Bohmian Zitterbewegung

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Abstract

A new Bohmian quantum-relativistic model, in which from the Klein-Gordon equation a generalization of the standard Zitterbewegung arises, is explored. It is obtained by introducing a new independent time parameter, whose relative motions are not directly observable but cause the quantum uncertainties of the observables. Unlike Bohm’s original theory, the quantum potential does not affect the observable motion, as for a normal external potential, but it only determines that one relative to the new time variable, of which the Zitterbewegung of a free particle is an example. The model also involves a relativistic revision of the uncertainty principle.

KEYWORDS: Quantum Mechanics; de Broglie-Bohm interpretation; Relativity; Klein-Gordon equation; Zitterbewegung; Uncertainty principle verification; Time

1 Introduction

In the de Broglie-Bohm interpretation of the non-relativistic quantum mechanics, determinism is recovered with the introduction of a specific quantum potential [1-5]. The difficulties or even the very existence of a coherent relativistic generalization of this approach, have been questioned by many researchers [7-15]. However, one of the main problems, namely the compatibility of the non-locality with the Lorentz covariance, has been judged solvable by some authors [16-23].

In this work we begin by showing that a relativistic generalization of the original de Broglie-Bohm theory is unsatisfactory as regards the very continuity of the quantum potential between the relativistic and non-relativistic cases. Another serious limitation of the current Bohmian relativistic theories is the inability to derive the Zitterbewegung [25-26] of a free particle. After, we show that both problems can be solved by introducing a new independent time parameter, whose relative motions - such as the Zitterbewegung - are not observable but are the cause of the uncertainties of the quantum observables. The quantum potential, unlike a normal external potential, does not influence the observable motion but only affects that one relative to the new time parameter. It turns out that the Zitterbewegung of a free particle is an example of this motion because it originates from the quantum potential. A relativist revision of the uncertainty principle also derives from the studied model.

The paper is organized as follows: in section 2 we recall the solutions of the de Broglie-Bohm model for a non-relativistic free particle. In the following 3 and 4 we show the problems of every possible relativistic generalization of these motions, adopting the original
Bohm’s model. In section 5 the new Bohmian model is defined and in section 6 it is applied to study the free particle, finding a generalization of both the standard Zitterbewegung and the uncertainty principle. The last section 7 collects final considerations.

2 Bohmian solutions for a non-relativistic free particle

The de Broglie-Bohm approach [1-5] consists in replacing the wave function:

\[ \psi(\vec{r}, t) = R(\vec{r}, t) e^{iS(\vec{r}, t)/\hbar} \]  

(1)

where \( R > 0 \) and \( S \) are real functions, into the Schrödinger equation:

\[ i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \nabla^2 \psi + V \psi \]  

(2)

Two equations are obtained:

\[ \frac{\partial S}{\partial t} + \frac{(\nabla S)^2}{2m} + V - \frac{\hbar^2}{2m} \nabla^2 R = 0 \]  

(3)

\[ \frac{\partial R^2}{\partial t} + \nabla \cdot (R^2 \nabla S/m) = 0 \]  

(4)

Now, by identifying \( S \) with the Hamilton’s principal function, we have \( \nabla S = \vec{p} \), and eq. (3) represents the Hamilton-Jacobi equation with Hamiltonian:

\[ H = \frac{p^2}{2m} + V - \frac{\hbar^2}{2m} \nabla^2 R \]  

(5)

where the last addend represents a potential energy that takes into account the quantum effects:

\[ V_Q = -\frac{\hbar^2}{2m} \nabla^2 R \]  

(6)

Eq. (4), on the other hand, shows that \( R^2 \), and \( R \) itself, is a wave that follows the particle with velocity \( \vec{v} \).

According to the Bohm’s original quantum interpretation, the motion equation of the particle is:

\[ \frac{d\vec{p}}{dt} = -\vec{\nabla} V - \vec{\nabla} V_Q \]  

(7)

In the case of a free particle (\( V = 0 \)), Bohm showed the possibility of non-stationary solutions, with \( \vec{p}, V_Q \) and \( H \) variable in time [6]. However, there are also stationary solutions. To find them, let’s assume that \( \vec{p} \) is constant, making coincide with \( \vec{x} \) the rectilinear trajectory. Let’s introduce the variable \( \ell \equiv x - vt \), denoting with a dot the derivative with respect to \( \ell \). Since \( \dot{x} = \dot{v} \), we obtain \( \vec{\nabla} R = \dot{R}(\ell) \dot{v} \) and, applying the divergence operator:

\[ \nabla^2 R = \dot{R}(\ell) \]  

(8)

Substituting into the eq. (6), we obtain:

\[ -\frac{2m}{\hbar^2} V_Q = \frac{\dot{R}}{R} \]  

(9)
The accomplishment of the eq. (7) requires that \( V_Q \) be constant. Placing \( k \equiv -\frac{2m}{\hbar} V_Q \), for \( k > 0 \), i.e. \( V_Q < 0 \), eq. (9) is satisfied for:

\[
R(\ell) = Ae^{\sqrt{k} \ell} + Be^{-\sqrt{k} \ell}
\]

with \( A \) and \( B \) arbitrary constants. Nonetheless, these solutions are physically unacceptable because they diverge for \( |\ell| \to \infty \).

