COMPARATIVE STUDY OF A CUBIC AUTOCATALYTIC REACTION VIA DIFFERENT ANALYSIS METHODS

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Abstract. In this paper we discuss an approximate solutions of the space-time fractional cubic autocatalytic chemical system (STFCACS) equations. The main objective is to find and compare approximate solutions of these equations found using Optimal \( q \)-Homotopy Analysis Method (O\( q \)-HAM), Homotopy Analysis Transform Method (HATM), Varitional Iteration Method (VIM) and Adomian Decomposition Method (ADM).

1. Introduction. If two chemicals, which we label \( A \) and \( B \), react through a mechanism known as cubic autocatalysis, we have the chemical reaction equation \[ A + 2B \rightarrow 3B, \quad \text{rate} \ k uv^2. \] (1)

Here \( k \) is the reaction rate constant and \( u \) and \( v \) are the concentrations of the two chemicals which are measured in moles. The chemical \( B \) is known as the auto-catalyst, since it catalyses its own production. The greater the concentration of \( B \), the faster it is produced by the reaction (1). If these two chemicals then react in a long thin tube, so that their concentrations only vary in the \( x \)-direction along the tube, the main physical processes that act, in the absence of any underlying fluid flow, are chemical reaction and one dimensional diffusion. The equation governing the chemical reaction (1) is the reaction-diffusion system

\[ \frac{\partial u}{\partial t} = D \frac{\partial^2 u}{\partial x^2} - kuv^2, \] (2)

\[ \frac{\partial v}{\partial t} = D \frac{\partial^2 v}{\partial x^2} + kuv^2. \] (3)

Here \( t \) is time and \( D \) the constant diffusivity of the chemicals, with the diffusivity of both species assumed equal.

The reaction-diffusion system (2)–(3) can be non-dimensionalised, so that \( D = 1 \) and \( k = 1 \). Furthermore, this system can be replaced by its equivalent space-time
fractional system by replacing \( u_t, v_t \) by \( u^\alpha_t, v^\alpha_t \) and \( u_{xx}, v_{xx} \) by \( u^{2\beta}_{xx}, v^{2\beta}_{xx} \), respectively, where \( 0 < \alpha, \beta \leq 1 \). We then obtain the space-time fractional derivative STFCACS

\[
\begin{align*}
\frac{u^\alpha_t}{(x,t)} - \frac{u^{2\beta}_{xx}}{(x,t)} + u(x,t)v^2(x,t) &= 0, \\
\frac{v^\alpha_t}{(x,t)} - \frac{v^{2\beta}_{xx}}{(x,t)} - u(x,t)v^2(x,t) &= 0.
\end{align*}
\]  

(4) (5)

We take the initial conditions

\[
\begin{align*}
u(x,0) &= 1 - \sum_{n=1}^{\infty} a \sin(0.5(n\pi) \cos (0.5(2\pi)(\mu_n)), \\
v(x,0) &= \sum_{n=1}^{\infty} b \sin(0.5(n\pi) \cos (0.5(2\pi)(\mu_n)),
\end{align*}
\]  

(6) (7)

where \( \mu_n = \frac{n\pi}{L}, 0 \leq L \leq L_0, L_0 > 0 \) and the boundary conditions

\[
u(0,t) = u(L,t) = 1, \quad v(0,t) = v(L,t) = 0.
\]  

(8)

The present paper is organized as follows: The second section is devoted to the basic idea of the fractional calculus. The third, fourth, fifth and sixth are devoted to applying the Oq-HAM, HATM, VIM and ADM on STFCACS respectively. The seventh section is devoted to the comparison analysis. In the last section, conclusion is presented.

2. Fractional calculus. Here we give some basic definitions and properties of fractional calculus theory [11, 27, 41, 45].

**Definition 2.1.** If \( f(t) \in L_1(a,b) \), the set of all integrable functions, and \( \alpha > 0 \), then the Rimann-Liouville fractional integral of order \( \alpha \), denoted by \( J^\alpha_a+ \) is defined by

\[
J^\alpha_a+ f(t) = \frac{1}{\Gamma(\alpha)} \int_{a}^{t} (t-\tau)^{\alpha-1} f(\tau)d\tau
\]  

(9)

**Definition 2.2.** For \( \alpha > 0 \), the Caputo fractional derivative of order \( \alpha \), denoted by \( C^D_{a+}^\alpha \), is defined by

\[
C^D_{a+}^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_{a}^{t} (t-\tau)^{n-\alpha-1} D^n f(\tau)d\tau,
\]  

(10)

where \( n \) is such that \( n-1 < \alpha < n \) and \( D = \frac{d}{dt} \).

