We study the $Q^2$ variation of the first moment of the nucleon’s spin-dependent structure function $G_1$. As $Q^2 \to 0$ the moment is determined by the low energy theorem for Compton scattering. In the deep-inelastic region the moment is calculated using twist expansion to order $1/Q^2$. Based on these limits, we construct a formula which smoothly interpolates between the two regions.
Recently, polarized deep-inelastic scattering has proven to be an excellent tool for studying the spin structure physics of the nucleon \[1-3\]. Supplemented with the operator product expansion analysis in Quantum Chromodynamics (QCD), experimental data at high-energy provides a direct measurement of the matrix elements of spin-dependent operators in the nucleon. A much discussed example in the current literature is the axial charge, or the forward matrix element of axial current, whose measurement by the EMC collaboration casts doubt on our traditional understanding of the proton’s spin structure \[1\].

A closely related question is can one learn anything about the nucleon’s spin structure from electro-production experiments away from the deep-inelastic limit? In particular, what insight do the spin structure functions \(G_1\) and \(G_2\) provide at low and moderate \(Q^2\)? Not long ago, Anselmino et al. \[4\] pointed out that at the real photon point \((Q^2 = 0)\), the first moment of \(G_1\) (called the sum rule in the following text) is related, via the celebrated Drell-Hearn-Gerasimov (DHG) sum rule \[5\], to the anomalous magnetic moment of the nucleon, and thus the physics of the \(G_1\) structure function again appears simple in the \(Q^2 \to 0\) limit. Together with knowledge from the deep-inelastic limit, the authors in Ref. \[4\] constructed a model for the sum rule at all \(Q^2\). This has motivated a number of proposals to measure \(G_1\) and \(G_2\) at low energy \[6,7\].

In Ref. \[8\], one of us pointed out that the analysis made in Ref. \[4\] excluded the nucleon’s elastic contribution to the moment, which in the \(Q^2 \to 0\) limit dominates the entire inelastic contribution calculated from the DHG sum rule. He argued that the moment has to include this contribution if it is to be analyzed in twist expansion in the deep-inelastic limit and its experimental measurement is to be used to extract the matrix elements of higher-twist operators.

In this Letter we study the \(Q^2\) variation of the sum rule by exploring the physics of the \(Q^2 \to 0\) and \(Q^2 \to \infty\) limits. In the first limit, we rely on the low energy result derived in Ref. \[8\], to calculate the exact value and the first derivative of the sum rule at \(Q^2 = 0\). In the second limit, we use a twist expansion appropriate for the deep inelastic region, focusing on the \(1/Q^2\) correction term. The matrix elements of higher twist operators are related to moments of the quark distributions functions \(g_{i1}(x)(i = 1, T, 3\text{ and } f = u, d, s...)\) through a novel use of the QCD equations of motion, which are in turn evaluated in the MIT bag model. As an application, we discuss the correction of higher twists to the Bjorken sum rule in the deep inelastic limit. Having obtained analytical results valid for the low and high ranges of \(Q^2\), we construct a simple parameterization to smoothly interpolate both limits, which should be checked experimentally.

To begin we consider the following fixed-mass sum rule,

\[
\Gamma(Q^2) = \frac{Q^2}{2M^2} \int_{Q^2/2}^{\infty} G_1(\nu, Q^2) \frac{d\nu}{\nu},
\]

where \(G_1(\nu, Q^2)\) is one of the nucleon’s spin dependent structure functions in the nucleon tensor,

\[
W_{\mu\nu} = -ie_{\mu\alpha\beta}q^\alpha \left[ S^\beta G_1 \frac{1}{M^2} + G_2 \frac{1}{M^4}(\nu S^3 - P^3 (S \cdot q)) \right].
\]

The lower integration limit in Eq. \(1\) implies the elastic contribution to \(G_1\) is also included. Here \(P\) and \(S\) are the nucleon’s momentum and polarization, \(q\) is the virtual photon momentum, \(M\) is the nucleon mass, \(\nu = P \cdot q\) and \(Q^2 = -q^2 (e^{0123} = 1)\). In deep-inelastic
limit, one defines scaling functions \( g_1(x, Q^2) = \nu/M^2 G_1 \) and \( g_2(x, Q^2) = (\nu/M^2)^2 G_2 \). The sum rule then becomes,

\[
\Gamma(Q^2) = \int_0^1 g_1(x, Q^2) dx, \tag{3}
\]

which is just the first moment of the scaling function.

