Dispersive contributions to $e^+p/e^-p$ cross section ratio in forward regime

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Two-photon exchange (TPE) contributions to elastic electron-proton scattering in the forward regime are considered. The imaginary part of TPE amplitude in these kinematics is related to the DIS nucleon structure functions. The real part of the TPE amplitude is obtained from the imaginary part by means of dispersion relations. We demonstrate that the dispersion integrals for the relevant elastic $ep$-scattering amplitude converge and do not need subtraction. This allows us to make clean prediction for the real part of the TPE amplitude at forward angles. We furthermore compare $e^+p$ and $e^-p$ cross sections which depends on the real part of TPE amplitude, and predict the positron cross section to exceed the electron one by a few per cent, with the difference ranging from 1.4% to 2.8% for electron lab energies in the range from 3 to 45 GeV. We furthermore predict that the absolute value of this asymmetry grows with energy, which makes it promising for experimental tests.

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I. INTRODUCTION

Recently, much attention has been attracted by the two-photon exchange (TPE) contribution to the elastic electron-proton scattering. On the one hand, the recent experimental data on $G_E$ to $G_M$ ratio at higher momentum transfers show significant discrepancy between the results obtained with the new polarization transfer technique and those obtained using Rosenbluth separation. Since the TPE contribution is the only largely unknown order $\alpha_{em}$ correction to the elastic $ep$-scattering, it was argued that a proper inclusion of these effects into the analysis of the $ep$-scattering observables would reconcile the two measurements. Recent theoretical calculations of the real part of the TPE amplitude in different models seem to support this idea, though at the qualitative level. At the moment, it is only the elastic (nucleon) intermediate state contribution together with the associated with it IR divergent part, needed for an analysis of the experimental data, that one can be confident about. The partonic model “handbag” calculation of Ref. though definitely represents important physics contribution at high energy and momentum transfer, suffers of unphysical IR divergencies which can be presumably cancelled by including the “cat ears” mechanism. As for the further inelastic contributions, only estimates with the $\Delta(1232)$ for the real part of the TPE amplitude exist in the literature.

Another experimental test of TPE effects is proposed at JLab and at VEPP-3 storage ring, where the $e^+p/e^-p$ cross section ratio will be studied in a wide angular range. A deviation from 1 originates to leading order in $\alpha_{em}$ from the interference between the one photon exchange contribution which is linear in lepton charge, and the real part of the TPE contribution, which is quadratic in lepton charge.

On the other hand, over past 5 years new observables become accessible experimentally, the single spin asymmetries with beam electron or target (recoil) proton polarized normally to the reaction plane. These observables have been first studied theoretically back in 1970’s and were shown to be directly sensitive to the imaginary part of the TPE amplitude. Furthermore, all IR divergent terms cancel in these asymmetries (unlike the cross section), which makes them especially attractive since they offer the most clean experimental tests of the TPE effects. There is a considerable interest to this class of observables from the theory point of view, as well.

The TPE amplitude is related to Compton scattering with two space-like photons (virtual-to-virtual Compton scattering - VVCS) which is largely unknown. For the inelastic intermediate states contributions, it is only the imaginary part of the VVCS amplitude in the forward direction that is well studied, since it can be expressed in terms of the DIS structure functions. So, the only kinematical point where a firm prediction for the TPE amplitude can be made is the exact forward limit. Unfortunately, all the observables which measure the TPE effects necessarily vanish at that point. It was proposed to combine the DIS input in forward direction with the phenomenological $t$-dependence taken from that of the Compton scattering differential cross section which is measured at high energies and low values of $t$. Modelling in this way the imaginary parts of the electron helicity-flipping $ep$-scattering amplitudes, it was possible to successfully describe the data on beam normal spin asymmetry at forward angles. The motivation of this work is to extend the phenomenological approach of the previous paper and compute the imaginary parts of the electron helicity-conserving amplitudes. The real parts which enter the expression for the elastic cross section and the charge asymmetry are obtained through dispersion relations. Combined with the phenomenological $t$-dependence which proved itself plausible for the imaginary part, it will allow us to predict the high energy dispersive contributions to the $e^+p/e^-p$ cross section ratio and the elastic cross section.

For the case of low electron energies, a dispersive ap-
proach was adopted in [14]. The authors of that work used once-subtracted dispersion relations in the annihilation channel $t$ at fixed energy $\approx 0$. At this energy, the hadronic part of the TPE graph may be parametrized through the nucleon polarizabilities. To calculate the subtraction constant, the authors notice that the imaginary part of the hadronic amplitude at $t = 0$ is given in terms of DIS structure functions. They furthermore use dispersion relations for these structure functions, and by doing the low energy expansion express the corresponding real parts as moments of the structure functions. After that, these real parts are embedded into the TPE amplitude.

Instead, our approach is designed for high electron energies, therefore no connection with the polarizabilities can be made, neither the low energy expansion can be performed to express the final result as an integral over the DIS structure functions’ momenta. First, we calculate the imaginary part of the TPE amplitude in terms of the electron phase space integral over DIS structure functions and obtain the imaginary parts of the invariant $ep$-scattering amplitudes. Instead of using dispersion relations for the hadronic amplitude separately, we proceed with the dispersion relations for the invariant amplitudes for elastic $ep$-scattering and demonstrate that they converge and no subtraction is needed. In this way, we obtain a clean prediction for the real parts of these amplitudes and the observables which they enter.

