STOCHASTICALLY PERTURBED SLIDING MOTION IN PIECEWISE-SMOOTH SYSTEMS

D. J. W. Simpson
Institute of Fundamental Sciences
Massey University
Palmerston North, New Zealand

R. Kuske
Department of Mathematics
University of British Columbia
Vancouver, BC, Canada

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Abstract. Sliding motion is evolution on a switching manifold of a discontinuous, piecewise-smooth system of ordinary differential equations. In this paper we quantitatively study the effects of small-amplitude, additive, white Gaussian noise on stable sliding motion. For equations that are static in directions parallel to the switching manifold, the distance of orbits from the switching manifold approaches a quasi-steady-state density. From this density we calculate the mean and variance for the near sliding solution. Numerical results of a relay control system reveal that the noise may significantly affect the period and amplitude of periodic solutions with sliding segments.

1. Introduction. Nonsmoothness and noise are two general features of a dynamical system that may be the cause of important qualitative behaviour. Hybrid and piecewise-smooth systems are utilized in a wide variety of fields to model phenomena that involve switching, impacts or other nonsmooth elements [1, 2, 3, 4, 5]. Recent analyses have revealed that nonsmooth features permit a variety of novel transitions between different dynamical regimes [6]. Moreover, parameter uncertainty, background vibrations and other sources of noise are ubiquitous in real systems. Studies of stochastic differential equation models have led to an understanding of noise-induced dynamics, such as stochastic resonance and coherence resonance in excitable systems [7, 8, 9, 10]. However, possibly because much of the theory of nonsmooth systems has been developed recently, and possibly because many general theorems regarding stochastic differential equations do not apply to nonsmooth systems, detailed analyses of systems that are both nonsmooth and involve noise are relatively uncommon. Given that nonsmooth systems exhibit a rich array of dynamics that is specific to this class of systems, we naturally expect the combination of nonsmoothness and noise to produce new and interesting phenomena. We
provide a further discussion of general aspects of stochastic nonsmooth systems in §2.

In the remaining sections we study piecewise-smooth, stochastic differential equations for which the underlying deterministic dynamics may be written in the general form:

\[
\dot{x} = F_i(x), \quad \text{for } x \in \Omega_i,
\]

where each \( \Omega_i \subset D \subset \mathbb{R}^N \) is open, nonempty and pairwise-disjoint, \( \cup_i \Omega_i = D \), and each \( F_i : \overline{\Omega}_i \to \mathbb{R}^N \) is a smooth function. Equation (1) is a Filippov system [11] and well-suited to model phenomena that alternate between different dynamical regimes, such as vibrating systems experiencing impacts or friction [12, 13, 14, 15, 16], and switching in electrical circuits [3, 4, 17]. Boundaries between the neighbouring subdomains, \( \Omega_i \), are codimension-one surfaces termed switching manifolds.

Often in applications, a section of a switching manifold has the property that on either side of the manifold the vector field points towards the manifold. In this case any orbit that reaches the switching manifold becomes trapped on the manifold for some time. The resulting motion on the switching manifold is known as sliding motion, Fig. 1-A. Formally this is achieved by Filippov’s method [6, 11, 18] which defines a vector field on the switching manifold by the unique convex combination of the two limiting vector fields on either side that is tangent to the switching manifold. Sliding motion corresponds to the sticking phase of stick-slip oscillators [19, 20] and the coalesced regime of a piecewise-linear relay control system [21, 22].

This paper represents a first look into the effects of noise on systems with sliding. We consider additive Gaussian noise of a small constant amplitude which represents general uncertainties throughout the system. With small noise, instead of sliding, trajectories are pushed off the switching manifold by the noise but large excursions are curbed by the deterministic component of the system, Fig. 1-B. Thus the motion is balanced by the competing actions of noise and drift. In this paper we explore this behaviour more carefully.

Here we summarize the basic effect of adding small noise to a smooth system so that below we can compare this to our results for the piecewise-smooth system (1). Consider an \( N \)-dimensional system

\[
\dot{x} = \Phi(x),
\]

Figure 1. Schematics of a Filippov system near a switching manifold that attracts orbits from both sides in the absence of noise, panel A, and with small amplitude additive noise, panel B.
where $\Phi$ is a smooth function. The addition of small amplitude, time-independent, white, Gaussian noise gives the stochastic differential equation
\begin{equation}
\frac{dx}{dt} = \Phi(x) + \sqrt{\varepsilon}B(x)\,dW(t),
\end{equation}
where $0 < \varepsilon \ll 1$, $B(x)$ is a non-degenerate $N \times N$ matrix with a smooth dependency on $x$, and $W(t)$ is a standard vector Brownian motion [23, 24]. Let $p_\varepsilon(x, t|x_0)$ denote the transitional probability density function (PDF) for the point $x(t)$, given $x(0) = x_0$. A straight-forward expansion in powers of $\sqrt{\varepsilon}$ reveals that the mean of $p_\varepsilon(x, t|x_0)$ differs from the deterministic solution (the solution to (2) with $x(0) = x_0$) by $O(\varepsilon)$, whereas deviations are $O(\sqrt{\varepsilon})$ [24, 25, 26]. Throughout this paper we let $O(\varepsilon^k) [o(\varepsilon^k)]$ denote terms that are order $\varepsilon^k$ or higher [higher than order $\varepsilon^k$]. More generally we let $O(k) [o(k)]$ denote terms that are order $k$ or higher [higher than order $k$] in all variables of a given expression.

In this paper we derive an analogous result for sliding motion for which the above method of expansion does not work because the vector field is discontinuous. Instead we analyze a one-dimensional, discontinuous stochastic differential equation for a quantity representing the distance from the switching manifold. We find that the mean solution differs from Filippov's deterministic sliding solution by $O(\varepsilon)$, and deviations are $O(\sqrt{\varepsilon})$, matching the smooth case. More interestingly, from explicit expressions for the leading order terms of the $O(\varepsilon)$ difference of the mean from the deterministic solution, and the $O(\sqrt{\varepsilon})$ standard deviation of solutions, we can identify parameter values for which the $O(\varepsilon)$ difference is larger than, or comparable to, the $O(\sqrt{\varepsilon})$ deviation despite $\varepsilon$ taking a relatively small value. In this scenario the deterministic solution lies outside a relatively large confidence region for the probability distribution of solutions, and thus the difference between the mean solution and the deterministic solution is, roughly speaking, significant. This may cause a change in the global dynamics that is not dominated by randomness constituting a novel noise-induced effect. For a relay control system that exhibits a periodic orbit with sliding motion, such an effect may appear as an observable change in the average amplitude or oscillation time, and represent a lack of reliability or robustness of the output signal of the control system.

Here we outline the remainder of the paper. In §2 we overview problems of noise in nonsmooth systems. We motivate our work in §3 by illustrating that noise may significantly reduce the amplitude and oscillation time of a periodic solution to a prototypical relay control model that involves segments of sliding motion. In §4 we introduce a system of two-dimensional stochastic differential equations, (8)-(11), that describes stochastically perturbed sliding motion near a linear switching manifold in the case that system is the same in directions tangent to the switching manifold. In this case, the equation for motion in the direction orthogonal to the switching manifold, $x$, is independent of the variable representing displacement tangent to the switching manifold, $y$, and for this reason is amenable to an exact analysis. We leave a description of more general scenarios for subsequent work. In §5 we analyze the stochastic differential equation for $x(t)$ and derive its quasi-steady-state distribution. In §6 we analyze the equation for $y(t)$ and obtain expressions for the mean and variance of $y(t)$. Derivations for this section are given in §7 and Appendix A. Conclusions are presented in §8.

2. Aspects of noise in nonsmooth systems. Over the past two decades a wide variety of new dynamical behaviour has been described and explained in deterministic nonsmooth systems; for details we refer the reader to [6, 27] and references
within. Examples include stable periodic orbits with sliding, complex dynamics near singularities of Filippov systems, period-adding and period-incrementing cascades born in grazing bifurcations, and novel routes to chaos. However, for the most part, theoretical aspects of nonsmooth systems have been analyzed in the absence of noise, yet randomness and uncertainty abound in the physical applications of these systems. Already some dynamical phenomena have been described that can only be explained with stochastic nonsmooth equations. An early example is the observation that small noise has a large effect near border-collision bifurcations at which multiple attractors converge to a point [28]. Yet many open problems remain. Here we overview basic problems of noise in nonsmooth systems and the mathematical approaches that have been used, or may need to be developed, to address them. Due to the diversity of nonsmooth dynamical systems and the range of ways by which randomness and uncertainty can be incorporated, both the problems and the relevant mathematical tools are similarly diverse.

