Degree zero Gromov–Witten invariants for smooth curves

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Abstract
For a smooth projective curve, we derive a closed formula for the generating series of its Gromov–Witten invariants in genus $g$ and degree zero. It is known that the calculation of these invariants can be reduced to that of the $\lambda_g$ and $\lambda_{g-1}$ integrals on the moduli space of stable algebraic curves. The closed formula for the $\lambda_g$ integrals is given by the $\lambda_g$ conjecture, proved by Faber and Pandharipande. We compute in this paper the $\lambda_{g-1}$ integrals via solving the degree zero limit of the loop equation associated to the complex projective line.

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1 INTRODUCTION

Let $X$ be a smooth projective variety of complex dimension $D$, and let $X_{g,n,\beta}$ be the moduli space of stable maps of degree $\beta \in H_2(X, \mathbb{Z})/\text{torsion}$ with target $X$ from curves of genus $g$ with $n$ distinct marked points. Here, $g, n \geq 0$. Choose a homogeneous basis $\phi_1 = 1, \phi_2, \ldots, \phi_l$ of the cohomology ring $H^*(X; \mathbb{C})$ with $\phi_\alpha \in H^{2q_\alpha}(X; \mathbb{C})$, $\alpha = 1, \ldots, l$, where $0 = q_1 < q_2 \leq q_3 \leq \cdots \leq q_{l-1} < q_l = D$. Note that $q_\alpha$ is a half integer if $\phi_\alpha$ is an odd degree class. We denote the Poincaré pairing on $H^*(X; \mathbb{C})$ by $\langle , \rangle$. The integrals

$$\int_{[X_{g,n,\beta}]^\text{virt}} c_1(L_1)^{i_1} \text{ev}_1^*(\phi_{\alpha_1}) \cdots c_1(L_n)^{i_n} \text{ev}_n^*(\phi_{\alpha_n}), \quad \alpha_1, \ldots, \alpha_n = 1, \ldots, l, \ i_1, \ldots, i_n \geq 0,$$

are called Gromov–Witten (GW) invariants of $X$ of genus $g$ and degree $\beta$. Here, $\text{ev}_a$, $a = 1, \ldots, n$, is the evaluation map, $L_a$ is the $a$th tautological line bundle on $X_{g,n,\beta}$, and $[X_{g,n,\beta}]^\text{virt}$ is the virtual

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fundamental class, which is an element in the Chow ring having the complex dimension
\[(1 - g)(D - 3) + n + \langle \beta, c_1(X) \rangle.\] (1.2)

The genus \( g \) free energy for the GW invariants of \( X \) is defined as the generating series
\[
F^X_g(t; q) = \sum_{n, \beta} \sum_{\alpha_1, \ldots, \alpha_n} \frac{q^\beta t^{\alpha_1 i_1} \cdots t^{\alpha_n i_n}}{n!} \int_{[X_{g,n,\beta}]^{\text{virt}}} c_1(L_1)^{i_1} ev_1^* (\phi_{\alpha_1}) \cdots c_1(L_n)^{i_n} ev_n^* (\phi_{\alpha_n}),
\] (1.3)
where \( t = (t^{\alpha, i})_{\alpha = 1, \ldots, l, i \geq 0} \) is an infinite vector of indeterminates, and
\[q^\beta = q_1^{m_1} \cdots q_r^{m_r} \] (for \( \beta = m_1 \beta_1 + \cdots + m_r \beta_r \))
is an element of the Novikov ring. Here, \((\beta_1, \ldots, \beta_r)\) is a basis of \( H_2(X; \mathbb{Z})/\text{torsion} \). The sum
\[
\sum_{g \geq 0} \varepsilon^{2g-2} F^X_g(t; \varepsilon, q) = F^X(t; \varepsilon, q)
\]
is called the free energy for the GW invariants of \( X \). It should be noted that the order of each monomial \( t^{\alpha_1 i_1} \cdots t^{\alpha_n i_n} \) in (1.3) is important because the odd cohomology classes are considered and that in some literature, the monomial \( t^{\alpha_n i_n} \cdots t^{\alpha_1 i_1} \) in (1.3) is ordered as \( t^{\alpha_1 i_1} \cdots t^{\alpha_n i_n} \).

The interest of this paper is on the calculation of degree zero GW invariants. In particular, we will focus on the case when the target is a smooth projective curve. Before specializing to curves, we will first review what is known about the degree zero invariants for a general smooth projective variety \( X \).

The moduli space \( X_{g,n,0} \) is isomorphic to \( \overline{M}_{g,n} \times X \), where \( \overline{M}_{g,n} \) denotes the Deligne–Mumford moduli space [2] of stable algebraic curves of genus \( g \) with \( n \) distinct marked points. With this identification,
\[
[X_{g,n,0}]^{\text{virt}} = e(E^\vee_{g,n} \boxtimes T_X) \cap [\overline{M}_{g,n} \times X],
\] (1.5)
where \( E_{g,n} \) is the Hodge bundle on \( \overline{M}_{g,n} \times X \) is the tangent bundle of \( X \), and \( e(E^\vee_{g,n} \boxtimes T_X) \) is the Euler class of the obstruction bundle on \( \overline{M}_{g,n} \times X \). Therefore, we have the following formula for the degree zero GW invariants of \( X \):
\[
\int_{[X_{g,n,0}]^{\text{virt}}} c_1(L_1)^{i_1} ev_1^* (\phi_{\alpha_1}) \cdots c_1(L_n)^{i_n} ev_n^* (\phi_{\alpha_n}) = \int_{\overline{M}_{g,n} \times X} \psi_1^{i_1} \cdots \psi_n^{i_n} \phi_{\alpha_1} \cdots \phi_{\alpha_n} \cup e(E^\vee_{g,n} \boxtimes T_X).
\] (1.6)
Here, \( \psi_a, a = 1, \ldots, n \), denotes the first Chern class of the \( a \)th tautological line bundle on \( \overline{M}_{g,n} \). We see from formula (1.6) that Hodge classes on \( \overline{M}_{g,n} \) enter into the story, and we use \( \lambda_j \) \( (j = 0, \ldots, g) \) denote the \( j \)th Chern class of \( E_{g,n} \). For more details and references about formula (1.6), see, for example, [25].
Denote by $F_{g, \deg=0}^X(t)$ the degree zero part of the genus $g$ free energy $F_{g}^X(t; q)$, that is,

