Generalization of the noise model for time-distance helioseismology

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ABSTRACT

Context. In time-distance helioseismology, information about the solar interior is encoded in measurements of travel times between pairs of points on the solar surface. Travel times are deduced from the cross-covariance of the random wave field. Here we consider travel times and also products of travel times as observables. They contain information about e.g. the statistical properties of convection in the Sun.

Aims. Using the travel time definition of Gizon & Birch (2004) we derive analytic formulae for the noise covariance matrix of travel times and products of travel times.

Methods. The basic assumption of the model is that noise is the result of the stochastic excitation of solar waves, a random process which is stationary and Gaussian. We generalize the existing noise model by dropping the assumption of horizontal spatial homogeneity. Using a recurrence relation, we calculate the noise covariance matrices for the moments of order 4, 6, and 8 of the observed wave field, for the moments of order 2, 3 and 4 of the cross-covariance, and for the moments of order 2, 3 and 4 of the travel times.

Results. All noise covariance matrices depend only on the expectation value of the cross-covariance of the observed wave field. For products of travel times, the noise covariance matrix consists of three terms proportional to 1/T, 1/T^2, and 1/T^3, where T is the duration of the observations. For typical observation times of a few hours, the term proportional to 1/T^2 dominates and

\[
\text{Cov}[\tau_1, \tau_2, \tau_3] \approx \text{Cov}[\tau_1, \tau_2] \text{Cov}[\tau_2, \tau_3] + \text{Cov}[\tau_1, \tau_3] \text{Cov}[\tau_2, \tau_3],
\]

where the \( \tau_i \) are arbitrary travel times. This result is confirmed for \( p_i \) travel times by Monte Carlo simulations and comparisons with SDO/HMI observations.

Conclusions. General and accurate formulae have been derived to model the noise covariance matrix of helioseismic travel times and products of travel times. These results could easily be generalized to other methods of local helioseismology, such as helioseismic holography and ring diagram analysis.

Key words. Sun: helioseismology – Sun: oscillations – Sun: granulation – convection – methods: statistical – methods: data analysis

1. Introduction

The purpose of time-distance helioseismology (Duvall et al. 1993; Gizon & Birch 2005, and references therein) is to infer the subsurface structure and dynamics of the Sun using spatial-temporal correlations of the random wave field observed at the solar surface. Wave travel times between pairs of points (denoted \( \tau \)) are measured from the cross-covariance function. Wave speed perturbations and vector flows are then obtained by inversion of the travel times (e.g. Kosovichev (1996); Jackiewicz et al. (2012)). Such inversions require knowledge of the noise covariance matrix \( \text{Cov}[\tau, \tau] \). Typically, noise is very high and strong correlations exist among travel times. Gizon & Birch (2004) studied the noise properties of travel times and derived a simple noise model that successfully explains the observations. The model is based on the assumption that the stochastic noise is stationary and horizontally spatially homogeneous, as a result of the excitation of waves by turbulent convection. In addition to time-distance helioseismology, this noise model has found applications in direct modeling inversions (Woodard 2006, 2009) and ring-diagram analysis (Birch et al. 2007).

Time-distance helioseismology has been successfully applied to map flow velocities, \( v_j \), at supergranulation scales (Kosovichev 1996; Duvall Jr. & Gizon 2000; Gizon et al. 2001; Jackiewicz et al. 2008). The statistical properties of convection can further be studied by computing horizontal averages of the turbulent velocities. For example, Duvall Jr. & Gizon (2001; Gizon et al. 2010) showed that the horizontal divergence and the vertical vorticity of the flows are correlated through the influence of the Coriolis force on convection. It would be highly desirable to extract additional properties of the turbulent velocities, for example the (anisotropic) Reynolds stresses \( \langle v_i v_j \rangle \) that control the global dynamics of the Sun (differential rotation and meridional circulation, see Kitchatinov & Rüdiger 2005). The noise associated with such measurement involves the fourth order moments of the travel times, \( \text{Cov}[\tau, \tau, \tau, \tau] \).

Alternatively, we would like to consider spatial averages of products of travel times \( \langle \tau \tau \rangle \) as the fundamental data from which to infer the Reynolds stresses (or other second-order moments of turbulence). Spatial averages are meaningful when turbulent flows are horizontally homogeneous over the averaging region. Inversions of average products of travel times are desirable since input data are fewer and less noisy. Once again, we need to know the noise covariance matrix \( \text{Cov}[\langle \tau \tau \rangle, \langle \tau \tau \rangle] \) in order to perform the inversion.

In this paper, we study the noise properties of travel times and products of travel times. In Section 2 the definitions for the...
2. Observables: cross-covariance function, travel times, and products of travel times

The fundamental observation in helioseismology is the filtered line-of-sight Doppler velocity \( \phi(x, t) \) at points \( x \) on the surface of the Sun and at times \( t \). The filter acts by multiplication in the Fourier domain. In this paper we will only consider the \( p_1 \)-ridge filter as an example. Note that all the results presented in this paper do not depend on the choice of the filter. The signal \( \phi(x, t) \) is recorded over a duration time \( T = (2N + 1)h_t \) where \( h_t \) is the temporal resolution at observation times \( t_n = nh_t \) for \( n = -N, \ldots, N \). The observed wavefield during the observation time \( T \) is denoted \( \phi_T \). We have \( \phi_T(x, t) = \phi(x, t) W_{\text{ref}}(t) \) where \( W_{\text{ref}} \) is a window function (equal to 1 if \( |t| \leq T/2 \) and 0 otherwise).

Helioseismic analysis is performed in Fourier space. Let us define the temporal Fourier transform of \( \phi_T \) by

\[
\phi_T(x, \omega) = \frac{2\pi}{T} \phi_T(x, \omega) \exp(i\omega t_n).
\]

The frequencies \( \omega \) are treated as continuous variables in the remainder of this paper in order to be able to take into account the frequency correlations (see Section 3.4). The cross-covariance function between two points at the surface of the Sun is a multiplication in the Fourier domain \cite{Duvall1993}.

\[
C(x_1, x_2, \omega) = \frac{2\pi}{T} \phi_T(x_1, \omega) \phi_T(x_2, \omega).
\] (1)

Working in Fourier space is faster (and easier). In the time-domain the cross-covariance becomes

\[
C(x_1, x_2, t_n) = \frac{1}{2N + 1} \sum_{j=\min(N-N-n)}^{\min(N+N-n)} \phi(x_1, t_j) \phi(x_2, t_{j+n}).
\] (2)

where \( t_n \) is the correlation time lag.

Cross-covariances are the basic data to compute the travel times. We denote \( \tau(x_1, x_2) \) the travel time for a wave packet traveling from point \( x_1 \) to point \( x_2 \) and \( \tau_0(x_1, x_2) \) the travel time for a wave packet traveling from \( x_1 \) to \( x_2 \). In the limit discussed by \cite{Gizon2004} the incremental travel times can be measured from the estimated cross-covariance using

\[
\tau(x_1, x_2) = h_t \sum_{n=-N}^{N} W_n(x_1, x_2, t_n) \times \left( C(x_1, x_2, t_n) - C^\text{ref}(x_1, x_2, t_n) \right)
\] (3)

where \( C^\text{ref} \) is a deterministic reference cross-covariance coming from spatial averaging or from a solar model and the weight function \( W_n \) are defined as

\[
W_n(x_1, x_2, t) = \frac{\pm f(x_1, t) f(x_2, t) C^\text{ref}(x_1, x_2, t)}{h_t \sum_{n=-N}^{N} \pm f(x_1, t) f(x_2, t) C^\text{ref}(x_1, x_2, t_n)}.
\] (4)

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cross-covariance function and the travel times are given. Section 4 presents the assumptions of the noise model generalizing the model of \cite{Gizon2004}. In Section 4 and in the Appendix, we derive analytical formulae for the noise covariance matrices of travel times and products of travel times. These formulae are confirmed in Section 5 by comparison to numerical Monte Carlo simulations and to SDO/HMI observations. The effects of horizontal spatial averaging are considered in Section 6.

denotes the type of travel time and the corresponding weight function \( W_\alpha \). The mean and difference travel times \( \tau_{\text{diff}} \) and \( \tau_{\text{mean}} \) can be obtained from the one way travel times by \( \tau_{\text{diff}} = \tau_+ - \tau_- \) and \( \tau_{\text{mean}} = (\tau_+ + \tau_-)/2 \).

In this paper, we are interested in the noise covariance matrix for travel times \( \tau_{\text{diff}}(x_1, x_2) \) and products of travel times \( \tau_{\text{diff}}(x_1, x_2) \tau_{\text{mean}}(x_1, x_2) \) where \( \tau \) is defined by Eq. (3). To simplify the notations, let

\[
\tau_1 := \tau_{\text{diff}}(x_1, x_2), \quad \tau_2 := \tau_{\text{mean}}(x_1, x_2),
\]

and more generally \( \tau_i := \tau_{\text{diff}}(x_{i-1}, x_i) \).

3. Generalization of the noise model

3.1. Assumptions

The basic assumption of the noise model is the following: The observations at the relevant spatial points \( x_1, \ldots, x_M \) are described by a vector-valued stationary Gaussian time series \( \phi(x_1, t), \ldots, \phi(x_M, t) \). For sake of simplicity, we can also assume without loss of generality that \( \mathbb{E}[\phi(x_n, t_n)] = 0 \) at each \( x_n \) for all \( n \in \mathbb{Z} \). This model is valid in the quiet Sun (away from evolving active regions) but does not assume that the noise is spatially homogeneous contrary to the model of \cite{Gizon2004} as detailed in Section 4.2. This assumption is supported by the observed distribution of the HMI Doppler velocity: Figure 1 shows the probability density of the filtered line-of-sight velocity. For a Gaussian distribution, the data should line up along a straight line. We can see a very good agreement for probabilities between 5% and 95%. The deviations in the tail of the plot (for probabilities smaller than 5%) may be due to statistical errors (as we have less realizations for these events).

One may also replace the spatial points by some spatial averages. Such averages are often used to improve the signal-to-noise ratio. We will denote \( \overline{C} \) the expectation value of the cross-covariance

\[
\overline{C}(x_\alpha, x_\beta, \omega) = \mathbb{E}[C(x_\alpha, x_\beta, \omega)] = \frac{2\pi}{T} \mathbb{E}[\phi_T(x_\alpha, \omega) \phi_T(x_\beta, \omega)].
\] (6)

3.2. Independence of the geometry

\cite{Gizon2004} assumed that the observations \( \phi(x_{ij}, t_n) \) are given on a Cartesian grid \( x_{ij} \) by an approximately flat patch of the Sun’s surface. The discrete Fourier transform of the finite dimensional signal was assumed to be of the form

\[
\phi(k_{ij}, \omega) = \sqrt{P(k_{ij}, \omega)} N_{ij}(\omega)
\]

where \( P \) is the power spectrum, \( \omega = 2\pi/\tau \), and \( N_{ij} \) are complex independent and identically distributed Gaussian variables.
with zero-mean and unit variance. In this case, the frequency correlations were ignored and

\[ \frac{2\pi}{T} \mathbb{E} \left[ \phi^*(x_a, \omega_1) \phi(x_b, \omega_2) \right] = \delta \mu \overline{C}_{GB}(x_b - x_a, \omega_2) \]  

was assumed. We have denoted \( \overline{C}_{GB} \) the expectation value of the cross-covariance used by Gizon & Birch (2004). Our assumption is more general as it does not require a planar geometry and allows a natural treatment of spatially averaged quantities. It means that all our results are valid in any geometry and it is in particular the case for the results presented in Gizon & Birch (2004).

### 3.3. On frequency correlations

As the observation time \( T \) is finite, the discrete Fourier transforms \( \phi_T(x, \omega) \) and \( \phi_T(x, \omega) \) for \( j \neq k \) are no longer uncorrelated because of the window function. The necessity of a correction term for finite \( T \) was discussed, but not further analyzed in Gizon & Birch (2004). It turns out that there is an explicit formula for this correction term in terms of the periodic Hilbert transform of \( \overline{C} \) and a smoothed version of \( \overline{C} \). The exact formula is given in Appendix A where it is also shown that the error made by considering a finite observation time can be bounded

\[ \sup_{j \neq k} \left| \frac{2\pi}{T} \mathbb{E} \left[ \phi_T(x_a, \omega_1) \phi_T(x_b, \omega_2) \right] - \delta \mu \overline{C}_{GB}(x_b - x_a, \omega_2) \right| \leq \frac{h_t}{4T} \sum_{k=1}^{2N} |h_k| \overline{C}(x_a, x_b, t_k). \]

Note that the right hand side of (9) depends only on \( T \) and on a quantity depending on the correlation length of the waves. This can be better seen using an analytic cross-covariance given by a Lorentzian of the form

\[ \overline{C}(x, x, \omega) = \frac{C_0}{1 + (\omega - \omega_0)^2/\gamma^2} \]  

where \( \gamma \) is the half width at half maximum of the Lorentzian centered at a frequency \( \omega_0 \). In this case, one can check that the bound in Eq. (9) is equal to \( 1/(4\pi^2\gamma) \). Therefore the correlations between frequencies should only be neglected when this bound is small, i.e. the observation time is long enough to represent correctly the mode.

