Many Greedy Cleaners in a Poisson Environment

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Abstract

We introduce a new “greedy cleaning” model where a star-like state space (containing \( N \) halflines connected by the origin) is covered by a homogeneous Poisson process of “dust particles”, and \( N^\alpha \) cleaners/workers proceed with cleaning in a “greedy” manner: each worker chooses the closest particle next. Assuming \( \alpha \in (0,1) \), we analyse the asymptotic behaviour of the workers, as \( N \to \infty \). We show that eventually all of them escape to infinity and that the way how do they do it depends on the value of \( \alpha \).

Keywords: Greedy cleaning, greedy service, Poisson dust, star-like space, many workers.

1 Introduction

Greedy cleaning models are known for a long time, and there is an increasing interest to the area within the recent years. An overview on the topic may be found in [1].

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The broadly known “greedy cleaning” problem may be presented as follows. There is given an unbounded connected closed subset \( \mathcal{X} \) of Euclidean space, having infinite Lebesgue measure. At time \( t = 0 \), countably many “dust” particles are placed there, as points of a Poisson process that has a constant intensity with respect to a natural measure associated with the state space. A single worker starts from a fixed location and removes these particles one-by-one as follows. There is another independent rate-1 Poisson process on the “time” halfline and, for \( i = 1, 2, \ldots, \) the worker waits for the \( i \)’th ring for exponential-1 random time \( \xi_i \), then moves/jumps to a new (the \( i \)’th) particle, instantaneously removes it and stays at this location until the next ring. So, by time \( S_n = \xi_1 + \ldots + \xi_n \), the worker cleans out \( n \) particles from the space. We are interested in the following “greedy-type” dynamics: each time the worker chooses the closest particle next. Then the basic question is: whether will the space \( \mathcal{X} \) be cleaned from all particles by time \( \infty \) or not?

Since the total number of dust particles is denumerable, we may arbitrarily numerate them and denote by \( T_k \) the time of particle \( k \) removal, so \( T_k = S_n \) for some \( n \) if particle \( k \) is eventually removed and \( T_k = \infty \), otherwise. Then one can formulate a more precise question: what is the probability \( P(T_k < \infty, \text{ for all } k) \) and when, in particular, this probability equals 0 or 1?

Note that in the case of a single worker the distribution of inter-ring times does not play any role and has been introduced for clarity of the process above. However, its exponentiality will play an important role later on in this paper.

The introduced model may be viewed as an example of a deterministic walk in a random environment, which may be opposed to random walks in either deterministic or random environments.

It appears that the formulated problem is either relatively easy or very hard to solve, depending on the state space. The case of a real line, \( \mathcal{X} = \mathbb{R} \) is easy, and one can use the 0-1-type laws to obtain that, for \( X(t) \) being the location of the server at time \( t \),

\[
P(X(t) \to \infty) = P(X(t) \to -\infty) = 1/2 \text{ (due to the space symmetry)},
\]

for any initial \( X(0) \), and, moreover, there is known the explicit distribution of \( \mathbf{P}(\inf_t X(t) \in \cdot | X(t) \to \infty) \) and of related characteristics (see [7]). The model becomes more complex when the Poisson process on the line is space-inhomogeneous. The case where its rate is a function of the distance from the origin has been analysed in [9], where the authors showed that, depending on the rate properties, either the whole space is cleaned out from
the dust or only a half of it. A similar problem has been analysed in the case where the state space contains two parallel lines, see [8]. In [13], the authors analysed a version of the original model of a homogeneous Poisson process on the line, but assume in addition that each dust point needs independently a random number of the worker’s visits (either 1, with probability \( p \), or 2, with probability \( 1 - p \)) to be cleaned out. If \( p = 1 \), then this is the original model, and only half a line is eventually cleaned out. However, as it is shown in [13], for any value \( p < 1 \), the whole line is eventually cleaned out from the dust.

In the 2D case, an answer to the key question is unknown, for any reasonable variant of the state space – either the whole plane \( \mathcal{X} = \mathbb{R}^2 \) or a cone or a slab.

There have been several attempts to solve the problem, both analytically and/or numerically. In particular, A. Holroyd [10] has run a number of simulations that do not conclusively support either answer to the question. There are many more advanced problems, related to the trajectory of the worker and, in particular, to hitting times of certain areas of the state space (see again [1]).

There is a belief that, by analogy to a random walk, in dimensions 3 or more it should typically be \( \mathbb{P}(T_k < \infty \text{ for all } k) = 0 \). However, we are unaware of any rigorous results here.

There is another class of models (we call them “greedy service” models) which is close in spirit to our “greedy cleaning” models. In a greedy service model, there are again (one or more )“greedy” workers/servers that clean/serve the dust, and, in addition, there is an independent temporal-spatial Poisson (or, more generally, renewal) input process of dust particles. The main problems here are to establish stability of a model (in the case of a bounded state space) or to analyse the asymptotic trajectory(ies) of the worker(s) (in the case of an unbounded state space). The first stability conjecture for continuous circle has been formulated in [2], see further [14] for a survey on the subject. A substantial progress in understanding discrete-space models has been obtained in the 90’s, see [3], [17], [4], [11] and references therein. Within the last decade, a number of new results have been obtained for continuous-space models, see [15], [12], [6], and finally the long-standing initial conjecture from [2] has been proven in [16] in the Markovian case, while there are still open questions in a broader generality.

One can place several workers to clean the space. Greedy service models with 2 and with 3 workers on the positive halflife have been analysed by Schmidt [18]. A model with any number of workers on the positive halfline and its
further extensions are considered in [3].

It appears that there is a significant progress in the studies of greedy service mechanisms, while only a little has been found in the greedy cleaning models.

This paper contributes to the latter direction. We introduce a new greedy cleaning model with several workers and with a “star-like” state space, which is a union of several halflines connected by the origin, and consider its asymptotic behaviour when both the number of workers and the number of halflines tend to infinity, with the latter to be significantly bigger than the former. We can prove that, with probability 1, each worker has eventually to choose a halfline on which they will stay. It is also clear that if the number of halflines is very large compared to the number of workers then the chance of two workers having the same halfline is small, while if the number of workers is strictly greater than the number of halflines then the pigeon principle demands that at least one line is shared. So the question arises as to how many workers are needed, as a function of the number of halflines for there to be a good chance of “doubling up”. We stress here that, as far as we can see, given \(N\), the number of halflines, it is not clear that the probability of ”doubling up” is monotone in the number of workers. Some monotonicity question are discussed in the appendix. This question can be extended to considering when it is possible for \(k\) workers to have the same ultimate direction.

A star-like state space is a very natural object. Such topological structures have been proven to be historically efficient (“All roads lead to Rome”!). They naturally appear in many areas: in (tele)communication networks (like call centres, multilayer networks), biology, physics, etc.

The rest of the paper is organised as follows. In Section 2 we introduce the model and formulate our main results. Then in Section 3 we obtain a number of preliminary results for the model with many halflines. In Section 4, we deal with results on times and prove the first statement of Theorem 1. Then in Section 5 we prove the second statement of Theorem 1 and in Section 6 the last statement. In Section 7 we comment on a possible proof of Theorem 2. Section 8 contains a number of results for an auxiliary model with a single halfline. In Appendix, we collect a few monotonicity properties of our model.
2 Model Description and Main Results

We consider a star-like state space: there are \( N \) halflines that start from the origin in different directions. So the state space is \( \mathcal{X} = \bigcup_{i=1}^{N} \mathcal{R}_i^+ \) where \( \mathcal{R}_i^+ \) is the \( i \)th halfline, and all halflines have a single common point (the origin).

We equip \( \mathcal{X} \) with the following \( L_1 \)-type distance: for two points from \( \mathcal{X} \), with one at distance \( x \) from the origin and the other at distance \( y \), the distance \( d \) between them is equal to \(|x - y|\) if they lie on the same halfline, and \( d = x + y \) if the points belong to different halflines.

Each halfline \( \mathcal{R}_i \) is covered by a homogeneous rate-1 Poisson process of “dust” particles \( \mathcal{K}_i \), and these processes are mutually independent. So \( \mathcal{K} = \bigcup_{i=1}^{N} \mathcal{K}_i \) is a homogeneous Poisson process on the state space \( \mathcal{X} \). In the case where a specific halfline \( \mathcal{R}_i^+ \) is being discussed, we may tacitly identify it with \( \mathcal{R}_i \), the real halfline and, in particular, identify the point on \( \mathcal{R}_i^+ \) at distance \( y \) from the origin with the positive value \( y \).

Let \( 0 < \alpha < 1 \). We defer discussions of \( \alpha \geq 1 \) for a companion article.

We assume there are \( N^\alpha \) “workers” that clean the space from the dust (to be more precise, \([N^\alpha]\) workers, where \([x]\) is the integer part of number \( x \) but in the following where referring to integer quantities we will drop the square brackets). Initially (at time 0) all the workers are located at the origin. Each worker \( j \) has his own clock process which is represented by an independent rate-1 Poisson process \( M_j(t) \). When \( j \)th clock rings, worker \( j \) instantaneously jumps to the closest existing dust particle, removes it and stays at that location until its clock rings again.

