l)
\]

\[
\times \ |[x_0, \ldots, x_l]|^{d-l-1} \lambda_{E\cap F}(dx_0) \cdots \lambda_{E\cap F}(dx_l) \mu_{d,F}(dE),
\]

where \( n_P(x) \) denotes the outer unit normal vector of \( P \) at \( x \) and \( P_E \) is the orthogonal projection onto the hyperplane \( E \).

The paper is organized as follows. In Sections 2 and 3 we present the proof of Theorem 1.1 and Theorem 1.2, then present the proof of Theorem 1.3 in Section 4. The proof of Theorem 1.4 is presented in Section 5.

## 2. Proof of Theorem 1.1

To prove Theorem 1.1 we need the notion of flag spaces.

Let \( p, q \in \{0, \ldots, d\} \), and let \( E \in A_{d,p} \) be a fixed \( p \)-dimensional affine subspace. As mentioned above, \( A_{E,q} \) is the space of all \( q \)-dimensional affine subspaces contained in \( E \) if \( q \leq p \). If \( q \geq p \) we denote by \( A_{E,q} \) the space of all \( q \)-dimensional affine subspaces containing \( E \). Denote by \( \mu_{E,q} \) the invariant measure on \( A_{E,q} \) (see [7], Section 7.1).

Consider pairs of affine subspaces:

\[
A(d, p, q) := \{(E, F) \in A_{d,p} \times A_{d,q} : E \subset F\}, \text{ if } p < q,
\]

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\]

We are going to apply the following Fubini-type theorem for flag spaces (see, e.g., [7], Theorem 7.1.2): if \( 0 \leq p < q \leq d - 1 \) and \( h : A(d, p, q) \to \mathbb{R} \) is a nonnegative measurable function, then

\[
\int_{A_{d,p}} \int_{A_{E,q}} h(E, F)\mu_{E,F}(dE)\mu_{d,q}(dF) = \int_{A_{d,p}} \int_{A_{E,q}} h(E, F)\mu_{E,q}(dF)\mu_{d,p}(dE).
\]
Now let us prove Theorem 1.1. It follows from (1.6) with \( l = 1 \) that for a non-negative measurable function \( h(x_1, x_2) \) we have

\[
(2.2) \quad \int_{A_{d,1}} h(x_1, x_2) \mu_{d,1}(dG) = \frac{1}{\omega_d} \int_{(\partial K)^2} h(x_1, x_2) \frac{\sin \alpha_1 \sin \alpha_2}{|x_1 - x_2|^{d-2}} \sigma(dx_1)\sigma(dx_2),
\]

where \( \alpha_1 \) and \( \alpha_2 \) are the angles between line \( \text{aff}(x_1, x_2) \) and tangent hyperplanes at points \( x_1 \) and \( x_2 \).

Using equation (2.1) for \( p = 1, q = 2 \) we get

\[
\int_{A_{d,1}} h(|G \cap K|) \mu_{d,1}(dG) = \int_{A_{d,1} A_{G,2}} h(|G \cap K|) \mu_{G,2}(dE) \mu_{d,1}(dG)
\]

\[
= \int_{A_{d,2} A_{E,1}} h(|G \cap K|) \mu_{E,1}(dG) \mu_{d,2}(dE).
\]

Applying Pleijel identity (1.3) to the inner integral and planar convex body \( K \cap E \) we have

\[
\int_{A_{E,1}} h(|G \cap K|) \mu_{E,1}(dG) = \int_{A_{E,1}} h'(|G \cap K|) |G \cap K| \cot \psi_1 \cot \psi_2 \mu_{E,1}(dG),
\]

where angles \( \psi_1 \) and \( \psi_2 \) are the angles between tangents to \( K \cap F \) at the endpoints of \( G \cap K \) and line \( G \) lying on the same side of \( G \).

