On Coupled Development of MHD Instabilities of Rayleigh-Taylor and Kelvin-Helmholtz Types in Nonuniform Gas-Plasmas Flows

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Abstract. The simultaneous development of the MHD instabilities of Rayleigh-Taylor and Kelvin-Helmholtz types at the interface between high-conducting plasmoid and surrounding non- or low-conducting gas is considered. The linear stage of the RTI development is studied analytically for incompressible and compressible fluids. The nonlinear stage of the individual development of the RTI and the coupled development of both instabilities has been investigated numerically. The time-dependent two-dimensional numerical model based on the solution of the Euler gasdynamic equations with body momentum and energy sources of MHD origin has been developed and used in calculations. A disturbance introducing in the background flow has been periodic with varied assignment type and wave length. Fundamental difference between the results of linear and nonlinear analysis has been revealed. In particular, the increment of the RTI development at nonlinear stage is one-two order of magnitude less than that predicted by linear theory and rather weakly depends on initial disturbance mode. In linear analysis the coupled development of the RTI and the KHI is determined by simple summing of the two effects in the expression of wave increment, whereas in nonlinear case the mutual influence of the instabilities leads to essential alterations in their development, main of which is the intensive “layer-by-layer” destruction of the plasmoid surface.

1. Introduction

The high-conducting plasmoid (i.e., localized plasma formation) intensively flowed over by non- or low-conducting compressible fluid is inherent in many MHD flows. Such flows may naturally arise (for instance, in astro- and magnetospheric physics [1, 2]) and can be realized in different plasmadynamic (PD) and magnetohydrodynamic (MHD) applications [3, 4]. The problem of hydrodynamic stability of plasmas often defines their life time and is important for the evolution of such flows.

At the interface between plasmoid and ambient gas conditions exist for the coupled development of MHD instabilities of Rayleigh-Taylor and Kelvin-Helmholtz types (hereafter referred to as “RTI” and “KHI” respectively). The RTI is induced by the nonuniformity of the ponderomotive force (in general case, overall body force) component normal to the interface, the KHI is initiated by the nonuniformity of the tangential velocity component. The interaction of these instabilities may produce essential alterations in the character of the development of each of them and in the flow in the whole.

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The peculiarities of the problem have not been adequately explored. Although the MHD RTI and to a smaller extend the KHI were considered before (see, for instance, review in monograph [5]), they were usually studied individually. The coupled development of instabilities and their interaction in nonlinear stage have been investigated rarely and in rather particular cases (for instance, [6]). The phenomena accompanying the instabilities development and occurring after the plasmoid disintegration were not studied. The results presented in this paper are the continuation of our work [7] in which the first attempt to consider the coupled development of these instabilities was taken. In the second paragraph of the paper the RTI linear stage is studied for incompressible and compressible fluids. The main goal of this analysis is to compare characteristic times of the instabilities development in linear and nonlinear formulations. Besides that a mode of maximum instability due to viscosity is estimated to obtain a lower limit of wave lengths such that the inviscid computational model used to investigate the nonlinear stage of instabilities development still remains valid.

The latter formulation is given in the third paragraph. The time-dependent two-dimensional model based on Euler gasdynamic equations is developed. The layered one-dimensional approximation of undisturbed flow structure is used. The velocity component tangential to the interface in the region upstream of the high-conducting layer is varied. Two types of artificial disturbance of the one-dimensional background flow are considered to provoke instabilities development. The results of the nonlinear analysis are discussed in the forth paragraph, and finally the conclusions of research are presented.

2. Linear analysis
Let us consider an isotropically conducting MHD flow containing a layer of elevated conductivity. It is supposed that the undisturbed flow velocity is uniform and parallel to the \( x \)-axis, i.e. \( \mathbf{U}_0=U_0 \mathbf{e}_x, \) \( u_0=\text{const} \). The pressure gradient \( \nabla p_0 \) is balanced by ponderomotive force \( f_{pi}=j_0 \mathbf{x} \mathbf{B} \), where \( \mathbf{B}=B \mathbf{e}_z, j_0=j_0 \mathbf{e}_z, \mathbf{E}=E \mathbf{e}_y, j_0=\sigma_0 (E-u_0 B) \), \( B=\text{const}, E=\text{const} \). Distributions of \( \rho_0, \sigma_0 \) are described below.