We consider asymptotic dynamic properties of the model, as \( N \) grows to infinity.

We use the following terminology: a jump of a worker is an \textit{advance} (along a halfline) if this is either a jump from the origin to the closest dust particle or a jump from one to another dust particle on the same halfline. Otherwise, this is a \textit{skip} (from one halfline to another), this is a jump from (a dust particle on) one halfline to (a dust particle on) another. In the case of a skip, we may also say that a worker returns to, or passes the origin.

We enumerate the lines \( l = 1, 2, \ldots, N \) and the workers \( w = 1, 2, \ldots, N^\alpha \) and let \( l_w(t) = i \) if worker \( w \) is located on line \( i \) at time \( t \) (we let \( l_w(t) = 0 \) if the worker is still at the origin at time \( t \)) and \( d_w(t) \) its distance from the origin at time \( t \).

Let \( \rho_t \) be the distance from the origin to the closest existing dust particle at
time $t$. Let $\sigma_x$ be the first time when some worker, say $w$, skips from one halfline to a dust point at another halfline at a distance at least $x$ from the origin. Clearly, $\sigma_x \geq \sup\{t : \rho_t \leq x\}$ a.s.

For $t \geq 0$, introduce event $A_2^{(t)} = \{l_w(s) \neq l(s), \text{ for all } w_1 \neq w_2 \text{ and all } s \geq t\}$, so the event occurs if all workers are located at different lines at all times $s$ starting from time $t$. Then let $A_2 = \cup_{t \geq 0} A_2^{(t)}$ be the event that eventually all workers stay at different halflines. Further, let $\theta = \theta_N$ be the time instant of the very last skip (of any worker).

Our main results are stated in the following two theorems.

**Theorem 1.** (I) For any $\alpha \in (0, 1)$ and $x > 0$,

$$P(\sigma_x = \infty) \to 1, \text{ as } N \to \infty. \quad (1)$$

Further, for any $\varepsilon > 0$,

$$P(\theta_N \leq N^{1-\alpha+\varepsilon}) \to 1, \text{ as } N \to \infty. \quad (2)$$

(II) If $2/3 < \alpha < 1$, then

$$P(A_2) \to 0, \text{ as } N \to \infty. \quad (3)$$

(III) If $0 < \alpha < 2/3$, then

$$P(A_2) \to 1, \text{ as } N \to \infty. \quad (4)$$

The approach offered is sufficiently robust to permit generalization. Let $A_m^{(t)} = \{\text{for all } s \geq t \text{ distinct } w_1, w_2, \ldots w_m: l_{w_1}(s) = l_{w_2}(s) = \cdots = l_{w_m}(s)\}$ and let $A_m = \cup_{t \geq 0} A_m^{(t)}$. Following the line of the proof of Theorem 1 with natural minor changes, one can obtain

**Theorem 2.** (I) If $(2m - 2)/(2m - 1) < \alpha < 1$, then

$$P(A_m) \to 0, \text{ as } N \to \infty. \quad (5)$$

(II) If $0 < \alpha < (2m - 2)/(2m - 1)$, then

$$P(A_m) \to 1, \text{ as } N \to \infty. \quad (6)$$
3 Preliminary results

The purpose of this section is to provide elementary results on the environment of “dust” particles.

We introduce the filtration \( \{ G_t \}_{t \geq 0} \) where

\[
G_t = \sigma((l_w(s), d_w(s)), s \leq t, 1 \leq w \leq N^\alpha)
\]

is the sigma-algebra generated by the moves of the \( N^\alpha \) workers by time \( t \) (we consider the process to be right continuous).

We now describe the conditional distribution of the Poisson processes on the halflines at times \( t \) (or at stopping times \( T \)). If a halfline \( l = R_i \) is fixed, then we can classify the “information” (for \( l \) given) by four types of jumps occurring at time \( s \) from point \( x \) to point \( y \). To this end we introduce a variable \( D_l(s) \) defined according to the four cases as below:

1) a jump to \( l \), this is either an advance along \( l \) or a skip to \( l \) from another halfline (this means that \( y \) belongs to \( l \) while \( x \) may be either on \( l \) or on any other halfline). Here we get the information \( D_l(s) = y \) (where \( y \) identified with its distance from the origin), this means : “no dust particles on \( l \) within distance \( y \) from the origin”.

2) an advance within \( j \neq l \). Here \( D_l(s) = (y - 2x)^+ \).

3) a skip from \( x \in j \neq l \) to \( y \in k \neq l \). Here \( D_l(s) = y \)

4) a skip from \( x \in l \) to \( y \in j \neq l \). Here \( D_l(s) = y + 2x \).

At time \( t \), let \( I_l(t) = \max_{s \leq t} D_l(s) \), the noneffaced dust particles on the halflines conditional upon \( \{ I_l(t) \}_{l} \) are conditionally independent Poisson processes on the halflines with, for any halfline \( l \), rate one on \([I_l(t), \infty)\) and rate zero on \((0, I_l(t))\).

In particular if \( \tau \) is, say, the \( j \)'th jump time for a given worker, then the environment “seen” from this worker on its halfline away from the origin, given \( G_\tau \), is simply a rate-one Poisson process. (Similarly – for other stopping times representing worker jumps).

Note that the process

\[
X(t) := ((l_w(t), d_w(t)), 1 \leq w \leq N^\alpha; I_l(t), 1 \leq l \leq N)
\]

is a continuous-time Markov process that possesses the strong Markov property.

In this and subsequent sections, we will introduce various “bad” events \( F_i \). Mostly, such an event can be described as \( F_i = \{ T_i \leq r_i \} \) or \( \{ T_i < \infty \} \),
for some fixed time $r_i$ and some stopping time $T_i$ (with respect to filtration $(\mathcal{G}_t)_{t \geq 0}$). When we will say (for time $t \leq r_i$) that $F_i$ “is not happened by time $t$”, we will mean that $T_i > t$.

By the superposition theorem for Poisson processes, the initial distribution of the distances of dust particles (from all halflines) to the origin constitutes a rate-$N$ Poisson process. This high intensity process will have many useful law of large numbers properties. We also wish to establish regularity properties holding on each of the $N$ halflines’ environments. In particular, we wish to show that each halfline cannot have “too many” dust particles close to the origin and that a certain large scale regularity may be assumed (see Proposition 3 below).

According to the introduced dynamics, workers jump (either advance or skip) at Poisson times. We can see that, with a high probability, all workers for a long time will only make skips after the first advance from the origin. The intuition behind this is that, for the first dust point on any halfline, the next dust point on this halfline is usually on the distance of order 1, while the closest point on the other halflines is of the order of two times the distance of the current dust particle from the origin, which will be small compared to the $O(1)$ distance along it current halfline. However we will show that unless the dust environment is negligibly “extreme”, a single isolated worker will stop skipping after having made $\log^3(N)$ consecutive advances.

In our model with many workers acting on the same halflines things can be more complicated. In principle a worker could travel far in a halfline but still return to the origin even if the dust environment is not extreme since a large number of other workers could move to this halfline. As a technical approach, we consider behaviour of workers until the random time when the position of the dust particle closest to the origin (over all halflines) becomes greater than 1. In fact we will see in the following section that this time will be infinity with probability tending to one, as $N$ tends to infinity. (See (i) of the statement of Theorem 1.)

Recall from Section 2 our definitions of $\rho_t$ and $\sigma_x$. In particular, for times before $\sigma_1$, no worker will enter a halfline whose closest dust particle is at distance greater than one from the origin. The value of this is that, as we already claimed, no worker will skip before time $\sigma_1$ after having made $\log^3 N$ consecutive advances. The time $\sigma_1$ will be shown to be infinite (see Section 4), with probability tending to one, so this conclusion, holding up to time $\sigma_1$ will hold forever.

The first two propositions concern the regularity of the dust particles on the
halflines and the regularity of Poisson processes. Since they rely on simple estimates on probabilities for extreme events for Poisson processes on the halfline, we leave the proofs to the reader.

**Proposition 3.** The following events on dust environments for $N$ halflines have probability tending to zero as $N$ tends to infinity:

$F_0 := \{\text{initially, there exists a halfline } l \text{ so that, for some } y \geq \log^2 N, \text{ the number of dust points on } l \text{ within distance } y \text{ from the origin, } M_l(y), \text{ is either less than or equal to } y/2 \text{ or greater than or equal to } 2y\};$

$F_1 := \{\text{initially, there exists a halfline } l \text{ with at least } \log N \text{ dust points within distance } 2 \text{ from the origin}\};$

$F_2 := \{\text{initially, there exists a halfline } l \text{ so that, for some } y \geq \log^2 N, \text{ the number of dust points in } [y, 5y/4] \text{ is outside of the interval (y/8, y/2)}\}.$

**Remark:** To reinforce our discussion of the way "event $A$ has not occurred by time $t$" is used, we say $F_0$ has not occurred by time $t$ (where $t$ is possibly a stopping time), if there does not exist an $l$ and $y \geq \log^2 N$ so that $N_l(t,y) \notin (y/2, 2y)$.

