CLASSIFYING $VII_0$ SURFACES WITH $b_2 = 0$ VIA GROUP THEORY

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Abstract. We give a new proof of Bogomolov’s theorem, that the only $VII_0$ surfaces with $b_2 = 0$ are, up to an étale cover, those constructed by Hopf and Inoue. The proof follows the strategy of the original one, but it is of purely group-theoretic nature.

Our aim is to provide a new proof of the well known:

Theorem 1. [Bo76, Bogomolov] Every $VII_0$ surface with $b_2 = 0$ has an étale cover that carries a holomorphic foliation. Therefore an étale cover of $X$ is either a Hopf or a Inoue surface.

The first part of the proof in op.cit., which is the content of our Section II, aims at constructing an affine structure on such surfaces, as well as establishing its uniqueness. Both the existence and uniqueness statements follow from the vanishing of certain cohomology groups, which could be naturally thought of as obstruction spaces. This structure provides two objects: a representation of the fundamental group of the surface into the affine group $\rho : \Gamma := \pi_1(X) \to \text{Aff}(2, \mathbb{C})$, with corresponding linearization $l\rho$ into $\text{GL}_2(\mathbb{C})$; and a $\rho$-equivariant local biholomorphism $p_{\text{aff}} : V := \tilde{X} \to \mathbb{C}^2$.

The automorphism group of the field $\mathbb{C}$ which we denote as $\text{Gal}(\mathbb{C}/\mathbb{Q})$ acts naturally on the set of flat vector bundles over $X$. It can be shown that indeed the tangent bundle is stable under this action. Such stability follows from the fact that these surfaces have very few vector bundles with at least one non-vanishing cohomology group, and that cohomology with locally constant coefficients is not altered by the Galois action. It follows that the linear representations, obtained by twisting $l\rho$ with field automorphisms, are conjugated to each other.

This observation imposes very strong arithmetic restrictions on such representation, namely that the image $l\rho(\Gamma)$ can be conjugated into $\text{GL}_2(\mathbb{Q})$ or into $H_2(F)$, where the latter is the unit group in a quaternion algebra over a quadratic field $F$. Crucially, however, the quadratic extension $F/\mathbb{Q}$ must be real. This follows from a very particular fact about
surfaces, namely that \( \dim(H^1(X, C(l\rho))) = 1 \) - where \( C(l\rho) \) is the flat bundle induced by \( l\rho \). This fact is also crucial in deriving that the image \( \rho(\Gamma) \) is contained in the real affine subgroup \( \text{Aff}(2, \mathbb{R}) \subset \text{Aff}(2, \mathbb{C}) \). The group \( \text{Aff}(2, \mathbb{R}) \) above leaves invariant a linear subspace \( R \cong \mathbb{R}^2 \subset \mathbb{C}^2 \), and there is a two-dimensional contractible Lie subgroup \( G_K \subset \text{Aff}(4, \mathbb{R}) \) of real affine transformations acting on \( \mathbb{C}^2 \), which is free on \( \mathbb{C}^2 \setminus R \) and fixes \( R \) pointwise. The quotient of \( \mathbb{C}^2 \setminus R \) by its action, is a Moebius band isomorphic to the space \( \text{Gr}(1, 2) \) of lines in \( \mathbb{R}^2 \), which carries a natural angle map \( s : \text{Gr}(1, 2) \to S^1 \). All those structures are inherited by \( V \) and since they are \( \Gamma \)-invariant, also by \( X \).

The pre-image of the real subspace \( R \) under the projection \( p_{aff} \) is a, possibly empty, closed 2-dimensional real submanifold \( V_R \subset V \). The action of \( G_K \) on \( V \setminus V_R \) is free, while trivial on \( V_R \). The image of \( V_R \) in \( X \) is a compact two-dimensional manifold with affine structure, i.e. a possibly empty finite union of compact tori or Klein bottles. Thus the quotient \( S := (V \setminus V_R)/G_K \) is an open Riemann surface with a \( \rho \)-equivariant local biholomorphism \( S \to \text{Gr}(1, 2) \). Diagram 15 consists of maps commuting with the action of \( \Gamma \), and summarizes the situation.

We thus have two possible radically different cases:

1. \( S \to \text{Gr}(1, 2) \) is a cyclic covering, which we call classical case.
2. \( S \to \text{Gr}(1, 2) \) is just a local isomorphism, hence rather wild, which we call pathological case.

After this was achieved the original strategy, first sketched in [Bo76] and then expanded on in [Bo82], aimed at deriving a contradiction profitting from the interplay between: the strong arithmetic constraints we have on the affine representation \( \rho \), and; the analysis of geometric information arising from the existence of a complex affine structure \( p_{aff} \) with the above properties. Unfortunately this derivation in [Bo82] is rather long, and contains complicated topological arguments. Our contribution is to replace such proof by a much simpler argument.

The case \( V_R = \emptyset \) contains the main innovation: a combination of a group-theoretic argument, together with an analysis of the tree structure of the topological Stein quotient \( S_s \) of \( \tilde{S} \), corresponding to the induced angle projection \( \tilde{S} \to \tilde{\text{Gr}}(1, 2) \to \mathbb{R} \). Though the Stein quotient \( S_s \) is potentially non-Hausdorff, we can use the fact that \( \Gamma \) acts densely on \( S \), to conclude that if \( V_R \) is empty, then the map \( \tilde{S} \to \tilde{\text{Gr}}(1, 2) \) must be an embedding. This implies that \( \Gamma \) is realized as a discrete subgroup of \( \tilde{A}(2, \mathbb{R}) \). A contradiction is derived by purely group theoretic means.
In case $V_R \neq \emptyset$, we venture into a detailed study of the induced affine structure on $V_R$, finally leading to an absurd.

It is interesting to remark that the key player, in the story, is our two dimensional Lie group $G_K$ acting on $V$. In fact, its orbits are holomorphic curves, but the resulting foliation is not holomorphic.

We should also remark that we can freely pass to finite-index subgroups of $\Gamma$, without loss of generality. This is particularly important once we know that $l_\rho$ has arithmetic values, so then by Selberg’s lemma, [Al87], we can assume $l_\rho(\Gamma)$ is torsion-free.

An analytic approach to this theorem appeared in [LYZ90], and was further completed in [Te94] and [LYZ94].

Our paper is structured as follows: in the first section we construct the affine structure, and prove its arithmetic properties. In the second section we study in details its geometry. In the third section we prove that the case $V_R = \emptyset$ is impossible, and in the fourth we prove that $V_R \neq \emptyset$ is impossible.

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**I. AFFINE STRUCTURES ON $\text{VII}_0$ SURFACES WITH $b_2 = 0$ AND NO HOLOMORPHIC FOLIATIONS**

The content of this section is the proof of:

**Proposition I.1.** Let $X$ be a $\text{VII}_0$ surface with $b_2 = 0$ and no holomorphic foliation. Then $X$ carries an affine structure, which defines a representation $\rho : \pi_1(X) \to \text{Aff}(2, \mathbb{C})$. Furthermore, the image of its linearization $l_\rho$ is conjugated to a subgroup of $\text{GL}_2(\mathbb{Q})$, or of the units $H_2(F)$ of a quaternion algebra over $F$, a real quadratic extension of $\mathbb{Q}$.

The structure of the section is as follows: the proof is split into four main technical statements, namely Propositions I.2, I.6, I.11 and I.13; each such Proposition is proved in a sequence of simple Lemmas.

Examples of $\text{VII}_0$ surfaces, i.e. Hopf and Inoue surfaces, are described in [Bo76], [BH75], [La076], [BH75].
and \([\text{In74}]\). We want to show that there are no other examples, which amounts to prove that surfaces with the following properties do not exist:

- \(c_1(X)\) is torsion - in particular \(c_1^2 = 0\) - and \(c_2(X) = 0\).
- \(b_2 = 0\), \(H^0(X, K_X^n) = 0\) for every \(n \in \mathbb{Z} \setminus \{0\}\).
- It has infinite fundamental group, moreover \(\dim H^1(X, \mathbb{C}) = 1\).
- It carries no holomorphic foliations, i.e. \(H^0(X, \mathcal{L} \otimes \Omega^1) = 0\) for every rank 1 coherent sheaf \(\mathcal{L}\).
- The kernel of \(\pi_1(X) \to \pi_1(X)/[\pi_1(X), \pi_1(X)]\) is infinitely generated (Bombieri \([\text{BH75}]\)).
- Every finite étale cover of \(X\) has all the previous properties.

We will denote \(\Gamma := \pi_1(X)\).

In this section we follow the strategy of \([\text{Bo76}]\) rather closely.

**Proposition I.2.** The tangent bundle \(TX\) admits a unique flat, torsion-free connection.

**Proof.** The following remark is used systematically in the proof: the triviality of Chern numbers implies that the Euler characteristic of any tensor bundle \(\mathcal{F}\) is trivial; hence the existence of a non-trivial element in \(H^1(X, \mathcal{F})\), implies the existence of a non-trivial holomorphic section of \(\mathcal{F}\) or \(K_X \otimes \mathcal{F}^\vee\).

Let us consider the obstructions to the existence of an affine structure on \(X\).

- The obstruction to the existence of a holomorphic connection on \(TX\) is in \(H^1(\text{End}(TX) \otimes \Omega^1)\)
- The obstruction to the flatness of the connection is in \(H^0(K_X \otimes \text{End}(TX))\)
- The obstruction to the uniqueness of the connection is in \(H^0(\text{End}(TX) \otimes \Omega^1)\)

This is a standard fact which is directly translated from differential geometry.

