TRUNCATED $t$-ADIC SYMMETRIC MULTIPLE ZETA VALUES AND DOUBLE SHUFFLE RELATIONS

MASATAKA ONO, SHIN-ICHIRO SEKI, AND SHUJI YAMAMOTO

Abstract. We study a refinement of the symmetric multiple zeta value, called the $t$-adic symmetric multiple zeta value, by considering its finite truncation. More precisely, two kinds of regularizations (harmonic and shuffle) give two kinds of the $t$-adic symmetric multiple zeta values, thus we introduce two kinds of truncations correspondingly. Then we show that our truncations tend to the corresponding $t$-adic symmetric multiple zeta values, and satisfy the harmonic and shuffle relations, respectively. This gives a new proof of the double shuffle relations for $t$-adic symmetric multiple zeta values, first proved by Jarossay. In order to prove the shuffle relation, we develop the theory of truncated $t$-adic symmetric multiple zeta values associated with 2-colored rooted trees. Finally, we discuss a refinement of Kaneko–Zagier’s conjecture and the $t$-adic symmetric multiple zeta values of Mordell–Tornheim type.

1. Introduction

1.1. Main results. For a tuple of non-negative integers $\mathbf{k} = (k_1, \ldots, k_r)$, we set $\text{wt}(\mathbf{k}) := k_1 + \cdots + k_r$ and $\text{dep}(\mathbf{k}) := r$, and call them the weight and the depth of $\mathbf{k}$, respectively. Such $\mathbf{k}$ is called an index if none of its entries is zero. In particular, there is a unique index of depth 0, which we call the empty index and denote by $\emptyset$. An index $\mathbf{k} = (k_1, \ldots, k_r)$ is said admissible if $r > 0$ and $k_r \geq 2$, or $\mathbf{k} = \emptyset$. For a non-empty admissible index $\mathbf{k} = (k_1, \ldots, k_r)$, the multiple zeta value (MZV) $\zeta(\mathbf{k})$ is defined by

$$
\zeta(\mathbf{k}) := \sum_{0 < n_1 < \cdots < n_r} \frac{1}{n_1^{k_1} \cdots n_r^{k_r}},
$$

while $\zeta(\emptyset)$ is set to be 1.

There are also various variants of MZV. One of such variants, called the $t$-adic symmetric multiple zeta value and denoted by $\zeta^S_{\mathbf{k}}$, is the main object of this paper. We also

2010 Mathematics Subject Classification. 11M32, 05C05.
Key words and phrases. $t$-adic symmetric multiple zeta values, double shuffle relation, Kaneko–Zagier’s conjecture, multiple zeta values of Mordell–Tornheim type.
This research was supported in part by JSPS KAKENHI Grant Numbers 26247004, 16J01758, 16H06336, 18J00151, 18K03221, 18H05233.
call it the $\hat{S}$-MZV for short. For the relationship between the $\hat{S}$-MZV and other variants, see the next subsection.

In order to define the $\hat{S}$-MZV, we first introduce two values $\zeta^*_S(k)$ and $\zeta^w_S(k)$. For tuples $k=(k_1,\ldots,k_r)$ and $l=(l_1,\ldots,l_r)$, we set

$$k + l := (k_1 + l_1,\ldots,k_r + l_r), \quad b\left(\frac{k}{l}\right) := \prod_{j=1}^{r} \frac{k_j + l_j - 1}{l_j},$$

$$\overline{\mathbb{K}} := (k_r,\ldots,k_1), \quad k[i] := (k_1,\ldots,k_i), \quad k[i] := (k_{i+1},\ldots,k_r) \quad (0 \leq i \leq r).$$

Regarding the depth 0 case, we understand $\emptyset + \emptyset = \emptyset$, $b(\overline{\emptyset}) = 1$, $\overline{\emptyset} = \emptyset$ and $k[0] = k[r] = \emptyset$. Let $\mathbb{Z}$ be the $\mathbb{Q}$-subalgebra of $\mathbb{R}$ generated by all MZVs, and set $\overline{\mathbb{Z}} := \mathbb{Z}/\pi^2\mathbb{Z}$ (recall that $\pi^2 = 6\zeta(2) \in \mathbb{Z}$). We denote by $\zeta^*$ and $\zeta^w$ the harmonic and shuffle regularized MZVs, respectively (see §2.1 below for the definitions).

**Definition 1.1.** For $\bullet \in \{\ast, w\}$ and an index $k=(k_1,\ldots,k_r)$, we define $\zeta^\bullet_S(k)$ as an element of $\mathbb{Z}/[t]$, with $t$ being an indeterminate, by

$$\zeta^\bullet_S(k) := \sum_{i=0}^{r} (-1)^{w(t)}(t)^{k[i]} \sum_{l \in \mathbb{Z} \geq 0} b\left(\frac{k[i]}{l}\right) \zeta^\bullet_S(k[i] + l)t^{w(t)}.$$

We will prove that $\zeta^*_S(k) - \zeta^w_S(k) \in \pi^2\mathbb{Z}[t]$ (Proposition 2.1). Thus, the following definition is independent of the choice of the regularization $\bullet \in \{\ast, w\}$.

**Definition 1.2 ($\hat{S}$-MZV).** For an index $k$, we define the $t$-adic symmetric multiple zeta value ($\hat{S}$-MZV) $\zeta^\bullet_S(k)$ by

$$\zeta_S(k) := \zeta^\bullet_S(k) \mod \pi^2 \in \overline{\mathbb{Z}}[t].$$

This definition of the $\hat{S}$-MZV with the expression (1.1) was personally communicated by Hirose to the second author. The same expression is also used by Rosen to define the motivic version of $\zeta_S(k)$ ([Ro2, Definition 2.4]). Independently, Jarossay introduced the $\Lambda$-adjoint multiple zeta value in terms of the Drinfeld associator. It turns out that the $\hat{S}$-MZV coincides up to sign with the $\Lambda$-adjoint MZV, where the indeterminate $\Lambda$ corresponds to our $t$.

Jarossay [J5] proved the double shuffle relation (DSR) for the $\Lambda$-adjoint MZVs. In terms of our $\hat{S}$-MZVs, it can be stated as follows:
Theorem 1.3 (DSR for \(\hat{S}\)-MZVs). For any indices \(k\) and \(l\), we have

\[
\zeta^e_{\hat{S}}(k \ast l) = \zeta^e_{\hat{S}}(k)\zeta^e_{\hat{S}}(l),
\]

(1.2)

\[
\zeta^w_{\hat{S}}(k \mathfrak{m} l) = (-1)^{wt(l)} \sum_{t' \in \mathbb{Z}_{\geq 0}^{d_{lep}(t)}} b\left(\frac{l}{l'}\right)\zeta^w_{\hat{S}}(k, l + l')\left(l'\right)^{wt(l')}.
\]

(1.3)

Here \(\zeta^e_{\hat{S}}(k \ast l)\) (resp. \(\zeta^w_{\hat{S}}(k \mathfrak{m} l)\)) is defined as \(\sum h \zeta^e_{\hat{S}}(h)\) (resp. \(\sum h' \zeta^w_{\hat{S}}(h')\)) when \(z_k \ast z_l = \sum h z_h\) (resp. \(z_k \mathfrak{m} z_l = \sum h' z_{h'}\)), respectively. See [21] for the definitions of these symbols. We use the same convention for other zeta values. In this paper, we give an alternative proof of Theorem 1.3. In fact, we refine the theorem by considering the following truncations.

Definition 1.4 (Truncated \(\hat{S}\)-MZVs). Let \(k = (k_1, \ldots, k_r)\) be an index and \(M\) a positive integer. We define the \(\ast\)-truncated \(\hat{S}\)-MZV \(\zeta^e_{\hat{S},M}(k)\) by

\[
\zeta^e_{\hat{S},M}(k) := \sum_{i=0}^r \sum_{0<n_1<\cdots<n_i<M} 1 \frac{n_{i+1}^{k_{i+1}} \cdots n_r^{k_r}}{n_1^{k_1} \cdots n_i^{k_i} (n_{i+1} + t) \cdots (n_r + t)^k_{r}} \in \mathbb{Q}[t]
\]

and the \(\mathfrak{m}\)-truncated \(\hat{S}\)-MZV \(\zeta^w_{\hat{S},M}(k)\) by

\[
\zeta^w_{\hat{S},M}(k) := \sum_{i=0}^r \sum_{0<n_1<\cdots<n_i<M} 1 \frac{n_{i+1}^{k_{i+1}} \cdots n_r^{k_r}}{n_1^{k_1} \cdots n_i^{k_i} (n_{i+1} + t) \cdots (n_r + t)^k_{r}} \in \mathbb{Q}[t].
\]

The first main result of this paper states that these truncated \(\hat{S}\)-MZVs tend to the corresponding \(\hat{S}\)-MZVs.

Theorem 1.5 (= Theorem 2.5). For any index \(k\) and \(\bullet \in \{\ast, \mathfrak{m}\}\), we have

\[
\zeta^e_{\hat{S}}(k) = \lim_{M \to \infty} \zeta^e_{\hat{S},M}(k),
\]

where the limit is taken coefficientwise as the power series in \(t\).

Naively speaking, this result says that

\[
\zeta^e_{\hat{S}}(k) = \sum_{i=0}^r \sum_{0<n_1<\cdots<n_i<M} \frac{1}{n_1^{k_1} \cdots n_i^{k_i} (n_{i+1} + t) \cdots (n_r + t)^k_{r}}.
\]

In practice, the specific choices of partial sums \(\zeta^e_{\hat{S},M}(k)\) and \(\zeta^w_{\hat{S},M}(k)\) give the limits \(\zeta^e_{\hat{S}}(k)\) and \(\zeta^w_{\hat{S}}(k)\), which are different in general. Nevertheless, we call Theorem 1.5 the series expression of \(\zeta^e_{\hat{S}}(k)\).
The second main result is the DSR for the truncated $\hat{S}$-MZVs, which is a refinement of Theorem 1.3 in view of Theorem 1.5.

**Theorem 1.6** (DSR for truncated $\hat{S}$-MZVs). For any indices $k$ and $l$ and any positive integer $M$, we have

\[
\zeta_{\hat{S},M}^{*}(k \ast l) = \zeta_{\hat{S},M}^{*}(k)\zeta_{\hat{S},M}^{*}(l),
\]

(1.4)

\[
\zeta_{\hat{S},M}^{\mathrm{m}}(k \, \mathrm{m} \, l) = (-1)^{\mathrm{wt}(l)} \sum_{l' \in \mathbb{Z}^r \geq 0 \atop \mathrm{dep}(l') \geq \mathrm{wt}(l')} b(l') \zeta_{\hat{S},M}^{\mathrm{m}}(k, l + l') \zeta_{\hat{S},M}^{\mathrm{m}}(l').
\]

(1.5)

1.2. **Background.** Our interest in the $\hat{S}$-MZVs is motivated by the work of Kaneko and Zagier [KZ] concerning two kinds of variants of MZVs, called $\mathcal{A}$-finite multiple zeta values ($\mathcal{A}$-MZVs) and symmetric multiple zeta values ($\mathcal{S}$-MZVs). First we recall the definitions of them.

Let $k = (k_1, \ldots, k_r)$ be any index. For a positive integer $M$, we define the truncated multiple zeta value $\zeta_M(k)$ by

\[
\zeta_M(k) := \sum_{0 < n_1 < \cdots < n_r < M} \frac{1}{n_1^{k_1} \cdots n_r^{k_r}},
\]

where we understand $\zeta_M(\emptyset) := 1$. Then the $\mathcal{A}$-MZV $\zeta_{\mathcal{A}}(k)$ and the $\mathcal{S}$-MZV $\zeta_{\mathcal{S}}(k)$ are defined by

\[
\zeta_{\mathcal{A}}(k) := (\zeta_p(k) \mod p)_p \in \mathcal{A},
\]

\[
\zeta_{\mathcal{S}}(k) := \zeta_{\mathcal{S}}^{*}(k) \mod \pi^2 \mathbb{Z} \in \overline{\mathbb{Z}}.
\]

Here $\mathcal{A}$ denotes the $\mathbb{Q}$-algebra

\[
\mathcal{A} := \prod_p \mathbb{Z}/p\mathbb{Z} \bigg/ \bigoplus_p \mathbb{Z}/p\mathbb{Z}
\]

where $p$ runs over the set of prime numbers, and $\zeta_{\mathcal{S}}^{*}(k)$ denotes the element of $\mathbb{Z}$ given by

\[
\zeta_{\mathcal{S}}^{*}(k) := \sum_{i=0}^{r} (-1)^{\mathrm{wt}(k[i])} \zeta^{*}(k[i])\zeta^{*}(k[i]),
\]

for $\bullet \in \{*, \mathrm{m}\}$. The definition of $\zeta_{\mathcal{S}}^{*}(k) \in \overline{\mathbb{Z}}$ is independent of the choice of $\bullet \in \{*, \mathrm{m}\}$, since it is shown that $\zeta_{\mathcal{S}}^{*}(k) \equiv \zeta_{\mathcal{S}}^{\mathrm{m}}(k) \mod \pi^2$ by Kaneko–Zagier (in fact, our Proposition 2.1 generalizes this congruence).

