Fractional Backward Stochastic Differential Equations
and Fractional Backward Variational Inequalities

Lucian Maticiuc · Tianyang Nie

Received: 3 November 2012 / Revised: 12 June 2013 / Published online: 22 August 2013
© Springer Science+Business Media New York 2013

Abstract In the framework of fractional stochastic calculus, we study the existence
and the uniqueness of the solution for a backward stochastic differential equation,
formally written as:

\[
\begin{align*}
- dY(t) &= f(t, \eta(t), Y(t), Z(t))dt - Z(t)\delta B^H(t), \quad t \in [0, T], \\
Y(T) &= \xi,
\end{align*}
\]

The work of Lucian Maticiuc was carried out at Faculty of Mathematics, “Alexandru Ioan Cuza”
University of Iași under project POSDRU/89/1.5/S/49944.

The work of Tianyang Nie was supported by the Marie Curie ITN Project, “Controlled Systems”, no. 213841.

L. Maticiuc
Faculty of Mathematics, “Alexandru Ioan Cuza” University, Carol I Blvd., no. 11, 700506 Iasi, Romania
e-mail: lucian.maticiuc@ymail.com

L. Maticiuc
Department of Mathematics, “Gheorghe Asachi” Technical University, Carol I Blvd., no. 11, 700506 Iasi, Romania

T. Nie (✉)
School of Mathematics, Shandong University, Jinan 250100, Shandong, China
e-mail: nietianyang@163.com

T. Nie
Laboratoire de Mathématiques, CNR-UMR 6205 Université de Bretagne Occidentale, 29285 Brest Cedex 3, France

T. Nie
School of Mathematics and Statistics, University of Sydney, Sydney, NSW 2006, Australia
where \( \eta \) is a stochastic process given by

\[
\eta(t) = \eta(0) + \int_0^t \sigma(s) \delta B^H(s), \quad t \in [0, T],
\]

and \( B^H \) is a fractional Brownian motion with Hurst parameter greater than \( 1/2 \). The stochastic integral used in above equation is the divergence-type integral. Based on Hu and Peng’s paper, Backward stochastic differential equation driven by fractional Brownian motion, SIAM J Control Optim (2009), we develop a rigorous approach for this equation. Moreover, we study the existence of the solution for the multivalued backward stochastic differential equation

\[
\begin{cases}
-dY(t) + \partial \varphi(Y(t))dt \ni f(t, \eta(t), Y(t), Z(t))dt - Z(t)\delta B^H(t), \quad t \in [0, T], \\
Y(T) = \xi,
\end{cases}
\]

where \( \partial \varphi \) is a multivalued operator of subdifferential type associated with the convex function \( \varphi \).

**Keywords** Backward stochastic differential equation · Fractional Brownian motion · Divergence-type integral · Malliavin calculus · Backward stochastic variational inequality · Subdifferential operator

**Mathematics Subject Classification (2010)** 60H10 · 60G22 · 47J20 · 60H05

1 Introduction

General backward stochastic differential equations (BSDEs) driven by a Brownian motion were first studied by Pardoux and Peng [18], where they also gave a probabilistic interpretation for the viscosity solution of semilinear partial differential equations (PDEs). Pardoux and Răşcanu [19] studied backward stochastic differential equations involving a subdifferential operator [which are often called backward stochastic variational inequalities (BSVI)]s), and they used them in order to generalize the Feynman–Kac type formula to represent the solution of multivalued parabolic PDEs [also called parabolic variational inequalities (PVIs)].

Backward stochastic differential equations driven by a fractional Brownian motion with Hurst parameter \( H \in (1/2, 1) \) were first considered by Biagini et al. [6], when they studied the stochastic maximal principle in the framework of a fractional Brownian motion. By adapting the four-step scheme introduced by Ma et al. [13] and the so-called \( S \)-transform, Bender [5] studied BSDEs driven by a fractional Brownian motion with Hurst parameter \( H \in (0, 1) \). Indeed, through a backward parabolic PDE, he constructed an explicit solution of a kind of linear fractional BSDE. Hu and Peng [12] were the first to study nonlinear BSDEs governed by a fractional Brownian motion (fBm). Our work, based on [12], has the objective to develop a rigorous approach for such BSDEs driven by a fBm and to extend the discussion to fractional BSVIs. Our paper is, to our best knowledge, the first one to study fractional BSVIs.

Let us recall that, for \( H \in (0, 1) \), a (one-dimensional) fBm \((B^H(t))_{t \geq 0}\) with Hurst parameter \( H \) is a continuous and centered Gaussian process with covariance

\[
\mathbb{E}\left[ B^H(t)B^H(s) \right] = \frac{1}{2}(t^{2H} + s^{2H} - |t - s|^{2H}), \quad t, s \geq 0.
\]
For $H = 1/2$, the fBm is a standard Brownian motion. If $H > 1/2$, then $B^H (t)$ has a long-range dependence, which means that for $r(n) := \text{cov}(B^H (1), B^H (n + 1) - B^H (n))$, we have $\sum_{n=1}^{\infty} r(n) = \infty$. Moreover, $B^H$ is self-similar, i.e., $B^H (at)$ has the same law as $a^H B^H (t)$ for any $a > 0$. Since there are many models of physical phenomena and finance which exploit the self-similarity and the long-range dependence, fBms are a very useful tool to characterize such type of problems.

However, since fBms are not semimartingales nor Markov processes when $H \neq 1/2$, we cannot use the classical theory of stochastic calculus to define the fractional stochastic integral. In essence, two different integration theories with respect to fractional Brownian motion have been defined and studied. The first one, originally due to Young [21], concerns the pathwise (with $\omega$ as a parameter) Riemann–Stieltjes integral, which exists if the integrand has Hölder continuous paths of order $\alpha > 1 - H$. But it turns out that this integral has the properties comparable to the Stratonovich integral, which leads to difficulties in applications.

The second one concerns the divergence operator (or Skorohod integral), defined as the adjoint of the derivative operator in the framework of the Malliavin calculus. This approach was introduced by Decreusefond and Üstünel [8], and it was very intensely studied, e.g., in Alòs and Nualart [3] for $H > 1/2$, and in Alòs et al. [1] for $H < 1/2$ as well as in Alòs et al. [2] for Gaussian processes.

An equivalent approach consists in defining, for $H \in (1/2, 1)$, the stochastic integral based on Wick product (introduced by Duncan et al. [9]), as the limit of Riemann sums. We mention that in contrast to the pathwise integral, the expectation of this integral is zero for a large class of integrands.

Concerning the study of BSDEs in the fractional framework, the major problem is the absence of a martingale representation type theorem with respect to a fBm. For the first time, Hu and Peng [12] overcome this problem, in the case $H > 1/2$. For this, they used the notion of quasi-conditional expectation $\hat{E}$ (introduced in Hu and Øksendal [11]). In our paper, we consider the BSDE

\[
\begin{cases}
-dY(t) = f(t, \eta(t), Y(t), Z(t))dt - Z(t)\delta B^H (t), & t \in [0, T], \\
Y(T) = g(\eta(T)),
\end{cases}
\]

(1)

driven by a fBm $B^H$ and governed by the process $\eta(t) = \eta(0) + \int_{0}^{t} \sigma(s)\delta B^H (s)$, $t \in [0, T]$, where $\sigma : [0, T] \rightarrow \mathbb{R}$ is deterministic, continuous function.

A special care has to be paid here to the stochastic integral in the BSDE (1). In [12], this stochastic integral is the Wick product one, but the Itô formula and the integration by part formula they used were established for the Itô–Skorohod type integral (see Definition 6.11 [10] ). In our approach, we use as stochastic integral the divergence operator.

Concerning the coefficient $\sigma$ of the driving process $\eta$, Hu and Peng [12] supposed that

there exists $c_0 > 0$ such that $\inf_{t \in [0,T]} \frac{\hat{\sigma}(t)}{\sigma(t)} \geq c_0$. 
for \( \hat{\sigma}(t) := \int_0^t \phi(t - r) \sigma(r) \, dr \), \( t \in [0, T] \). Here, in our manuscript, we work without such a condition. Let us mention that in [12], it is assumed that \( \eta(t) = \eta(0) + \int_0^t b(s) \, ds + \int_0^t \sigma(s) \delta B^H(s), \ t \in [0, T] \). In our paper, we adopt the form \( \eta(t) = \eta(0) + \int_0^t \sigma(s) \delta B^H(s), \ t \in [0, T] \), since we will use the proof of Theorem 3.8 from [12], especially the quasi-conditional expectation formula

\[
\hat{E}[f(\eta(T))|\mathcal{F}_t] = P_{\|\sigma\|_T^2 - \|\sigma\|_t^2} f(\eta(t)).
\]

Based on the above-described framework, we prove the existence and the uniqueness for BSDE (1). This approach includes, in particular, first a discussion of the equation

\[
Y(t) = g(\eta(T)) + \int_t^T f(s, \eta(s)) \, ds - \int_t^T Z(s) \delta B^H(s), \ t \in [0, T].
\]

After, the existence for BSDE (1) is proved by using a fixed point theorem over an appropriate Banach space.

Based on our results on BSDE driven by a fBm and on Pardoux and Rașcanu [19] on BSVI governed by a standard Brownian motion, we consider the following fractional BSVI

\[
\begin{cases}
-\, dY(t) + \partial \varphi(Y(t)) \, dt \ni f(t, \eta(t), Y(t), Z(t)) \, dt - Z(t) \delta B^H(t), \ t \in [0, T], \\
Y(T) = g(\eta(T)),
\end{cases}
\]

where \( \partial \varphi \) is the subdifferential of a convex lower semicontinuous (l.s.c.) function \( \varphi : \mathbb{R} \to (-\infty, +\infty] \). The existence of the solution will be proved.

Now, we give the outline of our paper: In Sect. 2, we recall some definitions and results about fractional stochastic integrals and the related Itô formula. We present the assumptions and some auxiliary results including the Itô formula w.r.t. the divergence-type integral in Sect. 3. Section 4 is devoted to prove the existence and the uniqueness result for BSDE driven by a fBm. In Sect. 5, we study the existence for fractional BSVI governed by a fBm. Finally, in the Appendix, we prove a more general Itô formula based on Theorem 8 [3] and an auxiliary lemma.

2 Preliminaries: Fractional Stochastic Calculus

In this section, we shall recall some important definitions and results concerning the Malliavin calculus, the stochastic integral with respect to a fBm, and Itô’s formula. For a deeper discussion, we refer the reader to [2, 3, 7, 9, 10] and [16].

Throughout our paper, we assume that the Hurst parameter \( H \) always satisfies \( H > 1/2 \). Define

\[
\phi(x) = H(2H - 1)|x|^{2H - 2}, \ x \in \mathbb{R}.
\]
Let us denote by $|\mathcal{H}|$ the Banach space of measurable functions $f : [0, T] \to \mathbb{R}$ such that

$$
\|f\|_{|\mathcal{H}|}^2 := \int_0^T \int_0^T \phi(u - v) |f(u)| |f(v)| \, du \, dv < +\infty.
$$

Given $\xi, \eta \in |\mathcal{H}|$, we put

$$
\langle \xi, \eta \rangle_T = \int_0^T \int_0^T \phi(u - v) \xi(u) \eta(v) \, du \, dv \quad \text{and} \quad \|\xi\|_T^2 := \langle \xi, \xi \rangle_T.
$$

Then $\langle \xi, \eta \rangle_T$ is a Hilbert scalar product. Let $\mathcal{H}$ be the completion of the space of step functions in $|\mathcal{H}|$ under this scalar product. We emphasize that the elements of $\mathcal{H}$ can be distributions. Moreover, from [14], we have the continuous embedding $L^2([0, T]) \subset L^1(\mathbb{R}(0, T)) \subset |\mathcal{H}| \subset \mathcal{H}$.

Let $B^H$ be a fractional Brownian motion defined on a complete probability space $(\Omega, \mathcal{F}, P)$, and that $\mathcal{F}$ is generated by $B^H$. Let $\mathcal{P}_T$ be the set of elementary random variables of the form

$$
F = f \left( \int_0^T \xi_1(t) dB^H(t), \ldots, \int_0^T \xi_n(t) dB^H(t) \right),
$$

where $f$ is a polynomial function of $n$ variables and $\xi_1, \xi_2, \ldots, \xi_n \in \mathcal{H}$. The Malliavin derivative $D^H_s$ of an elementary variable $F \in \mathcal{P}_T$ is defined by

$$
D^H_sF = \sum_{i=1}^n \frac{\partial f}{\partial x_i} \left( \int_0^T \xi_1(t) dB^H(t), \ldots, \int_0^T \xi_n(t) dB^H(t) \right) \xi_i(s), \ s \in [0, T].
$$

We denote by $\mathbb{D}_{1,2}$ the Banach space defined as the completion of $\mathcal{P}_T$ w.r.t. the following norm

$$
\|F\|_{1,2} = \left[ \mathbb{E}(|F|^2) + \mathbb{E} \left( \|D^H_sF\|_T^2 \right) \right]^{1/2}, \ F \in \mathcal{P}_T.
$$

Hence, $\mathbb{D}_{1,2}$ consists of all $F \in L^2(\Omega, \mathcal{F}, P)$ such that there exists a sequence $F_n \in \mathcal{P}_T, n \geq 1$, which satisfies

$$
F_n \to F \ \text{in} \ L^2(\Omega, \mathcal{F}, P), \quad \left( D^H_sF_n \right)_{n \geq 1} \ \text{is convergent in} \ L^2(\Omega, \mathcal{F}, P; \mathcal{H}).
$$

Moreover, from Proposition 1.2.1 [16], we have that $D^H_s = (D^H_s)_{s \in [0, T]}$ is a closable operator from $L^2(\Omega, \mathcal{F}, P)$ to $L^2(\Omega, \mathcal{F}, P; \mathcal{H})$. Thus, $D^H_sF = \lim_{n \to \infty} D^H_sG_n$ in
\[ L^2(\Omega, \mathcal{F}, P; \mathcal{H}), \] for every sequence \( G_n \in \mathcal{P}_T, n \geq 1, \) which satisfies
\[
G_n \longrightarrow F \text{ in } L^2(\Omega, \mathcal{F}, P), \\
(D^H G_n)_{n \geq 1} \text{ is convergent in } L^2(\Omega, \mathcal{F}, P; \mathcal{H}).
\]

Let us introduce also another derivative
\[ D_t^H F = \int_0^T \phi(t - v) D_v^H F dv, \quad t \in [0, T]. \tag{3} \]

We also need the adjoint operator of the derivative \( D^H \). This operator is called divergence operator, and it is denoted by \( \delta(\cdot) \) and represents the divergence-type integral with respect to a fBm (see, e.g., [2,3] and [7] for more details).

**Definition 1** We say that a process \( u \in L^2(\Omega, \mathcal{F}, P; \mathcal{H}) \) belongs to the domain \( \text{Dom}(\delta) \), if there exists \( \delta(u) \in L^2(\Omega, \mathcal{F}, P) \), such that the following duality relationship is satisfied
\[ \mathbb{E}(F\delta(u)) = \mathbb{E}(\langle D^H F, u \rangle_T), \quad \text{for all } F \in \mathcal{P}_T. \tag{4} \]

**Remark 2** In (4), the class \( \mathcal{P}_T \) can be replaced by \( \mathbb{D}^{1,2}_T \) (see [7] Definition 2.2.2 and 2.2.3). If \( u \in \text{Dom}(\delta) \), \( \delta(u) \) is unique, and we define the divergence-type integral of \( u \in \text{Dom}(\delta) \) w.r.t. fBm \( B^H \) by putting \( \int_0^T u(s) \delta B^H(s) := \delta(u) \).

Let us recall a result about a sufficient condition for the existence of the divergence-type integral. For this, we use the Itô–Skorohod type stochastic integral introduced in Definition 6.11 [10], which is defined in the spirit of the anticipative Skorohod integral w.r.t. Brownian motion in [17].

**Theorem 3** [Proposition 6.25, [10]] We denote by \( \mathbb{L}^{1,2}_H \) the space of all stochastic processes \( u : (\Omega, \mathcal{F}, P) \rightarrow \mathcal{H} \) such that
\[ \mathbb{E} \left( \|u\|_T^2 + \int_0^T \int_0^T |D^H_s u(t)|^2 ds dt \right) < \infty. \tag{5} \]

If \( u \in \mathbb{L}^{1,2}_H \), then the Itô–Skorohod type stochastic integral \( \int_0^T u(s) dB^H(s) \) defined by Definition 6.11 [10] exists and coincides with the divergence-type integral (see Theorem 6.23 [10]). Moreover,
\[
\begin{align*}
\mathbb{E} \left[ \int_0^T u(s) dB^H(s) \right] &= 0, \\
\mathbb{E} \left[ \int_0^T u(s) dB^H(s) \right]^2 &= \mathbb{E} \left( \|u\|_T^2 + \int_0^T \int_0^T D^H_s u(t) D^H_s u(s) ds dt \right). 
\end{align*}
\]
Let us finish this section by giving an Itô formula for the divergence-type integral. Due to Theorem 8 [3], the following Itô formula holds.

**Theorem 4** Let \( \psi \) be a function of class \( C^2(\mathbb{R}) \). Assume that the process \((u_t)_{t \in [0,T]}\) belongs to \( D_{2,2}^{1,2}([\mathcal{H}]) \) and that the integral \( X_t = \int_0^t u_s \delta B^H(s) \) is almost surely continuous. Assume that \( \mathbb{E}|u|^2 \) belong to \( \mathcal{H} \). Then, for each \( t \in [0, T] \), the following formula holds

\[
\psi(X_t) = \psi(0) + \int_0^t \frac{\partial}{\partial x} \psi(X_s) u_s \delta B^H(s) + H(2H - 1) \int_0^t \frac{\partial^2}{\partial x^2} \psi(X_s) |s - r|^{2H-2} \left( \int_0^s D_r u_0 \delta B^H(\theta) d\theta \right) ds
\]

\[
+ H(2H - 1) \int_0^t \frac{\partial^2}{\partial x^2} \psi(X_s) \left( \int_0^s u_\theta |s - \theta|^{2H-2} d\theta \right) ds.
\]

**Remark 5** The above theorem can be generalized, see Theorem 33 in the Appendix. In particular, Corollary 35 tells us:

Let \( f : [0, T] \to \mathbb{R} \) and \( g : [0, T] \to \mathbb{R} \) be deterministic continuous functions. If

\[
X_t = X_0 + \int_0^t g_s ds + \int_0^t f_s \delta B^H(s), \ t \in [0, T],
\]

and \( \psi \in C^{1,2}([0, T] \times \mathbb{R}) \), we have

\[
\psi(t, X_t) = \psi(0, X_0) + \int_0^t \frac{\partial}{\partial s} \psi(s, X_s) ds + \int_0^t \frac{\partial}{\partial x} \psi(s, X_s) dX_s
\]

\[
+ \frac{1}{2} \int_0^t \frac{\partial^2}{\partial x^2} \psi(s, X_s) \left( \frac{d}{ds} \|f\|^2 \right) ds, \ t \in [0, T].
\] (6)

### 3 Assumptions and Auxiliary Results

#### 3.1 Assumptions

Let us consider the Itô-type process

\[
\eta(t) = \eta(0) + \int_0^t \sigma(s) \delta B^H(s), \ t \in [0, T],
\] (7)
where the coefficients $\eta(0)$ and $\sigma$ satisfy:

\((H_1)\) $\eta(0) \in \mathbb{R}$ is a given constant;
\((H_2)\) $\sigma : \mathbb{R} \to \mathbb{R}$ is a deterministic continuous function such that $\sigma(t) \neq 0$, for all $t \in [0, T]$.

Let

$$\hat{\sigma}(t) := \int_0^t \phi(t - r)\sigma(r)dr, \ t \in [0, T]. \tag{8}$$

We recall that [see (2)]

$$\|\sigma\|^2_t = H(2H - 1) \int_0^t \int_0^t |u - v|^{2H-2}\sigma(u)\sigma(v)du dv.$$

**Remark 6** The function $\hat{\sigma}$ defined by (8) can be written in the following form:

$$\hat{\sigma}(t) = H(2H - 1)t^{2H-1} \int_0^1 (1 - u)^{2H-2}\sigma(tu)du, \ t \in [0, T].$$

Moreover, we observe that $\|\sigma\|^2_t$ is continuously differentiable with respect to $t$, and

\[(a)\] \[\frac{d}{dt}(\|\sigma\|^2_t) = 2\sigma(t)\hat{\sigma}(t) > 0, \ t \in (0, T],\]

\[(b)\] for a suitable constant $M > 0$, $\frac{1}{M}t^{2H-1} \leq \frac{\hat{\sigma}(t)}{\sigma(t)} \leq Mt^{2H-1}, \ t \in [0, T]. \tag{9}\]

Our objective is to study the following BSDE driven by the fBm $B^H$ and the above introduced stochastic process $\eta$:

$$\begin{cases}
-dY(t) = f(t, \eta(t), Y(t), Z(t))dt - Z(t)\delta B^H(t), \ t \in [0, T], \\
Y(T) = \xi.
\end{cases}$$

Here, the stochastic integral is understood as the divergence operator. We make the following assumptions on the function $f$ and the terminal condition $\xi$:

\((H_3)\) The function $f : [0, T] \times \mathbb{R}^3 \to \mathbb{R}$ belongs to the space $C^0_{pol}([0, T] \times \mathbb{R}^3) \tag{1}$, and there exists a constant $L$ such that, for all $t \in [0, T], x, y_1, y_2, z_1, z_2 \in \mathbb{R},$

$$|f(t, x, y_1, z_1) - f(t, x, y_2, z_2)| \leq L(|y_1 - y_2| + |z_1 - z_2|).$$

---

$C^k_{pol}([0, T] \times \mathbb{R}^m)$ is the space of all $C^k$-functions over $[0, T] \times \mathbb{R}^m$, which together with their derivatives, are of polynomial growth.

$\odot$ Springer
(H₄) \( \xi = g(\eta_T) \), where \( g : \mathbb{R} \to \mathbb{R} \) is a differentiable function with polynomial growth.

