Some results on co-recursive associated Laguerre and Jacobi polynomials

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Abstract

We present results on co-recursive associated Laguerre and Jacobi polynomials which are of interest for the solution of the Chapman-Kolmogorov equations of some birth and death processes with or without absorption. Explicit forms, generating functions, and absolutely continuous part of the spectral measures are given. We derive fourth-order differential equations satisfied by the polynomials with a special attention to some simple limiting cases.

Key words. Orthogonal polynomials, birth and death processes, hypergeometric functions.

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1 Introduction

Starting from a sequence of orthogonal polynomials \{P_n\}_{n \geq 0} defined by the recurrence relation
\[ P_{n+2}(x) = (x - \beta_{n+1})P_{n+1}(x) - \gamma_{n+1}P_n(x), \quad n \geq 0, \]
and the initial conditions
\[ P_0(x) = 1, \quad P_1(x) = x - \beta_0, \]
with \( \beta_n, \gamma_n \in \mathbb{C} \) and \( \gamma_n \neq 0 \), several modifications were considered:

- Associated polynomials arise when we replace \( n \) by \( n + c \) in the coefficients \( \beta_n \) and \( \gamma_n \) (keeping \( \gamma_n \neq 0 \)). If \( c \) is an integer \( k \) these polynomials are called associated of order \( k \). The associated polynomials of order one are the numerator polynomials.

- Co-recursive polynomials arise when we replace \( \beta_0 \) by \( \beta_0 + \mu \).

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- Perturbed polynomials arise when we replace $\gamma_1$ by $\lambda \gamma_1$, ($\lambda > 0$).
- Co-recursive of this perturbed polynomials arise when the two previous modifications are made together.
- Generalized co-recursive and perturbed polynomials arise when we change $\beta_n$ and/or $\gamma_n$ at any level $n$.

In the study of birth and death processes orthogonal polynomials, in particular all the hypergeometric families of the Askey scheme [1, 13] and their corresponding associated families, play a primordial role in the Karlin-McGregor solution of the Chapman-Kolmogorov equation [16, 17]

$$p_{mn}(t) \sim \int_0^\infty e^{-xt} P_m(x) P_n(x) d\phi(x).$$  \hspace{1cm} (3)

In certain birth and death processes, zero-related polynomials [12, 13, 14] arise in a natural way. Zero-related polynomials are special co-recursive associated polynomials.

More generally co-recursive and generalized co-recursive polynomials are involved in the solution of the Chapman-Kolmogorov equation of birth and death processes with absorption or killing [11, 17, 18].

The purpose of this paper is to present some results on co-recursive associated Laguerre and Jacobi polynomials which are of special interest in the study of birth and death processes with linear and rational rates respectively. The Laguerre polynomial families are involved in processes for which the birth and death rates are of the form [12]

$$\lambda_n = n + \alpha + c + 1, \hspace{0.5cm} \mu_{n+1} = n + c + 1, \hspace{0.5cm} n \geq 0, \hspace{0.5cm} \mu_0 = c - \mu.$$ \hspace{1cm} (4)

The Jacobi polynomial families are involved when the rates are of the form

$$\lambda_n = \frac{2(n+c+\alpha+\beta+1)(n+c+\beta+1)}{(2n+2c+\alpha+\beta+1)(2n+2c+\alpha+\beta+2)}, \hspace{0.5cm} n \geq 0,$$ \hspace{1cm} (5)

$$\mu_n = \frac{2(n+c)(n+c+\alpha)}{(2n+2c+\alpha+\beta)(2n+2c+\alpha+\beta+1)}, \hspace{0.5cm} n > 0,$$ \hspace{1cm} (6)

$$\mu_0 = \frac{2c(\alpha+c)}{(2c+\alpha+\beta)(2c+\alpha+\beta+1)} - \mu.$$ \hspace{1cm} (7)

In both cases $\mu_0 = 0$ correspond to the “honest” [24] linear processes (i.e. processes for which the sum of the probabilities $p_{mn}(t)$ is equal to 1). Cases $\mu_0 = \text{Const.} \neq 0$ correspond to processes with absorption and are not “honest”. However if $\mu_0 = c$ in the Laguerre case or $\mu_0 = \frac{2c(\alpha+c)}{(2c+\alpha+\beta)(2c+\alpha+\beta+1)}$ in the Jacobi case, the corresponding processes are simply solved using associated polynomials.

In section 2 we explain the method used by applying it to the Laguerre case. In 2.1 we give an explicit expression for the co-recursive associated Laguerre (CAL) polynomials. In 2.2 we derive a generating function of them and we found the absolutely continuous part of the spectral measure. The subsection 2.3 is devoted to the derivation, using the Orr’s method, of a fourth-order differential equation satisfied by the CAL polynomials. In 2.4 we present results obtain in some limiting cases among which a new simple case of associated Laguerre polynomials.
In section 3 we present briefly some results corresponding to the Jacobi case. In 3.1 we give an explicit expression for the co-recursive associated Jacobi (CAJ) polynomials. In 3.2 we present a generating function and in 3.3 we give the absolutely continuous component of the spectral measure. The subsection 3.5 is devoted to some limiting cases of CAJ polynomials for which we give fourth-order differential equations they satisfy. In section 3.6 are some concluding remarks.

We use the notation of [7] for the special functions used in this work. We don’t give validity conditions on the parameters of the used hypergeometric functions, analytic continuations or limiting processes giving, in general, valid formulas. We use Slater’s notation [29] for the product of $\Gamma$ functions

$$\Gamma\left(\alpha_1, \ldots, \alpha_p, \beta_1, \ldots, \beta_q\right) = \prod_{i=1}^{p} \Gamma(\alpha_i) / \prod_{i=1}^{q} \Gamma(\beta_i).$$

(8)

2 The case of Laguerre polynomials

Replacing $n$ by $n+c$ in the recurrence relation of the Laguerre polynomials we obtain the recurrence relation satisfied by the associated Laguerre polynomials

$$(2n+2c+\alpha+1-x)p_n = (n+c+1)p_{n+1} + (n+c+\alpha)p_{n-1}. \quad (9)$$

To complete the definition of the polynomials $L^n_\alpha(x; c)$ the initial condition

$$L_{-1}^\alpha(x; c) = 0, \quad L_0^\alpha(x; c) = 1, \quad (10)$$

has to be imposed. These polynomials are orthogonal with respect to a positive measure when $(n+c)(n+\alpha+c) > 0, \forall n > 0$. (See [2] for details).

Note that if we consider the monic polynomials $L_n^\alpha(x; c)$ defined by

$$L_n^\alpha(x; c) = (-1)^n(c+1)_nL_n^\alpha(x; c) \quad (11)$$

they satisfy the recurrence relation

$$(x - 2n - 2c - \alpha - 1)L_n^\alpha(x; c) = L_{n+1}^\alpha(x; c) + (n+c)(n+c+\alpha)L_{n-1}^\alpha(x; c). \quad (12)$$

We can see that this recurrence is invariant in the transformation $\mathcal{T}$ defined by

$$\mathcal{T}(c, \alpha) = (c + \alpha, -\alpha). \quad (13)$$

2.1 Explicit representation of the CAL polynomials

Equations (3) and (11) give for $L_1^\alpha(x; c)$

$$L_1^\alpha(x; c) = -\frac{1}{c+1}(x - 2c - \alpha - 1) \quad (14)$$

The CAL polynomials $L_n^\alpha(x; c, \mu)$ satisfy the same recurrence relation (3) with a shift $\mu$ on the monic polynomial of first degree, i.e.

$$L_1^\alpha(x; c, \mu) = -\frac{1}{c+1}(x + \mu - 2c - \alpha - 1). \quad (15)$$
To obtain $L_1^\alpha(x; c, \mu)$ with (13) we have to impose the initial condition for the CAL polynomials

$$L_{-1}^\alpha(x; c, \mu) = \frac{\mu}{c + \alpha}, \quad L_0^\alpha(x; c, \mu) = 0. \quad (16)$$

Even for $c + \alpha \to 0$ this initial condition in the recurrence relations (13) leads to (15).

