ON MULTIPLICITY IN RESTRICTION OF TEMPERED REPRESENTATIONS OF
P-ADIC GROUPS

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Abstract. We establish an equality between two multiplicities: one in the restriction of tempered representations of a p-adic group to its closed subgroup with the same derived group; and one occurring in their corresponding component groups in Langlands dual sides, so-called S-groups, under working hypotheses about the tempered local Langlands conjecture and the internal structure of tempered L-packets. This provides a formula of the multiplicity for p-adic groups by means of dimensions of irreducible representations of their S-groups.

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1. Introduction

From the construction of L-packets of the non-split inner form of SL_2 over a p-adic field F of characteristic 0 by Labesse and Langlands [30, 37], it is observed that the multiplicity in the restriction of irreducible smooth representations of the non-split inner form of GL_2 to the non-split inner form of SL_2 fails to be one. Due to the uniqueness of Whittaker models [32], on the other hand, the multiplicity is always one for the split form SL_n with any positive integer n [13, 38]. Extending the result of Labesse and Langlands to any inner form of SL_n over F, Hiraga and Saito discovered in [16] that the multiplicity in restriction from inner forms of GL_n to those of SL_n equals the dimension of an irreducible representation of a component group (so-called, S-group) in SL_n(ℂ) via the local Langlands correspondence for F-inner forms of SL_n. Adler and Prasad established some multiplicity-one theorems in the restriction from GU(V) to U(V) for the cases with no Whittaker model by means of Fourier-Jacobi models, where V is a finite-dimensional vector space over a local field of characteristic not 2 with a non-degenerate symmetric or skew-symmetric form [1]. B. Xu also studied the multiplicity for quasi-split groups under some assumptions [39]. Furthermore, the author’s work on the local Langlands correspondence for the F-inner form Sp_{1,1} of Sp_4 offers an interesting example [10, Section 7.7], where the multiplicity in restriction of an irreducible smooth representation from GSp_{1,1} to Sp_{1,1} no longer coincides with the dimension of the corresponding irreducible representation of the S-group in Sp_4(ℂ). The author and his collaborators also addressed the multiplicity for irreducible smooth representations of small
rank general spin groups $G\text{Spin}_4$, $G\text{Spin}_6$ and their inner forms in [6], and that for unitary principal series representations of $G\text{Spin}_n$ in [6, 7].

Inspired by such phenomena, we study the multiplicity in the restriction of irreducible smooth representations of $G(F)$ to $G(F)$ in a general setting, where $G$ is a connected reductive algebraic $F$-group and $G$ is its closed $F$-subgroup with the same derived group. The purpose of this paper is to establish an equality of the multiplicity in the restriction of tempered representations from $\tilde{G}(F)$ to $G(F)$ and the multiplicity in the restriction of corresponding representations from the associated $S$-group of $G$ to that of $\tilde{G}$, under the hypotheses, listed in Section 4.1, about the tempered local Langlands conjectures for $\tilde{G}$ and $G$ and the conjectural structures of their $L$-packets. This equality yields a formula of the multiplicity in the restriction that generalizes the previous results mentioned above.

To be precise, we let $\tilde{G}$ be a connected reductive group over $F$, and let $G$ be a closed $F$-subgroup of $\tilde{G}$ such that

$$G_{\text{der}} = \tilde{G}_{\text{der}} \subseteq G \subseteq \tilde{G},$$

where the subscript $\text{der}$ stands for the derived group. We write $G = G(F)$ and $\tilde{G} = \tilde{G}(F)$ for the groups of $F$-points. Given irreducible smooth representations $\sigma \in \text{Irr}(G)$ and $\pi \in \text{Irr}(\tilde{G})$, the multiplicity $\langle \sigma, \pi \rangle_G$ of $\sigma$ in the restriction $\text{Res}_{\tilde{G}}^G(\pi)$ of $\pi$ to $G$ is defined as follows:

$$\langle \sigma, \pi \rangle_G := \dim \text{Hom}_G(\sigma, \text{Res}_{\tilde{G}}^G(\pi)).$$

For any finite groups $H \leq \tilde{H}$, given $\delta \in \text{Irr}(H)$ and $\rho \in \text{Irr}(\tilde{H})$, we shall define the multiplicity $\langle \delta, \rho \rangle_H$ of $\delta$ in the restriction $\text{Res}_{\tilde{H}}^H(\rho)$ of $\rho$ to $H$ as follows:

$$\langle \delta, \rho \rangle_H := \dim \text{Hom}_H(\delta, \text{Res}_{\tilde{H}}^H(\rho)).$$

Let $\varphi$ be given in the set $\Phi_{\text{temp}}(G)$ of $\tilde{G}$-conjugacy classes of tempered $L$-parameters. Here $\tilde{G} = L^0G$ denotes the complex dual of $G$ [8]. Choose a lifting $\tilde{\varphi} \in \Phi_{\text{temp}}(\tilde{G})$ of $\varphi$ using Labesse’s theorem (See Theorem 2.1). We here assume that the local Langlands conjecture and the conjectural structure of $L$-packets are valid for tempered representations of $\tilde{G}$ and $G$ in [1, 1]. We refer the reader to Working Hypotheses 4.1, 4.3 and 4.6 for precise statements regarding these conjectures. Extending a lemma of Chao and Li [9], we have

$$1 \rightarrow S_{\tilde{\varphi}, \text{sc}}(\tilde{G}) \rightarrow S_{\varphi, \text{sc}}(\tilde{G}) \rightarrow X(\tilde{\varphi}) \rightarrow 1,$$

where $X(\tilde{\varphi})$ is the set of certain cohomological classes stabilizing $\tilde{\varphi}$. Given a tempered representation $\sigma \in \Pi_{\varphi}(G)$, we fix a tempered representation $\tilde{\sigma} \in \Pi_{\tilde{\varphi}}(G)$ such that $\sigma \subset \text{Res}_{\tilde{G}}^G(\tilde{\sigma})$. The internal structures of $L$-packets for $G$ and $\tilde{G}$ provide two finite representations $\rho \in \text{Irr}(S_{\varphi, \text{sc}}(\tilde{G}), \zeta_G)$ and $\bar{\rho} \in \text{Irr}(S_{\tilde{\varphi}, \text{sc}}(\tilde{G}), \bar{\zeta}_G)$ corresponding to $\sigma$ and $\tilde{\sigma}$, respectively. Here $\text{Irr}(S_{\varphi, \text{sc}}, \zeta_G)$ denotes the set of irreducible representations of $S_{\tilde{\varphi}, \text{sc}}$ with a certain requirement with respect to a character $\zeta_G$. The relevant definitions will be given in Section 4.1. Note that the condition (1.1) yields $\zeta_G = \zeta_{\tilde{G}}$. For simplicity of notation, we set $S_{\varphi, \text{sc}}(\tilde{G}) = S_{\varphi, \text{sc}}$ and $S_{\tilde{\varphi}, \text{sc}}(\tilde{G}) = S_{\tilde{\varphi}, \text{sc}}$. Based on (1.2), we construct a homomorphism

$$\Lambda : \tilde{G}/G \rightarrow (S_{\varphi, \text{sc}}/S_{\tilde{\varphi}, \text{sc}})^{\vee},$$

where $(S_{\varphi, \text{sc}}/S_{\tilde{\varphi}, \text{sc}})^{\vee}$ denotes the group of characters on the finite quotient group $S_{\varphi, \text{sc}}/S_{\tilde{\varphi}, \text{sc}}$. This homomorphism yields an isomorphism

$$\lambda : \tilde{G}/G_{\sigma} \simeq (S_{\varphi, \text{sc}}/S_{\tilde{\varphi}, \text{sc}})^{\vee}/I(\rho),$$

where $G_{\sigma} := \{ g \in \tilde{G} : g\sigma \simeq \sigma \}$, and $I(\rho) := \{ \chi \in (S_{\varphi, \text{sc}}/S_{\tilde{\varphi}, \text{sc}})^{\vee} : \rho\chi \simeq \rho \}$ (Theorem 4.13). For this bijection, we impose one further condition (Working Hypothesis 4.11) that the conjectural internal structure of $L$-packet of $G$ allows $g\sigma$ to correspond to $\rho\chi_g$, along with Working Hypotheses 4.1, 4.3 and 4.6. Here,
the character $\chi \in (S_{\varphi,sc}/S_{\tilde{\varphi},sc})^\vee$ is given by $\Lambda$. Furthermore, as an analogue of $\lambda$, we also have the following isomorphism
\begin{equation}
\tilde{\lambda}: (S_{\varphi,sc}/S_{\varphi,sc})_{\tilde{\rho}} \xrightarrow{\sim} X(\tilde{\varphi})/I(\tilde{\varphi}),
\end{equation}
where $(S_{\varphi,sc})_{\tilde{\rho}} = \{ s \in S_{\varphi,sc} : \tilde{s} \rho \simeq \tilde{\rho} \}$, and $I(\tilde{\varphi})$ denotes the image of $I(\tilde{\varphi}) := \{ \chi \in (G/G)^\vee : \tilde{\sigma} \simeq \tilde{\sigma} \chi \}$ in $X(\tilde{\varphi})$ via the maps $I(\tilde{\varphi}) \to \{ a \in H^1(W_F,(\tilde{G}/G)) : \tilde{a} \tilde{\varphi} \simeq \tilde{\varphi} \} \to X(\tilde{\varphi})$ (Theorem 4.17). Likewise, for this bijection, we need to assume Working Hypothesis 4.15 that the conjectural internal structure of $L$-packet of $G$ allows $\tilde{\sigma} \chi_s$ to correspond to $\tilde{s} \rho$, along with Working Hypotheses 4.14 and 4.16. Here, the character $\chi_s \in (G/G)^\vee$ is given by the composition of the isomorphism $X(\tilde{\varphi}) \simeq S_{\varphi,sc}/S_{\tilde{\varphi},sc}$ (see 4.16) and the map $X(\tilde{\varphi}) \to (\tilde{G}/G)^\vee$ (see 4.21).

Recalling that the $L$-packet $\Pi_\varphi(G)$ is parameterized by $\text{Irr}(S_{\varphi,sc},\varphi)$, we note that its subset $\Pi_{\tilde{\varphi}}(G)$ consisting of all irreducible inequivalent constituents in the restriction $\text{Res}^G_G(\tilde{\sigma})$ of $\tilde{\sigma}$ from $G$ to $G$ is controlled by the quotient $(S_{\varphi,sc}/S_{\tilde{\varphi},sc})^\vee$. We further remark that the quotient $S_{\varphi,sc}/S_{\varphi,sc}$ in (1.4) (hence, the set $\Pi_{\tilde{\varphi}}(S_{\varphi,sc})$) is in bijection with the $X(\tilde{\varphi})$-orbit, $\{ \tilde{\sigma} \chi_a : a \in X(\tilde{\varphi}) \}$, of $\tilde{\sigma}$ in $\Pi_{\tilde{\varphi}}(G)$ (see Remark 4.18). It should be mentioned that $\lambda$ and $\lambda$ both rely on the conjectural bijection (1.4) characterizing the internal structures of $L$-packets $\Pi_{\tilde{\varphi}}(G)$ and $\Pi_{\varphi}(G)$ (see Remarks 4.15 and 4.19).

Based on two bijections (1.3) and (1.4), we finally prove the following equality
(1.5)
\begin{equation}
\langle \sigma, \tilde{\varphi} \rangle_G = \langle \tilde{\rho} \rangle_{S_{\varphi,sc}}
\end{equation}
(Théorem 4.20). Using Clifford theory and some results in the restriction of representations of finite groups, the equality (1.5) yields
\begin{equation}
\langle \sigma, \tilde{\varphi} \rangle_G = \frac{\dim \rho}{\dim \tilde{\delta}} |\Pi_{\rho}(S_{\varphi,sc})|^{-1}
\end{equation}
for any $\tilde{\delta} \in \text{Irr}(S_{\varphi,sc})$ satisfying $\tilde{\delta} \subseteq \text{Res}^{S_{\varphi,sc}}_{S_{\tilde{\varphi},sc}}(\rho)$ (Theorem 4.22). Hence, the multiplicity in the restriction in $p$-adic groups is formulated by means of dimensions of irreducible representations of their $S$-groups. These $S$-groups are all finite but not necessarily abelian. We remark that, since $\tilde{\rho} \subseteq \text{Res}^{S_{\varphi,sc}}_{S_{\tilde{\varphi},sc}}(\rho)$, the formula (1.6) is the generalization of Hiraga and Saito’s related work in [16] for the case of $\tilde{G} = \text{GL}_m(D)$ and $G = \text{SL}_m(D)$ (see 4.29), where $\dim \tilde{\delta} = |\Pi_{\rho}(S_{\varphi,sc})| = 1$ and $\langle \sigma, \tilde{\varphi} \rangle_G = \dim \rho$ (see Section 4.3). Furthermore, the formula (1.6) extends relevant results in [24, 39] from some quasi-split settings to the non-quasi-split (see Remark 4.21).