Applying equation (2.1) once again yields

\[
(2.3) \quad \int_{A_{d,1}} h(|G \cap K|) \mu_{d,1}(dG)
\]

\[
= \int_{A_{d,1} A_{G,2}} h'(|G \cap K|) |G \cap K| \cot \psi_1 \cot \psi_2 \mu_{G,2}(dE) \mu_{d,1}(dG)
\]

\[
= \int_{A_{d,2}} h'(|G \cap K|) |G \cap K| \left( \int_{A_{G,2}} \cot \psi_1 \cot \psi_2 \mu_{G,2}(dE) \right) \mu_{d,1}(dG)
\]

\[
= \frac{1}{\omega_d} \int_{(\partial K)^2} \frac{h'(|x_1 - x_2|)}{|x_1 - x_2|^{d-2}} \sin \alpha_1 \sin \alpha_2 \left( \int_{A_{G,2}} \cot \psi_1 \cot \psi_2 \mu_{G,2}(dE) \right) \sigma(dx_1)\sigma(dx_2),
\]

where in the last equality we used (2.2).

Consider the inner integral in the last expression. Note that space \( A_{G,2} \) is parameterised by lines passing through the origin in the orthogonal complement of \( G \).

Denote by \( u_E \) the unit vector in the orthogonal complement of \( \text{aff}(x_1, x_2) \) corresponding to plane \( E \) (i.e. \( E \) passes through \( u_F \)). By \( t_1(E) \) and \( t_2(E) \) denote unit vectors collinear with tangents to \( E \cap K \) at \( x_1 \) and \( x_2 \) and lying in the same halfplane of \( E \) as \( u_E \). Denote by \( u_1 \) the normalised projection of \( n_{x_1} \) onto the orthogonal complement of \( \text{aff}(x_1, x_2) \) and by \( t_1 \) — normalised projection of vector \( x_2 - x_1 \) onto the tangent hyperplane at \( x_1 \). Vectors \( u_2 \) and \( t_2 \) are defined similarly.
Lemma 2.1.

\[ \cot \psi_1 = (u_1, u_E) \cdot \cot \alpha_1 \quad \text{and} \quad \cot \psi_2 = (u_2, u_E) \cdot \cot \alpha_2 \]

Proof. By definition,
\[ \cos \psi_1 = \frac{(x_2 - x_1, t_1(E))}{|x_2 - x_1|} \quad \text{and} \quad \sin \psi_1 = (u_E, t_1(E)) \]
\[ \cos \alpha_1 = \frac{(x_2 - x_1, t_1)}{|x_2 - x_1|} \quad \text{and} \quad \sin \alpha_1 = (u_1, t_1). \]

Hence we need to prove that
\[ \frac{(x_2 - x_1, t_1(E))}{(u_E, t_1(E))} = (u_1, u_E) \cdot \frac{(x_2 - x_1, t_1)}{(u_1, t_1)}. \]

Note that from definition of \( u_1 \) it follows that
\[ n_{x_1} = u_1 + \frac{x_1 - x_2}{|x_1 - x_2|}. \]

Thus,
\[ (x_2 - x_1, t_1) = |x_2 - x_1| \cdot (u_1 - n_{x_1}, t_1) = |x_2 - x_1| \cdot (u_1, t_1) \]
and
\[ (x_2 - x_1, t_1(E)) = |x_2 - x_1| \cdot (u_1 - n_{x_1}, t_1(E)) = |x_2 - x_1| \cdot (u_1, t_1(E)). \]

Moreover,
\[ t_1(E) = (t_1(E), \frac{x_2 - x_1}{|x_2 - x_1|}) \cdot \frac{x_2 - x_1}{|x_2 - x_1|} + (t_1(E), u_E) \cdot u_E, \]
hence
\[ (u_1, t_1(E)) = (t_1(E), u_E) \cdot (u_1, u_E) \quad \text{and} \quad (u_E, t_1(E)) = (t_1(E), u_E). \]