We know two linearly independent solutions of (13) in terms of confluent hypergeometric functions:

$$u_n = \frac{(c + \alpha + 1)_n}{(c + 1)_n} {}_1F_1\left( \begin{array}{c} -n - c \\ 1 + \alpha \end{array}; x \right) \quad \text{and} \quad v_n = {}_1F_1\left( \begin{array}{c} -n - c - \alpha \\ 1 - \alpha \end{array}; x \right). \quad (17)$$

Writing the polynomials $L_n^\alpha(x; c, \mu)$ as a linear combination

$$L_n^\alpha(x; c, \mu) = Au_n + Bv_n \quad (18)$$

and using the initial condition (16) we obtain

$$A = \frac{1}{\Delta} \left[ \frac{\mu}{c + \alpha} v_0 - u_1 \right] \quad \text{and} \quad B = -\frac{1}{\Delta} \left[ \frac{\mu}{c + \alpha} u_0 - u_1 \right], \quad (19)$$

where $\Delta$ may be calculated using contiguous relations of confluent hypergeometric functions [7, page 253–254]

$$\Delta = u_{-1}v_0 - u_0v_{-1} = -\frac{\alpha}{c + \alpha}e^x. \quad (20)$$

With the help of the relation

$$\gamma {}_1F_1\left( \begin{array}{c} 1 - \gamma \\ \alpha \end{array}; x \right) - \beta {}_1F_1\left( \begin{array}{c} -\gamma \\ \alpha \end{array}; x \right) = (\gamma - \beta) {}_2F_2\left( \begin{array}{c} -\gamma, \beta - \gamma + 1 \\ \alpha, \beta - \gamma \end{array}; x \right), \quad (21)$$

we can write the CAL polynomials as

$$L_n^\alpha(x; c, \mu) = \frac{e^{-x}}{(c + 1)_n} (1 + T)$$

$$\times \frac{\mu - c}{\alpha} \frac{(c + 1)_n}{(c + 1)} {}_2F_2\left( \begin{array}{c} -c, \mu - c + 1 \\ 1 + \alpha, \mu - c \end{array}; x \right) {}_1F_1\left( \begin{array}{c} -n - c - \alpha \\ 1 - \alpha \end{array}; x \right). \quad (22)$$

This representation is the one we use to derive the fourth-order differential equation in section 2.3. It is valid only for $\alpha \neq 0, \pm 1, \pm 2, \ldots$ but these restrictions can be removed by limiting processes.

Following the same way as in [2] we can find an explicit representation. We first transform the $1F_1$ in (14) using Kummer’s transformation [7, page 253]

$$1F_1\left( \begin{array}{c} a \\ c \end{array}; x \right) = e^x {}_1F_1\left( \begin{array}{c} c - a \\ c \end{array}; -x \right). \quad (23)$$

Then we use formula

$$1F_1\left( \begin{array}{c} a \\ b \end{array}; x \right) 1F_1\left( \begin{array}{c} c \\ d \end{array}; -x \right) = \sum_{k=0}^{\infty} \frac{(a)_k x^k}{k! (b)_k} {}_3F_2\left( \begin{array}{c} -k, 1 - k - b, c \\ 1 - k - a, d \end{array}; 1 \right), \quad (24)$$
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for the four products in [18].

Applying the three term relation [4, Eq. (2) page 15] to the four resulting $3F_2(1)$ gives a sum of two terms which we can group to obtain the explicit form

$$L_n^{\alpha}(x; c, \mu) = \frac{(c + \alpha + 1)_n}{n!} \sum_{k=0}^{n} \frac{(-n)_k}{(c + 1)_k(c + \alpha + 1)_k} \times 4F_3 \left( \begin{array}{c} k - n, c, c + \alpha, G + 1 \\ c + k + 1, c + \alpha + k + 1, G; 1 \end{array} \right) x^k,$$

where

$$G = \frac{c(c + \alpha)}{\mu}. \quad \text{(26)}$$

Take care of the limiting processes when $n = k$ and when $c$ or $c + \alpha = 0$. The representation (25) can also be proved using the generating function (30).

2.2 Generating function, spectral measure

Let $F(x, w)$ be a generating function of the CAL polynomials $L_n^{\alpha}(x; c, \mu)$

$$F(x, w) = \sum_{n=0}^{\infty} w^n L_n^{\alpha}(x; c, \mu). \quad \text{(27)}$$

The recurrence relation (9) and the initial condition (16) lead to the following differential equation for $F(x, w)$

$$w(1 - w)^2 \frac{\partial}{\partial w} F(x, w) + [(1 - w)(c - (c + \alpha + 1)w) + xw] F(x, w) = c - \mu w. \quad \text{(28)}$$

The function $F(x, w)$ is normalized by the condition $F(x, 0) = 1$ and due to the orthogonality of the $L_n^{\alpha}(x; c, \mu)$, we have the boundary condition

$$\int_{0}^{\infty} F(x, w) d\phi(x) = 1, \quad \text{(29)}$$

where $d\phi(x)$ is the spectral measure.

The solution of the differential equation (28) which is bounded at $w = 0$ is easily obtain, for $c > 0$, following the same method as in [12]

$$F(x, w) = w^{-c}(1 - w)^{-\alpha - 1} \exp \left[ -\frac{x}{1 - w} \right] \times \int_{0}^{w} u^{c-1}(1 - u)^{-\alpha - 1}(c - \mu u) \exp \left[ \frac{x}{1 - u} \right] du. \quad \text{(30)}$$

Changing variables according to

$$u = \frac{\tau}{1 + \tau}, \quad w = \frac{z}{1 + z}, \quad \text{(31)}$$

and integrating both side of (30) with respect to $d\phi(x)$, taking into account (29), leads to

$$z^{c}(1 + z)^{-1-\alpha-c} = \int_{0}^{\infty} \int_{0}^{z} \tau^{c-1}(1 + \tau)^{-1-\alpha-c}[c + \tau(c - \mu)] \exp [-x(z - \tau)] d\tau d\phi(x). \quad \text{(32)}$$
Taking the Laplace transform of the above identity we obtain for the Stieltjes transform of the measure \(d\phi(x)\) the relationship

\[
s(p) = \int_0^\infty \frac{d\phi(x)}{x+p} = \frac{\Psi(c+1, 1-\alpha; p)}{\Psi(c, -\alpha; p) + (c-\mu)\Psi(c+1, 1-\alpha; p)},
\]

which we can rewrite formally, using the expression of the Tricomi function \(\Psi\) in terms of generalized hypergeometric functions \([7, page 257]\), on the form

\[
s(p) = p^{c}\Psi(c+1, 1-\alpha; p)\binom{c}{c+\alpha, c}(c+\alpha, c)\mu + 1; -\frac{1}{p} \right)^{-1},
\]

where the principal branch of the \(3F_1\) is considered as that one of the \(\Psi\) function. The function \(\Psi(a, b; p)\) having no zeros for \(|\text{arg} p| \leq \pi\) the denominator of (33) has no zeros in this region at least for \(\mu \leq c\).

The CAL polynomials belong to the Laguerre-Hahn family of orthogonal polynomials and are of class zero \([22]\). It is easily verified that the Stieltjes transform \(s(p)\) of the measure, calculated in (33), is a solution of the Riccati equation

\[
ps'(p) = [\mu p - (c-\mu)(\alpha + c-\mu)]s^2(p) + [p + \alpha + 2(c-\mu)]s(p) - 1.
\]

The absolutely continuous part of the measure \(d\phi(x)\) can be computed using the Perron-Stieltjes inversion formula. Details of the method are in \([10]\) and we obtain

\[
\phi'(x) = \frac{1}{\Gamma(c+1)\Gamma(c+\alpha+1)} \frac{x^\alpha e^{-x}}{|\Psi(c, -\alpha; xe^{i\pi}) + (c-\mu)\Psi(c+1, 1-\alpha; xe^{i\pi})|^2},
\]

we can write formally

\[
\phi'(x) = \frac{x^{\alpha+2\alpha}e^{-x}}{\Gamma(c+1, c+\alpha+1)} \left|3F_1\left(c, c+\alpha, \frac{c+\alpha+\mu}{\mu}; 1; -\frac{e^{i\pi}}{x}\right)\right|^2.
\]

The CAL polynomials \(L_n^\alpha(x; c, \mu)\) satisfy the orthogonality relation, valid at least for \(\mu \leq c, c \geq 0, \alpha + c > -1,\)

\[
\int_0^\infty L_n^\alpha(x; c, \mu)L_m^\alpha(x; c, \mu)d\phi(x) = \frac{(c+\alpha+1)n}{(c+1)n}\delta_{mn}.
\]

**2.3 Fourth-order differential equation**

The CAL polynomials verify a fourth-order differential equation \([3, 22]\). One way to obtain this fourth-order differential equation is to start from their explicit form (22). The righthand side of (22) is a sum of two products of a \(1F_1\) times a \(2F_2\). The \(1F_1\) are solution of a second-order differential equation but for the \(2F_2\) a third-order one is expected. In fact the \(2F_2\) involved in (22) are of the form

\[
y(x) = 2F_2\left(\frac{b, e+1}{d, e}; x\right),
\]
we obtain the normal form of the differential equations (41) and (43) under this transformation. Let us notice that the second product being obtained by the transformation \( T \) of the first one, the fourth-order differential equation will have to be invariant under this transformation.

