Cremmer–Gervais $r$-Matrices and the Cherednik Algebras of Type $GL_2$

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Abstract. We give an interpretation of the Cremmer–Gervais $r$-matrices for $\mathfrak{sl}_n$ in terms of actions of elements in the rational and trigonometric Cherednik algebras of type $GL_2$ on certain subspaces of their polynomial representations. This is used to compute the nilpotency index of the Jordanian $r$-matrices, thus answering a question of Gerstenhaber and Giaquinto. We also give an interpretation of the Cremmer–Gervais quantization in terms of the corresponding double affine Hecke algebra.

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1. Introduction

Let $\mathfrak{sl}_n$ denote the Lie algebra of traceless $n \times n$ matrices having entries in a field $k$. Let $V$ denote the vector representation of $\mathfrak{sl}_n$ and let $r \in \mathfrak{sl}_n \otimes \mathfrak{sl}_n \subset End(V \otimes V)$ be a skew-symmetric linear operator. Define $r_{12} := r \otimes 1$, $r_{23} := 1 \otimes r$, and $r_{13} := P_{23}r_{12}P_{23}$ where $P_{23}$ is the permutation operator on $V \otimes 3$: $P_{23}(u \otimes v \otimes w) = u \otimes w \otimes v$. An important class of operators which arise in studying Lie bialgebras and Poisson–Lie groups are those satisfying the modified classical Yang–Baxter equation (MCYBE)

$$[r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}] - \lambda (P_{123} - P_{213}) = 0$$

for some $\lambda \in k$ (see [3,8] for more details). The solutions to the MCYBE are called classical $r$-matrices and fall into two classes: those satisfying the MCYBE for $\lambda$ nonzero (resp. zero) are called quasitriangular (resp. triangular).

In the early 1980s, Belavin and Drinfel’d successfully classified all quasitriangular $r$-matrices in the case when $k$ is the field of complex numbers [1]. This classification gives us a solution space which we view as a disjoint union of quasi-projective subvarieties of $\mathbb{P}(\mathfrak{sl}_n \otimes \mathfrak{sl}_n)$. In contrast, the triangular $r$-matrices are
more mysterious as there is not a constructive classification of them (except in the smaller cases of \( sl_2 \) and \( sl_3 \), see [3, Chapter 3]), only a homological interpretation exists (due to the work of Stolin [15]: for details see [3, Section 3.1.D.] and [8, Section 3.5]), and there are currently few known examples.

In the paper [9], Gerstenhaber and Giaquinto investigate the behavior along the boundaries of the aforementioned quasi-projective varieties and show that the boundary points are all triangular \( r \)-matrices. In the same paper, they construct the most general class of known examples of triangular \( r \)-matrices, the so-called generalized Jordanian \( r \)-matrices \( rJ, n \) (see [7,9]). They prove that the Jordanian \( r \)-matrices lie on the boundary of the component corresponding to the quasitriangular Cremmer–Gervais \( r \)-matrices (discussed in [6,7,9,11–13]) and conjecture that \( rJ, 3, n = 0 \).

In Theorem 4.3, we prove that the nilpotency index of \( rJ, n \) is quite different than conjectured. We do this by first interpreting the quantum Cremmer–Gervais \( R \)-matrix \( R \) in terms of the double affine Hecke algebra (DAHA) \( HH_{q,t} \) of type \( GL_2 \) (Theorem 2.2) and the classical Cremmer–Gervais \( r \)-matrix in terms of the degenerate DAHA \( HH'_c \) (Theorem 3.4). Using Suzuki's embedding [16] of the rational Cherednik algebra \( HH''_c \) into \( HH'_c \), we give a simple interpretation of both the Cremmer–Gervais and Jordanian \( r \)-matrices as operators on the polynomial representation of \( HH''_c \). Using the relations in \( HH''_c \), we find that the nilpotency index of \( rJ, n \) is \( n \) when \( n \) is odd, and \( 2n - 1 \) when \( n \) is even. The conceptual difference between the two cases has a representation theoretic origin: the polynomial representation of the rational Cherednik algebra \( HH'_c \) of type \( A_1 \) is reducible if and only if the deformation parameter \( c \) has the form \( n/2 \) for an odd positive integer \( n \) (a special case of Dunkl's theorem [2]).

