A New Achievable DoF Region for the 3-user $M \times N$ Symmetric Interference Channel

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Abstract—In this paper, the 3-user multiple-input multiple-output Gaussian interference channel with $M$ antennas at each transmitter and $N$ antennas at each receiver is considered. It is assumed that the channel coefficients are constant and known to all transmitters and receivers. A novel scheme is presented that spans a new achievable degrees of freedom region. For some values of $M$ and $N$, the proposed scheme achieve higher number of DoF than are currently achievable, while for other values it meets the best known upperbound. Simulation results are presented showing the superior performance of the proposed schemes to earlier approaches.

I. INTRODUCTION

The capacity of the interference channels has kept information theorists busy for more than three decades, e.g., [1], [2]. Their work has brought great understanding to interference channels, but full capacity characterization has not been found to the moment. In the absence of precise capacity characterizations, researchers have pursued asymptotic and/or approximate capacity characterizations. One of the most important capacity characterizations is the degrees of freedom (DoF) of the network [3], [13]. The DoF of wireless interference networks represent the rate of growth of network capacity with the log of the signal to noise ratio. Fortunately the number of DoF of an interference channel is equal to the number of interference-free signaling dimensions in the network.

The DoF of different forms of interference channels have recently been found, most of these results depend on interference alignment. Interference alignment refers to the overlapping of multiple interfering signals from different transmitters into a small subspace at each receiver so that the number of interference free dimensions remaining for the desired signal can be maximized. The idea of interference alignment first appeared in [3], [5] where it was used with dirty paper coding and successive decoding for proposing an achievable scheme for the X-network. Interference alignment was then independently used in [7], [10] by Sayed Jafar and Maddah-Ali respectively in studying the 2-user X network. Interference alignment has since been applied to a various types of networks; it was used by Cadambe and Sayed Jafar in finding the DoF of the $K$ user interference channel with time varying channel coefficients [9], [12], and in finding the DoF of X networks with of $M$ single antenna transmitters and $N$ single antenna receivers [8]. Motahari et. al settled the problem in general in [14], [15] with their new form of interference alignment which used results from Diophantine approximation in Number theory [16] to show that interference can be aligned based on the properties of rationals and irrationals. They showed that almost all $K$ user real Gaussian interference channel fading with constant coefficients have $K/2$ DoF [13]. They extended these results to the $K$ user multiple input multiple output (MIMO) interference channel when the number of transmit antennas is equal to the number of receive antennas. They showed that the total number of DoF is equal to $KM/2$ whether the channel has constant or time varying coefficients.

The $K$ user $M \times N$ Gaussian interference channel was first studied in [12] where it was shown that the total number of DoF of the $K$ user $M \times N$ time-varying MIMO Gaussian interference channel is equal to $K \min(M,N)$ if $K \leq R$ and $\min(M,N)KR/(R+1)$ when $R = \max(M,N)/\min(M,N)$ is equal to an integer. Also, it was shown that for the $G+2$ user MIMO Gaussian interference channel where each transmitter has $M$ antennas and each receiver has $GM$ antennas with constant channel coefficients, $GM + (GM/G^2 + 2G - 1)$ DoF can be achieved without channel extension. Ghasemi et. al [11] presented an achievable scheme that can achieve $KMN/(M+N)$ for constant channel with real coefficients. They showed that their scheme coincides with the new upperbound on the total number of DoF they found for $K > (M+N)/\gcd(M,N)$. It is worth noting that their scheme assumes no cooperation among transmit and/or receive antennas of each user. The previous results makes the DoF of the $K$ user $M \times N$ channel with complex constant coefficients almost an open problem, and for the channel with real constant coefficients, the results leave a gap when $K < (M+N)/\gcd(M,N)$ for which the number of DoF is unresolved. The gap of unknown DoF of different $(M,N)$ pairs is actually wide when the number of users is small. In this paper we provide achievable scheme for different cases along this gap. Our scheme is optimal for $M/N \geq 5/3$. It also achieves the beamforming upperbound for $M/N = (2L+3)/(2L+1)$ where $L = 1, 2, \ldots$ For other values of $M/N$, the proposed schemes do not coincide with the best known upperbound, nevertheless, they achieves more DoF than previously known.
II. 3-User M X N Symmetric Interference Channel

