TOwards the Modelling
of the Second Solar Spectrum

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Abstract. This paper addresses the modelling issue of the second solar spectrum. This is the name given to the linearly polarized spectrum which can be observed close to the solar limb using spectro-polarimeters of high polarimetric sensitivity (Stenflo and Keller, 1997). The second solar spectrum is due to scattering processes and offers a rich diagnostic potential for exploring solar magnetic fields via the Hanle effect. However, it is full of mysterious spectral features that cannot be understood with simplified polarization transfer theories, thus suggesting that the underlying scattering physics is more complex than previously thought. In this paper we argue that understanding the second solar spectrum requires the consideration of scattering processes in multilevel atomic models, taking fully into account the transfer of atomic polarization among all the levels involved.

To give support to this statement, we begin by pointing out the drastically different predictions, given by the standard resonance line polarization theory, with respect to the emergent polarization in three different line transitions. This standard theory neglects the atomic polarization of the lower level of the line transition under consideration, i.e. it assumes that there are no population imbalances among the lower-level sublevels. The density matrix polarization transfer theory is then applied to formulate the scattering polarization problem taking properly into account atomic polarization in both the upper and the lower line levels. The consideration of lower-level atomic polarization leads to coupled non-linear and non-local sets of equations, even for the two-level model atom case considered in this paper. The unknowns of these equations are the irreducible tensor components of the atomic density matrix whose self-consistent values have first to be obtained to be able to calculate the emergent Stokes profiles. To solve this non-LTE problem of the 2nd kind we present some iterative methods that are very suitable for developing a general multilevel scattering polarization code. With these numerical methods some model calculations are performed in order to demonstrate that the inclusion of lower-level atomic polarization leads to similar emergent linear polarization signals in such three different line transitions, as some observations show. After pointing out that the “Na solar paradox” (Landi Degl’Innocenti, 1998) might admit, in principle, a multilevel solution, the paper ends establishing a new solar paradox: “the Mg solar paradox”, for which no multilevel solution seems to be possible. This new result demonstrates that there indeed exists ground and metastable-level atomic polarization in the solar chromosphere and it suggests that the solution to these “solar paradoxes” is to be found by carefully revising our current ideas about the chromospheric magnetic field.

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1. Introduction

The “second solar spectrum” is a term adopted by Stenflo and Keller (1997) to refer to the remarkable observational discovery that the whole solar spectrum is linearly polarized, when observations are made close to the solar limb using novel spectro-polarimeters that allow the detection of very low amplitude polarization signals ($\sim 10^{-5}$ in the degree of linear polarization). This term is certainly adequate because, as pointed out by these authors, the linearly polarized solar spectrum has a structural richness that often exceeds that of the ordinary intensity spectrum. It is indeed as if the Sun has presented us with an entirely new spectrum to explore. In fact, the second solar spectrum contains a wealth of “inexplicable” spectral features which are the signature of physical processes that are presently challenging physicists working in the field.

This paper deals with the modelling issue of the second solar spectrum. This modelling requires the solution of a formidable numerical problem that is considered as one of the most challenging tasks of solar and stellar polarimetry. It consists in calculating, for multilevel atomic models, the excitation and ionization states of chemical species of given abundance that are consistent with the polarization properties of the radiation field produced by such species in any medium of given temperature, density, macroscopic velocity and magnetic field vector. Once this self-consistent atomic excitation is known along the line of sight, it is straightforward to solve the transfer equations for the Stokes parameters in order to calculate the emergent polarization profiles that are to be compared with spectro-polarimetric observations.

It is indeed a very complex problem because, in the polarization transfer case, one has to take into account that each level of total angular momentum value $X$ has associated with it $(2X+1)$ sublevels, with $\vec{X} = \vec{J} = \vec{L} + \vec{S}$ if fine structure due to the spin-orbit LS coupling is assumed, or with $\vec{X} = \vec{F} = \vec{J} + \vec{I}$ if hyperfine structure due to the nuclear angular momentum $\vec{I}$ is taken into account. The populations of these sublevels are sensitive to the polarization and anisotropy state of the radiation field at each point within the medium. Moreover, quantum interferences (or coherences) among the sublevels themselves may also appear, coherences that depend on the energy separation between the levels and on their splitting. These coherences must also be properly quantified to fully specify the excitation state. An additional complication stems from the fact that, in the polarized case, instead of the standard radiative transfer (RT) equation for the specific intensity, one has to solve, in general, a vectorial transfer equation for the four Stokes parameters.

Obviously, accounting for this complexity requires working within the
framework of a robust theory for the generation and transfer of polarized radiation. I believe that the most rigorous (and suitable) theoretical framework to work with is that developed by Landi Degl’Innocenti (1983), which is based on the irreducible tensor components ($\rho^K_Q$) of the atomic density matrix (see also Bommier and Sahal-Bréchet, 1978). According to this formalism, to each level of angular momentum value $X$, there correspond $(2X + 1)^2$ density-matrix elements. These $\rho^K_Q$-elements contain information about the populations of the atomic sublevels and about the coherences among them. For instance, for a level with total angular momentum $J=1$ we have that $\rho^0_0 = (N_1 + N_0 + N_{-1})/\sqrt{3}$, $\rho^1_0 = (N_1 - N_{-1})/\sqrt{2}$ and $\rho^2_0 = (N_1 - 2N_0 + N_{-1})/\sqrt{6}$, where $N_i$ (with $i=1, 0$ and $-1$) are the populations of the three magnetic sublevels. Thus, $\rho^0_0$ gives the total population of the level, while $\rho^1_0$ (the orientation coefficient) and $\rho^2_0$ (the alignment coefficient) inform us about the population differences among the sublevels. Finally, $\rho^K_Q$-terms with $Q \neq 0$ account for the coherences between Zeeman sublevels whose magnetic quantum numbers differ by $Q$.

One of the reasons that explain why the density matrix formalism is so suitable for dealing with the generation and transfer of polarized radiation is that, through an emission process, polarization in spectral lines can originate locally either by the splitting of the atomic levels (splitting that can in turn be due either to the Zeeman or the Stark effect) or by the presence of population differences and/or coherences among the sublevels. Thus, the $\rho^K_Q$ elements whose self-consistent values are sought at each spatial gridpoint, indeed provide the most suitable way of quantifying, at the atomic level, the information that we need to be able to calculate all the “sources” and “sinks” of polarization within the medium under consideration. The main criticism of this QED theory is that it is based on the approximation of complete frequency redistribution (CRD), i.e. on an assumption that is not adequate for modelling the polarization of several diagnostically important spectral lines. Fortunately, some very recent work has successfully started to incorporate the effects of partial redistribution (PRD) into the framework of the density matrix formalism (see Landi Degl’Innocenti et al., 1997; Bommier, 1997 a,b). These recent efforts to generalize the density matrix theory to PRD are truly important and should be continued because the CRD theory cannot be applied when coherences between non-degenerate levels are present unless the spectrum of the radiation is flat across a frequency interval $\Delta \nu$ centred on the line frequency and larger than the frequency separation between the two levels connected by the coherence (see Landi Degl’Innocenti et al., 1997).

