BREAKING IN THE WHITHAM EQUATION FOR SHALLOW WATER WAVES

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ABSTRACT. We prove wave breaking — bounded solutions with unbounded derivatives — in a nonlinear nonlocal equation, which combines the dispersion relation of water waves and a nonlinearity of shallow water equations, provided that the initial datum is sufficiently asymmetric, whereby solving a Whitham’s conjecture. We extend the result to equations of Korteweg de Vries type for a range of fractional dispersion.

1. INTRODUCTION

As Whitham emphasized in [Whi74, pp. 457], “the breaking phenomenon is one of the most intriguing long-standing problems of water wave theory.” The shallow water equations,

\begin{align*}
\partial_t h + \partial_x ((1 + h)u) &= 0, \\
\partial_t u + \partial_x h + u \partial_x u &= 0,
\end{align*}

approximate the physical problem in the long wavelength (but not necessarily small amplitude) regime and they explain breaking into bores. Here, \( t \in \mathbb{R} \) is proportional to elapsed time and \( x \in \mathbb{R} \) is the spatial variable in the predominant direction of wave propagation; \( h = h(x, t) \) is the surface displacement from the undisturbed fluid depth 1, say, and \( u = u(x, t) \) is the particle velocity at the rigid horizontal bottom; see [Lan13, Section 5.1.1.1], for instance, for the detail. The phase speed associated with the linear part of (1.1) is independent of the spatial frequency, whereas the phase speed of a plane wave with the spatial frequency \( \kappa \) near the quintessential state of water, after normalization of parameters, is

\begin{equation}
{c}^2_{WW}(\kappa) = \frac{\tanh \kappa}{\kappa}.
\end{equation}

In other words, (1.1) neglects the dispersion effects. When waves are long compared to the fluid depth so that \( \kappa \ll 1 \), by the way, one may expand the right side of (1.2) and find that

\begin{equation}
{c}^2_{WW}(\kappa) = \pm \left( 1 - \frac{1}{6} \kappa^2 \right) + O(\kappa^4).
\end{equation}

But the shallow water theory goes too far. It predicts that all solutions carrying an increase of elevation break. Observations have been long since established that some waves do not break! The missing dispersion effects seem to inhabit breaking.

Date: October 23, 2015.

2010 Mathematics Subject Classification. 35A20, 35B44, 35S10, 35F25, 76B15.

Key words and phrases. blow-up; wave breaking; Whitham equation; shallow water.
When gradients are no longer negligible, by the way, the long wavelength assumption, under which one derives the shallow water equations, is no longer adequate, and hence the solution of (1.1) loses relevance well before breaking sets in. Yet, as Whitham argued, “breaking certainly does occur and in some circumstances does not seem to be too far away from the description given by” the solution of (1.1).

But a simple theory incorporating some dispersion effects (see (1.3)), namely the Korteweg-de Vries (KdV) equation,

\[
\partial_t u + \left(1 + \frac{1}{6} \partial_x^2\right) \partial_x u + u \partial_x u = 0,
\]

in turn, goes too far and predicts that no solutions break. As a matter of fact, the global in time well-posedness for (1.4) was established in \cite{KPV91}, for instance, in \(H^1(\mathbb{R})\).

To conclude, one needs some dispersion effects to properly explain breaking but the dispersion of the KdV equation seems too strong for short wavelengths. It is not surprising since the phase speed \(1 - \frac{1}{6} \kappa^2\) associated with the linear part of (1.4) poorly approximates that of water waves (see (1.2)) when \(\kappa\) becomes large.

On the other hand, the KdV equation invites solitary and periodic traveling waves, which the shallow water equations do not. But it fails to explain sharp crests, which the water wave problem manifests. As Whitham noted, “it is intriguing to know what kind of simpler mathematical equation could include” breaking and peaking.

Whitham therefore in \cite{Whi67} (see also \cite[pp. 477]{Whi74}) put forward

\[
\partial_t u + \int_{-\infty}^{\infty} K(x-y) \partial_y u(y,t) \, dy + u \partial_x u = 0,
\]

where

\[
K(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} c(\kappa) e^{ix\kappa} \, d\kappa
\]

and \(c = c_{\text{WW}}\) (see (1.2)). In other words, \(K\) is the Fourier transform of the phase speed of water waves. The Whitham equation combines the full range of dispersion in the linear theory and a nonlinearity in the shallow water theory. Note that the KdV equation takes \(c(\kappa) = 1 - \frac{1}{6} \kappa^2\). Consequently (see (1.3)), it may be regarded as to approximate up to the quadratic order the dispersion of the Whitham equation, and hence the water wave problem, in the long wavelength regime. As a matter of fact, solutions of (1.4) and (1.5)-(1.6), where \(c = c_{\text{WW}}\), exist and they converge to those of the water wave problem up to the order of \(O(\kappa^2)\) for \(\kappa \ll 1\) during a relevant interval of time; see \cite[Section 7.4.5]{Lan13}, for instance, for the detail. But the Whitham equation may offer an improvement over the KdV equation for short and intermediately long waves. Whitham conjectured that (1.5)-(1.6), where \(c = c_{\text{WW}}\), would have the breaking effects.

For a broad class of \(c\) (real-valued), more generally, (1.5)-(1.6) makes a nonlinear dispersive equation, and it is of independent interest. It combines the nonlinearity, which compels singularities in short intervals of time, and dispersion, which instead acts to spread out waves and make them decay over time. Note that (1.5)-(1.6) is nonlocal unless \(c\) is a polynomial in \(i\kappa\). Examples include the KdV equation with
fractional dispersion
\begin{equation}
\partial_t u + \Lambda^{\alpha-1} \partial_x u + u \partial_x u = 0
\end{equation}
in the range $\alpha > 0$, where $\Lambda = \sqrt{-\partial_x^2}$ is a Fourier multiplier, defined via its symbol as
\[ \hat{\Lambda} f(\kappa) = |\kappa| \hat{f}(\kappa). \]

In the case of $\alpha = 3$, notably, (1.7) reduces to the KdV equation, after normalization of parameters. In the case of $\alpha = 2$ it becomes the Benjamin-Ono equation, and in the case of $\alpha = 1$ the Burgers equation. In the case of $\alpha = \frac{1}{2}$, moreover, the author argued in [Hur12] that (1.7) has relevance to water waves in two dimensions in the infinite depth. In particular, it shares in common with the physical problem the dispersion relation and scaling symmetry. We encourage the interested reader to [NS94], for instance, for more examples.

Seliger in [Sel68] made a rather ingenious argument, albeit formal, and showed that a sufficiently asymmetric solution of (1.5) would break in finite time, provided that $K$ be even, bounded, integrable and monotonically decrease to zero at infinity. Unfortunately it does not apply to the Whitham equation; as a matter of fact, $c_{WW}$ (see (1.2)) is not integrable. Later Constantin and Escher in [CE98] turned Seliger’s formal argument into a rigorous analytical proof. Naumkin and Shishmarev in [NS94] made an alternative argument of wave breaking, provided that $K$ and $K'$ be integrable and $K(x) \leq K_0 |x|^{-2/3}$ for some $K_0 > 0$ for $|x| \ll 1$. By wave breaking, we mean that the solution remains bounded but its slope becomes unbounded in finite time; physically, the wave profile steepens as it propagates until it develops a vertical slope and a multi-valued profile. Unfortunately the Fourier transform of $c_{WW}$ may not be written explicitly, and hence the assumptions in [NS94] are difficult to verify for the Whitham equation. (Moreover there seem some glitches in their argument.) To the best of the author’s knowledge, wave breaking is not yet answered for (1.7), where $c = c_{WW}$. By the way, Whitham claimed, without a proof, that the Fourier transform of $c_{WW}$ would behave like $|\cdot|^{-1/2}$ near zero and exponentially vanishes at infinity. Recently Ehrnström in [Ehr15] analytically confirmed it; see Section B.

Furthermore the kernel associated with the integral representation of $\Lambda^{\alpha-1}$ in the range $0 < \alpha < 1$ is $|\cdot|^{-\alpha}$ up to multiplication by a constant (see (2.6)), and hence the arguments in [NS94] and [CE98] do not apply to (1.7). Recently in [HT14], nevertheless, the author managed to prove wave breaking for $0 < \alpha < 1/2$, provided that the initial datum is in the Gevrey class (see [Hor83] pp. 335), for instance, and (1.10) below) and the slope is sufficiently negative. Here we take matters further and promote the result to $0 < \alpha < 2/3$.

**Theorem 1.1** (Wave breaking in (1.7) for $0 < \alpha < 2/3$). Let $0 < \alpha < \frac{21 - 10\epsilon}{3 + 4\epsilon}$ for $\epsilon > 0$ sufficiently small. If $u_0 \in H^\infty(\mathbb{R})$ satisfies that
\begin{equation}
\epsilon^2 (\inf_{x \in \mathbb{R}} u_0'(x))^2 > 1 + \|u_0\|_{H^3},
\end{equation}
\begin{equation}
\epsilon^3 (1 - \epsilon)^3 (\inf_{x \in \mathbb{R}} u_0'(x))^{3/4} > \frac{6}{\alpha} (1 + (1 + \epsilon^{1/\alpha}) b + b^{1/\alpha}),
\end{equation}
\begin{equation}
\epsilon^2 (\inf_{x \in \mathbb{R}} u_0'(x))^{1/4} > \frac{e}{1/\alpha - 1} \left( \frac{3}{2} \right)^{1/\alpha}
\end{equation}
for $u_0 \in H^\infty(\mathbb{R})$ satisfies that
and that
\begin{equation}
\|u_n^{(n)}\|_{L^\infty} \leq ((n-1)b)^{(n-1)/\alpha} \quad \text{for } n = 2, 3, \ldots
\end{equation}
for some \( b \geq 1 \), then the solution of the initial value problem associated with (1.7) and \( u(\cdot, 0) = u_0 \) exhibits wave breaking, i.e.
\[ |u(x, t)| < \infty \quad \text{for all } x \in \mathbb{R} \quad \text{for all } t \in [0, T) \]
but
\[ \inf_{x \in \mathbb{R}} (\partial_x u)(x, t) \to -\infty \quad \text{as } t \to T^- \]
for some \( T > 0 \). Moreover
\begin{equation}
- \frac{1}{1 + \epsilon \inf_{x \in \mathbb{R}} u_0'(x)} < T < - \frac{1}{(1 - \epsilon)^2 \inf_{x \in \mathbb{R}} u_0'(x)}.
\end{equation}
Furthermore we prove wave breaking for the Whitham equation, whereby solving his conjecture.

