DEFORMATIONS OF RIEMANN SURFACES

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Abstract. We prove that every Riemann surface not isomorphic to $\mathbb{P}^1$ admits an infinitesimal deformation of its complex structure. The proof is based on an investigation of the length of geodesics for the Kobayashi/Poincaré metric.

1. Summary

The purpose of this article is to prove the following statement:

On every orientable surface except the 2-sphere $S^2$ there exists at least two non-isomorphic complex structures.

For compact Riemann surfaces there is a classical theory of moduli spaces which includes a precise description of deformations of these compact Riemann surfaces. This theory easily implies the above statement for compact Riemann surfaces. This theory extends to complex algebraic curves, i.e., to Riemann surfaces which can be compactified by adding finitely many points. However, this theory does not extend to arbitrary non-compact Riemann surfaces whose topology can be quite complicated, e.g., the fundamental group may be not finitely generated. Thus to determine which Riemann surfaces admit deformations, we need a different approach.

The key idea in our approach is the following: Most Riemann surfaces are hyperbolic. Hence there is a canonical way to measure the length of a (real) curve.

Given a hyperbolic Riemann surface $X$, we look for a closed curve $\gamma : S^1 \to X$ for which there exists a bound $c > 0$ such that every curve homotopic to $\gamma$ has at least length $c$ where the length is calculated with respect to the Kobayashi pseudometric. Then we deform the complex structure on some neighbourhood of the image of $\gamma$ such that the length of $\gamma$ decreases below $c$. This yields a complex structure non-isomorphic to the original one.

In this way we obtain the following result.

Theorem 1. Let $(X, J)$ be a Riemann surface which is not biholomorphic to $\mathbb{P}^1$. Then there exists a continuous family of complex structures $J_t$ parametrized by $t \in [0, 1]$ such that $J_0 = J$ and such that $(X, J_1)$ is not biholomorphic to $(X, J)$.

From this we deduce a related result on discrete subgroups in $PSL_2(\mathbb{R})$:

Theorem 2. Let $F$ be free group (possibly not finitely generated) and let $\rho_0 : F \to G = PSL_2(\mathbb{R})$ be a group homomorphism which embeds $F$ into $G$ as a discrete subgroup.

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Then there exists a continuous family of group homomorphisms $\rho_t : F \to G$ ($t \in [0, 1]$) such that each $\rho_t$ embeds $F$ as a discrete subgroup in $G$ and such that $\rho_0$ is not conjugate to $\rho_1$. (I.e. there is no $g \in G$ such that $\rho_1(\gamma) = g \cdot \rho_0(\gamma) \cdot g^{-1}$ for all $\gamma \in F$.)

2. Classical Theory

Since the 19th century it is known that there is a moduli space for compact Riemann surfaces of a given genus $g$, and that the (complex) dimension of this moduli space equals $3g - 3$ for $g \geq 2$ and equals $g$ for $g \leq 1$.

This of course implies our statement for the special case of compact Riemann surfaces.

There is a generalization of this theory to algebraic complex curves, i.e., Riemann surfaces which can be compactified by adding finitely many points.

If $g$ denotes the genus of the compactification $\bar{X}$ of such a complex algebraic curve $X$ and if $d$ denotes the cardinality of the set $\bar{X} \setminus X$, then the respective moduli space has dimension $3g - 3 + d$ if $g \geq 2$ and every $d \in \mathbb{N} \cup \{0\}$. For $g = 1$ and $d \geq 1$ the dimension of the moduli space is $d$. For $g = 0$ it equals $\max\{0, d - 3\}$.

As a consequence, the only complex algebraic curves which do not admit any non-trivial deformation within the category of algebraic curves are $\mathbb{P}^1$, $\mathbb{P}^1 \setminus \{\infty\} = \mathbb{C}$, $\mathbb{C}^*$ and $\mathbb{P}^1 \setminus \{0, 1, \infty\}$. (However, as we will see below, the complex structure on $\mathbb{C}$ may be deformed to that of the unit disc. Hence $\mathbb{C}$ does admit a deformation in the category of Riemann surface although it does not within the category of complex algebraic curves. Similarly for $\mathbb{C}^*$ and $\mathbb{P}^1 \setminus \{0, 1, \infty\}$.)

For arbitrary non-compact Riemann surfaces there is no reasonable theory of moduli spaces.

Already the situation for complex structures on the orientable surface $\mathbb{R} \times S^1$ is quite intricate:

Every complex structure on $\mathbb{R} \times S^1$ defines a Riemann surface biholomorphic to

$$A(r, s) = \{ z \in \mathbb{C} : r < |z| < s \}$$

with $0 \leq r < s \leq +\infty$. The ratio $s/r$ is an invariant of the complex structure. In this sense the map $(r, s) \mapsto r/s \in [0, 1]$ yields almost a moduli space. However, for $r/s = 0$ (i.e. $r = 0$ or $s = \infty$) there are two inequivalent complex structures: that from $\mathbb{C}^* = A(0, \infty)$ and that from $\Delta^* = A(0, 1)$.

Thus one may say that the complex structures on $\mathbb{R} \times S^1$ are parametrized by a real-one dimensional non-Hausdorff space: $[0, 1]$ with the zero replaced by a double point.

For arbitrary non-compact Riemann surfaces (in particular those whose fundamental group is not finitely generated), there is no reasonable holomorphic classification theory.

(There does exists a topological classification of real surfaces though, cf. theorem of Kerékjártó., see [Ric63])

3. The special cases of $\mathbb{C}$ and $\Delta$

Here we consider some special cases where (as we will see later) our general methods can not be applied.

First we show that the complex plane $\mathbb{C}$ can be deformed to the unit disc $\Delta$ and vice versa.

For this purpose we consider some auxiliary functions.
We define
\[ \rho_t(r) = \left( r + t \arctan\left( \frac{\pi}{2} r \right) \right), \]
\[ \rho^*_t(r) = \arctan\left( \frac{\pi}{2} r \left( \frac{1}{1+t} \right) \right) \]
and
\[ \phi_t : z \mapsto z \frac{\rho_t(|z|)}{|z|}, \quad \phi^*_t : z \mapsto z \frac{\rho^*_t(|z|)}{|z|}. \]

We note that for \( t \neq 0 \), the unit disk \( \Delta \) is mapped bijectively onto \( \mathbb{C} \) by \( \phi_t \) and bijectively onto a bounded disc with some radius \( s(t) \in \mathbb{R}_+ \) by \( \phi^*_t \). Furthermore \( \phi_0 \) resp. \( \phi^*_0 \) maps the unit disk bijectively onto \( \Delta \) resp. \( \mathbb{C} \).

