MINIMUM SAMPLE SIZE ALLOCATION IN STRATIFIED SAMPLING UNDER CONSTRAINTS ON VARIANCE AND STRATA SAMPLE SIZES

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Abstract

We derive optimality conditions for the optimal sample allocation problem, formulated as the determination of the fixed strata sample sizes that minimize total sample size, under assumed level of the variance of the stratified $\pi$-estimator and one-sided upper bounds imposed on strata sample sizes. In this paper, this problem is considered in the context of general stratified sampling scheme that includes simple random sampling without replacement design within strata as a special case. Based on established optimality conditions, we create a new algorithm, the $LrNa$, that solves the allocation problem defined above. This new algorithm has its origin in popular recursive Neyman allocation procedure $rNa$, that is used to solve classical optimal sample allocation problem (i.e. minimization of the $\pi$-estimator’s variance under fixed total sample size) with only one-sided upper bounds constraints imposed on strata sample sizes (see e.g. Särndal, Swensson, and Wretman [1992, Remark 12.7.1, p. 466], or Wesołowski et al. [2021]). Ready-to-use R-implementation of the $LrNa$ is available on CRAN repository at https://cran.r-project.org/web/packages/stratallo.

Keywords: stratified sampling, optimal allocation, optimal allocation lower bounds, optimal allocation upper bounds, optimal allocation constant variance, minimum sample size allocation, recursive Neyman algorithm, convex optimization, Karush–Kuhn–Tucker conditions

1 INTRODUCTION

Let us consider a finite population consisting of $N$ distinct elements and denoted as $U = \{1, 2, \ldots, N\}$. Let the parameter of principal interest be the population total of a single study variable $Y$ in $U$, i.e.

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\[ t_Y = \sum_{k \in U} Y(k), \] where \( Y(k) \) denotes the value of \( Y \) for population element \( k \in U \). To estimate \( t_Y \), we consider the stratified sampling. Under this well-known technique, population \( U \) is stratified, i.e. \( U = \bigcup_{w \in W} U_w \), where \( U_w, w \in W \), called strata, are pairwise disjoint and non-empty, and \( W \) denotes a finite set of strata labels. The size of stratum \( U_w \) is denoted \( N_w, w \in W \) and clearly \( \sum_{w \in W} N_w = N \). A probability sample of size \( n_w \) is selected from each stratum \( U_w, w \in W \), according to a chosen sampling design, which is often the same in all strata. The resulting total sample will be of size \( n = \sum_{w \in W} n_w \).

In stratified sampling, the stratified \( \pi \)-estimator \( \hat{t}_{st} \) of the population total \( t_Y \) and the variance \( D^2_{st} \) of the estimator \( \hat{t}_{st} \), can be worked out for different sampling designs (see e.g. Särndal, Swensson, and Wretman [1992, Result 3.7.1, p. 102]). Suppose that the sampling designs used within strata have been decided and they are such that

\[ D^2_{st}(n_w, w \in W) = \sum_{w \in W} \frac{a^2_w}{n_w} - b, \tag{1} \]

where \( a_w > 0 \) and \( b \) do not depend on the \( n_w, w \in W \).

The problem of optimal sample allocation that we are interested in this paper is formulated as the determination of the sample allocation \( (n_w)_{w \in W} \) that minimizes total sample size \( n \), under assumed fixed level of the variance \( D^2_{st} \) of the stratified \( \pi \)-estimator and upper bounds imposed on the sample sizes. We phrase this problem in the language of mathematical optimization as follows.

**Problem 1.1.** Given numbers \( a_w > 0, M_w, \) such that \( 0 < M_w \leq N_w, w \in W \), and \( b, D > \sum_{w \in W} \frac{a^2_w}{M_w} - b > 0 \),

\[
\text{minimize } \quad n(x) = \sum_{w \in W} x_w \\
\text{subject to } \quad \sum_{w \in W} \frac{a^2_w}{x_w} - b = D \\
\quad x_w \leq M_w \quad \text{for all } w \in W,
\]

where \( x = (x_w)_{w \in W} \) is the optimization variable and \( \#W \) denotes cardinality of set \( W \).