For stochastic maps with bounded noise, bounds on invariant densities may be determined by considering the extremal values of the noise. In [29], this was achieved for a one-dimensional, piecewise-linear map with additive uniform noise. The results help to explain novel noise-induced transitions of piecewise-smooth systems of stochastic differential equations [30]. Alternatively, if randomness is incorporated only in the switching condition of a piecewise-smooth map, then the methods of iterated function systems may be applied [31]. In [31], Glendinning applies theorems that have been established for such systems to derive a bound on solutions to a two-dimensional, piecewise-linear map with bounded variability in the switching condition. Furthermore, numerical simulations indicate that if the map exhibits a dangerous bifurcation – for which there exists a unique stable fixed point everywhere except at the bifurcation [32, 33, 34, 35] – with randomness the range of parameter values over which there is no attractor widens to an interval [31]. Stochastic one-dimensional, piecewise-linear maps have been treated in other isolated investigations [36, 37, 38, 39].

For vibro-impacting systems, oscillations involving recurring low-velocity impacts are often well-modelled by a piecewise-smooth map with a square-root singularity [40, 41, 42, 43]. For this reason, in the presence of noise independent to the state of the system, the variability of motion may be proportional to the square-root of the noise amplitude, which for small noise represents an extreme effect [44]. There is a longer history of studies of vibro-impacting systems for which a wide range of impact velocities is expected [45, 46]. In this context, stochastic averaging and asymptotic expansions have been utilized to obtain quantitative results regarding the effects of noise [47, 48]. Averaging is performed over the natural period of the oscillator, which is fundamentally different to the averaging concepts that we have employed in §6. Also, Zhuravlev’s transformation [49] may be used be convert a stochastic equation of motion with an impact rule into a single discontinuous stochastic differential equation.

From the viewpoint of control, understanding stochasticity is particularly important in regards to the robustness of an output signal to noise. Moreover the governing equations will be nonsmooth if control is imposed via a switching rule, as is often the case because switching control is highly efficient and simple to implement [50]. Often, achievement of the control objective equates to stabilizing an equilibrium solution. In this case, robustness refers to the ability of the equilibrium solution to remain stable, in a generalized sense, despite the addition of
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Noise or uncertainty functions. Conditions for robustness have been derived for hybrid control systems [51, 52, 53, 54] and various new switching control laws with enhanced robustness have been created [55, 56, 57]. Lyapunov functions are an invaluable mathematical tool in this context [58]. Sliding mode control specifically utilizes sliding motion to achieve the control objective [59, 60]. In the presence of noise that vanishes at the equilibrium solution, conditions for maintained stability of the equilibrium solution by sliding mode control have been identified by relating stability to matrix inequalities [61, 62, 63, 64].

Coulomb’s discontinuous contact laws provide a simple and useful model of dry friction in mechanical systems [65, 66]. Experimental studies of objects with dry friction and random external forcing have revealed that in order to accurately model the rate of diffusion for the overall motion, Coulomb’s laws should not be neglected [67, 68]. Simple models of dry friction with noise were introduced by de Gennes and Hayakawa [69, 70]. In this context the transitional PDF and other aspects of the stochastic dynamics have been described by using eigenfunction expansions [71], a path integral methodology [72, 73], and Laplace transforms [74].

The notion of noise in piecewise-smooth systems with sliding raises several issues that have not been addressed. In this paper we perform an asymptotic expansion to determine the basic properties of stochastically perturbed sliding motion. This gives a first indication of how noise-induced effects in sliding motion can influence global dynamics. For a complete picture it is necessary to also analyze transitions between sliding motion and regular dynamics, as well as the dynamics away from the sliding region. An understanding of these components of the dynamics in the presence of noise requires separate asymptotic expansions, each with different scaling laws. A determination of the global stochastic dynamics then requires the computationally intensive task of combining the various asymptotic results; this is work in progress [75].

Sliding bifurcations [76] – at which a periodic orbit collides with a switching manifold and develops a sliding segment – remain to be explored in the presence of noise. An understanding of noisy sliding bifurcations requires a different approach to that of this paper because sliding segments near sliding bifurcations are short, whereas here we have analyzed sliding segments that are robust. Important advances can probably be achieved by analyzing stochastic versions of induced Poincaré maps for sliding bifurcations [77] in a manner similar to that recently performed for regular grazing bifurcations [44]. Since sliding motion is not permitted near regular grazing bifurcations and sliding bifurcations exhibit scaling laws different to the square-root scaling law of regular grazing, the effects of noise on the two bifurcations will be fundamentally different.

Sliding motion occurs near singularities of Filippov systems known as two-folds from which forward evolution is ambiguous [78, 79]. By adding small noise, the ambiguity is replaced by a probabilistic solution and it remains to determine properties of this solution. This is likely to be relatively difficult because at a two-fold the vector field is tangent to the switching manifold on both sides of the manifold and each tangency adds nonlinearity to the problem. In addition, at least three spatial dimensions are required for genericity [80].

It remains to investigate both large noise and large deviations due to small noise that may cause sample trajectories to miss the switching manifold altogether. In this paper we only consider small noise; large deviations require separate mathematical methods [81, 25]. Also the study of chaotic dynamics would likely require additional
considerations; for example, geometric calculations related to directions of dilation and contraction can play an important role in the influence of the noise [82, 83]. Overall the issues of sliding motion and noise are diverse, and given the range of mathematical techniques outlined above, it seems certain that their resolution necessitates a zoo of different methods far beyond those utilized in the present paper.

3. Periodic orbits with sliding and relay control. Broadly speaking, a relay control system is a system that aims to control a state variable using the measurements of an input signal via a switching action [84, 85, 86, 87]. Relay control systems are commonly modelled by

\[
\begin{align*}
\dot{x} &= Ax + Bu, \\
\varphi &= C^T x, \\
u &= -\text{sgn}(\varphi),
\end{align*}
\]  

where \( x \in \mathbb{R}^N \) represents the state of the system, \( \varphi \) is the signal measurement and \( u \) is the control response, [4, 6, 21]. The system (4) is a Filippov system with a single switching manifold, \( \{ x \mid C^T x = 0 \} \), on which sliding may occur. In this system sliding corresponds to the idealized scenario of discrete switching events occurring continuously in time. Periodic orbits of (4) that involve sliding are described in [22, 88, 89, 90]. Periodic orbits with sliding have also been identified in ecological models [91, 92, 93, 94] and stick-slip oscillators [20, 95].

As an example we consider the following canonical form, taken from [22] (also given in [6]),

\[
A = \begin{bmatrix} -20\zeta - \frac{1}{27} & 1 & 0 \\ -\zeta - 100 & 0 & 1 \\ -5 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix},
\]  

where \( \zeta \in \mathbb{R} \) is a parameter. Fig. 2 illustrates a stable periodic orbit of (4) with (5) involving sliding motion on the switching manifold. There are 12 separate sliding segments per period. These correspond to time intervals for which \( x_1 \) (the first component of the vector, \( x \)) is zero. The stability of periodic orbits with sliding may be determined by analyzing the Jacobian of a return map [22, 90, 96]. The robustness of periodic orbits with sliding has been investigated numerically by studying the size of the basin of attraction of the periodic orbit [96], and imposing a short time between consecutive switching events [97]. The effect of noise on more complicated structures involving sliding, such as chaotic attractors, has not been explored.

Fig. 3-A shows a typical orbit for the system when small noise is added to the control signal as

\[
dx = \left( Ax - B \text{sgn} \left( C^T x \right) \right) dt + \sqrt{\varepsilon} B dW(t).
\]  

By comparing with Fig. 2-B, this shows that the noise may dampen the large oscillations of \( x_3 \) and decrease the average oscillation time. Fig. 3-B shows that the average length of time per oscillation of \( x_3 \) decreases as the noise amplitude increases. We have observed similar behaviour for different values of \( \zeta \).

The motivation for the calculations that follow is a determination of the cause of this noise-induced phenomenon, where small noise causes a large change in the characteristics of the oscillations. Below we analyze segments of sliding motion and in §8 use the results to speculate on the mechanism driving this behaviour.
Figure 2. A stable periodic orbit of (4) with (5) and $\zeta = -0.06$. Here $x = (x_1, x_2, x_3)^T$. The periodic orbit exhibits sliding on the switching manifold, $x_1 = 0$.

