Principles for Verification Tools: Separation Logic

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Abstract. A principled approach to the design of program verification and construction tools is applied to separation logic. The control flow is modelled by power series with convolution as separating conjunction. A generic construction lifts resource monoids to assertion and predicate transformer quantales. The data flow is captured by concrete store/heap models. These are linked to the separation algebra by soundness proofs. Verification conditions and transformation laws are derived by equational reasoning within the predicate transformer quantale. This separation of concerns makes an implementation in the Isabelle/HOL proof assistant simple and highly automatic. The resulting tool is correct by construction; it is explained on the classical linked list reversal example.

1 Introduction

Separation logic yields an approach to program verification that has received considerable attention over the last decade. It is designed for local reasoning about a system’s states or resources, allowing one to isolate the part of a system that an action affects from the remainder, which is unaffected. This capability is provided by the idiosyncratic separating conjunction operator together with the frame inference rule, which makes local reasoning modular. A key application is program verification with pointer data structures [21, 19]; but the method has also been used for modular reasoning about concurrent programs [17, 23] or fractional permissions [10].

Separation logic is currently supported by a large number of tools; so large that listing them is beyond the scope of this paper. Implementations in higher-order interactive proof assistants [24, 22, 8, 12] are particularly relevant to this article. In comparison to automated tools or tools for decidable fragments, they can express more program properties, but are less effective for proof search. Ultimately, an integration seems desirable.

This article adds to this tool chain (and presents another implementation in the Isabelle/HOL theorem proving environment [16]). However, our approach is different in several respects. It focuses almost entirely on making the control flow layer as simple as possible and on separating it cleanly from the data flow layer. This supports the integration of various data flow models and modular reasoning about these two layers, with assignment laws providing and interface.

To achieve this separation of concerns, we develop a novel algebraic approach to separation logic, which aims to combine the simplicity of original logical approaches [19] with the abstractness and elegance of O’Hearn and Pym’s categorical logic of bunched implications [18] in a way suitable for formalisation in Isabelle. Our approach is based
on power series [7], as they have found applications in formal language and automata theory [4, 9]. Their use in the context of separation logic is a contribution in itself.

In a nutshell, a power series is a function \( f : M \to Q \) from a partial monoid \( M \) into a quantale \( Q \). Defining addition of power series by lifting addition pointwise from the quantale, and multiplication as convolution

\[
(f \otimes g) x = \sum_{x = y \circ z} f y \circ g z,
\]

it turns out that the function space \( Q^M \) of power series forms itself a quantale [7]. In the particular case that \( M \) is commutative (a resource monoid) and \( Q \) formed by the booleans \( \mathbb{B} \) (with \( \circ \) as meet), one can interpret power series as assertions or predicates over \( M \). Separating conjunction then arises as a special case of convolution, and, in fact, as a language product over resources. The function space \( \mathbb{E}^M \) is the assertion quantale of separation logic. The approach generalises to power series over program states modelled by store-heap pairs. This generalisation is needed for applications.

Using lifting results for power series again, we construct the quantale-like algebraic semantics of predicate transformers over assertion quantales. We characterise the monotone predicate transformers and derive the inference rules of Hoare logic (without assignment) in generic fashion within this subalgebra. Furthermore, we derive the frame rule of separation logic on the subalgebra of local monotone predicate transformers. We use these rules for automated verification condition generation with our tool. Formalising Morgan’s specification statement [15] on the quantale structures, we obtain tools for program construction and refinement with a frame law with minimal effort. The predicate transformer semantics for separation logic, instead of the usual state transformer ones [5], fits well into the power series approach and simplifies the development.

The formalisation of the algebraic hierarchy from resource monoids to predicate transformer algebras benefits from Isabelle’s excellent libraries for functions and, in particular, its integration of automated theorem provers and SMT-solvers via the Sledgehammer tool. These are optimised for equational reasoning and make the entire development mostly automatic. In addition, Isabelle’s reconstruction of proof outputs provided by the external tools makes our verification tools correct by construction.

At the data flow level, we currently use Isabelle’s extant infrastructure for the store, the heap and pointer-based data structures. An interface to the control flow algebra is provided by the assignment and mutation laws of separation logic and their refinement counterparts. Isabelle’s data flow models are linked formally with our abstract separation algebra by soundness proofs. Algebraic facts are then picked up automatically by Isabelle for reasoning in the concrete model. The verification examples in the last section of this article show that, at this layer, proofs may require some user interaction, but a Sledgehammer-style integration of optimised efficient provers and solvers for the data flow is an avenue of future work.

The entire technical development has been formalised in Isabelle; all proofs have been formally verified. For this reason we show only some example proofs which demonstrate the simplicity of algebraic reasoning. The complete executable Isabelle theories can be found online\(^1\).

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\(^1\) https://github.com/vborgesfer/sep-logic
2 Partial Monoids, Quantales and Power Series

This section presents the algebraic structures that underlie our approach to separation logic. Further details on power series and lifting constructions can be found in [7].