2. The Yang–Baxter Equations and the Double Affine Hecke Algebra

Let \( k \) be a field of characteristic 0 and let \( K = k(q, t^{1/2}) \). We begin with a

**DEFINITION 2.1.** (see [4,5]) The double affine Hecke algebra \( HH_{q,t} \) of type \( GL_2 \) is the associative \( K \)-algebra with generators \( X_1, X_2, Y_1, Y_2, T \) and relations

\[
X_j X^{-1}_j X_j = Y_j Y^{-1}_j Y_j = Y_j^{-1} Y_j = 1, \\
(T - t^{1/2})(T + t^{-1/2}) = 0, \quad TX_1 T = X_2, \quad TY_2 T = Y_1, \\
Y_2^{-1} X_1 Y_2 X_1^{-1} = T^2, \quad Y_1 Y_2 X_j = q X_j Y_1 Y_2, \\
Y_j X_1 X_2 = q X_1 X_2 Y_j, \quad [Y_1, Y_2] = 0, \quad [X_1, X_2] = 0, \\
\]

for \( j = 1, 2 \).

The \( K \)-vector space \( K[X_1, X_2] \) can be made into a \( HH_{q,t} \)-module, called the *polynomial representation*, defined as follows. Let (12) act on \( K[X_1, X_2] \) by swapping variables and let \( S = \frac{1 - (12)}{X_1 - X_2} \). For integers \( a, b \) define \( \Gamma_{a,b}.f(X_1, X_2) := \)
\( f(q^a X_1, q^b X_2) \). The double affine Hecke algebra \( \mathcal{H}_{q,t} \) acts faithfully on \( K[X_1^{\pm 1}, X_2^{\pm 1}] \) via

\[
T \mapsto t^{1/2}(12) - (t^{1/2} - t^{-1/2})X_2S,
Y_1 \mapsto T \Gamma_{0,1}(12), \quad Y_2 \mapsto \Gamma_{0,1}(12)T^{-1},
\]

and the \( X \)'s act via multiplication. For an operator \( R \in \text{End}_K(K[X^{\pm 1}, X^{\pm 1}, X^{\pm 1}]) \), let \( R_{12}, R_{13}, R_{23} \) denote the corresponding operators on \( K[X^{\pm 1}, X^{\pm 1}, X^{\pm 1}] \). In [10], Gerstenhaber and Giaquinto introduced the \textit{modified quantum Yang–Baxter equation} (MQYBE). An operator is called a \textit{modified quantum \( R \)-matrix} if it satisfies the MQYBE

\[
R_{12}R_{13}R_{23} - R_{23}R_{13}R_{12} - \lambda(P_{123}R_{12} - P_{213}R_{23}) = 0
\]

for some scalar \( \lambda \). Here, \( P_{ijk} \) denotes the permutation on the variables \( X_i \mapsto X_j \mapsto X_k \mapsto X_i \). Furthermore, \( R \) is called \textit{unitary} if \( R(12)R(12) = 1 \).

The classical analogue of the MQYBE is called the \textit{modified classical Yang–Baxter equation} (MCYBE). An operator \( r \) is called a \textit{classical \( r \)-matrix} if it satisfies the MCYBE

\[
[r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}] - \lambda(P_{123} - P_{213}) = 0
\]

for some scalar \( \lambda \) (for more details, see [3]). Furthermore, \( r \) is called \textit{skew-symmetric} if \( (12)r(12) = -r \). For shorthand, we will denote the left hand sides of the above equations by MQYBE\(_\lambda(R)\) and MCYBE\(_\lambda(r)\), respectively.

