DISTORTION GROWTH FOR ITERATIONS OF DIFFEOMORPHISMS OF THE INTERVAL

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Abstract. We obtain several results on the distortion asymptotics for the iterations of diffeomorphisms of the interval extending the recent work of Polterovich and Sodin.

1. Main results

We consider the groups Diff\(_N^1([0,1])\), 1 \(\leq N \leq +\infty\), of all \(C^N\)-smooth diffeomorphisms of the interval \([0,1]\) into itself fixing the endpoints 0, 1. We associate to every \(f \in \text{Diff}_0^1([0,1])\) its growth sequence,

\[
\Gamma_n(f) = \max \left( \max_{x \in [0,1]} |(f^n)'(x)|, \max_{x \in [0,1]} |(f^{-n})'(x)| \right), \quad n \in \mathbb{N},
\]

where \(f^n, n \geq 1\), is the \(n\)-th iteration of \(f\), and \(f^{-n}, n \geq 1\), is the \(n\)-th iteration of the inverse diffeomorphism \(f^{-1}\). The asymptotics of the growth sequence does not change under conjugations: for \(g \in \text{Diff}_0^1([0,1])\),

\[
c(g)\Gamma_n(g^{-1}fg) \leq \Gamma_n(f) \leq C(g)\Gamma_n(g^{-1}fg), \quad n \geq 1.
\]

This asymptotics is a basic dynamic invariant (see \[3\]). G. D’Ambra and M. Gromov proposed in \[1, 7.10.C\] to study the behavior of the growth sequences for various classes of diffeomorphisms on smooth manifolds. Recently, L. Polterovich and M. Sodin \[8\] obtained several interesting results on the growth sequences for diffeomorphisms in \(\text{Diff}_0^2([0,1])\). In particular, they established

The growth gap theorem (\[8\] Theorem 1.7]). If \(f \in \text{Diff}_0^2([0,1])\), then either

\[
\lim_{n \to \infty} \frac{\log \Gamma_n(f)}{n} > 0
\]

or

\[
\limsup_{n \to \infty} \frac{\Gamma_n(f)}{n^2} < \infty.
\]

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For other results on the growth sequences of diffeomorphisms see [7], [8], [2].

This paper is devoted to two problems related to the result of Polterovich and Sodin. First, we would like to get more information on the behavior of the growth sequences than that contained in the cited theorem, possibly for smoother diffeomorphisms \( f \), in terms of the local properties of \( f \). Another problem is to get analogs to the growth gap effect for diffeomorphisms of smoothness between \( C^1 \) and \( C^2 \).

To formulate our results we need to introduce a decomposition of the set of fixed points of a diffeomorphism. Let \( 1 \leq N \leq \infty \), and let \( f \in \text{Diff}^n_0([0,1]) \). Denote by \( E(f) \) the (closed) set of fixed points of \( f \); \( \{0,1\} \subset E(f) \). Consider the subsets of \( E(f) \):

\[
E_1(f) = \{ x \in E(f) : f'(x) \neq 1 \},
E_k(f) = \{ x \in E(f) : f'(x) = 1, f''(x) = \ldots = f^{(k-1)}(x) = 0, f^{(k)}(x) \neq 0 \}, \quad 1 < k \leq N.
\]

For \( N = \infty \) we consider also

\[
E_{\infty}(f) = \{ x \in E(f) : f'(x) = 1, f''(x) = \ldots = 0 \},
\]

and for \( N < \infty \) we consider

\[
E_N^0(f) = \{ x \in E(f) : f'(x) = 1, f''(x) = \ldots = f^{(N)}(x) = 0 \}.
\]

For \( N = \infty \) we have

\[
E(f) = \bigsqcup_{1 \leq s \leq \infty} E_s(f),
\]

and for \( N < \infty \) we have

\[
E(f) = \bigsqcup_{1 \leq s \leq N} E_s(f) \sqcup E_N^0(f),
\]

Set \( V = \max_{x \in E(f)} |\log f'(x)| \geq 0; V > 0 \) if and only if \( E_1(f) \neq \emptyset \).

We use the notation \( a(n) \sim b(n), n \to \infty \), if \( \lim_{n \to \infty} a(n)/b(n) = 1 \);

\( a(n) \asymp b(n), n \to \infty \), if \( 0 < c \leq a(n)/b(n) \leq C < \infty \).

Let \( f \in \text{Diff}^1_0([0,1]) \). It is known (see [8] Appendix) that if \( E_1(f) \neq \emptyset \), then

\[
\log \Gamma_n(f) \sim n V, \quad n \to \infty.
\]

Otherwise, if \( E_1(f) = \emptyset \), then

\[
\log \Gamma_n(f) = o(n), \quad n \to \infty.
\]
Furthermore (see [8]), for every \( f \in \text{Diff}^1_0([0,1]) \) different from the identity map,
\[
\sum_{n \geq 1} \frac{1}{\Gamma_n(f)} < \infty,
\]
and hence,
\[
\limsup_{n \to \infty} \frac{\log \Gamma_n(f)}{\log n} \geq 1.
\]

**Theorem 1.** Let \( f \in \text{Diff}^\infty_0([0,1]) \), and let \( E_1(f) = \emptyset \).

(A) If \( E_2(f) \neq \emptyset \), then
\[
\Gamma_n(f) \asymp n^2, \quad n \to \infty. \tag{1.1}
\]

(B) If \( E_2(f) = \emptyset \), \( E_\infty(f) \neq \emptyset \), then
\[
\Gamma_n(f) = o(n^2), \quad n \to \infty. \tag{1.2}
\]

(C) Finally, if \( k \geq 3 \), \( E_s(f) = \emptyset \), \( 1 < s < k \), \( E_\infty(f) = \emptyset \), \( E_k \neq \emptyset \), then
\[
\Gamma_n(f) \asymp n^{k/(k-1)}, \quad n \to \infty.
\]

Of course, any \( f \in \text{Diff}^\infty_0([0,1]) \) with \( E_1(f) = \emptyset \) satisfies one and only one condition among (A)–(C).

A version of this result for \( f \in \text{Diff}^2_0([0,1]) \) claims that if \( E_1(f) = \emptyset \), \( E_2(f) \neq \emptyset \), then (1.1) holds, and if \( E_1(f) = E_2(f) = \emptyset \), then (1.2) holds.

The principal part of Theorem 1 is the part (B); the parts (A) and (C) are rather standard; the lower estimates there follow, for example, from the description of the behavior of the iteration sequences \( \{f^n(x)\}_{n \geq 1} \) near a fixed point of \( f \) given in a book of Yu. Lyubich [5, Section 2.6]; the upper estimate in (A) follows from the growth gap theorem of [8].