For \( k < 0 \), i.e. \( V_Q > 0 \), we get instead:

\[
R(\vec{r}, t) = |A \cos \sqrt{\frac{2mV_Q}{\hbar}}(x - vt)|
\]

where \( A \) is an arbitrary constant, and we have imposed, just for simplicity, the symmetry \( R(\ell) = R(-\ell) \), assuming that at the initial instant, \( t = 0 \), the particle is in the origin. The solutions (11), although not normalizable, are limited and hence can fit in a satisfactory wave packet description.

We have therefore obtained solutions where \( \vec{p}, V_Q \) and hence \( H \) are constant, then close to that one of the standard quantum mechanics. But here \( R \) depends on space and time.

In the following sections, we will seek a relativistic generalization of the previous result, using the Klein-Gordon equation and limiting ourselves to consider only non-negative values for the Hamiltonian.

### 3 First attempt of relativistic generalization

In reference to eq. (1), let’s interpret \( S \) as the Hamilton’s principal function, by writing:

\[
\vec{\nabla} S = \vec{p} = m\gamma \vec{v}
\]

\[
\frac{\partial S}{\partial t} = -H = -m\gamma c^2 - V_Q
\]

where \( m \) is the rest mass, \( \gamma \) the Lorentz factor and we have introduced, for generality, a quantum potential \( V_Q \) to be determinated.

We will start with the previous stationary conditions: \( \vec{p}, V_Q \) and \( H \) constant.

From eq. (1), taking the gradient, then the divergence, and finally dividing by \( \psi \), we obtain:

\[
\frac{\nabla^2 \psi}{\psi} = \frac{\nabla^2 R}{R} + 2 \frac{i}{\hbar} \frac{\vec{\nabla} R}{R} \cdot \vec{p} - \frac{p^2}{\hbar^2}
\]

being \( \vec{\nabla} \cdot \vec{v} = 0 \). The comparison with the Klein-Gordon equation:

\[
\nabla^2 \psi = \frac{m^2 c^2}{\hbar^2} \psi + \frac{1}{c^2} \frac{\partial^2 \psi}{\partial t^2}
\]

provides:

\[
\frac{\nabla^2 R}{R} + 2 \frac{i}{\hbar} \frac{\vec{\nabla} R}{R} \cdot \vec{p} = \frac{m^2 \gamma^2 c^2}{\hbar^2} + \frac{1}{c^2 \psi} \frac{\partial^2 \psi}{\partial t^2}
\]

where we applied the identity \( p^2 + m^2 c^2 = m^2 \gamma^2 c^2 \).

On the other hand, by deriving eq. (1) successively with respect to time, we obtain:
\[ \frac{\partial \psi}{\partial t} = \frac{\partial R}{\partial t} e^{i\frac{s}{\hbar}} + \frac{i}{\hbar} \psi (-H) \]  

from which it can be inferred:

\[ \frac{1}{\psi} \frac{\partial^2 \psi}{\partial t^2} = \frac{1}{R} \frac{\partial^2 R}{\partial t^2} - 2 \frac{i}{\hbar} \frac{1}{R} \frac{\partial R}{\partial t} - \frac{H^2}{\hbar^2} \]  

By substituting eq. (19) into eq. (16) and matching the real and imaginary parts, we obtain:

\[ H^2 = m^2 \gamma^2 c^4 + \hbar^2 \left( \frac{s}{\hbar^2} \frac{\partial^2 R}{\partial t^2} - c^2 \frac{\nabla^2 R}{R} \right) \]  

\[ \vec{p} \cdot \vec{\nabla} R + \frac{H c}{c^2} \frac{\partial R}{\partial t} = 0 \]  

Now we suppose that \( R \) is a generic wave that follows the particle with velocity \( \vec{v} \). By limiting the study to the rectilinear trajectory, which we make coincide with \( \vec{x} \), let’s introduce the variable \( \ell \equiv x - vt \), obtaining as before:

\[ \nabla R = \dot{R} (\ell) \hat{v} \]  

\[ \nabla^2 R = \ddot{R} (\ell) \]  

The derivatives with respect to the time of \( R \) give:

\[ \frac{\partial R}{\partial t} = -\dot{R} (\ell) v \]  

\[ \frac{\partial^2 R}{\partial t^2} = v^2 \ddot{R} (\ell) \]  

