Generating mechanism and dynamic of the smooth positons for the derivative nonlinear Schrödinger equation

Wenjuan Song · Shuwei Xu · Maohua Li · Jingsong He

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Abstract Based on the degenerate Darboux transformation, the \( n \)-order smooth positon solutions for the derivative nonlinear Schrödinger equation are generated by means of the general determinant expression of the \( N \)-soliton solution, and interesting dynamic behaviors of the smooth positons are shown by the corresponding three-dimensional plots in this paper. Furthermore, the decomposition process, bent trajectory and the change of the phase shift for the positon solutions are discussed in detail. Additionally, three kinds of mixed solutions, namely (1) the hybrid of one-positon and two-positon solutions, (2) the hybrid of two-positon and two-positon solutions, and (3) the hybrid of one-soliton and three-positon solutions, are presented and their rather complicated dynamics are revealed.

Keywords DNLS equation · Positon · Degenerate Darboux transformation · Decomposition · Trajectory · Phase shift

1 Introduction

As a degenerate cases of Wronskian formula of solutions of the Korteweg–de Vries (KdV) equation, Matveev [1] firstly introduced the positons of the KdV equation which is a singular new solution of nonlinear evolution equations. Matveev also obtained positon and soliton–positon solutions of the KdV equation and summarized many significant properties of the positon solutions which is different from the soliton solutions. There exists a class of slowly decreasing reflectionless potential which is called potentials superreflectionless [2]. Compared with the soliton solutions with exponential decay, the positons were weakly localized. There are two reasons for the name called positon solutions that is generated by a positive spectral singularity embedded in the spectrum and always positive in a small enough neighborhood of the pole. About the other interacting objects, the positons are completely transparent. In particular, two-positons remain unchanged after mutual collision while during the positon and soliton collision, the soliton solution remain unchanged; however, the positon is affected of which the carrier wave and envelope produce the finite phase shift [3,4].

In Ref. [5], the authors established the connection of the solutions of the positons, soliton and breather. Inspiringly by the pioneering work of Matveev, such solutions have been extended to other well-known equations, such as the defocusing modified KdV (mKdV) equation [6,7], the sine-Gordon (sG) equation [8], the
Toda lattice [9] and the Hirota–Satsuma coupled KdV system [10]. It is crucial to get the smooth positons because the positon solutions above-mentioned are singular. Recently, the smooth positions of focusing mKdV equation [11], complex mKdV equation [12] and the second-type derivative nonlinear Schrödinger (DNLSII) equation [13] also have been constructed. Positon is a slowly decreasing analog of soliton, which is closed related to Wigner–von Neumann phenomenon [14]. It is known that the positon of the KdV, as one kind of potential in quantum mechanics, is expected to be realized [14] in practice using band engineering technique [15]. This interesting application of the positon inspires us to study the positon in other soliton equation with strong physical background.

The nonlinear Schrödinger (NLS) equation is one of the most significant equations in physics and mathematics, and it can be derived from the Ablowitz–Kaup–Newell–Segalur [16,17]. Various modifications of the equation have been investigated extensively and discussed intensively. One of these attempts is to study the effects of higher-order perturbations which has been proposed by various authors. Considering higher-order nonlinear effect, the derivative nonlinear Schrödinger (DNLS) equations with a polynomial spectral problem of arbitrary order [18] are regarded as the models in a wide variety of fields such as weekly nonlinear dispersive water waves [19]. The DNLS equations have three generic deformations, i.e., the DNLSI equation [20]

\[ i q_t - q_{xx} + i(q^2 q^*)_x = 0, \quad (1) \]

the DNLSII equation [21], which is also called the Chen–Lee–Liu (CLL) equation, of the form

\[ i q_t + q_{xx} + i q q^* q_x = 0, \quad (2) \]

and the DNLSIII equation (or the GI equation) [22]

\[ i q_t + q_{xx} - i q^2 q^* + \frac{1}{2} q^3 q^* = 0. \quad (3) \]

Equation (1) is also briefly called the DNLS equation, which is one of the most important integrable systems in the mathematics and physics, where \( \ast \) is the complex conjugation, and the subscript \( \times \) (or \( \ast \)) denotes the partial derivative with respect to \( x \) (or \( t \)). The DNLS equation is connected with the DNLSII equation by a simple gauge transformation [23,24], and the relationship of DNLS equation and DNLSIII equation is also discussed in Ref. [25].