Kaneko and Zagier conjectured that $\mathcal{A}$-MZVs and $\mathcal{S}$-MZVs satisfy exactly the same algebraic relations (see Conjecture 4.1 for the precise statement). For example, they showed that both $\mathcal{A}$-MZVs and $\mathcal{S}$-MZVs satisfy the following relations, called the double shuffle...
relation (the shuffle relation (1.7) was also proved by the first author [O, Corollary 4.1] for \( F = A \), and by Jarossay [J1, Théorème 1.7 i) and Hirose [Hi, Proposition 15] for \( F = S \)).

**Theorem 1.7** (DSR for \( A \)-MZVs and \( S \)-MZVs). Let \( F \in \{ A, S \} \). For any indices \( k \) and \( l \), we have

\[
(1.6) \quad \zeta_F(k \ast l) = \zeta_F(k) \zeta_F(l), \\
(1.7) \quad \zeta_F(k \mathbin{\#} l) = (-1)^{\text{wt}(l)} \zeta_F(k, l).
\]

We note that the harmonic relation (1.6) for each \( F \in \{ A, S \} \) follows from the same relation of truncated MZVs \( \zeta_M(k \ast l) = \zeta_M(k) \zeta_M(l) \). This is obvious for \( F = A \), while the series expression of \( \zeta_S^*(l) \) (Corollary 2.6 for \( \bullet = * \)) is needed for \( F = S \).

Rosen [Ro1] introduced a natural refinement of the \( A \)-MZV in the \( \mathbb{Q} \)-algebra

\[
\hat{A} := \lim_{n \to \infty} \left( \prod_p \mathbb{Z}/p^n \mathbb{Z} / \bigoplus_p \mathbb{Z}/p^n \mathbb{Z} \right).
\]

Slightly modifying his definition, the second author [S2] defined the \( \hat{A} \)-finite multiple zeta value (\( \hat{A} \)-MZV) by \( \zeta_{\hat{A}}(k) := ((\zeta_p(k) \mod p^n)_p)_n \in \hat{A} \). Note that \( \hat{A} \) is complete under the \( p \)-adic topology, where \( p \) denotes the element \( (p \mod p^n)_n \) of \( \hat{A} \), and \( \zeta_{\hat{A}}(k) \) is a lifting of \( \zeta_A(k) \) in the sense that \( \zeta_A(k) = \zeta_{\hat{A}}(k) \mod p \) under the canonical isomorphism \( A \simeq \hat{A}/p\hat{A} \). The DSR for \( A \)-MZVs are extended to the DSR for \( \hat{A} \)-MZVs as follows.

**Theorem 1.8** (DSR for \( \hat{A} \)-MZVs). For any indices \( k \) and \( l \), we have

\[
(1.8) \quad \zeta_{\hat{A}}(k \ast l) = \zeta_{\hat{A}}(k) \zeta_{\hat{A}}(l), \\
(1.9) \quad \zeta_{\hat{A}}(k \mathbin{\#} l) = (-1)^{\text{wt}(l)} \sum_{l' \in \mathbb{Z}_{\geq 0}^{\text{dep}(l)}} b(l') \zeta_{\hat{A}}(k, l + l') p^{\text{wt}(l')}.
\]

Again, the harmonic relation (1.8) follows immediately from the same relation for the truncated MZVs. The shuffle relation (1.9) was proved independently by the second author [S1, Theorem 6.4] and Jarossay [J4, Lemma 4.17].

From the perspective of the Kaneko–Zagier conjecture, it is natural to expect that there is also some complete algebra with residue ring \( \mathcal{Z} \) and a lifting of \( \zeta_S(k) \) in that algebra. The consistency between Theorem 1.3 and Theorem 1.8 strongly suggests that the expected lifting is \( \zeta_{\hat{S}}(k) \) in the \( t \)-adically complete algebra \( \mathcal{Z}[t] \). See §4 for further discussion on this extended correspondence.
1.3. Contents of this paper. This paper is organized as follows.

§2.1 provides some preliminaries including the definitions of the products $\ast$ and $\mathfrak{m}$ and the corresponding regularizations. We also prove the congruence $\zeta_{\hat{S}}(k) \equiv \zeta_{\hat{S}}^\mathfrak{m}(k) \mod \pi^2$ there. Then the series expressions of $\zeta_{\hat{S}}^\ast(k)$ (Theorem 1.5) and the harmonic relation for $\ast$-truncated $\hat{S}$-MZVs (Theorem 1.6 (1.4)) are proved in §2.2 and §2.3 respectively. In addition, we prove a relation of truncated $\hat{S}$-MZVs, which we call the reversal relation, in §2.4.

In §3 we build a theory of truncated $\hat{S}$-MZVs associated with 2-colored rooted trees. A 2-colored rooted tree is a combinatorial structure introduced by the first author [O] with applications to $\mathcal{A}$-MZVs including a proof of the shuffle relation (1.7). We show that a similar argument is applicable in the context of truncated $\hat{S}$-MZVs. In §3.1 we recall the notion of 2-colored rooted trees and indices on them, and define the associated truncated $\hat{S}$-MZVs. We see that they include the $\mathfrak{m}$-truncated $\hat{S}$-MZVs given in Definition 1.4. In §3.2 we prove some basic properties of the truncated $\hat{S}$-MZVs associated with 2-colored rooted trees. These properties are used in §3.3 to show the shuffle relation for $\mathfrak{m}$-truncated $\hat{S}$-MZVs (Theorem 1.6 (1.5)). In §3.4, we consider a fairly general class of truncated $\hat{S}$-MZVs associated with 2-colored rooted trees, and establish an algorithm for representing those values in terms of the $\mathfrak{m}$-truncated $\hat{S}$-MZVs in the sense of Definition 1.4.

In §4.1 we briefly discuss the Kaneko–Zagier conjecture on the correspondence between $\zeta_{\mathcal{A}}(k)$ and $\zeta_{\mathcal{S}}(k)$, and its refinement to that of $\zeta_{\mathcal{A}}^\ast(k)$ and $\zeta_{\mathcal{S}}(k)$. We recall Yasuda’s theorem that $\zeta_{\mathcal{S}}^\ast(k)$ generates $\mathbb{Z}$, and give a proof of an analogous result for $\zeta_{\hat{S}}^\ast(k)$ due to Jarossay. Finally, in §4.2 we introduce the $\hat{S}$-MZV of Mordell–Tornheim type which corresponds to $\hat{A}$-MZV of Mordell–Tornheim type under the refined Kaneko–Zagier conjecture. We prove some formulas on them by applying the theory developed in §3.

There are some recent works related with the contents of our paper. Hirose, Murahara and the first author [HMO] define the star-version of $\hat{S}$-MZVs ($\hat{S}$-MZSVs) and prove cyclic sum formulas for $\hat{S}$-MZ(S)Vs. The first and second authors [OS] calculate some special values and prove sum formulas and the Bowman–Bradley type theorem for $\hat{S}$-MZ(S)Vs modulo $t^n$ ($n = 2, 3$). Komori [Ko] defines and studies the unified multiple zeta functions which interpolate $\hat{S}$-MZVs. Bachmann, Takeyama and Tasaka [BTT] defines and studies the symmetric Mordell–Tornheim multiple zeta values in a way different from ours.

Acknowledgments. In summer of 2015, the second author learned the definition of the $\hat{S}$-MZV $\zeta_{\hat{S}}(k)$ from Hirose, who had been lead to the definition by Kontsevich’s idea underlying the definition of the $\mathcal{S}$-MZV $\zeta_{\mathcal{S}}(k)$ and a conjecture of Akagi–Hirose–Yasuda concerning $p$-adic multiple zeta values which was proved by Jarossay [J3, Theorem 1.2].
The authors would like to thank Dr. Minoru Hirose since this episode was the starting point of this work. They also would like to thank Dr. Yuta Suzuki for helpful comments and useful discussion on Proposition 4.4. They would like to express their sincere gratitude to Prof. Koji Tasaka and Dr. David Jarossay for informing us about Jarossay’s work for the Λ-adjoint multiple zeta values.

2. Series expressions of $\mathcal{S}$-MZVs

2.1. Preliminaries. Let $\mathcal{H} := \mathbb{Q}(x, y)$ be the non-commutative polynomial ring over $\mathbb{Q}$ with two variables $x$ and $y$. We define two $\mathbb{Q}$-subalgebras $\mathcal{H}^0$ and $\mathcal{H}^1$ of $\mathcal{H}$ by $\mathcal{H}^1 := \mathbb{Q} + y\mathcal{H} \supset \mathcal{H}^0 := \mathbb{Q} + y\mathcal{H}x$. Putting $z_k := yx^{k-1}$ for any positive integer $k$, we see that $\mathcal{H}^1$ has a basis $\{z_k \mid k: \text{index}\}$ consisting of monomials $z_k := z_{k_1} \cdots z_{k_r}$, for all indices $k = (k_1, \ldots, k_r)$, including $z_\emptyset := 1$. Note also that $\mathcal{H}^0$ is spanned by the subset $\{z_k \mid k: \text{admissible index}\}$.

The harmonic product $*$ on $\mathcal{H}^1$ and the shuffle product $\mathfrak{m}$ on $\mathcal{H}$ are defined as follows. First, we define the $\mathbb{Q}$-bilinear map $*: \mathcal{H}^1 \times \mathcal{H}^1 \rightarrow \mathcal{H}^1$ by the following rules:

(i) $w \ast 1 = 1 \ast w = w$ for all $w \in \mathcal{H}^1$,
(ii) $(w_1z_{k_1}) \ast (w_2z_{k_2}) = (w_1 \ast w_2z_{k_1})z_{k_2} + (w_1z_{k_1} \ast w_2)z_{k_2} + (w_1 \ast w_2)z_{k_1 + k_2}$ for all $w_1, w_2 \in \mathcal{H}^1$ and positive integers $k_1, k_2$.

We similarly define the $\mathbb{Q}$-bilinear map $\mathfrak{m}: \mathcal{H} \times \mathcal{H} \rightarrow \mathcal{H}$ by the following rules:

(i) $w \mathfrak{m} 1 = 1 \mathfrak{m} w = w$ for all $w \in \mathcal{H}$,
(ii) $(w_1u_1) \mathfrak{m} (w_2u_2) = (w_1 \mathfrak{m} w_2u_2)u_1 + (w_1 u_1 \mathfrak{m} w_2)u_2$ for all $w_1, w_2 \in \mathcal{H}$ and $u_1, u_2 \in \{x, y\}$.

It is known that $\mathcal{H}^1$ (resp. $\mathcal{H}$) becomes a commutative $\mathbb{Q}$-algebra with respect to the multiplication $*$ (resp. $\mathfrak{m}$), which is denoted by $\mathcal{H}^1_*$ (resp. $\mathcal{H}_*$). Then, the subspace $\mathcal{H}^0$ of $\mathcal{H}^1$ is closed under $*$ and becomes a $\mathbb{Q}$-subalgebra of $\mathcal{H}^1_*$. Similarly, the subspaces $\mathcal{H}^0$ and $\mathcal{H}^1$ of $\mathcal{H}$ are closed under $\mathfrak{m}$ and become $\mathbb{Q}$-subalgebras of $\mathcal{H}_*$. $\mathcal{H}^0_*, \mathcal{H}^0_\mathfrak{m}$ and $\mathcal{H}^1_\mathfrak{m}$ denote these subalgebras, respectively.

We define a $\mathbb{Q}$-linear map $Z: \mathcal{H}^0 \rightarrow \mathbb{R}$ by $Z(z_k) := \zeta(k)$. Similarly, for a positive integer $M$, we define a $\mathbb{Q}$-linear map $Z_M: \mathcal{H}^1 \rightarrow \mathbb{Q}$ by $Z_M(z_k) := \zeta_M(k)$. Then we have

$$Z(w_1 \ast w_2) = Z(w_1)Z(w_2) = Z(w_1 \mathfrak{m} w_2) \quad \text{for any } w_1, w_2 \in \mathcal{H}^0,$$
$$Z_M(w_1 \ast w_2) = Z_M(w_1)Z_M(w_2) \quad \text{for any } w_1, w_2 \in \mathcal{H}^1.$$

Let $T$ be an indeterminate. Since $\mathcal{H}^1_* \cong \mathcal{H}^0_\mathfrak{m}[y]$ and $\mathcal{H}^1_\mathfrak{m} \cong \mathcal{H}^0_\mathfrak{m}[y]$ (see [Ho] Theorem 3.1 and [Re] Theorem 6.1), we can uniquely extend the map $Z$ to $\mathbb{Q}$-algebra homomorphisms $Z^* : \mathcal{H}^1_* \rightarrow \mathcal{Z}[T]$ and $Z^\mathfrak{m} : \mathcal{H}^1_\mathfrak{m} \rightarrow \mathcal{Z}[T]$ satisfying $Z^*(y) = Z^\mathfrak{m}(y) = T$. For an index $k$
and \( \bullet \in \{*, \mathbb{m}\} \), we set \( \zeta^\bullet(k; T) := Z^\bullet(z_k) \) and \( \zeta^\bullet(k) := \zeta^\bullet(k; 0) \). We call \( \zeta^\bullet(k) \) (resp. \( \zeta^\mathbb{m}(k) \)) the harmonic (resp. shuffle) regularized MZV. See \( \text{[IKZ]} \) for details of the theory of regularization of multiple zeta values.

