Static spherically symmetric space-time: some remarks

Bijan Saha

Laboratory of Information Technologies
Joint Institute for Nuclear Research, Dubna
141980 Dubna, Moscow region, Russia
and
Institute of Physical Research and Technologies
RUDN University
Moscow, Russia

Within the scope of a spherically symmetric space-time we study the role of different types of matter in the formation of different configurations with spherical symmetries. Here we have considered matter with barotropic equation of state, scalar field, electromagnetic field and an interacting system of scalar and electromagnetic field as the source. Corresponding field equations are solved exploiting harmonic coordinates. An easy to handle method is proposed which allows one to have an idea about the possible behavior of the metric functions once the components of the EMT of the source field is known.

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I. INTRODUCTION

In order to describe simple isolated bodies and island-like configurations spherical symmetry is a natural choice \[1\]. Spherically symmetric space-times are invariant under spatial rotation. Metric functions in this case, generally, depend on the radial coordinate and the time coordinate. In case of a static space-time, metric functions do not depend on time.

Static spherically symmetric space-time is widely used in physics to obtain analytic and numerical solutions to the Einstein field equations in presence of different types of source fields. One of the most celebrated static spherically symmetric solutions to the Einstein equations is the Schwarzschild solution. Since the metric functions depend on only the radial coordinate the static spherically symmetric space-time gives rise to a simpler system of equations which is easier to analyze. This type of space-time is used to study different types of astrophysical objects such as black holes, wormholes, compact stars etc. Since there is a very large number of studies in this area, we mention just a few. Role of the spinor field in the formation of black hole and/or wormhole was investigated in \[2,3\]. The stability of static, spherically symmetric solutions of Rastall’s theory was studied in \[4\].

The aim of this paper is to analyze the static spherically symmetric space-time in the presence of different matters with typical energy-momentum tensor (EMT) widely exploited in literature. In

\*Electronic address: bijan@jinr.ru; URL: http://spinor.bijansaha.ru

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our view, the method, offered here, will be useful to solve the system of equations in question both analytically and numerically. It will allow the researchers to impose minimal additional conditions to obtain the solutions.

II. BASIC EQUATION

The action we choose in the form

$$\mathcal{S} = \int \sqrt{-g} \left[ \frac{R}{2\kappa} + L \right] d\Omega, \quad (2.1)$$

where $\kappa = 8\pi G$ is Einstein’s gravitational constant, $R$ is the scalar curvature and $L$ is the matter field Lagrangian. We don’t specify the Lagrangian right now. It might be given by the spinor, scalar, electromagnetic field or their interaction. We specify it in the course of our journey.

The spherically symmetric metric we choose in the form

$$ds^2 = e^{2\gamma}dt^2 - e^{2\alpha}du^2 - e^{2\beta}(d\vartheta^2 + \sin^2 \vartheta d\varphi^2), \quad (2.2)$$

where the metric functions $\gamma, \alpha, \beta$ depend on the spatial variable $u$ only. Since in order to describe the spherically symmetric gravitational field we need only two components of the metric tensor \[5\], then in (2.2) it is possible to choose explicitly one of the three metric functions $\gamma, \alpha, \beta$ or demand that all these functions satisfy one of the following coordinate conditions \[1, 5\]:

1. $\alpha = 0$, i.e. $e^\alpha = 1$ - the Gaussian normal coordinates;
2. $\alpha = \gamma$ - isometric or tortoise coordinates;
3. $\alpha = \beta$ - homogeneous coordinates;
4. $e^{2\beta} = e^{2\alpha}u^2$ - isotropic coordinates;
5. $e^\beta = r$ - curvature or Schwarzschild coordinates. $r$ is the radius of the sphere with $u = \text{const}$. In this case the metric \[2.2\] takes the form

$$ds^2 = e^{2\gamma(r)}dt^2 - e^{2\alpha(r)}du^2 - r^2(d\vartheta^2 + \sin^2 \vartheta d\varphi^2). \quad (2.3)$$

6. $\alpha = \gamma + 2\beta$ - harmonic coordinates;
7. $\alpha = -\gamma$ - quasiblobal coordinates.

In should be noted that since we consider the static spherically symmetric configuration, all the field functions should depend on the spatial variable $u$ only.

