Devinatz’s moment problem: a description of all solutions.

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1 Introduction.

We shall study the following problem: to find a non-negative Borel measure $\mu$ in a strip

$$\Pi = \{(x, \varphi): x \in \mathbb{R}, -\pi \leq \varphi < \pi\},$$

such that

$$\int_{\Pi} x^m e^{in\varphi} d\mu = s_{m,n}, \quad m \in \mathbb{Z}_+, n \in \mathbb{Z},$$

where $\{s_{m,n}\}_{m \in \mathbb{Z}_+, n \in \mathbb{Z}}$ is a given sequence of complex numbers. We shall refer to this problem as to the Devinatz moment problem.

A. Devinatz was the first who introduced and studied this moment problem [1]. He obtained the necessary and sufficient conditions of solvability for the moment problem (1) and gave a sufficient condition for the moment problem to be determinate [1, Theorem 4].

Our aim here is threefold. Firstly, we present a new proof of the Devinatz solvability criterion. Secondly, we describe canonical solutions of the Devinatz moment problem (see the definition below). Finally, we describe all solutions of the Devinatz moment problem. We shall use an abstract operator approach [2] and results of Godič, Lucenko and Shtraus [3, 4, Theorem 1, 5].

**Notations.** As usual, we denote by $\mathbb{R}, \mathbb{C}, \mathbb{N}, \mathbb{Z}, \mathbb{Z}_+$ the sets of real numbers, complex numbers, positive integers, integers and non-negative integers, respectively. For a subset $S$ of the complex plane we denote by $\mathcal{B}(S)$ the set of all Borel subsets of $S$. Everywhere in this paper, all Hilbert spaces are assumed to be separable. By $(\cdot, \cdot)_H$ and $\| \cdot \|_H$ we denote the scalar product and the norm in a Hilbert space $H$, respectively. The indices may be omitted in obvious cases. For a set $M$ in $H$, by $\overline{M}$ we mean the closure of $M$ in the norm $\| \cdot \|_H$. For $\{x_k\}_{k \in T}, x_k \in H$, we write $\text{Lin}\{x_k\}_{k \in T}$ for the set of linear combinations of vectors $\{x_k\}_{k \in T}$ and $\text{span}\{x_k\}_{k \in T} = \overline{\text{Lin}\{x_k\}_{k \in T}}$. Here $T := \mathbb{Z}_+ \times \mathbb{Z}$, i.e. $T$ consists of pairs $(m, n)$, $m \in \mathbb{Z}_+$, $n \in \mathbb{Z}$. The identity operator in $H$ is denoted by $E$. For an arbitrary linear operator $A$ in $H$, the operators $A^*, \overline{A}, A^{-1}$ mean its adjoint operator, its closure and its inverse (if they exist). By $D(A)$ and $R(A)$ we mean the domain and the range of the operator $A$. By $\sigma(A)$, $\rho(A)$ we denote the spectrum of $A$. 
and the resolvent set of $A$, respectively. We denote by $R_z(A)$ the resolvent function of $A$, $z \in \rho(A)$. The norm of a bounded operator $A$ is denoted by $\|A\|$. By $P_{H_1}^H = P_{H_1}$ we mean the operator of orthogonal projection in $H$ on a subspace $H_1$ in $H$. By $B(H)$ we denote the set of all bounded operators in $H$.

## 2 Solvability.

Let a moment problem (1) be given. Suppose that the moment problem has a solution $\mu$. Choose an arbitrary power-trigonometric polynomial $p(x, \varphi)$ of the following form:

$$
\sum_{m=0}^{\infty} \sum_{n=-\infty}^{\infty} \alpha_{m,n} x^m e^{in\varphi}, \quad \alpha_{m,n} \in \mathbb{C},
$$

where all but finite number of coefficients $\alpha_{m,n}$ are zeros. We can write

$$
0 \leq \int_{\Pi} |p(x, \varphi)|^2 d\mu = \int_{\Pi} \sum_{m=0}^{\infty} \sum_{n=-\infty}^{\infty} \alpha_{m,n} x^m e^{in\varphi} \sum_{k=0}^{\infty} \sum_{l=-\infty}^{\infty} \alpha_{k,l} x^k e^{i(l-n)\varphi} d\mu = \sum_{m,n,k,l} \alpha_{m,n} \alpha_{k,l} x^{m+k} e^{i(n-l)\varphi} d\mu = \sum_{m,n,k,l} \alpha_{m,n} \alpha_{k,l} s_{m+k,n-l}.
$$

Thus, for arbitrary complex numbers $\alpha_{m,n}$ (where all but finite numbers are zeros) we have

$$
\sum_{m=0}^{\infty} \sum_{n=-\infty}^{\infty} \alpha_{m,n} \alpha_{k,l} s_{m+k,n-l} \geq 0.
$$

Let $T = \mathbb{Z} \times \mathbb{Z}_+$ and for $t, r \in T$, $t = (m, n)$, $r = (k, l)$, we set

$$
K(t, r) = K((m, n), (k, l)) = s_{m+k,n-l}.
$$

Thus, for arbitrary elements $t_1, t_2, \ldots, t_n$ of $T$ and arbitrary complex numbers $\alpha_1, \alpha_2, \ldots, \alpha_n$, with $n \in \mathbb{N}$, the following inequality holds:

$$
\sum_{i,j=1}^{n} K(t_i, t_j) \alpha_i \overline{\alpha_j} \geq 0.
$$

The latter means that $K(t, r)$ is a positive matrix in the sense of E.H. Moore [6, p.344].
Suppose now that a Devinatz moment problem is given and conditions (3) (or what is the same conditions (5)) hold. Let us show that the moment problem has a solution. We shall use the following important fact (e.g. [7, pp.361-363]).

**Theorem 2.1** Let \( K = K(t,r) \) be a positive matrix on \( T = \mathbb{Z} \times \mathbb{Z}_+ \). Then there exist a separable Hilbert space \( H \) with a scalar product \((\cdot,\cdot)\) and a sequence \( \{x_t\}_{t \in T} \) in \( H \), such that

\[
K(t,r) = (x_t, x_r), \quad t, r \in T, \tag{6}
\]

and \( \text{span}\{x_t\}_{t \in T} = H \).

**Proof.** Consider an arbitrary infinite-dimensional linear vector space \( V \) (for example, we can choose a space of complex sequences \((u_n)_{n \in \mathbb{N}}, u_n \in \mathbb{C}\)). Let \( X = \{x_t\}_{t \in T} \) be an arbitrary infinite sequence of linear independent elements in \( V \) which is indexed by elements of \( T \). Set \( L_X = \text{Lin}\{x_t\}_{t \in T} \).

Introduce the following functional:

\[
[x,y] = \sum_{t,r \in T} K(t,r) a_t \overline{b_r}, \tag{7}
\]

for \( x, y \in L_X \),

\[
x = \sum_{t \in T} a_t x_t, \quad y = \sum_{r \in T} b_r x_r, \quad a_t, b_r \in \mathbb{C}.
\]

Here all but finite number of indices \( a_t, b_r \) are zeros.

The set \( L_X \) with \([\cdot,\cdot]\) will be a pre-Hilbert space. Factorizing and making the completion we obtain the required space \( H \) ([8, p. 10-11]). \( \square \)

By applying this theorem we get that there exist a Hilbert space \( H \) and a sequence \( \{x_{m,n}\}_{m \in \mathbb{Z}_+, n \in \mathbb{Z}}, \ x_{m,n} \in H \), such that

\[
(x_{m,n}, x_{k,l})_H = K((m,n),(k,l)), \quad m, k \in \mathbb{Z}_+, \ n, l \in \mathbb{Z}. \tag{8}
\]

Set \( L = \text{Lin}\{x_{m,n}\}_{(m,n) \in T} \). We introduce the following operators

\[
A_0 x = \sum_{(m,n) \in T} \alpha_{m,n} x_{m+1,n}, \tag{9}
\]

\[
B_0 x = \sum_{(m,n) \in T} \alpha_{m,n} x_{m,n+1}, \tag{10}
\]
where
\[ x = \sum_{(m,n) \in T} \alpha_{m,n} x_{m,n} \in L. \]  \hspace{1cm} (11)

We should show that these definitions are correct. Indeed, suppose that the element \( x \) in (11) has another representation:
\[ x = \sum_{(k,l) \in T} \beta_{k,l} x_{k,l}. \]  \hspace{1cm} (12)

We can write
\[
\left( \sum_{(m,n) \in T} \alpha_{m,n} x_{m+1,n}, x_{a,b} \right) = \sum_{(m,n) \in T} \alpha_{m,n} K((m + 1, n), (a, b))
\]
\[ = \sum_{(m,n) \in T} \alpha_{m,n} s_{m+1+a,n-b} \sum_{(m,n) \in T} \alpha_{m,n} K((m, n), (a + 1, b)) \]
\[ = \left( \sum_{(m,n) \in T} \alpha_{m,n} x_{m,n}, x_{a+1,b} \right) = (x, x_{a+1,b}), \]
for arbitrary \((a, b) \in T\). In the same manner we get
\[
\left( \sum_{(k,l) \in T} \beta_{k,l} x_{k+1,l}, x_{a,b} \right) = (x, x_{a+1,b}).
\]

Since \( \text{span}\{x_{a,b}\}_{(a,b) \in T} = H \), we get
\[ \sum_{(m,n) \in T} \alpha_{m,n} x_{m+1,n} = \sum_{(k,l) \in T} \beta_{k,l} x_{k+1,l}. \]

Thus, the operator \( A_0 \) is defined correctly.