By substituting the last four equations in eqs. (20) and (21), we finally obtain:

\[ H^2 = m^2 \gamma^2 c^4 + \hbar^2 (v^2 - c^2) \frac{\ddot{R} (\ell)}{R} \]  

\[ v \ddot{R} (\ell) (m \gamma c^2 - H) = 0 \]  

Since we are looking for non-trivial solutions, we will assume that \( m \gamma \) and \( v \) are non-zero; moreover that \( R \) is not constant, as we found in the non-relativistic case. But in such hypotheses, eq. (27) implies \( H = m \gamma c^2 \) and therefore \( V_Q = 0 \), while in the non-relativistic case we have seen that it is admitted that it is a non-zero constant. Now, it is true that this does not imply any absurdity, since \( H \) is defined up to an arbitrary additive constant; however, the fact remains that in the context of Bohm’s interpretation one cannot assign to the quantum potential a straightforward continuity between the relativistic and non-relativistic cases, and this is not very satisfactory.

For this reason it seems appropriate to look for an alternative.
4 No alternative in the Bohm’s standard interpretation

Keeping the hypothesis that $H$ is a constant of motion, let’s try to introduce a time dependence for the momentum of the particle. Eqs. (20) and (21) do not change. For $R$, we will continue to assume that it is a wave following the particle, and therefore an arbitrary function of $\ell \equiv x - \int_0^t v(\xi) \, d\xi$; but eqs. (22), (24) and, consequently, (27) remain unchanged, so one still gets the unsatisfactory solution $V_Q = 0$.

We therefore reach a first conclusion: if there is a more satisfactory relativistic generalization of the Bohmian quantum mechanics, in it the Hamiltonian cannot be a constant of motion$^1$.

If $H$ is a function of time, the case in which the moment of the particle remains constant is still impossible if we keep the hypothesis that $R$ is an arbitrary function of $\ell \equiv x - vt$. In fact, based on eq. (26), $V_Q$ is a function of $\frac{R(t)}{R}$ and therefore also a function of $\ell$. Then, from the equation of motion:

$$\frac{d\vec{p}}{dt} = -\vec{\nabla}V_Q = -\dot{V}_Q(\ell) \hat{v} = 0$$

(28)

it follows that $V_Q$ is constant. But then, from eq. (13), $H$ should also be constant.

Finally, let us explore the possibility that $H$ and $p$ are both variable, with $\dot{v}$ constant. By assuming for $R$ an arbitrary function of $\ell \equiv x - \int_0^t v(\xi) \, d\xi$, eqs. (22), (23) and (24) do not change, while eq. (25) transforms into:

$$\frac{\partial^2 R}{\partial t^2} = v^2 \ddot{R}(\ell) - \dot{R}(\ell) \, v'(t)$$

(29)

where the apostrophe indicates a derivative with respect to time. Substituting in eq. (20) we obtain:

$$H = \sqrt{m^2 \gamma^2 c^4 - \hbar^2 \frac{e^2}{\gamma^2} \frac{\dot{R}(\ell)}{R} - \hbar^2 v' \frac{\dot{R}}{R}}$$

(30)

On the other hand, the time dependence for $H$ implies the new addend $-\frac{i}{\hbar} \frac{\partial H}{\partial t}$ to the second member of eq. (19), so, in place of (27), we now have:

$$v \frac{\dot{R}}{R} (m \gamma c^2 - H) + \frac{1}{2} \frac{\partial H}{\partial t} = 0$$

(31)

Substituting eq. (30) into eq. (31), being $\gamma' = \frac{3}{c^2} v' v'$, we arrive at the following expression:

$$(H - m \gamma c^2 - \frac{\hbar^2 v'}{4m \gamma c^2 v'} \frac{4\dot{R}}{mvR} + \frac{3h^2}{m^2 \gamma c^2} (\frac{\dot{R}}{R^2} + \frac{\ddot{R}}{R}) + \frac{h^2}{m^2 \gamma^3 v'} \frac{3\dddot{R}}{R^2} + \frac{\dddot{R}}{R}) + 2\gamma^3 = 0$$

(32)

where we admit $v' \neq 0$, in addition to the previous hypotheses of non-triviality. Eq. (32) should hold whatever the rest mass $m$ is, but it is clear that this is impossible: when $m$ increases, all the addends become small, except the last one which remains a non-zero number.

This latter failure exhausts the possibilities, for Bohm’s original quantum model, of the existence in $[1+1]$ dimension of a relativistic generalization more satisfactory than that one we found in the previous section.