Let us conduct a linear analysis of stability of such a flow in relation to the disturbances occurring in the \( XZ \)-plane. In the analysis it is convenient to use the system of co-ordinates moving with velocity \( u_0 \). The variables can be presented in the form of a sum of the undisturbed value and a small disturbance:

\[
\mathbf{U}=\mathbf{U}_0+\delta \mathbf{U}, \quad p=p_0+\delta p, \quad \rho=\rho_0+\delta \rho, \quad \sigma=\sigma_0+\delta \sigma, \quad \mathbf{j}=j_0+\delta \mathbf{j}, \quad \mathbf{B}=B+\delta \mathbf{B}, \quad \mathbf{E}=E+\delta \mathbf{E},
\]

where electric field strength in moving co-ordinates \( E'=E-u_0 B \). Let the disturbances be plane waves propagating along magnetic field vector, wave vector \( \mathbf{k}=k \mathbf{e}_z \). The disturbances are presented in the form \( \delta \mathbf{E}=\delta \mathbf{E}(x) \exp(-i n x + i k z) \). The technique described by S. Chandrasekhar in monograph [8] has been used to obtain the dispersion relationships \( n(k) \). For real values of \( n \) the disturbances are immobile waves with amplitudes changing exponentially in time. At real \( n>0 \) the solution is characterised by the unlimited increase of the disturbance amplitude, i.e. the instability case takes place.

2.1. Incompressible fluid
Let us consider the flow in inviscid incompressible approximation as described above. Although the density and electric conductivity in every fluid particle are constant they may differ in neighbouring particles. The undisturbed distribution of \( \rho_0(x,z) \) is the following:

\[
\rho_0(x,z) = \begin{cases} 
\rho_1 & \text{at } x < 0, \ x > l \\
\rho_2 & \text{at } 0 < x < l 
\end{cases}
\]

where indices "1" and "2" refer to the low-conducting and high-conducting fluids, \( l \) is the layer thickness. The distributions of \( \sigma_0(x,z) \) and \( j_0(x,z) \) are characterized analogously. Omitting intermediate expressions the final dispersion relationship linking wave number and increment of instability development may be written as:

\[
\frac{k^2}{n^2} E^* B (\sigma_2 - \sigma_1) - (\rho_1^2 \alpha_1^2 + \rho_2^2 \alpha_2^2) - 2 \rho_1 \rho_2 \alpha_1 \alpha_2 \text{cth}(\alpha_2 l) = 0, \tag{1}
\]
where \( \alpha_1 = k \sqrt{1 + \frac{\sigma_1 B^2}{\rho_1 n}}, \quad \alpha_2 = k \sqrt{1 + \frac{\sigma_2 B^2}{\rho_2 n}}. \)

Let us consider several limiting cases:

a) thin layer: \( l \to 0. \)

Due to equation (1) one can obtain \( n = 0, \) i.e. the case of indifferent equilibrium is realized.

b) thick layer: \( l \to \infty. \)

In this case the undisturbed flow has two semi-infinite regions with different properties. Equation (1) takes the following form:

\[
2 \sqrt{\rho_1 \left(1 + \frac{\sigma_1 B^2}{\rho_1 n} + \rho_2 \left(1 + \frac{\sigma_2 B^2}{\rho_2 n}\right)\right)} = -k E^* B (\sigma_2 - \sigma_1)
\]

(2)

c) fast modes: \( \frac{\rho_1 n}{\sigma_1 B^2} \to \infty, \quad \frac{\rho_2 n}{\sigma_2 B^2} \to \infty. \)

In this case \( a_1 \to k, \ a_2 \to k. \) This means physically that a disturbance of velocity has weak influence on disturbances of current density and ponderomotive force. Writing (1) in the following form:

\[
n^2 = -k \frac{E^* B (\sigma_2 - \sigma_1)}{\rho_1 + \rho_2} \left(1 - e^{-2k l} \frac{1 - \frac{\rho_1 - \rho_2}{\rho_1 + \rho_2} e^{-2k l}}{1 - \frac{\rho_1 - \rho_2}{\rho_1 + \rho_2} e^{-2k l}}\right)^{1/2}
\]

(3)

one can see that the layered MHD flow may be unstable for both acceleration \( (E^*>0) \) and deceleration \( (E^*<0) \) regimes.

2.2. Compressible fluid

In the case of a gaseous medium it is necessary to consider three regions with different properties: non- or low-conducting gas at \( x < 0 \) (region 1) and at \( x > l \) (region 3), high-conducting gas at \( 0 < x < l \) (region 2).

