Spectral edge mode in interacting one-dimensional systems

O. Tsyplyatyev and A. J. Schofield
School of Physics and Astronomy, The University of Birmingham, Birmingham, B15 2TT, UK
(Dated: 12th March 2014)

A continuum of excitations in interacting one-dimensional systems is bounded from below by a spectral edge that marks the lowest possible excitation energy for a given momentum. We analyse short-range interactions between Fermi particles and between Bose particles (with and without spin) using Bethe-Ansatz techniques and find that the dispersions of the corresponding spectral edge modes are close to a parabola in all cases. Based on this emergent phenomenon we propose an empirical model of a free, non-relativistic particle with an effective mass identified at low energies as the bare electron mass renormalised by the dimensionless Luttinger parameter \( K \) (or \( K_\sigma \) for particles with spin).

The relevance of the Luttinger parameters beyond the low energy limit provides a more robust method for extracting them experimentally using a much wider range of data from the bottom of the one-dimensional band to the Fermi energy. The empirical model of the spectral edge mode complements the mobile impurity model to give a description of the excitations in proximity of the edge at arbitrary momenta in terms of only the low energy parameters and the bare electron mass. Within such a framework, for example, exponents of the spectral function are expressed explicitly in terms of only a few Luttinger parameters.

PACS numbers: 71.10.Pm, 03.75.Kk, 73.21.

I. INTRODUCTION

The low energy properties of interacting particles in one-dimension are well-described by the Tomonaga-Luttinger model based on the linear approximation to the spectrum of the excitation at the Fermi energy. In this framework various correlation functions, that involve a continuum of many-body excitations, can be evaluated explicitly resulting in a common power-law behaviour - in contrast to higher dimensions where the Fermi gas approximation with renormalised parameters (the Fermi liquid model) remains robust. In the last few decades different experimental realisations of one-dimensional geometries were developed: carbon nanotubes, cleaved edge or gated one-dimensional channels in semiconductor heterostructures, and cold atomic gases in cigar shaped optical lattices where the predictions of the low energy theory have already been observed and measurements of high-energy effects are already possible.

Recently, a new theoretical understanding of the behaviour at high energies was achieved by making a connection between the features of the dynamical response of the one-dimensional systems and the Fermi edge singularity in x-ray scattering in metals. Application of the mobile impurity model to the Tomonaga-Luttinger model gives a description of excitations at high energies incorporating dispersion of the spectral edge as an input parameter: the edge marks the smallest excitation energy at a fixed momentum. Within the resulting theory correlation functions exhibit a common power law behaviour where exponents are related to the curvature of the spectral edge and the Luttingers parameters. However, the theory for the edge mode itself remains an open problem.

In this paper we analyse fundamental models of Fermi and Bose particles with short-range interactions (with and without spin) in one dimension using the available diagonalisation methods based on Bethe-Ansatz. We investigate the edge mode of the spectral function - a dynamical response function that generalises the single particle spectrum to the many particle systems - and find that its dispersion is close to a parabola for all cases in the thermodynamic limit. It is exactly parabolic for...
fermions without spin and the biggest deviation ($\lesssim 20\%$) occurs for fermions with spin and a very large interaction potential. Based on this result we propose an empirical model of a free, non-relativistic particle for the spectral edge mode, which describes a charge wave in the spinless case and a spin wave in the spinful case, see a graphical representation of the spectral function in Fig. 1. The effective mass $m^*$ is identified at low energies as the bare electron mass $m$ strongly renormalised by the dimensionless Luttinger parameter; $m^*/m = K$ and $m^*/m = K_\sigma$ in the spinless and the spinful case respectively. The position of the edge of the spectral function in terms of this empirical model can be expressed as

$$\epsilon_{\text{edge}}(k) = \mu + \frac{k^2}{2m^*} - \frac{(k - k_0)^2}{2m^*},$$

where $\mu$ is the chemical potential, $k_F$ is the Fermi momentum, and $k_0 = 0 (k_F)$ for Fermi (Bose) particles.

The empirical model in Eq. (1) breaks down when the effective mass becomes infinite. At low energies $m^* = \infty$ is equivalent to zero sound velocity of the collective modes $v (v_\sigma)$. The characteristic threshold is given by the quantum of the momentum $v_1 = 2\pi/ (mL)$ in a system of a finite size $L$. For slower velocities $v (v_\sigma) \lesssim v_1$ the dispersion of the spectral edge mode is not parabola-like and is not universal.

The parabolic shape of the spectral edge mode, which emerges in microscopic calculation for different models, can be interpreted as an unusual manifestation of Galilean invariance. The kinetic energy of a single free particle is a parabolic function of its momentum, enforced by the translational symmetry. Finite system size discretises the boosts for changing inertial frames of reference in quanta of $2\pi/L$. For a system consisting of $N$ particles the minimal boost of $2\pi N/L$ corresponds to the $2k_F$-periodicity in the momentum space; note that interaction potentials are also Galilean invariant. However, the total momentum of the whole many-particle system is still quantised in the units of $2\pi/L$ that can be facilitated by giving a boost to only a fraction of the particles $j < N$. The state on the spectral edge with the momentum $k = 2\pi j/L$ corresponds to a hole left between $N - j$ particles in the rest frame and $j$ particles which have received the minimal boost, see section III for details. The effective mass of the hole-like quasiparticle is strongly renormalised by interactions since a partial boost is not a Galilean invariant transformation. However, the parabolic dependence of the hole energy on momentum - which is analogous to the kinetic energy of a free particle - is common for different microscopic models thus it is an emergent phenomenon.

Excitations above the spectral edge are well-described at high energies by the application of the mobile impurity model to the Tomonaga-Luttinger theory which incorporates the curvature of the spectral edge as an input parameter. The result in Eq. (1) removes this arbitrary input complementing the model above. Within such a framework, for example, the edge exponents of the spectral function are expressed explicitly in terms of only a few Luttinger parameters and the bare electron mass that provides a systematic way to classify them for a wide range of microscopic parameters.

The rest of the paper is organised as follows. Section II describes the model of one-dimensional particles interacting via short range-potentials, the corresponding spectral function, and discusses their general properties. In section III we evaluate momentum dependence of the spectral edge mode using the Bethe-Ansatz approach for Fermi particles in the fundamental region. Section IV contains the effective field theory for excitations above the spectral edge and calculates the edge exponents of the spectral functions using the dispersion of the spectral edge mode itself obtain in section III. In section V we show that Bose particles have the same parabolic dispersion, with the mass renormalised by the same Luttinger parameter $K$, of the spectral edge mode as the Fermi particles. In section VI we summarise the results and discuss experimental implications.