A partial monoid is a structure \((M, \cdot, 1, \bot)\) such that \((M, \cdot, 1)\) is a monoid and \(x \cdot \bot = \bot = \bot \cdot x\) holds for all \(x \in M \cup \{1, \bot\}\). We often do not mention \(\bot\) in definitions. A partial monoid \(M\) is commutative if \(x \cdot y = y \cdot x\) for all \(x, y \in M\). Henceforth \(\cdot\) is used for a general and \(*\) for a commutative multiplication.

A quantale is a structure \((Q, \leq, \cdot, 1)\) such that \((Q, \leq)\) is a complete lattice, \((Q, \cdot)\) is a monoid and the distributivity axioms

\[
(\sum_{i \in I} x_i) \cdot y = \sum_{i \in I} x_i \cdot y,
\]

\[
x \cdot (\sum_{i \in I} y_i) = \sum_{i \in I} (x \cdot y_i)
\]

hold, where \(\sum X\) denotes the supremum of a set \(X \subseteq Q\). Similarly, we write \(\prod X\) for the infimum of \(X\), and \(0\) for the least and \(U\) for the greatest element of the lattice. The monotonicity laws \(x \leq y \Rightarrow z \cdot x \leq z \cdot y\) and \(x \leq y \Rightarrow x \cdot z \leq y \cdot z\) follow from distributivity. A quantale is commutative and partial if the underlying monoid is. It is distributive if \(x \sqcap (\sum_{i \in I} y_i) = \sum_{i \in I} (x \sqcap y_i)\) and \(x + (\prod_{i \in I} y_i) = \prod_{i \in I} (x + y_i)\). In that case, the annihilation laws \(x \cdot 0 = 0 = 0 \cdot x\) follow from \(\sum_{i \in I} x_i = \sum \emptyset = 0\). A boolean quantale is a complemented distributive quantale. The boolean quantale \(\mathbb{B}\) of the booleans, where multiplication coincides with join, is an important example.

A power series is a function \(f : M \to Q\), from a partial monoid \(M\) into a quantale \(Q\). For \(f, g : M \to Q\) and a family of functions \(f_i : M \to Q, i \in I\) we define

\[
(f \cdot g) \cdot z = \sum_{x=y \cdot z} f y \cdot g z, \quad (\sum_{i \in I} f_i) \cdot x = \sum_{i \in I} f_i \cdot x.
\]

The composition \(f \cdot g\) is called convolution; the multiplication symbol is overloaded to be used on \(M, Q\) and the function space \(Q^M\). The idea behind convolution is simple: element \(x\) is split into \(y\) and \(z\), \(f y\) and \(g z\) are calculated in parallel, and their results are composed to form a value for the summation with respect to all possible splits of \(x\). Because \(x\) ranges over \(M\), the constant \(\bot \notin M\) is excluded as a value. Hence undefined splittings of \(x\) do not contribute to convolutions. In addition, \((f + g) \cdot z = f x + g x\) arises as a special case of the supremum. Finally, we define the power series \(0 : M \to Q\) as \(0 = 0\cdot x\). 0 and \(1 : M \to Q\) as \(1 = \lambda x. 1\). If \((x = 1)\) then \(1\) else \(0\).

The quantale structure lifts from \(Q\) to the function space \(Q^M\) of power series.

**Theorem 2.1** ([7]). Let \(M\) be a partial monoid. If \(Q\) is a boolean quantale, then so is \((Q^M, \leq, \cdot, 1)\). If \(M\) and \(Q\) are commutative, then so is \(Q^M\).

The power series approach generalises from one to \(n\) dimensions [7]. For separation logic, the two-dimensional case with power series \(f : S \times M \to Q\) from set \(S\) and partial commutative monoid \(M\) into the commutative quantale \(Q\) is needed. Now

\[
(f \cdot g) \cdot (x, y) = \sum_{y = y_1 \cdot y_2} f (x, y_1) \cdot g (x, y_2), \quad (\sum_{i \in I} f_i) \cdot (x, y) = \sum_{i \in I} f_i (x, y).
\]

The convolution \(f \cdot g\) acts solely on the second coordinate. Finally, we define two-dimensional units as \(0 = \lambda x. 0\) and \(1 = \lambda x. y\). If \((y = 1)\) then \(1\) else \(0\).
Theorem 2.2 ([7]). Let $S$ be a set and $M$ a partial commutative monoid. If $Q$ is a commutative boolean quantale, then so is $Q^{S \times M}$.

We have implemented partial monoids and quantale by using Isabelle’s type class and local infrastructure for engineering mathematical structures, building on existing libraries for monoids, quantales and complete lattices.

3 Assertion Quantales

In language theory, power series have been introduced for modelling formal languages. In that case, $S$ is the free monoid $X^*$ and $Q$ can be taken as a semiring $(Q, +, \cdot, 0, 1)$, because there are only finitely many ways of splitting words into prefix/suffix pairs in convolutions. In the particular case of the boolean semiring $\mathbb{B}$, where $\cdot$ is conjunction, power series $f : X^* \to \mathbb{B}$ are interpreted as characteristic functions or predicates that indicate whether or not a word is in a set. Since, in this case, sets are languages, convolution specialises to $(f \cdot g) x = \sum_{x=yz} f \sqcap g y$, hence, identifying predicates with their extensions, to the language product $p \cdot q = \{yz \mid y \in p \land z \in q\}$.