We can write
\[
\left\| \sum_{(m,n) \in T} (\alpha_{m,n} - \beta_{m,n}) x_{m+1,n+1} \right\| ^2
\]
\[ = \left( \sum_{(m,n) \in T} (\alpha_{m,n} - \beta_{m,n}) x_{m,n+1}, \sum_{(k,l) \in T} (\alpha_{k,l} - \beta_{k,l}) x_{k,l+1} \right) \]
\[ = \sum_{(m,n),(k,l) \in T} (\alpha_{m,n} - \beta_{m,n})(\alpha_{k,l} - \beta_{k,l}) K((m, n + 1), (k, l + 1)) \]
\[
= \sum_{(m,n),(k,l)\in T} (\alpha_{m,n} - \beta_{m,n})(\alpha_{k,l} - \beta_{k,l})K((m,n),(k,l))
\]
\[
= \left( \sum_{(m,n)\in T} (\alpha_{m,n} - \beta_{m,n})x_{m,n}, \sum_{(k,l)\in T} (\alpha_{k,l} - \beta_{k,l})x_{k,l} \right) = 0.
\]

Consequently, the operator \(B_0\) is defined correctly, as well.

Choose an arbitrary \(y = \sum_{(a,b)\in T} \gamma_{a,b}x_{a,b} \in L\). We have
\[
(A_0 x, y) = \sum_{m,n,a,b} \alpha_{m,n} \gamma_{a,b} (x_{m+1,n}, x_{a,b}) = \sum_{m,n,a,b} \alpha_{m,n} \gamma_{a,b} K((m+1,n),(a,b))
\]
\[
= \sum_{m,n,a,b} \alpha_{m,n} \gamma_{a,b} K((m,n),(a+1,b)) = \sum_{m,n,a,b} \alpha_{m,n} \gamma_{a,b} (x_{m,n}, x_{a+1,b}) = (x, A_0 y).
\]

Thus, \(A_0\) is a symmetric operator. Its closure we denote by \(A\). On the other hand, we have
\[
(B_0 x, B_0 y) = \sum_{m,n,a,b} \alpha_{m,n} \overline{\gamma}_{a,b} (x_{m,n+1}, x_{a,b+1}) = \sum_{m,n,a,b} \alpha_{m,n} \overline{\gamma}_{a,b} K((m,n+1),(a,b+1))
\]
\[
= \sum_{m,n,a,b} \alpha_{m,n} \overline{\gamma}_{a,b} K((m,n),(a,b)) = \sum_{m,n,a,b} \alpha_{m,n} \overline{\gamma}_{a,b} (x_{m,n}, x_{a,b}) = (x, y).
\]

In particular, this means that \(B_0\) is bounded. By continuity we extend \(B_0\) to a bounded operator \(B\) such that
\[
(Bx, By) = (x, y), \quad x, y \in H.
\]

Since \(R(B_0) = L\) and \(B_0\) has a bounded inverse, we have \(R(B) = H\). Thus, \(B\) is a unitary operator in \(H\).

Notice that operators \(A_0\) and \(B_0\) commute. It is straightforward to check that \(A\) and \(B\) commute:
\[
ABx = BAx, \quad x \in D(A).
\]

Consider the following operator:
\[
J_0 x = \sum_{(m,n)\in T} \overline{\alpha_{m,n}} x_{m,n},
\]
where
\[
x = \sum_{(m,n)\in T} \alpha_{m,n} x_{m,n} \in L.
\]
Let us check that this definition is correct. Consider another representation for $x$ as in (12). Then

$$
\sum_{(m,n) \in T} (\alpha_{m,n} - \beta_{m,n}) x_{m,-n} \geq 0
$$

Thus, the definition of $J_0$ is correct. For an arbitrary $y = \sum_{(a,b) \in T} \gamma_{a,b} x_{a,b} \in L$ we can write

$$(J_0 x, J_0 y) = \sum_{m,n,a,b} \alpha_{m,n} \gamma_{a,b} x_{m,-n, x_{a,-b}} = \sum_{m,n,a,b} \overline{\alpha_{m,n}} \gamma_{a,b} K((m, -n), (a, -b))$$

In particular, this implies that $J_0$ is bounded. By continuity we extend $J_0$ to a bounded antilinear operator $J$ such that

$$(Jx, y) = (y, x), \quad x, y \in H.$$

Moreover, we get $J^2 = E_H$. Consequently, $J$ is a conjugation in $H$ (9). Notice that $J_0$ commutes with $A_0$. It is easy to check that

$$AJx = JAx, \quad x \in D(A). \quad (16)$$

On the other hand, we have $J_0 B_0 = B_0^{-1} J_0$. By continuity we get

$$JB = B^{-1} J. \quad (17)$$
Consider the Cayley transformation of the operator $A$:

$$V_A := (A + iE_H)(A - iE_H)^{-1};$$

and set

$$H_1 := \Delta_A(i), \ H_2 := H \cap H_1, \ H_3 := \Delta_A(-i), \ H_4 := H \cap H_3. \tag{19}$$

**Proposition 2.1** The operator $B$ reduces subspaces $H_i$, $1 \leq i \leq 4$:

$$BH_i = H_i, \quad 1 \leq i \leq 4. \tag{20}$$

Moreover, the following equality holds:

$$BV_Ax = V_Ax, \quad x \in H_1. \tag{21}$$

**Proof.** Choose an arbitrary $x \in \Delta_A(z)$, $x = (A - zE_H)f_A$, $f_A \in D(A)$, $z \in \mathbb{C} \setminus \mathbb{R}$. By (13) we get

$$Bx = BAF_A - zBf_A = ABf_A - zBf_A = (A - zE_H)Bf_A \in \Delta_A(z).$$

In particular, we have $BH_1 \subseteq H_1$, $BH_3 \subseteq H_3$. Notice that $B_0^{-1}A_0 = A_0B_0^{-1}$. It is a straightforward calculation to check that

$$AB^{-1}x = B^{-1}Ax, \quad x \in D(A). \tag{22}$$

Repeating the above argument with $B^{-1}$ instead of $B$ we get $B^{-1}H_1 \subseteq H_1$, $B^{-1}H_3 \subseteq H_3$, and therefore $H_1 \subseteq BH_1$, $H_3 \subseteq BH_3$. Consequently, the operator $B$ reduces subspaces $H_1$ and $H_3$. It follows directly that $B$ reduces $H_2$ and $H_4$, as well.

Since

$$(A - iE_H)Bx = B(A - iE_H)x, \quad x \in D(A),$$

for arbitrary $y \in H_1$, $y = (A - iE_H)x_A$, $x_A \in D(A)$, we have

$$(A - iE_H)B(A - iE_H)^{-1}y = By;$$

$$B(A - iE_H)^{-1}y = (A - iE_H)^{-1}By, \quad y \in H_1,$$

and (21) follows. $\square$

Our aim here is to construct a unitary operator $U$ in $H$, $U \supset V_A$, which commutes with $B$. Choose an arbitrary $x \in H$, $x = x_{H_1} + x_{H_2}$. For an operator $U$ of the required type by Proposition 2.1 we could write:

$$BUx = BV_Ax_{H_1} + BUx_{H_2} = V_ABx_{H_1} + BUx_{H_2},$$
\[
UBx = UBx_{H_1} + UBx_{H_2} = VAx_{H_1} + UBx_{H_2}.
\]

So, it is enough to find an isometric operator \(U_{2,4}\) which maps \(H_2\) onto \(H_4\), and commutes with \(B\):

\[
BU_{2,4}x = U_{2,4}Bx, \quad x \in H_2.
\]

Moreover, all operators \(U\) of the required type have the following form:

\[
U = V_A \oplus U_{2,4},
\]

where \(U_{2,4}\) is an isometric operator which maps \(H_2\) onto \(H_4\), and commutes with \(B\).

