THE ELLIPTIC APOSTOL-DEDEKIND SUMS GENERATE ODD DEDEKIND SYMBOLS WITH LAURENT POLYNOMIAL RECIPROCITY LAWS

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Abstract. Dedekind symbols are generalizations of the classical Dedekind sums (symbols). There is a natural isomorphism between the space of Dedekind symbols with Laurent polynomial reciprocity laws and the space of modular forms. We will define a new elliptic analogue of the Apostol-Dedekind sums. Then we will show that the newly defined sums generate all odd Dedekind symbols with Laurent polynomial reciprocity laws. Our construction is based on Machide’s result [7] on his elliptic Dedekind-Rademacher sums. As an application of our results, we discover Eisenstein series identities which generalize certain formulas by Ramanujan [11], van der Pol [9], Rankin [12] and Skoruppa [14].

1. Introduction and statement of results

A Dedekind symbol is a generalization of the classical Dedekind sums ([10]), and is defined as a complex valued function $D$ on $V := \{(p, q) \in \mathbb{Z}^+ \times \mathbb{Z} \mid \gcd(p, q) = 1\}$ satisfying

$$D(p, q) = D(p, q + p).$$

The symbol $D$ is determined uniquely by its reciprocity law:

$$D(p, q) - D(q, -p) = R(p, q)$$

up to an additive constant. The function $R$ is defined on $U := \{(p, q) \in \mathbb{Z}^+ \times \mathbb{Z}^+ \mid \gcd(p, q) = 1\}$, and is called a reciprocity function associated with the Dedekind symbol $D$. The function $R$ necessarily satisfies the equation:

$$R(p + q, q) + R(p, p + q) = R(p, q).$$

When the reciprocity function $R$ is a Laurent polynomial in $p$ and $q$, the symbol $D$ is called a Dedekind symbol with Laurent polynomial reciprocity law. These symbols are particularly important because they naturally correspond to modular forms ([2]). The symbol $D$ is said to be even (resp. odd) if $D$ satisfies:

$$D(p, -q) = D(p, q) \quad \text{(resp. } D(p, -q) = -D(p, q)\text{)}.$$

To state our results, we need to review the relevant relationship between modular forms, Dedekind symbols and period polynomials (see [2] for details). Throughout
the paper, we assume that \( w \) is an even positive integer, and we use the following notation:

\[
M_{w+2} := \text{the vector space of modular forms on } SL_2(\mathbb{Z}) \text{ with weight } w + 2,
\]

\[
S_{w+2} := \text{the vector space of cusp forms on } SL_2(\mathbb{Z}) \text{ with weight } w + 2,
\]

\[
d_w := \begin{cases} 
\left\lfloor \frac{w+2}{12} \right\rfloor - 1 & \text{if } w \equiv 0 \pmod{12} \\
\frac{w+2}{12} & \text{if } w \not\equiv 0 \pmod{12}
\end{cases}
\]

where \( \lfloor x \rfloor \) denotes the greatest integer not exceeding \( x \in \mathbb{R} \). We note that \( \dim S_{w+2} = d_w \) and \( \dim M_{w+2} = d_w + 1 \).

Let \( B_k \) denote the \( k \)th Bernoulli number, and let \( g_w \) be a Laurent polynomial in \( p \) and \( q \) defined by

\[
g_w(p, q) := -\frac{1}{pq} \sum_{j=0}^{\frac{w}{2}+1} \frac{w! B_{2j} B_{w+2-2j}}{2(2j)! (w + 2 - 2j)!} p^{2j} q^{w+2-2j} + \frac{B_{w+2}}{2(w+2)}.
\]

We also use the following notation:

\[
V_{w} := \{ g | g \text{ is an odd homogeneous polynomial in } p \text{ and } q \text{ of degree } w \text{ which satisfies } g(p + q, q) + g(p, p + q) = g(p, q) \text{ and } g(p, q) = g(q, p) \}
\]

(an element of \( V_{w} \) is essentially an odd period polynomial),

\[
W_{w} := V_{w} \oplus \mathbb{C}(g_w) \quad (\mathbb{C}(g_w) \text{ is the vector space spanned by } g_w),
\]

\[
D_{w} := \{ D | D \text{ is an odd Dedekind symbol such that } D(p, q) - D(q, -p) \in W_{w} \}.
\]

First we will see that the three spaces \( M_{w+2}, D_{w} \) and \( W_{w} \) are naturally isomorphic. For a cusp form \( f \in S_{w+2} \) and \( (p, q) \in V \), we define \( D_f \) and \( D_f^{-} \) by

\[
D_f(p, q) := \int_{q/p}^{\infty} f(z)(pz - q)^w \, dz, \quad D_f^{-}(p, q) := \frac{1}{2}(D_f(p, q) - D_f(p, -q)).
\]

Then we can show \( D_f^{-} \) is an odd Dedekind symbol in \( D_{w} \) (\( D_f^{-} \) can be similarly defined for \( f \in M_{w+2} \), see [2] for further details). Hence we can define a map

\[
\alpha_{w+2}^{-} : M_{w+2} \to D_{w}^{-}
\]

by

\[
\alpha_{w+2}^{-}(f) = D_f^{-}.
\]

Next we define a map

\[
\beta_{w}^{-} : D_{w}^{-} \to W_{w}^{-}
\]

by

\[
\beta_{w}^{-}(D)(p, q) = D(p, q) - D(q, -p).
\]

In other words, \( \beta_{w}^{-}(D) \) is the reciprocity function of the Dedekind symbol \( D \).

It was shown in [2] that these two maps \( \alpha_{w+2}^{-} \) and \( \beta_{w}^{-} \) are isomorphisms and \( \beta_{w}^{-} \alpha_{w+2}^{-} \) can be identified with the Eichler-Shimura isomorphism. Indeed \( \beta_{w}^{-} \alpha_{w+2}^{-}(f)(p, q) \) gives the (homogeneous) odd period polynomial of \( f \).

These facts may be summarized in the following commutative diagram:
Next we also need to recall the generalized Dedekind sum defined by Apostol [1]. The first Dedekind symbol, after the classical Dedekind sum, was given by Apostol, which we call the Apostol-Dedekind sum to distinguish it from other generalized Dedekind sums. Let $k$ be a positive integer, and let $(p, q)$ be in $V$. The Apostol-Dedekind sum $s_k(q, p)$ is defined by

$$s_k(q, p) := \sum_{\mu=1}^{p-1} \frac{\mu}{p} \bar{B}_k\left( \frac{\mu q}{p} \right) .$$

Here $\bar{B}_k(x)$ denotes the $k$th Bernoulli function. That is, $\bar{B}_k(x)$ is given by the Fourier expansion

$$\bar{B}_k(x) := -k! \sum_{m=-\infty}^{+\infty} \frac{e^{2\pi i mx}}{(2\pi im)^k} .$$

It is well-known that for $0 \leq x < 1$, $\bar{B}_k(x)$ reduces to the $k$th Bernoulli polynomial $B_k(x)$.

If $k$ is even, it is easy to see that $s_k(q, p) = 0$. If $k$ is odd, a reciprocity law for the Apostol-Dedekind sums was obtained by Apostol [1, p.149]:

$$(1.5) \quad p^w s_{w+1}(q, p) + q^w s_{w+1}(p, q) = -2(w + 1) g_w(p, q) .$$

In [4] we have proposed an elliptic analogue of Apostol-Dedekind sums, say $\tilde{s}_{w+1}(q, p; \tau)$. Here $\tau \in \mathbb{H} := \{ z \in \mathbb{C} \mid \Im z > 0 \}$. These sums satisfy

$$\lim_{\tau \to i\infty} \tilde{s}_{w+1}(q, p; \tau) = s_{w+1}(q, p) .$$

However, they have two defects:

1. They are not real Dedekind symbols, instead they satisfy

   $$\tilde{s}_{w+1}(q + 2p, p; \tau) = \tilde{s}_{w+1}(q, p; \tau) ,$$

   (2) they are defined in two different ways depending on the parity condition of $p$ and $q$.

To rectify these defects, we introduce a new kind of the elliptic Apostol-Dedekind sum.

In what follows, $\sigma(z; \tau)$, $\varphi(z; \tau)$ and $\zeta(z; \tau)$ denote the Weierstrass sigma, pe and zeta functions, and $\zeta^{(k)}(z; \tau)$ denotes the $k$th derivative $\partial^k \zeta(z; \tau)/\partial z^k$ of $\zeta(z; \tau)$. Furthermore $E_k(\tau)$ denotes the $k$th Eisenstein series (details of these functions will be given in the section [4]).
**Definition 1.1.** For \((p, q) \in V, \tau \in \mathbb{H}\) and a positive integer \(n\), we define

\[
D_{2n}^{-}(p, q; \tau) := \frac{1}{(2\pi i)^2 p(2n)!} \sum_{\lambda, \mu = 0}^{p-1}_{(\lambda, \mu) \neq (0,0)} \zeta^{(2n)} \left( \frac{\lambda + \mu \tau}{p}; \tau \right)
\]

\[
\times \left[ \zeta \left( \frac{q(\lambda + \mu \tau)}{p}; \tau \right) - E_2(\tau) \frac{q(\lambda + \mu \tau)}{p} + 2\pi i q \mu \right].
\]

We call \(D_{2n}^{-}(p, q; \tau)\) the elliptic Apostol-Dedekind sum.

