The cosmic acceleration is one of the most significant cosmological discoveries over the last century. Following the more accurate data a more dramatic result appears: the recent analysis of the observation data (especially from SNe Ia) indicate that the time varying dark energy gives a better fit than a cosmological constant, and in particular, the equation of state parameter $w$ (defined as the ratio of pressure to energy density) crosses $-1$ at some low redshift region. This crossing behavior is a serious challenge to fundamental physics. In this article, we review a number of approaches which try to explain this remarkable crossing behavior. First we show the key observations which imply the crossing behavior. And then we concentrate on the theoretical progresses on the dark energy models which can realize the crossing $-1$ phenomenon. We discuss three kinds of dark energy models: 1. two-field models (quintom-like), 2. interacting models (dark energy interacts with dark matter), and 3. the models in frame of modified gravity theory (concentrating on brane world).
I. THE UNIVERSE IS ACCELERATING

Cosmology is an old and young branch of science. Every nation had his own creative idea about this subject. However, till the 1920s about the unique observation which had cosmological significance was a dark sky at night. On the other hand, we did not prepare a proper theoretical foundation till the construction of general relativity. Einstein’s 1917 paper is the starting point of modern cosmology [1]. The next milestone was the discovery of cosmic expansion, that was the recession of galaxies and the recession velocity was proportional to the distance to us.

Except some rare cases, our researches are always based on the cosmological principle, which says that the universe is homogeneous and isotropic. In the early time, this is only a supposition to simplify the discussions. Now we have enough evidences that the universe is homogeneous and isotropic at the scale larger than 100 Mpc. The cosmological principle requires that the metric of the universe is FRW metric,

\[ ds^2 = -dt^2 + a^2(t)(dr^2 + r^2d\Omega_2^2); \]  
\[ ds^2 = -dt^2 + a^2(t)(dr^2 + \sin(r)^2d\Omega_2^2); \]  
\[ ds^2 = -dt^2 + a^2(t)(dr^2 + \sinh(r)^2d\Omega_2^2), \]

depending on the spatial curvature, which can be Euclidean, spherical or pseudo-spherical. Here, \( t \) is the cosmic time, \( a \) denotes the scale factor, \( r \) represents the comoving radial coordinate of the maximal symmetric 3-space, and \( d\Omega_2^2 \) stands for a 2-sphere. Which geometry serves our space is decided by observations. FRW metric describes the kinetic evolution of the universe. To describe the dynamical evolution of the universe, that is, the function of \( a(t) \), we need the gravity theory which ascribes the space geometry to matter. The present standard gravity theory is general relativity. In 1922 and 1924, Friedmann found that there was no static cosmological solution in general relativity, that is to say, the universe is either expanding or contracting [2]. To get a static universe, Einstein introduce the cosmological constant. However, even in the Einstein universe, where the contraction of the dust is exactly counteracted by the repulsion of the cosmological constant, the equilibrium is only tentative since it is a non-stationary equilibrium. Any small perturbation will cause it to contract or expand. Hence, in some sense we can say that general relativity predicts an expanding (or contracting) universe, which should be regarded as one of the most important prediction of relativity.

In almost 70 years since the discovery of the cosmic expansion in 1929 [3], people generally believe that the universe is expanding but the velocity is slowing down. People try to understand
via observation that the universe will expand forever or become contracting at some stage. A striking result appeared in 1998, which demonstrated that the universe is accelerating rather than decelerating. Now we show how to conclude that our universe is accelerating. We introduce the standard general relativity,

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi G t_{\mu\nu},$$  \hspace{1cm} (4)

where $G_{\mu\nu}$ is Einstein tensor, $G$ is (4-dimensional) Newton constant, $t_{\mu\nu}$ denotes the energy-momentum tensor, $g_{\mu\nu}$ stands for the metric of a spacetime, $\mu, \nu$ run from 0 to 3. Throughout this article, we take a convention that $c = \hbar = 1$ without special notation. Define a new energy momentum $T_{\mu\nu}$,

$$8\pi G T_{\mu\nu} = 8\pi G t_{\mu\nu} - \Lambda g_{\mu\nu}.$$  \hspace{1cm} (5)

$T_{\mu\nu}$ has included the contribution of the cosmological constant, whose effect can not be distinguished with vacuum if we only consider gravity.

The 00 component of Einstein equation (4) is called Friedmann equation,

$$H^2 + \frac{k}{a^2} = \frac{8\pi G}{3} \rho,$$  \hspace{1cm} (6)

where $H = \dot{a}/a$ is the Hubble parameter, an overdot stands for the derivative with respect to the cosmic time, $k$ is the spatial curvature of the FRW metric, for (1), $k = 0$; for (2), $k = 1$; for (3), $k = -1$, $\rho = -T_0^0$. Throughout this article, we take the signature $(-, +, ..., +)$. The spatial component of Einstein equation can be replaced by the continuity equation, which is much more convenient,

$$\dot{\rho} + 3H(\rho + p) = 0,$$  \hspace{1cm} (7)

where $p = T_1^1 = T_2^2 = T_3^3$. Here we use a supposition that the source of the universe $T^\nu_\mu$ is in perfect fluid form. Using (6) and (7), we derive the condition for acceleration,

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3} (\rho + 3p).$$  \hspace{1cm} (8)

We see that the universe is accelerating when $\rho + 3p < 0$. We know that the galaxies and dark matter inhabit in the universe long ago. They are dust matters with zero pressure. Thus, if the universe is accelerating, there must exist an exotic matter with negative pressure or we should modify general relativity. The cosmological constant is a far simple candidate for this exotic matter, or dubbed dark energy. For convenience we separate the contribution of the cosmological constant (dark energy) from other sectors in the energy momentum, the Friedmann equation becomes,
\[
\frac{H^2}{H_0^2} = \Omega_{\Lambda 0} + \Omega_{m 0}(1 + z)^3 + \Omega_{k 0}(1 + z)^2,
\]

(9)

where \(z\) is the redshift, the subscript 0 denotes the present value of a quantity,

\[
\Omega_{\Lambda 0} = \frac{\Lambda}{8\pi H_0^2 G},
\]

(10)

\[
\Omega_{m 0} = \frac{\rho_{m 0}}{8\pi H_0^2 G},
\]

(11)

\[
\Omega_{k 0} = -\frac{k}{H_0^2 a_0^2}.
\]

(12)

The type Ia supernova is a most powerful tool to probe the expanding rate of the universe.

In short, type Ia supernova is a supernova which just reaches the Chandrasekhar limit (1.4 solar mass) and then explodes. Hence they have the same local luminosity since they have roughly the same mass and the same exploding process. They are the standard candles in the universe. We can get the distance of a type Ia supernova through its apparent magnitude. A sample of type Ia supernovae will generates a diagram of Hubble parameter versus distance, through which we get the information of the expanding velocity in the history of the universe. In 1998, two independent groups found that the universe is accelerating using the observation data of supernovae [4]. After that the data accumulate fairly quickly. The famous sample includes Gold04 [5], Gold06 [6], SNLS [7], ESSENCE [8], Davis07 [9], Union [10], Constitution [11]. Here, we show some results of one of the most recent sample, Union [10], which is plotted by \(\chi^2\) statistics.

From fig 1, we see that: 1. the universe is almost spatially flat, that is the curvature term \(\Omega_{k 0}\) is very small. 2. the present universe is dominated by cosmological constant (dark energy), whose partition is approximately 70%, and the partition of dust is 30%. We introduce a dimensionless parameter, the deceleration parameter \(q\),

\[
q = -\frac{\ddot{a}a}{\dot{a}^2}.
\]

(13)

A negative deceleration parameter denotes acceleration. In a universe with dust and cosmological constant (which is called ΛCDM model), by definition

\[
q = \frac{1}{2} \Omega_m - \Omega_\Lambda,
\]

(14)

whose present value \(q_0 = -0.55\). Hence the present universe is accelerating.
The previous result depends on a special cosmological model, $\Lambda$CDM model in frame of general relativity. How about the conclusion if we only consider the kinetics of the universe?

The simplest kinetic model is a sudden transition model, in which the deceleration parameter is a constant in some high redshift region and jumps to another constant at a critical redshift. The other simple choice is that the deceleration parameter is a linear function of $z$. We show the two kinetic models with the dynamic model $\Lambda$CDM in fig[2]. It is clear that the universe accelerates in the present epoch in all the three models. A more rigorous analyze shows that the evidence for an accelerating universe is fairly strong (more than 5 $\sigma$) [12]. So we should investigate it seriously.

$\Lambda$CDM is the most simple model for the acceleration, which is a concordance model of several observations. As we shown in fig[1] the counters of CMB, BAO, and SNe Ia have cross section, which almost laps over result of the joint fittings. However, $\Lambda$CDM has its own theoretical problems.
Furthermore, it is found that a dynamical dark energy model fits the observation data better. Especially, there are some evidences that the equation of state (EOS) of dark energy may cross \(-1\), which is a serious challenge to the foundation of theoretical physics.

In the next section we shall study some problems of \(\Lambda\)CDM model and display that a dynamical dark energy model is favored by observations. We’ll focus on the crossing behavior implied by the observation. In section III, we study 3 kinds of models with a crossing phantom divide dark energy. In section IV, we present the conclusion and more references of this topic.

II. A DARK ENERGY WITH CROSSING \(-1\) EOS IS SLIGHTLY FAVORED BY OBSERVATIONS

A. The problems of \(\Lambda\)CDM

\(\Lambda\)CDM has two famous theoretical problems.