II. MODEL

We consider particles in one-dimension interacting via a contact two-body potential, $U$, as

$$H = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} dx \left( -\frac{1}{2m} \psi_\alpha^{\dagger}(x) \Delta \psi_\alpha(x) - U L \rho(x)^2 \right)$$

where $\psi_\alpha(x)$ are the field operators of Fermi or Bose particles at point $x$ (with a spin $\alpha = \uparrow, \downarrow$ for spinful particles), $\rho(x) = \psi_\alpha^{\dagger}(x) \psi_\alpha(x)$ is the particle density operator, $L$ is the size of the system, and $m$ is the bare mass of a single particle. Below we consider periodic boundary conditions, $\psi_\alpha(x + L) = \psi_\alpha(x)$, to maintain the translational symmetry of the finite length system, restricting ourselves to repulsive interaction only, $U > 0$, and we assume $h = 1$.

The spectrum of excitations in the many-body case is given by the spectral function which describes the response of a strongly correlated system to a single particle excitation at energy $\varepsilon$ and momentum $k$. $A_\alpha(k, \varepsilon) = -\text{Im} G_{\alpha\alpha}(k, \varepsilon) \text{sgn}(\varepsilon - \mu)/\pi$, where $\mu$ is a chemical potential and $G_{\alpha\beta}(k, \varepsilon)$ is a Fourier transform of Green function at zero temperature. To be specific, we discuss particle like excitations, $\varepsilon > \mu$. The spectral function in this domain reads:

$$A_\alpha(k, \varepsilon) = \sum_f \left| \langle \psi_\alpha^{\dagger}(0)|0\rangle \right|^2 \delta (\omega - E_f + E_0) \delta (k - P_f),$$

where $E_0$ is the energy of the ground state $|0\rangle$, and $P_f$ and $E_f$ are the momenta and the eigenenergies of the eigenstate $|f\rangle$; all eigenstates are assumed normalised.
Galilean invariance defines a fundamental region for the spectrum of excitations on the momentum axis. A minimal boost for changing an inertial frame of reference for \( N \) particles is \( 2\pi N/L \) which is twice the Fermi momentum \( k_F = \pi N/L \). In momentum space this boost corresponds to \( 2k_F \)-periodicity. We choose the fundamental region as \(-k_F < k < k_F\) for the Fermi and as \( 0 < k < 2k_F\) for Bose particles.

Under a \( 2k_F \)-translation, the form factors in Eq. (3) do not change and the energies acquire simple shifts. The interaction term in Eq. (2) is invariant under the transformation \( x \rightarrow x + 2\pi tj/(mL)\), where \( j \) is the number of the translation quanta, since the latter can be absorbed into the a change of the integration variable. The transformation of the momentum operator, \(-i\nabla \rightarrow -i\nabla + 2\pi j/L\), in the the kinetic term results in a constant energy shift, \( E \rightarrow E + 2\pi j P/(mL) + 2\pi^2 j^2 N/(mL^2)\), of the Hamiltonian but keeps its matrix structure and, therefore, eigenstates unaltered. Thus the spectral function can be extended to arbitrary momenta by simultaneous translation of the momentum and of the energy variables starting from the fundamental region.

Here, we are concerned with a distinctive feature of the spectral function - the edge that marks the lowest possible excitation energy for a given momentum. To identify its location we need to obtain only the many-body spectrum of the model due to a singularity that guarantees large values of the form factors in the proximity of the spectral edge. The two \( \delta \)-functions in Eq. (3) directly map the total momenta \( P_l \) and the eigenenergies \( E_j \) of all many-body states \( |f\rangle \) into the points of the spectral function \( k \) and \( \epsilon \). We are going to identify the states that have the smallest energy for each momentum and study how the dispersion of the spectral edge mode, which they form, depends on the interaction strength.

III. FERMIONS

A. Spinless

The zero range profile of two-body interaction potential in the model in Eq. (2) has zero matrix elements for the Fermi particles without spin due to the Pauli exclusion principle. A model of interactions in this case requires a finite range of interactions which is usually introduced by the point-splitting technique developed to address the problem in the low energy limit. Here we will use a different approach of introducing a lattice with the next-neighbour interaction between particles. The lattice counterpart of the model in Eq. (2) is the Hamiltonian

\[
H = -\sum_{j=1}^{L/2} \left( \psi_j^\dagger \psi_{j+1} + \psi_j^\dagger \psi_{j-1} \right) / (2m) - U \sum_{j=1}^{L/2} \psi_j^\dagger \psi_j^\dagger + \psi_{j+1} \psi_{j+1} + \psi_{j-1} \psi_{j-1},
\]

where \( j \) is the site index on the lattice and the operators \( \psi_j \) obey the Fermi commutation relations \( \{ \psi_i, \psi_j \} = \delta_{ij} \).

The model above can be diagonalised using the Bethe Ansatz approach. In the coordinate basis a superposition of \( N \) plane waves, \( \Psi = \sum_{P,j_1<...<j_N} e^{i \sum_{l=1}^N k_{P,j_l}} e^{i \sum_{i<j} \varphi_i j_i} \psi_{j_1}^\dagger \psi_{j_2}^\dagger ... \psi_{j_N} |\text{vac}\rangle \), is an eigenstate, \( H \Psi = E \Psi \), with the corresponding eigenenergy

\[
E = \frac{1}{m} \sum_{j=1}^{N} (1 - \cos(k_j)).
\]

Here \( |\text{vac}\rangle \) is the vacuum state, the scattering phases are fixed by the two-body scattering problem,

\[
e^{i 2 \varphi_{P'}} = - e^{i (k_i + k_f)} + 1 - 2 m U e^{i k_i} - e^{i (k_i + k_f)} + 1 - 2 m U e^{i k_f},
\]

and \( \sum_P \) is a sum over all permutations of \( N \) quasimomenta. The periodic boundary condition quantises the set of \( N \) quasimomenta simultaneously,

\[
Lk_j - 2 \sum_{i \neq j} \varphi_{jil} = 2\pi I_j,
\]

where \( I_j \) are a set of non-equal integer numbers. The total momentum of \( N \) particles, \( P = \sum_{j} k_j \), is a conserved quantity.

