LARGE-STEP-SIZE INTEGRATORS WITH IMPROVED UNIFORM ACCURACY AND LONG TIME CONSERVATION FOR HIGHLY OSCILLATORY SYSTEMS WITH LARGE INITIAL DATA

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Abstract. In this paper, we are concerned with large-stepsize highly accurate integrators for highly oscillatory second-order differential equations with large initial data and a scaling parameter $0 < \epsilon \ll 1$. The highly oscillatory property of this model problem corresponds to the parameter $\epsilon$. We propose and analyze a novel class of highly accurate integrators which is based on some formulation approaches to the problem, Fourier pseudo-spectral method and exponential integrators. Two practical integrators up to order four are constructed by using the symmetric property and stiff order conditions of implicit exponential integrators. The convergence of the obtained integrators is rigorously studied, and it is shown that the accuracy is improved to be $O(\epsilon^2 h^r)$ in the absolute position error for the time stepsize $h$ and the order $r$ of the integrator. The near energy conservation over long times is established for the integrators with large time stepsizes. Numerical results show that the proposed integrators used with large stepsizes have improved uniformly high accuracy and excellent long time energy conservation.

Keywords: Large-stepsize integrators, Highly oscillatory problem, Highly accurate methods, Long time energy conservation.

AMS Subject Classification: 65L05, 65P10, 65L20, 65L70.

1. Introduction

A large amount of dynamics can be described by the following highly oscillatory second-order differential equations

$$
\ddot{q}(t) + \frac{1}{\epsilon^2} A q(t) = F(q(t)), \quad q(0) = \psi_1, \quad \dot{q}(0) := \psi_2, \quad t \in [0, T],
$$

where $q, p := \dot{q} : \mathbb{R} \to \mathbb{R}^d$ respectively represent position and velocity, $A \in \mathbb{R}^{d \times d}$ is a positive definite matrix of bounded norm independent of $\epsilon$, $0 < \epsilon \ll 1$ is inversely proportional to the spectral radius of $A$, and $F(q(t))$ is a nonlinear function with a Lipschitz constant bounded independently of $\epsilon$. Clearly, the solution of (1.1) becomes highly oscillatory when large frequencies are involved in the equation, which corresponds to the parameter $0 < \epsilon \ll 1$. Its computation represents major challenges because of the high oscillations. This model often arises in a variety of fields such as applied mathematics, quantum physics, classical mechanics, engineering, chemistry, molecular biology and so on. Some classical examples could be the Duffing equation, Fermi-Pasta-Ulam problem, Hénon-Heiles model, molecular dynamics and semi-linear wave equations, including Klein-Gordon equations, after appropriate spatial discretization [7, 22, 26, 31, 33]. If we further assume that $A$ is symmetric and $F(q) = -\nabla U(q)$ with a smooth potential $U$, the model (1.1) is a Hamiltonian system with the following Hamiltonian function [17]:

$$
H(q, p) = \frac{1}{2} p^T p + \frac{1}{2\epsilon^2} q^T A q + U(q),
$$

which is conserved along the exact solution of (1.1). It is noted that the scales of $\psi_1, \psi_2$ will lead to different properties of the system. In this paper, we consider the following scale of $\psi_1, \psi_2$:

large initial data: $\psi_1 = O(1), \quad \psi_2 = O(1) \to$ energy unbounded system (1.1).

Although this type of system can be transformed into energy bounded case by using a scaling to the variables, the large initial data case has more essential differences such as the stronger oscillation feature and this makes the construction and analysis of numerical methods be more difficult and challenging [22].
The time integration of such equation \( (1.1) \) is a basic algorithmic task and it has been received much attention in recent decades. Due to the high oscillations, the traditional methods such as symplectic/symmetric Runge–Kutta–Nyström (RKN) methods \([18, 34]\) or energy-preserving Runge-Kutta methods \([8, 36]\) often result in convergence problems. In order to get more competitive methods, Gauß-type trigonometric integrators were developed in \([28]\) and different kinds of trigonometric integrators were formulated and studied in \([19, 20, 24, 25, 37, 41]\). However, for these integrators, they at most have uniform second order accuracy in the absolute position error for trigonometric integrators were formulated and studied in \([19, 20, 24, 25, 37, 41]\). As a result, it is challenging to get trigonometric integrators of uniform higher-order accuracy in both position and velocity.

Recently, some new methods with uniform accuracy for highly oscillatory systems have been proposed and analysed such as two-scale formulation methods \([10, 11]\), uniformly accurate exponential-type integrators \([2]\), nested Picard iterative integrators \([9]\), multiscale time integrators \([2, 3, 12]\), multi-revolution composition methods \([32]\) and uniformly accurate methods with averaging \([13, 14]\).

In \([4]\), various uniformly accurate (UA) methods have been compared systematically for solving wave equations. Recently, time-splitting methods were proved to have improved uniform error bounds for solving nonlinear Klein-Gordon equation with weak nonlinearity \([1]\). Most of these uniformly accurate methods can be applied to the system \((1.1)\) and they have excellent uniform accuracy in both position and velocity. Unfortunately, most of them do not have good long time conservation behaviour when applied to conservative systems. More precisely, if a UA method is considered as the approximation of \((1.1)\), the numerical energy error will increase as time evolves. In a recent work \([15]\), the UA method named as pullback method is shown to hold the long time conservation by the numerical results but without rigorous analysis. In a more recent work \([40]\), the authors succeed in making the two-scale method with near conservation laws for first-order systems. However, only one-stage type methods are presented there and the equation \((1.1)\) does not share the form of the system considered in \([10]\), which means that the analysis of \([10]\) is no longer applicable for the system \((1.1)\) of this paper. Moreover, the second-order differential equation \((1.1)\) has its special structure which will be neglected if we rewrite it as a general highly oscillatory first-order differential equation \( \ddot{u}(t) = \frac{1}{\epsilon^2} K \dot{u}(t) + g(u(t)) \). Therefore, it is necessary and meaningful to design and analyze UA methods with uniform high order and good long time energy conservation for solving the second-order system \((1.1)\). It is worth pointing out that the integrators derived in this paper will be shown to have better accuracy than uniform accuracy, which is \(\mathcal{O}(\epsilon^2 h^r)\) for the time stepsize \(h\) and the order \(r\) of the integrator. This improved uniformly high accuracy is different from the existing UA methods \([2, 3, 4, 5, 10, 11, 12, 13, 14, 32, 40]\) and seems surprising at the first glance of \((1.1)\) due to the \(\mathcal{O}(1/\epsilon^2)\) commutator.

In this paper we are interested in using numerical integrators with time stepizes \(h\) that are much larger than the \(\epsilon\) of the system to obtain improved uniform accuracy and good long-time energy conservation when solving \((1.1)\). To obtain the integrators with more than improved uniform second order accuracy, we take advantage of two-scale formulation approach, spectral semi-discretisation and exponential integrators with more than one stage. However, this brings some challenges and difficulties in the achievement and analysis of long time energy conservation. a) The two-scale formulation approach results in a new system which has a completely different linear part and nonlinear function in comparison with the original problem \((1.1)\). This brings more difficulties in the proof of energy conservation of the original problem \((1.1)\). b) Long term analysis of high-order splitting integrators for Schrödinger equations was discussed in \([21]\) and it was shown that high-order splitting integrators have the same long-time behaviour as the low-order splitting, which is somewhat unreasonable for high order methods. One open question that whether high-order methods behave better than low-order methods was proposed in \([21]\). To achieve high order improved uniform accuracy and good long time behaviour, two-stage and three-stage exponential integrators are chosen in this paper. Unfortunately, however, for a method with more than one stage applied to highly oscillatory systems, long time analysis has not been done for any method so far. As pointed out in \([15]\), there is the technical difficulty coming from the identification of invariants in the corresponding modulation system. Thus, it remains a challenge to study long time behaviour for a method with more than one stage. c) The third challenge comes from the diversity of \((1.1)\) considered in this
paper. Large initial data case \(1.2\) usually leads to large bounds of the coefficient functions in the modulation Fourier expansion, which prevents the derivation of the long time conservation.

To overcome these difficulties and make the analysis go smoothly, a novel approach to the design of integrators is established. We first consider three transformations of the original system and then use symmetric exponential integrators which satisfy stiff order conditions of up to order four. The transformations of the system and stiff order conditions proposed in this paper can keep the accuracy of our integrators is improved to be \(O(\varepsilon^2 h^r)\), and as \(\varepsilon\) decreases, the integrators are more accurate. This improved uniformly high accuracy is very competitive in the numerical computation of highly oscillatory systems. Moreover, we should note that we managed to derive the long-time energy conservation for the two-stage and three-stage methods applied to the large initial value system \(1.2\), which is different form the existing long term analysis work \([15, 16, 17, 23, 25, 27, 38, 40]\), where one-stage type methods and small initial value are both necessary. From the results derived in this paper, it follows that the two-stage and three-stage methods we proposed here has better accuracy in the long time energy conservation than a one-stage method even they share the same accuracy order. This is unexpected and surprised, and is a brand-new angle to demonstrate the superiority of the methods with more than one stage.

The rest of this paper is organized as follows. In Section 2 we firstly present the construction process of the integrators, and then we construct two practical integrators by using the symmetry and stiff order conditions. The main results concerning the improved uniformly high accuracy process of the integrators, and then we construct two practical integrators by using the symmetry and stiff order conditions. We first make some transformations of the system and then consider the numerical integration. There are in all three steps in the process and we present them one by one.

**Step 1. Three transformations of the system.** Firstly, since the matrix \(A\) is symmetric and positive definite, there exist a unitary matrix \(P\) and an invertible diagonal matrix \(\Lambda\) such that \(A = P\Lambda^2 P^H\). Then letting \(y = P^H \tilde{y}\) transforms the original system \(1.1\) into

\[
\ddot{y}(t) + \frac{1}{\varepsilon} \Lambda^2 y(t) = f(y(t)), \quad y(0) = P^H \psi_1, \quad \dot{y}(0) = P^H \dot{\psi}_2, \quad t \in [0, T],
\]

where \(f(y) = -\nabla_y U(Py)\). This is still a conservative system and its energy is given by

\[
H_1(y, \dot{y}) = \frac{1}{2} (\dot{y})^T \dot{y} + \frac{1}{2\varepsilon} y^T \Lambda^2 y + U(Py).
\]

By using a scaling to the variables \(\tilde{q}(t) = y(t)\) and \(\tilde{p}(t) = \varepsilon \Lambda^{-1} \dot{y}(t)\), a new system is obtained immediately

\[
\dot{\tilde{q}}(t) = \frac{\Lambda}{\varepsilon} \tilde{p}(t), \quad \dot{\tilde{p}}(t) = -\frac{\Lambda}{\varepsilon} \tilde{q}(t) + \varepsilon \Lambda^{-1} f(\tilde{q}(t)), \quad \tilde{q}(0) = P^H \psi_1, \quad \tilde{p}(0) = \Lambda^{-1} P^H \psi_2.
\]

It can be verified that the solution of this system conserves the following energy

\[
H_2(\tilde{q}, \tilde{p}) := \frac{1}{2} \tilde{q}^T \frac{\Lambda^2}{\varepsilon^2} \tilde{q} + \frac{1}{2} \tilde{p}^T \frac{\Lambda^2}{\varepsilon^2} \tilde{p} + U(P\tilde{q}).
\]

Then letting

\[
\begin{align*}
\dot{q}(t) &= \cos(t\Lambda/\varepsilon) x(t) + \sin(t\Lambda/\varepsilon) v(t), \\
\dot{p}(t) &= -\sin(t\Lambda/\varepsilon) x(t) + \cos(t\Lambda/\varepsilon) v(t),
\end{align*}
\]

we obtain

\[
\begin{align*}
\dot{x}(t) &= -\varepsilon \Lambda^{-1} \sin(t\Lambda/\varepsilon) f\left( \cos(t\Lambda/\varepsilon)x(t) + \sin(t\Lambda/\varepsilon)v(t) \right), \quad x(0) = P^H \psi_1, \\
\dot{v}(t) &= \varepsilon \Lambda^{-1} \cos(t\Lambda/\varepsilon) f\left( \cos(t\Lambda/\varepsilon)x(t) + \sin(t\Lambda/\varepsilon)v(t) \right), \quad v(0) = \Lambda^{-1} P^H \psi_2.
\end{align*}
\]
Step 2. Two-scale formulation. By isolating the fast time variable $t/\varepsilon$ as another variable $\tau$ and denoting $X(t, \tau) = x(t)$, $V(t, \tau) = v(t)$, the two-scale pattern of (2.4) can be formulated as follows:

$$
\begin{align*}
\partial_\tau X(t, \tau) + \frac{1}{\varepsilon} \partial_t X(t, \tau) &= -\varepsilon\Lambda^{-1} \sin(\tau\Lambda)F\left(\cos(\tau\Lambda)X(t, \tau) + \sin(\tau\Lambda)V(t, \tau)\right), \\
\partial_\tau V(t, \tau) + \frac{1}{\varepsilon} \partial_t V(t, \tau) &= \varepsilon\Lambda^{-1} \cos(\tau\Lambda)F\left(\cos(\tau\Lambda)X(t, \tau) + \sin(\tau\Lambda)V(t, \tau)\right),
\end{align*}
$$

(2.5)

where $X(t, \tau)$ and $V(t, \tau)$ are periodic in $\tau$ on the torus $\mathbb{T} = (0, 2\pi)$. It is noted that this kind of two-scale equation with a modified initial data is analysed in [10, 14] to construct uniformly accurate approximations. Here we use the same strategy from [10, 14] to obtain the initial data for (2.5), which is presented briefly as follows.

