BI-LIPSCHITZ CHARACTERISTIC OF QUASICONFORMAL SELF-MAPPINGS OF THE UNIT DISK SATISFYING BI-HARMONIC EQUATION

SHAOLIN CHEN AND XIANTAO WANG *

Abstract. Suppose that $f$ is a $K$-quasiconformal self-mapping of the unit disk $D$, which satisfies the following: (1) the biharmonic equation $\Delta(\Delta f) = g$ ($g \in C(\overline{D})$), (2) the boundary condition $\Delta f = \varphi$ ($\varphi \in C(T)$ and $T$ denotes the unit circle), and (3) $f(0) = 0$. The purpose of this paper is to prove that $f$ is Lipschitz continuous, and, further, it is bi-Lipschitz continuous when $\|g\|_{\infty}$ and $\|\varphi\|_{\infty}$ are small enough. Moreover, the estimates are asymptotically sharp as $K \to 1$, $\|g\|_{\infty} \to 0$ and $\|\varphi\|_{\infty} \to 0$, and thus, such a mapping $f$ behaves almost like a rotation for sufficiently small $K$, $\|g\|_{\infty}$ and $\|\varphi\|_{\infty}$.

1. Preliminaries and main results

Let $\mathbb{C} \cong \mathbb{R}^2$ be the complex plane. For $a \in \mathbb{C}$ and $r > 0$, let $D(a, r) = \{z : |z - a| < r\}$, the open disk with center $a$ and radius $r$. For convenience, we use $D_r$ to denote $D(0, r)$ and $D$ the open unit disk $D_1$. Let $T$ be the unit circle, i.e., the boundary $\partial D$ of $D$ and $\overline{D} = D \cup T$. Also, we denote by $C^m(D)$ the set of all complex-valued $m$-times continuously differentiable functions from $D$ into $\mathbb{C}$, where $D$ is a subset of $\mathbb{C}$ and $m \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$. In particular, let $C(D) := C^0(D)$, the set of all continuous functions in $D$.

For a real $2 \times 2$ matrix $A$, we use the matrix norm

$$\|A\| = \sup \{|Az| : |z| = 1\}$$

and the matrix function

$$\lambda(A) = \inf \{|Az| : |z| = 1\}.$$ 

For $z = x + iy \in \mathbb{C}$, the formal derivative of a complex-valued function $f = u + iv$ is given by

$$D_f = \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix}.$$ 

Then,

$$\|D_f\| = |f_x| + |f_y| \quad \text{and} \quad \lambda(D_f) = ||f_x| - |f_y||.$$
where
\[ f_z = \frac{\partial f}{\partial z} = \frac{1}{2}(f_x - if_y) \quad \text{and} \quad f_\bar{z} = \frac{\partial f}{\partial \bar{z}} = \frac{1}{2}(f_x + if_y). \]

Moreover, we use
\[ J_f := \det Df = |f_z|^2 - |f_\bar{z}|^2 \]
to denote the Jacobian of \( f \).

For \( z, \zeta \in \mathbb{D} \) with \( z \neq \zeta \), let
\[ G(z, \zeta) = \log \left\{ \frac{1 - \bar{z}\zeta}{z - \zeta} \right\} \quad \text{and} \quad P(z, e^{it}) = \frac{1 - |z|^2}{|1 - ze^{-it}|^2} \]
be the Green function and the Poisson kernel, respectively, where \( t \in [0, 2\pi] \).

Let \( g \in L^1(\mathbb{D}) \) and \( f \in \mathcal{C}^4(\mathbb{D}) \). Of particular interest for our investigation is the following bi-harmonic equation:

\[ \Delta(\Delta f) = g \quad \text{in} \quad \mathbb{D} \]

with the following associated Dirichlet boundary value condition:

\[ \begin{cases} 
\Delta f = \varphi & \text{in} \quad \mathbb{T}, \\
 f = f^* & \text{in} \quad \mathbb{T},
\end{cases} \]

where \( f^*, \varphi \in \mathcal{C}(\mathbb{T}) \) and

\[ \Delta f := \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 4f_{\bar{z}z} \]
stands for the Laplacian of \( f \).

By [11, Theorem 1] (or [2, Theorem 1]), we see that all solutions to the equation (1.1) satisfying the condition (1.2) are given by

\[ f(z) = \mathcal{P}f^*(z) + G_1[\varphi](z) - G_2[g](z), \]
where

\[ \mathcal{P}f^*(z) = \frac{1}{2\pi} \int_0^{2\pi} P(z, e^{i\theta}) f^*(e^{i\theta}) d\theta, \]

\[ G_1[\varphi](z) = \frac{1}{8\pi} \int_0^{2\pi} (1 - |z|^2) \left[ 1 + \frac{\log(1 - ze^{-i\theta})}{ze^{-i\theta}} + \frac{\log(1 - \bar{z}e^{i\theta})}{\bar{z}e^{i\theta}} \right] \varphi(e^{i\theta}) d\theta, \]

\[ G_2[g](z) = \frac{1}{16\pi} \int_{\mathbb{D}} \left\{ 2|\zeta - z|^2 G(z, \zeta) + (1 - |z|^2)(1 - |\zeta|^2) \right. \]

\[ \times \left. \left[ \frac{\log(1 - \bar{z}\zeta)}{z\zeta} + \frac{\log(1 - \bar{z}\zeta)}{\bar{z}\zeta} \right] \right\} g(\zeta) d\sigma(\zeta), \]

and \( d\sigma \) denotes the Lebesgue area measure in \( \mathbb{D} \). We refer the reader to [13, 14, 31] etc for more discussions in this line.
Given a subset \( \Omega \) of \( \mathbb{C} \), a function \( \psi : \Omega \to \mathbb{C} \) is said to belong to the \textit{Lipschitz} space \( \Lambda(\Omega) \) if
\[
\sup_{z_1, z_2 \in \Omega, z_1 \neq z_2} \frac{|\psi(z_1) - \psi(z_2)|}{|z_1 - z_2|} < \infty.
\]
Further, a function \( \psi \in \Lambda(\Omega) \) is said to be \textit{bi-Lipschitz continuous} if there is a positive constant \( M \) such that for all \( z_1, z_2 \in \Omega \),
\begin{equation}
M |z_1 - z_2| \leq |\psi(z_1) - \psi(z_2)|. \tag{1.6}
\end{equation}

For a given domain \( \Omega \), we say that a function \( u : \Omega \to \mathbb{R} \) is \textit{absolutely continuous on lines}, ACL in brief, if for every closed rectangle \( R \subset \Omega \) with sides parallel to the axes \( x \) and \( y \), respectively, \( u \) is absolutely continuous on almost every horizontal line and almost every vertical line in \( R \). It is well-known that partial derivatives \( u_x \) and \( u_y \) exist almost everywhere in \( \Omega \).

The definition carries over to complex-valued functions.

**Definition 1.1.** Let \( K \geq 1 \) be a constant. A sense-preserving homeomorphism \( f : \Omega \to D \), between domains \( \Omega \) and \( D \) in \( \mathbb{C} \), is \( K \)-quasiconformal if \( f \) is ACL in \( \Omega \) and
\[
|f| \leq \frac{K-1}{K+1}|f_z| \quad \text{(or} \quad K^{-1}\|Df\|^2 \leq J_f \leq K\lambda^2(Df)\text{)}
\]
almost everywhere in \( \Omega \).

The following is the so-called Mori’s Theorem (cf. [6, 7, 22, 25, 32]).

**Theorem A.** Suppose that \( f \) is a \( K \)-quasiconformal self-mapping of \( \mathbb{D} \) with \( f(0) = 0 \). Then, there exists a constant \( Q(K) \), satisfying the condition \( Q(K) \to 1 \) as \( K \to 1 \), such that
\[
|f(z_2) - f(z_1)| \leq Q(K)|z_2 - z_1|^{\frac{1}{K}},
\]
where the notation \( Q(K) \) means that the constant \( Q \) depends only on \( K \).

We remark that in [35] it is proved
\[
1 \leq Q(K) \leq 16^{1-\frac{1}{K}} \min \left\{ \left( \frac{23}{8} \right)^{1-\frac{1}{K}}, \left( 1 + 2^{3-2K} \right)^{\frac{1}{K}} \right\}.
\]

A natural problem is that under which condition(s) a quasiconformal mapping is Lipschitz continuous. Recently, the study of this problem has been attracted much attention. For example, the Lipschitz characteristic of harmonic quasiconformal mappings has been discussed by many authors ([4, 18, 19, 21, 26, 30, 33, 34]). The Lipschitz continuity of \((K, K')\)-quasiconformal harmonic mappings has also been investigated in [3]. See, e.g., [8, 9, 12, 17, 24, 29, 34, 36, 37] for more discussions on the properties of harmonic quasiconformal mappings. On the study of the Lipschitz continuity of quasiconformal mappings satisfying certain elliptic PDEs, we refer to [1, 20, 22, 25]. The following result is from [25], which is a generalization of the main results of [33, 34].

**Theorem B.** ([25, Theorem 1.2]) Suppose that \( K \geq 1 \) is arbitrary and \( g \in C(\overline{\mathbb{D}}) \). Then, there exist constants \( N(K, g) \) and \( M(K) \) with \( \lim_{K \to 1} M(K) = 1 \) such that
if $f$ is a $K$-quasiconformal self-mapping of $\mathbb{D}$ satisfying the PDE: $\Delta f = g$ with $f(0) = 0,$ then for $z_1, z_2 \in \mathbb{D},$
\[
\left( \frac{1}{M(K)} - \frac{7||g||_\infty}{6} \right) |z_1 - z_2| \leq |f(z_1) - f(z_2)| \leq (M(K) + N(K, g)||g||_\infty)|z_1 - z_2|,
\]
where $||g||_\infty = \sup_{z \in \mathbb{D}} \{|g(z)|\}$.

The aim of this paper is to discuss the Lipschitz continuity of quasiconformal self-mapping of $\mathbb{D}$ satisfying the equation (1.1) with the boundary condition (1.2). Our result is as follows.

**Theorem 1.1.** Let $g \in C(\overline{\mathbb{D}}), \varphi \in C(\mathbb{T}),$ and let $K \geq 1$ be a constant. Suppose that $f$ is a $K$-quasiconformal self-mapping of $\mathbb{D}$ satisfying the equation (1.1) with $\Delta f = \varphi$ in $\mathbb{T}$ and $f(0) = 0$. Then, there are nonnegative constants $M_j(K)$ and $N_j(K, \varphi, g)$ ($j \in \{1, 2\}$) with
\[
\lim_{K \to 1} M_j(K) = 1 \text{ and } \lim_{||\varphi|| \to 0, ||g|| \to 0} N_j(K, \varphi, g) = 0
\]
such that for all $z_1$ and $z_2$ in $\mathbb{D},$
\[
(M_1(K) - N_1(K, \varphi, g))|z_1 - z_2| \leq |f(z_1) - f(z_2)| \leq (M_2(K) + N_2(K, \varphi, g))|z_1 - z_2|.
\]

**Remark 1.1.** By the discussions in Step 3 of the proof of Theorem 1.1 in Section 3, we see that the co-Lipschitz continuity coefficient $M_1(K) - N_1(K, \varphi, g)$ is positive for small enough norms $||g||_\infty$ and $||\varphi||_\infty$ (for example, if $||g||_\infty \leq \frac{60}{(25+61K^2)46^2(n-1)}$ and $||\varphi||_\infty \leq \frac{25}{(38+101K^2)46^2(n-1)}$ (see Corollary 3.1)). Example 4.1 shows that this condition for $f$ to be co-Lipschitz continuous cannot be replaced by the one that $\varphi$ and $g$ are arbitrary. In Section 4, another example is constructed to illustrate the possibility of $f$ from Theorem 1.1 to be bi-Lipschitz continuous.