More generally, we consider power series $S \to \mathbb{B}$ from a partial monoid $S$ into the boolean quantale $\mathbb{B}$ and set up the connection with separation logic. There, one is interested in modelling assertions or predicates over the memory heap. The heap can be represented abstractly by a resource monoid [5], which is simply a partial commutative monoid. In analogy to the language case, an assertion $p$ of separation logic is a boolean-valued function from a resource monoid $M$, hence a power series $p : M \to \mathbb{B}$. Then Theorem 2.1 applies.

Corollary 3.1. The assertions $\mathbb{B}^M$ over resource monoid $M$ form a commutative boolean quantale with convolution as separating conjunction.

The logical structure of the assertion quantale $\mathbb{B}^M$ is as follows. The predicate $\bot$ is a contradiction whereas $\top$ holds of the empty resource and is false otherwise. The operations $\Sigma$ and $\Pi$ correspond to existential and universal quantification; their finite cases yield conjunctions and disjunctions. The order $\leq$ is implication. Convolution becomes

$$(p \ast q) x = \sum_{x=yz} p y \sqcap q z.$$ 

By $x = y \ast z$, resource $x$ is separated into resources $y$ and $z$. By $p y \sqcap q z$, the value of predicate $p$ on $y$ is conjoined with that of $q$ on $z$. Finally, the supremum is true if one of the conjunctions holds for some splitting of $x$.

As for languages, one can again identify predicates with their extensions. Then

$$p \ast q = \{y \ast z \in M \mid y \in p \land z \in q\},$$

and separating conjunction becomes a language product over resources. The analogy to language theory is even more striking when considering multisets over a finite set $X$ as resources, which form the free commutative monoids over $X$.

Applications of separation logic, however, require program states which are store-heap pairs. Now Theorem 2.2 applies.
Corollary 3.2. The assertions $\mathbb{B}^{S \times M}$ over set $S$ (the store) and resource monoid $M$ form a commutative boolean quantale with convolution as separating conjunction. For all $p, q : S \times M \to \mathbb{B}, s \in S$ and $h \in M$,

$$(p * q) (s, h) = \sum_{h = h_1 + h_2} p (s, h_1) \sqcap q (s, h_2).$$

Quantales carry a rich algebraic structure. Their distributivity laws give rise to continuity or co-continuity properties. Therefore, many functions constructed from the quantale operations have adjoints as well as fixpoints, which can be iterated to the first limit ordinal. This is well known in denotational semantics and important for our approach to program verification. In particular, separating conjunction $*$ distributes over arbitrary suprema in $\mathbb{B}^M$ and $\mathbb{B}^{S \times M}$ and therefore has an upper adjoint: the magic wand operation $\Rightarrow$, which is widely used in separation logic. In the quantale setting, the adjunction gives us theorems for the magic wand for free.

One can think of the power series approach to separation logic as a simple account of the category-theoretical approach to O’Hearn and Pym’s logic of bunched implication [18] in which convolution generalises to coends and the quantale lifting is embodied by Day’s construction [6]. For the design of verification tools and our implementation in Isabelle, this simplicity is certainly an advantage.

4 Predicate Transformer Quantales

Our algebraic approach to separation logic is based on predicate transformers (cf. [3]). This is in contrast to previous state-transformer-based approaches and implementations [5, 12, 22]. First of all, predicate transformers are more amenable to algebraic reasoning [3]—simply because their source and target types are similar. Second, the approach is more coherent within our framework. Predicate transformers can be seen once more as power series and instances of Theorem 2.1 describe their algebras.

A state transformer $f_R : A \to 2^B$ is often associated with a relation $R \subseteq A \times B$ from set $A$ to set $B$ by defining $f_R a = \{ b \mid (a, b) \in R \}$. It can be lifted to a function $\langle R \rangle : 2^A \to 2^B$ defined by $\langle R \rangle X = \bigcup_{a \in X} f_R a$ for all $X \subseteq A$. More importantly, state transformers are lifted to predicate transformers $[R] : 2^B \to 2^A$ by defining

$$[R] Y = \{ x \mid f_R x \subseteq Y \}$$

for all $Y \subseteq B$. The modal box and diamond notation is justified by the correspondence between diamond operators and Hoare triples as well as box operators and weakest liberal precondition operators in the context of modal semirings [14]. In fact we obtain the adjunction $\langle R \rangle X \subseteq Y \iff X \subseteq [R] Y$ from the above definitions.

Predicate transformers in $\langle 2^A \rangle^{2^B}$ form complete distributive lattices [3]. In the power series setting, this follows from Theorem 2.1 in two steps, ignoring the monoidal structure. Since $\mathbb{B}$ forms a complete distributive lattice, so do $2^B \cong \mathbb{B}^B$ and $2^A \cong \mathbb{B}^A$ in the first step, and so does $\langle 2^A \rangle^{2^B}$ in the second one.
In addition, predicate transformers in \((2^A)^2\) form a monoid under function composition and the identity function as the unit. It follows that those predicate transformers form a distributive near-quantale, whereas the monotone predicate transformers in \((2^A)^2\), which satisfy \(p \leq q \Rightarrow f \cdot p \leq f \cdot q\), form a distributive pre-quantale [3].