The continuum model in Eq. (2) corresponds to the low density (long wave length) limit of the lattice model. In this limit the scattering phases in Eq. (5) are linear functions of quasimomentum, \( 2 \varphi_{P'} = (k_i - k_f) / (1 + (mU)^{-1}) + \pi \). And the non-linear systems of equations in Eq. (6) becomes linear. In the thermodynamic limit we solve it using perturbation theory and obtain an independent quantisation condition for each quasimomentum as solutions of the Bethe equations in the leading \( 1/N \)-order,

\[
k_j = \frac{2\pi I_j}{L - \frac{1}{1 + \frac{U}{2m}}.}
\]

Thus all \( N \)-particles eigenstates can be labeled by all possible sets of integers \( I_j \) similarly to Slater determinants for free fermions. The latter is possible as long as no bound states exist - which is the case for any value of interaction strength \( U \geq 0 \) in this limit.

The eigenstates contributing to the spectral function satisfy the number of particles constraint, \( i.e. \) fixed to
be \(N+1\). The lowest energy state for a fixed momentum \(-k_F + \Delta P\) is given by the set of integers in Fig. 2. At low energies the system is in the universality class of Luttinger liquids. Its properties are fully determined by the linear slope of the spectrum of excitations at \(\pm k_F\). Using the parameterisation in Fig. 2, the first Luttinger parameter (the sound velocity of the collective modes) is a discrete derivative \(v = \frac{L}{2}(E_2 - E_1) / (2\pi)\), where \(E_2\) and \(E_1\) are the energies of the states with \(\Delta P = 2\pi/L\) and \(\Delta P = 0\).

For Galilean invariant systems the product of the first and the second (dimensionless \(K\)) Luttinger parameters gives the Fermi velocity of the non-interacting system, \(v_F = \frac{\pi N}{(mL)}\). By a straightforward calculation of the eigenenergies in Eq. (4) using Eq. (7) for a pair states in Fig. 2 with \(\Delta P = 0\), \(2\pi/L\) we directly obtain the second Luttinger parameter,

\[
K = \left(1 - \frac{N}{L(1 + \frac{2\pi}{mL})}\right)^2.
\]  

(8)

The dispersion of the spectral edge mode is given by the energies and the momenta of all states in Fig. 2,

\[
\Psi = \int \int \ldots \int dx_1 \ldots dx_N \sum_{P,Q} A^{PQ} e^{i(Pk - Qk)} \psi^\dagger_{Q1}(x_1) \ldots \psi^\dagger_{QN}(x_N) |\text{vac}\rangle,
\]  

(9)

where the operators \(\psi_{\alpha}(x)\) obey the Fermi commutation rules \(\{\psi_{\alpha}(x), \psi^\dagger_{\beta}(x')\} = \delta(x - x') \delta_{\alpha\beta}\), \(\lambda_j\) are \(N\) quasimomenta, \(\sum_{P,Q}\) is a sum over all permutations of two independent sets of \(N\) integer numbers \((P\) and \(Q)\), and the coefficients \(A^{PQ}\) are chosen by a secondary use of the Bethe hypothesis,

\[
A^{PQ} = \text{sgn}(PQ) \sum_R \left( \prod_{1 \leq i < j \leq M} \frac{\lambda_{R_i} - \lambda_{R_j} - \text{im}U}{\lambda_{R_i} - \lambda_{R_j}} \right) \prod_{i=1}^M \frac{-\text{im}U}{\lambda_{R_i} - \frac{\lambda_{P_i} - \text{im}U}{2}} \prod_{j=1}^{z_1-1} \frac{\lambda_{R_i} - k_{P_j} - \frac{\text{im}U}{2}}{\lambda_{R_i} - k_{P_j} + \frac{\text{im}U}{2}}.
\]  

(10)

Here \(\lambda_i\) are spin degrees of freedom of \(M\) “up”-spins with respect to the reference ferromagnetic state of \(N\) “down”-spins, \(\sum_R\) is a sum over all permutations of \(M\) integer numbers, and \(z_1\) is position of the \(P^{th}\) spin \(\uparrow\) in permutation \(Q\). The eigenenergy corresponding to the eigenstate in Eq. (9) is

\[
E = \sum_{j=1}^N \frac{k_{P_j}^2}{2m}.
\]  

(11)

The periodic boundary condition quantises the set of \(N\) quasimomenta \(k_j\) (charge degrees of freedom) simultaneously,

\[
Lk_j - \sum_{l=1}^M \varphi_{jl} = 2\pi I_j,
\]  

(12)

Starting from the solutions for quasimomenta in Eq. (7) and repeating the same calculation as before, we directly obtain the parabolic function of momentum in Eq. (1), where \(m^*/m = K\) from Eq. (3). This calculation also gives the chemical potential in Eq. (1) as the bare electron mass renormalised by the Luttinger parameter \(K\), \(\mu = k_F^2 / (2mK)\).

B. Spinful

When Fermi particles have spin 1/2, the Pauli exclusion principle suppresses only the interaction between the particles with the same spin orientation in the model in Eq. (2). The remaining part of the density-density interaction term consists of a coupling between particles with opposite spin orientations.

This model can be diagonalised using the Bethe-Ansatz approach but the Bethe hypothesis has to be applied twice. In the coordinate basis, a superposition of plain waves is an eigenstate, \(H\Psi = E\Psi\), of the model in Eq. (2),

\[
\Psi = \int \int \ldots \int dx_1 \ldots dx_N \sum_{P,Q} A^{PQ} e^{i(Pk - Qk)} \psi^\dagger_{Q1}(x_1) \ldots \psi^\dagger_{QN}(x_N) |\text{vac}\rangle,
\]  

(10)

where the operators \(\psi_{\alpha}(x)\) obey the Fermi commutation rules \(\{\psi_{\alpha}(x), \psi^\dagger_{\beta}(x')\} = \delta(x - x') \delta_{\alpha\beta}\), \(\lambda_j\) are \(N\) quasimomenta, \(\sum_{P,Q}\) is a sum over all permutations of two independent sets of \(N\) integer numbers \((P\) and \(Q)\), and the coefficients \(A^{PQ}\) are chosen by a secondary use of the Bethe hypothesis,

\[
A^{PQ} = \text{sgn}(PQ) \sum_R \left( \prod_{1 \leq i < j \leq M} \frac{\lambda_{R_i} - \lambda_{R_j} - \text{im}U}{\lambda_{R_i} - \lambda_{R_j}} \right) \prod_{i=1}^M \frac{-\text{im}U}{\lambda_{R_i} - \frac{\lambda_{P_i} - \text{im}U}{2}} \prod_{j=1}^{z_1-1} \frac{\lambda_{R_i} - k_{P_j} - \frac{\text{im}U}{2}}{\lambda_{R_i} - k_{P_j} + \frac{\text{im}U}{2}}.
\]  