Let $U(t, \tau) = [X(t, \tau); V(t, \tau)]$ and

$$
U[0] = [p^{H_1}; \Lambda^{-1} p^{H_2}] + \varepsilon B_0^{k-1}[U[k-1]].
$$

(2.6)

Choose $U[0] = [p^{H_1}; \Lambda^{-1} p^{H_2}]$ and then we compute $U[k] = U[0] - \varepsilon B_0^{k-1}[U[k-1]]$. In such a way, the $j$-th order initial data for (2.5) is derived by

$$
U[j](\tau) = U[0] + \varepsilon B_0^{[j]} \left( \sum_{k=1}^{j} B_k^{[j]} \left( \sum_{i=1}^{k} \varepsilon B_0^{i-1}[U[i-1]] \right) \right).
$$

(2.7)

The $B_0^{[k-1]}$ appearing in (2.7) is computed by $B_0^{[0]} = 0$ and

$$
B_0^{[k-1]}(U) = \partial^{-1}_\tau (I - \Pi) f_\tau \left( U + \varepsilon B_0^{[k]}(U) \right) - \frac{1}{\varepsilon} \partial^{-1}_\tau \left( B_0^{[k]}(U + \varepsilon \Pi f_\tau \left( U + \varepsilon B_0^{[k]}(U) \right)) - B_0^{[k]}(U) \right),
$$

where $\Pi$ is the averaging operator defined by $\Pi v := \frac{1}{2\pi} \int_0^{2\pi} v(s)ds$ for some $v(\cdot)$ on $\mathbb{T}$.

Step 3. Fully discrete scheme. We now present the fully discrete scheme for (2.5) which is constructed by making use of the spectral semi-discretisation (see [34]) in $\tau$ and exponential integrators (see [30]) in time.

For the spectral semi-discretisation used in $\tau$, let $\mathcal{M} := \{-N_\tau/2, -N_\tau/2 + 1, \ldots, N_\tau/2\}$ with a positive integer $N_\tau > 1$ and $Y_M = \text{span}\{e^{ik\tau}, k \in \mathcal{M}, \, \tau \in [-\pi, \pi]\}$. For any periodic function $v(\tau)$ on $[-\pi, \pi]$, define the standard projection operator $P_M : L^2([-\pi, \pi]) \to Y_M$ and the trigonometric interpolation operator $I_M : C([-\pi, \pi]) \to Y_M$ respectively as

$$
(P_M v)(\tau) = \sum_{k \in \mathcal{M}} \hat{v}_k e^{ik\tau}, \quad (I_M v)(\tau) = \sum_{k \in \mathcal{M}} \hat{v}_k e^{ik\tau},
$$

where $i = \sqrt{-1}$, $\hat{v}_k$ for $k \in \mathcal{M}$ are the Fourier transform coefficients of the periodic function $v(\tau)$ and $\hat{v}_k$ are the discrete Fourier transform coefficients of the vector $\{v(\tau_k)\}_{\tau_k = \frac{2\pi k}{N_\tau}}$. For consistency reasons, we assume that the first term and the last one in the summation are taken with a factor $1/2$ here and after. Then the Fourier spectral method is given by finding the trigonometric polynomials

$$
X^M(t, \tau) = (X_j^M(t, \tau))_{j=1,2,\ldots,d}, \quad V^M(t, \tau) = (V_j^M(t, \tau))_{j=1,2,\ldots,d}
$$

with

$$
X_j^M(t, \tau) = \sum_{k \in \mathcal{M}} X_{k,j}(t)e^{ik\tau}, \quad V_j^M(t, \tau) = \sum_{k \in \mathcal{M}} \hat{V}_{k,j}(t)e^{ik\tau}, \quad (t, \tau) \in [0, T] \times [-\pi, \pi]
$$

such that

$$
\begin{align*}
\partial_\tau X^M(t, \tau) + \frac{1}{\varepsilon} \partial_t X^M(t, \tau) &= -\varepsilon\Lambda^{-1} \sin(\tau\Lambda)F\left(\cos(\tau\Lambda)X^M(t, \tau) + \sin(\tau\Lambda)V^M(t, \tau)\right), \\
\partial_\tau V^M(t, \tau) + \frac{1}{\varepsilon} \partial_t V^M(t, \tau) &= \varepsilon\Lambda^{-1} \cos(\tau\Lambda)F\left(\cos(\tau\Lambda)X^M(t, \tau) + \sin(\tau\Lambda)V^M(t, \tau)\right).
\end{align*}
$$

(2.8)

It follows from the orthogonality of the Fourier functions and collecting all the $\hat{X}_{k,j}, \hat{V}_{k,j}$ in $(N_\tau+1)$-periodic coefficient vectors $\hat{X}(t) = (\hat{X}_{k,j}(t)), \hat{V}(t) = (\hat{V}_{k,j}(t))$ that

$$
\begin{align*}
\frac{d}{dt}\hat{X}(t) &= i\Omega \hat{X}(t) - \varepsilon\Lambda^{-1} F\left(SF^{-1}\hat{X}(t) + SF^{-1}\hat{V}(t)\right), \\
\frac{d}{dt}\hat{V}(t) &= i\Omega \hat{V}(t) + \varepsilon\Lambda^{-1} F\left(CF^{-1}\hat{X}(t) + SF^{-1}\hat{V}(t)\right),
\end{align*}
$$

(2.8)
where the vectors \( \vec{X}, \vec{V} \) are in dimension \( D := d \times (N_\tau + 1) \), \( \mathcal{F} \) denotes the discrete Fourier transform, \( \mathbf{\Lambda}^{-1} = \mathbf{\Lambda}^{-1} \otimes I_{N_\tau + 1} \), \( \mathbf{S} = \text{diag}(\sin(\tau L))_{l=0,1,\ldots,N_\tau} \), \( \mathbf{C} = \text{diag}(\cos(\tau L))_{l=0,1,\ldots,N_\tau} \), \( \Omega = \text{diag}(\Omega_1, \Omega_2, \ldots, \Omega_d) \) with \( \Omega_1 = \Omega_2 = \ldots = \Omega_d := \frac{1}{\tau} \text{diag}(\frac{N_\tau}{2}, \frac{N_\tau}{2} - 1, \ldots, -\frac{N_\tau}{2}) \). Our analysis presented below will use the entries of \( \vec{X} \) and \( \vec{V} \), which are denoted by

\[
\vec{X} = ( \hat{X}_{-\frac{N_\tau}{2}}, \ldots, \hat{X}_{-\frac{N_\tau}{2}+1}, \hat{X}_{-\frac{N_\tau}{2}}, \ldots, \hat{X}_{-\frac{N_\tau}{2}d}, \ldots, \hat{X}_{\frac{N_\tau}{2}d} ),
\]

\[
\vec{V} = ( \hat{V}_{-\frac{N_\tau}{2}}, \ldots, \hat{V}_{-\frac{N_\tau}{2}+1}, \hat{V}_{-\frac{N_\tau}{2}}, \ldots, \hat{V}_{-\frac{N_\tau}{2}d}, \ldots, \hat{V}_{\frac{N_\tau}{2}d} ).
\]

The same notation is used for all the vectors and diagonal matrices with the same dimension as \( \vec{X} \). We also use the notations \( \vec{X}_{\cdot,l} = ( \hat{X}_{-\frac{N_\tau}{2},l}, \hat{X}_{-\frac{N_\tau}{2}+1,l}, \ldots, \hat{X}_{\frac{N_\tau}{2}d} ) \) for \( l = 0, 1, \ldots, N_\tau \) and \( \mathcal{F} \vec{X} \) denotes the discrete Fourier transform acting on each \( \vec{X}_{\cdot,l} \) of \( \vec{X} \). Then the fully discrete scheme \((\text{FS-F})\) can read

\[
X_{M,j}^{n,i}(\tau) = \sum_{k \in M} X_{k,j}^{n,i} e^{ik\tau}, \quad V_{M,j}^{n,i}(\tau) = \sum_{k \in M} V_{k,j}^{n,i} e^{ik\tau}, \quad i = 1, 2, \ldots, s,
\]

\[
X_{M,j}^{n+1,i}(\tau) = \sum_{k \in M} X_{k,j}^{n+1,i} e^{ik\tau}, \quad V_{M,j}^{n+1,i}(\tau) = \sum_{k \in M} V_{k,j}^{n+1,i} e^{ik\tau}, \quad n = 0, 1, \ldots,
\]

where we consider the following \( s \)-stage exponential integrators \((\text{SI})\) applied to \((2.8)\):

\[
\begin{align*}
X_{n,i} &= e^{c_{hM}n} \hat{X}_n - \varepsilon h \sum_{j=1}^{s} \hat{a}_{ij}(hM) \mathbf{\Lambda}^{-1} \mathcal{F} \left( \mathbf{S} \left( \mathbf{C} \mathcal{F}^{-1} \hat{X}_{nj} + \mathbf{S} \mathcal{F}^{-1} \hat{V}_{nj} \right) \right), \quad i = 1, 2, \ldots, s, \\
\hat{V}_{n,i} &= e^{c_{hM}n} \hat{V}_n + \varepsilon h \sum_{j=1}^{s} \hat{b}_{ij}(hM) \mathbf{\Lambda}^{-1} \mathcal{F} \left( \mathbf{C} \mathcal{F}^{-1} \hat{X}_{nj} + \mathbf{S} \mathcal{F}^{-1} \hat{V}_{nj} \right), \quad i = 1, 2, \ldots, s, \\
X_{n+1,i} &= e^{hM} \hat{X}_n - \varepsilon h \sum_{j=1}^{s} \hat{b}_{ij}(hM) \mathbf{\Lambda}^{-1} \mathcal{F} \left( \mathbf{S} \left( \mathbf{C} \mathcal{F}^{-1} \hat{X}_{nj} + \mathbf{S} \mathcal{F}^{-1} \hat{V}_{nj} \right) \right), \\
\hat{V}_{n+1,i} &= e^{hM} \hat{V}_n + \varepsilon h \sum_{j=1}^{s} \hat{b}_{ij}(hM) \mathbf{\Lambda}^{-1} \mathcal{F} \left( \mathbf{C} \mathcal{F}^{-1} \hat{X}_{nj} + \mathbf{S} \mathcal{F}^{-1} \hat{V}_{nj} \right).
\end{align*}
\]

(2.9)

Here \( M = i\Omega, c_i \) for \( i = 1, \ldots, s \) are constants belonging to \([0, 1]\), and \( \hat{a}_{ij}(hM), \hat{b}_{ij}(hM) \) are matrix-valued functions of \( hM \).

The above procedure, however, is unsuitable in practice because of the computation of Fourier transform coefficients. In order to find an efficient implementation, we now consider the discrete Fourier transform coefficients instead of Fourier transform coefficients. This gives the following fully discrete scheme.