The imaginary parts of helicity-flip amplitudes were shown to be dominated by the region of low virtualities of the exchanged photons, and it proved adequate to approximate the DIS structure function $W_1$ by its value at the real photon point, where it is related to the total photoabsorption cross section. However, this approach is not expected to work in the case of the helicity-conserving amplitudes, especially for their real parts where all the values of these virtualities are allowed. Therefore, we include in our calculation the phenomenological form of the $Q^2$ dependence of the virtual photon cross section obtained from low-$x$ data. For the moment, we will concentrate on the observables averaged over the nucleon spin, i.e. the cross section and the beam normal spin asymmetry. We leave the consideration of the target normal spin asymmetry to an upcoming work.

The paper is organized as follows. In Section II we overview the elastic $ep$-scattering amplitude and the kinematics of the reaction. The calculation of the imaginary parts of the invariant $ep$-scattering amplitudes is performed in Section III. These imaginary parts are then used as input for the dispersion relations to determine their real parts in Section IV and the convergence of these dispersion relations is discussed. In Section V we present the results of the calculation.

II. ELASTIC $ep$-SCATTERING AMPLITUDE

In this work, we consider elastic electron-proton scattering process $e(k) + p(p) \rightarrow e(k') + p(p')$ for which we define:

$$
P = \frac{p + p'}{2},$$

$$
K = \frac{k + k'}{2},$$

$$
q = k - k' = p' - p, \quad (1)$$

and choose the invariants $t = q^2 < 0$ and $\nu = (P\cdot K)/M$ as the independent variables. $M$ denotes the nucleon mass. They are related to the Mandelstam variables $s = (p + k)^2$ and $u = (p - k)^2$ through $s - u = 4M\nu$ and $s + u + t = 2M^2$. For convenience, we also introduce the usual polarization parameter $\varepsilon$ of the virtual photon, which can be related to the invariants $\nu$ and $t$ (neglecting the electron mass $m_e$):

$$
\varepsilon = \frac{\nu^2 - M^2\tau(1 + \tau)}{\nu^2 + M^2\tau(1 + \tau)}, \quad (2)
$$

with $\tau = -t/(4M^2)$. Elastic scattering of two spin 1/2 particles is described by six independent amplitudes. Three of them do not flip the electron helicity [3],

$$
T_{\text{no flip}} = \frac{e^2}{-t} \tilde{u}(k')\gamma_\mu u(k) \cdot \tilde{u}(p') \left( \hat{G}_M\gamma^\mu - \tilde{F}_2P^\mu + \tilde{F}_3K^\mu/M^2 \right) u(p), \quad (3)
$$

while the other three are electron helicity flipping and thus have in general the order of the electron mass $m_e$ [15]:

$$
T_{\text{flip}} = \frac{m_e}{M} \frac{e^2}{-t} \left[ \tilde{u}(k')u(k) \cdot \tilde{u}(p') \left( \hat{F}_4 + \tilde{F}_5 K/M \right) u(p) \\
+ \tilde{F}_6 \tilde{u}(k')\gamma_\mu u(k) \cdot \tilde{u}(p')\gamma_\mu u(p) \right] \quad (4)
$$

In the one-photon exchange (OPE) approximation, two of the six amplitudes match with the electromagnetic form factors,

$$
\hat{G}_M^{\text{Born}}(\nu, t) = G_M(t),
\hat{F}_2^{\text{Born}}(\nu, t) = F_2(t),
\tilde{F}_3^{\text{Born}}(\nu, t) = 0 \quad (5)
$$

where $G_M(t)$ and $F_2(t)$ are the magnetic and Pauli form factors, respectively. For further convenience we define

1 In elastic $ep$-scattering, the usual notation for the momentum transfer is $Q^2 = -q^2$ but we prefer the more general notation $t$ to avoid confusion with the incoming and outgoing photon virtualities.
also $G_E = G_M - (1 + \tau) \tilde{F}_2$ and $\tilde{F}_1 = G_M - \tilde{F}_2$ which in the Born approximation reduce to Sachs electric form factor $G_E$ and Dirac form factor $F_1$, respectively. It is useful to separate one- and two-photon exchange effects explicitly,

$$
\begin{align*}
\tilde{G}_M &= G_M + \delta \tilde{G}_M, \\
\tilde{G}_E &= G_E + \delta \tilde{G}_E, \\
\tilde{F}_2 &= F_2 + \delta \tilde{F}_2,
\end{align*}
$$

(6)

where $G_M$, $G_E$, and $F_2$ are the usual form factors, while the two photon effects are contained in the quantities $\delta \tilde{G}_M$, $\delta \tilde{G}_E$, $\delta \tilde{F}_2$.

The unpolarized cross section is

$$
\frac{d\sigma}{d\Omega_{Lab}} = \frac{\tau \sigma_R}{\varepsilon (1 + \tau)} \frac{d\sigma_0}{d\Omega_{Lab}},
$$

(7)

with the usual Rutherford cross section

$$
\frac{d\sigma_0}{d\Omega_{Lab}} = \frac{4\alpha^2 \cos^2 \Theta}{Q^2} \frac{E'^3}{E},
$$

(8)

where the electron Lab scattering angle and $E(E')$ the incoming (outgoing) electron Lab energy. The reduced cross section $\sigma_R$ is given by

$$
\sigma_R = G_M^2 + \frac{\varepsilon}{\tau} G_E^2 + 2G_M \text{Re} \left( \delta \tilde{G}_M + \varepsilon \frac{\nu}{M} \tilde{F}_3 \right) + \frac{2\varepsilon}{\tau} G_E \text{Re} \left( \delta \tilde{G}_E + \frac{\nu}{M} \tilde{F}_3 \right) + O(e^4)
$$

