A note on the global stochastic maximum principle for fully coupled forward-backward stochastic systems

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Abstract. Hu et. al [4] studied a stochastic optimal control problem for fully coupled forward-backward stochastic control systems with a nonempty control domain. By assuming a weakly coupled condition, they established an approach to obtain the first-order, second-order variational equations and the adjoint equations for the states $X$, $Y$ and $Z$ and deduced the global maximum principle. But it is well known that there are several different conditions such as monotonicity condition, weakly coupled condition and other conditions (see [6, 8–10, 13, 22, 24] and the references therein) which can guarantee the existence and uniqueness of the solution to (1.2). In this note, to overcome the limitations of assuming a specific condition, we propose two kinds of assumptions which can guarantee that the approach developed in [4] is still applicable. Under these two kinds of assumptions, we obtain the global stochastic maximum principle.

Key words. Backward stochastic differential equations, Nonconvex control domain, Stochastic recursive optimal control, Maximum principle, Spike variation.

AMS subject classifications. 93E20, 60H10, 35K15

1 Introduction

In 1990, Peng [14] obtained the global maximum principle for the classical stochastic optimal control problem. Since then, many researchers investigate this kind of optimal control problems for various stochastic systems (see [2, 3, 17, 18]). Peng [15] generalized the classical stochastic optimal control problem to the so-called stochastic recursive optimal control problem where the cost functional is defined by $Y(0)$. Here $(Y(\cdot), Z(\cdot))$
is the solution of the following backward stochastic differential equation (BSDE) (1.1):

\[ \begin{aligned}
- dY(t) &= f(t, X(t), Y(t), Z(t), u(t))dt - Z(t)dB(t), \\
Y(T) &= \phi(X(T)).
\end{aligned} \]  

(1.1)

In [13], the control domain is convex and a local stochastic maximum principle is established. The local stochastic maximum principles for other various problems were studied in (Dokuchaev and Zhou [1], Ji and Zhou [7], Peng [13], Shi and Wu [15], Xu [21], Meyer-Brandis, Øksendal and Zhou [11], see also the references therein). When the control domain is nonconvex, one encounters an essential difficulty when trying to derive the first-order and second-order variational equations for the BSDE (1.1) and it is proposed as an open problem in Peng [16]. Hu [3] studied this open problem and obtained a completely novel global maximum principle. Yong [23] studied a fully coupled controlled FBSDE with mixed initial-terminal conditions. In [23], Yong regarded \( Z(\cdot) \) as a control process and then applied the Ekeland variational principle to obtain an optimality variational principle which contains unknown parameters. Using the similar approach, Wu [20] studied a stochastic recursive optimal control problem.

In [4], the following optimal control problem was considered: minimize the cost functional

\[ J(u(\cdot)) = Y(0) \]

subject to the following fully coupled forward-backward stochastic differential equation (FBSDE):

\[ \begin{aligned}
dX(t) &= b(t, X(t), Y(t), Z(t), u(t))dt + \sigma(t, X(t), Y(t), Z(t), u(t))dB(t), \\
dY(t) &= -g(t, X(t), Y(t), Z(t), u(t))dt + Z(t)dB(t), \\
X(0) &= x_0, \quad Y(T) = \phi(X(T)),
\end{aligned} \]  

(1.2)

where the control variable \( u \) takes values in a nonempty subset of \( \mathbb{R}^k \) and the state variable \( X \) belongs to \( \mathbb{R} \). The authors systematically developed an approach to obtain the first-order, second-order variational equations and the adjoint equations for the states \( X, Y \) and \( Z \) and deduced the global maximum principle.

To guarantee the well-posedness of (1.2), a weakly coupled condition was assumed in [4]. But it is well known that there are several different conditions such as monotonicity condition, weakly coupled condition and other conditions (see [6, 8–10, 13, 22, 24] and the references therein) which can guarantee the existence and uniqueness of the solution to (1.2). Then it naturally leads to the following problem: is the approach established in [4] applicable to the other conditions except the weakly coupled condition? After careful analysis, we found that applying the approach in [4] to obtain the global maximum principle essentially depends on the following assumptions: (1) there exists a unique solution to FBSDE (1.2); (2) the solution to FBSDE (1.2) has \( L^p \)-estimates; (3) there exists a unique solution to the first-order adjoint equation. In other words, any assumptions which make the above three statements hold are sufficient to deduce the global maximum principle by the approach in [4].

In this paper, motivated by the above analysis, we give up assuming a specific condition (weakly coupled condition, monotonicity condition or other conditions in the related literatures) and directly propose the following two kind of assumptions. The first kind of assumptions is: (1) there exists a unique solution to
FBSDE (1.2); (2) there exists a unique bounded solution to the first-order adjoint equation. For this case, we can prove the $L^p$-estimates for the solution to FBSDE (1.2) hold. If the solution $q$ in the first-order adjoint equation is unbounded, then the optimal control problem becomes more complicated. So for this case, we propose the second kind of assumptions: (1) $\sigma$ is linear in $z$; (2) there exists a unique solution to FBSDE (1.2); (3) the solution to FBSDE (1.2) has $L^p$-estimates; (4) there exists a unique solution to the first-order adjoint equation. We prove that for both kinds of the assumptions, all the appropriate estimates for the solutions of the first-order and second-order variational equations hold. Thus, the global maximum principle can be deduced naturally. Beside this, we also generalize the state variable $X$ in (1.2) to multi-dimensional case in this paper.

The rest of the paper is organized as follows. In section 2, we give the preliminaries and formulation of our problem. A global stochastic maximum principle is obtained by spike variation method in section 3. In appendix, we give some results that will be used in our proofs.

## 2 Preliminaries and problem formulation

Let $(\Omega, \mathcal{F}, P)$ be a complete probability space on which a standard $d$-dimensional Brownian motion $B = (B_1(t), B_2(t), \ldots, B_d(t))_{0 \leq t \leq T}$ is defined. Assume that $\mathcal{F} = \{\mathcal{F}_t, 0 \leq t \leq T\}$ is the $P$-augmentation of the natural filtration of $B$, where $\mathcal{F}_0$ contains all $P$-null sets of $\mathcal{F}$. Denote by $\mathbb{R}^n$ the $n$-dimensional real Euclidean space and $\mathbb{R}^{k \times n}$ the set of $k \times n$ real matrices. Let $\langle \cdot, \cdot \rangle$ (resp. $|\cdot|$) denote the usual scalar product (resp. usual norm) of $\mathbb{R}^n$ and $\mathbb{R}^{k \times n}$. The scalar product (resp. norm) of $M = (m_{ij})$, $N = (n_{ij}) \in \mathbb{R}^{k \times n}$ is denoted by $\langle M, N \rangle = \text{tr} \{MN^\top\}$ (resp. $\|M\| = \sqrt{MM^\top}$), where the superscript $\top$ denotes the transpose of vectors or matrices.

We introduce the following spaces.

$L^p_{\mathcal{F}_T}(\Omega; \mathbb{R}^n)$ : the space of $\mathcal{F}_T$-measurable $\mathbb{R}^n$-valued random variables $\eta$ such that

$$||\eta||_p := (E[|\eta|^p])^{\frac{1}{p}} < \infty,$$

$L^\infty_{\mathcal{F}_T}(\Omega; \mathbb{R}^n)$: the space of $\mathcal{F}_T$-measurable $\mathbb{R}^n$-valued random variables $\eta$ such that

$$||\eta||_\infty := \text{ess sup}_{\omega \in \Omega} ||\eta|| < \infty,$$

$L^p([0, T]; \mathbb{R}^n)$: the space of $\mathcal{F}$-adapted and $p$-th integrable stochastic processes on $[0, T]$ such that

$$E \left[ \int_0^T |f(t)|^p dt \right] < \infty,$$

$L^\infty([0, T]; \mathbb{R}^n)$: the space of $\mathcal{F}$-adapted and uniformly bounded stochastic processes on $[0, T]$ such that

$$||f(\cdot)||_\infty = \text{ess sup}_{(t, \omega) \in [0, T] \times \Omega} |f(t)| < \infty,$$

$L^{p,q}_{\mathcal{F}}([0, T]; \mathbb{R}^n)$: the space of $\mathcal{F}$-adapted stochastic processes on $[0, T]$ such that

$$||f(\cdot)||_{p,q} = \left\{ E \left[ \left( \int_0^T |f(t)|^p dt \right)^{\frac{q}{p}} \right] \right\}^{\frac{1}{q}} < \infty,$$
$L^p_F(\Omega; C([0, T], \mathbb{R}^n))$: the space of $F$-adapted continuous stochastic processes on $[0, T]$ such that
$$\mathbb{E}\left[ \sup_{0 \leq t \leq T} |f(t)|^p \right] < \infty.$$

### 2.1 Problem formulation

Consider the following fully coupled stochastic control system:

\[
\begin{align*}
    dX(t) &= b(t, X(t), Y(t), Z(t), u(t))dt + \sigma(t, X(t), Y(t), Z(t), u(t))dB(t), \\
    dY(t) &= -g(t, X(t), Y(t), Z(t), u(t))dt + Z(t)dB(t), \\
    X(0) &= x_0, \quad Y(T) = \phi(X(T)),
\end{align*}
\]  

(2.1)

where

\[
\begin{align*}
    b : [0, T] \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R} \times U \to \mathbb{R}^n, \\
    \sigma : [0, T] \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R} \times U \to \mathbb{R}^{n \times 1}, \\
    g : [0, T] \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R} \times U \to \mathbb{R}, \\
    \phi : \mathbb{R}^n \to \mathbb{R}.
\end{align*}
\]

An admissible control $u(\cdot)$ is an $\mathbb{F}$-adapted process with values in $U$ such that

$$\sup_{0 \leq t \leq T} \mathbb{E}[|u(t)|^p] < \infty,$$

where the control domain $U$ is a nonempty subset of $\mathbb{R}^k$. Denote the admissible control set by $U[0, T]$.

Our optimal control problem is to minimize the cost functional

$$J(u(\cdot)) = Y(0)$$

over $U[0, T]$, that is

$$\inf_{u(\cdot) \in U[0, T]} J(u(\cdot)).$$  

(2.2)

### 3 Stochastic maximum principle

We derive maximum principle (necessary condition for optimality) for the optimization problem (2.2) in this section. For simplicity of presentation, we only study the case $d = 1$. In this section, the constant $C$ will change from line to line in our proof.

**Assumption 3.1** For $\psi = b, \sigma, g$ and $\phi$, we suppose

(i) $\psi, \psi_x, \psi_y, \psi_z$ are continuous in $(x, y, z, u)$; $\psi_x, \psi_y, \psi_z$ are bounded; there exists a constant $L > 0$ such that

$$|\psi(t, x, y, z, u)| \leq L \left( 1 + |x| + |y| + |z| + |u| \right),$$

$$|\sigma(t, 0, 0, z, u) - \sigma(t, 0, 0, z, u')| \leq L(1 + |u| + |u'|).$$

(ii) $\psi_{xx}, \psi_{xy}, \psi_{yy}, \psi_{xz}, \psi_{yz}, \psi_{zz}$ are continuous in $(x, y, z, u)$; $\psi_{xx}, \psi_{xy}, \psi_{yy}, \psi_{xz}, \psi_{yz}, \psi_{zz}$ are bounded.
Assumption 3.2 For any $u(\cdot) \in U[0,T]$ and $\beta \in [2,8]$, FBSDE (2.1) has a unique solution $(X(\cdot), Y(\cdot), Z(\cdot)) \in L^\beta_f(\Omega; C([0,T], \mathbb{R}^n)) \times L^\beta_f(\Omega; C([0,T], \mathbb{R})) \times L^\beta_f([0,T]; \mathbb{R})$.

Let $\bar{u}(\cdot)$ be optimal and $(\bar{X}(\cdot), \bar{Y}(\cdot), \bar{Z}(\cdot))$ be the corresponding state processes of (2.1). Since the control domain is not necessarily convex, we resort to spike variation method. For any $u(\cdot) \in U[0,T]$ and $0 < \epsilon < T$, define

$$ u'(t) = \begin{cases} \bar{u}(t), & t \in [0,T) \setminus E_\epsilon, \\ u(t), & t \in E_\epsilon, \end{cases} $$

where $E_\epsilon \subset [0,T]$ is a measurable set with $|E_\epsilon| = \epsilon$. Let $(X^\epsilon(\cdot), Y^\epsilon(\cdot), Z^\epsilon(\cdot))$ be the state processes of (2.1) associated with $u'(\cdot)$.

