Research Article

Finite-Time Simultaneous Stabilization for Stochastic Port-Controlled Hamiltonian Systems over Delayed and Fading Channels

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Received 17 June 2020; Accepted 31 July 2020; Published 29 September 2020

Guest Editor: Xiaodi Li

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In this paper, a finite-time simultaneous stabilization problem is investigated for a set of stochastic port-controlled Hamiltonian (PCH) systems over delayed and fading noisy channels. The feedback control signals transmitted via a communication network suffer from both constant transmission delay and fading channels which are modeled as a time-varying stochastic model. First, on the basis of dissipative Hamiltonian structural properties, two stochastic PCH systems are combined to form an augmented system by a single output feedback controller and then sufficient conditions are developed for the semiglobally finite-time simultaneous stability in probability (SGFSSP) of the resulting closed-loop systems. The case of multiple stochastic PCH systems is also considered and a new control scheme is proposed for the systems to save costs and achieve computational simplification. Finally, an example is provided to verify the feasibility of the proposed simultaneous stabilization method.

1. Introduction

Port-controlled Hamiltonian (PCH) systems are known as an important class of nonlinear systems ([1, 2]). Compared to the general nonlinear systems, an excellent benefit of PCH systems is that the Hamiltonian function in the systems can be used as a Lyapunov function candidate in stability analysis (see, for instance, [3–5]). Thanks to the special system structure and clear physical meaning, applications of PCH systems can be found in a variety of engineering systems including power systems, robotic systems, and irreversible thermodynamic systems ([6–10]). In recent years, stabilization as well as simultaneous stabilization problem has been extensively studied for PCH systems ([11–14]). In terms of PCH systems with disturbances, the above stabilization problem has been resolved in [11, 13]. Taking actuator saturation into account, the study in [14] has proposed an adaptive control strategy to simultaneously stabilize PCH systems with parameter uncertainties.

On the other hand, there usually exist stochastic components and random disturbances in practical control plants, which often result in performance degradation, as well as destabilization of the systems. In the last few decades, many researchers have made efforts to deal with the stabilization problem of stochastic systems ([15, 16]). For example, in [15], output feedback stabilization has been studied using the backstepping approach for Itô-type stochastic systems. As for stochastic PCH systems, the control problem has also captured public attentions ([17–20]). Exploiting an energy-based feedback control scheme, the authors of [17] have raised stochastic feedback stabilization results. In regard to time-varying stochastic PCH systems, the study in [18] has come up with a kind of stochastic generalized canonical transformations approach to stabilize stochastic PCH systems. In addition, the adaptive control topic for nonlinear stochastic Hamiltonian systems has been introduced in [19, 20]. Parameter uncertainty, randomness, and time delay are all considered in above references.

In many practical problems, the fast convergence within a fixed finite time interval plays an important role. Finite-time stabilization makes closed-loop systems enjoy fast convergence. In addition, disturbance rejection properties
and better robustness both can be reflected in the finite-time stabilization. Thus, many investigations about finite-time stabilization controller design have been carried out ([21–29]). For stochastic nonlinear systems which are written as Itô differential form, [23] has proposed a method to solve the finite-time stabilization problem. The finite-time stabilization of the Hamiltonian systems has been studied in [21, 25, 27, 28]. For instance, the finite-time feedback control manner is developed in [21] to deal with finite-time stabilization problem for PCH systems with nonvanishing disturbances.

Generally speaking, the phenomenon of fading channels as well as network-induced delay is very likely to occur in the networked control system, which can lead to various distortions and information constraints. By now, a considerable number of researches have been done for continuous and discrete systems over network-induced phenomenon ([30–38]). Under memoryless fading channels environment, discrete systems over network-induced phenomenon number of researches have been done for continuous and as well as network-induced delay is very likely to occur in the disturbances.

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[21, 25, 27, 28]. For instance, the finite-time feedback control stabilization controller design have been carried out
stabilization. fi´_hus, many investigations about finite-time and better robustness both can be reflected in the finite-time stabilization problem solved easily. fi´_hrough the Lyapunov stabilization for the systems. Utilizing the structural properties of dissipative Hamiltonian systems, the two stochastic PCH systems form an augmented stochastic PCH system, which makes the problem solved easily. Through the Lyapunov function and Itô differential formula, the closed-loop function will be SGFSSP. Besides, we will extend our approach to the case of multiple stochastic PCH systems over delayed and fading channels. A feedback control strategy is proposed. At last, the feasibility of the above method is illustrated by the simulation.

The contributions of this paper mainly lie in the following two aspects: (1) taking network-induced delay and fading noisy channels environment into consideration, a new single output feedback controller design method is raised to deal with the SGFSSP problem for stochastic PCH systems. In this way, the controller implementation costs can be greatly reduced, and the computational simplification of control can be achieved. (2) We make an in-depth study of the proposed method by extending the approach to the case of multiple PCH systems. SGFSSP result for multiple PCH systems over delayed and fading channels is given.

Notation: \( \mathbb{R}^n \) denotes the n-dimensional real column vectors and \( \mathbb{R}^{m \times n} \) is the real matrices with dimensions \( n \times m \). A real-valued function \( f(x) \in C^2 \) represents that \( f(x) \) is a continuously twice differentiable function. \( \| \cdot \| \) represents the 2-norm. \( \text{diag}(a_1, a_2, \ldots, a_m) \) represents diagonal matrix with \( a_1, a_2, \ldots, a_m \) as its diagonal elements. We denote \( \lambda_{\text{min}}(\cdot) \) as the smallest eigenvalue operator, \( \mathbb{E}\{\cdot\} \) as the expectation operator, and \( \text{Cov}(\cdot) \) as the covariance operator, respectively. For the probability space \((\Omega, \mathcal{F}, \mathcal{P})\), \( \Omega \) denotes the sample space, \( \mathcal{F} \) denotes the \( \sigma \)-algebra of the observable random events, and \( \mathcal{P} \) is the probability measure on \( \Omega \).

2. Problem Formulation and Preliminaries

Consider the following two stochastic PCH systems:

\[
\begin{align*}
\dot{x}(t) &= \left[ J(x(t)) - R(x(t)) \right] \nabla H_1(x(t)) dt + g_1 u(t) dt + h_1 d\omega(t), \\
y(t) &= g_2^T \nabla H_1(x(t)), \\
\end{align*}
\]

\[
\begin{align*}
\dot{\xi}(t) &= [\tilde{J}(\xi(t)) - \tilde{R}(\xi(t)) \nabla H_2(\xi(t))] dt + g_2 u(t) dt + h_2 d\tilde{\omega}(t), \\
\eta(t) &= g_2^T \nabla H_2(\xi(t)).
\end{align*}
\]

