M-ambiguity Sequences for Parikh Matrices and Their Periodicity Revisited

Ghajendran Poovanandran¹,² · Wen Chean Teh¹

Received: 6 January 2019 / Revised: 18 September 2019 / Published online: 6 December 2019
© Malaysian Mathematical Sciences Society and Penerbit Universiti Sains Malaysia 2019

Abstract
The introduction of Parikh matrices by Mateescu et al. in 2001 has sparked numerous new investigations in the theory of formal languages by various researchers, among whom is Şerbănuţă. Recently, a decade-old conjecture by Şerbănuţă on the M-ambiguity of words has disproved, leading to new possibilities in the study of such words. In this paper, we investigate how selective repeated duplications of letters in a word affect the M-ambiguity of the resulting words. The corresponding M-ambiguity of each of those words is then presented in sequences, which we term as M-ambiguity sequences. We show that nearly all patterns of M-ambiguity sequences are attainable. Finally, by employing certain algebraic approach and some underlying theory in integer programming, we show that repeated periodic duplications of letters of the same type in a word result in an M-ambiguity sequence that is ultimately periodic.

Keywords Injectivity problem · Subword · M-equivalence · Rational polyhedra · Periodic sequence

Mathematics Subject Classification 68R15 · 68R05 · 05A05

Communicated by Rosihan M. Ali.

Wen Chean Teh
dasmenteh@usm.my

Ghajendran Poovanandran
ghajendran@staffemail.apu.edu.my

¹ School of Mathematical Sciences, Universiti Sains Malaysia, 11800 USM Gelugor, Malaysia
² School of Mathematics, Actuarial and Quantitative Studies, Asia Pacific University of Technology & Innovation, Technology Park Malaysia, 57000 Bukit Jalil, Kuala Lumpur, Malaysia
1 Introduction

The classical Parikh theorem [7], which states that the Parikh vectors of all words from a context-free language form a semilinear set, established the Parikh mapping as a significant advancement in the theory of formal languages. The Parikh matrix mapping, introduced in [6], is a canonical generalization of the Parikh mapping. On top of dealing with the number of occurrences of individual letters (as in the case of Parikh vectors), the Parikh matrix of a word stores information on the number of occurrences of certain subwords in that word as well. The introduction of Parikh matrices has led to various new studies in combinatorics on words (for example, see [1–4,8–10,12,13,15–21]).

A word is $M$-ambiguous if and only if it shares the same Parikh matrix with another distinct word. In the pursuit of characterizing $M$-unambiguous words, Şerbănuţă proposed a conjecture in [16] that the duplication of any letter in an $M$-ambiguous word will result in another $M$-ambiguous word. The conjecture was, however, overturned in [19] by a counterexample from the quaternary alphabet.

In this work, we show that by duplicating certain letters in a word, it is possible to continuously change the $M$-ambiguity of the resulting words. In fact, such changes in the $M$-ambiguity of a word can occur in nearly any pattern. Given an infinite sequence of words, obtained by repeatedly duplicating certain letters in the first word, we present the corresponding $M$-ambiguity of those words in what we term as an $M$-ambiguity sequence. This work also proposes an algebraic way to determine the $M$-ambiguity of a word. This algebraic approach is then used together with some underlying theory in integer linear programming to show that if we repeatedly duplicate—in a periodic manner—the letters of the same type in a word, the corresponding $M$-ambiguity sequence is ultimately periodic.

The remainder of this paper is structured as follows: Sect. 2 provides the basic terminology and preliminaries. Section 3 highlights some previous results pertaining to the overturn of Şerbănuţă’s conjecture and serves the main motivation of this paper. After that, the central notion of our study, namely the $M$-ambiguity sequences, is introduced. It is then shown that nearly any pattern of $M$-ambiguity sequence can be realized. Section 4 mainly studies the periodicity of $M$-ambiguity sequences. First, an algebraic analysis to determine the $M$-ambiguity of a word is illustrated. Then, certain theories pertaining to rational polyhedra are used together with the algebraic approach to prove a main result on the periodicity of $M$-ambiguity sequences. Our conclusion follows after that.

2 Preliminaries

We denote as follows: $\mathbb{R}$ is the set of real numbers, $\mathbb{Q}$ is the set of rational numbers, $\mathbb{Z}$ is the set of integers, $\mathbb{Z}_+$ is the set of positive integers.

Suppose $\Sigma$ is a finite nonempty alphabet. The set of all words over $\Sigma$ is denoted by $\Sigma^*$. The unique empty word is denoted by $\lambda$. Given two words $v, w \in \Sigma^*$, the concatenation of $v$ and $w$ is denoted by $vw$. An ordered alphabet is an alphabet $\Sigma = \{a_1, a_2, \ldots, a_s\}$ with a total ordering on it. For example, if $a_1 < a_2 < \cdots < a_s$, then
we may write $\Sigma = \{a_1 < a_2 < \cdots < a_s\}$. Conversely, if $\Sigma = \{a_1 < a_2 < \cdots < a_s\}$, then $\{a_1, a_2, \ldots, a_s\}$ is the underlying alphabet. Frequently, we abuse notation and use $\Sigma$ to stand for both the ordered alphabet and its underlying alphabet. Suppose $\Gamma \subseteq \Sigma$. The projective morphism $\pi/\Gamma : \Sigma^* \to \Gamma^*$ is defined by

$$\pi/\Gamma(a) = \begin{cases} a, & \text{if } a \in \Gamma \\ \lambda, & \text{otherwise.} \end{cases}$$

A word $v$ is a scattered subword (or simply subword) of $w \in \Sigma^*$ if and only if there exist $x_1, x_2, \ldots, x_n, y_0, y_1, \ldots, y_n \in \Sigma^*$ (possibly empty) such that $v = x_1 x_2 \ldots x_n$ and $w = y_0 x_1 y_1 \ldots y_{n-1} x_n y_n$. The number of occurrences of a word $v$ as a subword of $w$ is denoted by $|w|_v$. Two occurrences of $v$ are considered different if and only if they differ by at least one position of some letter. For example, $|bcbcc|_{bc} = 5$ and $|aabcbc|_{abc} = 6$. By convention, $|w|_{\lambda} = 1$ for all $w \in \Sigma^*$.

For any integer $k \geq 2$, let $M_k$ denote the multiplicative monoid of $k \times k$ upper triangular matrices with nonnegative integral entries and unit diagonal.

**Definition 2.1** [6] Suppose $\Sigma = \{a_1 < a_2 < \cdots < a_k\}$ is an ordered alphabet. The Parikh matrix mapping with respect to $\Sigma$, denoted by $\Psi_\Sigma$, is the morphism:

$$\Psi_\Sigma : \Sigma^* \to M_{k+1},$$

defined such that for every integer $1 \leq q \leq k$, if $\Psi_\Sigma(a_q) = (m_{i,j})_{1 \leq i,j \leq k+1}$, then

- $m_{i,i} = 1$ for all $1 \leq i \leq k+1$;
- $m_{q,q+1} = 1$; and
- All other entries of the matrix $\Psi_\Sigma(a_q)$ are zero.

Matrices of the form $\Psi_\Sigma(w)$ for $w \in \Sigma^*$ are termed as Parikh matrices.

