Einstein Gravity in Almost Kähler Variables
and Stability of Gravity with Nonholonomic
Distributions and Nonsymmetric Metrics

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Abstract

We argue that the Einstein gravity theory can be reformulated in almost Kähler (nonsymmetric) variables with effective symplectic form and compatible linear connection uniquely defined by a (pseudo) Riemannian metric. A class of nonsymmetric theories of gravitation (NGT) on manifolds enabled with nonholonomic distributions is analyzed. There are considered some conditions when the fundamental geometric and physical objects are determined/modified by nonholonomic deformations in general relativity or by contributions from Ricci flow theory and/or quantum gravity. We prove that in such NGT, for certain classes of nonholonomic constraints, there are modeled effective Lagrangians which do not develop instabilities. It is also elaborated a linearization formalism for anholonomic NGT models and analyzed the stability of stationary ellipsoidal solutions defining some nonholonomic and/or nonsymmetric deformations of the Schwarzschild metric. We show how to construct nonholonomic distributions which remove instabilities in NGT. Finally we conclude that instabilities do not consist a general feature of theories of gravity with nonsymmetric metrics but a particular property of certain models and/or classes of unconstrained solutions.

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1 Introduction

In this article, we re-address the issue of nonsymmetric gravity theory (NGT) following three key ideas: 1) the general relativity theory can be written equivalently in terms of certain nonsymmetric variables; 2) nonsymmetric contributions to metrics and connections may be generated in
quasi–classical limits of quantum gravity and nonholonomic and/or noncommutative Ricci flow theory; 3) physically valuable solutions and their generalizations with nonsymmetric/ noncommutative / nonholonomic variables can be stabilized by corresponding classes of nonholonomic constraints on gravitational field and (geometric) evolution equations. This paper belongs to a series of three our works on gravity and spaces enabled with general symmetric nonsymmetric components metrics and related nonlinear and linear connection structures, see also partner articles [1, 2]. Our goal is to consider some knew applications in gravity physics and define the conditions when such gravitational “nonsymmetric” interactions can be modelled on Einstein spaces.

The Einstein gravity can be represented equivalently in almost Kähler (canonical almost symplectic) variables [3, 4], see a review of results in applications of the geometric formalism for constructing exact solutions in gravity [5] and modelling locally anisotropic interactions in standard theories of physics [6]. Following such an approach, the data for a (pseudo) Riemannian metric $g = \{g_{\mu\nu}\}$ and related Levi–Civita connection $\nabla[g] = \{\Gamma^\alpha_{\beta\gamma}[g]\}$ on a spacetime manifold $V$ (we shall write in brief $(g, \nabla)$) can be equivalently re–defined, in a unique form, into corresponding almost symplectic form $\theta = \{\theta_{\mu\nu}[g]\}$ and compatible symplectic connection $\nabla[D][\theta] = \{\nabla^D_\alpha[\theta_{\beta\gamma}]\}$ (we shall write in brief $(\theta, \nabla[D])$, for which $\nabla[D]\theta = \nabla[D]g = 0$. The almost symplectic/ Kähler connection $\nabla[D]$ is similar to the Cartan connection in Finsler–Lagrange geometry [7, 8, 9, 10, 11, 12, 13], but we emphasize that in this article we shall work only with geometric structures defined on nonholonomic (pseudo) Riemannian manifolds [2].

From a formal point of view, the Cartan’s almost symplectic connection contains nontrivial torsion components induced by the anholonomy coefficients [2]. But such a nonholonomically induced torsion is not similar to torsions from the Einstein–Cartan and/or string/gauge gravity theories, where certain additional field equations (to the Einstein equations) are considered for torsion fields. The almost Kähler variables are canonically defined for any 2+2 splitting (which allows us to define canonically an almost com-

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1 A pair $(V, N)$, where $V$ is a manifold and $N$ is a nonintegrable distribution on $V$, is called a nonholonomic manifold; we note that in our works we use left “up” and “low” symbols as formal labels for certain geometric objects and that the spacetime signature may be encoded into formal frame (vielbein) coefficients, some of them being proportional to the imaginary unity $i$, when $i^2 = -1$.

2 In mathematical and physical literature, there are used also some other equivalent terms like anholonomic, or non–integrable, restrictions/ constraints; we emphasize that in classical and quantum physics the field and evolution equations play a fundamental role but together with certain types of constraints and broken symmetries.
plex structure $\mathbf{J}$) in general relativity, and the induced "symplectic" torsion is completely determined by certain off–diagonal metric components under nonholonomic deformations of geometric structures. In such cases, $\theta(\mathbf{X}, \mathbf{Y}) \doteq g(\mathbf{JX}, \mathbf{Y})$, for any vectors $\mathbf{X}$ and $\mathbf{Y}$ on $\mathbf{V}$, and we can consider equivalently two linear connections subjected to a condition of type $\nabla[g] = \nabla[g] + Z[g]$, with the distorsion tensor $Z[g]$ completely defined by the original metric field $g$.

In classical general relativity, it is convenient to work with the variables $(g, \nabla)$ and, for instance, their tetradic or spinor representations. For different approaches to quantum gravity, there are considered $3 + 1$ spacetime decompositions (for instance, in the so–called Arnowit–Deser–Misner, ADM, formalism, Ashtekar variables and loop quantum gravity) or nonholonomic $2 + 2$ splittings, see a discussion of approaches and references in [14]. Even the almost symplectic variables $(\theta, \nabla)$ result in a more sophisticate form of gravitational field equations (similar situations exist for the the ADM and/or Ashtekar–Barbero representations of gravity), they allow us to apply directly the deformation quantization formalism and quantize general relativity following Fedosov’s methods [3, 4]. This is a rigorous mathematically quantization procedure which provides an alternative approach to quantum gravity (comparing to various loop, spin–networks methods, canonical quantization etc methods) even the problem of renormalization of gravity, if it exists also in a non–perturbative fashion, has not yet been approached in the Fedosov’s theory.

For the almost K"ahler representation of general relativity, the gravitational symplectic form is anti–symmetric, $\theta_{\mu\nu} = -\theta_{\nu\mu}$, and play the role of "anti–symmetric" metric. We can also obtain additional "nonsymmetric" metric contributions from the "de–quantization" procedure in deformation quantization of gravity, or (in a more straightforward form) from the theory of nonholonomic and/or noncommutative Ricci flows, see Refs. [15] [2] [16]. Such geometric quantum constructions and evolution models put in a new fashion the problem of gravity with nonsymmetric variables. There is already a long time history, beginning with A. Einstein [17] [18] and L. P. Eisenhart [19] [20], when the so–called nonsymmetric gravity theories have been elaborated in different modifications by J. Moffat and co–authors [21] [22] [23] [24] [25] [26] [27], see also a recent contribution in Ref. [28].

A series of works by T. Janssen and T. Prokopec [29] [30] [31] is devoted to the so–called "problem of instabilities" in NGT. The authors agreed that one can be elaborated a model of NGT with nonzero mass term for the nonsymmetric part of metric (treated as an absolutely symmetric torsion induced by an effective $B$–field like in string gravity, but in four dimensions).
That solved the problems formally created by absence of gauge invariance found by Damour, Deser and McCarthy \cite{32}, see explicit constructions and detailed discussions in \cite{23, 33}. It was also emphasized that, as a matter of principle, the Clayton’s effect \cite{34} (when, for a general relativity background, a small $B$–field for the nonsymmetric part quickly grows) may be stabilized by solutions with evolving backgrounds \cite{35} and/or introducing an extra Lagrange multiplier when the unstable modes dynamically vanish \cite{36}.

Nevertheless, the general conclusion following from works \cite{29, 30, 31} is that instabilities in NGT should not be seen as a relic of the linearized theory because certain nonlinearized NGT models with nontrivial Einstein background (for instance, on Schwarzschild spacetime) are positively unstable. Such solutions can not be stabilized by the former methods with dynamical solutions and, as a consequence, certain new models of NGT and methods of stabilizations should be developed.

It should be emphasized that the Janssen–Prokopek stability problem does not have a generic character for all models of gravity with nonsymmetric variables. As we emphasized above, the Einstein gravity can be represented equivalently in canonical almost symplectic variables and such a formal theory with nonsymmetric metric (nonholonomically transformed into components of a symplectic form) is stable under deformations of the Schwarzschild metric. But in such a representation, we have also certain nontrivial nonholonomic structures. So, it is important to study the problem of stability of physical valuable solutions in general relativity under nonholonomic deformations, which may keep the constructions in the framework of the Einstein theory (with certain classes of imposed non–integrable constraints), or may generalize the gravity theory to models with nontrivial contributions from Ricci flow evolution (for instance, under variation of gravitational constants) and/or from a noncommutative/quantum gravity theory.

The goal of this paper is to prove that stable configurations can be derived for various models of nonsymmetric gravity theories (NGT) \cite{21, 22, 23, 24, 25, 26, 27}. We shall use a geometric techniques elaborated in Refs. \cite{6, 2, 1, 37, 38} and show how nonholonomic frame constraints can be imposed in order to generate stable solutions in NGT. For vanishing nonsymmetric components of metrics such configurations can be reduced to nonholonomic\footnote{equivalently, there are used the terms anholonomic and/or nonitegrable} ones in general relativity (GR) theory and generalizations. We shall provide explicit examples of stationary solutions with ellipsoidal symmetry which can be constructed in NGT and GR theories; such metrics
are stable and transform into the Schwarzschild one for zero eccentricities.

In brief, the Janssen–Prokopec method proving that a full, nonlinearized, NGT may suffer from instabilities can be summarized in this form: One shows that there is only one stable linearized Lagrangian (see in Ref. [29] the formula (A26) which can be obtained from their formula (86); similar formulas, (52) and (54), are provided below in Section 3). Then, following certain explicit computations for different backgrounds in general relativity, one argues that for the Schwarzschild background the mentioned variant of stable Lagrangian cannot be obtained by linearizing NGT (because in such cases, the coefficient $\gamma$ in the mentioned formulas, can not be zero for the static spherical symmetric background in GR).

Generalizing the constructions from [29] in order to include certain types of nonholonomic distributions on (non) symmetric spacetime manifolds, we shall prove that stable Lagrangians can be generated by a superposition of nonholonomic transforms and linearization in general models of NGT with compatible (nonsymmetric) metrics and nonlinear and linear connection structures.

We argue that fixing from the very beginning an ansatz with spherical symmetry background (for instance, the Schwarzschild solution in gravity), one eliminates from consideration a large class of physically important symmetric and nonsymmetric nonlinear gravitational interactions. The resulting instability of such constrained to a given background solutions reflects the proprieties of some very special classes of solutions but not any intrinsic, fundamental, general characteristics of NGT. For instance, we shall construct explicit “ellipsoidal” stationary solutions in NGT to which a static Schwarzschild metric is deformed by very small nonsymmetric metric components and nonholonomic distributions and which seem to be stable for geometric distortions in Einstein gravity [39, 40] . Such metrics were constructed for different models of metric–affine, generalized Finsler on nonholonomic manifolds and noncommutative gravity [11, 5, 38] and can be included in NGT both by nonsymmetric metric components and/or as a nonholonomic symmetric background, see examples from Ref. [2].

The paper is organized as follows: In Section 2, we outline some basic results from the geometry of nonholonomic manifolds and NGT models on such spaces. The equivalent formulation of the Einstein gravity in canonical almost symplectic variables is provided. Section 3 is devoted to a method of nonholonomic deformations and linearization to backgrounds with symmet-

4 in this work, we can consider that the nonsymmetric components of a general metric induce such geometric and effective matter field distortions
ric metrics and nonholonomic distributions. We show how certain classes of nonsymmetric metric configurations can be stabilized by corresponding nonholonomic constraints. We present an explicit example in Section 4, when stable stationary solutions with nontrivial nonsymmetric components of metric and nonholonomic distributions are constructed as certain deformations of the Schwarzschild metric to an ellipsoidal nonholonomic background on which a constrained dynamics on nonsymmetric metric fields is modelled. Finally, in Section 5 we present conclusions and discuss the results. In Appendix, we provide some important formulas on torsion and curvature of linear connections adapted to a prescribed nonlinear connection structure.

2 Einstein Gravity in Almost Kähler Variables

In general relativity (GR), we consider a real four dimensional (pseudo) Riemannian spacetime manifold $V$ of signature $(-,+,+,+)$ and necessary smooth class. For a conventional $2+2$ splitting, the local coordinates $u = (x,y)$ on an open region $U \subset V$ are labelled in the form $u^\alpha = (x^i, y^a)$, where indices of type $i, j, k, \ldots = 1, 2$ and $a, b, c, \ldots = 3, 4$, for tensor like objects, will be considered with respect to a general (non-coordinate) local basis $e_\alpha = (e_i, e_a)$. One says that $x^i$ and $y^a$ are respectively the conventional horizontal/ holonomic (h) and vertical / nonholonomic (v) coordinates (both types of such coordinates can be time– or space–like ones). Primed indices of type $i', a'$, ... will be used for labelling coordinates with respect to a different local basis $e_\alpha' = (e_{i'}, e_{a'})$ or $e_\alpha'' = (e'_{i'}, e'_{a'})$, for instance, for an orthonormalized basis. For the local tangent Minkowski space, we chose $e_{0'} = i\partial/\partial u^0$, where $i$ is the imaginary unity, $i^2 = -1$, and write $e_{\alpha'} = (i\partial/\partial u^0, \partial/\partial u^1, \partial/\partial u^2, \partial/\partial u^3)$. To consider such formal Euclidean coordinates is useful for some purposes of analogous modelling of gravity theories as effective Lagrange mechanics geometries, but this does not mean that we introduce any complexification of classical spacetimes. In this section, we outline the constructions for classical gravity from [3, 4, 12].

