Universality in Dynamical Formation of Entanglement for Quantum Chaos

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Dynamical formation of entanglement is studied for quantum chaotic bi-particle systems. We find that statistical properties of the Schmidt eigenvalues for strong chaos are well described by the random matrix theory of the Laguerre ensemble. This implies that entanglement formation for quantum chaos has universal properties, and does not depend on specific aspects of the systems.

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At the dawn of the quantum mechanics, entangled states of spatially separated bi-particle systems triggered fierce discussions because of their paradoxical property. These discussions lead to the finding of the inequalities which measure how quantum systems deviate from the classical reality. Recently, a renewed interest has been paid to entanglement since it is recognized as an invaluable resource in quantum computation and quantum communication. Thus, the information theory and the circuit theory for entangled qubits has become fast growing fields.

On the other hand, in the quantum theory of open systems, dynamical formation of entanglement between a system and its environment is a target of the research. In these studies, it is supposed that uncontrolled formation of entanglement causes the loss of quantum coherence in the system and leads to the appearance of the classical reality. Then, the loss of quantum coherence is considered to provide the foundation of the statistical physics. Also in the quantum computation, dynamical formation of entanglement between qubits and their environment is a crucial subject since it can give rise to erroneous operations of quantum computers.

In the traditional study on the dynamical formation of entanglement, the system is supposed to interact with an environment composed of a large number of degrees of freedom. Then, the interaction with the environment causes complicated uncontrollable formation of entanglement, leading to the loss of coherence in the system.

However, interaction with an environment is not the only cause of complicated processes of forming entanglement. Actually, dynamical formation of entanglement by quantum chaos has been an active topic these days. In their studies, quantum chaos gives rise to increase of the von Neumann entropy (or a modified version of it) to the extent that the state approaches the maximally entangled state, which is an equally weighted superposition of entangled bases. Therefore, quantum chaotic systems can also be the origin of the loss of coherence, even when they are of a small number of degrees of freedom.

This indicates that there are several causes where complicated processes of forming entanglement takes place. Then, the next question is how we can classify, if possible, those processes of forming complicated entanglement. Are there any universality classes according to which we can differentiate seemingly similar processes of forming entanglement?

In this letter, we show that there actually exists a universal class in the dynamical processes of forming entangled states. Here, we investigate the time evolution of Schmidt eigenvalues for coupled systems of quantum chaos, and indicate that, for strongly chaotic systems, the distribution of Schmidt eigenvalues approaches a class of random matrices, i.e., the Laguerre unitary ensemble. So far, the Laguerre unitary ensemble has received only a little attention in the theory of random matrices. We will discuss possible implications of this universal class, and suggest its extensions for other cases.

We consider a wave function of a bi-particle system, ϕ(x1, x2). For simplicity, we assume that the dimensions of the Hilbert spaces for each degree of freedom are the same, and will be denoted by N. Then, the wave function is represented as

$$\Phi(x_1, x_2) = \sum_{i,j}^N A_{ij} |i >_1 \otimes |j >_2 . \tag{1}$$

Here {i >_1 |i = 1, 2, ..., N} and {j >_2 |j = 1, 2, ..., N} are the orthogonal bases for each degree of freedom, and A_{ij}(i, j = 1, 2, ..., N) are complex-valued coefficients. For the N × N matrix A whose components are A_{ij}, we...
perform the singular value decomposition \[ A = U \Lambda V^{-1} \] (2)

where \( U \) and \( V \) are \( N \times N \) unitary matrices, and \( \Lambda \) is a diagonal one whose components are \( \lambda_i (i = 1, 2, \ldots, N) \). Here, we can always make \( \lambda_i \geq \lambda_j \) for \( i < j \). We transform the bases from \( \{|i \rangle \} \) and \( \{ |j \rangle \} \) to \( \{|\tilde{i} \rangle \} \) and \( \{|\tilde{j} \rangle \} \) using \( U \) and \( V \), respectively so that the wave function is described as the Schmidt decomposition form,

\[ \Phi(x_1, x_2) = \sum_{i} \lambda_i |\tilde{i} \rangle >_1 \otimes |\tilde{i} \rangle >_2 . \] (3)

