BOUNDARY PROBLEM FOR LEVI FLAT GRAPHS

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Abstract. In [DTZ2] the authors provided general conditions on a real codimension 2 submanifold $S \subset \mathbb{C}^n$, $n \geq 3$, such that there exists a possibly singular Levi-flat hypersurface $M$ bounded by $S$.

In this paper we consider the case when $S$ is a graph of a smooth function over the boundary of a bounded strongly convex domain $\Omega \subset \mathbb{C}^{n−1} \times \mathbb{R}$ and show that in this case $M$ is necessarily a graph of a smooth function over $\Omega$. In particular, $M$ is non-singular.

1. Introduction

The problem of finding a Levi-flat hypersurface $M \subset \mathbb{C}^n$ with prescribed boundary $S$ (the complex analogue of the real Plateau’s problem), has been extensively studied for $n = 2$ (cf. [Bi, BeG, BeK, Kr, CS, Sh, ShT]). In [DTZ2] (announced in [DTZ1]) we addressed this problem for $n \geq 3$, where the situation is substantially different. In contrast to the case $n = 2$, for $n \geq 3$ the boundary $S$ has to satisfy certain compatibility conditions. Assuming those necessary conditions as well as the existence of complex points, their ellipticity and non-existence of complex subvarieties in $S$, we have constructed in [DTZ2] a (unique but possibly singular) solution to the above problem. An example was also provided in [DTZ2] showing that one may not always expect a smooth solution $M$ in general.

The purpose of this paper is to show that the solution $M$ is smooth if the given boundary has certain “graph form”. More precisely, in the coordinates $(z, u + iv) \in \mathbb{C}^{n−1} \times \mathbb{C}$, we assume that $S$ is the graph of a smooth function $g : b\Omega \to \mathbb{R}_v$, where $b\Omega$ is the smooth boundary of a strongly convex bounded domain $\Omega$ in $\mathbb{C}^{n−1} \times \mathbb{R}_u$ and $S$ satisfies the assumptions of [DTZ2] mentioned above. Let $M$ be the solution given by these theorems. Recall that it is obtained as a projection to $\mathbb{C}^n$ of a Levi-flat subvariety with negligible singularities in $[0, 1] \times \mathbb{C}^n$. Let $q_1, q_2 \in b\Omega$ be the projections of the complex points $p_1, p_2$ of $S$. Using

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a theorem of Shcherbina on the polynomial envelope of a graph in $\mathbb{C}^2$
(cf. [Sh]) we here prove (cf. Theorem 3.1) that
i) the solution $M$ is the graph of a Lipschitz function $f : \Omega \to \mathbb{R}$
with $f|_{b\Omega} = g$ which is smooth on $\Omega \setminus \{q_1, q_2\}$;
ii) $M_0 = \text{graph}(f) \setminus S$ is a Levi flat hypersurface in $\mathbb{C}^n$.
The regularity of $f$ at $q_1$ and $q_2$ remains an interesting open problem
closely related to the work of Kenig and Webster [KW1, KW2].

2. Preliminaries

In this section we collect some facts that will be used in the sequel.

2.1. Remarks about Harvey-Lawson theorem. Let $D$ be a strongly
pseudoconvex bounded domain in $\mathbb{C}^n$, $n \geq 3$, with boundary $bD$,
$\Sigma \subset bD$ a compact connected maximally complex $(2d-1)$-submanifold
with $d > 1$. Then, in view of the theorem of Harvey and Lawson
in [HL1, Theorem 10.4] (see also [HL2]), $\Sigma$ is the boundary of a uniquely
determined relatively compact subset $V \subset \overline{D}$ such that: $\nabla \setminus \Sigma$
is a complex analytic subset of $D$ with finitely many singularities of pure
dimension $d$ and, near $\Sigma$, $\nabla$ is a $d$-dimensional complex manifold with
boundary. We refer to $V = V_2$ as the solution of the boundary problem
corresponding to $\Sigma$. A simple consequence is the following:

Lemma 2.1. Let $D \subset \mathbb{C}^n$ be as above and $\Sigma_1, \Sigma_2$
connected, maximally complex $(2d-1)$-submanifolds of $bD$. Let $V_1, V_2$
be the corresponding solutions of the boundary problem. If $d > 1$, $2d > n$ and $\Sigma_1 \cap \Sigma_2 = \emptyset$,
then $V_1 \cap V_2 = \emptyset$.

