Numerical Relativistic Magnetohydrodynamics with ADER Discontinuous Galerkin methods on adaptively refined meshes.

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Abstract. We describe a new method for the solution of the ideal MHD equations in special relativity which adopts the following strategy: (i) the main scheme is based on Discontinuous Galerkin (DG) methods, allowing for an arbitrary accuracy of order $N+1$, where $N$ is the degree of the basis polynomials; (ii) in order to cope with oscillations at discontinuities, an "a-posteriori" sub-cell limiter is activated, which scatters the DG polynomials of the previous time-step onto a set of $2N+1$ sub-cells, over which the solution is recomputed by means of a robust finite volume scheme; (iii) a local spacetime Discontinuous-Galerkin predictor is applied both on the main grid of the DG scheme and on the sub-grid of the finite volume scheme; (iv) adaptive mesh refinement (AMR) with local time-stepping is used. We validate the new scheme and comment on its potential applications in high energy astrophysics.

1. Introduction
Discontinuous Galerkin (DG) methods, which have been applied to terrestrial physical problems for a long time, are still relatively unknown in astrophysics. The situation is now rapidly changing, and their usage is likely to increase in the next decade. Due to their robustness, high order of accuracy in smooth regions and very good scalability, these methods are attracting a lot of attention, particularly in the relativistic context [1, 2]. A particularly interesting field of research in high energy astrophysics is represented by the solution of the special relativistic magnetohydrodynamics (RMHD) equations, which govern the physics of peculiar systems such as extragalactic jets, gamma-ray bursts and magnetospheres of neutron stars. Unfortunately, Galerkin methods suffer from a strong limitation, which has prevented a widespread use due to the fact that they cannot escape the Gibbs phenomenon, thus producing oscillations at discontinuities. The common route to circumvent this problem is represented by artificial viscosity [3, 4], spectral filtering [5], (H)WENO limiting procedures [6, 7], and slope and moment limiting [8, 9]. However, an unavoidable consequence of these approaches is that they destroy the sub-cell resolution properties of the DG method. In [10] we have recently proposed a new solution to this longstanding problem, which is based on a sub-cell finite volume limiting approach, while preserving the high resolution capabilities of DG. If combined with Adaptive Mesh Refinement (AMR), this approach can guarantee un-precedented levels of accuracy [11] and their applications to the relativistic framework are very promising [12].
2. The numerical scheme

The equations of RMHD can be cast in conservative form as

\[ \partial_t u + \partial_i f^i = 0, \]  

(1)

where the definition of the conserved quantities \( u \) and of the corresponding fluxes \( f \) can be found in [13, 14]. The constraint on the divergence-free character of the magnetic field can also be written formally as in Eq. (1), according to the divergence-cleaning approach proposed by [15], that we also follow here. Concerning the recovering of the primitive variables from the conserved ones, we adopt the third method reported in Sect. 3.2 of [16]. We adopt Cartesian coordinates over a domain \( \Omega \), which is formed by elements \( T_i \) as

\[ \Omega = \bigcup_{i=1}^{N_E} T_i, \]  

(2)

where the index \( i \) ranges from 1 to the total number of elements \( N_E \). According to the DG philosophy, at any time \( t^n \), the numerical solution of Eq. (1) is represented by polynomials of maximum degree \( N \geq 0 \), namely

\[ u_h(x, t^n) = \sum_{l=0}^{N} \Phi_l(x) \hat{u}^n_l, \]  

(3)

where the coefficients \( \hat{u}^n_l \) are the so-called degrees of freedom. We can schematically describe our numerical scheme by distinguishing the following different steps:

- We apply a predictor step, in which Eq. (1) is solved within each element \( T_i \) through a locally implicit space-time discontinuous Galerkin scheme. In practice, the polynomials of Eq. (3) need to be evolved in time, locally for each cell, and subsequently used for the time integration of the numerical fluxes. This is the heart of the ADER approach, as formulated in the modern version by [17].

- We construct a one-step ADER discontinuous Galerkin (DG) scheme, which uses the information obtained by the predictor. This implies that we first multiply the governing equations (1) by a test function \( \Phi_k \) and then we integrate over the space-time control volume \( T_i \times [t^n; t^{n+1}] \). After integration by parts in space, we get

\[ \int_{t^n}^{t^{n+1}} \int_{T_i} \Phi_k \frac{\partial u_h}{\partial t} dxdt + \int_{t^n}^{t^{n+1}} \int_{\partial T_i} \Phi_k f(u_h) \cdot \mathbf{n} dSdt - \int_{t^n}^{t^{n+1}} \int_{T_i} \nabla \Phi_k \cdot f(u_h) dxdt = 0. \]  

(4)

If \( u_h \), expressed by (3), is inserted in the first term of Eq. (4), and if we use the result of the predictor step to compute the other terms of Eq. (4), we effectively obtain a one-step numerical scheme, which allows to obtain the new solution at time \( t^{n+1} \) with a high order of accuracy both in space and in time. We also note that the second term of Eq. (4) contains a surface integration, which involves the solution of a Riemann problem at the element boundary, thus guaranteeing that the final method is upwind. In our numerical tests we have used the simple Rusanov flux and the HLL solver.