Therefore pulling back the standard complex structure on \( \mathbb{C} \) via \( \phi_t \) resp. \( \phi^*_t \) to the unit disc yields a family of complex structures \( J_t \) resp. \( J^*_t \) on \( \Delta \) such that \( (\Delta, J_0) \) and \( (\Delta, J^*_0) \) \( (t \neq 0) \) are biholomorphic to the unit disc while \( (\Delta, J_t) \) \( (t \neq 0) \) are biholomorphic to \( \mathbb{C} \).

Hence both \( \Delta \) and \( \mathbb{C} \) admit non-trivial infinitesimal deformations of the complex structure.

Since \( \mathbb{C}^* = \mathbb{C} \setminus \{0\} \) and \( \Delta^* = \Delta \setminus \{0\} \) just by restricting these families of complex structures on \( \mathbb{C} \) resp. \( \Delta \) to the complement of \( \{0\} \) we see that \( \mathbb{C}^* \) and \( \Delta^* \) can be deformed as well.

Similarly we obtain that the standard complex structure on \( \mathbb{P}_1 \setminus \{0, 1, \infty\} \simeq \mathbb{C} \setminus \{0, \frac{1}{2}\} \) can be deformed to \( \Delta \setminus \{0, \frac{1}{2}\} \).

Thus we obtain:

**Proposition 1.** Let \((X, J)\) be a Riemann surface which is biholomorphic to \(\mathbb{C}, \mathbb{C}^*, \mathbb{P}_1 \setminus \{0, 1, \infty\}, \Delta\) or \(\Delta^*\).

Then there exists a continuous family of complex structures \(J_t\) parametrized by \(t \in [0, 1]\) such that \(J_0 = J\) and such that \((X, J_1)\) is not biholomorphic to \((X, J)\).

### 4. Ahlfors Schwarz Lemma

We will use a classical result which is known as Ahlfors-Schwarz lemma. The following states a version of the Ahlfors Schwarz lemma in the form most suitable for us.

**Theorem 3.** Let \(X\) be a Riemannian surface with hermitian metric \(h\). Assume that there is a constant \(C > 0\) such that the Gaussian curvature of \(h\) is bounded from above by \(-C\).

Then the inequality
\[ \|Df_p\| \leq \frac{1}{C} \]
holds for every holomorphic map \(f : \Delta \to X\) and every \(p \in \Delta\), where \(\|\|\) denotes the operator norm with respect to the hermitian metric \(h\) on \(X\) and the Poincaré metric on the unit disc \(\Delta\).

For a proof see e.g. Kobayashi.

### 5. Riemann Surfaces of Finite Type

**Definition 1.** A Riemann surface is called “of finite type” if its fundamental group is finitely generated.
Evidently every complex algebraic curve is a Riemann surface of finite type. But there are also non-algebraic Riemann surfaces of finite type, for example the unit disc and the annuli \( A(r, 1) = \{ z \in \mathbb{C} : r < |z| < 1 \} \) \( r \in [0, 1) \).

The following well-known result is a key tool.

**Theorem 4.** Every Riemann surface \( X \) of finite type admits an open embedding \( X \hookrightarrow \hat{X} \) into a compact Riemann surface \( \hat{X} \) such that every connected component of the complement of \( X \) in \( \hat{X} \) is either a point or isomorphic to a closed disc.

**Sketch of proof:**

Let \( \rho \) be a subharmonic Morse function on \( X \). Because \( \pi_1 \) is finitely generated, there are only finitely many critical points. Therefore \( X \) is diffeomorphic to \( X_c = \{ x \in X : \rho < c \} \) for some \( c > 0 \) and \( X \setminus \bar{X}_c \) is homotopic to \( \partial X_c \). Now \( \partial X_c \) is a compact real one-dimensional manifold. It follows that each connected component of \( X \setminus \bar{X}_c \) has an infinite cyclic group as fundamental group. Using the uniformization theorem, it follows that each connected component of \( X \setminus \bar{X}_c \) is biholomorphic to \( A(r, 1) \) for some \( 1 > r > 0 \). Using the natural embedding \( A(r, 1) \hookrightarrow \Delta \) we obtain the compactification \( X \subset \bar{X} \).

We introduce some notion.

**Definition 2.** Let \( X \) be a Riemann surface. A “standard compactification” of \( X \) is an open holomorphic embedding \( i : X \to Y \) into a compact Riemann surface \( Y \) such that for every connected component \( W \) of \( Y \setminus i(X) \) there exists an open neighborhood \( V \) of \( W \) in \( Y \), a biholomorphic map \( \phi : V \to \Delta \) and a real number \( 0 \leq r < 1 \) such that \( \phi(V) = \phi(V \setminus i(X)) = \{ z : |z| \leq r \} \).

The above result guarantees that every Riemann surface of finite type admits a standard compactification.

### 6. Definition of the Hyperbolic Length Spectrum

We recall the definition of the (infinitesimal) Kobayashi–Royden pseudometric \( F_X \): For a complex space \( X \) one defines \( (F_X)_p : T_p X \to \mathbb{R} \) as

\[
(F_X)_p(v) = \inf \{ |w| : \phi_* w = v, \exists \phi : (\Delta, 0) \to (X, p) \text{ holo.} \}
\]

The uniformization theorem implies that for a Riemann surface \( X \) either \( F_X \) vanishes identically or \( F_X \) is a complete Kähler metric of constant negative curvature.

Then one can define the hyperbolic length \( L(\gamma) \) of a differentiable path \( \gamma : S^1 = \mathbb{R}/\mathbb{Z} \to X \) as

\[
L(\gamma) = \int_0^1 F_X(\gamma'(t))dt.
\]

Given a manifold \( X \) (here always a Riemann surface), a *simple path* is injective and immersive differentiable map \( \gamma : S^1 \to X \).

Given a hyperbolic Riemann surface \( X \), let \( \Gamma_X \) be the set of free homotopy classes of simple paths. This can be regarded as a subset of the quotient space of the fundamental group \( \pi_1(X) \) by the equivalence relation given by inner automorphisms (i.e. conjugation) of \( \pi_1(X) \).

For every \( \gamma \in \Gamma_X \), we define a “stable hyperbolic length” \( \Lambda(\gamma) \) as the infimum of the hyperbolic length of all simple paths representing \( \gamma \). Let

\[
\Sigma_X = \{ \Lambda(\gamma) : \gamma \in \Gamma_X \}
\]
Σ_X is a countable subset of \( \mathbb{R}_+^+ \).

