Some Properties of Coefficients Kolchin Dimension Polynomial.

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Abstract. The article presents a formula expressing Macaulay constants of a numerical polynomial through its minimizing coefficients. From this, we have that Macaulay constants of Kolchin dimension polynomials do not decrease.

For the minimal differential dimension polynomial \( \omega_{G/F} \) (this concept was introduced by W.Sitt in [5]) we will prove a criterion for Macaulay constants to be equal. In this case, as the example [2] shows, there are no bounds from above to the Macaulay constants of the polynomial \( \omega_{\xi/F} \) for \( G = F \langle \xi \rangle \).

Keywords: differential algebra, differential polynomials, Kolchin dimension polynomial, minimizing coefficients.

1. Introduction

One of the basic objects of study in differential algebra is the differential dimension polynomial introduced by E.Kolchin [1]. This is an analogue of dimension in algebraic geometry, and its role is similar to that of the Hilbert polynomial in commutative algebra.

The dimension polynomial itself is not an invariant of the differential field, but contains some invariants, such as the degree and leading coefficient. In this article, we are considering another little studied invariant, introduced by W.Sit [5], namely a minimal dimension polynomial.

Note that since the 80s of the last century, interest in computer algebra has increased. Speaking of history, one of the first scientists whose work laid to the foundation of constructive theory of ideals in the ring of polynomials, is F. Macaulay (20s of the 20th century). He proved, in particular, the criterion for a numerical function to be equal to Hilbert function of a homogeneous ideal in the ring of commutative polynomials. In 1990 T.Dubé used in the proof of the bound of the degree of elements Gröbner basis numbers named in honor of Macaulay as Macaulay constants.

The Hilbert function for sufficiently large values becomes a polynomial, which is discussed in this article. Moreover, the set of all Hilbert polynomials of homogeneous ideals coincides with the set of

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Kolchin dimension polynomials. In [3] (proposition 2.4.10) a criterion for the equality of a numerical polynomial to some Hilbert polynomial is proved, which does not use Macaulay’s results. In this paper we present this result in terms of Macaulay constants.

2. PRELIMINARY FACTS.

One can find basic concepts and facts in [1, 3].

Denote the set of integers by \( \mathbb{Z} \), non-negative integers by \( \mathbb{N}_0 \). For \((i_1, \ldots, i_m) \in \mathbb{N}_0^m\), the order of \( e \) is defined by \( \text{ord } e = \sum_{k=0}^m j_k \). We also denote by \( \binom{n}{k} \) binomial coefficients, \( \frac{n!}{k!(n-k)!} \). Note that any numerical (i.e., taking integer values at integer points) polynomial \( v(s) \) can be written as \( v(s) = \sum_{i=0}^d a_i (s+i) \), where \( a_i \in \mathbb{Z} \). We will call the numbers \((a_d, \ldots, a_0)\) **standard coefficients** of the polynomial \( v(s) \). The following definition first appeared in [3].

**Definition 1.** (see [3], definition 2.4.9). Let \( \omega = \omega(s) \) be a numerical polynomial of degree \( d \) in variable \( s \). The sequence of **minimizing coefficients** of polynomial \( \omega \) is the vector \( b(\omega) = (b_d, \ldots, b_0) \in \mathbb{Z}^{d+1} \), defined inductively on \( d \) as follows. If \( d = 0 \) (i.e., \( \omega \) is a constant), then we set \( b(\omega) = (\omega) \). Let \( d > 0 \) and \((a_d, \ldots, a_0)\) are standard coefficients of polynomial \( \omega \); i.e., \( \omega(t) = \sum_{i=0}^d a_i (t+i) \). Denote by \( v(s) = \omega(s + a_d) - \binom{s+d+1+a_d}{d+1} - \binom{s+d+1}{d+1} \). Since \( \text{deg } v < d \), one can calculate the sequence of minimizing coefficients \( b(v) = (b_k, \ldots, b_0) \) \((0 \leq k < d)\) of polynomial \( v(s) \). Now, let \( b(\omega) = (a_d, 0, \ldots, 0, b_k, \ldots, b_0) \in \mathbb{Z}^{d+1} \).