$^1$This is possible if $V_Q$ is not a generalized potential, see e.g. [24].
5 The new Bohmian relativistic model

The attempt with variable $H$ and constant moment failed due to the Bohmian equation (28); one could then think of referring it to a new, independent, intrinsic moment $\vec{p}_i$, variable over time:

$$\frac{d\vec{p}_i}{dt} = -\nabla V_Q$$

(33)

This motion could coincide with the Zitterbewegung of a free particle, the famous sinusoidal motion of amplitude $\frac{\hbar}{2m\gamma c}$ and angular frequency $\frac{2m\gamma c^2}{\hbar}$ in any direction [25-26]. The Zitterbewegung emerges in the Heisenberg’s picture and the debate on its exact interpretation - and even observability - is not yet resolved [27-40]. To find this peculiar movement in a deterministic approach could clarify its nature.

First, the idea given by (33) does not work. In the sketched picture, the momentum of the particle would be:

$$\vec{p} \equiv \vec{p}_o + \vec{p}_i$$

(34)

where $\vec{p}_o$ is the constant moment (actually observed in mean value), to be substituted for $\vec{p}$ in all the equations written above, except those (7) and (28). In the one-dimensional simplification, a wave following the particle would then be an arbitrary function of $\ell \equiv x - v_o t - \int_0^t v_i(\xi) d\xi$; however, we have seen that for $R$ we must consider a dependence only on $x - v_o t$ to avoid getting the impossible eq. (32) again. But it is obvious that if $R$ does not also depend on $v_i$, eq. (33) is unable to account for the intrinsic motion. The idea of neglecting the intrinsic movement in derivatives with respect to time, due to the fact that it consists of very rapid oscillations, could work in many cases as an approximation but strictly, as we have seen at the end of the previous section, it is incorrect in $[1+1]$ dimension.

The solution here explored is the introduction of a new independent time parameter $\tau$ for the spatial coordinates $(X,Y,Z)$ of the particle. That is, we admit that they depend not only on commonly experienced time $t$, but also on another, totally independent, time parameter $\tau$. By referring, for simplicity, only to $X(t,\tau)$, let’s introduce two velocities:

$$v_o \equiv \frac{\partial X}{\partial t}$$

(35)

$$v_{ix} \equiv \frac{\partial X}{\partial \tau}$$

(36)

having so:

$$dX = v_o dt + v_{ix} d\tau$$

(37)

If we suppose that $v_o$ and $v_i$ depend, respectively, only on $t$ and $\tau$, by integrating, we get so:

$$X(t,\tau) = X(0,0) + \int_0^t v_o(\xi) d\xi + \int_0^\tau v_{ix}(\zeta) d\zeta$$

(38)

Due to the independence of $\tau$, we have to recognize that the intrinsic motion is not directly observable and is perceived as instantaneous with respect to time $t$. This does not at all mean, of course, that it does not have dramatic effects on the measurements.
of the observable quantities. As for the position, the eq. (38) allows to the particle, observed in its trajectory as a function of $t$, to jump instantly from one point of space to another, in principle arbitrarily distant, through the time dimension $\tau$. Actually, quantum mechanics is not new to this type of non-local peculiarity. In the Feynman’s "sum of histories" interpretation, for example, the physical characteristics of a displacement from $A$ to $B$, for a particle or photon, can be explained correctly only by admitting that it has traveled simultaneously all the possible trajectories from $A$ to $B$. In the rear, to make the described uncertainty in accordance with quantum mechanics, just recognize that the $\vec{r}(\tau)$ function should vary where $R^2$ is not null and we have, therefore, some probability to find the particle. In our case, in which the free particle is described by a single wave, we will have that $\vec{r}(\tau)$ has a Compton length order maximum amplitude, since this value represents the minimum spatial location. If, however, the particle is described by a wave packet, then the maximum excursion for the position will be of the order of $\lambda = \frac{\hbar}{p}$. From this point of view, hence, the motion in $\tau$ is direct cause of the quantum uncertainties.

In general, a generic wave that follows the particle will be a function of $\vec{\ell} \equiv \vec{r} - \int_0^\tau \vec{v}_o(\xi) \, d\xi - \int_0^\tau \vec{v}_i(\zeta) \, d\zeta$, setting: $\vec{r}(0,0) = 0$. The variable $\vec{\ell}$ includes the dependence on $t$ and $\tau$, having $\frac{\partial \vec{\ell}}{\partial t} = -\vec{v}_o$ and:

$$\frac{\partial \vec{\ell}}{\partial \tau} = -\vec{v}_i \quad (39)$$

For the intrinsic motion we will assume that the Restricted Relativity does not hold. This condition is consistent with the fact that, in this peculiar motion, the mass of the particle does not undergo any relativistic increase - other than that one due to the observable velocity - despite it reaches the speed $c$. That aside, the motion in $\tau$ obeys the classical laws of motion, such as conservation of energy and momentum. Considering that the mass involved in the generic intrinsic motion is $m\gamma_o$, energy conservation in $\tau$ reads:

$$V_Q + \frac{1}{2} m\gamma_o v_i^2 \equiv E_Q \quad (40)$$

independent on $\tau$. It represents the purely quantum total energy.

