Locally inequivalent four-qubit hypergraph states

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Abstract
Hypergraph states as real equally weighted pure states are important resources for quantum codes of non-local stabilizers. Using local Pauli equivalence and permutational symmetry, we reduce the 32,768 four-qubit real equally weighted pure states to 28 locally inequivalent hypergraph states and several graph states. The calculation of geometric entanglement supplemented with entanglement entropy confirms that further reduction is impossible for true hypergraph states.

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(Some figures may appear in colour only in the online journal)

1. Introduction
A real equally weighted pure state (REW) is a superposition of all basis states with real amplitudes and equal probabilities. Recently, it has been systematically proven that REWs have a one-to-one correspondence with the quantum hypergraph states or graph states [1, 2]. Thus, an REW can be described by a mathematical hypergraph, namely, a graph where at least one of its edges connects more than two vertices, or a ‘usual’ graph with two vertex edges. A hypergraph state can be described by a non-local stabilizer, in contrast to a usual graph state, which is described by a local stabilizer whose observables are simple tensor products of Pauli matrices. There are a large number of hypergraph states, even for a system of a few qubits. Therefore, the local equivalence of hypergraph states is an important problem in applying hypergraph states for quantum information processing. For the local equivalence of arbitrary multipartite pure quantum states, local polynomial invariants have been introduced [3] and necessary and sufficient conditions have been proposed [4]. However, a more practical method is to use the entanglement to characterize the local equivalence of states. In this paper, we will study the local equivalence of all the four-qubit hypergraph states and
classify them with their geometric measure of entanglement supplemented with bipartite entanglement entropy, both of which are easy to calculate.

2. Hypergraph states

A hypergraph $H = (V; E)$ is composed of a set $V$ of $n$ vertices and a set of hyperedges $E$. The hyperedge set $E$ consists of $k$-hyperedges (hyperedge connecting $k$ vertices) for $1 \leq k \leq n$. The rank of a hypergraph is the maximum cardinality of its hyperedges. The $k$-hyperedge neighborhood $N_k(i)$ of the vertex $i$ is defined as

$$N_k(i) = \{ [i, i_2, ..., i_{k-1}] | [i, i_1, i_2, ..., i_{k-1}] \in E \},$$

where $\{i_1, i_2, ..., i_{k-1}\}$ is the $k - 1$ hyperedge. The neighborhood $N(i)$ of the vertex $i$ is $N(i) = \bigcup_k N_k(i)$. We also use $N(i)$ to denote the corresponding set of neighbor hyperedges.

To associate the hypergraph state with the underlying mathematical hypergraph, we assign each vertex a qubit and initialize each qubit as the state $|0\rangle + |1\rangle$. Each $k$-hyperedge represents the $k$-body interaction among the corresponding $k$ qubits. The hypergraph state related to the hypergraph $H$ is defined as

$$|H\rangle = \prod_{k=1}^{\text{rank}(H)} \prod_{[i, i_2, ..., i_{k-1}] \in E} U_{i_1i_2...i_k} |0\rangle^\otimes n,$$

where $U_{i_1i_2...i_k}$ is the $k$-qubit control $Z$ gate such that $U_{i_1i_2...i_k} |11...1\rangle_{i_1i_2...i_k} = -|11...1\rangle_{i_1i_2...i_k}$, and leaves all the other components of the computational basis unchanged. The rank of a hypergraph state is the rank of its corresponding hypergraph. We will refer to a rank 3 or higher hypergraph state as a true hypergraph state (hereafter a hypergraph state refers to a true hypergraph state). Hypergraph states can also be put into a stabilizer formalism [2] as can graph states. The difference is that the stabilizer operators for a graph state are the product of local Pauli operators, while the set of stabilizer operators for a hypergraph state consists of non-local control phase gate operators. Given a general hypergraph, for any vertex $i$, the stabilizer operator is defined as

$$K_i = X_i \prod_{k=1}^{\text{rank}(H)} \prod_{[i, i_2, ..., i_{k-1}] \in N_k(i)} U_{i_1i_2...i_k},$$

