A unified non-interacting-electron theory of the integer and fractional quantum Hall conductance

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We generalize the Laughlin argument for the integer quantum Hall effect of non-interacting two-dimensional electrons by taking into account the full three-dimensional nature of gauge symmetry. This naturally leads to the prediction of the principal experimental results on the quantum Hall effect, including the integer and fractional values and plateau widths of quantized Hall conductance. The approach includes, in a fundamental way, the influence of spin, allowing the description of both single electrons and Cooper pairs. Not requiring the introduction of strongly correlated multi-particle states nor a key role for disorder, the unified analysis sheds light on the profound connection between dimensionality, symmetry, and quantum behavior.

At low temperatures, electrons bound to a plane and subject to a magnetic field normal to it manifest the so-called quantum Hall effect [1–8]. The Hall conductance shows strikingly precise plateaus versus magnetic field intensity at the values $G_{xy} = I/V = νe^2/h$, where $I$ is the current intensity along the plane, $V$ is the voltage difference measured across the plane in the direction normal to the current itself, and $ν$ is the so-called integer or fractional filling factor. While the integer effect, where $ν = n$ is an integer, can be interpreted using a general single-electron argument based on gauge invariance in a two-dimensional system [9], fractional values of $ν$ are presently believed to be connected to multi-electron strongly-correlated states [10–13]. The situation is further complicated by the fact that even-denominator values of $ν$, observed in very cold samples, do not easily fit into the strongly correlated electron picture [15–21].

In what follows, we deduce both integer and fractional values of the Hall conductance filling factor from a generalized version of the single-electron argument formulated by Laughlin. The key idea is to consider both electron spin and the three-dimensional nature of the underlying gauge symmetry. The model predicts, down to the very fine details, the observed integer and fractional values and plateau widths of quantized Hall conductance in a two-dimensional system [9], fractional values of $ν$ are presently believed to be connected to multi-electron strongly-correlated states [10–13]. The situation is further complicated by the fact that even-denominator values of $ν$, observed in very cold samples, do not easily fit into the strongly correlated electron picture [15–21].

We begin by recalling Laughlin’s single-electron argument to explain the observed integer Hall effect. The key idea is to represent the two-dimensional metal as a ribbon of length $ℓ$ bent into a loop, with the magnetic field $\mathbf{H}_0$ normal to the ribbon, the current $I$ flowing along the ribbon, and the Hall potential $V$ measured between its edges (see Fig.1A). Taking as a candidate vector potential $\mathbf{A} = -\mathbf{H}_0y\mathbf{u}_x$, the integer Hall effect then descends from the added constraint $Δ\mathbf{A} = n\hbar c/e\ell$ on the allowed values of the gauge transformation $\mathbf{A} \rightarrow \mathbf{A} + Δ\mathbf{A} \mathbf{u}_x$ that ensures the single-valued nature of extended wavefunctions around the loop itself. This leads to a condition on the magnetic flux $φ$ through the loop, that can change only by integer multiples of the flux quantum $Δφ = \hbar c/e$. The result is the quantization of the Hall conductance $G_{xy} = I/V = νe^2/h$, where the integer $n$ refers to the $μ_n$-Landau level [22].

We next note that the universality in the experimental results, where only fundamental constants appear, does not solely suggest the involvement of a general principle, as Laughlin put it, but also that any explanatory argument should not be connected to specific geometric or circumstantial constraints. More precisely, the closure condition on $A$ around the loop must be valid for any direction of the current $I$ on the plane normal to $\mathbf{H}_0$, for any specific value of $\mathbf{H}_0$, $I$, and $ℓ$, and even for any direction of $\mathbf{H}_0$ (Fig.1B). In short, the condition in the ribbon loop representation actually refers to a property of the entire set of all possible loop orientations, amounting to a statement of the fact that the extended electron spatial wavefunction must be single-valued under a full three-dimensional rotation. In these terms, the closure condition is, formally, equivalent to the quantization of the angular momentum of a point particle in 3D. Performing then a Hall conductance measurement, where the orientations of $\mathbf{H}_0$ and $I$ are actually fixed, implies, according to quantum mechanics, the projection of the wavefunction onto an a-priori unknown eigenstate of $\mathbf{J}^2$, $J_z$, where $\mathbf{J} = \mathbf{L} + \mathbf{S}$ is the sum of orbital angular momentum and spin (see Fig.1C).

