Using heat kernel techniques we show that the relation between Hawking temperature and radiation flux known from Einstein gravity in D dimensions can be reproduced from the spherically reduced action. A recent controversy regarding the $D = 2$ anomaly for that case is discussed. The generalized effective Polyakov action in the presence of a dilaton field is presented.

1 Introduction

Hawking radiation in $D = 4$ is regarded as a well-understood quantum theoretical feature of black holes (BH) or of other geometric backgrounds with an event horizon. On the other hand, the dilaton theory emerging after spherical reduction from Einstein gravity (SRG) and generalizations of those theories, only during the last years have been investigated in this connection. If the consideration of spherically reduced actions like

$$S_{SRG} = \int d^2 x \mathcal{L}_{SRG}$$

$$\mathcal{L}_{SRG} = \sqrt{-g} e^{-2\phi} \left\{ R + \frac{4(D-3)}{(D-2)} (\nabla \phi)^2 - e^{\frac{4\phi}{D-2}} \phi + \frac{1}{2} (\nabla f)^2 \right\}$$

obtained by the ansatz ($R$ and $\nabla$ refer to $g_{\mu\nu}(x)$ in 2D)

$$(ds)^2 = g_{\mu\nu} dx^\mu dx^\nu - \frac{e^{\frac{4\phi}{D-2}}}{(D-2)(D-3)} (d\Omega_{D-2})^2$$

from the D-dimensional Hilbert-Einstein action especially also in the quantum case should make any sense, it seems a necessary condition that this basic result is reproduced. The flux of radiation at infinity should be related correctly to the surface gravity at the horizon of the D-dimensional BH and the Hawking temperature following from the latter.

In fact, already Christenson and Fulling had shown by the simple use of energy momentum conservation $\nabla_\mu T^{\mu\nu} = 0$ in $D = 2$ minimal coupling of the scalars (i.e.
with the factor \( \exp(-2\phi) \) in front of \((\nabla f)^2\) removed) in (2) that, as expected, the flux to \( J_+ \) (\( T_{uu} \) refers to the appropriate light cone conformal coordinates, cf. (8) below)

\[
T_{uu}^{(\text{min})} \bigg|_{J_+} = \frac{\pi}{12} T_H^2
\]

(4)

in terms of the Hawking temperature (\( r_h \) is the radius of the horizon in asymptotic coordinates)

\[
T_H = \frac{D-3}{4 \pi r_h}
\]

(5)

precisely corresponds to the \( D = 2 \) Stefan-Boltzmann law. The only necessary input to that computation had been the 2D conformal anomaly (one-loop quantum contribution of \( f \) to the trace of \( T^{\mu\nu} \)) and the condition that \( T_{uu}|_{r_h = 0} \) which means finite flux in global (Kruskal) coordinates. However, in a complete derivation of this result for the S-wave part of \( D = 4 \) (non-minimal coupling of \( f \) in (1)) could not be achieved from the \( D = 4 \) anomaly, despite its close similarity to the case of minimal coupling in \( D = 2 \).

This problem lay dormant for many years until the authors of noticed a surprisingly difficult situation: although their approach did not proceed through integrating the energy momentum conservation as in 3, but through a (Polyakov-type) effective action, as determined from the anomaly, they found that by taking that piece of \( T_{uu}|_{J^+} \) alone no Stefan-Boltzmann flux, but even a negative flux was created. This originated from the large negative contribution of \( T_\phi^\mu \) are produced by the dilaton dependent part in the anomaly

\[
T_\phi^\mu = T^{(\text{min})\mu}_\mu + T^{(\phi)\mu}_\mu
\]

(6)

for the nonminimally coupled scalars as in (1). They realized that another conformally noninvariant contribution (caused by the \( \phi \)-field dependence) must be added. By perturbative methods they argued that then in the complete result the negative contribution would be completely cancelled yielding the expected result (3).