If \( \alpha \) is an integer, then this derivative takes the ordinary derivative

\[
C^D_{a+}^\alpha = D^n, \quad \alpha = 1, 2, 3, ...
\]  

(11)

Finally the Caputo fractional derivative on the whole space \( R \) is defined by

**Definition 2.3.** For \( \alpha > 0 \) the Caputo fractional derivative of order \( \alpha \) on the whole space, denoted by \( C^D_{a+}^\alpha \), is defined by

\[
C^D_{a+}^\alpha f(x) = \frac{1}{\Gamma(n-\alpha)} \int_{-\infty}^{x} (x-\xi)^{n-\alpha-1} D^n f(\xi)d\xi.
\]  

(12)
2.1. Basic idea of Oq-HAM. The principles of the HAM and its applicability for various kinds of differential equations are given in [4, 19, 33, 34, 37, 60, 64, 66]. Also, new results obtained into [2, 25, 47, 48, 49, 50, 53, 54, 55, 59, 62, 63] using the homotopy analysis method. For convenience, we will present a review of the HAM [34]. To describe the basic idea of the standard Oq-HAM [15, 51], we consider the nonlinear differential equation

\[ N[u(x, t)] = 0, \quad t \geq 0, \quad (13) \]

where \( N \) is nonlinear differential operator and \( u(x, t) \) is an unknown function. Liao [33] constructed the so-called zeroth-order deformation equation:

\[
(1 - nq)L[\phi(x, t; q) - u_0(t)] = qhH(x, t)N[\phi(x, t; q)],
\]

(14)

where \( q \in [0, \frac{1}{n}] \) is an embedding parameter, \( h \neq 0 \) is an auxiliary parameter, \( H(x, t) \neq 0 \) is an auxiliary function, \( L \) is an auxiliary linear operator, \( \phi(x, t; q) \) is an unknown function, and \( u_0(t) \) is an initial guess for \( u(x, t) \) which satisfies the initial conditions. It should be emphasized that one has great freedom in choosing the initial guess \( u_0(x, t), L, h \) and \( H(x, t) \). Obviously, when \( q = 0 \) and \( q = \frac{1}{n} \), the following relations hold respectively:

\[
\phi(x, t; 0) = u_0(x, t), \quad \phi(x, t; \frac{1}{n}) = u(x, t).
\]

Expanding \( \phi(x, t; q) \) in Taylor series with respect to \( q \), one has

\[
\phi(x, t; q) = u_0(x, t) + \sum_{m=1}^{\infty} u_m(x, t) q^m, \quad (15)
\]

where

\[
u_m(x, t) = \frac{1}{m!} \frac{\partial^m \phi(x, t; q)}{\partial q^m} \Big|_{q=0}. \quad (16)\]

We assume that the auxiliary parameter \( h \), the auxiliary function \( H(x, t) \), the initial approximation \( u_0(x, t) \) and the auxiliary linear operator \( L \) are selected such that the series (15) converges at \( q = \frac{1}{n} \), and one has

\[
u(x, t) = u_0(x, t) + \sum_{m=1}^{\infty} u_m(x, t) \left( \frac{1}{n} \right)^m. \quad (17)\]

We can deduced the governing equation from the zero order deformation equation by define the vector

\[
\vec{u}_n = \{ u_0(x, t), u_1(x, t), u_2(x, t), \ldots, u_n(x, t) \}. \quad (18)
\]

Differentiating (14) \( m \)-times with respect to \( q \), then setting \( q = 0 \) and dividing them by \( m! \), we have, using (16), the so-called \( m \)-th-order deformation equation

\[
L[u_m(x, t) - \chi_m u_{m-1}(x, t)] = hH(x, t)R_m(\vec{u}_{m-1}(t)), \quad m = 1, 2, 3, \ldots, n, \quad (19)
\]

where

\[
R_m(\vec{u}_{m-1}) = \frac{1}{(m-1)!} \frac{\partial^{m-1} N[\phi(x, t; q)]}{\partial q^{m-1}} \Big|_{q=0}, \quad (20)
\]

and

\[
\chi_m = \begin{cases} 
0, & m \leq 1 \\
\frac{1}{n}, & m > 1.
\end{cases} \quad (21)
\]
2.2. Basic idea of HATM. In this section, we introduce an approximate analytical method, namely the HATM, which is combination of the Laplace decomposition method (LDM) and the homotopy analysis method (HAM) [10, 20, 43, 56, 57]. This scheme is simple to apply to linear and nonlinear fractional differential equations and requires less computational effort compared with other existing methods.

2.2.1. Laplace transform. Let \( f(t) \) be defined for \( 0 \leq t < \infty \). Then, when the improper integral exists, the Laplace transform \( F(s) \) of \( f(t) \), written symbolically as \( F(s) = \mathcal{L}\{f(t)\} \), is defined by

\[
F(s) = \int_0^\infty e^{-st} f(t) dt.
\]

Lemma 2.4. If \( m-1 < \alpha \leq m, m \in \mathbb{N} \), then the Laplace transform of the fractional derivative \( D^\alpha f(t) \) is

\[
\mathcal{L}(D^\alpha f(t)) = s^\alpha F(s) - \sum_{k=0}^{m-1} f^{(k)}(0^+) s^{\alpha-k-1}, \quad t > 0,
\]

where \( F(s) \) is the Laplace transform of \( f(t) \).