where \( x(t), \xi(t) \in \mathbb{R}^n \) are the system state vectors, \( u(t) \in \mathbb{R}^m \) is the control input which satisfies \( \mathbb{E}\left\{ \int_0^t \|u(s)\|^2 ds \right\} < \infty \), and \( y(t), \eta(t) \in \mathbb{R}^m \) are the outputs of systems. The signals \( \omega(t) \) and \( \rho(t) \) are both \( \kappa \)-dimensional independent standard Wiener process defined on probability space \((\Omega, \mathcal{F}, \mathcal{P})\). We assume \( \mathbb{E}\{d\omega(t)\} = 0, \mathbb{E}\{d\rho(t)\} = 0, \mathbb{E}\{[d\omega(t)]^2\} = dt \), and \( \mathbb{E}\{[d\rho(t)]^2\} = dt \). \( \nabla H_1(x) \in \mathbb{R}^{n \times 1} \) is the gradient of the Hamilton function \( H_1(x) : \mathbb{R}^n \rightarrow \mathbb{R} \), which is defined as \( \nabla H_1(x) = (\partial H_1(x)/\partial x) \), and \( H_1(x) \geq 0, H_1(0) = 0 \), for all \( t \geq 0 \). \( J(\cdot) \in \mathbb{R}^{m \times n} \) and \( \tilde{J}(\cdot) \in \mathbb{R}^{m \times n} \) are both skew-symmetric structure matrices; \( R(\cdot) \in \mathbb{R}^{m \times m} \) and \( \tilde{R}(\cdot) \in \mathbb{R}^{m \times m} \) are positive definite strict dissipation matrices; \( g_1, g_2, h_1, h_2 \) are known real constant gain matrices. In addition, by setting \( f_1(x, u) = [J(x) - R(x)] \nabla H_1(x) + g_1 u, f_2(\xi, u) = [\tilde{J}(\xi) - \tilde{R}(\xi)] \nabla H_2(\xi) + g_2 u. \) Suppose that there exist constants \( k_F > 0 \) and \( k_C > 0 \) such that

\[
\begin{align*}
\|f_1(\theta_1, u) - f_1(\theta_2, u)\| & \leq k_F \|\theta_1 - \theta_2\|, \\
\|f_1(\theta_1, u)\| + \|h_1\| & \leq k_C (1 + \|\theta_1\| + \|u\|).
\end{align*}
\]

hold for all \( \theta_1, \theta_2 \in \mathbb{R}^n, u(t) \in \mathbb{R}^m, t \geq 0, \) and \( i = 1, 2 \).

For generalized PCH systems, it is shown in [13] that the two PCH systems can be simultaneously stabilized by a controller \( u = -K(y(t) - \eta(t)) \) over constraint conditions, where \( K \) is a gain matrix with appropriate dimension. Unfortunately, when it comes to the stochastic networked control system (NCS), the feedback control signals transmitted via a communication network may suffer from delayed and fading noisy channels.
Let us focus on the NCS as depicted in Figure 1. Suppose that the control signal $u(t)$ suffers both constant transmission delay $d > 0$ and signal attenuation in the closed-loop system. The transmission delay $d$ is caused by the message delivery from the controller to the actuator. The transmission of signal $u(t)$ is accomplished in a form of components through independent parallel channels. Then, the control signal $u(t)$ arriving at the actuator is modeled by the following multiple independent and memoryless forms:

$$\varepsilon(t)u(t - d) + q(t),$$

(4)

where $u(t) \in \mathbb{R}^m$ and $(\varepsilon(t)u(t - d) + q(t)) \in \mathbb{R}^m$ are the input and output of channels, respectively. $\varepsilon(t) \in \mathbb{R}^{m \times m}$ represents the multiplicative noise with the following form:

$$\varepsilon(t) = \text{diag}[\varepsilon_1(t), \varepsilon_2(t), \ldots, \varepsilon_m(t)].$$

(5)

$$E[\varepsilon_i(t)] = \mu_i, \text{Cov}(\varepsilon_i(t), \varepsilon_j(s)) = \sigma_i^2 \delta(t - s), \mu_i \neq 0, \sigma_i^2$$

is known power spectral density. $q(t) = [q_1(t), q_2(t), \ldots, q_m(t)]^T \in \mathbb{R}^m$ is an additive white Gaussian process noise with $E[q_i(t)] = 0$ and known power spectral density $g_i^2$, i.e., Cov $(q_i(t), q_i(s)) = g_i^2 \delta(t - s)$, $\delta$ denotes the Dirac delta function, $i = 1, \ldots, m$. We make the following assumption for $\varepsilon(t)$.

**Assumption 1.** (1) $\varepsilon_i(t)$ and $\varepsilon_j(t)$ are uncorrelated for $i \neq j$, i.e., $E[\varepsilon_i(t_1)\varepsilon_j(t_2)] = 0, \forall t_1, t_2 > 0$, and $i \neq j$

(2) $\varepsilon(t)$ is uncorrelated with $\omega(t)$ and $\rho(t)$

**Remark 2.** We consider interference channels noise in the systems and input channels noise. The conditions of Assumption 1 avoid the possible occurrence of noise coupling phenomenon.

Denote

$$M = \text{diag}[\mu_1, \mu_2, \ldots, \mu_m],$$

(6)

$$Q = \text{diag}[\sigma_1^2, \sigma_2^2, \ldots, \sigma_m^2].$$

Obviously, $M$ is nonsingular since $\mu_i \neq 0$. Without loss of generality, we assume $\mu_1^2 = 1$, i.e., $Q = I_m \in \mathbb{R}^{m \times m}$ for simplicity hereinafter, $i = 1, 2, \ldots, m$.

Substituting (4) into (1) and (2), we get

$$dx(t) = [f(x(t)) - R(x(t))]VH_1(x(t))dt$$

$$+ g_1\varepsilon(t)u(t - d)dt + g_1d\omega(t) + h_1d\omega(t),$$

(7)

$$\begin{cases}
    d\xi(t) = [\dot{f}(\xi(t)) - \dot{R}(\xi(t))]VH_2(\xi(t))dt \\
    + g_2\varepsilon(t)u(t - d)dt + g_2d\omega(t) + h_2d\omega(t),
\end{cases}$$

(8)

Then, we define

$$x(t) = VH_1(x(t)),$$

(9)

$$\begin{cases}
    \dot{\xi}(t) = [\dot{f}(\xi(t)) - \dot{R}(\xi(t))]VH_2(\xi(t))dt \\
    + g_2\varepsilon(t)u(t - d)dt + g_2d\omega(t) + h_2d\omega(t),
\end{cases}$$

**Lemma 1.** Consider the following Itô form stochastic system:

$$dx(t) = f(x)dt + g(x)dw(t).$$

(10)

Suppose $f(x)$ and $g(x)$ are locally Lipschitz continuous in $x$ and locally bounded, $f(0) = 0$, and $g(0) = 0$. If, for any $x_0 \in \mathbb{R}^n$, there exist class-$\mathcal{K}$ functions $\gamma_1$ and $\gamma_2$, real numbers $c > 0$, $0 < b < 1$, and a positive definite, function $V(x) \in \mathcal{C}^2$ such that

$$\gamma_1(||x||) \leq V(x) \leq \gamma_2(||x||),$$

(11)

then system (9) is SGFSSP. Furthermore, the compact set $\Omega$ is expressed as

$$\Omega = \left\{ x \mid V(x) \leq \frac{a}{c(1 - b)} \right\}, \quad \forall 0 < b < 1,$$

(12)

and the settling time of system (9) with respect to $x_0$ satisfies
\[ T^* = \frac{1}{bc(1-h)} \left\{ \lambda^{1-h}(x_0) - \left( \frac{a}{c(1-b)} \right)^{(1-h)/h} \right\}. \] (13)

**Lemma 2.** For any real number \( z_i, i = 1, \ldots, n \), and any positive real numbers \( \epsilon_1, \epsilon_2 \) which satisfy \( 0 < \epsilon_1 \leq \epsilon_2 \), it holds
\[
\left( \sum_{i=1}^{n} |z_i|^{\epsilon_1} \right)^{1/\epsilon_1} \leq \left( \sum_{i=1}^{n} |z_i|^{\epsilon_2} \right)^{1/\epsilon_2}. \] (14)

In Lemma 2, if \( \epsilon_1 = \beta \geq 1 \) and \( \epsilon_2 = 1 \), then
\[
\sum_{i=1}^{n} |z_i|^{1/\beta} \leq \left( \sum_{i=1}^{n} |z_i| \right)^{1/\beta}, \] (15)

for any real number \( z_i, i = 1, \ldots, n \).