Let us first consider the small \( Q^2 \) behavior of the sum rule. Introduce a spin-dependent virtual-photon Compton amplitude \( S_1(\nu, Q^2) \) whose imaginary part is proportional to \( G_1 \), and write down the unsubtracted dispersion relation,

\[
S_1(\nu, Q^2) = 4 \int_{Q^2/2}^{\infty} \frac{\nu' d\nu'}{\nu'^2 - \nu^2} G_1(\nu', Q^2). \tag{4}
\]

Through this, we relate the sum rule to the Compton amplitude at \( \nu = 0 \),

\[
\Gamma(Q^2) = \frac{Q^2}{8M^2} S_1(0, Q^2). \tag{5}
\]

At small \( \nu \) and \( Q^2 \), the dominant contribution to \( S_1 \) comes from the nucleon pole diagrams \[8\],

\[
S_1^{\text{pole}}(\nu, Q^2) = -2M^2 F_1(F_1 + F_2)[\frac{1}{2\nu - Q^2} - \frac{1}{2\nu + Q^2}] - F_2^2, \tag{6}
\]

where \( F_1 \) and \( F_2 \) are the Dirac and Pauli form factors of the nucleon. From this, we obtain for \( Q^2 \to 0 \),

\[
\Gamma(Q^2) = \frac{1}{2} F_1(F_1 + F_2) - \frac{1}{8M^2} F_2^2 Q^2. \tag{7}
\]

This result can be shown to be accurate up to the order of \( Q^2 \) in the small \( Q^2 \) region by explicitly evaluating Eq. \[8\]: The elastic contribution to \( G_1 \) is proportional to \( \delta(2\nu - Q^2) \), producing the first term in Eq. \[8\]; the integral over inelastic contributions is just the DHG sum rule in the limit of \( Q^2 \to 0 \) and the second term in Eq. \[7\] reproduces this in the same limit.

The elastic contribution vanishes identically at \( Q^2 = 0 \) because of energy-momentum conservation, and the DHG sum rule indicates \( \Gamma(0) = 0 \). Thus, due to the elastic contribution, \( \Gamma(Q^2) \) is non-analytic around \( Q^2 = 0 \), i.e.,

\[
\Gamma(Q^2 = 0) \neq \Gamma(Q^2 \to 0). \tag{8}
\]

To remedy this, one can take two approaches: The first approach subtracts away the elastic contribution from the sum rule for \( Q^2 \neq 0 \). The new sum, \( \tilde{\Gamma}(Q^2) = \Gamma(Q^2) - 1/2 F_1(F_1 + F_2) \), is a smooth extension of the DHG sum rule to virtual-photon scattering. The approach we take in this paper is to redefine \( \Gamma \) at \( Q^2 = 0 \),

\[
\Gamma(Q^2 = 0) \equiv \Gamma(Q^2 \to 0). \tag{9}
\]

This approach ensures that the sum rule at low \( Q^2 \) can be treated with the twist expansion that we will discuss below. The expansion is for moments of the \( g_1 \) structure function which include the integration limit \( x = 1 \), where the elastic contribution resides.
Since Eq. (7) is accurate up to the order of $Q^2$, we can determine $\Gamma(Q^2)$ and its first derivative in $Q^2 \to 0$ limit,

\[
\begin{align*}
\Gamma^{p}(0) &= 1.396, \\
\Gamma^{n}(0) &= 0, \\
\frac{d\Gamma^{p}(Q^2)}{dQ^2}|_{Q^2=0} &= -8.631 \text{GeV}^{-2}, \\
\frac{d\Gamma^{n}(Q^2)}{dQ^2}|_{Q^2=0} &= -0.479 \text{GeV}^{-2},
\end{align*}
\]