Apply the Laplace transform to the nonlinear differential operator \( N \), we can obtain HATM solutions by the similar procedure with Oq-HAM but with \( n = 1 \).

2.3. Basic idea of VIM. To illustrate the basic concept of the variational iteration method, we consider the following general nonlinear equation

\[
D^\alpha u(x,t) + Nu(x,t) = g(x,t), \quad \alpha > 0
\]

subject to the initial value

\[
u^{(k)}(0) = c_k, \quad k = 0, 1, 2, \ldots, n-1, \quad n-1 < \alpha < n
\]

Then successive approximations \( u_m(x,t), m = 0, 1, 2, \cdots \) follows immediately, and consequently the exact solution may be arrived since:

\[
u = \lim_{m \to \infty} u_m.
\]

2.4. Basic idea of ADM. We present the basic idea of the ADM [26] in this section by considering the following nonlinear partial differential equation

\[
D^\alpha u(x,t) + R(u(x,t)) + N(u(x,t)) = g(x,t), \quad \alpha > 0
\]

subject to the initial value

\[
u^{(k)}(0) = c_k, \quad k = 0, 1, 2, \cdots, n-1, \quad n-1 < \alpha < n
\]
where $R$ is the remaining linear operator, which might include other fractional derivatives operator $D^{\nu}(\nu < \alpha)$, $N$ represent a nonlinear operator and $g(x,t)$ is a given continuous function. Now, applying $J^\alpha$ to both the sides of (25), we get
\begin{equation}
  u(x,t) = \sum_{k=0}^{[\alpha]} c_k \frac{t^k}{k!} + J^\alpha g(x,t) - J^\alpha R(u(x,t)) - J^\alpha N(u(x,t)).
\end{equation}
(27)

We employ the Adomian decomposition method to solve equations (26)–(27). Let
\begin{equation}
  u = \sum_{m=0}^\infty u_m,
\end{equation}
(28)
and
\begin{equation}
  N(u) = \sum_{m=0}^\infty A_m,
\end{equation}
(29)
where $A_m$ are Adomian polynomials which depend upon $u$. In view of Equations (28)–(29), (27) takes the form
\begin{equation}
  \sum_{m=0}^\infty u_m = \sum_{k=0}^{[\alpha]} c_k \frac{t^k}{k!} + J^\alpha g(x,t) - J^\alpha R(u(x,t)) - J^\alpha \sum_{m=0}^\infty A_m(u).
\end{equation}
(30)

We set
\begin{equation}
  u_0(x,t) = \sum_{k=0}^{[\alpha]} c_k \frac{t^k}{k!} + J^\alpha g(x,t);
\end{equation}
(31)
\begin{equation}
  u_m = -J^\alpha R(u(x,t)) - J^\alpha \sum_{m=0}^\infty A_m(u),\ m = 0, 1, \cdots
\end{equation}
(32)

In order to determine the Adomian polynomials, we introduce a parameter $\lambda$ and (29) becomes
\begin{equation}
  N \left( \sum_{m=0}^\infty u_m \lambda^m \right) = \sum_{m=0}^\infty A_m \lambda^m.
\end{equation}
(33)

Let $u_\lambda(x,t) = \sum_{m=0}^\infty u_m \lambda^m$. Then
\begin{equation}
  A_m = \frac{1}{m!} \left[ \frac{d^m}{d\lambda^m} N_\lambda(u) \right]_{\lambda=0},
\end{equation}
(34)
where
\begin{equation}
  N_\lambda(y) = N(u_\lambda).
\end{equation}
(35)

In view of (34) and (35), we get
\begin{equation}
  A_m = \left[ \frac{1}{m!} \frac{d^m}{d\lambda^m} N(\sum_{m=0}^\infty u_m \lambda^m) \right]_{\lambda=0}.
\end{equation}
(36)

Hence, (31)–(32) and (36) lead to the following recurrence relations
\begin{equation}
  u_0(x,t) = \sum_{k=0}^{[\alpha]} c_k + J^\alpha g(x,t),\ u_m(x,t)
  = -J^\alpha R(u(x,t)) - J^\alpha \left[ \frac{1}{m!} \frac{d^m}{d\lambda^m} N(\sum_{m=0}^\infty u_m \lambda^m) \right]_{\lambda=0}.
\end{equation}
(37)
We can approximate the solution \( u \) by the truncated series
\[
\phi_k = \sum_{m=0}^{k-1} u_m, \quad \lim_{k \to \infty} \phi_k = u(x, t).
\]