More precisely, performing an experiment that corresponds to projecting an a priori unknown $\mathbf{J}$ in a specific direction identified by the associated orthogonal loop, will select one out of all possible eigenstates of $J_z$, from the maximum value $J$ to the minimum value, i.e., $(1/2)$ (in units of $\hbar$) for single electrons (negative values of $J_z$ are equivalent to the positive ones for obvious symmetry reasons). Consequently, limiting our analysis solely to the actual plane of the experiment and relative direction of the current, as done in the Laughlin argument (see Fig.1A), amounts to considering only cases in which $J_z$ is maximum, i.e., $J_z = J$. The resulting $n$-th
Landau level then corresponds to the integer Hall effect $I/V = n e^2 / h$. For a generic value of $J$, we note that the magnetic flux quantum $\Delta \phi = \hbar c / e$ refers to the loop orthogonal to the specific direction in which the act of measurement has projected $\mathbf{J}$. This is a necessary consequence of the single-valued nature of extended wavefunctions on every possible plane in space. Therefore, the apparent flux quantum through the actual loop that lies on the plane normal to $\mathbf{u}_z$ (orthogonal to $\mathbf{H}_0$) is enhanced with respect to $hc/e$, which refers to the tilted loop, i.e., $\Delta \phi(\mathbf{u}_z) = (J / J_z) \Delta \phi$ (see Fig.1D). The result is

$$I = \frac{ne^2 J_z}{h J},$$

that is, setting $\nu = nJ_z / J$.

$$I = \frac{\nu e^2}{h}.$$  

Equation (2) is the central result of this paper.

For an electron, with its half-integer spin, Eq.(2) implies

$$\nu = \frac{nJ_z}{J} = \frac{n q + 1/2}{p + 1/2},$$

with $q$ and $p$ integers ($q \leq p$). Hence, in conditions typical of a quantum Hall effect experiment, Eq.(3) with $J = 1/2$ leads to $J_z = 1/2$ and $\nu = n$; for $J = 3/2$, $J_z = 3/2$ leads to $\nu = n$, while $J_z = 1/2$ leads to $\nu = (1/3)n$; for $J = 5/2$, $J_z = 5/2$ gives $\nu = n$, while $J_z = 3/2$ gives $\nu = (3/5)n$ and $J_z = 1/2$ gives $\nu = (1/5)n$, and so on.

For an integer spin system, such as a Cooper pair, Eq.(2) implies

$$\nu = \frac{nJ_z}{J} = \frac{n q}{p},$$

with $q$ and $p$ integers (and again, $q \leq p$). Using Eq.(4) for $J = 1$ we have $J_z = 1$ and $\nu = n$; for $J = 2$, $J_z = 2$ leads to $\nu = n$, while $J_z = 1$ leads to $\nu = (1/2)n$; for $J = 3$, $J_z = 3$ leads to $\nu = n$, $J_z = 2$ gives $\nu = (2/3)n$, while $J_z = 1$ gives $\nu = (1/3)n$; for $J = 4$, $J_z = 4$ leads to $\nu = n$, $J_z = 3$ gives $\nu = (3/4)n$, $J_z = 2$ gives $\nu = (1/2)n$, while $J_z = 1$ leads to $\nu = (1/4)n$, and so on.

We begin our analysis of Eq.(3), that refers to non-interacting electron states, by reporting in Fig.2A an example of a sequence of $\nu$ with $n = 1, \ldots, n_{\text{max}}$ and $p = 0, \ldots, p_{\text{max}}$. A first important feature is the appearance of recurrent gaps in the spectrum, that is, regions for which no quantum Hall states form, this for an arbitrary $n_{\text{max}}$ and $p_{\text{max}}$. The paradigmatic example at $\nu = 1/2$ is reported in Fig.2B. The gaps are the result of a peculiar property of Eq.(3), that is, that $\nu \neq \nu_g = (2s + 1)/(2r)$ with $s$ and $r$ integers. This is because $\nu = (2s + 1)/(2r)$ would imply, through Eq.(3), $2nr(2q + 1) = (2s + 1)(2p + 1)$, that is manifestly impossible. Hence, for example, no Hall states and plateaus will be observed in and around $\nu_g = 1/2$ ($s = 0$, $r = 1), \nu_g = 1/4$ ($s = 0, r = 2), \nu_g = 3/4$ ($s = 1, r = 2).$ The width of the gaps depends specifically on $p_{\text{max}}$. To illustrate this, consider that Eq.(3) implies that the spectrum is explored in steps of $1/(2p + 1)$, with $p \leq p_{\text{max}}$. Since the spectrum necessarily includes the integer values $\nu = m$ as long as $n_{\text{max}} \geq m$, as follows from the fact that $n(2q + 1) = m(2p + 1)$ is always satisfied for