Proof. Suppose $V_1 \cap V_2 \neq \emptyset$. Then $2d > n$ implies $\dim V_1 \cap V_2 \geq 1$.
Since $V_1 \cap V_2$ is an analytic subset of $D$, its closure $\overline{V_1 \cap V_2}$
must intersect $bD$ and hence also $\Sigma_1 \cap \Sigma_2 \neq \emptyset$, which contradicts the assumption. ☐

2.2. Known results. First, we have the following: a real 2-codimensional
submanifold $S$ of $\mathbb{C}^n$, $n \geq 3$, which locally bounds a Levi flat
hypersurface must be nowhere minimal near a CR point, i.e. all local
CR orbits must be of positive codimension (cf. [DTZ2 Section 2]).
If $p \in S$ is a complex point, consider local holomorphic coordinates
$(z, w) \in \mathbb{C}^{n-1} \times \mathbb{C}$, vanishing at $p$, such that $S$ is locally given by the equation

$$w = Q(z) + O(|z|^3),$$

where $Q(z)$ is a complex valued quadratic form in the real coordinates
$(\text{Re} z, \text{Im} z) \in \mathbb{R}^{n-1} \times \mathbb{R}^{n-1}$. Observing that not all quadratic forms
$Q$ can appear when $S$ bounds a Levi flat hypersurface one comes to
the condition that \( p \) must be flat, i.e. \( Q(z) \in \mathbb{R} \) in suitable coordinates. A natural stronger condition is that of ellipticity which means by definition that \( Q(z) \in \mathbb{R}_+ \) for every \( z \neq 0 \) in suitable coordinates.

Assume that:

1. \( S \) is compact, connected and nowhere minimal at its CR points;
2. \( S \) has at least one complex point and every such point of is flat and elliptic;
3. \( S \) does not contain complex manifold of dimension \((n - 2)\).

Then in [DTZ2, Proposition 3.1] it was proved that

a) \( S \) is diffeomorphic to the unit sphere with two complex points \( p_1, p_2 \);

b) the CR orbits of \( S \) are topological \((2n - 3)\)-spheres that can be represented as level sets of a smooth function \( \nu : S \to \mathbb{R} \), inducing on \( S_0 = S \setminus \{ p_1, p_2 \} \) a foliation \( \mathcal{F} \) of class \( C^\infty \) with 1-codimensional compact leaves.

Next, by applying a parameter version of Harvey-Lawson’s theorem [HL1, Theorem 8.1], we obtained in [DTZ2, Theorem 1.3] a solution to the boundary problem as follows:

**Theorem 2.2.** Let \( S \subset \mathbb{C}^n \), \( n \geq 3 \) satisfy the above conditions. Then there exist a smooth submanifold \( \tilde{S} \) and a Levi flat \((2n - 1)\)-subvariety \( \tilde{M} \) in \( \mathbb{C}^n \times [0, 1] \) (i.e. \( \tilde{M} \) is Levi flat in \( \mathbb{C}^n \times \mathbb{C} \)) such that \( \tilde{S} = d \tilde{M} \) in the sense of currents and the natural projection \( \pi : \mathbb{C}^n \times [0, 1] \to \mathbb{C}^n \) restricts to a diffeomorphism between \( \tilde{S} \) and \( S \).