Among the stratified sampling designs that fall into the scheme of Problem 1.1 is Stratified Simple Random Sampling Without Replacement design, which is one of the most basic and commonly used stratified sampling designs. It yields: \( x_w = n_w, a_w = N_wS_w, w \in W \) and \( b = \sum_{w \in W} N_wS_w^2 \), where \( S_w, w \in W \), denote stratum standard deviations of study variable \( Y \) (see e.g. Särndal, Swensson, and Wretman [1992, Result 3.7.2, p. 103]). The upper bounds \( M_w \) imposed on \( x_w, w \in W \), are natural, since for instance the allocation with \( x_w > N_w \) for some \( w \in W \) is impossible. We also assume that
\[ D > \sum_{w \in W} \frac{a^2}{M_w} - b. \] If \( D < \sum_{w \in W} \frac{a^2}{M_w} - b, \) the problem is infeasible, and if \( D = \sum_{w \in W} \frac{a^2}{M_w} - b, \) the solution is simply \( x^* = (M_w)_{w \in W}. \) Because allocation vector \( x^* \) that solves Problem 1.1 is almost never an integer one, one might round it up in practice with the resulting variance (1) being possibly near \( D, \) instead of the exact \( D. \)

Optimum sample allocation Problem 1.1 is not only a theoretical problem, but it is also an issue of substantial practical importance. Usually, an increase in the number of samples entails greater costs of the data collection process. Thus, it is often demanded that total sample size \( n \) be somehow minimized. On the other hand, the minimization of the total sample size, should not cause significant reduction of the quality of the estimation, which is measured by the variance (1) in this case. Hence, Problem 1.1 arises very naturally in the planning of sample surveys, when it is necessary to obtain an estimator \( \hat{t}_{st} \) with some predetermined precision \( D, \) that ensures the required level of estimation quality, while keeping the total sample size \( n \) as small as possible. Problem 1.1 appears also in the context of optimum stratification and sample allocation between subpopulations in Lednicki and Wieczorkowski [2003]. The authors of this paper incorporate variance equality constraint into the objective function and then use numerical algorithms (for minimization of a non-linear multivariate function) to find the minimum of the objective function. If the solution found violates any of the inequality constrains, then the objective function is properly adjusted and the algorithm is re-run again. See also a related paper by Wright, Noble, and Bailer [2007], where the allocation under the constraint of the equal precision for estimation of the strata means was considered.

It is worth noting that in the optimal allocation problem under consideration, it is required that variance \( D^2_{st}, \) defined in (1), be equal to a certain fixed value, here denoted \( D, \) and not less than or equal to it. Taking into account the above-described practical context in which Problem 1.1 occurs, it might seem more favourable at first, to require that the variance \( D^2_{st}, \) be less than or equal to \( D. \) It is easy to see however, that the objective function \( n \) and the variance \( D^2_{st} \) are of such a form that the minimum of \( n \) is achieved for a value that yields the maximum allowable value of the variance, i.e. \( D. \) Thus, whether the variance constraint is an equality or an inequality constraint, the optimal solution will be the same.

Our approach to the allocation Problem 1.1 will be twofold. First, in Section 2 we make use of convex optimization methodology to establish necessary and sufficient conditions, so called optimality conditions, for a solution to slightly reformulated optimization Problem 1.1 In Section 3 based on the established optimality conditions, we propose an algorithm which solves Problem 1.1. We call the new
algorithm $LrNa$, to emphasize its descent from the recursive Neyman algorithm, or $rNa$. The $rNa$ solves classical optimum sample allocation problem with added one-sided upper bounds constrains imposed on strata sample sizes, i.e. Problem [B.2]. See Appendix [B] for more details.