The DNLS equation was proposed to describe Alfvén waves in plasma that is the wave of finite amplitude which propagate parallel to the magnetic field [20, 26]. On the one hand, the equation is used to describe large-amplitude magnetohydrodynamic (MHD) wave propagating in plasmas at moderate angles with respect to the equilibrium magnetic field [27]. Truncated DNLS equation [28] and DNLS with nonlinear Landau dumpling [29] are put into use in practical plasmas as well. More importantly, the equation is applied to nonlinear optics, such as the sub-picosecond or femtosecond pluses in single-mode optical fibers [30–32].

The DNLS equation is one of the rare several integrable nonlinear models that permit soliton solutions. Under vanishing boundary condition (VBC), Kaup and Newell [33] solved the appropriate inverse scattering problem and obtained the one-soliton solution. By using inverse scattering method, the soliton is examined analytically and numerically under vanishing boundary condition and nonvanishing boundary condition (NVBC) and introduces the “paired soliton” which generally pulsates with a period but degenerates to a stable “pure soliton” (bright and dark solitons) in a limited case [34]. An explicit expression for the \( N \)-soliton solution is expressed in terms of determinants by means of algebraic techniques that solving the reduced Zakharov–Shabat equations [35].

Furthermore, in order to avoid constructing Riemann sheets, an inverse scattering transform (IST) for the DNLS equation with NVBC is derived by introducing an affine parameter. A one-soliton solution which is a breather and degenerates to a bright or dark soliton as the discrete eigenvalue becomes purely imaginary [36], simpler than that in the literature [37], is obtained. And the bilinearization of a generalized derivative nonlinear schrödinger equation is also discussed, and the solitons solution is constructed as the quotients of the Wronski-type determinants [25].

Recently, \( N \)-soliton solution of two-component DNLS equation is investigated by the twofold Darboux transformation (DT) [38]. And Chan [39] presented the rogue waves of DNLS equation using a long-wave limit of breather. Later, the rogue wave of coupled DNLS equation by means of low-frequency limit of breather is derived [40]. In Ref. [41], the determinant representation of the \( n \)-fold DT and formulae of \( q^{[n]} \) and \( r^{[n]} \) is expressed. The complete classification of the
dark soliton, bright soliton as well as periodic solution is given and obtained rational traveling solution and rogue wave. As for mixed nonlinear Schrödinger (MNLS) equation, which an integrable equation with the nonlinear term in NLS equation and DNLS equation denote the effects of phase modulation (SPM) and self-steepening from the points of view mathematicics and physics, the rational solutions are also investigated in Ref. [42]. Those results have also been extended to other NLS-type equation with additional derivative terms [43,44]. It is worthwhile and natural to know whether or not it has other solutions or interesting dynamics. Inspired by the result above, the smooth positon solutions of DNLS equation and its dynamic behaviors are also worthwhile studied. To the best of our knowledge, the smooth positon solutions of DNLS equation have never been reported. The main aim of this paper is to obtain the multi-positon solution of the DNLS equation and studied its property of dynamics.

This paper is organized as follows. In Sect. 2, the explicit formula of smooth positons is obtained by means of DT and degenerate DT of the DNLS equation. In Sect. 3, with the special higher Taylor expansion, the positons are decomposed by modulus square, and then also discussed the trajectory and “phase shift” in detail. In Sect. 4, the combinations of solitons and positons are investigated. The conclusion is provided in Sect. 5.