For \((p, q) \in U, \tau \in \mathbb{H}\) and a positive integer \(n\), we also define

\[
R_{2n}^{-}(p, q; \tau) := \frac{1}{(2\pi i)^2 pq} \left[ \sum_{j=1}^{n} E_{2j}(\tau) E_{2n+2-2j}(\tau) p^{2j} q^{2n+2-2j}
\]

\[- E_{2n+2}(\tau) (p^{2n+2} + q^{2n+2}) - (2n + 1) E_{2n+2}(\tau) \right]

\[- \frac{1}{4\pi i n} \frac{\partial E_{2n}(\tau)}{\partial \tau} (p^{2n-1} q + pq^{2n-1}).
\]

Then this sum \(D_{2n}^{-}(p, q; \tau)\) is an odd Dedekind symbol and expressed without regard to the parities of \(p\) and \(q\). Furthermore, this sum is equipped with Laurent polynomial reciprocity law. We will formulate these findings more precisely as a theorem.

**Theorem 1.1.**

(1) For \((p, q) \in V, \tau \in \mathbb{H}\) and a positive integer \(n\), it holds that

\[
D_{2n}^{-}(p, q; \tau) = D_{2n}^{-}(p, q + p; \tau), \quad D_{2n}^{-}(p, -q; \tau) = -D_{2n}^{-}(p, q; \tau).
\]

(2) For \((p, q) \in U, \tau \in \mathbb{H}\) and a positive integer \(n\), \(D_{2n}^{-}(p, q; \tau)\) satisfies the following reciprocity law:

\[
D_{2n}^{-}(p, q; \tau) + D_{2n}^{-}(q, p; \tau) = R_{2n}^{-}(p, q; \tau).
\]

The sum has the following property:

\[
\lim_{\tau \to i\infty} D_{2n}^{-}(p, q; \tau) = -\frac{(2\pi i)^{2n}}{(2n + 1)!} p^{2n} s_{2n+1}(q, p).
\]

This means that \(D_{2n}^{-}(p, q; \tau)\) is an elliptic analogue of Apostol-Dedekind sums.

The most striking feature of the newly defined sum is that the sum “generates” all odd Dedekind symbols with Laurent polynomial reciprocity laws.

**Theorem 1.2.** There are \(\tau_0, \tau_1, \ldots, \tau_{d_w} \in \mathbb{H}\) such that \(D_{w}^{-}(p, q; \tau_i) (i = 0, 1, \ldots, d_w)\) form a basis of the space \(D_{w}^{-}\) of odd Dedekind symbols with Laurent polynomial reciprocity laws.

To prove Theorem 1.2, it is convenient to introduce the generating functions of \(D_{2n}^{-}(p, q; \tau)\) and \(R_{2n}^{-}(p, q; \tau)\).
Definition 1.2. For \((p, q) \in V, \tau \in \mathbb{H}\) and \(x \in \mathbb{R}\), we define
\[
D^-(p, q; \tau; x) := \frac{1}{(2\pi i)^2 p} \sum_{\lambda, \mu = 0}^{p-1} \left[ \zeta \left( \frac{\lambda + \mu \tau}{p} - x; \tau \right) - E_2(\tau) \left( \frac{\lambda + \mu \tau}{p} - x \right) + 2\pi i \frac{\mu}{p} \right] \times \left[ \zeta \left( \frac{q(\lambda + \mu \tau)}{p}; \tau \right) - E_2(\tau) \frac{q(\lambda + \mu \tau)}{p} + 2\pi i \frac{q\mu}{p} \right].
\]

For \((p, q) \in U, \tau \in \mathbb{H}\) and \(x \in \mathbb{R}\), we define
\[
R^-(p, q; \tau; x) := -\frac{1}{(2\pi i)^2} \left[ \zeta(px; \tau) - E_2(\tau)(px) \right] \left[ \zeta(qx; \tau) - E_2(\tau)(qx) \right] + \frac{q}{4\pi ip} \left[ 2 \frac{\partial \log \sigma(px; \tau)}{\partial \tau} - \frac{\partial E_2(\tau)}{\partial \tau} (px)^2 - \frac{1}{\pi i} E_2(\tau) \right] + \frac{p}{4\pi ip} \left[ 2 \frac{\partial \log \sigma(qx; \tau)}{\partial \tau} - \frac{\partial E_2(\tau)}{\partial \tau} (qx)^2 - \frac{1}{\pi i} E_2(\tau) \right] + \frac{1}{(2\pi i)^2 pq} \left[ \phi(x; \tau) + E_2(\tau) \right].
\]

Then we know that \(D^-(p, q; \tau; x)\) and \(R^-(p, q; \tau; x)\) are generating functions of \(D_{2n}^-(p, q; \tau)\) and \(R_{2n}^-(p, q; \tau)\), respectively. Our strategy of establishing Theorem 1.3 is first to prove Theorem 1.1 and then derive the assertion of Theorem 1.3 as its corollary.

Theorem 1.3. 

1. For \((p, q) \in V, \tau \in \mathbb{H}\) and \(x \in \mathbb{R}\), it holds that
\[
D^-(p, q; \tau; x) = D^-(p, q + p; \tau; x), \quad D^-(p, -q; \tau; x) = -D^-(p, q; \tau; x).
\]

2. For \((p, q) \in U, \tau \in \mathbb{H}\) and a sufficiently small real number \(x \neq 0\), \(D^-(p, q; \tau; x)\) satisfies the following reciprocity law:
\[
D^-(p, q; \tau; x) + D^-(q, p; \tau; x) = R^-(p, q; \tau; x) + C(\tau)
\]
where \(C(\tau)\) is a constant with respect to \(x\).

Remark 1.1. Using the function \(E_k(z; \tau)\) defined by
\[
E_k(z; \tau) := \sum_{\gamma \in \mathbb{Z} + \mathbb{Z} \tau} (\gamma + z)^{-k} |\gamma + z|^{-s}
\]
(see Sczech [13] for details of this function), we can express the sums \(D_{2n}^-(p, q; \tau)\) and \(D^-(p, q; \tau; x)\) in Definitions 1.1 and 1.2 as follows:
\[
D_{2n}^-(p, q; \tau) = \frac{1}{(2\pi i)^2 p} \sum_{\lambda, \mu = 0}^{p-1} E_{2n+1} \left( \frac{\lambda + \mu \tau}{p}; \tau \right) E_1 \left( \frac{q(\lambda + \mu \tau)}{p}; \tau \right),
\]
\[
D^-(p, q; \tau; x) = \frac{1}{(2\pi i)^2 p} \sum_{\lambda, \mu = 0}^{p-1} E_1 \left( \frac{\lambda + \mu \tau}{p} - x; \tau \right) E_1 \left( \frac{q(\lambda + \mu \tau)}{p}; \tau \right).
\]

As an application of our results, in the last section, we discover Eisenstein series identities (Theorem 1.1). In doing so, we rediscover the formulas by Ramanujan [11], van der Poll [9], Rankin [12] and Skoruppa [14].
2. Machide’s reciprocity laws

In this section we recall Machide’s result [7] on his elliptic Dedekind-Rademacher sums. His result will play an important role in proving Theorem 1.3. We will use some standard notation: $e(x) := \exp(2\pi i x)$, $q := e(\tau)$,

$$\theta(x; \tau) := \sum_{n=-\infty}^{\infty} e\left(\frac{1}{2}(n + \frac{1}{2})^2 \tau + (n + \frac{1}{2})(x + \frac{1}{2})\right).$$

We consider the following functions (refer to [6], [7], [16], [17])

$$F(\xi, \eta; \tau) := \frac{\theta'(0; \tau)\theta(\xi + \eta; \tau)}{\theta(\xi; \tau)\theta(\eta; \tau)},$$

$$F(x, y; \xi; \tau) := e(y\xi)F(-x + y\tau, \xi; \tau).$$

Set

$$F(x, y; \xi; \tau) = \sum_{m=0}^{\infty} \frac{B_m(x, y; \tau)}{m!} (2\pi i)^m \xi^{m-1}. \tag{2.1}$$

The function $B_m(x, y; \tau)$ is called Kronecker’s double series or the elliptic Bernoulli function. The following expansion of $B_m(x, y; \tau)$ will be used in the later section:

$$B_m(x, y; \tau) = m \left[ \sum_{j=1}^{\infty} (y - j)^{m-1} \frac{e(-y\tau)q^j}{e(-x) - e(-y\tau)q^j} - \sum_{j=1}^{\infty} (y + j)^{m-1} \frac{e(y\tau)q^j}{e(x) - e(y\tau)q^j} + y^{m-1} \frac{e(-x + y\tau)}{e(-x + y\tau) - 1} \right] + B_m(y). \tag{2.2}$$

Let $a, a', b, b', c, c'$ be positive integers, and $x, x', y, y', z, z'$ real numbers. Suppose that

$$a'z' - c'x' \notin \gcd(a', c')\mathbb{Z} \quad \text{and} \quad b'z' - c'y' \notin \gcd(b', c')\mathbb{Z}.$$

Set $(\vec{a}, \vec{b}, \vec{c}) := ((a', a), (b', b), (c', c))$ and $(\vec{x}, \vec{y}, \vec{z}) := ((x', x), (y', y), (z', z))$. Machide defined the elliptic Dedekind-Rademacher sum as:

$$S_{m,n}^\tau \begin{pmatrix} \vec{a} \\ \vec{x} \end{pmatrix} \begin{pmatrix} \vec{b} \\ \vec{y} \end{pmatrix} \begin{pmatrix} \vec{c} \\ \vec{z} \end{pmatrix} := \frac{1}{c'} \sum_{j \pmod{c'}} \sum_{(j') \pmod{c'}} B_m \left( \frac{a'j' + z'}{c'} - x', \frac{a\tilde{j} + z}{c} - x; \frac{a'}{a} \tau \right) \times B_n \left( \frac{b'j' + z'}{c'} - y', \frac{b\tilde{j} + z}{c} - y; \frac{b'}{b} \tau \right). \tag{2.3}$$

Furthermore he introduced a generating function for $S_{m,n}^\tau$ by

$$\mathcal{G}^\tau \begin{pmatrix} \vec{a} \\ X \end{pmatrix} \begin{pmatrix} \vec{b} \\ Y \end{pmatrix} \begin{pmatrix} \vec{c} \\ Z \end{pmatrix} := \sum_{m,n=0}^{\infty} \frac{1}{m!n!} S_{m,n}^\tau \begin{pmatrix} \vec{a} \\ \vec{x} \end{pmatrix} \begin{pmatrix} \vec{b} \\ \vec{y} \end{pmatrix} \begin{pmatrix} \vec{c} \\ \vec{z} \end{pmatrix} \left( \frac{X}{a} \right)^{m-1} \left( \frac{Y}{b} \right)^{n-1}.$$

where $Z$ is defined by $Z = -X - Y$.

