Leavitt path algebras are Bézout

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(joint work with F. Mantese and A. Tonolo)

Vietnam Institute for Advanced Study in Mathematics
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1. Definitions, examples and motivations.
Outline

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2. The main result, and some ideas used in the proof
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2. The main result, and some ideas used in the proof
3. Consequences
The algebra $L_K(E)$

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$\bullet s(e) \xrightarrow{e} \bullet r(e)$
The algebra $L_K(E)$

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Let $E = (E^0, E^1, s, r)$ be a directed graph. The extended graph of $E$ is the graph $\hat{E} = (E^0, E^1 \cup (E^1)^*, s', r')$, with

$$(E^1)^* = \{e^* \mid e \in E^1\},$$

$r'_{|E^1} = r, s'_{|E^1} = s, r'(e^*) = s(e), s'(e^*) = r(e)$.
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The Leavitt path algebra $L_K(E)$ of $E$ over $K$ is the $K$-path algebra $K\hat{E}$ modulo the relations:

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- $e^*e' = \delta_{e,e'}r(e)$ for any $e, e' \in E^1$
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Let $E = (E^0, E^1, s, r)$ be a directed graph. $\bullet s(e) \xrightarrow{e} \bullet r(e)$

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**The Leavitt path algebra** $L_K(E)$ of $E$ over $K$ is the $K$-path algebra $K\hat{E}$ modulo the relations:

- $e^*e' = \delta_{e,e'}r(e)$ for any $e, e' \in E^1$
- $\nu = \sum\{e \in E^1 \mid s(e) = \nu \} \ ee^*$ (for any $\nu \in E^0$ with $0 < |s^{-1}(\nu)| < \infty$.)

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Examples

Let $R_1$ be the graph

$$
\begin{array}{c}
\bullet \\
\uparrow \\
\downarrow
\end{array}
$$

The $L_K(R_1) \cong K[x, x^{-1}]$, the Laurent polynomial algebra.
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  $\xymatrix{e \ar@/^/[r] & \bullet \ar@/^/[l] v}$

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- Let $A_n$ be the graph

  $\xymatrix{v_1 \ar[r]^{e_1} & v_2 \ar[r]^{e_2} & \cdots \ar[r]^{e_{n-2}} & v_{n-1} \ar[r]^{e_{n-1}} & v_n}$

  Then $L_K(A_n) \cong M_n(K)$. 
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\[ A_n = \cdot \xrightarrow{v_1 \ x} \cdot \xrightarrow{v_2 \ x} \ldots \xrightarrow{v_{n-1} \ x} \cdot \xrightarrow{v_n \ x} \]

Then $L_K(A_n) \cong M_n(K)$.

Let $A_\mathbb{N}$ be the graph

\[ A_\mathbb{N} = \cdot \xrightarrow{v_1 \ x} \cdot \xrightarrow{v_2 \ x} \cdot \xrightarrow{v_3 \ x} \ldots \]

Then $L_K(A_\mathbb{N}) \cong FM_\mathbb{N}(K)$. 
Examples

■ (Most important example) Let $R_n$ be the graph

Then $L_K(R_n) \cong L_K(1, n)$, the Leavitt algebra of module type $(1, n)$.
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Then $L_K(R_n) \cong L_K(1, n)$, the Leavitt algebra of module type $(1, n)$. For this ring, denoting $S = L_K(R_n)$, then $S \cong S^n$ as left (or right) $S$-modules. In fact,

$$S^i \cong S^j \text{ if and only if } j - i \equiv 0 \pmod{n - 1} \text{ for any } i, j \geq 1.$$
Examples

- (Most important example) Let $R_n$ be the graph

$$
\begin{align*}
&\bullet & \downarrow & \downarrow & e_2 & \downarrow & \downarrow & \nearrow & \nearrow & e_n & \nearrow & \nearrow \\
& & & \downarrow & \downarrow & & & & & & \\
& & & & & v & & & & & \\
& & & \downarrow & \downarrow & & & & & & \\
& & & & & e_1 & & & & & \\
\end{align*}
$$

Then $L_K(R_n) \cong L_K(1, n)$, the Leavitt algebra of module type $(1, n)$. For this ring, denoting $S = L_K(R_n)$, then $S \cong S^n$ as left (or right) $S$-modules. In fact,

$$S^i \cong S^j \text{ if and only if } j - i \equiv 0 \pmod{n - 1} \text{ for any } i, j \geq 1.$$

So in particular we have that there is an epimorphism $S \twoheadrightarrow S^t$ for any $t \geq 1$. 
Let $E$ be the Toeplitz graph

\[ c \rightarrowarrow v \quad \rightarrowarrow f \rightarrowarrow w. \]

Then $L_K(E)$ is called the *Toeplitz algebra*. It is isomorphic to $K\langle X, Y \mid XY = 1 \rangle$, the *Jacobson algebra*.