For any given $N$, let $T^{(N)} = \{t^{(N)}_1 < t^{(N)}_2 < t^{(N)}_3 < \ldots \}$ be the ordered distances from the origin to the initial dust particles on the state space. As we mentioned, they form a rate-$N$ Poisson process.

**Proposition 4.** For $T^{(N)}$ as above, the following events have probability tending to zero as $N$ tends to infinity:

$F_3 := \{\text{there exists } i \leq \frac{N}{\log N} \text{ so that } t^{(N)}_{i+N^a} - t^{(N)}_i \notin (\frac{1}{2N^{1-a}}, \frac{1}{2N^{1-a}})\}.$

$F_4 := \{\text{there exists } i \in (N^{1/2}, N/\log N) \text{ so that } \left|\frac{N^{(N)}_i}{i} - 1\right| \geq 2/\log N\}.$

The next result is simple to prove but is useful in that it ensures a large supply of halflines which will be first visited at reasonably predictable times.

**Proposition 5.** For $0 < k_2 < k_1 < \infty$, let $W_N$ be the number of halflines whose closest to the origin dust point (at time 0) lies in the interval

$$(e^{-k_1\sqrt{\log N}}, e^{-k_2\sqrt{\log N}}).$$

As $N$ tends to infinity,

$$\frac{W_N}{Ne^{-k_2\sqrt{\log N}}} \to 1$$

in probability.
The next result addresses the worker process rather than merely the initial dust particle environment. We wish to show that when the process is reasonably advanced a worker after a skip (occurring definitely before time $\sigma_1$) has a reasonable possibility of advancing $\log^3 N$ times without skipping.

**Lemma 6.** Fix $k > 0$ and $0 < \varepsilon < k \wedge (1 - \alpha)/2$. Consider the worker/dust process on $N$ halflines and let $t$ be a stopping time at which worker $w$ skips to a dust particle on the halfline denoted $l$ at the distance from the origin $x \in \left[ e^{-k\sqrt{\log N}}, e^{-(k-\varepsilon)\sqrt{\log N}} \right]$.

Let $A$ be the event that, after time $t$, the worker $w$ advances at least $\log^3 N$ consequent jumps along $l$ without skipping and within $2\log^3 N$ units of time and that all other workers, who were on $l$ at the moment $t$ of the worker’s arrival (call them “black” workers), skip for other halflines without having advanced along $l$ by time $t + \log^3(N)$ and no further workers enter $l$. Let $A(\sigma_1)$ be the event that event $A$ has not been ruled out by time $\sigma_1$, which means that the selected worker $w$ has not skipped up to this time and if it has jumped $\log^3 N$ times then it has done so within time $2\log^3 N$ and every “black” worker is either continuing to remove its current dust particle or has left after that. Then, on the event $D(t, w)$,

$$P(A(\sigma_1) \setminus (F_1 \cup F_3 \cup F_4) | G_t) \geq \frac{1}{(\log N + 1)^2} \left( \frac{1}{N} \right)^{k^2(1+\varepsilon/k)} 2^{2\log 2} - 2e^{-\log^4 N},$$

for all $N$ large enough.

**Proof.** We first note that the fact that worker $w$ skips to a site distance $x < 1$ from the origin implies that $t < \sigma_1$. Given our definition of event $A$ and of event $F_1$, we can suppose that the number of workers present on halfline $l$ at time $t$ is bounded by $\log N - 1$ at all times less than $\sigma_1$.

We secondly note that, by the Markov property for Poisson processes, the conditional distribution of the dust points on $(x, \infty)$ given $G_t$ is simply a rate-1 Poisson process. Therefore we can apply Lemma 20 and use simple Markov properties of Poisson processes to conclude that, given $G_t$ and for all $N$ large enough, the conditional probability of the intersection of the following two events $E_1$ and $E_2$ is at least $\exp \left( -(1 + \varepsilon/200) \log^2 x/(2\log 2) \right)$. We define the events as follows: on the interval $(x, \infty)$ of halfline $l$,

$E_1 := \{ \text{there are a single dust particle on } [5x/3, 2x] \text{ and a single dust particle on } [25x/9, 10x/3] \text{ and there is no other dust particles on } [x, 10x/3] \}$, and

$E_2 := \{ \text{for a single half line model with only one worker at } x, \text{ given these two}$
dust particles the environment on $[10x/3, \infty)$ and with a single dust particle added at the origin, this worker will make $\log^3 N$ jumps to the right, without returning to the origin).

Given this dust environment event and assuming that event $F_1$ does not occur, the conditional probability that, among the workers present (worker $w$ and all “black” workers), the next two moves are made by the worker $w$ is at least $(\log N + 1)^{-2}$. So the probability that the two events above occur is at least $(\log N + 1)^{-2} \exp \left( -(1 + \varepsilon/200) \log^2 x/2 \log 2 \right)$.

Now, unless one of the following events occur:

a) $w$ takes time greater than $2 \log^3 N$ to make $\log^3 N$ moves or
b) the number of moves by any worker in the next $2 \log^3 N$ time units exceeds $4 \log^3 N$ or
c) either $F_3$ or $F_4$ is violated,

we have that $\rho_s$, the position of the closest dust particle to the origin does not change by more than $x/3$ in the time it takes for $w$ to make $\log^3 N$ jumps and so event $A$ occurs.

The above result will now be used to ensure that for any $k < \sqrt{2 \log 2(1 - \alpha)}$, there is an “appropriate” chance that a worker skipping into a new halfline at distance $x$ greater than $e^{-k\sqrt{\log N}}$ will then always advance along it until time $\sigma_1$.

**Lemma 7.** Fix a stopping time $T$ and worker $w$. Let $\lambda = \lambda(T, w)$ be the first time after $T$ that $w$ makes $\log^3 N$ consecutive advances without skipping. There exists universal strictly positive $c_0$ so that

$$P(w \text{ skips in } (\lambda, \sigma_1) \setminus (F_1 \cup F_2)) \leq e^{-c_0 \log^2 N}$$

(We interpret the event to be empty if $\sigma_1 \leq \lambda$).

**Remark:** Given the right-hand side upper bound, the lemma implies that the conclusion will hold for all workers, outside a set of small probability.

**Proof.** We let $l$ denote $l_w(\lambda)$ and $x_0$ denote $d_w(\lambda)$.

Under $F_2^c$ the distance $x_0$ must be at least $\log^2 N + \frac{\log^3 N - 2 \log^2 N}{4} \geq \frac{\log^3 N}{8}$ for $N$ large. Note that on $F_1^c$ from $\lambda$ up until time $\sigma_1$ no further workers may enter $l$ and the number of other workers on the halfl ine is less than or equal to $\log N - 1$.  

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For $i \geq 1$, we define $x_i = x_0(\frac{5}{4})^i$. We may assume, without loss of generality, that there is no dustpoints at any of these points. We write $J_i$ for the interval $(x_i, x_{i+1})$ and note that, on the complement of $F_2$, the number of dustpoints in $J_i$ must exceed $x_0(\frac{5}{4})^i/8$, for all $i$. We write $A_i$ for the event that $w$ cleans a point in $J_i$ before skipping and observe that, conditional upon event $A_i$, each dustparticle in $J_{i+1}$ has conditional probability (given $A_i$ and the cleaner of preceding particles) of being cleaned greater than $\frac{1}{\log N}$. Thus we have

$$P(A_{i+1} \cap A_1 \cap A_2 \cdots \cap A_i \cap F_1^c \cap F_2^c) \leq \left(1 - \frac{1}{\log N}\right)^{x_0(\frac{5}{4})^{i+1}/8}. $$

The result follows.

\[ \square \]

4 Results on Times

In what follows, we say that, from time $t$ on, a worker $w$ escapes, or escapes to infinity (along a certain halfline $l$), if the worker only advances along $l$ within time interval $(t, \infty)$. Then the escape time is the time of the last skip of the worker (or time 0 if there is no skip at all).

A major part of our work is understanding process $(\rho_s : s \geq 0)$. We first give an easy upper bound.

**Lemma 8.** Fix $k \in (0, \infty)$ and let $s_0 = N^{1-\alpha}e^{-k\sqrt{\log(N)}}$. For any $\delta > 0$

$$P(\rho_{s_0} \geq e^{-(k-\delta)\sqrt{\log(N)}} \setminus F_4) < e^{-N^{(1-\alpha)/2}},$$

for $N$ large.

**Proof.** Using the Chernoff bound, we may conclude that, by time $s_0$, outside probability bounded above by $N^\alpha e^{-bs_0}$, each worker has made less than $2s_0$ jumps, for some universal strictly positive $b$.