Under this notation Machide obtained the following reciprocity law for $\mathcal{G}^\tau$. 
Theorem 2.1 (Machide[7]). Let $X, Y, Z$ be variables with $X + Y + Z = 0$, and $a, a', b, b', c, c'$ positive integers, and $x, y, z$ real numbers. Let $x', y'$ and $z'$ be real numbers such that

\[(2.4) \quad a'y' - b'z' \notin \gcd(a', b')\mathbb{Z}, \quad a'z' - c'x' \notin \gcd(a', c')\mathbb{Z}, \quad b'z' - c'y' \notin \gcd(b', c')\mathbb{Z}, \]

and let $(\vec{a}, \vec{b}, \vec{c}) = ((a', a), (b', b), (c', c))$, $(\vec{x}, \vec{y}, \vec{z}) = ((x', x), (y', y), (z', z))$.

Suppose that the integers $a, b$ and $c$ (resp. $a', b'$ and $c'$) have no common factor.

Then we have

\[
\begin{align*}
S_{\tau} \left( \begin{array}{ccc}
\vec{a} & \vec{b} & \vec{c} \\
x & y & z \\
X & Y & Z
\end{array} \right) + S_{\tau} \left( \begin{array}{ccc}
\vec{b} & \vec{c} & \vec{a} \\
y & z & x \\
Y & Z & X
\end{array} \right) + S_{\tau} \left( \begin{array}{ccc}
\vec{c} & \vec{a} & \vec{b} \\
z & x & y \\
Z & X & Y
\end{array} \right) = 0.
\end{align*}
\]

3. Reciprocity laws derived from formulas of Machide and Sczech

In this section we prove the following proposition, from which we will deduce Theorem 1.3.

Proposition 3.1. For $(p, q) \in \mathbb{U}$, $\tau \in \mathbb{H}$ and a sufficiently small real number $s \neq 0$, it holds that

\[
\frac{1}{p} \sum_{\lambda, \mu=0}^{p-1} \left( \begin{array}{ccc}
\lambda & \mu & 0 \\
p & p & \tau
\end{array} \right) B_1 \left( \frac{\lambda}{p} - s, \frac{\mu}{p}; \tau \right) B_1 \left( \frac{q\lambda}{p}, \frac{q\mu}{p}; \tau \right)
\]

\[
+ \frac{1}{q} \sum_{\lambda, \mu=0}^{q-1} \left( \begin{array}{ccc}
\lambda & \mu & 0 \\
q & q & \tau
\end{array} \right) B_1 \left( \frac{\lambda}{q} - s, \frac{\mu}{q}; \tau \right) B_1 \left( \frac{p\lambda}{q}, \frac{p\mu}{q}; \tau \right)
\]

\[
= -B_1(ps, 0; \tau)B_1(qs, 0; \tau) + \frac{q}{2p}B_2(ps, 0; \tau) + \frac{p}{2q}B_2(qs, 0; \tau)
\]

\[
+ \frac{1}{2\pi i pq} \frac{\partial}{\partial s} B_1(s, 0; \tau) + C(\tau)
\]

where $C(\tau)$ is a constant with respect to $s$.

We will give two proofs for Proposition 3.1. The first proof is our original one which is derived from Machide's formula (Theorem 2.1). The second proof is the one proposed by the referee, and it is brief and elegant and is based on Sczech's reciprocity law for elliptic Dedekind sums (13). We believe that our original proof is still interesting in its own right, and it would be applicable to other problems related to generalized Dedekind sums.

The first proof of Proposition 3.1 rests on the following lemma.

Lemma 3.2. Under the notation and assumptions of Theorem 2.1, we have

\[
\begin{align*}
- \frac{c}{2b} S_{2, 0}^{\tau} \left( \begin{array}{ccc}
b & c & a \\
\tilde{y} & \tilde{z} & \tilde{x}
\end{array} \right) + \frac{c}{2a} S_{2, 2}^{\tau} \left( \begin{array}{ccc}
\tilde{c} & \tilde{a} & \tilde{b} \\
\tilde{z} & \tilde{x} & \tilde{y}
\end{array} \right) = 0,
\end{align*}
\]

\[
\begin{align*}
\frac{b}{2a} S_{2, 0}^{\tau} \left( \begin{array}{ccc}
a & b & c \\
\tilde{x} & \tilde{y} & \tilde{z}
\end{array} \right) - \frac{b}{2c} S_{2, 2}^{\tau} \left( \begin{array}{ccc}
b & \tilde{c} & \tilde{a} \\
\tilde{y} & \tilde{z} & \tilde{x}
\end{array} \right) = 0,
\end{align*}
\]

where $S_{2, j}^{\tau}$ denotes the Dedekind sum as in (2.1).
\[
\frac{a}{2b} S_{0,2}^r \left( \frac{\vec{a}}{\vec{x}}, \frac{\vec{b}}{\vec{y}}, \frac{\vec{c}}{\vec{z}} \right) - S_{1,1}^r \left( \frac{\vec{a}}{\vec{x}}, \frac{\vec{b}}{\vec{y}}, \frac{\vec{c}}{\vec{z}} \right) + \frac{b}{2a} S_{2,0}^r \left( \frac{\vec{a}}{\vec{x}}, \frac{\vec{b}}{\vec{y}}, \frac{\vec{c}}{\vec{z}} \right) \\
-(\frac{\vec{b}}{\vec{y}}, \frac{\vec{c}}{\vec{z}}, \frac{\vec{a}}{\vec{x}}) - \frac{c}{2b} S_{2,0}^r \left( \frac{\vec{b}}{\vec{y}}, \frac{\vec{c}}{\vec{z}}, \frac{\vec{a}}{\vec{x}} \right) + \frac{c}{a} S_{0,2}^r \left( \frac{\vec{c}}{\vec{z}}, \frac{\vec{a}}{\vec{x}}, \frac{\vec{b}}{\vec{y}} \right) - S_{1,1}^r \left( \frac{\vec{c}}{\vec{z}}, \frac{\vec{a}}{\vec{x}}, \frac{\vec{b}}{\vec{y}} \right) = 0.
\]

Proof. From Theorem 2.1 we have
\[
\sum_{m,n=0}^{\infty} \frac{1}{m!n!} S_{m,n}^r \left( \frac{\vec{a}}{\vec{x}}, \frac{\vec{b}}{\vec{y}}, \frac{\vec{c}}{\vec{z}} \right) \left( \frac{X}{a} \right)^{m-1} \left( \frac{Y}{b} \right)^{n-1} X + \sum_{m,n=0}^{\infty} \frac{1}{m!n!} S_{m,n}^r \left( \frac{\vec{b}}{\vec{y}}, \frac{\vec{c}}{\vec{z}}, \frac{\vec{a}}{\vec{x}} \right) \left( \frac{Y}{b} \right)^{m-1} \left( \frac{Z}{c} \right)^{n-1} X + \sum_{m,n=0}^{\infty} \frac{1}{m!n!} S_{m,n}^r \left( \frac{\vec{c}}{\vec{z}}, \frac{\vec{a}}{\vec{x}}, \frac{\vec{b}}{\vec{y}} \right) \left( \frac{Z}{c} \right)^{m-1} \left( \frac{X}{a} \right)^{n-1} X = 0.
\]

From this and the equation \( X = -Y - Z \), we know
\[
\sum_{m,n=0}^{\infty} \frac{1}{m!n!} S_{m,n}^r \left( \frac{\vec{a}}{\vec{x}}, \frac{\vec{b}}{\vec{y}}, \frac{\vec{c}}{\vec{z}} \right) (-1)^{m-1} a^{1-m} b^{1-n} (Y + Z)^{m-1} Y^{n-1} \]
\[
+ \sum_{m,n=0}^{\infty} \frac{1}{m!n!} S_{m,n}^r \left( \frac{\vec{b}}{\vec{y}}, \frac{\vec{c}}{\vec{z}}, \frac{\vec{a}}{\vec{x}} \right) (-1)^{n-1} b^{1-m} c^{1-n} Y^{m-1} Z^{n-1} (Y + Z) \]
\[
+ \sum_{m,n=0}^{\infty} \frac{1}{m!n!} S_{m,n}^r \left( \frac{\vec{c}}{\vec{z}}, \frac{\vec{a}}{\vec{x}}, \frac{\vec{b}}{\vec{y}} \right) (-1)^{n-1} c^{1-m} a^{1-n} Z^{m-1} (Y + Z)^n = 0.
\]

Now, taking the coefficients of \( Y^2 Z^{-1}, Z^2 Y^{-1} \) and \( Y \) in (3.5), we obtain the identities (3.2), (3.3) and (3.4), respectively.

Now we are ready to prove Proposition 3.1.