**Theorem 2.2** [6] Suppose $\Sigma = \{a_1 < a_2 < \cdots < a_s\}$ is an ordered alphabet and $w \in \Sigma^*$. The matrix $\Psi_\Sigma(w) = (m_{i,j})_{1 \leq i,j \leq s+1}$ has the following properties:

- $m_{i,i} = 1$ for each $1 \leq i \leq s+1$;
- $m_{i,j} = 0$ for each $1 \leq j < i \leq s+1$;
- $m_{i,j+1} = |w|_{a_i a_{i+1} \ldots a_j}$ for each $1 \leq i \leq j \leq s$. 
Example 2.3 Suppose $\Sigma = \{a < b < c < d\}$ and $w = abcdbc$. Then,

$$
\Psi_\Sigma(w) = \Psi_\Sigma(a)\Psi_\Sigma(b)\Psi_\Sigma(c)\Psi_\Sigma(d)\Psi_\Sigma(b)\Psi_\Sigma(c)
= \begin{pmatrix}
1 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1
\end{pmatrix}
\cdots
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1
\end{pmatrix}
= \begin{pmatrix}
1 & 1 & 2 & 3 & 1 \\
0 & 1 & 2 & 3 & 1 \\
0 & 0 & 1 & 2 & 1 \\
0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 1
\end{pmatrix}
= \begin{pmatrix}
1 & |w|_a & |w|_{ab} & |w|_{abc} & |w|_{abcd} \\
0 & 1 & |w|_b & |w|_{bc} & |w|_{bcd} \\
0 & 0 & 1 & |w|_c & |w|_{cd} \\
0 & 0 & 0 & 1 & |w|_d \\
0 & 0 & 0 & 0 & 1
\end{pmatrix}.
$$

Definition 2.4 Suppose $\Sigma$ is an ordered alphabet. Two words $w, w' \in \Sigma^*$ are $M$-equivalent, denoted by $w \equiv_M w'$, iff $\Psi_\Sigma(w) = \Psi_\Sigma(w')$. A word $w \in \Sigma^*$ is $M$-ambiguous if and only if it is $M$-equivalent to another distinct word. Otherwise, $w$ is $M$-unambiguous. For any word $w \in \Sigma^*$, we denote by $C_w$ the set of all words that are $M$-equivalent to $w$.

The following is a simple equivalence relation which involves the most evident rewriting rules that preserve $M$-equivalence (see [2]).

Definition 2.5 Suppose $\Sigma = \{a_1 < a_2 < \cdots < a_s\}$ is an ordered alphabet. Two words $w, w' \in \Sigma^*$ are $1$-equivalent, denoted by $w \equiv_1 w'$, iff $w'$ can be obtained from $w$ by applying finitely many rewriting rules of the following form:

$$
x a_k a_l y \rightarrow x a_l a_k y \text{ where } x, y \in \Sigma^* \text{ and } |k - l| \geq 2.
$$

The following notion and known result are needed for the proof of Theorem 3.11 later.

Definition 2.6 [20] Suppose $\Sigma$ is an alphabet and $v, w \in \Sigma^*$. The $v$-core of $w$, denoted by $\text{core}_v(w)$, is the unique subword $w'$ of $w$ such that $w'$ is the subword of shortest length which satisfies $|w'|_v = |w|_v$.

Proposition 2.7 [10] Suppose $\Sigma = \{a < b < c\}$ and $w \in \Sigma^*$ with $|w|_{abc} \geq 1$. Then, $w \equiv_1 u \text{core}_{abc}(w)v$ for some unique $u \in \{b, c\}^*$ and $v \in \{a, b\}^*$.

3 Attainable Patterns of $M$-ambiguity Sequences

The following conjecture was proposed by Şerbanuţă in [16] as an open problem pertaining to $M$-ambiguity of words.

Conjecture 3.1 Suppose $\Sigma$ is an ordered alphabet. For any $u, v \in \Sigma^*$ and $a \in \Sigma$, if $uav$ is $M$-unambiguous, then $uav$ is $M$-unambiguous as well. Equivalently, if $uav$ is $M$-ambiguous, then $uav$ is also $M$-ambiguous.
The above conjecture holds for the binary and ternary alphabets. (For exhaustive lists of $M$-unambiguous binary and ternary words, readers are referred to [5, Theorem 3] and [16, Theorem A.1], respectively.) On the contrary, for the quaternary alphabet, it was shown in [19] that the conjecture is invalid. The counterexample given was the $M$-ambiguous word $cbebabcdecbabcbe$ (which is $M$-equivalent to the word $beccabdbcbabc$). The following result was then proved, thus overturning the conjecture.

**Theorem 3.2** [19] The word $w = cbebabc^ndcbabcbe$ is $M$-unambiguous with respect to $\Sigma = \{a < b < c < d\}$ for every integer $n > 1$.

At this point, it is natural to ask Question 3.5, which is in a more general setting.

**Definition 3.3** [16] Suppose $\Sigma$ is an alphabet and $w \in \Sigma^*$. Suppose $w = a_1^{p_1}a_2^{p_2} \ldots a_n^{p_n}$ such that $a_i \in \Sigma$ and $p_i > 0$ for all $1 \leq i \leq n$ with $a_i \neq a_{i+1}$ for all $1 \leq i \leq n - 1$. The print of $w$, denoted by $pr(w)$, is the word $a_1a_2 \ldots a_n$.

**Definition 3.4** Suppose $\Sigma$ is an ordered alphabet and $w, w' \in \Sigma^*$. We write $w \not\sim w'$ iff $w = uav$ and $w' = uaav$ for some $u, v \in \Sigma^*$ and $a \in \Sigma$.

**Question 3.5** Suppose $\Sigma$ is an ordered alphabet. Consider an infinite sequence of words $w_i \in \Sigma^*, i \geq 0$ such that $pr(w_0) = w_0$ and

$$w_0 \not\sim w_1 \not\sim w_2 \not\sim \ldots.$$

In what patterns can the $M$-ambiguity of these words sequentially change?

In the spirit of answering the above question, we define the following notion.

**Definition 3.6** Suppose $\Sigma$ is an ordered alphabet. Let $\varphi = \{w_i\}_{i \geq 0}$ be a sequence of words over $\Sigma$ such that for all integers $i \geq 0$, we have $w_i \not\sim w_{i+1}$. We say that a sequence $\{m_i\}_{i \geq 0}$ is the $M$-ambiguity sequence corresponding to $\varphi$, denoted by $\Theta_\varphi$, if and only if for every integer $i \geq 0$, we have $m_i \in \{A, U\}$ such that if $m_i = A$, then $w_i$ is $M$-ambiguous; otherwise, if $m_i = U$, then $w_i$ is $M$-unambiguous.

By the above definition, one can see that Question 3.5 actually asks for the attainable patterns of $M$-ambiguity sequence, where the associated sequence of words starts with a print word. The following two examples, first presented in [11], provide a partial answer to this question.

For the remaining part of this section, we fix $\Sigma = \{a < b < c < d\}$. Whenever the $M$-ambiguity of a word is mentioned, it is understood that it is with respect to $\Sigma$.

**Example 3.7** For each integer $n \geq 1$, let $w_{n,m} = c^nbcabc^mdcbabcbe$. If $m = n$, then $w_{n,m}$ is $M$-ambiguous as it is $M$-equivalent to the word $bce^{n+1}abc^ndbcabc$. If $m = n + 1$, then $w_{n,m}$ is $M$-unambiguous [11, Theorem 3.6].

Therefore, if one wants to obtain a sequence $\varphi$ of words (where the first word is a print word) such that $\Theta_\varphi = A, U, A, U, A, U, \ldots$, the duplication of letters can be carried out in the following manner:

$$w_{1,1}, w_{1,2}, w_{2,2}, w_{2,3}, w_{3,3}, w_{3,4}, \ldots.$$
Example 3.8 The words $cbabcdcbabc$ and $cbabcdcbab^2c$ are $M$-unambiguous (computationally verified). By duplicating the first letter $b$ in the latter word, we obtain the $M$-ambiguous word $cbabcdcbabbc$ (it is $M$-equivalent to $bcabcdcbabcb$). For each integer $n \geq 1$, let $y_{n,m} = c^nbbabe^mdbcabbc$. If $m = n$, then $y_{n,m}$ is $M$-ambiguous as it is $M$-equivalent to the word $bc^nabc^nbdcbacb$. If $m = n + 1$, then $y_{n,m}$ is $M$-ambiguous [11, Theorem 3.7].

