Chen–Ruan Cohomology of $\mathcal{M}_{1,n}$ and $\overline{\mathcal{M}}_{1,n}$

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Abstract

In this work we compute the Chen–Ruan cohomology and the stringy Chow ring of the moduli spaces of smooth and stable $n$-pointed curves of genus 1. We suggest a definition for an Orbifold Tautological Ring in genus 1, which is both a subring of the Chen–Ruan cohomology and of the stringy Chow ring.

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1 Introduction

Motivated by physics, Chen–Ruan cohomology was introduced in the paper [7] in the analytic category, and in the two papers by Abramovich–Graber–Vistoli [1] and [2] in the algebraic category. This has produced two parallel objects: the Chen–Ruan cohomology and the stringy Chow ring, which provide the basis to develop the quantum cohomology ring of an orbifold. This cohomology ring recovers as a subalgebra the ordinary rational cohomology ring of the topological space that underlies the orbifold. As a vector space, the Chen–Ruan cohomology of $X$ is simply the cohomology of the Inertia of $X$. If $X$ is an orbifold, its Inertia $I(X)$ is constructed as the disjoint union, for $g$ in the stabilizer of some point $x$ of $X$, of the locus stabilized by $g$ in $X$ (see Definition 3.1). As an example, the orbifold $X$ itself appears as a connected component of $I(X)$, as the locus fixed by the identical automorphism, which is trivially in the stabilizer group of every point. All the other connected components of the Inertia $I(X)$ are usually called twisted sectors. In this paper we use the algebraic language, and whenever the word “orbifold” is mentioned, it stands for smooth Deligne–Mumford stack.

Among the first examples of smooth Deligne–Mumford stacks in the literature there are the moduli of smooth pointed curves $M_{g,n}$ and their compactifications $\overline{M}_{g,n}$. It seems thus interesting to study their Chen–Ruan cohomology. This has been done so far for $\overline{M}_{1,1}$ (a special case of weighted projective space) and for $M_2$ and $\overline{M}_2$ by Spencer [26] (see also [27]).

In the present work, we investigate the Chen–Ruan cohomology ring for $M_{1,n}$ and $\overline{M}_{1,n}$ with rational coefficients, assuming knowledge of the cohomology of $M_{1,n}$ and $\overline{M}_{1,n}$. We show how it is possible to describe the stringy Chow ring in a similar fashion. Indeed we show that for all the twisted sectors, the cycle map from the Chow ring to the cohomology is an isomorphism.

The main results of this paper are the complete description of the twisted sectors, and the explicit computation of the Chen–Ruan product as an extension of the usual cup product.

In Section 2 we recall some known results that we will use and fix notation. The complete description of the twisted sectors for $M_{1,n}$ and $\overline{M}_{1,n}$ is given in Section 3, where we prove the following result:

**Theorem 1.1.** (Theorem 3.24, Corollary 3.25) Each twisted sector of $\overline{M}_{1,n}$ is isomorphic to a product:

$$A \times \overline{M}_{0,n_1} \times \overline{M}_{0,n_2} \times \overline{M}_{0,n_3} \times \overline{M}_{0,n_4}$$

where $n_1, \ldots, n_4 \geq 3$ are integers and $A$ is in the set:

$$\{B_{13}, B_{14}, B_{16}, P(4,6), P(2,4), P(2,2)\}$$
where with $BG$ we denote the trivial gerbe banded by $G$ on a point (see Section 2.2), and $\mathbb{P}(a,b)$ is a weighted projective stack.

This result allows us to compute the generating series of the Chen–Ruan Poincaré polynomials for $\overline{M}_{1,n}$.

In Section 4, we compute the Chen–Ruan cohomology of $\mathcal{M}_{1,n}$ and $\overline{\mathcal{M}}_{1,n}$ as a graded vector space. To do so, we introduce the unconventional rational grading on the cohomology of the Inertia Stack, usually referred to as age, or degree shifting number, or again fermionic shift.

In Section 5, we describe the twisted sectors of the second Inertia Stack of $\mathcal{M}_{1,n}$ and $\overline{\mathcal{M}}_{1,n}$. Here a simplification occurs, indeed we show that every double twisted sector is canonically isomorphic to a twisted sector.

In Section 6, we compute all the excess intersection bundles for $\overline{\mathcal{M}}_{1,n}$ that we need, and their top Chern classes.

Finally, in Section 7, we determine the Chen–Ruan cup product and we prove the following result:

**Theorem 1.2.** (Theorem 7.2) The Chen–Ruan cohomology ring of $\overline{\mathcal{M}}_{1,n}$ is generated as $H^*(\overline{\mathcal{M}}_{1,n}, \mathbb{Q})$-algebra by the fundamental classes of the twisted sectors with explicit relations.

Note that the same result would hold verbatim if Chen–Ruan cohomology ring is substituted with stringy Chow ring, and $H^*$ is substituted with $A^*$. We advance a proposal for an Orbifold Tautological Ring for $\overline{\mathcal{M}}_{1,n}$ in Definition 7.16, encoding essentially all the interesting Chen–Ruan products, and then we compute it in the subsequent paragraphs. This can be interpreted as both an extension of the Tautological Ring $R^*$ and of its image in cohomology $RH^*$.

The final section is devoted to explicit examples and computations, in the cases with 1, 2, 3 and 4 marked points.

This paper is part of the PhD thesis [21], where the case of genus $g$ bigger than 1 is also discussed. However, we think that the genus 1 case can be conveniently described in a more explicit and elementary framework, mainly thanks to the fact that the automorphism groups of stable genus 1 marked curves are all abelian (in fact, cyclic).

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2 Foundation

2.a General notation

The generality we adopt for the category of schemes is schemes of finite type over $\mathbb{C}$. Although we treat only this case, our results can be easily extended to the case of an arbitrary field of characteristic different from 2 and 3.

In the paper, algebraic stack means Deligne–Mumford stack. The intersection theory on schemes is defined in [11], on Deligne–Mumford stacks it is defined in [28]. We refer to these texts for definitions and first properties of the Chow groups $A_*$. In particular, since all the spaces we consider are smooth, there is a standard identification (which could be taken as a definition) of $A^*$ with the dual of $A_*$. For the sake of simplicity, we present here the case of cohomology and Chow ring with rational coefficients.

We call $\mathbb{G}_m$ the group scheme of invertible multiplicative elements of $\mathbb{C}$. The discrete group subscheme of $\mathbb{G}_m$ of the $n$-th roots of 1 is called $\mu_n$. The generators of $\mu_2$, $\mu_4$ and $\mu_6$ are conventionally chosen to be respectively $-1$, $i$ and $\epsilon$. Since we work over the complex numbers, we can fix $\epsilon = e^{2\pi i/6}$.

We call $S_n$ the group of permutations on the set of the first $n$ natural numbers: $[n] := \{1, 2, \ldots, n\}$.

If $G$ is a finite abelian group, $G^\vee = \text{Hom}(G, \mathbb{C}^*)$ is the group of characters of $G$. We will call $BG$ the trivial gerbe over a point (the basic notions on gerbe we will need are presented in 2.b).

2.b Trivial gerbes

For a complete treatment on gerbes, we refer to the book [15], or to [6]. In this paper, we need only very few notions related to gerbes.

**Definition 2.1.** Let $G$ be a finite abelian group. We define $BG$ as the quotient stack $[\text{Spec}(\mathbb{C})/G]$.

Let $\pi : \text{Spec}(\mathbb{C}) \to BG$ be the canonical projection on the quotient. A vector bundle on $BG$ corresponds to a vector bundle on $\text{Spec} \mathbb{C}$ (i.e. a vector space) with a $G$-action. It follows that the Picard group of $BG$ is canonically isomorphic to the group $G^\vee$ of irreducible representations of $G$.

**Notation 2.2.** As a consequence, if $X$ is a scheme, the datum of a line bundle over $X \times BG$ is a pair $(L, \chi)$ where $L \in \text{Pic}(X)$ and $\chi \in G^\vee$.

2.c Notation for $\mathcal{M}_{g,n}$ and $\overline{\mathcal{M}}_{g,n}$

If $I$ is a finite set, we denote by $\overline{\mathcal{M}}_{g,I}$ the moduli stack of stable genus $g$ curves with marked points in the set $I$. If $I = \{i, \bullet\}$ then we Conventionally define $\overline{\mathcal{M}}_{g,[n]}$ as a point labeled by $i$. When the set $I$ is $[n]$, the set of the first $n$ natural numbers, we write $\mathcal{M}_{g,n}$ instead of $\mathcal{M}_{g,[n]}$ and $\overline{\mathcal{M}}_{g,n}$ instead of $\overline{\mathcal{M}}_{g,[n]}$.

If $I \subset J$, then $\pi_I : \mathcal{M}_{g,J} \to \mathcal{M}_{g,I}$ is the morphism that remembers only the sections inside $I$. We give the same name to the morphism $\pi_I : \overline{\mathcal{M}}_{g,J} \to \overline{\mathcal{M}}_{g,I}$ that forgets all the sections but the ones in $I$ and stabilizes. With this notation, let $s_i$ be the $i$-th section of $\pi_{[n]} : \overline{\mathcal{M}}_{g,n+1}$.
By definition $L_i$ is the line bundle $s_i^*(\omega_{\pi[n]})$, where $\omega_{\pi[n]}$ is the relative dualizing sheaf. We define $\psi_i = c_1(L_i)$.

Let $k > 0$ and let $(I_1, \ldots, I_k)$ be a partition of $[n]$. We define $j_{g,k}$ as the morphism gluing the marked points labeled with the same symbol (note that $j_{g,k}$ depends on the partition $I_1, \ldots, I_k$ and on the choice of $g_1, \ldots, g_k$, although this dependence is not explicit in its name):

$$j_{g,k} : \mathcal{M}_{g,1} \sqcup \mathcal{M}_{g_{1}, I_1 \sqcup \{•\}} \times \mathcal{M}_{g_{2}, I_2 \sqcup \{•\}} \times \cdots \times \mathcal{M}_{g_{k}, I_k \sqcup \{•\}} \to \mathcal{M}_{g+\sum g_i, n}$$

We also define:

$$j : \mathcal{M}_{g,n} \sqcup \mathcal{M}_{g+1,n}$$

as the morphism gluing the two bullets. In this paper, we will be dealing with the case of genus 1 curves. We will be using several times the map $j_{1,k}$, where all the $g_i$ are set equal to 0. We call this map simply $j_k$, so if $I_1, \ldots, I_k$ is a partition of $[n]$ we have the gluing map:

$$j_k : \mathcal{M}_{1, I_1 \sqcup \{•\}} \times \mathcal{M}_{0, I_2 \sqcup \{•\}} \times \cdots \times \mathcal{M}_{0, I_k \sqcup \{•\}} \to \mathcal{M}_{1,n}$$

The product space on the left admits projection maps onto each factor. We will call $p$ the projection map onto the first factor, and $p_i$ the projection maps on the genus 0 component with marked points in the set $I_i$.

### 2.4 The Tautological Ring

We recall the definition from [9].

**Definition 2.3.** The System of Tautological Rings $R^*(\mathcal{M}_{g,n})$ is defined to be the set of smallest $\mathbb{Q}$-subalgebras of the Chow rings:

$$R^*_{g,n} = R^*(\mathcal{M}_{g,n}) \subset A^*(\mathcal{M}_{g,n}, \mathbb{Q})$$

satisfying the following two properties:

1. The system is closed under push–forward via all maps forgetting marked points:

$$\pi_{[n]}_* : R^*(\mathcal{M}_{g,n+1}) \to R^*(\mathcal{M}_{g,n})$$

2. The system is closed under push–forward via the gluing maps:

$$j_* : R^*(\mathcal{M}_{g_1+n_1+\{•\}}) \boxtimes \mathbb{Q} R^*(\mathcal{M}_{g_2+n_2+\{•\}}) \to R^*(\mathcal{M}_{g_1+g_2,n_1+n_2})$$

$$j_* : R^*(\mathcal{M}_{g,n+\{•,•\}}) \to R^*(\mathcal{M}_{g+1,n})$$

**Remark 2.4.** Note that the system of Tautological Rings is closed under pull–back via the forgetful and the gluing maps. They are representations of $S_n$ via the action that permutes the points. We denote by $RH^*(\mathcal{M}_{g,n})$ the image of $R^*(\mathcal{M}_{g,n})$ under the cycle map to the ring of even cohomology classes.

**Definition 2.5.** We define $B^*_{g,n}$ to be the smallest system of subvector spaces of the Chow rings $A^*(\mathcal{M}_{g,n}, \mathbb{Q})$ that contain the fundamental classes, and that are stable under push–forward via all the gluing maps (see Definition 2.3 point 2). A boundary strata class is an element in $B_{g,n}$ that corresponds to a closed irreducible proper substack of $\mathcal{M}_{g,n}$.

Obviously, the Tautological Ring contains all the boundary strata classes.
**Notation 2.6.** We will call $D_I$ the closure of the substack of $\overline{M}_{1,n}$ of reducible nodal curves with two smooth components, where the marked points in the set $I$ are on the genus 0 component and the marked points on the genus 1 curve are in the complement, $[n] \setminus I$. We call $D_{irr}$ the closure of the substack of $\overline{M}_{1,n}$ of irreducible curves of geometric genus 0. We will sometimes indicate with $D_I$ also the class $[D_I] \in H^2(\overline{M}_{1,n})$ represented by the divisor $D_I$.

Analogously, given $I \subset [n]$, such that $|I| \geq 2$ and $|n| \setminus I \geq 2$, we will call $\delta_I = \delta_{[n] \setminus I}$ the sublocus of $M_{0,n}$ whose general element has two genus 0 components with marked points in $I$ in the first one and in $[n] \setminus I$ in the second one.

Clearly, these elements form basis respectively for $B^*_{1,n}$ and $B^*_{0,n}$.

In genus 0 we have

$$B^*_{0,n} = R^*_{0,n} = A^*(\overline{M}_{0,n}, Q) = H^{2*}(\overline{M}_{0,n}, Q)$$

by the work of Keel [13].

We use these results several times:

**Proposition 2.7.** The Tautological Ring $R^*(\overline{M}_{1,n})$ is spanned (additively generated) by boundary strata classes $2.5$, so that $B^*_{1,n} = R^*_{1,n}$.

**Proof.** This follows as a consequence of Theorem * [17].

**Proposition 2.8.** ([3, Theorem 3.1.1]) For $n \leq 10$ the following two equalities hold:

$$B^*_{1,n} = R^*_{1,n} = A^*(\overline{M}_{1,n}, Q) = H^{2*}(\overline{M}_{1,n}, Q)$$

It is well known that the eleventh cohomology group of $\overline{M}_{1,11}$ is non zero. From this, it follows that the second and third equality of the proposition above are no longer true for $n \geq 11$ (see for instance [16, p.2]).

**Claim 2.9.** ([13] second paragraph) The boundary strata classes of $\overline{M}_{1,n}$ span the even cohomology of $\overline{M}_{1,n}$.

**Claim 2.10.** ([13], second paragraph) The ideal of relations among the boundary cycles is generated by the genus 0 relations together with pull–backs to $\overline{M}_{1,n}$ of the relation in $H^4(\overline{M}_{1,4}, Q)$ which is stated in Lemma 1.1, [13].

In [22, Theorem 1], the relation in $H^4(\overline{M}_{1,4}, Q)$ is shown to be algebraic (Theorem 1).

The claims therefore split in the two ring isomorphisms:

$$R^*(\overline{M}_{1,n}) \cong RH^*(\overline{M}_{1,n}) \quad (2.11)$$

$$RH^*(\overline{M}_{1,n}) = H^{2*}(\overline{M}_{1,n}) \quad (2.12)$$

for $n \geq 1$.

For a part of this work, namely in the section of pull–back of the Tautological Ring to the twisted sectors, we assume Getzler’s claims (although our main results are independent of them).
3 The Chen–Ruan Cohomology of $\mathcal{M}_{1,n}$ and $\overline{\mathcal{M}}_{1,n}$ as Vector Space

3.a Definition of Chen–Ruan Cohomology as Vector Space

The following is a natural stack associated to a stack $X$, which points to where $X$ fails to be an algebraic space.

**Definition 3.1.** ([28, Definition 1.12]) Let $X$ be an algebraic stack. The **Inertia Stack** $I(X)$ of $X$ is defined as the fiber product $X \times_X X \times_X X$ where both morphisms $X \to X \times X$ are the diagonal ones. It comes with a natural map $f : I(X) \to X$.