We shall denote the operator \(B\) restricted to \(H_i\) by \(B_{H_i}, 1 \leq i \leq 4\). Notice that

\[
A^* Jx = JA^* x, \quad x \in D(A^*).
\]

Indeed, for arbitrary \(f_A \in D(A)\) and \(g_{A^*} \in D(A^*)\) we can write

\[
(\langle Af_A, Jg_{A^*} \rangle) = (\langle JAf_A, g_{A^*} \rangle) = (\langle Jf_A, A^* g_{A^*} \rangle) = (\langle f_A, JA^* g_{A^*} \rangle),
\]

and (25) follows.

Choose an arbitrary \(x \in H_2\). We have

\[
A^* x = -ix,
\]

and therefore

\[
A^* Jx = JA^* x = ix.
\]

Thus, we have

\[
JH_2 \subseteq H_4.
\]

In a similar manner we get

\[
JH_4 \subseteq H_2,
\]

and therefore

\[
JH_2 = H_4, \quad JH_4 = H_2.
\]

(26)

By the Godič-Lucenko Theorem ([3],[4, Theorem 1]) we have a representation:

\[
B_{H_2} = KL,
\]

(27)

where \(K\) and \(L\) are some conjugations in \(H_2\). We set

\[
U_{2,4} := JK.
\]

(28)
From (26) it follows that $U_{2,4}$ maps isometrically $H_2$ onto $H_4$. Notice that

$$U_{2,4}^{-1} := KJ.$$  \hspace{1cm} (29)

Using relation (17) we get

$$U_{2,4}B_{H_2}U_{2,4}^{-1} = J K K L K J x = J L K J x = J B_{H_2}^{-1} J x = J B_{H_2}^{-1} J x = B x = B_{H_4} x, \quad x \in H_4.$$ 

Therefore relation (23) is true.

We define an operator $U$ by (24) and define

$$A_U := i(U + E_H)(U - E_H)^{-1} = iE_H + 2i(U - E_H)^{-1}. \tag{30}$$

The inverse Cayley transformation $A_U$ is correctly defined since 1 is not in the point spectrum of $U$. Indeed, $V_A$ is the Cayley transformation of a symmetric operator while eigen subspaces $H_2$ and $H_4$ have the zero intersection.

Let

$$A_U = \int_{\mathbb{R}} s dE(s), \quad B = \int_{(-\pi,\pi)} e^{i\varphi} dF(\varphi), \tag{31}$$

where $E(s)$ and $F(\varphi)$ are the spectral measures of $A_U$ and $B$, respectively. These measures are defined on $\mathcal{B}(\mathbb{R})$ and $\mathcal{B}([-\pi,\pi])$, respectively (10). Since $U$ and $B$ commute, we get that $E(s)$ and $F(\varphi)$ commute, as well. By induction argument we have

$$x_{m,n} = A^m x_{0,n}, \quad m \in \mathbb{Z}_+, n \in \mathbb{Z},$$

and

$$x_{0,n} = B^n x_{0,0}, \quad n \in \mathbb{Z}.$$ 

Therefore we have

$$x_{m,n} = A^m B^n x_{0,0}, \quad m \in \mathbb{Z}_+, n \in \mathbb{Z}. \tag{32}$$

We can write

$$x_{m,n} = \int_{\mathbb{R}} s^m dE(s) \int_{[-\pi,\pi]} e^{in\varphi} dF(\varphi) x_{0,0} = \int_{\Pi} s^m e^{in\varphi} d(E \times F) x_{0,0},$$

where $E \times F$ is the product spectral measure on $\mathcal{B}(\Pi)$. Then

$$s_{m,n} = (x_{m,n}, x_{0,0})_H = \int_{\Pi} s^m e^{in\varphi} d((E \times F) x_{0,0}, x_{0,0})_H, \quad (m,n) \in T. \tag{33}$$

The measure $\mu := ((E \times F)x_{0,0}, x_{0,0})_H$ is a non-negative Borel measure on $\Pi$ and relation (33) shows that $\mu$ is a solution of the Devinatz moment problem.

Thus, we obtained a new proof of the following criterion.
Theorem 2.2 Let a Devinatz moment problem (1) be given. This problem has a solution if and only if conditions (3) hold for arbitrary complex numbers $\alpha_{m,n}$ such that all but finite numbers are zeros.

Remark. The original proof of Devinatz used the theory of reproducing kernels Hilbert spaces (RKHS). In particular, he used properties of RKHS corresponding to the product of two positive matrices and an inner structure of a RKHS corresponding to the moment problem. We used an abstract approach with the Gobič-Lucenko Theorem and basic facts from the standard operator theory.

3 Canonical solutions. A set of all solutions.

Let a moment problem (1) be given. Construct a Hilbert space $H$ and operators $A, B, J$ as in the previous Section. Let $\tilde{A} \supseteq A$ be a self-adjoint extension of $A$ in a Hilbert space $\tilde{H} \supseteq H$. Let $R_z(\tilde{A}), z \in \mathbb{C}\setminus\mathbb{R}$, be the resolvent function of $\tilde{A}$, and $E_{\tilde{A}}$ be its spectral measure. Recall that the function

$$R_z(A) := P_H^H R_z(\tilde{A}), \quad z \in \mathbb{C}\setminus\mathbb{R},$$

is said to be a generalized resolvent of $A$. The function

$$E_A(\delta) := P_H^H E_{\tilde{A}}(\delta), \quad \delta \in \mathcal{B}(\mathbb{R}),$$

is said to be a spectral measure of $A$. There exists a one-to-one correspondence between generalized resolvents and spectral measures established by the following relation (7):

$$(R_z(A)x, y)_H = \int_{\mathbb{R}} \frac{1}{t - z} d(E_Ax, y)_H, \quad x, y \in H.$$  

We shall reduce the Devinatz moment problem to a problem of finding of generalized resolvents of a certain class.

Theorem 3.1 Let a Devinatz moment problem (1) be given and conditions (3) hold. Consider a Hilbert space $H$ and a sequence $\{x_{m,n}\}_{m \in \mathbb{Z}, n \in \mathbb{Z}}$, $x_{m,n} \in H$, such that relation (3) holds where $K$ is defined by (4). Consider operators $A_0, B_0$ defined by (9), (10) on $L = \text{Lin}\{x_{m,n}\}_{(m,n) \in T}$. Let $A = \overline{A_0}$, $B = \overline{B_0}$. Let $\mu$ be an arbitrary solution of the moment problem. Then it has the following form:

$$\mu(\delta) = ((E \times F)(\delta)x_{0,0}, x_{0,0})_H, \quad \delta \in \mathcal{B}(\mathbb{R}),$$

10
where \( F \) is the spectral measure of \( B \), \( E \) is a spectral measure of \( A \) which commutes with \( F \). By \((\mathbf{E} \times F)(\delta_{x_0,0},\delta_{x_0,0})_H\) we mean the non-negative Borel measure on \( \mathbb{R} \) which is obtained by the Lebesgue continuation procedure from the following non-negative measure on rectangles

\[
((\mathbf{E} \times F)(I_x \times I_\varphi))_{x_0,0,0,0})_H := (\mathbf{E}(I_x)F(I_\varphi))_{x_0,0,0,0})_H, \tag{38}
\]

where \( I_x \subset \mathbb{R}, I_\varphi \subset [-\pi, \pi) \) are arbitrary intervals.

On the other hand, for an arbitrary spectral measure \( E \) of \( A \) which commutes with the spectral measure \( F \) of \( B \), by relation (37) it corresponds a solution of the moment problem (1).

Moreover, the correspondence between the spectral measures of \( A \) which commute with the spectral measure of \( B \) and solutions of the Devinatz moment problem is bijective.