As for the singularities of \( \tilde{M} \) we have the following results [DTZ2, Theorems 1.4]:

**Theorem 2.3.** The Levi-flat \((2n - 1)\)-subvariety \( \tilde{M} \) can be chosen with the following properties:

1. \( \tilde{S} \) has two complex points \( \tilde{p}_0 \) and \( \tilde{p}_1 \) with \( \tilde{S} \cap (\mathbb{C}^n \times \{ j \}) = \{ \tilde{p}_j \} \) for \( j = 0, 1 \); every other slice \( \mathbb{C}^n \times \{ x \} \) with \( x \in (0, 1) \), intersects \( \tilde{S} \) transversally along a submanifold diffeomorphic to a sphere that bounds (in the sense of currents) the (possibly singular) irreducible complex-analytic hypersurface \( (\tilde{M} \setminus \tilde{S}) \cap (\mathbb{C}^n \times \{ x \}) \);
2. the singular set \( \text{Sing} \tilde{M} \) is the union of \( \tilde{S} \) and a closed subset of \( \tilde{M} \setminus \tilde{S} \) of Hausdorff dimension at most \( 2n - 3 \); moreover each slice \( (\text{Sing} \tilde{M} \setminus \tilde{S}) \cap (\mathbb{C}^n \times \{ x \}) \) is of Hausdorff dimension at most \( 2n - 4 \);
3. there exists a closed subset \( \tilde{A} \subset \tilde{S} \) of Hausdorff \((2n - 2)\)-dimensional measure zero such that away from \( \tilde{A} \), \( \tilde{M} \) is a smooth
submanifold with boundary $\tilde{S}$ near $\tilde{S}$; moreover $\tilde{A}$ can be chosen such that each slice $\tilde{A} \cap (\mathbb{C}^n \times \{x\})$ is of Hausdorff $(2n - 3)$-dimensional measure zero.

3. The case of graph

From now on we assume that $S \subset \mathbb{C}^n$, $n \geq 3$, is a graph. Consider $\mathbb{C}^n = \mathbb{C}_z^{n-1} \times \mathbb{C}_w$ with complex coordinates $z = (z_1, \ldots, z_{n-1})$ and $w$ where $z_\alpha = x_\alpha + iy_\alpha, 1 \leq \alpha \leq n-1$, $w = u + iv$. We also write $\mathbb{C}^n = (\mathbb{C}_z^{n-1} \times \mathbb{R}_u) \times i\mathbb{R}_v$. Accordingly, a point of $\mathbb{C}^n$ will be denoted by $(z, u, v) = (z, u + iv)$.

Let $\Omega$ be a bounded strongly convex domain of $\mathbb{C}_z^{n-1} \times \mathbb{R}_u$ with smooth boundary $b\Omega$. By strong convexity here we mean that the second fundamental form of the boundary $b\Omega$ of $\Omega$ is everywhere positive definite. In particular, $\Omega \times i\mathbb{R}_v$ is a strongly pseudoconvex domain in $\mathbb{C}^n$.

Let $g : b\Omega \to \mathbb{R}_v$ be a smooth function, and $S \subset \mathbb{C}^n$ the graph of $g$. We assume that $S$ satisfies the conditions of [DTZ2, Theorem 1.3] and denote $q_1, q_2 \in b\Omega$ the natural projections of the complex points $p_1, p_2$ of $S$, respectively.

Our goal is to prove the following:

**Theorem 3.1.** Let $q_1, q_2 \in b\Omega$ be the projections of the complex points $p_1, p_2$ of $S$, respectively. Then, there exists a Lipschitz function $f : \Omega \to \mathbb{R}_v$ which is smooth on $\Omega \setminus \{q_1, q_2\}$ and such that $f|_{b\Omega} = g$ and $M_0 = \text{graph}(f) \setminus S$ is a Levi flat hypersurface of $\mathbb{C}^n$. Moreover, each complex leaf of $M_0$ is the graph of a holomorphic function $\phi : \Omega' \to \mathbb{C}$ where $\Omega' \subset \mathbb{C}^{n-1}$ is a domain with smooth boundary (that depends on the leaf) and $\phi$ is smooth on $\Omega'$.

The natural candidate to be the graph $M$ of $f$ is $\pi(\tilde{M})$ where $\tilde{M}$ and $\pi$ are as in Theorem 2.2. We prove that this is the case proceeding in several steps.

