On the number of P-invariant closed characteristics on partially symmetric compact convex hypersurfaces in $\mathbb{R}^{2n}$

Hui Liu$^1$, * Duanzhi Zhang$^2$, †

$^1$ School of Mathematical Sciences, University of Science and Technology of China, Hefei, Anhui 230026, P. R. China

$^2$ School of Mathematical Sciences and LPMC, Nankai University, Tianjin 300071, P. R. China

Abstract

In this paper, let $n \geq 2$ be an integer, $P = \text{diag}(-I_{n-\kappa}, I_\kappa, -I_{n-\kappa}, I_\kappa)$ for some integer $\kappa \in [0, n)$, and $\Sigma \subset \mathbb{R}^{2n}$ be a partially symmetric compact convex hypersurface, i.e., $x \in \Sigma$ implies $Px \in \Sigma$. We prove that if $\Sigma$ is $(r, R)$-pinched with $\frac{R}{r} < \sqrt{2}$, then there exist at least $n - \kappa$ geometrically distinct $P$-symmetric closed characteristics on $\Sigma$, as a consequence, $\Sigma$ carry at least $n$ geometrically distinct $P$-invariant closed characteristics.

Key words: Compact convex hypersurfaces, P-symmetric closed characteristics, Hamiltonian system.

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1 Introduction and main results

Let $\Sigma$ be a $C^2$ compact hypersurface in $\mathbb{R}^{2n}$, bounding a strictly convex compact set $U$ with non-empty interior, where $n \geq 2$. We denote the set of all such hypersurfaces by $\mathcal{H}(2n)$. Without loss of generality, we suppose $U$ contains the origin. We consider closed characteristics $(\tau, y)$ on $\Sigma$, which are solutions of the following problem

\begin{equation}
\begin{cases}
\dot{y}(t) = JN_\Sigma(y(t)), & y(t) \in \Sigma, \forall \ t \in \mathbb{R}, \\
y(\tau) = y(0),
\end{cases}
\end{equation}

*Partially supported by China Postdoctoral Science Foundation No.2013M540512. E-mail: huiliu@ustc.edu.cn

†Partially supported by NSF of China (10801078, 11171341, 11271200) and LPMC of Nankai University. E-mail: zhangdz@nankai.edu.cn
where \( J = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix} \). \( I_n \) is the identity matrix in \( \mathbb{R}^n \) and \( N_\Sigma(y) \) is the outward normal unit vector of \( \Sigma \) at \( y \) normalized by the condition \( N_\Sigma(y) \cdot y = 1 \). Here \( a \cdot b \) denotes the standard inner product of \( a, b \in \mathbb{R}^{2n} \). A closed characteristic \((\tau, y)\) is prime if \( \tau \) is the minimal period of \( y \). Two closed characteristics \((\tau, x)\) and \((\sigma, y)\) are geometrically distinct, if \( x(R) \neq y(R) \). We denote by \( \mathcal{T}(\Sigma) \) the set of all geometrically distinct closed characteristics on \( \Sigma \).

There is a long standing conjecture on the number of closed characteristics on compact convex hypersurfaces in \( \mathbb{R}^{2n} \)

\[
\# \mathcal{T}(\Sigma) \geq n, \quad \forall \Sigma \in \mathcal{H}(2n) \tag{1.2}
\]

Since the pioneering works \([\text{Rab1}]\) of P. Rabinowitz and \([\text{Wei1}]\) of A. Weinstein in 1978 on the existence of at least one closed characteristic on every hypersurface in \( \mathcal{H}(2n) \), the existence of multiple closed characteristics on \( \Sigma \in \mathcal{H}(2n) \) has been deeply studied by many mathematicians. In particular, I. Ekeland and J. Lasry proved (1.2) under a \( \sqrt{2} \)-pinched condition(cf. \([\text{EkL1}]\)), and under a \( \sqrt{3} \)-pinched condition(cf. \([\text{Gir1}]\)), M. Girardi proved that there exist \( n \) geometrically distinct symmetric closed characteristics on \( \Sigma \in \mathcal{H}(2n) \) which is symmetric with respect to the origin, i.e., \( x \in \Sigma \) implies \( -x \in \Sigma \). In 1987-1988 I. Ekeland-L. Lassoued, I. Ekeland-H. Hofer, and A. Szulkin (cf. \([\text{EkL2}]\), \([\text{EkH1}]\), \([\text{Szu1}]\)) proved

\[
\# \mathcal{T}(\Sigma) \geq 2, \quad \forall \Sigma \in \mathcal{H}(2n) \tag{1.3}
\]

In \([\text{LoZ1}]\) of 2002, Y. Long and C. Zhu proved

\[
\# \mathcal{T}(\Sigma) \geq \left\lceil \frac{n}{2} \right\rceil + 1, \quad \forall \Sigma \in \mathcal{H}(2n) \tag{1.4}
\]

In \([\text{WHL1}]\), the authors proved the conjecture for \( n = 3 \). In \([\text{LLZ1}]\), the the authors proved the conjecture when \( \Sigma \in \mathcal{H}(2n) \) is symmetric with respect to the origin, i.e., \( x \in \Sigma \) implies \( -x \in \Sigma \). In \([\text{DoL1}]\) of 2004, Y. Dong and Y. Long studied the multiplicity of closed characteristics on partially symmetric convex hypersurfaces in \( \mathbb{R}^{2n} \).