The problem of finding the self-consistent values of the irreducible tensor components of the atomic density matrix has been called “the non-LTE problem of the 2nd kind” (see Landi Degl’Innocenti, 1987). It requires solv-
ing jointly the statistical equilibrium (SE) equations for the density matrix elements associated with each level of the assumed atomic model and the Stokes-vector transfer equations for all the radiative transitions involved. This terminology also seems appropriate because, as I shall try to argue below, a better understanding of many of the “inexplicable” spectral features of the second solar spectrum can only be achieved by carefully formulating and solving multilevel non-LTE problems of the 2nd kind, taking fully into account the possibility of atomic polarization at all the levels of the chosen multilevel atomic model and including the depolarizing role of elastic collisions and magnetic fields.

The word “inexplicable” in the preceding paragraphs refers to the impossibility of explaining some of the spectral features of the second solar spectrum by means of theories based on the approximation of neglecting atomic polarization in the lower level of the line transition under consideration, i.e. theories based on the assumption that there are no significant differences in the populations of the lower-level sublevels or coherences among them. One example of a mysterious feature of the second solar spectrum that has triggered some recent theoretical work (see Trujillo Bueno and Landi Degl’Innocenti, 1997; Landi Degl’Innocenti, 1998) is the linear polarization pattern observed around the Na I D₂ and D₁ lines. In fact, in a recent letter in Nature that demonstrates the robustness of the density matrix polarization transfer theory including partial frequency redistribution effects, Landi Degl’Innocenti (1998) concluded that the observed Na D₂ and D₁ linear polarization pattern can be explained by assuming the presence of an amount of ground-level atomic polarization as important as (and indeed slightly larger than!) that of the upper level. However, because very small non-vertical magnetic fields (and/or elastic collisions) destroy the ground-level atomic polarization of Na, Landi Degl’Innocenti (1998) was forced to rule out in the solar chromosphere both elastic collisions and the existence of turbulent magnetic fields and of horizontal, canopy-like fields stronger than \( \sim 0.01 \text{ gauss} \). This has led to an exciting apparent paradox in solar physics because there are observational evidence for both turbulent fields of the order of 10 gauss and canopy-like horizontal fields (see Jones, 1984; Solanki and Steiner, 1990; Faurobert-Scholl, 1992; Bianda et. al, 1998; Stenflo et. al, 1998).

The argument in favour of the simplifying approximation of neglecting lower-level polarization is that the lower level of a line transition is generally long-lived, and that it must thus have plenty of time to be depolarized by elastic collisions and/or weak magnetic fields (Stenflo, 1994; 1997). However, this is expected to be the case concerning only the ground and metastable levels of atomic systems. It is indeed a major simplifying approximation because it implies that the scattered radiation can be expressed
linearly in terms of the incident radiation if stimulated emission processes are also neglected (see Landi Degl’Innocenti, 1984). In other words, the approximation of neglecting lower-level atomic polarization allows the study of scattering line polarization problems in terms of phase matrices that are decoupled from the SE equations. However, as we shall show below, the consideration of lower-level atomic polarization leads to a coupled system of non-linear equations, even for the simplest case of a two-level model atom.

My motivation for writing this paper is two-fold. Firstly, I would like to provide more arguments concerning the idea (already put forward in the paper by Trujillo Bueno and Landi Degl’Innocenti, 1997) that lower-level atomic polarization is an essential physical ingredient for understanding the second solar spectrum. Secondly, I aim to demonstrate that, contrary to some general beliefs, the density-matrix polarization transfer theory does have a suitable form for practical applications.

To these ends, I will consider here three types of line transitions in the solar atmosphere: (a) lines with $J_l = 0$ and $J_u = 1$, (b) lines with $J_l = 1$ and $J_u = 0$, and (c) lines with $J_l = 1$ and $J_u = 1$. Section 2 summarizes the predictions of the standard theory, which neglects lower-level atomic polarization. Section 3 is dedicated to outlining the formulation of the scattering line polarization problem taking into account atomic polarization in both the upper and the lower levels of such line transitions. Here I will show the self-consistent solution for the density matrix elements and the corresponding emergent fractional linear polarization that results from this more correct treatment. Section 4 discusses model calculations including the depolarizing role of elastic collisions.

As we shall see, my two-level atom scattering line polarization calculations suggest that, if we aim at understanding the second solar spectrum, we need to consider scattering processes in multilevel atomic models, taking fully into account the transfer of atomic polarization among all the atomic levels involved. Section 5 argues that, in principle, the above-mentioned “Na solar paradox” might admit a multilevel solution. However, Section 6 shows that the observed fractional linear polarization in the Mg b-lines can only be explained by invoking the presence of atomic polarization in the lower metastable levels of the Mg $b_1$ and $b_2$ lines, thus establishing a new paradox: the “Mg solar paradox”. Finally, Section 7 gives some concluding remarks after pointing out that there is no multilevel solution to this “Mg solar paradox”. The Appendix is dedicated to a brief description of some iterative methods for the solution of non-LTE problems of the 2nd kind, which I consider as “the road to be taken” for the development of a general multilevel scattering line polarization code.
2. Predictions of the Standard Theory

This and the following section deal with the scattering line polarization problem assuming a one-dimensional (1D), plane-parallel, non-magnetic, static solar atmospheric model and a two-level model atom neglecting stimulated emission processes. The only difference is that the results presented in this section are obtained with the standard theory, which neglects atomic polarization in the lower level, while those shown in the next section are the result of the application of the full theory that takes into account atomic polarization at both levels.

A detailed formulation of the standard resonance line polarization problem, together with some numerical results obtained applying novel iterative schemes, can be found in the paper by Trujillo Bueno and Manso Sainz (1999). In that paper it is clarified that in order to specify the excitation state of two-level atoms without ground-level atomic polarization it is enough to consider two density-matrix elements at each spatial grid-point: \( \rho_0^0(u) \) and \( \rho_2^0(u) \) (see also Landi Degl’Innocenti et al., 1990). The first one, i.e. \( \rho_0^0(u) \), measures the overall population of the upper level, while \( \rho_2^0(u) \) is the so-called alignment coefficient, which quantifies the degree of imbalance in the populations of the upper-level sublevels.

In the stellar atmospheric environment the main physical mechanism that leads to population differences among the sublevels of the atomic levels is the anisotropic illumination of the atoms. This is easy to understand by considering the academic case of a unidirectional unpolarized light beam that illuminates a gas of two-level atoms with \( J_l = 0 \) and \( J_u = 1 \) and that is propagating along the direction chosen as the quantization axis. Since these atoms must absorb \( \pm 1 \) units of angular momentum from the light beam, only the transitions corresponding to \( \Delta M = \pm 1 \) are effective, so that no transitions occur to the \( M = 0 \) sublevel of the upper level. Thus, in the absence of any relaxation mechanisms, the upper-level sublevels with \( M = 1 \) and \( M = -1 \) would be more populated than the \( M = 0 \) sublevel and the alignment coefficient \( \rho_2^0(u) = (N_1 - 2N_0 + N_{-1})/\sqrt{6} \) would have a positive value. Clearly, the amount of this atomic alignment in a stellar atmospheric environment is significantly smaller than in such an academic case due to the relaxation mechanisms present in a stellar atmosphere (e.g. depolarizing collisions and magnetic fields) and to the much weaker degree of the radiation field’s anisotropy.