3.1. Behaviour near $S$. Set $m_1 = \min_S g$, $m_2 = \max_S g$ and $r \gg 0$ such that

$$D = \Omega \times [m_1, m_2] \subset \mathbb{B}(r) \cap (\Omega \times i\mathbb{R}_v)$$

where $\mathbb{B}(r)$ is the ball $\{(z, w) | |(z, w)| < r\}$.

Let $\Sigma$ be a CR-orbit of the foliation of $S \setminus \{p_1, p_2\}$. Then, $\Sigma$ is a compact maximally complex $(2n-3)$-dimensional real submanifold of $\mathbb{C}^n$, which is contained in the boundary of the strongly pseudoconvex domain $\Omega \times i\mathbb{R}_v$ of $\mathbb{C}^n$. Let $V$ be the solution to the boundary problem corresponding to $\Sigma$, i.e. the complex-analytic subvariety of $\Omega \times i\mathbb{R}_v$
bounded by Σ. We refer to \( V \) as the leaf bounded by Σ. From Theorems 2.2 and 2.3 it follows that \( V \) is obtained as projection \( \pi(\tilde{V}) \), where \( \tilde{V} = (M \setminus S) \cap (\mathbb{C}^n \times \{x\}) \) for suitable \( x \in (0,1) \). In particular, if \( M \) denotes \( \pi(\tilde{M}) \), \( \pi|_V \) defines a biholomorphism \( \tilde{V} \simeq V \) and \( M \setminus S \subset D \).

Now let \( \Sigma_1 \) and \( \Sigma_2 \) be two distinct CR orbits of the foliation of \( S \setminus \{p_1, p_2\} \), and let \( \nabla_1, \nabla_2 \) be the corresponding leaves bounded by them. Then \( \nabla_1, \nabla_2 \) do not intersect by Lemma 2.1.

**Remark 3.1.** In the previous discussion, we only employed the fact that \( \Omega \times \mathbb{R}_v \) is a strongly pseudoconvex domain and \( S \) is contained in its boundary, without regarding the graph nature of \( S \). It can happen that the leaves have isolated singularities. We shall show that this cannot happen in our case.

**Lemma 3.2.** Let \( p \in S \) be a CR point. Then, near \( p \), \( M \) is the graph of a function \( \phi \) on a domain \( U \subset \mathbb{C}^{n-1}_x \times \mathbb{R}_u \), which is smooth up to the boundary of \( U \).

**Proof.** Near \( p \), \( S \) is foliated by local CR orbits. As a consequence of Theorem 2.2, each local CR orbit extends to a compact global CR orbit \( \Sigma \) that bounds a complex codimension 1 subvariety \( V_\Sigma \subset \Omega \times i\mathbb{R}_v \). Furthermore, near \( p \), each \( \Sigma \) is smooth and can be represented as the graph of a CR function over a strongly pseudoconvex hypersurface and \( V_\Sigma \) as the graph of the local holomorphic extension of this function. It follows from the Hopf Lemma that \( V \) is transversal to the strongly pseudoconvex hypersurface \( b\Omega \times i\mathbb{R}_v \) near \( p \). Hence the family of \( V_\Sigma \) near \( p \) forms a smooth real hypersurface with boundary on \( S \) that can be seen as the graph of a smooth function \( \phi \) from a relative open neighbourhood \( U \) of \( p \) in \( \Omega \) into \( \mathbb{R}_v \). Finally, Lemma 2.1 guarantees that this family does not intersect any other leaf \( V \) from \( M \). This completes the proof. \( \Box \)

**Corollary 3.3.** If \( p \in S \) is a CR point, each complex leaf \( V \) of \( M \), near \( p \), is the graph of a holomorphic function on a domain \( \Omega_V \subset \mathbb{C}^{n-1}_x \), which is smooth up to the boundary of \( \Omega_V \).