As the covariance between travel times is known to be of order \( 1/T \) (Gizon & Birch 2004), it is legitimate to wonder if the frequency correlations should be taken into account. It is shown below (see Eq. (13)) that considering frequency correlations will only lead to additional terms of order \( 1/T^2 \) that can be neglected for long observation times.

### 4. Model noise covariances

In this section and Appendices B–E we present explicit formulae for the covariance matrices of cross-covariances \( C \), travel times \( \tau \) and products of cross-covariances or travel-times:

- \( \operatorname{Cov}[\tau_1, \tau_2] \) and \( \operatorname{Cov}[C_1, C_2] \) which are linked to the fourth order moment of \( \phi_T \).
- \( \operatorname{Cov}[\tau_1, \tau_2] \) and \( \operatorname{Cov}[C_1, C_2] \) which requires the knowledge of the sixth order moment of \( \phi_T \) and is necessary to compute the moment of order four of \( C \).
- \( \operatorname{Cov}[\tau_1, \tau_2, \tau_3, \tau_4] \) and \( \operatorname{Cov}[C_1, C_2, C_3, C_4] \) which depend on the eighth order moment of \( \phi_T \).

For the covariance between two complex random variables \( X \) and \( Y \) we will use the convention

\[ \operatorname{Cov}[X, Y] = \mathbb{E}[XY^*] - \mathbb{E}[X] \mathbb{E}[Y^*]. \]  

In particular, as the mean value of the observables is zero, we have \( \mathbb{E}[x_1, x_2, \omega] = \frac{2\pi}{T} \mathbb{E}[\phi_T(x_2, \omega), \phi_T(x_1, \omega)] \).

We will show that all moments of cross-covariance functions depend on \( \overline{C} \) only. Because the travel time measurement procedure is linear in \( C \), the moments of the travel-times can be expressed in terms of \( \overline{C} \) and of the weight functions \( W_0 \) (see Eq. (4)).

#### 4.1. Covariance matrix for \( \tau \) and travel times

As a first step, we show in Appendix C that the covariance between two cross-correlations is given by

\[ \left( \frac{T}{2\pi} \right)^2 \operatorname{Cov}[C(x_1, x_2, \omega), C(x_3, x_4, \omega)] = \mathbb{E}[\phi_T(x_1, \omega_1) \phi_T(x_3, \omega_2)] \mathbb{E}[\phi_T(x_2, \omega_1) \phi_T(x_4, \omega_2)] + \mathbb{E}[\phi_T(x_1, \omega_1) \phi_T(x_4, \omega_2)] \mathbb{E}[\phi_T(x_2, \omega_1) \phi_T(x_3, \omega_2)]. \]  

For a comparison with and a small correction to the corresponding formula in Gizon & Birch (2004) we refer to Appendix B.

The covariance between two travel times is given by

\[ \operatorname{Cov}[\tau_1, \tau_2] = \frac{(2\pi)^3}{T} \int_{-\tau_{1h}}^{\tau_{1h}} d\omega W_0^\tau(x_1, x_2, \omega) \times \]

\[ \left( W_{0,\omega}(x_1, x_2, \omega) \overline{C}(x_1, x_3, \omega) \overline{C}(x_4, x_2, \omega) \right) \times \frac{X_3}{T^2} + O\left( \frac{1}{T^{m+1}} \right) \]  

where \( O\left(1/T^{m+1}\right) \) means that the additional terms decay at least as \( 1/T^{m+1} \) (\( m \) corresponds to the regularity, i.e. the number of
derivatives of the functions \( \overline{C} \) and \( W \). A good agreement between the leading order term in this formula and SOHO MDI measurements was found by Gizon & Birch (2004). An explicit formula for the second order term \( X_2 \) is derived in Appendices B and D. If the observation time \( T \) is so small that \( X_2/T^2 \) cannot be neglected, \( X_2 \) can easily be evaluated numerically.

4.2. Covariance matrix for products of travel times

In this section, we are interested in the covariance matrix for the travel times correlations i.e. to evaluate the quantity

\[
\text{Cov}[\tau_1(x_1, x_2)\tau_2(x_3, x_4), \tau_3(x_5, x_6)\tau_4(x_7, x_8)].
\]

(14)

This quantity is the most general we can evaluate for velocity correlations. It will be helpful to derive all the formulae in more specific frameworks. In general this quantity depends on the eight points \( x_i \) but it is of course possible to look at simpler cases. For example, we may be interested in the correlations between a East-West (EW) and North-South (NS) travel time as presented in Figure 4. This quantity can give us informations about the correlations between the velocities \( v_x \) and \( v_y \), velocities in the EW and NS directions respectively.

The formula for the product of cross-covariances is given in Appendix F (Eq. (C.17)) and will not be discussed in the text where we will focus on products of travel times. In Appendix F we derive the general formula for Eq. (14):

\[
\text{Cov}[\tau_1, \tau_2, \tau_3, \tau_4] = \frac{1}{T^2}Z_1 + \frac{1}{T^2}Z_2 + \frac{1}{T^2}Z_3 + O\left(\frac{1}{T^4}\right)
\]

(15)

where \( Z_1, Z_2 \) and \( Z_3 \) are given by Eqs. (15, 19) and (20) and will be detailed later after some general remarks on this formula. An important point is that all the terms in \( Z_i \) depend only on \( \overline{C} \) and on the weight functions \( W \). Thus, it is possible to estimate directly the noise covariance matrix via this formula instead of performing a large number of Monte-Carlo simulations. This strategy is much more efficient as we will see in Section 5.3 where we demonstrate the rate of convergence of the stochastic simulations.

The terms on the right hand side of the general formula Eq. (15) are of different orders with respect to the observation time. The behaviour of these terms is studied in Section 5.6.2.

Let us now give the expressions for the different terms \( Z_i \) in Eq. (15). The term of order \( T^{-1} \) is given by (for details, see Appendix F):

\[
\frac{1}{T}Z_1 = \overline{f}_1(\overline{f}_1\text{Cov}[\tau_1, \tau_3] + \overline{f}_3\text{Cov}[\tau_1, \tau_4]) + \overline{f}_1(\overline{f}_1\text{Cov}[\tau_2, \tau_3] + \overline{f}_3\text{Cov}[\tau_2, \tau_4])
\]

(16)

where the covariance between two travel times is given by Eq. (13) and \( f_1 \) is the expectation value of the travel time \( \tau_j \), for example,

\[
\overline{f}_1 = \int_{-\pi/\delta h}^{\pi/\delta h} d\omega W_1(x_1, x_2, \omega) (\overline{C}(x_1, x_2, \omega) - C_{\text{ref}}(x_1, x_2, \omega))
\]

(17)

As \( C_{\text{ref}} \) and \( \overline{C} \) are generally close or even equal it is possible that this quantity is close to 0 or even exactly 0. This simplification is discussed in Section 4.3. Note that the time dependence (in \( T^{-1} \)) in Eq. (16) is hidden on the right hand side in the covariance between two travel times (cf. Eq. (13)).

The term of order \( T^{-2} \) is given by:

\[
\frac{1}{T^2}Z_2 = \text{Cov}[\tau_1, \tau_3]\text{Cov}[\tau_2, \tau_4] + \text{Cov}[\tau_1, \tau_4]\text{Cov}[\tau_2, \tau_3] - \overline{f}_1(\text{Cov}[\tau_2, \tau_4] + \text{Cov}[\tau_1, \tau_3] + \text{Cov}[\tau_2, \tau_3]) - \overline{f}_3(\text{Cov}[\tau_1, \tau_4] + \text{Cov}[\tau_2, \tau_3]) - \overline{f}_1(\text{Cov}[\tau_1, \tau_3] + \text{Cov}[\tau_2, \tau_4]) - \overline{f}_3(\text{Cov}[\tau_1, \tau_4] + \text{Cov}[\tau_2, \tau_3]) - \overline{f}_1(\text{Cov}[\tau_1, \tau_3] + \text{Cov}[\tau_2, \tau_4]) - \overline{f}_3(\text{Cov}[\tau_1, \tau_4] + \text{Cov}[\tau_2, \tau_3])
\]

(18)

where the covariance involving three travel times is given in the Appendix E by Eq. (E.1) and the one between two travel times by Eq. (13). As we will see in Section 5.3, the first line of this term is dominant in most of the applications.

To write down the term \( Z_3 \) of order \( T^{-3} \) we introduce a function \( \Gamma_{a_1, a_2} \) such that

\[
\text{Cov}[\tau_1, \tau_2] = \frac{(2\pi)^3}{T} \int_{-\pi/\delta h}^{\pi/\delta h} d\omega \Gamma_{a_1, a_2}(x_1, x_2, x_3, x_4, \omega) + O(T^{-2}),
\]

i.e. according to Eq. (13)

\[
\Gamma_{a_1, a_2}(x_1, x_2, x_3, x_4, \omega) = W_{a_1}(x_1, x_2) (\overline{C}(x_1, x_3, \omega) - C_{\text{ref}}(x_1, x_3, \omega)) + W_{a_2}(x_3, x_4, \omega) (\overline{C}(x_3, x_2, \omega) - C_{\text{ref}}(x_3, x_2, \omega)).
\]

(19)

Then the term of order \( T^{-3} \) is given by

\[
Z_3 = \frac{(2\pi)^7}{T^3} \sum_{\mu \in \mathcal{M}} \Gamma_{a_1, a_2} (x_1, x_2, x_3, x_4, \omega)
\]

(20)

\[
\times \text{Cov}[\tau_1, \tau_3, \tau_4] + \text{Cov}[\tau_1, \tau_4, \tau_3] + \text{Cov}[\tau_2, \tau_3, \tau_4] + \text{Cov}[\tau_2, \tau_4, \tau_3] + \text{Cov}[\tau_3, \tau_4, \tau_2] + \text{Cov}[\tau_3, \tau_2, \tau_4] + \text{Cov}[\tau_4, \tau_3, \tau_2] + \text{Cov}[\tau_4, \tau_2, \tau_3].
\]

(21)

\( \mathcal{M} \) contains 12 elements, so the term \( Z_3 \) consists in a sum of 12 terms containing a product of the functions \( \Gamma \) defined by Eq. (13).

4.3. Important special cases

4.3.1. Case \( C_{\text{ref}} = \overline{C} \)

As \( C_{\text{ref}} \) is generally choosen as an average value of the observations, we have \( C_{\text{ref}} = \overline{C} \) or at least \( C_{\text{ref}} \approx \overline{C} \). If there is equality then we can simplify the formula given in the previous section because \( f_1 = 0 \). It follows that the term \( Z_1 \) is zero as are some elements of \( Z_2 \). Denoting by \( Z_2 \) the value of \( Z_2 \) when \( C_{\text{ref}} = \overline{C} \), we have

\[
\frac{1}{T^2}Z_2 = \text{Cov}[\tau_1, \tau_3]\text{Cov}[\tau_2, \tau_4] + \text{Cov}[\tau_1, \tau_4]\text{Cov}[\tau_2, \tau_3].
\]

(22)

This term is of order \( T^{-2} \) as each of the covariance in Eq. (22) are of order \( T^{-1} \). The noise covariance matrix is now given by the sum of two terms of order \( T^{-2} \) and \( T^{-3} \):

\[
\text{Cov}[\tau_1, \tau_2, \tau_3, \tau_4] = \frac{1}{T^2}Z_2 + \frac{1}{T^4}Z_3 + O\left(\frac{1}{T^4}\right)
\]

(23)
4.3.2. Case $C_{\text{eff}} \approx \overline{C}$

Suppose now that we do not have equality but $C_{\text{eff}} = (1 + \epsilon)\overline{C}$ where $\epsilon$ is a small parameter measuring the difference between the reference cross-covariance and their expectation value. In this case $Z_2$ is of order $\epsilon^2$ and the terms that cancelled out previously in $Z_2$ when $C_{\text{eff}} = \overline{C}$ are of order $\epsilon$. The numerical tests from Section 5.6.1 will confirm that these terms of order $\epsilon$ and $\epsilon^2$ can be neglected so that Eq. (23) can be used even if we just have $C_{\text{eff}} \approx \overline{C}$.

4.3.3. Simplified formula

We have now defined all the terms involved in Eq. (15) to compute the covariance of a product of travel times. As one term is of order $T^{-2}$ and the other one of order $T^{-3}$, it will follow that $Z_2$ will dominate for long observation times. In this case, we have the simplified formula:

$$
\text{Cov}[\tau_1, \tau_2, \tau_3] = \text{Cov}[\tau_1, \tau_3] \text{Cov}[\tau_2, \tau_4] + \text{Cov}[\tau_1, \tau_4] \text{Cov}[\tau_2, \tau_3].
$$

(24)

In the next section, we will show applications of this formula which will validate the model and the simplified formula. In particular, the numerical tests will tell us that Eq. (24) can be used if the observation time is more than roughly a few hours.

5. Examples and comparisons

5.1. SDO/HMI power spectrum for $p_1$ ridge

In this section we validate the analytic formulae for the noise by comparing with Monte Carlo simulations. We choose to use a homogeneous noise so the model depends only on the expectation value of the power spectrum, $P(k, \omega) = h_x \mathbb{E}[|\phi(k, \omega)|^2]$. This expectation value is computed in the Fourier domain in or-

5.2. Monte Carlo simulations

We use the expectation value of the observed power spectrum $P(k, \omega)$ defined above as input to the noise model. In order to validate the theoretical model, we run Monte Carlo simulations by generating many realizations of the wave field in Fourier space using Eq. (7). The normal distributions are generated with the ziggurat algorithm of MATLAB (Marsaglia & Tsang 1984). All realizations have the same dimensions as above, i.e. $h_x R_y = 24.5$ and $h_y / 2\pi = 34.7$ $\mu$Hz.