2 OPTIMALITY CONDITIONS

In this section, we establish a general form of the solution to (somewhat reformulated) Problem [1.1] so called optimality conditions. These optimality conditions can be derived reliably and efficiently using convex optimization methodology that studies convex optimization problems, i.e. problems of minimizing convex functions over convex sets (see e.g. Boyd and Vandenberghe [2004]). Problem [1.1] is however not a convex optimization problem as the equality constraint function $\sum_{w \in W} a_w^2 x_w - b - D$ of $x = (x_w)_{w \in W}$ is not affine, and hence the feasible set is not convex. Yet, it turns out that Problem [1.1] can easily be formulated as a convex optimization Problem [2.1] by a simple change of its optimization variable and the parameters, i.e.

\[
y_w := \frac{a_w^2}{x_w}, \quad w \in W,\\
m_w := \frac{a_w^2}{M_w}, \quad w \in W,\\
\tilde{D} := D + b.
\]

**Problem 2.1.** Given numbers $a_w > 0$, $m_w > 0$, $w \in W$, and $\tilde{D} > \sum_{w \in W} m_w$,

\[
\begin{align*}
\text{minimize} \quad & f(y) = \sum_{w \in W} a_w^2 y_w \\
\text{subject to} \quad & \sum_{w \in W} y_w = \tilde{D} \\
& m_w \leq y_w \quad \text{for all } w \in W,
\end{align*}
\]

where $y = (y_w)_{w \in W}$ is the optimization variable and $\#W$ denotes cardinality of set $W$.

The auxiliary optimization Problem [2.1] is indeed a convex optimization problem, since its objective function

\[
f(y) = \sum_{w \in W} a_w^2 y_w
\]

and inequality constraint functions

\[
g_w(y) = -y_w + m_w, \quad w \in W,
\]
are convex functions, and the equality constraint function
\[ h(y) = \sum_{w \in \mathcal{W}} y_w - \tilde{D} \] (5)
is affine (see Appendix A). Furthermore, minimization Problem 2.1 is feasible (due to \( \tilde{D} > \sum_{w \in \mathcal{W}} m_w \) requirement) and bounded with strictly convex objective function \( f \). Hence, \( f \) attains its strict global minimum on a feasible set at some \( y^* = (y_w^*)_{w \in \mathcal{W}} \). Consequently, due to (2), vector
\[ x^* = \left( \frac{a_w}{y_w^*} \right)_{w \in \mathcal{W}}, \] (6)
is a unique global optimal solution of Problem 1.1. For these reasons, for the remaining part of this work, we focus on methods and algorithms that provide a solution to Problem 2.1. We also note here that solution to Problem 2.1 is trivial in case of \( \tilde{D} = \sum_{w \in \mathcal{W}} m_w \), i.e. \( y^* = (m_w)_{w \in \mathcal{W}} \).

Before we establish necessary and sufficient optimality conditions for a solution to convex optimization Problem 2.1, we first define a set function \( s \) which greatly simplifies notation and many calculations in this and subsequent section. Following Problem 2.1 throughout the rest of this paper we assume that \( a_w > 0, m_w > 0, w \in \mathcal{W} \) and \( \tilde{D} > \sum_{w \in \mathcal{W}} m_w \) are given numbers. For any \( V \subseteq \mathcal{W} \) we denote \( V^c = \mathcal{W} \setminus V \).

**Definition 2.1.** Set function \( s : 2^\mathcal{W} \setminus \{\mathcal{W}\} \rightarrow (0, +\infty) \) is defined as
\[ s(V) = \frac{\tilde{D} - \sum_{w \in V} m_w}{\sum_{w \in V^c} a_w}, \quad V \subseteq \mathcal{W}. \] (7)

Below, we will introduce the notation of \( V \)-allocation vector.

**Definition 2.2.** Let \( V \subseteq \mathcal{W} \). Vector \( x^V = (x^V_w)_{w \in \mathcal{W}} \) with elements of the form
\[ x^V_w = \begin{cases} m_w & \text{for } w \in V, \\ a_w s(V) \neq m_w & \text{for } w \in V^c, \end{cases} \] (8)
will be refereed to as the \( V \)-allocation vector.

It appears that the solution of the Problem 2.1 is necessarily of the form (8).

**Theorem 2.1.** (Optimality conditions) \( V \)-allocation vector \( y^V \) solves Problem 2.1 if and only if
\[ V = \left\{ w \in \mathcal{W} : \frac{a_w}{m_w} s(V) \leq 1 \right\}. \] (9)
Proof. Solution to optimization Problem 2.1 can be identified through the Karush-Kuhn-Tucker (KKT) conditions, a first derivative tests for a solution in nonlinear programming to be optimal. The KKT conditions are not only necessary but also sufficient for convex optimization problem (see Appendix A for more details).