We shall prove that all these cohomology groups vanish.

**Lemma I.3.** For every \(n \in \mathbb{Z}\), \(n \neq 0\), we have \(H^0(X, K_X^n \otimes \text{End}(TX)) = 0\).

**Proof.** If \(s\) is a section, it defines a morphism \(s : TX \to TX \otimes K_X^n\). Its determinant is a section of \(K_X^{4n}\), which must be zero. Hence, if \(s\) is non-trivial, its kernel is generically of rank 1 and defines a holomorphic foliation, which is absurd. \(\square\)

**Lemma I.4.** \(H^0(\Omega^1 \otimes \text{End}(TX)) = 0\)
Proof. Let us assume it is non-zero, and hence there is a non-zero section. This tautologically defines a non-zero morphism \( s : TX \to \text{End}(TX) \). It is injective, otherwise the kernel defines a foliation. Composing \( s \) with the natural trace map \( \text{tr} : \text{End}(TX) \to \mathcal{O}_X \), we obtain a morphism \( \text{tr} \circ s : TX \to \mathcal{O}_X \), which must be trivial since otherwise its kernel would again define a foliation. Next, consider the composition of \( s \wedge s : K^{-1}_X = TX \wedge TX \to \text{End}(TX) \wedge \text{End}(TX) \) with the commutator \( c : \text{End}(TX) \wedge \text{End}(TX) \to \text{End}(TX) \). By Lemma I.3 it must be zero. It follows that the image of \( s \) is a rank 2, traceless subbundle, which generates a commutative sub-algebra of \( \text{End}(TX) \). Observe that a traceless endomorphism commuting with a nilpotent one must be a multiple of the latter. Therefore the image of \( s \), being of rank 2, cannot contain any nilpotent element. Hence it consists of simultaneously diagonalizable endomorphisms, and for the same reason it contains at least one which is not a homothety. In particular, \( TX \) is everywhere locally split by its eigenspaces. Such local splitting might not define a global splitting of \( TX \), but the obstruction lives in the corresponding monodromy representation \( \Gamma \to \mu_2 \), which can be killed on an étale double cover of \( X \). Since this double cover is again of type \( \boxed{\rm II} \) we have a contradiction. \( \square \)

At this point, we proved the vanishing of the last two obstructions mentioned above. On the other hand, since \( \chi(X) = 0 \), if \( h^1(\text{End}(TX) \otimes \Omega^1) \neq 0 \) then by Riemann-Roch and Serre duality, \( h^0(\text{End}(TX) \otimes \Omega^1) \) is non-zero (we are using \( K_X \otimes TX \cong \Omega^1 \) to identify \( h^0 \) and \( h^2 \)) which is impossible. \( \square \)

The connection just constructed defines an affine structure on \( X \), meaning a local biholomorphism from the universal cover

\[
p_{\text{aff}} : \check{X} \to \mathbb{C}^2
\]

which is equivariant for a representation \( \rho : \Gamma := \pi_1(X) \to \text{Aff}(2, \mathbb{C}) \), with corresponding linearization \( l\rho : \Gamma \to \text{GL}_2(\mathbb{C}) \). We are going to use systematically the fact that \( \rho(\Gamma) \)-invariant objects on \( \mathbb{C}^2 \) define naturally similar objects on the compact surface \( X \): first we lift them under \( p_{\text{aff}} \) to \( \Gamma \)-invariant objects on the universal covering \( V \), and then descend them to \( X \). The first example of implementation of this general principle is:

**Lemma I.5.** The action of \( \text{Im}(l\rho) \) is irreducible. In particular, such group is not cyclic.

**Proof.** If it were then there would be a direction, say \( (x = 0) \subset \mathbb{C}^2 \), which is invariant under \( \text{Im}(l\rho) \). Consider the 1-form \( dx \). It is clearly invariant under translations, and by
definition the action of \( \text{Im}(l\rho) \) on \( dx \) is defined by a character \( \chi \). Then the pull back \( p^*_\text{aff}(dx) \) defines a \( \Gamma \)-invariant holomorphic foliation on \( V \), which descends to \( X \) - contradiction. \( \square \)

We will extract informations by studying the orbit of \( l\rho \) under the natural action of \( \text{Gal}(\mathbb{C}/\mathbb{Q}) \), by which we mean the full automorphism group of \( \mathbb{C} \) as a field. Namely, if a bundle \( E \) is defined by a representation \( \eta : \Gamma \to \text{GL}_n(\mathbb{C}) \), we will denote by \( E^\sigma \) the one defined by \( \sigma \circ \eta \), for \( \sigma \in \text{Gal}(\mathbb{C}/\mathbb{Q}) \). For \( \mathcal{F} \) a locally constant coherent sheaf, we have exact sequences in the analytic topology:

\[
(2) \quad 0 \to \mathbb{C}(\mathcal{F}) \to \mathcal{O}(\mathcal{F}) \to d\mathcal{O}(\mathcal{F}) \to 0
\]

\[
(3) \quad 0 \to d\mathcal{O}(\mathcal{F}) \to \Omega^1(\mathcal{F}) \to K_X \otimes \mathcal{F} \to 0
\]

Where \( \mathbb{C}(\mathcal{F}) \) is the sheaf of locally constant sections, \( \mathcal{O}(\mathcal{F}) \) the sheaf of holomorphic sections, and \( d \) denotes the usual holomorphic differential.

**Proposition I.6.** [BH75, Bombieri] \( K^\sigma_X \xrightarrow{\sim} K_X \) for every \( \sigma \).

**Proof.** Let \( \mathcal{F} \) be a locally constant coherent sheaf.

**Lemma I.7.** If \( \mathcal{F} \neq \mathcal{O}_X, K_X \) is of rank 1, then \( h^i(X, \mathcal{O}(\mathcal{F})) = 0 \) for every \( i \).

**Proof.** This is the absence of compact curves, plus the triviality of Chern classes and Riemann-Roch. \( \square \)

**Lemma I.8.** If \( \mathcal{F} \) has rank 1, then \( h^0(d\mathcal{O}(\mathcal{F})) = 0 \).

**Proof.** This follows by taking global sections in the sequence 3. \( \square \)

**Lemma I.9.** If \( \mathcal{F} \neq \mathcal{O}_X, K_X^{-1} \) has rank 1 then \( h^i(d\mathcal{O}(\mathcal{F})) = 0 \) for every \( i \).

**Proof.** We can apply sequence 3 together with Lemma 7 for \( K_X \otimes \mathcal{F} \), to deduce \( h^i(d\mathcal{O}(\mathcal{F})) = h^i(\Omega^1(\mathcal{F})) \). If, for some \( i \) we have \( h^i(\Omega^1(\mathcal{F})) \neq 0 \), then by Riemann-Roch it holds for \( i = 0 \) or \( i = 2 \). But then we get a foliation, which is absurd. \( \square \)

**Lemma I.10.** \( h^1(\mathcal{C}(K_X)) = 1 \)

**Proof.** By Lemma 9 we know \( h^i(d\mathcal{O}(\mathcal{F})) = 0 \) for all \( i \). By the sequence 2 \( h^1(\mathcal{C}(K_X)) = h^1(\mathcal{O}(K_X)) = 1. \) \( \square \)
The above lemmas now merge to conclude the proof: if \( K_X^\sigma \neq K_X \), by Lemma [L7] \( h^i(\mathcal{O}(K_X^\sigma)) = 0 \). Then by the sequence [2] we get
\[
h^0(d\mathcal{O}(K_X^\sigma)) = h^1(C(K_X^\sigma)) = h^1(C(K_X)) = 1
\]
which is absurd by lemma [L8].

Now we turn our attention to the Galois action on \( TX \). As one imagines,

**Proposition I.11.** \( TX^\sigma \rightarrow TX \). In particular there exists a representation \( \xi : \text{Gal}(C/Q) \rightarrow \text{GL}_2(C) \) such that \( l\rho^\sigma = \xi_\sigma \cdot l\rho \cdot \xi_\sigma^{-1} \).

**Proof.** Let us start by observing that, for any locally constant coherent sheaf \( \mathcal{F} \), the existence of homotheties gives \( h^0(\mathcal{F} \otimes \mathcal{F}^\vee) = h^0(\text{End}(\mathcal{F})) \geq 1 \). We have, again, a series of lemmas.

**Lemma I.12.** \( h^1(C(TX)) = h^0(d\mathcal{O}(TX)) = 1 \)

**Proof.** The first equality follows from sequence [2] and the vanishing of \( h^1(\mathcal{O}(TX)) \) due to the absence of foliations and Riemann-Roch. Observe that, since \( TX \) is simple, \( h^0(\text{End}(TX)) = 1 \), and since \( h^0(K_X \otimes TX) = 0 \), we conclude via sequence [3].

Therefore: \( h^1(C(TX^\sigma)) = 1 \) for every \( \sigma \in \text{Gal}(C/Q) \). By the sequence [2] at least one of the following happens:

- \( h^0(d\mathcal{O}(TX^\sigma)) \geq 1 \)
- \( h^1(\mathcal{O}(TX^\sigma)) \geq 1 \)

The first option implies that, by the sequence [3] \( h^0(\Omega^1 \otimes TX^\sigma) \geq 1 \) so then there exists a morphism \( TX \rightarrow TX^\sigma \) which must be an isomorphism.

