Supersymmetry and Gravitational Duality

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Abstract

We study how the supersymmetry algebra copes with gravitational duality. As a playground, we consider a charged Taub-NUT solution of $D = 4$, $\mathcal{N} = 2$ supergravity. We find explicitly its Killing spinors, and the projection they obey provides evidence that the dual magnetic momenta necessarily have to appear in the supersymmetry algebra. The existence of such a modification is further supported using an approach based on the Nester form. In the process, we find new expressions for the dual magnetic momenta, including the NUT charge. The same expressions are then rederived using gravitational duality.
1 Introduction

Supersymmetry has been one of the major ingredients in providing evidence for dualities in the realm of string theories and M-theory. In particular, there is a very tight relation between U-duality [1], the most general duality encompassing electric-magnetic duality, S-duality and T-duality, and the existence of BPS bounds following from the most general maximally extended supersymmetry algebra. This relation follows from the fact that states (or supergravity solutions) which preserve some supersymmetries also saturate a BPS bound which takes the form:

\[ M = |Z|, \]

where \( Z \) is a U-duality invariant combination of all the possible charges arising in the specific theory one is considering. These charges, which correspond to possibly extended charged objects, arise in the supersymmetry algebra as central extensions [2-5], and this is the reason why they enter in the BPS bound.

It is however striking that U-duality acts only on the right hand side of the BPS equation (1), while it leaves the left hand side, \( M \), invariant. It is natural to ask whether there are more general duality transformations that also act on \( M \). Indeed, such a duality exists, at least in four dimensions. It is the gravitational electric-magnetic duality (see [4, 5, 6, 7, 8, 9] and references therein), which maps the mass \( M \) to a magnetic mass \( N \), usually called the NUT charge [10, 11] (see also [12, 13]). It is the purpose of this paper to study some aspects of this duality in relation to the BPS bound and the preservation of supersymmetries under it. In particular, in the context of \( D = 4, N = 2 \) supergravity we discuss how the BPS equation is generalized in presence of NUT charge to [14]:

\[ \sqrt{M^2 + N^2} = |Z|, \]

and in turn we want to understand how the superalgebra itself takes into account the possibility of turning on a NUT charge (or more generally a dual momentum). Some considerations on how the NUT charge transforms more generally under string dualities have appeared for instance in [15, 16].

The outline of the paper is as follows. In Section 2, we consider the Taub-NUT solution of \( N = 2 \) supergravity and find explicitly its Killing spinors, under the condition [2]. In Section 3, we inspect more closely the projection which defines the Killing spinors. For large radii, it takes a form which suggests the presence of a new term extending the supersymmetry algebra, which nevertheless fails to

\(^1\text{We stress that here we are concerned with the Lorentzian NUT charge. In contrast, the Euclidean NUT charge, also called the Kaluza-Klein monopole charge, is extensively discussed in the literature related to string dualities, where it appears on the same footing as the other \( p \)-form charges.}\)
pass the simple test of hermiticity. In Section 4, we take another route towards the superalgebra which consists in computing the variation of the supercharges when expressed in terms of surface integrals. We recover the same new extension in the r.h.s. of the superalgebra, but we recognize it now as a “topological” term violating the canonical association of a variation of a surface charge to the commutator of two such charges. In section 5, we rederive the expressions for the dual momenta that we obtained in the previous section, by demanding that they should be defined as the gravitational duals of the usual ADM momenta. We also show a way to correctly compute the NUT charge by writing the surface integrals in such a way that the integrand is free of string-like singularities. In section 6, we conclude by discussing how one could reconcile the presence of the dual momenta in the superalgebra with the theorems that prevent such terms to appear. In the Appendices we have relegated the conventions and all the computations that lead to the expression for the Killing spinor.

2 The BPS Taub-NUT charged solution in $\mathcal{N} = 2$ supergravity

In this section, we first recall the pure $\mathcal{N} = 2$ supergravity, and display the supersymmetric variation of the gravitini, which is essentially the Killing spinor equation. Then, we solve it for an already known solution, namely the black hole carrying not only mass but also NUT charge (also called “magnetic mass”), and both electric and magnetic Maxwell charges as well (see e.g. [17]). In the sense of the fall-off conditions used in [18], this black hole is asymptotically flat [6] (up to global issues involving time identifications [19, 12]).

The bosonic part of the $\mathcal{N} = 2$ supergravity Lagrangian is just the Einstein-Maxwell one:

$$\mathcal{L} = \sqrt{g} \left[ \frac{1}{4} R - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} \right]$$

(3)

where the signature used is $(-, +, +, +)$ and $F_{\mu\nu} = 2 \partial_{(\mu} A_{\nu)}$. We use greek letters where $\mu, \nu, \ldots = t, r, \theta, \phi$ for curved space indices and roman letters where $a, b, \ldots = 0, 1, 2, 3$ for flat indices. We essentially follow the conventions of [20].

One is usually focusing on supergravity solutions where all fermionic fields are set to zero. Then, the supersymmetries preserved by such a solution are simply given by any non-trivial solution to the Killing spinor equation, which is obtained by setting to zero the supersymmetric variation of the gravitino spin-3/2 field, which is a complex spinor in $\mathcal{N} = 2$ supergravity:

$$\delta \psi_\mu = \hat{\nabla}_\mu \epsilon = \hat{D}_\mu \epsilon + \frac{i}{4} F_{ab} \gamma^{ab} \gamma_\mu \epsilon = 0$$

(4)
where $\hat{\nabla}_\mu$ is called the super-covariant derivative, and the covariant derivative is $\hat{D}_\mu = \partial_\mu + \frac{1}{4} \omega_\mu^{ab} \gamma_{ab}$.

Here, we take the gamma matrices to be real and such that they satisfy $\{\gamma_a, \gamma_b\} = 2\eta_{ab}$. We also have $\gamma_{ab} = \frac{1}{2}[\gamma_a, \gamma_b]$. The parity matrix $\gamma_5$ is real and antisymmetric $\gamma_5 = \gamma_{0123}$.

A special solution to the equations of motion derived from the action (3) is a black hole solution carrying, besides mass, NUT charge and both electric and magnetic Maxwell charges. Such a solution is written as:

$$ds^2 = -\frac{r^2 - N^2 - 2Mr + Q^2 + H^2}{r^2 + N^2} (dt + 2N \cos \theta d\phi)^2 + \frac{r^2 + N^2}{r^2 - N^2 - 2Mr + Q^2 + H^2} dr^2 + (r^2 + N^2)(d\theta^2 + \sin^2 \theta d\phi^2),$$ (5)

$$A_t = \frac{Qr + NH}{r^2 + N^2}, \quad A_\phi = -\frac{H(r^2 - N^2) + 2NQr}{r^2 + N^2} \cos \theta.$$ (6)

It is easy to see that in the case $N = 0$, we recover the Reissner-Nordström black hole solution.

Defining $\lambda = r^2 - N^2 - 2Mr + Q^2 + H^2$ and $R^2 = r^2 + N^2$, the vielbein read:

$$e^0 = \frac{\sqrt{\lambda}}{R} (dt + 2N \cos \theta d\phi), \quad e^1 = \frac{R}{\sqrt{\lambda}} dr,$$

$$e^2 = Rd\theta, \quad e^3 = R \sin \theta d\phi.$$ (7)

It is obvious that the Killing spinor equations will have non-trivial solutions only if the operator acting on the supersymmetry parameter $\epsilon$ has vanishing eigenvalues, i.e. its determinant is zero. This will involve a relation among the constants $M, N, Q$ and $H$. This relation appears for instance when computing the integrability conditions of the Killing spinor equations [17]. In Appendix A, we provide an alternative derivation of the same condition.

