FREE FIELD REPRESENTATION OF $osp(2 \mid 1)$ AND $U_q(osp(2 \mid 1))$ AND N=1 (q-)SUPERSTRING CORRELATION FUNCTIONS

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Abstract

The free field realization of irreducible representations of $osp(2 \mid 1)$ is constructed, by a unified and systematic scheme. The $q$-analog of this unified scheme is used to construct $q$-free field realization of irreducible representations of $U_q(osp(2 \mid 1))$. By using these realization, the two point function of $N = 1$ superconformal (q-superconformal) model based on $osp(2 \mid 1)$ ($U_q(osp(2 \mid 1))$) symmetry have been calculated.
1 Introduction

In modern theoretical physics, free field realizations of Lie algebras drive an important role from technical point of view. Usually, this simplification allows to go much further with any kind of formal manipulations.

Free field representations of Lie (super)algebras and quantum (super)algebras [1-8] are of interest because, like quantum algebras they have many applications in (super)conformal field theories [9], inverse problems and integrable systems [10, 11] and statistical mechanics [12].

\(osp(2 \mid 1)\) as a Lie superalgebra plays a special rule, in part analogous to that played by \(sl(2)\) in simple Lie algebras. It is therefore of particular interest to construct the free field representation of this superalgebra. A natural question is whether such a realization exists in the case of \(U_q(osp(2 \mid 1))\) too.

There have been attempts to construct for a Heisenberg representation of \(osp(2 \mid 1)\) and \(U_q(osp(2 \mid 1))\) in ref. [13], by Schwinger method. Due to the method of construction, they are not generalizable to all irreducible representations and affinization.

There has been a lot of interest on class of superconformal models specially super-WZW models based on nonablian nonsemisimple Lie groups [14-19], particularly because they allow the construction of exact string background. One class of these superconformal models is \(N=1\) superconformal with \(osp(2 \mid 1)\) super-Virasoro symmetry. Correlation functions for such theories had been calculated by different methods [20, 21]. In this letter we give a very simple way of calculation, which is extendable to all other symmetrice groups. The same question arise if there exists any q-superconformal symmetry for such a theory, what will be then the q-analog correlation functions?

Our aim in this letter is to apply the unified and systematic scheme given in ref. [1] for \(A_n\) series, for \(osp(2 \mid 1)\) and \(U_q(osp(2 \mid 1))\) and use these realizations to calculate two point function of \(N = 1\) super\((q\text{-super})\)conformal model based on \(osp(2\mid 1)(U_q(osp(2\mid 1)))\) symmetry
The structure of this letter is as follows: In section 2, we construct the differential realization of $osp(2 | 1)$ for all irreducible representations. In section 3, we use the same method of section 2 to present $q$-difference operator realization of $U_q(osp(2 | 1))$, for any irreducible representation. In section 4, we will use the results of previous sections to calculate two point function for $N = 1$ super($q$-super)conformal model based on $osp(2 | 1)$ and $U_q(osp(2 | 1))$ symmetry group.

2 Free Field Representation of $osp(2 | 1)$

$osp(2 | 1)$ is a rank one Lie superalgebra. This algebra has three even generators $X_{\pm}$, $H$ of $sl(2)$ and two odd generators of $V_{\pm}$, which satisfy the following (anti)commutation relations:

$$
\begin{align*}
\{V_+, V_-\} &= H, \\
\{V_\pm, V_\pm\} &= 2X_\pm \\
[X_+, X_-] &= H, \\
[H, X_\pm] &= \mp 2X_\pm, \\
[H, V_\pm] &= \mp V_\pm.
\end{align*}
$$

Let $2h$ be any arbitrary highest weight,

$$
V_+|2h> = 0, \\
H|2h> = 2h|2h>.
$$

{\{V_+\}} and {\{H\}} are the isotropic subalgebras of state $|2h>$. According to ref. [1] the states in representation space are

$$
e^{\theta V_-}e^{zX_-}|2h>$$

where $\theta$ and $z$ are Grassmanian and complex variables, respectively. The basis vectors of this vector space are $\{|2h>, V_-|2h>, ..., (V_-)^d|2h>\}$.

