Two-photon Rabi–Stark model

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Abstract

The analytically exact solutions to the two-photon Rabi–Stark model are given by using the Bogoliubov operators approach. Transcendental functions responsible for the exact solutions are derived. The zeros of the transcendental functions reproduce completely the regular spectra. The first-order quantum phase transitions are also detected by the pole structure of the derived transcendental functions, similar to the one-photon Rabi–Stark model, different from the two-photon Rabi model where the energy levels for the ground-state and the first excited state do not cross. The spectra collapse characteristics resemble those in two-photon Rabi model. Some low lying levels can split off from the collapse energy, which is different from the one-photon Rabi–Stark model in which all levels collapse.

Keywords: two-photon Rabi–Stark model, Bogoliubov operators approach, analytic solutions

(Some figures may appear in colour only in the online journal)

1. Introduction

The most simple model in the light–matter interaction systems is the semi-classical Rabi model [1] which describes the interaction of a two-level atom and a classical light field. If the classical light field is treated as a quantized mode of an optical cavity, it was known as quantum Rabi model (QRM). However the QRM was hardly treated analytically at the early stage. To solve this model readily, one employs the rotating wave approximation (RWA) [2] so that the excitations of the atom and the photon number are conserved. Fortunately, this approximation was shown to be very effective in the earlier cavity quantum electrodynamics (QED) system [3] at the extremely weak coupling regime for nearly resonant interaction. The basic physics
realized and observed in the earlier experiments can be described by the QRM in the RWA, such as the phenomenon of vacuum Rabi splitting, and collapse and revivals in the population dynamics. However, in the past decade, with the progress of the advanced solid devices, such as a superconducting qubit-oscillator circuit and trapped ions, the ultrastrong coupling [4, 5], even deep strong coupling [6, 7] between the artificial atom and the quantum oscillator have been accessed, the consideration of the full QRM should be highly called for. On the other hand, the two-level system is just a qubit, which is the building block of quantum information science and quantum computations. Just motivated by the experimental developments and potential applications in quantum information technologies, the QRM without the RWA has attracted extensive attentions theoretically [8–17]. For a more complete review, please refer to [18–22].

The QRM and its variants are continuously employed to describe the physics phenomena observed in the recent experiments or quantum simulations. A nonlinear Stark coupling between the atom and the cavity was first generated in an atom interacting with a high finesse optical mode by applying two laser fields [23, 24]. The QRM with this Stark term was later named the quantum Rabi–Stark model (RSM) [25]. This model also attracts many attentions in recent years [25–29]. The nonlinear Stark coupling in the RSM can induce the spectra collapse as well as the first-order quantum phase transition (QPT).

On the other hand, the other nonlinear coupling in the form of the qubit and two photons is also a subject of interest for a long time [10, 30–39]. In contrast to its one-photon counterpart, some peculiar properties can be driven by the two-photon coupling, such as the spectra collapse. This so-called two-photon QRM (tpQRM) becomes a hot topic recently [40–52]. The more recent extensions include the two-photon Dicke model [53–57] and multiphoton and nonlinear dissipative spin-boson models [58].

Since both kinds of the nonlinear coupling can induce the spectra collapse, what is the cooperated effect? Can the nonlinear Stark coupling induce the first order QPT in the tpQRM? In this work, we add the Stark coupling terms to the two-photon Rabi model, which can be named two-photon RSM (tpRSM). The analytical solutions to tpRSM should be very interesting at the present stage. Some crucial issue mentioned above should be addressed.

The tpRSM would be possibly realized in the solid devices such as the trapped ions, circuit QED systems. Actually the implementation of the one-photon RSM using a single trapped ion has been demonstrated recently [29]. In this proposal, external laser beams are applied to induce an interaction between an electronic transition and the motional degree of freedom, thus the Stark term is generated. Since the tpQRM has been realized in the trapped ions [37], a route to simulate the tpRSM is also experimentally feasible with a laser driving in the same device.

The paper is structured as follows: in section 2, we extend the previous Bogoliubov operator approach to the tpRSM. A transcendent function is derived in a compact way, which can give the exact regular spectra of this model. The pole structure of the derived transcendent function is analyzed in section 3, and thus the level crossings and the energy spectra collapse are also discussed. The last section contains some concluding remarks.

2. Solutions to the two-photon Rabi–Stark model

The Hamiltonian of the tpRSM is given by

\[
H = -\left(\frac{1}{2}\Delta + Ua^\dagger a\right)\sigma_x + \omega a^\dagger a + g \left[(a^\dagger)^2 + a^2\right] \sigma_z,
\]  

(1)
where $\Delta$ is the qubit energy difference, $a^\dagger$ ($a$) is the photonic creation (annihilation) operator of the single-mode cavity with frequency $\omega$, $g$ is the two-photon coupling constants, $U$ is the nonlinear Stark coupling strength, and $\sigma_x (k = x, y, z)$ are the Pauli matrices.

The Hamiltonian (1) is also invariant under a discrete symmetry: $P_1 H P_1^\dagger = H$ where $P_1 = \exp(-\frac{1}{2} a^\dagger a) \sigma_z$, similar to the tpQRM. The operator $P_1$ generates the group $\mathbb{Z}_4$, therefore the Hilbert space separates into four invariant subspaces. Because $P_2^2 = 1$, with $P_2^2 = 1$, we have also a $\mathbb{Z}_2$-symmetry only in the bosonic Hilbert space. The invariance under $P_2$ is reflected in the presence of two inequivalent representations of $su(1, 1)$.

