Abstract
I explain why there are no Poincaré-invariant networks with a locally finite distribution of nodes and links in Minkowski-spacetime of any dimension.

Keywords: Lorentz-invariance, networks, discretization

1. Introduction
Discretization is a traditional cure for divergences. For this reason many scenarios have been proposed that try to deal with the non-renormalizability of quantum gravity by replacing the continuous spacetime of general relativity with a discrete background [1]. However, problems with recovering Lorentz-invariance in the suitable limits will inevitably occur in any discretization that makes use of networks or triangulations. Therefore, all approaches relying on a spacetime discretization have to address whether this Lorentz-invariance violation leads to conflicts with existing observations.

The following argument is probably intuitively clear to most people working in the field, but to my best knowledge it has not been put into writing before. These notes are to benefit newcomers and those not very familiar with the literature.

2. Invariance versus covariance
Before we go in media res, we first have to clarify what we mean with Poincaré-invariant because there is a common misunderstanding in the terminology. Any object that is specified by quantities which do not change under any Poincaré-transformations is Poincaré-invariant. This is the case for scalars and four-volumes (properly weighted), and combinations thereof.

A tensor of rank higher than zero is not invariant, it is covariant. This means it transforms from one coordinate system to the other by a well-defined procedure. A similar statement is true for tensor-densities. Any tensor known in one coordinate-system can be made into a...
Poincaré-covariant tensor by just defining it in all other coordinate systems as the transformation of the quantity in one coordinate system. Thus the question whether or not a tensor is Poincaré-invariant is only meaningful if the expression of the tensor is known in several coordinate systems.

Not any object that has four entries is a covariant vector. Take for example the diagonal entries of the stress–energy tensor. These do not transform like a vector. Likewise, not every $4 \times 4$ matrix is a second rank tensor. This is to say that the correct definition of physically meaningful quantities bears relevance, and in fact one of Einstein’s main achievement was to identify suitable combinations of physical quantities so that they would form tensors.

The expression of a tensor of rank larger than zero in one coordinate system can be obtained uniquely from its expression in another coordinate system, but the tensor entries themselves will in general be different and so these objects are not invariants. The difference between invariance and covariance is relevant if one is concerned with the question of Lorentz-invariance violation. We say Lorentz-invariance (and thereby Poincaré-invariance) is violated if one can construct a preferred spacetime slicing, or a preferred class of observers respectively.

What we mean here with ‘preferred slicing’ is any kind of procedure that allows arbitrary observers to determine a particular time-like vector-field $(\partial_t) \Sigma_t$ from possible measurements they can make. This vector field is covariant and one can construct a corresponding space-like slice $\Sigma_t$ at each point in time which is locally orthogonal to the vector field. Using the vector field and the corresponding slices constitutes then a specific $3 + 1$ split of spacetime. Typically, the laws giving rise to the measurements which the observers made will be particularly simple in the coordinates in which the vector field is just $(-1, 0, 0, 0)$ which is why this frame is said to be ‘preferred’.

It must be emphasized though that the existence of preferred frames does not mean that there are now quantities that do no longer transform under Lorentz-transformations from one inertial frame to the other.

**Definition.** Any structure that allows to construct a preferred slicing of spacetime or a preferred direction in space is Lorentz-invariance violating. A structure that does not allow such construction is Lorentz-invariant.

We often use preferred frames in our calculations. On Earth for example, a frame in rest to the Earth’s surface is preferred for calculations in the laboratory settings, and similarly a frame in rest with the cosmic microwave background is a preferred frame for cosmological observables. These frames are not worrisome because they arise from the dynamics and distribution of matter within space, but are not a fundamental ingredient of our theories independent of the matter. We know that on short distances the interactions of matter are described by the standard model and respect local Lorentz-invariance in all higher order corrections. The problem we are concerned with here is a fundamental vector field that is not merely an emergent field assigned to dynamical matter but is a property of spacetime itself. The effects of such a vector field will not normally go away on short distances. Instead, these fundamental vector fields can come to make unduly large contributions to standard model processes.