where $p, n$ refer to proton or neutron and the squares of the proton and neutron charge radii $\langle r^2_p\rangle_{\text{c.r.}} = (0.862 \text{ fm})^2$ and $\langle r^2_n\rangle_{\text{c.r.}} = -(0.342 \text{ fm})^2$ have been used. The initial slope of $\Gamma^{p}(Q^2)$ is primarily determined by the elastic contribution as the inelastic contribution, $-\kappa_p^2/8M^2 = -0.455 \text{GeV}^{-2}$, is only about 5% of the total. Therefore, one expects that for small $Q^2$, $\Gamma^{p}(Q^2)$ is mainly given by the elastic contribution. In contrast, due to a numerical coincidence, the elastic part of $\Gamma^{n}(Q^2)$ is negligible compared with the inelastic part.

In the limit of large $Q^2$ ($Q^2 \gg \Lambda^2_{\text{QCD}}$), $\Gamma(Q^2)$ can be calculated in terms of the twist expansion,

\[
\Gamma(Q^2) = \sum_{\tau=2,4,\ldots} \frac{\mu_{\tau}(Q^2)}{(Q^2)^{\frac{3}{2}}},
\]

where $\mu_{\tau}(Q^2)$ are matrix elements of quark-gluon operators which scale like $\Lambda^2_{\text{QCD}}$. The $Q^2$-dependence in $\mu_{\tau}$ are logarithmic and can be calculated in perturbative QCD. If the nucleon mass were zero, $\mu_{\tau}(Q^2)$ would contain only twist-$\tau$ operators. The effect of the nucleon mass is to induce contributions to $\mu_{\tau}(Q^2)$ from lower twist operators, as we shall illustrate below.

The leading term in Eq. (11) is well-known,

\[
\mu_2 = \frac{1}{2} \sum_{f=u,d,s\ldots} e_f^2 a_{0f},
\]

where the summation covers quarks of all flavors $f$ and $a_{0f}$ is the axial charge defined by the matrix element of axial current $A^\mu_f = \bar{\psi}_f \gamma^\mu \gamma^5 \psi_f$; $\langle PS|A^\mu_f|PS\rangle = 2a_{0f} S^u$. The QCD radiative corrections have been calculated to the first order in $\alpha_s(Q^2)$ [9] for the singlet contribution ($a_0^S = 2(a_{0u} + a_{0d} + a_{0s})/9$) and to the third order [10] for the non-singlet contribution ($a_0^{NS} = (2a_{0u} - a_{0d} - a_{0s})/9$). The proton-neutron difference of the moment defines the Bjorken sum rule,

\[
\mu_2^p(Q^2) - \mu_2^n(Q^2) = \frac{g_A}{6} \left[ 1 - \frac{\alpha_s(Q^2)}{\pi} - 3.58 \left( \frac{\alpha_s(Q^2)}{\pi} \right)^2 - 20.2 \left( \frac{\alpha_s(Q^2)}{\pi} \right)^3 + \ldots \right],
\]

where $g_A = a_{0u} - a_{0d} = 1.257$ is the neutron decay constant.

The 1/$Q^2$ power corrections to $\Gamma$ were first studied by Shuryak and Vainshtein (SV) [11]. Using the collinear expansion technique [12], one of us has calculated in Ref. [13] the entire 1/$Q^2$ corrections to the $g_1$ scaling function in terms of a few multi-parton distribution functions. Specializing to the first moment, we find
\[
\mu_4 = \frac{1}{9} \sum_f e_f^2 [a_{2f} + d_{2f} - 4f_{2f}] M^2 \\
= (A + D + F) M^2.
\] (14)

where \( A = \frac{1}{9} \sum_f e_f^2 a_{2f} \) comes from the twist-two contribution and \( a_{2f} \) is defined as,

\[
\langle PS|\bar{\psi}_f \gamma^{[\sigma_5 i} D^{\mu_1} i D^{\mu_2}] \psi_f | PS \rangle = 2a_{2f} S^{[\sigma P^{\mu_1} P^{\mu_2]},
\] (15)