For simplicity, for $\psi = b, \sigma, g, \phi$ and $\kappa = x, y, z$, denote

$$ \psi(t) = \psi(t, \bar{X}(t), \bar{Y}(t), \bar{Z}(t), \bar{u}(t)), $$

$$ \psi_x(t) = \psi_x(t, \bar{X}(t), \bar{Y}(t), \bar{Z}(t), \bar{u}(t)), $$

$$ \delta\psi(t) = \psi(t, \bar{X}(t), \bar{Y}(t), \bar{Z}(t), u(t)) - \psi(t), $$

$$ \delta\psi_x(t) = \psi_x(t, \bar{X}(t), \bar{Y}(t), \bar{Z}(t), u(t)) - \psi_x(t), $$

$$ \delta\psi(t, \Delta) = \psi(t, \bar{X}(t), \bar{Y}(t), \bar{Z}(t) + \Delta(t), u(t)) - \psi(t), $$

$$ \delta\psi_x(t, \Delta) = \psi_x(t, \bar{X}(t), \bar{Y}(t), \bar{Z}(t) + \Delta(t), u(t)) - \psi_x(t), $$

where $\Delta(\cdot)$ is an $\mathbb{F}$-adapted process. Moreover, denote the gradient of $\psi$ with respect to $x, y, z$ by $D\psi$, and $D^2\psi$ the Hessian matrix of $\psi$ with respect to $x, y, z$,

$$ D\psi(t) = D\psi(t, \bar{X}(t), \bar{Y}(t), \bar{Z}(t), \bar{u}(t)), $$

$$ D^2\psi(t) = D^2\psi(t, \bar{X}(t), \bar{Y}(t), \bar{Z}(t), \bar{u}(t)). $$

Let

$$ \xi^{1,\epsilon}(t) := X^\epsilon(t) - \bar{X}(t); \, \eta^{1,\epsilon}(t) := Y^\epsilon(t) - \bar{Y}(t); $$

$$ \zeta^{1,\epsilon}(t) := Z^\epsilon(t) - \bar{Z}(t); \, \Theta(t) := (X(t), \bar{Y}(t), \bar{Z}(t)); $$

$$ \Theta'(t) := (X^\epsilon(t), Y^\epsilon(t), Z^\epsilon(t)). $$

We have

$$ \begin{cases} d\xi^{1,\epsilon}(t) = \left[ \bar{b}_x(t)\xi^{1,\epsilon}(t) + \bar{g}_x(t)\eta^{1,\epsilon}(t) + \bar{g}_z(t)\zeta^{1,\epsilon}(t) + \delta b(t)I_{E_\epsilon}(t) \right] dt \\
+ \left[ \tilde{\sigma}_x(t)\xi^{1,\epsilon}(t) + \tilde{g}_x(t)\eta^{1,\epsilon}(t) + \tilde{g}_z(t)\zeta^{1,\epsilon}(t) + \delta g(t)I_{E_\epsilon}(t) \right] dB(t), \\
\xi^{1,\epsilon}(0) = 0, \end{cases} $$

$$ \begin{cases} d\eta^{1,\epsilon}(t) = - \left[ \tilde{g}_x(t)\xi^{1,\epsilon}(t) + \tilde{g}_z(t)\zeta^{1,\epsilon}(t) + \tilde{g}_y(t)\eta^{1,\epsilon}(t) + \delta g(t)I_{E_\epsilon}(t) \right] dt + \zeta^{1,\epsilon}(t)dB(t), \\
\eta^{1,\epsilon}(T) = \left[ \tilde{\sigma}_x(T), \xi^{1,\epsilon}(T) \right], \end{cases} $$

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where
\[ \tilde{b}_x(t) = \int_0^1 b_x(t, \Theta(t) + \theta(\Theta(t) - \Theta(t)), u^\varepsilon(t))d\theta \] (3.3)
and \( \tilde{b}_y(t), \tilde{b}_z(t), \tilde{\sigma}_x(t), \tilde{\sigma}_y(t), \tilde{\sigma}_z(t), \tilde{g}_x(t), \tilde{g}_y(t), \tilde{g}_z(t) \) and \( \tilde{\phi}_z(T) \) are defined similarly. Consider the following linear FBSDE
\[
\begin{align*}
\begin{cases}
\quad \quad d\tilde{X}(t) = \left[ \tilde{b}_x(t)\tilde{X}(t) + \tilde{b}_y(t)\tilde{Y}(t) + \tilde{b}_z(t)\tilde{Z}(t) + L_1(t) \right] dt + \left[ \tilde{\sigma}_x(t, \Delta)\tilde{X}(t) \right. \\
\quad \quad \quad \quad \quad \quad \quad \left. + \tilde{\sigma}_y(t, \Delta)\tilde{Y}(t) + \tilde{\sigma}_z(t, \Delta)\tilde{Z}(t) + L_2(t) \right] dB(t), \\
\quad \quad d\tilde{Y}(t) = -\left[ \tilde{g}_x(t)\tilde{Y}(t) + \tilde{g}_y(t)\tilde{Y}(t) + \tilde{g}_z(t)\tilde{Z}(t) + L_3(t) \right] dt + \tilde{Z}(t)dB(t), \\
\quad \quad \tilde{X}(0) = x_0, \quad \tilde{Y}(T) = \left\langle \tilde{\phi}_z(T), \tilde{X}(T) \right\rangle + \varsigma, 
\end{cases}
\end{align*}
\] (3.4)

where \( \tilde{b}_x(t), \tilde{b}_y(t), \tilde{b}_z(t), \tilde{g}_x(t), \tilde{g}_y(t), \tilde{g}_z(t), \tilde{\phi}_z(T) \) are defined as \( 3.3 \) and \( \tilde{\sigma}_x(t, \Delta) = \int_0^1 \sigma_x(t, \Theta(t, \Delta I_E(t))) + \theta(\Theta(t) - \Theta(t, \Delta I_E(t)))u^\varepsilon(t)d\theta \) for any given \( \Delta(\cdot) \), \( \tilde{\sigma}_y(t, \Delta), \tilde{\sigma}_z(t, \Delta) \) are defined similar to \( \tilde{\sigma}_x(t, \Delta), L_1(\cdot) \in L_2^{1,2}(\mathbb{F}; \mathbb{R}^n), \) \( L_2(\cdot) \in L_2^{1,2}(\mathbb{F}; [0,T]; \mathbb{R}^n), \) \( L_3(\cdot) \in L_2^{1,2}(\mathbb{F}; [0,T]; \mathbb{R}), \) \( \varsigma \in L_2^{2,2}(\mathbb{F}; \mathbb{R}). \) We impose the following assumption.

**Assumption 3.3** For any \( 0 < \epsilon < T, u^\varepsilon(\cdot) \in U[0,T] \) and \( \beta \in [2,8], \) the FBSDE \( \{ \tilde{X}(\cdot), \tilde{Y}(\cdot), \tilde{Z}(\cdot) \} \) has a unique solution \( \tilde{X}(\cdot), \tilde{Y}(\cdot), \tilde{Z}(\cdot) \in L_2^{\infty}(\Omega; C([0,T], \mathbb{R}^n)) \times L_2^{\infty}(\Omega; C([0,T], \mathbb{R})) \times L_2^{2,2}(\mathbb{F}; [0,T]; \mathbb{R}). \)

**Assumption 3.4** For any control \( u^\varepsilon(\cdot) \) the following BSDE:

\[
\begin{align*}
\begin{cases}
\quad \quad dp^\varepsilon(t) = -\left[ \tilde{g}_x(t)p^\varepsilon(t) + \tilde{g}_y(t)p^\varepsilon(t) + \tilde{g}_z(t)p^\varepsilon(t) \right] dt + \left[ \tilde{\sigma}_x(t, \Delta)q^\varepsilon(t) + \tilde{\sigma}_y(t, \Delta)q^\varepsilon(t) + \tilde{\sigma}_z(t, \Delta)q^\varepsilon(t) \right] dB(t), \\
\quad \quad p^\varepsilon(T) = \tilde{\phi}_z(T), 
\end{cases}
\end{align*}
\] (3.5)

where
\[ K_1(t) = (1 - \langle p(t), \sigma_x(t, \Delta) \rangle)^{-1} [\sigma_x(t)^T p^\varepsilon(t) + \langle p^\varepsilon(t), \sigma_y(t) \rangle q^\varepsilon(t) + \langle q^\varepsilon(t), \sigma_z(t, \Delta) \rangle q^\varepsilon(t)] dt + q^\varepsilon(t)dB(t) \]

has a unique solution \( \langle p^\varepsilon(\cdot), q^\varepsilon(\cdot) \rangle \in L_2^{\infty}(\Omega; C([0,T], \mathbb{R}^n)) \times L_2^{\infty}(\Omega; C([0,T], \mathbb{R})) \) such that \( |1 - \langle p^\varepsilon(t), \gamma_2(t) \rangle |^{-1} \) is uniformly bounded.

Note that \( \tilde{\sigma}_x(t, \Delta) = \tilde{\sigma}_x(t) \) when \( \Delta(\cdot) \equiv 0. \) Due to Assumption 3.3 there exists a unique solution \( (\xi^1, \eta^1, \zeta^1) \) to (3.1) and 3.2.

**Lemma 3.5** Suppose that Assumptions 3.1, 3.2, 3.3 and 3.4 hold. Then for any \( 2 \leq \beta \leq 8 \) we have

\[ \mathbb{E} \left[ \sup_{t \in [0,T]} \left( |X^\varepsilon(t) - \tilde{X}(t)|^\beta + |Y^\varepsilon(t) - \tilde{Y}(t)|^\beta \right) \right] + \mathbb{E} \left[ \left( \int_0^T |Z^\varepsilon(t) - \tilde{Z}(t)|^2 dt \right)^{\beta/2} \right] = O(\epsilon^2). \] (3.6)

**Proof.** Note that \( (\xi^1, \eta^1, \zeta^1) \) is the solution to (3.1) and (3.2), and

\[ \mathbb{E} \left[ \left( \int_{E_{\epsilon}} |u(t)| dt \right)^{\beta} \right] \leq \epsilon^{\beta-1} \mathbb{E} \left[ \int_{E_{\epsilon}} |u(t)|^2 dt \right]. \]
Then, by Lemma 5.2 in Appendix, we get

$$\mathbb{E} \left[ \sup_{t \in [0,T]} \left( |\xi^1(t)|^2 + |\eta^1(t)|^2 \right) + \left( \int_0^T |\xi^1(t)|^2 dt \right)^{\frac{2}{7}} \right]$$

$$\leq C E \left[ \left( \int_0^T (|\delta b(t)| I_{E_1}(t) + |\delta g(t)| I_{E_2}(t)) dt \right)^{\frac{2}{7}} + \left( \int_0^T |\delta \sigma(t)|^2 I_{E_3}(t) dt \right)^{\frac{2}{7}} \right]$$

$$\leq C E \left[ \left( \int_{E_1} (1 + |\hat{X}(t)| + |\hat{Y}(t)| + |\hat{Z}(t)| + |u(t)| + |\bar{u}(t)|) dt \right)^{\frac{2}{7}} \right.$$

$$\left. + \left( \int_{E_3} (1 + |\hat{X}(t)|^2 + |\hat{Y}(t)|^2 + |u(t)|^2 + |\bar{u}(t)|^2) dt \right)^{\frac{2}{7}} \right]$$

$$\leq C \left( 1 + \sup_{t \in [0,T]} E \left[ |\hat{X}(t)|^2 + |\hat{Y}(t)|^2 + |u(t)|^2 + |\bar{u}(t)|^2 \right] \right) + C \epsilon^{\frac{2}{7}} E \left[ \left( \int_0^T |\hat{Z}(t)|^2 dt \right)^{\frac{2}{7}} \right]$$

$$\leq C \epsilon^{\frac{2}{7}}.$$

### 3.1 First-order expansion

We introduce the following adjoint equation satisfied by \((p, q)\):

$$dp(t)$$

$$= - \{ g_x(t) + g_y(t)p(t) + g_z(t)K_1(t) + b_x(t) p(t) + \langle p(t), b_y(t) \rangle p(t) \} dt + \langle q(t), \sigma_y(t) \rangle p(t) + \langle q(t), \sigma_z(t) \rangle K_1(t) \} dt + q(t) dB(t),$$

$$p(T) = \phi_x(\hat{X}(T)),$$

where

$$K_1(t) = (1 - \langle p(t), \sigma_z(t) \rangle)^{-1} [\langle \sigma_z(t)^T p(t) \rangle + \langle p(t), \sigma_y(t) \rangle p(t) + \langle q(t), \sigma_z(t) \rangle K_1(t) \} dt + q(t) dB(t),$$

$$p(T) = \phi_x(\hat{X}(T)),$$

(3.7)

We study first the following algebra equation

$$\Delta(t) = p(t)(\sigma(t, \hat{X}(t), \hat{Y}(t), \hat{Z}(t) + \Delta(t), u(t)) - \sigma(t, \hat{X}(t), \hat{Y}(t), \hat{Z}(t), \bar{u}(t))), \quad t \in [0,T],$$

(3.9)

where \(u(\cdot)\) is a given admissible control.

**Assumption 3.6** Assume that equation (3.8) has a unique solution \(\Delta(\cdot)\), and it satisfies

$$|\Delta(t)| \leq C (1 + |\hat{X}(t)| + |\hat{Y}(t)| + |u(t)| + |\bar{u}(t)|), \quad t \in [0,T],$$

(3.10)

where \(C\) is a constant depending on \(\beta_0, L, \|\psi_x\|_\infty, \|\psi_y\|_\infty, \|\psi_z\|_\infty, T\).