By the action of $V_\pm$ and $H$ on (3) and using the above (anti)commutation relations we will have:

$$
V_-e^{\theta V_-}e^{zX_-}|2h> = V_-(1 + \theta V_-)e^{zX_-}|2h> = (\delta_\theta + \theta \partial_z)e^{\theta V_-}e^{zX_-}|2h>
$$

$$
V_+e^{\theta V_-}e^{zX_-}|2h> = (2h\theta + z\partial_\theta + z\theta \partial_z)e^{\theta V_-}e^{zX_-}|2h>
$$

(4)
Let us define

\[ v_+ = \partial_\theta + \theta \partial_z \]
\[ v_- = \theta 2h + z \partial_\theta + z \theta \partial_z \]
\[ H = 2h + \theta \partial_\theta + 2z \partial_z \]

where \( \partial_z = \frac{\partial}{\partial z}, \ \partial_\theta = \frac{\partial}{\partial \theta}, \ [\partial_z, z] = 1, \) and \( \{ \partial_\theta, \theta \} = 1. \) One can consider this, as the representation of \( osp(2 \mid 1) \) on the super-sub-space of analytic functions spanned by the monomials, \( \{ 1, z, z^2, ..., z^{4h}, \theta, \theta z, \cdots, \theta z^{4h} \}. \) We will find that this new operator realization, which satisfies the algebra of (1), is a finite-dimensional irreducible representation of \( osp(2 \mid 1). \)

Up to now, everywhere the covariant derivative in superspace was being defined by

\[ D := \partial_\theta + \theta \partial_z. \]

As we can see this is not definition but it is differential realization of \( v_+ \) on superspace of analytical function of \( f(z, \theta). \)

Geometrically, this realization describes the right action of the group on sections of a holomorphic line bundle over the flag manifold \( osp(2 \mid 1)/T, \) where \( T \) is isotropic subalgebras of state \( \mid 2h >. \)

Similar results for the left action on flag manifold with states on representation space,

\[ \langle -2h \mid e^{zX_-} e^{\theta V_-} \rangle, \quad \langle -2h \mid X_- = 0, \quad \langle -2h \mid H = \langle -2h \mid ( -2h ), \]

will obtain as follows

\[ v_- = \partial_\theta + \theta \partial_z \]
\[ v_+ = -z \partial_\theta - \theta z \partial_z - 2h \theta \]
\[ H = -2z \partial_z - \theta \partial_\theta - 2h. \]
3 \quad U_q(osp(2 \mid 1))

Let us first fix the notations. Consider \(q\)-exponential function:

\[
e^x_q := \sum_{n=0}^{\infty} x^n [n]!, \quad [n] := \frac{q^n - q^{-n}}{q - q^{-1}}
\]

and the \(q\)-difference operator

\[
D_x f(x) = \frac{f(qx) - f(q^{-1}x)}{(q - q^{-1})x} = \frac{1}{(q - q^{-1})x}(M_x^{+1} - M_x^{-1})f(x)
\]

where \(M_x\) is a translation operator, defined by \(M_x^n f(x) = f(q^n x)\).

Just as in the case of ordinary \(osp(2 \mid 1)\) for right action on flag manifold, we take the states, \(e^q_{\theta^1}, e^q_{\theta^2}, 2h\rangle\, \text{in the representation space. The algebra relations are}[22]:

\[
\{V_+, V_-\} = [H]_q, \quad \{V_\pm, V_\mp\} = 2X_\pm; \quad [H]_q := \frac{q^h - q^{-h}}{q - q^{-1}}
\]

\[
[H, X_\pm] = \mp 2X_\pm, \quad [H, V_\pm] = \mp V_\pm
\]

with the same procedure as ordinary case one will find the following \(q\)-difference operator realization for any arbitrary highest weight representation:

\[
v_+ = D_\theta + \theta D_z
\]

\[
v_- = \frac{q}{q+1} z(q^{2h} M^{+1} + q^{-2h-1} M^{-1}) D_\theta + \frac{1}{q - q^{-1}} (q^{2h} M^{+2} - q^{-2h M^{-2}}) \theta
\]

\[
+ \frac{q}{(q+1)(q-q^{-1})} (q^{2h-1} - q^{-2h} + q^{-2h M^{-2}} - q^{2h-1} M^{+2}) \theta
\]

\[
H = 2h + \theta \partial_\theta + 2z \partial_z
\]

where

\[
M_0^{\pm n} := T^{\pm n} = q^{\pm n \theta \partial_\theta} \sim 1 + (q^{\pm n} - 1) \theta \partial_\theta, \quad M_z^{\pm n} := M_z^{\pm n} = q^{z \partial_z}
\]

\[
[D_z, z] = \frac{q^{M^{+1}+M^{-1}}}{q+1}, \quad \{D_\theta, \theta\} = \frac{T^{+1}+qT^{-1}}{q+1}.
\]