On the basis of $\sigma_z$, the tpRSM Hamiltonian can be written in the following matrix form in units of $\omega = 1$

$$ H = \begin{pmatrix} a^\dagger a + g \left( (a^\dagger)^2 + a^2 \right) - \left( \frac{1}{2} \Delta + U a^\dagger a \right) \\ - \left( \frac{1}{2} \Delta + U a^\dagger a \right) a^\dagger a - g \left( (a^\dagger)^2 + a^2 \right) \end{pmatrix}. \quad (2) $$

We begin by performing the following Bogoliubov transformation on the bosonic degree of freedom,

$$ b = u a + v a^\dagger, \quad b^\dagger = u a^\dagger + v a, \quad (3) $$

where

$$ u = \sqrt{\frac{1 + \beta}{2 \beta}}, \quad v = \sqrt{\frac{1 - \beta}{2 \beta}}, $$

with $\beta = \sqrt{1 - 4 g^2}$ assumed to be nonnegative. In the tpQRM [38], we need set $\gamma = g$, so that one diagonal element of the Hamiltonian matrix can be transformed into the free particle form, facilitating the further study. In the tpRSM, due to the presence of the Stark coupling terms, this requirement might be relaxed. We will see that $\gamma$ is indeed not equal to $g$, and also depends on the nonlinear Stark coupling strength $U$ (cf equation (13) below).

In terms of the new operator $b^\dagger (b)$, the upper and lower diagonal matrix elements of the Hamiltonian can be written as

$$ H_{11} = (u^2 + v^2 - 4 g u v) b^\dagger b - \frac{\gamma - g}{\beta} \left( b^\dagger b + b b^\dagger \right) - 2 g u v + v^2, $$

$$ H_{22} = (u^2 + v^2 + 4 g u v) b^\dagger b - \frac{\gamma + g}{\beta} \left( b^\dagger b + b b^\dagger \right) + 2 g u v + v^2. $$

The off-diagonal matrix elements read

$$ H_{12} = H_{21} = - \frac{1}{2} \Delta - \frac{1}{2} U \left[ (u^2 + v^2) b^\dagger b - u v (b^\dagger b + b b^\dagger) + v^2 \right]. $$

The operators $b^\dagger b, (b^\dagger)^2, b^2$ provide a representation of the non-compact Lie algebra $su(1, 1)$ with

$$ K_0 = \frac{1}{2} \left( b^\dagger b + \frac{1}{2} \right), \quad K_+ = \frac{1}{2} (b^\dagger)^2, \quad K_- = \frac{1}{2} b^2, \quad (4) $$

we have

$$ [K_0, K_\pm] = \pm K_\pm, \quad [K_+, K_-] = -2 K_0. $$
The quadratic Casimir operator $C_2$ of the algebra is given by
\[
C_2 = K_+ K_- - K_0 (K_0 - 1).
\]
In terms of the $K_0, K_\pm$, the diagonal elements in the Hamiltonian read
\[
H_{11} = \frac{1 - 4g\gamma}{\beta} 2K_0 = \frac{\gamma - g}{\beta} 2(K_+ + K_-) - \frac{1}{2},
\]
\[
H_{22} = \frac{1 + 4g\gamma}{\beta} 2K_0 = \frac{\gamma + g}{\beta} 2(K_+ + K_-) - \frac{1}{2}.
\]
The off-diagonal matrix elements are
\[
H_{12} = -\Delta - U \frac{2U}{\beta} [K_0 - \gamma(K_+ + K_-)].
\]
(5)
The unitary oscillator representation 4 of $su(1,1)$ in the Hilbert space $\mathcal{H}$ generated by $b^\dagger$ on the state $|0\rangle_b$, defined as $b|0\rangle_b = 0$, decays into two irreducible representations, characterized by their lowest weight $q$: $K_0 q, 0 \rangle_b = q q, 0 \rangle_b$. For the even subspace $\mathcal{H}_\frac{1}{2} = \{b^{2n}|0\rangle_b, \ n = 0, 2, 4, \ldots \}$, we have $q = \frac{1}{4}$ and for $\mathcal{H}_\frac{3}{2} = \{b^{2n}|0\rangle_b, \ n = 1, 3, 5, \ldots \}$, $q = \frac{3}{4}$. $C_2 = \frac{3}{16}$ in both cases. $\mathcal{H}$ separates therefore into two subspaces, $\mathcal{H} = \mathcal{H}_{\frac{1}{2}} \oplus \mathcal{H}_{\frac{3}{2}}$. The Bargmann index $q$ allows us to deal with both cases independently. A basis of $\mathcal{H}_q$ is given by the normalized states
\[
|q, n\rangle_b = (b^\dagger)^{2(n+q-\frac{1}{4})} |0\rangle_b = \sqrt{\frac{1}{n!}}\ n^{n+q-\frac{1}{4}} q^{\frac{n}{4}}, \quad n = 0, 1, 2, \ldots \infty.
\]
The operators satisfy
\[
K_+ |q, n\rangle_b = \sqrt{\left( n + q + \frac{3}{4} \right) \left( n + q + \frac{1}{4} \right)} |q, n+1\rangle_b,
\]
\[
K_- |q, n\rangle_b = \sqrt{\left( n + q - \frac{1}{4} \right) \left( n + q - \frac{3}{4} \right)} |q, n-1\rangle_b,
\]
\[
K_0 |q, n\rangle_b = (n + q) |q, n\rangle_b.
\]
(6)