3. The Oq-HAM for STFCACS. In this section, we apply the Oq-HAM to solve STFCACS (4)–(5). According to the theory and procedure outlined in \([33, 34, 35]\) we have great freedom to choose the initial approximations \( u_0(x, t) \) and \( v_0(x, t) \), the auxiliary linear operator \( L \), the auxiliary function \( H(x, t) \) and the auxiliary parameter \( h \). For simplicity we use the auxiliary function \( H(x, t) = 1 \). Through using this method we have \( u_0(x, t) = u(x, 0) \) and \( v_0(x, t) = v(x, 0) \) and the \( m \)-th order deformations are then
\[
u_m(x, t) = \chi_m v_{m-1} + h J^\alpha [R_{2m}(\vec{v}_{m-1})],
\]
where
\[
R_{1m}(\vec{u}_{m-1}) = D_t^{\alpha} u_{m-1} - D_x^{2\beta} u_{m-1} + \sum_{j=0}^{m-1} u_{m-1-j} \sum_{i=0}^j v_i v_{j-i}
\]
and
\[
R_{2m}(\vec{v}_{m-1}) = D_t^{\alpha} v_{m-1} - D_x^{2\beta} v_{m-1} - \sum_{j=0}^{m-1} u_{m-1-j} \sum_{i=0}^j v_i v_{j-i}.
\]

For the first terms of the Oq-HAM series solution we take
\[
u_1(x, t) = \frac{h t^\alpha}{\Gamma(\alpha + 1)} \left( \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} a \sin(0.5(n\pi)) \mu_n^{2\beta} \cos(0.5(n\pi) - \mu_n x - \beta \pi) \right)
\]
\[
\times \cos(0.5(m\pi) - \mu_m x) - \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \sum_{r=1}^{\infty} b^2 \sin(0.5(n\pi)) \sin(0.5(m\pi)) \sin(0.5(r\pi)) \times \cos(0.5(n\pi) - \mu_n x) \cos(0.5(m\pi) - \mu_m x) \cos(0.5(r\pi) - \mu_r x) \right) (42)
\]
and
\[
u_1(x, t) = \frac{h t^\alpha}{\Gamma(\alpha + 1)} \left( -\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} b \sin(0.5(n\pi)) \mu_n^{2\beta} \cos(0.5(n\pi) - \mu_n x - \beta \pi) \right)
\]
\[
- \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} b^2 \sin(0.5(n\pi)) \sin(0.5(m\pi)) \cos(0.5(n\pi) - \mu_n x) \times \cos(0.5(m\pi) - \mu_m x) + \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \sum_{r=1}^{\infty} ab^2 \sin(0.5(n\pi)) \sin(0.5(m\pi)) \sin(0.5(r\pi)) \times \cos(0.5(n\pi) - \mu_n x) \cos(0.5(m\pi) - \mu_m x) \cos(0.5(r\pi) - \mu_r x) \right), (43)
\]
where \( \mu_n = \frac{n\pi}{L} \), \( \mu_m = \frac{m\pi}{L} \) and \( \mu_r = \frac{r\pi}{L} \). Hence, using equation (17), we obtain the approximate solution
Here and Repeating the procedure of [14, 18, 29, 30, 31, 32, 39, 42, 52, 61, 67] we obtain

\[ u(x, t) = u_0(x, t) + \sum_{m=1}^{\infty} u_m(x, t) \left( \frac{1}{n} \right)^m. \]  

(44)

Proceeding in this manner, the rest of the components can be obtained and the Oq-HAM is given by

\[ v(x, t) = v_0(x, t) + \sum_{m=1}^{\infty} v_m(x, t) \left( \frac{1}{n} \right)^m. \]  

(45)

The series (44)–(45) contains the auxiliary parameters \( h \) and \( n \). We evaluate the optimal values of the convergence-control parameters by the minimum of the averaged residual errors \([4, 5, 6, 7, 16, 36, 51, 66]\)

\[ E_u(h) = \frac{1}{N M} \sum_{s=0}^{N} \sum_{j=0}^{M} \left[ N \left( \sum_{k=0}^{m} u_k \left( \frac{100s}{N}, \frac{30j}{M} \right) \right) \right]^2, \]  

(46)

\[ E_v(h) = \frac{1}{N M} \sum_{s=0}^{N} \sum_{j=0}^{M} \left[ M \left( \sum_{k=0}^{m} v_k \left( \frac{100s}{N}, \frac{30j}{M} \right) \right) \right]^2, \]  

(47)

corresponding to a nonlinear algebraic equations

\[ \frac{dE_u(h)}{dh} = 0, \]  

(48)

\[ \frac{dE_v(h)}{dh} = 0. \]  

(49)

4. The HATM for STFCACS. In this section we apply the HATM to solve STFCACS (4)–(5). As in [14, 18, 29, 30, 31, 32, 39, 42, 52, 58, 61, 67], we take the Laplace transform of both sides of equations (4)–(5) to give

\[ \mathcal{L}(u(x, t)) - \frac{1}{s} u(x, 0) + \frac{1}{s^\alpha} \mathcal{L}[-D_x^{2\beta} u(x, t) + u(x, t)v^2(x, t)] = 0 \]  

(50)

\[ \mathcal{L}(v(x, t)) - \frac{1}{s} v(x, 0) + \frac{1}{s^\alpha} \mathcal{L}[-D_x^{2\beta} v(x, t) - u(x, t)v^2(x, t)] = 0. \]  