Before giving the definition of the solution for the above BSDE and investigating its wellposedness (see Sect. 4), we introduce, motivated by [12], the following space

\[ \mathcal{V}_T := \left\{ Y = \phi(\cdot, \eta(\cdot)) : \phi \in C_{pol}^{1,3}([0, T] \times \mathbb{R}) \text{ with } \frac{\partial \phi}{\partial t} \in C_{pol}^{0,1}([0, T] \times \mathbb{R}) \right\} \]

as well as its completion \( \bar{\mathcal{V}}_T \) under the following \( \alpha \)-norm

\[
\|Y\|_\alpha = \left( \int_0^T t^{2\alpha - 1} \mathbb{E}|Y(t)|^2 \, dt \right)^{1/2} = \left( \int_0^T t^{2\alpha - 1} \mathbb{E}|\phi(t, \eta(t))|^2 \, dt \right)^{1/2},
\]

where \( \alpha \geq 1/2 \). Let us study some auxiliary results concerning these spaces.

### 3.2 An Itô Formula

We begin with the following result concerning the space \( \mathcal{V}_T \).

**Lemma 7** We have \( \mathcal{V}_T \subset \mathbb{L}^{1,2}_H \subset \text{Dom}(\delta) \).

**Proof** Let \( u \in \mathcal{V}_T \). In order to show (5), we first prove that \( E\|u\|_T^2 < \infty \). From \( L^2([0, T]) \subset \mathcal{H} \) we see that it is sufficient to show that

\[
\mathbb{E}\int_0^T |u(s, \eta(s))|^2 \, ds < \infty,
\]

where the latter property can be deduced from \( \mathbb{E}|u(s, \eta(s))|^2 \leq C, \ s \in [0, T], \) for some suitable \( C \in \mathbb{R} \). Indeed, since \( u \in \mathcal{V}_T \), we have, for some \( C > 0, \ k \geq 1, \)

\[
\mathbb{E}|u(s, \eta(s))|^2 \leq C\mathbb{E}(1 + |\eta(s)| + \cdots + |\eta(s)|^k), \ s \in [0, T].
\]

On the other hand, from (7) and Theorem 7.10 [10], we see that for any \( p \geq 1 \), there exists \( C_p > 0 \) such that

\[
\mathbb{E}|\eta(s)|^p \leq C_{\mathbb{E}} \left( 1 + \left| \int_0^s \sigma(v) \delta B^H(v) \right|^p \right) \leq C_p + C_p \|\sigma\|_s^{p/2}.
\]

Hence, \( E\|u\|_T^2 < \infty \).

In a second step, we show that \( \mathbb{D}^H_s u(t, \eta(t)) \) exists for all \( s, t \in [0, T] \) and

\[
\mathbb{E}\int_0^T \int_0^T |\mathbb{D}^H_s u(t, \eta(t))|^2 \, ds \, dt < \infty.
\]
In fact, a straightforward computation shows that

$$D^H_s u(t, \eta(t)) = \frac{\partial}{\partial x} u(t, \eta(t)) \int_0^t \phi(s-v)\sigma(v)dv.$$  

From the polynomial growth of \( \frac{\partial u}{\partial x} \) and the continuity of \( \sigma \), we conclude

$$\mathbb{E}|D^H_s u(t, \eta(t))|^2 \leq C, \quad s, t \in [0, T],$$

for a suitable \( C \in \mathbb{R} \), which yields (12). Consequently, the process \( u \) satisfies (5) and belongs to \( \mathbb{L}^{1,2}_H \).

Let us give now a statement for the Itô formula in the framework of the divergence-type integral, which is for our purposes better adapted than Theorem 4.5 [9]. We mention that the formula in the following theorem is a particular case of our generalized Itô formula (64) (see Theorem 33 in the Appendix), but here, we use a different approach.

**Theorem 8** Let \( u \in \mathcal{V}_T \) and \( f \in C^0_{pol}([0, T] \times \mathbb{R}) \). We put \( f_s = f(s, \eta(s)), \ s \in [0, T] \). Then for the Itô process

$$X_t = X_0 + \int_0^t f_s ds + \int_0^t u_s \delta B^H(s), \ t \in [0, T],$$

we have

(i) \( uXI_{[0,t]} \in \text{Dom}(\delta), \ t \in [0, T], \)

(ii) \( X_s \in \mathbb{D}_{1,2}, \ s \in [0, T], \)

(iii) \( (D^H_s X_s)_{s \in [0,T]} \in L^2([0, T] \times \Omega) \)

and

$$X_t^2 = X_0^2 + 2 \int_0^t X_s f_s ds + 2 \int_0^t X_s u_s \delta B^H(s) + 2 \int_0^t u_s D^H_s X_s ds, \ a.s. \quad t \in [0, T].$$

Before proving this theorem, we give the following lemma.

**Lemma 9** Let \( u \in \mathcal{V}_T \) and \( X_t = \int_0^t u_s \delta B^H(s), \ t \in [0, T] \). Then \( uXI_{[0,t]} \in \text{Dom}(\delta), \ t \in [0, T], \) \( (D^H_s X_s)_{s \in [0,T]} \in L^2([0, T] \times \Omega), \) and

$$X_t^2 = 2 \int_0^t X_s u_s \delta B^H(s) + 2 \int_0^t u_s D^H_s X_s ds, \ t \in [0, T].$$
Proof Let \( F \in \mathcal{P}_T \). Then, since obviously \( X_t F \in \mathbb{D}_{1,2} \), we have from Definition 1

\[
\mathbb{E}[X_t^2 F] = \mathbb{E}[X_t(X_t F)] = \mathbb{E} \left[ \int_0^T u_s I_{[0,t]}(s) \delta B^H(s)(X_t F) \right]
\]

\[
= \mathbb{E} \left[ \int_0^T u_s I_{[0,t]}(s) \mathbb{D}_s^H(X_t F) ds \right] = \mathbb{E} \left[ \left( \int_0^t u_s \mathbb{D}_s^H F ds \right) X_t \right]
\]

\[
+ \mathbb{E} \left[ \left( \int_0^t u_s \mathbb{D}_s^H X_t ds \right) F \right].
\]

Here, we have used \( X \in \mathbb{L}^{1,2}_H \) and, hence also \( X F \in \mathbb{L}^{1,2}_H \). In particular, we observe that

\[
\mathbb{D}_s^H X_t = \int_0^t \phi(s-r)u_r dr + \int_0^t \mathbb{D}_s^H u_r \delta B^H(r), \quad s, t \in [0, T].
\]

Moreover, since \( \int_0^t u_s \mathbb{D}_s^H F ds \in \mathbb{D}_{1,2} \), we get again from Definition 1

\[
\mathbb{E} \left[ \left( \int_0^t u_s \mathbb{D}_s^H F ds \right) X_t \right] = \mathbb{E} \left[ \left( \int_0^t u_s \mathbb{D}_s^H F ds \right) \left( \int_0^t u_s \delta B^H(s) \right) \right]
\]

\[
= \mathbb{E} \left[ \int_0^t \int_0^t \mathbb{D}_r^H \left( u_s \mathbb{D}_s^H F \right) u_r dr ds \right].
\]

On the other hand, using (13), it follows

\[
\mathbb{E} \left[ \left( \int_0^t u_s \mathbb{D}_s^H X_t ds \right) F \right] = \mathbb{E} \left[ \int_0^t \int_0^t \phi(s-r)u_s u_r ds dr \cdot F \right]
\]

\[
+ \int_0^t \mathbb{E} \left[ \int_0^t \mathbb{D}_r^H (u_s F) \mathbb{D}_s^H u_r dr \right] ds.
\]

Therefore, by combining the above relations, we obtain

\[
\mathbb{E}[X_t^2 F] = \mathbb{E} \left[ \left( \int_0^t u_s \mathbb{D}_s^H F ds \right) X_t \right] + \mathbb{E} \left[ \left( \int_0^t u_s \mathbb{D}_s^H X_t ds \right) F \right]
\]
By noticing that the right-hand side of the above equality is symmetric in \((s, r)\), we deduce

\[
\begin{align*}
\mathbb{E}\left[X_t^2 \cdot F\right] &= 2\mathbb{E}\left[\int_t^s \mathcal{D}_r^H \left(u_s \mathcal{D}_s^H \cdot F\right) u_s \, dr \, ds\right] + 2\mathbb{E}\left[\int_0^t \mathcal{D}_r^H \left(u_s \mathcal{D}_s^H \cdot F\right) u_s \, ds \, dr \cdot F\right] \\
&+ 2\mathbb{E}\left[\int^s_0 \mathcal{D}_r^H \left(u_s \mathcal{D}_s^H \cdot F\right) u_r \, dr \, ds\right] := 2I_1 + 2I_2 + 2I_3. \quad (14)
\end{align*}
\]

Let us begin with the evaluation of \(I_1\). Obviously, by using that \(u_{I[0,s]} \in \mathbb{L}_H^{1,2} \subset \text{Dom}(\delta)\) and \(u_s \mathcal{D}_s^H \cdot F \in \mathbb{D}_{1,2}\), we have from Fubini’s Theorem and Definition 1

\[
I_1 = \mathbb{E}\left[\int_t^s \mathcal{D}_r^H \left(u_s \mathcal{D}_s^H \cdot F\right) u_r \, dr \, ds\right] = \int_t^s \mathbb{E}\left[\int_0^T \mathcal{D}_r^H \left(u_s \mathcal{D}_s^H \cdot F\right) u_r \, I_{[0,s](r)} \, dr\right] \, ds
\]

\[
= \int_t^s \mathbb{E}\left[\int_0^T u_r \, I_{[0,s](r)} \delta \mathcal{B}^H(r) u_s \mathcal{D}_s^H \cdot F\right] \, ds = \mathbb{E}\left[\int_0^t u_s X_s \mathcal{D}_s^H \cdot F \, ds\right]. \quad (15)
\]

On the other hand, since also \(\mathcal{D}_s^H u_{I[0,s]} \in \mathbb{L}_H^{1,2} \subset \text{Dom}(\delta)\) and \(u_s \cdot F \in \mathbb{D}_{1,2}, s \in [0, t]\), we obtain again from Fubini’s Theorem as well as Definition 1 that

\[
\mathbb{E}\left[\int_t^s \mathcal{D}_r^H \left(u_s F\right) \mathcal{D}_s^H u_r \, dr \, ds\right] = \int_t^s \mathbb{E}\left[\int_0^T \mathcal{D}_r^H \left(u_s F\right) \mathcal{D}_s^H u_r \, dr\right] \, ds
\]

\[
= \int_t^s \mathbb{E}\left[\int_0^T \mathcal{D}_s^H u_r \delta \mathcal{B}^H(r) u_s \, F\right] \, ds = \mathbb{E}\left[\int_0^t \left(\int_0^T \mathcal{D}_s^H u_r \delta \mathcal{B}^H(r)\right) u_s \, F \, ds\right].
\]

Thus, due to (13)

\[
I_2 + I_3 = \mathbb{E}\left[\int_0^t \int_0^s \phi(s - r) u_s u_r \, ds \, dr \cdot F\right] + \mathbb{E}\left[\int_0^t \int_0^s \mathcal{D}_r^H \left(u_s F\right) \mathcal{D}_s^H u_r \, dr \, ds\right].
\]
\[
\begin{align*}
&\mathbb{E} \left[ \int_0^t u_s \left( \int_0^s \phi(s-r)u_r \, dr + \int_0^s \mathcal{D}_s^H u_r \mathcal{B}^H(r) \right) ds \cdot F \right] \\
&= \mathbb{E} \left[ F \int_0^t u_s \mathcal{D}_s^H X_s ds \right].
\end{align*}
\] (16)

Consequently, from (14)–(16),

\[
\mathbb{E} \left[ 2 \int_0^T u_s X_s I_{[0,t]}(s) \mathcal{D}_s^H F ds \right]
= \mathbb{E} \left[ \left( X_t^2 - 2 \int_0^t u_s \mathcal{D}_s^H X_s ds \right) F \right],
\text{ for all } F \in \mathcal{P}_T.
\] (17)

On the other hand, from Theorem 7.10 [10] and the fact that \( u \in \mathcal{V}_T \), it follows that there exists \( C > 0 \) such that

\[
\mathbb{E} \left( \int_0^t u_s \delta \mathcal{B}^H(s) \right)^4 \leq C \mathbb{E} \|u\|_4^4 + C \mathbb{E} \left( \int_0^T \int_0^T \| \mathcal{D}_t^H u_s \|_2^2 ds \, dr \right)^2 \leq C \mathbb{E} \left( \int_0^t |u_s|^2 ds \right)^2 + C \mathbb{E} \int_0^T \int_0^T \| \mathcal{D}_t^H u_s \|^4 ds \, dr \leq C, \text{ for all } t \in [0, T],
\] as well as

\[
\mathbb{E} \left( \int_0^t \mathcal{D}_s^H u_r \delta \mathcal{B}^H(r) \right)^4 \leq C \mathbb{E} \| \mathcal{D}_s^H u \|_4^4 + C \mathbb{E} \left( \int_0^T \int_0^T \| \mathcal{D}_t^H (\mathcal{D}_s^H u_r) \|^2 ds \, dr \right)^2 \leq C \mathbb{E} \int_0^T \int_0^T |u_{xx}(r, \eta(r))|^4 dr \leq C, \text{ for all } t \in [0, T].
\]

Taking into account the definition of the process \( X \), we deduce from the above two estimates and Theorem 3 that

\[
u X \in L^2(\Omega, \mathcal{F}, P; \mathcal{H}) \text{ and } X_t^2 - 2 \int_0^t u_s \mathcal{D}_s^H X_s ds \in L^2(\Omega, \mathcal{F}, P).
\]
Therefore, from (17) and Definition 1, it follows that \( uX_{I[0,t]} \in \text{Dom}(\delta) \) and

\[
2 \int_0^t u_s X_s \delta^H(s) = X_t^2 - 2 \int_0^t u_s D_s^H X_s ds.
\]

\[ \Box \]

Proof of Theorem 8  Let

\[ Y_t := \int_0^t u_s \delta^H(s) \quad \text{and} \quad Z_t := X_0 + \int_0^t f_s ds, \quad t \in [0, T]. \]

From the previous lemma, we know that \( uY_{I[0,t]} \in \text{Dom}(\delta) \), for all \( t \in [0, T] \), and

\[
Y_t^2 = 2 \int_0^t u_s Y_s \delta^H(s) + 2 \int_0^t u_s D_s^H Y_s ds, \quad t \in [0, T].
\]

On the other hand, it is obvious that \( Z_t^2 = X_0^2 + 2 \int_0^t f_s Z_s ds, \quad t \in [0, T] \). Moreover, we assert that \( uZ_{I[0,t]} \in \text{Dom}(\delta) \), for all \( t \in [0, T] \), and

\[
Y_t Z_t = \int_0^t u_s Z_s \delta^H(s) + \int_0^t f_s Y_s ds + \int_0^t u_s D_s^H Z_s ds, \quad t \in [0, T].
\]

Indeed, since \( Z_t F \in \mathbb{D}_{1,2} \) and \( D_s^H(Z_t F) = D_s^H(Z_s F) + \int_s^t D_s^H(f_r F)dr, s \in [0, t] \), we have

\[
\mathbb{E}[Y_t Z_t F] = \mathbb{E} \left[ \left( \int_0^T u_s I_{[0,t]}(s) \delta^H(s) \right) Z_t F \right] = \mathbb{E} \left[ \int_0^t u_s D_s^H(Z_t F) ds \right]
\]

\[
= \mathbb{E} \left[ \int_0^t u_s D_s^H(Z_s F) ds \right] + \mathbb{E} \left[ \int_0^t u_s D_s^H(f_r F) dr ds \right]
\]

\[
= \mathbb{E} \left[ \int_0^t u_s D_s^H Z_s ds \cdot F \right] + \mathbb{E} \left[ \int_0^t u_s Z_s D_s^H F ds \right]
\]

\[
+ \int_0^t \mathbb{E} \left[ \int_0^r u_s D_s^H(f_r F) ds \right] dr
\]
\[ \begin{align*}
E \left[ \int_0^t u_s \mathbb{D}_s^H Z_s \delta_{B^H}(s) \right] &= E \left[ \int_0^t u_s Z_s \mathbb{D}_s^H F \, ds \right] + E \left[ \int_0^t Y_r f_r \, dr \right].
\end{align*} \]

Therefore,
\[ E \left[ \int_0^T u_s Z_s I_{[0,t]}(s) \mathbb{D}_s^H F \, ds \right] = E \left[ \left( Y_t Z_t - \int_0^t u_s \mathbb{D}_s^H Z_s \, ds - \int_0^t Y_r f_r \, dr \right) F \right], \quad F \in \mathcal{P}_T, \]

and since \( u Z_{[0,t]} \in L^2(\Omega, \mathcal{F}, P; \mathcal{H}) \) as well as \( Y_t Z_t - \int_0^t u_s \mathbb{D}_s^H Z_s \, ds - \int_0^t Y_s f_s \, ds \in L^2(\Omega, \mathcal{F}, P) \), we conclude from Definition 1 that \( u Z_{[0,t]} \in \text{Dom}(\delta) \) and
\[ \int_0^t u_s Z_s \delta_{B^H}(s) = Y_t Z_t - \int_0^t u_s \mathbb{D}_s^H Z_s \, ds - \int_0^t Y_s f_s \, ds. \]

Consequently, using the above notation as well as the linearity of \( \text{Dom}(\delta) \), we have \( X_t = Y_t + Z_t, u X_{[0,t]} \in \text{Dom}(\delta) \) and
\[ X_t^2 = Y_t^2 + 2Y_t Z_t + Z_t^2 = X_0^2 + 2 \int_0^t X_s f_s \, ds + 2 \int_0^t X_s u_s \delta_{B^H}(s) + 2 \int_0^t u_s \mathbb{D}_s^H X_s \, ds, \quad t \in [0, T]. \]

Emphasizing that the Itô–Skorohod integral and the divergence-type integral coincide for all \( u \in L^1_{\mathcal{H}} \). Then, from Hu and Peng Lemma 4.2 [12], the following lemma holds true:

**Lemma 10** Let \( a, b \in C_{pol}^{0,1}([0, T] \times \mathbb{R}) \). If
\[ \int_0^t b(s, \eta(s)) \, ds + \int_0^t a(s, \eta(s)) \delta_{B^H}(s) = 0, \quad \text{for all } t \in [0, T], \]

then
\[ b(s, x) = a(s, x) = 0, \quad \text{for all } t \in [0, T], \quad x \in \mathbb{R}. \]

### 3.3 Quasi-Conditional Expectation

In this subsection, we recall the quasi-conditional expectation which was introduced by Hu and Øksendal [11].
For any \( n \geq 1 \), we introduce the set \( \mathcal{H}^{\otimes n} \) of all real symmetric Borel functions \( f_n \) of \( n \) variables such that

\[
\| f_n \|^2_{\mathcal{H}^{\otimes n}} := \int_{\mathbb{R}_+^n \times \mathbb{R}_+^n} \prod_{i=1}^n \phi(s_i - r_i) f_n(s_1, \ldots, s_n) f_n(r_1, \ldots, r_n) ds_1 \ldots ds_n dr_1 \ldots dr_n < \infty.
\]

Then, one can define the iterated integral (see [7, 10, 11])

\[
I_n(f_n) = n! \int_{0 \leq t_1 < \ldots < t_n} f_n(t_1, \ldots, t_n) d\mathcal{B}^H(t_1) \cdots d\mathcal{B}^H(t_n).
\]

in the sense of Itô–Skorohod. For \( n = 0 \) and \( f = f_0 \) be a constant, we set \( I_0(f_0) = f_0 \) and \( \| f_0 \|^2_{\mathcal{H}^{\otimes 0}} = f_0^2 \).

We recall the following theorem, see Theorem 3.9.9 [7] or [9] (Theorem 6.11) or [11] (Theorem 3.22).

**Theorem 11** Let \( F \in L^2(\Omega, \mathcal{F}, P) \). Then there exist \( f_n \in \mathcal{H}^{\otimes n}, n \geq 0 \) such that

\[
F = \sum_{n=0}^{\infty} I_n(f_n).
\]

Moreover,

\[
\mathbb{E}|F|^2 = \sum_{n=0}^{\infty} n! \| f_n \|^2_{\mathcal{H}^{\otimes n}} < \infty.
\]

The convergence in this chaos expansion of \( F \) is understood in the sense of \( L^2(\Omega, \mathcal{F}, P) \).

Let \( \lbrace \mathcal{F}_t \rbrace_{t \geq 0} \) be the filtration generated by \( \mathcal{B}^H \). We now recall the definition of quasi-conditional expectation (see [11] and [12]):

**Definition 12** If \( F \in L^2(\Omega, \mathcal{F}, P) \), then the quasi-conditional expectation is defined as

\[
\hat{\mathbb{E}}[F|\mathcal{F}_t] = \sum_{n=0}^{\infty} I_n(f_n I^{\otimes n}_{[0,t]}), \ t \in [0, T],
\]

if the series on the right side converges in \( L^2(\Omega, \mathcal{F}, P) \). Here

\[
I^{\otimes n}_{[0,t]}(t_1, \ldots, t_n) = I_{[0,t]}(t_1) \cdots I_{[0,t]}(t_n).
\]

**Remark 13** \( \hat{\mathbb{E}} \left[ \hat{\mathbb{E}}[F|\mathcal{F}_t] | \mathcal{F}_s \right] = \hat{\mathbb{E}}[F|\mathcal{F}_s] \), for \( 0 \leq s \leq t \leq T \).
Lemma 14 (Lemma 3.3 [12]) For all $h \in L^{1,2}_H$ and all $t \in [0, T]$, $P$-a.s.

$$\hat{E} \left[ \int_t^T h(u) \delta B^H(u) \bigg| \mathcal{F}_t \right] = 0.$$ 

The following lemma is inspired by Theorem 3.9 [12].

Lemma 15 Let $F = h(\eta(T))$, where $h : \mathbb{R} \to \mathbb{R}$ is a continuous function of polynomial growth. Then $F \in L^2(\Omega, \mathcal{F}, P)$ and

$$\mathbb{E} \left[ \hat{E}[F|\mathcal{F}_t] \right] = \mathbb{E}F, \ t \in [0, T].$$

Proof First, from the polynomial growth of $f$ and

$$\mathbb{E}|\eta(T)|^p \leq C \mathbb{E} \left( 1 + \left| \int_0^T \sigma(v) dB^H(v) \right|^p \right) \leq C_p + C_p \|\sigma\|_{L^p}^{p/2} \leq M_p, \ p \geq 1,$$

we obtain $F \in L^2(\Omega, \mathcal{F}, P)$.

