Martingale Transforms, the Dyadic Shift and the Hilbert Transform: A Sufficient Condition for Boundedness Between Matrix Weighted Spaces

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Abstract. We give sufficient conditions on $N \times N$ matrix weights $U$ and $V$ for the dyadic martingale transforms to be uniformly bounded from $L^2(V)$ to $L^2(U)$. We also show that these conditions imply the uniform boundedness of the dyadic shifts as well as the boundedness of the Hilbert transform between these spaces.

1. Introduction

Much progress has been made recently on the two-weight problem for various important operators, for example the Sawyer type characterizations of F. Nazarov, S. Treil and A. Volberg, see e.g. [6], and the two-weight inequalities for maximal singular integrals by M. Lacey, E.T. Sawyer and I. Uriarte-Tuero [13]. This is currently an area of much activity and new proofs with broader scope and deeper insight are appearing. Little attention has been given so far to understanding two-weight problems on vector-valued function spaces (the work of C. M. Pereyra and N. H. Katz [8] being a notable exception), in contrast to the one-weight case, for which an analogue of the Hunt-Muckenhoupt-Wheeden characterization has been shown in [12] by S. Treil and A. Volberg. A sufficient condition for the operator weight case has been given by S. Pott in [11] and in [14] it is shown that the dyadic operator weight analogue of the matrix weight dyadic Hunt-Muckenhoupt-Wheeden condition does not imply the boundedness of the martingale transforms. We turn our attention firstly to conditions which imply the uniform boundedness of dyadic martingale transforms and then to other dyadic operators. The motivation here is that such dyadic operators can often be used as models for more general singular integral operators.

2. The Martingale Transform

Let $D$ denote the standard grid of dyadic subintervals of $\mathbb{R}$, $D = [k2^{-n}, (k+1)2^{-n})$ where $n$ and $k$ range over the integers. The Haar functions associated to a dyadic interval $I$ are defined as $h_I = \frac{1}{\sqrt{I}} (\chi_{I_+} - \chi_{I_-})$, where $I_-$ and $I_+$ are the largest proper dyadic subintervals of $I$, on the right and the left respectively. The $h_I$ form

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an orthonormal basis for $L^2(\mathbb{R})$. Let $L^2(\mathbb{R}, \mathbb{C}^n)$ denote the space of measurable functions

$$\left\{ f : \mathbb{R} \to \mathbb{C}^n : \int_{\mathbb{R}} \langle f(t), f(t)\rangle_{\mathbb{C}^n} dt < \infty \right\},$$

$$\|f\|_{L^2(\mathbb{R}, \mathbb{C}^n)} = \left( \int_{\mathbb{R}} \langle f(t), f(t)\rangle_{\mathbb{C}^n} dt \right)^{\frac{1}{2}}.$$

We consider the operator $T_\sigma$ on $L^2(\mathbb{R}, \mathbb{C}^n)$ defined by the mapping

$$T_\sigma f \mapsto \sum_{I \in D} \sigma(I) h_I f_I$$

where $f_I = \int_I f h_I$ is the Haar coefficient for $I$ and $\sigma(I) = \pm 1$. The $T_\sigma$ are dyadic martingale transforms and are unitary operators on $L^2(\mathbb{R}, \mathbb{C}^n)$. For a matrix valued function $V$ which is positive and invertible almost everywhere, let $L^2(\mathbb{R}, \mathbb{C}^n, V)$ be the space of measurable functions

$$\left\{ f : \mathbb{R} \to \mathbb{C}^n : \int_{\mathbb{R}} \langle V(t)^{\frac{1}{2}} f(t), V(t)^{\frac{1}{2}} f(t)\rangle dt < \infty \right\}$$

with norm

$$\|f\|_{L^2(\mathbb{R}, \mathbb{C}^n, V)} = \left( \int_{\mathbb{R}} \langle V(t)^{\frac{1}{2}} f(t), V(t)^{\frac{1}{2}} f(t)\rangle dt \right)^{\frac{1}{2}}.$$

This generalizes the notion of weighted $L^2$ spaces of scalar functions where a weight is a measurable almost everywhere positive function. We refer to matrix functions which are measurable, almost everywhere positive and invertible as matrix weights. The purpose of this paper is to find conditions on a pair of matrix weights, $U$ and $V$, which imply that the dyadic martingale transforms are uniformly bounded from $L^2(\mathbb{R}, \mathbb{C}^n, V)$ to $L^2(\mathbb{R}, \mathbb{C}^n, U)$. This is equivalent to showing that the operators $M_V^{-\frac{1}{2}} T_\sigma M_U^{\frac{1}{2}}$ are uniformly bounded on the unweighted space $L^2(\mathbb{R}, \mathbb{C}^n)$. The sufficient conditions we find on a pair of matrix weights are a joint $A_2$ condition, a matrix $A_\infty$ condition on one weight and a matrix reverse Hölder condition on the other weight. We can also as a corollary replace the matrix reverse Hölder condition by the matrix $A_\infty$ condition. The matrix $A_\infty$ and reverse Hölder condition will be discussed in the next section. In what follows we will denote $L^2(\mathbb{R}, \mathbb{C}^n, V)$ and $L^2(\mathbb{R}, \mathbb{C}^n, U)$ by $L^2(V)$ and $L^2(U)$. 

3. The $A_{2,0}$ condition and reverse Hölder

**Definition 3.1.** A matrix weight $U$ satisfies the dyadic reverse Hölder inequality if there exists constants $C > 0$ and $r > 2$ such that

$$\int_I \|U^{\frac{1}{2}}(x) (U)^{-\frac{1}{2}} y\|^r dx \leq C \|I\| \|y\|^r$$

holds for all dyadic intervals $I$ and nonzero vectors $y$.

Note that our definition of the reverse Hölder property is in general weaker the existing definition in the literature by Christ and Goldberg [2], but is equivalent for finite dimensional spaces. Our definition generalizes the scalar version and is in a form we find applicable.
**Definition 3.2.** We say that a matrix weight \( U \) is in the \( A_{2,0} \) class of weights if the following inequality holds uniformly over all intervals \( I \):

\[
\det (U)_I \leq C \exp\{\langle \log \det U \rangle_I\}.
\]

This \( A_{2,0} \) condition is a matrix analog of the scalar \( A_{\infty} \) condition, see [4] for discussion on this. Also see [1] for some reformulations and context of this property.

**Lemma 3.3.** If a matrix weight \( U \) satisfies the \( A_{2,0} \) condition, then it satisfies the reverse Hölder inequality.

