Berezin symbolic calculus for the \( n \)-sphere, \( n = 2, 3, 5 \)

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We consider the bounded linear operators with domain in the Hilbert space \( L^2(S^n) \), \( n = 2, 3, 5 \) and describe its symbolic calculus defined by the Berezin quantization. In particular, we derive an explicit formula for the composition of Berezin’s symbols and thus a noncommutative invariant star product.

I. INTRODUCTION AND SUMMARY

We start by recalling some results from Berezin’s theory that will be used below. See Ref. [4] for details. Let \( H \) be a Hilbert space endowed with the inner product \( \langle \cdot, \cdot \rangle \) and \( M \) be some set with the measure \( d\mu \). Let \( \{ K(\cdot, \alpha) \in H \mid \alpha \in M \} \) be a family of functions in \( H \) labelled by elements of \( M \), such that it satisfies the following properties:

i) Let \( \hat{f}(\alpha) = (f, K(\cdot, \alpha)) \), the map \( f \to \hat{f} \) is an embedding from \( H \) into \( L^2(M, d\mu) \).

ii) The family \( \{ K(\cdot, \alpha) \} \) is complete: that is, for any \( f, g \in H \), Parseval’s identity is valid

\[
(f, g) = \int_M (f, K(\cdot, \alpha))(K(\cdot, \alpha), g) d\mu(\alpha).
\]

The prototype of the space \( H \) is a reproducing kernel Hilbert space of complex-valued holomorphic functions on a complex domain \( M \), that is, for each \( \alpha \in M \), there exists an element \( K(\cdot, \alpha) \) of \( H \), such that \( (f, K(\cdot, \alpha)) = f(\alpha) \) for each \( f \in H \).

In this case, one can directly define the Berezin symbol of a bounded linear operator \( A \) on \( H \) as the complex-valued function \( \mathcal{B}(A) \) on \( M \) given by

\[
\mathcal{B}(A)(\alpha) = \frac{(AK(\cdot, \alpha), K(\cdot, \alpha))}{(K(\cdot, \alpha), K(\cdot, \alpha))}, \quad \alpha \in M,
\]

and thus a symbolic calculus on \( \mathcal{B}(H) \), the algebra of all bounded linear operators on \( H \) [4].

The Berezin map \( \mathcal{B} : A \to \mathcal{B}(A) \) has some nice properties: is a linear operator, the unit operator corresponds to the unit constant, Hermitian conjugation of operators corresponds to complex conjugation of symbols. Moreover, if we assume that the Berezin symbol may be extended in a neighbourhood of the diagonal \( M \times M \) to the function

\[
\mathcal{B}(A)(\alpha, \beta) = \frac{(AK(\cdot, \alpha), K(\cdot, \beta))}{(K(\cdot, \alpha), K(\cdot, \beta))},
\]

then we have the following formulas

\[
\hat{A} f(\alpha) = \int_M \hat{f}(\beta) \mathcal{B}(A)(\beta, \alpha)(K(\cdot, \beta), K(\cdot, \alpha)) d\mu(\beta), \quad f \in H. \tag{1}
\]

\[
\mathcal{B}(AB)(\alpha, \beta) = \int_M \mathcal{B}(A)(\gamma, \beta) \mathcal{B}(B)(\alpha, \gamma) \frac{(K(\cdot, \alpha), K(\cdot, \gamma))(K(\cdot, \gamma), K(\cdot, \beta))}{(K(\cdot, \alpha), K(\cdot, \beta))} d\mu(\gamma). \tag{2}
\]

A useful application of this symbolic calculus is that it allows us to build a star product. In [4] Berezin applied this method to Kähler manifold. In this case \( H \) is the Hilbert space of analytic functions in \( L^2(M, d\mu) \) so that the embedding from \( H \) into \( L^2(M, d\mu) \) is the inclusion, and the complete family \( \{ K(\cdot, \alpha) \} \) is obtained by freezing one variable in the reproducing kernel.

The main goal of the present paper is to introduce a symbolic calculus for the Hilbert space \( L^2(S^n) \), \( n = 2, 3, 5 \) of square integrable functions with respect to the normalized surface measure \( d\Omega \) on \( S^n \) endowed with the inner product

\[
\langle \phi, \psi \rangle_{S^n} = \int_{S^n} \phi(x) \overline{\psi(x)} d\Omega(x),
\]

and \( S^n = \{ x \in \mathbb{R}^{n+1} \mid |x| = 1 \} \) is the unit sphere immersed in \( \mathbb{R}^{n+1} \).
In order to introduce this symbolic calculus, we first need to find a family of functions in $L^2(S^n)$, $n = 2, 3, 5$ satisfying ii). Since it is not possible to define a reproducing kernel in $L^2(S^n)$, we can not use the same idea applied by Berezin to Kähler manifold. However, in general, the coherent states are a specific complete set of vectors in a Hilbert space satisfying a certain resolution of the identity condition, i.e. the coherent states satisfy ii).

In [14] Villegas-Blas introduced coherent states for $L^2(S^n)$, $n = 2, 3, 5$. In addition, he defined a unitary transform $B_{S^n}$ from $L^2(S^n)$, $n = 2, 3, 5$, to $F_m \subset B_{C^m}$, $m = 2, 4, 8$ respectively, where $B_{C^m}$ denotes the Bargmann space of all entire functions in $L^2(C^m, d\mu_m^h)$ for the Gaussian measure $d\mu_m^h = (\pi h)^{-m} e^{-|z|^2/h} dz d\overline{z}$, with $dz d\overline{z}$ denoting the Lebesgue measure on $C^m$ and $h$ the Planck’s constant. Moreover, the action of the Bargmann transform $B_{S^n}$ on a function $\psi$ in $L^2(S^n)$ is a function in $F_m$ whose evaluation in $z \in C^m$ is equal to the $L^2(S^n)$-inner product of $\psi$ with the coherent state labelled by $z$, i.e. the Bargmann transform $B_{S^n}$ satisfies i) (see section III for details on the notation and some general facts about the Bargmann transform and coherent states for $L^2(S^n)$, $n = 2, 3, 5$).

Starting from this and applying Berezin’s theory, in Sec. IV we describe the rules for symbolic calculus on $B(L^2(S^n))$, $n = 2, 3, 5$. In particular, we derive an explicit formula for the composition of Berezin’s symbols and thus a star product on the algebra which consists of Berezin symbols for bounded linear operators with domain in $L^2(S^n)$.

In Sec. IV we will prove that this noncommutative star product satisfies the usual requirement on the semiclassical limit; this result can be obtained by using Laplace’s method (see the Appendix for details of the way in which this method is used).

Finally, by the way in which Villegas-Blas introduced both the Bargmann transform and coherent states, we will prove in Sec. IV the invariance of our star product under the action of the group SU(2), SU(2) × SU(2) and SU(4) on $C^2$, $C^4$ and $C^8$ respectively.

It should be noted that, since $H = L^2(S^n)$ and $L^2(M, d\mu) = L^2(C^n, d\mu_m^h)$, in our construction there is no inclusion of $H$ into $L^2(M, d\mu)$. Furthermore, the functions of the complete family are not obtained by the reproducing kernel. This situation is thus slightly different Berezin’s situation.

Throughout the paper, we will use the following basic notation. For every $z, w \in C^k$, $z = (z_1, \ldots, z_k)$, $w = (w_1, \ldots, w_k)$, and for every multi-index $\ell = (\ell_1, \ldots, \ell_k) \in Z^k_+$ of length $k$, where $Z^k_+$ is the set of nonnegative integers, let

$$z \cdot w = \sum_{s=1}^k z_s \overline{w_s}, \quad |z| = \sqrt{z \cdot \overline{z}}, \quad |\ell| = \sum_{s=1}^k \ell_s, \quad \ell! = \prod_{s=1}^k \ell_s!, \quad z^\ell = \prod_{s=1}^k z_s^{\ell_s}.$$ 

Whenever convenient, we will abbreviate $\partial/\partial v_j, \partial/\partial \overline{v_j}$, etc., to $\partial_{v_j}, \partial_{\overline{v_j}}$, etc., respectively, and $\partial_{v_1} \partial_{v_2} \cdots \partial_{v_k}$ to $\partial_{v_1 v_2 \cdots v_k}$.

II. PRELIMINARIES

In this section, we review some results on the Bargmann transform and coherent states for $L^2(S^n)$, $n = 2, 3, 5$ introduced by Villegas-Blas in [14]. Here and in the sequel, the letters $n$ and $m$ will only denote integer numbers in the sets $\{2, 3, 5\}$ and $\{2, 4, 8\}$ respectively. Furthermore, whenever we write $(n, m)$ we mean the three possible cases $(n, m) = (2, 2), (3, 4), (5, 8)$ unless a particular value of $(n, m)$ is specified.

Both the Bargmann transform and coherent states for $L^2(S^n)$, $n = 2, 3, 5$ are based on the quantization of a canonical transformation which relates two different regularization of the $n$-dimensional Kepler problem, $n = 2, 3, 5$: what we call the harmonic oscillator and the Moser regularizations. The harmonic oscillator regularization is based on an extension of the Hopf fibration map. For the cases $n = 2$ and $n = 3$ this regularization was introduced by Levi-Civita [12] and by Kustaanheimo and Stiefel [10] respectively. As far as we know, the case $n = 5$ was first studied by Davtyan et al [5]. Moser [8] introduced a different type of regularization based on an extension of a stereographic projection in momentum space.

In [3] Kummer described the relationship between those two different ways to regularize the $n$-dimensional Kepler problem for $n = 2, 3$. Based on the work of Kummer, Villegas-Blas [14] regarded that relationship as a canonical transformation $C(n,m)$, which is defined by

$$C(n,m) = \sigma_n \circ \rho(n,m),$$

where the map $\sigma_n$ identifies the complex null quadric $Q^n := \{ \alpha \in \mathbb{C}^{n+1} | \alpha_1^2 + \cdots + \alpha_{n+1}^2 = 0 \}$ (with its origin
removed) with the cotangent bundle $T^*S^n$ (with its zero section removed)

$$\sigma_n : \mathbb{C}^n \setminus \{0\} \to T^*S^n \setminus \{0\},$$

$$\sigma_n(\alpha) = \left(\frac{\Re(\alpha)}{|\Im(\alpha)|}, -\Im(\alpha)\right),$$

and for $z \in \mathbb{C}^m$, the map $\rho_{(n,m)}(z) = (\rho_1(z), \ldots, \rho_{n+1}(z)) \in \mathbb{C}^n$ is defined by

Case $(n,m) = (2,2)$

$$\rho_1(z) = \frac{1}{2}(z^2_2 - z^2_1), \quad \rho_2(z) = \frac{1}{2}(z^2_1 + z^2_2), \quad \rho_3(z) = z_1z_2. \quad (3)$$

Case $(n,m) = (3,4)$

$$\rho_1(z) = z_1z_3 + z_2z_4, \quad \rho_2(z) = i(z_1z_3 - z_2z_4) \quad \rho_3(z) = i(z_1z_4 + z_2z_3), \quad \rho_4(z) = z_1z_4 - z_2z_3. \quad (4)$$

Case $(n,m) = (5,8)$

$$\rho_1(z) = i(-z_1z_6 + z_3z_8 + z_2z_5 - z_4z_7), \quad \rho_2(z) = z_1z_6 + z_3z_8 + z_2z_5 + z_4z_7,$$
$$\rho_3(z) = z_2z_6 + z_3z_7 - z_1z_5 - z_4z_8, \quad \rho_4(z) = i(-z_1z_5 + z_3z_8 - z_2z_6 + z_4z_7),$$
$$\rho_5(z) = i(-z_1z_8 - z_2z_7 - z_3z_6 - z_4z_5), \quad \rho_6(z) = z_1z_8 + z_2z_7 - z_3z_6 - z_4z_5. \quad (5)$$

Notice that the map $\rho_{(n,m)}$ is not one to one because it is invariant under the action of the group $G_m = \mathbb{Z}_2, S^1$ and $SU(2)$ on $\mathbb{C}^2, \mathbb{C}^4$ and $\mathbb{C}^8$, respectively, described by the following equation:

$$z' = T(g)z$$

with $T(g)$ given by

Case $m = 2$:

$$T(g) = \pm 1. \quad (6)$$

Case $m = 4$: For $\theta \in \mathbb{R}$ and therefore $g = \exp(i\theta) \in S^1$

$$T(g) = \begin{pmatrix}
\exp(-i\theta) & 0 & 0 & 0 \\
0 & \exp(-i\theta) & 0 & 0 \\
0 & 0 & \exp(i\theta) & 0 \\
0 & 0 & 0 & \exp(i\theta)
\end{pmatrix}. \quad (7)$$

Case $m = 8$: For $g \in SU(2)$

$$T(g) = L^\dagger V(g)L, \quad (8)$$

with

$$V(g) = \begin{pmatrix}
g & 0 & 0 & 0 \\
0 & g & 0 & 0 \\
0 & 0 & g & 0 \\
0 & 0 & 0 & g
\end{pmatrix}, \quad L = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}. \quad (9)$$

where all the entries in $V(g)$ are $2 \times 2$ matrices.