Here, near-quantale means a quantale where \(x \cdot (\sum_{i \in I} y_i) = \sum_{i \in I} (x \cdot y_i)\), the left distributivity law, need not hold. We call pre-quantale a near-quantale in which the left monotonicity law \(x \leq y \Rightarrow z \cdot x \leq z \cdot y\) holds. In these cases, the monoidal parts of the lifting are not obtained with the power series technique because function composition is not a convolution.

Adapting these results to separation logic requires the consideration of assertion quantales \(\mathbb{B}_M\) or \(\mathbb{B}_S \times M\) with store \(S\) and resource monoid \(M\) instead of the powerset algebra over a set \(A\). But these quantales can be lifted as boolean algebras—disregarding a convolution on predicate transformers, which is not needed for separation logic—and combined with the monoidal structure of function composition as previously. This yields the following result.

**Theorem 4.1.** Let \(S\) be a set, \(M\) a resource monoid and \(\mathbb{B}_S \times M\) an assertion quantale. The monotone predicate transformers over \(\mathbb{B}_S \times M\) form a distributive pre-quantale.

The proof requires showing that the predicate transformers over \(\mathbb{B}_S \times M\) form a near quantale and checking that the monotone predicate transformers form a subalgebra of this near-quantale. In fact, the unit predicate transformer is monotone—which is the case—and that the quantale operations of suprema, infima and composition preserve monotonicity. This is implied by properties such as

\[
[R \cup S] = [R] \cap [S], \quad [R; S] = [R] \cdot [S] = \lambda x. [R] ([S] x).
\]

Monotone predicate transformers are powerful enough to derive the standard inference rules of Hoare logic as verification conditions (Section 5) and the usual rules of Morgan’s refinement calculus (Section 6). Derivation of the frame rule of separation logic, however, requires a smaller class of predicate transformers defined as follows.

A state transformer \(f\) is local [5] if \(f(x*y) \leq (f x) \ast \{y\}\) for \(x*y \neq \bot\). Intuitively, this means that the effect of such a state transformer is restricted to a part of the heap; see [5] for further discussion. Analogously, we call a predicate transformer \(F\) local if

\[
(F p) \ast q \leq F (p \ast q)\cdot p
\]

It is easy to show that the two definitions are compatible.

**Lemma 4.2.** State transformer \(f_R\) is local iff predicate transformer \([R]\) is local.

The final theorem in this section establishes the local monotone predicate transformers as a suitable algebraic framework for separation logic.

**Theorem 4.3.** Let \(S\) be a set and \(M\) a resource monoid. The local monotone predicate transformers over the assertion quantale \(\mathbb{B}_S \times M\) form a distributive pre-quantale.

Once again it must be checked that the zero predicate transformer is local—which is the case—and that the quantale operations preserve locality and monotonicity.
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We have implemented the whole approach in Isabelle; all theorems have been formally verified, using mainly Theorem 2.1 for lifting to predicate transformers. An alternative Isabelle implementation of predicate transformers as lattices and Boolean algebras with operators is due to Preoteasa [20].

5 Verification Conditions

The pre-quantale of local monotone predicate transformers supports the derivation of verification conditions by equational reasoning. A standard set of such conditions are the inference rules of Hoare logic. For sequential programs, Hoare logic provides one inference rule per program construct. This suffices to eliminate the control structure of a program and generate verification conditions for the data flow.

The quantale setting also guarantees that the finite iteration $F^*$ of a predicate transformer is well defined. This supports a shallow algebraic embedding of a simple while language with the usual pseudocode for the verification of imperative programs.

First we lift predicates to predicate transformers [3]:

\[ [p] = \lambda q. p + q. \]

With predicates modelled as relational subidentities, this definition is justified by the lifting from the previous section: \((s, s) \in [p] q \text{ iff } (s, s) \in p \Rightarrow (s, s) \in q. \)

Second, we change notation, to use descriptive while program syntax for predicate transformers. We write $\text{skip}$ for the quantale unit (the identity function) and $;$ for function composition. We encode the semantics of conditionals and while loops as $\text{if } p \text{ then } F \text{ else } G \text{ fi } = [p] \cdot F + [p] \cdot G,$

$\text{while } p \text{ do } F \text{ od } = ([p] \cdot F)^* \cdot [\overline{p}].$

Third, we provide the standard assertions notation for programs via Hoare triple syntax:

\[ \{p\} F \{q\} \iff p \leq F q. \]

Box notation shows that $\{p\} [R] \{q\} \iff p \leq [R]q$ for relational program $R.$ Thus $[R]q = \text{wlp}(R, q)$ is the standard weakest liberal precondition of program $R$ and post-condition $q.$ It also explains our slight abuse of relational or imperative notation for predicate transformers: e.g. we write $[R]; [S]$ instead of $[R] \cdot [S]$ because the latter expression is equal to $[R; S],$ as indicated in the previous section.