For any \( s_i, t_i \in \mathbb{R}^{k_i} \) with \( i = 1, 2 \), we denote by \((s_1, t_1) \diamond (s_2, t_2) = (s_1, s_2, t_1, t_2)\). Fixing an integer \( \kappa \) with \( 0 \leq \kappa < n \), let \( P = \text{diag}(-I_{n-\kappa}, I_\kappa, -I_{n-\kappa}, I_\kappa) \) and \( \mathcal{H}_\kappa(2n) = \{ \Sigma \in \mathcal{H}(2n) \mid x \in \Sigma \text{ implies } Px \in \Sigma \} \). For \( \Sigma \in \mathcal{H}_\kappa(2n) \), let \( \Sigma(\kappa) = \{ z \in \mathbb{R}^{2\kappa} \mid 0 \odot z \in \Sigma \} \), where 0 is the origin in \( \mathbb{R}^{2n-2\kappa} \). As in \([\text{DoL1}]\), a closed characteristic \((\tau, y)\) on \( \Sigma \in \mathcal{H}_\kappa(2n) \) is \( P \)-asymmetric if \( y(R) \cap Py(R) = \emptyset \), it is \( P \)-symmetric if \( y(R) = Py(R) \) with \( y = y_1 \odot y_2 \) and \( y_1 \neq 0 \), or it is \( P \)-fixed if \( y(R) = Py(R) \) and \( y = 0 \odot y_2 \), where \( y_1 \in \mathbb{R}^{2(n-\kappa)}, y_2 \in \mathbb{R}^{2\kappa} \). We call a closed characteristic
$(\tau, y)$ is $P$-invariant if $y(R) = Py(R)$. Then a $P$-invariant closed characteristic is $P$-symmetric or $P$-fixed. Note that if we set $\kappa = 0$, then the $P$-symmetric closed characteristic is just symmetric and the $P$-fixed closed characteristics vanish.

It is very interesting to consider closed characteristics on hypersurfaces with special symmetries. Recently, W. Wang proved that there exist two symmetric closed characteristics on any compact convex hypersurface which is symmetric with respect to the origin (cf. [Wan1]), the first author of this paper proved that there exist at least two geometrically distinct $P$-invariant closed characteristics on any $\Sigma \in \mathcal{H}_\kappa(2n)$ (cf. [Liu1]) and the second author of this paper proved that there exist at least two geometrically distinct $P$-cyclic closed characteristics on $P$-cyclic compact convex hypersurfaces (cf. [Zha1]). Thus whether $\Sigma \in \mathcal{H}_\kappa(2n)$ carries more $P$-invariant closed characteristics becomes an interesting problem. Motivated by [EkL1], [Liu1] and [Zha1], we prove the following results in this paper.

**Theorem 1.1.** Assume $\Sigma \in \mathcal{H}_\kappa(2n)$ and $0 < r \leq |x| \leq R$, $\forall$ $x \in \Sigma$ with $\frac{R}{r} < \sqrt{2}$. Then there exist at least $n - \kappa$ geometrically distinct $P$-symmetric closed characteristics $(\tau_i, y_i)$ on $\Sigma$, where $\tau_i$ is the minimal period of $y_i$, and the actions $A(\tau_i, y_i)$ satisfy: $\pi r^2 \leq A(\tau_i, y_i) \leq \pi R^2$, $\forall 1 \leq i \leq n - \kappa$.

Combining Theorem 1.1 and the result proved by I. Ekeland and J. Lasry in 1980, we have a direct consequence:

**Theorem 1.2.** Assume $\Sigma \in \mathcal{H}_\kappa(2n)$ and $0 < r \leq |x| \leq R$, $\forall$ $x \in \Sigma$ with $\frac{R}{r} < \sqrt{2}$. Then $\Sigma$ carries at least $n$ geometrically distinct $P$-invariant closed characteristics $(\tau_i, y_i)$, where $\tau_i$ is the minimal period of $y_i$, and the actions $A(\tau_i, y_i)$ satisfy: $\pi r^2 \leq A(\tau_i, y_i) \leq \pi R^2$, $\forall 1 \leq i \leq n$.