Now we define the Kolchin dimension polynomial of a subset \( E \subset \mathbb{N}_0^m \). Regard the following partial order on \( \mathbb{N}_0^m \): the relation \((i_1, \ldots, i_m) \leq (j_1, \ldots, j_m)\) is equivalent to \( i_k \leq j_k \) for all \( k = 1, \ldots, m \). We consider a function \( \omega_E(s) \), that in a point \( s \) equals \( \text{Card } V_E(s) \), where \( V_E(s) \) is the set of points \( x \in \mathbb{N}_0^m \) such that \( \text{ord } x \leq s \) and for every \( e \in E \) the condition \( e \leq x \) isn’t true. Then (see for example, [1], p.115, or [3], theorem 5.4.1) function \( \omega_E(s) \) for all sufficiently large \( s \) is a numerical polynomial. We call this polynomial the **Kolchin dimension polynomial** of a subset \( E \) and denote \( \omega_E(s) \).

**Definition 2.** An operator \( \partial \) on a commutative ring \( \mathbb{K} \) with unit is called a **derivation** if it is linear \( \partial(a + b) = \partial(a) + \partial(b) \) and the Leibniz’s rule \( \partial(ab) = \partial(a)b + a\partial(b) \) holds for all elements \( a, b \in \mathbb{K} \).

A differential ring (or \( \Delta \)-ring) is a ring \( \mathbb{K} \) endowed with a set of derivations \( \Delta = \{\partial_1, \ldots, \partial_m\} \) which commute pairwise.

Let 
\[ 
\Theta = \Theta(\Delta) = \{\partial_1^{i_1} \cdots \partial_m^{i_m} \mid i_j \geq 0, \ 1 \leq j \leq m\} 
\]
and \( \theta = \partial_1^{i_1} \cdots \partial_m^{i_m} \). We define **order of derivative operator** \( \theta \):
\[ \text{ord}(\theta) = i_1 + \ldots + i_m \text{ and } \Theta(s) = \{\theta \in \Theta \mid \text{ord}(\theta) \leq s\} \.
Let
\[ R = \mathbb{K}\{y_j \mid 1 \leq j \leq n\} := \mathbb{K}[\theta y_j \mid \theta \in \Theta, 1 \leq j \leq n] \]
be a ring of commutative polynomials with coefficients in \( \mathbb{K} \) in the infinite set of variables \( \Theta Y = \Theta(y_j)_{j=1}^n \), and
\[ R_s = \mathbb{K}[\Theta(s)y_j], \quad s \geq 0. \]
A ring \( R \) is called a ring of differential polynomials in differential indeterminate \( y_1, \ldots, y_n \) over \( \mathbb{K} \).

**Definition 3.** Let \( F \) be a differential field with a set of derivations \( \Delta = \{\partial_1, \ldots, \partial_m\} \). The ring \( D = F[\partial_1, \ldots, \partial_m] \) of skew polynomials in indeterminates \( \partial_1, \ldots, \partial_m \) with coefficients in \( F \) and the commutation rules \( \partial_i \partial_j = \partial_j \partial_i \), \( \partial_i a = a\partial_i + \partial_i(a) \) for all \( a \in F \), \( \partial_i, \partial_j \in \Delta \) is called a (linear) differential (\( \Delta \)-) operator ring.

If derivation operators are trivial on \( F \), then \( D \) is isomorphic to the commutative polynomial ring.

Below we consider the case when \( \mathbb{K} \) is the differential field \( F \) and \( \text{char } F = 0 \) only. An ideal \( I \) in \( F\{y_1, \ldots, y_n\} \) is called differential, if \( \partial f \in I \) for all \( f \in I \) and \( \partial \in \Delta \). We will denote by \( \{I\} \) the minimal perfect differential ideal containing \( I \). By the theorem (II, p.126, Theorem 1), every perfect differential ideal can be represented as an intersection of finite number minimal prime differential ideals: \( \{I\} = \cap P_i \) (called the components of \( \{I\} \)).

Let \( G = F\langle \phi_1, \ldots, \phi_n \rangle \) is finitely generated \( \Delta \)-extension of the differential field \( F \). Define (Kolchin) differential dimension polynomial \( \omega_{\phi_1, \ldots, \phi_n}(s) \) if the following condition holds:
\[ \omega_{\phi_1, \ldots, \phi_n}(s) = \text{trdeg}_F(\Theta(s)\phi_1, \ldots, \Theta(s)\phi_n)/F \]
(here \( \text{trdeg} \) is the transcendence degree field extension). In other words, we add to the field \( F \) all derivations elements \( \phi_1, \ldots, \phi_k \) of order 1, \ldots, \( s \) and look for the cardinality of the maximum algebraically independent set of elements over \( F \) received field. For sufficiently large \( s \) this dependence is polynomial, or more precisely (see II, p. 115, Theorem 7), it is equal to the sum \( \sum_{j=1}^n \omega_{E_j}(s) \) of Kolchin polynomials of some subsets \( \mathbb{N}_0^m \). Note that the differential dimension polynomial is the Hilbert polynomial filtered module of differentials of the extension \( G \) over \( F \).