A. System model

We consider a 3 user MIMO interference channel where each transmitter and each receiver is equipped with M and N antennas, respectively. Let \( V_i \) denote the \( M \times d_i \) precoding matrix of the \( i \)th user where \( d_i \) is the number of streams, DoF, transmitted by the \( i \)th user. We can write the \( N \)-dimensional received signal at the \( i \)th receiver at the \( n \)th time instant as

\[
y_i(n) = \sum_{j=1}^{3} H_{i,j} V_j x_j(n) + z_i(n) \quad i = 1, \ldots, 3
\]

where \( H_{i,j} \) is the \( N \times M \) matrix containing the channel coefficients from transmitter \( j \) to receiver \( i \), \( z_i(n) \) is the \( N \times 1 \) additive white Gaussian noise vector at the \( i \)th receiver, \( x_j(n) \) is the \( d_j \times 1 \) vector of Gaussian coded symbols. The \( l \)th element of the vector \( x_j(n) \) represents the \( l \)th stream of the \( j \)th transmitter, which is transmit beamformed by the \( l \)th column of the matrix \( V_j \). The received signal vector at the \( i \)th receiver is linearly processed by the \( N \times d_i \) post-processing matrix \( U_i \) to extract the \( d_i \) streams sent by the \( i \)th transmitter.

B. Upperbounds on the DOF

1) General Upperbound: Following the footsteps of the proof of the upperbound on the DoF of the \( k \) user \( M \times N \) Interference Channel DoF in [11], the total number DoF is bounded above by

\[
\sum_{i=1}^{3} d_i \leq \min \{3M, 3N, \max\{2M, N\}, \max\{2N, M\}\}
\]

(2)

where the above upperbound can be derived by grouping two transmitters into one transmitter with \( 2M \) antennas and grouping the corresponding two receivers into a single receiver with \( 2N \) antennas. The upperbound in (2) is derived from the resulting two-user interference channel using the results in [6]. Note that the DoF of a 3-user \( M \times M \) interference channel is known to be \((3/2)M\). Since adding antennas to the transmitters/receivers cannot decrease the DoF, we can add more antennas at the transmitters or receivers to convert the system into an interference channel with \( \max\{M, N\} \) antennas at each transmitter and receiver. Hence, the total number DoF is also bounded above by \((3/2)\max\{M, N\}\). Hence, we can augment the upperbound in (2) as,

\[
\sum_{i=1}^{3} d_i \leq \min \{3M, 3N, \max\{2M, N\}, \max\{M, 2N\}, \frac{3}{2} \max\{M, N\}\}
\]

(3)

2) Beamforming Upperbound: In [17], it was shown that the total number DoF of a 3-user symmetric \( M \times N \) Interference Channel that can be achieved using beamforming only is bounded above by

\[
\sum_{i=1}^{3} d_i \leq \frac{3}{4} (M + N).
\]

(4)

C. Achievable Schemes

We assume-without loss of generality-that \( M \) is larger than \( N \). Note that in the case of \( M < N \), the precoding/decoding matrices of the transmitters/receivers can be obtained from the decoding/precoding matrices of the 3-user \( M \times N \) reciprocal channel because of the the reciprocity of the problem. Note that typically for \( M > N \), classical interference alignment is not enough for achieving the DoF of this network as it achieves at most \( N/2 \) DoF per user.

In the proposed schemes, the nullspaces of the channels to the non-intended receivers are used to mitigate the effect of interference. We show below that for \( M \geq 5N/3 \), there are achievable schemes that achieve the best known upperbound. For example, for \( 5N/3 \leq M < 2N \) half of the streams are sent in the nullspace of one receiver and are aligned with other interference streams at the other receiver. However, as the ratio between \( M \) and \( N \) get smaller, a smaller number of streams can be directed towards the nullspace and interference alignment schemes are needed for more streams.