In fact, if we want the map $\rho_{(n,m)}$ to be a bijection, its domain must be $\mathbb{C}^m/G_m$, where

$$\mathbb{C}^2 = \mathbb{C}^2,$$
$$\mathbb{C}^4 = \left\{z \in \mathbb{C}^4 : |z_1|^2 + |z_2|^2 = |z_3|^2 + |z_4|^2\right\}, \quad (10)$$
$$\mathbb{C}^8 = \left\{z \in \mathbb{C}^8 : \sum_{j=1}^4 |z_j|^2 = \sum_{j=1}^4 |z_{j+4}|^2, z_7\overline{z}_1 + z_3\overline{z}_3 = z_8\overline{z}_2 + z_6\overline{z}_4\right\}. \quad (11)$$
Furthermore, Villegas-Blas [14] observed that the function

\[ \rho_{n,m}(\mathbf{z}) \cdot \mathbf{x}, \quad \mathbf{z} \in \mathbb{C}^m, \mathbf{x} \in S^n, \]

is a generating function of the canonical transformation \( \mathcal{C}_{(n,m)} \). From which, he defined a Bargmann type transform
and coherent states for \( L^2(S^n) \), \( n = 2, 3, 5 \).

### A. Bargmann Transform for \( L^2(S^n) \) with \( n = 2, 3, 5 \)

The Bargmann transform \( \mathcal{B}_{S^n} \) is defined as a integral transform whose integral kernel is a power series in the generating function of the canonical transformation mentioned above. The Bargmann transform \( \mathcal{B}_{S^n} \) has the following expression:

\[
\mathcal{B}_{S^n} \psi(\mathbf{z}) = \int_{S^n} \left( \sum_{\ell=0}^{\infty} \frac{c_{\ell}}{\ell!} \left( \frac{\rho_{n,m}(\mathbf{z}) \cdot \mathbf{x}}{h} \right)^\ell \right) \psi(\mathbf{x}) d\Omega(\mathbf{x}), \quad \mathbf{z} \in \mathbb{C}^m,
\]

where \( h \) denotes the Planck’s constant (regarded as a parameter) and the coefficients \( c_{\ell} \) of the power series can be computed by requiring that \( \mathcal{B}_{S^n} \) should be an isometry. The result is

\[
c_{\ell} = \frac{\sqrt{2\ell + n - 1}}{\ell! \sqrt{n - 1}}.
\]

Then one has to prove that \( \mathcal{B}_{S^n} \) is a unitary transformation onto its range \( \mathcal{F}_m \subset \mathcal{B}_{\mathbb{C}^m} \), where \( \mathcal{B}_{\mathbb{C}^m} \) denotes the Bargmann space of complex value holomorphic functions on \( \mathbf{z} \in \mathbb{C}^m \) which are square integrable with respect to the following measure on \( \mathbb{C}^m \):

\[
d\mu^h_{m}(\mathbf{z}) = \frac{1}{(\pi h)^m} e^{-|\mathbf{z}|^2/h} d\mathbf{z},
\]

with \( |\mathbf{z}|^2 = |z_1|^2 + \ldots + |z_m|^2 \) and \( d\mathbf{z} \) the Lebesgue measure in \( \mathbb{C}^m \simeq \mathbb{R}^{2m} \). In fact, the space \( \mathcal{F}_m \) is defined by

- **Case** \( m = 2 \): Let \( \mathcal{F}_2 \) be the closed subspace of \( \mathcal{B}_{\mathbb{C}^2} \) generated by the monomials with even degree.
- **Case** \( m = 4 \): Let \( \mathcal{F}_4 \subset \mathcal{B}_{\mathbb{C}^4} \) be the kernel of the following operator:

\[
\mathcal{L} = v_1 \partial_{\nu_1} + v_2 \partial_{\nu_2} - v_3 \partial_{\nu_3} - v_4 \partial_{\nu_4}.
\]

The domain of \( \mathcal{L} \) is defined as \( \{ f \in \mathcal{B}_{\mathbb{C}^4} \mid \mathcal{L} f \in \mathcal{B}_{\mathbb{C}^4} \} \).

- **Case** \( m = 8 \): Let \( \mathcal{F}_8 \subset \mathcal{B}_{\mathbb{C}^8} \) be the intersection of the kernel of the following three operators:

\[
\mathcal{R}_1 = v_1 \partial_{\nu_1} + v_2 \partial_{\nu_2} + v_3 \partial_{\nu_3} + v_4 \partial_{\nu_4} - v_5 \partial_{\nu_5} - v_6 \partial_{\nu_6} - v_7 \partial_{\nu_7} - v_8 \partial_{\nu_8},
\]

\[
\mathcal{R}_2 = v_7 \partial_{\nu_1} - v_8 \partial_{\nu_2} + v_5 \partial_{\nu_3} - v_6 \partial_{\nu_4} - v_3 \partial_{\nu_5} + v_4 \partial_{\nu_6} - v_1 \partial_{\nu_7} + v_2 \partial_{\nu_8},
\]

\[
\mathcal{R}_3 = v_8 \partial_{\nu_1} - v_5 \partial_{\nu_2} + v_7 \partial_{\nu_3} - v_6 \partial_{\nu_4} + v_4 \partial_{\nu_5} - v_2 \partial_{\nu_6} + v_1 \partial_{\nu_7} - v_3 \partial_{\nu_8}.
\]

The domains of \( \mathcal{R}_\ell, \ell = 1, 2, 3 \) are defined in a similar way as the domain of \( \mathcal{L} \).

Notice that, since the operators \( \mathcal{L}, \mathcal{R}_\ell, \ell = 1, 2, 3 \) are closed (see Sec. 3d of Ref. [2]), the spaces \( \mathcal{F}_4 \) and \( \mathcal{F}_8 \) are actually Hilbert spaces. Even more, for \( m = 2, 4, 8 \), the elements of the Hilbert spaces \( \mathcal{F}_m \) are given by the invariant functions in \( \mathcal{B}_{\mathbb{C}^m} \) under the action of the group \( \mathcal{G}_m = \mathbb{Z}_2, S^1, SU(2) \) on \( \mathbb{C}^2, \mathbb{C}^4, \mathbb{C}^8 \) respectively (see Eqs. (6), (7) and (8)).

Moreover, \( \mathcal{F}_m \) enjoys the property of having a reproducing kernel \( Q^{(h)}_m(\mathbf{z}, \mathbf{w}) = Q^{(h)}_m(\mathbf{z}, \mathbf{w}) \). Namely, for all \( f \in \mathcal{F}_m \) we have

\[
f(\mathbf{z}) = (f, Q^{(h)}_m(\cdot, \mathbf{z}))_{\mathcal{F}_m} = \int_{\mathbb{C}^m} f(\mathbf{w}) Q^{(h)}_m(\mathbf{z}, \mathbf{w}) d\mu^h_{m}(\mathbf{z}), \quad \forall \mathbf{z} \in \mathbb{C}^m,
\]

where the reproducing kernel is given by the following expressions

\[
Q^{(h)}_2(\mathbf{z}, \mathbf{w}) = \frac{1}{2} \left( e^{\left( \frac{2z_1 w_1 + z_2 w_2}{h} \right)} + e^{\left( \frac{-2z_1 w_1 - z_2 w_2}{h} \right)} \right),
\]

\[
Q^{(h)}_4(\mathbf{z}, \mathbf{w}) = \frac{1}{2\pi} \int_{0}^{2\pi} \exp \left( \frac{1}{h} \mathbf{z} \cdot \mathbf{T}(\theta) \mathbf{w} \right) d\theta,
\]

\[
Q^{(h)}_8(\mathbf{z}, \mathbf{w}) = \int_{0}^{\pi/2} \int_{0}^{2\pi} \int_{\gamma=0}^{2\pi} \exp \left( \frac{1}{h} \mathbf{z} \cdot \mathbf{T}(\theta, \alpha, \gamma) \mathbf{w} \right) d\theta d\alpha d\gamma.
\]
with $T(g)$ given by the action of $G_m = \mathbb{Z}_2, S^1, SU(2)$ on $\mathbb{C}^2, \mathbb{C}^4, \mathbb{C}^8$ indicated in Eqs. \([5], [7]\) and \([8]\). We are using the notation $g = g(\psi) = \exp(i\psi) \in S^1$ for the $m = 4$ case and $g = g(\theta, \alpha, \gamma) \in SU(2)$ for the $m = 8$ case where

$$g(\theta, \alpha, \gamma) = \begin{pmatrix}
\cos(\theta) \exp(i\alpha) & \sin(\theta) \exp(i\gamma) \\
-\sin(\theta) \exp(-i\gamma) & \cos(\theta) \exp(-i\alpha)
\end{pmatrix},$$

(18)

with $\theta \in [0, \pi/2]$ and $\alpha, \gamma \in [-\pi, \pi]$. The measure appearing in the integral expression for $Q_s(z, w)$ is the Haar measure of $SU(2)$ under the parametrization of $SU(2)$ indicated in Eq. \([13]\): $dm(\theta, \alpha, \gamma) = \frac{1}{2\pi} \sin(\theta) \cos(\theta) d\theta d\alpha d\gamma$.

**Remark II.1.** The case of the 2-sphere and $h = 1$ was studied by Thomas and Wassel \([13]\). Their idea was based on the fact that the group $SU(2)$ (the covering group of $SO(3)$) has a representation as a group of operators acting in the Bargmann space of two complex variables $B_2$. The space $B_2$ gives us all of the irreducible representations of $SU(2)$. By choosing only those irreducible representations of even dimension, they basically considering a change of basis and the corresponding integral kernel. Following this approach we end up showing that the expression for $B_{S^2}$ given in Eq. \([12]\) holds (a double sum in the expression of the integral kernel is equal to a single sum of powers of the function $\rho_{(2,2)}(z) \cdot x$).

Villegas-Blas \([14]\) studied the case of the 3-sphere and $h = 1$ defining a Bargmann type transform $B_{S^3}$ in the analogous way to the one introduced by Thomas and Wassell.