The first is the finetune problem. The effect of the vacuum energy can not be distinguished from the cosmological constant in gravity theory. We can calculate the vacuum energy by a well-constructed theory, quantum field theory (QFT), which says that the vacuum energy should be larger than the observed value by 122 orders of magnitude, if QFT works well up to the Planck
scale. In supersymmetric (SUSY) theory, the vacuum energy of the Bosons exactly counteracts the vacuum energy of Fermions, such that we obtain a zero vacuum energy. However, SUSY must break at the electro-weak scale. At that scale, the vacuum energy is still large than the observed value by 60 orders of magnitude. So for getting a vacuum energy we observed, we should introduce a bare cosmological constant $\Lambda_{\text{bare}}$. The effective vacuum energy $\rho_{\text{effect}}$ then becomes,

$$\rho_{\text{effect}} = \frac{1}{8\pi G} \Lambda_{\text{bare}} + \rho_{\text{vacuum}}.$$  \hspace{1cm} (15)

$\frac{1}{8\pi G} \Lambda_{\text{bare}}$ and $\rho_{\text{vacuum}}$ have to almost counteract each other but do not exactly counteract each other, leaving a tiny tail which is smaller than the $\rho_{\text{vacuum}}$ by 60 orders of magnitude. Which mechanism can realize such a miraculous counteraction?

The second problem is coincidence problem, which says that the cosmological constant keeps a constant while the density of the dust evolves as $(1 + z)^3$ in the history of the universe, then why do they approximately equal each other at “our era”? Different from the first problem, the second problem says the present ratio of dark energy and dark matter is sensitively depends on the initial conditions. Essentially, the coincidence problem is the problem of an unnatural initial condition. The densities of different species in the universe redshift with different rate in the evolution of the universe, so if their densities coincidence in our era, their density ratio must be a specific, tiny number in the early universe. It is also a finetune problem, but a finetune problem of the initial condition.

Except the above theoretical problems, $\Lambda$CDM also suffers from observation problem, especially when faced to the fine structure of the universe, including galaxies, clusters and voids. Some specific observations differ from the predictions of $\Lambda$CDM (with standard partitions of dust and cosmological constant) at a level of $2\sigma$ or higher. Six observations are summarized in [13]: 1. scale velocity flows is much larger than the prediction of $\Lambda$CDM, 2. Type Ia Supernovae (Sne Ia) at High Redshift are brighter than what $\Lambda$CDM indicates, 3. the void seems more empty than what $\Lambda$CDM predicts, 4. the cluster haloes look denser than what $\Lambda$CDM says, 5. the density function of galaxy haloes is smooth, while $\Lambda$CDM indicates a cusp in the core, 6. there are too much disk galaxies than the prediction of $\Lambda$CDM. We do not fully understand the dynamics and galaxies and galaxy clusters, that is, the gravitational perturbation theory at the small scale. The agreement may approve when we advance our perturbation theory with a cosmological constant and the simulation methods at the small scale. However, in the cosmological scale, there are also some evidences that the dark energy is dynamical, including no. 2 of the previous 6 problems.
B. crossing $-1$

With data accumulation, observations which favor dynamical dark energy become more and accurate. Now we lose the condition that $p = -\rho$ for the exotic matter (dark energy) which accelerates the universe. We go beyond the $\Lambda$CDM model. We permit that the EOS of dark energy is not exactly equal to $-1$, but still a constant. The fitting results by different samples of SNe Ia are displayed in fig [3]. We see that although a cosmological constant is permitted, the dark energy whose EOS $< -1$ is favored by SNe Ia. The essence whose EOS is less than $-1$ is called phantom, which can be realized by a scalar field with negative kinetic term. The action for phantom $\psi$ is

$$S_{\text{ph}} = \int d^4x \sqrt{-g} \left( \frac{1}{2} \partial_{\mu} \psi \partial^{\mu} \psi - U(\psi) \right), \quad (16)$$

where $\sqrt{-g}$ is the determinate of the metric, the lowercase Greeks run from 0 to 3, $U(\psi)$ denotes the potential of the phantom. In an FRW universe, the density and pressure of the phantom reduce to

$$\rho = -\frac{1}{2} \dot{\psi}^2 + U, \quad (17)$$

$$p = -\frac{1}{2} \dot{\psi}^2 - U, \quad (18)$$

respectively. Now the EOS of the phantom $w$ is given by,

$$w = \frac{p}{\rho} = -\frac{\frac{1}{2} \dot{\psi}^2 - U}{\frac{1}{2} \dot{\psi}^2 + U}, \quad (19)$$

which is always less than $-1$ for a positive $U$. It seems that phantom is proper candidate for the dark energy whose EOS less than $-1$ [14]. It is very famous that a phantom field is unstable when quantized since the energy has no lower bound. It will transit to a lower and lower energy state. In this article we have no time to discuss this important topic for phantom dark energy. We would point out the basic idea for this issue, which requires the life time of the phantom is much longer than the age of the universe such that we still have no chance to observe the decay of the phantom, though it is fundamentally unstable. For references, see [16]. A completely regular quantum stress with $w < -1$ is suggested in [16].

If the dark energy really behaves as phantom at some low redshift region, it is an unusual discovery. But the dark energy may be more fantastic. In some model-independent fittings, the EOS of dark energy crosses $-1$, which is a really remarkable property and a serious challenge to our present theory of fundamental physics.
FIG. 3: The EOS of dark energy fitted by SNe Ia in a spatially flat universe, the contours display 68.3%, 95.4% and 99.7% confidence level on $w$ and $\Omega_m$ ($\Omega_m$ in the figure). The results of the Union set are shown as filled contours. The empty contours, from left to right, show the results of the Gold sample, Davis 07, and the Union without SCP nearby data. From [10].

FIG. 4: Panel (a): Uncorrelated band-power estimates of the EOS $w(z)$ of dark energy by SNe Ia (Gold set [5]). Vertical error bars show the 1 and 2-$\sigma$ error bars (in blue and green, respectively). The horizontal error bars denote the data bins used in [18]. Panel (b): The window functions for each bin from low redshift to high redshift. Panel (c): the likelihoods of $w(z)$ in the bins from low redshift to high redshift. From [18].

Pioneer results of the crossing $-1$ of EOS of dark energy appeared in [17, 18]. Fig [4] illuminates that the EOS of dark energy may cross $-1$ in some low redshift. In fig [4] the Gold04 data are applied, a uniform prior of $0.22 \leq \Omega_{m0} \leq 0.38$ is assumed, and a spatially flat universe is the working frame.

The perturbation of the dark energy will growth if its EOS is not exactly $-1$ in the evolution
FIG. 5: Constraints on the EOS of $w(z)$ by WMAP3 \cite{20} and Gold04 \cite{5}. The light grey region denotes 2 $\sigma$ constraint, while the dark grey for 1$\sigma$ constraint. The left panel shows the constraint with dark energy perturbation, while the right displays the result without dark energy perturbation. From \cite{19}.

history of the universe. Hence to fit a model with dynamical dark energy with observation, the perturbation of the dark matter should be considered in principle. Such a study was presented in \cite{19}, in which a parametrization of the EOS of the dark energy with two constant $w_0$, $w_1$ was applied,

$$w = w_0 + w_1 \frac{z}{1+z}.$$ \hfill (20)

The result is shown in fig 5. We see from fig 5 that there is a mild tendency that the EOS of the dark energy cross $-1$. For a more general parametrization of EOS for dark energy, see \cite{21}.

With more and accurate data, the possibility of crossing $-1$ (phantom divide) seems a little more specific, see for example \cite{22}. This crossing behavior is a significant challenge for theoretical physics. It was proved that the EOS of dark energy can not cross the phantom divide if 1. a dark energy component with an arbitrary scalar-field Lagrangian, which has a general dependence on the field itself and its first derivatives, 2. general relativity holds and 3. the spatially flat Friedmann universe \cite{23}, for a more detailed proof, see the appendix of \cite{24}. Thus realizing such a crossing is not a trivial work. In the next section we investigate the theoretical progresses for this extraordinary phenomenon.
III. THREE ROADS TO CROSS THE PHANTOM DIVIDE

To cross the phantom divide, we must break at least one of the conditions in [23]. Now that the dark energy behaves as quintessence at some stage, while evolves as phantom at the other stage, a natural suggestion is that we should consider a 2-field model, a quintessence and a phantom. The potential is carefully chosen such that the quintessence dominates the universe at some stage while the phantom dominates the universe at the other stage. It was invented a name for such 2-field model, “quintom”. There are also some varieties of quintom, such as hessence. We introduce these 2-field models in the first subsection. The next road is to consider an interacting model, in which the dark energy interacts with dark matter. The interaction can realize the crossing behavior which is difficult for independent dark energy. We shall study the interacting models in subsection B. The other possibility is that general relativity fails at the cosmological scale. The ordinary dark energy candidates, such as quintessence or phantom, can cross the phantom divide in a modified gravity theory. We investigate this approach in subsection C.