The second option, along with Riemann-Roch, leads to a further alternative:

- \( h^0(\mathcal{O}(TX^\sigma)) \geq 1 \)
- \( h^2(\mathcal{O}(TX^\sigma)) = h^0(K_X \otimes \mathcal{O}(TX^\sigma)^\vee) \geq 1 \)

In the first case, since \( h^0(C(TX^\sigma)) = 0 \), the sequence [2] implies \( h^0(d\mathcal{O}(TX^\sigma)) \geq 1 \), while the sequence [3] gives an isomorphism \( TX \rightarrow TX^\sigma \).

In the second case, by the sequence [2] and the identity \( (TX^\sigma)^\vee = (\Omega^1)^\sigma \), we have one more alternative:

- \( h^0(C(K_X \otimes (\Omega^1)^\sigma)) \geq 1 \)
\[ h^0(d\phi(K_X \otimes (\Omega^1)')) \geq 1 \]

In the first case Proposition I.6 gives
\[ h^0(C(K_X \otimes (\Omega^1)')) = h^0(C(K_X^2 \otimes \Omega^1)) = h^0(C(K_X \otimes \Omega^1)) \geq 1 \]
and hence a foliation.

In the second case, the sequence 3 gives
\[ h^0(K_X \otimes \Omega^1 \otimes (\Omega^1)') \geq 1, \]
which defines an isomorphism \( TX \xrightarrow{\sim} K_X \otimes (\Omega^1)' \). Taking determinants, however, gives \( K_X^2 \otimes K_X^2 \xrightarrow{\sim} K_X^3 - \) the last isomorphism being Bombieri’s I.6 - which is again absurd since \( K_X \) is not torsion.

This proves that \( TX \xrightarrow{\sim} TX^\sigma \) for every \( \sigma \). Since \( TX \) is built from \( \rho \), the representations \( l\rho^\sigma \) are isomorphic to each other, and the existence of \( \xi \) follows. \( \square \)

We wish to employ Proposition I.11 to derive concrete arithmetic restrictions on \( l\rho \).
Quite generally we have:

\textbf{Proposition I.13.} Let \( \eta : \Gamma \to \text{GL}_2(\mathbb{C}) \) be a representation, such that \( \eta^\sigma \) is conjugated to \( \eta \) for every \( \sigma \in \text{Gal}(\mathbb{C}/\mathbb{Q}) \). Then the image of \( \eta \) is either conjugated to a subgroup of \( \text{GL}_2(\mathbb{Q}) \), or to a subgroup of the units in a quaternion algebra \( H_2(\mathbb{Q}(\sqrt{d})) \).

\textbf{Proof.} The strategy is to compute the Galois action on a matrix in its Jordan form, and see what happens. This is possible, since our condition on the action of Galois is invariant under conjugation.

\textbf{Lemma I.14.} Either the image of \( \eta \) can be conjugated into \( \text{GL}_2(\mathbb{Q}) \), or the image of \( \eta \) contains a diagonalizable matrix which is not a multiple of the identity.

\textbf{Proof.} Let us assume that every matrix, in the image of \( \eta \), has two coincident eigenvalues.

\textbf{Claim I.15.} There exists a basis of \( \mathbb{C}^2 \) such that, all the matrices in the image of \( \eta \) are in their Jordan form in this basis.

\textbf{Proof.} We can assume \textit{wlog} that there exists \( M \in \text{Im}(\eta) \) which is not a multiple of the identity, otherwise the claim is trivial. Pick any \( N \in \text{Im}(\eta) \), and let’s represent everything in the Jordan basis for \( M \):
\[
M = \begin{pmatrix} a & b \\ 0 & a \end{pmatrix}; \quad N = \begin{pmatrix} x & y \\ z & w \end{pmatrix}
\]
We know that \( b \neq 0 \). We compute:
\[
MN = \begin{pmatrix} ax + bz & ay + bw \\ az & aw \end{pmatrix}
\]
We have identities:

(4) \[ 4 \det(N) = \tr(N)^2; \quad \tr(MN) = a \cdot \tr(N) + bz; \quad 4 \det(MN) = \tr(MN)^2. \]

Which can be merged together to obtain:

(5) \[ (a \cdot \tr(N))^2 = 4a^2 \det(N) = 4 \det(MN) = (a \cdot \tr(N) + bz)^2. \]

Hence \( bz = 0 \) and since \( b \neq 0 \) we get \( z = 0 \). The Claim follows, since \( N \) has coincident eigenvalues. \( \square \)

Let \( \sigma \) act, in this basis, via a matrix \( \Lambda = (\lambda_{ij}) \). Comparing the coefficients in the identity

(6) \[ \Lambda M^\sigma = M \Lambda \]

we get

(7) \[ a^\sigma = a; \quad \lambda_{21} = 0; \quad b^\sigma = \lambda_{22} \lambda_{11}^{-1} b. \]

It follows that, for any two matrices \( M, M' \), we have \( b/b' \in \mathbb{Q} \). In particular, after conjugating \( \eta \) with

\[
\begin{pmatrix}
    b & 0 \\
    0 & 1
\end{pmatrix}
\]

the image is in \( \text{GL}_2(\mathbb{Q}) \). This proves the Lemma. \( \square \)

We can now analyze what happens if there is a diagonal matrix which is not a multiple of the identity:

**Lemma I.16.** Let \( M \), belonging to the image of \( \eta \), have eigenvalues \( x_1 \neq x_2 \). Let \( \sigma \) act, in a basis where \( M \) is diagonal, by conjugation via a matrix \( \Lambda \). Then we have two cases:

- \( M \) is invariant under \( \sigma \), and \( \Lambda \) is diagonal, or
- \( M \) is not invariant, \( x_1^\sigma = x_2 \), \( \Lambda \) is anti-diagonal (its diagonal vanishes), and in particular \( \sigma^2 = 1 \).

In particular, the Galois stabilizer of \( M \) is a subgroup of a torus \( \mathbb{C}^* \times \mathbb{C}^* \), while the Galois orbit of \( M \) has at most two elements.

**Proof.** The two items follow from the analysis of the coefficients in the identity \( \square \). The structure of the stabilizer is clear, while the fact that \( \sigma^2 = 1 \) means that the eigenvalues \( x_1, x_2 \) live in a quadratic extension of \( \mathbb{Q} \) with Galois group generated by \( \sigma \). \( \square \)

We proceed, therefore, to analyze the action of Galois on the whole \( \Gamma \).
Lemma I.17. Assume the image of $\eta$ cannot be conjugated into a subgroup of $GL_2(\mathbb{Q})$. Then there exists a quadratic extension $F$ of $\mathbb{Q}$, such that the image of $\eta$ lies in $GL_2(F) \cap H_2(F)$, where $H_2(F) \subset M_2(F)$ is a quaternion algebra over $F$.

Proof. With the notation of Lemma I.16 we have two cases:

- $\sigma$ is in the stabilizer of $M$, and
- $\sigma$ is not.

Let $N$ be an element in the image of $\eta$, represented in the basis where $M$ is diagonal by a matrix

$$N = \begin{pmatrix} x & y \\ z & w \end{pmatrix}$$

and we consider the equation

$$\Lambda N^\sigma = N \Lambda$$

Assume $\sigma$ stabilizes $M$, then $\Lambda$ is diagonal and we have

$$x^\sigma = x; w^\sigma = w; \lambda_{11} y^\sigma = \lambda_{22} y; \lambda_{22} z^\sigma = \lambda_{11} z$$

Observe that the last two identities imply that $yz$ is $\sigma$-invariant. In particular, for any other

$$N' = \begin{pmatrix} x' & y' \\ z' & w' \end{pmatrix}$$

in the image of $\eta$, both $y/y'$ and $z/z'$ are $\sigma$-invariant. As such, after composing $\eta$ with conjugation by

$$\begin{pmatrix} 1 & 0 \\ 0 & y \end{pmatrix}$$

the action of $Stab(M) < Gal(C/\mathbb{Q})$ is trivial on $\eta$.

In case $\sigma$ does not stabilize $M$, then $\Lambda$ is anti-diagonal and

$$\sigma^2 = 1; x = w^\sigma; w = x^\sigma; \lambda_{21} y^\sigma = \lambda_{12} z; \lambda_{12} z^\sigma = \lambda_{21} y$$

Observe that the last two identities imply $q := yz$ is $\sigma$-invariant. Let $r := \lambda_{21}/\lambda_{12}$. We have

$$z = z'^{\sigma^2} = r^\sigma r^{-1} z$$
which implies \( r^\sigma = r \), and \( z = ry^\sigma \). It follows that the image of \( \eta \) is defined over a quadratic extension \( F \) of \( \mathbb{Q} \), with Galois group generated by \( \sigma \), and that its image, consisting of matrices of the form

\[
\begin{pmatrix}
x & y \\
y^\sigma & x^\sigma
\end{pmatrix}
\]

lies in a quaternion algebra \( H_2(F) \).

This concludes the proof of Proposition I.13.

By Proposition I.13, our linearization \( l_\rho \) has values in \( GL_2(\mathbb{Q}) \) or in the units of a quaternion algebra \( H_2(F) \) over a quadratic extension \( F \) of \( \mathbb{Q} \). We can now conclude the proof of Proposition I.1. Indeed we are left with the possibility that \( F \) is an imaginary quadratic field, which we proceed now to exclude.