A natural question is now whether additional smoothness conditions on \( f \in \text{Diff}^\infty_0([0,1]) \) may permit us to improve the asymptotics in the part (B) of Theorem 1. To answer this question in the negative, we fix a sequence \( \{\varepsilon_k\} \) of positive numbers tending to 0, and a non-quasianalytic Carleman class \( C\{M_n\} \),
\[
C\{M_n\} = \{ f \in C^\infty([0,1]) : |f^{(n)}(x)| \leq C(f)^n M_n, \ x \in [0,1] \},
\]
where \( f^{(n)} \) is the \( n \)-th derivative of \( f \), and \( \log M_n \) is an increasing convex sequence, \( \lim_{n \to \infty} M_n = +\infty \). Recall that such a class is non-quasianalytic if for every two closed subintervals \( I, J \) of \([0,1]\) with \( I \subset \text{Int} \ J \), there exists \( f \in C\{M_n\} \) with \( 0 \leq f \leq 1 \), \( f|I \equiv 1 \), and \( \text{supp} \ f \subset \).
J. The Denjoy–Carleman theorem (see, for example, [4, 6]) claims that the Carleman class $C\{M_n\}$ is non-quasianalytic if and only if
$$\sum_{n \geq 1} M_n^{-1/n} < \infty.$$

**Theorem 2.** There exists $f \in \text{Diff}_0^\infty([0, 1]) \cap C\{M_n\}$ such that $E(f) = E_\infty(f) = \{0, 1\}$, and
$$\Gamma_n(f) \geq \varepsilon_n n^2, \quad n \geq 1.$$

Thus, we can conclude that some predictions from the Outlook of [8] are true as demonstrated by Theorem 1; nevertheless, the “optimistic scenario” from the Outlook is disproved by Theorem 2.

**Question 1.** Suppose that $f \in \text{Diff}_0^\infty([0, 1])$, and that $E(f) = E_\infty(f) = \{0, 1\}$. What additional conditions should one impose on $f$ to guarantee that
$$\limsup_{n \to \infty} \frac{\log \Gamma_n(f)}{\log n} = 1? \quad (1.3)$$

The previous theorem shows that no additional smoothness conditions will work. On the other hand, a bounded oscillation condition is sufficient for the property (1.3) to hold. For simplicity, we consider here only a model case.

**Theorem 3.** Let $f \in \text{Diff}_0^\infty([0, 1])$ be different from the identity map, and let $E(f) = E_\infty(f) = \{0, 1\}$. Assume that $\varphi(x) = f(x) - x > 0$, $x \in (0, 1)$. Suppose that for every $\varepsilon > 0$ there exists $C_\varepsilon > 0$ such that
$$\varphi(y) \leq C_\varepsilon (\varphi(x))^{1-\varepsilon}, \quad 0 < y < x < \frac{1}{2}, \quad (1.4)$$
and the same inequality holds for $1/2 < x < y < 1$. Then
$$\limsup_{n \to \infty} \frac{\log \Gamma_n(f)}{\log n} = 1.$$

Note that the condition (1.4) guarantees, by a result of F. Sergeraert [9], that the germ of $f$ at 0 imbeds in a flow of germs of $C^\infty$-smooth diffeomorphisms $f_\sigma$, $\sigma > 0$, with $f_\sigma(x) - x$ flat at 0 (and an analogous imbedding holds at the point 1). However, such an imbedding by itself does not provide, apparently, any new information on the behavior of the growth sequence $\Gamma_n(f)$, see the remark at the end of Section 3.

One more natural question here is whether there is the growth gap effect for two (possibly non-commuting) diffeomorphisms $f, g$ in the
group $\text{Diff}^\infty_0([0, 1])$. Denote by $[f, g]$ the subgroup of $\text{Diff}^\infty_0([0, 1])$ generated by $f$ and $g$; given $h \in [f, g]$ denote by $|h|_w$ the distance from the identity map to $h$ in the word metric, that is the length of the shortest representation of $h$ by $f, f^{-1}, g, g^{-1}$. Consider the growth sequence

$$\Gamma_n(f, g) = \max_{h \in [f, g]} \max_{|h|_w \leq n, x \in [0, 1]} |h'(x)|, \quad n \in \mathbb{N}.$$ 

**Question 2.** Suppose that $\log \Gamma_n(f, g) = o(n), n \to \infty$. Is it possible that the value

$$\limsup_{n \to \infty} \frac{\log \log \Gamma_n(f, g)}{\log n}$$

is positive? is equal to $1$?

Next, we deal with diffeomorphisms of lower smoothness. The class $\text{Diff}^1_0([0, 1])$ is too large for any kind of the growth gap effect to be present. Namely, given a sequence $\{\varepsilon_k\}$ of positive numbers tending to $0$, one can construct $f \in \text{Diff}^1_0([0, 1])$ such that $E_1(f) = E^0_0(f) = \{0, 1\}$, and

$$\log \Gamma_n(f) \geq \varepsilon_n n, \quad n \geq 1.$$

The situation is different for the groupes

$$\text{Diff}^{1,\alpha}_0([0, 1]) = \text{Diff}^1_0([0, 1]) \cap C^{1,\alpha}, \quad 0 < \alpha \leq 1,$$

of diffeomorphisms with the derivative in the Lipschitz $\alpha$ class, where

$$C^{1,\alpha} = \{f \in C^1 : |f'(x) - f'(y)| \leq c(f)|x - y|^{\alpha}, \quad x, y \in [0, 1]\}.$$

(If $f \in \text{Diff}^{1,\alpha}_0([0, 1])$, then automatically $f^{-1} \in C^{1,\alpha}$, and hence $f^{-1} \in \text{Diff}^{1,\alpha}_0([0, 1])$.)

For $\alpha = 1$ we have just the growth gap of [8]: if $f \in \text{Diff}^{1,1}_0([0, 1])$, $E_1(f) = \emptyset$, then $\Gamma_n(f) = O(n^2), n \to \infty$.

In the case $0 < \alpha < 1$, we obtain a weaker form of the growth gap effect.

**Theorem 4.** Let $0 < \alpha < 1$. (A) If $f \in \text{Diff}^{1,\alpha}_0([0, 1])$, and if $E_1(f) = \emptyset$, then

$$\log \Gamma_n(f) = O(n^{1-\alpha}), \quad n \to \infty.$$

(B) There exists $f_\alpha \in \text{Diff}^{1,\alpha}_0([0, 1])$ such that $E_1(f_\alpha) = \emptyset$, and

$$\lim_{n \to \infty} \frac{\log \log \Gamma_n(f_\alpha)}{\log n} = 1 - \alpha. \quad (1.5)$$

The plan of the paper is as follows. We start Section 2 with an analysis of the behavior of the iteration sequences $\{f^n(x)\}_{n \geq 1}$ resembling the results contained in [3, Section 2.6]. In contrast to the asymptotical estimates of [5] we obtain global uniform estimates. Then, using
this analysis, we prove Theorems 4 and 3. Theorem 2 is proved in Section 3. Finally, in Section 4 we prove Theorem 4. Our proof of the part (A) of Theorem 4 imitates that of the original growth gap theorem of Polterovich and Sodin [8, Section 2].

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2. Proofs of Theorems 4 and 3

2.1. First, we make several simple remarks. Since $f$ is a diffeomorphism, we have

$$0 < c(f) \leq f'(x) \leq C(f),$$

$$-\infty < \log c(f) \leq \log f'(x) \leq \log C(f), \quad x \in [0, 1].$$

Let $x_1 \in (0, 1), x_k = f(x_{k-1}), 1 < k \leq n$. We are going to estimate the value

$$\Phi(n, x_1) = \Phi(n, x_1, f) = \sum_{k=1}^{n} \log f'(x_k),$$

(and the supremum of $|\Phi|$ for $x_1 \in (0, 1)$) as a function of $n$. This permits us to evaluate $a_n(f)$,

$$a_n(f) = \max_{[0, 1]} \log[(f^n)'(x)] \quad (2.1)$$

(and $a_n(f^{-1})$ is evaluated analogously).