**Proof.** If the weight \( U \) has the \( A_{2,0} \) condition, then by Lemma 3.2 and Lemma 3.3 of [1] we have that

\[
\left\{ \frac{1}{|I|} \int_I ||U^{\frac{1}{2}} x||^2 \right\}^{\frac{1}{r}} \leq C \exp\{\langle \log ||U^{\frac{1}{2}} x|| \rangle_I\},
\]

for all nonzero \( x \) and intervals \( I \). Consequently,

\[
\frac{1}{|I|} \int_I ||U^{\frac{1}{2}} x||^2 \leq C \exp\{\langle \log ||U^{\frac{1}{2}} x||^2 \rangle_I\}
\]

and thus the scalar weight \( ||U^{\frac{1}{2}} x||^2 \) satisfies the \( A_{\infty} \) condition and hence a reverse Hölder inequality;

\[
\left\{ \frac{1}{|I|} \int_I ||U^{\frac{1}{2}} x||^{2r} \right\}^{\frac{1}{r}} \leq C \left\{ \frac{1}{|I|} \int_I ||U^{\frac{1}{2}} x||^2 \right\}^{\frac{1}{2}}
\]

for some \( r > 1 \), all intervals \( I \) and all nonzero \( x \). Note that the index \( r \) does not depend on \( x \) because it only depends on the \( A_{\infty} \) constant \( C \) in (3.1), which is uniform for all \( x \). As this is true for all nonzero \( x \), we can replace \( x \) by \( \langle U \rangle_I^{-\frac{1}{2}} y \), where \( 0 \neq y \in \mathbb{C}^n \). Thus for all intervals \( I \in \mathbb{R} \) and \( y \in \mathbb{C}^n \)

\[
\left\{ \frac{1}{|I|} \int_I ||U^{\frac{1}{2}} \langle U \rangle_I^{-\frac{1}{2}} y||^{2r} \right\}^{\frac{1}{r}} \leq C \left\{ \frac{1}{|I|} \int_I ||U^{\frac{1}{2}} \langle U \rangle_I^{-\frac{1}{2}} y||^2 \right\}^{\frac{1}{2}} = C||y||.
\]

\( \square \)

## 4. Boundedness of the Martingale Transform

We are now in a position to state our main theorem concerning sufficient conditions for the boundedness of the dyadic martingale transforms:

**Theorem 4.1.** Let \( U \) and \( V \) be matrix weights satisfying the joint \( A_2 \) condition

\[
\langle V^{-1} \rangle_I^{\frac{1}{2}} \langle U \rangle_I \langle V^{-1} \rangle_I^{\frac{1}{2}} \leq C
\]

for all dyadic intervals \( I \), where \( C \) is a constant multiple of the identity. If \( V^{-1} \in A_{2,0} \) and \( U \) satisfies the matrix reverse Hölder inequality, then the dyadic martingale transforms are uniformly bounded from \( L^2(V) \) to \( L^2(U) \).

**Corollary 4.2.** Let \( U \) and \( V \) be matrix weights satisfying a joint \( A_2 \) condition

\[
\langle V^{-1} \rangle_I^{\frac{1}{2}} \langle U \rangle_I \langle V^{-1} \rangle_I^{\frac{1}{2}} \leq C
\]

for all dyadic intervals \( I \), where \( C \) is a constant multiple of the identity. If \( U \) and \( V^{-1} \) are also in \( A_{2,0} \), then the dyadic martingale transforms are uniformly bounded from \( L^2(V) \) to \( L^2(U) \).
Proof. By Lemma 3.3 Theorem 1.1 implies this corollary.

Note that the conditions on the matrix weights $U$ and $V^{-1}$ are symmetric in this corollary. Also in Theorem 6.1 of [3] the conditions and implications in Corollary 4.2 are stated but specifically for the scalar valued function space setting, this is also mentioned in [5].

5. Proof of Theorem 1.1 using a two-weighted dyadic square function

We introduce the operator $D_{V^{-1}}$ defined by

$$D_{V^{-1}}f = D_{V^{-1}} \sum_{I \in \mathcal{D}} f_I h_I(x) \mapsto \sum_{I \in \mathcal{D}} \langle V^{-1} \rangle_I^\frac{1}{2} f_I h_I(x)$$

for functions with finite Haar expansion.

Write $M_V^{-\frac{1}{2}} T_\sigma M_U^\frac{1}{2}$ as $M_V^{-\frac{1}{2}} T_\sigma D_{V^{-1}}^{-1} D_{V^{-1}} M_U^\frac{1}{2}$ and note that $T_\sigma$ and $D_{V^{-1}}^{-1}$ commute. This allows us to estimate the norm as

$$\|M_V^{-\frac{1}{2}} T_\sigma M_U^\frac{1}{2}\| = \|M_V^{-\frac{1}{2}} T_\sigma D_{V^{-1}}^{-1} D_{V^{-1}} M_U^\frac{1}{2}\| \leq \|M_V^{-\frac{1}{2}} D_{V^{-1}}^{-1} \| T_\sigma \| D_{V^{-1}} M_U^\frac{1}{2}\|.$$

We know that $T_\sigma$ is bounded on unweighted $L^2$ so we are interested in finding conditions on the matrix weights $U$ and $V^{-1}$ that imply the boundedness of the operators $M_V^{-\frac{1}{2}} D_{V^{-1}}^{-1}$ and $D_{V^{-1}} M_U^\frac{1}{2}$ on unweighted $L^2(\mathbb{R}, \mathbb{C}^n)$.

We deal with $D_{V^{-1}} M_U^\frac{1}{2}$, a two-weighted dyadic square function, using a stopping time argument and Cotlar’s Lemma.

**Theorem 5.1.** Let $U$ and $V^{-1}$ be matrix weights such that $U$ has the dyadic reverse Hölder inequality and such that for all dyadic intervals $I$,

$$\langle V^{-1} \rangle_I^\frac{1}{2} \langle (U)^{-1} \rangle_I^\frac{1}{2} < C.$$

Then the two-weighted square function $S = M_U^\frac{1}{2} D_{V^{-1}}$ is bounded on $L^2(\mathbb{R}, \mathbb{C}^n)$.

For $M_V^{-\frac{1}{2}} D_{V^{-1}}^{-1}$ we state without proof a theorem of Nazarov and Treil.

**Theorem 5.2.** Let $U$ be a matrix weight such that $W \in A_{2,0}$. Then $D_{W^{-1}} M_W^\frac{1}{2}$ is bounded on $L^2(\mathbb{R}, \mathbb{C}^n)$.

Proof. This is Theorem 7.8 of [3]. Note that the proof of this theorem uses a Bellman function technique. □

This theorem also applies to $M_V^{-\frac{1}{2}} D_{V^{-1}}^{-1}$ if we note that its adjoint is $D_{V^{-1}}^{-1} M_V^{-\frac{1}{2}}$.

