Connection between low energy effective Hamiltonians and energy level statistics

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We study the level statistics of a non-integrable one dimensional interacting fermionic system characterized by the GOE distribution. We calculate numerically on a finite size system the level spacing distribution \( P(s) \) and the Dyson-Mehta \( \Delta_3 \) correlation. We observe that its low energy spectrum follows rather the Poisson distribution, characteristic of an integrable system, consistent with the fact that the low energy excitations of this system are described by the Luttinger model. We propose this Random Matrix Theory analysis as a probe for the existence and integrability of low energy effective Hamiltonians for strongly correlated systems.

The idea of studying complex systems by analysis of the statistical properties of their energy levels goes back to the early Sixties when Wigner, Dyson, Mehta and others [1–3] proposed a new kind of statistical mechanics for the spectra of quantum Hamiltonians. Here one renounces exact knowledge on the nature of the system but proposes that the coarse statistical properties of the spectra depend only on the symmetries of the Hamiltonian and not on the detailed form of the interaction it describes. This statistical hypothesis then states that the distribution of \( n \) consecutive energy levels of a given system is statistically equivalent to the behavior of \( n \) consecutive eigenvalues chosen from an ensemble of random matrices with corresponding symmetries. The statistical theory of energy levels is the precise mathematical definition of these ensembles.

Using the language of random matrix theory (RMT) possible ensembles describing the fluctuations of the eigenvalues are defined [4,5]:

- when the Hamiltonian is invariant under rotation and time reversal, the corresponding ensemble is the GOE (Gaussian orthogonal ensemble, invariant under the orthogonal group); when time reversal invariance is broken, the GUE (invariant under the unitary group). Finally the Poisson distribution corresponds to uncorrelated energy levels.

This theory was first applied in Nuclear Physics and recently received great interest in studies of quantum billiards connected to the notion of quantum chaos [6]. It is observed that quantum systems whose classical analogue is fully chaotic give energy spectra with fluctuations described by a RMT ensemble, while classically integrable ones exhibit Poisson correlations.

Recently RMT has also been applied in the study of quantum Hamiltonians describing strongly correlated systems in the context of Condensed Matter Physics [7]. It was also observed that the level distribution is Poisson for integrable systems (e.g. by Bethe Ansatz), while typically changes to GOE for generic many-body systems [8,9]. This result emerges from the statistical analysis of energy levels obtained by exact numerical diagonalization of the Hamiltonian matrix for a small cluster; it can be used as a numerical test of integrability.

In this work we propose that an analysis of the distribution of low lying energy levels can provide information about the existence of an integrable effective Hamiltonian describing the low energy physics of the system. By integrable we mean that there exist an infinite number of conservation laws (hidden) as in Bethe ansatz systems or (obvious ones) as in a one-particle type Hamiltonian (e.g. Fermi or Luttinger liquid [10]).

It is a new tool to extract more information from exact diagonalization of small systems.

In order to test this idea we have studied a well known example of a quantum many-body system: spinless fermions in one dimension with nearest (\( V_1 \)) and next nearest neighbor interaction (\( V_2 \)) described by the Hamiltonian:

\[
H = -t \sum_i (c_i^\dagger c_{i+1} + h.c.) + V_1 \sum_i n_i n_{i+1} + V_2 \sum_i n_i n_{i+2} \tag{1}
\]

where \( c_i^\dagger (c_i) \) creates (annihilates) a spinless fermion at site \( i \) (running over an N site lattice with periodic boundary conditions) and \( t \) is the hopping matrix element.

For \( V_2=0 \) this model (equivalent to the anisotropic Heisenberg model) is integrable using the Bethe ansatz method and as it was previously shown its level statistics is Poisson like.

Introducing a \( V_2 \) interaction the model is no more integrable. The low energy effective Hamiltonian in the weak coupling limit and out of half-filling is the Luttinger model Hamiltonian as found by linearizing the spectrum around
the two Fermi points \cite{11}. It is exactly soluble using bosonization, the elementary excitations being density fluctuations. Actually the Luttinger Hamiltonian describes the low energy physics of a larger class of one dimensional interacting systems (in the metallic phase) coined Luttinger liquids by Haldane \cite{12}.

Therefore we expect that for $V_1, V_2 \neq 0$ the level distribution in the high energy regime will correspond to the GOE ensemble (the non-integrable case) while in the low energy part of the spectrum we expect a deviation from the GOE distribution towards the Poisson one. We are assuming that the spectrum generated by filling non-equidistant single particle levels with independent particles is characterized by Poisson statistics. Although no rigorous analysis of this assumption exists yet, we expect a behavior analogous to the case study of the anharmonic oscillator spectrum analyzed by Berry and Tabor \cite{4}. Actually our results, as we will show below, lend support for this assumption.