**Remark 3.2.** The construction of Chen–Ruan cohomology based on the definition of Inertia Orbifold was given for the first time in [7, Definition 3.2.3]. As observed in [1, Section 4.4], the latter is nothing but the coarse moduli space of the Inertia Stack we have just introduced. In [2, 7.3] the algebraic counterpart of Chen–Ruan cohomology is introduced, under the name of stringy Chow ring. It is built on the rigidification of the cyclotomic Inertia Stack introduced in [2, Section 3]. In this paper we work over $\mathbb{C}$, and all the cohomologies are taken with rational coefficients. Therefore, the cohomologies of the Inertia Stack, of the cyclotomic Inertia Stack, of the rigidified cyclotomic Inertia Stack and of the Inertia Orbifold are all canonically isomorphic, since all of them share the same coarse moduli space.

**Definition 3.3.** If $X$ is an algebraic stack, the connected component of the Inertia Stack associated with the identity automorphism is called the **untwisted sector** of the Inertia Stack. All the remaining connected components are called the **twisted sectors** of $I(X)$. The latter are sometimes called also the twisted sectors of $X$.

**Proposition 3.4.** [2, Corollary 3.1.4] Let $X$ be a smooth algebraic stack. Then the Inertia Stack $I(X)$ is smooth.

**Remark 3.5.** If $Y$ is a twisted sector of $I(X)$, then the map $f$ of Definition 3.1 restricts to a map $f|_Y : Y \to X$. In general $f|_Y : Y \to X$ is a composition of a stack covering and a closed embedding, as easily follows for instance from [28, Lemma 1.13], or from [7, 3.1.3]. In the present paper however, since all the stacks we consider are abelian, the map $f|_Y$ is a closed embedding. So if $Y$ is a twisted sector, it can be written as $Y = (Z, g)$, where $Z$ is a closed substack of $X$ and $g$ is an automorphism in the generic stabilizer of $Z$. From now on, by abuse of notation, we will refer to this twisted sector as $Y$, or $(Y, g)$.

**Definition 3.6.** Let $X$ be a smooth algebraic stack. Let $T$ be a set of indices in bijection with the twisted sectors of $I(X)$. We call the following equality:

$$I(X) = X \sqcup \bigsqcup_{i \in T} (X_i, g_i)$$

a **decomposition of the Inertia Stack of $X$ in twisted sectors** if each $(X_i, g_i)$ is a twisted sector.

**Notation 3.7.** In order to simplify the notation, if $(A, g), (A, g')$ are two twisted sectors, we shall write something like $(A, g/g')$ to denote the disjoint union of the two twisted sectors $(A, g)$ and $(A, g')$ in the Inertia Stack.

When we simply write $A$ we refer to the image of the closed embedding of the twisted sector inside the original stack $X$ (see Remark 3.5).
We can then define the Chen–Ruan cohomology vector space:

**Definition 3.8.** \([17\text{ Definition 3.2.3}]\) Let \(X\) be a smooth algebraic stack. Then:

\[
H^*_{\text{CR}}(X, \mathbb{Q}) := H^*(I(X), \mathbb{Q})
\]

as a rational vector space.

The Chen–Ruan cohomology decomposes as in Definition 3.6:

\[
H^*_{\text{CR}}(X, \mathbb{Q}) = H^*(X, \mathbb{Q}) \oplus \bigoplus_{i \in T} H^*(X_i, \mathbb{Q})
\]

### 3.b The Inertia Stack of \(\mathcal{M}_{1,n}\) and \(\overline{\mathcal{M}}_{1,n}\)

The twisted sectors in case \(n = 1\) are well known as a direct consequence of the Weierstrass Theorem. We refer to [25, III.1] for the basic material on this topic.

First of all, recall that every curve of the form:

\[
\tilde{C} = \{[x : y : z] \mid \Delta := 4a^3 + 27b^2 \neq 0, (a, b) \neq (0, 0) \} \subset \mathbb{P}^2
\]

is a smooth genus 1 curve. If, instead:

\[
\tilde{C} = \{[x : y : z] \mid \Delta := 4a^3 + 27b^2 = 0, (a, b) \neq (0, 0) \} \subset \mathbb{P}^2
\]

then \(\tilde{C}\) is a nodal curve of arithmetic genus 1, geometric genus 0 and one node. The following result describes all the genus 1 curves with a marked point this way.

**Theorem 3.9.** [25, III.1] (Weierstrass representation) Let \((C, P)\) be a nodal elliptic curve. Then there exist \((a, b) \in \mathbb{C}^2\) such that \((C, P)\) is isomorphic to \((C', [0 : 1 : 0])\), where

\[
C' := \{[x : y : z] \mid y^2 = x^3 + ax^2 + bx^3\} \subset \mathbb{P}^2
\]

If \(\alpha\) is an isomorphism of \((C, P)\) with \((D, Q)\) then, up to the isomorphism above, \(\alpha\) is:

\[
\alpha : \begin{cases} 
  a \to \lambda^4 a \\
  b \to \lambda^6 b \\
  x \to \lambda^2 x \\
  y \to \lambda^3 y \\
  z \to z
\end{cases}
\]

From this it follows that the moduli stack \(\overline{\mathcal{M}}_{1,1}\) is isomorphic to the weighted projective stack \(\mathbb{P}(4, 6)\).

**Notation 3.11.** There are two elements of \(\overline{\mathcal{M}}_{1,1}\) that are stabilized by the action of a group respectively isomorphic to \(\mu_4\) and \(\mu_6\), we call them respectively \(C_4\) and \(C_6\). These are classes of curves whose Weierstrass representation can be chosen respectively as:

\[
C_4 := \{[x : y : z] : y^2z = x^3 + xz^2\} \subset \mathbb{P}^2
\]

\[
C_6 := \{[x : y : z] : y^2z = x^3 + z^3\} \subset \mathbb{P}^2
\]

If \((C, P)\) is an elliptic curve, and \(G\) is its automorphism group, then it can be identified canonically with \(\mu_N\) for a certain \(N \in \{2, 4, 6\}\).
Notation 3.12. If \((C, P)\) is an elliptic curve, and \(G\) is its automorphism group, then \(G\) acts effectively on \(T^*_y(C)\), the cotangent space in \(P\) to \(C\), which is canonically isomorphic to \(\mathbb{C}\). Under this isomorphism, we can identify \(G\) with \(\mu_N\).

The decomposition of the Inertia Stack of \(\mathcal{M}_{1,1}\) and \(\overline{\mathcal{M}}_{1,1}\) in twisted sectors (Definition 3.6), is simply a way to summarize the well–known facts that we have exposed in this section:

**Corollary 3.13.** With the notation introduced in Notation 3.7 and 3.12, the decomposition of the Inertia Stack of \(\mathcal{M}_{1,1}\) in twisted sectors is:

\[
I(\mathcal{M}_{1,1}) = (\mathcal{M}_{1,1}, 1) \coprod (\mathcal{M}_{1,1}, -1) \coprod (\mathcal{C}_4, i/ -i) \coprod (\mathcal{C}_6, \epsilon^2/\epsilon^4/\epsilon^5)
\]

and that of \(\overline{\mathcal{M}}_{1,1}\) is:

\[
I(\overline{\mathcal{M}}_{1,1}) = (\overline{\mathcal{M}}_{1,1}, 1) \coprod (\overline{\mathcal{M}}_{1,1}, -1) \coprod (\mathcal{C}_4, i/ -i) \coprod (\mathcal{C}_6, \epsilon^2/\epsilon^4/\epsilon^5)
\]

3.b.1 **The case of \(\mathcal{M}_{1,n}\)**

We thus study the Inertia Stack of \(\mathcal{M}_{1,n}\). Note that if \(n > 4\), the objects of \(\mathcal{M}_{1,n}\) are rigid, and therefore in that range \(I(\mathcal{M}_{1,n}) = \mathcal{M}_{1,n}\).

A simple analysis of the fixed points of the action of \(\mu_3\), \(\mu_4\) and \(\mu_6\) on the curves \(\mathcal{C}_4\) and \(\mathcal{C}_6\) (see Notation 3.11) by Theorem 3.9 leads to the three special points:

Notation 3.14. We call \(C'_4\) the point in \(\mathcal{M}_{1,2}\) stabilized by \(i\) or \(−i\), \(C'_6\) the point in \(\mathcal{M}_{1,2}\) stabilized by \(\epsilon^2\) or \(\epsilon^4\), and finally \(C''_6\) the point in \(\mathcal{M}_{1,3}\) stabilized by \(\epsilon^2\) or \(\epsilon^4\).

To complete the study of the loci fixed by automorphisms in \(\mathcal{M}_{1,n}\), we shall need the loci fixed by the elliptic involution that according to Notation 3.12 writes as \((-1)\). We give a special name to them:

Definition 3.15. Let \(0 \leq i \leq 3\). We define \(A_i\) as the closed substack of \(\mathcal{M}_{1,i}\) whose objects \(A_i(S)\) are \(i\)-marked smooth genus 1 curves over \(S\) such that the sections are stabilized by the elliptic involution.

We shall see, as a consequence of Lemma 3.19, that \(A_i\) is connected for all \(i\) (note that \(A_1 = \mathcal{M}_{1,1}\)). What we have said so far, leads to the description of the Inertia Stacks of \(\mathcal{M}_{1,n}\):

**Corollary 3.16.** The decomposition of the Inertia Stack of \(\mathcal{M}_{1,n}\) (Notation 3.6, 3.7, 3.12) is:

- \(I(\mathcal{M}_{1,1}) = \mathcal{M}_{1,1} \coprod (\mathcal{M}_{1,1}, -1) \coprod (\mathcal{C}_4, i/ -i) \coprod (\mathcal{C}_6, \epsilon^2/\epsilon^4/\epsilon^5)\);
- \(I(\mathcal{M}_{1,2}) = \mathcal{M}_{1,2} \coprod (A_2, -1) \coprod (\mathcal{C}_4, i/ -i) \coprod (\mathcal{C}_6, \epsilon^2/\epsilon^4)\);
- \(I(\mathcal{M}_{1,3}) = \mathcal{M}_{1,3} \coprod (A_3, -1) \coprod (\mathcal{C}_6, \epsilon^2/\epsilon^4)\);
- \(I(\mathcal{M}_{1,4}) = \mathcal{M}_{1,4} \coprod (A_4, -1)\);
- \(I(\mathcal{M}_{1,n}) = \mathcal{M}_{1,n}\) if \(n \geq 5\).

We collect the twisted sectors of \(\mathcal{M}_{1,n}\) in the following table. Different rows correspond to different automorphisms, while the \(i\)-th column corresponds to the twisted sectors inside \(\mathcal{M}_{1,i}\). Remember that \(\mathcal{M}_{1,n}\) is a smooth scheme for \(n > 4\).
|   | 1  | 2  | 3  | 4  |
|---|----|----|----|----|
| $-1$ | $A_1$ | $A_2$ | $A_3$ | $A_4$ |
| $e^2/e^2$ | $C_0$ | $C_0'$ | $C_0''$ | $\varnothing$ |
| $i/-i$ | $C_4$ | $C_4'$ | $\varnothing$ | $\varnothing$ |
| $\epsilon/\epsilon^*$ | $C_0$ | $\varnothing$ | $\varnothing$ | $\varnothing$ |

If we want to compute the Chen–Ruan cohomology of $M_{1,n}$, we have to investigate the geometry of the spaces $A_i$ introduced in Definition 3.15.

**Remark 3.17.** Using analytic methods (see [24], [8, Chapter 3]), it is known that the coarse moduli space of $A_i$ is a genus 0 quasiprojective curve, and also how many points are needed to compactify it. In the literature, the coarse moduli space for $A_i$ is known under the name of $\Gamma_i(2) = \Gamma_0(2)$. The coarse moduli space for $A_i \cong A_4$ is usually called $\Gamma(2)$. We here want to give an algebraic and stack-theoretic description of those spaces, that we could not find anywhere.

**Definition 3.18.** We define $\overline{A}_1$ as the closure of the respective spaces $A_i$, in $\overline{M}_{1,11}$.

Notice that the stack $\overline{A}_1 \cong \overline{M}_{1,1}$ is isomorphic to $\mathbb{P}(4,6)$ as a consequence of Theorem 3.9. Following the same strategy that can be used to prove the latter isomorphism, we can obtain the following result:

**Lemma 3.19.** The stack $\overline{A}_3$ is isomorphic to the weighted projective space $\mathbb{P}(2,4)$. The stacks $\overline{A}_3$ and $\overline{A}_4$ are isomorphic to the weighted projective space $\mathbb{P}(2,2)$.

**Proof.** We first study the case of $\overline{A}_2$. Let us define the following space:

$$B_1 := \{((a, b), [x : y : z]) \mid (a, b) \neq (0, 0), \ yz^2 = x^3 + ax^2z + bz^3\} \subset \mathbb{P}_0^2 \times \mathbb{P}^2$$

The projection onto the first factor, with the section $\sigma_1(a, b) := ((a, b), [0 : 1 : 0])$, describe this space as an elliptic fibration over $\mathbb{A}^0_2$. So they determine a unique map $\phi : \mathbb{A}^0_2 \to \overline{M}_{1,1}$, making the following diagram cartesian (with $\overline{C}_{1,1}$ we indicate the universal curve):

$$\begin{array}{c}
B_1 \xrightarrow{\phi} \overline{C}_{1,1} \xrightarrow{\sim} \overline{M}_{1,2} \\
\downarrow \sigma_1 \downarrow \ x_1 \\
\mathbb{A}^0_2 \xrightarrow{\phi} \overline{M}_{1,1}
\end{array}$$

It is a well-known consequence of the Weierstrass theorem (11) made in families that the map $\phi$ factors via the quotient $[\mathbb{A}^0_2/G_m]$, where $G_m$ acts with weights 4 and 6, and that the resulting map $\widehat{\phi} : [\mathbb{A}^0_2/G_m] \to \overline{M}_{1,1}$ is an isomorphism of stacks. The loci in $B_1$ cut out by the equation $y = 0$ surjects onto $\overline{A}_2 \subset \overline{M}_{1,2}$. This locus is isomorphic to $\mathbb{A}^0_2$ with parameters $(a, x)$. Again as a consequence of the Weierstrass theorem, the action of $G_m$ with weights 4 and 2 can be factored out, to obtain an isomorphism of stacks $[\mathbb{A}^0_2/G_m] \to \overline{A}_2$.

Now we study the case of $\overline{A}_3$. Let:

$$B_2 := \{((a, x_1), [x : y : z]) \mid (a, x_1) \neq (0, 0), \ yz^2 = x^3 + ax^2z + (-ax_1 - x_1^3) z^3\} \subset \mathbb{A}^0_2 \times \mathbb{P}^2$$

We want to replace the role of $B_1$ with $B_2$. In this case however, the projection onto the first factor and the two sections $\sigma_1(a, x_1) = ((a, x_1), [0 : 1 : 0])$ and $\sigma_2(a, x_1) = ((a, x_1), [x_1 : 0 : 1])$
do not give a map to $\overline{M}_{1,2}$ since the image of the second section intersects the singular locus.

Therefore, let:

$$\Lambda := \{(a,x_1), [x:y:z] \mid x = x_1, y = 0, 4(a^3) + 27(-ax_1 - x_1^3)^2 = 0\} \subset B_2$$

Let $p : \tilde{B}_2 \rightarrow B_2$ be the blow-up of $B_2$ in $\Lambda$. Now the projection of $\tilde{B}_2$ onto $\mathbb{A}^3_0$ admits two distinct sections $\tilde{\sigma}_1$ and $\tilde{\sigma}_2$ that do not intersect the singular locus, and such that $p \circ \tilde{\sigma}_i = \sigma_i$.