**Theorem 1.2** (Wave breaking in the Whitham equation). If \( u_0 \in H^\infty(\mathbb{R}) \) satisfies that
\begin{equation}
\epsilon^2 (\inf_{x \in \mathbb{R}} u_0'(x))^2 > 1 + \|u_0\|_{H^3}
\end{equation}
for \( \epsilon > 0 \) sufficiently small and that
\begin{equation}
\|u_0^{(n)}\|_{L^\infty} \leq ((n-1)b)^2(n-1) \quad \text{for } n = 2, 3, \ldots
\end{equation}
for some \( b \geq 1 \), then the solution of the initial value problem associated with (1.5), where \( c = c_{\text{WW}} \) (see (1.2)), and \( u(\cdot, 0) = u_0 \) exhibits wave breaking for some \( T > 0 \). Moreover (1.12) holds.

The proofs follow along the same line as the argument in [HT14], studying ordinary differential equations for the solution and its derivatives of all orders along the characteristics, which by the way involve nonlocal forcing terms. Lemma 3.1 ensures that \( K \) (see (1.6)) associated with \( c = c_{\text{WW}} \) (see (1.2)) behaves like \(|\cdot|^{-1/2}\) near zero (and exponentially vanishes at infinity). Loosely speaking, therefore, the Whitham equation compares to (1.7) in the case of \( \alpha = 1/2 \), for which unfortunately the proof in [HT14] ceases to apply. Specifically, one loses controls of the second derivative of the solution along the characteristics. We overcome the difficulty by making strong use of the “smoothing effects” of the characteristics (see (2.1)), when the gradient of the solution is close to its minimum. It enables us to promote the result in [HT14] to \( 0 < \alpha < 2/3 \). We clarify several assertions in [NS94] in the course of the proof. Furthermore we relax the requirements on the initial datum in [HT14]. Consequently Theorem 1.1 offers an improvement over the result in [HT14] even when \( 0 < \alpha < 1/2 \).

The Whitham equation has recently gathered much attention from a number of vantage points. Numerical computations in [BKN13] and [MKD15] indicate that it approximates waves in water on par with or better than the KdV and other shallow water equations do in some respects outside the long wavelength regime. (The KdV equation seems a better approximation in the long wavelength regime, though.) Ehrnström announced in [Ehr15] sharp crests in the maximum amplitude, periodic traveling wave; a complete proof will be reported elsewhere. Moreover the author demonstrated in [HJ15] that a \( 2\pi/\kappa \)-periodic traveling wave of sufficiently
small amplitude of (1.5)-(1.6), where $c = c_{WW}$ (see (1.2)), be spectrally unstable to long wavelength perturbations if $\kappa > 1.145\ldots$, bearing out the Benjamin-Feir instability of Stokes waves. By the way, the Benjamin-Feir instability is another high frequency phenomenon in water waves, which the KdV and other shallow water equations do not manifest.

One may expect wave breaking for (1.7) in the range $0 \leq \alpha \leq 1$. In the case of $\alpha = 1$, as a matter of fact, (1.7) becomes the Burgers equation, which invites wave breaking. It will be interesting to promote the result in Theorem 1.1 to $0 \leq \alpha < 1$.

Moreover one may expect global regularity (in the energy space $H^{\alpha/2}$) for (1.7) in the range $\alpha > 3/2$ whereas finite time blowup in the range $0 < \alpha < 3/2$. In the case of $\alpha = 3/2$, observe that (1.7) is $L^2$-critical. But recent numerical experiments in [KS15] indicate that the blowup scenario for $1 < \alpha \leq 3/2$ be different from wave breaking. It will be interesting to analytically confirm blowup for (1.7) in the range $0 \leq \alpha < 3/2$ and to elucidate the blowup scenarios.

2. Proof of Theorem 1.1

We assume that the initial value problem associated with (1.7) and $u(\cdot, 0) = u_0$ possesses a unique solution in $C^\infty([0, T); H^\infty(\mathbb{R}))$ for some $T > 0$. Combining an a priori bound and a compactness argument, as a matter of fact, one may work out the local in time well-posedness for (1.7) in $H^{3/2+} \mathbb{R}$; see [Kat83], for instance, for the detail. We assume that $T$ is the maximal time of existence.

For $x \in \mathbb{R}$, let
\begin{equation}
\frac{dX}{dt}(t; x) = u(X(t; x), t) \quad \text{and} \quad X(0; x) = x.
\end{equation}

Since $u(x, t)$ is bounded and satisfies a Lipschitz condition in $x$ for all $x \in \mathbb{R}$ for all $t \in [0, T)$, it follows from the ODE theory that (2.1) possesses a unique solution in $C^1([0, T))$ for all $x \in \mathbb{R}$. Since $u(x, t)$ is smooth in $x$ for all $x \in \mathbb{R}$ for all $t \in [0, T)$, furthermore, $x \mapsto X(\cdot; x)$ is infinitely continuously differentiable throughout the interval $[0, T)$ for all $x \in \mathbb{R}$.

Let
\begin{equation}
v_n(t; x) = (\partial^nu)(X(t; x), t) \quad \text{for} \quad n = 0, 1, 2, \ldots.
\end{equation}

Differentiating (1.7) with respect to $x$ and evaluating along $x = X(t; x)$, we arrive at that
\begin{equation}
\frac{dv_n}{dt} + \sum_{j=1}^{n} \binom{n}{j} v_j v_{n+1-j} + \phi_n(t; x) = 0 \quad \text{for} \quad n = 2, 3, \ldots,
\end{equation}
\begin{equation}
\frac{dv_1}{dt} + v_1^2 + \phi_1(t; x) = 0
\end{equation}
and, moreover,
\begin{equation}
\frac{dv_0}{dt} + \phi_0(t; x) = 0
\end{equation}
throughout the interval \((0, T)\) for all \(x \in \mathbb{R}\). Here, \(\binom{n}{j}\) denotes a binomial coefficient and

\[
\phi_n(t; x) = (\Lambda^{n-1} \partial_x^{n+1} u)(X(t; x), t) \\
= \int_{-\infty}^{\infty} \frac{\text{sgn}(X(t; x) - y)}{|X(t; x) - y|^{1+\alpha}} ((\partial_x^n u)(X(t; x), t) - (\partial_x^n u)(X(t; x) - y, t)) \, dy \\
(2.6) \\
= \int_{-\infty}^{\infty} \frac{\text{sgn}(y)}{|y|^{1+\alpha}} ((\partial_x^n u)(X(t; x), t) - (\partial_x^n u)(X(t; x) - y, t)) \, dy
\]

up to multiplication by a constant for \(n = 0, 1, 2, \ldots\); see [Hur12], for instance, for the detail. Since \(u(x, t)\) is smooth and in \(L^2\) in \(x\) and smooth in \(t\) for all \(x \in \mathbb{R}\) for all \(t \in [0, T)\) and since \(X(t; x)\) is continuously differentiable in \(t\) and smooth in \(x\) for all \(t \in [0, T)\) for all \(x \in \mathbb{R}\), it follows that \(\phi_n(t; x)\) is continuously differentiable in \(t\) and smooth in \(x\) for all \(t \in [0, T)\) for all \(x \in \mathbb{R}\).

For \(\delta > 0\), we split the integral on the right side of (2.6) and perform an integration by parts to show that

\[
|\phi_n(t; x)| = \left| \left( \int_{|y|<\delta} + \int_{|y|>\delta} \right) \frac{\text{sgn}(y)}{|y|^{1+\alpha}} ((\partial_x^n u)(X(t; x), t) - (\partial_x^n u)(X(t; x) - y, t)) \, dy \right| \\
\leq \left| \frac{1}{\alpha} \delta^{-\alpha} ((\partial_x^n u)(X(t; x) - \delta, t) - (\partial_x^n u)(X(t; x) + \delta, t)) \right| \\
+ \left| \int_{|y|<\delta} \frac{1}{|y|^{1+\alpha}} (\partial_x^{n+1} u)(X(t; x) - y, t) \, dy \right| \\
+ \left| \int_{|y|>\delta} \frac{\text{sgn}(y)}{|y|^{1+\alpha}} ((\partial_x^n u)(X(t; x), t) - (\partial_x^n u)(X(t; x) - y, t)) \, dy \right| \\
(2.7) \\
\leq \frac{6}{\alpha} \delta^{-\alpha} \|v_n(t)\|_{L^\infty} + \delta^{1-\alpha} \|v_{n+1}(t)\|_{L^\infty}
\]

for \(n = 0, 1, 2, \ldots\) and for all \(t \in [0, T)\) for all \(x \in \mathbb{R}\). Here, the second inequality uses that \(\frac{\text{sgn}(y)}{|y|^{1+\alpha}} = \frac{1}{\alpha} \left( \frac{1}{|y|^\alpha} \right)\)' and the last inequality uses that \(0 < \alpha < 2/3\).