Now we introduce the first-order variational equation:

$$dX_1(t) = \left[ b_x(t)X_1(t) + b_y(t)Y_1(t) + b_z(t)(Z_1(t) + \Delta(t)I_{E_1}(t)) \right] dt$$

$$+ \left[ \sigma_x(t)X_1(t) + \sigma_y(t)Y_1(t) + \sigma_z(t)(Z_1(t) + \Delta(t)I_{E_1}(t)) + \delta(t, \Delta)I_{E_1}(t) \right] dB(t),$$

$$X_1(0) = 0,$$

(3.11)
and
\[
\begin{align*}
\begin{cases}
    dY_1(t) = -[g_x(t), X_1(t)] + g_y(t)Y_1(t) + g_z(t)(Z_1(t) - \Delta(t)I_{E_1}(t)) - \langle q(t), \delta\sigma(t, \Delta) \rangle I_{E_1}(t) \, dt \\
        + Z_1(t)dB(t), \\
    Y_1(T) = \langle \phi_x(\bar{X}(T)), X_1(T) \rangle.
\end{cases}
\end{align*}
\] (3.12)

By Assumption 5.3, the above FBSDE has a unique solution \((X_1(\cdot), Y_1(\cdot), Z_1(\cdot))\).

**Lemma 3.7** Suppose that Assumptions 3.1, 3.2, 3.3, 3.4 and 3.6 hold. Then we have
\[
\begin{align*}
Y_1(t) &= \langle p(t), X_1(t) \rangle, \\
Z_1(t) &= \langle K_1(t), X_1(t) \rangle + \Delta(t)I_{E_1}(t),
\end{align*}
\]

where \(p(\cdot)\) is the solution of (3.14) and \(K_1(\cdot)\) is given in (3.3).

**Proof.** From Lemma 5.1 in Appendix, we can obtain the desired results. \(\square\)

Let
\[
\begin{align*}
\xi^{2,\epsilon}(t) &:= X^\epsilon(t) - \bar{X}(t) - X_1(t); \\
\eta^{2,\epsilon}(t) &:= Y^\epsilon(t) - \bar{Y}(t) - Y_1(t); \\
\zeta^{2,\epsilon}(t) &:= Z^\epsilon(t) - \bar{Z}(t) - Z_1(t); \\
\Theta(t) &:= (\bar{X}(t), \bar{Y}(t), \bar{Z}(t)).
\end{align*}
\]

Then we have the following estimates.

**Lemma 3.8** Suppose Assumptions 3.1, 3.2, 3.3, 3.4 and 3.6 hold. Then for any \(2 \leq \beta \leq 8\), we have the following estimates
\[
\begin{align*}
\mathbb{E}\left[\sup_{t \in [0, T]} (|X_1(t)|^\beta + |Y_1(t)|^\beta)\right] + \mathbb{E}\left[\int_0^T |Z_1(t)|^2 \, dt\right]^{\beta/2} &= O(\epsilon^{\beta/2}), \\
\mathbb{E}\left[\sup_{t \in [0, T]} (|X^\epsilon(t) - \bar{X}(t) - X_1(t)|^2 + |Y^\epsilon(t) - \bar{Y}(t) - Y_1(t)|^2)\right] + \mathbb{E}\left[\int_0^T |Z^\epsilon(t) - \bar{Z}(t) - Z_1(t)|^2 \, dt\right] &= O(\epsilon^2), \\
\mathbb{E}\left[\sup_{t \in [0, T]} (|X^\epsilon(t) - \bar{X}(t) - X_1(t)|^4 + |Y^\epsilon(t) - \bar{Y}(t) - Y_1(t)|^4)\right] + \mathbb{E}\left[\int_0^T |Z^\epsilon(t) - \bar{Z}(t) - Z_1(t)|^2 \, dt\right]^2 &= o(\epsilon^2).
\end{align*}
\]

**Proof.** By Lemma 5.2 in Appendix, we have
\[
\begin{align*}
\mathbb{E}\left[\sup_{t \in [0, T]} (|X_1(t)|^\beta + |Y_1(t)|^\beta) + \left(\int_0^T |Z_1(t)|^2 \, dt\right)^{\beta/2}\right] &\leq C\mathbb{E}\left[\left(\int_0^T |\delta\sigma(t, \Delta) + \Delta(t)|^2 I_{E_1}(t) \, dt\right)^{\beta/2}\right] \\
&\leq C\mathbb{E}\left[\left(\int_{E_1} \left(1 + |\bar{X}(t)|^2 + |\bar{Y}(t)|^2 + |\bar{U}(t)|^2 + |\bar{U}(t)|^2\right) \, dt\right)^{\beta/2}\right] \\
&\leq C\epsilon^{\beta/2}.
\end{align*}
\]
We use the notations $\xi^{1,\epsilon}(t)$, $\eta^{1,\epsilon}(t)$ and $\zeta^{1,\epsilon}(t)$ in the proof of Lemma 5.3 and

$$\Theta(t, \Delta I_{E_1}) := (\hat{X}(t), \hat{Y}(t), \hat{Z}(t) + \Delta(t)I_{E_1}; \Theta^\epsilon(t) := (X^\epsilon(t), Y^\epsilon(t), Z^\epsilon(t)).$$

Note that

$$\delta \sigma(t, \Delta) I_{E_1}(t) = \sigma(t, \hat{X}(t), \hat{Y}(t), \hat{Z}(t) + \Delta(t)I_{E_1}, u^\epsilon(t)) - \sigma(t) = \sigma(t, \Theta(t, \Delta I_{E_1}(t)), u^\epsilon(t)) - \sigma(t).$$

We have

$$\begin{align*}
\sigma(t, \Theta^\epsilon(t), u^\epsilon(t)) - \sigma(t) - \delta \sigma(t, \Delta) I_{E_1}(t) \\
= \sigma(t, \Theta^\epsilon(t), u^\epsilon(t)) - \sigma(t, \Theta(t, \Delta I_{E_1}(t)), u^\epsilon(t)) \\
= \delta \sigma^\epsilon_x(t)(X^\epsilon(t) - \bar{X}(t)) + \delta \sigma^\epsilon_y(t)(Y^\epsilon(t) - \bar{Y}(t)) + \delta \sigma^\epsilon_z(t)(Z^\epsilon(t) - \bar{Z}(t) - \Delta(t)I_{E_1}(t)),
\end{align*}$$

where

$$\delta \sigma^\epsilon_x(t, \Delta) = \int_0^1 \sigma_x(t, \Theta(t, \Delta I_{E_1}(t))) + \theta(\Theta^\epsilon(t) - \Theta(t, \Delta I_{E_1}(t)), u^\epsilon(t)) d\theta,$$

and $\delta \sigma^\epsilon_x(t, \Delta), \delta \sigma^\epsilon_y(t, \Delta)$ are defined similarly.

Recall that $\delta^\epsilon_x(t), \delta^\epsilon_y(t), \delta^\epsilon_z(t), \bar{g}_x^\epsilon(t), \bar{g}_y^\epsilon(t)$ and $\bar{g}_z^\epsilon(T)$ are defined in Lemma 5.3. Then,

$$\begin{align*}
\left\{ \begin{array}{l}
d\xi_2^\epsilon(t) = \left[ \hat{b}_x^\epsilon(t) \xi_2^\epsilon(t) + \hat{b}_y^\epsilon(t) \eta_2^\epsilon(t) + \hat{b}_z^\epsilon(t) \zeta_2^\epsilon(t) + A_1^\epsilon(t) \right] dt \\
+ \left[ \hat{\sigma}_x^\epsilon(t, \Delta) \xi_2^\epsilon(t) + \hat{\sigma}_y^\epsilon(t, \Delta) \eta_2^\epsilon(t) + \hat{\sigma}_z^\epsilon(t, \Delta) \zeta_2^\epsilon(t) + B_1^\epsilon(t) \right] dB(t), \\
\xi_2^\epsilon(0) = 0,
\end{array} \right.
\end{align*}$$

(3.14)

$$\begin{align*}
d\eta_2^\epsilon(t) = - \left[ \langle \bar{g}_x^\epsilon(T), \xi_2^\epsilon(t) \rangle + \bar{g}_y^\epsilon(t) \eta_2^\epsilon(t) + \bar{g}_z^\epsilon(t) \zeta_2^\epsilon(t) + C_1^\epsilon(t) \right] dt + \xi_2^\epsilon(t) dB(t), \\
\eta_2^\epsilon(T) = \langle \bar{g}_x^\epsilon(T), \xi_2^\epsilon(T) \rangle + D_1^\epsilon(T),
\end{align*}$$

where

$$\begin{align*}
A_1^\epsilon(t) &= (\hat{b}_x^\epsilon(t) - b_x(t))X_1(t) + (\hat{b}_y^\epsilon(t) - b_y(t))Y_1(t) + (\hat{b}_z^\epsilon(t) - b_z(t))Z_1(t) \\
&+ b_z(t) \Delta(t) I_{E_1}(t) + \delta b(t) I_{E_1}(t), \\
B_1^\epsilon(t) &= (\hat{\sigma}_x^\epsilon(t, \Delta) - \sigma_x(t))X_1(t) + (\hat{\sigma}_y^\epsilon(t, \Delta) - \sigma_y(t))Y_1(t) + (\hat{\sigma}_z^\epsilon(t, \Delta) - \sigma_z(t)) \langle K_1(t), X_1(t) \rangle, \\
C_1^\epsilon(t) &= (\langle \bar{g}_x^\epsilon(t) - g_x(t), X_1(t) \rangle + \langle \bar{g}_y^\epsilon(t) - g_y(t), Y_1(t) \rangle + \langle \bar{g}_z^\epsilon(t) - g_z(t), Z_1(t) \rangle + \delta g(t) I_{E_1}(t) \\
&+ g_z(t) \Delta(t) I_{E_1}(t) + \langle q(t), \delta \sigma(t, \Delta) \rangle I_{E_1}(t), \\
D_1^\epsilon(T) &= \langle \bar{g}_x^\epsilon(T) - \phi_x(\hat{X}(T)), X_1(T) \rangle.
\end{align*}$$
By Lemma 5.2 in Appendix, we obtain
\[
\mathbb{E} \left[ \sup_{t \in [0,T]} \left( |\xi^2(t)|^2 + |\eta^2(t)|^2 + \int_0^T |\xi^2(t)|^2 dt \right) \right] \\
\leq C \mathbb{E} \left[ \left( \int_0^T |\mathcal{A}_1(t)| + |\mathcal{C}_1(t)| dt \right)^2 + \int_0^T |\mathcal{B}_1(t)|^2 dt + |\mathcal{D}_1(T)|^2 \right] \\
\leq C \mathbb{E} \left[ \left( \int_0^T |\mathcal{A}_1(t)| dt \right)^2 + \left( \int_0^T |\mathcal{C}_1(t)| dt \right)^2 + \int_0^T |\mathcal{B}_1(t)|^2 dt + |\mathcal{D}_1(T)|^2 \right].
\]

The following proof of the estimates are the same as in [4].

3.2 Second-order expansion

Noting that \(Z_1(t) = K_1(t)X_1(t) + \Delta(t)I_E_1(t)\) in Lemma 5.1, then we introduce the second-order variational equation as follows:

\[
\begin{aligned}
dX_2(t) &= \{ b_x(t)X_2(t) + b_y(t)Y_2(t) + b_z(t)Z_2(t) + \delta b(t, \Delta) I_E_1(t) \\
&\quad + \frac{1}{2} D^2 b(t) (X_1(t)^T, Y_1(t), (K_1(t), X_1(t))) \} dt \\
&\quad + \left\{ \sigma_x(t)X_2(t) + \sigma_y(t)Y_2(t) + \frac{1}{2} D^2 \sigma(t) (X_1(t)^T, Y_1(t), (K_1(t), X_1(t))) \right\} dt \\
&\quad + \sigma_z(t)Z_2(t) + [\delta \sigma_x(t, \Delta)X_1(t) + \delta \sigma_y(t, \Delta)Y_1(t)] I_E_1(t) + \delta \sigma_z(t, \Delta) (K_1(t), X_1(t)) I_E_1(t) dB(t),
\end{aligned}
\]