where $X_i$ is the Pauli $X$ operator (bit flip) of vertex $i$. The hypergraph state $|H\rangle$ is stabilized by an Abelian stabilizer group with generator set $\{K_i\}$ [2], namely,

$$K_i |H\rangle = |H\rangle.$$

Considering the local equivalence (denoted as $\equiv_{\text{Local}}$) of the hypergraph states, we have $X_i |H\rangle \equiv_{\text{Local}} |H\rangle$ since $X_i$ is a local operator. Using (2) and (3), we have

$$\prod_{k=1}^{\text{rank}(H)} \prod_{[i, i_2, ..., i_{k-1}] \in N_k(i)} U_{i_1i_2...i_k} |H\rangle \equiv_{\text{Local}} |H\rangle.$$

Hence applying all the control $Z$ operation contained in the neighborhood of a vertex to a hypergraph state, we obtain a locally equivalent hypergraph state. In the underlying mathematical hypergraph, the transform of the hyperedge set produced by $X_i$ is [1]
\[ \Delta \rightarrow \prime = E (i) \Delta E, \quad (5) \]

where \( \Delta \) is the symmetric difference, that is, \( E \Delta F = E \cup F - E \cap F \). The local equivalence (5) is a very useful tool in classifying the hypergraphs. As an example, suppose there is a four-vertex hypergraph \( H \) with \( E = \{ \{1, 2, 3, 4\}, \{1, 2, 3\} \} \); the local equivalent hypergraph \( \prime H \) with \( E' = N(4) \Delta E \) can be deduced by applying local operator \( X_4 \) of the fourth qubit to the hypergraph state. Then \( E' = \{ \{1, 2, 3, 4\} \} \). Hypergraph \( \prime H \) is shown as No.1 in figure 1. In fact, we can remove all the \( n - 1 \) hyperedges for an \( n \) vertex hypergraph of rank \( n \) by applying Pauli \( X_i \) operators.

We also have \( Z_i | H \rangle \equiv | H \rangle \) where \( Z_i \) is the Pauli \( Z \) operator (phase flip) of vertex \( i \). The transform of hyperedge set produced by \( Z_i \) is [1]

\[ E \rightarrow E' = \{ \{ i \} \} \Delta E, \quad (6) \]

that is, a loop is added to (removed from) vertex \( i \) when there is not (is) a loop. The unitary operator corresponds to a loop \( \{ \{ i \} \} \) on vertex \( i \) is \( U_i = Z_i \) [2]. We can also define hypergraph basis states \( | H_C \rangle = Z^C | H \rangle \), where \( C = (c_1, \ldots, c_n) \) is a bit string with \( c_i = 0, 1 \), and \( Z^C = \bigotimes_{i=1}^{n} Z_i^{c_i} \). Then all the local \( Z \) equivalent hypergraphs can be written in the form of \( H_C \).

### 3. Entanglement measures

The entanglement measures for multipartite quantum pure states are the Schmidt measure [5], the (logarithmic) geometric measure [6], the relative entropy of entanglement [7], the logarithmic robustness [8] and so on. The last three measures are equal for graph states [9]. Unfortunately, they are not equal for a (true) hypergraph state. Geometric measure is the easiest one to calculate among these three entanglement measures for multipartite entanglement. An iterative algorithm was derived for the entanglement of a graph state [10]. The algorithm can also be applied to a hypergraph state after a slight modification. Thus, we will...
use geometric measure to study the entanglement property of four-qubit hypergraph states and classify the states by their entanglement values.