(9)

The OPE contributions to the form factors depend linearly on the lepton’s charge, while the TPE contributions are quadratic in the lepton’s charge. Therefore, the interference terms between the OPE and TPE will enter $e^- p$ and the $e^+ p$ cross sections with opposite sign. The $e^+ p/ e^- p$ cross section ratio is thus given by

$$
R_{e^+ e^-} = \frac{\sigma_{e^+ p}}{\sigma_{e^- p}} = 1 - \frac{4}{G_M^2 + \frac{\varepsilon}{\tau} G_E^2} \left[ G_M \text{Re} \left( \delta \tilde{G}_M + \varepsilon \frac{\nu}{M} \tilde{F}_3 \right) + \frac{\varepsilon}{\tau} G_E \text{Re} \left( \delta \tilde{G}_E + \frac{\nu}{M} \tilde{F}_3 \right) \right]
$$

(10)

For a beam polarized normal to the scattering plane, one can define a single spin asymmetry,

$$
B_n = \frac{\sigma_{\uparrow} - \sigma_{\downarrow}}{\sigma_{\uparrow} + \sigma_{\downarrow}},
$$

(11)

where $\sigma_{\uparrow}$ ($\sigma_{\downarrow}$) denotes the cross section for an unpolarized target and for an electron beam spin parallel (antiparallel) to the normal polarization vector defined as

$$
S^\mu_n = \begin{pmatrix}
0, \\
[\hat{k} \times \hat{k}'], \\
|k \times k'|
\end{pmatrix},
$$

(12)

normalized to $(S \cdot S) = -1$. Similarly, one defines the target normal spin asymmetry $T_n$. It has been shown in the early 1970’s [10] that such asymmetries are directly related to the imaginary part of the $T$-matrix. Since the electromagnetic form factors and the one-photon exchange amplitude are purely real, $B_n$ obtains its finite contribution to leading order in the electromagnetic constant $\alpha_{em}$ from an interference between the Born amplitude and the imaginary part of the two-photon exchange amplitude. In terms of the amplitudes of Eqs. (7-12), the beam normal spin asymmetry is given by

$$
B_n = \frac{-m_e}{M} \sqrt{2(1 - \varepsilon) \sqrt{1 + \tau} \left( \tau G_M^2 + \varepsilon G_E^2 \right)} \cdot \left\{ \tau G_M \text{Im} \tilde{F}_3 + G_E \text{Im} \tilde{F}_4 + \frac{\nu}{M} \text{Im} \tilde{F}_5 \right\}
$$

(13)

III. IMAGINARY PART OF THE FORWARD ELASTIC $e\pi$-SCATTERING AMPLITUDE

The imaginary part of the diagram in Fig. 1 is given by the integral

$$
\text{Im} T_{2\gamma} = e^4 \int \frac{d^4 \tilde{F}_1}{(2\pi)^4} \frac{1}{\tilde{q}_1 \tilde{q}_2} \text{Im} W_{\mu\nu},
$$

(14)

where the leptonic tensor is given by

$$
l_{\mu\nu} = \bar{u}(k') \gamma_\nu (\tilde{k}_1 + m_e) \gamma_\mu u(k).
$$

(15)

The imaginary part of the spin-averaged part of the hadronic tensor is expressed in terms of the DIS structure functions $W_1$ and $W_2$. We will make use of Callan-Gross relation between them, thus we obtain

$$
\text{Im} W_{\mu\nu} = 2\pi W_1 \left\{ -g_{\mu\nu} + \frac{P^\mu q_1^\nu + P^\nu q_2^\mu}{PK} \right\} - \frac{(q_1 \cdot q_2)}{(P \cdot K)^2} \frac{P^\mu P^\nu}{2},
$$

(16)

with $q_1^\mu = k - k_1$ the incoming and $q_2^\mu = k' - k_1$ the outgoing photon momenta and their average $K = k + k'$. We next contract the leptonic and hadronic tensors and keep the lepton mass non-zero in order to cross-check the calculation with the previous work for the beam normal
dependence of the hadronic amplitude from its spin-spin asymmetry \[17\]. The result reads

\[
l_{\mu\nu} \cdot \text{Im} W^{\mu\nu} = 2\pi W_1 \left\{ m c u' u \frac{2P_k}{P K} \\
+ \bar{u'} P k \left[ \frac{2K^2 P_k}{(P K)^2} \left( \frac{P_k P K}{(P K)^2} + \frac{P K}{P K} - 2 \right) \right] + \frac{Q^2}{P K} \left( \frac{2P K}{P K} + \frac{P K}{P K} - 2 \right) \right\} \tag{17}
\]

In the above equation, we used the notation \( Q^2 = -q_1^2 = -q_2^2 \). Throughout the calculation, we neglect terms \( \sim \frac{K^2 K^2}{(P K)^2} \approx \frac{1-\epsilon}{1+\epsilon} \) which are small in forward kinematics.

The above formula represents the TPE amplitude in the nucleon helicity-conserving channel, averaged over the spin projections. The dependence on the electron spins is retained. Our goal is to obtain the expressions for the invariant ep-scattering amplitudes defined in Eqs. \[18\] \[19\]. For this, we will need to restore the dependence on the nucleon spin. It can be done as follows. There exist four independent scalars which can be formed from the initial and final nucleon spinors with the \( \gamma \)-matrices and the four-vectors that fix the external kinematics. They are \( \bar{N}' N, \bar{N}' K N, \bar{N}' \gamma_5 N \), and \( \bar{N}' \gamma_5 N \). However, the fourth structure vanishes in the exact forward limit and in the helicity conserving channel. Furthermore, the third structure drops after averaging over spins (that is, it is non-zero in the GDH sum rule-like cross section difference \( T_{1/2,1'} \), \( K_{1/2,1'} \), \( K_{1/2,1} \), but not in the sum).