Here the action of a closed characteristic $(\tau, y)$ is defined by (cf. P.190 of [Eke1])

$$A(\tau, y) = \frac{1}{2} \int_0^\tau (Jy \cdot \dot{y})dt.$$ 

In this paper, let $\mathbb{N}, \mathbb{N}_0, \mathbb{Z}, \mathbb{Q}, \mathbb{R}$ and $\mathbb{C}$ denote the sets of natural integers, non-negative integers, integers, rational numbers, real numbers and complex numbers respectively. Denote by $a \cdot b$ and $|a|$ the standard inner product and norm in $\mathbb{R}^{2n}$. Denote by $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$ the standard $L^2$-inner product and $L^2$-norm. For a set $A$, we denote by $\# A$ the number of elements in $A$. For an $S^1$-space $X$, we denote by $X_{S^1}$ the homotopy quotient of $X$ by $S^1$, i.e., $X_{S^1} = S^{\infty} \times_{S^1} X$, where $S^{\infty}$ is the unit sphere in an infinite dimensional complex Hilbert space.
2 A variational structure for P-invariant closed characteristics

In the rest of this paper, we fix a $\Sigma \in \mathcal{H}_n(2n)$. Note that a prime closed characteristic $(\tau, y)$ is P-symmetric if and only if it satisfies the problem

$$\begin{cases} \dot{y}(t) = JN_\Sigma(y(t)), y(t) \in \Sigma, \\ y(\tau) = Py(0), y(t) = y_1(t) \circ y_2(t), y_1 \neq 0, \end{cases}$$

(2.1)

and a prime closed characteristic $(\tau, y)$ is P-fixed if and only if it satisfies the problem

$$\begin{cases} \dot{y}(t) = JN_\Sigma(y(t)), y(t) \in \Sigma, \\ y(\tau) = y(0), y(t) = 0 \circ y_2(t), \end{cases}$$

(2.2)

where $y_1(t) \in \mathbb{R}^{2(n-\kappa)}, y_2(t) \in \mathbb{R}^{2\kappa}, \forall \ t \in \mathbb{R}$. Here we also note that even iterate $(2m\tau, y)$ of any prime P-symmetric closed characteristic $(\tau, y)$ does not satisfy the equation (2.1).

In this section, we transform the problems (2.1)-(2.2) into a fixed period problem of a Hamiltonian system and then study its variational structure.

As on Page 199 of [Eke1], we choose some $\alpha \in (1, 2)$ and associate with $U$ a convex function $H(x) = j(x)^\alpha, \forall x \in \mathbb{R}^{2n}$, where $j : \mathbb{R}^{2n} \to \mathbb{R}$ is the gauge function of $\Sigma$, that is, $j(\lambda x) = \lambda$ for $x \in \Sigma$ and $\lambda \geq 0$. Then we have the following:

Proposition 2.1. Consider the following problem

$$\begin{cases} \dot{x}(t) = JH'(x(t)), \\ x(\frac{\tau}{2}) = Px(0). \end{cases}$$

(2.3)

Then solutions of (2.3) are $x \equiv 0$ and $x = (\frac{\tau}{\alpha})^{1-\frac{1}{\alpha}} y(\tau t)$, where $(\tau, y)$ is a solution of (2.1), or $(\frac{\tau}{\alpha}, y)$ is a P-fixed closed characteristic. In particular, nonzero solutions of (2.3) are in one to one correspondence with P-invariant closed characteristics.

Proof. Clearly $x \equiv 0$ is the unique constant solution of (2.3). Suppose $x(t)$ is a nonconstant solution of (2.3), then $H(x(t)) = (j(x(t)))^\alpha = const$. Let $\rho = j(x(t))$ and $y(t) = \rho^{-1} x \left(\frac{\rho^{2-\alpha} t}{\alpha}\right)$. Then $j(y) = \rho^{-1} j(x) = \rho^{-1} \rho = 1$, hence $y(\mathbb{R}) \subset \Sigma$. Moreover, noticing the fact that $j'(y) = N_\Sigma(y)$ and $j(\lambda x) = \lambda j(x)$ for all $x \in \mathbb{R}^{2n} \setminus \{0\}$ and $\lambda \in \mathbb{R}^+$, we have $\dot{y}(t) = JN_\Sigma(y(t))$ and $y(\frac{\alpha}{\rho^{2-\alpha}}) = Py(0)$ by (2.3). Let $\tau = \frac{\alpha}{\rho^{-1}}$, then $(\tau, y)$ is a closed characteristic satisfying (2.1), or $(\frac{\tau}{\alpha}, y)$ is a P-fixed closed characteristic. The other side of the proposition can be proved similarly and thus is omitted.

In the following, we use the Clarke-Ekeland dual action principle to problem (2.3). As usual, let $G$ be the Fenchel transform of $H$ defined by $G(y) = \sup \{ x \cdot y - H(x) \mid x \in \mathbb{R}^{2n} \}$. Then $G$ is
strictly convex. As in Section 3 of [DoL2] (also Section 2 of [Liu1]), let

\[ L^2_\kappa\left(0, \frac{1}{2}\right) = \{u = u_1 \circ u_2 \in L^2((0, \frac{1}{2}), \mathbb{R}^{2n}) \mid u_1 \in L^2((0, \frac{1}{2}), \mathbb{R}^{2n-2\kappa}), \]  
\[ u_2 \in L^2((0, \frac{1}{2}), \mathbb{R}^{2\kappa}), u(\frac{1}{2}) = Pu(0), \int_0^{\frac{1}{2}} u_2(t)dt = 0\}. \quad (2.4) \]