Similar results for the left action on flag manifold with states on representation space, \(< -2h \mid e^q_{\theta^1} e^q_{\theta^2} \), is

\[
v_+ = D_\theta + \theta D_z
\]

\[
v_- = -\frac{q}{q+1} z(q^{2h} M^{+1} + q^{-2h-1} M^{-1}) D_\theta - \frac{1}{q - q^{-1}} (q^{2h} M^{+2} - q^{-2h M^{-2}}) \theta
\]

\[
- \frac{q}{(q+1)(q-q^{-1})} (q^{2h-1} - q^{-2h} + q^{-2h M^{-2}} - q^{2h-1} M^{+2}) \theta
\]

\[
H = - \theta \partial_\theta - 2z \partial_z - 2h.
\]
We find that these new operator realization which satisfy the algebra of (10), is a finite-dimensional irreducible representation of $U_q(osp(2\mid 1))$.

In the limit of $q \to 1$ all the above realization goes to that of the ordinary $osp(2\mid 1)$.

4 Correlation functions of $N = 1$ super(q-super)conformal models based on $osp(2\mid 1)$ and $U_q(osp(2\mid 1))$

The String theory which is invariant under the global superconformal group, $OSP(2|1)$, generated by $G_{\pm 1/2}$, $L_{\pm 1}$ and $L_0$, is $N = 1$ superstring theory. In what follows we set

$$v_{\pm} = G_{\pm 1/2}, \quad H = 2L_0$$

just to use the common notations for Virasoro algebra.

As we know the generators of super-Virasoro algebra of $osp(2\mid 1)$, $G_{\pm 1/2}$, $L_{\pm 1}$ and $L_0$, satisfy the following algebraic relations:

\[
\begin{align*}
[L_{+1}, L_{-1}] &= 2L_0 \\
[L_0, L_{\pm 1}] &= \mp L_{\pm} \\
\{G_{\pm 1/2}, G_{\pm 1/2}\} &= 2L_{\pm} \\
\{G_{1/2}, G_{-1/2}\} &= 2L_0 \\
[L_0, G_{\pm 1/2}] &= \mp G_{\pm 1/2}
\end{align*}
\]

(14)

In the above algebra, $L_{\pm 1}$ are bosonic generators and $G_{\pm 1/2}$ are fermionic generators.

In order to construct the Ward identity, we take following "co-product" and differential realization of (7) with $H = 2L_0$, to eliminate the two point function, which coincide with (15):

$$\Delta(g) = g \otimes I + I \otimes g; \quad g \in osp(2\mid 1)$$

(15)

where $I$ means the identity operator. Such a written form will be more visible in the case of $q$-analog.

Consider the quasi-primary superfield, $\Phi(z, \theta) = \phi(z) + \theta f(z)$, where $\phi(z)$ and $f(z)$ are bozonic and fermionic fields respectively.
Then the Ward identity is given by

\[ \Delta L_0 < \Phi(z_1, \theta_1) \Phi(z_2, \theta_2) >= \left( \sum_{i=1}^{2} (-z_i \partial_{z_i} - \frac{1}{2} \theta_i \partial_{\theta_i} - h_i) \right) < \Phi(z_1, \theta_1) \Phi(z_2, \theta_2) >= 0 \]

\[ \Delta G_{-1/2} < \Phi(z_1, \theta_1) \Phi(z_2, \theta_2) >= \left( \sum_{i=1}^{2} (\partial_{\theta_i} + \theta_i \partial_{z_i}) \right) < \Phi(z_1, \theta_1) \Phi(z_2, \theta_2) >= 0 \]

\[ \Delta G_{+1/2} < \Phi(z_1, \theta_1) \Phi(z_2, \theta_2) >= \left( \sum_{i=1}^{2} (\partial_{z_i} + \theta_i z_i \partial_{\theta_i} + 2h_i \theta_i) \right) < \Phi(z_1, \theta_1) \Phi(z_2, \theta_2) >= 0 \]

\[ \Delta L_- < \Phi(z_1, \theta_1) \Phi(z_2, \theta_2) >= \left( \sum_{i=1}^{2} (\partial_{z_i}) \right) < \Phi(z_1, \theta_1) \Phi(z_2, \theta_2) >= 0 \]

\[ \Delta L_+ < \Phi(z_1, \theta_1) \Phi(z_2, \theta_2) >= \left( \sum_{i=1}^{2} (z_i^2 \partial_{z_i} + \theta_i z_i \partial_{\theta_i} + h_i \partial_{z_i}) \right) < \Phi(z_1, \theta_1) \Phi(z_2, \theta_2) >= 0 \]  

(16)

where \( h_1 \) and \( h_2 \) are conformal weight of \( \Phi(z_1, \theta_1) \) and \( \Phi(z_2, \theta_2) \) respectively. If we solve the above Ward identity, we have the following expression for two point function which is well-known [20].

\[ < \Phi(z_1, \theta_1) \Phi(z_2, \theta_2) > \sim (z_1 - z_2 - \theta_1 \theta_2)^{-(h_1 + h_2)} = z_1^{-(h_1 + h_2)} \]

(17)

As we can see, this method which is easier and trustworthy than construction of combinations of \( osp(2|1) \) invariants [20], extendable to any correlation functions.