2 Positons of DNLS equation

The couple of the derivative nonlinear Schrödinger equations [33],

\[ q_t + iq_{xx} - (r q^2)_x = 0, \]
\[ r_t - i r_{xx} - (r^2)_x = 0, \]

is exactly lead to the DNLS equation for \( r = -q^* \), but the choice \( r = q^* \) would result in Eq. (1) with the sign of the nonlinear term changed. The * denotes the complex conjugation.

The Lax pairs of coupled DNLS equations (4) and (5) can be derived by the KN spectral problem [33]

\[ \partial_x \psi = (J \lambda^2 + Q \lambda) \psi = U \psi, \]
\[ \partial_t \psi = (2J \lambda^4 + V_3 \lambda^3 + V_2 \lambda^2 + V_1 \lambda) \psi = V \psi, \]

with

\[ \psi = \begin{pmatrix} \phi \\ \varphi \end{pmatrix}, \quad J = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad Q = \begin{pmatrix} 0 & q \\ r & 0 \end{pmatrix}, \]
\[ V_3 = 2Q, \quad V_2 = J q r, \]
\[ V_1 = \begin{pmatrix} 0 & -i q_x + q^2 r \\ i r_x + r^2 q & 0 \end{pmatrix}. \]

Here \( \lambda \) (an arbitrary complex number) is called the eigenvalue or spectral parameter and \( \psi \) is the eigenfunction associated with \( \lambda \) of the KN system. The zero curvature equation \( U_t - V_x + [U, V] = 0 \) infers the couple of DNLS equations (4) and (5).

The determinant representation of the \( N \)-fold DT for the DNLS equation is given in Ref. [41] which is similar to the determinant representation of the \( N \)-fold DT for the NLS equation [45]. The formulas of the soliton and positon of DNLS equation are obtained explicitly by setting seed solution \( q = 0 \), and the eigenfunction

\[ \psi_j = \psi(\lambda_j) = \begin{pmatrix} \phi_j(\lambda_j) \\ \varphi_j(\lambda_j) \end{pmatrix} = \begin{pmatrix} \exp(i(\lambda_j^2 x + 2\lambda_j^4 r)) \\ \exp(-i(\lambda_j^2 x + 2\lambda_j^4 r)) \end{pmatrix} \]

is associated with eigenvalue \( \lambda_j = \alpha_j + i\beta_j \) in Theorem 2 of Ref. [41], and then, an explicit form of the \( N \)-soliton of the DNLS equation is:

\[ q^{[n]} = \frac{\Omega_{11}^2 q + 2i \Omega_{12} \Omega_{21}}{\Omega_{11}^2 - 2i \Omega_{12} \Omega_{21}}, \quad r^{[n]} = \frac{\Omega_{22}^2 r - 2i \Omega_{21} \Omega_{22}}{\Omega_{11}^2 - 2i \Omega_{12} \Omega_{21}}. \]