Let

\[
m(t) = \inf_{x \in \mathbb{R}} v_1(t; x) = \inf_{x \in \mathbb{R}} (\partial_x u)(x, t) =: m(0) q^{-1}(t).
\]

Note that \(v_1(t, \cdot)\), and hence \(m(t)\), are continuous for all \(t \in [0, T)\). Note moreover that \(m(t) < 0\) for all \(t \in [0, T)\), \(q(0) = 1\) and \(q(t) > 0\) for all \(t \in [0, T)\).

We shall show that

\[
|\phi_1(t; x)| < \epsilon^2 m^2(t) \quad \text{for all } t \in [0, T) \quad \text{for all } x \in \mathbb{R}.
\]

It follows from (1.8) and the Sobolev inequality that

\[
|\phi_1(0; x)| = |\Lambda^{\alpha-1} u''(x)| \leq \|u_0\|_{H^{n+3/2+}} < \epsilon^2 m^2(0) \quad \text{for all } x \in \mathbb{R}.
\]

In other words, (2.4) holds at \(t = 0\). Suppose on the contrary that \(|\phi_1(T_1; x)| = \epsilon^2 m^2(T_1)\) for some \(T_1 \in (0, T)\) for some \(x \in \mathbb{R}\). By continuity, we may assume that

\[
|\phi_1(t; x)| < \epsilon^2 m^2(t) \quad \text{for all } t \in [0, T_1] \quad \text{for all } x \in \mathbb{R}.
\]

We seek a contradiction.
Lemma 2.1. For $0 < \gamma < 1$ and for $t \in [0, T_1]$, let

\begin{equation}
\Sigma_\gamma(t) = \{ x \in \mathbb{R} : v_1(t; x) \leq (1 - \gamma)m(t) \}.
\end{equation}

If $0 < \epsilon \leq \gamma < 1/2$ for $\epsilon > 0$ sufficiently small then $\Sigma_\gamma(t_2) \subset \Sigma_\gamma(t_1)$ whenever $0 \leq t_1 < t_2 < T_1$.

The proof extends that of \cite{HT14} Lemma 2.1. We present the detail in Appendix \[11\] for completeness.

Lemma 2.2. $0 < q(t) \leq 1$ and it is decreasing for all $t \in [0, T_1]$.

Proof. The proof is very similar to that of \cite{HT14} Lemma 2.2. Here we include the detail for future usefulness.

Let $x \in \Sigma_\gamma(T_1)$, where $0 < \epsilon < \gamma < 1/2$ for $\epsilon > 0$ sufficiently small. We suppress it to simplify the exposition. Note from (2.14) and Lemma 2.1 that

\begin{equation}
m(t) \leq v_1(t) \leq (1 - \gamma)m(t)(< 0) \quad \text{for all } t \in [0, T_1].
\end{equation}

Let’s write (2.11) as

\begin{equation}
v_1(t) = \frac{v_1(0)}{1 + v_1(0) \int_0^t (1 + (v_1^{-2})\phi_1(\tau)) \, d\tau} := m(0)r^{-1}(t).
\end{equation}

Clearly $r(t) > 0$ for all $t \in [0, T_1]$. Note from (2.12) and (2.10) that

$$(v_1^{-2}\phi_1)(t) < (1 - \gamma)^{-2}C < \epsilon \quad \text{for all } t \in [0, T_1]$$

for $\epsilon > 0$ sufficiently small. Therefore it follows from (2.13) that

\begin{equation}
(1 + \epsilon)m(0) \leq \frac{dr}{dt} \leq (1 - \epsilon)m(0) \quad \text{throughout the interval } (0, T_1).
\end{equation}

Consequently $r(t)$ and, hence, $v_1(t)$ (see (2.13)) are decreasing for all $t \in [0, T_1]$. Furthermore $m(t)$ and, hence, $q(t)$ (see (2.8)) are decreasing for all $t \in [0, T_1]$. This completes the proof. It follows from (2.8), (2.13) and (2.12) that

\begin{equation}
q(t) \leq r(t) \leq \frac{1}{1 - \gamma}q(t) \quad \text{for all } t \in [0, T_1].
\end{equation}

Lemma 2.3. For $s > 0$, $s \neq 1$, and for $t \in [0, T_1]$,

\begin{equation}
\int_0^t q^{-s}(\tau) \, d\tau \leq \frac{1}{s - 1} \frac{1}{(1 - \epsilon)^{1 + s}} \frac{1}{m(0)} \left( q^{1-s}(t) - \frac{1}{(1 - \epsilon)^{1-s}} \right).
\end{equation}

The proof is found in \cite{HT14} Lemma 2.3, for instance. Hence we omit the detail. See, instead, the proof of (2.31) below.

We shall show that

\begin{align}
\|v_0(t)\|_{L^\infty} &= \|u(t)\|_{L^\infty} < C_0, \\
\|v_1(t)\|_{L^\infty} &= \|\partial_x v(t)\|_{L^\infty} < C_1 q^{-1}(t), \\
\|v_n(t)\|_{L^\infty} &= \|\partial_x^n v(t)\|_{L^\infty} < C_2 ((n - 1)\epsilon)^{(n-1)/\sigma} q^{-1-\sigma} \frac{1}{(1 - \epsilon)^{1-s}}(t)
\end{align}

for $n = 2, 3, \ldots$ for all $t \in [0, T_1]$, where

\begin{align}
C_0 &= 2 \|u_0\|_{L^\infty} + \|u_0'\|_{L^\infty}, \quad C_1 = 2 \|u_0'\|_{L^\infty}, \quad C_2 = (-m(0))^{3/4}
\end{align}
and
\[(2.21) \quad \sigma = \frac{3}{2} + 6\epsilon \quad \text{so that} \quad \sigma \alpha < 1 - 10\epsilon \]
for \(\alpha\) and \(\epsilon\) in Theorem 1.3. Note from (1.8) that
\[\frac{1}{2} C_1 = \|u_0\|_{L^\infty} > C_2 > 1;\]
we tacitly exercise it throughout the proof. It follows from (2.20), (2.8) and (1.11), (1.8) that
\[(2.21) \quad |\alpha| \quad \text{for all} \quad x = 0\]
show that
\[(2.22) \quad T \quad \text{for some} \quad \epsilon > 0 \quad \text{at} \quad x = 0.\]
Suppose on the contrary that (2.17), (2.18) and (2.19) hold for all \(n = 0, 1, 2, \ldots\) at \(t = 0\). Suppose on the contrary that (2.17), (2.18) and (2.19) hold for all \(n = 0, 1, 2, \ldots\) throughout the interval \([0, T_2]\) but do not for some \(n > 0\) at \(t = T_2\) for some \(T_2 \in (0, T_1]\). By continuity, we find that
\[(2.22) \quad \|v_0(t)\|_{L^\infty} \leq C_0,\]
\[(2.23) \quad \|v_1(t)\|_{L^\infty} \leq C_1 q^{-1}(t),\]
\[(2.24) \quad \|v_2(t)\|_{L^\infty} \leq C_2 (n - 1) b^{(n-1)/\alpha} q^{-1-(n-1)\sigma}(t)\]
for \(n = 2, 3, \ldots\) for all \(t \in [0, T_2]\). We seek a contradiction.

**Proof of (2.17).** The proof is similar to that in [H11]. Here we include the detail for future usefulness.

It follows from (2.7), where \(\delta(t) = q(t),\) and (2.22), (2.23) that
\[(2.25) \quad |\phi_0(t; x)| \leq \frac{6}{\alpha} (C_0 q^{-\alpha}(t) + C_1 q^{-1}(t) q^{-1}(t)) = \frac{6}{\alpha} (C_0 + C_1) q^{-\alpha}(t)\]
for all \(t \in [0, T_2]\) for all \(x \in \mathbb{R}\). Integrating (2.25) over the interval \([0, T_2]\), we then show that
\[
|v_0(T_2; x)| \leq \|u_0\|_{L^\infty} + \int_0^{T_2} |\phi_0(t; x)| \, dt
\leq \frac{1}{2} C_0 + \frac{6}{\alpha} (C_0 + C_1) \int_0^{T_2} q^{-\alpha}(t) \, dt
\leq \frac{1}{2} C_0 - \frac{6}{\alpha} (C_0 + C_1) \frac{1}{(1 - \alpha)} \frac{1}{m(0)} \left( \frac{1}{(1 - \epsilon)^{1 - \alpha}} - q^{1 - \alpha}(T_2) \right)
\leq \frac{1}{2} C_0 - \frac{6}{\alpha(1 - \alpha)} (C_0 + C_1) \frac{1}{(1 - \epsilon)^2} \frac{1}{m(0)} < C_0
\]
for all \(x \in \mathbb{R}\). Therefore (2.17) holds throughout the interval \([0, T_2]\). Here, the second inequality uses (2.20) and (2.25), the third inequality uses (2.16), the fourth inequality uses Lemma 2.2 and the last inequality uses (1.12). Indeed
\[-(1 - \epsilon)^2 m(0) > \frac{12}{\alpha(1 - \alpha)} \left( 1 + \frac{C_1}{C_0} \right)\]
for \(\epsilon > 0\) sufficiently small.
Proof of (2.18). The proof is similar to that in [HT14]. Here we include the detail for future usefulness.