In 1997 this problem obtained some publicity because it was rediscovered by Boussa and Hawking. Unfortunately in their calculation of the anomaly in \( D = 2 \) these authors overlooked two essential points (namely, the dilaton dependence of the diffeomorphism invariant measure and certain contributions from the zero modes). Several contradicting results have been reported in the literature. It was even claimed that there was an inherent ambiguity in one term of the conformal anomaly anomaly. This discussion was essentially closed by the papers. It became a common practise to use the conformal anomaly derived in (4), (8) (see e.g. 12).

When the diffeomorphism invariant scalar product of matter fields in \( D \) dimensions with measure \( \sqrt{-g_{(D)}} \) is reduced to \( D = 2 \), from the factor of \((d \Omega_{D-2})^2\) in (2) the corresponding \( D = 2 \) relation becomes

\[
\langle f_1, f_2 \rangle = \int d^2 x \sqrt{-g} e^{-2\phi} f_1 f_2
\]

(7)

This dilaton dependence is nothing else but the necessary factor \( 1/r \) for S-waves.
Before the work of [1] was published the present authors together with H. Liebl [2] had given the general result for the anomaly, allowing arbitrary nonminimal coupling of the scalars \( \exp(-2\phi) \rightarrow \exp(-2\varphi(\phi)) \) in front of \((\nabla f)^2\) of (1) with arbitrary dilaton dependent measure \( \exp(-\varphi(\phi)) \) in (6). Employing the heat kernel technique the special case of SRG in [3] was confirmed and a well-defined anomaly had been obtained in contrast to [5] (cf. also [6]).

2 Stefan-Boltzmann flux for spherically reduced gravity

Still the question had not found an answer, how to arrive at the correct flux for SRG without relying on perturbative methods as in [4], when the present authors realized that an extended energy momentum argument, together with a novel application of the heat kernel technique for the missing piece in (6) allowed a complete (analytic) solution [7].

In the presence of a dilaton field the argument of [3] must be applied to a different energy momentum conservation law. The diffeomorphism invariance of the matter part (1) for the \( f \)-fields satisfying the e.o.m. \( \delta L^{(m)}/\delta f = 0 \) does not lead to \( \nabla_{\mu} T^{\mu\nu} = 0 \) but to

\[
\nabla_{\mu} T^{\mu\nu} = - (\partial_{\nu} \phi) \frac{1}{\sqrt{-g}} \frac{\delta L^{(m)}}{\delta \phi} \tag{8}
\]
a result also noted in [8]. Therefore, integrating (7) according to [9] for the quantum effects \( \mathcal{O}(\bar{\hbar}) \) on both sides also required information on the “dilaton anomaly” on the r.h.s. (with \( L^{(m)} \) replaced by the one-loop effective action \( W \)) beside the usual anomaly contribution to the l.h.s. .

In conformal gauge \((u = t - z, v = t + z)\)

\[
g_{\mu\nu} = e^{2\rho} \, du \, dv \tag{9}
\]
with \( z \) related to the radial variable \( r \) by \( dr/dz = \exp 2\rho \) we start from the well-known fact that the functional derivative of the active action \( W \) yields the anomaly of (6). For the effective action \( W \) (in Euclidean space)

\[
\exp W = \int (d\tilde{f} \sqrt{\hat{g}}) \exp \int d^2 x \, \sqrt{-\hat{g}} \, \tilde{f} \, A \, \tilde{f} \tag{10}
\]
with a redefined \( \tilde{f} = f e^{-\psi(\phi)} \) and where \( \hat{g}^{\mu\nu} \) is the resulting effective metric in

\[
\hat{A} = \hat{g}^{\mu\nu} \hat{D}_\mu \hat{D}_\nu - E \tag{11},
\]
the corresponding covariant derivative, the anomaly is extracted in \( \zeta \)-function regularization of the functional determinant from the (elliptic) operator \( A \) by a multiplicative change of \( \hat{A} \) (i.e. of \( \hat{g}^{\mu\nu} \) by \( \delta k(x) \))