where

\[ \Omega_{11} = \begin{vmatrix} \lambda_1^{2n-1} \varphi_1 & \lambda_1^{2n-2} \varphi_1 & \lambda_1^{2n-3} \varphi_1 & \cdots & \lambda_1 \varphi_1 & \varphi_1 \\ \lambda_2^{2n-1} \varphi_2 & \lambda_2^{2n-2} \varphi_2 & \lambda_2^{2n-3} \varphi_2 & \cdots & \lambda_2 \varphi_2 & \varphi_2 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \lambda_1^{2n-1} \varphi_1 & \lambda_1^{2n-2} \varphi_2 & \lambda_1^{2n-3} \varphi_3 & \cdots & \lambda_1 \varphi_2 & \varphi_3 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \lambda_2^{2n-1} \varphi_2 & \lambda_2^{2n-2} \varphi_2 & \lambda_2^{2n-3} \varphi_3 & \cdots & \lambda_2 \varphi_2 & \varphi_3 \end{vmatrix} \]
\[ \Omega_{12} = \begin{vmatrix} \lambda_1^{2n} \varphi_1 & \lambda_1^{2n-2} \varphi_1 & \lambda_1^{2n-3} \varphi_1 & \cdots & \lambda_1 \varphi_1 & \phi_1 \\ \lambda_2^{2n} \varphi_2 & \lambda_2^{2n-2} \varphi_2 & \lambda_2^{2n-3} \varphi_2 & \cdots & \lambda_2 \varphi_2 & \phi_2 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \lambda_1^{2n} \varphi_1 & \lambda_1^{2n-2} \varphi_2 & \lambda_1^{2n-3} \varphi_2 & \cdots & \lambda_1 \varphi_2 & \varphi_1 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \lambda_2^{2n} \varphi_2 & \lambda_2^{2n-2} \varphi_2 & \lambda_2^{2n-3} \varphi_2 & \cdots & \lambda_2 \varphi_2 & \varphi_2 \end{vmatrix} \]
\[ \Omega_{21} = \begin{vmatrix} \lambda_1^{2n} \varphi_1 & \lambda_1^{2n-2} \varphi_1 & \lambda_1^{2n-3} \varphi_1 & \cdots & \lambda_1 \varphi_1 & \phi_1 \\ \lambda_2^{2n} \varphi_2 & \lambda_2^{2n-2} \varphi_2 & \lambda_2^{2n-3} \varphi_2 & \cdots & \lambda_2 \varphi_2 & \phi_2 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \lambda_1^{2n} \varphi_1 & \lambda_1^{2n-2} \varphi_2 & \lambda_1^{2n-3} \varphi_2 & \cdots & \lambda_1 \varphi_2 & \varphi_1 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \lambda_2^{2n} \varphi_2 & \lambda_2^{2n-2} \varphi_2 & \lambda_2^{2n-3} \varphi_2 & \cdots & \lambda_2 \varphi_2 & \varphi_2 \end{vmatrix} \]
When \( n = 1 \), the explicit formula of one-soliton solution is follows:

\[
q_1 = 4\alpha_1 \beta_1 \left( (-\alpha_1 \cosh(4\alpha_1 \beta_1 H) + \beta_1 \sinh(4\alpha_1 \beta_1 H))^3 - (-\alpha_1^2 - \beta_1^2) \cosh(4\alpha_1 \beta_1 H)^2 + \beta_1^4 \right)^{\frac{1}{2}} \exp(2iH),
\]

\[
(10)
\]

where \( H = 4\alpha_1^2 t - 4\beta_1^2 t + x, \ h = -\beta_1^2 x + 2\beta_1^4 t - 12\alpha_1^2 \beta_1^2 t + \alpha_1^4 x + 2\alpha_1^4 t. \)

When \( \lambda_1 = \lambda_3 \), the denominator of two-soliton is

\[
-\beta_1^2 x + 2\beta_1^4 t + \alpha_1^4 x + 2\alpha_1^4 t.
\]

We only present the explicit form of the two-positon by letting \( n = 2 \) in Proposition 1 because of the tedious mathematical formulas. The explicit expression of two-positon solution is follows:

\[
q_2 = \frac{A_1 A_2}{(512\omega_3 - 512\omega_4 - (\alpha_1^2 + \beta_1^2)(\omega_5 \cosh(8\alpha_1 \beta_1 H) + \omega_6 \sinh(8\alpha_1 \beta_1 H)))^2},
\]

where

\[
N_{2n} = \Omega_{11} \Omega_{12}.
\]

\[
W_{2n} = \Omega_{21} \Omega_{21}.
\]

\[
N_{2n} = \left( \frac{\partial n_{2n-1}}{\partial \epsilon n_{2n-1}} \right)_{\epsilon=0}(N_{2n})j(j+1)(\lambda_1 + \epsilon)_{2n \times 2n}.
\]

\[
W_{2n} = \left( \frac{\partial n_{2n-1}}{\partial \epsilon n_{2n-1}} \right)_{\epsilon=0}(W_{2n})j(j+1)(\lambda_1 + \epsilon)_{2n \times 2n}.
\]

and \( n_i = \lfloor \frac{i+1}{2} \rfloor \), \( \lfloor \cdot \rfloor \) define the floor function of \( i \).