Let \( \tau_{\cdot,l} = \frac{\tau}{N_\tau} \) with \( l \in M \) and

\[
X_{k,j}^{n,i} \approx X_j(t_n + c_i h, \tau), \quad X_{k,j}^{n} \approx X_j(t_n, \tau), \quad V_{k,j}^{n,i} \approx V_j(t_n + c_i h, \tau), \quad V_{k,j}^{n} \approx V_j(t_n, \tau)
\]

for \( j = 1, 2, \ldots, d \). An exponential Fourier spectral discretization (FS-D) is defined as

\[
X_{k,j}^{n,i} = \sum_{k \in M} \tilde{X}_{k,j}^{n,i} e^{ik\tau_n}, \quad V_{k,j}^{n,i} = \sum_{k \in M} \tilde{V}_{k,j}^{n,i} e^{ik\tau_n}, \quad X_{k,j}^{n+1} = \sum_{k \in M} \tilde{X}_{k,j}^{n+1,i} e^{ik\tau_n}, \quad V_{k,j}^{n+1} = \sum_{k \in M} \tilde{V}_{k,j}^{n+1,i} e^{ik\tau_n},
\]

where

\[
\begin{align*}
\tilde{X}_{n,i} &= e^{c_{hM}n} \hat{X}_n - \varepsilon h \sum_{j=1}^{s} \hat{a}_{ij}(hM) \mathbf{\Lambda}^{-1} \mathcal{F} \left( \mathbf{S} \left( \mathbf{C} \mathcal{F}^{-1} \hat{X}_{nj} + \mathbf{S} \mathcal{F}^{-1} \hat{V}_{nj} \right) \right), \quad i = 1, 2, \ldots, s, \\
\tilde{V}_{n,i} &= e^{c_{hM}n} \hat{V}_n + \varepsilon h \sum_{j=1}^{s} \hat{b}_{ij}(hM) \mathbf{\Lambda}^{-1} \mathcal{F} \left( \mathbf{C} \mathcal{F}^{-1} \hat{X}_{nj} + \mathbf{S} \mathcal{F}^{-1} \hat{V}_{nj} \right), \quad i = 1, 2, \ldots, s, \\
\tilde{X}_{n+1,i} &= e^{hM} \hat{X}_n - \varepsilon h \sum_{j=1}^{s} \hat{b}_{ij}(hM) \mathbf{\Lambda}^{-1} \mathcal{F} \left( \mathbf{S} \left( \mathbf{C} \mathcal{F}^{-1} \hat{X}_{nj} + \mathbf{S} \mathcal{F}^{-1} \hat{V}_{nj} \right) \right), \\
\tilde{V}_{n+1,i} &= e^{hM} \hat{V}_n + \varepsilon h \sum_{j=1}^{s} \hat{b}_{ij}(hM) \mathbf{\Lambda}^{-1} \mathcal{F} \left( \mathbf{C} \mathcal{F}^{-1} \hat{X}_{nj} + \mathbf{S} \mathcal{F}^{-1} \hat{V}_{nj} \right).
\end{align*}
\]

(2.10)
We now obtain the numerical approximation of (2.2), i.e.,
\[ \tilde{q}^n = \cos(t_n A/\varepsilon)x^n + \sin(t_n A/\varepsilon)v^n, \quad \tilde{p}^n = -\sin(t_n A/\varepsilon)x^n + \cos(t_n A/\varepsilon)v^n, \]
where
\[ x_j^n = X^n_{n,j} = \sum_{\ell \in M} \bar{X}_{\ell,j}^n e^{i\ell n h/\varepsilon}, \quad v_j^n = V^n_{n,j} = \sum_{\ell \in M} \bar{V}_{\ell,j}^n e^{i\ell n h/\varepsilon}, \quad j = 1, 2, \ldots, d. \]
Finally, the numerical approximation of the original system (1.1) is given by
\[ q^n = P \tilde{q}^n \quad \text{and} \quad p^n = P \frac{\Lambda}{\varepsilon} \tilde{p}^n. \tag{2.11} \]

Remark 2.1. In the process described above, the matrix $M$ is required to be positive definite such that $\Lambda$ is invertible. Under this condition, the solution of (2.2) is periodic in $\tau$ on the torus $T = (0, 2\pi]$. If this requirement does not hold, we can formulate the methods as follows. Without loss of generality, assume that the first diagonal element of $\Lambda$ is zero and the others are not. Then the system (2.1) can be partitioned into $\bar{y}_1(t) = f_1(\bar{y}(t))$ and $\bar{y}(t) + \frac{1}{\varepsilon} A^2 \bar{y}(t) = f(\bar{y}(t))$ with $\bar{y} = [y_2; y_3; \ldots; y_d]$ and $\bar{y} = [y_1; \bar{y}]$. Then standard numerical discretizations can be used on $y_1$ and the integrators proposed in this section are applicable to $\bar{y}$.

2.2. Some practical integrators. The above procedure fails to be practical unless the coefficients $c_i$, $a_{ij}(hM)$ and $b_i(hM)$ appearing in (2.10) are determined. To this end, the symmetry and stiff order conditions of (2.10) are needed.

Proposition 2.2. (Symmetry) The $s$-stage implicit exponential integrator (2.10) is symmetric if and only if its coefficients satisfy
\[ c_i = 1 - c_{s+1-i}, \quad \bar{b}_i(hM) = e^{hM} \bar{b}_{s+1-i}(-hM), \quad i = 1, 2, \ldots, s, \]
\[ a_{ij}(hM) = e^{c_i hM} \bar{a}_{s+1-j}(-hM) - \bar{a}_{s+1-i}(-hM), \quad i, j = 1, 2, \ldots, s. \tag{2.12} \]

Proof. Under the conditions (2.12), it is trivial to verify that the method (2.10) remains the same after exchanging $n + 1 \leftrightarrow n$ and $h \leftrightarrow -h$. This completes the proof immediately. \hfill \qed

Proposition 2.3. (Stiff order conditions) Assume that the solution $q(t)$, $\varepsilon q(t)$ of (1.1) stays in a bounded open set of $K$ for $t \in [0, T]$. Let $F(q)$ be smooth and uniformly bounded on the closure of $K$ for all $\varepsilon$. Define
\[ \psi_j(z) = \varphi_j(z) - \sum_{k=1}^s \bar{b}_k(z) \frac{c_k^{j-1}}{(j-1)!} \] for $i = 1, 2, \ldots, s$,
where the notations $\varphi_k$ are defined by $\varphi_k(z) = \int_0^{1} \theta^{k-1}(1-\theta)^{r-1} d\theta$ for $k = 1, 2, \ldots$ \tag{30}.
The order conditions of Table 1 are assumed to be true up to order $r - 1$ for $2 \leq r \leq 4$. For the conditions of order $r$, we assume that the local error bounds satisfy the following inequalities
\[ \|
\bar{X}^n - X(t^n + c_i h)
\|_L^2 \leq C \varepsilon^2 h^r, \quad \|
\bar{V}^n - V(t^n + c_i h)
\|_L^2 \leq C \varepsilon^2 h^r, \quad i = 1, 2, \ldots, s, \]
\[ \|
\bar{X}^{n+1} - X(t^{n+1})
\|_L^2 \leq C \varepsilon^2 h^{r+1}, \quad \|
\bar{V}^{n+1} - V(t^{n+1})
\|_L^2 \leq C \varepsilon^2 h^{r+1}, \quad 0 \leq n \leq T/h, \]
where $C > 0$ is a constant independent of $\varepsilon$ and $h$.

Proof. The proof will be given in Section 4 combined with the analysis of convergence. \hfill \qed

The practical integrators presented below will be based on these symmetric conditions and stiff order conditions.

Second-order integrator. We first consider two-stage integrators, i.e., $s = 2$. Solving the order conditions $\psi_1(hM) = \psi_2(hM) = 0$ leads to
\[ \bar{b}_1(hM) = -c_2 \varphi_1(hM) + \varphi_2(hM) \quad \text{and} \quad \bar{b}_2(hM) = \frac{c_1 \varphi_1(hM) - \varphi_2(hM)}{c_1 - c_2}. \]
Then using some other order conditions \( \psi_{1,1}(hM) = \psi_{1,2}(hM) = \psi_{2,3}(hM) = 0 \) and a symmetric condition \( \bar{a}_{12}(hM) + \bar{a}_{21}(-hM) = \varphi_0(c_1 hM) \bar{b}_1(-hM) \), we get the results of \( \bar{a}_{ij} \) as

\[
\bar{a}_{21}(hM) = \frac{c_1^2 (-\varphi_1(c_2 hM) + \varphi_2(c_2 hM))}{c_1 - c_2}, \quad \bar{a}_{12}(hM) = -\bar{a}_{21}(-hM) + \varphi_0(c_1 hM) \bar{b}_1(-hM), \\
\bar{a}_{11}(hM) = -\bar{a}_{12}(hM) + c_1 \varphi_1(c_1 hM), \quad \bar{a}_{22}(hM) = -\bar{a}_{21}(hM) + c_2 \varphi_1(c_2 hM).
\]

On noticing \( c_1 = 1 - c_2 \), it can be verified that this class of integrators is symmetric and is of at least order two. As an example, we choose \( c_1 = 0 \) and denote the corresponding method accompanied by second order initial date as \( S2O2 \).

**Fourth-order integrator.** We now continue with three-stage integrators and obtain their coefficients by solving \( \psi_1(hM) = 0 \) and \( \psi_{3,i}(hM) = 0 \) for \( i, j = 1, 2, 3 \). The choice of

\[
c_1 = 1, \quad c_2 = 1/2, \quad c_3 = 0
\]

and the corresponding results

\[
\bar{a}_{31}(hM) = \bar{a}_{32}(hM) = \bar{a}_{33}(hM) = 0, \quad \bar{a}_{21}(hM) = -\frac{1}{4} \varphi_2(c_2 hM) + \frac{1}{2} \varphi_3(c_2 hM), \\
\bar{a}_{22}(hM) = \varphi_2(c_2 hM) - \varphi_3(c_2 hM), \quad \bar{a}_{23}(hM) = \frac{1}{2} \varphi_1(c_2 hM) - \frac{3}{4} \varphi_2(c_2 hM) + \frac{1}{2} \varphi_3(hM), \\
\bar{a}_{11}(hM) = \bar{b}_1(hM) = 4 \varphi_3(hM) - \varphi_2(hM), \quad \bar{a}_{12}(hM) = \bar{b}_2(hM) = 4 \varphi_2(hM) - 8 \varphi_3(hM), \\
\bar{a}_{13}(hM) = \bar{b}_3(hM) = \varphi_1(hM) - 3 \varphi_2(hM) + 4 \varphi_3(hM),
\]

determine this integrator. It is noted that this method satisfies all the stiff order conditions of order four and symmetric conditions. This integrator as well as fourth order initial date is referred as \( S3O4 \).

We end this section by noting that, with the same arguments stated above, higher-order integrators can be derived. This process can be succinctly presented by B-series and rooted trees. This is an issue which we will focus on in the future work.

### 3. Main results and numerical tests

In this section, we shall present the main results of this paper. The first one is about improved uniformly high accuracy and the second is long time energy conservation. To support these two results, two numerical experiments with numerical results are carried out in the second part of this section.
3.1. Main results.

Theorem 3.1. (Improved uniformly high accuracy) Under the conditions of Proposition 2.3 for the final numerical solutions $q^n$, $p^n$ produced by S2O2 or S3O4, the global errors are

$$S2O2: \|q^n - q(t_n)\|_{L^2} + \varepsilon \|p^n - p(t_n)\|_{L^2} \leq C(\varepsilon^2 h^2 + \delta x), \quad 0 \leq n \leq T/h,$$

$$S3O4: \|q^n - q(t_n)\|_{L^2} + \varepsilon \|p^n - p(t_n)\|_{L^2} \leq C(\varepsilon^2 h^4 + \delta x), \quad 0 \leq n \leq T/h,$$

where $C$ is independent of $n, h, \varepsilon$, and $\delta x$ denotes the projection error brought by the Fourier pseudospectral method.

The next theorem requires a non-resonance condition and to describe it, we introduce the notations

$$k = (k_{-\frac{N}{2}+1}, \ldots, k_{-\frac{N}{2}}, k_{-\frac{N}{2}+1}, \ldots, k_{\frac{N}{2}}, \ldots, k_{\frac{N}{2}}, \ldots),$$

$$|k| = \sum_{j=1}^d N \omega_{k,l} |k_j|, \quad \omega = (\text{diagonal elements of } \Omega), \quad k \cdot \omega = \sum_{j=1}^d \sum_{l=1}^d \omega_{k,l} k_j \omega_{j,l}.$$

Denote the resonance module by $\mathcal{M} = \{k \in \mathbb{Z}^d : k \cdot \omega = 0\}$. Let $(j)_l$ be the unit coordinate vector $(0, \ldots, 0, 1, 0, \ldots, 0)^T \in \mathbb{R}^d$ with the only entry 1 at the $(j, l)$-th position. Further let $\mathcal{K}$ be a set of representatives of the equivalence classes in $\mathbb{Z}^d / \mathcal{M}$. The set $\mathcal{K}$ is determined by two requirements. The first is that if $k \in \mathcal{K}$, we have $-k \in \mathcal{K}$. The other is to minimize the sum $|k|$ in the equivalence class $[k] = k + \mathcal{M}$ for each $k \in \mathcal{K}$. Meanwhile, denote

$$\mathcal{N} = \{k \in \mathcal{K} : \text{there exists an } l \in \{1, \ldots, d\} \text{ such that } |k_{j,l}| = |k| \text{ and } |k| \leq N\},$$

$$\mathcal{N}^* = \mathcal{N} \cup \{(0)_l l = 1, 2, \ldots, d\}$$

for a positive integer $N$.