The BPS condition reads

$$M^2 + N^2 = Q^2 + H^2.$$ (8)

Note that it is $r$-independent, and that it implies $\sqrt{\lambda} = r - M$. This is nothing else than the expression [2], which had already been derived several years ago in similar contexts [14, 17]. In order to analyze in more detail the implications of such a generalized BPS bound, let us introduce the following expressions:

$$r \pm \gamma_5 N = Re^{\pm \beta(r) \gamma_5}, \quad r \pm \gamma_5 N = Z e^{\pm \alpha \gamma_5}, \quad Q \pm \gamma_5 H = Z e^{\pm \alpha \gamma_5}.$$ (9) (10) (11)
where we have defined $Z^2 = M^2 + N^2 = Q^2 + H^2$ and

$$\tan \beta = \frac{N}{r}, \quad \tan \alpha_m = \frac{N}{M} \quad \tan \alpha_q = \frac{H}{Q}.$$  \hfill (12)

Then, the SUSY variations can be rewritten as:

$$\delta \psi_t = \partial_t \epsilon + \frac{r - M}{2R^3} Z \gamma_0 \{ 1 - i e^{(\beta + \alpha_m - \alpha_q) \gamma_5} \gamma_0 \} \epsilon,$$  \hfill (13)

$$\delta \psi_r = \partial_r \epsilon - \frac{Z}{2R(r - M)} i e^{(2\beta - \alpha_q) \gamma_5} \gamma_0 \epsilon,$$  \hfill (14)

$$\delta \psi_\theta = \partial_\theta \epsilon - \frac{1}{2} \gamma_{12} \epsilon + \frac{Z}{2R} \gamma_{12} e^{(\beta - \alpha_m) \gamma_5} \{ 1 - i e^{(\beta + \alpha_m - \alpha_q) \gamma_5} \gamma_0 \} \epsilon,$$  \hfill (15)

$$\delta \psi_\phi = \partial_\phi \epsilon - \frac{1}{2} (\sin \theta \gamma_{13} + \cos \theta \gamma_{23}) \epsilon +$$

$$+ \left[ \frac{Z}{2R} \sin \theta \gamma_{13} + \frac{NZ(r - M)}{R^3} \cos \theta \gamma_{01} \right] e^{(\beta - \alpha_m) \gamma_5} \{ 1 - i e^{(\beta + \alpha_m - \alpha_q) \gamma_5} \gamma_0 \} \epsilon.$$  \hfill (16)

We thus see that it is most natural to look for a Killing spinor which satisfies the factorization

$$\epsilon(t, r, \theta, \phi) = e^{\frac{i}{2} \gamma_{12} \theta} e^{\frac{i}{2} \gamma_{23} \phi} \epsilon_0(r),$$  \hfill (17)

where $\epsilon_0$ is independent on time and satisfies the projector equation

$$\left\{ 1 - i e^{(\beta + \alpha_m - \alpha_q) \gamma_5} \gamma_0 \right\} \epsilon = 0.$$  \hfill (18)

Note indeed that

$$\Pi = \frac{1}{2} \left\{ 1 - i e^{(\beta + \alpha_m - \alpha_q) \gamma_5} \gamma_0 \right\}$$  \hfill (19)

is a projector, satisfying $\Pi^2 = \Pi$. Moreover, since it verifies $\text{tr} \Pi = 2$, it has exactly two zero eigenvalues.

The above result (18) will be essentially enough for the rest of the discussion on the relation between the Killing spinor and the supersymmetry algebra. However for the sake of completeness, and in order to show that a solution indeed exists, we produce below the complete expression of the Killing spinor.

The only non trivial equation that remains to be solved is $\delta \psi_r = 0$. The final expression for the Killing spinor is (see Appendix B for the details):

$$\epsilon_0(r) = \left( \frac{r - M}{R} \right)^{\frac{1}{2}} \left( \frac{R [\frac{1}{2} (\beta(r) + \alpha_m - \alpha_q)] \bar{\epsilon}^1}{i R [\frac{1}{2} (\pi - \beta(r) - \alpha_m + \alpha_q)] \bar{\epsilon}^2} \right),$$  \hfill (20)

where

$$\bar{\epsilon} = \left( \begin{array}{c} \epsilon_1 \\ \epsilon_2 \end{array} \right)$$  \hfill (21)
is a constant two-component complex spinor and we have defined the rotation matrix
\[ R[\alpha] = \begin{pmatrix} \cos \alpha & -\sin \alpha \\ \sin \alpha & \cos \alpha \end{pmatrix}. \] (22)

We have thus shown that, provided the extremality condition (8) is satisfied, the metric has a Killing spinor, which actually depends on two complex numbers. The metric thus preserves half of the 8 supersymmetries, as expected from the arguments of [21] (see also [14, 17]).

As a last word, we could worry about the issue whether the Killing spinor is globally defined. Indeed, the metric has a coordinate singularity along the z axis, also known as the Misner string. One can remove the singularity along half of the axis by a coordinate transformation. Essentially, one obtains two completely regular patches on the upper and lower hemispheres, where the metric is the same as (5), but with \( \cos \theta \) replaced by \( \cos \theta \pm 1 \). It amounts to shift the time coordinate \( t \) by \( \pm 2N\phi \). Since the Killing spinor is \( t \)-independent, we can already see that it will be the same on the two patches. This can be verified by rederiving its expression as above in the regular metric in each patch. As expected one finds the same result as above.

3 The Killing spinor and its asymptotic projection

In this section we analyze in more detail the solution for the Killing spinor found in the previous section. In particular, we consider the projection that defines the Killing spinor and take its limit of large radius, where the metric is asymptotically flat. The projection can be recast in a form which is similar to the right hand side of the \( \mathcal{N} = 2 \) supersymmetry algebra. However, the term containing the NUT charge has the wrong hermiticity condition and thus does not seem to fit in any of the central (or else) extensions of the most general \( \mathcal{N} = 2 \) supersymmetry algebra.

The projection defining the four independent real components of the Killing spinor is given by:
\[ \left\{ 1 - ie^{(\beta(r)+\alpha_m-\alpha_q)\gamma_5\gamma_0} \right\} \epsilon = 0. \] (23)

We have emphasized that it is \( r \)-dependent. There are two observations one can make about this dependence. Recalling that \( \tan \beta(r) = N/r \) and that \( \tan \alpha_m = N/M \), we see that when the NUT charge is absent, both \( \beta = 0 \) and \( \alpha_m = 0 \). The projector becomes \( r \)-independent. However, even when \( N \neq 0 \), in the limit of large radius, \( r \to \infty \), we observe that \( \beta \to 0 \) and the \( r \)-dependence disappears. We are thus left with a constant asymptotic projector which depends on all of the four charges (where it is of course understood that they satisfy the BPS bound (8)).
Let us rewrite the projector in a more readable form. By setting $\beta = 0$ and multiplying by $e^{-\alpha m \gamma_5}$, we obtain:

$$\{ M - \gamma_5 N - i (Q - \gamma_5 H) \gamma_0 \} \epsilon = 0. \quad (24)$$

We now recall the $\mathcal{N} = 2$ superalgebra, including the scalar central charges (see e.g. [22]). Using Majorana supercharges $Q^I$, with $I = 1, 2$, it is:

$$\{Q^I, Q^J\} = \gamma^\mu C P_\mu \delta^{IJ} + C U^{IJ} + \gamma_5 C V^{IJ}, \quad (25)$$

where both $U^{IJ} = -U^{JI} \equiv U \varepsilon^{IJ}$ and $V^{IJ} = -V^{JI} \equiv V \varepsilon^{IJ}$, and $C$ is the charge conjugation matrix, which we take here to be $C \equiv \gamma_0$. In our conventions, Majorana spinors are real. Hence, we can define a single complex Dirac supercharge:

$$Q = \frac{1}{\sqrt{2}} (Q^1 + i Q^2). \quad (26)$$

The only non trivial relation of the superalgebra becomes:

$$\{Q, Q^*\} = \gamma^\mu C P_\mu - i (U + \gamma_5 V) C. \quad (27)$$

When there is a multiplet of BPS saturated states, some combinations of the supercharges have to be represented trivially, i.e. they have to vanish. This translates into the statement that the matrix $\{Q^I, Q^J\}$, or equivalently $\{Q, Q^*\}$, is not of maximal rank. This means that also the right hand side of (27) must have vanishing eigenvalues. In the present case, for a massive state at rest, we identify $P_0 \equiv M$. Further, if we set $U \equiv Q$ and $V \equiv H$, we see that we have preserved supersymmetries if the equation:

$$\{ M - i (Q - \gamma_5 H) \gamma_0 \} \epsilon = 0 \quad (28)$$

has solutions (note that we have multiplied the expression in (27) by $\gamma^0$ on the left and $C$ on the right).