In the above proposition, the reduction conditions are \( \lambda_{2j} = -\lambda_{2j+1}, \phi_{2j} = \phi_{2j+1}^* \), \( \varphi_j = \phi_j^* \), \( (j = 1, 2, 3, \ldots, n) \).

Proposition 1 The \( N \)-soliton solution with a “seed” solution \( q = 0 \) from \( N \)-fold DT in the degenerate limit \( \lambda_{2j+1} \rightarrow \lambda_1 \) generates a \( n \)-positon solution of DNLS equation, which is given by

\[
q_{n-p} = 2i \frac{N_n W_{2n}}{W_{2n}},
\]

(11)

\[
\text{where}
\]

\[
N_{2n} = \Omega_{11} \Omega_{12}.
\]

\[
W_{2n} = \Omega_{21} \Omega_{21}.
\]

\[
N_{2n} = \left( \frac{\partial n_{2n-1}}{\partial \epsilon n_{2n-1}} \right)_{\epsilon=0}(N_{2n})j(j+1)(\lambda_1 + \epsilon)_{2n \times 2n}.
\]

\[
W_{2n} = \left( \frac{\partial n_{2n-1}}{\partial \epsilon n_{2n-1}} \right)_{\epsilon=0}(W_{2n})j(j+1)(\lambda_1 + \epsilon)_{2n \times 2n}.
\]
\[ \omega_{34} = \left( \beta_1^2 t - \frac{X}{4} \right) \beta_1^4 \left( -1 + i \right) \beta_1^4 t + \frac{1}{4} (1 - i) \beta_1^2 x \alpha_1^2 
\]
\[ + \frac{1}{512} (-1 + i) \beta_1^4, \]
\[ \omega_4 = -\frac{1}{8} (1 + i) \alpha_1^2 \left( \alpha_1^4 t + (-6 \beta_1^2 t + (1/4) x) \alpha_1^2 \right) 
\]
\[ + \beta_1^2 \left( \beta_1^2 t - \frac{1}{4} x^2 \right), \]
\[ \omega_5 = (1 - i) (\alpha_1^2 - \beta_1^2), \]
\[ \omega_6 = -(2 + i) \alpha_1 \beta_1. \]

In Proposition 1, \( n \)-positon generated by the degenerated DT and the higher-order Taylor expansion is smooth and expressed by mixture of exponential functions and polynomials of \( x \) and \( t \), which is different substantially from the soliton expressed by exponential functions and the rogue wave represented by the polynomials.

We provide the three-dimensional evolutions of two-positons (Fig. 1a), three-positons (Fig. 2a), four positons (Fig. 3a). In order to see clearly their trajectories, the density plots of positons are given (Figs. 1b, 2b, 3b), respectively. The trajectory will be discussed in the following section in terms of exact and approximate ways.

3 Dynamics of the positons of DNLS

The dynamical properties of the positon solution of DNLS equation will be discussed in this section. It is clear that the \( q_{2-p} \) is not a traveling wave with a constant profile from the formula of the solution and the trajectory of the two-positon is a slowly changing curve instead of a straight line. By means of the higher Taylor expansion in Proposition 1, the trajectory, decomposition and the “phase shift” of the positons are introduced in the following proposition and we study the evolution of the positon solution of the DNLS equation with the decomposition of the modulus square.

**Proposition 2** As \(|t| \to \infty\), the modulus square of a two-positon solution of the DNLS equation is decomposed as following form:

\[ |q_{2-p}|^2 \approx |q_{1-s}(H + \frac{\ln(t^4)}{16\alpha_1 \beta_1})|^2 + |q_{1-s}(H - \frac{\ln(t^4)}{16\alpha_1 \beta_1})|^2 \]  