We will prove several auxiliary results in the next section, Section 2. The proof of Theorem 1.1 will be presented in Section 3, and in Section 4, two examples are constructed.

### 2. Preliminaries

In this section, we shall prove several lemmas which will be used later on. The first lemma is as follows.

**Lemma 2.1.** Suppose that $\varphi \in C(\mathbb{T})$ and $G_1[\varphi]$ is defined in (1.4). Then, the following statements hold:

1. For $z \in \mathbb{D},$
\[
\max \left\{ \left| \frac{\partial}{\partial z} G_1[\varphi](z) \right|, \left| \frac{\partial}{\partial \bar{z}} G_1[\varphi](z) \right| \right\} \leq \frac{||\varphi||_\infty}{4} \left[ \max_{x \in (0,1)} \{h(x)\} + \left( \frac{\pi^2}{3} - 1 \right)^{\frac{1}{2}} |z| \right],
\]
where
\[
h(x) = (1 - x) \left[ \sum_{n=2}^{\infty} \frac{(n-1)^2}{n^2} x^{n-2} \right]^{\frac{1}{2}} \text{ in } [0, 1].
\]
(2) Both $\frac{\partial}{\partial z}G_1[\varphi]$ and $\frac{\partial}{\partial \bar{z}}G_1[\varphi]$ have continuous extensions to the boundary, and further, for $t \in [0, 2\pi]$,

\begin{align}
(2.1) \quad &\frac{\partial}{\partial z}G_1[\varphi](e^{it}) = -\frac{e^{-it}}{8\pi} \int_0^{2\pi} \left[ 1 + \log \frac{1 - e^{i(t-\theta)}}{e^{i(t-\theta)}} + \frac{\log(1 - e^{i(t-\theta)})}{e^{i(t-\theta)}} \right] \varphi(e^{i\theta})d\theta, \\
(2.2) \quad &\frac{\partial}{\partial \bar{z}}G_1[\varphi](e^{it}) = -\frac{e^{-it}}{8\pi} \int_0^{2\pi} \left[ 1 + \log \frac{1 - e^{i(t-\theta)}}{e^{i(t-\theta)}} + \frac{\log(1 - e^{i(t-\theta)})}{e^{i(t-\theta)}} \right] \varphi(e^{i\theta})d\theta,
\end{align}

(2.3) \quad \left| \frac{\partial}{\partial z}G_1[\varphi](e^{it}) \right| \leq \frac{\|\varphi\|_\infty}{4} \left( \frac{\pi^2}{3} - 1 \right)^{\frac{1}{2}},

and

(2.4) \quad \left| \frac{\partial}{\partial \bar{z}}G_1[\varphi](e^{it}) \right| \leq \frac{\|\varphi\|_\infty}{4} \left( \frac{\pi^2}{3} - 1 \right)^{\frac{1}{2}}.

The proof of Lemma 2.1 needs the following result (cf. [25, Proposition 2.4]).

**Theorem C.** Suppose that $X$ is an open subset of $\mathbb{R}$, and $(\Omega, \mu)$ denotes a measure space. Suppose, further, that a function $F : X \times \Omega \to \mathbb{R}$ satisfies the following conditions:

1. $F(x, w)$ is a measurable function of $x$ and $w$ jointly, and is integrable with respect to $w$ for almost every $x \in X$.
2. For almost every $w \in \Omega$, $F(x, w)$ is an absolutely continuous function with respect to $x$. (This guarantees that $\partial F/\partial x$ exists almost everywhere.)
3. $\partial F/\partial x$ is locally integrable, that is, for all compact intervals $[a, b]$ contained in $X$,

$$\int_a^b \int_\Omega \left| \frac{\partial}{\partial x}F(x, w) \right| d\mu(w)dx < \infty.$$  

Then, $\int_\Omega F(x, w)d\mu(w)$ is an absolutely continuous function with respect to $x$, and for almost every $x \in X$, its derivative exists, which is given by

$$\frac{d}{dx} \int_\Omega F(x, w) d\mu(w) = \int_\Omega \frac{\partial}{\partial x}F(x, w) d\mu(w).$$

**Proof of Lemma 2.1.** To prove the first statement of the lemma, we only need to show the inequality:

$$\left| \frac{\partial}{\partial z}G_1[\varphi](z) \right| \leq \frac{\|\varphi\|_\infty}{4} \left( \max_{x \in [0, 1]} \{h(x)\} + \left( \frac{\pi^2}{3} - 1 \right)^{\frac{1}{2}} |z| \right),$$

since the proof for the other one is similar. Let

$$I_1(z) = \frac{(|z|^2 - 1)}{8\pi} \int_0^{2\pi} \left[ \frac{1}{z(1 - ze^{-i\theta})} + \frac{e^{i\theta} \log(1 - ze^{-i\theta})}{z^2} \right] \varphi(e^{i\theta})d\theta$$

and

$$I_2(z) = -\frac{z}{8\pi} \int_0^{2\pi} \left[ 1 + \frac{\log(1 - ze^{-i\theta})}{ze^{-i\theta}} + \frac{\log(1 - \bar{z}e^{i\theta})}{\bar{z}e^{i\theta}} \right] \varphi(e^{i\theta})d\theta.$$
First, we estimate \(|I_1(z)|\). Since
\[
|I_1(z)| \leq \frac{(1 - |z|^2)\|\varphi\|_\infty}{8\pi} \int_0^{2\pi} \left| \sum_{n=2}^{\infty} \frac{(n-1)(ze^{-i\theta})^{n-2}e^{-i\theta}}{n} \right| d\theta,
\]
and since Hölder’s inequality implies
\[
\frac{1}{2\pi} \int_0^{2\pi} \left| \sum_{n=2}^{\infty} \frac{(n-1)(ze^{-i\theta})^{n-2}e^{-i\theta}}{n} \right| d\theta \leq \left( \frac{1}{2\pi} \int_0^{2\pi} \left| \sum_{n=2}^{\infty} \frac{(n-1)(ze^{-i\theta})^{n-2}e^{-i\theta}}{n} \right|^2 d\theta \right)^{\frac{1}{2}},
\]
we see that
\[
|I_1(z)| \leq \frac{(1 - |z|^2)\|\varphi\|_\infty}{4} \left( \sum_{n=2}^{\infty} \frac{(n-1)^2}{n^2} |z|^{2(n-2)} \right)^{\frac{1}{2}}.
\]

Let
\[
h(x) = (1 - x) \left( \sum_{n=2}^{\infty} \frac{(n-1)^2}{n^2} x^{n-2} \right)^{\frac{1}{2}} \quad \text{in } [0, 1).
\]
It follows from
\[
h(x) \leq (1 - x) \left( \sum_{n=2}^{\infty} x^{n-2} \right)^{\frac{1}{2}} = (1 - x)^{\frac{1}{2}}
\]
that \(\max_{x \in [0,1]} \{h(x)\}\) does exist. Then, we obtain that for \(z \in \mathbb{D}\),
\[
(2.5) \quad |I_1(z)| \leq \frac{\|\varphi\|_\infty}{4} \max_{x \in [0,1]} \{h(x)\}.
\]

Next, we estimate \(|I_2(z)|\). Since
\[
|I_2(z)| \leq \frac{|z|\|\varphi\|_\infty}{8\pi} \int_0^{2\pi} \left| 1 + \frac{\log(1 - ze^{-i\theta})}{ze^{-i\theta}} + \frac{\log(1 - \overline{z}e^{i\theta})}{\overline{z}e^{i\theta}} \right| d\theta,
\]
we obtain from Hölder’s inequality that
\[
|I_2(z)| \leq \frac{|z|\|\varphi\|_\infty}{4} \left\{ \frac{1}{2\pi} \int_0^{2\pi} \left| 1 + \frac{\log(1 - ze^{-i\theta})}{ze^{-i\theta}} + \frac{\log(1 - \overline{z}e^{i\theta})}{\overline{z}e^{i\theta}} \right|^2 d\theta \right\}^{\frac{1}{2}}.
\]
Then, it follows from
\[
\left\{ \frac{1}{2\pi} \int_0^{2\pi} \left| 1 + \frac{\log(1 - ze^{-i\theta})}{ze^{-i\theta}} + \frac{\log(1 - \overline{z}e^{i\theta})}{\overline{z}e^{i\theta}} \right|^2 d\theta \right\}^{\frac{1}{2}} \leq \left( 2 \sum_{n=1}^{\infty} \frac{1}{n^2} - 1 \right)^{\frac{1}{2}}
\]
that
\[
(2.6) \quad |I_2(z)| \leq \frac{\|\varphi\|_\infty}{4} \left( \frac{\pi^2}{3} - 1 \right)^{\frac{1}{2}} |z|.
\]
Now, (2.5), (2.6) and Theorem C guarantee that
\[ \left| \frac{\partial}{\partial z} G_1[\varphi](z) \right| = \left| \sum_{j=1}^{2} I_j(z) \right| \leq \sum_{j=1}^{2} |I_j(z)| \]
\[ \leq \frac{\|\varphi\|_{\infty}}{4} \max_{x \in [0,1)} \{h(x)\} + \frac{\|\varphi\|_{\infty}}{4} \left( \frac{\pi^2}{3} - 1 \right)^{\frac{1}{2}} |z|, \]
as required.

It follows from the first statement of the lemma and the Vitali Theorem (cf. [10, Theorem 26.C]) that \( \frac{\partial}{\partial z} G_1[\varphi] \) has a continuous extension to the boundary, and thus,
\[ \lim_{r \to 1-} \frac{\partial}{\partial z} G_1[\varphi](re^{it}) = \lim_{r \to 1-} I_1(re^{it}) + \lim_{r \to 1-} I_2(re^{it}) = I_2(e^{it}), \]
which implies
\[ \left| \frac{\partial}{\partial z} G_1[\varphi](e^{it}) \right| \leq \frac{\|\varphi\|_{\infty}}{4} \left( \frac{\pi^2}{3} - 1 \right)^{\frac{1}{2}}. \]
These show that (2.1) and (2.3) hold.

Similarly, we see that (2.2) and (2.4) are also true. Hence, the proof of the lemma is complete. \( \square \)

The following result is useful for the proof of Lemma 2.2 below.

**Theorem D.** (cf. [28]) For \( z \in \mathbb{D} \), we have
\[ \frac{1}{2\pi} \int_{0}^{2\pi} \frac{d\theta}{|1 - ze^{i\theta}|^{2\alpha}} = \sum_{n=0}^{\infty} \left( \frac{\Gamma(n + \alpha)}{n!\Gamma(\alpha)} \right)^{2} |z|^{2n}, \]
where \( \alpha > 0 \) and \( \Gamma \) denotes the Gamma function.