Obviously, Σ_X is an invariant of the complex structure on X. We will use this invariant to distinguish non-equivalent complex structure. For this purpose we will show that Σ_X is non-trivial (i.e. Σ_X ≠ \{0\}) for almost all hyperbolic Riemann surfaces. Furthermore we will show that there is always a family of complex structures with changing Σ_X as soon as Σ_X ≠ \{0\}.

More precisely:

Σ_X ≠ \{0\} unless X is isomorphic to \( \Delta = \{ z : |z| < 1 \} \), \( \Delta \setminus \{0\} \) or \( \mathbb{P}_1 \setminus \{0, 1, \infty\} \).

7. Non-triviality of the length spectrum

Lemma 1. Let \( E \subset \Delta \) be a compact subset and let \( X = \Delta \setminus E \).

If E contains at least two points, there exists a simple path on X which cannot be deformed to a simple path of arbitrary small length.

Proof. Let \( p, q \in E \) with \( p \neq q \). Applying a suitable automorphism of the unit disc, we may assume \( p = 0 \). Then every simple path in X surrounding both \( p \) and \( q \) has a euclidean length of at least 2|q|. Now the euclidean distance is an lower bound for the Poincaré distance on the unit disc \( \Delta \) which in turn is an lower bound for the hyperbolic distance on \( X \subset \Delta \). Therefore every simple curve in X surrounding all of \( E \) has hyperbolic length at least 2|q|.

□

Lemma 2. Let \( X \) be a relatively compact smoothly bounded domain in a hyperbolic Riemann surface \( Y \).

Then there exists a number \( c > 0 \) such that every closed curve of hyperbolic length less than \( c \) is homotopic to a constant map.

Proof. Because \( X \) is relatively compact in \( Y \), we may choose a number \( r > 0 \) such that for all \( \rho \) \( < r \) and all \( x \in X \) the set \( \{ y \in Y : d_Y(x, y) < \rho \} \) is homeomorphic to an open ball. (Remember, that the infinitesimal Kobayashi pseudometric is actually a Riemannian metric for dim \( Y \) = 1. Hence we may use the exponential map for this argument.)

Because \( X \) is furthermore smoothly bounded, there is a number \( s > 0 \) such that \( W = \{ y \in: d_Y(y, X) < s \} \) is homotopically equivalent to \( X \).

Together, this two assertions imply that every closed curve whose length with respect to the hyperbolic metric of \( Y \) is less than min\{s, r\} must be homotopic to zero.

Since the inclusion map \( X \to Y \) is distance decreasing for the respective Kobayashi metrics, the desired assertion follows.

□

Lemma 3. Let \( X \) be a hyperbolic Riemann surface and let \( \Omega \) be a relatively compact open domain in \( X \) with smooth boundary. Assume \( \Omega \neq X \). Let \( \gamma : S^1 \to X \) be a simple path which can be deformed (in \( X \)) to a curve of arbitrarily small hyperbolic length. Then \( \gamma \) can be deformed to a curve \( \tilde{\gamma} \) with \( \tilde{\gamma}(S^1) \subset X \setminus \overline{\Omega} \).

Proof. The assertion is trivially true if \( \gamma \) is homotopic to the constant map. Hence we may and do assume that \( \gamma \) is not homotopic to the constant map. For \( \epsilon > 0 \) let \( \Omega_\epsilon = \{ x \in \Omega : d_X(x, \partial \Omega) > \epsilon \} \). Since \( \Omega \) is relatively compact with smooth boundary, for every sufficiently small \( \epsilon \) there exists a diffeomorphism \( \phi \) of \( X \) homotopic to the identity map with \( \phi(\Omega) = \Omega_\epsilon \). Next we choose a positive number \( \delta > 0 \), such that \( B_\delta(p) = \{ x \in X : d_X(x, p) < \delta \} \) is contained in a contractible subset of \( \Omega \) for every \( p \in \Omega_\epsilon \). This is possible, because \( \Omega_\epsilon \) is relatively compact. If \( \gamma : S^1 \to X \)
Lemma 4. Let \( \Omega \) be a Riemann surface with a finite subset \( E \) and let \( \rho : X \to X_1 \) be the quotient space obtained by collapsing \( E \) to a one point.

Then:

1. The natural group homomorphism \( \rho_* : \pi_1(X) \to \pi_1(X_1) \) is injective.
2. If \( \gamma : [0,1] \to X \) is a continuous path in \( X \) such that \( \gamma(0) \) and \( \gamma(1) \) are different points in \( E \), then \( \rho \circ \gamma \) defines a non-trivial element in \( \pi_1(X_1) \).

Proof. We prove the first statement for possibly singular Riemann surfaces. Then by induction it suffices to consider the case where \( E \) consists of two elements, say \( E = \{a,b\} \). Define \( Y_0 = X \times \mathbb{Z} \) and let \( Y \) denote the quotient obtained by identifying \((b,n)\) with \((a,n + 1)\) for each \( n \in \mathbb{N} \). Now we have a free and properly discontinuous \( \mathbb{Z} \)-action on \( Y \) induced by \( m : (x,n) \mapsto (x,n + m) \). The natural projection \( \pi : Y \to X_1 \) is an unramified Galois covering for this \( \mathbb{Z} \)-action. Therefore \( \pi_1(Y) \to \pi_1(X_1) \) is injective. We can embedd \( X \) into \( Y \) via \( i : x \mapsto [(x,0)] \). Then the natural projection \( \rho \) from \( X \) to \( X_1 \) is given as \( \rho = \pi \circ i \). Now \( \tau : Y \to X \) induced by the map \( \tau_0 : \Omega \to \Omega \) defined as

\[
\tau_0 : (x,n) \mapsto \begin{cases} 
a & \text{if } n < 0 \\
x & \text{if } n = 0 \\
b & \text{if } n > 0 
\end{cases}
\]

has the property \( \text{id}_X = \tau \circ i \). It follows that \( i_* : \pi_1(X) \to \pi_1(Y) \) is injective. Hence \( \rho_* = (\pi \circ i)_* : \pi_1(X) \to \pi_1(X_1) \) is injective as well. \qed

Proposition 2. Let \( \Omega \subset X \) be a smoothly bounded relatively compact domain in a Riemann surface and let \( \Omega \subset \overline{\Omega} \) be a standard compactification (in the sense of definition [3]).

Assume that the genus of \( \overline{\Omega} \) is at least one.

Then there exists a simple curve \( \gamma : S^1 \to X \) and a number \( c > 0 \) such that the hyperbolic length of every curve homotopic to \( \gamma \) is at least \( c \).