In this paper, our main goal is to make the two systems (1) and (2) with the delayed and fading noisy channels SGFSSP. More specifically, based on Lemma 1, we have an interest in designing a suitable output feedback controller \( u(t - d) \) such that systems (7) and (8) satisfy (10) and (11). Besides, we extend our results to multiple stochastic PCH systems.

For the above purpose, the following assumptions and lemmas are essential in the sequel.

**Assumption 2.** The Hamilton functions \( H_1(x) \) and \( H_2(\xi) \) are given as
\[
H_1(x) = \sum_{i=1}^{n} \left( x_i^2 \right)^{\alpha(2a-1)},
\]
\[
H_2(\xi) = \sum_{i=1}^{n} \left( \xi_i^2 \right)^{\alpha(2a-1)},
\] (16)

where \( \alpha > 1 \) is a real number.

**Assumption 3.** There exist constants \( c_1 > 0 \) and \( c_2 > 0 \) such that
\[
c_1 = \inf_{t \geq 0} \{ \lambda_{\min}(R(x(t))) \},
\]
\[
c_2 = \inf_{t \geq 0} \{ \lambda_{\min}(\tilde{R}(\xi(t))) \}. \] (17)

**Lemma 3.** For any matrices \( P_1, P_2 \in \mathbb{R}^{m \times n} \), it follows that
\[
P_1^T P_2 + P_2^T P_1 \leq P_1^T P_1 + P_2^T P_2. \] (18)

**3. SGFSSP of Two Stochastic PCH Systems and That of Multiple Stochastic PCH Systems**

In this section, we will give the analysis result that serves for the SGFSSP of two stochastic PCH systems.

**Theorem 1.** Consider systems (7) and (8). Assumptions 2 and 3 are satisfied. If there exist matrices \( K = K^T, L_1 = L_1^T, L_2 = L_2^T, \) and \( L_3 = L_3^T \) such that the following matrix inequality
\[
\begin{bmatrix}
A_{11} & A_{12} & 0 & 0 & A_{15} & -g_1Kg_1^T \\
* & -L_2 & 0 & 0 & A_{25} & 0 \\
* & * & A_{33} & A_{34} & g_2Kg_2^T & A_{36} \\
* & * & * & -\frac{1}{2}L_3 & 0 & A_{46} \\
* & * & * & * & -L_3 & 0 \\
\end{bmatrix} < 0
\] (19)

holds, where
\[
A_{11} = -2c_1 I_n - 2g_1Kg_1^T - L_1,
\]
\[
A_{12} = -\frac{1}{2} (L_1 + L_2),
\]
\[
A_{15} = g_1Kg_1^T + \frac{1}{2} (L_1 + L_3),
\]
\[
A_{25} = -\frac{1}{2} (L_2 + L_3),
\]
\[
A_{33} = -2c_2 I_n + 2g_2Kg_2^T - L_1,
\]
\[
A_{34} = -\frac{1}{2} (L_1 + L_2),
\]
\[
A_{36} = -g_2Kg_2^T + \frac{1}{2} (L_1 + L_3),
\]
\[
A_{46} = -\frac{1}{2} (L_2 + L_3),
\]

then systems (7) and (8) are SGFSSP under the output feedback control law
\[
u(t - d) = -M^{-1} K (y(t - d) - \eta(t - d)). \] (20)

**Proof.** First of all, substituting (21) into (7) and (8), we obtain
\[
\begin{align*}
\frac{dx(t)}{dt} &= [J(x(t)) - R(x(t))] \mathbf{H}_1(x(t)) dt \\
&\quad - g_1 \epsilon(t) M^{-1} K g_1^T \mathbf{H}_1(x(t - d)) dt \\
&\quad + g_1 \epsilon(t) M^{-1} K g_1^T \mathbf{H}_2(\xi(t - d)) dt \\
&\quad + g_1 d\omega(t) + h_1 d\omega(t),
\end{align*}
\] (22)

\[
\begin{align*}
\frac{d\xi(t)}{dt} &= [\tilde{J}(\xi(t)) - \tilde{R}(\xi(t))] \mathbf{H}_2(\xi(t)) dt \\
&\quad - g_2 \epsilon(t) M^{-1} K g_2^T \mathbf{H}_1(x(t - d)) dt \\
&\quad + g_2 \epsilon(t) M^{-1} K g_2^T \mathbf{H}_2(\xi(t - d)) dt \\
&\quad + g_2 d\omega(t) + h_2 d\omega(t).
\end{align*}
\] (23)

Applying Newton–Leibnitz formula, we have

\[
L_2 = L_2^T, \quad \text{and} \quad L_3 = L_3^T
\]
\[
\begin{align*}
\nabla H_1(x(t)) - \nabla H_1(x(t-d)) &= \int_{t-d}^{t} (\nabla H_1(x(s)))' ds, \\
\nabla H_2(\xi(t)) - \nabla H_2(\xi(t-d)) &= \int_{t-d}^{t} (\nabla H_2(\xi(s)))' ds.
\end{align*}
\]