**Remark.** The measure in (38) is non-negative. Indeed, for arbitrary intervals \( I_x \subset \mathbb{R}, I_\varphi \subset [-\pi, \pi) \), we can write

\[
(\mathbf{E}(I_x)F(I_\varphi))_{x_0,0,0,0})_H = (F(I_\varphi)\mathbf{E}(I_x)F(I_\varphi)_{x_0,0,0,0})_H
\]

\[
= (\mathbf{E}(I_x)F(I_\varphi))_{x_0,0,0,0})_H = (\widehat{E}(I_x)F(I_\varphi))_{x_0,0,0,0})_H \geq 0,
\]

where \( \widehat{E} \) is the spectral function of a self-adjoint extension \( \widehat{A} \supset A \) in a Hilbert space \( \widehat{H} \supset H \) such that \( E = P_{\widehat{H}}\widehat{E} \). The measure in (38) is additive.

If \( I_\varphi = I_{1,\varphi} \cup I_{2,\varphi}, I_{1,\varphi} \cap I_{2,\varphi} = \emptyset \), then

\[
(\mathbf{E}(I_x)F(I_\varphi))_{x_0,0,0,0})_H = (F(I_{1,\varphi} \cup I_{2,\varphi})\mathbf{E}(I_x)_{x_0,0,0,0})_H
\]

\[
= (F(I_{1,\varphi})\mathbf{E}(I_x)_{x_0,0,0,0})_H + (F(I_{2,\varphi})\mathbf{E}(I_x)_{x_0,0,0,0})_H.
\]

The case \( I_x = I_{1,x} \cup I_{2,x} \) is analogous. Moreover, repeating the standard arguments [11] Chapter 5, Theorem 2, p. 254-255] we conclude that the measure in (38) is \( \sigma \)-additive. Thus, it possesses the (unique) Lebesgue continuation to a (finite) non-negative Borel measure on \( \Pi \).

**Proof.** Consider a Hilbert space \( H \) and operators \( A,B \) as in the statement of the Theorem. Let \( F \) be the spectral measure of \( B \). Let \( \mu \) be an arbitrary solution of the moment problem (1). Consider the space \( \mathbb{L}_\mu^2 \) of complex functions on \( \Pi \) which are square integrable with respect to the measure \( \mu \). The scalar product and the norm are given by

\[
(f,g)_\mu = \int_\Pi f(x,\varphi)g(x,\varphi)d\mu, \quad \|f\|_\mu = \left( (f,f)_\mu \right)^{\frac{1}{2}}, \quad f, g \in \mathbb{L}_\mu^2.
\]
Consider the following operators:

\begin{align}
A_\mu f(x, \varphi) &= xf(x, \varphi), \quad D(A_\mu) = \{ f \in L^2_\mu : xf(x, \varphi) \in L^2_\mu \}, \\
B_\mu f(x, \varphi) &= e^{i\varphi}f(x, \varphi), \quad D(B_\mu) = L^2_\mu.
\end{align}

The operator \( A_\mu \) is self-adjoint and the operator \( B_\mu \) is unitary. Moreover, these operators commute and therefore the spectral measure \( E_\mu \) of \( A_\mu \) and the spectral measure \( F_\mu \) of \( B_\mu \) commute, as well.

Let \( p(x, \varphi) \) be a (power-trigonometric) polynomial of the form \( \Pi \) and \( q(x, \varphi) \) be a (power-trigonometric) polynomial of the form \( \Pi \) with \( \beta_{m,n} \in \mathbb{C} \) instead of \( \alpha_{m,n} \). Then

\[
(p, q)_\mu = \sum_{(m,n) \in T, (k,l) \in T} \alpha_{m,n} \beta_{k,l} \int_\Pi x^{m+k} e^{i(n-l)\varphi} d\mu
\]

On the other hand, we can write

\[
\left( \sum_{(m,n) \in T} \alpha_{m,n} x_{m,n}, \sum_{(k,l) \in T} \beta_{k,l} x_{k,l} \right)_H = \sum_{(m,n) \in T, (k,l) \in T} \alpha_{m,n} \beta_{k,l} (x_{m,n}, x_{k,l})_H
\]

\[
= \sum_{(m,n) \in T, (k,l) \in T} \alpha_{m,n} \beta_{k,l} K((m,n), (k,l)) = \sum_{(m,n) \in T, (k,l) \in T} \alpha_{m,n} \beta_{k,l} x_{m+k, n-l}.
\]

Therefore

\[
(p, q)_\mu = \left( \sum_{(m,n) \in T} \alpha_{m,n} x_{m,n}, \sum_{(k,l) \in T} \beta_{k,l} x_{k,l} \right)_H.
\]

Consider the following operator:

\[
V[p] = \sum_{(m,n) \in T} \alpha_{m,n} x_{m,n}, \quad p = \sum_{(m,n) \in T} \alpha_{m,n} x^m e^{in\varphi}.
\]

Here by \( [p] \) we mean the class of equivalence in \( L^2_\mu \) defined by \( p \). If two different polynomials \( p \) and \( q \) belong to the same class of equivalence then by \( \Pi \) we get

\[
0 = \|p-q\|_\mu^2 = (p-q, p-q)_\mu = \left( \sum_{(m,n) \in T} (\alpha_{m,n} - \beta_{m,n}) x_{m,n}, \sum_{(k,l) \in T} (\alpha_{k,l} - \beta_{k,l}) x_{k,l} \right)_H.
\]
Thus, the definition of $V$ is correct. It is not hard to see that $V$ maps a set of polynomials $P^2_{0, \mu}$ in $L^2_{\mu}$ on $L$. By continuity we extend $V$ to the isometric transformation from the closure of polynomials $P^2_{\mu} = \overline{P^2_{0, \mu}}$ onto $H$.

Set $H_0 := L^2_{\mu} \oplus P^2_{\mu}$. Introduce the following operator:

$$U := V \oplus E_{H_0},$$

which maps isometrically $L^2_{\mu}$ onto $H := H \oplus H_0$. Set

$$\tilde{A} := U A_{\mu} U^{-1}, \quad \tilde{B} := U B_{\mu} U^{-1}. \tag{43}$$

Notice that

$$\tilde{A} x_{m,n} = A_{\mu} U^{-1} x_{m,n} = A_{\mu} x^m e^{i\varphi} = x_{m+1,n},$$

$$\tilde{B} x_{m,n} = B_{\mu} U^{-1} x_{m,n} = B_{\mu} x^m e^{i\varphi} = x_{m,n+1}.\tag{44}$$

Therefore $\tilde{A} \supseteq A$ and $\tilde{B} \supseteq B$. Let

$$\tilde{A} = \int_{\mathbb{R}} s d\tilde{E}(s), \quad \tilde{B} = \int_{[-\pi, \pi]} e^{i\varphi} d\tilde{F}(\varphi), \tag{45}$$

where $\tilde{E}(s)$ and $\tilde{F}(\varphi)$ are the spectral measures of $\tilde{A}$ and $\tilde{B}$, respectively. Repeating arguments after relation (31) we obtain that

$$x_{m,n} = \tilde{A}^m \tilde{B}^n x_{0,0}, \quad m \in \mathbb{Z}_+, \ n \in \mathbb{Z}, \tag{46}$$

$$s_{m,n} = \int_{\mathbb{R}} s^m e^{i\varphi} d((\tilde{E} \times \tilde{F}) x_{0,0}, x_{0,0})_{\tilde{H}}, \quad (m, n) \in T, \tag{47}$$

where $(\tilde{E} \times \tilde{F})$ is the product measure of $\tilde{E}$ and $\tilde{F}$. Thus, the measure $\tilde{\mu} := ((\tilde{E} \times \tilde{F}) x_{0,0}, x_{0,0})_{\tilde{H}}$ is a solution of the Devinatz moment problem.

Let $I_x \subset \mathbb{R}$, $I_{\varphi} \subset [-\pi, \pi]$ be arbitrary intervals. Then

$$\tilde{\mu}(I_x \times I_{\varphi}) = ((\tilde{E} \times \tilde{F})(I_x \times I_{\varphi}) x_{0,0}, x_{0,0})_{\tilde{H}}$$

$$= (\tilde{E}(I_x) \tilde{F}(I_{\varphi}) x_{0,0}, x_{0,0})_{\tilde{H}} = (\tilde{P}^H_{\tilde{H}} \tilde{E}(I_x) \tilde{F}(I_{\varphi}) x_{0,0}, x_{0,0})_{\tilde{H}}$$

$$= (\tilde{E}(I_x) \tilde{F}(I_{\varphi}) x_{0,0}, x_{0,0})_{\tilde{H}},$$




where $E$ is the corresponding spectral function of $A$ and $F$ is the spectral function of $B$. Thus, the measure $\tilde{\mu}$ has the form \[37\] since the Lebesgue continuation is unique.