For the representation \( \Phi \), the von Neumann entropy for the reduced density matrix, in which one of the degree of freedom, \( x_1 \) or \( x_2 \) is traced out, is easily evaluated as

\[ S = - \sum_{i} \lambda_i^2 \ln(\lambda_i^2). \] (4)

We note that only the diagonal matrix \( \Lambda \) contributes to \( S \).

From now on, we construct a random matrix theory for the coefficient matrices, \( A \) of wave functions. First, we assume that the occurrence probability for the matrix \( A \) is a function of the independent components \( A_{ij} \). Moreover, we assume that the probability is invariant under local unitary transformations. These assumptions determine the joint probability function \( P\{A_{ij}\} \)

\[ P\{A_{ij}\} \prod_{i,j} dA_{ij} = C \exp[- \frac{1}{2 \alpha^2} \text{Tr}(A^1 A)] \prod_{i,j} dA_{ij}, \] (5)

where \( \alpha \) and \( C \) are constants. Transforming the arguments of \( P \) from \( A_{ij} \) into \( \lambda_i \) and the parameters which determine \( U \) and \( V \), and integrating the probability with respect to the variables other than \( \lambda_i \), we obtain the probability function with respect to the eigenvalues \( \varepsilon_i \),

\[ P_L(\varepsilon_i) \prod_i d\varepsilon_i \equiv C' \prod_i |\varepsilon_i - \varepsilon_j|^2 e^{-\sum_i \varepsilon_i} \prod_i d\varepsilon_i, \] (6)

where we introduced the new variables \( \varepsilon_i \equiv N^2 \lambda_i^2 \), which take the values from 0 to \( \infty \), and the normalization constant \( C' \) \[ \text{[14]} \]. From the joint probability \( P_L \), we can get the \( n \)-point correlation function as

\[ R_n(\varepsilon_1, \ldots, \varepsilon_n) \equiv \frac{N!}{(N-n)!} \int_0^\infty \cdots \int_0^\infty P_L(\varepsilon_1, \ldots, \varepsilon_N) d\varepsilon_{n+1} \cdots d\varepsilon_N \\
= \text{det}[K_N(\varepsilon_i, \varepsilon_j)]_{i,j=1, \ldots, n} , \] (7)

where

\[ K_N(\varepsilon_i, \varepsilon_j) \equiv \sum_{k=0}^{N-1} \varphi_k(\varepsilon_i) \varphi_k(\varepsilon_j). \] (8)

Here, we take \( \varphi_k(\varepsilon) \equiv e^{-\varepsilon/2} L_k(\varepsilon) \) with \( L_k(\varepsilon) \) being the \( k \)-th normalized Laguerre polynomial, hence we call the distribution \( \Phi \) the Laguerre unitary ensemble. From Eq. \( \text{[1]} \), we obtain the one-level density \[ \text{[14]} \] and the two-level cluster function \[ \text{[11]} \] as

\[ \sigma_N(\varepsilon) = R_1(\varepsilon) = K_N(\varepsilon, \varepsilon) \] (9)

and

\[ T_2(\varepsilon_1, \varepsilon_2) \equiv -R_2(\varepsilon_1, \varepsilon_2) + R_1(\varepsilon_1) R_1(\varepsilon_2) = [K_N(\varepsilon_1, \varepsilon_2)]^2, \] (10)

respectively. In order to discuss the statistical property of the level spacing, we need to unfold the level spectrum \[ \text{[10]} \]. Thus we introduce the rescaled level \( \omega(\varepsilon) \) as follows

\[ \omega(\varepsilon) = \int_0^{\varepsilon} \sigma_N(\varepsilon') d\varepsilon'. \] (11)