**Lemma 2.2.** Suppose \( g \in C(\mathbb{D}) \) and \( G_2[g] \) is defined in (1.5). Then, the following statements hold:

1. For \( z \in \mathbb{D} \),
\[ \max \left\{ \left| \frac{\partial}{\partial z} G_2[g](z) \right|, \left| \frac{\partial}{\partial \overline{z}} G_2[g](z) \right| \right\} \leq \|g\|_{\infty} \left[ \frac{1}{16} + \frac{(1 - |z|^2)^{\frac{1}{2}}}{60} + \frac{2\pi}{32} \left( \frac{1 + \pi^2}{6} \right)^{\frac{1}{2}} |z| \right]. \]

2. Both \( \frac{\partial}{\partial z} G_2[g] \) and \( \frac{\partial}{\partial \overline{z}} G_2[g] \) have continuous extensions to the boundary, and further, for \( \theta \in [0, 2\pi] \),
\[ \frac{\partial}{\partial z} G_2[g](e^{i\theta}) = \frac{1}{8\pi} \int_{\mathbb{D}} |\zeta - e^{i\theta}|^2 \frac{\partial}{\partial z} G(e^{i\theta}, \zeta) g(\zeta) d\sigma(\zeta) \]
\[ - \frac{e^{-i\theta}}{16\pi} \int_{\mathbb{D}} (1 - |\zeta|^2) \log(1 - e^{i\theta} \frac{\zeta)}{e^{i\theta} \zeta}) \]
\[ + \frac{\log(1 - e^{-i\theta} \zeta)}{e^{-i\theta} \zeta}) g(\zeta) d\sigma(\zeta), \]
\( \frac{\partial}{\partial z} G_2[g](e^{i\theta}) = \frac{1}{8\pi} \int_D |\zeta - e^{i\theta}|^2 \frac{\partial}{\partial \zeta} G(e^{i\theta}, \zeta) g(\zeta) d\sigma(\zeta) \\
- \frac{e^{i\theta}}{16\pi} \int_D (1 - |\zeta|^2) \left[ \frac{\log(1 - e^{i\theta} \zeta)}{e^{i\theta} \zeta} + \frac{\log(1 - e^{-i\theta} \zeta)}{e^{-i\theta} \zeta} \right] g(\zeta) d\sigma(\zeta), \)

\begin{equation}
\left| \frac{\partial}{\partial z} G_2[g](e^{i\theta}) \right| \leq \frac{\|g\|_{\infty}}{32} \left[ 1 + 2^\frac{1}{2} \left( 1 + \frac{\pi^2}{6} \right)^\frac{1}{2} \right],
\end{equation}

and

\begin{equation}
\left| \frac{\partial}{\partial \zeta} G_2[g](e^{i\theta}) \right| \leq \frac{\|g\|_{\infty}}{32} \left[ 1 + 2^\frac{1}{2} \left( 1 + \frac{\pi^2}{6} \right)^\frac{1}{2} \right].
\end{equation}

**Proof.** To prove the first statement, we only need to prove the inequality:

\[
\left| \frac{\partial}{\partial z} G_2[g](z) \right| \leq \|g\|_{\infty} \left[ \frac{1}{16} + \frac{(1 - |z|^2)^\frac{1}{2}}{60} + \frac{2^\frac{1}{2} \left( 1 + \frac{\pi^2}{6} \right)^\frac{1}{2}}{32} |z| \right]
\]

because the proof for the other one is similar. For this, let

\[
I_3(z) = \frac{1}{8\pi} \int_D (\bar{z} - \bar{\zeta}) G(z, \zeta) g(\zeta) d\sigma(\zeta),
\]

\[
I_4(z) = \frac{1}{8\pi} \int_D |\zeta - z|^2 \frac{\partial}{\partial \zeta} G(z, \zeta) g(\zeta) d\sigma(\zeta),
\]

\[
I_5(z) = -\frac{1}{16\pi} \int_D (1 - |\zeta|^2) \left[ \frac{\log(1 - z \bar{\zeta})}{z \zeta} + \frac{\log(1 + z \bar{\zeta})}{z \zeta} \right] g(\zeta) d\sigma(\zeta),
\]

and

\[
I_6(z) = -\frac{1}{16\pi} \int_D (1 - |\zeta|^2)(1 - |z|^2) \left[ \frac{1}{z(1 - z \bar{\zeta})} + \frac{\log(1 - z \bar{\zeta})}{z^2 \zeta^2} \right] g(\zeta) d\sigma(\zeta).
\]

We are going to estimate the norms of \( I_3(z), I_4(z), I_5(z), I_6(z) \), respectively. Before these estimates, we need some preparation. Set

\[
w = \frac{z - \zeta}{1 - \bar{\zeta}}.
\]

Then,

\begin{equation}
|J_\zeta(w)| = \frac{(1 - |z|^2)^2}{|1 - \bar{z}w|^4}, \quad \zeta - z = \frac{w(|z|^2 - 1)}{1 - \bar{z}w},
\end{equation}

and

\begin{equation}
1 - |\zeta|^2 = \frac{(1 - |w|^2)(1 - |z|^2)}{|1 - \bar{z}w|^2}.
\end{equation}
Firstly, we estimate $|I_3(z)|$. Since (2.11) and (2.12) guarantee that
\[ |I_3(z)| \leq \frac{\|g\|_{\infty}}{8\pi}(1 - |z|^2)^3 \int_{\mathbb{D}} \left( |w| \log \frac{1}{|w|} \right) \frac{d\sigma(w)}{|1 - z\bar{w}|^5}, \]
by letting $w = re^{i\theta}$, we obtain
\[ |I_3(z)| \leq \frac{\|g\|_{\infty}}{4}(1 - |z|^2)^3 \int_0^{2\pi} r^2 \log \frac{1}{r} \left( \frac{1}{2\pi} \int_0^{2\pi} \frac{d\theta}{|1 - zre^{i\theta}|^5} \right) dr. \]
Moreover, Hölder’s inequality and Theorem D show that
\[ \frac{1}{2\pi} \int_0^{2\pi} \frac{d\theta}{|1 - zre^{i\theta}|^5} \leq \left( \frac{1}{2\pi} \int_0^{2\pi} \frac{d\theta}{|1 - zre^{i\theta}|^4} \right)^{\frac{1}{2}} \left( \frac{1}{2\pi} \int_0^{2\pi} \frac{d\theta}{|1 - zre^{i\theta}|^6} \right)^{\frac{1}{2}}. \]
Since
\[ \left( \frac{1}{2\pi} \int_0^{2\pi} \frac{d\theta}{|1 - zre^{i\theta}|^4} \right)^{\frac{1}{2}} = \left[ \sum_{n=0}^{\infty} (n+1)^2 |zr|^{2n} \right]^{\frac{1}{2}}, \]
and
\[ \left( \frac{1}{2\pi} \int_0^{2\pi} \frac{d\theta}{|1 - zre^{i\theta}|^6} \right)^{\frac{1}{2}} = \left[ \sum_{n=0}^{\infty} (n+1)^2 (n^2 + 1) |zr|^{2n} \right]^{\frac{1}{2}}, \]
we see that
\[ \frac{1}{2\pi} \int_0^{2\pi} \frac{d\theta}{|1 - zre^{i\theta}|^5} \leq \frac{1}{4} \sum_{n=0}^{\infty} (n+1)^2 (n+2)^2 |zr|^{2n}, \]
which implies
\[ (2.13) \quad |I_3(z)| \leq \frac{\|g\|_{\infty}}{64}(1 - |z|^2)^3 \sum_{n=0}^{\infty} (n+1)(n+2)|z|^{2n} = \frac{\|g\|_{\infty}}{32}. \]
Secondly, we estimate $|I_4(z)|$. By Theorem D, we obtain that
\[ \frac{1}{2\pi} \int_0^{2\pi} \frac{d\theta}{|1 - zre^{i\theta}|^6} = \sum_{n=0}^{\infty} \frac{(n+1)^2 (n+2)^2}{4} |z|^{2n+2n}, \]
which implies
\[ (2.14) \quad \int_0^{1} \left( \frac{1}{2\pi} \int_0^{2\pi} \frac{r^2(1 - r^2)}{|1 - zre^{i\theta}|^6} d\theta \right) dr \leq \frac{1}{4(1 - |z|^2)^3}. \]
Moreover, by (2.11), (2.12), and by applying $w = re^{i\theta}$, we have
\[ \frac{1}{2\pi} \int_{\partial \mathbb{D}} \frac{|z - \zeta|(1 - |\zeta|^2)}{|1 - z\bar{\zeta}|} d\sigma(\zeta) = (1 - |z|^2)^3 \int_0^{1} \left( \frac{1}{2\pi} \int_0^{2\pi} \frac{r^2(1 - r^2)}{|1 - zre^{i\theta}|^6} d\theta \right) dr, \]
which, together with (2.14), yields
\[ (2.15) \quad |I_4(z)| \leq \frac{\|g\|_{\infty}}{16\pi} \int_{\partial \mathbb{D}} \frac{|z - \zeta|(1 - |\zeta|^2)}{|1 - z\bar{\zeta}|} d\sigma(\zeta) = \frac{\|g\|_{\infty}}{32}. \]
Next, we estimate $|I_5(z)|$. Let

$$A_1(z, \rho) = \frac{1}{2\pi} \int_0^{2\pi} \left| \sum_{n=1}^{\infty} \frac{(z \rho e^{-it})^{n-1}}{n} + \sum_{n=1}^{\infty} \frac{(\bar{z} \rho e^{it})^{n-1}}{n} \right| dt.$$ 

It follows from Hölder’s inequality that

$$A_1(z, \rho) \leq \left( \frac{1}{2\pi} \int_0^{2\pi} \left| \sum_{n=1}^{\infty} \frac{(z \rho e^{-it})^{n-1}}{n} + \sum_{n=1}^{\infty} \frac{(\bar{z} \rho e^{it})^{n-1}}{n} \right|^2 dt \right)^{\frac{1}{2}} \leq 2^{\frac{1}{2}} \left( 1 + \frac{\pi^2}{6} \right)^{\frac{1}{2}}.$$

By letting $\zeta = \rho e^{it}$, we get

$$|I_5(z)| \leq |z| \|g\|_{\infty} 8 \int_0^1 \rho (1 - \rho^2) A_1(z, \rho) d\rho \leq \frac{2^{\frac{1}{2}} \left( 1 + \frac{\pi^2}{6} \right)^{\frac{1}{2}} \|g\|_{\infty}}{32} |z|. \quad (2.16)$$

Finally, we estimate $|I_6(z)|$. Let

$$A_2(z) = \int_0^1 \rho (1 - \rho^2) \left[ \frac{1}{2\pi} \int_0^{2\pi} \left| \sum_{n=2}^{\infty} \frac{(n-1) z^{n-2} \rho^{n-1} e^{-it(n-1)}}{n} \right| dt \right] d\rho.$$