We now introduce the stopping time used in the proof of Theorem 5.1.

5.1. Stopping Time. Let $\lambda > 1$ and let $\mathcal{J}_\lambda(J)$ be the collection of maximal dyadic subintervals $I_\lambda$ of $J$ such that

$$\| \frac{1}{|I_\lambda|} \int_{I_\lambda} \langle V^{-1} \rangle_J^\frac{1}{2} U(x) \langle V^{-1} \rangle_J^\frac{1}{2} dx \| > \lambda$$

or

$$\| \frac{1}{|I_\lambda|} \int_{I_\lambda} \langle V^{-1} \rangle_J^\frac{1}{2} V^{-1}(x) \langle V^{-1} \rangle_J^\frac{1}{2} dx \| > \lambda$$

(5.1)

or

(5.2)
or

\[ \| \frac{1}{|I\lambda|} \int_{J_{1\lambda}} \langle U \rangle_I^{-1/2} U(x) \langle U \rangle_I^{-1/2} dx \| > \lambda. \]  

Then we define \( J_{\lambda,k}(J) \) as \( \cup_{J \in J_{\lambda,k-1}(J)} J_{\lambda,1}(I) \) for \( k > 1 \). Let \( F_{\lambda,1}(J) \) be the collection of those dyadic subintervals of \( J \) which are not a subinterval of any interval in \( J_{\lambda,1}(J) \). We likewise define \( F_{\lambda,k}(J) \) iteratively to be \( \cup_{J \in J_{\lambda,k-1}(J)} F_{\lambda,1}(I) \). Then \( \cup_{k} F_{\lambda,k}(I) \) forms a decomposition of the dyadic subintervals of \( I \).

**Lemma 5.3.** If \( U \) and \( V \) are matrix weights such that for some \( C > 0 \)

\[ \langle V^{-1} \rangle_I^\frac{1}{2} \langle U \rangle_I^\frac{1}{2} < C \]  

for all dyadic intervals \( I \), then \( J \) is a decaying stopping time for some \( \lambda > 1 \). By decaying stopping time, we mean that for a sufficiently large \( \lambda \), we have a constant \( 0 < \delta < 1 \) such that \( |J(I)_{\lambda,k}| \leq \delta^k |I| \) for all \( k \). We call this conditions on the two matrix weights the joint \( A_2 \) condition.

**Proof.** We first restrict ourselves to showing that \( J_{\lambda,k}(J) = \cup_{J \in J_{\lambda,k-1}(J)} J_{\lambda,1}(I) \) is a decaying stopping time when \( J_{\lambda,1}(I) \) is defined as the collection of maximal subintervals of \( I \) satisfying only \( (5.3) \) rather than all three conditions.

We have the following series of inequalities;

\[ |I| \geq \| \int_I \langle U \rangle_I^{-1/2} U(x) \langle U \rangle_I^{-1/2} dx \| \geq \sum_{J \in J_{\lambda,1}} \| \int_J \langle U \rangle_J^{-1/2} U(x) \langle U \rangle_J^{-1/2} dx \| \]

\[ \geq C_n \sum_{J \in J_{\lambda,1}} \| \int_J \langle U \rangle_J^{-1/2} U(x) \langle U \rangle_J^{-1/2} dx \|, \]

where \( C_n \) is a constant dependent on the matrix. This is possible due to the equivalence of all matrix norms and the additivity of the trace norm on positive matrices. By \( (5.3) \),

\[ C_n \sum_{J \in J_{\lambda,1}} \| \int_J \langle U \rangle_J^{-1/2} U(x) \langle U \rangle_J^{-1/2} dx \| \geq C_n \lambda \sum_{J \in J_{\lambda,1}} |J| \]

and hence

\[ \frac{1}{\lambda C_n} |I| \geq \sum_{J \in J_{\lambda,1}} |J| = |J_{\lambda,1}|. \]

Thus we can choose \( \lambda \) to be large enough such that \( \frac{1}{\lambda C_n} < 1 \) and we have \( |J_{\lambda,1}(I)| < \delta |I| \). Iteration now yields that \( |J_{\lambda,k}(I)| < \delta^k |I| \). We use a similar argument for \( 5.1 \) and \( 5.2 \) individually and then note that the finite union of decaying stopping times will also be a decaying stopping time, after a possible change of \( \lambda \).

\[ \square \]

**5.2. Proof of Theorem 5.1** This proof of this theorem is where the core of our proof takes place, it draws from Theorem 3.1 in [11]. We are presenting a generalization for the finite dimensional case.

**Proof.** We choose \( \lambda > 0 \) such that the condition \( J \) in \( (5.3) \) is a decaying stopping time. First note that almost everywhere on \( J \setminus \cup J(J) \):

\[ \langle V^{-1} \rangle_J^\frac{1}{2} U(x) \langle V^{-1} \rangle_J^\frac{1}{2} \leq \lambda \]
\[ \langle U \rangle_{J}^{\frac{1}{2}} U(x) \langle U \rangle_{J}^{\frac{1}{2}} \leq \lambda \]

and

\[ \langle V^{-1} \rangle_{J}^{\frac{1}{2}} V^{-1}(x) \langle V^{-1} \rangle_{J}^{\frac{1}{2}} \leq \lambda. \]

In this context \( \lambda \) stands for the identity matrix scaled by \( \lambda \), and the inequalities are matrix inequalities. Let us take \( f \in L^2(\mathbb{R}, \mathbb{C}^n) \) with finite Haar expansion. Assume without loss that \( f \) is supported in the unit interval. We write \( J_j \) and \( F_j \) for \( J_{\lambda, j}([0, 1]) \) and \( F_{\lambda, j}([0, 1]) \).

Define

\[ \triangle_j f = \sum_{K \in F_j} h_K f_K \]

and

\[ S_j f = S \triangle_j f = U^{\frac{1}{2}} \sum_{K \in F_j} \langle V^{-1} \rangle_{K}^{\frac{1}{2}} h_K f_K. \]

We can check that \( \sum_{j=1}^{\infty} \triangle_j f = f \) and also that

\[ \sum_{j=1}^{\infty} S_j f = S f. \]

We show that \( S \) is bounded using Cotlar's Lemma. First note that

\[ \|S_j f\|_{L^2} = \int_{\bigcup J_{j-1}} \|S_j f\|_{\mathbb{C}^n}^2 \, dx = \int_{\bigcup J_{j-1}\setminus\bigcup J_j} \|S_j f\|_{\mathbb{C}^n}^2 \, dx + \int_{\bigcup J_j} \|S_j f\|_{\mathbb{C}^n}^2 \, dx. \]