Thus the number of jumps by time $s_0$ made by $\rho$, the position of the closest dust particle to the origin, is bounded above by $2\exp\left(-k\sqrt{\log N}\right)$, outside the small probability given above. We have $\rho_{s_0}$ is not bigger than the $2\exp\left(-k\sqrt{\log N}\right)$th smallest value of the rate-$N$ Poisson process of dustparticles. Thus, outside of event $F_4$, the desired conclusion holds.

\[ \square \]

This proof (and similar “upper bound” results for $\rho$) simply rely on large deviations bounds for Poisson processes: large positive deviations of the worker
jump process, lower large deviation bounds for the dustparticle Poisson process. The “lower bounds” for \( \rho \) are not so immediate: indeed they are not true for \( k < c := \sqrt{2(1 - \alpha) \log 2} \) as we will see in this section. The basic problem is that while we must have a good number of worker jumps not all of them are skips (which necessarily result in a change in \( \rho \)). We now get some simple bounds on the jump rates for \( \rho \).

We will use the results in Section 8 for workers on a single halfline to analyze the evolution of our model.

We introduce a comparison model. Let \( (X^x_t : t \geq 0) \) be the position at time \( t \) of a worker \( w \) on \([0, \infty)\) starting at \( x \), where dustparticles are placed at 0 and on \((x, \infty)\) according to a rate 1 Poisson process. The position 0 is taken to be absorbing. This process is useful to compare with our multi worker process on \( X \). Here is a simple observation that follows from natural monotonicity properties (and does not need a proof).

**Proposition 9.** Consider a worker \( w \) who skips at time \( t \) to halfline \( l \) in \( X \) at distance \( y \) less than \( x \) from the origin. At time \( t \) we generate a process \( (X^x_s : s \geq 0) \) by using the Poisson process of jump times for \( w \) (shifted by \( t \)) as the jump times for \( X \) and the dust particles environment on \((3x, \infty)\) for \( X \) to be the dustparticles environment on \((y, \infty)\) (on \( l \) shifted by \( 3x - y \)). Then with this coupling, if \( X^x_s = 0 \) for \( s \leq \rho(2x - t) \), then \( w \) has skipped by time \( s \).

We now use Lemma 20 from the final Section to show the following

**Lemma 10.** Fix \( \delta > 0 \). Assume that worker \( w \) skips to a halfline \( l \) at random time \( t \leq \sigma_{e^{-k\sqrt{\log(N)}}} \). Let \( A \) be the event that \( w \) advances to distance \( 1/100 \) before skipping. Then \( P(A|F_1) \) is bounded above by \( N^{-(k^2 - \delta)/(2 \log(2))} \), for all \( N \) sufficiently large.

**Proof.** We apply Proposition 9 with \( x = e^{-k\sqrt{\log(N)}} \). We immediately have that the probability that \( w \) advances to 1/100 before skipping and before \( F_1 \) does not occur is less than

\[
P(X^x_{e^{-k\sqrt{\log(N)}}} \text{ reaches } 1/100) + P(\sigma_{e^{-k\sqrt{\log(N)}}} - \sigma_{e^{-k\sqrt{\log(N)}}} > Z)
\]

where \( Z \) is the time for \( w \) to make \( \log(N) \) additional jumps after random time \( t \). The first bound is less than \( N^{-(k^2 - \delta)/(2 \log(2))}/2 \) for \( N \) large by Lemma 20 and the second is of smaller order as \( N \) tends to infinity, by standard Poisson computations.

As noted, the conditional rates of the Poisson processes are known. Though \( \rho_t \) is not adapted to our filtration, the times of jumps of this process are
adapted: there is a jump (that is a change) in $\rho_t$ if at time $t$ a worker skips to another halfline. The jump rate for our process at time $t$ given the filtration $\{G_s\}_{s \geq 0}$,
\[
\lim_{h \to 0} P(\rho_{t+h} \neq \rho_t | G_t) / h,
\]
remains constant over time intervals free of worker jumps and on such intervals is at least the probability that the next jump is a skip (summed over all $N^\alpha$ workers). We similarly define the skipping rate to be
\[
\lim_{h \to 0} P( \text{a skip occurs in time interval } (t, t+h) | G_t) / h.
\]

Lemma 11. Let event $F_5$ be defined by
\[
F_5 := \{ \exists s \leq \frac{N^{1-\alpha}}{e^{-(c+\frac{2}{3})\sqrt{\log N}}} \text{ so that the total skipping rate is less than } 9N^\alpha/10 \}.
\]
Then $P(F_5) \to 0$ as $N$ tends to infinity.

Remark: The $9/10$ bound could be improved but is sufficient for our purposes.

Proof. As noted above, jumps of workers through the origin necessarily change $\rho$, so we only need to analyse workers on a halfline. It is easy to see that if a worker (at $x$ on halfline $l$) is at distance less than $1/100$ from the origin then, conditional upon jumping, they will skip unless either there is a dustparticle in $(x, x+2/100)$ on $l$ or $\rho_t \geq 1/100$. The latter is contained in the event
\[
\{ \rho \geq \frac{N^{1-\alpha}}{\exp((c+\frac{2}{3})\sqrt{\log N})} \geq 1/100 \}
\]
which, by Lemma 8, has probability tending to zero. So it remains to show that the number of workers that reach distance $1/100$ from the origin before time $\frac{N^{1-\alpha}}{e^{-(c+\frac{2}{3})\sqrt{\log N}}}$ is small compared to $N^\alpha$. Fix a worker $w$. By the Markov property, each time they skip to a point on a halfline at distance less than $e^{-(c+\frac{2}{3})\sqrt{\log N}}$ from the origin, their chance of advancing along the halfline to distance $1/100$ before skipping is, by Lemma 10, less than $N^{-(c+\frac{4}{3})^2/2 \log(2)} < N^{-(1-\alpha+b \epsilon)}$ (for $b = \frac{2}{3} \sqrt{\frac{1-\alpha}{2 \log(2)}}$ and $N$ large). Thus the expected number of workers to reach distance $1/100$ from the origin by time $\frac{N^{1-\alpha}}{e^{-(c+\frac{2}{3})\sqrt{\log N}}}$ is less than $N^{-\epsilon b}$, and the result follows. \qed
This bound and basic Poisson process bounds immediately yield

**Corollary 12.** Let \( a \geq 0, R \geq 1, b = a + R \) and let \( N(a,b) \) be the number of jumps of \( \rho \) in time interval \([a,b]\). There exists an event \( B_N \) of probability tending to one as \( N \) tends to infinity so that, for every \( 1 \leq R \leq N^{1-\alpha} \), the conditional probability \( \mathbb{P}(C(R)|B_N^c) \) of the event

\[
C(R) := \{ \exists a : \rho_b \leq \frac{1}{e^{-(c+\frac{\varepsilon}{2})\sqrt{\log N}} \text{ and } |N([a,b]) - RN^\alpha| > RN^\alpha/5 \}
\]

is bounded above by \( N/R e^{-kRN^\alpha} \), for some universal \( k > 0 \).

Another corollary is

**Corollary 13.** For any \( \varepsilon > 0 \),

\[
\mathbb{P}\left( \sigma_{e^{-(c+\varepsilon)\sqrt{\log N}}} < 2N^{1-\alpha} e^{-(c+\varepsilon)\sqrt{\log N}} \right) \to 1,
\]
as \( N \) tends to infinity.

**Proof.** We simply note that the complement to the event \( \sigma_{e^{-(c+\varepsilon)\sqrt{\log N}}} < 2N^{1-\alpha} e^{-(c+\varepsilon)\sqrt{\log N}} \) is a subset of a union of three events, \( F_5, F_6 \) and \( F_7 \), where \( F_5 \) was defined above,

\[
F_6 := \{ \text{the number of all dust points within distance } e^{-(c+\varepsilon)\sqrt{\log N}} \text{ from the origin is greater than } 3Ne^{-(c+\varepsilon)\sqrt{\log N}/2} \}
\]

and

\[
F_7 := \{ \text{the number of skips by time } 2Ne^{-(c+\varepsilon)\sqrt{\log N}} \text{ is less than } 3Ne^{-(c-\varepsilon)\sqrt{\log N}/2} \},
\]

and the probability of each of them tends to 0 when \( N \) grows, that of \( F_5 \) is simply Lemma 11, \( F_6 \) involves simple Poisson process bounds and that for \( F_7 \) is a consequence of lemma 11. \( \square \)

Our objective in the remainder of this section is to show that, in a crude sense, the dominant part of the workers escapes occur “around” time \( N^{1-\alpha} \exp(-c\sqrt{\log N}) \) and that, for any \( \varepsilon \in (0,c) \), we get \( \sigma_{\exp(-(c-\varepsilon)\sqrt{\log N})} = \infty \) with probability tending to one, as \( N \to \infty \).

We now address upper bounds on times beyond which all workers do not return to the origin (i.e. only advance along halflines).
Theorem 14. For the worker/dust process on \(N\) halflines, for each \(\varepsilon > 0\),

\[
P(\sigma e^{-(c - \varepsilon)\sqrt{\log N}} < \infty) \to 0
\]

as \(N\) tends to infinity, where as before \(c = \sqrt{2(1 - \alpha) \log 2}\).

Remark: This implies that the first statement of Theorem \(\Pi\) holds. In particular \(P(\sigma_1 < \infty)\) tends to zero as \(N\) tends to infinity. Thus all the preceding statements involving limiting probability of behaviour before time \(\sigma_1\) become strengthened to results over all time.