For a four-qubit hypergraph state, the iterative algorithm of the geometric measure is as follows: let the hypergraph state be $\psi$, its closest product state be $\Phi = \prod_i |\Phi_i\rangle$, with $|\Phi_i\rangle = x_i |0\rangle + y_i |1\rangle$ and $|x_i|^2 + |y_i|^2 = 1$. The overlap amplitude (inner product) of the hypergraph state and its closest product state is $f = \langle \psi | \Phi \rangle$. The Lagrange multiplier method of maximizing $f^2$ subject to the conditions $|x_i|^2 + |y_i|^2 = 1$ ($i = 1, \ldots, 4$) gives the iterative equations

$$x_i^* = N_i \frac{df}{dx_i},$$

$$y_i^* = N_i \frac{df}{dy_i},$$

where $N_i$ is the normalization. By solving the equations, we obtain the closest product state $|\Phi\rangle$. As far as $|\Phi\rangle$ is determined, it follows the overlap amplitude $f = \langle \psi | \Phi \rangle$ of a given

Figure 2. The locally inequivalent four-vertex hypergraphs without an overall four-vertex hyperedge. The three-vertex hyperedges are specified by closed curves, while the two-vertex hyperedges are simply denoted by edges as in [2].
hypergraph state $|\psi\rangle$, and the geometric measure of entanglement of $|\psi\rangle$ is

$$E_g = -\log_2 |f|^2.$$  \hspace{1cm} (9)

An exact expression of the entanglement values for some of the hypergraph states can also be obtained based on the numeric calculation of the closest product states. As an example, let us consider the No.12 (in figure 2) hypergraph state, which is the direct product of $|+\rangle$ with

$$|\psi_3\rangle = \frac{1}{\sqrt{8}} (|000\rangle + |001\rangle + |010\rangle + |011\rangle + |100\rangle + |101\rangle + |110\rangle - |111\rangle).$$  \hspace{1cm} (10)

The latter is the hypergraph state of a three-qubit hypergraph with a 3-hyperedge and without further two-vertex edges. The closest state of $|\psi_3\rangle$ is assumed be $\phi^\otimes 3$ due to the symmetry of the three-qubits, where $|\phi\rangle = x |0\rangle + y |1\rangle$ with normalization $|x|^2 + |y|^2 = 1$. Denote $z = y/x$, and then the iterative equation is

$$z^* = (1 + 2z - z^2)(1 + z^2).$$  \hspace{1cm} (11)

Numeric calculation shows that $z$ eventually converges to a real number in a few steps regardless of its randomly chosen initial complex value. Thus we arrive at the algebraic equation $z^3 + 3z^2 - z - 1 = 0$. The solution of this is $z = -1 - \frac{2\sqrt{3}}{3} \cos(\tau + \frac{\pi}{3}) \approx 0.6751$, where $\tau = \frac{1}{3} \arctan(\sqrt{37}/27)$. The geometric measure of the state $|\psi_3\rangle$ is

$$E_g = -\log_2 \left| \langle \psi_3 | \phi \rangle^\otimes 3 \right|^2 \approx 0.5647.$$  \hspace{1cm} (12)

The relative entropy of entanglement is $E_r = \min_{\sigma} -\log_2 \left( |\psi_3\rangle \log_2 |\sigma\rangle \right)$ for a pure state $|\psi_3\rangle$ where $\sigma$ belongs to the fully separable state set. However, the full separability for a generic three-qubit system is unknown. Hence, the relative entropy of entanglement is not available. The entanglement is lower bounded by the entanglement of a bipartition of the hypergraph state, which is due to fact that the fully separable state set is the subset of the biseparable state set. The minimization over a larger set will give a lower value. The bipartite relative entropy of entanglement $E_{\text{bip}}$ is simply the minimal entanglement entropy of all the bipartitions for the pure symmetric state $|\psi_3\rangle$. We have $E_{\text{bip}} = -\text{Tr} \rho \log_2 \rho$ with $\rho = \text{Tr}_{32} |\psi_3\rangle \langle \psi_3| = |\frac{1}{2} |0\rangle + |\frac{1}{2} |1\rangle\rangle$ being the reduced state by tracing the second and the third qubits, where $| - \rangle = \frac{1}{\sqrt{2}} (|0\rangle - |1\rangle)$. Thus,

$$E_r \geq E_{\text{bip}} \approx 0.8113.$$  \hspace{1cm} (13)

The relative entropy of entanglement is larger than the geometric measure for the three-qubit hypergraph state.