In Section 2, we recall basic notions and terminologies. Also, the local Langlands conjecture for a $p$-adic group and the conjectural structure of $L$-packets are reviewed and some useful arguments are discussed. In Section 3, we review some results of Gelbart-Knapp [13], Tadić [38], and Hiraga-Saito [16], regarding the restriction of representations of $p$-adic groups, and discuss some arguments related to the restriction of representations of finite groups, based on Clifford theory. Under working hypotheses about the local Langlands conjecture and the internal structure of $L$-packets, Section 4 provides two bijections 1.3 & 1.4. Furthermore, the equality 1.5 of multiplicities in the two sides is established and the multiplicity formula (1.6) is then formulated. We also observe that Hiraga and Saito’s work in [16] about the multiplicity in the case of $\text{GL}_m(D)$ and $\text{SL}_m(D)$ with a central division algebra $D$ is generalized to arbitrary connected reductive groups $\tilde{G}$ and $G$ in (1.1). In Appendix A, we provide some examples related to the results established in Section 3.

2. Basic notation and backgrounds

2.1. Notation and conventions. We take the following notation throughout the paper, unless otherwise specified. Let $p$ be a prime number. The field $F$ denotes a finite extension of $\mathbb{Q}_p$ with an algebraic closure $\overline{F}$. We denote by $W_F$ the Weil group of $F$ and by $\Gamma$ the absolute Galois group $\text{Gal}(\overline{F}/F)$.
Let $\mathbf{G}$ be a connected reductive algebraic group over $F$. We denote by $\mathfrak{A}_\mathbf{G}$ the split component, that is, the maximal $F$-split torus in the center of $\mathbf{G}$. We denote by $G = \mathbf{G}(F)$ the group of $F$-points. Fixing $\Gamma$-invariant splitting data, we define the $L$-group of $\mathbf{G}$ as a semi-direct product $L\mathbf{G} := \hat{G} \rtimes \Gamma$ (see [8] Section 2). For $i \in \mathbb{N}$ and $\Gamma$-module $A$, we denote by $H^i(F, A) := H^i(\Gamma, A)$ the Galois cohomology of $A$. Since any $\Gamma$-module $A$ can be given a $W_F$-module structure through the continuous homomorphism $W_F \to \Gamma$ with dense image, we have a restriction map $H^i(F, A) \to H^i(W_F, A)$ (see [29] [23]).

Let $\text{Irr}(G)$ denote the set of isomorphism classes of irreducible smooth complex representations of $G$. By abuse of notation, we identify an isomorphism class with its representative. We often write $V_\pi$ for the space of $\pi \in \text{Irr}(G)$. We denote by $\text{Irr}_{\text{disc}}(G)$ the subset of $\text{Irr}(G)$ consisting of discrete series representations, i.e., their central characters are unitary and the absolute values of their matrix coefficients are square-integrable modulo the center of $G$, and by $\text{Irr}_{\text{temp}}(G)$ the subset of tempered representations of $G$.

For any topological group $J$, Write $Z(J)$ for its center, and $J^\circ$ for the group $\text{Hom}(J, \mathbb{C}^\times)$ of all continuous characters. We denote by $\mathbb{I}$ the trivial character. The cardinality of a finite set $X$ is denoted by $|X|$. For two integers $x$ and $y$, $x|y$ means that $y$ is divisible by $x$.

### 2.2. Definitions of multiplicities in restriction

Let $\hat{\mathbf{G}}$ be a connected reductive group over $F$, and let $G$ be a closed $F$-subgroup of $\hat{\mathbf{G}}$ such that

\begin{equation}
\mathbf{G}_{\text{der}} = \hat{\mathbf{G}}_{\text{der}} \subseteq \mathbf{G} \subseteq \hat{\mathbf{G}},
\end{equation}

where the subscript der stands for the derived group. We write $G = \mathbf{G}(F)$ and $\hat{G} = \hat{\mathbf{G}}(F)$ for the groups of $F$-points. Given irreducible smooth representations $\sigma \in \text{Irr}(G)$ and $\pi \in \text{Irr}(\hat{G})$, the multiplicity $\langle \sigma, \pi \rangle_G$ of $\sigma$ in the restriction $\text{Res}_{\hat{G}}^G(\pi)$ of $\pi$ to $G$ is defined as follows:

$$
\langle \sigma, \pi \rangle_G := \dim_{\mathbb{C}} \text{Hom}_G(\sigma, \text{Res}_{\hat{G}}^G(\pi)).
$$

Let $\hat{H}$ be a finite group and let $H$ be its subgroup. We denote by $\text{Irr}(\hat{G})$ the set of isomorphism classes of irreducible complex representations of a finite group $\hat{\mathbf{G}}$ and identify an isomorphism class with its representative, by abuse of notation. Given $\hat{\delta} \in \text{Irr}(\hat{H})$ and $\hat{\rho} \in \text{Irr}(\hat{H})$, we shall define the multiplicity $\langle \hat{\delta}, \hat{\rho} \rangle_H$ of $\hat{\delta}$ in the restriction $\text{Res}_{\hat{H}}^\hat{G}(\hat{\rho})$ of $\hat{\rho}$ to $H$ as follows:

$$
\langle \hat{\delta}, \hat{\rho} \rangle_H := \dim_{\mathbb{C}} \text{Hom}_H(\hat{\delta}, \text{Res}_{\hat{H}}^\hat{G}(\hat{\rho})).
$$

(cf., Chapter 20 of [17]).

### 2.3. $L$-parameters

Given a connected reductive algebraic group $\mathbf{G}$ over $F$, we let $\Phi(G)$ denote the set of $\hat{\mathbf{G}}$-conjugacy classes of $L$-parameters, i.e., admissible homomorphisms

$$
\varphi : W_F \times \text{SL}_2(\mathbb{C}) \longrightarrow L\mathbf{G},
$$

(see [8] Section 8.2]). Here, we note from the definition of admissible homomorphisms that a condition of relevance is required for non quasi-split group $\mathbf{G}$, though any $L$-parameter for quasi-split groups is relevant to $\mathbf{G}$. Following [8], Sections 3 and 8], a parabolic subgroup of $L\mathbf{G}$ is relevant to $\mathbf{G}$ if it is equal to $L\mathbf{P}$ for some parabolic $F$-subgroup $\mathbf{P}$ of $\mathbf{G}$. We say that $\varphi \in \Phi(G)$ is relevant to $G$ if any parabolic subgroup of $L\mathbf{G}$ containing the image of $\varphi$ is relevant to $\mathbf{G}$.

Given $\varphi \in \Phi(G)$, we denote by $S_\varphi$ the centralizer in $\hat{\mathbf{G}}$ of the image of $\varphi$ and $Z(\hat{\mathbf{G}})^\Gamma$ the $\Gamma$-invariant group of $Z(\hat{\mathbf{G}})$. We say $\varphi \in \Phi(G)$ is tempered if $\varphi(W_F)$ is bounded, and $\varphi$ is discrete if it is tempered and the quotient group $S_\varphi/Z(\hat{\mathbf{G}})^\Gamma$ is finite. Let $\Phi_{\text{disc}}(G)$ and $\Phi_{\text{temp}}(G)$ denote the subsets of $\Phi(G)$ consisting of discrete and tempered $L$-parameters of $G$, respectively. We further say $\varphi$ is elliptic if $S_\varphi/Z(\hat{\mathbf{G}})^\Gamma$ is finite, equivalently if the image of $\varphi$ in $\hat{\mathbf{G}}$ is contained in no proper parabolic subgroup.
2.4. Labesse’s theorem. We discuss the following argument related to the local Langlands conjecture in a certain special setting, which will be used in Sections 4 and 4.3. Let \( G \) and \( \hat{G} \) be connected reductive algebraic groups over \( F \) satisfying the following exact sequence of connected components of \( L \)-groups

\[
(2.2) \quad 1 \longrightarrow \hat{S} \longrightarrow \hat{G} \xrightarrow{pr} \hat{G} \longrightarrow 1,
\]

where \( \hat{S} \) is a central torus in \( \hat{G} \), and \( pr \) is a surjective homomorphism which is compatible with \( \Gamma \)-actions on \( \hat{G} \) and \( \hat{G} \).

**Theorem 2.1.** (29 Théorème 8.1) With the above notation, for any \( \varphi \in \Phi(G) \), there exists \( \bar{\varphi} \in \Phi(\hat{G}) \) such that

\[
\varphi = \bar{\varphi} \circ pr.
\]

Such a parameter \( \bar{\varphi} \) is determined up to a 1-cocycle of \( W_F \) in \( \hat{S} \) (see [29 Section 7] and [3] Theorem 3.5.1). Theorem 2.1 tells us the existence of a lifting of a given \( L \)-parameter in the setting (2.2).

### 3. On restriction of representations

We recall some known results by Gelbart-Knapp in [13], Tadić in [38], and Hiraga-Saito in [16] about the restrictions of representations of \( p \)-adic groups. Further, similar arguments are also reviewed for representations of finite groups, based on Clifford theory (cf., see [17]).

3.1. **Restriction for \( p \)-adic groups.** We continue with connected reductive algebraic groups \( G \) and \( \hat{G} \) over \( F \) satisfying the condition (2.4). Given \( \sigma \in \text{Irr}(G) \), there exists \( \bar{\sigma} \in \text{Irr}(\hat{G}) \) such that

\[
\sigma \subset \text{Res}_{\hat{G}}^{G}(\bar{\sigma}),
\]

due to [13 Lemma 2.3] and [38 Proposition 2.2]. We use both \( \Pi_{\sigma}(G) \) and \( \Pi_{\bar{\sigma}}(G) \) for the set of equivalence classes of all irreducible constituents of \( \text{Res}_{\hat{G}}^{G}(\bar{\sigma}) \). It turns out that \( \Pi_{\sigma}(G) \) is finite and independent of the choice of the lifting \( \bar{\sigma} \in \text{Irr}(\hat{G}) \) (see [13 Lemma 2.1] and [38 Proposition 2.4 & Corollary 2.5]). Further, for any irreducible constituents \( \sigma_1 \) and \( \sigma_2 \) in \( \text{Res}_{\hat{G}}^{G}(\bar{\sigma}) \), it is clear that \( \Pi_{\sigma_1}(G) = \Pi_{\sigma_2}(G) \).

Given \( \bar{\sigma}_1, \bar{\sigma}_2 \in \text{Irr}(\hat{G}) \), we recall the following equivalent statements from [13 Lemma 2.4] and [38 Corollary 2.5]:

\[
(3.1) \quad \exists \chi \in (\hat{G}/G)^{\nu} \text{ such that } \bar{\sigma}_1 \simeq \bar{\sigma}_2 \chi \iff \Pi_{\bar{\sigma}_1}(G) \cap \Pi_{\bar{\sigma}_2}(G) \neq \emptyset \iff \Pi_{\bar{\sigma}_1}(G) = \Pi_{\bar{\sigma}_2}(G),
\]

where \( (\hat{G}/G)^{\nu} = \text{Hom}(\hat{G}/G, \mathbb{C}^\times) \) of all continuous characters of \( \hat{G}/G \), which are considered as continuous 1-dimensional representation of \( \hat{G} \) that are trivial on its subgroup \( G \). Since \( \text{Res}_{\hat{G}}^{G}(\bar{\sigma}) \) is completely reducible due to [13 Lemma 2.1] and [38 Lemma 2.1], we have the decomposition

\[
(3.2) \quad \text{Res}_{\hat{G}}^{G}(\bar{\sigma}) = \bigoplus_{\tau \in \Pi_{\sigma}(G)} \langle \tau, \bar{\sigma} \rangle_G \cdot \tau
\]

(cf. [16 Chapter 2]). We note that the multiplicity \( \langle \tau, \bar{\sigma} \rangle_G \) has the common value over \( \tau \in \Pi_{\sigma}(G) \) [13 Lemma 2.1(b)], that is, \( \langle \tau_1, \bar{\sigma} \rangle_G = \langle \tau_2, \bar{\sigma} \rangle_G \) for any \( \tau_1, \tau_2 \in \Pi_{\sigma}(G) \). We define

\[
(3.3) \quad I(\bar{\sigma}) := \{ \chi \in (\hat{G}/G)^{\nu} : \bar{\sigma} \simeq \bar{\sigma} \chi \}.
\]

Then we have the following equalities.