According to Nagao and Slevin \[ \text{[17]} \], we also introduce the renormalized two-level cluster function as

\[ \tilde{T}_2(\omega, \omega') = \frac{T_2(\varepsilon(\omega), \varepsilon(\omega'))}{\sqrt{R_1(\varepsilon(\omega)) R_1(\varepsilon(\omega'))}}. \] (12)

In comparing the Laguerre unitary ensemble with an ensemble of wavefunctions, we will divide the distribution of the Schmidt eigenvalues into the following three regions: The hard edge, i.e., those eigenvalues lying near zero, the soft edge, i.e., those eigenvalues lying near the largest one, and the bulk region, i.e., those lying away from the hard and soft edges. Note that specific features of the Laguerre unitary ensemble emerge only in the region where eigenvalues are near zero, i.e., the hard edge. For the other regions, it is known that characteristics of the Laguerre unitary ensemble are similar to those of the Gaussian unitary ensemble \[ \text{[12]} \].

In the following, we study the dynamical formation of entanglement for a coupled system of two degrees of freedom, and compare the Schmidt eigenvalues of wavefunctions to the Laguerre unitary ensemble. We give the Hamiltonian as

\[ H_T = H_1(x_1, p_1) + H_2(x_2, p_2) + H_{12}(p_1, p_2), \] (13)

where

\[ H_i(x_i, p_i) = 2 \sin^2(\frac{p_i}{2}) + k_i \sum_{n=1}^{\infty} \sin(x_i) \delta(t-n) \quad (i = 1, 2), \] (14)

\[ H_{12}(p_1, p_2) = 4c_{pp} \sin(\frac{p_1}{2}) \sin(\frac{p_2}{2}). \] (15)

Here \( k_i \ (i = 1, 2) \) are the parameters for the kick strength and \( c_{pp} \) is a coupling constant. In order to make the dimension of the Hilbert space finite \( N \), we impose the periodic boundary conditions not only for \( x_i \) but also for \( p_i \), i.e., \( 0 \leq x_i < 2\pi \) and \(-\pi < p_i < \pi \) \( (i = 1, 2) \). These conditions require that the Planck constant be \( h = \)
as follows. For strong chaos, their values are $k_1 = 3.0$, $k_2 = 2.5$ and, for weak chaos, they are $k_1 = 0.7$, $k_2 = 0.2$. In both cases, we choose $c_{pp} = 0.05$.

As seen in Fig. 1, the time evolution of $S$ by the strong chaos shows a temporal generation of entanglement between $x_1$ and $x_2$, where we choose as an initial state a product of two coherent states. We notice that the variation of the entropy consists of two stages. While the entropy increases rapidly in the initial stage, it starts to saturate for the latter stage. In the initial stage, the largest Schmidt eigenvalue decreases rapidly with the other ones growing from zero. For the latter stage, these Schmidt eigenvalues start to avoid each other [14]. The same behavior was found in a similar system [14]. Such avoided crossings remind us of the behavior of the energy eigenvalues for quantum chaotic systems. For weak chaos, saturation in the increase of entropy also takes place.

Both for weak chaos and strong chaos, we construct an ensemble of $10^6$ wavefunctions, respectively, by collecting them after the entropy starts to saturate. From the ensembles of the numerical data, we evaluate the one-level density $R_1(\varepsilon)$ and the two-level correlation function $R_2(\varepsilon, \varepsilon')$ directly, and estimate the renormalized two-level cluster function $\bar{T}_2(\omega, \omega')$ through Eq. (12).