We will present the proposed achievable schemes for different cases of the ratio \( M/N \).

1) \( M/N \) larger than or equal to 3: In this case, the number of DoF is upperbounded by \( 3N \). Furthermore, the zero forcing scheme of [12] can achieve the optimal number of DoF.

2) \( M/N \) larger than or equal to 2 and less than 3: In this case, it can be easily shown from (3) that the number of DoF is upperbounded by \( M \). The upperbound can be achieved by assigning each user \( d = M/3 \) DoF via choosing the precoding matrix of the \( i \)th user as

\[
V_i = \mathcal{N}\{U_i^H H_{j,i}\} \cap \mathcal{N}\{U_i^H H_{k,i}\}
\]

(5)

\[
= \mathcal{N}\left\{ \left[ U_i^H H_{j,i} \right] \right\}
\]

(6)

where \( \mathcal{N}\{A\} \) denotes the nullspace of the matrix \( A \), the indices \((l, j, i) = (1, 2, 3), (2, 3, 1), (3, 1, 2) \) , and the decoding matrices \( \{U_i\}_{i=1}^3 \) are selected randomly. Note that the dimension of the subspace in (6) is given by \( M - 2d \), and hence, \( d = M/3 \) streams can be transmitted from each user.

3) \( M/N \) smaller than 2: We here present the main scheme of the paper. Our scheme starts by dividing the precoding matrix of each transmitter into a number of parts where each part performs interference alignment independently. The alignment conditions of these parts are different, some parts totally align their signal such that they lie in the interference subspace at both non-intended receivers, others align their signal to lie in the interference subspace of one receiver and completely direct their interference in the nullspace of the other non-intended receiver.

Let us divide the \( M \times d_i \)-dimensional precoding matrix for the \( i \)th transmitter to \( L + 1 \) subblocks each of size \( M \times d \)

\[
V_i = \begin{bmatrix} V_i^{(1)} & V_i^{(2)} & \cdots & V_i^{(L+1)} \end{bmatrix}
\]

where the total number of streams transmitted by the \( i \)th user is given by \( d_i = d(L + 1) \) and the total number of DoF is given by \( d = 3d(L + 1) \). Our scheme can be described
by equations (9)–(16) at the bottom of next page where \( (\cdot)_n \)
denotes the modulo-\( n \) operator, \( (\cdot)_n = n \), and \( [x] \) denotes
the largest integer that is less than or equal to \( x \). The system
of equations in (9)–(16) is repeated three times where in the
first time \( (i, j_1, j_2, j_3) = (1, 1, 1, 1) \). Note that \( j_k - 1 \) indicates
the number of partitions of the matrix \( V_k \) that has appeared
in the previous, i.e., \( (i-1) \)th set of equations. Hence, for
the second set, we have \( i = 2 \) and \( j_k = [(L + 1)/3] + 1 \) if
\( k \leq (L + 1)/3 \) and \( [(L + 1)/3] + 1 \) else. Also, for the third
set, we have \( i = 3 \) and \( j_k \) is incremented by \( [(L + 1)/3] \) if
\( (k-1)/3 \leq (L + 1)/3 \) and by \( [(L + 1)/3] \) else.