Theorem 3.2. (Long time energy conservation) For the time stepsize $h$, we assume a lower bound $h/\sqrt{\varepsilon} = c_0 > 0$. It is further required that the numerical non-resonance condition $|\sin(\frac{h}{\sqrt{\varepsilon}} \omega_{j,1} - (k \cdot \omega))| \geq c_1 \sqrt{\varepsilon}$ holds for $c_1 > 0, j = -\frac{N}{2}, \ldots, \frac{N}{2}, k \in \mathcal{N} \neq \{j\}_l$ with $l = 1, 2, \ldots, d$ and some arbitrarily large $N \geq 3$. The bound of $\psi_1, \psi_2$ appeared in the initial value is denoted by $0 < \delta_0 := ||(\psi_1; \psi_2)|| < 1$. The non-linearity $F(q)$ is assumed to be smooth and satisfy $F(0) = 0$. Then the long time energy conservation of $q^n, p^n$ produced by S2O2 or S3O4 is estimated by

$$S2O2: \frac{\varepsilon^2}{\delta_0}(H(q^n, p^n) - H(q^0, p^0)) = \mathcal{O}(\varepsilon^5 \delta_0^2) + \mathcal{O}(\delta x), \quad 0 \leq nh \leq \frac{\varepsilon^3 \delta_0^{N+3}}{h},$$

$$S3O4: \frac{\varepsilon^2}{\delta_0}(H(q^n, p^n) - H(q^0, p^0)) = \mathcal{O}(\varepsilon^6 \delta_0^2) + \mathcal{O}(\delta x), \quad 0 \leq nh \leq \frac{\varepsilon^4 \delta_0^{N+3}}{h},$$

where the constants symbolized by $\mathcal{O}$ depend on $N, c_0, c_1$, but are independent of $n, h, \varepsilon$. Since $N$ can be arbitrarily large, the conservation law holds for a long time.

Remark 3.3. Although the smallness of the parameter $\delta_0$ is technically required, we should note here that $\delta_0$ is totally independent of $\varepsilon$. This means that the initial value of $\{1\}$ is large, which has not been considered yet in the long term analysis of any methods. In all the previous work on this topic, small initial data is required (see, e.g. 15, 16, 17, 23, 25, 27, 38, 40). Moreover, we only need the lower bound on the time stepsize $h \geq c_0/\sqrt{\varepsilon}$, which means that large stepsize can be used to keep the long time energy conservation. Moreover, the above two main results are easily extended to the integrators applied to the system (11) with small initial value $\delta_0 = \mathcal{O}(\varepsilon)$.

3.2. Numerical tests. Two numerical experiments are presented in this section to test the improved uniformly high accuracy and the long time energy conservation of the obtained integrators.

Problem 1. (Duffing equation) As the first numerical example, we consider the Duffing equation defined by

$$\ddot{q} + \frac{1}{\varepsilon} \dot{q} = 2k^2 q^3 - k^2 q, \quad q(0) = \psi_1, \quad \dot{q}(0) = \frac{\psi_2}{\varepsilon}$$

It is a Hamiltonian system with the Hamiltonian $H(q, \dot{q}) = \frac{1}{2} \dot{q}^2 + \frac{1}{\varepsilon^2} q^2 + \frac{k^2}{2}(q^2 - q^4)$. 
For comparison, we choose another two methods. The first one is the trigonometric integrator given in [25] and here we call it as improved Störmer-Verlet method (ISV). This method is directly used to solve (1.1) without taking the process given in Section 2. The second one is obtained by replacing S2-O2 with the following one-stage exponential integrator

\[ X_{n+1} = e^{hM/2}X_n - \varepsilon h/2A^{-1}F \left( SF(CF^{-1}X_n + SF^{-1}V_n) \right), \]
\[ V_{n+1} = e^{hM/2}V_n + \varepsilon h/2A^{-1}F \left( CF(CF^{-1}X_n + SF^{-1}V_n) \right), \]
\[ X_{n+1} = e^{hM}X_n - \varepsilon h\varphi_1(hM)A^{-1}F \left( SF(CF^{-1}X_n + SF^{-1}V_n) \right), \]
\[ V_{n+1} = e^{hM}V_n + \varepsilon h\varphi_1(hM)A^{-1}F \left( CF(CF^{-1}X_n + SF^{-1}V_n) \right). \]

This method is non-symmetric and do not satisfy the second-order stiff conditions. We shall denote it by NSM. For implicit methods, we use standard fixed point iteration as nonlinear solver in the practical computations. We set 10^{-16} as the error tolerance and 200 as the maximum number of each iteration.

**Accuracy.** In this test, \( N_r \) is chosen as 32 to discretize the \( \tau \) direction. The initial value is chosen as \( \psi_1 = 0.1 \) and \( \psi_2 = 0.2 \), and this is the large initial value case as \( \varepsilon \) decreases. Firstly the accuracy of all the methods is shown by displaying the global errors \( err = |q^n - q(t_n)| / |q(t_n)| + |\dot{q}^n - \dot{q}(t_n)| / |\dot{q}(t_n)| \) at \( t_n = 1 \) under different \( \varepsilon \), where \( h = 1/2^k \) with \( k = 1, 2, \ldots, 6 \).

\[ \text{Figure 1.} \text{ Duffing equation: the log-log plot of the temporal error } err = |q^n - q(t_n)| / |q(t_n)| + |\dot{q}^n - \dot{q}(t_n)| / |\dot{q}(t_n)| \text{ at } t_n = 1 \text{ under different } \varepsilon, \text{ where } h = 1/2^k \text{ with } k = 1, 2, \ldots, 6. \]
shown in Figure 2 for different $\varepsilon$ and large $h$. According to these numerical results, the following observations are made.

a) The energy $H$ is nearly preserved numerically by our integrators and ISV with large time stepsize $h$ over long times. S3O4 has an improved conservation accuracy than S2O2, and with large time stepsize $h$, the numerical error in the energy can be improved when $\varepsilon$ decreases. There observations agree with the results given in Theorem 3.2.

b) In contrast, NSM shows substantial drift in the energy quantity and thus it does not have long-term performance in the energy conservation. The reason is that it is not a symmetric method. This observation demonstrates that symmetry plays an important role in the numerical behaviour of energy conservation.

c) Both S2O2 and S3O4 are more accurate in the energy conservation than ISV. The reason is that ISV has $O(h)$ accuracy in the long time behaviour.


Efficiency. Compared with ISV applied to \((1.1)\) directly, the scheme of our integrators given in this paper is more complicated. For example, the two-scale method enlarges the dimension of the original system and this usually adds some computation cost. Fortunately, Fast Fourier Transform (FFT) techniques can be used in the integrators and we hope that the efficiency of our integrators is still acceptable even compared with methods applied to \((1.1)\) directly. To show this point, we solve this problem on the time interval \([0, 10]\) against CPU time. The efficiency of each integrator (measured by the log-log plot of the temporal error at \(t = 10\) against CPU time) is displayed in Figure 3. Clearly, our integrators are not expensive in terms of efficiency.

Problem 2. (Fermi-Pasta-Ulam model) The second illustrative numerical test is devoted to the Fermi-Pasta-Ulam problem, which can be written as a Hamiltonian system with the Hamiltonian

\[
H(q, p) = \frac{1}{2} \sum_{i=1}^{2m} (p_i)^2 + \frac{1}{\varepsilon^2} \sum_{i=1}^{2m} (q_i)^2 + \frac{1}{4} \left( |q_i - q_{m+i}|^4 + \sum_{i=1}^{m-1} (q_{i+1} - q_{m+i+1} - q_i - q_{m+i})^4 + (q_m + q_{2m})^4 \right),
\]

where \(q = [q_1, q_2, \ldots, q_{2m}]^T\) and \(p = [p_1, p_2, \ldots, p_{2m}]^T\). For large initial value case, we choose \(q(0) = [0.1; 0.2; 0.1; 0.4; 0.1; 0]\) and \(p(0) = [0.2; 0.2; 0.6; 0.2; 1.0; 0.2]/\varepsilon\). We apply the methods to solve this system with \(N_e = 32\) and \(m = 3\). The results of global errors \(err = |q^n - q(t_n)| / |q(t_n)| + |p^n - p(t_n)| / |p(t_n)|\), energy errors \(err_H = |H(q^n, p^n) - H(q^0, p^0)| / |H(q^0, p^0)|\) and efficiency are presented in Figures 4. All the numerical phenomena are similar to the first problem.

4. PROOF OF IMPROVED UNIFORMLY HIGH ACCURACY (THEOREM 3.1)

The convergence of the s-stage integrator of order \(r\) is studied in this section. To prove the improved uniformly accuracy, we will show that the transformed system formulated in Section 2 has an important and surprising property. Based on which and the stiff conditions of exponential integrators, local errors of integrators for solving transformed system can be improved to be

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\(^3\)This test is conducted in a sequential program in MATLAB R2020b on a laptop ThinkPad X1 nano (CPU: i7-1160G7 @ 1.20GHz 2.11 GHz, Memory: 16 GB, Os: Microsoft Windows 10 with 64bit).
Figure 5. Fermi-Pasta-Ulam model: energy errors $\text{err}_H = \frac{|H(q^n,p^n) - H(q^0,p^0)|}{|H(q^0,p^0)|}$ against $t$.

Figure 6. Fermi-Pasta-Ulam model: the log-log plot of the temporal error $\text{err} = |q^n - q(t_n)| / |q(t_n)| + |p^n - p(t_n)| / |p(t_n)|$ at $t_n = 10$ against CPU time, where $h = 1/2^k$ with $k = 1, 2, \ldots, 7$.

$O(\varepsilon^2 h^{r+1})$. Then the convergence of integrators will be derived for the transformed system and further for the original problem by taking the transformations of Section 2. Since the parameter $\varepsilon$ is very small in highly oscillatory problems, this improved uniformly accuracy makes the integrators very effective and competitive in solving highly oscillatory systems.

Proof. Step 1. We first study the transformed system (2.5) formulated in Section 2. Based on the Chapman-Enskog expansion, its solution can be formulated as

$$U(t, \tau) = \bar{U}(t) + \kappa(t, \tau) \quad \text{with} \quad \bar{U}(t) = \Pi U(t, \tau), \quad \Pi \kappa(t, \tau) = 0.$$ 

These two composers satisfy the differential equations

$$\dot{U}(t) = \Pi f_\tau(U(t, \tau)), \quad \partial_t \kappa(t, \tau) + \frac{1}{\varepsilon} \partial_{\tau} \kappa(t, \tau) = (I - \Pi) f_\tau(U(t, \tau)).$$
This yields that
\[ \partial_t U(t, \tau) + \frac{1}{\varepsilon} \partial_\tau U(t, \tau) = f_\varepsilon(U(t, \tau)). \]

This equation as well as the initial value (2.7) with \( k = 0 \) determines a unique solution satisfying \( U(t, \tau) = O(1) \) (see Proposition 2.3 of [10]). Then for the first derivative \( \partial_t U(t, \tau) \), it satisfies the equation
\[ \partial_t (\partial_t U(t, \tau)) + \frac{1}{\varepsilon} \partial_\tau (\partial_t U(t, \tau)) = \partial_t f_\varepsilon(U(t, \tau))(\partial_t U(t, \tau)). \]

From the initial value (2.7) with \( k = 1 \), it follows that the initial date of this system is
\[ \partial_t U(0, \tau) = \Pi f_0 + \varepsilon \Pi \partial_\tau f_0 \partial_\tau^{-1}(I - \Pi) f_0 + \varepsilon \partial_\tau^{-1}(I - \Pi) \partial_\tau f_0 \Pi f_0 + O(\varepsilon^2) = O(\varepsilon). \]

This result and the fact that \( \partial_t f_\varepsilon(U(t, \tau))(\partial_t U(t, \tau)) = O(\varepsilon) \) yield that
\[ \partial_t U(t, \tau) = O(\varepsilon). \]

For the second derivative \( \partial_t^2 U(t, \tau) \), the equation reads
\[ \partial_t (\partial_t^2 U(t, \tau)) + \frac{1}{\varepsilon} \partial_\tau (\partial_t^2 U(t, \tau)) = \partial_t^2 f_\varepsilon(U(t, \tau))(\partial_t U(t, \tau)) + \partial_t f_\varepsilon(U(t, \tau))(\partial_t^2 U(t, \tau)), \]
and the corresponding initial data obtained from (2.7) with \( k = 2 \) is
\[ \partial_t^2 U(0, \tau) = \Pi \partial_\tau f_0 \Pi f_0 + O(\varepsilon) = O(\varepsilon). \]

Therefore, one gets that
\[ \partial_t^2 U(t, \tau) = O(\varepsilon). \]

This procedure can be proceeded in an analogous way for \( \partial_t^3 U(t, \tau) \) and \( \partial_t^4 U(t, \tau) \). We summarize this important property as follows. For the two-scale system (2.5) with the constructed \( j \) (\( j \leq 4 \))-th order initial data (2.7), its solution as well as the derivatives with respect to \( t \) is uniformly bounded as
\[ U(t, \tau) = O(1), \quad \partial_t^k U(t, \tau) = O(\varepsilon), \quad k = 1, 2, 3, 4, \quad 0 \leq t \leq t_0 \quad \text{for some} \quad t_0 > 0. \quad (4.1) \]

**Step 2.** Then we prove Proposition 2.3 where the local errors of our integrators for the transformed system are stated.