Now the expression
\[
\zeta^\bullet_S(k) = \sum_{i=0}^r (-1)^{\text{wt}(k[i])} \zeta^\bullet(k[i]) \sum_{l \in \mathbb{Z}_{\geq 0}^{r-i}} b\left(\frac{k}{l}\right) \zeta^\bullet(k[i] + l) t^{\text{wt}(l)}
\]
in Definition 1.1 makes sense. To justify Definition 1.2 we need the following congruence.

**Proposition 2.1.** For any index \( k \), we have
\[
\zeta^\bullet_S(k) \equiv \zeta^\mathbb{m}_S(k) \mod \pi^2 \mathbb{Z}[t].
\]

**Proof.** First we note that, for \( \bullet \in \{*, \mathbb{m}\} \), the definition of \( \zeta^\bullet_S \) can be rewritten as
\[
\zeta^\bullet_S(k) = \sum_{l=\ldots} \sum_{s-t} \sum_{0 \leq i \leq r} \sum_{t=0} \sum_{l_0=0} (-1)^{\text{wt}(k[i])} \zeta^\bullet(k[i]) t^{\text{wt}(l)}.
\]
From this expression, we see that the coefficient of each \( t^n \) in the difference \( \zeta^\bullet_S(k) - \zeta^\mathbb{m}_S(k) \) is a \( \mathbb{Z} \)-linear combination of terms of the form
\[
(2.1) \quad \sum_{j=0}^s (-1)^{s-j} \left\{ \zeta^\bullet(h, 1, \ldots, 1) \zeta^\bullet(h', 1, \ldots, 1) - \zeta^\mathbb{m}(h, 1, \ldots, 1) \zeta^\mathbb{m}(h', 1, \ldots, 1) \right\}
\]
for some admissible indices \( h, h' \) and integer \( s \geq 0 \). To prove that such terms are contained in \( \pi^2 \mathbb{Z} \), we consider the generating functions
\[
f^\bullet(h, T, x) := \sum_{s=0}^\infty \zeta^\bullet(h, 1, \ldots, 1; T) x^s
\]
for any admissible index \( h \) and \( \bullet \in \{*, \mathbb{m}\} \). Then it suffices to show that
\[
f^\bullet(h, 0, x) f^\bullet(h', 0, -x) - f^\mathbb{m}(h, 0, x) f^\mathbb{m}(h', 0, -x) \in \pi^2 \mathbb{Z}[x],
\]
since the expression (2.1) is the coefficient of \( x^s \) in this power series.

By \( \text{[IKZ]} \) Theorem 1 and \( \text{[IKZ]} \) Proposition 10, we see that
\[
f^\bullet(h, T, x) = \rho^{-1}(f^\mathbb{m}(h, T, x)) = \rho^{-1}(e^{Tx} f^\mathbb{m}(h, 0, x)) = e^{-\gamma x} \Gamma(1 + x)^{-1} e^{Tx} f^\mathbb{m}(h, 0, x),
\]
where \( \rho \) is an \( \mathbb{R} \)-linear endomorphism of \( \mathbb{R}[T] \) determined by \( \rho(e^{T x}) = e^{\gamma x} \Gamma(1 + x) e^{T x} \) (see \( \text{[IKZ]} \) (2.2))). Thus we obtain
\[
f^\bullet(h, 0, x) f^\bullet(h', 0, -x) = \Gamma(1 + x)^{-1} \Gamma(1 - x)^{-1} f^\mathbb{m}(h, 0, x) f^\mathbb{m}(h', 0, -x),
\]

8
and the proof is complete since we have
\[ \Gamma(1 + x)^{-1} \Gamma(1 - x)^{-1} = \frac{\sin \pi x}{\pi x} \equiv 1 \mod \pi^2 \mathbb{Z}[x], \]
as is well-known. \qed

2.2. \( {}^* \) and \( x \)-truncated \( \hat{S} \)-MZVs and their limits. In this subsection, we prove Theorem 1.5 (= Theorem 2.5), which is our first main result. Recall the definitions of \( {}^* \)- and \( x \)-truncated \( \hat{S} \)-MZVs (Definition 1.4):

\[
\zeta_{\hat{S},M}^{{}^*}(k) = \sum_{i=0}^{r} \sum_{\substack{0 < n_1 < \cdots < n_i < M \\ -M < n_{i+1} < \cdots < n_r < 0}} \frac{1}{n_1^{k_1} \cdots n_i^{k_i} (n_{i+1} + t)^{k_{i+1}} \cdots (n_r + t)^{k_r}} \in \mathbb{Q}[t],
\]

\[
\zeta_{\hat{S},M}^{x}(k) = \sum_{i=0}^{r} \sum_{\substack{0 < n_1 < \cdots < n_i \\ n_{i+1} < \cdots < n_r < 0 \\ n_{i+1} - n_i < M}} \frac{1}{n_1^{k_1} \cdots n_i^{k_i} (n_{i+1} + t)^{k_{i+1}} \cdots (n_r + t)^{k_r}} \in \mathbb{Q}[t].
\]

Here and in what follows, the letter \( M \) denotes an arbitrary positive integer unless otherwise noted.

Definition 2.2. For an index \( k = (k_1, \ldots, k_r) \), a non-negative integer \( n \) and \( \bullet \in \{ {}^*, x \} \), we define an element \( w_{\hat{S},n}^\bullet(k) \) of \( \hat{S}^1 \) by

\[
w_{\hat{S},n}^\bullet(k) := \sum_{i=0}^{r} (-1)^{\text{wt}(k[i])} z_{k[i]} \cdot \sum_{l \in \mathbb{Z}^{r-1}_{\geq 0}, \text{wt}(l) = n} b\left(\frac{k[i]}{l}\right) z_{k[i] + l}.
\]

Proposition 2.3. The above \( w_{\hat{S},n}^\bullet(k) \) is always an element of \( \hat{S}^0 \).

Proof. First we consider the case of \( \bullet = {}^* \). For \( 0 < i < r \) and \( l = (l_{i+1}, \ldots, l_r) \in \mathbb{Z}^{r-i}_{\geq 0} \), we have

\[
z_{k[i]} * z_{k[i] + l} = \left(z_{k[i-1]} * z_{k[i] + l}\right) z_{k[i]} + \left(z_{k[i]} * z_{k[i+1] + l}\right) z_{k[i+1] + l},
\]

the second term of which belongs to \( \hat{S}^0 \) whenever \( l_{i+1} > 0 \). This implies a congruence

\[
z_{k[i]} \sum_{l \in \mathbb{Z}^{r-i}_{\geq 0}, \text{wt}(l) = n} b\left(\frac{k[i]}{l}\right) z_{k[i] + l} \equiv E_i + E_{i+1} \mod \hat{S}^0,
\]

where

\[
E_i := \sum_{l \in \mathbb{Z}^{r-i}_{\geq 0}, \text{wt}(l) = n} b\left(\frac{k[i]}{l}\right) \left(z_{k[i-1]} * z_{k[i] + l}\right) z_{k[i]}.
\]
This also holds for $i = 0$ and $i = r$ if we put $E_0 = E_{r+1} := 0$. Hence we have

$$w^*_S(n, k) = \sum_{i=0}^r (-1)^{\text{wt}(k_i)}(E_i + E_{i+1}) = \sum_{i=1}^r (-1)^{\text{wt}(k_i)}(1 + (-1)^{k_i})E_i.$$

Note that $E_i \in \mathfrak{S}^0$ if $k_i \geq 2$, while $1 + (-1)^{k_i} = 0$ if $k_i = 1$. This completes the proof for $\bullet = \ast$.

Next we treat the case of $\bullet = \mathfrak{m}$. First note that the sum

$$\sum_{l \in \mathbb{Z}_{\geq 0}^r, \text{wt}(l) = n} b\left(\frac{k}{l}\right) z^{k+l}$$

is obtained by expanding $y(x^{k_r-1}z_{k_r-1} \cdots z_{k_{i+1}} z^n)$ to a sum of monomials. Thus we can write $w^*_{S,n}(k)$ as

$$(2.2) \quad w^*_{S,n}(k) = \sum_{i=0}^r (-1)^{\text{wt}(k_i)}z_{k_i} \mathfrak{m} y(x^{k_r-1}z_{k_r-1} \cdots z_{k_{i+1}} z^n) = \sum_{i=0}^r (-1)^{\text{wt}(k_i)}(u_i + v_i).$$

Here we put

$$u_0 := y(x^{k_r-1}z_{k_r-1} \cdots z_k z^n), \quad v_0 := 0, \quad u_r := 0, \quad v_r := \begin{cases} z_k & (n = 0), \\ 0 & (n > 0), \end{cases}$$

and define $u_i$ (resp. $v_i$) ($0 < i < r$) to be the partial sum of the expansion of $z_{k_i} \mathfrak{m}$ $y(x^{k_r-1}z_{k_r-1} \cdots z_{k_{i+1}} z^n)$ consisting of monomials in which the rightmost $y$ in $z_{k_i}$ lies to the left (resp. the right) of the rightmost $y$ in $y(x^{k_r-1}z_{k_r-1} \cdots z_{k_{i+1}} z^n)$. Then we can check that

- $u_{i-1}, v_i \in \mathfrak{S}^0$ if $k_i \geq 2$,
- $u_{i-1} \equiv v_i \pmod{\mathfrak{S}^0}$ if $k_i = 1$

for $i = 1, \ldots, r$. Thus we obtain

$$w^*_{S,n}(k) = \sum_{i=1}^r (-1)^{\text{wt}(k_i)}((-1)^{k_i}u_{i-1} + v_i) \equiv 0 \pmod{\mathfrak{S}^0}.$$ 

We remark that Komori [Ko] also proved Proposition 2.3 for $\bullet = \ast$, independently.

**Lemma 2.4.** For any indices $k = (k_1, \ldots, k_r)$ and $l = (l_1, \ldots, l_s)$, we have

$$\sum_{0=m_0<n_1<\cdots<n_r, 0=n_0<n_1<\cdots<n_s} \frac{1}{m_1^{k_1} \cdots m_r^{k_r} n_1^{l_1} \cdots n_s^{l_s}} = Z_M(z_k \mathfrak{m} z_l).$$
Proof. This can be shown by applying the partial fraction decomposition

\[
\frac{1}{ab} = \frac{1}{a(a + b)} + \frac{1}{b(a + b)}
\]

repeatedly; see \cite{KMT} or \cite{O} for a similar argument. Alternatively, we may use an identity in \cite{Yam}, Lemma 2.1] as follows. Set \( k := k_1 + \cdots + k_r \) and \( l := l_1 + \cdots + l_s \), and

\[
\Sigma_{k,l} := \{ \sigma \in \mathfrak{S}_{k+l} \mid \sigma(1) < \cdots < \sigma(k), \sigma(k + 1) < \cdots < \sigma(k + l) \},
\]

where \( \mathfrak{S}_{k+l} \) denotes the group of permutations on the set \( \{1, \ldots, k + l\} \). If we write

\[
z_k = u_1 \cdots u_k \quad \text{and} \quad z_l = u_{k+1} \cdots u_{k+l}
\]

with \( u_i \in \{x, y\} \), then we have

\[
z_k \mathcal{M} z_l = \sum_{\sigma \in \Sigma_{k,l}} u_{\sigma^{-1}(1)} \cdots u_{\sigma^{-1}(k+l)}.
\]

Moreover, we consider a set \( P^M_{k,l} \) of tuples \((p_1, \ldots, p_{k+l})\) of non-negative integers such that \( p_1 + \cdots + p_{k+l} < M \) and \( p_1 > 0 \) if and only if \( u_i = y \). Then we see that

\[
\sum_{0=m_0<m_1<\ldots<m_r \atop 0=n_0<n_1<\ldots<n_s \atop m_r+n_s<M} \frac{1}{m_1^{k_1} \cdots m_r^{k_r} n_1^{l_1} \cdots n_s^{l_s}}
\]

\[
= \sum_{(p_1, \ldots, p_{k+l}) \in P^M_{k,l}} \frac{1}{p_1(p_1 + p_2) \cdots (p_1 + \cdots + p_k) \cdot p_{k+1}(p_{k+1} + p_{k+2}) \cdots (p_{k+1} + \cdots + p_{k+l})}
\]

and

\[
Z_M(u_{\sigma^{-1}(1)} \cdots u_{\sigma^{-1}(k+l)})
\]

\[
= \sum_{(p_1, \ldots, p_{k+l}) \in P^M_{k,l}} 1
\]

\[
= \sum_{\sigma \in \Sigma_{k,l}} p_{\sigma^{-1}(1)}(p_{\sigma^{-1}(1)} + p_{\sigma^{-1}(2)}) \cdots (p_{\sigma^{-1}(1)} + \cdots + p_{\sigma^{-1}(k+l)}).
\]