Define a linear operator \( \Pi : L^2_\kappa(0, \frac{1}{2}) \rightarrow L^2_\kappa(0, \frac{1}{2}) \) by

\[
(\Pi \kappa u)(t) = x_1(t) \circ x_2(t), \quad (2.5)
\]
\[
x_1(t) = \int_0^t u_1(\tau)d\tau - \frac{1}{2} \int_0^{\frac{t}{2}} u_1(\tau)d\tau, \quad (2.6)
\]
\[
x_2(t) = \int_0^t u_2(\tau)d\tau - 2 \int_0^{\frac{t}{2}} dt \int_0^t u_2(\tau)d\tau, \quad (2.7)
\]

for any \( u = u_1 \circ u_2 \in L^2_\kappa(0, \frac{1}{2}) \). Then by Lemma 2.4 of [Liu1], we know \( \Pi \kappa \) is a compact operator from \( L^2_\kappa(0, \frac{1}{2}) \) into itself and \( J\Pi \kappa \) is self-adjoint. Now the dual action functional on \( L^2_\kappa(0, \frac{1}{2}) \) is defined by

\[
\Psi(u) = \int_0^{\frac{1}{2}} \left(\frac{1}{2}Ju \cdot \Pi \kappa u + G(u)\right)dt. \quad (2.8)
\]

Since \( H \) is \( \alpha \)-homogeneous, we have \( H(x) \leq r^{-\alpha}|x|^\alpha \) for some positive constant, then by the definition of \( G \), we get \( G(y) \geq \frac{1}{\beta}(\frac{r}{\alpha})^{\beta-1}|y|^\beta \), where \( \beta > 2 \) satisfying \( \alpha^{-1} + \beta^{-1} = 1 \), cf. also (7) and (22) in Section V.2 of [Eke1]. Then by the same proofs of Propositions 2.5 and 2.6 in [Liu1], we have

**Lemma 2.2.** The functional \( \Psi \) is \( C^{1,1} \) and is bounded from below on \( L^2_\kappa(0, \frac{1}{2}) \) and satisfies the Palais-Smale condition. Suppose \( x \) is a solution of (2.3). Then \( u = \dot{x} \) is a critical point of \( \Psi \). Conversely, suppose \( u \) is a critical point of \( \Psi \). Then there exists a unique \( \xi \in \mathbb{R}^{2\kappa} \) such that \( \Pi \kappa u - 0 \circ \xi \) is a solution of (2.3). In particular, solutions of (2.3) are in one to one correspondence with critical points of \( \Psi \).

Note that we can identify \( L^2_\kappa(0, \frac{1}{2}) \) with the space \( \{u \in L^2(\mathbb{R}/\mathbb{Z}, \mathbb{R}^{2n}) \mid u_{(0,1/2)} \in L^2_\kappa(0, \frac{1}{2}), u(t+\frac{1}{2}) = Pu(t)\} \). Then we have a natural \( S^1 \)-action on \( L^2_\kappa(0, \frac{1}{2}) \) defined by \( \theta \ast u(t) = u(\theta + t) \), for all \( \theta \in S^1 \equiv \mathbb{R}/\mathbb{Z} \) and \( t \in \mathbb{R} \). Geometrically, under a partial reflection \( P \), \( u([0,1/2]) \) equals to \( u([\theta,1/2+\theta]) \) for all \( \theta \in S^1 \) and \( \Psi \) is invariant under the partial reflection \( P \), i.e., \( \Psi(Pu) = \Psi(u) \).

As Lemma 2.8 of [Liu1], we have

**Lemma 2.3.** The functional \( \Psi \) is \( S^1 \)-invariant.

**Definition 2.4.** Define the \( S^1 \)-orbit of a point \( u \in L^2_\kappa(0, \frac{1}{2}) \) by

\[
O(u) := \{\theta \ast u \mid \theta \in S^1\}.
\]
We shall say that the $S^1$-action is free at $u$ if the map $\theta \to \theta^* u$ is injective; that is, a homeomorphism of $S^1$ onto $O(u)$. If $\Omega \subset L^2_\kappa (0, \frac{1}{2})$ is an invariant subset, the $S^1$-action is free in $\Omega$ if it is free at every point $u \in \Omega$.

In our case, the $S^1$-action will be free at $u$ if and only if $u$ has minimal period 1. Note that by Lemma 2.3, the level sets $\Psi^\gamma := \{u \in L^2_\kappa (0, \frac{1}{2}) \mid \Psi(u) < \gamma \}$ are $S^1$-invariant.