2.1 N–anholonomic (pseudo) Riemannian manifolds

The coefficients of a general (pseudo) Riemannian metric on a spacetime $V$ are parametrized in the form:

$$g = g_{i' j'}(u)e^{i'} \otimes e^{j'} + h_{a' b'}(u)e^{a'} \otimes e^{b'},$$

$$e^{a'} = e^a - N^a_{b'}(u)e^{b'},$$

(1)
where the required form of vierbein coefficients $e'_\alpha$ of the dual basis

$$e'_\alpha = (e', e'^\prime) = e'_\alpha(u) du^\alpha,$$

(2)
defining a formal $2+2$ splitting, will be stated below.

On spacetime $V$, we consider any generating function $L(u) = L(x^i, y^a)$
(we may call it a formal Lagrangian if an effective continuous mechanical
model of GR is to be elaborated, see Refs. [37, 38]) with nondegenerate
Hessian

$$L h_{ab} = \frac{1}{2} \frac{\partial^2 L}{\partial y^a \partial y^b},$$

(3)
when $\det |L h_{ab}| \neq 0$. This function is useful for constructing in explicit form
a nonholonomic $2+2$ splitting for which a canonical almost symplectic model
of GR will be defined. We use $L$ as an abstract label and emphasize that
the geometric constructions are general ones, not depending on the type
of function $L(u)$ which states only a formal class of systems of reference
and coordinates. Working with such local fibrations, it is a more simple
procedure to define almost symplectic variables in GR. We introduce

$$L N_i^a = \frac{\partial G^a}{\partial y^{2+i}},$$

(4)
for

$$G^a = \frac{1}{4} L h_{2+i} \left( \frac{\partial^2 L}{\partial y^{2+i} \partial x^{2+k}} y^{2+k} - \frac{\partial L}{\partial x^i} \right),$$

(5)
where $L h_{ab}$ is inverse to $L h_{ab}$ and respective contractions of $h-$ and $v-$
indices, $i, j, ...$ and $a, b, ..$, are performed following the rule: we can write,
for instance, an up $v-$index $a$ as $a = 2 + i$ and contract it with a low index
$i = 1, 2$. Briefly, we shall write $y'$ instead of $y^{2+i}$, or $y^a$. The values (3), (4)
and (5) allow us to define

$$L g = L g_{ij} dx^i \otimes dx^j + L h_{ab} L e^a \otimes L e^b,$$

$$L e^a = dy^a + L N_i^a dx^i, L g_{ij} = L h_{2+i, 2+j}.$$

(6)

A metric $g$ (1) with coefficients $g_{\alpha'\beta'} = [g_{\alpha'j'}, h_{a'b'}]$ computed with re-
spect to a dual basis $e'^\alpha = (e'^i, e'^a)$ can be related to the metric
$L g_{\alpha\beta} = [L g_{ij}, L h_{ab}]$ (4) with coefficients defined with respect to a $N-$adapted dual
basis $L e^\alpha = (dx^i, L e^a)$ if there are satisfied the conditions

$$g_{\alpha'\beta'} e'_{\alpha} e'^{\beta'} = L g_{\alpha\beta}.$$  

(7)
Considering any given values $g_{\alpha'\beta'}$ and $L g_{\alpha\beta}$, we have to solve a system of quadratic algebraic equations with unknown variables $e'^{\alpha'}_\alpha$. How to define locally such coordinates, we discuss in Ref. [6,14]. For instance, in GR, there are 6 independent values $g_{\alpha'\beta'}$ and up till ten coefficients $L g_{\alpha\beta}$ which allows us always to define a set of vierbein coefficients $e'^{\alpha'}_\alpha$. Usually, a subset of such coefficients can be taken be zero, for given values $[g_{\nu'j'}, h_{\alpha'b'}, N'^a_i]$ and $[L g_{ij}, L h_{ab}, L N'^a_i]$, when

$$N'^a_i = e'^i_a e'^{\alpha'}_\alpha L N'^a_i$$

for $e'^i_i$ being inverse to $e'^{\alpha'}_i$.

For simplicity, in this work, we suppose that there is always a finite covering of $V^{2+2}$ (in brief, denoted $\mathbf{V}$) by a family of open regions $I^I U$, labelled by an index $I$, on which there are considered certain nontrivial effective Lagrangians $I L$ with real solutions $I e'^{\alpha'}_\alpha$ defining vierbein transforms to systems of so-called Lagrange variables. Finally, we solve the algebraic equations (7) for any prescribed values $g_{\nu'j'}$ (we also have to change the partition $I^I U$ and generating function $I L$ till we are able to construct real solutions) and find $I e'^i_i$ which, in its turn, allows us to compute $N'^a_i$ (8) and all coefficients of the metric $g$ (1) and vierbein transform (2). We shall omit for simplicity the left label $L$ if that will not result in a confusion for some special constructions.

A nonlinear connection (N–connection) structure $\mathbf{N}$ for $\mathbf{V}$ is defined by a nonholonomic distribution (a Whitney sum)

$$T \mathbf{V} = h \mathbf{V} \oplus v \mathbf{V}$$

into conventional horizontal (h) and vertical (v) subspaces. In local form, a N–connection is given by its coefficients $N'^a_i(u)$, when

$$\mathbf{N} = N'^a_i(u) dx^i \otimes \frac{\partial}{\partial y^a}. \quad (10)$$

A N–connection introduces on $\mathbf{V}^{n+n}$ a frame (vierbein) structure

$$e_\nu = \left( e_i = \frac{\partial}{\partial x^i} - N'^a_i(u) \frac{\partial}{\partial y^a}, e_a = \frac{\partial}{\partial y^a} \right), \quad (11)$$

and a dual frame (coframe) structure

$$e^\mu = \left( e^i = dx^i, e^a = dy^a + N'^a_i(u) dx^i \right). \quad (12)$$
The vielbeins (12) satisfy the nonholonomy relations

\[ [e_\alpha, e_\beta] = e_\alpha e_\beta - e_\beta e_\alpha = w^\gamma_{\alpha\beta} e_\gamma \]  

(13)

with (antisymmetric) nontrivial anholonomy coefficients \( w^b_{ia} = \partial_a N^b_i \) and \( w^a_{ji} = \Omega^a_{ij} \), where

\[ \Omega^a_{ij} = e_j (N^a_i) - e_i (N^a_j) \]  

(14)

are the coefficients of \( N \)-connection curvature (defined as the Neijenhuis tensor on \( V^{n+n} \)). The particular holonomic/ integrable case is selected by the integrability conditions \( w^\gamma_{\alpha\beta} = 0 \).

A \( N \)-anholonomic manifold is a (nonholonomic) manifold enabled with \( N \)-connection structure (9). The geometric properties of a \( N \)-anholonomic manifold are distinguished by some \( N \)-adapted bases (11) and (12). A geometric object is \( N \)-adapted (equivalently, distinguished), i.e. a \( d \)-object, if it can be defined by components adapted to the splitting (9) (one uses terms \( d \)-vector, \( d \)-form, \( d \)-tensor). For instance, a \( d \)-vector \( \mathbf{X} = X^\alpha e_\alpha = X^i e_i + X^a e_a \) and a one \( d \)-form \( \tilde{\mathbf{X}} \) (dual to \( \mathbf{X} \)) is \( \tilde{\mathbf{X}} = X^\alpha e_\alpha = X_i e^i + X_a e_a \).

2.2 Canonical almost symplectic structures in GR

Let \( e_{\alpha'} = (e_i, e'_b) \) and \( e^{\alpha'} = (e^i, e'^b) \) be defined respectively by (11) and (12) for the canonical \( N \)-connection \( L_N \) stated by a metric structure \( g = L g \) on \( V \). We introduce a linear operator \( \mathbf{J} \) acting on vectors on \( V \) following formulas

\[ \mathbf{J}(e_i) = -e_{2+i} \text{ and } \mathbf{J}(e_{2+i}) = e_i, \]

where and \( \mathbf{J} \circ \mathbf{J} = -\mathbf{I} \) for \( \mathbf{I} \) being the unity matrix, and construct a tensor field on \( V \),

\[ \mathbf{J} = J^\alpha_{\beta'} e_\alpha \otimes e^{\beta'} = J^\alpha_{\beta'} \frac{\partial}{\partial u^\alpha} \otimes du^\beta \]

(15)

\[ = J^\alpha_{\beta'} e_\alpha' \otimes \epsilon^{\beta'} = -e_{2+i} \otimes e^i + e_i \otimes e^2_{2+i} \]

\[ = -\frac{\partial}{\partial y^i} \otimes dx^i + \left( \frac{\partial}{\partial x^i} - \frac{L N^2_{1+j}}{L N^2_{1+j} dx^k} \right) \otimes \left( dy^i + \frac{L N^2_{1+k}}{L N^2_{1+k} dx^k} \right), \]

\[ \text{we use boldface symbols for spaces (and geometric objects on such spaces) enabled with } N \text{-connection structure} \]

\[ \text{We can redefine equivalently the geometric constructions for arbitrary frame and coordinate systems; the } N \text{-adapted constructions allow us to preserve the } h \text{- and } v \text{-splitting.} \]
defining globally an almost complex structure on \( V \) completely determined by a fixed \( L(x,y) \). Using vielbeins \( \mathbf{e}_\alpha^a \) and their duals \( \mathbf{e}_a^\alpha \), defined by \( \mathbf{e}^\alpha_a \) solving (7), we can compute the coefficients of tensor \( J \) with respect to any local basis \( e_\alpha \) and \( e^\alpha \) on \( V \), \( J_\beta^\alpha = e^\alpha_a J_\beta^a e_\beta^\alpha \). In general, we can define an almost complex structure \( J \) for an arbitrary \( N \)-connection \( N \), stating a nonholonomic 2 + 2 splitting, by using \( N \)-adapted bases (11) and (12).

The Neijenhuis tensor field for any almost complex structure \( J \) defined by a \( N \)-connection (equivalently, the curvature of \( N \)-connection) is

\[
J \Omega(\mathbf{X}, \mathbf{Y}) = -[\mathbf{X}, \mathbf{Y}] + [J\mathbf{X}, J\mathbf{Y}] - J[J\mathbf{X}, \mathbf{Y}] - J[\mathbf{X}, J\mathbf{Y}],
\]

(16)

for any \( d \)-vectors \( \mathbf{X} \) and \( \mathbf{Y} \). With respect to \( N \)-adapted bases (11) and (12), a subset of the coefficients of the Neijenhuis tensor defines the \( N \)-connection curvature, see details in Ref. [11],

\[
\Omega_{ij}^a = \frac{\partial N_j^a}{\partial x^i} - \frac{\partial N_i^a}{\partial y^j} + N_i^b \frac{\partial N_j^a}{\partial y^b} - N_j^b \frac{\partial N_i^a}{\partial y^b}.
\]

(17)

A \( N \)-anholonomic manifold \( V \) is integrable if \( \Omega_{ij}^a = 0 \). We get a complex structure if and only if both the \( h \)- and \( v \)-distributions are integrable, i.e. if and only if \( \Omega_{ij}^a = 0 \) and \( \frac{\partial N_i^a}{\partial y^b} - \frac{\partial N_j^a}{\partial y^b} = 0 \).

One calls an almost symplectic structure on a manifold \( V \) a nondegenerate 2–form

\[
\theta = \frac{1}{2} \theta_{\alpha\beta}(u) e^\alpha \wedge e^\beta.
\]

For any \( \theta \) on \( V \), there is a unique \( N \)-connection \( N = \{ N_i^a \} \) satisfying the conditions:

\[
\theta = (h \mathbf{X}, v \mathbf{Y}) = 0 \text{ and } \theta = h \theta + v \theta,
\]

(18)

for any \( \mathbf{X} = h \mathbf{X} + v \mathbf{X}, \mathbf{Y} = h \mathbf{Y} + v \mathbf{Y} \), where \( h \theta(\mathbf{X}, \mathbf{Y}) = \theta(h \mathbf{X}, h \mathbf{Y}) \) and \( v \theta(\mathbf{X}, \mathbf{Y}) \equiv \theta(v \mathbf{X}, v \mathbf{Y}) \).

For \( \mathbf{X} = \mathbf{e}_\alpha = (e_i, e_a) \) and \( \mathbf{Y} = \mathbf{e}_\beta = (e_i, e_b) \), where \( \mathbf{e}_\alpha \) is a \( N \)-adapted basis of type (11), we write the first equation in (18) in the form

\[
\theta = \theta(\mathbf{e}_i, e_a) = \theta(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial y^a}) - N_i^b \theta(\frac{\partial}{\partial y^b}, \frac{\partial}{\partial y^a}) = 0.
\]

We can solve this system of equations in a unique form and define \( N_i^b \) if \( \text{rank} |\theta(\frac{\partial}{\partial y^b}, \frac{\partial}{\partial y^a})| = 2 \). Denoting locally

\[
\theta = \frac{1}{2} \theta_{ij}(u) e^i \wedge e^j + \frac{1}{2} \theta_{ab}(u) e^a \wedge e^b,
\]

(19)
where the first term is for $h\theta$ and the second term is $v\theta$, we get the second formula in (18).

An almost Hermitian model of a (pseudo) Riemannian space $V$ equipped with a $N$–connection structure $N$ is defined by a triple $H^{2+2} = (V, \theta, J)$, where $\theta(X, Y) = g(JX, Y)$ for any $g$ (1). A space $H^{2+2}$ is almost Kähler, denoted $K^{2+2}$, if and only if $d\theta = 0$.