Then, systems (22) and (23) can be rewritten as
\[
\begin{align*}
\text{d}x(t) &= [J(x(t)) - R(x(t))]\nabla H_1(x(t)) dt \\
&\quad - g_1 \epsilon(t) M^{-1} K g_1^T \nabla H_1(x(t)) dt \\
&\quad + g_1 \epsilon(t) M^{-1} K g_2^T \nabla H_2(\xi(t)) dt \\
&\quad + g_1 \epsilon(t) M^{-1} K g_1^T \int_{t-d}^{t} (\nabla H_1(x(s)))' ds dt \\
&\quad - g_1 \epsilon(t) M^{-1} K g_2^T \int_{t-d}^{t} (\nabla H_2(\xi(s)))' ds dt \\
&\quad + g_1 \text{d}\omega(t) + h_1 \text{d}w(t),
\end{align*}
\]
\[
\begin{align*}
\text{d}\xi(t) &= -g_2 \epsilon(t) M^{-1} K g_1^T \nabla H_1(x(t)) dt \\
&\quad + \int \left( J(\xi(t)) - R(\xi(t)) \right) \nabla H_2(\xi(t)) dt \\
&\quad + g_2 \epsilon(t) M^{-1} K g_2^T \nabla H_2(\xi(t)) dt \\
&\quad + g_2 \epsilon(t) M^{-1} K g_1^T \int_{t-d}^{t} (\nabla H_1(x(s)))' ds dt \\
&\quad - g_2 \epsilon(t) M^{-1} K g_2^T \int_{t-d}^{t} (\nabla H_2(\xi(s)))' ds dt \\
&\quad + g_2 \text{d}\omega(t) + h_2 \text{d}w(t),
\end{align*}
\]
Defining the vectors \( \mathcal{X}(t) = [x^T(t) \quad \xi^T(t)]^T \), \( \zeta(t) = [\omega^T(t) \quad \rho^T(t)]^T \), the above equations can be further rewritten into an augmented Itô form stochastic PCH system described as
\[
\begin{align*}
\text{d}\mathcal{X}(t) &= [J(\mathcal{X}(t)) - R(\mathcal{X}(t))]\nabla H(\mathcal{X}(t)) dt \\
&\quad + G(t) \int_{t-d}^{t} \nabla H(\mathcal{X}(s))' ds dt \\
&\quad + g \text{d}\omega(t) + h \text{d}w(t),
\end{align*}
\]
where \( H(\mathcal{X}(t)) = H_1(\mathbf{x}(t)) + H_2(\xi(t)), \ R(\mathcal{X}(t)) = \text{diag} \left\{ R(\mathbf{x}(t)) + g_1 \epsilon(t) M^{-1} K g_1^T, R(\xi(t)) - g_1 \epsilon(t) M^{-1} K g_2^T \right\} \),
\[
J(\mathcal{X}(t)) = \begin{bmatrix}
j(\mathbf{x}(t)) & g_1 \epsilon(t) M^{-1} K g_2^T \\
g_1 \epsilon(t) M^{-1} K g_1^T & J(\xi(t)) \\
-\epsilon(t) M^{-1} K g_1^T & -\epsilon(t) M^{-1} K g_2^T \end{bmatrix},
\]
\[
G(t) = \begin{bmatrix}
g_1 \epsilon(t) M^{-1} K g_1^T \\
g_1 \epsilon(t) M^{-1} K g_2^T \\
-\epsilon(t) M^{-1} K g_1^T & -\epsilon(t) M^{-1} K g_2^T \end{bmatrix},
\]
\[
\int_{t-d}^{t} (\nabla H(\mathcal{X}(s)))' ds = \begin{bmatrix}
\int_{t-d}^{t} (\nabla H_1(\mathbf{x}(s)))' ds \\
\int_{t-d}^{t} (\nabla H_2(\xi(s)))' ds \\
g_1 \\
g_2 \\
h_1 & 0 \\
0 & h_2
\end{bmatrix}.
\]
Next, choosing the following Lyapunov function candidate:
\[
V(\mathcal{X}(t)) = 2H(\mathcal{X}(t)),
\]
and according to Itô differential formula, we have
\[
\begin{align*}
\text{d}V(\mathcal{X}(t)) &= \mathcal{L}V(\mathcal{X}(t)) dt + \frac{\partial V(\mathcal{X}(t))}{\partial \mathcal{X}(t)} g dt + h d\mathcal{W}(t),
\end{align*}
\]
where
\[
\begin{align*}
\mathcal{L}V(\mathcal{X}(t)) &= 2V^T(\mathcal{X}(t))[J(\mathcal{X}(t)) - R(\mathcal{X}(t))]\nabla H(\mathcal{X}(t)) \\
&\quad + 2V^T(\mathcal{X}(t))G(t) \int_{t-d}^{t} (\nabla H(\mathcal{X}(s)))' ds \\
&\quad + \text{tr} [g^T \text{Hess}(H(\mathcal{X})) g] \\
&\quad + \text{tr} [h^T \text{Hess}(H(\mathcal{X})) h].
\end{align*}
\]
Letting \( \lambda = \sup_{x, \xi} \| \text{Hess}(H_1(x) + H_2(\xi)) \|^2 \) and based on Lemma 3, we conclude that
\[
\begin{align*}
\text{tr} [g^T \text{Hess}(H(\mathcal{X})) g] \\
\leq \frac{1}{2} \text{tr} (g^T g) + \frac{1}{2} \text{tr} [g^T \text{Hess}(H(\mathcal{X})) \text{Hess}^T(\mathcal{X})) g] \\
\leq \frac{1}{2} (\lambda + 1) \text{tr} (g^T g)
\end{align*}
\]
where \( \bar{g}_{ij} \) and \( \tilde{g}_{ij} \) are the components of the matrices \( g_1 = (g_{ij})_{\text{nom}} \) and \( g_2 = (\tilde{g}_{ij})_{\text{nom}} \), respectively. Similarly, we have
\[
\begin{align*}
\text{tr} [g^T \text{Hess}(H(\mathcal{X})) h] \\
\leq \frac{1}{2} \text{tr} (h^T h) + \frac{1}{2} \text{tr} [h^T \text{Hess}(H(\mathcal{X})) \text{Hess}^T(\mathcal{X})) h] \\
\leq \frac{1}{2} (\lambda + 1) \text{tr} (h^T h)
\end{align*}
\]
\[
= \frac{1}{2} (\lambda + 1) \sum_{k=1}^{n} \sum_{i=1}^{m} \left( \bar{h}_{ik}^2 + \tilde{h}_{ik}^2 \right). 
\]
Then, denoting
\[
\tau := \sum_{j=1}^{m} \sum_{i=1}^{n} (\bar{g}_{ij}^2 + \bar{g}_{ij}^2) + \sum_{k=1}^{n} \sum_{i=1}^{m} (\bar{h}_{ik}^2 + \tilde{h}_{ik}^2),
\]
we obtain that
\[
\begin{align*}
\end{align*}
\]
\[
\begin{align*}
\text{tr}[g^T \text{Hess}(H(\mathcal{X} ))g + h^T \text{Hess}(H(\mathcal{X} ))h] \\
\leq \frac{1}{2} (\lambda + 1) \left[ \sum_{i=1}^{m} \sum_{j=1}^{n} (\eta_{ij}^2 + \tilde{g}_{ij}^2) + \sum_{k=1}^{k} \sum_{l=1}^{n} (\tilde{h}_{ik}^2 + \tilde{h}_{lk}^2) \right] \\
= \frac{1}{2} (\lambda + 1) r.
\end{align*}
\]

Thus, taking expectations of both sides of (32), we have
\[
\begin{align*}
E[\mathcal{L} V(\mathcal{X}(t))]
\leq 2E[\nabla^T H(\mathcal{X}(t))(J(\mathcal{X}(t)) - R(\mathcal{X}(t)))\nabla H(\mathcal{X}(t))]
+ 2E\left\{ \nabla^T H(\mathcal{X}(t))G(t) \int_{t-d}^{t} (\nabla H(\mathcal{X}(s)))' ds \right\}
+ \frac{1}{2} (\lambda + 1) r.
\end{align*}
\]