Thus our claim follows from the identity

\[
\frac{1}{p_1(p_1 + p_2) \cdots (p_1 + \cdots + p_k) \cdot p_{k+1}(p_{k+1} + p_{k+2}) \cdots (p_{k+1} + \cdots + p_{k+l})}
\]

\[
= \sum_{\sigma \in \Sigma_{k,l}} \frac{1}{p_{\sigma^{-1}(1)}(p_{\sigma^{-1}(1)} + p_{\sigma^{-1}(2)}) \cdots (p_{\sigma^{-1}(1)} + \cdots + p_{\sigma^{-1}(k+l)})},
\]

which is a special case of \cite{Yam}, Lemma 2.1].

\[\square\]

**Theorem 2.5 (\(= \text{Theorem 1.5}\). For any index \( k \) and \( \bullet \in \{\ast, \mathcal{M}\} \), we have

\[
\zeta_{\mathcal{S}}^\bullet(k) = \lim_{M \to \infty} \zeta_{\mathcal{S},M}^\bullet(k),
\]

where the limit is taken coefficientwise as the power series in \( t \).
Proof. First we prove

\[(2.3) \quad \zeta_{S,M}(k) = \sum_{n=0}^{\infty} Z_M(w_{S,n}(k)) t^n.\]

For \(\bullet = \ast\), by using

\[\frac{1}{(t-n)^k} = (-1)^k \sum_{l=0}^{\infty} \binom{k+l-1}{l} \frac{t^l}{n^{k+l}},\]

we can expand \(\zeta_{S,M}(k)\) as

\[\zeta_{S,M}(k) = \sum_{i=0}^{r} \sum_{0<n_1<\cdots<n_i<M} \frac{1}{n_{i}^{k_i} \cdots n_{i+r-1}^{k_{i+r-1}} (t-n_r)^{k_r} \cdots (t-n_{i+1})^{k_{i+1}}}.\]

Then the formula \((2.3)\) for \(\bullet = \ast\) follows from

\[\sum_{0<n_1<\cdots<n_i<M} \frac{1}{n_{i}^{k_i} \cdots n_{i+r-1}^{k_{i+r-1}} (t-n_r)^{k_r} \cdots (t-n_{i+1})^{k_{i+1}}} = Z_M(z_{k[i]}) Z_M(z_{k[i] + l}) = Z_M(z_{k[i]} * z_{k[i] + l})\]

for each \(0 \leq i \leq r\) and \(l = (l_{i+1}, \ldots, l_r) \in \mathbb{Z}_{\geq 0}^{r-i}\). Next we consider \(\bullet = \mathfrak{m}\). In this case, we have

\[\zeta_{S,M}(k) = \sum_{i=0}^{r} (-1)^{\text{wt}(k[i])} \sum_{l=(l_{i+1}, \ldots, l_r) \in \mathbb{Z}_{\geq 0}^{r-i}} b(k[i]) \sum_{0<n_1<\cdots<n_i<M} \frac{1}{n_{i}^{k_i} \cdots n_{i+r-1}^{k_{i+r-1}} (t-n_r)^{k_r} \cdots (t-n_{i+1})^{k_{i+1}}}.\]

For \(0 \leq i \leq r\), we have

\[\sum_{0<n_1<\cdots<n_i<M} \frac{1}{n_{i}^{k_i} \cdots n_{i+r-1}^{k_{i+r-1}} (t-n_r)^{k_r} \cdots (t-n_{i+1})^{k_{i+1}}} = Z_M(z_{k[i]} \mathfrak{m} z_{k[i] + l})\]

by Lemma 2.4. This shows the formula \((2.3)\) for \(\bullet = \mathfrak{m}\).

For any admissible index \(h\), we have \(\lim_{M \to \infty} Z_M(z_h) = Z(z_h) = Z^\bullet(z_h)\). Thus we also have \(\lim_{M \to \infty} Z_M(w_{S,n}^\bullet(k)) = Z^\bullet(w_{S,n}^\bullet(k))\), since \(w_{S,n}^\bullet(k) \in S^0\) by Proposition 2.3. Therefore, by \((2.3)\), we obtain

\[(2.4) \quad \lim_{M \to \infty} \zeta_{S,M}(k) = \sum_{n=0}^{\infty} Z^\bullet(w_{S,n}^\bullet(k)) t^n.\]
Finally, by using the fact that $Z^*$ is a $\mathbb{Q}$-algebra homomorphism with respect to the product $\cdot$, we calculate the right hand side of (2.4) as
\[
\sum_{n=0}^{\infty} Z^* \left( \left( \sum_{i=0}^{r} (-1)^{\text{wt}(k_{[i]})} z_{k_{[i]}} \right) \bullet \left( \sum_{l \in \mathbb{Z}_{\geq 0}^{r-i} \atop \text{wt}(l) = n} b \left( \frac{k_{[i]}}{l} \right) \right) \right) t^n
= \sum_{n=0}^{\infty} \sum_{i=0}^{r} (-1)^{\text{wt}(k_{[i]})} Z^* \left( z_{k_{[i]}} \right) \sum_{l \in \mathbb{Z}_{\geq 0}^{r-i} \atop \text{wt}(l) = n} b \left( \frac{k_{[i]}}{l} \right) Z^* \left( \frac{z_{k_{[i]}}}{l} \right) t^n
= \sum_{n=0}^{\infty} \sum_{i=0}^{r} (-1)^{\text{wt}(k_{[i]})} \zeta^* \left( k_{[i]} \right) \sum_{l \in \mathbb{Z}_{\geq 0}^{r-i} \atop \text{wt}(l) = n} b \left( \frac{k_{[i]}}{l} \right) \zeta^* \left( \frac{k_{[i]}}{l} + l \right) t^n.
\]
This is exactly the definition of $\zeta^*_S(k)$ (Definition 1.1). Now the proof is complete. \(\square\)

By taking the constant term of $\zeta^*_S(k)$, we obtain a series expression of $\zeta_S^*(k)$ as follows.

**Corollary 2.6.** For an index $k$ and $\bullet \in \{\ast, \cdot\}$, we have

\[
\lim_{M \to \infty} \zeta^*_S, M(k) = \zeta^*_S(k),
\]
where
\[
\zeta^*_S, M(k) := \sum_{i=0}^{r} \sum_{0 < n_1 < \cdots < n_i < M \atop -M < n_{i+1} < \cdots < n_r < 0} \frac{1}{n_1 \cdots n_r} \in \mathbb{Q},
\]
\[
\zeta^*_S, M(k) := \sum_{i=0}^{r} \sum_{0 < n_1 < \cdots < n_i \atop n_{i+1} < \cdots < n_r < 0 \atop n_1 - n_{i+1} < M} \frac{1}{n_1 \cdots n_r} \in \mathbb{Q}.
\]

### 2.3. The harmonic relation for (\ast-truncated) $\hat{S}$-MZVs.

In this subsection, we prove the identity (1.4), i.e., the harmonic relation for $\ast$-truncated $\hat{S}$-MZVs. Then we obtain the same relation for $\hat{S}$-MZVs by taking the limit.

**Theorem 2.7.** For any indices $k, l$, we have

\[
\zeta^*_S, M(k \ast l) = \zeta^*_S, M(k) \zeta^*_S, M(l).
\]

**Proof.** For a non-zero integer $n$, set

\[
n(t) := \begin{cases} n & (n > 0), \\ n + t & (n < 0). \end{cases}
\]
Then, we see that
\[ \zeta^*_{\tilde{S}, M}(k) = \sum_{n_1 < \cdots < n_r \atop 0 < |n_1|, \ldots, |n_r| < M} \frac{1}{n_1(t)^{k_1} \cdots n_r(t)^{k_r}}, \]
where \( \prec \) is Kontsevich’s order on the set \( \mathbb{Z} \setminus \{0\} \cup \{\infty = -\infty\} \) defined as
\[ 1 \prec 2 \prec \cdots \prec \infty = -\infty \prec \cdots \prec -2 \prec -1. \]
This is a natural generalization of \([\text{Kan}, (9.1)]\). By this expression, we can prove the harmonic relation for \(*\)-truncated \(\tilde{S}\)-MZVs in exactly the same way as the harmonic relation for truncated MZVs. \(\square\)

**Corollary 2.8.** For any indices \( k \) and \( l \), we have
\[ \zeta^*_S(k \ast l) = \zeta^*_S(k) \zeta^*_S(l). \]

**Proof.** Take the limit \( M \to \infty \) in (2.5) and use Theorem 2.5. \(\square\)

### 2.4. The reversal relation.
In this subsection, we prove a relation of \(\tilde{S}\)-MZVs, which we call the *reversal relation*. In fact, this relation holds for \(*\)- and \(m\)-truncated ones in exactly the same form. We also note that the \(m\)-truncated version is a special case of the shuffle relation (1.5), the proof of which is postponed to the next section.

**Proposition 2.9.** For an index \( k \) and \( \bullet \in \{*, m\} \), we have
\[
\zeta^*_{\tilde{S}, M}(\overline{k}) = (-1)^{\text{wt}(k)} \sum_{l \in \mathbb{Z}^{\text{dep}(k)} \geq 0} b \left( \frac{k}{l} \right) \zeta^*_{\tilde{S}, M}(k + l) t^{\text{wt}(l)}. \tag{2.7}
\]

**Proof.** By the definition (2.6) of \( n(t) \), we can easily check that
\[ (n(t)) = t - n(t) \]
for any non-zero integer \( n \). For \( k = (k_1, \ldots, k_r) \), we may rewrite the definition of \( \zeta^*_{\tilde{S}, M}(\overline{k}) \) as
\[
\zeta^*_{\tilde{S}, M}(\overline{k}) = \sum_{i=0}^{r} \sum_{0 < n_r < \cdots < n_i < |M| \atop -M < n_{i+1} < \cdots < n_1 < 0} \frac{1}{n_1(t)^{k_1} \cdots n_r(t)^{k_r}}.
\]
Then by using
\[
\frac{1}{(t - n(t))^k} = (-1)^k \sum_{l=0}^{\infty} \binom{k + l - 1}{l} \frac{t^l}{n(t)^{k+l}},
\]
we obtain the desired formula for \( \bullet = * \). The formula for \( \bullet = m \) is proved in the same way. \( \square \)

From Theorem 2.5 and Proposition 2.9, we deduce the reversal relation for \( \zeta^*_S \) and \( \zeta^m_S \).

**Corollary 2.10.** For an index \( k \) and \( \bullet \in \{ *, m \} \), we have

\[
\zeta^*_S(k) = (-1)^{wt(k)} \sum_{l \in Z_{\geq 0}} b\left( \frac{k}{l} \right) \zeta^*_S(k + l)t^{wt(l)}.
\]

### 3. TRUNCATED \( \hat{S} \)-MZV ASSOCIATED WITH A 2-COLORED ROOTED TREE

#### 3.1. Definition and example

In this subsection, we define the truncated \( \hat{S} \)-MZVs associated with 2-colored rooted trees. When the tree is linear, we recover the \( m \)-truncated \( \hat{S} \)-MZV \( \zeta^m_{S,M}(k) \) defined in Definition 1.4.

First, we recall the definition of 2-colored rooted trees.

**Definition 3.1** (2-colored rooted tree [O, Definition 1.2]). A 2-colored rooted tree is a quadruple \( X = (V, E, rt, V_\bullet) \) consisting of the following data:

(i) \( (V, E) \) is a finite tree with the set of vertices \( V \) and the set of edges \( E \). Note that \( \#V = \#E + 1 < +\infty \).

(ii) \( rt \in V \) is a vertex, called the root.

(iii) \( V_\bullet \) is a subset of \( V \) containing all terminals of \( (V, E) \). Here, a terminal vertex means a vertex of degree 1.

We call a tuple \( (k_e)_{e \in E} \in \mathbb{Z}^E \geq 0 \) an index on \( X \). Recall that \( t \) denotes an indeterminate.