Now by Proposition 2.1 and Lemma 2.2, we assume $u = \dot{x}$ is critical point of $\Psi$ corresponding to a solution $(\tau, y)$ of (2.1), or a P-fixed closed characteristic $(\tau^2, y)$, then $x = \Pi, u - 0 \circ \xi$ for some $\xi \in R^{2\kappa}$, the corresponding critical value of $\Psi$ is given by:

**Lemma 2.5.** $\Psi(u) = -\frac{1}{2}(1 - \frac{\alpha}{2})(\frac{r}{\alpha})^{-\frac{1}{2}}.$

**Proof.** Since $-Ju = -J\dot{x} = H'(x)$, then $G(-Ju) = x \cdot H'(x) - H(x)$. Thus by direct computations, we have

$$\Psi(u) = \int_0^{\frac{1}{2}} \left( \frac{1}{2} J\dot{x} \cdot (x + 0 \circ \xi) + G(-J\dot{x}) \right) dt$$

$$= \int_0^{\frac{1}{2}} \left( \frac{1}{2} J\dot{x} \cdot x + x \cdot H'(x) - H(x) \right) dt$$

$$= \int_0^{\frac{1}{2}} \left( \frac{1}{2} x \cdot H'(x) - H(x) \right) dt. \tag{2.9}$$

Note that $H(\lambda x) = \lambda^\alpha H(x)$ for all $\lambda \geq 0$, then $x \cdot H'(x) = \alpha H(x)$ and (2.9) becomes

$$\Psi(u) = \int_0^{\frac{1}{2}} \left( \frac{1}{2} \alpha - 1 \right) H(x) dt$$

$$= -\frac{1}{2}(1 - \frac{\alpha}{2})(\frac{r}{\alpha})^{-\frac{1}{2}}.$$

Here we used the fact that $x = (\frac{r}{\alpha})^{-\frac{1}{2}} y(\tau t)$.

### 3 Proofs of the main theorems

In this section, we assume $0 < r \leq |x| \leq R$, $\forall x \in \Sigma$ satisfying $\frac{R}{r} < \sqrt{2}$, $\alpha^{-1} + \beta^{-1} = 1$.

**Lemma 3.1.** $-\frac{1}{2}(1 - \frac{\alpha}{2})(\frac{r}{2\pi R^2})^{-\frac{\alpha}{2}} \leq \min \Psi \leq -\frac{1}{2}(1 - \frac{\alpha}{2})(\frac{R}{2\pi R^2})^{-\frac{\alpha}{2}}.$

**Proof.** Since $R^{-\alpha}|x|^{\alpha} \leq H(x) \leq r^{-\alpha}|x|^{\alpha}$, then by taking Fenchel conjugates we get

$$\frac{1}{\beta} \left( \frac{r}{\alpha} \right)^{\beta - 1} |y|^\beta \leq G(y) \leq \frac{1}{\beta} \left( \frac{R}{\alpha} \right)^{\beta - 1} |y|^\beta. \tag{3.1}$$

Hence, we have

$$\Psi_r(u) \leq \Psi(u) \leq \Psi_R(u),$$
\[ \Psi_r(u) := \int_0^\frac{1}{2} \left( \frac{1}{2} J u \cdot \Pi_\kappa u + \frac{1}{\beta} \left( \frac{r^\alpha}{\alpha} \right)^{\beta-1} |u|^\beta \right) dt \]

\[ \Psi_R(u) := \int_0^\frac{1}{2} \left( \frac{1}{2} J u \cdot \Pi_\kappa u + \frac{1}{\beta} \left( \frac{R^\alpha}{\alpha} \right)^{\beta-1} |u|^\beta \right) dt. \]

Then \( \text{Min} \Psi_r \leq \text{Min} \Psi \leq \text{Min} \Psi_R. \)

By the proof of Lemma 2.5 and replacing \( H(x) \) with \( r^{-\alpha}|x|^\alpha \), we obtain

\[ \text{Min} \Psi_r = -\frac{1}{2} (1 - \frac{\alpha}{2}) |\bar{x}|^{\alpha r^{-\alpha}}, \tag{3.2} \]

where \( \bar{x} \) satisfies \( \text{Min} \Psi_r = \Psi_r(\bar{x}) \) and is a solution of:

\[
\begin{cases}
\dot{x} = J^\alpha_r \frac{\bar{x}}{|x|^{\alpha r^{-\alpha}}}, \\
x(\frac{1}{2}) = P\bar{x}(0).
\end{cases} \tag{3.3}
\]

By Lemma 2.5, \( \bar{x} \) has minimal period 1 when \( \bar{x} \) corresponds to a \( P \)-symmetric closed characteristic or \( \bar{x} \) has minimal period \( \frac{1}{2} \) when \( \bar{x} \) corresponds to a \( P \)-fixed closed characteristic. By solving (3.3), \( \bar{x} \) has the form \( \bar{x}(t) = e^{J\omega t}\bar{x}(0) \). Then by (3.3), we have

\[ \omega = \frac{\alpha}{r^\alpha} |x|^{\alpha-2}, \tag{3.4} \]

and \( \omega = 2\pi \), or \( \omega = 4\pi \).

When \( \omega = 2\pi \), we have

\[ \Psi_r(\bar{x}) = -\frac{1}{2} (1 - \frac{\alpha}{2}) \left( \frac{\alpha}{2\pi r^\alpha} \right)^{\alpha r^{-\alpha}} \text{ by (3.2), (3.4).} \]

When \( \omega = 4\pi \), we have

\[ \Psi_r(\bar{x}) = -\frac{1}{2} (1 - \frac{\alpha}{2}) \left( \frac{\alpha}{4\pi r^\alpha} \right)^{\alpha r^{-\alpha}} \text{ by (3.2), (3.4).} \]

Note that \( \text{Min} \Psi_r = \Psi_r(\bar{x}) \), then we must have \( \omega = 2\pi \) and \( \Psi_r(\bar{x}) = -\frac{1}{2} (1 - \frac{\alpha}{2}) \left( \frac{\alpha}{2\pi r^\alpha} \right)^{\alpha r^{-\alpha}} \). Hence our Lemma holds.