We recognize the equation (24) for $N = 0$. So we see that for a Reissner-Nordström black hole, the projection on the Killing spinor in the extremal case maps directly to the right hand side of the $\mathcal{N} = 2$ superalgebra. Actually, we could have guessed the superalgebra (27) from the expression for the projector (28). It is thus tempting to do this for the case where $N \neq 0$. From (24), we see that $N$ must belong to a “charge” which carries a Lorentz index. The most straightforward guess is that $N \equiv K_0$ of a vectorial charge $K_\mu$ which enters the superalgebra as:

$$\{Q, Q^*\} \equiv \gamma^\mu C P_\mu + \gamma_5 \gamma^\mu C K_\mu - i (U + \gamma_5 V) C. \quad (29)$$
We see that the NUT charge $N$ seems to belong to an extension of the superalgebra which is not central in the sense that it is not a Lorentz scalar. Such extensions have been studied \cite{23}, and the most general $\mathcal{N} = 2$ superalgebra taking them into account has been written \cite{24, 25}. It is however straightforward to see that our term with $K_\mu$ is not part of any extension considered so far. The reason why Eq.\((29)\) is wrong is extremely simple: it violates hermiticity. Indeed, we have that $(\gamma_5 \gamma^\mu C) \dagger = -\gamma_5 \gamma^\mu C$, while any term on the right hand side must be hermitian since $\{Q, Q^\star\} \dagger = \{Q, Q^\star\}$. Before seeking a way to solve this puzzle, we will see in the following section that $K_\mu$ arises also through a different argument.

4 The superalgebra of charges at infinity and the Nester form

In this section we investigate an alternative approach to understand the existence of the magnetic gravitational charges. We first review the relation between the superalgebra and the variation of the supercharges when the latter are defined as surface integrals at spatial infinity \cite{26, 27, 28}. The bosonic charges appearing in the right hand side of the superalgebra then also appear as surface integrals at infinity. In this approach, the usual ADM momenta appear in their covariant formulation, i.e. in terms of the Nester form \cite{29, 30}, which is indeed closely related to the variation of the supercharges. Here we show that, analyzing carefully the Nester form, the ADM momenta appear together with the dual, magnetic, ADM momenta. These charges will appear to be related to “topological” terms in the algebra of the supercharges. The timelike component of the dual momenta is nothing else than the NUT charge discussed previously. Evaluated on the charged NUT black hole, the right hand side of the superalgebra reduces exactly to the asymptotic expression contained in the definition of the Killing spinor, discussed in the previous section.

Let us begin by showing how the Nester form \cite{30} is related to the variation of the supercharge expressed as a surface integral. We follow closely \cite{28}. Using the Noether method one computes the generator of supertranslations. It can be written as a volume integral, which in turn can be expressed as a surface integral:

$$\tilde{Q}[\epsilon, \tilde{\epsilon}] = \frac{i}{2\pi} \int \epsilon^{\mu\nu\rho\sigma} \tilde{\epsilon} \gamma_5 \gamma_\rho \hat{\nabla}_\sigma \psi d\Sigma_\mu + c.c.$$  

$$= -\frac{i}{4\pi} \int \epsilon^{\mu\nu\rho\sigma} \tilde{\epsilon} \gamma_5 \gamma_\rho \psi \sigma d\Sigma_{\mu\nu} + c.c., \quad (30)$$

where $\hat{\nabla}_\rho$ is the supercovariant derivative acting on a spin-3/2 field, $c.c.$ denotes complex conjugate, $\tilde{\epsilon} = \epsilon^1 C \equiv \epsilon^1 \gamma_0$ and we take the convention $\epsilon_{0123} = -\epsilon^{0123} = 1$. 

7
The charge $\tilde{Q}[\epsilon, \bar{\epsilon}]$ is bosonic, and it transforms the supergravity fields according to a supertranslation. When acting for instance on the bosonic fields, which are real, it generates a variation which is also real. We recall that in the present $\mathcal{N} = 2$ case, the gravitino $\psi_\mu$ is Dirac and hence complex. In terms of the fermionic Dirac supercharges defined in (26) we have:

$$\tilde{Q}[\epsilon, \bar{\epsilon}] = i(\bar{\epsilon}\bar{Q} + \tilde{Q}\epsilon)$$  \hspace{1cm} (31)

(note that $(\bar{\epsilon}\bar{Q})^* = -\bar{Q}\epsilon$).

It follows from the theory of surface charges (see for instance [31, 32]) that the variation of the supercharge should define its bracket in the usual way:

$$\delta_{\epsilon_1, \bar{\epsilon}_1} \tilde{Q}[\epsilon_2, \bar{\epsilon}_2] = i \left[ \tilde{Q}[\epsilon_1, \bar{\epsilon}_1], \tilde{Q}[\epsilon_2, \bar{\epsilon}_2] \right].$$  \hspace{1cm} (32)

In terms of the fermionic supercharges (26), using the expression (31), we would then obtain:

$$i \left[ \tilde{Q}[\epsilon_1, \bar{\epsilon}_1], \tilde{Q}[\epsilon_2, \bar{\epsilon}_2] \right] = i\bar{\epsilon}_2 \{Q, Q^*\} C\epsilon_1 - i\bar{\epsilon}_1 \{Q, Q^*\} C\epsilon_2.$$  \hspace{1cm} (33)

However we will see that our analysis will force us to consider a possible “topological extension” namely:

$$\delta_{\epsilon_1, \bar{\epsilon}_1} \tilde{Q}[\epsilon_2, \bar{\epsilon}_2] = i \left[ \tilde{Q}[\epsilon_1, \bar{\epsilon}_1], \tilde{Q}[\epsilon_2, \bar{\epsilon}_2] \right] + T.$$  \hspace{1cm} (34)

The crux of the matter is that $\delta_{\epsilon_1, \bar{\epsilon}_1} \tilde{Q}[\epsilon_2, \bar{\epsilon}_2]$ is not antisymmetric in the exchange of $\epsilon_1$ and $\epsilon_2$, as we now show.

Using (30) one finds for the bracket term and the “topological” term the following expressions

$$i \left[ \tilde{Q}[\epsilon_1, \bar{\epsilon}_1], \tilde{Q}[\epsilon_2, \bar{\epsilon}_2] \right] = \frac{1}{2} (\delta_{\epsilon_1, \bar{\epsilon}_1} \tilde{Q}[\epsilon_2, \bar{\epsilon}_2] - \delta_{\epsilon_2, \bar{\epsilon}_2} \tilde{Q}[\epsilon_1, \bar{\epsilon}_1])$$  \hspace{1cm} (35)

and

$$T \equiv \frac{1}{2} (\delta_{\epsilon_1, \bar{\epsilon}_1} \tilde{Q}[\epsilon_2, \bar{\epsilon}_2] + \delta_{\epsilon_2, \bar{\epsilon}_2} \tilde{Q}[\epsilon_1, \bar{\epsilon}_1])$$  \hspace{1cm} (36)

Note that obviously (35) is identically zero when $\epsilon_1 = \epsilon_2$ but $T$ is non-zero.

We now focus on the following expression which is the “building block” of the expressions appearing in (35)-(36):

$$\hat{E}^{\mu\nu} \equiv \frac{1}{4\pi} \varepsilon^{\mu\nu\rho\sigma} \hat{\nabla}_5 \hat{\nabla}_\rho (\epsilon_1 \gamma_5 \gamma_\sigma \epsilon_2 + \bar{\epsilon}_2 \gamma_5 \gamma_\sigma \bar{\epsilon}_1) d\Sigma^{\mu\nu}.$$  \hspace{1cm} (37)
This is precisely the expression presented by Nester \[30\] and generalized by Gibbons and Hull \[27\], albeit in its complex version\[2\] (recall that $\epsilon$ is Dirac in our setup). One can see that the (antisymmetric) bracket term \(35\) and the (symmetric) topological term \(36\) map respectively to the real and imaginary parts of the Nester form \(37\).

We are now going to use the expression \(37\) to obtain a linear combination of purely bosonic surface integrals, which correspond to space-time momenta and Maxwell charges. In order to proceed, we linearize gravity around Minkowski spacetime, in cartesian coordinates. As we have already seen, we consider space-time endowed with a NUT charge as asymptotically flat, at least as far as spacelike surface integrals are concerned \[6\].