Gradients of the objective function $f$ defined in (3) and constraint functions $h, g_w, w \in \mathcal{W}$, defined in (4) and (5) are as follows

$$\nabla f(y) = -\left(\frac{a_w^2}{y_w^2}\right)_{w \in \mathcal{W}}, \quad \nabla h(y) = 1, \quad \nabla g_w(y) = -\frac{1}{w}, \quad w \in \mathcal{W},$$

where $1$ is a vector with all entries 1 and $1_w$ is a vector with all entries 0 except the entry with the label $w$, which is 1. Consequently, the KKT conditions (21) assume the following form for the optimization Problem 2.1

$$-a_w^2 \frac{y_w^2}{y_v^2} + \lambda - \mu_w = 0, \quad (10)$$
$$\sum_{w \in \mathcal{V}} y_w^* - \tilde{D} = 0, \quad (11)$$
$$-y_w^* + m_w \leq 0, \quad (12)$$
$$\mu_w (-y_w^* + m_w) = 0, \quad \text{for all } w \in \mathcal{W}, \quad (13)$$

To prove Theorem 2.1 it suffices to show that there exist $\lambda \in \mathbb{R}$ and $\mu_w \geq 0$, $w \in \mathcal{W}$, such that (10) - (13) are all satisfied for $y^* = y^\mathcal{V}$ with $\mathcal{V}$ defined in (9). Let $\lambda = \frac{1}{s(\mathcal{V})}$, where $s(\mathcal{V}) > 0$ is defined in (7), and

$$\mu_w = \begin{cases} \lambda - \frac{a_w^2}{m_w} & \text{for } w \in \mathcal{V}, \\ 0 & \text{for } w \in \mathcal{V}^c. \end{cases} \quad (14)$$

Note that (9) yields $\mu_w \geq 0$ for all $w \in \mathcal{V}$. Then (10) and (11), i.e.

$$\sum_{w \in \mathcal{W}} y_w^\mathcal{V} = \sum_{w \in \mathcal{V}} m_w + \sum_{w \in \mathcal{V}^c} a_w s(\mathcal{V}) = \tilde{D},$$

are satisfied. Inequalities in (12) and equalities (13) are trivial for $w \in \mathcal{V}$ since $y_w^\mathcal{V} = m_w$. For $w \in \mathcal{V}^c$, inequalities (12) follow from (9), as $\frac{a_w}{m_w} s(\mathcal{V}) > 1$, and (13) hold true due to $\mu_w = 0$.

Theorem 2.1 gives the general form of the optimal allocation up to specification of the set $\mathcal{V} \subseteq \mathcal{W}$. The question, how to identify this set that corresponds to the optimal solution $y^* = y^\mathcal{V}$, is the subject of the remaining part of this paper.
3 RECURSIVE NEYMAN ALGORITHM UNDER LOWER BOUNDS CONSTRAINTS

In this section we make use of Theorem 2.1 to create an algorithm, termed \( LrNa \), solving Problem 2.1. The \( LrNa \) is inspired by the popular recursive Neyman algorithm, or \( rNa \), that solves classical optimal allocation problem with only one-sided upper bounds imposed on strata sample sizes (see Appendix B).

It turns out that the \( rNa \) can be easily adopted so that it provides a solution to the optimal allocation problem with one-sided lower bounds only, such as Problem 2.1. Below is the formal definition of the \( LrNa \).

**Algorithm 1 LrNa**

**Input:** \( \tilde{D}, W, (a_w)_{w \in W}, (m_w)_{w \in W} \).

**Require:** \( a_w > 0, m_w > 0, \forall w \in W, \tilde{D} > \sum_{w \in W} m_w \).

**Step 1:** Let \( V_1 = \emptyset, r = 1 \).

**Step 2:** Determine \( R_r = \{ w \in W \setminus V_r : a_w s(V_r) \leq m_w \} \), where set function \( s \) is defined in (7).

**Step 3:** If \( R_r = \emptyset \), set \( r^* = r \) and go to Step 4. Otherwise, set \( V_{r+1} = V_r \cup R_r, r = r + 1 \), and go to Step 2.