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**FIG. 1.** Generalized Laughlin argument. (a) Laughlin argument, the plane of the Hall experiment loop is identified by the unit vector $\mathbf{u}_m$ parallel to the direction of the electromotive Hall force $\mathbf{F} = (q / c) \mathbf{v} \wedge \mathbf{H}_0$, where $q = -e$ is the electron charge and $\mathbf{v}$ its drift velocity. (b) Quantum-mechanically possible loops associated to different directions of $\mathbf{q}$ and $\mathbf{H}_0$ and corresponding $S^2$ manifold for $\mathbf{u}_m$. (c) Act of measurement on a specific Hall experiment loop orientation $\mathbf{u}_m$. (d) Altered flux quantum $\Delta \phi(\mathbf{u}_m) = \Delta \phi / \cos \theta = (J / J_z)(hc/e)$ for $\mathbf{u}_m \neq \mathbf{u}_z$. 

$\Delta \phi(\mathbf{u}_z) = \Delta \phi / \cos \theta = (J / J_z)(hc/e)$ (see Fig.1D). The result is
An analogous similarity occurs for the mirror reflected self-similarity set of Eq.(4), in Fig.2E,F we report the spectrum 2 ≤ ν ≤ 3 for n_{max} = 20, p_{max} = 50, where the principal gap-center states are 15/7, 9/4, 16/7, 7/3, 12/5, 5/2, 18/7, 8/3, 14/5, 20/7 (see Fig.2F), while no gaps like the ones in Fig.2B are present, in agreement with the fact that in Eq.(4) p and q have no definite parity. Self-similarity holds for \nu' = \nu/l, with l integer, as long as the condition n'q' = nq is satisfied.

For electrons, Eq.(3) naturally contains the similarity between fractional states and those of the integer effect for an appropriately shifted magnetic field H_r = H_0 - H_0^r [21]. Indeed, since G_{xy} = C/H_0, with C = \rho c, \rho being the sample charge density, \nu = C/H_0 and \nu^* = C/(H_0 - H_0^r). Imposing \nu^* = n with \nu = n(2q+1)/(2p+1) (Eq.(3)) requires that H_0^r/C = (2p+1)/(n(2q+1)) - 1/n, an equality that holds only if q = 0 and p = nr, where r is an integer. Consequently, the correspondence holds for H_0^r = 2rC, \nu = n/(2nr+1), and \nu^* = n. In particular, for r = 1, we have the observed H_0^r = H(1/2) = 2C, \nu = n/(2n+1), and \nu^* = n. An analogous similarity occurs for the mirror reflected

FIG. 2. Spectrum sequence. (a) Illustration of the number sequence of Eq.(3) relative to electrons for n_{max} = 20 and p_{max} = 20. Zoom into (b) the gap at \nu_g = 1/2 and (c) the gap-center state at \nu_{gc} = 1. Illustration (d) of the approximate self-similarity as expressed by \nu' = \nu/(2l+1) for l = 1 in the spectral region 0 < \nu < 1 (top panel) and 0 < \nu' < 1/3 (bottom panel). (e) Illustration of the number sequence of Eq.(4) relative to bosons (Cooper pairs) for n_{max} = 20 and p_{max} = 50 and (f) rescaled histogram showing finer details.
\( \nu^* = -n \), in which case each state of spectrum \(-n\) is superimposed with the state \(n/(2m-1)\), a similarity typically reported for \(r = 1 \) \[21\]. The similarity does not, in turn, hold for Cooper pairs described by Eq.(4), since an equivalent superposition of the integer states with fractional ones would imply the condition \(1/n = p/nq - H_0/C\), that holds for any \(n\) only if \(p = q\), i.e., for the integer states themselves.