4. Four-qubit hypergraph states with a four-vertex hyperedge

For the four-qubit system, we have REW states

$$|\psi_{\text{REW}}\rangle = \frac{1}{4} \sum_{\mu} (-1)^{g(\mu)} |\mu\rangle,$$  \hspace{1cm} (14)

where $\mu$ is a four bit string and $g(\mu)$ is a Boolean function, i.e. $g: \{0, 1\}^4 \rightarrow \{0, 1\}$, the coefficient $(-1)^{g(\mu)}$ can be $\pm 1$ for each $\mu$. The state $|\psi_{\text{REW}}\rangle$ is uniquely defined by the function.
via the signs (either plus or minus) in front of each component \( \mu \) of the computational basis. It is clear that we have \( 2^{16} \) different expressions of the coefficient series. Up to the overall phase, the total number of REW states is \( 2^{15} \). There is a one-to-one correspondence between an REW state and a hypergraph state or graph state. The number of \( i \)-vertex hyperedges is 
\[
\binom{4}{i} = \frac{4!}{i!(4-i)!}
\]
The number of four-qubit hypergraph states with a four-vertex hyperedge (rank 4) is \( N_4 = 2^{14} \), which is half the total number of REW states. Applying the Pauli \( Z_i \) operator to a hypergraph state gives rise to a one-vertex hyperedge (namely, the loop) on the \( i \)th vertex of the underlying hypergraph. We then remove all the loops of a hypergraph by applying proper Pauli \( Z \) operators to obtain the standard hypergraph (hypergraph free of loops). On the other hand, all three-vertex hyperedges within the four-vertex hyperedge can be removed by applying the Pauli \( X_i \) operators to the hypergraph states. Hence, we will consider the standard hypergraphs with two-vertex edges and an overall four-vertex hyperedge for the four-qubit hypergraph states. The number of these hypergraphs is 64 up to local equivalence of the Pauli \( X_i \) operators and \( Z_i \) operators. Permutational symmetry of the vertices leads to further reduction of 64 hypergraphs to 11 hypergraphs as shown in figure 1. The geometric measure and bipartite entanglement are listed in table 1.

The exact expression of the entanglement values for some hypergraph states and further properties of the entanglement of hypergraph states with a four-qubit edge are shown in table 2.

5. Four-qubit hypergraph states without a four-vertex hyperedge

Unlike the four-qubit hypergraph states with a four-vertex hyperedge, hypergraph states without a four-vertex edge have a different locally equivalent classification with respect to Pauli \( X \) operators. As can be seen from No.12 hypergraph shown in figure 2, the locally equivalent states are generated by \( X_1, X_2, \) and \( X_3 \) but not \( X_4 \), where we denote the vertex outside the ellipse as the fourth vertex, while the vertices inside the ellipse are denoted as the first, second and third vertices, respectively. The hypergraph state with underneath hypergraph No.12 is an eigenstate of \( X_4 \). Thus the number of locally equivalent states produced by

| No. | \( m \) | \( GE \) | \( BE_2 \) | \( BE_1 \) |
|-----|-----|-----|-----|-----|
| 1   | 1   | 0.3043 | a,a,a | d,d,d,d |
| 2   | 6   | 0.8157 | a,b,b | e,e,d,d |
| 3   | 3   | 1.4891 | a,c,c | e,e,e,e |
| 4   | 12  | 0.8954 | b,b,b | e,e,d,e |
| 5   | 12  | 1.5261 | b,c,c | e,e,e,e |
| 6   | 4   | 0.8916 | b,b,b | e,e,e,e |
| 7   | 4   | 1.1360 | b,b,b | e,e,d,e |
| 8   | 3   | 1.1732 | c,b,c | e,e,e,e |
| 9   | 12  | 1.4316 | b,c,c | e,e,e,e |
| 10  | 6   | 1.1165 | c,b,c | e,e,e,e |
| 11  | 1   | 1.1726 | b,b,b | e,e,e,e |