\(X_2(0) = 0, \quad (3.15)\)

and

\[
\begin{aligned}
dY_2(t) &= \left\{ \langle g_x(t), X_2(t) \rangle + g_y(t)Y_2(t) + g_z(t)Z_2(t) + \langle \omega(t), \delta \sigma(t, \Delta) \rangle + \delta g(t, \Delta) \right\} I_E_1(t) \\
&\quad + \frac{1}{2} [X_2(t)^T, Y_2(t), (K_1(t), X_1(t))] D^2 g(t) [X_2(t)^T, Y_2(t), (K_1(t), X_1(t))]^T dt + Z_2(t) dB(t),
\end{aligned}
\]

\[Y_2(T) = \langle \phi_x(\bar{X}(T)), X_2(T) \rangle + \frac{1}{2} \langle \phi_{xx}(\bar{X}(T))X_1(T), X_1(T) \rangle, \quad (3.16)\]

where

\[
D^2 b(t) (X_1(t)^T, Y_1(t), (K_1(t), X_1(t))) = (\text{tr}[D^2 b^1(t) (X_1(t)^T, Y_1(t), (K_1(t), X_1(t))) (X_1(t)^T, Y_1(t), (K_1(t), X_1(t)))^T], \ldots, \\
\text{tr}[D^2 b^n(t) (X_1(t)^T, Y_1(t), (K_1(t), X_1(t))) (X_1(t)^T, Y_1(t), (K_1(t), X_1(t)))^T]^T)
\]

and \(D^2 \sigma(t) (X_1(t)^T, Y_1(t), (K_1(t), X_1(t)))^2\) is defined similarly. In the following lemma, we estimate the orders of \(X_2(\cdot), Y_2(\cdot), Z_2(\cdot), \text{ and } Y^c(0) - \bar{Y}(0) - Y_1(0) - Y_2(0).\) Let

\[
\begin{aligned}
\xi^3(\cdot) &:= X^c(\cdot) - \bar{X}(\cdot) - X_1(\cdot) - X_2(\cdot); \quad \eta^3(\cdot) := Y^c(\cdot) - \bar{Y}(\cdot) - Y_1(\cdot) - Y_2(\cdot); \\
\zeta^3(\cdot) &:= Z^c(\cdot) - \bar{Z}(\cdot) - Z_1(\cdot) - Z_2(\cdot); \quad \Theta(\cdot) := (\bar{X}(\cdot), \bar{Y}(\cdot), \bar{Z}(\cdot)).
\end{aligned}
\]
Lemma 3.9 Suppose that Assumption 3.1, 3.2, 3.3, 3.4, and 3.6 hold. Then for any \(2 \leq \beta \leq 4\) we have

\[
\begin{align*}
\mathbb{E} \left[ \sup_{t \in [0,T]} (|X_2(t)|^2 + |Y_2(t)|^2) \right] &+ \mathbb{E} \left[ \int_0^T |Z_2(t)|^2 dt \right] = O(\epsilon^2), \\
\mathbb{E} \left[ \sup_{t \in [0,T]} (|X_2(t)|^2 + |Y_2(t)|^2) \right] &+ \mathbb{E} \left[ \left( \int_0^T |Z_2(t)|^2 dt \right)^{\beta/2} \right] = o(\epsilon^2).
\end{align*}
\]

Proof. Let

\[
\begin{align*}
L_1(t) &= \delta b(t, \Delta) I_{E_\epsilon}(t) + \frac{1}{2} D^2 b(t) (X_1(t), Y_1(t), (K_1(t), X_1(t)))^2, \\
L_2(t) &= \frac{1}{2} D^2 \sigma(t) (X_1(t), Y_1(t), (K_1(t), X_1(t)))^2 + [\delta \sigma_x(t, \Delta) X_1(t) + \delta \sigma_y(t, \Delta) Y_1(t)] I_{E_\epsilon}(t) \\
&\quad + \delta \sigma_z(t, \Delta) (K_1(t), X_1(t)) I_{E_\epsilon}(t), \\
L_3(t) &= [(g(t), \delta \sigma(t, \Delta)) + \delta g(t, \Delta)] I_{E_\epsilon}(t) + \frac{1}{2} [X_1(t), Y_1(t), (K_1(t), X_1(t))] D^2 g(t) [X_1(t), Y_1(t), (K_1(t), X_1(t))]^\top,
\end{align*}
\]

\(\varsigma = \frac{1}{2} \langle \phi_{xx} (X(T)) X_1(T), X_1(T) \rangle\).

By Lemma 5.2 in Appendix, we have

\[
\begin{align*}
&\mathbb{E} \left[ \sup_{t \in [0,T]} (|X_2(t)|^2 + |Y_2(t)|^2) \right] + \mathbb{E} \left[ \int_0^T |Z_2(t)|^2 dt \right] \\
&\leq C \mathbb{E} \left[ |\varsigma|^2 + \left( \int_0^T (|L_1(t)| + |L_3(t)|) dt \right)^2 + \int_0^T |L_2(t)|^2 dt \right] \\
&\leq C \mathbb{E} \left[ \sup_{t \in [0,T]} |X_1(t)|^4 + \left( \int_{E_\epsilon} |\delta b(t, \Delta) + \delta \sigma(t, \Delta) + \delta g(t, \Delta)| dt \right)^2 \right] \\
&\quad + C \mathbb{E} \left[ \sup_{t \in [0,T]} |X_1(t)|^2 \int_{E_\epsilon} |\delta \sigma_x(t, \Delta) + \delta \sigma_y(t, \Delta) + \delta \sigma_z(t, \Delta)|^2 dt \right] \\
&\leq C \mathbb{E} \left[ \sup_{t \in [0,T]} |X_1(t)|^4 \right] + C \mathbb{E} \left[ \int_{E_\epsilon} (1 + |X(t)|^2 + |\dot{Y}(t)|^2 + |Z(t)|^2 + |u(t)|^2 + |\dot{u}(t)|^2) dt \right] \\
&\quad + C \mathbb{E} \left[ \sup_{t \in [0,T]} |X_1(t)|^2 \int_{E_\epsilon} |\delta \sigma_x(t, \Delta) + \delta \sigma_y(t, \Delta) + \delta \sigma_z(t, \Delta)|^2 dt \right] \\
&\leq C \epsilon^2
\end{align*}
\]
Define \( \epsilon \) in the proof of Lemma 3.5 and Lemma 3.8. Let

\[
B = o \leq \frac{E2}{\sup t \in [0,T] |X(t)|^2 + \sup t \in [0,T] |\xi(t)|^2 dt}^{\beta/2}.
\]

Now, we focus on the last estimate. We use the same notations \( \xi^{1,\epsilon}(t) \), \( \eta^{1,\epsilon}(t) \), \( \zeta^{1,\epsilon}(t) \), \( \xi^{2,\epsilon}(t) \), \( \eta^{2,\epsilon}(t) \) and \( \zeta^{2,\epsilon}(t) \) in the proof of Lemma 3.5 and Lemma 3.8. Let

\[
\Theta(t, \Delta I_{E_\epsilon}) = (\bar{X}(t), \bar{Y}(t), \bar{Z}(t) + \Delta(t)I_{E_\epsilon}(t)); \quad \Theta^\epsilon(t) := (X^\epsilon(t), Y^\epsilon(t), Z^\epsilon(t)).
\]

Define

\[
\widetilde{D}^2b(t) = 2 \int_0^1 \int_0^1 \theta D^2b(t, \Theta(t, \Delta I_{E_\epsilon}) + \lambda \theta(\Theta^\epsilon(t) - \Theta(t, \Delta I_{E_\epsilon})), u(t))d\theta d\lambda,
\]

and \( \widetilde{D}^2\sigma(t) \), \( \widetilde{D}^2g(t) \), \( \tilde{\phi}_{z_2}(T) \) are defined similarly. Then, we have

\[
\begin{aligned}
\left\{ \begin{array}{l}
d\xi^{3,\epsilon}(t) = \{ b_x(t)\xi^{3,\epsilon}(t) + b_y(t)\eta^{3,\epsilon}(t) + B_2(t) \} \ dt \\
+ \{ \sigma_x(t)\xi^{3,\epsilon}(t) + \sigma_y(t)\eta^{3,\epsilon}(t) + \sigma_z(t)\zeta^{3,\epsilon}(t) + C_2(t) \} \ dB(t), \\
\end{array} \right.
\xi^{3,\epsilon}(0) = 0,
\end{aligned}
\]

and

\[
\begin{aligned}
\left\{ \begin{array}{l}
d\eta^{3,\epsilon}(t) = -\{ g_x(t)\xi^{3,\epsilon}(t) + g_y(t)\eta^{3,\epsilon}(t) + g_z(t)\zeta^{3,\epsilon}(t) + C_2(t) \} \ dt - \zeta^{3,\epsilon}(t)dB(t), \\
\eta^{3,\epsilon}(T) = \langle \phi_x(\bar{X}(T)), \xi^{3,\epsilon}(T) \rangle + D_2(T),
\end{array} \right.
\end{aligned}
\]

where

\[
A_2(t) = \left[ \delta b_x(t, \Delta)\xi^{1,\epsilon}(t) + \delta b_y(t, \Delta)\eta^{1,\epsilon}(t) + \delta b_z(t, \Delta) \left( \xi^{1,\epsilon}(t) - \Delta(t)I_{E_\epsilon}(t) \right) \right] I_{E_\epsilon}(t)
+ \frac{1}{2} D^2\tilde{b}(t) \left( X_1(t)^T, Y_1(t), \langle K_1(t), X_1(t) \rangle \right)^2 - \frac{1}{2} D^2b(t) \left( X_1(t)^T, Y_1(t), \langle K_1(t), X_1(t) \rangle \right)^2,
\]

\[
B_2(t) = \left[ \delta \sigma_x(t, \Delta)\xi^{2,\epsilon}(t) + \delta \sigma_y(t, \Delta)\eta^{2,\epsilon}(t) + \delta \sigma_z(t, \Delta)\zeta^{2,\epsilon}(t) \right] I_{E_\epsilon}(t)
+ \frac{1}{2} D^2\sigma(t) \left( X_1(t)^T, Y_1(t), \langle K_1(t), X_1(t) \rangle \right)^2 - \frac{1}{2} D^2\sigma(t) \left( X_1(t)^T, Y_1(t), \langle K_1(t), X_1(t) \rangle \right)^2,
\]

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Indeed, (3.20) is due to the following estimates:

\[
C_z(t) = \left[ \delta g_x(t, \Delta), \xi^{1, \varepsilon}(t) + \delta g_x(t, \Delta) \eta^{1, \varepsilon}(t) + \delta g_x(t, \Delta) \left( \xi^{1, \varepsilon}(t) - (\Delta(t)I_E, (t)) \right) I_E(t) \right. \\
\left. + \frac{1}{2} \left[ \xi^{1, \varepsilon}(t)^T, \eta^{1, \varepsilon}(t), \xi^{1, \varepsilon}(t) - (\Delta(t)I_E, (t)) \right] \overline{D^2 g'}(t) \left[ \xi^{1, \varepsilon}(t)^T, \eta^{1, \varepsilon}(t), \xi^{1, \varepsilon}(t) - (\Delta(t)I_E, (t)) \right]^T \right] \\
- \frac{1}{2} \left[ X_1(t)^T, Y_1(t), K_1(t)X_1(t) \right] D^2 g(t) \left[ X_1(t)^T, Y_1(t), K_1(t)X_1(t) \right]^T,
\]

\[
D_z(T) = \frac{1}{2} \left\{ \hat{\phi}_{xx}(T) \xi^{1, \varepsilon}(T), \xi^{1, \varepsilon}(T) \right\} - \frac{1}{2} \left\{ \phi_{xx} (\tilde{X}(T))X_1(T), X_1(T) \right\},
\]

and \( \overline{D^2 b'}(t) (X_1(t)^T, Y_1(t), (K_1(t), X_1(t)))^2 \) is defined similar to \( D^2 b(t) (X_1(t)^T, Y_1(t), (K_1(t), X_1(t)))^2 \).

By Lemma 5.1 in Appendix,

\[
\eta^{3, \varepsilon}(t) = \langle p(t), \xi^{3, \varepsilon}(t) \rangle + \varphi(t), \\
\zeta^{3, \varepsilon}(t) = \langle K_1(t), \xi^{3, \varepsilon}(t) \rangle + (1 - \langle p(t), \sigma_z(t) \rangle)^{-1} \left[ \langle p(t), \sigma_y(t) \rangle \varphi(t) + \langle p(t), B_z(t) \rangle + \nu(t) \right].
\]

Then we have

\[
|\eta^{3, \varepsilon}(0)| = |E[\varphi(0)]| \leq CE \left[ |D_z(T)| + \int_0^T \left( |A_z(t)| + |B_z(t)| + |C_z(t)| \right) dt \right]. \tag{3.19}
\]

We estimate each term as follows.