**Theorem 2.2.** The operator \( R = (12)Y_2(12)Y_2^{-1} \) is unitary and satisfies the \textit{modified quantum Yang–Baxter equation}

\[
R_{12}R_{13}R_{23} - R_{23}R_{13}R_{12} = (1 - t^{-1})^2(P_{123}R_{12} - P_{213}R_{23}).
\]

**Proof.** It is obvious that \( R \) is unitary. To show \( R \) satisfies the MQYBE, we first set \( v = (X_1 + X_2)S \). This operator satisfies MQYBE\(_1(v) = 0\), MCYBE\(_1(v) = 0\), and \( v^2 = 0 \). Therefore, it follows that MQYBE\(_{1,2}(\exp(\lambda v)) = 0\) for all scalars \( \lambda \). Setting \( F = \Gamma_{0,-1} \) and \( \hat{R} = \exp((1 - t^{-1})v) \), we have \( R = F_{21}^{-1}\hat{R}F_{12} \). Furthermore

(i) \( F_{12}F_{13}F_{23} = F_{23}F_{13}F_{12} \),
(ii) \( \hat{R}_{12}F_{23}F_{13} = F_{13}F_{23}\hat{R}_{12} \),
(iii) \( \hat{R}_{23}F_{12}F_{13} = F_{13}F_{12}\hat{R}_{23} \).

So \( R \) is obtained by twisting \( \hat{R} \) by \( F \) and hence satisfies the MQYBE (cf. [14, Thm. 1]).
3. Semiclassical Limit

Setting \( q = e^{\hbar} \) and \( t = e^{\epsilon\hbar} \), we view \( K \) as a subfield of \( k[[c, \hbar]] \) and the above formulae for the \( Y \)'s in the polynomial representation become

\[
\begin{align*}
Y_1 &\mapsto 1 + \hbar \left( X_1 \frac{\partial}{\partial X_1} + cX_2 S + \frac{c}{2} \right) + O(\hbar^2) \\
Y_2 &\mapsto 1 + \hbar \left( X_2 \frac{\partial}{\partial X_2} - cX_2 S - \frac{c}{2} \right) + O(\hbar^2)
\end{align*}
\]

Define \( y_i \) as the coefficient of \( \hbar \) in \( Y_i \) in the above expressions. The operators \( y_1, y_2 \) obey commutation relations which motivate the following

**DEFINITION 3.1.** (see Cherednik [5]) The degenerate (or trigonometric) double affine Hecke algebra \( \mathcal{H}_c \) of type \( GL_2 \) is the \( k(c) \)-algebra having generators \( y_1, y_2, X_1^{\pm 1}, X_2^{\pm 1} \), and (12) and relations

\[
\begin{align*}
(12)^2 &= 1, \quad [X_1, X_2] = 0, \quad (12)X_1(12) = X_2, \\
(12)y_1 - y_2(12) &= c, \quad [y_1, y_2] = 0, \\
[y_i, X_j] &= \begin{cases} 
X_i + c(12)X_1 & \text{if } i = j \\
-c(12)X_1 & \text{if } i \neq j
\end{cases}
\end{align*}
\]

**Remark 3.2.** If MQYBE \( \lambda(1 + h + O(\hbar^2)) = 0 \), then \( \lambda \) is of the form \( \epsilon h^2 + O(h^3) \) for some scalar \( \epsilon \) and MCYBE \( \epsilon(r) = 0 \). Furthermore, if \( 1 + hr + O(h^2) \) is unitary, then \( r \) is skew-symmetric.

One can readily obtain the following

**LEMMA 3.3.** \( R = 1 + h(y_1 - y_2 - c(12)) + O(h^2) \)

Using Remark 3.2 together with Lemma 3.3, we have

**COROLLARY 3.4.** \( R, y_1, y_2 \) as above

(i) \( r := y_1 - y_2 - c(12) \) is skew-symmetric

(ii) \( MCYBE_{c^2}(r) = 0 \).

