A NOTE ON EXTENDED RECURRENT LORENTZIAN MANIFOLDS

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ABSTRACT. Extended recurrent pseudo-Riemannian manifolds were introduced by Mileva Prvanović. We reconsider her work in the light of recent results and show that the manifold is conformally flat, and it is a space of quasi-constant curvature. We also show that an extended recurrent Lorentzian manifold, with time-like associated covector, is a perfect fluid Robertson-Walker space-time. We obtain the equation of state; in $n = 4$ and if the scalar curvature is zero, a model for incoherent radiation is obtained.

Dedicated to the memory of Dr. Mileva Prvanović

1. INTRODUCTION

In 1999 Mileva Prvanović [22] introduced the following differential structure on a pseudo-Riemannian manifold, that she named “extended recurrent manifold”:

\[
\nabla_i R_{jklm} = A_i R_{jklm} + (\beta - \psi) A_i G_{jklm}
\]

\[
+ \frac{\beta}{2} [A_j G_{iklm} + A_k G_{jilm} + A_l G_{jkim} + A_m G_{jkl}] 
\]

$A_i$ is a closed one-form named “associated covector”, $\beta$ and $\psi$ are non vanishing scalar functions with $\nabla_j \psi = \beta A_j$. $G_{jklm} = g_{mj} g_{kl} - g_{mk} g_{jl}$. She proved that the associated covector is a concircular vector: $\nabla_s A_r = f g_{rs} + h A_r A_s$ with scalar functions $f$ and $h$, and showed that the metric has the warped form

\[
ds^2 = (dx^1)^2 + e^\eta g^*_{ab} dx^a dx^b
\]

where $g^*_{ab}$ are functions only of $x^\gamma$ ($\gamma = 2, \ldots, n$) and $\eta$ is a scalar function of $x^1$. These properties will be reviewed in Section 2, where we also derive some new ones. In particular we show that an extended recurrent pseudo-Riemannian manifold is conformally flat, and it is a space of quasi constant curvature, according the definition by K. Yano and B.-Y. Chen [5]. In Section 3 we focus on extended recurrent Lorentzian manifolds (space-times). Based on our recent study of Generalized Robertson Walker manifolds, to which the present model eventually belongs, we show that an extended recurrent space-time with time-like associated covector is a perfect fluid Robertson-Walker spacetime. The barotropic equation of state is

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obtained; in the particular case of vanishing scalar curvature, in 4 dimensions, we
obtain a model for incoherent radiation.

Throughout the paper we adopt the convention \( R_{ij} = R_{imjn} \) and \( R = R_{nm} \) for
the Ricci tensor and the scalar curvature, and use the notation \( v^2 = v^m v_m \).

2. General properties of extended recurrent
pseudo-Riemannian manifolds

We review some basic properties of extended recurrent pseudo-Riemannian man-
ifolds exposed in [22]. Furthermore, we prove some new characterizations of such
manifolds.