Let undisturbed distributions be the following:

\[
p_0(x,z) = \begin{cases} 
 p_1 & \text{at } x \leq 0 \\
 p_1 + E.B \sigma_0 x & \text{at } 0 < x < l \\
 p_1 + E.B \sigma_0 l & \text{at } x \geq l
\end{cases} \quad \rho_0(x,z) = \begin{cases} 
 \rho_0 / a_1^2 & \text{at } x \leq 0 \\
 \rho_0 / a_2^2 & \text{at } 0 < x < l \\
 \rho_0 / a_3^2 & \text{at } x \geq l
\end{cases}
\]

\[
T_0(x,z) = \begin{cases} 
 T_1 & \text{at } x \leq 0 \\
 T_2 & \text{at } 0 < x < l \\
 T_3 & \text{at } x \geq l
\end{cases} \quad \sigma_0(x,z) = \begin{cases} 
 \sigma_1 & \text{at } x \leq 0 \\
 \sigma_2 & \text{at } 0 < x < l \\
 \sigma_3 & \text{at } x \geq l
\end{cases}
\]

where \( a \) is sound velocity, \( T \) is temperature and \( \gamma \) is the specific heat ratio assigned in all calculations to be equal to 1.25, \( T_2 > T_1, T_3 > T_2, \sigma_2 > \sigma_1, \sigma_2 > \sigma_3. \) It is assumed that the Joule dissipation is negligibly small in the background flow and may be disregarded in studies of the linear stage of the instability development. The last assumption is valid when the energy dissipated by Joule heating during instability development is much less than the internal energy of high-conducting plasma. To determine initial parameters in region 3 it is supposed isentropic relation between them and parameters in region 1. It is also assumed that conductivity is pressure and temperature independent, i.e. remains constant in every fluid particle during the instability development.

In these assumptions the dispersion equation has been solved analytically. Its solution obtained in terms of hypergeometric functions is omitted here due to very cumbersome expressions. The dependence of the instability increment \( n \) on the wave number \( k \) has been calculated numerically. In connection with computational difficulties occurring at \( k > 20 \text{ m}^{-1} \) the simplified dispersion relation has been used in calculations for large values of \( k. \) The simplification has been made taking into account the inequality \( \rho n^2 \ll k^2 \gamma p, \) performed at \( k > 2 \text{ m}^{-1}. \)
2.3. Results of linear analysis

The dependences of $n(k)$ have been calculated at the following set of the background flow parameters: in the case of compressible fluid $p_1=1$ MPa, $a_1=a_3=800$ m/s, $a_2=1200$ m/s, $\sigma_1=\sigma_3=0$, $\sigma_2=500$ Sm/m, $B=4$ T, $E=-1200$ V/m, in the incompressible case the values of $p_1$, $\rho_1$, $\sigma_1$, $\sigma_2$, $B$ and $E$ are the same, the value of $\rho_2$ is equal to $\rho_0$ at $x=0$ from the compressible background flow. The analysis of results has showed that at layer thickness $l > 10^{-3}$ m its influence on the instability increment is negligible in wide range of background flow parameters both in the incompressible and compressible cases. In all calculations layer thickness has been assigned to be equal to 0.1 m.

The dispersion curves built for the cases of compressible and incompressible fluids are given in figure 1. One can see that at $k < 0.63$ or at disturbance wave length $\lambda > 10$ m ($\lambda=2\pi/k$) the dispersion equation for compressible fluid has three solutions. Two of them tend to zero at $k \to 0$ but the third monotonically increases with reduction of $k$ and in the limit approaches to finite value. The physical meaning of the upper branch of the solution (curve 2 in figure 1) is not clear completely. It is seems possible that this solution peculiarity is connected with the aforementioned constancy of conductivity in high-conducting plasma. In consequence of this some accidental expansion of the high-conducting layer leads to an increase of integral ponderomotive force acting on the layer. A force balance disturbance results in further layer expansion. If so, the upper branch of dispersion curve may disappear at inserting in model of real dependence of electric conductivity on thermodynamic parameters. This question will be considered later, for the moment the upper branch of solution is considered as unphysical.

In the incompressible case the increments are less and, correspondingly, the instability develops faster. This explains by the infinitely large sound velocity in the incompressible fluid. It is necessary to note the difference between increments quickly decreases with the growth of $k$ and at $k > 50$ becomes less than 1%.

The dispersion curve related to the compressible case in wide range of $k$ is presented in figure 2. One can see that the dispersion curve is rather smooth in the conjugation point at $k=20$ m$^{-1}$. It is shown that the aforementioned simplification of the dispersion relation is valid.