We take \( N = 1 \) superstring based on q-deformed Lie superalgebra \( U_q(osp(2 | 1)) \). In this case, in order to construct the q-Ward identity, we should define the co-product rule for the generators as follows:

\[ \Delta(L_0) = L_0 \otimes I + I \otimes L_0, \quad \Delta(G_{\pm 1/2}) = G_{\pm 1/2} \otimes q^{L_0} + q^{-L_0} \otimes G_{\pm 1/2}. \]

(18)

It is easy to show that this co-product fulfills the axiom of co-associatively;

\[ (I \otimes \Delta) \Delta = (\Delta \otimes I) \Delta. \]

(19)

The difference operator realization is been given by (13) with \( H = 2L_0 \). Then the q-Ward
where $x,$ if we solving the q-Ward identity, we have the following expression for the two point function

$$\Delta G_{1/2} < \Phi(z_1, \theta_1) \Phi(z_2, \theta_2) > =$$

$$[\partial_{\theta_1} + \frac{\theta_1}{z_1(q-x)}(M_1^{+1} - M_1^{-1})]q^h(1 + (q^{1/2} - 1)\theta_2 \partial_{\theta_2})M_2^{+1} < \Phi(z_1, \theta_1) \Phi(z_2, \theta_2) > + q^{-h}(1 + (q^{-1/2} - 1)\theta_1 \partial_{\theta_1})M_1^{+1} [\partial_{\theta_2} + \frac{\theta_2}{z_2(q-x)}(M_2^{+1} - M_2^{-1})] < \Phi(z_1, \theta_1) \Phi(z_2, \theta_2) > = 0$$

$$\Delta G_{1/2} < \Phi(z_1, \theta_1) \Phi(z_2, \theta_2) > =$$

$$= [\frac{q}{1+q}z_1 \partial_{\theta_1}(q^{2h}M_1^{+1} + q^{-2h}M_1^{-1}) + \frac{1}{q-q^1}\theta_1(q^{2h}M_1^{+2} - q^{-2h}M_1^{-2})$$

$$+ \frac{q}{q-1}(q^{2h} - q^{-2h} + q^{-2h}M_1^{+2} - q^{-2h}M_1^{-2})]$$

$$\times q^h(1 + (q^{1/2} - 1)\theta_2 \partial_{\theta_2})M_2^{+1} < \Phi(z_1, \theta_1) \Phi(z_2, \theta_2) > + q^{-h}(1 + (q^{-1/2} - 1)\theta_1 \partial_{\theta_1})M_1^{-1}$$

$$\times [\frac{q}{1+q}z_2 \partial_{\theta_2}(q^{2h}M_2^{+1} + q^{-2h}M_2^{-1}) + \frac{1}{q-q^1}\theta_2(q^{2h}M_2^{+2} - q^{-2h}M_2^{-2})$$

$$+ \frac{q}{q-1}(q^{2h} - q^{-2h} + q^{-2h}M_2^{+2} - q^{-2h}M_2^{-2})] < \Phi(z_1, \theta_1) \Phi(z_2, \theta_2) > = 0$$

(20)

If we solving the q-Ward identity, we have the following expression for the two point function

$$< \Phi(z_1, \theta_1) \Phi(z_2, \theta_2) > = z_1^{-2h} \frac{(q^{2h}x; q^2)}{(q^{-2h}x; q^2)} - \theta_1 \theta_2 q^{-1/2} [2h]z_1^{-2h-1} \frac{1}{1 - q^{-2h-1}x} \frac{(q^{2h+1}x; q^2)}{(q^{-2h+1}x; q^2)}$$

(21)

where $x = \frac{z_1}{z_2}$ and

$$(x; q^2) = \prod_{n=0}^{\infty} (1 - xq^{2n}).$$

(22)

In the limit $q \to 1,$ the q-correlation function reduces

$$< \Phi(z_1, \theta_1) \Phi(z_2, \theta_2) > \to (z_1 - z_2 + \theta_1 \theta_2)^{-2h}$$

(23)

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