(51)

Repeating the procedure of [14, 18, 29, 30, 31, 32, 39, 42, 52, 58, 61, 67] we obtain

\[ u_m(x, t) = \chi_m u_{m-1} + h \mathcal{L}^{-1}[R_{3m}(\tilde{u}_{m-1})] \]  

(52)

and

\[ v_m(x, t) = \chi_m v_{m-1} + h \mathcal{L}^{-1}[R_{4m}(\tilde{v}_{m-1})]. \]  

(53)

Here

\[ R_{3m}(\tilde{u}_{m-1}) = \mathcal{L}(u_{m-1}) - \frac{1}{s} (1 - \chi_m) u(x, 0) + \frac{1}{s^\alpha} \mathcal{L}[-D_x^{2\beta} u_{m-1} + \sum_{j=0}^{m-1} u_{m-1-j} \sum_{i=0}^{j} v_i v_{j-i}] \]  

(54)

and

\[ R_{4m}(\tilde{v}_{m-1}) = \mathcal{L}(v_{m-1}) - \frac{1}{s} (1 - \chi_m) v(x, 0) + \frac{1}{s^\alpha} \mathcal{L}[-D_x^{2\beta} v_{m-1} - \sum_{j=0}^{m-1} u_{m-1-j} \sum_{i=0}^{j} v_i v_{j-i}]. \]  

(55)

Consequently, the first terms of the HATM series approximate solution are \( u_0(x, t) = u(x, 0), \) \( v_0(x, t) = v(x, 0) \) and

\[ u_1(x, t) = \frac{h t^\alpha}{\Gamma(\alpha + 1)} \left( \sum_{n=1}^{\infty} a \sin(0.5(n\pi)) \mu_n^{2\beta} \cos(0.5(n\pi) - \mu_n x - \beta \pi) \right) \]
\[ + \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} b^2 \sin(0.5(n\pi)) \sin(0.5(m\pi)) \cos(0.5(n\pi) - \mu_n x) \times \cos(0.5(m\pi) - \mu_m x) \\
- \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \sum_{r=1}^{\infty} ab^2 \sin(0.5(n\pi)) \sin(0.5(m\pi)) \sin(0.5(r\pi)) \times \cos(0.5(n\pi) - \mu_n x) \cos(0.5(m\pi) - \mu_m x) \cos(0.5(r\pi) - \mu_r x)) \] (56)

and

\[ v_1(x, t) = \frac{ht^\alpha}{\Gamma(\alpha + 1)} \left( - \sum_{n=1}^{\infty} b \sin(0.5(n\pi)) \mu_n^{2\beta} \cos(0.5(n\pi) - \mu_n x - \beta\pi) \\
- \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} b^2 \sin(0.5(n\pi)) \sin(0.5(m\pi)) \cos(0.5(n\pi) - \mu_n x) \times \cos(0.5(m\pi) - \mu_m x) \\
+ \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \sum_{r=1}^{\infty} ab^2 \sin(0.5(n\pi)) \sin(0.5(m\pi)) \sin(0.5(r\pi)) \times \cos(0.5(n\pi) - \mu_n x) \cos(0.5(m\pi) - \mu_m x) \cos(0.5(r\pi) - \mu_r x)) \right). \] (57)

We therefore have the approximate solution of the STFCACS (4)–(5) derived using the HATM as

\[ u(x, t) = u_0(x, t) + \sum_{m=1}^{\infty} u_m(x, t) \] (58)

and

\[ v(x, t) = v_0(x, t) + \sum_{m=1}^{\infty} v_m(x, t). \] (59)

5. The VIM for STFCACS. In this section, we apply the VIM to solve the space-time fractional STFCACS (4)–(5). Based on the variational iteration method [3, 21, 22, 23, 38, 44, 46], expression (24) for (4)–(5) can be expressed as

\[ u_{n+1}(x, t) = u_n(x, t) + J_t^\alpha \lambda(\tau)[u^\alpha_{n,\tau} - u_{n,x}^{2\beta} + u_n v_n^2] \] (60)

and

\[ v_{n+1}(x, t) = v_n(x, t) + J_t^\alpha \lambda(\tau)[v^\alpha_{n,\tau} - v_{n,x}^{2\beta} - u_n v_n^2]. \] (61)

We can find the value of \( \lambda(\tau) \) as in [3, 21, 22, 23, 38, 44, 46]. This value is \( \lambda(\tau) = -1 \). Substituting this value of the Lagrange multiplier into the solution (60)–(61), the variational iteration formula gives

\[ u_{n+1}(x, t) = u_n(x, t) - J_t^\alpha [u^\alpha_{n,\tau} - u_{n,x}^{2\beta} + u_n v_n^2] \] (62)

and

\[ v_{n+1}(x, t) = v_n(x, t) - J_t^\alpha [v^\alpha_{n,\tau} - v_{n,x}^{2\beta} - u_n v_n^2]. \] (63)