1. \[
E \left| D_z(T) \right| \leq C \left\{ E \left[ |\hat{\phi}_{xx}(T) - \phi_{xx}(\tilde{X}(T))| \xi^{1, \varepsilon}(T)|^2 + |\xi^{2, \varepsilon}(T)| |\xi^{1, \varepsilon}(T) + X_1(T)| \right] \right\}
\]

\[
= o(\varepsilon).
\]

2. We estimate \[
E \left[ \int_0^T |A_z(t)| dt \right] = o(\varepsilon). \tag{3.20}
\]

Indeed, (3.20) is due to the following estimates:

\[
E \left[ \int_0^T |\delta b_z(t, \Delta)| (\xi^{1, \varepsilon}(t) - (\Delta(t)I_E, (t)) I_E(t) dt \right] \\
\leq E \left[ \int_{E_z} \left| \delta b_z(t, \Delta) \right| (|\zeta^{2, \varepsilon}(t)| + |(K_1(t), X_1(t))|) |\xi^{1, \varepsilon}(t) - (\Delta(t)I_E, (t)) dt \right] \\
\leq CE \left[ \int_{E_z} |\zeta^{2, \varepsilon}(t)| dt \right] + CE \left[ \sup_{t \in [0, T]} |X_1(t)| \int_{E_z} |\delta b_z(t, \Delta)| dt \right] \\
\leq CE \left\{ E \left[ \int_0^T |\zeta^{2, \varepsilon}(t)|^2 dt \right] \right\} + CE \left[ \sup_{t \in [0, T]} |X_1(t)| \right] \\
= o(\varepsilon),
\]
The other terms are similar.
(3) The estimate of $\mathbb{E} \left[ \int_0^T |B_{zz}^2(t)| dt \right]$
$$\mathbb{E} \left[ \int_0^T |\delta \sigma_z(t, \Delta) \zeta^{2, \epsilon}(t) I_{E_z}(t)| dt \right]$$
$$\leq C \mathbb{E} \left[ \int_0^T |\zeta^{2, \epsilon}(t)| dt \right]$$
$$\leq C \epsilon^{\frac{1}{2}} \left\{ \mathbb{E} \left[ \int_0^T |\zeta^{2, \epsilon}(t)|^2 dt \right] \right\}^{\frac{1}{2}}$$
$$= o(\epsilon),$$

$$\mathbb{E} \left[ \int_0^T |\tilde{\sigma}_{zz}^2(t) (\zeta^{1, \epsilon}(t) - \Delta(t) I_{E_z}(t))^2 - \sigma_{zz}^2(t) (K_1(t), X_1(t))^2 | dt \right]$$
$$\leq \mathbb{E} \left[ \int_0^T \left| \tilde{\sigma}_{zz}^2(t) (\zeta^{1, \epsilon}(t) - \Delta(t) I_{E_z}(t) + K_1(t) X_1(t)) \right| \zeta^{2, \epsilon}(t) | dt \right]$$
$$+ \mathbb{E} \left[ \int_0^T \left| \left( \tilde{\sigma}_{zz}^2(t) - \sigma_{zz}^2(t) \right) \left( K_1(t), X_1(t) \right)^2 \right| dt \right]$$
$$\leq C \left\{ \mathbb{E} \left[ \int_0^T |\zeta^{2, \epsilon}(t)|^2 dt \right] \right\}^{\frac{1}{2}} \left\{ \mathbb{E} \left[ \int_0^T \left| \left( \tilde{\sigma}_{zz}^2(t) - \sigma_{zz}^2(t) \right) \left( K_1(t), X_1(t) \right)^2 \right|^2 dt \right] \right\}^{\frac{1}{2}}$$
$$+ C \epsilon^{\frac{1}{2}} \left\{ \mathbb{E} \left[ \int_0^T |\zeta^{2, \epsilon}(t)|^2 dt \right] \right\}^{\frac{1}{2}}$$
$$= o(\epsilon).$$

The other terms are similar.
(4) The estimate of $\mathbb{E} \left[ \int_0^T |C_{zz}(t)| dt \right]$ is the same as the one of $\mathbb{E} \left[ \int_0^T |A_{zz}(t)| dt \right]$.
Finally, we obtain
$$Y^{\epsilon}(0) - \bar{Y}(0) = Y_1(0) - Y_2(0) = o(\epsilon).$$

The proof is complete. ■

In the above lemma, we only prove $Y^{\epsilon}(0) - \bar{Y}(0) - Y_1(0) - Y_2(0) = o(\epsilon)$ and have not deduced
$$\mathbb{E} \left[ \sup_{t \in [0, T]} \left| Y^{\epsilon}(t) - \bar{Y}(t) - Y_1(t) - Y_2(t) \right|^2 \right] = o(\epsilon^2).$$

The reason is
$$\mathbb{E} \left[ \int_0^T |\tilde{\sigma}_{zz}^2(t) (\zeta^{1, \epsilon}(t) - \Delta(t) I_{E_z}(t))^2 | \zeta^{2, \epsilon}(t)^2 | dt \right] = o(\epsilon^2)$$
may be not hold. But if
\[ \sigma(t, x, y, z, u) = A(t)z + \sigma_1(t, x, y, u) \]  
where \( A(t) \) is a bounded adapted process, then \( \sigma_{zz} \equiv 0 \). In this case, we can prove the following estimates.

**Lemma 3.10** Under the same Assumptions as in Lemma 3.9 and \( \sigma(t, x, y, z, u) = A(t)z + \sigma_1(t, x, y, u) \) where \( A(t) \) is a bounded adapted process. Then
\[
\begin{align*}
\mathbb{E} \left[ \sup_{t \in [0,T]} |X^\epsilon(t) - \bar{X}(t) - X_1(t) - X_2(t)|^2 \right] &= o(\epsilon^2), \\
\mathbb{E} \left[ \sup_{t \in [0,T]} |Y^\epsilon(t) - \bar{Y}(t) - Y_1(t) - Y_2(t)|^2 + \int_0^T |Z^\epsilon(t) - \bar{Z}(t) - Z_1(t) - Z_2(t)|^2 dt \right] &= o(\epsilon^2).
\end{align*}
\]

**Proof.** We use all notations in Lemma 3.9. By Lemma 5.2 in Appendix, we have
\[
\begin{align*}
\mathbb{E} \left[ \sup_{t \in [0,T]} |X^\epsilon(t) - \bar{X}(t) - X_1(t) - X_2(t)|^2 \right] &= o(\epsilon^2), \\
\mathbb{E} \left[ \sup_{t \in [0,T]} |Y^\epsilon(t) - \bar{Y}(t) - Y_1(t) - Y_2(t)|^2 + \int_0^T |Z^\epsilon(t) - \bar{Z}(t) - Z_1(t) - Z_2(t)|^2 dt \right] &= o(\epsilon^2).
\end{align*}
\]

By Lemma 5.2 in Appendix, we have
\[
\begin{align*}
\sup_{t \in [0,T]} \left( |\xi^3,\epsilon(t)|^2 + |\eta^{3,\epsilon}(t)|^2 \right) + \int_0^T |\xi^{3,\epsilon}(t)|^2 dt 
\leq C \mathbb{E} \left[ \left( \int_0^T |\xi^2(t)|^2 dt \right)^2 + \left( \int_0^T |C\xi^2(t)|^2 dt \right)^2 + \int_0^T |B^2(t)|^2 dt + |D^2(T)|^2 \right],
\end{align*}
\]
where \( A^2(t), C^2(t), D^2(T) \) are the same as Lemma 3.9 and
\[
B^2(t) = \left[ \xi^2(t) \xi^{3,\epsilon}(t) + \delta \eta^{3,\epsilon}(t) + \sigma(t) \eta^{2,\epsilon}(t) \right] I_{\epsilon,(t)} + \frac{1}{2} D^2 \sigma(t) \left( \xi^1,\epsilon(t)^T, \eta^{1,\epsilon}(t) \right)^2 - \frac{1}{4} D^2 \sigma(t) \left( X_1(t)^T, Y_1(t) \right)^2.
\]
Combining Lemmas 3.15 and 3.16 in [3], we can obtain the desired estimates. ■

### 3.3 Maximum principle

Note that \( Y_1(0) = 0 \), by Lemma 3.9, we have
\[
J(u^*(\cdot)) - J(\bar{u}(\cdot)) = Y^\epsilon(0) - \bar{Y}(0) = Y_2(0) + o(\epsilon).
\]
In order to obtain \( Y_2(0) \), we introduce the following second-order adjoint equation:
\[
\begin{align*}
- dP(t) &= \left\{ \left( D\sigma(t)[I_{n \times n}, p(t), K_1(t)] \right)^T P(t) D\sigma(t)[I_{n \times n}, p(t), K_1(t)] + P(t) D\sigma(t)[I_{n \times n}, p(t), K_1(t)]^T \\
+ (D\sigma(t)[I_{n \times n}, p(t), K_1(t)]^T P(t) + P(t) H_y(t) + Q(t) D\sigma(t)[I_{n \times n}, p(t), K_1(t)]^T \\
+ (D\sigma(t)[I_{n \times n}, p(t), K_1(t)]^T Q(t) + [I_{n \times n}, p(t), K_1(t)] D^2 H(t) [I_{n \times n}, p(t), K_1(t)]^T + H_z(t) K_2(t) \right\} dt \\
- Q(t) dB(t), \\
P(T) &= \phi_{xx}(\bar{X}(T)),
\end{align*}
\]  
(3.23)
where
\[ H(t, x, y, z, u, p, q) = g(t, x, y, z, u) + \langle p, b(t, x, y, z, u) \rangle + \langle q, \sigma(t, x, y, z, u) \rangle, \]
\[ K_2(t) = (1 - \langle p(t), \sigma_z(t) \rangle)^{-1} \{ \sigma_y(t)p(t)P(t) + (\sigma_x(t) + \sigma_y(t))p(t)^T + \sigma_z(t)K_1(t))^T P(t) \} \]
\[ + (1 - \langle p(t), \sigma_z(t) \rangle)^{-1} \left\{ P(t)(\sigma_x(t) + \sigma_y(t))p(t)^T + \sigma_z(t)K_1(t)^T \right\} + Q(t) + p(t)D^2\sigma(t) (I_{n \times n}, p(t), K_1(t))^2, \]
and \( p(t)D^2\sigma(t) (I_{n \times n}, p(t), K_1(t))^2 \in \mathbb{R}^{n \times n} \) such that
\[ \left\langle p(t)D^2\sigma(t) (I_{n \times n}, p(t), K_1(t))^2 X(t), X_1(t) \right\rangle = \left\langle p(t)D^2\sigma(t) (X_1(t)^T, Y_1(t), (K_1(t), X_1(t))^2 \right\rangle, \]
\( DH(t), D^2H(t) \) are defined similar to \( D\psi \) and \( D^2\psi \).

Lemma 3.11 Under the same Assumptions as in Lemma 3.9. Then we have
\[ Y_2(t) = \langle p(t), X_2(t) \rangle + \frac{1}{2} \langle P(t)X_1(t), X_1(t) \rangle + \hat{Y}(t), \]
\[ Z_2(t) = I(t) + \hat{Z}(t), \]
where \((\hat{Y}(\cdot), \hat{Z}(\cdot))\) is the solution to (3.24) and
\[ I(t) = (K_1(t), X_2(t)) + \frac{1}{2} \langle K_2(t)X_1(t), X_1(t) \rangle + (1 - \langle p(t), \sigma_z(t) \rangle)^{-1} \left\langle p(t), \sigma_y(t)\hat{Y}(t) + \sigma_z(t)\hat{Z}(t) \right\rangle \]
\[ + \langle P(t)\delta\sigma(t, \Delta), X_1(t) \rangle I_{E_1}(t) + (1 - \langle p(t), \sigma_z(t) \rangle)^{-1} \left\langle p(t), \delta\sigma(t, \Delta)X_1(t) \right\rangle I_{E_1}(t) \]
\[ + (1 - \langle p(t), \sigma_z(t) \rangle)^{-1} \left\langle [p(t), \delta\sigma(t, \Delta)] \langle p(t), X_1(t) \rangle + \delta\sigma(t, \Delta) \langle K_1(t), X_1(t) \rangle \right\rangle I_{E_1}(t). \]

Proof. Using the same method as in Lemma 3.7 we can deduce the above relationship similarly. ■

Consider the following equation:
\[
\begin{cases}
\frac{d\gamma(t)}{dt} = \gamma(t) \left[ H_y(t) + (1 - \langle p(t), \sigma_z(t) \rangle)g_z(t) \langle p(t), \sigma_y(t) \rangle \right] dt + \gamma(t) \left[ H_z(t) + (1 - \langle p(t), \sigma_z(t) \rangle)g_z(t) \langle p(t), \sigma_z(t) \rangle \right] dB(t), \\
\gamma(0) = 1.
\end{cases}
\] (3.25)

Applying Itô’s formula to \( \gamma(t)\hat{Y}(t) \), we obtain
\[ \hat{Y}(0) = \mathbb{E} \left\{ \int_0^T \gamma(t) \left[ \delta H(t, \Delta) + \frac{1}{2} \delta\sigma(t, \Delta)^T P(s)\delta\sigma(t, \Delta) \right] I_{E_1}(t) dt \right\}. \]
Define

\[\mathcal{H}(t, x, y, z, u, p, q, P) = \langle p, b(t, x, y, z + \Delta(t), u) \rangle + \langle q, \sigma(t, x, y, z + \Delta(t), u) \rangle + g(t, x, y, z + \Delta(t), u)\]

\[+ \frac{1}{2} \langle \sigma(t, x, y, z + \Delta(t), u) - \sigma(t, X(t), \bar{Y}(t), \bar{Z}(t), \bar{u}(t)) \rangle^T P \langle \sigma(t, x, y, z + \Delta(t), u) - \sigma(t, X(t), \bar{Y}(t), \bar{Z}(t), \bar{u}(t)) \rangle,\]

where \(\Delta(t)\) is defined in (3.9) corresponding to \(u(t) = u\). It is easy to check that

\[\delta \mathcal{H}(t, \Delta) + \frac{1}{2} \delta \sigma(t, \Delta)^T P(t) \delta \sigma(t, \Delta) = \mathcal{H}(t, \bar{X}(t), \bar{Y}(t), \bar{Z}(t), \bar{u}(t), p(t), q(t), P(t)) - \mathcal{H}(t, \bar{X}(t), \bar{Y}(t), \bar{Z}(t), \bar{u}(t), p(t), q(t), P(t)).\]

Noting that \(\gamma(t) > 0\) for \(t \in [0, T]\), then we obtain the following maximum principle.