For $g = Lg$ (6) and structures $L^N$ (4) and $J$ canonically defined by $L$, we define $L\theta(X, Y) = Lg(JX, Y)$ for any $d$–vectors $X$ and $Y$. In local $N$–adapted form form, we have

$$L\theta = \frac{1}{2} L\theta_{ij}(u)e^i \wedge e^j = \frac{1}{2} L\theta_{ij}(u)du^i \wedge du^j \quad (20)$$

$$= Lg_{ij}(x, y)e^{2+i} \wedge dx^j = Lg_{ij}(x, y)(dy^{2+i} + LN_{k}^{2+i}dx^k) \wedge dx^j.$$ Let us consider the form $L\omega = \frac{1}{2}\frac{\partial L}{\partial y^i}dx^i$. A straightforward computation shows that $L\theta = d L\omega$, which means that $d L\theta = dd L\omega = 0$, i.e. the canonical effective Lagrange structures $g = Lg$, $L^N$ and $J$ induce an almost Kähler geometry. We can express the 2–form (20) as

$$\theta = L\theta = \frac{1}{2} L\theta_{ij}(u)e^i \wedge e^j + \frac{1}{2} L\theta_{ab}(u)e^a \wedge e^b \quad (21)$$

$$= g_{ij}(x, y) \left[ dy^i + N^i_k(x, y)dx^k \right] \wedge dx^j,$$

see (19), where the coefficients $L\theta_{ab} = L\theta_{2+i, 2+j}$ are equal respectively to the coefficients $L\theta_{ij}$. It should be noted that for a general 2–form $\theta$ constructed for any metric $g$ and almost complex $J$ structures on $V$ one holds $d\theta \neq 0$. But for any $2 + 2$ splitting induced by an effective Lagrange generating function, we have $d L\theta = 0$. We have also $d \theta = 0$ for any set of 2–form coefficients $\theta_{\alpha', \beta'}e^{\alpha'}e^{\beta'} = L\theta_{\alpha', \beta'}$ (such a 2–form $\theta$ will be called to be a canonical one), constructed by using formulas (7).

We conclude that having chosen a generating function $L(x, y)$ on a (pseudo) Riemannian spacetime $V$, we can model this spacetime equivalently as an almost Kähler manifold

### 2.3 Equivalent metric compatible linear connections

A distinguished connection (in brief, d–connection) on a spacetime $V$,

$$D = (hD; vD) = \{ \Gamma^\alpha_{\beta \gamma} = (L^i_{jk}; vL^a_{bk}; C^i_{jc}; vC^a_{bc}) \},$$

is a linear connection which preserves under parallel transports the distribution (9). In explicit form, the coefficients $\Gamma^\alpha_{\beta \gamma}$ are computed with respect
to a N–adapted basis (11) and (12). A d–connection $D$ is metric compatible with a d–metric $g$ if $D_X g = 0$ for any d–vector field $X$.

If an almost symplectic structure $\theta$ is considered on a N–anholonomic manifold, an almost symplectic d–connection $\theta D$ on $V$ is defined by the conditions that it is N–adapted, i.e. it is a d–connection, and $\theta D X \theta = 0$, for any d–vector $X$. From the set of metric and/or almost symplectic compatible d–connections on a (pseudo) Riemannian manifold $V$, we can select those which are completely defined by a metric $g = L g$ (6) and an effective Lagrange structure $L(x, y)$:

There is a unique normal d–connection

$$\hat{D} = \begin{cases} h \hat{D} = (\hat{D}_k, v \hat{D}_k = \hat{D}_k) ; v \hat{D} = (\hat{D}_c, v \hat{D}_c = \hat{D}_c) \end{cases} \quad (22)$$

which is metric compatible, $\hat{D}_k L g_{ij} = 0$ and $\hat{D}_c L g_{ij} = 0$,

and completely defined by a couple of h– and v–components $\hat{D}_\alpha = (\hat{D}_k, \hat{D}_c)$, with N–adapted coefficients $\hat{\Gamma}_{\beta\gamma}^\alpha = (\hat{\Gamma}_{jk}^i, \hat{C}_{bc}^a)$, where

$$\hat{L}_{jk}^i = \frac{1}{2} L g^{ik}(e_k L g_{jh} + e_j L g_{hk} - e_h L g_{jk}), \quad (23)$$

$$\hat{C}_{jk}^i = \frac{1}{2} L g^{ik}\left(\frac{\partial L g_{jh}}{\partial y^k} + \frac{\partial L g_{hk}}{\partial y^j} - \frac{\partial L g_{jk}}{\partial y^h}\right).$$

In general, we can "foget" about label $L$ and work with arbitrary $g_{\alpha'\beta'}$ and $\hat{\Gamma}_{\beta'\gamma'}$ with the coefficients recomputed by frame transforms (2).

Introducing the normal d–connection 1–form

$$\hat{\Gamma}_{ij} = \hat{L}_{jk} e^k + \hat{C}_{jk}^i e^k,$$

we prove that the Cartan structure equations are satisfied,

$$d e^k - e^j \wedge \hat{\Gamma}_{ij}^k = -\hat{T}_i^j, \quad d e^k - e^j \wedge \hat{\Gamma}_{ij}^k = -v \hat{T}_i^j, \quad (24)$$

and

$$d \hat{\Gamma}_{ij}^k - \hat{\Gamma}_{ij}^h \wedge \hat{\Gamma}_{ih}^k = -\hat{R}_{ij}^k. \quad (25)$$

The h– and v–components of the torsion 2–form

$$\hat{T}_\alpha = \left(\hat{T}_i, v \hat{T}_i\right) = \hat{T}_{\alpha\beta} e^\alpha \wedge e^\beta$$
from (24) is computed with components

$$\hat{T}^i = \hat{C}_{jk} e^j \wedge e^k, \quad v \hat{T}^i = \frac{1}{2} L \Omega^i_{kj} e^k \wedge e^j + \left( \frac{\partial}{\partial y} L \Omega^i_{kj} \right) e^k \wedge e^j, \quad (26)$$

where $L \Omega^i_{kj}$ are coefficients of the curvature of the canonical N–connection $\tilde{N}_k^i$ defined by formulas similar to (17). The formulas (26) parametrize the h– and v–components of torsion $\hat{T}^i_{\beta \gamma}$ in the form

$$\hat{T}^i_{jk} = 0, \quad \hat{T}^i_{jc} = \hat{C}_{ijc}, \quad \hat{T}^i_{aj} = L \Omega^a_{ij}, \quad \hat{T}^i_{ab} = e_b \left( L N^a_i \right) - \hat{L}^a_{bi}, \quad (27)$$

It should be noted that $\hat{T}$ vanishes on h– and v–subspaces, i.e. $\hat{T}^i_{jk} = 0$ and $\hat{T}^i_{bc} = 0$, but certain nontrivial h–v–components induced by the nonholonomic structure are defined canonically by $g = L g$ and $L$. For convenience, in Appendix A we outline some important component formulas for the canonical d–connection which on spaces of even dimensions transform into those for the normal connection.

We compute also the curvature 2–form from (25),

$$\hat{R}^\tau_{\beta \gamma} = \hat{R}^\tau_{\gamma \alpha \beta} e^\alpha \wedge e^\beta$$

$$= \frac{1}{2} \hat{R}^i_{jkhe} e^k \wedge e^h + \hat{P}^i_{jka} e^k \wedge e^a + \frac{1}{2} \hat{S}^i_{jcd} e^c \wedge e^d, \quad (28)$$

where the nontrivial N–adapted coefficients of curvature $\hat{R}^\alpha_{\beta \gamma \tau}$ of $\hat{D}$ are

$$\hat{R}^i_{hjk} = e_k \hat{L}^i_{hk} - e_j \hat{L}^i_{hk} + \hat{L}^i_{hk} \hat{L}^j_{mk} - \hat{L}^i_{hk} \hat{L}^j_{m_k} - \hat{C}_h^i L \Omega^a_{kj}$$

$$\hat{P}^i_{jka} = e_a \hat{L}^i_{jk} - \hat{D}_k \hat{C}^i_{ja},$$

$$\hat{S}^a_{bcd} = e_d \hat{C}^a_{bc} - e_c \hat{C}^a_{bd} + \hat{C}^e_{bd} \hat{C}^a_{ed} - \hat{C}^e_{bc} \hat{C}^a_{ec}. \quad (29)$$

Contracting the first and forth indices $\hat{R}_{\beta \gamma} = \hat{R}_{\beta \gamma \alpha}$, we get the N–adapted coefficients for the Ricci tensor

$$\hat{R}_{\beta \gamma} = \left( \hat{R}_{ij}, \hat{R}_{ia}, \hat{R}_{ai}, \hat{R}_{ab} \right). \quad (30)$$

The scalar curvature $L R = \hat{R}$ of $\hat{D}$ is

$$L R = L g^{\gamma \beta} \hat{R}_{\beta \gamma} = g^{\beta \gamma'} \hat{R}_{\beta \gamma'}. \quad (31)$$

The normal d–connection $\tilde{D}$ (22) defines a canonical almost symplectic d–connection, $\tilde{D} \equiv \theta \tilde{D}$, which is N–adapted to the effective Lagrange and,
related, almost symplectic structures, i.e. it preserves under parallelism the splitting \( \tilde{\Theta} \), \( \theta \tilde{D}_X \frac{L}{g} \theta \) \( = \theta \tilde{D}_X \theta = 0 \) and its torsion is constrained to satisfy the conditions \( \tilde{T}_{jk} = \tilde{T}_{kj} = 0 \).

In the canonical approach to the general relativity theory, one works with the Levi Civita connection \( \nabla = \{ \Gamma^\alpha_{\beta\gamma} \} \), which is uniquely derived following the conditions \( \mathcal{T} = 0 \) and \( \nabla g = 0 \). This is a linear connection but not a d–connection because \( \nabla \) does not preserve (9) under parallelism. Both linear connections \( \nabla \) and \( \hat{D} \equiv \theta \hat{D} \) are uniquely defined in metric compatible forms by the same metric structure \( g \) (1). The second one contains nontrivial d–torsion components \( \hat{T}^\alpha_{\beta\gamma} \), induced effectively by an equivalent Lagrange metric \( g = Lg \) (6) and adapted both to the N–connection \( LN \), see (4) and (9), and almost symplectic \( L\theta \) (20) structures.

Any geometric construction for the normal d–connection \( \hat{D}(\theta) \) can be re–defined by the Levi Civita connection, and inversely, using the formula

\[
\Gamma^\gamma_{\alpha\beta}(\theta) = \hat{\Gamma}^\gamma_{\alpha\beta}(\theta) + \frac{1}{2} \hat{Z}^\gamma_{\alpha\beta}(\theta),
\]

where the both connections \( \Gamma^\gamma_{\alpha\beta}(\theta) \) and \( \hat{\Gamma}^\gamma_{\alpha\beta}(\theta) \) and the distortion tensor \( \hat{Z}^\gamma_{\alpha\beta}(g) \) with N–adapted coefficients (for the normal d–connection \( \hat{T}^\gamma_{\alpha\beta}(g) \) is proportional to \( T^\gamma_{\alpha\beta}(g) \) (27)), see formulas (A.7). In this work, we emphasize if it is necessary the functional dependence of certain geometric objects on a d–metric \( (g) \), or its canonical almost symplectic equivalent \( \theta \) for tensors and connections completely defined by the metric structure.

If we work with nonholonomic constraints on the dynamics/geometry of gravity fields in deformation quantization, it is more convenient to use a N–adapted and/or almost symplectic approach. For other purposes, it is preferred to use only the Levi–Civita connection. Introducing the distortion relation (32) into respective formulas (27), (29) and (30) written for \( \hat{\Gamma}^\gamma_{\alpha\beta} \), we get deformations

\[
\frac{1}{2} T^\gamma_{\alpha\beta}(g) = \hat{T}^\gamma_{\alpha\beta}(g) + \frac{1}{2} \hat{Z}^\gamma_{\alpha\beta}(g), \quad R^\gamma_{\beta\gamma}(g) = \hat{R}^\gamma_{\beta\gamma}(g) + \hat{Z}^\gamma_{\beta\gamma}(g),
\]

see Refs. [37, 38] for explicit formulas for distortions of the torsion, curvature, Ricci tensors, i.e. for \( T^\gamma_{\alpha\beta}(g) \), \( \hat{T}^\gamma_{\alpha\beta}(g) \) and \( \hat{Z}^\gamma_{\beta\gamma}(g) \), which are completely defined by a metric structure \( g = Lg \) with a nonholonomic 2+2 splitting induced by a prescribed regular \( L \). Such formulas can be re–defined equivalently for \( T^\gamma_{\alpha\beta}(\theta) \), \( \hat{T}^\gamma_{\alpha\beta}(\theta) \) and \( \hat{Z}^\gamma_{\beta\gamma}(\theta) \), written only in terms of the canonical almost symplectic from \( \theta \) (21).

\[\text{\footnotesize 7see Appendix on similar deformation properties of fundamental geometric objects}\]
2.4 An almost symplectic formulation of GR

Having chosen a canonical almost symplectic d–connection, we compute the Ricci d–tensor \( \hat{R}_{\beta\gamma} \) (30) and the scalar curvature \( \hat{L} \) (31). Then, we can postulate in a straightforward form the field equations

\[
\hat{R}_{\beta}^\alpha - \frac{1}{2} (L + \lambda) e_{\beta}^\alpha = 8\pi G T_{\beta}^\alpha,
\]

where \( \hat{R}_{\beta}^\alpha = e_\gamma^\alpha \hat{R}_{\gamma\beta} \), \( T_{\beta}^\alpha \) is the effective energy–momentum tensor, \( \lambda \) is the cosmological constant, \( G \) is the Newton constant in the units when the light velocity \( c = 1 \), and the coefficients \( e_\beta^\alpha \) of vierbein decomposition \( e_\beta = e_\beta^\alpha \partial/\partial u^\alpha \) are defined by the N–coefficients of the N–elongated operator of partial derivation, see (11). But the equations (34) for the canonical \( \hat{\Gamma}_{\alpha\beta}^\gamma(\theta) \) are not equivalent to the Einstein equations in GR written for the Levi–Civita connection \( \bar{\Gamma}_{\alpha\beta}^\gamma(\theta) \) if the tensor \( T_{\beta}^\alpha \) does not include contributions of \( \bar{Z}_{\alpha\beta}^\gamma(\theta) \) in a necessary form.