Proof. We fix \(\varepsilon > 0\) small compared to \(c\) and define two stopping times for our process:

\[
T_1 = \sigma \exp\left((-\varepsilon)\sqrt{\log N}\right) \quad \text{and} \quad T_2 = \sigma \exp\left((-\varepsilon)\sqrt{\log N}\right).
\]

We wish to show that \(T_2\) equals infinity with probability tending to one. In fact both stopping times can well be infinity and in fact our proof (and the arbitrariness of \(\varepsilon\)) will show that with probability tending to one as \(N\) tends to infinity, this is so. Our first point is that on \(T_1 < \infty\), \(T_2 - T_1 \geq N^{\frac{1}{8}}\log N\) outside an event of probability tending to zero as \(N\) tends to infinity. This follows from Proposition 5 (applied with \(k_2 = (c - \varepsilon)\) and \(k_1 = (c - \varepsilon)\)) and the fact that the corresponding dust particles are removed at rate bounded by \(N^\alpha\). Let \(A_1\) be the (bad) event that this lower bound does not hold.

Again, on the event \(\{T_1 \text{ is finite}\}\), we have by elementary large deviations bounds on rate-1 Poisson processes that (outside an event of probability tending to zero as \(N\) tends to infinity) every worker will make at least \(N^{1-\alpha}e^{-(c - \varepsilon)\sqrt{\log N}}/8\) jumps in time interval \((T_1, T_1 + N^{1-\alpha}e^{-(c - \varepsilon)\sqrt{\log N}}/6)\). The event that these laws of large numbers are not respected will be denoted by \(A_2\).

A third bad event, \(A_3\), is that for at least one of the \(N^\alpha\) workers, say worker \(w\), skips in time interval \((\lambda(T_1), \sigma_1)\) (where the stopping time \(\lambda(T_1)\) is understood to be specific to worker \(w\)). By Lemma 7 and the following Remark, this has probability tending to zero as \(N\) becomes large.

Finally let \(A_4\) be the event that, for some worker \(w\), \(\lambda'(T_1)\) is more than \(T_1 + N^{1-\alpha}e^{-(c - \varepsilon)\sqrt{\log N}}/6\) where \(\lambda'(T_1)\) is the time of the last skip preceding \(\lambda(T_1)\) as defined in Lemma 7. If \(A_2\) does not occur this would imply that the worker made

\[
N^{1-\alpha}e^{-(c - \varepsilon)\sqrt{\log N}}/(8 \log^3 N)
\]
skips before time $T_1 + N^{1-\alpha} e^{-(c-\frac{\epsilon}{4})\sqrt{\log N}}/6$ without one of the skip times being $\lambda'(T_1)$. By Lemma 6 and the Markov property, $P(A_4)$ tends to zero as $N$ tends to infinity.

The result now follows by noting that on the intersection of the complements of the $A_i$ we have, by definition of $A_3$, that every worker does not skip on interval $(\lambda(T_1), \sigma_1)$ and hence not on $(\lambda'(T_1), \sigma_1)$. Furthermore by the definition of $A_4$ and $A_2$ we also have that no worker will skip on interval $(T_1 + N^{1-\alpha} e^{-(c-\frac{\epsilon}{4})\sqrt{\log N}}/6, \sigma_1)$. But this and $A_4$ imply together that $T_2$ must be infinite.

We now wish to relate this to actual times. We note that, by Lemma 6, for $\varepsilon > 0$ and $N$ large, after time $\sigma_{\exp(-(c+\varepsilon)\sqrt{\log N})}$ every time a worker changes halfl ine (and thus moves to a closest dustpoint) it has a chance $\frac{1}{N^{(1-\alpha+3\varepsilon)/d}}$ of coming to a halfl ine which it will not leave before time $\sigma_1$.

The chance that a worker can make $N^{1-\alpha+3d\varepsilon}$ visits without the above event occurring does not exceed $e^{-N\varepsilon}$. So with probability tending to zero as $N$ becomes large none of the $N^\alpha$ workers satisfies this.

Let $s_w$ be the number of skips of worker $w$ by time $\sigma_1$. Let event $F_8^c$ be that, for each worker and for each of his first $\min(s_w, N^{1+2\varepsilon})$ skips to another halfl ine, all the numbers of advances between consecutive skips do not exceed $\log^3 N$.

Given Lemma 6 we can say that $P(F_8)$ tends to zero as $N$ becomes large. But unless event $F_1$ occurs, each jump to a new halfl ine after time $\sigma_{\exp(-(c+\varepsilon)\sqrt{\log N})}$ gives a chance at least

$$\frac{K}{N^{1-\alpha+3\varepsilon/d} \log^2 N}$$

of escape, for universal $K$. So the probability that there exists a worker $w$ which has not escaped by time $N^{1-\alpha+3\varepsilon/d} \log^5 N$ is bounded by

$$P(F_1) + P(F_8) + \left(1 - \frac{K}{N^{1-\alpha+3\varepsilon/d} \log^2 N}\right)^{N^{(1-\alpha+3\varepsilon/d) \log^2 N}}$$

for some universal $K$. This implies statement (2) of Theorem 1 since all three probabilities above tend to 0 when $N$ grows.
5 Proof of existence of “double” halflines

In this section we prove Theorem 1 for the case of $\alpha > 2/3$. That is with probability tending to one as $N$ tends to infinity, there will be a halfline on which a pair of workers will eventually travel. We fix $\varepsilon < (\alpha - 2/3)/100$.

We note that, by Corollary 13 for $\varepsilon > 0$ fixed with probability tending to one (as $N$ tends to infinity), $\sigma_{\exp(-(c+\varepsilon/2)\sqrt{\log N})} < \infty$. Furthermore by Proposition 5, with probability close to one, we have at least $N \exp(-(c+\varepsilon/2)\sqrt{\log N})/2$ halflines having the property that their first dustparticle is (at time 0) at a distance from the origin in the interval

$$\left(\exp(-(c+\varepsilon)\sqrt{\log N}), \exp(-(c+\varepsilon/2)\sqrt{\log N})\right).$$

We denote by $V$ the set of such halflines and by $V(t)$ the subset of halflines in $V$ which have not been visited by any worker by time $t$.

So we may conclude that if we define the stopping times $s_i$ recursively by $s_0 = \sigma_{\exp(-(c+\varepsilon)\sqrt{\log N})}$ and $s_{i+1} = \inf\{t > s_i : \text{a worker skips to a halfline in } V(s_i)\}$, then, with probability tending to one as $N$ tends to infinity,

$$s_N \exp(-(c+\varepsilon/2)\sqrt{\log N})/2 < \sigma_{\exp(-(c+\varepsilon)\sqrt{\log N})}.$$

We denote by $l_i$ and $x_i$ the halfline and dustparticle position associated with $s_i$.

The advantage of considering just the stopping times $s_i$ is that at time $s_i$ there will be a unique worker on $l_i$ and we will have, conditional upon $G_{s_i}$, a rate-1 Poisson dust environment on $(x_i, \infty)$.

We say that stopping time $s_i$ is bold if

- (I) the position $x_i$ is within the distance
  $$[e^{-(c+\varepsilon)\sqrt{\log N}}, e^{-(c+\varepsilon/2)\sqrt{\log N}}]$$
  from the origin;
- (II) the second dustparticle $x_i(2)$ on $l_i$ is within distance $N^{-(1-\alpha+\varepsilon)}$ of $x_i$;
- (III) the third dustparticle $x_i(3)$ on $l_i$ is of distance at least $\frac{5x_i}{4}$ from the origin;
- (IV) the environment on the line $l_i$ is “$2\varepsilon$ good”. That is: if we consider an auxiliary model of a single halfline $l_i$ with dust particles located at $x_i, x_i(2)$ and $x_i(3)$, plus the given dustparticle environment within $(x_i(3), \infty)$ and an extra dust particle at the origin, and if we assume that there are only two
workers, who are initially located at $x_i$ and $x_i(2)$, then both workers have chance at least $\frac{1}{N^{\epsilon/2}}$ of escaping the origin.

It is clear from Lemma 23 and basic properties of the Poisson process that the probability that $s_i$ is bold is at least $\frac{K}{N^{1-\alpha+\epsilon}} \times \frac{1}{N^{2(1-\alpha+\epsilon)}}$, for universal $K$.

We say that $s_i$ is a success if it is bold and, in addition,  

(i) a second worker arrives at the second nearest dust point within time $N^{-\epsilon/2}$ and the first worker at $x_i$ does not move within this time interval and 

(ii) the first and second workers do not pass the origin again, having made $\log N$ jumps in next $N^{\epsilon/2}$ units of time.

As a useful comparison process for jump time $s_i$ if (i) above occurs at time $\tau_i$, then $Y(l_i, \tau_i)$ will denote the 2 worker process on $l_i$ with initial worker positions being those on $l_i$ at time $\tau_i$ with the given dustparticle environment on $l_i$ augmented by a dustparticle at the origin and with the two jump Poisson processes given by the two workers on $l_i$ in the larger model.