A. 2-field model

A typical 2-field model is the quintom model, which was proposed in [25], and was widely investigated later [26]. Generally, the action of a universe with quintom dark energy $S$ is

$$S = \int d^4x \sqrt{-g} \left( \frac{R}{16\pi G} + \mathcal{L}_{\text{stuff}} \right), \quad (21)$$

where $R$ is the Ricci scalar, $\mathcal{L}_{\text{stuff}}$ encloses all kinds of the stuff in the universe, for instance the dust matter, radiation, and quintom. At the late universe, the radiation can be negligible. So, often we only consider the dark energy, here quintom $\mathcal{L}_{\text{quintom}}$, and dust matter $\mathcal{L}_{\text{dm}},$ 

$$\mathcal{L}_{\text{quintom}} = -\frac{1}{2} \partial_\mu \phi \partial^\mu \phi + \frac{1}{2} \partial_\mu \psi \partial^\mu \psi - W(\phi, \psi). \quad (22)$$

In (22) the first term is the kinetic term of an ordinary scalar, the second term is the kinetic term of a phantom, and $W(\phi, \psi)$ is an arbitrary function of $\phi$ and $\psi$. In an FRW universe, the density and pressure of the quintom are

$$\rho = \frac{1}{2} \dot{\phi}^2 - \frac{1}{2} \dot{\psi}^2 + W, \quad (23)$$

$$p = \frac{1}{2} \dot{\phi}^2 - \frac{1}{2} \dot{\psi}^2 - W. \quad (24)$$
Hence, the EOS of the quintom $w$ is

$$w = \frac{1}{2} \dot{\phi}^2 - \frac{1}{2} \dot{\psi}^2 - W, \quad (25)$$

$w = -1$ requires

$$\dot{\phi}^2 = \dot{\psi}^2. \quad (26)$$

We see that in a quintom model, we do not require a static field (a field with zero kinetic term or a field at ground state) to get a cosmological constant. We only need that $\psi$ and $\phi$ evolves in the same step. $w < -1$ implies,

$$\dot{\phi}^2 - \dot{\psi}^2 < 0, \quad (27)$$

if

$$\frac{1}{2} \dot{\phi}^2 - \frac{1}{2} \dot{\psi}^2 + W > 0; \quad (28)$$

and

$$\dot{\phi}^2 - \dot{\psi}^2 > 0, \quad (29)$$

if

$$\frac{1}{2} \dot{\phi}^2 - \frac{1}{2} \dot{\psi}^2 + W < 0. \quad (30)$$

(30) yields an unnatural physical result, that is, the density of dark energy is negative. However, this is not as serious as the first glance, since we have little knowledge of the dark energy besides its effect of gravitation. Several evidences imply that we should go beyond the standard model of the particle physics when we describe dark energy. There are a few dark energy models permit density of the dark energy, or a component of it is negative (at the same time keep the total density positive), for example, see [27, 28]. But, for a model with only two components, a dust and a quintom, it is difficult to set a negative density dark energy. In that case we need too much dust than we observed or a big curvature term. In the following text of this section, we only consider a dark energy with positive density. So $w > -1$ implies,

$$\dot{\phi}^2 - \dot{\psi}^2 > 0. \quad (31)$$

In summary, if the kinetic term of the quintessence dominates that of phantom, the quintom behaves as quintessence; else it behaves as phantom. We should select a proper potential to make quintessence and phantom dominate alternatively such that we can realize the crossing behavior.
A simple choice of the potential is that the quintessence and the phantom do not interact with each other, which requires, \( W(\phi, \psi) = V(\phi) + U(\psi) \). The exponential potential is an important example which can be solved exactly in the quintessence model (a toy universe only composed by quintessence). In addition, we know that such exponential potentials of scalar fields occur naturally in some fundamental theories such as string/M theories. We introduce a model with such potentials in [29], in which the potential \( V(\phi, \psi) \) is given by

\[
W(\phi, \psi) = V(\phi) + U(\psi) = A_\phi e^{-\lambda_\phi \kappa \phi} + A_\psi e^{-\lambda_\psi \kappa \psi},
\]

where \( A_\phi \) and \( A_\psi \) are the amplitude of the potentials, \( \kappa^2 = 8\pi G \), \( \lambda_\phi \) and \( \lambda_\psi \) are two constants. Since there is no direct couple between the quintessence and the phantom, the equations of motion of the quintessence and the phantom are two independent equations,

\[
\ddot{\phi} + 3H \dot{\phi} + \frac{dV}{d\phi} = 0,
\]

\[
\ddot{\psi} + 3H \dot{\psi} - \frac{dU}{d\psi} = 0.
\]

The continuity equation of the dust reads,

\[
\rho_{\text{dust}} + 3H \rho_{\text{dust}} = 0,
\]

where \( \rho_{\text{dust}} \) denotes the density of the dust. The method of dynamical system has been widely used in cosmology. This method can offer a clear history of the cosmic evolution, especially the final states of the university. For applying this method, first we define the following dimensionless variables,

\[
x_\phi \equiv \frac{\kappa \dot{\phi}}{\sqrt{6} H}, \quad y_\phi \equiv \frac{\kappa \sqrt{V_\phi}}{\sqrt{3} H},
\]

\[
x_\psi \equiv \frac{\kappa \dot{\psi}}{\sqrt{6} H}, \quad y_\psi \equiv \frac{\kappa \sqrt{V_\psi}}{\sqrt{3} H},
\]

\[
z \equiv \frac{\kappa \sqrt{\rho_{\text{dust}}}}{\sqrt{3} H},
\]

(36)
the evolution equations (33)-(35) become,

$$x' = -3x \left(1 + x^2 - x^2 - \frac{1}{2}z^2 \right) + \lambda \sqrt{6} y^2,$$

$$(37)$$

$$y' = 3y \left(-x^2 + x^2 + \frac{1}{2}z^2 - \lambda \sqrt{6} x \right),$$

$$(38)$$

$$x' = -3x \left(1 + x^2 - x^2 - \frac{1}{2}z^2 \right) - \lambda \sqrt{6} y^2,$$

$$(39)$$

$$y' = 3y \left(-x^2 + x^2 + \frac{1}{2}z^2 - \lambda \sqrt{6} x \right),$$

$$(40)$$

$$z' = 3z \left(-x^2 + x^2 + \frac{1}{2}z^2 - \frac{1}{2} \right),$$

$$(41)$$

in which a prime denotes derivative with respect to ln $a$. Generally, $z$ in the above set will not be confused with redshift. The five equations in this system are not independent. They are constrained by Friedemann equation,

$$H^2 = \frac{\kappa^2}{3} \left( \frac{1}{2} \dot{\phi}^2 + V - \frac{1}{2} \dot{\psi}^2 + U + \rho_{dust} \right),$$

$$(42)$$

which becomes

$$x^2 + y^2 - x^2 + y^2 + z^2 = 1.$$  

$$(43)$$

with the dimensionless variables defined before. The critical points dwell at $x' = y' = x' = y' = z' = 0$. We present the result in table I. For detailed discussion of the critical points, see \[29\]. We would like to show a numerical example in which the EOS of the quintom crosses the phantom divide. Fig 6 illuminates that the EOS crosses $-1$.

The previous quintom model includes two fields, which are completely independent and rather arbitrary. We can impose some symmetry in the quintom model. An interesting model with an internal symmetry between the two fields which work as dark energy is hessence \[30\]. Rather than
two uncorrelated fields, we consider one complex scalar field with internal symmetry between the real and the imaginary parts,

\[ \Phi = \phi_1 + i\phi_2, \]  

(44)

with a Lagrangian density

\[ \mathcal{L}_{\text{hess}} = -\frac{1}{4} \left[ (\partial_\mu \Phi)^2 + (\partial_\mu \Phi^*)^2 \right] - V(\xi, \Phi^*) = -\frac{1}{2} \left[ (\partial_\mu \xi)^2 - \xi^2 (\partial_\mu \theta)^2 \right] - V(\xi), \]  

(45)

which is invariant under the transformation,

\[ \phi_1 \rightarrow \phi_1 \cos \alpha - i\phi_2 \sin \alpha, \]  

(46)

\[ \phi_2 \rightarrow -i\phi_1 \sin \alpha + \phi_2 \cos \alpha, \]  

(47)

if the potential is only a function of \( \Phi^2 + (\Phi^*)^2 \). For convenience, in (45) we have introduced two new variables \((\xi, \theta)\),

\[ \phi_1 = \xi \cosh \theta, \quad \phi_2 = \xi \sinh \theta, \]  

(48)

which are defined by

\[ \xi^2 = \phi_1^2 - \phi_2^2, \quad \coth \theta = \frac{\phi_1}{\phi_2}. \]  

(49)

The equations of motion of \( \xi \) and \( \theta \) are

\[ \ddot{\xi} + 3H \dot{\xi} + \xi \dot{\theta}^2 + \frac{dV}{d\xi} = 0, \]  

(50)
\[ \xi^2 \ddot{\theta} + (2\xi \dot{\xi} + 3H\xi^2) \dot{\theta} = 0. \] (51)

Clearly, \( \xi \) and \( \theta \) couple to each other. The pressure and density of the hessence read,

\[ p_{\text{hess}} = \frac{1}{2} \left( \dot{\xi}^2 - \xi^2 \dot{\theta}^2 \right) - V(\xi), \] (52)

\[ \rho_{\text{hess}} = \frac{1}{2} \left( \dot{\xi}^2 - \xi^2 \dot{\theta}^2 \right) + V(\xi), \] (53)

respectively. The EOS of hessence, playing as dark energy,

\[ w = \frac{\frac{1}{2} \left( \dot{\xi}^2 - \xi^2 \dot{\theta}^2 \right) - V(\xi)}{\frac{1}{2} \left( \dot{\xi}^2 - \xi^2 \dot{\theta}^2 \right) + V(\xi)}. \] (54)

Qualitatively, hessence evolves as quintessence when \( \dot{\xi}^2 \geq \xi^2 \dot{\theta}^2 \), while as phantom when \( \dot{\xi}^2 < \xi^2 \dot{\theta}^2 \).