By using Hölder’s inequality, we obtain that

$$A_2(z) \leq \int_0^1 \rho (1 - \rho^2) \left[ \frac{1}{2\pi} \int_0^{2\pi} \left| \sum_{n=2}^{\infty} \frac{(n-1) z^{n-2} \rho^{n-1} e^{-it(n-1)}}{n} \right|^2 dt \right]^{\frac{1}{2}} d\rho.$$

Since

$$\frac{1}{2\pi} \int_0^{2\pi} \left| \sum_{n=2}^{\infty} \frac{(n-1) z^{n-2} \rho^{n-1} e^{-it(n-1)}}{n} \right|^2 dt = \rho^2 \sum_{n=2}^{\infty} \frac{(n-1)^2}{n^2} (|z|^2 \rho)^2(n-2) \leq \rho^2 \sum_{n=2}^{\infty} |z|^{2(n-2)},$$

we see that

$$A_2(z) \leq \int_0^1 \rho^2 (1 - \rho^2) \left( \sum_{n=0}^{\infty} |z|^{2n} \right)^{\frac{1}{2}} d\rho = \frac{2}{15 (1 - |z|^2)^{\frac{1}{2}}} \left( \sum_{n=0}^{\infty} |z|^{2n} \right)^{\frac{1}{2}},$$

which implies

$$|I_6(z)| \leq \frac{(1 - |z|^2) \|g\|_{\infty}}{8} A_2(z) \leq \frac{\|g\|_{\infty} (1 - |z|^2)^{\frac{1}{2}}}{60}. \quad (2.17)$$

Therefore, by (2.13), (2.15), (2.16), (2.17) and Theorem C, we conclude that
\[
\left| \frac{\partial}{\partial z} G_2[g](z) \right| \leq \sum_{j=3}^{6} |I_j(z)| \leq \|g\|_{\infty} \left[ \frac{1}{16} + \frac{(1-|z|^2)^{\frac{1}{2}}}{60} + \frac{2^\frac{4}{2} \left(1 + \frac{|z|^2}{6} \right)^{\frac{1}{2}}}{32} |z| \right],
\]

as required.

It follows from the first statement of the lemma, along with the Vitali Theorem (cf. [10, Theorem 26.C]), that \( \frac{\partial}{\partial z} G_2[g] \) has a continuous extension to the boundary.

Since for \( \zeta \in \mathbb{D} \), we have

\[
\lim_{|z| \to 1^{-}} G(z, \zeta) = \lim_{|z| \to 1^{-}} \log \left| \frac{1-z \overline{\zeta}}{z - \zeta} \right| = 0,
\]

which gives

\[
\lim_{|z| \to 1^{-}} I_3(z) = 0,
\]

and because (2.17) leads to

\[
\lim_{|z| \to 1^{-}} I_6(z) = 0,
\]

we have

\[
\lim_{r \to 1^{-}} \frac{\partial}{\partial z} G_2[g](re^{i\theta}) = I_4(e^{i\theta}) + I_5(e^{i\theta}).
\]

Then, (2.9) easily follows from (2.15) and (2.16).

Similarly, we know that (2.8) and (2.10) are also true. Hence, the lemma is proved. \( \square \)

**Lemma 2.3.** For \( \varphi \in \mathcal{C}(\mathbb{T}) \) and \( g \in \mathcal{C}(\overline{\mathbb{D}}) \), suppose that \( f \) is a sense-preserving homeomorphism from \( \mathbb{D} \) onto itself satisfying (1.1) and \( \Delta f = \varphi \) in \( \mathbb{T} \), and suppose that \( f \) is Lipschitz continuous in \( \mathbb{D} \). Then, for almost every \( e^{i\theta} \in \mathbb{T} \), the following limits exist:

\[
(2.18) \quad D_f(e^{i\theta}) := \lim_{z \to e^{i\theta}, z \in \mathbb{D}} D_f(z) \text{ and } J_f(e^{i\theta}) := \lim_{z \to e^{i\theta}, z \in \mathbb{D}} J_f(z).
\]

Further, we have

\[
(2.19) \quad J_f(e^{i\theta}) \leq \frac{\eta'(\theta)}{2\pi} \int_{0}^{2\pi} \frac{|f(e^{it}) - f(e^{i\theta})|^2}{|e^{it} - e^{i\theta}|^2} dt + \frac{\eta'(\theta)\|\varphi\|_{\infty}}{2} \left( \frac{\pi^2}{3} - 1 \right)^{\frac{1}{2}}
\]

\[
+ \frac{\eta'(\theta)\|g\|_{\infty}}{16} \left[ 1 + 2^\frac{1}{2} \left(1 + \frac{\pi^2}{6} \right)^{\frac{1}{2}} \right]
\]

and

\[
(2.20) \quad J_f(e^{i\theta}) \geq \frac{\eta'(\theta)}{2\pi} \int_{0}^{2\pi} \frac{|f(e^{it}) - f(e^{i\theta})|^2}{|e^{it} - e^{i\theta}|^2} dt - \frac{\eta'(\theta)\|\varphi\|_{\infty}}{2} \left( \frac{\pi^2}{3} - 1 \right)^{\frac{1}{2}}
\]

\[
- \frac{\eta'(\theta)\|g\|_{\infty}}{16} \left[ 1 + 2^\frac{1}{2} \left(1 + \frac{\pi^2}{6} \right)^{\frac{1}{2}} \right],
\]
where \( f(e^{i\theta}) = e^{i\eta(\theta)} \) and \( \eta(\theta) \) is a real-valued function in \([0, 2\pi]\).

Before the proof of Lemma 2.3, let us recall the following result (cf. [25, Lemma 2.1]).

**Theorem E.** Suppose that \( f \) is a harmonic mapping defined in \( \mathbb{D} \) and its formal derivative \( D_f \) is bounded in \( \mathbb{D} \) (or equivalently, according to Rademacher’s theorem, suppose that \( f \) itself is Lipschitz continuous in \( \mathbb{D} \)). Then, there exists a mapping \( A \in L^\infty(\mathbb{T}) \) such that 
\[
D_f(z) = P A(z)
\]
and for almost every \( e^{i\theta} \in \mathbb{T} \),
\[
\lim_{r \to 1^-} D_f(re^{i\theta}) = A(e^{i\theta}).
\]

Moreover, the function \( F(e^{i\theta}) := f(e^{i\theta}) \) is differentiable almost everywhere in \([0, 2\pi]\) and
\[
A(e^{i\theta}) = \frac{\partial}{\partial \theta} F(e^{i\theta}).
\]

**Proof of Lemma 2.3.** We first prove the existence of the two limits in (2.18). By Lemmas 2.1 and 2.2, we get that for any \( e^{i\theta} \in \mathbb{D} \),
\[
\lim_{z \to e^{i\theta}, z \in \mathbb{D}} D_{G_1[\varphi]}(z) = D_{G_1[\varphi]}(e^{i\theta}) \quad \text{and} \quad \lim_{z \to e^{i\theta}, z \in \mathbb{D}} D_{G_2[g]}(z) = D_{G_2[g]}(e^{i\theta}).
\]

Again, by Lemmas 2.1 and 2.2, we know that
\[
\|D_{G_1[\varphi]}\| < \infty \quad \text{and} \quad \|D_{G_2[g]}\| < \infty,
\]
which implies the Lipschitz continuity of \( G_1[\varphi] \) and \( G_2[g] \) in \( \mathbb{D} \). Since \( f \) is Lipschitz continuous in \( \mathbb{D} \), we see that \( \|D_f\| \) is bounded in \( \mathbb{D} \). Thus, it follows from (1.3) that \( P_f \) is also Lipschitz continuous in \( \mathbb{D} \), where \( f^* = f|_T \). Now, we conclude from Theorem E that for almost every \( e^{i\theta} \in \mathbb{T} \),
\[
\lim_{z \to e^{i\theta}, z \in \mathbb{D}} D_{P_f}(z)
\]
does exist, which, together with (1.3) and (2.21), guarantees that for almost every \( \theta \in [0, 2\pi] \),
\[
\lim_{z \to e^{i\theta}, z \in \mathbb{D}} D_f(z)
\]
also exists.

Since \( J_f(z) = \det D_f(z) \), obviously, we see that
\[
\lim_{z \to e^{i\theta}, z \in \mathbb{D}} J_f(z)
\]
exists for almost every \( \theta \in [0, 2\pi] \).

Next, we demonstrate the estimates in (2.19) and (2.20). For convenience, in the rest of the proof of the lemma, let
\[
D_f(e^{i\theta}) = \lim_{z \to e^{i\theta}, z \in \mathbb{D}} D_f(z) \quad \text{and} \quad J_f(e^{i\theta}) = \lim_{z \to e^{i\theta}, z \in \mathbb{D}} J_f(z).
\]
By Lebesgue Dominated Convergence Theorem, the boundedness of $\|D_f\|$, and by letting $z = re^{i\theta} \in \mathbb{D}$, we see that for any fixed $\theta \in [0, 2\pi]$,

\begin{equation}
\tag{2.22}
f(e^{i\theta}) = \lim_{r \to 1^-} f(re^{i\theta}) = \lim_{r \to 1^-} \int_0^\theta \frac{\partial}{\partial t} f(re^{it}) dt + f(e^{i\theta_1})
= \int_0^\theta \lim_{r \to 1^-} [ir(f_z(re^{it})e^{it} - f_r(re^{it})e^{-it})] dt + f(e^{i\theta_1}),
\end{equation}

which implies that $f(e^{i\theta})$ is absolutely continuous. Let $\eta(\theta)$ be a real-valued function in $[0, 2\pi]$ such that

\[ e^{i\eta(\theta)} = f(e^{i\theta}). \]

Then,

\begin{equation}
\tag{2.23}
f'(e^{i\theta}) = i\eta'(\theta)e^{i\eta(\theta)}
\end{equation}

holds almost everywhere in $[0, 2\pi]$.