For all \( h \geq 0 \), it is true that \( \| \varphi_0(hM) \|_{L^2} = 1 \), and \( \| \varphi_j(hM) \|_{L^2} \) for \( j = 1, 2, \ldots, \) are uniformly bounded. Therefore, the coefficients of exponential integrators are bounded as
\[ \| \tilde{a}_{ij}(hM) \|_{L^2} \leq C, \quad \| \tilde{b}_j(hM) \|_{L^2} \leq C \quad \text{for} \quad i, j = 1, 2, \ldots, s, \]
where the constant \( C \) is independent of \( h, \varepsilon \).

Define the error functions by
\[ e^X_n(\tau) := X(t_n, \tau) - I_M X^n, \quad e^n_{X}(\tau) := X(t_n + c_i h, \tau) - I_M X^{ni}, \]
\[ e^V_n(\tau) := V(t_n, \tau) - I_M V^n, \quad e^n_{V}(\tau) := V(t_n + c_i h, \tau) - I_M V^{ni}, \]
and the projected errors as
\[ e^X_{M,X}(\tau) := P_M X(t_n, \tau) - X^n_M, \quad e^n_{X,M}(\tau) := P_M X(t_n + c_i h, \tau) - X^{ni}_M, \]
\[ e^V_{M,V}(\tau) := P_M V(t_n, \tau) - V^n_M, \quad e^n_{V,M}(\tau) := P_M V(t_n + c_i h, \tau) - V^{ni}_M. \]

By the triangle inequality and estimates on projection error [35], one has
\[ \| e^X_{M,X} \|_{L^2} \leq \| e^X_n(\tau) \|_{L^2} + \| X^n_M - I_M X^n \|_{L^2} + \| X(t_n, \tau) - P_M X(t_n, \tau) \|_{L^2} \]
\[ \lesssim \| e^X_n(\tau) \|_{L^2} + \delta_f, \]
\[ \| E^n_{X,M} \|_{L^2} \leq \| e^n_{X,M}(\tau) \|_{L^2} + \| X(t_n + c_i h, \tau) - P_M X(t_n + c_i h, \tau) \|_{L^2} \]
\[ \lesssim \| e^n_{X,M}(\tau) \|_{L^2} + \delta_f, \]
and similar results for \( e^V_{M,V}, \quad e^n_{V,M} \). Therefore, the estimations for \( e^X_n, \quad e^n_V \) and \( E^n_{X,M}, \quad E^n_{V,M} \) could be converted to the estimations for \( e^X_{M,X}, \quad e^n_{M,V} \) and \( E^n_{M,X}, \quad E^n_{M,V} \).
Since the scheme (2.3) is implicit, iterative solutions are needed, and we consider the following iterative pattern

\[
(\widetilde{X}^{ni})^{[0]} = e^{c_{i}hM}X^{n} - \varepsilon h \sum_{j=1}^{s} \bar{a}_{ij}(hM)\Lambda^{-1} F\left( SF(CF^{-1}X^{n} + SF^{-1}V^{n}) \right),
\]

\[
(\widetilde{V}^{ni})^{[0]} = e^{c_{i}hM}V^{n} + \varepsilon h \sum_{j=1}^{s} \bar{a}_{ij}(hM)\Lambda^{-1} G\left( CF(CF^{-1}X^{n} + SF^{-1}V^{n}) \right),
\]

\[
(\widetilde{X}^{ni})^{[j+1]} = e^{c_{i}hM}X^{n} - \varepsilon h \sum_{j=1}^{s} \bar{a}_{ij}(hM)\Lambda^{-1} F\left( SF(CF^{-1}(\widetilde{X}^{ni})^{[j]} + SF^{-1}(\widetilde{V}^{ni})^{[j]}) \right), \quad j = 0, 1, \ldots,
\]

\[
(\widetilde{V}^{ni})^{[j+1]} = e^{c_{i}hM}V^{n} + \varepsilon h \sum_{j=1}^{s} \bar{a}_{ij}(hM)\Lambda^{-1} G\left( CF(CF^{-1}(\widetilde{X}^{ni})^{[j]} + SF^{-1}(\widetilde{V}^{ni})^{[j]}) \right), \quad j = 0, 1, \ldots,
\]

in this paper. Then according to the property of \(e^{c_{i}hM}\) and the boundedness of the coefficients, the following result can be proved. That is, there exists a small constant \(0 < \varepsilon < 1\) such that when \(0 < h \leq \varepsilon\) and \(\widetilde{X}^{0} \in L^{2}\) with \(\|\widetilde{X}^{0}\|_{L^{2}} \leq K_{1}\), we can obtain \(\widetilde{X}^{n} \in L^{2}\), \((\widetilde{X}^{ni})^{[\text{stopped num.}]} \in L^{2}\) as well as their bounds \(\|\widetilde{X}^{n}\|_{L^{2}} \leq C_{0}, \|\widetilde{X}^{ni}\|_{[\text{stopped num.}]} \|_{L^{2}} \leq C_{0}\), where the constant \(C_{0}\) depends on \(K_{1}\). The same bounds can be derived for \(\widetilde{V}^{n} \in L^{2}\), \((\widetilde{V}^{ni})^{[\text{stopped num.}]} \in L^{2}\).

The error system of FS-F method is to find \(e_{M,X}^{n}(\tau), e_{M,V}^{n}(\tau)\) and \(E_{M,X}^{n}(\tau), E_{M,V}^{n}(\tau)\) in the space \(X_{M}\), i.e., we have

\[
e_{M,X}^{n+1}(\tau) = \sum_{k \in M} (\widetilde{e}_{M,X}^{n+1})_{k} e^{ik\tau}, \quad E_{M,X}^{n}(\tau) = \sum_{k \in M} (\widetilde{E}_{M,X}^{n})_{k} e^{ik\tau},
\]

\[
e_{M,V}^{n+1}(\tau) = \sum_{k \in M} (\widetilde{e}_{M,V}^{n+1})_{k} e^{ik\tau}, \quad E_{M,V}^{n}(\tau) = \sum_{k \in M} (\widetilde{E}_{M,V}^{n})_{k} e^{ik\tau},
\]

where

\[
\widetilde{e}_{M,X}^{n+1} = e^{hM}\widetilde{e}_{M,X}^{n} + h\varepsilon \sum_{j=1}^{s} \bar{b}_{j}(hM)\Delta f_{X}^{n+j} + \varepsilon_{X}^{n}, \quad \widetilde{E}_{M,X}^{n} = e^{c_{i}hM}\widetilde{E}_{M,X}^{n} + h\varepsilon \sum_{j=1}^{s} \bar{a}_{ij}(hM)\Lambda^{-1} f_{X}^{n+j} + \Delta_{X}^{n},
\]

\[
\widetilde{e}_{M,V}^{n+1} = e^{hM}\widetilde{e}_{M,V}^{n} + h\varepsilon \sum_{j=1}^{s} \bar{b}_{j}(hM)\Delta f_{V}^{n+j} + \varepsilon_{V}^{n}, \quad \widetilde{E}_{M,V}^{n} = e^{c_{i}hM}\widetilde{E}_{M,V}^{n} + h\varepsilon \sum_{j=1}^{s} \bar{a}_{ij}(hM)\Lambda^{-1} f_{V}^{n+j} + \Delta_{V}^{n},
\]

and

\[
\Delta f_{X}^{n} = -\Lambda^{-1}FS(G(P_{M}X(t_{n} + c_{j}h, \tau), P_{M}V(t_{n} + c_{j}h, \tau)) - G(X_{M}^{n+j}, V_{M}^{n+j})),
\]

\[
\Delta f_{V}^{n} = -\Lambda^{-1}FC(G(P_{M}X(t_{n} + c_{j}h, \tau), P_{M}V(t_{n} + c_{j}h, \tau)) - G(X_{M}^{n+j}, V_{M}^{n+j})),
\]

and \(G(\widetilde{X}, \widetilde{V}) = F(CF^{-1}\widetilde{X} + SF^{-1}\widetilde{V})\). Here the remainders \(\varepsilon_{X}^{n}, \Delta_{X}^{n}\) and \(\varepsilon_{V}^{n}, \Delta_{V}^{n}\) can be bounded by inserting the exact solution of (2.3) into the numerical approximation, i.e.,

\[
X(t_{n} + c_{j}h) = e^{c_{i}hM}\widetilde{X}(t_{n}) - \varepsilon h \sum_{j=1}^{s} \bar{a}_{ij}(hM)\Lambda^{-1} FSG(X(t_{n} + c_{j}h), V(t_{n} + c_{j}h)) + \Delta_{X}^{n},
\]

\[
V(t_{n} + c_{j}h) = e^{c_{i}hM}\widetilde{V}(t_{n}) + \varepsilon h \sum_{j=1}^{s} \bar{a}_{ij}(hM)\Lambda^{-1} CG(X(t_{n} + c_{j}h), V(t_{n} + c_{j}h)) + \Delta_{V}^{n},
\]

\[
X(t_{n} + h) = e^{hM}\widetilde{X}(t_{n}) - \varepsilon h \sum_{j=1}^{s} \bar{b}_{j}(hM)\Lambda^{-1} FSG(X(t_{n} + c_{j}h), V(t_{n} + c_{j}h)) + \varepsilon_{X}^{n},
\]

\[
V(t_{n} + h) = e^{hM}\widetilde{V}(t_{n}) + \varepsilon h \sum_{j=1}^{s} \bar{b}_{j}(hM)\Lambda^{-1} CG(X(t_{n} + c_{j}h), V(t_{n} + c_{j}h)) + \varepsilon_{V}^{n}.
\]
By the Duhamel principle and Taylor expansions, the remainders \( \delta^{n+1}_X \) and \( \delta^{n+1}_V \) can be represented as

\[
[\delta^{n+1}_X; \delta^{n+1}_V] = h\varepsilon \int_0^1 \left( e^{(1-\xi)c_i M} \otimes \text{diag}(1, 1) \right) \sum_{j=0}^{\infty} \frac{(\xi c_i h_j)^j}{j!} \frac{d^j}{d\xi^j} G(t_n) d\xi
\]

\[
- h\varepsilon \sum_{k=1}^s \left( \hat{b}_k (hM) \otimes \text{diag}(1, 1) \right) \sum_{j=0}^{\infty} \frac{c_k h_j^j}{j!} \frac{d^j}{d\xi^j} G(t_n)
\]

\[
= h\varepsilon \sum_{j=0}^s h^j \left( \psi_j (hM) \otimes \text{diag}(1, 1) \right) \frac{d^j}{d\xi^j} G(t_n),
\]

where we take the notation \( G(t) = [\Lambda^{-1} FSG(\tilde{X}(t), \tilde{V}(t)); \Lambda^{-1} FCG(\tilde{X}(t), \tilde{V}(t))] \) and the Kronecker product \( \otimes \). Using the stiff order condition presented in Proposition 2.4.3, the bound 4.4.1 and the bootstrap type argument of 29 modified for implicit exponential integrators, we deduce that these remainders are bounded by \( O(\varepsilon^2 h^{r+1}) \). Similarly, it is obtained that \( \tilde{\delta}^n_X \) and \( \tilde{\delta}^n_V \) are bounded by \( O(\varepsilon^2 h^r) \) for any \( i = 1, 2, \ldots, s \). The proof of Proposition 2.4.3 is immediately complete.

**Step 3.** We are now in position to derive the error bounds in a standard way. To make the analysis be more compact and concise, here we only present main points but without details.