The first proof of Proposition 3.1. From the three identities (3.2), (3.3) and (3.4) we have
\[
\frac{a}{2b} S_{0,2}^r \left( \frac{\vec{a}}{\vec{x}}, \frac{\vec{b}}{\vec{y}}, \frac{\vec{c}}{\vec{z}} \right) - S_{1,1}^r \left( \frac{\vec{a}}{\vec{x}}, \frac{\vec{b}}{\vec{y}}, \frac{\vec{c}}{\vec{z}} \right) + \frac{b}{2a} S_{2,0}^r \left( \frac{\vec{a}}{\vec{x}}, \frac{\vec{b}}{\vec{y}}, \frac{\vec{c}}{\vec{z}} \right) \\
-(\frac{\vec{b}}{\vec{y}}, \frac{\vec{c}}{\vec{z}}, \frac{\vec{a}}{\vec{x}}) - \frac{c}{2b} S_{2,0}^r \left( \frac{\vec{b}}{\vec{y}}, \frac{\vec{c}}{\vec{z}}, \frac{\vec{a}}{\vec{x}} \right) + \frac{c}{a} S_{0,2}^r \left( \frac{\vec{c}}{\vec{z}}, \frac{\vec{a}}{\vec{x}}, \frac{\vec{b}}{\vec{y}} \right) - S_{1,1}^r \left( \frac{\vec{c}}{\vec{z}}, \frac{\vec{a}}{\vec{x}}, \frac{\vec{b}}{\vec{y}} \right) = 0.
\]

We set
\[
\vec{a} = (1, 1), \quad \vec{b} = (p, p), \quad \vec{c} = (q, q) \quad \text{and} \quad \vec{x} = (s, 0), \quad \vec{y} = (pt, 0), \quad \vec{z} = (−qt, 0).
\]
Note that the conditions (2.74) are satisfied in this setting. From (3.6) and (2.3) we have

\[
0 = \frac{a}{2b} S_{0,2}^\tau \left( \frac{\bar{a}}{\bar{x}}, \frac{\bar{b}}{\bar{y}}, \frac{\bar{c}}{\bar{z}} \right) - S_{1,1}^\tau \left( \frac{\bar{a}}{\bar{x}}, \frac{\bar{b}}{\bar{y}}, \frac{\bar{c}}{\bar{z}} \right) + \frac{b}{2c} S_{0,2}^\tau \left( \frac{\bar{b}}{\bar{y}}, \frac{\bar{c}}{\bar{z}}, \frac{\bar{a}}{\bar{x}} \right) - S_{1,1}^\tau \left( \frac{\bar{b}}{\bar{y}}, \frac{\bar{c}}{\bar{z}}, \frac{\bar{a}}{\bar{x}} \right) + \frac{c}{2b} S_{2,0}^\tau \left( \frac{\bar{b}}{\bar{y}}, \frac{\bar{c}}{\bar{z}}, \frac{\bar{a}}{\bar{x}} \right) - S_{1,1}^\tau \left( \frac{\bar{c}}{\bar{z}}, \frac{\bar{a}}{\bar{x}}, \frac{\bar{b}}{\bar{y}} \right) = \frac{1}{2pq} \sum_{\substack{\lambda \equiv q \, (\text{mod } q) \\ \mu \equiv q \, (\text{mod } q)}} B_2 \left( \frac{\lambda}{q} - 2pt, \frac{\mu}{q}; \tau \right) - \frac{1}{q} \sum_{\substack{\lambda \equiv q \\ \mu \equiv q \, (\text{mod } q)}} B_1 \left( \frac{\lambda}{q} - t - s, \frac{\mu}{q}; \tau \right) B_1 \left( \frac{\lambda}{q} - 2pt, \frac{\mu}{q}; \tau \right) + \frac{p}{2q} B_2 \left( qs + qt, 0; \tau \right) - B_1 \left( ps - pt, 0; \tau \right) B_1 \left( qs + qt, 0; \tau \right) + \frac{q}{2p} B_2 \left( ps - pt, 0; \tau \right) - \frac{1}{p} \sum_{\substack{\lambda \equiv p \, (\text{mod } p) \\ \mu \equiv p \, (\text{mod } p)}} B_1 \left( \frac{\lambda}{p} + 2qt, \frac{\mu}{p}; \tau \right) B_1 \left( \frac{\lambda}{p} + t - s, \frac{\mu}{p}; \tau \right) .
\]

Now we will take the limit of the last expression in (3.7) as \( t \) tends to 0. Extra care should be taken for the terms involving \( \lambda = \mu = 0 \), as a priori, \( B_1(0, 0; \tau) \) is not defined. To go around this difficulty, we will make use of the following expansion of \( B_1(x, 0; \tau) \) at \( x = 0 \) (this will be proved later in Lemma 4.1):

\[
B_1(x, 0; \tau) = -\frac{1}{2\pi i} \left[ \frac{1}{x} - E_2(\tau)x - E_4(\tau)x^3 - \cdots \right] .
\]

We have

\[
-\frac{1}{q} B_1(-t - s, 0; \tau) B_1(-2pt, 0; \tau) = -\frac{1}{q} B_1(t + s, 0; \tau) B_1(2pt, 0; \tau)
= \frac{1}{2\pi i} \left[ B_1(s, 0; \tau) + \frac{\partial B_1(s, 0; \tau)}{\partial s} t + \cdots \right] \left[ \frac{1}{2pt} - E_2(\tau)(2pt) - \cdots \right]
\]

and

\[
-\frac{1}{p} B_1(2qt, 0; \tau) B_1(t - s, 0; \tau) = -\frac{1}{p} B_1(2qt, 0; \tau) B_1(-t + s, 0; \tau)
= -\frac{1}{2\pi i} \left[ \frac{1}{2qt} - E_2(\tau)(2qt) - \cdots \right] \left[ B_1(s, 0; \tau) + \frac{\partial B_1(s, 0; \tau)}{\partial s} (-t) + \cdots \right] .
\]

Hence we know

\[
\lim_{t \to 0} \left[ -\frac{1}{q} B_1(-t - s, 0; \tau) B_1(-2pt, 0; \tau) - \frac{1}{p} B_1(2qt, 0; \tau) B_1(t - s, 0; \tau) \right]
= \frac{1}{2\pi i pq} \frac{\partial B_1(s, 0; \tau)}{\partial s} .
\]
From this we know that the last expression in (3.7) converges to

\[ \frac{1}{2pq} \sum_{\mu \equiv \lambda \pmod{q}} B_2 \left( \frac{p\lambda}{q}, \frac{p\mu}{q}; \tau \right) \]

\[ - \frac{1}{q} \sum_{\lambda, \mu = 0 \atop (\lambda, \mu) \neq (0, 0)} B_1 \left( \frac{\lambda}{q} - s, \frac{\mu}{q}; \tau \right) B_1 \left( \frac{p\lambda}{q}, \frac{p\mu}{q}; \tau \right) \]

\[ + \frac{p}{2q} B_2 (qs, 0; \tau) - B_1 (ps, 0; \tau) B_1 (qs, 0; \tau) + \frac{q}{2p} B_2 (ps, 0; \tau) \]

\[ - \frac{1}{p} \sum_{\lambda, \mu = 0 \atop (\lambda, \mu) \neq (0, 0)} B_1 \left( \frac{q\lambda}{p}, \frac{q\mu}{p}; \tau \right) B_1 \left( \frac{\lambda}{p} - s, \frac{\mu}{p}; \tau \right) \]

\[ + \frac{1}{2\pi ipq} \frac{\partial B_1 (s, 0; \tau)}{\partial s} \]

when \( t \) tends to 0.

Finally, setting

\[ C(\tau) = \frac{1}{2pq} \sum_{\mu \equiv \lambda \pmod{q}} B_2 \left( \frac{p\lambda}{q}, \frac{p\mu}{q}; \tau \right), \]

we obtain the identity (3.11). This completes the proof. \( \square \)

Now we will give the second proof, which was kindly communicated to us by the referee.

**The second proof of Proposition 3.1.** We recall the identity (2.1)

\[ \mathcal{F}(x, y; \xi; \tau) = \sum_{m=0}^{\infty} \frac{B_m(x, y; \tau)}{m!} (2\pi i)^m \xi^{m-1}. \]

According to a classical result of Kronecker (refer to Weil [16]), the left hand side above admits the following partial fraction decomposition

(3.8) \[ \mathcal{F}(x, y; \xi; \tau) = \lim_{M \to \infty} \sum_{m=-M}^{M} \left( \lim_{N \to \infty} \sum_{n=-N}^{N} \frac{\chi(w \bar{z})}{w + \xi} \right) \]

where \( z = -x + y \tau, w = m \tau + n \) and \( \chi(t) = \exp(2\pi i \Im(t)/\Im(\tau)) \). Expanding the right hand side of (3.8) into a power series in \( \xi \), we have

(3.9) \[ \mathcal{F}(x, y; \xi; \tau) = \sum_{k=0}^{\infty} (-1)^{k-1} C_k(z) \xi^{k-1} \]

where

(3.10) \[ C_k(z) = \lim_{M \to \infty} \sum_{m=-M}^{M} \left( \lim_{N \to \infty} \sum_{n=-N}^{N} \frac{\chi(w \bar{z})}{w^k} \right). \]

Therefore, from (2.1) and (3.9), we know

(3.11) \[ B_k(x, y; \tau) = \frac{(-1)^{k-1} k!}{(2\pi i)^k} C_k(-x + y \tau). \]
In what follows, we use the notation $\mathcal{E}(z)$ and $\mathcal{E}_k(z)$ in place of $E(z)$ and $E_k(z)$ in Sczech [13] to distinguish them from the Eisenstein series. First we note that

\begin{equation}
C_1(z) = \mathcal{E}_1(z), \quad C_2(z) = \mathcal{E}(z).
\end{equation}

Now we apply Satz 1 in Sczech [13, p. 530], setting $c_1 = p$, $c_2 = q$, $c_3 = 1$, $z_1 = z_2 = 0$, $z_3 = x$. This gives the following reciprocity law

\begin{equation}
q \sum_{k=0}^{\infty} q^{-k} \sum_{r \in \mathbb{L}/pL} \mathcal{E}_k(px) + \sum_{k=0}^{\infty} q^{k-1} \sum_{r \in \mathbb{L}/qL} \mathcal{E}_k(qx) + \frac{1}{pq} \mathcal{E}(0) + \sum_{k=0}^{\infty} q^{-k} \sum_{r \in \mathbb{L}/pL} \mathcal{E}_k(px) = 0
\end{equation}

where $L$ denotes the lattice $\mathbb{Z}\tau + \mathbb{Z}$.