The Einstein tensor corresponding to the metric \[2.2\] possesses only diagonal elements, hence the Einstein equations in this case takes the form

$$\begin{align*}
(2\gamma'\beta' + \beta'^2) - e^{2(\alpha - \beta)} &= -\kappa T_1^1, \quad (2.4a) \\
(\gamma^2 + \gamma'\beta' - \gamma'\alpha' + \beta'^2 - \beta'\alpha' + \gamma'' + \beta'') &= -\kappa T_2^2, \quad (2.4b) \\
(3\beta'^2 - 2\beta'\alpha' + 2\beta'') - e^{2(\alpha - \beta)} &= -\kappa T_0^0. \quad (2.4c)
\end{align*}$$
Subtraction of \((2.4a)\) from \((2.4c)\) gives
\[
\beta'' + \beta'^2 - \alpha' \beta' - \gamma' \beta' = -\frac{\kappa}{2} [T^0_0 - T^1_1]. \tag{2.5}
\]
Subtraction of \((2.5)\) from \((2.4b)\) yields
\[
\gamma'' + \gamma'^2 - \alpha' \gamma' + 2\gamma' \beta' = -\frac{\kappa}{2} [2T^2_2 - T^0_0 + T^1_1]. \tag{2.6}
\]
For numerical study it is convenient to rewrite the equations \((2.5)\) and \((2.6)\) in the Cauchy form:
\[
\beta' = \nu, \tag{2.7a}
\gamma' = \tau, \tag{2.7b}
\nu' + \nu^2 - \alpha' \nu - \nu \tau = -\frac{\kappa}{2} [T^0_0 - T^1_1], \tag{2.7c}
\tau' + \tau^2 - \alpha' \tau + 2\nu \tau = -\frac{\kappa}{2} [2T^2_2 - T^0_0 + T^1_1]. \tag{2.7d}
\]
To solve this system we have to know the concrete form of energy-momentum tensor and some additional relation which is known as coordinate condition. One can exploit one of the coordinate conditions listed above. Note that in many problems considered within the scope of static spherically symmetric space-time we deal with the EMT such that
\[
T^0_0 = T^2_2 = T^3_3, \\
T^1_1 = T^2_2 = T^3_3, \\
T^0_0 = T^1_1 = -T^2_2 = -T^3_3.
\]
So in any of those cases listed above the task becomes even easier.
In what follows we consider the harmonic radial coordinate such that \(\alpha = \gamma + 2\beta\). In view of it the Eqns. \((2.7c)\) and \((2.7d)\) can be written as
\[
\nu' = \nu^2 + 2\nu \tau - \frac{\kappa}{2} [T^0_0 - T^1_1], \tag{2.8a}
\tau' = -\frac{\kappa}{2} [2T^2_2 - T^0_0 + T^1_1]. \tag{2.8b}
\]
Now, to find the metric functions we have to know the energy-momentum tensor (EMT) \(T^\nu_\mu\) of the material field. Depending on the source fields it may vary. Further we consider a few cases those are widely used in literature both listed above and beyond.

**Perfect fluid**

The first case we consider is the one when the source field is given by \(T^\nu_\mu = \text{diag}(\varepsilon, -p, -p, -p)\), i.e. \(T^1_1 = T^2_2 = T^3_3\). In this case from the conservation law
\[
T^\nu_\mu ;\nu = T^\nu_\mu + \Gamma^\mu_\rho_\mu T^\rho_\nu - \Gamma^\rho_\nu T^\mu_\rho = 0, \tag{2.9}
\]
in view of the fact that \(\varepsilon\) and \(p\) depends on \(u\) we find
\[
p' + (p + \varepsilon) \gamma' = 0. \tag{2.10}
\]
The perfect fluid obeys the barotropic equation of state (EOS) such that

\[ p = W \varepsilon, \tag{2.11} \]

where \( W \) is a constant and known as EOS parameter. For \( W \geq 0 \) it describes a perfect fluid such as dust, radiation, hard Universe and stiff matter. In case of \( W < 0 \) we have dark energy such as quintessence, \( \Lambda \) term and phantom matter.

In this case from the (2.10) one finds

\[ \varepsilon = \left( Ce^{-\gamma} \right)^{(1+1/W)}. \tag{2.12} \]

Here \( C \) is the integration constant and can be taken to be unity. In this case we have the following system of equations

\[ \beta' = \nu, \tag{2.13a} \]

\[ \gamma' = \tau, \tag{2.13b} \]

\[ \nu' = \nu^2 + 2\nu\tau - \frac{\kappa}{2}(1-W)e^{-(1+1/W)\gamma}, \tag{2.13c} \]

\[ \tau' = -(3W - 1)e^{-(1+1/W)\gamma}. \tag{2.13d} \]

We solve the system (2.13) numerically. For simplicity we set \( \kappa = 1 \). The initial values are taken to be \( \tau(0) = 0.2, \nu(0) = 0.2, \gamma(0) = 0.3, \beta(0) = 0.3 \). In Figs. 1 and 2 we have plotted the behavior of metric functions for \( W = 1/3 \) which corresponds to radiation and \( W = -2/3 \) that corresponds to quintessence, respectively. In the first case \( \beta(u) \) is growing faster than \( \gamma(u) \), whereas in the second case \( \beta(u) \) is decreasing as \( u \) increase while \( \gamma(u) \) is still increasing.