with \((\cdots)\) denotes symmetrizing the indices and subtracting the trace; \( D = \frac{1}{9} \sum_f e_f^2 d_{2f} \) comes from the twist-three contribution and \( d_{2f} \) is defined as,

\[
\langle PS|g_{\bar{\psi}} F^{\sigma(\mu_1, \mu_2)} | PS \rangle = 2d_{2f} S^{[\sigma P^{\mu_1} P^{\mu_2]},
\] (16)

with \([\cdots]\) denotes anti-symmetrizing the indices and \( F^{\sigma\mu_1} = \frac{1}{2} \epsilon^{\sigma\mu_1\alpha\beta} F_{\alpha\beta} \) is the dual of the gluon field tensor; \( F = \frac{1}{9} \sum_f e_f^2 f_{2f} \) comes from the twist-four contribution and \( f_{2f} \) is defined as,

\[
\langle PS|g_{\bar{\psi}} F^{\mu_\alpha \gamma_{\nu}} | PS \rangle = 2f_{2f} M^2 S^\mu.
\] (17)

We note that the result quoted for \( \mu_4 \) in Ref. [11] is \((2A + 2D + F) M^2\).

To study the QCD radiative corrections to \( \mu_4 \), one has to consider operator-mixing from gluon operators, the anomalous dimensions of which are not currently available and their matrix elements are difficult to estimate. Therefore, in the following discussion, we neglect entirely the scale dependence of \( \mu_4 \).

The higher twist operators in Eqs. (16) and (17) depend explicitly on gauge fields. To calculate their matrix elements we need a wave function of the nucleon containing gluon components. However, for special types of higher twist operators such as the present case, we can eliminate the gluons in terms of the “bad” components of quark fields using the QCD equations of motion [13]. Then the higher twist matrix elements can be related to moments of parton distributions with no explicit gluon fields. Indeed, by defining in the light-cone gauge \((A \cdot n = 0)\),

\[
g_{(1,T,3)f}(x) = \frac{1}{2} \int \frac{d\lambda}{2\pi} e^{i\lambda x} \langle PS|\bar{\psi}_f Q_{(1,T,3)} \gamma_5 \psi_f(\lambda n)| PS \rangle,
\] (18)

where \( Q_1 = n, Q_T = -S_T/M, Q_3 = -2p/M^2 \) and \( n \) and \( p \) are two null vectors \((n^2 = p^2 = 0 \text{ and } p \cdot n = 1)\), we find,

\[
d_{2f} = \frac{1}{2} \int x^2 (3g_{2f}(x) + 2g_{1f}(x)) dx,
\]

\[
f_{2f} = \frac{1}{2} \int x^2 (7g_{1f}(x) + 12g_{2f}(x) - 9g_{3f}(x)) dx,
\] (19)

where \( g_{2f} = g_{Tf} - g_{1f} \). These relations are exact in QCD.

We choose to estimate the \( 1/Q^2 \) corrections to the sum rule in the simplest version of the MIT bag model, in which the bag boundary simulates gluon confinement [14,15]. Using \( g_{if}(x) \) \((i = 1, T, 3)\) calculated in this model, we obtain for the proton, \( A^p = 0.0065, D^p = 0.0092, \) and \( F^p = 0.0155 \). Inserting them into Eq. (14), we have,
\[
\mu_n^4(\text{Bag}) = 0.031 M^2. 
\] (20)

Compared with the size of \( \mu_p^2 = 0.126 \pm 0.025 \) from the EMC data or 0.175 from the Ellis-Jaffe sum rule, the bag \( 1/Q^2 \) power correction is about 10\% at \( Q^2 = 2 \text{ GeV}^2 \) and about 2\% at \( Q^2 = 10 \text{ GeV}^2 \). Assuming there are no abnormal twist-six or higher contributions, we conclude that most \( Q^2 \) variations of the proton sum rule occur below 1 GeV\(^2\). For the neutron, the bag model predicts, \( A^n = D^n = F^n = 0 \), and the \( 1/Q^2 \) correction vanishes:

\[
\mu_n^4(\text{Bag}) = 0. 
\] (21)

This follows from the SU(6) structure of the bag wave function, which also predicts \( \mu_n^2 = 0 \).