First of all, following \[27\], we rewrite the complex Nester form as:

$$\hat{E}^{\mu\nu} = E^{\mu\nu} + H^{\mu\nu},$$

where

$$E^{\mu\nu} = \frac{1}{4\pi} \varepsilon^{\mu\nu\rho\sigma} \varepsilon_{\gamma\delta\mu} \hat{D}_\sigma \epsilon,$$

$$H^{\mu\nu} = \frac{i}{16\pi} \varepsilon^{\mu\nu\rho\sigma} F_{ab} \bar{\varepsilon} \gamma_5 \gamma_\rho \gamma^a \gamma^b \gamma_\sigma \epsilon. \quad (39)$$

One can readily check that $H^{\mu\nu}$ is actually real, hence any surprise will necessarily come from the purely gravitational term $E^{\mu\nu}$.

In the following, we will express everything in terms of the linearized spin connection $\omega_{\mu\rho}$. Hence the covariant derivative on a spinor becomes (note that we no longer distinguish between flat and curved indices, since they are the same at first order):

$$\hat{D}_\mu \epsilon = \partial_\mu \epsilon + \frac{1}{4} \omega_{\nu\rho\mu} \gamma^{\nu\rho} \epsilon. \quad (40)$$

We now plug back this expression in $E^{\mu\nu}$. Note that in the surface integral, the piece proportional to $\partial_\mu \epsilon$ will drop out as explained in detail in \[29\]\[28\]. The spinors will henceforth be identified with the constant value that they take asymptotically\[3\]. Hence we restrict to:

$$E^{\mu\nu} = \frac{1}{16\pi} \varepsilon^{\mu\nu\rho\sigma} \omega_{\alpha\beta\sigma} \varepsilon_{\gamma\delta\mu} \gamma_5 \gamma_\rho \gamma^\alpha \gamma^\beta \epsilon. \quad (41)$$

Using the relation:

$$\gamma_\mu \gamma_\lambda = \eta_{\mu\lambda} \gamma_T - \eta_{\mu\sigma} \gamma_\lambda = \varepsilon_{\rho\lambda\sigma} \gamma^\rho \gamma_5 \gamma_\sigma,$$

we thus obtain:

$$E^{\mu\nu} = \varepsilon \gamma^{\lambda} \epsilon \left( \omega^{\mu\nu}_{\lambda} + \delta^{\mu}_{\lambda} \omega^{\nu\rho}_{\rho} - \delta^{\nu}_{\lambda} \omega^{\mu\rho}_{\rho} \right) + \frac{1}{8\pi} \varepsilon \gamma_5 \gamma_\lambda \gamma_\delta \epsilon \varepsilon^{\mu\nu\rho\sigma} \omega_{\lambda\rho\sigma}. \quad (43)$$

\[2\] In references \[30\] and \[27\], they indeed considered $\hat{E}^{\mu\nu} + (\hat{E}^{\mu\nu})^*$.

\[3\] Indeed, we can actually take the spinors to be the Killing spinors of flat space in cartesian coordinates.
Note that the first term above is real while the second is imaginary.

Integrating the above 2-form at spatial infinity, we select the $E^0_i$ component, with $i = 1, 2, 3$. We can then reexpress the integral in terms of purely bosonic surface integrals as:

$$\oint E^0_i d\hat{\Sigma}_i = \bar{\epsilon} \gamma^\lambda P_\lambda \epsilon + \bar{\epsilon} \gamma_5 \gamma^\lambda K_\lambda \epsilon,$$

(44)

where we obtain the following expressions for the ADM momenta and the dual magnetic momenta:

$$P_\lambda = \frac{1}{8\pi} \oint (\omega^0_\lambda + \delta^0_\lambda \omega^i_\rho - \delta^i_\lambda \omega^0_\rho) d\hat{\Sigma}_i,$$

(45)

$$K_\lambda = \frac{1}{8\pi} \oint \epsilon^{ijk} \omega_{\lambda jk} d\hat{\Sigma}_i.$$

(46)

Note that $\epsilon^{0ijk} = -\epsilon^{ijk}$.

One can show that the above momenta are such that $P_0 = M$ and $K_0 = N$ for the solution (43). We defer to the next section the discussion of the subtleties of this evaluation along with the gravitational duality existing between $P_\lambda$ and $K_\lambda$.

At last, we can also address the second term of the generalized Nester form, which is treated as in [27]. By writing:

$$H^{\mu\nu} = \frac{i}{32\pi} \varepsilon^{\mu\nu\rho\sigma} F_{\lambda\sigma} \bar{\epsilon} \gamma_5 (\gamma_\rho \gamma_\lambda \gamma_\sigma - \gamma_\sigma \gamma_\lambda \gamma_\rho) \epsilon,$$

(47)

and using

$$\gamma_\rho \gamma_\lambda \gamma_\sigma - \gamma_\sigma \gamma_\lambda \gamma_\rho = 2\bar{\epsilon} \gamma_5 (\eta_\rho \lambda \sigma + 2(\eta_\rho \lambda \sigma - \eta_\rho \sigma \lambda \sigma)),$$

(48)

we obtain:

$$H^{\mu\nu} = \frac{i}{4\pi} F^{\mu\nu} \epsilon \bar{\epsilon} + \frac{i}{8\pi} \varepsilon^{\mu\nu\rho\sigma} F_{\rho\sigma} \bar{\epsilon} \gamma_5 \epsilon.$$

(49)

The surface integral then becomes:

$$\oint H^{0i} d\hat{\Sigma}_i = -i\bar{\epsilon} U \epsilon - i\bar{\epsilon} \gamma_5 V \epsilon,$$

(50)

with the central charges defined by:

$$U = -\frac{1}{4\pi} \oint F^{0i} d\hat{\Sigma}_i,$$

(51)

$$V = \frac{1}{8\pi} \oint \epsilon^{ijk} F_{jkd} d\hat{\Sigma}_i.$$

(52)

It can be checked that $U = Q$ and $V = H$ on our solution.

Summing up all the terms, we have:

$$\oint \hat{E}^{0i} d\hat{\Sigma}_i = \bar{\epsilon} \gamma^\lambda P_\lambda \epsilon + \bar{\epsilon} \gamma_5 \gamma^\lambda K_\lambda \epsilon - i\bar{\epsilon} U \epsilon - i\bar{\epsilon} \gamma_5 V \epsilon.$$

(53)
It is clear that the above expression cannot be equated to $\bar{\epsilon}\{Q, Q^*\}C\epsilon$, which would then result in the “wrong” superalgebra (29). But now we see that the obstruction to do so is precisely the presence of the topological term $T$ in (34).

Using the definitions of $T$ (36) and of the complex Nester form (37) we see that

$$T(\epsilon, \bar{\epsilon}) = -i \oint (\hat{E} - \hat{E}^*)$$

(54)

Using then the result (53) we finally indeed find:

$$T(\epsilon, \bar{\epsilon}) = -2i\bar{\epsilon}\gamma_5\gamma^\lambda K_\lambda\epsilon.$$  

(55)

To sum up, we see that a refined analysis of the Nester form in its complex version permits to recover precisely the additional term which was guessed from the asymptotic projection acting on the Killing spinor. In this context, we see that this additional term is actually violating the relation (32) and corresponds to a “topological” term leading to the bosonic algebra (34). It would be interesting, but beyond the scope of this note, to understand better under the lines of [32], the appearance of such topological terms.

5 The dual magnetic momenta and a generalization of the ADM formula

In this section we derive the expressions for the dual magnetic ADM momenta, containing as the timelike component the NUT charge. The derivation is based on a straightforward application of the usual ADM argument (see e.g. [33]) to the dual Riemann tensor, in its linearized form. We stress that we express all the quantities in terms of the linearized spin connection, so that the Bianchi identities are not automatically satisfied. Eventually we reformulate the classical treatment of [6] (where the magnetic charge would be obtained from contributions of the metric and the Misner string) by using the gauge-variance of the spin connection. We rewrite the integrals in terms of the vielbein in a fixed gauge so as to express the surface charges in terms of a regular spin connection, i.e. without string-like singularities.

In electromagnetism, when magnetic charges are considered, one has to add a magnetic current to the Bianchi identity. The conserved magnetic charge is calculated using (52). Obviously this charge would be trivially zero if the field strength verified $F = dA$ but one has to write $F = dA + C$ where $C$ represents the contribution from the Dirac string of the monopole to obtain the magnetic charge. As explained in [6], the situation looks quite similar in gravity. The
Bianchi identities can be rewritten in terms of the dual Riemann tensor defined by
\[
\tilde{R}_{\mu \nu \rho \sigma} = \frac{1}{2} \varepsilon_{\mu \nu \alpha \beta} R^{\alpha \beta}_{\rho \sigma} \quad (56)
\]
in the form
\[
\tilde{G}^{\mu \nu} = 8\pi \Theta^{\mu \nu} \quad (57)
\]
where \(\Theta^{\mu \nu}\) is the conserved magnetic stress-energy tensor.