Therefore, if one wants to obtain a sequence $\varphi$ of words (where the first word is a print word) such that $\Theta_\varphi = U, U, A, U, A, U, A, \ldots$, the duplication of letters can be carried out in the following manner:

$$cbabcdcbabc, cbabcdcbabbc, y_{1,1}, y_{1,2}, y_{2,2}, y_{2,3}, y_{3,3}, y_{3,4}, \ldots$$

Remark 3.9 Examples 3.7 and 3.8 shows that $M$-ambiguity sequences with alternating $A$ and $U$ are attainable. In contrast to the word $w_{1,1}$ in Example 3.7, the word $y_{1,1}$ in Example 3.8 is not a print word. That is why we needed the word $cbabcdcbabc$ to begin the sequence, followed by $cbabcdcbabbc$, before we reach $y_{1,1}$.

We now generalize the words used in Examples 3.7 and 3.8 to provide a nearly complete picture—almost any pattern of $M$-ambiguity sequence is attainable. For that, we need the following observations and theorems as a basis.

Observation 3.10 For all positive integers $n$ and $p$, the word $c^nbcababc^mdcbabcbcp$ is $M$-ambiguous as it is $M$-equivalent to the word $bc^{n+1}abc^nbdcbaccbc^p-1$.

The proof of the following result closely resembles that of Theorem 3.6 in [11], yet we include it here for completeness.

Theorem 3.11 The word $w = c^nbcababc^mdcbabcbcp$ is $M$-unambiguous for all integers $n \geq 1$, $p \geq 1$ and $m \geq n + 1$.

Proof We argue by contradiction. Fix integers $n \geq 1$, $p \geq 1$ and $m \geq n + 1$. Assume that $w$ is $M$-ambiguous. Then, $w \equiv_M w'$ for some $w' \in \Sigma^*$ such that $w' \neq w$. It follows that $\pi_{[a,b]}(w) \equiv_M \pi_{[a,b]}(w')$, $\pi_{[b,c]}(w) \equiv_M \pi_{[b,c]}(w')$ and $\pi_{[c,d]}(w) \equiv_M \pi_{[c,d]}(w')$. Note that $\pi_{[c,d]}(w) = c^{n+m+1}dc^{p+2}$ is $M$-ambiguous; thus, $\pi_{[c,d]}(w) = \pi_{[c,d]}(w')$. Meanwhile, $\pi_{[a,b]}(w) = bbababb$. Thus, $\pi_{[a,b]}(w')$ is either $bbababb$, $babababb$, $bbababb$, or $abababab$. Write $w = c^nbcababc^mdcbabcbcp$ and $w' = v_1dv_2'$, where $v_1, v_2' \in \{a, b, c\}^*$.

(Note that $v_1$ and $v_2$ are both $M$-unambiguous$^1$ as this fact is be needed later in this proof.) Since $|w|_x = |w'|_x$ for every $x \in \{abcd, bcd, cd\}$, it follows that $|v_1|_y = |v_1'|_y$ for every $y \in \{abc, bc, c\}$. Furthermore, since $|v_1|_c + |v_2|_c = |w|_c = |w'|_c = |v_1'|_c + |v_2'|_c$, we have $|v_2|_c = |v_2'|_c$.

Note that $|w'|_{bc} = |v_1'|_{bc} + |v_1'|_{bc} + |v_2'|_{bc} = |v_1|_{bc} + |v_1'|_{bc} \cdot (p + 2) + |v_2'|_{bc}$. At the same time, $|w'|_{bc} = |v_1|_{bc} + |v_1|_{bc} + |v_2|_{bc} + |v_2|_{bc} = |v_1|_{bc} + 3 \cdot (p + 2) + (3p + 2) = |v_1|_{bc} + 6p + 8$. Thus,

$$|v_1'|_{bc} \cdot (p + 2) + |v_2'|_{bc} = 6p + 8. \quad (3.11.1)$$

$^1$ See Theorem A.1 in [16].
Meanwhile, we have $|v_1'|_b \leq |w'|_b = |w|_b = 6$. If $|v_1'|_b = 6$, then $|v_2'|_{bc} = -4$, which is impossible. Thus, $|v_1'|_b \leq 5$. Also, since $|v_1'|_{abc} = |v_1|_{abc} = m \geq n + 1$, it follows that $|\text{core}_{abc}(v_1')|_b \geq 1$.

**Case 1** $\pi_{(a,b)}(w') = bbababb.$

Since $|v_1'|_b \leq 5$, $|\text{core}_{abc}(v_1')|_b \geq 1$ and $\pi_{(a,b)}(w') = \pi_{(a,b)}(v_1')\pi_{(a,b)}(v_2')$, it follows that $\pi_{(a,b)}(v_1') \in \{bbabb, bbabba, bbabbb\}$. Assume $\pi_{(a,b)}(v_1') = bbabba$. Then $|v_1'|_{ab} = 4$. Furthermore, as $|v_1'|_b = 5$, it holds by (3.11.1) that $|v_2'|_{bc} = p - 2$. Note that $\pi_{(a,b)}(v_2') = b$; therefore, $|v_2'|_a = 0$, and consequently, $|v_2'|_{abc} = 0$. Thus,

$$|w'|_{abc} = |v_1'|_{abc} + |v_1'|_{ab}|v_2'|_c + |v_1'|_a|v_2'|_{bc} + |v_2'|_{abc}$$

$$= |v_1'|_{abc} + |v_1'|_{ab}|v_2'|_c + |v_1'|_a|v_2'|_{bc}$$

$$= m + 4 \cdot (p + 2) + 2 \cdot (p - 2)$$

$$= m + 6p + 4.$$ 

That is to say, $|w'|_{abc} = m + 6p + 4 < m + 6p + 5 = |w|_{abc}$, which is a contradiction.

Assume $\pi_{(a,b)}(v_1') \in \{bbabb, bbabba\}$. Then, $|v_1'|_{ab} = 2$. Furthermore, as $|v_1'|_b = 4$, it holds by (3.11.1) that $|v_2'|_{bc} = 2p$. Note that if $\pi_{(a,b)}(v_1') = bbabb$, then $\pi_{(a,b)}(v_2') = abb$. Consequently, $|v_2'|_a = 1$, and therefore, $|v_2'|_{abc} = |v_2'|_{bc}$. Otherwise if $\pi_{(a,b)}(v_1') = babba$, then $\pi_{(a,b)}(v_2') = bb$. Consequently, $|v_2'|_a = 0$, and therefore, $|v_2'|_{abc} = 0$ as well. In both cases, we have $|v_2'|_{abc} = |v_2'|_a|v_2'|_{bc}$. Thus,

$$|w'|_{abc} = |v_1'|_{abc} + |v_1'|_{ab}|v_2'|_c + |v_1'|_a|v_2'|_{bc} + |v_2'|_{abc}$$

$$= |v_1'|_{abc} + |v_1'|_{ab}|v_2'|_c + |v_1'|_a|v_2'|_{bc}$$

$$= m + 2 \cdot (p + 2) + 2 \cdot 2p$$

$$= m + 2p + 4 + 4p$$

$$= m + 6p + 4.$$ 

Similar to the case $\pi_{(a,b)}(v_1') = bbababb$, we have $|w'|_{abc} = m + 6p + 4 < m + 6p + 5 = |w|_{abc}$, which is a contradiction.