\[
\delta W = -\frac{1}{2} \int d^2 x \, \sqrt{\hat{g}} \, \delta k(x) \, T^\mu_\mu = \\
= -\frac{1}{48} \Tr \int d^2 x \, \sqrt{\hat{g}} \, \delta k(x) \, (\hat{R} + 6\hat{E}) \tag{12},
\]
where the second line is the standard result in the heat kernel technique\textsuperscript{13}. In this way for a general theory (cf. the last paragraph of Section 1)

\[ T_\mu^\nu = \frac{1}{24\pi} (R - 6(\nabla \varphi)^2 + 4\Box \varphi + 2\Box \psi) \]  

(13)
can be computed, and in conformal gauge (9) with \( \delta k = -2\delta \rho \)

\[ \frac{\delta W}{\delta \rho} = -\sqrt{-g} T_\mu^\mu, \]

(14)

where \( \sqrt{-g} R = -2\Delta \rho \) \( (\Delta = \eta_{\mu\nu} \partial_\mu \partial_\nu) \) and \( \sqrt{-g} \Box = \Delta \) may be used.

The computation of the dilaton anomaly (r.h.s. of (7) with \( L(m) \rightarrow W \)) starts from the identity in conformal gauge

\[ \frac{\delta W}{\delta \phi} = \int_0^\rho d\rho' \frac{\delta^2 W(\rho', \phi)}{\delta \rho' \delta \phi} + \frac{W_0(\phi)}{\delta \phi}, \]

(15)

where in the (functional) integral over \( \rho \) in the first term, eq. (13) can be used. The last term with

\[ \frac{\delta W_0}{\delta \phi} = \frac{\delta W_0}{\delta \varphi} \frac{d\varphi}{d\phi} + \frac{\delta W_0}{\delta \psi} \frac{d\psi}{d\phi} \]

(16)

refers to flat space. It can be made amenable to the multiplicative variation needed for the heat kernel technique by replacing the determinant of the differential operator \( A_0 \) in the effective action by one half of the contribution from a “fermionic” operator related to (commuting) Majorana fields \( \chi \) as (again in Euclidean space)

\[ W_0 = \frac{1}{4} \ln \int (d\chi) \exp(-\int d^2 x \chi D\tilde{D}\chi), \]

(17)

because after partial integration

\[ \int_x \chi D\tilde{D}\chi = \int_x f A_0 f \]

(18)

with \( D = i\gamma^\mu \epsilon^\psi \partial_\mu e^{-\varphi} \) and \( \tilde{D} = D(\psi \leftrightarrow -\varphi) \). In this form in the trace of the \( \xi \)-regularization multiplicative changes by \( \delta \varphi \) and \( \delta \psi \) result from the corresponding variation, and the step analogous to the one from (11) to (12) is applicable (for details we refer to\textsuperscript{13}).

In the resulting functional differential equations for \( W \)

\[ -12\pi \frac{\delta W}{\delta \varphi} = 6\eta^{\mu\nu} \partial_\nu(\rho \partial_\mu \varphi) + 2\Delta \rho - 2\Delta \psi - \Delta \varphi, \]

\[ -12\pi \frac{\delta W}{\delta \varphi} = \Delta \rho - 2\Delta \varphi - \Delta \psi, \]

\[ -12\pi \frac{\delta W}{\delta \rho} = -\Delta \rho - 3\eta^{\mu\nu} (\partial_\mu \varphi)(\partial_\nu \varphi) + 2\Delta \varphi + \Delta \psi, \]