Due to the fact that
\[
E[\nabla^T H(\mathcal{X}(t))J(\mathcal{X}(t))\nabla H(\mathcal{X}(t))]
= \nabla^T H(\mathcal{X}(t))\tilde{J}(\mathcal{X}(t))\nabla H(\mathcal{X}(t)),
\]
where
\[
\tilde{J}(\mathcal{X}(t)) = \begin{bmatrix}
J(x(t)) & g_1 K g_2^T \\
- g_2 K g_1^T & \tilde{J}(\xi(t))
\end{bmatrix}
= \begin{bmatrix}
J(x(t)) & g_1 K g_2^T \\
- (g_1 K g_2^T)^T & \tilde{J}(\xi(t))
\end{bmatrix},
\]
and the fact that \( \tilde{J}(\mathcal{X}(t)) = -T(\mathcal{X}(t)) \), we get
\[
\nabla^T H(\mathcal{X}(t))\tilde{J}(\mathcal{X}(t))\nabla H(\mathcal{X}(t)) = 0.
\]

Furthermore, the following inequality holds:
\[
\begin{align*}
E[\mathcal{L} V(\mathcal{X}(t))]
\leq - \nabla^T H(\mathcal{X}(t)) \left( \bar{R}(\mathcal{X}(t)) + \bar{R}^T (\mathcal{X}(t)) \right) \nabla H(\mathcal{X}(t))
+ 2\nabla^T H(\mathcal{X}(t))\bar{G} \int_{t-d}^{t} (\nabla H(\mathcal{X}(s)))' ds \\
+ \frac{1}{2} (\lambda + 1) r,
\end{align*}
\]
where \( \bar{R}(\mathcal{X}(t)) = \text{diag}[R(x(t)) + g_1 K g_1^T, \tilde{R}(\xi(t)) - g_2 K g_2^T] \),
\[
\bar{G} = \begin{bmatrix}
g_1 K g_1^T & -g_1 K g_2^T \\
g_2 K g_1^T & -g_2 K g_2^T
\end{bmatrix}.
\]

Since Assumption 3 holds, the following inequalities
\[
- \nabla^T H_1(x(t)) R(x(t)) \nabla H_1(x(t))
\leq - \lambda_{\text{min}} R(x(t)) \nabla^T H_1(x(t)) \nabla H_1(x(t))
\leq - c_1 \nabla^T H_1(x(t)) \nabla H_1(x(t))
= - \nabla^T H_1(x(t))(c_1 I_n) \nabla H_1(x(t)),
\]
\[
- \nabla^T H_2(\xi(t)) \tilde{R}(\xi(t)) \nabla H_2(\xi(t))
\leq - \lambda_{\text{min}} \tilde{R}(\xi(t)) \nabla^T H_2(\xi(t)) \nabla H_2(\xi(t))
\leq - c_2 \nabla^T H_2(\xi(t)) \nabla H_2(\xi(t))
= - \nabla^T H_2(\xi(t))(c_1 I_n) \nabla H_2(\xi(t)),
\]
are true. Then, (41) becomes
\[
\begin{align*}
E[\mathcal{L} V(\mathcal{X}(t))]
\leq - \nabla^T H(\mathcal{X}(t)) \left( \bar{R} + \bar{R}^T \right) \nabla H(\mathcal{X}(t))
+ 2\nabla^T H(\mathcal{X}(t))\bar{G} \int_{t-d}^{t} (\nabla H(\mathcal{X}(s)))' ds \\
+ \frac{1}{2} (\lambda + 1) r,
\end{align*}
\]
where \( \bar{R} = \text{diag}[c_1 I_n + g_1 K g_1^T, c_1 I_n - g_2 K g_2^T] \).
Assume that there exist matrices \( L_1 = L_1^T, L_2 = L_2^T \), and \( L_3 = L_3^T \) such that
\[
(\nabla H(\mathcal{X}(t)) - \nabla H(\mathcal{X}(t-d)))
- \left( \int_{t-d}^{t} (\nabla H(\mathcal{X}(s)))' ds \right)^T
\cdot (-\text{diag}[L_1, L_1] \nabla H(\mathcal{X}(t)) + \text{diag}[L_2, L_2] \nabla H(\mathcal{X}(t-d))
+ \text{diag}[L_3, L_3] \int_{t-d}^{t} (\nabla H(\mathcal{X}(s)))' ds) = 0.
\]

Combining (45) and (46), we deduce that
\[
\begin{align*}
E[\mathcal{L} V(\mathcal{X}(t))]
\leq - \nabla^T H(\mathcal{X}(t)) \left( \bar{R} + \bar{R}^T \right) \nabla H(\mathcal{X}(t))
+ 2\nabla^T H(\mathcal{X}(t))\bar{G} \int_{t-d}^{t} (\nabla H(\mathcal{X}(s)))' ds \\
+ (\nabla H(\mathcal{X}(t)) - \nabla H(\mathcal{X}(t-d)))
- \left( \int_{t-d}^{t} (\nabla H(\mathcal{X}(s)))' ds \right)^T
\cdot (-\text{diag}[L_1, L_1] \nabla H(\mathcal{X}(t))
+ \text{diag}[L_2, L_2] \nabla H(\mathcal{X}(t-d))
+ \text{diag}[L_3, L_3] \int_{t-d}^{t} (\nabla H(\mathcal{X}(s)))' ds) \\
+ \frac{1}{2} (\lambda + 1) r,
\end{align*}
\]
\[
= v^T(t) \Pi_1 v(t) + \frac{1}{2} (\lambda + 1) r,
\]
According to Assumption 2 and Lemma 2, we have
\[
\begin{align*}
\text{E} \{ \mathcal{L} V(\mathcal{X}) \} &\leq -\lambda_{\min}(-\Pi_1)(\Psi^T H_1(x(t))\Psi H_1(x(t))) \\
&\quad + \lambda_{\min}(-\Pi_1)\nabla^T H_2(\xi(t))\nabla H_2(\xi(t)) \\
&\quad + \frac{1}{2} (\lambda + 1) r \\
&= -\lambda_{\min}(-\Pi_1)\nabla^T H(\mathcal{X})\nabla H(\mathcal{X}) \\
&\quad + \frac{1}{2} (\lambda + 1) r.
\end{align*}
\] (49)

According to Assumption 2 and Lemma 2, we have

\[
\begin{align*}
\nabla^T H(\mathcal{X})\nabla H(\mathcal{X}) &= \left( \frac{2\alpha}{2\alpha - 1} (x_1^{(1-a)/2a-1} x_1, \ldots, x_n^{(1-a)/2a-1} x_n, \\
&\quad \frac{2\alpha}{2\alpha - 1} (\xi_1^{(1-a)/2a-1} \xi_1, \ldots, \xi_n^{(1-a)/2a-1} \xi_n) \\
&\quad \frac{2\alpha}{2\alpha - 1} (x_1^{(1-a)/2a-1} x_1, \ldots, x_n^{(1-a)/2a-1} x_n, \\
&\quad \frac{2\alpha}{2\alpha - 1} (\xi_1^{(1-a)/2a-1} \xi_1, \ldots, \xi_n^{(1-a)/2a-1} \xi_n) \right)^T \\
&= \left( \frac{2\alpha}{2\alpha - 1} \right)^2 \left( \sum_{i=1}^n (x_i^{1/(2a-1)})^2 + \sum_{i=1}^n (\xi_i^{1/(2a-1)})^2 \right) \\
&= \left( \frac{2\alpha}{2\alpha - 1} \right)^2 \sum_{i=1}^{2n} \left( \mathcal{X}_i \right)^{1/(2a-1)} \\
&= \left( \frac{2\alpha}{2\alpha - 1} \right)^2 \left( \sum_{i=1}^{2n} \left( \mathcal{X}_i \right)^{1/(2a-1)} \right)^{1/a} \\
&\geq \left( \frac{2\alpha}{2\alpha - 1} \right)^2 \left( \sum_{i=1}^{2n} \left( \mathcal{X}_i \right)^{1/(2a-1)} \right)^{1/a} \\
&= \left( \frac{2\alpha}{2\alpha - 1} \right)^2 V^{1/a}(\mathcal{X}).
\end{align*}
\] (50)