5.3. Rate of convergence toward the analytic formula

To show the importance of having an explicit formula for the noise, we look at the convergence of Monte Carlo simulations to the analytic formula. For that, we define the following measure of the error:

$$
\text{Err}_1(n) = \frac{\text{Var}[	au] - \text{Var}_{\text{MC}}[	au]}{\text{Var}[\tau]},
$$

(25)

where $\text{Var}[\tau]$ is the theoretical variance for travel times computed by Eq. (13) and $\text{Var}_{\text{MC}}[\tau]$ is the variance obtained by Monte Carlo simulations with $n$ realisations. Similarly, we define

$$
\text{Err}_2(n) = \frac{\text{Var}[\tau^2] - \text{Var}_{\text{MC}}[\tau^2]}{\text{Var}[\tau^2]},
$$

(26)

where $\text{Var}[\tau^2]$ is the theoretical variance for a product of travel times computed by Eq. (15).

Figure 3 shows the errors $\text{Err}_1(n)$ for $\text{Var}[\tau_{\text{diff}}]$ and $\text{Err}_2(n)$ for $\text{Var}[\tau^2_{\text{diff}}]$ for travel times between two points separated by a distance $\Delta = 10$ Mm. As expected we have

$$
\text{Err}_1(n) \approx \text{const} n^{-\frac{3}{2}}
$$

(27)

with constants depending on the type of measurement. Even if the rate of convergence is the same for $\tau_{\text{diff}}$ or $\tau^2_{\text{diff}}$, the constant is much smaller for a travel time than for a product of travel times. The variance of a product of travel times converges much slower than the travel time variance. For example, an accuracy of 5% is reached with about $n = 1000$ realisations for $\tau_{\text{diff}}$ but around $n = 5000$ for $\tau^2_{\text{diff}}$. This underlines the importance of having an analytic formula to obtain the correct limit when $n \rightarrow \infty$, especially in the case of products of travel times.

5.4. Noise of travel times: comparison with Monte-Carlo simulations and SDO/HMI observations

To show the level of noise in the data, we compare the noise matrix with HMI data from 6 April 2012 until 14 May 2012. The point to point travel times are obtained for a distance $\Delta = 10$ Mm in the $x$ and $y$ direction so that we can compare $\text{Cov}[\tau_x(x_1, x_3), \tau_y(x_2, x_4)]$ in the configuration given by Figure 4. The comparison between the data, Monte Carlo simulation and the explicit formula is given in Figure 5. As expected, data contain mainly noise as we are looking only at point-to-point travel-times and a good agreement is found between stochastic simulations and the analytic formula.

5.5. Noise of products of travel times: comparison with Monte-Carlo simulations and SDO/HMI observations

We show in the previous section that the data are dominated by noise in the case of point to point travel times so it is legitimate to ask if there is information in a product of travel times. We look at the covariance between two products of EW and NS travel times $\text{Cov}[	au_x(x_1, x_5), \tau_y(x_2, x_6), \tau_y(x_3, x_7), \tau_x(x_4, x_8)]$ as presented in Figure 6. The results are given in Figure 7. As previously we note a good agreement between the analytic formula and the Monte Carlo simulation. In this case, one can see the differences between the noise and the data which are separated by 2$\sigma$. To confirm that this difference is due to the presence of physical signal (supergranulation) and not to a problem in the model, we
For both lattitudes, the correlation length is identical, equal to $\lambda/4$ where $\lambda = 7$ Mm is the dominant wavelength of the filtered wave field. This is half of the correlation length for travel times as one can see with the simplified formula Eq. (24).

5.6. Test of simplified formula for products of travel times using Monte Carlo simulations

We have shown in Section 4.3 that some simplifications can be made to the analytic formula for the noise covariance matrix if $C^{\text{ref}} = \overline{C}$. In this section, we show numerically that these simplifications can be done even if we do not have equality and that Eq. (24) is a good approximation for the noise covariance matrix.

5.6.1. Sensitivity to choice of $C^{\text{ref}}$

Let us first consider a fixed observation time ($T = 8$ h for the numerical examples) and look at the dependence on the term $C^{\text{ref}}$. This dependence is due to the term $Z_1$ and one part of $Z_2$ which depends on $\Psi$. Figure 8 makes this comparison for a product of travel times $\tau^2_{\text{diff}}$ between points separated by a distance $\Delta = 10$ Mm. The dashed lines have a slope of 1/2 and shows that the error decays as $n^{-1/2}$.

show in Figure 2 the same covariance but at the equator instead of at a latitude of $40^\circ$. In this case, data, analytic formula and Monte Carlo simulations fit perfectly. Since the product $\langle \tau_1 \tau_2 \rangle$ (configuration #2 with $d = 0$) measures the Reynolds stress $\langle v'_x v'_y \rangle$, it is expected to be zero at the equator and non-zero away from the equator (as we observe).

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Fig. 5: Cov[τ₁, τ₂] (in s²) for a p₁-ridge at a latitude of 40° with an observation time T = 8 h in the configuration #1 given by Figure 4. τₚ is used for τ₁ and τ₂. Left: SDO/HMI observations, middle: Monte Carlo simulation, right: analytic formula.

Fig. 6: Geometrical configuration #2: geometry used for the covariance between a product of EW and NS travel times Cov[τ₁, τ₂, τ₃] where τᵢ = τᵢₕ(x₂₋₁, x₃) are defined in Eq. (5). The travel distance between pairs of points is Δ = 10 Mm.

The term Z₁ is initially kept to ensure that the dependence on the observation time will not make this term become significant. As previously, we suppose that we have no knowledge about a reference cross-covariance (Cᵣᵢ = 0). Figure 10 makes this comparison for the variance in the configuration #1 and the covariance in the configuration #2 as a function of T (with Δ = 20 Mm). We see that the contribution of the term Z₁ is almost zero so this term can be neglected independently of the observation time. In the first configuration, the term Z₁ is always at least two decades smaller than Z₂ and so only this last term can be kept. The situation is slightly different for the second configuration. When T is smaller than one hour, then the standard deviation varies as T⁻¹ and the term Z₁ is dominant. When the observation time is greater than four hours then it varies as T⁻² and Z₂ is dominant. If T is very long then the variations should be in T⁻¹. This area happens theoretically for observation time longer than two months which is not realistic for solar applications and is thus not shown in Figure 10. The intersection between both terms is given by Tₑ = Z₁/Z₂. For this test case, a good approximation can be found in the far field as presented in Appendix F where it is shown that Tₑ ≈ 100 min which is confirmed numerically in Figure 10. These comparisons of the different terms are extremely important as it implies that we can use the approximation given by Eq. (22) if we consider observation times of a few hours which is generally the case. If the observation time is shorter, Z₂ is still a good approximation and gives a good estimate of the noise even if the amplitude is not exact. It is certainly sufficient to use Z₂ as noise covariance matrix in order to perform an inversion but numerical tests still have to be performed.

6. Spatial averages

We define the average value of a quantity q over an area A as follows:

\[ \langle q \rangle_A = \frac{1}{A} \sum_{x \in A} q(x). \]  

The noise covariance matrix for averaged travel times and products of travel times can be obtained by integrating respectively Eq. (13) and Eq. (15). Averaging data has the advantage of increasing the signal-to-noise ratio and allows to deal with fewer data. Table 1 shows the accuracy of the analytic formula and the importance of the averaging. It compares the value of the variance for a product between EW and NS travel times (configuration #2 with d = 0) and the same variance when the quantities are averaged over a domain A = \( l^2 \) with \( l = 18 \) Mm. First of all, we note a good agreement between the analytic formula and the Monte Carlo simulations. Second, the value of the variance is reduced of a factor 100 when we average the product of travel times over the spatial domain. As expected the variance decreases with the number of independent realisations which is the area A divided by square of the correlation length \( \lambda/4 \) (see Section 5.5) i.e. \( 18^2/(7/4)^2 = 105 \). Finally, the signal to noise ratio increases with the averaging and we can see a difference due to physical signal between the observations and the noise model.

7. Conclusions

In this paper we presented two main generalizations of the noise model of Gizon & Birch (2004) for helioseismic travel times.

|                  | Var(τ₁, τ₂) (s²) | Var((τ₁, τ₂, A)) (s²) |
|------------------|------------------|------------------------|
| SDO/HMI Observations | 5.7.10⁶          | 6.2.10⁴                |
| Monte Carlo simulations | 5.4.10⁶          | 5.0.10⁴                |
| Analytic formula     | 5.4.10⁶          | 5.0.10⁴                |

Table 1: Var[τ₁, τ₂] and Var[(τ₁, τ₂, A)] (in s²) with \( l = 18 \) Mm for the product of a EW and NS travel time (configuration #2 with \( d = 0 \). Comparison of SDO/HMI observations, analytic formula and Monte Carlo simulations for a p₁-ridge at 40° latitude and for an observation time \( T = 8 \) h.
First, the assumption of spatial homogeneity has been dropped. This is useful to model noise in regions of magnetic activity (sunspots and active regions) where oscillation amplitudes are significantly reduced and also to model noise across the solar disk as at different center-to-limb distances. Second, we generalized the noise model to higher-order moments of the travel times, in particular products of travel times. We showed that the covariance matrix for products of travel times consists of three terms that scale like $1/T$, $1/T^2$, $1/T^3$, where $T$ is the total observation time. For standard applications of time-distance helioseismology, we showed that the term in $1/T^2$ is dominant:

$$\text{Cov}[\tau_1 \tau_2, \tau_3 \tau_4] = \text{Cov}[\tau_1, \tau_3] \text{Cov}[\tau_2, \tau_4] + \text{Cov}[\tau_1, \tau_4] \text{Cov}[\tau_2, \tau_3].$$

This very simple formula links the noise covariance of products of travel times to the covariance of travel times and depends only on the expectation value of the cross-covariance $C(x, \omega)$ and can be obtained directly from the observations. The model is accurate and computationally efficient. It compares very well with Monte Carlo simulations and SDO/HMI observations. The analytic formulae presented in this paper can be used to compute the noise covariance matrices for averaged quantities and thus increase the signal to noise ratio. Finally we would like to emphasize that our results (moments of order 4, 6, and 8 of the wavefield $(x, \omega)$) can be extended to modelling noise for other methods of local helioseismology such as ring-diagram analysis, holography, or far-side imaging.

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Fig. 10: Left: Var[$\tau_{\text{diff}}^2$] as a function of the observation time with $\Delta = 20$ Mm. Right: Comparison of the three terms in Eq. (15) for the variance between a product of EW and NS travel time with $\Delta = 20$Mm.
Appendix A: On frequency correlations for the observables

In this appendix we study the frequency correlations in frequency space that result from a finite observation duration $T$. First we collect some definitions.

Since observations are discrete and to avoid some technical difficulties, we only consider discrete time points $t_j = h_j, j \in \mathbb{Z}$ in this paper. As a consequence, the frequency variable $\omega$ is $2\pi/h_\ell$-periodic. However, our definitions of the discrete Fourier transform and its inverse are chosen such that we obtain the time-continuous case in the limit $h_\ell \to 0$:

$$\mathcal{P}(\omega) = \frac{h_\ell}{2\pi} \sum_{k=-\infty}^{\infty} e^{i\omega t_k} \mathcal{P}(t_k), \quad \mathcal{P}(t_k) = \int_{-\pi/h_\ell}^{\pi/h_\ell} e^{-i\omega t_k} \mathcal{P}(\omega) \, d\omega. \quad \text{(A.1)}$$

We will need the orthogonal projection $D_N$ of $L^2([-\pi/h_\ell, \pi/h_\ell])$ onto the space $\Pi_N$ of $2\pi/h_\ell$-periodic trigonometric polynomials of degree $\leq N$ with the Dirichlet kernel $D_N$, the Fejér smoothing operator $F_N : L^2([-\pi/h_\ell, \pi/h_\ell]) \to \Pi_N$ with the Fejér kernel $\mathcal{F}_N$, and the projected periodic Hilbert transform $H_N : L^2([-\pi/h_\ell, \pi/h_\ell]) \to \Pi_N$ with kernel $\mathcal{H}_N$, which are defined by

$$D_N(\omega) = \sum_{k=-N}^{N} \exp(ik\omega) = \frac{\sin((2N+1)\omega/2)}{(2N+1)} \left\{ \begin{array}{ll} \sin((N+1)\omega/2), & \omega \neq 0 \\ 0, & \omega = 0 \end{array} \right., \quad (D_N\mathcal{P})(\omega) := \frac{h_\ell}{2\pi} \int_{-\pi/h_\ell}^{\pi/h_\ell} D_N(h_\ell(\omega - \tilde{\omega}))\mathcal{P}(\tilde{\omega}) \, d\tilde{\omega},$$

$$\mathcal{F}_N(\omega) = \frac{N+1}{N+1} \exp(ik\omega) = \frac{1}{N+1} \left[ \frac{\sin((N+1)\omega/2)}{\sin(\omega/2)} \right], \quad (F_N\mathcal{P})(\omega) := \frac{h_\ell}{2\pi} \int_{-\pi/h_\ell}^{\pi/h_\ell} \mathcal{F}_N(h_\ell(\omega - \tilde{\omega}))\mathcal{P}(\tilde{\omega}) \, d\tilde{\omega},$$

$$\mathcal{H}_N(\omega) = \frac{N+1}{N+1} \exp(ik\omega) = \frac{1}{N+1} \left[ \frac{\cos((N+1)\omega/2) - \cos(\omega/2)}{\sin(\omega/2)} \right], \quad (H_N\mathcal{P})(\omega) := \frac{h_\ell}{2\pi} \int_{-\pi/h_\ell}^{\pi/h_\ell} \mathcal{H}_N(h_\ell(\omega - \tilde{\omega}))\mathcal{P}(\tilde{\omega}) \, d\tilde{\omega}.$$