In this way, we obtain the following cartesian diagram:

So we obtain a smooth map $\tilde{\psi} : \tilde{B}_2 \rightarrow \overline{M}_{1,3}$. Let $F$ be the locus in $B_2$ cut out by the equation $y = 0$ and $\tilde{F}$ its strict transform under $p : \tilde{B}_2 \rightarrow B_2$. The map $\tilde{\psi}$ restricted to $\tilde{F}$ surjects onto $A_3$. There is an isomorphism from $\mathbb{A}^2_0$ (parameters $(x_2, x_1)$) to $F$: $\lambda : (x_2, x_1) \rightarrow ((-x_1^2 - x_1x_2 - x_2^2, x_1), [x_2 : 0 : 1])$. Since $F$ is smooth, the restriction of the map $p : \tilde{F} \rightarrow F$ is an isomorphism, and therefore $\lambda$ lifts to an isomorphism $\tilde{\lambda} : A^2_0 \rightarrow \tilde{F}$. So we have a surjection:

$$\tilde{\psi} \circ \tilde{\lambda} : \mathbb{A}^2_0 \rightarrow \overline{A}_3$$

Again, as a consequence of Weierstrass theorem, this map factors via the quotient $[\mathbb{A}^2_0/G_m]$ where the action has weights 2 and 2, thus inducing an isomorphism of stacks $[\mathbb{A}^2_0/G_m] \rightarrow \overline{A}_3$.

To conclude the proof, we observe that the restriction of the forgetful map: $\overline{A}_4 \rightarrow \overline{A}_3$ is an equivalence of categories. Indeed, when three among the four $2-$torsion points of an elliptic curve have been chosen, there is only one possible way to choose the fourth one.

Note that as a consequence of the proof, we have a description of the forgetful maps $\overline{A}_3 \rightarrow \overline{A}_2 \rightarrow A_1$ in terms of maps of weighted projective spaces.

Note moreover that there are two points in $\overline{A}_2 \setminus A_2$:

Figure 1: The two points that compactify $A_2$

And there are three points in $\overline{A}_3 \setminus A_3$:

Figure 2: The three points that compactify $A_3$
Definition 3.21. Let \( Z \) be a twisted sector inside \( M_{1,1} \). We call \( Z \) a base twisted sector. We define \( Z^{(I_1, \ldots, I_k)} \) as:

\[
Z^{(I_1, \ldots, I_k)} := j_k(Z \times \overline{M}_{0, I_1 \cup \ast_1} \times \ldots \times \overline{M}_{0, I_k \cup \ast_k})
\]

We call \( Z \) the base twisted sector associated with \( Z^{(I_1, \ldots, I_k)} \).

Theorem 3.22. If \((Z, \alpha)\) is a twisted sector in \( M_{1,k} \), and \((I_1, \ldots, I_k)\) is a partition of \([n]\), then \((Z^{(I_1, \ldots, I_k)}, \alpha)\) is a twisted sector of the Inertia Stack of \( M_{1,n} \).

Proof. The automorphism \( \alpha \) lifts to an automorphism \( \alpha' \) of \( Z^{(I_1, \ldots, I_k)} \) that acts as \( \alpha \) on the base and as the identity on the components \( \overline{M}_{0, i} \). We can call with the same name \( \alpha \) and \( \alpha' \), and represent them by the same element in \( \mu_4 \) (see Notation 3.12). It is easy to check that \( Z^{(I_1, \ldots, I_k)} \) is a connected component of the Inertia Stack of \( M_{1,n} \).

Notation 3.23. Let \( \sigma \in S_k \). Then \( Z^{(I_1, \ldots, I_k)} = Z^{(I_{\sigma(1)}, \ldots, I_{\sigma(k)})} \). The twisted sector is identified up to isomorphism by \( Z \) and the partition \( \{I_1, \ldots, I_k\} \) where the ordering of the \( I_i \)'s does not matter. From now on we will simply denote this twisted sector in \( M_{1,n} \) as \( Z^{(I_1, \ldots, I_k)} \): the elements of the set of parameters for the twisted sectors whose base space is \( Z \) is given by the set of the \( k \) partitions of \([n]\). To simplify notation, we will usually write \( Z^{(I_1, \ldots, I_k)} \) to mean \( Z^{(I_1, \ldots, I_k)} \).

Note also that \( Z \) is identified with \( Z^{(I_1, \ldots, (k)}) \) for every \( Z \) twisted sector of \( M_{1,k} \).

With the notation introduced, we can state and prove the fundamental result of this section:
Theorem 3.24. The decomposition of \( \mathcal{M}_{1,n} \) in twisted sectors is (see Notation 3.13):

\[
(\mathcal{M}_{1,n}, 1) \prod (A_1^{[n]}, -1) \prod (A_2^{1:1_2}, -1) \prod (A_3^{1:1_2:1_3}, -1) \\
\prod (A_4^{1:1_2:1_3:1_4}, -1) \prod (C_4^{[n]}, i - i) \prod (C_3^{1:1_2:1_3}, \epsilon^2 / \epsilon^3) \prod (C_6^{1:1_2:1_3, \epsilon^2 / \epsilon^4}) \prod (C_6^{[n]}, \epsilon^2 / \epsilon^4 / \epsilon^5)
\]

where all the decompositions of the set \([n]\) are made of non empty subsets.

Proof. We have just seen in Theorem 3.22 that all the components that appear in the decomposition are twisted sectors of \( \mathcal{M}_{1,n} \). We have to prove that there are no further connected components in the Inertia Stack of \( \mathcal{M}_{1,n} \). To see that there are no further twisted sectors, one can work by induction using the fact that:

\[
\pi_{[n]} : \mathcal{M}_{1,n+1} \to \mathcal{M}_{1,n}
\]

is the universal curve.

From this, we obtain the following corollary, which describes all the possible twisted sectors of \( \mathcal{M}_{1,n} \) stack-theoretically:

Corollary 3.25. Each twisted sector of \( \mathcal{M}_{1,n} \) is isomorphic to a product:

\[
A \times \mathcal{M}_{0,n_1} \times \mathcal{M}_{0,n_2} \times \mathcal{M}_{0,n_3} \times \mathcal{M}_{0,n_4}
\]

where \( n_1, \ldots, n_4 \geq 3 \) are integers and \( A \) is in the set:

\[
\{ B_{1\mu}, B_{2\mu}, B_{3\mu}, P(4, 6), P(2, 4), P(2, 2) \}
\]

Proof. It is a consequence of Theorem 3.24, Lemma 3.19, and the fact that \( C_4 \cong B_{1\mu}, C_6 \cong B_{3\mu}, C_6' \cong C_6'' \cong B_{3\mu} \) (as a consequence of Theorem 3.3).

3.c The cohomology of the Inertia Stack of \( \mathcal{M}_{1,n} \)

We can use the results established in Theorem 3.24 and Corollary 3.22 to compute the dimension of the vector space \( H_{CR}(\mathcal{M}_{1,n}, \mathbb{Q}) \) (see Definition 3.8). We write the formula for the dimension as a function of the dimension of \( H^*(\mathcal{M}_{0,n}) \), which is well known after Keel [18]. Then, let:

\[
h(n) := \dim H^*(\mathcal{M}_{0,n+1}, \mathbb{Q}) = \sum_k a_k(n)
\]

(the latter notation is the one of [18] p. 550) shifted by 1).

Corollary 3.26. The dimension of the Chen–Ruan cohomology vector space of \( \mathcal{M}_{1,n} \) is:

\[
\dim (H_{CR}(\mathcal{M}_{1,n}, \mathbb{Q})) = \dim (H^*(\mathcal{M}_{1,n}, \mathbb{Q})) + 8h(n) + 3 \sum \binom{n}{i, j} h(i)h(j) + \\
+ \frac{2}{3} \sum \binom{n}{i, j, k} h(i)h(j)h(k) + \frac{1}{12} \sum \binom{n}{i, j, k, l} h(i)h(j)h(k)h(l)
\]

Where the sum is over indices \( 1 \leq i, j, k, l \leq n \) such that their sum is \( n \).
Proof. This result is obtained from Theorem 3.24 and Corollary 7.2, using the fact that the dimension of the cohomology of a point is 1 and the dimension of the cohomology of the projective line is 2.

If we introduce the generating polynomials:

\[ P_0(s) := \sum_{n=0}^{\infty} \frac{Q_0(n)}{n!} s^n \]  (3.27)

\[ P_1(s) := \sum_{n=0}^{\infty} \frac{Q_1(n)}{n!} s^n \]  (3.28)

\[ P^{CR}_1(s) := \sum_{n=0}^{\infty} \frac{Q^{CR}_1(n)}{n!} s^n \]  (3.29)

where:

\[ Q_0(n) := \dim H^*(\overline{M}_{0,n+1}) = h(n) \]

\[ Q_1(n) := \dim H^*(\overline{M}_{1,n}) \]

\[ Q^{CR}_1(n) := \dim H^{CR}(\overline{M}_{1,n}) \]

with the convention that when the right hand side is not defined, the left hand side equals 1.

Formula 3.26 can now be written compactly.

**Theorem 3.30.** The following equality between power series relates the dimensions of the cohomology group of \( \overline{M}_{0,n} \) and \( \overline{M}_{1,n} \) with the dimension of the Chen–Ruan cohomology group of \( \overline{M}_{1,n} \).

\[ P^{CR}_1(s) = P_1(s) + 8P_0(s) + 3P_0(s)^2 + \frac{2}{3}P_0(s)^3 + \frac{1}{12}P_0(s)^4 \]  (3.31)

**3.d The twisted sectors as linear combinations of products of divisors**

In this section, we want to express the classes \([Y] \) for all \( Y \) a twisted sector of \( \overline{M}_{1,n} \), as linear combinations of elements in \( R^*(\overline{M}_{1,n}) \). In fact it is possible to express them as linear combinations of products of divisor classes in \( \overline{M}_{1,n} \). This is due to the fact that there are base twisted sectors (Definition 3.21) in genus 1 only up to 4 marked points, and:

**Theorem 3.32.** \([3]\) The Chow ring of \( \overline{M}_{1,n} \) is generated by the divisor classes when \( n \leq 5 \).

**Notation 3.33.** If \( Y \) is a base twisted sector \((3.21)\), we can write \([Y] \in A^*(\overline{M}_{1,n}) = R^*(\overline{M}_{1,n}) = H^{\text{ev}}(\overline{M}_{1,n}, \mathbb{Q}) \) (these equalities hold when \( n \leq 5 \)). If \( i : Y \to \overline{M}_{1,n} \) is the restriction of the map from the Inertia Stack, \([Y] \) is the push–forward via \( i \) of the fundamental class of the twisted sector \( Y \) (see \( 3.5 \)).

We stress that our result will be symmetric under the action of \( S_n \). We use the notation for the divisors introduced in Notation 2.6.

**Theorem 3.34.** Let \( Y \) be a base twisted sector of \( \overline{M}_{1,n} \) (Definition 3.21). We express its class in the cohomology ring as a linear combination of products of divisor classes.
• Base space classes coming from $\overline{\mathcal{M}}_{1,1}$:
  1. $[A_1] = 1$, the fundamental class of $\overline{\mathcal{M}}_{1,1}$;
  2. $[C_4] = \frac{1}{\pi} D_{irr}$
  3. $[C_6] = \frac{1}{\pi} D_{irr}$.

• Base space classes coming from $\overline{\mathcal{M}}_{1,2}$:
  1. $[A_2] = \frac{1}{\pi} D_{irr} + 3D_{\{1,2\}}$
  2. $[C_4'] = \frac{1}{\pi} D_{irr} D_{\{1,2\}}$
  3. $[C_6'] = \frac{2}{\pi} D_{irr} D_{\{1,2\}}$

• Base space classes coming from $\overline{\mathcal{M}}_{1,3}$:
  1. $[A_3] = \frac{4}{\pi} D_{irr} \left( \sum_{\{i,j\}\subseteq \{1,2,3\}} D_{\{i,j\}} \right) + \frac{1}{\pi} D_{irr} D_{\{1,2,3\}} + 2D_{\{1,2,3\}} \left( \sum_{\{i,j\}\subseteq \{1,2,3\}} D_{\{i,j\}} \right)$
  2. $[C_6''] = \frac{2}{\pi} D_{irr} D_{\{1,2,3\}} \left( \sum_{\{i,j\}\subseteq \{1,2,3\}} D_{\{i,j\}} \right)$

• Base space classes coming from $\overline{\mathcal{M}}_{1,4}$:
  1. $[A_4] = 2D_{\{1,2,3,4\}} \left( \sum_{\{i,j,k,l\}\subseteq \{1,2,3,4\}} D_{\{i,j\}} D_{\{k,l\}} \right) + \frac{1}{12} \sum_{\{i,j,k\}\subseteq \{1,2,3,4\}} D_{\{i,j,k\}} + \left( \sum_{\{i,j,m\}\subseteq \{i,j,k,l\}} D_{\{i,j\}} \right) + \frac{1}{\pi} D_{irr} D_{\{1,2,3,4\}} \left( \sum_{\{i,j\}\subseteq \{1,2,3,4\}} D_{\{i,j\}} \right)$

Proof. For the classes of the points the result is trivial. There is a little care involved in writing $[C_6'']$ as a linear combination that is $S_n$-invariant. We show how to obtain the result for the classes of the spaces $\overline{A}_i$.

We refer to [3] for all the bases of the Chow groups of $\overline{\mathcal{M}}_{1,n}$ that we use in the following. We have modified the bases that Belorousski finds in such a way that the sets of the elements of the bases are closed under the action of $S_n$.

First of all, a basis of $A^1(\overline{\mathcal{M}}_{1,2})$ is given by $D_{irr}$ and $D_{\{1,2\}}$. Therefore:

$$[A_2] = a D_{irr} + b D_{\{1,2\}} \quad (3.35)$$

taking the push–forward via $\pi_1: \overline{\mathcal{M}}_{1,2} \to \overline{\mathcal{M}}_{1,1}$, and using that the forgetful morphism restricted to $A_2$ is $3 : 1$ [3.19], gives that $b = 3$. Now taking on both sides of (3.35) the intersection product with $D_{\{1,2\}}$, and using that $D_{\{1,2\}} D_{\{1,2\}} = \frac{1}{\pi}$, we obtain $a = \frac{1}{4}$.

A basis of $A^1(\overline{\mathcal{M}}_{1,3})$ is given by:

$$D_{irr} D_{\{1,2\}}, \ D_{irr} D_{\{1,3\}}, \ D_{irr} D_{\{2,3\}}, \ D_{irr} D_{\{1,2,3\}}, \ D_{\{1,2\}} \left( \sum_{\{i,j\}\subseteq \{1,2,3\}} D_{\{i,j\}} \right)$$

therefore $[A_3]$ can be uniquely written as:

$$[A_3] = a D_{irr} D_{\{1,2\}} + b D_{irr} D_{\{1,3\}} + c D_{irr} D_{\{2,3\}} + d D_{irr} D_{\{1,2,3\}} + e D_{\{1,2\}} \left( \sum_{\{i,j\}\subseteq \{1,2,3\}} D_{\{i,j\}} \right) \quad (3.36)$$

Taking the push-forwards via $\pi_{\{1,2\}}, \pi_{\{1,3\}}, \pi_{\{2,3\}}$, and using that these forgetful morphisms restricted to $A_3$ are $2 : 1$ (Lemma [3.19], gives:
We can write it in an unique way:

\[ a + b = a + c = b + c = \frac{1}{2}, \quad e = 2 \]

Now to determine \( d \), intersect both sides of \( \eqref{3.39} \) with \( D_{\{1,2\}} \) to find \( d = c \).

To conclude, we have to work out the case of \( \overline{A}_4 \). A basis for \( A^1(\overline{M}_{1,4}) \) is given by:

\[
D_{\{1,2,3,4\}} \left( \sum_{(i,j,k,l) = \{1,2,3,4\}} D_{\{i,j\}} D_{\{k,l\}} \right) D_{\text{irr}} \left( \sum_{(l,m) \subset \{i,j,k\}} D_{\{l,m\}} \right) \{i,j,k\} \subset [4]
\]

\[
D_{\text{irr}} D_{\{1,2,3,4\}} D_{\{i,j,k\}} \{i,j,k\} \subset [4] \quad D_{\text{irr}} D_{\{1,2,3,4\}} \left( \sum_{(i,j) \subset \{1,2,3,4\}} D_{\{i,j\}} \right)
\]

This set of classes is not closed under the action of \( S_4 \). Since the last two coordinates of \( \overline{A}_4 \) with respect to this basis will turn out to be zero, our result will again be symmetric under \( S_4 \).

We can write it in an unique way:

\[
[\overline{A}_4] = \sum_{i=1}^{12} b_i v_i
\]

where the \( v_i \)'s are the vectors of the basis just introduced, taken in the order of the previous list. Observe that \( v_2, \ldots, v_5 \) do not have a precise position in the list, and nor do \( v_6, \ldots, v_{10} \).