Proof. By assumption \( \pi_1(\overline{\Omega}) \neq \{e\} \). Since \( \overline{\Omega} \setminus \Omega \) can be contracted to a finite set, it is clear that every element in \( \gamma_0 \in \pi_1(\overline{\Omega}) \setminus \{e\} \) can be represented by a curve with image inside \( \Omega \) and that there exists a simple curve \( \gamma : S^1 \to \Omega \) which is not homotopic to a constant map as a map from \( S^1 \) to \( \overline{\Omega} \).

Now let \( \Omega' \) be the one-point-compactification of \( \Omega \). Due to lemma [4] we know that \( \gamma \) gives us a non-trivial element in \( \pi_1(\Omega') \). Since we have a natural continuous map from \( X \) to \( \Omega' = \Omega \cup \{\infty\} \) given by

\[
x \mapsto \begin{cases} 
x & \text{if } x \in \Omega \\
\infty & \text{if } x \notin \Omega \n\end{cases},
\]

it follows that \( \gamma \) can not be deformed in \( X \) to a curve whose image is contained in \( X \setminus \Omega \). Now the assertion follows from lemma [3]. \qed
8. The case of genus 0

Lemma 5. Let $\Omega_n$ be an increasing sequence of relatively compact smoothly bounded domains in a Riemann surface $X$ such that $\cup_n \Omega_n = X$.

Assume that every $\Omega_n$ admits a standard compactification $\hat{\Omega}_n$ of genus $g = 0$.

Then for every $n$ the inclusion $\partial \Omega_n \to X \setminus \Omega_n$ induces a bijective correspondence $\pi_0(\partial \Omega_n) \simeq \pi_0(X \setminus \Omega_n)$ between the sets of connected components.

Proof. We have to show: If $p,q$ are points in two distinct connected components of $\partial \Omega_n$, then they can not be connected by a path inside $X \setminus \Omega_n$. We assume the contrary. Then $p,q$ can be connected by a path inside $\Omega_m \setminus \Omega_n$ for some sufficiently large $m > n$. We concatenate this path with a path connecting $q$ and $p$ inside $\Omega_n$ and obtain an element $\gamma \in \pi_1(\Omega_m)$. Due to lemma 4 (ii) $\gamma$ projects onto a non-trivial element in $\pi_1(\Omega'_n)$ where $\Omega'_n$ denotes the one-point-compactification of $\Omega_n$. Since the identity map of $\Omega_n$ extends in an obvious way to a continuous map from $\hat{\Omega}_m$ to $\hat{\Omega}'_n = \Omega_n \cup \{\infty\}$ by mapping every point in $\hat{\Omega}_m \setminus \Omega_n$ to $\infty$, it follows that $\gamma$ defines a closed curve in $\hat{\Omega}_m$ which is not homotopic to a constant map. But this contradicts our assumption that each $\hat{\Omega}_m$ has genus zero. \hfill \Box

Corollary 1. Under the above assumptions, let $\gamma$ be a closed curve in $\Omega_n$ which can be deformed into a closed curve in $X \setminus \Omega_n$.

Then $\gamma$ can be deformed to a closed curve in $\partial \Omega_n$.

Proof. Let $W$ be a connected component of $X \setminus \Omega_n$ such that $\gamma$ can be deformed to a closed curve inside $W$. The above proposition implies that $W \cap \Omega_n$ is a connected component of $\partial \Omega_n$. Therefore the Seifert-van-Kampen theorem may be applied to $\hat{\Omega}_n \cup W$. It follows that $\gamma$ can be deformed to a closed curve in $W \cap \hat{\Omega}_n \subset \partial \Omega_n$. \hfill \Box

Corollary 2. Under the assumptions of the proposition, assume that every simple path in $X$ may be deformed to a closed curve of arbitrarily small hyperbolic length.

Then for every $n$, the natural linear map $i_* : H_1(\Omega_n, \mathbb{R}) \to H_1(X, \mathbb{R})$ induced by the inclusion map $\Omega_n \subset X$ has rank at most 2.

Proof. Let $r$ be the cardinality of the set of connected components of $\partial \Omega_n$. Then $\Omega_n$ is homeomorphic to $\mathbb{P}^1 \setminus \{1,2,\ldots,r\}$. Let $\gamma_s$ for $s \in I = \{1,\ldots,r\}$ denote a small cycle around $s$. Then $H_1(\Omega_n, \mathbb{Z})$ is the $\mathbb{Z}$-module generated by the $\gamma_s$ subject to the relation $\sum_s \gamma_s = 0$. Since every permutation of $\{1,2,\ldots,r\}$ extends to a homeomorphism of $\mathbb{P}^1$, it is clearly that for every subset $B \subset \{1,\ldots,r\}$ the homology class $\sum_{s \in B} [\gamma_s] \in H_1(\Omega_n, \mathbb{Z})$ can be realized by a simple curve in $\Omega_n$.

Next, consider injections $\Omega_n \subset \Omega_m$ for $m > n$. To each $s \in I$ there corresponds a connected component $a_s \in \pi_0(\partial \Omega_n) \simeq \pi_0(X \setminus \Omega_n)$. (for the latter equivalence see lemma 5).

Given an index $s \in I$, the cycle $\gamma_s$ maps to a non-zero homology class in $H_1(\Omega_n, \mathbb{Z})$ if and only if there is a connected component of $\partial \Omega_m$ included in the connected component of $X \setminus \Omega_n$ corresponding to $s$.

Now let $I_0$ denote the set of all $s \in I$ such that for all $m > n$ there is a connected component of $X \setminus \Omega_m$ included in the connected component of $X \setminus \Omega_n$ corresponding to $s$.

Then the induced map $i_* : H_1(\Omega_n; \mathbb{Z}) \to H_1(X; \mathbb{Z})$ maps $\gamma_s$ to zero if and only if $s \notin I_0$ and moreover for the elements $s \in I_0$ the homology classes of $\gamma_s$ in $H_1(X, \mathbb{Z})$ are non-zero and subject only to the relation $\sum_{s \in I} [\gamma_s] = 0$. As a consequence, if
$I_0$ contains at least four elements, say 1, 2, 3, 4, we may take two of them, say 1, 2 and there exists a simple path $\gamma \in \Omega_n$ such that the homology class of $[\gamma]$ equals $[\gamma_1] + [\gamma_2]$.

Thus in this case there exists a simple path in $\Omega_n$ which in $X$ is not homologous to any multiple of one of the $\gamma_i$ ($i \in I$), i.e., not homologous to any closed curve inside $\partial \Omega_n$.

Then by corollary [1], we found a simple path in $X$ which cannot be deformed to a curve of arbitrarily small hyperbolic length in $X$.