**Proof.** Since \( M \) is the graph of a smooth function near \( p \), its tangent space at every point near \( p \) is transversal to \( i\mathbb{R}_v \). Hence the complex tangent space of \( M \) at every point near \( p \) is transversal to \( \mathbb{C}_w \). Since the tangent spaces of the complex leaves of \( M \) coincide with the complex tangent spaces of \( M \), it follows that each leaf \( V \) projects immersively to \( \mathbb{C}^{n-1}_x \) and the conclusion follows. \( \Box \)
3.2. \textit{M is the graph of a Lipschitz function.} Assume as before that \( \Omega \) is strongly convex. We have the following

**Proposition 3.4.** \textit{M is the graph of a Lipschitz function} \( f : \overline{\Omega} \to \mathbb{R}_u \).

**Proof.** We fix a nonzero vector \( a \in \mathbb{C}^{n-1} \) and for a given point \((\zeta, \xi) \in \Omega\) denote by \( H(\zeta, \xi) \subset \mathbb{C}^{n-1} \times \{\xi\} \) the complex line through \((\zeta, \xi)\) in the direction of \((a, 0)\). Furthermore, we set

\[
L(\zeta, \xi) = H(\zeta, \xi) + \mathbb{R}(0, 1), \quad \Omega(\zeta, \xi) = L(\zeta, \xi) \cap \Omega, \quad S(\zeta, \xi) = (H(\zeta, \xi) + \mathbb{C}(0, 1)) \cap S
\]

Then \( S(\zeta, \xi) \) is contained in the strongly convex cylinder

\[
(H(\zeta, \xi) + \mathbb{C}(0, 1)) \cap (b\Omega \times i\mathbb{R}_u)
\]

over \( H(\zeta, \xi) + \mathbb{C}(0, 1) \simeq \mathbb{C}^2 \) and it is the graph of \( g|_{b\Omega(\zeta, \xi)} \).

Since \( \Omega(\zeta, \xi) = \Omega \cap L(\zeta, \xi) \), in view of the main theorem of [Sh], the polynomial hull \( \hat{S}(\zeta, \xi) \) of \( S(\zeta, \xi) \) is a continuous graph over \( \overline{\Omega}(\zeta, \xi) \). Consider \( M = \pi(\hat{M}) \) and set

\[
M(\zeta, \xi) = (H(\zeta, \xi) + \mathbb{C}(0, 1)) \cap M.
\]

Since \( M \) is a union of irreducible analytic subvarieties of codimension 1 in \( \mathbb{C}^n \) with boundary in the graph \( S \), each intersection \( M(\zeta, \xi) \) is the union of a family \( \mathcal{A} \) of 1-dimensional analytic subsets. Clearly, the boundary of a connected component of any such analytic set is contained in \( S(\zeta, \xi) \). It follows that \( M(\zeta, \xi) \) is contained in the polynomial hull \( \hat{S}(\zeta, \xi) \) of \( S(\zeta, \xi) \). In view of the main theorem of Shcherbina [Sh], \( \hat{S}(\zeta, \xi) \) is a graph over \( \overline{\Omega}(\zeta, \xi) = \overline{\Omega} \cap L(\zeta, \xi) \), foliated by analytic discs, so \( M(\zeta, \xi) \) is a graph over a subset \( U \) of \( \overline{\Omega}(\zeta, \xi) \).

On the other hand, every analytic disc \( \Delta \) of \( \hat{S}(\zeta, \xi) \) has its boundary on \( S(\zeta, \xi) \subset S \). Since all elliptic complex points are isolated, the boundary of \( \Delta \) contains a CR point \( p \) of \( S \). In view of Lemma 3.2 near \( p \), \( M(\zeta, \xi) \) is also a graph over \( \Omega(\zeta, \xi) \). Thus, near \( p \), we must have \( M(\zeta, \xi) = \hat{S}(\zeta, \xi) \).

In particular, near \( p \), \( \Delta \) is contained in \( M(\zeta, \xi) \), and therefore in a leaf \( V_\Sigma \) of \( M \). Since \( V_\Sigma \) is a closed analytic subset in \( \mathbb{C}^n \setminus S \), the whole disc \( \Delta \) is contained in \( V_\Sigma \) and hence in \( M \). Moreover, \( \Delta \subset H(\zeta, \xi) + \mathbb{C}(0, 1) \) thus we conclude that \( \Delta \subset M(\zeta, \xi) \). Therefore, every analytic disc of \( \hat{S}(\zeta, \xi) \) is contained in \( M(\zeta, \xi) \), consequently \( M(\zeta, \xi) \) and \( \hat{S}(\zeta, \xi) \) coincide. It follows that \( M \) is the graph of a function \( f : \overline{\Omega} \to \mathbb{R}_u \).