Using the Parseval’s identity and the formulae of (4.2), it is arrived at

\[
\| c^{n+1}_{MK} \|_{L^2} \leq \| c^{n+1}_{MK} \|_{L^2} + h\varepsilon \sum_{j=1}^s \| \Delta f_{X}^{n_j} \|_{L^2} + C\varepsilon^2 h^{r+1},
\]

\[
\| c^{n+1}_{MK} \|_{L^2} \leq \| c^{n+1}_{MK} \|_{L^2} + h\varepsilon \sum_{j=1}^s \| \Delta f_{V}^{n_j} \|_{L^2} + C\varepsilon^2 h^{r+1}.
\]

The formulation (4.3) and the Lipschitz condition of \( F \) contribute with

\[
\| \Delta f_{X}^{n_j} \|_{L^2} \leq C(\| E_{X, MK}^{n_j} \|_{L^2} + \| E_{V, MK}^{n_j} \|_{L^2}), \quad \| \Delta f_{V}^{n_j} \|_{L^2} \leq C(\| E_{X, MK}^{n_j} \|_{L^2} + \| E_{V, MK}^{n_j} \|_{L^2}),
\]

and based on which, we further have

\[
\| c^{n+1}_{MK} \|_{L^2} \leq \| c^{n+1}_{MK} \|_{L^2} + h\varepsilon \sum_{j=1}^s (\| E_{X, MK}^{n_j} \|_{L^2} + \| E_{V, MK}^{n_j} \|_{L^2}) + C\varepsilon^2 h^{r+1},
\]

\[
\| c^{n+1}_{MK} \|_{L^2} \leq \| c^{n+1}_{MK} \|_{L^2} + h\varepsilon \sum_{j=1}^s (\| E_{X, MK}^{n_j} \|_{L^2} + \| E_{V, MK}^{n_j} \|_{L^2}) + C\varepsilon^2 h^{r+1}.
\]

Similar result can be obtained for \( E_{X, MK}^{n_j} \) and \( E_{V, MK}^{n_j} \) as follows:

\[
\| E_{X, MK}^{n_j} \|_{L^2} \leq \| c^{n+1}_{MK} \|_{L^2} + h\varepsilon \sum_{j=1}^s (\| E_{X, MK}^{n_j} \|_{L^2} + \| E_{V, MK}^{n_j} \|_{L^2}) + C\varepsilon^2 h^r,
\]

\[
\| E_{V, MK}^{n_j} \|_{L^2} \leq \| c^{n+1}_{MK} \|_{L^2} + h\varepsilon \sum_{j=1}^s (\| E_{X, MK}^{n_j} \|_{L^2} + \| E_{V, MK}^{n_j} \|_{L^2}) + C\varepsilon^2 h^r.
\]

Combined with the above results, it arrives at

\[
\sum_{j=1}^s (\| E_{X, MK}^{n_j} \|_{L^2} + \| E_{V, MK}^{n_j} \|_{L^2}) \leq 2s \| c^{n+1}_{MK} \|_{L^2} + 2sh\varepsilon \sum_{j=1}^s (\| E_{X, MK}^{n_j} \|_{L^2} + \| E_{V, MK}^{n_j} \|_{L^2}) + C\varepsilon^2 h^r.
\]

If the stepsize \( h \) satisfies \( h\varepsilon \leq \frac{1}{L r} \), we confirm that

\[
\sum_{j=1}^s (\| E_{X, MK}^{n_j} \|_{L^2} + \| E_{V, MK}^{n_j} \|_{L^2}) \leq 4s \| c^{n+1}_{MK} \|_{L^2} + C\varepsilon^2 h^r.
\]

Inserting this into (4.4) and using Gronwall inequality leads to

\[
\| P_M X(t_n, \tau) - X^n_M \|_{L^2} \leq C \varepsilon^2 h^r, \quad \| P_M V(t_n, \tau) - V^n_M \|_{L^2} \leq C \varepsilon^2 h^r.
\]
It is noted that this result will lose a factor of \( \varepsilon \) in the transformations back to the numerical solutions of the original system \((1.1)\), which gives the accuracy \( \mathcal{O}(\varepsilon h^r) \) for the integrators applied to \((1.1)\). The proof of Theorem 3.1 is complete. \( \square \)

5. Proof of Long Time Energy Conservation (Theorem 3.2)

Long time energy conservation is proved mainly based on the technology of modulated Fourier expansions, which was firstly developed in [25] and was used for the long-term analysis of many methods [15, 16, 17, 23, 27, 38, 39, 40]. The main differences and contributions of long term analysis given in this section involve in two aspects. We prove the long-time result for both two-stage and three-stage methods, and show that the three-stage method has better conservation behaviour. Moreover, in contrast to the existing work, we neither assume bounded energy, nor assume small initial value for the considered system. In the proof, the result is derived for the methods applied to the energy unbounded system \((1.1)\) with large initial value.

Proof. Step 1. Modulated Fourier expansions. We first derive the modulated Fourier expansions of the numerical integrators. Use the notations

\[
\Phi(t) = \sum_{k \in \mathbb{N}^*} e^{i(k \cdot \omega)t} \alpha_k(t), \quad \Psi(t) = \sum_{k \in \mathbb{N}^*} e^{i(k \cdot \omega)t} \beta_k(t),
\]

with smooth coefficient functions \( \alpha_k \) and \( \beta_k \) for \( t = nh \). The proof will firstly construct the coefficient functions \( \alpha_k \) and \( \beta_k \) such that when \( \Phi(t), \Psi(t) \) are inserted into the numerical scheme S2O2, there are only small defects \( R_{X,N} \) and \( R_{\tilde{V},N} \).

According to the symmetry and stiff order conditions of the scheme S2O2, it follows that for \( i = 1, 2 \), we have

\[
\tilde{X}^n_i = \Phi(t + c_i h), \quad \tilde{V}^n_i = \Psi(t + c_i h).
\]

Inserting \((5.1)\) into \((2.10)\) and defining the operator

\[
\mathcal{L}(h\mathcal{D}) = (\varepsilon h^2 - e^{hM})(\bar{b}_1(hM)e^{c_1 h\mathcal{D}} + \bar{b}_2(hM)e^{c_2 h\mathcal{D}})^{-1},
\]

one has

\[
\mathcal{L}(h\mathcal{D})\Phi(t) = -\varepsilon h \Lambda^{-1}\mathcal{F}\mathcal{S}\mathcal{F}(\Upsilon(t)), \quad \mathcal{L}(h\mathcal{D})\Psi(t) = \varepsilon h \Lambda^{-1}\mathcal{F}\mathcal{C}\mathcal{F}(\Upsilon(t)).
\]

Here \( \mathcal{D} \) is referred to the differential operator [see 26] and we use the following notation

\[
\Upsilon(t) := \mathcal{C}\mathcal{F}^{-1}\Phi(t) + \mathcal{S}\mathcal{F}^{-1}\Psi(t) = \sum_{k \in \mathbb{N}^*} e^{i(k \cdot \omega)t} \gamma_k(t)
\]

with \( \gamma_k(t) = \mathcal{C}\mathcal{F}^{-1}\alpha_k(t) + \mathcal{S}\mathcal{F}^{-1}\beta_k(t) \). Expanding the nonlinearity into its Taylor series yields

\[
\mathcal{L}(h\mathcal{D})\Phi(t) = -\varepsilon h \Lambda^{-1}\mathcal{F}\sum_{k \in \mathbb{N}^*} e^{i(k \cdot \omega)t} \sum_{m \geq 2} \frac{F^{(m)}(0)}{m!} \sum_{k_1 + \ldots + k_m = k} \left[ \gamma_1^{k_1} \ldots \gamma_m^{k_m} \right](t),
\]

\[
\mathcal{L}(h\mathcal{D})\Psi(t) = \varepsilon h \Lambda^{-1}\mathcal{F}\mathcal{C}\sum_{k \in \mathbb{N}^*} e^{i(k \cdot \omega)t} \sum_{m \geq 2} \frac{F^{(m)}(0)}{m!} \sum_{k_1 + \ldots + k_m = k} \left[ \gamma_1^{k_1} \ldots \gamma_m^{k_m} \right](t).
\]

We remark that the assumption \( F(0) = 0 \) and \( F'(0) = 0 \) are used here since the linearization of \( F(u) \) leads to a new nonlinearity \( \tilde{F}(u) = F(u) - F'(0)u \) with the property \( \tilde{F}'(0) = 0 \). Inserting the ansatz \((5.1)\) into these expressions and comparing the coefficients of \( e^{i(k \cdot \omega)t} \), we obtain

\[
\mathcal{L}(h\mathcal{D} + i(k \cdot \omega)h)\alpha_k(t) = -\varepsilon h \Lambda^{-1}\mathcal{F}\sum_{m \geq 2} \frac{F^{(m)}(0)}{m!} \sum_{k_1 + \ldots + k_m = k} \left[ \gamma_1^{k_1} \ldots \gamma_m^{k_m} \right](t),
\]

\[
\mathcal{L}(h\mathcal{D} + i(k \cdot \omega)h)\beta_k(t) = \varepsilon h \Lambda^{-1}\mathcal{C}\sum_{m \geq 2} \frac{F^{(m)}(0)}{m!} \sum_{k_1 + \ldots + k_m = k} \left[ \gamma_1^{k_1} \ldots \gamma_m^{k_m} \right](t).
\]

This is the modulation system for the coefficients \( \alpha_k(t) \) and \( \beta_k(t) \) of the modulated Fourier expansion.
In the light of the coefficients of S2O2, the Taylor expansions of \( L(hD) \) are given by

\[
L(hD) = -h\Omega + hD - \frac{1}{2}h\Omega^{-1}( -2I + h\Omega \cot(h\Omega/2))iD^2 + \ldots, \tag{5.2}
\]

\[
L(hD + ih(k \cdot \omega)) = -h^2\Omega^2 \left( h\Omega - \cot \zeta + \cos \zeta \csc \zeta \right)^{-1}i + \ldots,
\]

with \( \zeta := \frac{1}{2}h(\Omega - (k \cdot \omega)I) \). Some particular components we need are expressed as

\[
(L(hD + ih(j \cdot \omega)))_{j,l} = \frac{h^2\omega_j^2}{4\sin^2(\omega_{j,l}/2)}hD - \frac{h^2\omega_j^2}{16\sin^2(\omega_{j,l}/2)}(\omega_{j,l} - \sin(\omega_{j,l}))(hD)^2 + \ldots. \tag{5.3}
\]

With these Taylor expansions, the following ansatz of the modulated Fourier functions \( \alpha^k(t) \) is derived:

\[
\alpha^{(j)}_{j,l}(t) = \frac{4\pi \sin^2(\omega_{j,l}/2)}{h\omega_{j,l}} F^1_{j0}(\cdot) + \ldots, \quad j = -\frac{N}{2}, \ldots, \frac{N}{2}, \tag{5.4}
\]

\[
\alpha^k(t) = -h \epsilon \frac{\Omega^2}{h^2}, \quad h\Omega - \cot \zeta + \cos \zeta \csc \zeta \right)^i (F^1_{j0}(\cdot) + \ldots), \quad k \neq (j)_l,
\]

where the dots mean the power series in \( h \) and \( F^k \) and so on stand for formal series. The coefficients \( \beta^k(t) \) have the same ansatz as \( \alpha^k(t) \). We truncate the series after the \( \mathcal{O}(h^{N+N_0}) \) terms for arbitrary positive integer \( N_0 \) since they often diverge. The initial values for the differential equations appeared in the ansatz are determined by \( \Psi(0) = \hat{X}^0 \). We thus get \( \hat{X}^0_{j,l} = \alpha^{(j)}_{j,l}(0) + \mathcal{O}(h\epsilon^2\delta^2_{0}) \), which yields \( \alpha^{(j)}_{j,l}(0) = \mathcal{O}(\delta_0) \).