This polynomial contains some \( \Delta \)-invariants of the field \( G \) over \( F \) (in particular, the degree and leading coefficient), but the polynomial can change when another system of generators is chosen \( G = F\langle \psi_1, \ldots, \psi_l \rangle \). For example, polynomial \( \omega_{F(\psi_1, \ldots, \psi_n)}(s+1) = \omega_{F(\Theta(1)\psi_1, \ldots, \Theta(1)\psi_n)}(s) \).

One of the invariants is introduced by W.Sit in [5] minimal dimension polynomial.
Definition 4. (see [5]). The polynomial \( \omega_{\eta_1, \ldots, \eta_n} / F(s) \) is called minimal dimension differential polynomial for \( \Delta \)-extension \( G = F(\eta_1, \ldots, \eta_n) \) if for any system of \( \Delta \)-generators \( G = F(\psi_1, \ldots, \psi_k) \) we have \( \omega_{\psi_1, \ldots, \psi_k} / F(s) \geq \omega_{\eta_1, \ldots, \eta_n} / F(s) \) for all sufficiently large \( s \). In this case the polynomial \( \omega_{\eta_1, \ldots, \eta_n} / F(s) \) will be denoted by \( \omega_{G/F}(s) \).

W. Sit ([5], proposition 5) proves the existence \( \omega_{G/F}(s) \).

3. Basic results.

3.1. Macaulay constants of a numerical polynomial. The article [4] proves a criterion for the equality of a numerical polynomial to some Kolchin polynomial \( \omega_E(s) \). Denote by \( W \) the set of all possible polynomials \( \{ \omega_E(s) : E \subset \mathbb{N}^m, m = 1, \ldots \} \).

Theorem 1. ([3], proposition 2.4.10) A numerical polynomial \( \omega(s) \) belongs to \( W \), iff its sequence of minimizing coefficients consists only of non-negative integers.

Note (see [4]) that the set of Kolchin polynomials is closed under additions:

Theorem 2. ([3], proposition 2.4.13.) Let \( \omega_1(s), \omega_2(s) \in W \). Then the polynomial \( \omega(s) = \omega_1(s) + \omega_2(s) \) also belongs to \( W \).

As follows from these theorems, the differential dimension polynomial \( \omega_{\eta_1, \ldots, \eta_n} / F(s) \) has non-negative minimizing coefficients.

Theorem 3. Let \( \omega(s) \) is a numerical polynomial of degree \( d \), and \( b(\omega) = (b_d, \ldots, b_0) \in \mathbb{Z}^{d+1} \) is a sequence of its minimizing coefficients. Then

\[
\omega(s) = \binom{s+d+1}{d+1} - \sum_{i=0}^{d+1} \binom{s+i-1-c_i}{i}
\]

(1)

where

\[
c_i = \sum_{j=i}^{d} b_j, \quad i = 0, \ldots, d + 1,
\]

(the value of \( c_0 \) can be set arbitrarily).

Proof. We will prove this theorem by induction on \( d \). If \( d = 0 \), \( \omega(s) = b_0 \), we need to check the equality \( b_0 = (s+1) - (\binom{s-1-c_0}{0} + \binom{s-c_1}{1}) = (s+1) - (1 + (s-c_1)) \), which follows from the definition \( c_1 = b_0 \).