The Lagrangian (45) does not include \( \theta \), hence the canonical momentum \( \pi^\mu_{\theta} \) corresponding to the cyclic coordinate \( \theta \) are conserved quantities,

\[ \pi^\mu_{\theta} = \partial (L_{\text{hess}} \sqrt{-g}) / \partial (\partial^\mu \theta). \] (55)

In an FRW universe, only \( \pi^0_{\theta} \) exists. We define a conserved quantity \( Q \) which is proportional to \( \pi^0_{\theta} \),

\[ Q = a^3 \xi^2 \dot{\theta}. \] (56)

With this conserved quantity, the EOS becomes,

\[ w = \frac{\frac{1}{2} \dot{\xi}^2 - \frac{Q^2}{2\xi \dot{\xi}^2} - V(\xi)}{\frac{1}{2} \dot{\xi}^2 - \frac{Q^2}{2\xi \dot{\xi}^2} + V(\xi)}, \] (57)

which is only a function of \( \xi \). The Friedmann equations read as

\[ H^2 = \frac{8\pi G}{3} \left[ \rho_{\text{dust}} + \frac{1}{2} \left( \dot{\xi}^2 - \xi^2 \dot{\theta}^2 \right) + V(\xi) \right], \] (58)

where \( \rho_{\text{dust}} \) is the energy density of dust. The continuity equation of dust is (35). The continuity equations of hessence are identical to the equations of motion (50) and (51). Then the system is closed and we present a numerical example in fig 7. Evidently, the EOS of hessence, playing the role of dark energy, crosses \(-1\) at about \( a = 0.95 \) (\( z = 0.06 \)).

After the presentation of hessence model, several aspects of this model have been investigated, including to avoid the big rip [31], attractor solutions for general hessence [32], reconstruction of hessence by recent observations [33], dynamics of hessence in frame of loop quantum cosmology [34], and holographic hessence model [35].
FIG. 7: The EOS of hessence \( w \) as a function of scale factor with the potential \( V(\xi) = \lambda \xi^4 \). The parameters for this plot are as follows: \( \Omega_{\text{m}0} = \rho_{\text{m}0}/(3H_0^2) = 0.3 \), \( \lambda = 5.0 \), \( Q = 1.0 \), \( a_0 = 1 \) and the unit \( 8\pi G = 1 \). From [30].

B. interacting model

Two-field model is a natural and obvious construction to realize the crossing \(-1\) behavior of dark energy. However, there are two many parameters in the set-up, though we can impose some symmetries to reduce the parameters to a smaller region. One symmetry decrease one parameter, but we have little clue to impose the symmetries since we have no evidence in the ground labs.

Interaction is a universal phenomenon in the physics world. An interaction term is helpful to cross the phantom divide. To illuminate this point, we first carefully analyze the previous observations which imply the crossing. Both the results of [18] and [19], which are shown in fig 4 and fig 5 respectively, are derived with a presupposition, that is, the dark energy evolves freely. In fact, what we observed is the effective EOS of the dark energy in the sense of gravity at the cosmological scale. When we suppose it evolves freely, we find that its EOS may cross the phantom divide. We can demonstrate for an essence with (local) EOS\(< -1\), the cosmological effective EOS can cross \(-1\) by aids of an interacting term. For the case with interaction, the continuity equation for dark energy becomes,

\[
\dot{\rho}_{\text{de}} + 3H(\rho_{\text{de}} + p_{\text{de}}) = -\Gamma, \tag{59}
\]

or

\[
\dot{\rho}_{\text{de}} + 3H(\rho_{\text{de}} + p_{\text{de}} + \frac{\Gamma}{3H}) = 0. \tag{60}
\]

Here \( \rho_{\text{de}} \) is the density of dark energy, \( p_{\text{de}} \) denotes the local pressure measured in the lab (if we can measure), \( \Gamma \) stands for the interaction term, and \( p_{\text{eff}} = p_{\text{de}} + \frac{\Gamma}{3H} \) is the effective pressure in the
cosmological sense. In a universe without expanding or contracting, $H = 0$, the interaction does no effect on the continuity equation, or energy conservation law, and thus does not yield surplus pressure \cite{71}. Two special cases are interesting: 1. $\frac{\Gamma}{3H}$ is a constant, under which the interaction term contributes a constant pressure throughout the history of the universe. 2. $\frac{\Gamma}{3H\rho_{de}}$ is a constant, under which the interaction term contributes a constant EOS in the history of the universe. In frame of a quintessence or phantom dark energy, the interaction term $\frac{\Gamma}{3H\rho_{de}}$ only shifts the EOS up or down by a constant distance in the $w - z$ plane, without changing the profile of the curve of $w$. While the term $\frac{\Gamma}{3H}$ shifts the pressure, which can change the EOS significantly since the density $\rho_{de}$ is a variable in the history of the universe.

If the dark energy can couple to some stuff of the universe, the dark matter is the best candidate. Although non-minimal coupling between the dark energy and ordinary matter fluids is strongly restricted by the experimental tests in the solar system \cite{37}, due to the unknown nature of the dark matter as part of the background, it is possible to have non-gravitational interactions between the dark energy and the dark matter components, without conflict with the experimental data. The continuity equation for dust-like dark matter reads,

$$\dot{\rho}_{dm} + 3H\rho_{dm} = \Gamma. \quad (61)$$

Based on the previous discussion, we assume a most simple case

$$\Gamma = H\delta \rho_{dm}, \quad (62)$$

where $\delta$ is constant \cite{38, 39, 40}. This interaction term shifts a constant to the EOS of dark matter, that is, it is no longer evolving as $(1 + z)^3$. We uniformly deal with quintessence and phantom, which are often labeled by $X$, with a constant EOS $w_X$. So, the continuity equation of dark energy can be written as,

$$\dot{\rho}_X + 3H(\rho_X + w_X\rho_X) = -H\delta \rho_{dm}. \quad (63)$$

Integrating (61), we derive

$$\rho_{dm} = \rho_{dm0}a^{-3+\delta} = \rho_{dm0}(1 + z)^{3-\delta}. \quad (64)$$

Substituting to (63), we reach

$$\rho_X = \rho_{X0}(1 + z)^{3(1+w_X)} + \rho_{dm0}\frac{\delta}{\delta + 3w_X} \left[(1 + z)^{3(1+w_X)} - (1 + z)^{3-\delta}\right]. \quad (65)$$
Only from the above equation, we can extract the effective EOS of the dark energy. To see this point, we make a short discussion. In a dynamical universe with interaction, the effective EOS of dark energy reads,

$$w_{de} = \frac{p_{de}}{\rho_{de}} = \frac{p_{de} + \Gamma/3H}{\rho_{de}} = -1 + \frac{1}{3} \frac{d \ln \rho_{de}}{d \ln (1+z)}.$$  \hspace{1cm} (66)

Clearly, if \(\frac{d \ln \rho_{de}}{d \ln (1+z)}\) is greater than 0, dark energy evolves as quintessence; if \(\frac{d \ln \rho_{de}}{d \ln (1+z)}\) is less than 0, it evolves as phantom; if \(\frac{d \ln \rho_{de}}{d \ln (1+z)}\) equals 0, it is just cosmological constant. In a more intuitionistic way, if \(\rho_{de}\) decreases and then increases with respect to redshift (or time), or increases and then decreases, which implies that EOS of dark energy crosses phantom divide. So, some time we directly use the evolution of density of dark energy to describe the EOS of it. There is a more important motivation to use the density directly: the density is more closely related to observables, hence is more tightly constrained for the same number of redshift bins used [41].

The derivative of \(\rho_X\) with respect to \((1+z)\) reads,

$$\frac{d \rho_X}{d(1+z)} = 3(1+w_X)\rho_X(1+z)^2 + \rho_{dm0} + \frac{\delta}{3w_X} \left[3(1+w_X)(1+z)^2 - (3-\delta)(1+z)^{2-\delta}\right].$$  \hspace{1cm} (67)

If \(\frac{d \rho_X}{d(1+z)} = 0\) at some redshift \(z = z_c\), the effective EOS crosses \(-1\). The result is illuminated by fig 8, in which we set \(z_c = 0.3\) as an example. This figure displays the corresponding \(w_X\) when one fixes a \(\delta\), or vice versa if we require the EOS crosses \(-1\) at \(z_c = 0.3\). This is an original figure plotted for this review article.

Then the Friedmann equation reads,

$$\frac{H^2}{H_0^2} = \Omega_{X0}(1+z)^3(1+w_X) + \frac{1 - \Omega_{X0}}{\delta + 3w_X} \left[\delta(1+z)^3(1+w_X) + 3w_X(1+z)^{3-\delta}\right].$$  \hspace{1cm} (68)
FIG. 9: Constraints of \((w_X, \delta)\) by SNLS data at 68.3%, 95.4% and 99.7% confidence levels marginalized over \(\Omega_{X0}\) with priors \(\Omega_{X0} = 0.72 \pm 0.04\) and \(\delta < 3\). From [38]

where \(\Omega_{X0} = \frac{\kappa^2 \rho_{X0}}{(3H_0^2)}\), and we have used \(\Omega_{dm0} + \Omega_{X0} = 1\). Thus we need to constrain the three parameters \(\delta, \Omega_{X0}, w_X\). The constraint result by SNLS data is shown in fig 9. \(\delta = 0\) and \(w_X = -1\) are indicated by the horizontal and vertical dashed lines, which represent the non-interacting XCDM model and interacting \(\Lambda\)CDM model, respectively.

From fig 8 and 9, we see that the observations leave enough space for the parameters \((\delta, w_X)\) to cross the phantom divide.

In the previous interacting model, we consider a phenomenological interaction, which is put in “by hand”. We should find a more sound physical foundation for the interactions. We will deduce an interaction term from the low energy limit of string/M theory in the scenario of the interacting Chaplygin gas model [27].