To recover the expression for the NUT charge, let us first begin by briefly recalling how to quickly obtain the expressions for the ADM momenta. In this section, all the curvature tensors are to be considered as the linearized ones. By considering the higher order terms as belonging to the stress-energy tensor, one arrives at the definition:
\[
P_\mu = \frac{1}{8\pi} \int G_{0\mu} d^3V, \quad (58)
\]
which involves a volume integral.

The linearized Riemann tensor is written in terms of the spin connection:
\[
R_{\mu \nu \rho \sigma} = \partial_\rho \omega_{\mu \nu \sigma} - \partial_\sigma \omega_{\mu \nu \rho}. \quad (59)
\]
We thus have the following expressions:
\[
G_{00} = \frac{1}{2} R_{0ijj} = \partial_i \omega_{0jj} \quad (60)
\]
\[
G_{0i} = R_{0ijj} = \partial_i \omega_0 jj = \partial_j \omega_0 ij. \quad (61)
\]
Using the definition (58) and rewriting the \(P_\mu\) in terms of surface integrals using (60) and (61) one recovers the usual expression (45) for the ADM momenta.

Looking at (57) and (58) it is natural to consider that the definition of the conserved magnetic charge is:
\[
K_\mu = \frac{1}{8\pi} \int \tilde{G}_{0\mu} d^3V. \quad (62)
\]
The dual Ricci tensor is:
\[
\tilde{R}_{\mu \rho} = \eta^{\nu \sigma} \tilde{R}_{\mu \nu \rho \sigma} = \frac{1}{2} \eta^{\nu \sigma} \varepsilon_{\mu \nu \alpha \beta} R^{\alpha \beta}_{\rho \sigma}. \quad (63)
\]
The dual Ricci scalar and dual Einstein tensor are defined just as \(\tilde{R} = \eta^{\mu \rho} \tilde{R}_{\mu \rho}\) and \(\tilde{G}_{\mu \rho} = \tilde{R}_{\mu \rho} - \frac{1}{2} \eta_{\mu \rho} \tilde{R}\). We thus have the following expressions:
\[
\tilde{G}_{00} = -\frac{1}{2} \varepsilon_{ijk} R_{0ijk} = \varepsilon_{ijk} \partial_i \omega_{0jk}, \quad (64)
\]
\[
\tilde{G}_{0i} = \frac{1}{2} \varepsilon_{jkl} R_{klij} = \frac{1}{2} \varepsilon_{jkl}(\partial_i \omega_{klj} - \partial_j \omega_{kli}) = \varepsilon_{jkl} \partial_i \omega_{lij}. \quad (65)
\]
In the last equality of (65) we have used the identity \( \partial_i [\omega_{jkl}] = 0 \). Note also that \( \tilde{G}_{0i} \neq \tilde{G}_{i0} \) for an arbitrary (i.e. off-shell) spin connection. Using now the definition (62) for the dual momenta and using (61) and (65) we recover exactly (46). We have thus gained confidence that the expressions that we obtained through the complex Nester form are indeed what one would expect from a canonical definition of the dual ADM momenta.

We now want to express our formulas in function of the vielbein. Here this should be done carefully. Indeed, there is a big difference with respect to the case of electromagnetism where the quantity entering the surface integral is \( F_{\mu\nu} \), a gauge invariant quantity. Since the field strength for a magnetic monopole is related by duality to the field strength of an electric charge it will never contain string contributions, this being obviously true in whatever gauge. For this reason, the calculation of \( F_{\mu\nu} \) can also quickly be done by taking derivatives of the gauge potential away from the singularities. As we have just seen, in General Relativity, the conserved magnetic charge is expressed in terms of a spin connection which is a gauge variant object. To treat it correctly, one should then express the spin connection as a function of the metric and the string contributions, as shown in [6]. The calculation of the NUT charge will then involve contributions of the Misner string at infinity. However, if we look at (56) and (59) we see that:

\[
\tilde{\omega}_{\mu\nu\sigma} = \frac{1}{2} \varepsilon_{\mu\alpha\beta} \omega^{\alpha\beta\sigma}.
\]

(66)

Because the Schwarzschild metric has no singularities at infinity, this means that there exists a fixed gauge where the dual metric (the NUT metric) has a regular spin connection. From now on, all expressions will be written by supposing that we are in the gauge where the spin connection is regular at infinity, and derivatives are taken away from the singularity.

The linearization of the vielbein is:

\[
e^\mu = dx^\mu + \frac{1}{2} \eta^{\mu\nu}(h_{\nu\rho} + v_{\nu\rho})dx^\rho,
\]

(67)

where \( h_{\nu\rho} = h_{\rho\nu} \) and \( v_{\nu\rho} = -v_{\rho\nu} \). We recall that the linearized vielbein has 16 independent components, while the linearized metric has only 10, precisely the \( h_{\nu\rho} \) above. The extra 6 components \( v_{\nu\rho} \) are of course related to the local Lorentz invariance introduced by the tetrad formalism. The spin connection then reads:

\[
\omega_{\mu\nu} = \omega_{\mu\nu\rho} e^\rho, \quad \omega_{\mu\nu\rho} = \frac{1}{2} (\partial_\nu h_{\mu\rho} - \partial_\mu h_{\nu\rho} + \partial_\rho v_{\nu\mu}).
\]

(68)

We now rewrite the charges (45) and (46) using the above expression. This leads to the following generalized ADM and dual ADM formulae:

\[
P_0 = \frac{1}{16\pi} \oint (\partial_i h_{li} - \partial_i h_{ii} + \partial_i v_{il})d\Sigma_l,
\]

(69)
\[ P_k = \frac{1}{16\pi} \oint (\partial_0 h_{ik} - \partial_i h_{0k} + \delta^k_i \partial_0 h_{0i} - \delta^k_i \partial_0 h_{ii} + \partial_k v_{0i} + \delta^k_i \partial_i v_{0i}) d\hat{\Sigma}_l, \quad (70) \]

\[ K_0 = \frac{1}{16\pi} \oint \epsilon^{lij} (\partial_i h_{0j} + \partial_j v_{0i}) d\hat{\Sigma}_l, \quad (71) \]

\[ K_k = \frac{1}{16\pi} \oint \epsilon^{lij} (\partial_i h_{kj} + \partial_j v_{ik}) d\hat{\Sigma}_l, \quad (72) \]

where the gauge is fixed such that the spin connection is regular at infinity and string contributions disappear from the surface integrals. In order to evaluate the NUT charge, let us first display the metric as a first order correction to the flat metric in cartesian coordinates.\(^4\) For simplicity, and since the different perturbations are independent, we set the mass parameter \( M \) to zero:

\[ ds^2 = -dt^2 + dx^2 + dy^2 + dz^2 + 4N^2 \frac{z r}{r^2 + y^2} dx dy dt + \mathcal{O}(N^2). \quad (73) \]

Following (7), a natural choice for the vielbein is the triangular one:

\[ e^t = dt - 2N \frac{zy}{r(x^2 + y^2)} dx + 2N \frac{zx}{r(x^2 + y^2)} dy, \quad (74) \]

\[ e^x = dx, \quad e^y = dy, \quad e^z = dz. \quad (75) \]

In terms of the tensors \( h_{\mu\nu} \) and \( v_{\mu\nu} \), we have:

\[ h_{tx} = v_{tx} = 2N \frac{zy}{r(x^2 + y^2)}, \quad h_{ty} = v_{ty} = -2N \frac{zx}{r(x^2 + y^2)}. \quad (76) \]

Note that when \( M \neq 0 \), the tensor \( v_{\mu\nu} \) only depends on \( N \) at the linear level (i.e., the \( M \)-dependent perturbation of the vielbein is symmetric). Hence its presence is not going to affect the computation of the ADM mass \( P_0 \). On the other hand, in the expression giving \( P_i \) it can be checked that its presence makes the integrand vanishing. The charges \( K_i \) also straightforwardly vanish. We are left with computing \( K_0 \).