It follows from (2.7), where \(\delta(t) = q^\sigma(t)\), and (2.20) and (2.24) that

\[
|\phi_1(t; x)| \leq \frac{6}{\alpha}(C_1 q^{-1} q^{-\sigma \alpha}(t) + C_2 b^{1/\alpha} q^{-\sigma - \alpha}(t) q^{-1 - \sigma}(t))
\]

\[
= \frac{6}{\alpha}(C_1 + C_2 b^{1/\alpha}) q^{-1 - \sigma \alpha}(t)
\]

for all \(t \in [0, T_2]\) for all \(x \in \mathbb{R}\). Suppose for now that \(u_1(T_2; x) \geq 0\). Note from (2.2) that

\[
\frac{dv_1}{dt}(t; x) = -v_1(t; x) - \phi_1(t; x) \leq |\phi_1(t; x)|
\]

for all \(t \in (0, T_2)\) for all \(x \in \mathbb{R}\). Integrating this over the interval \([0, T_2]\), we then show that

\[
v_1(T_2; x) \leq \|u_0\|_{L^\infty} + \int_0^{T_2} |\phi_1(t; x)| dt
\]

\[
\leq \frac{1}{2} C_1 + \frac{6}{\alpha}(C_1 + C_2 b^{1/\alpha}) \int_0^{T_2} q^{-2}(t) dt
\]

\[
\leq \frac{1}{2} C_1 - \frac{6}{\alpha}(C_1 + C_2 b^{1/\alpha}) \frac{1}{(1 - \epsilon)^3} \frac{1}{m(0)} (q^{-1}(T_2) - (1 - \epsilon))
\]

\[
< \frac{1}{2} C_1 q^{-1}(T_2) - \frac{6}{\alpha}(C_1 + C_2 b^{1/\alpha}) \frac{1}{(1 - \epsilon)^3} m(0) q^{-1}(T_2)
\]

for \(\epsilon > 0\) sufficiently small.

Suppose on the other hand that \(v_1(T_2; x) < 0\). We may assume, without loss of generality, that \(\|u_0\|_{L^\infty} = -m(0)\); we take \(-u\) otherwise. It then follows from (2.8) and (2.20) that

\[
v_1(T_2; x) \geq m(T_2) = m(0) q^{-1}(T_2) > -C_1 q^{-1}(T_2).
\]

Therefore (2.18) holds throughout the interval \([0, T_2]\).

Proof of (2.19) for \(n \geq 3\). The proof is similar to that in [HT14]. Here we include the detail for future usefulness.

For \(n \geq 2\), it follows from (2.7), where \(\delta(t) = (nb)^{-1/\alpha} q^\sigma(t)\), and (2.21) that

\[
|\phi_n(t; x)| \leq \frac{6}{\alpha}((nb)C_2 (n-1)b)^{(n-1)/\alpha} q^{-\sigma \alpha}(t) q^{-1-(n-1)\sigma}(t)
\]

\[
+ (nb)^{1-1/\alpha} C_2 (nb)^{n/\alpha} q^{-\sigma - \alpha} q^{-1-n \sigma}(t)
\]

\[
= \frac{6}{\alpha}(nb)C_2 ((n-1)b)^{(n-1)/\alpha} (1 + \left(\frac{n}{n-1}\right)^{(n-1)/\alpha}) q^{-1-\sigma - (n-1)\sigma}(t)
\]

\[
< \frac{6}{\alpha}(1 + e^{1/\alpha})(nb)C_2 ((n-1)b)^{(n-1)/\alpha} q^{-1-\sigma - (n-1)\sigma}(t)
\]

for all \(t \in [0, T_2]\) for all \(x \in \mathbb{R}\).
For \( n \geq 2 \), furthermore, let

\[
(2.28) \quad v_1(T_3; x) = m(T_3) \quad \text{and} \quad m(t) \leq v_1(t; x) \leq \frac{1}{(1 + \epsilon)^{1/(2 + (n-1)\alpha)}} m(t)
\]

for all \( t \in [T_3, T_2] \), for some \( T_3 \in (0, T_2) \) and for some \( x \in \mathbb{R} \). As a matter of fact, since \( v_1 \) and \( m \) are uniformly continuous throughout the interval \([0, T_2]\) we may find \( T_3 \) close to \( T_2 \) so that \((2.28)\) holds. We rerun the argument in the proof of Lemma 2.2 to arrive at that

\[
(2.29) \quad (1 + \epsilon) m(0) \leq \frac{dr}{dt} \leq (1 - \epsilon) m(0) \quad \text{throughout the interval} \quad (T_3, T_2)
\]

for \( \epsilon > 0 \) sufficiently small and

\[
(2.30) \quad q(t) \leq r(t) \leq (1 + \epsilon)^{1/(2 + (n-1)\sigma)} q(t) \quad \text{for all} \quad t \in [T_3, T_2].
\]

It then follows that

\[
\begin{align*}
\int_{T_3}^{T_2} q^{-2-(n-1)\sigma}(t) \, dt \\
&\leq (1 + \epsilon) \int_{T_3}^{T_2} r^{-2-(n-1)\sigma}(t) \, dt \\
&\leq \frac{1 + \epsilon}{1 - \epsilon m(0)} \int_{T_3}^{T_2} r^{-2-(n-1)\sigma}(t) \frac{dr}{dt}(t) \, dt \\
&= -\frac{1}{1 + (n-1)\sigma} \frac{1}{1 - \epsilon m(0)} (r^{-1-(n-1)\sigma}(T_2) - r^{-1-(n-1)\sigma}(T_3)) \\
&\leq -\frac{1}{1 + (n-1)\sigma} \frac{1}{1 - \epsilon m(0)} (q^{-1-(n-1)\sigma}(T_2) - q^{-1-(n-1)\sigma}(T_3)).
\end{align*}
\]

This offers an improvement over \((2.16)\) when \( T_3 \) and \( T_2 \) are close. As a matter of fact, the right side decreases in \( n \). Here, the first inequality uses \((2.30)\), the second inequality uses \((2.29)\), and the last inequality uses \((2.30)\) and \((2.28)\).

**Lemma 2.4.** For \( n \geq 3 \),

\[
(2.32) \quad \sum_{j=2}^{n-1} \binom{n}{j} (j-1)^{(j-1)/\alpha} (n-j)^{(n-j)/\alpha} \leq \frac{\epsilon}{1/\alpha - 1} \left(\frac{3}{2}\right)^{1/\alpha-1} n(n-1)^{(n-1)/\alpha}.
\]

The proof is in \([\text{NS94}], \text{Lemma 2.6.1}\), for instance. We include the detail in Appendix A for completeness.

For \( n \geq 3 \), let \( |v_n(T_2; x_n)| = \max_{x \in \mathbb{R}} |v_n(T_2; x)| \). We may assume, without loss of generality, that \( v_n(T_2; x_n) > 0 \). We choose \( T_3 \) close to \( T_2 \) so that

\[
(2.33) \quad v_n(t; x_n) \geq 0 \quad \text{for all} \quad t \in [T_3, T_2].
\]
We necessarily choose $T_3$ closer to $T_2$ so that (2.28) holds for some $x \in \mathbb{R}$. Consequently (2.31) holds. It follows from (2.3) that

$$\frac{dv_n}{dt}(t; x_n)$$

$$= - (n + 1)v_1(t; x_n)v_n(t; x_n) - \sum_{j=2}^{n-1} \binom{n}{j} v_j(\cdot; x_n)v_{n+1-j}(t; x_n) - \phi_n(t; x_n)$$

$$\leq - (n + 1)m(0)C_2((n - 1)b)^{(n-1)/\alpha}q^{-1}(t)q^{1-(n-1)\sigma}(t)$$

$$+ \frac{1}{\alpha - 1} \left( \frac{3}{2} \right) \frac{1}{(n - 1)b)^{(n-1)/\alpha}q^{-2-(n-1)\sigma}(t)}$$

$$\leq \left( - m(0)(n + 1) + \left( \frac{1}{\alpha - 1} \right) \left( \frac{3}{2} \right) \frac{1}{(n - 1)b)^{(n-1)/\alpha}q^{-1-(n-1)\sigma}(t) \int_{T_3}^{T_2} q^{-2-(n-1)\sigma}(t) \, dt \right)$$

$$\leq C_2((n - 1)b)^{(n-1)/\alpha}q^{-1-(n-1)\sigma}(T_3)$$

$$< C_2((n - 1)b)^{(n-1)/\alpha}q^{-1-(n-1)\sigma}(T_3)$$

Integrating this over the interval $[T_3, T_2]$, we then show that

$$v_n(T_2; x_n) \leq v_n(T_3; x_n)$$

$$+ \left( - m(0)(n + 1) + \left( \frac{1}{\alpha - 1} \right) \left( \frac{3}{2} \right) \frac{1}{(n - 1)b)^{(n-1)/\alpha}q^{-1-(n-1)\sigma}(T_3) \int_{T_3}^{T_2} q^{-2-(n-1)\sigma}(t) \, dt \right)$$

$$\leq C_2((n - 1)b)^{(n-1)/\alpha}q^{-1-(n-1)\sigma}(T_3)$$

$$+ \left( - m(0)(n + 1) + \left( \frac{1}{\alpha - 1} \right) \left( \frac{3}{2} \right) \frac{1}{(n - 1)b)^{(n-1)/\alpha}q^{-1-(n-1)\sigma}(T_3) \int_{T_3}^{T_2} q^{-2-(n-1)\sigma}(t) \, dt \right)$$

$$\leq C_2((n - 1)b)^{(n-1)/\alpha}q^{-1-(n-1)\sigma}(T_3)$$

$$+ \frac{1 + \epsilon}{1 + (n - 1)\sigma} C_2((n - 1)b)^{(n-1)/\alpha}q^{-1-(n-1)\sigma}(T_3)$$

$$\leq C_2((n - 1)b)^{(n-1)/\alpha}q^{-1-(n-1)\sigma}(T_3)$$

$$+ \frac{4}{2\sigma + 1} \frac{1 + \epsilon}{1 - \epsilon} C_2((n - 1)b)^{(n-1)/\alpha}q^{-1-(n-1)\sigma}(T_3)$$

$$< C_2((n - 1)b)^{(n-1)/\alpha}q^{-1-(n-1)\sigma}(T_2)$$

$$< C_2((n - 1)b)^{(n-1)/\alpha}q^{-1-(n-1)\sigma}(T_2).$$
Therefore (2.19) holds for \( n = 3, 4, \ldots \) throughout the interval \([0, T_2]\). Here, the second inequality uses (2.24) and (2.31), the third inequality uses (1.8), (1.9) and (1.10). Indeed

\[
-\epsilon m(0) > \frac{e}{1-\alpha} \left( \frac{3}{2} \right)^{1/\alpha - 1} C_2 + \frac{6}{\alpha} (1 + e^{1/b})b
\]

for \( \epsilon > 0 \) sufficiently small. The fourth inequality uses (2.21) and that \( \frac{(1+\epsilon)n+1}{n\sigma+1-\sigma} \) deceases in \( n \geq 3 \), and the last inequality uses (2.21) and Lemma 2.2. Indeed

\[
0 < \frac{4 + 3\epsilon + e}{2\sigma + 1} < 1.
\]