The Chaplygin gas model was suggested as a candidate of a unified model of dark energy and dark matter [42]. The Chaplygin gas is characterized by an exotic equation of state

\[
p_{ch} = -\frac{A}{\rho_{ch}},
\]

where \(A\) is a positive constant. The above equation of state leads to a density evolution in the form

\[
\rho_{ch} = \sqrt{A + \frac{B}{a^6}},
\]

where \(B\) is an integration constant. The attractive feature of the model is that it naturally unifies both dark energy and dark matter. The reason is that, from (70), the Chaplygin gas behaves as dust-like matter at early stage and as a cosmological constant at later stage.
Though Chaplygin gas has such a nice property, it is a serious flaw when one studies the fluctuation growth in Chaplygin gas model. It is found that Chaplygin gas produces oscillations or exponential blowup of the matter power spectrum, which is inconsistent with observations. So we turn to a model that the Chaplygin gas only plays the role of dark energy. To cross the phantom divide we consider a model in which the Chaplygin gas couples to dark matter.

Although non-minimal coupling between the dark energy and ordinary matter fluids is strongly restricted by the experimental tests in the solar system, due to the unknown nature of the dark matter as part of the background, it is possible to have non-gravitational interactions between the dark energy and the dark matter components, without conflict with the experimental data. Thus, the observation constrain the only proper candidate to be coupled to Chaplygin gas is dark matter.

We consider the original Chaplygin gas, whose pressure and energy density satisfy the relation, $p_{ch} = -A/\rho_{ch}$. By assuming the cosmological principle the continuity equations are written as

$$\dot{\rho}_{ch} + 3H\gamma_{ch}\rho_{ch} = -\Gamma,$$  \hspace{1cm} (71)

and

$$\dot{\rho}_{dm} + 3H\gamma_{dm}\rho_{dm} = \Gamma,$$  \hspace{1cm} (72)

where the subscript $dm$ denotes dark matter, and $\gamma$ is defined as

$$\gamma = 1 + \frac{p}{\rho} = 1 + w,$$  \hspace{1cm} (73)

in which $w$ is the parameter of the state of equation, and $\gamma_{dm} = 1$ throughout the evolution of the universe, whereas $\gamma_{ch}$ is a variable.

$\Gamma$ is the interaction term between Chaplygin gas and dark matter. Since there does not exist any microphysical hint on the possible nature of a coupling between dark matter and Chaplygin gas (as dark energy), the interaction terms between dark energy and dark matter are rather arbitrary in literatures. Here we try to present a possible origin from fundamental field theory for $\Gamma$.

Whereas we are still lack of a complete formulation of unified theory of all interactions (including gravity, electroweak and strong), there at present is at least one hopeful candidate, string/M theory. However, the theory is far away from mature such that it is still not known in a way that would enable us to ask the questions about space-time in a general manner, say nothing of the properties of realistic particles. Instead, we have to either resort to the effective action approach which takes into account stringy phenomena in perturbation theory, or we could study some special classes of string solutions which can be formulated in the non-perturbative regime. But the latter approach
is available only for some special solutions, most notably the BPS states or nearly BPS states in the string spectrum: They seems to have no relation to our realistic Universe. Especially, there still does not exist a non-perturbative formulation of generic cosmological solutions in string theory. Hence nearly all the investigations of realistic string cosmologies have been carried out essentially in the effective action range. Note that the departure of string-theoretic solutions away from general relativity is induced by the presence of additional degrees of freedom which emerge in the massless string spectrum. These fields, including the scalar dilaton field, the torsion tensor field, and others, couple to each other and to gravity non-minimally, and can influence the dynamics significantly. Thus such an effective low energy string theory deserve research to solve the dark energy problem. There a special class of scalar-tensor theories of gravity is considered to avoid singularities in cosmologies in [44]. The action is written below,

\[
S_{st} = \int d^4x \sqrt{-g} \left[ \frac{1}{16\pi G} R - \frac{1}{2} \partial_{\mu} \phi \partial^{\mu} \phi + \frac{1}{q(\phi)^2} L_{dm}(\xi, \partial \xi, q^{-1} g_{\mu \nu}) \right],
\]

(74)

where \( G \) is the Newton gravitational constant, \( \phi \) is a scalar field, \( L_{dm} \) denotes Lagrangian of matter, \( \xi \) represents different matter degrees of matter fields, \( q \) guarantees the coupling strength between the matter fields and the dilaton. With action (74), the interaction term can be written as follow [44],

\[
\Gamma = H \rho_{dm} \frac{d \ln q'}{d \ln a}.
\]

(75)

Here we introduce new variable \( q(a)' \triangleq q(a)^{(3w_{n}-1)/2} \), where \( a \) is the scale factor in standard FRW metric. By assuming

\[
q'(a) = q_0 e^{3 \int c(\rho_{dm} + \rho_{\xi})/\rho_{dm} d \ln a},
\]

(76)

where \( \rho_{dm} \) and \( \rho_{\xi} \) are the densities of dark matter and the scalar field respectively, one arrive at the interaction term,

\[
\Gamma = 3Hc(\rho_{dm} + \rho_{\phi}).
\]

(77)

With this interaction form we study the equation set (71) and (72). Set \( s = -\ln(1 + z) \), \( \Gamma = 3Hc(\rho_{ch} + \rho_{dm}) \), \( u = (3H_0^2)^{-1}(3\mu^2)^{-1}\rho_{dm} \), \( v = (3H_0^2)^{-1}(3\mu^2)^{-1}\rho_{ch} \), \( A' = A(3H_0^2)^{-2}(3\mu^2)^{-2} \), where \( c \) is a constant without dimension. Using these variables, (71) and (72) reduce to

\[
\frac{du}{ds} = -3u + 3c(u + v),
\]

(78)

\[
\frac{dv}{ds} = -3(v - A'/v) - 3c(u + v).
\]

(79)
We note that the variable time does not appear in the dynamical system (78) and (79) because time has been completely replaced by redshift $s = -\ln(1 + z)$. The critical points of dynamical system (78) and (79) are given by
\[
\frac{du}{ds} = \frac{dv}{ds} = 0.
\] (80)

The solution of the above equation is
\[
u_c = \frac{c}{1 - c} v_c, \quad v_c^2 = (1 - c) A'.
\] (81) (82)

We see the final state of the model contains both Chaplygin gas and dark matter of constant densities if the singularity is stationary. The final state satisfies perfect cosmological principle: the universe is homogeneous and isotropic in space, as well as constant in time. Physically $\Gamma$ in (72) plays the role of matter creation term $C$ in the theory of steady state universe at the future time-like infinity. Recall that $c$ is the coupling constant, may be positive or negative, corresponds the energy to transfer from Chaplygin gas to dark matter or reversely. $A'$ must be a positive constant, which denotes the final energy density if $c$ is fixed. Also we can derive an interesting and simple relation between the static energy density ratio
\[
\frac{c}{1 + r_s} = \frac{r_s}{r_s - 1}
\] (83)

where
\[
r_s = \lim_{z \to -1} \frac{\rho_{dm}}{\rho_{ch}}.
\] (84)

To investigate the properties of the dynamical system in the neighbourhood of the singularities, impose a perturbation to the critical points,
\[
\frac{d(\delta u)}{ds} = -3\delta u + 3c(\delta u + \delta v), \quad \frac{d(\delta v)}{ds} = -3(\delta v + A' v_c^2 \delta v) - 3c(\delta u + \delta v).\] (85) (86)

The eigen equation of the above linear dynamical system $(\delta u, \delta v)$ reads
\[
\left(\frac{\lambda}{3}\right)^2 + (2 + \frac{1}{1 - c})\lambda + 2 - 2c^2 = 0,
\] (87)

whose discriminant is
\[
\Delta = [(1 - c)^4 + (3/2 - c)^2]/(1 - c)^2 \geq 0.
\] (88)
FIG. 10: The plane $v$ versus $u$. (a) left panel: We consider the evolution of the universe from redshift $z = e^2 - 1$. The initial condition is taken as $u = 0$, $v = 400$; $u = 50$, $v = 350$; $u = 100$, $v = 300$; $u = 120$, $v = 280$ on the four orbits, from the left to the right, respectively. It is clear that there is a stationary node, which attracts most orbits in the first quadrant. At the same time the orbits around the neighbourhood of the singularity is not shown clearly. (b) right panel: Orbit distributions around the node $u_c = v_c/(1 - c)$, $v_c = \sqrt{(1 - c)A'}$. From [27]

Therefore both of the two roots of eigen equation (87) are real, consequently centre and focus singularities can not appear. Furthermore only $r_s \in (0, \infty)$, such that $c \in (0, 1)$, makes physical sense. Under this condition it is easy to show that both the two roots of (87) are negative. Hence the two singularities are stationary. However it is only the property of the linearized system (85) and (86), or the property of orbits of the neighbourhoods of the singularities, while global Poincare-Hopf theorem requires that the total index of the singularities equals the Euler number of the phase space for the non-linear system (78) and (79). So there exists other singularity except for the two nodes. In fact it is a non-stationary saddle point at $u = 0$, $v = 0$ with index $-1$. This singularity has been omitted in solving equations (78) and (79). The total index of the three singularities is 1, which equals the Euler number of the phase space of this plane dynamical system. Hence there is no other singularities in this system. From these discussions we conclude that the global outline of the orbits of this non-linear dynamical system (78) and (79) is similar to the electric fluxlines of two negative point charges. Here we plot figs 10 and 11 to show the properties of evolution of the universe controlled by the dynamical system (78) and (79). As an example we set $c = 0.2$, $A' = 0.9$ in figs 10-12.