**Proof** It is well known that the \( N \)-soliton solution can be decomposed into \( N \) single soliton with the “phase shift” when \(|t| \gg 0\). This fact stimulates strongly us to consider a similar decomposition about the multi-positon because the multi-positon is the degenerate limit of a multi-soliton. In this proposition, we start from two-positon solution in terms of the decomposition of the modulus square, i.e.,

\[ |q_{2-p}|^2 \approx |q_{1-s}(H + c_{11})|^2 + |q_{1-s}(H - c_{11})|^2 \]  

when \(|t| \to \infty\), and in which

\[ q_{1-s}(\theta) = 4i\alpha_1 \beta_1 \frac{(-i\alpha_1 \cosh(4\alpha_1 \beta_1 \theta) + \beta_1 \sinh(4\alpha_1 \beta_1 \theta))^3}{(-i\alpha_1^2 - \beta_1^2)(\cosh(4\alpha_1 \beta_1 \theta)^2 + \beta_1^4)} \exp(2ith), \]

where \( \theta = H \pm c_{11} \).

It is easy to see that Eq. (15) is a one-soliton in Eq. (10) with a “phase shift” \( c_{11} \) and worth mentioning that the “phase shift” for usual two-solitons is a constant, but “phase shift” \( c_{11} \) is the undetermined function of \( x \) and \( t \) which will be given later. In order to get the “phase shift” \( c_{11} \), substituting (15) in (14) by a simple calculation and just considering the corresponding approximation of this equation in the neighborhood of \( H = 0 \) when \(|t| \to \infty\), then it yields

\[ 16777216\alpha_1^8 \beta_1^8 (\alpha_1^2 + \beta_1^2)^4 t^4 + e^{16\alpha_1 \beta_1 c_{11}} \]
\[ + e^{-16\alpha_1 \beta_1 c_{11}} - 16\alpha_1^8 + 6\alpha_1^4 - 4\alpha_1^2 \beta_1^2 + 6\beta_1^4 \approx 0. \]

Solving the above equation, then \( c_{11} \approx \frac{\ln(t^4)}{16\alpha_1 \beta_1} \). So Eq. (14) also holds when \( c_{11} \) is replaced by a simplified form of approximation \( c_{11} = \frac{\ln(t^4)}{16\alpha_1 \beta_1} \) as \(|t| \to \infty\), which implies Eq. (13).  

**Remark 3.1** In Proposition 2, the “phase shift” of positon \( c_{11} \) is a function of \( t \) which is different from the “phase shift” of soliton solution that is usually a constant. It is connected with \( \ln(t^4) \), differing from the focusing mKdV equation (see Ref. [11]) and the complex mKdV equation (see Ref. [12]). The “phase shift” of the focusing mKdV equation and the complex mKdV equation is related to the \( \ln(t^2) \).
Generally, as for the larger $n$, the more complicated and interesting positon solution is obtained by similar method. In the next, because of the complexity of exact form of the four-positon soliton, we do not write down its explicit expression, but we have plotted it in Fig. 3 and the higher order will not display because of intricacy.

**Proposition 3** As $|t| \to \infty$, the modulus square of a three-positon solution of the DNLS equation is decomposed as following form:

$$|q_3 - p|^2 \approx |q_1 - s\left(H + \frac{\ln(t^{16})}{16\alpha_1\beta_1}\right)|^2$$

$$+ |q_1 - s(H)|^2 + |q_1 - s\left(H - \frac{\ln(t^{16})}{16\alpha_1\beta_1}\right)|^2$$

A more precise approximate trajectory are three curves defined by $H \pm \frac{\ln(t^{16})}{16\alpha_1\beta_1} = 0$ and $H = 0$ ($H = 4\alpha_1^2t - 4\beta_1^2t + x$).