Following the procedure in [22], by contracting (1) with \( g^{im} \) it is
\[
(3) \quad \nabla_i R_{kl} = A_i[R_{kl} - g_{kl}(n\beta - (n - 1)\psi)] - \frac{\beta}{2}(n - 2)(A_k g_{il} + A_l g_{ik}).
\]
Contracting again (3) with \( g^{kl} \) we obtain
\[
(4) \quad \nabla_i R = A_i[R - (n^2 + n - 2)\beta + n(n - 1)\psi].
\]
On the other hand, by the second Bianchi identity for the Riemann tensor it is
\[
A_i(R_{jklm} - \psi G_{jklm}) + A_j(R_{kilm} - \psi G_{kilm}) + A_k(R_{ijlm} - \psi G_{ijlm}) = 0.
\]
Contracting this with \( g^{im} \) it is
\[
(5) \quad R_{jklm} A^m = A_k[R_{jl} + \psi(n - 2)g_{jl}] - A_j[R_{kl} + \psi(n - 2)g_{kl}].
\]
and contracting (5) with \( g^{kl} \) we obtain
\[
(6) \quad R_{jm} A^m = \frac{1}{2}A_j[R + \psi(n - 2)(n - 1)].
\]
The components of the Weyl conformal curvature tensor are [19]:
\[
(7) \quad C_{ijkl}^m = R_{ijkl}^m + \frac{1}{n-2}(g_{jm} R_{kl} - g_{km} R_{jl} + R_{jim} g_{kl} - R_{km} g_{jl})
- \frac{g_{jm} g_{kl} - g_{km} g_{jl}}{(n-1)(n-2)} R
\]
By taking the covariant derivative of (7) and inserting (1) and (3) we infer that
\[
(8) \quad \nabla_i C_{jklm} = A_i C_{jklm}
\]
Now, (5), (6) are used to evaluate \( A_m C_{ijkl}^m \):
\[
(9) \quad A_m C_{ijkl}^m = \frac{n-3}{n-2} A_k \left[ A_j \left( R_{jl} - \frac{R - \psi(n-1)(n-2)}{2(n-1)} g_{jl} \right) \right.
- A_j \left( R_{kl} - \frac{R - \psi(n-1)(n-2)}{2(n-1)} g_{kl} \right) \right]
\]
Next, consider Lovelock’s identity [14] page 289):
\[
\nabla_i \nabla_m R_{ijkl}^m + \nabla_j \nabla_m R_{iklm} + \nabla_k \nabla_m R_{ijlm} = -R_{im} R_{jkl}^m - R_{jm} R_{iklm} - R_{km} R_{ijlm}^m
\]
The evaluation of \( \nabla_i \nabla_m R_{ijkl}^m + \nabla_j \nabla_m R_{iklm} + \nabla_k \nabla_m R_{ijlm}^m \) with the aid of (3)
gives zero, therefore it is \( R_{im} R_{jkl}^m + R_{jm} R_{iklm} + R_{km} R_{ijlm}^m = 0 \). By taking the
covariant derivative \( \nabla_s \) of the last expression and contracting with \( g^{is} \), after long
calculations, it is inferred that (provided \( \beta \neq 0 \) and \( n > 3 \))
\[
(10) \quad A_j \left[ R_{jl} - g_{jl} \frac{R - \psi(n-1)(n-2)}{2(n-1)} \right] = A_k \left[ R_{kl} - g_{kl} \frac{R - \psi(n-1)(n-2)}{2(n-1)} \right]
\]
From (10) and (9) immediately it is $\nabla_mC_{jkl}^m = A_mC_{jkl}^m = 0$.

The second Bianchi identity for the Weyl tensor is (see [1])

$$\nabla_i C_{jkl}^m + \nabla_j C_{ikl}^m + \nabla_k C_{ijl}^m = \frac{1}{n-3} \left[ \delta^m_p \nabla_p C_{kil}^p + \delta^m_k \nabla_p C_{ijl}^p \right] + a^m_i \nabla_p C_{jkl}^p + g_{kl} \nabla_p C_{ijl}^m + g_{il} \nabla_p C_{kjm} + g_{jl} \nabla_p C_{ikm}^p$$

For a conformally recurrent manifold it becomes

$$(11) \quad A_i C_{jkl} + A_j C_{kil} + A_k C_{ijl} = \frac{A^p}{n-3} \left[ g_{mj} C_{kil}^p + g_{mk} C_{ijp} + g_{ml} C_{ikj} + g_{jl} C_{ikm} \right] = 0$$

because $A_p C_{jkl}^p = 0$. Thus in our case it is $A_i C_{jkl} + A_j C_{kil} + A_k C_{ijl} = 0$ from which $A^2 C_{jkl} = 0$. Therefore, if $A^2 \neq 0$, the manifold is conformally flat: $C_{jkl} = 0$. Moreover if $A^2 \neq 0$ eq. (10) readily rewrites as:

$$(12) \quad 2(n-1) R_{kl} - g_{kl} (R - \psi (n-1)(n-2)) = \frac{A_i A_j}{A^2} (n-2) [R + \psi n(n-1)]$$

and shows that the space is quasi-Einstein (see for example [5, 10, 11, 12, 20]):

$$(13) \quad R_{kl} = a g_{kl} + b \frac{A_i A_j}{A^2}, \quad a = \frac{R - \psi (n-1)(n-2)}{2(n-1)}, \quad b = \frac{n - 2}{2(n-1)} [R + \psi n(n-1)]$$

Inserting this in (7) with $C_{jklm} = 0$ gives the Riemann tensor:

$$(14) \quad R_{ijkl} = \frac{b}{n-2} \left[ - g_{jm} \frac{A_i A_j}{A^2} + g_{km} \frac{A_j A_l}{A^2} - g_{kl} \frac{A_j A_m}{A^2} + g_{jl} \frac{A_k A_m}{A^2} \right] + \psi (g_{jm} g_{kl} - g_{jl} g_{km})$$

Eq. (14) characterizes the “manifolds of quasi constant curvature”, introduced by Chen and Yano in 1972 [5]. We thus proved the following

**Theorem 2.1.** An $n \geq 3$ dimensional extended recurrent pseudo-Riemannian manifold is conformally flat and is a space of quasi-constant curvature.

Note that the hypothesis $\nabla_j \psi = A_j \beta$ is not used in the proof of Theorem 2.1

As shown in [22], the covariant derivative $\nabla_s$ of (12) and the condition $\nabla_j \psi = A_j \beta$ imply that

$$\nabla_s A_r = f g_{rs} + \omega_s A_r$$

where $f = -\frac{(n-1)\beta}{R+n(n-1)\psi} A^2$, $\omega_s = h A_s$, $h = A^2 A^2 \nabla_s A_i + \frac{(n-1)\beta}{R+n(n-1)\psi}$. By showing $\nabla_s h = \mu A_s$ it follows that $\omega_s$ is closed (i.e. $A_j$ is a proper concircular vector). Based on the works [20, 30] by Yano, Prvanovic in [22] concluded that the metric has the warped form (2).

3. Extended recurrent space-times

In this section we consider extended recurrent Lorentzian manifolds (i.e. space-times) with a time-like associated covector ($A^2 < 0$). We prove it that it is a Robertson-Walker space-time. For this, we need a generalization of such spaces:
An $n \geq 3$ dimensional Lorentzian manifold is named generalized Robertson-Walker space-time (for short GRW) if the metric may take the shape:

\begin{equation}
   ds^2 = -(dx^1)^2 + q(x^1)^2 g_{\alpha \beta}^* (x^2, \ldots, x^n) dx^\alpha dx^\beta,
\end{equation}

A GRW space-time is thus the warped product $1 \times q^2 M^*$ \cite{2, 3, 25, 26} where $M^*$ is a $(n - 1)$-dimensional Riemannian manifold. If $M^*$ is a 3-dimensional Riemannian manifold of constant curvature, the space-time is called Robertson-Walker space-time. GRW space-times are thus a wide generalization of Robertson-Walker space-times on which standard cosmology is modelled and include the Einstein-de Sitter space-time, the Friedman cosmological models, the static Einstein space-times, and the de Sitter space-time. They are inhomogeneous space-times admitting an isotropic radiation (see Sánchez \cite{25}). We refer to the works by Romero et al. \cite{23, 24}, Sánchez \cite{25} and Gutiérrez and Olea \cite{13} for an exhaustive presentation of geometric and physical properties.

Recently, perfect fluids with the condition $\nabla_m C_{jklm} = 0$ were studied in \cite{15} and \cite{16}, where the authors showed that such spaces are GRW space-times. The following deep result was proved by Bang Yen Chen, in ref.\cite{4} (for similar results see also the works by Yano \cite{29, 30}, Prvanović \cite{21}, and the recent paper \cite{9}).