The function \( y = e^{-x} F_1 \left( \frac{-n - c - \alpha}{1 - \alpha} ; x \right) \) is solution of

\[
x y'' + (1 - \alpha + x)y' + (1 + n + c)y = 0
\]

(41)

and the function \( z = 2 F_2 \left( \frac{-c, \mu - c + 1}{1 + \alpha, \mu - c} ; x \right) \) of

\[
x[\mu x + (c - \mu)(c + \alpha - \mu)]z''(x) - \{\mu x^2 + [(c - \mu)(c + \alpha - \mu) - \alpha \mu]x
\]

\[
-(\alpha + 1)(c - \mu)(c + \alpha - \mu) \} z'(x) + c[(\mu x + (c - \mu + 1)(c + \alpha - \mu)]z(x) = 0.
\]

(42)

Changing the functions \( y \) and \( z \) to \( y = fv \) and \( z = gw \) with

\[
f = x^\frac{\mu - c}{2} e^\frac{x}{2},
\]

(43)

\[
g = [(c - \mu)(c + \alpha - \mu) + \mu x]^{\frac{1}{2}} x^{-\frac{\mu + 1}{2}} e^\frac{x}{2},
\]

(44)

we obtain the normal form of the differential equations (44) and (45)

\[
v'' + Iv = 0, \quad w'' + Jw = 0.
\]

(45)

The product \( u = vw \) is solution of the fourth-order differential equation (see [31, page 146])

\[
\frac{d}{dx} \left[ \frac{u'' + 2(I + J)u' + (I' + J')u}{I - J} \right] = -(I - J)u
\]

(46)

Finally we obtain the needed equation for the CAL polynomials setting \( y(x) = f gu \).

Details of this calculations are very difficult to write explicitly and were achieved with the help of the MAPLE computer algebra [3]. Although with this help the fourth-order differential equation for the CAL polynomials is not easy to find. We give it as a curiosity:

\[
c_4 y^{(4)}(x) + c_3 y^{(3)}(x) + c_2 y^{(2)}(x) + c_1 y^{(1)}(x) + c_0 y(x) = 0,
\]

(47)

with

\[
c_4 = x^2(2Ax^2 + Bx + 2C), \\
c_3 = 2x(3Ax^2 + 2Bx + 5C), \\
c_2 = -2Ax^4 + Dx^3 + Ex^2 + Fx + G, \\
c_1 = -4Ax^3 + Hx^2 - 4Ax^3 + Ix + J, \\
c_0 = n(n + 1)(2Ax^2 + Kx + 2L),
\]

(48) - (52)
where

\[ A = \mu^2(1 + 2n), \]
\[ B = \mu(2(1 + 4n)\mu^2 - (4(1 + 2n)(2c + \alpha) - 1)\mu + 2c(3 + 4n)(c + \alpha)), \]
\[ C = (c - \mu)(c + \alpha - \mu)(2n\mu^2 - ((1 + 2n)(2c + \alpha) + 1)\mu + 2c(n + 1)(c + \alpha)), \]
\[ D = -\mu(2(1 + 4n)\mu^2 - (2(1 + 2n)(8c + 4\alpha + 1 + 2n) - 1)\mu + 2c(3 + 4n)(c + \alpha)), \]
\[ E = -4n\mu^4 + ((1 + 4n)(4n + 12c + 6\alpha + 3) + 3\mu^3 - 2(1 + 2n)((2c + \alpha)\]
\[ \times(4n + 12c + 6\alpha + 2) - 2c(c + \alpha - 1)\mu^2 + c(c + \alpha)((3 + 4n)\]
\[ \times(4n + 12c + 6\alpha + 1) + 3)\mu - 4c^2(c + \alpha)^2(n + 1), \]
\[ F = 8n(2c + \alpha + n + 1)\mu^4 \]
\[ + 2((1 + 4n)((2c + \alpha)(4n + 12c + 6\alpha + 3) - 8c(c + \alpha) + 1/2) + 10c + 5\alpha + 5/2)\mu^3 \]
\[ + (1 + 4n)((\alpha^2 + 6c(c + \alpha)(2n + 8c + 4\alpha + 3/2) - (2c + \alpha)(12c(c + \alpha) + 1/2)) \]
\[ + (2\alpha^2 + 6c(c + \alpha) - 1/4)(4c + 2\alpha + 23/6) - 25/6\alpha^2 + 71/4\mu^2 \]
\[ - 2c(c + \alpha)((1 + 4n)((2c + \alpha)(2n + 4c + 2\alpha + 3/2) + 2\alpha^2 - 1/4) \]
\[ + (2c + \alpha)(8c + 4\alpha + 5/2) + 2\alpha^2 - 7/4)\mu + 8c^2(c + \alpha)^2(n + 1)(n + 2c + \alpha), \]
\[ G = -2(\alpha - 2)(\alpha + 2)(c - \mu)(c + \alpha - \mu)(2(c - \mu)(c + \alpha - \mu)n \]
\[ - 2(2c + \alpha + 1)\mu + 2c(c + \alpha)), \]
\[ H = 2\mu(-2(5n + 1)\mu^2 + ((1 + 2n)(n + 12c + 6\alpha + 1/2) - 3/2)\mu - 2c(c + \alpha)(5n + 4)), \]
\[ I = -12n\mu^4 + 2((n + 1/5)(8n + 40c + 20\alpha + 7/5) + 93/25)\mu^3 \]
\[ + 2(-2(1 + 2n)((2c + \alpha)(4n + 12c + 6\alpha + 2) + 10c(c + \alpha) - 1) - 6c - 3\alpha)\mu^2 \]
\[ + 2c(c + \alpha)((4/5 + n)(8n + 40c + 20\alpha + 33/5) + 93/25)\mu - 12c(c + \alpha)^2(n + 1), \]
\[ J = 4(\mu - \alpha)(\mu - c - \alpha)(3n + 2c + \alpha + 1)\mu^2 + (-1/4(1 + 2n)((2c + \alpha)(6n + 8c \]
\[ + 4\alpha + 3) + 8c(c + \alpha) + 8) - 9/2c - 9/4\alpha)\mu + 3c(c + \alpha)(n + 1)(n + 2c + \alpha)), \]
\[ K = \mu(2(4n - 1)\mu^2 - (4(1 + 2n)(2c + \alpha) - 3)\mu + 2c(4n + 5)(c + \alpha)), \]
\[ L = (c - \mu)(c + \alpha - \mu)(2(n - 1)\mu^2 - ((1 + 2n)(2c + \alpha) + 6)\mu + 2c(n + 2)(c + \alpha)). \]

Note the invariance of the differential equation \[ (47) \] by the transformation \( T \) defined in \[ (3) \].

2.4 Particular cases

We now give the different results corresponding to limiting cases of special interest.

2.4.1 Limit \( c = 0 \)

In this limit we obtain from \[ (22) \] the co-recursive Laguerre polynomials.

\[ L_n^\alpha(x; 0, \mu) = \frac{e^{-x}}{\alpha} \left\{ \frac{(\alpha - \mu)(\alpha + 1)_n}{n!} \right\} _2F_2 \left( \begin{array}{c} -\alpha, \mu - \alpha + 1 \\ 1 - \alpha, \mu - \alpha \end{array}; x \right) _1F_1 \left( \begin{array}{c} -n \\ 1 + \alpha \end{array}; x \right) + \mu _1F_1 \left( \begin{array}{c} -n - \alpha \\ 1 - \alpha \end{array}; x \right) \right\}. \] (53)
The limit $\mu = 0$ in (53) leads back to the classical Laguerre polynomials. An explicit form is
\[
L_n^{\alpha}(x; 0, \mu) = \frac{(\alpha + 1)_n}{n!} \sum_{k=0}^{n} \frac{(-n)_k}{k!(1 + \alpha)_k} \times \left[ 1 + \frac{\mu(k - n)}{(1 + k)(1 + \alpha + k)} \right]^{3} F_{2} \left( \begin{array}{c} 1 + k - n, 1 + \alpha, 1 \\ k + \alpha + 2, k + 2 \end{array} ; 1 \right) \right] x^k, \quad (54)
\]
and the corresponding absolutely continuous part of the measure is given for $\mu \leq 0, \alpha > -1,$ by
\[
\phi'(x) = \frac{x^\alpha e^{-x}}{\Gamma(1 + \alpha)} |1 - \mu \Psi(1, 1 - \alpha; xe^{-i\pi})|^{-2}, \quad (55)
\]
where the limit $\mu = 0$ is also straightforward.