Since

\[ J_f(re^{i\theta}) = |f_z(re^{i\theta})|^2 - |f_r(re^{i\theta})|^2 = -\text{Re} \left( \frac{\partial f}{\partial r} \frac{i \partial f}{\partial \theta} \right), \]

we infer from (2.23) that

\begin{equation}
\tag{2.24}
J_f(e^{i\theta}) = \lim_{r \to 1^-} J_f(re^{i\theta}) = -\lim_{r \to 1^-} \text{Re} \left( \frac{\partial f}{\partial r} \frac{i \partial f}{\partial \theta} \right) = I_7 - I_8 + I_9,
\end{equation}

where

\[ I_7 = \lim_{r \to 1^-} \text{Re} \left( \frac{f(e^{i\theta}) - Pf(re^{i\theta})}{1 - r} \cdot \eta'(\theta)f(e^{i\theta}) \right), \]

\[ I_8 = \lim_{r \to 1^-} \text{Re} \left( \frac{G_1(\varphi)(re^{i\theta})}{1 - r} \cdot \eta'(\theta)f(e^{i\theta}) \right), \]

and

\[ I_9 = \lim_{r \to 1^-} \text{Re} \left( \frac{G_2(\varphi)(re^{i\theta})}{1 - r} \cdot \eta'(\theta)f(e^{i\theta}) \right). \]

Now, we are going to prove (2.19) and (2.20) by estimating the quantities $I_7$, $|I_8|$ and $|I_9|$, respectively. We start with the estimate of $I_7$. Since

\[ \text{Re}(f(e^{i\theta}), f(e^{i\theta}) - f(e^{it})) = \text{Re}[f(e^{i\theta})(f(e^{i\theta}) - f(e^{it}))] = \frac{1}{2} |f(e^{it}) - f(e^{i\theta})|^2 \]

and

\[ I_7 = \lim_{r \to 1^-} \text{Re} \left( \frac{1}{2\pi} \int_0^{2\pi} \frac{1 + r}{1 - r e^{i(\theta-t)}} \langle \eta'(\theta)f(e^{i\theta}), f(e^{i\theta}) - f(e^{it}) \rangle dt \right), \]

where $\langle \cdot, \cdot \rangle$ denotes the inner product, it follows that

\begin{equation}
\tag{2.25}
I_7 = \eta'(\theta) \frac{1}{2\pi} \int_0^{2\pi} \frac{|f(e^{it}) - f(e^{i\theta})|^2}{|e^{it} - e^{i\theta}|^2} dt.
\end{equation}
Next, we estimate $|I_8|$. For this, let
\[
\mathcal{A}_3(r) = \frac{1}{2\pi} \int_0^{2\pi} \left| 1 + \frac{\log(1 - re^{i(\theta - \ell)})}{re^{i(\theta - \ell)}} + \frac{\log(1 - re^{i(l - \theta)})}{re^{i(l - \theta)}} \right| dt.
\]
By Hölder’s inequality, we have
\[
\lim_{r \to 1^-} \mathcal{A}_3(r) \leq \lim_{r \to 1^-} \left( \frac{1}{2\pi} \int_0^{2\pi} \left| 1 - \sum_{n=1}^{\infty} \frac{(re^{i(l - \theta)})^n}{n} - \sum_{n=1}^{\infty} \frac{(re^{i(l - \theta)})^{n-1}}{n} \right|^2 dt \right)^{\frac{1}{2}} = \left( \frac{\pi^2}{3} - 1 \right)^{\frac{1}{2}},
\]
which yields
\[
|I_8| = \left| \lim_{r \to 1^-} \Re \left[ \frac{1}{8\pi} \int_0^{2\pi} \left( 1 + \frac{\log(1 - re^{i(\theta - \ell)})}{re^{i(\theta - \ell)}} + \frac{\log(1 - re^{i(l - \theta)})}{re^{i(l - \theta)}} \right) \right] \right|
\times (1 + r) \left( \eta'(\theta) f(e^{i\theta}), \varphi(e^{i\theta}) \right) dt
\leq \frac{\eta'(\theta)\|\varphi\|_{\infty}}{2} \lim_{r \to 1^-} \mathcal{A}_3(r).
\]

Thus,
\[
(2.26) \quad |I_8| \leq \frac{\eta'(\theta)\|\varphi\|_{\infty}}{2} \left( \frac{\pi^2}{3} - 1 \right)^{\frac{1}{2}}.
\]

Finally, we estimate $|I_9|$. To reach this goal, let
\[
\mathcal{A}_4 = \lim_{z \to e^{i\theta}, \zeta \in \mathbb{D}} \frac{1}{8\pi} \int_{\mathbb{D}} \frac{|\zeta - z|^2}{1 - |z|} (G(z, \zeta) - G(e^{i\theta}, \zeta)) \left( \eta'(\theta) f(e^{i\theta}), g(\zeta) \right) d\sigma(\zeta)
\]
and
\[
\mathcal{A}_5 = \lim_{z \to e^{i\theta}, \zeta \in \mathbb{D}} \frac{1}{16\pi} \left| \int_{\mathbb{D}} (1 + |z|)(1 - |\zeta|^2) \left[ \frac{\log(1 - z\overline{\zeta})}{z\overline{\zeta}} + \frac{\log(1 - \overline{\zeta}\zeta)}{\overline{\zeta}\zeta} \right] \left( \eta'(\theta) f(e^{i\theta}), g(\zeta) \right) d\sigma(\zeta) \right|.
\]
Since
\[
\lim_{z \to e^{i\theta}, \zeta \in \mathbb{D}} \frac{G(z, \zeta)}{1 - |z|} = \lim_{z \to e^{i\theta}, \zeta \in \mathbb{D}} \frac{G(z, \zeta) - G(e^{i\theta}, \zeta)}{1 - |z|} = P(\zeta, e^{i\theta}),
\]
we deduce that
\[
\mathcal{A}_4 \leq \frac{\eta'(\theta)\|g\|_{\infty}}{8\pi} \int_{\mathbb{D}} |\zeta - z|^2 P(\zeta, e^{i\theta}) d\sigma(\zeta) = \frac{\eta'(\theta)\|g\|_{\infty}}{16}
\]
and
\[
\mathcal{A}_5 \leq \frac{\eta'(\theta)\|g\|_{\infty}}{8\pi} \int_{\mathbb{D}} (1 - |\zeta|^2) \left| \frac{\log(1 - z\overline{\zeta})}{z\overline{\zeta}} + \frac{\log(1 - \overline{\zeta}\zeta)}{\overline{\zeta}\zeta} \right| d\sigma(\zeta).
\]
Now, we conclude from (1.5) and (2.16) that

\[(2.27) \quad |I_9| \leq A_4 + A_5 \leq \frac{3}{16} \left[ 1 + 2^\frac{1}{2} \left( 1 + \frac{\pi^2}{6} \right)^\frac{1}{2} \right].\]

Hence, (2.19) and (2.20) follow from the inequalities (2.24) \(\sim\) (2.27) along with the following chain of inequalities:

\[I_7 - |I_8| - |I_9| \leq J_f(e^{i\theta}) \leq I_7 + |I_8| + |I_9|.

The proof of the lemma is complete. \(\square\)

The following is the so-called Heinz-Theorem.

**Theorem F.** ([15, Theorem]) Suppose that \(f\) is a harmonic homeomorphism of \(\mathbb{D}\) onto itself with \(f(0) = 0\). Then, for \(z \in \mathbb{D}\),

\[|f_z(z)|^2 + |f_{\overline{z}}(z)|^2 \geq \frac{1}{\pi^2}.

Our next lemma is a generalization of Theorem F.

**Lemma 2.4.** Suppose that \(f\) is a harmonic homeomorphism of \(\mathbb{D}\) onto itself with \(f(a) = 0\), where \(a \in \mathbb{D}\). Then, for \(z \in \mathbb{D}\),

\[|f_z(z)|^2 + |f_{\overline{z}}(z)|^2 \geq \frac{(1 - |a|^2)^2}{\pi^2(1 + |a|)^2}.

**Proof.** For \(\zeta \in \mathbb{D}\), let

\[\phi(\zeta) = \frac{\zeta + a}{1 + \overline{a}\zeta}.

Then,

\[\phi'(\zeta) = \frac{1 - |a|^2}{(1 + \overline{a}\zeta)^2}.

Let

\[\mathcal{F}(\zeta) = f(\phi(\zeta)).\]

Then, \(\mathcal{F}\) is also a harmonic homeomorphism of \(\mathbb{D}\) onto itself with \(\mathcal{F}(0) = 0\). By Theorem F, we have

\[\left( |f_w(\phi(\zeta))|^2 + |f_{\overline{w}}(\phi(\zeta))|^2 \right) \left( \frac{1 - |a|^2}{|1 + \overline{a}\zeta|^4} \right) = |\mathcal{F}_\zeta(\zeta)|^2 + |\mathcal{F}_{\overline{\zeta}}(\zeta)|^2 \geq \frac{1}{\pi^2},

which implies

\[|f_w(\phi(\zeta))|^2 + |f_{\overline{w}}(\phi(\zeta))|^2 \geq \frac{1}{\pi^2} \frac{|1 + \overline{a}\zeta|^4}{(1 - |a|^2)^2} \geq \frac{(1 - |a|^2)^2}{\pi^2(1 + |a|)^2},

where \(w = \phi(\zeta)\). This is what we need. \(\square\)

The following result is a direct consequence of Lemma 2.4.
Corollary 2.1. Suppose that $f$ is a harmonic homeomorphism of $\mathbb{D}$ onto itself. Then

$$\inf_{z \in \mathbb{D}} |f_z(z)| > 0.$$ 

3. THE PROOF OF THEOREM 1.1

The purpose of this section is to prove Theorem 1.1. The proof consists of three steps. In the first step, the Lipschitz continuity of the mappings $f$ is proved, the co-Lipschitz continuity of $f$ is demonstrated in the second step, and in the third step, the Lipschitz and co-Lipschitz continuity coefficients obtained in the first two steps are shown to have bounds with the forms as required in Theorem 1.1.

Before the proof, let us recall a result due to Kalaj and Mateljević, which is used in the discussions of the first step.

Theorem G. ([22, Theorem 3.4]) Suppose that $f$ is a quasiconformal $C^2$ diffeomorphism from the plane domain $\Omega$ with $C^{1,\alpha}$ compact boundary onto the plane domain $\Omega^*$ with $C^{2,\alpha}$ compact boundary. If there exist constants $a_1$ and $b_1$ such that

$$|\Delta f(z)| \leq a_1\|Df(z)\|^2 + b_1$$

in $\Omega$, then $f$ has bounded partial derivatives. In particular, it is a Lipschitz mapping in $\Omega$.

Step 3.1. Lipschitz continuity.

We start the discussions of this step with the following claim.

Claim 3.1. The limits

$$\lim_{z \to \xi \in \mathcal{T}, z \in \mathbb{D}} Df(z) \text{ and } \lim_{z \to \xi \in \mathcal{T}, z \in \mathbb{D}} Jf(z)$$

exist almost everywhere in $\mathcal{T}$.

We are going to verify the existence of these two limits by applying Theorem G and Lemma 2.3. For this, we need to get an upper bound of the quantity $|\Delta f(z)|$ as stated in (3.1) below. By the formula (1.3) in [25] (see also [16, pp. 118-120]), we have that for $z \in \mathbb{D}$,

$$\Delta f(z) = \mathcal{P}_\varphi(z) - \frac{1}{2\pi} \int_\mathbb{D} G(z, \zeta) g(\zeta) d\sigma(\zeta).$$

Since Theorem D implies

$$\frac{1}{2\pi} \int_0^{2\pi} \frac{dt}{|1 - zre^{it}|^4} = \sum_{n=0}^{\infty} (n + 1)^2 |z|^{2n} r^{2n},$$

by letting $w = \frac{z-\zeta}{1-\bar{z}\zeta} = re^{it}$, we obtain

$$\frac{1}{2\pi} \int_\mathbb{D} G(z, \zeta) d\sigma(\zeta) = \frac{1}{2\pi} \int_\mathbb{D} \left( \log \frac{1}{|w|} \right) \frac{(1 - |z|^2)^2}{|1 - \bar{w}w|^4} d\sigma(w) = \frac{(1 - |z|^2)^2}{4},$$
and so, we get

\[
|\Delta f(z)| \leq |P_\varphi(z)| + \frac{\|g\|_\infty}{2 \pi} \int_D G(z, \zeta) d\sigma(\zeta) \leq \|\varphi\|_\infty + \frac{\|g\|_\infty}{4}.
\]

Now, the existence of the limits

\[
D_f(\xi) = \lim_{z \to \xi \in \mathbb{T}, z \in \mathbb{D}} D_f(z) \quad \text{and} \quad J_f(\xi) = \lim_{z \to \xi \in \mathbb{T}, z \in \mathbb{D}} J_f(z)
\]

almost everywhere in $\mathbb{T}$ follows from Theorem G and Lemma 2.3.