Here $\text{sgn}(k) := 1$ and $\text{sgn}(-k) := -1$ for $k \in \mathbb{N}$, and $\text{sgn}(0) := 0$. $H_N$ is related to the standard periodic Hilbert transform $H$ with convolution kernel $\mathcal{H}(\omega) = \cot(\omega/2)$ by $H_N = HD_N = D_N H$. With our convention for the Fourier transform the Fourier convolution theorem is $h_{\ell_1} \sum_{k=-\infty}^{\infty} f(t_k) \mathcal{P}(t_k) e^{i\omega t_k} = \int_{-\pi/h_\ell}^{\pi/h_\ell} f(\omega - \tilde{\omega})\mathcal{P}(\tilde{\omega}) \, d\tilde{\omega}$. In particular (with $f(\omega) = \mathcal{F}_N(h_\ell \omega)$ and $f(t_k) = \mathcal{F}_N(\omega)$, etc.) we have

$$(D_N\mathcal{P})(\omega) = \frac{h_\ell}{2\pi} \sum_{k=-N}^{N} \mathcal{P}(t_k) e^{i\omega t_k}, \quad (F_N\mathcal{P})(\omega) = \frac{h_\ell}{2\pi} \sum_{k=-N}^{N} \frac{N+1-|k|}{N+1} \mathcal{P}(t_k) e^{i\omega t_k}, \quad (H_N\mathcal{P})(\omega) = \frac{h_\ell}{2\pi} \sum_{k=-N}^{N} \frac{\text{sgn}(k)}{i} \mathcal{P}(t_k) e^{i\omega t_k}. \quad \text{(A.2)}$$

To simplify the notations, the cross-covariance (resp. its expectation value) $C(x_\ell, x_\ell, \omega)$ (resp. $\overline{C}(x_\ell, x_\ell, \omega)$) will be simply written as $C_{ab}(\omega)$ (resp. $\overline{C}_{ab}(\omega)$) and similarly the weight functions $W(x_\ell, x_\ell, \omega)$ will be $W_{ab}(\omega)$. We will show the following theorem on the correlation function

$$\mathcal{C}_{ab}(\omega_1, \omega_2) := \frac{2\pi}{T} \mathbb{E}[\phi^*(x_\ell, \omega_1)\phi(x_\ell, \omega_2)] : \quad \text{(A.3)}$$

The covariance between the wavefield at two frequencies $\omega_1$ and $\omega_2$ can be expressed as

$$\mathcal{C}_{ab}(\omega_1, \omega_2) = \begin{cases} I_{ab}(\omega_1, \omega_2) + I_{ab}(\omega_1, \omega_2) & \text{for } \omega_1 \neq \omega_2, \\
(\overline{F_{2N}C}_{ab})(\omega_1) & \text{otherwise}, \end{cases} \quad \text{(A.4)}$$

where

$$I_{ab}(\omega_1, \omega_2) := \frac{h_{\ell_1}}{2T} D_N(h_\ell(\omega_2 - \omega_1)) \left( \left( D_{2N} \overline{C}_{ab} \right)(\omega_2) + \left( D_{2N} \overline{C}_{ab} \right)(\omega_1) \right) \quad \text{(A.5)}$$

$$I_{ab}(\omega_1, \omega_2) := \frac{h_{\ell_1}}{2T} \cos(T(\omega_2 - \omega_1)/2) \left( H_{2N} \overline{C}_{ab} \right)(\omega_1) - \left( H_{2N} \overline{C}_{ab} \right)(\omega_2). \quad \text{(A.6)}$$

The second term is bounded by

$$|I_{ab}(\omega_1, \omega_2)| \leq \frac{h_{\ell_1}}{4} \left[ \frac{2N}{T} \right] \mathcal{C}_{ab}(t_k) \left. \right| . \quad \text{(A.7)}$$

For stationary Gaussian time series the error of the approximate noise model Eq. (5) in [Gizon & Birch 2004] is bounded by

$$\left| \mathcal{C}_{ab}(h_{\ell_1}j, \omega_2) - \delta_{jl} \left( D_{2N} \overline{C}_{ab} \right) \left( \frac{2\pi j l}{T} \right) \right| \leq \frac{h_{\ell_1}}{4} \left[ \frac{2N}{T} \right] \mathcal{C}_{ab}(t_k) \left. \right| \quad \text{for } j, l \in \mathbb{Z}, |j|, |l| \leq N. \quad \text{(A.8)}$$

The proof of the above theorem is given below.
For $\omega D (\omega m, m = -1, \tau - i \omega (t - t_2)^2)$, and $\omega_1 \neq \omega_2$.

For $\omega_1 = \omega_2$, we have $g_{a,j} (j) = 2N + 1 - j |t_2|$, so Eq. (A.2) for this case follows from Eq. (A.2).

We now consider the case $\omega_1 \neq \omega_2$. For $j > 0$ we have

$$g_{a,j} (j) = \sum_{n = -N}^{N-j} e^{i \omega_2 m} = e^{i \omega_2 0} \frac{1 - e^{-i \omega_2 (2N - 1)}}{1 - e^{-i \omega_2}} = e^{i \omega_2 (N+1/2)} \frac{1 - e^{-i \omega_2 (2N - 1)}}{e^{i \omega_2/2} - e^{-i \omega_2/2}} = e^{i \omega_2/2} \frac{\sin (\omega_2 (2N - j + 1)/2)}{\sin (\omega_2/2)}.$$  

If $|t_2| < 0$, then $g_{a,j} (j) = g_{a,j} (-j)$. Inserting the expression for $g_{a,j}$ in Eq. (A.9), using the identity $\sin(x - y) = \sin x \cos y - \cos x \sin y$ for $x = T(\omega_2 - \omega_1)/2$, and $y = T(\omega_2 - \omega_1)/2$, and finally using Eq. (A.2) leads to

$$\mathcal{C}_{ab} (\omega_1, T) = \frac{h_1^2}{4\pi T} \sin (h_1 (\omega_2 - \omega_1)/2) \left( \sin \left( \frac{\omega_2 - \omega_1}{2} T \right) \sum_{n = -N}^{N} \mathcal{C}_{ab} (t_n) \left( e^{i \omega_2 t_n} + e^{i \omega_1 t_n} \right) - \cos \left( \frac{\omega_2 - \omega_1}{2} T \right) \sum_{n = -N}^{N} \mathcal{C}_{ab} (t_n) \frac{\text{sgn}(j)}{i} e^{i \omega_2 t_n} - e^{i \omega_1 t_n} \right)$$

To bound $I_{ab}$ we may assume without loss of generality that $|\omega_2 - \omega_1| \leq \pi/h_1$ due to $2\pi/h_1$-periodicity. Using the mean value theorem and Eq. (A.2) and the inequality $\frac{|l|}{\sin(|l|/2)} \leq \frac{\pi}{2}$ for $|l| \leq \pi$ we obtain

$$\left| \frac{H_{2N} \mathcal{C}_{ab} (\omega_2) - H_{2N} \mathcal{C}_{ab} (\omega_1)}{\sin (h_1 (\omega_2 - \omega_1)/2)} \right| \leq \frac{|\omega_2 - \omega_1|}{\sin (h_1 (\omega_2 - \omega_1)/2)} \sup_{\omega} \left| H_{2N} \mathcal{C}_{ab} \right| \left( |\omega| \right)$$

This yields Eq. (A.7). It also implies Eq. (A.8) for $j \neq l$ since $D_{2N} \left( \frac{2N-j-l}{2N+1} \right) = 0$, i.e. $I_{ab} (h_a j, h_a l) = 0$. To show Eq. (A.8) for $j = l$ we use the bound

$$\left| \left( D_{2N} \mathcal{C}_{ab} - F_{2N} \mathcal{C}_{ab} \right) (\omega) \right| \leq h_1 \left( \frac{1}{\pi} \sum_{k = -2N}^{2N} \left| k \mathcal{C}_{ab} (t_k) \right| \right) \leq \frac{h_1}{4\pi} \left( \sum_{k = -2N}^{2N} \left| k \mathcal{C}_{ab} (t_k) \right| \right) \leq \frac{h_1}{4\pi} \left( \sum_{k = -2N}^{2N} \left| k \mathcal{C}_{ab} (t_k) \right| \right).$$

**Appendix B: On frequency correlations for the travel times**

In this appendix we derive the noise covariance matrix for the cross-covariance function $C$ and for the travel time $\tau$ when the frequency correlations are taken into account. Appendix A has shown that taking into account the frequency correlations leads to an additional term of order $1/T$ in the covariance of the observables at the grid points. As the covariance between two travel times is also of order $1/T$ it is of interest to look if this correction should be taken into consideration. This appendix proves that the extra term in $1/T$ of the observable covariance will only lead to an additional term in $1/T^2$ for the travel times. We also underline the main difficulties that will occur when computing higher order moments of $C$ and $\tau$.

Since with our convention Eq. (A.1) the Fourier transform is unitary up to the factor $\sqrt{2\pi/\tau}$, it follows from definition (3) that

$$\tau_1 (x_1, x_2) = 2\pi \int_{-\pi/h_1}^{\pi/h_1} W^{12}_{12} (\omega_1) \left( C_{12} (\omega_1) - \tilde{C}^{\text{cov}}_{12} (\omega_1) \right) d\omega_1.$$  

Therefore,

$$\text{Cov} \left( \tau_1 (x_1, x_2), \tau_2 (x_3, x_4) \right) = (2\pi)^2 \int d\omega_1 \int d\omega_2 \left( W^{12}_{12} (\omega_1) W^{34}_{34} (\omega_2) \text{Cov} \left( C_{12} (\omega_1), C_{34} (\omega_2) \right) \right).$$  

(B.1)
The first difficulty is to evaluate the quantity \( \text{Cov}[C_{12}(\omega_1), C_{34}(\omega_2)] \). For higher order moment we will also need to evaluate \( \text{Cov}[C_{12}(\omega_1)C_{34}(\omega_2), C_{56}(\omega_1)] \) and \( \text{Cov}[C_{12}(\omega_1)C_{34}(\omega_2), C_{56}(\omega_3)C_{78}(\omega_4)] \). The way to deal with these terms is presented in Appendix C where it is shown that

\[
\text{Cov}[C_{12}(\omega_1), C_{34}(\omega_2)] = \epsilon_2(\omega_1, \omega_2)\epsilon_{42}(\omega_2, \omega_1) + \epsilon_{14}(\omega_1, -\omega_2)\epsilon_{12}(-\omega_2, \omega_1). \tag{B.2}
\]

It leads to

\[
\text{Cov}[\tau(x_1, x_2), \tau(x_3, x_4)] = (2\pi)^3 \int \int \int \int \omega_1 \omega_2 W_2^{12}(\omega_1, \omega_2)W_2^{34}(\omega_2, \omega_1)\left(\epsilon_2(\omega_1, \omega_2)\epsilon_{42}(\omega_2, \omega_1) + \epsilon_{14}(\omega_1, -\omega_2)\epsilon_{12}(-\omega_2, \omega_1)\right). \tag{B.3}
\]

The second difficulty comes from the evaluation of these integrals i.e. the evaluation of linear functionals of the expectation value of the cross-covariance \( \epsilon \) given by the weight functions \( W \). Similarly, for higher order moments, we will need to be able to evaluate

\[
\int \int \int \int \int \omega_1 \omega_2 \omega_3 \omega_4 W_2^{12}(\omega_1, \omega_2)W_2^{34}(\omega_2, \omega_1)\epsilon_{34}(\omega_2, \omega_3)\epsilon_{42}(\omega_3, \omega_2)\epsilon_{24}(\omega_1, \omega_3)\epsilon_{12}(\omega_3, \omega_1)\epsilon_{56}(\omega_1, \omega_4)\epsilon_{12}(\omega_4, \omega_1). \tag{B.4}
\]

The method to compute these terms is presented in Appendix D. Applying the result for the second order moment presented in Appendix D leads to the result:

The travel-time covariance for finite \( T \) is given by the travel-time covariance for infinite observation time (Eq. (13)) plus a correction that decreases as \( 1/T^2 \)

\[
\text{Cov}[\tau(x_1, x_2), \tau(x_3, x_4)] = \frac{(2\pi)^3}{T} \int \int \int \omega_1 \omega_2 W_2^{12}(\omega_1, \omega_2)W_2^{34}(\omega_2, \omega_1)\left(\epsilon_2(\omega_1, \omega_2)\epsilon_{42}(\omega_2, \omega_1) + \epsilon_{14}(\omega_1, -\omega_2)\epsilon_{12}(-\omega_2, \omega_1)\right) + \frac{1}{T^2} \left(\mathcal{Y}(W_2^{12}, W_2^{34}, \mathcal{C}_{12}, \mathcal{C}_{34}) + \mathcal{Y}(W_2^{12}, W_2^{34}, \mathcal{C}_{14}, \mathcal{C}_{32}) + \mathcal{O}\left(\frac{1}{T^m+1}\right)\right), \tag{B.6}
\]

where \( \mathcal{Y}(W_1, W_2, f, g) = - \frac{(2\pi)^3}{T} \int \int \int \omega_1 \omega_2 W_1(\omega_1)W_2(\omega_2)\frac{H_{2N}f(\omega_2) - H_{2N}f(\omega_1)}{\sin\left(\frac{\omega_2 - \omega_1}{2}\right)}\frac{H_{2N}g(\omega_2) - H_{2N}g(\omega_1)}{\sin\left(\frac{\omega_2 - \omega_1}{2}\right)}\right). \tag{B.7}

Remark concerning the setting of Gizon & Birch (2004).