The fact that the construction of \( \overline{A}_4 \) is \( S_4 \)-equivariant means that this is not important, because for any possible choice of an ordering of these \( v_i \)'s it turns out that:

\[
b_2 = b_3 = b_4 = b_5 \quad b_6 = b_7 = b_8 = b_9 = b_{10}
\]

Using the same trick as before, taking the four push-forwards via \( \pi_{\{1,2,3,4\}} \cdot \pi_{\{1,2,4\}} \cdot \pi_{\{1,3,4\}} \cdot \pi_{\{2,3,4\}} \cdot \pi \), plus the fact that the forgetful morphism is an isomorphism when restricted to \( \overline{A}_4 \) \( \eqref{3.19} \), one can determine \( b_1, b_2, b_3, b_4, b_5, b_{11}, b_{12} \). Moreover in this way one finds relations like:

\[
b_i + 3b_{10} = \frac{1}{4}, \quad 6 \leq i \leq 9
\]

To finish the computation, it is enough to intersect everything with \( D_{\{1,2,3,4\}} \). This gives \( b_{10} = \frac{1}{12} \), and therefore concludes the proof of the last equality of the statement. \( \square \)

**Corollary 3.37.** Let \((Y,g)\) be a twisted sector of \( \overline{M}_{1,n} \). Then \([Y]\) is in the subalgebra generated by the divisors of \( \overline{M}_{1,n} \).

**Proof.** As a consequence of Theorem \( \ref{5.23} \), every twisted sector class is \( j_k \cdot p^*([Z]) \), where \( Z \) is a base twisted sector in \( \overline{M}_{1,k} \) (whose class in the Chow ring was studied in Theorem \( \ref{5.34} \)) and the maps fit into the diagram:

\[
\begin{array}{ccc}
\overline{M}_{1,k} \times \overline{M}_{0,l_1+1} \times \ldots \times \overline{M}_{0,l_k+1} & \xrightarrow{f_k} & \overline{M}_{1,n} \\
\downarrow p & & \downarrow \\
\overline{M}_{1,k} & & \\
\end{array}
\]

where \( j_k \) is the gluing map defined in \( \ref{5.25} \) and \( p \) is the projection onto the first factor. From this one can compute explicitly the twisted sectors as linear combinations of products of divisors. \( \square \)
4 The Age Grading

4.a Definition of Chen–Ruan degree

Let $X$ be a smooth algebraic stack of dimension $n$, and $x \in X$ a point. Let $T$ be the tangent bundle of $X$. For any $g$ in the stabilizer group of $x$ of order $k$, there is a basis of $T_x(X)$ consisting of eigenvectors for the $g$–action. In terms of such a basis, the $g$–action is given by a diagonal matrix $M = \text{diag}(\xi^{a_1}, \ldots, \xi^{a_n})$ where $\xi = e^{2\pi i/k}$ and $a_i < k$. For any pair $(x, g)$, define $a(x, g) := \frac{1}{k} \sum a_i$. This function is nonnegative and takes rational values. This function is constant on each connected component of the Inertia Stack ([7, Chapter 3.2]).

**Definition 4.1.** ([7, Chapter 3.2]) Let $X$ be an algebraic smooth stack. The age of a twisted sector $X_{(g)}$ is defined to be $a(x, g)$ for any point $(x, g) \in X_{(g)}$.

Note that this definition is also referred to as degree shifting number, or fermionic shift. The algebraic definition of age was given in [1, 7.1] and [2, 7.1].

**Proposition 4.2.** Let $(Y, g)$, $(Y, g^{-1})$ be two connected components of the Inertia Stack of an algebraic stack $X$ which are exchanged by the involution of the Inertia Stack. Then:

$$a((Y, g)) + a((Y, g^{-1})) = \text{codim}(Y, X)$$

**Remark 4.3.** If $i: Y \to X$ is a twisted sector, and $x \in Y$ is a point, then the following splitting holds:

$$T_x X = T_x Y \oplus N_{Y|X}$$

If $G := \langle g \rangle$ is in the stabilizer group of $x$, then $T_x X$ is a representation of $G$ which splits as a sum of two representations: $T_x Y$ and $N_{Y|X}$. The first of such representations is trivial by definition of the twisted sector. Therefore what is needed in order to compute the age of a twisted sector is to study the action of $G$ on $N_{Y|X}$.

In conclusion of this subsection we define the orbifold degree.

**Definition 4.4.** We define the $d$–th degree Chen–Ruan cohomology group as follows:

$$H^d_{CR}(X, \mathbb{Q}) := \bigoplus H^{d-2a(X, g_i)}(X, \mathbb{Q})$$

where the sum is over all the twisted sectors. If $\alpha$ is an element of this vector space, we define its orbifold degree as $d$.

4.b Age of $\mathcal{M}_{1,n}$ and $\overline{\mathcal{M}}_{1,n}$

We start our computations with the smooth case. The result for the age of $\mathcal{M}_{1,1}$ is well known, and we compute the other cases.

If $\phi_n$ is a generator for $\mu_n$ (and therefore of $\mu_n^\vee$ since we work over $\mathbb{C}$), we will call $\langle \phi_n^k \rangle$ the one dimensional complex vector space with the action of $\mu_n$, where $\phi_n$ acts as the multiplication by $\phi_n^k$. Recall that if $G = \text{Aut}(C, P)$ is the automorphism group of an elliptic curve, it is canonically identified with $\mu_n$ for a certain $n$ (Notation see 3.12).

**Lemma 4.5.** Let $Z \subset \mathcal{M}_{1,k}$ be the closed embedding of a twisted sector $(Z, \alpha)$ of $\mathcal{M}_{1,k}$, with $Z \in \{C_4, C'_4, C_6, C'_6, C''_6\}$. The normal bundle $N_Z \mathcal{M}_{1,k}$ is a representation of $\mu_n$ on a $k$–dimensional vector space:
• $N_{C_4}M_{1,1}$ is isomorphic as a representation of $\mu_4$ to $\langle i^2 \rangle$,
• $N_{C_3}M_{1,2}$ is isomorphic as a representation of $\mu_4$ to $\langle i^2 \rangle \oplus \langle i^3 \rangle$,
• $N_{C_0}M_{1,1}$ is isomorphic as a representation of $\mu_6$ to $\langle e^4 \rangle$,
• $N_{C_0}M_{1,2}$ is isomorphic as a representation of $\mu_3$ (generated by $e^2$) to $\langle e^2 \rangle \oplus \langle e^4 \rangle$,
• $N_{C_0}M_{1,3}$ is isomorphic as a representation of $\mu_3$ (generated by $e^2$) to $\langle e^2 \rangle \oplus \langle e^3 \rangle \oplus \langle e^4 \rangle$.

Proof. The age for the twisted sectors of $M_{1,1}$ is known from the description of it as an open substack of $\mathbb{P}(4,6)$ (see for instance [19]). It is easily checked that $i$ acts on the normal bundle as the multiplication by $i^6 = i^2$ and $\epsilon$ acts as multiplication by $e^4$.

We study the tangent to $C'_4$ in $M_{1,2}$, and the other cases will follow through. Since the forgetful morphism $\overline{M}_{1,2} \to \overline{M}_{1,1}$ is the universal curve, we have that the following diagram is cartesian (see [3.11] for the definition of $C_4$):

\[
\begin{array}{ccc}
C'_4 & \xrightarrow{g} & C_4 \\
\downarrow & & \downarrow \\
C_4 & \xrightarrow{} & \overline{M}_{1,1} \\
\end{array}
\]

The normal bundle $N_{C'_4}M_{1,2}$ is therefore isomorphic as a representation of $\mu_4$ to the direct sum: $g^*(N_{C_4/\mu_4}M_{1,2}) \oplus N_{C'_4}C_4$, since $C_4 \to [C_4/\mu_4]$ is a finite étale map. The first term in the direct sum is equivariantly isomorphic to $N_{C'_4}M_{1,1}$ (since the diagram is cartesian and the forgetful map is flat). The normal bundle $N_{C'_4}C_4$ is equivariantly isomorphic to the tangent space to $C_4$ in the second marked point. In the Weierstrass representation of $C_4$ (3.11), this is the point with projective coordinates: $[0 : 1 : 0]$. The tangent space in these coordinates is then parametrized by $[0 : t : 1]$ and on it $i$ acts as the multiplication by $i^3$ (since in Theorem 3.11 the weight of the action on the variable $y$ is 3).

With this result, it is straightforward to compute the age for all the twisted sectors of $M_{1,n}$, using the fact that the twisted sector corresponding to an involution automorphism have age equal to half the codimension (see Proposition 4.2). We do not write here this result explicitly, because it will be part of the more general table given in Corollary 4.8.

Remark 4.6. We observe that, in the proof of the above proposition, we had to use a different argument for the case $n = 1$ and $n > 1$. This is due to the fact that there is no forgetful map from $M_{1,1}$. In particular, if $(C, p) \in M_{1,1}$, and $\mu_n = \text{Aut}(C, P)$ then the actions of $\mu_n$ on $N(C, p)$ is equivariantly isomorphic to the normal bundle $N_{M_{1,1}}$ on $M_{1,1}$, and on $T_pC$ do not necessarily coincide.

In the following Proposition, we give a description of the normal bundle $N_Z\overline{M}_{1,n}$ where $Z$ is a twisted sector. We restrict to the cases where the base space of $Z$ (Definition 3.21) has dimension 0, since this simplifies the notation. Hence, let $Z$ be in $\{C_4, C'_4, C_6, C'_6, C''_6\}$. The normal bundle $N_{Z\overline{M}_{1,k}}$ is a $k$-dimensional vector space with an action of $\mu_n$ on it (where $n$ can be 3, 4 or 6) (which was studied in Lemma 4.3). For the sake of simplicity, we identify $Z^{t_1, \ldots, t_k}$ with $Z \times \overline{M}_{0, t_1+1} \times \ldots \times \overline{M}_{0, t_k+1}$ (see Definition 3.21). We denote with $p, p_1, \ldots, p_k$ the projections on each factor (see for instance the end of Section 2.6). With $\underline{C}$ we indicate the trivial bundle of rank 1.
Proposition 4.7. Let $I_1, \ldots, I_k$ be a partition of $[n]$ in $k \leq 3$ subsets. Suppose that $|I_i| > 1$ for all $i$'s. Then the normal bundle $N_{\mathcal{M}_{1,n}}$ is of rank $2k$ and splits as a direct sum of line bundles.

1. $N_{C_{[n]}^1} \mathcal{M}_{1,n}$ is isomorphic to: $(i^2, \mathbb{C}) \oplus (i^3, p_1^1(\mathbb{L}_n^{\vee}_{n+1}))$,

2. $N_{C_{[n]}^1,2} \mathcal{M}_{1,n}$ is isomorphic to: $(i^2, \mathbb{C}) \oplus (i^3, \mathbb{C}) \oplus (i^3, p_1^1(\mathbb{L}_1^{\vee}_{1+1})) \oplus (i^3, p_2^2(\mathbb{L}_2^{\vee}_{2+1}))$,

3. $N_{C_{[n]}^3} \mathcal{M}_{1,n}$ is isomorphic to: $(i^4, \mathbb{C}) \oplus (i^3, p_1^1(\mathbb{L}_n^{\vee}_{n+1}))$,

4. $N_{C_{[n]}^3,1,2} \mathcal{M}_{1,n}$ is isomorphic to: $(\epsilon^2, \mathbb{C}) \oplus (\epsilon^3, \mathbb{C}) \oplus (\epsilon^4, p_1^1(\mathbb{L}_1^{\vee}_{1+1})) \oplus (\epsilon^4, p_2^2(\mathbb{L}_2^{\vee}_{2+1}))$,

5. $N_{C_{[n]}^3,1,2,3} \mathcal{M}_{1,n}$ is isomorphic to: $(\epsilon^2, \mathbb{C}) \oplus (\epsilon^3, \mathbb{C}) \oplus (\epsilon^3, \mathbb{C}) \oplus (\epsilon^4, p_1^1(\mathbb{L}_1^{\vee}_{1+1})) \oplus (\epsilon^4, p_2^2(\mathbb{L}_2^{\vee}_{2+1})) \oplus (\epsilon^4, p_3^3(\mathbb{L}_3^{\vee}_{3+1}))$.

If some of the $I_i$'s has cardinality 1, the normal bundle has the same description after deleting the corresponding component $p_i^i(\mathbb{L}_i^{\vee}_{i+1})$.

We postpone the proof of the proposition, in order to immediately see that as a consequence of it we can compute the age of all the twisted sectors of $\mathcal{M}_{1,n}$. We use the convention that $\delta(I) = \delta_{1,|I|}$, the Kronecker delta.

Corollary 4.8. (of Proposition 4.7) The following table gives the age of all the twisted sectors of $\mathcal{M}_{1,n}$:

| Comp          | Aut | Codimension | Age                     |
|---------------|-----|-------------|-------------------------|
| $A_{[n]}^1$   | −1  | 1           | $\frac{1}{2}(3 - \delta(I_1) - \delta(I_2))$ |
| $A_{[n]}^2$   | −1  | 2           | $\frac{3}{2}(5 - \delta(I_1) - \delta(I_2) - \delta(I_3))$ |
| $A_{[n]}^3$   | −1  | 3           | $\frac{4}{7}(7 - \delta(I_1) - \delta(I_2) - \delta(I_3) - \delta(I_4))$ |
| $C_{[n]}^1$   | i   | 2           | $\frac{5}{2}$            |
| $C_{[n]}^2$   | −i  | 2           | $\frac{3}{2}$            |
| $C_{[n]}^3$   | i   | 4           | $\frac{11}{7} - \frac{3}{2}(\delta(I_1) + \delta(I_2))$ |
| $C_{[n]}^4$   | −i  | 4           | $\frac{11}{7} - \frac{3}{2}(\delta(I_1) + \delta(I_2))$ |
| $C_{[n]}^5$   | $\epsilon^2$ | 6           | $\frac{11}{7} - \frac{3}{2}(\delta(I_1) + \delta(I_2) + \delta(I_3))$ |
| $C_{[n]}^6$   | $\epsilon^3$ | 6           | $\frac{11}{7} - \frac{3}{2}(\delta(I_1) + \delta(I_2) + \delta(I_3))$ |
| $C_{[n]}^7$   | $\epsilon^4$ | 2           | $\frac{1}{2}$            |
| $C_{[n]}^8$   | $\epsilon^5$ | 2           | $\frac{1}{2}$            |

Proof. The age of the sectors with the automorphism $−1$ is easily computed as one half the codimension. They are all the twisted sectors whose associated base twisted sector $\mathcal{M}_{1,1}$ has dimension 1. For the twisted sectors whose associated base twisted sector has dimension 0, we simply apply Proposition 4.7.

1 see Notation 22 for a line bundle on a trivial gerbe
To prove Proposition 4.7, we will use the following classical result, which is attributed to Mumford [20].

Lemma 4.9. Let $I_1, \ldots, I_k$ be a partition of $[n]$, and $j_k$ the gluing map defined in Section 2.c.

$$j_k : \overline{M}_{1,1}^{\times 1} \times \overline{M}_{0,I_1 \cup \ast} \times \cdots \times \overline{M}_{0,I_k \cup \ast} \rightarrow \overline{M}_{1,n}$$

Let $p$ be the projection onto the first factor, and $p_1, \ldots, p_k$ the projections onto the moduli spaces of genus 0 curves. Then the normal bundle of the map $j_k$ is isomorphic to:

$$N_{j_k} = \bigoplus_{i=1}^{k} p^*(L_{\lambda i}^\vee) \otimes p_i^*(L_{\lambda i}^\vee)$$

where $L_{\lambda i}$ are the line bundles defined in Section 2.c, and the first one is on $\overline{M}_{1,1}^{\times 1} \cup \ast$, while the second one is on $\overline{M}_{0,I_i \cup \ast}^\bullet$.