**Step 4:** Return \( V_{r^*} \).

**Theorem 3.1.** The \( LrNa \) determines a set of strata \( V_{r^*} \subseteq W \), such that the \( V \)-allocation vector (8) with \( V = V_{r^*} \) is the optimal solution of Problem 2.1.

Before we prove Theorem 3.1, we first reveal the following monotonicity property of set function \( s \), defined in (7), that will be essential to the proof of this theorem.

**Lemma 3.2.** Let \( A \subseteq B \subseteq W \). Then

\[
s(A) \geq s(B) \iff s(A) \sum_{w \in B \setminus A} a_w \leq \sum_{w \in B \setminus A} m_w.
\]  

**Proof.** Clearly, for any \( \alpha, \beta, \delta \in \mathbb{R}, \gamma > 0 \), such that \( \gamma + \delta > 0 \), we have

\[
\frac{\alpha + \beta}{\gamma + \delta} \geq \frac{\alpha}{\gamma} \iff \frac{\alpha + \beta}{\gamma + \delta} \leq \beta.
\]
To prove (15), take
\[ \alpha = \tilde{D} - \sum_{w \in B} m_w, \quad \beta = \sum_{w \in B \setminus A} m_w, \]
\[ \gamma = \sum_{w \in B^c} a_w, \quad \delta = \sum_{w \in B \setminus A} a_w. \]
Then, \( \frac{\alpha}{\gamma} = s(B), \frac{\alpha + \beta}{\gamma + \delta} = s(A), \) and hence (15) holds as an immediate consequence of (16).

**Proof of Theorem 3.1.** According to Theorem 2.1, in order to prove that \( V^\triangledown \)-allocation vector \( y_{V^\triangledown}^* \) is the optimal solution to Problem 2.1, we need to show that for all \( w \in W \)
\[ w \in V^\triangledown \Leftrightarrow \frac{a_w}{m_w} s(V^\triangledown) \leq 1. \]
(17)

For \( r^* = 1 \) we have \( V_{r^*} = \emptyset \) and \( R_1 = \emptyset \), i.e. (17) trivially holds. Let \( r^* > 1 \), then:

**Sufficiency:** First, assume that \( \frac{a_w}{m_w} s(V^\triangledown) \leq 1 \) and \( w \notin V^\triangledown \). Then, Step 3 of LrNa yields \( \frac{a_w}{m_w} s(V^\triangledown) > 1 \) for \( w \notin V^\triangledown \), which is a contradiction.

**Necessity:** By Step 2 of LrNa, we have \( a_w s(V_r) \leq m_w, w \in R_r \), for every \( r \in \{1, \ldots, r^* - 1\} \). Summing these inequalities over \( w \in R_r \), we get the second inequality in (15) with \( A = V_r \) and \( B = V_r \cup R_r \). Then, by Lemma 3.2 the first inequality in (15) follows. Consequently,
\[ s(V_1) \geq \ldots \geq s(V^\triangledown). \]
(18)

Now, assume that \( w \in V^\triangledown \). Thus, \( w \in R_r \) for some \( r \in \{1, \ldots, r^* - 1\} \), and again, using Step 2 of LrNa, we get \( \frac{a_w}{m_w} s(V^\triangledown) \leq 1 \). Consequently, (18) yields \( \frac{a_w}{m_w} s(V^\triangledown) \leq 1 \).

Following Theorem 3.1 and given (2) and (6), the optimal solution of Problem 1.1 is vector \( x^* = (x^*_w)_{w \in W} \) with elements of the form
\[ x^*_w = \begin{cases} M_w & \text{for } w \in V^\triangledown, \\ \frac{\sum_{v \in V^\triangledown} a_v}{D + b - \sum_{v \in V^\triangledown} \frac{a_v}{m_w}} & \text{for } w \in V^\triangledown. \end{cases} \]
(19)

where \( V^\triangledown \subseteq W \) is determined by the LrNa.

Ready-to-use R-implementation of the LrNa is included in stratallo package, which is available on CRAN repository at https://cran.r-project.org/web/packages/stratallo.
4 FINAL REMARKS AND CONCLUSIONS

As already stressed out in Section 1, Problem 1.1 is a sample allocation problem of great practical importance. The fact that it can equivalently be expressed as Problem 2.1, that is essentially the classical optimum sample allocation Problem B.1 with added one-sided lower bound constraints, has few advantages.