**Proposition 3.1.** With the above notation, we have

\[
|I(\bar{\sigma})| = \dim_{\mathbb{C}} \text{Hom}_{G}(\text{Res}_{\hat{G}}^{G}(\bar{\sigma}), \text{Res}_{\hat{G}}^{G}(\bar{\sigma})) = |\Pi_{\sigma}(G)| \cdot \langle \sigma, \bar{\sigma} \rangle_G^{2}.
\]

**Proof.** The first equality is a consequence of [38 Proposition 2.4]. The second equality follows from the decomposition (3.2) and Schur’s lemma. Indeed, for any \( \tau_1, \tau_2 \subset \text{Res}_{\hat{G}}^{G}(\bar{\sigma}) \), we have \( \text{Hom}_{G}(\tau_1, \tau_2) = 0 \) unless \( \tau_1 \simeq \tau_2 \), in which case \( \text{Hom}_{G}(\tau_1, \tau_2) \simeq \mathbb{C} \). Thus, the proof is complete. \( \square \)
We define the stabilizer of $\sigma$ in $\widetilde{G}$

$$G_\sigma := \{g \in \widetilde{G} : g\sigma \simeq \sigma\}.$$ 

It is known [28 Corollary 2.3] that $G_\sigma$ is an open normal subgroup of $\widetilde{G}$ of finite index and satisfies

$$Z(\widetilde{G}) \cdot G \subseteq G_\sigma \subseteq \widetilde{G}.$$ 

Since $\widetilde{G}$ and $G$ share the same derived group by the condition (2.1), we also note that $Z(\widetilde{G}) \cdot G$ is an open normal subgroup of $\widetilde{G}$ of finite index.

**Proposition 3.2.** ([13 Lemma 2.1(c)]) The quotient $\widetilde{G}/G_\sigma$ acts by conjugation on the set $\Pi_\sigma(G)$ simply and transitively. In particular, the set $\Pi_\sigma(G)$ is in one-to-one correspondence with the quotient $\widetilde{G}/G_\sigma$. □

We review some results in [10 Chapter 2] (see also Section 4.3). Given $\tilde{\sigma} \in \text{Irr}(\widetilde{G})$ and $\chi \in I(\tilde{\sigma})$, there is a non-zero endomorphism $\delta_\chi \in \text{Aut}_C(V_{\tilde{\sigma}})$ such that $\delta_\chi \circ (\tilde{\sigma}\chi) = \tilde{\sigma} \circ I_\chi$. The scalar endomorphism $\tilde{\delta} \mapsto z \cdot \tilde{\delta}$ for $v \in V_{\tilde{\sigma}}$ and $z \in \mathbb{C}^\times$ is defined and denoted by $z \cdot \text{id}_{V_{\tilde{\sigma}}}$. Thus, the subgroup consisting of $z \cdot \text{id}_{V_{\tilde{\sigma}}}$ in $\text{Aut}_C(V_{\tilde{\sigma}})$ is identified with $\mathbb{C}^\times$. Define $\mathcal{A}(\tilde{\sigma})$ as the subgroup of $\text{Aut}_C(V_{\tilde{\sigma}})$ generated by $\{I_\chi : \chi \in I(\tilde{\sigma})\}$ and $\mathbb{C}^\times$. Then the map $I_\chi \mapsto \chi$ induces the following exact sequence

$$1 \rightarrow \mathbb{C}^\times \rightarrow \mathcal{A}(\tilde{\sigma}) \rightarrow I(\tilde{\sigma}) \rightarrow 1.$$

Following [16, p.11], we denote by $\text{Irr}(\mathcal{A}(\tilde{\sigma}), \text{id})$ the set of isomorphism classes of irreducible representations of the group $\mathcal{A}(\tilde{\sigma})$ such that $z \cdot \text{id}_{V_{\tilde{\sigma}}} \in \mathbb{C}^\times$ acts as the scalar $z$. Recall the following isomorphism from [16, Corollary 2.10]

$$V_{\tilde{\sigma}} \simeq \bigoplus_{\chi \in \text{Irr}(\mathcal{A}(\tilde{\sigma}), \text{id})} \xi \boxtimes \sigma_\chi$$

as representations of the direct product $\mathcal{A}(\tilde{\sigma}) \times G$. There is thus a one-to-one correspondence

$$\text{Irr}(\mathcal{A}(\tilde{\sigma}), \text{id}) \simeq \Pi_\tilde{\sigma}(G),$$

sending $\xi \mapsto \sigma_\xi$. By $\xi_\sigma$ we denote the inverse of $\sigma$ via the correspondence (3.7). The isomorphism (3.6) also yields

$$\langle \sigma, \tilde{\sigma} \rangle_G = \dim \xi_\sigma.$$  

**Remark 3.3.** Given $\tilde{\sigma} \in \text{Irr}_{\text{disc}}(\widetilde{G})$, since the multiplicity is common, the equality (3.8) implies that $\dim \xi_{\sigma_1} = \dim \xi_{\sigma_2}$ for any $\sigma_1, \sigma_2 \in \Pi_\tilde{\sigma}(G)$.

3.2. **Restriction for finite groups.** Given two finite groups $H$ and $\bar{H}$ satisfying the property that $H$ is a normal subgroup of $\bar{H}$ with abelian quotient, we shall review some results on the restriction of representations from $\bar{H}$ to $H$ along with Clifford theory. We mainly refer to [17] (20.7 & 20.8). In principle, all the arguments in the restriction for $p$-adic groups in Section 3.1 hold for finite groups as follows. Given $\delta \in \text{Irr}(H)$, there exists $\bar{\delta} \in \text{Irr}(\bar{H})$ such that

$$\delta \in \text{Res}_{\bar{H}}^H(\bar{\delta}),$$

and we employ both $\Pi_\delta(H)$ and $\Pi_{\bar{\delta}}(H)$ for the set of equivalence classes of all irreducible constituents of $\text{Res}_{\bar{H}}^H(\bar{\delta})$. Further, we have $\Pi_{\delta_1}(H) = \Pi_{\delta_2}(H)$ for any irreducible constituents $\delta_1$ and $\delta_2$ in $\text{Res}_{\bar{H}}^H(\bar{\delta})$, and the decomposition

$$\text{Res}_{\bar{H}}^H(\bar{\delta}) = \bigoplus_{\theta \in \Pi_\delta(H)} \langle \theta, \tilde{\delta} \rangle_{\bar{H}} \cdot \theta.$$ 

The multiplicity $\langle \theta, \tilde{\delta} \rangle_{\bar{H}}$ here is also common over $\theta \in \Pi_\delta(H)$.

We define

$$I(\tilde{\delta}) := \{\eta \in (\bar{H}/H)^\vee : \tilde{\delta} \simeq \tilde{\eta}\},$$
where \((\bar{H}/H)^\vee = \text{Hom}(\bar{H}/H, \mathbb{C}^\times)\) of all characters of \(\bar{H}/H\), which are considered as 1-dimensional representation of \(\bar{H}\) that are trivial on its subgroup \(H\). Choose \(\delta \in \Pi_{\delta}(H)\), and define the stabilizer of \(\delta\) in \(\bar{H}\)

\[
\bar{H}_\delta := \{ h \in \bar{H} : h\delta = \delta \}.
\]

It is obvious that \(\bar{H}_\delta\) satisfies \(Z(\bar{H}) \cdot H \subseteq \bar{H}_\delta \subseteq \bar{H}\) and is a normal subgroup of \(\bar{H}\). Further, the quotient \(\bar{H}/\bar{H}_\delta\) acts by conjugation on the set \(\Pi_{\delta}(H)\) simply and transitively. We also have the equality \(|I(\delta)| = |\Pi_{\delta}(H)| \cdot \langle \delta, \delta \rangle^2_H\), since \(\bar{H}/H\) is abelian.

4. Multiplicity in restriction for tempered representations

We continue with the notation in Sections 2 and 3. Given two connected reductive algebraic groups \(G\) and \(G\) over \(F\) satisfying (2.1), under working hypotheses about the tempered local Langlands conjecture and the internal structure of tempered \(L\)-packets, we construct two crucial bijections in Section 4.1. We then apply these bijections to establish an equality of multiplicities in \(p\)-adic groups and their \(S\)-groups, and formulate the multiplicity in the restriction in \(p\)-adic groups by means of dimensions of irreducible representations of their \(S\)-groups in Section 4.2.

4.1. Two bijections. This section is devoted to constructing two bijections between finite abelian groups occurring in \(p\)-adic groups and their \(S\)-groups: one between two quotients in \(G\) and a character group under Working Hypothesis 4.1, 4.3, 4.6, and 4.11 (see Theorem 4.13); and the other one between two quotients in \(S\)-groups and a cohomological group under Working Hypothesis 4.1, 4.3, 4.6, and 4.11 (see Theorem 4.17).

To begin with, the local Langlands conjecture predicts that there is a surjective, finite-to-one map from \(\text{Irr}(G)\) to \(\Phi(G)\), which is expected to satisfy various natural properties. The map preserves \(\gamma\)-factors, \(L\)-factors, and \(\epsilon\)-factors, if they are available in both sides (cf. [14, 15]). Furthermore, through the map, the subsets \(\Phi_{\text{disc}}(G)\) and \(\Phi_{\text{temp}}(G)\) in \(\text{Irr}(G)\) are characterized by the subsets \(\text{Irr}_{\text{disc}}(G)\) and \(\text{Irr}_{\text{temp}}(G)\) in \(\Phi(G)\), respectively.

The first working hypothesis is the local Langlands conjecture for tempered representations of \(p\)-adic groups as follows.

Working Hypothesis 4.1. The local Langlands conjecture is true for irreducible tempered representations of \(G\), i.e., there is a surjective, finite-to-one map

\[
\mathcal{L}_G : \text{Irr}_{\text{temp}}(G) \longrightarrow \Phi_{\text{temp}}(G).
\]

Remark 4.2. We shall need the temperedness condition so as to state Shahidi’s conjecture (Working Hypothesis 4.3 below) which will be related to other working hypotheses and used in Section 2. We may further note that the study of \(\text{Irr}(G)\) can be reduced to its subset \(\text{Irr}_{\text{temp}}(G)\), due to the Langlands classification theorem (see [23, Section 3] for a proof and survey for the theorem). Furthermore, we shall refer the reader to [8, Section 10] for several properties that \(\mathcal{L}_G\) is expected to satisfy.

Given a tempered \(L\)-parameter \(\varphi \in \Phi_{\text{temp}}(G)\), each \(L\)-packet \(\Pi_{\varphi}(G) := \mathcal{L}_G^{-1}(\varphi)\) is conjectured to be parameterized in terms of the group of connected components of a certain centralizer in the \(L\)-group. We recall such conjectural parametrization in detail mainly based on [20, Section 4.2] and [21, Section 5.4] as follows.

Let \(G^*\) be a quasi-split form of \(G\) over \(F\) (\(G^*\) is \(G\) itself when \(G\) is quasi-split over \(F\)) i.e., \(G^*\) is a connected reductive quasi-split group over \(F\) and there is an \(F\)-isomorphism \(\xi : G \xrightarrow{\sim} G^*\) such that \(\xi \circ \tau(\xi)^{-1}\) is an inner automorphism \((g \mapsto xgyx^{-1})\) defined over \(F\) for all \(\tau \in \text{Gal}(\bar{F}/F)\) (see [8, 2.4(3)] or [28, p.280]). Such \(\xi\) is called an inner twist. Fix a Whittaker datum \(m\). Here \(m\) stands for a \(G^*\)-conjugacy class of pairs \((B^*, \psi^*)\), where \(B = T^*U^*\) is a Borel subgroup of \(G^*\) defined over \(F\) with maximal \(F\)-torus \(T^*\) and unipotent radical \(U^*\), and \(\psi^*\) is a generic character of \(U^*\). There is a one-to-one correspondence between the set of \(T^*\)-orbits of the generic characters and the quotient \(G^*_{\text{ad}}(F)/\text{Im}(G^*(F) \to G^*_{\text{ad}}(F))\) that is embedded into \(H^1(F, Z(G^*))\). Given a Whittaker datum \(m = (B^*, \psi^*)\), an irreducible admissible
representation \( \pi \in \operatorname{Irr}(G^*) \) is called \( \mathfrak{m} \)-generic if there exists a non-zero \( \psi^* \)-generic Whittaker functional for \( \pi \), equivalently if \( \operatorname{Hom}_{U^*}(\pi, \psi^*) \neq 0 \) (for the details, see [36, Chapter 3]).

The second working hypothesis is a strong form of Shahidi’s conjecture in [35] Section 9 as follows.