Lemma I.18. \( l_\rho \) has values in \( GL_2(\mathbb{Q}) \) or in the units of a quaternion algebra \( H_2(F) \) over a real quadratic extension \( F \) of \( \mathbb{Q} \).

Proof. Consider the composite \( \Gamma \to GL_4(\mathbb{R}) \), of \( l_\rho \) followed by the canonical inclusion \( GL_2(\mathbb{C}) \to GL_4(\mathbb{R}) \). We keep denoting its image by \( G \), and we let \( \mathbb{R}[G] \subset M_4(\mathbb{R}) \) be the real sub-algebra it generates. Quite generally, for a \( \mathbb{R} \)-algebra \( R \) and a \( \mathbb{R} \)-module \( M \), we have maps

\[
R \to \text{End}_R(M) = C_{\text{End}_R(M)}(R) \subset \text{End}_R(M)
\]

where \( C \) means centralizer. On the other hand, the centralizer of \( \mathbb{R}[G] \subset M_4(\mathbb{R}) \) acts on \( H^1(\Gamma, \mathbb{C}^2_{l_\rho}) \) non-trivially by definition, and therefore \( \mathbb{R}[G] \) itself acts on \( H^1(\Gamma, \mathbb{C}^2_{l_\rho}) \) non-trivially. Observe that, by Lemma I.12, \( h^1(\mathbb{C}(TX)) = 1 \) while we have a natural isomorphism

\[
H^1(\Gamma, \mathbb{C}^2_{l_\rho}) \cong H^1(X, \mathbb{C}(TX))
\]

So that \( \dim \mathbb{R} H^1(\Gamma, \mathbb{C}^2_{l_\rho}) = 2 \). For a quaternion algebra over a quadratic field \( F \), we have \( H_2(F) \otimes _{\mathbb{Q}} \mathbb{R} = M_2(\mathbb{R}) \) iff \( F \) is real, while \( H_2(F) \otimes _{\mathbb{Q}} \mathbb{R} = H_4(\mathbb{R}) \) iff \( F \) is imaginary. Since the real quaternions \( H_4(\mathbb{R}) \) have no non-trivial 2-dimensional representation, we deduce that \( F \) cannot be imaginary.

This concludes the proof of Proposition I.1.
II. Geometry of affine structures

In order to reach a contradiction using our affine structure, it will be important to study in detail the geometry that governs it. This is the aim of the present section. A substantial part of the presentation is inspired by [Bo82, chapter 3].

Recall that the affine structure on $X$ is defined by a morphism $\tilde{V} := \tilde{X} \to \mathbb{C}^2$, equivariant with respect to a representation $\rho : \Gamma \to \text{Aff}(2, \mathbb{C})$. This representation is uniquely defined by its linearization $l\rho$, whose image we denote by $G := \text{Im}(l\rho)$, along with a cocycle $\omega \in H^1(\Gamma, \mathbb{C}^2_{l\rho})$.

**Remark II.1.** The affine structure is not linear, since otherwise $X$ carries a holomorphic foliation coming from the vector field commuting with linear action of $\Gamma$. In particular the cocycle $\omega$ is non-trivial.

We already know the structure of $l\rho$, but something more can be said about $\rho$:

**Lemma II.2.** There exists a real 2-dimensional subspace $R \subset \mathbb{C}^2$, invariant under $\rho(\Gamma)$.

**Proof.** First we set up some notation. There is a natural embedding $e : \text{Aff}(2, \mathbb{C}) \to \text{Aff}(4, \mathbb{R})$ obtained by considering $\mathbb{C}$ as a real vector space, along with a linearized analogue $le : \text{GL}_2(\mathbb{C}) \to \text{GL}_4(\mathbb{R})$. A trivial inspection reveals that the centralizer of $e(\text{GL}_2(\mathbb{R}))$ is naturally isomorphic to $\text{GL}_2(\mathbb{R})$, and will be referred to as $\text{GL}_2(\mathbb{R})_K$ in the sequel. Indeed, we can represent the elements of $e(\text{GL}_2(\mathbb{R}))$, in some basis, as matrices

$$
\begin{pmatrix}
aI & bI \\
cI & dI
\end{pmatrix}
$$

where $I$ is the $2 \times 2$ identity, and $a, b, c, d \in \mathbb{R}$, while those elements of $\text{GL}_2(\mathbb{R})_K$ can be written, in the same basis, as

$$
\begin{pmatrix}
a & 0 \\
0 & A
\end{pmatrix}
$$

with $A \in \text{GL}_2(\mathbb{R})$. The intersection $\text{GL}_2(\mathbb{R})_K \cap e(\text{GL}_2(\mathbb{C}))$ is the center of $\text{GL}_2(\mathbb{C})$.

Now we can start the proof. By Proposition [1.1] the image of $l\rho$ is inside $\text{GL}_2(\mathbb{R})$, and therefore the composite $le \circ l\rho : \Gamma \to \text{GL}_4(\mathbb{R})$ is a reducible real representation, which splits as a direct sum $\rho_1 \oplus \rho_2$ of isomorphic representations $\rho_i : \Gamma \to \text{GL}_2(\mathbb{R})$. Observe that $\text{GL}_2(\mathbb{R})_K$ acts naturally on the set of such splittings of $le \circ l\rho$. We want to improve this, to show that the whole $e \circ \rho$ can be split into a sum of: a real affine representation valued
in Aff(2, \mathbb{R}), and a linear one valued in GL_2(\mathbb{R}).

If we fix an e(GL_2(\mathbb{R}))-invariant splitting \mathbb{R}^4 = \mathbb{R}^2 \oplus \mathbb{R}^2 then we have a decomposition

(14) \quad H^1(\Gamma, \mathbb{C}_p^2) \cong H^1(\Gamma, \mathbb{R}_p^2) \oplus H^1(\Gamma, \mathbb{R}_p^2)

where, by Lemma I.12, each space on the right is real 1-dimensional; correspondingly we obtain that the cocycle \omega, defining \rho, splits in a pair \omega_1 \oplus \omega_2 of real-valued cocycles.

The center of GL_2(\mathbb{C}) acts by scalar multiplication on \mathbb{H}^1(\Gamma, \mathbb{C}_p^2), and hence the whole GL_2(\mathbb{R})_K acts transitively on \mathbb{H}^1(\Gamma, \mathbb{R}_p^2) \oplus \mathbb{H}^1(\Gamma, \mathbb{R}_p^2) \setminus \{(0,0)\}. In particular, since \omega \neq 0, for some e(GL_2(\mathbb{R}))-invariant splitting of \mathbb{R}^4 we have \omega_1 = 1 and \omega_2 = 0, so then the action of \Gamma on \mathbb{R}^4/\mathbb{R}_p^2 is linear. The existence of \mathbb{R} follows. \hfill \Box

We continue by inspecting those elements commuting with the image of \rho, in the whole affine group. From the existence of \mathbb{R}, indeed by the previous proof, it follows that the image of \rho lies in the extension of \mathbb{e}(GL_2(\mathbb{R})) by the subgroup of translations preserving \mathbb{R}. This extension is clearly isomorphic to Aff(2, \mathbb{R}). We denote its image in Aff(2, \mathbb{C}) by Aff(2, \mathbb{R})_\Delta, and by G_K its centralizer in Aff(2, \mathbb{C}).

**Lemma II.3.** \( G_K \) is the subgroup of GL_2(\mathbb{R})_K acting trivially on \( \mathbb{R} \). It is isomorphic to Aff(1, \mathbb{R}).

**Proof.** \( G_K \) acts on \( \mathbb{R} \), and since Aff(2, \mathbb{R}) has no center, this action must be trivial. This implies that \( G_K < GL_2(\mathbb{R})_K \). Moreover, any \( g \in GL_2(\mathbb{R})_K \) commutes with the linear part of any \( h \in Aff(2, \mathbb{R})_\Delta \) by definition. Hence \( t := ghg^{-1}h^{-1} \) is a translation. If moreover \( g \) acts trivially on \( \mathbb{R} \), also \( t \) does, so then \( t = 0 \) and \( g \in G_K \). That \( G_K \) is isomorphic to Aff(1, \mathbb{R}) is clear at this point, since it consists of matrices that can be written, in the above basis, as

\[
\begin{pmatrix}
1 & a \\
0 & b \\
0 & 1 \\
0 & b
\end{pmatrix}
\]

and of course \( \mathbb{R} \) is spanned by \((1,0,0,0)\) and \((0,0,1,0)\). \hfill \Box
We have a commutative diagram of groups with exact rows and columns:

\[
\begin{array}{c}
\text{1} \\
\downarrow \\
\text{1} \\
\end{array}
\quad A \subset \mathbb{R}^2
\quad
\begin{array}{c}
\downarrow \\
\end{array}
\quad
\begin{array}{c}
1 \\
\text{1} \\
\end{array}
\quad
\begin{array}{c}
\longrightarrow \\
K \\
\end{array}
\quad
\Gamma \xrightarrow{\rho} G_A \subset \text{Aff}(2, \mathbb{R})
\quad
\begin{array}{c}
\downarrow \\
\end{array}
\quad
\begin{array}{c}
1 \\
\text{1} \\
\end{array}
\quad
\begin{array}{c}
\longrightarrow \\
K_A \\
\end{array}
\quad
\Gamma \xrightarrow{l_\rho} G \subset \text{GL}_2(\mathbb{R})
\quad
\begin{array}{c}
\downarrow \\
\end{array}
\quad
\begin{array}{c}
A \\
\text{1} \\
\end{array}
\quad
\begin{array}{c}
\downarrow \\
\end{array}
\quad
\begin{array}{c}
1 \\
\end{array}
\end{array}
\]

where: $G_A$ is the image of $\rho$ in $\text{Aff}(2, \mathbb{R})_\Delta$, which is identified with $\text{Aff}(2, \mathbb{R})$; $A$ is the subgroup of translations inside $G_A$.