We estimate \( \int_{\bigcup J_{j-1}\setminus\bigcup J_j} \|S_j f\|_{\mathbb{C}^n}^2 \, dx \) and then \( \int_{\bigcup J_j} \|S_j f\|_{\mathbb{C}^n}^2 \, dx \).

\[
\begin{align*}
\int_{\bigcup J_{j-1}\setminus\bigcup J_j} \|S_j f\|_{\mathbb{C}^n}^2 \, dx &= \sum_{J \in J_{j-1}} \int_{J \setminus \bigcup J_j} \|S_j f\|_{\mathbb{C}^n}^2 \, dx \\
&= \sum_{J \in J_{j-1}} \int_{J \setminus \bigcup J_j} \|U^{\frac{1}{2}}(x) \sum_{K \in F(J)} \langle V^{-1} \rangle_{K}^{\frac{1}{2}} h_K(x) f_K \|_{\mathbb{C}^n}^2 \, dx \\
&= \sum_{J \in J_{j-1}} \int_{J \setminus \bigcup J_j} \|U^{\frac{1}{2}}(x) \langle V^{-1} \rangle_{J}^{\frac{1}{2}} \langle V^{-1} \rangle_{J}^{-\frac{1}{2}} \sum_{K \in F(J)} \langle V^{-1} \rangle_{K}^{\frac{1}{2}} h_K(x) f_K \|_{\mathbb{C}^n}^2 \, dx \\
&\leq \sum_{J \in J_{j-1}} \int_{J \setminus \bigcup J_j} \lambda \| \langle V^{-1} \rangle_{J}^{-\frac{1}{2}} \sum_{K \in F(J)} \langle V^{-1} \rangle_{K}^{\frac{1}{2}} h_K(x) f_K \|_{\mathbb{C}^n}^2 \, dx \\
&\leq \sum_{J \in J_{j-1}} \int_J \lambda \sum_{K \in F(J)} \| \langle V^{-1} \rangle_{J}^{-\frac{1}{2}} \langle V^{-1} \rangle_{K}^{\frac{1}{2}} \|^2 \| f_K \|_{\mathbb{C}^n}^2 \, dx \\
&\leq \sum_{J \in J_{j-1}} \int_J \lambda \sum_{K \in F(J)} \| f_K \|_{\mathbb{C}^n}^2 \, dx \\
&\text{since for } K \in F(J), \\
\langle V^{-1} \rangle_{J}^{\frac{1}{2}} \langle V^{-1} \rangle_{K}^{\frac{1}{2}} \langle V^{-1} \rangle_{J}^{-\frac{1}{2}} \langle V^{-1} \rangle_{K}^{-\frac{1}{2}} &= \frac{1}{|K|} \int_K \langle V^{-1} \rangle_{J}^{\frac{1}{2}} \langle V^{-1} \rangle_{K}^{-\frac{1}{2}} V^{-1}(x) \langle V^{-1} \rangle_{J}^{-\frac{1}{2}} \langle V^{-1} \rangle_{K}^{-\frac{1}{2}} \leq \lambda.}
\end{align*}
\]
Thus
\[ ||S_j f||^2_{L^2} \leq \sum_{J \in \mathcal{J}_{j-1}} \int_J \lambda \sum_{K \in \mathcal{F}(J)} \lambda ||f_K||^2_{\mathcal{L}^2} dx \]

\[ = \sum_{J \in \mathcal{J}_{j-1}} \lambda^2 \sum_{K \in \mathcal{F}(J)} \int_J ||f_K||^2_{\mathcal{L}^2} dx = \lambda^2 ||\Delta_j f||_{L^2}^2. \]

We now consider
\[ \int_{\cup J_j} ||S_j f||^2_{\mathcal{L}^2} dx = \sum_{J \in \mathcal{J}_{j-1}} \sum_{I \in \mathcal{J}(J)} \int_I ||U^\frac{1}{2}(x)|| \sum_{K \in \mathcal{F}(J)} \langle V^{-\frac{1}{2}} fkh_K(x) ||^2_{\mathcal{L}^2} dx \]

As $h_K$ is constant on $I \in \mathcal{J}(J)$ for $K \in \mathcal{F}(J)$, this is equal to
\[ \sum_{J \in \mathcal{J}_{j-1}} \sum_{I \in \mathcal{J}(J)} \int_I \langle U \rangle^\frac{1}{2} \sum_{K \in \mathcal{F}(J)} \langle V^{-\frac{1}{2}} fkh_K \rangle ||^2_{\mathcal{L}^2} dx \]

\[ \leq \sum_{J \in \mathcal{J}_{j-1}} \sum_{I \in \mathcal{J}(J)} \int_I \langle U \rangle^\frac{1}{2} \langle U \rangle^{-\frac{1}{2}} ||^2_{\mathcal{L}^2} \langle U \rangle^\frac{1}{2} \sum_{K \in \mathcal{F}(J)} \langle V^{-\frac{1}{2}} fkh_K \rangle ||^2_{\mathcal{L}^2} dx \leq 2 \lambda^2 ||\Delta_j f||_{L^2}^2. \]

We have shown that there is a constant $C$ such that $||S_j f||^2 \leq C ||\Delta_j f||^2$. Let us now show that there exists a constant $C'$ and $0 < d < 1$ such that for $k > j$,
\[ \int_{\cup J_{k-1}} ||S_j f||^2_{L^2} dx \leq C'd^{k-j} ||\Delta_j f||^2. \]

Cotlar’s Lemma (see [7]) then implies that $S = \sum S_j$ is bounded. Note that
\[ \int_{\cup J_{k-1}} ||S_j f||^2_{L^2} dx = \sum_{J \in \mathcal{J}_j} \sum_{I \in \mathcal{J}(I)} \int_I ||U^\frac{1}{2}(x)|| \sum_{K \in \mathcal{F}(I)} \langle V^{-\frac{1}{2}} fkh_K \rangle ||^2_{\mathcal{L}^2} dx \]