Considering the numerical limitations we have chosen to diagonalize the Hamiltonian matrix of a system with $N=21$ sites and $M=7$ fermions, a 1/3 filling. In order to apply the RMT the Hamiltonian matrix must be diagonalised in a subspace of the total Hilbert space in which no symmetries are left (energy levels in disconnected subspaces are not correlated). So using the translational symmetry we block diagonalized the Hamiltonian in $k$-momentum labeled subspaces thus removing all obvious symmetries. We so obtain 10 independent subspaces corresponding to momenta $k$ (in units of $\frac{2\pi}{N}$) $k = 1, 10$ (we omitted the $k = 0$ subspace as it possesses an extra symmetry under reflection). We diagonalize the matrices (at most of dimension $D=5539$) using the Lanczos technique \cite{13}.

As the number of levels available for analysis in the low energy part is rather limited and after having verified that the results obtained are the same for every $k$-subspace, we averaged the level distributions obtained for the different $k$-subspaces. This corresponds to considering the independent $k$-subspaces as independent realizations of the system.

Having total momentum $k$ different from zero our matrices are complex as in Hamiltonians with broken time reversal symmetry ($T$). However, due to the invariance of our system under reflection symmetry, $I : R \to -R$ the Hamiltonian is still invariant under $T \times I$ and this invariance leads again to GOE spectral fluctuations instead of GUE \cite{5}.

To characterize the fluctuations of $n$ levels with energies $\{E_i\}$ ($i = n_0 + 1, n_0 + n$) starting from level $n_0$, in a given $k$-subspace, we study the probability density $P(s)$ of spacings between consecutive ordered levels and the Dyson-Mehta $\Delta_3$ correlation. As a standard procedure before analysing the fluctuations we have “unfolded” the spectrum. This procedure amounts to removing the smooth irrelevant part of the density function $\langle N(E)_{av} \rangle$. In practice we consider the new variables:

$$\delta_i = N(E_i) - \langle N_{av}(E_i) \rangle$$

where $N(E_i)$ is the number of levels with energy less than $E_i$ and $\langle N_{av}(E_i) \rangle$ is constructed by fitting the $n$ level spectrum with a second order polynomial. Given this new set of ordered levels we define the spacing $s_i = \delta_{i+1} - \delta_i$.

This variable is then rescaled to correspond to a normalized probability distribution function with average $< s >= 1$. This allows us a direct comparison with the ideal Poisson $P(s) = \exp\{-s\}$ or the so called GOE distribution function:

$$P(s) = \frac{s}{\pi} \exp\{-\frac{s^2}{2}\}$$

Notice that the GOE distribution with $P(s) \to 0$ for $s \to 0$, characterizes the level repulsion present in correlated spectra, while in the Poisson distribution the largest probability is for $s \to 0$ corresponding to level crossings characteristic of uncorrelated spectra.

Now we will describe the results of our study: first to emphasize the difference between an integrable and a non-integrable case for our model system we show in Fig.1 the $P(s)$ for $V_1 = 2t, V_2 = 0$ (integrable) and $V_1 = t, V_2 = 0.5t$ (non-integrable) case. It is clear that indeed the first follows a Poisson while the second follows a GOE distribution. In the inset we also show the $N(E_i)$ for $V_1 = t, V_2 = 0.5t$ as a guide for our further choice of high and low energy windows where we will perform the partial analysis of the distribution function.

In Fig.2 we present $P(s)$ as estimated from two different energy windows, the one centered at the low energy part of the spectrum ($n_0 = 10$), the other at the middle part (see inset Fig.1): we observe a definite displacement towards the Poisson distribution for the low energy window, although the fluctuations in the estimation of $P(s)$ are considerable due the small number of levels used. The deviation from the GOE distribution is manifested by an increased weight of $P(s)$ for $s \to 0$, characteristic of the absence of level repulsion, but also by an interesting shift at large $s$ towards the Poisson distribution tail (seen clearly in Fig.3). This deviation we attribute to the existence of an integrable low energy effective Hamiltonian as the Luttinger liquid Hamiltonian proposed for this model \cite{13}. We should stress that with our finite size system and limited low energy window we do not observe a genuine Poisson distribution but only a shift from the GOE one. We cannot really say from these data if the generic low energy level distribution for a macroscopic system is the Poisson or some other intermediate distribution.