Finally, the exact solution is obtained using

\[ u(x, t) = \lim_{n \to \infty} u_n(x, t) \] (64)

and

\[ v(x, t) = \lim_{n \to \infty} v_n(x, t). \] (65)
With the initial approximations $u_0(x,t) = u(x,0)$ and $v_0(x,t) = v(x,0)$ the iteration (62)–(63) leads to the solution. The first components of the solutions $u(x,t)$ and $v(x,t)$ using (62)–(63) are given by

$$u_1(x,t) = u_0(x,t) - \frac{t^\alpha}{\Gamma(\alpha + 1)} \left( \sum_{n=1}^{\infty} a \sin(0.5(n\pi)) \mu_n^{2\beta} \cos(0.5(n\pi) - \mu_n x) ight)$$

$$+ \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} b^2 \sin(0.5(n\pi)) \sin(0.5(m\pi)) \cos(0.5(n\pi) - \mu_n x)$$

$$\times \cos(0.5(n\pi) - \mu_n x - \beta\pi)$$

$$- \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \sum_{r=1}^{\infty} a b^2 \sin(0.5(n\pi)) \sin(0.5(m\pi)) \sin(0.5(r\pi))$$

$$\times \cos(0.5(n\pi) - \mu_n x) \cos(0.5(m\pi) - \mu_m x) \cos(0.5(r\pi) - \mu_r x)) \quad (66)$$

and

$$v_1(x,t) = v_0(x,t) - \frac{t^\alpha}{\Gamma(\alpha + 1)} \left( \sum_{n=1}^{\infty} b \sin(0.5(n\pi)) \mu_n^{2\beta} \cos(0.5(n\pi) - \mu_n x) \right)$$

$$- \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} b^2 \sin(0.5(n\pi)) \sin(0.5(m\pi)) \cos(0.5(n\pi) - \mu_n x)$$

$$\times \cos(0.5(n\pi) - \mu_n x - \beta\pi)$$

$$+ \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \sum_{r=1}^{\infty} a b^2 \sin(0.5(n\pi)) \sin(0.5(m\pi)) \sin(0.5(r\pi))$$

$$\times \cos(0.5(n\pi) - \mu_n x) \cos(0.5(m\pi) - \mu_m x) \cos(0.5(r\pi) - \mu_r x)) \quad (67)$$

6. **The ADM for STFCACS.** In this section, we apply the ADM [1, 8, 9, 12, 13, 17, 40] to solve the STFCACS (4)–(5). Applying the operator $J^\alpha$ to both sides of Eq. (4)–(5) yields

$$u(x,t) = f(x) - J^\alpha \left[ -u_x^{2\beta} + uv^2 \right] \quad (68)$$

and

$$v(x,t) = g(x) - J^\alpha \left[ -v_x^{2\beta} - uv^2 \right]. \quad (69)$$

Now the ADM solutions and the nonlinear functions can be represented as the infinite series

$$u(x,t) = u_0(x,t) + \sum_{m=1}^{\infty} u_m(x,t) \quad (70)$$

and

$$v(x,t) = v_0(x,t) + \sum_{m=1}^{\infty} v_m(x,t). \quad (71)$$

In addition,

$$N(u(x,t)) = u(x,t)v^2(x,t) = \sum_{m=0}^{\infty} A_m \quad (72)$$

$$N(v(x,t)) = u(x,t)v^2(x,t) = \sum_{m=0}^{\infty} B_m. \quad (73)$$
where
\[ A_m = \frac{1}{m!} \left[ \frac{d^m}{d\lambda^m} N(\sum_{k=0}^{m} \lambda^k u_k(x,t)) \right]_{\lambda=0} \] (74) and
\[ B_m = \frac{1}{m!} \left[ \frac{d^m}{d\lambda^m} N(\sum_{k=0}^{m} \lambda^k v_k(x,t)) \right]_{\lambda=0} . \] (75)

\( A_m \) and \( B_m \) are called Adomian polynomials. Furthermore, the components \( u_m(x,t) \) and \( v_m(x,t) \) of the solutions \( u(x,t) \) and \( v(x,t) \) are determined from the initial approximations
\[ u_0(x,t) = u(x,0) \] (76) and
\[ v_0(x,t) = v(x,0) \] (77) and the recurrence relations
\[ u_{m+1}(x,t) = -J^\alpha [-u_{2,m}^{2\beta} + A_m] \] (78) and
\[ v_{m+1}(x,t) = -J^\alpha [-v_{2,m}^{2\beta} - B_m]. \] (79)