The isomorphism is explicit: $X \leftrightarrow c^* + f^*$, $Y \leftrightarrow c + f$. 
Examples

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In $L_K(E)$, if $I$ is the two-sided ideal $\langle w \rangle$ generated by $w$, then as $K$-algebras

\[ I \cong FM_N(K) \quad \text{via (roughly)} \quad E_{i,j} \leftrightarrow c_i f f^* (c^*)^j. \]

Also, $L_K(E)/I \cong K[x, x^{-1}]$ via $x \leftrightarrow c + l$, $x^{-1} \leftrightarrow c^* + l$. 

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Properties of $L_K(E)$

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- If $e$ is an edge with $r(e) = w$, then
  \[ L_K(E)ee^* = L_K(E)e^* \cong L_K(E)w \]
as left ideals.

\[ (ree^* \mapsto ree^* e = re \in L_K(E)w, \text{ and } rw \mapsto re^* = re^* ee^* \in L_K(E)ee^*). \]

More generally, if $p$ is a path in $E$ with $r(p) = w$, then
\[ L_K(E)pp^* = L_K(E)p^* \cong L_K(E)w. \]
Properties of $L_K(E)$

There are many results of the form:

$L_K(E)$ has some specified algebraic property $\iff$ $E$ has some specified graph-theoretic property

- simple
- purely infinite simple
- finite dimensional
- prime
- primitive
- von Neumann regular
- semiprime
- hereditary
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A ring $R$ is called \textit{left Bézout} in case every finitely generated left ideal of $R$ is principal. (’principal’ = ’cyclic’).

Rephrased, $R$ is left Bézout in case for any finite set $x_1, x_2, \ldots, x_t$ of elements in $R$, there exists $x \in R$ with $\sum_{i=1}^{t} Rx_i = Rx$. 

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Bézout rings

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A ring $R$ is called **Bézout** in case every finitely generated one-sided (left and right) ideal of $R$ is principal (see, for instance, Kaplansky, Cohn, Warfield...)

This is a weaker condition than being a **(left) principal ideal ring (p.i.r.)**
Some properties of Bézout rings

- Warfield showed that if $R$ is Bézout, then so is any finite matrix ring over $R$. (Not easy.)
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- The Bézout property need not pass to corners, not even to full corners. (So $R$ Bézout and $e = e^2 \in R$ and $ReR = R$ does not necessarily give that $eRe$ is Bézout.) In particular, the Bézout property is not a Morita invariant.
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- Intuitively, the Bézout property allows us to do some “Number Theory”, since for each \( a, b \in R \), there exists \( c \in R \) for which \( Ra + Rb = Rc \).
Some remarks about Bézout rings

- Of course $K$ is Bézout for any field $K$. So is $\mathbb{Z}$.
- $K[x, y]$ is not Bézout; $\langle x, y \rangle$ is finitely generated but not principal.
- $R = U_2(K)$ is not Bézout; $Re_{11} \oplus Re_{12}$ is not principal.
- The Bézout property does not in general pass to *modules*: clearly there are finitely generated $\mathbb{Z}$-modules which are not cyclic. Keep in mind the example $\mathbb{Z} \oplus \mathbb{Z}$. 
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Key observation: Any direct summand of a principal left ideal is principal.
The classical examples of Leavitt path algebras

- Any $n \times n$ matrix ring $\mathbb{M}_n(K)$ is Bézout

(Idea of proof ...)

So actually $\mathbb{M}_n(K)$ is in fact a p.i.r.
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- Any $n \times n$ matrix ring $M_n(K)$ is Bézout
  
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  So actually $M_n(K)$ is in fact a p.i.r.

- The Laurent polynomial ring $K[x, x^{-1}]$ is Bézout. (Essentially the same proof as for $K[x]$.)
  
  $K[x, x^{-1}]$ is also a p.i.r.
The classical examples of Leavitt path algebras

- The (nonunital) ring $R = M_N(K)$ is Bézout, but it is not a p.i.r.
The classical examples of Leavitt path algebras

- The (nonunital) ring $R = M_N(K)$ is Bézout, but it is not a p.i.r.

Note that $R R$ itself is not finitely generated (so can’t be principal).

But we can use the same idea as in the finite matrix case to get that any *finitely generated* left $R$-ideal is principal.
The classical examples of Leavitt path algebras

- The Leavitt algebras $L_K(1, n)$ are Bézout.
The classical examples of Leavitt path algebras

The Leavitt algebras $L_K(1, n)$ are Bézout.

**Proof.** Let $I$ be a finitely generated left ideal of $L_K(1, n)$.
The classical examples of Leavitt path algebras

The Leavitt algebras $L_K(1, n)$ are Bézout.