(10)

where scattering phases \(\varphi_{jl} = \log \left(\frac{(\lambda_i - k_j - \frac{\text{im}U}{2})}{(\lambda_i - k_j + \frac{\text{im}U}{2})}\right) / i\) depend on the quasimomenta of both kinds \((k_j\) and \(\lambda_i)\), \(I_j\) are a set of \(N\) non-equal integer numbers, and \(M\) quasimomenta \(\lambda_i\) (spin degrees of freedom) satisfy another set of non-linear equations,

\[
\prod_{j=1}^N \frac{\lambda_i - k_j - \frac{\text{im}U}{2}}{\lambda_i - k_j + \frac{\text{im}U}{2}} = \prod_{m=1}^N \frac{\lambda_{I_m} - \lambda_{P_i} + \frac{\text{im}U}{2}}{\lambda_{I_m} - \lambda_{P_i} - \frac{\text{im}U}{2}}.
\]  

(13)

The sum \(P = \sum_{j=1}^N k_j\) is a conserved quantity - the total momentum of \(N\) particles.

The system of non-linear equations Eqs. (12) (13) can be solved explicitly in the limit of infinite repulsion \(U = \infty\). The quasimomenta \(\lambda_i\) diverge in this limit. Under the substitution of \(\lambda_i = mU \tan y_i/2\), the second system
Figure 3: Parametrisation of many-body states for fermions with spin using the $U = \infty$ limit in Eqs. (13, 14). Charge-like excitations correspond to different sets of $J_j$ and spin-like excitation correspond to different sets of $J_L$.

of equations Eq. (13) becomes independent of the first system of equations Eq. (12) in leading $1/U$-order,

$$e^{iN\lambda l} = (-1)^N + M - 1 \prod_{l' = 1, \neq l}^M e^{i(y_l + y_{l'})} + 1 + 2e^{i\lambda y_l}.$$ (14)

The above Bethe equations are identical to that of a Heisenberg anti-ferromagnet where the number of particles $N$ plays the role of the system size. In one dimension a spin chain is mapped into the model of interacting Fermi particles by the Jordan-Wigner transformation. Eq. (13) are identical to Eqs. (5, 6) where the interaction strength is set to $mU = -1$. Thus all solutions of Eq. (13) can be labelled by all sets of $M$ non-equal integer numbers $J_j$ similarly to the case of Fermi particles without spin (see Fig. 2). The system of equations for the quasimomenta $k_j$ in Eq. (12) in the $U = \infty$ limit decouples into a set of single particles quantisation conditions,

$$Lk_j = 2\pi I_j + \frac{1}{2} - \frac{(-1)^M}{2} + \sum_{l = 1}^M y_l.$$ (15)

Note that the independent magnetic subsystem, where quasimomenta $y_l$ satisfy Eq. (14), is translationally invariant thus $\sum_{l = 1}^M y_l = 2\pi \sum_{l = 1}^M J_l/N$ (as can be checked explicitly by solving Eq. (14) for all $y_l$). Therefore, the quantisation condition in Eq. (15) depends only on two sets of integer number $I_j$ and $J_l$.

All solutions of the original system of equations Eqs. (12, 13) can be labelled by all sets of $N + M$ integer numbers $I_j$ and $J_l$, see Fig. 3. The values of $k_j$ and $\lambda l$ that correspond to these integers can be obtained in two steps. Firstly, the spin degrees of freedom $y_l$ that correspond to a set of $J_l$ are adiabatically continued under a smooth deformation of Eq. (6) from $U = 0$, which is the free particle limit, to $U = -1/m$, which coincides with Eq. (14). Note that the long wavelength solution in Eq. (7) cannot be used here because the most interesting case of zero polarisation for spinful fermions corresponds to half-filling of the band in the model in Eqs. (12, 13) which is outside of the limits of applicability of the low density regime. The values $k_j$ that correspond to a set of $J_j$ and $J_l$ are obtained directly from Eq. (15). Secondly, the known values of $k_j$ and $\lambda l$ in the $U = \infty$ limit are adiabatically continued under a smooth deformation of Eq. (12, 13) to arbitrary value of the interaction strength $U$.

The interaction effects are controlled by a single dimensionless parameter that can be defined using the $1/U$ corrections in the large $U$ limit. Power series expansion of the Eqs. (12, 13) up to the first subleading $1/U$-order, $\lambda_l = mU \tan y_l/2 + y_l^{(1)}$ and $k_j = k_j^{(0)} + 2k_j^{(1)}/(mU)$, where $y_l$ and $k_j^{(0)}$ are the solutions of Eqs. (14, 15), yields

$$\sum_{j = 1}^N (k_j^{(0)} - y_l^{(1)}) \cos^2 y_l = -2 \sum_{l' = 1, \neq l}^M \frac{y_{l'}^{(1)} - y_l^{(1)}}{(\tan y_l - \tan y_{l'})^2 + 4}$$

and

$$k_j^{(1)} = 2 \sum_{l = 1}^M \left( k_j^{(0)} - y_l^{(1)} \right) \cos^2 y_l.$$ (17)

The first order coefficients $y_l^{(1)}$ can be expressed from Eq. (16) in terms of zeroth order coefficients $k_j^{(0)}$ and $y_l$. Then, in the thermodynamic limit, the first order corrections to the quasimomenta $k_j$ in Eq. (14) become $k_j^{(1)} = 2k_j^{(0)} \sum_{m = 1}^M \cos y_l/L$. This gives a condition of validity for the $1/U$-expansion of the Bethe equations, $2k_j^{(1)}/(mUk_j^{(0)})$, which is independent of both indices $j$ and $l$.

