On General Conditions for Uniqueness and Robustness of Structured Matrix Signal Reconstruction

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Abstract. This paper investigates the problem of reconstructing \( n \)-by-\( n \) structured matrix signal via convex optimization. The traditional vector signal model is extended to matrix signal model. We establish fundamental conditions on the measurement operator to guarantee strong properties of sparse and flat matrix signal reconstruction from noisy measurements, i.e., conditions to guarantee uniqueness, support and sign stability as well as value-error robustness. In comparison with other works, these conditions are more general and our method is more suitable to be generalized to dealing with high-order tensor signals. These theoretical results, together with the auxiliary results in the argument, are heuristic for developing more effective algorithms in, e.g., wide-band communications and related signal processing applications.

Keywords: Inverse Problem; Signal Reconstruction; Matrix Signal; Uniqueness; Robustness.

1. Introduction
Compressive sensing theory develops effective methods to reconstruct signals accurately or approximately from accurate or noisy measurements by exploiting a priori knowledge about the signal[1,2]. So far in most works the signals are modeled as vectors of high ambient dimension. However, there are lots of applications in which signals are matrices or even tensors of high orders, particularly in big data processing applications. For example, in highly data-intensive radar system[3], measurements are modeled as \( y_{kl} = \sum_{ij} \Phi_{kl,ij} X_{ij} + e_{kl} \) where each \( y_{kl} \) is the echo sampled at specific time \( k \) and specific receiver element \( l \) in a linear or planar array; \( \Phi_{kl,ij} \) is the coefficient of a linear processor; \( e_{kl} \) is the intensity of noise and clutter; signal \( X_{ij} \) is the scattering intensity of a target detected in state cell \((i,j)\), e.g., a target at specific distance and radial speed, or at specific distance and direction, etc. In applications related to sparse and wide-band signal sampling/reconstruction, multivariable functions in a linear space spanned by given basis, e.g., \( \{\psi_{\mu}(u)\phi_{\nu}(v)\}_{\mu,\nu} \), are sampled as \( s(u,v) = \sum_{\mu,\nu} \psi_{\mu}(u)\phi_{\nu}(v)\chi_{\mu,\nu} \) where \( \chi_{\mu,\nu} \) are the signal’s Fourier coefficients to be reconstructed from the samples. These are typical problems of matrix signals reconstruction and many of them can be naturally extended to even more general tensor signal models.

In comparison with traditional methods in most literatures, which mainly deal with vector signals, the problem of reconstructing matrix, or more generally, tensor signals are more challenging. One difficulty is that such signals have richer and more complicated structures than vector signals. When solving the reconstruction problem via convex programming, it is important to select the appropriate matrix norm (regularizer) for specific signal structure. For example, \( L_1 \)-norm is suitable for general sparsity, nuclear norm is suitable for singular-value-sparsity, and other regularizers are needed for more special or more fine-grained structures, e.g., column-wise sparsity, row-wise sparsity or some hybrid structure. Appropriate regularizer determines the reconstruction’s performance.
So far the works on matrix or tensor signal reconstruction are relatively few, among which typical works include low-rank matrix recovery[1,4], matrix completion, Kronecker compressive sensing[5,6], etc. Low-rank matrix recovery deals with how to reconstruct the matrix signal with sparse singular values from linear measurements using nuclear norm (sum of singular values) as the regularizer. Kronecker compressive sensing reconstructs the matrix signal from matrix measurements via matrix $L_1$-norm $\sum \|X\|$ as the regularizer, dealing with the measurement operator in tensor-product form.

In this paper we deal with the challenging problem of reconstructing $n$-by-$n$ matrix signal $X=(x_1,\ldots,x_n)$ by convex optimization. Signal’s structural features in concern are sparsity and flatness, i.e., each column $x_j$ is a vector of $s$-sparsity and all columns have the same $l_1$-norm. Such signals naturally appear in some important applications, e.g., radar waveform space-time analysis, which will be dealt with as an application in subsequent papers. The regularizer to be used is matrix norm $\|X\|_1:=\max \|x\|$ where $\|\|$ is the $l_1$-norm on column vector space.