We switch our attention to the action of $G_K$ on $\mathbb{C}^2$. We fix a splitting $\mathbb{C}^2 = R \oplus iR$. For a point $x \in \mathbb{C}^2 \setminus R$, there exists a unique line $L_x \subset R$ such that: it is a translation of a line contained in $iR$; it contains $x$; it intersects $R$. Such line has a complexification $C_x := \mathbb{R}C_x$, which intersects $R$ along $R_x$.

**Lemma II.4.** The orbit of $x$ under $G_K$ is $L_x^C \setminus R_x$.

**Proof.** Say $x = (x_1, x_2, x_3, x_4)$ and $g \in G_K$ is as in the previous proof, then $g(x) = (x_1 + ax_2, bx_2, x_3 + ax_4, bx_4)$. We claim: $L_x = \{(x_1, \lambda x_2, x_3, \lambda x_4), \lambda \in \mathbb{R}\}$. Indeed it is a translation by $(x_1, 0, x_3, 0)$ of a line in $iR$; it contains $x$; it intersects $R$. Such line goes through $x$ at $\lambda = 1$. Finally, its complexification is clearly $\{(x_1 + \mu x_2, \lambda x_2, x_3 + \mu x_4, \lambda x_4), \lambda, \mu \in \mathbb{R}\}$.

The previous lemma can be rephrased by saying that the quotient space of $\mathbb{C}^2 \setminus R$ under the action of $G_K$ is canonically the Grassmannian manifold, $Gr(1, 2)$, of affine lines in $R$. There is a fiber bundle structure $s : Gr(1, 2) \to S^1$, where each fiber parametrizes a family of parallel lines in $R$. $Gr(1, 2)$ is non-orientable and homeomorphic to a Moebius band. The quotient of $\mathbb{C}^2 \setminus R$ by $G_K^+$, a connected component of $G_K$, is a cylinder mapping to $Gr(1, 2)$ as its canonical orientable double covering. We denote such covering by $Gr(1, 2)^o$.
to lift this setup on $V(=\tilde{X})$ under the unramified developing map, introduced in equation $\Pi$ $p_{aff}: V \to \mathbb{C}^2$, it is natural to introduce the closed submanifold $V_R := p_{aff}^* R$. A crucial observation is that, since $\rho(\Gamma)$ and $G_K$ commute, the action of $\rho(\Gamma)$ descends naturally to the orbit space $Gr(1,2)$. Moreover, since $G_K^+$ is connected and simply connected, its action lifts to $V \setminus V_R$, and commutes with $\Gamma$. Since $G_K^+$ acts freely on $\mathbb{C}^2 \setminus R$, it acts freely on $V \setminus V_R$. Let $S$ denote the quotient of $V \setminus V_R$ by $G_K^+$. The projection $S \to Gr(1,2)^o$ is a local diffeomorphism, and $S$ is an open real surface. Since $V_R$ is closed in $V$ and $\Gamma$-invariant, its image $X_R$ in $X$ is a finite union of compact, real surfaces. We deduce that $X \setminus X_R$ is an Eilenberg-MacLane space. Observe that the factorization $S \to S/K \to Gr(1,2)^o$ gives $S/K$ a natural structure of open orientable topological surface. In particular $K$ acts discontinuously on $S$.

We summarize in a commutative diagram:

$$
\begin{array}{ccc}
V \setminus V_R & \xrightarrow{G_K^+} & S \\
\| & & \| \\
V \setminus V_R & \xrightarrow{p_{aff}} & S/K \\
\| & q & \| \\
\mathbb{C}^2 \setminus R & \xrightarrow{G_K^+} & Gr(1,2)^o \\
\downarrow & s & \downarrow \\
S^1 & \leftarrow & \mathbb{R}
\end{array}
$$

(15)

Where the rightmost horizontal arrows are universal coverings.

**Lemma II.5.** The intersection of $\rho(G) \subset A(2,\mathbb{R})$ with the normal subgroup of translation, is either:

1. *Trivial*, or
2. *Infinitely generated dense subgroup of $\mathbb{R}^2$*

**Proof.** The crucial observation is that the character corresponding to the canonical bundle $K_X$ is a non-trivial morphism $\Gamma/[\Gamma,\Gamma] \to \mathbb{R}^*$, whose image contains an infinite cyclic subgroup of finite index. Hence there is an element $s \in l \rho(\Gamma)$ with $\det(s)^2 = p/q < 1$ and $q \neq 1$. Since the intersection of $\rho(\Gamma)$ with translations $\mathbb{R}^2$ is invariant under $l \rho(\Gamma)$, if it is non-trivial, then it contains a group isomorphic to $\mathbb{Z}_{[\frac{1}{q}]}$. Since the representation is irreducible, it must be dense as well. \qed
Consider the case in which the intersection $\rho(\Gamma) \cap \mathbb{R}^2 = (0)$, namely $l_{\rho}(\Gamma) \sim_{\rightarrow} \rho(\Gamma)$.

**Lemma II.6.** Assume that $l_{\rho}(\Gamma)$ contains a non-trivial homothety $h$. Then the intersection $\rho(\Gamma) \cap \mathbb{R}^2$ has to be infinitely generated.

**Proof.** In this case $\rho(\Gamma)$ is isomorphic to $l_{\rho}(\Gamma)$ and hence commutes with $h$. Thus the image $\rho(\Gamma)$ is contained in $\text{Centralizer}(h) \subset A(2, \mathbb{R})$ which is $\text{GL}_2(\mathbb{R})_h \subset A(2, \mathbb{R})$. Hence the representation $\rho$ is linear, contradicting remark II.1. □

We proceed to a detailed study of our surface $S$ and its map to $\text{Gr}(1,2)^o$. Observe that the pre-image of any point in $\mathbb{R}$ under $s$ is a real line, corresponding, in Grassmannian, to a family of parallel lines in $\mathbb{R}^2$. Therefore $s$ is the universal covering of the angle map. Moreover, the pre-image of a point in $\mathbb{R}$ under $s \circ \tilde{q}$ is a union of open segments, half-lines, or complete lines, and each of them embeds naturally into the corresponding line in Grassmannian.

Thus we can split our study into two cases:

1. The map $V \setminus V_R \to \mathbb{C}^2 \setminus \mathbb{R}$ is an isomorphism and hence $\pi_1(V \setminus V_R)$ is discrete subgroup of $\tilde{A}(2, \mathbb{R})$ and the quotient $(V \setminus V_R)/G_K^+ = \tilde{G}(1,2)^o$, i.e cyclic non-ramified covering of the Grassmanian.
2. The map $V \setminus V_R \to \mathbb{C}^2 \setminus \mathbb{R}$ is not an isomorphism and hence the map $(V \setminus V_R)/G_K^+ = S \to \tilde{G}(1,2)^o$ is a $\Gamma$-equivariant map of open surfaces, but not a covering map.

We will call the first case classical, and the second case pathological.

**III. $V_R = \emptyset$**

In this section we exclude the possibility that $V_R = \emptyset$. Let us show how general steps of the proof.

In the classical case group $\Gamma$ must be a discrete subgroup of $\tilde{A}(2, \mathbb{R})$. If $V_R = \emptyset$ then the universal covering of $X$ is equal to $\mathbb{C}^2 \setminus \mathbb{R}$ and contractible. Hence cohomological dimension of $X$ which is equal to 4 must be equal to cohomological dimension of $\Gamma < \tilde{A}(2, \mathbb{R})$. The following Theorem [III.1] which is of independent interest, is implemented in Proposition [III.5] to find a contradiction.

In the non-classical case, the induced angle map $S \to \mathbb{R}$ must have some disconnected fibers. A careful study of their connected components, along with the geometry of corresponding stabilizers in $\Gamma$, lead to the required contradiction.
Theorem III.1. Let $\Gamma$ be a finitely generated discrete subgroup of $\tilde{A}(2,K) \subset \tilde{A}(2,R)$, $K$ a number field. Assume that:

1. The cohomological dimension of $\Gamma$ is 4.
2. The linearization $l(\Gamma) \subset GL(2,R)$ has no invariant subspace in $C^2$.
3. For any subgroup of finite index $\Gamma' < \Gamma$, the image of the map $\det : l(\Gamma') \to R^*$ is infinite.

Then the group contains a finite index subgroup $\Gamma'$ which intersects the center $Z \subset \tilde{A}(2,R)$ non-trivially. Moreover the quotient group $l(\Gamma') \subset GL(2,R)$ sits in an extension:

$$1 \to \pi_g \to l(\Gamma) \to Z \to 1$$

where $\pi_g$ denotes the fundamental group of a compact Riemann surface of genus $g \geq 2$

Proof. We start with a:

Lemma III.2. The above condition 2) implies that the intersection $l(\Gamma)_1 := l(\Gamma) \cap SL(2,R) \subset GL(2,R)$ is discrete.