Let us prove that \( f \) is a continuous function. Choose \((\zeta, \xi) \in \Omega\) and a complex line \( H(\zeta, \xi) \) as before. Consider a neighborhood \( U \) of \((\zeta, \xi) \) in \( \mathbb{C}^{n-1} \times \mathbb{R}_u \). For \( q \in U \), let \( H_q \) be the translated of \( H(\zeta, \xi) \) which passes through \( q \). With the notation corresponding to the one employed above, we can state the following. For a small enough neighborhood
V \subset U of p in \mathbb{C}_z^{n-1} \times \mathbb{R}_u$, let \( \hat{S}_q \) be the polynomial hull of \( S_q \) in \( H_q + \mathbb{C}(0,1) \), and let
\[ S_U = \bigcup_{q \in U} \hat{S}_q; \]
then \( S_U \) is the graph of a continuous function. Indeed let \( \bar{q} \) be a point in \( V \), and let \( \{q_m\}_{m \in \mathbb{N}} \) be a sequence of points such that \( q_m \to \bar{q} \). Then, obviously, the sets \( \hat{S}_q_m \) converge to the set \( \hat{S}_\bar{q} \) in the Hausdorff metric as \( n \to \infty \). Moreover, it is also clear that \( \hat{\Omega}_q_m \to \hat{\Omega}_{\bar{q}} \) for \( n \to \infty \). Then, by [Shi Lemma 2.4] it follows that \( \hat{S}_q_m \to \hat{S}_\bar{q} \) as \( m \to \infty \). Since every \( \hat{S}_q \) is a continuous graph, this allows to prove easily that \( S_U \) is a continuous graph as a whole.

Thus, \( f \) is continuous on \( \Omega \), whence on \( \overline{\Omega} \setminus \{q_1, q_2\} \) in view of Lemma 3.2. Continuity at \( q_1 \) is proved as follows. Let \( \{a_m\}_{m \in \mathbb{N}} \subset \Omega \) be a sequence of points which converges to \( q_1 \). Each point \( (a_m, f(a_m)) \) belongs to a complex leaf \( V_{\Sigma_m} \) of \( M \) which is bounded by a compact CR orbit \( \Sigma_m \) of the foliation of \( S \setminus \{p_1, p_2\} \) (cf. Section 2). By the maximum principle, for every \( m \in \mathbb{N} \) there exists a point \( (b_m, g(b_m)) \) in \( \Sigma_m \) such that
\[ |(q_1, g(q_1)) - (a_m, f(a_m))| \leq |(q_1, g(q_1)) - (b_m, g(b_m))|. \]
We claim that
\[ |(q_1, g(q_1)) - (b_m, g(b_m))| \to 0 \]
as \( m \to \infty \). If not there exists an open \( B = B(q_1, r) \cap \mathbb{R}_u \) centered at \( q_1 \) such that \( b_m \notin \overline{B} \) for all \( m \). It follows that
\[ \Sigma_m \cap \pi^{-1}(\overline{B}) = \emptyset \]
for all \( m \) and
\[ V_{\Sigma_m} \cap \pi^{-1}(B) \neq \emptyset \]
for \( m \gg 0 \). This violates the Kontinuitätsatz since \( \Omega \times i\mathbb{R}_u \) is a domain of holomorphy.

Continuity at \( q_2 \) is proved in a similar way.

Thus \( f \) is continuous on \( \overline{\Omega} \) and smooth near \( b\Omega \setminus \{q_1, q_2\} \).