Note that the vacuum with respect to the original boson operators $a, a^\dagger$, $|0\rangle_a$, with the property $a|0\rangle_a = 0$, may be expressed in terms of $|\frac{1}{4}, n\rangle_b$ as
\[
|0\rangle_a = \sum_{n=0}^{\infty} z_n\left(\frac{1}{4}\right)\frac{1}{4}, n \right)_b,
\]
because the decomposition $\mathcal{H} = \mathcal{H}_{\frac{1}{2}} \oplus \mathcal{H}_{\frac{3}{2}}$ is left invariant by the Bogoliubov transformation (3). We can write therefore $|0\rangle_a = |\frac{1}{4}, 0\rangle_a$. The condition $a|0\rangle_a = 0$ gives
\[ z_n \propto \frac{\sqrt{(2n)!}}{n!} \left( \frac{v}{2u} \right)^n. \] (7)

The lowest lying state (with respect to the \( a \)-operators) in \( H_{\frac{3}{4}} \) reads then

\[ |\frac{3}{4}, 0\rangle_a = a^\dagger |\frac{1}{4}, 0\rangle_a = (ub^\dagger - vb) \sum_{n=0}^\infty z_n^{(\frac{1}{4})} |\frac{1}{4}, n\rangle_b = \sum_{n=0}^\infty z_n^{(\frac{1}{4})} |\frac{3}{4}, n\rangle_b, \]

where

\[ z_n^{(\frac{1}{4})} \propto \frac{\sqrt{(2n+1)!}}{n!} \left( \frac{v}{2u} \right)^n. \] (8)

In summary,

\[ z_n^{(q)} \propto \frac{\sqrt{[2(n+q+\frac{1}{4})]!}}{n!} \left( \frac{v}{2u} \right)^n. \] (9)

An eigenfunction \( |\psi, E\rangle^{(q)} \) of \( H \) with eigenvalue \( E \) may be expanded in terms of the \( b \)-operators as

\[
|\psi, E\rangle^{(q)} = \left( \sum_{m=0}^\infty \sqrt{\left[ 2 \left( m+q+\frac{1}{4} \right) \right]!} e_m^{(q)} |q, m\rangle_b \right) \left( \sum_{m=0}^\infty \sqrt{\left[ 2 \left( m+q+\frac{1}{4} \right) \right]!} f_m^{(q)} |q, m\rangle_b \right),
\]

where \( e_m^{(q)} \) and \( f_m^{(q)} \) are coefficients to be determined later. The Schrödinger equations are given by

\[
\sum_{m=0}^\infty \left[ \frac{1 - 4g^2}{\beta} \left( n + q \right) - \frac{1}{2} - E \right] \sqrt{\left[ 2 \left( m + q + \frac{1}{4} \right) \right]!} e_m^{(q)} |q, m\rangle_b
- \frac{\gamma - g}{\beta} \sum_{m=0}^\infty \left\{ e_m^{(q)} + \left[ 2 \left( m + 1 + q - \frac{1}{4} \right) \right] \left[ 2 \left( m + 1 + q - \frac{3}{4} \right) \right] e_{m+1}^{(q)} \right\}
\sqrt{\left[ 2 \left( m + q + \frac{1}{4} \right) \right]!} |q, m\rangle_b
- \sum_{m=0}^\infty \left[ \frac{\Delta - U}{2} + 2U \left( n + q \right) \right] \sqrt{\left[ 2 \left( m + q + \frac{1}{4} \right) \right]!} f_m^{(q)} |q, m\rangle_b
+ \frac{U\gamma}{\beta} \sum_{m=0}^\infty \left\{ f_m^{(q)} + \left[ 2 \left( m + 1 + q - \frac{1}{4} \right) \right] \left[ 2 \left( m + 1 + q - \frac{3}{4} \right) \right] f_{m+1}^{(q)} \right\}
\sqrt{\left[ 2 \left( m + q + \frac{1}{4} \right) \right]!} f_m^{(q)} |q, m\rangle_b
= 0,
\]
for the upper level, and

\[-\sum_{m=0} \left[ \frac{\Delta - U}{2} + \frac{2U}{\beta} (n + q) \right] \sqrt{\frac{2}{2}} (m + q - \frac{1}{4}) \] \[\left| e_{m}^{(q)} | q, m \rangle \right|_{b} \]

\[\frac{U_{\gamma}}{\beta} \sum_{m=0} \left\{ e_{m-1}^{(q)} + \left[ 2 \left( m + 1 + q - \frac{1}{4} \right) \right] \left[ 2 \left( m + 1 + q - \frac{3}{4} \right) \right] e_{m+1}^{(q)} \right\} \]

\[\sqrt{2 \left( m + q - \frac{1}{4} \right)} \] \[\left| e_{m}^{(q)} | q, m \rangle \right|_{b} \]

\[U_{\gamma} \sum_{m=0} \left\{ e_{m-1}^{(q)} + \left[ 2 \left( m + 1 + q - \frac{1}{4} \right) \right] \left[ 2 \left( m + 1 + q - \frac{3}{4} \right) \right] e_{m+1}^{(q)} \right\} \]

\[\sqrt{2 \left( m + q - \frac{1}{4} \right)} \] \[\left| f_{m}^{(q)} | q, m \rangle \right|_{b} \]