Let now \( d > 0 \), \( b(\omega) = (b_d, \ldots, b_0) \in \mathbb{Z}^{d+1} \). According to the definition (1), we denote

\[
v(s) = \omega(s + b_d) - \left( \frac{s+d+1+b_d}{d+1} \right) + \left( \frac{s+d+1}{d+1} \right)
\]

(2)

Then \( \deg v < d \), \( b(v) = (b_{d-1}, \ldots, b_0) \in \mathbb{Z}^d \) and by induction we can assume that
\[ v(s) = \left( s + \frac{d'}{d} \right) - \sum_{i=0}^{d} \binom{s + i - 1 - c'_i}{i}, \quad (3) \]

where
\[ c'_i = \sum_{j=i-1}^{j=d-1} b_j, \quad i = 0, \ldots, d. \]

Let’s make the substitution \( s' = s + b_d \) in the expression (2):
\[ v(s' - b_d) = \omega(s') - \binom{s' + 1 + d}{d + 1} + \binom{s' - b_d + d + 1}{d + 1}. \]

Then, taking into account the formula (3) we will have:
\[ \omega(s') = \binom{s' + d + 1}{d + 1} - \binom{s' - b_d + d + 1}{d + 1} + v(s' - b_d) = \]
\[ \left( \begin{array}{c} s' + 1 + d \\ d + 1 \end{array} \right) - \left( \begin{array}{c} s' - b_d + d + 1 \\ d + 1 \end{array} \right) + \left( \begin{array}{c} s' - b_d + d \\ d \end{array} \right) - \sum_{i=0}^{d} \binom{s' - b_d + i - 1 - c'_i}{i} = \]
\[ \left( \begin{array}{c} s' + d + 1 \\ d + 1 \end{array} \right) - \left( \begin{array}{c} s' - b_d + d + 1 \\ d + 1 \end{array} \right) - \left( \begin{array}{c} s' - b_d + d \\ d \end{array} \right) - \sum_{i=0}^{d} \binom{s' - b_d + i - 1 - c'_i}{i}. \]

From equality \( \binom{k+1}{l+1} - \binom{k}{l+1} = \binom{k}{l} \) we have
\[ \omega(s') = \binom{s' + d + 1}{d + 1} - \binom{s' - b_d + d}{d + 1} - \sum_{i=0}^{d} \binom{s' - b_d + i - 1 - c'_i}{i} = \]
\[ = \binom{s' + d + 1}{d + 1} - \binom{s' + d - c_{d+1}}{d + 1} - \sum_{i=0}^{d} \binom{s' - b_d + i - 1 - c'_i}{i} = \]
\[ \left( \begin{array}{c} s' + d + 1 \\ d + 1 \end{array} \right) - \sum_{i=0}^{d+1} \binom{s' + i - 1 - c_i}{i}, \quad \text{т.к. } c'_i + b_d = c_i \text{ для } i = 0, \ldots, d, \ c_{d+1} = b_d. \]

The values of \( c_i \) in the expression (1) are called Macaulay constants (see, for example, [2], formula (*), p. 768). The arbitrary value \( c_0 \) for \( w \in W \) can be considered equal to the smallest number, for which the value of the Hilbert function is equal to the value of the polynomial. For \( E = (e_{ij}) \subset N_0^n, i = 1, \ldots, n \) this number equals to \( \sum_{i=1}^{m} \max_{j=1,\ldots,n} e_{ij}. \)

So, the theorem (3) establishes a relationship between the values of the minimizing coefficients polynomial and its Macaulay constants.

From here and from the theorem (1) we immediately have

**Corollary 1.** A numerical polynomial is a Kolchin dimension polynomial if and only if the sequence of its Macaulay constants is non-decreasing.
Later in this article, since we will not use the value of Macaulay’s constant $c_0$, we will consider only the values $c_{d+1}, \ldots, c_1$ and, for convenience, number them, as well as the minimizing coefficients, starting from zero: $(c_d, \ldots, c_0)$, where $d$ is polynomial degree.

**Example 1.** Consider the following system of linear differential equations:

\[
\begin{align*}
\partial_1^2 \partial_2 \xi &= 0 \\
\partial_1 \partial_2^2 \xi &= 0 \\
\partial_1 \partial_2 \partial_3^2 \xi &= 0 \\
\vdots \\
\partial_1 \partial_2 \partial_3 \ldots \partial_m^2 \xi &= 0
\end{align*}
\]

Let’s find the dimension polynomial of this system and its Macaulay constants.

Since the equations are linear and form a Gröbner basis in the ring of differential operators, we need to calculate the dimension polynomial of the lower triangular matrix $E \subset \mathbb{N}_m$, with 2 on the diagonal, and 1 below it:

\[
\begin{array}{cccc}
200 & \cdots & 0 \\
120 & \cdots & 0 \\
\vdots & \ddots & \ddots & \ddots \\
11 & \cdots & 12
\end{array}
\]

To calculate a dimension polynomial we will apply induction and the formula polynomial changes when adding an element, (see [3], theorem 2.2.10).