In order to show that \( f \) is Lipschitz we now observe that, as it is easily proved, \( f_{\Omega} \) is a weak solution of the Levi-Monge-Ampère operator defined in [SIT] with smooth boundary value, so, in view of [SIT] Theorems 2.4, 4.4, 4.6], it is Lipschitz. This concludes the proof of Proposition 3.4.

\[ \square \]

**Remark 3.2.** \( M \) is the envelope of holomorphy of \( S \).
3.3. Regularity. In order to prove that $M \setminus \{p_1, p_2\}$ is a smooth manifold with boundary we need the following:

**Lemma 3.5.** Let $U$ be a domain in $\mathbb{C}^{n-1}_z \times \mathbb{R}_u$, $n \geq 2$, $f : U \to \mathbb{R}_v$ a continuous function. Let $A \subset \text{graph}(f)$ be a germ of complex analytic set of codimension 1. Then $A$ is a germ of a complex manifold, which is a graph over $\mathbb{C}^{n-1}_z$.

**Proof.** The idea of the proof (here is slightly modified) is due to Jean-Marie Lion cfr. [L].

Let us denote by $z_1, \ldots, z_{n-1}, w = u + iv$, the complex coordinates in $\mathbb{C}^{n-1}_z \times \mathbb{C}_w$. We may suppose that $A$ is a germ at 0. Let $h \in \mathcal{O}_{n+1}$ be a non identically vanishing germ of holomorphic function such that $A = \{h = 0\}$. Let $D_\varepsilon$ be the disc $\{z = 0\} \cap \{|w| < \varepsilon\}$. Then, for $\varepsilon << 1$, we have either $A \cap D_\varepsilon = \{0\}$ or $A \cap D_\varepsilon = D_\varepsilon$. The latter is not possible since $D_\varepsilon$ is not contained in any graph over $\mathbb{C}^{n-1}_z \times \mathbb{R}_u$. It follows that $A \cap D_\varepsilon = \{0\}$, i.e. $A$ is $w$-regular. Let us denote by $\pi$ the projection $\mathbb{C}^n_z \to \mathbb{C}^{n-1}_z$. Then, by the local parametrization theorem for analytic sets there exists $d \in \mathbb{N}$ such that

- for some neighborhood $U$ of 0 in $\mathbb{C}^{n-1}_z$, there exists an analytic set $\Delta \subset U$ such that $A_\Delta = A \cap ((U \setminus \Delta) \times D_\varepsilon)$ is a manifold;
- $\pi : A_\Delta \to U \setminus \Delta$ is a $d$-sheeted covering.

We claim that the covering $\pi : H_\Delta \to U \setminus \Delta$ is trivial. Otherwise, there would exist a closed loop $\gamma : [0, 1] \to U \setminus \Delta$ whose lift $\tilde{\gamma}$ to $A_\Delta$ is not closed. We extend $\gamma$ to $\mathbb{R}$ by periodicity and extend $\tilde{\gamma}$ to $\mathbb{R}$ as lift of $\gamma$. Define $\alpha = u \circ \tilde{\gamma} = u \circ \gamma$, $\beta = v \circ \tilde{\gamma}$. Since $\alpha$ is continuous and bounded, there exists $\theta \in \mathbb{R}$ such that $\alpha(\theta) = \alpha(\theta + 1)$. But then $\beta(\theta) = \beta(\theta + 1)$ since by the assumption, $\beta(\theta) = f(\alpha(\theta), \alpha(\theta))$. Hence $\tilde{\gamma}(\theta) = \tilde{\gamma}(\theta + 1)$, a contradiction with the assumption that $\tilde{\gamma}$ is not closed.