Furthermore, it was shown in [13] (using results of Hecke) that

\begin{equation}
\mathcal{E}_1(z) = \frac{1}{\tau} - z \mathcal{E}_2(\tau), \quad 2\mathcal{E}(z) = \varphi(z; \tau) - C_1(z)^2,
\end{equation}

$\mathcal{E}(0) = E_2(\tau) - \frac{\pi i}{3\tau}$,

$\mathcal{E}_0(z) = \begin{cases} -1 & z \in \mathbb{L} \\ 0 & \text{otherwise} \end{cases}$

where $\varphi(z; \tau)$ and $\zeta(z; \tau)$ are the Weierstrass pe and zeta functions and $E_2(\tau)$ is the Eisenstein series of weight two.

Now we take $x$ to be a sufficiently small and $x \neq 0$ so that $px$ and $qx$ are not rational integers. Then the equation (3.13) combined with the identities (3.12) and (3.14) produces the following formula

\begin{align*}
(2\pi i)^2 [D^{-}(p, q; \tau; x) + D^{-}(q, p; \tau; x)]
&= -C_1(qx)C_1(px) + \frac{\varphi(x)}{pq} - \frac{p}{q}C_2(qx) - \frac{q}{p}C_2(px).
\end{align*}

This formula and the identities (3.11) imply Proposition 3.4.

4. **Weierstrass elliptic functions and elliptic Bernoulli functions**

In this section we study the relationship between the Weierstrass elliptic functions and the elliptic Bernoulli functions.
For \( z \in \mathbb{C} \) and \( \tau \in \mathbb{H} \), the Weierstrass sigma, zeta and \( \wp \) functions are given as follows:

\[
\sigma(z;\tau) := z \prod_{\gamma \in \mathbb{Z} + \tau, \gamma \neq 0} \left(1 - \frac{z}{\gamma}\right) \exp\left(\frac{z}{\gamma} + \frac{1}{2}\left(\frac{z}{\gamma}\right)^2\right),
\]

\[
\zeta(z;\tau) := \frac{1}{z} + \sum_{\gamma \in \mathbb{Z} + \tau, \gamma \neq 0} \left(\frac{1}{z - \gamma} + \frac{1}{\gamma} + \frac{z}{\gamma^2}\right),
\]

\[
\wp(z;\tau) := \frac{1}{z^2} + \sum_{\gamma \in \mathbb{Z} + \tau} \left(\frac{1}{(z - \gamma)^2} - \frac{1}{\gamma^2}\right).
\]

It is known that these functions have the following expansions at \( z = 0 \):

\[
\log \sigma(z;\tau) = \log z - \sum_{n=2}^{\infty} \frac{1}{2n} E_{2n}(\tau) z^{2n},
\]

\[
\zeta(z;\tau) = \frac{1}{z} - \sum_{n=2}^{\infty} E_{2n}(\tau) z^{2n-1},
\]

\[
\wp(z;\tau) = \frac{1}{z^2} + \sum_{n=2}^{\infty} (2n - 1) E_{2n}(\tau) z^{2n-2}
\]

where \( E_{2n}(\tau) \) is the Eisenstein series of weight \( 2n \), namely,

\[
E_{2n}(\tau) := \sum_{\gamma \in \mathbb{Z} + \tau, \gamma \neq 0} \frac{1}{\gamma^{2n}}.
\]

It is also known that \( E_{2n}(\tau) \) have the following expansion:

\[
E_{2n}(\tau) = 2\zeta(2n) + \frac{(2\pi i)^{2n}}{(2n-1)!} \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} k^{2n-1} q^{kj}
\]

\[
= 2\zeta(2n) + \frac{(2\pi i)^{2n}}{(2n-1)!} \sum_{k=1}^{\infty} k^{2n-1} \frac{q^k}{1-q^k}
\]

\[
= 2\zeta(2n) + \frac{(2\pi i)^{2n}}{(2n-1)!} \sum_{k=1}^{\infty} \sigma_{2n-1}(k) q^k \quad (\sigma_\ell(k) = \sum_{d|k} d^\ell)
\]

where \( \zeta(z) \) denotes the Riemann zeta function, and it holds that

\[
2\zeta(2n) = -\frac{(2\pi i)^{2n} B_{2n}}{(2n)!}.
\]

Now it is easy to see that these functions have the following relation:

\[
\frac{\partial \log \sigma(z;\tau)}{\partial z} = \zeta(z;\tau), \quad \frac{\partial \zeta(z;\tau)}{\partial z} = -\wp(z;\tau).
\]

The function \( \zeta(z;\tau) \) is subject to the following identities ([15] p. 84)):

\[
(4.1) \quad \zeta(z + 1;\tau) = \zeta(z;\tau) + E_2(\tau), \quad \zeta(z + \tau;\tau) = \zeta(z;\tau) + E_2(\tau)\tau - 2\pi i.
\]

Next we express the elliptic Bernoulli functions of lower degrees in terms of the Weierstrass elliptic functions and the Eisenstein series.
Lemma 4.1. For sufficiently small real numbers \( x, y \) and \( \tau \in \mathbb{H} \), it holds that

\[
B_1(x, y; \tau) = \frac{1}{2\pi i} \left[ \frac{1}{x - y\tau} - \sum_{n=1}^{\infty} E_{2n}(\tau)(x - y\tau)^{2n-1} \right] + y
\]

(4.2)

\[
= -\frac{1}{2\pi i} \left[ \zeta(x - y\tau; \tau) - E_2(\tau)(x - y\tau) \right] + y,
\]

(4.3)

\[
B_2(x, 0; \tau) = -\frac{1}{\pi i} \left[ \sum_{n=1}^{\infty} \frac{1}{2n} \frac{\partial E_{2n}(\tau)}{\partial \tau} (x - y\tau)^{2n} + \frac{1}{2\pi i} E_2(\tau) \right]
\]

(4.4)

\[
= \frac{1}{2\pi i} \left[ 2\frac{\partial \log \sigma(x; \tau)}{\partial \tau} - \frac{\partial E_2(\tau)}{\partial \tau} x^2 - \frac{1}{\pi i} E_2(\tau) \right],
\]

(4.5)

\[
\frac{\partial B_1(x, 0; \tau)}{\partial x} = \frac{1}{2\pi i} \left[ \frac{1}{x^2} + \sum_{n=1}^{\infty} (2n - 1) E_{2n}(\tau)x^{2n-2} \right]
\]

(4.6)

\[
= \frac{1}{2\pi i} \left[ \varphi(x; \tau) + E_2(\tau) \right].
\]

(4.7)

Proof. The proof is based upon the expansion (2.2) and direct calculations:

\[
B_1(x, y; \tau) = \sum_{j=1}^{\infty} \frac{e(-yr)q^j}{e(-x) - e(-yr)q^j} - \sum_{j=1}^{\infty} \frac{e(yr)q^j}{e(x) - e(yr)q^j} + \frac{e(-x + y\tau)}{e(-x + y\tau) - 1} + B_1(y)
\]

\[
= \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} \left[ e(k(x - y\tau))q^{k j} - e(-k(x - y\tau))q^{k j} \right]
\]

\[
- \frac{1}{2\pi i} \sum_{n=0 \text{ even}} \frac{B_n}{n!} (2\pi i(x - y\tau))^n + y
\]

(4.8)

\[
= 2 \sum_{n=0, n \text{ odd}} \frac{1}{n!} \left[ \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} k^n q^{k j} \right] (2\pi i(x - y\tau))^n - \sum_{n=0 \text{ even}} \frac{B_n}{n!} (2\pi i(x - y\tau))^{n-1} + y
\]

(4.9)

\[
= 2 \sum_{n=0, n \text{ odd}} \frac{1}{n!} \left[ \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} k^n q^{k j} \right] (2\pi i(x - y\tau))^n - \sum_{n=0 \text{ even}} \frac{B_n}{n!} (2\pi i(x - y\tau))^{n-1} + y
\]

(4.10)

\[
= \frac{1}{2\pi i} \zeta(x - y\tau; \tau) - E_2(\tau)(x - y\tau)\]

(4.11)

\[
= \frac{1}{2\pi i} \zeta(x - y\tau; \tau) - E_2(\tau)(x - y\tau) + y,
\]

(4.12)

\[
B_2(x, 0; \tau) = 2 \left[ \sum_{j=1}^{\infty} \frac{(-j)^q q^j}{e(-x) - q^j} - \sum_{j=1}^{\infty} j \frac{q^j}{e(x) - q^j} \right] + B_2
\]

(4.13)

\[
= -2 \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} j [e(kx)q^{k j} + e(-kx)q^{k j}] + B_2
\]

(4.14)
\[
-4 \sum_{n=0}^{\infty} \frac{1}{n!} \left[ \sum_{j=1}^{\infty} \sum_{k=1}^{j} jk^n q^k \right] (2\pi i x)^n + B_2
\]

\[
= -4 \sum_{n=1}^{\infty} \frac{1}{(2n)!} \left[ \sum_{j=1}^{\infty} \sum_{k=1}^{j} jk^{2n} q^k \right] (2\pi i x)^{2n} - 4 \sum_{j=1}^{\infty} \frac{j q^j}{1 - q^j} + B_2
\]

\[
= -\frac{1}{\pi i} \sum_{n=1}^{\infty} \frac{1}{2n} \frac{\partial E_{2n}(\tau)}{\partial \tau} x^{2n} - \frac{1}{2 (2\pi i)^2} E_2(\tau)
\]

\[
\frac{\partial B_1(x, 0; \tau)}{\partial x} = \frac{1}{2\pi i} \left[ \frac{1}{x^2} + \sum_{n=1}^{\infty} (2n - 1) E_{2n}(\tau) x^{2n-2} \right]
\]

\[
= \frac{1}{2\pi i} \left[ \psi(x; \tau) + E_2(\tau) \right].
\]

These give the identities from (4.2) to (4.7).