**Proposition 5.1.** Let $p, q, r, p', q' \in \mathbb{P}^{S \times M}$ be predicates. Let $F, G, H$ be monotone predicate transformers over $\mathbb{P}^{S \times M}$, with $H$ being local. Then the rules of propositional Hoare logic (no assignment rule) and the frame rule of separation logic are derivable.

\[
\begin{align*}
\{p\} \text{skip} \{p\}, \\
p \leq p' \land q' \leq q & \land \{p'\} F \{q'\} \Rightarrow \{p\} F \{q\}, \\
\{p\} F \{r\} \land \{r\} G \{q\} & \Rightarrow \{p\} F; G \{q\}, \\
\{p \land r\} F \{G\} \land \{p \land \overline{r}\} G \{q\} & \Rightarrow \{p\} \text{if } r \text{ then } F \text{ else } G \text{ fi } \{q\}, \\
\{p \land q\} F \{p\} & \Rightarrow \{p\} \text{while } q \text{ do } F \text{ od } \{\overline{q} \land p\}, \\
\{p\} H \{q\} & \Rightarrow \{p \land r\} H \{q \land r\}. \\
\end{align*}
\]
Proof. We derive the frame rule as an example. Suppose \( p \leq H q \). Then, by isotonicity of * and locality, \( p * r \leq (H q) * r \leq H(q * r) \).

The remaining derivations are equally simple and fully automatic in Isabelle.

6 Refinement Laws

To demonstrate the power of the predicate transformer approach to separation logic we now outline its applicability to local reasoning in program construction and transformation. More precisely, we show that the standard laws of Morgan’s refinement calculus [15] plus an additional framing law for resources can be derived with little effort. It only requires defining one single additional concept—Morgan’s specification statement—which already exists in every predicate transformer quantale.

Formally, for predicates \( p, q \in B^{S \times M} \), we define the specification statement as

\[
[p, q] = \sum \{ F \mid p \leq F q \}.
\]

It models the most general predicate transformer or program which links postcondition \( q \) with precondition \( p \). It is easy to see that \( \{ p \} F \{ q \} \iff p \leq F q \), which entails the characteristic properties \( \{ p \} [p, q] \{ q \} \) and \( \{ p \} F \{ q \} \Rightarrow F \leq [p, q] \) of the specification statement: program \([p, q]\) relates precondition \( p \) with postcondition \( q \) whenever it terminates; and it is the largest program with that property. It is easy to check that specification statements over the pre-quantale of local monotone predicate transformers are themselves local and monotone.

Like Hoare logic, Morgan’s basic refinement calculus provides one refinement law per program construct. Once more we ignore assignments at this stage. We also switch to standard refinement notation with refinement order \( \preceq \) being the converse of \( \leq \).

Proposition 6.1. For \( p, q, r, p', q' \in B^{S \times M} \), and predicate transformer \( F \) the following refinement laws are derivable in the algebra of local monotone predicate transformers.

\[
\begin{align*}
& p \leq q \Rightarrow [p, q] \preceq \text{skip}, \\
& p' \leq p \land q \leq q' \Rightarrow [p, q] \preceq [p', q'], \\
& [0, 1] \preceq F, \\
& F \preceq [1, 0], \\
& [p, q] \preceq [p, r] \land [r, q], \\
& [p, q] \preceq \text{if } b \text{ then } [b \land p, q] \text{ else } [\overline{b} \land p, q] \text{ fi,} \\
& [p, \overline{b} \land p] \preceq \text{while } b \text{ do } [b \land p, p] \text{ od,} \\
& [p * r, q * r] \preceq [p, q].
\end{align*}
\]

Proof. Using the frame rule, we derive the framing law as an example:

\[
\{ p \} [p, q] \{ q \} \Rightarrow \{ p * r \} [p, q] \{ q * r \} \iff [p * r, q * r] \preceq [p, q]
\]

The proofs of the other refinement laws are equally simple, using the corresponding Hoare rules in their proofs. They are fully automatic in Isabelle.
7 Principles of Tool Design

The previous sections have introduced a new algebraic approach to separation logic based on power series and quantales with separating conjunction modelled algebraically as convolution, that is, a language product over resources. The theory hierarchy from partial monoids to predicate transformer quantales has been formalised in Isabelle/HOL, much of which was highly automatic and required only a moderate effort. It benefits, to a large extent, from Isabelle’s integrated first-order theorem proving, SMT-solving and counterexample generation technology. These tools are highly optimised for equational reasoning, interacting efficiently with the algebraic layer.

Another important feature is that the mathematical structures formalised in Isabelle are all polymorphic—their elements can have various types. Isabelle’s type classes and locales, which have been used for implementing mathematical hierarchies, then allow us to link these abstract algebras with various concrete models, that is, quantales with predicate transformers, predicate transformers with binary relations and functions which update program states. In particular, abstract resource monoids are linked with various concrete models for resources, including the heap. By formalising these soundness results in Isabelle, theorems are automatically propagated across classes and models. Those proved for quantales, for instance, become available automatically for predicate transformers over concrete detailed store-heap models.

Finally, our development can build on excellent Isabelle libraries and decades-long experience in reasoning with functions and relations, all sorts of data structures and data types. In particular, for program construction and verification with separation logic, Isabelle provides support for reasoning with pointers and the heap [13, 24].