**Proof.** Let $I$ be a finitely generated left ideal of $L_K(1, n)$. So there is an epimorphism $L_K(1, n)^T \to I$ for some $T \geq 1$. But we saw above that there is an epimorphism $L_K(1, n) \to L_K(1, n)^T$. Indeed, every finitely generated left module over $L_K(1, n)$ is principal. But $L_K(1, n)$ is NOT a p.i.r. It is easy to find a not-finitely-generated left ideal in $L_K(1, n)$. (There are infinite orthogonal sets of nonzero idempotents in $L_K(1, n)$.)
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But $L_K(1, n)$ is NOT a p.i.r. It is easy to find a not-finitely-generated left ideal in $L_K(1, n)$.

(There are infinite orthogonal sets of nonzero idempotents in $L_K(1, n)$.)

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Leavitt path algebras are Bézout
The Jacobson algebra \( R = K\langle X, Y \mid XY = 1 \rangle \) is Bézout. (Shown directly by Gerritzen, 2000. This was hard work!)

**Sketch of a proof, from the point of view of Leavitt path algebras.** Let \( E \) be the Toeplitz graph

\[ c \quad \bullet^v \quad \overset{f}{\rightarrow} \quad \bullet^w. \]

Then \( R \cong L_K(E) \).
The Toeplitz algebra is Bézout

Let $S$ denote $\langle w \rangle$. (It is well-known that $S = \text{Soc}(L_K(E))$.)

- As mentioned above, $S \cong FM_N(K)$ as $K$-algebras, and $L_K(E)/S \cong K[x, x^{-1}]$ as $K$-algebras.

- As left ideals,

\[
S = L_K(E)w \oplus (\oplus_{i=0}^{\infty} L_K(E)c^i ff^*(c^*)^i)
\]

\[
= L_K(E)w \oplus (\oplus_{i=0}^{\infty} L_K(E)f^*(c^*)^i).
\]

Here each $L_K(E)f^*(c^*)^i \cong L_K(E)w$, and each is a simple left $L_K(E)$-module.
Case 1. Suppose \( I \) is any left ideal of \( L_K(E) \) properly containing \( S \). Then \( I = L_K(E)p(c) + S \) for some nonzero \( p(x) \in K[x, x^{-1}] \).

Let \( p(x) = \sum_{i=m}^{M} k_i x^i \), where \( m \leq M \in \mathbb{Z} \).

Note:

\[
L_K(E)c^t p(c) = L_K(E)p(c)
\]

for any \( t \in \mathbb{N} \).

\( \subseteq \) is clear, and the equation \( p(c) = (c^*)^t c^t p(c) \) gives \( \supseteq \).

In particular, we may assume that \( m = 0 \), so that \( p(x) = \sum_{i=0}^{M} k_i x^i \) for some \( k_i \in K \), \( k_0 \neq 0 \); and by multiplying by \( k_0^{-1} \), we may assume that \( k_0 = 1 \).
The Toeplitz algebra is Bézout

For each $n \geq 0$ let $e_n$ denote the idempotent
\[ w + ff^* + cff^*c^* + \cdots + c^n ff^* (c^*)^n. \]
(It is clear that the set of summands appearing in $e_n$ is a set of orthogonal idempotents, so $e_n$ itself is an idempotent.)

Since $S = \langle w \rangle$ we see that $e_n \in S$. For each $n \in \mathbb{N}$ let $S_n$ denote the left ideal
\[ S_n = L_K(E)e_n = L_K(E)w \oplus (\bigoplus_{i=0}^{n} L_K(E)c^i ff^* (c^*)^i) \]
\[ = L_K(E)w \oplus (\bigoplus_{i=0}^{n} L_K(E)f^* (c^*)^i). \]
The Toeplitz algebra is Bézout

Claim:

\[ L_K(E)p(c) + S = L_K(E)p(c) + S_{M-1}. \]

The idea is that expressions of the form \( f^*(c^*)^j \) for \( j \geq M \) are in \( L_K(E)p(c) + S_{M-1} \). For example:

\[ f^*(c^*)^M p(c) = f^*(c^*)^M (1 + k_1 c + \cdots + k_M c^M), \]

so that

\[ f^*(c^*)^M = f^*(c^*)^M p(c) - k_1 f^*(c^*)^{M-1} - \cdots - k_M f^* \in L_K(E)p(c) + S_{M-1}. \]

So we have that

\[ I = L_K(E)p(c) + L_K(E)e, \]

where \( p(x) = \sum_{i=0}^{M} k_i x^i \) is some monic element of \( K[x] \) and

\[ e = e_{M-1} = w + ff^* + \cdots + c^{M-1}ff^*(c^*)^{M-1}. \]
The Toeplitz algebra is Bézout

We claim that $I$ is principal.