Proof. (of Proposition 4.7) If $j_k$ denotes as usual the morphism defined in Section 2.c, the following diagram is cartesian by definition of $Z_{I_1, \ldots, I_k}$ ($p$ is the projection onto the first factor):

$$\begin{array}{ccc}
Z_{I_1, \ldots, I_k} & \xrightarrow{\lambda} & \overline{M}_{1,k} \times \overline{M}_{0,I_1 \cup \ast} \times \cdots \times \overline{M}_{0,I_k \cup \ast} \\
\downarrow p & \xrightarrow{j_k} & \overline{M}_{1,n} \\
Z & \xrightarrow{\lambda} & \overline{M}_{1,k}
\end{array}$$

In this case the following isomorphism holds:

$$p^*(N_{Z, \overline{M}_{1,k}}) \cong N_{Z^{I_1, \ldots, I_k}, \overline{M}_{1,k}} \times \overline{M}_{0,I_1 \cup \ast} \times \cdots \times \overline{M}_{0,I_k \cup \ast} = N_{\lambda}$$

since the diagram is cartesian and $p$ is flat. This is the trivial bundle with a certain representation of $\mu_n$, that we studied in Lemma 4.5.

Note that the map $j_k \circ \lambda$ is the restriction to $Z^{I_1, \ldots, I_k}$ of the map $I(\overline{M}_{1,n}) \rightarrow \overline{M}_{1,n}$. Therefore, the normal bundle $N_{Z^{I_1, \ldots, I_k}, \overline{M}_{1,n}}$ is isomorphic to the direct sum: $\lambda^* N_{j_k} \oplus N_{\lambda}$. To conclude the proof, we have now to study the first term $\lambda^* N_{j_k}$.

According to Lemma 4.9 we have:

$$\lambda^* N_{j_k} = \bigoplus_{i=1}^{k} p_i^*(L_{\lambda i}^\vee) \otimes p_i^*(L_{\lambda i}^\vee)$$

The term $p_i^*(L_{\lambda i}^\vee)_{|Z^{I_1, \ldots, I_k}}$ is the trivial bundle whose constant fiber is the tangent space to $Z$ in the $i$-th marked point. It carries the representation of $\mu_n$, as the latter group acts on the tangent space to $Z$ in the $i$-th marked point (this action can be computed explicitly, see Lemma 4.5). On the other hand, the term $p_i^*(L_{\lambda i}^\vee)_{|Z^{I_1, \ldots, I_k}}$ carries the trivial representation of $\mu_n$, but is non trivial as a line bundle. This last observation concludes the proof.  

\[\Box\]
Since we now know the grading, we want now to write a formula analogous to (4.10) adding a new variable to separate the different degrees in the Chen–Ruan cohomology of $\overline{M}_{1,n}$. We define:

$$P_0(s, t) := \sum_{n=0}^{\infty} \frac{Q_0(n, m)}{n!} s^n t^m$$

(4.10)

$$P_1(s, t) := \sum_{n=0}^{\infty} \frac{Q_1(n, m)}{n!} s^n t^m$$

(4.11)

$$P_{1,\alpha}^{CR}(s, t) := \sum_{n=0}^{\infty} \frac{Q_{1,\alpha}^{CR}(n, m)}{n!} s^n t^m$$

(4.12)

where:

$$Q_0(n, m) := \dim H^{2m}(\overline{M}_{0,n+1}) = a^m(n)$$

$$Q_1(n, m) := \dim H^{2m}(\overline{M}_{1,n})$$

$$Q_{1,\alpha}^{CR}(n) := \dim H^{2m+\alpha}_{CR}(\overline{M}_{1,n})$$

The relevant values of $\alpha$ are $\{0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \frac{9}{4} \}$. Summing over $\alpha$ and shifting by $\alpha$ the degree of $t$, we have the generating series of the Chen–Ruan Poincaré polynomials:

$$P_1^{CR}(s, t) := \sum_{\alpha \in A} t^\alpha P_{1,\alpha}^{CR}(s, t)$$

In other words, the coefficient of degree $n$ in the variable $s$ of $P_1^{CR}$ is the Chen–Ruan Poincaré polynomial of $\overline{M}_{1,n}$ divided by $n!$.

We recall that the power series of (4.10) are described in [12, Theorem 5.9], [14, Theorem 2.6]. There, the author describes the cohomology of the moduli of genus 0 and genus 1, $n$-pointed stable curves as a representation of $S_n$.

**Theorem 4.13.** The following equality between power series relates the dimension of the $m$-th Chen–Ruan cohomology group of $\overline{M}_{1,n}$ with the dimensions of the $m$-th cohomology group of $\overline{M}_{0,n}$ and $\overline{M}_{1,n}$.

$$P_{1,0}^{CR}(s, t) = P_0 + (t + t^2)P_0 + 3s(t^2 + t^3)P_0^2 + \frac{1}{27}(t^3 + t^4)P_0^3 + 2s(t + t^2)\frac{\partial}{\partial s}(sP_0) + \frac{1}{6}(t^2 + t^3)\frac{\partial^2}{\partial s^2}(s^3P_0)$$

$$P_{1,\frac{1}{2}}^{CR}(s, t) = tP_0 + \frac{t^2}{2}P_0^2$$

$$P_{1,\frac{1}{2}}^{CR}(s, t) = \frac{t^3}{2}P_0^2 + \frac{3}{4}tP_0^3 + ts\frac{\partial}{\partial s}(sP_0) + \frac{2t^3}{3}\frac{\partial^2}{\partial s^2}(s^2P_0)$$

$$P_{1,\frac{1}{4}}^{CR}(s, t) = 2(1 + t)P_0 + \frac{t^2}{2}P_0^2 + \frac{t^3}{6}P_0^3 + \frac{t^4}{24}P_0^4 + \frac{t^5}{2} + t^2\frac{\partial^4}{\partial s^4}(s^2P_0)$$

$$P_{1,\frac{3}{4}}^{CR}(s, t) = \frac{t^2}{2}P_0^2 + \frac{t^3}{2}P_0^3 + st\frac{\partial}{\partial s}(sP_0) + \frac{t^2}{2}t\frac{\partial^2}{\partial s^2}(s^2P_0)$$

$$P_{1,\frac{1}{2}}^{CR}(s, t) = P_0 + \frac{t^2}{2}P_0^2$$

5 The Second Inertia Stack

The definition of the Chen–Ruan product involves the second Inertia Stack.
**Definition 5.1.** Let $X$ be an algebraic stack. The *second Inertia Stack* $I_2(X)$ is defined as:

$$I_2(X) = I(X) \times_X I(X)$$

**Remark 5.2.** As for the Inertia Stack, the second Inertia Stack is smooth (cfr. [3.4](#) see also [4](#) p. 15], after noticing that $\kappa_{0,3}(X, 0) \cong I_2(X)$).

**Remark 5.3.** A point in $I_2(X)$ is a triplet $(x, g, h)$ where $x$ is a point of $X$ and $g, h \in \text{Aut}(x)$. It can be equivalently given as $(x, g, h, (gh)^{-1})$.

**Remark 5.4.** $I_2(X)$ comes with three natural morphisms to $I(X)$: $p_1$ and $p_2$, the two projections of the fiber product, and $p_3$ which acts on points sending $(x, g, h)$ to $(x, gh)$.

This gives the following diagram, where $(Y, g, h, (gh)^{-1})$ is a double twisted sector and $(X_1, g), (X_2, h), (X_3, (gh))$ are twisted sectors:

\[
\begin{array}{c}
(x, g) \\
\downarrow p_1 \\
(x, g, h) \\
\downarrow p_2 \\
(x, h) \\
\downarrow p_3 \\
(x, gh)
\end{array}
\]

\[
\begin{array}{c}
(X_1, g) \\
\downarrow p_1 \\
(Y, g, h) \\
\downarrow p_2 \\
(X_2, h) \\
\downarrow p_3 \\
(X_3, gh)
\end{array}
\]

Let us now study the double twisted sectors for the case when $X = \overline{\mathcal{M}}_{1,n}$. From now on we focus on the compact case, since the case of $\mathcal{M}_{1,n}$ follows through analogously and much more simply.

**Remark 5.6.** We label each sector of $I_2(X)$ via the triplet $(g, h, (gh)^{-1})$. There are two automorphism groups acting on $I_2(X)$: an involution sending a sector labeled with $(g, h, (gh)^{-1})$ into $(g^{-1}, h^{-1}, (gh))$, and $S_3$ which permutes the three automorphisms. Up to permutations and involution, the following are all the possible labels of the sectors in $I_2(X)$ (that correspond to nonempty connected sectors):

- $(1, 1, 1)$, generated group $\mu_1$;
- $(1, -1, -1)$, generated group $\mu_2$;
- $(\epsilon^2, \epsilon^2, \epsilon^2)$, generated group $\mu_3$;
- $(1, \epsilon^2, \epsilon^4)$, generated group $\mu_3$;
- $(1, i, -i)$ generated group $\mu_4$;
- $(i, i, -1)$, generated group $\mu_4$;
• \((\epsilon, \epsilon, \epsilon^4)\), generated group \(\mu_6\);
• \((\epsilon, \epsilon^2, -1)\), generated group \(\mu_6\);
• \((1, \epsilon, \epsilon^5)\), generated group \(\mu_6\).

We now describe the sectors of the double Inertia Stack. We do so up to the automorphisms described in the previous remark, and up to the permutations of the marked points.

**Proposition 5.7.** Up to permutation of the automorphisms, and up to involution, the following are the twisted sectors of \(I_2(\overline{\mathcal{M}}_{1,n})\):

\[
\left(\overline{A}_i[6], (1, -1, -1)\right), \left(\overline{A}_2^{i_1,i_2}, (1, -1, -1)\right), \left(\overline{A}_3^{i_1,i_2,i_3}, (1, -1, -1)\right), \left(\overline{A}_4^{i_1,i_2,i_3,i_4}, (1, -1, -1)\right)
\]

\[
\left(C_6^{i_1,i_2}, (1, \epsilon^2, \epsilon^4)/(\epsilon^2, \epsilon^2, \epsilon^2)\right), \left(C_6^{i_1,i_2,i_3}, (1, \epsilon^2, \epsilon^4)/(\epsilon^2, \epsilon^2, \epsilon^2)\right),
\]

\[
\left(C_4^{[i], (1, i, -i)/(i, i, -1)}\right), \left(C_4^{i_1,i_2}, (1, i, -i)/(i, i, -1)\right), \left(C_6^{i_1,i_2,i_3}, (1, \epsilon, \epsilon^5)/(\epsilon, \epsilon, \epsilon^4)\right)
\]

\[
\left(C_6^{i_1,i_2,i_3,i_4}, (1, \epsilon, \epsilon^5)/(\epsilon, \epsilon, \epsilon^4)/(\epsilon^2, \epsilon^2, -1)/(\epsilon^2, \epsilon^2, \epsilon^2)\right)
\]

**Proof.** This follows from Theorem 3.24 once one observes that no point in \(\overline{\mathcal{M}}_{1,n}\) is stable under the action of both \(\epsilon\) and \(i\).

From this, a very easy consideration follows:

**Corollary 5.8.** Let \((Z, g, h, (gh)^{-1})\) be a double twisted sector of \(\overline{\mathcal{M}}_{1,n}\). Then either \((Z, g)\) or \((Z, h)\) or \((Z, (gh)^{-1})\) is a twisted sector of the Inertia Stack of \(\overline{\mathcal{M}}_{1,n}\).

### 6 The Excess Intersection Bundle

#### 6.a Definition of the Chen–Ruan product

We review the definition of the excess intersection bundle over \(I_2(X)\), for \(X\) an algebraic smooth stack. Let \((Y, g, h, (gh)^{-1})\) be a twisted sector of \(I_2(X)\). Let \(H := \langle g, h \rangle\) be the group generated by \(g\) and \(h\).

**Construction 6.1.** Let \(\gamma_0, \gamma_1, \gamma_\infty\) be three small loops around \(0, 1, \infty \subset \mathbb{P}^1\). Any map \(\pi_1(\mathbb{P}^1 \setminus \{0, 1, \infty\}) \to H\) corresponds to an \(H\)–principal bundle on \(\mathbb{P}^1 \setminus \{0, 1, \infty\}\). Let \(\pi^0 : C^0 \to \mathbb{P}^1 \setminus \{0, 1, \infty\}\) be the \(H\)–principal bundle which corresponds to the map \(\gamma_0 \to g, \gamma_1 \to h, \gamma_\infty \to (gh)^{-1}\). It can uniquely be extended to a ramified \(H\)–Galois covering \(C \to \mathbb{P}^1\) (see [10 Appendix]), where \(C\) is a smooth compact curve. Note that \(H\) acts on \(C\) and, by definition, \(\mathbb{P}^1\) is the quotient \(C/H\) as varieties, and hence on \(H^1(C, \mathcal{O}_C)\).

Let \(f : Y \to X\) be the restriction of the canonical map \(I_2(X) \to X\) to the twisted sector \(Y\). Note that \(H\) acts on \(f^*(T_X)\).

**Definition 6.2.** With the same notation as in the previous paragraph, the **excess intersection bundle** over \(Y\) is defined as:

\[
E_Y = \left(H^1(C, \mathcal{O}_C) \otimes \mathcal{O} f^*(T_X)\right)^H
\]

i.e. the \(H\)-invariant subbundle of the expression between parenthesis.
Remark 6.3. Since $H^1(C, \mathcal{O}_C)^H = 0$, it is the same to consider in the previous definition:

$$(H^1(C, \mathcal{O}_C) \otimes N_Y X)^H$$

where $N_Y$ is the coker of $T_Y \to f^*(T_X)$.

We now review the definition of the Chen–Ruan product.

Definition 6.4. Let $\alpha \in H^*_{CR}(X)$, $\beta \in H^*_{CR}(X)$. We define:

$$\alpha *_{CR} \beta = p_3^* (p_1^*(\alpha) \cup p_2^*(\beta) \cup c_{top}(E))$$

We sometimes use the notation $\alpha \beta$ instead of $\alpha *_{CR} \beta$.

Theorem 6.5. ([7]) With the age grading defined in the previous section, $(H^*_{CR}(X, \mathbb{Q}), *_{CR})$ is a graded $(H^*(X, \mathbb{Q}), \cup)$-algebra.

Theorem 6.5 allows us to compute the rank of the excess intersection bundle in terms of the already computed age grading. If $(Y, (g, h, (gh)^{-1}))$ is a sector of the second Inertia Stack, the rank of the excess intersection bundle is (here we stick to the notation introduced in Remark 5.4):

$$\text{rk}(E_i(Y, g, h)) = a(X_1, g) + a(X_2, h) + a(X_3, (gh)^{-1}) - \text{codim}(Y) \quad (6.6)$$

where the codimension is taken in $X$.

Corollary 6.7. The excess intersection bundle over double twisted sectors when either $g, h$, or $(gh)^{-1}$ is the identity, is the zero bundle.

One other useful formula that follows from Proposition 4.2 relates the rank of the excess bundle over a double twisted sector and the rank of the excess bundle over the double twisted sector obtained inverting the automorphisms that label the sector:

$$\text{rk}(E_{i}(g^{-1}, h^{-1})) = \text{codim}(X_1) + \text{codim}(X_2) + \text{codim}(X_3) - 2 \text{codim}(Y) - \text{rk}(E_i(Y, g, h)) \quad (6.8)$$

Let $(Y, (g, h, (gh)^{-1}))$ be a double twisted sector in $I_2(\overline{M}_{1, n})$, and let $H$ be the group generated by $(g, h, (gh)^{-1})$. We want to study $N_Y X$ and $H^1(C, \mathcal{O}_C)$ as representations of $H$.

6.6 The Excess Intersection Bundle for $\overline{M}_{1, n}$

We have seen in Remark 6.6 what are all the possible couples of automorphisms that correspond to non empty connected substacks of $I_2(\overline{M}_{1, n})$. Thanks to Corollary 6.7, the double twisted sectors whose excess intersection bundles have non zero rank are those labelled by:

$$\langle \epsilon^2, \epsilon, \epsilon^2, (i, i, -1), (\epsilon, \epsilon, \epsilon^4), (\epsilon, \epsilon, \epsilon^2, -1) \rangle \quad (6.9)$$

up to permutation and involution. The top Chern classes of the excess intersection bundles for $\mathcal{M}_{1, n}$ are always 0 or 1, since the coarse moduli spaces of the double twisted sectors labeled by these automorphisms are points.

The rank of the excess intersection bundles for the twisted sectors labeled by 6.9 can be given thanks to formulas 6.6 and 6.8.