Thus,
\[ \frac{(x_2 - x_1, t_1(E))}{(u_E, t_1(E))} = |x_2 - x_1| \cdot (u_1, t_1(E)) = \frac{|x_2 - x_1|(t_1(E), u_E) \cdot (u_1, u_E)}{(t_1(E), u_E)} \]
\[ = |x_2 - x_1| \cdot (u_1, u_E) \cdot \frac{(x_2 - x_1, t_1)}{(u_1, t_1)}. \]

Proof of the second part of lemma is the same. \( \square \)

Lemma 2.1 yields
\[ (2.4) \quad \int_{A_{G,2}} \cot \psi_1 \cot \psi_2 \mu_{G,2}(dE) = \int_{A_{G,2}} \cot \alpha_1 \cot \alpha_2 \cdot (u_1, u_E) \cdot (u_2, u_E) \mu_{G,2}(dE) \]
\[ = \cot \alpha_1 \cot \alpha_2 \cdot \int_{S^{d-2}} (u_1, u_E) \cdot (u_2, u_E) \tilde{\sigma}(du_E), \]
where \( \tilde{\sigma} \) is the uniform measure on \( S^{d-2} \) normalised to be probabilistic.
Substituting (2.4) in (2.3) yields

\[\int_{A_{d,1}} h(|G \cap K|) \mu_{d,1}(dG) = \frac{1}{\omega_d} \int_{(\partial K)^2} \frac{h(|x_1 - x_2|)}{|x_1 - x_2|^{d-2}} \cos \alpha_1 \cos \alpha_2 \left( \int_{\mathbb{S}^{d-2}} (u_1, u_E) \cdot (u_2, u_E) \tilde{\sigma}(d u_E) \right) \sigma(dx_1) \sigma(dx_2),\]

To finish the proof of the main result it suffices to prove that

\[\int_{\mathbb{S}^{d-2}} (u_1, u_E) \cdot (u_2, u_E) \tilde{\sigma}(d u_E) = \frac{1}{d-1} (u_1, u_2),\]

since \((u_1, u_2) = \cos(\phi_0)\).

The latter statement follows from the fact that if point \(u_F\) is uniformly distributed on the unit sphere \(\mathbb{S}^{n-1}\) \((n = d - 1)\) than vector formed by its first \(n - 2\) coordinates is uniformly distributed on the unit ball \(\mathbb{B}^{n-2}\).

Change the coordinate system so that first \(n - 2\) coordinates of \(u_1\) and \(u_2\) are zero. Then

\[(2.5)\]

\[\int_{\mathbb{S}^{n-1}} (u_1, u_E) \cdot (u_2, u_E) \tilde{\sigma}(d u_E) = \int_{\mathbb{B}^{n-2}} \int_{\mathbb{S}^{1}} (x + z, u_1) \cdot (x + z, u_2) \tilde{\sigma}(dx) \tilde{\lambda}(dz) = \int_{\mathbb{B}^{n-2}} \int_{\mathbb{S}^{1}} (1 - |z|^2) (x, u_1) \cdot (x, u_2) \tilde{\sigma}(dx) \tilde{\lambda}(dz) = \int_{\mathbb{S}^{1}} (x, u_1) \cdot (x, u_2) \tilde{\sigma}(dx) \cdot \int_{\mathbb{B}^{n-2}} (1 - |z|^2) \tilde{\lambda}(dz) = \frac{2\pi}{2\pi} \int_{\mathbb{B}^{n-2}} \cos(x) \cos(x - \phi_0) dx \int_{\mathbb{S}^{n-2}} (1 - |z|^2) \tilde{\lambda}(dz).\]