Let us show that $\tilde{\mu} = \mu$. Consider the following transformation:

$$S : (x, \varphi) \in \Pi \mapsto \left( \text{Arg} \frac{x - i}{x + i}, \varphi \right) \in \Pi_0,$$

(48)

where $\Pi_0 = [-\pi, \pi) \times [-\pi, \pi)$ and $\text{Arg} e^{iy} = y \in [-\pi, \pi)$. By virtue of $V$ we define the following measures:

$$\mu_0(VG) := \mu(G), \quad \bar{\mu}_0(VG) := \tilde{\mu}(G), \quad G \in \mathcal{B}(\Pi),$$

(49)

It is not hard to see that $\mu_0$ and $\bar{\mu}_0$ are non-negative measures on $\mathcal{B}(\Pi_0)$. Then

$$\int_{\Pi} \left( \frac{x - i}{x + i} \right)^m e^{in\varphi} d\mu = \int_{\Pi_0} e^{in\varphi} e^{im\varphi} d\mu_0,$$

(50)

$$\int_{\Pi} \left( \frac{x - i}{x + i} \right)^m e^{in\varphi} d\tilde{\mu} = \int_{\Pi_0} e^{in\varphi} e^{im\varphi} d\tilde{\mu}_0, \quad m, n \in \mathbb{Z};$$

(51)

and

$$\int_{\Pi} \left( \frac{x - i}{x + i} \right)^m e^{in\varphi} d\mu = \int_{\Pi} \left( \frac{x - i}{x + i} \right)^m e^{in\varphi} d((\tilde{\mathcal{E}} \times \tilde{\mathcal{F}}) x_{0,0}, x_{0,0}) \tilde{H}$$

$$= \left( \int_{\Pi} \left( \frac{x - i}{x + i} \right)^m e^{in\varphi} d((\tilde{\mathcal{E}} \times \tilde{\mathcal{F}}) x_{0,0}, x_{0,0}) \right) \tilde{H}$$

$$= \left( \int_{\mathbb{R}} \left( \frac{x - i}{x + i} \right)^m d\tilde{\mathcal{E}} \int_{[-\pi, \pi)} e^{in\varphi} d\tilde{\mathcal{F}} x_{0,0}, x_{0,0} \right) \tilde{H}$$

$$= \left( \left( (\tilde{A} - i\tilde{E}_\mu)(\tilde{A} + i\tilde{E}_\mu)^{-1} \right)^m \tilde{B}^n x_{0,0}, x_{0,0} \right) \tilde{H}$$

$$= \left( (A_{\mu} - iE_{L_{\mu}^2})(A_{\mu} + iE_{L_{\mu}^2})^{-1} \right)^m B^n U_{1,1} \mu$$

$$= \left( (A_{\mu} - iE_{L_{\mu}^2})(A_{\mu} + iE_{L_{\mu}^2})^{-1} \right)^m B^n U_{1,1} \mu$$

(52)

By virtue of relations \[50\], \[51\] and \[52\] we get

$$\int_{\Pi_0} e^{im\varphi} e^{in\varphi} d\mu_0 = \int_{\Pi_0} e^{im\varphi} e^{in\varphi} d\tilde{\mu}_0, \quad m, n \in \mathbb{Z}.$$

(53)
By the Weierstrass theorem we can approximate any continuous function by exponentials and therefore

\[ \int_{\Pi_0} f(\psi)g(\varphi)d\mu_0 = \int_{\Pi_0} f(\psi)g(\varphi)d\tilde{\mu}_0, \]

for arbitrary continuous functions on \( \Pi_0 \). In particular, we have

\[ \int_{\Pi_0} \psi^n \varphi^m d\mu_0 = \int_{\Pi_0} \psi^n \varphi^m d\tilde{\mu}_0, \quad n, m \in \mathbb{Z}_+. \]

However, the two-dimensional Hausdorff moment problem is determinate \((12)\) and therefore we get \( \mu_0 = \tilde{\mu}_0 \) and \( \mu = \mu_0 \). Thus, we have proved that an arbitrary solution \( \mu \) of the Devinatz moment problem can be represented in the form \((37)\).

Let us check the second assertion of the Theorem. For an arbitrary spectral measure \( E \) of \( A \) which commutes with the spectral measure \( F \) of \( B \), by relation \((37)\) we define a non-negative Borel measure \( \mu \) on \( \Pi \). Let us show that the measure \( \mu \) is a solution of the moment problem \((1)\).

Let \( \hat{A} \) be a self-adjoint extension of the operator \( A \) in a Hilbert space \( \hat{H} \supseteq H \), such that

\[ E = P_{\hat{H}} \hat{E}, \]

where \( \hat{E} \) is the spectral measure of \( \hat{A} \). By \((32)\) we get

\[ x_{m,n} = A^m B^n x_{0,0} = \hat{A}^m B^n x_{0,0}, \]

\[ = \left( \lim_{a \to +\infty} \int_{[-a,a]} x^m \hat{E} \right) \int_{[-\pi,\pi]} e^{i\varphi} dF x_{0,0} = \left( \lim_{a \to +\infty} \int_{[-a,a]} x^m dE \right) \int_{[-\pi,\pi]} e^{i\varphi} dF x_{0,0}, \]

\[ \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad m \in \mathbb{Z}_+, \quad n \in \mathbb{Z}, \]

where the limits are understood in the weak operator topology. Then we choose arbitrary points

\[ -a = x_0 < x_1 < \ldots < x_N = a; \]

\[ \max_{1 \leq i \leq N} |x_i - x_{i-1}| =: d, \quad N \in \mathbb{N}; \]

\[ -\pi = \varphi_0 < \varphi_1 < \ldots < \varphi_M = \pi; \]

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\[
\max_{1 \leq j \leq M} |\varphi_j - \varphi_{j-1}| =: r; \quad M \in \mathbb{N}.
\]

Set

\[
C_a := \int_{[-a,a]} x^m dE \int_{[-\pi,\pi]} e^{in\varphi} dF = \lim_{d \to 0} \sum_{i=1}^{N} x_{i-1}^m E([x_{i-1}, x_i])
\]

\[
\ast \lim_{r \to 0} \sum_{j=1}^{M} e^{in\varphi_j} F([\varphi_{j-1}, \varphi_j]),
\]

where the integral sums converge in the strong operator topology. Then

\[
C_a = \lim_{d \to 0} \lim_{r \to 0} \sum_{i=1}^{N} \sum_{j=1}^{M} x_{i-1}^m E([x_{i-1}, x_i]) e^{in\varphi_j} F([\varphi_{j-1}, \varphi_j])
\]

\[
= \lim_{d \to 0} \lim_{r \to 0} \sum_{i=1}^{N} \sum_{j=1}^{M} x_{i-1}^m e^{in\varphi_j} E([x_{i-1}, x_i]) F([\varphi_{j-1}, \varphi_j]),
\]

where the limits are understood in the strong operator topology. Then

\[
(C_a x_0, x_0)_{H} = \left( \lim_{d \to 0} \lim_{r \to 0} \sum_{i=1}^{N} \sum_{j=1}^{M} x_{i-1}^m e^{in\varphi_j} E([x_{i-1}, x_i]) F([\varphi_{j-1}, \varphi_j]) x_0, x_0 \right)_{H}
\]

\[
= \lim_{d \to 0} \lim_{r \to 0} \sum_{i=1}^{N} \sum_{j=1}^{M} x_{i-1}^m e^{in\varphi_j} (E([x_{i-1}, x_i]) F([\varphi_{j-1}, \varphi_j]) x_0, x_0)_{H}
\]

\[
= \lim_{d \to 0} \lim_{r \to 0} \sum_{i=1}^{N} \sum_{j=1}^{M} x_{i-1}^m e^{in\varphi_j} ((E \times F)([x_{i-1}, x_i] \times [\varphi_{j-1}, \varphi_j]) x_0, x_0)_{H}
\]

\[
= \lim_{d \to 0} \lim_{r \to 0} \sum_{i=1}^{N} \sum_{j=1}^{M} x_{i-1}^m e^{in\varphi_j} (\mu([x_{i-1}, x_i] \times [\varphi_{j-1}, \varphi_j]) x_0, x_0)_{H}.
\]

Therefore

\[
(C_a x_0, x_0)_{H} = \lim_{d \to 0} \lim_{r \to 0} \int_{[-a,a] \times [-\pi,\pi]} f_{d,r}(x, \varphi) d\mu,
\]

where \( f_{d,r} \) is equal to \( x_{i-1}^m e^{in\varphi_j} \) on the rectangular \([x_{i-1}, x_i] \times [\varphi_{j-1}, \varphi_j], 1 \leq i \leq N, 1 \leq j \leq M.\)
If \( r \to 0 \), then the simple function \( f_{d,r} \) converges uniformly to the function \( f_d \) which is equal to \( x_{i-1}^m e^{in\varphi} \) on the rectangular \([x_{i-1}, x_i) \times [\varphi_{j-1}, \varphi_j), 1 \leq i \leq N, 1 \leq j \leq M \). Then

\[
(C_a x_{0,0}, x_{0,0})_H = \lim_{d \to 0} \int_{[-a,a] \times [-\pi,\pi]} f_d(x, \varphi) d\mu.
\]

If \( d \to 0 \), then the function \( f_d \) converges uniformly to the function \( x^m e^{in\varphi} \).