Since $\pi : A_\Delta \to U \setminus \Delta$ is a trivial covering, we may define $d$ holomorphic functions $\tau_1, \ldots, \tau_d : U \setminus \Delta \to \mathbb{C}$ such that $A_\Delta$ is a union of the graphs of the $\tau_j$’s. By Riemann’s extension theorem, the functions $\tau_j$ extend as holomorphic functions $\tau_j \in \mathcal{O}(U)$. The desired conclusion will follow from the fact that all the $\tau_j$ coincide. Indeed, suppose, by contradiction, $\tau_1 \neq \tau_2$; then for some disc $D \subset U$ centered at 0 we have $\tau_1|_D \neq \tau_2|_D$ and then, after shrinking $D$, $(\tau_1 - \tau_2)|_D$ vanishes only at 0. But, by virtue of the hypothesis, $\{\text{Re}(\tau_1 - \tau_2) = 0\} \subset \{\tau_1 - \tau_2 = 0\} = \{0\}$, when restricted to $D$. The latter is not possible since $(\tau_1 - \tau_2)|_D \neq 0$ is holomorphic and thus an open map (whose image must include a segment of the imaginary axis). $\square$
**Proof of Theorem 3.1** Consider the foliation on $S \setminus \{p_1, p_2\}$ given by the level sets of the smooth function $\nu: S \to [0, 1]$ as in Section 2 and set $L_t = \{\nu = t\}$ for $t \in (0, 1)$. Let $V_t \subset \overline{\Omega} \times i\mathbb{R}_v \subset \mathbb{C}^n$ be the complex leaf of $M$ bounded by $L_t$ and $\pi: \mathbb{C}^{n-1} \times \mathbb{C}_w \to \mathbb{C}^{n-1}$ denote the natural projection. We have:

- by Proposition 3.4, $M$ is the graph of a continuous function over $\Omega$ and by Lemma 3.5, each leaf $V_t$ is a complex hypersurface and $\pi|_{V_t}$ is a submersion.
- Since $\Omega$ is strongly convex, an argument completely analogous to that of [Sh, Lemma 3.2] shows that $\pi|_{V_t}$ is one-to-one, then, by Corollary 3.3, $\pi$ sends $V_t$ onto a domain $\Omega_t \subset \mathbb{C}_z^{n-1}$ with smooth boundary.

If 

$$
\pi_u: (\mathbb{C}_z^{n-1} \times \mathbb{R}_u) \times i\mathbb{R}_v \to \mathbb{R}_u,
$$

$$
\pi_v: (\mathbb{C}_z^{n-1} \times \mathbb{R}_u) \times i\mathbb{R}_v \to \mathbb{R}_v
$$

denote the natural projections then $\pi_u|_{L_t} = a_t \circ \pi|_{L_t}$ and $\pi_v|_{L_t} = b_t \circ \pi|_{L_t}$, where $a_t$ and $b_t$ are smooth functions in $\partial \Omega_t$. Furthermore, the boundary $\partial \Omega_t$ and $a_t$, $b_t$ depend smoothly on $t$ for $t \in (0, 1)$. The latter property means that one has a local parametrization of $\partial \Omega_t$ smoothly depending on $t$ and such that the functions $a_t$, $b_t$ also depend smoothly on $t$ when composed with this parametrization. It follows that

- if $(z_t, w_t) \in M$, then $w_t = u_t + iv_t$ is varying in $V_t$, so $u_t + iv_t$ is the holomorphic extension to $\Omega_t$ of $a_t + ib_t$. In particular, $u_t$ and $v_t$ are smooth functions in $(z, t)$, e.g. as a consequence of the Martinelli-Bochner formula.
- The derivative $\partial u_t/\partial t$ is defined and harmonic in $\Omega_t$ for each $t$, and has a smooth extension to the boundary $\partial \Omega_t$. Moreover, it follows from Lemma 3.2 and Corollary 3.3 that $\partial u_t/\partial t$ does not vanish on $\partial \Omega_t$. Since the CR orbits $L_t$ are connected in view of Theorem 2.2, the boundary $\partial \Omega_t$ is also connected and hence $\partial u_t/\partial t$ has constant sign on $\partial \Omega_t$. Then, by the maximum principle, $\partial u_t/\partial t$ has constant sign in $\Omega_t$ and, in particular, does not vanish. The latter implies the $M \setminus S$ is the graph of a smooth function over $\Omega$, which extends smoothly to $\overline{\Omega} \setminus \{q_1, q_2\}$.
- It furthermore follows from Proposition 3.4 that $M$ is the graph of a Lipschitz function over $\overline{\Omega}$. This completes the proof of Theorem 3.1.
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