Note that the parameter \( L \) completely specifies the above
system of equations. The number of equations in (9)–(16) is
given by \( L + 2 \). Equation (9) constrains the transmission of
the matrix \( V_{(i)3}^{(j)(i)3} \) to lie in the nullspace of \( H_{(i+1)3,(i)3} \), and
hence, it does not cause any interference on the \( (i+1) \)th receiver.
The next \( L \) equations align the interference from the preceding
submatrices of each two transmitters at the interference
subspace of the third receiver. For example, equation (10)
aligns the interference caused by the transmission of \( V_{(i+1)3}^{(j)(i)3} \)
at the \( (i+2) \)th receiver with the interference caused by the
transmission of \( V_{(i)3}^{(j)(i)3} \) at the same receiver. The final equation
constrains the transmission of the matrix \( V_{(L+1)3}^{(j)(i)3} \)
to lie in the nullspace of \( H_{(L+i-1)3,(L+i)3} \), and hence, it
does not cause any interference on the \( (L+i-1) \)th receiver.
Note that the system of equations in (9)–(16) constrains the
transmission of \( L + 1 \) subblocks of the preceding matrices of
the three users that are transmitting a total number of \( (L+1)d \)
streams in the signal subspaces of the 3 receivers. Furthermore,
these transmissions are constrained to lie in an interference
subspace of dimension \( Ld \) distributed among the three
receivers. Hence, the total number of dimensions consumed at
the 3 receivers (interference+signal) by the complete system of
equations corresponding to \( i = 1, 2, 3 \) is given by \( 3(2L+1)d \).
Since the total number of receive dimensions is given by \( 3N \),
and the total number of streams transmitted by the \( i \)th user is
given by \( d_i = d(L+1) \), we have the following upperbound
on the total number of achievable DoF of our scheme
\[
d \leq \frac{3L+3}{2L+1}N. \tag{16}
\]

Later in this section, we will show the relationship between
the parameter \( L \) and the number of transmit and receive antennas
such that there is no interference at any receiver.

Before explaining how to obtain a solution of the above
system of equations, and presenting the conditions on the
number of achievable DoF, we present the case which has
only one alignment equation, i.e., \( L = 1 \). In this case, the
complete system of equations is given by
\[
\begin{align*}
V_{1}^{(1)} & \in \mathcal{N}(H_{2,1}) \tag{17} \\
H_{3,2}V_{2}^{(1)} & = H_{3,1}V_{1}^{(1)} \tag{18} \\
V_{2}^{(1)} & \in \mathcal{N}(H_{1,2}) \tag{19} \\
V_{2}^{(2)} & \in \mathcal{N}(H_{3,2}) \tag{20} \\
H_{1,3}V_{3}^{(1)} & = H_{1,2}V_{2}^{(1)} \tag{21} \\
V_{3}^{(1)} & \in \mathcal{N}(H_{2,3}) \tag{22} \\
V_{3}^{(2)} & \in \mathcal{N}(H_{1,3}) \tag{23} \\
H_{2,1}V_{1}^{(2)} & = H_{2,3}V_{3}^{(2)} \tag{24} \\
V_{1}^{(2)} & \in \mathcal{N}(H_{3,1}) \tag{25} \\
\end{align*}
\]

We will show how we can find \( V_1, V_2, \) and \( V_3 \) that satisfy
the system of equations, and find the number of achievable
DoF. Let \( \Xi, A \) denote the \( M \times N \) matrix whose columns
span nullspace of the matrix \( H_{i,j} \). Let us consider the first
group of equations, i.e., the first \( L+2 \) equations. From (17),
(18) and (19) respectively, we can write
\[
\begin{align*}
V_{1}^{(1)} & = \Xi_{2,1}A_{2,1} \tag{26} \\
V_{2}^{(1)} & = H_{3,2}^{†}H_{3,1}V_{1}^{(1)} + \Xi_{3,2}A_{3,2} \tag{27} \\
V_{1}^{(2)} & = \Xi_{1,2}A_{1,2} \tag{28}
\end{align*}
\]
where \( A^\dagger \) denotes the pseudo-inverse of the matrix \( A \) and the dimensions of the matrices \( A_{2,1}, A_{2,2}, \) and \( A_{2,3} \) are \( M - N \times d \). Substituting with (26) and (28) in (27), we get
\[
\Xi_{1,2}A_{1,2} = H_i^{\dagger}H_x^{\dagger}\Xi_{1,2}A_{2,1} + \Xi_{3,2}A_{3,2} \tag{29}
\]
and hence, we can write
\[
\tilde{\Xi}_1\tilde{A}_1 = 0_{M \times d} \tag{30}
\]
where the \( 3(M - N) \times d \) matrix \( \tilde{A} = \left( A_{1,2}^T, A_{2,1}^T, A_{3,2}^T \right)^T \) and the \( M \times 3(M - N) \) matrix \( \tilde{\Xi}_1 \) is given by
\[
\tilde{\Xi}_1 = \left(-\Xi_{1,2}, H_1^{\dagger}H_3^{\dagger}\Xi_{1,2}, \Xi_{3,2} \right) \tag{31}
\]
Since the dimension of the nullspace of \( \tilde{\Xi}_1 \) is given by \( 2M - 3N \), we obtain the following bound on the number of streams
\[
d \leq (2M - 3N)^+ \tag{32}
\]
where \((x)^+ = \max\{x, 0\}\).