Since \(|f_d| \leq A^m\), by the Lebesgue theorem we get

\[
(C_a x_{0,0}, x_{0,0})_H = \int_{[-a,a] \times [-\pi,\pi]} x^m e^{in\varphi} d\mu. \tag{59}
\]

By virtue of relations (56) and (59) we get

\[
s_{m,n} = (x_{m,n}, x_{0,0})_H = \lim_{a \to +\infty} (C_a x_{0,0}, x_{0,0})_H
\]

\[
= \lim_{a \to +\infty} \int_{[-a,a] \times [-\pi,\pi]} x^m e^{in\varphi} d\mu = \int_{\Pi} x^m e^{in\varphi} d\mu. \tag{60}
\]

Thus, the measure \( \mu \) is a solution of the Devinatz moment problem.

Let us prove the last assertion of the Theorem. Suppose to the contrary that two different spectral measures \( E_1 \) and \( E_1 \) of \( A \) commute with the spectral measure \( F \) of \( B \) and produce by relation \( (67) \) the same solution \( \mu \) of the Devinatz moment problem. Choose an arbitrary \( z \in \mathbb{C} \setminus \mathbb{R} \). Then

\[
\int_{\Pi} \frac{x^m e^{in\varphi}}{x-z} d\mu = \int_{\Pi} \frac{x^m e^{in\varphi}}{x-z} ((E_k \times F)(\delta)x_{0,0}, x_{0,0})_H
\]

\[
= \lim_{a \to +\infty} \int_{[-a,a] \times [-\pi,\pi]} \frac{x^m e^{in\varphi}}{x-z} ((E_k \times F)(\delta)x_{0,0}, x_{0,0})_H, \quad k = 1, 2. \tag{61}
\]

Consider arbitrary partitions of the type \( (57), (58) \). Then

\[
D_a := \int_{[-a,a] \times [-\pi,\pi]} \frac{x^m e^{in\varphi}}{x-z} ((E_k \times F)(\delta)x_{0,0}, x_{0,0})_H
\]

\[
= \lim_{d \to 0} \lim_{r \to 0} \int_{[-a,a] \times [-\pi,\pi]} g_{z,d,r}(x, \varphi) d((E_k \times F)(\delta)x_{0,0}, x_{0,0})_H.
\]

Here the function \( g_{z,d,r}(x, \varphi) \) is equal to \( \frac{x_{i-1}^{m-1} e^{in\varphi_{j-1}}}{x_{i-1}-z} \) on the rectangular \([x_{i-1}, x_i) \times [\varphi_{j-1}, \varphi_j), 1 \leq i \leq N, 1 \leq j \leq M \). Then

\[
D_a = \lim_{d \to 0} \lim_{r \to 0} \sum_{i=1}^{N} \sum_{j=1}^{M} \frac{x_{i-1}^{m-1} e^{in\varphi_{j-1}}}{x_{i-1}-z} (E_k([x_{i-1}, x_i]) F([\varphi_{j-1}, \varphi_j]) x_{0,0}, x_{0,0})_H
\]

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\[\lim_{d \to 0} \lim_{r \to 0} \left( \sum_{i=1}^{N} \frac{x_{i-1}^m}{x_i - z} E_k([x_{i-1}, x_i]) \sum_{j=1}^{M} e^{i\varphi_j} F(\varphi_{j-1}, \varphi_j)x_{0,0}x_{0,0} \right)_H \]

= \left( \int_{[-a,a]} \frac{x^m}{x - z} dE_k \int_{[-\pi, \pi]} e^{i\varphi} dFx_{0,0}x_{0,0} \right)_H.

Let \( n = n_1 + n_2, n_1, n_2 \in \mathbb{Z} \). Then we can write:

\[D_a = \left( B^{n_1} \int_{[-a,a]} \frac{x^m}{x - z} dE_k B^{n_2}x_{0,0}x_{0,0} \right)_H \]

= \left( \int_{[-a,a]} \frac{x^m}{x - z} dE_k x_{0,n_2}, x_{0,-n_1} \right)_H.

By (61) we get

\[\int_{\Pi} \frac{x^m}{x - z} e^{i\varphi} d\mu = \lim_{a \to +\infty} D_a = \lim_{a \to +\infty} \left( \int_{[-a,a]} \frac{x^m}{x - z} d\widehat{E}_k x_{0,n_2}, x_{0,-n_1} \right) \hat{H}_k \]

= \left( \int_{\mathbb{R}} \frac{x^m}{x - z} d\widehat{E}_k x_{0,n_2}, x_{0,-n_1} \right) \hat{H}_k = \left( \hat{A}^{m_2} R_z(\hat{A}_k) \hat{A}^{m_1} x_{0,n_2}, x_{0,-n_1} \right) \hat{H}_k

= \left( R_z(\hat{A}_k)x_{m_1,n_2}, x_{m_2,-n_1} \right)_H, \quad (62)

where \( m_1, m_2 \in \mathbb{Z}_+: m_1 + m_2 = m \), and \( \hat{A}_k \) is a self-adjoint extension of \( A \) in a Hilbert space \( \hat{H}_k \supset H \) such that its spectral measure \( \widehat{E}_k \) generates \( E_k \):

\[E_k = P^H_\hat{H}_k \widehat{E}_k; k = 1, 2.\]

Relation (62) shows that the generalized resolvents corresponding to \( E_k \), \( k = 1, 2 \), coincide. That means that the spectral measures \( E_1 \) and \( E_2 \) coincide. We obtained a contradiction. This completes the proof. \( \square \)

**Definition 3.1** A solution \( \mu \) of the Devinatz moment problem (1) we shall call canonical if it is generated by relation (37) where \( E \) is an orthogonal spectral measure of \( A \) which commutes with the spectral measure of \( B \). Orthogonal spectral measures are those measures which are the spectral measures of self-adjoint extensions of \( A \) inside \( H \).

Let a moment problem (1) be given and conditions (3) hold. Let us describe canonical solutions of the Devinatz moment problem. In the proof of Theorem 2.2 we have constructed one canonical solution, see relation (33). Let
\( \mu \) be an arbitrary canonical solution and \( \mathbf{E} \) be the corresponding orthogonal spectral measure of \( A \). Let \( \tilde{A} \) be the self-adjoint operator in \( H \) which corresponds to \( \mathbf{E} \). Consider the Cayley transformation of \( \tilde{A} \):

\[
U\tilde{A} = (\tilde{A} + iE_H)(\tilde{A} - iE_H)^{-1} \supset V_A,
\]

where \( V_A \) is defined by (18). Since \( \mathbf{E} \) commutes with the spectral measure \( F \) of \( B \), then \( U\tilde{A} \) commutes with \( B \). By relation (24) the operator \( U\tilde{A} \) has the following form:

\[
U\tilde{A} = V_A \oplus \tilde{U}_{2,4},
\]

where \( \tilde{U}_{2,4} \) is an isometric operator which maps \( H_2 \) onto \( H_4 \), and commutes with \( B \). Let the operator \( U_{2,4} \) be defined by (28). Then the following operator

\[
U_2 = U_{2,4}^{-1}\tilde{U}_{2,4}.
\]

is a unitary operator in \( H_2 \) which commutes with \( B_{H_2} \).

Denote by \( S(B;H_2) \) a set of all unitary operators in \( H_2 \) which commute with \( B_{H_2} \). Choose an arbitrary operator \( \hat{U}_2 \in S(B;H_2) \). Define \( \tilde{U}_{2,4} \) by the following relation:

\[
\tilde{U}_{2,4} = U_{2,4}\hat{U}_2.
\]

Notice that \( \tilde{U}_{2,4} \) commutes with \( B_{H_2} \). Then we define a unitary operator \( U = V_A \oplus \tilde{U}_{2,4} \) and its Cayley transformation \( \hat{A} \) which commute with the operator \( B \). Repeating arguments before (33) we get a canonical solution of the Devinatz moment problem.

