EQUIVALENCE OF THE ROTHBERGER AND 2-ROTHBERGER GAMES FOR HAUSDORFF SPACES

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ABSTRACT. We prove that in any Hausdorff space, the Rothberger game is equivalent to the k-Rothberger game, i.e. the game in which player II chooses k open sets in each move. This result follows from a more general theorem in which we show these games are equivalent to a game we call the restricted Menger game. In this game I knows immediately in advance of playing each open cover how many open sets II will choose from that open cover. This result illuminates the relationship between the Rothberger and Menger games in Hausdorff spaces. The equivalence of these games answers a question posed by Aurichi, Bella, and Dias [1], at least in the context of Hausdorff spaces.

1. Introduction

Let X be a topological space. Let O denote the collection of open covers of X. The Menger game on X is the two player game where at each round n of the game player I first plays an open cover U_n ∈ O of X, and player II responds by playing a finite subset U_0^0, . . . , U_n^{k_n−1} of U_n. Player II wins the run of the game if X = ∪_n U_n. We denote the Menger game by G_∞_inf(O,O). The notation reflects the facts that I is playing from O, II is trying to build an element of O, and II is picking a finite subset from I’s moves at each round. The Rothberger game [2], G_1(O,O), on X is the game where player I plays at round n an open cover U_n ∈ O and player II plays a single U_n ∈ U_n. Again, player II wins the run of the game if X = ∪_n U_n. The k-Rothberger game G_k(O,O) is the variation of the Rothberger game where player II plays k sets from I’s cover at each round. A natural extension of this is the game G_f(O,O) where f: Ω → 2_0. In this game, at each round n player II plays f(n) sets from player I’s move U_n. A still further extension of the games is the restricted Menger game G^*_inf(O,O), which we define precisely below, where player II decides at the start of each round n how many sets he will get to choose from I’s play U_n.

It is clear that

II wins G^*_inf(O,O) ⇒ ∀f II wins G_f(O,O) ⇒ ∀k II wins G_k(O,O) ⇒ II wins G_1(O,O)

Our main result, Theorem [1], is that for all T_2 spaces X, the above games are all equivalent. Recall two games are said to be equivalent if whenever one of the players has a winning strategy in one of the games, then that same player has a winning strategy in the other game. We note that the equivalence of the above games for arbitrary spaces is no stronger than the equivalence for T_0 spaces (by considering the T_0 quotient of an arbitrary space). On the other hand, it is well known that the full Menger game G_∞_inf(O,O) is not equivalent to the above mentioned games. For example, player II wins the Menger game on R, or any σ-compact space, while I has a winning strategy in G_1(O,O) on R (I can easily play to ensure that λ(∪ U_n) < ε for any given ε > 0).

The games mentioned above are closely related to selection principles on the space X. These types of covering games and selection principles were extensively studied by Scheepers and others, see for example [3]. Recall that X has the Menger property, denoted S_∞(O,O), if whenever {U_n} ∈ ω is a sequence of open covers of X, then there is a sequence {F_n}_{n ∈ ω}, where each F_n is a finite subset of U_n, such that X = ∪_n F_n. Similarly, X has the Rothberger property, denoted S_1(O,O), if whenever {U_n} ∈ ω is a sequence of open covers of X, then there is a sequence U_n ∈ U_n such that X = ∪_n U_n. There are two theorems which relate the games with the corresponding selection principles. One theorem, due to Hurewicz [4] (see also [5]), says that for any space X the selection principle S_∞(O,O) (i.e., X having the
Menger property) is equivalent to I not having a winning strategy in $G^*_{\text{fin}}(\mathcal{O}, \mathcal{O})$. Another theorem, due to Pawlikowski [3], says that for any space $X$ the selection property $\mathcal{S}_1(\mathcal{O}, \mathcal{O})$ (i.e., $X$ having the Rothberger property) is equivalent to I not having a winning strategy in $G_1(\mathcal{O}, \mathcal{O})$. The equivalence of $\mathcal{S}_k(\mathcal{O}, \mathcal{O})$ (where $k \in \omega$) and $\mathcal{S}_1(\mathcal{O}, \mathcal{O})$ was shown in [3] and noted by the authors of [1].

The Rothberger game $G_1(\mathcal{O}, \mathcal{O})$, for any space $X$, has a dual version called the point-open game. In this game, I plays at each round $n$ a point $x_n \in X$, and II then plays an open set $U_n$ with $x_n \in U_n$. Player I wins the run of the game iff $X = \bigcup_n U_n$. A theorem of Galvin [2] says that (for any $X$) these games are dual, that is, one of the players has a winning strategy in one of the games iff the other player has a winning strategy in the other game. A natural variation of the point-open game is the finite-open game, where I plays at each round $n$ a finite set $F_n \subseteq X$, and II plays an open set $U_n$ with $F_n \subseteq U_n$. Player I again wins the run iff $X = \bigcup_n U_n$. It is easy to see that for any $X$ that the point-open game is equivalent to the finite-open game.