Recall that the action of a closed characteristic \((\tau, y)\) is defined by (cf. P.190 of [Eke1])

\[ A(\tau, y) = \frac{1}{2} \int_0^\tau (Jy \cdot \dot{y}) dt. \]

Note that \( A(\tau, y) \) is a geometric quantity depending only on how many times one runs around the closed characteristic. In fact, we can compute it as follows

\[
A(\tau, y) = \frac{1}{2} \int_0^\tau (Jy \cdot JN_{\Sigma}(y)) dt \\
= \frac{1}{2} \int_0^\tau 1 dt \\
= \frac{\tau}{2}. \tag{3.5}
\]

Here we used (1.1) and the fact that \( N_{\Sigma}(y) \cdot y = 1 \).

Now we state a lemma dues to C. Croke and A. Weinstein which is useful for our proofs:
Lemma 3.2. (cf. Theorem V.1.4 of [Eke1]) Assume Σ is a $C^1$ hypersurface bounding a convex compact set $U$. Suppose there is a point $x_0 \in \mathbb{R}^{2n}$ such that

$$0 < r \leq |x - x_0|, \forall x \in \Sigma$$

Then, if $(\tau, y)$ is a closed characteristic on $\Sigma$, we have $A(\tau, y) \geq \pi r^2$.

Now we restrict $\Psi$ on the subspace $X \equiv \{u = 0 \circ u_2 \in L^2((0, \frac{1}{2}), \mathbb{R}^{2n}) \mid u_2 \in L^2((0, \frac{1}{2}), \mathbb{R}^{2n}), u\left(\frac{1}{2}\right) = Pu(0), \int_0^{\frac{1}{2}} u_2(t)dt = 0\}$ of $L^2_k\left(0, \frac{1}{2}\right)$. Then by Lemma 2.2, $\Psi$ is bounded from below on $X$, let $\Psi(\bar{u}) = \min\{\Psi(u) \mid u \in X\}$, then $\bar{u}$ corresponds to a $P$-fixed closed characteristic via Proposition 2.1 and Lemma 2.2, precisely, $\bar{u} = \dot{x}$, $x = (\frac{\tau}{2} - \frac{1}{2}) y(\tau t)$, where $(\frac{\tau}{2}, y)$ is a $P$-fixed closed characteristic. Hence, by Lemma 2.5 and (3.5), we obtain

$$\Psi(\bar{u}) = \frac{1}{2}(1 - \frac{\alpha}{2}) \left(\frac{4A(\frac{\tau}{2}, y)}{\alpha}\right)^{-\frac{\alpha}{4\alpha}}. \quad (3.6)$$

By Lemma 3.2, we have

$$A(\frac{\tau}{2}, y) \geq \pi r^2. \quad (3.7)$$

Then (3.6) and 3.7 imply $\Psi(\bar{u}) \geq -\frac{1}{2}(1 - \frac{\alpha}{2}) \left(\frac{4\pi r^2}{\alpha}\right)^{-\frac{\alpha}{4\alpha}}$. Thus

$$\Psi(u) \geq -\frac{1}{2}(1 - \frac{\alpha}{2}) \left(\frac{4\pi r^2}{\alpha}\right)^{-\frac{\alpha}{4\alpha}} \quad (3.8)$$

for every $u \in X$. Now as Lemma V.2.5 of [Eke1], we can prove the following:

Lemma 3.3. When $\gamma \leq -\frac{1}{2}(1 - \frac{2}{\alpha}) \left(\frac{4\pi r^2}{\alpha}\right)^{-\frac{\alpha}{4\alpha}}$, the $S^1$-action is free in $\Psi^\gamma$.

Proof. Firstly, by (3.8), we have $\Psi^\gamma \cap X = \emptyset$. Let $u \in L^2_k\left(0, \frac{1}{2}\right) \setminus X$ has minimal period $\frac{1}{k}$ for some integer $k \geq 2$. Note that by (2.4), $k$ must be odd, thus $k \geq 3$. We have to prove that

$$\Psi(u) \geq -\frac{1}{2}(1 - \frac{\alpha}{2}) \left(\frac{4\pi r^2}{\alpha}\right)^{-\frac{\alpha}{4\alpha}} \quad (3.9)$$