Proposition 6.10. In the following table we list the ranks for the excess intersection bundles over all the double twisted sectors $(Z, g, h)$ of $\overline{M}_{1, n}$, such that $g, h$ and $gh \neq 1$:

\begin{center}
\begin{tabular}{|c|c|c|c|c|c|}
\hline
$(Z, g, h)$ & $a(X_1, g)$ & $a(X_2, h)$ & $a(X_3, (gh)^{-1})$ & $\text{codim}(Y)$ & $\text{rk}(E_{i}(g, h))$ \\
\hline
$(\epsilon^2, \epsilon, \epsilon^2, i, i, -1)$ & 1 & 1 & 1 & 2 & 1 \\
\hline
$(\epsilon, \epsilon, \epsilon^4, i, i, -1)$ & 1 & 1 & 1 & 2 & 1 \\
\hline
$(\epsilon, \epsilon, \epsilon^2, i, i, -1)$ & 1 & 1 & 1 & 2 & 1 \\
\hline
\end{tabular}
\end{center}
We work out the

\[ \phi \]

write

For the reduced building data (Proposition 2.1) we write:

Hence

\[ L \]

Remark 6.11. Corollary 6.7 together with the proposition above, tells us that a lot of top Chern classes of excess intersection bundles are 1, namely all the top Chern classes of excess intersection bundles whose rank is 0. Moreover it is obvious, after Proposition 4.11 that all the rank 3 or 5 bundles must contain at least one trivial subbundle of rank 1, this implying that the top Chern classes of them must be zero.

Now we want to compute explicitly the excess intersection bundles and their top Chern classes for \( \mathfrak{M}_{1,n} \). The following two propositions give the result we need in this direction.

Firstly, we give the decomposition of \( H^1(C, \mathcal{O}_C) \) as a representation of \( H \) in the cases corresponding to non 0 ranks in Proposition 6.10. Here \( \phi_n \) is a generator for \( \mu_n \), the same chosen in Section 4. As in that section, we indicate with \( \langle \phi_n^i \rangle \) the one dimensional complex vector space endowed with the action of \( \phi_n \in \mu_n \) given by the product by \( \phi_n^i \).

**Proposition 6.12.** Let \( H \) be generated by two elements \( g, h \) as in Proposition 6.10. Let \( C \to \mathbb{P}^1 \) be the \( H \)-covering associated with the generators \( g, h \) (see Construction 6.7). We study \( H^1(C, \mathcal{O}_C) \) as an \( H \)-representation:

- \( H = \mu_3: g = e^2, h = e^2 \), then \( H^1(C, \mathcal{O}_C) = \langle e^2 \rangle \),
- \( H = \mu_3: g = e^4, h = e^4 \), then \( H^1(C, \mathcal{O}_C) = \langle e^4 \rangle \),
- \( H = \mu_4: g = i, h = i \), then \( H^1(C, \mathcal{O}_C) = \langle i \rangle \),
- \( H = \mu_6: g = e, h = e \), then \( H^1(C, \mathcal{O}_C) = \langle e \rangle \),
- \( H = \mu_6: g = e, h = e^2 \), then \( H^1(C, \mathcal{O}_C) = \langle e^2 \rangle \).

**Proof.** This is a direct computation which uses the tools developed in 23. We work out the case of \( g = i, h = i, (gh)^{-1} = -1 \). We use the notation introduced in 23. In particular we write \( \phi_i \) for the one dimensional representation that corresponds to the multiplication by \( i \). For the reduced building data (23 Proposition 2.1) we write:

\[ 4L_{\phi_4} = D_{\phi_4}^{(\mu_4, \phi_4)} + 3D_{\phi_4}^{(\mu_4, \phi_4^2)} + 2D_{\phi_4}^{(\mu_2, \phi_4^2)}. \]

Hence \( L_{\phi_4} = O(1) \).

Now we have to compute \( L_{\phi_4^2} \). Again, with the same notation as in section 2 of 23 we get:

\[ m_{\phi_4}^{\mu_4, \phi_4} = 1; \quad m_{\phi_4}^{\mu_4, \phi_4^2} = 3; \quad m_{\phi_4}^{\mu_2, \phi_4^2} = 1. \]

And so:

\[ e_{\phi_4}^{\mu_4, \phi_4} = 0; \quad e_{\phi_4}^{\mu_4, \phi_4^2} = 1; \quad e_{\phi_4}^{\mu_2, \phi_4^2} = 1. \]
In this way we can then compute:

\[ L_{\phi_2} = L_{\phi_4} + L_{\phi_4} - D_{\phi_4}^{(\mu_4, \phi_4^2)} - D_{\phi_4}^{(\mu_2, \phi_4^2)} \]

So \( L_{\phi_2} = L_{\phi_4} = \mathcal{O}(1) \).

Finally, we have to compute \( L_{\phi_3} \). Thus we write:

\[ m_{\phi_4^3}^{\mu_4, \phi_4} = 1; \quad m_{\phi_4^3}^{\mu_4, \phi_4^3} = 3; \quad m_{\phi_4^3}^{\mu_2, \phi_4^2} = 0. \]

and:

\[ \epsilon_{\phi_4, \phi_4^3}^{\mu_4, \phi_4^2} = 0; \quad \epsilon_{\phi_4, \phi_4^3}^{\mu_4, \phi_4^3} = 1; \quad \epsilon_{\phi_4, \phi_4^2}^{\mu_2, \phi_4^2} = 0. \]

So that we can finally compute:

\[ L_{\phi_3} = L_{\phi_4} + L_{\phi_4} - D_{\phi_4}^{(\mu_4, \phi_4^2)} \]

So \( L_{\phi_3} = \mathcal{O}(2) \).

As a corollary of Lemma 4.3 in [23] we obtain the following \( \mu_4 \)-equivariant decomposition:

\[ H_1(C, \mathcal{O}_C) \cong H^0(C, K_C) = H^0(\mathbb{P}^1, K_{\mathbb{P}^1}) \oplus \bigoplus_{\phi \neq 1} H^0(\mathbb{P}^1, K_{\mathbb{P}^1} + L_{\phi^{-1}}) \]

where \( \mu_4 \) acts on \( H^0(\mathbb{P}^1, K_{\mathbb{P}^1} + L_{\phi^{-1}}) \) via the character \( \phi \). From the computations given above, the genus of \( C \) is 1 and the character of the 1-dimensional representation is \( \phi_4 \). Hence it is a one dimensional vector space, where \( i \) acts as the multiplication by \( i \) itself.

With all this, and thanks to Proposition 4.7, we can compute the excess intersection bundles and their respective top Chern classes. We know already that, among the list of couples of automorphisms of Proposition 6.10, the rank 3 and 5 bundles have top Chern class zero (see 6.11). Among the vector bundles having rank greater than zero, we can prove:

**Corollary 6.13.** In the table 6.10, the top Chern classes of all the excess intersection bundles (which are all line bundles) corresponding to the couple \((\epsilon^4, \epsilon^4)\) are zero. The top Chern class of the excess intersection bundle that corresponds to the couple \((\epsilon, \epsilon)\) (of rank 1), is zero too.

**Proof.** From Proposition 4.7 and Proposition 6.12, it is straightforward to see that all the excess bundles mentioned in the statement contain a trivial subbundle, forcing their top Chern class to be zero.

So we are now left with three rank 1 excess intersection bundles, whose top Chern class in non zero nor 1. In the following diagram and in the following lemma, we identify the isomorphic spaces in order to simplify the notation for the projection maps.

![Diagram](image)

Where \( a \) can be 4 or 6. Remember that \( \bullet \) is the gluing point.
Corollary 6.14. The only top Chern classes of the excess intersection bundles over double twisted sectors of $\overline{M}_{1,n}$ which are not 0 nor 1 are:

1. $\left(C_{0}^{[n]}(, \epsilon^{2}, \epsilon^{2}, \epsilon^{2}) \right) \cong B_{\mu_{0}} \times \overline{M}_{0,n, \emptyset}$, where the top Chern class of the excess intersection bundle is $-p_{1}(\psi_{\bullet}) = -p_{1}(\psi_{n+1})$;
2. $\left(C_{0}^{[n]}(, i, i, -1) \right) \cong B_{\mu_{4}} \times \overline{M}_{0,n, \emptyset}$, where the top Chern class of the excess intersection bundle is $-p_{1}(\psi_{\bullet}) = -p_{1}(\psi_{n+1})$;
3. $\left(C_{0}^{[n]}(, \epsilon, \epsilon^{2}, -1) \right) \cong B_{\mu_{6}} \times \overline{M}_{0,n, \emptyset}$, where the top Chern class of the excess intersection bundle is $-p_{1}(\psi_{\bullet}) = -p_{1}(\psi_{n+1})$.

Proof. The fact that all the other top Chern classes are zero or one follows from all the considerations in this Section. In particular, we have observed in the beginning of the section that the excess intersection bundle that may have top Chern class different from 1 are listed in 6.10. In Remark 6.11 and in Corollary 6.13 we have computed the top Chern class of all the remaining cases to be zero or 1.

The result stated then follows as a consequence of Proposition 4.7, Proposition 6.12, and the definition of the $\psi$ classes that we gave in Section 2.c.

Note that when $n = 2$ the top Chern classes in Corollary above are 0 too, because the sectors involved are all points.

To conclude, we summarize the result we have obtained in this section:

Theorem 6.15. All the top Chern classes of excess intersection bundles over all double twisted sectors are explicitly given. They can be:

1. either 1, for all the sectors listed in Proposition 5.4 such that one of the three automorphisms of the labeling is 1,
2. or again 1, for some of the sectors in the list 6.10 mentioned in Remark 6.11,
3. or 0, for some of the sectors listed 6.10 as discussed in Remark 6.11 and in Corollary 6.13,
4. or a pullback of a $\psi$ class over a component $\overline{M}_{0,n}$, for the the remaining elements of the list 6.10, as in Corollary 6.14.

7 Pull–Backs and Push–Forwards of Strata to the Twisted Sectors

In order to compute the Chen–Ruan product, one has to compute pull–backs from the twisted sectors to the double twisted sectors and push–forwards from the double twisted sectors to the twisted sectors. Thanks to Corollary 6.14 it is necessary and sufficient to compute the push–forward and the pull–back between twisted sectors of the Inertia Stack.

In this section we fix $n$ and call $X := \overline{M}_{1,n}$. Let $(Y, g)$ be a twisted sector of $X$, and $f : Y \to X$ be the closed embedding of the twisted sector.

Lemma 7.1. The cycle map:

$$A^\ast(Y, \mathbb{Q}) \to H^{2\ast}(Y, \mathbb{Q})$$

is a graded ring isomorphism. Moreover the Chow ring of all the twisted sectors is generated by divisors.
Proof. All the factors of all the twisted sectors have Chow ring isomorphic to the even cohomology. The cohomology ring of $\overline{M}_{0,n}$ is generated by divisors due to the work of Keel [18]. The spaces $\overline{M}_{1,1}, \overline{M}_{2,2}, \overline{M}_{3,3}, \overline{M}_{4,4}$ all have coarse moduli space isomorphic to $\mathbb{P}^1$, hence their cohomology is generated by divisors. \hfill \Box

We can now state and prove the result announced in the introduction. For some of the results needed in the proof we refer to the following two subsections on pull–back and push–forward.

**Theorem 7.2.** The Chen–Ruan cohomology ring of $\overline{M}_{1,n}$ is generated as an $H^*(\overline{M}_{1,n}, \mathbb{Q})$-algebra by the fundamental classes of the twisted sectors with explicit relations.

Proof. We first show how the algebra is generated by the fundamental classes of the twisted sectors.

We will prove in Corollary 7.7 that $f^*$ is surjective. Let now $(X_1, g), (X_2, h)$ be two twisted sectors of $\overline{M}_{1,n}$. Then, as a consequence of Corollary 6.8, $(X_1, g) \times_X (X_2, h)$ is connected, and hence a double twisted sector of $I_2(X)$. Let $(X_3, gh)$ be the twisted sector of $I(X)$ that corresponds to $gh$, and we call $f_i$ the closed embeddings of $X_i$ in $X$. Let $\alpha_1 \in H^*(\mathbb{X}, \mathbb{Q}), \alpha_2 \in H^*((X_2, h), \mathbb{Q})$. We want to compute $\alpha_1 *_{CR} \alpha_2$. We call $\tilde{\alpha}_i$ two liftings of $\alpha_i$ to $H^*(X, \mathbb{Q})$ obtained using the surjectivity of $f_i$. Let $p_3 : (X_1, g) \times_X (X_2, h) \to (X_3, gh)$ be the third projection of the double twisted sector as in Formula 5.3. Let $E$ be the excess intersection bundle on $(X_1, g) \times_X (X_2, h)$, and $\gamma := p_{3*}(c_{top}(E))$. Then we have:

$$\alpha_1 *_{CR} \alpha_2 = (\tilde{\alpha}_1 * \tilde{\alpha}_2) * 1_{(X_1, g)} * 1_{(X_2, h)} = f_3^*(\tilde{\alpha}_1 \cup \tilde{\alpha}_2) \cup \gamma$$

(7.3)

and this already proves the generation claim. Now we explain how the relations are found.

In 7.19 we fix a candidate, for every couple $X_1, X_2$ of twisted sectors, of a cohomology class $\beta = \beta((X_1, g), (X_2, h)) \in H^*(X, \mathbb{Q})$ such that $p_{3*}(c_{top}(E)) = f_3^*(\beta)$. Finally, we obtain the formula for the Chen–Ruan product:

$$\alpha_1 *_{CR} \alpha_2 = f_3^*(\tilde{\alpha}_1 \cup \tilde{\alpha}_2 \cup \beta) = \tilde{\alpha}((X_1, g), (X_2, h)) *_{CR} 1_{(X_3, gh)}$$

(7.4)

where we have posed $\tilde{\alpha}((X_1, g), (X_2, h)) := \tilde{\alpha}_1 \cup \tilde{\alpha}_2 \cup \beta((X_1, g), (X_2, h))$. Therefore, the task of finding all the relations is accomplished once a class $\beta \in H^*(X, \mathbb{Q})$ is fixed for all the couples $(X_1, g), (X_2, h)$ (see 7.19) and the association:

$$(X_1, g), (X_2, h) \to (X_3, gh)$$

is known (see 7.3).

\hfill \Box

We observe that the same identical result would hold for the stringy Chow ring (see [1], [2]) if we would have developed its definition instead of that of Chen–Ruan cohomology. Moreover, assuming Getzler’s claims 2.11 and 2.12 one can obtain a description of $H_{CR}^*(\overline{M}_{1,n}, \mathbb{Q})$ as a $\mathbb{Q}$-algebra as a consequence of Theorem 7.11.
7.a Pull–Backs

Let now \((Y, g) = (Z^{i_1, ..., i_k}, g)\) be a twisted sector of \(\overline{\mathcal{M}}_{1,n}\) (see Theorem 3.22). Let \(f : Y \to \overline{\mathcal{M}}_{1,n}\) the restriction to \(Y\) of the natural map from the Inertia Stack to \(\overline{\mathcal{M}}_{1,n}\).

We want to study the pull–back morphism:

\[ f^* : H^*(\overline{\mathcal{M}}_{1,n}, \mathbb{Q}) \to H^*(Y, \mathbb{Q}) \]

The main results of this section, are:

1. the explicit description of the pull–back of the divisor classes of \(\overline{\mathcal{M}}_{1,n}\) (Theorem 7.10);
2. the pull–back morphism \(f^*\) is determined by its restriction to the subalgebra of the cohomology generated by the divisors (Theorem 7.11). This last result depends upon Getzler’s claims 2.11 and 2.12.

Note that while point (1) is enough for proving our main result 7.3, as a consequence of point (2) we can actually describe \(H^*_CR(\overline{\mathcal{M}}_{1,n}, \mathbb{Q})\) as a \(\mathbb{Q}\)-algebra.

Let us first compute the pull–back morphism for the divisor classes. Here we identify the twisted sector \(\overline{Z}^{i_1, ..., i_k}\) with the product of \(\overline{\mathcal{Z}} \times \overline{\mathcal{M}}_{0,t_1+1} \times \cdots \times \overline{\mathcal{M}}_{0,t_k+1}\), and therefore \(A^*(\overline{Z}^{i_1, ..., i_k}) = H^{2*}(\overline{\mathcal{Z}}) \times A^*(\overline{\mathcal{M}}_{0,t_1+1}) \times \cdots \times A^*(\overline{\mathcal{M}}_{0,t_k+1})\). As usual (see 2.6), the projections onto the factors are called \(p, p_1, ..., p_k\). The notation for the divisors in \(\overline{\mathcal{M}}_{1,n}\) is explained in Notation 2.6.