Now we assume that $f$ is $C^2$-smooth, $E_1(f) = \emptyset$. Replacing, if necessary, $f$ by $x \mapsto \Delta f(x/\Delta)$, (and changing the domain $D = [0, 1]$ to $D = [0, \Delta]$, we can guarantee that

$$\max_{D} |\varphi''| \leq 1. \quad (2.2)$$

Next we consider the set $A$ of the closed subintervals $I$ of $D$ such that $E(f) \cap I = \partial I$. If $I \in A$, $\{x_k\}_{1 \leq k \leq n} \cap I \neq \emptyset$, then $\{x_k\}_{1 \leq k \leq n} \subset I$. We have $f' = 1$ at the endpoints of $I$.

2.2. In the following three lemmas for simplicity of notations we assume that the left end point of $I$ is 0; without loss of generality we assume that $f(x) \geq x, x \in I$. Set $\varphi(x) = f(x) - x, x \in I = [0, b]$. We have $\varphi(x) > 0$ for $x \in (0, b), \varphi(0) = \varphi'(0) = \varphi(b) = \varphi'(b) = 0, \max_I \varphi \leq \Delta, \max_I |\varphi''| \leq 1,$

$$-1 < c(f) \leq \min_I \varphi' \leq C(f).$$

Hence,

$$|\varphi'(x) - \log(1 + \varphi'(x))| \leq c(f)[\varphi'(x)]^2, \quad x \in I. \quad (2.3)$$
Furthermore, for $x, y \in I$,
\[
|\log f(x) - \log f(y)| = \left| \log \frac{1 + \varphi'(x)}{1 + \varphi'(y)} \right| \\
\leq c(\varphi)|\varphi'(x) - \varphi'(y)| \leq c(\varphi)|x - y|. \quad (2.4)
\]
Since $\varphi(x) = \varphi'(x) = 0$, $x \in \{0, b\}$, by (2.2) we have
\[
\varphi(x) \leq \min\{x^2/2, (b - x)^2/2\}, \quad x \in I. \quad (2.5)
\]

**Lemma 1.** (A) In this situation, we have
\[
\left| \log \frac{\varphi(x_n)}{\varphi(x_1)} - \Phi(n - 1, x, x_1) \right| \leq C(f)|I|,
\]
where $|I|$ is the length of $I$. (B) If $H > 1$, and we have $\varphi < 1/100$, $H^{-1} - 1 < \varphi' < H$ on $I$, then we can choose $C(f) = C(H)$ in the previous inequality.

Thus, to estimate $\Phi(n, x_1)$, we need only to know the asymptotics of $\varphi(x_n)$.

**Proof.** (A) As a consequence of (2.3) we have
\[
0 \leq x - (\varphi(x))^{1/2} \leq x + (\varphi(x))^{1/2} \leq b, \quad x \in I.
\]
First we verify that for every $x \in I$, $\theta \in [0, 1]$,
\[
|\varphi'(x + \theta(\varphi(x))^{1/2})| \leq 3(\varphi(x))^{1/2}. \quad (2.6)
\]
Otherwise, (2.2) would imply that for some $x \in I$, the function $\varphi'$ is of constant sign on the interval $I_x = [x - (\varphi(x))^{1/2}, x + (\varphi(x))^{1/2}]$, and
\[
|\varphi'(y)| > (\varphi(x))^{1/2}, \quad y \in I_x.
\]
If $\varphi'(x)$ is positive, then $\varphi(x - (\varphi(x))^{1/2}) < 0$, and if $\varphi'(x)$ is negative, then $\varphi(x + (\varphi(x))^{1/2}) < 0$, which is impossible. Thus, (2.6) is proved.

Inequality (2.6) implies that
\[
|\varphi'(x)| \leq 3(\varphi(x))^{1/2}, \quad x \in I, \quad (2.7)
\]
and that
\[
\frac{\varphi(x + \theta(\varphi(x))^{1/2})}{\varphi(x)} \leq 4, \quad x \in I, \quad \theta \in [0, 1]. \quad (2.8)
\]

Fix $x \in I$. Suppose that $\varphi(x) < 1/100$. For some $\theta, \theta_1 \in [0, 1]$,
\[
\int_x^{x + \varphi(x)} \frac{\varphi'(t)}{\varphi(t)} dt = \frac{\varphi'(x + \theta \varphi(x))}{\varphi(x + \theta \varphi(x))} = \frac{\varphi(x)}{\varphi(x) + \theta \varphi(x) \varphi'(x + \theta \theta_1 \varphi(x))}.
\]
By \((2.7)\) and \((2.8)\) we obtain
\[
|\varphi'(x + \theta \varphi(x))| \leq 3(\varphi(x + \theta \varphi(x)))^{1/2} \leq 6(\varphi(x))^{1/2} \leq \frac{3}{5}.
\]
Furthermore,
\[
\left| \frac{\varphi(x)}{\varphi(x) + \theta \varphi(x) \varphi'(x + \theta \varphi(x))} - 1 \right| = \left| \frac{1}{1 + \theta \varphi'(x + \theta \varphi(x))} - 1 \right| \leq \frac{5}{2} \theta |\varphi'(x + \theta \varphi(x))| \leq 15(\varphi(x))^{1/2}.
\]
Hence,
\[
\left| \int_x^{x+\varphi(x)} \frac{\varphi'(t)}{\varphi(t)} dt - \varphi'(x + \theta \varphi(x)) \right| \leq 90 \varphi(x),
\]
and for some \(\theta_1 \in [0, 1]\),
\[
\left| \int_x^{x+\varphi(x)} \frac{\varphi'(t)}{\varphi(t)} dt - \varphi'(x) \right| \leq 90 \varphi(x) + |\varphi''(x + \theta \varphi(x))| \cdot \theta \varphi(x) \leq 91 \varphi(x). \quad (2.9)
\]
Suppose now that \(x \in \mathcal{E} = \{y \in D : |\varphi(y)| \geq 1/100\}\). The set \(\mathcal{E}\) is a compact subset of \(D\) disjoint with \(E(f)\). Furthermore, for \(y \in \mathcal{E}\), the interval \([y, y + \varphi(y)]\) does not intersect \(E(f)\). Hence, the function \(y \mapsto \varphi(y + \varphi(y))\) does not vanish on \(\mathcal{E}\), and as a consequence,
\[
|\varphi(y + \varphi(y))| \geq A(f) > 0, \quad y \in \mathcal{E}.
\]
Therefore,
\[
\left| \int_x^{x+\varphi(x)} \frac{\varphi'(t)}{\varphi(t)} dt - \varphi'(x) \right| \leq \left| \log \frac{\varphi(x + \varphi(x))}{\varphi(x)} \right| + |\varphi'(x)| \leq \max\{\log \max_D |100\varphi|, \log \max_D |\varphi/A(f)|\} + \max_D |\varphi'| \leq c(f). \quad (2.10)
\]
Next, for \(x = x_1, \ldots, x_{n-1}\) we sum up the inequalities \((2.9)\) (if \(x_k \notin \mathcal{E}\)) or \((2.10)\) (if \(x_k \in \mathcal{E}\)). Since
\[
\sum_{k \geq 1} \varphi(x_k) \leq |I|,
\]
we have
\[
\text{card} \left\{ \{x_k\}_{1 \leq k \leq n} \cap \mathcal{E} \right\} \leq 100|I|.
\]
Using (2.3) we obtain
\[
\left| \log \frac{\varphi(x_n)}{\varphi(x_1)} - \Phi(n - 1, x_1) \right| \leq \sum_{k=1}^{n-1} \left| \int_{x_k}^{x_k + \varphi(x_k)} \frac{\varphi'(t)}{\varphi(t)} dt - \log(1 + \varphi'(x_k)) \right|
\]
\[
\leq c(f) \sum_{k=1}^{n-1} |\varphi'(x_k)|^2 + c(f) \sum_{k=1}^{n-1} \varphi(x_k) + c(f) |I|.
\]
Again using (2.7), we conclude that
\[
\left| \log \frac{\varphi(x_n)}{\varphi(x_1)} - \Phi(n - 1, x_1) \right| \leq C(f) |I|.
\]
(2.11)