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1 Since Lorentz-transformations are just a special case of coordinate transformations in Minkowski-space, the attempt to do physics with laws of nature that do not transform like tensors is extremely problematic and will in the following not be considered.
Violations of Lorentz-invariance by the presence of a preferred frame or a preferred direction are highly constrained by existing experiments because one would expect such a vector field to couple to the Standard Model and make itself noticeable in particle interactions. Moreover, one expects that in an effective limit any Lorentz-invariance violating fundamental spacetime structure gives rise to such a preferred vector field. Or if it does not, this at least necessitates a strong argument as for why a fundamental Lorentz-invariance violation should vanish in the effective limit. It was pointed out already in [2] that Lorentz-invariance violation induced by quantum gravity causes a fine-tuning problem, and the worry that even small Lorentz-invariance violations can lead to large effects was further fleshed out in [3, 4].

The strength of existing constraints depends of course on exactly which particles the field couples to. Constraints are particularly strong if the vector field that defines the preferred frame couples to electrons or photons. In the case where the field couples only gravitationally, constraints are comparably weak. The purpose of this paper is not to elaborate on experimental constraints on Lorentz-invariance violation and its parameterization, but just to explain that it has to be paid attention to in any approach that replaces spacetime with a discrete structure. For more details and constraints on Lorentz-invariance violation, please refer to [5].

To conclude this section I should be fair and admit that much of the confusion around this comes about because in the literature the distinction between Lorentz-invariance and Lorentz-covariance is not always properly made. I too have almost certainly sometimes sloppily said ‘invariance’ when what I really meant was ‘covariance’. The phrase ‘observer-independence’ can for this reason also be ambiguous because it might either mean that all observers really see the same (invariant) or that they can transform their observations into each other’s by a well-defined procedure (covariant). However, the exact physical meaning often reveals itself from the given context.

3. Theorem

Any regular spacetime lattice, that is a network with nodes connected by links, violates Poincaré-invariance. This is so to begin with because the lattice is not translationally invariant under continuous displacements, but in a sense this rather uninteresting. One could argue that as long as one looks at large enough scales, this discretization of translations would be unobservable. The more severe problem is that Lorentz-boosts make the very notion of what is a ‘large’ scale meaningless and so these violations cannot easily be limited—if one just boosts sufficiently much, even a wavelength similar to the Planck-length can be redshifted to a kilometer. A regular lattice is defined by its lattice vectors that do not differ from one node to the next. These vectors are covariant, but not invariant. Their (scalar) spacetime interval is invariant, but their spatial components can become arbitrarily large or arbitrarily small. One can use them to construct a preferred frame for example by defining it to be the frame in which the spatial spacing is just exactly a Planck length. Pictorially speaking, the lattice is not invariant because its spacing is stretched and squeezed under Lorentz boosts. The same holds true for non-periodic quasi-crystals because the tiles have typical widths.

We will assume here that the discrete spacetime network approximates continuous spacetime and that it is possible to create a distance measure on that emergent spacetime which corresponds to the metric. We can then assign relative distances to the nodes of the network. The Poincaré transformations act on these distances.

It has been known for some while however that there are Poincaré-invariant random distributions of points in Minkowski-space that are locally finite [6]. Locally finite means that there are finitely many points in any finite spacetime volume, which is what one needs for
discretization to be meaningful. These invariant distributions can be explicitly constructed by a Poincaré-process. They are invariant in the stochastical sense, so that averaged over a large number of repetitions (or a large sample of volumes respectively) they are invariant, even though any single point distribution on its own is not.

We will not need the explicit expression of this Poisson-process for the following, but its mere existence raises the question whether not one can use this distribution of points to make it into a network that is also both Poincaré-invariant (on the average) and locally finite. With a locally finite network we will mean the following:

**Definition.** A locally finite network is a network in which the number of nodes is finite in any finite spacetime volume and the number of links pinching the surface of each volume is finite.