Introducing the absolute antisymmetric tensor \( \epsilon_{\alpha\beta\gamma\delta} \) and the effective source 3–form \( T_{\gamma} = T_{\beta}^\alpha \epsilon_{\alpha\beta\gamma\delta} du^\beta \wedge du^\gamma \wedge du^\delta \)

and expressing the curvature tensor \( \hat{R}_{\gamma}^\alpha = \hat{R}_{\gamma\alpha\beta} e^\alpha \wedge e^\beta \) of \( \hat{\Gamma}_{\alpha\beta}^\gamma \) = \( \bar{\Gamma}_{\alpha\beta}^\gamma - \bar{Z}_{\alpha\beta}^\gamma \) as \( \hat{R}_{\gamma}^\alpha = \bar{R}_{\gamma}^\alpha - \bar{Z}_{\alpha\beta}^\gamma \) where \( \bar{R}_{\gamma}^\alpha = \bar{R}_{\gamma}^\alpha - \bar{Z}_{\alpha\beta}^\gamma \), we derive the equations (34) (varying the action on components of \( e_\beta \), see details in Ref. [14]). The gravitational field equations are represented as 3–form equations,

\[
\epsilon_{\alpha\beta\gamma\tau} \left( e^\alpha \wedge \hat{R}_{\beta\tau}^\gamma + \lambda e^\alpha \wedge e^\beta \wedge e^\tau \right) = 8\pi G T_{\tau},
\]

when

\[
T_{\tau} = m T_{\tau} + Z T_{\tau},
\]

\[
m T_{\tau} = m T_{\tau}^\alpha \epsilon_{\alpha\beta\gamma\delta} du^\beta \wedge du^\gamma \wedge du^\delta,
\]

\[
Z T_{\tau} = (8\pi G)^{-1} \hat{Z}_{\alpha\beta}^\gamma \epsilon_{\alpha\beta\gamma\delta} du^\delta \wedge du^\gamma \wedge du^\tau,
\]

where \( m T_{\tau}^\alpha \) is the matter tensor field. The above mentioned equations are equivalent to the usual Einstein equations for the Levi–Civita connection \( \bar{\nabla} \),

\[
\hat{R}_{\beta}^\alpha - \frac{1}{2} (L + \lambda) e_{\beta}^\alpha = 8\pi G m T_{\beta}^\alpha.
\]
The vacuum Einstein equations with cosmological constant, written in terms of the canonical N–adapted vierbeins and normal d–connection, are

\[ \epsilon_{\alpha\beta\gamma\tau} (e^\alpha \wedge \widehat{R}^{\beta\gamma} + \lambda e^\alpha \wedge e^\beta \wedge e^\gamma) = 8\pi G Z\widehat{T}_\tau, \]

with effective source \( Z\widehat{T}_\tau \) induced by nonholonomic splitting by the metric tensor and its off–diagonal components transformed into the N–connection coefficients or, in terms of the Levi–Civita connection

\[ \epsilon_{\alpha\beta\gamma\tau} \left( e^\alpha \wedge \bar{\bar{\Gamma}}^{\beta\gamma} + \lambda e^\alpha \wedge e^\beta \wedge e^\gamma \right) = 0. \]

Such formulas expressed in terms of canonical almost symplectic form \( \theta \) \((21)\) and normal d–connection \( \bar{D} \equiv \theta \bar{D} \) \((22)\) are necessary for encoding the vacuum field equations into cohomological structure of quantum almost Kähler models of the Einstein gravity, see \([3, 4, 12, 13, 14]\).

If former geometric constructions in GR were related to frame and coordinate form invariant transforms, various purposes in geometric modelling of physical interactions and quantization request application of more general classes of transforms. For such generalizations, the linear connection structure is deformed (in a unique/canonical form following well defined geometric and physical principles) and there are considered nonholonomic spacetime distributions. All geometric and physical information for any data

\begin{enumerate}
\item \( [g, \Gamma^\gamma_{\alpha\beta}(g)] \) are transformed equivalently for canonical constructions with
\item \( [g = Lg, N, \widehat{\Gamma}^\gamma_{\alpha\beta}(g)] \), which allows us to provide an effective Lagrange interpretation of the Einstein gravity, or
\item \( [\theta = L\theta, \theta \widehat{\Gamma}^\gamma_{\alpha\beta} = \widehat{\Gamma}^\gamma_{\alpha\beta}, J(\theta)] \),
\end{enumerate}

for an almost Kähler model of general relativity. The canonical almost symplectic form \( \theta \) \((21)\) represents the "original" metric \( g \) \((1)\) equivalently in a "nonsymmetric" form. Any deformations of such structures, in the framework of GR or quantized models and generalizations, result in more general classes of nonsymmetric metrics.

### 3 NGT with Nonholonomic Distributions

In this section, we follow the geometric conventions and results from Ref. [1]. The aim is to outline some basic definitions, concepts and formulas from the geometry of nonholonomic manifolds enabled with nonlinear connection and general nonsymmetric structure and introduce a general Lagrangian for NGT and corresponding nonholonomic distributions.
3.1 Preliminaries: geometry of N–anholonomic manifolds

In this paper, we also consider gravity models on spaces \((\hat{g}_{ij}, V^{n+n}, N)\) when the h–subspace is enabled with a nonsymmetric tensor field (metric) \(\hat{g}_{ij} = g_{ij} + a_{ij}\), where the symmetric part \(g_{ij} = g_{ji}\) is nondegenerated and \(a_{ij} = -a_{ji}\). A d–metric \(\hat{g}_{ij}(x, y)\) is of index \(k\) if there are satisfied the properties: 1. \(\det |g_{ij}| \neq 0\) and 2. \(\text{rank} |a_{ij}| = n - k = 2p\), for \(0 \leq k \leq n\). By \(g^{ij}\) we note the reciprocal (inverse) to \(g_{ij}\) d–tensor field. The matrix \(g_{ij}\) is not invertible unless for \(k = 0\).

We write by \(g^{ij}\) the reciprocal (inverse) to \(g_{ij}\) d–tensor field. The matrix \(g_{ij}\) is not invertible unless for \(k = 0\). For \(k > 0\) and a positive definite \(g_{ij}(x, y)\), on each domain of local chart there exists \(k\) d–vector fields \(\xi^i_{\prime j}\), where \(i = 1, 2, ..., n\) and \(i' = 1, ..., k\) with the properties

\[
a_{ij} \xi^i_{\prime j} = 0 \quad \text{and} \quad g_{ij} \xi^i_{\prime j} \xi^j_{\prime i} = \delta_{i'j'}.\]

If \(g_{ij}\) is not positive definite, we shall assume the existence of \(k\) linearly independent d–vector fields with such properties.

The metric properties on \(V^{n+n}\) are supposed to be defined by d–tensor

\[
\begin{align*}
\tilde{g} &= g + a = \tilde{g}_{\alpha\beta}e^\alpha \otimes e^\beta = \bar{g}_{ij}e^i \otimes e^j + \bar{g}_{ab}e^a \otimes e^b, \\
g &= g_{\alpha\beta}e^\alpha \otimes e^\beta = g_{ij}e^i \otimes e^j + g_{ab}e^a \otimes e^b, \\
a &= a_{ij}e^i \wedge e^j + a_{cb}e^c \wedge e^b,
\end{align*}
\]

where the \(v\)–components \(\tilde{g}_{ab}\) are defined by the same coefficients as \(\bar{g}_{ij}\). With respect to a coordinate local cobasis \(du^\alpha = (dx^i, dy^a)\), we have equivalently

\[
g = g_{\alpha\beta}du^\alpha \otimes du^\beta,
\]

where

\[
g_{\alpha\beta} = \begin{bmatrix} g_{ij} + N^i_{\ i}N^j_{\ j}g_{ab} & N^e_{\ j}g_{ae} \\ N^i_{\ e}g_{be} & g_{ab} \end{bmatrix}.
\]

A h–v–metric on a N–anholonomic manifold is a second rank d–tensor of type \(37\). We can define the local d–covector fields \(\eta^i_{\prime j} = g_{ij}\xi^i_{\prime j}\) and the d–tensors of type \((1,1), \ell^i_{\ j}\) and \(m^i_{\ j}\), satisfying the conditions

\[
\begin{align*}
\ell^i_{\ j} &= \xi^i_{\prime j} \quad \text{and} \quad \ell^i_{\ j} = \delta^i_{\ j} - \xi^i_{\prime j}n^i_{\ j}, \quad \text{for} \ i' = 1, ..., k; \\
m^i_{\ j} &= 0 \quad \text{and} \quad m^i_{\ j} = \delta^i_{\ j}, \quad \text{for} \ k = 0.
\end{align*}
\]

One considers the matrices

\[
\begin{align*}
\tilde{g} &= (g_{ij}), \tilde{a} = (a_{ij}), \tilde{\xi} = (\xi^i_{\prime j}), \tilde{l} = (\ell^i_{\ j}), \\
\tilde{\eta} &= (\eta^i_{\prime j}), \tilde{m} = (m^i_{\ j}), \tilde{\delta} = (\delta^i_{\ j}).
\end{align*}
\]

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The next step is to extend the matrix \( \hat{a} \) to a nonsingular skew symmetric one of dimension \((n + k, n + k)\),

\[
\tilde{a} = \begin{bmatrix}
\hat{a} & -t \varphi \\
\varphi & 0
\end{bmatrix}.
\]

The inverse matrix \( \tilde{a}^{-1} \), satisfying the condition \( \tilde{a}^{-1} = \hat{a} \delta \), has the form

\[
\tilde{a}^{-1} = \begin{bmatrix}
\tilde{a} & \tilde{\xi} \\
\varphi & 0
\end{bmatrix},
\]

where the matrix \( \tilde{a} = (\tilde{a}_{ij}) \) does not depend on the choice of \( \tilde{\xi} \) and it is uniquely defined by \( \hat{a} = t \tilde{\varphi} \) and \( \hat{a} = 0 \), i.e. this matrix is uniquely defined on \( V^{n+n} \).

In general, the concept of linear connection (adapted or not adapted to a N–connection structure) is independent from the concept of metric (symmetric or nonsymmetric). A distinguished connection (d–connection) \( D \) on \( V \) is a N–adapted linear connection, preserving by parallelism the vertical and horizontal distribution \([3]\). In local form, \( D = (hD, vD) \) is given by its coefficients \( \Gamma_{\alpha\beta}^\gamma = (L_{jk}^i, L_{bk}^a, C_{jc}^i, C_{bc}^a) \), where \( hD = (L_{jk}^i, L_{bk}^a) \) and \( vD = (C_{jc}^i, C_{bc}^a) \) are respectively the covariant h– and v–derivatives. For any d–connection, we can compute the torsion, curvature and Ricci tensors and scalar curvature, see Appendix.

A normal d–connection \( nD \) is compatible with the almost complex structure \( J \) \([15]\), i.e. satisfies the condition

\[
nD_X J = 0,
\]

for any d–vector \( X \) on \( V^{n+n} \). The operator \( nD \) is characterized by a pair of local coefficients \( n\Gamma_{\alpha\beta}^\gamma = (nL_{jk}^i, nC_{bc}^a) \) defined by conditions

\[
nD_{e_k}(e_j) = nL_{jk}^i e_i, \quad nD_{e_k}(e_a) = nL_{ak}^b e_b : \\
\text{for } j = a, i = b, \quad nL_{jk}^i = nL_{ak}^b,
\]

\[
nD_{e_c}(e_j) = nC_{jc}^i e_i, \quad nD_{e_c}(e_a) = nC_{ac}^b e_b : \\
\text{for } j = a, i = b, \quad nC_{jc}^i = nC_{ac}^b.
\]

Here we emphasize that the normal d–connection \( nD \) is different from \( \hat{D} \) \([22]\) (the first one is defined for a space with nonsymmetric metrics, but for the second one the metrics must be symmetric).
A d–connection \( D = \{ \Gamma_{\alpha\beta}^{\gamma} \} \) is compatible with a nonsymmetric d–metric \( \tilde{g} \) if

\[
D_k \tilde{g}_{ij} = 0 \quad \text{and} \quad D_a \tilde{g}_{ij} = 0.
\]

For a d–metric (37), the equations (43) are

\[
D_k \tilde{g}_{ij} = 0, D_a \tilde{g}_{bc} = 0, D_k a_{ij} = 0, D_e a_{bc} = 0.
\]

The set of d–connections \( \{ D \} \) satisfying the conditions \( D X g = 0 \) for a given \( g \) is defined by formulas

\[
L_{jk}^i = L_{jk}^i + -O_{km}^i X_{ej}^m, \quad L_{bk}^a = \tilde{L}_{bk}^a + -O_{bd}^a Y_{ck}^d,
\]

\[
C_{jc}^i = \tilde{C}_{jc}^i + +O_{jk}^m X_{mc}^k, \quad C_{bc}^a = \tilde{C}_{bc}^a + +O_{bd}^a Y_{ec}^d,
\]

where

\[
\pm O_{jk}^a = \frac{1}{2} (\delta_j^i \delta_k^a \pm g_{jk} g^{ih}), \quad \pm O_{bd}^a = \frac{1}{2} (\delta_b^c \delta_d^a \pm g_{bd} g^{ca})
\]

are the so–called the Obata operators; \( X_{ej}^m, X_{mc}^k, Y_{ck}^d \) and \( Y_{ec}^d \) are arbitrary d–tensor fields and \( \Gamma_{\alpha\beta}^{\gamma} = \left( \tilde{L}_{jk}^i, \tilde{L}_{bk}^a, \tilde{C}_{jc}^i, \tilde{C}_{bc}^a \right) \), with

\[
\tilde{L}_{jk}^i = \frac{1}{2} g^{ir} (e_k g_{jr} + e_j g_{kr} - e_r g_{jk}),
\]

\[
\tilde{L}_{bk}^a = e_b (N^a_j) + \frac{1}{2} g^{ac} \left( e_k g_{be} - g_{de} e_b N_k^d - g_{db} e_c N_k^d \right),
\]

\[
\tilde{C}_{jc}^i = \frac{1}{2} g^{ik} e_c g_{jk}, \quad \tilde{C}_{bc}^a = \frac{1}{2} g^{ad} (e_c g_{bd} + e_c g_{cd} - e_d g_{bc})
\]

is the canonical d–connections uniquely defined by the coefficients of d–metric \( g = [g_{ij}, g_{ab}] \) and N–connection \( N = \{ N^a_i \} \) in order to satisfy the conditions \( D X g = 0 \) and \( \tilde{T}_{jk}^i = 0 \) and \( \tilde{T}_{bk}^a = 0 \) but \( \tilde{T}_{ja}^i, \tilde{T}_{ji}^a \) and \( \tilde{T}_{bi}^a \) are not zero (on definition of torsion, see formulas (A.2) in Appendix; we can compute the torsion coefficients \( \tilde{T}_{\alpha\beta}^{\gamma} \) by introducing d–connection coefficients (46)).