We use the latter to define a single parameter,

$$\gamma_{YG} = \frac{mL}{2N} \frac{U}{\left( 1 + \frac{1}{N} \sum_{l = 1}^M \cos y_l \right)},$$ (18)

that characterises the degree of repulsion between fermions. When $\gamma_{YG} \gg 1$ the particles with opposite spin orientations scatter strongly off each other and when $\gamma_{YG} \ll 1$ they interact weakly with each other. For example, this is manifested in a change of degeneracy of the quasimomenta $k_j$ that correspond to the ground state of unpolarized Fermi particles, $M = N/2$. We account for the degree of double degeneracy with respect to spin-1/2 using

$$D = 2 - \frac{L \sum_{j = 1}^{N-1} (k_{j+1} - k_j)}{\pi N}.$$ (19)

This quantity is $D = 1$ when each momentum state of free fermions is doubly occupied ($U = 0$) and is $D = 0$ when each momentum state is occupied by a single particle ($U = \infty$). The crossover from one regime to another occurs at $\gamma_{YG} = 1$ where $D$ crosses the value of 1/2, see inset in Fig. 4.
The ground state of the model in Eq. (2) has zero spin polarisation when the external magnetic field is absent, $M = N/2$. To be specific we consider the ground states with even values of $N$ and $M$. Excited states contributing to the spectral function satisfy the number of particles being constrained to be $N + 1$. In the $U = \infty$ limit the lowest energy eigenstates for a fixed momentum $P = -k_F + \Delta P$ are given by a set of integers in Fig. 3 with $\Delta k = 0$ and $\Delta P = \Delta \lambda$. In the opposite limit of free fermions, the lowest energy eigenstates for a fixed momentum $P = -k_F + \Delta P$ are doubly degenerate with respect to spin-$1/2$ and are given by the set of integers in Fig. 3 for each spin orientation. The quasimomenta in both limits are smoothly connected under adiabatic deformation of Eqs. (12, 13) from $U = \infty$ to $U = 0$ marking the edge of the spectral function in Fig. 3 for arbitrary $U$.

At low energies the eigenstate are strongly mixed in the spin sector due to spin-charge separation\textsuperscript{1} implying that $A_{\pm}(k, \varepsilon) = A_{\pm}(k, \varepsilon)$. The excitations of the system are spinons and holons which are well approximated by the spinful generalisation of the Tomonaha-Luttinger model with only four free parameters $v_{\rho,\sigma}$ and $K_{\rho,\sigma}$. The pair of velocities are the slopes of the linearised dispersions of the charge and spin excitations at $\pm k_F$. Using the representation of the eigenstates in Fig. 3 they are

\begin{align*}
  v_{\rho} &= \frac{L(E_2 - E_1)}{2\pi}, & v_{\sigma} &= \frac{L(E_3 - E_1)}{2\pi},
\end{align*}

where $E_1$, $E_2$, and $E_3$ correspond to the energies of the states with $(\Delta k = 0, \Delta \lambda = 0)$, $(\Delta k = 2\pi/L, \Delta \lambda = 0)$, and $(\Delta k = 0, \Delta \lambda = 2\pi/L)$ respectively\textsuperscript{24}. The numerical evaluation of $v_{\rho,\sigma}$ as a function of the interaction parameter $\gamma_{YG}$\textsuperscript{22} is presented in Fig. 4. For $\gamma_{YG} = 0$ both velocities coincide $v_{\rho} = v_{\sigma} = v_F$. For large $\gamma_{YG} \gg 1$ the holon velocity doubles $v_{\rho} \approx 2v_F$ due to strong repulsion between particles with opposite spin orientations\textsuperscript{22} and the spinon velocity becomes zero $v_{\sigma} = 0$ since it vanishes as $\sim 1/(m^2U)$ in this limit\textsuperscript{22}. The other pair of Luttinger parameters can be obtained directly for Galilean invariant systems using $v_{\rho,\sigma}$ and the Fermi velocity, $K_{\rho,\sigma} = v_{\rho,\sigma}/v_0$, where $v_0 = \pi M/L$, without the need of a second observable such as compressibility\textsuperscript{17}.

Beyond the linear regime the position of the edge of the spectral function is given by following of the low energy spinon mode. Numerical evaluation shows that $\varepsilon_{\text{edge}}(k) = E_k - E_0$, where $E_k$ corresponds to the states in Fig. 5 with $\Delta k = 0$ and $k = -k_F + \Delta \lambda$, is close to a parabola for all values of $\gamma_{YG}$, see Fig. 5. For $\gamma_{YG} = 0$ the shape of the spectral edge mode is exactly parabolic following the dispersion of free Fermi particles. For $\gamma_H \gg 1$ deviations from a parabola are largest. We
quantify them by comparing the effective mass $m^*$, obtained by the best fit of Eq. (1) at all energies, with the spinon velocity $v_σ$ from Eq. (20), obtained at low energy. The deviation $(v_σ - k_F/m^*)/v_σ$ decreases as the number of particles $N$ grows but it saturates at a finite value of $\sim 0.2$ in the limit $N \to \infty$; see inset in Fig. 5.

The edge of the spectral function in the complementary part of the fundamental range, $k_F < k < 3k_F$, also has a parabolic shape. The eigenstates with the smallest eigenenergies for a fixed momentum $k$ in this range are connected with their counterparts in the $-k_F < k < k_F$ range by a shift of the spin variables $\lambda_j \to \lambda_j + 2\pi/L$. Repeating the same numerical procedure as before for $\varepsilon_{edge}(k) = E_k - E_0$, where $E_k$ corresponds to the states in Fig. 3 with $\Delta k = 0$ and $k = k_F + \Delta \lambda$, we obtain the result in Eq. (1) with $k_0 = 2k_F$. In the “hole region” $\varepsilon < \mu$, the position of the edge of the spectral function is obtained by reflection of $\varepsilon_{edge}(k)$ with respect to the line $\varepsilon = \mu$.

The parabola-like behaviour of the edge mode breaks down in finite sized systems in the ultra-strong interaction regime when the spinon velocity $v_σ$ becomes smaller than its own quantum set by the finite size of the system $v_1 = 2\pi/(mL)$, see the dashed line in Fig. 4. Correspondingly, the threshold for entering this regime becomes $\gamma_{YL} \to \infty$ in the thermodynamic limit, as observed in Fig. 4 when $v_1 \to 0$. When $v_σ < v_1$, the behaviour of the system is dominated by doubling of the period in the momentum space from $2k_F$ to $4k_F$ which can be seen explicitly from Eqs. (11, 15) in the $U = \infty$ limit. The doubling in the spinful case is a direct consequence of Galilean invariance of the model in Eq. (2). However, it does not manifest itself in the thermodynamic limit for finite spinon velocities $v_σ > v_1$, for which the edge of the spectral function is still $2k_F$-periodic.