**Definition 3.2.** Let \( X = (V, E, rt, V_\bullet) \) be a 2-colored rooted tree, \( u \) an element of \( V_\bullet \) and \( k = (k_e)_{e \in E} \in \mathbb{Z}^E \geq 0 \) an index on \( X \). Then we define

\[
\zeta_M(X, u; k) := \sum_{(m_v)_{v \in V_\bullet} \in I_M(V_\bullet, u)} \prod_{e \in E} \left( \sum_{v \in V_\bullet \text{ s.t. } e \in P(rt, v)} (m_v + \delta_{u,v}t) \right)^{-k_e} \in \mathbb{Q}[t].
\]

Here \( P(rt, v) \) is the path from the root \( rt \) to \( v \), \( \delta_{u,v} \) is the Kronecker delta and

\[
I_M(V_\bullet, u) := \left\{ (m_v)_{v \in V_\bullet} \in \mathbb{Z}^{V_\bullet} \mid m_v > 0 \ (v \neq u), \ -M < m_u < 0, \ \sum_{v \in V_\bullet} m_v = 0 \right\}.
\]

Moreover, we define the truncated \( \hat{S} \)-MZV associated with \( X \) by

\[
\zeta_{S,M}(X; k) := \sum_{u \in V_\bullet} \zeta_M(X, u; k).
\]
In the following, we denote by \( L_e(X, u; (m_v)_{v \in V_e}) \) the factor appearing in the definition of \( \zeta_M(X, u; k) \), that is,

\[
L_e(X, u; (m_v)_{v \in V_e}) := \sum_{v \in V_e \text{ s.t. } e \in P(rt, v)} (m_v + \delta_{u,v} t).
\]

We use diagrams to indicate 2-colored rooted trees. For \( X = (V, E, rt, V_\bullet) \), the symbol \( \bullet \) (resp. \( \circ \)) denotes a vertex in \( V_\bullet \) (resp. \( V_\circ := V \setminus V_\bullet \)) which is not the root. We use the symbol \( \Box \) or \( \square \) to denote the root according to whether the root belongs to \( V_\bullet \) or not. We also use the symbol \( \times \) as the wild-card, i.e., to indicate a vertex which may or may not belong to \( V_\bullet \) and may or may not be the root. If endpoints of an edge \( e \) are \( v \) and \( v' \), then we will sometimes express \( e \) by the set \( \{v, v'\} \).

**Example 3.3.** For an integer \( r \geq 0 \), let us consider the linear tree with \( r + 1 \) vertices \( v_1, \ldots, v_{r+1} \) and \( r \) edges \( e_a := \{v_a, v_{a+1}\} \) \((a = 1, \ldots, r)\). We define the 2-colored rooted tree \( X = (V, E, rt, V_\bullet) \) by setting \( rt := v_{r+1} \) and \( V_\bullet := V = \{v_1, \ldots, v_{r+1}\} \). We identify an index \( k = (k_1, \ldots, k_r) \) in the usual sense with an index on \( X \) by setting \( k_a := k_{e_a} \). This situation is indicated by the diagram:

For \( (m_v)_{v \in V_\bullet} \in \mathbb{Z}^{V_\bullet} \), we put \( m_i := m_{v_i} \) \((i = 1, \ldots, r + 1)\) and

\[
(3.1) \quad M_{i,j} := m_i + \cdots + m_j, \quad M_j := M_{1,j}
\]

for \( 1 \leq i \leq j \leq r + 1 \). Then we have

\[
L_{e_a}(X, v_i; (m_v)_{v \in V_e}) = \begin{cases} 
M_a & (a < i), \\
M_a + t & (a \geq i)
\end{cases}
\]
for $1 \leq a \leq r$ and $1 \leq i \leq r + 1$. Thus we obtain

$$\zeta_M(X, v_i; k)$$

$$= \sum_{m_1, \ldots, m_{i-1}, m_{i+1}, \ldots, m_r > 0} \frac{1}{M_1^{k_1} \cdots M_{i-1}^{k_{i-1}} (M_i + t)^{k_i} \cdots (M_r + t)^{k_r}}$$

$$= \sum_{m_1, \ldots, m_{i-1}, m_{i+1}, \ldots, m_r > 0, M_1 + \cdots + M_r + 1 < M} \frac{1}{M_1^{k_1} \cdots M_{i-1}^{k_{i-1}} (t - M_{i+1} + 1)^{k_i} \cdots (t - M_{r+1} + 1)^{k_r}}$$

$$= \sum_{0 < n_1 < \cdots < n_i-1 < M} \frac{1}{n_1^{k_1} \cdots n_{i-1}^{k_{i-1}} (n_i + t)^{k_i} \cdots (n_r + t)^{k_r}},$$

and hence

$$\zeta_{\hat{S}, M}(X; k) = \sum_{i=1}^{r+1} \zeta_M(X, v_i; k) = \zeta_{\hat{S}, M}^m(k).$$

Therefore, the truncated $\hat{S}$-MZV associated with a 2-colored rooted tree generalizes the $m$-truncated $\hat{S}$-MZV.

3.2. Basic properties. In this subsection, we give basic properties for truncated $\hat{S}$-MZVs associated with 2-colored rooted trees. The proofs proceed in almost the same way as those in [O].

**Proposition 3.4.** Let $X = (V, E, rt, V_\bullet)$ be a 2-colored rooted tree and $k = (k_e)_{e' \in E}$ an index on $X$. Assume that there exists an edge $e = \{a, b\} \in E$ such that $b \in V_\circ \setminus \{rt\}$ and $k_e = 0$. Let $(V', E')$ be the tree obtained by contracting the edge $e$, as represented by the following figure:

![Diagram](image)

Identifying $V'$ with $V \setminus \{b\}$ and $E'$ with $E \setminus \{e\}$, we define a 2-colored rooted tree $X' := (V', E', rt, V_\bullet)$ and an index $k' := (k_{e'})_{e' \in E' \setminus \{e\}}$ on $X'$. Then we have

$$\zeta_{\hat{S}, M}(X; k) = \zeta_{\hat{S}, M}(X'; k').$$
Proof. It follows from the equality
\[
\zeta_M(X, u; k)
= \sum_{(m_v)\in V_\bullet \in I_M(V_\bullet, u)} L_e(X, u; (m_v)_{v\in V_\bullet})^{-k_e} \prod_{e'\in E\setminus\{e\}} L_{e'}(X, u; (m_v)_{v\in V_\bullet})^{-k_{e'}}
\]
\[
= \sum_{(m_v)\in V_\bullet \in I_M(V_\bullet, u)} \prod_{e\in E} L_{e'}(X', u; (m_v)_{v\in V_\bullet})^{-k_{e'}}
= \zeta_M(X', u; k'),
\]
which holds for any \( u \in V_\bullet \).

\[\square\]

**Proposition 3.5.** Let \( X = (V, E, rt, V_\bullet) \) be a 2-colored rooted tree and \( k = (k_e)_{e\in E} \) an index on \( X \). Assume that there is a vertex \( b \in V_\circ \setminus \{rt\} \) of degree 2, and let \( e_1 = \{a, b\} \) and \( e_2 = \{b, c\} \) be the edges incident on \( b \). By setting \( V' := V \setminus \{b\}, e_{12} := \{a, c\} \) and \( E' := (E \setminus \{e_1, e_2\}) \cup \{e_{12}\} \), we define a 2-colored rooted tree \( X' := (V', E', rt, V_\bullet) \). Moreover, we define an index \( k' = (k'_e)_{e\in E'} \) on \( X' \) by putting \( k'_{e_{12}} := k_{e_1} + k_{e_2} \) and \( k'_e := k_e \) for other edges \( e \in E' \setminus \{e_{12}\} \). This situation can be represented by the following figure:

Then we have
\[
\zeta_{\hat{S}, M}(X; k) = \zeta_{\hat{S}, M}(X'; k').
\]

Proof. Since \( b \) is a vertex in \( V_\circ \setminus \{rt\} \), we have
\[
\{v \in V_\bullet \mid e_1 \in P(rt, v)\} \cap \{v \in V_\bullet \mid e_2 \in P(rt, v)\} = \{v \in V_\bullet \mid e_1 \in P(rt, v)\}.
\]
Therefore, for \( u \in V_\bullet \), we obtain
\[
\zeta_M(X, u; k)
= \sum_{(m_v)\in V_\bullet \in I_M(V_\bullet, u)} \prod_{e\in E} L_e(X, u; (m_v)_{v\in V_\bullet})^{-k_e}
\]
\[
= \sum_{(m_v)\in V_\bullet \in I_M(V_\bullet, u)} L_{e_1}(X, u; (m_v)_{v\in V_\bullet})^{-k_{e_1}+k_{e_2}} \prod_{e\in E\setminus\{e_1, e_2\}} L_e(X, u; (m_v)_{v\in V_\bullet})^{-k_e}
\]
\[
= \sum_{(m_v)\in V_\bullet \in I_M(V_\bullet, u)} L_{e_{12}}(X', u; (m_v)_{v\in V_\bullet})^{-k'_{e_{12}}} \prod_{e\in E\setminus\{e_{12}\}} L_e(X', u; (m_v)_{v\in V_\bullet})^{-k_e}
= \zeta_M(X', u; k').
\]
Thus we complete the proof by taking the sum over all \( u \in V_\bullet \).
\[\square\]
Proposition 3.6. Let $X_1$ and $X_2$ be 2-colored rooted trees which are distinct only in their roots, written as $X_i = (V, E, v_i, V_\bullet)$ ($i = 1, 2$). Then, for any index $k = (k_e)_{e \in E}$ on $X_1$, we have

\[(3.2) \quad \zeta_{S,M}(X_1; k) = (-1)^{\sum_{e \in P(v_1, v_2)} k_e} \sum_{l = (l_e) \in \mathbb{Z}^E_{\geq 0}} \left[ \prod_{e \in P(v_1, v_2)} \binom{k_e + l_e - 1}{l_e} \right] \times \zeta_{S,M}(X_2; k \oplus l) t^{\sum_{e \in P(v_1, v_2)} l_e}.\]

Here $k \oplus l$ denotes an index on $X_2$ whose $e$-component is given by

\[
\begin{cases} 
  k_e + l_e & (e \in P(v_1, v_2)), \\
  k_e & (e \not\in P(v_1, v_2)),
\end{cases}
\]

and we use the convention

\[
\binom{l - 1}{l} = \begin{cases} 
  1 & (l = 0), \\
  0 & (l > 0).
\end{cases}
\]

Proof. Take $u \in V_\bullet$ and $e \in E$. First we consider the case $e \in P(v_1, v_2)$. This situation can be illustrated as follows:

\[
\begin{array}{c}
\times v_1 \\
\otimes \\
\times v_2
\end{array}
\]

Since

\[(3.3) \quad V_\bullet = \{ v \in V_\bullet \mid e \in P(v_1, v) \} \sqcup \{ v \in V_\bullet \mid e \in P(v_2, v) \},\]

we have

\[
L_e(X_1, u; (m_v)_{v \in V_\bullet}) + L_e(X_2, u; (m_v)_{v \in V_\bullet}) = \sum_{v \in V_\bullet \text{ s.t. } e \in P(v_1, v)} (m_v + \delta_{u,v}t) + \sum_{v \in V_\bullet \text{ s.t. } e \in P(v_2, v)} (m_v + \delta_{u,v}t) = \sum_{v \in V_\bullet} m_v + \sum_{v \in V_\bullet} \delta_{u,v}t = t.
\]

Next we consider the case $e \not\in P(v_1, v_2)$, which can be illustrated as follows:

\[
\begin{array}{c}
\times v_1 \\
\otimes \\
\times v_2
\end{array}
\]

Since

\[
\{ v \in V_\bullet \mid e \in P(v_1, v) \} = \{ v \in V_\bullet \mid e \in P(v_2, v) \},
\]

we have

\[
L_e(X_1, u; (m_v)_{v \in V_\bullet}) = \sum_{v \in V_\bullet \text{ s.t. } e \in P(v_1, v)} (m_v + \delta_{u,v}t).
\]
we obtain

\[
L_e(X_1, u; (m_v)_{v \in V^*}) = \sum_{v \in V^* \text{ s.t. } e \in P(v_1, v)} (m_v + \delta_{u,v}t) = \sum_{v \in V^* \text{ s.t. } e \in P(v_2, v)} (m_v + \delta_{u,v}t) = L_e(X_2, u; (m_v)_{v \in V^*}).
\]

Thus we have

\[
\zeta_M(X_1, u; k) = \sum_{(m_v)_{v \in V^*}} \prod_{e \in P(v_1, v_2)} \left( t - L_e(X_2, u; (m_v)_{v \in V^*}) \right)^{-k_e} \times \prod_{e \in E \setminus P(v_1, v_2)} L_e(X_2, u; (m_v)_{v \in V^*})^{-k_e}.
\]

Therefore, we obtain the desired formula by expanding the factors \((t - L_e)^{-k_e}\) as

\[
\frac{1}{(t - L_e)^{k_e}} = (-1)^{k_e} \sum_{l_e = 0}^{\infty} \binom{k_e + l_e - 1}{l_e} \frac{t^{l_e}}{L_e^{k_e + l_e}}
\]

and taking the sum over \(u \in V^*\). \qed

**Lemma 3.7.** Consider a 2-colored rooted tree \(X = (V, E, rt, V^*)\) and an index \(k = (k_e)_{e \in E}\) on \(X\) of the following shape:

![Tree Diagram](https://via.placeholder.com/150)

Here, \(s\) and \(l_i := k_{e_i}\) \((1 \leq i \leq s)\) are positive integers and \(k' := k_{e'}\) is a non-negative integer, where \(\{e_i\}\) and \(e'\) are the corresponding edges in \(E\). \(T_0, \ldots, T_s\) are subtrees of \((V, E)\). Moreover, for \(1 \leq i \leq s\), let \(h_i\) be the index on \(X\) whose \(e\)-component is

\[
\begin{cases} 
  l_i - 1 & (e = e_i), \\
  k' + 1 & (e = e'), \\
  k_e & \text{(otherwise)}.
\end{cases}
\]