Thus, $\pi_{(a,b)}(v_1') = babb$. We have $\pi_{(a,b)}(v_1') = \pi_{(a,b)}(v_1)$; therefore, $|v_1'|_y = |v_1|_y$ for every $y \in \{a, b, ab\}$. As we already know that $|v_1'|_y = |v_1|_y$ for every $y \in \{abc, bc, c\}$, it follows that $v_1' \equiv_M v_1$ with respect to $\{a < b < c\}$. However, $v_1$ is $M$-unambiguous; thus, $v_1' = v_1$. Consequently, $v_2' \equiv_M v_2$ with respect to $\{a < b < c\}$ by the left invariance of $M$-equivalence. Similarly, $v_2$ is $M$-unambiguous; thus, $v_2' = v_2$. Therefore, $w' = w$, which is a contradiction.

**Case 2** $\pi_{ab}(w') = babbbbab.$

By similar reasoning as in Case 1, we have $\pi_{(a,b)}(v_1') = \{bab, babb, babbb, babbb\}$. In all four cases, $|v_1'|_a = 1$. Also, note that $|v_1'|_c = |v_1|_c = n + m + 1$, $|v_1'|_{bc} = |v_1|_{bc} = 3m + 1$ and $|v_1'|_{abc} = |v_1|_{abc} = m$. 

 Springer
By Proposition 2.7, it holds that $v'_1 \equiv_1 u_1 \text{core}_{abc}(v'_1)u_2$ for some unique
$u_1 \in \{b, c\}^*$ and $u_2 \in \{a, b\}^*$. Since $\pi_{ab}(v'_1) \in \{bab, babb, babbb, babbbb\}$ and
$a$ is a prefix of $\text{core}_{abc}(v'_1)$, it follows that $|u_1|_b = 1$. Also, note that $|v'_1|_{bc} =
|u_1|_{bc} + |u_1|_b \text{core}_{abc}(v'_1)_c + |\text{core}_{abc}(v'_1)|_{bc}$. Since $|\text{core}_{abc}(v'_1)|_{abc} = |v'_1|_{abc} = m$,
a is a prefix of $\text{core}_{abc}(v'_1)$, and that is the only $a$ in $\text{core}_{abc}(v'_1)$, it follows that
$|\text{core}_{abc}(v'_1)|_{bc} = m$. Additionally, since $|u_1|_{bc} + |u_1|_c = |u_1|_b |u_1|_c$, it follows that
$|u_1|_{bc} \leq |u_1|_b |u_1|_c$. Therefore, $|v'_1|_{bc} \leq |u_1|_c + |\text{core}_{abc}(v'_1)|_c + m = |v'_1|_c + m = n + m + 1 + m = n + 2m + 1$. Consequently, $3m + 1 = |v'_1|_{bc} = |v'_1|_{bc} \leq n + 2m + 1$,
which reduces to $m \leq n$. Thus, a contradiction occurs.

Case 3 $\pi_{ab}(w') = bbbaabbb$. This case is impossible. Observe that $|v'_1|_b = 3 + |\text{core}_{abc}(v'_1)|_b$. Since $|v'_1|_b \leq 5$ and
$|\text{core}_{abc}(v'_1)|_b \geq 1$, it follows that $\pi_{ab}(v'_1) \in \{bbbaab, bbbaabb\}$.
If $\pi_{ab}(v'_1) = bbbaab$, then $|v'_1|_b = 5$ and consequently $|v'_2|_{bc} = p - 2$ due to
(3.11.1). Correspondingly, we have $|w'|_{abc} = |v'_1|_{abc} + |v'_1|_{ab} |v'_2|_c + |v'_1|_a |v'_2|_b + |v'_2|_{abc} = m + 4 \cdot (p + 2) + 2 \cdot (p - 2) + 0 = m + 6p + 4$. On the other hand, if
$\pi_{ab}(v'_1) = bbbaabb$, then $|v'_1|_b = 4$, and consequently, $|v'_2|_{bc} = 2p$ due to (3.11.1).
Correspondingly, we have $|w'|_{abc} = |v'_1|_{abc} + |v'_1|_{ab} |v'_2|_c + |v'_1|_a |v'_2|_b + |v'_2|_{abc} = m + 2 \cdot (p + 2) + 2 \cdot (2p) + 0 = m + 6p + 4$ as well. In both cases, $|w'|_{abc} < m + 6p + 5 = |w|_{abc}$, which is a contradiction.

Case 4 $\pi_{ab}(w') = abbbbbba$. This case is trivially impossible. Note that $|v'_1|_{bc} = |v'_1|_{bc} = 3m + 1$. Consequently, $|v'_1|_{abc} = 1 \cdot |v'_1|_{bc} = 3m + 1$. However, $|v'_1|_{abc} = |v'_1|_{abc} = m$, thus a contradiction.

Observation 3.10 and Theorem 3.11 allow us to generate sequences of words (starting with a print word) such that the first word is $M$-ambiguous and the $M$-ambiguity of the remaining words sequentially changes in an arbitrary pattern. This is illustrated by the following example.

Example 3.12 Consider the Fibonacci sequence 0, 1, 1, 2, 3, 5, 8, 13, . . . . Suppose we want to generate a sequence of words realizing the $M$-ambiguity sequence where the number of terms $U$ between two consecutive terms $A$ follows the Fibonacci sequence—i.e, $A, A, U, A, U, A, U, A, . . . .$

For all integers $n, m, p \geq 1$, let $w_{n,m,p} = \varepsilon^n bcbabc^m dcbabcb^p$. By Observation 3.10, if $m = n$, then $w_{n,m,p}$ is $M$-ambiguous for any $p \geq 1$. By Theorem 3.11, if $m = n + 1$, then $w_{n,m,p}$ is $M$-unambiguous for any $p \geq 1$. Thus, it remains to duplicate the letters in the following manner:

\[
w_{1,1,1}, w_{1,1,2}, w_{1,2,1}, w_{2,2,1}, w_{2,3,1}, w_{3,3,1}, w_{3,4,1}, w_{3,4,2}, w_{4,4,2}, \ldots
\]

Notice that in the above, whenever we need to retain the preceding term in the sequence, we increase the power $p$ by one—that is, to duplicate the last letter $c$.

On the other hand, to generate similar sequences of words such that the first word is $M$-unambiguous, we need the following observation and result.
Observation 3.13 For all positive integers \(n\) and \(p\), the word \(c^nbbabc^n dcbabbc^p\) is \(M\)-ambiguous as it is \(M\)-equivalent to the word \(bc^nabc^nbdbcbacbc^p\).

Theorem 3.14 The word \(c^nbbabe^m dcbabbc^p\) is \(M\)-unambiguous for all integers \(n \geq 1, p \geq 1\) and \(m \geq n + 1\).

Proof Argue similarly as in the proof of Theorem 3.11. \(\Box\)

Remark 3.15 When \(n = m = p = 1\), in contrast to the word in Theorem 3.11, the word \(c^nbbabc m dcbabbc p\) is a print word. Thus, similarly as in Example 3.8, we need the \(M\)-unambiguous words \(cbabcdcbabc\) and \(cbabcdcbabbc\) on top of Observation 3.13 and Theorem 3.14 to realize \(M\)-ambiguity sequences starting with \(U\). However, this forces the first three terms to be \(U, U, A\) before we can change the terms arbitrarily.

4 Periodicity of \(M\)-ambiguity Sequences

Consider the word \(cbcbabc\) over the ordered alphabet \(\{a < b < c < d\}\). By Theorem 3.2, it holds that every duplication of the underlined letter \(c\) in that word gives rise to an \(M\)-unambiguous word. Thus, for the sequence of words \(\varphi = \{w_i\}_{i \geq 1}\) such that \(w_i = cbcbabc^i dcbabbc\), we have \(\Theta_{\varphi} = A, U, U, U, \ldots\).

We see that the sequence \(\Theta_{\varphi}\) is ultimately periodic with its period being one. Thus, we seek to know whether an \(M\)-ambiguity sequence is always periodic in the case of duplicating a single letter in a word. We formulate this question formally as follows.