These features suggest a principled approach to program verification and construction in Isabelle in which the control flow layer is cleanly separated from the data flow layer. The control flow is modelled in a lightweight way within suitable algebras, which makes tool design fast, simple and automatic, including the development of verification conditions, transformation and refinement laws. Their application can then be automated by programming Isabelle tactics. The data flow can, by and large, rely on existing Isabelle libraries, which have previously been designed for verification purposes. The interface between these layers is provided, at an abstract level, by soundness theorems
and, at the concrete level, by assignment laws and similar laws that link data and control. The approach has previously been applied to simple while programs [2] and the rely-guarantee approach [1]. Its basic features are summarised in Figure 1.

For separation logic, the algebraic structures used are partial monoids, quantales and power series. The intermediate semantics is provided by predicate transformers over assertion quantales based on stores and resource monoids. The concrete models yield detailed descriptions of the store and heap. These are explained in the remaining sections of this article.

8 Integration of Data Flow

This section describes the integration of the data flow layer into our Isabelle tools for program verification and construction.

As previously mentioned, program states are store-heap pairs \((s, h)\). Program stores are implemented in Isabelle as records of program variables, each of which has a retrieve and an update function. This approach is polymorphic and supports variables of any Isabelle type. For instance, Isabelle’s built-in list data type and list libraries can be used to reason about list-based programs. Heaps have been modelled in Isabelle as partial functions on \(\mathbb{N}\) [13, 24]; they therefore have type \(\text{nat} \to \text{nat} \to \text{option}\).

We implement assignments first as functions from states to states,

\[
(x := e) = \lambda(s, h). (x \circ \text{update} s e, h),
\]

where \(x\) is a program variable, \(x \circ \text{update}\) the update function for \(x\), \((s, h)\) a state and \(e\) an evaluated expression of the same type as \(x\).

Separation logic also requires a notion of mutation or heap update. In Isabelle, it is first implemented in similar fashion as

\[
(@e := e') = \lambda(s, h). (s, h[e \to e']),
\]

where \(e\) and \(e'\) are expressions that evaluate to natural numbers, and \(h[e \to e']\) is the function that maps \(e\) to \(e'\) and is the same as \(h\) for the remaining expressions.

Secondly, we lift assignment and mutation functions to predicate transformers as

\[
[f] = \lambda q. q \circ f,
\]

where \(\circ\) denotes function composition, as usual. This is consistent with the definition of lifting in Section 4. As previously, we generally do not write the lifting brackets explicitly, identifying program pseudocode with predicate transformers to simplify verification notation.

With this infrastructure in place we can prove Hoare’s assignment rule and Reynolds’ mutation rule for separation logic in the concrete heap model.

**Proposition 8.1.** The following rules are derivable in the concrete store-heap model:

\[
p \leq q[e/x] \Rightarrow \{p\} (x := e) \{q\}, \quad \{(e \to -) \ast r\} (@e := e') \{(e \to e') \ast r\},
\]

writing \(q[e/x]\) for the substitution of variable \(x\) by expression \(e\) in \(q\) as well as \(e \to e'\) and \(e \to -\) for the singleton heaps mapping \(e\) to \(e'\) and to any value, respectively.
The resulting set of inference rules for separation logic allows us to implement the Isabelle proof tactic \texttt{hoare}, which generates all verification conditions automatically and eliminate the entire control structure when the invariants of while loops are annotated.

One can use the assignment rule to derive its refinement counterparts:

\begin{align*}
  p \leq q[e/x] & \Rightarrow [p, q] \sqsubseteq (x := e), \\
  q' \leq q[e/x] & \Rightarrow [p, q] \sqsubseteq [p, q'] : (x := e), \\
  p' \leq p[e/x] & \Rightarrow [p, q] \sqsubseteq (x := e); [p', q].
\end{align*}

The second and third laws are called the \textit{following} and \textit{leading} refinement law for assignments \cite{DBLP:journals/jar/Reynolds92}. They are useful for program construction. We have derived analogous laws for mutation. We have also implemented the tactic \texttt{refinement}, which automatically tries to apply all the rules of this refinement calculus in construction steps of pointer programs.

\section{Examples}

To show our approach at work, we present the obligatory correctness proof of the classical in situ linked-list reversal algorithm. The post-hoc verification of this algorithm in Isabelle has been considered before \cite{DBLP:journals/corr/abs-1806-09311, DBLP:journals/tcs/Reynolds91}. However, we follow Reynolds \cite{DBLP:journals/jar/Reynolds91}, who gave an informal annotated proof, and reconstruct his proof step-by-step in refinement style. As usual for verification with interactive theorem provers, functional specifications are related to imperative data structures. The former are defined recursively and hence amenable to proof by induction.

First, we define two inductive predicates on the heap. The first one creates a contiguous heap from a position \(e\) using Isabelle’s functional lists as its representation. That is, by induction on the structure of the list,

\begin{align*}
  e \mapsto [] & = \text{emp}, \\
  e \mapsto (t \# ts) & = (e \mapsto t) \ast (e + 1 \mapsto ts),
\end{align*}

where \([]\) denotes the empty list, \(t \# ts\) denotes concatenation of element \(t\) with list \(ts\), \(e \mapsto t\) is again a singleton heap predicate and \texttt{emp} states that the heap is empty.