Given the nullspace of the matrix \( \tilde{\Xi}_1 \), we can compute the precoding matrices \( V_1^{(1)} \) and \( V_2^{(1)} \) using (26) and (28). The same procedure can be applied on equations (26) to evaluate the matrices \( V_2^{(2)} \) and \( V_3^{(1)} \) and on equations (23)–(25) to evaluate the matrices \( V_1^{(2)} \) and \( V_3^{(2)} \). Hence, the total number of DoF that can be achieved in the network is upperbounded by
\[
d \leq (12M - 18N)^+ \tag{33}
\]
In addition, we have the upperbound on the achieved DoF by the proposed scheme in (16). Therefore, the total number of achieved DoF obtained by solving the sets of equations for \( L = 1 \) is upperbounded by
\[
d \leq \min\{(12M - 18N)^+, 2N\}. \tag{34}
\]
For \( 5/3 \leq M/N < 2 \), the general upperbound on the DoF in \( \text{(3)} \) is equal to \( 2N \), and is tighter than the beamforming upperbound in \( \text{(4)} \). In this region, the number of DoF obtained by solving the sets of equations for \( L = 1 \) is given by \( 2N \) since \( \min\{(12M - 18N)^+, 2N\} = 2N \). Hence, the proposed scheme with \( L = 1 \) can achieve the maximum number of DoF available in the network and therefore is DoF-optimal. Also, note that at \( M/N = 5/3 \), the two general and the beamforming upperbounds in \( \text{(3)} \) and \( \text{(4)} \) are equal and are achieved by the proposed scheme with \( L = 1 \).

Let us return to the general scheme with \( 3(L + 2) \) equations in (9)–(16). We can write
\[
V_j^{(L+i)3} = \Xi_{(i+1)3}A_{(i+1)i} \tag{35}
\]

\[
V_j^{(i+1)3} = \Xi_{(i+1)3}A_{(i+1)i} + H_i^{(i+1)3}(i+1) \tag{36}
\]

\[
V_j^{(L+i+2)3} = \Xi_{(i+1)L+2}A_{(i+1)i} + H_i^{(L+i+2)3}(i+1) \tag{37}
\]

\[
V_j^{(i+1)3} = \xi_{(i+1)3}A_{(i+1)i} \tag{38}
\]

Therefore, we can write
\[
\tilde{\Xi}_i\tilde{A}_i = 0_{M \times d} \tag{39}
\]
where \( i = 1, 2, 3 \) and the dimensions of the matrices \( \tilde{\Xi}_i \) and \( \tilde{A}_i \) are given by \( M \times (L+2)(M-N) \) and \( (L+2)(M-N) \times d \), respectively. Since the dimension of the nullspace of \( \tilde{\Xi}_i \) is \( (L+1)M - (L+2)N \), we can achieve the following DoF
\[
d \leq 3(L+1)((L+1)M - (L+2)N)^+. \tag{40}
\]
Combining the above upperbound on the achievable DoF by the proposed scheme with that in (16), we get
\[
d \leq \min\{3(L+1)((L+1)M - (L+2)N)^+ + 3L+3, 2L+1\}. \tag{41}
\]
Note that when \( M/N \leq (2L+3)/(2L+1) \), we have \( ((L+1)M - (L+2)N)^+ \leq N/(2L+1) \), and hence, the total number of DoF achieved by the proposed scheme is given by
\[
d = 3(L+1)((L+1)M - (L+2)N)^+. \tag{42}
\]
Furthermore, at \( M/N = (2L+3)/(2L+1) \), the number of achieved DoF meets the beamforming upperbound in (4) which is tighter than the general upperbound in (3) in this case. Hence, the proposed scheme achieves the maximum DoF available through beamforming when \( M/N = (2L+3)/(2L+1) \). Fig. 1 shows the number of achievable DoF by the proposed scheme at different values of the parameter \( L \) versus the ratio \( M/N \) and their relationship to the general and beamforming upperbounds.