By choosing a particular vielbein, we fixed the gauge. Evaluating the linearized spin connection, using for instance (68) one finds:

\[ \omega_{0ij} = -\omega_{ij0} = \frac{\xi_{ijk} N x^k}{r^3}. \quad (77) \]

\(^4\)One might be worried by higher order corrections which could cease being subleading near the string-like singularity. Such terms are quadratic or higher in the charge \( N \) (and possibly \( M \), \( Q \) and \( H \)), and hence must cancel among themselves in the field equations, since the sources are linear in the charges. As a result, they do not contribute to the surface integrals, as can be checked explicitly in the case below if we were to retain also the higher order terms in the metric.
An additional check of the validity of this particular gauge comes from the fact that
one recovers the same spin connection if calculated using the dual of the linearized
spin connection for the Schwarzschild metric by means of (66). In other words,
we have used the local Lorentz gauge freedom of the vielbein to obtain a regular
spin connection. In some sense, the Misner string has been gauged away, and the
expressions for the surface charges given above become completely reliable. At
last, using the above expression it is straightforward to get:

$$ K_0 = \frac{1}{8\pi} \oint \varepsilon^{ijk} \omega_{0jk} d\hat{\Sigma}_i = \frac{N}{4\pi} \oint \frac{x^i}{r^3} d\hat{\Sigma}_i = N. $$

(78)

We have thus shown that the surface charges computing $M$ and $N$ for the
(charged) Taub-NUT metric are indeed $P_0$ and $K_0$ respectively, and that they
can be both obtained from the Nester form and independently from an ADM-like
argument involving the dual Riemann tensor.

6 Discussion

In this final section we discuss the results we have derived in the previous sections.
Taub-NUT spaces are notoriously problematic for the time identifications that they
imply [12], and for the presence of the Misner strings [19], which are gauge-variant
singularities. It has been suggested that these pathologies are enough to conclude
that such spacetimes are not globally supersymmetric [34], even though they have
locally (and globally as well) Killing spinors. However from the point of view of
the surface integrals that define both the bosonic and the fermionic charges of
the superalgebra, the spacetime with NUT charge is asymptotically flat according
to the simplest definition [6]. If we were to assume that the presence of Killing
spinors implies that the spacetime is supersymmetric, we would be faced with
the challenge of including the NUT charge in the superalgebra. The (asymptotic)
projection acting on the Killing spinor must be the same as the projection acting
on the supercharges which are represented trivially on a BPS multiplet. However
as we have shown the NUT charge enters in a term which cannot be part of the
superalgebra because of its wrong hermiticity. Below, we suggest a tentative path
to trivialize this problem.

A logical possibility is to write the corrected variation of the supercharge (34)
in a different form, by introducing a new supercharge $\tilde{Q}'$:

$$ \delta_{\epsilon_1,\epsilon_1} \tilde{Q}[\epsilon_2, \bar{\epsilon}_2] = i \left[ \tilde{Q}'[\epsilon_1, \bar{\epsilon}_1], \tilde{Q}[\epsilon_2, \bar{\epsilon}_2] \right]. $$

(79)

The above expression is not antisymmetric under the exchange of $\epsilon_1$ and $\epsilon_2$, which
is another way of encoding the presence of the (symmetric) topological term. In
terms of the fermionic supercharges $Q$ and $Q'$, (79) reads:

$$\delta_{\epsilon_1, \bar{\epsilon}_1} \hat{Q}[\epsilon_2, \bar{\epsilon}_2] = i \bar{\epsilon}_2 \{Q, Q^*\} C \epsilon_1 - i \epsilon_1 \{Q', Q^*\} C \bar{\epsilon}_2,$$

(80)

where we have supposed that $\{Q, Q'\} = 0$. Then, equating the above to the expression obtained through the Nester form, we get:

$$\{Q, Q^*\} = \gamma^\mu C P_\mu + \gamma_5 \gamma^\mu C K_\mu - i(U + \gamma_5 V) C,$$

(81)

Now the l.h.s. is no longer hermitian, so there are no obstructions to having the antihermitian term containing $K_\mu$ in the r.h.s. The question is of course what is $Q'$. It must be related to $Q$ otherwise we would be doubling the number of supercharges. We now show that it is related to $Q$ through an “axial” phase shift.

Let us rewrite for definiteness the relation (81) on our particular static massive, charged states with NUT charge:

$$\{Q, Q'^*\} = M + \gamma_5 N - i(Q + \gamma_5 H) \gamma_0.$$

(82)

Using the angles defined in Section 2, it can be rewritten as:

$$\{Q, Q'^*\} = \sqrt{M^2 + N^2} e^{\alpha_m \gamma_5} - i Z e^{\alpha_m \gamma_5} \gamma_0.$$

(83)

If the charge $Q'$ is related to $Q$ by a simple phase rotation:

$$Q'^* = Q^* e^{\alpha_m \gamma_5},$$

(84)

then eq. (82) takes a more standard, hermitian form:

$$\{Q, Q^*\} = M' - i(Q' + \gamma_5 H') \gamma_0,$$

(85)

with

$$M' = \sqrt{M^2 + N^2}, \quad Q' = \frac{QM - HN}{\sqrt{M^2 + N^2}}, \quad H' = \frac{HM + QN}{\sqrt{M^2 + N^2}}.$$

(86)

Hence, through a non-linear redefinition of the charges, we obtain the relation (85) that in the new variables defines an hermitian superalgebra. Actually, the new variable $M'$ is precisely the result of a gravitational duality rotation that eliminates the NUT charge, namely:

$$\begin{pmatrix} \cos \alpha_m & \sin \alpha_m \\ -\sin \alpha_m & \cos \alpha_m \end{pmatrix} \begin{pmatrix} M \\ N \end{pmatrix} = \begin{pmatrix} M' \\ 0 \end{pmatrix}.$$

(87)

Notice the similarity with pseudo-supersymmetry (see e.g. [35, 36]), where one is also dealing with non-hermitian relatives of superalgebras.
Note that also $Q'$ and $H'$ are obtained from $Q$ and $H$ through an electromagnetic duality rotation of the same angle.

The phase rotation (84) depends on dynamical quantities, such as $N$ and $M$. The latter however commute with the supercharges for consistency of the superalgebra, hence for instance we are assured that $\{Q, Q'\} = 0$. Moreover, one could wonder what modified supersymmetry variation is induced by $Q'$. This clearly deserves to be investigated, though for consistency we anticipate that we should not find any modification in the transformation laws of the elementary fields.

In a more general case where both ordinary and NUT momenta $P_i$ and $K_i$ are non zero the situation is a bit subtler. Indeed, focusing only on the “gravitational” part, we would have:

$$\{Q, Q'\} = P_0 + \gamma_5 K_0 + (P_i + \gamma_5 K_i)\gamma^i\gamma_0.$$

(88)

After a rotation similar to (84) we would get:

$$\{Q, Q'\} = \sqrt{P_0^2 + K_0^2} + \frac{1}{\sqrt{P_0^2 + K_0^2}} [P_i P_0 + K_i K_0 + \gamma_5 (K_i P_0 - P_i K_0)] \gamma^i\gamma_0.$$

(89)

We thus still have an offending anti-hermitian term, which is however proportional to $K_i P_0 - P_i K_0$ and is thus not present when $K_\mu$ is parallel to $P_\mu$. Now, under a general gravitational duality rotation [6] we have that:

$$\left( \begin{array}{cc} \cos \alpha & \sin \alpha \\ -\sin \alpha & \cos \alpha \end{array} \right) \left( \begin{array}{c} P_\mu \\ K_\mu \end{array} \right) = \left( \begin{array}{c} P'_\mu \\ K'_\mu \end{array} \right),$$

(90)

and a NUT 4-momentum $K_\mu$ can be completely eliminated only if it is parallel to $P_\mu$. We thus seem to be able to make sense out of a superalgebra in the presence of NUT charges only when the latter can be eliminated by a gravitational duality rotation.

When this is not possible, we do not seem to be able to define a superalgebra. Note that we are not aware of solutions with non-aligned $K_\mu$ and $P_\mu$ charges. Actually, it can be shown on simple examples that the r.h.s. of (81) does not have vanishing eigenvalues when $K_\mu$ and $P_\mu$ are non parallel.