**Working Hypothesis 4.3.** Assume Working Hypothesis [4.1] is true for \( G^* \). Let an \( L \)-parameter \( \varphi \in \Phi_{\text{temp}}(G^*) \) be given. For each Whittaker datum \( \mathfrak{m} \), there is a unique \( \mathfrak{m} \)-generic member in the \( L \)-packet \( \Pi_{\mathfrak{m}}(G^*) \) associated to \( \varphi \).

The next hypothesis is related to the internal structure of \( L \)-packets. We shall introduce two relevant formulations in order. The first formulation suggested by Arthur [2] utilizes a finite group \( S_{\varphi, \text{sc}} \), which is described as follows. By \( \hat{G}_{\text{sc}} \) we denote the simply connected cover of the derived group \( \hat{G}_{\text{der}} \) of \( \hat{G} \). Let \( \hat{G}_{\text{ad}} \) be the adjoint group \( \hat{G}/Z(\hat{G}) \) of \( \hat{G} \). Then the quotient
\[
S_{\varphi}(\hat{G}) := C_{\varphi}(\hat{G})/Z(\hat{G})^{\Gamma},
\]
is considered as a subgroup in \( \hat{G}_{\text{ad}} \). Via the isogeny \( \hat{G}_{\text{sc}} \to \hat{G}_{\text{ad}} \), we get the full pre-image of \( S_{\varphi}(G) \), which is denoted by \( S_{\varphi, \text{sc}}(\hat{G}) \). We then have an exact sequence
\[
1 \to Z(\hat{G}_{\text{sc}}) \to S_{\varphi, \text{sc}}(\hat{G}) \to S_{\varphi}(\hat{G}) \to 1. \tag{4.2}
\]
We put
\[
S_{\varphi}(\hat{G}) := \pi_0(S_{\varphi}(\hat{G})),
\]
\[
S_{\varphi, \text{sc}}(\hat{G}) := \pi_0(S_{\varphi, \text{sc}}(\hat{G})),
\]
\[
\hat{Z}_{\varphi, \text{sc}}(G) := Z(\hat{G}_{\text{sc}})/(Z(\hat{G}_{\text{sc}}) \cap S_{\varphi, \text{sc}}(\hat{G})),
\]
and have an exact sequence
\[
1 \to \hat{Z}_{\varphi, \text{sc}}(G) \to S_{\varphi, \text{sc}}(\hat{G}) \to S_{\varphi}(\hat{G}) \to 1. \tag{4.3}
\]

Consider again an inner twist \( \xi : G^* \to G \) with \( G^* \) quasi-split form of \( G \). Choose a character \( \zeta_G \) of \( Z(\hat{G}_{\text{sc}}) \) whose restriction to \( Z(\hat{G}_{\text{sc}})^{\Gamma} \) corresponds to the class of the \( F \)-inner form \( G \) of \( G^* \) via the Kottwitz isomorphism [27, Theorem 1.2] (note here that \( \hat{G} = G^* \)). Following [3] p.549-550], we denote by \( \operatorname{Irr}(S_{\varphi, \text{sc}}(\hat{G}), \zeta_G) \) the set of irreducible representations of \( S_{\varphi, \text{sc}}(\hat{G}) \) that is equivariant under the pullback of \( \zeta_G \) to \( \hat{Z}_{\varphi, \text{sc}}(G) \) (cf., [22] Section 4.6), [2] p.209].

Given a tempered \( L \)-parameter \( \varphi \) for \( G \) and a tempered \( L \)-packet \( \Pi_{\varphi}(G) \) associated to \( \varphi \), it is conjectured [2] Section 3] that there is a one-to-one correspondence
\[
\Pi_{\varphi}(G) \xrightarrow{\sim} \operatorname{Irr}(S_{\varphi, \text{sc}}(\hat{G}), \zeta_G). \tag{4.4}
\]
This conjecture provides the internal structure of \( \Pi_{\varphi}(G) \). In particular, when \( G^* = G \), the character \( \zeta_G \) equals the trivial character \( 1 \) and it is clear that \( \operatorname{Irr}(S_{\varphi, \text{sc}}(\hat{G}), 1) = \operatorname{Irr}(S_{\varphi}(\hat{G})) \).

The other formulation suggested by Kaleska [20] utilizes a finite group \( S_{\varphi}^+ \), which is described as follows. Fix a finite subgroup \( \mathfrak{Z} \subset Z(G^*) \) defined over \( F \) and set \( \hat{G}^* := G^*/\mathfrak{Z} \). We then have an isogeny \( \hat{G}^* \to \hat{G}^* \) dual to the isogeny \( G^* \to G^* \). For our purpose, we only consider the case that \( \mathfrak{Z} = Z(G_{\text{der}}^*) \) from now on, though the following notions are available in the general case of \( \mathfrak{Z} \). According to [20] Section 3] and [21] Section 5], the new cohomology set, denoted by \( H^1(u \to W_\mathfrak{Z} \to G^*) \), has the property that there is a surjective map
\[
H^1(u \to W_\mathfrak{Z} \to G^*) \to H^1(F, G^*/Z(G^*)�).
\]
We note from [20] Corollary 3.8] that the map \( \mathfrak{Z} \to u \to W_\mathfrak{Z} \to G^* \) is bijective when \( G^* \) is split. Following [21] Section 5.1] and [20] Section 4], a rigid inner twist \( (\xi', \zeta') \) is defined as a pair consisting of an inner twist \( \xi' : G' \to G^* \) and an element \( \zeta' \in Z^1(u \to W_\mathfrak{Z} \to G^*) \) such that for all \( \tau \in \Gamma \), we have \( \xi'^{-1} \tau(\xi') = \operatorname{Ad}(\zeta'(\tau)) \), where \( \bar{\zeta}' \in \bar{Z}^1(F, G^*/Z(G^*)) \) is the image of \( \zeta' \) under the map [15.5].
Remark 4.4. Since the map (4.3) is surjective, we have $z' \in Z^1(u \to W, Z \to G^*)$ for every inner twist $\xi' : G' \to G^*$ such that $(\xi', z')$ is a rigid inner twist (see [21 Section 5.4]). Thus, since $G^*$ is a quasi-split form of $G$ over $F$, there is a corresponding element in $Z^1(u \to W, Z \to G^*)$ to the given connected reductive group $G$ over $F$ and inner twist $\xi : G \to G^*$.

Let $\varphi \in \Phi_{\text{temp}}(G^*)$ be a tempered $L$-parameter. For each rigid inner twist $(\xi', z') : G' \to G^*$ with $z' \in Z^1(u \to W, Z \to G^*)$, we let $\Pi_\varphi((\xi', z'))$ be the $L$-packet $\Pi_\varphi(G')$ under Working Hypothesis 4.1. We note that $\Pi_\varphi((id, 1)) = \Pi_\varphi(G^*)$. Also, due to the relevance condition, we have $\Phi_{\text{temp}}(G') \subset \Phi_{\text{temp}}(G^*)$ and $\Pi_\varphi(G')$ may be empty if $\varphi$ is not relevant to $G'$, i.e., $\varphi \notin \Phi_{\text{temp}}(G')$ (see [21 Section 5.4]). Recalling the centralizer $C_\varphi$ in $\hat{G}^*$ of the image of $\varphi$, we denote by $S^+_\varphi$ the preimage of $C_\varphi$ in $\hat{G}^*$ under the isogeny $\hat{G}^* \to \hat{G}^*$. We let denote $S^+_\varphi$ the group $\pi_0(S^+_\varphi)$ of connected components of $S^+_\varphi$, which is finite, and let $\text{Irr}(S^+_\varphi)$ denote the set of irreducible representations of $\pi_0(S^+_\varphi)$. Given a tempered $L$-parameter $\varphi \in \Phi_{\text{temp}}(G^*)$ and a Whittaker datum $m$, there exists a bijective map

$$
\iota_m : \bigsqcup_{(\xi', z')} \Pi_\varphi((\xi', z')) \to \text{Irr}(S^+_\varphi),
$$

such that the image of the unique $m$-generic member of $\Pi_\varphi((id, 1))$ is the trivial representation of $S^+_\varphi$. Let a rigid inner twist $(\xi', z') : G' \to G^*$ be given. Recalling the notation in [20 Section 4.1], we consider a refined endoscopic quadruple $\check{\xi} = (G^*, G^*, s^\xi, \eta^\xi)$ for $G^*$. For any $\Delta[m, \check{\xi}, z']$-matching functions $f^\xi \in C_c^\infty(G^*)$ and $f' \in C_c^\infty(G')$, we have the equality

$$
\Theta^\xi_{\varphi^\xi}(f^\xi) = e(G') \sum_{\pi\in \Pi_\varphi((\xi', z'))} \langle \pi^\xi, s^\xi \rangle \Theta_\pi^\xi(f'),
$$

where $\varphi^\xi \in \Phi_{\text{temp}}(G^*)$ is such that $\varphi = L^\xi \circ \varphi^\xi$ and $\langle \pi^\xi, - \rangle = \text{tr}(\iota_m(\pi^\xi)(-))$.

It is important to discuss the relationship between two formulations: one is in terms of $S^+_\varphi$ and the other is $S^+_{\varphi, \text{sc}}$. Following [20 Section 4.6] and [22 Section 4.6], the character $\zeta_\varphi$ of $Z(G^*_\text{sc})$ that is described below Equation (4.3), determines a cohomology class in $H^1(u \to W, Z(G^*_\text{sc}) \to G^*_\text{sc})$ (due to being in the non-archimedean case) whose image in $H^1(u \to W, Z(G^*_\text{der}) \to G^*_\text{der})$ gives a character, denoted by $\zeta_G^\text{sc}$, of $\pi_0(Z(G^*_\text{der})^+)$. Here $Z(G^*_\text{der})^+$ is the preimage in $\hat{G}^*$ under the isogeny $\hat{G}^* \to \hat{G}^*$ of the diagonalizable group $Z(G^*)^\text{der}$. It is also noted that the character $\zeta_G^\text{sc}$ is the pull-back of $\zeta_G$ under the map $\pi_0(Z(G^*_\text{der})^+) \to Z(G^*_\text{sc})$, since we work with the case $Z = Z(G^*_\text{der})$. We then have the following bijection

$$
\text{Irr}(S^+_{\varphi, \text{sc}}(\hat{G}^*), \zeta_G) \overset{1:1}{\longrightarrow} \text{Irr}(S^+_\varphi(\hat{G}^*), \zeta_G).
$$

We refer the reader to [20 p.244] and [22 (4.6)] for further differences between these two groups.

Remark 4.5. The bijection (4.4) depends on a certain choice of data. For quasi-split $G$ with a tempered $L$-parameter $\varphi$, the choice of Whittaker datum yields a unique bijection in (4.4) so that a unique generic representation in $\Pi_\varphi(G)$ with respect to the Whittaker datum corresponds to the trivial representation of $\text{Irr}(S^+_{\varphi, \text{sc}}(\hat{G}^*))$. For non-quasi-split $G$ with a tempered $L$-parameter $\varphi$, despite the absence of the Whittaker datum, Kaletha carried out the dependence and made the bijection (4.4) canonical by means of a certain cohomological group related to Galois gerbes in [21]. To be precise, we consider an inner twist $\xi : G^* \to G$ with $G^*$ quasi-split form of $G$, and an element $z$ of $Z^1(u \to W, Z(G^*_\text{sc}) \to G^*_\text{sc})$ that lifts the element of $Z^1(F, G^*_\text{der})$ given by $\xi^{-1}\tau(\xi)$ for $\tau \in \Gamma$ (see [22 Section 4] for more details). Then, fixing a Whittaker datum for $G^*$, the datum of a pair $(\xi, z)$ turns out to make the bijection (4.4) canonical. In particular, if $G$ is quasi-split and $G^* = G$, then the pair $(\xi, z)$ can be taken to be the trivial pair $(id, 1)$, so that the fixed Whittaker datum for $G$ alone is sufficient to make the bijection (4.4) canonical, as explained in the case of quasi-split $G$ above. The author would like to thank Kaletha for valuable communications on this remark.

The third working hypothesis is the internal structure of tempered $L$-packets as follows.
Working Hypothesis 4.6. Let $G$ be a connected reductive algebraic group over $F$. Consider an inner twist \( \xi : G^* \rightarrow G \) with \( G^* \) quasi-split form of $G$ and an element $z$ of $\mathbb{Z}^1(u \rightarrow W, Z(G_{\text{ad}}^*)) \rightarrow G_{\text{ad}}^*$ that lifts the element of $\mathbb{Z}^1(F, G_{\text{ad}}^*)$ given by $\xi^{-1}(\sigma(\xi))$. Assume Working Hypotheses 4.4 and 4.5. Given a tempered $L$-parameter $\varphi \in \Phi_{\text{temp}}(G^*)$ and a Whittaker datum $\mathbf{m}$, the bijection (4.6) holds for $G$, so that the bijection (4.1) is canonical, due to Remark 4.5 and the bijection (4.7).