Having clarified all necessary expressions, we can then make precise the statement we want to prove:

**Theorem.** There exist no Poincaré-invariant networks in Minkowski-space that are locally finite.

3.1. Proof in 1 + 1 dimensional Minkowski-space

We will lead this proof by contradiction. For simplicity we first do this in 1 + 1 dimensional Minkowski-space. The generalization to higher dimensions is then straight-forward.

Suppose there exists a locally finite Poincaré-invariant network. This network must have a finite amount of nodes in any finite volume. We thus take the 1 + 1 dimensional Minkowski-space and tile it up into causal diamonds of some fixed spatial and temporal extension, see figure 1. This tiling is not Lorentz-invariant, but this does not matter in the following. Relevant is only that the tiles all have the same volume and thus, due to homogeneity must, on the average, contain the same finite number of nodes.

Next we pick one of these tiles (marked gray in figure 1) and one node in this tile and ask what can be the network neighbors of this node so that the network connections are invariant on the average. Well, first the node might be entirely unconnected, which is of course Lorentz-invariant, but if all nodes were like this it would be a pretty uninteresting type of network, so we will not further consider this possibility. Consider then the node has at least one neighbor; Where is it?

The node and its neighbor define a spacetime distance $(\Delta t, \Delta x)$. The spacetime interval defined by the proper length $\alpha := -\Delta t^2 + \Delta x^2$ is Lorentz-invariant and we need not worry about it. The spatial and temporal component however depend on the reference frame and could be used to construct a preferred frame, unless they are uniformly distributed over all values with the same $\alpha$. This means that the probability for the position of the neighboring point must be uniformly distributed on the hyperbolae defined by $\alpha$.

The Lorentz-group is non-compact and so the boost parameters cover an infinite range, corresponding to an infinite length of the hyperbolae. Uniform distributions on non-compact support are awkward—the mathematician would say they are ill-defined and would go on to now introduce all kinds of epsilons and deltas to handle the issue. The physicist who is more comfortable with infinities just concludes that the neighboring node is with probability one in the infinite spatial and temporal distance, see figure 2.

For a uniform probability distribution of points on an infinite interval, the probability to find the point in any finite interval is zero. If we pick for example the top right quadrant, this
Figure 1. First we divide up spacetime into tiles of equal volume. Dots are nodes of the network. This is a sketch—the shown distribution is not actually Poincaré-invariant on the average.

Figure 2. Next we try to find out where a node in the center tile might have its neighbors. Since the probability for the endpoints must be uniformly distributed on each hyperbola at constant proper length from the origin, the neighbors are almost certainly located at an arbitrarily large spatial distance, i.e. arbitrarily close to the lightcone.
means the neighboring node is to be found at \( \lim_{\Delta t \to -\infty} (\sqrt{\Delta t^2 - \alpha}, \Delta x) \). Another way to say this is that the only vectors invariant under a Lorentz-transformation are \((0, 0)\) and \((\pm \infty, \pm \infty)\), understood as a limiting case. The nodes in the center tile might have several more neighbors for which the same conclusion applies, but taking these into account is not necessary for the following.

We could now select whether we want the node to be only in the forward or in the backward lightcone, depending on whether we have a directed or undirected network, but this is not so relevant for the following. So let us just assume that with probability \(1/2\) the neighbor is either in the future or the past and with probability \(1/2\) either left or right.

At this point one may already question what the physical meaning is of such a network that aligns its links along lightcones, but it seems too early to give up. So let us then ask now how many links go through the center node. Since there are infinitely many points with links arbitrarily close to the lightcone, there are infinitely many links passing through the center tile, and thus through every tile.

![Figure 3](image.png)

Finally we count all the links that go through the center node. Since there are infinitely many points with links arbitrarily close to the lightcone, there are infinitely many links passing through the center tile, and thus through every tile.