As mentioned above, in order to calculate the fractional linear polarization emerging from a given solar atmospheric model we have first to find the self-consistent values of \( \rho_0^0(u) \) and \( \rho_2^0(u) \) by solving the SE and the RT equations (see Trujillo Bueno and Manso Sainz, 1999). Figure 1a shows the variation with the line integrated optical depth of the self-consistent values.
of $\rho^2(u)/\rho^0(u)$ for the three types of line transitions mentioned in the introduction, calculated for the case of zero depolarizing rate due to elastic collisions.

Figure 1. Predictions of the standard theory (which neglects lower-level polarization) for the zero depolarizing rate case in three different line transitions. The left panel (Fig. 1a) shows the variation with line integrated optical depth of $\rho^2(u)/\rho^0(u)$. The right panel (Fig. 1b) shows the emergent fractional linear polarization at $\mu = 0.1$, with positive Q defined along the radial direction through the observed point. The input atmospheric model is an isothermal atmosphere with the solar effective temperature. Inelastic collisions are taken into account by assuming a collisional destruction probability $\epsilon = 10^{-4}$.

As seen in Fig. 1a, the largest values for the upper-level alignment are obtained for line transitions with $J_l = 0$ and $J_u = 1$, i.e. for triplet lines. Alignment values $\sim 8\%$ are found in the surface layers, while $\rho^2(u)$ becomes negligible for optical depths $\tau_{\text{line}} > 1$, because in these high-opacity regions the radiation field approaches isotropy. The alignment of the upper level is found to be zero for lines with $J_l = 1$ and $J_u = 0$, as expected from the fact that levels with $J=0$ have no sublevel structure. For lines with $J_l = J_u = 1$ we have $\rho^2(u)/\rho^0(u) \sim -3\%$, i.e. alignment absolute values that are smaller than for the case of triplet lines.

Figure 1b shows the corresponding emergent fractional linear polarization ($Q/I$) for simulated observations at $\mu = \cos \theta = 0.1$, with $\theta$ being the angle between the line of sight and the normal to the stellar surface. In order to understand these results we point out that, for this standard resonance line polarization case where population differences among the lower-level
sublevels are neglected, one has the following RT equations (see, e.g., Trujillo Bueno and Manso Sainz, 1999):

\[
\frac{d}{ds} I = \epsilon_I - \eta_I I, \tag{1}
\]

\[
\frac{d}{ds} Q = \epsilon_Q - \eta_Q Q, \tag{2}
\]

where \( s \) is the geometrical distance along the ray and where the line contributions to the I and Q components of the emission vector are:

\[
\epsilon_{I,\text{line}} = \left( \frac{\hbar \nu}{4 \pi} \right) A_{ul} N \sqrt{2J_u + 1} \phi_x \rho_0^0(u) + \frac{\omega}{2\sqrt{2}} (3\mu^2 - 1) \rho_0^2(u), \tag{3}
\]

\[
\epsilon_{Q,\text{line}} = \left( \frac{\hbar \nu}{4 \pi} \right) A_{ul} N \sqrt{2J_u + 1} \phi_x \frac{3 \omega}{2\sqrt{2}} (\mu^2 - 1) \rho_0^2(u), \tag{4}
\]

with \( N \) the total number of atoms per unit volume, \( \nu \) the line frequency, \( A_{ul} \) the Einstein spontaneous emission coefficient, \( \phi_x \) the normalized line profile (where \( x \) is the frequency measured from the line centre in units of the Doppler width), and \( \omega \) is equal to the \( w_{J_u,J_l}^{(2)} \) coefficient introduced by Landi Degl’Innocenti (1984), whose values for transitions \( J_l \to J_u \) are (see Table 1 in Landi Degl’Innocenti, 1984):

\[
\omega = 1 \text{ for } 0 \to 1
\]

\[
\omega = 0 \text{ for } 1 \to 0
\]

\[
\omega = -1/2 \text{ for } 1 \to 1
\]

The above transfer equations show that, for this standard resonance line polarization case, the absorption matrix is diagonal, i.e. \( K = \eta I \), where the line contribution to the I-component of the absorption matrix is given by

\[
\eta_{I,\text{line}} = \left( \frac{\hbar \nu}{4 \pi} \right) B_{lu} N \sqrt{2J_l + 1} \phi_x \rho_0^0(l), \tag{5}
\]

where \( B_{lu} \) is the Einstein coefficient for the absorption process and \( \rho_0^0(l) \) is simply proportional to the total population of the lower level.

Therefore, from Eqs. (2) and (4), one sees that the emissivity in Q at each point within the atmosphere is directly proportional to the alignment coefficient of the upper level, weighted by a number (\( \omega \)) that depends on the total angular momentum of the lower and upper levels. The conclusion is
that the standard theory predicts the largest fractional linear polarization for triplet lines ($\sim 6\%$ at the line centre at $\mu = 0.1$), zero linear polarization for lines with $J_l = 1$ and $J_u = 0$, while for lines with $J_l = J_u = 1$ we have $Q/I \sim 1\%$ at $\mu = 0.1$.

3. The Effect of Lower-level Depopulation Pumping

Consider again the above-mentioned academic case of an unpolarized light beam incident on a gas of two-level atoms, but now with $J_l = 1$ and $J_u = 0$. In this case no transitions can occur out of the $M = 0$ sublevel of the lower level, since only transitions corresponding to $\Delta M = \pm 1$ are effective. On the other hand, the spontaneous de-excitation from the upper level populates with equal probability the three sublevels ($M = -1, 0, +1$) of the lower level. In the absence of any relaxation mechanisms, the final result of this optical-pumping cycle is that all atoms will eventually be pumped into the $M = 0$ sublevel of the lower level, and the medium will become transparent.

In the quantum optics literature this is known as depopulation pumping (Happer, 1972; see also Landolfi and Landi Degl’Innocenti, 1986). As we have seen, depopulation pumping occurs when certain lower-level sublevels absorb light more strongly than others. As a result, an excess population tends to build up in the weakly absorbing sublevels. In the above two-level atom example depopulation pumping leads to a negative lower-level alignment coefficient $\rho_{0}^{2}(l) = -2N_l/\sqrt{6}$, with $N_l$ the total population of the lower level. The physical mechanism that is responsible of these population differences between the lower-level sublevels is again the anisotropic illumination of the atoms. Thus, if the anisotropy of the solar radiation field is capable of inducing significant population differences among the upper-level sublevels, why should it not be capable of producing similar lower-level alignment coefficient values? In fact, it does, as demonstrated by Trujillo Bueno and Landi Degl’Innocenti (1997), who formulated this scattering line polarization problem and solved the ensuing non-local and non-linear set of equations. For instance, for a line transition with $J_l = 1$ and $J_u = 0$ in a two-level model atom the rate equation that governs the temporal evolution of $\rho_{0}^{2}(l)$ is

$$\frac{d}{dt} \rho_{0}^{2}(l) = -B_{lu} J_{0}^{2} \rho_{0}^{2}(l) + \frac{B_{lu}}{\sqrt{2}} J_{0}^{2} \rho_{0}^{2}(l) - B_{lu} J_{0}^{2} \rho_{0}^{2}(l) - (C_{lu} + D_{l}) \rho_{0}^{2}(l) = 0,$$

where $C_{lu}$ is the upward inelastic collisional rate and $D_{l}$ the lower-level depolarizing rate due to elastic collisions. In this equation $J_{0}^{2}$ is given by a frequency and angular average of the Stokes-I parameter weighted by the line absorption profile, while $J_{0}^{2}$ is given by a frequency and angular integral
of the Stokes I and Q parameters, additionally weighted by some angle-dependent factors. These quantities ($J_0^0$ and $J_2^0$) are two of the spherical tensor components of the radiation field (see Landi Degl’Innocenti, 1983) and they quantify, respectively, the average intensity of the radiation field and mainly the degree of its anisotropy.