**Theorem 3.12** Under the same Assumptions as in Lemma 3.9. Let \(\bar{u}(-) \in \mathcal{U}(0, T)\) be optimal and \((\bar{X}(-), \bar{Y}(-), \bar{Z}(-))\) be the corresponding state processes of (3.27). Then the following stochastic maximum principle holds:

\[\mathcal{H}(t, \bar{X}(t), \bar{Y}(t), \bar{Z}(t), \bar{u}(t), p(t), q(t), P(t)) \geq \mathcal{H}(t, X(t), \bar{Y}(t), \bar{Z}(t), \bar{u}(t), p(t), q(t), P(t)), \quad \forall u \in \mathcal{U} \text{ a.e., a.s.},\]

(3.27)

where \((p(-), q(-))\), \((P(-), Q(-))\) satisfy (3.26), (3.24) respectively, and \(\Delta(-)\) satisfies (3.14).

### 4 The case when \(q\) is unbounded

In this section, we consider the case when \(q\) is unbounded and propose the second kind of assumptions.

The relations \(Y_1(t) = \langle p(t), X_1(t) \rangle\) and \(Z_1(t) = \langle K_1(t), X_1(t) \rangle + \Delta(t)I_{E_1}(t)\) in Lemma 3.7 is the key point to derive the maximum principle (3.26). Note that to prove Lemma 3.7 we need Assumption 3.4 which implies

\[\mathbb{E} \left[ \sup_{t \in [0, T]} |X_1(t)|^2 \right] < \infty.\]

(4.1)

However, under the following assumption, combing Theorems 3.3 we can obtain the relations \(Y_1(t) = \langle p(t), X_1(t) \rangle\) and \(Z_1(t) = \langle K_1(t), X_1(t) \rangle + \Delta(t)I_{E_1}(t)\) without the Assumption \(q(-)\) is bounded.

**Assumption 4.1** \(\sigma(t, x, y, z, u) = A(t)z + \sigma_1(t, x, y, u)\).

**Assumption 4.2** For any \(u'(-) \in \mathcal{U}(0, T)\) and \(\beta \in [2, 8]\), the FBSDE (3.4) has a unique solution \((\bar{X}(-), \bar{Y}(-), \bar{Z}(-)) \in L^2_F(\Omega; C([0, T], \mathbb{R}^n)) \times L^2_F(\Omega; C([0, T], \mathbb{R})) \times L^2_F([0, T]; \mathbb{R})\). Moreover, we assume that the following estimate for FBSDE (3.4) holds, that is,

\[\| (\bar{X}, \bar{Y}, \bar{Z}) \|^2_\beta = \mathbb{E} \left\{ \sup_{t \in [0, T]} \left( |\bar{X}(t)|^\beta + |\bar{Y}(t)|^\beta \right) + \left( \int_0^T |\bar{Z}(t)|^2 dt \right)^{\frac{\beta}{2}} \right\} \leq C \mathbb{E} \left\{ \left( \int_0^T \|L_1(t)\|dt \right)^2 + \left( \int_0^T \|L_2(t)\|^2 dt \right)^{\frac{\beta}{2}} + \|\psi_x\|_{\infty}^2 + \|\psi_y\|_{\infty}^2 + \|\psi_z\|_{\infty}^2 + c_1 \right\},\]

where \(C\) depends on \(T, \beta, \|\psi_x\|_{\infty}, \|\psi_y\|_{\infty}, \|\psi_z\|_{\infty}, c_1\).
In this case, the first-order adjoint equation becomes

\[
\begin{align*}
    dp(t) &= -\{g_x(t) + y(t)p(t) + g_z(t)K_1(t) + b_z(t)p(t) + \{b_y(t), p(t)\}p(t) + \{b_z(t), K_1(t)\}p(t) \\
    &\quad + \sigma_x(t)q(t) + \{\sigma_y(t), p(t)\}q(t) + \{A(t), K_1(t)\}q(t)\} \ dt + q(t)dB(t), \\
    p(T) &= \phi_x(\bar{X}(T)),
\end{align*}
\]

where

\[K_1(t) = (1 - \langle p(t), A(t) \rangle)^{-1} \sigma_x(t)p(t) + \langle \sigma_y(t), p(t) \rangle p(t) + q(t)\].

Assumption 4.3 Assume the BSDEs (4.2) have a unique solution \((p(\cdot), q(\cdot)) \in L^\infty(\Omega; C([0, T], \mathbb{R}^n)) \times L^2_\mathbb{F}([0, T]; \mathbb{R}^n)\) such that \(|1 - (p(t), \gamma_2(t))|^{-1}\) is bounded.

The first-order variational equation becomes

\[
\begin{align*}
    dX_1(t) &= [b_z(t)X_1(t) + b_y(t)Y_1(t) + b_z(t)(Z_1(t) - \Delta(t)I_{E_1}(t))] \ dt \\
    &\quad + [\sigma_x(t)X_1(t) + \sigma_y(t)Y_1(t) + A(t)(Z_1(t) - \Delta(t)I_{E_1}(t)) + \delta \sigma(t, \Delta)I_{E_1}(t)] dB(t), \\
    X_1(0) &= 0,
\end{align*}
\]

and

\[
\begin{align*}
    dY_1(t) &= -\{g_x(t), X_1(t)\} + g_y(t)Y_1(t) + g_z(t)(Z_1(t) - \Delta(t)I_{E_1}(t)) - \langle q(t), \delta \sigma(t, \Delta)I_{E_1}(t) \rangle \ dt + Z_1(t)dB(t), \\
    Y_1(T) &= \phi_x(\bar{X}(T))X_1(T),
\end{align*}
\]

where

\[\Delta(t) = (1 - \langle p(t), A(t) \rangle)^{-1} \langle p(t), \sigma_1(t, \bar{X}(t), \bar{Y}(t), u(t)) - \sigma_1(t, \bar{X}(t), \bar{Y}(t), u(t)) \rangle\].

Assumption 4.4 Suppose the following SDE

\[
\begin{align*}
    d\bar{X}_1(t) &= \left[b_z(t)\bar{X}_1(t) + b_y(t)\left\langle p(t), \bar{X}_1(t) \right\rangle + b_z(t)\left\langle K_1(t), \bar{X}_1(t) \right\rangle \right] \ dt \\
    &\quad + \left[\sigma_x(t)\bar{X}_1(t) + \sigma_y(t)\left\langle p(t), \bar{X}_1(t) \right\rangle + A(t)\left\langle K_1(t), \bar{X}_1(t) \right\rangle + \delta \sigma(t, \Delta)I_{E_1}(t) \right] dB(t), \\
    \bar{X}_1(0) &= 0,
\end{align*}
\]

has a unique solution \(\bar{X}_1(\cdot) \in L^2(\Omega; C([0, T], \mathbb{R}^n))\).

By Theorem 5.3 we have the following relationship.

Lemma 4.5 Suppose that Assumptions 4.1, 4.2, 4.3, 4.4, 4.5 and 4.6 hold. Then we have

\[
\begin{align*}
    Y_1(t) &= \langle p(t), X_1(t) \rangle, \\
    Z_1(t) &= \langle K_1(t), X_1(t) \rangle + \Delta(t)I_{E_1}(t),
\end{align*}
\]

where \(p(\cdot)\) is the solution of (4.2).
Lemma 4.6 Under the same Assumptions as in Lemma 4.3 for any $2 \leq \beta < 8$, we have the following estimates

$$
E \left[ \sup_{t \in [0,T]} (|X(t)|^\beta + |Y(t)|^\beta) \right] + E \left[ \left( \int_0^T |Z(t)|^2 dt \right)^{\beta/2} \right] = O(\epsilon^{\beta/2}),
$$

(4.4)

$$
E \left[ \sup_{t \in [0,T]} (|X^x(t) - X(t) - X_1(t)|^4 + |Y^x(t) - Y(t) - Y_1(t)|^4) \right] + E \left[ \left( \int_0^T |Z^x(t) - Z(t) - Z_1(t)|^2 dt \right)^2 \right] = o(\epsilon^2).
$$

Proof. Applying the $L^\beta$-estimates for $(X_1(\cdot), Y_1(\cdot), Z_1(\cdot))$, $(\xi^2, \eta^2, \zeta^2)$ and following the same steps as Lemma 3.23 in [4], we can obtain the desired estimates.

The second-order variational equation becomes

$$
\left\{ \begin{align*}
\quad X_2(t) &= \{ b_x(t)X_2(t) + b_y(t)Y_2(t) + b_z(t)Z_2(t) + \delta b(t, \Delta)I_{E_2}(t) \\
&\quad + \frac{1}{2} D^2b(t) (X_1(t))^T, Y_1(t), (K_1(t), X_1(t)))^2 \} dt \\
&\quad + \{ \sigma_x(t)X_2(t) + \sigma_y(t)Y_2(t) + A(t)Z_2(t) + [\delta \sigma_x(t)X_1(t) + \delta \sigma_y(t)Y_1(t)] I_{E_2}(t) \\
&\quad + \frac{1}{2} D^2 \sigma_1(t) (X_1(t))^T, Y_1(t))^2 \} dB(t), \\
X_2(0) &= 0,
\end{align*} \right.
$$

(4.5)

$$
\left\{ \begin{align*}
\quad Y_2(t) &= \{ g_x(t, X_2(t)) + g_y(t)Y_2(t) + g_z(t)Z_2(t) + \langle q(t), \delta \sigma(t, \Delta) \rangle I_{E_2}(t) + \delta g(t, \Delta)I_{E_2}(t) \\
&\quad + \frac{1}{2} (X_1(t))^T, Y_1(t), (K_1(t), X_1(t)))) D^2g(t) (X_1(t))^T, Y_1(t), (K_1(t), X_1(t)))^T \} dt + Z_2(t)dB(t), \\
Y_2(T) &= \langle \phi_x(\hat{X}(T)), X_2(T) \rangle + \frac{1}{2} \langle \phi_{xx}(\hat{X}(T))X_1(T), X_1(T) \rangle.
\end{align*} \right.
$$

(4.6)

The following second-order estimates hold.

Lemma 4.7 Under the same Assumptions as in Lemma 4.3, we have the following estimates

$$
E \left[ \sup_{t \in [0,T]} |X^x(t) - \hat{X}(t) - X_1(t) - X_2(t)|^2 \right] = o(\epsilon^2),
$$

$$
E \left[ \sup_{t \in [0,T]} |Y^x(t) - \hat{Y}(t) - Y_1(t) - Y_2(t)|^2 \right] + E \left[ \int_0^T |Z^x(t) - \hat{Z}(t) - Z_1(t) - Z_2(t)|^2 dt \right] = o(\epsilon^2).
$$

Proof. We use the same notations $A^2(t) C^2(t)$ and $D^2(t)$ as in Lemma 3.20. The only different term is

$$
B^2(t) = \delta \sigma_x(t) \xi^2(t) I_{E_2}(t) + \delta \sigma_y(t) \eta^2(t) I_{E_2}(t) + \frac{1}{2} D^2 \sigma(t) (\xi^1, \eta^1)^2 \\
- \frac{1}{2} D^2 \sigma(t) (X_1(t)^T, Y_1(t))^2.
$$

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Then, we have that
\[
\begin{align*}
\begin{cases}
\displaystyle d\xi^{3,\epsilon}(t) &= \left[ b_x(t)\xi^{3,\epsilon}(t) + b_y(t)\eta^{3,\epsilon}(t) + b_z(t)\xi^{3,\epsilon}(t) + A_2(t) \right] dt \\
&
\quad + \left[ \sigma_x(t)\xi^{3,\epsilon}(t) + \sigma_y(t)\eta^{3,\epsilon}(t) + A(t)\xi^{3,\epsilon}(t) + B_2(t) \right] dB(t),
\end{cases}
\end{align*}
\]
(4.7)
and
\[
\begin{align*}
\begin{cases}
\displaystyle d\eta^{3,\epsilon}(t) &= - \left[ \langle g_x(t),\xi^{3,\epsilon}(t) \rangle + g_y(t)\eta^{3,\epsilon}(t) + g_z(t)\xi^{3,\epsilon}(t) + C_2(t) \right] dt + \xi^{3,\epsilon}(t)dB(t),
\end{cases}
\end{align*}
\]
(4.8)
By Assumption 12,
\[
\mathbb{E} \left[ \sup_{t \in [0,T]} (|\xi^{3,\epsilon}(t)|^2 + |\eta^{3,\epsilon}(t)|^2) + \int_{0}^{T} |\xi^{3,\epsilon}(t)|^2 dt \right] 
\leq \mathbb{E} \left[ \left( \int_{0}^{T} |A_2(t)|dt \right)^2 + \left( \int_{0}^{T} |C_2(t)|dt \right)^2 + \int_{0}^{T} |B_2(t)|^2 dt + |D_2(T)|^2 \right].
\]
We can estimate term by term by the same steps in Lemma 3.24 in 14. Thus completes the proof. ■