$$F_{g, \deg=0}^X(t) = \sum_n \sum_{\alpha_1, \ldots, \alpha_n \neq 0} \frac{t^{\alpha_1} \cdots t_{i_1, \ldots, i_n}}{n!} \int_{[X_{g,n},0]^\text{virt}} c_1(L_1)^{\alpha_1} \cdots c_n(L_n)^{\alpha_n} \psi_1^{\alpha_1} \cdots \psi_n^{\alpha_n}. \quad (1.7)$$

Following [6, 21, 25], let us apply formula (1.6) for the computation of $F_{g, \deg=0}^X(t)$.

In genus zero, we have

$$\int_{[X_{0,n},0]^\text{virt}} \psi_1^{\alpha_1} \cdots \psi_n^{\alpha_n} = \frac{(n-3)!}{i_1! \cdots i_n!}, \quad (1.8)$$

Together with the well-known formula $\int_{[X_{0,n},0]^\text{virt}} \psi_1^{\alpha_1} \cdots \psi_n^{\alpha_n} = \frac{(n-3)!}{i_1! \cdots i_n!}$, one obtains

$$F_{g=0, \deg=0}^X(t) = \sum_{n \geq 0} \frac{1}{n(n-1)(n-2)} \sum_{i_1 + \cdots + i_n = n-3} \int_X t_{i_1} \cdots t_{i_n}. \quad (1.9)$$

where $t_i := \sum_\alpha t^{\alpha} \phi_\alpha$, $i \geq 0$, are the cohomology-valued times.

In genus one, we know from [25] that

$$e(E_{1,n}^V \boxtimes T_X) = c_D(E_{1,n}^V \boxtimes T_X) = c_D(X) - \lambda_1 c_{D-1}(X), \quad n \geq 1, \quad (1.10)$$

and so,

$$\int_{[X_{1,n},0]^\text{virt}} c_1(L_1)^{\alpha_1} \cdots c_n(L_n)^{\alpha_n} \psi_1^{\alpha_1} \cdots \psi_n^{\alpha_n} = \int_{[X_{1,n},0]^\text{virt}} \psi_1^{\alpha_1} \cdots \psi_n^{\alpha_n} c_D(X) - \int_{[X_{1,n},0]^\text{virt}} \psi_1^{\alpha_1} \cdots \psi_n^{\alpha_n} \lambda_1 c_{D-1}(X). \quad (1.11)$$

Combined with explicit computations given in, for example, [6–8], it follows that

$$F_{g=1, \deg=0}^X(t) = \frac{1}{24} \left( c_D(X), \log V_1(t) \right) - \frac{1}{24} \left( c_D(X), V(t) \right). \quad (1.12)$$

Here and below, for $T = (T_0, T_1, T_2, \ldots)$,

$$V(T) := \sum_{n \geq 1} \frac{1}{n} \sum_{i_1 + \cdots + i_n = n-1} T_{i_1} \cdots T_{i_n}, \quad V_k(T) := \frac{\partial^k V(T)}{\partial T_0^k} (k \geq 0), \quad (1.13)$$

and $V_k(T), k \geq 0$, are understood as replacing $T_i$ by $t_i, i \geq 0$.

In genus bigger than or equal to two, we divide the consideration into several cases. For the case when the dimension $D$ is bigger than 3, the virtual dimension (1.2) is negative and $F_{g, \deg=0}^X(t)$ vanishes. For the case when $D = 3$, that is, $X$ is a threefold, Getzler and Pandharipande [21] obtain

$$e(E_{g,n}^V \boxtimes T_X) = (-1)^g \left( c_3(X) - c_2(X)c_1(X) \lambda_{g-1} \lambda_{g-2} \right), \quad g \geq 2. \quad (1.14)$$
Using this formula and the well-known formula (cf. [16, 17]),
\[
\int_{\mathcal{M}_{g,0}} \lambda_g \lambda_{g-1} \lambda_{g-2} = \frac{1}{2(2g-2)!} \frac{|B_{2g-2}| |B_{2g}|}{2g-2} \lambda_g \lambda_{g-1} \lambda_{g-2}, \quad g \geq 2,
\]
the following closed expression for \( P_{g, \text{deg}=0}^X (t) \) is obtained in [7]:
\[
P_{g, \text{deg}=0}^X (t) = \frac{(-1)^g}{2(2g-2)!} \frac{|B_{2g-2}| |B_{2g}|}{2g} \left\langle c_3(X) - c_2(X)c_1(X), V_1(t)^{2g-2} \right\rangle, \quad g \geq 2.
\]

Here and below, \( B_k \) denote the Bernoulli numbers. When \( D = 0 \), that is, \( X \) is a point, the GW invariants of \( X \) can be uniquely determined by the celebrated Witten–Kontsevich theorem (cf. [24, 30]). The remaining cases are the \( D = 1 \) case and the \( D = 2 \) case, that is, \( X \) is a curve or a surface. The focus of this paper is the curve case, and we leave the surface case yet to a future publication.