5. Proofs of Theorems 1.1 and 1.3

In this section we give proofs of Theorems 1.1 and 1.3.

Proof of Theorem 1.1. For \( \lambda, \mu \) such that \((\lambda, \mu) \neq (0, 0)\), the identity (4.3) gives

\[
B_1 \left( \frac{\lambda}{p} - x, -\frac{\mu}{p}; \tau \right) = -\frac{1}{2\pi i} \left[ \zeta \left( \frac{\lambda + \mu \tau}{p} - x; \tau \right) - E_2(\tau) \left( \frac{\lambda + \mu \tau}{p} - x \right) + 2\pi i \frac{\mu}{p} \right],
\]

\[
B_1 \left( \frac{q \lambda}{p}, -\frac{q \mu}{p}; \tau \right) = -\frac{1}{2\pi i} \left[ \zeta \left( \frac{q(\lambda + \mu \tau)}{p}; \tau \right) - E_2(\tau) \frac{q(\lambda + \mu \tau)}{p} + 2\pi i \frac{q \mu}{p} \right].
\]

Hence we have

(5.1)

\[
\frac{1}{p} \sum_{\lambda, \mu = 0}^{p-1} B_1 \left( \frac{\lambda}{p} - x, -\frac{\mu}{p}; \tau \right) B_1 \left( \frac{q \lambda}{p}, -\frac{q \mu}{p}; \tau \right)
\]

\[
= \frac{1}{p} \sum_{\lambda, \mu = 0}^{p-1} B_1 \left( \frac{\lambda}{p} - x, -\frac{\mu}{p}; \tau \right) B_1 \left( \frac{q \lambda}{p}, -\frac{q \mu}{p}; \tau \right)
\]

\[
(\text{since } B_1(x, y + 1; \tau) = B_1(x, y; \tau))
\]

\[
= \frac{1}{(2\pi i)^2 p} \sum_{\lambda, \mu = 0}^{p-1} \left[ \zeta \left( \frac{\lambda + \mu \tau}{p} - s; \tau \right) - E_2(\tau) \left( \frac{\lambda + \mu \tau}{p} - s \right) + 2\pi i \frac{\mu}{p} \right]
\]

\[
\times \left[ \zeta \left( \frac{q(\lambda + \mu \tau)}{p}; \tau \right) - E_2(\tau) \frac{q(\lambda + \mu \tau)}{p} + 2\pi i \frac{q \mu}{p} \right]
\]

\[
= D^{-}(p, q; \tau; x).
\]
Moreover the identities (4.3), (4.5) and (4.7) give us

\begin{align}
- B_1(px, 0; \tau)B_1(qx, 0; \tau) + \frac{q}{2p} B_2(px, 0; \tau) + \frac{p}{2q} B_2(qx, 0; \tau) + \frac{1}{2\pi ipq} \frac{\partial B_1(x, 0; \tau)}{\partial x} \\
= - \frac{1}{(2\pi i)^2} [\zeta(px; \tau) - E_2(\tau)(px)] [\zeta(qx; \tau) - E_2(\tau)(qx)] \\
+ \frac{q}{4\pi ip} \left[ 2 \frac{\partial \log \sigma(px; \tau)}{\partial \tau} - \frac{\partial E_2(\tau)}{\partial \tau} (px)^2 - \frac{1}{\pi i} E_2(\tau) \right] \\
+ \frac{p}{4\pi iq} \left[ 2 \frac{\partial \log \sigma(qx; \tau)}{\partial \tau} - \frac{\partial E_2(\tau)}{\partial \tau} (qx)^2 - \frac{1}{\pi i} E_2(\tau) \right] \\
+ \frac{1}{2\pi ipq} \frac{1}{2\pi i} [\varphi(x; \tau) + E_2(\tau)] \\
= R^-(p, q; \tau, x).
\end{align}

Now the identities (5.1) and (5.2) together with Proposition 3.1 give (2) of Theorem 1.3.

The assertion (1) follows easily from (4.1) and the fact that \( \zeta(z; \tau) \) is an odd function with respect to \( z \). This completes the proof.

Next we give a proof of Theorem 1.1.

**Proof of Theorem 1.1.** Let us consider the Taylor expansion of

\[ D^-(p, q; \tau, x) = \frac{1}{(2\pi i)^2 p(2n)!} \sum_{\lambda, \mu = 0}^{p-1} \sum_{\lambda, \mu = 0}^{p-1} \zeta(\frac{\lambda + \mu \tau}{p}; \tau) - E_2(\tau) \left( \frac{\lambda + \mu \tau}{p} - x \right) + 2\pi i \frac{\mu}{p} \]

at \( x = 0 \). Then we see that the coefficient of \( x^{2n} \) in this expansion is equal to

\[ \frac{1}{(2\pi i)^2 p(2n)!} \sum_{\lambda, \mu = 0}^{p-1} \sum_{\lambda, \mu = 0}^{p-1} \zeta(2n) \left( \frac{\lambda + \mu \tau}{p}; \tau \right) \]

\[ \times \left[ \zeta \left( \frac{q(\lambda + \mu \tau)}{p}; \tau \right) - E_2(\tau) \frac{q(\lambda + \mu \tau)}{p} + 2\pi i \frac{q\mu}{p} \right]. \]

This is nothing but \( D_{2n}^-(p, q; \tau). \)

Next, applying (4.2), (4.4) and (4.6), we expand \( R^-(p, q; \tau, x) \) as follows:

\[ R^-(p, q; \tau, x) = - \frac{1}{(2\pi i)^2} [\zeta(px; \tau) - E_2(\tau)(px)] [\zeta(qx; \tau) - E_2(\tau)(qx)] \\
+ \frac{q}{4\pi ip} \left[ 2 \frac{\partial \log \sigma(px; \tau)}{\partial \tau} - \frac{\partial E_2(\tau)}{\partial \tau} (px)^2 - \frac{1}{\pi i} E_2(\tau) \right] \\
+ \frac{p}{4\pi iq} \left[ 2 \frac{\partial \log \sigma(qx; \tau)}{\partial \tau} - \frac{\partial E_2(\tau)}{\partial \tau} (qx)^2 - \frac{1}{\pi i} E_2(\tau) \right] \\
+ \frac{1}{2\pi ipq} \frac{1}{2\pi i} [\varphi(x; \tau) + E_2(\tau)]. \]
\[- \frac{1}{(2\pi i)^2} \left[ \frac{1}{p} - \sum_{n=1}^{\infty} E_{2n}(\tau)(px)^{2n-1} \right] \left[ \frac{1}{q} - \sum_{n=1}^{\infty} E_{2n}(\tau)(qx)^{2n-1} \right] \]
\[- \frac{q}{2\pi ip} \left[ \sum_{n=1}^{\infty} \frac{1}{2n} \frac{\partial E_{2n}(\tau)}{\partial \tau}(px)^{2n} + \frac{1}{2\pi i} E_2(\tau) \right] \]
\[- \frac{p}{2\pi iq} \left[ \sum_{n=1}^{\infty} \frac{1}{2n} \frac{\partial E_{2n}(\tau)}{\partial \tau}(qx)^{2n} + \frac{1}{2\pi i} E_2(\tau) \right] \]
\[+ \frac{1}{2\pi ipq} \left[ \frac{1}{x^2} + \sum_{n=1}^{\infty} (2n-1)E_{2n}(\tau)x^{2n-2} \right].\]

Thus we know that the coefficient of \(x^{2n}\) in this expansion is equal to

\[- \frac{1}{(2\pi i)^2} \left[ \sum_{j=1}^{n} E_{2j}(\tau)E_{2n+2-2j}(\tau)p^{2j-1}q^{2n+1-2j} - E_{2n+2}(\tau)\frac{p^{2n+2} + q^{2n+2}}{pq} \right] \]
\[- \frac{q}{2\pi ip} \left[ \frac{1}{2n} \frac{\partial E_{2n}(\tau)}{\partial \tau}p^{2n} - \frac{p}{2\pi iq} \frac{1}{2n} \frac{\partial E_{2n}(\tau)}{\partial \tau}q^{2n} + \frac{1}{2\pi i} (2n+1)E_{2n+2}(\tau)\frac{1}{pq} \right].\]

This is nothing but \(R_w^{-n}(p, q; \tau)\).

Hence, from the reciprocity laws (2) in Theorem 1.3, we obtain the reciprocity laws (2) in Theorem 1.1.