**Scalar field**

Let us consider a scalar field with the Lagrangian

\[ L_{sc} = \frac{1}{2} \varphi, \alpha \varphi^{,\alpha} - V(\varphi). \tag{2.14} \]

If \( V(\varphi) = (1/2)m^2\varphi^2 \) the foregoing Lagrangian leads to the Klein-Gordon equation. The corresponding EMT reads

\[ T^\mu_{sc\nu} = \varphi,\nu \varphi^{,\mu} - \delta^\mu_\nu \left( \frac{1}{2} \varphi, \alpha \varphi^{,\alpha} - V(\varphi) \right). \tag{2.15} \]

If the scalar field depends on only redial coordinate \( u \) we find \( T^0_0 = T^2_2 = T^3_3 = e^{-2\alpha}\varphi'^2 + V(\varphi) \) and \( T^1_1 = -e^{-2\alpha}\varphi'^2 + V(\varphi) \).

The corresponding scalar field equation

\[ \partial_\mu \left( \sqrt{-g}g^{\mu\nu} \varphi,\nu \right) + \sqrt{-g}V_\varphi = 0, \quad V_\varphi = \partial V/\partial \varphi. \tag{2.16} \]

On account of coordinate condition in this case we finally have the following system of gravitational and scalar field equations:
The system (2.17) we solve numerically. As in previous case we set $\kappa = 1$. The initial values are set to be $\tau(0) = 0.05$, $v(0) = 0.05$, $\gamma(0) = 0.5$, $\beta(0) = 0.5$, $\varphi(0) = 0.1$, $\varphi'(0) = 0.2$. In Fig. 3 we have drawn the picture of the metric functions when the source is given by the scalar field. In Fig. 4 the corresponding scalar field is demonstrated. As one sees, in this case $\gamma(u)$ decreases with the growth of $u$, while $\beta(u)$ slowly increases.

Note that in case of spinor field we also obtain $T^0_0 = T^2_2 = T^3_3$ and the corresponding solutions will be similar to the one found above.

**Electromagnetic field**

Let us now consider the case with electromagnetic field with the Lagrangian

$$L_{\text{em}} = \frac{1}{4} F_{\mu \nu} F^{\mu \nu}.$$ (2.18)

The corresponding EMT reads
Since we consider the static spherically-symmetric configuration, all the field functions should depend on the spatial variable \( u \) only. Herewith

\[ F_{10}(u) = -F_{01}(u) = A', \quad \text{(2.20)} \]

with all other components \( F_{\mu \nu} \equiv 0 \). Here we assume that the vector potential has only one non-trivial component \( A_\mu = (A, 0, 0, 0) \).

The electromagnetic field equation

\[ \frac{1}{\sqrt{-g}} \frac{\partial}{\partial x^\nu} (\sqrt{-g} F^{\nu \mu}) = 0, \quad \text{(2.21)} \]

in this case takes the form

\[ \frac{\partial}{\partial u} (e^{2\alpha} F^{10}) = 0, \quad \text{(2.22)} \]

with the solution

\[ F^{10} = q e^{-2\alpha} = q e^{-2(\gamma + 2\beta)}, \quad q = \text{const.} \quad \text{(2.23)} \]

The foregoing equations leads to

\[ A' = -q e^{2\gamma}, \quad \text{(2.24)} \]

The components of the EMT in this case takes the form

\[ T_0^0 = T_1^1 = -T_2^2 = -T_3^3 = -\frac{1}{2} q^2 e^{-4\beta}. \quad \text{(2.25)} \]
In this case the Einstein system reads

\begin{align}
\beta' &= \nu, \\
\gamma' &= \tau, \\
\nu' &= \nu^2 + 2\nu\tau, \\
\tau' &= -\kappa q^2 e^{-4\beta}.
\end{align}

(2.26)

The system (2.26) we solve numerically. \(\kappa\) is taken to be unity, whereas the initial values are set to be \(\tau(0) = 0.3, \nu(0) = 0.2, \gamma(0) = 0.3, \beta(0) = 0.3, A(0) = 1\). In Fig. 5 we have drawn the picture of the metric functions when the source is given by the electromagnetic field. In Fig. 6 the corresponding vector potential is demonstrated. In this case \(\gamma(u)\) and \(\beta(u)\) behave like the one with radiation illustrated in Fig. 1.