Finally, we are left with only two structures \( \bar{N}' N, \bar{N}' K N \). By a direct calculation, it can be shown that in the forward limit,

\[
1 = \frac{1}{2} \sum_{\text{spins}} \left\{ -\frac{M}{s - M^2} \bar{N}' N + \frac{s - M^2}{(s - M^2)^2} \bar{N}' K N \right\} \tag{18}
\]

Therefore, we now modify the hadronic tensor by substituting

\[
W_1 \to W_1 \left\{ -\frac{M}{s - M^2} \bar{N}' N + \frac{s + M^2}{(s - M^2)^2} \bar{N}' K N \right\} \tag{19}
\]

This procedure allows one to restore the nucleon spin dependence of the hadronic amplitude from its spin-averaged part in a correct way, as long as only cross section channel is used as the phenomenological input.

We can now identify the invariant amplitudes \( F_i, i = 1, \ldots, 6 \) comparing Eqs. \[17\] \[19\]:

\[
\begin{align*}
\text{Im} \tilde{G}_M & = 0 \\
\text{Im} \tilde{F}_2 & = -2\pi^2 \frac{M^2}{s - M^2} \int_A \left\{ W_1 (w^2, q_1^2, q_2^2) \right\} \\
\text{Im} \tilde{F}_3 & = \frac{s + M^2}{s - M^2} \text{Im} F_2 \\
\text{Im} \tilde{F}_4 & = 2\pi \frac{M^2}{s - M^2} \int_B W_1 (w^2, q_1^2, q_2^2) \\
\text{Im} \tilde{F}_5 & = \frac{s + M^2}{s - M^2} \text{Im} F_4 \\
\text{Im} \tilde{F}_6 & = 0,
\end{align*}
\]

where we introduced shorthands:

\[
\begin{align*}
&\int_A \left\{ W_1 (w^2, q_1^2, q_2^2) \right\} = \int \frac{d^4 E_1}{(2\pi)^4} \frac{1}{2E_1} Q_1^2 Q_2^2 \\
&\times \left[ 2K^2 P_k \left( \frac{P_k P K}{(P K)^2} + \frac{P K}{P K} - 2 \right) \right] W_1 (w^2, Q_1^2, Q_2^2) \\
&+ \frac{Q^2}{P K} \left( \frac{2P K}{P K} + \frac{P K}{P K} - 2 \right) W_1 (w^2, Q_1^2, Q_2^2) \\
&\int_B W_1 (w^2, Q_1^2, Q_2^2) = \int \frac{d^4 E_1}{(2\pi)^4} \frac{1}{2E_1} Q_1^2 Q_2^2 \\
&\times \frac{2P_k}{P K} W_1 (w^2, Q_1^2, Q_2^2).
\end{align*}
\tag{21}
\]

Note that the fact that \( \text{Im} G_M \) vanishes implies that \( \text{Im} F_1 = -\text{Im} F_2 \). In the above formula, the first argument of \( W_1 \) is the invariant mass squared of the \( \gamma^* p \) system, \( w^2 = (P + K)^2 \). In the integral \( \int_A \), we have to keep the term \( \sim K^2 = -\frac{4}{3} \) till the end, since it is needed to cancel the \( 1/t \) behaviour of the integral \( I_1 \) introduced below.

The next step is to perform the phase space integrals in Eq. \[21\]. First, we express the nucleon structure function \( W_1 \) through the virtual photon cross section \( \sigma_{\gamma^* p}(w^2, Q^2) \):

\[
W_1 = \frac{w^2 - M^2}{2\pi e^2} \sigma_{\gamma^* p}(w^2, Q^2).
\tag{22}
\]

We will need an explicit form of energy and \( Q^2 \) dependence of this cross section. We will use the phenomenological form proposed in Ref. \[16\] which amounts in factorization of the virtual photon cross section into purely \( w^2 \)-dependent part, and the low-\( x \) scaling variable \( \eta \)-dependent part:

\[
\sigma_{\gamma^* p}(w^2, Q^2) = \sigma_{\gamma^* p}^{\text{Regge}}(w^2) \frac{I_1(\eta, \Delta^2(w^2))}{I_0(\eta, \eta_0)},
\tag{23}
\]

where \( \sigma_{\gamma^* p}^{\text{Regge}}(w^2) \) is the total photoabsorption cross section in Regge regime and is conveniently parametrized in terms of two Regge trajectories,

\[
\sigma_{\gamma^* p}^{\text{Regge}}(w^2) = A_\rho (w^2)^{\alpha_\rho} + A_P (w^2)^{\alpha_P}.
\tag{24}
\]
with \( A_\rho = (145\pm 2)\mu b, \) \( A_P = (63.5\pm 0.9)\mu b, \) \( \alpha_\rho = 0.5 \) and \( \alpha_P = 1.097 \pm 0.002 \) the parameters of the \( \rho \) and pomeron trajectories, respectively. In the above formula, \( w \) should be taken in GeV. The scaling variable \( \eta \) is defined as

\[
\eta = \frac{Q^2 + m_q^2}{\Lambda^2(w^2)},
\]