**D. Combinatorics**

We have see from the previous subsection that if \( M/N \neq (2L+3)/(2L+1) \) for some integer \( L \), the proposed scheme is not DoF-optimal. Nevertheless, we cannot design our precoding matrices using more than one value of \( L \) to achieve more DoF than those achieved by using only one value of \( L \). Note that we can utilize any value of \( L \) as long as \( (L+1)M - (L+2)N \) is larger than or equal to 1, so that we can find a solution to the system of equations in (9)–(16). We also notice that for each \( L + 1 \) streams sent in the network we consume \( 2L+1 \) receiver dimensions. It is hence more efficient.
to use smaller values of $L$. However, the streams sent using a certain $L$ might not “fill” all the receiver dimensions. Hence, an efficient scheme would start with the lowest possible value of $L$, send the highest possible number of streams using this value. If there is unutilized receiver dimensions, we design precoding matrices for extra streams using the next value of $L$ and so on.

In order to illustrate this design technique, Let us assume that $M = 30$ and $N = 19$. Setting $L = 1$, we can design $d_L = (L+1)(2M−3N) = 6$ streams per user while consuming $(2L+1)(2M−3N) = 9$ receiver dimensions per user. Since $N = 19$, we still have 19 − 9 = 10 receive dimensions per user to exploit. Hence we then use $L = 2$. Since, $3M − 4N = 14$, we can design up to $14(L + 1)$ streams per user, but we will be limited by the receiver dimensions. For $L = 2$, each $(L + 1) = 3$ streams sent per user utilize $(2L + 1) = 5$ receiver dimensions. Hence, we can send 6 more streams per user bringing the total number of streams to 12 streams per user or 36 streams for the overall system which is close to the beamforming upperbound of $3(M + N)/4 = 36.75$.

We have assumed throughout this paper that all users use the same number of streams. It can be actually shown that in some cases, different users should use different number of streams. To find the optimal solution, i.e. one that maximizes the total number of system streams, the problem can be posed as an integer programming problem. We will explore this in an extended version of this paper.

III. NUMERICAL RESULTS

In order to illustrate what our scheme really add, we give the total number of achievable DoF for $N = 5$ and $M = 1, 2, ..., 16$. Fig. 2 compares the new DoF region we obtained to the best known DoF region found [11] and to the best known upperbound. For $N > M$ the number of DoF are equal to the DoF of the reverse link and Fig. 2 shows the DoF with number of antennas at both the transmitter and the receiver exchanged. The fractional DoF is obtained using finite symbol extension in time. For $M = 4$, $N = 5$, the scheme is used with $L = 4$ over 9 time slots. This scheme wastes 4 dimensions for interference for every 5 streams sent, a total of 9 dimensions needed at the receiver for every 5 DoF, so 9 time slots is used to make the total number of resources a multiple of 9 and maximize the total number of DoF. For $M = 8$, the scheme is used twice with $L = 1, 2$ over 5 time slots. First the scheme is used with $L = 1$ to obtain 2 DoF per user per time slot, then it is used with $L = 2$ to obtain 1.2 DoF per user per time slot. We achieve more DoF than known before for $M = 2, 3, 7, 8, 9$. Our region actually meets the upperbound at $M = 2, 3, 9$. This leaves the number of DoF for $M = 4, 6, 7, 8$ unknown.

IV. CONCLUSION

We have provided a new achievable interference alignment scheme for 3-user MIMO Gaussian interference channel with constant channel coefficients. We have shown that the proposed scheme spans a new achievable DoF region. We have showed that for some values of $M$ and $N$ it meets the best known upperbound.

Fig. 2. Number of DoF that can be obtained by new schemes

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