It is easy to see that the differential equation (47) satisfied by the co-recursive Laguerre polynomials can be factorized in the limit $c = 0$ to obtain the fourth-order factorized $(2+2)$
differential equation
\[
\left[ x A(x) D^2 + \left\{ (2 + \alpha - x) A(x) - x B(x) \right\} D + (n - 1) A(x) + (x - \alpha - 1) B(x) + C(x) \right] \\
\times \left[ x D^2 + (x + 1 - \alpha) D + n + 1 \right] L_n^{\alpha}(x; 0, \mu) = 0, \quad (56)
\]
where $D \equiv d/dx$ and
\[
A(x) = 3x + 2(x - \alpha + \mu) \{ 2n(x - \alpha + \mu) + x - \alpha - 1 \}, \quad (57)
\]
\[
B(x) = 1 + 2x - 2\alpha + 2(1 + 4n)(x - \alpha + \mu), \quad (58)
\]
\[
C(x) = (1 + 2x - 2\alpha) \{ 1 + \alpha - x - 2n(x - \alpha + \mu) \} + 3x(1 + 4n). \quad (59)
\]
The comparison with the differential equation given in [27, Eq. 34–35] requires some attention because of a few misprints.

2.4.2 Limit $c = -\alpha$

In this limit we obtain a special class of CAL polynomials corresponding to the associated Laguerre polynomials for which
\[
L_n^{\alpha}(x; -\alpha) = \frac{n!}{(1 - \alpha)_n} L_n^{-\alpha}(x). \quad (60)
\]
We can write the $L_n^{\alpha}(x; -\alpha, \mu)$
\[
L_n^{\alpha}(x; -\alpha, \mu) = \frac{n!}{(1 - \alpha)_n} e^{-x} \left\{ (-\alpha - \mu)(1 - \alpha)_n \right\}^{-2} F_{2} \left( \begin{array}{c} \alpha, \mu + \alpha + 1 \\ 1 + \alpha, \mu + \alpha \end{array} ; x \right) \right]^{1} F_{1} \left( \begin{array}{c} -n \\ 1 - \alpha \end{array} ; x \right) + \mu \right] F_{1} \left( \begin{array}{c} -n + \alpha \\ 1 + \alpha \end{array} ; x \right). \quad (61)
\]
which gives (60) in the limit $\mu = 0$. Except for the global factor $\frac{n!}{(1 - \alpha)_n}$, (61) is obtained from (53) changing $\alpha$ to $-\alpha$. The corresponding measure and differential equation are obtained.
in the same way from 2.4.1. An explicit form is
\[
L_n^\alpha(x; -\alpha, \mu) = \sum_{k=0}^{n} \frac{(-n)_k}{k!(1-\alpha)_k} \times \left[ 1 + \frac{\mu(k-n)}{(1+k)(1-\alpha+k)^3} \right] x^k. \tag{62}
\]

2.4.3 Limit \(\mu = 0\)

In this limit we obtain the associated Laguerre polynomials studied in [3] and [12]
\[
L_n^\alpha(x, c) = \frac{e^{-x}}{(c+1)^n} \left( 1 + c + \alpha \right)^{n+1}_\alpha \left[ F_1 \left( 1 - c - \alpha ; x \right) \right] \left[ F_1 \left( -n - c ; x \right) \right], \tag{63}
\]
with the measure
\[
\phi'(x) = \frac{x^{\alpha} e^{-x}}{\Gamma(1+c, 1+c+\alpha)} \left| \Psi(c, 1-\alpha; xe^{-i\pi}) \right|^2. \tag{64}
\]
An explicit form is
\[
L_n^\alpha(x; c) = \frac{(c + \alpha + 1)_n}{n!} \sum_{k=0}^{n} \frac{(-n)_k}{(c+1)_k(c+\alpha+1)_k} \times \left[ F_2 \left( \frac{k-n, c, c+\alpha}{c+k+1, c+\alpha+k+1}; 1 \right) \right] x^k, \tag{65}
\]
The limit \(c = 0\) leads back to the Laguerre polynomial case. The coefficients of the differential equation (67) satisfied by the associated Laguerre polynomials are now very simple
\[
c_4 = x^2, \ c_3 = 5x, \ c_2 = -x(x-2F) - \alpha^2 + 4, \ c_1 = 3(F-x), \ c_0 = n(n+2), \tag{66}
\]
with \(F = n + 2c + \alpha\). This differential equation was first given by Hahn [3, Eq. 22]. See [23, 27] for the special factorizable case \(c = 1\) and [3, 29] when \(c\) is an integer.

2.4.4 Limit \(\mu = c\)

In this limit we obtain the so called zero related Laguerre polynomials studied in [12]. Note the symmetry \(\mathcal{T}\) of the monic polynomials is now broken.
\[
L_n^\alpha(x, c) = e^{-x} \left\{ \frac{(c + \alpha + 1)_n}{(c+1)^n} \left[ F_1 \left( -c - \alpha ; x \right) \right] \left[ F_1 \left( -n - c ; x \right) \right] \right. \\
- \frac{c}{\alpha(c+1)} x \left[ F_1 \left( 1 - c - \alpha ; x \right) \right] \left[ F_1 \left( -n - c - \alpha ; x \right) \right], \tag{67}
\]
and the measure
\[
\phi'(x) = \frac{x^{\alpha} e^{-x}}{\Gamma(1+c, 1+c+\alpha)} \left| \Psi(c, -\alpha; xe^{-i\pi}) \right|^2. \tag{68}
\]
Again the limit \(c = 0\) leads back to the Laguerre polynomial case.
The explicit form \( (25) \) simplify in
\[
L_n^\alpha(x; c, \mu) = \frac{(c + \alpha + 1)_n}{n!} \sum_{k=0}^{n} \frac{(-n)_k}{(c + 1)_k(c + \alpha + 1)_k} \binom{3F_2}{k-n, c, c+\alpha+1; c+k+1, c+\alpha+k+1; 1} x^k.
\]  

The coefficients of the differential equation satisfied by the zero related Laguerre polynomials are
\[
\begin{align*}
c_4 &= x^2(2(2n+1)x + D), \\
c_2 &= -2(2n+1)x^3 + (8n(F+1) + 8c + D)x^2 \\
&\quad- (4(\alpha^2 - 1)n - 2\alpha^2 - D(4c + 1) - 1)x - 1/4D(D^2 - 9), \\
c_1 &= 8(2n+1)x^2 - 4(2n(F+1) + 2c - D)x - 2D(n(D+3) + 6c + 2D), \\
c_0 &= n(n+1)(2(2n+1)x + 3D),
\end{align*}
\]
with
\[
F = n + 2c + \alpha, \quad D = 1 + 2\alpha,
\]
and are no longer invariant under the transformation \( \mathcal{T} \).

### 2.4.5 Limit \( \mu = c + \alpha \)

This is a new simple case of CAL polynomials lacking in [12].
\[
\mathcal{L}_n^\alpha(x, c) = e^{-x} \left\{ \binom{1F_1}{-c; \alpha} \binom{1F_1}{-n-c-\alpha; 1-\alpha; x} \\
+ \frac{(c+\alpha)(c+\alpha+1)_n}{\alpha(1-\alpha)(c+1)_n} x \binom{1F_1}{1-c-\alpha; 2-\alpha; x} \binom{1F_1}{-n-c; 1+\alpha; x} \right\},
\]
and the measure is obtained using [4, (10) page 258]
\[
\phi'(x) = \frac{x^{\alpha-1}e^{-x}}{\Gamma(1+c, 1+c+\alpha)} \left| \Psi(c+1, 2-\alpha; xe^{-ix}) \right|^2.
\]
In this case the limit \( c = 0 \) does not lead back to the Laguerre polynomial case but to the co-recursive Laguerre one with \( \mu = \alpha \).