However, as mentioned in the previous section, a quantitative calculation of the expected oscillation time requires both analyzing the transition from stochastically perturbing sliding motion to regular dynamics (distant from the switching manifold), and deriving first passage statistics for return to the switching manifold, and is work in progress.

4. A simple set of equations for stochastically perturbed sliding motion.

We are interested in the effects of noise on the dynamics of (1) near a switching manifold. In this section we introduce a simple set of equations that approximates (1) near a switching manifold, then formulate the inclusion of noise.

If a switching manifold of (1) is locally $C^K$ we can choose a coordinate system such that in a neighbourhood of the origin, $x = 0$, the switching manifold is simply $e_1^T x = O(K)$ [49]. In this paper we are not concerned with effects due to nonsmoothness in the switching manifold and for this reason suppose that the switching manifold is the coordinate plane, $e_1^T x = 0$. Two smooth subsystems govern the nearby flow, thus, locally, we may write the deterministic system as

$$
\dot{x} = \begin{cases} 
F^{(L)}(x), & e_1^T x < 0 \\
F^{(R)}(x), & e_1^T x > 0 
\end{cases},
$$

(7)
Figure 3. Panel A shows a time series of (6) with $\varepsilon = 0.001$, using the same parameter values as in Fig. 2. Panel B shows the median, upper quartile and lower quartile values of 1000 numerically computed times for an oscillation of $x_3$, for several values of $\varepsilon$. To obtain each oscillation time, we computed an orbit up to $t = 100$ and identified the last three instances at which the value of $x_3$ changed sign (discounting rapid sign changes over a handful of grid points near this value due to noise), then subtracted the first time from the third time. Orbits were computed with the Euler-Maruyama method of fixed step size, $\Delta t = 0.0001$.

where $F^{(L)}$ and $F^{(R)}$ are, say, $C^1$.

Suppose there exists a section of the switching manifold, call it $\Sigma$, for which $e_1^T F^{(L)}(x) > 0$ and $e_1^T F^{(R)}(x) < 0$, as in Fig. 1-A. In this scenario, forward orbits arrive at $\Sigma$ from either side. We use Filippov’s definition to define dynamics constrained to $\Sigma$ [6, 11, 18]. $\Sigma$ is known as an attracting sliding region and evolution on $\Sigma$ is referred to as sliding motion.

For simplicity, for this paper, which is a first detailed analysis of sliding motion with noise, we ignore the dependency of $F^{(L)}$ and $F^{(R)}$ on components of $x$ parallel to $\Sigma$ as this enables us to reduce mathematical problems to one dimension but still capture what seems to be the essence of stochastically perturbed sliding motion. Moreover, this provides a useful approximation to the general case over short time-frames. With general $F^{(L)}$ and $F^{(R)}$ we expect the magnitude and direction
of perturbations due to spatially independent noise to differ across the attracting sliding region. We leave such complications for future work.

Given that \( F^{(L)} \) and \( F^{(R)} \) are functions of only \( e_1^T x \), the remaining \( N - 1 \) components of \( x \) may be treated identically and for this reason it suffices to study a two-dimensional system. We write \( x = (x, y) \) and add small amplitude, white, Gaussian noise independent to the state of the system. For simplicity we assume that the noise in \( x \) is independent to the noise in \( y \). (Additional effects regarding the strength of the noise response can arise if the noise is correlated, but we ignore these here.) The resulting stochastic differential equation may be written as

\[
\frac{dx}{dy} = \left[ \begin{array}{c} \phi(x) \\ \psi(x) \end{array} \right] dt + \sqrt{\varepsilon} \left[ \begin{array}{c} dW_1(t) \\ \sqrt{\kappa} dW_2(t) \end{array} \right],
\]

(8)

where \( W_1(t) \) and \( W_2(t) \) are independent Brownian motions, \( 0 < \varepsilon \ll 1 \) and \( \kappa > 0 \) are constants, and \( \phi \) and \( \psi \) are piecewise-\( C^1 \) that for small \( |x| \) are given by

\[
\phi(x) = \left\{ \begin{array}{ll} a_L + c_L x + o(|x|), & x < 0 \\ -a_R + c_R x + o(|x|), & x > 0 \end{array} \right.,
\]

(9)

\[
\psi(x) = \left\{ \begin{array}{ll} b_L + d_L x + o(|x|), & x < 0 \\ b_R + d_R x + o(|x|), & x > 0 \end{array} \right..
\]

(10)

We assume

\[
a_L, a_R > 0,
\]

(11)

to ensure that in the absence of noise the switching manifold \((x = 0)\) is an attracting sliding region. Since \( \phi \) and \( \psi \) are independent of \( \varepsilon \), their coefficients are \( O(1) \). Consequently, for \( x(0) \) near zero, orbits of (8) likely remain near \( x = 0 \) for relatively long periods of time, as shown in §5.1, and for this reason we do not specify the behaviour of \( \phi \) and \( \psi \) for large \( |x| \).

The existence and uniqueness of solutions to stochastic differential equations with an irregular drift coefficient, such as (8), is a complex and active area of research. Different classes of solutions (e.g., weak solutions, strong solutions, Sobolev diffeomorphisms) are known to exist under various general assumptions [98, 99, 100, 101, 102, 103, 104]. In particular, if \( \phi \) and \( \psi \) are also bounded, (8) has a unique strong solution.

5. Properties of \( x(t) \). Since (8) lacks dependency on \( y \), the equation for \( dx \) is decoupled from \( y \):

\[
dx = \phi(x) \, dt + \sqrt{\varepsilon} \, dW_1(t)
\]

(12)

Given \( x(0) = x_0 \), let \( p_r(x, t | x_0) \) denote the transitional PDF for the value of \( x(t) \), as governed by (12). Despite the discontinuity at \( x = 0 \), \( p_r(x, t | x_0) \) is unique and continuous. For \( x \neq 0 \) and \( t > 0 \), the PDF satisfies the Fokker-Planck equation

\[
\frac{\partial p_r}{\partial t} = -\frac{\partial (\phi p_r)}{\partial x} + \varepsilon \frac{\partial^2 p_r}{\partial x^2},
\]

(13)

with the initial condition \( p_r(x, 0 | x_0) = \delta(x - x_0) \), [23, 24, 105]. In §7 we provide an explicit expression for \( p_r(x, t | x_0) \) in the special case that \( \phi \) is piecewise-constant. For the remainder of this section we use (13) to determine the long time behaviour of \( x(t) \).
If \( \phi(x) > 0 \) for all \( x < 0 \), and \( \phi(x) < 0 \) for all \( x > 0 \), then (12) has a steady-state density on \( \mathbb{R} \) centred about the origin. Otherwise, \( \phi(x) = 0 \) for some \( x \neq 0 \), and with nonzero probability orbits may cross this value of \( x \) and undergo dynamics far from the origin not described by the expansion (9). However, regardless of the global nature of \( \phi \), since \( \varepsilon \) is small the local attraction to the origin is relatively strong. Thus we expect orbits to remain near the origin for long periods of time and be distributed by a quasi-steady-state distribution for large but finite \( t \). In §5.1 we determine the mean escape time of orbits from an \( O(1) \) neighbourhood of the origin. In §5.2 we use (13) to derive the quasi-steady-state probability density function asymptotically.