Because all the links are arbitrarily close to the lightcones, we must count all the points in the volume-tiles that can send light-signals to or receive light signals from the volume we considered (see figure 3). Each of these points has a probability of \(1/4\) to have a link that goes through the center volume. Since there are infinitely many tiles, there are infinitely many links going through the surface of the center volume and thereby, due to homogeneity, through every other volume. So we conclude that the network is not locally finite.

3.2. Minkowski-space of higher dimension

In the case of more than one spatial dimension the argument of the previous subsection has to be modified so as to take into account that the probability for the endpoint is now distributed over all spatial dimensions, and also that the volume from which links can go through the
center tile is larger. These two factors cancel each other: the probability that a link from one of
the volume tiles in the light-like past or future of the center tile goes through the center tile
drops with $1/D^2$, where $D$ is the spatial distance (in the chosen coordinate system), while the
number of tiles from which a link can reach the center tile increases with $1/D^2$. This means if
one integrates over all links the result is still infinite. (And in this infinite limit it is inde-
pendent of the reference frame. If one cuts off the integral at any finite slice, then the result
will depend on how that slice was chosen.)
If this sounds familiar, it is probably because the same argument leads to Olber’s
paradox, according to which the night sky should be infinitely bright in an infinite Universe
that is homogeneously and isotropically populated with stars. In this case it is the luminosity
of the star that falls with $1/D^2$, while the number of stars at a given distance increases
with $D^2$.

3.3. Partitions by hypersurfaces

The problem of infinitely dense networks carries over to partitions of spacetime by two-
dimensional hypersurfaces, like for example triangulations. These cannot be made Poincaré-
invariant while being locally finite for the same reason as networks, because these surfaces,
like one-dimensional lines but unlike points and spacetime volumes, are merely covariant but
not invariant under coordinate transformations. The easiest way to see this is to take the
intersection of such a partition with an arbitrary timelike slice of Minkowski-space. This will
result in a network on the slice which should also be Poincaré-invariant, and thus the above
considerations apply.

3.4. Links that are not straight

It should be added for completion that the above conclusion cannot be avoided if the links of
the network are not straight lines. If the network (the nodes and links, for example described
by a matrix) is thought to be fundamental, such a notion does not make much sense, but one
may think of this as a generalized kind of network, for example like the string-network
considered in [7].
If the connections are continuous but not straight lines, a volume enclosing them must
have a finite maximal spatial width\(^2\). For this width to be Lorentz-invariant it thus again can
either be zero (getting us back to the straight line) or infinite (which cannot be because
otherwise the curve would not connect the links). Thus, considering curves rather than
straight lines just makes the problem worse because it introduces more quantities whose
Lorentz-invariance has to be assured.

4. Discussion

Now that we have seen just what the problem is with making networks Poincaré-invariant let
us discuss some ways to circumvent these problems.
First, one can avoid links and surfaces and only deal with points and volumes. This is
exactly the idea of the causal set approach [8]. The causal set is a set of points with an order
relation; it is not a network and thus does not suffer from the above mentioned problem. In the
causal set approach, neighbors of points are only constructed to the end of allowing propa-
gation of particles and fields, giving rise to ‘chains’ on which particles move [9]. This

\(^2\) And a non-continuous connection seems an oxymoron.
propagation however depends on initial values (a momentum for example) that singles out a preferred direction. One does not expect the propagation of a particle with some initial momentum to be invariant, but merely covariant, and consequently there is no problem with picking neighbors to construct these chains.

It was argued in [10] that the causal set cannot be considered nodes of a network on reasoning similar to the one put forward here but for causal sets specifically. In [10] it was also not emphasized that the homogeneity of the distribution is essential to be able to draw the conclusion.