**Proof** Similar to Proposition 2, we take a similar decomposition of three-positon solution into account, i.e., suppose

$$|q_3 - p|^2 \approx |q_1 - s(H + c_{22})|^2 + |q_1 - s(H)|^2$$

$$+ |q_1 - s(H - c_{22})|^2$$

Fig. 1 The evolution of two-positon $|q_2 - p|^2$ with the parameter $\alpha_1 = 1$ and $\beta_1 = 0.8$ of the DNLS equation on $(x, t)$-plane. a 3D plot, b the density plot, c the trajectory and the decomposition of two-positon solution with $c_{11}$. Three blue dots line in c denote extreme values, two maxima denote the trajectories, but one minima (middle) is not. Two dashed lines (green, upper; red, lower) denote approximate trajectories. (Color figure online)
is correct when $|t| \to \infty$. Certainly, “phase shift” $c_{22}$ is an undetermined function of function $x$ and $t$ as before, and $q_{1-s}(H \pm c_{22})$ is obtained by a simple replacement of $c_{11}$ in Eq. (15). Substituting $q_{1-s}(H + c_{22}), q_{1-s}(H)$ and $q_{1-s}(H + c_{22})$ in Eq. (18) and considering the corresponding approximation in the neighborhood of $H = 0$, then it yields

$$e^{(16\alpha_1 \beta_1 c_{22})} + e^{-16\alpha_1 \beta_1 c_{22}} - \beta_1^{32} \alpha_1^3 (\alpha_1^2 + \beta_1^2)^{14} 16 \approx 0$$

After that, $c_{22} \approx \ln \left( \frac{\ln(\alpha_1^{16})}{16\alpha_1 \beta_1} \right)$ is an approximate solution of the above equation by simple calculation.

Substituting a simple form of $c_{22}$, i.e., $c_2 = \frac{\ln(\alpha_1^{16})}{16\alpha_1 \beta_1}$, in Eq. (18), infers Eq. (17).

**Remark 3.2** From Proposition 3, it is easy to see the “phase shift” is equivalent to $\ln(\alpha_1^{16})$ when $t \to \infty$, differing from the focusing mKdV equation (see Ref. [11]) and the complex mKdV equation (see Ref. [12]). The “phase shift” of the focusing mKdV equation and the complex mKdV equation is related to the $\ln(t)$.

It is trivial to know that the “phase shift” in Proposition 2 and Proposition 3 is equivalent to $\ln(t)$ that are different from the ones of the focusing mKdV equation and the complex mKdV equation which are equivalent to $\ln(t^2)$ when $t \to \infty$. Furthermore, even $t$ is small, the more precise forms of the approximations about trajectories of positons are provided, i.e., $c_{ij}$, there is very explicit expression in Figs. 1c and 2c.

The trajectories of positon are given precisely in Figs. 1c and 2c, and the line of the extreme maximum
is plotted by blue dots. Moreover, in process of the decomposition, the approximate trajectories of positons are defined by $H \pm c_{ij} = 0$ which are plotted in Figs. 1c and 2c by green and red dashed lines. It is worth mentioning that the black line is plotted in Fig. 2c in the middle which is also trajectory of the three-positon solution that is $H = 0$. More importantly, the formulas of “phase shift” $c_{ij}$ have been calculated explicitly in Propositions 2 and 3.

4 Combinations of solitons and positons

In this section, we will discuss the hybrid of the soliton solutions and the positons solutions of DNLS equation. We omit the formulas which are indeed calculated complexly and the discussion about higher-order mixed solutions of the DNLS equation.

The $n$-positon solution is obtained after performing the higher-order Taylor expansion with $\lambda_j \to \lambda_1$ in the $N$-soliton solution. And two-soliton can degenerate a positon; if some of $\lambda_j \to \lambda_1$ and others keep the original forms, namely not performing limit, then we can get mixed solutions about positons and solutions. Let us start from the lower order:

Case 1: $n = 3$, we can get combination of one-soliton and two-positon solutions (see Fig. 4) with $\lambda_3 \to \lambda_1$ and $\lambda_5$ remain unchanged.

Case 2: $n = 4$, the combinations of two-positon and two-positon solution (see Fig. 5) can be obtained with $\lambda_3 \to \lambda_1$, $\lambda_7 \to \lambda_5$, but the mixed solutions of one-soliton and three-positon solution are obtained (see Fig. 6) with $\lambda_3 \to \lambda_1$, $\lambda_5 \to \lambda_1$ and $\lambda_7$ remain unchanged.