By direct computations, we can check that for any given d–connection \( \Gamma_{\alpha\beta}^{\gamma} = \left( \circ L_{jk}^i, \circ C_{bc}^a \right) \) and nonsymmetric d–metric \( \tilde{g} = g + a \) on \( V \) the d–connection \( \star \Gamma_{\alpha\beta}^{\gamma} = \left( \star L_{jk}^i, \star C_{bc}^a \right) \), where

\[
\star L_{jk}^i = \circ L_{jk}^i + \frac{1}{2} g^{ir} \circ D_k g_{rj} + \pm O_{sj}^i (\hat{a}^{st} \circ D_k a_{tr} + 3 l^s_l d_k l^t_l - \circ D_k l^s_l),
\]

\[
\star C_{bc}^a = \circ C_{bc}^a + \frac{1}{2} g^{ah} \circ D_c g_{hb} + \pm O_{eb}^{ah} (\hat{a}^{ed} \circ D_c a_{dh} + 3 l^e_l d_c l^d_l - \circ D_c l^e_l),
\]
is $d$–metric compatible, i.e. satisfies the conditions $\mathcal{D}\tilde{g} = 0$.

The set of $d$–connections $\mathcal{D} = \mathcal{D} + \mathcal{B}$ being generated by deformations of an arbitrary fixed $d$–connection $\mathcal{D}$ in order to be compatible with a given nonsymmetric $d$–metric $\tilde{g} = g + a$ on $V$ is defined by distorsion $d$–tensors $\mathcal{B} = (B, v, B, n, D, \hat{D})$ which can be computed in explicit form, see Ref. [1].

In this paper, for simplicity, we shall work with a general $d$–connection $\mathcal{D}$ which is compatible to $\tilde{g}$, i.e. satisfies the conditions (44), or (43), and can be generated by a distorsion tensor $\mathcal{B}$ from $\hat{D}$ (46), or from $n, D$ (42). We note for certain canonical constructions the $d$–objects $\mathcal{D}, \mathcal{D}, \hat{D}, n, D$ and $\mathcal{B}$ are completely defined by the coefficients of a $d$–metric $\tilde{g} = g + a$ and $N$ on $V$.

Finally, it should be emphasized that because $\mathcal{D}\Gamma_{\beta\gamma} = (\mathcal{D}L_{jk}, \mathcal{D}C^a_{bc})$ is an arbitrary $d$–connection, it can be chosen to be an important one for certain physical or geometrical problems. In this work, we shall consider certain exact solutions in gravity with nonholonomic variables defining a corresponding $\mathcal{D}\Gamma_{\beta\gamma}$ and then deformed to nonsymmetric configurations following formulas (47).

### 3.2 General NGT models with $d$–connections

The goal of this section is to analyze $N$–adapted nonholonomic NGT models completely defined by a $N$–connection $\mathcal{N} = \{N^a_i\}$, $d$–metric $\tilde{g} = g + a$ (37) and a metric compatible $d$–connection $\Gamma^\alpha_{\mu\nu}$.

We follow a $N$–adapted variational calculus, when instead of partial derivatives there are used the ”$N$–elongated” partial derivatives $e^\rho$ (11), varying independently the $d$–fields $\tilde{g} = g + a$ and $\Gamma^\alpha_{\beta\gamma}$. In this case, $\tilde{a} = (\tilde{a}^i)$ does not depend on the choice of fields $\tilde{\xi}$, see (11), and we can write $\tilde{g}^{\rho\sigma} = \tilde{a}^{\rho\sigma} = [\tilde{a}^i, \tilde{a}^j]$, where $\tilde{a}^i = -\tilde{a}^j$ and $\tilde{a}^{ij} = -\tilde{a}^{ji}$. We shall work with $d$–connections,

$$W^\alpha_{\mu\nu} \doteq \frac{1}{2} \left( \Gamma^\alpha_{\mu\nu} - \frac{2}{3} \delta^\alpha_{\mu} \right) W^\mu_{\nu},$$

where $W^\mu_{\nu} = \frac{1}{2} \left( W^\alpha_{\mu\lambda} - W^\alpha_{\lambda\mu} \right)$, which means that $\Gamma^\alpha_{\beta\gamma} = 0$. This defines a covariant derivative of type

$$wD_{\gamma} \tilde{g}_{\alpha\beta} = e_{\gamma} \tilde{g}_{\alpha\beta} - W^\tau_{\alpha\gamma} \tilde{g}_{\tau\beta} - W^\tau_{\beta\gamma} \tilde{g}_{\alpha\tau}.$$ 

We also can compute

$$P^\mu_{\nu} \doteq wR^\alpha_{\lambda\mu\nu} = e_{\mu} W^\lambda_{\lambda\nu} - e_{\nu} W^\lambda_{\lambda\mu}.$$ 

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where $W R_{\lambda \mu \nu}$ is computed following formulas (A.6) with d–connection $W$ instead of $\Gamma$. The corresponding to $W$ and $\Gamma$ Ricci d–tensors, are related by formulas

$$W R_{\mu \nu} = r R_{\mu \nu} + \frac{2}{3} e_{[\nu} W_{\mu]} ,$$

where $r R_{\mu \nu}$ are given by formulas (30). The variables of this generalized theory, with gravitational constant $(16 \pi G_N)^{-1} = 1$, are parametrized:

1. $\tilde{g}_{\mu \nu} = g_{\mu \nu} + a_{\mu \nu} + ...$, full, nonsymmetric d–metric;
2. $\tilde{g}_{(\mu \nu)} = \frac{1}{2} (\tilde{g}_{\mu \nu} + \tilde{g}_{\nu \mu}) \approx g_{\mu \nu}$, symmetric d–metric;
3. $\tilde{g}_{[\mu \nu]} = \frac{1}{2} (\tilde{g}_{\mu \nu} - \tilde{g}_{\nu \mu}) \approx a_{\mu \nu}$, antisymmetric d–metric;
4. $\tilde{g}_{\mu \alpha} \tilde{g}_{\mu \beta} = \tilde{g}_{\alpha \mu} \tilde{g}_{\beta \mu} = \delta_{\beta}^{\alpha} \neq \tilde{g}_{\alpha \mu} \tilde{g}_{\mu \beta}$;
5. $W^\alpha_{\beta \gamma} = W^\alpha_{\beta \gamma}$, full, nonsymmetric d–connection;
6. $W^\alpha_{\beta} = W^\alpha_{\beta} [\beta \alpha]$, full, nonsymmetric d–connection;
7. $W^\alpha_{\beta} = W^\alpha_{\beta}$.

We shall use a nonholonomic generalization of the Lagrangian from [32],

$$L = \sqrt{-\tilde{g}}^\mu_{\nu} [ W R_{\mu \nu} + a_1 P_{\mu \nu} + a_2 e_{[\mu} W_{\nu]} + b_1 W D_\gamma W^\gamma_{\mu \nu} ] + b_2 W^\lambda_{[\mu \alpha]} W^\alpha_{\lambda \nu} + b_3 W^\lambda_{[\mu \nu]} W^\lambda_{\lambda \nu} + \tilde{g}^\lambda_{[\alpha \beta]} (c_1 W^\alpha_{\mu \lambda} W^\beta_{\nu \delta} + c_2 W^\alpha_{\mu \nu} W^\beta_{\lambda \delta} + c_3 W^\alpha_{\mu \delta} W^\beta_{\nu \lambda} + d_1 W^\alpha_{\mu \lambda} W^\beta_{\nu \delta} + 2 \Lambda) ,$$

where the parameters $a_1, a_2$, etc. are certain constants and $\Lambda$ is the cosmological constant. One should fix certain values of such constants and take $W^\alpha_{\beta \gamma}$ to be defined by a general affine (in particular, Levi–Civita) connection, in order to get different Moffat or other models of NGT.

4 Linearization to Symmetric Anholonomic Backgrounds

In this Section, we shall prove that for general nonsymmetric metrics defined on nonholonomic manifolds and corresponding nonholonomic deformations and linearization a class of general Lagrangians for NGT can be transformed into stable Lagrangians similar to those used in $\sigma$–model and anholonomic and/or noncommutative corrections to general relativity. We follow the geometric formalism elaborated in [1, 37, 38] and reconsider the results of

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works [29] [30] for nonholonomic spaces enabled both with nonlinear connection and nonsymmetric metric structures.

Let us consider an expansion of the Lagrangian (49) for $\tilde{g} = g + a$ around a background spacetime defined by a symmetric d–connection $g = \{g_{\alpha \beta}\}$ and a metric compatible d–connection $b\Gamma^\alpha_{\beta \gamma}$ defined by $N$ and $g$ (it can be a normal, canonical d–connection, Cartan or another one) and denote $\tilde{g}_{[\alpha \beta]} = a_{\alpha \beta}$. We use decompositions of type

\[
\tilde{g}_{\alpha \beta} = g_{\alpha \beta} + 1a_{\alpha \beta} + \ldots, \quad a_{\alpha \beta} = 1a_{\alpha \beta} + 2a_{\alpha \beta} + \ldots, \quad (50)
\]

\[
\Gamma^\alpha_{\beta \gamma} = b\Gamma^\alpha_{\beta \gamma} + 1\Gamma^\alpha_{\beta \gamma} + \ldots, \quad W_\mu = 1W_\mu + 2W_\mu + \ldots
\]

when, re–defining $1a_{\alpha \beta} \rightarrow a_{\alpha \beta}$, $2a_{\alpha \beta} \sim a \ldots a$, one holds

\[
\tilde{g}_{\mu \nu} = g_{\mu \nu} + a_{\mu \nu} + (1 - \rho)a_{\mu \alpha}a_{\nu}^\alpha + \sigma a^2g_{\mu \nu} + O(a^3), \quad (51)
\]

\[
\tilde{g}^{\mu \nu} = g^{\mu \nu} + a^{\mu \nu} + (1 - \rho)a^{\mu \alpha}a_{\nu}^\alpha + \sigma a^2g^{\mu \nu} + O(a^3),
\]

which implies that

\[
\sqrt{|\tilde{g}_{\mu \nu}|} = \sqrt{|g_{\mu \nu}|} \left(1 + \frac{1}{2}(\frac{1}{2} - \rho + 4\sigma)a^2\right),
\]

for $a^2 = a_{\mu \nu}a^{\mu \nu}$, where $g_{\mu \nu}$ and its inverse $g^{\mu \nu}$ are used to raise and lower indices. Following a N–adapted calculus with "N–elongated" partial differential and differential operators (see (11) and (12)) instead of usual partial derivatives and local coordinate (co) bases, similarly to constructions in Appendix to Ref. [1], we get from (49) (up to the second order approximations on $a$) the effective Lagrangian

\[
\mathcal{L} = \sqrt{-\tilde{g}}\left(\bar{\mathcal{R}} + 2\Lambda - \frac{1}{12}H^2 + \left(\frac{1}{4}\mu^2 + \beta \bar{\mathcal{R}}\right)a^2\right) (52)
\]

\[
-\sigma R_{\mu \nu}a^{\mu \alpha}a_{\alpha}^\nu - \gamma R_{\mu \nu \alpha \beta}a^{\mu \alpha}a^{\nu \beta} + O(a^3),
\]

where the effective gauge field (absolutely symmetric torsion) is

\[
H_{\alpha \beta \gamma} = e_\alpha a_{\beta \gamma} + e_\beta a_{\gamma \alpha} + e_\gamma a_{\alpha \beta}, \quad (53)
\]

with an effective mass for $a_{\beta \gamma}$, $\mu^2 = 2\Lambda(1 - 2\rho + 8\sigma)$,when the curvature d–tensor $R_{\mu \nu \alpha \beta}$, Ricci d–tensor $R_{\mu \nu}$ and scalar curvature $\mathcal{R}$ are correspondingly computed following formulas (A.6), (30) and (31), and the constants from (49) and (51) are re–defined following formulas (A.8) in Appendix.

If in the effective Lagrangian (52) we take instead of a metric compatible d–connection $\Gamma^\alpha_{\beta \gamma}$ the Levi–Civita connection $\bar{\Gamma}^\alpha_{\beta \gamma}$, we get the formula (A29) in [29] for nonsymmetric gravitational interactions modelled on
a (pseudo) Riemannian background. It exists a theorem proven by van Nieuwenhuizen [42] stating that in flat space the only consistent action for a massive antisymmetric tensor field is of the form

\[ f!L = -\frac{1}{12}H^2 + \frac{1}{4}\mu^2a^2 + O(a^3), \]

for \( a^2 = a^{\mu\nu}a_{\mu\nu} \). A rigorous study provided in [29] proves that \( \gamma = 0 \), see (52), is not allowed in NGT extended nearly a Schwarzschild background because in such a case it is not possible to solve in a compatible form the conditions (A.9) for \( \gamma = \Xi = 0 \).