First of all, Problem 2.1 might be itself interesting for practitioners, when one treats \( y_w \) as stratum sample size \( n_w \) with imposed lower bound \( m_w < N_w, w \in W \), and \( \tilde{D} \) as a fixed total sample size \( n \). In practice, often the population strata variances \( S^2_w, w \in W \) are rarely known a priori. If they are to be estimated from the sample, it is required that at least \( n_w \geq 2, w \in W \).

Furthermore, it turns out that the existing recursive Neyman algorithm, or rNa, that is dedicated to classical optimum allocation problem with one-sided upper bound constraints, i.e. Problem B.2, can easily be modified to LrNa, which provides an optimal solution to Problem 2.1. Such modification is particularly desirable, given the popularity of the rNa, its simplicity along with relatively high computational efficiency as well as the formal proof of its optimality, which was given only very recently in Wesolowski et al. [2021]. Most of the alternative approaches that could potentially be used to solve Problem 2.1 is based on black-box iterative methods of non-linear programming. For example, capacity scaling algorithm proposed in Friedrich at al. [2015], which gives integer-valued allocation for Problem 2.1 with added upper bound constraints. Nevertheless, as pointed in Friedrich at al. [2015], computational efficiency of these allocation algorithms becomes an issue for cases with "many strata or when the optimal allocation has to be applied repeatedly, such as in iterative solutions of stratification problems."

For other, integer-valued optimal solutions to Problem 2.1 with added upper bound constraints, see e.g. Wright [2017], Wright [2020].

Finally, we would like to emphasize that the optimality conditions established in Theorem 2.1 can be used as a baseline for any new algorithms. These can be derived for instance from existing SGa or coma, dedicated for Problem B.2 (see Wesolowski et al. [2021] or Stenger and Gabler [2005]).
A APPENDIX: CONVEX OPTIMIZATION SCHEME AND THE KKT CONDITIONS

A convex optimization problem is an optimization problem in which the objective function is a convex function and the feasible set is a convex set. In standard form it can be written as

\[
\begin{align*}
\text{minimize} & \quad f(x) \\
\text{subject to} & \quad h_i(x) = 0, \quad i = 1, \ldots, r \\
& \quad g_j(x) \leq 0, \quad j = 1, \ldots, s,
\end{align*}
\]

where \( x \in (0, \infty)^k \) is an optimization variable, the objective function \( f : (0, \infty)^k \to \mathbb{R} \) is convex, constraint functions \( h_i : (0, \infty)^k \to \mathbb{R}, i = 1, \ldots, r \) are affine and \( g_j : (0, \infty)^k \to \mathbb{R}, j = 1, \ldots, s \) are convex. It is well known, see e.g. the monograph [Boyd and Vandenberghe 2004], that if the optimization problem is feasible, an optimal solution can be identified through the set of equations and inequalities known as Karush-Kuhn-Tucker (KKT) conditions, which for convex optimization problems, are not only necessary but also sufficient.

**Theorem A.1** (Karush-Kuhn-Tucker conditions for convex optimization problem). A point \( x^* \in (0, \infty)^k \) is a solution to the convex optimization problem (20) if and only if there exist numbers \( \lambda_i \in \mathbb{R}, i = 1, \ldots, r \) and \( \mu_j \geq 0, j = 1, \ldots, s \), called KKT multipliers, such that

\[
\nabla f(x^*) + \sum_{i=1}^{r} \lambda_i \nabla h_i(x^*) + \sum_{j=1}^{s} \mu_j \nabla g_j(x^*) = 0
\]

\[
\begin{align*}
& \quad h_i(x^*) = 0, \quad i = 1, \ldots, r \\
& \quad g_j(x^*) \leq 0, \quad j = 1, \ldots, s \\
& \quad \mu_j g_j(x^*) = 0, \quad j = 1, \ldots, s.
\end{align*}
\]

For any feasible, bounded, and convex optimization problem with strictly concave objective function, there exists one, globally unique optimal solution.