Let $m = 2$, $e = (1, 0)$,

\[
\omega_E(s) = \omega_{E,e}(s) + \omega_H(s - 1),
\]

where $H$ is a matrix subtracted from each row $E$ of vector $e$, ord $e = 1$. We have: $\omega_E(s) = \omega_e(s) + \omega_{10}(s - 1) = (s + 1) + 1$. In particular, the sequence of minimizing coefficients equals $b(\omega_E) = (1, 1)$.

Now suppose $m > 2$. Let $e = (1, 0, \ldots, 0)$ and apply the same formula: $\omega_E(s) = \omega_e(s) + \omega_H(s - 1)$, where $H$ is a matrix:

\[
\begin{array}{cccc}
100 & \cdots & 0 \\
020 & \cdots & 0 \\
\vdots & \ddots & \ddots & \ddots \\
01 & \cdots & 12
\end{array}
\]

It is clear that the dimension polynomial of the matrix $H$ is equal to the dimension polynomial of the matrix $E' \subset \mathbb{N}_m$, for which, by the inductive hypothesis, holds $b(\omega_{E'}) = (1, \ldots, 1) \subset \mathbb{N}_m$.

By definition of minimizing coefficients (1), from the condition $\omega_e(s) = \binom{s + m - 1}{m - 1}$ we get $b(\omega_E) = (1, \ldots, 1) \subset \mathbb{N}_m$. From the theorem (3), the Macaulay constants are: $(1, 2, \ldots, m)$. 
Let us now calculate the standard coefficients of the dimension polynomial. Let for \( E \subset \mathbb{N}_0^n \) the standard coefficients be the numbers \( a(\omega_E) = (a_{m-1}, \ldots, a_0) \) which means representation

\[
\omega_E(s) = \sum_{i=0}^{m-1} a_i \binom{s + i}{i}.
\]

Denote the dimension polynomial of the matrix \( m \times m \) by \( \omega_m(s) \). As shown above, for \( m = 2 \) the dimension polynomial equals \((s + 1) + 1\), so it’s sequence of standard coefficients is \((1, 1)\). Let \( m > 2 \). Denote by \( \nabla \) the difference operator acting on polynomials: \( \nabla f(s) = f(s) - f(s - 1) \).

Taking into account the formula \( \nabla \binom{s + i}{i} = \binom{s + i - 1 - 1}{i} \) and theorem \([3]\) about the representation of a polynomial in terms of its Macaulay constants, we see that \( \nabla \omega_m(s) \) is a polynomial whose Macaulay constants are equal to \( c_{m-1}, \ldots, c_1 \) and we can use the inductive assumption to calculate standard coefficients \( \omega_m(s) \), \( a_i(\omega_m) = a_{i-1}(\omega_{m-1}) \), \( i = m - 1, \ldots, 1 \). It remains for us to calculate the lowest standard coefficient, \( a_0(\omega_m) \). We use the formula \([4]\) \( \omega_m(s) = \binom{s + m - 1}{m} + \omega_{m-1}(s - 1) \).

Substitute in this formula \( s = -1 \), from the equality \( \binom{s + i}{i} \mid_{s = -1} = 0 \) for \( i > 0 \) we get \( a_0(\omega_m) = \sum_{i=0}^{m-2} a_i(\omega_{m-1}) \binom{-1 + i}{i} \).

Compute \( \binom{s + i - 1}{i} \mid_{s = -1} = \binom{-i - 2}{i} = \frac{(-i - 2)(i - 3)\ldots}{i!} \), where there are exactly \( i \) factors in the product from above. It’s clear that if \( i > 1 \), this product is equal to zero. We have \( a_0(\omega_m) = a_1(\omega_{m-1})(1 - 2) + a_0(\omega_{m-1}) = -a_2(\omega_m) + a_1(\omega_m) \).

We have a recursive formula, and can calculate all the standard coefficients of the polynomial \( \omega_m \): for \( m = 3 \) it is \((1, 1, 0)\), for \( m = 4 \) it is \((1, 1, 0, -1)\), then \((1, 1, 0, -1, 0, 1, \ldots)\).

