Numerical investigation of the threshold for primordial black hole formation

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Abstract. First results of a numerical investigation of primordial black hole formation in the radiation dominated phase of the Early Universe are presented. The simulations follow the gravitational collapse of three different families of high-amplitude density fluctuations imposed at the time of horizon crossing. The threshold for black hole formation, \( \delta_c \approx 0.7 \), is found to be nearly identical for all perturbation families if the control parameter, \( \delta \), is chosen as the total excess mass within the initial horizon volume. Furthermore, we demonstrate that the scaling of black hole mass with distance from the formation threshold, known to occur in near-critical gravitational collapse, applies to primordial black hole formation.

1. Introduction

Every quantitative analysis of the primordial black hole (PBH) number and mass spectrum [1] requires knowledge of the threshold parameter, \( \delta_c \), separating perturbations that form black holes from those that do not, and the resulting black hole mass, \( M_{bh} \), as a function of distance from the threshold. In order to determine \( \delta_c \) and \( M_{bh} \) for various initial conditions, we performed one-dimensional, general relativistic simulations of the hydrodynamics of PBH formation in the radiation-dominated phase of the early universe. Three families of perturbation shapes were chosen to represent “generic” classes of initial data, reflecting the lack of information about the specific shape of primordial fluctuations. The numerical technique is sketched in Section 2. Defined as the excess mass within the horizon sphere at the onset of the collapse, we find \( \delta_c \approx 0.7 \) for all three perturbation shapes, indicating that the threshold value may indeed be universal (Section 3). A numerical confirmation of the previously suggested power-law scaling of \( M_{bh} \) with \( \delta - \delta_c \) [3], related to the well-known behavior of collapsing space–times at the critical point of black hole formation [3], is presented in Section 4. In this framework, the PBH mass spectrum is determined by the dimensionless coefficient, \( K \), the scaling exponent, \( \gamma \), and the initial horizon mass, \( M_h \), such that

\[
M_{bh} = KM_h(\delta - \delta_c)\gamma.
\]

We provide numerical results for \( K \) and \( \gamma \) for the three perturbation families. A more detailed description of the numerical technique and further results will be presented in a forthcoming publication [4].
2. Numerical technique

The dynamics of collapsing density perturbations in the Early Universe are determined by the general relativistic equations of motion for a perfect fluid, the field equations, the first law of thermodynamics, and a radiation-dominated equation of state. The assumption of spherical symmetry is well justified for large fluctuations in a Gaussian distribution [5], reducing the problem to one spatial dimension.

For our simulations, we chose the formulation of the hydrodynamical equations by Hernandez and Misner [6] as implemented by Baumgarte et al. [7]. Based on the original equations by Misner and Sharp [8], Hernandez and Misner proposed to exchange the Schwarzschild time variable, $t$, with the outgoing null coordinate, $u$. In so doing, the hydrodynamical equations retain the Lagrangian character of the Misner–Sharp equations but avoid crossing into the event horizon of a black hole once it has formed. Covering the entire space–time outside while asymptotically approaching the event horizon, the Hernandez–Misner equations are perfectly suited to follow the evolution of a black hole for long times after its formation without encountering coordinate singularities. This allowed us, in principle, to study the accretion of material onto newly formed PBHs for arbitrarily long times (in contrast with earlier calculations [9]). Since the expanding outer regions of our simulated piece of the universe are most conveniently tracked in a comoving numerical reference frame, the Lagrangian form of the Hernandez–Misner equations is their second major asset. It also provides a simple prescription for the outer boundary condition, which is defined to match the exact solution of the Friedmann equations for a radiation dominated flat universe. Hence, the pressure follows the analytic solution

$$P = P_0 \left( \frac{\tau}{\tau_0} \right)^{-2},$$

where $\tau$ is the proper time of the outermost fluid element (corresponding to the cosmological time $t$ in a Friedmann–Robertson–Walker (FRW) universe) and $P_0$ and $\tau_0$ are the initial values for pressure and proper time.