In fact, define $v$ by

$$v(t) := k^{\frac{\alpha - 1}{2\pi r^2}} u\left(\frac{t}{k}\right).$$

Then $v \in L^2_k\left(0, \frac{1}{2}\right)$. Computing $\Psi(v)$, and using the $\beta$-homogeneous (since $H$ is $\alpha$-homogeneous), we get

$$\Psi(v) = k^{\frac{\alpha}{2\pi r^2}} \Psi(u).$$

Since $\Psi(v) \geq \min\Psi$, then by Lemma 3.1, we have

$$k^{\frac{\alpha}{2\pi r^2}} \Psi(u) \geq -\frac{1}{2}(1 - \frac{\alpha}{2}) \left(\frac{\alpha}{2\pi r^2}\right)^{\frac{\alpha}{2\pi r^2}}. \quad (3.10)$$
Then

\[ \Psi(u) \geq -\frac{1}{2}(1 - \frac{\alpha}{2}) \left( \frac{2k \pi r^2}{\alpha} \right)^{-\frac{\alpha}{2}} \geq -\frac{1}{2}(1 - \frac{\alpha}{2}) \left( \frac{4 \pi r^2}{\alpha} \right)^{-\frac{\alpha}{2}}. \]

We complete our proof.

In the following, we shall take:

\[ \gamma > -\frac{1}{2}(1 - \frac{\alpha}{2}) \left( \frac{\alpha}{2} \pi R \right)^{2/n}. \tag{3.10} \]

By Lemma 3.1, we have \( \gamma > \text{Min} \Psi \), so \( \Psi^\gamma \) is not empty. Since \( 2r^2 > R^2 \), we may also assume that

\[ \gamma < -\frac{1}{2}(1 - \frac{\alpha}{2}) \left( \frac{\alpha}{4 \pi r^2} \right)^{2/n} \tag{3.11} \]

so the \( S^1 \)-action on \( \Psi^\gamma \) is free by Lemma 3.3. At this point we prove \( \Psi^\gamma \) contains a copy of the Euclidean sphere \( S^{2n-2\kappa-1} = \{ x \in \mathbb{R}^{2n-2\kappa} \mid |x|^2 = 1 \} \) with the standard \( S^1 \)-action. The standard \( S^1 \)-action on \( S^{2n-2\kappa-1} \) is defined by:

\[ \theta * x = e^{2\pi J_{2n-2\kappa} \theta} x. \]

We call a map \( f : \mathbb{R}^{2n-2\kappa} \to L^2_\kappa (0, \frac{1}{2}) \) equivariant if

\[ f(e^{2\pi J_{2n-2\kappa} \theta} x) = \theta * f(x). \]

**Lemma 3.4.** There is an equivariant isometry \( f : \mathbb{R}^{2n-2\kappa} \to L^2_\kappa (0, \frac{1}{2}) \) and a number \( \rho > 0 \) such that \( f(\rho S^{2n-2\kappa-1}) \subset \Psi^\gamma \).

**Proof.** We define \( f(\xi) = u_\xi \) by \( u_\xi(t) = (\sqrt{2} e^{2\pi J_{2n-2\kappa} t} \xi) \circ 0, \forall \xi \in \mathbb{R}^{2n-2\kappa} \). Then

\[ \|f(\xi)\| = \left( \int_0^{1/2} \sqrt{2} e^{2\pi J_{2n-2\kappa} t} |\xi|^2 dt \right)^{1/2} = |\xi|, \]

that is, \( f \) is an isometry. Since

\[ f(e^{2\pi J_{2n-2\kappa} \theta} \xi)(t) = (\sqrt{2} e^{2\pi J_{2n-2\kappa} t} e^{2\pi J_{2n-2\kappa} \theta} \xi) \circ 0 = (\sqrt{2} e^{2\pi J_{2n-2\kappa} (t+\theta)} \xi) \circ 0 = \theta * f(\xi), \]

then \( f \) is equivariant.

By direct computations, we have \( \Pi_\kappa u_\xi(t) = (-\sqrt{2} e^{2\pi J_{2n-2\kappa} t} e^{2\pi J_{2n-2\kappa} \theta} \xi) \circ 0 \), then

\[ \frac{1}{2} \langle Ju_\xi, \Pi_\kappa u_\xi \rangle = -\frac{1}{4\pi} |\xi|^2. \]
and using (3.1), we have

\[ \Psi(u_\xi) = \int_0^\frac{1}{2} \left( \frac{1}{2}Ju_\xi \cdot \Pi_\kappa u_\xi + G(-Ju_\xi) \right) dt \]

\[ = -\frac{1}{4\pi}|\xi|^2 + \int_0^\frac{1}{2} G(-J(\sqrt{2}\pi J_{2n-2})\xi \circ 0) dt \]

\[ \leq -\frac{1}{4\pi}|\xi|^2 + \frac{1}{2} \cdot \frac{1}{\beta} \left( \frac{R^\alpha}{\alpha} \right)^{\beta-1} |\sqrt{2}\xi|^{\beta} \]

\[ = \frac{1}{2} \left( -\frac{1}{4\pi}|\sqrt{2}\xi|^2 + \frac{1}{\beta} \left( \frac{R^\alpha}{\alpha} \right)^{\beta-1} |\sqrt{2}\xi|^\beta \right). \quad (3.12) \]