The main contribution in this paper is a group of innovative conditions on the measurement operator to guarantee strong properties of sparse and flat matrix signal reconstruction from noisy measurements, i.e., conditions to guarantee uniqueness, support and sign stability as well as value-error robustness. We take the convex optimization approach in deterministic setting. In comparison with representative works in current literatures[7,8,9,10], our conditions are more general and our method is more suitable to be generalized to dealing with high-order tensor signals. In particular, the results guaranteed by these conditions include the error robustness in terms of any matrix norm, support stability and non-zero components’ sign stability of the reconstructed matrix signal. Further more, the reconstruction error has linear convergence rate which is asymptotically optimal[10]. These central results are presented and proved in the fundamental theorem.

2. Foundations

2.1. Conventions and Notations

Any vector $x$ is regarded as column vector, $x^T$ denotes its transpose (row vector). For vectors $x$ and $y$, $\langle x,y \rangle$ denotes the scalar product. For matrices $X$ and $Y$, $\langle X,Y \rangle$ denotes the scalar product $tr(X^TY)$. In particular, the Frobenius norm $\|X\|^2_F$ is denoted as $\|X\|$.

For a positive integer $s$, $\Sigma(n,s)$ denotes the set of $n$-by-$n$ matrices which column vectors are all of sparsity $s$, i.e., the number of non-zero components of each column vector is at most $s$. Let $S=S_1 \cup \ldots \cup S_s$ be a subset of $\{(i,j): i,j=1,\ldots,n\}$ where each $S_i$ is a subset of $\{(i,j): i=1,\ldots,n\}$ and its cardinality $|S_i| \leq s$, $\Sigma(n,s;S)$ denotes the set of $n$-by-$n$ matrices $\{M: M_{ij}=0$ if $(i,j)$ not in $S\}$. $S$ is called the matrix signal’s $s$-sparsity pattern.

A matrix $M=(m_1,\ldots,m_n)$ is called $l_1$-column-flat if all its columns’ $l_1$-norms $\|m_i\|$ have the same value. If $X_k$ is a group of random variables and $p(x)$ is some given probability distribution, then $X_k \sim p(x)$ denotes that all these $X_k$’s are identically and independently sampled under this distribution.

2.2. Basic Problems

We investigate the problem of reconstructing $n$-by-$n$ matrix signal $X=(x_1,\ldots,x_n)$ with $s$-sparse and $l_1$-flat column vectors $x_1,\ldots,x_n$ (i.e., $\|x\|_1=\|x\|_1$ for all $j$) by solving the following convex programming problems. The regularizer is matrix norm $\|X\|_1:=\max \|x\|$.

$$
Problem \ MP_x^{(\alpha,\Phi,\eta)}: \inf_{\|Z\|_1} \|y-\Phi(Z)\|_0 \ s.t. \ |y-\Phi(Z)| \leq \eta.
$$

In this setting $y$ is a measurement vector in $R^n$ with some vector norm $\|\cdot\|_0$ defined on it, e.g., $\|\cdot\|_0$ being the $l_2$-norm. $\Phi$ is a linear operator and there is a matrix $X$ (the real signal) satisfying $y=\Phi(X)+e$ where $\|e\|_0 \leq \eta$.

2.3. Related Concepts

For brevity the following basic concepts are only presented in form of vectors, however, their generalization to the form of matrices is straightforward.

A cone $C$ is a subset in $R^n$ such that $tC$ is a subset of $C$ for any $t>0$. For a subset $K$ in $R^n$, its polar dual $K^*:=\{y: \langle x,y \rangle \leq 0 \text{ for all } x \in K\}$. $K^*$ is always a convex cone.
For a proper convex function $F(x)$, there are two important and related sets\cite{2}:

$$D(F, x) = \{ y : F(x + \nu y) \leq F(x) \text{ for some } \nu > 0 \},$$
and an important relation is $D(F, x)^c = \text{the union of } t \partial F(x)$ with all $t > 0$.

Let $|.|$ be some vector norm and $|.|^*$ be its conjugate norm, i.e., $|u|^* = \sup\{<u, v> : |v| \leq 1\}$ (e.g., $|||X|||_1^* = \sum |x_j|$).

Now present an important fact which proof is in the author’s full version preprint \cite{11}.