= 0,

for the lower level, where the operator properties (6) have been used. Projecting both sides of the Schrödinger equation onto b | q, n \rangle yields

\[\left[ \frac{1 - 4g_{\gamma}}{\beta} 2 (n + q) - \frac{1}{2} - E \right] e_{n}^{(q)} - \frac{\gamma - g}{\beta} \Lambda_{n} + \frac{U_{\gamma}}{\beta} F_{n} = 0, \quad (11)\]

\[\left[ \frac{\Delta - U}{2} + \frac{2U}{\beta} (n + q) \right] f_{n}^{(q)} = 0,\]

\[\left[ \frac{\Delta - U}{2} + \frac{2U}{\beta} (n + q) \right] e_{n}^{(q)} + \frac{U_{\gamma}}{\beta} \Lambda_{n} - \frac{\gamma + g}{\beta} F_{n} = 0, \quad (12)\]

where

\[\Lambda_{n} = e_{n-1}^{(q)} + \left[ 2 \left( n + 1 + q - \frac{1}{4} \right) \right] \left[ 2 \left( n + 1 + q - \frac{3}{4} \right) \right] e_{n+1}^{(q)}, \]

\[F_{n} = f_{n-1}^{(q)} + \left[ 2 \left( n + 1 + q - \frac{1}{4} \right) \right] \left[ 2 \left( n + 1 + q - \frac{3}{4} \right) \right] f_{n+1}^{(q)}.\]

Multiplying equation (11) by \( \frac{U_{\gamma}}{\beta} \) and equation (12) by \( \frac{U_{\gamma}}{\beta} \), we have

6
\[
\frac{\gamma + g}{\beta} \left[ \frac{1 - 4g\gamma}{\beta} 2(n + q) - \frac{1}{2} - E \right] e_n^{(q)} = \frac{\gamma + g}{\beta} - \frac{\gamma + g}{\beta} \Lambda_n
\]
\[
+ \frac{U \gamma + g}{\beta} F_n - \frac{\gamma + g}{\beta} \left[ \frac{\Delta - U}{2} + \frac{2U}{\beta} (n + q) \right] f_n^{(q)} = 0,
\]
\[
- \frac{U \gamma}{\beta} \left[ \frac{\Delta - U}{2} + \frac{2U}{\beta} (n + q) \right] e_n^{(q)} + \left( \frac{U \gamma}{\beta} \right)^2 \Lambda_n - \frac{U \gamma + g}{\beta} f_n
\]
\[
+ \frac{U \gamma}{\beta} \left[ \frac{1 + 4g\gamma}{\beta} 2(n + q) - \frac{1}{2} - E \right] f_n^{(q)} = 0.
\]

The summation of the above two equations gives
\[
\frac{\gamma + g}{\beta} \left[ \frac{1 - 4g\gamma}{\beta} 2(n + q) - \frac{1}{2} - E \right] e_n^{(q)} - \frac{U \gamma}{\beta} \left[ \frac{\Delta - U}{2} + \frac{2U}{\beta} (n + q) \right] e_n^{(q)}
\]
\[
+ \left[ \left( \frac{U \gamma}{\beta} \right)^2 - \frac{\gamma + g}{\beta} \right] \Lambda_n - \frac{\gamma + g}{\beta} \left[ \frac{\Delta - U}{2} + \frac{2U}{\beta} (n + q) \right] f_n^{(q)}
\]
\[
+ \frac{U \gamma}{\beta} \left[ \frac{1 + 4g\gamma}{\beta} 2(n + q) - \frac{1}{2} - E \right] f_n^{(q)} = 0.
\]

If set
\[
\gamma = \sqrt{\frac{g^2}{1 - U^2}},
\]
we can remove \( \Lambda_n \), and obtain a linear relation between the coefficients \( e_n^{(q)} \) and \( f_n^{(q)} \) below
\[
e_n^{(q)} = \frac{\Omega_n^{(q)}}{f_n^{(q)}},
\]
where
\[
\Omega_n^{(q)} = \frac{(\gamma + g) \left[ \frac{\Delta - U}{2} + \frac{2U}{\beta} (n + q) \right] - U \gamma \left[ \frac{1 + 4g\gamma}{\beta} 2(n + q) - \frac{1}{2} - E \right]}{(\gamma + g) \left[ \frac{1 - 4g\gamma}{\beta} 2(n + q) - \frac{1}{2} - E \right] - U \gamma \left[ \frac{\Delta - U}{2} + \frac{2U}{\beta} (n + q) \right]}
\]
(15)

In either equation (11) or (12), we replace \( e_n^{(q)} \) by \( f_n^{(q)} \) through the relation (14), and then arrive at a three-term recurrence relation for \( f_n^{(q)} \).
\[
f_{n+1}^{(q)} = \frac{a_n^{(q)} f_n^{(q)} + b_n^{(q)} f_n^{(q)}}{c_n^{(q)}},
\]
(16)

where
\[
a_n^{(q)} = \frac{U \gamma}{\beta} - \frac{\gamma + g}{\beta} \Omega_n^{(q)},
\]
(17)
\[
b_n^{(q)} = \left[ 2 - \frac{1 - 4g\gamma}{\beta} (n + q) - \frac{1}{2} - E \right] \Omega_n^{(q)} - \left[ \frac{\Delta - U}{2} + \frac{2U}{\beta} (n + q) \right],
\]
(18)
\[
c_n^{(q)} = 4 \left( \frac{\gamma + g}{\beta} \Omega_n^{(q)} - \frac{U \gamma}{\beta} \right) \left( n + q + \frac{1}{4} \right) \left( n + q + \frac{3}{4} \right).
\]
(19)
Figure 1. $G(E)$ for tpRSM at $\Delta = 0.5, g = 0.25, U = 0.25, q = 1/4$ (left) and $q = 3/4$ (right).