Thus, all canonical solutions of the Devinatz moment problem are generated by operators \( \hat{U}_2 \in S(B;H_2) \). Notice that different operators \( U', U'' \in S(B;H_2) \) produce different orthogonal spectral measures \( \mathbf{E}', \mathbf{E} \). By Theorem 3.1 these spectral measures produce different solutions of the moment problem.

Recall some definitions from [10]. A pair \((Y, \mathfrak{A})\), where \( Y \) is an arbitrary set and \( \mathfrak{A} \) is a fixed \( \sigma \)-algebra of subsets of \( A \) is said to be a measurable space. A triple \((Y, \mathfrak{A}, \mu)\), where \((Y, \mathfrak{A})\) is a measurable space and \( \mu \) is a measure on \( \mathfrak{A} \) is said to be a space with a measure.

Let \((Y, \mathfrak{A})\) be a measurable space, \( \mathbf{H} \) be a Hilbert space and \( \mathcal{P} = \mathcal{P}(\mathbf{H}) \) be a set of all orthogonal projectors in \( \mathbf{H} \). A countably additive mapping \( E : \mathfrak{A} \to \mathcal{P}, E(Y) = E_{\mathbf{H}} \), is said to be a spectral measure in \( \mathbf{H} \). A set \((Y, \mathfrak{A}, H, E)\) is said to be a space with a spectral measure. By \( S(Y, E) \) one means a set of all \( E \)-measurable \( E \)-a.e. finite complex-valued functions on \( Y \).
Let \((Y, \mathfrak{A}, \mu)\) be a separable space with a \(\sigma\)-finite measure and to \(\mu\)-everyone \(y \in Y\) it corresponds a Hilbert space \(G(y)\). A function \(N(y) = \dim G(y)\) is called the dimension function. It is supposed to be \(\mu\)-measurable. Let \(\Omega\) be a set of vector-valued functions \(g(y)\) with values in \(G(y)\) which are \(\mu\)-everywhere and are measurable with respect to some base of measurability. A set of (classes of equivalence) of such functions with the finite norm
\[
\|g\|^2_H = \int |g(y)|^2_{G(y)} d\mu(y) < \infty
\] (67)
form a Hilbert space \(H\) with the scalar product given by
\[
(g_1, g_2)_H = \int (g_1, g_2)_{G(y)} d\mu(y).
\] (68)
The space \(H = \mathcal{H}_{\mu,N} = \int_Y \oplus G(y) d\mu(y)\) is said to be a direct integral of Hilbert spaces. Consider the following operator
\[
X(\delta)g = \chi_\delta g, \quad g \in H, \ \delta \in \mathfrak{A},
\] (69)
where \(\chi_\delta\) is the characteristic function of the set \(\delta\). The operator \(X\) is a spectral measure in \(\mathcal{H}\).

Let \(t(y)\) be a measurable operator-valued function with values in \(B(G(y))\) which is \(\mu\)-a.e. defined and \(\mu - \sup \|t(y)\|_{G(y)} < \infty\). The operator
\[
T : g(y) \mapsto t(y)g(y),
\] (70)
is said to be decomposable. It is a bounded operator in \(\mathcal{H}\) which commutes with \(X(\delta), \forall \delta \in \mathfrak{A}\). Moreover, every bounded operator in \(\mathcal{H}\) which commutes with \(X(\delta), \forall \delta \in \mathfrak{A}\), is decomposable [10]. In the case \(t(y) = \varphi(y)E_{G(y)}\), where \(\varphi \in S(Y, \mu)\), we set \(T =: Q\varphi\). The decomposable operator is unitary if and only if \(\mu\)-a.e. the operator \(t(y)\) is unitary.

Return to the study of canonical solutions. Consider the spectral measure \(F_2\) of the operator \(B_{H_2}\) in \(H_2\). There exists an element \(h \in H_2\) of the maximal type, i.e. the non-negative Borel measure
\[
\mu(\delta) := (F_2(\delta)h, h), \quad \delta \in \mathcal{B}([-\pi, \pi]),
\] (71)
has the maximal type between all such measures (generated by other elements of \(H_2\)). This type is said to be the spectral type of the measure \(F_2\). Let \(N_2\) be the multiplicity function of the measure \(F_2\). Then there exists a unitary transformation \(W\) of the space \(H_2\) on \(\mathcal{H} = \mathcal{H}_{\mu,N_2}\) such that
\[
WB_{H_2}W^{-1} = Q_{e^{iy}}, \quad WF_2(\delta)W^{-1} = X(\delta).
\] (72)
Notice that $\hat{U}_2 \in S(B; H_2)$ if and only if the operator

$$V_2 := W\hat{U}_2 W^{-1},$$

(73)
is unitary and commutes with $X(\delta), \forall \delta \in [-\pi, \pi)$. The latter is equivalent to the condition that $V_2$ is decomposable and the values of the corresponding operator-valued function $t(y)$ are $\mu$-a.e. unitary operators. A set of all decomposable operators in $\mathcal{H}$ such that the values of the corresponding operator-valued function $t(y)$ are $\mu$-a.e. unitary operators we denote by $D(B; H_2)$.

**Theorem 3.2** Let a Devinatz moment problem (1) be given. In conditions of Theorem 3.1 all canonical solutions of the moment problem have the form (37) where the spectral measures $E$ of the operator $A$ are constructed by operators from $D(B; H_2)$. Namely, for an arbitrary $V_2 \in D(B; H_2)$ we set

$$U_2 = W^{-1}V_2 W, \quad \hat{U}_2 = U_2, \quad \hat{U}_2 = \hat{U}_2,$$

$$U = V_A \oplus \hat{U}_2, \quad \hat{A} = i(U + E_H)(U - E_H)^{-1},$$

and then $E$ is the spectral measure of $\hat{A}$.

Moreover, the correspondence between $D(B; H_2)$ and a set of all canonical solutions of the Devinatz moment problem is bijective.

**Proof.** The proof follows directly from the previous considerations. \(\square\)

Consider a Devinatz moment problem (1) and suppose that conditions (3) hold. Let us turn to a parameterization of all solutions of the moment problem. We shall use Theorem 3.1. Consider relation (37). The spectral measure $E$ commutes with the operator $B$. Choose an arbitrary $z \in \mathbb{C} \setminus \mathbb{R}$. By virtue of relation (36) we can write:

$$\langle BR_z(A)x, y \rangle_H = \langle R_z(A)x, B^*y \rangle_H = \int_{\mathbb{R}} \frac{1}{t - z} d(E(t)x, B^*y)_H$$

$$\int_{\mathbb{R}} \frac{1}{t - z} d(BE(t)x, y)_H = \int_{\mathbb{R}} \frac{1}{t - z} d(E(t)Bx, y)_H, \quad x, y \in H; \quad (74)$$

$$\langle R_z(A)Bx, y \rangle_H = \int_{\mathbb{R}} \frac{1}{t - z} d(E(t)Bx, y)_H, \quad x, y \in H, \quad (75)$$

where $R_z(A)$ is the generalized resolvent which corresponds to $E$. Therefore we get

$$R_z(A)B = BR_z(A), \quad z \in \mathbb{C} \setminus \mathbb{R}. \quad (76)$$

On the other hand, if relation (76) holds, then

$$\int_{\mathbb{R}} \frac{1}{t - z} d(EBx, y)_H = \int_{\mathbb{R}} \frac{1}{t - z} d(BEx, y)_H, \quad x, y \in H, \quad z \in \mathbb{C} \setminus \mathbb{R}. \quad (77)$$

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By the Stieltjes inversion formula [12], we obtain that $E$ commutes with $B$. We denote by $M(A, B)$ a set of all generalized resolvents $R_z(A)$ of $A$ which satisfy relation (76).

Recall some known facts from [5] which we shall need here. Let $K$ be a closed symmetric operator in a Hilbert space $H$, with the domain $D(K)$, $D(K) = H$. Set $N_{\lambda} = N_{\lambda}(K) = H \ominus \Delta_K(\lambda), \lambda \in \mathbb{C}\setminus \mathbb{R}$.