In Gizon & Birch (2004), it was supposed that

\[
\epsilon_{12}(\omega_1, \omega_2) = \delta_{\omega_1,\omega_2} \mathcal{C}(x_2 - x_1, \omega_1), \tag{B.8}
\]

so the covariance of \( C \) is

\[
\text{Cov}[C_{12}(\omega_1), C_{34}(\omega_2)] = \delta_{\omega_1,\omega_2} \mathcal{C}(x_2 - x_1, \omega_1)\mathcal{C}(x_2 - x_3, \omega_2) + \delta_{\omega_1,\omega_2} \mathcal{C}(x_2 - x_3, \omega_1)\mathcal{C}(x_2 - x_1, \omega_1). \tag{B.9}
\]

Note that Eq. (B.9) is exact. It differs slightly from Eq. (C.8) in Gizon & Birch (2004) which incorrectly contained an additional term. It leads to the covariance between travel times

\[
\text{Cov}[\tau_1, \tau_2] = \frac{(2\pi)^3}{T} \int \int \omega_1 \omega_2 W_1(\omega_1)W_2(\omega_2)\left(W_2^{12}(x_2 - x_1, \omega_1)\mathcal{C}(x_2 - x_3, \omega_2)\mathcal{C}(x_3 - x_1, \omega_1) + W_2^{12}(x_2 - x_3, \omega_1)\mathcal{C}(x_3 - x_1, \omega_2)\mathcal{C}(x_2 - x_1, \omega_1)\right). \tag{B.10}
\]

Note that Eq. (B.10) is identical to Eq. (28) in Gizon & Birch (2004) as the extra term in the covariance of \( C \) was actually neglected by the authors. Taking into account the frequency correlations, Eq. (B.8) is no longer valid and correction terms have to be added to Eqs. (B.9)(B.10). These correction terms are given in the previous result.

Appendix C: Noise covariance matrix for high order cross-covariances

In this section we present the way to compute the noise covariance matrices for the cross-covariance function \( C \)

\[
\text{Cov}[C_{12}(\omega_1), C_{34}(\omega_2)] \tag{C.1}
\]

\[
\text{Cov}[C_{12}(\omega_1)C_{34}(\omega_2), C_{56}(\omega_1)] \tag{C.2}
\]

\[
\text{Cov}[C_{12}(\omega_1)C_{34}(\omega_2), C_{56}(\omega_3)C_{78}(\omega_4)] \tag{C.3}
\]

Using in Eqs. (C.1), (C.2) and (C.3) that the cross-covariance function can be written as a function of the observables

\[
C_{12}(\omega) = \frac{2\pi}{T} \delta_1(\omega)\phi_2(\omega) \quad \text{where} \quad \phi_j(\omega) := \phi(x_j, \omega) \tag{C.4}
\]

we see that the moments of 4, 6 and 8 of the observables have to be computed. In the next section we present a formula to compute high order moment of Gaussian variables. Then, we will apply this formula to compute Eqs. (C.1), (C.2) and (C.3).
Appendix C.1: Expectation value of high-order products of Gaussian random variables

We have seen that the moments of order 4, 6 and 8 of the observables have to be computed in order to find the noise covariance matrix for cross-covariances and products of cross-covariances. A formula to compute the \((2J)^n\)th-order moment of a multivariate complex normal distribution with zero-mean can be found in Isserlis (1918):

\[
\mathbb{E} \left[ \prod_{i=1}^{2J} z_i \right] = \sum_{(\mu, \nu) \in M'} \prod_{i=1}^{J} \mathbb{E} \left[ z_{\mu_i} z_{\nu_i} \right],
\]

where \(\mu\) and \(\nu\) have distinct values in \(\{1, 2J\}\) and the set \(M'\) is defined by

\[
M' = \{(\mu, \nu) | \mu, \nu \in \{1, 2J\}, \text{s.t. } \mu < \nu\}.
\]

Here, we used the notation \(\{1, 2J\}\) for the set of all integers between 1 and 2J. In order to better understand Eq. (C.5) let us explain it for the case \(J = 2\). In this case, Eq. (C.5) can be written as

\[
\mathbb{E}[z_1 z_2 z_3 z_4] = \sum_{i,j,k,l} \mathbb{E}[z_iz_j]\mathbb{E}[z_3z_4],
\]

where the indices \(i, j, k, l\) must satisfy \(i < j, i < k, k < l\) according to Eq. (C.6). This enforces that \(i = 1\). Then, we can have \(k = 2\) or \(k = 3\). If \(k = 3\), then \(l = 4\) and \(j = 2\). If \(k = 2\) then we have again two possibilities: \(j = 3\) and so \(j = 4\) or \(l = 4\) and \(j = 3\). So three combinations are possible: (1, 2, 3, 4), (1, 4, 2, 3) and (1, 3, 2, 4). It leads to

\[
\mathbb{E}[z_1 z_2 z_3 z_4] = \mathbb{E}[z_1 z_2] \mathbb{E}[z_3 z_4] + \mathbb{E}[z_1 z_3] \mathbb{E}[z_2 z_4] + \mathbb{E}[z_1 z_4] \mathbb{E}[z_2 z_3].
\]

In particular, we have

\[
\text{Cov}(z_1 z_2, z_3 z_4) = \mathbb{E}[z_1 z_2] \mathbb{E}[z_3 z_4] - \mathbb{E}[z_1 z_3] \mathbb{E}[z_2 z_4] - \mathbb{E}[z_1 z_4] \mathbb{E}[z_2 z_3],
\]

which is the formula required to compute the moment of order 4 in Eq. (C.1). For \(J = 3\), Eq. (C.5) becomes

\[
\mathbb{E}[z_1 z_2 z_3 z_4 z_5 z_6] = \sum_{i,j,k,m,n} \mathbb{E}[z_iz_j] \mathbb{E}[z_iz_k] \mathbb{E}[z_mz_n],
\]

where the indices \(i, j, k, l, m, n\) must satisfy \(i < k < m\) (since the sequence (\(\mu_i\)) must increase) and \(i < j, k < l\) and \(m < n\) (since \(\mu_i < \nu_i\)) according to Eq. (C.6). Hence we obtain

\[
\text{Cov}(z_1 z_2 z_3 z_4 z_5 z_6) = \mathbb{E}[z_1 z_2 z_3 z_4 z_5 z_6] - \mathbb{E}[z_1 z_2 z_3 z_4] \mathbb{E}[z_5 z_6] - \mathbb{E}[z_1 z_2 z_3 z_5] \mathbb{E}[z_4 z_6] - \mathbb{E}[z_1 z_2 z_3 z_6] \mathbb{E}[z_4 z_5] - \mathbb{E}[z_1 z_2 z_4 z_5] \mathbb{E}[z_3 z_6] - \mathbb{E}[z_1 z_2 z_4 z_6] \mathbb{E}[z_3 z_5] - \mathbb{E}[z_1 z_2 z_5 z_6] \mathbb{E}[z_3 z_4] - \mathbb{E}[z_1 z_3 z_4 z_5] \mathbb{E}[z_2 z_6] - \mathbb{E}[z_1 z_3 z_4 z_6] \mathbb{E}[z_2 z_5] - \mathbb{E}[z_1 z_3 z_5 z_6] \mathbb{E}[z_2 z_4] - \mathbb{E}[z_2 z_3 z_4 z_5] \mathbb{E}[z_1 z_6] - \mathbb{E}[z_2 z_3 z_4 z_6] \mathbb{E}[z_1 z_5] - \mathbb{E}[z_2 z_3 z_5 z_6] \mathbb{E}[z_1 z_4] - \mathbb{E}[z_2 z_4 z_5 z_6] \mathbb{E}[z_1 z_3] - \mathbb{E}[z_3 z_4 z_5 z_6] \mathbb{E}[z_1 z_2] - \mathbb{E}[z_3 z_4 z_6] \mathbb{E}[z_1 z_2 z_5] - \mathbb{E}[z_3 z_5 z_6] \mathbb{E}[z_1 z_2 z_4] - \mathbb{E}[z_4 z_5 z_6] \mathbb{E}[z_1 z_2 z_3].
\]

A problem is that the cardinality of the set \(M'\) is \((4J)!/((2J)!4J)!\) Isserlis (1918) increases exponentially. The sum in Eq. (C.5) contains 3 terms for \(J = 2\) and 15 for \(J = 3\) as shown above. Unfortunately for \(J = 4\) it leads to 105 terms so it is not convenient to write them down explicitly and we will just list the main guidelines in Section C.4.

Appendix C.2: Second order moment of C

In the original paper, the fourth order moment of the observables was guessed after looking at all the possible cases in the Fourier domain. Using the formula Eq. (C.9) and the definitions Eqs. (C.2) and (C.3) of \(C_{ab}\) and \(C'_{ab}\) and recalling that \(\phi_J(\omega) = \phi_J(-\omega)\) as \(\phi_J(t)\) is real-valued, the covariance matrix between two cross-covariances is readily computed as follows:

\[
\text{Cov}[C_{12}(\omega_1), C_{34}(\omega_2)] = \frac{2\pi}{T} \left[ \mathbb{E}[\phi_J(\omega_1)\phi_J(\omega_2)] \mathbb{E}[\phi_J(\omega_1)\phi_J(\omega_2)] \right] - \frac{2\pi}{T} \left( \mathbb{E}[\phi_J(\omega_1)\phi_J(\omega_2)] \mathbb{E}[\phi_J(\omega_1)\phi_J(\omega_2)] + \mathbb{E}[\phi_J(\omega_1)\phi_J(\omega_2)] \mathbb{E}[\phi_J(\omega_1)\phi_J(\omega_2)] \right) = C_{13}(\omega_1, \omega_2)C_{32}(\omega_2, \omega_1) + C_{14}(\omega_1, -\omega_2)C_{32}(-\omega_2, \omega_1).
\]
Appendix C.3: Third order moment of C

In this section we compute the sixth order moment of the observables defined by Eq. (C.2). After writing the cross-correlations as a function of the observables, we need to compute the moment of order 6 of the observables. This can be done using Eq. (C.11) with \( z_1 = \phi_1(\omega_1), z_2 = \phi_2(\omega_2), z_3 = \phi_3(\omega_3), z_4 = \phi_4(\omega_4), z_5 = \phi_5(\omega_5), \) and \( z_6 = \phi_6(\omega_6). \) It will turn out that after integration against weight functions the order of the different terms in \( 1/T \) depends on their degree of separability. Therefore, we denote by \( \Lambda^i_N \) the sum of the terms which can be written as product of at most \( N \) functions of disjoint subsets of the set of variables \( \{\omega_1, \omega_2, \omega_3\}. \) Then

\[
\text{Cov}[C_{12}(\omega_1)C_{34}(\omega_2), C_{56}(\omega_3)] = \Lambda^1_3(\omega_1, \omega_2, \omega_3) + \Lambda^2_3(\omega_1, \omega_2, \omega_3),
\]

where

\[
\Lambda^1_3 = \left( C_{13}(\omega_1, \omega_3)C_{24}(\omega_2, \omega_4)C_{56}(\omega_5, \omega_6) + C_{14}(\omega_1, \omega_2)C_{65}(\omega_5, \omega_6)C_{34}(\omega_3, \omega_4) \right)
\]

\[
+ \left( C_{13}(\omega_1, \omega_3)C_{56}(\omega_5, \omega_6)C_{24}(\omega_2, \omega_4) + C_{14}(\omega_1, \omega_2)C_{34}(\omega_3, \omega_4)C_{56}(\omega_5, \omega_6) \right)
\]

\[
+ \left( C_{13}(\omega_1, \omega_3)C_{65}(\omega_5, \omega_6)C_{24}(\omega_2, \omega_4) + C_{14}(\omega_1, \omega_2)C_{56}(\omega_5, \omega_6)C_{34}(\omega_3, \omega_4) \right)
\]

\[
+ \left( C_{13}(\omega_1, \omega_3)C_{34}(\omega_3, \omega_4)C_{56}(\omega_5, \omega_6) + C_{14}(\omega_1, \omega_2)C_{24}(\omega_2, \omega_4)C_{65}(\omega_5, \omega_6) \right),
\]

and

\[
\Lambda^2_3 = \left( C_{12}(\omega_1, \omega_2)C_{34}(\omega_3, \omega_4)C_{56}(\omega_5, \omega_6) \right)
\]

\[
+ \left( C_{13}(\omega_1, \omega_3)C_{24}(\omega_2, \omega_4)C_{56}(\omega_5, \omega_6) \right)
\]