Define $q(c) = c^M p(c)$. Then $L_K(E)p(c) = L_K(E)q(c)$ by the previous observation, so $I = L_K(E)q(c) + L_K(E)e$. 
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Since every nonzero term in the polynomial $q(c)$ has degree at least $M$, we have that

$$c^M(c^*)^M q(c) = q(c).$$
The Toeplitz algebra is Bézout

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Since every nonzero term in the polynomial $q(c)$ has degree at least $M$, we have that

$$c^M(c^*)^M q(c) = q(c).$$

On the other hand, $e = e_{M-1} = w + ff^* + \cdots + c^{M-1}ff^*(c^*)^{M-1}$, so that

$$c^M(c^*)^M e = 0.$$
The Toeplitz algebra is Bézout

Claim:

\[ L_K(E)q(c) + L_K(E)e = L_K(E)(q(c) + e). \]

\( \supseteq \) is clear.

But \( c^M(c^*)^M(q(c) + e) = q(c) + 0 = q(c) \) by the above computation, so

\[ q(c) \in L_K(E)(q(c) + e). \]

And this is sufficient: \( q(c) \in L_K(E)(q(c) + e) \) and (obviously) \( q(c) + e \in L_K(E)(q(c) + e) \), so

\[ e = (q(c) + e) - q(c) \in L_K(E)(q(c) + e). \]
Case 2. Suppose $I$ is not contained in $S$, and $I$ does not contain $S$. Consider the left ideal $A = I + S$. Then $A$ properly contains $S$, so we may apply the Case 1 analysis to $A$ (recall that we did not need to assume that the ideal in Case 1 was finitely generated), so that $A$ is principal.
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Since $S$ is a direct sum of simple left $L_K(E)$-modules, we have $S = (I \cap S) \oplus B$ for some left ideal $B$ of $L_K(E)$. 
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**Case 2.** Suppose $I$ is not contained in $S$, and $I$ does not contain $S$. Consider the left ideal $A = I + S$. Then $A$ properly contains $S$, so we may apply the Case 1 analysis to $A$ (recall that we did not need to assume that the ideal in Case 1 was finitely generated), so that $A$ is principal.

Since $S$ is a direct sum of simple left $L_K(E)$-modules, we have $S = (I \cap S) \oplus B$ for some left ideal $B$ of $L_K(E)$.

Then it is straightforward to show that $A = I \oplus B$. So $I$ is a direct summand of the principal left ideal $A$, so that $I$ is principal as well.
Case 3. Suppose \( I \) is finitely generated, and \( I \) is contained in \( S \). But \( S \) is Bézout as a ring (it’s isomorphic to \( \mathbb{F}M_N(K) \)), which gives easily that any finitely generated left \( L_K(E) \)-module contained in \( S \) is principal.
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But \( S \) is Bézout as a ring (it’s isomorphic to \( \mathbb{F}M_N(K) \)), which gives easily that any finitely generated left \( L_K(E) \)-module contained in \( S \) is principal.

So the three cases have been established, and thus the Toeplitz algebra \( L_K(E) \) is Bézout. \( \square \)
Bézout rings

Note: $R = L_K(E)$ is not a p.i.r.
Bézout rings

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For example, the left $R$-ideal

$$
\bigoplus_{n \in \mathbb{N}} R(c^n (c^*)^n - c^{n+1} (c^*)^{n+1})
$$

is not principal (since it’s not finitely generated).
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Note: $R = L_K(E)$ is not a p.i.r.

For example, the left $R$-ideal

$$\bigoplus_{n \in \mathbb{N}} R(c^n(c^*)^n - c^{n+1}(c^*)^{n+1})$$

is not principal (since it’s not finitely generated).

Remark: None of these summands is 0, because otherwise

$$c^n(c^*)^n - c^{n+1}(c^*)^{n+1} = 0$$

would give

$$(c^n(c^*)^n - c^{n+1}(c^*)^{n+1})c^nf = 0,$$

which would give $c^nf = 0$, a contradiction.
Question:

Is every Leavitt path algebra a Bézout ring?
Bézout rings

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Note: $L_K(E)$ is ring-isomorphic to its opposite ring $L_K(E)^{op}$, so “Bézout” and “left (or right) Bézout” mean the same thing in this context.
**One particular motivation** for asking this question:

Starting from a particular class of simple modules (the *Chen simple modules*), we are studying some ”Prüfer-type” modules over $L_K(E)$. (These are constructed as direct limits of indecomposable uniserial modules of finite length, with all the composition factors isomorphic to the same Chen simple module).
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If the finitely generated left ideals of $L_K(E)$ are principal, then Baer’s Criterion is equivalent to testing the division of the elements of the Prüfer-type modules by the elements of $L_K(E)$. 
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If the finitely generated left ideals of $L_K(E)$ are principal, then Baer’s Criterion is equivalent to testing the division of the elements of the Prüfer-type modules by the elements of $L_K(E)$.

And we have a division algorithm for these Prüfer-type modules.