IV. EFFECTIVE FIELD THEORY

Eigenmodes above the spectral edge can be described by “the mobile impurity model" with two different types of fields that account for all possible low energy excitations with respect to a state on the spectral edge with a given momentum $k$ in Figs. 2 and 3. One field is responsible for bosonic excitations around $\pm k_F$ whose behavior is well-approximated by the Tomonaga-Luttinger model. Another field models the dynamics of the hole-like degree of freedom, as observed in Fig. 2 for a large $\Delta P$. For a $k$ away from $\pm k_F$ creation of a second or removal the existing hole-like excitation is associated with a significant energy cost thus the corresponding field describes a single Fermi particle.

The interaction between the deep hole and the excitations at $\pm k_F$ is of the density-density type since their corresponding energy bands are separated by a large barrier. Bosonisation of the excitations at $\pm k_F$ leaves two unknown coupling constants between a pair of the canonically conjugated variables of the Tomonaga-Luttinger model and a fermionic field of the deep hole that can be identified by considering two different physical properties. One is translation invariance of the hybrid system that can be represented as a motion of a fermionic excitation in a bosonic fluid with the velocity $v = \langle \nabla \theta \rangle /m$. Another is an observable that corresponds to the change of the total energy with respect long-range variations of the density, which for the hybrid systems is given by $\delta \rho = -\langle \nabla \varphi \rangle /\pi$. Here $\varphi$ and $\nabla \theta$ are the canonically conjugated variables of the Tomonaga-Luttinger model that correspond to the density and the current of the hydrodynamic modes respectively.

For a fixed value of $k$, the dynamics of the free Bose-like and the free Fermi-like fields can be linearised for states close to the spectral edge. Using the dispersion in Eq. (1) for the Fermi-like field and the Luttinger parameters for the Bose-like field, the mobile impurity model reads

$$H = \int dx \left[ \frac{v}{2\pi} \left( K (\nabla \theta)^2 + \frac{(\nabla \varphi)^2}{K} \right) + \left( \frac{k(K-1)}{m^*} \right) \nabla \theta + \frac{v(K+1)}{K} \nabla \varphi \right] d^1\theta + d^1\varphi \left( \frac{k^2}{2m^*} - \frac{i\nabla}{m^*} \right) \right]$$

where $-k_F < k < k_F$ is the total momentum of the system - an input parameter of the model, $m^* = mK$ is the effective mass of the deep hole, $v$ and $K$ are the Luttinger parameters defined at $\pm k_F$, the fields $\theta$ and $\varphi$ are the canonically conjugated variables $[\varphi(x), \nabla \theta(y)] = i\pi \delta(x-y)$ of the Tomonaga-Luttinger model, and the field $d$ obey the Fermi commutation rules $\{d(x), d^\dagger(y)\} = \delta(x-y)$.

The Hamiltonian in Eq. (21) can be diagonalised by a unitary transformation, $e^{-iUH}$, where $U = \int dy \left[ C_{\pm} \left( \sqrt{K} \nabla \theta + \varphi/\sqrt{K} \right) + C_{\pm} \left( \sqrt{K} \nabla \theta - \varphi/\sqrt{K} \right) \right] d^1\theta$ and $C_{\pm} = \left( 2\sqrt{K} \right)^{-1} \left[ k(K-1) \pm k_F(K+1) \right] / (k \pm k_F)$, eliminates the coupling term between
the fields turning Eq. (21) into a pair of free harmonic models. Then, the observables can be calculated in a straightforward way as averages over free fields only.

The spectral function in Eq. (3) can calculated using the effective field model. The original operators $\psi^\dagger(x)$ of Fermi particles of the model in Eq. (2) correspond to a composite excitation consisting of two bosons and one fermion in the field language of the model in Eq. (21) (see the state in Fig. 2). The fermionic excitation gives a dominant contribution to the spectral weight $\langle|f|\psi^\dagger(0)|0\rangle^2$, thus at leading order in $|\varepsilon - \varepsilon_{\text{edge}}(k)|$ close to the spectral edge the spectral function reads $A(k, \varepsilon) = \int dtdx e^{i(kx-\varepsilon t)} \langle d^\dagger(x, t) d(0, 0) \rangle$ where $d(x, t) = e^{-iHt}d(x) e^{iHt}$ and $\langle \ldots \rangle$ is the zero temperature expectation value with respect to the model in Eq. (21). In the diagonal basis the average is evaluated over free fields by standard means. Following the steps of Ref. 10 we obtain $A(\varepsilon, k) \sim \theta (\varepsilon - \varepsilon_{\text{edge}}(k)) / |\varepsilon - \varepsilon_{\text{edge}}(k)|^\alpha$ where the exponent depends only on the Luttinger parameter $K$,

$$\alpha = 1 - \frac{K}{2} \left( \frac{1}{1 - \frac{K}{2}} \right)^2 . \quad (22)$$

This result is the same for the particle and the hole parts of the spectrum. Here $K$ is given by the analytic result in Eq. (5).

Excitations above the spectral edge for Fermi particles with spin can be described using the mobile impurity model in an analogous way. The number of the bosonic fields doubles due to the two spin orientations. Bosonisation of the modes at $\pm k_F$ gives a diagonal Tomonaga-Luttinger model in the basis of spin and charge fields. Here there are four unknown coupling constants between two pair of the canonically conjugated variables of the Tomonaga-Luttinger model and the Fermi-like field of the deep hole. One pair of the constants that corresponds to the coupling to spinon modes are zero due to the symmetry with respect to the spin orientation in the original microscopic model in Eq. (2), where the external magnetic field is zero. Another pair of the constants that correspond to the coupling to holon modes can be identified by considering the same physical properties as for the Fermi particles without spin.