B APPENDIX: RECURSIVE NEYMAN ALLOCATION

The classical optimum sample allocation problem can be formulated in the language of mathematical optimization as follows.
Problem B.1. Given numbers $a_w > 0$, $w \in \mathcal{W}$, $b$, $n < \sum_{w \in \mathcal{W}} N_w$, 
\[
\begin{align*}
\text{minimize} & \quad f(x) = \sum_{w \in \mathcal{W}} \frac{a_w^2}{x_w} - b \\
\text{subject to} & \quad \sum_{w \in \mathcal{W}} x_w = n
\end{align*}
\]
where $x = (x_w)_{w \in \mathcal{W}}$ is the optimization variable and $\# \mathcal{W}$ denotes cardinality of set $\mathcal{W}$.

The solution to Problem B.1 is given by
\[
x_w = n \frac{a_w}{\sum_{v \in \mathcal{W}} a_v}, \quad w \in \mathcal{W}, \quad (22)
\]

It was established by Tchuprov (Tchuprov [1923]) and Neyman (Neyman [1934]) for Stratified Simple Random Sampling Without Replacement design, for which $a_w = N_w S_w$, $w \in \mathcal{W}$.

The recursive Neyman allocation algorithm, denoted here $rNa$, is a well-established allocation procedure that finds a solution to the classical optimum sample allocation Problem B.1 with only one-sided upper bound constraints, i.e. Problem B.2.

Problem B.2. Given numbers $a_w > 0$, $M_w$, such that $0 < M_w \leq N_w$, $w \in \mathcal{W}$ and $b$, $n < \sum_{w \in \mathcal{W}} M_w$, 
\[
\begin{align*}
\text{minimize} & \quad f(x) = \sum_{w \in \mathcal{W}} \frac{a_w^2}{x_w} - b \\
\text{subject to} & \quad \sum_{w \in \mathcal{W}} x_w = n \\
& \quad x_w \leq M_w \quad \text{for all } w \in \mathcal{W},
\end{align*}
\]
where $x = (x_w)_{w \in \mathcal{W}}$ is the optimization variable and $\# \mathcal{W}$ denotes cardinality of set $\mathcal{W}$.

Let set function $t : 2^{\mathcal{W}} \setminus \{\mathcal{W}\} \to (0, +\infty)$ be defined as
\[
t(\mathcal{V}) = \frac{n - \sum_{w \in \mathcal{V}} M_w}{\sum_{w \in \mathcal{V}} a_w}, \quad \mathcal{V} \subset \mathcal{W}, \quad (23)
\]
where $a_w$, $M_w$, $w \in \mathcal{W}$, $n$ are given numbers as in Problem B.2. The $rNa$ is then defined as follows.
Algorithm 2 rNa

Input: \( n, \mathcal{W}, (a_w)_{w \in \mathcal{W}}, (M_w)_{w \in \mathcal{W}} \).

Require: \( a_w > 0, M_w > 0, \forall w \in \mathcal{W}, n < \sum_{w \in \mathcal{W}} M_w \).

Step 1: Let \( V_1 = \emptyset, r = 1 \).

Step 2: Determine \( R_r = \{ w \in \mathcal{W} : a_w t(V_r) \geq M_w \} \), where set function \( t \) is defined in (23).

Step 3: If \( R_r = \emptyset \), set \( r^* = r \) and go to Step 4. Otherwise, set \( V_{r+1} = V_r \cup R_r, r = r + 1 \), and go to Step 2.

Step 4: Return \( V_{r^*} \).

The optimal solution of Problem B.2 is \( x^* = (x^*_w)_{w \in \mathcal{W}} \) with elements of the form

\[
x^*_w = \begin{cases} 
M_w & \text{for } w \in V_{r^*}, \\
a_w t(V_{r^*}) & \text{for } w \in V_{r^*}^c, 
\end{cases}
\]

where \( V_{r^*} \) is obtained from the rNa and set function \( t \) is defined in (23). See also Särndal, Swensson, and Wretman [1992, 3.7.3, p. 104] for more information on classical optimum sample allocation problem, Särndal, Swensson, and Wretman [1992, Remark 12.7.1, p. 466] for more information on the recursive Neyman procedure, and Wesołowski et al. [2021] for the proof of its optimality.

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