Theorem 5. Let $X$ be a hyperbolic Riemann surface. Assume that at least one of the following conditions is fulfilled:

(1) There exists a smoothly bounded relative compact domain $\Omega$ whose standard compactification has genus at least 1.

(2) $b_1 = \dim H_1(X, \mathbb{R}) \geq 3$.

(3) There exists an open embedding of $X$ as a relatively compact domain into another hyperbolic Riemann surface.

Then there exists a simple path which cannot be deformed to a path of arbitrarily small hyperbolic length.

Proof. (1) see proposition [2].

(2) Let $\Omega_n$ be an increasing sequence of relatively compact smoothly bounded domains in $X$ exhausting all of $X$. Then $H_1(X, \mathbb{R}) = \lim H_1(\Omega_n, \mathbb{R})$. Hence $b_1(X) \geq 3$ implies that there exists a number $n$ such that $i_* : H_1(\Omega_n, \mathbb{R}) \to H_1(X, \mathbb{R})$ has at least rank three. Furthermore, thanks to $(i)$ we may now assume that all the $\Omega_n$ have a standard compactification of genus 0. Thus the assertion follows from corollary [2].

(3) This is lemma [2].

Corollary 3. Let $X$ be a hyperbolic Riemann surface. Assume that every simple curve in $X$ can be deformed to a curve of arbitrarily small hyperbolic length.

Then $X$ is biholomorphic to one of the following:

(1) the unit disc $\Delta = \{z \in \mathbb{C} : |z| < 1\}$,

(2) the punctured unit disc $\Delta^* = \{z \in \mathbb{C} : 0 < |z| < 1\}$,

(3) $\mathbb{P}_1 \setminus \{0, 1, \infty\}$,

Proof. By the above theorem [3](ii) we have $b_1(X) \leq 2 < +\infty$. Hence $X$ is of finite type and admits a “standard compactification” $X \hookrightarrow \hat{X}$. Furthermore $\hat{X} \cong \mathbb{P}_1$ (Theorem [3](i)). By the definition of a standard compactification each connected component of $\hat{X} \setminus X$ is a point or a closed disc. There are at most three connected components of $\hat{X} \setminus X$ because of $b_1(\hat{X}) \leq 2$. If there are only isolated points and no closed discs, then there are three points in $\hat{X} \setminus X$ (otherwise $X$ would not be hyperbolic) and $X \cong \mathbb{P}_1 \setminus \{0, 1, \infty\}$ (because $PSL_2(\mathbb{C})$ acts triply transitive on $\mathbb{P}_1$). This leaves the case where one of the connected components is a closed disc. Let $B$ be this component. Then $\hat{X} \setminus B$ is isomorphic to the unit disc. Thus $X$ is isomorphic to the complement of a compact set $K$ (possibly empty) in the unit disc. Due to lemma [1] the set $K$ contains at most one point. Therefore either $X \cong \Delta$ or $X \cong \Delta^*$. □
Lemma 6. Let \( \gamma \) be a simple curve in a Riemann surface \((X, J)\). Let \( \epsilon > 0 \).

Then there exists a smooth family of complex structures \( J_t \) \( (t \in [0,1]) \) such that

1. the hyperbolic length of \( \gamma \) with respect to the complex structure \( J_1 \) is less than \( \epsilon \).
2. \( J_0 \) equals the given complex structure \( J \).
3. There is a compact subset \( K \) of \( X \) such that \( (J_t)_x = (J_s)_x \) for all \( x \notin K \), \( s, t \in [0,1] \), i.e., all the complex structures \( J_t \) agree outside \( K \).

Proof. We choose real constants \( r > r' > 1 \). Then we choose an open neighbourhood \( V \) of \( \gamma \) which admits a diffeomorphism \( \phi \) to the annulus \( A(1/r, r) \) taking \( \gamma \) to the unit circle \( S^1 = \{ z : |z| = 1 \} \). Define \( W = \phi^{-1}(A(1/r', r')) \). Choose a smooth real function \( \chi : X \rightarrow [0,1] \) such that

1. \( \chi|_W = 1 \).
2. \( \chi \) is constant zero in some open neighbourhood of \( X \setminus V \).

Next we choose a hermitian metric \( h \) on \( X \) and a hermitian metric \( \rho \) on \( A(r, s) \). We define \( H_t = (1-t\chi)h + t\chi \phi^*\rho \). This is a Riemannian metric on \( X \) which determines a complex structure \( J_t \) on \( X \) with \( J_0 \) being the original complex structure on \( X \). By construction \( (W, J_1) \) is biholomorphic to \( A(1/r', r') \). Since the choice of \( r > r' > 1 \) was arbitrary, the value of \( r' \) may be as large as desired. Then the hyperbolic length of \( \gamma \) with respect to \((W, J_1)\) becomes as small as desired (corollary \( \Box \)). Since the injection of \((W, J_1)\) into \((X, J_1)\) is distance-decreasing, the claimed assertion follows. \( \Box \)

10. Some preparation

Lemma 8. Let \( M \) be a (real) differentiable manifold and let \( f_n : M \rightarrow M \) be a sequence of smooth self-maps \( C^1 \)-converging to the identity map.

Let \( K \subset K' \) be compact subsets of \( M \). Assume that \( K \) is connected and contained in the interior of \( K' \).
Then there exists a number $N$ such that the following assertions hold for all $n \geq N$:

1. The restricted map $f_n|_{K'}$ is injective.
2. The image $f_n(K')$ contains $K$.

Proof. Assume that there are arbitrary large numbers $n$ for which $f_n|_{K'}$ is not injective. Then there exists sequences $p_n, q_n \in K$ such that (after passing to a suitably chosen subsequence) $p_n \neq q_n$, but $f_n(p_n) = f_n(q_n)$. We may assume that $p_n$ and $q_n$ are both convergent. Then there is a point $p \in K'$ such that

$$\lim p_n = \lim f_n(p_n) = \lim f_n(q_n) = \lim q_n.$$

Locally, i.e., in a neighbourhood of $p$, we may embed everything in the euclidean space. Then the above implies that there are vectors of unit length $v_n$ and points $\xi_n$ on the segment between $p_n$ and $q_n$ such that

$$D_{v_n}(f_n)\xi_n = \lim_{t \to 0} \frac{f_n(\xi_n + tv_n) - f_n(\xi_n)}{t} = 0$$

Taking the limit, we obtain that $\lim f_n$ has at $p$ a zero directional derivative in some direction. This contradicts the assumption that $f_n \to id_M$ in $C^1$-topology.