**Lemma 1** For $n$-by-$n$ matrix $X = X_1, \ldots, X_s$, the subdifferential of matrix norm $|||X|||_1 = \max_j |x_j|$ is

$$\partial |||X|||_1 = \{(\lambda_1 \xi_1, \ldots, \lambda_s \xi_s) : \xi_j \in \partial |x_j| \text{ and } \lambda_j \geq 0 \text{ for all } j, \lambda_1 + \ldots + \lambda_s = 1 \text{ and } \lambda_j = 0 \text{ for all } j : |x_j| < \max_j |x_i| \}$$

3. Conditions on Uniqueness and Robustness of Reconstruction: Auxiliary Results

In this and next section, we establish fundamental conditions on the measurement operator $\Phi$ for some strong properties on sparse and flat matrix signal reconstruction from noisy measurements, e.g., conditions to guarantee uniqueness, support and sign stability as well as value-error robustness.

At first we note the basic fact that $X = \arg \inf |||Z|||_1$ s.t. $|y - \Phi(Z)|_2 \leq \eta$ if and only if it’s a multiplier $\gamma > 0$ (dependent on $X^*$ in general) such that $X^*$ is a minimizer of the unconstrained convex programming $\inf |||Z|||_1 + (1/2) \gamma |y - \Phi(Z)|_2^2$. For a preparation, consider the unconstrained convex programming with given parameter $\gamma > 0$ (value of $\gamma$ is independently set) in this section:

$$\text{Problem MLP}_{\gamma, \Phi}(\gamma) \quad \inf \|||Z|||_1 + (1/2)\gamma |y - \Phi(Z)|_2^2. \quad (2)$$

On basis of some critical properties for the minimizer of this unconstrained optimization, we establish conditions for robustness, support and sign stability in signal reconstruction via solving problem (1) in next section.

In the following context, for given positive integer $s$, sparsity pattern $S = S_1 \cup \ldots \cup S_n$ where $|S_j| \leq s$ for all $j$ and the linear operator $\Phi$ if $\Phi S_1 S_2$ is a bijection $\Sigma(n,s;S) \rightarrow \Sigma(n,s;S)$ then we denote the pseudo-inverse $(\Phi S_1 S_2^{-1})^\dagger \Phi S_2^{-1}$ as $\Phi S_2^{-1}$.

**Lemma 2** Given $y$, positive integer $s$ and sparsity pattern $S = S_1 \cup \ldots \cup S_n$ where $|S_j| \leq s$ for all $j$, suppose the linear measurement operator $\Phi$ satisfies:

1. $\Phi S^\dagger(z)$ does not have any 0-column for $z \neq 0$;
2. $\Phi S^\dagger \Phi$ is a bijection;
3. $\gamma \sup \{\gamma \Phi S^\dagger(\Phi S^\dagger) \} \mathbb{H} : |||\mathbb{H}|||_1 = 1 \} + \sup \{\Phi S^\dagger(\Phi S^\dagger) \} \mathbb{M}, \mathbb{H} : |||\mathbb{H}|||_1 = 1, |||\mathbb{M}|||_1 \leq 1 \} < 1 \quad (3)$

Let $X^* = \arg \inf \sup \{s|||Z|||_1 + (1/2)\gamma |y - \Phi(Z)|_2^2 \} \text{ for all } s$, i.e.,

$$X^* = \arg \inf \|||Z|||_1 + (1/2)\gamma |y - \Phi(Z)|_2^2 \text{.} \quad (4)$$

Then there are the following conclusions:

1. $X^*$ is the unique minimizer of problem (4) and is $l_1$-column-flat;
2. $X^*$ is also the unique minimizer of problem (2), i.e., the (global) minimizer of (2) is unique and is $X^*$.

3. Let $Y^* = \Phi S_1^{-1}(x)$ in $\Sigma(n,s;S)$, then $X^* \neq 0$ for all $(i,j)$: $|Y^*_{ij}| > \gamma^{-1} N((\Phi S_1 S_2^{-1}) \dashv \dashv M)_{ij}$. Where $N((\Phi S_1 S_2^{-1}) \dashv \dashv M)$ denotes $(\Phi S_1 S_2^{-1})$‘s operator norm and matrix norm $|||M|||_{\max} = \max_j |M|_{ij}$.

4. With the same notations as the above, if

$$\min \{\gamma \Phi S_1 S_2^{-1} \} \dashv \dashv M_{ij} > \gamma^{-1} N((\Phi S_1 S_2^{-1}) \dashv \dashv M)_{ij} \quad (5)$$

then $\text{sgn}(X^*_{ij}) = \text{sgn}(Y^*_{ij})$ for all $(i,j)$ in $S$.