We see easily that the probability that $s_i$ is a success is greater than or equal to $cN^{3(1-\alpha+\epsilon)}$, for universal $c$. Let $A_i$ be the event that $s_i$ is a success. The $A_i$ are (slightly) dependent, so we need to attend to some technical issues to complete the proof.

Let us define event $F_{10}$ to be that for some Poisson jump time $t$ in interval $(\sigma(e^{c+\epsilon/2}), \sigma(e^{c+\epsilon/2}+\sqrt{\log N}))$ we have that there are less than $\frac{N^{\alpha-\epsilon/2}}{2}$ skips by workers in time interval $(t, t+N^{-\epsilon/2})$ or there are for the same interval more than $2N^{\alpha-\epsilon/2}$ worker jumps or event $F_1$ occurs. We have by Corollary 13 and Lemma 11 and basic Poisson bounds that $P(F_{10})$ tends to zero as $N$ tends to infinity.

It has to be noted, for $N$ large, that if $F_{10}$ does not occur then for each $s_i$ provided the associated worker does not move in the succeeding time interval of length $N^{\epsilon/2}$, then condition (i) of success will automatically be satisfied and that (provided $F_2$ does not occur) then the required movement of the two workers on $l$ implies that no further workers arrive on $l$. Thus if we consider $\tau_i$ to be the arrival of the second worker on $l$, then $s_i$ is a success if and only if process $Y(l_i, \tau_i)$ satisfies corresponding conditions. These considerations yield the following result which clearly implies statement (iii) of Theorem 1.

Given these facts and results we have

**Lemma 15.** There is a coupling between the worker/dust process and a collection of indicator random variables $\{K_i\}_{i \leq N \exp(-c+\epsilon/2)\sqrt{\log N}/2}$ so that
• The $K_i$ are i.i.d. with $P(K_i = 1) = N^{-3(1-\alpha)-4\epsilon}$.

• On event $B_N^c$ where $P(B_N) \to 0$ as $N \to \infty$, we have $K_i = 1 \Rightarrow s_i$ is a success.

From this the desired result is shown.

6 Nonexistence of double escape

To show that if $\alpha < \frac{2}{3}$, we do not obtain any halflines on which two workers escape, we choose $\varepsilon > 0$ so that $100\varepsilon < \frac{2}{3} - \alpha$.

We will need the following result which has affinities with Proposition 5

**Lemma 16.** Fix $k > 0$. With probability tending to one as $N$ tends to infinity, for all $r \leq e^{-k\sqrt{\log(N)}}$,

$$\sigma_{r+N^{-(1-\alpha)}} \geq \sigma_r + 1/4.$$  

**Proof.** We fix $r \leq e^{-k\sqrt{\log(N)}}$. The event

$B_r = \{ \sigma_{r+N^{-(1-\alpha)/2}} \leq \sigma_r + 1/4 \}$ is contained in the union of events 

$B_{r,1} = \{ \text{number of worker jumps between } \sigma_r \text{ and } \sigma_r + 1/4 \text{ is at least } 3N^\alpha/7 \}$

and

$B_{r,2} = \{ \text{number of halflines so that the first dust particle is in } (r, r + N^{-(1-\alpha)/2}) \text{ is at most } 3/7N^\alpha \}.$

By elementary Poisson and Binomial tail probability bounds we find that $P(B_r) \leq e^{-c_1N^\alpha}$, for some universal $c_1 > 0$ and uniformly in $r$, for $N$ large enough. Taking $r_i = (i - 1)N^{-(1-\alpha)/10}$ for $i = 1, 2, \ldots, 10N^{1-\alpha}e^{-k\sqrt{\log(N)}}$, we get

$$P(\bigcup_i B_{r_i}) \to 0$$

as $N$ tends to infinity, which gives the result. \qed

For worker $u$ we write $D'(u)$ for the event that there exists another worker $v$ so that $u$ and $v$ escapes to infinity on the same halfline and that the final arrival time of $u$ to this halfline precedes that of $v$. In general, other workers may also escape along the same halfline and their final arrival times may occur before that of $u$. For technical reasons we will work with event $D(u) = D'(u) \cap (\bigcup_{i=1}^{10} F_i)^c$ where $F_i$ are “bad” events of probability tending to zero as $N$ becomes large. Most of the $F_i$ have already been introduced,
some remain to be defined.

Let \( F_{12} \) be the union of the following three events, \( \{ \sigma_{e - (c - \varepsilon) \sqrt{\log N}} < \infty \} \) and \( \{ \text{there is a worker which has not escaped by time } N^{1-\alpha + \varepsilon} \} \) and \( \{ \text{there is a worker making } 2N^{1-\alpha + \varepsilon} \text{ jumps by time } N^{1-\alpha + \varepsilon} \} \).

By the first statement of Theorem 1 (proven at the end of Section 4) and Theorem 14 and elementary Poisson bounds, the probability \( P(F_{12}) \) tends to zero as \( N \) tends to infinity.

Let \( F_{13} \) the union of two events, \( \{ \text{there is a worker } u \text{ so that for some jump (of } u \text{) while } t \leq N^{1-\alpha + \varepsilon}, \text{ the worker makes less than } N^{\varepsilon/2} \text{ jumps in the next } N^{\varepsilon/2} \text{ time units } \} \) and \( \{ \text{the conclusion of Lemma 16 fails to hold} \} \).

It is clear \( P(F_{13}) \to 0 \) as \( N \) becomes large. We wish to analyze \( D(u) \). We first note that \( D'(u) = \bigcup_j D'(u, j) \) where \( D'(u, j) \) is the event that after the \( j \)'th jump of worker \( u, \tau^u_j \), but before the \( (j + 1) \)'st a different worker skips to the current halfline of \( u \) and that thereafter the two workers do not skip.

An advantage of working with the events \( D(u) \) is that since \( D(u) \subset F_{12} \) for each \( u \)

\[
D(u) = \bigcup_{j=1}^{2N^{1-\alpha + \varepsilon}} D(u, j)
\]

where \( D(u, j) = D'(u, j) \cap D(u) \). Another advantage of using the events \( D(u, j) \) is the following

**Lemma 17.** Event \( D(u, j) \) is contained in the event that after the \( j \)'th jump of \( u \) there is a dustparticle on the same halfline within distance \( N^{1-\alpha + \varepsilon} \) of \( u \), as \( N \) becomes large.

**Proof.** If \( F_{13} \) does not occur, then worker \( u \) in the time interval \( (\tau^u_j, \tau^u_j + N^{\varepsilon/2}) \) must make at least \( N^{\varepsilon/2}/2 \gg \log(N) \) jumps. So if \( F_1 \) does not occur, then in this time interval either \( u \) skips or all dustparticle up to distance 1 are cleaned. But, again if \( F_{13} \) does not occur so the conclusions of Lemma 16 hold, then unless there is a dustparticle within distance \( N^{1-\alpha + \varepsilon} \) of \( u \) at time \( \tau^u_j \), no particles can skip onto the halfline of \( u \) before \( \sigma_1 \) which is infinity if \( F_{12} \) does not occur.

In what follows, we need the following notion. An array of points \( P \) on interval \( (x, \infty) \) is said to be \( k- (\delta) \) blocking if there exists \( (y, 2y] \subset (\delta, 1) \) so that \( (P - x) \cap (y, 2y] \) has fewer than \( k \) points. Here \( P - x \) is simply the translation of \( P \) for \( x \) units to the left. This is discussed further in Section 8.
Lemma 18. If a worker $w$ skips to a halfline $l$ at random time $t \leq \sigma e^{-(c-\varepsilon)\sqrt{\log(N)}}$, where $u$ is also located then if the dustparticle environment to the right of $w$ is $2 - (2e^{-(c-\varepsilon)\sqrt{\log(N)}})$ blocking then it is not possible that two of the workers currently on the halfline escapes on $l$ (without subsequent skipping) before time $\sigma e^{-(c-\varepsilon)\sqrt{\log(N)}}$.

The above result, the Markov property, Corollary 22 (and the definition of event $D(u)$ immediately imply

Lemma 19. For all $u$ and $j \leq 2N^{1-\alpha+\varepsilon}$ and for all $N$ sufficiently large,
\[ P(D(u, j)) < N^{-3(1-\alpha)+3\varepsilon}. \]

Proof. This simply follows since by Lemmas 17 and 18. This gives that for $D(u, j)$ to occur there must be a dustparticle within $N^{1-\alpha+\varepsilon}$ of $u$ at time $\tau_j$ and that and the dustparticle environment to the right of this dustparticle cannot be $2-$blocking. The bound now follows from the Markov property and Corollary 22.

Proof of Theorem 1 (III) The bound provided by Lemma 19 is sufficient to show part (III) of Theorem 1. We have that the probability of two workers escaping on the same halfline is less than
\[ P(\bigcup_{i=1}^{13} F_i) + P(\cup_u D(u)) = P(\bigcup_{i=1}^{13} F_i) + P(\cup_u \cup_{j=1}^{2N^{1-\alpha+\varepsilon}} D(u, j)) \]
\[ \leq P(\bigcup_{i=1}^{13} F_i) + N^\alpha 2N^{1-\alpha+\varepsilon} N^{-3(1-\alpha)+3\varepsilon} \]
which is $P(\bigcup_{i=1}^{13} F_i) + 2N^{\alpha - 2(1-\alpha)+4\varepsilon} = o(1) + 2N^{-(2-3\alpha)+4\varepsilon}$ which tends to zero by our assumption on $\varepsilon$.