Remark 4.7. We shall use $S_{\varphi,sc}$ in this paper rather than $S_{\varphi}^+$, under Working Hypothesis 4.6 due to the complexity of the group structure of $S_{\varphi}^+$.

Now we construct two bijections in Theorems 4.13 and 4.17 below. From now on, we assume that Working Hypotheses 4.1, 4.3, and 4.6 are valid for $G$ and $G$. Due to [26, Section 1], the exact sequence given by the condition (2.1)

\[(4.11) \quad 1 \rightarrow G \rightarrow \widehat{G} \rightarrow \widehat{G}/G \rightarrow 1\]

yields (see also [11, Remark 2.4]). Thus, the kernel $\tilde{S}$ in (2.2) is now $\widehat{(G/G)}$.

Let $\varphi \in \Phi_{\text{temp}}(G)$ be given, and choose a lifting $\tilde{\varphi} \in \Phi_{\text{temp}}(\widehat{G})$ of $\varphi$, namely $\varphi = pr \circ \tilde{\varphi}$, as in Theorem 2.1. The surjective map $pr : \tilde{\varphi} \rightarrow \widehat{G}$ in (4.9) induces a natural map between centralizers

\[pr_C : C_{\tilde{\varphi}}(\widehat{G}) \rightarrow C_{\varphi}(\widehat{G}).\]

The kernel of $pr_C$ then equals the subgroup $\left(\frac{\widehat{G}}{G}\right)^T$ of $\Gamma$-invariants in the kernel of $pr$ in (4.9). Applying (4.8) to [26, (1.8.1)] and taking $\Gamma$-invariants, we have

\[(4.10) \quad 1 \rightarrow \left(\frac{\widehat{G}}{G}\right)^T \rightarrow Z(\widehat{G})^T \rightarrow Z(\widehat{G})^T \rightarrow H^1(F, \frac{\widehat{G}}{G}) \rightarrow H^1(F, Z(\widehat{G})) \rightarrow \cdots .\]

As noted in Section 2.1, the restriction map $H^1(F, \frac{\widehat{G}}{G}) \rightarrow H^1(W_F, \frac{\widehat{G}}{G})$ yields the image, $\text{Im} \left( Z(\widehat{G})^T \rightarrow H^1(F, Z(\widehat{G})) \right)$, and define the following quotient

\[(4.11) \quad X(\tilde{\varphi}) := \{ a \in H^1(W_F, \left(\frac{\widehat{G}}{G}\right)) : a\tilde{\varphi} \simeq \tilde{\varphi} \text{ in } \widehat{G} \} / \text{Im} \left( Z(\widehat{G})^T \rightarrow H^1(F, Z(\widehat{G})) \right).\]

Remark 4.8. The set $\{ a \in H^1(W_F, \left(\frac{\widehat{G}}{G}\right)) : a\tilde{\varphi} \simeq \tilde{\varphi} \text{ in } \widehat{G} \}$ is a finite abelian group (cf., [9, p.74]), so is $X(\tilde{\varphi})$. Furthermore, in [9, Lemma 5.3.4], since $Z(\widehat{G})^T$ is the identity of the quotient group $S_{\tilde{\varphi},\text{ad}}$ therein (note that their notation $G, \phi, S_{\phi, \text{ad}}, G^*, \phi^*$, and $S_{\phi, \text{ad}}$ respectively correspond to our notation $\widehat{G}, \tilde{\varphi}, S_{\tilde{\varphi}}(\widehat{G}), G, \varphi$, and $S_{\varphi}(G)$), it seems that their group $X^G(\phi)$ needs to take quotient by $\text{Im} \left( Z(\widehat{G})^T \rightarrow H^1(W_F, \widehat{G}^T) \right)$, though their lemma remains unchanged, as suggested from (4.13) below (see also [39, (3.4) and (3.5)]). We further refer the reader to [13, Theorem 4.3] and [12, Proposition 2.9] where some other groups than $X(\tilde{\varphi})$ were discussed in different situation.

In the following lemma, we slightly enhance [9, Lemma 5.3.4] by adding the injective argument.

Lemma 4.9. ([9, Lemma 5.3.4]) With the above notation, given $\varphi \in \Phi_{\text{temp}}(G)$ and $\tilde{\varphi} \in \Phi_{\text{temp}}(\widehat{G})$ with $\varphi = \tilde{\varphi} \circ pr$ as in Theorem 2.1, we have an exact sequence of finite groups

\[(4.12) \quad 1 \rightarrow S_{\tilde{\varphi}}(\widehat{G}) \rightarrow S_{\varphi}(\widehat{G}) \rightarrow X(\tilde{\varphi}) \rightarrow 1.\]

Proof. From [9, Lemma 5.3.4] and Remark 4.8, we have an exact sequence

\[(4.12) \quad S_{\tilde{\varphi}}(\widehat{G}) \rightarrow S_{\varphi}(\widehat{G}) \rightarrow X(\tilde{\varphi}) \rightarrow 1.\]
It now remains to show that \( \ker(S_{\hat{\varphi}}(\hat{G}) \to S_{\varphi}(\hat{G})) \) is trivial. Combining (4.10) and (4.12), we have the following commutative diagram:

\[
\begin{array}{ccc}
1 & 1 & 1 \\
\downarrow & & \downarrow \\
1 & \longrightarrow & (\hat{G}/G)^G & \longrightarrow & Z(\hat{G})^G & \longrightarrow & Z(\hat{G})^G & \longrightarrow & H^1(F, \hat{G}/G) \\
\downarrow & & \cap & \downarrow & \cap & \downarrow & \cap & \downarrow \\
1 & \longrightarrow & S_{\hat{\varphi}}(\hat{G}) & \longrightarrow & S_{\varphi}(\hat{G}) & \longrightarrow & X(\hat{\varphi}) & \longrightarrow & 1 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
1 & 1 & 1
\end{array}
\]

The exactness on the bottom of (4.13):

\[
1 \longrightarrow S_{\hat{\varphi}}(\hat{G}) \longrightarrow S_{\varphi}(\hat{G})
\]

and the connectedness argument (cf., [10, Lemma 4.8]) thus verify that

\[
\ker\left(S_{\hat{\varphi}}(\hat{G}) \to S_{\varphi}(\hat{G})\right) = \{1\}.
\]

Therefore, the proof of Lemma 4.9 is complete. \( \square \)

**Remark 4.10.** The continuous homomorphism \( W_F \to \Gamma \) has dense image (see Section 2.1), the exact sequence (4.10) can be replaced by

\[
1 \longrightarrow (\hat{G}/G)^G \longrightarrow Z(\hat{G})^G \longrightarrow Z(\hat{G})^G \longrightarrow H^1(W_F, \hat{G}/G) \longrightarrow H^1(W_F, Z(\hat{G})) \longrightarrow \cdots
\]

Accordingly, \( H^1(F, -) \) in the diagram (4.13) can be replaced by \( H^1(W_F, -) \) (cf., [23, Section 4] and [39, Section 3.2]).

We first consider the case that \( \varphi \) is elliptic (so is \( \hat{\varphi} \)), i.e., \( C_{\varphi}(\hat{G})/Z(\hat{G})^G \) is finite. Then, \( S_{\varphi}(\hat{G}) \) itself equals \( S_{\hat{\varphi}}(\hat{G}) \) and \( \hat{Z}_{\varphi,sc}(\hat{G}_{sc}) \) is identical with \( Z(\hat{G}_{sc}) \). Hence, the exact sequence (4.13) is equal to (1.2). Combining Lemma 4.9 and the exact sequence (4.12) for \( \hat{G} \) and \( G \), we have the following commutative diagram:

\[
\begin{array}{ccc}
1 & 1 \\
\downarrow & & \downarrow \\
1 & \longrightarrow & Z(\hat{G}_{sc}) & \longrightarrow & S_{\hat{\varphi},sc}(\hat{G}) & \longrightarrow & S_{\hat{\varphi}}(\hat{G}) & \longrightarrow & 1 \\
\downarrow & & \cap & \downarrow & \cap & \downarrow & \cap & \downarrow \\
1 & \longrightarrow & S_{\varphi,sc}(G) & \longrightarrow & S_{\varphi}(G) & \longrightarrow & X(\hat{\varphi}) & \longrightarrow & 1 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
1 & 1
\end{array}
\]
The bottom isomorphism comes from the Snake Lemma (cf., [33 Corollary 6.12]) which is applied to the other two horizontal exact sequences. We thus have the following exact sequence (the middle vertical one):
\[ 1 \rightarrow S_{\̂\varphi,sc}(\̂G) \rightarrow S_{\varphi,sc}(G) \rightarrow X(\̂\varphi) \rightarrow 1. \]

For tempered L-parameters, from [10 Lemma 4.8], we have \( S_{\varphi,sc}(G)^{\rho} = S_{\varphi,sc}(\̂G)^{\rho} \), which implies that \( Z_{\varphi,sc}(G) = Z_{\varphi,sc}(\̂G) \) by definition. Thus, it follows from the same way as in (4.15) that
\[ (4.16) \quad 1 \rightarrow S_{\̂\varphi,sc}(\̂G) \rightarrow S_{\varphi,sc}(G) \rightarrow X(\̂\varphi) \rightarrow 1. \]

Given \( \sigma \in \Pi_{\varphi}(G) \), we fix a lifting \( \tilde{\sigma} \in \Pi_{\̂\varphi}(\̂G) \). The internal structure (4.4) for \( G \) and \( \̂G \) provides two finite representations \( \rho \in \text{Irr}(S_{\varphi,sc}(G), \zeta_G) \) and \( \tilde{\rho} \in \text{Irr}(S_{\̂\varphi,sc}(\̂G), \zeta_{\̂G}) \) corresponding to \( \sigma \) and \( \tilde{\sigma} \), respectively. Note that the condition (2.1) yields \( \zeta_G = \zeta_{\̂G} \). We fix \( \sigma \) corresponding to \( \rho \) and \( \tilde{\sigma} \) corresponding to \( \tilde{\rho} \), under the conjectural bijection (4.1) (cf., Remark 4.5). For simplicity of notation, we set \( S_{\varphi,sc}(G) = S_{\varphi,sc} \) and \( S_{\̂\varphi,sc}(\̂G) = S_{\̂\varphi,sc} \).

Applying Galois cohomology to (4.8), we have the following exact sequence
\[ (4.17) \quad 1 \rightarrow G \rightarrow \̂G \rightarrow (G/G)(F) \rightarrow H^{1}(F,G) \rightarrow H^{1}(F,\̂G) \rightarrow H^{1}(F,\̂G/G). \]

Since the cokernel of \( G \rightarrow \̂G \) in (4.17) is embedded into the \( F \)-points \((G/G)(F)\) of \( \̂G/G \), we have
\[ (4.18) \quad 1 \rightarrow \̂G/G \rightarrow (G/G)(F) \rightarrow (\̂G/G)(F)/(\̂G/G) \rightarrow 1. \]

Taking the dual \((-)^\vee = \text{Hom}(-, \mathbb{C}^\times)\) and using the local Langlands correspondence for \( \̂G/G \) (cf., [10]), we have
\[ (4.19) \quad 1 \rightarrow ((G/G)(F)/(\̂G/G))^\vee \rightarrow ((G/G)(F))^\vee \simeq H^{1}(W_{F},\̂G/G) \rightarrow (G/G)^\vee \rightarrow 1. \]

Furthermore, the exact sequence (4.17) and the bijection \( H^{1}(F,G) \simeq \pi_{0}(Z(\̂G)^{F})^{\vee} \) ([26 Proposition 6.4]) yield
\[ 1 \rightarrow (G/G)(F)/(\̂G/G) \rightarrow \pi_{0}(Z(\̂G)^{F})^{\vee} \rightarrow \pi_{0}(Z(\̂G)^{F})^{\vee}/\ker\left(\pi_{0}(Z(\̂G)^{F})^{\vee} \rightarrow \pi_{0}(Z(\̂G)^{F})^{\vee}\right) \]
and we then have a surjective map
\[ \pi_{0}(Z(\̂G)^{F}) \twoheadrightarrow ((G/G)(F)/(\̂G/G))^\vee. \]

Using (4.18) and the surjection \( Z(\̂G)^{F} \twoheadrightarrow \pi_{0}(Z(\̂G)^{F}) \), we then have
\[ (4.19) \quad Z(\̂G)^{F} \twoheadrightarrow \ker\left(((G/G)(F))^\vee \rightarrow (G/G)^\vee\right). \]