All coefficients $f_n^{(q)}$ can be calculated with initial conditions $f_0^{(q)} = 1, f_{-1}^{(q)} = 0$.

Because $H$ is invariant with respect to the action of $\hat{P}_1$ in each space $\mathcal{H}_q \otimes \mathbb{C}^2$, we can project the wavefunction $|\psi, E\rangle^{(q)}$ onto $|\uparrow\rangle_q |0\rangle_a$ and $|\downarrow\rangle_q |0\rangle_a$, respectively, to define the $G$-function as

$$G_{\pm}^{(q)}(E) = \langle \uparrow |_a \langle q, 0 | \psi, E \rangle^{(q)} - \Pi \langle \downarrow |_a \langle q, 0 | \psi, E \rangle^{(q)}$$

$$= \sum_{n=0}^{\infty} \left[f_n^{(q)} - \Pi e^{(q)}_n\right] \frac{2(n + q - 1)!}{n!} \left(\frac{v}{2\mu}\right)^n,$$

where equation (9) has been used, $\Pi = \pm 1$, corresponding to positive(negative) parity. The similar procedures for the derivation of the $G$-function within the Bogoliubov operators approach have been outlined in the tpQRM [38, 42] as well as the one-photon RSM [27].

We see that each eigenstate can be labeled by three quantum numbers, the index $n = 0, 1, 2, \ldots$ corresponding to the continuous (bosonic) degree of freedom, the Bargmann index $q = 1, 3, 5, \ldots$ and the parity $\Pi = \pm 1$, each taking two values. The numbers $q, \Pi$ label the four (infinite-dimensional) invariant subspaces of $\mathcal{H}$, obtained from the $\mathbb{Z}_4$-symmetry.

The $G$-function (20) is a well-defined transcendental function. The zeros of this function should give the regular spectrum. We plot the $G$-functions for $\Delta = 0.5, g = 0.25, U = 0.25$ in figure 1. The zeros are just corresponding to the regular eigenenergies, which can be confirmed by the numerical exact diagonalizations in the truncated original Fock space.

We then discuss the validity of the tpRSM G-function. Note that the maximum value of $\gamma$ is $1/2$. According to equation (13), the maximum coupling strength is

$$g_m = \sqrt{1 - U^2}.$$
3. Level crossings and spectra collapse

The exact solutions in the QRM and related models can be obtained in different ways, even superficially in the analytical way. To the best of our knowledge, only the so-called G-function technique allows to analyze the levels distribution, level crossings, and spectra collapse, because the roots are pinched in between the poles. These interesting issues might be hardly treated by other analytical and numerical approaches [37]. The G-function derived above indeed has the well defined pole structure. From figure 1, one can easily observe that the G-curves diverge at some energy \( E \), which are the typical characteristics of pole structures. Below, we will give their positions in a closed-form.

In equation (16), if \( c^{(q)}_n \) vanishes, i.e. the denominator of \( f^{(q)}_{n+1} \) is zero, the G-function (20) diverges, corresponding to the \( n \)th pole of the \( G(E) \). So the \( n > 0 \) pole position is

\[
E^{\text{q/pole}}_n = 2\sqrt{(1 - U^2)(1 - U^2 - 4g^2)(n + q)} - \frac{1 + U\Delta - U^2}{2}. \tag{22}
\]

For the \( n = 0 \) pole, we need inspect \( e^{(q)}_0 \) and \( f^{(q)}_0 \). In our construction of the G-function, we have set \( f^{(q)}_0 = 1 \), so it cannot diverge. Only \( e^{(q)}_0 \) in G-function (20) can diverge if the denominator of \( \Omega^{(q)}_0 \) vanishes, thus we have the zeroth pole of the G-function as

\[
E^{\text{q/pole}}_0 = 2q\sqrt{1 - U^2 - 4g^2} - \frac{\Delta}{2U} - \frac{1 - \Delta/U}{2}\sqrt{1 - U^2}. \tag{23}
\]

So equations (22) and (23) compose the whole poles in the G-function of the tpRSM.

First-order quantum phase transitions: when both the numerator and the denominator of \( \Omega^{(q)}_0 \) vanish, we have

\[
\beta_c = \frac{1 - \Delta/U}{4q}. \tag{24}
\]

Inserting into equation (23) gives the energy without specified parity

\[
E^{\text{cross}}_0 = -\frac{\Delta}{2U}, \tag{25}
\]

which is independent of \( q \). The lowest energy levels for both parities thus intersect at \( \beta_c \). It is a doubly degenerate state, corresponding to a Juddian solution [59]. The level crossing of the ground-state and the first excited state just demonstrates a first-order QPT.

By equation (24) we can obtain the condition for the occurrence of the first-order QPT as

\[
1 - 4q < \Delta/U < 1, \tag{26}
\]

because \( 1 > \beta > 0 \). We write critical coupling strength at the crossing point explicitly for later use

\[
g_c = g_m \sqrt{1 - \frac{(1 - \frac{\Delta}{2})^2}{16q^2}}. \tag{27}
\]

To demonstrate the first-order QPT, we plot the spectra at \( \Delta = 0.5 \) for three typical values of \( U = -0.3, 0.2, 0.9 \) and \( q = \frac{1}{4} \) in the upper panel, for \( U = -0.8, -0.3, 0.75 \) and \( q = \frac{1}{4} \) in
Figure 2. Spectra for $q = \frac{1}{4}$ (upper panel) and $\frac{3}{4}$ (lower panel) for different values of $U$. $\Delta = 0.5$. The first level crossing points are marked by open circles.