Using the result in Eq. (1) and the Luttinger parameters, the mobile impurity model reads

$$H = \int dx \left[ \sum_{\alpha=\rho, \sigma} \frac{v_\alpha}{2\pi} \left( K_\alpha (\nabla \theta_\alpha)^2 + \frac{(\nabla \varphi_\alpha)^2}{K_\alpha} \right) + \frac{v_a}{\sqrt{2}} \frac{k}{m^*} (K_\sigma \nabla \theta_\rho + \nabla \varphi_\rho) d^\dagger d + \frac{k^2}{2m^*} - \frac{i\nabla}{m^*} \right] , \quad (23)$$

where $k$ is the total momentum of the system - an input parameter of the model, $m^* = mK_\sigma$ is the effective mass of the deep hole, $v_\rho, K_\rho, v_\sigma, K_\sigma$ are the four Luttinger parameters for the spin and the charge modes, the bosonic fields $\varphi_\rho, \varphi_\sigma, \theta_\rho, \theta_\sigma$ are canonically conjugated variables $[\varphi_\rho(x), \varphi_\sigma(y)] = i\pi \delta(x-y)$ of the Tomonaga-Luttinger model, and the field $d$ obey the Fermi commutation rules $\{d(x), d^\dagger(y)\} = \delta(x-y)$.

The diagonalisation of the Hamiltonian in Eq. (23) can be done by a unitary transformation in a very similar fashion to the spinless case. The rotation $e^{-iU}He^{iU}$, where $U = \int dx \left[ C_+ \left( \sqrt{K_\rho} \varphi + \varphi / \sqrt{K_\rho} \right) + C_- \left( \sqrt{K_\rho} \varphi - \varphi / \sqrt{K_\rho} \right) \right] d^\dagger d$ and $C_\pm = \mp \sqrt{K_\rho} e^{-\frac{\pm i}{2} (k - k_F) (K_\rho^{-1} \mp K_\sigma^{-1}) / (k/K_\rho \pm K_F/K_\sigma)}$, removes the coupling term in the Hamiltonian in Eq. (23) allowing straightforward calculations of the observables.

The spectral function in Eq. (3) can be evaluated within the framework of the effective field model in the same way. The original Fermi operators $\psi^\dagger_\alpha(x)$ in the form factor $\langle|f|\psi^\dagger_\alpha(0)|0\rangle^2$ correspond to composite excitation consisting of two bosons (one for spin and one for charge) and one fermion in the field language of the model in Eq. (23), see the state in Fig. 3. The fermionic part gives the dominant contribution to the spectral weight, thus the spectral function reads $A(k, \varepsilon) = \int dtdx e^{i(kx-\varepsilon t)} \langle d^\dagger(x, t) d(0, 0) \rangle$ where $d(x, t) = e^{-iHt}d(x) e^{iHt}$ and $\langle \ldots \rangle$ is the zero temperature expectation value with respect to the model in Eq. (23). In the diagonal basis the average is evaluated over free fields by standard means. Following the steps of Ref. 12 we obtain in proximity of the edge $A(\varepsilon, k) \sim \theta (\varepsilon - \varepsilon_{\text{edge}}(k)) / |\varepsilon - \varepsilon_{\text{edge}}(k)|^{-\alpha}$ where the exponent depends only on a pair of the dimensionless Luttinger parameters and the momentum along the spectral edge,
The result is different for the particle (+) and the hole (−) sectors. The values of the Luttinger parameters obtained numerically using Eq. (20), see Fig. 4 give divergent values of $0 < \alpha < 1$ in the particle sector and cusp-like positive powers $-1 < \alpha < 0$ in the hole sector.

V. BOSONS

While our primary interest lies in Fermi particles, for completeness and to test the generality of our result we consider Bose particles without spin. In this case the application of Bethe-Ansatz approach is very similar to the case of Fermi particles without spin.\textsuperscript{22}

We closely follow the approach of Lieb and Liniger in Ref. \textsuperscript{28}. In the coordinate basis a superposition of cusp-like positive powers\textsuperscript{29} $e^{i\sum_{j<k} \varphi_{jk} \psi_{j}^{\dagger}(x_{k}) \psi_{k}(x_{j})}$, is an eigenstate, $H\Psi = E\Psi$, of the model in Eq. (2) with the corresponding eigenenergy $E = \sum_{j=1}^{N} k_{j}^{2}/(2m)$. Here the operators $\psi(x)$ obey the Bose commutation rules $[\psi(x), \psi^{\dagger}(y)] = \delta(x-y)$. $\sum_{P}$ is a sum over all permutation of $N$ quasimomenta $k_{j}$, and the scattering phases $2\varphi_{jl}$ are fixed by the two-body scattering problem.

The periodic boundary condition quantises a set of $N$ quasimomenta simultaneously

$$k_{j}L - \sum_{i=1,j \neq j}^{N} 2\varphi_{ji} = 2\pi I_{j},$$

where $I_{j}$ are a set of non-equal integer numbers. The total momentum of $N$ particles, $P = \sum_{j} k_{j}$, is a conserved quantity.

The non-linear system of equations Eq. (25) can be solved explicitly in the limit of infinite repulsion. The hard-core bosons in this limit are identical to free fermions\textsuperscript{28} which decouples Eq. (25) into a set of plain wave quantisation conditions, $k_{j} = 2\pi I_{j}/L$. The corresponding eigenstates are Slater determinants whose classification is identical to that of free fermions - all many-body states correspond to all sets of $N$ non-equal integer numbers. These values of quasimomenta $k_{j}$ can be adiabatically continued under a smooth deformation of Eq. (25) by varying the interaction strength from $U = \infty$ to arbitrary value of $U$.

The single parameter that controls the behaviour of interacting bosons can be obtained from the Bogoliubov theory in the weak interaction regime.\textsuperscript{29} This theory is valid when the interaction length is smaller than the kinetic energy of particles, e.g. the high density limit. The same parameter can be generalised to arbitrary interaction strengths\textsuperscript{28}

$$\gamma_{LL} = \frac{2mUL}{N}.$$
have an almost perfect parabola dispersion in all cases. Based on this emergent phenomenon, the spectral edge mode can be described empirically by a free, non-relativistic particle with effective mass identified from the low energy theory as a free electron mass strongly renormalised by interactions via the dimensionless Luttinger parameter $K$ ($K_\sigma$ for particles with spin). However, unlike a free particle, the spectral edge mode is not protected by a symmetry thus deviations from the quadratic dispersion may develop - the biggest discrepancy ($\lesssim 20\%$) occurs for Fermi particles with spin and a very large interaction strength. The empirical model remains robust for finite sound velocities of the collective modes at low energies $v(v_*) > v_1$, where $v_1 = 2\pi/(mL)$ is the quantum of momentum.