The right-hand side of (3.12) achieves its minimum when \(-\frac{1}{2\pi}|\sqrt{2}\xi| + \left( \frac{R^\alpha}{\alpha} \right)^{\beta-1} |\sqrt{2}\xi|^\beta = 0\), which yields:

\[ |\sqrt{2}\xi| = \left( \frac{1}{2\pi} \right)^{\frac{\alpha-1}{2-\alpha}} \left( \frac{\alpha}{2\pi R^\alpha} \right)^{\frac{1}{2-\alpha}}. \quad (3.13) \]

Hence, by (3.10) we have

\[ \Psi(u_\xi) \leq -\frac{1}{2}(1 - \frac{\alpha}{2}) \left( \frac{\alpha}{2\pi R^\alpha} \right)^{\frac{1}{2-\alpha}} < \gamma, \quad (3.14) \]

when (3.13) holds. Let \( \rho = \frac{1}{2\sqrt{2}} \left( \frac{\alpha}{2\pi R^\alpha} \right)^{\frac{1}{2-\alpha}} \), we complete the proof of the lemma.

Recall that for a principal \( U(1) \)-bundle \( E \to B \), the Fadell-Rabinowitz index (cf. [FaR1]) of \( E \) is defined to be \( \text{sup}\{ k \mid c_1(E)^k \neq 0 \} \), where \( c_1(E) \in H^2(B, \mathbb{Q}) \) is the first rational Chern class. For a \( U(1) \)-space, i.e., a topological space \( X \) with a \( U(1) \)-action, the Fadell-Rabinowitz index \( \hat{I}(X) \) is defined to be the index of the bundle \( X \times S^\infty \to X \times_{U(1)} S^\infty \), where \( S^\infty \to CP^\infty \) is the universal \( U(1) \)-bundle.

It is well known that \( \hat{I}(S^{2n-2\kappa-1}) = n - \kappa \), then by monotonicity of the Fadell-Rabinowitz index \( \hat{I} \) and Lemma 3.4, we have

\[ \hat{I}(\Psi^\gamma) \geq n - \kappa. \quad (3.15) \]

Now we define

\[ c_i = \inf \{ \delta \in \mathbb{R} \mid \hat{I}(\Psi^\delta) \geq i \}. \]

Then

\[ \text{Min}_\psi = c_1 \leq c_2 \leq \cdots \leq c_{n-\kappa} \leq \gamma < 0. \quad (3.16) \]

As Proposition V.3.3 in P.218 of [Eke1], we have

**Proposition 3.5.** Every \( c_i \) is a critical value of \( \Psi \). If \( c_i = c_j \) for some \( i < j \leq n - \kappa \), then there are infinitely many \( S^1 \)-orbits of critical points on the level \( c_i = c_j \).
Proof. For the reader’s convenience, we sketch a brief proof here and refer to Sections V.2 and V.3 of [Eke1] for related details.

By the proof of Theorem V.2.9 of [Eke1], if we replace $L^\beta$ and $\psi$ by $L^2(0, \frac{1}{2})$ and $\Psi$ respectively, the Theorem V.2.9 of [Eke1] also works. Since the Fadell-Rabinowitz index $\hat{I}$ has the properties of monotonicity, subadditivity, continuity which are the only three properties of $I$ used in the proof of Proposition V.2.10 of [Eke1], then the proof carries over verbatim of that of Proposition V.2.10 of [Eke1].

Now we prove the main theorems.

Proof of Theorem 1.1. By (3.16) and Proposition 3.5, we find at least $n - \kappa$ distinct $S^1$-orbits of critical points $u_1, \cdots, u_{n-\kappa}$ of $\Psi$ with $\Psi(u_i) < \gamma$. Using Lemma 3.1 and taking the infimum of $\gamma$ in formula (3.10), we obtain:

$$-\frac{1}{2}(1 - \frac{\alpha^2}{2})\frac{\alpha^2}{2\pi r^2} \leq \Psi(u_i) \leq -\frac{1}{2}(1 - \frac{\alpha^2}{2})\frac{\alpha^2}{2\pi R^2}.$$  

(3.17)

By (3.8) and Lemma 3.3, we know that every $u_i$ must corresponds to a prime $P$-symmetric closed characteristic, denote it by $(\tau_i, y_i)$. Moreover by Lemma 2.5 and (3.5), (3.17), we have

$$\pi r^2 \leq A(\tau_i, y_i) \leq \pi R^2.$$  

(3.18)

The proof is complete.

Proof of Theorem 1.2. By the main result of [EkL1] (cf. also Theorem V.2.1 of [EkL1]), there exist at least $\kappa$ geometrically distinct closed characteristics on $\Sigma(\kappa)$, where $\Sigma(\kappa) = \{ z \in \mathbb{R}^{2\kappa} | 0 \diamond z \in \Sigma \}$, $0$ is the origin in $\mathbb{R}^{2n-2\kappa}$, then we obtain at least $\kappa$ geometrically distinct $P$-fixed closed characteristics $\{(\tau_i, y_i) | n - \kappa + 1 \leq i \leq n \}$ on $\Sigma$ satisfying $\pi r^2 \leq A(\tau_i, y_i) \leq \pi R^2$. Together with Theorem 1.1, it proves Theorem 1.2.

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