We now put $p_t(x) = \frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}}, \ t \in (0, T], x \in \mathbb{R},$ and

$$P_t h(x) = \int_\mathbb{R} p_t(x-y) h(y) dy.$$

Applying (6) to $P_{\|\sigma\|_2^2 - \|\sigma\|_T^2} h(\eta(t))$ and noticing that $\frac{\partial}{\partial t} P_t h(x) = \frac{1}{2} \frac{\partial^2}{\partial x^2} P_t h(x)$, we have

$$h(\eta(T)) = P_{\|\sigma\|_T^2 - \|\sigma\|_T^2} h(\eta(t)) + \int_t^T \frac{\partial}{\partial x} P_{\|\sigma\|_T^2 - \|\sigma\|_T^2} h(\eta(s)) \sigma(s) \delta B^H(s), \quad (19)$$

and hence,

$$\mathbb{E} h(\eta(T)) = \mathbb{E} \left\{ P_{\|\sigma\|_T^2 - \|\sigma\|_T^2} h(\eta(t)) \right\}. \quad (20)$$

On the other hand, from the proof of Theorem 3.8 [12], it follows that

$$\hat{E}[F|\mathcal{F}_t] = P_{\|\sigma\|_T^2 - \|\sigma\|_T^2} h(\eta(t)). \quad (21)$$
For the reader’s convenience, we give a justification for (21) here. By taking \( t = 0 \) in (19), we obtain

\[
h(\eta(T)) = P_{\|\sigma\|_T^2} h(\eta(0)) + \int_0^T \frac{\partial}{\partial x} P_{\|\sigma\|_T^2 - \|\sigma\|_T^2} h(\eta(s)) \sigma(s) \delta B^H(s).
\]

Thus, due to Lemma 14 and Remark 4.10 [11],

\[
\hat{E}\left[ F | \mathcal{F}_t \right] = P_{\|\sigma\|_T^2} h(\eta(0)) + \int_0^t \frac{\partial}{\partial x} P_{\|\sigma\|_T^2 - \|\sigma\|_T^2} h(\eta(s)) \sigma(s) \delta B^H(s). \tag{22}
\]

On the other hand, by applying (6) to \( P_{\|\sigma\|_T^2 - \|\sigma\|_T^2} h(\eta(s)) \) over time interval \([0, t]\), we get

\[
h(\eta(t)) = P_{\|\sigma\|_T^2} h(\eta(0)) + \int_0^t \frac{\partial}{\partial x} P_{\|\sigma\|_T^2 - \|\sigma\|_T^2} h(\eta(s)) \sigma(s) \delta B^H(s).
\]

Thus, from semigroup property

\[
P_{\|\sigma\|_T^2 - \|\sigma\|_T^2} h(x) = P_{\|\sigma\|_T^2 - \|\sigma\|_T^2} P_{\|\sigma\|_T^2 - \|\sigma\|_T^2} h(x), \quad 0 \leq s \leq t \leq T
\]

and

\[
\frac{\partial}{\partial x} P_{\|\sigma\|_T^2 - \|\sigma\|_T^2} h(x) = P_{\|\sigma\|_T^2 - \|\sigma\|_T^2} \frac{\partial}{\partial x} P_{\|\sigma\|_T^2 - \|\sigma\|_T^2} h(x),
\]

then, we get form (22) that

\[
\hat{E}\left[ F | \mathcal{F}_t \right] = P_{\|\sigma\|_T^2} h(\eta(0)) + \int_0^t \frac{\partial}{\partial x} P_{\|\sigma\|_T^2 - \|\sigma\|_T^2} h(\eta(s)) \sigma(s) \delta B^H(s)
\]

\[
= P_{\|\sigma\|_T^2 - \|\sigma\|_T^2} \left[ \int_0^t \frac{\partial}{\partial x} P_{\|\sigma\|_T^2 - \|\sigma\|_T^2} h(\eta(s)) \sigma(s) \delta B^H(s) \right]
\]

\[
= P_{\|\sigma\|_T^2 - \|\sigma\|_T^2} h(\eta(t)).
\]

This means that we have proved (21). Consequently, from (20) and (21), we have

\[
\mathbb{E}\left\{ \hat{E}[F | \mathcal{F}_t] \right\} = \mathbb{E}\left\{ P_{\|\sigma\|_T^2 - \|\sigma\|_T^2} h(\eta(t)) \right\} = \mathbb{E}F.
\]

\(\Box\)
4 BSDEs Driven By $B^H$

The objective of this section is to study the BSDE
\begin{equation}
\begin{aligned}
-\frac{dY(t)}{dt} &= f(t, \eta(t), Y(t), Z(t))dt - Z(t)\delta B^H(t), \quad t \in [0, T], \\
Y(T) &= \xi.
\end{aligned}
\end{equation}

We now give the definition of the solution for the above BSDE.

**Definition 16** A pair $(Y, Z)$ is called a solution of BSDE (23), if the following conditions are satisfied:

(a1) $Y \in \bar{V}_T^{1/2}$ and $Z \in \bar{V}_T^H$ (Recall (10));

(a2) $Y(t) = \xi + \int_{t}^{T} f(s, \eta(s), Y(s), Z(s)) \, ds - \int_{t}^{T} Z(s)\delta B^H(s), \text{ a.s., } t \in (0, T].$

Let us begin by discussing the existence of a solution for BSDE (23).

4.1 Existence

We begin with considering the following equation:
\begin{equation}
Y(t) = \xi + \int_{t}^{T} f(s, \eta(s), \chi(s, \eta(s)), \psi(s, \eta(s))) \, ds - \int_{t}^{T} Z(s)\delta B^H(s), \quad t \in [0, T],
\end{equation}

where $\chi, \psi \in C^{1,3}_{pol}([0, T] \times \mathbb{R})$ with $\frac{\partial \chi}{\partial t}, \frac{\partial \psi}{\partial t} \in C^{0,1}_{pol}([0, T] \times \mathbb{R}).$ Observe that (24) is a special case of BSDE (23).

In Proposition 4.5 [12], the existence problem of a solution for an equation of type (24) was not explicitly specified. Therefore, we shall give the following proposition:

**Proposition 17** Under the assumptions $(H_1)$–$(H_4)$, BSDE (24) has a unique solution $(Y, Z) \in V_T \times \bar{V}_T.$ This solution has the form

(i) $Y(t) = u(t, \eta(t)), \quad Z(t) = v(t, \eta(t)),$

(ii) $v(t, x) = \sigma(t) \frac{\partial}{\partial x} u(t, x).$

where $u, v \in C^{1,3}_{pol}([0, T] \times \mathbb{R})$ with $\frac{\partial u}{\partial t}, \frac{\partial v}{\partial t} \in C^{0,1}_{pol}([0, T] \times \mathbb{R}).$

Before giving the proof, we show the following auxiliary result:
Lemma 18 Assume that \( h \in C^0_{pol}([0, T] \times \mathbb{R}) \) and put \( h_s = h(s, \eta(s)), s \in [0, T] \). Then

\[
\mathbb{E} \left[ \int_t^T h_s ds \bigg | \mathcal{F}_t \right] = \int_t^T \mathbb{E} [h_s | \mathcal{F}_t] ds, \quad P - a.s., \quad t \in [0, T].
\]

Proof Since \( h \in C^0_{pol}([0, T] \times \mathbb{R}) \), we know that \( h_s \in L^2(\Omega, \mathcal{F}, P) \), for all \( s \in [0, T] \). From Theorem 11, there exist \( h_{n,s} \in \mathcal{H}^\otimes n, n \geq 0 \) such that \( h_s = \sum_{n=0}^{\infty} I_n(h_{n,s}) \), with \( I_0(h_{0,s}) = \mathbb{E} h_s \). Recall that the series converges in \( L^2(\Omega, \mathcal{F}, P) \). From the proof of Theorem 3.9.9 [7], we deduce that \( h_{n,s} \) is measurable w.r.t. \( s \), for \( n \geq 0 \). Similarly, for \( G := \int_t^T h_s ds \in L^2(\Omega, \mathcal{F}, P) \), there exist \( G_n \in \mathcal{H}^\otimes n, n \geq 0 \), such that, \( G = \sum_{n=0}^{\infty} I_n(G_n) \) with \( I_0(G_0) = \mathbb{E} G \). Let us show that we can choose \( G_n = \int_t^T h_{n,s} ds, \quad n \geq 0 \). For this, we observe that

\[
\mathbb{E} \int_t^T \sum_{n=0}^N I_n(h_{n,s})^2 ds = \mathbb{E} \int_t^T \sum_{n=0}^N |I_n(h_{n,s})|^2 ds \leq \mathbb{E} \int_t^T \sum_{n=0}^{\infty} |I_n(h_{n,s})|^2 ds
\]

\[
= \mathbb{E} \int_t^T |h_s|^2 ds < \infty,
\]

and hence \( I_n(h_{n,s}) \) is square integrable w.r.t. \( s \).

Now, for arbitrary \( F \in L^2(\Omega, \mathcal{F}, P) \) with the chaos expansion \( F = \sum_{n=0}^{\infty} I_n(l_n) \), we have from Fubini’s Theorem

\[
\mathbb{E} [F \cdot I_n(G_n)] = \mathbb{E} [I_n(l_n) \cdot I_n(G_n)] = \mathbb{E} \left[ I_n(l_n) \cdot \int_t^T h_s ds \right] = \int_t^T \mathbb{E} [I_n(l_n) \cdot h_s] ds
\]

\[
= \int_t^T \mathbb{E} [I_n(l_n) \cdot I_n(h_{n,s})] ds = \int_t^T \mathbb{E} [F \cdot I_n(h_{n,s})] ds
\]

\[
= \mathbb{E} \left[ F \cdot \int_t^T I_n(h_{n,s}) ds \right].
\]

It follows that \( I_n(G_n) = \int_t^T I_n(h_{n,s}) ds, \quad n \geq 0 \), and the stochastic Fubini Theorem (see Theorem 1.13.1 [15]) yields \( I_n(G_n) = \int_t^T I_n(h_{n,s}) ds = I_n \left( \int_t^T h_{n,s} ds \right) \). Thus, we can indeed choose \( G_n = \int_t^T h_{n,s} ds, \quad n \geq 0 \) and \( I_0^{\otimes n} G_n = I_0^{\otimes n} \int_t^T h_{n,s} ds \).
Consequently, we have
\[
\lim_{N \to \infty} \sum_{n=0}^{N} I_n \left( \int_{t}^{T} h_{n,s} ds \right) = \lim_{N \to \infty} \sum_{n=0}^{N} I_n(G_n) = G = \int_{t}^{T} h_s ds, \text{ in } L^2(\Omega, \mathcal{F}, P).
\]

Now, we are going to show that \( \hat{E} \left[ \int_{t}^{T} h_s ds \bigg| \mathcal{F}_t \right] = \int_{t}^{T} \hat{E} \left[ h_s \bigg| \mathcal{F}_t \right] ds \). In fact
\[
\hat{E} \left[ \int_{t}^{T} h_s ds \bigg| \mathcal{F}_t \right] = \hat{E} \left[ \sum_{n=0}^{\infty} I_n(G_n) \bigg| \mathcal{F}_t \right] = \sum_{n=0}^{\infty} I_n(I_{[0,t]}^n G_n)
\]
\[
= \sum_{n=0}^{\infty} I_n(I_{[0,t]}^n \int_{t}^{T} h_{n,s} ds) = \sum_{n=0}^{\infty} I_n \left( \int_{t}^{T} I_{[0,t]}^n h_{n,s} ds \right)
\]
\[
= \sum_{n=0}^{\infty} \int_{t}^{T} I_n(I_{[0,t]}^n h_{n,s}) ds,
\]
where, for the later equality in the above equation, we have used the stochastic Fubini Theorem. Moreover, from (21), we know that for \( s \geq t \), \( \hat{E} \left[ h_s \big| \mathcal{F}_t \right] \) exists and \( \hat{E} \left[ h_s \mathcal{F}_t \right] = \sum_{n=0}^{\infty} I_n(I_{[0,t]}^n h_{n,s}) \). Then from \( h \in C_{pol}^{0,1}([0, T] \times \mathbb{R}) \), Theorem 3.9 of [12] and dominate convergence theorem, we have
\[
\sum_{n=0}^{\infty} \int_{t}^{T} I_n(I_{[0,t]}^n h_{n,s}) ds = \int_{t}^{T} \hat{E} \left[ h_s \big| \mathcal{F}_t \right] ds, \quad P-a.s.
\]
which completes our proof. \( \square \)

After the above auxiliary result, we can now prove our Proposition 17.

**Proof of Proposition 17** Using similar arguments to those of the proof of Lemma 7, we get
\[
g(\eta(T)) + \int_{0}^{T} f(s, \eta(s), \chi(s, \eta(s)), \psi(s, \eta(s))) ds \in L^2(\Omega, \mathcal{F}, P).
\]

Then, we define
\[
M(t) = \hat{E} \left[ g(\eta(T)) + \int_{0}^{T} f(s, \eta(s), \chi(s, \eta(s)), \psi(s, \eta(s))) ds \bigg| \mathcal{F}_t \right], \quad t \in [0, T].
\]
Using (21) and Lemma 18, we obtain

\[
M(t) = P_{||\sigma||^2_2 - ||\sigma||^2_1} g(\eta(t)) + \int_0^T P_{||\sigma||^2_2 - ||\sigma||^2_1} f(u, \eta(t), \chi(u, \eta(t)), \psi(u, \eta(t))) du \\
+ \int_0^t f(u, \eta(u), \chi(u, \eta(u)), \psi(u, \eta(u))) du.
\]

Moreover, \( \hat{E}[M(t)|F_s] = M(s), 0 \leq s \leq t \leq T \) (see Remark 13). Recall the definition of the Malliavin derivative, it follows that if \( F \in \mathbb{D}_{1,2} \) is \( F_t \)-measurable, then \( D^H_s F = 0, ds\text{-a.e. on } [t, T] \). Therefore, for \( s \in [0, t] \)

\[
D^H_s M(t) = \sigma_s P_{||\sigma||^2_2 - ||\sigma||^2_1} \hat{g}(\eta(t)) + \sigma_s \int_0^T P_{||\sigma||^2_2 - ||\sigma||^2_1} \hat{g}(u, \eta(t)) du + \sigma_s \int_s^t \hat{g}(u, \eta(u)) du
\]

where

\[
\hat{g}(u, x) = \frac{\partial}{\partial x} f(u, x, \chi(u, x), \psi(u, x)) + \frac{\partial}{\partial y} f(u, x, \chi(u, x), \psi(u, x)) \chi_x(u, x)
\]

\[
+ \frac{\partial}{\partial z} f(u, x, \chi(u, x), \psi(u, x)) \psi_x(u, x).
\]

Moreover, (21) and the semigroup property of \( P_u \) yields

\[
\hat{E}[D^H_s M(t)|F_s] = \sigma_s P_{||\sigma||^2_2 - ||\sigma||^2_1} P_{||\sigma||^2_2 - ||\sigma||^2_1} \hat{g}'(\eta(s))
\]

\[
+ \sigma_s \int_0^T P_{||\sigma||^2_2 - ||\sigma||^2_1} P_{||\sigma||^2_2 - ||\sigma||^2_1} \hat{g}(u, \eta(s)) du + \sigma_s \int_s^t \hat{g}(u, \eta(s)) du
\]

\[
= \sigma_s P_{||\sigma||^2_2 - ||\sigma||^2_1} \hat{g}'(\eta(s)) + \sigma_s \int_s^T \hat{g}(u, \eta(s)) du. \tag{25}
\]

Now, we are going to prove that \( \mathbb{E}|M(t)|^2 < \infty \). Indeed,

\[
\mathbb{E}|M(t)|^2 \leq 3 \mathbb{E} \left| \hat{E}[g(\eta(T))|F_T] \right|^2 + 3 \mathbb{E} \left| \hat{E} \left[ \int_0^T f(s, \eta(s), \chi(s, \eta(s)), \psi(s, \eta(s))) ds | F_t \right] \right|^2
\]

\[
+ 3 \mathbb{E} \left| \int_0^t f(s, \eta(s), \chi(s, \eta(s)), \psi(s, \eta(s))) ds \right|^2.
\]
and, similarly to Theorem 3.9 [12], we obtain

\[ \mathbb{E} \left| \hat{E}[g(\eta(T))|\mathcal{F}_t] \right|^2 \leq \mathbb{E}|g(\eta(T))|^2 \]

On the other hand, from Lemma 18, we have

\[- \hat{\mathbb{E}} \int_t^T \hat{f}(s, \eta(s), \chi(s, \eta(s)), \psi(s, \eta(s)))ds |\mathcal{F}_t \]

\[ = \mathbb{E} \left( \int_t^T \hat{E} \left[ f(s, \eta(s), \chi(s, \eta(s)), \psi(s, \eta(s))) |\mathcal{F}_t \right] ds \right)^2 \]

\[ \leq T \int_t^T \mathbb{E} \left[ \hat{E} \left[ f(s, \eta(s), \chi(s, \eta(s)), \psi(s, \eta(s))) |\mathcal{F}_t \right] \right]^2 ds \]

Consequently, \( \mathbb{E}|M(t)|^2 < \infty \). Then by using fractional Clark formula (see [10] and [11]), we get

\[ M(t) = \mathbb{E}M(t) + \int_0^t \hat{E}[D_s^HM(t)|\mathcal{F}_s]d\delta B^H(s). \tag{26} \]

From Lemmas 15 and 18, we have \( \mathbb{E}\left\{ \hat{E}[g(\eta(T))|\mathcal{F}_t] \right\} = \mathbb{E}g(\eta(T)) \) and

\[ \mathbb{E} \left\{ \hat{E} \left[ \int_t^T f(s, \eta(s), \chi(s, \eta(s)), \psi(s, \eta(s))) ds |\mathcal{F}_t \right] \right\} \]

\[ = \int_t^T \mathbb{E} \left\{ \hat{E} \left[ f(s, \eta(s), \chi(s, \eta(s)), \psi(s, \eta(s))) |\mathcal{F}_t \right] \right\} ds \]

\[ = \mathbb{E} \int_t^T f(s, \eta(s), \chi(s, \eta(s)), \psi(s, \eta(s))) ds. \]
Consequently, from the definition of $M(t)$ and Remark 4.10 [11], we have

$$M(T) = g(\eta(T)) + \int_0^T f(s, \eta(s), \chi(s, \eta(s)), \psi(s, \eta(s))) ds$$

and

$$\mathbb{E}M(t) = \mathbb{E}\left\{ \hat{E}[g(\eta(T))|\mathcal{F}_T] \right\} + \mathbb{E} \int_0^t f(s, \eta(s), \chi(s, \eta(s)), \psi(s, \eta(s))) ds$$

$$+ \mathbb{E} \left\{ \hat{E} \left[ \int_t^T f(s, \eta(s), \chi(s, \eta(s)), \psi(s, \eta(s))) ds | \mathcal{F}_T \right] \right\}$$

$$= \mathbb{E}g(\eta(T)) + \mathbb{E} \int_0^T f(s, \eta(s), \chi(s, \eta(s)), \psi(s, \eta(s))) ds = \mathbb{E}M(T).$$

(27)

On the other hand, from (25), we obtain

$$\hat{E}[D_s^HM(t)|\mathcal{F}_s] = \hat{E}[D_s^HM(T)|\mathcal{F}_s], \quad P - a.s., \quad s \in [0, t].$$

(28)

We deduce from (26), (27), and (28) that

$$M(t) = \mathbb{E}M(T) + \int_0^t \hat{E}[D_s^HM(T)|\mathcal{F}_s] \delta B^H(s).$$

Let us now introduce the process $Z(s) = \hat{E}[D_s^HM(T)|\mathcal{F}_s], \quad s \in [0, T]$. Similarly as Proposition 4.5 [12], using the property of the operator $P_{\|\sigma\|_T^2 - \|\sigma\|_2^2}$, we can prove that $Z \in \mathcal{V}_T$. Moreover, in virtue of the latter relation, we have

$$M(t) = \mathbb{E}M(T) + \int_0^t Z(s) \delta B^H(s), \quad P - a.s. \quad t \in [0, T].$$

Now, we define

$$Y(t) = M(t) - \int_0^t f(s, \eta(s), \chi(s, \eta(s)), \psi(s, \eta(s))) ds, \quad t \in [0, T].$$
Then,
\[
Y(T) - Y(t) = M(T) - M(t) - \int_t^T f(s, \eta(s), \chi(s, \eta(s)), \psi(s, \eta(s))) \, ds
\]
\[
= \int_t^T Z(s) \, dB^H(s) - \int_t^T f(s, \eta(s), \chi(s, \eta(s)), \psi(s, \eta(s))) \, ds.
\]

Moreover, it’s not hard to check that
\[
Y(T) = M(T) - \int_0^T f(s, \eta(s), \chi(s, \eta(s)), \psi(s, \eta(s))) \, ds = g(\eta(T)) = \xi,
\]
so we have
\[
Y(t) = \xi + \int_t^T f(s, \eta(s), \chi(s, \eta(s)), \psi(s, \eta(s))) \, ds - \int_t^T Z(s) \delta B^H(s).
\]

Moreover, from Remark 4.10 [11] and (21)
\[
Y(t) = \hat{E}[Y(t)|\mathcal{F}_t] = \hat{E}
\left[
\left.
\begin{array}{l}
g(\eta(T)) + \int_t^T f(s, \eta(s), \chi(s, \eta(s)), \psi(s, \eta(s))) \, ds
\end{array}
\right|\mathcal{F}_t
\right]
\]
\[
= P_{\|\sigma\|_T^2 - \|\sigma\|_T^2} g(\eta(t)) + \int_t^T P_{\|\sigma\|_T^2 - \|\sigma\|_T^2} f(s, \eta(t), \chi(s, \eta(t)), \psi(s, \eta(t))) \, ds,
\]
so that it can be easily shown that also \( Y \in \mathcal{V}_T \). Consequently, we have constructed a solution \((Y, Z) \in \mathcal{V}_T \times \mathcal{V}_T\) for BSDE (24). Moreover, \( Y \) is continuous since \( Y \in \mathcal{V}_T \).