The explicit form is
\[
L_n^\alpha(x; c, \mu) = \frac{(c + \alpha + 1)_n}{n!} \sum_{k=0}^{n} \frac{(-n)_k}{(c + 1)_k(c + \alpha + 1)_k} \binom{3F_2}{k-n, c, c+\alpha+1; c+k+1, c+\alpha+k+1; 1} x^k.
\]

The coefficients of the differential equation \( (47) \) satisfied by the polynomials \( \mathcal{L}_n^\alpha(x, c) \) are obtained from \( (71) \) changing only \( D = 1 + 2\alpha \) by \( D = 1 - 2\alpha \).
3 The case of Jacobi polynomials

We now present some results on the CAJ polynomials. The recurrence relation of the associated Jacobi polynomials $P_{n}^{\alpha,\beta}(x; c)$ is \[31\]

\[
(2n + 2c + \alpha + \beta + 1) [(2n + 2c + \alpha + \beta + 2)(2n + 2c + \alpha + \beta)x + \alpha^2 - \beta^2]p_n
= 2(n + c + 1)(n + c + \alpha + \beta + 1)(2n + 2c + \alpha + \beta)p_{n+1} \tag{78}
+ 2(n + c + \alpha)(n + c + \beta)(2n + 2c + \alpha + \beta + 2)p_{n-1}.
\]

We can note the invariance of the recurrence relation (79) under the transformation $T'$ defined by

\[T'(c, \alpha, \beta) = (c + \alpha + \beta, -\alpha, -\beta). \tag{79}\]

All polynomials satisfying (79), for which the initial conditions are symmetric in $\alpha$ and $\beta$ and invariant under $T'$ have the property

\[
p_{n}^{\alpha,-\beta}(x, c + \alpha + \beta) = p_{n}^{\alpha,\beta}(x, c). \tag{80}\]

As in [31] we use the more convenient shifted polynomials defined as

\[
R_{n}^{\alpha,\beta}(x; c) = P_{n}^{\alpha,\beta}(2x - 1; c). \tag{81}\]

Due also to the properties of the recurrence relation (79) we have

\[
R_{n}^{\alpha,\beta}(x; c) = (-1)^{n}R_{n}^{\beta,\alpha}(1 - x; c). \tag{82}\]

3.1 Explicit representation for the CAJ polynomials

A solution of the recurrence relation satisfied by the $R_{n}^{\alpha,\beta}(x; c)$ in terms of the hypergeometric function is [21, page 280]

\[
u_{n} = \frac{(c + \alpha + 1)_{n}}{(c + 1)_{n}}{2F1}_{1}(\frac{-n - c, n + c + \alpha + \beta + 1}{1 + \alpha}; 1 - x), \tag{83}\]

and another linearly independent solution is given by

\[
v_{n} = T'u_{n} = \frac{(c + \beta + 1)_{n}}{(c + \alpha + \beta + 1)_{n}}{2F1}_{1}(\frac{-n - c - \alpha - \beta, n + c + 1}{1 - \alpha}; 1 - x). \tag{84}\]

The functions $u_{n}$ and $x^{-\beta}(1 - x)^{-\alpha}v_{n}$ are two independent solutions of the second-order differential equation

\[
x(1 - x)y''(x) + [1 + \beta - (\alpha + \beta + 2)x]y'(x) + (n + c)(n + c + \alpha + \beta + 1)y(x) = 0. \tag{85}\]

The associated Jacobi polynomials are defined by (79) and the initial condition

\[
P_{-1}^{\alpha,\beta}(x; c) = 0, \quad P_{0}^{\alpha,\beta}(x; c) = 1. \tag{86}\]
This gives for $P_{1}^{\alpha, \beta}(x; c)$

$$P_{1}^{\alpha, \beta}(x; c) = \frac{(2c + \alpha + \beta + 1)(2c + \alpha + \beta + 2)}{2(c + 1)(c + \alpha + \beta + 1)} \times \left[ x + \frac{\alpha^2 - \beta^2}{(2c + \alpha + \beta)(2c + \alpha + \beta + 2)} \right]. \quad (87)$$

The CAJ polynomials $P_{n}^{\alpha, \beta}(x; c, \mu)$ satisfy the recurrence relation (79) with a shift $\mu$ on the first monic polynomial. This corresponds to the initial condition on the shifted CAJ polynomials $R_{n}^{\alpha, \beta}(x; c, \mu)$

$$R_{-1}^{\alpha, \beta}(x; c, \mu) = D = -\frac{(2c + \alpha + \beta)(2c + \alpha + \beta + 1)}{2(c + \alpha)(c + \beta)} \mu, \quad R_{0}^{\alpha, \beta}(x; c, \mu) = 1. \quad (88)$$

If $c + \alpha \to 0$ or $c + \beta \to 0$ this initial condition in (79) leads nevertheless to a shift $\mu$ on the value of $x$ in $P_{1}^{\alpha, \beta}(x; c)$.

As in section 2.1 writing $R_{n}^{\alpha, \beta}(x; c, \mu) = A u_{n} + B v_{n}$

and using (88) we obtain

$$A = \frac{1}{\Delta} [Dv_{0} - v_{-1}] \quad \text{and} \quad B = -\frac{1}{\Delta} [Du_{0} - u_{-1}], \quad (90)$$

where $\Delta$ is easily calculated using the fact that $u_{n}$ and $x^{-\beta}(1 - x)^{-\alpha} v_{n}$ are two independent solutions of (85),

$$\Delta = u_{-1}v_{0} - u_{0}v_{-1} = -\frac{\alpha(2c + \alpha + \beta)}{(c + \alpha)(c + \beta)}. \quad (91)$$

The condition $\Delta \neq 0$ leads to $\alpha \neq 0$ and $2c + \alpha + \beta \neq 0$. We can note the invariance of $D$ under $T'$ and that $B = T'A$.

Grouping the two $\text{2F1}$ involved in the expression (90) of $A$ gives

$$A = \frac{c + \alpha + \beta - D(c + \beta)}{\alpha(2c + \alpha + \beta)} _{3F2} \left( -c - \alpha - \beta, c, F + 1 ; 1 - x \right), \quad (92)$$

with

$$F = \frac{c[D(c + \beta) - c - \alpha - \beta]}{D(c + \beta) + c}, \quad (93)$$

and the CAJ polynomials could be writed

$$R_{n}^{\alpha, \beta}(x; c, \mu) = (1 + T') \frac{c + \alpha + \beta - D(c + \beta)}{\alpha(2c + \alpha + \beta)} \frac{(c + \alpha)(c + \alpha + 1)n}{(c + 1)n} \times _{3F2} \left( -c - \alpha - \beta, c, F + 1 ; 1 - x \right) \times _{2F1} \left( -n - c, n + c + \alpha + \beta + 1 ; 1 - x \right). \quad (94)$$

We will use this expression of the CAJ polynomials in 3.4 to obtain a fourth-order differential equation of them.
Transforming the $2F_1(1-x)$ in (89) by [4, Eq. 1, page 108] one obtains with little algebra

$$R_n^{\alpha,\beta}(x; c, \mu) = (1 + T') \frac{(-1)^n c(c + \alpha)}{\beta(2c + \alpha + \beta)}$$

$$\times \frac{(c + \alpha + 1)_n}{(c + \alpha + \beta + 1)_n} 2F_1 \left( \begin{array}{c} -n - c - \alpha - \beta, n + c + 1 \\ 1 - \beta \end{array} ; x \right)$$

$$\times \left[ \frac{c + \beta}{c} D_2F_1 \left( \begin{array}{c} -c, c + \alpha + \beta + 1 \\ 1 + \beta \end{array} ; x \right) - 2F_1 \left( \begin{array}{c} 1 - c, c + \alpha + \beta \\ 1 + \beta \end{array} ; x \right) \right].$$

(95)

This formula generalizes the one of [31, Eq. 28] to the case of the CAJ polynomials. As the explicit form of the CAL polynomials (22) for the associated one. Let $G$ as in [31] for the product of $\alpha, \beta$ for each product of $\alpha, \beta$ and $\beta \neq 0, \pm 1, \pm 2 \ldots$ but can be extended by limiting processes.