Using these dual games (specifically the finite-open game) simplifies the presentation of our main result. This observation was noted by R. Dias, whom we thank.

2. Equivalence of Restricted Menger and Rothberger Games

We define a variation of the Menger game which we call the restricted Menger game, denoted by $G^*_{\text{fin}}(\mathcal{O}, \mathcal{O})$. The rounds of this game are as in the Menger game except that at the start of round $n$ player II will make an initial move, which must be a positive integer $k_n$, which is a declaration of how many open sets II intends to select this round. As in the Menger game, I will then play an open cover $\mathcal{U}_n \in \mathcal{O}$, and II will then respond by choosing $k_n$ of the sets from $\mathcal{U}_n$, which we denote $U_{n, 1}, \ldots, U_{n, k_n - 1}$. Player II wins the run of the game iff $X = \bigcup_n \bigcup_{i < k_n} U_{n, i}$.

![Figure 1. An illustration of the game $G^*_{\text{fin}}(\mathcal{O}, \mathcal{O})$.](image)

**Theorem 2.1.** Let $X$ be a $T_2$ space. Then the restricted Menger game $G^*_{\text{fin}}(\mathcal{O}, \mathcal{O})$ is equivalent to the Rothberger game $G_1(\mathcal{O}, \mathcal{O})$.

**Proof.** It is clear that if I wins $G^*_{\text{fin}}(\mathcal{O}, \mathcal{O})$ then I wins $G_1(\mathcal{O}, \mathcal{O})$. It is also clear that if II wins $G_1(\mathcal{O}, \mathcal{O})$, then II wins $G^*_{\text{fin}}(\mathcal{O}, \mathcal{O})$.

If I wins $G_1(\mathcal{O}, \mathcal{O})$, then by [3], $X$ does not satisfy the selection principle $\mathcal{S}_1(\mathcal{O}, \mathcal{O})$. Thus, there is a sequence $\{V_n\}$ of open covers of $X$ such that there is no sequence $V_n \in \mathcal{V}_n$ with $X = \bigcup_n V_n$. Then I has a winning strategy in $G^*_{\text{fin}}(\mathcal{O}, \mathcal{O})$ by playing as follows. If II first plays the integer $k_0$, then I plays the common refinement $\mathcal{U}_0 = V_0 \wedge \cdots \wedge V_{k_0 - 1}$. II will end the round by picking $k_0$ of the sets $U_{0, 1}, \ldots, U_{0, k_0 - 1}$ from $\mathcal{U}_0$. Player I continues in this manner. Because of the refining property of the $\mathcal{U}_n$, there is a sequence $V_n \in \mathcal{V}_n$ with $\bigcup_n \bigcup_{i < k_n} U_{n, i} \subseteq \bigcup_n V_n$. Since $\bigcup_n V_n \neq X$, I has won this run of $G^*_{\text{fin}}(\mathcal{O}, \mathcal{O})$.

Assume now that II has a winning strategy $\tau$ in $G^*_{\text{fin}}(\mathcal{O}, \mathcal{O})$. We let $\tau(\mathcal{U}_0, \ldots, \mathcal{U}_n)$ denote the response of $\tau$ when I plays open covers $\mathcal{U}_0, \ldots, \mathcal{U}_n$ (we are suppressing II’s moves according to $\tau$ in this notation). So, $\tau(\mathcal{U}_0, \ldots, \mathcal{U}_n)$ is a finite subset of $\mathcal{U}_n$. We let $\tau'(\mathcal{U}_0, \ldots, \mathcal{U}_n)$ denote the integer that $\tau$ plays at the start of the next round, immediately after $\tau(\mathcal{U}_0, \ldots, \mathcal{U}_n)$ was played. By $\cup \tau(\mathcal{U}_0, \ldots, \mathcal{U}_n)$ we mean the union of the (finitely many) open sets in $\tau(\mathcal{U}_0, \ldots, \mathcal{U}_n)$. Note that according to this notation $|\tau(\mathcal{U}_0, \ldots, \mathcal{U}_n)| = \tau'(\mathcal{U}_0, \ldots, \mathcal{U}_{n - 1})$.

We define a strategy $\sigma$ for I in the finite open game on $X$. We begin by explicitly describing $\sigma$ on the first round. Let $k_0$ be $\tau$’s first (integer) move in $G^*_{\text{fin}}(\mathcal{O}, \mathcal{O})$. Define

$$C_0 = \bigcap_{U \in \mathcal{O}} \cup \tau(U).$$


The next Lemma is the only point in the proof where we use the assumption that $X$ is $T_2$.