The second predicate indicates whether or not a heap, starting from position \(i\), contains the linked list represented as a functional list:

\begin{align*}
  \text{list } i \ [ ] & = (i = 0) \land \text{emp}, \\
  \text{list } i \ (j \# js) & = (i \neq 0) \land (\exists k. i \mapsto [j, k] \ast \text{list } k \ js).
\end{align*}

This is Reynolds’ definition, it uses separating conjunction instead of plain conjunction.

We can now reconstruct Reynolds’ proof relative to the standard recursive function \texttt{rev} for functional list reversal. The initial specification statement is

\[ [\text{list } 'i \ A_0, \text{list } 'j \ (\text{rev } A_0)] \],

where \(A_0\) is the input list and \('i\) and \('j\) are pointers to the head of the list on the heap.
Fig. 2. In situ list reversal by refinement. The first block shows the refinement up to the introduction of the while-loop. The second block shows the refinement of the body of that loop.

\[
\begin{align*}
\text{list } 'i A_0, \text{ list } 'j (\text{rev } A_0) & \quad \Box \quad (1) \\
'j' & := 0; \\
\exists A. (\text{list } 'i A \ast \text{ list } 'j B) \land (\text{rev } A_0) = (\text{rev } A) \otimes B, \text{ list } 'j (\text{rev } A_0) & \quad \Box \quad (2) \\
'j' & := 0; \\
\text{while } 'i \neq 0 \text{ do} & \\
\exists A. (\text{list } 'i A \ast \text{ list } 'j B) \land (\text{rev } A_0) = (\text{rev } A) \otimes B \land 'i \neq 0, \\
\exists A. (\text{list } 'i A \ast \text{ list } 'j B) \land (\text{rev } A_0) = (\text{rev } A) \otimes B & \\
\od
\end{align*}
\]

\[
\begin{align*}
\exists A. (\text{list } 'i A \ast \text{ list } 'j B) \land (\text{rev } A_0) & = (\text{rev } A) \otimes B \land 'i \neq 0, \\
\exists A. (\text{list } 'i A \ast \text{ list } 'j B) \land (\text{rev } A_0) & = (\text{rev } A) \otimes B & \\
\exists A. (\text{list } 'i A \ast \text{ list } 'j B) \land (\text{rev } A_0) & = (\text{rev } A) \otimes B & \\
'k' & := @('i + 1); \\
\exists A. (\text{list } 'i A \ast \text{ list } 'j B) \land (\text{rev } A_0) & = (\text{rev } A) \otimes B \land 'i \neq 0, \\
\exists A. (\text{list } 'i A \ast \text{ list } 'j B) \land (\text{rev } A_0) & = (\text{rev } A) \otimes B & \\
'k' & := @('i + 1); \\
\exists A. (\text{list } 'i A \ast \text{ list } 'j B) \land (\text{rev } A_0) & = (\text{rev } A) \otimes B \land 'i \neq 0, \\
\exists A. (\text{list } 'i A \ast \text{ list } 'j B) \land (\text{rev } A_0) & = (\text{rev } A) \otimes B & \\
'k' & := @('i + 1); \\
\exists A. (\text{list } 'i A \ast \text{ list } 'j B) \land (\text{rev } A_0) & = (\text{rev } A) \otimes B \land 'i \neq 0, \\
\exists A. (\text{list } 'i A \ast \text{ list } 'j B) \land (\text{rev } A_0) & = (\text{rev } A) \otimes B & \\
'k' & := @('i + 1); \\
\exists A. (\text{list } 'i A \ast \text{ list } 'j B) \land (\text{rev } A_0) & = (\text{rev } A) \otimes B \land 'i \neq 0, \\
\exists A. (\text{list } 'i A \ast \text{ list } 'j B) \land (\text{rev } A_0) & = (\text{rev } A) \otimes B & \\
'k' & := @('i + 1); \\
\exists A. (\text{list } 'i A \ast \text{ list } 'j B) \land (\text{rev } A_0) & = (\text{rev } A) \otimes B \land 'i \neq 0, \\
\exists A. (\text{list } 'i A \ast \text{ list } 'j B) \land (\text{rev } A_0) & = (\text{rev } A) \otimes B & \\
'k' & := @('i + 1); \\
'j' & := 'i; \\
'k' & := 'k
\end{align*}
\]
The main idea behind Reynolds’ proof is to split the heap into two lists, initially $A_0$ and an empty list, and then iteratively swing the pointer of the first element of the first list to the second list. The full proof is shown in Figure 2; we now explain its details.

In (1), we strengthen the precondition, splitting the heap into two lists $A$ and $B$, and inserting a variable $j$ initially assigned to 0 (or null). The equation $(\text{rev } A_0) = (\text{rev } A) @ B$ then holds of these lists, where @ denotes the append operation on linked lists. Justifying this step in Isabelle requires calling the refinement tactic from Section 7, which applies the leading law for assignment. This obliges us to prove that the lists $A$ and $B$ de facto exist, which is discharged automatically by calling Isabelle’s force tactic. In fact, 8 out of the 10 proof steps in our construction are essentially automatic: they only require calling refinement followed by Isabelle’s force or auto provers.