In terms of the spectrum, each of the possible states associated to an allowed value of \(J_z\) leads to an equivalent Landau-level harmonic oscillator spectrum \(E(n, J_z/J) = \hbar(J_z/J)\omega_c(n + 1/2)\), where \(\omega_c = eH_0/mc\) and \(m\) is the effective mass of the electrons. It follows that as \(H_0\) is increased, different extended states with ever lower values of \(n\) and \((J_z/J)\) reach the electron Fermi level \(E_F\). The structure of the spectrum, with its characteristic gap-center states and gaps (Fig.2), emerges for a finite value of \(p_{max}\), which is fixed by the maximum available \(J_z\), \(J_z^{max} = p_{max} + 1/2\) for electrons (Eq.(3)), and \(J_z^{max} = p_{max}\) for Cooper-pairs (Eq.(4)). The \(J_z^{max}\) is, in turn, fixed by the uncertainty principle for a given sample, since \((1/2)(J_z^{max}) \leq \sigma_{J_x}\sigma_{J_y}\) \[22\], where the maximum value of the uncertainties \(\sigma_{J_x}\) and \(\sigma_{J_y}\) in the in-plane components of \(J\) is determined by how strong the electrons and pairs are confined to the metallic plane. For a given set of explored values \(H_0 \leq H_0^{max}\), the actual features of the spectrum that are observed depend on the spread of the electron distribution around \(E_F\), i.e., on the electron temperature \(T\). As \(T\) is lowered, the first observable effects will be associated to widest gap-center states. As \(H_0\) causes the gap-center state spectrum to sweep through the electron Fermi level, the states above and below the gaps that flank the gap-center state form the equivalent of band structures, implying that the Fermi level automatically coincides with the gap-center state itself. It follows that for all values of \(H_0\) that correspond to values of \(\nu\) that would fall inside the gap-center spectrum, \(R_{xy} = \nu_0 e^2/h\), a plateau whose width \(\Delta H_0\) then provides a direct measurement of \(p_{max}\) for the given sample. For \(m\) integer, since the plateau forms from \(\nu_- = m - 1/(2p_{max} + 1)\) to \(\nu_+ = m + 1/(2p_{max} + 1)\), then \(\Delta H_0 = (2C/(2p_{max} + 1))(m^2 - 1/(2p_{max} + 1)^2)^{-1}\).
In turn, for these values of $H_0$, since the electrons are pinned to the highly degenerate extended gap-center state, $R_{xx} = 0$. We note that the electron temperature at which plateaus will emerge is determined by the condition that $k_BT$ be smaller than the energy spread that corresponds to the specific plateau. For $\Delta H_0(\nu_{gc} = 1)$, considering the energy of the corresponding Landau levels $E(n, J_z)$, the condition becomes $k_BT < \hbar e\Delta H_0(1)/(mc(2\nu_{max} + 1)) \approx 2\hbar eC/(mc(2\nu_{max} + 1))$ (corresponding to a temperature of several degrees K in typical experiments). It goes without saying that to observe the spectrum of Eq.(4), the generally stronger constraint that $k_BT$ be smaller than the Cooper pair gap must be met. Incidentally, for values of $T$ and $H_{max}$ that allow the observation of the $\nu = \nu_{gc} = 1$ plateau (at $R_{xy} = e^2/h$), the value of $C$, that depends on the density of charge carriers of the specific sample, can be experimentally determined from the value of magnetic field $H_0(1)$ at which the $\nu_{gc} = 1$ plateau begins to develop as $T$ is lowered. For a given $p_{max}$ and a sufficiently low $T$, universal curves will emerge plotting $R_{xy}$ versus $H_0 = H_0/H_0(1)$ ($\nu = 1/h_0$). The situation at the gaps at $\nu_{gc}$ is fundamentally different. Here no extended state is present in the spectrum at the Fermi level, so that the material behaves like a standard metal, $R_{xy}$ increasing in proportion to $H_0$, and $R_{xx}$ being approximately constant, apart from the inevitable changes associated to standard magnetoresistance. In turn, for temperatures low enough as to allow Cooper pairs to form, Eq.(4) implies that a gap-center state right at the center of the single-electron gap forms. Since the Cooper pair gap is pinned to form below the single-electron Fermi level, as long as the superconducting gap is much smaller than the Hall spectrum gap of Eq.(4), the Cooper pair superconducting state will coincide with the gap-center state for the whole development of the gap, implying that also in this case we expect a plateau.