Question 4.1 Suppose \(\Sigma\) is an ordered alphabet. Let \(\varphi = \{w_i\}_{i \geq 1}\) be any sequence of words over \(\Sigma\) such that for every integer \(k \geq 1\), we have \(w_k = ua^kv\) for some \(u, v \in \Sigma^*\) and \(a \in \Sigma\). Must the sequence \(\Theta_{\varphi}\) be ultimately periodic?

In the spirit of answering the above question, we first present a way to determine the \(M\)-ambiguity of a word—by transforming it to a problem of solving systems of linear equalities. To illustrate this, we analyze the word considered in Theorem 3.2 and deduce that it is \(M\)-unambiguous for every integer \(n > 1\).

Let \(\Sigma = \{a < b < c < d\}\) and consider the word \(w = cbcbabc^n dcbabbc\), where \(n\) is a positive integer. If a word \(w' \in \Sigma^*\) is \(M\)-equivalent to \(w\), then \(\pi_{\{a,b,d\}}(w') \equiv_M \pi_{\{a,b,d\}}(w)\) with respect to \(\{a < b < d\}\). Since \(\pi_{\{a,b,d\}}(w) = bbabdbabbb\), it follows that for such a word \(w'\), the projection \(\pi_{\{a,b,d\}}(w')\) must be one of the followings:

\[
\begin{align*}
&\text{dbbabbb, bdabbbab, bdbdabbb, bbabdbab, bbadbdab, } \text{bbabbdab, babbdbab, bdabbdab, bdbdab, bdababdb, bbababdb, bdbdbadb, bdabbdab, bdabdbdb, bdabdbdb, bdabdbdb, bdabdbdb, bdabdbdb, bdabdbdb, bdabdbdb, bdabdbdb, bdabdbdb}, \\
&\text{abdbdbdb, abdbdbdb, abdbdbdb, abdbdbdb, abdbdbdb, abdbdbdb, abdbdbdb, abdbdbdb.}
\end{align*}
\]
Consider the scenario \( \pi\{a, b, d\}(w') = \pi\{a, b, d\}(w) = bbabdbabb \). Then,

\[
w' = c^{x_1}bc^{x_2}bc^{x_3}ac^{x_4}bc^{x_5}d^{x_6}bc^{x_7}ac^{x_8}bc^{x_9}bc^{x_{10}}
\]

for some nonnegative integers \( x_i \) (1 \( \leq i \leq 10 \)). Since \( w' \equiv_M w \), it follows that

\[
\begin{align*}
x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 + x_8 + x_9 + x_{10} &= |w'|_c = |w|_c = n + 5, \\
x_2 + 2x_3 + 2x_4 + 3x_5 + 3x_6 + 4x_7 + 4x_8 + 5x_9 + 6x_{10} &= |w'|_{bc} = |w|_{bc} = 3n + 15, \\
x_1 + x_2 + x_3 + x_4 + x_5 &= |w'|_{cd} = |w|_{cd} = n + 2, \\
x_5 + x_6 + 2x_7 + 2x_8 + 4x_9 + 6x_{10} &= |w'|_{abc} = |w|_{abc} = n + 11, \\
x_2 + 2x_3 + 2x_4 + 3x_5 &= |w'|_{bcd} = |w|_{bcd} = 3n + 1, \\
x_5 &= |w'|_{abcd} = |w|_{abcd} = n.
\end{align*}
\]

Solving the above system of linear equalities, we obtain the solution set

\[
\begin{align*}
x_1 &= 1 + x_3 + x_4, \\
x_2 &= 1 - 2x_3 - 2x_4, \\
x_5 &= n, \\
x_6 &= 1, \\
x_7 &= -1 - x_8 + x_{10}, \\
x_9 &= 3 - 2x_{10}.
\end{align*}
\]

By imposing the constraints \( x_i \geq 0 \) (1 \( \leq i \leq 10 \)) and \( n \geq 1 \), we now have the additional system of linear inequalities

\[
\begin{align*}
x_3 + x_4 &\geq -1, \\
2x_3 + 2x_4 &\leq 1, \\
x_8 - x_{10} &\leq -1, \\
2x_{10} &\leq 3,
\end{align*}
\]

\[ (**) \]

\[
x_3, x_4, x_8, x_{10} \geq 0, \quad n \geq 1.
\]

From the above system of linear inequalities, notice that the only possible value of \( x_{10} \) is 1, and therefore, \( x_8 = 0 \). Also, observe that it can only be the case that \( x_3 = x_4 = 0 \). By the system of linear equations before that, it follows that \( x_1 = x_2 = x_6 = x_9 = 1, x_5 = n \) and \( x_7 = 0 \). As a result, for each positive integer \( n \), we have \( w' = w \), and thus, it follows that no \( w' \) distinct from \( w \) with \( \pi_{a,b,d}(w') = bbabdbabb \) is \( M \)-equivalent to \( w \).

Next, consider the scenario \( \pi_{(a,b,d)}(w') = babbbdbbab \). Analyzing similarly as above, we obtain the solution set
\[ x_1 = 1 - n, \]
\[ x_2 = 1 - x_3 + x_5 + n, \]
\[ x_4 = -2x_5 + n, \]
\[ x_6 = -2 + x_8 + x_9, \]
\[ x_7 = 5 - 2x_8 - 2x_9, \]
\[ x_{10} = 0, \]

and the additional system of linear inequalities

\[ n \leq 1, \]
\[ x_3 - x_5 \leq 1, \]
\[ 2x_5 - n \leq 0, \]
\[ x_8 + x_9 \geq 2, \]
\[ 2x_8 + 2x_9 \leq 5, \]
\[ x_3, x_5, x_8, x_9 \geq 0, \]
\[ n \geq 1. \]

By some simple analysis, one can see that integral solutions with \( n = 1 \) exist for the above system—each of them gives rise to a word \( w' \) that is distinct from \( w \). This implies that when \( n = 1 \), the word \( w \) is \( M \)-ambiguous. However, when \( n > 1 \), there are no integral solutions, with such \( n \), satisfying the system.

Arguing like this, one can see that each possibility of \( \pi_{\{a,b,d\}}(w') \) in (\( * \)) leads to a system of linear equations and inequalities. Every such system can then be analyzed similarly as in above (thus, we omit the details of the remaining computations). In our case here, when \( n > 1 \), all the remaining 34 systems lead to no solutions. Thus, we conclude that the word \( w = cbcbabc^n dcbabcbc \) is \( M \)-unambiguous for all integers \( n > 1 \).

**Remark 4.2** Suppose \( \Sigma \) is an ordered alphabet. For a general word,

\[ w = x_1a^kx_2a^k \ldots x_{j-1}a^kx_j \]

where \( x_1, x_2, \ldots, x_j \in \Sigma^* \), \( a \in \Sigma \), and \( k \geq 1 \) is an integer, the above algebraic analysis can be used to determine the values of \( k \) such that the word \( w \) is \( M \)-ambiguous. Each possibility of \( \pi_{\Sigma\setminus\{a\}}(w) \) gives rise to a solution set, described by a system of linear equalities and inequalities. Furthermore, these finitely many systems are rational due to the nature of elementary row operations applied in order to obtain them. Note that we need this observation for the proof of Theorem 4.6 later.

Next, we need the following notion and known result, which in turn will be used to prove a lemma necessary for our purpose.

**Definition 4.3** Suppose \( n \) is a positive integer. A set \( P \subseteq \mathbb{R}^n \) is a **rational polyhedron** if and only if \( P = \{ x \in \mathbb{R}^n \mid Ax \geq b \} \) for some matrix \( A \in \mathbb{Q}^{m \times n} \) and vector \( b \in \mathbb{Q}^m \), where \( m \) is a positive integer.
The following result was deduced as Equation 19 in Chapter 16 of [14]. We do not state the underlying details that lead to this result here as they are not essential for our purpose.