On the other hand, in presence of a conservative potential $V$, the following energy will be constant:

$$m\gamma_o c^2 + V = \text{const} \equiv E_C \quad (41)$$

that is the classic total energy.

Taking the $\frac{\partial}{\partial \tau}$ of eq. (40), being $\frac{\partial V_Q}{\partial \tau} = -\vec{\nabla} V_Q \cdot \vec{v}_i$ as wave in $\tau$, one finds:

$$-m\gamma_o \frac{d\vec{v}_i}{d\tau} = m\gamma_o \frac{\partial^2 \vec{\ell}}{\partial \tau^2} = -\vec{\nabla} V_Q(\vec{\ell}) \quad (42)$$

On the other hand, from eq. (41), one gets:

$$\frac{d\vec{p}_o}{dt} = -\vec{\nabla} V \quad (43)$$

So, (42) and (43) are the equations of motion to replace Bohm’s original one (7).

\(^2\)Doing everything that was possible to do, such as emitting or absorbing an arbitrary number of photons (if it is an electron) and interacting with every other particle in all possible ways, even exceeding $c$.

\(^3\)This latter is not valid because it derives from admitting $\frac{dV_Q}{dt} = \vec{\nabla} V_Q \cdot \vec{v}_o$, but, considering $V_Q$ as a wave, now we have $\frac{\partial V_Q}{\partial t} = \vec{\nabla} V_Q \cdot \vec{v}_o + \frac{\partial V_Q}{\partial t} = 0$. 
The new Bohmian model can thus be so summarized:

1. The spatial coordinates of the particle have to vary as a function of two independent temporal parameters, \( t \) and \( \tau \). Motion in \( \tau \) is omnidirectional, not directly observable and responsible for quantum uncertainties.

2. The particle is represented by the wavefunction \( \psi(\vec{r}, t, \tau) = R(\vec{r}, t, \tau) e^{iS(\vec{r}, t, \tau)} \), where \( \vec{\nabla}S = \vec{p}_o = m\gamma_o \vec{v}_o \) and \( \frac{\partial S}{\partial t} = -H = -m\gamma_o c^2 - V_Q - V \). It obeys the Klein-Gordon equation, generalized for presence of potentials.

3. The wave function module \( R \) - and therefore also \( V_Q \) and \( H \) - must be considered as wave following the particle, i.e. as function of \( \ell \). Consequentially, the Bohmian law of motion (7) must be replaced by eq. (43) for motion in \( t \) and by eq. (42) for motion in \( \tau \), for which the Restricted Relativity is not valid.

We also observe that taking the gradient of \( \psi \) and applying \( \vec{\nabla}S = \vec{p}_o \) one can easily find the guidance equation:

\[ \vec{p}_o = \hbar \text{Im} \left( \frac{\vec{\nabla}\psi}{\psi} \right) \]

6 Zitterbewegung in the new Bohmian model

Let’s study finally a free particle in the newly introduced Bohmian model. From eq. (43) we deduce that \( \vec{p}_o \) is constant. The equations referring to \( t \) time, when \( H \) is variable, were previously found:

\[
H = \sqrt{m^2\gamma_o^2 c^4 - \hbar^2 c^2 \frac{\dot{R}}{\gamma_o R}} \tag{44}
\]

\[
v_o \frac{\dot{R}}{R} (m\gamma_o c^2 - H) + \frac{1}{2} \frac{\partial H}{\partial t} = 0 \tag{45}
\]

where we remember that \( v_o \) is the observed (on average) velocity.

Placing \( \lambda^2 \equiv \frac{\hbar^2}{m^2 c^2 \gamma_o^4} \) - the square of the relativistic Compton length divided again by a Lorentz factor - eqs. (44) and (45) become:

\[
H = m\gamma_o c^2 \sqrt{1 - \lambda^2 \frac{\dot{R}}{R}} \tag{46}
\]

\[
v_o \frac{\dot{R}}{R} \left( \frac{H}{m\gamma_o c^2} - \frac{H^2}{m^2\gamma_o^2 c^4} \right) + \frac{1}{4} \frac{m^2\gamma_o^2 c^4}{\dot{R}} \frac{\partial H^2}{\partial t} = 0 \tag{47}
\]