**B. Coherent states for $L^2(S^n)$, $n = 2, 3, 5$**

In \([16]\) Villegas-Blas introduced the coherent states as the complex conjugate of the integral kernel defining the Bargmann transform $B_{S^n}$ (see Eq. \([12]\)). For $z \in \mathbb{C}^n - \{0\}$, let us define

$$\Phi_{\rho(n,m)}^h(x) = \sum_{\ell=0}^{\infty} c_{\ell} \left( \frac{x \cdot \rho(n,m)}{\hbar} \right)^{\ell}, \quad x \in S^n.$$  

(19)

Note that $\Phi_{\rho(n,m)}^h(x)$ is in $L^2(S^n)$ because it is a bounded function. Moreover, the action of the Bargmann transform $B_{S^n}$ on a function $\Psi$ in $L^2(S^n)$ is a function in $F_m$ whose evaluation in $z \in \mathbb{C}^n$ is equal to the $L^2(S^n)$-inner product of $\Psi$ with the coherent state labeled by $\rho(n,m)(z)$ (see Eq. \([12]\)). This is

$$B_{S^n} \Psi(z) = \langle \Psi, \Phi_{\rho(n,m)}^h(z) \rangle_{S^n}.$$  

(20)

The Bargmann transform $B_{S^n}$ of a coherent state $\Phi_{\rho(n,m)}^h(w)$ is given by the following expressions

$$B_{S^n} \Phi_{\rho(2,2)}^h(w)(z) = \sum_{k=0}^{\infty} \frac{1}{(2k)!} \frac{1}{h^{2k}} (z_1 \bar{w}_1 + z_2 \bar{w}_2)^{2k},$$

$$B_{S^n} \Phi_{\rho(3,4)}^h(w)(z) = \sum_{k=0}^{\infty} \frac{1}{(k)!} \frac{1}{h^{2k}} (z_1 \bar{w}_1 + z_2 \bar{w}_2)^{k} (z_3 \bar{w}_3 + z_4 \bar{w}_4)^{k},$$

$$B_{S^n} \Phi_{\rho(5,8)}^h(w)(z) = \sum_{k=0}^{\infty} \frac{1}{k(k+1)!} \frac{1}{h^{2k}} (\varrho(z, w))^{k},$$

(21)

with

$$\varrho(u, v) = [u_1 \bar{v}_1 + u_2 \bar{v}_2 + u_3 \bar{v}_3 + u_4 \bar{v}_4] [u_5 \bar{v}_5 + u_6 \bar{v}_6 + u_7 \bar{v}_7 + u_8 \bar{v}_8] + [u_7 \bar{v}_7 - u_8 \bar{v}_8 + u_5 \bar{v}_5 - u_6 \bar{v}_6] [u_4 \bar{v}_4 - u_3 \bar{v}_3 + u_2 \bar{v}_2 - u_1 \bar{v}_1].$$

(22)

It can be shown that the Bargmann transform $B_{S^n}$ of a coherent state is equal to the reproducing kernel of the space $F_m$, i.e

$$B_{S^n} \Phi_{\rho(n,m)}^h(w)(z) = Q_m^h(z, w), \quad z, w \in \mathbb{C}^n.$$  

(23)

Using this equality, we can express the reproducing kernel $Q_m^h(z, w)$ in terms of the modified Bessel functions of the first kind of order $v$, $I_v$ (see Secs. 8.4 and 8.5 of Ref. \([3]\) for definition and expressions for this special function) as the following proposition establishes it:
Proposition II.2. For \( z, w \in \mathbb{C}^m - \{0\} \):

\[
Q_m^{(h)}(z, w) = \Gamma \left( \frac{n-1}{2} \right) \left( \frac{\alpha \cdot \beta}{2h^2} \right)^{\frac{3-n}{2}} I_{\frac{n-3}{2}} \left( \frac{\sqrt{2\alpha \cdot \beta}}{h} \right),
\]

with \( \alpha = \rho_{(n,m)}(z) \) and \( \beta = \rho_{(n,m)}(w) \). We are taking the branch of the square root function defined by \( \sqrt{z} = |z|^{1/2} \exp(i \theta/2) \), where \( \theta = \text{Arg}(z) \) and \( -\pi < \theta < \pi \).

**Proof.** Using the explicit expression definition of \( \rho_{(n,m)} \) (see Eqs. (3), (4) and (13)) we obtain the following relations

\[
\begin{align*}
\rho_{(2,2)}(z) \cdot \rho_{(2,2)}(w) &= \frac{1}{2} (z \cdot w)^2, \\
\rho_{(3,4)}(z) \cdot \rho_{(3,4)}(w) &= 2(z_1 \overline{w}_1 + z_2 \overline{w}_2)(z_3 \overline{w}_3 + z_4 \overline{w}_4), \\
\rho_{(5,8)}(z) \cdot \rho_{(5,8)}(w) &= 2 \rho(z, w),
\end{align*}
\]

with \( \rho(z, w) \) defined in Eq. (22). From Eqs. (24), (26) and using the following expression by \( I_\nu \) (see formula 9.6.10 of Ref. [1])

\[
I_\nu(\omega) = \left( \frac{\omega}{2} \right)^\nu \sum_{\ell=0}^\infty \frac{\left( \frac{\omega^2}{4} \right)^\ell}{\ell! \Gamma(\nu + \ell + 1)}
\]

we obtain, for \( \alpha = \rho_{(n,m)}(z) \) and \( \beta = \rho_{(n,m)}(w) \),

\[
B_{S^n} \Phi_{\rho_{(n,m)}}^{(h)}(w; z) = \Gamma \left( \frac{n-1}{2} \right) \left( \frac{\alpha \cdot \beta}{2h^2} \right)^{\frac{3-n}{2}} I_{\frac{n-3}{2}} \left( \frac{\sqrt{2\alpha \cdot \beta}}{h} \right).
\]

From Eq. (23) we conclude the proof of Proposition II.2. \( \square \)

Let \( \alpha = \rho_{(n,m)}(z) \) and \( \beta = \rho_{(n,m)}(w) \), with \( z, w \in \mathbb{C}^m - \{0\} \), since the Bargmann transform \( B_{S^n} \) is a unitary transformation, we obtain from Eq. (23), the reproducing property of \( Q_m^{(h)} \) (see Eq. (10)) and the expression for the reproducing kernel \( Q_m^{(h)} \) given in Eq. (24)

\[
\left\langle \Phi_\beta^{(h)}, \Phi_\alpha^{(h)} \right\rangle_{S^n} = Q_m^{(h)}(z, w)
\]

\[
= \Gamma \left( \frac{n-1}{2} \right) \left( \frac{\alpha \cdot \beta}{2h^2} \right)^{\frac{3-n}{2}} I_{\frac{n-3}{2}} \left( \frac{\sqrt{2\alpha \cdot \beta}}{h} \right).
\]

From Eq. (27) and the fact that the modified Bessel function \( I_\vartheta, \vartheta \in \mathbb{R} \), has the following asymptotic expression when \( |\omega| \to \infty \) (see formula 8.451-5 of Ref. [4])

\[
I_\vartheta(\omega) = \frac{e^{\omega}}{\sqrt{2\pi\omega}} \sum_{k=0}^\infty \frac{(-1)^k}{(2\omega)^k} \frac{\Gamma(\vartheta + k + \frac{1}{2})}{k! \Gamma(\vartheta - k + \frac{1}{2})}, \quad |\text{Arg}(\omega)| < \frac{\pi}{2},
\]

we can obtain the asymptotic expansion for the inner product of two coherent states, as the following proposition establishes it:

Proposition II.3. Let \( z, w \in \mathbb{C}^m - \{0\} \) and \( \alpha = \rho_{(n,m)}(z), \beta = \rho_{(n,m)}(w) \). Assume \( \alpha \cdot \beta \neq 0 \) and \( |\text{Arg}(\alpha \cdot \beta)| < \pi \). Then

\[
\left\langle \Phi_\beta^{(h)}, \Phi_\alpha^{(h)} \right\rangle_{S^n} = \Gamma \left( \frac{n-1}{2} \right) \left( \frac{h^2}{2\alpha \cdot \beta} \right)^{\frac{n-2}{2}} \frac{2^{n-4}}{\sqrt{\pi}} \exp \left( \frac{\sqrt{2\alpha \cdot \beta}}{h} \right) [1 + O(h)].
\]

We end this section by showing that the family of coherent states is complete

Proposition II.4. The family of coherent states forms a complete system in \( L^2(S^n) \).

**Proof.** Let \( \phi, \psi \in L^2(S^n) \), since the Bargmann transform \( B_{S^n} \) is unitary and Eq. (20)

\[
\left\langle \phi, \psi \right\rangle_{S^n} = \left\langle B_{S^n} \phi, B_{S^n} \psi \right\rangle_{S^n} = \int_{\mathbb{C}^m} \left\langle \phi, \Phi_{\rho_{(n,m)}}^{(h)}(z) \right\rangle_{S^n} \left\langle \Phi_{\rho_{(n,m)}}^{(h)}(z), \psi \right\rangle_{S^n} d\mu^h_m(z).
\]

\( \square \)
III. BEREZIN SYMBOLIC CALCULUS

According to Berezin’s theory (see Ref. [4]), since the Bargmann transform is a unitary operator, Eq. (20) and Proposition III.2, we may consider the following

**Definition III.1.** The Berezin symbol of a bounded linear operator $A$ with domain in $L^2(S^n)$ is defined, for every $z \in \mathbb{C}^m$, by

$$\mathfrak{B}_{(n,m)}^{(h)}(A)(z) = \langle A\Phi_{\rho_{(n,m)}(z)}^{(h)}, \Phi_{\rho_{(n,m)}(z)}^{(h)} \rangle_{S^n}. \quad (29)$$

From Eq. (27) we have

$$||\Phi_{\rho_{(n,m)}(z)}^{(h)}||_{S^n}^2 = \Gamma \left( \frac{n - 1}{2} \right) \left( \frac{||\rho_{(n,m)}(z)||}{2\hbar^2} \right)^{\frac{3-n}{2}} \int_{\mathbb{R}^{2n}} \left( \frac{\sqrt{2||\rho_{(n,m)}(z)||}}{\hbar} \right)^{2} > 0,$$

hence the functions $\Phi_{\rho_{(n,m)}(z)}^{(h)}$ are continuous, i.e. the map $z \mapsto ||\Phi_{\rho_{(n,m)}(z)}^{(h)}||$ is continuous. Therefore, if $A : L^2(S^n) \to L^2(S^n)$ is a bounded linear operator, its Berezin symbol can be extended uniquely to a function defined on a neighbourhood of the diagonal in $\mathbb{C}^m \times \mathbb{C}^m$ in such a way that it is holomorphic in the first factor and anti-holomorphic in the second. In fact, such an extension is given explicitly by

$$\mathfrak{B}_{(n,m)}^{(h)}(A)(w, z) := \langle A\Phi_{\rho_{(n,m)}(w)}^{(h)}, \Phi_{\rho_{(n,m)}(w)}^{(h)} \rangle_{S^n}. \quad (30)$$

**Remark III.2.** By Eq. (27), the extended Berezin symbol has singularities at the zeros of the modified Bessel function $I_{-\frac{3-n}{2}}(z)$, which are well known (see Ref. [14] Sec. 5.13) and at $\rho_{(n,m)}(w) \cdot \rho_{(n,m)}(z) = 0$.

We now give the rules for symbolic calculus

**Proposition III.3.** Let $A, B$ be bounded linear operators with domain in $L^2(S^n)$. Then for $z, w \in \mathbb{C}^m$ and $\phi \in L^2(S^n)$ we have

$$\mathfrak{B}_{(n,m)}^{(h)}(\text{Id}) = 1, \text{ with Id denoting the identity operator},$$

$$\mathfrak{B}_{(n,m)}^{(h)}(A^*)(z, w) = \overline{\mathfrak{B}_{(n,m)}^{(h)}(A)(w, z)},$$

$$\mathfrak{B}_{(n,m)}^{(h)}(A\phi)(z) = \int_{\mathbb{C}^m} \mathfrak{B}_{(n,m)}^{(h)}(A)(z, \mu) Q_m^{(h)}(z, \mu) d\mu^h_m(\mu). \quad (31)$$

$$\mathfrak{B}_{(n,m)}^{(h)}(AB)(z, w) = \int_{\mathbb{C}^m} \mathfrak{B}_{(n,m)}^{(h)}(B)(u, w) \mathfrak{B}_{(n,m)}^{(h)}(A)(z, u) Q_m^{(h)}(u, z)Q_m^{(h)}(u, w) d\mu^h_m(\mu), \quad (32)$$

**Proof.** This is a direct consequence of the formulas in Ref. [4] (see Eqs. (11), (22)), the unitarity of the Bargmann transform $B_{S^n}$, Eq. (23) and the reproducing property of $Q_m^{(h)}$ (see Eq. (110)). \qed

**Corollary III.4.** Let $A : L^2(S^n) \to L^2(S^n)$ be a bounded linear operator, and define the function on $\mathbb{C}^m \times \mathbb{C}^m$,

$$\mathfrak{R}_A(z, w) := \langle A\Phi_{\rho_{(n,m)}(w)}^{(h)}, \Phi_{\rho_{(n,m)}(z)}^{(h)} \rangle_{S^n}. \quad (33)$$

Then $B_{S^n}^{-1}A(B_{S^n}^{-1})^*$ is an integral operator on $\mathcal{F}_m$ with Schwartz kernel $\mathfrak{R}_A$. Namely, for all $f \in \mathcal{F}_m$, $z \in \mathbb{C}^m$ we have

$$B_{S^n}^{-1}A(B_{S^n}^{-1})^*f(z) = \int_{\mathbb{C}^m} f(w)\mathfrak{R}_A(z, w) d\mu^h_m(w). \quad (33)$$

**Proof.** Let $f \in \mathcal{F}_m$, and $\phi = (B_{S^n}^{-1})^*f$. We obtain Eq. (33) from Eqs. (31), (30) and (26). \qed

Now we show some properties of the extended Berezin symbol that we will use in the next section to obtain the asymptotic expansion of the star-product.
Proposition III.5. Let $A$ be a bounded linear operator on $L^2(S^n)$, $n = 2, 3, 5$ and $z, w \in \mathbb{C}^m$, $m = 2, 4, 8$, respectively. Then

$$\mathfrak{B}^{(h)}_{(n,m)}(A)(T(\theta)w, T(\theta)z) = \mathfrak{B}^{(h)}_{(n,m)}(A)(w, z), \quad \forall g, \tilde{g} \in G_m$$

where $T(\theta), T(\tilde{\theta})$ are given by the action of $G_m$ on $\mathbb{C}^m$ (see Eqs. (7), (8) and (10)).