We can see that the lowest two levels cross at $g_c$ exactly given by equation (27). For example, the predicted $g_c = 0.195$ for $\Delta = 0.5$, $U = 0.9$, $q = 1/4$ is exactly shown in the upper right panel with an open circle.

Juddian solutions for doubly degenerate states: as usual, if both $c_n^{(q)}$ and $b_n^{(q)}$ in equation (16) vanish simultaneously, one obtains the Juddian solutions for doubly degenerate states. In this case, two adjacent energy levels with the positive and negative parity can simultaneously intersect with one pole line in the energy spectra. Below we will only discuss the special Juddian solutions with the same crossing energy, and skip the discussions on the whole Juddian solutions, which are in fact similar to the previous ones in both the one-photon QRM [9] and tpQRM [10, 38].

Actually, for any $n$, if both the denominator and the numerator of $\Omega_n^{(q)}$ in equation (15) vanish, $\Omega_n^{(q)}$ is analytic, leading to analytic coefficients $e_n^{(q)}$ and $f_n^{(q)}$. The proof is given in the following. $\Omega_n^{(q)} = x_n/y_n$ is analytic for both $x_n = 0$ and $y_n = 0$. The denominator of $f_n^{(q)}$ in equation (16) is proportional to

$$(\gamma - g) \Omega_n^{(q)} = \frac{(\gamma - g) x_n - U \gamma y_n}{y_n}.$$ 

Thus both the denominator and the numerator of $f_n^{(q)}$ are also zero, leading to analytic coefficients $f_n^{(q)}$, and therefore analytic coefficients $e_n^{(q)}$, for the real physical systems. It can be easily found that the limit of $\Omega_n^{(q)}$ in equation (15) is just $-U\gamma/(\gamma + g)$.

The condition that both the denominator and the numerator of $\Omega_n^{(q)}$ in equation (15) vanish simultaneously yields the crossing coupling strength

$$\beta_n^a = \frac{1 - \frac{\Delta}{U}}{4(n + q)},$$

which includes the first crossing point for $n = 0$ discussed above. Obviously, these Juddian solutions exist only for $\frac{\Delta}{U} < 1$ because $\beta > 0$ is assumed. Intriguingly, the corresponding
energies at $\beta'_c$ for any $n$ are the same

$$E^\text{cross}_n = -\frac{\Delta}{2U} > -\frac{1}{2}, \quad (29)$$

Analogous to the one-photon RSM model [27], the lowest crossing energy is also independent of the coupling constant. Here it is even independent of the Bargmann index $q$. Surprisingly, its value is the same as that in the one-photon RSM, indicating the independence of the detailed atom–cavity coupling. The Juddian solutions above $E = -\frac{\Delta}{2U}$ are also exhibited in the spectra of figure 2, and are not discussed here.

Then, we turn to the number of the crossing points situated at the same energy $E^\text{cross}_n$ (29) before a given coupling constant $g$ in the energy spectra. By equation (28), we have maximum value of $n$ for the given $g$

$$n_{\text{max}} = \left[ \frac{1 - \frac{\Delta}{2U} - q}{4\sqrt{1 - \frac{4q^2}{1 - U^2}}} \right],$$

where the bracket $[\ldots]$ denotes the Gaussian step function. Including the crossing point for $n = 0$, there are $n_{\text{max}} + 1$ level crossings at the same energy $-\frac{\Delta}{2U}$ in the coupling interval $[0, g]$. The energy levels that connect these crossing points should go below the energy $E^\text{cross}_n = -\frac{\Delta}{2U}$ at the given $g$. This is to say, at the coupling constant $g$ in the spectra, $n_{\text{max}} + 1$ pairs of levels are located under the energy $-\frac{\Delta}{2U}$. Particularly, when $g \rightarrow g_m$, $n_{\text{max}} \rightarrow \infty$, there are possibly an infinite number of levels staying below the energy $-\frac{\Delta}{2U}$. It is very challenging to discern these crossings without analytical reasonings outlined here, if $g$ is very close to $g_m$.

For $U = 0.9, \Delta = 0.5$, we have $g_m = 0.21794, E^\text{cross}_n = -0.27778$. If $g \leq 0.21$, $n_{\text{max}} = [0.1653] = 0$, we should have one crossing point for $n = 0$. This is clearly seen in the upper right panel of figure 2. If $g \leq 0.2178$, $n_{\text{max}} = [2.7971] = 2$, we should have three crossing points on a horizontal line $E = -0.27778$, which is however hardly visible. The situation becomes more serious if further approaching $g_m$, so the analytical study performed above is highly called for in this case.