By squaring eq. (46) and substituting in eq. (47) we get:

\[
v_o \frac{\dot{R}}{R} (\sqrt{\beta} - \beta) = -\frac{1}{4} \frac{\partial \beta}{\partial t} \tag{48}
\]

where \( \beta(\ell) \equiv 1 - \lambda^2 \frac{\dot{R}}{R} \). Since it is \( \frac{\partial \beta(\ell)}{\partial t} = -\dot{\beta} (\ell) v_o \), we obtain:

\[
\frac{\dot{R}(\ell)}{R} = \frac{1}{4} \frac{\dot{\beta}}{\sqrt{\beta} - \beta} \tag{49}
\]
Equation (49) can be integrated member by member after multiplying by $d\ell$. After easy steps, one finds:

$$1 + \frac{c_1}{R^2} = \sqrt{\beta}$$

(50)

where $c_1$ is an arbitrary constant. Recall that we are limiting ourselves to consider only non-negative values for the Hamiltonian, so discarding the possibility of having $-\sqrt{\beta}$ instead of $\sqrt{\beta}$ in eqs. (48-50). On the basis of eq. (46) we therefore obtain:

$$H = m\gamma_o c^2 (1 + \frac{c_1}{R^2})$$

(51)

and so the quantum potential is: $V_Q = m\gamma_o c^2 \frac{c_1}{R^2}$. Compatibility with the stationary non-relativistic case requires $c_1$ be positive, as we have seen. By squaring eq. (50) and substituting $\beta$, we find this differential equation for $R$:

$$(1 + \frac{c_1}{R^2})^2 = 1 - \lambda^2 \frac{\dddot{R}}{R}$$

(52)

Recalling that $R>0$, a first integration provides:

$$\dot{R} = \pm \frac{1}{\lambda} \sqrt{\frac{c_1}{R^2} - 4c_1 \ln R + c_2}$$

(53)

where $c_2$ is an arbitrary constant. Equation (53) is compatible with a $R(\ell)$ even and with a maximum in the origin, which we denote by $R_M$; so, we will have sign - for $\ell > 0$ and + for $\ell < 0$. In this way we determine $c_2$, getting:

$$\dot{R} = \pm \frac{\sqrt{c_1}}{\lambda} \sqrt{\frac{c_1}{R^2} - \frac{c_1}{R_M^2} + 4 \ln \frac{R_M}{R}}$$

(54)

Placing $f \equiv \frac{c_1}{R_M^2} > 0$, the quantum potential is rewritten:

$$V_Q = m\gamma_o c^2 f \frac{R_M^2}{R^2}$$

(55)

and its minimum value is in the origin, where $R = R_M$; it is: $V_{Qm} = m\gamma_o c^2 f$. $R$ is determined by integrating eq. (54):

$$\int \frac{dR}{\sqrt{f(\frac{R^2}{R_M^2} - 1) + 4 \ln \frac{R_M}{R}}} = - \frac{R_M \sqrt{f}}{\lambda} \frac{\ell}{|\ell|} + c_3$$

(56)

which cannot be simplified by means of standard functions.

In a small neighborhood of the origin, i.e. for $R \to R_M$, eq. (56) gives for $R$ a quadratic dependence on $\ell$. Here, in fact, the rooting at first member is approximated by $2(f + 2)(1 - \frac{R}{R_M})$, so by integrating we have:

$$\sqrt{1 - \frac{R}{R_M}} \simeq \frac{\sqrt{f(f + 2)}}{\sqrt{2} \lambda} \frac{\ell}{|\ell|} + c_4$$

(57)

The arbitrary constant $c_4$ is null by imposing $\ell = 0$ for $R = R_M$. We thus obtain:

$$R \simeq R_M (1 - \frac{f(f + 2)}{2\lambda^2} \ell^2)$$

(58)
\[
\frac{1}{R^2} \simeq \frac{1}{R_M^2} \frac{1}{1 - \frac{(f+2)}{\lambda_r^2}} \simeq \frac{1}{R_M^2} \left( 1 + \frac{f(f+2)}{\lambda_r^2} \right)^2
\]  
(59)

Therefore, from eq. (55), the quantum potential around the origin can be approximated by the potential of a harmonic oscillator:

\[
V_Q \simeq m\gamma_0 c^2 f \left( 1 + \frac{f(f+2)}{\lambda_r^2} \right)^2
\]  
(60)

By replacing it into the equation of motion in \(\tau\) (42) and imposing \(\ell(\tau) = 0\), we finally get the solutions referred to the motion not directly observable:

\[
\ell(0,0,\tau) \simeq -A \sin \left( \frac{c}{\lambda_r} f \sqrt{2(f+2)} \right) \tau
\]  
(61)

\[
v_i(\tau) = -\frac{\partial \ell}{\partial \tau} \simeq A \frac{c}{\lambda_r} f \sqrt{2(f+2)} \cos \left( \frac{c}{\lambda_r} f \sqrt{2(f+2)} \right) \tau
\]  
(62)

The angular frequency obtained is more general than that one of the classical Zitterbewegung. This latter, \(\frac{2m\gamma_0 c^2}{\hbar}\), we can get for \(A = \frac{\hbar}{2m\gamma_0 c}\) and \(v_{m,AX} = c\). With these values, we also obtain: \(\gamma_0 f \sqrt{2(f+2)} = 2\), that, for \(\gamma_0 = 1\), provides: \(f \simeq 0.839\).