All the rates in Eq. (6) are relaxation rates. We point out that the most important radiative rates here are the first and the third, which are due, respectively, to the anisotropic ($J_2^0$) and to the isotropic ($J_0^0$) components of the radiation field tensor, acting on the unpolarized and polarized components of the lower-level density matrix, respectively. Neglecting in Eq. (6) the collisional rates and taking into account the weakly anisotropic nature of the solar radiation field (i.e. that $J_2^0/J_0^0 \ll 1$) one finds that $\rho_2^0/\rho_0^0 \sim -J_2^0/J_0^0$. Figure 2a shows the self-consistent solution for $\rho_2^0(l)/\rho_0^0(l)$, for the zero depolarizing rate case, but taking into account inelastic collisions.

The question now is whether the presence of population differences in the lower level can by itself lead to local sources of linear polarization. The answer is affirmative. To understand this we have to write down the transfer equations for the Stokes I and Q parameters that result from the application of the density-matrix theory taking into account the possibility of atomic polarization in both levels (Trujillo Bueno and Landi Degl’Innocenti, 1997):

$$\frac{d}{ds} I = \epsilon_I - \eta_I I - \eta_Q Q, \quad (7)$$

$$\frac{d}{ds} Q = \epsilon_Q - \eta_Q I - \eta_I Q, \quad (8)$$

where the line contribution to $\eta_I$ and the full Q-component of the absorption matrix (which is not diagonal now !) are given by

$$\eta_{\text{line}} = (\hbar \nu/4\pi) B_{lu} N \sqrt{2J_l + 1} \phi_x [\rho_0^0(l) + \frac{Z}{2\sqrt{2}} (3\mu^2 - 1) \rho_2^0(l)], \quad (9)$$

$$\eta_Q = (\hbar \nu/4\pi) B_{lu} N \sqrt{2J_l + 1} \phi_x \frac{3 Z}{2\sqrt{2}} (\mu^2 - 1) \rho_2^0(l), \quad (10)$$

where $Z$ should not be confused with the $\omega$-symbol of Eqs. (3) and (4), since $Z$ is equal to the symbol $w^{(2)}_{J_l,J_u}$ introduced by Landi Degl’Innocenti (1984), while $\omega = w^{(2)}_{J_u,J_l}$. The values of $Z$ for transitions $J_l \rightarrow J_u$ are:

$$Z = 0 \text{ for } 0 \rightarrow 1$$
\[ Z = 1 \text{ for } 1 \rightarrow 0 \]

\[ Z = -1/2 \text{ for } 1 \rightarrow 1 \]

Equation (4) shows that \( Z_{\text{line}} = 0 \) for \( 1 \rightarrow 0 \) transitions, as it must be, because for a level with \( J = 0 \) the alignment is zero. Thus, for \( 1 \rightarrow 0 \) transitions the term \( -\eta_Q I \) in Eq. (8) is the only one that plays the role of the emissivity in \( Q \) at each point in the atmosphere. In other words, Eq. (10) shows that the existence of differences in the populations of the magnetic sublevels of the lower level leads to non-zero values of \( \eta_Q \), i.e. it introduces dichroism in the stellar atmosphere and a coupling of the intensity of the radiation beam with the Stokes Q-parameter that is due to the absorption process. Note that the larger the absolute value of the alignment coefficient \( \rho_2(l) \), the larger the expected Q/I signal (Trujillo Bueno and Landi Degl’Innocenti, 1997). Figure 2b shows, for the zero depolarizing rate case, the emergent fractional linear polarization at \( \mu = 0.1 \) that corresponds to the self-consistent solution given in Fig. 2a. In conclusion, the prediction of the correct two-level atom theory for line transitions with \( J_l = 1 \) and \( J_u = 0 \) is that the emergent fractional linear polarization is as important as that corresponding to triplet lines.

Figure 2. Predictions of the full theory for line transitions in a two-level model atom with \( J_l = 1 \) and \( J_u = 0 \). The left panel (Fig. 2a) shows the variation with line integrated optical depth of \( \rho_2(l)/\rho_0(l) \). The right panel (Fig. 2b) shows the emergent fractional linear polarization at \( \mu = 0.1 \), which is to be compared with the dotted-line of Fig. 1b.
Finally, we ask how important can be the fractional linear polarization in lines with $J_l = J_u = 1$ when atomic alignment is taken into account in both the upper and the lower levels. For two-level atoms with these angular momentum values four quantities are needed to fully specify their excitation state: $\rho_0^0(u)$, $\rho_2^0(u)$, $\rho_0^0(l)$ and $\rho_2^0(l)$. These quantities at all the spatial grid-points are the unknowns whose self-consistent values are sought. To this end, one has to derive the SE equations applying the density-matrix theory.

Now, corresponding to each spatial grid-point, one has four equations with four unknowns. A detailed presentation of these non-linear equations will be published elsewhere. For our purposes here we simply write down the rate equation that governs the temporal evolution of $\rho_0^2(u)$:

$$\frac{d}{dt} \rho_0^2(u) = -A_{ul} \rho_0^2(u) - \frac{B_{lu}}{2} \bar{J}_0^2 \rho_0^0(l) - \frac{B_{lu}}{2} \bar{J}_0^0 \rho_2^0(l) - \frac{B_{lu}}{\sqrt{2}} \bar{J}_2^0 \rho_0^0(l) + C_{lu}^{(2)} \rho_0^2(l) - (C_{ul} + D_u) \rho_0^2(u) = 0.$$  

In this equation the first and last terms are relaxation rates, while the remaining ones are transfer rates. We point out that, besides the usual rate $T_2 = -B_{lu} \bar{J}_2^0 \rho_0^0(l)/2$, which results from the anisotropic component ($\bar{J}_2^0$) of the radiation field, we now have three extra rates that are due to transfer of atomic polarization from the lower level to the upper level. The most important of these three rates is the transfer rate due to the isotropic component ($\bar{J}_0^0$) of the radiation field, i.e. $T_3 = -B_{lu} \bar{J}_0^0 \rho_2^0(l)/2$. The rate $T_5 = C_{lu}^{(2)} \rho_0^2(l)$ is to be interpreted as transfer of atomic alignment from the lower to the upper level by inelastic collisions, where $C_{lu}^{(2)}$ is the $K = 2$ multipole component of the inelastic collisional rate. These three extra rates (but mainly $T_3$!) produce an additional contribution to the atomic alignment of the upper level which, in turn, implies an extra contribution to the linear polarization emitted by the atoms at each spatial point. It is also important to point out that there is a further mechanism that contributes to the emergent linear polarization, and that it also arises from the presence of atomic alignment in the lower level. This is due to the term $-\eta Q I$ in Eq. (8), with $\eta Q$ given by Eq. (10), i.e. to the differential absorption due to the unequally populated sublevels of the lower level. As discussed above, this is the only mechanism that leads to linear polarization in $1 \rightarrow 0$ transitions.