Proof. Assume that $l(\Gamma)_1$ is dense in $SL(2,R)$, then it contains two non-commuting conjugated elements $h_1, h_2$ of infinite order, whose complex eigenvalues $\lambda, \bar{\lambda}$ are not roots of unity, and satisfy $|\lambda| = 1$. This follows easily from the fact that the collection of roots of unity contained in a quadratic extension of $K$ is finite. In particular, since $h_1, h_2$ have infinite order, if $h_1^N = h_2^N$ for some $N$ then $h_1, h_2$ correspond to the same invariant metric on $R^2$ and represent rotations with coincident centers, in other words they commute. Thus for any $\varepsilon > 0$, we can find an integer $n = n(\varepsilon)$ such that $h_1^n = e^\varepsilon g_1^n, h_2^n = e^\varepsilon g_2^n$, where $e^\varepsilon \in Z \subset \tilde{A}(2,R)$ is a central element, and both $g_1^n$ are in an $\varepsilon$-neighborhood of identity in $\tilde{A}(2,R)$. Note that since $h_1^n, h_2^n$ do not commute, the product $h_1^n h_2^{-n} = g_1^n (g_2^n)^{-1} \in \Gamma$. However, the elements $g_1^n (g_2^n)^{-1}$ converge to identity, when $\varepsilon \to 0$, in $\tilde{A}(2,R)$, which contradicts discreteness of $\Gamma \subset \tilde{A}(2,R)$. This proves that $l(\Gamma)_1$ is not dense.

Assume now that the closure of $l(\Gamma)_1$ in $SL(2,R)$ contains a proper non-trivial connected Lie subgroup $G_0$. Then the representation induced on $G_0$ into $C^2$ contains an invariant 1-dimensional complex subspace, which contradicts the assumption of the theorem. \(\square\)

Remark III.3. In our proof we used the fact that $l(\Gamma)$ is contained in $GL(2,K)$ for some number field, but a similar result holds without this assumption.
Moreover, an analogue of this result holds for any discrete subgroup of the universal covering
$\tilde{\text{ASp}}(2n, \mathbb{R})$ of affine extension $\text{ASp}(2n, \mathbb{R})$ of symplectic linear group. Indeed, rotations constitute an open subset in $\text{ASp}(2n, \mathbb{R})$.

**Lemma III.4.** $\Gamma$ is central $\mathbb{Z}$-extension of the fundamental group of a three-dimensional compact manifold, which fibers over $S^1$ with a compact Riemann surface as fiber.

**Proof.** Since the image of $\text{det}$ is non-trivial, the intersection of $\Gamma$ with the subgroup of translations $\sim \mathbb{R}^2$ must be trivial. Indeed $\Gamma \cap \mathbb{R}^2$ is invariant under $l(\Gamma)$, and the action of $\text{det}(l(\Gamma))$ shows that $\Gamma$ cannot be discrete unless $\Gamma \cap \mathbb{R}^2 = 0$. Since $l(\Gamma)_1$ is discrete by the previous Lemma, it is either free or $\pi_g$ with $g \geq 2$. Since the kernel of the projection $l(\Gamma) \to l(\Gamma)_1$ is non-trivial, cohomological dimensions satisfy $\text{cd}(l(\Gamma)) = \text{cd}(l(\Gamma)_1) + 1$, thus $\text{cd}(l(\Gamma))$ is either 2 or 3. Since $\text{cd}(\Gamma) = 4$, necessarily we have: $l(\Gamma)_1 = \pi_g$ and $\Gamma$ has non-trivial cyclic kernel under composite projection $\tilde{A}(2, \mathbb{R}) \to A(2, \mathbb{R}) \to \text{GL}(2, \mathbb{R})$. Thus the only possibility for $\Gamma$ satisfying the assumptions of the theorem is a central $\mathbb{Z}$-extension of the fundamental group of three-dimensional compact manifold, fibered over $S^1$, with a compact Riemann surface as a fiber. \hfill \Box

This concludes the proof of the Theorem. \hfill \Box

Now we can exclude the classical case in which $\Gamma$ is discrete.

**Proposition III.5.** There are no $VII_0$ surfaces with $V_R = \emptyset$, and affine complex structure inducing an affine representation of the fundamental group which satisfies the assumptions of the previous Theorem.

**Proof.** Let $X$ be such a surface. Then it is an Eilenberg-Maclane space and its fundamental group $\Gamma$ satisfies the conclusion of the previous Theorem. Consider the action of the cyclic quotient of $\Gamma$ on the surface group $\pi_g$. Then a generator, $h$, acts naturally as an element of $Sp(2g, \mathbb{Z})$. Given any eigenvalue $\lambda$ of $h$, also $\bar{\lambda}$, $\lambda^{-1}$ and $\bar{\lambda}^{-1}$ are eigenvalues of $h$. Moreover, each of them is an algebraic integer. For such $\lambda$ we obtain a character $\lambda : \mathbb{Z} \cdot h \to \mathbb{C}^*$, and we have $\dim(H^1(\Gamma, \mathcal{C}_\lambda)) = \dim(H^1(X, \mathcal{O}(\mathcal{C}_\lambda))) = 1$ Hence $\mathcal{O}(\mathcal{C}_\lambda)$ is equal either $\mathcal{O}_X$ or $\mathcal{O}(K_X)$ by lemma I.7.

**Claim III.6.** We have $\lambda = 1$ and $\mathcal{O}(\mathcal{C}_\lambda) = \mathcal{O}$.

**Proof.** Since $K_X$ is defined by a non-trivial rational character, if $\mathcal{O}(\mathcal{C}_\lambda) = \mathcal{O}(K_X)$ then $\lambda \in \mathbb{Q} \setminus \{\pm 1\}$, which implies that $\mathcal{O}(\mathcal{C}_\lambda) = \mathcal{O}(K_X) = \mathcal{O}(\mathcal{C}_{\lambda^{-1}})$, absurd. \hfill \Box
Thus all eigenvalues of $h$ are 1, and there is a non-trivial map $\pi_g \to \mathbb{Z}$ which extends to a map $\Gamma \to \mathbb{Z}$. In particular $\text{rk}(H^1(X, \mathbb{Q})) \geq 2$ which contradicts the properties of $VII_0$ surfaces. This proves that the image $\tilde{\rho} : \pi_1(X) \subset \tilde{AGL}(2, \mathbb{R})$ cannot be a discrete subgroup in $\tilde{AGL}(2, \mathbb{R})$. □

**Remark III.7.** The above argument was earlier elaborated by Bombieri, who proved that the commutator $[\pi_1(X), \pi_1(X)]$ is infinitely generated.

In case $\rho(\Gamma)$ is not discrete inside $\tilde{A}(2, \mathbb{R})$, we study the structure of $\Gamma$ via the Stein factorization of the induced angle map $s : S \to \mathbb{R}$. This means, by definition, that there exists a topological space $S_s$ and a factorization $S \xrightarrow{s_1} S_s \xrightarrow{s_2} \mathbb{R}$ such that: the composite $s_2 \circ s_1 = s$; the map $s_1$ has connected fibers; the fibers of the map $s_2$ are in bijection with the set of connected components of the corresponding fibers of $s$.

Observe that, since the angle map $s$ is not proper, the topology of $S_s$ can be very strange, e.g. it might fail to be Hausdorff. However, on the positive side $S_s$ contains no cycles since $S$ is simply connected, and moreover $\Gamma$ acts naturally on $S_s$, each orbit being a locally finite tree.

We proceed to understand the structure of $\Gamma$ by carefully studying its action on $S_s$. Observe that if $\Gamma$ acts freely on one of its orbits, then it is free. Hence we can assume that there is $x \in S_s$ whose stabilizer $\text{Stab}_x$ is non-trivial. Since $x$ corresponds to an interval in $Gr(1, 2)$ which is contracted to a point by the angle map, necessarily $l\rho$ embedds $\text{Stab}_x$ into upper triangular matrices. In order to move on, we need a couple of simple definitions:

**Definition III.8.** Let $x, y \in S_s$. Then we say $x < y$ if there exists a continuous map $f : [0, 1] \to S_s$ with $f(0) = x$, $f(1) = y$, and the induced $s_2 \circ f : [0, 1] \to \mathbb{R}$ increasing. If none among $x < y$ and $y < x$ holds, then we write $x \not> y$.

Since $S_s$ contains no cycles, the relation $<$ defines a partial ordering on $S_s$.

**Definition III.9.** A point $x \in S_s$ is called a ramification point if, for every $\epsilon > 0$ and every neighborhood $U_x \subset S_s$ of $x$, there exist two points $y, z \in U_x$ with $y \not> z$ and $s_2(y), s_2(z) \in (s_2(x), s_2(x) + \epsilon)$.

We have two cases:

1. There exists $x \in S_s$ which is not ramification;
2. Any point in $S_s$ is ramification.
Let us deal with the first case. There is a maximal open segment $I \subset S_x$ which contains $x$ and no ramification points. Its pre-image $U \subset V$ is an open subvariety of $V$, with the property that for any $\gamma \in \Gamma$, either $\gamma U = U$ or $\gamma U \cap U = \emptyset$. Hence the image of $U/\text{Stab}_U$ is an open subvariety of $X$, whose compact topological boundary in $X$ is equal to the quotient $\partial U/\text{Stab}_U$ of the topological boundary of $U$ in $V$.