5.1. **Escape from a neighbourhood of** \( x = 0 \). For the function \( \phi(x) \), (9), with (11), in the case that \( \phi(x) = 0 \) for some \( x \neq 0 \), it is necessary to identify a value \( x_b > 0 \), independent of \( \varepsilon \), such that

\[
\min_{|x| \leq x_b} |\phi(x)| \geq \frac{1}{2} \min(a_L, a_R). \tag{14}
\]

Then in the interval \([−x_b, x_b]\), the drift of (12) is towards \( x = 0 \). With \( \varepsilon > 0 \) and any initial condition \( x_0 \in (−x_b, x_b) \), \( x(t) \) will eventually escape \([−x_b, x_b] \) with probability 1. Calculating the time to escape is a standard problem in the context of a single potential well, where the potential function is given by

\[
U(x) = −\int_0^x \phi(y) \, dy. \tag{15}
\]

The mean escape time, \( \mathcal{T}(x_0) \), may be found exactly [24, 26, 106]. Via Laplace’s method of asymptotic evaluation of integrals [107], it follows that whenever \( |x_0| < x_b \) there exist \( \varepsilon \)-independent constants \( \alpha_1 \) and \( \alpha_2 \) such that

\[
\mathcal{T}(x_0) \sim \varepsilon \alpha_1 e^{\alpha_2}. \tag{16}
\]

For instance if \( U(x_b) < U(−x_b) \) (other cases are similar),

\[
\mathcal{T}(x_0) \sim \frac{\varepsilon (a_L + a_R)}{2a_L a_R U'(x_b)} e^{\frac{2U(x_b)}{\varepsilon}}. \tag{17}
\]

5.2. **The quasi-steady-state probability density function.** In §5.1 we showed that the mean escape time from an \( \varepsilon \)-independent neighbourhood \([−x_b, x_b]\) is exponentially large in \( \frac{1}{\varepsilon} \). Consequently, we can assume that the probability an orbit escapes \([−x_b, x_b]\) within the polynomial time \( \frac{1}{\varepsilon M} \), for any fixed \( M > 0 \), is extremely small. Our choice of polynomial time is primarily for convenience. The assumptions imposed in §4 diminish the utility of the model for time-frames longer than \( O(1) \). We let

\[
\tilde{t} = \varepsilon^M t, \tag{18}
\]

represent the long time scale, and look for a solution to the Fokker-Planck equation (13) as a function of \( x \) and \( \tilde{t} \). By substituting (18) with the WKB-type expansion [23, 24],

\[
p_\varepsilon(x, \tilde{t}|x_0) = e^{\frac{\ddot{q}_\varepsilon(x_\tilde{t}|x_0)}{\varepsilon}}, \tag{19}
\]

into (13), we arrive at

\[
\varepsilon^M \frac{\partial p_\varepsilon}{\partial \tilde{t}} = \left( \frac{\partial q_\varepsilon}{\partial x} + \varepsilon \frac{\partial}{\partial x} \left( \frac{1}{2} \frac{\partial q_\varepsilon}{\partial x} - \phi \right) \right). \tag{20}
\]
Therefore
\[
\frac{\partial q_\varepsilon(x, \varepsilon^{-M}\hat{t}|x_0)}{\partial x} = 2\phi(x) + O(\varepsilon^M), \tag{21}
\]
and so by integrating \(\phi(x)\) we obtain
\[
q_\varepsilon(x, \varepsilon^{-M}\hat{t}|x_0) = r(\hat{t}) + \begin{cases}
2a_L x + c_L x^2 + o(x^2), & x < 0 \\
-2a_R x + c_R x^2 + o(x^2), & x > 0
\end{cases} + O(\varepsilon^M), \tag{22}
\]
for an \(\varepsilon\)-independent function \(r\). As a function of \(x\) and \(t\), the dependence of this solution on \(t\) and \(x_0\) appears only in \(O(\varepsilon^M)\) terms which may be ignored. Consequently we treat the solution as solely a function of \(x\) and refer to it as the quasi-steady-state solution, \(\rho_{\text{qss}}(x)\). Specifically \((19)\) and \((22)\) combine to give
\[
\rho_{\text{qss}, \varepsilon}(x) = \begin{cases}
\frac{K_{\varepsilon}}{\varepsilon}e^{-\frac{1}{2}(2a_L x + c_L x^2 + o(x^2)) + O(\varepsilon^{M-1})}, & x < 0 \\
\frac{K_{\varepsilon}}{\varepsilon}e^{-\frac{1}{2}(-2a_R x + c_R x^2 + o(x^2)) + O(\varepsilon^{M-1})}, & x > 0
\end{cases}, \tag{23}
\]
where we must have
\[
K_{\varepsilon} = \frac{2a_L a_R}{a_L + a_R} - \frac{a_L^2 c_R + a_R^2 c_L}{a_L a_R (a_L + a_R)^2} \varepsilon + o(\varepsilon), \tag{24}
\]
to ensure \(\rho_{\text{qss}, \varepsilon}\) is normalized.

For small \(\varepsilon\) and \(x_0\), the transitional PDF of \((12)\), \(p_\varepsilon(x,t|x_0)\), quickly settles to \((23)\). The scaling
\[
\hat{x} = \frac{x}{\varepsilon}, \quad \hat{t} = \frac{t}{\varepsilon}, \tag{25}
\]
transforms \((12)\) to
\[
d\hat{x} = \phi(\hat{x}) \, d\hat{t} + dW_1(\hat{t}), \tag{26}
\]
from which we infer that \(p_\varepsilon(x,t|x_0)\) approaches \((23)\) on an \(O(\varepsilon)\) time-scale, when \(x_0 = O(\varepsilon)\). Furthermore, for times in the range \(\varepsilon^{1-\delta} \leq t \leq \varepsilon^{-M}\), where \(\delta > 0\), it is reasonable to suppose \(x \sim \rho_{\text{qss}, \varepsilon}\), in which case
\[
\langle \text{sgn}(x) \rangle = \frac{a_L - a_R}{a_L + a_R} + \frac{a_L^2 c_R - a_R^2 c_L}{a_L a_R (a_L + a_R)^2} \varepsilon + o(\varepsilon), \tag{27}
\]
\[
\langle x \rangle = \frac{a_L - a_R}{2a_L a_R} \varepsilon + O(\varepsilon^2), \tag{28}
\]
\[
\langle x \, \text{sgn}(x) \rangle = \frac{a_L^2 + a_R^2}{2a_L a_R (a_L + a_R)} \varepsilon + O(\varepsilon^2), \tag{29}
\]
which are useful in the next section.

6. Moments of \(y(t)\). In this section we compute the mean of \(y(t)\) and conjecture the leading order term of its variance. We assume \(x \sim \rho_{\text{qss}, \varepsilon}\) at all times under consideration which greatly simplifies calculations. We begin by deriving \(y(t)\) when \(\varepsilon = 0\).

6.1. Deterministic sliding motion. When \(\varepsilon = 0\), \((8)\) is the Filippov system:
\[
\begin{bmatrix}
\dot{x} \\
\dot{y}
\end{bmatrix} = \begin{bmatrix}
\phi(x) \\
\psi(x)
\end{bmatrix}. \tag{30}
\]
As in \([18, 11, 6]\), we define a vector field for sliding motion on the switching manifold \((x = 0)\) by the unique convex combination of the two vector fields at the manifold that is tangent to the manifold. That is,
\[
\begin{bmatrix}
\dot{x} \\
\dot{y}
\end{bmatrix}_{\text{slide}} = (1 - q) \begin{bmatrix}
a_L \\
b_L
\end{bmatrix} + q \begin{bmatrix}
a_R \\
b_R
\end{bmatrix}, \tag{31}
\]
for the unique scalar, \( q \in (0, 1) \), for which \( \dot{x}_{\text{slide}} = 0 \). Solving \( \dot{x}_{\text{slide}} = 0 \) gives

\[
q = \frac{a_L}{a_L + a_R}
\]

and therefore

\[
\dot{y}_{\text{slide}} = \frac{a_R b_L + a_L b_R}{a_L + a_R}.
\]  

(32)

Consequently, if \((x(0), y(0)) = (0, y_0)\), then \(x_{\text{slide}}(t) \equiv 0\) and

\[
y_{\text{slide}}(t) = y_0 + \frac{a_R b_L + a_L b_R}{a_L + a_R} t.
\]  

(33)

6.2. The mean of \( y(t) \). From (8) and (10) we have

\[
\begin{aligned}
dy &= \left( \frac{b_L + b_R}{2} - \frac{b_L - b_R}{2} \text{sgn}(x(t)) + \frac{d_L + d_R}{2} x(t) \\
&\quad - \frac{d_L - d_R}{2} x(t) \text{sgn}(x(t)) + o(|x(t)|) \right) dt + \sqrt{\varepsilon R} dW_2(t).
\end{aligned}
\]  

(34)

This is the same as

\[
\begin{aligned}
y(t) &= y_0 + \frac{b_L + b_R}{2} t - \frac{b_L - b_R}{2} \int_0^t \text{sgn}(x(s)) \, ds + \frac{d_L + d_R}{2} \int_0^t x(s) \, ds \\
&\quad - \frac{d_L - d_R}{2} \int_0^t x(s) \text{sgn}(x(s)) \, ds + \int_0^t o(|x(s)|) \, ds + \sqrt{\varepsilon R} W_2(t),
\end{aligned}
\]

and therefore

\[
\begin{aligned}
\langle y(t) \rangle &= y_0 + \frac{b_L + b_R}{2} t - \frac{b_L - b_R}{2} \int_0^t \langle \text{sgn}(x(s)) \rangle \, ds + \frac{d_L + d_R}{2} \int_0^t \langle x(s) \rangle \, ds \\
&\quad - \frac{d_L - d_R}{2} \int_0^t \langle x(s) \text{sgn}(x(s)) \rangle \, ds + \int_0^t \langle o(|x(s)|) \rangle \, ds.
\end{aligned}
\]

If \( x \sim p_{\text{qss}, \varepsilon} \), by substituting (27)-(29) we obtain

\[
\langle y(t) \rangle = y_{\text{slide}}(t) + \frac{(a_L^2 d_R - a_R^2 d_L)(a_L + a_R) - (a_L^2 c_R - a_R^2 c_L)(b_L - b_R)}{2a_L a_R (a_L + a_R)^2} \varepsilon t + o(\varepsilon),
\]  

(35)

where the \( \varepsilon \)-independent terms have combined to form \( y_{\text{slide}}(t) \), (33).