The assertion (1) easily follows from that of Theorem 1.3. This completes the proof. \(\square\)

**Remark 5.1.** A direct calculation shows that

\[\lim_{\tau \to i\infty} R_w^{-n}(p, q; \tau) = \frac{2(2\pi i)^w}{w!} g_w(p, q).\]

From this and (1.5), we obtain (1.6) which shows that \(D_w^{-}(p, q; \tau)\) is an elliptic analogue of the Apostol-Dedekind sum.

### 6. A proof of Theorem 1.2

In this section we give a proof of Theorem 1.2. We first set up some notation. Let \(f\) be an element of \(S_{w+2}\). Then \(n\)th period of \(f\), \(r_n(f)\), is defined by

\[r_n(f) := \int_0^{\infty} f(z)z^ndz \quad (n = 0, 1, \ldots, w).\]

Furthermore, the period polynomial \(r(f)\) and the odd period polynomial \(r^{-}(f)\) of \(f\) in the variables \(p\) and \(q\) is defined by

\[r(f)(p, q) := \int_0^{\infty} f(z)(pz-q)^wdz \quad \text{and} \quad r^{-}(f)(p, q) := \frac{1}{2}[r(f)(p, q) - r(f)(p, -q)].\]

It is clear that \(r^{-}(f)(p, q)\) has the following expression:

\[r^{-}(f)(p, q) = - \sum_{n=0}^{w} \binom{w}{n} r_{w-n}(f)p^{w-n}q^n.\]

Here and hereafter \(\binom{w}{n}\) denotes a binomial coefficient.
Let $G_{2n}$ be a normalized Eisenstein series:

$$G_{2n}(\tau) := -\frac{B_{2n}}{4n} + \sum_{k=1}^{\infty} \frac{\sigma_{2n-1}(k)}{k} q^k.$$ 

Notice that

$$E_{2n} = \frac{2(2\pi i)^{2n}}{(2n-1)!} G_{2n}.$$ 

To prove Theorem 1.2, we need the following lemma.

**Lemma 6.1.** Set $w = 2n$ and let $\{f_j\}_{j=1}^{d_w}$ be a basis of normalized eigenforms of $S_{w+2}$. Then it holds that

$$\sum_{j=1}^{n} \left[ G_{2j} G_{2n+2-2j} + (\delta_{j,1} + \delta_{j,n}) \frac{1}{8\pi in} \frac{\partial G_{2n}}{\partial \tau} + \frac{B_{2j}}{2j} \frac{B_{2n+2-2j}}{2n+2-2j} \frac{2n+2}{B_{2n+2}} G_{2n+2} \right] \times \left( \frac{2n}{2j-1} \right) p^{2j-1} q^{2n+1-2j}$$

$$= -\frac{1}{(2i)^{2n+1}} \sum_{j=1}^{d_w} r_{2n} (f_j) r_2 (p, q) \frac{r_2 (f_j)}{(f_j, f_j)} f_j$$

where $\delta_{i,j}$ is the Kronecker delta symbol, and $(f, g)$ denotes the Petersson inner product of $f$ and $g$.

**Proof.** We use the following Rankin’s identity (refer to Kohnen-Zagier [5] noting that their notation of $r_n(f)$ differs from ours by a factor $(n+1)^{i}$): for a normalized eigenform $f$ of $S_{w+2}$,

$$(f, G_{2j} G_{2n+2-2j} + (\delta_{j,1} + \delta_{j,n}) \frac{1}{8\pi in} \frac{\partial G_{2n}}{\partial \tau}) = \frac{1}{(2i)^{2n+1}} r_{2n} (f) r_{2j-1} (f)$$

where $j = 1, 2, \ldots, n$.

We set $g$ and $h$ to be the left and right hand sides of (6.1), respectively. We note that both $g$ and $h$ are cusp forms of $S_{w+2}$. Then, for any $f_i$ ($i = 1, 2, \ldots, d_w$), we have

$$(f_i, g) = \sum_{j=1}^{n} \left( f_i, G_{2j} G_{2n+2-2j} + (\delta_{j,1} + \delta_{j,n}) \frac{1}{8\pi in} \frac{\partial G_{2n}}{\partial \tau} \right) \left( 2n \right) \frac{2n}{2j-1} p^{2j-1} q^{2n+1-2j}$$

(since $(f_i, G_{2n+2}) = 0$)

$$= \sum_{j=1}^{n} \frac{1}{(2i)^{2n+1}} r_{2n} (f_i) r_{2j-1} (f_i) \left( 2n \right) \frac{2n}{2j-1} p^{2j-1} q^{2n+1-2j}$$

(by (6.2))

$$= -\frac{1}{(2i)^{2n+1}} r_{2n} (f_i) r^- (f_i) (p, q)$$

$$= (f_i, h).$$

This implies (6.1). □

Now we are ready to prove Theorem 1.2.
Proof of Theorem 1.3. Set \( w = 2n \) and let \( \{ f_j \}_{j=1}^{d_w} \) be a basis of normalized eigenforms of \( S_{2n+2} \).

We use the formulas ([17] pp. 453–454)

\[
  r_{2n}(G_{2n+2}) = \frac{(2n)! \zeta(2n+1)}{2(2\pi i)^{2n+1}},
\]

\[
  (G_{2n+2}, G_{2n+2}) = \frac{(2n)!}{(4\pi)^{2n+1}} \frac{B_{2n+2}}{2(2n+2)} \zeta(2n+1)
\]

and the formula (3)

\[
  r^-(G_{2n+2})(p, q) = -\frac{1}{pq} \left\{ \sum_{j=0}^{n+1} \frac{(2n)!}{2(2j)! (2n+2-2j)!} p^{2j} q^{2n+2-2j} + \frac{B_{2n+2}}{4(n+1)} \right\},
\]

to reformulate \( R_{2n}^{-}(p, q; \tau) \) as follows:

(6.3)

\[
  R_{2n}^{-}(p, q; \tau) = -\frac{1}{(2\pi i)^2 pq} \left[ \sum_{j=1}^{n} E_{2j}(\tau) E_{2n+2-2j}(\tau) p^{2j} q^{2n+2-2j} - E_{2n+2}(\tau) (p^{2n+2} + q^{2n+2}) - (2n+1) E_{2n+2}(\tau) \right]
\]

\[
  -\frac{1}{4\pi in} \frac{\partial E_{2n}(\tau)}{\partial \tau} \left( q^{2n-1} - pq^{2n-1} \right)
\]

\[
  = -\frac{4(2\pi i)^{2n+2}}{(2\pi i)^2 (2n)!} \left\{ \sum_{j=1}^{n} G_{2j}(\tau) G_{2n+2-2j}(\tau) \left( \delta_{j,1} + \delta_{j,n} \right) \frac{1}{8\pi in} \frac{\partial G_{2n}(\tau)}{\partial \tau} \right\}
\]

\[
  \times \left( \frac{2n}{2j-1} \right) q^{2n+1-2j}
\]

\[
  + \frac{2(2\pi i)^{2n}}{(2n+1)!} G_{2n+2}(\tau) q^{2n+2} + \frac{2(2\pi i)^{2n}}{(2n)!} G_{2n+2}(\tau) \frac{1}{pq}
\]

\[
  = \frac{4(2\pi i)^{2n}}{(2n)!} \frac{1}{(2i)^{2n+1}} \sum_{j=1}^{d_w} \frac{r_{2n}(f_j) r^-(f_j)(p, q)}{(f_j, f_j)} f_j(\tau)
\]

\[
  + \frac{2(2\pi i)^{2n}}{(2n)!} \frac{2n + 2}{B_{2n+2}} (2n)! \sum_{j=0}^{n+1} \left[ \frac{B_{2j}}{(2j)!} \frac{B_{2n+2-2j}}{(2n+2-2j)!} q^{2n+1-2j} + \frac{2n+1}{(2n+2)!} \frac{B_{2n+2}}{pq} \right] G_{2n+2}(\tau)
\]

(by Lemma 6.1)

\[
  = \frac{4(2\pi i)^{2n}}{(2n)!} \frac{1}{(2i)^{2n+1}} \sum_{j=1}^{d_w} \frac{r_{2n}(f_j) r^-(f_j)(p, q)}{(f_j, f_j)} f_j(\tau)
\]