Proof. Since the Bargmann transform of the operator $B_{S^n}$ is unitary, we obtain from Eqs. (30) and (23)

$$\mathfrak{B}^{(h)}_{(n,m)}(A)(w, z) = \frac{\langle B_{S^n} A(B_{S^n})^{-1} Q^{(h)}_{m} (\cdot, z), Q^{(h)}_{m} (\cdot, w) \rangle_{F_m}}{\langle Q^{(h)}_{m} (\cdot, z), Q^{(h)}_{m} (\cdot, w) \rangle_{F_m}}.$$  

Therefore, from Eqs. (37), (39) and (55) we obtain Eq. (34). $\square$

Corollary III.6. Let $A$ be a bounded linear operator on $L^2(S^n)$, $n = 2, 3, 5$, then for $z \in \mathbb{C}^m$ fixed, $m = 2, 4, 8$, respectively, the extended Berezin symbol $\mathfrak{B}^{(h)}_{(n,m)}(A)(w, z)$ is invariant under the action of the group $G_m$ on $\mathbb{C}^m$ (see Eqs. (9), (10) and (11)), i.e.

$$\mathfrak{B}^{(h)}_{(n,m)}(A)(T(g)w, z) = \mathfrak{B}^{(h)}_{(n,m)}(A)(w, z), \quad \forall w \in \mathbb{C}^m, \forall g \in G_m.$$  

A similar result is obtained by freezing the first variable in the extended Berezin symbol.

Proof. Let $g_0$ be the identity element in $G_m$. From the explicit expression for $T(\theta)$ (see Eqs. (9), (10) and (11)), we obtain $T(g_0)v = v$ for all $v \in \mathbb{C}^m$. Thus, taking $\tilde{g} = g_0$ in Eq. (34), we obtain Eq. (35).

Similarly, we obtain the second part of this Corollary. $\square$

In addition to having the property indicated in Proposition III.5, the extended Berezin symbol belongs to the kernel of the operator $L$ (for the $(n, m) = (3, 4)$ case) and the kernel of the operators $R_j$, $j = 1, 2, 3$ (for the $(n, m) = (5, 8)$ case), as the following proposition establishes it:

Proposition III.7. Let $A$ be a bounded linear operator on $L^2(S^n)$, $n = 3, 5$, and $w, v \in \mathbb{C}^m$, $m = 4, 8$ respectively. Assume $\rho_{(n,m)}(v) \cdot \rho_{(n,m)}(w) \neq 0$ and $\rho_{(n,m)}(v) \cdot \rho_{(n,m)}(w) < \pi$. Then

For $(n, m) = (3, 4)$ : $L \mathfrak{B}^{(h)}_{(3,4)}(A)(v, w) = 0$, 

For $(n, m) = (5, 8)$ : $R_j \mathfrak{B}^{(h)}_{(5,8)}(A)(v, w) = 0$, $j = 1, 2, 3$

with $L$, $R_j$, $j = 1, 2, 3$, defined in Eqs. (14), (15) and where we think of $\mathfrak{B}^{(h)}_{(n,m)}(A)(v, w)$ as a function of $v$ for $w$ fixed.

Proof. The case $(n, m) = (3, 4)$. Let $g(\theta) = \exp(i\theta)$ and $T(g(\theta))$ given in Eq. (11). From Corollary III.6

$$\mathfrak{B}^{(h)}_{(3,4)}(A)(T(g(\theta))v, w) = \mathfrak{B}^{(h)}_{(3,4)}(A)(v, w).$$  

By considering the partial derivative of both sides in Eq. (29) with respect to $\theta$ and evaluating the resulting equation at the point $\theta = 0$, we obtain that $\mathfrak{B}^{(h)}_{(3,4)}(A)(z, w)$ must belong to the kernel of the operator $L$.

The case $(n, m) = (5, 8)$. Let $g(\theta, \alpha, \gamma) \in SU(2)$ and $T(g(\theta, \alpha, \gamma))$ given in Eq. (18). From Corollary III.6

$$\mathfrak{B}^{(h)}_{(5,8)}(A)(T(g(\theta, \alpha, \gamma))v, w) = \mathfrak{B}^{(h)}_{(5,8)}(A)(v, w).$$  

We consider the expression for $g(\theta, \alpha, \gamma) \in SU(2)$ given in Eq. (18). In a similar way to the case for $n = 3$, we can prove that $\mathfrak{B}^{(h)}_{(5,8)}(z, w)$ is in the kernel of the operators $R_1$, $R_2$ and $R_3$ by considering the partial derivatives of both sides in Eq. (10) with respect to $\theta, \alpha, \gamma$, respectively, and then evaluating at the point $(\theta, \alpha, \gamma) = (0, 0, 0)$ (we actually need to take the limit $\theta \to 0$ in the last case). $\square$
IV. THE STAR PRODUCT

In Ref. [4], Berezin showed that the formula (32) will allow us to define a star product, which will be denoted by $*_{m}$, on the algebra $A_{(n,m)}^{(h)}$ which consists of Berezin symbols for bounded linear operators with domain in $L^2(S^n)$. See Ref. [3] for the standard definition of star product. Thus, we have the following

**Definition IV.1.** $A_{(n,m)}^{(h)} = \left\{ \mathfrak{B}_{(n,m)}^{(h)}(A) \mid A \in B(L^2(S^n)) \right\}$.

**Definition IV.2.** For $f_1, f_2 \in A_{(n,m)}^{(h)}$,

$$ (f_1 *_{m} f_2)(z) = \int_{\mathbb{C}^m} f_1(z, u) f_2(u, z) \frac{|Q_m^{(h)}(u, z)|^2}{Q_m^{(h)}(z, z)} d\mu_m^{(h)}(u), \quad (41) $$

where the functions $f_j(v, w), j = 1, 2$, are the analytic continuation of $f_j(v)$ to $\mathbb{C}^m \times \mathbb{C}^m$ (see Eq. (30)).

**Remark IV.3.** From Eqs. (34) and (35), it follows that the above star product is $G_m$-invariant in the sense that

$$ (f_1 \circ T(g)) *_{m} (f_2 \circ T(g)) = (f_1 *_{m} f_2) \circ T(g), \quad \forall g \in G_m, \forall f_1, f_2 \in A_{(n,m)}^{(h)}, \quad (42) $$

where $T(g)$ is given by the action of $G_m$ on $\mathbb{C}^m$ (see Eqs. (3), (7) and (8)). Even more, in the next section we will prove that the star product $*_{m}$ is $\varphi_m$-invariant, where $\varphi_2 = SU(2)$, $\varphi_4 = SU(2) \times SU(2)$ and $\varphi_8 = SU(4)$.

In this section we verify that this noncommutative star product $*_{m}$ satisfies the usual requirement on the semiclassical limit, i.e. as $\hbar \to 0$

$$ f_1 *_{m} f_2(z) = f_1(z) f_2(z) + \hbar B(f_1, f_2)(z) + O(\hbar^2), \quad z \in \mathbb{C}^m, f_1, f_2 \in A_{(n,m)}^{(h)}, $$

where $B(\cdot, \cdot)$ is a certain bidifferential operator of the first order.

**Theorem IV.4.** Let $(n, m) = (2, 2), (3, 4), (5, 8)$. The star product $*_{m}$ (see Eq. (47)) satisfies

a) $f *_{m} 1 = 1 *_{m} f = f$, for all $f \in A_{(n,m)}^{(h)}$,

b) $*_{m}$ is associative, and

c) for $f_1, f_2 \in A_{(n,m)}^{(h)}$ and $z \in \mathbb{C}^m$, we have the following asymptotic expression when $\hbar \to 0$

$$ f_1 *_{m} f_2(z) = f_1(z) f_2(z) + \hbar \sum_{j=1}^{m} \left( \partial_{u_j} f_2(u, z) \partial_{\overline{u}_j} f_1(z, \overline{u}) \right)_{u=z} + O(\hbar^2), \quad (43) $$

where the functions $f_j(z, u), j = 1, 2$, are the analytic continuation of $f_j(z)$ to $\mathbb{C}^m \times \mathbb{C}^m$ (see Eq. (30)).

**Proof.**

a) Let $f \in A_{(n,m)}^{(h)}$, then $f = \mathfrak{B}_{(n,m)}^{(h)}(A) \in B(L^2(S^n))$. From Eqs. (20), (30) and Proposition 1.4 we have

$$ f *_{m} 1(z) = \int_{\mathbb{C}^m} \langle A \Phi_{(n,m)}^{(h)}(w), \Phi_{(n,m)}^{(h)}(z) \rangle_{S^n} \langle \Phi_{(n,m)}^{(h)}(z), \Phi_{(n,m)}^{(h)}(w) \rangle_{S^n} d\mu_m^{(h)}(w) = f(z). $$

Analogously, $1 *_{m} f = f$.

b) The associativity follows from the fact that the composition in the algebra of all bounded linear operator on $L^2(S^n)$ is associative.

c) Let us first assume that $z \neq 0$. The case $z = 0$ can be easily studied, see below.

Case $(n, m) = (2, 2)$: From Eq. (17)

$$ \frac{|Q_2^{(h)}(u, z)|^2}{Q_2^{(h)}(z, z)} e^{-|u|^2/h} = \frac{\exp(-|u-z|^2/h)}{2(1 + \exp(-2|z|^2/h))} + \frac{\exp(-|u+z|^2/h)}{2(1 + \exp(-2|z|^2/h))} $$

$$ + \frac{\cos(2\text{Im}(u \cdot z)/h) \exp(-|u|^2/h) \exp(-|z|^2/h)}{1 + \exp(-2|z|^2/h)}. \quad (44) $$
Since the last term in Eq. (44) is $O(h^\infty)$, where $O(h^\infty)$ denotes a quantity tending to zero faster than any power of $h$, we have from Eqs. (41) and (43)

$$(f_1 *_2 f_2)(z) = \frac{I(z) + I(-z)}{2(1 + \exp(-2|z|^2/h))} + O(h^\infty) \tag{45}$$

where

$$I(v) := \frac{1}{(\pi h)^2} \int \beta_z(u, \overline{w}) \exp \left(-\frac{1}{h}|u - v|^2\right) \, du \, d\overline{w} \tag{46}$$

with $\beta_z(u, \overline{w}) = f_1(z, u)f_2(u, z)$. Notice however that $I(v)$ is just the standard formula for the solution at time $t = h/4$ of the heat equation on $\mathbb{C}^2 = \mathbb{R}^4$ with initial data $\beta_z$; i.e.

$$I(v) = \sum_{\ell=0}^{\infty} h^\ell (\partial_{u\overline{w}})^\ell \beta_z \bigg|_{u=v} \tag{47}$$

where $\partial_{u\overline{w}} = \sum_{j} \partial_{u_j \overline{w}_j}$ denotes the Laplace operator. Eq. (47) can be obtained using the stationary phase method (see Eq. (A11)), however we do not include its description in order not to make this proof too long.

Thus, from Eqs. (41), (46) and (47)

$$(f_1 *_2 f_2)(z) = \left(1 + h^2 \partial_{u\overline{w}} \right) \left[ \frac{1}{2} \beta_z(u, \overline{w}) \bigg|_{u=z} + \frac{1}{2} \beta_z(u, \overline{w}) \bigg|_{u=-z} \right] + O(h^2) \tag{48}$$

where we have used $1 + \exp(-2|z|^2/h) = 1 + O(h^\infty)$.

Note that for $v \in \mathbb{C}^2$, the function $\beta_z(v, \overline{v}) = f_1(z, v)f_2(v, z)$ satisfies $\beta_z(v, \overline{v}) = \beta_z(-v, -\overline{v})$ because $f_1, f_2 \in A_{2,2}^{(h)}$ and Corollary (11.6).