**Non-degenerate solutions:** we can also discuss the other kind of the exceptional solution, so called the non-degenerate solution [26]. The non-degenerate solution is present only if one level crosses the pole line alone, which is essentially different from the doubly degenerate solutions discussed above. One may note that the first two levels are higher than the zeroth pole line in the upper right panel of figure 2. For more detail, we plot $G$-functions around $g_c$ in figure 3. For $g < g_c$, the first two zeros locate at the both sides of the zeroth pole line as shown in the left $G$-curve. Just after $g_c$, as indicated in the middle $G$-curve, the two zeros again go to the both sides of the zeroth pole line in a reversed way. If $g$ increases further, as exhibited in the left $G$-curve, both the first two zeros are located above the zeroth pole. This is to say, after $g_c$, the level with the negative parity must cross the zeroth pole line alone, while the level with the positive parity does not cross at the same point. This crossing point in the energy spectra just corresponds to the non-degenerate solution, which can be located in the same way as in the one-photon RSM model [27].

**Spectra collapse:** for the finite Stark coupling, i.e. $U \neq 0$, when $g \rightarrow g_m$, for both $q = 1/4$ and $q = 3/4$, all the $n > 0$ poles in equation (22) become the same

$$E_c \rightarrow -\frac{1 + U\Delta - U^2}{2}. \quad (30)$$
Therefore all zeros of $G$-function (energy levels) are pinched in between the poles within the vanishing intervals. It follows that they will collapse towards the same value of $E_c$ when $g \to g_m$, which are just demonstrated in figure 2 for all cases.

When $g \to g_m$, the zeroth pole described by equation (23) becomes

$$E_0 = \frac{-\Delta}{2U} - \frac{1 - \Delta/U}{2\sqrt{1 - U^2}}. \tag{31}$$

Obviously, if $U = 0$, all poles in both equations (30) and (31) tend to $-1/2$, as $g \to 1/2$, which is well known in the tpQRM. Usually the $n = 0$ pole is less than $n > 0$ ones. But in the presence of the Stark coupling, one can find that $E_0$ is even larger than $E_c$ if $0 < U < \Delta$. In the limit of $g \to g_m$, some energy levels could separate from the collapse energy $E_c$, as exhibited in figure 2. We find that the ground state energy always separate from the collapse energy $E_c$ at $g_m$. The energy gap does not close, indicating no continuous QPT in this model.

When $0 < U < \Delta$, the collapse issue is somehow challenging. In the limit of $g \to g_m$, $E_0$ can move above the collapse energy $E_c$. If there are some energy levels lies in between $E_c$ and $E_0$, even above $E_0$, these levels should not collapse. Since the analytical solution at $g = g_m$ is still lacking at the moment, whether there are some levels escaped from the collapse energy ($E_c$) from above remains a open question.

4. Conclusion

In this work, we have derived the $G$-function for the tpRSM in a compact way by using the Bogoliubov operators approach. Zeros of the $G$-function determine the regular spectra. The first-order QPT is detected analytically by the pole structure of $G$-functions. The critical coupling strength of the phase transition is obtained analytically. The occurrence of the first-order QPT is originated from the Stark coupling, because the tpQRM does not experience the first-order QPT. Very interestingly, the lowest level crossing energy for $U > \Delta$ is even exactly the same as that in the one-photon RSM, independent of the detailed dipole coupling between the atom and cavity.

The energy spectral collapse is also found and analyzed based on the poles of the derived $G$-function. Similar to the tpQRM, some energy levels could escape from the collapse energy because the first pole is separated from all other poles. The collapse characteristics is different from the one-photon RSM where all levels collapse without exceptions, but resemble the tpQRM. The infinite discrete upper spectra found in the one-photon RSM are absent in the tpRSM because the terminated coupling vanishes in the limit of $U \to 1$. 

Figure 3. $G$-curves for $g = 0.19$ (left), 0.20 (middle), and 0.21 (right), at $\Delta = 0.5, U = 0.9, q = 1/4$. 

J. Phys. A: Math. Theor. 53 (2020) 315301 J Li and Q-H Chen
The exotic spectra observed in the tpRSM may be of fundamental importance if the nonlinear Stark coupling is introduced to the extended two-photon systems involving multiple levels and multiple bosonic modes, and even the relevant open quantum systems.