(B) In this case, \( I \cap \mathcal{E} = \emptyset \), \( c(f) = c(H) \) in (2.3), and the above argument shows that \( C(f) = C(H) \) in (2.11). \( \square \)

If we have more information on the size of \( \varphi'' \), then we can extend the estimate (2.8) to bigger intervals.

**Lemma 2.** Let \( 0 < \delta \leq 1 \). Fix \( x \in I \). Suppose that either (A) \( x \leq b/2 \) and
\[
\max_{[0,2x]} |\varphi''(y)| \leq \delta,
\]
or (B) \( x > b/2 \) and
\[
\max_{[2x-b,b]} |\varphi''(y)| \leq \delta.
\]
Then \( I_x = [x - \delta^{-1/2}(\varphi(x))^{1/2}, x + \delta^{-1/2}(\varphi(x))^{1/2}] \subset I \), and
\[
\varphi(y) \leq 4\varphi(x), \quad y \in [x, x + \delta^{-1/2}(\varphi(x))^{1/2}].
\]

**Proof.** The arguments in the cases (A) and (B) are analogous, and we restrict ourselves to the case (A). By (2.12),
\[
\varphi(y) \leq \delta y^2/2, \quad y \in I,
\]
and hence, \( I_x \subset [0,2x] \subset I \). Let us verify that
\[
\varphi'(t) \leq 3\delta^{1/2}(\varphi(x))^{1/2}, \quad t \in I_x.
\]
(2.13)
Otherwise, we would obtain that
\[
\varphi'(y) > \delta^{1/2}(\varphi(x))^{1/2}, \quad y \in I_x,
\]
and hence \( \varphi(x - \delta^{-1/2}(\varphi(x))^{1/2}) < 0 \), which is impossible.
Inequality (2.13) implies our assertion:
\[
\varphi(y) - \varphi(x) \leq (y - x) \max_{x \leq t \leq y} \varphi'(t) \leq \delta^{-1/2}(\varphi(x))^{1/2} \cdot 3\delta^{1/2}(\varphi(x))^{1/2} = 3\varphi(x), \quad x \leq y \leq x + \delta^{-1/2}(\varphi(x))^{1/2}.
\]

The following lemma shows that \(x_n\) as a function of \(n\) behaves asymptotically as the function inverse to the integral of \(1/\varphi\). For related results see [5, Appendix to Chapter 2, Theorem 3].

**Lemma 3.** Suppose that
\[
\max_{[x_1, x_n]} |\varphi'(x)| \leq \frac{1}{2},
\]
Then
\[
\frac{2}{3} \leq \frac{1}{n-1} \int_{x_1}^{x_n} \frac{dt}{\varphi(t)} \leq 2.
\]

**Proof.** For every \(x \in [x_1, x_{n-1}], \theta \in [0, 1]\), there exists \(\theta_1 \in [0, 1]\) such that
\[
\left| \frac{\varphi(x + \theta \varphi(x))}{\varphi(x)} - 1 \right| = \theta |\varphi'(x + \theta \theta_1 \varphi(x))| \leq \frac{1}{2}.
\]
Since \(x_{k+1} = x_k + \varphi(x_k)\), we get
\[
\frac{2}{3} \leq \int_{x_k}^{x_{k+1}} \frac{dt}{\varphi(t)} \leq 2.
\]
Summing up for \(k = 1, \ldots, n - 1\) we obtain the assertion. \(\square\)

**2.3.** We return to the analysis of the behavior of our sequence \({x_k}_{1 \leq k \leq n}\) on the interval \(I = [a, b] \in \mathfrak{A}\). If \(|f' - 1| < 1/2\) on \(I\), then we put \(J(I) = \emptyset\). Otherwise, we choose the minimal closed subinterval \(J(I) = [c, d] \subset I\) such that
\[
|f'(x) - 1| < \frac{1}{2}, \quad x \in I \setminus J(I) = I^l \cup I^r,
\]
where
\[
I^l = [a, c], \quad I^r = (d, b].
\]
Since \(E_1(f) = \emptyset\), the set
\[
J = \bigcup_{I \in \mathfrak{A}} J(I)
\]
is a compact subset of \(\mathcal{D}\) having empty intersection with \(E(f)\). Indeed, if \(y \in \partial J\), then there are \(y_k \in \partial J(I_k), I_k \in \mathfrak{A}\), such that \(y_k \to y\) as \(k \to \infty\). Hence, \(|f'(y) - 1| \geq 1/2\), and \(y \notin E(f)\). Therefore, for some
Since $f(x) - x$ is of constant sign on $I$, and $|x_{k+1} - x_k| \geq \rho(f)$ for $x_k \in J(I)$, our sequence $\{x_k\}_{1 \leq k \leq n}$ may contain at most $N(f) = 1 + \Delta/\rho(f)$ points of $J(I)$. 