Thus, from the chain rule and Eq. (48)

$$(f_1 *_2 f_2)(z) = f_1(z) f_2(z) + h \sum_{\ell=1}^{2} \left[ \partial_{u\overline{w}} f_2(u, z) \partial_{\overline{w}} f_1(z, u) \right]_{u=z} + O(h^2),$$

where we have used that the extended Berezin symbol is holomorphic in the first factor and anti-holomorphic in the second.

Case $(n, m) = (3, 4)$: From Eqs. (41), (47) and (43) we have

$$(f_1 *_4 f_2)(z) = \frac{e^{|z|^2/h}}{Q^4_{\psi}(z, \psi)} \frac{1}{4 \pi^2 h^2} \int_\psi=0 \int_\psi=0 \int_{\mathbb{R}^4} \int_{\mathbb{R}^4} \beta_z(u, \overline{w}) \exp \left(\frac{1}{h} p_\psi(u, \overline{w}, \theta) \right) \, du \, d\overline{w} \, d\psi, \tag{49}$$

where $\beta_z(u, \overline{w}) = f_1(z, u)f_2(u, z)$ and the phase function $p_\psi$ is

$$p_\psi(u, \overline{w}, \theta) = i \left(|u|^2 + |z|^2 - u \cdot T(g(\psi))z - u \cdot T(g(\theta))z \right),$$

with $g(\psi) = \exp(i\psi)$ and $g(\theta) = \exp(i\theta)$. To obtain the asymptotic expansion (43), we can use the stationary phase method with complex phase, see Eq. (A11), in the integral appearing on the right hand side of Eq. (49).

For our purpose, we need to consider the gradient and Hessian matrix of the function $p_\psi$ with respect to the nine variables $\theta$, $x_j = \Re(u_j)$, $y_j = \Im(u_j)$, $j = 1, 2, 3, 4$. It is actually more convenient to consider the derivatives of $p_\psi$ with respect to the variables $\theta$, $u_j$, $\overline{w}_j$, $j = 1, 2, 3, 4$. Namely,

i) the condition $\nabla_{x, y, \theta} p_\psi = 0$ is equivalent to $\nabla_{u, \overline{w}, \theta} p_\psi = 0$ with $x = (x_1, \ldots, x_4)$, $y = (y_1, \ldots, y_4)$, $u = (u_1, \ldots, u_4)$, and

ii) to obtain the Hessian matrix of $p_\psi$ with respect to the variables $x, y, \theta$ we use the following equalities: $\partial_{x, x_k} = \partial_{u_k \overline{w}_k} + \partial_{\overline{w}_k} + \partial_{\overline{w}_k} + \partial_{\overline{w}_k}, \partial_{x, y_k} = \partial_{u_k \overline{w}_k} + \partial_{u_k \overline{w}_k} + \partial_{\overline{w}_k} + \partial_{\overline{w}_k}, \partial_{y, y_k} = -\partial_{u_k \overline{w}_k} + \partial_{u_k \overline{w}_k} + \partial_{u_k \overline{w}_k} - \partial_{\overline{w}_k}, \partial_{x, \theta} = \partial_{u_k \theta} + \partial_{w_k \theta}, \partial_{y, \theta} = \partial_{u_k \theta} - \partial_{w_k \theta}.$
Notice that $\Im p_\psi \geq 0$ because for all $\psi$ and $\theta$
\[
|\Re(z_j \bar{\psi} e^{i\psi} + \bar{\psi} u_j e^{-i\theta})| \leq 2|z_j| |u_j| \quad \text{with} \quad j = 1, 2, 3, 4.
\]

Given $\psi$ fixed, $\psi \in (0, 2\pi)$, the gradient $\nabla u, \overline{u}, \partial \psi$ is zero if and only if $\theta = \psi$, and $u = T(g(\psi)z)$ (i.e. $u_j = z_j \exp(-i\psi)$, $j = 1, 2$ and $u_3 = z_3 \exp(i\psi)$, $j = 3, 4$), where we have used that $z \in \mathbb{C}^m$, i.e. $z$ satisfies the condition $|z_1|^2 + |z_2|^2 = |z_3|^2 + |z_4|^2$ (see Eq. (11)). Moreover, the Hessian matrix of $p_\psi$ evaluated at the critical point $x_0 + iy_0 = u_0 = T(g(\psi)z)$, $\theta_0 = \psi$ is equal to
\[
A := p''_\psi(u_0, \overline{u}_0, \theta_0) = \begin{pmatrix} 2iI_4 & 0_4 & B_4u_0 \\ 0_4 & 2iI_4 & -iB_4u_0 \\ (B_4u_0)^t & -(B_4u_0)^t & i|\psi|^2 \end{pmatrix}
\]
where $I_s$ and $0_s$ denote the identity matrix and zero matrix of size $s$ respectively, $A^t$ denotes the transpose matrix of a given matrix $A$ and
\[
B_{2\ell} = \begin{pmatrix} -I_\ell & 0_\ell \\ 0_\ell & I_\ell \end{pmatrix}, \quad \ell \in \mathbb{N}.
\]

Then from Eqs. (10) and (A2), with $D = i|\psi|^2$, we have
\[
\det(p''_\psi(u_0, \overline{u}_0, \theta_0)) = (2\pi)^s|\psi|^2.
\]

From the stationary phase method we obtain that
\[
(f_1 * f_2)(z) = \frac{e^{i|\psi|^2/\hbar}}{Q_4^{(h)}(z, z)} \frac{\sqrt{2\pi}}{2\pi |\psi|} \int_0^{2\pi} \sum_{\ell \leq k} h^\ell M_{\ell} \beta_\psi |_{\psi = cp} \theta \cdot \psi) \frac{1}{s!(s-\ell)!} \left[ (-A^{-1}) \dot{D} \cdot \dot{D} \right]^s \beta_\psi(p_\psi)^s |_{\psi = cp}.
\]

with $g |_{\psi = cp}$ denoting the evaluation at the critical point $u_0, \theta_0$ of a given function $g$,
\[
p_{\psi = cp}(u, \overline{u}, \theta) = i (T(g(\theta)z + u_0 + i(\theta - \psi)B_4u_0) \cdot u - \frac{1}{2} |\psi|^2 (\theta - \psi)^2, (53)
\]
and $\dot{D}$ the column vector of size 9 whose entries are defined by: $(\dot{D})_{j} = \partial_{x_j}$, $(\dot{D})_{j+4} = \partial_{y_j}$, $j = 1, \ldots, 4$, and $(\dot{D})_9 = \partial_{\theta}$. The last Eq. (53) is a consequence of equalities $|u_0| = |z|$ and $p'_{\psi}(u_0, \overline{u}_0, \theta_0) = 0$.

In order to estimate $M_{\ell} \beta_\psi |_{\psi = cp}$, we first need to obtain the inverse of the matrix $A$. From Eqs. (10) and (A2), with $D = i|\psi|^2$,
\[
A^{-1} = \begin{pmatrix} \frac{1}{2i}I_4 - \frac{1}{4|z|^2}B_4u_0u_0^tB_4 & \frac{1}{4|z|^2}B_4u_0u_0^tB_4 & \frac{1}{2|z|^2}B_4u_0 \\ \frac{1}{4|z|^2}B_4u_0u_0^tB_4 & \frac{1}{2i}I_4 + \frac{1}{4|z|^2}B_4u_0u_0^tB_4 & \frac{1}{2|z|^2}B_4u_0 \\ \frac{1}{2|z|^2}u_0^tB_4 & \frac{1}{2|z|^2}u_0^tB_4 & \frac{1}{4|z|^2}u_0^t \end{pmatrix}.
\]

Using the following equalities: $B_4u_0u_0^tB_4p \cdot \overline{p} = (B_4u_0)^t|p|^2$ for all $p \in \mathbb{C}^4$ and $\partial_{u_j} = (\partial_{x_j} - i\partial_{y_j})/2$, and easy linear algebra manipulations, we can show
\[
-(A^{-1}) \dot{D} \cdot \dot{D} = 2i\partial_{u_\overline{u}} \cdot \frac{1}{|\psi|^2} \left[ (B_4u_0)^t \partial_{u} - i\partial_{\theta} \right]^2.
\]

where $\partial_{u_\overline{u}} = \sum_{j=1}^4 \partial_{u_j} \overline{u}_j$ and $\partial_{u}$ denote the Laplace operator and the column vector of size 4 whose $j$ entry is $\partial_{u_j}$ (i.e. $\partial_{u} = \partial_{u_j}$) respectively.
From Eqs. 52 and 53
\[
M_0 \beta_u \bigg|_{cp} = \beta_u (u_0, \overline{u}_0),
\]
\[
M_1 \beta_u \bigg|_{cp} = \left[ \partial_{\overline{u}_0} - \frac{1}{2|z|^2} \left( (B_4 u_0)^{t} \partial_u \right)^2 - \frac{1}{2|z|^2} u_0 \partial_u + \frac{1}{2|z|^2} \right] \beta_u (u_0, \overline{u}_0).
\] (56)

where we have used the equality $B_4 u_0 \cdot u_0 = 0$ to obtain Eq. 56.

Notice that the right side of equality $B_4 u_0 \cdot u_0 = 0$ still depend on the variable $\psi$ because $u_0 = T(g(\psi))z$.

Thus, from Eqs. 55, 56 and 57
\[
(f_{1} \ast_{1} f_{2})(z) = \frac{e^{i|z|^2/\hbar}}{Q_{u}^{(k)}(z, z)} \frac{\sqrt{2\pi}}{2\pi} \left\{ \int_{0}^{2\pi} \left[ 1 + \hbar \left( \partial_{\overline{u}_0} - \frac{1}{2|z|^2} \left( (B_4 u_0)^{t} \partial_u \right)^2 \right. \right.
\]
\[
- \frac{1}{2|z|^2} u_0 \partial_u + \frac{1}{2|z|^2} \left\}\right] \beta_u (u_0, \overline{u}_0) d\psi + O(h^2) \right\}. \] (57)

Note that for $v \in \mathbb{C}^4$ and $g(\theta) \in S^1$, the function $\beta_z (v, \nabla) = f_1 (z, v) f_2 (v, z)$ satisfies
\[
\beta_z (v, \nabla) = \beta_z (T(g(\theta)) v, \overline{T(g(\theta)) v})
\]
because $f_1, f_2 \in A_{(3,4)}^{(k)}$ and Corollary III.6.

Then, from the chain rule
\[
\begin{align*}
\partial_v \beta_u (z, v) \bigg|_{\theta = \psi} &= u_0 \partial_\theta \beta_u (u_0, \overline{u}_0), \\
\partial_\nabla \beta_u (z, v) \bigg|_{\theta = \psi} &= \partial_{\overline{u}_0} \beta_u (u_0, \overline{u}_0),
\end{align*}
\]
(58)

Thus, from Eqs. 57, 58, Proposition III.2, the asymptotic expression of the modified Bessel function $I_\nu$ (see Eq. 25) and the relation $\sqrt{2|\theta (3,4)(z)|} = |z|^2$ we have
\[
(f_{1} \ast_{4} f_{2})(z) = \left[ 1 + \hbar \left( \partial_{\overline{u}_0} - \frac{1}{2|z|^2} (u_0 B_4 \partial_\theta) \right)^2 \right] \beta_u (u, \overline{u}) \bigg|_{u = z} + O(h^2),
\]
(59)

Using the equality $u_0 B_4 \partial_\theta f_2 (u, z) = 0$ (see Proposition III.7) and that the extended Berezin symbol is holomorphic in the first factor and anti-holomorphic in the second we finally obtain Eq. 43.