(1.14)

\[
= \text{Cov}[C_{12}(\omega_1, \omega_2)]C_3(\omega_3, \omega_4)C_{56}(\omega_5, \omega_6)
\]

(1.15)

Appendix C.4: Fourth order moment of C

This section is devoted to the computation of the eighth order moment of the observables defined by Eq. (C.3). Writing the cross-correlations as a function of the observables leads to:

\[
\text{Cov}[C_{12}(\omega_1)C_{34}(\omega_2), C_{56}(\omega_5)C_{78}(\omega_6)] = \left( \frac{2\pi}{T} \right)^4 \text{Cov}[\phi_1\phi_2\phi_3\phi_4, \phi_5\phi_6\phi_7\phi_8].
\]

(1.16)

Here and in the following we omit the argument \( \omega_j \) of the observables \( \phi_{2j-1} = \phi_{2j-1}(\omega_j) \) and \( \phi_{2j} = \phi_{2j}(\omega_j) \). As for the moments of order 4 and 6, we can calculate this expression. But as explained in Section C.1 the moment of order 8 contains 105 terms, so we will not write explicitly all the terms. As for the moments of order 6 we arrange the terms as

\[
\text{Cov}[C_{12}(\omega_1)C_{34}(\omega_2), C_{56}(\omega_5)C_{78}(\omega_6)] = \left( \Lambda^4_6 + \Lambda^1_4 + \Lambda^3_4 \right)(\omega_1, \omega_2, \omega_5, \omega_6).
\]

(1.17)

where \( \Lambda^N_4 \) is the sum of all terms which can be written as product of at most \( N \) functions of disjoint subsets of the set of variables \( \{\omega_1, \omega_2, \omega_5, \omega_6\} \). The three terms \( \Lambda^N_4 \) will be computed below.

Expression for \( \Lambda^4_6 \)

These terms are the ones from the subset given by Eq. (C.6) from which in two expectation values, the observables use the same frequencies, for example \( \mathbb{E}[\phi_1^\prime\phi_2]\mathbb{E}[\phi_3^\prime\phi_4] \). It leads to the following formula:

\[
\left( \frac{T}{2\pi} \right)^4 \Lambda^4_6 = \mathbb{E}[\phi_1^\prime\phi_2]\mathbb{E}[\phi_3^\prime\phi_4]\mathbb{E}[\phi_5^\prime\phi_6]\mathbb{E}[\phi_7^\prime\phi_8] + \mathbb{E}[\phi_1^\prime\phi_2]\mathbb{E}[\phi_3^\prime\phi_4]\mathbb{E}[\phi_5^\prime\phi_6]\mathbb{E}[\phi_7^\prime\phi_8] + \mathbb{E}[\phi_1^\prime\phi_2]\mathbb{E}[\phi_3^\prime\phi_4]\mathbb{E}[\phi_5^\prime\phi_6]\mathbb{E}[\phi_7^\prime\phi_8] + \mathbb{E}[\phi_1^\prime\phi_2]\mathbb{E}[\phi_3^\prime\phi_4]\mathbb{E}[\phi_5^\prime\phi_6]\mathbb{E}[\phi_7^\prime\phi_8]
\]

(1.18)

Calculating all the expectation values implies

\[
\Lambda^4_6 = \mathbb{C}_{34}(\omega_2)\mathbb{C}_{78}(\omega_4)\left( \mathbb{C}_{13}(\omega_1, \omega_3)\mathbb{C}_{56}(\omega_5, \omega_6) + \mathbb{G}_{16}(\omega_1, -\omega_6)\mathbb{G}_{52}(\omega_3, -\omega_6) \right)
\]

\[
\mathbb{C}_{34}(\omega_2)\mathbb{C}_{65}(\omega_5)\left( \mathbb{C}_{12}(\omega_1, \omega_2)\mathbb{C}_{34}(\omega_3, \omega_4) + \mathbb{G}_{16}(\omega_1, -\omega_6)\mathbb{G}_{72}(\omega_2, -\omega_6) \right)
\]

\[
\mathbb{C}_{12}(\omega_1)\mathbb{C}_{34}(\omega_3)\left( \mathbb{G}_{15}(\omega_1, \omega_3)\mathbb{G}_{62}(\omega_5, \omega_6) + \mathbb{G}_{16}(\omega_1, -\omega_6)\mathbb{G}_{54}(\omega_3, -\omega_6) \right)
\]

\[
\mathbb{C}_{12}(\omega_1)\mathbb{C}_{65}(\omega_5)\left( \mathbb{G}_{15}(\omega_1, \omega_3)\mathbb{G}_{62}(\omega_5, \omega_6) + \mathbb{G}_{16}(\omega_1, -\omega_6)\mathbb{G}_{54}(\omega_3, -\omega_6) \right),
\]

which can be written in terms of the covariance between two cross-covariance functions

\[
\Lambda^4_6 = \mathbb{C}_{34}(\omega_4)\mathbb{C}_{78}(\omega_4)\text{Cov}[C_{12}(\omega_1), C_{56}(\omega_5)] + \mathbb{C}_{12}(\omega_1)\text{Cov}[C_{34}(\omega_2), C_{56}(\omega_5)]
\]

\[
+ \mathbb{C}_{65}(\omega_5)\mathbb{C}_{34}(\omega_2)\text{Cov}[C_{12}(\omega_1), C_{78}(\omega_4)] + \mathbb{C}_{12}(\omega_1)\text{Cov}[C_{34}(\omega_2), C_{78}(\omega_4)].
\]
Expression for $\Lambda_4$

Two kinds of products in Eq. (C.5) will lead to terms with only two frequency integrals:

- in two expectation values, the constraints on $\omega$ are the same, for example $\mathbb{E}[\phi_1^* \phi_4] \mathbb{E}[\phi_2^* \phi_5]$ (they will lead to the first two terms in Eq. (C.19))
- in one expectation value, the observables use the same frequencies, for example $\mathbb{E}[\phi_1^* \phi_2]$

Computing all the terms, one can show that

$$\Lambda_4^* = \text{Cov}[C_{12}(\omega_1), C_{34}(\omega_1)] \text{Cov}[C_{34}(\omega_2), C_{78}(\omega_2)] + \text{Cov}[C_{12}(\omega_1), C_{78}(\omega_1)] \text{Cov}[C_{34}(\omega_2), C_{56}(\omega_2)]$$

$$+ \text{Cov}[C_{12}(\omega_1) \text{Cov}[C_{34}(\omega_2), C_{78}(\omega_2)] + \text{Cov}[C_{12}(\omega_1), C_{34}(\omega_2) \text{Cov}[C_{34}(\omega_2), C_{56}(\omega_2)].$$

The terms $\text{Cov}[C, C]$ and $\text{Cov}[CC, C]$ appearing in this expression can be computed using Eqs. (C.12, C.13).

Expression for $\Lambda_4^*$

All the other terms will lead to terms that contains only one frequency integral in the covariance of the product of travel times. After reorganizing all the terms, one can show that $\Lambda_4^*$ can be written as

$$\Lambda_4^* = \left\{ \begin{aligned}
&\text{Cov}[C_{13}(\omega_1, -\omega_2) \text{Cov}[C_{32}(\omega_1, -\omega_1)] + \text{Cov}[C_{15}(\omega_1, \omega_3) \text{Cov}[C_{32}(\omega_2, \omega_1)]
&\text{Cov}[C_{13}(\omega_1, -\omega_2) \text{Cov}[C_{32}(\omega_1, -\omega_1)] + \text{Cov}[C_{15}(\omega_1, \omega_3) \text{Cov}[C_{32}(\omega_2, \omega_1)]
&\text{Cov}[C_{13}(\omega_1, -\omega_2) \text{Cov}[C_{32}(\omega_1, -\omega_1)] + \text{Cov}[C_{15}(\omega_1, \omega_3) \text{Cov}[C_{32}(\omega_2, \omega_1)]
&\text{Cov}[C_{13}(\omega_1, -\omega_2) \text{Cov}[C_{32}(\omega_1, -\omega_1)] + \text{Cov}[C_{15}(\omega_1, \omega_3) \text{Cov}[C_{32}(\omega_2, \omega_1)]
&\text{Cov}[C_{13}(\omega_1, -\omega_2) \text{Cov}[C_{32}(\omega_1, -\omega_1)] + \text{Cov}[C_{15}(\omega_1, \omega_3) \text{Cov}[C_{32}(\omega_2, \omega_1)]
\end{aligned} \right\}$$

Appendix D: Evaluation of separable linear functionals of nonseparable products of $\mathcal{C}_{ab}$’s

In this section we will derive asymptotic expansions of the terms

$$\int \text{d}\omega_1 \int \text{d}\omega_2 \ W_{12}(\omega_1) W_{34}(\omega_2) \mathcal{C}_{12}(\omega_1, \omega_2) \mathcal{C}_{34}(\omega_1, \omega_2)$$

$$\int \text{d}\omega_1 \int \text{d}\omega_2 \ W_{12}(\omega_1) W_{34}(\omega_2) W_{56}(\omega_3) \mathcal{C}_{12}(\omega_1, \omega_2) \mathcal{C}_{34}(\omega_1, \omega_3) \mathcal{C}_{56}(\omega_2, \omega_3)$$

$$\int \text{d}\omega_1 \int \text{d}\omega_2 \int \text{d}\omega_3 \ W_{12}(\omega_1) W_{34}(\omega_2) W_{56}(\omega_3) W_{78}(\omega_4) \mathcal{C}_{12}(\omega_1, \omega_2) \mathcal{C}_{34}(\omega_1, \omega_3) \mathcal{C}_{56}(\omega_2, \omega_3) \mathcal{C}_{78}(\omega_4, \omega_1)$$

in $1/T$ as $T \to \infty$ and explicit formulae for the leading order terms. Recall that $\mathcal{C}$ defined in Eq. (A.3) depends on $T$ although this is suppressed in our notation.

Appendix D.1: Functions of nonseparable products of two $\mathcal{C}_{ab}$ functions

In this subsection we will show that

$$\left(2\pi\right)^3 \int \text{d}\omega_1 \int \text{d}\omega_2 \ W_1(\omega_1) W_2(\omega_2) \mathcal{C}_{12}(\omega_1,\omega_2) \mathcal{C}_{34}(\omega_1,\omega_2) = \frac{(2\pi)^3}{T} \int \text{d}\omega_1 W_1(\omega) W_2(\omega) \overline{C}_{12}(\omega) \overline{C}_{34}(\omega)$$

$$+ \frac{Y(W_1, W_2, \overline{C}_{12}, \overline{C}_{34})}{T^2} + O\left(\frac{1}{T^{m+1}}\right),$$

(D.4)
where $\mathbf{Y}$ is defined by Eq. (B.7) if $\overline{C}_{12}$ and $\overline{C}_{34}$ have $m$ derivatives and $W_{12}$ and $W_{34}$ have $m - 1$ derivatives.

Plugging Eq. (A.4) into the left hand side of Eq. (D.4) we arrive at a sum $(2\pi)^3(X + 2Y + Z)$ involving the following three terms:

$$X := \int d\omega_1 \int d\omega_2 W_1(\omega_1)W_2(\omega_2)I_{12}(\omega_1, \omega_2)I_{34}(\omega_1, \omega_2)$$

$$Y := \int d\omega_1 \int d\omega_2 W_1(\omega_1)W_2(\omega_2)I_{12}(\omega_1, \omega_2)H_{34}(\omega_1, \omega_2)$$

$$Z := \int d\omega_1 \int d\omega_2 W_1(\omega_1)W_2(\omega_2)H_{12}(\omega_1, \omega_2)H_{34}(\omega_1, \omega_2).$$

We will repeatedly use the following transformation of variables formula for functions $f(\omega_1, \omega_2)$ which are $2\pi/h_1$-periodic in both variables:

$$\int_{-\pi/h_1}^{\pi/h_1} d\omega_1 \int_{-\pi/h_1}^{\pi/h_1} d\omega_2 f(\omega_1, \omega_2) = \int_{-\pi/h_2}^{\pi/h_2} d\omega_1 \int_{-\pi/h_2}^{\pi/h_2} d\omega_2 f(\tilde{\omega}_1 - \omega_2, \tilde{\omega}_1 + \omega_2),$$

$$\left(\tilde{\omega}_1 \middle/ \omega_2\right) = \frac{1}{2} \left(\omega_1 + \omega_2\right), \quad \left(\tilde{\omega}_1 \middle/ \omega_2\right) = \left(\tilde{\omega}_1 - \omega_2\right).$$

(D.8)

Note that even though the Jacobian of this transformation of variables is $1/2$, no factor appears since on the right hand side we integrate over a domain which can be reassembled to two periodicity cells.