**Theorem 4.4** [14] Suppose $n$ is a positive integer. For any rational polyhedra $P \subseteq \mathbb{R}^n$, there exist vectors $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_r, \mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_s \in \mathbb{Z}^n$ such that

\[
\{ \mathbf{x} \in P \cap \mathbb{Z}^n \} = \{ \lambda_1 \mathbf{x}_1 + \cdots + \lambda_r \mathbf{x}_r + \mu_1 \mathbf{y}_1 + \cdots + \mu_s \mathbf{y}_s \mid \lambda_1, \ldots, \lambda_r, \mu_1, \ldots, \mu_s \text{ are nonnegative integers with } \lambda_1 + \cdots + \lambda_r = 1 \}.
\]

We are now ready to prove our main lemma.

**Lemma 4.5** Suppose $n$ is a positive integer and $P \subseteq \mathbb{R}^n$ is a rational polyhedron. For arbitrary integer $1 \leq k \leq n$, let

\[
Q = \{ p \in \mathbb{Z}_+ \mid p \text{ is the } k\text{th component of some } \mathbf{x} \in P \cap \mathbb{Z}^n \}.
\]

If the set $Q$ is infinite, then for some positive integer $d$ and nonempty set $T \subseteq [0, d) \cap \mathbb{Z}$, there exists a positive integer $N$ such that

\[
\{ p \in Q \mid p \geq N \} = \{ p \in \mathbb{Z}_+ \mid p \geq N \text{ and } p = dq + t \text{ for some } t \in T \text{ and integer } q \}.
\]

**Proof** Since $P$ is a rational polyhedron, by Theorem 4.4, it holds that there exist vectors $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_r, \mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_s \in \mathbb{Z}^n$ such that

\[
\{ \mathbf{x} \in P \cap \mathbb{Z}^n \} = \{ \lambda_1 \mathbf{x}_1 + \cdots + \lambda_r \mathbf{x}_r + \mu_1 \mathbf{y}_1 + \cdots + \mu_s \mathbf{y}_s \mid \lambda_1, \ldots, \lambda_r, \mu_1, \ldots, \mu_s \text{ are nonnegative integers with } \lambda_1 + \cdots + \lambda_r = 1 \}.
\]

Fix an arbitrary integer $1 \leq k \leq n$. Let $x[i]$ denote the $i$th component of a vector $\mathbf{x}$. Then, for an arbitrary $p \in Q$, it holds that

\[
p = \lambda_1 x_1[k] + \cdots + \lambda_r x_r[k] + \mu_1 y_1[k] + \cdots + \mu_s y_s[k] \text{ for some nonnegative integers } \lambda_1, \ldots, \lambda_r, \mu_1, \ldots, \mu_s \text{ with } \lambda_1 + \cdots + \lambda_r = 1.
\]

Suppose the set $Q$ is infinite. Assume $y_i[k]$ is nonpositive for all integers $1 \leq i \leq s$. Note that $p$ is a positive integer and $\lambda_1, \ldots, \lambda_r$ are nonnegative integers with $\lambda_1 + \cdots + \lambda_r = 1$. If $x_i$ is nonpositive for all integers $1 \leq i \leq r$, then $p \leq 0$, thus a contradiction. On the other hand, if $x_i$ is positive for some integer $1 \leq i \leq r$, then $p \leq \max\{x_i[k] \mid 1 \leq i \leq r \}$. However, such values of positive integers $p$ are only finitely many, thus again contradicting the supposition that $Q$ is infinite. Therefore, $y_i[k]$ is positive for some integer $1 \leq i \leq s$.

Choose an integer $1 \leq h \leq s$ such that $y_h[k]$ is positive. Let

\[
T = \{ t \in [0, y_h[k]) \cap \mathbb{Z} \mid t = p - y_h[k] \cdot q \text{ for some } p \in Q \text{ and integer } q \}.
\]
For every $t \in T$, let

$$p_t^* = \min\{ p \in Q \mid p = y_h[k] \cdot q + t \text{ for some integer } q \}.$$

Suppose $t \in T$ and integer $j \geq 0$ are arbitrary. Clearly, $p_t^* + j \cdot y_h[k]$ is positive. Since $p_t^* \in Q$, it follows that $p_t^* = x[k]$ for some vector $x \in P \cap \mathbb{Z}^n$. Then, by (4.5.1), it holds that $x + j \cdot y_h \in P \cap \mathbb{Z}^n$. Therefore,

For every $t \in T$ and integer $j \geq 0$, we have $p_t^* + j \cdot y_h[k] \in Q$. (4.5.2)

Let $N = \max\{p_t^* \mid t \in T\}$ and $d = y_h[k]$. Clearly, for every $p \in Q$ with $p \geq N$, we have $p \in \mathbb{Z}_+$ and $p = y_h[k] \cdot q + t$ for some $t \in T$ and integer $q$. To see that the converse holds, fix an arbitrary $p \in \mathbb{Z}_+$ with $p \geq N$ such that $p = y_h[k] \cdot q + t$ for some $t \in T$ and integer $q$. Let $q_t^*$ be the integer such that $p_t^* = y_h[k] \cdot q_t^* + t$. Note that $p_t^* \leq N \leq p$; thus, $q_t^* \leq q$. Notice that

$$p = y_h[k] \cdot q + t = y_h[k] \cdot (q + q_t^* - q_t^*) + t = y_h[k] \cdot (q_t^* + (q - q_t^*)) + t = y_h[k] \cdot q_t^* + t + y_h[k] \cdot (q - q_t^*) = p_t^* + y_h[k] \cdot (q - q_t^*).$$

It remains to see that since $q - q_t^* \geq 0$, by (4.5.2), it holds that $p \in Q$. Thus, our conclusion holds.

Theorem 4.6 Suppose $\Sigma$ is an ordered alphabet. Let $\varphi = \{w_k\}_{k \geq 1}$ be a sequence of words over $\Sigma$ such that for every integer $k \geq 1$, we have

$$w_k = v_1 a_k v_2 a_k \ldots v_{j-1} a_k v_j$$

for some $v_1, v_2, \ldots, v_j \in \Sigma^*$ and $a \in \Sigma$. Then, $\Theta_\varphi$ is ultimately periodic.

Proof The idea presented in this proof follows Remark 4.2. Let $\Gamma = \Sigma \setminus \{a\}$. Write the general word $v_1 a^k v_2 a^k \ldots v_{j-1} a^k v_j$ in the form $a^{x_1} u_1 a^{x_2} u_2 \ldots u_n a^{x_{n+1}}$ for some positive integer $n$, (fixed) integers $x_i \geq 0$ ($1 \leq i \leq n + 1$) and $u_i \in \Gamma$ ($1 \leq i \leq n$).\footnote{It will be clear why each $u_i$ is a letter instead of a word when the reader reaches the second sentence of the next paragraph.} Note that $u_1 u_2 \ldots u_n = \pi_\Gamma(w_k)$ regardless of the value of $k$. Suppose there exists $w' \in \Sigma^*$ such that $w' \equiv_M w_k$. Then, $\pi_\Gamma(w') \equiv_M \pi_\Gamma(w_k)$, and thus, $\pi_\Gamma(w') \in C_{u_1 u_2 \ldots u_n}$. Each possibility of the projection $\pi_\Gamma(w')$ gives rise to a rational system of linear inequalities as in (**) with $k$ being a variable in it due to the constraint $k \geq 1$. Each such system, when solved for nonnegative integral solutions, contains the positive integral values of $k$ such that $w_k$ is $M$-equivalent to some $w'$ with that projection—in other words, the positive integral values of $k$ such that $w_k$ is $M$-ambiguous.
Let \( \pi \Gamma (w') = \pi \Gamma (w_k) = u_1 u_2 \ldots u_n \). Then \( w' = a^y_1 u_1 a^y_2 u_2 \ldots a^y_n u_n a^{y_n+1} \) for some integers \( y_i \geq 0 \) (\( 1 \leq i \leq n + 1 \)). If \( y_i = x_i \) for every integer \( 1 \leq i \leq n + 1 \), then \( w' = w_k \). To exclude this scenario, we impose on top of the system of linear inequalities obtained as in (\( \ast \ast \)), the condition \( y_i < x_i \) or \( y_i > x_i \) for some integer \( 1 \leq i \leq n + 1 \). Thus, for every integer \( 1 \leq i \leq n + 1 \), we consider two distinct systems of linear inequalities, each of them with one of the conditions \( y_i < x_i \) or \( y_i > x_i \)—this gives a total of \( 2(n + 1) \) systems. (For any positive integer \( k \), the word \( w_k \) is \( M \)-ambiguous if there exist nonnegative integral solutions with that \( k \) in the union of the solution sets corresponding to the \( 2(n + 1) \) systems.)