**Proposition 7.5.** The pull–back \(f^*(D_{a})\) is zero when the base space is \(C_a\), for \(a = 4\) or \(a = 6\).

1. It is \(\frac{1}{4}\)pt \(\times \overline{\mathcal{M}}_{0,n+1}\), when the space is \(\overline{C}^4\);
2. It is \(\frac{3}{8}\)pt \(\times \overline{\mathcal{M}}_{0,t_1+1}\), when the space is \(\overline{C}^2\);
3. It is \(3\)pt \(\times \overline{\mathcal{M}}_{0,t_1+1}\), when the space is \(\overline{A}_3\);
4. It is \(3\)pt \(\times \overline{\mathcal{M}}_{0,t_1+1}\), when the space is \(\overline{A}_4\).

**Proof.** Here the intersection of the loci inside \(\overline{\mathcal{M}}_{1,n}\) is transversal, and therefore the intersection is the set theoretic intersection. Another way to compute this, is by using Theorem 3.33.

Let \(M \subset [n]\) with \(|M| \geq 2\). We describe the pull–back \(f^*([D_M])\) (see 2.6 for the notation). If \(M \subset [I_1]\) we call \(\Delta_M \in A^1(\overline{\mathcal{M}}_{0,t_1})\) the divisor whose general element is curve with two genus 0 components and \(M\) marked points on one component and \(I_1 \setminus M\) on the other.

**Proposition 7.6.** The pullback \(f^*\) is zero whenever \(M\) is not contained in any of the \(I_i\)’s. If it is contained in (wlog) \(I_1\), then there are two cases. If \(M\) is a proper subset of \(I_1\), then

\[ f^*([D_M]) \cong [Z] \times \Delta_M \times \overline{\mathcal{M}}_{0,t_2+1} \times \cdots \times \overline{\mathcal{M}}_{0,t_k+1} \]

Otherwise, if \(I = M\), then:

\[ f^*([D_M]) = p_1^*(-\psi_{I+1}) \]

**Proof.** One simply observes that when \(M\) is strictly contained in \(I_1\), then the intersection is proper. When \(M = I_1\) there is an excess intersection whose result is obtained thanks to Lemma 4.8. Another way to see the same result is by using Theorem 3.34.

\(^3\)see Lemma 4.11 for the next equality.
Now a corollary of our description of the pull–back morphism, gives us a very important theoretical result:

**Corollary 7.7.** The morphisms $f^*: R^*(\overline{M}_{1,n}, \mathbb{Q}) \to A^*(Y, \mathbb{Q})$ are surjective. The same holds for the induced map in cohomology.

**Proof.** Thanks to Lemma 7.1, it is sufficient to prove that the morphism $f^*: R^1(\overline{M}_{1,n}, \mathbb{Q}) \to A^1(Y, \mathbb{Q})$ is surjective. The Kunneth formula allows us to reduce the problem to proving that one can obtain all divisors of each single factor of each twisted sector by pull–back from $R^1(\overline{M}_{1,n})$. The set of the divisor classes described in Proposition 7.6 surjects onto $A^1(Y, \mathbb{Q})$.

Another way to express this result is:

**Corollary 7.8.** If $Y$ is a twisted sector, then the cohomology $H^*(Y, \mathbb{Q})$ is an $H^*(\overline{M}_{1,n}, \mathbb{Q})$-module generated by the fundamental class $[Y]$. Indeed $H^*(Y, \mathbb{Q})$ is cyclic also as an $R^*(\overline{M}_{1,n})$-module, or as a module over the subring of the Tautological Ring generated by the divisors.

Now we deal with point (2) (see the beginning of this section). Getzler’s claims 2.11 and 2.12 imply the following:

**Claim 7.9.** (Getzler) If we write the cohomology ring $H^*(\overline{M}_{1,n}) = H^{even}(\overline{M}_{1,n}) \oplus H^{odd}(\overline{M}_{1,n})$, then:

$$R^*(\overline{M}_{1,n}) \cong RH^*(\overline{M}_{1,n}) = H^{even}(\overline{M}_{1,n})$$

**Remark 7.10.** The difference between the Chow ring and the cohomology ring here is mainly that in the second case we have a canonical candidate for a splitting of the ring into two parts: a tautological one and a “purely non tautological” one. Namely, the decomposition is simply the decomposition into even and odd parts. This is the main reason why we choose to describe everything in terms of cohomology instead of using the Chow ring.

It is clear that the pull–back morphism $f^*$ is zero when restricted to the odd cohomology classes. Hence to study $f^*$, it is enough to study its restriction to the Tautological Ring.

Now we state the following theorem:

**Theorem 7.11.** (depends upon claims 2.11 and 2.12) The pull–back morphism $f^*$ is determined by its restriction to the subalgebra of $H^*(\overline{M}_{1,n}, \mathbb{Q})$ generated by the divisors.

We want a description of the cycles that are not in the subalgebra of the Tautological Ring which is generated by divisor classes. Let $B$ be the class of the point of $\overline{M}_{1,2}$ given by the following picture. Note that $B$ is a boundary strata class:

![Figure 3: Necklace cycle in $\overline{M}_{1,2}$](image)

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Definition 7.12. Let $I_1 \sqcup I_2$ be a partition of $[n]$. We define the 2-necklace $B^{I_1,I_2}$, following \cite[p.2]{14}, by:

$$j_{1,2*}(p^*(B))$$

where the maps fit in the following diagram:

$$
\begin{array}{ccc}
\mathcal{M}_{1,2} \times \mathcal{M}_{0,I_1+1} \times \mathcal{M}_{0,I_2+1} & \xrightarrow{j_{1,2}} & \mathcal{M}_{1,n} \\
p & & \\
\mathcal{M}_{1,2} & & 
\end{array}
$$

and $j_{1,2}$ is the gluing map defined in Section 2.2. (Note that in \cite{3} it is called banana cycle)

Lemma 7.13. A boundary strata class that is not in the subalgebra of the cohomology generated by the divisors is a closed substacks of a 2-necklace cycle.

Proof. See \cite{3} for a proof of this.

Lemma 7.14. The square of $D_{irr}$ (Notation 2.6) is zero in the Tautological Ring of $\mathcal{M}_{1,n}$, for every $n$.

Proof. (of Theorem 7.11) Thanks to Lemma 7.13 all that we have to prove is that the product of a necklace cycle $[B^{I_1,I_2}]$ and $[Y]$ vanishes. We use Corollary 3.37. There, we have expressed explicitly $Y$ as a linear combination of product of divisor classes in $\mathcal{M}_{1,n}$. The product of $B^{I_1,I_2}$ with all the summands that contain a factor $D_{irr}$ is zero, thanks to Lemma 7.14 (because $B^{I_1,I_2}$ is a substack of the $D_{irr}$ in $\mathcal{M}_{1,n}$). All the other summands are easily checked to have product zero with the necklace cycle, because the set theoretic intersection of the substacks of $\mathcal{M}_{1,n}$ that they describe is empty.

7.b Push–Forwards

We now start studying the push–forward morphism. Let:

$$g : Z \to Y \quad f : Y \to X$$

be respectively the inclusion of a double twisted sector in a twisted sector and of the latter twisted sector inside $X = \overline{\mathcal{M}}_{1,n}$. We do not write the automorphisms that label each twisted sector.

We study the induced push–forward morphism. We start with the case $Y = \overline{\mathcal{M}}_{1,n}$ itself.

Lemma 7.15. The push–forward morphism $f_* : A^*(Z, \mathbb{Q}) \to A^*(\overline{\mathcal{M}}_{1,n}, \mathbb{Q})$ has image in the Tautological Ring. The same holds for the induced map in cohomology.

Proof. Thanks to Corollary 7.8 one has only to prove that the fundamental classes of the twisted sectors belong to the Tautological Ring.

The fact that the base spaces (see Definition 3.24) $Z$ belong to the Tautological Ring was proved in 3.34. Now to prove that all the twisted sectors $\overline{\mathcal{Z}}^{I_1,...,I_k}$ belong to the Tautological Ring, one observes that (a similar argument was used in Corollary 3.37)
1. Let \( j_k \) be the morphism described in Section 2.c. Then \( j_k(\overline{M}_{1,k} \times \overline{M}_{0,I_1} \times \ldots \times \overline{M}_{0,I_{k+1}}) \) is in the Tautological Ring by the very definition of it.

2. Suppose without loss of generality that \( 1 \in I_1, \ldots, k \in I_k \), and \( \pi_{\{1,\ldots,k\}} \) be the forgetful map. Then \( \pi_{\{1,\ldots,k\}}(\overline{M}_{1,k}) \) is in the Tautological Ring, since the latter is closed under pull–back via the map \( \pi \) (Remark 2.4).

3. The Tautological Ring is by definition closed under the cup product, and \( Z_{I_1,\ldots,I_k} \) is exactly the product of the two classes constructed in the points 1 and 2.

Now, the two lemmas 7.17 and 7.18 give meaning to the definition:

**Definition 7.16.** We define the Orbifold Tautological Ring of \( \overline{M}_{1,n} \) as:

- \( R_{CR}(\overline{M}_{1,n}) := R^\ast(\overline{M}_{1,n}) \oplus \bigoplus H^\ast((X_i, g_i), \mathbb{Q}) \) as \( \mathbb{Q} \)-vector space, where \( X_i \) are all the twisted sectors;
- the graduation is inherited by \( H_{CR}(\overline{M}_{1,n}, \mathbb{Q}) \);
- the product is the product \( \ast_{CR} \) restricted to the previously defined rationally graded \( \mathbb{Q} \)-algebra.

We want to show how \( g_\ast([Z]) \) can be obtained as a pull–back of a class in \( X \) in a canonical way.

**Definition 7.17.** If \( a = 4 \) or \( a = 6 \), we define \( C^\ast_a \) via the following pull–back diagram:

\[
\begin{array}{ccc}
C^\ast_a & \overset{\pi_1}{\longrightarrow} & \overline{M}_{1,1} \\
\downarrow & & \downarrow \pi_1 \\
C^\ast_a & \overset{\pi_1}{\longrightarrow} & \overline{M}_{1,n}
\end{array}
\]

Note that the equality:

\[
[C^\ast_a] = \frac{2}{a} D_{irr}
\]

holds in the Tautological Ring of \( \overline{M}_{1,n} \).

**Proposition 7.18.** With the notation introduced in this section, for \( Z \) a double twisted sector and \( Y \) a twisted sector, there is a canonical choice of \( W \) closed substack of \( \overline{M}_{1,n} \), such that \( g_\ast([Z]) = f_\ast([W]) \).

**Proof.** The only cases are, thanks to Proposition 5.7:

1. it happens that \( Z = Y \) or \( Y = X \). In all these cases we choose \([W] = [X]\);
2. either \( Z = C^a_4 \) for \( a = 4,6 \) and \( Y = \overline{A}_1^{(n)} \). In these cases we choose \([W] := [C^a_4]\);
3. or \( Z = C^{1,1}_4 \) and \( Y = \overline{A}_2^{1,1}_2 \). In these cases we choose \([W] = [C^a_4]\).

One can easily check that these are all the cases that occur, and that all intersections are transversal.

We have just fixed the cohomology classes that represent via pull–back the push–forward of all the fundamental classes. This choice determines the top Chern class of the excess intersection bundles via projection formula.
Corollary 7.19. Let now $E$ be the excess intersection bundle over the double twisted sector $Z$. Once the choice of $c_{top}(E)$ is fixed, a cohomology class $\beta$ on $\overline{M}_{1,n}$ is determined such that:

$$g_*(c_{top}(E)) = f^*(\beta)$$

Proof. If the top Chern class of $E$ is zero, we choose $\beta := 0$. When the top Chern class is 1, the choice of Proposition 7.13 determines the class $\beta$ of this corollary too. The list of non trivial top Chern classes of excess intersection bundles (non zero and non 1), is given in 6.14. So, if the top Chern class is a $\psi$ class, there are only two possibilities: either the double twisted sector $Z$ is isomorphic to the twisted sector $Y$ (case 1 of Corollary 6.14), or $Z = C_{6}^{[n]}$ and $Y = \overline{M}_{1,n}$ (cases 2 and 3 of Corollary 6.14). In the first case, we choose $\beta := D_{[n]}$ (see 2.6 for the latter), and in the second case we choose $\beta := D_{[n]} \cup [C_{6}^{n}]$ where $[C_{6}^{n}]$ was defined above.

\[ \square \]

7.6 Products of the fundamental classes of the twisted sectors

If $X_{i}, X_{j}$ are twisted sectors, we have understood in Theorem 3.25 that, in order to determine the Chen–Ruan product structure, we need to compute all the products $1_{X_{i}} *_{CR} 1_{X_{j}}$.

Remark 7.20. An explicit computation of all intersections of twisted sectors, shows that besides the orbifold intersections of the kinds $(1_{X_{i}}, \alpha) * (1_{X_{j}}, \beta)$, and besides the trivial products $1_{\overline{M}_{1,n}} * 1_{X_{j}}$, the only pairs of twisted sectors whose fundamental classes give rise to non zero Chen–Ruan products are in the following list:

1. $((\overline{A}_{1}^{[n]}, -1), (C_{4}^{[n]}, i - i))$;
2. $((\overline{A}_{1}^{[n]}, -1), (C_{6}^{[n]}, \epsilon^{2}/\epsilon^{4}/\epsilon^{5}))$;
3. $((\overline{A}_{1}^{[n]}, -1), (C_{4}^{[n]}, i - i))$.

We now compute the products of the pairs just described. Here if $(X, \alpha)$ is a twisted sector, we write $H^{*}((X, \alpha), \mathbb{Q})$, which is a direct summand of $H^{*}_{CR}(\overline{M}_{1,n}, \mathbb{Q})$ with its own graduation. In other words, we assume implicitly the inclusion

$$i : H^{*}((X, \alpha), \mathbb{Q}) \subset H^{*}_{CR}(\overline{M}_{1,n}, \mathbb{Q})$$

shifts the degree by twice the age of $(X, \alpha)$. As usual, $-p_{1}(\psi)$ is the Chern class of Corollary 6.14 and Theorem 6.15.

Corollary 7.21. With our usual notation for the twisted sectors, and with the notation introduced above, here is the explicit result of all Chen–Ruan products described in Remark 7.20.

1. $[C_{4}^{[n]}, i] *_{CR} ((\overline{A}_{1}^{[n]}, -1)) = p_{1}(-\psi) \in H^{2}((C_{4}^{[n]}, -i), \mathbb{Q})$;
2. $[C_{4}^{[n]}, -i] *_{CR} ((\overline{A}_{1}^{[n]}, -1)) = [C_{4}^{[n]}] \in H^{0}((C_{4}^{[n]}, i), \mathbb{Q})$;
3. $[C_{6}^{[n]}, \epsilon] *_{CR} ((\overline{A}_{1}^{[n]}, -1)) = p_{1}(-\psi) \in H^{2}((C_{6}^{[n]}, \epsilon^{4}), \mathbb{Q})$;
4. $[C_{6}^{[n]}, \epsilon^{2}] *_{CR} ((\overline{A}_{1}^{[n]}, -1)) = p_{1}(-\psi) \in H^{2}((C_{6}^{[n]}, \epsilon^{5}), \mathbb{Q})$;
5. $[C_{6}^{[n]}, \epsilon^{3}] *_{CR} ((\overline{A}_{1}^{[n]}, -1)) = [C_{6}^{[n]}] \in H^{0}((C_{6}^{[n]}, \epsilon), \mathbb{Q})$;
6. $[C_{6}^{[n]}, \epsilon^{4}] *_{CR} ((\overline{A}_{1}^{[n]}, -1)) = [C_{6}^{[n]}] \in H^{0}((C_{6}^{[n]}, \epsilon^{2}), \mathbb{Q})$.