### 3.2. Macaulay constants of a minimal dimension polynomial.

According to Sit’s theorem \([5]\), Proposition 5), set \( W \) is well-ordered with respect to the above (definition \([4]\)) order. Note that this order is equivalent to the lexicographic order on a sequence of minimizing coefficients (from highest to lowest) and on a sequence of Macaulay constants. If you need to compare sequences of different lengths, the shorter sequence must be left-completed with zeros.

The simplest example of a differential field extension, for which succeed specify the minimal dimension polynomial is an extension, given by one equation.

**Theorem 4.** Let \( F \in \mathcal{F}\{y_1\} \). Then for any prime differential component \( P \) of ideal \( \{F\} \) holds

\[
\omega_P(s) = \binom{s + m}{m} - \binom{s + m - d_j}{m}, \tag{5}
\]

and the sequence of Macaulay constants of the minimal dimension polynomial for each component is constant (may be different for different
prime components). Conversely, if the Macaulay constants of the differential dimension polynomial of the field extension \( \omega_{\xi_1, \dots, \xi_n/F} \) are the same and \( \deg(\omega_{\xi_1, \dots, \xi_n/F}) = |\Delta| - 1 \), then this polynomial is minimal \( \omega_G/F \), and a prime ideal, defining the field \( G \), is the general component one differential polynomial \( F \in F \{y_1\} \).

Proof. By the components theorem ([1], p.185, theorem 5), every minimal prime differential ideal containing \( \{F\} \), may be represented as \( P_j = [G_j] : H_{G_j}^\infty \), where \( G_j \in F \{y_1\} \) is an irreducible differential polynomial. Denote \( d_j = \text{ord} G_j \) the order of this polynomial. We obtain

\[
\omega_{\xi_1}(s) = \left( s + m \right) \frac{m}{s + m - d_j},
\]

where \( \xi_1 \) is the common zero of \( P_j \), therefore \( \omega_{P_j}(s) \leq \left( \frac{s + m}{m} \right) - \left( \frac{s + m - d_j}{m} \right) \).

Sequence of minimizing coefficients polynomial \( [3] \) is \( (d, 0, \ldots, 0) \), so inequality \( \omega_{P_j} \geq \left( \frac{s + m}{m} \right) - \left( \frac{s + m - d_j}{m} \right) \) will follow from the differential type and typical \( |\Delta| \)-dimensions invariance theorem (see, for example, [3], corollary 5.4.7).

According to the theorem \( [3] \) the Macaulay constants of the polynomial \( [1] \) are equal to \( (d_1, \ldots, d_j) \).

Conversely, let all the Macaulay constants of the minimum dimension polynomial \( \omega_G/F \) be the same and equal to \( d \), differential type \( G \) over \( F \) is equal to \( m - 1 \) (this condition means that the polynomial \( \omega_G/F \) has degree \( m - 1 \), see [1], p. 118) and \( \xi_1, \ldots, \xi_n - \text{common zero} \), for which \( \omega_{\xi_1, \ldots, \xi_n/F}(s) = \omega_G/F(s) = \left( \frac{s + m}{m} \right) - \left( \frac{s + m - d_j}{m} \right) \).

By Theorem ([1], p. 115, Theorem 7) \( \omega_{\xi_1, \ldots, \xi_n/F} = \sum_{j=1}^n \omega_{E_j} \), and since all the minimizing coefficients of \( \omega_{E_j} \) polynomials are non-negative, from the condition \( \omega_{\xi_1, \ldots, \xi_n/F} = \left( \frac{s + m}{m} \right) - \left( \frac{s + m - d_j}{m} \right) \) it follows that \( \omega_{E_j} = 0 \) for all \( j \), except for the total one (we can assume that for all \( j > 1 \)). Because \( \omega_{E_j} = 0 \) for \( j > 1 \), the elements \( \xi_2, \ldots, \xi_n \) are algebraic over \( F \langle \xi_1 \rangle \) and \( F \langle \xi_1, \ldots, \xi_n \rangle = F \langle \xi_1 \rangle \). Since \( E_1 \) consists of one element, a prime ideal defining \( F \{y_1\} \), is the general component of some differential polynomial.