From (4.14) that the set \( \text{Im}(Z(\̂G)^{F} \twoheadrightarrow H^{1}(W_{F},\̂G/G)) \) vanishes in \( H^{1}(W_{F},Z(\̂G)) \), and it induces the trivial character on \( \̂G \) via the map, \( H^{1}(W_{F},Z(\̂G)) \rightarrow (\̂G)^{\vee} \), described in [8 Section 10.2] (see also [39 Appendix A] for quasi-split cases). From this argument, (4.18) and (4.19), it follows that the image of \( Z(\̂G)^{F} \) in \( H^{1}(W_{F},(G/G)) \simeq ((G/G)(F))^\vee \) equals the kernel of the surjective map
\[ (4.20) \quad ((G/G)(F))^\vee \twoheadrightarrow (G/G)^\vee \]
(see also [18 Section 4], [19 Section 5.5], and [39 Section 3] for similar arguments in a different setting). Recalling the definition of \( X(\̂\varphi) \) in (4.11), we then have
\[ (4.21) \quad X(\̂\varphi) \leftrightarrow \left( H^{1}(W_{F},(G/G))/\text{Im}(Z(\̂G)^{F} \twoheadrightarrow H^{1}(W_{F},\̂G/G)) \right) \xrightarrow{\sim} (G/G)^{\vee}. \]
Combining the isomorphism $X(\tilde{\varphi}) \simeq S_{\varphi,sc}/S_{\tilde{\varphi},sc}$ from (4.14), we have
\[ \tilde{G}/G \cong \left( H^1(W_F, (\tilde{G}/G)) \cap \text{Im} \left( Z(\tilde{G})^\Gamma \to H^1(W_F, \tilde{G}/G) \right) \right)^\vee \to \left( S_{\varphi,sc}/S_{\tilde{\varphi},sc} \right)^\vee. \]

Hence, we obtain the following surjective homomorphism
\[ \Lambda : \tilde{G}/G \to (S_{\varphi,sc}/S_{\tilde{\varphi},sc})^\vee. \]

The fourth working hypothesis proposes a certain correspondence, related to the map $\Lambda$ in (4.23), which will be assumed for Theorem 4.13.

**Working Hypothesis 4.11.** Let $g \in \tilde{G}/G$ be given. Denote by $\chi_g \in (S_{\varphi,sc}/S_{\tilde{\varphi},sc})^\vee$ the image of $g$ under the map $\Lambda$. We assume that $g\sigma$ corresponds to $\rho\chi_g$ via the bijection (4.3) for $G$.

**Remark 4.12.** Working Hypothesis 4.11 is discussed for quasi-split groups in a different setting, and deduced by means of the conjectural endoscopic character identity (see [39, Theorem 2.6 and Lemma 3.13]).

From Section 3 we recall the set $I(\rho) = \{ \chi \in (S_{\varphi,sc}/S_{\tilde{\varphi},sc})^\vee : \rho\chi \simeq \rho \}$, and consider the quotient group $(S_{\varphi,sc}/S_{\tilde{\varphi},sc})^\vee/I(\rho)$. The first bijection is as follows.

**Theorem 4.13.** Suppose that Working Hypotheses 4.1, 4.3, and 4.6 are valid for $G$ and $\tilde{G}$, and further that Working Hypothesis 4.11 holds. Then we have the following isomorphism of finite abelian groups
\[ \lambda : \tilde{G}/G_{\sigma} \cong (S_{\varphi,sc}/S_{\tilde{\varphi},sc})^\vee/I(\rho). \]

**Proof.** Consider the following homomorphism
\[ \tilde{\Lambda} : \tilde{G}/G \to (S_{\varphi,sc}/S_{\tilde{\varphi},sc})^\vee/I(\rho). \]

Since $\tilde{\Lambda}$ is surjective due to the definition of $\Lambda$ in (4.23), it suffices to show that $\ker \tilde{\Lambda} = \tilde{G}_{\sigma}/G$. To this end, for any $g \in \tilde{G}/G$ with $\Lambda(g) = \chi_g \in I(\rho)$, due to Working Hypothesis 4.11, $g\sigma$ corresponds to $\rho\chi_g$ via (4.4). Since $\sigma$ corresponds to $\rho$ via (4.3), it follows that $g\sigma \simeq \sigma$, and thus $g \in \tilde{G}_{\sigma}/G$. On the other hand, for any $g \in \tilde{G}_{\sigma}/G$, $g\sigma$ corresponds to $\rho\chi_g$ via (4.4). Since $\sigma \simeq g\sigma$, we have $\rho\chi_g \simeq \rho$, and thus $\chi_g \in I(\rho)$. \qed

**Remark 4.14.** Since $\tilde{G}/G_{\sigma}$ simply transitively acts on $\Pi_{\tilde{\sigma}}(G)$ (Proposition 3.2), Theorem 4.13 immediately implies that there is a bijection between $\Pi_{\tilde{\sigma}}(G)$ and $(S_{\varphi,sc}/S_{\tilde{\varphi},sc})^\vee/I(\rho)$, sending $\sigma' \in \Pi_{\tilde{\sigma}}(G)$ with $\sigma' \simeq g\sigma$ for some $g \in \tilde{G}_{\sigma}/G$ to $\chi_g \in (S_{\varphi,sc}/S_{\tilde{\varphi},sc})^\vee/I(\rho)$.

Combining the map (4.20) and the isomorphism $H^1(W_F, (\tilde{G}/G)) \cong \left( (\tilde{G}/G)(F) \right)^\vee$, it is noted that $\chi \in I(\tilde{\sigma})$ (see the definition (3.3)) gives a 1-cocycle $a_\chi \in H^1(W_F, \tilde{G}/G)$ such that $\tilde{\varphi}a_\chi \simeq \tilde{\varphi}$. We then have the following embedding
\[ I(\tilde{\sigma}) \to \{ a \in H^1(W_F, (\tilde{G}/G)) : a\tilde{\varphi} \simeq \tilde{\varphi} \text{ in } \tilde{G} \} / \text{Im} \left( Z(\tilde{G})^\Gamma \to H^1(W_F, \tilde{G}/G) \right) = X(\tilde{\varphi}). \]

Note that this injection can also follow from the desiderata of the local Langlands correspondence (cf., [K, Section 10]). We denote by $\tilde{I}(\sigma)$ the image of $I(\tilde{\sigma})$ in $X(\tilde{\varphi})$ via (4.24), and consider the quotient group $X(\tilde{\varphi})/\tilde{I}(\sigma)$. Using the isomorphism $X(\tilde{\varphi}) \simeq S_{\varphi,sc}/S_{\tilde{\varphi},sc}$ in (4.14), we consider the following surjective homomorphism
\[ \tilde{\Lambda} : S_{\varphi,sc}/S_{\tilde{\varphi},sc} \to X(\tilde{\varphi})/\tilde{I}(\sigma). \]

The last working hypothesis propose a counterpart of Working Hypothesis 4.11 related to the map $\tilde{\Lambda}$ in (4.25), which will be assumed for Theorem 4.14.

**Working Hypothesis 4.15.** Let $s \in S_{\varphi,sc}/S_{\tilde{\varphi},sc}$ be given. Denote by $\chi_s \in (\tilde{G}/G)^\vee$ the image of $s$ under the composition of the isomorphism $X(\tilde{\varphi}) \simeq S_{\varphi,sc}/S_{\tilde{\varphi},sc}$ in (4.10) and the maps in (4.21). We assume that $\tilde{\sigma}\chi_s$ corresponds to $\tilde{\varphi}$ via the bijection (4.4) for $G$. 
Remark 4.16. Working Hypothesis 4.15 is discussed for quasi-split groups in a different setting (see Sections 6.2 and 6.3). In addition, this hypothesis implies that the representation $\tilde{\sigma} \chi_s$ lies in $\Pi_{\tilde{G}}$ and $\Pi_{\tilde{G}}(G) = \Pi_{\tilde{G} \chi_s}(G) \subset \Pi_{\tilde{G}}(G)$. In fact, one can notice that the character $\chi_s$ is unitary, since it is given by an element in the finite abelian group $\{ a \in H^1(W_F, (G/G)) : a \tilde{\varphi} \simeq \tilde{\varphi} \text{ in } \tilde{G} \}$ corresponding to $s$ through (4.21) where the local Langlands correspondence for $(G/G)$ is applied.

As above, recalling the set $(S_{\varphi, sc})_{\tilde{\varphi}} = \{ s \in S_{\varphi, sc} : \cdot \tilde{\varphi} \simeq \tilde{\varphi} \}$, we consider the quotient $S_{\varphi, sc}/(S_{\varphi, sc})_{\tilde{\varphi}}$. The other bijection is as follows.

**Theorem 4.17.** Suppose that Working Hypotheses 4.1, 4.3, and 4.6 are valid for $G$ and $\tilde{G}$, and further that Working Hypothesis 4.15 holds. Then, there is an isomorphism of finite abelian groups

$$\hat{\lambda} : S_{\varphi, sc} / (S_{\varphi, sc})_{\tilde{\varphi}} \cong X(\tilde{\varphi}) / \tilde{I}(\tilde{\sigma}).$$

**Proof.** It suffices to show that the kernel of $\hat{\lambda}$ in (4.26) equals $(S_{\varphi, sc})_{\tilde{\varphi}} / S_{\varphi, sc}$. Let $s \in S_{\varphi, sc} / S_{\varphi, sc}$ be given such that $\hat{\lambda}(s) = 1$. Then, recalling the image $\chi_s$ in $(G/G)^\vee$ of $s$ as described in Working Hypothesis 4.15, we note that $\tilde{\sigma} \chi_s \simeq \tilde{\sigma}$. Since $\tilde{\varphi}$ corresponds to $\tilde{\sigma}$ under the bijection (4.4), Working Hypothesis 4.15 yields $\tilde{\varphi} \simeq \cdot \tilde{\varphi}$, which implies that $s \in (S_{\varphi, sc})_{\tilde{\varphi}} / S_{\varphi, sc}$. On the other hand, since $\tilde{\varphi} \simeq \cdot \tilde{\varphi}$ for any element $s \in (S_{\varphi, sc})_{\tilde{\varphi}} / S_{\varphi, sc}$, we have $\tilde{\sigma} \chi_s \simeq \tilde{\sigma}$ and $\chi_s \in I(\tilde{\sigma})$ due to Working Hypotheses 4.6 and 4.15. It follows from (4.24) that $\hat{\lambda}(s) = 1$. This completes the proof. □

Remark 4.18. We note from Theorem 4.13 that the subset $\Pi_{\tilde{G}}(G) \subset \Pi_{\tilde{G}}(G)$ is controlled by the quotient $(S_{\varphi, sc} / S_{\varphi, sc})^\vee$. Furthermore, the set $X(\tilde{\varphi})$ acts on the $L$-packet $\Pi_{\tilde{G}}(\tilde{G})$ by the character twisting via the maps in (4.27), i.e., for given $a \in X(\tilde{\varphi})$ and $\tilde{\sigma} \in \Pi_{\tilde{G}}(\tilde{G})$, the pair $(a, \tilde{\sigma})$ maps to $\tilde{\sigma} \chi_a$ for $\chi_a \in (G/G)^\vee$. Therefore, the quotient $X(\tilde{\varphi}) / \tilde{I}(\tilde{\sigma})$ in Theorem 4.17 (hence, the set $\Pi_\rho(S_{\varphi, sc})$ which is in bijection with $S_{\varphi, sc} / (S_{\varphi, sc})_{\tilde{\varphi}}$) is in bijection with the $X(\tilde{\varphi})$-orbit, $\{ \tilde{\sigma} \chi_a : a \in X(\tilde{\varphi}) \}$, of $\tilde{\sigma}$ in $\Pi_{\tilde{G}}(\tilde{G})$.

Remark 4.19. It should be mentioned, that from their definitions, both $\lambda$ and $\hat{\lambda}$ rely on the bijection (4.24), which is conjectured to characterize the internal structures of $L$-packets $\Pi_{\tilde{G}}(\tilde{G})$ and $\Pi_\rho(G)$. Thus, both maps depend on the choices of $\sigma \in \Pi_\rho(G)$ corresponding to $\rho \in \text{Irr}(S_{\varphi, sc}, \chi_G)$ and $\tilde{\sigma} \in \Pi_{\tilde{G}}(\tilde{G})$ corresponding to $\tilde{\rho} \in \text{Irr}(S_{\varphi, sc}, \chi_G)$. Specially, if $G$ and $\tilde{G}$ are quasi-split and if $L$-parameters $\varphi$ and $\tilde{\varphi}$ are tempered, the canonical choices are made by fixing a Whittaker datum so that $\rho$ and $\tilde{\rho}$ are taken to be the trivial character $1$ (see also Remark 4.5).