Now if we take the initial values \( u_0(x,0) = u(x,0) \) and \( v_0(x,0) = v(x,0) \), we obtain the first iterates
\[ u_1(x,t) = -\frac{t^\alpha}{\Gamma(\alpha+1)} \left( \sum_{n=1}^{\infty} a \sin(0.5(n\pi)) \mu_n^{2\beta} \cos(0.5(n\pi)) - \mu_n x - \beta \pi \right) \]
\[ + \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} b^2 \sin(0.5(n\pi)) \sin(0.5(m\pi)) \cos(0.5(n\pi)) - \mu_n x \]
\[ \times \cos(0.5(m\pi) - \mu_m x) \]
\[ - \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \sum_{r=1}^{\infty} ab^2 \sin(0.5(n\pi)) \sin(0.5(m\pi)) \sin(0.5(r\pi)) \]
\[ \times \cos(0.5(n\pi) - \mu_n x) \cos(0.5(m\pi) - \mu_m x) \cos(0.5(r\pi) - \mu_r x) \] (80)

\[ v_1(x,t) = -\frac{t^\alpha}{\Gamma(\alpha+1)} \left( -\sum_{n=1}^{\infty} b \sin(0.5(n\pi)) \cos(0.5(n\pi)) - \mu_n x - \beta \pi \right) \]
\[ - \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} b^2 \sin(0.5(n\pi)) \sin(0.5(m\pi)) \cos(0.5(n\pi)) - \mu_n x \]
\[ \times \cos(0.5(m\pi) - \mu_m x) \]
\[ + \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \sum_{r=1}^{\infty} ab^2 \sin(0.5(n\pi)) \sin(0.5(m\pi)) \sin(0.5(r\pi)) \]
\[ \times \cos(0.5(n\pi) - \mu_n x) \cos(0.5(m\pi) - \mu_m x) \cos(0.5(r\pi) - \mu_r x) \] (81)

Proceed in this way, the rest of the components can be obtained. The general form of the approximations of the STFCACS (4)–(5) derived using the ADM as (70)–(71).
7. Comparison analysis. In this section we compare the solutions obtained above using the four methods for fractional differential equations with the numerical solutions of the fractional space-time STFCACS obtained using the command NDSolve of Mathematica 9. After substituting the initial values for \( u(x,0) \) and \( v(x,0) \) into the space-time fractional STFCACS (4)–(5), we obtain the first approximation of the VIM, which are the same as the first two terms of the Oq-HAM, HATM and ADM for (4)–(5). So the errors of each method are the same and we need more iterations to find differences between the methods. A comparison between numerical solutions and solutions obtained using the Oq-HAM, HATM, VIM and ADM methods are shown in Figures 1–4 for \( \alpha = 1, \beta = 1, a = 0.001 \) and \( b = 0.001 \). The same comparisons are shown in Figures 7–10, but with \( \alpha = 0.9 \) and \( \beta = 0.99 \). It can be seen from Figures 1–4 and Figures 7–10 that the order of the maximum errors of these methods is approximately \( 10^{-5} \).

We shall now compare the results using these approximate methods with numerical solutions as a function of \( x \). Figures 5–6 show these comparisons. Due to the periodic initial conditions for our problem, the errors depend on \( x \). In Figures 5–6(b), (c), (d) and (e), showing the ADM solution for \( 0 < t < 25, 0 < t < 70, 0 < t < 65 \) and \( 0 < t < 40 \) respectively. It can be seen that the ADM solution converges more rapidly than the Oq-HAM, HATM and VIM solutions. However, the errors displayed in Figures 5–6(a) and (f) are uniform. Furthermore, these figures show that the Oq-HAM converges more rapidly than the HATM, VIM and ADM. Also, we noted from Figures 5–6(a), (e) and (f) that the errors of the VIM and ADM are identical and of the same order. The solutions obtained using the Oq-HAM, HATM, VIM and ADM for the fractional space-time STFCACS (4)–(5) with \( \alpha = 0.4, 0.7, 0.99 \) and \( \beta = 0.7, 0.9, 0.99 \) and \( t = 10, a = 0.4, b = 0.2, n = 5, h_{Oq-HAM} = -3.00 \) and \( h_{HATM} = -0.64 \) are plotted in Figures 11–13. These figures show the convergence of the solutions using these methods. The solutions show the same behaviour for these different values of \( \alpha \) and \( \beta \). Figures 14–17 show the solutions obtained using the Oq-HAM, HATM, VIM and ADM for different values of \( \alpha \) and \( \beta \). It can be seen from this figure, that the solutions based on the four methods approach the numerical solution as \( \alpha \to 1 \) and \( \beta \to 1 \).

8. Conclusion. In this paper the Oq-HAM, HATM, VIM and ADM methods have been applied to efficiently obtain approximate solutions of the space-time fractional STFCACS. It was show that the Oq-HAM, HATM, VIM and ADM can be successfully applied to the STFCACS.

The main advantage of the four methods over mesh points methods is that they do not require discretization of the variables, i.e. time and space, and thus they are not affected by computation round off errors and one is not faced with the necessity of large computer memory and time. The four methods provide the solutions in terms of convergent series with easily computable components. The main disadvantage of these methods is they only give a good approximation of the true solution in a restricted region. The first two terms of the Oq-HAM, HATM and ADM solutions and the first approximation of the VIM are identical. Hence, we computed the first three terms of the Oq-HAM, HATM and ADM and the second approximation for the VIM. The efficiency and accuracy of these methods are clear from the comparisons with numerical solutions displayed in the figures. Comparisons of solutions obtained using the Oq-HAM, HATM, VIM and ADM methods with numerical results obtained using Mathematica show the efficiency of the methods. Finally, we
found that the Oq-HAM has more rapid convergence than the HATM, VIM and ADM.