Remark Figure 4a displays the hybrid of one-soliton and two-positon solutions, and Fig. 4b is the density plots of solution.
plots of solution; Fig. 5a displays the hybrid of two-positon and two-positon solutions, and Fig. 5b is the density plots of solution; and Fig. 6a displays the hybrid of one-soliton and three-positon solutions, and Fig. 6b is the density plots of solution. From the picture, the main features of these new solutions are the following: The solutions are smooth, and after the collision, two-positons do not suffer a change of shape nor experience any asymptotic shift of phase. Asymptotically, the soliton comes out of the collision with positon without any “phase shift,” but the positon gains two different phase shifts expressed in terms of the spectral parameters. The study of the solution will certainly enrich the theory of the DNLS equation. It is reasonable to suspect that the smooth mixed solutions of the positons and solitons of nature will likely exist for other nonlinear evolution equations as well.

5 Conclusions

From our study, the $n$-order smooth positon solutions of DNLS equation is provided explicitly by means of Taylor expansion in the corresponding determinant representation of the multi-soliton solution. Furthermore, we analyzed the crucial properties of positon solutions of DNLS equation from the following three points of view: the decomposition, the approximate trajectories and the phase shift. From Figs. 1, 2 and 3, it is easy to see that the two-positon, three-positon, four-positon are not traveling wave, and the trajectory of positons is not a straight line, that is to say, it is a slowly changing curve.

It is worth mentioning that the “phase shift” of the decomposition is different from focusing mKdV equation and complex mKdV equation. In the early
stage of research on positons for complex mKdV equation, the distance of two peaks in two-positon of the complex mKdV equation is $2c_{11} \approx \frac{1}{2} \ln \left( \frac{\ln(4\alpha^2)}{2} \right)$. As for focusing mKdV equation, the distance of two peaks in two-positon of the focusing mKdV equation is $2c_{11} \approx \frac{1}{2} \ln \left( \frac{\ln(4\alpha^2)}{2} \right)$. While the distance of two peaks in two-positon of the DNLS equation is $2c_{11} \approx \frac{1}{8\alpha_1^2 \beta_1^2 (\alpha_1^2 + \beta_1^2)^2}$ which is different from them. In our paper, the “phase shift” is equivalent to $\ln(t^4)$ when $t$ is very large, while the “phase shift” of focusing mKdV equation and complex mKdV equation is equivalent to $\ln(t^2)$. It is new finding and has never reported in studies before.

It is easy to see that the smooth $n$-positon soliton is a expression which is a mixture of polynomials and hyperbolic functions where the similarity is about the multi-pole solutions of the mKdV equation [46–50] and the NLS equation [51], which were reported by means of the Hirota method and the classical inverse scattering method in the past three decades. Comparing our findings with the previous results, we come to the following conclusions: (1) The expression of $n$-positon solution is obtained simply and accurately in Eq. (11); (2) it provides delicate and direct process for the decomposition of the modulus square in Eqs. (13) and (17), i.e., decomposing the multi-positon solutions into single solutions and the formulas of “phase shift” and trajectories are shown precisely as well. The corresponding soliton trajectories are revealed clearly in Figs. 1 and 2.

We also discuss the combination of the solutions and positons. The interaction of positons and solitons can been seen from Figs. 4, 5 and 6. As for two-positons, in the process of its interaction, the positons are not affected by the shape change or they do not suffer a change of any asymptotic shift of phase, which is a known characteristic to solitons. Positons are completely transparent to solitons, and vice versa, the positons are slightly altered by the solitons in a predictable way. During the soliton–positon collision, the soliton remains unchanged, while the carrier wave and the envelop of positon experience “phase shift,” which is keeping with the result of the Refs. [2,3]. A detailed analysis of the dynamical evolution of the degenerate solution of NLS has already been developed in Ref. [52]. Wang et al. [53] investigate a special kind of breather solution of the NLS equation, called breather-positon (or b-positon). The methods are similar to the means of the positons and worthy to study the b-positon solution of DNLS equation in the near future.

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**Compliance with ethical standards**

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