However, based on eq. (55), dependence of \(f\) on \(\gamma_0\) appears forced in our model. By only imposing \(v_{m,AX} = c\), we get \(A = \frac{\lambda_r}{\sqrt{2(f+2)}}\), where we will assume that \(f\) is of the order of unity. Recalling that these values constitute the uncertainties on the corresponding observable quantities, let us reconsider the uncertainty principle:

\[
\frac{\lambda_r}{f \sqrt{2(f+2)}} \times m\gamma_0 c = \frac{\hbar}{f \sqrt{2(f+2)} \gamma_0} \sim \frac{\hbar}{\gamma_0}
\]  
(63)

As a novelty, there is the Lorentz factor in the denominator. Looking at eq. (44) this seems correct: there is a \(\gamma_0\) which increases the relativistic mass and another \(\gamma_0\) which decreases the influence of \(\hbar\). At speeds close to \(c\) the quantum effects in the direction of motion must be negligible.

The maximum value of the quantum potential, \(V_{QM}\), can be found by applying energy conservation to the motion in \(\tau\):

\[
\frac{1}{2} m\gamma_0 c^2 + m\gamma_0 c^2 f = 0 + V_{QM}
\]  
(64)

that provides: \(V_{QM} = m\gamma_0 c^2 (f + \frac{1}{2})\); it coincides with the purely quantum energy \(E_Q\).

\(V_Q\), and so also \(H\), oscillates with an amplitude of \(\frac{1}{2} m\gamma_0 c^2\); this value represents the quantum uncertainty in a measure of the energy for a free particle with a minimum location given by \(\frac{\lambda_r}{f \sqrt{2(f+2)}}\). Let’s find again the uncertainty principle by multiplying by the half-period of the oscillation:

\[
\frac{1}{2} m\gamma_0 c^2 \times \frac{\pi \lambda_r}{f \sqrt{2(f+2)} c} = \frac{\pi \hbar}{2 f \sqrt{2(f+2)} \gamma_0} \sim \frac{\hbar}{\gamma_0}
\]  
(65)

In correspondence of \(V_{QM}\) we obtain the minimum value for \(R\):

\[
R_m = \frac{R_M}{\sqrt{1 + \frac{1}{2f}}}
\]  
(66)
For a numerical estimate, we will now impose \( A = \frac{\lambda r}{2} \), that is \( f \sqrt{2(f + 2)} = 2 \Rightarrow f \simeq 0.839 \), obtaining: \( R_m \simeq 0.79 R_M \). Yet, using eq. (58) for an estimate of the maximum value of \( \ell \) we have:

\[
\ell_M \simeq \lambda r \sqrt{\frac{f}{1 - \frac{1}{2f}}} \simeq 0.42\lambda r
\]

close to \( \frac{\lambda r}{2} \), where the velocity is zero based on the harmonic solution.

Finally, a normalization of \( R^2 \) (numerical, given the non-simplification by means of standard functions of (56)), can determine \( R_M \).

6.1 Zitterbewegung in arbitrary direction

We have got the intrinsic motion in the direction of motion \( \hat{v}_o = \hat{x} \). The generalization in an arbitrary direction \( \hat{s} \), which forms an angle \( \vartheta \) with \( \hat{v}_o \), is obtained by considering the variable \( \ell_s = s - v_s t - \int_0^\tau v_{is}(\zeta) \, d\zeta \), where \( v_s = v_o \cos \vartheta \). Eqs. (44) and (45) are generalized by:

\[
H = \sqrt{m^2 \gamma_o^2 c^4 - \hbar^2 c^2 \gamma_s^2 \frac{\ddot{R}}{R}}
\]

\[
v_s \frac{\dot{R}}{R} (m \gamma_o c^2 - H) + \frac{1}{2} \frac{\partial H}{\partial t} = 0
\]

Where the dot denotes derivative over \( \ell_s \). Placing: \( \lambda_r^2 \equiv \frac{\hbar^2}{m \gamma_o^2 c^2} \), we get back the same equations we have considered. The Hamiltonian and the quantum potential, given by (51) and (55), do not change, not even the amplitude of their oscillations. By imposing the equation of motion:

\[
m \gamma_o \frac{d v_{is}}{d\tau} = \dot{V}_Q
\]

we obtain an intrinsic harmonic motion described again by (61) and (62), but with the new value of \( \lambda_r \). The consequence on the uncertainty principle - equations (63) and (65) - is the presence of \( \gamma_s \) instead of \( \gamma_o \) in the denominator. Therefore, in the particular case of \( \hat{s} \) perpendicular to \( \hat{v}_o \), \( \gamma_s = 1 \) and the standard uncertainty principle, independent of speed, is re-established.