Then inequality in (49) becomes

\[
\begin{align*}
\text{E} \{ \mathcal{L} V(\mathcal{X}) \} &\leq -\lambda_{\min}(-\Pi_1)(\frac{2\alpha}{2\alpha - 1})^2 V^{1/a}(\mathcal{X}) \\
&\quad + \frac{1}{2} (\lambda + 1) r \\
&= -\tau V^{h_i}(\mathcal{X}) + a_i,
\end{align*}
\] (51)
where \( \tau = \lambda_{\min}(-\Pi_1)(2a/(2a - 1))^2 > 0 \), \( 0 < h_i = (1/a) < 1 \) and \( a_i = (1/2)(\lambda + 1)r > 0 \). Thus, we obtain that inequality (51) satisfies (11) in Lemma 1. In addition, there exist two class-$\mathcal{L}_\infty$ functions \( \gamma_1(\|\mathcal{X}\|) = 2\|\mathcal{X}\| \) and \( \gamma_2(\|\mathcal{X}\|) = 2\|\mathcal{X}\|^2 \) such that (10) in Lemma 1 holds.

Eventually, in view of Lemma 1, we arrive at a conclusion that systems (7) and (8) are SGFSSP under the output feedback controller (21). Furthermore, the settling time \( T^*_i \) is obtained and satisfies

\[
T^*_i = \frac{1}{b_i \tau(1 - h_i)} V^{1-h_i}(\mathcal{X}_0) \\
= \frac{1}{b_i \tau(1 - h_i)} \left( \frac{a_i}{\tau(1 - b_i)} \right)^{(1-h_i)b_i},
\] (52)
where \( 0 < b_i < 1 \). In addition, the compact set \( \Omega_i \) is expressed as

\[
\Omega_i = \left\{ x \mid V^{h_i}(\mathcal{X}) \leq \frac{a_i}{\tau(1 - b_i)} \right\}.
\] (53)

The proof of this theorem is now completed. \( \square \)

Remark 3. In [33], the channel is modeled as a cascade of a multiplicative noise and an additive white Gaussian noise. Based on this channel, we take the constant transmission delay into consideration. Thus, the channel model in this paper is more general. In addition, [33] proposes a state feedback controller design strategy to stabilize linear systems. Meanwhile, this paper deals with the output feedback simultaneous stabilization problem for stochastic PCH systems in finite time.

Remark 4. Under Lemma 1, how to choose a suitable Lyapunov function is an essential difficulty during the research. Accordingly, we have overcome this difficulty by taking \( H(x) \) as a Lyapunov function, and \( H(x) \) has a concrete form which is given in (19) in Assumption 2.

Remark 5. Through the proof of Theorem 1, we can see that even if the dimensions of \( x(t) \) are not the same as that of \( \xi(t) \), the result of Theorem 1 still holds. Thus, the design strategy of controllers in Theorem 1 can be extended to multiple systems. Thus, we have the following analysis about SGFSSP of multiple stochastic PCH systems.

Next, consider the following multiple stochastic PCH systems:
\[
\left\{ \begin{array}{l}
\frac{dx_j(t)}{dt} = [J_j(x_j(t)) - R_j(x_j(t))] \nabla H_j(x_j(t)) dt
\end{array} \right.
\]
\[+ g_j u(t) dt + h_j d\omega_j(t), \]
\[y_j(t) = g_j^T(x_j) \nabla H_j(x_j), \quad j = 1, 2, \ldots, V,
\]
where \( V \) is the number of stochastic systems, \( x_j(t) \in \mathbb{R}^n \) is the plant state vector, \( y_j(t) \in \mathbb{R}^m \) is the output of the plant, and the signal \( \omega_j(t) \in \mathbb{R}^s \) is the independent scalar Wiener process with \( E[\omega_j(t)] = 0 \), \( E[(d\omega_j(t))^2] = dt \). \( \nabla H_j(x_j) \) is the gradient of the Hamilton function \( H_j(x_j) : \mathbb{R}^n \rightarrow \mathbb{R} \), which is defined as \( \nabla H_j(x_j) = (\partial H_j(x_j)/\partial x_j) \), and \( H_j(x_j(t)) \geq 0, H_j(0) = 0, \) for all \( t \geq 0 \). \( J_j(x_j) \) is a skew-symmetric structure matrix; \( R_j(x) \in \mathbb{R}^{n \times m} \) is a positive definite strict dissipation matrix; \( g_j \) and \( h_j \) are known real constant gain matrices. In addition, \( J_j(x_j), R_j(x_j), g_j, \) and \( h_j \) satisfy locally Lipschitz condition.

**Assumption 4.** The Hamilton functions \( H_j(x_j) \) are given as
\[
H_j(x_j) = \sum_{i=1}^{n_j} (x_j^i)^{a/(2a-1)}.
\]

**Assumption 5.** There exist constants \( c_3 > 0 \) and \( c_4 > 0 \) such that
\[
c_3 = \inf_{t \in \mathbb{R}} \left\{ \lambda \min \left\{ R_j(x_j(t)), \ldots, R_j(x_j(t)) \right\} \right\},
\]
\[
c_4 = \inf_{t \in \mathbb{R}} \left\{ \lambda \min \left\{ R_{j_{i+1}}(x_{j_{i+1}}(t)), \ldots, R_{j_{i+1}}(x_{j_{i+1}}(t)) \right\} \right\}.
\]

Assume that we can find out an arbitrary permutation \((j_1, j_2, \ldots, j_V)\) from the positive integer set \( \{1, 2, \ldots, V\} \) and that \( S \) is a positive integer which satisfies \( 1 \leq S \leq V - 1 \). In addition, taking \( V_1 = n_{j_1} + \cdots + n_{j_S}, V_2 = n_{j_{S+1}} + \cdots + n_{j_V}, \)
\( r_1 = r_{j_1} + \cdots + r_{j_S}, \) and \( r_2 = r_{j_{S+1}} + \cdots + r_{j_V} \), we divide the \( V \) stochastic PCH systems into two parts: \( \{j_1, \ldots, j_S\} \) and \( \{j_{S+1}, \ldots, j_V\} \).