with its minimal value \( \eta_0 = \eta(Q^2 = 0) = \frac{m_q^2}{\Lambda^2(w^2)} \). Furthermore, the function of energy \( \Lambda(w^2) \) is determined by a fit to DIS data which are well reproduced by

\[
\Lambda^2(w^2) = C_1(w^2 + w_0^2)C_2,
\]

with \( C_1 = 0.34 \pm 0.05, C_2 = 0.27 \pm 0.01, w_0^2 = 882 \pm 246 \) GeV\(^2\), and \( m_0^2 = 0.16 \pm 0.01 \) GeV\(^2\). The general form of the function \( I(\eta, \eta_0) \) can be found in [10]. Here, we will use its asymptotic form for low values of \( \eta \),

\[
I(\eta, \eta_0)_{\eta<} = \ln \left( \frac{1}{\eta} \right).
\]

Furthermore, we divide the virtual photon cross section into two parts, the \( Q^2 \) independent and the rest,

\[
\sigma_{\gamma P}(w^2, Q^2) = \sigma_{\gamma P}^{\text{Regge}}(w^2) + (\sigma_{\gamma P}(w^2, Q^2) - \sigma_{\gamma P}^{\text{Regge}}(w^2))
\]

\[
= \sigma_{\gamma P}^{\text{Regge}}(w^2) - \sigma_{\gamma P}^{\text{Regge}}(w^2) \ln \left( \frac{1 + Q^2/m_0^2}{\ln \frac{\Lambda^2}{m_q^2}} \right)
\]

\[\sigma_{\gamma P}(w^2, Q^2) \approx \sigma_{\gamma P}^{\text{Regge}}(w^2) = \sigma_{\gamma P}^{\text{Regge}}(w^2) \ln \left( \frac{1 + Q^2/m_0^2}{\ln \frac{\Lambda^2}{m_q^2}} \right)
\]

A. Electron helicity-flip amplitudes

We start with the integral \( I_B \). Introducing the dimensionless variable \( z = \frac{E}{E_z} \) and consistently neglecting terms \( \sim t/s \), we can write the integral \( I_B \) as

\[
\int_B W_{1}(w^2, q_1^2, q_2^2) = \frac{(s - M^2)E^2}{16\pi^4e^2}
\]

\[
\times \int_{\frac{m_h}{E_z}}^{\frac{E_z}{m_h}} dz z^2(1 - z) \sigma_{\gamma P}(M^2 + (s - M^2)(1 - z))
\]

\[
\times \int d\Omega_1 \frac{d\Omega_2}{Q_1^2Q_2^2} \left[ 1 - \frac{\ln \left( 1 + Q^2/m_0^2 \right)}{\ln \frac{\Lambda^2}{m_q^2}} \right]
\]

The integral corresponding to the first term in the square brackets was performed before (see [12, 17]). In order to obtain the correct \( t \)-dependence, we keep the photon indices in the denominator: then, the first term in the square bracket leads to the leading \( 1/t \) behaviour. For the second term, the denominator can be written as \( 1/(Q^2) \), as this corresponds to neglecting the order \( t^2 \) corrections to the leading term. The second term can be related to the table integral \( \int \frac{\ln(1 + x)}{x^2} dx = \ln x - \frac{1}{x} \ln(1 + x) \).

The result for the electron helicity-flip amplitudes then reads

\[
\text{Im} F_4 = -\frac{M^2}{4\pi^2} \times \int_{\frac{m_p}{E_z}}^{\frac{E_z}{m_h}} dz (1 - z) \sigma_{\gamma P}(w^2) \left\{ \ln \left( \frac{z^2 - t}{(1 - z^2)m^2} \right) + \frac{zt}{2m_0^2 \ln \left( \frac{\Lambda^2}{m_q^2} \right)} \left[ \ln \left( \frac{2Ez}{m_e} + \frac{1}{2} - \ln \left( \frac{1 + 4E^2z}{m_0^2} \right) \right) \right] \right\}
\]

\[\text{Im} F_3 = \frac{s + M^2}{s - M^2} \text{Im} F_4
\]

Putting these expressions into Eq. (30), we obtain our result for the beam normal spin asymmetry,

\[
B_n = -\frac{m_e\sqrt{-t}}{4\pi^2} \frac{F_1(t)}{F_1^2(t) + \tau F_2^2(t)} \times \int_{\frac{m_p}{E_z}}^{\frac{E_z}{m_h}} dz (1 - z) \sigma_{\gamma P}(w^2) \left\{ \ln \left( \frac{z^2 - t}{(1 - z^2)m^2} \right) + \frac{zt}{2m_0^2 \ln \left( \frac{\Lambda^2}{m_q^2} \right)} \left[ \ln \left( \frac{2Ez}{m_e} + \frac{1}{2} - \ln \left( \frac{1 + 4E^2z}{m_0^2} \right) \right) \right] \right\}
\]

Before presenting the results of the numerical integration, we recall that in the previous work, the cross section was assumed to be independent of the photon virtuality and roughly constant as function of \( w^2 \), \( \sigma_{\gamma P}(w^2, Q^2) \approx \sigma_T = \text{const} \). In that case, the expression for \( B_n \) was obtained [13]:

\[
B_n = -\frac{m_e\sqrt{-t}\sigma_T}{4\pi^2} \frac{F_1(t)}{F_1^2(t) + \tau F_2^2(t)} \left( \ln \frac{\sqrt{-t}}{m_e} - 1 \right)
\]

In Fig. 2 we compare the results for the beam normal spin asymmetry of Eq. (30) with the approximative formula of Eq. (33). Comparing the thick and the thin lines, we see that the numerical impact of the subleading terms in \( t \) is very small, and the leading \( t \) approximation indeed dominates.