A quite general solution of the problem of instability in NGT found by Janssen and Prokopec is to compensate the term with \( \gamma \neq 0 \) in (52). To do this, we can constrain such a way the nonholonomic frame dynamics \( \tilde{\gamma} \) when we get for decompositions of NGT with respect to any general relativity background an effective Lagrangian without coupling of spacetime curvature tensors with nonsymmetric tensor \( b^{\mu\alpha} \) (i.e. without a term of type \( \gamma \tilde{\Gamma}_{\alpha\beta\gamma}b^{\mu\alpha}b^{\alpha\beta} \)),

\[ E.L = \sqrt{-g}[R + 2\lambda - \frac{1}{12}H^2 + \left( \frac{1}{4}\mu^2 + \beta R \right)b^2 - \alpha R_{\mu\nu}b^{\mu\alpha}b^{\nu\beta} + O(b^3)]. \]

In this formula \( R \) and \( R_{\mu\nu} \) are respectively the scalar curvature and the Ricci tensor computed for \( \tilde{\Gamma}_{\alpha\beta\gamma} \), see formulas (33) in Appendix and \( \lambda \) is an effective cosmological constant with possible small polarizations depending on \( u^a \).

We show how for a class of nonholonomic deformations of general relativity backgrounds, we get effective Lagrangians which seem to have a good flat spacetime limit of type (54):

Let us consider \( N^a_i \approx \varepsilon^2n^a_i \) and \( a^{\mu\alpha} \approx \varepsilon b^{\mu\alpha} \) and take \( \tilde{\Gamma}^{\gamma}_{\alpha\beta} = \tilde{\Gamma}_{\alpha\beta\gamma} \) in decomposition for \( \Gamma \)-connection (50), where \( \varepsilon \) is a small parameter, which results (following formulas (32), (53), (11) and (33)) in deformations of type

\[ \tilde{\Gamma}^{\gamma}_{\alpha\beta} = \tilde{\Gamma}^{\gamma}_{\alpha\beta} + \varepsilon^2\tilde{z}^{\gamma}_{\alpha\beta}(n^a_i)\ldots, \]

\[ ^eR = \tilde{R} + \varepsilon^2\tilde{z}(n^a_i)\ldots, H^2 = \varepsilon^2H(b^{\mu\alpha}), \]

where \( \tilde{H}(b^{\mu\alpha}) \) is computed by formula (53) with \( e_{\alpha} \rightarrow \partial_{\alpha} \) and \( a_{\beta\gamma} \rightarrow b_{\beta\gamma} \) and the functionals \( \varepsilon^2\tilde{z}^{\gamma}_{\alpha\beta}(g_{ij},g_{ab},n^a_i) \) and \( \varepsilon^2\tilde{z}(g_{ij},g_{ab},n^a_i) \) can be computed.

\[ ^8 \text{in explicit form, we have to impose certain constraints on coefficients } N^a_i \text{ from (39) and (35), see the end of this Section} \]
by introducing \( \Lambda \approx \Lambda \), into respective formulas for connections and scalar curvature. Introducing values \( 56 \) into \( 54 \) and identifying
\[
\Lambda \approx \Lambda, \tag{57}
\]
we get that \( L \rightarrow E \) if and only if
\[
i_{\cdot} R_{\mu \nu \alpha \beta} b^{\mu \nu} b^{\alpha \beta}.
\tag{58}
\]
The left part of this equation is defined by the quadratic \( \dot{\varepsilon}^2 \) deformation of scalar curvature, from \( \bar{R} \) to \( \hat{R} \), relating algebraically the coefficients \( g_{ij}, h_{ab} \) and \( \bar{n}^a_i \) and their partial derivatives. We do not provide in this work the cumbersome formula for \( i_{\cdot} R_{\mu \nu \alpha \beta} b^{\mu \nu} b^{\alpha \beta} \) in the case of general nonholonomic or Einstein gravity backgrounds, but we shall compute it explicitly and solve the equation \( 58 \) for an ellipsoidal background in next section. Here we emphasize that in theories with zero cosmological constant we have to consider \( \Lambda \approx \Lambda = 0 \).

We conclude that we are able to generate stable NGT gravity models on backgrounds with small nonholonomic frame and nonsymmetric metric deformations if the conditions \( 58 \) are satisfied. This induces a small locally anisotropic polarization of the cosmological constant, see \( 57 \). Having stabilized the gravitational interactions with the nonsymmetric components of metric, for certain gravitational configurations with another small parameter \( \varepsilon \to 0 \), we get certain backgrounds in general relativity (for instance, the Schwarzschild one). For generic nonlinear theories, such as NGT and the Einstein gravity, the procedures of constraining certain nonlinear solutions in order to get stable configurations and taking smooth limits on a small parameter resulting in holonomic backgrounds are not commutative.

Finally, we note that we can use similar decompositions of type \( 56 \) to transform an arbitrary metric compatible d–connection \( \Gamma^\gamma_{\alpha \beta} \) to \( \hat{\Gamma}^\gamma_{\alpha \beta} \), and/or to introduce two small parameters consider deformations of type \( \Gamma^\gamma_{\alpha \beta} \to \hat{\Gamma}^\gamma_{\alpha \beta} \to \bar{\Gamma}^\gamma_{\alpha \beta} \). We shall use this approach in the next section.

5 Stability of Stationary Ellipsoidal Solutions

The effective gravitational field equations for nonsymmetric metrics on symmetric nonholonomic backgrounds are derived. We also analyze a class of solutions in NGT on a nonholonomic ellipsoidal background. For vanishing eccentricity, such solutions have nontrivial limits to Schwarzschild configurations.
5.1 Field equations with nonholonomic backgrounds

The field equations derived from an effective Lagrangian \( \mathcal{L} \) for a d–connection \( \Gamma_{\alpha\beta}^\gamma \) are

\[
\left( \sqrt{|g_{\mu\nu}|} \right)^{-1} e_\alpha(\sqrt{|g_{\mu\nu}|} H^{\alpha\beta\nu}) + (\mu^2 + 4\beta \, s R) a^{\beta\nu} + 4\alpha a^{\alpha(^\nu R)} + 4\gamma a^{\alpha\tau} R^{\beta}_{\alpha\tau} + O(a^2) = 0,
\]

\[
R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} s R - \Lambda g_{\mu\nu} + O(a^2) = 0.
\]

We shall work with two–parameter, deformations of nonlinear and linear connections, respectively of

\[
N_i^a \approx \varepsilon n_i^a + \varepsilon^2 n_i^a + ... a^{\mu\alpha} \approx \varepsilon b_{\mu\alpha}
\]

and

\[
\hat{\Gamma}_{\alpha\beta}^\gamma = \Gamma_{\alpha\beta}^\gamma + \varepsilon^2 \hat{\varepsilon}_{\alpha\beta}(n_i^a) + ... , \quad \hat{\Gamma}_{\alpha\beta}^\gamma = \hat{\Gamma}_{\alpha\beta} + \varepsilon \hat{\varepsilon}_{\alpha\beta}(n_i^a) + ... ,
\]

for

\[
s R = \hat{s R} + \varepsilon^2 \hat{z}(g_{ij}, g_{ab}, n_i^a) + \ldots, \quad H^2 = \varepsilon^2 \hat{H}(b^{\mu\alpha}),
\]

\[
s \hat{R} = \hat{s R} + \varepsilon \hat{z}(g_{ij}, g_{ab}, n_i^a) + \ldots,
\]

where

\[
\Lambda \approx \varepsilon^2 \hat{\Lambda}, \quad (59)
\]

we transform \( \mathcal{L} \) into

\[
\mathcal{\hat{L}} = \sqrt{-g} [s \hat{R} + 2\hat{\Lambda} - \frac{\hat{H}^2}{12} + \left( \frac{\mu^2}{4} + \beta s R \right) b^2 - \alpha \hat{R}_{\mu\nu} b^{\mu\alpha} b^{\alpha\nu} + O(b^3)] \quad (60)
\]

if and only if

\[
\hat{z}(g_{ij}, g_{ab}, n_i^a) = \gamma \hat{R}_{\mu\alpha\nu\beta} b^{\mu\alpha} b^{\alpha\beta}. \quad (61)
\]

The N–adapted variational field equations derived from (60) are

\[
\frac{e_\alpha(\sqrt{|g_{\mu\nu}|} H^{\alpha\beta\nu})}{\sqrt{|g_{\mu\nu}|}} + (\mu^2 + 4\beta s R) a^{\beta\nu} + 4\alpha a^{\alpha(\nu R)} + 4\gamma a^{\alpha\tau} R^{\beta}_{\alpha\tau} + O(a^2) = 0, \quad (62)
\]

\[
\hat{R}_{\mu\nu} - \frac{1}{2} g_{\mu\nu} s R - \varepsilon^2 \hat{\Lambda} g_{\mu\nu} + O(b^2) = 0, \quad (63)
\]

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where \( \tilde{R}^\beta_{\alpha\tau\nu}, \tilde{R}_{\mu\nu} \) and \( \tilde{R} \) are computed respectively by introducing the coefficients (46) into formulas (A.6), (30) and (31). We can see that to the order \( \mathcal{O}(b^2) \) the fields equations decouple on the symmetric and non-symmetric parts of d–metrics which allows us to consider a nonholonomic symmetric background defined by \( (g_{\mu\nu}, N_i^a, \tilde{\Gamma}^{\gamma}_{\alpha\beta}) \) and to reduce the problem to the study of constrained dynamics of the antisymmetric d–field \( a^\beta_{\nu} \) on this background.

5.2 Solutions with ellipsoidal symmetry

The simplest class of solutions for the system (62) and (62) can be constructed in the approximation that \( \tilde{\varepsilon}^2 \Lambda \sim 0 \) and \( \mu^2 \sim 0 \). For the ansatz

\[
\tilde{H}_{\alpha\beta\nu} = b^\lambda \sqrt{|g_{\mu\nu}|} \epsilon_{\alpha\beta\nu}, \tag{64}
\]

where \( b^\lambda = \text{const} \) and \( \epsilon_{\alpha\beta\nu} \) being the complete antisymmetric tensor, and any (vacuum) solution for

\[
\tilde{R}_{\mu\nu} = 0, \tag{65}
\]

we generate decoupled solutions both for the symmetric and non-symmetric part of metric. The nonsymmetric field \( b_{\beta\gamma} \) is any solution of

\[
b^\lambda \sqrt{|g_{\mu\nu}|} \epsilon_{\alpha\beta\nu} = e_\alpha b_{\beta\gamma} + e_\beta b_{\gamma\alpha} + e_\gamma b_{\alpha\beta}, \tag{66}
\]

which follows from formulas (53) and (64).

5.2.1 Anholonomic deformations of the Schwarzschild metric

Let us consider a primary quadratic element

\[
\delta s^2_{[1]} = -d\xi^2 - r^2(\xi) \, d\vartheta^2 - r^2(\xi) \sin^2 \vartheta \, d\varphi^2 + \varpi^2(\xi) \, dt^2, \tag{67}
\]

where the local coordinates and nontrivial metric coefficients are parametrized in the form

\[
x^1 = \xi, x^2 = \vartheta, x^3 = \varphi, x^4 = t, \tag{68}
\]

\[
\bar{y}_1 = -1, \quad \bar{y}_2 = -r^2(\xi), \quad \bar{y}_3 = -r^2(\xi) \sin^2 \vartheta, \quad \bar{y}_4 = \varpi^2(\xi),
\]

As a matter of principle, we can consider solutions with nonzero values of mass \( \mu \), but this will result in more sophisticated configurations for the non-symmetric components of metrics which is not related to the problem of nonholonomic stabilization of NGT; see Chapter 3 in Ref. [38], for similar details on constructing static black ellipsoid solutions in gravity with nonholonomic completely antisymmetric metric defined as a Proca field, and [41], for complex generalizations of such solutions to noncommutative gravity.
\[ \xi = \int dr \left| 1 - \frac{2m_0}{r} + \frac{\varepsilon}{r^2} \right|^{1/2} \quad \text{and} \quad \varpi^2(r) = 1 - \frac{2m_0}{r} + \frac{\varepsilon}{r^2}. \]

For the constants \( \varepsilon \to 0 \) and \( m_0 \) being a point mass, the element \( (67) \) defines the Schwarzschild solution written in spacetime spherical coordinates \((r, \vartheta, \varphi, t)\). The parameter \( \varepsilon \) should not be confused with the square of the electric charge \( e^2 \) for the Reissner–Nordström metric. In our further considerations, we treat \( \varepsilon \) as a small parameter, for instance, defining a small deformation of a circle into an ellipse (eccentricity).