From now on, we assume that \( X \) is a smooth projective curve of genus \( h \). Getzler and Pandharipande [21] show that
\[
e(\mathbb{E}_{g,n} \boxtimes T_X) = (-1)^g (\lambda_g - \lambda_{g-1} c_1(X)), \quad g \geq 1
\]
(also true for \( g = 1 \) being compared with (1.10)). So, for all \( g \geq 1 \), we have
\[
(-1)^g \int_{\mathcal{M}_{g,n,0}} \psi_1^{i_1} \cdots \psi_n^{i_n} \lambda_g \int_X \phi_{a_1} \cdots \phi_{a_n} - \int_{\mathcal{M}_{g,n}} \sum_{i_1, \ldots, i_n} \psi_1^{i_1} \cdots \psi_n^{i_n} \lambda_{g-1} \int_X \phi_{a_1} \cdots \phi_{a_n} c_1(X).
\]

In terms of generating series, we have
\[
(-1)^g P_{g, \text{deg}=0}^X (t) = \left\langle 1, H_g (\lambda_g; t) \right\rangle - \left\langle c_1(X), H_g (\lambda_{g-1}; t) \right\rangle, \quad g \geq 1.
\]

Here, for an element \( \varphi \in H^* (\overline{\mathcal{M}_{g,n}}; \mathbb{C}) \), and for \( T = (T_0, T_1, T_2, \ldots) \),
\[
H_g (\varphi; T) := \sum_n \sum_{i_1, \ldots, i_n} \int_{\mathcal{M}_{g,n}} \psi_1^{i_1} \cdots \psi_n^{i_n} \varphi \frac{T_{i_1} \cdots T_{i_n}}{n!},
\]
and \( H_g (\varphi; t), g \geq 1 \), are understood by replacing \( T_i \) by \( t^i, i \geq 0 \). We note that the power series \( V(T) \) given in (1.13) is just \( \partial^2 H_0 (1; T) / \partial T_0^2 \). It remains to compute \( H_g (\lambda_g; T) \) and \( H_g (\lambda_{g-1}; T) \); in other words, we need to compute
\[
\int_{\mathcal{M}_{g,n}} \psi_1^{i_1} \cdots \psi_n^{i_n} \lambda_g, \quad \int_{\mathcal{M}_{g,n}} \psi_1^{i_1} \cdots \psi_n^{i_n} \lambda_{g-1}.
\]

It follows from the \( \lambda_g \) conjecture [21] that the Hodge integrals in (1.21) with \( \lambda_g \) have the explicit expression:
\[
\int_{\mathcal{M}_{g,n}} \psi_1^{i_1} \cdots \psi_n^{i_n} \lambda_g = \binom{2g+n-3}{i_1, \ldots, i_n} \frac{2^{2g-1} - 1}{2^{2g-1}} \frac{|B_{2g}|}{(2g)!}, \quad g \geq 1.
\]
The $\lambda_g$ conjecture is proved in [18]. In [8], it is shown that formula (1.22) yields

$$H_g(\lambda_g; T) = \frac{2^{2g-1} - 1}{2^{2g-1}} |B_{2g}| \cdot V_{2g-2}(T), \quad g \geq 1.$$  

(1.23)

Computation for the Hodge integrals in (1.21) with $\lambda_{g-1}$ is more involved. In [6, 7], closed expressions for $H_g(\lambda_{g-1}; T)$ for the first few values of $g$ are found, for example,

$$H_1(\lambda_0 = 1; T) = \frac{1}{24} \log V_1, \quad H_2(\lambda_1; T) = \frac{1}{480} V_3 - \frac{11}{5760} V_2^2,$$

(1.24)

$$H_3(\lambda_2; T) = \frac{-19V_4^2}{53760V_1^4} + \frac{151}{207360} V_2^2 V_3 - \frac{61V_3^2}{322560V_1^2} - \frac{373V_2 V_4}{1451520V_1^2} + \frac{41V_5}{580608V_1^4}.$$  

(1.25)

Note that on the right-hand sides, we omitted the arguments $T$ in $V_m(T)$.

Before proceeding, we recall some notations. A partition is a nonincreasing infinite sequence of nonnegative integers $\mu = (\mu_1, \mu_2, ...)$, the number of nonzero components of $\mu$ is called the length of $\mu$, denoted by $\ell(\mu)$. The sum $\sum_{\mu} \mu_i$ is called the weight of $\mu$, denote by $|\mu|$. The set of all partitions is denoted by $\mathcal{P}$, and the set of partitions of weight $k$ is denoted by $\mathcal{P}_k$. A partition $\mu$ of weight $k$ is also called a partition of $k$. If the length of the partition $\mu$ is positive, it is often denoted by $\mu = (\mu_1, \ldots, \mu_{\ell(\mu)})$; otherwise, it can be denoted either as $(0)$ or as $(\cdot)$. Denote $\mu + 1 = (\mu_1 + 1, \ldots, \mu_{\ell(\mu)} + 1)$ if $\ell(\mu) > 0$, and $(\cdot) + 1 = (\cdot)$ otherwise. The expression $m_i(\mu)$ will denote the multiplicity of $i$ in $\mu$, $i \geq 1$, and denote $m(\mu)! = \prod_{i=1}^{\infty} m_i(\mu)!$. For a sequence of indeterminates $(y_0, y_1, y_2, ...)$, $y_\mu := \prod_{i=1}^{\ell(\mu)} y_{\mu_i}$ (clearly, $y_{(\cdot)} = 1$).

According to [7, 8], for $g \geq 1$, there exist functions $W_g(V_1, \ldots, V_{2g-1})$ of $(2g - 1)$ variables of the form

$$W_1(V_1) = \frac{\log V_1}{24}, \quad W_g(V_1, \ldots, V_{2g-1}) = \sum_{\mu \in \mathcal{P}_{2g-1}} e^g_{\mu} \frac{V_{\mu + 1}}{V_1^\ell(\mu)} (g \geq 2),$$

(1.26)

such that

$$H_g(\lambda_{g-1}; T) = W_g(V_1(T), \ldots, V_{2g-1}(T)), \quad g \geq 1.$$  

(1.27)

Here, $e_{\mu}^g$ are constants. For example, $W_2(V_1, V_2, V_3) = \frac{1}{480} V_3 - \frac{11}{5760} V_2^2$. It then follows from formula (1.19) the following proposition.