7 Comments on the Proof of Theorem 2

The arguments given in the two preceding sections readily generalize to the case of more than two workers escaping together. So we provide only a sketch of the proof of Theorem 2.

To show that event $A_m$, as defined in Section 2, is likely for $\frac{2m-2}{2m-1} < \alpha$ as $N$ becomes large, we fix $0 < \varepsilon \ll \alpha - \frac{2m-2}{2m-1}$ and operate on times less than $e^{-(c+\varepsilon)\sqrt{\log(N)}}$ for which the evolution of $\rho$, is predictable and governed by laws of large numbers. Proposition 5 implies that we will be visiting many
halflines for the first time at a near deterministic rate. We simply modify the definition of bold given in section 5: (II) is changed to require that beyond the first dust particle there are \(m - 1\) dust particles within distance \(N^{1-\alpha-\varepsilon}\) and (IV) is changed to require that the dust particle environment is such that \(m\) particles can escape with “reasonable” probability.

To show that for \(\alpha < \frac{2m-2}{2m-1}\) event \(A_m\) has small probability for \(N\) large we first argue, as in Section 6, that this event is essentially the event that there is a halfline with a dust particle close to the origin so that there are \(m - 1\) other dust particles within \(N^{1-\alpha+\varepsilon}\) of the first (for \(0 < \varepsilon \ll \frac{2m-2}{2m-1} - \alpha\)) and that the environment is not \(m\)-blocking (rather than 2-blocking). Thereafter the argument is the same.

8 Auxiliary results – Cleaning process on the halfline

Lemma 20. Consider an auxiliary model, with a single halfline and a single worker which is initially located at distance \(x > 0\) from the origin. Assume that initially there are infinitely many dust particles on the halfline, with one locating at the origin and all the others at points of a rate-1 Poisson process on the set \((x, \infty)\). The worker always jumps to the closest existing dust particle and removes it. Let \(D(x)\) be the event that the dust particle at the origin will be never removed and let \(P_1(x)\) be its probability. Then

\[
\lim_{x \to 0} \frac{\log P_1(x)}{\log^2 x} = -\frac{1}{2 \log 2}.
\]

Here is an extension of Lemma 20 onto the case of \(k\) workers located at the same point.

Lemma 21. Consider a single halfline and \(k\) workers located initially at the same point at distance \(x > 0\) from the origin. Assume there are dust particles that are located at points of a rate-1 Poisson process at \((x, \infty)\) and at the origin. Each worker has its own Poisson rate-1 clock, and the clocks ring independently of each other. The workers always jump to the closest dust particle and remove it. Let \(D_k(x)\) be the event that the dust particle at the origin will be never removed and \(P_k(x)\) its probability. Then

\[
\lim_{x \to 0} \frac{\log P_k(x)}{\log^2 x} = -\frac{k}{2 \log 2}.
\]
We need now the notation of $k-(\delta)$ blocking array that has been introduced close to the end of Section 6. The proof of Lemma 21 immediately implies

**Corollary 22.** For any $x > 0$ and $\gamma > 0$ and for all $\delta$ sufficiently small, the probability that a rate one Poisson process of points on $(x, \infty)$ is not $k-(\delta)$ blocking is less than

$$e^{-(\log(\delta))^2k(1-\gamma)/2\log(2)}$$

**Remark:** in the above result the variable $x$ plays no role, but it is formulated as it is for application in Section 6.

Here is a further extensions of Lemma 21 onto the case where the initial locations of $k$ workers may differ, in general.

**Lemma 23.** Let $\delta \in (0,1)$. Assume that, in conditions of Lemma 21, the initial locations of $k$ workers on the halfline are

$$0 < x = x_{-k+1} \leq x_{-k+2} \leq \ldots \leq x_0$$

where $x_0 \leq (2-\delta)x$. Let $x = (x_{-k+1}, x_{-k+2}, \ldots, x_0)$ and $P_k(x)$ be the probability that, with this initial configuration, the dust particle at the origin will be never removed. Then

$$\lim_{x \to 0} \frac{\log P_k(x)}{\log^2 x} = \frac{k}{2\log 2},$$

(9)

where the convergence is uniform in $x_{-k+i}/x \in [1, (2-\delta))$. The following useful corollary shows that the logarithmic tail asymptotics for the probability for all workers to escape to infinity without visiting the origin is mostly due to the corresponding dust and clock environments, and given that, the probability to escape is a power function of the initial value $x_0$.

**Corollary 24.** In conditions of the previous Lemma 23, let $Z$ be the conditional (upon the dust environment) probability that the $k$ workers will never return to the origin. Then there exists a constant $M > 0$ such that, for any small $\delta$ and uniformly in $x_{-k+i}/x \in [1, (2-\delta))$,

$$\mathbb{P}(Z > x_0^M) \geq e^{-k(1+\delta)\log^2(x_0)/2\log 2},$$

for all $x_0$ small enough.
Proof of Lemma 20. We obtain separately the upper and the lower bounds
that are logarithmically equivalent.

Write, for short, \( P(x) = P_1(x) \). Clearly, \( P(x) \) is an increasing function that
tends to 1 if \( x \to \infty \), and to 0 if \( x \to 0 \). By the total probability formula,

\[
P(x) = \mathbb{P}(D(x)) = \int_0^x \mathbb{P}(\psi \in dy) \mathbb{P}(D(x) \mid \psi = y) = \int_0^x e^{-y} P(x + y) dy
\]

where \( \psi \) is the distance from \( x \) to the first dust particle on the right. Indeed,
for the event of interest to happen, there should be at least one dust particle
within \( (x, 2x) \). Recall that, for \( x \in (0, 1) \), we have
\[
x/2 < 1 - e^{-x} < x. \tag{11}
\]

Upper bound. By monotonicity of \( P \) (see Appendix) and by (10) and (11), \( P(x) \leq \min(1, x) P(2x) \). Let \( m = \min \{ n : 2^n x \geq 1/2 \} \), then \( m = -\log x / \log 2 + O(1) \), as \( x \to 0 \). Using the induction argument, we get:

\[
P(x) \leq x \cdot P(2x) \leq x \cdot 2x \cdot \ldots \cdot 2^{m-1} x \cdot P(2^m x) \leq \frac{x^{m} 2^{m(m-1)/2}}{m}
= \exp \left( m \log x + (1 + o(1)) \log 2 \cdot m(m-1)/2 \right)
= \exp \left( -(1 + o(1)) \frac{\log^2 x}{2 \log 2} \right),
\]
as \( x \to 0 \).

Lower bound. We again use the monotonicity property of \( P \) and (10) and (11). For any \( 0 < x, \varepsilon < 1/2 \).

\[
P(x) \geq \int_{x(1-2\varepsilon)}^x e^{-y} P(x + y) dy
\geq 2\varepsilon e^{-2x(1-\varepsilon)} P(2x(1 - \varepsilon)).
\]

Let \( \varepsilon_n = \frac{c}{n^\gamma} \) be a decreasing to 0 sequence, where \( 0 < c < 1/2 \) and \( \gamma > 2 \).
Using consequently \( \varepsilon_1, \varepsilon_2, \ldots \) in place of \( \varepsilon \), we get:

\[
P(x) \geq 2\varepsilon_1 e^{-2x(1-\varepsilon_1)} P(2x(1 - \varepsilon_1))
\geq 2\varepsilon_1 e^{-2x(1-\varepsilon_1)} \cdot 4\varepsilon_2 x(1 - \varepsilon_1)e^{-4x(1-\varepsilon_1)(1-\varepsilon_2)} P(4x(1 - \varepsilon_1)(1 - \varepsilon_2)) \geq \ldots
\geq \left( \prod_1^m \varepsilon_i \right) x^m 2^{m(m+1)/2} A_{m-1} \exp(-x \sum_1^m 2^i B_i) P(2^m B_m x)
\geq A \frac{c^m}{(m!)^\gamma} \exp(m \log x + \log 2 \cdot m(m-1)/2 - x 2^{m+1}) P(2^m B x),
\]
where \( A_n = \prod_1^n (1 - \varepsilon_i)^i \), \( A = \prod_1^\infty (1 - \varepsilon_i)^i \), \( B_n = \prod_1^n (1 - \varepsilon_i) \) and \( B = \prod_1^\infty (1 - \varepsilon_i) \) are strictly positive numbers. Letting again \( m \) be the integer part of \( \frac{\log x}{\log 2} \), we get, as \( x \to 0 \),

\[
P(x) \geq A \frac{P(B/2) e^{-2} \exp \left( -\frac{\log^2 x}{2 \log 2} (1 + o(1)) \right)}{\exp \left( -\frac{\log^2 x}{2 \log 2} (1 + o(1)) \right)}
\]

since \( 2^m x = O(1) \), \( \varepsilon^m = e^{O(|\log x|)} = e^{o(\log^2 x)} \) and

\[
(m!)^\gamma = \exp(\gamma |\log x| \log |\log x|(1 + o(1))) = \exp(o(\log^2 x)).
\]