B. Electron helicity-conserving amplitudes

We next turn to the calculation of the imaginary parts of the electron helicity conserving amplitudes \( F_2 \) and \( F_4 \). This amounts in calculating the electron phase space integral \( I_A \). As in the case of \( I_B \), we start with the angular integration. There are four independent integrals over the solid angle in terms of which all other integrals can
be expressed. They are

$$I_1 = \int \frac{d\Omega_1}{Q_1^2 Q_2^2} \left[ 1 - \frac{\ln \left( \frac{1 + Q^2/m_0^2}{m_e^2} \right)}{\ln \left( \frac{m^2}{m_0^2} \right)} \right],$$

$$I_2 = \int \frac{d\Omega_1}{Q^2} \left[ 1 - \frac{\ln \left( \frac{1 + Q^2/m_0^2}{m_e^2} \right)}{\ln \left( \frac{m^2}{m_0^2} \right)} \right],$$

$$I_3 = \int \frac{d\Omega_1}{w^2 - M^2 + Q^2} \left[ 1 - \frac{\ln \left( \frac{1 + Q^2/m_0^2}{m_e^2} \right)}{\ln \left( \frac{m^2}{m_0^2} \right)} \right],$$

$$I_4 = \int \frac{d\Omega_1}{(w^2 - M^2 + Q^2)^2} \left[ 1 - \frac{\ln \left( \frac{1 + Q^2/m_0^2}{m_e^2} \right)}{\ln \left( \frac{m^2}{m_0^2} \right)} \right].$$

The integrals $I_1$ and $I_4$ do not contain any singularity, while one has to be careful with the first two integrals in Eq. (34). In the limit of small electron mass, one has for the first terms in the square brakets (for details, see [13, 17])

$$I_1 \approx \frac{2\pi}{-t E_1^2} \ln \left( \frac{E_1^2}{(E_1 - E_1)^2 m_e^2} \right),$$

$$I_2 \approx \frac{\pi}{E E_1} \ln \left( \frac{4E_1^2}{m_e^2} \right).$$

One can notice that in the limit of zero mass of the electron the first two integrals become singular. In the case of the beam asymmetry $B_n$, this singularity is cancelled by the overall factor $m_e$, thus leading to the dominant term $\sim m_e \ln \frac{4}{m_e^2}$. In the case of helicity conserving amplitudes, no such singularity may occur since it does not contain the overall factor $m_e$, which would cancel the chiral singularity. Therefore, the divergent parts of the integrals $I_1$ and $I_2$ should cancel exactly in the integral $I_A$. The integral $I_A$ can be written in terms of these four integrals as

$$I_A \left\{ W_1(w^2,0,0) \right\} = \int \frac{E_1 dE_1}{16\pi^2} \frac{w^2 - M^2}{2\pi^2} \sigma_{\gamma p}(w^2)$$

$$\times \left\{ \frac{4}{w^2 - M^2} \frac{E_1}{2E - E_1} \left[ I_2 - 2K^2 \frac{E_1}{E} I_1 \right] - \frac{4E}{E - E_1} \left[ I_4 + \frac{1}{w^2 - M^2} \frac{E_1}{E} I_3^2 \right] \right\}.$$  \hspace{1cm} (36)

Indeed, in the combination $I_2 - 2K^2 \frac{E_1}{E} I_1$, the dependence on $m_e$ cancels,

$$I_2 - 2K^2 \frac{E_1}{E} I_1 = \frac{\pi}{EE_1} \ln \frac{4(E - E_1)^2}{Q^2}.$$  \hspace{1cm} (37)

Combining all the terms, we obtain:

$$I_A \left\{ W_1(w^2,q_1^2,q_2^2) \right\}$$

$$= \frac{1}{8\pi^2} \int_{Q}^{E_1} \frac{dz}{1 - z} \sigma_{\gamma p}(w^2)$$

$$\times \left\{ (1 + z^2) \ln \frac{2E}{Q} + (1 + z + z^2) \ln(1 - z) - 2E \frac{z}{\sqrt{s}} \ln \left( 1 - \frac{M^2}{s} \right) \right. $$

$$\left. + \frac{1}{\ln \frac{\Delta}{m_0^2}} \left[ \frac{1 + z + z^2}{2} \ln \left( \frac{4E_1^2}{m^2} \right) \right] \right\}.$$  \hspace{1cm} (38)

At high energies, the upper limit of the integration goes as $\frac{E_1}{E} \approx 1 - \frac{m_e}{E_1}$, $\to 1$. Consequently, the integrand has a rather singular behaviour for photon energies just above the pion production threshold. This singularity is compensated by the vanishing of the cross section at pion production threshold, as it is dictated by unitarity. In the high energy approach used here, to ensure the correct threshold behaviour we multiply the cross section by a function which vanishes smoothly at the threshold and goes to 1 at higher energies, such that the high energy region is not affected. Here, we will adopt the threshold regulator of the following form:

$$f_{th}(z) = \sqrt{\frac{E_1}{E} - z},$$  \hspace{1cm} (39)
which obeys the properties required above. An introduction of such a threshold regulator does, of course, lead to a certain model dependence in the low energy part of the integral. It is worthwhile to treat low photon energy region more accurately, for example, by using some phenomenological model for the resonant region and match it to the high energies contribution, and we leave this improvement to an upcoming work. For the moment, we concentrate on the high energies-motivated calculation. In this framework, the presented approach corresponds to accounting for the non-resonant background which originates from the quark-hadron duality picture. Since vanishing of this background function at the pion production threshold is required by unitarity, the introduced threshold function is dictated by physics, though not unique.