Finally, using that \( Y, Z \in \mathcal{V}_T \), we can find \( u, v \in C^{1,3}_{pol}([0, T] \times \mathbb{R}) \) with \( \frac{\partial u}{\partial t}, \frac{\partial v}{\partial t} \in C^{0,1}_{pol}([0, T] \times \mathbb{R}) \) such that \( Y(t) = u(t, \eta(t)), Z(t) = v(t, \eta(t)), t \in [0, T] \). Then \( v(t, x) = \sigma(t) \frac{\partial}{\partial x} u(t, x) \). Indeed, by applying (6), we have
\[
du(t, \eta(t)) = \frac{\partial}{\partial t} u(t, \eta(t)) \, dt + \sigma(t) \frac{\partial}{\partial x} u(t, \eta(t)) \, dB^H(t) + \frac{1}{2} \tilde{\sigma}(t) \frac{\partial^2}{\partial x^2} u(t, \eta(t)) \, dt
\]
\[
= \left[ \frac{\partial}{\partial t} u(t, \eta(t)) + \frac{1}{2} \tilde{\sigma}(t) \frac{\partial^2}{\partial x^2} u(t, \eta(t)) \right] \, dt
\]
\[
+ \sigma(t) \frac{\partial}{\partial x} u(t, \eta(t)) \, dB^H(t),
\]
where $\tilde{\sigma}(t) := \frac{d}{dt}(\|\sigma\|_t^2)$. Consequently,

$$u(t, \eta(t)) = \xi - \int_t^T \left[ \frac{\partial}{\partial s} u(s, \eta(s)) + \frac{1}{2} \tilde{\sigma}(s) \frac{\partial^2}{\partial x^2} u(s, \eta(s)) \right] ds - \int_t^T \sigma(s) \frac{\partial}{\partial x} u(s, \eta(s)) \delta B^H(s).$$

From (24), it can be concluded that

$$\int_t^T \left[ \frac{\partial}{\partial s} u(s, \eta(s)) + \frac{1}{2} \tilde{\sigma}(s) \frac{\partial^2}{\partial x^2} u(s, \eta(s)) \right] ds + \int_t^T \sigma(s) \frac{\partial}{\partial x} u(s, \eta(s)) \delta B^H(s) = - \int_t^T f(s, \eta(s), \chi(s, \eta(s)), \psi(s, \eta(s))) ds + \int_t^T v(s, \eta(s)) \delta B^H(s).$$

Using Lemma 10, we deduce that

$$v(t, x) = \sigma(t) \frac{\partial}{\partial x} u(t, x), \text{ for all } t \in [0, T], \ x \in \mathbb{R}. \quad (29)$$

It remains to prove that the above solution is the unique one in $\mathcal{V}_T \times \mathcal{V}_T$ for BSDE (24). Indeed, we suppose that there is another solution $(\tilde{Y}, \tilde{Z}) \in \mathcal{V}_T \times \mathcal{V}_T$. Then, by applying Theorem 8, using (29) (which yields $D^H_t Y(t) = \frac{\hat{\sigma}(t)}{\sigma(t)} Z(t)$) and taking expectation, we have

$$\mathbb{E}|Y(t) - \tilde{Y}(t)|^2 + \frac{2}{M} \int_t^T s^{2H-1} \mathbb{E}|Z(s) - \tilde{Z}(s)|^2 ds$$

$$\leq \mathbb{E}|Y(t) - \tilde{Y}(t)|^2 + 2 \int_t^T \frac{\hat{\sigma}(s)}{\sigma(s)} \mathbb{E}|Z(s) - \tilde{Z}(s)|^2 ds$$

$$= 2\mathbb{E} \int_t^T \left( Y(t) - \tilde{Y}(t) \right) \left( Z(t) - \tilde{Z}(t) \right) \delta B^H(s) = 0, \text{ for all } t \in [0, T],$$

where the latter equality follows the fact that $(Y - \tilde{Y})(Z - \tilde{Z}) \in \mathcal{V}_T$. Therefore, taking into account the continuity of $Y - \tilde{Y}$, the uniqueness follows. \hfill \Box

**Proposition 19** Let the assumptions $(H_1)$–$(H_4)$ be satisfied. For $(U, V) \in \mathcal{V}_T \times \mathcal{V}_T$, let $(Y, Z) \in \mathcal{V}_T \times \mathcal{V}_T$ be the unique solution of the following BSDE
\[ Y(t) = \int_0^T f(s, \eta(s), U(s), V(s)) \, ds - \int_0^T Z(s) \delta B^H(s), \quad t \in [0, T]. \]

Then, for all \( \beta > 0 \), there exists \( C(\beta) \in \mathbb{R} \) (depending also on \( L \) and \( T \)) such that

\[
\sup_{t \in [0,T]} e^{2\beta t} \mathbb{E} |Y(t)|^2 + \int_0^T e^{2\beta s} \mathbb{E} |Y(s)|^2 \, ds + \int_0^T s^{2H-1} e^{2\beta s} \mathbb{E} |Z(s)|^2 \, ds \\
\leq C(\beta) \left( \int_0^T e^{2\beta s} \mathbb{E} |U(s)|^2 \, ds + \int_0^T s^{2H-1} e^{2\beta s} \mathbb{E} |V(s)|^2 \, ds \\
+ \int_0^T e^{2\beta s} |f(s, \eta(s), 0, 0)|^2 \, ds \right). \tag{30}
\]

Moreover, \( C(\beta) \) can be chosen such that \( \lim_{\beta \to \infty} C(\beta) = 0 \).

**Proof** From Proposition 17, it is not hard to check that \( D^H_t Y(t) = \hat{\sigma}(t) Z(t) \). Then, from Theorem 8, we deduce that for \( t \in [0, T] \),

\[
\mathbb{E} |Y(t)|^2 + \frac{2}{M} \int_0^T s^{2H-1} \mathbb{E} |Z(s)|^2 \, ds \\
\leq 2 \int_0^T \mathbb{E} \left[ Y(s) f(s, \eta(s), U(s), V(s)) \right] \, ds \\
\leq 2 \int_0^T \mathbb{E} \left[ |Y(s)| (L|U(s)| + L|V(s)| + |f(s, \eta(s), 0, 0)|) \right] \, ds \\
\leq 2 \int_0^T \left[ \mathbb{E} |Y(s)|^2 \right]^{1/2} \left[ \mathbb{E} (L|U(s)| + L|V(s)| + |f(s, \eta(s), 0, 0)|)^2 \right]^{1/2} \, ds.
\]

Let \( x(t) = \mathbb{E} |Y(t)|^2 \), \( t \in [0, T] \). Then

\[
x^2(t) \leq 2 \sqrt{3} \int_0^T x(s) \left[ \mathbb{E} (L^2|U(s)|^2 + L^2|V(s)|^2 + |f(s, \eta(s), 0, 0)|^2) \right]^{1/2} \, ds,
\]

\( t \in [0, T] \). \( \Box \)
To estimate \( x(t) \), we will apply the following inequality:

**Lemma 20** Let \( a, \alpha, \beta : [0, T] \rightarrow \mathbb{R}_+ \) be three nonnegative Borel functions such that \( a \) is decreasing and \( \alpha, \beta \in L^1_{loc}([0, \infty)) \). If \( x : [0, T] \rightarrow \mathbb{R}_+ \) is a continuous function such that

\[
x^2(t) \leq a(t) + 2 \int_0^T \alpha(s) x(s) \, ds + 2 \int_0^T \beta(s) x^2(s) \, ds, \quad t \in [0, T],
\]

then

\[
x(t) \leq \sqrt{a(t)} \exp \left( \int_0^T \beta(s) \, ds \right) + \int_0^T \alpha(s) \exp \left( \int_t^s \beta(r) \, dr \right) \, ds, \quad t \in [0, T].
\]

**Remark 21** For this lemma the reader is referred to Corollary 6.61 [20].

Now from (31) and the above lemma, by setting

\[
a(t) = 0, \quad \beta(s) = 0, \\
\alpha(s) = \sqrt{3} \left[ \mathbb{E} \left( L^2 |U(s)|^2 + L^2 |V(s)|^2 + |f(s, \eta(s), 0, 0)|^2 \right) \right]^{1/2}, \quad s \in [0, T],
\]

we have

\[
x(t) \leq \sqrt{3} \int_0^T \left[ \mathbb{E} \left( L^2 |U(s)|^2 + L^2 |V(s)|^2 + |f(s, \eta(s), 0, 0)|^2 \right) \right]^{1/2} \, ds, \quad t \in [0, T],
\]

and, hence, for any \( \beta > 0 \),

\[
\left[ \mathbb{E} |Y(t)|^2 \right]^{1/2} \leq \sqrt{3} \int_0^T \left( L \left[ \mathbb{E} |U(s)|^2 \right]^{1/2} + L \left[ \mathbb{E} |V(s)|^2 \right]^{1/2} \right) \, ds
\]

\[
\leq \sqrt{3} L \int_0^T \left( e^{-\beta s} \left[ e^{2\beta s} \mathbb{E} |U(s)|^2 \right]^{1/2} + e^{-\beta s} \left[ e^{2\beta s} \mathbb{E} |V(s)|^2 \right]^{1/2} \right) \, ds
\]

\[
+ \sqrt{3} \int_0^T e^{-\beta s} \left[ e^{2\beta s} \mathbb{E} |f(s, \eta(s), 0, 0)|^2 \right]^{1/2} \, ds
\]

\[
\leq \sqrt{3} L \left( \int_0^T e^{-2\beta s} \, ds \right)^{1/2} \left( \int_0^T e^{2\beta s} \mathbb{E} |U(s)|^2 \, ds \right)^{1/2}
\]
\[ + \sqrt{3} L \left( \int_t^T \frac{e^{-2\beta s}}{s^{2H-1}} \, ds \right)^{1/2} \left( \int_t^T s^{2H-1} e^{2\beta s} \mathbb{E}[V(s)]^2 \, ds \right)^{1/2} \]

\[ + \sqrt{3} \left( \int_t^T e^{-2\beta s} \, ds \right)^{1/2} \left( \int_t^T e^{2\beta s} \mathbb{E}[f(s, \eta(s), 0, 0)]^2 \, ds \right)^{1/2}. \]  

(32)

Let us use the following notations:

\[ A_t := \left( \int_t^T e^{2\beta s} \mathbb{E}[U(s)]^2 \right)^{1/2}, \quad B_t := \left( \int_t^T s^{2H-1} e^{2\beta s} \mathbb{E}[V(s)]^2 \, ds \right)^{1/2} \]

and \[ C_t = \left( \int_t^T e^{2\beta s} \mathbb{E}[f(s, \eta(s), 0, 0)]^2 \right)^{1/2}, \quad t \in [0, T]. \]

Since \[ \int_t^T e^{-2\beta s} \, ds = \frac{1}{2\beta} (e^{-2\beta t} - e^{-2\beta T}) \]
we have for \( \alpha > 0 \) with \( 0 < \alpha < 2 - 2H < 1 \) and \( \beta > 0 \),

\[ e^{2\beta t} \int_t^T \frac{e^{-2\beta s}}{s^{2H-1}} \, ds \leq \int_t^T \frac{(2\beta(s-t))^{-\alpha}}{s^{2H-1}} \, ds \leq \frac{1}{(2\beta)^\alpha} \int_0^T \frac{1}{s^{\alpha+2H-1}} \, ds < \infty. \]

This allows to conclude from (32) that

\[ e^{2\beta t} \mathbb{E}[|Y(t)|^2] \leq \frac{9L^2}{2\beta} A_t^2 + \frac{9L^2}{(2\beta)^\alpha} \int_t^T \frac{(s-t)^{-\alpha}}{s^{2H-1}} \, ds B_t^2 + \frac{9}{2\beta} C_t^2. \]

(33)

Consequently, there exists \( C(\beta) \) with \( \lim_{\beta \to \infty} C(\beta) = 0 \), s.t.

\[ e^{2\beta t} \mathbb{E}[|Y(t)|^2] \, dt \leq C(\beta) \left( A_t^2 + B_t^2 + C_t^2 \right), \quad t \in [0, T]. \]

(34)

Applying the Itô formula to \( |Y(t)|^2 \), taking the expectation \( \mathbb{E}[|Y(t)|^2] \) and then determining the function \( d \left( e^{2\beta t} \mathbb{E}[|Y(t)|^2] \right) \) and using \( \mathbb{D}^H Y(t) = \frac{\hat{\sigma}(t)}{\sigma(t)} Z(t) \), the Lipschitz property of \( f \) as well as (33) we obtain (Recall (9) for the definition of \( M \))

\[ e^{2\beta t} \mathbb{E}[|Y(t)|^2] + 2\beta \int_t^T e^{2\beta s} \mathbb{E}|Y(s)|^2 \, ds + \frac{2}{M} \int_t^T s^{2H-1} e^{2\beta s} \mathbb{E}|Z(s)|^2 \, ds \]
\[
\leq 2 \int_t^T e^{2\beta_s} \mathbb{E} \left[ |Y(s)| (L|U(s)| + L|V(s)| + |f(s, \eta(s), 0, 0)|) \right] ds
\]
\[
\leq 2L \int_t^T \left[ \mathbb{E} \left( e^{2\beta_s} |Y(s)|^2 \right) \right]^{1/2} \left[ \mathbb{E} \left( e^{2\beta_s} |U(s)|^2 \right) \right]^{1/2} ds
\]
\[
+ 2L \int_t^T \left[ \mathbb{E} \left( e^{2\beta_s} |Y(s)|^2 \right) \right]^{1/2} \left[ \mathbb{E} \left( e^{2\beta_s} s^{2H-1}|V(s)|^2 \right) \right]^{1/2} ds
\]
\[
+ 2\int_t^T \left[ \mathbb{E} \left( e^{2\beta_s} |Y(s)|^2 \right) \right]^{1/2} \left[ \mathbb{E} \left( e^{2\beta_s} |f(s, \eta(s), 0, 0)|^2 \right) \right]^{1/2} ds
\]
\[
\leq 2L \int_t^T \left[ C(\beta) \left(A_s^2 + B_s^2 + C_s^2\right) \right]^{1/2} \left[ \mathbb{E} \left( e^{2\beta_s} |U(s)|^2 \right) \right]^{1/2} ds
\]
\[
+ 2L \int_t^T \left[ \frac{1}{s^{2H-1}} C(\beta) \left(A_s^2 + B_s^2 + C_s^2\right) \right]^{1/2} \left[ \mathbb{E} \left( e^{2\beta_s} s^{2H-1}|V(s)|^2 \right) \right]^{1/2} ds
\]
\[
+ 2\int_t^T \left[ C(\beta) \left(A_s^2 + B_s^2 + C_s^2\right) \right]^{1/2} \left[ \mathbb{E} \left( e^{2\beta_s} |f(s, \eta(s), 0, 0)|^2 \right) \right]^{1/2} ds
\]
\[
\leq 2L\sqrt{C(\beta)} (A_t + B_t + C_t) \left( \sqrt{T-t} A_t + \sqrt{\frac{T^2-2H-t^2-2H}{2-2H}} B_t + \sqrt{T-t} C_t \right).
\]

Thus, the above inequality and (34) allow to conclude inequality (30). \(\square\)

**Theorem 22** Let the assumptions \((H_1)-(H_4)\) be satisfied. Then the BSDE

\[
Y(t) = \xi + \int_t^T f(s, \eta(s), Y(s), Z(s)) ds - \int_t^T Z(s)\delta B^H(s), \quad t \in [0, T]
\]

has a solution \((Y, Z) \in \hat{V}_{\mathcal{T}}^{1/2} \times \hat{V}_{\mathcal{T}}^H\).

**Remark 23** Let us mention that it is not clear here, if the solution \(Y\) has continuous paths or not. Indeed, since \(Z\) does not necessarily belong to \(\mathbb{L}_{1,2}^H\), the divergence integral \(\int_t^T Z(s)\delta B^H(s)\) can eventually be discontinuous in \(t\).

**Proof** The existence of the solution is obtained by the Banach fixed point theorem. Let us consider the mapping \(\Gamma : \mathcal{V}_T \times \mathcal{V}_T \rightarrow \mathcal{V}_T \times \mathcal{V}_T\) given by \((U, V) \rightarrow \Gamma (U, V) = \)
First, we remark that $\Gamma$ is well defined (see Proposition 17).

Let us show that $\Gamma$ is a contraction w.r.t. the norm $\| (u, v) \|_{1/2, H} := \|u\|_{1/2} + \|v\|_H$, for $(u, v) \in \tilde{V}_{T}^{1/2} \times \tilde{V}_{T}^{H}$ (for the definition of $\| \cdot \|_a$, see (10)).

For $(U, V), (U', V') \in \tilde{V}_{T} \times \tilde{V}_{T}$ and $(Y, Z) = \Gamma(U, V), (Y', Z') = \Gamma(U', V')$, we set $\Delta Y = Y - Y', \Delta Z = Z - Z', \Delta U = U - U'$ and $\Delta V = V - V'$. Using Proposition 19, we know that there exists $C(\beta)$ which can depend on $L$ and $T$, such that $\lim_{\beta \to \infty} C(\beta) = 0$, and

$$
\sup_{t \in [0, T]} e^{2\beta t} \mathbb{E} |\Delta Y(t)|^2 + \int_0^T e^{2\beta s} \mathbb{E} |\Delta Y(s)|^2 \, ds + \int_0^T s^{2H-1} e^{2\beta s} \mathbb{E} |\Delta Z(s)|^2 \, ds
\leq C(\beta) \left( \int_0^T e^{2\beta s} \mathbb{E} |\Delta U(s)|^2 \, ds + \int_0^T s^{2H-1} e^{2\beta s} \mathbb{E} |\Delta V(s)|^2 \, ds \right),
$$

(35)

Taking $\beta$ large enough such that $C(\beta) \leq 1/2$, then $\Gamma$ becomes a strict contraction on $\tilde{V}_{T} \times \tilde{V}_{T}$ w.r.t. the norm $\| (\cdot, \cdot) \|_{1/2, H}$.

Now, we define $\{(Y_k, Z_k)\}_{k \in \mathbb{N}}$ recursively by putting $Y_0 = \chi(t, \eta(t))$, $Z_0 = \psi(t, \eta(t))$ for $\chi, \psi \in C_{1,3}^1([0, T] \times \mathbb{R})$ with $\frac{\partial \chi}{\partial t}, \frac{\partial \psi}{\partial t} \in C_{pol}^0([0, T] \times \mathbb{R})$, and by defining $(Y_{k+1}, Z_{k+1}) \in \tilde{V}_{T} \times \tilde{V}_{T}$ through the BSDE

$$
Y_{k+1}(t) = \xi + \int_t^T f(s, \eta(s), Y_k(s), Z_k(s)) \, ds - \int_t^T Z_{k+1}(s) dB^H(s), \quad t \in [0, T],
$$

(36)

$k \geq 0$. From Proposition 17, we know that for all $k \geq 0$, $Y_k(t) = u_k(t, \eta(t)), Z_k(t) = v_k(t, \eta(t))$, for suitable $u_k, v_k \in C_{pol}^1([0, T] \times \mathbb{R})$ with $\frac{\partial u_k}{\partial t}, \frac{\partial v_k}{\partial t} \in C_{pol}^0([0, T] \times \mathbb{R})$ such that $Z_k(t) = \sigma(t) \frac{\partial}{\partial x} u_k(t, \eta(t)), t \in [0, T]$.

Since $\Gamma$ is contraction, which means that $\{(Y_k, Z_k)\}_{k \in \mathbb{N}}$ is a Cauchy sequence in $\tilde{V}_{T}^{1/2} \times \tilde{V}_{T}^{H}$, then there exists $(Y, Z) \in \tilde{V}_{T}^{1/2} \times \tilde{V}_{T}^{H}$ such that $(Y_k, Z_k)$ converges to $(Y, Z)$ in $\tilde{V}_{T}^{1/2} \times \tilde{V}_{T}^{H}$. Moreover, we can prove that $(Y, Z)$ satisfies

$$
Y(t) = \xi + \int_t^T f(s, \eta(s), Y(s), Z(s)) \, ds - \int_t^T Z(s) dB^H(s), \quad t \in (0, T].
$$
Indeed, from (35)

\[
\lim_{k \to \infty} \mathbb{E}|Y_k(t) - Y(t)|^2 = 0, \quad t \in [0, T], \quad \text{and}
\]

\[
\lim_{k \to \infty} \mathbb{E} \int_0^T |Y_k(s) - Y(s)|^2 \, ds + \mathbb{E} \int_0^T s^{2H-1}|Z_k(s) - Z(s)|^2 \, ds = 0.
\]

(37)

Then, for arbitrary \(\rho > 0\) and for all \(t \in [\rho, T]\),

\[
\lim_{k \to \infty} \left\{ -Y_{k+1}(t) + \xi + \int_t^T f(s, \eta(s), Y_k(s), Z_k(s)) \, ds \right\}
\]

\[
= -Y(t) + \xi + \int_t^T f(s, \eta(s), Y(s), Z(s)) \, ds := \theta(t), \quad \text{in } L^2(\Omega, \mathcal{F}, P),
\]

(38)

and \(Z_k1_{[t, T]} \to Z1_{[t, T]}\) in \(L^2(\Omega, \mathcal{F}, P; \mathcal{H})\). Therefore, using Definition 1, (36) and (38), we see that for all \(F \in \mathcal{P}_T\),

\[
\mathbb{E} \left( \left\langle D^H F, Z(\cdot)1_{[t, T]}(\cdot) \right\rangle_T \right) = \lim_{k \to \infty} \mathbb{E} \left( \left\langle D^H F, Z_{k+1}(\cdot)1_{[t, T]}(\cdot) \right\rangle_T \right)
\]

\[
= \lim_{k \to \infty} \mathbb{E} \left( F \int_t^T Z_{k+1}(s)\delta B^H(s) \right) = \mathbb{E}(F \theta(t)).
\]

From the definition of the divergence operator \(\delta\), it follows that \(Z1_{[t, T]} \in Dom(\delta)\) and \(\delta(Z1_{[t, T]}) = \theta(t)\). Consequently, we have

\[
Y(t) = \xi + \int_t^T f(s, \eta(s), Y(s), Z(s)) \, ds - \int_t^T Z(s)\delta B^H(s), \quad a.s., \quad \text{for all } t \in [\rho, T].
\]

Considering that \(\rho\) is arbitrary, we complete our proof. \(\square\)

**Proposition 24** Let \((Y, Z) \in \mathcal{V}_T \times \mathcal{V}_T\) be the solution of BSDE (23) constructed in the proof of Theorem 22. Then for almost \(t \in (0, T]\),

\[
D^H_0 Y(t) = \frac{\dot{\sigma}(t)}{\sigma(t)} Z(t).
\]

**Proof** From (36) we know that \((Y_k, Z_k) \in \mathcal{V}_T \times \mathcal{V}_T\) satisfies

\[
Y_{k+1}(t) = \xi + \int_t^T f(s, \eta(s), Y_k(s), Z_k(s)) \, ds - \int_t^T Z_{k+1}(s)\delta B^H(s), \quad t \in [0, T], \quad k \geq 1.
\]
We recall that
\[ Y_k(t) = u_k(t, \eta(t)), \quad Z_k(t) = v_k(t, \eta(t)), \quad t \in [0, T] \]
and \[ Z_k(t) = \sigma(t) \frac{\partial}{\partial x} u_k(t, \eta(t)). \] Since \((Y_k, Z_k) \to (Y, Z)\) in \(\tilde{V}^{1/2}_T \times \tilde{V}^H_T\), there exists a subsequence, by convenience still denoted by \(\{(Y_k, Z_k)\}_{k \in \mathbb{N}}\), such that for arbitrary \(\rho > 0\), we have that
\[
\lim_{k \to \infty} \mathbb{E}|Y_k(s) - Y(s)|^2 = 0 \quad \text{and} \quad \lim_{k \to \infty} \mathbb{E}|Z_k(s) - Z(s)|^2 = 0, \quad \text{for almost all } s \in [\rho, T].
\]
As a process with the parameter \(r\),
\[
Dr Y_k(t) = \frac{\partial}{\partial x} u(t, \eta(t)) \sigma(r) 1_{[0,t]}(r)
\]
\[
= \frac{\sigma(r)}{\sigma(t)} Z_k(t) 1_{[0,t]}(r) \xrightarrow{L^2([0,T] \times \Omega)} \frac{\sigma(t)}{\sigma(t)} Z(t) 1_{[0,t]}(r),
\]
as \(k \to \infty\), for almost all \(t \in [\rho, T]\).