Furthermore, if $J(I) = \emptyset$, then condition (2.14) holds for our sequence $\{x_k\}_{1 \leq k \leq n} \subset I$. Otherwise, we have one of the following four possibilities:

(I) either $\{x_k\}_{1 \leq k \leq n} \subset I^l$ or $\{x_k\}_{1 \leq k \leq n} \subset I^r$, and condition (2.14) holds for the sequence $\{x_k\}_{1 \leq k \leq n}$;

(II) either $\{x_k\}_{1 \leq k \leq n} \subset I^l \cup J(I)$ or $\{x_k\}_{1 \leq k \leq n} \subset J(I) \cup I^r$. Then dropping at most $N(f)$ points of the sequence $\{x_k\}$ we return to the situation in (I) without changing the asymptotics of $\Phi$;

(III) $\{x_k\}_{1 \leq k \leq n} \cap I^l \neq \emptyset$, $\{x_k\}_{1 \leq k \leq n} \cap I^r \neq \emptyset$, $\{x_k\}_{1 \leq k \leq n} \cap J(I) \neq \emptyset$. Once again we assume that $f(x) \geq x$ on $I$. Applying Lemma 1, we get

$$\left| \log \frac{\varphi(x_n)}{\varphi(x_k)} - \Phi(n-k, x_k) \right| \leq c(f), \quad 1 \leq k \leq n.$$ 

If $j$ is the minimal index such that $x_j \in J(I)$, then

$$\log \frac{\varphi(x_n)}{\varphi(x_j)} \leq \log \frac{\max_D \varphi}{\rho(f)} \leq c(f),$$

and we are able to drop all the points $x_k, j \leq k \leq n$, and return to (I) without worsening the asymptotics of $\Phi$;

(IV) $\{x_k\}_{1 \leq k \leq n} \cap I^l \neq \emptyset$, $\{x_k\}_{1 \leq k \leq n} \cap I^r \neq \emptyset$, $\{x_k\}_{1 \leq k \leq n} \cap J(I) = \emptyset$. Replacing $x_1$ by a suitable $x'_1 \in [x_1, x_2]$, and defining $x'_{k+1} = f(x'_k)$, $k \geq 1$, we can guarantee that $\{x'_k\}_{1 \leq k \leq n} \cap J(I) \neq \emptyset$. Now, the sequence $\{x'_k\}_{1 \leq k \leq n}$ satisfies the conditions of (II) or (III), and

$$|\Phi(n, x_1) - \Phi(n, x'_1)| \leq c(f)|I|.$$ 

(Here we use (2.4).)

Thus, from now on we may assume that the assertion of Lemma 3 holds for $\{x_k\}_{1 \leq k \leq n}$.

2.4. Proof of Theorem. The parts (A) and (C) are rather standard; the lower estimates follow, for example, from the description of the behavior of the iteration sequences $\{f^n(x)\}_{n \geq 1}$ near a fixed point of $f$ given in [5, Section 2.6]; to get the upper estimates we can use an argument similar to that in the part (B).
(B) Fix $A \geq 1$. Let $\{x_k\}_{1 \leq k \leq n} \subset I = [a, b] \in A$, suppose that $\varphi \geq 0$ on $I$, and let $\{x_k\}_{1 \leq k \leq n}$ and $I$ satisfy the conditions of Lemma 3. Using Lemma 2 and the fact that $\varphi''$ vanishes on $\partial I$ and is uniformly continuous on $D$, we obtain that for $\text{dist}(x, \partial I) < \varepsilon(A, f)$,

$$\varphi(t) \leq 4 \varphi(x), \quad x \leq t \leq x + A(\varphi(x))^{1/2}. \quad (2.15)$$

If $\text{dist}(x_1, \partial I) \geq \varepsilon(A, f)$, then $\varphi(x_1) \geq \beta(A, f) > 0$, and then, by Lemma 1,

$$\Phi(n - 1, x_1) \leq c(A, f).$$

Otherwise, $(2.15)$ holds for $x = x_1$. Now, if $x_n \leq x_1 + A(\varphi(x_1))^{1/2}$, then $\varphi(x_n) \leq 4 \varphi(x_1)$, and again by Lemma 1,

$$\Phi(n - 1, x_1) \leq c f.$$

Finally, if $x_n > x_1 + A(\varphi(x_1))^{1/2}$, then by Lemma 3 and by $(2.15)$,

$$2n \geq \int_{x_1}^{x_n} \frac{dt}{\varphi(t)} \geq \int_{x_1}^{x_1 + A(\varphi(x_1))^{1/2}} \frac{dt}{\varphi(t)} \geq \frac{A}{4(\varphi(x_1))^{1/2}}.$$

Hence,

$$\log \frac{\varphi(x_n)}{\varphi(x_1)} - 2 \log n \leq \log \left( \frac{64 \varphi(x_n)}{A^2} \right) \leq \log \left( \frac{64}{A^2} \max_I \varphi \right).$$

It remains to apply once again Lemma 1 to conclude that

$$\Phi(n - 1, x_1) \leq 2 \log n + c(f) - 2 \log A.$$

Since $A$ is arbitrary, our proof is completed. \hfill \Box

2.5. Proof of Theorem 3 We argue as in the previous proof. Instead of Lemma 2, we use the following result:

**Lemma 4.** If $\varphi$ vanishes at 0 with all its derivatives, $N \geq 1$, $x > 0$, and

$$\varphi(y) \leq (\varphi(x))^{1 - 1/(2N)} \leq 1, \quad 0 < y < x, \quad (2.16)$$

then

$$\varphi(t) \leq C(N, \varphi) \varphi(x), \quad x \leq t \leq x + (\varphi(x))^{1/N}. \quad (2.17)$$

Thus, condition $(2.16)$ permits us to extend the estimate of Lemma 2 to much bigger intervals.
Proof. The Gorny-Cartan inequalities (see, for example, [6, 6.4.IV]) claim that for every \(1 \leq k \leq M\), and \(F \in C^M[0,1]\), there exists \(C(M)\) such that

\[
|F^{(k)}(t)| \leq C(M) \max_{[0,1]} |F(s)|^{1-k/M} \times \max \left[ \max_{[0,1]} |F(s)|, \max_{[0,1]} |F^{(M)}(s)| \right]^{k/M}, \quad 0 \leq t \leq 1,
\]

where \(F^{(k)}\) is the \(k\)-th derivative of \(F\).

Applying these inequalities to \(F(t) = \varphi(xt)\), \(M = 4N^2\), \(1 \leq k \leq N\), and using (2.16) and the fact that for some \(C(N, \varphi)\),

\[
|\varphi^{(4N^2)}(t)| \leq C(N, \varphi), \quad 0 \leq t \leq 1,
\]

we conclude that

\[
\varphi^{(k)}(x) = x^{-k}|F^{(k)}(1)| \leq C(N, \varphi)x^{-k}(\varphi(x))^{(1-1/(2N))(1-N/(4N^2))} \leq C_1(N, \varphi) \cdot (\varphi(x))^{1-1/N}, \quad 1 \leq k \leq N,
\]

for some \(C_1(N, \varphi) \geq 1\).

If \(\varphi(t) > C\varphi(x)\) for some \(x \leq t \leq x + (\varphi(x))^{1/N}\), then by induction we get a sequence of points \(x \leq t_{k+1} \leq t_k \leq t_0 = t\), \(1 \leq k < N\), with

\[
\varphi^{(k)}(t_k) > [C - kC_1(N, \varphi)](\varphi(x))^{1-k/N}, \quad 1 \leq k \leq N.
\]

Fix \(C(N, \varphi) = NC_1(N, \varphi) + \max_{D} |\varphi^{(N)}|\). Then we get

\[
\varphi^{(N)}(t_N) > \max_{D} |\varphi^{(N)}|,
\]

which is impossible. This contradiction proves (2.17) with our choice of \(C(N, \varphi)\).