Note that $\sum_{L \in \mathcal{J}(I)} \sum_{K \in \mathcal{F}(L)} \langle V^{-\frac{1}{2}} fkh_K \rangle$ is constant on $J \in \mathcal{J}_j$, and denote this constant by $M_j f$. The above expression is equal to
\[ \sum_{J \in \mathcal{J}_j} \sum_{I \in \mathcal{J}(I)} ||I|| \langle U \rangle^\frac{1}{2} M_j f ||^2_{\mathcal{L}^2} \]

\[ = \sum_{J \in \mathcal{J}_j} \sum_{I \in \mathcal{J}(I)} \sum_{J' \in \mathcal{J}(I')} ||I|| \langle U \rangle^\frac{1}{2} M_j f ||^2_{\mathcal{L}^2} \]

\[ = \sum_{J' \in \mathcal{J}(I')} \sum_{I \in \mathcal{J}(I)} \left\langle \sum_{I'} ||I'|| \langle U \rangle^\frac{1}{2} \right\rangle \langle U \rangle^\frac{1}{2} M_j f, ||\langle U \rangle^\frac{1}{2} \langle U \rangle^\frac{1}{2} M_j f \rangle \]

\[ = \sum_{J' \in \mathcal{J}(I')} \sum_{I \in \mathcal{J}(I)} \left\langle \sum_{I'} ||I'|| \langle U \rangle^{-\frac{1}{2}} \langle U \rangle^\frac{1}{2} \langle U \rangle^\frac{1}{2} M_j f, ||\langle U \rangle^{-\frac{1}{2}} \langle U \rangle^\frac{1}{2} \langle U \rangle^\frac{1}{2} M_j f \rangle \right\rangle \]

\[ = \sum_{J' \in \mathcal{J}(I')} \sum_{I \in \mathcal{J}(I)} \left\langle \sum_{I'} ||I'|| \langle U \rangle^{-\frac{1}{2}} \langle U \rangle^\frac{1}{2} \langle U \rangle^\frac{1}{2} M_j f, ||\langle U \rangle^{-\frac{1}{2}} \langle U \rangle^\frac{1}{2} \langle U \rangle^\frac{1}{2} M_j f \rangle \right\rangle \]

\[ = \sum_{J' \in \mathcal{J}(I')} \sum_{I \in \mathcal{J}(I)} \int_{J_{k-1}(I)} ||U^\frac{1}{2}(x)|| \langle U \rangle^\frac{1}{2} \langle U \rangle^\frac{1}{2} M_j f ||^2_{L^2} dx \]
We now apply Hölder’s inequality with $p$ such that $2p$ is the $r$ from our reverse Hölder inequality on $U$. Then the above expression is less than or equal to

$$\sum_{J' \in J_{\ell-1}} \sum_{J \in J(J')} \left( \int J_{\ell-1}(J) \left\| \langle U \rangle_{\frac{p}{2}} (U)_{J'} \right\|^2 \left\| \langle U \rangle_{\frac{p}{2}} M_J f \right\|^2 \langle J_{\ell-1}(J) \right\|^2. $$

We now use the fact that we are working with a decaying stopping time to see that this is less than or equal to

$$\sum_{J' \in J_{\ell-1}} \sum_{J \in J(J')} \left( \int J_{\ell-1}(J) \left\| \langle U \rangle_{\frac{p}{2}} (U)_{J'} \right\|^2 \left\| \langle U \rangle_{\frac{p}{2}} M_J f \right\|^2 \langle J_{\ell-1}(J) \right\|^2 \right)^{\frac{1}{p}} d_{\ell-1-\frac{1}{p}} |J|^{\frac{1}{p}}$$

where $0 < d < 1$.

Now we apply the reverse Hölder inequality \[\text{with vector } \langle U \rangle_{\frac{p}{2}} M_J f \text{ to obtain this is less than or equal to}

$$\sum_{J' \in J_{\ell-1}} \sum_{J \in J(J')} \left\| \langle U \rangle_{\frac{p}{2}} M_J f \right\|^2 C^|J|^{\frac{k-1}{q}} \sum_{J' \in J_{\ell-1}} \left( \int J(J') \left\| U(x)_{\frac{p}{2}} M_J f \right\|^2 \right)^{\frac{1}{p}} = d_{\ell-1-\frac{1}{p}} C^|J|^{\frac{k-1}{q}} \sum_{J' \in J_{\ell-1}} \left( \int J(J') \right)^{\frac{1}{p}} \left\| S_j f \right\|^2. $$

This is our core estimate.

To apply Cotlar’s Lemma, consider

$$\langle S_k S_j f, g \rangle_{L^2} = \langle S_j f, S_k g \rangle_{L^2} = \int_{\cup J_{\ell-1}} \langle S_j f(x), S_k g(x) \rangle \, dx \lesssim \int_{\cup J_{\ell-1}} \langle S_j f(x) \rangle \langle S_k g(x) \rangle \, dx \lesssim \left\{ \int_{\cup J_{\ell-1}} \left\| S_j f(x) \right\|^2 \right\}^{\frac{1}{2}} \left\{ \int_{\cup J_{\ell-1}} \left\| S_k g(x) \right\|^2 \right\}^{\frac{1}{2}} $$

This is true as the support of $S_k f$ is contained in $J_{\ell-1}$ and by Cauchy-Schwartz. We have just dealt with the relevant bounds for the two factors at the end of this chain of inequalities.

Also note that

$$\langle S_k S_j f, g \rangle_{L^2} = \langle S_j f, S_k g \rangle_{L^2} = \langle (S \triangle_j)^* f, (S \triangle_k)^* g \rangle_{L^2} = \langle \triangle_j S^* f, \triangle_k S^* g \rangle_{L^2} = 0$$

as the $\triangle_i$ are self adjoint orthogonal projections. This finishes the proof of Theorem 5.1.

**Remark 5.4.** The proof of Corollary 4.2 also follows from the embedding theorem of Nazarov and Treil. Theorem 5.2. Ideally we would like to prove this independently of their theorem however we were unable to do this. $M_V^{1/2} T_\sigma M_U^{1/2}$ can be written as

$$M_V^{1/2} V_{-1} D_{V-1} T_\sigma D_U D_U^{-1} M_U^{1/2}. $$

Note that $T_\sigma$ commutes with $D_{V-1}$ and we can estimate the norm as follows

$$\| M_V^{1/2} T_\sigma M_U^{1/2} \| \leq \| M_V^{1/2} D_{V-1} T_\sigma D_U D_U^{-1} M_U^{1/2} \| \leq \| M_V^{1/2} D_{V-1} T_\sigma \| \| D_{V-1} D_U \| \| D_U^{-1} M_U^{1/2} \|. $$

We need conditions on $U$ and $V$ that imply that the operators $M_V^{1/2} V_{-1} D_{V-1}$, $D_{V-1} D_U$ and $D_U^{-1} M_U^{1/2}$ are bounded. Theorem 5.2 immediately gives us that $D_U^{-1} M_U^{1/2}$ is
bounded. This theorem also applies to \( M_{V}^{\#} D_{V}^{-1} \), if we note that its adjoint is \( D_{V}^{-1} M_{V}^{\#} \). All we need to show now is that under the hypothesis \( D_{V}^{-1} D_{U} \) is a bounded operator. This follows from the joint \( A_{2} \) condition.

6. APPLICATION TO THE HILBERT TRANSFORM

As well as showing that the martingale transforms are uniformly bounded under the conditions of the two main theorems we can also show that the dyadic shift, III, defined below, will be bounded and hence the Hilbert transform by way of S.