On the other hand, since \(L^2([0, T]) \subset \mathcal{H}\), we conclude that the convergence also holds in \(L^2(\Omega, \mathcal{F}, P; \mathcal{H})\). Consequently, in \(L^2(\Omega, \mathcal{F}, P; \mathcal{H})\)
\[
Dr Y(t) = \lim_{k \to \infty} Dr Y_k(t) = \lim_{k \to \infty} \frac{\sigma(r)}{\sigma(t)} Z_k(t) 1_{[0,t]}(r) = \frac{\sigma(t)}{\sigma(t)} Z(t) 1_{[0,t]}(r), \quad \text{a.e. } t \in [\rho, T],
\]
and, thus,
\[
\mathbb{D}^H_t Y(t) = \int_0^T \phi(t - r) Dr Y(t) dr = \frac{\hat{\sigma}(t)}{\sigma(t)} Z(t), \quad \text{a.e. } t \in [\rho, T],
\]
where \(\hat{\sigma}(t)\) is defined by (8). Considering that \(\rho > 0\) is arbitrary, we have
\[
\mathbb{D}^H_t Y(t) = \frac{\hat{\sigma}(t)}{\sigma(t)} Z(t), \quad \text{a.e. } t \in (0, T],
\]
which completes the proof. \(\square\)

4.2 Uniqueness

Before giving our uniqueness result, we introduce the following spaces:
\[
\mathcal{M} = \left\{ X \big| X(t) = X(0) - \int_0^t u_s ds - \int_0^t u_s \delta B^H(s), \quad t \in [0, T] \right\}
\]
with \(u \in \mathcal{V}_T, \quad v_s = v(s, \eta(s)), \quad \text{where } v \in C^{0,1}_{pol}([0, T] \times \mathbb{R})\)
and \(S_f\), the set of \((Y, Z) \in \tilde{V}^{1/2}_T \times \tilde{V}^H_T\) such that, for \(t \in (0, T]\),
Theorem 25 Let the assumptions \((H_1)-(H_4)\) be satisfied. Then BSDE

\[
Y(t) = \xi + \int_t^T f(s, \eta(s), Y(s), Z(s)) \, ds - \int_t^T Z(s) \delta B^H(s)
\]

has a unique solution \((Y, Z) \in S_f\).

Proof We show first that the solution \((Y, Z)\) we constructed in the proof of Theorem 22 belongs to \(S_f\). Indeed, the sequence \(\{(Y_k, Z_k)\}_{k \in \mathbb{N}}\) introduced in the proof of Theorem 22 is in \(\mathcal{V}_T \times \mathcal{V}_T\) and converges to \((Y, Z)\) in \(\tilde{\mathcal{V}}_T^{1/2} \times \tilde{\mathcal{V}}_T^H\). Applying the Itô formula to \(Y_{k+1}^2\) (see Theorem 8, using \(\mathbb{D}_t^H Y(t) = \frac{\hat{\sigma}(t)}{\sigma(t)} Z(t)\)) and taking the expectation, we have

\[
\mathbb{E}|Y_{k+1}(t)|^2 + 2\mathbb{E} \int_t^T \frac{\hat{\sigma}(s)}{\sigma(s)} |Z_{k+1}(s)|^2 \, ds = \mathbb{E}|\xi|^2
\]

\[
+ 2\mathbb{E} \int_t^T Y_{k+1}(s) f(s, \eta(s), Y_k(s), Z_k(s)) \, ds. \quad (39)
\]

Moreover, from (37), we know

\[
\lim_{k \to \infty} \mathbb{E}|Y_k(t) - Y(t)|^2 = 0, \ t \in [0, T], \ and
\]

\[
\lim_{k \to \infty} \mathbb{E} \int_t^T |Y_k(s) - Y(s)|^2 \, ds + \mathbb{E} \int_t^T s^{2H-1} |Z_k(s) - Z(s)|^2 \, ds = 0, \ t \in [0, T].
\]

Letting \(k \to \infty\) in (39), it follows that for arbitrary \(\rho > 0\),

\[
\mathbb{E}|Y(t)|^2 + 2\mathbb{E} \int_t^T \frac{\hat{\sigma}(s)}{\sigma(s)} |Z(s)|^2 \, ds = \mathbb{E}|\xi|^2
\]

\[
+ 2\mathbb{E} \int_t^T Y(s) f(s, \eta(s), Y(s), Z(s)) \, ds, \ t \in [\rho, T]. \quad (40)
\]
On the other hand, for any $X \in \mathcal{M}$, we deduce from Theorem 8,

$$
\mathbb{E}\left[ Y_{k+1}(t)X(t) \right] + \mathbb{E}\int_t^T \left[ u_s \mathbb{D}_s^H Y_{k+1}(s) + Z_{k+1}(s) \mathbb{D}_s^H X(s) \right] ds
$$

$$
= \mathbb{E}\left[ \xi X(T) \right] + \mathbb{E}\int_t^T \left[ Y_{k+1}(s)v_s + X(s) f(s, \eta(s), Y_k(s), Z_k(s)) \right] ds.
$$

Letting $k \to \infty$ in the above equation and recalling that $\mathbb{D}_s^H Y_{k+1}(s) = \frac{\hat{\sigma}(s)}{\sigma(s)} Z_{k+1}(s)$, we obtain for arbitrary $\rho > 0$,

$$
\mathbb{E}\left[ Y(t)X(t) \right] + \mathbb{E}\int_t^T \left[ \frac{\hat{\sigma}(s)}{\sigma(s)} u_s + \mathbb{D}_s^H X(s) \right] Z(s) ds
$$

$$
= \mathbb{E}\left[ \xi X(T) \right] + \mathbb{E}\int_t^T \left[ Y(s)v_s + X(s) f(s, \eta(s), Y_k(s), Z_k(s)) \right] ds, \quad t \in [\rho, T].
$$

(41)

Consequently, (40) and (41) yield that $(Y, Z) \in \mathcal{S}_f$.

Now, it remains to show the uniqueness in the class $\mathcal{S}_f$. We suppose that $(\tilde{Y}, \tilde{Z}) \in \mathcal{S}_f$ is another solution of BSDE (23). Then, for arbitrary $\rho > 0$,

$$
\mathbb{E} |\tilde{Y}(t)|^2 + 2\mathbb{E}\int_t^T \frac{\hat{\sigma}(s)}{\sigma(s)} |\tilde{Z}(s)|^2 ds = \mathbb{E}|\xi|^2
$$

$$
+ 2\mathbb{E}\int_t^T \tilde{Y}(s) f(s, \eta(s), Y_k(s), Z_k(s)) ds, \quad t \in [\rho, T].
$$

(42)

and

$$
\mathbb{E}\left[ \tilde{Y}(t) Y_{k+1}(t) \right] + \mathbb{E}\int_t^T \frac{2 \hat{\sigma}(s)}{\sigma(s)} Z_{k+1}(s) \tilde{Z}(s) ds
$$

$$
= \mathbb{E}|\xi|^2 + \mathbb{E}\int_t^T \left[ \tilde{Y}(s) f(s, \eta(s), Y_k(s), Z_k(s))
$$

$$
+ Y_{k+1}(s) f(s, \eta(s), \tilde{Y}(s), \tilde{Z}(s)) \right] ds, \quad t \in [\rho, T].
$$
and letting $k \to \infty$, we have
\[
\mathbb{E} \left[ \tilde{Y}(t)Y(t) \right] + \mathbb{E} \int_t^T \frac{\hat{\sigma}(s)}{\sigma(s)} Z(s) \tilde{Z}(s) \, ds = \mathbb{E}|\xi|^2 + \mathbb{E} \int_t^T \left[ \tilde{Y}(s)f(s, \eta(s), Y(s), Z(s)) + Y(s)f(s, \eta(s), \tilde{Y}(s), \tilde{Z}(s)) \right] \, ds, \quad t \in [\rho, T].
\]

Thus, from (40), (42), and (43) as well as $\mathbb{E}|Y(t) - \tilde{Y}(t)|^2 = \mathbb{E}|Y(t)|^2 - 2\mathbb{E} \left[ Y(t)\tilde{Y}(t) \right] + \mathbb{E}|\tilde{Y}(t)|^2$ we have, for $t \in [\rho, T]$,
\[
\begin{align*}
\mathbb{E}|Y(t) - \tilde{Y}(t)|^2 &+ 2\mathbb{E} \int_t^T \frac{\hat{\sigma}(s)}{\sigma(s)} |Z(s) - \tilde{Z}(s)|^2 \, ds \\
&= \mathbb{E} \int_t^T \left[ Y(s) - \tilde{Y}(s) \right] \left[ f(s, \eta(s), Y(s), Z(s)) - f(s, \eta(s), \tilde{Y}(s), \tilde{Z}(s)) \right] \, ds \\
&\leq \mathbb{E} \int_t^T \left( L|Y(s) - \tilde{Y}(s)|^2 + L^2 Ms^{1-2H} |Y(s) - \tilde{Y}(s)|^2 + \frac{1}{M} s^{2H-1} |Z(s) - \tilde{Z}(s)|^2 \right) \, ds.
\end{align*}
\]

where $M$ is the constant introduced in Remark 6. Then, using Remark 6
\[
\begin{align*}
\mathbb{E}|Y(t) - \tilde{Y}(t)|^2 &+ \frac{1}{M} \mathbb{E} \int_t^T s^{2H-1} |Z(s) - \tilde{Z}(s)|^2 \, ds \\
&\leq \mathbb{E} \int_t^T (L + L^2 Ms^{1-2H}) |Y(s) - \tilde{Y}(s)|^2 \, ds
\end{align*}
\]
and Gronwall's inequality yields that
\[
\mathbb{E}|Y(t) - \tilde{Y}(t)|^2 + \frac{1}{M} \mathbb{E} \int_t^T s^{2H-1} |Z(s) - \tilde{Z}(s)|^2 \, ds = 0, \quad t \in [\rho, T].
\]

Since $\rho > 0$ is arbitrary, our proof is complete now. □

5 Fractional Backward Stochastic Variational Inequality

Let us now consider the following BSVI driven by a fBm:
\[
\begin{align*}
\begin{cases}
-dY(t) + \partial \varphi(Y(t)) \, dt \ni f(t, \eta(t), Y(t), Z(t)) \, dt - Z(t) \delta B^H(t), \quad t \in [0, T] \\
Y(T) = \xi.
\end{cases}
\end{align*}
\]

Springer
where the coefficients satisfy \((H_1)-(H_4)\), and \(\partial \varphi\) is the subdifferential of the function \(\varphi : \mathbb{R} \to (-\infty, +\infty]\) satisfying

\((H_5)\) \(\varphi\) is a convex lower semicontinuous (l.s.c.) function with \(\varphi(x) \geq \varphi(0) = 0\), for all \(x \in \mathbb{R}\) and \(\mathbb{E}|\varphi(\xi)| < \infty\) (Recall that \(\xi = g(\eta(T))\)).

Let us introduce the following notations:

\[
\begin{align*}
\text{Dom} \varphi &= \{ u \in \mathbb{R} : \varphi(u) < \infty \}, \\
\partial \varphi(u) &= \{ u^* \in \mathbb{R} : u^*(v - u) + \varphi(u) \leq \varphi(v), \text{ for all } v \in \mathbb{R} \}, \\
\text{Dom} (\partial \varphi) &= \{ u \in \mathbb{R} : \partial \varphi(u) \neq \emptyset \}, \\
(u, u^*) \in \partial \varphi &\iff u \in \text{Dom} (\partial \varphi), u^* \in \partial \varphi(u).
\end{align*}
\]

We know that the multivalued subdifferential operator \(\partial \varphi\) is a monotone operator, i.e.,

\[
(u^* - v^*)(u - v) \geq 0, \quad \text{for all } (u, u^*), (v, v^*) \in \partial \varphi.
\]

Now, we give the definition of the solution for BSVI \((44)\).

**Definition 26** A triple \((Y, Z, U)\) is a solution for BSVI \((44)\), if:

\[
\begin{align*}
(a_1) \quad & Y, U \in \mathcal{V}^H_T \text{ and } Z \in \mathcal{V}_T^{2H - 1/2}, \\
(a_2) \quad & (Y(t), U(t)) \in \partial \varphi, \text{ d}P \otimes dt \text{ a.e. on } \Omega \times [0, T], \\
(a_3) \quad & Y(t) + \int_t^T U(s) ds = \xi + \int_t^T f(s, \eta(s), Y(s), Z(s)) ds - \int_t^T Z(s) \delta B^H(s), \text{ a.s., } t \in (0, T].
\end{align*}
\]

In this section, our objective is to show the following existence result:

**Theorem 27** Let the assumptions \((H_1)-(H_5)\) be satisfied. There exists a solution of BSVI \((44)\).

5.1 A Priori Estimates

We consider the penalized BSDE by using the Moreau–Yosida approximation of \(\varphi\):

\[
Y^\varepsilon(t) + \int_t^T \nabla \varphi_\varepsilon(Y^\varepsilon(s)) ds = \xi + \int_t^T f(s, \eta(s), Y^\varepsilon(s), Z^\varepsilon(s)) ds - \int_t^T Z^\varepsilon(s) \delta B^H(s) .
\]

Recall that the regularization \(\varphi_\varepsilon\) of \(\varphi\) is defined by:

\[
\varphi_\varepsilon(u) := \inf \left\{ \frac{1}{2\varepsilon} |u - v|^2 + \varphi(v) : v \in \mathbb{R} \right\}, \quad u \in \mathbb{R}, \quad \varepsilon > 0.
\]
It is well known that $\varphi_\varepsilon$ is a convex function of class $C^1$ on $\mathbb{R}$ and its gradient $\nabla \varphi_\varepsilon$ is a Lipschitz function with Lipschitz constant $1/\varepsilon$. Let

$$J_\varepsilon u = u - \varepsilon \nabla \varphi_\varepsilon(u), \ u \in \mathbb{R}.$$ 

For all $u, v \in \mathbb{R}$ and $\varepsilon, \delta > 0$, the following properties hold true (see [4] and [19]).

\begin{enumerate}[(a)]  
  
  \item $\varphi_\varepsilon(u) = \frac{\varepsilon}{2} |\nabla \varphi_\varepsilon(u)|^2 + \varphi(J_\varepsilon u)$,
  
  \item $|J_\varepsilon u - J_\varepsilon v| \leq |u - v|$, 
  
  \item $\nabla \varphi_\varepsilon(u) \in \partial \varphi(J_\varepsilon u)$, 
  
  \item $0 \leq \varphi_\varepsilon(u) \leq u \nabla \varphi_\varepsilon(u)$,
  
  \item $(\nabla \varphi_\varepsilon(u) - \nabla \varphi_\varepsilon(v)) (u - v) \geq - (\varepsilon + \delta) \nabla \varphi_\varepsilon(u) \nabla \varphi_\varepsilon(v)$.
\end{enumerate}

**Theorem 28** Let the assumptions $(H_1)$–$(H_5)$ be satisfied. Then, for all $\varepsilon > 0$, the penalized BSDE (45) has a solution $(Y^\varepsilon, Z^\varepsilon) \in \bar{V}^{1/2}_T \times \bar{V}^H_T$ such that, for $t \in (0, T]$,

$$\mathbb{E}|Y^\varepsilon(t)|^2 + 2\mathbb{E} \int_t^T \frac{\hat{\sigma}(s)}{\sigma(s)} |Z^\varepsilon(s)|^2 \, ds = \mathbb{E}|\xi|^2 + 2\mathbb{E} \int_t^T Y^\varepsilon(s) f(s, \eta(s), Y^\varepsilon(s), Z^\varepsilon(s)) \, ds - 2\mathbb{E} \int_t^T Y^\varepsilon(s) \nabla \varphi_\varepsilon(Y^\varepsilon(s)) \, ds. \quad (46)$$

**Proof** In order to use Theorem 25, we mollify $\nabla \varphi_\varepsilon$ in a standard way:

$$(\nabla \varphi_\varepsilon)^\alpha(x) := \int_\mathbb{R} \nabla \varphi_\varepsilon(x - au) \lambda(u) \, du, \ x \in \mathbb{R}, \text{ where } \lambda(u) = \frac{1}{\sqrt{2\pi}} e^{-\frac{u^2}{2}}, \ u \in \mathbb{R}.$$ 

Considering that $\varphi_\varepsilon$ is convex and $\nabla \varphi_\varepsilon$ is Lipschitz continuous with Lipschitz constant $1/\varepsilon$, $(\nabla \varphi_\varepsilon)^\alpha$ has the following properties for $x_1, x_2 \in \mathbb{R}$ and $\alpha, \alpha_1, \alpha_2 > 0$:

\begin{enumerate}[(i)]  
  
  \item $(\nabla \varphi_\varepsilon)^\alpha$ belongs to $C^1_{pol}(\mathbb{R})$, and is convex;
  
  \item $|((\nabla \varphi_\varepsilon)^{\alpha_1}(x_1) - (\nabla \varphi_\varepsilon)^{\alpha_2}(x_2))| \leq \frac{1}{\varepsilon} |x_1 - x_2| + \frac{1}{\varepsilon} \sqrt{\frac{2}{\pi}} |\alpha_1 - \alpha_2|.$
\end{enumerate}

Now, we consider the following mollified BSDE

$$Y^{\varepsilon, \alpha}(t) + \int_t^T (\nabla \varphi_\varepsilon)^\alpha (Y^{\varepsilon, \alpha}(s)) \, ds = \xi + \int_t^T f(s, \eta(s), Y^{\varepsilon, \alpha}(s), Z^{\varepsilon, \alpha}(s)) \, ds - \int_t^T Z^{\varepsilon, \alpha}(s) \delta B^H(s). \quad (47)$$
From Theorem 25, we obtain that (47) admits a unique solution \((Y^{e,\alpha}, Z^{e,\alpha})\) in \(S_{f,\epsilon,\alpha} := S_{f,\epsilon,\alpha}^T \subset \mathcal{V}_T \times \mathcal{V}_T\). This solution \((Y^{e,\alpha}, Z^{e,\alpha})\) can be approximated by the sequence \((Y^{k,e,\alpha}, Z^{k,e,\alpha}) \in \mathcal{V}_T \times \mathcal{V}_T, k \geq 0\) constructed by the following method: Define \((Y^{k,e,\alpha}, Z^{k,e,\alpha}), k \geq 0\) recursively: \(Y^{0,e,\alpha} = \chi(t, \eta(t)), Z^{0,e,\alpha} = \psi(t, \eta(t))\) for \(\chi, \psi \in C_{pol}^{1,3}(0, T) \times \mathbb{R}\) with \(\frac{\partial \chi}{\partial t}, \frac{\partial \psi}{\partial t} \in C_{pol}^{0,1}(0, T) \times \mathbb{R}\), and let \((Y^{k+1,e,\alpha}, Z^{k+1,e,\alpha}) \in \mathcal{V}_T \times \mathcal{V}_T\) be the unique solution of the BSDE

\[
Y^{k+1,e,\alpha}(t) + \int_t^T (\nabla \varphi)(Y^{k,e,\alpha}(s))ds = \xi + \int_t^T f(s, \eta(s), Y^{k,e,\alpha}(s), Z^{k,e,\alpha}(s))ds \\
- \int_t^T Z^{k+1,e,\alpha}(s)\delta B^H(s), \ t \in [0, T].
\]

(48)

Similar to (37), we have

\[
\lim_{k \to \infty} \mathbb{E}|Y^{k,e,\alpha}(t) - Y^{e,\alpha}(t)|^2 = 0, \ t \in [0, T], \text{ and }
\]

\[
\lim_{k \to \infty} \mathbb{E}\int_0^T |Y^{k,e,\alpha}(s) - Y^{e,\alpha}(s)|^2 ds + \mathbb{E}\int_0^T s^{2H-1}|Z^{k,e,\alpha}(s) - Z^{e,\alpha}(s)|^2 ds = 0.
\]