The geometric measure and bipartite entanglement of four-qubit rank 4 hypergraph states. Here, \( BE_2 \) is the vector of bipartite entanglement of partitions \( 12|34, 13|24, 14|23 \). \( BE_1 \) is the vector of bipartite entanglement of partitions \( 11234, 2134, 3124, 4123 \). \( m \) is the degeneracy with respect to permutational symmetry. The locations of qubits 1,...,4 are specified as in figure 1. \( a = 0.6561, b = 1.2624, c = 1.6773, d = 0.5436, e = 0.9544 \).
Pauli X operators is eight. This is also true for all the other hypergraphs shown in figure 2. Together with the Pauli Z local equivalence, we have 128 locally equivalent states for each hypergraph in figure 2. The permutational symmetry of vertices gives rise to the degeneracy \( m \) in table 3. The total number of hypergraph states with three-vertex hyperedges (rank 3) are \( N_3 = 128 \times \sum m = 128 \times 120 = 15 \times 2^{10} \). The rank 2 hypergraphs are just graphs. For

### Table 2. The properties of four-qubit rank 4 hypergraph states. Here, ‘PE’ represents the properties of the geometric entanglement, it can be either exact or numerical. ‘D’ represents the degeneracies of the closest product states. The closest product state types \( |\psi_1\rangle \cdots |\psi_4\rangle \) are denoted with ‘4’, ‘2,2’, ‘1,2,1’, and ‘1,1,1’, respectively. ‘R/C’ means the stable values of \( z_i \) are real or imaginary.

| No. | PE       | D   | R/C |
|-----|----------|-----|-----|
| 1   | Closed form | 4   | R   |
| 2   | Numerical | 2,2 | R   |
| 3   | Numerical | 2,2 | R   |
| 4   | Numerical | 1,2,1 | R |
| 5   | \( 3 + 2 \log_2 \frac{3}{5} \) | 1,1,1,1 | R |
| 6   | Numerical | 2,2 | R   |
| 7   | Numerical | 2,2 | C   |
| 8   | Closed form | 4   | R   |
| 9   | Numerical | 1,2,1 | R |
| 10  | Numerical | 2,2 | R   |
| 11  | \( 5 \log_2 (9 - 3 \sqrt{3}) \) | 4   | C   |

### Table 3. The geometric measure of entanglement and bipartite entanglement of four-qubit rank 3 hypergraph states. Here, \( BE_2 \) is the vector of the bipartite entanglement of partitions 12|34, 13|24, 14|23. \( BE_1 \) is the vector of the bipartite entanglement of partitions 12|34, 21|34, 31|24, 41|23. The locations of qubits 1, ..., 4 are specified as in figure 1. \( r = 0.8113 \), \( s = 1.5 \), \( t = 1.2238 \), \( u = 1.6009 \).

| No. | \( m \) | GE    | \( BE_2 \) | \( BE_1 \) |
|-----|-------|-------|------------|------------|
| 12  | 4     | 0.5647 | r,r,r      | r,r,r,r,0   |
| 13  | 12    | 1.5417 | s,s,s      | 1,r,1,1    |
| 14  | 12    | 1      | s,s,r      | 1,r,1,1    |
| 15  | 4     | 1.5261 | s,s,s      | 1,1,1,1    |
| 16  | 6     | 0.6115 | r,t,t      | r,r,r      |
| 17  | 6     | 1.2284 | r,u,u      | 1,1,r,r    |
| 18  | 12    | 1      | s,t,t      | r,r,r      |
| 19  | 12    | 1.4150 | s,u,u      | 1,1,r,1    |
| 20  | 6     | 1.4569 | s,t,t      | r,r,1,1    |
| 21  | 6     | 1.4569 | s,u,u      | 1,1,1,1    |
| 22  | 4     | 1      | t,t,t      | 1,r,r,r    |
| 23  | 12    | 0.6781 | t,t,t      | r,r,r      |
| 24  | 12    | 1.3173 | u,u,t      | 1,1,1,r    |
| 25  | 4     | 1.4150 | u,u,t      | r,1,1,r    |
| 26  | 1     | 1.2230 | t,t,t      | 1,1,1,1    |
| 27  | 6     | 1.2767 | t,u,u      | r,r,1,1    |
| 28  | 1     | 0.8301 | t,t,t      | r,r,r      |