- We introduce an a-posteriori sub-cell limiter. Since the scheme resulting from Eq. (4) is unlimited, it will necessarily produce oscillations at discontinuities. In order to solve this problem, we introduce a special kind of limiter, which acts a posteriori in the following way. First we identify a particular cell as troubled if it either violates physical conditions or it
violates a simple version of the discrete maximum principle [10]. In each troubled cell, a subgrid composed of $2N + 1$ sub-cells (along each spatial direction) is built and the DG polynomial at the previous time (when pathologies had not manifested yet) is projected over the subgrid in such a way to provide a piecewise constant data representation at the subgrid level. Having done that, it is possible to evolve the cell averages at the subgrid level via standard finite volume, or even a TVD scheme.

- We combine the new scheme with Adaptive Mesh Refinement (AMR) and local time-stepping. Our AMR approach can be referred to as a "cell-by-cell" refinement, according to which every cell $T_i$ is individually refined with no creation of grid patches. The refinement factor along each direction is denoted by $r$ and the maximum level of refinement is $\ell_{\text{max}}$. In particular, the two AMR operations of projection and averaging need to involve the sub-cell averages of the solution on the sub-grids. Further details can be found in [11, 18].

In the numerical tests that follow we refer to the new scheme as ADER-DG-$P_N$ where $N$ is the degree of the DG polynomial, and we effectively use a TVD sub-cell limiter.

3. Numerical tests

The convergence of the new ADER scheme, with a single step for the time update, has been verified by [12] up to the 5-th order. Here we show the performances of the new scheme by solving two representative and well known RMHD tests. In particular, we solve the first shock tube problem in the sample proposed by [13], using a coarse grid of $40 \times 5$ cells, which is adaptively refined in space and time according to $r = 3$ and $\ell_{\text{max}} = 2$. Fig. 1 reports the results of the calculation, for which we have used the ADER-DG-$P_3$ scheme and a Courant factor CFL = 0.5. In the two top panels of the figure, where 3D-views of the magnetic field $B_y$ and of the rest mass density are shown, we have highlighted in red those cell which required the activation of the limiter, while the unlimited cells are colored in cyan. Our ADER-DG numerical scheme can capture all the significant waves of this Riemann problem, while suppressing all the spurious oscillations.

The second test that we consider is the cylindrical expansion of a blast wave in a plasma with an initially uniform magnetic field. The details about it can be found in [14]. The initial conditions assume that, within a radius $R = 1.0$, the rest-mass density and the pressure are $\rho = 0.01$ and $p = 1$. On the contrary, $\rho = 10^{-4}$ and $p = 5 \times 10^{-4}$ outside the cylinder. There is not velocity to start with, and the magnetic field is constant and oriented along the $x$-direction. In Fig. 2, we show the results at the final time $t = 4.0$ obtained for $B_x = 0.5$, which makes this test particularly severe. We have solved this problem over the computational domain $\Omega = [-6, 6] \times [-6, 6]$, with $40 \times 40$ elements on the coarsest refinement level, $r = 3$ and $\ell_{\text{max}} = 2$. Moreover, the Rusanov Riemann solver has been chosen. The wave pattern of the configuration is formed by a weak external circular fast shock, visible in the magnetic pressure, and by a reverse shock which surrounds the region where the magnetic field is confined. The two bottom panels show the AMR grid and the limiter map, from which we deduce that only a relatively small number of cells required the activation of the sub-cell TVD scheme.

4. Conclusions

We have proposed a modification of the pure discontinuous Galerkin (DG) method that solves the longstanding problem of Gibbs oscillation with an a-posteriori limiter. The limiter acts on a sub-grid of each single troubled cell, and it recomputes the solution via a robust WENO or TVD finite volume scheme. This strategy, combined with AMR, allows for an unprecedented level of accuracy. The new method can solve successfully the equations of special relativistic magnetohydrodynamics and it is likely to contribute significantly in the numerical modeling of various high energy astrophysics phenomena. It has been already adopted in the relativistic
Figure 1. Solution of the first Riemann problem by [13] at $t_{\text{final}} = 0.55$, obtained with the ADER-DG-$p^3$ scheme supplemented with the ADER-TVD sub-cell limiter. Top panels: 3D views of the magnetic field $B_y$ (left) and of the rest-mass density (right). Bottom panels: one-dimensional profiles compared to the exact solution.

context by other groups [1] and it is likely to be applied also for the solution of the Einstein equations.

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Figure 2. RMHD blast wave problem with $B_x = 0.5$ at time $t = 4.0$, obtained with the ADER-DG $P_3$ scheme and the a-posteriori second order TVD sub-cell limiter. Top panels: gas pressure (left) and magnetic pressure (right). Bottom panels: AMR grid (left) and limiter map (right) with troubled cells marked in red and regular unlimited cells marked in cyan.

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