(49)

Moreover, analogously to (40), we show that for arbitrary \(\rho > 0\),

\[
\mathbb{E}|Y^{e,\alpha}(t)|^2 + 2\mathbb{E}\int_t^T \frac{\hat{\sigma}(s)}{\sigma(s)}|Z^{e,\alpha}(s)|^2 ds = \mathbb{E}||\xi||^2 - 2\mathbb{E}\int_t^T Y^{e,\alpha}(s) (\nabla \varphi)(Y^{e,\alpha}(s))ds \\
+ 2\mathbb{E}\int_t^T Y^{e,\alpha}(s) f(s, \eta(s), Y^{e,\alpha}(s), Z^{e,\alpha}(s))ds, \ t \in [\rho, T],
\]

(50)

and

\[
\mathbb{E}||Y^{e,\alpha_1,\alpha_2}(t)||^2 + 2\mathbb{E}\int_t^T \frac{\hat{\sigma}(s)}{\sigma(s)}||Z^{e,\alpha_1,\alpha_2}(s)||^2 ds = 2\mathbb{E}\int_t^T \Delta Y^{e,\alpha_1,\alpha_2}(s) \Delta f^{e,\alpha_1,\alpha_2}(s) ds \\
- 2\mathbb{E}\int_t^T \Delta Y^{e,\alpha_1,\alpha_2}(s) ( (\nabla \varphi_{e_1})(Y^{e,\alpha_1}(s)) - (\nabla \varphi_{e_2})(Y^{e,\alpha_2}(s)) ) ds, \ t \in [\rho, T],
\]

where \(\Delta Y^{e,\alpha_1,\alpha_2}(s) = Y^{e,\alpha_1}(s) - Y^{e,\alpha_2}(s), \ \Delta Z^{e,\alpha_1,\alpha_2}(s) = Z^{e,\alpha_1}(s) - Z^{e,\alpha_2}(s)\) and \(\Delta f^{e,\alpha_1,\alpha_2}(s) = f(s, \eta(s), Y^{e,\alpha_1}(s), Z^{e,\alpha_1}(s)) - f(s, \eta(s), Y^{e,\alpha_2}(s), Z^{e,\alpha_2}(s)), \ s \in \mathbb{R}\).
\[ [0, T]. \text{ Then} \]

\[
\mathbb{E}|\Delta Y^{\varepsilon, \alpha_1, \alpha_2}(t)|^2 + 2\mathbb{E}\int_t^T \frac{\hat{\sigma}(s)}{\sigma(s)} |\Delta Z^{\varepsilon, \alpha_1, \alpha_2}(s)|^2 \, ds
\]

\[
\leq \mathbb{E}\int_t^T (2L + L^2 Ms^{1-2H}) |\Delta Y^{\varepsilon, \alpha_1, \alpha_2}(s)|^2 \, ds
\]

\[
+ \mathbb{E}\int_t^T \frac{1}{M} s^{2H-1} |\Delta Z^{\varepsilon, \alpha_1, \alpha_2}(s)|^2 \, ds + \frac{3}{\varepsilon} \mathbb{E}\int_t^T |\Delta Y^{\varepsilon, \alpha_1, \alpha_2}(s)|^2 \, ds
\]

\[
+ \frac{2T}{\pi} |\alpha_1 - \alpha_2|^2, \quad t \in [\rho, T],
\]

(by using the property for \((\nabla \varphi_\varepsilon)^\alpha\) where \(M\) is the constant given by Remark 6. Then, using (9), we obtain

\[
\mathbb{E}|\Delta Y^{\varepsilon, \alpha_1, \alpha_2}(t)|^2 + \frac{1}{M} \mathbb{E}\int_t^T s^{2H-1} |\Delta Z^{\varepsilon, \alpha_1, \alpha_2}(s)|^2 \, ds
\]

\[
\leq \frac{2T}{\pi} |\alpha_1 - \alpha_2|^2 + \mathbb{E}\int_t^T (2L + \frac{3}{\varepsilon} + L^2 Ms^{1-2H}) |\Delta Y^{\varepsilon, \alpha_1, \alpha_2}(s)|^2 \, ds, \quad t \in [\rho, T],
\]

and Gronwall’s inequality yields that

\[
\mathbb{E}|\Delta Y^{\varepsilon, \alpha_1, \alpha_2}(t)|^2 + \frac{1}{M} \mathbb{E}\int_t^T s^{2H-1} |\Delta Z^{\varepsilon, \alpha_1, \alpha_2}(s)|^2 \, ds \leq M_{\varepsilon, L, T} |\alpha_1 - \alpha_2|^2, \quad t \in [\rho, T],
\]

where \(M_{\varepsilon, L, T}\) is a constant depending only on \(\varepsilon, L, T\) but independent of \(\rho > 0\). Consequently, taking into account the arbitrariness of \(\rho > 0\), there exists a couple of processes \((Y^\varepsilon, Z^\varepsilon)\) with \(Y^\varepsilon 1_{[\rho, T]} \in \tilde{\mathcal{V}}^{1/2}_T\), \(Z^\varepsilon 1_{[\rho, T]} \in \tilde{\mathcal{V}}^H_T\) for all \(\rho > 0\), such that

\[
\lim_{\alpha \to 0} \mathbb{E}|Y^{\varepsilon, \alpha}(t) - Y^\varepsilon(t)|^2 = 0, \quad \text{for all} \ t \in [\rho, T],
\]

\[
\lim_{\alpha \to 0} \mathbb{E}\int_t^T |Y^{\varepsilon, \alpha}(s) - Y^\varepsilon(s)|^2 \, ds + \mathbb{E}\int_t^T s^{2H-1} |Z^{\varepsilon, \alpha}(s) - Z^\varepsilon(s)|^2 \, ds = 0,
\]

\[
\text{for all} \ t \in [\rho, T], \quad (51)
\]
Now, let \( \alpha \to 0 \), and by using (47) and a similar discussion as in Theorem 22, we obtain that \( Z^\varepsilon 1_{[t,T]} \in \text{Dom}(\delta) \), \( t \in (0, T] \), and

\[
Y^\varepsilon(t) + \int_t^T \nabla \varphi^\varepsilon(Y^\varepsilon(s))ds = \xi + \int_t^T f(s, \eta(s), Y^\varepsilon(s), Z^\varepsilon(s))ds
- \int_t^T Z^\varepsilon(s)\delta B^H(s), \ t \in (0, T].
\]

Moreover, taking \( \alpha \to 0 \) in (50) yields that

\[
\mathbb{E}|Y^\varepsilon(t)|^2 + 2\mathbb{E}\int_t^T \frac{\dot{\sigma}(s)}{\sigma(s)}|Z^\varepsilon(s)|^2ds = \mathbb{E}|\xi|^2 + 2\mathbb{E}\int_t^T Y^\varepsilon(s)f(s, \eta(s), Y^\varepsilon(s), Z^\varepsilon(s))ds
- 2\mathbb{E}\int_t^T Y^\varepsilon(s)\nabla \varphi^\varepsilon(Y^\varepsilon(s))ds, \ t \in (0, T].
\]

The next three propositions provide a priori estimates for the sequence \((Y^\varepsilon, Z^\varepsilon)\), \( \varepsilon > 0 \).

**Proposition 29** Let the assumptions \((H_1)\)–\((H_5)\) be satisfied. Let \((Y^\varepsilon, Z^\varepsilon)\) be the solution constructed in the proof of Theorem 28. Then there exists a positive constant \(C\) independent of \(\varepsilon > 0\), such that, for all \(t \in [0, T]\),

\[
\mathbb{E}|Y^\varepsilon(t)|^2 + \mathbb{E}\int_t^T s^{2H-1}|Z^\varepsilon(s)|^2ds \leq C \Gamma_1(T),
\]

where \( \Gamma_1(T) = \mathbb{E}[|\xi|^2 + \int_0^T |\eta(s)|^2ds + \int_0^T |f(s, 0, 0, 0)|^2ds]\).

**Proof** From (46), (9-b) and \(u \nabla \varphi_e(u) \geq 0\), for all \(u \in \mathbb{R}\), we have, for \(t \in (0, T]\),

\[
\mathbb{E}|Y^\varepsilon(t)|^2 + \frac{2}{M}\mathbb{E}\int_t^T s^{2H-1}|Z^\varepsilon(s)|^2ds \leq \mathbb{E}|\xi|^2
+ 2\int_t^T \mathbb{E}\left[Y^\varepsilon(s)f(s, \eta(s), Y^\varepsilon(s), Z^\varepsilon(s))\right]ds.
\]
On the other hand, from assumption \((H_3)\) and Schwartz’s inequality, we obtain

\[
2Y^\varepsilon(s) f(s, \eta(s), Y^\varepsilon(s), Z^\varepsilon(s)) 
\leq 2|Y^\varepsilon(s)| \left( |f(s, 0, 0, 0)| + L |\eta(s)| + L |Y^\varepsilon(s)| + L |Z^\varepsilon(s)| \right)
\]

\[
\leq |f(s, 0, 0, 0)|^2 + |\eta(s)|^2 + \left( 1 + L^2 + 2L + L^2 M \frac{1}{s^{2H-1}} \right) |Y^\varepsilon(s)|^2 + \frac{1}{M} s^{2H-1} |Z^\varepsilon(s)|^2.
\]

Then,

\[
\mathbb{E} |Y^\varepsilon(t)|^2 + \frac{1}{M} \mathbb{E} \int_t^T s^{2H-1} |Z^\varepsilon(s)|^2 \, ds 
\leq \mathbb{E} |\xi|^2 + \int_t^T \mathbb{E} |f(s, 0, 0, 0)|^2 \, ds 
+ \int_t^T \mathbb{E} |\eta(s)|^2 \, ds + \int_t^T \left( 1 + L^2 + 2L + L^2 M \frac{1}{s^{2H-1}} \right) \mathbb{E} |Y^\varepsilon(s)|^2 \, ds.
\]

Therefore, by Gronwall’s inequality, we deduce that

\[
\mathbb{E} |Y^\varepsilon(t)|^2 + \frac{1}{M} \mathbb{E} \int_t^T s^{2H-1} |Z^\varepsilon(s)|^2 \, ds 
\leq \Gamma_1(T) \exp \left[ \left( 1 + L^2 + 2L \right) (T - t) + L^2 M \frac{T^{2-2H} - t^{2-2H}}{2 - 2H} \right],
\]

which completes the proof.

Proposition 30 Let the assumptions \((H_1)\)–\((H_5)\) be satisfied. Then there exists a positive constant \(C\) such that, for all \(t \in [0, T]\),

\[
(i) \quad \mathbb{E} \int_t^T s^{2H-1} |\nabla \varphi_{\varepsilon}(Y^\varepsilon(s))|^2 \, ds \leq C \Gamma_2(T), \\
(ii) \quad t^{2H-1} \mathbb{E} [\varphi(J_{\varepsilon}(Y^\varepsilon(t)))] \leq C \Gamma_2(T), \\
(iii) \quad t^{2H-1} \mathbb{E} \left[ |Y^\varepsilon(t) - J_{\varepsilon}(Y^\varepsilon(t))|^2 \right] \leq \varepsilon C \Gamma_2(T),
\]

where \(\Gamma_2(T) = \mathbb{E} \left[ |\xi|^2 + \varphi(\xi) \right] + \int_0^T |\eta(s)|^2 \, ds + \int_0^T |f(s, 0, 0, 0)|^2 \, ds\).

In order to obtain the above proposition, it is essential to use the following fractional stochastic subdifferential inequality:

Lemma 31 Let \(\psi : \mathbb{R} \to \mathbb{R}_+\) be a convex \(C^1\) function which derivative \(\nabla \psi\) is a Lipschitz function (with Lipschitz constant denoted by \(K\)). Then, for all \(t \in (0, T]\),
\begin{align*}
\mathbb{P}\text{-}a.s.
&\quad t^{2H-1} \mathbb{E}\left[ \psi\left( Y^\varepsilon(t) \right) \right] + \mathbb{E} \int_t^T s^{2H-1} \nabla \psi \left( Y^\varepsilon(s) \right) \nabla \varphi^\varepsilon(Y^\varepsilon(s)) ds \\
&\quad \leq T^{2H-1} \mathbb{E}\left[ \psi(\xi) \right] + \mathbb{E} \int_t^T s^{2H-1} \nabla \psi \left( Y^\varepsilon(s) \right) f(s, \eta(s), Y^\varepsilon(s), Z^\varepsilon(s)) ds.
\end{align*}

(52)

where \((Y^\varepsilon, Z^\varepsilon)\) is the solution constructed in the proof of Theorem 28.

**Proof.** We first show that

\begin{align*}
\quad t^{2H-1} \mathbb{E}\left[ \psi\left( Y^{k+1,\varepsilon,\alpha}(t) \right) \right] + \mathbb{E} \int_t^T s^{2H-1} \nabla \psi \left( Y^{k+1,\varepsilon,\alpha}(s) \right) (\nabla \varphi^\varepsilon)^\alpha(Y^{k,\varepsilon,\alpha}(s)) ds \\
&\quad \leq T^{2H-1} \mathbb{E}\left[ \psi(\xi) \right] + \mathbb{E} \int_t^T s^{2H-1} \nabla \psi \left( Y^{k+1,\varepsilon,\alpha}(s) \right) f(s, \eta(s), Y^{k,\varepsilon,\alpha}(s), Z^{k,\varepsilon,\alpha}(s)) ds,
\end{align*}

where \((Y^{k,\varepsilon,\alpha}, Z^{k,\varepsilon,\alpha}) \in \mathcal{V}_T \times \mathcal{V}_T\) is defined through (48). We mollify the function \(\psi\) by setting, for \(\theta > 0\),

\[\psi^\theta(x) := \int_{\mathbb{R}} \psi(x - \theta u) \lambda(u) du, \quad x \in \mathbb{R},\]

where \(\lambda(u) = \frac{1}{\sqrt{2\pi}} e^{-\frac{u^2}{2}}, \quad u \in \mathbb{R}\). From the convexity of \(\psi\), it follows that \(\psi^\theta\) is convex. Moreover \(\psi^\theta \geq 0\). The generalized Itô formula [see (64) in Remark 34] yields

\begin{align*}
T^{2H-1} \psi^\theta(\xi) &= t^{2H-1} \psi^\theta(Y^{k+1,\varepsilon,\alpha}(t)) + (2H - 1) \int_t^T s^{2H-2} \psi^\theta(Y^{k+1,\varepsilon,\alpha}(s)) ds \\
&\quad - \int_t^T s^{2H-1} \nabla \psi^\theta(Y^{k+1,\varepsilon,\alpha}(s)) f(s, \eta(s), Y^{k,\varepsilon,\alpha}(s), Z^{k,\varepsilon,\alpha}(s)) ds \\
&\quad + \int_t^T s^{2H-1} \nabla \psi^\theta(Y^{k+1,\varepsilon,\alpha}(s)) (\nabla \varphi^\varepsilon)^\alpha(Y^{k,\varepsilon,\alpha}(s)) ds \\
&\quad + \int_t^T s^{2H-1} \nabla \psi^\theta(Y^{k+1,\varepsilon,\alpha}(s)) Z^{k+1,\varepsilon,\alpha}(s) \delta B^H(s) \\
&\quad + \int_t^T s^{2H-1} \mathcal{D}^2_{xx} \psi^\theta(Y^{k+1,\varepsilon,\alpha}(s)) Z^{k+1,\varepsilon,\alpha}(s) \mathcal{D}^H_X Y^{k+1,\varepsilon,\alpha}(s) ds.
\end{align*}

(53)
Now taking the expectation in (53), by using $\psi^\theta \geq 0$, the convexity of $\psi^\theta$ and the fact that $\mathbb{D}_s^H Y^{k+1,\varepsilon,\alpha}(s) = \frac{\hat{\sigma}(s)}{\sigma(s)} Z^{k+1,\varepsilon,\alpha}(s)$, we have

$$
t^{2H-1} \mathbb{E} \left[ \psi^\theta (Y^{k+1,\varepsilon,\alpha}(t)) \right] + \mathbb{E} \int_t^T s^{2H-1} \nabla \psi^\theta (Y^{k+1,\varepsilon,\alpha}(s)) (\nabla \varphi_\varepsilon)^\alpha (Y^{k,\varepsilon,\alpha}(s)) ds
\leq T^{2H-1} \mathbb{E} \psi^\theta (\xi) + \mathbb{E} \int_t^T s^{2H-1} \nabla \psi^\theta (Y^{k+1,\varepsilon,\alpha}(s)) f \left( s, \eta(s), Y^{k,\varepsilon,\alpha}(s), Z^{k,\varepsilon,\alpha}(s) \right) ds.
$$

(54)

Considering that

$$
\left| \nabla \psi^\theta (x) - \nabla \psi (x) \right| = \left| \nabla \int_x^\infty (x - \theta u) \lambda(u) du - \nabla \psi (x) \right|
\leq \int_\mathbb{R} \left| \nabla \psi (x - \theta u) - \nabla \psi (x) \right| \lambda(u) du \leq \sqrt{\frac{2}{\pi}} K |\theta|,
$$

we have

$$
\mathbb{E} \left| \int_t^T s^{2H-1} \nabla \psi^\theta (Y^{k+1,\varepsilon,\alpha}(s)) (\nabla \varphi_\varepsilon)^\alpha (Y^{k,\varepsilon,\alpha}(s)) ds \right|
\leq T^{2H-1} \sqrt{\frac{2}{\pi}} K |\theta| \mathbb{E} \int_t^T (\nabla \varphi_\varepsilon)^\alpha (Y^{k,\varepsilon,\alpha}(s)) ds \to 0, \text{ as } \theta \to 0.
$$

Similarly, we get

$$
\mathbb{E} \left| \int_t^T s^{2H-1} \nabla \psi^\theta (Y^{k+1,\varepsilon,\alpha}(s)) f \left( s, \eta(s), Y^{k,\varepsilon,\alpha}(s), Z^{k,\varepsilon,\alpha}(s) \right) ds \right|
\to \mathbb{E} \int_t^T s^{2H-1} \nabla \psi (Y^{k+1,\varepsilon,\alpha}(s)) f \left( s, \eta(s), Y^{k,\varepsilon,\alpha}(s), Z^{k,\varepsilon,\alpha}(s) \right) ds \to 0, \text{ as } \theta \to 0.
$$
Moreover, using Fatou’s Lemma (recalling that \( \psi \geq 0 \)), we obtain

\[
\mathbb{E} \left[ \psi(Y^{k+1,\epsilon,\alpha}(t)) \right] = \mathbb{E} \left[ \lim_{\theta \to 0} \psi^\theta(Y^{k+1,\epsilon,\alpha}(t)) \right] \leq \lim_{\theta \to 0} \mathbb{E} \left[ \psi^\theta(Y^{k+1,\epsilon,\alpha}(t)) \right].
\]

On the other hand, we know that \( \psi \) is quadratic growth; therefore, there exists a suitable constant \( C \), such that

\[
|\psi^\theta(x)| \leq \int \psi(x - \theta u)\lambda(u)du \leq C(1 + x^2 + \theta^2).
\]

From \((H_4)\) and \((11)\), it follows \( \sup_{\theta \leq 1} \mathbb{E} [\psi^\theta(\xi)] < \infty \), which implies that \( \{\psi^\theta(\xi)\}_{\theta \leq 1} \) is uniformly integrable. Then, considering that \( \psi^\theta(\xi) \xrightarrow{\theta \to 0} P-a.s. \psi(\xi) \), we have \( \mathbb{E}[\psi^\theta(\xi)] \to \mathbb{E}[\psi(\xi)] \). Consequently, letting \( \theta \to 0 \) in \((54)\), we have

\[
T^{2H-1}\mathbb{E} \left[ \psi(Y^{k+1,\epsilon,\alpha}(t)) \right] + \mathbb{E} \int_0^T s^{2H-1} \nabla \psi(Y^{k+1,\epsilon,\alpha}(s)) (\nabla \varphi_\epsilon)^\alpha(Y^{k,\epsilon,\alpha}(s)) ds
\]

\[
\leq T^{2H-1} \mathbb{E}[\psi(\xi)] + \mathbb{E} \int_0^T s^{2H-1} \nabla \psi(Y^{k+1,\epsilon,\alpha}(s)) f(s, \eta(s), Y^{k,\epsilon,\alpha}(s), Z^{k,\epsilon,\alpha}(s)) ds.
\]

(55)

Recalling that \( \psi(Y^{k,\epsilon,\alpha}(t) - Y^{\epsilon,\alpha}(t))^2 = 0 \), \( t \in [0, T] \), and

\[
\lim_{k \to \infty} \mathbb{E} \int_0^T \left| Y^{k,\epsilon,\alpha}(s) - Y^{\epsilon,\alpha}(s) \right|^2 ds + \mathbb{E} \int_0^T s^{2H-1} \left| Z^{k,\epsilon,\alpha}(s) - Z^{\epsilon,\alpha}(s) \right|^2 ds = 0,
\]

it follows that \( Y^{k,\epsilon,\alpha}(t) \xrightarrow{k \to \infty} P-a.s. Y^{\epsilon,\alpha}(t) \), for all \( t \in [0, T] \) and then also \( \psi(Y^{k,\epsilon,\alpha}(t)) \xrightarrow{k \to \infty} P-a.s. \psi(Y^{\epsilon,\alpha}(t)) \), for all \( t \in [0, T] \). Thus, using Fatou’s Lemma once again (recalling that \( \psi \geq 0 \)), we obtain

\[
\mathbb{E} \left[ \psi(Y^{\epsilon,\alpha}(t)) \right] \leq \lim_{k \to \infty} \mathbb{E} \left[ \psi(Y^{k+1,\epsilon,\alpha}(t)) \right].
\]