Under the above analysis and the conditions given in Theorem 3.2, the numerical result \( \hat{X}^n \) and \( \hat{V}^n \) obtained from S2O2 can be expressed by the following modulated Fourier expansions at \( t = nh \)

\[
\hat{X}^n = \sum_{k \in \mathbb{N}^n} e^{i(k \cdot \omega)t} \alpha^k(t) + R_{\hat{X},N}(t), \quad \hat{V}^n = \sum_{k \in \mathbb{N}^n} e^{i(k \cdot \omega)t} \beta^k(t) + R_{\hat{V},N}(t). \tag{5.5}
\]

From the construction of the coefficient functions, it follows that it is reasonable to assume \( \alpha^k_{0,l} = 0 \) if \( k \neq (0)_l \) and \( \alpha^k_{i,m} = 0 \) if \( |k| > 0 \) and \( i \neq m \). The coefficient functions and their derivatives have the following bounds

\[
\alpha^{(j)}_{j,l}(t) = \mathcal{O}(\delta_0), \quad \alpha^{(j)}_{j,l}(t) = \mathcal{O}(\frac{\delta^2}{\sqrt{h}}), \quad \alpha^k(t) = \mathcal{O}(\epsilon^2 \delta^0_{0}), \quad k \neq (j)_l, \tag{5.6}
\]

\[
\beta^{(j)}_{j,l}(t) = \mathcal{O}(\delta_0), \quad \beta^{(j)}_{j,l}(t) = \mathcal{O}(\frac{\delta^2}{\sqrt{h}}), \quad \beta^k(t) = \mathcal{O}(\epsilon^2 \delta^0_{0}), \quad k \neq (j)_l,
\]

where \( l = 1, \ldots, d \), and \( \nu = \frac{1}{c_1\nu^a} \). These bounds of \( \alpha \) are obtained from the ansatz \( \alpha \) with the derived initial values and the bounds of \( \beta \) are deduced similarly. Considering the fact \( \hat{X}^n, \hat{V}^n \in \mathbb{R}^D \) yields that \( \alpha^{-k}_{-l,j} = \alpha_{l,j}^{-k} \) and \( \beta^{-k}_{-l,j} = \beta_{l,j}^{-k} \). The constants symbolised by the notation \( \mathcal{O} \) are independent of \( h, \epsilon, N \), but depend on \( c_0, c_1 \). The remainders are bounded by

\[
R_{\hat{X},N}(t) = \mathcal{O}(th^2 \epsilon^3 \delta^3_{0} + 1), \quad R_{\hat{V},N}(t) = \mathcal{O}(th^2 \epsilon^3 \delta^3_{0} + 1), \tag{5.7}
\]

which are shown as follows.

First insert \( \Phi(t), \Psi(t) \) into the numerical scheme \( \Phi_{10} \) and then the corresponding discrepancies are

\[
d_1(t) = \Phi(t + c_1 h) - e^{c_1 h M} \Phi(t) - h \{ \bar{a}_{11} \Lambda^{-1} FSF(\Upsilon(t + c_1 h)) + \bar{a}_{12} \Lambda^{-1} FSF(\Upsilon(t + c_2 h)) \},
\]

\[
d_2(t) = \Phi(t + c_2 h) - e^{c_2 h M} \Phi(t) - h \{ \bar{a}_{21} \Lambda^{-1} FSF(\Upsilon(t + c_1 h)) + \bar{a}_{22} \Lambda^{-1} FSF(\Upsilon(t + c_2 h)) \},
\]

\[
d_3(t) = \Psi(t + c_1 h) - e^{c_1 h M} \Psi(t) + h \{ \bar{a}_{11} \Lambda^{-1} FCF(\Upsilon(t + c_1 h)) + \bar{a}_{12} \Lambda^{-1} FCF(\Upsilon(t + c_2 h)) \},
\]

\[
d_4(t) = \Psi(t + c_2 h) - e^{c_2 h M} \Psi(t) + h \{ \bar{a}_{21} \Lambda^{-1} FCF(\Upsilon(t + c_1 h)) + \bar{a}_{22} \Lambda^{-1} FCF(\Upsilon(t + c_2 h)) \},
\]

\[
d_5(t) = \Phi(t + h) - e^{h M} \Phi(t) - h \{ \bar{b}_1 \Lambda^{-1} FSF(\Upsilon(t + c_1 h)) + \bar{b}_2 \Lambda^{-1} FSF(\Upsilon(t + c_2 h)) \},
\]

\[
d_6(t) = \Psi(t + h) - e^{h M} \Psi(t) + h \{ \bar{b}_1 \Lambda^{-1} FCF(\Upsilon(t + c_1 h)) + \bar{b}_2 \Lambda^{-1} FCF(\Upsilon(t + c_2 h)) \}.
\]

There are two aspects in the bounds of these discrepancies: \( \mathcal{O}(h\epsilon^2 \delta^3_{0} + 1) \) in the truncation of modulated Fourier expansions and \( \mathcal{O}(h^{N+N_0}) \) in the truncation of the ansatz \( \alpha \). Therefore, discrepancies are bounded by

\[
d_j(t) = \mathcal{O}(h^{N+N_0}) + \mathcal{O}(h^2 \epsilon^3 \delta^3_{0} + 1) = \mathcal{O}(h^2 \epsilon^3 \delta^3_{0} + 1) \text{ for } j = 1, 2, \ldots, 6
\]
on the basis of the arbitrarily large $N_0$. Then define the errors
\[ e_n^\alpha = \hat{X}^n - \Phi(t_n), \quad e_n^\alpha = \hat{X}^{ni} - \Phi(t_n + c_i h), \quad e_n^\gamma = \hat{V}^n - \Psi(t_n), \quad e_n^\gamma = \hat{V}^{ni} - \Psi(t_n + c_i h). \]

They satisfy the error recursion
\[
\begin{align*}
E_n^{\alpha} &= e_n^{\alpha}h^M e_n^{\alpha} - h \varepsilon \sum_{j=1}^{s} a_{ij} (hM) \Lambda^{-1} FS(G(\hat{X}^{ni}), \hat{V}^{ni}) - F(\Psi(t_n + c_i h))) + d_i(t_n), \quad i = 1, 2, \\
E_n^{\gamma} &= e_n^{\gamma}h^M e_n^{\gamma} + h \varepsilon \sum_{j=1}^{s} b_{ij} (hM) \Lambda^{-1} FC(G(\hat{X}^{ni}), \hat{V}^{ni}) - F(\Psi(t_n + c_i h))) + d_i(t_n), \quad i = 1, 2, \\
e_{n+1}^{\alpha} &= e_n^{\alpha}h^M e_n^{\alpha} - h \varepsilon \sum_{j=1}^{s} \hat{b}_{ij} (hM) \Lambda^{-1} FS(G(\hat{X}^{ni}), \hat{V}^{ni}) - F(\Psi(t_n + c_i h))) + d_5(t_n), \\
e_{n+1}^{\gamma} &= e_n^{\gamma}h^M e_n^{\gamma} + h \varepsilon \sum_{j=1}^{s} \hat{b}_{ij} (hM) \Lambda^{-1} FC(G(\hat{X}^{ni}), \hat{V}^{ni}) - F(\Psi(t_n + c_i h))) + d_6(t_n).
\end{align*}
\]

Taking the Lipschitz condition of $F$ into account, we obtain
\[
\begin{align*}
\left\| F(CF^{-1}\hat{X}^{ni} + SF^{-1}\hat{V}^{ni}) - F(\Psi(t_n + c_i h)) \right\| &\leq L \left\| CF^{-1}\hat{X}^{ni} + SF^{-1}\hat{V}^{ni} - \Psi(t_n + c_i h) \right\|, \\
&\leq L \left\| CF^{-1}\hat{X}^{ni} + SF^{-1}\hat{V}^{ni} - CF^{-1}\Phi(t_n + c_i h) \right\| \leq L \left\| E_n^{\alpha} \right\| + L \left\| E_n^{\gamma} \right\|.
\end{align*}
\]

The application of the Gronwall inequality now shows the bounds of the defects \([5, 7]\).

**Step 2. Almost-invariant.** We have derived the modulated Fourier expansions of S2O2 integrator and its long time energy conservation will be studied on the basis of an almost-invariant of the coefficient functions of modulated Fourier expansions.

By collecting the coefficient functions of \([5, 7]\) in the new vectors $\vec{\alpha} = (\alpha^k)_{k \in \mathbb{N}^*}$ and $\vec{\beta} = (\beta^k)_{k \in \mathbb{N}^*}$. The proof of part 1 immediately results in
\[
\begin{align*}
L(hD)X_h(t) &= -h \varepsilon \Lambda^{-1} FSF(\Psi(t)) + O(h^2 \varepsilon^3 \delta_0^N + 1), \\
L(hD)V_h(t) &= h \varepsilon \Lambda^{-1} FCF(\Psi(t)) + O(h^2 \varepsilon^3 \delta_0^N + 1),
\end{align*}
\]
where the following notations are used here and after
\[
X_h(t) = \sum_{k \in \mathbb{N}^*} X_h^k(t), \quad V_h(t) = \sum_{k \in \mathbb{N}^*} V_h^k(t), \quad Y_h(t) = \sum_{k \in \mathbb{N}^*} Y_h^k(t),
\]
with
\[
X_h^k(t) = e^{i(k \omega t)} \alpha^k(t), \quad V_h^k(t) = e^{i(k \omega t)} \beta^k(t), \quad Y_h^k(t) = e^{i(k \omega t)} \gamma^k(t).
\]

Switching the formulation to $X_h$ and $V_h$, we obtain
\[
\begin{align*}
L(hD)X_h(t) &= -h \varepsilon \Lambda^{-1} \nabla_{\vec{v}} \mathcal{V}(\vec{Y}(t)) + O(h^2 \varepsilon^3 \delta_0^N + 1), \\
L(hD)V_h(t) &= h \varepsilon \Lambda^{-1} \nabla_{\vec{v}} \mathcal{V}(\vec{Y}(t)) + O(h^2 \varepsilon^3 \delta_0^N + 1),
\end{align*}
\]
where the vector $\vec{Y}(t)$ is defined as $\vec{Y}(t) = (Y_h^k(t))_{k \in \mathbb{N}^*}$ with $Y_h(t) = [X_h(t); V_h(t)]$, and the function $\mathcal{V}(\vec{Y}(t))$ is given by
\[
\mathcal{V}(\vec{Y}(t)) = \sum_{m=1}^{N} \frac{V^{(m+1)}(0)}{(m+1)!} \sum_{k^1+\ldots+k^{m+1}=0} (Y_h^k)^1 \ldots (Y_h^k)^{m+1}(t) \quad (5.8)
\]
with the potential
\[
\mathcal{V}(\vec{Y}(t)) = \frac{1}{d(N_\tau + 1)} \sum_{l=1}^{d} \sum_{k=-d}^{d} U(PCF^{-1}\hat{X}(t) + P\tilde{S}F^{-1}\hat{V}(t))_{k,l} \text{ with } \vec{Y}(t) = [\hat{X}(t); \hat{V}(t)].
\]

With this reformulation, we further switch to the quantities $\alpha^k(t)$ and $\beta^k(t)$, and then get
\[
\begin{align*}
L(hD + ih(k \cdot \omega)) \alpha^k(t) &= -h \varepsilon \Lambda^{-1} \nabla_{\vec{v}} \mathcal{V}(\vec{\varphi}(t)) + O(h^2 \varepsilon^3 \delta_0^N + 1), \\
L(hD + ih(k \cdot \omega)) \beta^k(t) &= h \varepsilon \Lambda^{-1} \nabla_{\vec{v}} \mathcal{V}(\vec{\varphi}(t)) + O(h^2 \varepsilon^3 \delta_0^N + 1). \quad (5.9)
\end{align*}
\]
Here $\alpha^k$ and $\beta^k$ determine a new vector $\vec{\varphi} = [\alpha^k; \beta^k]$ and then compose $\vec{\varphi} = (\alpha^k)_{k \in \mathbb{N}^*}$. 