Next we deal with the second assertion. By enlarging $K$ (if necessary) we may assume that $K$ has non-empty interior. Let $q \in \operatorname{int}(K)$. We fix some metric on $X$ defining the topology. Choose $\epsilon > 0$ such that $\epsilon$ is smaller that the distance between $K$ and $X \setminus K'$ and furthermore smaller than the distance between $q$ and $\partial K$.

Then we can choose a number $N$ such that for all $n \geq N$ we have

$$\sup_{K'} d(x, f_n(x)) < \epsilon \quad \forall n \geq N, \forall x \in K'$$

In addition, we may and do require that $(Df_n)_x$ is invertible for all $x \in K'$ and $n \geq N$.

We fix a number $n \geq N$ and consider the set $A = K \cap f_n(K')$. By construction $f_n(q) \in A$, hence $A$ is not empty. The set $A$ is closed, because $K$ and $K'$ are compact. On the other hand, if $p \in K'$ with $f_n(p) \in K$, then $d(p, f_n(p)) < \epsilon$ implies $d(p, K) < \epsilon$, which in turn implies that $p$ is in the interior of $K'$. Now $f_n$ is locally a diffeomorphism, since $(Df_n)_x$ is invertible for all $x \in K'$. Therefore, if $p \in K'$ with $f_n(p) \in K$, then $f_n(p)$ admits an open neighbourhood in $X$ which is also contained in the image $f(K')$. As a consequence, the set $A = K \cap f(K')$ is both closed and open in $K$. Since $A$ is non-empty and $K$ is connected, it follows that $A = K$, i.e., it follows that $K \subset f_n(K')$.

Corollary 5. Under the above assumptions for $n \geq N$ there is a unique inverse map $g_n : K \to K'$, i.e., a unique map $g_n : K \to K'$ with $f_n \circ g_n = id_K$.

11. Continuity of the Kobayashi pseudodistance

Theorem 6. Let $S$ be an orientable real surface with a smooth family of complex structures $J_t$. Assume that there is a relatively compact subset $\Omega$ such that all the complex structures $J_t$ agree outside $\Omega$.

Then the map $F : TS \times I \to \mathbb{R}_0^+$ given by the Kobayashi-Royden pseudometric on $TS$ with respect to $J_t$ $(t \in I)$ is continuous on $TS \times I$. 
Remark. It is important that we deform the complex structure only inside some fixed relatively compact subset. Without this assumption the statement is not true. For example, we have seen that there is a family of complex structure \( J_t \) on \( D = \{ z : |z| < 1 \} \) such that \( (D, J_0) \) is biholomorphic to the unit disc while \( (D, J_t) \cong \mathbb{C} \) for every \( t \neq 0 \). Evidently for this family \( F \) is not continuous, since it vanishes for \( t \neq 0 \) and is non-zero for \( t = 0 \).

Proof. We will need the result only for the case where \( X \) is hyperbolic. However, it is easy to see that the statement holds if \( X \) is not hyperbolic: \( X \) is not hyperbolic precisely if and only if one of the following conditions are fulfilled:

1. \( X \) is compact and \( b_1(X) \leq 2 \).
2. There exists a complex analytic compactification \( X \to \bar{X} \) such that \( \bar{X} \) is compact and simply-connected and \( \bar{X} \setminus X \) contains at most two points.

In this formulation it is clear that the property of not being hyperbolic can not be changed by modifying the complex structure only inside some fixed compact set. Therefore \( d_{(X, J_t)} \) vanish for all \( t \) if \( d_{(X, J_0)} \) vanishes. In particular, in this case the Kobayashi pseudometric depends continuously on \( t \) (because it is constantly zero).

Thus we may from now on assume that \((X, J_0)\) is hyperbolic.

\[ \Box \]

Proposition 3. Let \( X \) be an orientable real surface with a smooth family of complex structures \( J_t \). Assume that there is a relatively compact subset \( \Omega \) such that all the complex structures \( J_t \) agree outside \( \Omega \).

Assume that \((X, J_0)\) is hyperbolic.

Then for every \( \varepsilon > 0 \) there exists a \( \delta > 0 \) such that for all \( x \in X \), \( v \in T_x X \) and \( t \in [0, \delta] \) the inequality

\[ F_{(X, J_t)}(v) \leq (1 + \varepsilon)F_{(X, J_0)}(v) \]

holds.

In words: The Kobayashi pseudometric is uniformly upper semi-continuous with respect to \( t \).

Proof. Let \( h_t \) be a family of hermitian metrics for \((X, J_t)\) varying smoothly with \( t \). Let \( g \) be the complete hermitian metric of constant Gaussian curvature \(-1\) on \((X, J_0)\) (which is unique and exists because \((X, J_0)\) is hyperbolic). Then there is a positive function \( \rho : X \to \mathbb{R} \) such that \( h_0 = \rho \cdot g \). We define \( g_t = \rho \cdot h_t \). Let \( G_t \) denote the Gaussian curvature for \( g_t \). Then \( G(t, x) = G_t(x) \) is continuous and equals \(-1\) on \( \{0 \} \times X \cup ([0,1] \times (X \setminus K)) \). Since \( K \) is compact, we can find a number \( c > 0 \) and a constant \( 1 > C > 0 \) such that \( G_t \leq -C \) for all \( t \leq c \) and at every point of \( X \). Furthermore, once again by compactness of \( K \), there is an other constant \( C' \) such that \( g_0 = C' g \). Using Ahlfors-Schwarz lemma it now follows that

\[ ||\phi^* v||_g \leq \frac{C'}{C} ||v|| \]

for every \( v \in T_0 \Delta \) and every holomorphic map \( \phi : \Delta \to (X, J_t) \) \((t \in [0, c])\). The family of all such maps \( \phi \) is therefore equicontinuous. Due to the theorem of Arzela Ascoli they form a normal family. Now fix \( t, p \in X \) and let \((v_n, t_n, x_n)\) be a sequence such that \( \lim x_n = x \), \( \lim t_n = t \) and \( \lim v_n = v \in T_x X \). Let
\( \alpha = \limsup_n (F_{(X,J_n)})_{x_n}(v_n) \). By the definition of the infinitesimal Kobayashi-Royden there is a sequence of holomorphic maps

\[ \phi_n : \Delta \to (X, J_n), \quad \phi_n(0) = x_n, (\phi_n) \alpha n \frac{\partial}{\partial z} = v_n \]

with \( \lim \alpha_n = \alpha \). Due to the normal family property the sequence \( \phi_n \) admits a convergent subsequence, i.e., there is a holomorphic map

\[ \phi : \Delta \to (X, J_t), \quad \phi(0) = x, (\phi) \alpha \frac{\partial}{\partial z} = v \]

It follows that

\[ F_{(X,J_t)}(v) \leq \alpha \liminf_n (F_{(X,J_n)})_{x_n}(v_n) \]

This establishes lower semicontinuity. Upper semicontinuity follows from proposition \( \Box \) below.