**Proposition 1.** For $g \geq 1$, the genus $g$ free energy of $X$ of degree zero has the expression:

$$F_{g, \text{deg}=0}(t) = (-1)^g 2^{2g-1} - 1 |B_{2g}| \cdot (1, V_{2g-2}(t)) - (-1)^g (2 - 2h) W_g(V_1(P), \ldots, V_{2g-1}(P)).$$

(1.28)

Here, $P := (t_1^1, t_1^1, \ldots)$.

The goal of this paper is to give a closed formula for $W_g(V_1, \ldots, V_{2g-1})$. 
Introduce some more notations. The Lagrange number \( L(\mu) \) is defined by
\[
L(\mu) = \frac{(|\mu| + \ell(\mu))!(-1)^{\ell(\mu)}}{m(\mu)! \prod_{j \geq 1} (j + 1)!^{m_j(\mu)}}. \tag{1.29}
\]
For \( g \geq 1 \), introduce the rational function \( B_g(\lambda; \mathbf{V}) \) as follows:
\[
B_g(\lambda; \mathbf{V}) = \frac{2^{2g-1} - 1}{2^{2g-1}} \left( \frac{1}{(2g)!} \right)^2_{2g-2} (2g-2) \sum_{k=1}^{g-2} \binom{2g-2}{k} \left( \frac{1}{\lambda - V} \right)_{k-1} \left( \frac{1}{\lambda - V} \right)_{2g-1-k} - \frac{1}{2} \sum_{g_1 + g_2 = g} \frac{2^{2g_1-1} - 1}{2^{2g_1-1}} \left( \frac{1}{(2g_1)!} \right)^2_{2g_1-1} \left( \frac{1}{(2g_1)!} \right)^2_{2g_2-1} \left( \frac{1}{\lambda - V} \right)_{g_1} \left( \frac{1}{\lambda - V} \right)_{g_2-1}, \tag{1.30}
\]
where \( \mathbf{V} = (V_0 = V, V_1, V_2, \ldots) \), and for a function \( f(\mathbf{V}) \) and \( r \geq 0 \), \( (f(\mathbf{V}))_r \) means \( \partial^r (f(\mathbf{V})) \) with \( \partial := \sum_k V_k \partial / \partial V_k \). Denote \( B_{g,j} = \text{Coef} \left( \frac{(\lambda - V)^{-j-1}}{2g} \right) \). We will prove in Section 3 the following theorem.

**Theorem 1.** For \( g \geq 2 \), we have
\[
W_g(V_1, \ldots, V_{2g-1}) = \frac{1}{2g-2} \sum_{k=2}^{2g-1} (k-1) V_k \sum_{j=1}^{2g-1} c_{k,j} B_{g,j} \tag{1.31}
\]
with
\[
c_{i,j} := \frac{1}{j!} \sum_{\mu \in P_{j-i}} \binom{\ell(\mu) + j - 1}{i - 1} L(\mu) \frac{V_{\mu + 1}}{V_1^{\ell(\mu) + j}}. \tag{1.32}
\]

Let us briefly describe the idea of the proof. Observe from (1.28) that the functions \( W_g \) in (1.28) are independent of \( X \), so we can compute \( W_g \) by taking \( X = \mathbb{P}^1 \). The partition function
\[
Z^{\mathbb{P}^1}(t; \varepsilon, q) := \exp \left( T^{\mathbb{P}^1}(t; \varepsilon, q) \right)
\]
is deeply connected with integrable systems [12, 28, 29], so one can apply the theory of integrable systems (Lax pairs, Virasoro constraints, etc.) to get closed expressions of them; for example, explicit \( n \)-point functions are obtained in [9, 20] by using the Toda lattices. In this paper, we will employ the loop equation [12] for \( \mathbb{P}^1 \).

As a special case of Theorem 1, we will give in Section 4 a simple proof of the following theorem.

**Theorem A [17, 21].** For \( g \geq 1 \), the following formula holds:
\[
\int_{\mathcal{M}_{0,1}} \psi_1^{2g-1} \lambda^{2g-1} = \frac{2^{2g-1} - 1}{2^{2g-1} (2g)!} \sum_{k=1}^{2g-1} \frac{1}{k} - \frac{1}{2^{2g-1} (2g - 1)!} \sum_{g_1 + g_2 = g \atop g_1, g_2 > 0} (2^{2g_1-1} - 1) (2^{2g_2-1} - 1) \frac{|B_{2g_1}|}{2g_1} \frac{|B_{2g_2}|}{2g_2}. \tag{1.33}
\]
The paper is organized as follows. In Section 2, we review the loop equation for GW invariants of \( \mathbb{P}^1 \). In Section 3, we prove Theorem 1. Some applications are discussed in Section 4.

## 2 | LOOP EQUATION FOR GW INVARIANTS OF \( \mathbb{P}^1 \)

In this section, we review the loop equation for GW invariants of the complex projective line, which can be derived using the structure of the associated Frobenius manifold as a result of the following three properties:

1. Virasoro constraints for the genus zero free energy.
2. Existence of jet-variable representation for the higher genera free energies.
3. Virasoro constraints for the all-genus free energy.