Then, we have

\[
(3.4) \quad \zeta_{\tilde{G}, M}(X; k) = \sum_{i=1}^{s} \zeta_{\tilde{G}, M}(X; h_i).
\]
Proof. For a tree $T$, we denote by $V(T)$ (resp. $E(T)$) the set of vertices (resp. edges) of $T$. By definition, we have

$$
\zeta_{\hat{S}, M}(X; k) = \sum_{u \in V_{\star}} \sum_{(m_v)_{v \in V_\bullet}} \prod_{i=1}^{s} \frac{1}{L_{e_i}(X, u; (m_v)_{v \in V_\bullet})}^{k_{e_i}} \times \frac{1}{L_{e'}(X, u; (m_v)_{v \in V_\bullet})}^{\sum_{i=0}^{s} \prod_{v \in E(T_i)} L_{e}(X, u; (m_v)_{v \in V_\bullet})^{k_e}}.
$$

If we abbreviate as $L_{e'} = L_{e'}(X, u; (m_v)_{v \in V_\bullet})$ and $L_i = L_{e_i}(X, u; (m_v)_{v \in V_\bullet})$ ($i = 1, \ldots, s$), we have $L_{e'} = L_1 + \cdots + L_s$ by definition, and hence

$$
\frac{1}{L_1 \cdots L_s} = \frac{1}{L_{e'}} \sum_{i=1}^{s} \frac{L_i}{L_1 \cdots L_s}.
$$

This implies (3.3). \hfill \Box

We define a $\mathbb{Q}$-linear map $Z_{\hat{S}, M}^w : \hat{S}^1 \to \mathbb{Q}[t]$ by $Z_{\hat{S}, M}^w(\hat{z}_k) := \zeta_{\hat{S}, M}^w(k)$. We prove that a truncated $\hat{S}$-MZV associated with a 2-colored rooted tree of a certain shape can be written explicitly as a value of $Z_{\hat{S}, M}^w$.

**Theorem 3.8.** Consider a 2-colored rooted tree $X = (V, E, rt, V_\bullet)$ and an index $k = (k_e)_{e \in E}$. We allow $r = 0$.
and \( k' = \emptyset \). Then, we have

\[
\zeta_{S,M}(X; k) = Z_{S,M}^w \left( (z_{k_1} \cdots z_{k_s}) x^{k'} z_{k'} \right).
\]

(3.5)

**Proof.** We prove this theorem by induction on \( \ell := \sum_{i=1}^s \sum_{j=1}^{r_i} k_{i,j} \). When \( \ell = 1 \), the statement reduces to Example 3.3 via Proposition 3.5.

Assume \( \ell > 1 \) and that the theorem holds for the case of \( \ell - 1 \). From Lemma 3.7, we have

\[
\zeta_{S,M}(X; k) = \sum_{i=1}^s \zeta_{S,M}(X; h_i),
\]

(3.6)

where \( h_i \) is the index on \( X \) whose \( e \)-component is

\[
\begin{aligned}
&k_{i,r_i} - 1 & (e = e_{i,r_i}), \\
k' + 1 & (e = e'), \\
k_e & (\text{otherwise}).
\end{aligned}
\]

We calculate \( \zeta_{S,M}(X; h_i) \) for each \( i \). If \( k_{i,r_i} \geq 2 \), we have

\[
\zeta_{S,M}(X; h_i) = Z_{S,M}^w \left( \prod_{a=1}^{s} z_{k_a} \prod z_{k_{i,1}} \cdots z_{k_{i,r_i-1}} (z_{k_{i,r_i}} - 1) x^k z_{k'} \right)
\]

by the induction hypothesis. Next we consider the case \( k_{i,r_i} = 1 \). In this case, we construct a 2-colored rooted tree \( X' \) and an index \( h'_i \) as indicated in the following diagram:

![Diagram](https://example.com/diagram.jpg)

Here the edge \( e_{i,r_i} \) of \( X \) is contracted, and a new vertex \( v_0 \) is inserted between \( w \) and \( v_1 \). The components of the index \( h'_i \) on the edges \( \{w, v_0\} \) and \( \{v_0, v_1\} \) are set to be zero and
According to Proposition [3.4] and the induction hypothesis, we have
\[ \zeta_{\tilde{S}, M}(X; h_i) = \zeta_{\tilde{S}, M}(X'; h_i') \]
which appears in Theorem 3.8 may be regarded as an analogue of MZVs studied by Umezawa [U].

**Remark 3.9.** The value \( \zeta_{\tilde{S}, M} \) which appears in Theorem 3.8 may be regarded as an analogue of MZVs studied by Umezawa [U].

### 3.3. The shuffle relation for \( (m\text{-truncated}) \tilde{S}\text{-MZVs} \)

In this subsection, we give a proof of the identity (1.5), i.e., the shuffle relation for \( (m\text{-truncated}) \tilde{S}\text{-MZVs} \) as an application of the basic properties in the previous subsection.

By specializing at \( t = 0 \), this also gives a new proof of the shuffle relation for \( S\text{-MZVs} \), which was proved by Kaneko–Zagier [KZ], Jarossay [J1, Théorème 1.7 i)] and Hirose [Hi, Proposition 15]. In fact, this is completely parallel to the proof of the shuffle relation for \( A\text{-MZVs} \) given by the first author [O].

**Theorem 3.10.** For indices \( k, l \), we have
\[ \zeta_{\tilde{S}, M}^{\text{m}}(k \boxplus l) = (-1)^{\text{wt}(l)} \sum_{t' \in \mathbb{Z}_{\geq 0}} b(l') \zeta_{\tilde{S}, M}^{\text{m}}(k, l + l') t'^{\text{wt}(l')}. \]  

**Proof.** We write \( k = (k_1, \ldots, k_r) \), \( l = (l_1, \ldots, l_s) \), and consider two 2-colored rooted trees \( X_1 \) and \( X_2 \) and the index \( k' = (k'_r) \) on them defined as in the following diagrams:

\[ X_1 \quad X_2 \]
Since they are distinct only in their roots, Proposition 3.6 shows

\begin{equation}
\zeta_{S,M}(X_1; k') = (-1)^{\sum_{e \in P(v,v')} k'_e} \sum_{l'=(l'_e) \in \mathbb{Z}_{\geq 0}^{P(v,v')}} \prod_{e \in P(v,v')} \left( k'_e + l'_e + 1 \right) \times \zeta_{S,M}(X_2; k' \oplus l') t^{\sum_{e \in P(v,v')} l'_e}.
\end{equation}

By using Proposition 3.4 and applying Theorem 3.8 to \((X_1, k')\), we have

\begin{equation}
\zeta_{S,M}(X_1; k') = \zeta_{S,M}(k \in l).
\end{equation}

On the other hand, by Example 3.3 we see that the right hand side of (3.10) coincides with

\begin{equation}
(-1)^{\text{wt}(l)} \sum_{l' \in \mathbb{Z}_{\geq 0}^{\text{dep}(l)}} b\left(\frac{l}{l'}\right) \zeta_{S,M}(k, l + l') t^{\text{wt}(l')}.
\end{equation}

By combining (3.10) and (3.11) with (3.12), we have the desired formula. \(\Box\)

Corollary 3.11. For any indices \(k\) and \(l\), we have

\begin{equation}
\zeta_{S}(k \in l) = (-1)^{\text{wt}(l)} \sum_{l' \in \mathbb{Z}_{\geq 0}^{\text{dep}(l)}} b\left(\frac{l}{l'}\right) \zeta_{S}(k, l + l') t^{\text{wt}(l')}.
\end{equation}

Proof. Take the limit \(M \to \infty\) in (3.9) and use Theorem 2.5. \(\Box\)

3.4. Representation algorithm. In this subsection, we prove the following:

Theorem 3.12. Let \(X = (V, E, \text{rt}, V_\ast)\) be a 2-colored rooted tree such that \(\text{rt} \in V_\ast\) and \(k = (k_e)_{e \in E}\) an essentially positive index on \(X\). Then, there exists \(w \in \mathcal{S}^1\) such that

\[\zeta_{S,M}(X; k) = Z_{S,M}^w(w)\]

holds.

Here the essential positivity of an index is defined as follows.

Definition 3.13. Let \(X = (V, E, \text{rt}, V_\ast)\) be a 2-colored rooted tree. An index \(k\) on \(X\) is said essentially positive when the sum \(\sum_{e \in P(v_1,v_2)} k_e\) is positive for any two distinct vertices \(v_1, v_2 \in V_\ast\).

We will give an algorithm to construct the element \(w \in \mathcal{S}^1\) in Theorem 3.12. First let us suppose a stronger condition.
Definition 3.14. Let \( X = (V, E, rt, V_\bullet) \) be a 2-colored rooted tree and \( k = (k_e)_{e \in E} \) an index on \( X \). The pair \((X, k)\) is called harvestable if the following conditions hold:

(H1): The root \( rt \) is a terminal of \((V, E)\). In particular, \( rt \) is in \( V_\bullet \).

(H2): All elements of \( V_\circ \) are branched points.

(H3): All elements of \( V_\bullet \) are not branched points.

(H4): If \( v \in V_\circ \) is the parent of \( w \in V \), then \( k_{\{v,w\}} \) is positive.

(H5): If \( v, w \in V_\bullet \) and \( \{v, w\} \in E \), then \( k_{\{v,w\}} \) is positive.

Here, a branched point is a vertex of degree at least 3 and the parent of a vertex \( v \neq rt \) is the unique vertex \( p \) satisfying \( \{v, p\} \in P(rt, v) \).

Remark 3.15. The notions of harvestability and essential positivity are introduced by the first author \([O]\), but we add the condition (H5) here. Note that, for a pair \((X, k)\) satisfying the conditions from (H1) to (H4), it satisfies (H5) if and only if \( k \) is essentially positive. In particular, a harvestable pair satisfies the condition of Theorem 3.12.

Definition 3.16. For a harvestable pair \((X, k)\), we define an element \( w(X, k) \) in \( \hat{H}^1 \) recursively as follows.

(i) For \((X, k)\) of the following form, we define \( w(X, k) := z_{k_1} \cdots z_{k_r} \).

(ii) Let \((X, k)\) (resp. \((X_j, k_j)\)) be given by the left hand side (resp. the right hand side) of the following diagrams.

Assume that \( w_j = w(X_j, k_j) \) for \( j = 1, \ldots, s \) are already defined. Then we define

\[
w(X, k) := (w_1 \ast \cdots \ast w_s)x^{k'}z_{k_1} \cdots z_{k_r}.
\]
In fact, we see that this procedure exhausts all harvestable pairs \((X, k)\) by induction on the cardinal of \(V_\circ\).

The next theorem generalizes Theorem 3.8 to the harvestable case.

**Theorem 3.17.** For any harvestable pair \((X, k)\), we have

\[
\hat{\zeta}_{\mathcal{S},\mathcal{M}}(X; k) = Z^w_{\mathcal{S},\mathcal{M}}(w(X, k)).
\]

*Proof.* The equation is given in Example 3.3 if \(X\) has no branched point, and in Theorem 3.8 if there is just one branched point. In the general case, one obtains the equation by making a computation similar to the proof of Theorem 3.8 for each branched point. □

The following proposition says that any 2-colored rooted tree with the root in \(V_\bullet\) and an essentially positive index can be transformed to a harvestable pair without changing the value of the associated truncated \(\hat{\mathcal{S}}\)-MZV.

**Proposition 3.18.** Let \(X = (V, E, rt, V_\bullet)\) be a 2-colored rooted tree such that \(rt \in V_\bullet\) and \(k\) an essentially positive index on \(X\). Then there exists a harvestable pair \((X_h, k_h)\) such that

\[
\hat{\zeta}_{\mathcal{S},\mathcal{M}}(X; k) = \hat{\zeta}_{\mathcal{S},\mathcal{M}}(X_h; k_h).
\]

Explicitly, we obtain such \((X_h, k_h)\) by the following procedures:

(i) In \((X, k)\), if there exists an edge \(e\) satisfying the condition in Proposition 3.4, then contract \(e\) according to the proposition. Repeat this until we obtain a pair \((X_1, k_1)\) without such \(e\).

(ii) In \((X_1, k_1)\), if there exists a pair of edges satisfying the condition in Proposition 3.5, then joint them according to the proposition. Repeat this until we obtain a pair \((X_2, k_2)\) without such edges.

(iii) In \((X_2, k_2)\), if there exists a black branched point \(v \neq rt\), then insert a new white vertex \(v'\) together with an edge \(\{v, v'\}\) at the location of \(v\), and replace the edges \(\{v, w\}\) by \(\{v', w\}\) for vertices \(w\) whose parent is \(v\) in the original tree. Set the component of the index on the new edge \(\{v, v'\}\) to be zero. Note that this is the inverse operation of the contraction according to Proposition 3.4. Repeat this until we obtain a pair \((X_3, k_3)\) without such \(v\).