We construct a generic off–diagonal vacuum solution \(^{10}\) by using nonholonomic deformations,
\[
\begin{align*}
    g_{ij} &= \eta_{ij} \tilde{g}_{ij} \\
    h_{ab} &= \eta_{ab} \tilde{h}_{ab},
\end{align*}
\]

where \((\tilde{g}_{ij}, \tilde{h}_{ab})\) are given by data \( (68) \), when the new ansatz (target metric),
\[
\delta s^2_{[de]} = -\eta_1(\xi) d\xi^2 - \eta_2(\xi) r^2(\xi) \sin^2 \vartheta \frac{\partial}{\partial \varphi}^2 + \eta_4(\xi, \vartheta, \varphi) \varpi^2(\xi) \frac{\partial}{\partial t}^2,
\]

\[
\begin{align*}
    \delta \varphi &= d\varphi + w_1(\xi, \vartheta, \varphi) d\xi + w_2(\xi, \vartheta, \varphi) d\vartheta, \\
    \delta t &= dt + n_1(\xi, \vartheta) d\xi + n_2(\xi, \vartheta) d\vartheta,
\end{align*}
\]
is supposed to solve the equation \( (65) \). In formulas \( (69) \) there are used 3D spacial spherical coordinates, \((\xi(r), \vartheta, \varphi)\) or \((r, \vartheta, \varphi)\). The details on determining certain classes of coefficients for the target metric solving the vacuum Einstein equations for the canonical d–connection can be found in Refs. \[2, 39, 41, 5\] and Part II in \[38\]. Here we summarize the results which can be verified by direct computations:

The functions \( \eta_3 \) and \( \eta_4 \) can be generated by a function \( b(\xi, \vartheta, \varphi) \) following the conditions
\[
-\eta_3(\xi, \vartheta, \varphi) r^2(\xi) \sin^2 \vartheta \text{ and } b^2(\xi, \vartheta, \varphi) \varpi^2(\xi),
\]
for
\[
|\eta_3| = (h_0)^2 |\tilde{h}_3/\tilde{h}_3| \left[ \left( \sqrt{|\eta_4|} \right)^* \right]^2, \tag{70}
\]
with \( h_0 = \text{const} \), where \( \tilde{h}_a \) are stated by the Schwarzschild solution for the chosen system of coordinates and \( \eta_4 \) can be any function satisfying the condition \( \eta_4^* = \partial \eta_4/\partial \varphi \neq 0 \). We can compute the polarizations \( \eta_1 \) and \( \eta_2 \), when \( \eta_1 = \eta_2 r^2 = e^\psi(\xi, \vartheta) \) with \( \psi \) solving
\[
\frac{\partial^2 \psi}{\partial \xi^2} + \frac{\partial^2 \psi}{\partial \vartheta^2} = 0.
\]

\(^{10}\)it can not diagonalized by coordinate transforms
The nontrivial values of N–connection coefficients $N_i^3 = w_i(\xi, \vartheta, \varphi)$ and $N_i^4 = n_i(\xi, \vartheta, \varphi)$, when $i = 1, 2$, for vacuum configurations with the Levi–Civita connection $\nabla$ are given by

$$w_1 = \partial_\xi (\sqrt{|\eta_4|})/ \left(\sqrt{|\eta_4|}\right)^* \varpi, \quad w_2 = \partial_\vartheta (\sqrt{|\eta_4|})/ \left(\sqrt{|\eta_4|}\right)^*$$

and any $n_{1,2} = 1 n_{1,2}(\xi, \vartheta)$ for which $\partial_\vartheta(1 n_1) - \partial_\xi(1 n_2) = 0$, when, for instance $\partial_\xi = \partial/\partial\xi$. In a more general case, when $\nabla \neq \hat{D}$, but the nonholonomic vacuum equation (65) is solved, we have to take

$$n_{1,2}(\xi, \vartheta, \varphi) = 1 n_{1,2}(\xi, \vartheta) + 2 n_{1,2}(\xi, \vartheta) \int d\varphi \left(\sqrt{|\eta_4|}\right)^3,$$

for $1 n_{1,2}(\xi, \vartheta)$ and $2 n_{1,2}(\xi, \vartheta)$ being certain integration functions to be defined from certain boundary conditions, or constrained additionally to solve certain compatibility equations in some limits.

Putting the defined values of the coefficients in the ansatz (69), we construct a class of exact vacuum solutions of the Einstein equations for the canonical d–connection (in particular, for the Levi–Civita connection) defining stationary nonholonomic deformations of the Schwarzschild metric,

$$\delta s^2 = -e^\psi (d\xi^2 + d\vartheta^2) - h_0^2 \left(\sqrt{|\eta_4|}\right)^2 \varpi^2 \delta \varphi^2 + \eta_4 \varpi^2 \delta t^2, \quad (71)$$

$$\delta \varphi = d\varphi + \frac{\partial_\xi (\sqrt{|\eta_4|})}{\left(\sqrt{|\eta_4|}\right)^*} d\xi + \frac{\partial_\vartheta (\sqrt{|\eta_4|})}{\left(\sqrt{|\eta_4|}\right)} d\vartheta,$$

$$\delta t = dt + n_1 d\xi + n_2 d\vartheta.$$
5.2.2 Solutions with small nonholonomic polarizations

The class of solutions (75) is defined in a very general form. Let us extract a subclass of solutions related to the Schwarzschild metric. We consider decompositions on a small parameter \(0 < \varepsilon < 1\) in (71), when
\[
\sqrt{\eta_3} = q_0^3(\xi, \vartheta, \varphi) + \varepsilon q_1^3(\xi, \vartheta, \varphi) + \varepsilon^2 q_2^3(\xi, \vartheta, \varphi) \ldots,
\]
\[
\sqrt{\eta_4} = 1 + \varepsilon q_1^4(\xi, \vartheta, \varphi) + \varepsilon^2 q_2^4(\xi, \vartheta, \varphi) \ldots,
\]
where the "hat" indices label the coefficients multiplied to \(\varepsilon, \varepsilon^2, \ldots\).

The conditions (70) are expressed in the form
\[
\varepsilon h_0 \sqrt{h_4/h_3} (q_4^1)^* = q_0^3, \quad \varepsilon^2 h_0 \sqrt{h_4/h_3} (q_4^2)^* = \varepsilon q_1^3, \ldots
\]
We take the integration constant, for instance, to satisfy the condition \(\varepsilon h_0 = 1\) (choosing a corresponding distribution and system of coordinates). This condition will be important in order to get stable solutions for certain \(\varepsilon \neq 0\), but small, i.e. \(0 < \varepsilon < 1\). For such small deformations, we prescribe a function \(q_0^0\) and define \(q_1^1\) integrating on \(\varphi\) (or inversely, prescribing \(q_1^1\), then taking the partial derivative \(\partial_\varphi\), to compute \(q_0^0\)). In a similar form, there are related the coefficients \(q_3^3\) and \(q_3^2\). An important physical situation arises when we select the conditions when such small nonholonomic deformations define rotoid configurations.

This is possible, for instance, if
\[
2q_1^1 = \frac{q_0(r)}{4m_0^2} \sin(\omega_0 \varphi + \varphi_0) - \frac{1}{r^2}, \quad (72)
\]
where \(\omega_0\) and \(\varphi_0\) are constants and the function \(q_0(r)\) has to be defined by fixing certain boundary conditions for polarizations. In this case, the coefficient before \(\delta t^2\) is
\[
\eta_4 \omega^2 = 1 - \frac{2m_0}{r} + \varepsilon \left(\frac{1}{r^2} + 2q_1^1\right). \quad (73)
\]
This coefficient vanishes and defines a small deformation of the Schwarzschild spherical horizon into an ellipsoidal one (rotoid configuration) given by
\[
r_+ \simeq \frac{2\mu}{1 + \varepsilon \frac{q_0(r)}{4m_0^2} \sin(\omega_0 \varphi + \varphi_0)}.
\]
Such solutions with ellipsoid symmetry seem to define static black ellipsoids which are stable (they were investigated in details in Refs. [39, 40]).
ellipsoid configurations were proven to be stable under perturbations and transform into the Schwarzschild solution far away from the ellipsoidal horizon. In general relativity, this class of vacuum metrics violates the conditions of black hole uniqueness theorems [43] because the ”surface” gravity is not constant for stationary black ellipsoid deformations.

We can construct an infinite number of ellipsoidal locally anisotropic black hole deformations. Nevertheless, they present physical interest because they preserve the spherical topology, have the Minkowski asymptotic and the deformations can be associated to certain classes of geometric spacetime distortions related to generic off–diagonal metric terms. Putting
\[ \phi_0 = 0, \]
we get \[ q_1^0 \rightarrow 0 \] in (72). To get a smooth limit to the Schwarzschild solution we have to state the limit \( q_3^0 \rightarrow 1 \) for \( \varepsilon \rightarrow 0 \).

Let us summarize the above presented approximations for ellipsoidal symmetries: For (73), we have
\[ h_4 = \eta_4(\xi, \vartheta, \varphi) \omega^2(\xi) = 1 - \frac{2m_0}{r} + \varepsilon \frac{q_0(r)}{4m_0} \sin(\omega_0 \varphi + \varphi_0) + O(\varepsilon^2) \]
and
\[ h_3 = \eta_3(\xi, \vartheta, \varphi) r^2(\xi) \sin^2 \vartheta = h_0^2 \left( \sqrt{|\eta_4|} \right)^2 \omega^2(\xi) = (\varepsilon h_0)^2 \left( q_4^1 \right)^2, \]
which results in
\[ h_3 = (\varepsilon h_0)^2 \frac{q_0(r)}{16m_0} \cos^2(\omega_0 \varphi + \varphi_0) + O(\varepsilon^3), \]
where we must preserve the second order on \( \varepsilon^2 \) if \( \varepsilon h_0 \sim 1 \). To get a smooth limit of off–diagonal coefficients in solutions to the Schwarzschild metric (67), we state that after integrations one approximates the N–connection coefficients as \( N_a^i \sim \varepsilon n^a_i \). Putting together all decompositions of coefficients on \( \varepsilon \) in (71), we get a family of ellipsoidal solution of equations (65) decomposed on eccentricity \( \varepsilon \),
\[
\delta s_{[1]}^2 = -e^{2\psi} \left( d\xi^2 + d\vartheta^2 \right) - (\varepsilon h_0)^2 \frac{q_0(r)}{16m_0} \cos^2(\omega_0 \varphi + \varphi_0) \delta \varphi^2 \quad (74)
\]
\[
+ \left[ 1 - \frac{2m_0}{r} + \varepsilon \frac{q_0(r)}{4m_0} \sin(\omega_0 \varphi + \varphi_0) + O(\varepsilon^2) \right] \delta t^2,
\]
\[
\delta \varphi = d\varphi + \varepsilon \frac{\partial_\xi (\sqrt{|\eta_4|} \omega)}{\sqrt{|\eta_4|} \omega} d\xi + \varepsilon \frac{\partial_\vartheta (\sqrt{|\eta_4|} \omega)}{\sqrt{|\eta_4|} \omega} d\vartheta,
\]
\[
\delta t = dt + \varepsilon n_1 d\xi + \varepsilon n_2 d\vartheta.
\]
One can be defined certain more special cases when $q_1^2$ and $q_3^1$ (as a consequence) are of solitonic locally anisotropic nature. In result, such solutions will define small stationary deformations of the Schwarzschild solution embedded into a background polarized by anisotropic solitonic waves.

Now, we show how we can solve the problem of stability related to the condition (61): Let us consider a small cosmological constant of type (59) stated only in the horizontal spacetime distribution $h \Lambda \approx \Delta \varepsilon^2 h \Lambda$, but $v \Lambda = 0$. By straightforward computations, we can verify that the symmetric part of the ansatz

$$\delta s^2 = -e^{\psi + \varepsilon^2 \delta} (d\xi^2 + d\vartheta^2) - (\varepsilon h_0)^2 q_0(r) \omega_0^2 \cos^2(\omega_0 \varphi + \varphi_0) \delta \varphi^2 + \left[1 - \frac{2m_0}{r} + \varepsilon q_0(r) \sin(\omega_0 \varphi + \varphi_0) + O(\varepsilon^2)\right] \delta t^2 + \varepsilon b_{\alpha \beta} e^\alpha \wedge e^\beta,$$

solves the equations

$$R_{ij} = \tilde{R}_{ij} + \varepsilon^2 \tilde{z}_{ij}, \quad \text{for } \tilde{z}_{ij} = -h \Lambda e^{\psi + \varepsilon^2 \delta} \delta_{ij}, \quad \tilde{R}_{ij} = 0 \quad (76)$$

$$R_{ia} = \tilde{R}_{ia} = 0, \quad R_{ai} = \tilde{R}_{ai} = 0, \quad R_{ab} = \tilde{R}_{ab} = 0,$$

see formulas for $R_{\beta \gamma}$ (30), where $e^\alpha = (d\xi, d\vartheta, \delta \varphi, \delta t)$, $\psi$ is the solution of

$$\frac{\partial^2 \psi}{\partial \xi^2} + \frac{\partial^2 \psi}{\partial \vartheta^2} = h \Lambda.$$

If the nonsymmetric part of (75) is with $b_{\alpha \beta}$ being a solution of (66), the rest of coefficients are constrained to satisfy above mentioned conditions, we generate a class of both nonholonomic and nonsymmetric metric deformations of the Schwarzschild metric which defines a family of two parametric nonholonomic solutions in NGT (when the gravitational field equations are approximated by (62) and (62)). The stability conditions (61) result in

$$\tilde{z} = g^{ij} \tilde{z}_{ij}(g_{ij}, g_{\alpha \beta}, n^i_\alpha) = h \Lambda = \gamma \tilde{R}_{\mu \alpha \nu \beta} b^{\mu \nu} b^{\alpha \beta}.$$

The techniques presented in Refs. [2, 39, 38, 5] allows us to construct solutions for nontrivial values $h \Lambda$, but this would result in modifications of the formulas for the vertical part of d–metric and N–connection coefficients, which is related to a more cumbersome calculus; in this work, we analyze the simplest examples.
This imposes a constraint of the generating function \( q_1^4(\xi, \vartheta, \varphi) \) and integration functions and constants, of type \( q_0(r) \) and \( n_1(\xi, \vartheta) \) and \( n_2(\xi, \vartheta) \), which selects of subspace in the integral variety of solutions of (76). We have \( \dot{z} = 0 \) and \( \bar{R}_{\mu\alpha\nu\beta} \), for any nonzero \( \gamma \), in the case of teleparallel nonholonomic manifolds, see Chapter 1 in Ref. [38] (we note that for such configurations the Riemann curvature for the Levi–Civita connection, in general, is not zero).

We conclude that the presence of a small cosmological constant \( h\Lambda \approx \varepsilon^2 h\Lambda \) may stabilize additionally the solutions but stability can be obtained also for vanishing cosmological constants. Constructing such solutions we considered, for simplicity, that the mass of effective gauge fields is very small. In a more general case, we can generate nonsymmetric metrics with effective Proca fields with nonzero mass and nonzero cosmological constants, see more sophisticated constructions in Refs. [41, 38, 37].