Considering that \( \nabla \psi \) and \( (\nabla \varphi_\epsilon)^\alpha \) are Lipschitz with the Lipschitz constant \( K \) and \( 1/\epsilon \), respectively, we get
Indeed, using that the functions $\nabla \psi$ and $(\nabla \varphi_\varepsilon)^\alpha$ are of linear growth, $\mathbb{E} \int_t^T |\nabla \varphi_\varepsilon|^2 ds + \mathbb{E} \int_t^T |Y^{k,\varepsilon,\alpha}(s)|^2 ds \leq C, k \geq 1$, and $\lim_{k \to \infty} \mathbb{E} \int_0^T |Y^{k,\varepsilon,\alpha}(s) - Y^{\varepsilon,\alpha}(s)|^2 ds = 0$, we can obtain the above convergence with the help of Hölder inequality. Similarly, we show

\[
\mathbb{E} \left[ \int_t^T s^{2H-1} \nabla \psi(Y^{k+1,\varepsilon,\alpha}(s)) (\nabla \varphi_\varepsilon)^\alpha(Y^{k,\varepsilon,\alpha}(s)) ds \right] - \int_t^T s^{2H-1} \nabla \psi(Y^{\varepsilon,\alpha}(s)) (\nabla \varphi_\varepsilon)^\alpha(Y^{\varepsilon,\alpha}(s)) ds \\
\leq T^{2H-1} \mathbb{E} \int_t^T |\nabla \psi(Y^{k+1,\varepsilon,\alpha}(s)) - \nabla \psi(Y^{\varepsilon,\alpha}(s))| (\nabla \varphi_\varepsilon)^\alpha(Y^{k,\varepsilon,\alpha}(s)) ds \\
+ T^{2H-1} \mathbb{E} \int_t^T |\nabla \psi(Y^{\varepsilon,\alpha}(s))| (\nabla \varphi_\varepsilon)^\alpha(Y^{k,\varepsilon,\alpha}(s)) - (\nabla \varphi_\varepsilon)^\alpha(Y^{\varepsilon,\alpha}(s)) ds \\
\leq K T^{2H-1} \mathbb{E} \int_t^T \left| Y^{k+1,\varepsilon,\alpha}(s) - Y^{\varepsilon,\alpha}(s) \right| (\nabla \varphi_\varepsilon)^\alpha(Y^{k,\varepsilon,\alpha}(s)) ds \\
+ \frac{1}{\varepsilon} T^{2H-1} \mathbb{E} \int_t^T \left| \nabla \psi(Y^{\varepsilon,\alpha}(s)) \right| \left| Y^{k,\varepsilon,\alpha}(s) - Y^{\varepsilon,\alpha}(s) \right| ds \to 0, \quad \text{as } k \to \infty.
\]

Consequently, letting $k \to \infty$ in (55) yields that

\[
T^{2H-1} \mathbb{E} \left[ \psi(Y^{\varepsilon,\alpha}(t)) \right] + \mathbb{E} \int_t^T s^{2H-1} \nabla \psi(Y^{\varepsilon,\alpha}(s)) (\nabla \varphi_\varepsilon)^\alpha(Y^{\varepsilon,\alpha}(s)) ds \\
\leq T^{2H-1} \mathbb{E} \psi(\xi) + \mathbb{E} \int_t^T s^{2H-1} \nabla \psi(Y^{\varepsilon,\alpha}(s)) f(s, \eta(s), Y^{\varepsilon,\alpha}(s), Z^{\varepsilon,\alpha}(s)) ds, \quad t \in [0, T].
\]
A similar argument allows to take the limit $\alpha \to 0$ in (56) [using (51)], it follows
\[
t^{2H-1}E\left[\psi\left(Y^\varepsilon(t)\right)\right] + E\int_t^T s^{2H-1} \nabla \psi\left(Y^\varepsilon(s)\right) \nabla \varphi_\varepsilon\left(Y^\varepsilon(s)\right) ds
\]
\[\leq T^{2H-1}E\left[\psi(\xi)\right] + E\int_t^T s^{2H-1} \nabla \varphi_\varepsilon\left(Y^\varepsilon(s)\right) f(s, \eta(s), Y^\varepsilon(s), Z^\varepsilon(s)) ds, \ t \in (0, T],\]
and the statement is proven. \hfill \Box

Now, we are able to give the proof of Proposition 30.

**Proof of Proposition 30** We consider $\psi(x) = \varphi_\varepsilon(x), \ x \in \mathbb{R}$, and applying (52) we have
\[
t^{2H-1}E\left[\varphi_\varepsilon\left(Y^\varepsilon(t)\right)\right] + E\int_t^T s^{2H-1} \left| \nabla \varphi_\varepsilon\left(Y^\varepsilon(s)\right) \right|^2 ds
\]
\[\leq T^{2H-1}E\left[\varphi_\varepsilon(\xi)\right] + E\int_t^T s^{2H-1} \left| \nabla \varphi_\varepsilon\left(Y^\varepsilon(s)\right) f(s, \eta(s), Y^\varepsilon(s), Z^\varepsilon(s)) ds, \ s \in [0, T].\]

Since $0 \leq \varphi_\varepsilon(u) \leq \varphi(u), \ u \in \mathbb{R}$ we obtain
\[
t^{2H-1}E\left[\varphi_\varepsilon\left(Y^\varepsilon(t)\right)\right] + E\int_t^T s^{2H-1} \left| \nabla \varphi_\varepsilon\left(Y^\varepsilon(s)\right) \right|^2 ds
\]
\[\leq T^{2H-1}E\left[\varphi(\xi)\right] + \frac{1}{2} \int_t^T s^{2H-1} \left| \nabla \varphi_\varepsilon\left(Y^\varepsilon(s)\right) \right|^2 ds
\]
\[+ \frac{1}{2} \int_t^T s^{2H-1} \left| f(s, \eta(s), Y^\varepsilon(s), Z^\varepsilon(s)) \right|^2 ds
\]
\[\leq T^{2H-1}E\left[\varphi(\xi)\right] + \frac{1}{2} E\int_t^T s^{2H-1} \left| \nabla \varphi_\varepsilon\left(Y^\varepsilon(s)\right) \right|^2 ds
\]
\[+ 2E\int_t^T s^{2H-1} \left( |f(s, 0, 0, 0)|^2 + L^2 |\eta(s)|^2 + L^2 |Y^\varepsilon(s)|^2 + L^2 |Z^\varepsilon(s)|^2 \right) ds.
\]

Considering Proposition 29, we see that
\[
t^{2H-1}E\left[\varphi_\varepsilon\left(Y^\varepsilon(t)\right)\right] + E\int_t^T s^{2H-1} \left| \nabla \varphi_\varepsilon\left(Y^\varepsilon(s)\right) \right|^2 ds \leq C \Gamma_2(T).
\]
Therefore, since $\varphi(J_\varepsilon u) \leq \varphi_\varepsilon(u)$, $u \in \mathbb{R}$, we have proven (i) and (ii) of the proposition.

Finally, in order to obtain (iii), it is suffices to remark that $|u - J_\varepsilon(u)|^2 = |\nabla \varphi_\varepsilon(u)|^2 \leq 2\varepsilon \varphi_\varepsilon(u)$, $u \in \mathbb{R}$.

**Proposition 32** Let the assumptions $(H_1)$–$(H_5)$ be satisfied. Then there exists a positive constant $C$ such that, for all $\varepsilon, \delta > 0$,

$$
\sup_{s \in [0, T]} s^{2H-1} \mathbb{E} \left| Y_\varepsilon(s) - Y_\delta(s) \right|^2 ds + \int_0^T s^{2(2H-1)} \mathbb{E} \left| Z_\varepsilon(s) - Z_\delta(s) \right|^2 ds \\
\leq (\varepsilon + \delta) C \Gamma_2(T).
$$

**Proof** Similarly to (46), we have, for $t \in (0, T]$,

$$
\mathbb{E} \left| \Delta Y(t) \right|^2 + 2\mathbb{E} \int_t^T \frac{\hat{\sigma}(s)}{\sigma(s)} |\Delta Z(s)|^2 ds = -2\mathbb{E} \int_t^T \Delta Y(s)[\nabla \varphi_\varepsilon(Y_\varepsilon(s)) - \nabla \varphi_\delta(Y_\delta(s))] ds \\
+ 2\mathbb{E} \int_t^T \Delta Y(s)[f(s, \eta(s), Y_\varepsilon(s), Z_\varepsilon(s)) - f(s, \eta(s), Y_\delta(s), Z_\delta(s))] ds
$$

(57)

where $\Delta Y(t) = Y_\varepsilon(t) - Y_\delta(t)$ and $\Delta Z(t) = Z_\varepsilon(t) - Z_\delta(t)$. Then from

$$
d \left( s^{2H-1} \mathbb{E} \left| \Delta Y(s)^2 \right| \right) = s^{2H-1} d \mathbb{E} \left[ \left| \Delta Y(s)^2 \right| \right] + (2H - 1)s^{2H-2} \mathbb{E} \left[ \left| \Delta Y(s)^2 \right| \right]
$$

as well as (57), we deduce that for $t \in (0, T]$,

$$
t^{2H-1} \mathbb{E} \left| \Delta Y(t) \right|^2 + \mathbb{E} \int_t^T (2H - 1)s^{2H-2} |\Delta Y(s)|^2 ds \\
+ \mathbb{E} \int_t^T 2 \frac{\hat{\sigma}(s)}{\sigma(s)} s^{2H-1} |\Delta Z(s)|^2 ds \\
= -2\mathbb{E} \int_t^T s^{2H-1} \Delta Y(s)[\nabla \varphi_\varepsilon(Y_\varepsilon(s)) - \nabla \varphi_\delta(Y_\delta(s))] ds \\
+ 2\mathbb{E} \int_t^T s^{2H-1} \Delta Y(s)[f(s, \eta(s), Y_\varepsilon(s), Z_\varepsilon(s)) - f(s, \eta(s), Y_\delta(s), Z_\delta(s))] ds.
$$

(58)
Since
\[
2s^{2H-1}|\Delta Y(s)||f(s, \eta(s), Y^\varepsilon(s), Z^\varepsilon(s)) - f(s, \eta(s), Y^\delta(s), Z^\delta(s))| \\
\leq 2Ls^{2H-1}|\Delta Y(s)|^2 + 2Ls^{2H-1}|\Delta Z(s)| \\
\leq \left(2L + L^2M\frac{1}{s^{2H-1}}\right)s^{2H-1}|\Delta Y(s)|^2 + \frac{1}{M}s^{2(2H-1)}|\Delta Z(s)|^2,
\]
(58) yields [Recall (9-b)]
\[
t^{2H-1}\mathbb{E}|\Delta Y(t)|^2 + \frac{2}{M}\mathbb{E}\int_t^T s^{2(2H-1)}|\Delta Z(s)|^2 ds + \mathbb{E}\int_t^T (2H - 1)s^{2H-2}|\Delta Y(s)|^2 ds \\
\leq \mathbb{E}\int_t^T \left(2L + L^2M\frac{1}{s^{2H-1}}\right)s^{2H-1}|\Delta Y(s)|^2 ds + \frac{1}{M}\mathbb{E}\int_t^T s^{2(2H-1)}|\Delta Z(s)|^2 ds \\
-2\mathbb{E}\int_t^T s^{2H-1}\Delta Y(s)[\nabla \varphi_\varepsilon(Y^\varepsilon(s)) - \nabla \varphi_\delta(Y^\delta(s))]ds. \tag{59}
\]
By using the following inequality [see (46-e)]
\[(\nabla \varphi_\varepsilon(u) - \nabla \varphi_\delta(v))(u - v) \geq -(\varepsilon + \delta) \nabla \varphi_\varepsilon(u)\nabla \varphi_\delta(v),\]
and Proposition 30 (i) as well as Gronwall's inequality, we conclude from (59) that there exists \(C > 0\) such that
\[
t^{2H-1}\mathbb{E}|\Delta Y(t)|^2 + \mathbb{E}\int_t^T s^{2(2H-1)}|\Delta Z(s)|^2 ds \\
\leq C(\varepsilon + \delta)\mathbb{E}\int_t^T s^{2H-1}|\nabla \varphi_\varepsilon(Y^\varepsilon(s))||\nabla \varphi_\delta(Y^\delta(s))|ds \\
\leq C(\varepsilon + \delta)\mathbb{E}\int_t^T \left(s^{2H-1}|\nabla \varphi_\varepsilon(Y^\varepsilon(s))|^2 + s^{2H-1}|\nabla \varphi_\delta(Y^\delta(s))|^2\right) ds \\
\leq C(\varepsilon + \delta)\Gamma_2(T),
\]
and the proof is complete. \(\square\)

5.2 Proof of the Existence of the Solution

Proof of Theorem 27 For arbitrary \(\rho > 0\), by Proposition 32, there exist \((Y, Z) \in \hat{V}_T^H \times \hat{V}_T^{2H-1/2}\) such that
\[
\sup_{s \in [0, T]} s^{2H-1} \mathbb{E}[|Y^\varepsilon(s) - Y(s)|^2] ds \to 0 \quad \text{and} \quad \int_0^T s^{2(2H-1)} \mathbb{E}[|Z^\varepsilon(s) - Z(s)|^2] ds \to 0,
\]
(60)

From Proposition 30 (iii), we deduce that
\[
\lim_{\varepsilon \to 0} \mathbb{E} \left| Y^\varepsilon(t) - J_{\varepsilon} \left( Y^\varepsilon(t) \right) \right|^2 = 0 \quad \text{for all} \quad t \in [0, T],
\]
\[
\lim_{\varepsilon \to 0} J_{\varepsilon}(Y^\varepsilon) = Y \text{ in } \overline{\mathcal{V}}_T^H.
\]
(61)

For each \( \varepsilon > 0 \), let \( U^\varepsilon(t) = \nabla \varphi_{\varepsilon} \left( Y^\varepsilon(t) \right) \), \( t \in [0, T] \). The process \( U^\varepsilon \) belongs to the space \( \overline{\mathcal{V}}_T^H \) (see Lemma 36 in the Appendix). From Proposition 30 (i), we obtain that
\[
\left\| U^\varepsilon \right\|_H^2 = \mathbb{E} \int_0^T s^{2H-1} |U^\varepsilon(s)|^2 ds \leq C \Gamma_2(T), \quad \varepsilon > 0.
\]

Hence, there exists a subsequence \( \varepsilon_n \to 0 \) and a process \( U \in \overline{\mathcal{V}}_T^H \) such that
\[
U^\varepsilon \xrightarrow{\varepsilon_n \to 0} U, \text{ weakly in the Hilbert space } \overline{\mathcal{V}}_T^H.
\]
(62)

Consequently,
\[
\mathbb{E} \int_0^T s^{2H-1} |U(s)|^2 ds \leq \liminf_{\varepsilon_n \to 0} \mathbb{E} \int_0^T s^{2H-1} |U^{\varepsilon_n}(s)|^2 ds \leq C \Gamma_2(T).
\]

From Eq. (45) we have
\[
\int_0^T \int_t^T Z^{\varepsilon_n}(s) \delta B^H(s) = -Y^{\varepsilon_n}(t) - \int_t^T U^{\varepsilon_n}(s) ds + \xi + \int_t^T f \left( s, \eta(s), Y^{\varepsilon_n}(s), Z^{\varepsilon_n}(s) \right) ds
\]
\[
:= \theta^{\varepsilon_n}(t), \quad t \in (0, T], \quad n \geq 1.
\]

On the other hand, since \( Z^{\varepsilon_n}1_{[\rho, T]} \) converges to \( Z1_{[\rho, T]} \) in \( L^2(\Omega, \mathcal{F}, \mathbb{P}; \mathcal{H}) \) for all \( \rho > 0 \) and \( Z^{\varepsilon_n}1_{[\rho, T]} \in \text{Dom}(\delta) \), then we apply Definition 1 and we obtain for all \( F \in \mathcal{P}_T \),
\[
\mathbb{E} \left( \left( D^H F, Z(\cdot)1_{[\rho, T]}(\cdot) \right|_T \right) = \lim_{n \to \infty} \mathbb{E} \left( \left( D^H F, Z^{\varepsilon_n}(\cdot)1_{[\rho, T]}(\cdot) \right|_T \right)
\]
\[
= \lim_{n \to \infty} \mathbb{E} \left( F \int_0^T Z^{\varepsilon_n}(s) \delta B^H(s) \right) = \lim_{n \to \infty} \mathbb{E}(F \theta^{\varepsilon_n}(\rho)) = \mathbb{E}(F \theta(\rho)),
\]
where it follows from (60) and (62) that

$$\theta(t) = -Y(t) - \int_{t}^{T} U(s)ds + \xi + \int_{t}^{T} f(s, \eta(s), Y(s), Z(s)) ds$$

is the weak limit in $L^2(\Omega, \mathcal{F}, P)$ of $\theta^{en}(t)$ as $n \to \infty$, $t \in (0, T]$. From the definition of the divergence operator, it follows that $Z_{1_{[\rho, T]}} \in \text{Dom}(\delta)$ and $\delta(Z_{1_{[\rho, T]}}) = \theta(\rho)$, $P$-a.s. Consequently, since $\rho > 0$ is arbitrarily chosen, we have

$$Y(t) + \int_{t}^{T} U(s)ds = \xi + \int_{t}^{T} f(s, \eta(s), Y(s), Z(s)) ds$$

$$- \int_{t}^{T} Z(s)\delta B^H(s), \text{ for all } t \in (0, T]. \quad (63)$$

Moreover, since $U^\varepsilon(t) \in \partial \varphi(J_{\varepsilon}(Y^\varepsilon(t)), \text{ for } t \in [0, T], we have

$$U^\varepsilon(t)(V(t) - J_{\varepsilon}(Y^\varepsilon(t))) + \varphi(J_{\varepsilon}(Y^\varepsilon(t))) \leq \varphi(V(t)), \text{ for all } V \in \bar{V}_{T}^H, t \in [0, T],$$

and we deduce that for all $A \times [a, b] \subset \Omega \times [0, T]$, $A \in \mathcal{F}$,

$$E\left(\int_{a}^{b} s^{2H-1}1_A U^\varepsilon(t)(V(t) - J_{\varepsilon}(Y^\varepsilon(t)))dt\right) + E\left(\int_{a}^{b} s^{2H-1}1_A \varphi(J_{\varepsilon}(Y^\varepsilon(t)))dt\right)$$

$$\leq E\left(\int_{a}^{b} s^{2H-1}1_A \varphi(V(t))dt\right).$$

Considering that $\varphi$ is a proper convex l.s.c. function; hence, (61) and (62) yield that

$$E\left(\int_{a}^{b} s^{2H-1}1_A U(t)(V(t) - Y(t))dt\right) + E\left(\int_{a}^{b} s^{2H-1}1_A \varphi(Y(t))dt\right)$$

$$\leq E\left(\int_{a}^{b} s^{2H-1}1_A \varphi(V(t))dt\right), \text{ for all } A \times [a, b] \subset \Omega \times [0, T].$$

Therefore,

$$U(t)(V(t) - Y(t)) + \varphi(Y(t)) \leq \varphi(V(t)) \, dP \otimes dt \text{ a.e. on } \Omega \times [0, T],$$
which means that

\[(Y(t), U(t)) \in \partial \varphi, \quad dP \otimes dt \text{ a.e. on } \Omega \times [0, T].\]

This together with (63) complete the proof. 

\[\square\]

Acknowledgments The authors wish to express their thanks to Rainer Buckdahn, Yaozhong Hu, Shige Peng, and Aurel Răşcanu for their useful suggestions and discussions.

Appendix

Theorem 33 Let \(\psi\) be a function of class \(C^{1,2}([0, T] \times \mathbb{R})\). Assume that \(u\) is a process in \(V_T\) and \(f \in C^{0,1}_{pol}([0, T] \times \mathbb{R})\). Let

\[X_t = X_0 + \int_0^t f(s, \eta(s))ds + \int_0^t u_s \delta B^H(s), \quad s \in [0, T]\]

Then for all \(t \in [0, T]\), the following formula holds

\[
\psi(t, X_t) = \psi(0, X_0) + \int_0^t \frac{\partial}{\partial s} \psi(s, X_s)ds + \int_0^t \frac{\partial}{\partial x} \psi(s, X_s)f(s, X_s)ds \\
+ \int_0^t \frac{\partial}{\partial x} \psi(s, X_s)u_s \delta B^H(s) + \int_0^t \frac{\partial^2}{\partial x^2} \psi(s, X_s)u_s \mathbb{D}_s^H X_s ds. \tag{64}
\]

Remark 34 Since \(u \in V_T\), we know that \(u\) is adapted. Then, due to the definition of \(\mathbb{D}_s^H\) [see (3)], we have

\[
\mathbb{D}_s^H X_s = \int_0^s \phi(s - r) \left( \int_0^r D_r f(\theta, \eta(\theta))d\theta \right) d\theta + \left( \int_0^s \phi(s - r) D_r u_\theta d\theta \right) \delta B^H(\theta)
\]

\[
+ \int_0^s \phi(s - \theta) u_\theta d\theta
\]

\[
= \int_0^T \phi(s - r) \left( \int_0^r D_r f(\theta, \eta(\theta))d\theta \right) d\theta + \int_0^T \phi(s - r) \left( \int_0^s D_r u_\theta \delta B^H(\theta) \right) d\theta \\
+ \int_0^s \phi(s - \theta) u_\theta d\theta.
\]

Proof of Theorem 33 We follow the similar discussion as in the proof of Theorem 8 [3]. Here, we only give the sketch of the proof.
First, we mention that since \( u \in \mathcal{V}_T \), we have \( \mathbb{E}|u_s|^2 + \mathbb{E}|\mathcal{D}^H u_s|^2 + \mathbb{E}|\mathcal{D}^H \mathcal{D}^H u_s|^2 \leq C \), for all \( s, \tau, \tau_1, \tau_2 \), where \( C \) is a suitable constant.