Now we fix \(N\) and \(x_1 \in D\), and obtain as in the proof of Theorem \(\Pi\) (B) that either

\[
\Phi(n-1, x_1) \leq c(N, f)
\]

or

\[
2n \geq \int_{x_1}^{x_n} \frac{dt}{\varphi(t)} \geq \int_{x_1}^{x_1 + (\varphi(x_1))^{1/N}} \frac{dt}{\varphi(t)} \geq \frac{1}{C(N, \varphi)(\varphi(x_1))^{1-1/N}}.
\]

In the latter case,

\[
\Phi(n-1, x_1) \leq \log \frac{\varphi(x_n)}{\varphi(x_1)} + c(\varphi) \leq \frac{N}{N-1} \log n + c_1(N, \varphi).
\]

Since \(N\) is arbitrary, our proof is completed. \(\Box\)
3. Proof of Theorem 2

Without loss of generality we assume that \( \{\varepsilon_k\} \) is a decreasing sequence. We are going to construct a function \( f, f(x) = x + \varphi(x) \) with non-negative \( \varphi \in \mathcal{C}\{M_n\} \) vanishing at 0 with all its derivatives, and points \( x_1^{(n)} \in [0, 1] \) such that

\[
\Phi(n, x_1^{(n)}) \geq \log(\varepsilon_n n^2), \quad n \geq 1. \tag{3.1}
\]

The function \( \varphi \) will be of the form \( t \mapsto \gamma_k(t - z_k)^2 + \omega_k \) on disjoint intervals \( J_k \) tending to 0 with \( \omega_k \ll \gamma_k \to 0 \). In this way, we can make \( \max_{x \in [0,1]} \Phi(n, x) \) grow almost as fast as for \( \varphi(x) = x^2 \), and still keep \( \varphi \) smooth and flat at 0.

We choose closed intervals \( J_k \), and numbers \( z_k, w_k \) such that

\[
[z_k, z_k + w_k] \subset \text{Int} J_k \subset J_k \subset I_k = \left( \frac{1}{2k}, \frac{1}{2k - 1} \right).
\]

Then we find non-negative \( u_k \in \mathcal{C}\{M_n\} \), supp \( u_k \subset I_k \), with \( u_k \mid J_k \equiv 1 \). Choose \( 1/100 > \gamma_k \searrow 0 \), \( k \to \infty \), such that for every sequence \( \{\theta_k\}, \theta_k \in [0, 1] \), the sums

\[
\sum_{k \geq 1} \gamma_k (\cdot - z_k)^2 u_k
\]

and

\[
\sum_{k \geq 1} \theta_k \gamma_k u_k
\]

belong to \( \mathcal{C}\{M_n\} \) and vanish at 0 with all their derivatives. Finally, we choose \( n_k \) such that

\[
\gamma_k w_k \geq \varepsilon_{n_k}, \quad n_k \geq \frac{3}{\gamma_k w_k}. \tag{3.2}
\]

Now we put \( \varphi_k(x) = \gamma_k(x - z_k)^2, f_k(x) = x + \varphi_k(x) \). The functions \( f_k \) satisfy the conditions of Lemma 11 (B) with \( H = 2 \). We start with \( x_1^{(k),m} = z_k + m, 0 < m \leq w_k/3 \), and continue by \( x_{s+1}^{(k),m} = x_s^{(k),m} + \varphi_k(x_s^{(k),m}) \), until \( x_{N(m)+1}^{(k),m} > z_k + 2w_k/3 \). Since \( \gamma_k < 1/100 \), we have

\[
x_{N(m)+1}^{(k),m} \leq z_k + \frac{2w_k}{3} + \frac{1}{100} \left( \frac{2w_k}{3} \right)^2 < z_k + w_k.
\]
Using Lemma 3 we obtain that
\[
\frac{2}{3} N(m) \leq \int_m^{x_{N(m)+1} - z_k} \frac{dt}{\gamma_k t^2} \leq \int_m^{w_k} \frac{dt}{\gamma_k t^2} = \frac{1}{\gamma_k} \left[ \frac{1}{m} - \frac{1}{w_k} \right] \leq \frac{2}{3} \frac{1}{\gamma_k m}. \tag{3.4}
\]

In particular, by (3.3),
\[
N(w_k/3) \leq \frac{3}{\gamma_k w_k} \leq n_k.
\]

By continuity of \( \varphi_k \), for every \( n \geq n_k \) there exists \( m = m(n) \in (0, w_k/3] \) such that \( n = N(m(n)) \). Furthermore, by Lemma 1 (B),
\[
\Phi(n, x_1^{(k), m(n)}, f_k) \geq \log \frac{\gamma_k (x_1^{(k), m(n)} - z_k)^2}{\gamma_k (x_1^{(k), m(n)} - z_k)^2} + C \geq \log \frac{\gamma_k w_k^2}{\gamma_k (m(n))^2} + C_1
\]
(by (3.2) and (3.4))
\[
\geq \log[\varepsilon_{nk}^2 (N(m(n)))^2] + C_1 = \log[\varepsilon_{nk}^2 n^2] + C_1, \quad n \geq n_k,
\]
with \( C, C_1 \) independent of \( k \) and \( \{ \varepsilon_n \} \).

Next we choose \( 0 < \omega_k \leq \gamma_k \) such that for \( f_k^*, f_k^*(x) = \omega_k + \gamma_k (x - z_k)^2 \),
we still have
\[
\Phi(n, x_1^{(k), m(n)}, f_k^*) \geq \log (\varepsilon_{nk}^2 n^2) + C_1 - 1, \quad n_k \leq n \leq n_{k+1}.
\]

It remains to define
\[
\varphi_0(x) = \sum_{k \geq 1} \left[ \gamma_k (x - z_k)^2 + \omega_k \right] u_k(x),
\]
and add to \( \varphi_0 \) a non-negative function in \( C\{M_n\} \) vanishing at 0 with all its derivatives, with support on
\[
(0, 1) \setminus \bigcup_{k \geq 1} [z_k, z_k + w_k],
\]
which is strictly positive on
\[
(0, 1) \setminus \bigcup_{k \geq 1} J_k,
\]
to get \( \varphi \). Now, (3.1) is verified for all sufficiently big \( n \). Finally we change \( \varphi \) on a small interval inside \( I_1 \) to get (3.1) for all \( n \geq 1 \).

Remark. We can use the above construction to produce a flow \( g^1 \) of germs of \( C^\infty \)-smooth diffeomorphisms with \( g^1(x) - x \) flat at 0, such that
\[
\Gamma_n(g^1) \geq \varepsilon_n n^2, \quad n \geq 1. \tag{3.5}
\]
To do this, consider the equation

\[
\begin{cases}
\frac{\partial F}{\partial t}(t,x) = \varphi(F(t,x)), & x, t \geq 0, \\
F(0,x) = x, & x \geq 0.
\end{cases}
\]

Then \(g^t = F(t, \cdot), t \geq 0\), are the germs of \(C^\infty\)-smooth diffeomorphisms, and the germs \(g^t(x) - x\) are flat at 0. Put \(g = g^1\). An easy argument shows that for \(x > 0, n \geq 1\), such that \(g^n(x)\) is sufficiently small, we have

\[
\int_0^{g^n(x)} \frac{dt}{\varphi(t)} = \int_{F(0,x)}^{F(n,x)} \frac{dt}{\varphi(t)} = \int_0^n \frac{\partial_s F(s,x)}{\varphi(F(s,x))} ds = n.
\]

and hence

\[
(g^n)'(x) = \frac{\varphi(g^n(x))}{\varphi(x)}.
\]

Starting from these equalities, and using the same argument as in the previous proof, we conclude that (3.5) holds.