If degree \( \omega_{\psi_1, \ldots, \psi_n/F}(s) \) is less than \( m \), and the field \( F \) contains \( \mathbb{C}(x_1, \ldots, x_m) \) (field of rational functions in unknowns \( x_1, \ldots, x_m \)), then (see [1], part II, §8, assertion 9) \( G \) has a primitive element \( \xi \) (i.e. such that \( G = F \langle \xi \rangle \)). Moreover, \( \xi \) can be chosen in the form \( \xi = \sum_{j=1}^n \lambda_j \psi_j, \lambda_j \in F, j = 1, \ldots, n \). With such a changing, the differential dimension polynomial does not increase, i.e. \( \omega_{\xi/F}(s) \leq \omega_{\psi_1, \ldots, \psi_n/F}(s) \) for all sufficiently large \( s \) and so the problem of finding the minimal differential dimension polynomial is related to the search for a primitive element fields (although it doesn’t solve it, see [3], Example 5.7.7). Both the problem of calculating the polynomial \( \omega_{\psi_1, \ldots, \psi_n/F}(s) \), and the searching \( \xi \) are "in principle" solved using the algorithm Rosenfeld-Grebsner, included in the diffag package of the system Computer Algebra Maple.
Suppose a primitive element is found in the extension, \( b(\omega_{\xi/F}) = (b_d, b_{d-1}, \ldots, b_0) \) and \( b_i \neq 0 \) for some \( i < d \). We are interested in whether there are such values \( b_i \) that \( b(\omega_G/F)(s)) = (b_d, 0, \ldots, 0) \) is impossible.

The following example gives a negative answer to this question.

**Example 2.** Let \( F = \mathbb{C}(x_1, x_2) \), \( k \geq 2 \) and \( \Delta \)-extension \( G \) be given by the system

\[
\begin{align*}
\partial_1 \partial_2^2 \varphi &= 0, \\
x_1 x_2 \partial_1^k \partial_2 \varphi - x_1 \partial_1^k \varphi + x_2 \partial_1^{k-1} \partial_2 \varphi + \partial_1 \partial_2 \varphi - \partial_1^{k-1} \varphi &= 0.
\end{align*}
\]

These equations form the Gröbner basis, so \( \omega_{\varphi/F}(t) = \omega_{(k, 1)}(s) = 2s + k \) and \( b(\omega_{\varphi/F}) = (2, k-1) \). Let \( \psi = \varphi - x_1 x_2 (\partial_1^{k-1} \varphi - x_2 \partial_1^{k-1} \partial_2 \varphi) \).

This change of variables is invertible, since \( \varphi = (1 + x_1 x_2 \partial_1^{k-1}) \psi \). At the same time \( \partial_1 \partial_2 \psi = 0 \), hence \( \omega_{\psi/F}(s) = \omega_{G/F}(s) = 2s + 1 \) and \( b(\omega_{G/F}) = (2, 0) \).

In the example (2), the field contains elements that are not constants. This condition, as the following statement shows, is essential.

**Theorem 5.** Let \( G \) be defined by the system \( \Sigma \) of linear differential equations in one variable with coefficients, which are constants of the field \( F \). If \( \omega_{\xi/F}(s)! = \binom{s+m}{m} - \binom{s+m-d}{m} \), then \( \omega_{G/F}(s)! = \binom{s+m}{m} - \binom{s+m-d}{m} \).

**Proof.** Consider \( \Omega_{G/F} \), the module of differentials (see, for example, [3], p.38), and let \( J = [\Sigma] \) be the ideal of the ring \( D = G[\Delta] \) of differential operators. We have the exact sequence of \( D \)-modules:

\[
0 \to D J \to D \to \Omega_{G/F} \to 0.
\]

Suppose \( \omega_{G/F}(s) = \binom{s+m}{m} - \binom{s+m-d}{m} \). By Theorem 5.7.8 ([3]), there exists this exact sequence:

\[
0 \to D J \to D \to \Omega_{G/F} \to 0.
\]

It follows from these representations, that \( D J \oplus D = D \oplus D \). Hence the ideal \( J \) is a projective \( D \)-module. Under the conditions of the theorem, we can assume that the ring \( D \) is a ring of commutative polynomials over a field, and then \( J \) must be free (principal ideal in a polynomial ring). Then, by the theorem (1), holds \( \omega_{\xi/F} = \binom{s+m}{m} - \binom{s+m-d}{m} \). This is contrary to the condition. \( \square \)

As follows from the proof, the example (2) represents the projective ideal in non-commutative ring \( D = \mathbb{C}(x_1, x_2)[\partial_1, \partial_2] \), which is not free.
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