On the other hand, if \( \pi \Gamma (w') \neq \pi \Gamma (w_k) \), then it is impossible for \( w' \) to be the same word as \( w_k \). Thus, for each such possibility of \( \pi \Gamma (w') \), we consider just the system of linear inequalities obtained as in (\( \ast \ast \)) without any additional conditions—this gives a total of \( |C_{u_1 u_2 \ldots u_n}| - 1 \) systems. In total, there are \( 2(n + 1) + |C_{u_1 u_2 \ldots u_n}| - 1 \) systems of linear inequalities.

Let \( N = 2(n + 1) + |C_{u_1 u_2 \ldots u_n}| - 1 \). Let integers \( 1 \leq i \leq N \) enumerate the above systems of linear inequalities and write each of them in the form \( A_i y \geq b_i \) for some matrix \( A_i \) and vector \( b_i \) (the dimensions of \( A_i \), \( y \) and \( b_i \) depend on the system of linear inequalities being represented). For every integer \( 1 \leq i \leq N \), let \( P_i \) be the set of solution vectors \( y \) to the \( i \)th system of linear inequalities. Note that by Remark 4.2 and Definition 4.3, every \( P_i \) (\( 1 \leq i \leq N \)) is a rational polyhedron. Also, due to the nonnegativity constraints imposed to obtain the systems of linear inequalities (see (\( \ast \ast \))), all components of the solution vectors \( y \) are nonnegative.

For every integer \( 1 \leq i \leq N \), let \( \tau_i \) be the index such that the \( \tau_i \)th component of \( y \) corresponds to the variable \( k \), and define the set

\[
P_i^* = \{ p \in \mathbb{Z}_+ \mid p \text{ is the } \tau_i \text{th component of some integral } y \in P_i \}.
\]

Notice that \( P_i^* \) is the set containing all positive integral values of \( k \) such that \( w_k \) is \( M \)-ambiguous in relation to the projection associated with the \( i \)th system. Hence, in a complete picture,

The word \( w_k \) is \( M \)-ambiguous if and only if \( k \in \bigcup_{1 \leq i \leq N} P_i^* \). \hspace{1cm} (4.6.1)

**Case 1** The set \( P_i^* \) is finite for every integer \( 1 \leq i \leq N \).

Then, the set \( \bigcup_{1 \leq i \leq N} P_i^* \) is finite as well. By (4.6.1), the word \( w_k \) is \( M \)-ambiguous for only finitely many values of \( k \). For every integer \( k > \max \{ j \mid w_j \text{ is } M \text{-ambiguous} \} \), the word \( w_k \) is \( M \)-unambiguous. Therefore, \( \Theta_\Psi \) is ultimately periodic (with its period being one).

**Case 2** The set \( P_i^* \) is infinite for some integer \( 1 \leq i \leq N \).

Let \( I = \{ 1 \leq i \leq N \mid \text{the set } P_i^* \text{ is infinite} \} \). For every integer \( 1 \leq i \leq N \), \( P_i \) is a rational polyhedron. Thus, for every integer \( i \in I \), by Lemma 4.5, it follows that for some positive integer \( d_i \) and nonempty set \( T_i \subseteq [0, d_i) \cap \mathbb{Z} \), there exists a positive integer \( M_i \) such that

\[
\{ p \in P_i^* \mid p \geq M_i \} = \{ p \in \mathbb{Z}_+ \mid p \geq M_i \text{ and } p = d_i q + t \text{ for some } t \in T_i \text{ and integer } q \}.
\]
Let $M' = \max(\{M_i \mid i \in I\} \cup \{p \in P_i^* \mid P_i^* \text{ is finite}\})$. Then, by (4.6.1), it follows that

for every integer $k > M'$, the word $w_k$ is $M$-ambiguous if and only if there exists $i \in I$ such that $k = d_i q + t$ for some $t \in T_i$ and integer $q$. (4.6.2)

Let $d' = \prod_{i \in I} d_i$. By some simple argument, one can see that for any $i \in I$ and integer $k$, we have $k = d_i q + t$ for some $t \in T_i$ and integer $q$ if and only if $k + d' = d_i q + t$ for some $t \in T_i$ and integer $q$. Therefore, by (4.6.2), it holds that for every integer $k > M'$, the $M$-ambiguity of the words $w_{k+d'}$ and $w_k$ are the same. That is to say, the sequence $\Theta_\varphi$ is ultimately periodic.

In both cases, our conclusion holds. □

Finally, the following generalization holds as a consequence of the above theorem.

**Corollary 4.7** Suppose $\Sigma$ is an ordered alphabet. Let $\varphi = \{w_n\}_{n \geq 0}$ be a sequence of words over $\Sigma$ such that for every integer $n \geq 0$, we have

$$w_n = v_1 a_{(1)}^{k_1(n)} v_2 a_{(2)}^{k_2(n)} \ldots v_j a_{(j)}^{k_j(n)} v_{j+1}$$

for some $v_1, v_2, \ldots, v_{j+1} \in \Sigma^*$ and $a \in \Sigma$ where

- $k_{(i)}^0 = 1$ for every integer $1 \leq i \leq j$;
- For every integer $1 \leq i \leq j$, let $e_i$ denote the $j$-tuple with $1$ in the $i$th coordinate and $0$ elsewhere, and for every integer $n \geq 1$, let $\alpha_n \in \{e_i \mid 1 \leq i \leq j\}$ and

$$(k_{(1)}^n, k_{(2)}^n, \ldots, k_{(j)}^n) = (k_{(1)}^{n-1}, k_{(2)}^{n-1}, \ldots, k_{(j)}^{n-1}) + \alpha_n.$$

If the sequence $\{\alpha_n\}_{n \geq 1}$ is periodic, then the sequence $\Theta_\varphi$ is ultimately periodic.

**Proof** Suppose the sequence $\{\alpha_n\}_{n \geq 1}$ is periodic, with a period $p$. Then, for all integers $1 \leq n \leq p$ and $t \geq 0$, we have $\alpha_{n+t} = \alpha_n$. Let integers $d_i(1 \leq i \leq j)$ be such that $(d_1, d_2, \ldots, d_j) = \sum_{n=1}^p \alpha_n$. Next, we need the following observation. (The validity of the following claim can be easily verified by the reader; thus, we omit its technical proof.)

**Claim 4.8** For every integer $1 \leq n \leq p$, let $\alpha_n^j$ be the $j$-tuple such that $\alpha_n^j = \sum_{i=1}^n \alpha_i$ (the addition of tuples is defined element-wise). For all integers $1 \leq n \leq p$ and $1 \leq i \leq j$, let $\mu_{n,i}$ be the value in the $i$th coordinate of $\alpha_n^j$. Then, for all integers $1 \leq n \leq p$, $1 \leq i \leq j$ and $t \geq 0$, we have $k_{(j)}^{n+t} = d_i t + \mu_{n,i} + 1$. 