4. Proof of Theorem 4

(A) We use the scheme proposed in [8]. First we prove two lemmas: a Denjoy-type statement and a convex analysis result.

Lemma 5. Let \(f \in \text{Diff}^{1,\alpha}_0([0,1])\). If \(J \subset [0,1]\) is a closed interval such that \(f(J) \cap J = \emptyset\), then for every \(n \in \mathbb{N}\) and every \(x, y \in J\),

\[
\left| \log \frac{(f^n)'(x)}{(f^n)'(y)} \right| \leq c(f)n^{1-\alpha}. \tag{4.1}
\]

Proof. Since \(0 < \min_{[0,1]} f' \leq \max_{[0,1]} f' < \infty, f \in C^{1,\alpha}\), we have

\[
\left| \log \frac{(f^n)'(x)}{(f^n)'(y)} \right| \leq c(f) \sum_{k=1}^n |f'(x_k) - f'(y_k)| \leq c_1(f) \sum_{k=1}^n |x_k - y_k|^{\alpha},
\]

where \(x_1 = x, y_1 = y, x_k = f(x_{k-1}), y_k = f(y_{k-1}), k > 1\). Next, the intervals \([x_k, y_k]\) are disjoint, and hence \(\sum_{k=1}^n |x_k - y_k| \leq 1\). By the Hölder inequality we obtain

\[
\sum_{k=1}^n |x_k - y_k|^{\alpha} \leq \left( \sum_{k=1}^n |x_k - y_k| \right)^\alpha \left( \sum_{k=1}^n 1 \right)^{1-\alpha} \leq n^{1-\alpha},
\]

and (4.1) follows. \(\square\)
Lemma 6 (compare to Lemma 2.3 of [8]). Let \( \{a_n\}_{n \geq 0} \) be a sequence of non-negative numbers. Suppose that the following almost convexity inequality

\[
2a_n - a_{n-1} - a_{n+1} \leq K \exp[-a_n + K_1 n^{1-\alpha}], \quad n \geq 1,
\]

holds for some positive \( K, K_1 \), \( a_0 = 0 \), and

\[
\liminf_{n \to \infty} \frac{a_n}{n} = 0.
\]

Then

\[
a_n \leq An^{1-\alpha}, \quad n \geq 1,
\]

where \( A = A(K, K_1) \).

Proof. First we fix \( A \) so large that for every \( n \geq 1 \),

\[
A[(n+1)^{1-\alpha} - n^{1-\alpha}] \geq AC(\alpha)n^{-\alpha} \geq 2K \sum_{k>n} \exp[-\left(\frac{A}{2} - K_1\right)k^{1-\alpha}],
\]

(4.5)

If (4.5) does not hold, then we could find the smallest integer \( n \) such that

\[
a_n \leq An^{1-\alpha}, \quad a_{n+1} > A(n+1)^{1-\alpha},
\]

(4.6)

and hence,

\[
a_{n+1} - a_n > A[(n+1)^{1-\alpha} - n^{1-\alpha}].
\]

(4.7)

Then either

\[
a_k \geq \frac{A}{2}k^{1-\alpha}, \quad k > n,
\]

(4.8)

or we could find the smallest integer \( m > n \) such that

\[
a_m \geq \frac{A}{2}m^{1-\alpha}, \quad a_{m+1} < \frac{A}{2}(m+1)^{1-\alpha}.
\]

(4.9)

In the latter case, by (4.2),

\[
(a_{n+s+1} - a_{n+s}) - (a_{n+s+2} - a_{n+s+1}) \leq
\]

\[
K \exp[-\left(\frac{A}{2} - K_1\right)(n + s + 1)^{1-\alpha}], \quad 0 \leq s < m - n.
\]

(4.10)

Now, by (4.5) and by (4.7),

\[
a_{n+s+2} - a_{n+s+1} \geq
\]

\[
A[(n+1)^{1-\alpha} - n^{1-\alpha}] - K \sum_{k>n} \exp[-\left(\frac{A}{2} - K_1\right)k^{1-\alpha}] \geq
\]

\[
\frac{A}{2}[(n+1)^{1-\alpha} - n^{1-\alpha}], \quad 0 \leq s < m - n,
\]

(4.10)
and

\[ a_{m+1} = a_{n+1} + \sum_{k=n+1}^{m} [a_{k+1} - a_k] \]

\[ \geq A(n + 1)^{1-\alpha} + (m - n) \frac{A}{2} [(n + 1)^{1-\alpha} - n^{1-\alpha}] . \]

Since the function \( x \mapsto x^{1-\alpha} \) is concave, we obtain

\[ a_{m+1} \geq \frac{A}{2} (n + 1)^{1-\alpha} + \frac{A}{2} \sum_{k=n+1}^{m} [(k + 1)^{1-\alpha} - k^{1-\alpha}] = \frac{A}{2} (m + 1)^{1-\alpha}, \]

and we get a contradiction to (4.9). Thus, (4.8) is established. Arguing as above we derive from (4.10) that

\[ a_k \geq \frac{A}{2} [(n + 1)^{1-\alpha} - n^{1-\alpha}] + o(1), \quad k \to \infty, \]

that contradicts to our condition (4.3). Thus, (4.6) does not hold, and (4.4) is proved. □

Now, the assertion (A) of Theorem 4 follows just as in [8]. To make our proof self-contained, we repeat here the argument from [8].

Consider the sequence \( a_n(f) \) defined by (2.1). Fix \( n \geq 1 \), and choose \( x_1 \in [0, 1] \) such that for \( x_{k+1} = f(x_k), 1 \leq k \leq n \), we have

\[ a_n(f) = \log[(f^n)'(x_2)] = \sum_{k=2}^{n+1} \log f'(x_k). \]

Then

\[ a_{n+1}(f) \geq \sum_{k=1}^{n+1} \log f'(x_k), \]

\[ a_{n-1}(f) \geq \sum_{k=3}^{n+1} \log f'(x_k), \]

and as a result,

\[ 2a_n(f) - a_{n+1}(f) - a_{n-1}(f) \leq \log f'(x_2) - \log f'(x_1) \]

\[ \leq c(f) |x_2 - x_1|^\alpha \leq c(f) \left| \frac{x_2 - x_1}{x_{n+2} - x_{n+1}} \right|^\alpha = \frac{c(f)}{|(f^n)'(y)|^\alpha} \]

for some \( y \) between \( x_1 \) and \( x_2 \). By Lemma 5

\[ |(f^n)'(y)| \geq |(f^n)'(x_2)| \exp[-c(f)n^{1-\alpha}] = \exp[a_n(f) - c(f)n^{1-\alpha}], \]
and we conclude that the sequence $a_n(f)$ satisfies the condition $[1.2]$. Since $E_1(f) = 0$, this sequence satisfies also $[1.3]$, and we can apply Lemma $[6]$ to complete the proof of the part (A) of the theorem.