The transfer equations for the Stokes I and Q parameters are given by Eqs. (7) and (8), with the components of the emission vector and of the absorption matrix given by Eqs. (3),(4),(9) and (10). (Note that, for the present case of $1 \rightarrow 1$ transitions, the coefficients $\omega$ and $\bar{Z}$ are such that $\omega = \bar{Z} = -0.5$).
Figure 3. Predictions of the full theory for line transitions in a two-level model atom with \( J_l = J_u = 1 \). The left panel (Fig. 3a) shows the variation with line integrated optical depth of \( \rho_2^0(l)/\rho_0^0(l) \) (dashed line) and of \( \rho_2^0(u)/\rho_0^0(u) \) (solid line). The right panel (Fig. 3b) shows the emergent fractional linear polarization at \( \mu = 0.1 \), which is to be compared with the dashed-line of Fig. 1b.

Figure 3a shows, for the zero depolarizing rate case of \( 1 \to 1 \) transitions, the self-consistent solution for \( \rho_2^0(u)/\rho_0^0(u) \) (solid line) and \( \rho_2^0(l)/\rho_0^0(l) \) (dashed line). As with the previous example, I solved this non-linear and non-local non-LTE problem of the 2\textsuperscript{nd} kind by applying the iterative methods described in the Appendix. The ensuing emergent fractional linear polarization at \( \mu = 0.1 \) is given in Figure 3b. We see, again, that the prediction of the correct two-level atom theory for line transitions with \( J_l = J_u = 1 \) is that the emergent linear polarization profile is as important as that corresponding to triplet lines. In conclusion, in the absence of relaxation mechanisms, three different types of line transitions (which according to the standard theory should show up very different degrees of linear polarization) turn out to lead to similar amounts of linear polarization when the effect of lower-level atomic alignment is taken into account.

4. The Effect of Depolarizing Elastic Collisions

Elastic collisions have a depolarizing role, i.e. they reduce the alignment of the atomic levels. The rate of level depolarization is expected to be comparable with the ordinary collisional broadening rate. Figure 4 shows a rough estimate of the variation with height in the VAL-C solar atmospheric model (Vernazza, Avrett and Loeser, 1981) of the depolarizing rates corresponding...
to the lower and upper levels of the Mg I $b_2$ line, whose Einstein coefficient for spontaneous emission is $A_{ul} = 10^8 \text{s}^{-1}$. The approximate formulae for the calculation of depolarizing rates used here are similar to those presented by Lamb and Ter Haar (1971), which only take into account elastic collisions with neutral hydrogen atoms and are based on the short range of the van der Waals interaction. For reference purposes the dotted line gives the height variation of the temperature in the VAL-C solar model atmosphere. As seen in the figure the rough estimates of the depolarizing rates for the chromospheric Mg $b_2$ line vary between $10^8 \text{s}^{-1}$ in the solar photosphere and $10^3 \text{s}^{-1}$ in the upper chromosphere, with a value of about $10^7 \text{s}^{-1}$ at a height of 500 km.

Figure 4. A rough estimate of the height variation of the depolarizing rates in the VAL-C solar model atmosphere (see the dotted line for its temperature profile) for the lower level (dashed line) and the upper level (solid line) of the Mg I $b_2$ line, which has $A_{ul} = 10^8 \text{s}^{-1}$.

Figure 5 shows, for the case of line transitions with $J_l = J_u = 1$, the emergent fractional linear polarization at $\mu = 0.1$ for the $D_u = D_l = 0$ depolarizing rate case (solid line), and for increasing values of $\delta = D/A_{ul}$, with $D = D_u = D_l$ and $A_{ul}$ the Einstein coefficient for spontaneous emission. As expected, increasing $\delta$-values produce correspondingly decreasing emergent fractional linear polarization signals. Note that for the case with $\delta = 10^{-2}$ (see the long-dashed line) the resulting Q/I is similar to the result provided by the standard theory, which neglects lower-level polarization (see Fig. 1b). In other words, for $\delta = 10^{-2}$ the alignment of the ground level has been already destroyed, while the amount of upper-level alignment is still important and very similar to that predicted by the standard theory. The conclusion is that the alignment of the ground level of optical resonance-line
Figure 5. The emergent fractional linear polarization for line transitions with \(J_l = J_u = 1\), and for various depolarizing rate values \(\delta\) (measured in units of the Einstein \(A_{ul}\) coefficient). The dashed lines show the emergent Q/I profiles for \(\delta = 10^{-5}\), \(\delta = 10^{-4}\), \(\delta = 10^{-2}\), \(\delta = 10^{-1}\) and \(\delta = 1\), from the largest to the smallest plotted linear polarization signals, respectively. The solid line refers to the zero depolarizing rate case.

transitions is very sensitive to depolarizing elastic collisions and that only if \(\delta < 10^{-5}\) can we expect to have the maximum Q/I signal possible.

5. A Possible Multilevel Solution to the Na Solar Paradox

The explanation of the linear polarization pattern around the Na D\(_2\) and D\(_1\) lines proposed by Landi Degl’Innocenti (1998) implies the absence of depolarizing effects due to magnetic fields and/or to elastic collisions. As pointed out by Landi Degl’Innocenti (1998) this apparently leads to a paradox because, on the one hand, his two-level atom modelling, characterized by an amount of ground-level atomic polarization larger than that of the upper level, beautifully explains the polarization profile of the Na doublet. On the other hand, this ground-level polarization cannot survive in the presence of turbulent or canopy-like horizontal fields stronger than 0.01 gauss, which contradicts previous evidence (see Jones, 1984; Solanki and Steiner, 1990; Faurobert-Scholl, 1992; Bianda et. al, 1998; Stenflo et. al, 1998). In principle, the presence of atomic polarization in the ground level of Na I also seems difficult to understand because, as shown above, even depolarizing collisional rates with \(\delta\)-values as small as \(\delta = 10^{-4}\) would significantly reduce the amount of ground-level atomic alignment that is needed to fit the observations according to Landi Degl’Innocenti’s (1998) modelling.

Can one think of a multilevel scenario for Na that might help resolve this “Na Solar Paradox”? The answer is affirmative. The following multilevel
scenario for Na is motivated by the two-level atom calculations presented in Section 3 and also by multilevel Na modelling without polarization physics. In fact, it is known that the two-level atom approximation does not apply to minority species like Na I, because the difference in coupling to the continuum reservoir between the two levels affects the populations strongly (Bruls, Rutten and Shchukina, 1992). The two-level atom rate Eqs. (6) and (11) can also be invoked to point out that the amount of atomic alignment of the upper level of a line transition in a multilevel atomic model is expected to be different from that predicted by a two-level atom approach.