This lemma’s proof can be found in the author’s full version preprint \cite{11}.

4. Conditions on Uniqueness and Robustness of Reconstruction: Fundamental Results

Now consider matrix signal reconstruction via solving the constrained convex programming:

$$\text{MP}_{\gamma, \Phi, \eta} : \quad X^* = \arg \inf \|||Z|||_1 \text{ s.t. } |y - \Phi(Z)|_2 \leq \eta. \quad (6)$$
Lemma 3 Given y, positive integer s and sparsity pattern S = S₁ ∪ ... ∪ Sₙ where |Sᵢ| ≤ s for all i, suppose the linear measurement operator Φ satisfies:
(1) ΦSᵢᵀ(z) does not have any 0-column for z ≠ 0;
(2) ΦSᵢΦS is a bijection;
(3) sup {<ΦSᵢ(ΦSᵢΦSᵢ⁻¹(y) – y), H> : ||H||₁ = 1} < η Aₘᵢᵢ(ΦSᵢ⁻¹) (1 – N(ΦSᵢ⁻¹ΦS: ||||₁ → ||||₂)).
where Aₘᵢᵢ(ΦSᵢ⁻¹) := inf {|||ΦSᵢᵀ(z)|||₁ : |z|₂ = 1}.
Then there are the following conclusions:
(1) As the minimizer of problem MPᵢ, y, X* is unique, S-sparse and l₁-column-flat;
(2) Let Y* = ΦS⁻¹(y) in Σ(n, s, R), then for all (i,j) in S:

\[ X*_{ij} ≠ 0 \text{ and } sgn(X*_{ij}) = sgn(Y*_{ij}). \]

This lemma’s proof is in the author’s full version preprint [11]. In applications, the support S of the signal is of course unknown so lemma 3 cannot be applied directly. However, on basis of this lemma a stronger and uniform sufficient condition can be established to guarantee uniqueness and robustness of the signal reconstruction by solving MPᵢ, y, Φ, η.

Theorem Given positive integer s and the linear measurement operator Φ, suppose Φ satisfies the following conditions for any s-sparsity pattern S = S₁ ∪ ... ∪ Sₙ where |Sᵢ| ≤ s for all i:
(1) ΦSᵢᵀ(z) does not have any 0-column for z ≠ 0;
(2) ΦSᵢΦS is a bijection;
(3) N(ΦSᵢ⁻¹(ΦSᵢΦS⁻¹ – Iₙ): l₂ → ||||₂) < η Aₘᵢᵢ(ΦSᵢ⁻¹) (1 – N(ΦSᵢ⁻¹ΦS: ||||₁ → ||||₂)) or equivalently
\[ N((ΦSᵢ⁻¹)ᵀΦSᵢ⁻¹(ΦSᵢ⁻¹ – Iₙ)ΦS: ||||₁ → l₁) < η Aₘᵢᵢ(ΦSᵢ⁻¹) (1 – N(ΦSᵢ⁻¹(ΦSᵢ⁻¹):(||||₁ → ||||₂))). \]
where Aₘᵢᵢ(ΦSᵢ⁻¹) := inf {|||ΦSᵢᵀ(z)|||₁*: |z|₂ = 1} and Iₙ := the identical mapping on Σ(n, s; S).
Then for the minimizer X* of problem MPᵢ, y, Φ, η where y = Φ(X) + e with noise |e|₂ ≤ η and a real flat signal X in Σ(n, s; R) of some s-sparsity pattern R, there are the following conclusions:
(1) Sparsity, flatness and support stability, namely:
X* in Σ(n, s; R) and is l₁-column-flat and the unique minimizer of MPᵢ, y, Φ, η;
(2) Robustness: For any given matrix norm |||₀||| there holds:

\[ |X* – X₀|₀ ≤ 2η N(ΦR⁻¹: l₂ → |||₀). \]

(3) Sign Stability: sgn(X*₀) = sgn(X₀) for (i,j) in R such that:

\[ |X₀_{ij}| > η (N(ΦR⁻¹: l₂ → |||₀) + N(ΦR⁻¹: l₂ → |||₀)  N((ΦR⁻¹(ΦR): ||||₁ → ||||₂))). \]