In the case $K_\mu = \lambda P_\mu$, we have $\lambda = N/M = \tan \alpha_m$ and performing the rotation (90) with $\alpha = \alpha_m$, the relation (89) becomes the usual superalgebra:

$$\{Q, Q'\} = \gamma^\mu CP'_\mu.$$

(91)

Note that $K_\mu$ is always parallel to $P_\mu$ if the spatial components $K_i$ and $P_i$ are obtained by boosting a static object with $K_0$ and $P_0$ charges. We show in [37] that boosting a pure Taub-NUT solution, one indeed obtains solutions with
\( K_i \neq 0 \), and that in the infinite boost limit, one recovers the magnetic dual of the usual \( pp \)-wave, which is moreover half-BPS. This latter fact lends support to the presence of the dual magnetic momenta even in the \( \mathcal{N} = 1 \) superalgebra, along the same lines as above.

We could thus sum up in the following way the answer to the question that motivated this work, namely how does the NUT charge enter in the supersymmetry algebra. When \( K_\mu \) is parallel to \( P_\mu \), which seems to be the only situation where we have Killing spinors, by a gravitational duality rotation (90) we can eliminate \( K_\mu \). The superalgebra then incorporates the NUT charges through the (duality invariant) combination \( P'_\mu \). Alternatively, we can define a generalization of the superalgebra (81) where the NUT charges appear on the r.h.s. but where we have to define a new supercharge through the axial phase rotation (84). It is this latter generalized superalgebra that can be directly related to the complex Nester form. Nevertheless, both alternatives give the same BPS bound and projection on the supercharges, and are hence compatible with the projection on the Killing spinor. In conclusion, this is evidence that backgrounds which are obtained through gravitational duality rotations from ordinary BPS solutions, such as Reissner-Nordström black holes, are indeed supersymmetric.

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### A Computation of the variation of the gravitino

In this Appendix, we compute the variation of the gravitino, which is a complex Dirac spinor in \( \mathcal{N} = 2 \) supergravity:

\[
\delta \psi_\mu = \hat{\nabla}_\mu \epsilon = \hat{D}_\mu \epsilon + \frac{i}{4} F_{ab} \gamma^{ab} \gamma_\mu \epsilon = 0 \tag{92}
\]

where we recall that \( \hat{\nabla}_\mu \) is the super-covariant derivative and \( \hat{D}_\mu = \partial_\mu + \frac{1}{4} \omega^{ab}_\mu \gamma_{ab} \).

We take the gamma matrices to be real and such that they satisfy \( \{ \gamma_a, \gamma_b \} = 2 \eta_{ab} \). We also denote \( \gamma_{ab} = \frac{1}{2} [\gamma_a, \gamma_b] \). \( \gamma_5 = \gamma_{0123} \) is real and antisymmetric. For
definiteness, we list below a choice of real gamma matrices:

\[
\begin{align*}
\gamma_0 &= \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} & \quad \gamma_1 &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \\
\gamma_2 &= \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} & \quad \gamma_3 &= \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix}
\end{align*}
\]

(93)

We use conventions where \( C = \gamma_0 \), \( \bar{\epsilon} = \epsilon^\dagger C \) and \( \epsilon_{0123} = -\epsilon^{0123} = 1 \).

Using the definitions \( \lambda = r^2 - N^2 - 2Mr + Q^2 + H^2 \) and \( R^2 = r^2 + N^2 \), the charged Taub-NUT solution that we study is:

\[
ds^2 = -\frac{\lambda}{R^2} (dt + 2N \cos \theta d\phi)^2 + \frac{R^2}{\lambda} dr^2 + R^2 (d\theta^2 + \sin^2 \theta d\phi^2), \quad (94)
\]

\[
A_t = \frac{Qr + NH}{R^2}, \quad A_\phi = -\frac{H(r^2 - N^2) + 2N Qr}{R^2} \cos \theta. \quad (95)
\]

We choose the vielbein to be:

\[
\begin{align*}
e^0 &= \frac{\sqrt{\lambda}}{R} (dt + 2N \cos \theta d\phi), & \quad e^1 &= \frac{R}{\sqrt{\lambda}} dr, \\
e^2 &= Rd\theta, & \quad e^3 &= R \sin \theta d\phi.
\end{align*}
\]

We also list below the non-trivial components of the spin connection:

\[
\begin{align*}
\omega_{t}^{01} &= \frac{\lambda'}{2R^2} \frac{R'}{R^3} R' & \omega_{t}^{12} &= -\frac{\sqrt{\lambda}}{R} R'
\end{align*}
\]

\[
\begin{align*}
\omega_{\phi}^{13} &= -\frac{\sqrt{\lambda}}{R} R' \sin \theta & \omega_{\phi}^{23} &= -\cos \theta \left( 1 + \frac{2\lambda N^2}{R^4} \right)
\end{align*}
\]

\[
\begin{align*}
\omega_{\phi}^{02} &= -\frac{\sqrt{\lambda}}{R^2} N \sin \theta & \omega_{\phi}^{03} &= \frac{\sqrt{\lambda}}{R^2} N
\end{align*}
\]

\[
\begin{align*}
\omega_{t}^{23} &= -\frac{\lambda}{R^4} N & \omega_{\phi}^{01} &= 2N \cos \theta \left( \frac{\lambda'}{2R^2} - \frac{\lambda}{R^3} R' \right).
\end{align*}
\]

The non-zero components of \( F_{ab} \) are:

\[
\begin{align*}
F_{01} &= \frac{1}{R^4} (Q(r^2 - N^2) + 2HN r) = -\frac{Q}{R^2} + 2r \frac{Qr + NH}{R^4} \\
F_{23} &= \frac{1}{R^4} (H(r^2 - N^2) - 2Q N r) = \frac{H}{R^2} - 2N \frac{Qr + NH}{R^4}
\end{align*}
\]
so that
\[
F_{ab}\gamma^{ab} = -2F_{01}\gamma_{01} + 2F_{23}\gamma_{23}
\]
\[
= -\frac{2}{R^4}\gamma_{01}(r + \gamma_5N)^2(Q - \gamma_5H).
\]
(96)

We now compute the expressions for \(\omega_{ab}^{\gamma_{ab}}\):
\[
\omega_{t}^{\gamma_{ab}} = \frac{2}{R^4}\gamma_{01} \left[ (r - M)R^2 - \lambda(r + \gamma_5N) \right],
\]
(97)
\[
\omega_{r}^{\gamma_{ab}} = 0,
\]
(98)
\[
\omega_{\theta}^{\gamma_{ab}} = -2\frac{\sqrt{\lambda}}{R^2}\gamma_{12}(r + \gamma_5N),
\]
(99)
\[
\omega_{\phi}^{\gamma_{ab}} = -2\frac{\sqrt{\lambda}}{R^2}\sin \theta \gamma_{13}(r + \gamma_5N) - 2\cos \theta \gamma_{23}
\]
\[
+ 4N \cos \theta \frac{1}{R^4}\gamma_{01} \left[ (r - M)R^2 - \lambda(r + \gamma_5N) \right].
\]
(100)

Taking also into account that
\[
\gamma_t = \frac{\sqrt{\lambda}}{R}\gamma_0, \quad \gamma_r = \frac{R}{\sqrt{\lambda}}\gamma_1, \quad \gamma_\theta = R\gamma_2, \quad \gamma_\phi = R\sin \theta \gamma_3 + 2N\frac{\sqrt{\lambda}}{R}\cos \theta \gamma_0,
\]
(101)
we finally arrive at the SUSY variations
\[
\delta \psi_t = \partial_t \epsilon + \frac{1}{2R^4}\gamma_{01} \left[ (r - M)R^2 - \lambda(r + \gamma_5N) - i(r + \gamma_5N)^2(Q - \gamma_5H)\frac{\sqrt{\lambda}}{R}\gamma_0 \right] \epsilon,
\]
\[
\delta \psi_r = \partial_r \epsilon - i\frac{1}{2R^4}\gamma_{01}(r + \gamma_5N)^2(Q - \gamma_5H)\frac{R}{\sqrt{\lambda}}\gamma_1 \epsilon,
\]
\[
\delta \psi_\theta = \partial_\theta \epsilon + \frac{1}{2R^4} \left\{ -\sqrt{\lambda}R^2\gamma_{12}(r + \gamma_5N) - i\gamma_{01}(r + \gamma_5N)^2(Q - \gamma_5H)\gamma_2 \right\} \epsilon,
\]
\[
\delta \psi_\phi = \partial_\phi \epsilon + \frac{1}{2R^4} \left\{ -\sqrt{\lambda}R^2\sin \theta \gamma_{13}(r + \gamma_5N) - R^4\cos \theta \gamma_{23}
\]
\[
+ 2N \cos \theta \gamma_{01} \left[ (r - M)R^2 - \lambda(r + \gamma_5N) \right]
\]
\[
- i \gamma_{01}(r + \gamma_5N)^2(Q - \gamma_5H) \left( R\sin \theta \gamma_3 + 2N\frac{\sqrt{\lambda}}{R}\cos \theta \gamma_0 \right) \right\} \epsilon.
\]
(102)