3. Threshold for black hole formation

We studied the spherically symmetric evolution of three families of curvature perturbations. Initial conditions were chosen to be perturbations of the energy density, $\epsilon = \rho_0 \epsilon_0$, in unperturbed Hubble flow specified at horizon crossing. The first family of perturbations is described by a Gaussian-shaped overdensity that asymptotically approaches the FRW solution at large radii. The other two families of initial conditions involve a spherical Mexican Hat function and a fourth order polynomial. These functions are characterized by rarefaction regions outside of the horizon radius, $R_h$, that identically compensate for the additional mass of the overdensities inside the horizon volume, so that the mass derived from the total integrated density profile is equal to that of an unperturbed FRW solution. In our numerical experiments, the amplitude, $A$, of the perturbations is used to tune the initial conditions to sub- or supercriticality with respect to black hole formation. The critical amplitude, $A_c$, is strongly shape-dependent, varying between $A_c = 3.04$ for Mexican-Hat-shaped perturbations and $A_c = 2.05$ for the Gaussian curve. If, however, we define the control parameter $\delta$ as the additional mass inside $R_h$ in units of the horizon mass, we find strikingly similar values — $\delta_c = 0.67$ (Mexican Hat), $\delta_c = 0.70$ (Gaussian
curve), and $\delta_c = 0.71$ (polynomial) — for all three families of initial data in our study. This number is considerably greater than the previously employed threshold $\delta_c = 1/3$ following from analytical estimates [1]. Given the qualitative difference of the functional forms of the different perturbation families, the result that $\delta_c$ of the Gaussian perturbation lies in between the critical values of the mass compensated functions is surprising. Our results suggest that $\delta_c \approx 0.7$ (with $\delta$ defined as above) is a universal statement, i.e., true for all perturbation shapes in the radiation-dominated regime. This remains to be verified by means of additional experiments.

4. Scaling of PBH masses with distance to the threshold

For a variety of matter models, it is well-known that the dynamics of near-critical collapse exhibit continuous or discrete self-similarity and power-law scaling of the black hole mass with the offset from the critical point \{Eq. (1) [3, 10]\}. In particular, Evans and Coleman [11] found self-similarity and mass scaling in numerical experiments of a collapsing radiation fluid. They numerically determined the scaling exponent $\gamma \approx 0.36$, followed by a linear perturbation analysis of the critical solution by Koike et al. [12] that yielded $\gamma \approx 0.3558$. Until recently, it was believed that entering the scaling regime requires a degree of fine-tuning of the initial data that is unnatural for any astrophysical application. It was noted [12] that fine-tuning to criticality occurs naturally in the case of PBHs forming from a steeply declining distribution of primordial density fluctuations, as generically predicted by inflationary scenarios. In the radiation-dominated cosmological epoch, the only difference with the fluid collapse studied by Evans and Coleman [11] is the expanding, finite-density-background space–time of a FRW universe. Assuming that self-similarity and mass scaling are consequences of an intermediate asymptotic solution that is independent of the asymptotic boundary conditions, Eq. (1) is applicable to PBH masses, allowing the derivation of a universal PBH initial mass function [2]. Figure (1) presents numerical evidence for mass scaling according to Eq. (1) in black hole collapse in an asymptotic FRW space–time. All three perturbation families give rise to scaling solutions with a scaling exponent $\gamma \approx 0.36$.

5. Conclusions

This work discusses numerical collapse simulations of three generic families of energy density perturbations, one with a finite total excess mass with respect to the unperturbed FRW solution and two mass compensated ones. Among various possible definitions for the collapse control parameter $\delta$, the total excess gravitational mass of the perturbed space–time with respect to the unperturbed FRW background enclosed in the initial horizon volume is the only one that gives rise to a similar threshold value for all three shape families, $\delta_c \approx 0.7$. Whether this result is an indication for universality of $\delta_c$ in this specific definition needs to be verified with the help of additional simulations using a larger sample of initial perturbation shapes.

The previously suggested [2] scaling relation between $M_{bh}$ and $\delta - \delta_c$, based on the analogy with critical phenomena observed in near-critical black hole collapse in asymptotically non-expanding space–times [13], is confirmed numerically for an asymptotic FRW background. For the smallest black holes in this investigation, the scaling exponent is $\gamma \approx 0.36$, which is identical to the non-expanding numerical and analytical results [11, 12] within our numerical accuracy.
Figure 1. Black hole masses as a function of $\delta - \delta_c$ for three different perturbation shape families. The best fit parameters to equation (1) are: $\gamma = 0.36$, $K = 2.85$, $\delta_c = 0.6745$ (Mexican hat perturbation, triangles); $\gamma = 0.37$, $K = 2.39$, $\delta_c = 0.7122$ (polynomial perturbation, crosses); $\gamma = 0.34$, $K = 11.9$, $\delta_c = 0.7015$ (Gaussian curve perturbation, diamonds).

The parameter $K$ of Eq. (1), needed in addition to $\gamma$ to evaluate the two-parameter PBH IMF derived in [2], ranges from $K \approx 2.4$ to $K \approx 12$.

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