On the one hand, extra contributions to \( \rho_0^2(u) \) (with \( u \) the upper level of the line transition being considered) can come from the transfer of atomic alignment from lower levels “i” with excitation energy \( E_i < E_u \), including also the contribution due to the \( K = 2 \) multipole component of the upward inelastic collisional rate \( (C^{(2)}_{1,u}) \). On the other hand, \( \rho_0^2(u) \) can be modified via the radiative rates that are due to absorption to higher levels “j” with \( E_j > E_u \) and also because, in a multilevel model atom, we can have transfer of alignment from higher levels “j” to the level “u”, including also contributions that can arise via the \( K = 2 \) multipole component of the downward inelastic collisional rate \( (C^{(2)}_{j,u}) \).

For the Na D\(_2\) and D\(_1\) lines their lower level \( ^2S_{1/2} \) is the ground level and in the multilevel scenario being suggested here there is no atomic polarization in this ground level because it is completely destroyed by both magnetic fields and elastic collisions. The extra contributions to the alignment of the hyperfine components of the upper levels \( ^2P_{3/2} \) and \( ^2P_{1/2} \) of the D\(_2\) and D\(_1\) lines, which is needed to explain the observations, can then only come from the transfer of alignment from higher levels and from the relaxation rates due to absorption processes from these two upper levels to higher ones. By having a detailed look at multilevel atomic models for Na it is easy to see that there are radiative transitions from the levels \( ^2P_{3/2} \) and \( ^2P_{1/2} \) to higher levels in the Grotrian diagram that are probably influenced by optical pumping processes. In doing this complicated modelling for the Na atom it will be also of interest to take into account the transfer of atomic alignment, among the hyperfine components of the two levels \( ^2P_{3/2} \) and \( ^2P_{1/2} \), due to the multipole components of inelastic collisional rates. Since the lifetime of non-ground atomic levels is about two orders of magnitude smaller than the ground-level lifetime, one finds that neither low-gauss magnetic fields nor elastic collisions with rates not larger than \( A_{ul} \) can destroy the atomic polarization of such excited levels. Note that in this multilevel scenario there is no contribution to the emitted linear polarization due to dichroism (i.e. to the term \( -\eta Q I \) of Eq. 8), simply because we are saying that the Na ground-level atomic polarization is practically destroyed by depolarizing mechanisms.
Speculations apart, what it is really important to emphasize is that a better understanding of the observed linear polarization pattern around the Na D lines (and of the second solar spectrum in general!) can only be achieved after carefully formulating and solving multilevel non-LTE problems of the 2nd kind, taking fully into account the possibility of atomic polarization at all the atomic levels and considering the depolarizing role of elastic collisions and magnetic fields. To this end, it is imperative to develop a general multilevel scattering line polarization code. For this, we had first to develop some suitable iterative methods for the solution of non-LTE problems of the 2nd kind that are briefly described in the Appendix.

6. Observational Search for Lower-level Atomic Polarization and the Mg Solar Paradox

In an observational search for lower-level atomic polarization the first choice of lines to look at are those suggested in the paper of Trujillo Bueno and Landi Degl’Innocenti (1997), i.e. lines with $J_l = 1$ and $J_u = 0$ and ideally corresponding to atoms devoid of hyperfine structure. These are “null” lines because the standard theory predicts zero linear polarization for them and the only mechanism capable of producing non-zero line polarization values is the possible presence of lower-level atomic polarization. A useful list with this type of line transition can easily be obtained by selecting only the unblended lines from a long list of spectral lines with $J_l = 1$ and $J_u = 0$ that can be automatically found by computer search in the whole spectrum. Jorge Sánchez Almeida and I followed this strategy and, in September 1997, we made an observing run using Semel’s technique (Semel, 1994; see also Bianda et. al., 1998) with the Gregory-Coudé Telescope operated by Göttingen University at the Spanish Observatorio del Teide (Tenerife, Spain). Unfortunately, we could not reach any definitive conclusion because, due to non-linearities in the CCD camera used, we failed to reach the good polarimetric sensitivity that should be achievable with Semel’s technique. A similar observational programme is being pursued independently by other colleagues using the polarimeter ZIMPOL attached to the McMath solar telescope at Kitt Peak Observatory (Stenflo 1998; private communication). This is an instrumental set-up that can reach very high polarimetric sensitivity and it is likely that their search will soon lead to some useful results.

Perhaps a good idea for exploring whether lower-level atomic polarization is at work in the solar photosphere is that of performing spectropolarimetric observations in two lines of the same multiplet, having very similar line formation properties, but one having $J_l = 1$ and $J_u = 0$ (i.e. a “null” line) and the other being a triplet line (i.e. with $J_l = 0$ and $J_u = 1$). An example can be found in the Si I 5701.11 and the Si I 5665.55 lines, respec-
tively. It is also of interest to mention that, in our list, there is no “null” line whose lower level is the ground level. The lower level is always an excited level whose lifetime is not much larger than the lifetime of the upper level. Although the observation of linear polarization in such lines would be an irrefutable proof of the existence of lower-level atomic polarization, the detection of Hanle depolarization in the same lines would imply the existence of magnetic fields of the order of a few gauss. The situation is different from the case analyzed by Landi Degl’Innocenti (1998) because, for the sodium D lines, the lower level is the ground level and a magnetic field of the order of few mgauss is sufficient to destroy its atomic polarization.

One can also try to obtain some observational hints about the importance of lower-level atomic polarization for understanding the second solar spectrum by using different line transitions. For instance, in this paper we have also paid some attention to lines with \( J_l = J_u = 1 \). A very interesting line with these total angular momentum values is the Mg I \( b_2 \) line at 5172.68 Å, which belongs to the multiplet \(^3\)P\(^o\) \(-\) \(^3\)S. In this same multiplet we find the Mg I \( b_1 \) line at 5183.60 Å (which is a \( J_l = 2 \rightarrow J_u = 1 \) transition) and the Mg I \( b_4 \) line at 5167.32 Å (which is a \( J_l = 0 \rightarrow J_u = 1 \) transition). We point out that 90% of Mg has zero nuclear spin.

These three lines share the same upper level (\(^3\)S\(_1\)) that has \( J_u = 1 \). Assume that, at each point in the solar atmosphere, we know the exact value of the fractional alignment coefficient \( \beta_u = \rho_0^2(u)/\rho_0^0(u) \) of this upper level. If one now calculates at the line centre the emergent fractional linear polarization \((Q/I)\) assuming that there is no atomic polarization in their corresponding lower levels one finds values proportional to \(-0.1, +0.5\) and \(-1\) for the Mg \( b_1, b_2 \) and \( b_4 \) lines, respectively, and where the minus sign indicates that the electric vector is parallel to the nearest solar limb, while the plus sign refers to the perpendicular direction. However, this prediction of the standard polarization transfer theory is not correct because old polarimetric observations (Stenflo et. al., 1983) and recent ZIMPOL observations of the second solar spectrum (Stenflo, 1998; private communication) show that the emergent fractional linear polarization in the Mg \( b_1, b_2 \) and \( b_4 \) lines are similar. It is informative to mention that all the workshop participants could see the \( Q/I \) plot of the \( J_l = 0 \rightarrow J_u = 1 \) Mg I \( b_4 \) line stamped on our workshop identification cards.