Similar to the discussion as in the proof of Theorem 8 [3], we can assume that \( \psi, \frac{\partial \psi}{\partial t}, \frac{\partial \psi}{\partial x}, \frac{\partial^2 \psi}{\partial x^2} \) are bounded. Set \( t_i = \frac{it}{n}, 0 \leq i \leq n \). Then

\[
\psi(t, X_t) - \psi(0, X_0) = \sum_{i=0}^{n-1} \left[ \psi(t_{i+1}, X_{t_{i+1}}) - \psi(t_i, X_{t_{i+1}}) + \psi(t_i, X_{t_{i+1}}) - \psi(t_i, X_t) \right]
\]

\[
= \sum_{i=0}^{n-1} \left[ \frac{\partial}{\partial t} \psi(t_i, X_{t_{i+1}})(t_{i+1} - t_i) + \frac{\partial}{\partial x} \psi(t_i, X_{t_{i+1}})(X_{t_{i+1}} - X_t) \right]
\]

\[
+ \frac{1}{2} \sum_{i=0}^{n-1} \frac{\partial^2}{\partial x^2} \psi(t_i, X_{t_{i+1}})(X_{t_{i+1}} - X_t)^2
\]

\[
= \sum_{i=0}^{n-1} \left[ \frac{\partial}{\partial t} \psi(t_i, X_{t_{i+1}})(t_{i+1} - t_i) + \frac{\partial}{\partial x} \psi(t_i, X_{t_{i+1}}) \int_{t_i}^{t_{i+1}} f(s, \eta(s))ds \right]
\]

\[
+ \frac{1}{2} \sum_{i=0}^{n-1} \frac{\partial^2}{\partial x^2} \psi(t_i, X_{t_{i+1}})(X_{t_{i+1}} - X_t)^2
\]

where \( \tilde{t}_i \in [t_i, t_{i+1}] \) and \( \tilde{X}_{t_i} \) denotes a random intermediate point between \( X_{t_i} \) and \( X_{t_{i+1}} \). Since

\[
\frac{\partial}{\partial x} \psi(t_i, X_{t_{i+1}}) \int_{t_i}^{t_{i+1}} u_s \delta \mathcal{H}^H (s) = \int_{t_i}^{t_{i+1}} \frac{\partial}{\partial x} \psi(t_i, X_{t_{i+1}})u_s \delta \mathcal{H}^H (s)
\]

\[
+ \langle \mathcal{D}^H \left( \frac{\partial}{\partial x} \psi(t_i, X_{t_{i+1}}) \right), u1_{[t_i, t_{i+1}]} \rangle_T
\]

[for \( \langle \cdot, \cdot \rangle_T \), see (2)]. Observe that from our assumption, \( \frac{\partial}{\partial x} \psi(t_i, X_{t_{i+1}})u \in \mathbb{L}^{1,2}(\mathcal{H}) \) and all the terms in the above inequality are square integrable. Moreover,

\[
\langle \mathcal{D}^H \left( \frac{\partial}{\partial x} \psi(t_i, X_{t_{i+1}}) \right), u1_{[t_i, t_{i+1}]} \rangle_T = \langle \frac{\partial^2}{\partial x^2} \psi(t_i, X_{t_{i+1}}) \int_{0}^{t_i} \mathcal{D}^H f(\theta, \eta(\theta))d\theta, u1_{[t_i, t_{i+1}]} \rangle_T
\]

\[
+ \langle \frac{\partial^2}{\partial x^2} \psi(t_i, X_{t_{i+1}})u1_{[0, t_i]}, u1_{[t_i, t_{i+1}]} \rangle_T + \langle \frac{\partial^2}{\partial x^2} \psi(t_i, X_{t_{i+1}}) \int_{0}^{t_i} \mathcal{D}^H u_\theta \delta \mathcal{H}^H (\theta), u1_{[t_i, t_{i+1}]} \rangle_T.
\]

Now, we use the following several steps to proof our theorem.
Step 1 The term
\[
\frac{1}{2} \sum_{i=0}^{n-1} \frac{\partial^2}{\partial x^2} \psi(t_i, \bar{X}_{t_i})(X_{t_{i+1}} - X_{t_i})^2
\]
converges to 0 in \(L^1(\Omega, \mathcal{F}, P)\) as \(n \to \infty\). In fact,
\[
\left| \frac{1}{2} \sum_{i=0}^{n-1} \frac{\partial^2}{\partial x^2} \psi(t_i, \bar{X}_{t_i})(X_{t_{i+1}} - X_{t_i})^2 \right|
\leq \left| \sum_{i=0}^{n-1} \frac{\partial^2}{\partial x^2} \psi(t_i, \bar{X}_{t_i}) \left( \int_{t_i}^{t_{i+1}} f(\theta, \eta(\theta)) d\theta \right)^2 \right|
+ \sum_{i=0}^{n-1} \left| \frac{\partial^2}{\partial x^2} \psi(t_i, \bar{X}_{t_i}) \left( \int_0^{t_i} u_\theta \delta B^H(\theta) \right)^2 \right|.
\]
Then, our result holds because of Proposition 7 \([3]\) as well as
\[
\mathbb{E} \sum_{i=0}^{n-1} \left( \int_{t_i}^{t_{i+1}} f(\theta, \eta(\theta)) d\theta \right)^2 \leq \frac{t}{n} \int_0^t \mathbb{E} |f(\theta, \eta(\theta))|^2 d\theta \to 0, \text{ as } n \to \infty.
\]

Step 2 The term
\[
\sum_{i=0}^{n-1} \frac{\partial}{\partial t} \psi(t_i, X_{t_{i+1}})(t_{i+1} - t_i) \to \int_0^t \frac{\partial}{\partial t} \psi(s, X_s) ds, \text{ in } L^1(\Omega, \mathcal{F}, P) \text{ as } n \to \infty,
\]
by the dominate convergence theorem and the continuity of \(\frac{\partial \psi}{\partial t}\).

Step 3 The term
\[
\sum_{i=0}^{n-1} \frac{\partial}{\partial x} \psi(t_i, X_{t_i}) \int_{t_i}^{t_{i+1}} f(s, \eta(s)) ds
\]
\[
\to \int_0^t \frac{\partial}{\partial x} \psi(s, X_s) f(s, \eta(s)) ds, \text{ in } L^1(\Omega, \mathcal{F}, P) \text{ as } n \to \infty,
\]
by the dominate convergence theorem and the continuity of \(\frac{\partial \psi}{\partial x}\).

Step 4 The term
\[
\sum_{i=0}^{n-1} \left( \frac{\partial^2}{\partial x^2} \psi(t_i, X_{t_i}) \int_0^{t_i} D^H f(\theta, \eta(\theta)) d\theta, u_1_{[t_i, t_{i+1}]} \right)_T
\]
\[
\to \int_0^t \frac{\partial^2}{\partial x^2} \psi(s, X_s) u_s \left( \int_0^T \phi(s - r) \left( \int_0^s D_r f(\theta, \eta(\theta)) d\theta \right) dr \right) ds
\]
in $L^1(\Omega, \mathcal{F}, P)$ as $n \to \infty$. Indeed

$$
\int_0^{t_i} \langle \int_0^{t_i} D^H f(\theta, \eta(\theta)) d\theta, u_{1_{[t_i, t_{i+1}]}} \rangle_T = \int_0^{t_i} \left( \int_0^{t_i} D^H f(\theta, \eta(\theta)) d\theta \right) u_s \phi(s - \mu) ds d\mu.
$$

Then

$$
\mathbb{E} \left[ \sum_{i=0}^{n-1} \left( \frac{\partial^2}{\partial x^2} \psi(t_i, X_{t_i}) \right) \int_0^{t_i} D^H f(\theta, \eta(\theta)) d\theta, u_{1_{[t_i, t_{i+1}]}} \right]_T
$$

$$
- \int_0^{t_i} \frac{\partial^2}{\partial x^2} \psi(s, X_s) u_s \left( \int_0^{t} \phi(s - r) \left( \int_0^T D^H f(\theta, \eta(\theta)) d\theta \right) d\mu \right) ds
$$

$$
= \mathbb{E} \left[ \sum_{i=0}^{n-1} \int_0^{t_i} \frac{\partial^2}{\partial x^2} \psi(t_i, X_{t_i}) u_s \left( \int_0^T \phi(s - r) \left( \int_0^T D^H f(\theta, \eta(\theta)) d\theta \right) d\mu \right) ds
$$

$$
- \sum_{i=0}^{n-1} \int_0^{t_i} \frac{\partial^2}{\partial x^2} \psi(s, X_s) u_s \left( \int_0^T \phi(s - r) \left( \int_0^T D^H f(\theta, \eta(\theta)) d\theta \right) d\mu \right) ds \right]
$$

$$
\to 0, \text{ as } n \to \infty
$$

by the dominate convergence theorem and the continuity of $\frac{\partial^2 \psi}{\partial x^2}$. Observing that

$$
\sum_{i=0}^{n-1} \int_0^{t_i} \frac{\partial^2}{\partial x^2} \psi(t_i, X_{t_i}) u_s \left( \int_0^T \phi(s - r) \left( \int_0^T D^H f(\theta, \eta(\theta)) d\theta \right) d\mu \right) ds
$$

$$
\leq M \int_0^T \left| u_s \right| \left( \int_0^T \phi(s - r) \left( \int_0^T D^H f(\theta, \eta(\theta)) d\theta \right) d\mu \right) ds
$$

and

$$
\mathbb{E} \left[ \int_0^T \left| u_s \right| \left( \int_0^T \phi(s - r) \left( \int_0^T D^H f(\theta, \eta(\theta)) d\theta \right) d\mu \right) ds \right] < \infty
$$

**Step 5** Analogously to the Step 4, we get

$$
\sum_{i=0}^{n-1} \int_0^{t_i} \frac{\partial^2}{\partial x^2} \psi(t_i, X_{t_i}) u_{1_{[0, t_i]}}, u_{1_{[t_i, t_{i+1}]}} \rangle_T \to \int_0^T \frac{\partial^2}{\partial x^2} \psi(s, X_s) u_s \left( \int_0^s u_\theta \phi(s - \theta) d\theta \right) ds,
$$
in $L^1(\Omega, \mathcal{F}, P)$ as $n \to \infty$.

**Step 6** The term

$$
\sum_{i=0}^{n-1} \left( \frac{\partial^2}{\partial x^2} \psi(t_i, X_{t_i}) \int_0^{t_i} D^H u_0 \delta B^H(\theta), u_{1_{[t_i, t_{i+1}]}} \right)_T
$$

\[
\to \int_0^t \frac{\partial^2}{\partial x^2} \psi(s, X_s) u_s \left( \int_0^T \phi(s-r) \left( \int_0^s D^H r u_0 \delta B^H(\theta) \right) dr \right) ds
\]

in $L^1(\Omega, \mathcal{F}, P)$, as $n \to \infty$. Indeed, we can adapt the discussion of *step 3* in the proof of Theorem 8 [3], by using $E|u_s|^2 + E|D^H u_s|^2 + E|D^H u_1|^2 \leq C$, for all $s, \tau, \tau_1, \tau_2$, where $C$ is a suitable constant.

**Step 7** The term

$$
\sum_{i=0}^{n-1} \frac{\partial}{\partial x} \psi(t_i, X_{t_i}) u_{1_{[t_i, t_{i+1}]}} \to \frac{\partial}{\partial x} \psi(\cdot, X) u \quad \text{in} \quad L^1(\Omega, \mathcal{F}, P; |\mathcal{H}|).
$$

In fact

$$
E \left\| \sum_{i=0}^{n-1} \frac{\partial}{\partial x} \psi(t_i, X_{t_i}) u_{1_{[t_i, t_{i+1}]}} - \frac{\partial}{\partial x} \psi(\cdot, X) u \right\|^2_{|\mathcal{H}|}
$$

$$
= E \sum_{i=0}^{n-1} \sum_{j=0}^{t_{i+1}-t_i} \int_{t_i}^{t_{i+1}} \int_{t_j}^{t_{i+1}} \left| \left( \frac{\partial}{\partial x} \psi(t_i, X_{t_i}) - \frac{\partial}{\partial x} \psi(s, X_s) \right) u_s \right| \left( \frac{\partial}{\partial x} \psi(t_i, X_{t_i}) - \frac{\partial}{\partial x} \psi(r, X_r) \right) u_r \phi(s-r) ds dr.
$$

which converges to 0 by the dominated convergence theorem and the continuity of $\frac{\partial \psi}{\partial x}$.

Note that for $u \in \mathcal{V}_T$ and $\frac{\partial \psi}{\partial x}, \frac{\partial^2 \psi}{\partial x^2}$ bounded, we have $\frac{\partial \psi}{\partial x} u \in D^{1,2}(|\mathcal{H}|) \subset Dom(\delta)$. Consequently, for $F \in \mathcal{T}_T$, we have

$$
\lim_{n \to \infty} E \left[ F \sum_{i=0}^{t_{i+1}} \frac{\partial}{\partial x} \psi(t_i, X_{t_i}) u_s \delta B^H(s) \right] = E \left[ F \int_0^t \frac{\partial}{\partial x} \psi(s, X_s) u_s \delta B^H(s) \right]
$$

On the other hand, from steps 1–6, we know that $\sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} \frac{\partial}{\partial x} \psi(t_i, X_{t_i}) u_s \delta B^H(s)$ converges in $L^1(\Omega, \mathcal{F}, P)$ to

\[\circ\text{ Springer}\]
\[
\psi(t, X_t) - \psi(0, X_0) + \int_0^t \frac{\partial}{\partial s} \psi(s, X_s) ds + \int_0^t \frac{\partial}{\partial x} \psi(s, X_s) f(s, X_s) ds \\
+ H(2H - 1) \int_0^t \frac{\partial^2}{\partial x^2} \psi(s, X_s) u_s \left( \int_0^T |s - r|^{2H-2} \left( \int_0^s D_r f(\vartheta, \eta(\vartheta)) d\vartheta \right) dr \right) ds \\
+ H(2H - 1) \int_0^t \frac{\partial^2}{\partial x^2} \psi(s, X_s) u_s \left( \int_0^T |s - r|^{2H-2} \left( \int_0^s D_r u_{\theta} \delta B^H(\theta) d\vartheta \right) dr \right) ds \\
+ H(2H - 1) \int_0^t \frac{\partial^2}{\partial x^2} \psi(s, X_s) u_s \left( \int_0^s u_{\theta} |s - \theta|^{2H-2} d\theta \right) ds,
\]

as \( n \to \infty \), which allows to complete the proof. \(\square\)

In particular, we have the following corollary.

**Corollary 35** Let \( f : [0, T] \to \mathbb{R} \) and \( g : [0, T] \to \mathbb{R} \) be deterministic continuous functions. If

\[
X_t = X_0 + \int_0^t g_s ds + \int_0^t f_s \delta B^H(s), \ t \in [0, T],
\]

and \( \psi \in C^{1,2}([0, T] \times \mathbb{R}) \), then we have

\[
\psi(t, X_t) = \psi(0, X_0) + \int_0^t \frac{\partial}{\partial s} \psi(s, X_s) ds + \int_0^t \frac{\partial}{\partial x} \psi(s, X_s) dX_s \\
+ \frac{1}{2} \int_0^t \frac{\partial^2}{\partial x^2} \psi(s, X_s) \left( \frac{d}{ds} \| f \|^2_s \right) ds, \ t \in [0, T].
\]

**Proof** Since \( f, g \) are deterministic function, they satisfy the condition of Theorem 33. Then from (64) and Remark 34, we have

\[
\psi(t, X_t) = \psi(0, X_0) + \int_0^t \frac{\partial}{\partial s} \psi(s, X_s) ds + \int_0^t \frac{\partial}{\partial x} \psi(s, X_s) dX_s \\
+ \int_0^t \frac{\partial^2}{\partial x^2} \psi(s, X_s) f_s \left( \int_0^s f_{\theta} \phi(s - \theta) d\theta \right) ds.
\]
On the other hand
\[
\frac{d}{ds} \|f\|_2^2 = 2 \int_0^s \int_0^s \phi(u - v) f_u f_v dv du = 2 \int_0^s \phi(u - s) f_u du,
\]

which completes our proof. \(\square\)

Lemma 36 If \(Y \in \tilde{V}_T^\alpha\) and \(\psi\) is a Lipschitz function, then \(\psi(Y) \in \tilde{V}_T^\alpha\).

Proof We remark first that, since \(Y \in \tilde{V}_T^\alpha\), there exists a sequence \(\{Y_n\}_{n=1}^\infty \subset V_T\) such that
\[
\lim_{n \to \infty} \int_0^T \frac{d}{dt} \left( \frac{1}{2} \alpha^{-1} E |Y_n(t) - Y(t)|^2 \right) dt = 0.
\]

Let us set
\[
\psi\delta(x) = \int_{\mathbb{R}} \psi(x - \delta u) \rho(u) du, \quad x \in \mathbb{R},
\]

where \(\delta > 0\) and \(\rho(u) = \frac{1}{\sqrt{2\pi}} e^{-u^2/2}, \quad u \in \mathbb{R}\). We know that
\[
|\psi\delta(x) - \psi(x)| \leq \int_{\mathbb{R}} |\psi(x - \delta u) - \psi(x)| \rho(u) du \leq \int_{\mathbb{R}} K \delta |u| \rho(u) du \leq K \delta \sqrt{2/\pi},
\]

where \(K\) is a Lipschitz constant of \(\psi\). Now, we have
\[
\int_0^T \frac{d}{dt} \left( \frac{1}{2} \alpha^{-1} E |\psi\delta(Y_n(t)) - \psi(Y(t))|^2 \right) dt
\]
\[
\leq 2 \int_0^T \frac{d}{dt} \left( \frac{1}{2} \alpha^{-1} E |\psi\delta(Y_n(t)) - \psi(Y_n(t))|^2 \right) dt + 2 \int_0^T \frac{d}{dt} \left( \frac{1}{2} \alpha^{-1} E |\psi(Y_n(t)) - \psi(Y(t))|^2 \right) dt
\]
\[
\leq \frac{4}{\pi} \int_0^T K^2 \delta^2 t^{2\alpha-1} dt + 2 K^2 \int_0^T t^{2\alpha-1} E |Y_n(t) - Y(t)|^2 dt.
\]

Consequently,
\[
\int_0^T \frac{d}{dt} \left( \frac{1}{2} \alpha^{-1} E |\psi\delta(Y_n(t)) - \psi(Y(t))|^2 \right) dt \to 0, \quad \text{for } n \to \infty, \quad \delta \to 0,
\]

and the proof is completed by showing that \(\psi\delta(Y_n) \in V_T\), for all \(n\) and \(\delta > 0\).
First, it is obvious that $\psi_\delta \in C^\infty$ and
\[ |\psi_\delta(x) - \psi_\delta(y)| \leq \int |\psi(x - \delta u) - \psi(y - \delta u)| \rho(u) du \leq K |x - y|, \]
which implies that \( |\frac{d}{dx} \psi_\delta(x)| \leq K \). Moreover, a straightforward computation shows
that the derivatives \( \frac{d^2}{dx^2} \psi_\delta(x) \) and \( \frac{d^3}{dx^3} \psi_\delta(x) \) have polynomial growth. Recalling that
$Y_n$ belongs to $\mathcal{V}_T$, we complete the proof. \qed

References

1. Alòs, E., Mazet, O., Nualart, D.: Stochastic calculus with respect to fractional Brownian motion with
Hurst parameter lesser than $\frac{1}{2}$. Stoch. Process. Appl. 86, 121–139 (2000)
2. Alòs, E., Mazet, O., Nualart, D.: Stochastic calculus with respect to Gaussian processes. Ann. Probab.
29, 766–801 (2001)
3. Alòs, E., Nualart, D.: Stochastic integration with respect to the fractional Brownian motion. Stochastics
75, 129–152 (2003)
4. Barbu, V.: Nonlinear Semigroups and Differential Equations in Banach Spaces. Noordhoff International
Publishing, Leiden (1976)
5. Bender, C.: Explicit solutions of a class of linear fractional BSDEs. Syst. Control Lett. 54, 671–680
(2005)
6. Biagini, F., Hu, Y., Øksendal, B., Sulem, A.: A stochastic maximal principle for processes driven by
fractional Brownian motion. Stoch. Process. Appl. 100, 233–253 (2002)
7. Biagini, F., Hu, Y., Øksendal, B., Zhang, T.: Stochastic Calculus for Fractional Brownian Motion and
Applications. Springer, London (2006)
8. Decreusefond, L., Üstünel, A.S.: Stochastic analysis of the fractional Brownian motion. Potential Anal.
10, 177–214 (1998)
9. Duncan, T.E., Hu, Y., Pasik-Duncan, B.: Stochastic calculus for fractional Brownian motion I. Theory.
SIAM J. Control Optim. 38, 582–612 (2000)
10. Hu, Y.: Integral Transformations and Anticipative Calculus for Fractional Brownian Motions, vol. 175.
Memoirs of the American Mathematical Society (2005)
11. Hu, Y., Øksendal, B.: Fractional white noise calculus and applications to finance. Infin. Dimens. Anal.
Quantum Probab. Relat. Top. 6, 1–32 (2003)
12. Hu, Y., Peng, S.: Backward stochastic differential equation driven by fractional Brownian motion.
SIAM J. Control Optim. 48, 1675–1700 (2009)
13. Ma, J., Protter, P., Yong, J.: Solving forward–backward stochastic differential equations explicitly—a
four step scheme. Probab. Theor. Relat. Fields 98, 339–359 (1994)
14. Memin, J., Mishura, Y., Valkeila, E.: Inequalities for the moments of Wiener integrals with respect to
fractional Brownian motions. Stat. Probab. Lett. 55, 421–430 (2001)
15. Mishura, Y.: Stochastic Calculus for Fractional Brownian Motion and Related Processes. Springer,
Berlin (2007)
16. Nualart, D.: The Malliavin Calculus and Related Topics, 2nd edn. Springer, Berlin (2006)
17. Nualart, D., Pardoux, E.: Stochastic calculus with anticipating integrands. Probab. Theor. Relat. Fields
78, 535–581 (1988)
18. Pardoux, E., Peng, S.: Backward stochastic differential equations and quasilinear parabolic partial
differential equations. In: Rozovski, B.L., Sowers, R.B. (eds.) Stochastic PDE and Their Applications.
LNCIS 176, pp. 200–217. Springer, Berlin (1992)
19. Pardoux, E., Răşcanu, A.: Backward stochastic differential equations with subdifferential operators and
related variational inequalities. Stoch. Process. Appl. 76, 191–215 (1998)
20. Pardoux, E., Răşcanu, A.: Stochastic differential equations, Backward SDEs, Partial differential
equations (to appear)
21. Young, L.C.: An inequality of the Höder type connected with Stieltjes integration. Acta Math. 67,
251–282 (1936)