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Leavitt path algebras are Bézout
Two recent results

1) **K. M. Rangaswamy (2014)**: Let $E$ be an arbitrary graph. Let $I$ be a two-sided ideal of $R = L_K(E)$ which is finitely generated (i.e., there exists $x_1, x_2, \ldots, x_n \in R$ for which $I = \sum_{i=1}^{n} Rx_i R$). Then $I$ is principal (i.e., there exists $x \in R$ with $I = RxR$.)
Two recent results

A vertex $v$ in $E$ is a source vertex in case $|r - 1(v)| = 0$.

A cycle $c$ in $E$ is a source cycle in case $|r - 1(v)| = 1$ for every vertex $v$ appearing in $c$. (i.e., $c$ is a source cycle if the vertices in the cycle receive no edges other than those which are already in the cycle $c$).

Example: In the Toeplitz graph

\[ \begin{array}{c}
  v \\
  \uparrow \\
  f \\
  \rightarrow \\
  \uparrow \\
  w
\end{array} \]

$c$ is a source cycle.
Two recent results

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Gene Abrams  University of Colorado Colorado Springs  (joint work with F. Mantese and A. Tonolo)  Leavitt path algebras are Bézout
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A-, T.G. Nam, and N.T. Phuc (2015): If $E$ is a finite graph which contains neither source vertices nor source cycles, then $\mathcal{L}_K(E)$ is Bézout.

Idea of proof: Some hard work shows that the assumption on $E$ implies that $\mathcal{L}_K(E)$ fails to have the Unbounded Generator Number property. That is, there are positive integers $m < n$ and an epimorphism $\mathcal{L}_K(E)_m \twoheadrightarrow \mathcal{L}_K(E)_n$, which can then be used (by a deep property of Leavitt path algebras) to get an epimorphism of left $\mathcal{L}_K(E)$-modules $\mathcal{L}_K(E) \twoheadrightarrow \mathcal{L}_K(E)^2$.

So the same argument as for $\mathcal{L}_K(1, n)$ can be used.

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Leavitt path algebras are Bézout
Two recent results

2) **A.-, T.G. Nam, and N.T. Phuc (2015):** If \( E \) if a finite graph which contains neither source vertices nor source cycles, then \( L_K(E) \) is Bézout.
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So the same argument as for $L_K(1, n)$ can be used.
A subset $V$ of $E$ is called a source subset if either $V = \{v\}$ where $v$ is a source vertex in $E$, or $V$ is the set of the vertices of a source cycle $c$ in $E$. 

In case $E$ is finite, we denote $W = E_0 \setminus V$ and $\omega = \sum_{w \in W} w$. 

$T(V)$ denotes the set of those vertices in $E$ which are the range vertex of some path whose source vertex is in $V$. 

Define $W' = T(V) \cap W$, and $\omega' = \sum_{w \in W'} w$. 

Define $W'' = W \setminus W'$, and $\omega'' = \sum_{w \in W''} w$. 

So $\omega = \omega' + \omega''$. 

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Leavitt path algebras are Bézout
source elimination

Let $E$ be a finite graph and $V \subseteq E$ be a source subset. The source elimination graph $E_W$ is the subgraph of $E$ from which we have eliminated all of the vertices in $V$, and all of the edges having a source vertex in $V$. (We can view $V_E$ as the “restriction” subgraph of $E$ to $W$).
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Proposition: The set $W = E^0 \setminus V$ is “hereditary”, and

$$\omega L_K(E) \omega = L_K(E_W)$$
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**Proposition:** The set $W = E^0 \setminus V$ is “hereditary”, and

$$\omega L_K(E)\omega = L_K(E_W)$$

**Note:** The graph $E_W$ has fewer vertices than $E$. 

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Leavitt path algebras are Bézout
The ideal $J$

We denote by $J$ the two-sided ideal of $L_K(E)$ generated by $\omega$:

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$$J = L_K(E)\omega L_K(E) = \langle w \mid w \in W \rangle.$$

**Proposition**: Let $E$ be a finite graph and $V \subseteq E$ a source subset.

1) If $V = \{v\}$ is a non-isolated source vertex, then $J = L_K(E)$.

2) Otherwise, there is an isomorphism of $K$-algebras

$$L_K(E)/J \cong M_n(K[x, x^{-1}]).$$

In particular, $L_K(E)/J$ is Bézout.

Moreover, any left ideal of $L_K(E)$ properly containing $J$ is principal.
(Use the same idea as shown above for the Toeplitz algebra.)
The strategy

The idea is to study how the Bézout property passes from various factor rings and subalgebras of $L_K(E)$ back to $L_K(E)$. We then use the source elimination and other graph constructions to reduce to a smaller graph for which the associated algebra is Bézout.

(For instance, the smaller graph might not contain source cycles or source vertices, so that we can use the A- / Nam / Phuc result.)
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**Remark:** By the previous specific example, it’s likely that the structure of $J$ as a left ideal will play a key role:

$$J = L_K(E)\omega \oplus (\bigoplus_{f \in \Delta(V); \epsilon \in \Theta(V)} L_K(E)f^*\epsilon^*)$$

where

$$\Delta(V) = \{ e \in E^1 \mid s(e) \in V, r(e) \in W \}$$ and

$\Theta(V)$ is the set of paths in $E$ which both start and end in $V$. 