Petermichl’s averaging techniques \([9,10]\).

Definition 6.1. The dyadic shift III with respect to the standard dyadic grid is the operator given by

\[
III f = \sum_{I \in \mathcal{D}} f_{I} h_{I} = \sum_{I \in \mathcal{D}} (f, h_{I_{+}} - h_{I_{-}}) h_{I},
\]

where \( f \) is supported on the unit interval and has finite Haar expansion.

Definition 6.2. Define the operator \( D_{V}^{+} \) as

\[
D_{V}^{+} : f = \sum_{I \in \mathcal{D}} f_{I} h_{I} \mapsto \sum_{I \in \mathcal{D}} \langle V \rangle_{I_{+}}^{\#} f_{I} h_{I}
\]

and the operator \( D_{V}^{-} \) as

\[
D_{V}^{-} : f = \sum_{I \in \mathcal{D}} f_{I} h_{I} \mapsto \sum_{I \in \mathcal{D}} \langle V \rangle_{I_{-}}^{\#} f_{I} h_{I},
\]

for \( f \in L^{2}(\mathbb{C}^{n}) \) with finite Haar expansion.

If we split the shift operator into a sum of two operators, each of which is bounded,

\[
III f = (III_{1} + III_{2}) f = \sum_{I \in \mathcal{D}} \langle f, h_{I_{+}} \rangle h_{I} - \sum_{I \in \mathcal{D}} \langle f, h_{I_{-}} \rangle h_{I}
\]

We can then check that \( D_{V}^{-1} III_{1} D_{V}^{-1} = III_{1} \) and \( D_{V}^{-1} III_{2} D_{V}^{-1} = III_{2} \). As before we can estimate \( ||M_{U}^{\#} III M_{V}^{\#}|| \),

\[
||M_{U}^{\#} (III_{1} + III_{2}) M_{V}^{\#}|| = ||M_{U}^{\#} (D_{V}^{-1} III_{1} D_{V}^{-1} + D_{V}^{-1} III_{2} D_{V}^{-1}) M_{V}^{\#}||
\]

\[
\leq \left( ||M_{U}^{\#} (D_{V}^{-1} III_{1})|| ||III_{1}|| \right) ||M_{V}^{\#} (D_{V}^{-1} III_{2})|| ||III_{2}||
\]

We have already dealt with the boundedness of the third operator and it is known that \( III_{1} \) and \( III_{2} \) are bounded on unweighted \( L^{2} \). This leaves the operators \( M_{U}^{\#} D_{V}^{-1} \) and \( M_{U}^{\#} D_{V}^{-1} \).
boundedness of $M$ and $V$ for all intervals.

Theorem 6.4. The results from [9] and [10] of the Hilbert transform in terms of these translated and dilated Haar shifts using $M$ and taking adjoints where appropriate. For larger interval. The second last equality is due to the fact that $|I_{+}| = \frac{1}{2}|I|$. The boundedness of $M_{\psi}^{2} D_{V_{-1}}^{2}$ follows from our previous bounding of $M_{\psi}^{2} D_{V_{-1}}^{2}$ and taking adjoints where appropriate. For $M_{\psi}^{2} D_{V_{-1}}^{2}$ the proof is similar.

Definition 6.3. Instead of this canonical dyadic grid we can define the shift operator, $\Pi\beta,r$, on the grid $D_{r,\beta} = \{ r2^{m} \mid [0, 1) + l + \sum_{n < m} 2^{m-n} \beta_{1} \} \}, l,m \in \mathbb{Z}$:

$$\Pi\beta,r f = \Pi\beta,r \sum_{l \in \mathbb{D}^{0},r} f_{l} h_{l} = \sum_{l \in \mathbb{D}^{0},r} \langle f, h_{l} - h_{1} \rangle h_{l},$$

The shift operators defined with respect to these dyadic grids will be bounded $L^{2}(V) \to L^{2}(U)$ given the joint $A_{2}$ condition is satisfied, $U$ satisfies the reverse Hölder condition and $V^{-1}$ the $A_{2,0}$ condition, all on this new lattice. The resulting estimate for the norm will be independent of the lattice.

Assuming the joint $A_{2}$ condition, that $U$ satisfies the reverse Hölder condition and $V$ the $A_{2,0}$ condition, all on arbitrary intervals, allows us to estimate the norm of the Hilbert transform in terms of these translated and dilated Haar shifts using the results from [9] and [10].

Theorem 6.4. Let $U$ and $V$ be matrix weights satisfying the joint $A_{2}$ condition

$$\langle V^{-1} \frac{1}{I} \langle U \rangle_{I} \langle V^{-1} \frac{1}{I} \rangle \rangle < C$$

for all intervals $I$, where $C$ is a constant multiple of the identity. If $V^{-1} \in A_{2,0}$ and $U$ satisfies the matrix reverse Hölder inequality, then the Hilbert Transform is bounded from $L^{2}(V)$ to $L^{2}(U)$.

Proof.

$$\left| \left< M_{U}^{\frac{1}{2}} H M_{V}^{\frac{1}{2}} f, g \right> \right| = C \left| \left< M_{U}^{\frac{1}{2}} \int_{[0,1]^2} \int_{I} \Pi_{\beta,r} \int_{r} \| \Pi_{\beta,r} M_{V}^{\frac{1}{2}} f \| dr d\mu(\beta), g \right> \right|$$

$$= C \left| \int_{[0,1]^2} \int_{I} \left| \left< M_{U}^{\frac{1}{2}} \Pi_{\beta,r} M_{V}^{\frac{1}{2}} f, g \right> \right| dr d\mu(\beta) \right|$$

$$\leq C \left| \int_{[0,1]^2} \int_{I} \left| \left< M_{U}^{\frac{1}{2}} \Pi_{\beta,r} M_{V}^{\frac{1}{2}} f, g \right> \right| dr d\mu(\beta) \right|$$

$$\leq CC^{*} \int_{[0,1]^2} \int_{I} \| \langle f, g \rangle \| dr d\mu(\beta) \leq CC^{*} ||f|| ||g||,$$

where $C$ is the proportion of the Hilbert Transform to the average of the shift operators and $C^{*}$ is the uniform operator norm of the shift operators.

The heuristic for adapting our main argument to the case of the dyadic shift can be applied to a more general class of operators, band operators.
7. Application to band operators and certain singular integral operators

Definition 7.1. A band operator $T$ is a bounded operator on $L^2$ such that there exists an integer $r > 0$ for which $\langle Th_I, h_J \rangle = 0$ for all Haar functions $h_I, h_J$ where $J$ is at least a distance of $r$ away from $I$. By distance we mean tree distance between dyadic intervals where the tree is formed by connecting each interval with its parent and children intervals.