Recall that the Frobenius manifold \([5]\) associated to the GW invariants of \( \mathbb{P}^1 \) has the potential \[
F = \frac{1}{2} (v^1)^2 v^2 + q e^{v^2}.
\] (2.1)

Here \((v^1, v^2)\) is a system of flat coordinates for the invariant flat metric of this Frobenius manifold with \(\partial_x\) being the unity vector field, and we will also use the notation \(v = v^1, u = v^2\). The principal hierarchy associated to this Frobenius manifold is a hierarchy of commuting evolutionary partial differential equations (PDEs) given by

\[
\frac{\partial v^\alpha}{\partial t^\beta} = \sum_{\rho} \eta^\alpha\rho \partial_x \left( \frac{\partial \theta_{\beta,i+1}(v,u;q)}{\partial v^\rho} \right), \quad i \geq 0, \alpha, \beta = 1, 2,
\] (2.2)

where \(\eta^\alpha\rho = \delta_{\alpha+\rho,3},\) and \((\theta_{\alpha,k}(v^1,v^2;q))_{\alpha=1,2,k\geq0}\) are holomorphic functions that can be defined via the generating series \([11, 12]\]

\[
\theta_1(v,u;q;z) := \sum_{k \geq 0} \theta_{1,k}(v,u;q) z^k = -2e^{zu} \sum_{m \geq 0} \left( \gamma - \frac{1}{2} u + \psi(m+1) \right) q^m e^{mu} z^{2m} \frac{m!^2}{m!^2},
\] (2.3)

\[
\theta_2(v,u;q;z) := \sum_{k \geq 0} \theta_{2,k}(v,u;q) z^k = z^{-1} \left( \sum_{m \geq 0} q^m e^{mu+zv} \frac{z^{2m}}{m!^2} - 1 \right),
\] (2.4)

where \(\gamma\) is the Euler constant and \(\psi\) denotes the digamma function. The reader who is familiar with the theory of Frobenius manifolds recognizes that the above \((\theta_1(v,u;z), \theta_2(u,v;z))\) give a system of the deformed flat coordinates for the Dubrovin connection of the Frobenius manifold under consideration. It is easy to observe that the \(\partial / \partial t^{1,0}\) flow coincides with \(\partial_x\); therefore, we identify \(t^{1,0}\) with \(x\). We also remind the reader that the integrability of the principal hierarchy (2.2) is guaranteed by the Frobenius manifold structure. Since (2.2) is integrable, one can solve equations in (2.2) together, yielding solutions of the form \((v = v(t; q), u = u(t; q))\). Following [5], define the genus zero two-point correlation functions \(\Omega^{[0]}_{\alpha,i;\beta,j}(v, u)\) by means of the generating series as follows:

\[
\sum_{i,j \geq 0} \Omega^{[0]}_{\alpha,i;\beta,j}(v,u;q) z^i y^j = \frac{1}{z + y} \left( \sum_{\rho,\sigma} \frac{\partial \theta_{\alpha}(v,u;q;z)}{\partial v^\rho} \frac{\partial \theta_{\beta}(v,u;q;y)}{\partial v^\sigma} - \eta_{\alpha\beta} \right), \quad \alpha, \beta = 1, 2.
\] (2.5)
Solutions \((v = v(t; q), u = u(t; q))\) to the principal hierarchy (2.2) are characterized by their initial values

\[
(v(t; q), u(t; q))|_{\alpha,j=x^{\alpha-1,\delta,0}, \alpha=1,2, i \geq 0}.
\] (2.6)

The topological solution \((v_{\text{top}}(t; q), u_{\text{top}}(t; q))\) to the principal hierarchy is defined as the unique solution subjected to the initial value

\[
(x, 0).
\] (2.7)

It is straightforward to show that this solution satisfies

\[
v_{\text{top}}(t; q)|_{t \alpha, i = 0, \alpha = 1, 2, i > 0} = t^{1,0}, \quad u_{\text{top}}(t; q)|_{t \alpha, i = 0, \alpha = 1, 2, i > 0} = t^{2,0}.
\] (2.8)

According to [5], the topological solution \((v_{\text{top}}(t; q), u_{\text{top}}(t; q))\) can alternatively be determined by the following genus zero Euler–Lagrange equation:

\[
\sum_{\alpha} \sum_{i} \tilde{t}_{\alpha, i} \frac{\partial}{\partial v_{\beta}} (v_{\text{top}}(t; q), u_{\text{top}}(t; q); q) = 0, \quad \beta = 1, 2,
\] (2.9)

where \(\tilde{t}_{\alpha, i} := t_{\alpha, i} - \delta_{\alpha, 1} \delta_{i, 1}\).

Recall [5] (cf. [10, 12]) that the genus zero free energy of GW invariants of \(\mathbb{P}^1\) has the expression

\[
\mathcal{P}_{\text{top}}(t; q) = \frac{1}{2} \sum_{\alpha, \beta} \sum_{i, j} \tilde{t}_{\alpha, i} \tilde{t}_{\beta, j} \Omega_{0}^{\alpha, i; \beta, j}(v_{\text{top}}(t; q), u_{\text{top}}(t; q); q),
\] (2.10)

which satisfies the following genus zero Virasoro constraints [10] (cf. [14, 22, 26]):

\[
e^{-\epsilon^{-2} \mathcal{P}_{\text{top}}(t; q)} L_m \left( e^{-\epsilon^{-2} \mathcal{P}_{\text{top}}(t; q)} \right) = O(1) \quad (\epsilon \to 0), \quad m \geq -1,
\] (2.11)

where \(L_m, m \geq -1,\) are the linear operators given by

\[
L_{-1} = \sum_{k \geq 1} \sum_{\alpha} \tilde{t}_{\alpha, k} \frac{\partial}{\partial t_{\alpha, k-1}} + \frac{t^{1,0} t^{2,0}}{\epsilon^2},
\] (2.12)

\[
L_m = \epsilon^2 \sum_{k=1}^{m-1} k!(m-k)! \frac{\partial^2}{\partial t^{2,k-1} \partial t^{2,m-k-1}} + \sum_{k \geq 1} (m+k)! \left( \frac{t^{1,k}}{(k-1)!} \frac{\partial}{\partial t^{1,m+k}} + \frac{t^{2,k-1}}{(k-1)!} \frac{\partial}{\partial t^{2,m+k-1}} \right)
\]

\[
+2 \sum_{k \geq 0} A_m(k) \frac{t^{1,k}}{(m+k)!} \frac{\partial}{\partial t^{2,m+k-1}} + \delta_{m,0} \frac{t^{1,0}}{\epsilon^2}, \quad m \geq 0
\] (2.13)

with

\[
A_m(0) = m!, \quad A_m(k) = \frac{(m+k)!}{(k-1)!} \sum_{j=k}^{m+k} \frac{1}{j} (k > 0).
\] (2.14)