What emergent \( Q/I \) signal would we find if in the lower levels of the Mg \( b_1 \) and \( b_2 \) lines (which have \( J_l = 2 \) and \( J_l = 1 \), respectively) we had an amount of atomic alignment larger than the one of the upper level? As discussed above, the presence of atomic alignment in the lower level of a line transition leads to an extra contribution to the linear polarization originating at each atmospheric point that comes from the term \(-\eta_Q I\) of Eq. (8), with \( \eta_Q \sim Z \rho_0^0(l) \) (see Eq. 10). This contribution adds to the usual
one \((\epsilon_Q)\) that is proportional to \(\omega \rho_0^2(u)\) (see Eq. 4). It is not difficult to show that the following expression holds at \(\tau_\nu \approx 1\) for the purpose of estimating the emergent fractional linear polarization:

\[
\frac{\eta_Q I}{\epsilon_Q} \approx \frac{Z}{\omega} \frac{\beta_l}{\beta_u},
\]

(12)

where \(\beta_l = \rho_0^3(l)/\rho_0^3(0)\) and \(\beta_u = \rho_0^2(u)/\rho_0(0)\). Using this expression and Eqs. (3), (4) and (10) we find that, close to the limb, the emergent fractional linear polarization at the line centre is approximately given by

\[
\frac{Q_I}{I} \approx -\omega \beta_u + Z \beta_l,
\]

(13)

where the second contribution is that due to dichroism. Taking into account, from Table I of Landi Degl’Innocenti (1984), that \(Z = \omega = -0.5\) for the \(1\rightarrow 1\) \(b_2\) line and that \(Z \approx 0.6\) and \(\omega = 0.1\) for the \(2\rightarrow 1\) \(b_1\) line, and assuming that \(|\beta_l| > |\beta_u|\) for the Mg \(b_1\) and \(b_2\) lines, one arrives at the conclusion that the presence of a sizable amount of lower-level atomic alignment would imply that the emergent fractional linear polarization in the Mg \(b_1\), \(b_2\) and \(b_4\) lines would be similar, as the above-mentioned observations show.

Finally, it is necessary to note that the lower levels of the Mg \(b\) lines are metastable. The \(^3P_1\) level is indeed connected to the ground level \(^1S_0\) by a forbidden transition at 4571 Å, but its lifetime for spontaneous de-excitations is of the order of \(5 \times 10^{-3}\) s, whereas the lifetime of the other two levels \((^3P_0\) and \(^3P_2\)) is even larger. Thus, if levels \(^3P_2\) and \(^3P_1\) (the lower levels of the \(b_1\) and \(b_2\) lines, respectively) turn out to have an amount of atomic polarization that is larger (in absolute value) than that of the upper level, we would end up with a conclusion similar to that found by Landi Degl’Innocenti (1998) using the Na D lines argument, i.e. that magnetic fields stronger than about a few mgauss, either in the form of volume-filling turbulent fields or in the form of canopy-like, horizontal fields, cannot exist in the solar chromosphere. Although the line formation regions of the Mg \(b\) lines are much higher than those of the Mg 4571 Å line, I think that careful spectropolarimetric limb observations in the Mg 4571 \(J_l = 0 \rightarrow J_u = 1\) line should be carried out urgently. This may tell us whether the lower level of the Mg \(b_2\) line is polarized at the atmospheric heights of formation of the Mg 4571 Å line (\(~ 400\) km). One should keep in mind, however, that this line is formed much closer to LTE than other lines of similar strength because it is an optically forbidden intercombination line. In any case, a new paradox seems to exist: the “Mg solar paradox”. 

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7. Concluding Remarks

There is a crucial difference between the Na and Mg “solar paradoxes”. For Na there was still the chance that the multilevel scenario outlined above might help to solve it. However, there is no similar multilevel solution for the “Mg solar paradox”, which leads to the conclusion that there must indeed exist ground-level and metastable-level atomic polarization in the solar chromosphere. The three Mg $b$ lines share the same upper level and what cannot be understood by means of the standard transfer theory (which neglects lower-level polarization) is that the observed linear polarization in these three Mg $b$ lines turns out to be similar. Thus, even if one chooses multilevel atomic models for Mg to calculate the self-consistent values of the density-matrix elements, and then solves the Stokes transfer equations to get the emergent fractional linear polarization, but neglecting the dichroism contribution that comes from the atomic alignment of the lower metastable levels, one would find again that the ensuing prediction is wrong. The only way I see for increasing the emitted polarization in the Mg $b_1$ and $b_2$ lines, so as to bring it to the same level of that corresponding to the Mg $b_4$ line (that has $J_l = 0$), is via the dichroism contribution (i.e. the term $-\eta_{QI}$ of Eq. 8). As discussed in Section 3, this dichroism can only arise if the magnetic sublevels of the $J_l = 2$ and $J_l = 1$ metastable lower-levels of the $b_1$ and $b_2$ lines are unequally populated. The same explanation can be given for other groups of lines belonging to other atoms, and arising from a similar multiplet ($3^3P^o - 3^3S$), with the ensuing appearance of extra “paradoxes” for other chemical elements.

As we have seen, the anisotropic illumination of the atoms in a stellar atmosphere can lead to large population imbalances among the lower-level sublevels of many spectral lines. The modelling of the second solar spectrum requires the reliable calculation of the atomic polarization of the lower and upper levels corresponding to the line transitions of interest. To this end, it is crucial to be able to consider multilevel atomic models. This goal can presently be achieved by formulating the problems of interest within the framework of the density matrix polarization transfer theory (see Landi Degl’Innocenti, 1983) and by numerically solving the ensuing non-linear and non-local equations with the iterative methods presented in the Appendix. Only when we know the self-consistent values of the alignment coefficients of the Na and Mg atomic levels in several solar atmospheric models shall we be able to figure out a possible solution to such “solar paradoxes”.

Obviously, we are facing a complex problem here, from the observational, theoretical and modelling viewpoints. But it is a highly interesting one, not only because of the fascinating physics that it involves, but mainly because in trying to clarify it we may learn something new about the sun.
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APPENDIX

Iterative methods for the non-LTE problem of the 2\textsuperscript{nd} kind

In general, both for two-level and multilevel atomic models, with or without hyperfine structure included, but taking fully into account atomic polarization in all the atomic levels, the above-mentioned non-linear and non-local system of equations can be symbolically represented as

\[ \mathbf{A} \mathbf{x} = \mathbf{b}, \]

with \( \mathbf{b} \) a known vector and \( \mathbf{x} \) the unknown vector formed by the density matrix elements at all the spatial grid-points, and where \( \mathbf{A} \) is an operator which depends on collisional rates (both inelastic and elastic) and on radiation field tensors that are given by weighted frequency and angular averages of the Stokes parameters. In these SE equations one finds non-linear terms of the form \( \bar{J}^{K}_{Q} \rho^{k}_{q} \) (see Eqs. 6 and 11). They are non-linear because the radiation field tensors \( \bar{J}^{K}_{Q} \) depend implicitly on the density matrix elements \( \rho^{k}_{q} \) via the RT equations. This non-linearity means that the operator \( \mathbf{A} \) depends implicitly on the unknown \( \mathbf{x} \). The solution of non-linear problems necessarily requires the application of iterative methods. Here, at each iterative step, one has to manage to set up and solve a suitable linear system of equations whose solution leads to approximate corrections to the unknowns (see Socas Navarro and Trujillo Bueno, 1997). To this end, it is very important that the approximations one introduces for achieving linearity at each iterative step adequately treat the coupling between transitions and the non-locality of the problem.