For convenience, in the following, let

\[
C_2(K, \varphi, g) = \sup_{z \in \mathbb{D}} \|D_f(z)\|.
\]

Since for almost all $z_1$ and $z_2 \in \mathbb{D}$,

\[
|f(z_1) - f(z_2)| = \left| \int_{[z_1, z_2]} f_z dz + f_\varphi dz \right| \leq C_2(K, \varphi, g)|z_1 - z_2|,
\]

we see that, to prove the Lipschitz continuity of $f$, it suffices to estimate the quantity $C_2(K, \varphi, g)$. To reach this goal, we first show that the quantity $C_2(K, \varphi, g)$ satisfies an inequality which is stated in the following claim.

**Claim 3.2.** $C_2(K, \varphi, g) \leq (C_2(K, \varphi, g))^{1 - \frac{1}{K}} \mu_1 + \mu_2$, where

\[
\mu_1 = \frac{K(Q(K))}{2 \pi} \int_0^{2 \pi} \left| 1 - e^{it} \right|^{-1 + \frac{1}{K}} dt,
\]

$Q(K)$ is from Theorem A, $\mu_2 = \mu_3 + \mu_4$,

\[
\mu_3 = \frac{K\|\varphi\|_\infty}{2} \left( \frac{\pi^2}{3} - 1 \right) \frac{1}{2} + \frac{K\|g\|_\infty}{16} \left[ 1 + 2^\frac{1}{2} \left( 1 + \left( \frac{\pi^2}{6} \right)^\frac{1}{2} \right) \right],
\]

and

\[
\mu_4 = \frac{\|\varphi\|_\infty}{2} \left[ \max_{x \in [0, 1)} \{ h(x) \} + 2 \left( \frac{\pi^2}{3} - 1 \right) \right] + \|g\|_\infty \left[ \frac{53}{240} + \frac{2^\frac{1}{2} \left( 1 + \left( \frac{\pi^2}{6} \right)^\frac{1}{2} \right)}{8} \right].
\]

To prove the claim, we need the following preparation. Firstly, we prove that for almost every $\theta \in [0, 2\pi]$,

\[
\|D_f(e^{i\theta})\| \leq \frac{K}{2 \pi} \int_0^{2 \pi} \frac{|f(e^{it}) - f(e^{i\theta})|^2}{|e^{it} - e^{i\theta}|^2} dt + \mu_3.
\]

Since $f$ is a $K$-quasiconformal self-mapping of $\mathbb{D}$, we see that $f$ can be extended to the homeomorphism of $\overline{\mathbb{D}}$ onto itself. For $\theta \in [0, 2\pi]$, let

\[
f(e^{i\theta}) = f^*(e^{i\theta}) = e^{i\eta(\theta)}.
\]

Then, by (2.22), we see that $f(e^{i\theta})$ is absolutely continuous. It follows that

\[
\eta'(\theta)e^{i\eta(\theta)} = \frac{d}{d\theta} f(e^{i\theta}) = \lim_{r \to 1^-} \frac{\partial}{\partial \theta} f(re^{i\theta}) = \lim_{r \to 1^-} \left[ ir \left( f_\varphi(re^{i\theta})e^{i\theta} - f_\varphi(re^{i\theta})e^{-i\theta} \right) \right].
\]
which implies
\[
\frac{1}{K} \|D_f(e^{i\theta})\| \leq \lim_{r \to 1^-} \lambda(D_f(r e^{i\theta})) \leq \eta'(\theta) \leq \lim_{r \to 1^-} \|D_f(r e^{i\theta})\| = \|D_f(e^{i\theta})\|
\]
earm almost everywhere in \([0, 2\pi]\), where \(r \in [0, 1]\).

Since the existence of the two limits \(D_f(e^{i\theta}) = \lim_{z \to e^{i\theta}, z \in D} D_f(z)\) and \(J_f(e^{i\theta}) = \lim_{z \to e^{i\theta}, z \in D} J_f(z)\) almost everywhere in \([0, 2\pi]\) guarantees that
\[
\|D_f(e^{i\theta})\|^2 \leq K J_f(e^{i\theta}),
\]
we deduce from (2.19) and (3.4) that
\[
\|D_f(e^{i\theta})\|^2 \leq K \|D_f(e^{i\theta})\| \left\{ \frac{1}{2\pi} \int_0^{2\pi} \frac{|f(e^{it}) - f(e^{i\theta})|^2}{|e^{it} - e^{i\theta}|^2} dt + \mu_3 \right\},
\]
from which the inequality (3.3) follows.

Secondly, we show that for any \(\epsilon > 0\), there exists \(\theta_\epsilon \in [0, 2\pi]\) such that
\[
C_2(K, \varphi, g) \leq (1 + \epsilon) \|D_f(e^{i\!\theta})\| + \mu_4.
\]
For the proof, let \(t \in [0, 2\pi]\), and let
\[
H_t(z) = \frac{\partial}{\partial z} P_{f_*}(z) + e^{it} \frac{\partial}{\partial \zeta} P_{f_*}(z)
\]
in \(\mathbb{D}\).

Since \(P_{f_*} = f - G_1[\varphi] + G_2[g]\) is harmonic, we see that \(H_t\) is analytic in \(\mathbb{D}\), and thus,
\[
|H_t(z)| \leq \text{esssup}_{\theta \in [0, 2\pi]} |H_t(e^{i\theta})| \leq \text{esssup}_{\theta \in [0, 2\pi]} \|D_{P_{f_*}}(e^{i\theta})\|.
\]
Then, the facts
\[
\|D_{P_{f_*}}(z)\| = \max_{t \in [0, 2\pi]} |H_t(z)|
\]
and
\[
\|D_{P_{f_*}}(z)\| = \left| \frac{\partial f}{\partial z} - \frac{\partial G_1[\varphi]}{\partial z} + \frac{\partial G_2[g]}{\partial z} \right| + \left| \frac{\partial f}{\partial \zeta} - \frac{\partial G_1[\varphi]}{\partial \zeta} + \frac{\partial G_2[g]}{\partial \zeta} \right|
\]
ensure
\[
\|D_{P_{f_*}}(z)\| \leq \text{esssup}_{\theta \in [0, 2\pi]} \|D_f(e^{i\theta})\| + \text{esssup}_{\theta \in [0, 2\pi]} \|D_{G_1[\varphi]}(e^{i\theta})\| + \text{esssup}_{\theta \in [0, 2\pi]} \|D_{G_2[g]}(e^{i\theta})\|,
\]
which, together with Lemmas 2.1 and 2.2, guarantees that for all \(z \in \mathbb{D}\),
\[
\|D_f(z)\| \leq \|D_{P_{f_*}}(z)\| + \|D_{G_1[\varphi]}(z)\| + \|D_{G_2[g]}(z)\| \leq \text{esssup}_{\theta \in [0, 2\pi]} \|D_f(e^{i\theta})\| + \mu_4,
\]
from which the inequality (3.5) follows.

Let
\[
\nu = \frac{1}{2\pi} \int_0^{2\pi} \frac{|f(e^{it}) - f(e^{i\theta})|^2}{|e^{it} - e^{i\theta}|^2} dt.
\]
Finally, we need the following estimate of \(\nu\):
\[
(3.6) \quad \nu \leq \frac{(C_2(K, \varphi, g))^{1-\frac{1}{k}}(Q(K))^{\frac{1}{k}+1}}{2\pi} \int_0^{2\pi} |e^{it} - e^{i\theta}|^{-1+\frac{1}{k}} dt.
\]
Since it follows from (3.2) that for almost all \( \theta_1, \theta_2 \in [0, 2\pi] \),
\[
|f(e^{i\theta_1}) - f(e^{i\theta_2})| \leq C_2(K, \varphi, g) |e^{i\theta_1} - e^{i\theta_2}|,
\]
we infer that
\[
\nu \leq \left( \frac{C_2(K, \varphi, g)}{2\pi} \right)^{1-\frac{1}{K}} \int_0^{2\pi} \left| e^{it} - e^{i\theta_1} \right|^{-1+\frac{1}{K}} \left| f(e^{it}) - f(e^{i\theta_1}) \right|^{1+\frac{1}{K}} dt,
\]
from which, together with Theorem A, the inequality (3.6) follows.

Now, we are ready to finish the proof of the claim. It follows from (3.5) that
\[
C_2(K, \varphi, g) \leq (1 + \epsilon) \|Df(e^{i\theta_1})\| + \mu_4,
\]
and so, (3.3) and (3.6) give
\[
C_2(K, \varphi, g) \leq \left( C_2(K, \varphi, g) \right)^{1-\frac{1}{K}} \mu_1 (1 + \epsilon) + \mu_2 (1 + \epsilon) + \mu_4.
\]
Moreover, by [17, Lemma 1.6], we know that
\[
\int_0^{2\pi} \left| e^{it} - e^{i\theta_1} \right|^{-1+\frac{1}{K}} dt < \infty,
\]
which shows \( \mu_1 < \infty \).

By letting \( \epsilon \to 0^+ \), we get from (3.8) that
\[
C_2(K, \varphi, g) \leq \left( C_2(K, \varphi, g) \right)^{1-\frac{1}{K}} \mu_1 + \mu_2,
\]
as required.

The following is a lower bound for \( C_2(K, \varphi, g) \).

**Claim 3.3.** \( C_2(K, \varphi, g) \geq 1 \).

Since
\[
\int_0^{2\pi} \eta'(\theta)d\theta = \eta(2\pi) - \eta(0) = 2\pi,
\]
we conclude that
\[
\text{esssup}_{\theta \in [0, 2\pi]} \lim_{t \to \theta} \left| \frac{f(e^{i\theta}) - f(e^{it})}{e^{i\theta} - e^{it}} \right| = \text{esssup}_{\theta \in [0, 2\pi]} \eta'(\theta) \geq 1.
\]
Then, it follows from (3.7) and the following fact
\[
\text{esssup}_{\theta \in [0, 2\pi]} \lim_{t \to \theta} \left| \frac{f(e^{i\theta}) - f(e^{it})}{e^{i\theta} - e^{it}} \right| \leq \text{esssup}_{0 \leq \theta \neq t \leq 2\pi} \left| \frac{f(e^{i\theta}) - f(e^{it})}{e^{i\theta} - e^{it}} \right|
\]
that
\[
C_2(K, \varphi, g) \geq 1.
\]
Hence, the claim is true.

An upper bound of \( C_2(K, \varphi, g) \) is established in the following claim.
Claim 3.4. If \( \frac{(K-1)}{K} \mu_1 < 1 \), then
\[
C_2(K, \varphi, g) \leq \mu_5,
\]
where \( \mu_5 = \frac{\frac{1}{K-1} + \mu_2}{1 - \mu_1(1 - \frac{1}{K})} \).