Using Eq. (1.7.3) and noting that $D_N(\omega)^2 = (2N + 1)f_{2N}(\omega) = (T/h)F_{2N}(\omega)$, the first term can be written as

$$X = \frac{h_1}{2T} \int d\tilde{\omega}_1 \int d\omega_2 W_1(\tilde{\omega}_1 - \omega_2)W_2(\tilde{\omega}_1 + \omega_2)F_{2N}(2h_1\omega_2)\left(D_{2N}\overline{C}_{12}(\tilde{\omega}_1 - \omega_2) + D_{2N}\overline{C}_{34}(\tilde{\omega}_1 + \omega_2)\right) \times \left(D_{2N}\overline{C}_{34}(\tilde{\omega}_1 - \omega_2) + D_{2N}\overline{C}_{34}(\tilde{\omega}_1 + \omega_2)\right),$$

We want interpret the inner product as a convolution with $F_{2N}$ evaluated at 0. First note that by a change of variables

$$\int d\tilde{\omega}_1 F_{2N}(2h_1\omega_2)g(\omega_2) = \int d\tilde{\omega}_1 F_{2N}(h_1\omega_2)\frac{1}{2}\left[g(\omega_2) + g(\omega_2 + \pi/h_1)\right].$$

Let $f(\omega_1, \omega_2)$ be $2\pi/h_1$-periodic in both arguments and define

$$\tilde{f}(\tilde{\omega}_1, \omega_2) := f(\tilde{\omega}_1, \omega_2), \tilde{\omega}_1 + \omega_2))$$

and hence

$$\int d\tilde{\omega}_1 \int d\omega_2 F_{2N}(2\omega_2)\tilde{f}(\tilde{\omega}_1, \omega_2) = \frac{1}{2} \int d\tilde{\omega}_1 \left[F_{2N}(\tilde{f})(\tilde{\omega}_1, 0) + (F_{2N}\tilde{f})(\tilde{\omega}_1 + \frac{\pi}{h_1}, 0)\right] = \int d\tilde{\omega}_1 \left(F_{2N}\tilde{f}\right)(\tilde{\omega}_1, 0)$$

where $F_{2N}$ always acts on the second argument. As $F_{2N}f = D_{2N}f - \frac{1}{h}H_{2N}f'$, it follows that

$$X = \frac{2\pi}{T} \int d\omega_1 D_{2N}\left(W_1 W_2(D_{2N}\overline{C}_{12})(D_{2N}\overline{C}_{34})(\omega_1)\right) - \frac{2\pi}{T^2} \int d\omega_1 H_{2N}\left(W_1 W_2(D_{2N}\overline{C}_{12})(D_{2N}\overline{C}_{34})\right)'(\tilde{\omega}_1).$$

Since $|D_{2N}\overline{C}_{ab} - C_{ab}| = O(T^{-m})$, we get an additional $O(T^{-m})$ if we omit the orthogonal projections $D_{2N}$ in the last equation.

To bound $Y$ (Eq. (D.6), we again apply the change of variables in Eq. (D.8) to obtain

$$Y = \frac{h_1^2}{4T^2} \int d\tilde{\omega}_1 \int d\omega_2 \sin(\tilde{\omega}_2T)f(\tilde{\omega}_1, \tilde{\omega}_2) = \frac{h_1^2}{8T^2} \int d\tilde{\omega}_1 \int d\omega_2 \sin(2\tilde{\omega}_2T)f(\tilde{\omega}_1, \tilde{\omega}_2),$$

where $f$ has uniformly bounded derivatives of order $m - 1$. When $T$ tends to infinity this corresponds to a high order Fourier coefficient and thus can be made as small as desired. In particular, by repeated partial integration

$$\int f(\tilde{\omega}_1, \tilde{\omega}_2)\sin(2\tilde{\omega}_2T)d\tilde{\omega}_2 \leq \frac{1}{(2T)^m-1} \int d\omega_2 \frac{\partial^{m-1}f}{\partial\omega_2^{m-1}}(\tilde{\omega}_1, \tilde{\omega}_2).$$

(D.10)

The term $Z$ (Eq. (D.7)) can be transformed on the same way and after using that $\cos^2(\tilde{\omega}_2T)$ = $(1 - \cos(2\tilde{\omega}_2T))/2$, we find that

$$Z = \frac{h_1^2}{8T^2} \int d\omega_1 \int d\omega_2 W_1(\omega_1)W_2(\omega_2)\left(H_{2N}\overline{C}_{12}(\omega_2) - H_{2N}\overline{C}_{12}(\omega_1)\right)\left(H_{2N}\overline{C}_{34}(\omega_2) - H_{2N}\overline{C}_{34}(\omega_1)\right) + O\left(\frac{1}{T^{m+1}}\right)$$

where the higher order term comes from $\cos(2\tilde{\omega}_2T)$ in analogy to Eq. (D.10). As $\lim_{T\to\infty} H_{2N}f = Hf$ and all the terms in the integrals are bounded it follows that $X$ is of order $1/T^2$. Gathering the expressions for the three terms $X$, $Y$, $Z$ leads to Eq. (D.4).
Appendix D.2: Functionals of nonseparable products of three $\delta_{ab}$ functions

Let $\mathcal{O}$ be defined by Eq. (A.3) and $W_i$ representing some functions of $\omega$. Then, we have the following expension:

\[
(2\pi)^3 \int d\omega_1 \int d\omega_2 \int d\omega_3 W_1(\omega_1)W_2(\omega_2)W_3(\omega_3)\delta_{i1}(\omega_1, \omega_2)\delta_{i2}(\omega_1, \omega_2)\delta_{i3}(\omega_1, \omega_2) = \\
\frac{(2\pi)^3}{T^2} \int d\omega \ W_1(\omega)W_2(\omega)W_3(\omega)\overline{C}_{i1}(\omega)\overline{C}_{i2}(\omega)\overline{C}_{i3}(\omega) + O\left(\frac{1}{T^3}\right).
\]

(D.11)

Using Eq. (A.4) in the left hand side of Eq. (D.11), four different types of terms have to be studied

\[
X := \int d\omega_1 \int d\omega_2 \int d\omega_3 W_1(\omega_1)W_2(\omega_2)W_3(\omega_3)I_{12}(\omega_1, \omega_2)I_{34}(\omega_1, \omega_3)I_{56}(\omega_2, \omega_3) \\
Y_1 := \int d\omega_1 \int d\omega_2 \int d\omega_3 W_1(\omega_1)W_2(\omega_2)W_3(\omega_3)I_{12}(\omega_1, \omega_2)I_{34}(\omega_1, \omega_3)II_{56}(\omega_2, \omega_3) \\
Y_2 := \int d\omega_1 \int d\omega_2 \int d\omega_3 W_1(\omega_1)W_2(\omega_2)W_3(\omega_3)I_{12}(\omega_1, \omega_2)I_{34}(\omega_1, \omega_3)II_{56}(\omega_2, \omega_3) \\
Z := \int d\omega_1 \int d\omega_2 \int d\omega_3 W_1(\omega_1)W_2(\omega_2)W_3(\omega_3)I_{12}(\omega_1, \omega_2)I_{34}(\omega_1, \omega_3)II_{56}(\omega_2, \omega_3)
\]

where the expressions $I$ and $II$ are given respectively by Eqs. (A.5) (A.6).

We will use the change of variables

\[
\int_Q d\omega f(\omega) = \int_Q d\omega \overline{f}(\omega(\overline{\omega})), \quad \overline{\omega} = \frac{1}{3} \begin{pmatrix} 1 & 1 & 1 \\ -1 & 0 & 1 \\ -1 & -1 & 2 \end{pmatrix} \omega, \quad \omega = \frac{1}{3} \begin{pmatrix} 1 & 1 & 1 \\ -1 & 0 & 1 \end{pmatrix} \overline{\omega}, \quad Q := [-\pi/h_i, \pi/h_i]^3
\]

where the Jacobian 1/3 does not appear for the same reason as in Eq. (D.8). Applying this to $X$ we obtain

\[
X = \left(\frac{h_i}{2T}\right)^3 \int d\overline{\omega}_1 \int d\overline{\omega}_2 \int d\overline{\omega}_3 W_1(\overline{\omega}_1)W_2(\overline{\omega}_2)W_3(\overline{\omega}_3)D_{2N}(3h_i\overline{\omega}_1)D_{2N}(3h_i\overline{\omega}_2)D_{2N}(3h_i\overline{\omega}_3 - \overline{\omega}_2) \times
\left(D_{2N}\overline{C}_{i1}(\omega_1) + D_{2N}\overline{C}_{i2}(\omega_2)\right)\left(D_{2N}\overline{C}_{i3}(\omega_1) + D_{2N}\overline{C}_{i4}(\omega_3)\right)\left(D_{2N}\overline{C}_{i5}(\omega_2) + D_{2N}\overline{C}_{i6}(\omega_3)\right)
\]

where $\omega_i$ can be replaced by the corresponding value in $\overline{\omega}_i$. The role of the Fejér kernel is played by the function

\[
\mathcal{F}_{2N}^{df}(h_i\overline{\omega}_2, h_i\overline{\omega}_3) = \frac{h_i}{T} \sum_{m,l=0}^{2N} \exp(ih_i(\overline{\omega}_2 - l + \overline{\omega}_3))
= \frac{h_i}{T} \sum_{m,l=0}^{2N} \exp(ih_i(m\overline{\omega}_2 + n\overline{\omega}_3)) = \sum_{m,l=0}^{2N} \left(1 - \frac{\max(|m|, |n|, |m-n|)}{2N+1}\right) \exp(ih_i(m\overline{\omega}_2 + n\overline{\omega}_3))
\]

where we have used the change of variables $m = j - l$, $n = k + l$, $o = l$. If $\mathcal{F}_{2N}^{df}$ denotes the corresponding convolution operator and

\[
(D_{2N}^{df}(\omega, \omega) := \frac{h_i}{(2T)^2} \sum_{m,l=0}^{2N} f(t_m, t_n) \exp(i\omega t_m + i\omega t_n)\right)
\]

the two-dimensional orthogonal projection, we can use the inequality

\[
\left|\mathcal{F}_{2N}^{df}(\omega_1, \omega_2) - \mathcal{F}_{2N}^{df}(\omega_1, \omega_2)\right| \leq \frac{h_i^2}{(2\pi)^2(2N+1)} \left|\sum_{m,l=0}^{2N} f(t_m, t_n) \left|\exp(i\omega t_m + i\omega t_n)\right|\right| = \frac{1}{T} \left|\frac{\partial D_{2N}^{df}(\omega, 0, 0)}{\partial \omega_1} + \frac{\partial D_{2N}^{df}(\omega, 0, 0)}{\partial \omega_2}\right|.
\]

(D.17)

If $f(\omega_1, \omega_2, \omega_3)$ is $2\pi/h_i$-periodic in all its arguments and $\overline{f}(\omega) = f(\omega(\overline{\omega}))$, we find in analogy to section D.1 that

\[
\int d\overline{\omega} \int d\overline{\omega}_1 \mathcal{F}_{2N}^{df}(3\overline{\omega}_2, 3\overline{\omega}_3)\overline{f}(\overline{\omega}_1, \overline{\omega}_2, \overline{\omega}_3) = \frac{1}{3} \sum_{k,l=0}^{2N} \left(\frac{F_{2N}^{df}(\overline{\omega}_1, \overline{\omega}_2, \overline{\omega}_3)}{3h_i}\right) \left(\overline{\omega}_1 - \frac{2\pi}{3h_i}(k + l), 0, 0\right)
\]

and hence

\[
\int d\overline{\omega}_1 \int d\overline{\omega}_2 \int d\overline{\omega}_3 \mathcal{F}_{2N}^{df}(3\overline{\omega}_2, 3\overline{\omega}_3)\overline{f}(\overline{\omega}_1, \overline{\omega}_2, \overline{\omega}_3) = \int d\overline{\omega}_1 \left(F_{2N}^{df}(\overline{\omega}_1, 0, 0)\right) \left(\overline{\omega}_1 - \frac{2\pi}{3h_i}(k + l), 0, 0\right).
\]

Together with Eq. (D.17) we obtain

\[
X = \frac{(2\pi)^3}{T^2} \int d\omega W_1(\omega)W_2(\omega)W_3(\omega)\overline{C}_{i1}(\omega)\overline{C}_{i2}(\omega)\overline{C}_{i3}(\omega) + O\left(\frac{1}{T^3}\right).
\]

(D.18)

The terms $Y_1$ is proved to be of very high order using the same method than in Section D.1. The term $Y_2$ is treated in the same way as it also contains a cosine that oscillates with $T$. Finally, $Y_3$ is of order $1/T^3$ using a similar demonstration than in Section D.1.
Appendix D.3: Functionals of nonseparable products of four \(C_{ab}\) functions

Let \(\mathcal{C}\) be defined by Eq. (A.3) and \(W_i\) representing some functions of \(\omega\). Then, we have the following expansion:

\[
(2\pi)^3 \int d\omega_1 \int d\omega_2 \int d\omega_3 \int d\omega_4 \ W_1(\omega_1)W_2(\omega_2)W_3(\omega_3)W_4(\omega_4)\mathcal{C}_{1234}(\omega_1, \omega_2)\mathcal{C}_{5678}(\omega_3, \omega_4) = \\
\frac{(2\pi)^7}{T^5} \int d\omega \ W_1(\omega)W_2(\omega)W_3(\omega)W_4(\omega)\bar{\mathcal{C}}_{1234}(\omega)\bar{\mathcal{C}}_{5678}(\omega) + O\left(\frac{1}{T^2}\right).
\]