These operators \(L_m, m \geq -1,\) satisfy the Virasoro commutation relations:

\[
[L_{m_1}, L_{m_2}] = (m_1 - m_2) L_{m_1 + m_2}, \quad m_1, m_2 \geq -1.
\] (2.15)
Let us proceed with higher genera. According to [4, 11, 12, 15, 19, 22], the higher genus free energies $F_g^{p_1}(t; q)$, $g \geq 1$, have the $(3g - 2)$ property, namely, there exist functions
\[ F_g = F_g(v, u, v_1, u_1, \ldots, v_{3g-2}, u_{3g-2}; q), \quad g \geq 1, \]
(2.16)
such that, for $g \geq 1$,
\[
F_g^{p_1}(t; q) = F_g\left(v_{\text{top}}(t; q), u_{\text{top}}(t; q), \frac{\partial v_{\text{top}}(t; q)}{\partial x}, \frac{\partial u_{\text{top}}(t; q)}{\partial x}, \ldots, \frac{\partial^{3g-2} v_{\text{top}}(t; q)}{\partial x^{3g-2}}, \frac{\partial^{3g-2} u_{\text{top}}(t; q)}{\partial x^{3g-2}}; q\right).
\]
(2.17)
Moreover, the partition function of the GW invariants of $p_1$ satisfies the Virasoro constraints [22, 27] (cf. [11–13]):
\[ L_m(Z^{p_1}(t; \varepsilon, q)) = 0, \quad m \geq 0. \]
(2.18)
Here $L_m$, $m \geq -1$, are the above-defined operators.

Denote
\[
\Delta F = \Delta F(v, u, v_1, u_1, v_2, u_2, \ldots; \varepsilon, q) := \sum_{g \geq 1} \varepsilon^{2g-2} F_g(v, u, v_1, u_1, \ldots, v_{3g-2}, u_{3g-2}; q).
\]
(2.19)
Based on formulae (2.10), (2.11), the $(3g - 2)$ jet representation (2.17), and the above Virasoro constraints (2.18), Dubrovin and Zhang derive [11, 12] the loop equation for GW invariants of $p_1$, given in the following theorem. (The derivation is also revisited and slightly simplified in [31].)

**Theorem B** (Dubrovin–Zhang [11, 12]). *The function $\Delta F$ satisfies the following loop equation:*
\[
\sum_{r \geq 0} \left( \frac{\partial \Delta F}{\partial v_r} \left( \frac{v - \lambda}{D} \right)_r - 2 \frac{\partial \Delta F}{\partial u_r} \left( \frac{1}{D} \right)_r \right)
+ \sum_{r \geq 1} \sum_{k=1}^r \binom{r}{k} \left( \frac{1}{\sqrt{D}} \right)^{r-k} \left( \frac{\partial \Delta F}{\partial v_r} \left( \frac{v - \lambda}{\sqrt{D}} \right)_{r-k+1} - 2 \frac{\partial \Delta F}{\partial u_r} \left( \frac{1}{\sqrt{D}} \right)_{r-k+1} \right)
+ \frac{q e^{\varepsilon t}}{(\lambda - v)^2 - 4q e^{\varepsilon t})^2}
+ \varepsilon^2 \sum_{k, l \geq 0} \left( \frac{1}{4} S(\Delta F, v_k, v_l) \left( \frac{\lambda - v}{\sqrt{D}} \right)_{k+1} \left( \frac{1}{\sqrt{D}} \right)_{l+1}
+ S(\Delta F, v_k, u_l) \left( \frac{1}{\sqrt{D}} \right)_{k+1} \left( \frac{1}{\sqrt{D}} \right)_{l+1}
+ S(\Delta F, u_k, v_l) \left( \frac{1}{\sqrt{D}} \right)_{k+1} \left( \frac{1}{\sqrt{D}} \right)_{l+1}
\right)
+ \frac{\partial \Delta F}{\partial v_k} \left( q e^{t} 4q e^{\varepsilon t} (v - \lambda) v_1 - ((v - \lambda)^2 + 4q e^{\varepsilon t}) v_1 \right)_{k+1}
+ \frac{\partial \Delta F}{\partial u_k} \left( q e^{t} 4(v - \lambda) v_1 - ((v - \lambda)^2 + 4q e^{\varepsilon t}) u_1 \right)_{k+1} = 0,
\]
(2.20)
where $D = (v - \lambda)^2 - 4qeu$, $S(f, a, b) := \frac{\partial^2 f}{\partial a \partial b} + \frac{\partial f}{\partial a} \frac{\partial f}{\partial b}$, and $f_r$ stands for $\partial^r(f)$ with

$$\partial := \sum_{\alpha} \sum_{m} v^\alpha_{m+1} \frac{\partial}{\partial v^\alpha_m}. \quad (2.21)$$

Moreover, the solution to the above loop equation (2.20) is unique up to a sequence of additive elements $c_1, c_2, \ldots \in \mathbb{C}[[q]]$ that can be determined by the value

$$F_1 = \frac{1}{24} \log(v^2_e - 4qu^2) - \frac{1}{24} u \quad (2.22)$$

and the dilation equation

$$\sum_{\alpha} \sum_{m} m v^\alpha_m \frac{\partial F_g}{\partial v^\alpha_m} = (2g - 2)F_g \quad (g \geq 2). \quad (2.23)$$

We note that the identity (2.20) should be understood to be valid identical in $\lambda$ for $\lambda$ large. In particular, the branch of $\sqrt{D}$ is taken so that $\sqrt{D} \sim \lambda$ for $\lambda$ large.