Proof (1) Note that in case of X in Σ(n, s; R) and y = Φ(X) + e = ΦR(X) + e, |e|₂ ≤ η, one has

\[ ΦRΦR⁻¹(y) – y = (ΦRΦR⁻¹ – Iₙ)e. \]

It’s straightforward to verify that in this situation condition (3) in this theorem leads to condition (3) in lemma 3: sup {<ΦSᵢ⁻¹(ΦSᵢΦS⁻¹(y) – y), H> : ||H||₁ = 1} < η Aₘᵢᵢ(ΦSᵢ⁻¹) (1 – N(ΦR⁻¹ΦR: ||||₁ → ||||₂)) for any η. As a result, X* in Σ(n, s; R) and is l₁-column-flat and the unique minimizer of MPᵢ, y, Φ, η.
(2) For Y* = ΦR⁻¹(y) in Σ(n, s; R) and by lemma 3(4), we obtain |X* – Y*|₀ ≤ η N(ΦR⁻¹: l₂ → |||₀) for any given matrix norm |||₀. On the other hand, Y* = Φᵢ⁻¹(y) implies ΦR⁻¹(ΦR(Y*) – y) = O then condition (1) leads to ΦR(Y*) = y, hence ΦR(Y*) = y = Φ(X) + e = ΦR(X) + e, namely ΦR⁻¹(ΦR(Y*)) = ΦR⁻¹(ΦR(X) + ΦR⁻¹(e),

\[ Y* – X = (ΦR⁻¹(ΦR): l₂ → |||₀) = ΦR⁻¹(e). \]

(10)
Since |e|₂ ≤ η, we get |Y* – X₀|₀ ≤ η N(ΦR⁻¹: l₂ → |||₀) for any given matrix norm |||₀. Combining with |X* – Y*|₀ ≤ η N(ΦR⁻¹: l₂ → |||₀) we get the reconstruction error bound |X* – X₀|₀ ≤ 2η N(ΦR⁻¹: l₂ → |||₀).
(3) By the first-order optimization condition on minimizer X* with the fact sup(X*) = R, we have the equation X* = ΦR⁻¹(y) – γ⁻¹(ΦR⁻¹(ΦR): l₂ → |||₀)(M*) = Y* – γ⁻¹(ΦR⁻¹(ΦR): l₂ → |||₀)(M*) where M* is in c|||X*|||₀, namely:
\[ X^* - Y^* = -\gamma^{*-1}(\Phi_R^T\Phi_R)^{-1}(M^*). \]  

Combining with (10), we get \[ X^* - X = \Phi_R^{*-1}(e) - \gamma^{*-1}(\Phi_R^T\Phi_R)^{-1}(M^*). \]  

Since \( sgn(X^*_{ij}) = sgn(X_{ij}) \) \( iff \) \( |X_{ij} - X^*_{ij}| = |\Phi_R^{*-1}(e) - \gamma^{*-1}(\Phi_R^T\Phi_R)^{-1}(M^*)_{ij}| \), in particular, if \( X_{ij} \) can satisfy \( |X_{ij}| > max_{ij} |\Phi_R^{*-1}(e)_{ij} - \gamma^{*-1}(\Phi_R^T\Phi_R)^{-1}(M^*)_{ij}| \), then the former inequality is true and as a result \( sgn(X^*_{ij}) = sgn(X_{ij}) \). It’s straightforward to verify that the condition (9) just provides a guarantee for this.

5. Summary

The main contribution in this paper is a group of sufficient conditions on the measurement operator to guarantee strong properties of sparse and flat matrix signal reconstruction from noisy measurements, i.e., conditions to guarantee uniqueness, support and sign stability as well as value-error robustness. In comparison with representative current works, our conditions are more general and our method is more suitable to be generalized to dealing with high-order tensor signals. In particular, the results guaranteed by these conditions include the error robustness in terms of any matrix norm, support stability and non-zero components’ sign stability of the reconstructed matrix signal. Further more, the reconstruction error has linear convergence rate which is asymptotically optimal.

This paper is focused on theoretical analysis. In future woks, these results will be applied to design effective algorithms and be generalized to dealing with more challenging and more general problem of tensor signal reconstruction, which is emerging as one of critical problems in wide-band communications and related non-linear signal processing techniques.

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