4.2. An equality and multiplicity formula. Using two bijections constructed in Theorems 4.13 and 4.17, we obtain an equality of multiplicities in two sides (Theorem 4.20) and a general formulation of the multiplicity (Theorem 4.22).

Fix a lifting $\tilde{\varphi} \in \text{Irr}_{\text{temp}}(\tilde{G})$ of $\sigma \in \text{Irr}_{\text{temp}}(G)$ and $\tilde{\rho} \in \text{Irr}(S_{\varphi, sc})$ with $\tilde{\rho} \subset \text{Res}_{S_{\varphi, sc}}(\rho)$. Theorems 4.13 and 4.17 yield the following theorem.

**Theorem 4.20.** Suppose that Working Hypotheses 4.1, 4.3, and 4.6 are valid for $G$ and $\tilde{G}$, and further that Working Hypotheses 4.15 and 4.17 hold. With the above notation, we have

$$\langle \sigma, \tilde{\rho} \rangle_G = \langle \tilde{\rho}, \rho \rangle_{S_{\varphi, sc}}.$$ 

**Proof.** We first recall that

$$|I(\tilde{\sigma})| = |\Pi_{\tilde{G}}(G)| \cdot (\sigma, \tilde{\sigma})_{\tilde{G}}^2$$

and $|X(\tilde{\varphi})| = |S_{\varphi, sc} / S_{\varphi, sc}|$. Then, since $I(\tilde{\sigma}) \simeq I(\tilde{\varphi})$ as in (4.24), the bijection in Theorem 4.13 yields

$$\frac{|I(\tilde{\varphi})|}{(\sigma, \tilde{\sigma})_{\tilde{G}}} = \frac{|I(\rho)|}{|\Pi(\rho)|} = \frac{|I(\rho)|}{|\Pi(\rho)|} = \frac{|S_{\varphi, sc} / S_{\varphi, sc}|}{|I(\rho)|}.$$
Note that Theorem 4.17 provides
\[
\frac{\langle\overline{\sigma},\tilde{\sigma}\rangle_G}{|\overline{\sigma}|} = \frac{\dim \rho}{|\rho|} \frac{|\Pi_\rho(S_{\tilde{\varphi},sc})|}{|\rho|^2 S_{\tilde{\varphi},sc}}.
\]
Due to (4.26) and (4.27), we thus have
\[
\frac{|\overline{\sigma}|}{\langle\overline{\sigma},\tilde{\sigma}\rangle_G} = \frac{|\overline{\sigma}|}{\langle\overline{\sigma},\tilde{\sigma}\rangle_G},
\]
which implies that \(\langle\overline{\sigma},\tilde{\sigma}\rangle_G = \langle\overline{\sigma},\rho\rangle^2 S_{\tilde{\varphi},sc}\). This completes the proof. \(\square\)

**Remark 4.21.** The equality in Theorem 4.20 has been earlier discussed for unitary principal series of quasi-split unitary groups in [24, Section 4] as well as for quasi-split groups with some other hypotheses in [39, Section 6].

Next, we obtain the following formula for the multiplicity in the restriction in terms of some notions in Langlands dual side, using Clifford’s theorem (see [17, Proposition 20.8]).

**Theorem 4.22.** Assume as in Theorem 4.20. For any \(\rho \in \text{Irr}(S_{\tilde{\varphi},sc})\) with \(\rho \subseteq \text{Res}_{S_{\tilde{\varphi},sc}}(\rho)\),
\[
\langle\sigma,\tilde{\sigma}\rangle_G = \frac{\dim \rho}{\dim \rho} |\Pi_\rho(S_{\tilde{\varphi},sc})|^{-1}.
\]

**Proof.** Due to Theorem 4.20, it suffices to show that
\[
\langle\overline{\rho},\rho\rangle S_{\tilde{\varphi},sc} = \frac{\dim \rho}{\dim \rho} |\Pi_\rho(S_{\tilde{\varphi},sc})|^{-1}.
\]
This is a consequence of Clifford’s theorem which applies to the situation that \(S_{\tilde{\varphi},sc}\) is a normal subgroup of a finite group \(S_{\tilde{\varphi},sc}\) whose factor group is abelian. \(\square\)

**Remark 4.23.** We note from Clifford’s theorem that \(\dim \overline{\rho}_1 = \dim \overline{\rho}_2\) for any \(\overline{\rho}_1, \overline{\rho}_2 \subseteq \text{Res}_{S_{\tilde{\varphi},sc}}(\rho)\). Recall that \(\rho \in \text{Irr}(S_{\tilde{\varphi},sc}, \zeta_G)\) and \(\rho \subseteq \text{Irr}(S_{\tilde{\varphi},sc}, \tilde{\zeta}_G)\) correspond to \(\sigma\) and \(\tilde{\sigma}\), respectively, due to the internal structure (4.3) for \(G\) and \(\tilde{G}\). Since \(\overline{\rho} \subseteq \text{Res}_{S_{\tilde{\varphi},sc}}(\rho)\), it is clear that
\[
\langle\sigma,\tilde{\sigma}\rangle_G = \frac{\dim \rho}{\dim \rho} |\Pi_\rho(S_{\tilde{\varphi},sc})|^{-1}.
\]

**Remark 4.24.** When \(S_{\tilde{\varphi},sc}\) is abelian, Theorem 4.22 yields that \(\langle\sigma,\tilde{\sigma}\rangle_G\) is always equal to 1. Further, if \(\tilde{G}\) is quasi-split (so is \(G\)) and if \(S_{\tilde{\varphi}}\) is abelian, then it also follows that \(\langle\sigma,\tilde{\sigma}\rangle_G\) is always equal to 1.

### 4.3. Hiraga and Saito’s work for SL\(_m(D)\) revisited

In this section, we shall observe that Theorem 4.22 generalizes the Hiraga and Saito’s work in [16] on the multiplicity in restriction for the case of inner forms of \(GL_n\) and \(SL_n\) toward the case of arbitrary connected reductive algebraic groups \(\tilde{G}\) and \(G\) with the condition (2.1).

Denote by \(D\) a central division algebra of dimension \(d^2\) over \(F\) (possibly \(D = F\), in which case \(d = 1\)). Let \(GL_m(D)\) denote the group of all invertible elements of \(m \times m\) matrices over \(D\), and \(SL_m(D)\) the subgroup of elements in \(GL_m(D)\) with reduced norm equal to 1. By abuse of notation, we write \(GL_m(D)\) and \(SL_m(D)\) for their algebraic groups over \(F\) as well. Let \(n = md\). Then, any \(F\)-inner forms of \(GL_n\) and \(SL_n\) are of the form \(GL_m(D)\) and \(SL_m(D)\), respectively (see [31, Sections 2.2 & 2.3]). We have \(GL_m(D) = PGL_n(C)\) and \(SL_m(D) = PGL_n(C)\), since \(\Gamma\) acts trivially. Due to [5, 13, 14, 15, 16, 34], the local Langlands conjecture in [11] and the conjectural internal structure of \(L\)-packets in [13] for all inner forms of \(SL_n\) and \(GL_n\) are known. In particular, from [16, Chapter 12] and [5, Section 3], given \(\varphi \in \Phi_{\text{temp}}(SL_m(D))\) and its lifting \(\tilde{\varphi} \in \Phi_{\text{temp}}(GL_m(D))\) such that \(\Pi_{\tilde{\varphi}}(GL_m(D)) = \{\tilde{\varphi}\}\), we have a bijection
\[
\Pi_{\varphi}(SL_m(D)) \xrightarrow{\text{1-1}} \text{Irr}(S_{\tilde{\varphi},sc}(SL_m(D)), \zeta_{SL_m(D)}).
\]
such that the isomorphism
\[ V_{\tilde{\sigma}} \cong \bigoplus_{\rho \in \text{Irr} \left( S_{\varphi,sc}(SL_m(D)), \zeta_{SL_m(D)} \right)} \rho \boxtimes \sigma_{\rho} \]
as representations of \( S_{\varphi,sc}(SL_m(D)) \times SL_m(D) \) holds, where \( \sigma_{\rho} \) denotes the image of \( \rho \) via the bijection \( \Phi \) (see [16, Lemma 12.6]). It then follows from (3.8) and [16, p.5] that
\[ \langle \sigma, \tilde{\sigma} \rangle_{SL_m(D)} = \dim \xi_{\sigma} = \dim \rho_{\sigma}. \]
Further, for any \( \sigma_1, \sigma_2 \in \Pi_{\varphi}(SL_m(D)) \), we have \( \dim \rho_{\sigma_1} = \dim \rho_{\sigma_2} \) (see also Remark 3.3).

Therefore, Theorem 4.22 generalizes the Hiraga and Saito’s work in (4.29) to the case of arbitrary connected reductive algebraic groups \( \tilde{G} \) and \( G \) with the condition (2.1). To be precise, when \( G = GL_m(D) \) and \( \tilde{G} = SL_m(D) \), we always have
\[ \dim \tilde{\rho} = 1 \quad \text{and} \quad |\Pi_{\varphi}(S_{\varphi,sc})| = 1. \]
It follows from Theorem 4.22 and Remark 4.23 that
\[ \langle \sigma, \tilde{\sigma} \rangle_{SL_m(D)} = \dim \rho, \]
which coincides with (4.29).

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Appendix A. Examples

We present some examples related to the results established in Section 4. We continue with the notation in the previous sections.

Example A.1. This example is based on [10] Sections 7.6 and 7.7. Let \( \tilde{G} = GSp_{1,1} \) be the non-quasi-split inner form of \( GSp_4 \), and let \( G = Sp_{1,1} \) be the non-quasi-split inner form of \( Sp_4 \). Let \( \tilde{\varphi} = \varphi_0 \oplus (\varphi_0 \otimes \chi) \in \Phi(GSp_{1,1}) \) be given, where \( \chi \) is a quadratic character, \( \varphi_0 \in \Phi(GL_2) \) is primitive (by definition, \( \varphi_0 \) is not of the form \( \text{Ind}^{GSp_4}_{W_E} \tau \) for any non-trivial finite extension \( E/F \) and irreducible representation \( \tau \)), and \( \varphi_0 \neq \varphi_0 \otimes \chi \).

We have
\[ \text{Res}^{GSp_{1,1}}_{Sp_{1,1}}(\tilde{\sigma}_1') = \text{Res}^{GSp_{1,1}}_{Sp_{1,1}}(\tilde{\sigma}_2') = \{ \sigma' \}, \]
and \( \tilde{\sigma}_2' \cong \tilde{\sigma}_1' \chi \). Moreover, from the fact that \( X(\tilde{\varphi}) \simeq \{ 1, \chi \} \) (see [12] Proposition 6.3(iii)(b))), it follows that
\[ I(\tilde{\sigma}_1') = I(\tilde{\sigma}_2') = \{ 1 \}. \]
Thus, the \( L \)-packet \( \Pi_{\varphi}(Sp_{1,1}) \) of \( Sp_{1,1} \) attached to the \( L \)-parameter \( \varphi \) is \( \{ \sigma' \} \). We recall from [10] Sections 7.6 and 7.7 that
\[ 1 \rightarrow \mu_2(\mathbb{C}) \rightarrow S_{\tilde{\varphi},sc}(GSp_{1,1}) \simeq (\mathbb{Z}/2\mathbb{Z})^2 \rightarrow S_{\varphi}(GSp_{1,1}) \simeq \mathbb{Z}/2\mathbb{Z} \rightarrow 1, \]
\[ 1 \rightarrow \mu_2(\mathbb{C}) \rightarrow S_{\varphi,sc}(Sp_{1,1}) \simeq D_8 \rightarrow S_{\varphi}(Sp_{1,1}) \simeq (\mathbb{Z}/2\mathbb{Z})^2 \rightarrow 1, \]
where \( D_8 \) denotes the dihedral group of order 8. Further, we note that \( \text{Irr}(D_8) \) consists of four 1-dimensional characters and one 2-dimensional irreducible representation. We denote by \( \rho' \) the 2-dimensional irreducible representation. Setting \( \text{Irr}(\mu_2(\mathbb{C})) = \{ 1, \text{sgn} \} \), the map \( \sigma' \mapsto \rho' \) from \( \Pi_{\varphi}(Sp_{1,1}) \) to \( \text{Irr}(S_{\varphi,sc}(Sp_{1,1}), \text{sgn}) \) provides an equality
\[ \dim \rho' = 2. \]
while the multiplicity $\langle \sigma', \tilde{\sigma}' \rangle_{\text{Sp}_{1,1}}$ in $\text{Res}_{\text{Sp}_{1,1}}^{\text{GSp}_{1,1}}(\tilde{\sigma}')$ for $i = 1, 2$ satisfies

\[ \langle \sigma', \tilde{\sigma}' \rangle_{\text{Sp}_{1,1}} = 1. \]

We now consider $(\text{GSp}_{1,1}(F))_{\sigma'}$ which turns out to be equal to $\text{GSp}_{1,1}(F)$ and $\widetilde{G}/G_{\sigma'} = \{1\}$.