It is also very important to point out that iterative schemes that require the construction and inversion of large matrices are useless. Thus, my first step towards the modelling of the second solar spectrum has been the development of two effective iterative methods that are indeed capable of
solving non-linear polarization transfer multilevel problems of the 2nd kind without having to build and invert large matrices at each iterative step. I have called these methods the DALI and DEGAS iterative schemes.

DALI, besides the name of the famous Spanish painter, is an acronym for Density-matrix ALI method (which is based on Jacobi iteration), while DEGAS, besides the name of the fine French painter, refers here to my Density-matrix Gauss-Seidel iterative scheme.

As pointed out above the non-linear terms appearing in the SE equations are of the form $\vec{J}_K^k \vec{Q}_q \rho_{kq}$. A detailed presentation of these iterative methods will be published elsewhere. Here I simply give a “numerical recipe” for implementing the DALI and DEGAS methods. It consists in making the following changes for achieving linearity in the SE equations at each iterative step:

If $K \neq 0$,

$$\bar{J}_Q^k \rho_q^k \rightarrow \bar{J}_Q^k \rho_q^{old} \rho_q^{new} \quad (15)$$

If $K = 0$,

$$\bar{J}_0^k \rho_q^k \rightarrow \bar{J}_0^k \rho_q^{new} + \Lambda_0^0(i,i) \left[ \rho_q^{old} \rho_0^{new}(u) - \rho_0^{old}(u) \rho_q^{new} \right], \quad (16)$$

where $\Lambda_0^0(i,i)$ ("i" being the spatial grid-point under consideration) is the diagonal element of a $\Lambda-$ operator that arises in the definition of $\bar{J}_0^k$, and where "old" is meant to take the value of the previous iterative step, while "new" simply indicates the density-matrix elements that are to be obtained at the current iterative step by simply solving the resulting linear system of equations. In the DALI method we take $\bar{J}_0^* = \bar{J}_0^{old}$, while for DEGAS $\bar{J}_0^* = \bar{J}_0^{old and new}$, with these two quantities (and also $\Lambda_0^0(i,i)$) calculated, at each iterative step, as explained by Trujillo Bueno and Fabiani Bendicho (1995) (see also Trujillo Bueno and Manso Sainz, 1999).

Figure 6 shows an example of the convergence rate of the DALI method for a calculation initialized using the LTE density-matrix $\rho$-values. With the DEGAS multilevel iterative scheme the number of iterations (and the computing time) required to achieve convergence is substantially smaller.

For the solution of scattering line polarization problems using realistic multilevel atoms it is better to initialize the calculation using the $\rho_0$ self-consistent values corresponding to the unpolarized case. Since these $\rho_0$ values can be directly obtained from the atomic level populations such an initialization can be found easily by using any of the fast RT multilevel codes that are now available (see, e.g., Socas Navarro and Trujillo Bueno,
Figure 6. The convergence rate of the DALI iterative method applied for numerically solving the $J_1 = J_u = 1$ scattering line polarization problem taking into account atomic alignment in both levels. The solid line gives the variation with the iterative step of the maximum relative change in $\rho_0^u(u)$ and $\rho_0^l(l)$, the dotted line in $\rho_0^u(u)$ and the dashed line in $\rho_0^l(l)$.

1997; Fabiani Bendicho and Trujillo Bueno, 1999). With this initialization given, it is possible to obtain the self-consistent solution of multilevel scattering line polarization problems formulated with the density matrix theory applying one of the three following methods: $\Lambda$-iteration, DALI iteration or DEGAS iteration, with increasing improvements in the resulting convergence rate.

References

Bianda, M., Solanki, S.K., Stenflo, J.O. (1998), Astron. Astrophys., 331, 760
Bommier, V. (1997a), Astron. Astrophys., 328, 706
Bommier, V. (1997b), Astron. Astrophys., 328, 726
Bommier, V., Sahal-Bréchet (1978), Astron. Astrophys., 69, 57
Bruils, J., Rutten, R.J., Shechukina, N.G. (1992), Astron. Astrophys., 265, 237
Fabiani Bendicho, P., Trujillo Bueno, J. (1999), in Solar Polarization, edited by K.N. Nagendra & J.O. Stenflo, Kluwer Academic Publishers, 1999. (Astrophysics and Space Science Library ; V. 243), p. 219-230
FauRobert-Scholl, M. (1992), Astron. Astrophys., 258, 521
Happer, W. (1972), Rev. Mod. Phys., 44, 169
Jones, H. P. (1984), in “Chromospheric Diagnostics and Modelling”, B. Lites (ed.), National Solar Observatory
Lamb, F. K., Ter Haar, D. (1971), Physics Reports, 2, No. 4, 253
Landi Degl’Innocenti, E. (1983), Solar Phys., 85, 3
Landi Degl’Innocenti, E. (1984), Solar Phys., 91, 1
Landi Degl’Innocenti, E. (1987), in Numerical Radiative Transfer, W. Kalkofen (ed.),
Landi Degl’Innocenti, E. (1998), *Nature*, **392**, 256
Landi Degl’Innocenti, E., Bommier, V., Sahal-Bréchot, S. (1990), *Astron. Astrophys.*, **235**, 459
Landi Degl’Innocenti, E., Landi Degl’Innocenti, M., Landolfi, M. (1997), in *Science with Themis*, N. Mein and Sahal-Bréchot (eds.), Paris Obs. Publ., p. 59
Landolfi, M., Landi Degl’Innocenti, E. (1986), *Astron. Astrophys.*, **167**, 200
Semel, M. (1994), in *Solar Surface Magnetism*, R.J. Rutten and C.J. Schrijver (eds.), Kluwer, p. 509
Socas Navarro, H., Trujillo Bueno, J. (1997), *Astrophysical Journal*, **490**, 383
Solanki, S., Steiner, O. (1990), *Astron. Astrophys.*, **234**, 519
Stenflo, J.O. (1994) *Solar Magnetic Fields. Polarized Radiation Diagnostics*. Kluwer Academic Publishers, Dordrecht.
Stenflo, J.O. (1997), *Astron. Astrophys.*, **324**, 344
Stenflo, J.O., Twerenbold, D., Harvey, J. W., Brault, W. (1983), *Astron. Astrophys. Suppl. Ser.*, **54**, 505
Stenflo, J.O., Keller, C.U. (1997), *Astron. Astrophys.*, **321**, 927
Stenflo, J.O., Keller, C.U., Gandorfer, A. (1998), *Astron. Astrophys.*, **329**, 319
Trujillo Bueno, J. and Landi Degl’Innocenti, E. (1997), *Astrophysical Journal Letters*, **482**, 183
Trujillo Bueno, J., Mano Sainz, R. (1999), *Astrophys. Journal*, **516**, 436
Vernazza, J., Avrett, E., Loeser, R. (1981), *Astrophys. J. Suppl. Series*, **45**, 635