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7. \([\overline{M}^{c1,-2}_{1}; -1]\) \(*_{CR} [C_{4}^{c1,-2}, i] = 0 \in H^2((C_{4}^{c1,-2}, -i), \mathbb{Q})
8. \([\overline{M}^{c1,-2}_{2}; -1]\) \(*_{CR} [C_{4}^{c1,-2}, i] = [C_{4}^{c1,-2}] \in H^2((C_{4}^{c1,-2}, i), \mathbb{Q})

Corollary 7.22. With our usual notation for the twisted sectors, and with the notation introduced above, here we have all the products of the kind \((1_{X}, \alpha) \ast (1_{X}, \beta)\):

1. \([X, \alpha]) \ast_{CR} [X, \alpha^{-1})] = [X] \in H^*([\overline{M}_{1, n}], \mathbb{Q})
2. \([C_{0}^{n}, i]) \ast_{CR} [C_{0}^{n}, i]) = p_{1}(\psi_{\bullet}) \cap [C_{0}^{n}] \in H^4([\overline{M}^{c1}_{1}], -1), \mathbb{Q})
3. \([C_{0}^{n}, i]) \ast_{CR} [C_{0}^{n}, i]) = [C_{0}^{n}] \in H^2([\overline{M}^{c0}_{1}], -1), \mathbb{Q})
4. \([C_{4}^{c1,-2}, i]) \ast_{CR} [C_{4}^{c1,-2}, i]) = 0 \in H^4([\overline{M}^{c1,-2}_{2}], -1), \mathbb{Q})
5. \([C_{4}^{c1,-2}, i]) \ast_{CR} [C_{4}^{c1,-2}, i]) = [C_{4}^{c1,-2}] \in H^2([\overline{M}^{c1,-2}_{2}], -1), \mathbb{Q})
6. \([C_{0}^{n}, \epsilon]) \ast_{CR} [C_{0}^{n}, \epsilon]) = 0 \in H^4((C_{0}^{n}, \epsilon^2), \mathbb{Q})
7. \([C_{0}^{n}, \epsilon]) \ast_{CR} [C_{0}^{n}, \epsilon]) = p_{1}(\psi_{\bullet}) \cap [C_{0}^{n}] \in H^4([\overline{M}^{c1}_{1}], -1), \mathbb{Q})
8. \([C_{0}^{n}, \epsilon]) \ast_{CR} [C_{0}^{n}, \epsilon]) = 0 \in H^4((C_{0}^{n}, \epsilon^5), \mathbb{Q})
9. \([C_{0}^{n}, \epsilon^2]) \ast_{CR} [C_{0}^{n}, \epsilon^2]) = p_{1}(\psi_{\bullet}) \in H^2((C_{0}^{n}, \epsilon^4), \mathbb{Q})
10. \([C_{0}^{n}, \epsilon^2]) \ast_{CR} [C_{0}^{n}, \epsilon^5]) = p_{1}(\psi_{\bullet}) \in H^2((C_{0}^{n}, \epsilon), \mathbb{Q})
11. \([C_{0}^{n}, \epsilon^4]) \ast_{CR} [C_{0}^{n}, \epsilon^4]) = 0 \in H^2((C_{0}^{n}, \epsilon^2), \mathbb{Q})
12. \([C_{0}^{n}, \epsilon^4]) \ast_{CR} [C_{0}^{n}, \epsilon^5]) = [C_{0}^{n}] \in H^2([\overline{M}^{c0}_{1}], -1), \mathbb{Q})
13. \([C_{0}^{n}, \epsilon^5]) \ast_{CR} [C_{0}^{n}, \epsilon^5]) = [C_{0}^{n}] \in H^0((C_{0}^{n}, \epsilon^4), \mathbb{Q})
14. \([C_{0}^{c1,-2,13}, \epsilon^2]) \ast_{CR} [C_{0}^{c1,-2,13}, \epsilon^2]) = 0 \in H^4((C_{0}^{c1,-2,13}, \epsilon^4), \mathbb{Q})
15. \([C_{0}^{c1,-2,13}, \epsilon^4]) \ast_{CR} [C_{0}^{c1,-2,13}, \epsilon^4]) = 0 \in H^2((C_{0}^{c1,-2,13}, \epsilon^2), \mathbb{Q})

Moreover, the product of two fundamental classes of twisted sectors that do not belong to this list, nor to the one of Corollary [7.21] is zero.

8. Examples

8.a The Chen–Ruan cohomology ring of \(\overline{M}_{1, 1}\)

With our notation, the product in \(\overline{M}_{1, 1}\) becomes:

| \(\ast_{CR}\) | \(\overline{M}_{1, 1}\) | \(\overline{M}_{1, 1}\) | \(\overline{M}_{1, 1}\) | \(\overline{M}_{1, 1}\) | \(\overline{M}_{1, 1}\) | \(\overline{M}_{1, 1}\) | \(\overline{M}_{1, 1}\) | \(\overline{M}_{1, 1}\) | \(\overline{M}_{1, 1}\) |
|---|---|---|---|---|---|---|---|---|---|
| \((\overline{M}_{1, 1}, -1)\) | \((C_{4}, i)\) | \((C_{4}, i)\) | \((C_{4}, i)\) | \((C_{4}, i)\) | \((C_{4}, i)\) | \((C_{4}, i)\) | \((C_{4}, i)\) | \((C_{4}, i)\) | \((C_{4}, i)\) |
| \((C_{4}, -i)\) | \((C_{4}, -i)\) | \((C_{4}, -i)\) | \((C_{4}, -i)\) | \((C_{4}, -i)\) | \((C_{4}, -i)\) | \((C_{4}, -i)\) | \((C_{4}, -i)\) | \((C_{4}, -i)\) | \((C_{4}, -i)\) |
| \((C_{0}, \epsilon)\) | \((C_{0}, \epsilon)\) | \((C_{0}, \epsilon)\) | \((C_{0}, \epsilon)\) | \((C_{0}, \epsilon)\) | \((C_{0}, \epsilon)\) | \((C_{0}, \epsilon)\) | \((C_{0}, \epsilon)\) | \((C_{0}, \epsilon)\) | \((C_{0}, \epsilon)\) |
| \((C_{0}, \epsilon^2)\) | \((C_{0}, \epsilon^2)\) | \((C_{0}, \epsilon^2)\) | \((C_{0}, \epsilon^2)\) | \((C_{0}, \epsilon^2)\) | \((C_{0}, \epsilon^2)\) | \((C_{0}, \epsilon^2)\) | \((C_{0}, \epsilon^2)\) | \((C_{0}, \epsilon^2)\) | \((C_{0}, \epsilon^2)\) |
| \((C_{0}, \epsilon^4)\) | \((C_{0}, \epsilon^4)\) | \((C_{0}, \epsilon^4)\) | \((C_{0}, \epsilon^4)\) | \((C_{0}, \epsilon^4)\) | \((C_{0}, \epsilon^4)\) | \((C_{0}, \epsilon^4)\) | \((C_{0}, \epsilon^4)\) | \((C_{0}, \epsilon^4)\) | \((C_{0}, \epsilon^4)\) |

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Here $A < B$ means the fundamental class $[A]$ of $A$ inside the cohomology of the space $B$.

We can explicitly write the inclusion:

$$H^*(\mathcal{M}_{1,1}, \mathbb{Q}) \to H^*_C(\mathcal{M}_{1,1}, \mathbb{Q})$$

as:

$$\frac{\mathbb{Q}[t]}{(t^2)} \to \frac{\mathbb{Q}[x_0, y_0, z_0]}{(x_0^2 - 1, 2y_0^2 - 3z_0^2, y_0z_0)} \to 2x_0y_0^2z_0$$

Where $x_0 = [A_1^{-1}]$, $y_0 = [C_4, -i]$, and $z_0 = [C_6, e]$.

8.b The Chen–Ruan cohomology ring of $\mathcal{M}_{1,2}$

We first review the result for the age of the sectors in the Inertia Stack:

| Sector | Age |
|--------|-----|
| $(\mathcal{M}_{1,1}^{-1}, -1)$ | $\frac{1}{2}$ |
| $(A_2, -1)$ | $\frac{1}{2}$ |
| $(C_4^0, i)$ | $\frac{1}{2}$ |
| $(C_4^0, -i)$ | $\frac{1}{2}$ |
| $(C_4, i)$ | $\frac{1}{2}$ |
| $(C_4, -i)$ | $\frac{1}{2}$ |
| $(A_4, 0)$ | $\frac{1}{2}$ |
| $(C_6^0, e^2)$ | $\frac{1}{2}$ |

Note that in $\mathcal{M}_{1,2}$ the double twisted sectors either involve the identical automorphism, or are of dimension 0. As a consequence, the top Chern class of the excess intersection bundle can be either 1 or 0, respectively when the rank of the bundle is 0 or greater than 0.

In the following table we denote $X := \mathcal{M}_{1,2}$.
We can explicitly write the inclusion:

\[ H^*(\mathcal{M}_{1,2}; \mathbb{Q}) \to H^*_{CR}(\mathcal{M}_{1,2}; \mathbb{Q}) \]

as:

\[
\begin{align*}
\mathbb{Q}[t_0, t_1] & \to \mathbb{Q}[x_0, y_0, z_0, x_1, y_1, w] \\
(t_0^2, t_0t_1 + 12t_1^2) & \mapsto \frac{Q[t_0, t_1]}{I} \\
t_0 & \mapsto -12x_0^2 + 4x_1^2 \\
t_1 & \mapsto x_0^2
\end{align*}
\]

where \( I \) is the ideal defined as:

\[
I := (x_0^2y_0, 6x_0^3 + y_0^2, x_0^2z_0, 3z_0^3 - 2y_0^2, y_0z_0,
\]

\[
x_1^2y_1, 2x_1^3 - 3y_1^2, 9x_0^4 + x_1^4,
\]

\[
x_0x_1, x_0y_1, x_1y_0, y_0y_1, z_0x_1, z_0y_1
\]

\[
wx_0, wy_0, wz_0, wx_1, wy_1, w^3 + 4x_0^4)
\]

The generators for the ordinary cohomology are taken to be \( t_0 := \delta_{irr} \) and \( t_1 := \delta_{1,1,2} \).

The generators for the Chen–Ruan cohomology are taken to be \( x_0 = [A_2], y_0 = [A_2], y_1 = [C_4^i - i] \) and \( w = [C_6^j, \epsilon] \).

8.c The Chen–Ruan cohomology ring of \( \overline{M}_{1,3} \)

From now on we use a uniform notation for the spaces \( Z^i_1, Z^i_2, Z^i_3 \) (i.e. all the base twisted sectors of Theorem 3.16 and Definition 3.21 will be referred to using the notation of Definition 3.21, see Remark 3.23). To simplify the notation, we write (for example) \( C_6^{i_1,i_2,i_3} \) for \( C_6^{i_1,i_2,i_3} \). We make an analogue convention for all the twisted sectors, since all the ones collected under the same name act in a similar fashion (for instance, they have same age).

There is no ambiguity nor loss of information in the tables, since two sectors with different superscripts have empty intersection (example: \( C_4^{i_1,i_2,i_3} \cap C_4^{i_1,i_2,i_3} = \emptyset \)). Moreover, the same general rule holds for the intersection of two different kinds of sectors. We present an example. With the contracted notation, the expression:

\[ \overline{A}_2^{i,2} \cap C_4^{i,2} = C_4^{i,2} \]

means:

- \( \overline{A}_2^{i,1} \cap C_4^{i,1} = C_4^{i,1} \), where \( \{i, j, k\} = \{1, 2, 3\} \);

- \( \overline{A}_2^{i,1} \cap C_4^{i,1} = \emptyset \) whenever \( \{i, j\} \neq \{i', j'\} \).
We adopt the same notation as in the case of \( \overline{\mathcal{M}}_{1,3} \) for the \(<\) and we define \( X = \overline{\mathcal{M}}_{1,3} \). We need the following divisor class several times: \( \psi_4 \cap [\overline{\mathcal{M}}_{0,4}] \). If

\[
i : X_i \to \overline{\mathcal{M}}_{0,4} \times A
\]

is an isomorphism (\( X_i \) is a twisted sector), then we call \( \theta \) the class \( i^*(-\psi_4 \cap [\overline{\mathcal{M}}_{0,4}] \times [A]) \). Note that \( \theta \in A^2(X_i) \).

| \( \mathcal{M}_{1,3} \) | \( A_{3,4} \times \mathbb{C}^2 \) | \( \mathbb{C}^2 \) | \( \mathbb{C}^2 \) | \( \mathbb{C}^2 \) | \( \mathbb{C}^2 \) |
|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|
| \( \mathcal{M}_{1,3} \) | \( A_{3,4} \times \mathbb{C}^2 \) | \( \mathbb{C}^2 \) | \( \mathbb{C}^2 \) | \( \mathbb{C}^2 \) | \( \mathbb{C}^2 \) |

8.d The Chen–Ruan cohomology ring of \( \overline{\mathcal{M}}_{1,4} \)

We use all the conventions adopted in the previous section for \( \overline{\mathcal{M}}_{1,3} \). We firstly review the result for the age of the sectors in the Inertia Stack:
| Sector Age | Sector Age | Sector Age | Sector Age |
|------------|------------|------------|------------|
| \((A_4^1, -1)\) | \((C_6^4, \epsilon)\) | \((C_4^4, i)\) | \((C_{3, 3, 3}^4, \epsilon^4)\) |
| \((A_2^1, -1)\) | \((C_6^4, \epsilon^2)\) | \((C_4^4, -i)\) | \((C_{6, 2}^2, \epsilon^2)\) |
| \((A_2^2, -1)\) | \((C_6^4, \epsilon^4)\) | \((C_4^4, i)\) | \((C_{6, 2}^2, \epsilon^4)\) |
| \((A_4^1, -1)\) | \((C_6^4, \epsilon^3)\) | \((C_4^4, -i)\) | \((C_{6, 2}^1, \epsilon^2)\) |
| \((A_4^1, -1)\) | \((C_6^4, \epsilon^3)\) | \((C_4^4, i)\) | \((C_{6, 2}^1, \epsilon^4)\) |

To write down the table of the product, we split it into four parts, all the other products being zero due to empty intersection of the twisted sectors:

1. products of \(\overline{A}_i^{i_1, i_2}\) and \(C_4^{i_2, i_3}\) by themselves where \(i = 1, 2\);
2. products of \(\overline{A}_i^{n}\) by all \(C_6^{[4]}\);
3. products of \(\overline{A}_i^{i_1, i_2}\) by themselves, where \(i = 3, 4\);
4. products of \(C_6^{[4]}\) by themselves.

We write \(X := \overline{M}_{1, 4}\). We need the following divisor class several times: \(\psi_5 \cap [\overline{M}_{0, 5}]\). If

\[ i : X_i \to \overline{M}_{0, 5} \times A \]

is an isomorphism (\(X_i\) is a twisted sector), then we call \(\theta\) the class \(i^*(-\psi_5 \cap [\overline{M}_{0, 5}] \times [A])\).

Note that \(\theta \in A^1(X_i)\).

| 1 | \((\overline{A}_1^{[4]}, -1)\) | \((\overline{A}_2^{[4]}, -1)\) | \((\overline{A}_2^{[4]}, -1)\) | \((\overline{A}_1^{[4]}, -1)\) | \((\overline{A}_2^{[4]}, -1)\) | \((\overline{A}_2^{[4]}, -1)\) |
|---|---|---|---|---|---|---|
| \((C_6^4, \epsilon)\) | \((C_6^4, \epsilon^2)\) | \((C_6^4, \epsilon^3)\) | \((C_6^4, \epsilon^4)\) | \((C_6^4, \epsilon)\) | \((C_6^4, \epsilon^2)\) | \((C_6^4, \epsilon^4)\) |
| \((C_4^4, i)\) | \((C_4^4, \epsilon)\) | \((C_4^4, \epsilon)\) | \((C_4^4, \epsilon)\) | \((C_4^4, i)\) | \((C_4^4, \epsilon)\) | \((C_4^4, \epsilon)\) |

\[\theta < \overline{M}_1^{[4]}\]

\[c_4^{[4]} < \overline{A}_1^{[4]}\]

\[0 < c_4^{[3]} < X\]

\[c_4^{[2]} < X\]

\[c_4^{[2]} < \overline{A}_2^{[2]}\]

\[c_4^{[2]} < \overline{A}_2^{[2]}\]

\[\overline{M}_{1, 4}\]

\[\theta < \overline{C}_6^{[4]}, \theta < \overline{C}_6^{[4]}, \theta < \overline{C}_6^{[4]}, \theta < \overline{C}_6^{[4]}\]

\[\overline{A}_4^{[4]}, \overline{A}_4^{[4]}, \overline{A}_4^{[4]}, \overline{A}_4^{[4]}\]
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