- if $s_2(I) = \mathbb{R}$ then the strip $I \times \mathbb{R}$ is a connected component of, hence equal to, $S$. This implies that $S$ maps homeomorphically to an open subset of $\tilde{\text{Gr}}(1, 2)$, and hence $\Gamma \subset \tilde{A}(2, \mathbb{R})$ is discrete - contradiction.
- If $s_2(I) \neq \mathbb{R}$ and the boundary of $s_1^{-1}I$ is empty, then the previous argument also works.
- If $s_2(I) \neq \mathbb{R}$ and $s_1^{-1}I$ has non-empty boundary, which is disconnected by assumption. Hence the pre-image $s_1^{-1}(I)$ has non-empty boundary consisting of the disjoint union of at least two connected segments, or half-lines, inside $R_x$ - where $x$ is the boundary of $s_2(I)$. Let $J$ be one such connected component. Then its stabilizer $\text{Stab}_J \subset \rho(G)$ cannot contain any translations, hence it is fully contained in the upper-triangular group. Moreover, its pre-image $G^+_K \times J \subset V$ projects onto a compact real 3-manifold in $X$. We show that $\text{Stab}_J$, being upper triangular, cannot act co-compactly on $G^+_K \times J$.

The group of upper-triangular matrices $UT$ has a normal series $T \subset N \subset UT$ where $T$ is a group of translations and $N$ is kernel of projections to diagonal $2 \times 2$ matrices. Since our discrete subgroup $\text{Stab}_J$ of $UT$ injects, under projection $p : UT \to UT/T$, into $UT/T$ - which is $2 \times 2$ upper-triangular group - there are the following options:

1. The intersection $p(\text{Stab}_J) \cap N/T$ is nontrivial;
2. The intersection above is trivial.

In the first case the intersection above is infinitely generated unless the projection to $UT/N$ consists of scalar matrices only. In the latter case $\text{Stab}_J$ is abelian, and its action is non-discrete on $R \times G^+_K$ unless $\text{Stab}_J$ is contained in $\mathbb{Z} + \mathbb{Z}$. But then $\text{cd}(\text{Stab}_J) \leq 2$, contradicting the assumption on $\text{Stab}_J$.

In the former case the intersection with $N/T$ is infinitely generated, and since the corresponding intersection is abelian it again contradicts discreteness.

In the second case, namely the intersection with $N/T$ is trivial, then $\text{Stab}_J$ embeds
into diagonal group $UT/N$ and hence is abelian and simultaneously diagonalizable. Thus again the action is non-discrete unless $\text{Stab}_J \subset \mathbb{Z} + \mathbb{Z}$, contradicting assumption on cohomological dimension.

We can now focus on the second case, namely every point in $S_s$ is ramification. The space $S_s$ is one-dimensional, non-Hausdorff generalized tree. Note that using Serre’s argument on groups acting on tree, we obtain that if the stabilizer $\Gamma_s$ of any vertical segment $I_s \subset S$ has cohomological dimension $\text{cd}(\Gamma_s) \leq 2$, then $\text{cd}(\Gamma) \leq \text{cd}(\Gamma_s) + 1 \leq 3$. In particular, there are segments $I_s$ with $\text{cd}(\Gamma_s) \leq 3$. The group $\Gamma_s$ is a discrete subgroup of the group $UT_s$ of upper-triangular matrices mapping the three-dimensional manifold $I_s \times G_K^+$ into itself. The assumption on the $\text{cd}(\Gamma)$ implies that $(I_s \times G_K^+)/\Gamma_s$ is a compact three-dimensional real manifold and we can use the methods of the previous argument to describe the geometry of the action.

Let $T_s \subset N_s \subset UT_s$ be subgroups of translations, nilpotent elements and upper-triangular matrices mapping $I_s$ into itself. Note that $UT_s \subset UT$, and the groups $T_s \subset N_s$ correspond to the intersection of $UT_s$ with the corresponding members of the filtration $T \subset N$ in $UT$. If $\Gamma_s$ contains a translation, there is an interval $I_s \subset S$ such that $I_s \times \mathbb{R} \subseteq S$, and hence ramification points are not dense in $S_s$, absurd.

If $\Gamma_s$ does not contain translations then it embeds into $D_s/T_s$ under projection $p : UT_s \to UT_s/T_s$. If $p(\Gamma_s) \cap N_s$ is nonempty then it is invariant under the action of the image of $\Gamma_s$ in $UT_s/N_s$. If this action is non-trivial then the intersection of $\Gamma_s$ with $N_s$ is a non-discrete subgroup of $D_s$, contradiction. If the action is trivial then $\Gamma_s$ is abelian and hence $\Gamma_s$ is contained in a maximal abelian subgroup of $UT_s$. All such subgroups have cohomological dimension two, again contradiction.

There is also a set-theoretic argument to deal with the case $S_s$ consists of ramification points. Such assumption indeed implies that at least one fiber of $S_s \to \mathbb{R}$ is an uncountable set. This can be sketched as follows. Let $x \in S_s$ be any point and $U$ an open neighborhood. Then for any $x_1 \in U$, away from a subset of measure zero, with $s_2(x_1) > s_2(x)$, necessarily $s_2^{-1}s_2(x) \cap U$ contains at least two points. Similarly if $x_2 \in U$, away from a subset of measure zero, with $s_2(x_2) > s_2(x_1)$, then $s_2^{-1}s_2(x_2) \cap U$ contains at least four points. Inductively we can construct a sequence $\{x_n\} \subset U$ with $s_2(x_n)$ increasing, and $s_2^{-1}s_2(x_n)$ containing at least $2^n$ points. If such sequence $\{x_n\}$ is sufficiently generic, i.e. away from a subset of measure zero in the space of Cauchy sequences in $S_s$, then any accumulation point $x_\infty$ is such that $s_2^{-1}s_2(x_\infty)$ is uncountable.
IV. \( V_R \neq \emptyset \)

In this section we exclude the possibility that \( V_R \neq \emptyset \), thereby proving the Main Theorem. Before getting into the proof, let us recall that the only real surfaces carrying an affine structure have vanishing Euler characteristic, hence are the torus, and the Klein bottle, which is covered by a torus. The affine structures on a torus are classified in [NY74]. In fact they correspond to discrete rank-2 abelian subgroups inside connected abelian Lie subgroup of the universal cover \( \tilde{A}(2, \mathbb{R}) \) of the affine group. There are exactly five distinct affine structures on 2-dimensional real tori:

(i) A lattice in \( \mathbb{R}^2 \)
(ii) A discrete cyclic group in \( \mathbb{R}^2 \setminus \{0\} \)
(iii) A rank-2 discrete subgroup of \( \mathbb{C} \subset \tilde{A}(2, \mathbb{R}) \). Under the universal cover map \( \tilde{A}(2, \mathbb{R}) \to A(2, \mathbb{R}) \), the image of such rank 2 discrete subgroup to a non-discrete rank-2 subgroup of \( \mathbb{C}^* \)
(iv) A lattice in \( \mathbb{R}^* \times \mathbb{R}^* \)
(v) A lattice in \( \mathbb{R} \times \mathbb{R}^* \)

There is a well defined notion of line in a real torus \( T \) with affine structure. In general such lines are affinely isomorphic either to complete lines, or half lines, or closed intervals in \( \mathbb{R}^2 \). Note that in case (i) all lines in \( T \) correspond to complete lines in \( \mathbb{R}^2 \). Similarly in cases (ii), (iii) all lines in \( T \) which correspond to lines not passing through \( 0 \in \mathbb{R}^2 \) are affinely isomorphic to complete lines, and hence this property is violated only by a codimension 1 subset of lines in \( Gr(1, 2) \). In case (iv) we have complete lines and half lines, while in case (v) we have half lines and closed intervals.

Now we set up some notation we need in the proof. Denote by \( V_R \) the pre-image of \( R \) inside \( V \), and by \( X_R \) its image in \( X \). \( X_R \) is a disjoint union of a finite number of real, compact surfaces that inherit an affine structure from \( X \). We can assume therefore that every component of \( X_R \) is a torus. Correspondingly, each connected component of \( V_R \) maps homeomorphically to \( R \), and the image can be \( R, R \setminus 0 \), an open half-space \( H \), or a quadrant \( Q \) - by the above remark on affine structures on tori.

Denote by \( V_R^i \) the connected components of \( V_R \) - which we can assume to be real 2-tori - and for each \( i \) let \( U_i = \{ x \in V \text{ s.t. } G_K \cdot x \cap V_R^i \neq \emptyset \} \). This is a \( G_K \)-invariant open neighborhood of \( V_R^i \).
Lemma IV.1. Let \( x \in V \setminus V_R \) be any point, and \( G_K^+ \cdot x \) its orbit. Then the boundary of the closure of \( G_K^+ \cdot x \) is a disjoint union of segments, which are simultaneously embedded into \( R \subseteq \mathbb{C}^2 \) under \( p_{\text{aff}} \).

Proof. Indeed the restriction of \( p_{\text{aff}} \) on any \( G_K^+ \)-orbit is an embedding, while the action of \( G_K^+ \) on \( V \setminus V_R \) is free, and hence the boundary of such orbits is contained in \( V_R \). Such boundary embeds into the boundary of the closure of \( G_K^+ \cdot p_{\text{aff}}(x) \subseteq \mathbb{C}^2 \), which is the line \( R_{p_{\text{aff}}(x)} \) corresponding to the point \( p_{\text{aff}}(x) \).