**Theorem 3.1** (Chen). Let $(M, g)$ an $n \geq 3$ dimensional Lorentzian manifold. The space-time is a generalized Robertson-Walker space-time if and only if it admits a time-like vector of the form $\nabla_k X_j = \rho g_{kj}$.

In the previous section we reviewed Prvanovic’s result that the associate covector is concircular, $\nabla_j A_k = f g_{jk} + \omega_j A_k$, with $\omega_j = h A_j$ being a closed one-form. In this case $\omega_j = \nabla_j \sigma$ for a suitable scalar function.

If the associated covector is time-like, i.e. $A^2 < 0$ (with Lorentzian signature), then it can be rescaled to a time-like vector $X_k = A_k e^{-\sigma}$ such that $\nabla_j X_k = \rho g_{kj}$. In fact it is $\nabla_j X_k = (\nabla_j A_k - \omega_j A_k) e^{-\sigma} = (f e^{-\sigma}) X_k$. By Chen’s theorem 3.1 the space is a GRW space-time (see \cite{15} \cite{16}).

Thus for $A^2 < 0$ Prvanović’s model \cite{1} is a quasi-Einstein GRW space-time with $C_{jklm} = 0$. It is well known (see \cite{7}) that in this case the fiber is a space of constant curvature and the GRW space-time reduces to an ordinary Robertson-Walker model. Moreover in the region $A^2 < 0$, on defining $u_k = A_k / \sqrt{-A^2}$, it is $u^2 = -1$ and the Ricci tensor \cite{13} becomes $R_{kl} = a g_{kl} - b u_k u_l$. With this form of the Ricci tensor, a Lorentzian manifold is named perfect fluid space-time \cite{14}.

**Theorem 3.2.** An $n > 3$ dimensional extended recurrent Lorentzian manifold with $A^2 < 0$ is a Robertson-Walker space-time.

**Remark 3.3.** In \cite{17}, we proved that for a GRW space-time the condition $\nabla_m C_{jklm} = 0$ is equivalent to have $R_{kl} = a g_{kl} + b X_k X_l$, where $X_j$ is the concircular vector of Chen’s theorem. Prvanović’s model matches these conditions.

Some physical consequences are now outlined. Let $(M, g)$ be an $n$-dimensional Lorentzian manifold equipped with Einstein’s field equations without cosmological constant,

\begin{equation}
   R_{kl} - \frac{1}{2} R g_{kl} = \kappa T_{kl}
\end{equation}

$\kappa = 8\pi G$ is Einstein’s gravitational constant (in units $c = 1$) and $T_{kl}$ is the stress-energy tensor describing the matter content of the space-time (see for example
Eq. (16) is used to evaluate $T_{kl}$ obtaining:

$$\kappa T_{kl} = -\frac{n-2}{2(n-1)}[R + \psi(n-1)] (g_{kl} - u_k u_l)$$

We recognize a perfect fluid stress-energy tensor $T_{kl} = (p + \mu) u_k u_l + p g_{kl}$, being $p$ the isotropic pressure, $\mu$ the energy density and $u_j$ the fluid flow velocity. It is

$$\kappa p = -\frac{n-2}{2(n-1)}[R + \psi(n-1)], \quad \kappa \mu = -\frac{1}{2} \psi(n-1)(n-2)$$

One reads that the (non constant) function $\psi$ controls the energy density of the perfect fluid (then it must be negative). An equation of state can be written:

$$p = \frac{\mu}{n-1} - \frac{n-2}{2(n-1)} \frac{R}{\kappa}$$

In $n = 4$ dimensions with the particular choice $R = 0$, we have a model for incoherent radiation: $p = \mu/3$ [27] (a superposition of waves of a massless field with random propagation directions).

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