In Fig.3 we analyze specific experimental results that have been amply presented in literature. We begin in Fig.3A with the data of Fig. 1 in Ref.[15]. Direct inspection indicates that the $R_{xy}$ predicted on the basis of the spectrum described by Eq.(3) with $p_{max} = 10$ is able to reproduce the principal experimental findings. The spectrum, reported in the bottom panel of Fig.3A in the form of the normalized state degeneracy versus $1/\nu$, manifests a sequence of gap-center states (blue shade) and gaps (yellow shade). As shown in the top panel of Fig.3A, the corresponding predicted and detected behavior are found to be in agreement. Increasing the value of $p_{max}$ does not alter the picture in any fundamental way, aside from reducing progressively the actual width of the plateaus and causing the full observed spectrum, that includes the $\nu_{gc} = 3/7$ state, to emerge. An illustration of this can be found in the inset of Fig.3A for $R_{xy}$ with $p_{max} = 18$. An analogous situation holds for $\nu_{gc} = 2/5$, that manifests rigorously a gap-center state only for $p_{max} \geq 10$, while for the $p_{max} = 10$, this plateau is only a very good approximation (the gap forms from $\nu_- = 5/13$ to $\nu_+ = 7/17$, so that the $(\nu_- + \nu_+)/2 = 88/221$ that differs from $\nu_{gc} = 2/5$ by 0.002). The implication of this is two-fold: first, $p_{max}$ may not be constant throughout the $H_0$ scan, a fact that would then question the idea that $p_{max}$ is principally connected to the specific fabrication features of the sample and its ability to confine the electrons to the plane. Second, that when we assume that an extended state is populated, we cannot, in all cases, wholly neglect the role of thermal agitation, that will cause the Fermi level to be affected, even though very weakly, by details of the degeneracy of the spectrum.

While the analysis detailed in Fig.3A can be repeated for a large part of experimental scans reported in literature, some very well-established results cannot be accommodated into the electron picture of Eq.(3). For example, in Fig.3B we report the data from Fig.1 of Ref.[25]. It is immediately apparent that the behavior of $R_{xy}$ does not include segments corresponding to the classical Hall effect, i.e., for which $R_{xy} = 1/\nu$ (apart from the asymptotic $\nu \to 0$ regime), as in turn occurs for the data analyzed in Fig.3A. The absence of these classical-like regimes means that the system does not manifest the gap states $\nu_g$ that characterize the electron spectrum. The data, in turn, can be analyzed in terms of Cooper-pairs obeying Eq.(4). The predicted $R_{xy}$ on the basis of the Cooper-pair spectrum (central panel in Fig.3B) for increasing and decreasing values of $H_0$ (bottom and top curves respectively), as reported in Fig.3B, is in reasonable agreement with data. We should note that the behavior is expected, as discussed above, in regimes of specifically low temperatures ($50$mK in the experiment analyzed), and the Cooper pair gap levels follow the Fermi level of the corresponding condensing electrons (bottom panel of Fig.3B). Since two electrons form a Cooper pair, the correspondence between the electron spectrum and that of the pairs, reported in the central panel of Fig.3B, is such that each electron state at $\nu$ is associated to a Cooper pair state at $\nu' = \nu/4$. Hysteresis is the product of the basic difference in how electrons and Cooper pairs participate in transport. Specifically, as $H_0$ is increased so that the corresponding electron level is pinned to the center of the gap at a given $\nu_g$ (yellow shaded regions), the fact that no extended electron Hall state exists implies that no corresponding Cooper-pair state exists, so that the populated boson state that participates in transport refers to the higher $\nu_+$ boundary state (or lower $1/\nu_-$ state in the plots). If, on the other hand, $H_0$ is decreased to the electron $\nu_g$ gap, the corresponding populated boson state is the lower $\nu_-$ boundary state (i.e., the corresponding higher $1/\nu_-$ state in the plots). The correspondence discussed in Fig.3B is for an electron spectrum with $p_{max} = 1$ and a Cooper pair spectrum with $p_{max} = 6$ ($2(2\nu_{max} + 1) = \nu_{max}$). We note that while hysteresis is now known to occur in specific low-temperature $H_0$ scans, original papers did not explicitly mention this behavior. One might speculate that Cooper pairs could condense from a virtual electron state, even though this state does not exist, a mechanism that could lead to the
absence of hysteresis and the formation of extra plateaus. We show how the physical underpinnings of quantum Hall effects lie more with quantum-mechanically projecting 3D gauge symmetry in 2D than with many-body correlations. Apart from allowing the understanding of a large variety of effects and results that are not all explained in the particle-interaction picture, the shift brings the entire field back into the folds of how symmetry and topology affect quantum behavior [27, 28]. Apart from clarifying the fundamental relationship between fractional and integer states [29], findings provide a more profound assessment of how 2D physics can play a role in applicative scenarios, such as the emerging field of topological quantum computing [8, 30]. This said, our analysis is here focused on the comndrum of Hall effects in cold and ultra-cold semiconductors. In different 2D systems, such as graphene, behavior will also depend on the specific charge carrier physics [31, 32].

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