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Leavitt path algebras are Bézout
Proposition: Let \( V \) be a source in \( E \). Let \( I \) be a finitely generated left ideal of \( L_K(E) \) with \( I \leq L_K(E)\omega \) or \( I \leq L_K(E)f^*\epsilon^* \) (as above). Suppose \( \omega L_K(E)\omega \) is Bézout. Then \( I \) is principal, and is isomorphic to a direct summand of \( L_K(E)\omega \).

In other words, if the finitely generated left ideal \( I \) is contained in ONE of the direct summands of \( J \), then \( I \) can be shown to be principal.
The first steps

**Proposition:** Let $V$ be a source in $E$. Let $I$ be a finitely generated left ideal of $L_K(E)$ with $I \leq L_K(E)\omega$ or $I \leq L_K(E)f^*\epsilon^*$ (as above). Suppose $\omega L_K(E)\omega$ is Bézout. Then $I$ is principal, and is isomorphic to a direct summand of $L_K(E)\omega$.

In other words, if the finitely generated left ideal $I$ is contained in ONE of the direct summands of $J$, then $I$ can be shown to be principal.

**Sketch of proof:** Not too bad; uses that $L_K(E)$ is hereditary.
The main result

**Theorem:** If $E$ is a finite graph, then $R = L_K(E)$ is a Bézout ring
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**Theorem**: If $E$ is a finite graph, then $R = L_K(E)$ is a Bézout ring.

**Proof**: The proof is by induction on $n = |E^0|$. If $|E^0| = 1$, then there are three possibilities: either $E$ has no edges, exactly one edge, or $m \geq 2$ edges. The Leavitt path algebras of such graphs are then, respectively, $K$, $K[x, x^{-1}]$, and $L_K(1, m)$, which are Bézout. So we assume that $|E^0| = n$, and that $L_K(F)$ is Bézout for any graph $F$ having fewer than $n$ vertices. We show that $L_K(E)$ is Bézout. If $E$ has neither source vertices nor source cycles, then we are done (A-/Nam/Phuc result). So we assume that $E$ contains a source vertex or a source cycle.
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Leavitt path algebras are Bézout
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By the previous Proposition, since $\omega R\omega$ is Bézout by the induction hypothesis, we get that any finitely generated left ideal contained in $R\omega$ or in a summand of the form $Rf^*e^*$ is principal.

Next we consider the finitely generated left ideals which are contained in $J$, but are not contained in $R\omega$, nor in a summand of the form $Rf^*e^*$. (This is the hard part.) Then show the left ideals properly containing $J$ are principal. Finally, combine these ideas to show that the "mixed" case also works.
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General outline of the proof

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Ideas for the proof

We need to show that the left ideals contained in $J$, but not in $R\omega$ nor in $Rf^*e^*$, are principal. We have two cases:

1) Case of a source vertex $\{v\}$.
   
   Case 1a) Show the result holds when $v$ emits exactly one edge. This uses an induction argument in which we assume that the Leavitt path algebra of any graph having less than $|E_0|$ edges is Bézout.
   
   Case 1b) Use Case 1a) as the basis step of another induction proof, where one argues on $|s - 1(v)|$. The argument here requires the use of the "outsplitt graph."
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Leavitt path algebras are Bézout
Ideas for the proof

2) Case of a source cycle.

Ideas for the proof
Ideas for the proof

2) Case of a source cycle. Here we can see the interesting idea of how to “capture” a finitely generated left ideal as a summand of the regular module.

Recall

\[ J = L_K(E)\omega \oplus \left( \bigoplus_{f \in \Delta(V); \epsilon \in \Theta(V)} L_K(E)f^*\epsilon^* \right) \]

where \( \Theta \) is the set of paths which both start and end in \( V \) and

\[ \Delta = \Delta(V) = \{ e \in E^1 \mid s(e) \in V, r(e) \in W \} \].
Ideas for the proof

Since $I$ is finitely generated we have that $I$ lives inside a direct sum of finitely many summands in $J$.

If $I$ lives inside a single summand, then the previous Proposition can be invoked.