The new precondition generated then becomes the loop invariant of the algorithm. It allows us to refine our specification statement to a while loop in step (2), where we iterate `i` until it becomes 0. Calling the refinement tactic applies the while law for refinement. From step (3) to (10), we refine the inside part of the while loop and do not display the outer part of the program.

Because now `$i \neq 0$, the list $A$ has at least an element $a$. We can thus expand the definition of list in step (3). Next, we assign the value pointed to by `$i + 1` to `$j`—our first list now starts at `$j` and `$i` points to `$a`, `$j`. Isabelle then struggles to discharge the generated proof goal automatically. In this predicate, the heap is divided in three parts. One needs to prove first that `$i + 1` really points to the same value when considering just the first part of the heap or the entire heap. After that, the proof is automatic.

Step (5) performs a mutation on the heap, changing the cell `$i + 1` to `$j`, consequently `$i` points now to `$a`, `$j`. Because $\ast$ is commutative, we can strengthen the precondition accordingly in step (6). We now work backwards, folding the definition of list in step (7) and removing the existential of $a$ in step (8). This step requires again interaction: we need to indicate to Isabelle how to properly split the heap. Lastly, to establish the invariant, we only need to swap the pointers `$j` to `$i` and `$i` to `$j` in steps (9) and (10). The resulting algorithm is highlighted in Figure 2.

Using our tool we have also post-hoc verified this algorithm with separation logic in two different ways. The first one, previously taken by Weber [24], uses Reynolds’ list predicate, as we have used it in the above refinement proof. The second one follows Nipkow in using separating conjunction in the pre- and postcondition, but not in the definition of the list predicate. Since our approach is modular with respect to the underlying data model, it was straightforward to replay all the steps of Nipkow’s proof in our setting. The degree of proof automation with our tool is comparable for both proofs.

Interestingly, however, none of the list reversal proofs have used the frame rule or its refinement counterpart. We have therefore tested this rule separately on a small example, where a verification without the frame rule would be difficult. The following Isabelle code fragment shows the Hoare triple used for verification.

$$
\vdash \{ x \mapsto [\cdot, j] \ast \text{list } j \text{ as } \} \ast x := a \{ x \mapsto [a, j] \ast \text{list } j \text{ as } \}
$$

Calling the hoare tactic for verification condition generation was sufficient for proving the correctness of this simple example automatically. Internally, the frame and the mutation rule have been applied. The Isabelle code for all these proofs is available online.
In sum, our approach supports the program construction and verification of pointer-based programs with separation logic, but more case studies need to be performed to assess the performance of our tool. In the future, a Sledgehammer-style integration of optimised provers and solvers for the data level seems desirable for increasing the general degree of automation.

10 Conclusion

A principled approach to the design of program verification and construction tools for separation logic with the Isabelle theorem proving environment has been presented. This approach has been used previously for implementing tools for the construction and verification of simple while programs [2] and rely-guarantee based concurrent programs [1]. It aims at a clean separation of concern between the control flow and the data flow of programs and focusses on developing a lightweight algebraic layer from which verification conditions or transformation and refinement laws can be developed by simple equational reasoning. In the case of while programs, this layer is provided by Kleene algebras with tests; in the rely-guarantee case, new algebraic foundations based on concurrent Kleene algebras were required.

Our approach to separation logic uses a conceptual reconstruction of separation logic beyond a mere implementation as well, which forms a contribution in its own right. Though strongly inspired by abstract separation logic [5] and the logic of bunched implications [18], we aim at a different combination of simplicity and mathematical abstraction. In contrast to the logic of bunched implication, we use power series instead of categories, and in contrast to abstract separation logic we use predicate transformers instead of state transformers. These design choices allow us to use power series, quantales and generic lifting constructions throughout the approach, which leads to a small and highly automated Isabelle implementation. The main contribution of this approach is probably the view on separating conjunction as a notion of convolution and language product over resources.

Our tool prototype has so far allowed us to verify some simple pointer-based program with a relatively high degree of automation. It is certainly a useful basis for educational and research purposes, but extension and optimisation beyond the mere proof of concept are desirable. This includes the consideration of error states [5] or of the \texttt{cons} and \texttt{dispose} operations, the development of more sophisticated proof tactics, and the integration of tools for automatic data-level reasoning in Sledgehammer style.

Other opportunities for future work lie in the consolidation with previous approaches to predicate transformers in Isabelle [20], in a further abstraction of the control flow layer by defining modal Kleene algebras over assertion quantales [14] for which some Isabelle infrastructure already exists [11], in a combination with our rely-guarantee tool into RGSep-style tools for concurrency verification [23], and in the exploration of the language connection of separating conjunction in terms of representability and decidability results.

Acknowledgements. We are grateful for support by EPSRC grant EP/J003727/1 and the CNPq. The third author would like to thank Tony Hoare, Peter O’Hearn and Matthew Parkinson for discussions on separation logic.
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