**Proposition 4.** Let \( (J_t)_{0 \leq t \leq 1} \) be a smooth family of complex structures on a orientable real surface \( S \). Let \( K \) be a compact subset of \( S \), \( K \neq S \). Then there is a number \( c > 0 \) such that for all \( t \in [0, c] \) there exists a holomorphic injective map \( \phi_t : K^c \to (S, J_t) \) such that \( \phi_t \) converges uniformly to the identity map \( \text{id}_S \) for \( t \to 0 \).

**Proposition 5.** Let \( (J_t)_{0 \leq t \leq 1} \) be a smooth family of complex structures on an orientable real surface \( S \).

The Kobayashi-Royden pseudodistance is upper-semicontinuous as a function on \( TS \times [0, 1] \).

**Proof.** Let \( v \in T_pS, p \in S \). Assume \( F_{(S,J_0)}(v) = c \in \mathbb{R} \). Fix \( \epsilon > 0 \). Then there is a holomorphic map \( f : (\Delta, 0) \to (X, p) \) with \( f_* \frac{\partial}{\partial z} = \lambda v, |\lambda| > c - \epsilon \). Define \( F(z) = f((1 + \epsilon)z) \). Then \( F_* \frac{\partial}{\partial z} = (1 + \epsilon)\lambda v \). By construction the image \( F(\Delta) \) is now contained in the compact set

\[ K_0 = f \left( \left\{ z : |z| \leq \frac{1}{1 + \epsilon} \right\} \right). \]

The family of complex structures \( (J_t) \) endow the product \( S \times [0, 1] \) with the structure of a CR-manifold which is foliated by complex leaves, namely the \((S, t)\). Hence we have a Levi-flat real three-dimensional CR-hypersurface. Such a CR-manifold can be embedded into a complex manifold \( Y \) (HLNOS). Now \( S_0 = (S, 0) \) is a closed complex Stein submanifold of \( Y \). Let \( \Omega \) be a Stein open neighbourhood of \( S_0 \) in \( Y \) (which exists due to [Su77]). After shrinking \( \Omega \) if necessary there exists a holomorphic retraction \( \rho : \Omega \to S_0 \) (see e.g. [DLS84], lemma 2.1).

Let \( K \) be a compact subset of \( S \) containing \( K_0 \) in its interior.

For every \( t \in [0, 1] \) we consider \( K_t = \{(x,t) : x \in K \} \subset Y \). There is a bound \( r > 0 \) such that \( K_t \subset \Omega \) for all \( t < r \) due to compactness of \( K \) and openness of \( \Omega \).

Let \( \zeta_t \) denote the map given by

\[ \zeta_t : x \mapsto (x, t) \in Y. \]

Now \( \rho \circ \zeta_t \) converges uniformly on \( K \) to the identity map in \( C^1 \)-topology for \( t \to 0 \). Using corollary \( \Box \) we may deduce that for \( t \) sufficiently close to 0 there is a map \( g_t : K_0 \to K \) with \( \rho \circ \zeta_t \circ g_t = \text{id}_{K_0} \). Now we define (for sufficiently small \( t \)) a map \( F_t : \Delta \to S_t \) via

\[ F_t = \zeta_t \circ g_t \circ F. \]
We obtained a family of holomorphic maps $F_t : \Delta \rightarrow (S, J_t)$ with $\lim F_t = F$. This yields the desired semicontinuity.

**Proposition 6.** Let $X$ be an orientable surface and $J_t$ a smooth family of complex structures which agree outside a compact subset $K \subset X$. Let $\gamma \in \pi_1(S)$ and for each $t$ define $\lambda(t)$ as the infimum of the hyperbolic length (with respect to the complex structure $J_t$) of closed simple curves (freely) homotopic to $\gamma$.

Then $t \mapsto \lambda(t)$ is continuous.

**Proof.** Let $\xi$ be a fixed closed curve freely homotopic to $\gamma$.

By theorem 6 we know that its hyperbolic length $L_{(X,J_t)}(\xi) = \int F_{(X,J_t)}(\xi')ds$ is continuous in $t$.

The infimum of a family of continuous functions is always upper-semicontinuous. Thus it suffices to show that $\lambda(t)$ is lower-semicontinuous. Due to proposition 3 we know that for every simple closed curve $\zeta$, every $t_0$ and every $\varepsilon > 0$ there exists a real number $\delta > 0$ such that

$$L_t(\zeta) > L_{t_0}(\zeta)/(1 + \varepsilon)$$

for all $t$ with $|t - t_0| < \delta$. This implies immediately

$$\lambda(t) \geq \lambda(t_0)/(1 + \varepsilon)$$

for all $t$ with $|t - t_0| < \delta$. Hence the desired lower semicontinuity.

**12. Proof of the Main Theorem**

Here we prove the Main Theorem.

**Proof.** We distinguish three cases:

1. $X$ is not hyperbolic.
2. $X$ is hyperbolic, but the hyperbolic length spectrum is trivial.
3. $X$ is hyperbolic and the hyperbolic length spectrum is not trivial.

Case (1):

If $X$ is compact and non-hyperbolic, then $X$ is biholomorphic to $\mathbb{P}_1$ or an elliptic curve and the result follows from the classical theory of moduli spaces of compact Riemann surfaces.

If $X$ is non-compact and non-hyperbolic, then $X \simeq \mathbb{C}$ or $X \simeq \mathbb{C}^*$ and we refer to proposition 1.

Case (2):

In this case we know from corollary 3 that $X \simeq \Delta$, $X \simeq \Delta^*$ or $X \simeq \mathbb{P}_1 \setminus \{0, 1, \infty\}$ and the statement follows from proposition 1.

Case (3):

In this case $X$ is hyperbolic and there exists a simple closed curve $\gamma$ and a constant $c > 0$ such that every simple closed curve homotopic to $\gamma$ has hyperbolic length at least $c$. Fix such a curve $\gamma$ and let $\lambda(\gamma) = c$ denote its “stable hyperbolic length”, i.e., the infimum of the hyperbolic length $L(\tilde{\gamma})$ where the infimum is taken over all simple closed curves homotopic to $\gamma$.