Note that

\[(2.6)\]

\[\int_{0}^{2\pi} \cos(x) \cos(x - \phi_0) dx = \int_{0}^{2\pi} \frac{1}{2} (\cos(\phi_0) + \cos(2x - \phi_0)) dx = \pi \cos(\phi_0)\]

and

\[(2.7)\]

\[\int_{\mathbb{B}^{n-2}} (1 - |z|^2) \tilde{\lambda}(dz) = \frac{1}{\kappa_{n-2}} \int_{\mathbb{S}^{n-2}} (1 - |z|^2) \lambda(dz) = \frac{1}{\kappa_{n-2}} \int_{\mathbb{S}^{n-3}} \int_{0}^{2\pi} (1 - r^2) r^{n-3} dr \sigma(d\phi) = (n - 2) \left( \frac{1}{n - 2} - \frac{1}{n} \right) = \frac{2}{n} = \frac{2}{d - 1}.\]

Substituting (2.6) and (2.7) in (2.5) finishes the proof.
3. PROOF OF THEOREM 1.2

Assume that \( P \) is a convex polytope in \( \mathbb{R}^d \). Similarly as in the proof of Theorem 1.1 we get

\[
\int_{A_{d,1}} h(|G \cap P|) \mu_{d,1}(dG) = \int_{A_{d,2} A_{E,1}} h(|G \cap P|) \mu_{E,1}(dG) \mu_{d,2}(dE).
\]

Applying the Ambartzumian-Pleijel identity to the planar polygon \( P \cap E \) yields

\[
\int_{A_{E,1}} h(|G \cap P|) \mu_{E,1}(dG) = \int_{A_{E,1}} h'(|G \cap P|) |G \cap P| \cot \psi_1 \cot \psi_2 \mu_{E,1}(dG) + \sum_{i=1}^{N} \int_0 a_i h(t) dt,
\]

where \( \psi_1 \) and \( \psi_2 \) are the angles between tangents to \( P \cap E \) at the endpoints of \( G \cap P \) and line \( G \) lying on the same side of \( G \) and \( a_i \) are the lengths of the sides of \( E \cap P \).

The second term in the right-hand side of (3.2) can be rewritten the following way:

\[
\sum_{i=1}^{N} \int_0 a_i h(t) dt = \sum_{F \in \mathcal{F}(P)} \int_0 h(t) dt.
\]

Substituting (3.2) and (3.3) in (3.1) and applying equation (2.1) implies

\[
\int_{A_{d,1}} h(|G \cap P|) \mu_{d,1}(dG) = \int_{A_{d,2} A_{E,1}} h'(|G \cap P|) |G \cap P| \cot \psi_1 \cot \psi_2 \mu_{E,1}(dG) + \sum_{i=1}^{N} \int_0 a_i h(t) dt.
\]

To calculate \( \int_{A_{G,2}} \cot \psi_1 \cot \psi_2 \mu_{G,2}(dE) \) we can apply Lemma 2.1 which is true for \( \mu_{d,2} \)-almost every line \( G \) intersecting \( P \) (more precisely for those lines \( G \) which meet \( \partial P \) in the relative interior of two different facets).

Hence, for \( \mu_{d,2} \)-almost every line \( G \) we have:

\[
\int_{A_{G,2}} \cot \psi_1 \cot \psi_2 \mu_{G,2}(dE) = \cot \alpha_1 \cot \alpha_2 \cdot \int_{S^{d-2}} (u_1, u_E) \cdot (u_2, u_E) \hat{\sigma}(d u_E)
\]

\[
= \frac{1}{d-1} (u_1, u_2) \cot \alpha_1 \cot \alpha_2.
\]

Consider the second term of (3.1). Note that for a fixed facet \( F \),

\[
\int_{A_{d,2}} \int_0 h(t) dt \mu_{d,2}(dE) = \int_{A_{F,1}} \int_0 h(t) dt \mu_{F,1}(dG) = \int_{A_{F,1}} H(|G \cap F|) \mu_{F,1}(dG).
\]

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Substituting (3.5) and (3.6) in (3.4) yields
\[ \int h(|G \cap P|) \mu_{d,1}(dG) = \frac{1}{d-1} \int h'(|G \cap P|) |G \cap P| \cot \alpha_1 \cot \alpha_2 \cos(\phi_0) \mu_{d,1}(dG) \]
\[ + \sum_{F \in \mathcal{F}(p)} \int H(|G \cap F|) \mu_{F,1}(dG), \]
which finishes the proof.