Lemma 2.2. $|C_\emptyset| \leq k_\emptyset$.

Proof. Suppose towards a contradiction that $x_0, \ldots, x_{k_\emptyset}$ are $k_\emptyset + 1$ distinct points in $C_\emptyset$. Since $X$ is a $T_2$ space, there are open sets $U_0, \ldots, U_{k_\emptyset}$ in $X$ with $x_i \in U_i$ for all $i \leq k_\emptyset$ and with the sets $\{U_i\}$ pairwise disjoint. For each $x \in X \setminus \{x_i\}_{i \leq k_\emptyset}$ let $U_x$ be an open set containing $x$ such that $U_x$ is disjoint from a neighborhood of $\{x_i\}_{i \leq k_\emptyset}$ (using $T_2$ again). Let $\mathcal{U} = \{U_x : x \notin \{x_i\}_{i \leq k_\emptyset} \cup \{U_i\}_{i \leq k_\emptyset}\}$, so $\mathcal{U}$ is an open cover of $X$. $\tau(\mathcal{U})$ consists of $k_\emptyset$ of the sets from $\mathcal{U}$. There is an $i \leq k_\emptyset$ such that $U_i \notin \tau(\mathcal{U})$. Then $x_i \notin \cup \tau(\mathcal{U})$, a contradiction to $x_i \in C_\emptyset$.

Then let $\sigma$’s first move in the finite open game be $C_\emptyset$. Say $\Pi$ responds with $V_0$. Before we continue, we need to define some auxiliary sets which correspond to the position $\{C_\emptyset, V_0\}$. If $V_0$ was legal, then we note that $X \setminus V_0 \subseteq X \setminus C_\emptyset$, and thus for each $x \in X \setminus V_0$, there is some $U \in \mathcal{O}$ such that $x \in X \setminus \cup \tau(U)$. These sets form an open cover of $X \setminus V_0$, which is a closed subspace of $X$, and thus is Lindelöf, and so we fix $\mathcal{U}_{\emptyset}(V_0) = \{U_{\emptyset}(m)\}_{m \in \omega}$ such that $\{X \setminus \cup \tau(U_{\emptyset}(m))\}_{m \in \omega}$ is a cover of $X \setminus V_0$.

To define $\sigma$ in subsequent rounds, we need to dovetail various moves on subsequences, using the previously defined open covers $\mathcal{U}_s$ for $s \in \omega^{<\omega}$, and for this purpose we fix any bijection $\varphi: \omega^{<\omega} \rightarrow \omega$ with the property that if $s \leq t$ then $\varphi(s) \leq \varphi(t)$. For $s \in \omega^{<\omega}$ we let $lh(s)$ denote the length of $s$. Now in general, suppose we are at round $n$ in the finite open game, and the moves $C_0, V_0, \ldots, C_{n-1}, V_{n-1}$ have been played, where $|C_i| = k_i$ for $i < n$. Assume in addition that for each $j < n$ we have defined open covers $\mathcal{U}_{\varphi^{-1}(j)-m}$ for all $m \in \omega$ (which depend on the $V_j$ played thus far). Furthermore, assume that the $C_j, V_j, \mathcal{U}_{\varphi^{-1}(j)-m}$ for $j < n$ satisfy the following. Let $s = \varphi^{-1}(j)$, then:

1. $C_j = \bigcap_{i \leq \omega} \cup \tau(U_{\varphi(i)}, U_{\varphi(i+1)}, \ldots, U_{\varphi(lh(s))}, U)$,
2. $\{X \setminus \cup \tau(U_{\varphi(i)}, U_{\varphi(i+1)}, \ldots, U_{\varphi(lh(s))}, U_{\varphi^{-1}(m))})\}_{m \in \omega}$ is a cover of $X \setminus \cup_{i \leq lh(s)} V_{\varphi(s(i))}$.

Note that property 2 for $j$ is possible since the space $X \setminus \cup_{i \leq lh(s)} V_{\varphi(s(i))}$ is Lindelöf and $X \setminus \cup_{i \leq lh(s)} V_{\varphi(s(i))} \subseteq X \setminus \bigcup_{i \leq lh(s)} C_{\varphi(s(i))}$, and using property 1 for the $C_i$ for $i \leq j$.