**Proof of (2.19) for \( n = 2 \).** Let \( |v_2(T_2; x_2)| = \max_{x \in \mathbb{R}} |v_2(T_2; x)| \). We may assume, without loss of generality, that \( v_2(T_2; x_2) > 0 \). We choose \( T_3 \) close to \( T_2 \) so that

\[
v_2(t; x_2) > 0 \quad \text{for all } t \in [T_3, T_2].
\]

We necessarily choose \( T_3 \) closer to \( T_2 \) so that (2.28) and, hence, (2.31) hold.

Suppose for now that \( x_2 \notin \Sigma_{1/3}(T_2) \), i.e. \( v_1(T_2; x_2) > \frac{2}{3} m(T_2) \) (see (2.11)). We may necessarily choose \( T_3 \) closer to \( T_2 \) so that

\[
v_1(t; x_2) \geq \frac{2}{3} m(t) \quad \text{for all } t \in [T_3, T_2].
\]

As a matter of fact, \( v_1 \) and \( m \) are uniformly continuous throughout the interval \([0, T_2]\). The proof is similar to that for \( n \geq 3 \). Specifically, it follows from (2.3) that

\[
\frac{d v_2}{d t}(t; x_2) = -3 v_1(t; x_2)v_2(t; x_2) - K_2(t; x_2)
\]

\[
\leq -2m(0)C_2 b^{1/\alpha} q^{-1}(t) q^{-1-\sigma}(t) + \frac{6}{\alpha} (1 + e^{1/\alpha})(2b) C_2 b^{1/\alpha} q^{-1-\sigma-\sigma}(t)
\]

\[
\leq 2 \left( -m(0) + \frac{6}{\alpha} (1 + e^{1/\alpha})b \right) C_2 b^{1/\alpha} q^{-2-\sigma}(t)
\]

for all \( t \in (T_3, T_2) \). The first inequality uses (2.35), (2.21), (2.2), and (2.27), and the second inequality uses Lemma 2.2 and (2.21). Integrating this over the interval \([T_3, T_2]\), we then show that

\[
v_2(T_2; x_2) < v_2(T_3; x_2) + 2 \left( -m(0) + \frac{6}{\alpha} (1 + e^{1/\alpha})b \right) C_2 b^{1/\alpha} \int_{T_3}^{T_2} q^{-2-\sigma}(t) \, dt
\]

\[
\leq C_2 b^{1/\alpha} q^{-1-\sigma}(T_3) - 2 \left( -m(0) + \frac{6}{\alpha} (1 + e^{1/\alpha})b \right) \times \frac{1}{1 + \epsilon} \frac{1}{\frac{1 + \epsilon}{\sigma} + \frac{1}{1 - \epsilon} m(0)} C_2 b^{1/\alpha} (q^{-1-\sigma}(T_2) - q^{-1-\sigma}(T_3))
\]

\[
\leq C_2 b^{1/\alpha} q^{-1-\sigma}(T_3) + \frac{2}{1 + \epsilon} (1 + \epsilon) C_2 b^{1/\alpha} (q^{-1-\sigma}(T_2) - q^{-1-\sigma}(T_3))
\]

\[
= \left( 1 - \frac{2}{1 + \epsilon} (1 + \epsilon) \right) C_2 b^{1/\alpha} q^{-1-\sigma}(T_3) + \frac{2}{1 + \epsilon} (1 + \epsilon) C_2 b^{1/\alpha} q^{-1-\sigma}(T_2)
\]

\[
< C_2 b^{1/\alpha} q^{-1-\sigma}(T_2).
\]
The second inequality uses (2.24) and (2.31), and the third inequality uses (1.8) and (1.9). Indeed
\[-\epsilon m(0) > \frac{6}{\alpha}(1 + e^{1/\alpha}b)\]
for \(\epsilon > 0\) sufficiently small. The last inequality uses (2.21) and Lemma 2.2. Indeed
\[0 < \frac{2}{1 + \sigma} \frac{(1 + \epsilon)^2}{1 - \epsilon} < 1.\]

Suppose on the other hand that \(x_2 \in \Sigma_{1/3}(T_2)\). It follows from Lemma 2.1 that
\[(2.36)\quad v_1(t; x_2) \leq \frac{2}{3} m(t) < 0 \quad \text{for all } t \in [0, T_2].\]

We shall explore the “smoothing effects” of the solution of (2.1).

Differentiating (2.1) with respect to \(x\) and recalling (2.2), we arrive at that
\[(2.37)\quad \frac{d}{dt}(\partial_x X) = v_1(\partial_x X), \quad (\partial_x X)(0; x) = 1\]
and
\[(2.38)\quad \frac{d}{dt}(\partial_x^2 X) = v_2(\partial_x X)^2 + v_1(\partial_x^2 X), \quad (\partial_x^2 X)(0; x) = 0,\]
\[(2.39)\quad \frac{d}{dt}(\partial_x^3 X) = v_3(\partial_x X)^3 + 3v_2(\partial_x X)(\partial_x^2 X) + v_1(\partial_x^3 X), \quad (\partial_x^3 X)(0; x) = 0.\]

Integrating (2.35), moreover, we show that
\[v_0(t; x) = u_0(x) - \int_0^t \phi_0(t; x) \, dt.\]

Differentiating it with respect to \(x\) and recalling (2.2), we then arrive at that
\[(2.40)\quad (v_2(\partial_x X)^2 + v_1(\partial_x^2 X))(t; x) = u_0''(x) - I_2(t; x),\]
\[(2.41)\quad (v_3(\partial_x X)^3 + 3v_2(\partial_x X)(\partial_x^2 X) + v_1(\partial_x^3 X))(t; x) = u_0'''(x) - I_3(t; x),\]
where
\[(2.42)\quad I_2(t; x) = \int_0^t (\phi_2(\partial_x X)^2 + \phi_1(\partial_x^2 X))(\tau; x) \, d\tau,\]
\[(2.43)\quad I_3(t; x) = \int_0^t (\phi_3(\partial_x X)^3 + 3\phi_2(\partial_x X)(\partial_x^2 X) + \phi_1(\partial_x^3 X))(\tau; x) \, d\tau.\]

Note from (2.39) and (2.41) that
\[(2.44)\quad \frac{d}{dt}(\partial_x^2 X)(\cdot; x) = u_0'''(x) - I_3(\cdot; x), \quad (\partial_x^2 X)(0; x) = 0.\]

We claim that
\[(2.45)\quad \frac{1}{2} q^{1+2r}(t) \leq (\partial_x X)(t; x_2) \leq 2q^{1-r}(t) \quad \text{for all } t \in [0, T_2].\]

As a matter of fact, it follows from (2.1), (2.37) and (2.13), (2.14) that
\[
\frac{1}{1 - \epsilon} \frac{dr/dt}{r} \leq \frac{d(\partial_x X)/dt}{\partial_x X} \leq \frac{1}{1 + \epsilon} \frac{dr/dt}{r}.
\]
throughout the interval \((0, T_2)\). Integrating this over the interval \([0, t]\) and recalling (2.37), we then show that

\[
\left( \frac{r(t)}{r(0)} \right)^{1/(1-\epsilon)} \leq (\partial_x X)(t; x_2) \leq \left( \frac{r(t)}{r(0)} \right)^{1/(1+\epsilon)}
\]

for all \(t \in [0, T_2]\). Therefore (2.45) follows from (2.15).

To proceed, we shall show that

\[
(\partial_x X)(t; x_2) < -\frac{8}{m(0)} C_2 b^{1/\alpha} q^{2-\sigma-2\epsilon}(t)
\]

and

\[
(\partial_x X)(t; x_2) < \frac{\epsilon}{m^2(0)} C_2^2 (2b)^{2/\alpha} q^{3-2\sigma+7\epsilon}(t)
\]

for all \(t \in [0, T_2]\). It follows from (2.35) and (2.39) that (2.46) and (2.47) hold at \(t = 0\). Suppose on the contrary that (2.46) and (2.47) hold throughout the interval \([0, T_4]\) but do not at \(t = T_4\) for some \(T_4 \in (0, T_2]\). By continuity, we find that

\[
(\partial_x X)(t; x_2) < \frac{8}{m(0)} C_2 b^{1/\alpha} q^{2-\sigma-2\epsilon}(t)
\]

and

\[
(\partial_x X)(t; x_2) \leq \frac{\epsilon}{m^2(0)} C_2^2 (2b)^{2/\alpha} q^{3-2\sigma+7\epsilon}(t)
\]

for all \(t \in [0, T_4]\). We seek a contradiction.