In [11] it was pointed out that if one does not consider the full Lorentz-group but only a subgroup whose boost parameter is discretized, then at least in $1 + 1$ dimensional Minkowski-space there exist distributions of points that are regular and invariant under this subgroup. These point distributions were named ‘Lorentzian lattices’ in [11], but note that they do not have connections between the points, ie do not have lattice vectors. If one tries to add connections between the points of these distributions, then one sees easily that that these define a preferred frame which is not invariant under the subgroup.

The second possibility to deal with the absence of Poincaré invariant networks is to make sure that all arising observables that are covariant rather than invariant depend on a physical process that gives rise to a preferred frame locally. This consideration was used in [12, 13], in which the distribution of spacetime defects is pointlike and thus can be (on the average) Poincaré-invariant, while the scattering on the defects depends on the momentum of the incoming particle and is thus not invariant, but covariant. The network constructed in [14] is locally finite, but observer-dependent.

The third possibility is to avoid the infinitely dense network by making spacetime non-homogeneous or time-dependent, or both. After all, our Universe has a preferred frame, which could be taken for example as the restframe of the cosmic microwave background. The expansion of the Universe providing us with the preferred frame also means that stars have not lived forever and why, last time you looked, the night sky was not infinitely bright.

The challenge with this possibility of just using a preferred frame is to make sure that the violations of Poincaré-invariance are unobservable at scales where spacetime should to good precision be locally flat. Since this is the main point for writing this note, it is worth reflecting on it for a bit. If it was possible to construct Poincaré-invariant networks in Minkowski-space, then there would be no reason to doubt that a network in a Friedmann–Robertson–Walker space would reinstall Lorentz-invariance to good precision when spacetime curvature can be neglected. But since we have seen now it is not possible to construct such networks, it must be checked in each case just what happens to the preferred frame in local interactions. The worry is, as mentioned previously, that particles will locally couple to the vector field defining the frame.

It should be added that one may discard homogeneity if one is dealing with momentum-space rather than spacetime. One can thus in principle discretize momentum space while maintaining Lorentz-invariance. However, since momenta are the generators of translations, the both spaces are related to each other and it must be ensured that no contradictions arise.

Fourth, one may try to keep a network that has a Poincaré-invariant distribution of nodes, but have connections that are not Lorentz-invariant, such as suggested in [7]. Such networks are easy to create based on the causal set’s Poisson-sprinkling. We can for example put a tile of arbitrary but fixed volume around each node and connect that node with all points in that tile. This breaks Lorentz-invariance because the tiling was not invariant—the resulting network will have a typical spatial length of connections that depends on the restframe. In this case too one is then however left with the question whether the Lorentz-invariance violation gives rise to any observables.
Fifth, one might start out with a network that violates Poincaré-invariance but eventually take a limit in which it becomes infinitely dense and so one recovers Poincaré-invariance in this limit. This is similar to taking the limit of zero lattice spacing. While this has much promise to reinstall local Lorentz-invariance when quantities are suitably defined, it raises the question whether in this limit one will still enjoy the benefits of introducing a spacetime discretization to begin with, or if not one will just get back the problems with perturbatively quantized gravity.

Sixth, one could try to just accept that the network is not locally finite. As with the previously listed probability, in this case the question is whether that has any benefits over just using continuous spacetime to begin with.

Finally, I want to emphasize that this does not mean that network-based discretizations of spacetime cannot work as a basis of quantum gravity because they necessarily lead to Lorentz-invariance violations that conflict with observations. That all locally finite networks must violate Poincaré-invariance merely means that one has reason to suspect that they will give rise to an observable preferred frame, and one must find a convincing argument why the inevitable Lorentz-invariance violation of the underlying structure is not noticeable by us. Just referring to the existence of Poincaré-invariant point-distributions is not sufficient to make the problem go away.

5. Summary

I have demonstrated here why there cannot be Poincaré-invariant networks in Minkowski-space that do not have an infinitely dense coverage by links, and have discussed the challenges this poses for discrete approaches to quantum gravity.

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