Note that in flat space we still have non trivial equations:
\[
\partial_t \epsilon = 0
\]
\[
\partial_r \epsilon = 0
\]
\[
\partial_\theta \epsilon = \frac{1}{2}\gamma_{12} \epsilon
\]
\[
\partial_\phi \epsilon = \frac{1}{2}(\sin \theta \gamma_{13} + \cos \theta \gamma_{23}) \epsilon
\]
(103)
The general expression for the Killing spinor satisfying equations (103) is:
\[ \epsilon(t, r, \theta, \phi) = e^{\frac{1}{2} \gamma_{12} \theta} e^{\frac{i}{2} \gamma_{23} \phi} \epsilon_0 \]  
(104)

where \( \epsilon_0 \) is a constant spinor.

In our more general case, let us suppose that all the dependence in \( \theta \) and \( \phi \) factorizes as above. Hence, we look for a Killing spinor with the form (104) where however \( \epsilon_0 \) depends on \( r \) and possibly \( t \).

Let us first look at the expression for \( \delta \psi_\theta \). It becomes an algebraic condition on \( \epsilon_0 \), which can be rewritten as:
\[ \left[(r + \gamma_5 N)(\sqrt{\lambda} - r - \gamma_5 N)\gamma_0 - i(Q + \gamma_5 H)R \right] \epsilon_0 \equiv P \epsilon_0 = 0, \]  
(105)

where we have used \( R^2 = r^2 + N^2 = (r + \gamma_5 N)(r - \gamma_5 N) \).

The Killing spinor equations will have non-trivial solutions only if the operator \( P \) above has vanishing eigenvalues, i.e. its determinant is zero. It appears however easier to just compute the square of the operator \( P \):
\[ P^2 = (-2iQR)P - R^2 \left[ (\sqrt{\lambda} - r)^2 + N^2 - Q^2 - H^2 \right]. \]  
(106)

The coefficients are just complex numbers, so that the eigenvalues of \( P \) must satisfy the same equation, with two solutions. Therefore, the operator \( P \) will have zero eigenvalues (and be proportional to a projector) only if \( (\sqrt{\lambda} - r)^2 + N^2 - Q^2 - H^2 = 0 \), which translates into
\[ M^2 + N^2 = Q^2 + H^2, \]  
(107)

an \( r \)-independent condition. Note that another way to state the above BPS condition is to write \( \sqrt{\lambda} = r - M \). It is this expression that we will substitute back into the SUSY variations. This is done in Section 2.

**B Computation of the Killing spinor**

In this section we compute the explicit expression of the Killing spinor, using the results obtained in Section 2, namely that the Killing spinor has to satisfy the projection
\[ \left\{1 - i e^{(\beta + \alpha_m - \alpha_q)\gamma_5 \gamma_0} \right\} \epsilon = 0. \]  
(108)

The only non trivial equation that remains to be solved is \( \delta \psi_r = 0 \):
\[ \partial_r \epsilon = \frac{Z}{2R(r - M)} i e^{(2\beta - \alpha_q)\gamma_5 \gamma_0} \epsilon. \]  
(109)

The strategy we adopt is straightforward. We just solve the projector equation above in components, and then plug back the components into the first order differential equation.
Let us call
\[ c \equiv \cos(\beta + \alpha_m - \alpha_q), \quad s \equiv \sin(\beta + \alpha_m - \alpha_q). \]  
(110)

Then the solution to (108) can be written in the form (104) with
\[ \epsilon_0 = \epsilon_1(r) \begin{pmatrix} 1 \\ 0 \\ is \\ ic \end{pmatrix} + \epsilon_2(r) \begin{pmatrix} 0 \\ 1 \\ -ic \\ is \end{pmatrix}. \]  
(111)

The equation \( \delta \psi_r = 0 \) becomes then
\[ \partial_r \epsilon_0 = \frac{Z}{2R(r - M)} (\cos(\beta - \alpha_m) + \sin(\beta - \alpha_m)\gamma_5) \epsilon_0. \]  
(112)

In computing \( \partial_r \epsilon_0 \), one has to recall that \( \partial_r \beta = -N/R^2 \). We have 4 equations for 2 functions \( \epsilon_1(r) \) and \( \epsilon_2(r) \). It is fairly straightforward to see that the equations for the two lower components of \( \epsilon_0 \) are automatically satisfied once the equations for the upper two components are satisfied.

The two equations to be solved are
\[ \partial_r \epsilon_1 = \frac{Z}{2R(r - M)} (\cos(\beta - \alpha_m)\epsilon_1 - \sin(\beta - \alpha_m)\epsilon_2), \]  
(113)
\[ \partial_r \epsilon_2 = \frac{Z}{2R(r - M)} (\sin(\beta - \alpha_m)\epsilon_1 + \cos(\beta - \alpha_m)\epsilon_2). \]  
(114)

We can clearly write the 2 functions \( \epsilon_1(r) \) and \( \epsilon_2(r) \) in terms of a common scalar function and a phase:
\[ \begin{pmatrix} \epsilon_1(r) \\ \epsilon_2(r) \end{pmatrix} = h(r) \begin{pmatrix} \cos \hat{\alpha}(r) & \sin \hat{\alpha}(r) \\ -\sin \hat{\alpha}(r) & \cos \hat{\alpha}(r) \end{pmatrix} \begin{pmatrix} \hat{\epsilon}_1 \\ \hat{\epsilon}_2 \end{pmatrix} \]  
(115)

where \( \hat{\epsilon}_1 \) and \( \hat{\epsilon}_2 \) are constants.

We obtain the two equations
\[ \partial_r h = \frac{Z}{2R(r - M)} \cos(\beta - \alpha_m)h, \quad \partial_r \hat{\alpha} = -\frac{Z}{2R(r - M)} \sin(\beta - \alpha_m). \]  
(116)

They can be rewritten as
\[ \partial_r h = \frac{N^2 + rM}{2R^2(r - M)} h, \quad \partial_r \hat{\alpha} = \frac{N}{2R^2} \equiv -\frac{1}{2} \partial_r \beta. \]  
(117)
The solution is thus:

$$\begin{pmatrix} \epsilon_1(r) \\ \epsilon_2(r) \end{pmatrix} = \left( \frac{r - M}{R} \right)^{\frac{1}{2}} \begin{pmatrix} \cos \frac{1}{2} \beta(r) & -\sin \frac{1}{2} \beta(r) \\ \sin \frac{1}{2} \beta(r) & \cos \frac{1}{2} \beta(r) \end{pmatrix} \begin{pmatrix} \hat{\epsilon}_1 \\ \hat{\epsilon}_2 \end{pmatrix}.$$  \hspace{1cm} (118)

We can define the rotation matrix

$$R[\alpha] = \begin{pmatrix} \cos \alpha & -\sin \alpha \\ \sin \alpha & \cos \alpha \end{pmatrix}.$$  \hspace{1cm} (119)

Then by performing an additional constant rotation of the spinors

$$\begin{pmatrix} \hat{\epsilon}_1 \\ \hat{\epsilon}_2 \end{pmatrix} \equiv \tilde{\epsilon} = R[\frac{1}{2} (\alpha_m - \alpha_q)] \hat{\epsilon}$$  \hspace{1cm} (120)

we can write the final expression for the Killing spinor as:

$$\epsilon_0(r) = \left( \frac{r - M}{R} \right)^{\frac{1}{2}} \begin{pmatrix} R[\frac{1}{2} (\beta(r) + \alpha_m - \alpha_q)] \tilde{\epsilon} \\ iR[\frac{1}{2} (\pi - \beta(r) - \alpha_m + \alpha_q)] \tilde{\epsilon} \end{pmatrix}.$$  \hspace{1cm} (121)

This is the expression presented in Section 2.

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