We use (2.42) to compute that

\[
I_2(t; x_2) \leq \int_0^t \left( \frac{4}{\alpha} (1 + e^{1/\alpha})(2b)C_2 b^{1/\alpha} q^{1-\sigma-\sigma}(\tau) q^{2-2\epsilon}(\tau) - \frac{8}{m(0)} C_2 b^{1/\alpha} q^{1-\sigma}(\tau) q^{2-2\epsilon}(\tau) \right) d\tau
\]

\[
\leq \frac{48}{\alpha} \left( (1 + e^{1/\alpha})b + 2 \left( 1 + \frac{C_2^2}{C_1} b^{1/\alpha} \right) \right) C_2 b^{1/\alpha} \int_0^t q^{-\sigma+8\epsilon}(\tau) d\tau
\]

\[
\leq - \frac{48}{\alpha} \left( (1 + e^{1/\alpha})b + 2 \left( 1 + \frac{C_2^2}{C_1} b^{1/\alpha} \right) \right) \times \frac{1}{\sigma - 1 - 8\epsilon} \frac{1}{m(0)} C_2 b^{1/\alpha} q^{1-\sigma+8\epsilon}(t) - (1 - \epsilon)^{\sigma-1-8\epsilon}
\]

(2.50)

\[
< \epsilon C_2 b^{1/\alpha} q^{1-\sigma+8\epsilon}(t)
\]

for all \(t \in [0, T_4]\). The first inequality uses (2.27), (2.45) and (2.26), (2.48), and the second inequality uses Lemma (2.22) and (2.21). Here one may assume, without loss of generality, that \(\|\phi''\|_{L^\infty} = -m(0)\). The third inequality use (2.16), and the last inequality uses (1.9) and (2.21). Indeed

\[
-\epsilon(1 - \epsilon)^{\sigma+1-8\epsilon} m(0) > \frac{48}{\alpha} \frac{1}{\sigma - 1 - 8\epsilon} \left( (1 + e^{1/\alpha})b + 2 \left( 1 + \frac{C_2^2}{C_1} b^{1/\alpha} \right) \right)
\]
for \( \epsilon > 0 \) sufficiently small. Evaluating (2.40) at \( t = T_4 \) and \( x = x_2 \), we then show that

\[
|\partial_x^2 X(T_4; x_2)| = |v_1^{-1}(T_4; x_2)|I_2(T_4; x_2) - v_2(T_4; x_2)(\partial_x X)(T_4; x_2)^2| \\
< - \frac{3}{2} \frac{1}{m(0)} q(T_4)(b^{1/\alpha} + \epsilon C_2 b^{1/\alpha} q^{-1-\sigma + 8\epsilon}(T_4) + 4 C_2 b^{1/\alpha} q^{-1-\sigma}(T_4) q^{2-2\epsilon}(T_4)) \\
\leq - \frac{3}{2} (5 + \epsilon) \frac{1}{m(0)} C_2 b^{1/\alpha} q^{2-\sigma - 2\epsilon}(T_4) \\
< - \frac{8}{m(0)} C_2 b^{1/\alpha} q^{2-\sigma - 2\epsilon}(T_4).
\]

Therefore (2.40) holds throughout the interval \([0, T_2]\). Here, the first inequality uses (2.27), (2.45), (2.48) and (2.26), (2.49), the second inequality uses (1.8), (2.20) and Lemma 2.2 (2.21), and the last inequality follows for \( \epsilon > 0 \) sufficiently small.

Similarly, we use (2.43) to compute that

\[
|I_3(t; x_2)| < \int_0^t \left( 6 \frac{m(0)}{\alpha} (1 + e^{1/\alpha})(3b) C_2 (2b)^{2/\alpha} q^{-1-\sigma - 2\epsilon}(\tau) q^{3-3\epsilon}(\tau) \\
- 48 \frac{m(0)}{\alpha} (1 + e^{1/\alpha})(2b) C_2 (2b)^{2/\alpha} \frac{1}{m(0)} q^{-1-\sigma - \sigma - \epsilon}(\tau) q^{1-\epsilon}(\tau) q^{2-\sigma - 2\epsilon}(\tau) \\
+ \frac{6}{\alpha} (C_1 + C_2 b^{1/\alpha}) \frac{\epsilon}{m^2(0)} C_2 (2b)^{2/\alpha} q^{-1-\sigma - \sigma - \epsilon}(\tau) q^{3-2\sigma + \tau\epsilon}(\tau) \right) d\tau \\
\leq \frac{12}{\alpha} \left( 12(1 + e^{1/\alpha}) b \left( \frac{1}{C_2} - \frac{1}{m(0)} \right) (1 + \frac{C_2}{C_1} b^{1/\alpha}) \frac{\epsilon}{m(0)} \right) \\
\times C_2 (2b)^{2/\alpha} \int_0^t q^{1-2\sigma + \tau\epsilon}(\tau) d\tau \\
\leq \frac{12}{\alpha} \left( 12(1 + e^{1/\alpha}) b \left( \frac{1}{C_2} - \frac{1}{m(0)} \right) (1 + \frac{C_2}{C_1} b^{1/\alpha}) \frac{\epsilon}{m(0)} \right) \\
\times \frac{1}{2\sigma - 2 - 7\epsilon} \frac{1}{m(0)} C_2 (2b)^{2/\alpha} (q^{2-\sigma - \tau\epsilon}(t) - (1 - \epsilon) q^{2-2\sigma - 2\epsilon} - 7\epsilon) \\
(2.51) \\
< - \frac{\epsilon^2}{m(0)} C_2 (2b)^{2/\alpha} q^{2-\sigma - \tau\epsilon}(t)
\]

for all \( t \in [0, T_4] \). The first inequality uses (2.27), (2.45), (2.48) and (2.26), (2.49), and the second inequality uses that (2.21) implies that \( 2 - \sigma \alpha - 2\sigma - 3\epsilon > 1 - 2\sigma + 7\epsilon \). Here one may assume, without loss of generality, that \( \|\phi\|_{L_\infty} = -m(0) \). The third inequality uses (2.16), and the last inequality uses (1.9). Indeed

\[
-\epsilon^2 (1 - \epsilon)^{2\sigma - 7\epsilon} m(0) > \frac{12}{\alpha} \frac{1}{2\sigma - 2 - 7\epsilon} \left( 12(1 + e^{1/\alpha}) b ((-m(0))^{1/4} + 2^{1/2} + \epsilon (1 + \frac{C_2}{C_1} b^{1/\alpha}) \right)
\]

WA VE BREAKING IN THE WHITHAM EQUATION 15
for $\epsilon > 0$ sufficiently small. Integrating (2.41) over the the interval $[0, T_4]$, we then show that

$$
|\langle \partial_x^3 X(T_4; x_2) \rangle| \leq \int_0^{T_4} \left( |\phi'''(x_2)| + |I_3(t; x_2)| \right) dt \\
< \int_0^{T_4} \left( (2b)^{2/\alpha} - \frac{\epsilon^2}{m(0)} C_2^2 (2b)^{2/\alpha} q^{2-2\sigma+7\epsilon}(t) \right) dt \\
\leq \left( \frac{1}{C_2^2} - \frac{\epsilon^2}{m(0)} \right) \frac{1}{2\sigma - 3 - 7\epsilon} \frac{1}{(1 - \epsilon)^{2\sigma - 1 - 7\epsilon}} m(0) \\
\times C_2^2 (2b)^{2/\alpha} q^{3-2\sigma+7\epsilon}(T_4) - (1 - \epsilon)^{2\sigma-3-7\epsilon}) \\
< \frac{\epsilon}{m^2(0)} C_2^2 (2b)^{2/\alpha} q^{3-2\sigma+7\epsilon}(T_4).
$$

Therefore (2.47) holds throughout the interval $[0, T_2]$. Here, the second inequality uses (1.14) and (2.51), the third inequality uses (2.16), and the last inequality uses (2.45), (2.36), (2.8), (2.49), (2.24) and (1.11), (2.50), (2.51), the last inequality uses (2.17), (2.18) and (2.19) hold for all $n = 0$ sufficiently small.

To summarize, a contradiction proves that (2.17), (2.18) and (2.19) hold for all $n = 0, 1, 2, \ldots$ throughout the interval $[0, T_1]$. 

To proceed, it follows from (2.15), (2.8) and (1.8) that
\[ |K_1(t; x)| \leq \frac{6}{\alpha}(C_1 + C_2b^{1/\alpha})m^{-2}(0)m^2(t) \leq \epsilon^2m^2(t) \]
for all \( t \in [0, T_1] \) for all \( x \in \mathbb{R} \). As a matter of fact, one may assume, without loss of generality, that \( \|\phi\|_{L^\infty} = -m(0) \) and
\[ -\epsilon^2m(0) > \frac{12}{\alpha}(1 + \frac{C_2}{C_1}b^{1/\alpha}) \]
for \( \epsilon > 0 \) sufficiently small. A contradiction therefore proves (2.9). Furthermore (2.17), (2.18), (2.19) hold for all \( n = 0, 1, 2, \ldots \) throughout the interval \([0, T']\) for all \( T' < T \).