6.2 Non-relativistic limit; singularity in \( v_o = 0 \)

For the non-relativistic (\( v_o \ll c \)) and \( v_o \to 0 \) cases it is sufficient to approximate \( \gamma_o \) in the previous equations (respectively with \( 1 + \frac{\beta^2}{2\gamma} \) and 1), obtaining still high frequency oscillatory motions in \( \tau \). These results cannot be obtained starting from the Schrödinger equation because from it, for a \( \psi \) described by eq. (1), the non-relativistic approximation of eq. (69) cannot be deduced.

In \( v_o = 0 \) there is a singularity; in fact, from equation (45) we have that \( H \), and therefore \( V_Q \), is constant. So, \( v_i = \text{const} \), contrary to the case \( v_o \to 0 \).

Finally, we see how it would be possible to achieve the non-relativistic potential of Bohm (6) from eq. (68), or:

\[
H = m \gamma_o c^2 \sqrt{1 - \frac{\lambda_r^2}{R}}
\]
with $\lambda^2 \equiv \frac{\hbar^2}{m c^2 \gamma_o c^2}$. The first step is to consider small the second addend in the square root; but this approximation is not non-relativistic: if true, it holds - *a fortiori* - also for relativistic speeds. One would get:

$$H \simeq m \gamma_o c^2 \left(1 - \frac{\hbar^2}{2m^2 \gamma_o c^2} \frac{\dot{R}}{R} \right)$$

and finally, for $v_o \ll c$, and taking into account the eq. (8):

$$H = mc^2 + \frac{1}{2} m v^2 - \frac{\hbar^2}{2m} \frac{\nabla^2 R}{R}$$

in agreement with eq. (5).

However, *this approximation is never legitimate for a free particle*. In fact, in this case it has been found that the maximum value of the quantum potential is *always* at least of the order of $m \gamma_o c^2$: it does not depend on $\hbar$! So the second addend in the square root is never small compared to the first one. What happens as the rest mass increases is that the amplitude of the motion's oscillation decreases, until it becomes negligible for sufficiently large masses.

### 6.3 Standard Zitterbewegung

The standard Zitterbewegung can be obtained interpreting the oscillatory motion in $\tau$ as occurring in $t$. First, in the standard view $R$ is considered time independent, so the second side of eq. (19) is simply $-\frac{H^2}{\hbar}$, with $H = \text{const}$. In the new theory this means demanding

$$\frac{1}{R} \frac{\partial^2 R}{\partial t^2} - 2i \frac{\hbar}{\hbar} \frac{\partial R}{\partial t} = 0,$$

that provides $R \propto e^{\frac{\hbar}{\hbar} \hat{W} t}$ and thence $V_Q \propto e^{-\frac{\hbar}{\hbar} \hat{W} t}$. Therefore, referring eq. (42) at $t$ time and conveniently determining the arbitrary constants, one gets $v_i \propto c e^{-\frac{\hbar}{\hbar} \hat{W} t}$, as found in the Heisenberg picture.

### 7 Final remarks

The new Bohmian model is obtained by introducing a new independent temporal dimension, $\tau$, whose relative movements are not directly observable but constitute the uncertainties themselves of all the observables of quantum mechanics. Unlike Bohm’s original theory, the quantum potential does not affect the observable motion in $t$, as occurs for a normal external potential, but it only determines the intrinsic one in $\tau$. This peculiarity can be understood considering the "retroactive" nature of the quantum potential, which originates from the same wave function of the particle.

The intrinsic motion found for a free particle, so, is due to an interaction of the particle with its own wave (without the need to invoke the antiparticle) and although is more general than the Zitterbewegung discovered by Schrödinger, can be well approximated with it. This movement is caused by the quantum potential and gives rise to the uncertainty of a measure of the particle’s position. The Hamiltonian itself is not constant, but is a wave that follows the particle, function of both temporal variables; the variability in $\tau$ causes an oscillation which, like for any other observable, produces its indeterminacy.

The evanescence of the uncertainty principle in the direction of motion, for velocities close to $c$, becomes clear just by observing the general equation (68): in it, not only we have a Lorentz factor that multiplies the rest mass, but also another one - equal to 1 only

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4 Often it is said, incorrectly, that this approximation is obtained for $c \to \infty$. 

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in perpendicular direction to the direction of motion - which directly reduces the quantum potential, which is the cause of motion in \( \tau \) and hence of quantum uncertainties.

The consideration of a new independent time parameter is an unusual and certainly not intuitive idea; but its mathematical simplicity is disarming. Traveling in an independent time dimension, a particle or photon can instantly (in relation to the usual time) interact with more slits (as in a Davisson-Germer diffraction), it can run across all the possible paths between two points, also exceeding the speed of light (in relation to the usual time), it can admit a non-local relationship with another particle ... All things that it actually does based on experiments.

In the studied model all these peculiarities are explained within a deterministic picture, along with the prediction of a modification of the uncertainty principle that could be tested in high-energy accelerators.

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