Now we introduce the second-order adjoint equation:
\[
\begin{align*}
\begin{cases}
\quad & -dP(t) \quad = \langle (D\sigma(t)[I_{n\times n},p(t),K_1(t)]^\top P(t)D\sigma(t)[I_{n\times n},p(t),K_1(t)]^\top + P(t)Db(t)[I_{n\times n},p(t),K_1(t)]^\top \\
&
\quad \quad + (Db(t)[I_{n\times n},p(t),K_1(t)]^\top P(t) + P(t)H_y(t) + Q(t)D\sigma(t)[I_{n\times n},p(t),K_1(t)]^\top \\
&
\quad \quad \quad + (D\sigma(t)[I_{n\times n},p(t),K_1(t)]^\top Q(t) + [I_{n\times n},p(t),K_1(t)] D^2H(t)[I_{n\times n},p(t),K_1(t)]^\top + H_z(t)K_2(t) \rangle dt \\
&
\quad \quad \quad - Q(t)dB(t),
\end{cases}
\end{align*}
\]
(4.9)
where
\[
H(t,x,y,z,u,p,q) = \langle g(t,x,y,z,u) + \langle p, b(t,x,y,z,u) \rangle + \langle g, \sigma(t,x,y,z,u) \rangle, K_2(t) = (1 - \langle p(t), A(t) \rangle)^{-1} \left\{ \sigma_y(t)p(t)^\top P(t) + (\sigma_x(t) + \sigma_y(t)p(t))^\top + A(t)K_1(t)^\top \right\} P(t) + Q(t) \right\} + (1 - \langle p(t), A(t) \rangle)^{-1} \left\{ (P(t) (\sigma_x(t) + \sigma_y(t)p(t))^\top + A(t)K_1(t)^\top) + p(t)D^2\sigma(t) (I_{n\times n},p(t),K_1(t))^2 \right\} \right\}.
\]
(4.9) is a linear BSDE with non-Lipschitz coefficient for P(·). Then, (4.9) has a unique pair of solution according to Theorem 5.21 in 12. By the same analysis as in Lemma 3.11, we introduce the following auxiliary equation:
\[
\begin{align*}
\hat{Y}(t) = \int_{t}^{T} \left\{ \langle H_y(s) + g_z(s) \langle \sigma_y(s),p(s) \rangle (1 - \langle p(s), A(t) \rangle)^{-1} \rangle \hat{Y}(s) \\
\quad \quad + \langle H_z(s) + g_z(s) \langle \sigma_z(s),p(s) \rangle (1 - \langle p(s), A(t) \rangle)^{-1} \rangle \hat{Z}(s) \\
\quad \quad + \delta H(s,\Delta) + \frac{1}{2} \delta \sigma(s,\Delta)^\top P(s)\delta \sigma(s,\Delta) \right\} dB(s),
\end{align*}
\]
(4.10)
where \( \delta H(s, \Delta) := \langle p(s), \delta b(s, \Delta) \rangle + \langle y(s), \delta \sigma(s, \Delta) \rangle + \delta g(s, \Delta) \). We obtain the following relationship.

**Lemma 4.8** Suppose the same Assumptions as in Lemma 4.5 hold. Furthermore, we suppose the following SDE

\[
\begin{align*}
    dX_2(t) &= \left\{ b_x(t)X_2(t) + b_y(t) \left( \langle p(t), X_2(t) \rangle + \frac{1}{2} X_1(t)^T P(t) X_1(t) + \bar{Y}(t) \right) + b_z(t) \left( I(t) + \bar{Z}(t) \right) \\
    &+ \delta b(t, \Delta) I_{E_1}(t) + \frac{1}{2} D^2 b(t) \left( X_1(t)^T, Y_1(t), \langle K_1(t), X_1(t) \rangle \right)^2 \right\} dt \\
    X_2(0) &= 0,
\end{align*}
\]

has a unique solution \( X_2(\cdot) \in L^2_T(\Omega; C([0, T], \mathbb{R}^n)) \) and \( \langle p(t), X_2(t) \rangle + \frac{1}{2} X_1(t)^T P(t) X_1(t) + \bar{Y}(t) \in L^2_T(\Omega; C([0, T], \mathbb{R})) \).

\( I(t) + \bar{Z}(t) \in L^2_T([0, T]; \mathbb{R}) \), where \((\bar{Y}(\cdot), \bar{Z}(\cdot))\) is the solution to (4.10) and

\[
I(t) = \langle K_1(t), X_2(t) \rangle + \frac{1}{2} \langle K_2(t) X_1(t), X_1(t) \rangle + \left( 1 - \langle p(t), A(t) \rangle \right)^{-1} \langle p(t), \sigma_y(t) \bar{Y}(t) + A(t) \bar{Z}(t) \rangle + \langle P(t) \sigma(t, \Delta), X_1(t) \rangle I_{E_1}(t) + \left( 1 - \langle p(t), \sigma_z(t) \rangle \right)^{-1} \langle p(t), \delta \sigma(t, \Delta) X_1(t) \rangle I_{E_1}(t) + \left( 1 - \langle p(t), \sigma_z(t) \rangle \right)^{-1} \langle p(t), \delta \sigma(t, \Delta) \sigma_y(t) \bar{Y}(t) \rangle I_{E_1}(t) I_{E_1}(t).
\]

Then the solution to FBSDE \((4.3)\) has the following relationship

\[
Y_2(t) = \langle p(t), X_2(t) \rangle + \frac{1}{2} X_1(t)^T P(t) X_1(t) + \bar{Y}(t), \\
Z_2(t) = I(t) + \bar{Z}(t).
\]

**Proof.** Applying the techniques in Lemma 3.3 we can deduce the above relationship similarly. □

Combing the estimates in Lemma 4.7 and the relationship in Lemma 4.8 we deduce that

\[
Y^*(0) - \bar{Y}(0) = Y_1(0) + Y_2(0) + o(\epsilon) = \bar{Y}(0) + o(\epsilon) \geq 0.
\]

Define

\[
H(t, x, y, z, u, p, q, P) = \langle p, b(t, x, y, z + \Delta(t), u) + q, \sigma(t, x, y, z + \Delta(t), u) \rangle + g(t, x, y, z + \Delta(t), u) + \frac{1}{2} \left( \sigma(t, x, y, z + \Delta(t), u) - \sigma(t, \bar{X}(t), \bar{Y}(t), \bar{Z}(t), \bar{u}(t)) \right)^T P(\sigma(t, x, y, z + \Delta(t), u) - \sigma(t, \bar{X}(t), \bar{Y}(t), \bar{Z}(t), \bar{u}(t))).
\]

By the same analysis as in Theorem 3.12 we obtain the following maximum principle.

**Theorem 4.9** Under the same Assumptions as in Lemma 4.8. Let \( \bar{u}(\cdot) \in U(0, T] \) be optimal and \((\bar{X}(\cdot), \bar{Y}(\cdot), \bar{Z}(\cdot))\) be the corresponding state processes of \((2.7)\). Then the following stochastic maximum principle holds:

\[
H(t, \bar{X}(t), \bar{Y}(t), \bar{Z}(t), u, p(t), q(t), P(t)) \geq H(t, \bar{X}(t), \bar{Y}(t), \bar{Z}(t), \bar{u}(t), p(t), q(t), P(t)), \quad \forall u \in U \text{ a.e., a.s.}
\]

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5 Appendix

5.1 $L^\beta$-estimate for FBSDE

We introduce the following lemmas. Consider the controlled forward-backward stochastic differential equation

\[
\begin{aligned}
\frac{d\hat{X}(t)}{dt} &= \left[\alpha_1(t)\hat{X}(t) + \beta_1(t)\hat{Y}(t) + \gamma_1(t)\hat{Z}(t) + L_1(t)\right] dt + \left[\alpha_2(t)\hat{X}(t) + \beta_2(t)\hat{Y}(t) + \gamma_2(t)\hat{Z}(t) + L_2(t)\right] dB(t), \\
\frac{d\hat{Y}(t)}{dt} &= -\left[\alpha_3(t)\hat{X}(t) + \beta_3(t)\hat{Y}(t) + \gamma_3(t)\hat{Z}(t) + L_3(t)\right] dt + \hat{Z}(t) dB(t), \\
\hat{X}(0) &= x_0, \quad \hat{Y}(T) = \langle \kappa, \hat{X}(T) \rangle + \varsigma,
\end{aligned}
\]

where $\alpha_i(\cdot), \beta_i(\cdot), \gamma_i(\cdot), i = 1, 2, 3,$ are bounded adapted processes, $\alpha_1(\cdot), \alpha_2(\cdot) \in \mathbb{R}^{n \times n}, \alpha_3(\cdot), \beta_1(\cdot), \beta_2(\cdot), \gamma_1(\cdot), \gamma_2(\cdot) \in \mathbb{R}^n, \beta_3(\cdot), \gamma_3(\cdot) \in \mathbb{R}, L_1(\cdot) \in L^2_F([0, T]; \mathbb{R}^n), L_2(\cdot) \in L^2_F([0, T]; \mathbb{R}), L_3(\cdot) \in L^2_F([0, T]; \mathbb{R}^n),$ $\varsigma \in L^2_F(\Omega; \mathbb{R}^n)$ for some $\beta \in [2, 8], \kappa \in \mathbb{R}^n$ is a $F_T$-measurable random variable. Suppose that the solution to (5.1) has the following relationship

\[
\hat{Y}(t) = \left\langle p(t), \hat{X}(t) \right\rangle + \varphi(t),
\]

where $p(t), \varphi(t)$ satisfies

\[
\begin{aligned}
\frac{dp(t)}{dt} &= -A(t) dt + q(t) dB(t), \\
p(T) &= \kappa, \\
\frac{d\varphi(t)}{dt} &= -C(t) dt + \nu(t) dB(t), \\
\varphi(T) &= \varsigma,
\end{aligned}
\]

$A(t)$ and $C(t)$ will be determined later. Applying Itô’s formula to $\left\langle p(t), \hat{X}(t) \right\rangle + \varphi(t)$, we have

\[
\begin{aligned}
d \left( \left\langle p(t), \hat{X}(t) \right\rangle + \varphi(t) \right) &= \left\langle p(t), \alpha_1(t)\hat{X}(t) + \beta_1(t)\hat{Y}(t) + \gamma_1(t)\hat{Z}(t) + L_1(t) \right\rangle dt + \left\langle p(t), \alpha_2(t)\hat{X}(t) + \beta_2(t)\hat{Y}(t) + \gamma_2(t)\hat{Z}(t) + L_2(t) \right\rangle dB(t) \\
&+ \left\langle q(t), \alpha_3(t)\hat{X}(t) + \beta_3(t)\hat{Y}(t) + \gamma_3(t)\hat{Z}(t) + L_3(t) \right\rangle dt + \left\langle q(t), \hat{X}(t) \right\rangle dB(t).
\end{aligned}
\]

Comparing with the equation satisfied by $\hat{Y}(t)$, one has

\[
\begin{aligned}
\hat{Z}(t) &= \left\langle p(t), \alpha_2(t)\hat{X}(t) + \beta_2(t)\hat{Y}(t) + \gamma_2(t)\hat{Z}(t) + L_2(t) \right\rangle + \left\langle q(t), \hat{X}(t) \right\rangle + \nu(t), \\
&= \left\langle p(t), \alpha_1(t)\hat{X}(t) + \beta_1(t)\hat{Y}(t) + \gamma_1(t)\hat{Z}(t) + L_1(t) \right\rangle dt + \left\langle q(t), \hat{X}(t) \right\rangle dB(t) \\
&+ \left\langle q(t), \alpha_2(t)\hat{X}(t) + \beta_2(t)\hat{Y}(t) + \gamma_2(t)\hat{Z}(t) + L_2(t) \right\rangle - C(t).
\end{aligned}
\]
From equation (5.5), we have the form of \( \hat{Z}(t) \) as
\[
\hat{Z}(t) = (1 - \langle p(t), \gamma_2(t) \rangle)^{-1} \left[ \langle p(t), \alpha_2(t) \hat{X}(t) + \beta_2(t) \hat{Y}(t) + L_2(t) \rangle + \langle q(t), \hat{X}(t) \rangle + \nu(t) \right]
\]
\[
= (1 - \langle p(t), \gamma_2(t) \rangle)^{-1} \left[ \langle \alpha_2(t)p(t) + \langle p(t), \beta_2(t) \rangle p(t) + q(t), \hat{X}(t) \rangle + \langle p(t), \beta_2(t) \rangle \varphi(t) + \langle p(t), L_2(t) \rangle + \nu(t) \right].
\]