We can significantly improve on the evaluation of $P(s)$, as is shown in Fig.3, where results for $P(s)$ for the same windows are presented but now averaged over all independent $k$-subspaces $(\langle P(s) \rangle_k)$. These results are generic to our system as we also obtained them for the $N=15$ ($M=5$, $D=201$) and $N=18$ ($M=6$, $D=1038$) systems. On the other hand we observed no such deviations on a study of a test random matrix of finite size, so these results are not due to finite size or density of states effects.
At this point we should mention that we observed similar deviations from the GOE distribution at the highest energy part of the spectrum which we can also attribute to a simple level structure, characteristic of an effective Hamiltonian description obtained by a unitary transformation ($\tilde{c}_l = c_l e^{-i\pi l}$) which maps $H(V_1, V_2) \rightarrow -H(-V_1, -V_2)$.

To further study this smooth transition away from the GOE distribution we studied the $P(s)$ for a group of levels weighted by a Boltzmann factor which amounts to introducing a soft cutoff procedure in the window of levels we are studying. Introducing a fictitious temperature $T$, $P(s)$ is calculated as: $P(s) \simeq \sum_i e^{-\beta E_i} \delta(s - s_i) (\beta = 1/T)$. For $T \rightarrow \infty$ this cutoff procedure is equivalent to a finite energy window as before; the results are shown in Fig.4 for different values of $\beta$. We find the same trend in the results as before, the results being qualitatively independent of the procedure used (note that introducing a Boltzmann weight does not affect a spectrum with pure GOE or Poisson level distribution). This method can be used for a consistent comparison of level fluctuations between different size systems.

Finally another probe of level fluctuations introduced by Dyson and Mehta \cite{DysonMehta63} is the correlation $\Delta_3$ which we calculate as described by Bohigas and Giannoni \cite{BohigasGiannoni84}:

$$\Delta_3 = \frac{1}{2L \min_{A,B}} \int_{-L}^{L} [N(E) - AE - B]dE$$

(3)

In Fig. 5 we present the results again for energy windows at different parts of the spectrum; the Poisson distribution takes the value $L/15$ while the asymptotic behavior of the GOE one is $(\ln L)/\pi^2$. The same behavior as for $P(s)$ characterizes $\Delta_3$, a similarity to the GOE behavior for high energies and a deviation towards the Poisson one as the energy is lowered. The results shown are averaged over $k$ subspaces; similar ones where obtained for every $k$ subspace although with poorer statistics.

In conclusion we studied the level statistics of a non-integrable system of interacting spinless fermions in one dimension: we find that although the overall spectrum is characterized by the GOE distribution, its low energy part exhibits a clear tendency towards the Poisson distribution characteristic of an integrable system. We attributed this change to the existence of an integrable effective Hamiltonian describing the low lying excitations. So far we see two classes of Hamiltonians which will give rise to uncorrelated levels, characterized by a Poisson distribution: Bethe ansatz systems possessing a macroscopic number of conservation laws and single particle Hamiltonians describing practically independent quasiparticles (which notice, might have nothing to do with the original bare particles as is the case in the Luttinger model). From these observations we propose to use this Random Matrix Theory analysis to probe the existence and integrability of low energy effective Hamiltonians for strongly correlated systems. Unfortunately the study of these simple level correlations seems too crude a probe to provide information on the nature of the quasiparticle description.

It is even worth considering in future studies the question if all Hamiltonians describing physical systems possess a low energy quasiparticle description.

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FIG. 1. $P(s)$ for $V_1 = 2t$ and $V_2 = 0$ (circles), $V_1 = t$ and $V_2 = 0.5t$ (squares), $k = 4$ ; continuous lines are the ideal Poisson and GOE distributions (in the inset $N(E_i)$ for $V_1 = t$, $V_2 = 0.5t$).

FIG. 2. $P(s)$ for $V_1 = t$ and $V_2 = 0.5t$, $k = 4$, from $n = 150$ levels; at low energies, $n_0 = 10$ (black dots) and at medium energies (squares).

FIG. 3. $\langle P(s) \rangle_k$ (average over $k$-momenta) for the same parameters as in Figure 2.

FIG. 4. $\langle P(s) \rangle_k$ for $V_1 = t$ and $V_2 = 0.5t$, ($n_0 = 10$, $n = 2000$) introducing a Boltzmann weight with $\beta = 0.001, 0.005, 0.01$.

FIG. 5. $\langle \Delta_3(L) \rangle_k$ for $V_1 = t$, $V_2 = 0.5t$; $n = 150$, energy windows at $n_0 = 30$ (triangles), $n_0 = 200$ (black dots), $n_0 = 3000$ (squares). Continuous lines: $(L - 2)/15$ (Poisson), $\ln(L - 2)/\pi^2$ (GOE)