In this situation, \( r \) is called the **semiclassical limit** of \( R \). Since \( r \) is homogeneous, it follows that for any natural number \( n \), we can restrict the action of \( r \) to the subspace \( k(c)^{n,n} \) of \( k(c)[X_1^{\pm 1}, X_2^{\pm 1}] \) spanned by the monomials \( X_1^a X_2^b \) with \( 0 \leq a, b \leq n - 1 \). Doing this yields

\[
r_n = 2 \sum_{1 \leq k < l \leq n} (k - l - c)e_{kk} \wedge e_{ll} + 2c \sum_{1 \leq k < l \leq n} e_{kl} \wedge e_{lk} + 4c \sum_{1 \leq i < k < j \leq n} e_{i+j-k,k} \wedge e_{ki}.
\]

Here, \( e_{ij} \wedge e_{kl} \) is the operator

\[
X_1^a X_2^b \mapsto \frac{1}{2} \left( \delta_{j, a+1} \delta_{l, b+1} X_1^{i-1} X_2^{k-1} - \delta_{j, a+1} \delta_{l, b+1} X_1^{k-1} X_2^{i-1} \right).
\]

Remark 3.5. The formula for $r_n$ above suggests that we can view it as being in $\mathfrak{gl}_n \wedge \mathfrak{gl}_n$. Thus, we have a one-parameter family of solutions to the MCYBE over $\mathfrak{gl}_n$. Setting $c = -\frac{n}{2}$ is the only instance that $r_n$ will be in $\mathfrak{sl}_n \wedge \mathfrak{sl}_n$.

As mentioned in the introduction, the skew-symmetric solutions to the MCYBE

$$[r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}] - \lambda(P_{123} - P_{213}) = 0$$

fall into two classes: those solutions satisfying the MCYBE with $\lambda$ nonzero (resp. zero) are called quasitriangular (resp. triangular) $r$-matrices. The quasitriangular $r$-matrices (over $\mathbb{C}$) were classified in the early 1980’s by Belavin and Drinfel’d using combinatorial objects on the Dynkin graph, called BD-triples [1]. The classification tells us that the solution space of quasitriangular $r$-matrices may be viewed as a disjoint union of quasi-projective subvarieties of $\mathbb{P}(\mathfrak{sl}_n \wedge \mathfrak{sl}_n)$: the aforementioned subvarieties are indexed by the BD-triples. In this case $r_n$ corresponds to the maximal BD-triple obtained by deleting an extremal node. It is the so-called Cremmer–Gervais $r$-matrix (discussed in [6,7,9,11–13]).

THEOREM 3.6. When $c = -\frac{n}{2}$, $r_n$ is the Cremmer–Gervais $r$-matrix.

Proof. Apply the Lie algebra automorphism $e_{ij} \mapsto e_{n+1-j,n+1-i}$ of $\mathfrak{gl}_n$ to the formula for $r_n$ above, then multiply the result by $-\frac{1}{n}$, and finally set $c = -\frac{n}{2}$; we obtain the same formula for the Cremmer—Gervais $r$-matrix $r_{CG}$ as it appears in [9].

4. Connections with the Rational Cherednik Algebra

We begin this section by recalling a

DEFINITION 4.1. (see Cherednik [5]) The rational Cherednik algebra (over $k(c)$) $\mathcal{H}'_c$ of type $GL_2$ has generators $(12), x_1, x_2, u_1, u_2$ and relations

$$(12)^2 = 1, \quad (12)x_1(12) = x_2, \quad (12)u_1(12) = u_2,$$

$$[x_1, x_2] = 0, \quad [u_1, u_2] = 0,$$

$$[u_i, x_j] = \begin{cases} 1 - c(12) & \text{if } i = j \\ c(12) & \text{if } i \neq j \end{cases}$$

The polynomial representation $k(c)[x_1, x_2]$ of $\mathcal{H}'_c$ is defined where $(12)$ permutes the variables, $x_1$ and $x_2$ act via multiplication, and the $u_i$ act by the Dunkl operators

$$u_i \mapsto \frac{\partial}{\partial x_i} + c(-1)^i S.$$
In [16], Suzuki shows that there is an algebra embedding $\psi : \mathcal{H}'_{c} \rightarrow \mathcal{H}'_{c}$ defined on generators by

$$(12) \mapsto (12)$$

$x_1 \mapsto X_1$

$x_2 \mapsto X_2$

$u_1 \mapsto X_1^{-1} \left(y_1 + \frac{c}{2} - c(12)\right)$

$u_2 \mapsto X_2^{-1} \left(y_2 + \frac{c}{2}\right)$

Using this algebra embedding, we see that the Cremmer–Gervais $r$-matrix has an interpretation in the rational Cherednik algebra. Here, $r_n$ corresponds to $x_1u_1 - x_2u_2 \in \mathcal{H}'_{c}.$