From the equation (5.6), and utilizing the form of \( \hat{Y}(t) \) and \( \hat{Z}(t) \), we derive that
\[
A(t) = \alpha_3(t) + \beta_3(t)p(t) + \gamma_3(t)K_1(t) + \alpha_1(t)p(t) + \langle p(t), \beta_1(t) \rangle p(t) + \langle p(t), \gamma_1(t) \rangle K_1(t),
\]
where
\[
K_1(t) = (1 - \langle p(t), \gamma_2(t) \rangle)^{-1} [\alpha_2(t)p(t) + \langle p(t), \beta_2(t) \rangle p(t) + q(t)],
\]
and
\[
C(t) = \left[ \beta_3(t) \varphi(t) + \gamma_3(t) (1 - \langle p(t), \gamma_2(t) \rangle)^{-1} [\langle p(t), \beta_2(t) \rangle \varphi(t) + \langle p(t), L_2(t) \rangle + \nu(t)] + L_3(t) \right]
\]
\[
+ \left\{ \langle p(t), \beta_1(t) \rangle \varphi(t) + \gamma_1(t) (1 - \langle p(t), \gamma_2(t) \rangle)^{-1} [\langle p(t), \beta_2(t) \rangle \varphi(t) + \langle p(t), L_2(t) \rangle + \nu(t)] + L_4(t) \right\}
\]
\[
+ \left\{ \langle q(t), \beta_1(t) \rangle \varphi(t) + \gamma_2(t) (1 - \langle p(t), \gamma_2(t) \rangle)^{-1} [\langle p(t), \beta_2(t) \rangle \varphi(t) + \langle p(t), L_2(t) \rangle + \nu(t)] + L_5(t) \right\}.
\]

**Lemma 5.1** Assume (5.2) has a unique solution \((p(\cdot), q(\cdot)) \in L^2_F(\Omega; C([0, T], \mathbb{R}^n)) \times L^2_F([0, T]; \mathbb{R}^n)\) such that \((1 - \langle p(t), \gamma_2(t) \rangle)^{-1}\) is bounded. Then

(i) BSDE (5.3) has a unique solution in \(L^2_F(\Omega; C([0, T], \mathbb{R})) \times L^2_F([0, T]; \mathbb{R})\);

(ii) FBSDE (5.1) has a solution \((\hat{X}(\cdot), \hat{Y}(\cdot), \hat{Z}(\cdot)) \in L^2_F(\Omega; C([0, T], \mathbb{R}^n)) \times L^2_F(\Omega; C([0, T], \mathbb{R})) \times L^2_F([0, T]; \mathbb{R}),\)

where \(\hat{X}(\cdot)\) is the solution to
\[
\begin{aligned}
d\hat{X}(t) &= \left\{ \begin{array}{l}
\alpha_1(t) \hat{X}(t) + \beta_1(t) \langle p(t), \hat{X}(t) \rangle + \gamma_1(t) \langle K_1(t), \hat{X}(t) \rangle + \beta_1(t) \varphi(t) + L_1(t) \\
+ \gamma_1(t) (1 - \langle p(t), \gamma_2(t) \rangle)^{-1} [\langle p(t), \beta_2(t) \rangle \varphi(t) + \langle p(t), L_2(t) \rangle + \nu(t)] \end{array} \right\} dt \\
+ \left\{ \begin{array}{l}
\alpha_2(t) \hat{X}(t) + \beta_2(t) \langle p(t), \hat{X}(t) \rangle + \gamma_2(t) \langle K_1(t), \hat{X}(t) \rangle + \beta_2(t) \varphi(t) + L_2(t) \\
+ \gamma_2(t) (1 - \langle p(t), \gamma_2(t) \rangle)^{-1} [\langle p(t), \beta_2(t) \rangle \varphi(t) + \langle p(t), L_2(t) \rangle + \nu(t)] \end{array} \right\} dB(t),
\end{aligned}
\]
\[
\hat{X}(0) = x_0,
\]

and
\[
\hat{Y}(t) = \langle p(t), \hat{X}(t) \rangle + \varphi(t),
\]
\[
\hat{Z}(t) = \langle K_1(t), \hat{X}(t) \rangle + (1 - \langle p(t), \gamma_2(t) \rangle)^{-1} [\langle p(t), \beta_2(t) \rangle \varphi(t) + \langle p(t), L_2(t) \rangle + \nu(t)].
\]

**Proof.** The result can be obtained by applying Itô’s formula. ■

According to above Lemma, we have the following result which describes the estimate of the solution \((\hat{X}(\cdot), \hat{Y}(\cdot), \hat{Z}(\cdot))\). Before that, we need impose the following assumption.

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Lemma 5.2 Suppose that the same assumptions in Lemma 5.1 hold. Furthermore, suppose the FBSDE (5.1) has a unique solution in \( L^2_T(\Omega; C([0,T],\mathbb{R}^n)) \times L^2_T(\Omega; C([0,T],\mathbb{R})) \times L^2_T([0,T];\mathbb{R}) \). Then
\[
\mathbb{E} \left[ \sup_{t \in [0,T]} (|\hat{X}(t)|^\beta + |\hat{Y}(t)|^\beta) \right] + \mathbb{E} \left[ \left( \int_0^T |\hat{Z}(t)|^2 dt \right)^{\frac{\beta}{2}} \right] \\
\leq C \mathbb{E} \left[ |x_0|^\beta + |\varsigma|^\beta + \left( \int_0^T (|L_1(t)| + |L_2(t)| + |\varphi(t)| + |\nu(t)|) dt \right)^\beta + \left( \int_0^T (|L_2(t)|^2 + |\nu(t)|^2) dt \right)^{\frac{\beta}{2}} \right].
\]

Proof. The equation satisfied by \((\varphi(\cdot),\nu(\cdot))\) is a linear BSDE with bounded coefficients. By standard estimate of BSDE, we have
\[
\mathbb{E} \left[ \sup_{t \in [0,T]} |\varphi(t)|^\beta \right] + \mathbb{E} \left[ \left( \int_0^T |\nu(t)|^2 dt \right)^{\frac{\beta}{2}} \right] \\
\leq C \mathbb{E} \left[ |\varphi|^\beta + \left( \int_0^T (|L_1(t)| + |L_2(t)| + |\varphi(t)| + |\nu(t)|) dt \right)^\beta + \left( \int_0^T (|L_2(t)|^2 + |\nu(t)|^2) dt \right)^{\frac{\beta}{2}} \right].
\]
From the result of above Lemma and the relation between \((\hat{Y}(\cdot),\hat{Z}(\cdot))\) and \(\hat{X}(\cdot)\), we obtain the estimate of \(\hat{X}(\cdot)\) as follows
\[
\mathbb{E} \left[ \sup_{t \in [0,T]} |\hat{X}(t)|^\beta \right] \\
\leq C \mathbb{E} \left[ |x_0|^\beta + \left( \int_0^T (|L_1(t)| + |L_2(t)| + |\varphi(t)| + |\nu(t)|) dt \right)^\beta + \left( \int_0^T (|L_2(t)|^2 + |\nu(t)|^2) dt \right)^{\frac{\beta}{2}} \right] \\
\leq C \mathbb{E} \left[ |x_0|^\beta + \left( \int_0^T (|L_1(t)| + |L_2(t)| + |\varphi(t)| + |\nu(t)|) dt \right)^\beta + \left( \int_0^T (|L_2(t)|^2 + |\nu(t)|^2) dt \right)^{\frac{\beta}{2}} \right].
\]
Since the relation (5.4), we can obtain
\[
\mathbb{E} \left[ \sup_{t \in [0,T]} |\hat{Y}(t)|^\beta \right] \leq C \mathbb{E} \left[ |x_0|^\beta + |\varsigma|^\beta + \left( \int_0^T (|L_1(t)| + |L_2(t)| + |\varphi(t)| + |\nu(t)|) dt \right)^\beta + \left( \int_0^T (|L_2(t)|^2 + |\nu(t)|^2) dt \right)^{\frac{\beta}{2}} \right],
\]
\[
\mathbb{E} \left[ \left( \int_0^T |\hat{Z}(t)|^2 dt \right)^{\frac{\beta}{2}} \right] \leq C \mathbb{E} \left[ \left( \int_0^T \left( |\hat{X}(t)|^2 + |L_2(t)|^2 + |\varphi(t)|^2 + |\nu(t)|^2 \right) dt \right)^{\frac{\beta}{2}} \right] \\
\leq C \mathbb{E} \left[ \sup_{t \in [0,T]} \left( |\hat{X}(t)|^\beta + |\varphi(t)|^\beta \right) + \left( \int_0^T \left( |L_2(t)|^2 + |\nu(t)|^2 \right) dt \right)^{\frac{\beta}{2}} \right] \\
\leq C \mathbb{E} \left[ |x_0|^\beta + |\varsigma|^\beta + \left( \int_0^T (|L_1(t)| + |L_2(t)| + |L_2(t)| + |\varphi(t)| + |\nu(t)|) dt \right)^\beta + \left( \int_0^T (|L_2(t)|^2 + |\nu(t)|^2) dt \right)^{\frac{\beta}{2}} \right].
\]
This completes the proof. \(\blacksquare\)
5.2 FBSDE with non-Lipschitz coefficients

Lemma 5.3 Suppose BSDE \((\cdot, \cdot, \cdot, \cdot)\) has a unique solution \((p(\cdot), q(\cdot)) \in L^2_{\mathcal{F}}(\Omega; C([0, T], \mathbb{R}^n)) \times L^2_{\mathcal{F}}([0, T]; \mathbb{R})\) such that \(1 - \langle p(t), \gamma_2(t) \rangle^{-1}\) is bounded. Let

\[
\begin{aligned}
    d\tilde{X}(t) &= \left\{ \alpha_1(t)\tilde{X}(t) + \beta_1(t)\langle p(t), \tilde{X}(t) \rangle + \gamma_1(t)\langle K_1(t), \tilde{X}(t) \rangle + L_1(t) \\
    &\quad + \gamma_1(t)(1 - \langle p(t), \gamma_2(t) \rangle)^{-1}\langle p(t), L_2(t) \rangle \right\} dt \\
    &\quad + \left\{ \alpha_2(t)\tilde{X}(t) + \beta_2(t)\langle p(t), \tilde{X}(t) \rangle + \gamma_2(t)\langle K_1(t), \tilde{X}(t) \rangle + L_2(t) \right\} dB(t), \quad t \in [0, T],
\end{aligned}
\]

(5.10)

Assume \(\tilde{X}(\cdot) \in L^2_{\mathcal{F}}(\Omega; C([0, T], \mathbb{R}^n))\) and

\[ p(t)L_1(t) + q(t)L_2(t) + L_3(t) + (\gamma_1(t)p(t) + \gamma_2(t)q(t) + \gamma_3(t)(1 - p(t)\gamma_2(t))^{-1}p(t)L_2(t) = 0. \]

Then \((\tilde{X}(\cdot), \tilde{Y}(\cdot), \tilde{Z}(\cdot)) \in L^2_{\mathcal{F}}(\Omega; C([0, T], \mathbb{R}^n)) \times L^2_{\mathcal{F}}(\Omega; C([0, T], \mathbb{R})) \times L^2_{\mathcal{F}}([0, T]; \mathbb{R})\) is the unique solution to FBSDE \((\cdot, \cdot, \cdot, \cdot)\), where

\[
\begin{aligned}
    \tilde{Y}(t) &= \langle p(t), \tilde{X}(t) \rangle, \\
    \tilde{Z}(t) &= \langle K_1(t), \tilde{X}(t) \rangle + (1 - \langle p(t), \gamma_2(t) \rangle)^{-1}\langle p(t), L_2(t) \rangle.
\end{aligned}
\]

(5.11)

Proof. Due to \(p(\cdot) \in L^2_{\mathcal{F}}(\Omega; C([0, T], \mathbb{R}^n))\), we have \(\tilde{Y}(\cdot) \in L^2_{\mathcal{F}}(\Omega; C([0, T], \mathbb{R}))\). On the other hand, from Theorem 5.2 in [4], we can obtain \(q(\cdot) \in L^2_{\mathcal{F}}([0, T]; \mathbb{R})\). Combining with \(\tilde{X}(t) \in L^2_{\mathcal{F}}(\Omega; C([0, T], \mathbb{R}^n))\), we have

\[
\begin{aligned}
    \mathbb{E}\left[ \int_0^T |\langle K_1(t), \tilde{X}(t) \rangle|^2 dt \right] &\leq C\mathbb{E}\left[ \sup_{t \in [0, T]} |\tilde{X}(t)|^2 \int_0^T (1 + |q(t)|^2) dt \right] \\
    &\leq C\mathbb{E}\left[ \sup_{t \in [0, T]} |\tilde{X}(t)|^2 \right] + C \left\{ \mathbb{E}\left[ \sup_{t \in [0, T]} |\tilde{X}(t)|^4 \right] \right\}^{\frac{1}{2}} \left\{ \mathbb{E}\left[ \int_0^T |q(t)|^2 dt \right] \right\}^{\frac{1}{2}} < \infty.
\end{aligned}
\]

This completes the proof. ■

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