One crucial fact is that, for each $r$ there are only a finite number of Haar basis elements $h_J$ less than tree distance $r$ from $h_I$. Suppose there are $m$ Haar basis elements less than $r$ away from each $h_I$ and we label these basis elements $h_{Ii}$ for $i = 1..m$. Then our band operator $T$ will be of the form

$$f \mapsto \sum_{I \in \mathcal{D}} \sum_{i=1}^m \phi(I, I_i) \langle f, h_I \rangle h_{Ii},$$

where $\phi$ is a function from $\mathcal{D} \bigoplus \mathcal{D}$ to $\mathbb{C}$.

Lemma 7.2. If we have a band operator $T$, written in the form

$$f \mapsto \sum_{I \in \mathcal{D}} \sum_{i=1}^m \phi(I, I_i) \langle f, h_I \rangle h_{Ii},$$

then the function $\phi : \mathcal{D} \bigoplus \mathcal{D} \to \mathbb{C}$ is bounded.

Proof. Suppose that $\phi$ is unbounded, as $T$ is a bounded operator we can choose $I$ and $I_i$ such that $\phi(I, I_i) > ||T||$. Then we can see that

$$||Th_I|| = ||\sum_{I \in \mathcal{D}} \sum_{i=1}^m \phi(I, I_i) \langle h_I, h_{Ii} \rangle|| = ||\sum_{i=1}^m \phi(I, I_i) ||h_{Ii}|| > ||T||,$$

contradicting our hypothesis that $T$ is bounded. \qed

\begin{figure}[h]
\centering
\begin{tikzpicture}
\draw (0,0) -- (3,0) -- (3,1) -- (0,1) -- cycle;
\draw (0,0.5) -- (1,0.5) -- (1,0) -- (0,0);
\draw (1,0.5) -- (2,0.5) -- (2,0) -- (1,0);
\draw (2,0.5) -- (3,0.5) -- (3,0) -- (2,0);
\end{tikzpicture}
\caption{A dyadic interval $I$ together with first and second generation subintervals.}
\end{figure}

Theorem 7.3. Let $U$ and $V$ be matrix weights satisfying the dyadic joint $A_2$ condition

$$\langle V^{-1} \rangle_I \langle U \rangle_I \langle V^{-1} \rangle_I < C$$
for all dyadic intervals \( I \), where \( C \) is a constant multiple of the identity. If \( V^{-1} \in A_{2,0} \) and \( U \) satisfies the dyadic matrix reverse H"older inequality, then any band operator \( T \) is bounded from \( L^2(V) \) to \( L^2(U) \). If \( r \) is the maximum distance associated to the band operator then the bound will depend only on \( r \), the \( L^2 \to L^2 \) norm of the operator and the \( A_2, A_{2,0} \) and reverse H"older constants associated to the weights.

Proof. Again we note that

\[
Tf = \sum_{I \in \mathcal{D}} \sum_{i=1}^{m} \phi(I, I_i) \langle f, h_{I_i} \rangle h_{I_i},
\]

where \( \phi \) is a function from \( \mathcal{D} \oplus \mathcal{D} \) to \( \mathbb{C} \). \( I \) and \( I_i \) will always share an ancestor less than \( r \) generations away for each \( i = 1..m \). In the case that \( I_i \) is a descendant of \( I \) then \( I \) will be the common ancestor. In the case where \( I_i \) is an ancestor of \( I \) then \( I_i \) will be the common ancestor. It is also possible to be in a situation where neither of these are true but the intervals still share a common ancestor.

We can split \( T \) into a sum of \( m \) bounded operators

\[
T = \sum_{i=1}^{m} T_i,
\]

where \( T_i \) is the operator

\[
f \mapsto \sum_{I \in \mathcal{D}} \phi(I, I_i) \langle f, h_{I_i} \rangle h_{I_i}.
\]

This sum is constructed so that for each summand \( T_i \) and Haar basis element \( h_I \) there is exactly one Haar coefficient, \( \langle f, h_I \rangle \), being mapped to \( h_I \). Due to the nature of the band operator there are at most \( m \) Haar coefficients being mapped to each basis element and thus it is possible to decompose \( T \) into a finite sum of these operators.

We proceed to estimate \( \| M_{-1}^{\frac{3}{2}} T M_{-1}^{\frac{1}{2}} \| \). Note that

\[
TD_{V^{-1}} = \left( \sum_{i=1}^{m} T_i \right) D_{V^{-1}} = \sum_{i=1}^{m} D_{V^{-1}}^{i} T_i,
\]
where $D^{\frac{1}{2}}_{V^{-1}}$ is the operator

$$f \mapsto \sum_{i \in D} \langle V^{-1} \rangle_{I_i}^\frac{1}{2} f_i h_i.$$ 

So

$$||M^{\frac{1}{2}}_T M^{\frac{1}{2}}_{V^{-1}}|| = ||M^{\frac{1}{2}}_T D_{V^{-1}} D^{\frac{1}{2}}_{V^{-1}}, M^{\frac{1}{2}}_{V^{-1}}|| = ||M^{\frac{1}{2}}_T \left( \sum_{i=1}^{m} D^{\frac{1}{2}}_{V^{-1}} T_i \right) D^{\frac{1}{2}}_{V^{-1}}, M^{\frac{1}{2}}_{V^{-1}}||$$ 

$$\leq \left( \sum_{i=1}^{m} ||M^{\frac{1}{2}}_T D^{\frac{1}{2}}_{V^{-1}} T_i|| \right) ||D^{\frac{1}{2}}_{V^{-1}}, M^{\frac{1}{2}}_{V^{-1}}||$$ 

$$\leq \left( \sum_{i=1}^{m} ||M^{\frac{1}{2}}_T D^{\frac{1}{2}}_{V^{-1}}|| ||T_i|| \right) ||D^{\frac{1}{2}}_{V^{-1}}, M^{\frac{1}{2}}_{V^{-1}}||$$ 

We know that each $T_i$ is bounded and we have already dealt with the boundedness of $D^{\frac{1}{2}}_{V^{-1}}, M^{\frac{1}{2}}_{V^{-1}}$. So it remains to bound each $M^{\frac{1}{2}}_T D^{\frac{1}{2}}_{V^{-1}}$.