Therefore as \( \varepsilon \to 0 \), the mean of \( y(t) \) limits to Filippov’s sliding solution, \( y_{\text{slide}}(t) \). This is non-trivial because Filippov’s method, to obtain (33), and standard stochastic dynamical systems definitions, to obtain (35), are not immediately related. Analogous results have been shown in more general settings [108, 109].

The perturbation of \( \langle y(t) \rangle \) from \( y_{\text{slide}}(t) \) is order \( \varepsilon \), mirroring the result for smooth systems, see §1. The explicit expression for the coefficient of the \( O(\varepsilon) \)-term in (35) is particularly useful. For instance, we can see that if \( c_L = c_R \) and \( d_L = d_R \), then we require the asymmetry \( a_L \neq a_R \) in order for the \( O(\varepsilon) \)-term to be nonzero. The above calculations are a form of stochastic averaging [7, 25, 110] where the value of \( x(t) \) changes rapidly and is averaged according to the density, \( p_{\text{qss}, \varepsilon} \). A more sophisticated implementation of stochastic averaging may be used to address more general systems of equations.

6.3. The variance of \( y(t) \). The variance of \( y(t) \) may be computed via

\[
\text{Var}(y(t)) = \langle y(t)^2 \rangle - \langle y(t) \rangle^2,
\]  

(36)

however this requires knowledge of \( p_e(x, t|x_0) \), for which we have not been able to obtain a useful expression in the case of general \( \phi \). We conjecture that the leading order terms of \( \text{Var}(y(t)) \) are independent of non-constant terms in \( \phi \) and \( \psi \).
because $x(t) = O(\varepsilon)$ with high probability. Indeed this is consistent with numerical simulations, Fig. 4. In view of the result for the case that $\phi$ and $\psi$ are piecewise-constant (Theorem 6.1, given below), we propose the following result:

**Conjecture 1.** Consider (8) with (11). Suppose $x(0)$ is random with PDF, $p_{qss,\varepsilon}$, and $y(0) = y_0$. Then for any $\delta > 0$, whenever $\varepsilon^{1 - \delta} \leq t \leq \varepsilon^{-M}$ we have

$$\text{Var}(y(t)) = \varepsilon \kappa t + \frac{(b_L - b_R)^2}{(a_L + a_R)^2} \varepsilon t + O(\varepsilon^2).$$

(37)

The above results can quantitatively describe a substantial perturbation of the mean of the stochastic solution, $\langle y(t) \rangle$, from the deterministic solution, $y_{\text{slide}}$. As an example we take the functions $\phi$ and $\psi$ to be piecewise-linear, using the same parameter values as in Fig. 4, and $(c_L, c_R, d_L, d_R) = (0, 0, 0, 1)$. By (35), the difference, $\langle y(t) \rangle - y_{\text{slide}}$, is equal to $\frac{1}{2} + O(\varepsilon)$, and (37) gives $\text{Std}(y(t)) = \frac{\sqrt{2}}{\varepsilon} + O(\varepsilon)$. The numerical computation of $10^6$ orbits of this system with $\varepsilon = 0.1$ gives $\langle y(t) \rangle - y_{\text{slide}} = 0.0397$ and $\text{Std}(y(t)) = 0.0879$ to three significant figures (repetitions of this numerical experiment yield similar values). We conclude that the formulas (35) and (37) provide the moments of $y(t)$ to reasonable accuracy in an example for which the mean of $y(t)$ is perturbed noticeably from $y_{\text{slide}}$.

From this point in the paper until the concluding section, § 8, we study the case that $\phi$ and $\psi$ are piecewise-constant, i.e.

$$\phi(x) = \begin{cases} a_L, & x < 0 \\ -a_R, & x > 0 \end{cases}$$

(38)

and

$$\psi(x) = \begin{cases} b_L, & x < 0 \\ b_R, & x > 0 \end{cases}$$

(39)

**Theorem 6.1.** Consider (8) with (11) and suppose $\phi$ and $\psi$ are given by (38)-(39). Suppose $x(0)$ is random with PDF, $p_{qss,\varepsilon}$, and $y(0) = y_0$. Then for any $\delta > 0$, whenever $t \geq \varepsilon^{1 - \delta}$ we have

$$\text{Var}(y(t)) = \varepsilon \kappa t + \frac{(b_L - b_R)^2}{(a_L + a_R)^2} \varepsilon t + O(\varepsilon^2).$$

(40)

We prove this result in the next section.

7. **Two-valued drift and a proof of Theorem 6.1.** The stochastic differential equation (12) with (38):

$$dx = \begin{cases} a_L, & x < 0 \\ -a_R, & x > 0 \end{cases} dt + \sqrt{\varepsilon} dW_1(t),$$

(41)

has been referred to as Brownian motion with two-valued drift. Below we state an expression for transitional PDF of (41) which was first derived by Karatzas and Shreve in [111] for $\varepsilon = 1$ via Girsanov’s theorem [112, 113, 114] and the trivariate PDF of Brownian motion, its positive occupation time, and its local time about zero. The result for $\varepsilon \neq 1$ follows simply from the scaling (25). We then use this PDF to prove Theorem 6.1. (Our expression for the PDF, (42), is equivalent to the slightly different expression given in [111] (equation 5.7), as may be shown by a brief algebraic manipulation.)
7.1. The transitional probability density function for \( x(t) \). The transitional PDF for (41) is given by

\[
p_c(x,t|x_0) = \begin{cases} 
\frac{2}{\pi} e^{\frac{2aLx}{\pi} t} \int_0^\infty h_c(t, b, a_R) * h_c(t, b - x, a_L) db & x_0 \leq 0, x \leq 0 \\
\frac{2}{\pi} e^{-\frac{2aLx}{\pi} t} \int_0^\infty h_c(t, b + x, a_R) * h_c(t, b - x, a_L) db & x_0 \leq 0, x \geq 0 \\
\frac{2}{\pi} e^{\frac{2aLx}{\pi} t} \int_0^\infty h_c(t, b + x + x_0, a_R) * h_c(t, b, a_L) db & x_0 \geq 0, x \leq 0 \\
\frac{2}{\pi} e^{-\frac{2aLx}{\pi} t} \int_0^\infty h_c(t, b + x + x_0, a_R) * h_c(t, b, a_L) db & x_0 \geq 0, x \geq 0 
\end{cases}
\]

(42)

where

\[
h_c(t, x_0, \mu) = \frac{|x_0|}{\sqrt{2\pi t}} e^{-\frac{(x_0-\mu)^2}{2t}}.
\]

(43)

is the PDF for the first passage time to zero of Brownian motion with constant drift,

\[
G_{\text{absorb},c}(x, t, \mu|x_0) = \frac{1}{\sqrt{2\pi t}} e^{\frac{(x-x_0-\mu)^2}{2t}} - e^{\frac{-2ax_0}{\pi} t} \frac{1}{\sqrt{2\pi t}} e^{-\frac{(x+x_0-\mu)^2}{2t}}.
\]

(44)

is the transitional PDF for Brownian motion with constant drift and an absorbing boundary condition at zero, and

\[
f_1(t) * f_2(t) = \int_0^t f_1(\tau) f_2(t - \tau) d\tau,
\]

(45)

is the convolution relating to Laplace transforms. Fig. 5 shows (42) at different times. The PDF (42) has arisen in other contexts. In [115, 116], the authors derive \( p_c(0,t|x_0) \) and use this to bound PDFs for a large class of scalar stochastic differential equations. In [117], \( p_c(x, t|0) \) is derived for \( a_L = a_R = -1 \). In addition, Zhang [118] derived an expression for the transitional PDF of Brownian motion.
with a general bounded piecewise-continuous drift function. We speculate that this could be used to analyze the PDF of (12) with general \( \phi \) asymptotically.