$\Theta_\varphi$ Springer
For all integers $1 \leq n \leq p$ and $t \geq 0$, we have
\[
w_{n+tp} = v_1a(d_{1+1}^{(1)})v_2a(d_{2+1}^{(2)}) \ldots v_\j a(d_{\j+1}^{(j)})v_{\j+1}
= v_1a(d_{1+\mu_{1,1}+1})v_2a(d_{2+\mu_{1,2}+1}) \ldots v_\j a(d_{\j+\mu_{\j,1}+1})v_{\j+1}
= v_1a_1^{(1)} \times a_2^{(1)} \times a_\j^{(1)} \times v_2a_1^{(2)} \times a_2^{(2)} \times a_\j^{(2)} \times \ldots \times v_\j a_1^{(\j)} \times a_2^{(\j)} \times a_\j^{(\j)}v_{\j+1}
\]
(4.9.1)

where the second equality holds by Claim 4.8.

For all integers $0 \leq n < p$, define the sequence of words $\varphi_n = \{w_{n+tp}\}_{t \geq 0}$ and let $\{\theta_{n,t}\}_{t \geq 0} = \Theta_{\varphi_n}$ (the term $\theta_{n,t}$ corresponds to the $M$-ambiguity of the word $w_{n+tp}$). Then, for every integer $0 \leq n < p$, it follows by (4.9.1) and Theorem 4.6 that the sequence $\Theta_{\varphi_n}$ is ultimately periodic. That is to say, for every integer $0 \leq n < p$, there exist positive integers $T_n$ and $P_n$ such that for all integers $t \geq T_n$ and $m \geq 0$, we have $\theta_{n,t+mP_n} = \theta_{n,t}$. Let $T = \max\{T_n \mid 0 \leq n < p\}$, then clearly
\[
\forall n \leq p, \quad t \geq T \; \text{and} \; m \geq 0, \quad \theta_{n,t+mP_n} = \theta_{n,t}. \quad (4.9.2)
\]

Let $P = p \cdot \prod_{n=0}^{p-1} P_n$ and $\{\vartheta_t\}_{t \geq 0} = \Theta_{\varphi}$ (the term $\vartheta_t$ corresponds to the $M$-ambiguity of the word $w_t$). To see that the sequence $\Theta_{\varphi}$ is ultimately periodic, we show that for every integer $t \geq T$, we have $\vartheta_{t+p} = \vartheta_t$. Fix an arbitrary integer $t \geq T$. Let integers $q$ and $0 \leq r < p$ be such that $t = pq + r$. Then, by definition, we have $\vartheta_t = \theta_{r,q}$ and $\vartheta_{t+p} = \theta_{r,q+p}$. It remains to see that since $\frac{p}{p} = \prod_{n=0}^{p-1} P_n$, it follows by (4.9.2) that $\vartheta_{t+p} = \theta_{r,q+p} = \theta_{r,q} = \vartheta_t$. Thus, our conclusion holds. □

5 Conclusion

Unlike the case of binary and ternary alphabets, for larger alphabets, duplication of letters in a word can continuously alter the $M$-ambiguity of the resulting words. In fact, by using the main observations and results in Sect. 3, we have seen that nearly any pattern of $M$-ambiguity sequence is attainable.

As implied in Remark 3.15, we are yet to find a print word such that selective repeated duplications of letters in that word could give rise to arbitrary $M$-ambiguity sequences starting with the term $U$. We believe that by further investigation, this would be achievable as well. However, we leave it as an open problem.

The final result in Sect. 4 shows that repeated duplications of letters of the same type in a word, when done in a periodic manner, give rise to a periodic $M$-ambiguity sequence. It remains to see if periodic duplications of different types of letters in a word would lead to the same conclusion. The main complexity would be that the associated systems consist of nonlinear equations and inequalities.

Acknowledgements The authors would like to thank the anonymous referees for their careful reading of the original version of this paper. Their comments and suggestions have significantly improved the clarity.
of the work presented here. The authors also gratefully acknowledge support for this research by a Research University Grant No. 1001/PMATHS/8011019 of Universiti Sains Malaysia. This study is an extension of the work in [11].

References

1. Atanasiu, A.: Binary amiable words. Int. J. Found. Comput. Sci. 18(2), 387–400 (2007)
2. Atanasiu, A., Atanasiu, R., Petre, I.: Parikh matrices and amiable words. Theor. Comput. Sci. 390(1), 102–109 (2008)
3. Atanasiu, A., Teh, W.C.: A new operator over Parikh languages. Int. J. Found. Comput. Sci. 27(6), 757–769 (2016)
4. Mahalingam, K., Bera, S., Subramanian, K.G.: Properties of Parikh matrices of words obtained by an extension of a restricted shuffle operator. Int. J. Found. Comput. Sci. 29(3), 403–3143 (2018)
5. Mateescu, A., Salomaa, A.: Matrix indicators for subword occurrences and ambiguity. Int. J. Found. Comput. Sci. 15(2), 277–292 (2004)
6. Mateescu, A., Salomaa, A., Salomaa, K., Yu, S.: A sharpening of the Parikh mapping. Theor. Inform. Appl. 35(6), 551–564 (2001)
7. Parikh, R.J.: On context-free languages. J. Assoc. Comput. Mach. 13, 570–581 (1966)
8. Poovanandran, G., Teh, W.C.: Strong 2 · t and strong 3 · t transformations for strong M-equivalence. Int. J. Found. Comput. Sci. 29(1), 123–137 (2018)
9. Poovanandran, G., Teh, W.C.: Elementary matrix equivalence and core transformation graphs for Parikh matrices. Discrete Appl. Math. 251, 276–289 (2018)
10. Poovanandran, G., Teh, W.C.: On M-equivalence and strong M-equivalence for Parikh matrices. Int. J. Found. Comput. Sci. 29(1), 123–137 (2018)
11. Poovanandran, G., Teh, W.C.: Parikh matrices and M-ambiguity sequence. In: Journal of Physics: Conference Series, vol. 1132, p. 012012 (2018)
12. Salomaa, A.: Criteria for the matrix equivalence of words. Theor. Comput. Sci. 411(16), 1818–1827 (2010)
13. Salomaa, A., Yu, S.: Subword occurrences, Parikh matrices and Lyndon images. Int. J. Found. Comput. Sci. 21(1), 91–111 (2010)
14. Schrijver, A.: Theory of Linear and Integer Programming. Wiley, Hoboken (1998)
15. Șerbănuță, V.N.: On Parikh matrices, ambiguity, and prints. Int. J. Found. Comput. Sci. 20(1), 151–165 (2009)
16. Șerbănuță, V.N., Șerbănuță, T.F.: Injectivity of the Parikh matrix mappings revisited. Fundam. Inform. 73(1), 265–283 (2006)
17. Teh, W.C.: Parikh matrices and Parikh rewriting systems. Fundam. Inform. 146, 305–320 (2016)
18. Teh, W.C., Atanasiu, A.: On a conjecture about Parikh matrices. Theor. Comput. Sci. 628, 30–39 (2016)
19. Teh, W.C., Atanasiu, A., Poovanandran, G.: On strongly M-unambiguous prints and Șerbănuță’s conjecture for Parikh matrices. Theor. Comput. Sci. 719, 86–93 (2018)
20. Teh, W.C., Kwa, K.H.: Core words and Parikh matrices. Theor. Comput. Sci. 582, 60–69 (2015)
21. Teh, W.C., Subramanian, K.G., Bera, S.: Order of weak M-relation and Parikh matrices. Theor. Comput. Sci. 743, 83–92 (2018)

Publisher’s Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.