So we assume that if $I \leq A_1 \oplus A_2 \oplus \cdots \oplus A_{m-1}$ (where each $A_t$ is of the form $Rf^*\epsilon^*$, and possibly $A_t = R\omega$ for some (unique) $t$), then $I$ is principal. We show that if $I \leq A_1 \oplus A_2 \oplus \cdots \oplus A_m$, where each $A_t$ is of the form $Rf^*\epsilon^*$, and possibly $A_t = R\omega$ for some (unique) $t$, then $I$ is principal as well.
Ideas for the proof

If $I \cap A_t = \{0\}$ for some $t$ then $I$ is isomorphic (via the projection along $A_t$) to an ideal which is contained in a direct sum of $m - 1$ $A_i's$, and so we are done by induction. Hence we may assume that $I \cap A_t \neq \{0\}$ for all $1 \leq t \leq m$. So in particular we have $I \cap A_j \neq \{0\}$ for some $A_j = Rf^*\epsilon^*$. 
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Then

\[
\frac{I}{I \cap A_j} \cong \frac{I + A_j}{A_j} \leq A_1 \oplus \cdots \oplus A_{j-1} \oplus A_{j+1} \oplus \cdots \oplus A_m.
\]

So by induction we get \( I/I \cap A_j \) is isomorphic to a principal left ideal.
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So by induction we get \( I/I \cap A_j \) is isomorphic to a principal left ideal.

That is, since \( R \) is hereditary and hence any left ideal is projective, \( I/I \cap A_j \) is isomorphic to a direct summand of \( R = R_\nu \oplus R_\omega \).
Ideas for the proof

Moreover, using again that $I/I \cap A_j$ is projective, the short exact sequence

$$0 \to I \cap A_j \to I \to I/I \cap A_j \to 0$$

splits, and so

$$I \cong (I \cap A_j) \oplus (I/I \cap A_j).$$

So, as a direct summand of $I$, in particular we get that $B := I \cap A_j$ is both finitely generated and projective.
Ideas for the proof

Recall $W' = T(V) \cap W$ (which also equals $T(r(f))$ for any $f \in \Delta(V)$); and $\omega'$ denotes the sum of the vertices in $W'$. Then $\omega'R\omega'$ is Bézout by the induction hypothesis.

So we may apply the previous Proposition to get that $I \cap A_j$ is principal, and is isomorphic to a direct summand of $R\omega'$.

So we have

$$I \cong (I \cap A_j) \oplus (I/I \cap A_j),$$

which is then isomorphic to a direct summand of $R\omega' \oplus R\nu \oplus R\omega$. 

WARNING: We are not finished. The problem is that this isomorphism is as left $R$-modules. We cannot view $R\omega' \oplus R\nu \oplus R\omega$ as a left ideal of $R$, because $\omega'$ and $\omega''$ are not orthogonal.
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Ideas for the proof

Now consider the cycle $c$ for which $c^0 = V$. Denote $V$ by $v_1, v_2, \ldots, v_\ell$, and the edges of $c$ by $h_1, h_2, \ldots, h_\ell$, with $s(h_i) = v_i$ and $r(h_i) = v_{i+1}$.

By the CK2 relation we have $v_1 = h_1 h_1^* + \sum_{f \in \Delta(V), s(f) = v_1} ff^*$. So

$$Rv_1 \cong Rh_1 h_1^* \oplus (\bigoplus_{f \in \Delta(V), s(f) = v_1} Rff^*).$$

But $Rv_2 \cong Rh_1 h_1^*$, so that

$$Rv_1 \cong Rv_2 \oplus (\bigoplus_{f \in \Delta(V), s(f) = v_1} Rff^*).$$

Using this idea at each subsequent vertex of the cycle, we eventually return back to $v_1$, and we get

$$Rv_1 \cong Rv_1 \oplus (\bigoplus_{f \in \Delta(V)} Rff^*).$$
We denote $\bigoplus_{f \in \Delta(V)} Rf^*$ by $N$. So we have shown that

\[ Rv_1 \cong Rv_1 \oplus N. \]
Ideas for the proof

(repeating ...) \( R_{v_1} \cong R_{v_1} \oplus (\oplus_{f \in \Delta(V)} R_{ff^*}) \)

We denote \( \oplus_{f \in \Delta(V)} R_{ff^*} \) by \( N \). So we have shown that

\[ R_{v_1} \cong R_{v_1} \oplus N. \]

But by repeated substitution, this then gives

\[ R_{v_1} \cong R_{v_1} \oplus N^s \]

for any positive integer \( s \).
Ideas for the proof

(repeating ...) \( Rv_1 \cong Rv_1 \oplus (\bigoplus_{f \in \Delta(V)} Rff^*) \)

We denote \( \bigoplus_{f \in \Delta(V)} Rff^* \) by \( N \). So we have shown that

\[ Rv_1 \cong Rv_1 \oplus N. \]

But by repeated substitution, this then gives

\[ Rv_1 \cong Rv_1 \oplus N^s \]

for any positive integer \( s \). Similarly, \( Rv_i \cong Rv_i \oplus N^s \) for all \( 1 \leq i \leq \ell \) and all positive integers \( s \). So

\[ \bigoplus_{i=1}^{\ell} Rv_i \cong (\bigoplus_{i=1}^{\ell} Rv_i) \oplus N^{s\ell} \]

for any positive integer \( s \); rephrased,

\[ R_v \cong R_v \oplus N^{s\ell} \text{ for any positive integer } s. \]
Ideas for the proof