7.2. Proof of Theorem 6.1. In the case of two-valued drift (38), the quasi-steady-state density (23) is a true steady-state defined for all \( x \in \mathbb{R} \):

\[
p_{\text{ss}, \varepsilon}(x) = \begin{cases} 
K \varepsilon e^{2aL \varepsilon x}, & x < 0 \\
K \varepsilon e^{-2aR \varepsilon x}, & x > 0
\end{cases}, \quad K = \frac{2 a_L a_R}{a_L + a_R}.
\]  

(46)

With the notation

\[
\frac{\partial p^\pm_x}{\partial x}(0, t \mid x_0) \equiv \lim_{\Delta \to 0^\pm} \frac{\partial p_x}{\partial x}(\Delta, t \mid x_0),
\]

(47)

we have the following expression for the probability that \( x(t) > 0 \), given \( x_0 = 0 \).

**Lemma 7.1.** For any \( t > 0 \),

\[
\int_{0}^{\infty} p_x(x, t \mid x_0) \, dx = \frac{1}{2} - \int_{0}^{t} a_R p_x(0, s \mid 0) + \varepsilon \frac{\partial p^+_x}{\partial x}(0, s \mid 0) \, ds.
\]

(48)

**Proof.** For \( x > 0 \), in the case of two-valued drift (38), the Fokker-Planck equation (13) is

\[
\frac{\partial p_x}{\partial t} = \frac{\partial}{\partial x} \left( a_R p_x + \varepsilon \frac{\partial p^+_x}{\partial x} \right).
\]

(49)

Integration over \( x \in [\Delta, \infty) \) yields

\[
\frac{\partial}{\partial t} \int_{\Delta}^{\infty} p_x(x, t \mid 0) \, dx = - \left( a_R p_x(\Delta, t \mid 0) + \varepsilon \frac{\partial p^+_x}{\partial x}(\Delta, t \mid 0) \right),
\]

(50)

for any \( \Delta > 0 \). Then integrating with respect to \( t \) and taking \( \Delta \to 0 \) produces (48) where we also use

\[
\lim_{t \to 0^+} \int_{0}^{\infty} p_x(x, t \mid 0) \, dx = \frac{1}{2},
\]

(51)

which may be demonstrated by noting that as \( t \to 0^+ \), \( p_x(x, t \mid 0) \) is well-approximated by a zero-mean Gaussian.

Proofs of the following two lemmas are given in Appendix A. Theorem 6.1 is an immediate consequence of Lemma 7.3 combined with (27) and (36).

**Figure 5.** The probability density function of \( x(t) \), (42), with \( x_0 = 0.02, a_L = 2, a_R = 1 \) and \( \varepsilon = 0.01 \) at four different times.
Lemma 7.2. For the density (42),
\begin{align}
\int_0^\infty a_R p_\varepsilon(0, t; 0) + \frac{\varepsilon}{2} \frac{\partial p_\varepsilon^+}{\partial x}(0, t; 0) \, dt &= \frac{-(a_L - a_R)}{2(a_L + a_R)}, \\
\int_0^\infty t \left( a_R p_\varepsilon(0, t; 0) + \frac{\varepsilon}{2} \frac{\partial p_\varepsilon^+}{\partial x}(0, t; 0) \right) \, dt &= \frac{-\varepsilon(a_L - a_R)}{2a_L a_R(a_L + a_R)}. 
\end{align}

Lemma 7.3. Consider (41) and suppose \( a_L, a_R > 0 \) and \( x(0) \) is random with PDF, \( p_{\text{ss}, \varepsilon} \). Then for any \( \delta > 0 \), if \( t \geq \varepsilon^{1-\delta} \), we have
\begin{equation}
\int_0^t \int_0^t \langle \text{sgn}(x(s)) \text{sgn}(x(u)) \rangle \, ds \, du = \frac{(a_L - a_R)^2 t^2}{(a_L + a_R)^2} + \frac{4 \varepsilon t}{(a_L + a_R)^2} + O(\varepsilon^2). \tag{54}
\end{equation}

In the special case, \( a_L = a_R = a \), we can write \( \text{Var}(y(t)) \) exactly. In this case, by symmetry, \( \int_0^\infty p_\varepsilon(x, t; 0) \, dx = \frac{1}{a} \), thus by Lemma 7.1, \( a p_\varepsilon(0, t; 0) + \varepsilon \frac{\partial p_\varepsilon^+}{\partial x}(0, t; 0) \equiv 0 \). Then from equations given in the proof of Lemma 7.3, we obtain
\begin{equation}
\text{Var}(y(t)) = \frac{(b_L - b_R)^2}{4} \left( \frac{\varepsilon^2}{a^2} - \frac{\varepsilon^2}{a^4} + \frac{\sqrt{2\pi}}{\sqrt{\varepsilon}} \left( \frac{\varepsilon^2}{a^3} - \frac{2 \varepsilon t}{a^3} - \frac{at^2}{3} \right) + \left( \frac{\varepsilon^2}{a^2} - \frac{\varepsilon t}{a^2} + t^2 + \frac{a t^3}{3\varepsilon} \right) \text{erfc} \left( \frac{a \sqrt{t_0}}{\sqrt{2\varepsilon}} \right) \right), \tag{55}
\end{equation}
which is consistent with (40).

8. Conclusions. A variety of novel dynamical behaviour has recently been described in piecewise-smooth systems [6]. The effects of noise and model uncertainties on these dynamics is an almost completely unexplored topic and in §2 we highlighted several possible directions for future research. To our knowledge this paper is the first detailed study of stochastically perturbed sliding motion. With noise, orbits no longer slide along an attracting sliding section of a switching manifold. Instead, with high probability, orbits follow a random path near the switching manifold, Fig. 1-B. The average size of deviations from the switching manifold is governed by the strength of the noise relative to the magnitude of the vector field in a direction orthogonal to the switching manifold.

The general \( N \)-dimensional stochastic differential equation formed by adding noise to the Filippov system (7) is particularly difficult to analyze due to the discontinuity and unspecified dimensionality. For this reason we made the supposition that the system is invariant along the switching manifold. This prevents an exploitation into the effects of noise on orbits that reach the end of an attracting sliding region (which we anticipate requires a completely different analysis), but enables calculations to be reduced to one dimension. Moreover, with this reduction directions parallel to the switching manifold may be treated identically and hence it is sufficient to study the two-dimensional system (8).

For the system (8), \( x(t) \) denotes the displacement from the switching manifold and is governed by (12) with (9). Sample paths of (12) settle to the quasi-steady-state PDF, \( p_{\text{qss}, \varepsilon} \) (23), on an \( O(\varepsilon) \) time scale, where \( \sqrt{\varepsilon} \) is the noise amplitude. This PDF is not a true steady-state because orbits escape a neighbourhood of zero. However, since this escape occurs on an exponentially long time scale, see §5.1, it is suitable to assume \( x(t) \) is distributed by \( p_{\text{qss}, \varepsilon} \) at times in a given range.
\[ \varepsilon^{1-\delta} \leq t \leq \varepsilon^{-M}, \]
for any \( \delta, M > 0 \), where we may take small \( \delta \) and large \( M \), such that this is a long time interval. This assumption has the benefit of significantly simplifying our calculations.

Equation (34) is the stochastic differential equation for \( y(t) \), which represents displacement along the switching manifold. In the limit, \( \varepsilon \to 0 \), the mean of \( y(t) \), (35), is Filippov’s sliding solution, \( y_{\text{slide}}(t) \). For \( \varepsilon \neq 0 \), the perturbation of \( \langle y(t) \rangle \) from \( y_{\text{slide}}(t) \) is \( O(\varepsilon) \), as in the case of a generic smooth system. In order to gauge the effect of this perturbation on the overall dynamics, it is necessary to compare it to the standard deviation of \( y(t) \). In the case that \( \phi \) and \( \psi \) are piecewise-constant, \( \text{Var}(y(t)) \) is \( O(\varepsilon) \), see Theorem 6.1. For general \( \phi \) and \( \psi \), we conjectured that the leading order terms of \( \text{Var}(y(t)) \) are unchanged and consequently deviations of \( y(t) \) from \( \langle y(t) \rangle \) are \( O(\sqrt{\varepsilon}) \). Therefore, assuming \( \varepsilon \ll 1 \), we expect deviations in \( y(t) \) from \( \langle y(t) \rangle \) to dominate the difference between \( \langle y(t) \rangle \) and the deterministic solution, \( y_{\text{slide}}(t) \).