We define \( R^i_x \) the segment of line \( R_{p_{\text{aff}}(x)} \) which is contained in \( U_i \). If \( x \) is contained in several such \( U_i \)'s, then the disjoint union of \( R^i_x \) embeds into \( R_x \) under \( p_{\text{aff}} \).

Corollary IV.2. Assume that \( g(U_i) \cap U_i \) is nonempty and \( g(U_i) \neq U_i \). Let \( x \in g(U_i) \cap U_i \) then the closure \( G_K^+ \cdot x \subseteq V \) intersects both \( V_R^i \) and \( g(V_R^i) \). Moreover the union of the corresponding line segments \( R^i_x, R^g(i)_x \subseteq g(V_R^i) \) maps homeomorphically onto a strictly proper subset of \( R_x \).

The proof follows from the fact that the closure of \( G_K^+ \cdot x \) contains both the closure of \( G_K^+ \cdot x \) in \( V_R^i \) and \( g(V_R^i) \) which adds up nonintersecting segments \( R^i_x \) and \( R^g(i)_x \) embedded under \( p_{\text{aff}} \) into the same real line \( R_x \).

We have to consider several cases, and we are going to do it using the following simple statements.

Lemma IV.3. The action of \( \Gamma \) on \( S = (V \setminus V_R)/G_K^+ \) is nowhere discrete.

Proof. Assume there is a domain \( \Omega \subseteq S \) which has trivial intersection with its \( \Gamma \)-orbit. For any \( x \in \Omega \), denote by \( x^* \in V \setminus V_R \) any point in \( p_{\text{aff}}^{-1}(x) \). Then the image of \( G_K^+ \cdot x^* \) under the cover \( u: V \setminus V_R \to X \setminus X_R \) in \( X \setminus X_R \) is closed and hence the closure of \( u(G_K^+ \cdot x^*) \) in \( X \) is contained in \( u(G_K^+ \cdot x^*) \cup X_R \). This is clearly impossible if the stabilizer of \( x \), \( \text{Stab}_x < \Gamma \) is trivial. But such stabilizer is certainly trivial for \( x \) outside a countable subset of \( \Omega \).

We also need the following statement:

Lemma IV.4. Assume that \( U \) is a domain in \( V \) such that \( p_{\text{aff}} \) restricts to a covering map \( U \to U' := p_{\text{aff}}(U) \) with the property that \( \dim(\partial U') < 3 \). Then \( V = \hat{U} \).

Proof. If \( \hat{U} \subseteq V \), then \( \partial U \) had dimension 3, and hence the same holds for \( \partial U' \) since \( p_{\text{aff}} \) is a local isomorphism on \( U \).
We can now start to exclude the various possibilities for affine structures on $X_R$. We have:

**Definition IV.5.** An open domain $U \subset V$ is called $\Gamma$-fundamental if for any $g \in \Gamma$, we have: either $U \cap g(U) = \emptyset$ or $g(U) = U$.

**Corollary IV.6.** In cases (i),(ii),(iii) $U_i$ is $\Gamma$-fundamental.

This follows from the previous Corollary [IV.2] and the fact that all lines, but perhaps a codimension 1 subset of such, map isomorphically onto the full line $R_x \subset R \subset \mathbb{C}^2$ under $p_{\text{aff}}$ in cases (i),(ii),(iii).

**Lemma IV.7.** The affine structure on $V_R^i$ cannot be of type (i),(ii) or (iii).

**Proof.** Note that $U_i$ in the case (i) maps isomorphically to $\mathbb{C}^2$. Similarly in case (ii) $U_i$ maps isomorphically onto $\mathbb{C}^2 \setminus \{0\}$, and in case (iii) $U_i$ maps as a universal covering of $\mathbb{C}^2 \setminus \{0\}$, and hence is an isomorphism. From Lemma [IV.4] we deduce that in all these cases $U_i = V$. But then $\Gamma$ stabilizes $V_R^i$, hence $\Gamma$ is almost abelian, i.e. contains an abelian subgroup of finite index coming from the action on $V_R^i$. Hence $l\rho$ is reducible on a finite étale cover of $X$, contradicting Lemma [I.5].

Now we handle cases (iv) and (v). The issue is that $U_i$ may not be $\Gamma$-fundamental. Consider the corresponding geometry in $\mathbb{C}^2$. In case (iv) we can assume that $V_R^i \subset R$ is given in real coordinates $x_1, x_2$ as a the first quadrant $Q = \{x_1 > 0, x_2 > 0\}$. Then $U_i$ embeds into $\mathbb{C}^2$ by $p_{\text{aff}}$ and the image $p_{\text{aff}}(U_i)$ can be identified with the union of $Q$, and the open subset obtained as a union of complex lines $C_x \setminus R_x$ - where $C_x$ denotes the complexification of the real line $R_x \subset R$, and the union is extended to those points $x$ such that $R_x \cap Q \neq \emptyset$. In particular, $U_i$ contains $T_i := \{\text{Re}(z_1) > 0, \text{Re}(z_2) > 0\} \supseteq Q \times \mathbb{R}^2$ as a subdomain. The domain $T_i$ is invariant under the action of the subgroup of finite index $\mathbb{Z} + \mathbb{Z} \to \Lambda_i \subset \text{Stab} U_i$. Such action is co-compact on $Q$ and discrete on $T_i$. Indeed $\Lambda_i$ acts discretely on $Q$ and $T_i/\Lambda_i$ is an $\mathbb{R}^2$-fiber bundle over $V_R^i/\Lambda_i$.

In case (v) we can assume that $V_R^i \subset R$ is given in real coordinates $x_1, x_2$ as a half-plane $Q = \{x_2 > 0\}$. We have $p_{\text{aff}}(U_i) = \mathbb{C}^2 \setminus \bigcup C_x$, where $C_x$ is complexification of $R_x$ given by equation $\{x_2 = c\}$ for $c \in \mathbb{R}_{\leq 0}$. Then $U_i$ contains a subdomain $T_i \supseteq Q \times \mathbb{R}^2$ given by equation $\{\text{Re}(z_2) > 0\}$. The domain $T_i$ is invariant under the action of a subgroup $\mathbb{Z} + \mathbb{Z} \to \Lambda_i \subset \text{Stab} U_i$ which acts co-compactly on $Q$. The action of $\Lambda_i$ on $T_i$ is discrete.
with $T_i/\Lambda_i = V_R/\Lambda_i \times \mathbb{R}^2$. Up to replacing $\Gamma$ by a subgroup of finite index, we can assume that $\Lambda_i = \text{Stab} U_i$.

**Lemma IV.8.** There is a neighborhood $V^i_R/\Lambda_i \subset T_i/\text{Stab} U_i$ which is isomorphic to the neighborhood of $V^i_R/\Lambda_i = Q/\Lambda_i$ in $T_i/\Lambda_i$.

**Proof.** Indeed the map $T_i/\Lambda_i \to X$ is a local isomorphism which restricts to a global isomorphism $V^i_R/\Lambda_i \to X^i_R$ as well as on their normal bundles. Since $X^i_R$ is compact we obtain a local isomorphism of small neighborhoods $U(X^i_R) \subset X$ and $U(V^i_R/\Lambda_i) \subset T_i/\Lambda_i$.

**Corollary IV.9.** Let $\tilde{U}(V^i_R/\Lambda_i) \subset T_i$ denote the pre-image of $U(V^i_R/\Lambda_i)$. It contains $V^i_R$. If for some $x \in \tilde{U}(V^i_R/\Lambda_i)$ and $g \in \Gamma$ we also have $g \cdot x \in U(V^i_R/\Lambda_i)$, then $g \in \Lambda_i$.

Indeed since $U(V^i_R/\Lambda_i)$ embeds into $X$, we see that $g(\tilde{U}(V^i_R)) \cap \tilde{U}(V^i_R) \neq \emptyset$ only if $g \in \Lambda_i$.

**Lemma IV.10.** Assume that for $x \in T_i$ and $g \in \Gamma$ we have $g(x) \in T_i$. Then $g \in \Lambda_i$.

**Proof.** Let $R^*_i$ be subgroup of $G_K$ consisting of diagonal matrices $A := \text{diag}(a,1), a \in \mathbb{R}^*$, acting on the complexification $C_x$ of the real line $R_x \subset R$. It acts on coordinates $z_i = x_i + iy_i, i = 1, 2$ as $A(z_i) = x_i + iay_i$. The action of $R^*_i$ commutes with the action of $\Gamma$ on $V$ and $R^*_i$ maps $T_i$ into itself, with the points of $V^i_R$ being stable under this action. Hence if $x, g(x) \in T_i$ then $A(x), g(A(x)) \in T_i$ for arbitrary $A \in R^*_i$. For sufficiently small $a$, we can assume that $\text{Im}(A(x)), \text{Im}(x) < \epsilon$ for arbitrary small $\epsilon$, while $\text{Re}(x) = \text{Re}(A(x))$. Thus we can find such a pair $x, g(x) \in \tilde{U}(V^i_R)$, and hence conclude that $g \in \Lambda_i$ by the previous Corollary.

**Corollary IV.11.** Cases (iv) and (v) don’t occur.

Indeed we obtain that a domain $T_i/\Lambda_i \to X_R \times \mathbb{R}^2$ embeds into the quotient $X = V/\Gamma$ and hence $X$ cannot be compact. This finishes the proof of the Main Theorem. 

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