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Figure 1. The absolute error between the 3-terms of Oq-HAM solutions and numerical method using Mathematica for (4)–(5) with $\alpha = 1, \beta = 1, a = 0.001, b = 0.001, h = -3.055, n = 5$.

Figure 2. The absolute error between the 3-terms of HATM solutions and numerical method using Mathematica for (4)–(5) with $\alpha = 1, \beta = 1, a = 0.001, b = 0.001, h = -0.64$.

Figure 3. The absolute error between the second approximation by VIM and the numerical method using Mathematica for (4)–(5) with $\alpha = 1, \beta = 1, a = 0.001, b = 0.001$.

Figure 4. The absolute error between the 3-terms of ADM and the numerical method using Mathematica for (4)–(5) with $\alpha = 1, \beta = 1, a = 0.001, b = 0.001$. 
Figure 5. The comparison of Oq-HAM, HATM, VIM and ADM for (4)–(5) with numerical method in Mathematica for \( x = 0.1, 5, 20, 40, 100 \) respectively and \( \alpha = 1, \beta = 1, a = 0.001, b = 0.001, n = 5, h_{Oq-HAM} = -3.055, h_{HATM} = 0.64 \). Dash - dotted line (Oq-HAM), dotted line (HATM), dash line (VIM), and solid line (ADM).
Figure 6. The comparison of Oq-HAM, HATM, VIM and ADM for (4)–(5) with numerical method in Mathematica for $x = 0.1, 5, 20, 40, 100$ respectively and $\alpha = 1, \beta = 1, a = 0.001, b = 0.001, n = 5, h_{Oq-HAM} = -3.055, h_{HATM} = 0.64$. Dash-dotted line (Oq-HAM), dotted line (HATM), dash line (VIM), and solid line (ADM).

Figure 7. The absolute error between the 3-terms of Oq-HAM solutions and numerical method using Mathematica for (4)–(5) with $\alpha = 0.9, \beta = 0.99, a = 0.001, b = 0.001, h = -1.9, n = 5$. 
Figure 8. The absolute error between the 3-terms of HATM solutions and numerical method using Mathematica for (4)–(5) with \( \alpha = 0.9, \beta = 0.99, a = 0.001, b = 0.001, h = -0.64 \).

Figure 9. The absolute error between the second approximation by VIM and the numerical method using Mathematica for (4)–(5) with \( \alpha = 0.9, \beta = 0.99, a = 0.001, b = 0.001 \).

Figure 10. The absolute error between the 3-terms of ADM and the numerical method using Mathematica for (4)–(5) with \( \alpha = 0.9, \beta = 0.99, a = 0.001, b = 0.001 \).

Figure 11. The plot of Oq-HAM, HATM, VIM and ADM for (4)–(5) with \( \alpha = 0.4, \beta = 0.7, a = 0.4, b = 0.2, n = 5, h_{Oq-HAM} = -3.00, h_{HATM} = -0.64 \). Dash - dotted line (Oq-HAM), dotted line (HATM), dash line (VIM), and solid line (ADM).
Figure 12. The plot of Oq-HAM, HATM, VIM and ADM for (4)–(5) with $\alpha = 0.7, \beta = 0.9, a = 0.4, b = 0.2, n = 5, h_{Oq-HAM} = -3.00, h_{HATM} = -0.64$. Dash - dotted line (Oq-HAM), dotted line (HATM), dash line (VIM), and solid line (ADM).

Figure 13. The plot of Oq-HAM, HATM, VIM and ADM for (4)–(5) with $\alpha = 0.99, \beta = 0.99, a = 0.4, b = 0.2, n = 5, h_{Oq-HAM} = -3.00, h_{HATM} = -0.64$. Dash - dotted line (Oq-HAM), dotted line (HATM), dash line (VIM), and solid line (ADM).

Figure 14. The surface of Oq-HAM for (4)–(5) with $\alpha = 0.5, 0.8, 1.00, \beta = 0.75, 0.90, 1.00$ and $a = 0.4, b = 0.2, n = 5, h_{Oq-HAM} = -3.00$. 
Figure 15. The surface of HATM for (4)–(5) with $\alpha = 0.5, 0.8, 1.00, \beta = 0.75, 0.90, 1.00$ and $a = 0.4, b = 0.2, h_{HATM} = -0.64$.

Figure 16. The surface of VIM for (4)–(5) with $\alpha = 0.5, 0.8, 1.00, \beta = 0.75, 0.90, 1.00$ and $a = 0.4, b = 0.2$.

Figure 17. The surface of ADM for (4)–(5) with $\alpha = 0.5, 0.8, 1.00, \beta = 0.75, 0.90, 1.00$ and $a = 0.4, b = 0.2$. 