Further, to compare with observation data we need the explicit forms of $u(x)$ and $v(x)$, especially $v(x)$. We need the properties of $\gamma_{ch}$ in our model, which is contained in $v(x)$, to compare with
we have to include some "unphysical" initial conditions, such as $u = -100$, $v = -300$, except for physical initial conditions which have been shown in figure 10. (b) right panel: Orbits distributions around the nodes. The two nodes $u_c = v_c c / (1 - c)$, $v_c = \sqrt{(1 - c) A'}$ and $u_c = v_c c / (1 - c)$, $v_c = -\sqrt{(1 - c) A'}$ keep reflection symmetry about the original point. Just as we have analyzed, we see that the orbits of this dynamical system are similar to the electric fluxlines of two negative point charges. From

observations. Eliminate $u(x)$ by using (78) and (79) we derive

$$\frac{1}{3c} \frac{d^2v}{ds^2} + \left[1 + \frac{(1 + A' / v^2)}{c}\right] \frac{dv}{ds} + 3cv + 3(1 - c)\left\{v + \left[\frac{dv}{ds} + 3(v - A' / v)\right] \right\} = 0, \quad (89)$$

which has no analytic solution. We show some numerical solutions in figure 12. We find that for proper region of parameter spaces, the effective equation of state of Chaplygin gas crosses the phantom divide successfully.

Up to now all of our results do not depend on Einstein field equation. They only depend on the most sound principle in physics, that is, the continuity principle, or the energy conservation law. Different gravity theories correspond to different constraints imposed on our previous discussions. Our improvements show how far we can reach without information of dynamical evolution of the universe.

(78) illuminates that the dark matter in this interacting model does not behaves as dust. Qualitatively, the dark matter gets energy from dark energy for a positive $c$, and becomes soft, ie, its energy density decreases slower than $(1 + z)^3$ in an expanding universe. The parameter which carries the total effects of cosmic fluids is the deceleration parameter $q$. From now on we introduce the Friedmann equation of the standard general relativity. As a simple case we study the evolution of $q$ in a spatially flat universe. So $q$ reads
FIG. 12: $v$ versus $s$. The evolution of $v$ with different initial conditions $u(-2) = 0$, $v(-2) = 400$; $u(-2) = 50$, $v(-2) = 350$; $u(-2) = 100$, $v(-2) = 300$ reside on the blue, red, and yellow curves, respectively. Obviously the energy density of Chaplygin gas rolls down and then climbs up in some low redshift region. So the Chaplygin gas dark energy can cross the phantom divide $w = -1$ in a fitting where the dark energy is treated as an independent component to dark matter. From [27], this figure has been re-plotted.

\[ q = \frac{-\ddot{a}a}{\dot{a}^2} = \frac{1}{2} \left( \frac{u + v - 3A'/v^2}{u + v} \right), \]  

(90)

and density of Chaplygin gas $u$ and density of dark matter $v$ should satisfy

\[ u(0) + v(0) = 1. \]  

(91)

And then Friedmann equation ensures the spatial flatness in the whole history of the universe. Before analyzing the evolution of $q$ with redshift, we first study its asymptotic behaviors. When $z \to \infty$, $q$ must go to 1/2 because both Chaplygin gas and dark matter behave like dust, while when $z \to -1$ $q$ is determined by

\[ \lim_{z \to -1} q = \frac{1}{2} \left( \frac{u_c + v_c - 3A'/v_c^2}{u_c + v_c} \right). \]  

(92)

One can finds the parameters $c = 0.2$, $A' = 0.9$ are difficult to content the previous constraint Friedmann constraint (91). Here we carefully choose a new set of parameter which satisfies Friedmann constraint (91), say, $A' = 0.4$, $c = 0.06$. Therefore we obtain

\[ \lim_{z \to -1} q = -1.95, \]  

(93)

by using (81) and (82). Then we plot figure 13 to clearly display the evolution of $q$. One can check $u(0) = 0.25$, $v(0) = 0.75$; $u(0) = 0.28$, $v(0) = 0.72$; $u(0) = 0.3$, $v(0) = 0.7$, respectively on
FIG. 13: $q$ versus $s$. The evolution of $q$ with different initial conditions $u(-2) = 0, v(-2) = 273; u(-2) = 15, v(-2) = 250; u(-2) = 25, v(-2) = 233$, reside on the blue, red, and yellow curves, respectively. Evidently the deceleration parameter $q$ of Chaplygin gas rolls down and crosses $q = 0$ in some low redshift region. The transition from deceleration phase to acceleration phase occurs at $z = 0.18; z = 0.21; z = 0.23$ to the curves $u(-2) = 0, v(-2) = 273; u(-2) = 15, v(-2) = 250; u(-2) = 25, v(-2) = 233$, respectively. One finds $-q \approx 0.5 \sim 0.6$ at $z = 0$, which is well consistent with observations. From [27], this figure has been re-plotted.

The curves $v(-2) = 273; v(-2) = 250; v(-2) = 233$. One may find an interesting property of the deceleration parameter displayed in fig 13: the bigger the proportion of the dark energy, the smaller the absolute value of the deceleration parameter. The reason roots in the extraordinary state of Chaplygin gas [69], in which the pressure $p_{ch}$ is inversely proportional to the energy density $\rho_{ch}$.

Also we note that maybe an FRW universe with non-zero spatial curvature fits deceleration parameter better than spatially flat FRW universe. This point deserves to research further.

After the presentation of the original interacting Chaplygin gas model, there are several generalizations. For details of these generalizations, see [45].

C. model in frame of modified gravity

The judgement that there exists an exotic component with negative pressure, or dark energy, which accelerates the universe, is derived in frame of general relativity. The validity of general relativity has been well tested from the scale of millimeter to the scale of the solar system. Beyond this scale, the evidences are not so sound. So we should not be surprised if general relativity fails at the scale of the Hubble radius. Surely, any new gravity theory must reduce to general relativity at
the scale between millimeter to the solar system. In frame of the new gravity theories, the cosmic acceleration may be a natural result even we only have dust in the universe.

There are various suggestions on how to modify general relativity. In this brief review we concentrates on the brane world theory. Inspired by the developments of string/M theory, the idea that our universe is a 3-brane embedded in a higher dimensional spacetime has received a great deal of attention in recent years. In this brane world scenario, the standard model particles are confined on the 3-brane, while the gravitation can propagate in the whole space. In this picture, the gravity field equation gets modified at the left hand side (LHS) in (4), while the dark energy is a stuff put at the right hand side (RHS) in (4). In the modified gravity model, the surplus geometric terms respective to the Einstein tensor play the role of the dark energy in general relativity.

We consider a 3-brane imbedded in a 5-dimensional bulk. The action includes the action of the bulk and the action of the brane,

\[ S = S_{\text{bulk}} + S_{\text{brane}}. \] (94)

Here

\[ S_{\text{bulk}} = \int_{\mathcal{M}} d^5X \sqrt{-g_5} \mathcal{L}_{\text{bulk}}, \] (95)

where \( X = (t, z, x^1, x^2, x^3) \) is the bulk coordinate, \( x^1, x^2, x^3 \) are the coordinates of the maximally symmetric space. \( \mathcal{M} \) denotes the bulk manifold. The bulk Lagrangian can be

\[ \mathcal{L}_{\text{bulk}} = \frac{1}{2\kappa_5^2} [R_5 + \alpha F(R_5)] + \mathcal{L}_m + \Lambda_5, \] (96)

where \( g_5, \kappa_5, R_5, \mathcal{L}_m, \) denote the bulk manifold, the determinant of the bulk metric, the 5-dimensional Newton constant, the 5-dimensional Ricci scalar, and the bulk matter Lagrangian, respectively. \( F(R_5) \) denotes the higher order term of scalar curvature \( R_5 \), the Ricci curvature \( R_{5AB} \), the Riemann curvature \( R_{5ABCD} \).

There are too much possibilities and rather arbitrary to choose the higher order terms. Generally the resulting equations of motion of such a term give more than second derivatives of metric and the resulting theory is plagued by ghosts. However there exists a combination of quadratic terms, called Gauss-Bonnet term, which generates equation of motion without the terms more than second derivatives of metric and the theory is free of ghosts \[^{[46]}\]. Another important property of Gauss-Bonnet term is that, just like Hilbert Lagrangian is a pure divergence in 2 dimensions and Einstein tensor identifies zero in 1 and 2 dimensions, we have that in 4 or less dimension the Gauss-Bonnet Lagrangian is a pure divergence. We see the dilemma of quadratic term in 4 dimensional theory:
if we include it with non pure divergence we shall confront ghosts; if we want to remove ghosts we get a pure divergence term. So only in theories in more than 4 dimensional Gauss-Bonnet combination provides physical effects. Moreover the Gauss-Bonnet term also appears in both low energy effective action of Bosonic string theory \[47\] and low energy effective action of Bosonic modes of heterotic and type II super string theory \[48\]. An investigation into the effects of a Gauss-Bonnet term in the 5 dimensional bulk of brane world models is therefore well motivated. The Gauss-Bonnet term in 5 dimension reads,

\[
F(R_5) = R^2_5 - 4R_{5AB}R^{AB}_5 + R_{5ABCD}R^{ABCD}_5. \tag{97}
\]

The action of the brane can be written as,

\[
S_{brane} = \int_M d^4x \sqrt{-g} \left( \kappa_5^{-2}K + L_{brane} \right), \tag{98}
\]

where \(M\) indicates the brane manifold, \(g\) denotes the determinant of the brane metric, \(L_{brane}\) stands for the Lagrangian confined to the brane, and \(K\) marks the trace of the second fundamental form of the brane. \(x = (\tau, x^1, x^2, x^3)\) is the brane coordinate. Note that \(\tau\) is not identified with \(t\) if the the brane is not fixed at a position in the extra dimension \(z = \text{constant}\). We will investigate the cosmology of a moving brane along the extra dimension \(z\) in the bulk, and such that \(\tau\) is different from \(t\).