To evaluate the maximum instability mode taking place due to viscous effects the approximation of incompressible fluid has been used. The estimation has been obtained on the basis of the classic study performed in [8]. For an MHD flow with a plane discontinuity one can apply a ponderomotive force instead of a gravity force. This leads to the following expression of the wave length of the mode of maximum instability:

$$\lambda_{\text{max}} \approx 4\pi \nu^{2/3} \frac{(\rho_1 + \rho_2)}{|f_{\text{pm1}} - f_{\text{pm2}}|} \frac{1}{n},$$

where $\nu$ is kinematic viscosity. Substituting typical magnitudes of parameters in (4) one can obtain that $\lambda_{\text{max}}$ is less than $(10^{-2} - 10^{-3})$ m. Taking into account this estimation the wave length $\lambda \sim 10^{-4}$ m has been chosen as basic for inviscid computational modelling of the nonlinear stage of instabilities development. One can suppose that in the vicinity of basic wave length the inviscid model is valid for the disturbances evolution description.

Figure 1. The dispersion curves for incompressible (1) and compressible (2, 3) fluids

Figure 2. The dispersion curve 3 for compressible fluid in wide range of $k$
3. Nonlinear analysis: problem formulation

A model MHD flow in the $XZ$-plane is formed by two semi-infinite regions of non-conducting gas ($x \leq 0$ and $x \geq l$) separated by high-conducting layer of thickness $l$. The viscosity, the heat conductivity and the induced magnetic fields are neglected, the perfect non-radiative gas approximation is used. It is assumed that electric field $E$ and current density $j$ are directed normally to the aforementioned plane. The velocity of the undisturbed flow is uniform and has $x$-component $u_0$ only, the flow inhomogeneities in $y$-direction are absent. The distributions of $E$ and $B$ are uniform and constant throughout the whole process. The flow is considered in the system of co-ordinates moving with velocity $u_0$.

With these assumptions the flow can be described by a two-dimensional system of non-steady-state gasdynamic equations taking into account momentum and energy body sources of MHD origin:

$$\frac{\partial F}{\partial t} + \frac{\partial G}{\partial t} + \frac{\partial H}{\partial t} = R,$$  \hspace{1cm} (5)

where the total energy density $e=c_vT+\frac{(u^2+w^2)}{2}$. The system of equations is written here in the dimensionless form. For normalization of the variables the following parameters are used: the sound velocity $a_{1d}$ for velocity components and sound velocity (indices "l" and "d" specify the left non-conducting region and the dimension form of a parameter, correspondingly), $\rho_{1d}$ for density, $a_{1d}^2$ for pressure, $a_{1d}^2$ for energy density, the assigned wave length of disturbance $\lambda_{d}$ for spatial co-ordinates, $\sigma_{d}$ for electrical conductivity ($\sigma_d$ is the conductivity in high-conducting gas near upstream surface of layer, the dependence of conductivity on thermodynamic parameters used in this work is the same as in [7]), $\sigma_d a_{1d} B_d$ for current density, $a_{1d} B_d$ for electrical field.

The parameter $S_2$ having meaning of Stuart number is determined as $\lambda_{d} \sigma_d B_d^2/\rho_{1d} a_{1d}$. The system of equations is closed by dimensionless equation of state in form $p = \rho T$.

The initial distributions of variables are assigned by the following way. The undisturbed state of the flow has been determined from condition that ponderomotive force within the layer is balanced by pressure gradient, in the non-conducting regions pressure is uniform. At $x > 0$ transversal velocity component $w$ is zero, at $x < 0$ $w$ has a constant value varied in different versions of calculations from zero to 0.2. As a rule, it was supposed that $u_0 = a_{10}$. It is supposed that parameters in the left and right non-conducting regions are related isentropically, temperature within high-conducting layer $T_{20}$ is uniform.

To study instability development these distributions are disturbed. Two types of disturbances have been used (figure 3). The first type (figure 3a) consists of the variation of the temperature distribution in the high-conducting layer in accordance with the expression $T = T_2(1 + T_0 e^{x/d} \sin(2\pi z))$, where values of $T_0$ and $d$ are assigned (hereafter $x$-axis is horizontal and $z$-axis is vertical).

Figure 3. The disturbance types of initial flow structure (a – the temperature field, b – the interface shape)
The second type of disturbances (figure 3b) results in distortion of the interface between upstream non-conducting region and high-conducting layer. The distorted configuration of the interface is given by a sinusoid with assigned amplitude.

On the left and right boundaries of calculation region «soft» conditions ($\frac{\partial V}{\partial x} = 0$) have been used, on the lower and upper boundaries the periodic conditions have been realized. The modified Godunov-Kolgan method described in [4] has been used in calculations.