We also note that

\[ I(\rho') = \{ \eta \in (D_8/(\mathbb{Z}/2\mathbb{Z}))^\vee : \rho' \eta \simeq \rho' \} = (D_8/(\mathbb{Z}/2\mathbb{Z}))^\vee, \]

which implies that

\[ (S_{\varphi,sc}/S_{\varphi,sc})^\vee/I(\rho'). \]

Hence, this coincides with Theorem 4.13. Moreover, from the bijection $\Pi_{\varphi}$ we have

\[ \text{Res}^D_H(\rho') = \{ \chi_1, \chi_2 \}, \]

where $\chi_1$ and $\chi_2$ are distinct 1-dimensional characters of $H$, since the trace of $\rho'$ vanishes outside such subgroup $H$ (see (20.13)(2)]. Fix $\chi_1$ corresponding to $\tilde{\sigma}_1'$ via the bijection between $\Pi_{\varphi}(\text{GSp}_{1,1})$ and $\text{Irr}(S_{\varphi,sc}(\text{GSp}_{1,1}), \text{sgn})$. Computing the stabilizer $(D_8)_{\chi_1} = \{ s \in D_8 : s \chi_1 = \chi_1 \}$, we have $D_8/(D_8)_{\chi_1} \simeq \mathbb{Z}/2\mathbb{Z}$. Hence, this coincides with Theorem 4.14.

Lastly, for the multiplicity in the restriction, given a lifting $\tilde{\sigma}' \in \Pi_{\varphi}(\text{GSp}_{1,1})$ of $\sigma' \in \Pi_{\varphi}(\text{Sp}_{1,1})$, we have

\[ \langle \sigma', \tilde{\sigma}' \rangle_G = \frac{\dim \rho'}{\dim \chi_1} |\Pi_{\varphi}(S_{\varphi,sc})|^{-1} = \frac{2}{1} 2^{-1} = 1, \]

which coincides with (A.3). Hence, this coincides with Proposition 4.22. This applies to all the others in $\Pi_{\varphi}(\text{Sp}_{1,1})$.

**Example A.2.** Let $\bar{G} = \text{GSp}_4$, $G = \text{Sp}_4$, and $\bar{\varphi}$ be given as above in Example A.1. From [10] Sections 7.6 and 7.7, we have

\[ S_{\varphi}(\text{Sp}_4) \simeq \mathbb{Z}/2\mathbb{Z}, \quad S_{\bar{\varphi}}(\text{Sp}_4) \simeq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}, \]

\[ \text{Res}_{\text{Sp}_4}^{\text{GSp}_4}(\tilde{\sigma}_1) = \{ \sigma_1^+, \sigma_1^- \}, \quad \text{Res}_{\text{Sp}_4}^{\text{GSp}_4}(\tilde{\sigma}_2) = \{ \sigma_2^+, \sigma_2^- \}, \]

\[ \Pi_{\varphi}(\text{GSp}_4) = \{ \tilde{\sigma}_1, \tilde{\sigma}_2 \}, \quad \Pi_{\bar{\varphi}}(\text{Sp}_4) = \{ \sigma_1^+, \sigma_1^-, \sigma_2^+, \sigma_2^- \}. \]

From the bijection $\Pi_{\varphi} \xrightarrow{\sim} \text{Irr}(S_{\varphi}(\text{Sp}_4))$, we correspond $\sigma = \sigma_1^+$ to $\rho = 1$. Recall that there is a bijection $\text{GSp}_4(F)/(\text{GSp}_4(F))_{\sigma} \xrightarrow{\sim} \Pi_{\bar{\varphi}}(\text{Sp}_4)$. Then we note that

\[ \{ \eta \in (S_{\varphi}(\text{Sp}_4)/S_{\bar{\varphi}}(\text{GSp}_4))^\vee : \eta 1 = 1 \} = \{ 1 \} \]

and we have

\[ \text{GSp}_4(F)/(\text{GSp}_4(F))_{\sigma} \simeq \mathbb{Z}/2\mathbb{Z}, \]

\[ (S_{\varphi} \simeq \mathbb{Z}/2\mathbb{Z}, (S_{\bar{\varphi}}(\text{Sp}_4)/S_{\varphi}(\text{GSp}_4))^{\vee}/\{ \eta \in (S_{\varphi}(\text{Sp}_4)/S_{\bar{\varphi}}(\text{GSp}_4))^\vee : \eta 1 = 1 \} \simeq \mathbb{Z}/2\mathbb{Z}. \]

Hence, this coincides with Theorem 4.13. Moreover, from the bijection $\Pi_{\bar{\varphi}} \xrightarrow{\sim} \text{Irr}(S_{\bar{\varphi}}(\text{GSp}_4))$, we correspond $\tilde{\sigma} = \tilde{\sigma}_1$ to $\bar{\rho} = 1$. Then we have

\[ X(\bar{\varphi})/I(\tilde{\sigma}) = \{ 1 \}, \quad S_{\bar{\varphi}}(\text{GSp}_4)/S_{\bar{\varphi}}(\text{GSp}_4)_{1} = \{ 1 \}. \]

Hence, this coincides with Theorem 4.17.

Lastly, for the multiplicity in the restriction, given a lifting $\tilde{\sigma}$ of $\sigma$, we have

\[ \langle \sigma, \tilde{\sigma} \rangle_G = \frac{\dim 1}{\dim \Pi_{\bar{\varphi}}(S_{\varphi,sc})} |\Pi_{\bar{\varphi}}(S_{\varphi,sc})|^{-1} = \frac{1}{1} 1^{-1} = 1. \]
Hence, this coincides with Proposition 4.22. This applies to all the others in $\Pi_\varphi(Sp_4)$.

**Example A.3.** Let $\tilde{G} = GL_2$, $G = SL_2$, and $\tilde{\varphi} \in \Phi(\tilde{G})$ be dihedral with respect to three quadratic extensions. Then we have

$$S_{\tilde{\varphi}}(\tilde{GL}_2) = \{1\}, \quad S_{\varphi}(SL_2) \simeq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z},$$

$$\Pi_{\tilde{\varphi}}(GL_2) = \{\tilde{\sigma}\}, \quad \text{Res}^{GL_2}_{SL_2}(\tilde{\sigma}) = \Pi_\varphi(SL_4) = \{\sigma_1, \sigma_2, \sigma_3, \sigma_4\}.$$ 

From the bijection $\Pi_{\tilde{\varphi}} \overset{\sim}{\longrightarrow} \text{Irr}(S_{\tilde{\varphi}}(\tilde{SL}_2))$, we correspond $\sigma = \sigma_1$ to $\rho = 1$. Note that there is a bijection $GL_2(F)/(GL_2(F))_{\sigma} \overset{\sim}{\longrightarrow} \Pi_{\tilde{\varphi}}(SL_2)$. We then have

$$GL_2(F)/(GL_2(F))_{\sigma} \simeq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z},$$

$$(S_{\tilde{\varphi}}(\tilde{SL}_2)/S_{\tilde{\varphi}}(\tilde{SL}_2))^{\vee}/\{\eta \in (S_{\tilde{\varphi}}(\tilde{SL}_2)/S_{\tilde{\varphi}}(\tilde{SL}_2))^{\vee} : \eta = 1\} \simeq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}.$$ 

Hence, this coincides with Theorem 4.13. Moreover, from the bijection $\Pi_{\tilde{\varphi}} \overset{\sim}{\longrightarrow} \text{Irr}(S_{\tilde{\varphi}}(\tilde{GL}_2))$, we correspond $\tilde{\sigma} = \tilde{\sigma}_1$ to $\tilde{\rho} = 1$. Then we have

$$X(\tilde{\varphi})/I(\tilde{\sigma}) = \{1\}, \quad S_{\tilde{\varphi}}(\tilde{GL}_2)/S_{\tilde{\varphi}}(\tilde{GL}_2)_{1} = \{1\}.$$ 

Hence, this coincides with Theorem 4.17

Lastly, for the multiplicity in the restriction, given a lifting $\tilde{\sigma}$ of $\sigma$, we have

$$\langle \sigma, \tilde{\sigma} \rangle_C = \frac{\dim 1}{\dim \Pi_1(S_{\tilde{\varphi}, sc})^{-1}} = \frac{1}{1}1^{-1} = 1.$$ 

Hence, this coincides with Proposition 4.22. This applies to all the others in $\Pi_\varphi(SL_2)$.

**Remark A.4.** From Example A.3 we note that two sizes $|I(\rho)| = 1$ and $|I(\tilde{\sigma})| = 4$ does not necessarily equal each other. This implies that $|\Pi_{\tilde{\varphi}}(G)| = 4$ does not need to be identical with $|\Pi_\varphi(S_{\tilde{\varphi}, sc})| = 1$.

**Example A.5.** Let $\tilde{G} = GL_1(D)$, $G = SL_1(D)$, where $D$ is the quaternion division algebra over $F$, and $\tilde{\varphi} \in \Phi(\tilde{G})$ be as in Example A.3. Then we have

$$1 \longrightarrow \mu_2(\mathbb{C}) \longrightarrow S_{\varphi,sc}(GL_1(D)) \simeq \mathbb{Z}/2\mathbb{Z} \longrightarrow S_{\tilde{\varphi}}(GL_1(\tilde{D})) = 1 \longrightarrow 1,$$

$$1 \longrightarrow \mu_2(\mathbb{C}) \longrightarrow S_{\varphi,sc}(SL_1(D)) \simeq Q_8 \longrightarrow S_{\varphi}(SL_1(D)) \simeq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \longrightarrow 1,$$

where $Q_8$ denotes the quaternion group of order 8. Recall that

$$\Pi_{\tilde{\varphi}}(GL_1(D)) = \{\tilde{\sigma}'\}, \quad \text{Res}^{GL_1(D)}_{SL_1(D)}(\tilde{\sigma}') = \Pi_\varphi(SL_1(D)) = \{\sigma'\},$$

$$\text{Irr}(Q_8) = \{\chi_1, \chi_2, \chi_3, \chi_4, \rho'\},$$

where $\chi_i$’s are distinct 1-dimensional representations, and $\rho'$ is the 2-dimensional representation of $Q_8$. From the bijection $\Pi_{\tilde{\varphi}}(SL_1(D)) \overset{\sim}{\longrightarrow} \text{Irr}(S_{\varphi,sc}(SL_1(D)), sgn) = \{\rho'\}$, we correspond $\sigma'$ to $\rho'$. Note that there is a bijection $GL_1(D)/(GL_1(D))_{\sigma'} \overset{\sim}{\longrightarrow} \Pi_{\tilde{\varphi}}(SL_1(D))$. We then have

$$GL_1(D)/(GL_1(D))_{\sigma'} = \{1\}, \quad (Q_8/(\mathbb{Z}/2\mathbb{Z}))^{\vee}/\{\eta \in (Q_8/(\mathbb{Z}/2\mathbb{Z}))^{\vee} : \rho' \eta = \rho'\} \simeq \{1\},$$

since $\{\eta \in (Q_8/(\mathbb{Z}/2\mathbb{Z}))^{\vee} : \rho' \eta = \rho'\} = \{\chi_1, \chi_2, \chi_3, \chi_4\}$. Hence, this coincides with Theorem 4.13. Moreover, from the bijection $\Pi_{\tilde{\varphi}} \overset{\sim}{\longrightarrow} \text{Irr}(S_{\varphi,sc}(GL_1(D)), sgn)$, we correspond $\tilde{\sigma}'$ to $sgn$. Then we have

$$X(\tilde{\varphi})/I(\tilde{\sigma}') = \{1\}, \quad Q_8/(Q_8)_{\rho'} = \{1\}.$$ 

Hence, this coincides with Theorem 4.17

Lastly, for the multiplicity in the restriction, given a lifting $\tilde{\sigma}'$ of $\sigma'$, we have

$$\langle \sigma', \tilde{\sigma}' \rangle_C = \frac{\dim \rho'}{\dim 1} \frac{\dim 1}{\dim \Pi_1(S_{\tilde{\varphi}, sc})^{-1}} = \frac{2}{1}1^{-1} = 2.$$ 

Hence, this coincides with Proposition 4.22. This applies to all the others in $\Pi_\varphi(SL_1(D))$. 


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