Although our above calculations are for differential equations that are static along the switching manifold, we can infer results for more general systems such as the relay control system, (4) with (5). A key observation for this system is that \( \frac{\partial x_2}{\partial x_1} \approx -100 \) is an extremely large value, relatively speaking. Since \( x_1 \) represents displacement from the switching manifold and \( x_2 \) is a direction parallel to the switching manifold, in the context of (8)-(10), \( \frac{\partial x_2}{\partial x_1} \) corresponds to the values of \( d_L \) and \( d_R \). These values influence, to lowest order, the perturbation of the mean from the deterministic solution, (35), but not the deviation of sample paths from the mean, (37). Consequently, for moderately small values of \( \varepsilon \), such as those Fig. 3-B, the average perturbation from the deterministic solution may in fact be greater than the deviation of sample paths. This is a possible explanation for the reduction in the oscillation times evident in the figure. A quantitative determination of the oscillation time requires detailed calculations relating to both escape from the proximity of the switching manifold and a return to the switching manifold. These involve asymptotic expansions with different scaling laws and is work in progress [75].

Appendix A. Proofs.

A.1. Proof of Lemma 7.2.

Proof. By Lemma 7.1 and (46),
\[
\int_0^\infty a_Rp_c(0,t(0)) + \varepsilon \frac{\partial p_c^+}{\partial x} (0,t(0)) dt = \frac{1}{2} - \int_0^\infty p_{ss,\varepsilon}(x) dx = \frac{-a_L - a_R}{2(a_L + a_R)},
\]
which verifies (52). To obtain (53), we multiply (50) by \( t \) and take \( \Delta \to 0 \) to obtain
\[
t\frac{\partial}{\partial t} \int_0^\infty p_c(x,t(0)) dx = -t \left( a_Rp_c(0,t(0)) + \frac{\varepsilon}{2} \frac{\partial p_c^+}{\partial x}(0,t(0)) \right).
\]
Integration by parts yields
\[
\int_0^T t \left( a_Rp_c(0,t(0)) + \frac{\varepsilon}{2} \frac{\partial p_c^+}{\partial x}(0,t(0)) \right) dt = \int_0^T \int_0^\infty p_c(x,t(0)) - p_c(x,T(0)) dx dt,
\]
for any \( T > 0 \). Taking \( T \to \infty \) gives
\[
\int_0^\infty t \left( a_Rp_c(0,t(0)) + \frac{\varepsilon}{2} \frac{\partial p_c^+}{\partial x}(0,t(0)) \right) dt = \int_0^\infty \int_0^\infty p_c(x,t(0)) - p_{ss,\varepsilon}(x) dx dt.
\]
To evaluate the double integral we introduce Laplace transforms. For $x > 0$,

$$P_{ss,\varepsilon}(x, \lambda) \equiv \mathcal{L}[p_{ss,\varepsilon}(x)] = \int_0^\infty e^{-\lambda t} K e^{-\frac{2a_R x}{\varepsilon}} dt = \frac{2a_L a_R e^{-\frac{2a_R x}{\varepsilon}}}{\varepsilon(a_L + a_R)\lambda}. \quad (60)$$

The Laplace transform of $h_{\varepsilon}$, (43), is

$$H_{\varepsilon}(h, z, \mu) \equiv \mathcal{L}[h_{\varepsilon}(t, z, \mu)] = e^{\frac{1}{2} \left( \mu^2 - \sqrt{\mu^2 + 2x\lambda z} \right)}, \quad (61)$$

for $\lambda > 0$. Using (42) and expanding about $\lambda = 0$:

$$P_{\varepsilon}(x, \lambda|0) \equiv \mathcal{L}[p_{\varepsilon}(x, t|0)] = \frac{2e^{-\frac{2a_R x}{\varepsilon}}}{\varepsilon} \int_0^\infty H_{\varepsilon}(\lambda, b + x, a_R)H_{\varepsilon}(\lambda, b, a_L) db$$

$$= \frac{2e^{-\frac{2a_R x}{\varepsilon}}}{\varepsilon} \left( \frac{a_L a_R}{(a_L + a_R)\lambda} + \frac{\varepsilon(a_L^3 + a_R^3)}{2a_L a_R(a_L + a_R)^2} - 2a_L a_R(a_L + a_R)x + O(\lambda) \right). \quad (62)$$

By (60) and (62),

$$P_{\varepsilon}(x, \lambda|0) - P_{ss,\varepsilon}(x, \lambda) = \frac{\varepsilon(a_L^3 + a_R^3)}{\varepsilon a_L a_R(a_L + a_R)^2} - \frac{2a_L a_R(a_L + a_R)x}{\varepsilon} + O(\lambda), \quad (63)$$

and thus from (59)

$$\int_0^\infty t \left( a_R p_{\varepsilon}(0, t|0) + \frac{\varepsilon}{2} \frac{\partial p_{\varepsilon}}{\partial x}(0, t|0) \right) dt = \int_0^\infty P_{\varepsilon}(x, 0|0) - P_{ss,\varepsilon}(x, 0) dx$$

$$= \int_0^\infty \frac{\varepsilon(a_L^3 + a_R^3)}{\varepsilon a_L a_R(a_L + a_R)^2} - \frac{2a_L a_R(a_L + a_R)x}{\varepsilon} + O(\lambda) dx$$

$$= \frac{-\varepsilon(a_L - a_R)}{2a_L a_R(a_L + a_R)},$$

as required.

**A.2. Proof of Lemma 7.3.**

**Proof.** Since (41) is time-dependent and $x(0)$ has a steady-state density, $p_{ss,\varepsilon}$, we have,

$$\int_0^t \int_0^t \langle \text{sgn}(x(s)) \text{sgn}(x(u)) \rangle ds du = \int_0^t \int_0^t \langle \text{sgn}(x(0)) \text{sgn}(x(|u - s|)) \rangle ds du$$

$$= 2 \int_0^t (t - u) \langle \text{sgn}(x(0)) \text{sgn}(x(u)) \rangle du. \quad (64)$$

Furthermore,

$$\langle \text{sgn}(x(0)) \text{sgn}(x(u)) \rangle = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \text{sgn}(x)p_{ss,\varepsilon}(x) \text{sgn}(y)p_{\varepsilon}(y, u|x) dy dx$$

$$= -\int_{-\infty}^{\infty} Ke^{\frac{2a_R x}{\varepsilon}} \int_{-\infty}^{\infty} \text{sgn}(y)p_{\varepsilon}(y, u|x) dy dx$$

$$+ \int_{-\infty}^{0} Ke^{-\frac{2a_R x}{\varepsilon}} \int_{-\infty}^{\infty} \text{sgn}(y)p_{\varepsilon}(y, u|x) dy dx. \quad (65)$$
To evaluate (64) using (65), we reorder the integrals of $y$, $x$ and $u$, but to do this we must first transfer the $u$-dependence from the integrand to the limits of integration. Writing, for instance, "$x < 0$" to indicate that the given value of $x$ is negative, we have

$$
\int_{-\infty}^{\infty} \text{sgn}(y)p_c(y,u|x < 0)\,dy = 2 \int_{0}^{\infty} p_c(y, u|x < 0)\,dy - 1
$$

$$
= 2 \left( \int_{0}^{u} h_c(s, x, -a_L) \int_{0}^{\infty} p_c(y, u - s)\,dy\,ds \right) - 1 ,
$$

(66)

where in the second line we conditioned over the first passage time, $s$, of (41) from $x < 0$ to 0. By Lemma 7.1, and since $\int_{0}^{\infty} h_c(s, x, -a_L)\,ds = 1$, for all $x < 0$, we obtain

$$
\int_{-\infty}^{\infty} \text{sgn}(y)p_c(y,u|x < 0)\,dy = -\int_{u}^{\infty} h_c(s, x, -a_L)\,ds
$$

$$
- 2 \int_{0}^{u} h_c(s, x, -a_L) \int_{s}^{u-s} a_R p_c(0, v|0) + \frac{\varepsilon}{2} \frac{\partial p_c^+}{\partial x}(0, v|0)\,dv\,ds .
$$