We obtain an explicit formula following the same way as in [31]. We first use [7, Eq. 1, page 108] for each product of $2F_1$ in (95) to obtain four series involving gamma functions and a $4F_3$. For two of them we use [4, Eq. 1, page 56]. The next step is to use for each $4F_3$ twice [4, Eq. 3, page 62]. After numerous cancellations only two series of $4F_3$ remains we can group to obtain the following explicit form

$$R_n^{\alpha,\beta}(x; c, \mu) = (-1)^n \frac{(2c + \alpha + \beta + 1)_n(c + \alpha + \beta)_n}{n!(c + \alpha + \beta + 1)_n} \sum_{k=0}^{n} \frac{(-n)_k(n + 2c + \alpha + \beta + 1)_k}{(c + 1)_k(c + \beta + 1)_k}$$

$$\times 5F_4 \left( \begin{array}{c} k - n, n + k + 2c + \alpha + \beta + 1, c + \beta, G + 1 \\ c + k + 1, c + \beta + k + 1, 2c + \alpha + \beta + 1, 1 \end{array} ; 1 \right) x^k,$$

(96)

where

$$G = \frac{2c(c + \beta)(2c + \alpha + \beta)}{2c(c + \beta) + \mu(2c + \alpha + \beta)(2c + \alpha + \beta + 1)}.$$ 

(97)

### 3.2 Generating function

One can obtain a generating function of the CAJ polynomials following the same strategy as in [31] for the associated one. Let $G(x, w)$ be a generating function of $R_n^{\alpha,\beta}(x; c, \mu)$

$$G(x, w) = \sum_{n=0}^{\infty} \frac{(c + 1)_n(c + \alpha + \beta + 1)_n}{n!(c + \alpha + \beta + 2)_n} w^n R_n^{\alpha,\beta}(x; c, \mu).$$

(98)

Starting from the form (94) for the $R_n^{\alpha,\beta}(x; c, \mu)$ it follows

$$G(x, w) = (1 + T') \frac{c + \alpha + \beta - D(c + \beta)}{\alpha(2c + \alpha + \beta)} (c + \alpha)_3F_2 \left( \begin{array}{c} -c - \alpha - \beta, c, F + 1 \\ 1 - \alpha, F \end{array} ; 1 - x \right)$$

$$\times \sum_{n=0}^{\infty} \frac{(c + \alpha + 1)_n(c + \alpha + \beta + 1)_n}{n!(2c + \alpha + \beta + 2)_n} w^n 2F_1 \left( \begin{array}{c} -n - c, n + c + \alpha + \beta + 1 \\ 1 + \alpha \end{array} ; 1 - x \right),$$

(99)

and using [31, Th. 4] we obtain

$$G(x, w) = (1 + T') \frac{c + \alpha + \beta - D(c + \beta)}{\alpha(2c + \alpha + \beta)}$$
Co-recursive associated Laguerre and Jacobi polynomials

\[ \times (c + \alpha) \left[ \frac{2}{w(Z + 1)} \right]^{c + \alpha + \beta + 1} \times_2 F_1 \left( \frac{1}{1 - \alpha, F} \right) \]

where

\[ Z_1 = \frac{1 - \sqrt{(1 + w)^2 - 4wx}}{w}, \quad Z_2 = \frac{1 + \sqrt{(1 + w)^2 - 4wx}}{w}, \]

which generalize the already exotic generating function [31, Eq. 75].

### 3.3 Spectral measure

The Stieltjes transform of the measure of the shifted associated Jacobi polynomials is [31, Eq. 63–64]

\[ s(p) = \frac{1}{p} \times_2 F_1 \left( \frac{c + 1, c + \beta + 1 + \frac{1}{p}}{2c + \alpha + \beta + 2} \right) \times_2 F_1 \left( \frac{c + \beta + \frac{1}{p}}{2c + \alpha + \beta + 2} \right)^{-1}. \]

The CAJ polynomials \( R_n^{\alpha,\beta}(x; c, \mu) \) satisfy the same recurrence relations as the \( R_n^{\alpha,\beta}(x; c) \) with a shift \( \mu \) on the first monic polynomials

\[ R_1^{\alpha,\beta}(x; c, \mu) - R_1^{\alpha,\beta}(x; c) = \frac{(2c + \alpha + \beta + 1)(2c + \alpha + \beta + 2)}{2(1 + c)(c + \alpha + \beta + 1)} \mu. \]

Using continued J-fractions [14, 8, 28] whose denominators are \( R_n^{\alpha,\beta}(x; c, \mu) \) and \( R_n^{\alpha,\beta}(x; c) \) we can derive for the Stieltjes transform of the measure of the CAJ polynomials

\[ s(p; \mu) = s(p) \left( 1 + \frac{\mu}{2} s(p) \right)^{-1} \]

that we can also write using contiguous relations

\[ s(p; \mu) = \times_2 F_1 \left( \frac{c + 1, c + \beta + 1 + \frac{1}{p}}{2c + \alpha + \beta + 2} \right) \times_2 F_1 \left( \frac{c + \beta + \frac{1}{p}}{2c + \alpha + \beta + 2} \right)^{-1}. \]

Grouping the \( x_2 F_1 \) in (105) gives the compact formula

\[ s(p; \mu) = \times_3 F_1 \left( \frac{c + 1, c + \beta + 1 + \frac{1}{p}}{2c + \alpha + \beta + 2} \right) \times_3 F_1 \left( \frac{c + \beta + \frac{1}{p}}{2c + \alpha + \beta + 2} \right)^{-1}. \]
where $G$ is given by (97). A sufficient condition for the positivity of the denominator in (106) on $(1, \infty)$ is

$$c \geq 0, \quad c > -\beta, \quad \alpha > -1, \quad \mu \geq -\frac{2c(c + \beta)}{(2c + \alpha + \beta)(2c + \alpha + \beta + 1)},$$

but other conditions are possible.

To obtain the absolutely continuous part of the spectral measure we need to evaluate $s^+(p; \mu) - s^-(p; \mu)$ where $s^\pm$ are the values of $s$ above and below the cut $[0,1]$. Using the analytic continuation [7, Eq. 2, page 108] for each $2F_1$ in (104) we find for the spectral measure of the CAJ polynomials

$$\phi'(x) = (1 - x)^{\frac{2c}{x}} \left| _{2F_1} \right. \left( \frac{c, c + \beta + 1}{2c + \alpha + \beta + \frac{e^{i\pi}}{x}} \right)^2 + \frac{\mu}{2x} \left| _{2F_1} \right. \left( \frac{c + 1, c + \beta + 1}{2c + \alpha + \beta + 2 + \frac{e^{i\pi}}{x}} \right)^2,
\quad (107)$$

valid at least under the conditions (107).

### 3.4 Fourth-order differential equation

The method used to obtain the differential equation satisfied by the $R_{\alpha, \beta}^n(x; c, \mu)$ is the same as in 2.3. In (94) the hypergeometric function $2F_1$ is solution of the equation (85) and the $3F_2$ is of the form $3F_2\left( a, b, e + 1; \frac{d}{c} \right)$ which is also solution of the second-order differential equation

$$x(x - 1) \left[ (a - e)(b - e)x + e(d - e - 1) \right] y''(x)$$

$$+ \left\{ (a - e)(b - e)(a + b + 1)x^2 + [e(a + b + 1)(2d - e - 2) - d(ab + e^2) + ab] x + dc(e - d + 1) \right\} y'(x) + ab [(a - e)(b - e)x + (e + 1)(d - e - 1)] y(x) = 0.$$

We don’t write here the fourth-order differential equation hardly obtained by symbolic MAPLE computation. The coefficients are at most of degree eight in $x$ and it would take several pages to write them. We give the results only in the following limiting cases.