Adachi and Iida [14] compared the one-level density of Schmidt eigenvalues in the coupled standard map with the prediction of the random matrix theory, and found that they agree well with each other. We confirmed their results for our system [15]. In order to see higher order statistics, we compare, in Fig. 2, the renormalized two-level cluster functions $\bar{T}_2(\omega, \omega')$ between the Laguerre unitary ensemble and the ensembles produced by the time evolution. For the hard edge, $\bar{T}_2(0, \omega)$ is depicted (Fig. 2(a)). For the strong chaos, the renormalized two-level cluster function agrees with that of the random matrix theory, although the fluctuations are not negligible. For the weak chaos, it behaves similarly to that of the random matrix theory, while the deviation from the random matrix theory is remarkable in $\omega \lesssim 1$. For the bulk region, in Fig. 2(b), $\bar{T}_2(\omega', \omega' + \omega)$ is averaged over $\omega'$ in the bulk region. For the strong chaos, the cluster function agrees well with that of the random matrix theory. However, the cluster function for the weak chaos takes negative values after it falls. This indicates that the Schmidt eigenvalues show weaker avoided crossings. For the soft edge, $\bar{T}_2(N - \omega, N)$ is depicted (Fig. 2(c)). For both the strong and weak chaos, the renormalized two-level cluster functions do not agree with that of the random matrix theory. In particular, an oscillation with large amplitude occurs for the weak chaos.

In Fig. 3, the distribution of level spacing for the bulk region is shown for the Gaussian unitary ensemble, the ensemble of the strong chaos, and that of the weak chaos. Here, we can see that the ensemble of the strong chaos agrees well with the Gaussian unitary ensemble. On the other hand, for the weak chaos, the level spacing distribution shows smaller level spacings than for the strong chaos. This is consistent with the results of the renormalized cluster function showing weaker avoided crossings.

We have shown that, when the coupled system exhibits strong quantum chaos, the set of the Schmidt eigenvalues approaches the Laguerre unitary ensemble. When the coupled system shows weak quantum chaos, the set drastically deviates from the Laguerre unitary ensemble. This indicates that the Schmidt eigenvalues for coupled systems exhibiting quantum chaos belong to a universality class of the Laguerre unitary ensemble.

The universality of the Schmidt eigenvalues implies that their distribution will be insensitive to how the coupled system is divided into two subsystems interacting with each other. In other words, the boundary between subsystems has the flexibility that the properties of subsystems do not depend significantly on how we define the boundary. This is exactly what we expect when the coupled system behaves in a classically statistical way. Thus, the universality of the Schmidt eigenvalues can be considered as one of the crucial features of the effects called decoherence.

However, the approach to the the Laguerre unitary ensemble is not complete even for the coupled system exhibiting strong quantum chaos. This is shown by the slight deviation around the soft edge. This means that dynamics within the Hilbert space spanned by these Schmidt eigenvectors still behave in a nonstatistical manner. Then, the following questions are the next target of our study: How does the boundary between those behaving statistically and those behaving in a nonstatistical way change in time? Is there a natural boundary where we can separate the coupled system into the two subsystems? How does the boundary differ depending on the choice of initial conditions?

There exist a couple of ways to extend the idea of uni-
FIG. 2: Renormalized two-level cluster function $\bar{T}_2(\omega_1, \omega_2)$. (a) Hard edge: $\bar{T}_2(0, \omega)$ is shown. (b) Bulk region: the average of $\bar{T}_2(\omega', \omega' + \omega)$ over the region $10 \leq \omega' \leq N - 10$ is shown. (c) Soft edge: $\bar{T}_2(N - \omega, N)$ is shown.

FIG. 3: Level spacing distribution. The distributions for the level spacing $S_L$ in the bulk region are plotted.

One possibility is to consider the coupled system where the existence of the symmetry will lead to other universality classes. Another way of extending our idea is to consider the coupled system where the sizes of the Hilbert spaces for the subsystems differ significantly [14]. This situation corresponds to the cases where the traditional studies of the statistical physics and the measurement theory consider. One possibility is that the universality classes can be described by the extended Laguerre ensemble. If so, we can think of the following problem: How does the decoherence effects vary between the cases when the system interacts with the other one of the same size and those when it interacts with the environment?

The study into the above questions is under progress and the results will be published elsewhere.

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