Case $(n, m) = (5, 8)$: This case is similar to the case when $(n, m) = (3, 4)$ but the computations are more involved. First note that from Eqs. III.1, III.7 and III.8
\[
(f_{1} \ast_{8} f_{2})(z) = \frac{e^{i|z|^2/\hbar}}{Q_{u}^{(k)}(z, z)} \left( \frac{\pi h}{i} \right)^{k} \int_{\theta, \alpha, \gamma \in \mathbb{C}^n} \int_{\tilde{\theta}, \tilde{\alpha}, \tilde{\gamma} \in \mathbb{C}^n} \exp \left( \frac{i}{h} \tilde{\theta}_0, \alpha, \gamma (u, \overline{u}, \tilde{\theta}, \tilde{\alpha}, \tilde{\gamma}) \right) \beta_u (u, \overline{u}) d\theta \alpha \gamma d\tilde{\theta} \tilde{\alpha} \tilde{\gamma},
\]
(59)

with $\beta_z (u, \overline{u}) = f_1 (z, u) f_2 (u, z)$ and the phase function $p_{\theta, \alpha, \gamma}$ is given by
\[
p_{\theta, \alpha, \gamma} (u, \overline{u}, \tilde{\theta}, \tilde{\alpha}, \tilde{\gamma}) = i \left( |u|^2 + |z|^2 - u \cdot T(g(\theta, \alpha, \gamma)) z - u \cdot \overline{T(g(\theta, \alpha, \gamma)) z} \right),
\]
where we are considering the expression for $g(\theta, \alpha, \gamma), g(\tilde{\theta}, \tilde{\alpha}, \tilde{\gamma}) \in SU(2)$ given in Eq. 18. For $\theta, \alpha, \gamma$ fixed, $\theta \in (0, \pi/2)$ and $\alpha, \gamma \in (-\pi, \pi)$, the equations $\partial_\theta p_{\theta, \alpha, \gamma} = 0$ and $\partial_{\overline{u}_0} p_{\theta, \alpha, \gamma} = 0$, $\ell = 1, \ldots, 8$ imply $u = T(g(\theta, \alpha, \gamma)) z = T(g(\tilde{\theta}, \tilde{\alpha}, \tilde{\gamma}) z$, which in turn implies (see Eq. 8)
\[
V \left( g^{-1} (\tilde{\theta}, \tilde{\alpha}, \tilde{\gamma}) g(\theta, \alpha, \gamma) \right) L_z = L_z.
\] (60)
where $V(g)$ and $L$ are defined in Eq. (63). Since $z \neq 0$ then $Lz \neq 0$, therefore we obtain from Eq. (60) that $g^{-1}(\hat{\theta}, \hat{\alpha}, \hat{\gamma})g(\theta, \alpha, \gamma)$ must be the identity matrix, which in turn implies $(\hat{\theta}, \hat{\alpha}, \hat{\gamma}) = (\theta, \alpha, \gamma)$.

Even more, we claim that
\[
\frac{\partial^2 p_{\theta, \alpha, \gamma}}{\partial \vartheta} \bigg|_{\vartheta = \hat{\theta}, \hat{\alpha}, \hat{\gamma}} = 0, \quad \text{for } \vartheta = \theta, \alpha, \gamma, \tag{61}
\]
where $\frac{\partial^2 p_{\theta, \alpha, \gamma}}{\partial \vartheta} \bigg|_{\vartheta = \hat{\theta}, \hat{\alpha}, \hat{\gamma}}$ denotes $\frac{\partial^2 p_{\theta, \alpha, \gamma}}{\partial \vartheta}$ evaluated at $u = T(g(\theta, \alpha, \gamma))z$ and $(\hat{\theta}, \hat{\alpha}, \hat{\gamma}) = (\theta, \alpha, \gamma)$. To show this fact, note that from the explicit expression of the function $p_{\theta, \alpha, \gamma}$ and Eq. (5), we have
\[
\frac{\partial^2 p_{\theta, \alpha, \gamma}}{\partial \vartheta} \bigg|_{\vartheta = \hat{\theta}, \hat{\alpha}, \hat{\gamma}} = -iV(g^{-1}(\theta, \alpha, \gamma))\partial \vartheta V(g(\theta, \alpha, \gamma))Lz \cdot Lz, \quad \vartheta = \theta, \alpha, \gamma.
\]

From the expression for $g = g(\theta, \alpha, \gamma)$ given in Eq. (68) we find
\[
\frac{\partial^2 p_{\theta, \alpha, \gamma}}{\partial \vartheta} \bigg|_{\vartheta = \hat{\theta}, \hat{\alpha}, \hat{\gamma}} = e^{i(\gamma - \alpha)}\left[ z_{7}\overline{z}_{1} + z_{5}\overline{z}_{3} - z_{6}\overline{z}_{4} - z_{8}\overline{z}_{2} \right],
\]
\[
\frac{\partial^2 p_{\theta, \alpha, \gamma}}{\partial \vartheta} \bigg|_{\vartheta = \hat{\theta}, \hat{\alpha}, \hat{\gamma}} = \cos^2 \theta \left[ |z_1|^2 + |z_2|^2 + |z_3|^2 + |z_4|^2 - |z_5|^2 - |z_6|^2 - |z_7|^2 - |z_8|^2 \right] + 2i\Re \left( \sin(\theta) \cos(\theta)e^{i(\gamma - \alpha)}[z_5\overline{z}_3 - z_6\overline{z}_4 + z_7\overline{z}_1 - z_8\overline{z}_2] \right),
\]
\[
\frac{\partial^2 p_{\theta, \alpha, \gamma}}{\partial \vartheta} \bigg|_{\vartheta = \hat{\theta}, \hat{\alpha}, \hat{\gamma}} = \sin^2 \theta \left[ |z_1|^2 + |z_2|^2 + |z_3|^2 + |z_4|^2 - |z_5|^2 - |z_6|^2 - |z_7|^2 - |z_8|^2 \right] + 2i\Re \left( e^{i(\gamma - \alpha)} \sin(\theta)[z_5\overline{z}_3 - z_6\overline{z}_4 + z_7\overline{z}_1 - z_8\overline{z}_2] \right).
\]

Since $z \in \mathcal{C}^\circ$ (see Eq. (11)), then Eqs. (61) hold. Thus, the critical point is
\[
u_0 = T(g(\theta, \alpha, \gamma))z, \quad (\hat{\theta}_0, \hat{\alpha}_0, \hat{\gamma}_0) = (\theta, \alpha, \gamma). \tag{62}
\]

One can also check that $\Im p_{\theta, \alpha, \gamma} \geq 0$ on the domain of $p_{\theta, \alpha, \gamma}$ and that $p_{\theta, \alpha, \gamma} = 0$ at the critical point. Moreover, the Hessian matrix of $p_{\theta, \alpha, \gamma}$ with respect to the variables $x, y, \hat{\theta}, \hat{\alpha}, \hat{\gamma}$ (with $x = \Re u$ and $y = \Im u$) evaluated at the critical point is equal to

\[
A := \begin{pmatrix}
2iI_8 & 0_8 & -iT_\theta(g)z & -iT_\alpha(g)z & -iT_\beta(g)z \\
0_8 & 2iI_8 & -T_\theta(g)z & -T_\alpha(g)z & -T_\beta(g)z \\
-i(T_\theta z)^t & -(T_\theta z)^t & i|z|^2 & 0 & 0 \\
-i(T_\alpha z)^t & -(T_\alpha z)^t & 0 & i|z|^2 \cos^2 \theta & 0 \\
-i(T_\beta z)^t & -(T_\beta z)^t & 0 & 0 & i|z|^2 \sin^2 \theta
\end{pmatrix}, \tag{63}
\]

where $T_\phi(g)z := \partial_\phi T(g(\theta, \alpha, \gamma))z = L^t \partial_\phi V(g(\theta, \alpha, \gamma))Lz$, $\vartheta = \theta, \alpha, \gamma$ (see Eq. (5)).

From Eqs. (63) and (A2), with $D$ the diagonal matrix
\[
D = \text{diag}(iz|z|^2, iz|z|^2 \cos^2 \theta, iz|z|^2 \sin^2 \theta),
\]
we have that $\text{det}(A) = (2\pi)^{16} \delta^3 |z|^6 \cos^2 \theta \sin^2 \theta$.

Thus, from the stationary phase method
\[
(f_1 \ast_s f_2)(z) = \frac{O(|z|^2/h)}{Q_S^h(z, z) (\pi h)^8} \int_{\vartheta, \alpha, \gamma} \left( \frac{2^3(\pi h)^{19}}{|z|^6 \cos^2 \theta \sin^2 \theta} \right)^{1/2} \frac{1}{2\pi^2} \sum_{k \leq \ell} h^\ell M_\ell \left( \cos \vartheta \sin \vartheta z \right) \bigg|_{cp} + O(h^k) \, dm(\theta, \alpha, \gamma) \tag{65}
\]

where
\[
M_\ell \left( \cos \vartheta \sin \vartheta z \right) \bigg|_{cp} = \sum_{s = \ell}^{3\ell} \frac{1}{s! (s - \ell)!} \left[ (-A^{-1}) \hat{D} \cdot \hat{D} \right]^s \cos \vartheta \sin \vartheta z \bigg|_{cp} \tag{66}
\]

and $z \neq 0$. Therefore, we obtain from Eq. (60) that $g^{-1}(\hat{\theta}, \hat{\alpha}, \hat{\gamma})g(\theta, \alpha, \gamma)$ must be the identity matrix, which in turn implies $(\hat{\theta}, \hat{\alpha}, \hat{\gamma}) = (\theta, \alpha, \gamma)$.
with \( g_{\mid \text{cp}} \) denoting the evaluation at the critical point \( u_0, \theta_0, \alpha_0, \gamma_0 \) of a given function \( g \),

\[
p_{\text{cp}} = p_{\text{cp}}(u, \mathbf{u}, \tilde{\theta}, \tilde{\alpha}, \tilde{\gamma}) = i (u_0 - T(g(\tilde{\theta}, \tilde{\alpha}, \tilde{\gamma})) \mathbf{z} + (\tilde{\theta} - \theta)T_{\theta}(g) \mathbf{z} + (\tilde{\alpha} - \alpha)T_{\alpha}(g) \mathbf{z} + (\tilde{\gamma} - \gamma)T_{\gamma}(g) \mathbf{z}) \cdot \mathbf{u} - \frac{i}{2} |\mathbf{z}|^2 \left[ (\tilde{\theta} - \theta)^2 + \cos^2 \theta (\tilde{\alpha} - \alpha)^2 + \sin^2 \theta (\tilde{\gamma} - \gamma)^2 \right],
\]

(67)

and \( \tilde{D} \) the column vector of size 19 whose entries are defined by: \( (\tilde{D})_j = \partial z_j \), \( (\tilde{D})_{8+j} = \partial y_j \), \( j = 1, \ldots, 8 \), \( (\tilde{D})_{17} = \partial \tilde{\theta} \), \( (\tilde{D})_{18} = \partial \tilde{\alpha} \) and \( (\tilde{D})_{19} = \partial \tilde{\gamma} \). The last Eq. (67) is a consequence of equalities \( |u_0| = |\mathbf{z}| \) and \( T_{\theta}(g) \mathbf{z} \cdot u_0 = 0 \), \( \theta = \theta, \alpha, \gamma \).

In a similar way as we did for the case \( n = 3 \), we obtain the inverse of the matrix \( A \) (see Eq. (68)) using Eq. (A3), with \( D \) the diagonal matrix given in Eq. (71). By considering the explicit expression for the inverse matrix \( A^{-1} \), using the equality \( T_{\theta}(g) \mathbf{z} (T_{\theta}(g) \mathbf{z})^t \mathbf{p} \cdot \mathbf{p} = (T_{\theta}(g) \mathbf{z} \cdot \mathbf{p})^2 \), \( \theta = \theta, \alpha, \gamma \), for all \( \mathbf{p} \in \mathbb{C}^8 \), and easy linear algebra manipulations, we can show

\[
-(A^{-1}) \cdot \tilde{D} = 2i \partial_{\mathbf{u} \mathbf{u}^t} + \frac{i}{|\mathbf{z}|^2} \left[ \frac{1}{\cos^2 \theta} \left( (T_{\alpha}(g) \mathbf{z})^t \partial_{\mathbf{u}} + \partial_{\tilde{\alpha}} \right)^2 + \frac{1}{\sin^2 \theta} \left( (T_{\gamma}(g) \mathbf{z})^t \partial_{\mathbf{u}} + \partial_{\tilde{\gamma}} \right)^2 + \left( (T_{\theta}(g) \mathbf{z})^t \partial_{\mathbf{u}} + \partial_{\tilde{\theta}} \right)^2 \right],
\]

(68)

where \( \partial_{\mathbf{u} \mathbf{u}^t} = \sum_{j=1}^{8} \partial u_j \partial u_j \) and \( \partial_{\mathbf{u}} \) denote the Laplace operator and the column vector of size 8 whose \( j \) entry is \( \partial_{u_j} \) (i.e. \( \partial_{u_j} = \partial_{u_j} \)) respectively.