### 3.5 Particular cases

#### 3.5.1 Laguerre case limit

The limit giving the CAL polynomial case is obtained by the replacement

$$x \rightarrow 1 - \frac{2x}{\beta}, \quad \mu \rightarrow \frac{2\mu}{\beta}$$

in $P_{\alpha, \beta}^n(x; c, \mu)$. The representation (94) is the more suitable to obtain the form of the CAL polynomials (22) using the Kummer’s transformation (23) for one of the confluent
3.5.2 Limit \( c = 0 \)

In this limit we obtain the co-recursive Jacobi polynomials. An explicit form is

\[
R_n^{\alpha,\beta}(x; \mu) = (-1)^n (\beta + 1)_n \frac{n!}{n!} \left\{ \frac{(-n)_k (n + \alpha + \beta + 1)_k}{k!(\beta + 1)_k} x^k \right\} + \mu(k - n)(n + k + \alpha + \beta + 1) \right( \frac{1}{2(1 + (\beta + 1)(\beta + k + 1)} \right) \right\},
\]

and the spectral measure is given by

\[
\phi'(x) = (1 - x)^{\alpha} x^{\beta} \left| 1 + \frac{\mu}{2x^2} F_1 \left( 1, \beta + 1 : \frac{e^{i\pi}}{x} \right) \right|^{-2},
\]

The limit \( \mu = 0 \) leads back to the Jacobi polynomials.

The fourth-order differential equation satisfied by the co-recursive Laguerre polynomials can be factorized in the limit \( c = 0 \) to obtain as in [27] the factorized (2+2) differential equation

\[
0 = \left[ (1 - x^2) A(x) D^2 + \left\{ (\beta - \alpha - (\alpha + \beta + 4)x) A(x) - (1 - x^2) B(x) \right\} D \\
+ \{ n(n + \alpha + \beta + 1) - (\alpha + \beta + 2) \right\} A(x) + \{ \beta - \alpha - (\alpha + \beta + 2) \right\} B(x) + C(x) \right] (114)
\]

where

\[
A(x) = 2(\alpha + \beta)^2(2n + 1)(n + \alpha + \beta + 1/2)x^2 + 2(\alpha + \beta)(4n(n + \alpha + \beta + 1)
\times(-\mu(1 + \alpha + \beta + \alpha - \beta) + (1 + \alpha + \beta)(-\mu(\alpha + \beta + 2) + 2\alpha - 2\beta))x
+ 4n(n + \alpha + \beta + 1)(-\mu(1 + \alpha + \beta) + \alpha - \beta)^2 - (\alpha + \beta)(-2\mu(1 + \alpha + \beta)
\times(\beta - \alpha) - 2(\beta - \alpha)^2 - 3\alpha - 3\beta)
\]

\[
B(x) = -(\alpha + \beta)((\alpha + \beta)(8n(n + \alpha + \beta + 1) + 3\alpha + 3\beta)x + 8n(n + \alpha + \beta + 1)
\times(-\mu(1 + \alpha + \beta) + \alpha - \beta) - 2(\alpha + \beta + 2)(1 + \alpha + \beta)\mu + (\alpha - \beta)(3\alpha + 3\beta + 4))
\]

\[
C(x) = -(\alpha + \beta)((\alpha + \beta)(\alpha + \beta + 2)(2n(n + \alpha + \beta + 1) + \alpha + \beta - 1)x^2
+ 2(n(n + \alpha + \beta + 1)(-\mu(\alpha + \beta + 1)(\alpha + \beta - 4) + 2(\alpha - \beta)(\alpha + \beta - 2))
+(\alpha + \beta)(\alpha - \beta)(\alpha + \beta - 1)x - 2n(\alpha + \alpha + \beta + 1)(-\mu(\alpha + \beta + 1)(\beta - \alpha)
-(\alpha - \beta)^2 + 6\alpha + 6\beta) + (\alpha + \beta)((\alpha - \beta)^2 - 3\alpha - 3\beta - 6))
\]
3.5.3 Limit $c = -\alpha - \beta$

Due to the $T'$ invariance of (79) we obtain in this limit the special case of CAJ polynomials for which

$$R_n^{\alpha,\beta}(x; -\alpha - \beta, \mu) = R_n^{-\alpha,-\beta}(x; \mu).$$

All the results are obtained from 3.5.2 by changing $\alpha$ to $-\alpha$ and $\beta$ to $-\beta$.

3.5.4 Limit $c = -\beta$

The explicit form (96) simplifies in the same way as in the case $c = 0$. One obtain

$$R_n^{\alpha,\beta}(x; -\beta, \mu) = (-1)^n \frac{(\alpha + 1)_n}{(\alpha - 1)_n} \frac{(-n)_n}{k!(1-\beta)_n} x^k \sum_{k=0}^{n} \frac{(-n)_k (n + \alpha - \beta + 1)_k}{k!(1-\beta)_k}$$

$$\times \left\{ 1 + \frac{\mu(k-n)(n+k+\alpha-\beta+1)}{2(k+1)(1-\beta+k)} \right\}.$$ (116)

Comparing this form with the explicit form of the co-recursive Jacobi polynomials (112) one see

$$R_n^{\alpha,\beta}(x; -\beta, \mu) = \frac{n!(\alpha - \beta + 1)_n}{(\alpha + 1)_n(1-\beta)_n} R_n^{-\alpha,-\beta}(x; \mu),$$

(117)

of course the spectral measure and the fourth-order differential equation are obtained from (112) and (114) changing $\beta$ to $-\beta$.

3.5.5 Limit $c = -\alpha$

This case is the $T'$ transform of the preceding case. All the results are obtained from 3.5.4 by changing $\alpha$ to $-\alpha$ and $\beta$ to $-\beta$.

3.5.6 Limit $\mu = 0$

In this limit we obtain the associated Jacobi polynomials studied in [31]. The form [31, Eq. 28] is obtained directly using (95) but a form slightly different is

$$R_n^{\alpha,\beta}(x; c) = (1 + T') \frac{(c + \alpha)(c + \alpha + \beta)(c + \alpha + 1)_n}{\alpha(2c + \alpha + \beta)(c + 1)_n}$$

$$\times \frac{(c + \alpha + \beta)_n}{(c + 1)_n} x^k \sum_{k=0}^{n} \frac{(-n)_k (n + \alpha - \beta + 1)_k}{k!(1-\beta)_k}$$

$$\times \left\{ 1 + \frac{\mu(k-n)(n+k+\alpha-\beta+1)}{2(k+1)(1-\beta+k)} \right\}.$$ (118)

The explicit form [31, Eq. 19] is easily obtained starting from (96) with $G = 2c + \alpha + \beta$, the $5F_4$ reducing to a $4F_3$. Obviously the limit $c = 0$ leads back to the Jacobi polynomials.

The coefficients of the differential equation (17) satisfied by the associated Jacobi polynomials are

$$c_4 = x^2(x-1)^2,$$
$$c_3 = 5x(x-1)(2x-1),$$
$$c_2 = \left(24 - (n+1)^2 - A\right)x(x-1) - Bx - \beta^2 + 4,$$
$$c_1 = -3/2 \left(3A + (n+3)(n-1))(2x-1) + B\right),$$
$$c_0 = n(n+2)A,$$

(119) (120) (121) (122)
with
\[ A = (C + n + 1)(C + n - 1), \quad B = (\alpha - \beta)(\alpha + \beta), \quad C = 2c + \alpha + \beta. \] (123)
This result was also first given by Hahn [3, Eq. 20]. Note the \( T' \) invariance of \( A, B \) and \( C \) leading to the invariance of the \( c_i \) more obvious than in [31, Eq. 47–48].

### 3.5.7 Limit \( \mu = \frac{2c(c+\alpha)}{(2c+\beta+\alpha)(2c+\beta+\alpha+1)} \)

In this limit the symmetry \( T' \) is broken. We obtain the zero-related Jacobi polynomials studied in [13]. An explicit form is

\[
\mathcal{R}_n^{\alpha,\beta}(x; c) = (-1)^n \frac{(2c+\alpha+\beta+1)_n(\beta+c+1)_n}{n!(c+\alpha+\beta+1)_n} \sum_{k=0}^{n} \frac{(-n)_k(n+2c+\alpha+\beta+1)_k}{(c+1)_k(c+\beta+1)_k} \times 4F_3 \left( \begin{array}{c} k-n, n+k+2c+\alpha+\beta+1, c, c+\beta+1 \\ c+k+1, c+\beta+k+1, 2c+\alpha+\beta+1 \end{array} ; 1 \right) x^k. \] (124)

The limit \( c = 0 \) leads back to the Jacobi polynomials and the limit defined in [110] gives the zero-related Laguerre polynomials \( (2.4.4) \), using Kummer’s transformations. The spectral measure is

\[
\phi'(x) = (1-x)^{\alpha}x^{\beta+2c} \left| 2F_1 \left( \begin{array}{c} c, c+\beta+1 \\ 2c+\alpha+\beta+1 \end{array} ; \frac{e^{i\pi}}{x} \right) \right|^2. \] (125)