4. PROOF OF THEOREM 1.3

The proof of Theorem 1.3 is similar to one in [6], but since it covers the case when some of the points are inside the body, we present it here.

Consider the functional
\[ \mathcal{I}(h) = \int_{(\partial K)^k \setminus (K)^{l-k+1}} h(x_0, \ldots, x_l) \prod_{j=l-k+1}^l \| \text{aff}(x_0, \ldots, x_j)(n_K(x_j)) \| dx_0 \ldots dx_{l-k-1} \sigma(dx_{l-k}) \ldots \sigma(dx_l). \]

Applying Steiner’s formula yields
\[ \mathcal{I}(h) = \lim_{\varepsilon \to 0} \left( \frac{1}{\varepsilon} \right)^k \int_{(K \setminus K)^k \setminus (K)^{l-k+1}} h(x_0, \ldots, x_l, x_{l-k+1}, \ldots, x_l) \prod_{j=l-k+1}^l \| \text{aff}(x_0, \ldots, x_j)(n_K(x_j)) \| dx_0 \ldots dx_l, \]
where \( K^\varepsilon \) is the \( \varepsilon \)-neighbourhood of \( K \) and for a point \( x \notin K \) we denote by \( \overline{x} \) the point in \( K \), nearest to \( x \) (the metric projection).

The Blaschke–Petkantschin formula (see (1.5)) implies that
\[ \mathcal{I}(h) = (l!)^{d-1} b_{d,l} \lim_{\varepsilon \to 0} \left( \frac{1}{\varepsilon} \right)^k \int_{A_{d,l} (E \cap K)^k \setminus (E \cap K)^{l-k+1}} h(x_0, \ldots, x_l, x_{l-k+1}, \ldots, x_l) \prod_{j=l-k+1}^l \| \text{aff}(x_0, \ldots, x_j)(n_K(x_j)) \| \cdot |x_0, \ldots, x_l|^{d-1} dx_0 \ldots dx_l \mu_{d,l}(dE). \]

Point \( x \in E \cap (K^\varepsilon \setminus K) \) is determined by the point in \( E \cap K \) nearest to \( x \) and the distance \( t(x) \) from \( x \) to \( E \cap \partial K \). Then \( 0 \leq t(x) \leq h_E(x) \), where \( h_E(x) \) is the length of intersection of \( K^\varepsilon \setminus K \) with the line in \( \partial E \) passing through \( x \) and orthogonal to \( E \cap \partial K \). Generalization of Steiner’s formula applied to \( E \cap (K^\varepsilon \setminus K) \) and continuity of integrand implies
\[ \mathcal{I}(h) = (l!)^{d-1} b_{d,l} \lim_{\varepsilon \to 0} \int_{A_{d,l} (E \cap K)^k \setminus (E \cap K)^{l-k+1}} h(x_0, \ldots, x_l) |x_0, \ldots, x_l|^{d-1} \]
\[ \times \prod_{j=l-k+1}^l \| \text{aff}(x_0, \ldots, x_j)(n_K(x_j)) \| \cdot |x_0, \ldots, x_l|^{d-1} \int_0^{h_E(x)} dt \lambda_E(dx_0) \ldots \lambda_E(dx_{l-k}) \]
\[ \times \sigma_{E \cap \partial K}(dx_{l-k+1}) \ldots \sigma_{E \cap \partial K}(dx_l) \mu_{d,l}(dE). \]

Applying
\[ h_E(x) \leq \varepsilon \| \text{aff}(x_0, \ldots, x_l)(n_K(x)) \|^{-1} \]
and

$$\lim_{\varepsilon \to 0} \varepsilon^{-1} h_E(x) = \|P_{aff(x_0,\ldots,x_l)}(n_K(x))\|^{-1}$$

yields

$$\mathcal{I}(h) = (l!)^{d-l}b_{d,l}\int_{\partial P(l)\cap \partial K} h(x_0,\ldots,x_l)\sigma_E(dx_0)\ldots\sigma_E(dx_l)$$

$$\times \lambda_E(dx_0)\ldots\lambda_E(dx_{l-k})\sigma_{E\cap \partial K}(dx_{l-k+1})\ldots\sigma_{E\cap \partial K}(dx_l)\mu_{d,l}(dE),$$

which finishes the proof.