6 Conclusions and Discussion

In this article we developed a new method of stabilization in nonsymmetric gravity theories (NGT) and spacetimes provided with nonholonomic distributions and canonically induced anholonomic frames with associated nonlinear connection (N–connection) structures. For general effective Lagrangians modelling NGT on (non)holonomic backgrounds, we shown how to construct stable and nonstable solutions. We argued that the corresponding systems of field equations possess different types of gauge like and nonholonomically deformed symmetries which may stabilize, or inversely, evolve into instabilities which depends on the type of imposed constraints and ansatz for the symmetric and nonsymmetric components of metric and related N–connection and linear connection structures.

The N–connection geometry and the formalism of parametric nonholonomic frame transforms are the key prerequisites of the so-called anholonomic frame method of constructing exact and approximate solutions in Einstein gravity and various generalizations to (non)symmetric metrics, metric–affine, noncommutative, string like and Lagrange–Finsler gravity models, see reviews and explicit examples in Refs. [5, 37, 38, 41]. Such geometric methods allow us to generate very general classes of solutions of nonlinear field and constraints equations, depending on three and four variables and on infinite number of parameters, and solve certain stability problems in various models of gravity. For simplicity, in this paper we consider the nonholonomic stabilization method for a class of solutions with ellipsoidal
symmetries which transform into the Schwarzschild background for small eccentricities and small nonsymmetry (of metrics) parameters.

The idea to use Lagrange multipliers and dynamical constraints proposed and elaborated in Ref. [36], in order to solve instabilities discovered in NGT by Clayton [33, 34], contains already a strong connection to the nonholonomic geometry and field dynamics. This work develops that dynamical constraint direction to the case of nonholonomic parametric deformations following certain results from Refs. [2, 1] (on the geometry of generalized spaces and Ricci flows constrained to result in nonholonomic and (non)symmetric structures). This way we can solve the Janssen–Prokopec stability problem in NGT [29, 30, 31] and develop a new (nonholonomic) direction in (non) symmetric gravity and related spacetime geometry. Here we also note that nonsymmetric components of metrics arise naturally as generalized almost symplectic structures in deformation quantization of gravity [4, 13] when corresponding almost Kähler models are elaborated for quantum models. It was proved how general relativity (GR) can be represented equivalently in nonsymmetric almost symplectic variables for a canonical model on a corresponding almost Kähler spaces. For such a model of ”nonsymmetric” gravity/ general relativity, the questions on stability of solutions is to be analyzed as in GR, together with additional considerations for nonholonomic constraints.

Following the above–mentioned results, we have to conclude that nonsymmetric metrics and connections are defined naturally from very general constructions in modern geometry, nonlinear functional analysis and theoretical methods in gravity and particle physics. Such nonsymmetric generalizations of classical and quantum gravity models can not be prohibited by some examples when a gauge symmetry or stability scenario fail to be obtained for a fixed flat or curved background like in Refs. [32, 33, 34, 29, 30, 31]. It is almost sure that certain nonlinear mathematical techniques always can be provided in order to construct stable, or un–stable, solutions, with evolutions of necessary type; as well one can be elaborated well defined physical scenario and alternatives. This is typical for generic nonlinear theories like GR and NGT.

Of course, there exists the so–called generality problem in NGT when a guiding principle has to be formulated in order to select from nine and more constants and extra terms in generalized Lagrangians (at least by 11 undetermined parameters come from the full theory and the decomposition of the metric tensor). It may be that (non)symmetric corrections to metrics and connections can be derived following certain geometric principles in Ricci flow and/or deformation quantization theories, not only from the
variational principle for generalized field interactions and imposed nonholonomic constraints. One also has to be exploited intensively certain variants of selection from different theories following existing and further experimental data like in [25, 26, 27, 31], see also references therein. At this moment, there are none theoretical and experimental prohibitions for nonsymmetric metrics which would be established in modern cosmology, astrophysics and experimental particle physics.

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A Torsion and Curvature of d–Connections

The torsion $T$ of a d–connection $D$ is defined

$$T(X, Y) = D_X Y - D_Y X - [X, Y],$$

for any d–vectors $X = hX + vX = hX + vX$ and $Y = hY + vY$, with a corresponding N–adapted decomposition into

$$T(X, Y) = \{ hT(hX, hY), hT(hX, vY), hT(vX, hY), hT(vX, vY), vT(hX, hY), vT(hX, vY), vT(vX, hY), vT(vX, vY) \}. \tag{A.1}$$

The nontrivial N–adapted coefficients,

$$T = \{ T^\alpha_{\beta\gamma} = -T^\alpha_{\gamma\beta} = (T^i_{jk}, T^i_{ja}, T^a_{jk}, T^b_{ja}, T^b_{ca}) \}, \tag{A.2}$$

can be computed by introducing $X = e_{\alpha}$ and $Y = e_{\beta}$ into (A.1), see details in Refs. [1, 37].

The curvature of a d–connection $D$ is defined

$$R(X, Y) = D_X D_Y - D_Y D_X - D_{[X,Y]}, \tag{A.3}$$

The formulas for local N–adapted components and their symmetries, of the d–torsion and d–curvature, can be computed by introducing $X = e_{\alpha}$, $Y = e_{\beta}$ and $Z = e_{\gamma}$ in (A.3). The nontrivial N–adapted coefficients

$$R = \{ R^\alpha_{\beta\gamma\delta} = (R^i_{hjk}, R^i_{hja}, R^i_{bja}, R^i_{hba}, R^i_{bea}) \} \tag{A.4}$$

are given by formulas (A.6), see details in Ref. [1, 37].

The simplest way to perform computations with d–connections is to use N–adapted differential forms like $\Gamma^\alpha_{\beta\gamma} = \Gamma^\alpha_{\beta\gamma} e^\gamma$ with the coefficients defined
with respect to (12) and (11). For instance, the N–adapted coefficients of
torsion (A.1), i.e. d–torsion, is computed in the form
\[ T^\alpha \triangleq \text{De}^\alpha = d\varepsilon^\alpha + \Gamma_\beta^\alpha \wedge \varepsilon^\beta, \]
where
\[
\begin{align*}
T_{jk} &= L^j_{i k} - L^i_{j k}, \\
T_{ja} &= C^j_{i a}, \\
T_{ji} &= \Omega^a_{ji}, \\
T_{bi} &= \frac{\partial N^a_i}{\partial y^b} - L^a_{ib}, \\
T_{bc} &= C^a_{bc} - C^a_{cb},
\end{align*}
\]
where \( \Omega^a_{ji} \) is the curvature of N–connection (14).

By a straightforward d–form calculus, we can find the N–adapted com-
ponents of the curvature (A.3) of a d–connection \( D \),
\[ R_\beta^\alpha = d\Gamma_\beta^\alpha - \Gamma_\gamma^\beta \wedge \Gamma_\alpha^\gamma = R^\alpha_\beta \varepsilon^\gamma \wedge \varepsilon^\delta, \]
i.e. the d–curvature,
\[
\begin{align*}
R^i_{hjk} &= \varepsilon_k (L^i_{h j}) - \varepsilon_j (L^i_{h k}) + L^m_{h j} L^i_{m k} - L^m_{h k} L^i_{m j} - C^i_{h a} \Omega^a_{kJ}, \\
R^a_{bjk} &= \varepsilon_k (L^a_{b j}) - \varepsilon_j (L^a_{b k}) + L^c_{b j} L^a_{c k} - L^c_{b k} L^a_{c j} - C^a_{b c} \Omega^c_{kJ}, \\
R^i_{jka} &= e_a L^i_{j k} - D_k C^i_{ja} + C^i_{jb} T^h_{ka}, \\
R^c_{bka} &= e_a L^c_{b k} - D_k C^c_{ba} + C^c_{bd} T^a_{ka}, \\
R^i_{jbc} &= e_c C^i_{jb} - e_b C^i_{jc} + C^i_{jb} C^i_{hc} - C^i_{jc} C^i_{hb}, \\
R^a_{bcd} &= e_d C^a_{bc} - e_c C^a_{bd} + C^a_{bc} C^a_{ed} - C^a_{bd} C^a_{ec}.
\end{align*}
\]
Contracting the first and forth indices \( R_{\beta \gamma} = R^a_{\beta \gamma a} \), one gets the N–
adapted coefficients for the Ricci tensor
\[ R_{\beta \gamma} = \left\{ R_{\beta \gamma} = (R_{ij}, R_{ia}, R_{ai}, R_{ab}) \right\}. \]
see explicit formulas in Ref. [37]. It should be noted here that for general
d–connections the Ricci tensor is not symmetric, i.e. \( R_{\beta \gamma} \neq R_{\gamma \beta} \).

Finally, we note that there are two scalar curvatures, \( s R \) and \( s \tilde{R} \), of a
d–connection defined by formulas
\[ s R = g^{\beta \gamma} R_{\beta \gamma} \quad \text{and} \quad s \tilde{R} = \tilde{g}^{\beta \gamma} R_{\beta \gamma}. \]
Both geometric objects can be considered in generalized gravity theories.

Similar formulas holds true, for instance, for the Levi–Civita linear con-
nection \( \nabla = \{ \Gamma^\alpha_{\beta \gamma} \} \) is uniquely defined by the symmetric metric structure
by the conditions $\mathcal{J} = 0$ and $\nabla g = 0$. It should be noted that this connection is not adapted to the distribution \( \mathcal{D} \) because it does not preserve under parallelism the $h$- and $v$-distribution. Any geometric construction for the canonical $d$–connection $\hat{\mathbf{D}}$ can be re-defined by the Levi–Civita connection by using the formula

$$\Gamma^\gamma_{\alpha\beta} = \hat{\Gamma}^\gamma_{\alpha\beta} + Z^\gamma_{\alpha\beta},$$

where the both connections $\Gamma^\gamma_{\alpha\beta}$ and $\hat{\Gamma}^\gamma_{\alpha\beta}$ and the distorsion tensor $Z^\gamma_{\alpha\beta}$ with $N$–adapted coefficients where

\begin{align*}
Z_{jk} &= 0, \quad Z_{jk}^a = -C_{jk}^a g_{ik} h^{ib} - \frac{1}{2} \Omega_{jk}^a, \quad Z_{bk}^i = \frac{1}{2} \Omega_{jk}^c h_{cb} g^{ji} - \Xi_{jk}^i C_{hb}^j, \\
Z_{bk}^a &= \Xi_{cd}^a L_{bk}^c, \quad Z_{lk}^i = \frac{1}{2} \Omega_{jk}^a h_{cb} g^{ji} + \Xi_{jk}^i C_{hb}^j, \\
Z_{jb}^a &= -\Xi_{cd}^a L_{dj}^c, \quad Z_{bc}^i = 0, \quad Z_{ab}^i = \frac{g^{ij}}{2} \left[ L_{aj}^c h_{cb} + L_{bj}^c h_{ca} \right], \\
\Xi_{jk}^i &= \frac{1}{2} (\delta^i_j \delta_k^h - g_{jk} g^{ih}), \quad \Xi_{cd}^{ab} = \frac{1}{2} (\delta_c^a \delta_d^b + h_{cd} h^{ab}),
\end{align*}

(A.7)

for $L_{aj}^c = L_{aj}^c - e_a (N_j^c)$, are defined by the generic off–diagonal metric \( \mathcal{D} \), or (equivalently) by $d$–metric \( \mathcal{D} \) and the coefficients of $N$–connection \( \mathcal{D} \). If we work with nonholonomic constraints on the dynamics/geometry of gravity fields, it is more convenient to use a $N$–adapted approach. For other purposes, it is preferred to use only the Levi–Civita connection.

\section*{B Redefinition of Constants}

In order to get a convenient form of effective Lagrangian, the constants from \( \mathcal{D} \) and \( \mathcal{D} \) are re-defined in the form:

$$\alpha = \rho + \Xi = 1, \quad \beta = \frac{1}{2}(\beta - \rho + 2\sigma), \quad \gamma = \Xi = 3 \Sigma^2 \theta^2 \left[ 2(c_1 + c_3) + 1 - b_2 \right] + \Sigma^2 \left[ d - \frac{b_2}{3} + \frac{2}{3}(c_1 + c_3) - \frac{3}{8} L^2 \right]
$$

\begin{align*}
+ \Sigma \{ (2 \theta(1 + b_1) - \frac{2}{3} - \frac{8a_1}{3} + \frac{a_2}{2} - \frac{b_1}{3} - (a_1 + 1) \frac{L^2}{2}) - \\
\theta(\phi + \xi) \left[ 4(c_1 + c_3) - 2b_2 + 2 \right]\}
\end{align*}

(A.8)
for

\[ \Sigma = \frac{3a_2}{b_2 - 2(c_1 + c_3) - 3d_1} - \frac{b_1}{2} - 8a_1 - \frac{8}{3}a_2 - b_1, \]

\[ \mathcal{K} = \frac{3(a_1 - a_2/4)}{(1 + 2a_1)} \mathcal{L} + d_1 - \frac{b_2}{3} + \frac{2}{3}(c_1 + c_3), \quad \theta \equiv \frac{2\mathcal{K} + \mathcal{L}}{\mathcal{A} - \mathcal{B}}, \]

\[ \xi \equiv \frac{(A + 3B)(b_1 + 1)}{A^2 + AB - B^2}, \quad \phi = \psi = \frac{(A + B)(b_1 + 1)}{A^2 + AB - B^2}, \]

\[ A = 2(1 - b_2 + c_1 + c_2), \quad B = -2(c_1 + c_3), \]

\[ \Psi = (b_2 - 1)(\xi - \phi)^2 + 2\phi(1 + b_1), \quad \Phi = \frac{1}{3}(\phi^2 + 2\xi\phi)(b_2 + c_1 + c_3 - 1)^2, \]

\[ \Omega = (c_1 + c_3)(\xi - \phi)^2 + \xi(1 + b_1), \]

where the conditions

\[ \Omega + 3\Phi = -\frac{1}{4} \quad \text{and} \quad \Xi = \Psi - 2\Omega \quad (A.9) \]

have to be imposed in order to get a stable effective Lagrangian in the flat space limit.

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