To derive an almost-invariant of the coefficient functions, we consider the vector functions \( \tilde{\theta}(\lambda, t) = (e^{i(k \cdot \mu)} \varphi^k(t))_{k \in \mathbb{N}^n} \) for any real sequence \( \mu \). Then it follows from the definition (5.8) that \( k^1 + \ldots + k^{n+1} = 0 \). This shows that the function \( \mathcal{V}(\tilde{\theta}(\lambda, t)) \) is independent of \( \lambda \). Therefore, it is clear that

\[
0 = \frac{d}{d\lambda} \mathcal{V}(\tilde{\theta}(\lambda, t)) = \sum_{k \in \mathbb{N}^n} \left[i(k \cdot \mu)(\varphi^{-k}(t))^\top \nabla_{\varphi^{-k}} \mathcal{V}(\tilde{\theta}(t))\right] = \sum_{k \in \mathbb{N}^n} \left[i(k \cdot \mu)(\alpha^k; \beta^k)^\top [\nabla_{\alpha^{-k}} \mathcal{V}(\tilde{\theta}(t)); \nabla_{\beta^{-k}} \mathcal{V}(\tilde{\theta}(t))].\right]
\]

Combining with (5.9), it arrived that

\[
0 = \frac{1}{h} \sum_{k \in \mathbb{N}^n} i(k \cdot \mu)(\varphi^{-k}(t))^\top J [L(h \mathcal{D} + ih(k \cdot \omega)) \alpha^k; L(h \mathcal{D} + ih(k \cdot \omega)) \beta^k] + O(h \varepsilon^2 \delta_0^{n+1}) \tag{5.10}
\]

with \( J = \begin{pmatrix} 0 & \Lambda \\ -\Lambda & 0 \end{pmatrix} \). By using the properties \( \alpha_{-k,i,j} = \overline{\alpha_{k,i,j}} \) and \( \beta_{-k,i,j} = \overline{\beta_{k,i,j}} \), (5.10) can be reformulated as

\[
0 = \frac{1}{h} \sum_{k \in \mathbb{N}^n} i(k \cdot \mu)(\varphi^{-k}(t))^\top SJ L(h \mathcal{D} + ih(k \cdot \omega)) \varphi^k(t) + O(h \varepsilon^2 \delta_0^{n+1}),
\]

where \( S = \text{diag}(S_1, S_2, \ldots, S_{2d}) \) with

\[
S_1 = S_2 = \ldots = S_{2d} := \begin{pmatrix} 0 & \cdots & 0 & 1 \\ 0 & \cdots & 1 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 1 & \cdots & 0 & 0 \end{pmatrix}_{(N_r+1) \times (N_r+1)}.
\]

It follows from the skew-symmetry of \( SJ \) that the matrix decomposition \( SJ = iP^H \tilde{\Lambda} P \) is clear, where \( P \) is a unitary matrix and \( \tilde{\Lambda} \) is a diagonal matrix of the form \( \tilde{\Lambda} = \text{diag}(-I, I) \). These formulations lead to

\[
\frac{1}{h} \sum_{k \in \mathbb{N}^n} (k \cdot \mu)(\varphi^{-k}(t))^\top \tilde{\Lambda} L(h \mathcal{D} + ih(k \cdot \omega)) \varsigma^k = O(h \varepsilon^2 \delta_0^{n+1}), \tag{5.11}
\]

where \( \varsigma^k = (P \varphi^k) \) and \( \tilde{L}_{j,l} = L_{-j,l} \) for \( j = -N_r/2, \ldots, N_r/2 \) and \( l = 1, \ldots, d \). It is worth pointing out that \( L \) has another possible scheme \( \tilde{L}_{j,l} = L_{j,l} \) and this does not affect the proof described below.

The following analysis is mainly based on the arguments of [25, 26, 40] and thus we only present the main points here. With the Taylor expansions of \( L(h \mathcal{D}) \), it is easy to get a function \( \mathcal{I} \) which satisfies \( \frac{d}{dt} \mathcal{I}[\mu](t) = O(h \varepsilon^2 \delta_0^{n+1}) \) and then

\[
\mathcal{I}[\mu](t) = \mathcal{I}[\mu](0) + O(th \varepsilon^2 \delta_0^{n+1})
\]

by an integration. In the light of (5.2) and (5.3) with the notations

\[
\tau_1 = \frac{h^2 \omega_{j,l}^2}{4 \sin^2(h \omega_{j,l}/2)} h, \quad \tau_2 = -\frac{h^2 \omega_{j,l}^2}{16 \sin^4(h \omega_{j,l}/2)} (h \omega_{j,l} - \sin(h \omega_{j,l})) h^2,
\]

the construction of \( \mathcal{I} \) is obtained as follows

\[
\mathcal{I}(t) = \sum_{l=1}^{d} \sum_{j=-N_r/2}^{N_r/2} \Lambda^2(l, l) \left| \alpha_{j,l}^{(j)} \right|^2 (t) + \sum_{l=1}^{d} \sum_{j=-N_r/2}^{N_r/2} \frac{\tau_2/\tau_1 \text{Re} \left( \eta_{j,l}^{(j)} \Sigma_{j,l}^{(j)} \right)}{(\tau_2/\tau_1 \varepsilon h/\tau_1 \varepsilon \delta_0^4)} (t) + O(\varepsilon^6 \delta_0^4)
\]

\[
= \sum_{l=1}^{d} \sum_{j=-N_r/2}^{N_r/2} \Lambda^2(l, l) \left| \alpha_{j,l}^{(j)} \right|^2 (t) + O(\varepsilon^6 \delta_0^4) + O(\varepsilon^6 \delta_0^4)
\]

\[
= \sum_{l=1}^{d} \sum_{j=-N_r/2}^{N_r/2} \Lambda^2(l, l) \left| \alpha_{j,l}^{(j)} \right|^2 (t) + O(\varepsilon^4 \delta_0^4),
\]
where $\mu$ is chosen as $\mu_{j,l} = (-1)^{|\frac{2l+1}{2}|} \Lambda^2(l, l) \sin^2 \left( \frac{\omega_{j,l}}{2} \right) \delta_{j,l}$ in the analysis. By multiplying (5.3) with $(\dot{\varphi}^k) \dot{\varphi}$ and summing up the terms, one gets
\[
\mathcal{O}(h \varepsilon^N_{0}^{N+1}) = \frac{1}{h} \sum_{k \in \mathbb{N}^n} (\dot{\varphi}^k)^T \mathbf{JL}(h \mathbf{D} + i h (k \cdot \mathbf{w})) \dot{\varphi}^k + \frac{d}{dt} \mathcal{V}(\varphi(t))
\]
\[
= \frac{1}{h} \sum_{k \in \mathbb{N}^n} (\dot{\varphi}^k)^T i \omega \mathbf{L}(h \mathbf{D} + i h (k \cdot \mathbf{w})) \dot{\varphi}^k + \frac{d}{dt} \mathcal{V}(\varphi(t)).
\]
Then it can be shown that there exists a function $\mathcal{H}_1$ of the form
\[
\mathcal{H}_1(t) = V(\varphi(t)) + \frac{d}{dt} \mathcal{V}(\varphi(t)) + \frac{1}{e^2} \mathcal{I} + \mathcal{H}_1,
\]
such that $\frac{d}{dt} \mathcal{H}_1(\varphi(t)) = \mathcal{O}(h^2 \varepsilon^N_{0}^{N+1})$. We are now in a position to give the almost-invariant by defining
\[
\mathcal{H} = \frac{1}{e^2} \mathcal{I} + \mathcal{H}_1.
\]
It has an important property that
\[
\varepsilon^2 \mathcal{H}(t) = \mathcal{O}(\varepsilon^2) + \mathcal{O}(h \varepsilon^N_{0}^{N+1}).
\]
Moreover, the relationship between this almost-invariant and the result $H_2$ of the numerical method is derived as
\[
\varepsilon^2 \mathcal{H}(t_n) = \mathcal{I}(t_n)^{\varepsilon^2} V(\tilde{\mathbf{U}}^n) + \mathcal{O}(\varepsilon^5 \delta_0^4) + \mathcal{O}(\delta_\mathcal{H}) = \varepsilon^2 H_2(u^n) + \mathcal{O}(\varepsilon^5 \delta_0^4) + \mathcal{O}(\delta_\mathcal{H}).
\]
With the above results, it is easy to get
\[
\varepsilon^2 H_2(u^n) = \varepsilon^2 \mathcal{H}(t_n) + \mathcal{O}(\varepsilon^5 \delta_0^4) + \mathcal{O}(\delta_\mathcal{H}) = \varepsilon^2 \mathcal{H}(t_{n-1}) + h \mathcal{O}(h \varepsilon^N_{0}^{N+1}) + \mathcal{O}(\varepsilon^5 \delta_0^4) + \mathcal{O}(\delta_\mathcal{H})
\]
\[
= \ldots = \varepsilon^2 \mathcal{H}(t_0) + nh \mathcal{O}(h \varepsilon^N_{0}^{N+1}) + \mathcal{O}(\varepsilon^5 \delta_0^4) + \mathcal{O}(\delta_\mathcal{H}) = \varepsilon^2 H_2(u^0) + \mathcal{O}(\varepsilon^5 \delta_0^4) + \mathcal{O}(\delta_\mathcal{H})
\]
as long as $nh^2 \varepsilon^2 \delta_0^N + \delta^4_\mathcal{H}$ is small. Then using the connection of $H_2$ with $H$ completes the proof of Theorem 3.1 for $S2O2$.

**Step 3. Proof for $S3O4$.** For the integrator $S3O4$, it follows from its coefficients that
\[
\tilde{X}^{n1} = \Phi(t + c_1 h), \quad \tilde{X}^{n3} = \Phi(t + c_3 h), \quad \tilde{X}^{n2} = \Phi(t + c_2 h) + \mathcal{O}(h^2),
\]
and similar results hold for $\tilde{V}^{n1}, \tilde{V}^{n2}, \tilde{V}^{n3}$. Considering the fact that $F((\mathbf{C} \mathbf{F}^{-1} \tilde{X}^{n2} + \mathbf{S} \mathbf{F}^{-1} \tilde{V}^{n2})) = F(\tilde{T}(t + c_2 h) + \mathcal{O}(h^2))$, we define the operator
\[
\mathcal{L}_S(h \mathbf{D}) = (e^{h \mathbf{D}} - e^{h M}) (\tilde{b}_1 (h M) e^{c_1 h \mathbf{D}} + \tilde{b}_2 (h M) (e^{c_2 h \mathbf{D}} + \mathcal{O}(h^2)) + \tilde{b}_3 (h M) e^{c_3 h \mathbf{D}})^{-1}
\]
for $S3O4$. Then the rest proceeds similarly to that stated above for $S2O2$. It should be noted that the bounds of the coefficients in modulated expansions are changed into
\[
\alpha^{(j)}_{j,l}(t) = \mathcal{O}(\delta_0), \quad \dot{\alpha}^{(j)}_{j,l}(t) = \mathcal{O}(\varepsilon^5 \delta_0^4), \quad \alpha^{k}_{j,l}(t) = \mathcal{O}(\varepsilon^4 \delta_0^4), \quad \mathbf{k} \neq (j,l),
\]
\[
\beta^{(j)}_{j,l}(t) = \mathcal{O}(\delta_0), \quad \dot{\beta}^{(j)}_{j,l}(t) = \mathcal{O}(\varepsilon^5 \delta_0^4), \quad \beta^{k}_{j,l}(t) = \mathcal{O}(\varepsilon^4 \delta_0^4), \quad \mathbf{k} \neq (j,l),
\]
where Lemma 3.1 of [29] is used here. For $S3O4$, the notations $t_1$ and $t_2$ appeared in the above proof are bounded by $t_1 = \mathcal{O}(h^4/\varepsilon^3)$ and $t_2 = \mathcal{O}(h^5/\varepsilon^3)$ based on the Taylor expansion of $\mathcal{L}_S$ and Lemma 3.1 of [29]. Therefore, the energy at $S3O4$ has the following relation with the almost-invariant $\mathcal{H}$:
\[
\varepsilon^2 \mathcal{H}(t_n) = \mathcal{I}(t_n)^{\varepsilon^2} V(\tilde{\mathbf{U}}^n) + \mathcal{O}(t_2/t_1 \varepsilon/\sqrt{h} e^3 \delta_0^4) + \mathcal{O}(\varepsilon^5 \delta_0^4) + \mathcal{O}(\delta_\mathcal{H})
\]
\[
= \mathcal{I}(t_n)^{\varepsilon^2} V(\tilde{\mathbf{U}}^n) + \mathcal{O}(\varepsilon^7/\sqrt{h} \delta_0^4) + \mathcal{O}(\varepsilon^6 \delta_0^4) + \mathcal{O}(\delta_\mathcal{H})
\]
\[
= \varepsilon^2 H_2(u^n) + \mathcal{O}(\varepsilon^6 \delta_0^4) + \mathcal{O}(\delta_\mathcal{H}).
\]
This result yields the estimate of $S3O4$ given in Theorem 3.2.
6. Conclusion

In this paper, we have designed and analyzed numerical integrators for a class of highly oscillatory second-order differential equations with large initial values. Using three transformations of the original system, two-scale formulation approach, spectral semi-discretisation and general $s$-stage exponential integrators, a class of large-stepsize highly accurate integrators was formulated as the numerical approximation of (1.1) with large initial data. Stiff order conditions and symmetric property were used in the construction of practical methods. The proposed integrators were shown to have not only improved uniformly high accuracy but also good long-term energy conservation even for large time step sizes. The numerical results of two numerical experiments supported the properties of the obtained integrators.

Last but not least, we point out that the main contribution of this paper is that we have established a new framework to design improved uniform higher-order trigonometric integrators with long time behavior for solving highly oscillatory differential equations with large initial values. We believe that the methodology presented in this paper can be extended to a range of nonlinear Hamiltonian PDEs such as the wave equation and Dirac equation. The rigorous analysis on this topic will be considered in our next work. Another issue for future exploration is the study of uniform higher-order integrators with exact structure conservation such as symplecticity and energy.

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