Defining the vectors \( \mathcal{X}_1(t) = [x_{j_1}^T(t), \ldots, x_{j_S}^T(t)]^T \in \mathbb{R}^{V_1}, \mathcal{W}_1(t) = [\omega_{j_1}^T(t), \ldots, \omega_{j_S}^T(t)]^T \in \mathbb{R}^{V_1}, \mathcal{X}_2(t) = [x_{j_{S+1}}^T(t), \ldots, x_{j_V}^T(t)]^T \in \mathbb{R}^{V_2}, \mathcal{W}_2(t) = [\omega_{j_{S+1}}^T(t), \ldots, \omega_{j_S}^T(t)]^T \in \mathbb{R}^{V_2}, \) then system (54) becomes
\[
\left\{ \begin{array}{l}
\frac{d\mathcal{X}_1(t)}{dt} = [\mathcal{J}_1(\mathcal{X}_1(t)) - \mathcal{R}_1(\mathcal{X}_1(t))] \nabla \mathcal{H}_1(\mathcal{X}_1(t)) dt
\end{array} \right.
\]
\[+ \tilde{g}_1 u(t) dt + \tilde{h}_1 dW_1(t), \]
\[Y_1(t) = \tilde{g}_1^T \nabla \mathcal{H}_1(\mathcal{X}_1(t)), \]
\[
\frac{d\mathcal{X}_2(t)}{dt} = [\mathcal{J}_2(\mathcal{X}_2(t)) - \mathcal{R}_2(\mathcal{X}_2(t))] \nabla \mathcal{H}_2(\mathcal{X}_2(t)) dt
\]
\[+ \tilde{g}_2 u(t) dt + \tilde{h}_2 dW_2(t), \]
\[Y_2(t) = \tilde{g}_2^T \nabla \mathcal{H}_2(\mathcal{X}_2(t)),
\]
where
\[
\mathcal{J}_1(\mathcal{X}_1) = -\mathcal{J}_1(\mathcal{X}_1),
\]
\[
= \text{diag}(\mathcal{J}_{j_1}(x_{j_1}), \ldots, \mathcal{J}_{j_S}(x_{j_S})) \in \mathbb{R}^{V_1 \times V_1},
\]
\[
\mathcal{R}_1(\mathcal{X}_1) = \text{diag}(\mathcal{R}_{j_1}(x_{j_1}), \ldots, \mathcal{R}_{j_S}(x_{j_S})) > 0,
\]
\[
\tilde{g}_1 = [g_{j_1}, \ldots, g_{j_S}]^T \in \mathbb{R}^{V_1 \times m},
\]
\[
\tilde{h}_1 = \text{diag}(h_{j_1}, \ldots, h_{j_S}) \in \mathbb{R}^{V_1 \times r_1},
\]
\[
\mathcal{J}_2(\mathcal{X}_2) = -\mathcal{J}_2(\mathcal{X}_2),
\]
\[
= \text{diag}(\mathcal{J}_{j_{S+1}}(x_{j_{S+1}}), \ldots, \mathcal{J}_{j_S}(x_{j_S})) \in \mathbb{R}^{V_2 \times V_2},
\]
\[
\mathcal{R}_2(\mathcal{X}_2) = \text{diag}(\mathcal{R}_{j_{S+1}}(x_{j_{S+1}}), \ldots, \mathcal{R}_{j_S}(x_{j_S})) > 0,
\]
\[
\tilde{g}_2 = [g_{j_{S+1}}, \ldots, g_{j_S}]^T \in \mathbb{R}^{V_2 \times m},
\]
\[
\tilde{h}_2 = \text{diag}(h_{j_{S+1}}, \ldots, h_{j_S}) \in \mathbb{R}^{V_2 \times r_2},
\]
\[
\mathcal{H}_1(\mathcal{X}_1) = \sum_{k=1}^{S} \mathcal{H}_{j_{k}}(\mathcal{X}_{j_{k}}), \mathcal{H}_2(\mathcal{X}_2),
\]
\[
Y_1 = y_{j_1} + \cdots + y_{j_S},
\]
\[
Y_2 = y_{j_{S+1}} + \cdots + y_{j_S}.
\]

Substituting (4) into systems (57) and (58), we have
\[
\left\{ \begin{array}{l}
\frac{d\mathcal{X}_1(t)}{dt} = [\mathcal{J}_1(\mathcal{X}_1(t)) - \mathcal{R}_1(\mathcal{X}_1(t))] \nabla \mathcal{H}_1(\mathcal{X}_1(t)) dt
\end{array} \right.
\]
\[+ \tilde{g}_1 u(t) dt + \tilde{h}_1 dW_1(t), \]
\[Y_1(t) = \tilde{g}_1^T \nabla \mathcal{H}_1(\mathcal{X}_1(t)),
\]
\[
\frac{d\mathcal{X}_2(t)}{dt} = [\mathcal{J}_2(\mathcal{X}_2(t)) - \mathcal{R}_2(\mathcal{X}_2(t))] \nabla \mathcal{H}_2(\mathcal{X}_2(t)) dt
\]
\[+ \tilde{g}_2 u(t) dt + \tilde{h}_2 dW_2(t), \]
\[Y_2(t) = \tilde{g}_2^T \nabla \mathcal{H}_2(\mathcal{X}_2(t)).
\]

Furthermore, we can obtain the following corollary.

**Corollary 1.** Consider systems (60) and (61). Assumptions 4 and 5 are satisfied. If there exist matrices \( K = K^T, N_1 = N_1^T, N_2 = N_2^T, N_3 = N_3^T, \) an arbitrary permutation \((j_1, j_2, \ldots, j_V)\) of \( \{1, 2, \ldots, V\} \) and a positive integer \( S(1 \leq S \leq V) \) such that
Complexity

$$\Pi_2 = \begin{bmatrix} B_{11} & B_{12} & 0 & 0 & B_{15} & -\bar{g}_1K\bar{g}_2^T \\ * & -N_2 & 0 & 0 & B_{25} & 0 \\ * & * & B_{33} & B_{34} & \bar{g}_2K\bar{g}_1^T & B_{36} \\ * & * & * & -\frac{1}{2}N_2 & 0 & B_{46} \\ * & * & * & * & -N_3 & 0 \\ * & * & * & * & * & -N_3 \end{bmatrix} < 0$$

(62)

holds, where

$$B_{11} = -2\alpha_3I_{V_1} - 2\bar{g}_1K\bar{g}_1^T - N_1,$$

$$B_{12} = \frac{1}{2}(N_1 + N_2),$$

$$B_{15} = \bar{g}_1K\bar{g}_1^T + \frac{1}{2}(N_1 + N_3),$$

$$B_{25} = \frac{1}{2}(N_2 + N_3),$$

$$B_{33} = -2\alpha_4I_{V_2} + 2\bar{g}_2K\bar{g}_2^T - N_1,$$

$$B_{34} = \frac{1}{2}(N_1 + N_2),$$

$$B_{36} = -\bar{g}_2K\bar{g}_2^T + \frac{1}{2}(N_1 + N_3),$$

$$B_{46} = \frac{1}{2}(N_2 + N_3),$$

then systems (60) and (61) are SGFSSP under the output feedback control law:

$$u(t - d) = -M^{-1}K(Y_1(t - d) - Y_2(t - d)).$$  (64)