So for any $f \in L^2$

$$||M^{\frac{1}{2}}_T (D^{\frac{1}{2}}_{V^{-1}} f)||^2 = (M^{\frac{1}{2}}_T (D^{\frac{1}{2}}_{V^{-1}})^2 M^{\frac{1}{2}}_T f, f) = (\langle D^{\frac{1}{2}}_{V^{-1}} \rangle, M^{\frac{1}{2}}_T f, M^{\frac{1}{2}}_T f)$$

$$= \left( \sum_{i \in D} \langle V^{-1} \rangle_{I_i} (U^{\frac{1}{2}} f)_i h_i, M^{\frac{1}{2}}_T f \right) = \sum_{i \in D} \left( \langle V^{-1} \rangle_{I_i} (U^{\frac{1}{2}} f)_i h_i, M^{\frac{1}{2}}_T f \right)$$

$$= \sum_{i \in D} \frac{1}{|I_i|} \int_{I_i} \langle V^{-1} \rangle (U^{\frac{1}{2}} f)_i h_i, M^{\frac{1}{2}}_T f \rangle dx \leq \sum_{i \in D} \frac{2^r}{|I_i|} \int_{I_i} \langle V^{-1} \rangle (U^{\frac{1}{2}} f)_i h_i, M^{\frac{1}{2}}_T f \rangle dx$$

where $I'$ is the common ancestor of $I$ and $I_i$. This is true because each term

$$\langle V^{-1} \rangle (U^{\frac{1}{2}} f)_i h_i, M^{\frac{1}{2}}_T f \rangle = \langle V^{-1} (U^{\frac{1}{2}} f)_i h_i, V^{-\frac{1}{2}} (U^{\frac{1}{2}} f)_i h_i \rangle$$

is positive.

We have seen before that if a matrix weight $U$ satisfies the dyadic $A_{2.0}$ condition then for any vector $\gamma$ the scalar weight $||U^{\frac{1}{2}} \gamma||^2$ will satisfy the scalar dyadic $A_{\infty}$ condition. So if we have a dyadic interval $I$ and a dyadic interval $J$ contained in $I$ such that the tree distance between these two is less than $r$, i.e. $|I| \leq 2^r |J|$ then one of the standard properties of $A_{\infty}$, see [15] page 196, tells us that

$$\beta \int_{I} ||U^{\frac{1}{2}} \gamma||^2 \leq \int_{J} ||U^{\frac{1}{2}} \gamma||^2$$

for some $0 < \beta < 1$ bounded away from 0, with the bound dependent only on $r$ and the $A_{\infty}$ constant.

Using our hypothesis that $V^{-1} \in A_{2.0}$ we can see that

$$\sum_{i \in D} \frac{2^r}{|I_i|} \int_{I_i} \langle V^{-1} \rangle (U^{\frac{1}{2}} f)_i h_i, M^{\frac{1}{2}}_T f \rangle dx \leq \sum_{i \in D} \frac{2^r}{|I_i|} \int_{I_i} \langle V^{-\frac{1}{2}} (U^{\frac{1}{2}} f)_i h_i, V^{-\frac{1}{2}} (U^{\frac{1}{2}} f)_i h_i \rangle dx$$

$$\leq \sum_{i \in D} \frac{2^r}{|I_i|} \int_{I_i} ||V^{-\frac{1}{2}} (U^{\frac{1}{2}} f)_i h_i||^2 dx \leq \sum_{i \in D} \frac{2^r}{|I_i|} \int_{I_i} ||V^{-\frac{1}{2}} (U^{\frac{1}{2}} f)_i||^2 dx \int_{I_i} ||h_i||^2 dx$$

$$= \sum_{i \in D} \frac{2^r}{|I_i|} \int_{I_i} ||V^{-\frac{1}{2}} (U^{\frac{1}{2}} f)_i||^2 dx \leq \sum_{i \in D} \frac{2^r}{|I_i|} \int_{I_i} ||V^{-\frac{1}{2}} (U^{\frac{1}{2}} f)_i||^2 dx$$
This reduces the estimate of each $D^i_{V^{-1}} M_{U}^f$ to $D_{V^{-1}} M_{U}^f$ which was dealt with in Theorem 5.1.

If $K$ is a function from $\mathbb{R} \setminus \{0\}$ to $\mathbb{R}$ that is twice differentiable and such that the function $x^3 K(x)$ is almost everywhere bounded and the limit as $x \to \infty$ of both $K(x)$ and the first derivative $K'(x)$ are 0 then the following theorem due to Vaghshakyan’s allows us to apply our hypothesis to singular integral operators of convolution type with such kernels $K$. Vaghshakyan’s theorem models singular integral operators with such kernels in terms of translations and dilations of band operators.

**Theorem 7.4 (Vaghshakyan).** If $T$ is a singular integral operator of convolution type with kernel $K$ as defined above, then $T$ is a positive multiple of the following operator

$$ f \mapsto \int_{(0,1)^2} \int_1^2 B^{\beta,r} f \frac{dr}{r} d\mathcal{P}(\beta), $$

where $B^{\beta,r}$ is a band operator defined in terms of the dyadic grid $\mathcal{D}_{\beta,r}$ exactly as they are defined for the canonical dyadic grid.

**Theorem 7.5.** Let $U$ and $V$ be matrix weights satisfying the joint $A_2$ condition

$$ \langle V^{-1} \rangle_I^{\frac{1}{2}} \langle U \rangle_I^{\frac{1}{2}} < C $$

for all intervals $I$, where $C$ is a constant multiple of the identity. If $V^{-1} \in A_{2,0}$ and $U$ satisfies the matrix reverse Hölder inequality, then the singular integral operator of convolution type with kernel $K$ is bounded from $L^2(V)$ to $L^2(U)$.

Proof.

$$ \left| \left< M_U^f T M_{V^{-1}}^{\frac{1}{2}} f, g \right> \right| = \hat{C} \left| \left< M_U^f \int_{(0,1)^2} \int_1^2 B^{\beta,r} M_{V^{-1}}^{\frac{1}{2}} f \frac{dr}{r} d\mathcal{P}(\beta), g \right> \right| $$

$$ = \hat{C} \left| \int_{(0,1)^2} \int_1^2 \left< M_U^f B^{\beta,r} M_{V^{-1}}^{\frac{1}{2}} f, g \right> \frac{dr}{r} d\mathcal{P}(\beta) \right| $$

$$ \leq \hat{C} \int_{(0,1)^2} \int_1^2 \left| \left< M_U^f B^{\beta,r} M_{V^{-1}}^{\frac{1}{2}} f, g \right> \right| \frac{dr}{r} d\mathcal{P}(\beta) $$

$$ \leq \hat{C} C^* \int_{(0,1)^2} \int_1^2 \|f, g\| \frac{dr}{r} d\mathcal{P}(\beta) \leq \hat{C} C^* ||f|| ||g||, $$

where $\hat{C}$ is the constant multiple of the singular integral operator corresponding to the average of the band operators and $C^*$ is the operator norm of the band operators. Note by uniform norm we mean that a particular band operator then defined with respect to different dyadic grids will have the same operator norm. \(\square\)
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