\[
  - \frac{4(2\pi i)^{2n}}{(2n)!} \frac{2n + 2}{B_{2n+2}} r^-(G_{2n+2})(p, q) G_{2n+2}(\tau)
\]
the identity (6.3) can be rewritten as
\[
\frac{4(2\pi i)^{2n}}{(2n)!} \frac{1}{(2\pi i)^{2n+1}} \sum_{j=1}^{d_w} r_{2n}(f_j) r^-(f_j)(p,q) f_j(\tau)
\]
\[
+ \frac{4(2\pi i)^{2n}}{(2n)!} \frac{1}{(2\pi i)^{2n+1}} \sum_{j=1}^{d_w} r_{2n}(G_{2n+2}) r^-(G_{2n+2})(p,q) G_{2n+2}(\tau).
\]
By setting
\[
f_0(\tau) := G_{2n+2}(\tau),
\]
the identity (6.3) can be rewritten as
\[
R_{2n}(p,q;\tau) = -\frac{2i\pi 2^n}{(2n)!} \sum_{j=0}^{d_w} r_{2n}(f_j) r^-(f_j)(p,q) f_j(\tau).
\]
Now, since \( f_j \) \((j = 0,1,\cdots,d_w)\) form a basis of \( M_{w+2} \), thus they are linearly independent. Hence there are \( \tau_0, \tau_1, \ldots, \tau_{d_w} \in \mathbb{H} \) such that
\[
f_0(\tau_0) \quad f_0(\tau_1) \quad \cdots \quad f_0(\tau_{d_w}) \quad f_1(\tau_0) \quad f_1(\tau_1) \quad \cdots \quad f_1(\tau_{d_w})
\]
\[
\cdots \quad \cdots \quad \cdots \quad \cdots
\]
\[
f_{d_w}(\tau_0) \quad f_{d_w}(\tau_1) \quad \cdots \quad f_{d_w}(\tau_{d_w})
\]
\[
\neq 0.
\]
On the other hand, since \( \beta_w \alpha_{w+2} \) is an isomorphism, and \( \beta_w \alpha_{w+2}(f_j) = r^-(f_j) \), we can deduce that
\[
\{r^-(f_j)(p,q)\}_{j=0}^{d_w}
\]
is a basis of \( \mathcal{W}^-_w \). Therefore, noting that \( r_{2n}(f_j) \neq 0 \), we know that
\[
\left\{ \frac{r_{2n}(f_j) r^-(f_j)(p,q)}{(f_j,f_j)} \right\}_{j=0}^{d_w}
\]
is also a basis of \( \mathcal{W}^-_w \) and, by (6.5), we conclude that
\[
\left\{ \frac{\sum_{j=0}^{d_w} r_{2n}(f_j) r^-(f_j)(p,q)}{(f_j,f_j)} f_j(\tau_i) \right\}_{i=0}^{d_w}
\]
is also a basis of \( \mathcal{W}^-_w \). This together with the identity (6.4) imply that
\[
\{R_{2n}^-(p,q;\tau)\}_{i=0}^{d_w}
\]
is again a basis of \( \mathcal{W}^-_w \).
Finally, from the fact that \( \beta_w \) is an isomorphism, and that \( \beta_w(D_{2n}^-(p,q;\tau_i)) = R_{2n}^-(p,q;\tau_i) \), we deduce that
\[
\{D_{2n}^-(p,q;\tau_i)\}_{i=0}^{d_w}
\]
is a basis of \( \mathcal{D}^-_w \). This establishes what we are after.

\textbf{Remark 6.1.} It should be remarked that \( R_{2n}^-(p,q;\tau) \) “generates” not only odd period polynomials but also modular forms. In other words, it follows that there are \( (p_i, q_i) \in \mathbb{C}^2 \) \((i = 0,1,\ldots,d_w)\) such that \( \{R_{2n}^-(p_i,q_i;\tau)\}_{i=0}^{d_w} \) is a base of \( M_{2n+2} \).
7. An application to Eisenstein series identities

In this section we will give an application of Theorem 1.1. Let \( S_{2n}^-(p, q; \tau) \) and \( T_{2n}^-(p, q; \tau) \) be defined by

\[
S_{2n}^-(p, q; \tau) := R_{2n}^-(p, q; \tau) - \frac{1}{(2\pi i)^2} (2n + 1) E_{2n+2}(\tau) \frac{1}{pq}
\]

\[
= - \frac{1}{(2\pi i)^2} \left[ \sum_{j=1}^{\infty} E_{2j}(\tau) E_{2n+2-2j}(\tau) p^{2j-1} q^{2n+1-2j} - E_{2n+2}(\tau) \frac{p^{2n+2} + q^{2n+2}}{pq} \right]
\]

\[
- \frac{q}{2\pi ip} \frac{1}{2n} \frac{\partial E_{2n}(\tau)}{\partial \tau} p^{2n} - \frac{p}{2\pi iq} \frac{1}{2n} \frac{\partial E_{2n}(\tau)}{\partial \tau} q^{2n},
\]

\[
T_{2n}^-(p, q; \tau) := (2\pi i)^2 pq S_{2n}^-(p, q; \tau).
\]

By Theorem 1.1 \( R_{2n}^-(p, q; \tau) \) satisfies the equation (1.3). Since \( 1/pq \) also satisfies the equation (1.3), this can be carried over to \( S_{2n}^-(p, q; \tau) \):

\[
S_{2n}^-(p + q, q; \tau) + S_{2n}^-(p, p + q; \tau) = S_{2n}^-(p, q; \tau).
\]

Hence it follows that \( T_{2n}^-(p, q; \tau) \) satisfies the equation

\[
p T_{2n}^-(p + q, q; \tau) + q T_{2n}^-(p, p + q; \tau) = (p + q) T_{2n}^-(p, q; \tau).
\]

Now we set

\[
c_j := \begin{cases} E_{2n+2}(\tau), & j = 0, n + 1 \\ -E_2(\tau) E_{2n}(\tau) - \frac{\pi i \frac{\partial E_{2n}(\tau)}{\partial \tau}}{n}, & j = 1, n \\ -E_{2j}(\tau) E_{2n+2-2j}(\tau), & \text{otherwise} \end{cases}
\]

so that \( T_{2n}^-(p, q; \tau) \) can be expressed as

\[
T_{2n}^-(p, q; \tau) = \sum_{j=0}^{n+1} c_j p^{2j} q^{2n+2-2j}.
\]

This gives rise to the following Eisenstein series identities:

**Theorem 7.1.** For positive integers \( n \) and \( k \) with \( 1 \leq k \leq 2n + 2 \), it holds that

\[
\sum_{i=0}^{n+1} \binom{2i}{k-1} c_i + \sum_{i=0}^{n+1} \binom{2n + 2 - 2i}{2n + 2 - k} c_i = \begin{cases} \frac{c_{k-1}}{2}, & k \text{ odd} \\ \frac{c_k}{4}, & k \text{ even} \end{cases}
\]

**Proof.** From (7.1) and (7.2), we have

\[
p \sum_{i=0}^{n+1} c_i \sum_{j=0}^{2i} \binom{2i}{j} p^i q^{2i-j} q^{2n+2-2i} + q \sum_{i=0}^{n+1} c_i p^{2i} \sum_{j=0}^{2n+2-2i} \binom{2n + 2 - 2i}{j} p^{2n+2-2i-j} q^j
\]

\[
= p \sum_{i=0}^{n+1} c_i p^{2i} q^{2n+2-2i} + q \sum_{i=0}^{n+1} c_i p^{2i} q^{2n+2-2i}.
\]

By comparing the coefficients of \( p^k q^{2n+3-k} \) in the both sides of the equation above, we obtain the identities (7.3). □
If we take \( k = 1 \) in Theorem 7.1 we rediscover the formulas
\[
\frac{2\pi i}{n} \frac{\partial E_{2n}(\tau)}{\partial \tau} = - \sum_{j=1}^{n} E_{2j}(\tau)E_{2n+2-2j}(\tau) + (2n + 3)E_{2n+2}(\tau) \quad (n \geq 1)
\]
which were proved by van der Pol [9, p. 266] and Rankin [12, Theorem 3] (originated with Ramanujan [11, p. 142]). Furthermore, Skoruppa [14] discussed a method to produce such identities for given \( n \) and showed the first few of them. On the other hand, our result (7.3) gives explicit formulas for any \( n \).

**Note added.** We are informed by Machide that his new result [8, Lemma 6.3] implies that \( C(\tau) = -E_2(\tau)/(2\pi i)^2pq \) for the constant \( C(\tau) \) in Theorem 1.3.

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**References**

[1] Apostol, T. M.: Generalized Dedekind sums and transformation formulae of certain Lambert series, Duke Math. J. 17 (1950), 147–157.
[2] Fukuhara, S.: Modular forms, generalized Dedekind symbols and period polynomials, Math. Ann. 310 (1998), 83–101.
[3] Fukuhara, S.: Generalized Dedekind symbols associated with the Eisenstein series, Proc. Amer. Math. Soc. 127 (1999), 2561–2568.
[4] Fukuhara, S., Yui, N.: Elliptic Apostol sums and their reciprocity laws, Trans. Amer. Math. Soc. 356 (2004), 4237–4254.
[5] Kohnen, W., Zagier, D.: Modular forms with rational periods, In: Rankin, R. A. (ed.): Modular Forms, pp. 197–249, Horwood, Chichester, 1984.
[6] Levin, A.: Elliptic polylogarithms: an analytic theory, Compositio Math. 106 (1997), 267–282.
[7] Machide, T.: An elliptic analogue of the generalized Dedekind-Rademacher sums, J. Number Theory 128 (2008), 1060–1073.
[8] Machide, T.: Elliptic Dedekind-Rademacher sums and transformation formulae of certain infinite series, preprint.
[9] van der Pol, B.: On a non-linear partial differential equation satisfied by the logarithm of the Jacobian theta-functions, with arithmetical applications, I, II, Indag. Math. 13 (1951), 261–271, 272–284.
[10] Rademacher, H., Grosswald, E.: Dedekind sums (Carus Math. Mono. No. 16), Math. Assoc. Amer., Washington D.C., 1972.
[11] Ramanujan, S: On certain arithmetical functions, In: Collected papers of Srinivasa Ramanujan, pp. 136–162, AMS Chelsea Publishing, Providence, RI, 2000.
[12] Rankin, R. A.: Elementary proofs of relations between Eisenstein series, Proc. Roy. Soc. Edinburgh Sect. A 76 (1976), 107–117.
[13] Sczech, R.: Dedekindsummen mit elliptischen Funktionen, Invent. Math. 76 (1984), 523–551.
[14] Skoruppa, N.-P.: A quick combinatorial proof of Eisenstein series identities, J. Number Theory 43 (1993), 68–73.
[15] Walker, P.: Elliptic functions, John Wiley & Sons, Chichester, 1996.
[16] Weil, A.: Elliptic functions according to Eisenstein and Kronecker, Springer-Verlag, Berlin-New York, 1976.
[17] Zagier, D.: Periods of modular forms and Jacobi theta functions, Invent. Math. 104 (1991), 449–465.