From Eqs. (69) and (68)

\[
M_0 \left( \cos \tilde{\theta} \sin \tilde{\beta} \delta_{\mathbf{z}} \right) \mid_{\text{cp}} = \cos \theta \sin \theta \beta_{\mathbf{z}}(u_0, \overline{u_0}), \quad (69)
\]

\[
M_1 \left( \cos \tilde{\theta} \sin \tilde{\beta} \delta_{\mathbf{z}} \right) \mid_{\text{cp}} = \cos \theta \sin \theta \left[ 2|\mathbf{z}|^2 \partial_{\mathbf{u} \mathbf{u}^t} - (T_{\alpha}(g) \mathbf{z})^t \partial_{\mathbf{u}} + \left[ (T_{\theta}(g) \mathbf{z})^t \partial_{\mathbf{u}} + \partial_{\tilde{\theta}} \right)^2 + \left[ (T_{\gamma}(g) \mathbf{z})^t \partial_{\mathbf{u}} + \partial_{\tilde{\gamma}} \right)^2 \right] \sin^2 \theta - \left[ (T_{\alpha}(g) \mathbf{z})^t \partial_{\mathbf{u}} + \partial_{\tilde{\alpha}} \right]^2 \cos^2 \theta + \left[ (T_{\gamma}(g) \mathbf{z})^t \partial_{\mathbf{u}} + \partial_{\tilde{\gamma}} \right]^2 \cos^2 \theta - \left( \cos \theta - \sin \theta \right)^2 \cos^2 \theta + \left( \sin \theta - \cos \theta \right)^2 \sin^2 \theta \right], \quad (70)
\]

Notice that the right side of Eqs. (69) and (70) still depend on the variables \( \theta, \alpha, \gamma \), because \( u_0 = T(g(\theta, \alpha, \gamma)) \mathbf{z} \) (see Eq. (62)).

On the other hand, since \( f_1, f_2 \in P^{(h)}_{(5,8)} \), we obtain from Corollary [16] that, for \( \mathbf{v} \in \mathbb{C}^8 \) and \( g(\tilde{\theta}, \tilde{\alpha}, \tilde{\gamma}) \in SU(2) \), the function \( \beta_{\mathbf{v}}(\mathbf{v}, \mathbf{v}) = f_1(\mathbf{z}, \mathbf{v}) \mathbf{v} \) satisfies

\[
\beta_{\mathbf{v}}(\mathbf{v}, \mathbf{v}) = \beta_{\mathbf{v}}(T(g(\theta, \alpha, \gamma))) \mathbf{v}, \quad (T(g(\theta, \alpha, \gamma))) \mathbf{v}).
\]

Then, from the chain rule

\[
\nabla^t \partial_{\mathbf{v} \mathbf{v}^t} \beta_{\mathbf{v}}(\mathbf{v}, \mathbf{v}) \mid_p = u_0^t \partial_{\mathbf{u} \mathbf{u}^t} + (T_{\theta}(g) \mathbf{z})^t \partial_{\mathbf{u} \mathbf{u}^t} + \left[ (T_{\alpha}(g) \mathbf{z})^t \partial_{\mathbf{u} \mathbf{u}^t} + \partial_{\tilde{\alpha}} \right]^2 \cos^2 \theta + \left[ (T_{\gamma}(g) \mathbf{z})^t \partial_{\mathbf{u} \mathbf{u}^t} + \partial_{\tilde{\gamma}} \right]^2 \cos^2 \theta - \left( \cos \theta - \sin \theta \right)^2 \cos^2 \theta + \left( \sin \theta - \cos \theta \right)^2 \sin^2 \theta \right] \beta_{\mathbf{v}}(u_0, \overline{u_0}),
\]

(71)

\[
-2v^t \partial_{\mathbf{v} \mathbf{v}^t} \beta_{\mathbf{v}}(\mathbf{v}, \mathbf{v}) \mid_p = \left( \frac{B_8 T_{\alpha}(g) \mathbf{z})^t \partial_{\mathbf{u} \mathbf{u}^t}}{i \cos^2 \theta} - \frac{B_8 T_{\gamma}(g) \mathbf{z})^t \partial_{\mathbf{u} \mathbf{u}^t}}{i \sin^2 \theta} \right) \beta_{\mathbf{v}}(u_0, \overline{u_0}) \]

where for a given function \( g \) we denote by \( g \mid_p \) the evaluation of \( g \) at \( \mathbf{v} = \mathbf{z}, (\tilde{\theta}, \tilde{\alpha}, \tilde{\gamma}) = (\theta, \alpha, \gamma) \) and \( \mathbf{R}_\ell, \ell = 1, 2, 3 \) are give in Eq. (15).

Thus, from Proposition [12] the asymptotic expression of the modified Bessel function \( I_\nu \) (see Eq. (23)), Eqs. (65), (69), (70), (71), and the relation \( \sqrt{2}|\mathbf{R}_{(5,8)}(\mathbf{z})| = |\mathbf{z}|^2 \) we have
\[(f_1 \ast f_2)(z) = \left(1 + \hbar \left[ \partial_{\text{u}} \beta_{z}(\text{u}, \text{u}) \right]_{\text{u} = z} \right) + O(\hbar^2).\]

Then, from Proposition III.7 (applying to \(f_2\)) and using that the extended Berezin symbol is holomorphic in the first factor and anti-holomorphic in the second we obtain Eq. (43).

Finally, suppose we have the case \(|z| = 0\). Then \(\rho_{(n,m)}(z) = 0\) and therefore the coherent state \(\Phi^{(h)}_{\rho_{(n,m)}}(z)\) is the constant function 1 on the whole sphere with \(L^2(S^n)\) norm equal to one. Thus, in a similar way as we did for the case \(n = 2\) (see Eqs. (46) and (47)) we conclude from the stationary phase method:

\[
\begin{align*}
(f_1 \ast f_2)(z) &= \frac{1}{(\pi\hbar)^m} \int_{\mathbb{C}^m} f_1(0, \text{u}) f_2(\text{u}, 0) \exp \left( - \frac{|\text{u}|^2}{\hbar} \right) d\text{u} d\text{u} \\
&= f_1(0) f_2(0) + \hbar \sum_{\ell=1}^{m} \left[ \partial_{u_{\ell}} f_2(\text{u}, 0) \partial_{\overline{u}_{\ell}} f_1(0, \text{u}) \right]_{\text{u} = 0} + O(\hbar^2).
\end{align*}
\]

\[\Box\]

V. INVARIANCE OF STAR-PRODUCT \(\ast_m\)

In this section we show that the star product \(\ast_p\) defined on the algebra \(A^{(h)}_{(n,m)}\) is invariant under the group \(\mathfrak{g}_m = SU(2), SU(2) \times SU(2), SU(4)\) for \(m = 2, 4, 8\) respectively, i.e. it satisfy

\[(f_1 \circ L(g)) \ast_m (f_2 \circ L(g)) = (f_1 \ast_m f_2) \circ L(g), \quad \forall g \in \mathfrak{g}_m, \forall f_1, f_2 \in A^{(h)}_{(n,m)},\]

where \(L(g)\) denotes the action of the group \(\mathfrak{g}_m\) on \(\mathbb{C}^m\) and it is defined by the following equation

\[z' = L(g)z, \quad g \in \mathfrak{g}_m\]

with \(L(g)\) given by

The case \(m = 2\). For \(g \in SU(2)\)

\[L(g) = g\]

The case \(m = 4\). For \(g = (V, W) \in SU(2) \times SU(2)\)

\[L(g) = \begin{pmatrix} V & 0_2 \\ 0_2 & W \end{pmatrix}, \]

where \(0_\ell\) denotes the zero matrix of size \(\ell\).

The case \(m = 8\). For \(g \in SU(4)\)

\[L(g) = \begin{pmatrix} U & 0_4 \\ 0_4 & EUE \end{pmatrix}, \]

where \(E\) is the following orthogonal matrix

\[
E = \begin{pmatrix}
0 & 0 & -1 & 0 \\
0 & 0 & 0 & 1 \\
-1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0
\end{pmatrix}.
\]

In order to prove that the star product \(\ast_m\) is \(\mathfrak{g}_m\)-invariant, we first give the explicit relation between the \(SO(n+1)\) action of rotations on the quadric \(Q^n\), \(n = 2, 3, 5\) (whose elements are expressed in term of the map \(\rho_{(n,m)}\)) and the action of the group \(\mathfrak{g}_m\) on \(\mathbb{C}^m\), \(m = 2, 4, 8\), respectively.
Proposition V.1. Let \( g \in \mathfrak{g}_m \) and \( L(g) \) the action defined above, \( m = 2, 4, 8 \), then exist \( R \in SO(n + 1) \), \( n = 2, 3, 5 \) respectively, such that

\[
R \rho_{(n,m)}(z) = \rho_{(n,m)}(L(g)z), \quad \forall z \in \mathbb{C}^m;
\]

where for \( w \in \mathbb{C}^{n+1} \), \( Rw = R\Re(w) + iR\Im(w) \) with \( R\Re(w) \) and \( R\Im(w) \) denoting the usual action of \( R \) on the real and imaginary part of \( w \), respectively (regarded as elements of \( \mathbb{R}^{n+1} \)).

Proof. The main idea is the same for the three cases \( n = 2, 3, 5 \). We describe in detail the most complicated case \( n = 5 \). The cases \( n = 2, 3 \) follow in a similar way and we will only sketch the structure of the proof.

The case \( n = 5 \): let us write the map \( \rho_{(5,8)}(z) = (\rho_1(z), \rho_2(z), \ldots, \rho_6(z)) \) in matrix form

\[
\rho_\ell(z) = (z_1, z_2, z_3, z_4) A_\ell \begin{pmatrix} z_5 \\ z_6 \\ z_7 \\ z_8 \end{pmatrix}, \quad z = (z_1, \ldots, z_8), \quad \ell = 1, \ldots, 6,
\]

where the matrices \( A_\ell \) are defined as follows:

\[
A_1 = \begin{pmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & -i & 0 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad A_3 = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix},
\]

\[
A_4 = \begin{pmatrix} -i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & 0 & 0 & i \end{pmatrix}, \quad A_5 = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & -i & 0 \\ 0 & -i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix}, \quad A_6 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}.
\]

From Eqs. (23), (26) and (28)

\[
\rho_\ell(z') = (z_1, z_2, z_3, z_4) U^t A_\ell U E E E \begin{pmatrix} z_5 \\ z_6 \\ z_7 \\ z_8 \end{pmatrix}.
\]

Let \( V \) denote the real vector space generated by the matrices \( A_\ell, \ell = 1, \ldots, 6 \). The vector space \( V \) is the set of complex matrices of the form

\[
\begin{pmatrix} -\vartheta & \mu & 0 & \gamma \\ \mu & -\vartheta & \gamma & 0 \\ 0 & -\gamma & \vartheta & \mu \\ -\gamma & 0 & \mu & -\vartheta \end{pmatrix}, \quad \text{with } \mu, \vartheta, \gamma \in \mathbb{C}.
\]

We now claim that, for \( U \in SU(4) \), the matrix \( U^t A_\ell U E E \) is in the vector space \( V \). To prove this fact, let us denote by \( U_{jk} \) the matrix elements of \( U \). Since \( \det(U) = 1 \) and \( U^t = U^{-1} \) then by considering the explicit expression for the inverse matrix \( U^{-1} \) we find that the matrix elements of \( U \) must satisfy the following relations:

\[
U_{11} U_{23} - U_{21} U_{13} = U_{42} U_{34} - U_{32} U_{44}, \quad U_{11} U_{43} - U_{41} U_{13} = U_{32} U_{24} - U_{22} U_{34},
\]

\[
U_{11} U_{33} - U_{31} U_{13} = U_{22} U_{44} - U_{42} U_{24}, \quad U_{21} U_{43} - U_{41} U_{23} = U_{12} U_{34} - U_{32} U_{14},
\]

\[
U_{31} U_{23} - U_{21} U_{33} = U_{12} U_{44} - U_{42} U_{14}, \quad U_{41} U_{33} - U_{31} U_{43} = U_{12} U_{24} - U_{22} U_{14},
\]

Then by using the last equalities and computing the explicit expression for the matrix \( U^t A_\ell U E E \), we find that \( U^t A_\ell U E E \) has the form indicated in Eq. (76).