We set the Lagrangian confined to the brane as follows,

\[
L_{brane} = \frac{1}{16\pi G} R - \lambda + L_m, \tag{99}
\]

where \(\lambda\) is the brane tension and \(L_m\) denotes the ordinary matter, such as dust and radiation, located at the brane. \(R\) denotes the 4 dimensional scalar curvature term on the brane, which is an important one except a Gauss-Bonnet term in the bulk. This induced gravity correction arises because the localized matter fields on the brane, which couple to bulk gravitons, can generate via quantum loops a localized four-dimensional world-volume kinetic term for gravitons \[49\].

Assuming there is a mirror symmetry in the bulk, we have the Friedmann equation on the brane \[50\], see also \[51\],

\[
\frac{4}{r_c^2} \left[ 1 + \frac{8}{3\alpha} \left( H^2 + \frac{k}{a^2} + \frac{U}{2} \right) \right]^2 \left( H^2 + \frac{k}{a^2} - U \right) = \left( H^2 + \frac{k}{a^2} - \frac{8\pi G}{3} (\rho + \lambda) \right)^2, \tag{100}
\]

where

\[
U = -\frac{1}{4\alpha} \pm \frac{1}{4\alpha} \sqrt{1 + 4\alpha \left( \frac{\Lambda_5}{6} + \frac{M\kappa_5^2}{4\pi a^4} \right)}, \tag{101}
\]
Here $M$ is a constant, standing for the mass of bulk black hole. For various limits of (100), see [52].

For convenience, we introduce the following new variables and parameters,

\[
x \equiv \frac{H^2}{H_0^2} + \frac{k}{a^2 H_0^2} = \frac{H^2}{H_0^2} - \Omega_{k0}(1 + z)^2,
\]

\[
u \equiv \frac{8\pi(4)G}{3H_0^2}(\rho + \lambda) = \Omega_{m0}(1 + z)^3 + \Omega_{\lambda},
\]

\[
m \equiv \frac{8}{3}\alpha H_0^2,
\]

\[
n \equiv \frac{H^2r_c^2}{H_0^2},
\]

\[
y \equiv \frac{1}{2} U H_0^{-2} = \frac{1}{3m} \left(-1 + \sqrt{1 + \frac{4\alpha\Lambda_5}{6} + \frac{8\alpha M(5)G}{a^4}}\right)
\]

\[
= \frac{1}{3m} \left(-1 + \sqrt{1 + m\Omega_{\Lambda_5} + m\Omega_{M0}(1 + z)^4}\right),
\]

(103)

and we have assumed that there is only pressureless dust in the universe. As before, we have used the following notations

\[
\Omega_{k0} = -\frac{k}{a_0^2 H_0^2}, \quad \Omega_{m0} = \frac{8\pi G \rho_{m0}}{3 H_0^2}, \quad \Omega_{\lambda} = \frac{8\pi G \lambda}{3 H_0^2}, \quad \Omega_{\Lambda_5} = \frac{3\Lambda_5}{8H_0^2}, \quad \Omega_{M0} = \frac{3M\kappa_5^2}{8\pi a_0^4 H_0^2}.
\]

(104)

With these new variables and parameters, (100) can be rewritten as

\[
4n(x - 2y)[1 + m(x + y)]^2 = (x - u)^2.
\]

(105)

This is a cubic equation of the variable $x$. According to algebraic theory it has 3 roots. One can explicitly write down three roots. But they are too lengthy and complicated to present here. Instead we only express those three roots formally in the order given in Mathematica

\[
x_1 = (y, u|m, n),
\]

\[
x_2 = (y, u|m, n),
\]

\[
x_3 = (y, u|m, n),
\]

(106)

where $y$ and $u$ are two variables, $m$ and $n$ stand for two parameters. The root on $x$ of the equation (105) gives us the modified Friedmann equation on the Gauss-Bonnet brane world with induced gravity. From the solutions given in (106), this model seems to have three branches. In addition, note that all parameters introduced in (103) and (104) are not independent of each other. According
to the Friedmann equation (106), when all variables are taken current values, for example, \( z = 0 \), the Friedmann equation will give us a constraint on those parameters,

\[
1 = f(\Omega_{k0}, \Omega_{m0}, \Omega_{M0}, \Omega_{\Lambda}, \Omega_{\lambda}, m, n). \tag{107}
\]

To compare with observation, we introduce the concept “equivalent dark energy” or “virtual dark energy” in the modified gravity models, since almost all the properties of dark energy are deduced in the frame of general relativity with a dark energy.

The Friedmann equation in the four dimensional general relativity can be written as

\[
H^2 + \frac{k}{a^2} = \frac{8\pi G}{3}(\rho + \rho_{de}), \tag{108}
\]

where the first term of RHS of the above equation represents the dust matter and the second term stands for the dark energy. Generally speaking the Bianchi identity requires,

\[
\frac{d\rho_{de}}{dt} + 3H(\rho_{de} + p_{de}) = 0, \tag{109}
\]

we can then express the equation of state for the dark energy as

\[
w_{de} = \frac{p_{de}}{\rho_{de}} = -1 - \frac{1}{3} \frac{d\ln \rho_{de}}{d\ln a}. \tag{110}
\]

Note that we can rewrite the Friedmann equation (106) in the form of (108) as

\[
xH_0^2 = \frac{8\pi G}{3} \rho + \left( H_0^2 x(y, u|m, n) - \frac{8\pi G}{3} \rho \right) = \frac{8\pi G}{3}(\rho + Q), \tag{111}
\]

where \( \rho \) is the energy density of dust matter on the brane and the term

\[
Q \equiv \frac{3H_0^2}{8\pi G} x(y, u|m, n) - \rho \tag{112}
\]

corresponds to \( \rho_{de} \) in (108).

In fig 14 we show the equation of state for the virtual dark energy when we take \( m = 1.036 \) and \( n = 0.04917 \). In this case, from the constraint equation (128), one has \( \Omega_{M0} = 2.08 \). From the figure we see that \( w_{eff} < -1 \) at \( z = 0 \) and

\[
\frac{dw_{eff}}{dz} \bigg|_{z=0} < 0.
\]

Therefore the equation of state for the virtual dark energy can indeed cross the phantom divide \( w = -1 \) near \( z \sim 0 \).

Fig 14 illuminates that the behavior of the virtual dark energy seems rather strange. However we should remember that it is only virtual dark energy, not actual stuff. The whole evolution of the
FIG. 14: The equation of state $w_{\text{eff}}$ with respect to the red shift $1 + z$, with $\Omega_m = 0.28$ and $\Omega_A = 2.08$. From [52].

$w_{\text{eff}} = 1.036, \ n = 0.04917$

FIG. 15: $H^2/H_0^2$ versus $1 + z$, with $\Omega_m = 0.28$ and $\Omega_A = 2.08$. From [52].

$H^2 = H_0^2$

The universe is described by the Hubble parameter. We plot the Hubble parameter $H$ corresponding to fig.14 in fig.15.

Fig.15 displays that the universe will eventually becomes a de Sitter one. For more figures with different parameters, see [52]. The constraint of this brane model with induced scalar term on the brane and Gauss-Bonnet term in the bulk has been investigated in [53].

The above is an example of “pure geometric” dark energy. We can also consider some mixed dark energy model, ie, the cosmic acceleration is driven by an exotic matter and some geometric effect in part. Why such an apparently complicated suggestion? There are many interesting models are proposed to explain the cosmic acceleration, including dark energy and modified gravity.
models. However, several influential and hopeful models, such as quintessence and DGP model, fundamentally can not account for the crossing $-1$ behavior of dark energy. By contrast, some hybrid model of the dark energy and modified gravity may realize such a crossing. As an example we study the quintessence and phantom in frame of DGP \[54\].

Our starting point is still action (94). In a DGP model with a scalar, (96) becomes a pure Einstein-Hilbert action,

$$L_{\text{bulk}} = \frac{1}{2\kappa_5^2} R_5,$$

and we add a scalar term in (99),

$$L_{\text{brane}} = \frac{1}{16\pi G} R + L_m + L_{\text{scalar}}.$$

Here the scalar term can be ordinary scalar (quintessence) or phantom (scalar with negative kinetic term). The Lagrangian of a quintessence reads,

$$L_{\phi} = -\frac{1}{2} \partial_{\mu} \phi \partial^{\mu} \phi - V(\phi),$$

and for phantom,

$$L_{\psi} = \frac{1}{2} \partial_{\mu} \psi \partial^{\mu} \psi - U(\psi).$$

In an FRW universe we have

$$\rho_{\phi} = \frac{1}{2} \dot{\phi}^2 + V(\phi),$$

$$p_{\phi} = \frac{1}{2} \dot{\phi}^2 - V(\phi).$$

The exponential potential is an important example which can be solved exactly in the standard model. Also it has been shown that the inflation driven by a scalar with exponential potential can exit naturally in the warped DGP model \[55\]. It is therefore quite interesting to investigate a scalar with such a potential in late time universe on a DGP brane. Here we set

$$V = V_0 e^{-\lambda_1 \frac{\phi}{\mu}}.$$

Here $\lambda_1$ is a constant and $V_0$ denotes the initial value of the potential.