Lemma 7 implies that there is a compact subset $K \subset X$ and a smooth family of complex structures $J_t$ on $X$ such that all these complex structures agree outside...
Let $K$, and such that the hyperbolic length $L_{(X,J_t)}(\gamma)$ of $\gamma$ with respect to the complex structure $J_t$ fulfills the inequality $L_{(X,J_t)}(\gamma) < c$. For each $t$ we define $\lambda(t)$ as the infimum of the hyperbolic length (with respect to the complex structure $J_t$) of closed simple curves (freely) homotopic to $\gamma$. Now $t \mapsto \lambda(t)$ is continuous due to proposition $\Box$ and furthermore non-constant by construction ($\lambda(0) = c > \lambda(1)$).

We recall the definition of $\Sigma_{i(X,J_t)}$ as in $\Box$. We observe that $\Sigma_t = \Sigma_{(X,J_t)}$ is a countable subset of $\mathbb{R}$ for every $t$. Since $\lambda$ is continuous and non-constant, there exists a parameter $s$ such that $\lambda(s) \notin \Sigma_0$. Then $\Sigma_0 \neq \Sigma_1$ and therefore $(X, J_0)$ is not biholomorphic to $(X, J_s)$.

\[ \Box \]

13. Deformations of discrete subgroups of $PSL_2(\mathbb{R})$

Our result on deformations of complex structures on Riemann surfaces can be translated into a result on deforming discrete subgroups of $PSL_2(\mathbb{R})$.

**Theorem 7.** Let $F$ be free group (possibly not finitely generated) and let $\rho_0 : F \to G = PSL_2(\mathbb{R})$ be a group homomorphism which embeds $F$ into $G$ as a discrete subgroup.

Then there exists a continuous family of group homomorphisms $\rho_t : F \to G$ $(t \in [0,1])$ such that each $\rho_t$ embeds $F$ as a discrete subgroup in $G$ and such that $\rho_0$ is not conjugate to $\rho_1$. (I.e. there is no $g \in G$ such that $\rho_1(\gamma) = g \cdot \rho_0(\gamma) \cdot g^{-1}$.)

**Lemma 9.** Let $S$ be a hyperbolic Riemann surface, $I = [0,1]$, $M = S \times I$.

Let $A$ be a closed subset of $M$. Assume that $A$ equals the closure of its interior. For $t \in I$, let $X_t = \{ p \in S : (p,t) \notin A \}$.

Then the Kobayashi-pseudodistance on $X_t$ is continuous in $t$.

**Proof.** Let $t \in I$ and $p \in X_t$ and let $(p_n, t_n) \in M \setminus A$ with $\lim(p_n, t_n) = (p, t)$. Let $\phi_n : (\Delta, 0) \to (X_{t_n}, p_n)$ be a sequence of holomorphic maps. Since $S$ is hyperbolic, there is a subsequence converging to a holomorphic map $\phi : (\Delta, 0) \to (S, p)$. By the open mapping theorem the image $\phi(\Delta)$ is open in $S$. On the other hand $\phi(\Delta) \times \{t\}$ can not intersect the interior of $A$, since $\phi = \lim \phi_n$. In combination with our assumption on $A$ it follows that $\phi(\Delta) \subset X_t$.

Conversely let $\psi : (\Delta, t) \to (X_t, p)$ be a holomorphic map. For every $\epsilon > 0$ the set $\psi(\Delta_{1-\epsilon})$ is compact. Since $M \setminus A$ is open, it follows that for every $\epsilon > 0$ there exists a $\delta > 0$ such that the map $\psi_\epsilon : \Delta \to S$ defined by

$$ z \mapsto \psi((1-\epsilon)z) $$

has its image contained in $X_s$ for all $s$ with $|s-t| < \delta$.

These two considerations imply that the Kobayashi pseudodistance is both upper- and lower semicontinuous, hence continuous in $t$.

**Corollary 6.** Let $X$ be biholomorphic to $\Delta^*$ or $\mathbb{P}_1 \setminus \{0,1,\infty\}$.

Then there exists a deformation of the complex structure such that the Kobayashi pseudodistance varies continuously.

**Proof.** Note that $\mathbb{P}_1 \setminus \{0,1,\infty\} \simeq \mathbb{C}^* \setminus \{2\}$. We use the lemma with $S = \Delta$ resp. $S = \mathbb{C}^* \setminus \{2\}$ and $A = \{(z,t) : |z| \leq t\}$.

**Corollary 7.** Let $X$ be a hyperbolic Riemann surface which is not simply-connected.

Then there exists a non-trivial deformation family of complex structures on $X$ for which the Kobayashi pseudodistance varies continuously.
Proof. This follows from the preceding corollary if $X$ is isomorphic to $\Delta^*$ or $\mathbb{P}_1 \setminus \{0,1,\infty\}$ and from $X$ in all other cases.

Now we can prove the theorem.

Proof. There is a non-trivial family of complex structures $J_t$ on $X$ such that the Kobayashi-pseudodistance varies continuously. Let $\tilde{X}$ denote the universal covering. We obtain a family of complex structures $\tilde{J}_t$ such that the Kobayashi pseudodistance varies continuously and the fundamental group $\pi_1(X)$ acts on $\tilde{X}$ by “deck transformations” which are holomorphic with respect to each $\tilde{J}_t$. We fix a point $p \in \tilde{X}$ and a real tangent direction $\xi$, i.e., a real half-line in the real tangent space (or equivalently, a tangent vector of length 1). Since $\tilde{X}$ is hyperbolic and simply-connected, for each $t$ there is a unique biholomorphic map $\phi_t$ from $(\tilde{X}, \tilde{J}_t)$ to the unit disc mapping $p$ to 0 and $\xi$ to the positive real half line. The continuity of the Kobayashi pseudodistance implies that these maps $\phi_t$ as well as there inverse maps $\phi_t^{-1}$ are continuous in $t$. For each $\gamma \in \pi_1(X)$ let $\alpha_\gamma$ denote the deck transformation acting on $\tilde{X}$. Then

$$\pi_1(X) \times I \ni (\gamma, t) \mapsto \phi_t \circ \alpha_\gamma \circ \phi_t^{-1} \in \text{Aut}(\Delta) \simeq \text{PSL}_2(\mathbb{R})$$

defines a continuous family of group homomorphisms.

For arbitrary Lie groups there is a weaker result, cf. [Win02].

Proposition 7. Let $G$ be a real Lie group which contains a non-compact semisimple Lie subgroup. Let $k \in \mathbb{N}$. Then there exists a non-trivial family of injective group homomorphism $\rho_t$ $(t \in I)$ from the free group $F_k$ with $k$ generators into $G$ such that $\rho_t(F_k)$ is discrete for every $t$.

Proof. In [Win02] we proved that there exists a non-empty open subset $W \subset G^k$ such that for every $(g_1, \ldots, g_k) \in W$ the subgroup of $G$ generated by $g_1, \ldots, g_k$ is free and discrete.

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