We define $\sigma$’s response to this position, and the necessary sets $\mathcal{U}_{\varphi^{-1}(j)-m}$, in a similar manner to the base step. Let $t = \varphi^{-1}(n)$ and define $\sigma$’s response to be

$$C_n = \bigcap_{U \in \mathcal{O}} \cup \tau(U_{\varphi(i)}, U_{\varphi(i+1)}, \ldots, U_{\varphi(lh(s))}, U),$$

which clearly maintains property 1. Note also that $C_n$ is finite, and in fact has size at most $|C_n| \leq \tau(\mathcal{U}_{0}, \ldots, \mathcal{U}_{\varphi^{-1}(n)})$, by the same proof of Lemma 2.2.

Similarly to the base step, define $\mathcal{U}_{\varphi^{-1}(n)-m}$ to be a countable collection of open covers such that $\{X \setminus \cup \tau(U_{\varphi(i)}, U_{\varphi(i+1)}, \ldots, U_{\varphi(lh(s))}, U_{\varphi^{-1}(m))})\}_{m \in \omega}$ covers $X \setminus \bigcup_{i \leq lh(t)} V_{\varphi(t(i))}$. Of course, this uses the fact that $X \setminus \bigcup_{i \leq lh(t)} V_{\varphi(t(i))}$ is Lindelöf and that it is contained in $X \setminus C_n$. This completes the definition of $\sigma$. To show that $\sigma$ is winning, we suppose that $C_0, V_0, C_1, V_1, \ldots$ is a full run of the finite open game which is consistent with $\sigma$. Note that since this run is consistent with $\sigma$, we can recover the tree of open covers $\{U_s\}_{s \in \omega^{<\omega}}$ associated to this run which satisfies the properties 1 and 2 above. Suppose that $X \neq \bigcup_{n} V_n$, and let $x \in X \setminus \bigcup_{n} V_n$. In particular, $x \in X \setminus V_0$. Now we use property 2 to obtain $i_0$ such that $x \notin \cup \tau(U_{i_0})$. In general, supposing we have $i_0, i_1, \ldots, i_{n-1}$ where $x \notin \cup \tau(U_{i_0}, \ldots, U_{i_0 \ldots i_{n-1}})$ for any $k < n$, then use the fact that $x \in X \setminus \bigcup_{s \leq \varphi^{-1}(n)} V_{\varphi(s)}$ and property 2 to obtain $i_n$ so that $x \notin \cup \tau(U_{i_0}, \ldots, U_{i_0 \ldots i_{n-1}}, U_{i_0 \ldots i_n})$. This builds a branch through the tree of open covers $\{U_s\}_{s \in \omega^{<\omega}}$, associated to this run, which has the property that $x$ is not in any of the closures of $\tau$’s moves in response to this branch. This contradicts the assumption that $\tau$ was a winning strategy.

Corollary 2.3. For any $T_2$ space $X$ and any $f: \omega \rightarrow \omega$, the games $G_f(O,O)$ and $G_f(O,O)$ are equivalent.

In particular, we have the following corollary which answers Problem 4.5 of [1] for $T_2$ spaces.

Corollary 2.4. For any $T_2$ space $X$ and any $n \in \omega$, the games $G_1(O,O)$ and $G_n(O,O)$ are equivalent.
3. Open Questions

A natural question is whether we can drop the assumption that $X$ is $T_2$ from the hypothesis of Theorem 2.1. In fact, the authors of [1] originally asked if for any topological space the games $G_1(\mathcal{O}, \mathcal{O})$ and $G_2(\mathcal{O}, \mathcal{O})$ are equivalent. Our Theorem 2.1 shows these games are equivalent for any $T_2$ space, but the $T_2$ assumption seems necessary for the argument. We are not aware of any space (with no assumptions on the space) for which these games are not equivalent. Since the determinacy of these games is not guaranteed in $ZF$, it is possible even that the equivalence for arbitrary spaces is independent of $ZF$.

**Question 3.1.** Can we weaken the hypotheses of Theorem 2.1 from $T_2$ to $T_1$, or even remove it entirely? That is, can we prove in $ZFC$ that the games $G_1(\mathcal{O}, \mathcal{O})$ and $G_2(\mathcal{O}, \mathcal{O})$ are equivalent for any space $X$?

One possibility for a negative answer to Question 3.1 would be to construct in $ZFC$ a space for which the games are not equivalent (in this case the game $G_1(\mathcal{O}, \mathcal{O})$ is not determined, and II must win the other game). It is also possible that the existence of a space for which the games are not equivalent is independent of $ZFC$. So we ask:

**Question 3.2.** Is it consistent with $ZFC$ that there is a space $X$ for which the games $G_1(\mathcal{O}, \mathcal{O})$ and $G_2(\mathcal{O}, \mathcal{O})$ are not equivalent. Is the existence of such a space consistent with $ZF$? In particular are the games equivalent in models of determinacy?

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