4. Results and discussion
Within the framework of this formulation five versions of calculations have been carried out. They have been obtained at the following set of initial dimensionless data fixed for all versions: $\gamma = 1.2$, $\rho_f = 1$, $\rho_l = 0.833$, $T_f = 0.833$, $T_l = 2.315$, $\rho_1 = 0.836$, $\rho_3 = 0.672$, $T_3 = 0.804$, $B = 1$, $E_a = -0.4$, $\lambda = 1$, vertical size of computational region $L_y$ is equal to 2. The dimensional thickness of the high-conducting layer $l_{hd}$ has been fixed and equal to 0.15 m, accordingly, its dimensionless value has been changed with alternation of the dimensional wave length.

The initial three versions of calculations have been conducted to study features of the RTI (i.e. at $w_l = 0$, and KHI is absent). The first version is characterized by $S_o = 0.268$ corresponding to the following dimension parameters: $\omega_d = 0.1$ m, $\sigma_d = 500$ Sm/m, $B_d = 4$ T, $\rho_{ld} = 0.926$ kg/m$^3$. The horizontal region size $L_x$ is equal to 4, high-conducting layer is bounded by co-ordinates $x_{h1} = 2$ and $x_{h2} = 3.5$. The first type of the flow disturbance has been used with parameters $T_a = 0.2$ and $d = 1$. The results obtained are shown in figure 4 for two moments of time.

![Figure 4](image1.png)

**Figure 4.** The level lines of temperature for two moments of time (the 1st version of calculations)

One can see a rather classical structures characterized by occurrence of «jets» with extended «heads» of more dense non-conducting medium and «bubbles» of less dense high-conducting gas. The dimensional break time of the layer $\tau_{lb}$ is near to $1.2 \times 10^{-3}$ s.

![Figure 5](image2.png)

**Figure 5.** The level lines of temperature for two moments of time (the 2nd version of calculations)

The analysis of these two versions reveals weak variation of the width of jets, the speed of head extension and so on. It is interesting that the values of $\tau_{lb}$ observed in these calculations are very close.
To obtain the third version of solution the second type of the flow disturbance has been used with amplitude of interface distortion being equal to 0.2. The other determining parameters are the same as in the first version. The results are illustrated by figure 6.

![Figure 6. The level lines of temperature for two moments of time (the 3rd version of calculations)](image)

It is likely that the observable difference of this version from the previous ones is due to more strong initial disturbance, practical absence of the linear stage and fast transition to slower nonlinear stage of the instability development. Apparently this fact has resulted in a rather higher $t_{db}$ (about $2 \cdot 10^{-3}$ s in the dimensional form).

The presence of transversal velocity in the left non-conducting region leads to fundamental changes of results (in the 4th and 5th versions of calculations the transversal velocity $w_1$ has been assigned to be equal to 0.2, the first and second types of initial disturbance have been used correspondingly). As viewed in figures 7 and 8, in this case the development of disturbances is specific and essentially differ from the solutions corresponding to the individual development of the RTI discussed above and the KHI with well known effect of "cat eyes" creation.

![Figure 7. The level lines of temperature for two moments of time (the 4th version of calculations)](image)

![Figure 8. The level lines of temperature for two moments of time (the 5th version of calculations)](image)
The destruction of the disturbed surface of the high-conducting layer is accompanied by formation and subsequent separation of its fragments. Although explicitly expressed break of the layer by the non-conducting gas is not observed it is clear that layer-by-layer destroy of the plasmoid surface introduces new aspects to the problem of the plasmoid stability. On the other hand, this process is relatively slow. In calculations the reduction of the layer in half required about $1.5 \cdot 10^{-3}$ s.

5. Conclusions
The study conducted has shown the following. First, the increments of the MHD instability of RT type arising from nonlinear analysis are one order of magnitude less than those predicted by linear theory. Second, at the same type of the initial disturbances the time of the break of high-conducting layer rather weakly depends on wave length. Third, the presence in upstream non-conducting region of velocity component parallel to the interface provokes the interaction of the MHD instabilities of the Rayleigh-Taylor and Kelvin-Helmholtz types. Manifestation of this interaction introduces essential alterations in the development of the RTI.

It is necessary to note that the effect of destruction of the surface of the high-conducting layer with separation of its fragments may induce difficulties in the solution of the stability problem considered. On the other hand, this process is relatively long in time. The estimation of its importance requires to compare the characteristic times (for instance, of this process and of a residence of plasmoid in a channel of some device). Apparently this question calls for special consideration.

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