Finally, we can write down the analytical part of the result for \( \text{Im} F_2 \),

\[
\text{Im} \tilde{F}_2 = \frac{M^2 Q^2}{8\pi^2 (s - M^2)} \times \left\{ \text{Im} \left[ \int_0^{E_M/E} \frac{dE}{z(E_M^2 + (s - M^2)(1 - z))} \sqrt{\frac{E_M}{1 - z}} \times \left[ (1 + z^2) \ln \frac{2E}{Q} + (1 + z + z^2) \ln (1 - z) \right] \right] \right. \\
\left. \times \left[ \frac{2E}{\sqrt{s} \sqrt{1 - \frac{M^2}{s}}} - z \ln \left( 1 - \frac{M^2}{s} \right) \right] \right. \\
\left. + \frac{1}{\ln \frac{s}{m_0^2}} \left[ \frac{1 + z^2}{2} S_p \left( -\frac{4E^2}{m_0^2} z \right) \right. \right. \\
\left. \left. + \left[ 1 + z \ln \left( 1 + \frac{4E^2}{m_0^2} z \right) \right] \ln \frac{1 - \frac{M^2}{s}}{1 - z} \right. \\
\left. - \frac{2E}{\sqrt{s} \sqrt{1 - \frac{M^2}{s}}} \ln \left( 1 + \frac{4E^2}{m_0^2} z \right) \right. \\
\left. + \frac{z}{\sqrt{s} \sqrt{1 - \frac{M^2}{s}}} \right] \right\} \\
\end{align*}

\[
\text{Im} \tilde{F}_3 = \frac{s + M^2}{s - M^2} \text{Im} \tilde{F}_2
\]

\[
(40)
\]

\[
(41)
\]

IV. DISPERSE CALCULATION FOR REAL PARTS OF \( F_{1,2,3} \)

In the previous section we obtained imaginary parts of the amplitudes \( F_1, F_2 \) and \( F_3 \). Real parts of these amplitudes enter the expression of the elastic \( ep \) cross section. To calculate these, we use dispersion relations at fixed \( t = -Q^2 \). Dispersion relations capitalize on the idea of analyticity of the scattering amplitude, which states that the same scattering amplitude describe the reaction in all three distinct channels related by partial \( CP \)-transformation, the so-called crossing. \( s \)-channel \( e(k) + N(p) \rightarrow e(k') + N(p') \), \( u \)-channel \( e(-k') + N(p) \rightarrow e(-k) + N(p') \), and \( t \)-channel \( e(k) + e(-k') \rightarrow N(-p) + N(p') \).

For elastic \( eN \) scattering, the invariant amplitudes possess definite symmetry properties between \( s \)- and \( u \)-channels. These properties become more transparent if introducing the explicitly crossing antisymmetric variable \( \nu = \frac{m_p - m_N}{m_p + m_N} \) which is positive in the \( s \)-channel kinematics and negative in the \( u \)-channel. At zero momentum transfer this variables reduces to the lab energy of the electron. It can be shown that the amplitudes \( \tilde{F}_{1,2} \) are odd functions of \( \nu \), while \( \tilde{F}_3 \) is an even function. As a result, these amplitudes obey the dispersion relations of two different forms,

\[
\text{Re} \tilde{F}_{1,2}(\nu, Q^2) = \frac{2\nu}{\pi} \int \frac{d\nu'}{\nu'^2 - \nu^2} \text{Im} \tilde{F}_{1,2}(\nu', Q^2),
\]

\[
\text{Re} \tilde{F}_3(\nu, Q^2) = \frac{2}{\pi} \int \frac{\nu' d\nu'}{\nu'^2 - \nu^2} \text{Im} \tilde{F}_3(\nu', Q^2),
\]

\[
(42)
\]

where \( \text{Im} \) indicates that imaginary parts are calculated in the \( s \)-channel. We next discuss the convergence of these dispersion integrals. As can be easily derived from Eq. (18), at very high energies,

\[
\text{Im} \tilde{F}_{1,2}(\nu \rightarrow \infty, Q^2) \sim \frac{1}{\nu} T_{\lambda' \lambda' ; \lambda \lambda},
\]

\[
\text{Im} \tilde{F}_3(\nu \rightarrow \infty, Q^2) \sim \frac{1}{\nu^2} T_{\lambda' \lambda' ; \lambda \lambda},
\]

\[
(43)
\]

where \( T_{\lambda' \lambda' ; \lambda \lambda} \) stands for the helicity amplitudes for elastic \( ep \)-scattering with initial (final) electron (proton) helicities indicated. In Regge-regime, the asimptotic energy dependence of the latter in the helicity conserving channel is given by the pomeron trajectory,

\[
T_{\lambda' \lambda' ; \lambda \lambda} \sim \nu^{\alpha_P},
\]

\[
(44)
\]

with \( \alpha_P \approx 1.08 \). Inserting these asimptotics in the corresponding dispersion relations for the invariant amplitudes, we see that the dispersion integrals converge, and no subtractions are needed. We therefore can obtain clean predictions for the real parts of the invariant amplitudes from a dispersive calculation which capitalizes on the unitarity and analyticity of the invariant amplitudes.