The coefficients of the differential equation \( (17) \) satisfied by the polynomials \( \mathcal{R}_n^{\alpha,\beta}(x; c) \) are

\[
c_4 = x^2(x-1)^2(Ax+D), \quad (126)
\]

\[
c_3 = x(x-1) \left( 8Ax^2 - 3(A-3D)x - 4D \right), \quad (127)
\]

\[
c_2 = -1/2A(A+2C^2-29)x^3 + \left( 1/2A(A+2C^2-2B-23) - D(C^2-19) \right) x^2 \] \( -1/4 \left( A(D+1)(D-3) - 2D(2C^2-2B+D-35) \right) x - 1/4D(D^2-9), \]

\[
c_1 = -A(A+2C^2-5)x^2 + 1/4 \left( A(A+2C^2-2B-5D-5) - 3D(4C^2-11) \right) x \] \( +1/4D \left( (D+3)A + 6C^2 - 6B + 3D - 15 \right), \]

\[
c_0 = 2n(n+1)(C+n)(C+n+1)(Ax+3D), \quad (130)
\]

where \( B \) and \( C \) are defined in [123] and

\[
A = (2n+1)(1+2C+2n), \quad D = 1+2\beta. \] (131)

### 3.5.8 Limit \( \mu = \frac{2c+\beta(c+\alpha+\beta)}{(\beta+\alpha+2c)(\beta+\alpha+2c+1)} \)

This case is the \( T' \) transform of the case [5,5,4]. The explicit form is

\[
\mathcal{R}_n^{\alpha,\beta}(x; c) = (-1)^n \frac{(2c+\alpha+\beta+1)_n(\alpha+c+1)_n}{n!(c+\alpha+\beta+1)_n} \sum_{k=0}^{n} \frac{(-n)_k(n+2c+\alpha+\beta+1)_k}{(c+\alpha+\beta+1)_k(c+\alpha+1)_k} \times 4F_3 \left( \begin{array}{c} k-n, n+k+2c+\alpha+\beta+1, c+\alpha+\beta, c+\alpha+1 \\ c+\alpha+\beta+k+1, c+\alpha+k+1, 2c+\alpha+\beta+1 \end{array} ; 1 \right) x^k. \] (132)
The limit \((110)\) leads back to the Laguerre case \(2.4.5\) and the limit \(c = 0\) to the co-recursive Jacobi polynomials with \(\mu = \frac{2\beta}{\alpha + \beta + 1}\). The spectral measure is

\[
\phi'(x) = (1 - x)^\alpha x^{\beta+2c} \left| 2F_1 \left( \frac{c + 1, c + \beta}{2c + \alpha + \beta + 1} ; \frac{e^{i\pi}}{x} \right) \right|^{-2}.
\]

The coefficients of the differential equation \((17)\) satisfied by the polynomials \(R^{\alpha,\beta}_n(x; c)\) are obtained from \((126)\) changing only \(D = 1 + 2\beta\) by \(D = 1 - 2\beta\).

### 3.5.9 Limit \(\mu = -\frac{2\epsilon(c+\beta)}{(2\epsilon+\alpha+\beta)(2\epsilon+\alpha+\beta+1)}\)

The symmetry \(T'\) is also broken. We obtain a new simple case of CAJ polynomials. An explicit form is

\[
\tilde{R}^{\alpha,\beta}_n(x; c) = (-1)^n \frac{(2c + \alpha + \beta + 1)_n (\beta + c + 1)_n}{n! (\alpha + \beta + 1)_n} \sum_{k=0}^{n} \frac{(-n)_k (n + 2c + \alpha + \beta + 1)_k}{(c + 1)_k (\beta + 1)_k} \times {}_4F_3 \left( \begin{array}{c} k - n, n + k + 2c + \alpha + \beta + 1, c, c + \beta \\ c + k + 1, c + \beta + k + 1, 2c + \alpha + \beta + 1 \end{array} ; 1 \right) x^k.
\]

and the spectral measure

\[
\phi'(x) = (1 - x)^\alpha x^{\beta+2c} \left| 2F_1 \left( \frac{c, c + \beta}{2c + \alpha + \beta + 1} ; \frac{e^{i\pi}}{x} \right) \right|^{-2}.
\]

The coefficients of the differential equation \((17)\) satisfied by the polynomials \(\tilde{R}^{\alpha,\beta}_n(x; c)\) are

\[
c_4 = x^2 (x - 1)^2 (A(x - 1) - D),
\]

\[
c_3 = x(x - 1) \left( 8Ax^2 - (13A + 9D)x + 5A + 5D \right),
\]

\[
c_2 = -1/2A(A + 2C^2 - 29)x^3 + \left( A(A + 2C^2 - B - 32) + D(C^2 - 19) \right) x^2
\]

\[
-1/2 \left( A(A + 2C^2 + 2B^2 - 2B - 43) + D(2C^2 - 2B - D - 41) \right) x
\]

\[
+ (B^2 - 4)(A + D),
\]

\[
c_1 = -2A(A + 2C^2 - 5)x^2
\]

\[
+ 1/2 \left( A(7A + 14C^2 - 2B + 5D - 35) + D(3C^2 - 33) \right) x
\]

\[
- A(3/2A + 3C^2 - B - 2A^2 - 7) - 3D(C^2 - B - 1/2D - 3),
\]

\[
c_0 = n(n + 1)(C + n)(C + n + 1)(A(x - 1) - 3D),
\]

where \(B\) and \(C\) are defined in \((123)\) and

\[
A = (2n + 1)(1 + 2C + 2n), \quad D = 1 + 2\alpha.
\]
3.5.10 Limit $\mu = -\frac{2(c+\alpha)(c+\alpha+\beta)}{(2c+\alpha+\beta)(2c+\alpha+\beta+1)}$

This case is the $T'$ transform of the case 3.5.9. The explicit form is

$$\tilde{R}_n^{\alpha,\beta}(x; c) = (-1)^n \frac{(2c + \alpha + \beta + 1)_n (\alpha + c + 1)_n}{n!(c + 1)_n} \sum_{k=0}^{n} \frac{(-n)_k (n + 2c + \alpha + \beta + 1)_k}{(c + \alpha + \beta + 1)_k (c + \alpha + 1)_k} \times {}_4F_3 \left( \begin{array}{c} k - n, n + k + 2c + \alpha + \beta + 1, c + \alpha + \beta, c + \alpha + \beta + 1 \\ c + \alpha + \beta + k + 1, c + \alpha + k + 1, 2c + \alpha + \beta + 1 \end{array} ; 1 \right) x^k. \quad (142)$$

and the spectral measure

$$\phi'(x) = (1 - x)^{\alpha - 2} x^{\beta + 2c + 2} \left| {}_2F_1 \left( \begin{array}{c} c + 1, c + \beta + 1 \\ 2c + \alpha + \beta + 1 \end{array} ; \frac{e^{ix}}{x} \right) \right|^2. \quad (143)$$

The coefficients of the differential equation (17) satisfied by the polynomials $\tilde{R}_n^{\alpha,\beta}(x; c)$ are obtained from (136–141) by changing only $D = 1 + 2\alpha$ by $D = 1 - 2\alpha$.

3.6 Conclusion

We end by brief remarks. In this article we have studied properties of the co-recursive associated Laguerre and Jacobi polynomials which are of interest in the resolution of some birth and death processes with and without absorption. For few values of the co-recursivity parameter we obtain polynomial families for which the results are of the same complexity as the corresponding associated polynomials. For the CAL polynomials we find the two expected cases corresponding to $\mu = \mu_0$ (zero-related polynomials) and the new dual case $\mu = \lambda_{-1}$ [20]. For the CAJ polynomials, due to the properties (80) and (82) we have two more cases corresponding to $\mu = -T''\mu_0$ and $\mu = -T''\lambda_{-1}$ where the transformation $T''$ is defined by

$$T''(c, \alpha, \beta) = (c + \alpha + \beta, -\beta, -\alpha). \quad (144)$$

In some cases the fourth-order differential equations satisfied by the polynomials studied above are factorizable (co-recursive and associated of order one) but we don’t find factorization either for the co-recursive associated polynomials or for the associated one. Of course this is not a proof that the conjectures on this factorizability made in [27] are wrong.

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