To conclude, let \( x \in \Sigma_\epsilon(t) \) for \( t \in [0, T] \). It follows from (2.13) and (2.14) that
\[ m(0)(v_1^{-1}(0; x) + (1 + \epsilon)t) \leq r(t; x) \leq m(0)(v_1^{-1}(0; x) + (1 - \epsilon)t). \]
Moreover it follows from Lemma 2.1 that \( m(0) < v_1(0; x) \leq (1 - \epsilon)m(0). \) Hence
\[ 1 + m(0)(1 + \epsilon)t \leq r(t) \leq \frac{1}{1 - \epsilon} + m(0)(1 - \epsilon)t \]
Furthermore it follows from (2.15) that
\[ (1 - \epsilon) + m(0)(1 - \epsilon^2)t \leq q(t) \leq \frac{1}{1 - \epsilon} + m(0)(1 - \epsilon)t. \]

Since the function on the left side decreases to zero as \( t \to -\frac{1}{m(0)} \frac{1}{1 + \epsilon} \) and since the function on the right side decreases to zero as \( t \to -\frac{1}{m(0)} \frac{1}{(1 - \epsilon)^2} \), therefore, \( q(t) \to 0 \) and, hence (see (2.8)), \( m(t) \to -\infty \) as \( t \to T^- \), where \( T \) satisfies (1.12). On the other hand, (2.17) dictates that \( v_0(t; x) \) remains bounded for all \( t \in [0, T'] \), \( T' < T \), for all \( x \in \mathbb{R} \). In other words, \( \inf_{x \in \mathbb{R}} \partial_x u(x, t) \to -\infty \) as \( t \to T^- \) but \( u(x, t) \) is bounded for all \( x \in \mathbb{R} \) for all \( t \in [0, T] \). This completes the proof.

3. Proof of Theorem 1.2

We assume that the initial value problem associated with (1.5) - (1.6), where \( c = c_{W} \) (see (1.2)), and \( u(\cdot, 0) = u_0 \) possesses a unique solution in \( C^\infty([0, T]; H^\infty(\mathbb{R})) \) for some \( T > 0 \). As a matter of fact, one may work out the local in time well-posedness in \( H^{3/2+}(\mathbb{R}) \). Without recourse to the dispersion effects, the proof is nearly identical to that for (1.7), and hence we omit the detail. We assume that \( T \) is the maximal time of existence.

Note that \( K(x) \) is even and vanishes as \( |x| \to \infty \) faster than any polynomial. Since its Fourier transform \( c_{W}(\kappa) = \sqrt{\text{tanh} \kappa / \kappa} \) behaves like \( |\kappa|^{-1/2} \) as \( |\kappa| \to \infty \), Whitham in [Whi74] heuristically argued that \( K(x) \) would behave like \( |x|^{-1/2} \) as \( |x| \to 0 \). One may, indeed, analytically confirm it.

Lemma 3.1. It follows that
\[ K(x) \sim \frac{1}{\sqrt{2\pi|x|}} \quad \text{and} \quad K'(x) \sim \frac{-\text{sgn}(x)}{2\sqrt{2\pi|x|^3}} \]
as \( |x| \to 0 \).
We include the proof by Ehrnström in Appendix [B] for completeness.

Therefore
\begin{equation}
K(x) \leq \frac{K_0}{\sqrt{|x|}} \quad \text{and} \quad K'(x) \leq \frac{K_0}{|x|^3}
\end{equation}
for some $K_0 > 0$ a constant and
\begin{equation}
\int_{\delta_0}^{\infty} |K'(x)| \, dx \leq K_\infty
\end{equation}
for some $K_\infty > 0$ a constant for some $0 < \delta_0 < 1$.

Recall the notation of the Section [2] where
\begin{equation}
\phi_n(t; x) = \int_{-\infty}^{\infty} K(y)(\partial_x^{n+1}u)(X(t; x) - y, t) \, dy
\end{equation}
instead of (2.6). Since $u(x, t)$ is smooth and in $L^2$ in $x$ and smooth in $t$ and $X(t; x)$ is continuously differentiable in $t$ and smooth in $x$ for all $x \in \mathbb{R}$ for all $t \in [0, T)$ and since $K$ is in $L^2$, it follows that $\phi_n(t, x)$ is continuously differentiable in $t$ and smooth and uniformly bounded in $x$ for all $t \in [0, T)$ for all $x \in \mathbb{R}$.

For $0 < \delta < \delta_0$, we split the integral and perform an integration by parts to show that
\begin{equation}
|\phi_n(t; x)| \leq \left| \left( \int_{|y| < \delta} + \int_{|y| > \delta} \right) K(y)(\partial_x^{n+1}u)(X(t; x) - y, t) \, dy \right|
\leq \int_{|y| < \delta} K(y)(\partial_x^{n+1}u)(X(t; x) - y, t) \, dy
\quad + \left| K(\delta)(\partial_x^n u)(X(t; x) - \delta, t) - K(-\delta)(\partial_x^n u)(X(t; x) + \delta, t) \right|
\quad + \int_{|y| > \delta} K'(y)(\partial_x^n u)(X(t; x) - y, t) \, dy
\leq \left( \int_{|y| < \delta} \frac{K_0}{\sqrt{|y|}} \, dy \right) \|v_{n+1}(t)\|_{L^\infty} + 2|K(\delta)|\|v_n(t)\|_{L^\infty}
\quad + \left( \int_{\delta < |y| < \delta_0} \frac{K_0}{\sqrt{|y|}^3} \, dy + \int_{|y| > \delta_0} |K'(y)| \, dy \right) \|v_n(t)\|_{L^\infty}
\leq 4K_0\delta^{1/2}\|v_{n+1}(t)\|_{L^\infty} + 2K_0\delta^{-1/2}\|v_n(t)\|_{L^\infty}
\quad + (4K_0(\delta^{-1/2} - \delta_0^{-1/2}) + 2K_\infty)\|v_n(t)\|_{L^\infty}
\leq C(\delta^{-1/2}\|v_n(t)\|_{L^\infty} + \delta^{1/2}\|v_{n+1}(t)\|_{L^\infty})
\end{equation}
for some $C > 0$ a constant for $n = 0, 1, 2, \ldots$ and for all $t \in [0, T)$ for all $x \in \mathbb{R}$. The first inequality uses (3.1), the second inequality uses (3.2), and the last inequality uses that $0 < \delta_0 < 1$. We merely pause to remark that $K$ is integrable near zero, although $K(0)$ does not exist (and hence the arguments in [Sel68] and [CE98] do not apply) and $K'$ is not integrable near zero (and hence the argument in [NS94] may not apply).

We are done if
\begin{equation}
|\phi_1(t; x)| < c^2 m^2(t) \quad \text{for all } t \in [0, T) \quad \text{for all } x \in \mathbb{R}
\end{equation}
for $\epsilon > 0$ sufficiently small. It follows from (1.13) and the Sobolev inequality that
\[
|\phi_1(0; x)| = \left| \int_{-\infty}^{\infty} K(x - y)u_0'(y) \, dy \right| \leq \|u_0\|_{H^2} < \epsilon^2 m^2(0) \quad \text{for all } x \in \mathbb{R}.
\]
In other words, (3.4) holds at $t = 0$. Suppose on the contrary that $|\phi_1(T_1; x)| = \epsilon^2 m^2(T_1)$ for some $T_1 \in (0, T)$ for some $x \in \mathbb{R}$. By continuity, we may assume that
\[
|\phi_1(t; x)| < \epsilon^2 m^2(t) \quad \text{for all } t \in [0, T_1] \quad \text{for all } x \in \mathbb{R}.
\]
Under the assumption, we rerun the argument in the previous section to show that Lemma 2.1, Lemma 2.2, Lemma 2.3 hold.

We claim (2.17), (2.18) and (2.19) hold, where $\sigma$ is in (2.19) but $\alpha = 1/2$. It follows from (2.20), (2.3) and (1.14) that (2.17), (2.18) and (2.19) hold for all $n = 0, 1, 2, \ldots$ at $t = 0$. Suppose on the contrary that (2.17), (2.18) and (2.19) hold for all $n = 0, 1, 2, \ldots$ throughout the interval $[0, T_2]$ but do not for some $n \geq 0$ at $t = T_2$ for some $T_2 \in (0, T_1]$. By continuity, (2.22), (2.23) and (2.24) hold for all $n = 0, 1, 2, \ldots$ for all $t \in [0, T_2]$.

For $n = 0$, it follows from (3.3), where $\delta(t) = \delta_0 q(t)$, and (2.22), (2.23) that
\[
|\phi_0(t; x)| \leq C(C_0 \delta_0^{-1/2} q^{-1/2}(t) + C_1 \delta_1^{1/2} q^{1/2}(t) q^{-1}(t) < C\delta_0^{-1/2}(C_0 + C_1) q^{-1}(t)
\]
for all $t \in [0, T_2]$ for all $x \in \mathbb{R}$. The last inequality uses that $0 < \delta_0 < 1$. For $n = 1$, similarly, it follows from (3.3), where $\delta(t) = \delta_0 q^\sigma(t)$, and (2.23), (2.24), where $\alpha = 1/2$, that
\[
|\phi_1(t; x)| \leq C(C_1 \delta_0^{-1/2} q^{-\sigma/2}(t) q^{-1}(t) + C_2 b^{1/2} q^{\sigma/2}(t) q^{-1-\sigma}(t))
\leq C\delta_0^{-1/2}(C_1 + C_2 b^{1/2}) q^{-1-\sigma/2}(t)
\]
for all $t \in [0, T_2]$ for all $x \in \mathbb{R}$. The last inequality, similarly, uses that $0 < \delta_0 < 1$. For $n \geq 2$, moreover, it follows from (3.3), where $\delta(t) = \delta_0(n b)^{-2} q^{\sigma}(t)$, and (2.24), where $\alpha = 1/2$, that
\[
|\phi_n(t; x)| \leq C(C_1 \delta_0^{-1/2} C_2 (n - 1) b^{2(n-1)} q^{-\sigma/2}(t) q^{-1-(n-1)\sigma}(t)
\leq C\delta_0^{-1/2}(1 + \epsilon^2)(nb) C_2 ((n - 1) b)^{2(n-1)} q^{-1-\sigma/2-(n-1)\sigma}(t))
\]
for all $t \in [0, T_2]$ for all $x \in \mathbb{R}$. The last inequality, similarly, uses that $0 < \delta_0 < 1$.