(67)

A similar calculation with $x > 0$ produces

$$
\int_{-\infty}^{\infty} \text{sgn}(y)p_c(y,u|x > 0)\,dy = \int_{u}^{\infty} h_c(s, x, a_R)\,ds
$$

$$
- 2 \int_{0}^{u} h_c(s, x, a_R) \int_{s}^{u-s} a_R p_c(0, v|0) + \frac{\varepsilon}{2} \frac{\partial p_c^+}{\partial x}(0, v|0)\,dv\,ds .
$$

(68)

By combining (64), (65), (67) and (68) we arrive at

$$
\int_{0}^{t} \int_{0}^{t} \langle \text{sgn}(x(s)) \rangle \text{sgn}(x(u))\,ds\,du = 2 \int_{0}^{t} (t - u) \int_{-\infty}^{0} K e^{2as} \int_{u}^{\infty} h_c(s, x, -a_L)\,ds\,dx\,du
$$

$$
+ 4 \int_{0}^{t} (t - u) \int_{-\infty}^{0} K e^{2as} \int_{0}^{\infty} h_c(s, x, -a_L) \int_{0}^{u-s} a_R p_c(0, v|0) + \frac{\varepsilon}{2} \frac{\partial p_c^+}{\partial x}(0, v|0)\,dv\,ds\,dx\,du
$$

$$
+ 2 \int_{0}^{t} (t - u) \int_{0}^{\infty} K e^{2as} \int_{0}^{\infty} h_c(s, x, a_R)\,ds\,dx\,du
$$

$$
- 4 \int_{0}^{t} (t - u) \int_{0}^{\infty} K e^{2as} \int_{0}^{\infty} h_c(s, x, a_R) \int_{0}^{u-s} a_R p_c(0, v|0) + \frac{\varepsilon}{2} \frac{\partial p_c^+}{\partial x}(0, v|0)\,dv\,ds\,dx\,du .
$$

(69)

To simplify the four terms of (69) we define

$$
Q_c(t, a) \equiv \int_{0}^{t} (t - u) \int_{0}^{\infty} K e^{2as} \int_{0}^{u} h_c(s, x, a)\,ds\,dx\,du ,
$$

(70)

so that the first term of (69) may be written as

$$
2 \int_{0}^{t} (t - u) \int_{-\infty}^{0} K e^{2as} \int_{u}^{\infty} h_c(s, x, -a_L)\,ds\,dx\,du = \frac{a_R}{a_L + a_R} t^2 - 2Q_c(t, a_L) .
$$

Similarly the third term is

$$
2 \int_{0}^{t} (t - u) \int_{0}^{\infty} K e^{2as} \,dx\,du - 2Q_c(t, a_R) = \frac{a_L}{a_L + a_R} t^2 - 2Q_c(t, a_R) .
$$

To the second term of (69) we reorder the integration such that $dv$ is the outermost integral instead of the inner-most integral. This step is straight-forward but
We are now able to write (72) as
\[
\int_0^t \int_0^u \langle \text{sgn}(x(s)) \text{sgn}(x(u)) \rangle \, ds \, du = t^2 - 2(Q_\varepsilon(t, a_L) + Q_\varepsilon(t, a_R))
\]
\[
+ 4\int_0^t \left( a_{RP_\varepsilon}(0, v|0) + \frac{\varepsilon \partial p^+}{2 \partial x}(0, v|0) \right) Q_\varepsilon(t - v, a_L) - Q_\varepsilon(t - v, a_R) \, dv .
\]
Via (42), it may be demonstrated that \( a_{RP_\varepsilon}(0, v|0) + \frac{\varepsilon \partial p^+}{2 \partial x}(0, v|0) \) decays exponentially to zero as \( v \to \infty \) on an \( O(\varepsilon) \) time scale. For this reason it is helpful to apply the substitution \( \tilde{v} = \frac{v}{\varepsilon} \) to obtain
\[
\int_0^t \int_0^u \langle \text{sgn}(x(s)) \text{sgn}(x(u)) \rangle \, ds \, du = \frac{t^2}{2} - 2Q_\varepsilon(t, a_L) + Q_\varepsilon(t, a_R)
\]
\[
+ 4\varepsilon\int_0^t \left( a_{RP_\varepsilon}(0, \varepsilon\tilde{v}|0) + \frac{\varepsilon \partial p^+}{2 \partial x}(0, \varepsilon\tilde{v}|0) \right) \left( Q_\varepsilon(t - \varepsilon\tilde{v}, a_L) - Q_\varepsilon(t - \varepsilon\tilde{v}, a_R) \right) \, d\tilde{v} ,
\]
and expand \( Q_\varepsilon(t - \varepsilon\tilde{v}, a_L) - Q_\varepsilon(t - \varepsilon\tilde{v}, a_R) \) in \( \varepsilon \) such that the integral on the right-hand side of (71) has the form
\[
\sum_i \sum_j \alpha_{ij}\varepsilon^j \int_0^t \tilde{v}^j \left( a_{RP_\varepsilon}(0, \varepsilon\tilde{v}|0) + \frac{\varepsilon \partial p^+}{2 \partial x}(0, \varepsilon\tilde{v}|0) \right) \, d\tilde{v} ,
\]
for some coefficients \( \alpha_{ij} \). To obtain the coefficients, we first evaluate (70) via multiple applications of integration by parts:
\[
Q_\varepsilon(t, a) = K \left( \frac{\varepsilon a^3}{4a} - \frac{\varepsilon^2 t}{4a^3} + \frac{\varepsilon^3}{4a^5} \right) + \frac{K \sqrt{7}}{\sqrt{2\pi}} \left( \frac{\sqrt{\varepsilon} t^2}{6} + \frac{\varepsilon^2}{3a^2} - \frac{\varepsilon^3}{2a^3} \right) e^{-\frac{\varepsilon a^3}{3}}
\]
\[
- K \left( \frac{a t^3}{12} + \frac{\varepsilon^2 t}{4a} - \frac{\varepsilon^2}{4a^3} + \frac{\varepsilon^3}{4a^5} \right) \text{erfc} \left( \frac{a\sqrt{7}}{\sqrt{2\varepsilon}} \right) .
\]
In view of (71), we use (73) to obtain
\[
Q_\varepsilon(t, a_L) + Q_\varepsilon(t, a_R) = \frac{t^2}{2} - \frac{(a_L^3 + a_R^3)t}{2a_L^2 a_R^2 (a_L + a_R)} + O(\varepsilon^2) ,
\]
\[
Q_\varepsilon(t - \varepsilon\tilde{v}, a_L) - Q_\varepsilon(t - \varepsilon\tilde{v}, a_R) = \frac{(a_L - a_R)^2}{2(a_L + a_R)} + \frac{(a_L^3 - a_R^3)\varepsilon}{2a_L^2 a_R^2 (a_L + a_R) + (a_L - a_R)\varepsilon t} + O(\varepsilon^2) .
\]
Using Laplace’s method to asymptotically evaluate (72), the substitution of (74) and (75) into (71) yields

\[
\int_0^t \int_0^t \langle \text{sgn}(x(s)) \text{sgn}(x(u)) \rangle ds \, du = \frac{(a_L^3 + a_R^3)\varepsilon t}{a_L^2 a_R^2(a_L + a_R)} \\
+ \frac{2(a_L - a_R)t^2}{(a_L + a_R)} + \frac{2(a_L^2 - a_R^2)\varepsilon t}{a_L^2 a_R^2(a_L + a_R)} \int_0^\infty \left( a_R p_\varepsilon(0, v|0) + \frac{\varepsilon}{2} \frac{\partial p_\varepsilon^+}{\partial x}(0, v|0) \right) dv \\
+ \frac{4(a_L - a_R)t}{(a_L + a_R)} \int_0^\infty \left( a_R p_\varepsilon(0, v|0) + \frac{\varepsilon}{2} \frac{\partial p_\varepsilon^+}{\partial x}(0, v|0) \right) dv + O(\varepsilon^2) .
\]

(76)

By applying Lemma 7.2 and simplifying we finally arrive at (54).  

\[\square\]

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E-mail address: d.j.w.simpson@massey.ac.nz
E-mail address: rachel@math.ubc.ca