The vector spaces \( V \) is endowed with the real valued inner product

\[
\langle A, B \rangle_V = \frac{1}{2} \left( \text{trace}(AB^t) + \text{trace}(BA^t) \right).
\]
The set of matrices \( \{ \frac{1}{2} A_\ell \mid \ell = 1, \ldots, 6 \} \) gives an orthonormal basis for the space \( \mathcal{V} \). Thus \( U^\dagger A_\ell EUE \) must be the following linear combination of the basis elements (summation over repeated indexes):

\[
U^\dagger A_\ell EUE = \frac{1}{4} \langle U^\dagger A_\ell EUE, A_k \rangle_{\mathcal{V}} A_k, \quad \ell = 1, \ldots, 6.
\]

Therefore we have

\[
\rho_\ell (z') = (z_1, z_2, z_3, z_4) R_{\ell k} A_k \begin{pmatrix}
  z_5 \\
  z_6 \\
  z_7 \\
  z_8
\end{pmatrix}
= R_{\ell k} \rho_k(z),
\]

with the real numbers \( R_{\ell k}, \ell, k = 1, \ldots, 6 \), given by

\[
R_{\ell k} = \frac{1}{4} \langle U^\dagger A_\ell EUE, A_k \rangle_{\mathcal{V}}.
\]

Thus to each element \( U \in SU(4) \) we associated a \( 6 \times 6 \) matrix \( R \) whose matrix elements are \( R_{\ell k} \) given by Eq. (77). Since the matrix \( R \) satisfies the relation \( R_{jk} R_{sk} = T_{js} \) (thus \( R \) must be an orthogonal matrix) then we have a continuous map \( U \mapsto R \) from \( SU(4) \) into \( O(6) \). Since \( SU(4) \) is a connected manifold and the identity element of \( SU(4) \) goes to the identity element of \( O(6) \) (see Eq. (77)), then the image of \( SU(4) \) under the map we are considering must be the connected component of the identity matrix in \( O(6) \). Thus \( R \in SO(6) \).

For the cases \( n = 2, 3 \), let us write the maps \( \rho_{(n,m)}(z) = (\rho_1(z), \ldots, \rho_{n+1}(z)) \) in the following matrix form

For the case \( n = 2 \):

\[
\rho_\ell (z) = (z_1, z_2) B_\ell \begin{pmatrix}
  z_1 \\
  z_2
\end{pmatrix}
= (z_1, z_2), \quad \ell = 1, 2, 3,
\]

where the matrices \( B_\ell \) are defined as follows:

\[
B_1 = \frac{1}{2} \begin{pmatrix}
  -1 & 0 \\
  0 & 1
\end{pmatrix}, \quad B_2 = \frac{1}{2} \begin{pmatrix}
  0 & 1 \\
  1 & 0
\end{pmatrix}, \quad B_3 = \frac{1}{2} \begin{pmatrix}
  0 & 1 \\
  1 & 0
\end{pmatrix}.
\]

For the case \( n = 3 \):

\[
\rho_\ell (z) = (z_1, z_2) C_\ell \begin{pmatrix}
  z_3 \\
  z_4
\end{pmatrix}
= (z_1, z_2), \quad \ell = 1, 2, 3, 4,
\]

where the matrices \( C_\ell \) are defined as follows:

\[
C_1 = \begin{pmatrix}
  1 & 0 \\
  0 & 1
\end{pmatrix}, \quad C_2 = \begin{pmatrix}
  i & 0 \\
  0 & -i
\end{pmatrix}, \quad C_3 = \begin{pmatrix}
  0 & i \\
  i & 0
\end{pmatrix}, \quad C_4 = \begin{pmatrix}
  0 & 1 \\
  -1 & 0
\end{pmatrix}.
\]

In a similar way we did for the case \( n = 5 \) we need to prove, for the case \( n = 2 \), that the matrix \( U^\dagger B_\ell U \) (for all \( U \in SU(2) \)) is in the vector space generated by matrices \( B_\ell, \ell = 1, 2, 3 \), which is not difficult to prove using the parametrization of \( SU(2) \) indicated in Eq. (18). And for the case \( n = 3 \) we need to prove that the matrix \( V^\dagger C_\ell W \) (for all \( V, W \in SU(2) \)) is in the vector space generated by matrices \( C_\ell, \ell = 1, \ldots, 4, \) which is a consequence of \( C_\ell \in SU(2), \ell = 1, \ldots, 4 \).

The rest of the proof is similar to the case \( n = 5 \), therefore we will omit it. \( \square \)

**Theorem V.2.** The star product \(*_p\) defined on the algebra \( A^{(h)}_{(n,m)} \) is \( \mathcal{F}_m \)-invariant in the sense that

\[
(f_1 \circ L(g)) *_m (f_2 \circ L(g)) = (f_1 *_m f_2) \circ L(g), \quad g \in \mathcal{F}_m, \forall f_1, f_2 \in A^{(h)}_{(n,m)},
\]

where \( L(g) \) is given by the action of the group \( \mathcal{F}_m = SU(2), SU(2) \times SU(2), SU(4) \) on \( \mathbb{C}^2, \mathbb{C}^4, \mathbb{C}^8 \) respectively, indicated in Eq. (73).

**Proof.** Given \( \hat{R} \in SO(n+1) \), define the operator \( T_{\hat{R}} : L^2(S^n) \rightarrow L^2(S^n) \) by \( T_{\hat{R}} \psi(x) = \psi(\hat{R}^{-1}x) \). Let \( A \) be a bounded linear operator with domain in \( L^2(S^n), n = 2, 3, 5 \) and \( g \in \mathcal{F}_m, m = 2, 4, 8 \) respectively. Let \( R \in SO(n+1) \) be the orthogonal matrix mentioned in the hypothesis of Proposition V.1 associated to \( g \).
From the expression for the coherent states (see Eq. (39) and Eq. (74) we obtain
\[ \Phi_{\rho(n,m)}^{(h)}(L(g)z) = \mathcal{T}_R \Phi_{\rho(n,m)}^{(h)}(z), \quad \forall z \in \mathbb{C}^m. \] (78)

Then from Eqs. (29) and (78) we have
\[ \mathfrak{B}_{(n,m)}^{(h)}(A) \circ L(g)(z) = \mathfrak{B}_{(n,m)}^{(h)}(\mathcal{T}_R^{-1} AT_R)^{-1}(z), \quad \forall z \in \mathbb{C}^m. \] (79)

Thus \( \mathfrak{B}_{(n,m)}^{(h)}(A) \circ L(g) \) can be expressed as the Berezin symbol of the bounded linear operator \( \mathcal{T}_R^{-1} AT_R \) and from Eqs. (30) and (78) its extended Berezin symbol is
\[ \mathfrak{B}_{(n,m)}^{(h)}(\mathcal{T}_R^{-1} AT_R)(w, z) = \mathfrak{B}_{(n,m)}^{(h)}(A)(L(g)w, L(g)z). \] (80)

Moreover, from Eqs. (26), (78) and the unitary of \( B_{S_n} \)
\[ Q_{m}^{(h)}(u, L(g)z) = \left< \mathcal{T}_R \Phi_{\rho(n,m)}^{(h)}(z), \Phi_{\rho(n,m)}^{(h)}(u) \right>_{S_n} = Q_{m}^{(h)}(L(g^{-1})u, z), \] (81)

where we have used that the orthogonal matrix associated to \( g^{-1} \) which makes Eq. (74) holds is \( R^{-1} \).

Thus we conclude from Eqs. (11) and (81)
\[ (f_1 \ast_m f_2)(L(g)z) = \int_{\mathbb{C}^m} f_1(L(g)z, w)f_2(L(g)z) \frac{|Q_{m}^{(h)}(w, L(g)z)|^2}{Q_{m}^{(h)}(L(g)z, L(g)z)} \frac{d\mu_{m}^{(h)}(w)}{Q_{m}^{(h)}(L(g)z, L(g)z)} \]
\[ = \int_{\mathbb{C}^m} f_1(L(g)z, L(g)u)f_2(L(g)u, L(g)z) \frac{|Q_{m}^{(h)}(u, z)|^2}{Q_{m}^{(h)}(u, z)} \frac{d\mu_{m}^{(h)}(u)}{Q_{m}^{(h)}(u, z)} = (f_1 \circ L(g)) \ast_m (f_2 \circ L(g))(z) \]

where we have made the change of variables \( u = L(g^{-1})w \), used the invariance of the Gaussian measure \( d\mu_{m}^{(h)} \) with respect to the action of \( L(g^{-1}) \) and Eqs. (29) and (30).

\[ \square \]

**Appendix A: Necessary results: Stationary phase method and partitioned matrix**

In this appendix, we mention two known results needed to prove Theorem [IV.4] We start with the stationary phase method which, for our purpose, we apply in the following way (see Theorem 7.7.5 of Ref. [7] for details):

**Theorem A.1 (Stationary Phase Method).** Let \( p \) and \( \beta \) be two smooth complex valued functions defined on \( \mathbb{R}^d \) with \( d \) any positive integer. Assume that \( \beta \) has compact support, \( \Im(p) \geq 0, \) \( p \) has a critical point at \( x_0 \) and \( p'(x) \neq 0 \) for \( x \neq x_0 \) (where \( p' \) denotes the gradient of \( p \)). Moreover, assume that \( \Im(p(x_0)) = 0 \) and \( \det(p''(x_0)) \neq 0 \) (where \( p''(x_0) \) denotes the Hessian matrix of \( p \) evaluated at the critical point \( x_0 \)). Then
\[ \int e^{ip(x)/\hbar} \beta(x) dx = e^{ip(x_0)/\hbar} \left[ \det \left( \frac{p''(x_0)}{2\pi i\hbar} \right) \right]^{-\frac{d}{2}} \left[ \sum_{\ell \leq k} \hbar^\ell M_\ell \beta(x_0) + O(\hbar^k) \right] \] (A1)

where \( dx \) denotes the Lebesgue measure on \( \mathbb{R}^d \) and
\[ M_\ell \beta(x_0) = \sum_{s=\ell}^{3\ell} \frac{s!(-2)^s}{\ell!(s-\ell)!} \left[ (-\langle p''(x_0) \rangle^{-1}) \hat{D} \cdot \hat{D} \right]^s \beta(p_{x_0}) \left| \frac{\partial^s}{\partial x_0} \right|_{x=x_0}, \]

with \( (p''(x_0))^{-1} \) the inverse of the matrix \( p''(x_0) \), \( \hat{D} \) the column vector of size \( d \) whose \( j \) entry is \( \partial / \partial x_j \), and
\[ p_{x_0}(x) = p(x) - p(x_0) - \frac{1}{2} p''(x_0) (x-x_0) \cdot (x-x_0). \]

The second result we mentioned is used in Theorem [IV.4] to avoid laborious calculations by obtaining both the determinant and the inverse matrix of a partitioned matrix.
Lemma A.2. Let $A, B, C, D$ be matrices of $\ell \times \ell$, $\ell \times s$, $s \times \ell$, $s \times s$ respectively, and $A$ invertible. Then

$$\det \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \det(A) \det(D - CA^{-1}B).$$  \hfill (A2)

Furthermore, if we assume that $D$ is nonsingular, $B^tB = 0$ and $A = I_\ell$ where $I_\ell$ denotes the identity matrix of size $\ell$. Then,

$$\begin{pmatrix} A & B \\ B^t & D \end{pmatrix}^{-1} = \begin{pmatrix} I_\ell + BD^{-1}B^t & BD^{-1} \\ -D^{-1}B^t & D^{-1} \end{pmatrix}^{-1}. $$  \hfill (A3)

Proof. The first part of this Lemma (Eq. (A2)) is a direct consequence from proposition 2.8.3 of Ref. [17]. To prove the second part of this Lemma (Eq. (A3)), we used the following analytical inversion formula for a partitioned matrix, provided that $A - BD^{-1}B^t$ is nonsingular (see proposition 2.8.7 of Ref. [17])

$$\begin{pmatrix} A & B \\ B^t & D \end{pmatrix}^{-1} = \begin{pmatrix} (A - BD^{-1}B^t)^{-1} & - (A - BD^{-1}B^t)^{-1}BD^{-1} \\ -D^{-1}B^t(A - BD^{-1}B^t)^{-1} & D^{-1} + D^{-1}B^t(A - BD^{-1}B^t)^{-1}BD^{-1} \end{pmatrix},$$

and the fact that $(I_\ell - BD^{-1}B^t)^{-1} = I_\ell + BD^{-1}B^t$.  \hfill $\square$

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