The proof of this claim easily follows from [25, Lemma 2.9].

Now, we are ready to finish the discussions in this step. By Claims 3.2 and 3.3, we obtain
\[
1 \leq C_2(K, \varphi, g) \leq \mu_6,
\]
where \( \mu_6 = (\mu_1 + \mu_2)^K \).

By letting
\[
C_3 = \begin{cases} 
\mu_6, & \text{if } \frac{(K-1)}{K} \mu_1 \geq 1, \\
\min\{\mu_5, \mu_6\}, & \text{if } \frac{(K-1)}{K} \mu_1 < 1,
\end{cases}
\]
we infer that
\[
(3.9) \quad 1 < C_2(K, \varphi, g) \leq C_3.
\]
Then, the Lipschtz continuity of \( f \) easily follows from these estimates of \( C_2(K, \varphi, g) \).

Step 3.2. Co-Lipschtz continuity.

We begin the discussions of this step with some preparation which consists of the following two claims.

Claim 3.5. \( \lambda(D_{P_r}(e^{i\theta})) \geq \frac{\mu_7}{K^2} - \left(1 + \frac{1}{K^2}\right) \mu_8 \) almost everywhere in \([0, 2\pi]\), where
\[
(3.10) \quad \mu_7 = \max\{\mu'_7, \mu''_7\}, \quad \mu'_7 = (Q(K))^{-2K} \frac{1}{2\pi} \int_0^{2\pi} |e^{it} - e^{i\theta}|^2 K^{-2} dt,
\]
and
\[
(3.11) \quad \mu_8 = \frac{\|\varphi\|_{\infty}}{2} \left(\frac{\pi^2}{3} - 1\right)^\frac{1}{2} + \frac{\|g\|_{\infty}}{16} \left[1 + 2^\frac{1}{2} \left(1 + \frac{\pi^2}{6}\right)^\frac{1}{2}\right],
\]

By (3.4), we have
\[
\frac{\eta'(\theta)}{K} \leq \|D_f(e^{i\theta})\| \leq \lambda(D_f(e^{i\theta})) \leq \lambda(D_{P_r}(e^{i\theta})) + \|D_g(e^{i\theta})\| + \|D_{g_{1[g]}}(e^{i\theta})\|,
\]
which, together with Lemmas 2.1 and 2.2, implies
\[
(3.12) \quad \lambda(D_{P_r}(e^{i\theta})) \geq \frac{\eta'(\theta)}{K} - \|D_g(e^{i\theta})\| - \|D_{g_{1[g]}}(e^{i\theta})\| \geq \frac{\eta'(\theta)}{K} - \mu_8.
\]
Then, we know from (3.12) that, to prove the claim, it suffices to show that
\[
(3.13) \quad K \eta'(\theta) \geq \mu_7.
\]
Again, it follows from (3.4) that
\[
\frac{J_f(e^{i\theta})}{\eta'(\theta)} \leq \frac{J_f(e^{i\theta})}{\lambda(D_f(e^{i\theta}))} \leq K \lambda(D_f(e^{i\theta})) \leq K \eta'(\theta),
\]
and thus, (2.20) gives
\[
K \eta'(\theta) \geq \frac{1}{2\pi} \int_0^{2\pi} \frac{|f(e^{it}) - f(e^{i\theta})|^2}{|e^{it} - e^{i\theta}|^2} dt - \mu_8.
\]
This implies that, to prove (3.13), we only need to verify the validity of the following
inequality:
\[
(3.14) \quad \frac{1}{2\pi} \int_0^{2\pi} \frac{|f(e^{it}) - f(e^{i\theta})|^2}{|e^{it} - e^{i\theta}|^2} dt \geq \mu_7.
\]
We now prove this inequality. On the one hand, since \( f^{-1} \) is a \( K \)-quasiconformal
mapping, it follows from Theorem A that for any \( z_1, z_2 \in \mathbb{D} \),
\[
(Q(K))^{-K}|z_1 - z_2|^K \leq |f(z_1) - f(z_2)|,
\]
which implies
\[
(3.15) \quad \frac{1}{2\pi} \int_0^{2\pi} \frac{|f(e^{it}) - f(e^{i\theta})|^2}{|e^{it} - e^{i\theta}|^2} dt \geq \mu_7'.
\]
On the other hand, since \( f(0) = 0 \) and \( x^2 \log \frac{1}{x} - (1 - x^2) \leq 0 \) for \( x \in (0, 1] \), we see from
\[
|G_2[f](0)| \leq \frac{\|g\|_{\infty}}{8\pi} \int_0^1 \left| \log \frac{1}{|\zeta|} - (1 - |\zeta|^2) \right| d\sigma(\zeta) = \frac{3 \|g\|_{\infty}}{64}
\]
that
\[
(3.16) \quad |P_{f^*}(0)| \leq |G_1[\varphi](0)| + |G_2[f](0)| \leq \frac{\|\varphi\|_{\infty}}{4} + \frac{3 \|g\|_{\infty}}{64}.
\]
Then, we infer from (3.16) and the following fact:
\[
\frac{1}{2\pi} \int_0^{2\pi} \frac{|f(e^{it}) - f(e^{i\theta})|^2}{|e^{it} - e^{i\theta}|^2} dt \geq \frac{1}{4\pi} \int_0^{2\pi} \left[ 1 - \text{Re}(f(e^{it})f(e^{i\theta})) \right] dt \geq \frac{1 - |P_{f^*}(0)|}{2}
\]
that
\[
(3.17) \quad \frac{1}{2\pi} \int_0^{2\pi} \frac{|f(e^{it}) - f(e^{i\theta})|^2}{|e^{it} - e^{i\theta}|^2} dt \geq \mu_7'.
\]
Obviously, the inequality (3.14) follows from (3.15) and (3.17), and so, the claim
is proved.

**Claim 3.6.** For \( z \in \mathbb{D} \), \( \lambda(D_{P_{f^*}}(z)) \geq \frac{\mu_7'}{\eta^2} - \left( 1 + \frac{1}{K^2} \right) \mu_8. \)

By the Choquet-Radó-Kneser theorem (see [5]), we see that \( P_{f^*} \) is a sense-
preserving harmonic diffeomorphism of \( \mathbb{D} \) onto itself. Then, by Corollary 2.1, we can let
\[
p_1(z) = \frac{\partial}{\partial z} P_{f^*}(z) \quad \text{and} \quad p_2(z) = \left( \frac{\mu_7}{K^2} - \mu_8 \right) \frac{1}{\eta^2} P_{f^*}(z)
\]
in $D$, and for $\tau \in [0, 2\pi]$, let
\[
q_{\tau}(z) = p_1(z) + e^{i\tau}p_2(z).
\]

Then, by Corollary 2.1,
\[
(3.18) \quad \sup_{z \in D} |q_{\tau}(z)| < \infty
\]
for all $\tau \in [0, 2\pi]$.

By Claim 3.5, we have
\[
(3.19) \quad |q_{\tau}(e^{i\theta})| \leq |p_1(e^{i\theta})| + |p_2(e^{i\theta})| = \left| \frac{\partial}{\partial z} P_{f^*}(e^{i\theta}) \right| + (1 + \frac{1}{K^2}) \mu_8 \leq 1
\]
almost everywhere in $[0, 2\pi]$. Let
\[
E = \{ \theta \in [0, 2\pi] : \lim_{z \to e^{i\theta}} q_{\tau}(z) \text{ exists} \}.
\]

Then measure of $[0, 2\pi] \setminus E$ is zero. Hence it follows from (3.18) and (3.19) that, for all $\tau \in [0, 2\pi]$,
\[
|q_{\tau}(z)| \leq \frac{1}{2\pi} \int_E P(z, e^{i\theta})|q_{\tau}(e^{i\theta})|d\theta \leq 1,
\]
from which, together with the arbitrariness of $\tau \in [0, 2\pi]$, the claim follows.

Step 3.3. Bounds of the Lipschitz continuity coefficients $C_1(K, \varphi, g)$ and $C_2(K, \varphi, g)$.

The discussions of this step consists of the following two claims.

Claim 3.7. There are constants $M_2(K)$ and $N_2(K, \varphi, g)$ such that

1. $C_2(K, \varphi, g) \leq M_2(K) + N_2(K, \varphi, g)$;
(2) \( \lim_{K \to 1} M_2(K) = 1 \), and
(3) \( \lim_{\|\varphi\|_\infty \to 0, \|g\|_\infty \to 0} N_2(K, \varphi, g) = 0 \).

From (3.9), we see that
\[
1 \leq C_2(K, \varphi, g) \leq C_3,
\]
where
\[
C_3 = \begin{cases} 
(\mu_1 + \mu_2)^K, & \text{if } \frac{(K-1)}{K} \mu_1 \geq 1, \\
\min \left\{ (\mu_1 + \mu_2)^K, \frac{1}{1 - \mu_1 (1 - \frac{1}{K})} \right\}, & \text{if } \frac{(K-1)}{K} \mu_1 < 1.
\end{cases}
\]

Then, we have
\[
C_3 = \begin{cases} 
M_2'(K) + N_2(K, \varphi, g), & \text{if } \frac{(K-1)}{K} \mu_1 \geq 1, \\
\min \{ M_2'(K) + N_2'(K, \varphi, g), M_2''(K) + N_2''(K, \varphi, g) \}, & \text{if } \frac{(K-1)}{K} \mu_1 < 1,
\end{cases}
\]
where
\[
M_2'(K) = \mu_1^K, \quad M_2''(K) = \frac{\mu_2}{K-\mu_1(K-1)}, \quad N_2'(K, \varphi, g) = (\mu_1 + \mu_2)^K - \mu_1^K, \quad \text{and} \quad N_2''(K, \varphi, g) = \frac{\mu_2}{1 - \mu_1 (1 - \frac{1}{K})}.
\]

Let
\[
M_2(K) = \max \{ M_2'(K), M_2''(K) \}
\]
and
\[
N_2(K, \varphi, g) = \max \{ N_2'(K, \varphi, g), N_2''(K, \varphi, g) \}.
\]

It follows from the facts
\[
\lim_{K \to 1} M_2(K) = 1 \quad \text{and} \quad \lim_{\|\varphi\|_\infty \to 0, \|g\|_\infty \to 0} N_2(K, \varphi, g) = 0
\]
that these two constants are what we need, and so, the claim is proved.

**Claim 3.8.** There are constants \( M_1(K) \) and \( N_1(K, \varphi, g) \) such that

1. \( C_1(K, \varphi, g) \geq M_1(K) - N_1(K, \varphi, g) \);
2. \( \lim_{K \to 1} M_1(K) = 1 \), and
3. \( \lim_{\|\varphi\|_\infty \to 0, \|g\|_\infty \to 0} N_1(K, \varphi, g) = 0 \).