For
Table 4. The properties of four-qubit rank 3 hypergraph states. Here ‘PE’ represents the properties of the geometric entanglement, which can be either exact or numerical. ‘D’ represents the degeneracies of the closest product states. The closest product state types \( |\psi_1\rangle, |\psi_2\rangle, |\psi_3\rangle, |\psi_4\rangle\) are denoted with ‘\(1\)' and ‘\(1,3\)' respectively. R/C means the stable values of \(z_i\) are real or imaginary.

| No. | PE       | D       | R/C |
|-----|----------|---------|-----|
| 12  | Closed form | 1,3     | R   |
| 13  | Numerical | 1,2,1   | R   |
| 14  | 1        | 1,3     | R   |
| 15  | 3 + 2 \log_{2.5} 3 | 1,3     | R   |
| 16  | 4 – 2 \log_2(1 + \sqrt{5}) | 2,2 | R   |
| 17  | 2.5 – \log_2(1 + \sqrt{2}) | 2,2 | C   |
| 18  | 1        | 1,3     | R   |
| 19  | 3 – \log_3 | 1,2,1   | R   |
| 20  | 4 – 2 \log_2(1 + \sqrt{2}) | 1,2,1 | R   |
| 21  | 4 – 2 \log_2(1 + \sqrt{2}) | 2,2 | R   |
| 22  | 1        | 1,3     | R   |
| 23  | 3 – \log_5 | 1,3     | R   |
| 24  | Numerical | 1,2,1   | R   |
| 25  | 3 – \log_3 | 1,2,1   | R   |
| 26  | 6 – 2 \log_2(3 + \sqrt{5}) | 4 | R   |
| 27  | Numerical | 2,2     | R   |
| 28  | 4 – 2 \log_3 | 4 | R   |

The geometric measure and bipartite entanglement are listed in table 3. The exact expression of the entanglement values for some hypergraph states and further properties of the entanglement of hypergraph states with three-qubit edges are shown in table 4.

6. Discussion and conclusion

Each of the rank 4 hypergraphs in section 4 has a characteristic value of geometric entanglement. Hence these 11 hypergraph states are confirmed to be all locally inequivalent due to their different values of geometric entanglement. The entanglement entropy alone is not a good indication for local inequivalence even considering all possible bipartitions. We can see that No.4 and No.7 hypergraph states are not discriminated by the series of bipartite entanglement entropy. The spectra of bipartite entanglement entropy for No.5, No.8, No.9 and No.10 hypergraph states are identical, or identical under the permutation of qubits. The geometric entanglement of No.5 hypergraph state coincides with that of No.15; however, they are discriminated by their entanglement entropy. The rank 3 hypergraphs in section 5 are all...
different by their geometric entanglement values except for three cases where further consideration of entanglement entropy is necessary. The three cases are that the values of geometric entanglement of No.19 and No.25 hypergraph states are equal; the values of geometric entanglement of No.20 and No.21 hypergraph states are equal; the values of geometric entanglement of No.14, No.18 and No.22 hypergraph states are also equal. Further discriminations are attributed to their bipartition entanglement entropy.

We conclude that there are 28 locally inequivalent four-qubit hypergraph states, 11 of which are rank 4, and 17 of which are rank 3. They can be discriminated by geometric entanglement supplemented with bipartition entanglement entropy. Local Pauli equivalence together with permutation symmetry are enough to discriminate inequivalent four-qubit hypergraph states (this may not be true for hypergraph states with more qubits). Further local Clifford equivalence, such as local complementary transform, is not necessary (it is also limited to four-qubit hypergraph states), although it is a powerful tool in discriminating locally inequivalent graph states. We have given a complete classification of the REW four-qubit pure states.

Note added: After submission of this work, we became aware of a recent preprint by O Guhne et al [11], which shows a similar result.

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