(iv) In \((X_3, k_3)\), if the root is not terminal, then insert a new white vertex and an edge at location of the root, in the same way as (iii). The result is \((X_h, k_h)\) we want to construct.

26
The following diagrams illustrate these procedures.

![Diagrams](image)

**Proof.** It is sufficient to check that the pair \((X_h, k_h)\) obtained by the above procedures is harvestable since it is obvious that the equality (3.14) holds by two propositions.

After performing (i) (resp. (ii), resp. (iii) and (iv)), the condition (H4) (resp. (H2), resp. (H3) and (H1)) is satisfied. In particular, note that the conditions (H2) and (H4) are not violated after performing (iii) and (iv) since the new white color vertex is branched and \(k\) is essentially positive on \(X\). Moreover, each procedure keeps the essential positivity of the index, and hence the result \((X_h, k_h)\) satisfies (H5). Therefore \((X_h, k_h)\) is harvestable.

\[\square\]

Even if the condition \(rt \in V_+\) in Theorem 3.12 does not hold, we can still describe \(\zeta_{\tilde{S}, M}(X; k)\) for any essentially positive index \(k\) on \(X\) in terms of values of the map \(Z_{\tilde{S}, M}^u\) as

\[\zeta_{\tilde{S}, M}(X; k) = \sum_{n=0}^{\infty} Z_{\tilde{S}, M}^u(w_n)t^n, \quad w_n \in \mathcal{S}_1^1.\]

To obtain such an expression, we first use Proposition 3.6 to change the root to some terminal vertex, and then apply Theorem 3.12 to each \((X_2, k \oplus l)\) appearing in the right hand side of (3.2). In particular, the limit in the following definition exists by virtue of Theorem 2.5.

**Definition 3.19.** Let \(X = (V, E, rt, V_+)\) be a 2-colored rooted tree and \(k = (k_e)_{e \in E}\) be an essentially positive index on \(X\). We define the \(\tilde{S}\)-MZV \(\zeta_{\tilde{S}}(X; k)\) associated with \(X\) as

\[\zeta_{\tilde{S}}(X; k) := \lim_{M \to \infty} \zeta_{\tilde{S}, M}(X; k),\]

where the limit is taken coefficientwise in \(\mathbb{Z}[t]\).
4. ON A REFINEMENT OF THE CONJECTURE OF KANEKO AND ZAGIER

4.1. The Kaneko–Zagier conjecture and its refinement. The Kaneko–Zagier conjecture states that $A$-MZVs and $S$-MZVs satisfy exactly the same algebraic relations. More precisely:

**Conjecture 4.1 ([KZ]).** Let $Z_A$ be the $Q$-subalgebra of $A$ generated by all $A$-MZVs. Then there is a $Q$-algebra isomorphism from $Z_A$ onto $\mathbb{Z}$ which sends $\zeta_A(k)$ to $\zeta_S(k)$.

If the above conjecture is true, then $Z$ is generated by $S$-MZVs. In fact, Yasuda proved the following result without assuming the conjecture.

**Theorem 4.2 ([Yas, Theorem 6.1]).** For $\bullet \in \{*, \mathfrak{m}\}$, let $Z^*_S$ be the $Q$-subalgebra of $R$ generated by all $\zeta^*_S(k)$. Then we have $Z^*_S = Z$.

As the $\hat{A}$-version of $Z_A \subset A$, we define

$Z_{\hat{A}} := \left\{ \sum_{n=1}^{\infty} a_n \zeta_{\hat{A}}(k_n)p^{bn} \in \hat{A} \middle| a_n \in Q, \quad k_n: \text{index,} \quad b_n \in \mathbb{Z}_{\geq 0} \text{ with } b_n \to \infty \ (n \to \infty) \right\}.$

We equip $Z_{\hat{A}}$ and $\mathbb{Z}[t]$ with the $p$-adic and $t$-adic topology, respectively. Then, Kaneko–Zagier’s conjecture (Conjecture 4.1) is refined as follows:

**Conjecture 4.3 (cf. [Ro2, Conjecture 2.3]).** There is a topological $Q$-algebra isomorphism from $Z_{\hat{A}}$ onto $\mathbb{Z}[t]$ which sends $\zeta_{\hat{A}}(k)$ to $\zeta^*_S(k)$ and $p$ to $t$.

We give the $t$-adic version of Yasuda’s theorem related to the above conjecture. This is due to Jarossay ([J2, Proposition 3, Theorem 3.12], cf. [Ro2, Theorem 3.3]), but we exhibit a proof for the convenience of the reader.

**Proposition 4.4.** For $\bullet \in \{*, \mathfrak{m}\}$, we define

$Z^*_{\hat{S}} := \left\{ \sum_{n=1}^{\infty} a_n \zeta^*_S(k_n)t^{bn} \in \mathbb{R}[t] \middle| a_n \in Q, \quad k_n: \text{index,} \quad b_n \in \mathbb{Z}_{\geq 0} \text{ with } b_n \to \infty \ (n \to \infty) \right\}.$

Then we have $Z^*_{\hat{S}} = Z[t]$.

**Proof.** The inclusion $Z^*_{\hat{S}} \subset Z[[t]]$ is obvious. To prove the opposite inclusion, it suffices to show that $Z \subset Z^*_{\hat{S}}$, since $Z^*_{\hat{S}}$ is $t$-adically complete. Moreover, again by the completeness of $Z^*_{\hat{S}}$, it suffices to show the following claim:

For any $n \geq 1$ and any $\xi \in Z$, there exists $\tilde{\xi}_n \in Z_{\hat{S}}^*$ such that $\xi \equiv \tilde{\xi}_n \mod t^n$. 28
We prove it by induction on \( n \).

First we consider \( n = 1 \). By virtue of Theorem 4.2, we may assume \( \xi = \zeta^*_n(k) \) with some index \( k \). Then \( \tilde{\xi}_1 = \zeta^*_n(k) \) works. Next, let \( n > 1 \). By the induction hypothesis, we have \( \tilde{\xi}_{n-1} \in \mathcal{Z}_S \) such that \( \xi \equiv \tilde{\xi}_{n-1} \mod t^{n-1} \). If we denote by \( \eta \) the coefficient of \( t^{n-1} \) in \( \tilde{\xi}_{n-1} \), we have \( \xi \equiv \tilde{\xi}_{n-1} - \eta t^{n-1} \mod t^n \). Since \( \eta \in \mathcal{Z} \), there is some \( \tilde{\eta}_1 \in \mathcal{Z}_S \) such that \( \eta \equiv \tilde{\eta}_1 \mod t \), by the claim for \( n = 1 \) proved above. Then \( \tilde{\xi}_n := \tilde{\xi}_{n-1} - \tilde{\eta}_1 t^{n-1} \) satisfies the desired property. This completes the proof. \( \square \)

4.2. \( \hat{S} \)-MZVs of Mordell–Tornheim type. In this subsection, we define and study the \( \mathcal{S} \) and \( \hat{S} \)-MZVs of Mordell–Tornheim type, which correspond to Kamano’s \( \mathcal{A} \)-MZV of Mordell–Tornheim type \([\text{Kam}]\) and its natural lift to \( \hat{\mathcal{A}} \) via Conjecture 4.1 and Conjecture 4.3, respectively.

First let us recall the definitions of the (\( \mathcal{A} \)-finite) multiple zeta values of Mordell–Tornheim type. Let \( k_1, \ldots, k_r, k_{r+1} \) be non-negative integers. Suppose that at least \( r \) numbers of them are positive. Then the \textit{multiple zeta value of Mordell–Tornheim type} \( \zeta^\text{MT}(k_1, \ldots, k_r; k_{r+1}) \) (MZV of MT type, for short) is defined by the following infinite series (cf. \([M, T]\)):

\[
\zeta^\text{MT}(k_1, \ldots, k_r; k_{r+1}) := \sum_{m_1, \ldots, m_r > 0} \frac{1}{m_1^{k_1} \cdots m_r^{k_r} (m_1 + \cdots + m_r)^{k_{r+1}}}.
\]

As the truncation of this series, we also consider

\[
\zeta^\text{MT}_M(k_1, \ldots, k_r; k_{r+1}) := \sum_{m_1, \ldots, m_r > 0} \frac{1}{m_1^{k_1} \cdots m_r^{k_r} (m_1 + \cdots + m_r)^{k_{r+1}}}.\]

By using this truncation, Kamano \([\text{Kam}]\) introduced the \textit{\( \mathcal{A} \)-finite multiple zeta value of Mordell–Tornheim type} (\( \mathcal{A} \)-MZV of MT type) as

\[
\zeta^\text{MT}_\mathcal{A}(k_1, \ldots, k_r; k_{r+1}) := (\zeta^\text{MT}_p(k_1, \ldots, k_r; k_{r+1}) \mod p)_p \in \mathcal{A},
\]

and proved that

\[
\zeta^\text{MT}_\mathcal{A}(k_1, \ldots, k_r; k_{r+1}) = \begin{cases} 
\mathcal{Z}_\mathcal{A}(\cdots z_{k_i} \cdots z_{k_r}) x^{k_{r+1}} & (k_1, \ldots, k_r > 0), \\
\mathcal{Z}_\mathcal{A}(\cdots \tilde{z}_{k_i} \cdots \tilde{z}_{k_r}) (z_{k_1} \cdots z_{k_r}) & (k_i = 0), 
\end{cases}
\]

where the symbol \( \tilde{z}_{k_i} \) means that the factor \( \tilde{z}_{k_i} \) is skipped (see \([\text{Kam}]\) Theorem 2.1]). Here and in the following, we use the rule that the symbol \( \mathcal{Z} \) with some suffixes denotes the \( \mathbb{Q} \)-linear map determined by the corresponding zeta values with the same suffixes. Thus \( \mathcal{Z}_\mathcal{A} \) denotes the \( \mathbb{Q} \)-linear map from \( \mathcal{A} \) to \( \mathcal{A} \) defined by \( \mathcal{Z}_\mathcal{A}(z_k) = \zeta^\mathcal{A}_\mathcal{A}(k) \).
In fact, if we define the $\hat{A}$-finite multiple zeta value of Mordell–Tornheim type ($\hat{A}$-MZV of MT type) by
\[
\zeta^\text{MT}_\hat{A}(k_1, \ldots, k_r; k_{r+1}) := \left(\left(\zeta^\text{MT}_p(k_1, \ldots, k_r; k_{r+1}) \mod p^n\right)_n\right) \in \hat{A},
\]
then Kamano’s proof also gives
\[
\zeta^\text{MT}_\hat{A}(k_1, \ldots, k_r; k_{r+1}) = \begin{cases} 
Z_{\hat{A}}((z_{k_1} \ldots z_{k_r})x^{k_{r+1}}) & (k_1, \ldots, k_r > 0), \\
Z_{\hat{A}}((z_{k_1} \ldots z_{k_r} \hat{z}_{k_1} \ldots \hat{z}_{k_r} z_{k_{r+1}})) & (k_i = 0).
\end{cases}
\]
Therefore, in view of Conjecture 4.1 and Conjecture 4.3 the $S$- and $\hat{S}$-counterparts of these values should be defined as follows:

**Definition 4.5.** Let $k_1, \ldots, k_{r+1}$ be non-negative integers which are positive with at most one exception. Then we set
\[
\zeta^\text{MT}_S(k_1, \ldots, k_r; k_{r+1}) := \begin{cases} 
Z_S^\mu((z_{k_1} \ldots z_{k_r})x^{k_{r+1}}) & (k_1, \ldots, k_r > 0), \\
Z_S^\mu((z_{k_1} \ldots z_{k_r} \hat{z}_{k_1} \ldots \hat{z}_{k_r} z_{k_{r+1}})) & (k_i = 0).
\end{cases}
\]
Its reduction modulo $\pi^2$ is called the $t$-adic symmetric multiple zeta value of Mordell–Tornheim type ($\hat{S}$-MZV of MT type) and denoted by $\zeta_S^\text{MT}(k_1, \ldots, k_r; k_{r+1})$.

The truncated values $\zeta^\text{MT}_S(k_1, \ldots, k_r; k_{r+1}) \in \mathbb{Q}[t]$ are similarly defined, and we have
\[
\lim_{M \to \infty} \zeta^\text{MT}_S(k_1, \ldots, k_r; k_{r+1}) = \zeta^\text{MT}_S(k_1, \ldots, k_r; k_{r+1}) \text{ by Theorem 2.5.}
\]

The $S$-versions of these values, $\zeta^\text{MT}_S(k_1, \ldots, k_r; k_{r+1})$ etc., are defined in the same way. They are also obtained by substituting $t = 0$ in the corresponding $\hat{S}$-values. In particular, $\zeta^\text{MT}_S(k_1, \ldots, k_r; k_{r+1}) \in \mathbb{Z}$ is called the symmetric multiple zeta value of Mordell–Tornheim type ($S$-MZV of MT type).