3 | PROOF OF THEOREM 1

In this section, we give a proof of Theorem 1 by applying the loop equation (2.20).

Proof of Theorem 1. To simplify the notations, let us denote $P_i = t^{1,i}, Q_i = t^{2,i}, i \geq 0$, and

$$P = (P_0, P_1, \ldots), \quad Q = (Q_0, Q_1, \ldots).$$

By (1.19) and (1.23), we know that for $g \geq 1$,

$$(-1)^g F_{g, \text{deg}=0}(t; \varepsilon) = \frac{2^{2g-1} - 1}{2^{2g-1}} \frac{|B_{2g}|}{(2g)!} \left(1, V_{2g-2}(t)\right) - 2H_g(\lambda_{g-1}; P). \quad (3.1)$$

Here, we recall that $t = (t^{\alpha,i})_{\alpha=1,2, i \geq 0}$. We also note that $t^i = P_i 1 + Q_i [pt], i \geq 0$, are the cohomology-valued times for $p^1$. Define a power series

$$U(P, Q) = \sum_{i \geq 0} Q_i \frac{\partial V(P)}{\partial P_i}, \quad (3.2)$$

and put

$$U_m(P, Q) = \frac{\partial^m U(P, Q)}{\partial P^m_0}, \quad m \geq 0.$$

By using Proposition 1, we can write (3.1) in terms of the two power series $V(P)$ and $U(P, Q)$ as follows:

$$(-1)^g F_{g, \text{deg}=0}(t; \varepsilon) = \frac{2^{2g-1} - 1}{2^{2g-1}} \frac{|B_{2g}|}{(2g)!} U_{2g-2}(P, Q) - 2W_g(V_1(P), \ldots, V_{2g-1}(P)), \quad g \geq 1. \quad (3.3)$$
(Observe that the power series \((V(P), U(P, Q))\) is actually the degree zero part of the topological solution to the principal hierarchy \((2.2)\) of the \(\mathbb{P}^1\)-Frobenius manifold.)

Taking the degree zero part (coefficient of \(q^0\)) in the Dubrovin–Zhang loop equation \((2.20)\), we obtain

\[
\sum_{r=1}^{2g-1} \frac{\partial W_g(V_1, \ldots, V_{2g-2})}{\partial V_r} \left( \frac{1}{\lambda - V} \right)_r = B_g(\lambda; V). 
\] (3.4)

Comparing the coefficients of \((\lambda - V)^{-2}, \ldots, (\lambda - V)^{-2g}\), we get

\[
M \cdot \left( \frac{\partial W_g}{\partial V_1}, \ldots, \frac{\partial W_g}{\partial V_{2g-1}} \right)^T = C_g,
\] (3.5)

where \(M\) is an upper-triangular and nondegenerate matrix and \(C_g = (B_{g,1}, \ldots, B_{g,2g-1})^T\) with \(B_{g,j} = \text{Coef}((\lambda - V)^{-j-1}, B_g(\lambda; V))\) as in Section 1. Since \(M\) is upper-triangular, it is not difficult to calculate the explicit expression of its inverse, and we find

\[
(M^{-1})_{ij} = \sum_{\mu \in P_{j-i}} \frac{1}{j!} \binom{\ell(\mu) + j - 1}{i - 1} L(\mu) \frac{V_{\mu+1}}{V_1} \frac{1}{V^{\ell(\mu)+j}}.
\] (3.6)

The theorem is then proved by noticing from \((1.26)\) that \(\sum_{m=2}^{2g-1} (m - 1) V_m \frac{\partial W_g}{\partial V_m} = (2g - 2) W_g. \) □

We note that it is also clear from \((1.26)\) that \((1.31)\) can be equivalently written as

\[
W_g(V_1, \ldots, V_{2g-2}) = \frac{1}{2g - 2} \sum_{k=1}^{2g-1} k V_k \sum_{j=1}^{2g-1} c_{k,j} B_{g,j}. 
\] (3.7)

Remark 1. In our proof, we used the degree zero part of the loop equation (or say of Virasoro constraints) together with \((1.22)\). We note that it is also possible to use the Virasoro constraints themselves as the uniqueness in Theorem B suggests (of course, existence of jet representation is also needed here), and the point is that more constraints on the degree zero invariants come from the degree one part of the Virasoro constraints.

\section*{4 APPLICATIONS}

In this section, we give some applications of Proposition 1 and Theorem 1.

Let \(X\) be a smooth projective curve of genus \(h\). The notations will be the same as in Section 1. For a partition \(\lambda = (\lambda_1, \ldots, \lambda_n)\) with length \(n\), denote

\[
C^X_{\lambda_1, \ldots, \lambda_n}(q) := \sum_{d, g \geq 0} q^d \int_{[X]_{g,n,d}}^{\text{vir}} c_1(\mathcal{L}_1)^{\lambda_1} e_1^*(\text{pt}) \cdots c_1(\mathcal{L}_n)^{\lambda_n} e_n^*(\text{pt}). 
\] (4.1)

We note that the integrals appearing in the above generating function are often called GW invariants in the stationary sector.
Consider the case that $X = E$ is an elliptic curve. The following closed formula for the genus $g$, $g \geq 1$, free energy (in all degrees) $\mathcal{F}_g^E(t; q)$ for the elliptic curve $E$ is obtained by Buryak [1]:

$$\mathcal{F}_g^E(t; q) = \sum_{\lambda \in \mathcal{P}_{2g-2}} \frac{U_\lambda(t)}{\prod_{j \geq 1} m_j(\lambda)!} C_\lambda^E(qe^{U(t)}) - \frac{U(t)}{24} \delta_{g,1},$$