In Fig. 5: Plot of metric functions for electromagnetic field

In Fig. 6: Plot of vector potential \(A(u)\)

**Interacting scalar and electromagnetic fields**

Let us finally consider the interacting scalar and spinor field given by the Lagrangian [6]

\[
L = \frac{1}{2} \Phi_{,\alpha} \Phi^{,\alpha} - \frac{1}{4} F_{\alpha\beta} F^{\alpha\beta} \Psi(\varphi), \quad \Psi(\varphi) = 1 + \lambda \Phi(\varphi).
\]

(2.27)

Here \(\lambda\) is the coupling constant between the scalar and electromagnetic fields. Setting \(\lambda = 0\) one obtains the case with minimal coupling.

The corresponding equations

\[
\partial_{\nu} (\sqrt{-g} g^{\mu\nu} \Phi_{,\mu}) + \frac{1}{2} \sqrt{-g} F_{\alpha\beta} F^{\alpha\beta} \Psi_{,\varphi} = 0,
\]

(2.28)
\[
\partial_\nu \left( \sqrt{-g} F^{\mu \nu} \Psi(\phi) \right) = 0,
\]

(2.29)

where \( \Psi_\phi = d\Psi/d\phi \).

The EMT in this case is

\[
T^\mu_\nu = \left[ \varphi_\nu \varphi^\mu - F_{\mu \beta} F^{\nu \beta} \Psi(\phi) \right] - \frac{1}{4} \delta^\mu_\nu \left[ 2 \varphi_\alpha \varphi^\alpha - F_{\alpha \beta} F^{\alpha \beta} \Psi(\phi) \right].
\]

(2.30)

The corresponding field equations read

\[
\left( \sqrt{-g} g^{11} \phi' \right)' + \frac{1}{2} \sqrt{-g} F_{10} F^{10} \Psi = 0,
\]

(2.31)

and

\[
\left( \sqrt{-g} F^{10} \Psi(\phi) \right)' = 0,
\]

(2.32)

where \( \sqrt{-g} = \sqrt{-g}/\sin \vartheta \).

The solution to the (2.32) can be written as

\[
F^{10} = q P(\phi) / \sqrt{-g}, \quad P(\phi) = 1/\Psi(\phi), \quad q = \text{const}.
\]

(2.33)

From (2.33) one finds

\[
A' = -q e^{2\gamma} P(\phi),
\]

(2.34)

On account of (2.33) the scalar field equation (2.31) now looks

\[
\phi'' = \frac{q^2}{2} e^{2\gamma} P(\phi).
\]

(2.35)

The components of EMT in this case read

\[
T^\mu_\nu = \frac{1}{2} e^{-2\alpha} \varphi^2 \text{diag} (+1, -1, +1, +1) + \frac{q^2}{2} e^{2(\gamma - \alpha)} P^2 \text{diag} (+1, +1, -1, -1).
\]

(2.36)

So in this case we have the following system of equations

\[
\begin{align*}
\beta' &= \nu, \quad (2.37a) \\
\gamma' &= \tau, \quad (2.37b) \\
\nu' &= \nu^2 + 2\nu \tau + \frac{\kappa}{2} e^{-2(\gamma + 2\beta)} \frac{1}{P^2}, \quad (2.37c) \\
\tau' &= \frac{\kappa}{2} q^2 e^{-4\beta} P^2, \quad (2.37d) \\
\phi' &= \psi, \quad (2.37e) \\
\psi' &= \frac{q^2}{2} e^{\gamma} P(\phi). \quad (2.37f)
\end{align*}
\]

As in previous cases we solve the system (2.37) numerically. Again we set \( \kappa = 1 \) and \( \tau(0) = 0.5, \nu(0) = 0.5, \gamma(0) = 0.2, \beta(0) = 0.3, \phi(0) = 1, \phi'(0) = 0.2, A(0) = 0.2 \). In Figs. 7 and 8 we we have plotted the metric functions and \( \gamma(u) \), respectively for \( P = J^{2-4/\sigma} \left( 1 - J^{2/\sigma} \right) \) with \( J = \lambda \varphi \),
\[ \sigma = 2n + 1 \text{ and } n = 1, 2, 3, \ldots. \] Here we set \( n = 1 \) and \( \lambda = 1 \). Comparing the Figs. 3, 5 and 7 one sees, in this case the the behavior of the metric functions determined by the electromagnetic field.

In Figs. 9 and 10 corresponding picture of the scalar and electromagnetic fields are illustrated.

### III. CONCLUSION

We have considered a static spherically symmetric space-time and analyzed it in presence of different types of matter. The system was transformed in such a way that some idea about the qualitative solutions to the Einstein field equations can be made looking at the type of source field, precisely the interrelation between the components of EMT. For simplicity we have considered only harmonic coordinate. We plan to extend our study for the other cases as well as for some other realistic source fields in near future.

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FIG. 9: Plot of scalar functions for interacting scalar and electromagnetic fields
FIG. 10: Plot of vector potential for interacting scalar and electromagnetic fields

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