**Remark 4.6.** Here we do not define $\zeta^\text{MT}_S(k_1, \ldots, k_r; k_{r+1})$; the authors are not sure what definition is good, if any. An obvious candidate is to replace $Z^\mu_S$ by $Z^\ast_S$ in the above definition, but we have no non-trivial result on it except its congruence to $\zeta^\text{MT}_S(k_1, \ldots, k_r; k_{r+1})$ modulo $\pi^2$, which follows from Proposition 2.1.

In what follows, we study the m-truncated $\hat{S}$-MZV of MT type as an application of our theory developed in 4.3. Let $X$ be the 2-colored rooted tree in the figure below:
We regard an $r+1$-ple $k = (k_1, \ldots, k_{r+1})$ of non-negative integers as an index on $X$ as indicated in the figure. Note that $k$ is essentially positive if and only if the integers $k_1, \ldots, k_{r+1}$ are positive with at most one exception.

**Proposition 4.7.** Let $k = (k_1, \ldots, k_{r+1})$ be an essentially positive index on $X$. Then we have

$$
\zeta^\text{MT, M}_{\tilde{S}, M}(k_1, \ldots, k_r; k_{r+1}) = \zeta_{\tilde{S}, M}(X; k),
$$

and hence

$$
\zeta^\text{MT, M}_{\tilde{S}}(k_1, \ldots, k_r; k_{r+1}) = \zeta_{\tilde{S}}(X; k).
$$

**Proof.** If $k_1, \ldots, k_r > 0$, we can apply Theorem 3.17 or Theorem 3.8 to obtain the formula. If $k_i = 0$ for some $i \in \{1, \ldots, r\}$, we first apply the algorithm of Proposition 3.18 with $r_t = r_t = v_{r+1}$, and then use Theorem 3.17 or Theorem 3.8. \hfill \Box

From the definition of $\zeta_{\tilde{S}, M}(X; k)$, we can compute the $t$-adic expansion of the $\tilde{S}$-truncated $\tilde{S}$-MZV of MT type. The coefficients are described in terms of the truncated MZVs of MT type as follows.

**Proposition 4.8.** Let $k_1, \ldots, k_{r+1}$ be non-negative integers which are positive with at most one exception. Then, we have

$$
\zeta^\text{MT, M}_{\tilde{S}, M}(k_1, \ldots, k_r; k_{r+1}) = \zeta^\text{MT}_{M}(k_1, \ldots, k_r; k_{r+1}) + \sum_{i=1}^{r} (-1)^{k_i+k_{r+1}} \sum_{l,l' = 0}^{\infty} \binom{k_i + l - 1}{l} \binom{k_{r+1} + l' - 1}{l'} \times \zeta^\text{MT}_{M}(k_1, \ldots, \tilde{k}_i, \ldots, k_r, k_{r+1} + l'; k_i + l) t^{l+l'}.
$$

By taking the limit $M \to \infty$, we also have the same identity without truncation.

**Proof.** Set $m_i := m_{v_i}$ ($1 \leq i \leq r+1$), $M_{i,j} := m_i + \cdots + m_j$ and $M_j := M_{1,j}$ as in (3.1), and let $e_a$ denotes the edge $\{w, v_a\}$. Then we have

$$
L_{e_a}(X, v_i; (m_v)_{v \in V^*}) = \begin{cases} 
m_a & (a \neq i), \\
m_a + t & (a = i) 
\end{cases}
$$

for $1 \leq a \leq r$ and

$$
L_{e_{r+1}}(X, v_i; (m_v)_{v \in V^*}) = \begin{cases} 
M_r & (r+1 = i), \\
M_r + t & (r+1 \neq i). 
\end{cases}
$$
Thus we obtain
\[ \zeta_M(X, v_{r+1}; k) = \sum_{m_1, \ldots, m_r > 0, m_r < M \land m_i < 0} \frac{1}{m_1^{k_1} \cdots m_r^{k_r} M^{k_{r+1}}} = \zeta_M^{\text{MT}}(k_1, \ldots, k_r; k_{r+1}) \]
and, for \(1 \leq i \leq r\),
\[ \zeta_M(X, v_i; k) = \sum_{m_1, \ldots, m_i-1, m_{i+1}, \ldots, m_{r+1} \geq 1} \frac{1}{m_1^{k_1} \cdots m_{i-1}^{k_i-1} (m_i + t)^{k_i} m_{i+1}^{k_{i+1}} \cdots m_r^{k_r} (M + t)^{k_{r+1}}} \]
\[ = \sum_{m_1, \ldots, m_{i-1}, m_{i+1}, \ldots, m_{r+1} \geq 1} \frac{1}{m_1^{k_1} \cdots m_{i-1}^{k_i-1} (t - M_{i-1} - M_{i+1, r+1})^{k_i} m_{i+1}^{k_{i+1}} \cdots m_r^{k_r} (t - m_{r+1})^{k_{r+1}}} \]
By expanding \((t - M_{i-1} - M_{i+1, r+1})^{-k_i}\) and \((t - m_{r+1})^{-k_{r+1}}\) in \(t\), we see that
\[ \zeta_M(X, v_i; k) = (-1)^{k_i + k_{r+1}} \sum_{l, l' = 0}^{\infty} \binom{k_i + l - 1}{l} \binom{k_{r+1} + l' - 1}{l'} \times \zeta_M^{\text{MT}}(k_1, \ldots, \tilde{k}_i, \ldots, k_r, k_{r+1} + l'; k_i + l) t^{l+l'} \]
Therefore, the summation on \(i = 1, \ldots, r+1\) gives the desired formula. \(\square\)

**Remark 4.9.** By setting \(t = 0\) in Proposition 4.8 we have
\[ \zeta_S^{\text{MT}, m}(k_1, \ldots, k_r; k_{r+1}) = \sum_{i=1}^{r+1} (-1)^{k_i + k_{r+1}} \zeta_M^{\text{MT}}(k_1, \ldots, \tilde{k}_i, \ldots, k_r; k_{r+1}; k_i) \]
This is equal to \((-1)^{k_{r+1}}\Omega(k)\) with \(k = (k_1, \ldots, k_{r+1})\), where \(\Omega(k)\) denotes the symmetric MZV of Mordell–Tornheim type recently introduced by Bachmann–Takeyama–Tasaka [BTT §4.5].

Proposition 3.6 gives the following relation among the \(m\)-truncated \(\hat{S}\)-MZVs of MT type. Note that this involves an infinite series and is not a linear relation in the algebraic sense.

**Proposition 4.10.** Let \(k_1, \ldots, k_r, k_{r+1}\) be non-negative integers which are positive with at most one exception. Then we have
\[ \zeta_S^{\text{MT}, m}(k_1, \ldots, k_r; k_{r+1}) = (-1)^{k_1 + k_{r+1}} \sum_{l, l' \geq 0} \binom{k_1 + l - 1}{l} \binom{k_{r+1} + l' - 1}{l'} \times \zeta_S^{\text{MT}, m}(k_{r+1} + l', k_2, \ldots, k_r; k_1 + l) t^{l+l'} \]
(4.1)
Proof. By swapping the root $v_{r+1}$ with $v_1$ according to Proposition 3.6, we have the desired formula.

By writing down both sides of (4.1) in terms of $Z_{S,M}^n$, we obtain an identity of series involving infinitely many $m$-truncated $\hat{S}$-MZVs. In particular, by setting $t = 0$, this gives a linear relation among $m$-truncated $S$-MZVs. This is an analogue of [Kam, Theorem 3.2].

Corollary 4.11. Let $k_1, \ldots, k_r$ be positive integers. Then we have

$$Z_{S,M}^m(z_{k_1} \cdots z_{k_r}) = (-1)^{k_1} Z_{S,M}^m((z_{k_2} \cdots z_{k_r})z_{k_1})$$

and

$$Z_{S,M}^m((z_{k_1} \cdots z_{k_r})x^l) = (-1)^{k_1 + l} Z_{S,M}^m((z_{k_2} \cdots z_{k_r} \cdots z_l)x^{k_1})$$

for any integer $l \geq 1$.

References

[BTT] H. Bachmann, Y. Takeyama, K. Tasaka, Finite and symmetric Mordell-Tornheim multiple zeta values, to appear in J. Math. Soc. Japan.

[Hi] M. Hirose, Double shuffle relations for refined symmetric multiple zeta values, Doc. Math. 25 (2020), 365–380.

[HMO] M. Hirose, H. Murahara, M. Ono, On variants of symmetric multiple zeta-star values and the cyclic sum formula, preprint, arXiv:2001.03832.

[Ho] M. E. Hoffman, The algebra of multiple harmonic series, J. Algebra 194 (1997), 477–495.

[IKZ] K. Ihara, M. Kaneko, D. Zagier, Derivation and double shuffle relations for multiple zeta values, Compositio Math. 142 (2006), 307–338.

[J1] D. Jarossay, Double mélange des multiséta finis et multiséta symétrisé, C. R. Math. Acad. Sci. Paris 352 (2014), 767–771.

[J2] D. Jarossay, Algebraic relations, Taylor coefficients of hyperlogarithms and images by Frobenius - II : Relations with other motives and the Taylor period map, preprint (2016), arXiv:1601.01158v1.

[J3] D. Jarossay, Algebraic relations, Taylor coefficients of hyperlogarithms and images by Frobenius - I: The prime multiple harmonic sum motive, preprint (2016), arXiv:1412.5099v2.

[J4] D. Jarossay, An explicit theory of $\pi_1^{un-crys}(\mathbb{P}^1 - \{0, \mu, \infty\})$-II-1: Standard algebraic equations of prime weighted multiple harmonic sums and adjoint multiple zeta values, preprint (2017), arXiv:1412.5099v3.

[J5] D. Jarossay, Adjoint cyclotomic multiple zeta values and cyclotomic multiple harmonic values, preprint (2019), arXiv:1412.5099v5.

[Kam] K. Kamano, Finite Mordell-Tornheim multiple zeta values, Funct. Approx. Comment. Math. 54 (2016), 65–72.

[Kan] M. Kaneko, An introduction to classical and finite multiple zeta values, Publications mathématiques de Besançon, no. 1 (2019), 103–129.
[KZ] M. Kaneko, D. Zagier, *Finite multiple zeta values*, in preparation.

[Ko] Y. Komori, *Finite multiple zeta values, symmetric multiple zeta values and unified multiple zeta functions*, preprint.

[KMT] Y. Komori, K. Matsumoto, H. Tsumura, *Shuffle products for multiple zeta values and partial fraction decompositions of zeta-functions of root systems*, Math. Z. **268** (2011), 993–1011.

[M] K. Matsumoto, *On the analytic continuation of various multiple zeta- functions*, Number theory for the millennium, II (Urbana, IL, 2000), 417–440, A K Peters, Natick, MA, 2002.

[O] M. Ono, *Finite multiple zeta values associated with 2-colored rooted trees*, J. Number Theory **181** (2017), 99–116.

[OS] M. Ono, S. Seki, *A note on \(A_n\)-finite and \(S_n\)-symmetric multiple zeta values*, in preparation.

[Re] C. Reutenauer, *Free Lie Algebras*, Oxford Science Publications, Oxford, 1993.

[Ro1] J. Rosen, *Asymptotic relations for truncated multiple zeta values*, J. Lond. Math. Soc. (2) **91** (2015), 554–572.

[Ro2] J. Rosen, *The completed finite period map and Galois theory of supercongruences*, Int. Math. Res. Not. IMRN 2019, no. 23, 7379–7405.

[S1] S. Seki, *Finite multiple polylogarithms*, Doctoral dissertation in Osaka University, 2017.

[S2] S. Seki, *The p-adic duality for the finite star-multiple polylogarithms*, Tohoku Math. J. **71** (2019), 111–122.

[T] H. Tsumura, *On Mordell-Tornheim zeta values*, Proc. Amer. Math. Soc. **133** (2005), 2387–2393.

[U] R. Umezawa, *On an analog of the Arakawa-Kaneko zeta function and relations of some multiple zeta values*, Tsukuba J. Math. **42** (2018), 259–294.

[Yam] S. Yamamoto, *A sum formula of multiple L-values*, Int. J. Number Theory **11** (2015), 127–137.

[Yas] S. Yasuda, *Finite real multiple zeta values generate the whole space \(Z\)*, Int. J. Number Theory **12** (2016), 787–812.

**Multiple Zeta Research Center, Kyushu University, 744, Motooka, Nishiku, Fukuoka, 819-0395, Japan**

*E-mail address*: m-ono@math.kyushu-u.ac.jp

**Mathematical Institute, Tohoku University, 6-3, Aoba, Aramaki, Aoba-Ku, Sendai, 980-8578, Japan**

*E-mail address*: shinichiro.seki.b3@tohoku.ac.jp

**Department of Mathematics, Faculty of Science and Technology, Keio University, 3-14-1 Hiyoshi, Kouhoku-ku, Yokohama, 223-8522, Japan**

*E-mail address*: yamashu@math.keio.ac.jp

---

34