Proof: the proof of this corollary is similar to the proof of Theorem 1, and we give some main analysis. First, similar to the proof of Theorem 1, we can obtain an augmented Itô form stochastic PCH system with state $\mathcal{X}(t) = [\mathcal{X}_1(t) \mathcal{X}_2(t)]^T$. Next, choose the following Lyapunov function candidate:

$$V(\mathcal{X}(t)) = 2(F_{11}(\mathcal{X}_1(t)) + F_{12}(\mathcal{X}_2(t))).$$  (65)

Let $\lambda_1 = \sup_{x \in \mathbb{R}^n} \|\text{Hess}(\sum_{j=1}^V g_j^Tg_j + \sum_{j=1}^V h_j^Th_j)\|^2$ and $\tau_1 = \text{tr}(\sum_{j=1}^V g_j^Tg_j + \sum_{j=1}^V h_j^Th_j)$. Then, we have

$$\mathbb{E}\{\mathcal{L}V(\mathcal{X})\} \leq -\lambda_1(-\Pi_2)\left(\frac{2\alpha}{2\lambda - 1}\right)^2V^{1/\alpha}(\mathcal{X})$$

$$+ \frac{1}{2}(\lambda_1 + 1)\tau_1$$

$$= -\bar{\alpha}V^{1/\alpha}(\mathcal{X}) + \bar{\alpha},$$

where $\bar{\alpha} = \lambda_1(-\Pi_2)(2\alpha/(2\alpha - 1))^2 > 0$ and $\bar{\alpha} = (1/2)(\lambda_1 + 1)\tau_1 > 0$. In the end, under Lemma 1, we can see that systems (60) and (61) are SGFSSP under the output feedback controller (64). Furthermore, the settling time $T_2^*$ satisfies

$$T_2^* = \frac{1}{bc(1-h_1)}\left(\frac{\bar{a}}{c(1-b)}\right)^{(1-h_1)/h_1},$$

(67)

where $0 < b < 1$. In addition, we have

$$\Omega_2 = \left\{x | V^{1/\alpha}(\mathcal{X}) \leq \frac{\bar{a}}{c(1-b)}\right\}.$$  (68)

The proof of this corollary is now completed.

4. Illustrative Example

In this section, a numerical example is performed to illustrate the stabilization scheme for stochastic PCH systems subject to delayed and fading channels.

The considered systems are two stochastic PCH systems:

$$\begin{cases}
\frac{dx}{dt} = [J(x) - R(x)]VH_1(x)dt + g_1u(t)dt \\
\quad + h_1d\omega(t), \\
y(t) = g_1^T\nabla H_1(x(t))
\end{cases}$$

(69)

$$\begin{cases}
\frac{d\xi}{dt} = [\bar{J}] \quad (\xi)dt - n\bar{R}(\xi)\nabla H_2(\xi)dt + g_2u(t)dt + H_2d\phi(t), \\
\eta(t) = g_2^T\nabla H_2(\xi(t))
\end{cases}$$

(70)

where $x = [x_1 \ x_2]^T \in \mathbb{R}^2$, $H_1(x) = x_1^{\alpha/3} + x_2^{\alpha/3}$ ($\alpha = 2$), $H_2(\xi) = \xi_1^{\alpha/3} + \xi_2^{\alpha/3}$ ($\alpha = 2$),

$$J(x) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix},$$

$$R(x) = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix},$$

$$\bar{J}(\xi) = \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix},$$

$$\bar{R}(\xi) = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix},$$

(71)

$$g_1 = [0.025 \ 0.05]^T,$$

$$h_1 = [0.05 \ 0.05]^T,$$

$$g_2 = [0.05 \ 0.05]^T,$$

$$h_2 = [0.05 \ 0.05]^T.$$

We take the constant transmission delay $d = 0.25$. The control input signal $u(t - d)$ is sent through the delayed and fading noisy channels to the actuator, so a single controller
for systems (69) and (70) is modeled as (4). Through statistical experiments, the probability density function of multiplicative noise $\epsilon(t)$ with mean $\mu = 0.9$ and variance $\sigma^2 = 0.065$ is listed as

$$p_t (s) = \begin{cases} 0.05, & s = 0, \\ 0.10, & s = 0.5, \\ 0.85, & s = 1. \end{cases}$$  (72)

Obviously, $M = \mu = 0.9$.

Substituting (4) into (69) and (70) leads to

$$dx(t) = [J(x) - R(x)]\nabla H_1(x)dt + g_1\epsilon(t)u(t - 0.25)dt + g_1d\omega(t) + h_1d\varphi(t),$$  (73)

$$d\xi(t) = [\tilde{J}(\xi) - \tilde{R}_2(\xi)]\nabla H_2(\xi)dt + g_2\epsilon(t)u(t - 0.25)dt + g_2d\omega(t) + h_2d\varphi(t).$$  (74)

Choose $L_1 = \begin{bmatrix} 2 & 1 \\ 1 & -1 \end{bmatrix}$, $L_2 = \begin{bmatrix} 1 & 1 \\ 1 & 6 \end{bmatrix}$, and $L_3 = \begin{bmatrix} 3 & 1 \\ 1 & 1 \end{bmatrix}$.

Then we can use the LMI toolbox of MATLAB to obtain

$$K = 4.447.$$  (75)

To sum up, all the conditions of Theorem 1 hold. Consequently, from Theorem 1, systems (73) and (74) are SGFSSP under the output feedback control law:

$$u(t - d) = -0.16x_1^{1/3}(t - d) - 0.33x_2^{1/3}(t - d) + 0.33\xi_1^{1/3}(t - d) + 0.33\xi_2^{1/3}(t - d).$$  (76)

In the simulation, we choose the initial states of the systems as $x(0) = [-0.3 \ 0.3]^T$, $\xi(0) = [0.6 \ 0.4]^T$, and set the parameter $b_i = 0.6$. Then, considering (52) in Theorem 1, it is easy to obtain that the settling time $T^{*}_{11} = 0.32s$. The state trajectories of $x$ and $\xi$ are shown in Figures 2 and 3, respectively.

From Figures 2 and 3 in the simulation, we can see that a single controller (76) can simultaneously stabilize systems (69) and (70) in finite time. The settling time $T^{*}_{11}$ is consistent with that of (52), and the states converge to the origin in 0.32 s. In summary, the output feedback controller proposed in Theorem 1 performs well in the SGFSSP of systems (69) and (70).

5. Conclusion

In this paper, the finite-time simultaneous stabilization in probability of stochastic PCH systems over delayed and fading channels has been investigated. On the basis of the dissipative Hamiltonian structural properties and Lyapunov functional technique, a single output feedback controller has been designed for two stochastic PCH systems, which guarantees the SGFSSP of the closed-loop Hamiltonian systems. The case of multiple stochastic PCH systems also has been studied. Sufficient conditions for the existence of the stabilization controllers have been derived in consideration of the phenomenon of delayed and fading channels. At last, a numerical example has highlighted the effectiveness of the stabilization technology proposed in this paper. As for network-based control, sometimes the states of systems are not fully measured, so a possible future research will be involved in observer-based simultaneous stabilization of a set of stochastic PCH systems over delayed and fading channels.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Acknowledgments

This work was supported by the National Natural Science Foundation of China under grant 62073189.
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