The higher twist matrix elements have also been calculated using the QCD sum rule (QSR) technique by Balitsky et al \[16\]. Their most recent result in terms of our notation is

\[
\mu_p^2(\text{QSR}) = -(0.023 \pm 0.015) M^2, \\
\mu_n^2(\text{QSR}) = -(0.006 \pm 0.004) M^2. 
\] (22)

Thus, the QCD sum rule calculation gives a power correction the same size as the bag calculation with the correction for the neutron being significantly smaller than for the proton. However, the sign of the correction differs from the bag result in Eq. (20). This difference has a large effect on the Bjorken sum rule at small \( Q^2 \).

The Bjorken sum rule has recently been extracted from the data on the proton \[3\] and neutron \[2\] \( g_1 \) structure functions,

\[
\int_0^1 g_1^{p-n}(x, 2 \text{ GeV}^2) dx = 0.146 \pm 0.021 [2], \\
= 0.152 \pm 0.025 [17]. 
\] (23)

In QCD, the Bjorken sum rule at low \( Q^2 \) is contaminated by higher twist corrections discussed above. If the QSR result is used for the correction, one obtains a theoretical prediction at the same \( Q^2 \),

\[
\int_0^1 g_1^{p-n}(x, 2 \text{ GeV}^2) dx = 0.160 [17], 
\] (24)

On the other hand, the bag result produces,

\[
\int_0^1 g_1^{p-n}(x, 2 \text{ GeV}^2) dx = 0.182. 
\] (25)

While Eq. (24) gives a corrected Bjorken sum rule within the experiment errors, the bag calculation disagrees with the extraction of the sum rule by the E142 collaboration by 1.7\( \sigma \) and with the extraction by Ellis-Karliner by 1.2\( \sigma \). In our opinion, a deviation from the Bjorken sum rule means either a measurement of higher twist matrix elements, or that the data are inconsistent.

From the high and low \( Q^2 \) knowledge of the sum rule, we propose a model for \( \Gamma^p(Q^2) \) in the entire \( Q^2 \) region,

\[
\Gamma^p(Q^2) = \frac{1}{2} F_1(F_1 + F_2)(1 - \lambda_1 \frac{Q^2}{M^2}) + \lambda_2 \frac{1 + \lambda_3 M^2/Q^2}{1 + \lambda_4 M^4/Q^4}, 
\] (26)
where the first term is the elastic contribution with its derivative modified by the $Q^2$ term. The second term is basically $a + b/Q^2$ and the denominator serves to suppress the contribution at small $Q^2$. From the EMC data and the various constraints derived above, we determine all $\lambda_i$ except $\lambda_4$, which controls the size of the twist-six contribution. The solid and upper-dashed curves shown in Fig. 1 are our parameterization with the bag and QSR higher twist matrix elements, respectively. [We choose $\lambda_4 = 0.3$, which gives a $\mu_6 \sim -0.03$.] The dotted curve represents the result of the twist expansion to order $1/Q^2$ and the dot-dashed curve represents the elastic contribution. As can be seen from the figure, the different choices for higher twist matrix elements result in about 15% difference in $\Gamma$ in the $Q^2 = 0.5$ to 1.0 GeV$^2$ region. A similar interpolation is made for the neutron, and the result is shown as the lower-dashed curve.

Thus it appears that the $Q^2$ variation of the $\Gamma(Q^2)$ sum rule is quite simple. Nevertheless, its experimental measurement is interesting, particularly around $Q^2 = 0.5$ GeV$^2$. If we know $\Gamma(Q^2)$ in an extended $Q^2$ region, we can fit data with a parameterization similar to the one used in Eq. (26). Then by expanding in a $1/Q^2$ power series, we can extract the higher-twist matrix elements, such as $f_2f$, which shall provide valuable insight into the spin structure of the nucleon.

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FIGURE CAPTION

Fig. 1: A model for the sum rule $\Gamma(Q^2)$ at all $Q^2$. The meaning of the curves is explained in the text.