In [9], Gerstenhaber and Giaquinto provide the largest known class of examples of triangular $r$-matrices, the so-called generalized Jordanian $r$-matrices (also discussed in [7]). They demonstrate that the Jordanian $r$-matrices lie on the boundary of the orbit $SL_n r_n,$ where $SL_n$ acts via the adjoint action. One can translate this into the setting of the rational Cherednik algebra. Here,

$$e^{\tau \cdot ad(u_1 + u_2)}(x_1u_1 - x_2u_2) = x_1u_1 - x_2u_2 + \tau(u_1 - u_2).$$

Therefore, we have the following

**COROLLARY 4.2.** $u_1 - u_2 \in \mathcal{H}'_{c}$ is a boundary solution to the MCYBE (in particular, $MCYBE_0(u_1 - u_2) = 0$). Restricting its action to the linear subspace $k(e)^{n,n}$ and setting $c = -n/2$ corresponds to the Jordanian $r$-matrix $r_{J,n}.$

As an operator on the polynomial representation of $\mathcal{H}'_{c},$

$$u_1 - u_2 = \frac{\partial}{\partial x_1} - \frac{\partial}{\partial x_2} - 2cS.$$

This gives us a one-parameter family of triangular $r$-matrices.

In [9], Gerstenhaber and Giaquinto conjecture that $r_{J,n}^3 = 0.$ We use the above interpretation to show that this is not true and to compute the nilpotency index.

**THEOREM 4.3.** *(Gerstenhaber–Giaquinto Conjecture)* The nilpotency index of $r_{J,n} \in \mathcal{H}'_{n/2}$ is $n$ when $n$ is odd and $2n - 1$ when $n$ is even.

**Proof.** Set $x = \frac{1}{2}(x_1 - x_2),\ x' = \frac{1}{2}(x_1 + x_2).$ We have $[u_1 - u_2, x'] = 0$ and $(u_1 - u_2)x^m = (m - n[[m]])x^{m-1}.$ Here $[[m]] = 1$ if $m$ is odd and $[[m]] = 0$ otherwise. Observe that

$$x_1^i x_2^j = (-1)^i x^{i+j} + (-1)^i (i - j) x^{i+j-1} x' + \cdots.$$

One computes that in the case when $n$ is odd, we have

$$(u_1 - u_2)^{n-1} x_2^{n-1} = (n - 1)(-2)(n - 3)(-4) \cdots (3 - n)(2)(1 - n) \neq 0.$$
and for all $0 \leq i, j \leq n-1$, $(u_1 - u_2)^n(x_i^1 x_2^j) = 0$. In the case when $n$ is even, one computes
\[(u_1 - u_2)^{2(n-1)}(x_1^{n-1} x_2^{n-1}) = (2n-2)(n-3)(2n-4)(n-5) \cdots (3-n)(2-1-n) \neq 0
\]
and for all $1 \leq i, j \leq n-1$, $(u_1 - u_2)^{2n-1}(x_i^1 x_2^j) = 0$.

**Remarks.** For $n \geq 2$, the Cremmer–Gervais $r$-matrix $r_n$ is not nilpotent except when $c = -1/2$ and $n = 2$. One can see this by viewing $r_n$ as the operator
\[X_1 \frac{\partial}{\partial X_1} - X_2 \frac{\partial}{\partial X_2} + c (X_1 + X_2) S
\]
and verifying $r_n^2(X_1 - X_2) = (1 + 2c)(X_1 - X_2)$ and $r_n^2(X_1^2 - X_2^2) = 4(1 + c)(X_1^2 - X_2^2)$. So in this particular case, nilpotency is only a boundary condition.

The conceptual difference between the even and odd cases in Theorem 4.3 has a representation theoretic origin: the polynomial representation of the rational Cherednik algebra $\mathcal{H}_c''$ of type $A_1$ is reducible if and only if the deformation parameter $c$ has the form $n/2$ for an odd positive integer $n$ (a special case of Dunkl’s theorem [2]).

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