The Friedmann equation (100) becomes

$$H^2 + \frac{k}{a^2} = \frac{1}{3\mu^2} \left[ \rho + \rho_0 + \theta \rho_0 (1 + \frac{2\rho}{\rho_0})^{1/2} \right],$$

(120)
where
\[ \rho_0 = \frac{6\mu^2}{r_c^2}. \] (121)

Similar to the previous case, we derive the virtual dark energy by comparing (120) and (108),
\[ \rho_{de} = \rho_\phi + \rho_0 + \theta \rho_0 \left[ \rho + \rho_0 + \theta \rho_0 (1 + \frac{2\rho}{\rho_0})^{1/2} \right]. \] (122)

From (110), we calculate the derivation of effective density of dark energy with respective to \( \ln(1 + z) \) for a ordinary scalar,
\[ \frac{d\rho_{de}}{d\ln(1 + z)} = \frac{3(\dot{\phi}^2 + \theta (1 + \frac{\dot{\phi}^2 + 2V + 2\rho_{dm}}{\rho_0})^{-1/2} (\dot{\phi}^2 + \rho_{dm}))}{2}. \] (123)

If \( \theta = 1 \), both terms of RHS are positive, hence it never goes to zero at finite time. But if \( \theta = -1 \), the two terms of RHS carry opposite sign, therefore it is possible that the EOS of dark energy crosses phantom divide. In a scalar-driven DGP, we only consider the case of \( \theta = -1 \).

For convenience, we define some dimensionless variables,
\[ y_1 \triangleq \frac{\dot{\phi}}{\sqrt{6\mu H}}, \] (124)
\[ y_2 \triangleq \frac{\sqrt{V}}{\sqrt{3\mu H}}, \] (125)
\[ y_3 \triangleq \frac{\sqrt{\rho_m}}{\sqrt{3\mu H}}, \] (126)
\[ y_4 \triangleq \frac{\sqrt{\rho_0}}{\sqrt{3\mu H}}. \] (127)

The Friedmann equation (120) becomes
\[ y_1^2 + y_2^2 + y_3^2 + y_4^2 \left( 1 + 2y_1^2 + y_2^2 + y_3^2 \right)^{1/2} = 1. \] (128)

The stagnation point, that is, \( d\rho_{de}/d\ln(1 + z) = 0 \) dwells at
\[ \frac{y_4}{\sqrt{2 + y_4}} \left( 2 + \frac{y_3^2}{y_1^2} \right) = 2, \] (129)
which can be derived from (123) and (128).

One concludes from the above equation that a smaller \( r_c \), a smaller \( \Omega_{m0} \) (Recall that it is defined as the present value of the energy density of dust matter over the critical density), or a larger \( \Omega_{k0} \) (which is defined as the present value of the kinetic energy density of the scalar over the critical density) is helpful to shift the stagnation point to lower redshift region. We show a
concrete numerical example of this crossing behaviours in fig [16]. For convenience we introduce the
dimensionless density and rate of change with respect to redshift of dark energy as below,
\[
\beta = \frac{\rho_{de}}{\rho_c} = \frac{\Omega_{rc}}{y^2} \left[ y_1^2 + y_2^2 + y_3^2 - y_4^2 (1 + 2 \frac{y_1^2 + y_2^2 + y_3^2}{y_4^2})^{1/2} \right],
\] (130)
where \( \rho_c \) denotes the present critical density of the universe, and
\[
\gamma = \frac{1}{\rho_c \Omega_{rc}} \frac{d\rho_{de}}{ds} = 3 \left[ (1 + 2 \frac{y_1^2 + y_2^2 + y_3^2}{y_4^2})^{-1/2} (2y_1^2 + y_3^2) - 2y_1^2 \right].
\] (131)

A significant parameters from the viewpoint of observations is the deceleration parameter \( q \),
which carries the total effects of cosmic fluids. We plot \( q \) in these figures for corresponding density
curve of dark energy. In the fig [16] we set \( \Omega_m = 0.3 \). \( \Omega_{rc} \) is defined as the present value of the
energy density of \( \rho_0 \) over the critical density \( \Omega_{rc} = \rho_0/\rho_c \).

Now, we turn to the evolution of a universe with a phantom (116) in DGP. In an FRW universe
the density and pressure of a phantom can be written as (17), (18).
To compare with the results of the ordinary scalar, here we set a same potential as before,
\[
U = U_0 e^{-\lambda \frac{\psi}{\mu}}.
\] (132)

The ratio of change of density of virtual dark energy with respective to \( \ln(1 + z) \) becomes,
\[
\frac{d\rho_{de}}{d \ln(1 + z)} = 3[ -\dot{\psi}^2 + \theta (1 + \frac{-\dot{\psi}^2 + 2U + 2\rho_{dm}}{\rho_0})^{-1/2} ( -\dot{\psi}^2 + \rho_{dm})].
\] (133)
To study the behaviour of the EOS of dark energy, we first take a look at the signs of the terms of RHS of the above equation. \((-\dot{\psi}^2 + \rho_{dm})\) represents the total energy density of the cosmic fluids, which should be positive. The term \((1 + \frac{-\dot{\psi}^2 + 2\dot{\theta} + 2\rho_{dm}}{\rho_0})^{-1/2}\) should also be positive. Hence if \(\theta = -1\), both terms of RHS are negative: it never goes to zero at finite time. Contrarily, if \(\theta = 1\), the two terms of RHS carry opposite sign: the EOS of dark energy is able to cross phantom divide.

In the following of the present subsection we consider the branch of \(\theta = 1\).

Now the Friedmann constraint becomes
\[
- y_1^2 + y_2^2 + y_3^2 + y_4^2 + y_4^2 \left(1 + 2\frac{-y_1^2 + y_2^2 + y_3^2}{y_4^2}\right)^{1/2} = 1. \tag{134}
\]
Again, one will see that in reasonable regions of parameters, the EOS of dark energy crosses \(-1\), but from below \(-1\) to above \(-1\).

The stagnation point of \(\rho_{de}\) inhabits at
\[
\frac{y_4}{\sqrt{2 - y_4}} \left(-2 + \frac{y_3^2}{y_1^2}\right) = 2, \tag{135}
\]
which can be derived from (133) and (134). One concludes from the above equation that a smaller \(r_c\), a smaller \(\Omega_m\), or a larger \(\Omega_{k_i}\) is helpful to shift the stagnation point to lower redshift region, which is the same as the case of an ordinary scalar. Then we show a concrete numerical example of the crossing behaviour of this case in fig 17. The dimensionless density and rate of change with respect to redshift of dark energy become,
\[
\beta = \frac{\rho_{de}}{\rho_c} = \frac{\Omega_r}{b^2} \left[ -y_1^2 + y_2^2 + y_4^2 + y_4^2 (1 + 2\frac{-y_1^2 + y_2^2 + y_3^2}{y_4^2})^{1/2}\right], \tag{136}
\]
and
\[
\gamma = 3 \left[ -(1 + 2\frac{-y_1^2 + y_2^2 + y_3^2}{y_4^2})^{-1/2}(-2y_1^2 + y_4^2) + 2y_1^2\right]. \tag{137}
\]
Similarly, the deceleration parameter is plotted in the figure for corresponding density curve of dark energy. In this figures we also set \(\Omega_m = 0.3\).

Fig 17 explicitly illuminates that the EOS of virtual dark energy crosses \(-1\), as expected. At the same time the deceleration parameter is consistent with observations.

IV. SUMMARY

The recent observations imply that the EOS of dark energy may cross \(-1\). This is a remarkable phenomenon and attracts much theoretical attention.
We review three typical models for the crossing behavior. They are two-field model, interacting model, and modified gravity model.

There are several other interesting suggestions in or beyond the three categories mentioned above. We try to list them here for the future researches. We are apologized for this incomplete reference list on this topic.

Almost in all dark energy models the dark energy is suggested as scalar. However, 3 orthogonal vectors can also play this role. For the interacting vector dark energy and phantom divide crossing, see [56]. For the suggestion of crossing the phantom divide with a spinor, see [57]. Multiple k-essence sources are helpful to fulfil the condition for phantom divide crossing [58]. Phantom divide crossing can be realized by non-minimal coupling and Lorentz invariance violation [59]. An exact solution of a two-field model for this crossing has been found in [60].

For previous interacting X (quintessence or phantom) models with crossing −1, see [61]. The cosmology of interacting X in loop gravity has been studied in [62].

Interacting holographic dark energy is a possible mechanism for the phantom divide crossing [63]. And the thermodynamics of interacting holographic dark energy with phantom divide crossing is investigated in [64]. An explicit model of $F(R)$ gravity in which the dark energy crosses the phantom divide is reconstructed in [65]. The phantom-like effects in a DGP-inspired $F(R, \phi)$ gravity model is investigated in [66]. Based on the recent progress in studies of source of Taub space [67], a new braneworld in the sourced-Taub background is proposed [68], while the previous braneworld models are imbedded in AdS (RS) or Minkowski (DGP). In this model the EOS for
the virtual dark energy of a dust brane in the source region can cross the phantom divide. For other suggestions in brane world model, see [69].

Similar to the coincidence problem of dark energy, we can ask why the EOS crosses $-1$ recently? This problem is studied in [70].

On the observational side, the present data only mildly favor the crossing behavior. We need more data to confirm or exclude it.

Theoretically, we should find more natural model which has less parameters. We must go beyond the standard model of particle physics. The problem cosmic acceleration is a pivotal problem to access new physics. To study the problem of crossing $-1$ EOS will impel the investigation to the new Laws of nature.

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[71] One may think that $\Gamma/H$ is meaningless when $H = 0$. But in fact, in most realistic cases, we always assume that $\Gamma$ is proportional to $H$. 
S. region

V. region

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