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References

[1] Rabi I I 1937 Phys. Rev. 51 652
[2] Jaynes E T and Cummings F W 1963 Proc. IEEE 51 89
[3] Scully M O and Zubairy M S 1997 Quantum Optics (Cambridge: Cambridge University Press)
Orszag M 2007 Quantum Optics Including Noise Reduction, Trapped Ions, Quantum Trajectories, and Decoherence (Berlin: Springer)
[4] Niemczyk T et al 2010 Nat. Phys. 6 772
[5] Form-Díaz P et al 2010 Phys. Rev. Lett. 105 237001
[6] Form-Díaz P, García-Ripoll J J, Peropadre B, Orgiazzi J-L, Yurtalan M A, Belyansky R, Wilson C M and Lupascu A 2016 Nat. Phys. 13 39
[7] Yoshihara F,Fuse T, Ashhab S, Kakuyanagi K, Saito S and Semba K 2016 Nat. Phys. 13 44
[8] Casanova J, Romero G, Lizuain I, Garcia-Ripoll J J and Solano E 2010 Phys. Rev. Lett. 105 263603
[9] Braak D 2011 Phys. Rev. Lett. 107 100401
[10] Chen Q H, Wang C, He S, Liu T and Wang K L 2012 Phys. Rev. A 86 023822
[11] He S, Wang C, Chen Q-H, Ren X-Z, Liu T and Wang K-L 2012 Phys. Rev. A 86 033837
[12] Zhong H-H, Xie Q-T, Batchelor M and Lee C-H 2013 J. Phys. A 46 415302
[13] He S, Zhao Y and Chen Q-H 2014 Phys. Rev. A 90 053848
[14] Ying Z-J, Liu M-X, Luo H-G, Lin H-Q and You J Q 2015 Phys. Rev. A 92 053823
[15] Hwang M-J, Puebla R and Plenio M B 2015 Phys. Rev. Lett. 115 180404
[16] Liu M X, Chesi S, Ying Z-J, Chen X S, Luo H-G and Lin H-Q 2017 Phys. Rev. Lett. 119 220601
[17] Moroz A 2012 Europhys. Lett. 100 60010
[18] Braak D, Chen Q-H, Batchelor M and Solano E 2016 J. Phys. A: Math. Gen. 49 300301
[19] Xie Q T et al 2017 J. Phys. A: Math. Theor. 50 113001
[20] Gu X et al 2017 Phys. Rep. 718–719 1
[21] Form-Diaz P, Lamata L, Rico E, Kono J and Solano E 2019 Rev. Mod. Phys. 91 25005
[22] Le Bosté A 2020 Adv. Quantum Tech. 3 1900140
[23] Grimsmo A L and Parkins S 2013 Phys. Rev. A 87 033814
[24] Grimsmo A L and Parkins S 2014 Phys. Rev. A 89 033802
[25] Eckle H-P and Johannesson H 2017 J. Phys. A: Math. Theor. 50 294004
[26] Maciejewski A J, Przybylska M and Stachowiak T 2014 Phys. Lett. A 378 3445
Maciejewski A J, Przybylska M and Stachowiak T 2015 Phys. Lett. A 379 1503
[27] Xie Y-F, Duan L W and Chen Q-H 2019 J. Phys. A: Math. Theor. 52 245304
[28] Xie Y-F and Chen Q-H 2019 Commun. Theor. Phys. 71 623
[29] Cong L, Felicetti S, Casanova J, Lamata L, Solano E and Arrazola I 2020 Phys. Rev. A 101 032350
[30] Toor A H and Zubairy M S 1992 Phys. Rev. A 45 4951
[31] Peng J S and Li G X 1993 Phys. Rev. A 47 3167
[32] Ng K M, Lo C F and Liu K L 1999 Eur. Phys. J. D 6 119
[33] Emary C and Bishop R F 2002 J. Math. Phys. 43 3916
[34] Dolya S N 2009 J. Math. Phys. 50 033512
[35] Albert V V, Scholes G D and Brumer P 2011 Phys. Rev. A 84 042110
[36] Travènec I 2012 Phys. Rev. A 85 043805
Maciejewski A J, Przybylska M and Stachowiak T 2015 Phys. Rev. A 91 037801
Travènec I 2015 Phys. Rev. A 91 037802
[37] Felicetti S, Pedernales J S, Egusquiza I L, Romero G, Lamata L, Braak D and Solano E 2015 Phys. Rev. A 92 033817
[38] Duan L W, Xie Y-F, Braak D and Chen Q-H 2016 J. Phys. A: Math. Theor. 49 464002
[39] Zhang Y Z 2013 J. Math. Phys. 54 102104
Zhang Y Z 2016 Ann. Phys. 375 460
[40] Lv Z G, Zhao C J and Zheng H 2017 J. Phys. A: Math. Theor. 50 074002
Maciejewski A J and Stachowiak T 2017 J. Phys. A: Math. Theor. 50 244003
[41] Cui S, Cao J P, Fan H and Amico L 2017 J. Phys. A: Math. Theor. 50 204001
[42] Peng J et al 2017 J. Phys. A: Math. Theor. 50 174003
[43] Felicetti S, Rossatto D Z, Rico E, Solano E and Forn-Díaz P 2018 Phys. Rev. A 97 013851
[44] Cong L, Sun X M, Liu M X, Ying Z J and Luo H G 2019 Phys. Rev. A 99 013815
[45] Casanova J, Puebla J, Moya-Cessa H and Plenio M 2018 npj Quantum Inf. 4 47
[46] Malekakhlagh M and Rodriguez A 2019 Phys. Rev. Lett. 122 043601
[47] Lupo E, Napoli A and Messina A 2019 Sci. Rep. 9 4156
[48] Maldonado-Villamizar F H, Alderete C H and Rodriguez-Lara B M 2019 Phys. Rev. A 100 013811
[49] Maciejewski A J and Stachowiak T 2019 J. Phys. A: Math. Theor. 52 485303
[50] Dodonov A V, Napoli A and Militello B 2019 Phys. Rev. A 99 033823
[51] Villas-Boas C J and Rossatto D Z 2019 Phys. Rev. Lett. 122 123604
[52] Garbe L, Egusquiza I L, Solano E, Ciuti C, Coudreau C, Milman O and Felicetti S 2017 Phys. Rev. A 95 053854
[53] Chen X Y and Zhang Y Y 2018 Phys. Rev. A 97 053821
[54] Chen X Y and Zhang Y Y 2018 Phys. Rev. A 97 053821
[55] Felicetti S, Hwang M-J and Le Boitě A 2018 Phys. Rev. A 98 053859
[56] Wang S, Chen S and Jing J L 2019 Phys. Rev. E 100 022207
[57] Cui S F, Hebert F and Gremaud B 2019 Phys. Rev. A 100 033608
[58] Puebla R, Casanova J, Houhou O, Solano E and Paternostro M 2019 Phys. Rev. A 99 032303
[59] Judd B R 1979 J. Phys. C: Solid State Phys. 12 1685