We may rerun the argument in the previous section but we use (3.5), (3.6), (3.7) instead of (2.25), (2.26), (2.27), respectively, and necessarily choose $\epsilon > 0$ smaller in various places, to draw a contradiction. The detail is nearly identical to that in the previous section. Hence we omit the detail. To conclude, (2.17), (2.18) and (2.19) hold for all $n = 0, 1, 2, \ldots$ throughout the interval $[0, T_1]$, where $\alpha = 1/2$.

To proceed, we necessarily choose $\epsilon$ smaller and it follows from (3.6) and (2.8) that
\[
|\phi_1(t; x)| \leq C\delta_0^{-1/2}(C_1 + C_2 b^2) m^{-2}(0) m^2(t) < \epsilon^2 m^2(t)
\]
for all $t \in [0, T_1]$ for all $x \in \mathbb{R}$. A contradiction therefore proves (3.4).
The remainder of the proof is nearly identical to that in the previous section. Hence we omit the detail.

**Acknowledgment.** The author thanks Mats Ehrnström for helpful discussions. She is supported by the National Science Foundation grant CAREER DMS-1352597, an Alfred P. Sloan research fellowship, and a Beckman fellowship of the Center for Advanced Study at the University of Illinois at Urbana-Champaign. She thanks the Mathematisches Forschungsinstitut Oberwolfach for its hospitality during the workshop “Mathematical Theory of Water Waves,” where part of the research was carried out.

**APPENDIX A. ASSORTED PROOFS OF LEMMAS**

**Proof of Lemma 2.1.** Suppose on the contrary that $x_1 \notin \Sigma_\gamma(t_1)$ but $x_1 \in \Sigma_\gamma(t_2)$ for some $x_1 \in \mathbb{R}$ for some $0 \leq t_1 \leq t_2 \leq T_1$, i.e.

(A.1) \[ v_1(t_1; x_1) > (1 - \gamma)m(t_1) \quad \text{and} \quad v_1(t_2; x_1) \leq (1 - \gamma)m(t_2) < \frac{1}{2}m(t_2). \]

We may choose $t_1$ and $t_2$ close so that

\[ v_1(t; x_1) < \frac{1}{2}m(t) \quad \text{for all } t \in [t_1, t_2]. \]

As a matter of fact, $v_1(\cdot; x_1)$ and $m$ are uniformly continuous throughout the interval $[0, T_1]$. Let

(A.2) \[ v_1(t_1; x_2) = m(t_1) < \frac{1}{2}m(t_1). \]

We may necessarily choose $t_2$ closer to $t_1$ so that

\[ v_1(t; x_2) < \frac{1}{2}m(t) \quad \text{for all } t \in [t_1, t_2]. \]

For $\epsilon > 0$ sufficiently small, it follows from (2.10) that

\[ |\phi_1(t; x_j)| \leq \epsilon^2 m^2(t) \leq 4 \epsilon^2 v_1^2(t; x_j) < \frac{2}{2}v_1^2(t; x_j) \quad \text{for all } t \in [t_1, t_2] \text{ and } j = 1, 2. \]

To proceed, note from (2.3) that

\[ \frac{dv_1}{dt}(\cdot; x_1) = -v_1^2(\cdot; x_1) - \phi_1(\cdot; x_1) \geq \left( -1 - \frac{\gamma}{2} \right) v_1^2(\cdot; x_1) \]

and

\[ \frac{dv_1}{dt}(\cdot; x_2) \leq \left( -1 + \frac{\gamma}{2} \right) v_1^2(\cdot; x_2) \]

throughout the interval $(t_1, t_2)$. Integrating them over the interval $[t_1, t_2]$, we arrive at that

\[ v_1(t_2; x_1) \geq \frac{v_1(t_1; x_1)}{1 + (1 + \frac{\gamma}{2})v_1(t_1; x_1)(t_2 - t_1)} \quad \text{and} \quad v_1(t_2; x_2) \leq \frac{v_1(t_1; x_2)}{1 + (1 - \frac{\gamma}{2})v_1(t_1; x_2)(t_2 - t_1)}. \]

The latter inequality and (A.2) imply that

\[ m(t_2) \leq \frac{m(t_1)}{1 + (1 - \frac{\gamma}{2})m(t_1)(t_2 - t_1)}. \]
The former inequality and (A.1), on the other hand, imply that
\[
v_1(t_2; x_1) > \frac{(1 - \gamma)m(t_1)}{1 + (1 + \frac{1}{2})(1 - \gamma)m(t_2 - t_1)} > \frac{(1 - \gamma)m(t_1)}{1 + (1 - \frac{7}{2})m(t_2 - t_1)} \geq (1 - \gamma)m(t_2).
\]

A contradiction therefore completes the proof. □

Proof of Lemma 2.4. We use Stirling’s inequality to compute that
\[
\sum_{j=2}^{n-1} \binom{n}{j} (j-1)^{(1-1)/\alpha}(n-j)^{(n-j)/\alpha} \\
\leq \sum_{j=2}^{n-1} \frac{n^n}{j^j(n-j)^{n-j}} (j-1)^{(1-1)/\alpha}(n-j)^{(n-j)/\alpha} \\
= n \left( \frac{n}{n-1} \right)^{n-1} (n-1)^{(n-1)/\alpha} \sum_{j=2}^{n-1} \frac{1}{j} \left( \frac{j-1}{n-1} \right)^{1/\alpha-1} \\
\leq en(n-1)^{(n-1)/\alpha} \sum_{j=2}^{n-1} \frac{1}{j} \left( \frac{j-1}{n-1} \right)^{1/\alpha-1} \\
\leq en(n-1)^{(n-1)/\alpha} \left( \frac{1}{n-1} \right)^{1/\alpha-1} \int_1^n y^{1/\alpha-2} dy \\
\leq \frac{e}{1/\alpha - 1} n(n-1)^{(n-1)/\alpha} \left( \frac{n}{n-1} \right)^{1/\alpha-1}.
\]
The last inequality uses that 0 < α < 2/3. □

APPENDIX B. PROOF OF LEMMA 3.1

The proof below is by Ehrnström.

Let’s write
\[
K(x) = \frac{1}{2\pi} \lim_{\epsilon \to 0} \int_{-1/\epsilon}^{1/\epsilon} \sqrt{\frac{\tan \kappa}{\kappa}} e^{ix\kappa} d\kappa.
\]
For x > 0, we make a straightforward calculation to show that
\[
\sqrt{2\pi x} K(x) = \sqrt{\frac{x}{2\pi}} \lim_{\epsilon \to 0} \int_{-1/\epsilon}^{1/\epsilon} \sqrt{\frac{\tan \kappa}{\kappa}} e^{ix\kappa} d\kappa = \sqrt{\frac{2}{\pi}} \lim_{\epsilon \to 0} \int_{0}^{1/\epsilon} \sqrt{\frac{\tan \kappa}{\kappa}} \cos(x\kappa) d\kappa
\]
\[
= - \frac{1}{\sqrt{2\pi x}} \lim_{\epsilon \to 0} \int_{0}^{1/\epsilon} \frac{1}{\sqrt{\kappa}} \left( \frac{2\kappa}{\sinh(2\kappa)} - 1 \right) \sqrt{\tan \kappa} \sin(x\kappa) d\kappa
\]
\[
= \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} \frac{\sin \zeta}{\sqrt{\zeta}} \frac{2}{\zeta} f_0(\zeta/x) d\zeta,
\]
where
\[ f_0(z) = \left(1 - \frac{2z}{\sinh(2z)}\right) \sqrt{\tanh z}. \]

Since \( f_0(z) \) is bounded for all \( z \in \mathbb{R} \) and \( f_0(z) \to 1 \) as \( z \to \infty \) and since it is well known that
\[ \int_0^\infty \frac{\sin z}{\sqrt{z^3}} \, dz = \sqrt{2\pi}, \]
it follows from Lebegues’ dominated convergence theorem that
\[ \lim_{x \to 0^+} \sqrt{2\pi x} K(x) = 1. \]

For \( x > 0 \), moreover,
\[ K(x) = \frac{1}{2\pi} \frac{1}{\sqrt{x}} \int_0^\infty \frac{\sin \zeta}{\sqrt{\zeta}} f_0(\zeta/x) \, d\zeta = \frac{1}{2\pi} \frac{1}{\zeta^2} \int_0^\infty \frac{\sin \zeta}{\sqrt{\zeta}} f_1(\zeta/x) \, d\zeta. \]

We then make another straightforward calculation to show that
\[ \sqrt{x^3} K'(x) = -\frac{1}{2\pi} \int_0^\infty \frac{\sin \zeta}{\sqrt{\zeta}} f_1(\zeta/x) \, d\zeta, \]
where \( f_1(z) = \sqrt{z} (\sqrt{z} f(z))' \). Since \( f_1(z) \) is bounded for all \( z \in \mathbb{R} \) and \( f_1(z) \to 1/2 \) as \( z \to \infty \), similarly, it follows from Lebegues’ dominated convergence theorem that
\[ \lim_{x \to 0^+} \sqrt{2\pi x^3} K'(x) = -1/2. \] This completes the proof.

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