By (3.21), we have
\[
C_1(K, \varphi, g) \geq M_1(K) - N_1(K, \varphi, g),
\]
where
\[
M_1(K) = K^{-2} (Q(K))^{-2K} \frac{1}{2\pi} \int_0^{2\pi} |e^{it} - e^{i\theta}|^{2K-2} dt
\]
and
\[
N_1(K, \varphi, g) = \frac{\|\varphi\|_\infty}{2} \left[ \max_{x \in [0, 1]} \{ h(x) \} + \left( 2 + \frac{1}{K^2} \right) \left( \frac{\pi^2}{3} - 1 \right)^{\frac{1}{2}} \right]
+ \|g\|_\infty \left[ \frac{1}{16K^2} + \frac{53}{240} + \frac{2^\frac{3}{2} (1 + 2K^2) \left( 1 + \frac{\pi^2}{6} \right)^{\frac{1}{2}}}{16K^2} \right].
\]
The following facts
\[
\lim_{K \to 1} M_1(K) = 1 \quad \text{and} \quad \lim_{\|\varphi\|_\infty \to 0, \|g\|_\infty \to 0} N_1(K, \varphi, g) = 0
\]
show that these two constants are what we want, and thus, the claim is true.

Now, by the discussions of Steps 3.1 \sim 3.3, we see that the theorem is proved. □

As a direct consequence of Claim 3.8, we have the following result.

**Corollary 3.1.** Under the assumptions of Theorem 1.1, if, further, \(\|g\|_\infty \leq a_1(K)\) and \(\|\varphi\|_\infty \leq a_2(K)\), then \(f\) is co-Lipschitz continuous, and so, it is bi-Lipschitz continuous, where \(a_1(K) = \frac{60}{(25+61K^2)46^{2(K-1)}}\) and \(a_2(K) = \frac{25}{(38+101K^2)46^{2(K-1)}}\).

4. **TWO EXAMPLES**

The purpose of this section is to construct two examples. The first example shows that the co-Lipschitz continuity of \(f\) from Theorem 1.1 is invalid for arbitrary \(g\) and \(\varphi\), and the second one illustrates the possibility of \(f\) to be bi-Lipschitz continuous.

**Example 4.1.** For \(z \in \overline{D}\), let
\[
f(z) = \beta|z|^\gamma z,
\]
where \(\gamma\) and \(\beta\) are constants with \(\gamma > 3\) and \(|\beta| = 1\). Then, \(f\) is a four times continuously differentiable and \(K\)-quasiconformal self-mapping of \(D\) with \(f(0) = 0\) and \(K = 1 + \gamma\). Furthermore, for \(z \in \overline{D}\),
\[
g(z) = \Delta(\Delta f(z)) = \beta \gamma^2(\gamma^2 - 4)\frac{|z|^{\gamma-2}}{2},
\]
and for \(\xi \in T\),
\[
\varphi(\xi) = \beta \gamma(2 + \gamma)\xi \quad \text{and} \quad f^*(\xi) = \beta \xi.
\]
Then \(\|g\|_\infty = \gamma^2(\gamma^2 - 4)\) and \(\|\varphi\|_\infty = \gamma(2 + \gamma)\). However, \(f\) is not co-Lipschitz continuous (i.e., it does not satisfy (1.6)) because
\[
\lambda(Df(0)) = |f_z(0)| - |f^*_z(0)| = 0.
\]

**Example 4.2.** For \(z \in \overline{D}\), let
\[
f(z) = z + \frac{1}{200}(|z|^2 - |z|^4).
\]
Then, we have
\[
g(z) = \Delta(\Delta f(z)) = -\frac{8}{25}
\]
in \(D\) and
\[
\varphi(\xi) = -\frac{3}{50}, \quad f^*(\xi) = \xi
\]
in \(T\).

Moreover, it is not difficult to know that \(f\) is a \(K\)-quasiconformal self-mapping of \(D\) with
\[
K = \max_{z \in \overline{D}} \left\{ \frac{|1 + \overline{z}(1 - 2|z|^2)M| + |Mz(1 - 2|z|^2)|}{|1 + \overline{z}(1 - 2|z|^2)M| - |Mz(1 - 2|z|^2)|} \right\} = \frac{100}{99},
\]
where $M = \frac{1}{200}$. Since elementary computations lead to

$$a_1(K) = \frac{60}{(25 + 61K^2)46^{2(K-1)}} > 0.63 \quad \text{and} \quad a_2(K) = \frac{25}{(38 + 101K^2)46^{2(K-1)}} > 0.16,$$

we know that

$$\|g\|_{\infty} < a_1(K) \quad \text{and} \quad \|\varphi\|_{\infty} < a_2(K),$$

where $a_i(K)$ ($i = 1, 2$) are from Corollary 3.1. Now, it follows from Corollary 3.1 that $f$ is co-Lipschitz continuous, and so, it is bi-Lipschitz continuous.

**References**

1. K. Astala, T. Iwaniec and G. Martin, *Elliptic partial differential equations and quasiconformal mappings in the plane*, in: Princeton Mathematical Series, vol. 48, Princeton University Press, Princeton, NJ, 2009, p. xviii+677.
2. H. Begehr, *Dirichlet problems for the bi-harmonic equation*, Gen. Math., 13 (2005), 65–72.
3. J. Chen, P. Li, S. K. Sahoo and X. Wang, On the Lipschitz continuity of certain quasiregular mappings between smooth Jordan domain, Israel J. Math., 220 (2017), 453–478.
4. Sh. Chen, S. Ponnusamy and X. Wang, On planar harmonic Lipschitz and planar harmonic Hardy classes, Ann. Acad. Sci. Fenn. Math., 36 (2011), 567–576.
5. G. Choquet, Sur un type de transformation analytique généralisant la représentation conforme et définie au moyen de fonctions harmoniques, Bull. Sci. Math., 69 (1945), 156–165.
6. G. Cui and Zh. Li, A note on mori’s theorem of $K$-quasiconformal mappings, Acta Math. Sinica (N. S.), 9 (1993), 55–62.
7. R. Fehlmann and M. Vuorinen, Mori’s theorem for $n$-dimensional quasiconformal mappings, Ann. Acad. Sci. Fenn. Math., 69 (1945), 156–165.
8. S. I. Goldberg and T. Ishihara, Harmonic quasiconformal mappings of Riemannian manifolds, Bull. Amer. Math. Soc., 80 (1974), 225–240.
9. S. I. Goldberg and T. Ishihara, Harmonic quasiconformal mappings of Riemannian manifolds, Amer. J. Math., 98 (1976), 225–240.
10. P. R. Halmos, Measure theory, D. Van Nostrand Company, Inc., New York, N. Y., 1950.
11. W. K. Hayman and B. Horenblum, Representation and uniqueness theorems for polyharmonic functions, J. Anal. Math., 60 (1993), 113–133.
12. W. Hengartner and G. Schober, Harmonic mappings with given dilatation, J. London Math. Soc., 33 (1986), 473–483.
13. H. Hedenmalm, A computation of Green functions for the weighted bi-harmonic Green functions $\Delta|z|^{-2\alpha}\Delta$, Duke Math. J., 75 (1994), 51–78.
14. H. Hedenmalm, S. Jakobsson and S. Shimorin, A bi-harmonic maximum principle for hyperbolic surfaces, J. Reine Angew. Math., 550 (2002), 25–75.
15. E. Heinz, On one-to-one harmonic mappings, Pacific J. Math., 9 (1959), 101–105.
16. L. Hörmander, *Notions of convexity*, Progress in Mathematics, Vol. 127, Birkhäuser Boston Inc, Boston 1994.
17. D. Kalaj, On harmonic quasiconformal self-mappings of the unit ball, Ann. Acad. Sci. Fenn. Math., 33 (2008), 261–271.
18. D. Kalaj, Quasiconformal and harmonic mappings between Jordan domains, Math. Z., 260 (2008), 237–252.
19. D. Kalaj, On quasiregular mappings between smooth Jordan domains, J. Math. Anal. Appl., 38 (2010), 58–63.
20. D. Kalaj, On the quasiconformal self-mappings of the unit ball satisfying the Poisson differential equation, Ann. Acad. Sci. Fenn. Math., 36 (2011), 177–194.
21. D. Kalaj and M. Mateljević, Inner estimate and quasiconformal harmonic maps between smooth domains, J. Anal. Math., 100 (2006), 117–132.
22. D. Kalaj and M. Mateljević, On certain nonlinear elliptic PDE and quasiconformal maps between Euclidean surfaces, *Potential Anal.*, **34** (2011), 13–22.

23. D. Kalaj and M. Mateljević, \((K, K')\)-quasiconformal harmonic mappings, *Potential Anal.*, **36** (2012), 117–135.

24. D. Kalaj and M. Pavlović, Boundary correspondence under harmonic quasiconformal diffeomorphisms of a half-plane, *Ann. Acad. Sci. Fenn. Math.*, **30** (2005), 159–165.

25. D. Kalaj and M. Pavlović, On quasiconformal self-mappings of the unit disk satisfying Poisson’s equation, *Trans. Amer. Math. Soc.*, **363** (2011), 4043–4061.

26. M. Knežević and M. Mateljević, On the quasi-isometries of harmonic quasiconformal mappings, *J. Math. Anal. Appl.*, **334** (2007), 404–413.

27. H. Lewy, On the non-vanishing of Jacobian in certain one-to-one mappings, *Bull. Amer. Math. Soc.*, **42** (1936), 689–692.

28. P. Li and S. Ponnusamy, Representation formula and bi-Lipschitz continuity of solutions to inhomogeneous bi-harmonic Dirichlet problems in the unit disk, *J. Math. Anal. Appl.*, **456** (2017), 1150–1175.

29. O. Martio, On harmonic quasiconformal mappings, *Ann. Acad. Sci. Fenn. Ser. A I*, **425** (1968), 3–10.

30. M. Mateljević and M. Vuorinen, On harmonic quasiconformal quasi-isometries, *J. Inequal. Appl.*, 2010, Art. ID 178732.

31. S. Mayboroda and V. Maz’ya, Boundedness of gradient of a solution and wiener test of order one for bi-harmonic equation, *Invent. Math.*, **175** (2009), 287–334.

32. A. Monr, On an absolute constant in the theory of quasiconformal mappings, *J. Math. Soc. Japan*, **8** (1956), 156–166.

33. M. Partyka and K. Sakan, On bi-Lipschitz type inequalities for quasiconformal harmonic mappings, *Ann. Acad. Sci. Fenn. Math.*, **32** (2007), 579–594.

34. M. Pavlović, Boundary correspondence under harmonic quasiconformal homeomorphisms of the unit disk, *Ann. Acad. Sci. Fenn. Math.*, **27** (2002), 365–372.

35. S. Qiu, On Mori’s theorem in quasiconformal theory, *Acta Math. Sinica (N.S.)*, **13** (1997), 35–44.

36. Luen-Fai Tam and Tom Y.-H. Wan, Quasiconformal harmonic diffeomorphism and the universal Teichmüller space, *J. Differential Geom.*, **42** (1995), 368–410.

37. Luen-Fai Tam and Tom Y.-H. Wan, On quasiconformal harmonic maps, *Pacific J. Math.*, **182** (1998), 359–383.

Sh. Chen, College of Mathematics and Statistics, Hengyang Normal University, Hengyang, Hunan 421008, People’s Republic of China.

E-mail address: mathechen@126.com

X. Wang, Department of Mathematics, Shantou University, Shantou, Guangdong 515063, People’s Republic of China.

E-mail address: xtwang@stu.edu.cn