Consider an arbitrary bounded linear operator $C$, which maps $N_i$ into $N_{-i}$. For $g = f + C\psi - \psi, \quad f \in D(K), \psi \in N_i$, (78) we set

$$K_C g = Kf + iC\psi + i\psi. \quad (79)$$

Since an intersection of $D(K), N_i$ and $N_{-i}$ consists only of the zero element, this definition is correct. Notice that $K_C$ is a part of the operator $K^*$. The operator $K_C$ is said to be a quasiself-adjoint extension of the operator $K$, defined by the operator $K$.

The following theorem can be found in [5, Theorem 7]:

**Theorem 3.3** Let $K$ be a closed symmetric operator in a Hilbert space $H$ with the domain $D(K), D(K) = H$. All generalized resolvents of the operator $K$ have the following form:

$$R_\lambda(K) = \begin{cases} (K_{F(\lambda)} - \lambda E_H)^{-1}, & \text{Im} \lambda > 0 \\ (K_{F^*(\lambda)} - \lambda E_H)^{-1}, & \text{Im} \lambda < 0 \end{cases}, \quad (80)$$

where $F(\lambda)$ is an analytic in $\mathbb{C}_+$ operator-valued function, which values are contractions which map $N_i(A) = H_2$ into $N_{-i}(A) = H_4$ ($\|F(\lambda)\| \leq 1$), and $K_{F(\lambda)}$ is the quasiself-adjoint extension of $K$ defined by $F(\lambda)$.

On the other hand, for any operator function $F(\lambda)$ having the above properties there corresponds by relation (80) a generalized resolvent of $K$.

Notice that the correspondence between all generalized resolvents and functions $F(\lambda)$ in Theorem [5,3] is bijective [5].

Return to the study of the Devinatz moment problem. Let us describe the set $M(A, B)$. Choose an arbitrary $R_\lambda \in M(A, B)$. By (80) we get

$$R_\lambda = (A_{F(\lambda)} - \lambda E_H)^{-1}, \quad \text{Im} \lambda > 0, \quad (81)$$

where $F(\lambda)$ is an analytic in $\mathbb{C}_+$ operator-valued function, which values are contractions which map $H_2$ into $H_4$, and $A_{F(\lambda)}$ is the quasiself-adjoint extension of $A$ defined by $F(\lambda)$. Then

$$A_{F(\lambda)} = R_\lambda^{-1} + \lambda E_H, \quad \text{Im} \lambda > 0.$
By virtue of relation (76) we obtain
\[ BA_{F(\lambda)}h = A_{F(\lambda)}Bh, \quad h \in D(A_{F(\lambda)}), \ \lambda \in \mathbb{C}_+. \] (82)

Consider the following operators
\[ W_\lambda := (A_{F(\lambda)} + iE_H)(A_{F(\lambda)} - iE_H)^{-1} = E_H + 2i(A_{F(\lambda)} - iE_H)^{-1}, \] (83)
\[ V_A = (A + iE_H)(A - iE_H)^{-1} = E_H + 2i(A - iE_H)^{-1}, \] (84)
where \( \lambda \in \mathbb{C}_+ \). Notice that (5)
\[ W_\lambda = V_A \ominus F(\lambda), \quad \lambda \in \mathbb{C}_+. \] (85)

The operator \((A_{F(\lambda)} - iE_H)^{-1}\) is defined on the whole \( H \), see [5, p.79]. By relation (82) we obtain
\[ B(A_{F(\lambda)} - iE_H)^{-1}h = (A_{F(\lambda)} - iE_H)^{-1}Bh, \quad h \in H, \ \lambda \in \mathbb{C}_+. \] (86)

Then
\[ BW_\lambda = W_\lambda B, \quad \lambda \in \mathbb{C}_+. \] (87)

Recall that by Proposition 2.1 the operator \( B \) reduces the subspaces \( H_j \), \( 1 \leq j \leq 4 \), and \( BV_A = V_A B \). If we choose an arbitrary \( h \in H_2 \) and apply relations (87), (85), we get
\[ BF(\lambda) = F(\lambda)B, \quad \lambda \in \mathbb{C}_+. \] (88)

Denote by \( F(A,B) \) a set of all analytic in \( \mathbb{C}_+ \) operator-valued functions which values are contractions which map \( H_2 \) into \( H_4 \) and which satisfy relation (88). Thus, for an arbitrary \( R_\lambda \in M(A,B) \) the corresponding function \( F(\lambda) \in F(A,B) \). On the other hand, choose an arbitrary \( F(\lambda) \in F(A,B) \). Then we derive (87) with \( W_\lambda \) defined by (83). Then we get (86), (82) and therefore
\[ BR_\lambda = R_\lambda B, \quad \lambda \in \mathbb{C}_+. \] (89)

Calculating the conjugate operators for the both sides of the last equality we conclude that this relation holds for all \( \lambda \in \mathbb{C} \).

Consider the spectral measure \( F_2 \) of the operator \( B_{H_2} \) in \( H_2 \). We have obtained relation (72) which we shall use one more time. Notice that \( F(\lambda) \in F(A,B) \) if and only if the operator-valued function
\[ G(\lambda) := WF(\lambda)U_{2,4}^{-1}W^{-1}, \quad \lambda \in \mathbb{C}_+, \] (90)
is analytic in \( C_+ \) and has values which are contractions in \( \mathcal{H} \) which commute with \( X(\delta) \), \( \forall \delta \in [-\pi, \pi) \).

This means that for an arbitrary \( \lambda \in C_+ \) the operator \( G(\lambda) \) is decomposable and the values of the corresponding operator-valued function \( t(y) \) are \( \mu \)-a.e. contractions. A set of all decomposable operators in \( \mathcal{H} \) such that the values of the corresponding operator-valued function \( t(y) \) are \( \mu \)-a.e. contractions we denote by \( T(B; H_2) \). A set of all analytic in \( C_+ \) operator-valued functions \( G(\lambda) \) with values in \( T(B; H_2) \) we denote by \( G(A, B) \).

**Theorem 3.4** Let a Devinatz moment problem (1) be given. In conditions of Theorem 3.1 all solutions of the moment problem have the form (37) where the spectral measures \( E \) of the operator \( A \) are defined by the corresponding generalized resolvents \( R_\lambda \) which are constructed by the following relation:

\[
R_\lambda = (A_{F(\lambda)} - \lambda E_H)^{-1}, \quad \text{Im} \lambda > 0,
\]  

where \( F(\lambda) = W^{-1}G(\lambda)WU_{2,4} \), \( G(\lambda) \in G(A, B) \).

Moreover, the correspondence between \( G(A, B) \) and a set of all solutions of the Devinatz moment problem is bijective.

**Proof.** The proof follows from the previous considerations. \( \square \)

Consider an arbitrary non-negative Borel measure \( \mu \) in the strip \( \Pi \) which has all finite moments (1). What can be said about the density of power-trigonometric polynomials (2) in the corresponding space \( L_2^\mu \)? The measure \( \mu \) is a solution of the corresponding moment problem (1). Thus, \( \mu \) admits a representation (37) where \( F \) is the spectral measure of \( B \) and \( E \) is a spectral measure of \( A \) which commutes with \( F \) (the operators \( A \) and \( B \) in a Hilbert space \( H \) are defined as above).

Suppose that (power-trigonometric) polynomials are dense in \( L_2^\mu \). Repeating arguments from the beginning of the Proof of Theorem 3.1 we see that in our case \( H_0 = \{0\} \) and \( \tilde{A}, \tilde{B} \) are operators in \( H \). Moreover, we have \( \mu = ((\tilde{E} \times \tilde{F})x_{0,0}, x_{0,0})_H \), where \( \tilde{E} \) is the spectral measure of \( \tilde{A} \), \( \tilde{F} = F \). Consequently, \( \mu \) is a canonical solution of the Devinatz moment problem.

The converse assertion is more complicated and will be studied elsewhere.

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Devinatz’s moment problem: a description of all solutions.

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In this paper we study Devinatz’s moment problem: to find a non-negative Borel measure $\mu$ in a strip $\Pi = \{(x, \varphi) : x \in \mathbb{R}, -\pi \leq \varphi < \pi\}$, such that $\int_{\Pi} x^m e^{im\varphi} d\mu = s_{m,n}$, $m \in \mathbb{Z}^+$, $n \in \mathbb{Z}$, where $\{s_{m,n}\}_{m \in \mathbb{Z}^+, n \in \mathbb{Z}}$ is a given sequence of complex numbers. We present a new proof of the Devinatz solvability criterion for this moment problem. We obtained a parameterization of all solutions of Devinatz’s moment problem. We used an abstract operator approach and results of Godič, Lucenko and Shtraus.

Key words: moment problem, measure, generalized resolvent.

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