(D.19)

As in the previous proof, different terms have to be treated. The terms with combinations of the expressions \(I\) and \(II\) can be bounded by the same methods as in Section [D.2] and the term involving only expressions \(II\) can be bounded as in Section [D.1]. The only different term is

\[
X := \int d\omega_1 \int d\omega_2 \int d\omega_3 \int d\omega_4 \ W_1(\omega_1)W_2(\omega_2)W_3(\omega_3)W_4(\omega_4)I_{12}(\omega_1, \omega_2)I_{34}(\omega_3, \omega_4)I_{78}(\omega_3, \omega_4).
\]

(D.20)

Here small adaptions of the argument in Section [D.2] with the change of variables

\[
\int_Q d\omega f(\omega) = \int \tilde{\omega} f(\tilde{\omega}(\omega)), \quad \tilde{\omega} = \frac{1}{4} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix} \omega, \quad \omega = \begin{pmatrix} 1 & -1 & -1 & -1 \\ 1 & -1 & -1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & 1 & 1 & -1 \end{pmatrix} \tilde{\omega}, \quad Q := [-\pi/h, \pi/h]^4.
\]

(D.21)

lead to the formula

\[
X = \frac{(2\pi)^3}{T^5} \int d\omega \ W_1(\omega)W_2(\omega)W_3(\omega)W_4(\omega)\bar{\mathcal{C}}_{1234}(\omega)\bar{\mathcal{C}}_{5678}(\omega) + O\left(\frac{1}{T^2}\right).
\]

(D.22)

Appendix E: Noise covariance matrix for products of travel times

Appendix E.1: Third order moment of the travel times

Using the definition of the travel times, we obtain that the covariance for the product of travel times is given by:

\[
\text{Cov}[\tau_1(x_1, x_2)\tau_2(x_3, x_4), \tau_3(x_5, x_6)] = (2\pi)^3 \int d\omega_1 \int d\omega_2 \int d\omega_3 \int d\omega_4 \ W^*_{12}(\omega_1)W^*_{34}(\omega_2)W_{56}(\omega_3)\times \\
\times \left[\text{Cov}[C_{12}(\omega_1)C_{34}(\omega_2), C_{56}(\omega_3)] - C_{12}^{ref}(\omega_1)\text{Cov}[C_{34}(\omega_2), C_{56}(\omega_3)] - C_{34}^{ref}(\omega_2)\text{Cov}[C_{12}(\omega_1), C_{56}(\omega_3)]\right].
\]

Using Eq. (C.13) and the two results presented in Sections [D.1] and [D.2] we can express the covariance for three travel-times as

\[
\text{Cov}[\tau_1(x_1, x_2)\tau_2(x_3, x_4), \tau_3(x_5, x_6)] = \frac{(2\pi)^5}{T^5} \int d\omega \ W^*_{12} \left[ W_{56}(C_{123}C_{456} + C_{142}C_{56}) + W_{56}(C_{142}C_{56} + C_{142}C_{56}) \right] \\
\times (\text{Cov}[C_{12}(\omega_1)C_{34}(\omega_2), C_{56}(\omega_3)] - C_{12}^{ref}(\omega_1)\text{Cov}[C_{34}(\omega_2), C_{56}(\omega_3)] - C_{34}^{ref}(\omega_2)\text{Cov}[C_{12}(\omega_1), C_{56}(\omega_3)]) \\
- \bar{T}_1\text{Cov}[\tau_2(x_3, x_4), \tau_3(x_5, x_6)] - T_2\text{Cov}[\tau_1(x_1, x_2), \tau_3(x_5, x_6)] + O\left(\frac{1}{T^2}\right).
\]

(E.1)

where \(\bar{T}_j\) is the expectation value of \(\tau_j\) and the covariance involving two travel times can be computed with Eq. (13).

Appendix E.2: Analytic formula for the covariance matrix for products of travel times

In this section, we derive the main result of this paper. It gives an analytic expression for the covariance matrix between a product of travel times. Using the definition of the travel times, one can show that the covariance of the product of travel times is given by:

\[
\text{Cov}[\tau_1\tau_2, \tau_3\tau_4] = (2\pi)^4 \int d\omega_1 \int d\omega_2 \int d\omega_3 \int d\omega_4 \ W^*_{12}(\omega_1)W^*_{34}(\omega_2)W_{56}(\omega_3)W_{78}(\omega_4)\times \\
\left\{ \text{Cov}[C_{12}(\omega_1)C_{34}(\omega_2), C_{56}(\omega_3)C_{78}(\omega_4)] \right. \\
- C_{78}^{ref}(\omega_2)\text{Cov}[C_{12}(\omega_1)C_{34}(\omega_2), C_{56}(\omega_3)] - C_{56}^{ref}(\omega_2)\text{Cov}[C_{12}(\omega_1)C_{34}(\omega_2), C_{78}(\omega_4)] \\
- C_{34}^{ref}(\omega_2)\text{Cov}[C_{12}(\omega_1), C_{56}(\omega_3)C_{78}(\omega_4)] - C_{12}^{ref}(\omega_1)\text{Cov}[C_{34}(\omega_2), C_{56}(\omega_3)C_{78}(\omega_4)] \\
+ C_{34}^{ref}(\omega_2)\text{Cov}[C_{12}(\omega_1), C_{56}(\omega_3)C_{78}(\omega_4)] + C_{56}^{ref}(\omega_2)\text{Cov}[C_{12}(\omega_1), C_{78}(\omega_4)] \\
+ C_{12}^{ref}(\omega_1)\left\{ C_{78}^{ref}(\omega_4)\text{Cov}[C_{34}(\omega_2), C_{56}(\omega_3)] + C_{56}^{ref}(\omega_3)\text{Cov}[C_{34}(\omega_2), C_{78}(\omega_4)] \right\} \right\}.
\]

(E.2)
In Appendix D we have shown that not all the terms will lead to the same number of frequency integrals. It implies that the covariance given by Eq. (E.2) has terms of different order with respect to the observation time $T$. The terms containing 3 integrals in $\omega$ are of order $T^{-1}$ while the other ones are of order $T^{-2}$ and $T^{-3}$. We write the covariance as the sum between three terms for the different orders:

$$\text{Cov}[\tau_1, \tau_2, \tau_3, \tau_4] = \frac{1}{T} Z_1 + \frac{1}{T^2} Z_2 + \frac{1}{T^3} Z_3 + O\left(\frac{1}{T^4}\right).$$

(E.3)

The terms of order $1/T^4$ come from the correlation between the frequencies in the frequency domain as detailed in Section D for the covariance between travel times. The other terms are detailed below.

**Term $Z_1$ of order $T^{-1}$**

Looking at Eq. (E.2) one can see that this term is composed of

- all the terms involving $\text{Cov}[C, C]$,
- the terms with two integrals in $\omega$ for the terms with $\text{Cov}[CC, C]$ (term $\Lambda^2_1$),
- the terms with three integrals in $\omega$ for the terms with $\text{Cov}[CC, CC]$ (term $\Lambda^3_1$)

where $C$ is a generic cross-covariance. Reorganizing terms leads to the formula Eq. (16) for $Z_1$.

**Term $Z_2$ of order $T^{-2}$**

Looking at Eq. (E.2) one can see that this term is composed of

- the terms with one integral in $\omega$ for the terms with $\text{Cov}[CC, C]$ (term $\Lambda^1_2$)
- the terms with two integrals in $\omega$ for the terms with $\text{Cov}[CC, CC]$ (term $\Lambda^2_2$).

Reorganizing terms leads to the formula Eq. (18) for $Z_2$.

**Term $Z_3$ of order $T^{-3}$**

The terms of order $T^{-3}$ come for the terms with only one integral in $\omega$ in $\text{Cov}[CC, CC]$ (term $\Lambda^3_2$). This yields Eq. (20) for $Z_3$.

### Appendix F: Far-field approximation for $\text{Var}[^{\text{diff}}_\text{2} \tau^2 (\Delta)]$

In this section we give approximate expressions for the different terms composing Eq. (15) for $\text{Var}[^{\text{diff}}_\text{2} \tau^2 (\Delta)]$ in the far field ($\Delta \rightarrow \infty$). We start with the definitions of $Z_1$, $Z_2$ and $Z_3$:

$$\frac{1}{T} Z_1 = 4 \pi^2 \text{Var}[\tau(\Delta)],$$

$$\frac{1}{T^2} Z_2 = 2(\text{Var}[\tau(\Delta)])^2 - 4 \pi^2 \left(\frac{2\pi}{T}\right)^2 \int d\omega |W(\Delta, \omega)|^2 \overline{C}(\Delta, \omega) \overline{C}(0, \omega) \times \left(W(\Delta, \omega) \overline{C}(0, \omega) + W^*(\Delta, \omega) \overline{C}(\Delta, \omega)\right),$$

$$\frac{1}{T^3} Z_3 = 3 \left(\frac{2\pi}{T}\right)^3 \int d\omega |W(\Delta, \omega)|^2 \left(W(\Delta, \omega) \overline{C}(0, \omega)^2 + W^*(\Delta, \omega) \overline{C}(\Delta, \omega)^2\right).$$

In the far field, we have $\overline{C}(\Delta, \omega) \ll \overline{C}(0, \omega)$. If we suppose that $C^{\text{ref}} = (1 + \epsilon) \overline{C}$ then the global behaviour of the four terms is:

$$\frac{1}{T} Z_1 \sim 4(2\pi)^3 \left(\int d\omega W^*(\Delta, \omega) \overline{C}(\Delta, \omega)^2 \right)^2 \int d\omega |W(\Delta, \omega)|^2 \overline{C}(0, \omega)^2,$$

$$\frac{1}{T^2} Z_2 \sim 2(2\pi/\epsilon)^2 \left(\int d\omega |W(\Delta, \omega)|^2 \overline{C}(0, \omega)^2 \right)^2 + 4(2\pi)^6 \epsilon \left(\int d\omega |W(\Delta, \omega)|^2 W(\Delta, \omega) \overline{C}(0, \omega)^2 \overline{C}(\Delta, \omega) \right) \times \int d\omega W^*(\Delta, \omega) \overline{C}(\Delta, \omega),$$

$$\frac{1}{T^3} Z_3 \sim 3 \left(2\pi\epsilon\right)^3 \int d\omega |W(\Delta, \omega)|^4 \overline{C}(0, \omega)^4.$$

(F.1)

We can thus see that the global behaviour of the terms is

$$\frac{1}{T} Z_1 \sim \epsilon^2 \overline{C}(\Delta, \omega)^2 \overline{C}(0, \omega)^2,$$

$$\frac{1}{T^2} Z_2 \sim \frac{1}{T^2} \overline{C}(0, \omega)^4 + \epsilon \overline{C}(0, \omega)^4 \overline{C}(\Delta, \omega),$$

$$\frac{1}{T^3} Z_3 \sim \frac{1}{T^3} \overline{C}(0, \omega)^4.$$
As $C(\Delta, \omega) \ll C(0, \omega)$ we can conclude that in this case the first term in $Z_2$ and the one in $Z_3$ are dominant. We can go further to see for which observation time $T_c$ these two last terms intersect in the case of difference travel times. If the window function $f(t)$ in the definition of $W_{\text{diff}}$ defined by Eq. (4) is a Heavyside function then we have (Gizon & Birch 2004)

$$W_{\text{diff}}(\Delta, \omega) = \frac{2i\omega C^{\text{ref}}(\Delta, \omega)^*}{2\pi h \omega \sum \omega^2 |C^{\text{ref}}(\Delta, \omega')|^2}.$$  (F.2)

For a $p$–mode ridge $\kappa_r = \kappa_r(\omega)$ the function $C(\Delta, \omega)$ can be written in the far field as (Gizon & Birch 2004)

$$C(\Delta, \omega) \approx \sqrt{\frac{2}{\pi \kappa_r \Delta}} C(0, \omega) e^{-\kappa_r \Delta} \cos \left( \kappa_r \Delta - \frac{\pi}{4} \right)$$  (F.3)

where $\kappa$ is the imaginary part of the wavenumber at resonance and represents attenuation of the waves. The sums in Eq. (F.1) can be approximated using the fact that the cosine in Eq. (F.3) oscillates many times within the frequency width $\xi$ of the envelope of $C(0, \omega)$ such that

$$\frac{1}{T^2} Z_2 \approx 2 \left( \frac{2\pi \kappa_r \Delta e^{2\kappa_r \Delta}}{\xi \omega_0^2} \right)^2$$  and  $$\frac{1}{T^3} Z_3 \approx 3 \left( \frac{2\pi}{T^3} \right)^7 \frac{\kappa_r^2 \Delta^2 e^{4\kappa_r \Delta}}{\pi^2 \omega_0^6 \xi^3}.$$  

Using the numerical value $\xi/2\pi = 1 \text{mHz}$, the observation time $T_c$ at which the two terms are equal is

$$T_c = T \frac{Z_2}{Z_3} \approx \frac{12\pi}{\xi} = 100 \text{min}.$$  (F.4)

For $T > T_c$, $Z_2/T^2$ is the dominant term. As the observation time is traditionally of at least eight hours in helioseismology, the term of order $1/T^3$ can be neglected.