5. Proof of Theorem 1.4

In (1.6) we have $\|P_{aff(x_0,\ldots,x_l)}(n_P(x_k))\| = 0$ if and only if all points $x_k$ lie on the same facet of $P$. Therefore for measurable function $h$ we have

$$\int_{(\partial P)^{l+1}} h(x_0,\ldots,x_l)\sigma(dx_0)\ldots\sigma(dx_l)$$

$$= \int_{(\partial P)^{l+1}} h(x_0,\ldots,x_l)\mathbb{1}_{\{x_1,\ldots,x_l\text{ are not all in the same facet}\}}\sigma(dx_0)\ldots\sigma(dx_l)$$

$$+ \sum_{F \in \mathcal{F}(P)_{l+1}} \int_{(\partial P)^{l+1}} h(x_0,\ldots,x_l)\lambda_F(dx_0)\ldots\lambda_F(dx_l).$$

Applying (1.6) to the first term and (1.5) to the second term we get the following:
\[ \int_{(\partial P)^{l+1}} h(x_0, \ldots, x_l) \sigma(dx_0) \ldots \sigma(dx_l) \]

\[ = (l!)^d b_{d,l} \int_{A_{d,l}} \int_{(E \cap \partial P)^{l+1}} h(x_0, \ldots, x_l) \mathbb{1}_{\{x_1, \ldots, x_l \text{ are not all in the same facet}\}} \times |[x_0, \ldots, x_l]|^{d-1} \prod_{j=0}^{l} |P_E((n_P(x_j)))|^{-1} \sigma_{E \cap \partial P}(dx_0) \ldots \sigma_{E \cap \partial P}(dx_l) \mu_{d,l}(dE) \]

\[ + (l!)^{d-l-1} b_{d-1,l} \sum_{F \in \mathcal{F}(P)_{A_{d,l}}(E \cap F)^{l+1}} h(x_0, \ldots, x_l) \times |[x_0, \ldots, x_l]|^{d-l-1} \lambda_{E \cap F}(dx_0) \ldots \lambda_{E \cap F}(dx_l) \mu_{F,l}(dE) \]

\[ = (l!)^{d-l-1} b_{d-1,l} \sum_{\{F_0, \ldots, F_l\} \in \mathcal{F}^{l+1}(P)_{A_{d,l}}(E \cap F_0)^{l+1} \exists F_j \neq F_i} \int_{E \cap F_0} \ldots \int_{E \cap F_l} h(x_0, \ldots, x_l) \times |[x_0, \ldots, x_l]|^{d-l} \prod_{j=0}^{l} |P_E((n_P(x_j)))|^{-1} \lambda_{E \cap F_0}(dx_0) \ldots \lambda_{E \cap F_l}(dx_l) \mu_{d,l}(dE) \]

\[ + (l!)^{d-l-1} b_{d-1,l} \sum_{F \in \mathcal{F}(P)_{A_{d,l}}(E \cap F)^{l+1}} h(x_0, \ldots, x_l) \times |[x_0, \ldots, x_l]|^{d-l-1} \lambda_{E \cap F}(dx_0) \ldots \lambda_{E \cap F}(dx_l) \mu_{F,l}(dE). \]

This proves Theorem 1.4.

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TATIANA MOSEEEVA, LEONHARD EULER INTERNATIONAL MATHEMATICAL INSTITUTE, RUSSIA

Email address: polezina@yandex.ru