(4.2)

where

$$U(t) := \frac{\partial^2 \mathcal{F}_0^E(t)}{\partial t_1,0 \partial t_1,0}.$$  

(4.3)

Note that we have omitted the argument $q$ in $\mathcal{F}_g^E(t; q)$ because it actually does not depend on $q$, and that $C_\lambda^E(q)$ vanishes if $|\lambda|$ is odd. Explicitly, $U(t)$ has the expression:

$$U(t) = \sum_i Q_i \frac{\partial V(P)}{\partial P_i} + \sum_{i,j} t_{2,i} t_{3,j} \frac{\partial^2 V(P)}{\partial P_i \partial P_j},$$

(4.4)

which can be obtained from (1.9). Here $Q_i = t^{4,i}$, $P_i = t^{1,i}$, and $V(P)$ is the power series defined by (1.13) as before.

By taking the degree zero limit in Buryak’s formula (4.2), we obtain that

$$\mathcal{F}_g^E, \text{deg}=0(t) = \sum_{\lambda \in \mathcal{P}_{2g-2}} \frac{U_\lambda(t)}{\prod_{j \geq 1} m_j(\lambda)!} C_\lambda^E(0) - \frac{U(t)}{24} \delta_{g,1}, \quad g \geq 1.$$  

(4.5)

We have the following corollary.

**Corollary 1.** For $g \geq 1$ and for a partition $\lambda \in \mathcal{P}_{2g-2}$,

$$C_\lambda^E(0) = \begin{cases} 
(-1)^g \frac{2^{2g-1} - 1}{2^{2g-1} (2g)!} \frac{|B_{2g}|}{(2g)!} + \frac{1}{24} \delta_{g,1}, & \lambda = (2g - 2), \\
0, & \text{otherwise}. 
\end{cases}$$

(4.6)

**Proof.** From Proposition 1 and formula (4.4), we know that

$$\mathcal{F}_g^E, \text{deg}=0(t) = (-1)^g \frac{2^{2g-1} - 1}{2^{2g-1} (2g)!} \frac{|B_{2g}|}{(2g)!} U_{2g-2}(t).$$

(4.7)

The corollary is then proved by comparing this expression with (4.5).  

□

**Example 1.** Denote

$$\sigma_s(d) = \sum_{a \mid d} a^s, \quad s \geq 0,$$

$$E_k(q) = \frac{\zeta(1-k)}{2} + \sum_{d \geq 1} \sigma_{k-1}(d) q^d, \quad k = 2, 4, 6.$$  

(4.8)

(4.9)

According to [28] (cf. [1, 3, 23]),

$$C_{(0)}^E(q) = E_2(q) - \frac{\zeta(-1)}{2}, \quad C_{(2)}^E(q) = \frac{E_4(q)}{12} + \frac{E_2(q)^2}{2},$$

$$C_{(1,1)}^E(q) = \frac{7E_6(q)}{180} + \frac{2}{3} E_2(q) E_4(q) - \frac{8}{3} E_2(q)^3.$$  

(4.10)

(4.11)

The constant values of these functions agree with (4.6).
Actually, by taking the stationary sector part (i.e., by taking all $t = 0$ except for $t^{2h+2,i}$) of formula (1.28) in Proposition 1, we find that formula (4.6) with $E$ replaced by $X$ is true for an arbitrary smooth curve $X$. This general formula was obtained in [28] from the GW/Hurwitz correspondence (see pages 529–530 therein). In particular, formula (4.6) is covered by the GW/Hurwitz correspondence, while we used the $\lambda_g$ conjecture.

It is clear that, by our formulation, on the contrary, the formula of $C_E\lambda(0)$ and Buryak’s formula (4.2) lead to a proof of the $\lambda_g$ conjecture.

Let us proceed to consider the $\lambda_{g-1}$ integrals and give a new proof of Theorem A.

**Proof of Theorem A.** From the definition (1.30), we know that $B_g(\lambda; V)$ is a rational function of $\lambda$, which has a $(2g)$th order pole at $\lambda = V$. More precisely, we deduce from (1.30) that

$$B_g(\lambda; V) = \frac{2^{2g-1} - 1}{2^{2g-1}} \frac{|B_{2g}|}{(2g)!} \frac{V_1^{2g-2}}{(\lambda - V)^{2g}}$$

$$+ \frac{2^{2g-1} - 1}{2^{2g-1}} \frac{|B_{2g}|}{(2g)!} \sum_{k=1}^{2g-2} \frac{V_1^{2k-2}}{(\lambda - V)^{2k}}$$

$$- \frac{1}{2} \sum_{g_1 + g_2 = g} \frac{2^{2g_1 - 1} - 1}{2^{2g_1 - 1}} \frac{|B_{2g_1}|}{(2g_1)!} \frac{2^{2g_2 - 1} - 1}{2^{2g_2 - 1}} \frac{|B_{2g_2}|}{(2g_2)!} (2g_1 - 1)(2g_2 - 1) \frac{V_1^{2g-2}}{(\lambda - V)^{2g}}$$

Here, $\sim$ means keeping only the most singular terms. Therefore, using (1.31), we find

$$\text{Coef}(V_{2g-1}, W_g(V_1, ..., V_{2g-1})) = \left( \frac{2^{2g-1} - 1}{2^{2g-1}} \frac{|B_{2g}|}{(2g)!} \sum_{k=1}^{2g-1} \frac{V_1^{2g-2}}{(\lambda - V)^{2g}} \right) 1.$$

By noticing that $V_k(0, ..., 0, T_{2g-1}, 0, ...) = T_{2g-1} \delta_{k, 2g-1} + \delta_{k, 1} (g \geq 2, 1 \leq k \leq 2g - 1)$, the theorem is proved.

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