V. RESULTS

We use a numerical routine for the principal value integrals, and present our results for the real and imaginary parts of the elastic \( ep \)-scattering amplitudes \( F_2 \) and \( F_3 \) for 5 different beam energies and as functions of the elastic momentum transfer \( Q^2 \). To present our results in the off-forward kinematics, we need phenomenological input in order to reproduce the Compton amplitude correctly at finite values of \( t \). We use the exponential fit to Compton data,

\[
\frac{d\sigma}{dt} \approx \left( \frac{d\sigma}{dt} \right)_{t=0} \times e^{Bt},
\]

\[
(45)
\]
which provides a good description for $B \approx 7$ GeV$^{-2}$ for values of $-t$ up to $\approx 0.8$ GeV$^2$. Since $\frac{d\sigma}{dt}$ is related to the amplitude squared, while the total cross section to the amplitude’s imaginary part, we conclude that

$$\sigma_T(t) = \sigma_T(t_0) \times e^{\frac{-t}{2}}.$$  \hspace{1cm} (46)

In Figs. 3 and 4 we display results for the amplitudes $\tilde{F}_2$ and $\tilde{F}_3$, respectively, in the forward regime using the model described above. The imaginary parts of the two amplitudes are related in a simple way, see Eq. (41), and therefore are roughly of the same size and of the same sign. However, $\text{Re} \tilde{F}_2$ is only about $10\%$ of $\text{Re} \tilde{F}_3$. This is due to the different forms of the dispersion relations which obey the two amplitudes. In the dispersion integral for $\tilde{F}_2$, $\mathcal{P} \int \frac{d\omega}{\omega^2 - \nu^2} \text{Im} \tilde{F}_2$, the low values of $\nu'$, where the denominator is negative, have more impact than in that for $\tilde{F}_3$, thus we observe a substantial cancellation in this integral. Instead, for $\tilde{F}_3$, the extra power of the energy in the numerator, $\mathcal{P} \int \frac{d\omega}{\omega^2 \nu'} \text{Im} \tilde{F}_3$ shifts the emphasis on higher values of energy and suppresses low energies, therefore the cancellation between the two regions is only very moderate. In Fig. 3 we show the relative contribution of the TPE effects to the form factors weighted to the OPE values, $\frac{\text{Re}\delta \tilde{F}_2}{\tilde{F}_2}$ and $Y_{2\nu} = \frac{\nu}{M} \frac{\text{Re}\delta \tilde{F}_3}{\tilde{F}_3}$. It can be seen that the contribution to $\tilde{F}_2$ is fairly small. The combination $Y_{2\nu}$ is quite small (0.1%) at intermediate $\epsilon$’s, as well, but grows rapidly with $\epsilon$ approaching 1, i.e. at higher energies. In this plot, we observe a very steep $\epsilon$ dependence of the generalized form factors at large $\epsilon$ values. As one goes to higher momentum transfers, though, this behaviour becomes smoother. At very high energies, the TPE contributions should decrease, according to the high energy asymptotics discussed above. Since we are interested in low values of $t$, such energies would correspond to the values of $\epsilon$ extremely close to 1 and cannot be displayed on this plot.

Finally, we turn to the $e^+/e^-$ cross section ratio. In the forward kinematics and recalling that in forward regime $\delta G_M = 0$, we obtain

$$R_{e^+/e^-} = 1 - \frac{4}{F_1^2 + \tau F_2^2} \text{Re} \left[ F_1 \nu \tilde{F}_3 - G_E \delta \tilde{F}_2 \right].$$  \hspace{1cm} (47)

The numerical results are shown in Fig. 5. In the forward regime, the positron cross section is expected to exceed that for electrons by a few percents: from $1.4\%$ for $E_{\text{lab}} = 3$ GeV to $2.8\%$ for $E_{\text{lab}} = 45$ GeV. If going to even higher beam energies, the ratio drops due to the high energy asymptotics of the invariant amplitudes. At the lab energies as high as several tens GeV, the dispersive contributions presented here are expected to be the dominant effect, while the elastic contribution (nucleon intermediate state in the blob in Fig. 1) should be relatively small. So, the above results should give a correct estimate of the TPE contribution to the elastic cross section in forward regime.
VI. CONCLUSIONS

We presented a dispersive calculation for the real parts of the electron helicity-conserving amplitudes \( \tilde{F}_{1,2,3} \) for elastic \( e^+p \)-scattering in forward regime. Their imaginary parts in this regime can be related to the DIS structure functions. We used the phenomenological \( w^2 \) and \( Q^2 \) dependence of the structure functions. The imaginary parts of the \( ep \) scattering amplitudes are IR finite, and we demonstrated that their high energy asymptotics ensure the convergence of the unsubtracted dispersive integrals for the corresponding imaginary parts. Due to different forms of the dispersion relations for the amplitudes \( F_2 \) and \( F_3 \), the resulting real parts have very different values: the dispersive effects contribute only about 0.05\% of the low \( Q^2 \)-value of \( F_2 \), while they account for a half to few per cents in the case of \( Y_{2\gamma} \), see text for the definition) is displayed in the lower panel.

![Graph](image)

FIG. 5: Real part of \( \tilde{F}_2 \) normalized to the OPE value of \( F_2 \) at three different values of \( Q^2 \) and as function of \( \epsilon \) (upper panel). The same for the \( 2\gamma \)-amplitude \( Y_{2\gamma} \) (see text for the definition) is displayed in the lower panel.

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FIG. 6: $e^+/e^-$ asymmetry in the forward regime for five different values of the lab beam energy as function of $Q^2$. 