The relevance of the Luttinger (low energy) parameters beyond the low energy limit implies that they can be extracted using a much wide range of experimental data using the whole energy window from the bottom of the band to the Fermi energy. However, the dispersion of the spectral edge mode itself can not be used as a qualitative feature to rule out interaction effects since the interactions between particles do not change the parabolic shape of the single particle dispersion. The biggest deviations could be observed for strongly interacting spinful fermions ($K_\sigma \gtrsim 10$), e.g. electrons in semiconductors at low densities or cold Fermi atoms in a 1D trap that would require a good resolution of the experiment.

The main result of this paper Eq. (1) complements the mobile impurity model which was developed by Glazman and co-workers as a description of one-dimensional systems above the spectral edge at high energies. Our explicit expression for the dispersion of the edge mode removes an arbitrary input parameter (curvature of the dispersion) that leaves only the few Luttinger parameters and the bare electron mass as a minimal set of necessary ingredients to model excitations above the spectral edge at arbitrary energies. Within such a framework, for example, exponents of the spectral functions are expressed explicitly in terms of only a few Luttinger parameters. The results in Eqs. (22, 24) provide a systematic way to classify the edge exponents for wide range of microscopic parameters.

VI. CONCLUSIONS

In this work, we have analysed the spectral edge mode for a variety of one-dimensional models with short-range interactions that bounds from below a continuum of many-body excitations. Explicit diagonalisation by means of Bethe-Ansatz techniques shows this mode to breaks down in the ultra-weak interaction regime when the sound velocity of collective modes at low energies mode becomes comparable with its own quantum set by the finite size of the system $v_1 = 2\pi/(mL)$, see the dashed line in Fig. 8. Correspondingly, the threshold for entering this regime becomes $\gamma_{LL} \to 0$ in the thermodynamic limit, as observed in Fig. 8 when $v_1 \to 0$. When $v \sim v_1$ the edge of the spectral function is linear at all energies, including the high energy domain, with the slope that is governed by the kinetic energy of a single free Bose particle.

Acknowledgments

We thank EPSRC for the financial support through Grant No. EP/J016888/1.

1. S. Tomonaga, Prog. Theor. Phys. 5, 544 (1950); J.M. Luttinger, J. Math. Phys. 4, 1154 (1963).
2. P. Nozieres, Theory of Interacting Fermi Systems (Addison-Wesley, 1997).
3. M. Bockrath, D. H. Cobden, J. Lu, A. G. Rinzler, R. E. Smalley, L. Balents, and P. L. McEuen, Nature 397, 598 (1999).
4. O. M. Auslaender, A. Yacoby, R. de Picciotto, K. W. Bald-
win, L. N. Pfeiffer, and K. W. West, Science 295, 825 (2002).
5 Y. Jompol, C. J. B. Ford, J. P. Griffiths, I. Farrer, G. A. C. Jones, D. Anderson, D. A. Ritchie, T. W. Silk, A. J. Schofield, Science 325, 597 (2009).
6 B. Paredes, A. Widera, V. Murg, O. Mandel, S. Folling, I. Cirac, G. V. Shlyapnikov, T. W. Hansch, and I. Bloch, Nature 429, 277 (2004).
7 T. Giamarchi, Quantum Physics in One Dimension (Clarendon Press, Oxford, 2003); A.O. Gogolin, A.A. Nersesyan, and A.M. Tsvelik, Bosonization and Strongly Correlated Systems (Cambridge University Press, 1998).
8 M. Pustilnik, M. Khodas, A. Kamenev, and L. I. Glazman, Phys. Rev. Lett. 96, 196405 (2006); M. Khodas, M. Pustilnik, A. Kamenev, and L. I. Glazman, Phys. Rev. B 76, 155402 (2007).
9 P. Nozieres and C. T. De Dominicis, Phys. Rev. 178, 1097 (1969).
10 Adilet Imambekov and Leonid I. Glazman, Phys. Rev. Lett. 102, 126405 (2009).
11 R. G. Pereira, S. R. White, and I. Affleck, Phys. Rev. Lett. 100, 027206 (2008).
12 T. L. Schmidt, A. Imambekov, and L. I. Glazman, Phys. Rev. Lett. 104, 116403 (2010).
13 In this paper we restrict ourselves to only repulse interactions. An attractive potential admits formation of bound states that can alter the properties of the system completely.
14 A. Imambekov, T. L. Schmidt, and L. I. Glazman, Rev. Mod. Phys. 84, 1253 (2012).
15 A. A. Abrikosov, L. P. Gorkov, and I. E. Dzyaloshinskii, Methods of Quantum Field Theory in Statistical Physics (Dover Publications, New York, 1975).
16 J. von Delft and H. Schoeller, Ann. Phys. 7, 225 (1998).
17 F.D.M. Haldane, Phys. Lett. A 81, 153 (1981).
18 O. Tsyplyatyev and A. J. Schofield, Phys. Rev. B 88, 115142 (2013).
19 C. N. Yang, Phys. Rev. Lett. 19, 1312 (1967); E. H. Lieb and F. Y. Wu, Phys. Rev. Lett. 21, 192 (1968).
20 M. Ogata and H. Shiba, Phys. Rev. B 41, 2326 (1990).
21 V. E. Korepin, N. M. Bogoliubov, and A. G. Izergin, Quantum Inverse Scattering Methods and Correlation Functions (Cambridge University Press, Cambridge, UK, 1993).
22 The definition of $\gamma_YG$ does not change for excited states around the edge of the spectral function in the thermodynamic limit. The average energy per magnon in Eq. (13) is still set by the ground state properties which are not altered by a few excitations within the band of $\lesssim E_F$ in a large system.
23 The biggest deviation of 20% quantifies the discrepancy between the best by Eq. (1) at high energy and the low energy velocity of the collective modes obtained using Eq. (20). The fit by Eq. (1) on its own is in fact more accurate: the coefficient of determination is already $R^2 = 0.9934$ for $N = 80$ in the strong interaction regime with $\gamma_YG = 6.88$.
24 C. F. Coll, Phys. Rev. B 9, 2150 (1974).
25 H. J. Schulz, Phys. Rev. Lett. 64, 2831 (1990).
26 F. H. L. Essler, Phys. Rev. B 81, 205120 (2010).
27 A. Kamenev and L. I. Glazman, Phys. Rev. A 80, 011603(R) (2009).
28 E. H. Lieb and W. Liniger, Phys. Rev. 130, 1605 (1963).
29 L. Tonks, Phys. Rev. 50, 955 (1936); M. Girardeau, J. Math. Phys. 1, 516 (1960).
30 N. N. Bogoliubov, J. Phys. (USSR) 11, 23 (1947).