(19)
the variations $\delta \varphi$ and $\delta \psi$ are still independent. Integrating (8) in the conformal
gauge ($T^\mu_\rho = 4e^{-2\rho}T_{\mu\rho}$, the only nonzero elements of the affine connections are
$\Gamma_{\nu v}^\rho = 2\partial_\nu \rho, \Gamma_{\nu u}^u = 2\partial_\nu \rho$, for a background $\rho = \rho(z)$ and stationary flux $\partial_u =
-\partial_v = -\frac{1}{2}\partial_z$) one finds that the new piece for the physically most interesting
application $\varphi = \psi = \phi$ of SRG exactly cancels the contribution from the $\phi$-dependent
part of the conformal anomaly, apart from a total divergence, yielding

$$T_{uu} = T_{uu}^{(min)} + \frac{1}{16\pi} \left[ 2(\partial_z \phi)(\partial_z \rho) + 2\rho (\partial_z \phi)^2 + (\partial_z \phi)^2 - \partial_z^2 \phi \right]_{r = r_h} \ ,$$

(20)

where, according to the Unruh vacuum condition $T_{uu}$ has been assumed to vanish
at the horizon $r_h$. Eq. (20) holds for a general background $e^{2\rho(r)} = K(r)$, $dr/dz =
K(r)$, with $L(r_h) = 0$ and dilaton field $\phi(r)$ and for all $r \geq r_h$. Thus for a D-
dimensional BH with

$$K_{BH} = 1 - \left( \frac{r_h}{r} \right)^{D-3}$$

(21)

and

$$T_{uu} = T_{uu}^{(min)} + \frac{1}{16\pi} \frac{K^2}{r^2} \ln K/\mu$$

(22)

at $\mathcal{J}_+(r \to \infty)$ only the (expected) result (4) remains. In (22) also the renormalization
contribution (with factor $\ln \mu$) has been added as another piece in the second term. In global coordinates with factor $K^{-2}$ the latter yields a logarithmic
divergence at the horizon which, nevertheless, still implies an integrable flux.

Finally it should be emphasized that the functionally integrated effective action can be obtained by inspection from (19) (making it covariant by $\Delta \varphi \to \sqrt{-g} \Box$, $\Delta \rho \to \sqrt{-g} R/2$)

$$W = -\frac{1}{24\pi} \int d^2x \sqrt{-g} \left[ -\frac{1}{4} R \Box^{-1} R + 3(\nabla \varphi)^2 \Box^{-1} R - R(\psi + 2\varphi) +
+(\nabla \psi)^2 + (\nabla \varphi)^2 + 4(\nabla^\mu \psi)(\nabla_\mu \varphi) \right] + W^{(ren)}$$

(23)

For a generally nonminimally coupled scalar field $\varphi(\phi), \psi = \psi(\phi)$ it represents
the (exact) generalization of the Polyakov action [14] for $\psi = \varphi = 0$. However, we did
not use this action in our argument, because it is derived from local (UV) quantum
effects. To directly calculate a flux at $\mathcal{J}_+$ from (23), as in the approach of [13] in our
opinion introduces the need for further input for its asymptotic behavior (e.g. for
$\Box^{-1}$).

3 Conclusion

Our result passes the main test, the relation between Hawking temperature and
flux at $\mathcal{J}_+$. The presence of the (mild) logarithmic singularity at horizon has been
considered unphysical in [14]. However, it seems closely related to the renormalization
procedure in $D = 2$, and it is also not forbidden by any $D = 4$ calculation along
the lines of our approach (which is sadly missing so far). Another problem is
the transition to the spinor operator $\bar{D}D$ in (18), a step which lacks complete
mathematical rigor, although it gave in the end the completely acceptable central
result. Thus further work will be needed, also in view of the “dimensional reduction anomaly”, which in general may cause unphysical effects after spherical reduction [19], although we believe them not to be obviously relevant in our example, because so far it has passed all consistency checks.

Acknowledgements

The authors are grateful for the support by Fonds zur Förderung der Wissenschaftlichen Forschung (Austrian Science Foundation), Project P12815-TPH, of the Alexander von Humboldt Foundation and of DFG project Bo 1112/11-1.

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