**Proof of Lemma 21.**

**Upper bound.** For the event of interest to occur, we need to have at least \( k \) points of the Poisson dust process to be within \( (x, 2x) \) and then, by the monotonicity property 2 (see Appendix),

\[
P_k(x) \leq P(\xi_k \leq x) P_k(2x).
\]

Here \( \xi_k \) is the distance from \( x \) to the \( k \)th point of the Poisson dust process on \( (x, \infty) \), so \( \xi_n \) has Gamma distribution with parameters \( (k, 1) \). Let \( p_k(x) = P(\xi_k \leq x) \). Then, for all \( x \),

\[
x^k/k! \geq p_k(x) \geq x^k e^{-x}/k!.
\]

Then we may use the monotonicity properties 1 and 2 (see Appendix) to get, with \( m \) as before,

\[
P_k(x) \leq \frac{x^k}{k!} P_k(2k) \leq \ldots \leq x^{km} 2^{\frac{km(m-1)}{2}}/(k!)^m = \exp \left( -\frac{k \log^2 x}{2 \log 2} (1 + o(1)) \right).
\]

**Lower bound.** We define \( \varepsilon_n \) as in the proof of Lemma 20. Let \( D_k(x) \) be the event the interest and

\[
G(x, x+c) = \{\text{no particles of the dust process within interval } (x, x+c)\}.
\]

Introduce the following events, for \( n = 1, 2, \ldots \):

(i) let

\[
E_n = \{\text{there is no dust particles in the interval } (2^{n-1} B_{n-1} x, 2^n B_n x)\},
\]

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the probability of this event is not smaller than \( \exp(-2^n B_n x) \);

(ii) let

\[ H_n = \{ \text{there are exactly } k \text{ dust particles in the interval } (2^n B_n x, 2^n B_{n-1} x) \} \]

the probability of this event is equal to \( x^k K_n^k e^{-K_n x} / k! \) where \( K_n = 2^n B_{n-1} \varepsilon_n \);

(iii) let

\[ J_n = \{ \text{the clock of each of the } k \text{ workers rings exactly once within } k \]

consecutive rings numbered \( k(n - 1) + 1, k(n - 1) + 2, \ldots, k_n \} \)

(we number the rings in order of their appearance), the probability of this event is not smaller that the probability

\[ p := \mathbf{P}(X_k \leq Y_k) > 0 \]

where \( X_k \) is the maximum of \( k \) i.i.d. exponential-1 r.v.'s, \( Y_k \) the minimum of other \( k \) i.i.d. exponential-1 r.v.'s, and \( X_k \) and \( Y_k \) do not depend on each other.

Taking \( m \) as before, we get:

\[
P_k(x) = \mathbf{P}(D_k(x)) \\
\geq \mathbf{P}(\bigcap_{i=1}^{m} (E_i \cap H_n \cap J_n)) \cdot \mathbf{P}(D_k(2^m B_m x) \cap G(2^m B_m x, 2^m B_{m-1} x)) \\
\geq e^{-x \sum_{i=1}^{m} 2^n B_n} \cdot \frac{x^{km} 2^{m(m-1)/2} B^{km} (\prod_{i=1}^{m} \varepsilon_n)^k e^{-\sum_{i=1}^{m} 2^n B_{n-1} \varepsilon_n x}}{(k!)^m} \\
\times p^{m} (\mathbf{P}(B/2) - 1 + e^{-2^m \varepsilon_m x}) .
\]

Here

(1) We apply the first monotonicity property from Appendix: we observe that after removing \( mk \) particles, all \( k \) workers are located within the interval \((2^m B_m x, 2^m B_{m-1} x)\) and there are dust particles only at the origin and to the right of this interval. Therefore, in we move all workers to the smallest point of this interval, the probability of interest becomes smaller, so the first inequality follows.

(2) Given that, we have all \( k \) points at \( 2^m B_m x \) and are interested in the probability of escaping of all workers to infinity given there is no dust particles within \((2^m B_m x, 2^m B_{m-1} x)\). Then we apply basic inequalities: for events \( A, B \) of positive probabilities,

\[ \mathbf{P}(A | B) \geq \mathbf{P}(AB) \geq \mathbf{P}(A) - \mathbf{P}(\bar{B}) \]

where \( \bar{B} \) is the complement of \( B \).
Similarly to that in the proof of the lower bound in the previous lemma, we have a number of inequalities:

\[
\exp(-x \sum_{1}^{m} 2^n B_n) \geq \exp(-2B) > 0,
\]

\[(k!)^m = \exp(O(|\log x| \log |\log x|)) = \exp(o(\log x)^2),\]

\[B^{km} = \exp(O(|\log x|)),\]

\[(\prod_{1}^{m} \varepsilon_n)^k = \exp(O(|\log x| \log |\log x|)) = \exp(o(\log x)^2),\]

\[p^m = \exp(O(|\log x|))\]

and then

\[x^{km} 2^{km(m-1)/2} = \exp(-\log^2 x/\log 2 + \log^2 x/2 \log 2(1 + o(1))) = \exp(-(1 + o(1)) \log^2 x/2 \log 2)\]

and, for any monotone function \(h(n) \to \infty, h(n) = o(n),\)

\[\sum_{1}^{m} 2^n B_{n-1} \varepsilon_n x \leq \sum_{1}^{m} 2^n x + \varepsilon_{h(m)} 2^{m+1} x \leq 2^{h(m)+1} x + 2\varepsilon_{h(m)} \to 0.\]

Since

\[e^{-2^m \varepsilon_m x} \geq e^{-\varepsilon_m/2} \to 1,\]

the result follows.

**Proof of Lemma 23.** The probability of interest is upper-bounded by \(P_{k}(2x),\) so we may use the upper bound from the previous lemma. The proof of the lower bound differs only in the first step where the event \(E_1\) is replaced by a bigger event, which makes the probability bigger.

**Proof of Corollary 24** follows from the proofs of the lemmas 21 and 23, since the events \(G, E_n\) and \(H_n\) relate to the dust environment and events \(J_n\) to the clock environment.

## Appendix

### Monotonicity properties

Consider the following deterministic model. Assume the state space is halfline \([0, \infty)\) and there are \(k\) workers there, located initially at points \(0 < x_{-k+1} \leq \ldots \leq x_{-2} \leq x_{-1} \leq x_0 \leq x_1 \leq \ldots \leq x_{n-1} \leq x_n \leq x_{n+1} \leq \infty.\)
Let $x = (x_{-k+1}, \ldots, x_0)$, with worker 1 located at point $x_{-k+1}$, worker 2 at point $x_{-k+2}$, etc., worker $k$ at point $x_0$.

Assume next that there are infinitely many dust particles located at points $0 = x_0 < x_1 < x_2 < \ldots$ where $x_1 > x_0$, so all workers are located initially between 0 and $x_1$. Let $d = (d_1, d_2, \ldots)$ be an infinite vector (that does not include $d_0 = 0$).

Assume further that there is given a fixed (predefined) order of moves of workers $\{w_n\}_{n \geq 1}$ that says that, first, worker $w_1$ jumps to its closest dust particle and removes it, then worker $w_2$ jumps and removes its closest particle etc. If a worker finds two particles at the same location, it chooses any of them. If a worker finds that he is located, say, at point $y$ and there are two closest particles at points 0 and 2$y$, then it chooses point 0. We assume that each number 1, 2, $\ldots$, $k$ appears in the sequence $\{w_n\}$ infinitely often.

Assume it takes a unit of time per move. Let $D(x, d)$ represent the event that all workers escape to infinity without visiting 0 (in other words, that by time $\infty$ all dust particles but the one at the origin have been removed).

Then we have the following elementary monotonicity properties.

**Monotonicity property 1.** Let $\hat{x} = (\hat{x}_{-k+1}, \ldots, \hat{x}_0)$ be another initial location of $k$ workers. Assume that $\hat{x} \succeq x$ component-wise and that $\hat{x}_0 < d_1$. Then if event $D(x, d)$ occurs, then event $D(\hat{x}, d)$, occurs too.

Indeed, when any worker, say, $j$, moves first time, it finds that it is more likely to move to the right from location $\hat{x}_{-k+j}$ than from location $x_{-k+j} \leq \hat{x}_{-k+j}$.

**Monotonicity property 2.** Let $c > 0$. Let $\bar{x}_{-k+j} = x_{-k+j} + c$ and $\bar{d}_i = d_i + c$, for all $1 \leq j \leq k$ and all $i \geq 1$. Then if event $D(\bar{x}, \bar{d})$ occurs, then event $D(x, d)$ occurs too.

This property can be easily verified step by step. As a corollary, we have the following. Let $P_k(x)$ be defined as in the previous Section. Then

$$P_k(x) \leq P_k(y), \quad \text{for all } \ 0 < x \leq y \text{ and } k = 1, 2, \ldots$$

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