(B) First we fix $\beta > 0$, and consider the function

$$\varphi(x) = \varphi(\beta, x) = (x^{-1/\beta} - 1)^{-\beta} - x - x^{(\alpha+1)(\beta+1)/\beta} \sin \frac{2\pi}{x^{1/\beta}}, \quad 0 < x < 1.$$  

Since $\alpha > 0$, we have

$$\varphi(x) \sim \beta x^{(\beta+1)/\beta}, \quad x \to 0.$$

Fix a positive integer $N$. If $x_1 = N^{-\beta}$, then

$$x_2 = x_1 + \varphi(x_1) = (N - 1)^{-\beta},$$

and by induction we obtain

$$x_k = x_{k-1} + \varphi(x_{k-1}) = (N + 1 - k)^{-\beta}, \quad 1 \leq k \leq N.$$

Furthermore,

$$\varphi'(x) = (1 - x^{1/\beta})^{-\beta} - 1 - \frac{1}{\beta} x^1 (\alpha+1)(\beta+1)/\beta - 1 \sin \frac{2\pi}{x^{1/\beta}} + \frac{2\pi}{\beta} x^{\alpha(\beta+1)/\beta} \cos \frac{2\pi}{x^{1/\beta}}, \quad x > 0,$$

and

$$\varphi'(k^{-\beta}) = \left(1 - \frac{1}{k}\right)^{-\beta} - 1 - \frac{2\pi}{\beta} k^{-\alpha(\beta+1)}, \quad k \in \mathbb{N}.$$

If $\beta + 1 < 1/\alpha$, then

$$\varphi'(k^{-\beta}) \sim \frac{2\pi}{\beta} k^{-\alpha(\beta+1)}, \quad k \in \mathbb{N}, \quad k \to \infty, \quad (4.11)$$

$$\varphi'\left((k + \frac{1}{2})^{-\beta}\right) \sim -\frac{2\pi}{\beta} k^{-\alpha(\beta+1)}, \quad k \in \mathbb{N}, \quad k \to \infty.$$

In particular, we obtain that $\varphi'$ vanishes on a sequence of points $y_k$,

$$\left(k + \frac{1}{2}\right)^{-\beta} < y_k < k^{-\beta}, \quad k > 1.$$  

The formula $[4.11]$ implies that

$$\sum_{k=1}^{N} \log(1 + \varphi'(x_k)) \sim \frac{2\pi}{\beta(1 - \alpha(\beta+1))} N^{1 - \alpha(\beta+1)}, \quad N \to \infty. \quad (4.12)$$  

Next we verify that $\varphi'$ belongs to the Lipschitz $\alpha$ class.
Lemma 7. If $p > 0$, $0 < \alpha < 1$, $b \geq (p + 1)\alpha$, then the functions $g, h$, 

\[
g(x) = x^b \sin x^{-p}, \quad h(x) = x^b \cos x^{-p},
\]

belong to $\text{Lip}_\alpha[0,1]$. 

Proof. Let $0 \leq y < x \leq 1$. If $0 < x - y \leq x^{p+1}$, then 

\[
|g(x) - g(y)| \leq (x - y) \cdot \max_{y \leq t \leq x} |g'(t)| \leq c(x - y)x^{b-p-1} \leq c_1(x - y)^\alpha,
\]

where $c, c_1$ depend only on $b, p, \alpha$. Otherwise, if $y < x - x^{p+1}$, then 

\[
|g(x) - g(y)| \leq 2 \max_{y \leq t \leq x} |g(t)| \leq 2x^b \leq 2(x - y)^\alpha.
\]

The same argument works for the function $h$. \qed

Since the function $x \mapsto (1 - x^{1/\beta})^{-\beta-1}$ is $C^1$-smooth on $[0,1/2]$, we conclude that 

\[
\|\varphi'\|_{\text{Lip}_\alpha[0,1/2]} \leq K(\beta).
\]

Finally, we use the functions $\varphi(\cdot, \cdot)$, $\beta > 0$, to construct $f$ with 

\[
\lim_{N \to \infty} N^{\alpha (\beta + 1) - 1} \max_{x \in [0,1]} \Phi(N, x, f) > 0
\]

for every $\beta > 0$.

Fix $\psi \in C^\infty([0,1])$, $0 \leq \psi \leq 1$, with $\text{supp} \; \psi \subset [0,1)$, $\text{supp} \; (1 - \psi) \subset (0,1]$. Choose a sequence $1 - \beta_k \to 0$, and a sequence of disjoint intervals $I_k = [a_k, b_k] \subset [0,1]$, $k \geq 1$. For every $k \geq 1$ we define $\varphi(\cdot) = \varphi(\beta_k, \cdot)$, $\Delta = |I_k|(K(\beta_k))^{-1/\alpha}$. Given $0 < \delta < \Delta/2$ such that $\varphi'(\delta) = 0$, define 

\[
\varphi_\delta(x) = \begin{cases} 
\varphi(x), & 0 < x \leq \delta, \\
\varphi(\delta), & \delta < x \leq \Delta/2, \\
\varphi(\delta) \cdot \psi((2x - \Delta)/\Delta), & \Delta/2 \leq x \leq \Delta.
\end{cases}
\]

Then 

\[
\|\varphi_\delta'\|_{\text{Lip}_\alpha[0,\Delta]} \leq K(\beta_k) + c|\varphi(\delta)|\Delta^{-1-\alpha},
\]

and for sufficiently small $\delta$ with $\varphi'(\delta) = 0$, 

\[
\|\varphi_\delta'\|_{\text{Lip}_\alpha[0,\Delta]} \leq 2K(\beta_k).
\]

Fix such a value $\delta = \delta(k)$, and define 

\[
\varphi_k^*(x) = \frac{|I_k|}{\Delta} \varphi_\delta \left( \frac{\Delta x}{|I_k|} \right), \quad 0 \leq x \leq |I_k|.
\]

Then 

\[
\|\varphi_k^*\|_{\text{Lip}_\alpha[0,|I_k|]} \leq 2 \frac{\Delta^\alpha}{|I_k|^\alpha} K(\beta_k) \leq 2.
\]
Put
\[ f(x) = \begin{cases} 
  x + \varphi_k^*(x - a_k), & x \in I_k, \ k \geq 1, \\
  x & \text{elsewhere}.
\end{cases} \]

Then
\[ \|f'\|_{\text{Lip}, \alpha, [0,1]} \leq 4, \]
and for every \( k \geq 1 \), by (4.12) we obtain
\[ \Phi(N, a_k + N^{-\beta_k}, f) \sim \frac{2\pi}{\beta_k(1 - \alpha(\beta_k + 1))} N^{1-\alpha(\beta_k+1)}, \quad N \to \infty. \]

Hence, for every \( k \geq 1 \),
\[ \liminf_{N \to \infty} \frac{\log \log \Gamma_N(f)}{\log N} \geq 1 - \alpha(\beta_k + 1). \]

Applying the result of the part (A) we obtain that \( f \) satisfies (1.5), and the proof of Theorem 4 is completed.

**Remark.** Note that \( \varphi \) and \( \varphi' \) constructed in the part (B) of the previous proof may vanish at 0 more rapidly than any preassigned power if we take sufficiently small \( \beta > 0 \). However, \( \varphi'' \) is always unbounded near the point 0.

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