Now consider any vertex \( u \in T(V) \setminus V = W' \). Then there is a path from \( V \) to \( u \), which necessarily must have as its initial edge an element \( f \in \Delta(V) \). So using the same argument as in the previous paragraphs, by the CK2 relations we have that, for each \( u \in W' \),

\[
Ru \text{ is isomorphic to a direct summand of } Rff^*
\]

for some \( f \in \Delta(V) \). Let \( p = |W'| \). Then

\[
\bigoplus_{u \in W'} Ru \text{ is isomorphic to a direct summand of } N^p.
\]

that is, \( R\omega' \) is isomorphic to a direct summand of \( N^p \). So

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R\omega' \oplus R\omega' \text{ is isomorphic to a direct summand of } N^{2p}.
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So \( R\nu \oplus R\omega' \oplus R\omega' \) is isomorphic to a direct summand of \( R\nu \oplus N^{2p} \), which is isomorphic to a direct summand of \( R\nu \oplus N^{s \ell} \) for some \( s \) sufficiently large.
Ideas for the proof

But $R\nu \oplus N^{s}\ell$ in turn is isomorphic to $R\nu$, by the previous observation. In other words,

$$R\nu \oplus R\omega' \oplus R\omega'$$
is isomorphic to a direct summand of $R\nu$.

BUT REMEMBER, we have

$$I$$
is isomorphic to a direct summand of $R\omega' \oplus R\omega \oplus R\nu$.

So writing $\omega = \omega' + \omega''$, we then have $I$ is isomorphic to a direct summand of

$$R\omega' \oplus R\omega' \oplus R\omega'' \oplus R\nu \cong R\nu \oplus R\omega' \oplus R\omega' \oplus R\omega''$$,

which in turn (above display) gives that

$$I$$
is isomorphic to a direct summand of $R\nu \oplus R\omega''$.

But this last module is actually a left ideal (a direct summand of $R$), so that $I$ is isomorphic to a direct summand of $R$, i.e., $I$ is principal.

Gene Abrams University of Colorado Colorado Springs

Leavitt path algebras are Bézout

(Joint work with F. Mantese and A. Tonolo)
Ideas for the proof

So the hard part is done. And the final two steps (ideals properly containing $J$, and the "mixed" case), are established in a way similar to that used in the proof for the Toeplitz algebra. □
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Consequences

**Theorem:** $L_K(E)$ is Bézout for any graph $E$. 
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Proof: $L_K(E)$ is the direct limit of unital subalgebras, each of which is isomorphic to the Leavitt path $K$-algebra of a finite graph. So each of these unital subalgebras is Bézout. Hence every finite set of elements of $L_K(E)$ is contained in a unital Bézout subring of $L_K(E)$. Now an easy general ring-theoretic Lemma applies.
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Theorem: Let $E$ be a finite graph. Then $L_K(E)$ is a principal ideal ring if and only if no cycle in $E$ has an exit.
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**Theorem:** Let $E$ be a finite graph. Then $L_K(E)$ is a principal ideal ring if and only if no cycle in $E$ has an exit.

**Proof:** If no cycle in $E$ has an exit then $L_K(E)$ is known to be isomorphic to a direct sum of algebras of the form $M_n(K)$ or $M_n(K[x, x^{-1}])$, and so is a principal ideal ring.

If on the other hand the cycle $c$ has an exit, then similar to above one can show that the left $L_K(E)$-ideal

$$
\bigoplus_{n \in \mathbb{N}} L_K(E)(c^n(c^*)^n - c^{n+1}(c^*)^{n+1})
$$

is not finitely generated, and so can’t be principal. (The existence of the exit gives that each of the summands is nonzero.)
The corresponding result for graph $C^*$-algebras?

For any graph $E$ there is an intimate relationship between $L_C(E)$ and the “graph $C^*$ algebra” $C^*(E)$. There are many theorems of the form:

$L_C(E)$ has algebraic property $\mathcal{P} \iff C^*(E)$ has analytic property $\mathcal{P}$
The corresponding result for graph $C^*$-algebras?

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$L_C(E)$ has algebraic property $\mathcal{P} \iff C^*(E)$ has analytic property $\mathcal{P}$

but the proofs are not direct! They all are based on showing that the two properties are both equivalent to

$$E \text{ has graph property } Q.$$

Why this happens is still a mystery.
The corresponding result for graph $C^*$-algebras?

But:

**Theorem.** Every (closed) left ideal of a graph $C^*$-algebra is principal.

That is, any graph $C^*$-algebra is a principal ideal ring (with respect to closed one-sided ideals).

So the Bézout property is special in this context.
Questions?

Thank you.

Gene Abrams  University of Colorado Colorado Springs
Leavitt path algebras are Bézout